Large Deviation Multifractal Analysis of a Process Modeling TCP CUBIC

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Abstract. Multifractal characteristics of the Internet traffic have been discovered and discussed in several research papers so far. However, the origin of this phenomenon is still not fully understood. It has been proven that the congestion control mechanism of the Internet transport protocol, i.e., the mechanism of TCP Reno can generate multifractal traffic properties. Nonetheless, TCP Reno does not exist in today’s network any longer, surprisingly traffic multifractality has still been observed. In this paper we give the theoretical proof that TCP CUBIC, which is the default TCP version in the Linux world, can generate multifractal traffic. We give the multifractal spectrum of TCP CUBIC traffic and compare it with the multifractal spectrum of TCP Reno traffic. Moreover, we present the multifractal spectrum for a more general model, where TCP CUBIC and TCP Reno are special cases. Our results also show that TCP CUBIC produces less bursty traffic than TCP Reno.

1. Introduction

The extensive measurements and analyses of network traffic in the previous three decades have revealed rich and complex traffic properties highlighting scale invariant features and fractal characteristics. These properties have helped to understand the most striking feature of network traffic: its burstiness. Burstiness refers to the inherent nature of network traffic meaning that packets are transmitted in short uneven spurts. A kind of burstiness manifests itself over long periods identified as self-similarity and long-range dependence (LRD), which have been studied intensively since their first discovery in the research of Leland et al. \cite{13}. LRD can be captured well by monofractal models like Fractional Brownian Motion (FBM) \cite{21}. Temporal burstiness, which is the variation of traffic intensity on small time-scales, has also been explored. However, comprehensive research has shown that simple monofractal scaling cannot describe traffic burstiness at this scale and more sophisticated multifractal models are needed \cite{7, 21}.

Multifractal analysis has been found to be a useful tool to explore temporal burstiness and multifractal characteristics of network traffic at small time scales \cite{21}. Several research studies have revealed that the multifractality nature of traffic is mainly due to the TCP (Transmission Control Protocol) \cite{18, 21}, which carries more than 80% of network traffic \cite{1}. Moreover, it has also been shown that there are significant performance implications of the multifractality of network traffic regarding queueing performance \cite{5, 6}. As a consequence, it is vital to explore the behavior of TCP traffic in order to characterize its multifractal features and this is exactly the motivation for our work.

Our methodology in this paper has been inspired by the work of Lévy-Véhel and Rams \cite{16}, where a Large Deviation Multifractal Analysis has been carried out. Their work focuses on the analysis of a simplified TCP model of TCP Reno and presents its multifractal spectra. This important result has brought forth theoretical proof that TCP
Reno dynamics itself (i.e., the additive increase multiplicative decrease (AIMD) mechanism) can lead to multifractal behavior.

In this paper we make a step further in discovering the multifractal nature of network traffic. We have been motivated by the surprising fact that TCP Reno does not exist in today’s network any longer traffic multifractality has still been observed. Compared to the TCP Reno model in [16] we choose a realistic model for today’s network, i.e., the TCP CUBIC, which is the default TCP version in the Linux world and analyze its multifractal spectrum. Moreover, we also present the multifractal spectrum for a more general case where the functions used in TCP CUBIC and TCP Reno are special cases. We also compare our results to the results in [16] and provide the theoretical proof of showing that TCP CUBIC generates less bursty traffic than TCP Reno.

Our proves follow the line of the proofs of Lévy-Véhel and Rams paper [16]. However, in this much more general setup, the details became much more cumbersome and we needed to face with technical difficulties that did not appear in the case of TCP Reno.

The rest of the paper is organized as follows. Section 2 discusses the related work in the field of TCP multifractality. In Section 3 we present the TCP CUBIC model with its multifractal spectrum. The comparison of the multifractal spectrum of TCP CUBIC and TCP Reno is discussed in Section 4. In Section 5 we present the more general model and its multifractal spectrum, with TCP CUBIC being a special case. The detailed proof of our results is given in Section 6—Section 10. Finally, Section 11 concludes the paper.

2. Related Work

Traffic burstiness has been investigated for a long time in the teletraffic research [9,11,2] and found to be one of the key characteristics of network traffic from network design, dimensioning and performance evaluation point of view. Several advanced burstiness measures have been proposed because simple measures like Peak-To-Mean ratio (PMR) and Coefficient of Variation were found to be inadequate [9, 11, 2]. These measures includes, for example, Hurst parameter, Index of dispersion for intervals (IDI), Index of dispersion for counts (IDC), the peakedness functional, etc. see [9, 11, 2]. In addition, recently it has been found that burstiness is even more complex and requires a full high-order and correlation characterization and a more appropriate burstiness characterization was proposed via the multifractal analysis of the traffic [2,21]. In our paper we chose the multifractal analysis of TCP CUBIC traffic and the first time gave a mathematical proof based on the multifractal spectrum why TCP CUBIC burstiness is smaller than TCP Reno. The only related paper in this regard is [4] but in this paper both the goal and the methodology are different, i.e., focusing on the rate variation metric by convex ordering and measured by the Coefficient of Variation capturing only the second order properties of traffic variability. In contrast, we focus on a much richer characterization of burstiness by the Large Deviation Multifractal Analysis and measured by the full multifractal spectrum.

Multifractal characteristics of network traffic was first published by Riedi and Lévy-Véhel [18,17]. From this discovery several research studies have been carried out to understand the multiscale nature of network traffic, see [2] for an excellent overview. From traffic modeling purpose different model classes have been developed, e.g., multiplicative cascades [7,20], Fractional Brownian Motion in multifractal time [17], α-stable processes [22] and other general multifractal models [15].

Regarding the reason why multifractality is present and observed in the Internet we still have a lack of clear understanding. In order to find explanations for the traffic multifractality Feldmann et al. [7] presented a cascade framework that allowed for a plausible physical explanation of the observed multifractal scaling properties of network traffic.
They applied wavelet-based analysis and obtained a detailed description of multifractality. Their main findings is that the cascade paradigm over small time scales appears to be a traffic invariant for wide area network traffic.

However, there is no physical evidence that TCP traffic actually behaves as a cascading or multiplicative process. Lévy-Véhel and Rams [16] showed that adding sources managed by TCP can lead to multifractal behavior. This result demonstrates that there is no need for assuming any multiplicative structure but multifractality is simply due to the interactions of additive increase multiplicative decrease (AIMD) mechanism of TCP and to the random non-synchronous transmission of the sources. The result was proved for a simplified TCP model capturing the main features of TCP Reno.

Nevertheless, TCP Reno is not used in recent networks and due to the inefficient performance of old TCPs (e.g., TCP Reno or TCP Vegas) a fast development of several new TCP versions have been triggered (e.g., FAST [23], HSTCP [8], STCP [12], BIC [24], etc.). Among these versions the TCP CUBIC [19] is widespread since it is implemented and used by default in Linux kernels 2.6.19 and above. This was the main motivation for us focusing on TCP CUBIC in this paper.

3. Multifractal Analysis of TCP CUBIC

In this Section we introduce the model of TCP CUBIC with some basic definitions and properties used in the analysis. Furthermore, we present our results for the multifractal spectrum of TCP CUBIC.

3.1. The Model of TCP CUBIC. TCP CUBIC is a successful transport protocol in the evolution of TCP versions where the congestion control method is optimized to high bandwidth networks [19]. TCP CUBIC is similar to the standard TCP Reno algorithm regarding the additive increase and multiplicative decrease behavior but there are also major differences. For instance, TCP CUBIC increases its sending rate according to a cubic function [19] instead of linear increase, that was implemented in TCP Reno. In the following model we have captured the main characteristics of TCP CUBIC.

The aggregated TCP CUBIC traffic is modelled by the infinite sum of independent random functions

\[ Z(t) := \sum_{j=1}^{\infty} Z_j(t), \]

where \( Z_j(t) \) are piecewise deterministic functions on random time intervals representing the TCP CUBIC traffic from source \( j \). The main idea is the following: events of losses of packages occur at a sequence of random points in time, that we denote by a random sequence \( T^j_1, T^j_2, \ldots \) for each \( j \in \mathbb{N} \). Then, \( Z_j(t) \) is deterministic and monotone increasing on each random time interval \( [T^j_{k-1}, T^j_k) \) for \( k = 1, 2, \ldots \). This corresponds to the fact that the TCP protocol increases its sending rate when no loss occurs. First we define the random time of losses \( (T^j_k)_{k=1,2,\ldots} \), then we describe the deterministic rule which gives \( Z_j(t) \) on \( [T^j_{k-1}, T^j_k) \).

Each \( Z_j(t) \) has an intensity parameter \( \lambda_j \). Without loss of generality we may assume that \( 1 \leq \lambda_1 < \lambda_2 < \cdots \), and we also assume that the sequence \( (\lambda_j)_{j=1}^{\infty} \) is regular (see Definition [2]). Moreover, we require that

\[ \sum_{j=1}^{\infty} \left( \frac{\log \lambda_j}{\lambda_j} \right)^3 < \infty. \]
Then the sequence of losses, \( (T^j_k)_{k=0}^\infty \), are independent Poisson(\( \lambda_j \)) processes for different \( j \)'s. Hence,

\[
T^{(j)}_k := \sum_{j=0}^{k} \tau^{(j)}_k ,
\]

where the inter-event times \( (\tau^{(j)}_k)_{k=1}^\infty \) are Exp(\( \lambda_j \)) random variables such that \( (\tau^{(j)}_{j,k}) \) are independent.

We will define the functions \( Z_j(t) \) in a right-continuous way. First we define \( Z_j(0) \) in an arbitrary way such that \( \sum_{j=1}^{\infty} Z_j(0) < \infty \). Then we assume that \( Z_j(t) \) is already defined for all \( t < T^{(j)}_{k-1} \). So, in particular \( Z_j(T^{(j)}_{k-1}) \), the left hand side limit of \( Z_j \) at \( T^{(j)}_{k-1} \) has already been defined. (See Figure 2.) Then we define \( Z_j(t) \) for a \( t \in [T^{(j)}_{k-1}, T^{(j)}_k) \) by

\[
Z_j(t) := g_{Z_j(T^{(j)}_{k-1})} \left( t - T^{(j)}_{k-1} \right),
\]

where

\[
g_w(t) := C \left( t - \sqrt[3]{\frac{wb}{C}} \right)^3 + w,
\]

where we set \( b = 0.7 \) and \( C = 0.4 \) in the TCP CUBIC model since this is the setting in most of the Linux kernels, see Figure 1.

![Figure 1](image1.png)

**Figure 1.** We compare \( g_1(t) \), \( g_{1.7}(t) \) and \( g_{2.3}(t) \) in the TCP CUBIC model.

One can see that the family \( \{g_w(\cdot)\}_{w>0} \) of functions have self-affine property:

\[
g_w(t) = \lambda^3 g_{\frac{w}{\lambda^3}} \left( \frac{t}{\lambda} \right).
\]

Set \( \lambda_0 := 1 \) and we define the reference process \( Z_0 \) by [4], see Figure 2.
We define $\Phi_k := Z_0(T^{(0)}_k)$, the value of $Z_0(t)$ right before the $k$th loss happens. Due to the fact that the function $g_w$ is nonlinear, $Z_0(t)$ is not a Markov process. However, $\{\Phi_k\}$ is a Markov chain.

3.2. Large Deviation Multifractal Spectrum $f_g(\alpha)$. We define the Large Deviation Multifractal Spectrum, the main object of our analysis. Recall the function $Z(t)$ from (1).

**Definition 1** (Large deviation multifractal spectrum). First in we define the Large deviation multifractal spectrum for increments then for the oscillations. We write

$$
\Delta_k^\ell Z := Z\left(\frac{k + 1}{2^\ell}\right) - Z\left(\frac{k}{2^\ell}\right)
$$

We define $\Delta_k^\ell Z_j, j \geq 1$ in an analogous way. We denote the number of level-$\ell$ binary intervals on which the increment of $Z$ is approximately $(1/2^{\ell(\alpha + \varepsilon)})$ by $N_\varepsilon(\alpha)$:

$$
N_\varepsilon(\alpha) := \#J_{x,\ell,\varepsilon},
$$

where

$$
J_{x,\ell,\varepsilon} := \left\{0 \leq k < 2^\ell : |\Delta_k^\ell Z| \in \left(\frac{1}{2^{\ell(\alpha + \varepsilon)}}, \frac{1}{2^{\ell(\alpha - \varepsilon)}}\right)\right\}
$$

In general $N_\varepsilon(\alpha) \ll 2^\ell$. The large deviation multifractal spectrum $f_g(\alpha)$ is the smallest exponent for which for (in a very vague sense)

$$
N_\varepsilon(\alpha) \lesssim (2^\ell)^{f_g(\alpha)}.
$$

More precisely,

$$
f_g(\alpha) := \lim_{\varepsilon \to 0} \lim_{\ell \to \infty} \frac{\log N_\varepsilon(\alpha)}{\log 2^\ell}.
$$

(2) The large deviation multifractal spectrum for oscillation $f^O_g(\alpha)$, is determined by the oscillations of $Z$. That is, in Definition 1, we change $\Delta_k^\ell Z$ in (7) to

$$
O_k^\ell Z := \sup_{x \in I_k^\ell} Z(x) - \inf_{x \in I_k^\ell} Z(x),
$$
and then define $J_{\alpha,\ell,\varepsilon}^O$ and consecutively $N_{\varepsilon}^{\alpha,\ell}(\alpha)$ similarly as in (9) and (8) by replacing $\Delta_k Z$ by $O_k^\ell Z$. Finally, we obtain $f_g^O(\alpha)$ as in (10) but with $N_{\varepsilon}^{\alpha,\ell}(\alpha)$ instead of $N_{\varepsilon}^\ell(\alpha)$.

3.3. Blumenthal-Getoor Index and the Regularity of $(\lambda_j)_{j=1}^\infty$. The notation here are used later in the more general setting. Let $\theta \geq 1$ be an arbitrary real number. (In the case of TCP CUBIC $\theta = 3$.) Blumenthal-Getoor index of an infinite sequence $(\lambda_j)_{j \geq 1}$ and exponent $\theta$ is defined as

$$\beta_\theta := 1 + \inf \left\{ \gamma \geq 0 : \sum_j \frac{1}{\lambda_j^{\theta/\gamma}} < \infty \right\}.$$  

Clearly $\beta_\theta$ depends on $(\lambda_j)_{j \geq 1}$ and $\theta$. It follows from (23) that $\beta_\theta \in (1, 2]$.

To obtain an equivalent definition first we fix a natural number $L > 2$. The powers of $L$ are denoted sometimes by $L^k := L^k$. We write

$$M_k := \# \left\{ i : \lambda_i^\theta < L^k \right\} \quad \text{and} \quad N_k := \# \left\{ i : \lambda_i^\theta \in (L^{k-1}, L^k] \right\}.$$  

It is easy to check that

$$\beta_\theta = 1 + \limsup_{n \to \infty} \frac{\log N_k}{k \log L}.$$  

By the definitions (11) and (13) it is obvious that the following three statements hold:

$$\forall \varepsilon_0 > 0 \exists K_1(\varepsilon_0), \forall k : N_k \leq K_1(\varepsilon_0) \cdot L_k^{\beta_\theta - 1 + \varepsilon_0},$$  

$$\forall \varepsilon_0 > 0 \exists K_2(\varepsilon_0), \forall k : M_k \leq K_2(\varepsilon_0) \cdot L_k^{\beta_\theta - 1 + \varepsilon_0},$$  

$$\forall \varepsilon_0 > 0 \exists a_k \uparrow \infty, \forall k : N_{a_k} \geq L_{a_k}^{\beta_\theta - 1 - \varepsilon_0}.$$  

In particular, the last statements (15) yields the definition of a sequence $(a_k)_{k \geq 1}$: the $k$th element of the sequence $a_k$ is defined as the $k$th index of $N$ for which there are enough $\lambda_j$ falling in the interval $N_{a_k}$. We are ready to define the regularity of the sequence $(\lambda_j)_{j \geq 1}$. 

\textbf{Figure 3.} The increment and the oscillation, where $0 < \eta < 1$ is a constant (defined in the supplement).
Definition 2. We say that the sequence \((\lambda_j)_{j=1}^\infty\) is regular if \(\forall \varepsilon_0 > 0\) the sequence 
\((a_k - a_{k-1})_{k \geq 2}\) is bounded. We denote 
\(A := A(\varepsilon) = \max_{k \geq 1} (a_k - a_{k-1})\)

In what follows we always assume that \((\lambda_j)_{j=1}^\infty\) is regular.

3.4. The Multifractal Spectrum of TCP CUBIC. We present here our main results of the multifractal spectrum of TCP CUBIC.

Let us define the following two regions in the first quadrant in the \(\alpha, \beta\) plane:

\[ R_1 := \{ (\alpha, \beta) : 1 \leq \beta \leq 2 \text{ and } 0 \leq \alpha \leq \frac{1}{\beta - 2/3} \}, \]
\[ R_2 := \{ (\alpha, \beta) : 1 \leq \beta \leq 2 \text{ and } \frac{1}{\beta - 2/3} \leq \alpha \leq 1 + \frac{1}{\beta - 2/3} \}. \]

Furthermore, we partition \(R_2\) into the lower and upper part \(R_2^L\) and \(R_2^U\).

\[ R_2^L := \{ (\alpha, \beta) : \beta \in [1, 2], \alpha \in [\frac{1}{\beta - 2/3}, \frac{\beta}{\beta - 2/3}] \}. \]
\[ R_2^U := \{ (\alpha, \beta) : \beta \in [1, 2], \alpha \in \left[ \frac{\beta}{\beta - 2/3}, 1 + \frac{1}{\beta - 2/3} \right] \}. \]

Theorem 3. Assume that \((\lambda_j)_{j=1}^\infty\) is regular. Then we have the following estimates on the multifractal spectrum of TCP CUBIC for the increments \((f^{(C)}(\alpha))\) and for the oscillations \(f^{O(C)}(\alpha)\):

(a): \(f^{(C)}(\alpha) = \alpha(\beta_3 - 2/3)\) on \(R_1\),
(b): \(f^{(C)}(\alpha) \leq 1 + \frac{1}{\beta_3 - 2/3} - \alpha\) on \(R_2^L\).
(c): \(f^{O(C)}(\alpha) \leq 1 + \frac{1}{\beta_3 - 2/3} - \alpha\) on \(R_2^U\).

where \(\beta_3\) is as in (11).

The proof of this result follows from the proof of a more general result, Theorem 8 presented in Section 5.

4. Comparison of the Multifractal Spectra of TCP CUBIC and TCP Reno

In this Section we compare our results, i.e., the multifractal spectrum of TCP CUBIC in Theorem 3 and the multifractal spectrum of TCP Reno obtained by Lévy-Véhel, Rams [16, Theorem III.4] that we cite here for the readers convenience. Recall the definition of \(\beta_g\) from equation (11).

Theorem 4 (Lévy-Véhel, Rams). Assume that \((\lambda_j)_{j=1}^\infty\) is regular and

\(\sum_{j=1}^\infty \frac{1}{\lambda_j} < \infty\)

The large deviation multifractal spectra for TCP Reno for \(\beta_1 \in (1, 2)\) is

\[ f^{(R)}_g(\alpha) = \begin{cases} 
\beta_1 \alpha, & \text{if } \alpha \in [0, 1/\beta_1]; \\
1 + 1/\beta_1 - \alpha & \text{if } \alpha \in [1/\beta_1, 1 + 1/\beta_1]; \\
-\infty & \text{otherwise.}
\end{cases} \]
It is elementary to see from the definition of $\beta_\theta$ in (11) that

$$\beta_1 = 3 \left( \beta_3 - \frac{2}{3} \right).$$

Using this identity, we obtain that under condition (17), $\beta_3 \in \left(1, \frac{4}{3}\right)$. Now we fix an arbitrary $\beta_3 \in \left(1, \frac{4}{3}\right)$ and vary $\alpha$. This implies that the region considered by Lévy-Véhel and Rams in Theorem 4 is contained in the region $R_1$, see also Figure 4. We remind the reader that in region $R_1$ we have a complete result (i.e., matching upper ad lower bounds, see Theorem 3). Based on the given parameter sequence $(\lambda_j)_{j=1}^\infty$ of our model, $\beta_\theta$ for all $\theta \geq 1$ are determined by (11). For these $\beta_1$ and $\beta_3$ (c.f. (18)) we compare the multifractal spectra of TCP Reno ($f_R^{(\alpha)}(\alpha)$) and TCP CUBIC ($f_C^{(\alpha)}(\alpha)$).

This comparison means that we move $\alpha$ upwards on the dashed vertical line in Figure 4, starting from $(\beta_3, 0)$ (point A) all the way up to the point $E$ which is the intersection between the dashed vertical line $\beta_3 = \text{const}$ and the upper boundary of region $R_1$. The

**Figure 4.** Parameter space of TCP Reno and TCP CUBIC for $\beta_1 = 3 \left( \beta_3 - \frac{2}{3} \right)$. 
behavior of the large deviation multifractal spectra on this dashed line is shown in Figure 5. We obtain the following corollary:

**Corollary 5.** When comparing the multifractal spectrum of TCP RENO, \( f_g^{(R)}(\alpha) \) and that of TCP CUBIC \( f_g^{(C)}(\alpha) \), we have the following inequalities:

(a): For \( 1 \leq \beta_3 \leq 1.244 \) and for \( \alpha \leq 1 \) we have \( f_g^{(C)}(\alpha) < f_g^{(R)}(\alpha) \).

(b): For \( \beta_3 \in (1.244, \frac{4}{3}) \) and for \( \alpha < \frac{\beta_3 - 1/3}{(\beta_3 + 1/3)(\beta_3 - 2/3)} \) we again have \( f_g^{(C)}(\alpha) < f_g^{(R)}(\alpha) \). In particular this happens for all \( \beta < 4/3 \) when \( \alpha < \frac{9}{10} \).

(c): If \( \beta_3 \in \left(1.244, \frac{4}{3}\right) \) then for \( \alpha > \frac{\beta_3 - 1/3}{(\beta_3 + 1/3)(\beta_3 - 2/3)} \) we have \( f_g^{(C)}(\alpha) \geq f_g^{(R)}(\alpha) \).

In the rest of this section we use the notation of Section 3.2. Recall that \( J_{\alpha,\ell,\varepsilon} \) stands for those level-\( \ell \) diadic intervals on which the increment of \( Z \) on is approximately, \( |2^\ell|^{\alpha} \). Further, \( N^\ell(\alpha) = \#J_{\alpha,\ell,\varepsilon} \) is the number of these diadic intervals and finally \( f_g(\alpha) \sim \log N^\ell(\alpha) \). Note that the smaller the \( \alpha \), the larger the increment and hence a larger multifractal spectrum function \( f \) for the case \( \alpha < 1 \) is of great importance to traffic analysis point of view since in this case the traffic is bursty. The multifractal spectra of TCP Reno and TCP CUBIC shows that both TCP versions generate bursty traffic, however, we see in Corollary 5 that for almost all cases \( f_g^{(C)}(\alpha) < f_g^{(R)}(\alpha) \), i.e., TCP Reno is more bursty than TCP CUBIC. It can happen only for \( \beta_3 \in (1.244, \frac{4}{3}) \) and \( \frac{9}{10} < \alpha \) that \( f_g^{(R)}(\alpha) < f_g^{(C)}(\alpha) \). This exceptional region is the small black triangle with one side aligned with the \( \beta_3 = 4/3 \) line with right upper vertex \( Y \) in Figure 4. However, in this special case the contribution to burstiness is not large since it comes only from \( \alpha > 9/10 \), thus the increments of \( Z \) in this region are smaller than in the case of small \( \alpha \)’s. In other words, the increments with small \( \alpha \) values dominate the traffic burstiness. As a general observation we can conclude that the traffic of TCP CUBIC is less bursty than the traffic of TCP Reno.

In the above comparison we discussed the behavior of traffic for those \( (\lambda_j)_{j \geq 1} \) sequences for which both the TCP Reno solution and the TCP CUBIC exists, i.e., \( \beta_3 \in \left(1, \frac{4}{3}\right) \). However, for \( \beta_3 > \frac{4}{3} \), we have solution for TCP CUBIC, see Theorem 3 and Figure 4. So, we have the following result.

**Corollary 6.**

(a): For \( \frac{4}{3} < \beta_3 \leq \frac{5}{3} \) and for \( \alpha \leq 1 \) we have \( f_g^{(C)}(\alpha) = \alpha(\beta_3 - 2/3) \).

(b): For \( \frac{5}{3} < \beta_3 \leq 2 \) and for \( \alpha \leq \frac{1}{\beta_3 - 2/3} \) we again have \( f_g^{(C)}(\alpha) = \alpha(\beta_3 - 2/3) \).

(c): For \( \frac{5}{3} < \beta_3 \leq 2 \) and for \( \frac{1}{\beta_3 - 2/3} < \alpha \leq 1 \) we have \( f_g^{(C)}(\alpha) \leq 1 + \frac{1}{\beta_3 - 2/3} - \alpha \).

5. **Generalization of the TCP CUBIC Process**

We shall prove the results stated in the previous section for a more general family of random processes. This general family includes not only both the TCP CUBIC and TCP Reno as special cases but many other stochastic processes which are infinite sums of random functions. The main point of the generalization is that we replace the very specific family, \( \{g_x(t)\}_{x \geq 0} \) defined in [3], with a much more general family of functions. This generalization is carried out based on the self-affine property [3] of \( \{g_x(t)\}_{x \geq 0} \) (cf. [21]).

5.1. **Heuristic description of the generalization with an example.** In the general case, we also consider the infinite sum

\[
Z(t) = \sum_{j=1}^{\infty} Z_j(t),
\]

where \( Z(t) \) is the total traffic received up to time \( t \). The function \( Z(t) \) is defined as the sum of the contributions of the individual packets sent during the time interval \( [0,t] \). Each packet is characterized by its size \( Z_j(t) \) and arrival time \( t_j \). The function \( Z(t) \) is a random variable that depends on the stochastic process \( g_x(t) \) and the arrival times \( t_j \). The generalization of the TCP CUBIC process is achieved by considering a more general family of functions \( g_x(t) \) that satisfy the self-affine property.
where $Z_j(t)$ is defined in a way which is similar to the case of TCP CUBIC model:

- $Z_j(t)$ increases according to a deterministic rule in between two consecutive random points of losses.
- The random points of losses of $Z_j(t)$ are chosen according to a Poisson process of intensity $\lambda_j$ with $1 = \lambda_1 < \lambda_2 < \ldots$
- The deterministic rule of growth between the consecutive points of losses are governed by a self-affine family of functions like the one in (6) with the exponent 3 in (6) replaced by a general $\theta \geq 1$.

We remark that the $\theta = 1$ case is essentially settled by Lévy-Véhel, Rams [16, Theorem III.4].

To highlight the meaning of the abstract definition of $\{g_x(t)\}_{x>0}$ given below in Section 5.2, as an intermediate step, first we give an example which is included in the general case.

**Example 1.** Let $\{g_x(t)\}_{x>0}$ be defined as follows:

\[
(19) \quad g_x(t) = x \cdot g_1 \left( \frac{t}{x^{1/\theta}} \right), \quad x > 0, t > 0,
\]

where $g_1(t)$ is an arbitrary polynomial satisfying:

(a): The order of $g_1$ is $\theta$,
(b): $g'_1(t) \geq 0$ for every $t \in \mathbb{R}^+$ and $g'_1(0) > 0$,
(c): $g_1(0) \in (0,1)$.

Example 1 covers both TCP CUBIC and TCP Reno. Namely, we get the TCP CUBIC model with the choice of $\theta = 3$ and

\[
(20) \quad g_x(t) = x + C \left( t - \sqrt[3]{\frac{b}{C}} \cdot x \right)^3 \quad \text{with} \quad b = 0.7, \quad C = 0.4.
\]

Similarly, the TCP Reno is included in Example 1 with $\theta = 1$ and $g_x(t) := x/\mu + t$ for a constant $\mu > 1$.

**5.2. The definition of $Z(t)$ in the general case.** The most general definition of the family $\{g_x(t)\}_{x>0}$ given below differs from the one in Example 1 in the following way: We preserve the self-affine property by assuming (A1) below. Although we no longer require that $\theta$ is an integer, we would still like to preserve some properties of order $\theta$ polynomial $g_1(t)$ in Example 1. This is why we assume (A2) and (A3) below.

**Definition 7.** For every $x > 0$, $g_x : (0, \infty) \to (0, \infty)$ such that $(x,t) \mapsto g_x(t)$ is a $C^\infty$ function satisfying the following assumptions: There exists a $\theta \geq 1$ exponent such that
(A1): Self-affine property: For every $0 < r, t < \infty$ we have

\[(21) \quad \frac{1}{r^\theta} g_x(t) = g_x/r^\theta \left( \frac{t}{r} \right)\]

The properties (A2) and (A3) guarantee that $g_1(t)$ behaves similar to the polynomial in Example 7.

(A2): Growth properties:

(A2a): The derivative of $g_1^{1/\theta}(t)$ is a bounded function on $[0, \infty)$. That is,

\[\exists \psi > 0, \quad \frac{d}{dt} \left( g_1(t)^{1/\theta} \right) < \psi, \quad \forall t \geq 0.\]

(A2b): There exists $c_1 > 0$ such that

\[(22) \quad g_1(t) \geq c_1 t^\theta.\]

(A3): Regularity property: We assume that $g_1'(t)$ has finitely many zeros, $g_1'(t) \geq 0$ that is $g_1(t)$ is increasing and

\[g_1'(0) > 0\]

and $\eta := g_1(0) \in (0, 1)$.

(A4): The sequence of the intensities $(\lambda_j)_{j=0}^\infty$ of the independent Poisson point processes $(T^{(j)}_k)$ (defined in Section 3.1) satisfy:

(1) $1 = \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$

(2) $(\lambda_j)_{j=1}^\infty$ is regular in the sense of Definition 2

(3) We assume that

\[(23) \quad \sum_{j=1}^\infty \frac{\log \lambda_j}{\lambda_j^\theta} < \infty.\]

Note that (21) means that for $0 < a$ we have

\[\left( \begin{array}{cc} a & 0 \\ 0 & a^\theta \end{array} \right) \cdot \text{graph}(g_x) = \text{graph}(g_{a^\theta x}).\]

This is why we call the family $\{g_x(t)\}_{x>0}$ self-affine. The definition of the random function $Z_j(t)$ in the general case is the same as in Section 3.1 with the only modification that we use in (4) the previously defined more general version of $\{g_x(t)\}_{x>0}$. That is, $Z_j(t)$ is defined in a right-continuous way:

\[Z_j(t) := g_{Z_j(T^{(j)}_k-1_{k-1})}(t - T^{(j)}_{k-1}).\]

Observe that by the self-affine property of $g_x(t)$ we have the distributional identity

\[(24) \quad \frac{1}{\lambda_j^\theta} Z_1(\lambda_j t) \overset{d}{=} Z_j(t),\]

This completes the definition of $Z(t) = \sum_{j=1}^\infty Z_j(t)$. See Corollary 18 that $Z(t) < \infty$, $\forall t \in [0, 1]$. 

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5.3. Our result in the general settings. From now on we always write $f_g(\alpha) := f_g^\theta(\alpha)$ for the Large Deviation Multifractal spectrum of $Z(t)$ (see Definition 7). Observe that it follows from the definition of $f_g(\alpha)$ and from (24) that $f_g(\alpha)$ remains the same if we change from the sequence of intensities $(\lambda_j)_{j \geq 1}$ to $(\text{const} \cdot \lambda_j)_{j \geq 1}$. So, without loss of generality we may assume that $\lambda_1 = 1$.

First we define the regions of the $\alpha, \beta$ plane

$$R_1 := \left\{ (\alpha, \beta): \beta \in [1, 2], \alpha \in \left[0, \frac{1}{\beta - (1 - 1/\theta)} \right] \right\},$$

and

$$R_2 := \left\{ (\alpha, \beta): \beta \in [1, 2], \alpha \in \left[\frac{1}{\beta - (1 - 1/\theta)}, \frac{1}{\beta - (1 - 1/\theta)} + 1 \right] \right\}.$$

Furthermore, we partition $R_2$ into the lower and upper part $R_2^l$ and $R_2^u$.

$$R_2^l := \left\{ (\alpha, \beta): \beta \in [1, 2], \alpha \in \left[\frac{1}{\beta - (1 - 1/\theta)}, \frac{\beta}{\beta - (1 - 1/\theta)} \right] \right\},$$

and

$$R_2^u := \left\{ (\alpha, \beta): \beta \in [1, 2], \alpha \in \left[\frac{\beta}{\beta - (1 - 1/\theta)}, 1 + \frac{1}{\beta - (1 - 1/\theta)} \right] \right\}.$$

See Figure 6.

![Figure 6](image)

**Figure 6.** Here $\beta = \beta_\theta$ defined in (11).

As a generalization of Theorem 8 we state:

**Theorem 8.** Let $f_g(\alpha)$ be the large deviation multifractal spectrum for the increments and $f_g^0$ for the oscillations of the random function $Z(t)$ defined in Section 5.1. We assume that the sequence of intensities $(\lambda_j)_{j \geq 1}$ satisfies the assumption (A4) in Definition 7. Then we have

(a): $f_g(\alpha) = \alpha(\beta_\theta - (1 - 1/\theta))$ on $R_1$,

(b): $f_g(\alpha) \leq 1 + \frac{1}{\beta_\theta - (1 - 1/\theta)} - \alpha$ on $R_2^l$,

(c): $f_g^0(\alpha) \leq 1 + \frac{1}{\beta_\theta - (1 - 1/\theta)} - \alpha$ on $R_2$. 

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The rest of the paper is devoted to the proof of Theorem 8. First we give a heuristic argument to show why it is natural to expect that at least the first part of this theorem holds.

5.4. Heuristic proof for Part (a) of Theorem 8. Fix a $\beta \in [1, 2]$ and choose $\alpha$ such that

$$\alpha (\beta - 1 + 1/\theta) < 1,$$

that is we are on the region $R_1$. Fix a small $h = 2^{-\ell} > 0$. We would like to compute the number of $2^{-\ell}$-mesh intervals on which the magnitude of the increment of $Z$ is approximately $h^\alpha$. To do so, we consider the $L$-block $(L^{k-1}, L^k)$ which 'corresponds to' $h$ and $\alpha$. That is we define $k$ such that

$$L^{k-1} < h^{-\alpha} \leq L^k.$$

We say that an index $j$ is $\alpha$-good if

$$\lambda_j^\theta \in (L^{k-1}, L^k).$$

We will see that, roughly speaking, a typical jump for the process $Z_j$ is of magnitude $\lambda_j^\theta$ and these jumps happen roughly once in a time interval of length $1/\lambda_j$. We will prove that

- if $\lambda_j$ is greater than the $\alpha$-good parameters, then although the process $Z_j(t)$ jumps frequently, these jumps are too small to influence the outcome as far as we count increments of magnitude $h^\alpha$.
- On the other hand, if $\lambda_j$ is smaller than the $\alpha$-good parameters, then although the jumps of the process $Z_j(t)$ are big but these jumps happen rarely, so their effect is not significant for counting the jumps of magnitude $h^\alpha$.

So, we can focus on the $\alpha$-good indices $j$. Observe that

- Each process $Z_j$ with $\alpha$-good $j$ jumps approximately $\lambda_j \sim h^{-\alpha/\theta}$ times
- There are approximately $h^{-\alpha(\beta\theta-1)}$ good indices $j$ for $\alpha$. Namely, By the definition of the Blumetal Gétoor index $\beta_\theta$ and the regularity of $\{\lambda_j\}_{j=1}^\infty$ (see Section 3.3) we have approximately $L^{k(\beta\theta-1)}$ indices $j$ such that $\lambda_j^\theta \in (L^{k-1}, L^k)$. That is the number of good indices is approximately $L^{k(\beta\theta-1)}$. However, by (27) $L^k \sim h^{-\alpha}$. That is we have approximately $h^{-\alpha(\beta\theta-1)}$ good indices.

So, the combined effect of these two points above suggests that there should be approximately

$$h^{-\alpha/\theta} \cdot h^{-\alpha(\beta\theta-1)} = h^{-\alpha(\beta\theta-1)+1/\theta)}$$

$2^{-\ell}$-mesh intervals on which the magnitude of the increments is $h^\alpha$. Here we used that for different indices $j$ which are $\alpha$-good the majority of the corresponding mesh-intervals are different and well-separated from each other. This follows from the assumption (26). Observe that by (10) and by $h = 2^{-\ell}$, formula (28) is actually part (a) of Theorem 8.

6. Stationary measure of the Markov Chain associated to $Z_j$

Now we turn to the technical details of the proofs. Fix a $j \geq 1$. It is easy to see that $Z_j(t)$ is not a continuous time Markov process if $\theta > 1$, that is, $g$ is not an affine function. Therefore we consider $\Phi_k^{(j)}$, the values of $Z_j$ just before the the $k$-th loss (see Figure 2). Then for every $j \geq 1$, $\Phi^j := (\Phi_k^{(j)})_{k=1}^\infty$ is a discrete time continuous state space Markov chain. As the Meyn, Tweedie book [14] is a major reference book of this field we use its terminology in this paper. Due to the self-affine property, it is enough to study the
Markov chain $\Phi$ which corresponds to a reference process $Z_0$ that is defined exactly as $Z_j$, for $\lambda := \lambda_1 := 1$.

6.1. Geometric ergodicity of $\Phi$. In this section we study the scalar non-linear discrete time, continuous state space Markov chain model (following the terminology in [14]):

$$\Phi_k = g_{\Phi_{k-1}}(T_k - T_{k-1}),$$

where the function $g_x : [0, \infty) \to \mathbb{R}^+$ is defined for all $x > 0$ and satisfies the assumptions (A1)-(A3) of Definition 7 and $(T_k)_{k \geq 1}$ is Poisson(1) point process.

Let $P$ be the probability kernel of the time-homogeneous Markov Chain $\Phi = (\Phi_k)_{k \geq 1}$.

That is, for a set $A \subset \mathbb{R}^+$,

$$P(x, A) := P(\Phi_{k+1} \in A | \Phi_k = x).$$

We prove Theorem 9.

For the Markov chain $\Phi$ described above

(a): there exists a unique stationary state $\pi$.

(b): $\int V(x) d\pi(x) < \infty$ for

$$V(x) := \exp \left\{ \tilde{c} \cdot x^{1/\theta} \right\},$$

where $\tilde{c}$ is positive constant defined in (47).

(c): (Geometric Ergodicity) There exists constants $r > 1$ and $R < \infty$ such that

$$\sum_n r^n \| P^n(x, \cdot) - \pi \| V \leq R \cdot V(x),$$

where by definition for a non-negative function $f$, and the $f$-norm of a measure $\nu$ is defined as $\| \nu \|_f := \sup_{h : |h| \leq f} | \nu(h) |$, $|h| \leq f$ is meant pointwise and $\nu(f)$ is the integral of $f$ with respect to the measure $\nu$.

We prove Theorem 9 in Section 6.3. Part (b) of Theorem 9 immediately implies the following corollary:

Corollary 10. There exists a constant $K_3$ such that for every $k \geq 1$ we have

$$\pi \left[ \left[ k^\theta, (k+1)^\theta \right] \right] \leq K_3 \cdot e^{-\tilde{c}k},$$

where $\tilde{c}$ is the constant in the exponent of $V(x)$, defined in (47) below.

In the rest of this section our aim is to prove Theorem 9. The assertions of Theorem 9 follow from two theorems ([14, Theorem 14.0.1], [14, Theorem 15.0.2]) from Meyn, Tweedie book. The conditions of these two theorems are as follows:

(i) Drift Condition (Proposition 12),

(ii) $\Phi$ is $\psi$-irreducible (Lemma 13),

(iii) $\Phi$ is recurrent (Proposition 16),

(iv) $\Phi$ is strongly aperiodic, (Lemma 17),

for the terminology see [14]. In Section 6.2 as a preparation for the proof of Theorem 9, we study the self-affine family $\{g_x(t)\}$. In Section 6.3 we verify (i) above. In Section 6.4 we prove (ii)-(iv) above and as a consequence of these we also prove Theorem 9. Finally in Section 6.5 we describe some properties of the density of the kernel.
6.2. Properties of $g_x(t)$. Here we frequently use the following notation (see Figure 7):

$$\eta := g_1(0) \quad \text{and} \quad \xi := g_1^{-1}\left(\frac{1 + \eta}{2}\right) \quad \text{and} \quad \tau := g_1^{-1}(1),$$

where $g_1^{-1}()$ is the inverse function of $g_1$. We will use the following properties of $g_x(t)$:

**Remark 11.** Here we mention a few properties of the function $g_x(t)$.

1. Substituting $r = x^{1/\theta}$ in (A1) in Definition 7 above, we obtain

$$g_x(t) = x \cdot g_1\left(\frac{t}{x^{1/\theta}}\right) \quad \text{and} \quad g_x^{-1}(t) = x^{1/\theta} g_1^{-1}\left(\frac{t}{x}\right).$$

Hence

$$g'_x(t) = x^{1-1/\theta} g'_1\left(\frac{t}{x^{1/\theta}}\right).$$

Combining the first equation in (33) with (22) we obtain

$$g_x(t) \geq c_1 \cdot t^\theta \quad \forall x > 0, t > 0.$$

2. With $\psi$ as in in Definition 7 (A2a), the assumption (A2a) implies that

$$g_1(t) < \psi \theta (g_1(t))^{1-1/\theta}.$$

Combining this inequality first with (34) then with (33) yields that for $x > 0$

$$g'_x(t) \leq \psi \cdot \theta \cdot g_x(t)^{1-1/\theta}.$$
We say that $g_x^{(n)}(t_1, \ldots, t_n)$ is the associated control system of the Markov chain $\Phi_k$ (defined in (29)) driven by $g_x(t)$. By definition (see [14, p. 141]) this means that the associated control system is forward accessible.

The next fact is simple but is important, so we list it separately.

**Fact 1.** Suppose that $\{g_x(t)\}_{t \geq 0}$ satisfies the assumptions in Definition 7. Then there exists $c_2 > 0$ such that

$$
(38) \quad g_1(t) \leq c_2 t^\theta, \quad \text{for } t \geq \xi.
$$

**Proof.** By the assumption (A2a) in Definition 7, Lagrange Theorem implies that

$$
(39) \quad (g^{-1}_x(t))' \geq c_3 t^{-(1-1/\theta)}, \quad \forall x > 0, \forall t > x\eta.
$$

This implies that (38) holds.

**Fact 2.** We need the following properties of $g^{-1}_x$, the inverse function of $g_x$.

(a): There exists a constant $c_3 > 0$ such that

$$
(40) \quad g^{-1}_x(t) \leq \left(\frac{t}{c_1}\right)^{1/\theta}, \quad \forall x > 0, \forall t > x\eta.
$$

**Proof.** Part (a): Using that $(g^{-1}_x)'(t) = 1/g_x(g^{-1}_x(t))$, it follows from (33) that for every $x, t > 0$ we have

$$
(41) \quad g_x(g^{-1}_x(t)) = x^{1-1/\theta}g_1(g^{-1}_x(t/x)).
$$

If we apply (36) with $t$ replaced by $g^{-1}_1(t/x)$ then we get the assertion of the lemma with $c_3 := (\psi\theta)^{-1}$.

Part (b): Using assumption (A2b) in Definition 7 and second part of (33) we obtain that (40) holds.

**6.3. The verification of the drift condition.** In this section we prove Proposition 12 that implies that the so-called Drift Condition holds. This is the first step towards proving Theorem 9; see point (i) in the argument below Corollary 10.

We frequently use the drift operator $\Delta$ that is defined for any measurable function $f : (0, \infty) \to (0, \infty)$ by

$$
(41) \quad \Delta f(x) := \int P(x, dy) f(y) - f(x) = \int_{y=0}^{\infty} p(x, y) f(y) dy - f(x), \quad x > 0,
$$

where $p(x, y)$ is the density of the kernel $P(x, dy)$. For the Markov Chain $\Phi_k$, by (29), elementary calculation using the density of the exponential distribution yields that this density kernel is given by

$$
(42) \quad p(x, y) := \exp \left( -g^{-1}_x(y) \right) g_x'(g^{-1}_x(y)).
$$

Now we state and prove that the drift condition holds.

**Proposition 12.** Recall the definition of $V(x)$ from (31). There exists a $K > 0$ such that

$$
(43) \quad \Delta V(x) < -\frac{V(x)}{2} + 2 \cdot 1_{(0,K)}, \quad \forall x > 0,
$$

where $1_{(0,K)}$ is the indicator function of the interval $(0, K]$. 

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Proof of Proposition 12. Let \( r := \xi \cdot x^{1/\theta} \). Using the formula for the density kernel (42) and the definition of the operator \( \Delta \) in (41), we calculate

\[
\Delta V(x) = \int_{y=\eta}^{\infty} V(y) \exp \left[ -g_x^{-1}(y) \right] \frac{dy}{g_x'(g_x^{-1}(y))}
\]

\[
= \int_{u=0}^{r} V(g_x(u)) e^{-u} du + \int_{u=r}^{r} V(g_x(u)) e^{-u} du - V(x),
\]

where we applied the substitution \( u = g_x^{-1}(y) \). First we estimate

\[
I_1 - V(x) \leq \int_{u=0}^{r} \left( e^{\tilde{c} - x^{1/\theta}(u)} - e^{\bar{c} - x^{1/\theta}} \right) e^{-u} du.
\]

Using (33) and the fact that \( t \mapsto g_x(t) \) is increasing we obtain that \( g_x^{1/\theta}(u) \leq \eta^{1/\theta} \cdot x^{1/\theta} \) holds for all \( u \in (0, r) \), where \( \eta_1 := g_1(\xi) \in (\eta, 1) \). From this inequality and (44) follows that

\[
I_1 - V(x) \leq e^{\tilde{c} - x^{1/\theta}} \left[ e^{\bar{c} - x^{1/\theta}(\eta^{-1}) - 1} - 1 \right] = V(x) \left[ e^{\bar{c} - x^{1/\theta}(\eta^{-1}) - 1} - 1 \right].
\]

Note that since \( \eta < 1 \), the exponent on the right hand side is negative. Hence \( I_1 - V(x) < 0 \) and tends to \( -\infty \) as \( x \to \infty \). To estimate \( I_2 \) we combine (33) again with (38) to obtain that \( g_x^{1/\theta}(u) \leq c_2^{1/\theta} u \) if \( u \geq r \). Hence

\[
I_2 \leq \int_{u=r}^{\infty} e^{\tilde{c} g_x^{1/\theta} - u} du \leq \int_{u=r}^{\infty} e^{(\tilde{c} c_2^{1/\theta} - 1) - u} du.
\]

Now we fix

\[
\tilde{c} := \frac{1}{2c_2^{1/\theta}}.
\]

Then by (46), we have \( I_2 \leq 2 \).

Combining this and (45) we obtain from (43) that

(a): \( \Delta V(x) < 2 \) for all \( x > 0 \).

(b): We can choose a \( K \) such that \( \Delta V(x) \leq -\frac{1}{2} V(x) \) if \( x \geq K \).

\[\square\]

6.4. Recurrence, irreducibility and strong aperiodicity of \( \Phi \). It is a most fundamental property of \( \Phi \) that it is a \( \psi \)-irreducible chain (for the definition see [14]).

Lemma 13. The chain \( \Phi \) is \( \psi \)-irreducible.

Proof. Our chain \( \Phi \) is a so-called scalar nonlinear model which clearly satisfies the conditions SNSS1-SNSS3 of the book [14]. Also the associated control system is forward accessible (see part (5) of Remark [11]). Then by [14] Proposition 7.1.2 \( \Phi \) is a so-called \( T \)-chain. Let

\[
L(x, A) := \mathbb{P}_x (\Phi \text{ ever enters } A)
\]

According to [14] Theorem 6.0.1 the chain \( \Phi \) is a \( \psi \)-irreducible chain if

\[
L(x, O) > 0, \quad \forall x > 0 \text{ and open set } O \subset (0, \infty).
\]

However, it is obvious from the construction that (48) holds in our case. \[\square\]
The purpose of the next two technical lemmas are to prove that the interval $(0, K]$ is a petite set (see [14, Section 5.5]) for all $K > 0$. Roughly speaking, a set $A$ is a petite set if there is a function that serves as a uniform lower bound on the density of the transition kernel of the Markov Chain from an arbitrary $x \in A$ to the complement of $A$. We apply this for a $K$ which satisfies Proposition 12. This yields a petite set $(0, K]$ out of $\Delta V < 0$.

Then by [14, Theorem 8.0.2 (ii)] the chain is recurrent.

**Lemma 14.** Recall the constants $c_1$ from Definition 7, (A2b) and $c_3$ from (39) from Fact 2. For $t > 0$ let us define the function

$$h(t) := c_3 \exp \left( - \frac{t}{c_1} \right) \cdot t^{-\frac{1}{\theta}}.$$  

Then

$$\left| \frac{d}{dt} \exp \left( -g_x^{-1}(t) \right) \right| > h(t), \text{ if } t > \eta \cdot x.$$  

**Proof.** The function $t \mapsto \exp \left( -g_x^{-1}(t) \right)$ monotone decreasing. By chain rule:

$$- \frac{d}{dt} \exp \left( -g_x^{-1}(t) \right) = \exp \left( -g_x^{-1}(t) \right) \cdot \frac{d}{dt} g_x^{-1}(t)$$

We estimate the first term from below by the inequality $g_x^{-1}(t) \leq \frac{t}{c_1}$ for $x > 0$ and $t > x \cdot \eta$ and second term by (39). This yields the proof. \hfill $\square$

In the rest of this section we use the terminology of the book [14]. For an arbitrary $K > 0$ we define the measure $\nu^{(K)}$ supported on $[K, \infty)$ by

$$\nu^{(K)}([\alpha, \beta]) := \int_{t=\alpha}^{\beta} h(t) dt,$$

where $K \leq \alpha < \beta$.

**Lemma 15.** For every $K > 0$ the set $(0, K]$ is a small set (see Section 14, Section 5.2 for the definition.) and then by [14, Proposition 5.5.3] $(0, K]$ is also a petite set for all $K > 0$.

**Proof.** Let $K < \alpha < \beta$. Then for an $x \in (0, K]$ we have

$$P(x, [\alpha, \beta]) = \int_{t=\alpha}^{\beta} p(x, t) dt = \int_{t=\alpha}^{\beta} \exp \left( -g_x^{-1}(t) \right) \frac{d}{dt} g_x^{-1}(t) dt$$

$$= \exp \left( -g_x^{-1}(\alpha) \right) - \exp \left( -g_x^{-1}(\beta) \right) \geq \int_{\alpha}^{\beta} h(t) dt.$$

That is for every $x \in (0, K]$ we have $P(x, [\alpha, \beta]) \geq \nu^{(K)}([\alpha, \beta])$. This implies that the interval $(0, K]$ is a small set. Then $(0, K]$ is also a petite set by [14, Proposition 5.5.3]. \hfill $\square$

**Corollary 16.** The chain $\Phi$ is recurrent.

**Proof.** Using Lemma 15 and Proposition 12 the assertion immediately follows from [14, Theorem 8.0.2 (ii)]. \hfill $\square$

**Lemma 17.** The chain $\Phi := \{\Phi_n\}$ is strongly aperiodic (for the terminology see [14, p. 114]).

**Proof.** Let $A := [\eta, 1]$ and let $\nu := \nu^{(\eta)}$ as defined in (49). Then $\nu(A) > 0$ and for every $x \in A$ we have $P(x, B) \geq \nu(B)$ holds for every $B \subset (0, \infty)$. \hfill $\square$
Now we are ready to prove the main result of the section.

**Proof of Theorem 9.** Part (a) and (b): We know that $\Phi$ is a $\psi$-irreducible and aperiodic chain (see Lemmas 13, 17). We have also verified that for any $K$ the set $\{0, K\}$ is a petite set. Hence by Proposition 12 the conditions of (iii) of [14, Theorem 15.0.2] hold. It follows from this theorem that $\Phi$ is positive recurrent with unique stationary measure $\pi$ satisfying (32).

Part (c) immediately follows from [14, Theorem 14.0.1] since condition (iii) of [14, Theorem 14.0.1] holds with the choice of $V(x) = f(x)$.

6.5. **The density** $p_j(x, y)$. For an $A \subset \mathbb{R}^+ := (0, \infty)$ let $P_j(x, A) := \mathbb{P}_x (Z_j \in A)$ and let $p_j(x, y)$ be the density of $P_j(x, dy)$. When $j = 1$ then we suppress the index. Similarly to (42), using the density of an exponential random variable with parameter $\lambda_j$, we have that

\[
 p_j(x, y) = \frac{\lambda_j \exp(-\lambda_j g^{-1}_x(y))}{g_x'(g^{-1}_x(y))}.
\]

Combining (21) and the second part of (33) we obtain the scaling property

\[
 p_j \left( \frac{x}{\lambda_j^\theta}, \frac{y}{\lambda_j^\theta} \right) = \lambda_j^\theta \cdot p_1(x, y).
\]

Let $\pi_j$ be the stationary distribution for the chain $\Phi^{(j)}$. That is $\pi_j$ is defined as the unique finite measure satisfying

\[
 \pi_j(A) = \int P_j(x, A)d\pi_j(x) = \int \mathbb{1}_A(y) \cdot p_j(x, y)dyd\pi_j(x).
\]

It follows from Theorem 9 that $\pi_j$ exists and absolute continuous. Let $\varphi_j$ be its density. Then

\[
 \int \varphi_j(x)p_j(x, y)dx = \varphi_j(y).
\]

It follows from this identity and (51) that for all $j$ and $u \in \mathbb{R}^+$ and $A \subset \mathbb{R}^+$ that

\[
 \varphi_j(u) = \varphi(u\lambda_j^\theta) \cdot \lambda_j^\theta \quad \text{and} \quad \pi_j(A) = \pi_0(\lambda_j^\theta \cdot A).
\]

From now on we always assume that $\Phi_0^{(j)}$ is chosen according to $\pi_j$.

7. **Some large deviation and variance estimates**

7.1. **Some large deviation results.** Now we estimate the number and length of losses of $Z_j(t)$ on the interval $[0, 1]$, using some large deviation results. The numbers of losses on the interval $[0, 1]$ of $Z_j(t)$ for $j = 1, 2, \ldots$ are given by the independent Poisson($\lambda_j$) processes. Let $N_j(t)$ be the number of losses of $Z_j(t)$ on $[0, t]$. We write

\[
 N_j := N_j(1)
\]

Then $N_j \sim \text{Poi}(\lambda_j)$. Here we define four events that are likely to happen in these Poisson processes. Recall that $T_k^{(j)}$ stands for the time of the $k$th loss while $\tau_k^{(j)} = T_k^{(j)} - T_{k-1}^{(j)}$ is
the $k$th inter-event time.

\[ E_1^{(j)} := \left\{ \frac{\lambda_j}{2} < N_j \leq 2\lambda_j \right\} \tag{56} \]

\[ E_2^{(j)} := \left\{ \# \left\{ k : k \leq \lfloor \lambda_j/2 \rfloor, \tau_k^{(j)} > \frac{1}{2\lambda_j} \right\} \geq \frac{\lambda_j}{4} \right\}, \tag{57} \]

\[ E_3^{(j)} := \left\{ \# \left\{ k : k \leq 2\lambda_j, \frac{1}{100\lambda_j} < \tau_k^{(j)} < \frac{5}{\lambda_j} \right\} \geq 2\lambda_j \frac{98}{100} \right\} \tag{58} \]

Further, we define

\[ E_4^{(j)} := \left\{ \# \left\{ k : k \leq N_j, \frac{1}{100\lambda_j} < \tau_k^{(j)} < \frac{5}{\lambda_j} \right\} > 0.97N_j \right\} \tag{59} \]

Finally we set

\[ E^{(j)} := E_1^{(j)} \cap E_2^{(j)} \cap E_3^{(j)} \cap E_4^{(j)}. \tag{60} \]

**Fact 3.** For almost all realizations $\omega$ of the Poisson point processes with intensities $\lambda_j$, $j \geq 1$, there exists a $j_0 = j_0(\omega)$ such that for all $j > j_0$ the event $E^{(j)}$ holds.

**Proof.** It follows from a standard Chernoff bound for Poisson random variables (see e.g. [10, Theorem 2.19, Exercise 2.221]) that

\[ \mathbb{P}(N_j < \lambda_j/2) < \exp(-\lambda_j/8) \quad \text{and} \quad \mathbb{P}(N_j > 2\lambda_j) < \exp(-\lambda_j \cdot 3/8). \]

By (23) both of these series are summable in $j$. Thus, using Borel-Cantelli Lemma we obtain that the event $E_1^{(j)}$ holds for all sufficiently large $j$. To estimate the probability of $E_2^{(j)c}$, the complement of $E_2^{(j)}$, from above note that

\[ E_2^{(j)c} = \left\{ \sum_{k=1}^{\lambda_j/2} X_k > \lambda_j/4 \right\}, \]

where $X_k = 1_{\{\tau_k^{(j)} < 1/(2\lambda_j)\}}$ is the indicator of the event $\{\tau_k^{(j)} < 1/(2\lambda_j)\}$, a Bernoulli random variable with parameter $p := \mathbb{E}[X_k] = \mathbb{P}(\tau^{(1)} < 1/2) < 0.4$. Then again, from a usual Chernoff bound (see again [10, Theorem 2.19]) we obtain that

\[ \mathbb{P}(E_2^{(j)c}) \leq \exp \left( -\lambda_j \cdot \frac{(0.5 - p)^2}{2(p + [0.5 - p]/3)} \right). \]

This is also summable by (23). Similarly, using the same argument for the lower and upper bounds we obtain that

\[ \mathbb{P}(E_3^{(j)c}) \leq 2 \exp(-3\lambda_j/100) \tag{61} \]

Finally, in the exact same way as above one can easily see that $\mathbb{P}(E_4^{(j)c})$ is also summable. Then we use Borel-Cantelli Lemma again to complete the proof of the fact. \qed

**Fact 4.** Let

\[ i_j := \frac{\theta + 1}{\hat{c}} \cdot \frac{\log \lambda_j}{\lambda_j}, \tag{62} \]

where the constant $\hat{c} > 0$ comes from (47). Further let

\[ A_k^{(j)} := \left\{ \Phi_k^{(j)} < i_j^\theta \right\} \quad \text{and} \quad A^{(j)} := \bigcap_{k=1}^{2\lambda_j} A_k^{(j)}. \]
Then there exists a \( j_1 \) such that for every \( j > j_1 \) the event \( A^{(j)} \) happens.

Note that if \( E^{(j)}_1 \cap A^{(j)} \) happens then for all \( 0 \leq t \leq 1, Z_j(t) < t_j^\theta \) holds.

**Proof.** We compute using the scaling property in (54) of the stationary measure \( \pi_j \)

\[
\mathbb{P} \left( A^{(j)}_k \right) = \mathbb{P} \left( \Phi_k^{(j)} > t_j^\theta \right) = \pi_j \left( t_j^\theta, \infty \right) = \pi_0 \left( \lambda_j^{\theta} t_j, \infty \right).
\]

Decomposing the right hand side into intervals \((k^\theta, (k + 1)^\theta)\) we estimate

\[
\mathbb{P} \left( A^{(j)}_k \right) \leq \sum_{k \geq \lambda_j t_j^\theta} \pi_0 \left( k^\theta, (k + 1)^\theta \right) \leq \sum_{k \geq \lambda_j t_j^\theta} K_3 \exp \left( -c\lambda_j t_j \right) \leq C \exp \left( -c\lambda_j t_j \right),
\]

for some constant \( C > 0 \), where we used Corollary 10. Using this estimate we obtain

\[
\mathbb{P} \left( A^{(j)} \right) = \mathbb{P} \left( \bigcup_{k=1}^{2\lambda_j} A^{(j)}_k \right) \leq 2\lambda_j C \exp \left( -c\lambda_j t_j \right) < C \frac{1}{\lambda_j^\theta},
\]

again for some constant \( C > 0 \). By (23) the rhs is summable. Applying Borel-Cantelli lemma finishes the proof. \( \square \)

In what follows we combine Fact 3 and Fact 4 to obtain

**Corollary 18.** Almost surely, there exists a \( j_2 \) (which depends on the representation) such that

\[
\forall j > j_2, \forall t \in [0, 1], Z_j(t) \leq t_j^\theta.
\]

In the next two sections we frequently use the following fact: For \( r \geq 1 \) there exists a constant \( c_{12} > 0 \) such that for any \( h \geq 0 \)

\[
\int_{t=0}^{\infty} (t + h)^r e^{-t} dt = \begin{cases} c_{12} h^r, & \text{if } h \geq 1; \\ c_{12}, & \text{if } 0 < h < 1. \end{cases}
\]

7.2. **Expected increments, assuming no loss.** In this and the next sections we study the increments of \( Z_j \)

\[
\Delta_{[a,a+h]} Z_j(t) := Z_j(a + h) - Z_j(a)
\]

on a given interval \([a, a + h]\). Since the process is stationary,

\[
\mathbb{E} \left[ \Delta_{[a,a+h]} Z_j(t) \right] = 0.
\]

However, here we assume that there is no event of loss on \([a, a + h]\). Under this condition in Proposition 19 below we give an effective upper bound on the expected increment.

**Proposition 19.** For an arbitrary \( a > 0 \), and \( j \) such that \( \lambda_j h < 1 \) we have for some constant \( C > 0 \) that

\[
\mathbb{E} \left[ \Delta_{[a,a+h]} Z_j \mid Z_j \text{ has no loss on } (a, a + h) \right] \leq C \cdot \frac{\lambda_j h}{\lambda_j^\theta}.
\]

If \( Z_1(t) \) has no loss on the interval \([a, a + h]\), then the distribution of the time of the last loss before \( a \) is exponential. Assuming that the value of \( Z_1 \) right before the last loss preceding \( a \) was \( x \) then the conditional expectation of the increment of \( Z_1 \) on this interval is

\[
\int_{t=0}^{a} (g_x(t + h) - g_x(t)) e^{-t} dt,
\]

assuming that there was a loss before \( a \) at all. Otherwise the increment is \( g_x(a + h) - g_x(a) \), where \( x = Z_1(0) \) and this happens with probability \( e^{-a} \).

This observation motivates the following two lemmas about the first and second moment of the (conditional) increment.
Lemma 20. Let $\ell = 1, 2$ and we define

$$I_{x,h,\ell} := \int_{t=0}^{\infty} (g_x(t+h) - g_x(t))^\ell e^{-t} dt$$

Then there exists a constant $c_{17} > 0$ such that

(68) $$I_{x,h,\ell} \leq c_{17} \max \left\{ x^{(1-1/\theta)+1/\theta}, 1 \right\} \cdot \max \left\{ h^\ell, h^\ell \theta \right\}$$

Proof. Fix $x, h$ and $\ell$. By the mean value theorem, for every $t$ we pick a $t' \in \left( \frac{t}{x^{1/\theta}}, \frac{t+h}{x^{1/\theta}} \right)$ such that

(69) $$g_1 \left( \frac{t+h}{x^{1/\theta}} \right) - g_1 \left( \frac{t}{x^{1/\theta}} \right) = g_1'(t') \cdot \frac{h}{x^{1/\theta}}$$

Then we can write

$$I_{x,h,\ell} = \int_{t=0}^{\infty} x^\ell \left( g_1 \left( \frac{t+h}{x^{1/\theta}} \right) - g_1 \left( \frac{t}{x^{1/\theta}} \right) \right)^\ell e^{-t} dt$$

where we have decomposed the integral into the integral on three disjoint intervals.

The estimate of $I_1$: For $t \leq \xi x^{1/\theta} - h$ we have $t' \leq \frac{t+h}{x^{1/\theta}} < \xi$. Let $R := \max_{u \in [0, \xi]} g_1'(u)$. So, it is easy to see that for some constant $C > 0$,

$$I_1 \leq x^\ell R h^\ell x^{-\ell/\theta} \xi x^{(1-1/\theta)+1/\theta} \leq C \cdot h^\ell x^{(1-1/\theta)+1/\theta}.$$ 

Thus, $I_1$ is at most the right hand side of (68).

The estimate of $I_2$: Using (69) first we can apply the upper bound on $g_1'$ in (36) to obtain:

$$I_2 \leq \int_{\xi x^{1/\theta} - h}^{\xi x^{1/\theta}} x^\ell \left( \frac{h}{x^{1/\theta}} \right) e^{-t} dt \leq x^{(1-1/\theta)} h^\ell \psi \theta \int_{\xi x^{(1-1/\theta)/(\theta-1)}}^{\xi x^{1/\theta}} g_1(t') e^{-t} dt.$$ 

Now we use that $g_1$ is increasing, thus we can use the upper bound on $t'$ from before (69) and then apply (38) to obtain

$$I_2 \leq x^{(1-1/\theta)} h^\ell \psi \theta \int_{\xi x^{1/\theta} - h}^{\xi x^{1/\theta}} g_1(t') e^{-t} dt \leq h^\ell \psi \theta \int_{\xi x^{1/\theta} - h}^{\xi x^{1/\theta}} C(t+h)^{\ell-1} e^{-t} dt.$$ 

By (64) we have shown

$$I_2 \leq \begin{cases} c_{17} h^\ell, & \text{if } h \geq 1; \\ c_{17} h^\ell, & \text{if } h \in (0, 1). \end{cases}$$

Note that the right hand side of (73) is an upper bound on the rhs.
Lemma 21. Let \( \ell = 1, 2 \) and \( b > 0 \) be arbitrary. We define
\[
J_{x,h,\ell,b} := (g_x(b + h) - g_x(b))^\ell \cdot e^{-b}.
\]
Then there exists a constant \( c_{18} > 0 \) such that
\[
J_{x,h,\ell,b} \leq c_{18} \max \left\{ x^{\ell(1-1/\theta)}, 1 \right\} \cdot \left\{ \begin{array}{ll}
h^\ell, & \text{if } h < 1; \\
h^\ell \cdot \theta, & \text{if } h > 1.
\end{array} \right.
\]

Proof. This can be proved exactly as we proved Lemma 20. More precisely, we separate three cases according to \( b < \xi x^{1/\theta} - h \), \( \xi x^{1/\theta} - h < b < \xi x^{1/\theta} \) and \( b > \xi x^{1/\theta} \). Using the same arguments as above, we obtain that there exists a constant \( c_{30} > 0 \) such that
\[
(g_x(b + h) - g_x(b))^\ell \cdot e^{-b} \leq c_{30} \max \left\{ x^{\ell(1-1/\theta)}, 1 \right\} \cdot \left\{ \begin{array}{ll}
h^\ell, & \text{if } h < 1; \\
h^\ell \cdot \theta, & \text{if } h > 1.
\end{array} \right.
\]

We are ready to prove Proposition 19. 

Proof of Proposition 19. Recall that we write \( \phi_j \) for the density of the stationary distribution of \( Z_j(t) \), and that the time of the last loss before time \( a \) has a truncated exponential distribution. Let us write \( P_a^{(j)} \) for the number of losses (points of the underlying Poisson point process) of \( Z_j \) on the interval \([a, a + h]\). We obtain
\[
\mathbb{E} \left[ \Delta_{[a,a+h]} Z_j | P_a^{(j)} \right] = 0 \leq \int_{x=0}^{\infty} \int_{t=0}^{a} (g_x(t + h) - g_x(t))^\ell e^{-\lambda_xt} dt \phi_j(x) dx \\
= I_1 + \int_{x=0}^{\infty} (g_x(a + h) - g_x(a))^\ell e^{-\lambda_{a+h} \phi_j(x)} dx \]

We apply \([34]\) on \( \phi_j \) and then \([21]\) in this order. Then we make the change or variables in \( I_1 \) in the most natural way to obtain the upper bound for \( I_1 \) from Lemma 20 and Part (b) of Theorem 9. We obtain the upper bound for \( I_2 \) immediately by Lemma 21 and also Part (b) of Theorem 9.

\[ \square \]
7.3. Variance of the increments. In this section our aim is to compute the variance of the increments. For the model of TCP RENO this was done in [3]. Then we reformulate a very simple but useful inequality which was introduced [16, Lemma VI.1]. We simply call it Markov inequality and just as in [16] we use it frequently later on.

**Proposition 22.** For every \( a > 0 \) we have

\[
\text{Var} \left( \Delta_{[a,a+h]} Z_j(t) \right) \leq K_0 \min \left\{ \frac{1}{\lambda_j^{2\theta}}, \frac{h}{\lambda_j^{2\theta-1}} \right\}.
\]

**Proof.** Recall that \( P_{a}^{(j)} \) stands for the number of losses (points of the underlying Poisson process) of \( Z_j \) on the interval \([a, a + h]\). Let \( R_j := \{P_{a}^{(j)} \geq 1\} \). Since the process \( Z_j \) is stationary, we can write

\[
\text{Var} \left( \Delta_{[a,a+h]} Z_j \right) = \mathbb{E} \left[ \Delta_{[a,a+h]} Z_j^2 \right] = \mathbb{E} \left[ (Z_j(a + h) - Z_j(a))^2 \right]
\]

\[
\leq \mathbb{E} \left[ (Z_j(a + h) - Z_j(a))^2 \mid R_j \right] \mathbb{P}(R_j) + \mathbb{E} \left[ (Z_j(a + h) - Z_j(a))^2 \mid R_j^c \right] \mathbb{P}(R_j^c).
\]

By the definition of \( R_j \):

\[
\mathbb{P}(R_j) = 1 - \mathbb{P}(R_j^c) = 1 - e^{-\lambda_j h} \leq h \lambda_j.
\]

Using (24) we can bound term \( A \) as follows:

\[
A \leq \frac{1}{\lambda_j^{2\theta}} \cdot \mathbb{E} \left[ Z_1(\lambda_j(a + h))^2 + Z_1(\lambda_j a)^2 \mid R_j \right] \cdot \mathbb{P}(R_j)
\]

First we observe that Theorem 9 implies that there is a constant \( K_6 \) such that for every \( t \geq 0 \) we have

\[
\mathbb{E} \left[ Z_1(t)^2 \right] < K_6.
\]

Hence, \( A \leq 2K_6/(\lambda_j^{2\theta}) \min \{1, h \lambda_j\} \). So we have verified that

\[
A \leq 2K_6 \min \left\{ \frac{1}{\lambda_j^{2\theta}}, \frac{h}{\lambda_j^{2\theta-1}} \right\}
\]

To estimate term \( B \) we introduce \( F_{C_j}(t) \) which is the cumulative distribution function of the current lifetime \( C_j(a) \) (the time between \( a \) and the last loss before \( a \)) for a Poisson(\( \lambda_j \)) process, a truncated exponential distribution. That is,

\[
F_{C_j}(t) := \mathbb{P}(C_j(a) \leq t) = \begin{cases} 1 - e^{-\lambda_j t}, & \text{if } t < a; \\ 1, & \text{if } t \geq a. \end{cases}
\]

Set \( u(x, t, h) := (g_x(t + h) - g_x(t))^2 \). Using that \( \varphi_j(x) \) is the density of the stationary distribution \( \pi_j(x) \).

\[
B = \mathbb{P}(R_j^c) \cdot \mathbb{E} \left[ u(x, t, h) \mid R_j^c \right] = e^{-\lambda_j h} \int \int u(x, t, h) dF_{C_j}(t) d\pi_j(x)
\]

\[
= e^{-\lambda_j h} \int_0^\infty \left[ \int_0^a u(x, t, h) e^{-\lambda_j t} dt + u(x, a, h) e^{-\lambda_j a} \right] \varphi(x) \lambda_j^\theta dx.
\]
We switch to the stationary density \( \varphi := \varphi_1 \) using (54) as well as use the self-similar property of \( g_c(t) \) as in (33) to transform \( u \) we write
\[
B = e^{-\lambda_jh} \int_{x=0}^{a} \left[ u \left( \frac{y}{\lambda_j^2}, t, h \right) e^{-\lambda_j^1 dt} + e^{-\lambda_j^a u} \left( \frac{y}{\lambda_j^2}, t, h \right) \right] \varphi(y)dy
\]
\[
eq e^{-\lambda_jh} \int_{y=0}^{a} \left[ u(y, \lambda_j^t, \lambda_jh) e^{-\lambda_j^1 dt} + e^{-\lambda_j^a u (y, \lambda_ja, \lambda_jh)} \right] \varphi(y)dy
\]
\[
\leq e^{-\lambda_jh} \int_{y=0}^{a} e^{\lambda_j^a u} \left[ u(y, \lambda_j^t, \lambda_jh) e^{-\lambda_j^1 dt} + e^{-\lambda_j^a u (y, \lambda_ja, \lambda_jh)} \right] \varphi(y)dy
\]
Using the notation of Lemmas 20 and 21 and then using the assertions of Lemmas 20 and 21 we continue as follows
\[
B = e^{-\lambda_jh} \frac{1}{\lambda_j^2 \lambda_j^\min} \cdot \int_{y=0}^{a} \left( I_{y, \lambda_jh, 2} + J_{y, \lambda_jh, 2, \lambda_ja} \right) d\pi(y)
\]
\[
\leq e^{-\lambda_jh} \frac{1}{\lambda_j^2 \lambda_j^\min} \cdot \int_{y=0}^{a} \left( c_{19} \max \left\{ y^{2(1/\theta)+1/\theta}, 1 \right\} \max \left\{ (\lambda_jh)^{\hat{\ell}}, (\lambda_jh)^{\theta} \right\} \right) d\pi(y)
\]
\[
\leq \frac{1}{\lambda_j^2 \lambda_j^\min} \cdot \lambda_j^h \cdot e^{-\lambda_j^h} \cdot \left( (\lambda_jh) + (\lambda_jh)^{2\theta-1} \right)
\]
\[
\leq K_{23} \cdot \min \left\{ \frac{1}{\lambda_j^2 \lambda_j^\min}, \frac{h}{\lambda_j^{2\theta-1}} \right\},
\]
where \( K_{23} \) is \( \max_x q_j(x) \) if \( \lambda_jh \leq 1 \) and \( K_{23} \) is \( \max_{x \geq 1} x q_j(x) \) if \( \lambda_jh > 1 \). We choose \( K_0 \) as the maximum of \( 2K_6 \) and \( K_{23} \) to complete the proof.

An immediate corollary of Markov’s inequality is the following assertion that we will call Markov inequality in the note.

**Lemma 23.** [Markov’s inequality] Given \( n \in \mathbb{N} \) and events \( A_1, \ldots, A_n \) such that for all \( i = 1, \ldots, n \) we have
\[
P(A_i) \leq \hat{p}
\]
for some \( \hat{p} \in [0, 1] \). Then for any \( N > 1 \)
\[
P(\# \{ k \leq n : A_k \text{ happens} \} > N \cdot n\hat{p}) < \frac{1}{N}.
\]

8. Technical Lemmas

Fix an \( \ell \) and a \( 0 \leq k < 2^\ell \). Here as well as throughout the paper we write \( h := 2^{-\ell} \). For an \( r > 0 \) to be specified later, we divide the increments \( \Delta^k_r Z_j \) of \( Z_j \) on the \( 2^\ell \)-mesh intervals (defined in (7)) into two groups depending on their intensity:
\[
T^k_r := \sum_{\lambda_j^\theta < r} \Delta^k_r Z_j, \quad \tilde{T}^k_r := \sum_{\lambda_j^\theta \geq r} \Delta^k_r Z_j.
\]
To simplify the notation we suppress the super indexes when we consider \( k \) and \( \ell \) fixed. We remark that the first sum is the combined effect of increments that are generally small and the second one is the combined effect of increments that are expected to be large. Namely, the typical magnitude of \( Z_j \) is \( \lambda_j^\theta \) and its typical increments are also \( \lambda_j^\theta \) if there
In this section we always assume that
\( A \) is a loss on the interval under consideration. More precisely, it follows from (22) and (33) that
\[
g_{x}(u) \geq c_{1}u^{\theta}, \quad \forall u > 0, t > 0.
\]
The following events will appear frequently the sequel:
\[
A_{\text{sum}}^{\alpha}(k, \ell) := \{|\Delta_{k}^{R}Z| \geq h^{\alpha}\}
\]
and
\[
A^{\alpha}(k, \ell) := \left\{|\Gamma_{h-\alpha}^{k,\ell}| \leq \frac{h^{\alpha}}{2}\right\} \text{ and } \overline{A}^{\alpha}(k, \ell) := \left\{|\Gamma_{h-\alpha}^{k,\ell}| \leq \frac{h^{\alpha}}{2}\right\}.
\]

**Fact 5.** Let \( \varepsilon_{0} > 0 \). Then there is a positive constant \( c(\varepsilon_{0}) \) such that for every \( k < 2^{\ell} \)
\[
\text{Var}\left(\Gamma_{h-\alpha}^{k,\ell}\right) \leq c(\varepsilon_{0}) \cdot h^{1+\alpha(2+(1-1/\theta)-\beta-\varepsilon_{0})}.
\]

**Proof.** Recall that \( L_{k} = L^{k} \) for some constant \( L > 1 \). Since the processes \( Z_{j} \) are independent of each other for different values of \( j \), we can write
\[
\text{Var}\left(\Gamma_{h-\alpha}^{k,\ell}\right) \leq \sum_{k : L_{k} > h-\alpha} \sum_{\lambda_{j}^{\alpha} \in [L_{k-1}, L_{k})} \text{Var}(\Delta_{k}^{R}Z_{j})
\]
Then, recall that \( N_{k} \) stands for the number of indices \( j \) with \( \lambda_{j}^{\alpha} \) in the interval \([L_{k-1}, L_{k})\), as in (12). We estimate \( N_{k} \) using (14) to obtain
\[
\text{Var}\left(\Gamma_{h-\alpha}^{k,\ell}\right) \leq \sum_{k : L_{k} > h-\alpha} N_{k} \cdot \max_{\lambda_{j}^{\alpha} \geq L_{k-1}} \text{Var}(\Delta_{k}^{R}Z_{j})
\]
\[
\leq \sum_{k : L_{k} > h-\alpha} K_{1}(\varepsilon_{0})K_{0}L_{k}^{\beta-1+\varepsilon_{0}}hL_{k-1}^{2-1/\theta}.
\]
After rearranging terms we obtain that the exponent of \( L_{k} \) is \( \beta - 2 + \varepsilon_{0} - (1 - 1/\theta) < 0 \), thus the sum can be estimated by the first term. We obtain
\[
\text{Var}\left(\Gamma_{h-\alpha}^{k,\ell}\right) \leq \sum_{k : L_{k} > h-\alpha} K_{1}(\varepsilon_{0})K_{0}L_{k}^{3-1/\theta}hL_{k}^{\beta-2+\varepsilon_{0}-(1-1/\theta)}
\]
\[
\leq c(\varepsilon_{0}) \cdot h^{1+\alpha(2+(1-1/\theta)-\beta-\varepsilon_{0})}.
\]
This finishes the proof. \( \square \)

### 8.1. Estimates on the region \( R_{1} \)
In this section we make some preparation to determine the multifractal spectrum on the region \( R_{1} \) as defined in [25]. That is, we assume that
\[
\alpha < \frac{1}{\beta - (1 - 1/\theta)}.
\]
In this section we always assume that \( \varepsilon_{0} > 0 \) satisfies that
\[
\alpha(\beta - (1 - 1/\theta)) + \varepsilon_{0}(\alpha + 1) < 1.
\]
To understand the aim of the following assertions recall that on region \( R_{1} \) our aim is to verify that
\[
\# \left\{ k : |\Delta_{k}^{R}Z| \sim h^{\alpha} \right\} \approx h^{-\alpha(\beta-(1-1/\theta))}.
\]
Recall the event \( A^{\alpha}(k, \ell) \) from (84). First note that it follows from Chebyshev’s inequality, Fact 5 and (66) that for every \( 0 \leq k < 2^{\ell} \) we have
\[
\text{P}(A^{\alpha}(k, \ell)) \geq 1 - 4c(\varepsilon_{0}) \cdot h^{1-\alpha(\beta-(1-1/\theta)+\varepsilon_{0})}.
\]
To shorten the presentation we introduce the event

$$F := \{ j : \lambda_j^\theta \leq h^{-\alpha}, Z_j \text{ has loss in the interval } I_k^j \},$$

in other words \( F \) is the event that one of the processes in \( T_{h^{-\alpha}} \) has a jump (an event of loss) in the interval \( I_k^j \). Recall \( A_{\text{sum}}^\alpha(k, \ell) \) from (83). Using (89) we can estimate \( \mathbb{P}(A_{\text{sum}}^\alpha(k, \ell)) \) by conditioning on \( A^\alpha(k, \ell) \). Namely,

**Fact 6.** Assume that \( \varepsilon_0 > 0 \) satisfies (87). Let \( c(\varepsilon_0) \) be the constant introduced in Fact 5. Then for every \( k, \ell \):

\[
\mathbb{P}(A_{\text{sum}}^\alpha(k, \ell)) \leq 7c(\varepsilon_0)h^{1-\alpha(\beta-(1-1/\theta))-\varepsilon_0(\alpha+1)}
\]

**Proof.** We fix an \( \alpha \) and \( 0 \leq k < 2^\ell \) and we suppress them below, that is, we write \( A_{\text{sum}} := A_{\text{sum}}^\alpha(k, \ell) \) and \( A := A^\alpha(k, \ell) \).

\[
\mathbb{P}(A_{\text{sum}}) \leq \mathbb{P}(A^c) + \mathbb{P}(A_{\text{sum}} \cap A) \leq 4c(\varepsilon_0) \cdot h^{1-\alpha(\beta-(1-1/\theta))-\varepsilon_0} + \mathbb{P}(A_{\text{sum}} \cap A).
\]

It is left to estimate \( \mathbb{P}(A_{\text{sum}} \cap A) \). Clearly, \( |\Delta Z| = |T_{h^{-\alpha}} + T_{h^{-\alpha}}| \). Hence

\[
\mathbb{P}(A_{\text{sum}} \cap A) \leq \mathbb{P}(A^c) = \mathbb{P}(T_{h^{-\alpha}} > h^{\alpha}/2) \leq \mathbb{P}(F) + \mathbb{P}(T_{h^{-\alpha}} > h^{\alpha}/2 | F^c).
\]

First we estimate \( \mathbb{P}(F) \) in (92). We decompose the indices \( j \) according to which exponential interval \( [L_{k-1}, L_k) \) their parameter \( \lambda_j^\theta \) falls in, then count the indices \( j \) in each interval using the estimate on \( N_k \) as in (25). We also use that \( 1 - e^{-x} \leq x \) to obtain that

\[
\mathbb{P}(F) \leq \sum_{\lambda_j^\theta < h^{-\alpha}} (1 - e^{-h\lambda_j}) \leq \sum_{k: L_{k-1} < h^{-\alpha}} \frac{h\lambda_j}{L_k} \sum_{\lambda_j^\theta \in (L_{k-1}, L_k)} h\lambda_j \leq \sum_{k: L_{k-1} < h^{-\alpha}} N_k h L_k^{-\alpha/\theta} \leq \sum_{k: L_{k-1} < h^{-\alpha}} K_1(\varepsilon_0) L_k^{\beta-1+\varepsilon_0} h L_k^{1/\theta}.
\]

Since the exponent of \( L_k \) is \( \beta - 1 + \varepsilon_0 > 0 \), the terms in the sum grow exponentially, and hence the sum can be estimated from above by some constant times the last term. Thus we arrive at

\[
\mathbb{P}(F) \leq C(\varepsilon_0)h^{1-\alpha(\beta-(1-1/\theta))-\alpha\varepsilon_0}.
\]

Now we turn to estimate \( \mathbb{P}(T_{h^{-\alpha}} > h^{\alpha}/2 | F^c) \) in (92). We write \( F_j^c := Z_j \text{ has no loss on } I_k^j \) below. Applying Markov’s inequality, we have the upper bound

\[
\mathbb{P}(T_{h^{-\alpha}} > h^{\alpha}/2 | F^c) \leq \frac{\mathbb{E}[T_{h^{-\alpha}} | F^c]}{h^{\alpha}/2} \leq \frac{\sum_{\lambda_j^\theta < h^{-\alpha}} \mathbb{E}[\Delta_z Z_j | F^c]}{h^{\alpha}/2}.
\]

Here, we would like to use Proposition 19. For this we need to use that \( \lambda_j h < \lambda_j h^{\alpha/\theta} < 1 \) to be able to apply Proposition 19. This is the place where we actually use that \( \alpha < 1/\theta \) (which is always the case when \( \alpha(\beta - (1 - 1/\theta)) < 1 \) that is we are in \( R_1 \)). Decomposing
the indices $j$s again,

$$
\mathbb{P}\left( T_{h^{-\alpha}} > \frac{h^{\alpha}}{2} F^c \right) \leq \frac{\sum_{k: \text{ } L_{k-1} < h^{-\alpha} \lambda_j \in (L_{k-1}, L_k \wedge h^{-\alpha})} \sum_{j} K_j \lambda_j^{1-\theta} h}{h^{\alpha}/2} \leq \frac{\sum_{k: \text{ } L_{k-1} < h^{-\alpha}} K_1(h_0) L_k^{\beta-1} \sum_{j} L_k^{1/\theta-1} h/L}{h^{\alpha}/2}.
$$

(96)

Note that here the exponent of $L_k$ is $\beta - 1 + \varepsilon_0 + (1/\theta - 1) > 0$. Thus again, the summands form a geometric series with mean greater than 1, so the sum is constant times the last element.

$$
\mathbb{P}\left( T_{h^{-\alpha}} > \frac{h^{\alpha}}{2} | F^c \right) \leq C(h_0) h^{1-\alpha(\beta - (1-1/\theta)) - \varepsilon_0}.
$$

(97)

Combining the estimates (94) and (97), combined with (92) and then with (91) finishes the proof of Fact 6. □

An immediate consequence is the following

**Corollary 24.** Almost surely, for all $\ell$ large enough we have

$$
\# \left\{ k \leq 2^\ell : |\Delta^k h h Z | \geq h^{\alpha} \right\} < 7c(h_0) h^{-\alpha(\beta - (1-1/\theta)) - \varepsilon_0(1+\alpha)},
$$

(98)

and

$$
\# \left\{ k \leq 2^\ell : \frac{|T_{h^{-\alpha}}^k| > h^{\alpha}/2 (\Delta^k_{0(1+\alpha)})}{(\Delta^k_{0(1+\alpha)})} \right\} < 4c(h_0) h^{-\alpha(\beta - (1-1/\theta)) - \varepsilon_0(1+\alpha)},
$$

(99)

finally,

$$
\# \left\{ k \leq 2^\ell : \frac{|T_{h^{-\alpha}}^k| > h^{\alpha}/2 (\Delta^k_{0(1+\alpha)})}{(\Delta^k_{0(1+\alpha)})} \right\} < 3c(h_0) h^{-\alpha(\beta - (1-1/\theta)) - \varepsilon_0(1+\alpha)}.
$$

(100)

**Proof.** First we apply Lemma 23 with $N = h^{-\varepsilon_0}$, $\hat{p} = r c(\varepsilon) h^{1-\alpha(\beta - (1-1/\theta)) - \varepsilon_0}$, where $r = 7$, $r = 4$ and $r = 3$ in the first, second and third case respectively. Since $\{h^{-\varepsilon_0} = 2^{\varepsilon_0\cdot \ell}\}$ is summable we obtain the assertions from Borel-Cantelli lemma. □

9. $f_\gamma$ on region $R_1$

In this section we are ready to prove the upper and lower bound on the large deviation multifractal spectrum $f_\gamma$ on the region $R_1$.

9.1. Upper bound on $f_\gamma$ on the region $R_1$.

**Fact 7.** The large deviation spectrum $f_\gamma(\alpha)$ (defined in (10)) satisfies

$$
f_\gamma(\alpha) \leq \alpha(\beta - (1 - 1/\theta))
$$

(101)
Proof. Fix a small \( \varepsilon_0 > 0 \) satisfying (87). By (8) we have
\[
N^{e_0}_\ell(\alpha) \leq N^{e_0}_\ell(\alpha) := \# \{ 0 \leq k < 2^\ell : |\Delta^k Z| > h^{-(\alpha+\varepsilon_0)} \}.
\]
Now we replace \( \alpha \) with \( \alpha + \varepsilon_0 \) in (92). This yields that for almost all realizations, there is an \( \ell^* \) such that for \( \ell > \ell^* \) we have
\[
N^{e_0}_\ell(\alpha) < 2^{\theta((\alpha+\varepsilon_0)(1-1/\theta)+\varepsilon_0(1+\alpha))}
\]
Taking logarithm of both sides we have for all \( \ell > \ell^* \):
\[
\frac{\log N^{e_0}_\ell(\alpha)}{\log 2^\ell} < (\alpha + \varepsilon_0)(\beta - (1 - 1/\theta)) + \varepsilon_0(1 + \alpha).
\]
This implies that (101) holds. \( \square \)

9.2. The lower bound on \( f_\alpha \) on \( R_1 \). In this case we need to assume regularity (introduced in Definition 2) of the sequence \( (\lambda_j)_{j \geq 1} \). Let \( \varepsilon_0 \) be fixed satisfying (87) and let \( A \) be an upper bound on \( a_{i+1} - a_i \) from (16). That is
\[
0 < a_{i+1} - a_i < A \quad \text{for all } i \geq 2.
\]
Our aim is to prove
\[
f_\alpha(\alpha) \geq a(\beta - (1 - 1/\theta)) \quad \text{if } (\alpha, \beta) \in R_1.
\]
Further, we always assume that \( \ell \) is so large that for \( h = 2^{-\ell} \) we have
\[
L^{A+1} h^{-(\alpha-\varepsilon_0)} \ll h^{-\alpha}.
\]
**Assumption 1.** Let \( \ell_0 \) be so large that besides (105) the following statements hold for all \( \ell > \ell_0 \): whenever \( \lambda_0^\beta > 2^{\ell(\alpha-\varepsilon_0)} = h^{-(\alpha-\varepsilon_0)} \) then we have:
\[
\text{(a): } j > \max \{ j_0, j_1, j_2 \}, \text{ where } j_0, j_1, j_2 \text{ were defined in Section 7.1}
\]
\[
\text{(b): } \nu_j < \left( \frac{1 - \theta}{10 c_{31}} \right)^{\alpha/(\theta - \alpha)}, \text{ where } \nu_j \text{ was defined in (62) and } c_{31} := \psi \left( \frac{(w+1)^\theta L^{A+1}}{c_1} \right)^{1/(2\alpha)}
\]
\[
\text{and } w \text{ is defined below in (113)}.
\]
\[
\text{(c): } h^{e_0} < \frac{\theta}{2}(1 - \eta) (w+1)^{2\alpha} L^{A+1}.
\]
Fix an \( \ell > \ell_0 \). Recall that the sequence \( (\lambda_j)_{j \geq 1} \) is regular, as in Definition 2. By the definition of \( A \) in (16) there exists an \( r \) such that \( L_{ar} \in \{ L \cdot h^{-(\alpha-\varepsilon_0)}, L^{A+1} h^{-(\alpha-\varepsilon_0)} \} \). That is for
\[
J_\ell := \left\{ j : \lambda_j^\theta \in \left( h^{-(\alpha-\varepsilon_0)}, L^{A+1} h^{-(\alpha-\varepsilon_0)} \right) \right\}
\]
we have
\[
\# J_\ell > h^{-(\alpha+\varepsilon_0)(\beta-1-\varepsilon_0)}.
\]
In (89) we verified that for \( A = A^\alpha(k, \ell) \)
\[
\mathbb{P}(A^c) = \mathbb{P} \left( |T_{h^{-\alpha}}| > \frac{h^{\alpha}}{2} \right) \leq 4C(\varepsilon_0)h^{1-(\alpha-1/\theta)-\varepsilon_0(1+\alpha)}.
\]
Recall the definition \( A = A^\alpha(k, \ell) \) from (84). It is immediate from the bounds following (92) that for any \( j \in J_\ell \) and
\[
B_j := B_j(k, \ell) := \left\{ \left| T_{h^{-\alpha}} - \Delta^k Z_j \right| < \frac{h^{\alpha}}{2} \right\}.
\]
we have
\[
\mathbb{P} \left( B_j^c \right) \leq 3C(\varepsilon) h^{1-(\alpha-1/\theta)-\varepsilon_0(\alpha+1)}
\]
Therefore exactly as in Corollary 24 we get
Fact 8. Almost surely, there exists an \( \ell_1 \geq \ell_0 \) (depending on the realization) such that for all \( \ell > \ell_1 \) and \( j \in J_\ell \) we have

\[
\# \left\{ k < 2^\ell : A(k, \ell) \cap \overline{B}_j(k, \ell) \right\} > h^{-1} - 7c(\varepsilon_0)h^{-\alpha-(1-1/\theta)-\varepsilon_0(\alpha+1)}. \tag{110}
\]

Recall that \( T_u^{(j)} \) is the time of the \( u \)-th jump in the Poisson process with intensity \( \lambda_j \). Fix an \( \ell > \ell_1 \) and \( j \in J_\ell \). Let \( k := k_j(u) \) be the index of the \( 2^\ell \)-mesh interval that contains \( T_u^{(j)} \):

\[
k_j(u) := k \text{ if } T_u^{(j)} \in I_k^\ell.
\]

Let

\[
Q_j := \left\{ k : \exists u \in \mathbb{N} \text{ with } k = k_j(u) \text{ s.t. } T_u^{(j)} < 1 \right\}
\]

be the set of the \( 2^\ell \)-mesh intervals where the Poisson process with intensity \( \lambda_j \) has jumps. Recall that \( N_j = \#Q_j \), where \( N_j \) was defined in (55). Further we collect the indices \( k \) such that we can find a process \( Z_j \), \( j \in J_\ell \) in such a way that in \( I_k^\ell \) there is 'possibly large' loss of \( Z_j \), reflected in the fact that the inter-event time \( \tau_u^{(j)} = T_u^{(j)} - T_u^{(j-1)} \) is sufficiently large:

\[
I_u^w := \left\{ k \in Q_j : \exists u \in \mathbb{N} \text{ with } k = k_j(u) \text{ s.t. } \tau_u^{(j)} > \frac{1}{w \lambda_j} \right\},
\]

where \( w \) is so big that

\[
e^{-1/w} > 0.99. \tag{113}
\]

Then it follows from the Large Deviation Theorem and Borel Cantelli lemma that for all \( \ell \) large enough we have

\[
\# \left\{ r \leq N_j : \tau_r^{(j)} > \frac{1}{w \lambda_j} \right\} = \#I_u^w > N_j e^{-1/(2w)}. \tag{114}
\]

Using (110), also for all \( \ell \) large enough we have

\[
\# \left\{ k : A(k, \ell) \cap \overline{B}_j(k, \ell) \text{ holds} \right\} \geq h^{-1} \cdot e^{-1/(2w)}. \tag{115}
\]

Observe that the events that \( k \in I_j \) and the event that \( A^\alpha(k, \ell) \cap \overline{B}_j(k, \ell) \) holds are independent (see (84) and (108) for the definitions). This is so, because \( B_j(k, \ell) \) excludes the contribution of \( Z_j \), while in \( A^\alpha(k, \ell) \) only processes from \( I^{k,\ell}_{h-\alpha} \) can contribute. On the other hand, we have assumed that \( j \in J_\ell \) and thus \( j \) does not belong to \( I^{k,\ell}_{h-\alpha} \).

Combining (114) and (115) for

\[
\tilde{I}_j^w := \left\{ k \in I_j^w : A(k, \ell) \cap \overline{B}_j(k, \ell) \text{ holds} \right\}
\]

yields that for an \( \ell \) large enough,

\[
\#\tilde{I}_j^w \geq N_j \cdot e^{-1/w}. \tag{116}
\]

Note that we expect that for any \( k \in \tilde{I}_j^w \) will have a sufficiently large increment of \( \Delta Z \), since (1) the process \( Z_j \) did not jump for a while already, thus it had enough time to increase and thus sustain a sufficiently large loss (2) the increment coming from processes with small intensities is rather small, i.e., \( A \) holds (see (99)) (3) the processes with relatively large intensities, excluding \( Z_j \), have a small increments, i.e., \( B_j \) also holds (see (108)). The next fact makes the heuristics of (1) precise: any \( k \in \tilde{I}_j^w \) will actually have a large loss of \( Z_j \):

Fact 9. Fix an arbitrary \( k \in \tilde{I}_j^w \). Then

\[
\Delta_k^\ell Z_j < -2 \cdot h^\alpha. \tag{118}
\]
Proof of Fact 9. We use the notation of Figure 8.

Since we are on region $R_1$, $\alpha < \theta$, therefore

\begin{equation}
(119) \quad h \ll h^{(\alpha-\varepsilon_0)/\theta} L^{-(A+1)/\theta} < \lambda_j^{-1} < h^{(\alpha-\varepsilon_0)/\theta}.
\end{equation}

So, the length of interval $U = k/2^\ell - T_{u-1}^{(j)}$ satisfies $|U| > \frac{1}{4\lambda_j}$ (see Figure 8). Using this, (35) and (119), we obtain that $x$, the position of the process $Z_j$ right before the loss at time $T_{u}^{(j)}$ satisfies

\begin{equation}
(120) \quad x \geq \frac{c_1}{4^\theta L^{A+1}} h^{\alpha-\varepsilon_0}.
\end{equation}

We denote $b := Z_j(k/2^\ell)$ and $a := Z_j(k + 1/2^\ell)$. Then $\Delta_k Z_j$ can be estimated as follows

\begin{equation}
(121) \quad \Delta_k Z_j = a - b = -x + a + (x - b)
\end{equation}

We have already estimated $-x$ from above in (120).

To estimate $a$: Note that by (33) $a \leq g_x(h) = x g_1 \left( h/x^{1/\theta} \right)$. Note, by (120),

\begin{equation}
 h/x^{1/\theta} \leq c_1 L^{(A+1)/\theta} h^{1-\alpha/\theta+\varepsilon_0/\theta}.
\end{equation}

This expression is sufficiently small if $\ell$ is large, that is $h = 2^{-\ell}$ is small. Using that $g_1(0) = \eta$ for $\ell$ large enough we get

\begin{equation}
 a \leq x \left( \eta + \frac{1-\eta}{10} \right).
\end{equation}

To estimate $x - b$: Observe that by (37) we have

\begin{equation}
(122) \quad \max_{x \in V} \frac{d}{dt} g_r(t) \leq \psi\theta x^{1-1/\theta},
\end{equation}

where the interval $V$ is defined on Figure 8. Combining this with (120) we obtain that

\begin{equation}
(123) \quad x - b \leq h \cdot \max_{x \in V} \frac{d}{dt} g_r(t) \leq \text{const} \cdot x^{1-1/\theta+1/(\alpha-\varepsilon_0)}
\end{equation}

Using Assumption 1 (b) we get

\begin{equation}
(124) \quad x - b \leq x \cdot \frac{1-\eta}{10}.
\end{equation}
Combining the three estimates on \(-x, a\) and \(x - b\) and using Assumption 1(c) we obtain that (118) holds.

Combining (83), (108) and (118) yields

\[(125) \quad \forall j \in J_\ell, \forall k \in \tilde{T}_j^w, \quad \Delta^k Z < -h^\alpha.\]

Now, \(\tilde{T}_j\) is the set of \(2^{-\ell}\)-mesh intervals for the process \(Z_j, j \in J_\ell\) with a certain good property. Now we take the union of these intervals for different \(j\)-s to obtain all the \(2^{-\ell}\)-mesh intervals that contain this good property for some \(j \in J_\ell\). Let \(K_\ell\) be their number, that is,

\[(126) \quad K_\ell := \# \bigcup_{j \in J_\ell} \tilde{T}_j^w.\]

Fact 10. If \(\ell\) is large enough then for \(h = 2^{-\ell}\) we have

\[K_\ell \geq \frac{1}{16} h^{-(\alpha + \varepsilon_0)\beta - 1^{-\varepsilon_0}} \cdot h^{-\alpha/\theta + \varepsilon_0/\theta} = \frac{1}{16} h^{-(\beta - (1-1/\theta))} \cdot h^{\varepsilon_0(\beta - (1-1/\theta)) - \varepsilon_0^2}. \]

We postpone to prove Fact 10 and we finish the proof of the lower bound on \(f_\beta(\alpha)\) given Fact 10.

Proof of (104). Fix a small \(\varepsilon > 0\). Choose an \(\varepsilon_0\) satisfying (87) and

\[(127) \quad \varepsilon_0 < \varepsilon \cdot \frac{\beta - (1 - 1/\theta)}{1 + \alpha + \beta - (1 - 1/\theta)}.\]

By Fact 10 and by (125)

\[\# \left\{ k < 2^\ell : |\Delta^k Z| > h^\alpha \right\} \geq \frac{1}{16} h^{-(\beta - (1-1/\theta))} \cdot h^{\varepsilon_0(\beta - (1-1/\theta)) - \varepsilon_0^2}. \]

Apply Corollary 24 with replacing \(\alpha\) by \(\alpha - \varepsilon\). Then using (127) we obtain

\[\# \left\{ k < 2^\ell : |\Delta^k Z| > h^{\alpha - \varepsilon} \right\} < 6c_6(\varepsilon_0) h^{-(\beta - (1-1/\theta)) - \varepsilon_0 + \varepsilon(\beta - (1-1/\theta))} \leq \frac{1}{32} h^{-(\beta - (1-1/\theta))} \cdot h^{\varepsilon_0(\beta - (1-1/\theta)) - \varepsilon_0^2} \]

whenever \(\ell\) is large enough. This yields that

\[\# \left\{ k < 2^\ell : |\Delta^k Z| \in \left( h^\alpha, h^{\alpha - \varepsilon} \right) \right\} > \frac{1}{32} h^{-(\beta - (1-1/\theta))} \cdot h^{\varepsilon_0(\alpha + \beta - (1-1/\theta)) - \varepsilon_0^2}. \]

This immediately implies that (104) holds.

Proof of Fact 10. Recall that we defined the set \(Q_j\) in (111) as the \(2^\ell\)-mesh intervals where \(Z_j\) has a loss and that \(J_\ell\) defined in (106) collects those processes that have the right intensity for our purpose. First we prove that

\[(128) \quad \# \bigcup_{j \in J_\ell} Q_j > \frac{1}{2} S_\ell, \]

holds almost surely for a sufficiently large \(\ell\), where

\[S_\ell := \sum_{j \in J_\ell} \#N_j. \]
Basically (128) means that the intervals where the processes in \( J_\ell \) jump do not have too much overlap, that is, at least half of the total number of jumps are kept when taking the union. On the other hand, by (117) (see also (116))

\[
\# \left( \bigcup_{j \in J_\ell} Q_j \setminus \bigcup_{j \in J_\ell} \tilde{I}_j^w \right) \leq \# \bigcup_{j \in J_\ell} \left( Q_j \setminus \tilde{I}_j^w \right) \\
\leq \sum_{j \in J_\ell} \# \left( Q_j \setminus \tilde{I}_j^w \right) \\
\leq (1 - e^{-1/w}) \sum_{j \in J_\ell} N_j \\
= S_\ell \cdot (1 - e^{-1/w}).
\]

(129)

Recall \( K_\ell \) from (126). Using the identity

\[
\bigcup_{j \in J_\ell} \tilde{I}_j^w = \bigcup_{j \in J_\ell} Q_j \setminus \left[ \bigcup_{j \in J_\ell} Q_j \setminus \bigcup_{j \in J_\ell} \tilde{I}_j^w \right],
\]

in combination with (128) and (129) yields that

\[
K_\ell \geq \left( \frac{1}{2} - (1 - e^{-1/w}) \right) \cdot S_\ell \geq \frac{S_\ell}{4} \geq \frac{1}{4} \# J_\ell \cdot \min_{j \in J_\ell} N_j
\]

(131)

Further, (107) and (56) as well as (106) yield that

\[
K_\ell \geq \frac{1}{4} h^{-\left(\alpha - \varepsilon_0\right)\left(\beta - 1 - \varepsilon_0\right)} \cdot \frac{\lambda_j}{2} \geq \frac{1}{8} h^{-\left(\alpha - \varepsilon_0\right)\left(\beta - 1 - \varepsilon_0\right)} \cdot h^{-\left(\alpha - \varepsilon_0\right)/\theta}.
\]

(132)

So, to complete the proof of Fact 10 we only need to verify that (128) holds. To do this, imagine that we have \( S_\ell \) balls that we throw into \( 2^\ell \) urns independently. The number \( B_\ell \) of non-empty urns stochastically dominates \( \# \bigcup \tilde{I}_j^w \) (Since in \( \tilde{I}_j^w \) we have the extra restriction that the previous jump had to happen relatively long ago). Using that \( S_\ell \ll 10^{-6} \cdot 2^\ell \) (implying that the probability that two balls fall into the same urn is small) it is easy to see that

\[
\sum_{\ell} \mathbb{P}\left\{ B_\ell < \frac{S_\ell}{4} \right\} < \infty.
\]

So, by Borel-Cantelli Lemma for \( \ell \) large enough, the assertion of Fact 10 holds. \( \square \)

10. Upper bound for \( f_g \) on region \( R_2 \)

10.1. The upper bound. Let us write \( \beta' := \beta - \left(1 - \frac{1}{\beta}\right) \). We work now on region \( R_2 \). That is

\[
\frac{1}{\beta'} \leq \alpha < 1 + \frac{1}{\beta'}.
\]

(133)

Our aim is to verify that on this region we have

\[
f_g(\alpha) \leq 1 + \frac{1}{\beta'} - \alpha.
\]

(134)

In this section first we show that on region \( R_2 \) we have

\[
f_{g}^0(\alpha) \leq 1 + \frac{1}{\beta'} - \alpha,
\]

(135)

then we quickly derive the same estimate for \( f_g(\alpha) \) on the lower part of this region, on \( R_2^\ell \).
We start defining

\[ J_{h,\varepsilon} := \left( -h^{\alpha - \varepsilon}, -h^{\alpha + \varepsilon} \right) \cup \left( h^{\alpha + \varepsilon}, h^{\alpha - \varepsilon} \right), \]

the sizes of increments/oscillations that have the right absolute value when counting the \( N_\ell^g(\alpha) \) for \( f_g(\alpha) \) or \( N_\ell^{\varepsilon,O}(\alpha) \) for \( f_g^O(\alpha) \), respectively. First we prove that for every \( \ell \) big enough and \( k < 2^\ell \) we have

\[ \mathbb{P} \left( O_\ell^k Z \in J_{h,\varepsilon} \right) \sim h^{\alpha - 1/\beta'}. \]

Then by Lemma 23 we conclude that \( N_\ell^{\varepsilon,O}(\alpha) \) (defined similarly as in (8), but with \( O_\ell^k Z \) instead) satisfies

\[ N_\ell^{\varepsilon,O}(\alpha) \lesssim h^{-1/\beta'}, \]

where as always \( h = 2^\ell \). Take logarithm on both sides and divide by \( \log h^{-1} \) to obtain that (134) holds.

In the rest of the section we make these heuristics precise and prove (138). The line of the proof is the same as the one of the corresponding statement in (16).

The rest of the section is organized as follows: First in Section 10.1.1 we introduce some notation used in this section. In particular we define the notion of a good loss. Then in Section 10.1.2 we explain the heuristics of the proof. As an important technical step of the proof, in Section 10.1.3 we verify that for every net interval \( I_\ell^k \) there exists a process \( Z_j \) with intensity that satisfies \( \lambda_j^\theta \sim h^{-\frac{1}{\beta'}} \) and \( Z_j \) has a good loss in \( I_\ell^k \). Finally we present the proof of (134) in Section 10.1.4.

For technical reasons, we need to divide \( R_2 \) into a lower part \( R_2^l \) (defined as \( \alpha \leq \beta/\beta' \)) and an upper part \( R_2^u \) (defined as \( \alpha > \beta/\beta' \)). We have the upper bound for the oscillations \( O_\ell^k Z \) and thus obtain that \( f_g^O(\alpha) \) satisfies (135).

On the lower part \( R_2^l \), we derive estimates along the lines so that it will be easy to see that \( O_\ell^k Z - |\Delta_\ell^k Z| \) is relatively small compared to \( h^\alpha \), hence, we immediately obtain that on this region, \( f_g(\alpha) \) satisfies the same upper bound as \( f_g^O(\alpha) \) does, thus we obtain (134).

Unfortunately, on the upper part of the region \( R_2^u \), the difference between the oscillation and the increment is typically bigger than \( h^\alpha \), that is, we have \( O_\ell^k Z - |\Delta_\ell^k Z| \gg h^\alpha \). This makes it impossible to transfer our result on \( f_g^O(\alpha) \) to \( f_g(\alpha) \) on this region.

We remark however that the ‘sizes of bursts’ are rather determined by the oscillations, not by the increments of \( Z \), thus, we believe that this upper bound on \( f_g^O(\alpha) \) is just as relevant for practical purposes as the bound would have been on \( f_g(\alpha) \).
10.1.1. Good losses. Fix a $j$ and $r$ such that $T_{r}^{(j)}$, the time of $r$-th loss of $Z_j$ is in $(0,1)$. The variables defined below are dependent on $j$ and $r$ but in order to simplify the notation we suppress them. Recall that the order of magnitude of a process $Z_j$ with intensity $\lambda_j$ is $\lambda_j^{-\theta}$. Thus it is reasonable to scale back and define

$$X := \lambda_j^\theta \Phi_{r-1}^{(j)}$$

Moreover,

$$A := \lambda_j^\theta Z_j \left( \frac{k + 1}{2^l} \right)$$

Finally, let

$$I_1 := Y - B$$

while the oscillation of $Z_j$ is simply given by

$$OZ_j = \lambda_j^{-\theta} (1 - \eta) Y.$$
(iv): Moreover we also require that
\[ \frac{\tau}{X^{1/\theta}} \notin G_{\delta_0}. \]
If (i)-(iv) hold then we also say that \( Z_j \) has a \((K, \delta_0)\) loss in \( I_k^\ell \).

Fact 11. We use the notation of Definition 25. For any \( v \in (0, 1) \) we can select a vector \( K = K^{(v)} \) with sufficiently large components and a sufficiently small positive number \( \delta_0 = \delta^{(v)}_0 \) such that
\[ \mathbb{P} \left( T^{(j)}_\tau \text{ is not } (K^{(v)}, \delta^{(v)}_0)-\text{regular} \right) < v. \]

The proof of Fact 11 is immediate from the definitions.

Definition 26. If \( T^{(j)}_\tau \) is called a \( v \)-good loss if \((K^{(v)}, \delta^{(v)}_0)\)-regular.

The next fact analyses the properties of the increment of \( Z_j \) on a \( 2^\ell \)-mesh interval given that \( Z_j \) has a \((K, \delta_0)\)-regular loss there.

Fact 12. Let \( k < 2^\ell \) and assume that \( Z_j \) has a \((K, \delta_0)\)-regular loss on \( I_k^\ell \). Let \( X, Y \) be as in (139). Then
\[ X, Y \geq \frac{c_1}{K_1^\theta}. \]
and
\[ Y \leq K_{66} := K_4 \cdot g_1 \left( \frac{K_2 \cdot K_1^\theta}{c_1} \right). \]

Proof. To verify (149) first we introduce \( x_{-1} := \lim_{t \uparrow T^{(j)}_\tau} Z_j(t) \). Then we apply (35) and (144) in this order to obtain the sequence of inequalities
\[ X \cdot \lambda_j^{-\theta} = g_{x_{-1}} \left( \tau^{(j)}_{\tau-1} \right) \geq c_1 \left( \tau^{(j)}_{\tau-1} \right)^\theta \geq \frac{c_1}{K_1^\theta} \lambda_j^{-\theta}. \]
Applying the same for \( \tau^{(j)}_{\tau-1} \) and \( Y \) instead of \( \tau^{(j)}_{\tau-1} \) and \( X \) we complete the proof of (149).

Now we prove that (150) holds. Using (149) and the fact that \( g_1 \) is monotone increasing we get as well as the self-similarity property (33) we obtain that
\[ \lambda_j^{-\theta} Y = g_{X \lambda_j^{-\theta}} \left( \tau \lambda_j^{-\theta} \right) = X \lambda_j^{-\theta} g_1 \left( \frac{\tau \lambda_j^{-1}}{X^{1/\theta} \lambda_j^{-1}} \right) \]
Using (144), (145) to estimate the right hand side yields the upper bound
\[ \lambda_j^{-\theta} Y \leq \lambda_j^{-\theta} K_4 g_1 \left( \frac{K_2 \cdot K_1^\theta}{c_1} \right), \]
which finishes the proof. \( \square \)

10.1.2. Heuristics of the proof of (134) and (135). In this section we describe the intuitive idea behind the proof of the upper bound of the multifractal spectrum of \( Z \) on the region \( R_2 \).

First, we determine the smallest magnitude of the intensity \( \lambda_j \), for which at least one of the \( Z_j \)'s having intensity of this magnitude, still has a loss in every net interval \( I_k^\ell \). Recall that \( \beta' = \beta - (1 - 1/\theta) \), where \( \beta \) is as in (11). We claim that this happens when we focus on those \( j \) for which
\[ \lambda_j^\theta \sim h^{-1/\beta'}. \]
Namely, if we choose \( q \) such that \( h^{-1/\beta'} \in (L^q, L^{q+1}) \) but \( h^{-1/\beta'} \sim L^q \) then by the definition of \( \beta \) we have

\[
\# \left\{ j : \lambda_j^0 \in (L^q, L^{q+1}) \right\} \sim h^{-(\beta-1)/\beta'}.
\]

This indeed follows from the definition (11) or (13) of \( \beta \), see also (14) and (15). For each of these \( j \) the process \( Z_j \) has approximately \( \lambda_j^0 \sim h^{-1/(\theta\beta')} \) losses on \((0, 1)\). So the total number of losses of each of these \( j \) is the product

\[
h^{-(\beta-1)/\beta'} \cdot h^{-1/(\theta\beta')} = h^{-1}.
\]

This shows that the total number of the losses of the union of \( Z_j \) with intensity satisfying (151) is approximately \( h^{-1} \). We actually prove that for each net interval \( I_k^\ell \) we can find a \( Z_j \) satisfying (151) which has a sufficiently regular loss in \( I_k^\ell \). Then the proof continues as follows: we decompose the oscillation \( O_{\ell}^Z \) into the oscillation of this \( Z_j \) plus the oscillation of all the other \( Z_i, i \neq j \). Respectively, on the lower part of the region \( R_2 \) we decompose the increment \( \Delta_{\ell}^Z \) into the increment of this \( Z_j \) plus the increment of all the other \( Z_i, i \neq j \). From here, since we have a good control on the oscillation/increment of this particular \( Z_j \), we can show that (138) holds whatever the oscillations/increments of the remaining \( Z_i \)'s are. We use the following notion for the rest of this section.

Fix a pair \((\beta, \alpha) \in R_2 \). Recall that an \( \varepsilon_0 \) comes from the regularity of the sequence \((\lambda_j)_{j \geq 1} \), see (15). For the rest of the proof we choose an \( \varepsilon, \varepsilon_1, \varepsilon_2 > 0 \) in such a way that they satisfy the following inequalities:

\[
3\varepsilon < \alpha - 1/\beta',
\]

\[
\frac{\varepsilon_0}{\beta' - \varepsilon_0} < \varepsilon_1 < \varepsilon_2 < 3\theta^2\varepsilon_0 < \varepsilon.
\]

Note that this choice is possible since we assume (133) on \( R_2 \) and \( \beta'\theta^2 > 1 \) holds when \( \beta > 1 \) and when \( \beta = 1 \) then \( \beta' = 1/\theta \) and \( \theta \geq 1 \).

10.1.3. The existence of an appropriate \( Z_j \) for an arbitrary net intervals. In this section we prove that

**Lemma 27.** Assume the sequence \((\lambda_j)_{j \geq 1} \) is regular. Then, if \( \ell \) is large enough then for every \( k < 2^\ell \) there exists a \( j \) such that \( T_{r_j^\ell}(k, \ell) \) is a \( 2^{-2\ell\varepsilon} \)-good loss.

Fix \( c > 0 \) as in the statement of the lemma. Let us pick a constant \( C > 0 \) large enough and define the interval with \( h = 2^{-\ell} \)

\[
[M_1, M_2] := \left[ h^{-(1+\varepsilon_1)/\beta'}, C\ell h^{-(1+\varepsilon_1)/\beta'} \right]
\]

in such a way that (15) is satisfied \( c\ell \) many times. That is, \( C\ell \) is so large that in the interval \([2^{\ell(1+\varepsilon_1)/\beta'}, C\ell 2^{\ell(1+\varepsilon_1)/\beta'}] \) there is at least \( c\ell \) many regular intervals \([L^q, L^{q+1}] \) with at least \( L^q(\beta-1-\varepsilon_0) \) many \( \lambda_j^\theta \)'s falling in each of them. Note that when \( \ell \) is so large that \( C\ell < 2^{(\varepsilon_2-\varepsilon_1)/\beta'} \) holds for some \( \varepsilon_2 \geq \varepsilon_1 \), then we have the bound

\[
M_2 \leq h^{-(1+\varepsilon_2)/\beta'}
\]

that we shall use later on. We use now that \( L^q \geq 2^{\ell(1+\varepsilon_1)/\beta'} \) to obtain that

\[
\# \left\{ j : \lambda_j^0 \in [M_1, M_2] \right\} \geq c\ell 2^{\ell(1+\varepsilon_1)(\beta-1-\varepsilon_0)/\beta'}
\]

Let \( \Gamma_{\ell} \) be the elements of the set in formula (155), and let us partition \( \Gamma_{\ell} \) into disjoint sets immediately as \( \Gamma_{\ell} = \bigcup_{i=1}^{\ell} \Gamma_{\ell}^{(i)} \) according to which regular interval \( \lambda_j^\theta \) is falling in (j's
outside the regular intervals can be assigned arbitrarily to one of these sets). For an \( j \in \Gamma^{(i)}_{\ell} \), \( i \leq \ell \) and \( k \leq 2^\ell \) we define the events

\[
E^j_k := \{ Z_j \text{ has a loss on } I^j_k \}, \quad E^{(i)}_k := \bigcap_{k=1}^{2^\ell} \bigcup_{j \in \Gamma^{(i)}_{\ell}} E^j_k.
\]

**Fact 13.** There exists \( \ell_8 \) such that for all \( \ell > \ell_8 \), the event \( \bigcap_{k \leq \ell} E^{(i)}_k \) holds almost surely. That is, for every \( k \leq 2^\ell \), there is at least \( \ell \) many processes \( Z_j \) with intensity in \([M_1, M_2] \) that have a loss on \( I^j_k \).

**Proof.** Note that the union of independent Poisson processes is Poisson process with intensity as the sum of the intensities. Hence

\[
\mathbb{P}\left( \left( \bigcup_{j \in \Gamma^{(i)}_{\ell}} E^j_k \right)^c \right) = \exp \left\{- \sum_{j \in \Gamma^{(i)}_{\ell}} \lambda_j h \right\} \leq \exp \left\{ 2^{\ell(1+1/j)(\delta-1/\theta)} 2^{\ell(1+1/j)\theta^2} 2^{-\ell} \right\}.
\]

where we used that the cardinality of \( \Gamma^{(i)}_{\ell} \) is at least as in (155) and that the intensity \( \lambda_j \) is at least \( M_1^{1/\theta} \). Using now that \( \beta - 1 = \beta' - 1/\theta \), the exponent of \( 2^\ell \) on the rhs of the previous formula simplifies and becomes \( \beta'\theta \varepsilon_1 - \theta \varepsilon_0 (1+\varepsilon_1) := \delta \), which is positive as long as \( \varepsilon_1 > \varepsilon_0 / (\beta' - \varepsilon_0) \), that we precisely assumed in (152). To continue the proof, we apply a union bound again to obtain that

\[
\mathbb{P} \left( \left( E^{(i)}_{j,k} \right)^c \right) = \mathbb{P} \left( \left( \bigcap_{k=1}^{2^\ell} \bigcup_{j \in \Gamma^{(i)}_{\ell}} E^j_k \right)^c \right) \leq 2^\ell \cdot \exp \left\{ -2^\delta \ell \right\}.
\]

Finally, we apply a union bound again

\[
\mathbb{P} \left( \left( \bigcup_{i=1}^{\ell} E^{(i)}_{\ell} \right)^c \right) \leq \ell \cdot 2^\ell \cdot \exp \left\{ -2^\delta \ell \right\}.
\]

The right hand side is summable. Now we apply Borel-Cantelli Lemma to complete the proof of the Fact 13.

**Proof of Lemma 27.** Fix an arbitrary \( k \leq 2^\ell \). By Fact 13 there is at least \( \ell \) many \( j \)s with respective losses \( T^{(j)}_{r_j} \) \((k, \ell) \) that fall in \( I^j_k \). By Fact 11 the probability that \( T_{r_j}^{(j)}(k, \ell) \) is not a \( 2^{\ell - 2/c} \)-good loss is less than \( 2^{\ell - 2/c} \). Note that these processes are independent. Hence the probability that for all \( \ell \) processes, the loss in \( I^j_k \) is not a \( 2^{\ell - 2/c} \)-good loss is \( 2^{-2/c} \). So, by a union bound, the probability that there exists at least one \( k \leq 2^\ell \) that does not have a process with a \( 2^{\ell - 2/c} \)-good loss on it, is less than \( 2^\ell \cdot 2^{-2\ell} = 2^{-\ell} \). This is summable in \( \ell \), so, from Borel-Cantelli Lemma we obtain that for a sufficiently large \( \ell_0 \), for all \( \ell > \ell_0 \) we have

\[
(156) \quad \forall k < 2^\ell, \exists j \in \Gamma_{\ell}, \ T^{(j)}_{r_j}(k, \ell) \text{ is a } 2^{\ell - 2/c} \text{-good loss.}
\]

This finishes the proof.

10.1.4. Deriving the upper bound on \( f^O_q(\alpha) \) and \( f_q(\alpha) \). To verify (137) first we fix an \( \ell \) large enough and an arbitrary \( k < \ell \). Using Lemma 27 we choose an \( j \in \Gamma_{\ell} \) such that for an appropriate \( j = j(k, \ell) \) and \( r = r(k, \ell, j) \), \( Z_j \) has a \( 2^{\ell - 2/c} \)-good loss \( T^{(j)}_{r}(k, \ell) \) in \( I^j_k \), where \( c \) as in Lemma 27. For \( v = 2^{\ell - 2/c} \) we write

\[
(157) \quad (K, \delta_0) := (K^v, \delta_0^v).
\]
First we decompose $\Delta^k Z$ as follows

$$
\Delta^k Z := \Delta^k Z_j + \Delta^k Z_{\neq j},
$$

where $\Delta^k Z_{\neq j} := \sum_{q \neq j} \Delta^k Z_q$. On the other hand, for the oscillations we can get upper and lower bounds: a jump of the process $Z_j$ already produces an oscillation of size $O Z_j(k, \ell)$, while triangle inequality yields the upper bound:

$$
(158) \quad O_Z^k Z_j \leq O^k Z \leq O^k Z_j + O^k Z_{\neq j},
$$

where similarly $O^k Z_{\neq j} := O^k (\sum_{q \neq j} Z_q)$, i.e., this latter is the oscillation of the sum of all the other processes. From now on we often suppress $(k, \ell)$ in the rest of the proof. Further, we write $F_{\neq j}$ for the CDF of $\Delta Z_{\neq j}$, and $F_{\neq j}$ for the CDF of $O Z_{\neq j}$.

We handle the increments first. Recall that $φ_j$ denotes the density function of $\Delta Z_j$. Recall the notation $J_{h,ε}$ from (136). Clearly, by a simple convolution,

$$
(159) \quad \mathbb{P} \left( \Delta^k Z \in J_{h,ε} \right) = \mathbb{P} \left( \Delta^k Z_j + \Delta^k Z_{\neq j} \in J_{h,ε} \right) = \int_{a \in \mathbb{R}} \int_{\mathbb{R}} z_j(y) dy dF_{\neq j}(a)
$$

$$
\leq 2 \max_{a} \int_{a}^{a+h^{α-ε}} z_j(y) dy = 2 \max_{a} \mathbb{P} \left( \Delta Z_j \in a + \left( 0, h^{α-ε} \right) \right).
$$

Now we assume that

$$
(160) \quad 1 \leq α \cdot β' \leq β.
$$

This defines the sub-region of $R_2$ that we denote by $R_2^c$, see Figure 6. Clearly, the right hand side of (160) always holds when $α < 1$. Recall the notations $I_1, I_2$ from (140).

Using (37), Lagrange Mean Value Theorem, (160) and (149), after a somewhat longish but elementary calculation we obtain that

$$
(161) \quad λ_{j}^{-θ} (I_1 + I_2) \leq h^{β/β'} \ll h^α \ll λ_{j}^{-θ} Y.
$$

This yields an estimate on the second term in (141). Hence, using (141) combined with (161), we obtain that

$$
(162) \quad \mathbb{P} \left( \Delta Z_j \in a + \left( 0, h^{α-ε} \right) \right) \approx \mathbb{P} \left( \left( 1 - \eta \right) λ_{j}^{-θ} Y \in a + \left( 0, h^{α-ε} \right) \right).
$$

By the fact that $λ_{j}^{-θ} \in (M_1, M_2)$ from (153) with the bound on $M_2$ as in (154) we obtain

$$
(163) \quad \mathbb{P} \left( \Delta Z \in J_{h,ε} \right) \leq 2 \max_{a} \mathbb{P} \left( Y \in a + \left( 0, \frac{2}{1 - \eta} λ_{j}^{θ} h^{α-ε} \right) \right)
$$

$$
\leq 2 \max_{a} \mathbb{P} \left( Y \in a + \left( 0, h^{α-\frac{1}{θ}(1+3θ^2ε_0)-ε} \right) \right)
$$

$$
= 2 \max_{a} \mathbb{P} \left( g_X(τ) \in a + \left( 0, h^{α-\frac{1}{θ}(1+3θ^2ε_0)-ε} \right) \right),
$$

where $τ$ is a truncated Exp(1) random variable taking values from the interval $\left( \frac{1}{K_1}, K_2 \right)$ and the distribution of $X$ is $π(· | (K_1^{-θ}, K_4))$. This is so, because we assumed that $Z_j$ has a good loss in $I^k_1$. Let $\tilde{φ}(x)$ and $\tilde{f}(t)$ be the density functions of $X$ and $τ$ respectively. Moreover, for a fixed $x$ let $y \mapsto ψ_x(y)$ be the density function of $Y = g_x(τ)$ conditioned on $X = x$. That is by (34)

$$
ψ_x(y) := \tilde{f}(g_x^{-1}(y)) / g_x'(g_x^{-1}(y)) = \tilde{f}(g_x^{-1}(y)) x^{1-1/θ} \cdot g_x' \left( \frac{y}{x^{1/θ}} \right)
$$
By the choice of $j$ we know that on the one hand, $g'_1\left(\frac{\tau}{x^{1/\theta}}\right) > \delta_0$ on the other hand, $x > K_1^\theta$. From these we obtain that

$$\psi_x(y) \leq \frac{2K_1^{\theta-1}}{\delta_0}. \quad (164)$$

Then the right hand side of (163) without the maximum can be estimated as follows

$$\int_{x=K_1^{-\theta}}^{K_1} \mathbb{P}\left(g_x(\tau) \in a + \left(0, h^{\alpha-\frac{1}{\beta}}(1+3\theta^2\varepsilon_0) - \varepsilon\right)\right) \tilde{\varphi}(x) dx \quad (165)$$

Our goal is to verify that there is a constant $c_{55}$ such that

$$\mathbb{P}\left(\Delta Z \in J_{h,\varepsilon}\right) \leq c_{55} \cdot h^{\alpha-\frac{1}{\beta}(1+\varepsilon)-\varepsilon} \quad (166)$$

This immediately follows from (164) and the lower bound on $\varepsilon$ in (152). Note that this estimate is independent of $x$, $a$ and that $\tilde{\varphi}_x$ in (165) is a density function thus it integrates to 1. Thus we obtain the upper bound

$$\mathbb{P}\left(\Delta Z \in J_{h,\varepsilon}\right) \leq c_{55} \cdot h^{\alpha-\frac{1}{\beta}(1+\varepsilon)-\varepsilon} \quad (167)$$

This establishes the upper bound on $f_{g}(\alpha)$ in the region $R_2^\ell$.

We remark that we used the crucial estimate (161) that is only valid on $R_2^u$ but not on $R_2^\ell$. This made it possible to move from the estimate on $Y$ to the estimate on $\Delta Z$ in (162).

Now our goal is to modify the calculations above to hold for the oscillations, for the whole region $R_2^\ell$. Again, we emphasize that on $R_2^\ell$ we have $\alpha > \beta/\beta'$, thus the estimate (161) is not valid, and as a result (162) fails to hold. Thus we cannot control the increments, resulting in the lack of a bound on $f_{g}(\alpha)$ on this region.

On the other hand, for the oscillations we have the inequalities in (158). Even though the convolution argument is no longer valid, by a similar argument as in (159), one can still show using (158) that

$$\mathbb{P}\left(OZ \in J_{h,\varepsilon}\right) \leq 2 \max_{a} \mathbb{P}\left(OZ_j \in a + \left(0, h^{\alpha-\varepsilon}\right)\right). \quad \text{(from here on, the calculation is similar as for the increments: one notes that $OZ_j = (1-\eta)Y\lambda_\eta^{\theta}$ (see (142)), and the calculation in (163), (165) and (166) also remain valid for $\mathbb{P}(OZ \in J_{h,\varepsilon})$.}$$

Thus we obtain as in (167) that

$$\mathbb{P}\left(OZ \in J_{h,\varepsilon}\right) \leq c_{55} \cdot h^{\alpha-\frac{1}{\beta}(1+\varepsilon)-\varepsilon} \quad (168)$$

This establishes the upper bound on $f_{g}^{O}(\alpha)$ in the region $R_2$ (both on $R_2^u$ as well as on $R_2^\ell$).

11. IMPLICATIONS OF THE RESULTS

In this paper we provided the multifractal spectra for one of the most widespread TCP versions of the Internet, i.e., for the TCP CUBIC, which is the default TCP version in the Linux world. We have also compared our results with the results obtained for TCP Reno in [16]. Based on our results the following conclusions can be made from the point of view of Internet traffic theory.
The multifractal spectrum \( f(\alpha) \) can provide a rich characterization of traffic burstiness. Intuitively, \( f(\alpha) \) captures how frequently a value \( \alpha \) is found. Heuristically speaking, \( \alpha \) describes the magnitude of the burst as a power of the time it lasts, on a small time scale. Hence, values for \( \alpha < 1 \) indicate bursty behavior. As a consequence, the values and the shape of \( f(\alpha) \) in the range of \( \alpha < 1 \) have the primary importance for the evaluation of traffic burstiness.

The first conclusion follows from Theorem 3 is that TCP CUBIC traffic is a bursty traffic since \( f(\alpha) > 0 \) for all values where \( \alpha < 1 \). This finding is in line with the analysis of TCP CUBIC traces measured in Internet. The importance of our result is that we have provided the theoretical proof why TCP CUBIC traffic is bursty.

Our second conclusion can be made if we compare the multifractal spectrum of TCP CUBIC with the result obtained for TCP Reno in [16], see Theorem III.4 in [16] as we have carried out in Section 4. From Corollary 5 we see that \( f_{\text{Reno}}(\alpha) > f_{\text{CUBIC}}(\alpha) \) for \( \alpha < \frac{9}{10} \) and \( f_{\text{Reno}}(\alpha) = 3 f_{\text{CUBIC}}(\alpha) \) for \( \alpha < \frac{1}{2} \). Please note that this effect is dramatic from the point of view of burstiness since this difference means that the number of dyadic intervals of size \( \Delta X \) behaves as \( (\Delta X)^{-f(\alpha)} \). As a practical conclusion we can say that the importance of this observation is that we have theoretically proved that TCP CUBIC traffic is less bursty than TCP Reno. It is also a good indication why besides many other reasons (fairness, scalability, etc.) TCP CUBIC has been a good choice for being the default version in the Linux world.

As a performance implication of our results regarding queueing performance of TCP CUBIC traffic we refer to our earlier results where we gave the queue tail asymptotic of a single queueing model with general multifractal input [5].

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