For an integer $n \geq 2$ denote by $\mathbb{R}^n$ the $n$ dimensional Euclidean space. For $p \in \mathbb{R}^n$ and $r \in \mathbb{R}$ denote by

$$S_r(p) = \{ x \in \mathbb{R}^n : \|x - p\| = r \}$$

the hypersphere in $\mathbb{R}^n$ with the center at $p$ and radius $r$. Let $S^{n-1}$ be the unit sphere in $\mathbb{R}^n$ and for every $\psi \in S^{n-1}$ denote by $S^{-1}(\psi)$, the great $n-2$ dimensional subsphere of $S^{n-1}$ which is orthogonal to $\psi$.

$$S^{-1}(\psi) = \{ x \in S^{n-1} : \langle x, \psi \rangle = 0 \}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^n$. For an integer $k$ satisfying $1 \leq k \leq n-2$ denote by $H(n,k)$ and by $\mathcal{H}(n,k)$ respectively the families of all $k$ dimensional planes (planes for short) and $k$ dimensional subspaces (i.e., planes passing through the origin) in $\mathbb{R}^n$.

For a continuous function $f$, defined in $\mathbb{R}^n$, define the $k$ dimensional Radon transform $\mathcal{R}_{k}f$ of $f$ by

$$\mathcal{R}_{k}f : H(n,k) \to \mathbb{R}, \quad \mathcal{R}_{k}f(\Sigma) = \sum_{\gamma \in \Sigma} f(\gamma),$$

where $\Sigma$ denotes the standard infinitesimal area measure on the $k$ plane $\Sigma$.

The main problem and known results

The aim of this paper is to find inversion methods for the $k$ dimensional Radon transform which is defined on a family of $k$ dimensional planes in $\mathbb{R}^n$ where $1 \leq k \leq n-2$. For these values of $k$ the dimension of the set $H(n,k)$, of all $k$ dimensional planes in $\mathbb{R}^n$, is greater than $n$ and thus in order to obtain a well-posed problem one should choose proper subsets of $H(n,k)$. We present inversion methods for some prescribed subsets of $H(n,k)$ which are of dimension $n$.

Key Words: Integer; Euclidean space; Hyperplanes; Subset

Revisiting the problem of recovering functions in $\mathbb{R}^n$ by integration on $k$ Dimensional planes

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ABSTRACT

The aim of this paper is to present inversion methods for the classical Radon transform which is defined on a family of $k$ dimensional planes in $\mathbb{R}^n$ where $1 \leq k \leq n-2$. For these values of $k$ the dimension of the set $H(n,k)$, of all $k$ dimensional planes in $\mathbb{R}^n$, is greater than $n$ and thus in order to obtain a well-posed problem one should choose proper subsets of $H(n,k)$. We present inversion methods for some prescribed subsets of $H(n,k)$ which are of dimension $n$.

Key Words: Integer; Euclidean space; Hyperplanes; Subset

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The first inversion method is obtained for a special subset $\mathcal{H}' \subset \mathcal{H}(n,k)$ which is determined by fixing $k+1$ points $x_1, \ldots, x_{k+1}$ in $\mathbb{R}^n$ in general position. Each $k$ plane in $\mathcal{H}'$ is determined by taking a point $p$ in $\mathbb{R}^n$, which is in general position with the other fixed points, and then taking the intersection of the $k+1$ plane passing through $p$, $x_1, \ldots, x_{k+1}$, and the unique hyperplane which passes through $p$ and its normal is in the direction $op$. We prove that our inversion problem is well posed and provide an inversion method for each function, whose $k$ dimensional Radon transform $\mathcal{R}_k f$ is defined on this subset $\mathcal{H}'$, and which satisfies a decaying condition at infinity. The proof of the reconstruction method is purely geometrical and does not use any analytical tools. We find a family of $k+1$ planes such that the union of these planes is dense in $\mathbb{R}^n$ and such that for every $k+1$ plane $\sum_i$ in this family the set of $k$ planes in $\mathcal{H}'$, which are contained in $\sum_i$, is in fact the set of all $k$ planes in $\sum_i$. Thus, we can use known inversion formulas for the classical Radon transform, on each such $k+1$ plane, in order to reconstruct the function $f$ in question.

The second inversion method is obtained for the subset $\mathcal{H}' \subset \mathcal{H}(3,1)$ of all lines in $\mathbb{R}^3$ which are at equal distance from a fixed given point $p$. We show that our problem is well posed (i.e., in this case $\dim \mathcal{H}' = 3$) and obtain an inversion method using expansion into spherical harmonics and exploiting the fact that $\mathcal{H}'$ is invariant under rotations with respect to the point $p$. Observe that $\mathcal{H}'$ can be also described as the set of lines which are tangent to a given sphere with center at $p$.

The third inversion method is obtained for the subset $\mathcal{H}' \subset \mathcal{H}(3,1)$ of all lines in $\mathbb{R}^3$ which have equal distances from two given fixed points $x$ and $y$ in $\mathbb{R}^3(x \neq y)$. Again, we show that our problem is well posed and for the inversion method we exploit the fact that $\mathcal{H}'$ is invariant under rotations with respect to the line $l$ passing through the points $x$ and $y$. Using this invariance property of $\mathcal{H}'$ we take any hyperplane $H$, which is orthogonal to $l$, and expand the restriction $f|_l$ of the function $f$ in question, into Fourier series with respect to the angular variable in $H$.

We would like again to emphasize that the problem of recovering functions from integration on $k$ planes, which belong to an $n$ dimensional manifold $\mathcal{H}' \subset \mathcal{H}(n,k)$, is not new and the manifolds of lines, corresponding to the second and third inversion problems described above, were already considered in (5,6). However, the inversion methods presented in this paper are new and can provide new insights on developing other reconstruction methods for more complex subsets $\mathcal{H}'$ of $k$ planes in $\mathcal{H}(n,k)$.

**Exact formulations and proofs of the main results**

In this section we formulate and prove the inversion methods for the problem of recovering a function $f$ from integration on any subset of $k$ planes from the three subsets $\mathcal{H}' \subset \mathcal{H}(n,k)$ which were described in the previous section. For the first subset $\mathcal{H}'$ of $k$ planes we do not give an explicit inversion formula but instead we just assert that every function $f$ in question, can be reconstructed from integration on $k$ planes, which are at equal distance from two given fixed points $x$ and $y$. Theorem 2.1 states that since $x_0 \in \mathcal{H}'$, it follows that its closest point to the origin is $x'$ and in the same way $x''$ is the closest point of $\mathcal{H}'$ to the origin. Hence, if $\sum_i \cap \mathcal{H}' \neq \emptyset$, it follows that both $x'$ and $x''$ are the closest points of a $k$ plane to the origin and they are both different. This obviously leads to a contradiction.

To show that for each $x \in \mathbb{R}^n$ the plane $\sum x$ is of dimension $k$ we first observe that since $x_0 \in \mathcal{H}'$, then $\sum x$ is a nonempty intersection of a hyperplane and a $k$-plane in $\mathbb{R}^n$. Hence, $\dim \sum x$ is equal to $k$ or $k+1$ where the latter case occurs when $H(x_0, x_1, \ldots, x_{k+1})$ is contained in $\mathcal{H}'$. This will imply in particular that $x_1, \ldots, x_{k+1} \in \mathcal{H}_w$ and thus $(x_0 - x_1, \ldots, x_0 - x_{k+1}) = 0$ which is a contradiction to the assumption that $x \notin \{x_1, \ldots, x_{k+1}\}$.

Now, for every function $f$, defined in $\mathbb{R}^n$, which decays to zero fast enough at infinity, our aim is to recover $f$ from $\mathcal{R}_k f$ which is now restricted to the smaller set $\mathcal{H}'$ and thus our problem is well posed. For this we have the following result.

**Theorem 2.1:** Let $f$ be a function in $C^\infty(\mathbb{R}^n)$ (the space of infinitely differentiable functions in $\mathbb{R}^n$) which satisfies the decaying condition $|f(x)| \leq \exp(-\|x\|^2)$ for some $N > k+1$. Then, $f$ can be recovered from its $k$ dimensional Radon transform $\mathcal{R}_k f$ restricted to the set $\mathcal{H}'$ of $k$ dimensional planes.

**Proofs:** From here and after we will assume, without loss of generality, that $(x_1, x_2, \ldots, x_k)$ is an orthonormal set. Since this is not the case then by equation (2.1) it follows that there exists orthogonal set of vectors $(y_1, y_2, \ldots, y_k)$ such that $\langle y_i, y_j \rangle = \delta_{ij}$ if and only if $\langle x_i, x_j \rangle = \delta_{ij}$. Hence, we can define $x'_i = y_i - x_i$ then it is easily verified that $H(x_1, \ldots, x_k) = H(x'_1, \ldots, x'_k)$. Hence, for the set $(x_1, \ldots, x_k)$ we obtain the same set of $k$ planes, i.e.,

$H_{x_1, \ldots, x_k} = H_{x'_1, \ldots, x'_k}$

and we also have that $(x'_1, \ldots, x'_k)$ is an orthonormal set.

The proof of Theorem 2.1 is divided into 5 parts.

In the first part we find necessary and sufficient conditions on a point $x_0$ in $\mathbb{R}^n$, such that the $k$ plane $\sum x_0$ is contained in a given $k+1$ plane $\sum_i$.

In the second part we define a special subset $\mathcal{H}(n, k+1) \subset \mathcal{H}(n, k+1)$
of k+1 planes.

In the third part, we find, with the help of the first part, necessary and sufficient conditions on a point \( x_i \) in \( \mathbb{R}^n \), \( \{H(x_{i+1},...,x_{n+1})\} \) for the k plane \( \sum_i \) to be contained in a given k+1 plane in \( \mathcal{H}(n,k+1) \).

In the fourth part we show that the union of all the k planes in \( \mathcal{H}(n,k+1) \) is the whole space \( \mathbb{R}^n \).

Lastly, in the fifth part we show that the k planes in \( \mathcal{H}' \), which are contained in a given k+1 plane \( \sum_i \) in \( \mathcal{H}(n,k+1) \), are in fact all the k planes in \( \sum_i \) and thus the function in Theorem 2.1, can be recovered on \( \sum_i \). Since, by the fourth part, the union of the k planes in \( \mathcal{H}(n,k+1) \) is the whole space \( \mathbb{R}^n \) it follows that f can be recovered in \( \mathbb{R}^n \). This will finish the proof of Theorem 2.1.

Finding conditions on the point \( x_i \) so that the k plane \( \sum_i \) is contained in a given k+1 plane \( \sum_j \). For \( x_i \in \mathbb{R}^n \) \( \{H(x_{i+1},...,x_{n+1})\} \) observe that the k plane \( \sum_i \) in \( \mathcal{H}' \) can be parameterized as follows

\[
\sum_i = \{ \omega_i x_i | \omega_i \in \mathbb{R} \} \]

where \( \omega_i \) are parameters.

From the above parameterization we obtain for \( \sum_i \) that it follows for every \( \omega \in \mathbb{R}^n \) \( \{x_i \} \) and \( \{x_i' \} \) the k plane \( \sum_i \) is contained in the hyperplane \( (x_i,\omega) = 0 \) if and only if the following two conditions are satisfied

1. \( (x_i,\omega) = 0 \)
2. \( (x_i,\omega) = 0 \)

Now, for a given k plane \( \sum_{i'} \) we would like to nd necessary and sufficient conditions on \( x_i \) so that the k plane \( \sum_i \) is contained in \( \sum_{i'} \). Let us assume that \( \sum_i \) is contained by the following system of equations

\[
(x_i,\omega) = 0, \quad i = 1,...,k+1
\]

where \( \omega_i \) are parameters.

From the above analysis it follows that the k plane \( \sum_i \) is contained in a k+1 plane \( \sum_{i'} \) given by the intersection of all hyperplanes given by \( \sum_i \), if and only if the following two conditions are satisfied

1. \( (x_i,\omega_i) = 0, \quad i = 1,...,k+1 \)
2. \( (x_i,\omega_i) = 0, \quad i = 1,...,k+1 \)

Defining the set \( \mathcal{H}(n,k+1) \) of k+1 planes. Now, let us look only on k+1 planes given as the intersection of the n-k-1 hyperplanes given by \( \sum_i \) where \( \omega_i \) are parameters.

\[
(x_i,\omega) = 0, \quad i = 1,...,k+1
\]

and where the parameters \( \omega_i \) are parameters.

\[
(x_i,\omega) = 0, \quad i = 1,...,k+1
\]

Denote this family of k+1 planes by \( \mathcal{H}(n,k+1) \). That is,

\[
\mathcal{H}(n,k+1) = \{ x_i | (x_i,\omega) = 0, i = 1,...,k+1 \}
\]

Finding conditions on the point \( x_i \) so that the k plane \( \sum_i \) is contained in the k+1 plane \( \sum_{i'} \) in \( \mathcal{H}(n,k+1) \). Let us take a k plane \( \sum_i \) in \( \mathcal{H}(n,k+1) \). Then, for a k plane \( \sum_i \) to be contained in \( \sum_{i'} \) the point \( x_i \) in \( \mathbb{R}^n \) \( \{H(x_{i+1},...,x_{n+1})\} \) needs to satisfy only condition (i') since then condition (ii') is also satisfied. Indeed, since \( x_i \in \mathbb{R}^n \) \( x_{i+1},...,x_{n+1} \) there exists an index i such that \( x_i = x_{i+1} \). Using the fact that for \( \sum_i \) we have, from equation [2.3], that \( (x_i,\omega_i) = 0 \), it follows that

\[
(x_i,\omega_i) = (x_{i},\omega_i) = (x_{i+1},\omega_i) = \mu_i \neq 0
\]

where in the second passage we used condition (i') and in the last passage we used equation [2.4]. In the same way we can show that \( (x_{i+1},\omega_i) = 0 \) and thus condition (ii') is also satisfied.

Thus, it follows that the k plane \( \sum_i \) is contained in the k+1 plane \( \sum_{i'} \) in \( \mathcal{H}(n,k+1) \) if and only if

\[
(x_i,\omega_i) = (x_{i+1},\omega_i), \quad i = 1,...,k+1
\]

The k+1 planes in \( \mathcal{H}(n,k+1) \) cover the whole space \( \mathbb{R}^n \). Our aim is to reconstruct \( f \) on every k+1 plane \( \sum_{i'} \) in \( \mathcal{H}(n,k+1) \). If this can be done then \( f \) can be recovered on the whole of \( \mathbb{R}^n \). Indeed, all we need to do is that for every \( x \in \mathbb{R}^n \) there exists a k+1 plane \( \sum_{i'} \) in \( \mathcal{H}(n,k+1) \) such that \( \sum_{i'} \). That is, we need to nd n-k+1 orthonormal vectors \( \omega_1,...,\omega_n \) such that each of them is orthogonal to \( x_{i+1},...,x_{n+1} \) and such that the following equations are satisfied

\[
(x_i,\omega_i) = (x_{i+1},\omega_i), \quad i = 1,...,n-k+1
\]

That is, the orthonormal system \( \omega_1,...,\omega_n \) needs to be orthogonal to \( \sum_{i'} \). As we previously mentioned, we can assume that \( (x_i,\omega_i) = (x_{i+1},\omega_i) \) is an orthonormal set. Hence, since the vectors \( \omega_i,...,\omega_n \) form an orthonormal set then from equation [2.3] it follows that

\[
(x_i,\omega_i) = (x_{i+1},\omega_i), \quad i = 1,...,n-k+1
\]

The assumption that \( x_i \) is not in the unique k plane which passes through \( x_{i+1},...,x_{n+1} \) is equivalent to the assumption that rank \( \{x_i,...,x_{n+1}\} = k+1 \) or equivalently that rank \( \{x_i,...,x_{n+1}\} = k+1 \) and from equation [2.6] this is equivalent to \( x_{i+1},...,x_{n+1} \). The assumption that \( x_i \in \mathbb{R}^n \) \( x_{i+1},...,x_{n+1} \) is equivalent to the assumption that there exists an index i such that

\[
(x_i,\omega_i) = 0 \quad \text{or equivalently that} \quad (x_{i+1},\omega_i) = 0
\]

where we denote \( x_i \).

Hence, the k plane \( \sum_{i'} \) is contained in the k+1 plane \( \sum_{i'} \) if and only if \( x_i \in \mathbb{A} \) has the form

\[
x_i = (u \omega_i), \quad u \in \mathbb{R}^{n-k+1}
\]

and which satisfies the following condition

\[
u_{i'} = (x_{i+1},\omega_i), \quad i = 1,...,n-k+1
\]

or equivalently the condition

\[
u_{i'} = (x_{i+1},\omega_i), \quad i = 1,...,n-k+1
\]

Let us recall again that the k+1 plane \( \sum_{i'} \) is given by the system of equations

\[
(x_i,\omega_i) = 0, \quad i = 1,...,n-k
\]

or equivalently by the system of equations

\[
(x_i,\omega_i) = 0, \quad i = 1,...,n-k
\]

Hence, the k+1 plane \( \sum_{i'} \) is contained in the k+1 plane \( \sum_{i'} \) if and only if \( x_i \in \mathbb{A} \) has the form

\[
x_i = (u \omega_i), \quad u \in \mathbb{R}^{n-k+1}
\]

and for this plane we define its origin to be the point \( (0,u) \). Now, we claim that for each point \( x_i \), which is of the form [2.7] and satisfies condition (ii) the closest point of the k plane \( \sum_{i'} \) to the origin \( (0,u) \) is obtained at \( x_i \).

Indeed, the closest point of \( \sum_{i'} \) to the origin is \( x_i \) (since \( \sum_{i'} \in \mathbb{H}(n,k) \)) and thus the closest point of \( \sum_{i'} \) to the origin is \( x_i \). Now, if there exists a point \( x_i \) \( \sum_{i'} \in \sum_{i'} \) whose distance to the origin \( (0,u) \) in \( \sum_{i'} \) is smaller than that of \( x_i \) then we can assume that
with $|x|=|u|$. This will imply that $|x|\in[x',x]$ and so $x'$ is a point in $A_{\sum}$, whose distance to the origin in $\mathbb{R}^k$ is smaller than that of $x$. This is obviously a contradiction.

Hence, for a point $x$ such that $x, \omega A_\nu$ is of the form $(u_\nu, u_\nu)$, where $u_\nu$ satisfies condition (ii), we have that $A_{\sum} \subset A_{\sum}'$, and that its closest point to the origin in $A_{\sum}$ is attained at $x_\nu$. This means that if we project the $k+1$ plane $\Sigma^{k+1}_u = \{u_\nu\}$ to $\mathbb{R}^k$ then the closest point, of the projection of $A_{\sum}'$, to the origin $\mathbf{0}$ in $\mathbb{R}^{k+1}$ is obtained at the point $u$.

Thus, it follows that the projection of $A_{\sum}'$ to $\mathbb{R}^k$ is the following $k$ dimensional hyperplane
$$
\|x-u\| \in \mathbb{R}^k.
$$

Now, the restriction that the point $u$ must satisfy condition (iii) can be omitted since the set of points $u$ which do not satisfy this condition is of co dimension 1 in $\mathbb{R}^{k+1}$. Thus, by passing to limits when preforming integration we can obviously obtain the integrals on any hyperplane of the form [2.8]. Thus, we can assume that $x \in \mathbb{R}^k$. Hence, if $u=\sigma \mathbf{0}$ where $\sigma>0$ then the plane $\Sigma'_{u}$ is given by
$$
\Sigma'_{u} = \{x \in \mathbb{R}^k : \|x-u\| = \sigma \}\mathbf{0} \in \mathbb{R}^k.
$$

Observe that every $k$ dimensional hyperplane in $\mathbb{R}^{k+1}$ can be written as $\Sigma_{x}$ for some fixed $x \in \mathbb{R}^{k+1}$. Every function in $C^1(\Sigma_{x})$ which decays to zero faster than $|x|^{-\alpha}$, $\alpha>1$, at infinity can be recovered from its classical radial transform (see 7, Chap. 1.3, Theorem 3.1) which integrates the function on every $k$ dimensional hyperplane in $\mathbb{R}^{k+1}$. From our assumption on the function $f$ it follows that the restriction of $g$ to the $k+1$ plane $\Sigma_{x}$ satisfies these conditions and thus it can be recovered on this plane. Equivalently, this means that $f$ can be recovered on $\Sigma_{x}$. Thus, Theorem 2.1 is proved.

The case of integration on lines in $\mathbb{R}^3$ with a fixed distance from a given point
For $\rho \neq 0$, a point $p$ in $\mathbb{R}^3$ and a compactly supported continuous function $f$, defined on $\mathbb{R}^3$, we consider the following problem. Suppose that the integrals of $f$ are given on each line whose distance from the point $p$ is equal to $\rho$ and we would like to recover $f$ from this family $[\mathbb{L}]$ of lines. Since the intersection of the interior of the sphere $S^3$ with any line in $[\mathbb{L}]$ is empty then obviously $f$ cannot be recovered inside $S^3$ and thus we can assume that the function in question vanishes inside $S^3$. Without loss of generality we can assume that $p=0$, $\rho=1$. That is, our family $[\mathbb{L}]$ consists of all the lines which are tangent to the unit sphere $S^3$ and our aim is to recover a continuous compactly supported function, which vanishes inside $S^3$, given its integrals on each line in $[\mathbb{L}]$. Since $S^3$ is of dimension two and since, for each point $p \in S^3$, the family of lines which are tangent to $S^3$ and pass through $p$ is one dimensional it follows that $dim[\mathbb{L}]=3$. Hence, our problem is well-posed.

Observe that for a point $x \in S^3$, satisfying $|x| \geq 1$, the family of all lines which are tangent to the unit sphere $S^3$, and let $\mathbb{A}$ be the expansion of $f$ into spherical harmonics. Denote
$$
\hat{G}_{\nu,k} (\lambda) = \int_{\mathbb{A}} G (\lambda, \lambda') \hat{T}_{\nu,k} (\lambda') d\lambda',
$$
and proving Theorem 2.3 we introduce some notations and definitions.

Remark 2.4: In Theorem 2.3 the parameter $p$, $|p|$ is always the same parameter which satisfies $Y_{\nu} (\phi) \neq 0$. Also, observe that by Theorem 2.3 we can recover the function $f_{\nu,k}$ and thus we can also recover the function $f_{\nu,k}$ which is equivalent of recovering $f_{\nu,k}$ on $[0,1]$. Since, by our assumption, $f$ vanishes inside $S^3$ it follows that $\hat{f}_{\nu,k}$ vanishes in $[0,1]$ and thus we can extract $f_{\nu,k}$ on the whole line $\mathbb{R}$. Hence, since we can obtain $f_{\nu,k}$ for every $m \geq 1$, $|k|$ we obviously the function $f_{\nu,k}$ can be recovered.
Revisiting the problem of recovering functions in $\mathbb{R}^n$ by integration on $k$ Dimensional planes

Using the orthogonality relations for the family of functions $\mathcal{T}_{\alpha,\beta}^{p,\beta}$, $m \geq 0$, $|k| \leq p$, $m \leq \infty$, we have that

$$\int_{\mathbb{S}^2} \mathcal{G}(A, \lambda) \mathcal{T}_{\alpha,\beta}^{p,\beta}(A) dA = \frac{1}{2m+1} \sum_{k=-m}^{m} f_{m,k}(r) Y_\nu^p(\omega).$$

Hence, we obtained that

$$\int_{\mathbb{S}^2} G(A, \lambda) \mathcal{T}_{\alpha,\beta}^{p,\beta}(A) dA = \frac{1}{2m+1} \sum_{k=-m}^{m} f_{m,k}(r) Y_\nu^p(\omega)$$

where $f_{m,k}$ is an even function defined on $-\pi/2 \leq \lambda \leq \pi/2$.

Then, from equation [2.13] we have

$$\mathcal{G}_{m,p}(\lambda) = \int_{\mathbb{S}^2} G(A, \lambda) \mathcal{T}_{\alpha,\beta}^{p,\beta}(A) dA$$

and

$$\mathcal{G}_{m,p}(\lambda) = \frac{\lambda}{2m+1} \sum_{k=-m}^{m} f_{m,k}(r) Y_\nu^p(\omega)$$

where $f_{m,k}$ is an even function defined on $-\pi/2 \leq \lambda \leq \pi/2$.

Now, observe that there exists an integer $p$ satisfying $|p| \leq m$ such that $Y_\nu^p(\omega) \neq 0$. Otherwise, we can just choose a spherical harmonic $Y_\nu^p$ of degree $m$ and choose a point $\omega \in \mathbb{S}^2$ such that $Y_\nu^p(\omega) \neq 0$. Then, if $A$ is a rotation such that $A \omega = \omega$, then $K(A)Y_\nu^p(\omega \omega) = Y_\nu^p(\omega \omega)$ will also be a spherical harmonic of degree $m$ such that $K(\omega \omega)$.

Now, since $K$ can be expanded into spherical harmonics from the family $Y_\nu^p(\omega)$, then if $Y_\nu^p(\omega) \neq 0$ for every $|p| \leq m$ then we will also have $K_\nu(\omega \omega)$ which is a contradiction.

Let us choose $p$, $|p| \leq m$ such that $Y_\nu^p(\omega) \neq 0$. Then, from equation [2.13] we have

$$\mathcal{G}_{m,p}(\lambda) = \int_{\mathbb{S}^2} G(A, \lambda) \mathcal{T}_{\alpha,\beta}^{p,\beta}(A) dA$$

and

$$\mathcal{G}_{m,p}(\lambda) = \frac{\lambda}{2m+1} \sum_{k=-m}^{m} f_{m,k}(r) Y_\nu^p(\omega).$$

Extracting the variable $t$ from equation [2.15] we have

$$\mathcal{G}_{m,p}(\lambda) = \int_{\mathbb{S}^2} G(A, \lambda) \mathcal{T}_{\alpha,\beta}^{p,\beta}(A) dA$$

and

$$\mathcal{G}_{m,p}(\lambda) = \frac{\lambda}{2m+1} \sum_{k=-m}^{m} f_{m,k}(r) Y_\nu^p(\omega).$$

Observe that the function $u(t) = \sqrt{x^2 + (1 + t^2) + 2\lambda \sqrt{t^2 - 1}}, t \in \mathbb{R}$

is injective on the intervals $(-\infty, -\sqrt{x^2 - 1}/\lambda)$ and $(-\sqrt{x^2 - 1}/\lambda, \infty).$

Hence, we can choose the following variable changes

$$\int_{-\infty}^{-\infty} f_{m,k}(r) Y_\nu^p(\omega) dt = \int_{\mathbb{S}^2} G(A, \lambda) \mathcal{T}_{\alpha,\beta}^{p,\beta}(A) dA$$

and

$$\mathcal{G}_{m,p}(\lambda) = \frac{\lambda}{2m+1} \sum_{k=-m}^{m} f_{m,k}(r) Y_\nu^p(\omega).$$

Observe that the right hand side of equation [2.19] is given for $0 \leq |\beta| \leq m,$ but using the change of variables $\beta \rightarrow -\beta$ and the evenness of $f_{m,k}$ it follows that the right hand side of equation [2.19] is an even function of $\beta$. Therefore, we have

$$\mathcal{G}_{m,p}(\lambda) = \frac{\lambda}{2m+1} \sum_{k=-m}^{m} f_{m,k}(r) Y_\nu^p(\omega).$$

Observe that $f_{m,k}(r)$ is an even function denoted on $-\pi/2 \leq \beta \leq \pi/2.$ Since the sum of the Gegenbauer polynomials in the integral [2.18] is an even function of $\beta$ it follows that

$$\mathcal{G}_{m,p}(\lambda) = \frac{\lambda}{2m+1} \sum_{k=-m}^{m} f_{m,k}(r) Y_\nu^p(\omega).$$

Observe that $f_{m,k}(r)$ is an even function denoted on $-\pi/2 \leq \beta \leq \pi/2.$ Since the sum of the Gegenbauer polynomials in the integral [2.18] is an even function of $\beta$ it follows that

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\[ \text{dim } [\beta] = 3 \] (in case where \( p = \tilde{0} \) then we can take all the lines which pass through \( p \)).

Our aim in this section is to recover a continuous function \( f \), defined in \( \mathbb{R}^2 \), in case where we are given the integrals of \( f \) on the set of all lines in \( \mathbb{R}^2 \). Since \( \text{dim } [\beta] = 3 \) our problem is well posed. In order to guarantee that inversion formulas are obtainable, when considering integration on lines in \( \mathbb{R}^2 \), we must restrict our family of functions since any function \( f = f(x_1, x_2, x_3) \) which is radial with respect to the variable \( x^* = (x_1, x_2) \) and is odd with respect to the variable \( x_3 \) produces no signals.

Before formulating Theorem 2.5 we will need first to parameterize the family of lines \( [\beta] \) and introduce the Mellin transform. Observe that since \( [\beta] \) is invariant with respect to rotations which leave the vector \( e_1 \) fixed it follows that it is enough to parameterize the subset of lines in \( [\beta] \) which are parallel to the YZ plane (and then we take rotations of these lines with respect to the Z axis to obtain the whole set \( [\beta] \)). If a line \( l \) in this subset has a distance \( \lambda \) from the origin and its projection to the YZ plane forms an angle \( \theta \), \(-\pi/2 < \theta < \pi/2\), with the Y axis then \( l \) has the following parametrization:

\[ \Pi = \{(\sin \theta, \cos \theta, \phi) \mid \phi \in [\pi/2, \pi/2], \theta \in [\pi/2, \pi/2] \}. \]

For a function \( F \), defined in \( \mathbb{R}^2 \), define the Mellin transform \( \mathcal{M} F \) of \( F \) by

\[ (\mathcal{M} F)(s) = \int_0^\infty r^{s-1} F(y) dy, \Re s > 0 \]

where it should be noted that the above integral might not converge for every complex number \( s \) satisfying \( \Re s > 0 \). For the Mellin transform we have the following inversion formula (see [10], Chap. 8.2).

\[ F(r) = r^{-s} \mathcal{M}^{-1} (\mathcal{M} F)(s) ds, \quad \Re s > 0 \]

where \( F \) is a function of two variables then we denote by \( \mathcal{M} F_i \) the Mellin transform of \( F \) with respect to its \( i \)th variable. Finally, for every integer \( n \) and complex numbers \( \xi_k \) define the following integral

\[ I_n(\xi_1, \xi_2) = \int_0^\infty (1 + t^2)^{-s \xi_1/2} (1 + t^2)^{-s \xi_2/2} dt. \]

\[ \text{Theorem 2.1:} \quad \mathcal{M} \mathcal{M}^{-1} (\mathcal{M} F)(s) = a^{-s} (\mathcal{M} F)(s), \quad \text{for } a > 0 \]

where \( a \) is arbitrary. For any domain of \( \mathcal{M} F \) and \( F \) we have to justify the existence of these transforms, in the corresponding domain, each time this scaling property is used (11). If \( F \) is a function of two variables then we denote by \( \mathcal{M} F_i \) the Mellin transform of \( F \) with respect to its \( i \)th variable. Finally, for every integer \( n \) and complex numbers \( \xi_k \) define the following integral

\[ I_n(\xi_1, \xi_2) = \int_0^\infty (1 + t^2)^{-s \xi_1/2} (1 + t^2)^{-s \xi_2/2} dt. \]
be the Fourier expansions of $f$ and $G$ respectively in the variables $\varphi$ and $\theta$. Define

$$G_\nu(\lambda,s) = G_\nu(\lambda,\arctan(s)/\sqrt{1+s^2}, \lambda, s \geq 0),$$

$$G^*_\nu(\lambda,\xi) = \lambda^{-\nu-1} \gamma(\lambda,\xi), \lambda \geq 0, 0 < \Re(\xi) < 1,$$

$$M(\lambda,\zeta) = M(\lambda,\zeta), \Re(\zeta) > n', 0 < \Re(\xi) < 1$$

where $n' = \max(0, n+1)$, then

$$f_n(r,T) = r^n \cdot \rho_m(\Pi(\xi,\zeta))(r,T), r,T \geq 0.$$

**Remark 2.6:** When using the Mellin inversion formula [2.24] in equation [2.28] then when using $M^{-1}$, for which integration is taken with respect to the variable $\xi$, the point $Q$ must be taken from the ray $[n',\infty)$ whereas when using $M^{-1}$, for which integration is taken with respect to the variable $\xi$, the point $Q$ must be taken from the interval $[0,1]$ (13). This is to ensure that integration is taken over the domain of definition of $\Pi(\zeta,\xi)$ for which the integral $I_n$ converges.

**Proof of Theorem 2.5:** In the domain of definition of the function $G$ let us assume for now that the variable $\theta$ is restricted to the interval $[0,\pi/2]$ (14). Then, integrating $f$ on the line $I_{n\theta}$ with its parametrization given in equation (2.21) and using the expansion (2.22) of $f$ we obtain

$$G(\lambda,\theta,\phi) = \int f dx = \sum_{n=-\infty}^{\infty} \int f_{n\theta}(\sqrt{\lambda^2 + t^2 \cos^2 \theta}, t \sin \theta) e^{int\phi} dt$$

where

$$\frac{\lambda \cos \phi - \rho \sin \phi}{\sqrt{\lambda^2 + t^2 \cos^2 \theta}} = \frac{\rho \sin \phi - \lambda \cos \phi}{\sqrt{\lambda^2 + t^2 \cos^2 \theta}}$$

and where in the last passage of equation (2.29) we used the fact that $\sqrt{t^2}$ is supported on the half-space $t^2 > 0$ and that $\sin \theta = 0$ if $0 \leq \theta < \pi/2$. The following relation

$$G(\lambda,\theta,\phi) = G(\lambda,-\theta,-\phi)$$

can be easily checked and thus, assuming that $\lambda > 0$ and $-\pi < \theta < \pi$, negative values of $\theta$ do not give any new information on $f$ and hence we will assume from now on that $\theta$ is given only in the interval $[0,\pi/2]$. Now, define the variable so that

$$\cos \alpha(\lambda, t, \theta) = \frac{\lambda}{\sqrt{\lambda^2 + t^2 \cos^2 \theta}}, \sin \alpha(\lambda, t, \theta) = \frac{t \cos \theta}{\sqrt{\lambda^2 + t^2 \cos^2 \theta}}$$

then we have

$$\cos \phi = \cos \alpha \cos \phi - \sin \alpha \sin \phi = \cos (\phi + \alpha),$$

$$\sin \phi = \cos \alpha \sin \phi + \sin \alpha \cos \phi = \sin (\phi + \alpha).$$

Hence,

$$e^{int\phi} = e^{int\alpha} e^{int\phi}$$

and thus we have

$$G(\lambda,\theta,\phi) = \sum_{n=-\infty}^{\infty} e^{int\phi} \int f_{n\theta}(\sqrt{\lambda^2 + t^2 \cos^2 \theta}, t \sin \theta) e^{int\phi} dt$$

from which we obtain that

$$G_n(\lambda,\theta) = \frac{1}{2\pi} \int f_{n\theta}(\lambda,\theta,\phi) e^{-int\phi} d\phi$$

$$= \int f_{n\theta}(\sqrt{\lambda^2 + t^2 \cos^2 \theta}, t \sin \theta) e^{int\phi} dt$$

$$= \int f_{n\theta}(\sqrt{\lambda^2 + t^2 \cos^2 \theta}, t \sin \theta) (\lambda + it \cos \theta)^n dt$$

where we denote

$$g_n(r,T) = r^n f_n(r,T).$$

Making the change of variables $t' = \cos \theta$ in the right hand side of equation (2.30) we obtain that

$$G_n(\lambda,\theta) = \int g_n(\sqrt{\lambda^2 + (t')^2}, t' \tan \theta) (\lambda + it')^n \frac{dt'}{\cos \theta}$$

if we denote $s = \tan \theta$ then since $0 < \Re(\xi) < 1$ it follows that $s \geq 0$ and, since

$$\cos \theta = \frac{1}{\sqrt{1 + \tan^2 \theta}}$$

in this domain of $\theta$, we can write

$$G_n(\lambda,\arctan(s)/\sqrt{1+s^2}) = \int g_n(\sqrt{\lambda^2 + (t')^2}, t') (\lambda + it')^n dt', s, \lambda \geq 0.$$

Hence, if we define $G_n$ by the following relation

$$G_n(\lambda,\arctan(s)/\sqrt{1+s^2}) = \int g_n(\sqrt{\lambda^2 + (t')^2}, t') (\lambda + it')^n dt', s, \lambda \geq 0$$

we have

$$G_n(\lambda,\arctan(s)/\sqrt{1+s^2}) = \int g_n(\sqrt{\lambda^2 + (t')^2}, t') (\lambda + it')^n dt', s, \lambda \geq 0.$$
and one can use the Mellin inversion formula \([2.24]\) for \(Q_c\) in order to reconstruct \(F\).

Proof: Let us evaluate the Mellin transform of \(F\) on the line \(l_c\). Assume that \(F\) is supported inside the interval \([0, a]\) where \(a > 0\), then
\[
(MF)(c + it) = \int_0^a y^{-a-1} F(y) dy = \left[ y = e^{-x}, dy = e^{-x} dx \right]
\]
\[
= \int_{-\ln a}^\infty e^{-i(t+c)x} e^x F(e^{-x}) e^{-x} dx = \int_{-\ln a}^\infty e^{-ix} e^{-c} F(e^{-x}) dx
\]
\[
= \int_{-\infty}^\infty e^{-ix} F^*(x) dx
\]

Where
\[
F^*(x) = \int_{0, x < -\ln a}^{e^{-x}F(e^{-x}), x > -\ln a} f(x) dx
\]

Since \(c > 0\) it follows that \(F^*\) is in \(L_1\, \mathbb{R}\) and thus its Fourier transform exists in \(\mathbb{R}\) which is equivalent to the fact that the Mellin transform of \(F\) exists on the line \(l_c\). Inverting \(F\) from its Mellin transform \([2.23]\) is equivalent to inverting \(F^*\) from its Fourier transform and since \(F\) is continuous it follows that \(F^*\) is also continuous and since it is also in \(L_1, \mathbb{R}\) then it is well known that one can use the Fourier inversion formula in order to recover \(F^*\). This finishes the proof of Lemma 2.7.

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