Defect of characters of the symmetric group

Jean-Baptiste Gramain

École Polytechnique Fédérale de Lausanne
Lausanne, Switzerland

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Abstract
Following the work of B. Külshammer, J. B. Olsson and G. R. Robinson on generalized blocks of the symmetric groups, we give a definition for the \(\ell\)-defect of characters of the symmetric group \(S_n\), where \(\ell > 1\) is an arbitrary integer. We prove that the \(\ell\)-defect is given by an analogue of the hook-length formula, and use it to prove, when \(n < \ell^2\), an \(\ell\)-version of the McKay Conjecture in \(S_n\).

1 Introduction

B. Külshammer, J. B. Olsson and G. R. Robinson gave in \cite{6} a definition of generalized blocks for a finite group. Let \(G\) be a finite group, and denote by \(\text{Irr}(G)\) the set of complex irreducible characters of \(G\). Take a union \(\mathcal{C}\) of conjugacy classes of \(G\) containing the identity. Suppose furthermore that \(\mathcal{C}\) is closed, that is, if \(x \in \mathcal{C}\), and if \(y \in G\) generates the same subgroup of \(G\) as \(x\), then \(y \in \mathcal{C}\). For \(\chi, \psi \in \text{Irr}(G)\), we define the \(\mathcal{C}\)-contribution \(\langle \chi, \psi \rangle_{\mathcal{C}}\) of \(\chi\) and \(\psi\) by

\[
\langle \chi, \psi \rangle_{\mathcal{C}} := \frac{1}{|G|} \sum_{g \in \mathcal{C}} \chi(g)\psi(g^{-1}).
\]

The fact that \(\mathcal{C}\) is closed implies that, for any \(\chi, \psi \in \text{Irr}(G)\), \(\langle \chi, \psi \rangle_{\mathcal{C}}\) is a rational number.

We say that \(\chi, \psi \in \text{Irr}(G)\) belong to the same \(\mathcal{C}\)-block of \(G\) if there exists a sequence of irreducible characters \(\chi_1 = \chi, \chi_2, \ldots, \chi_n = \psi\) of \(G\) such that \(\langle \chi_i, \chi_{i+1} \rangle_{\mathcal{C}} \neq 0\) for all \(i \in \{1, \ldots, n-1\}\). The \(\mathcal{C}\)-blocks define a partition of \(\text{Irr}(G)\) (the fact that \(1 \in \mathcal{C}\) ensures that each irreducible character of \(G\) belongs to a \(\mathcal{C}\)-block). If we take \(\mathcal{C}\) to be the set of \(p\)-regular elements of \(G\) (i.e. whose order is not divisible by \(p\)), for some prime \(p\), then the \(\mathcal{C}\)-blocks are just the “ordinary” \(p\)-blocks (cf for example \cite{8}, Theorem 3.19).
Let $CF(G)$ be the set of complex class functions of $G$, and $\langle \ldots \rangle$ be the ordinary scalar product on $CF(G)$. For any $\chi \in \text{Irr}(G)$, we define $\chi^C \in CF(G)$ by letting
\[\chi^C(g) = \begin{cases} 
\chi(g) & \text{if } g \in C \\
0 & \text{otherwise} 
\end{cases} .\]

Then, for $\chi \in \text{Irr}(G)$, we have $\chi^C = \sum_{\psi \in \text{Irr}(G)} \langle \chi^C, \psi \rangle \psi = \sum_{\psi \in \text{Irr}(G)} \langle \chi, \psi \rangle c \psi$. Since $\langle \chi, \psi \rangle_c \in \mathbb{Q}$ for all $\psi \in \text{Irr}(G)$, there exists $d \in \mathbb{N}$ such that $d \chi^C$ is a generalized character of $G$. We call the smallest such positive integer the $C$-defect of $\chi$, and denote it by $d_C(\chi)$.

It is easy to check that $\chi \in \text{Irr}(G)$ has $C$-defect 1 if and only if $\chi$ vanishes outside $C$. This is also equivalent to the fact that $\{\chi\}$ is a $C$-block of $G$.

Writing $1_G$ for the trivial character of $G$, we see that, for any $\chi \in \text{Irr}(G)$, $d_C(1_G) \chi^C = \chi \otimes (d_C(1_G) 1_G^C)$ is a generalized character, so that $d_C(\chi)$ divides $d_C(1_G)$. In particular, $1_G$ has maximal $C$-defect.

Note that, if $C$ is the set of $p$-regular elements of $G$ ($p$ a prime), then, for all $\chi \in \text{Irr}(G)$, we have (cf for example [8], Lemma 3.23) $d_C(\chi) = \left( \frac{|G|}{|\chi|^2} \right)_p = p^{d(\chi)}$, where $d(\chi)$ is the ordinary $p$-defect of $\chi$.

One key notion defined in [6] is that of generalized perfect isometry. Suppose $G$ and $H$ are finite groups, and $C$ and $D$ are closed unions of conjugacy classes of $G$ and $H$ respectively. Take $b$ a union of $C$-blocks of $G$, and $b'$ a union of $D$-blocks of $G$. A generalized perfect isometry between $b$ and $b'$ (with respect to $C$ and $D$) is a bijection with signs between $b$ and $b'$, which furthermore preserves contributions. That is, $I: b \rightarrow b'$ is a bijection such that, for each $\chi \in b$, there is a sign $\varepsilon(\chi)$, and such that
\[\forall \chi, \psi \in b, \quad \langle I(\chi), I(\psi) \rangle_D = \langle \varepsilon(\chi) \chi, \varepsilon(\psi) \psi \rangle_C.\]

In particular, one sees that a generalized perfect isometry $I$ preserves the defect, that is, for all $\chi \in b$, we have $d_C(\chi) = d_D(I(\chi))$.

Note that, if $C$ and $D$ are the set of $p$-regular elements of $G$ and $H$ respectively, then this notion is a bit weaker than that of perfect isometry introduced by M. Broué (cf [1]). If two $p$-blocks $b$ and $b'$ are perfectly isometric in Broué’s sense, then there is a generalized perfect isometry (with respect to $p$-regular elements) between $b$ and $b'$. It is however possible to exhibit generalized perfect isometries in some cases where there is no perfect isometry in Broué’s sense (cf [3]).

Külshammer, Olsson and Robinson defined and studied in [6] the $\ell$-blocks of the symmetric group, where $\ell \geq 2$ is any integer. They did this by taking $C$ to be the set of $\ell$-regular elements, that is which have no cycle (in their canonical cycle decomposition) of length divisible by $\ell$ (in particular, if $\ell$ is a prime $p$, then the $\ell$-blocks are just the $p$-blocks).

In section 2, we find the $\ell$-defect of the characters of the symmetric group $S_n$. It turns out (Theorem [2,6]) that it is given by an analogue of the hook-length formula (for the degree of a character). In section 3, we then use this to prove, when $n < \ell^2$, an $\ell$-analogue of the McKay Conjecture in $S_n$ (Theorem [3,4]).
2 Hook-length formula

2.1 $\ell$-blocks of the symmetric group

Take two integers $1 \leq \ell \leq n$, and consider the symmetric group $\mathfrak{S}_n$ on $n$ letters. The conjugacy classes and irreducible complex characters of $\mathfrak{S}_n$ are parametrized by the set $\{\lambda \vdash n\}$ of partitions of $n$. We write $\text{Irr}(\mathfrak{S}_n) = \{\chi_{\lambda}, \lambda \vdash n\}$. An element of $\mathfrak{S}_n$ is said to be $\ell$-regular if none of its cycles has length divisible by $\ell$. We let $\mathcal{C}$ be the set of $\ell$-regular elements of $\mathfrak{S}_n$. The $\mathcal{C}$-blocks of $\mathfrak{S}_n$ are called $\ell$-blocks, and they satisfy the following

**Theorem 2.1 (Generalized Nakayama Conjecture).** ([10], Theorem 5.13) Two characters $\chi_\lambda, \chi_\mu \in \text{Irr}(\mathfrak{S}_n)$ belong to the same $\ell$-block if and only if $\lambda$ and $\mu$ have the same $\ell$-core.

The proof of this goes as follows. If $\langle \chi_\lambda; \chi_\mu \rangle \neq 0$, then an induction argument using the Murnaghan-Nakayama Rule shows that $\lambda$ and $\mu$ must have the same $\ell$-core. In particular, the partitions labeling the characters in an $\ell$-block all have the same $\ell$-weight, and we can talk about the $\ell$-weight of an $\ell$-block.

Conversely, let $B$ be the set of irreducible characters of $\mathfrak{S}_n$ labeled by those partitions of $n$ which have a given $\ell$-core, $\gamma$, say, and $\ell$-weight $w$. It is a well known combinatorial fact (cf for example [5], Theorem 2.7.30) that the characters in $B$ are parametrized by the $\ell$-quotients, which can be seen as the set of $\ell$-tuple of partitions of $w$. For $\chi_\lambda \in B$, the quotient $\beta_\lambda$ is a sequence $(\lambda^{(1)}, \ldots, \lambda^{(\ell)})$ such that, for each $1 \leq i \leq \ell$, $\lambda^{(i)}$ is a partition of some $k_i$, $0 \leq k_i \leq w$, and $\sum_{i=1}^\ell k_i = w$ (the quotient $\beta_\lambda$ “stores” the information about how to remove $w$ $\ell$-hooks from $\lambda$ to get $\gamma$). We write $\beta_\lambda \vdash w$. To prove that $B$ is an $\ell$-block of $\mathfrak{S}_n$, Kühlhammer, Olsson and Robinson use a generalized perfect isometry between $B$ and the wreath product $\mathbb{Z}_\ell \wr \mathfrak{S}_w$ (where $\mathbb{Z}_\ell$ denotes a cyclic group of order $\ell$).

The conjugacy classes of $\mathbb{Z}_\ell \wr \mathfrak{S}_w$ are parametrized by the $\ell$-tuples of partitions of $w$ as follows (cf [5], Theorem 4.2.8). Write $\mathbb{Z}_\ell = \{g_1, \ldots, g_\ell\}$ the cyclic group of order $\ell$. The elements of the wreath product $\mathbb{Z}_\ell \wr \mathfrak{S}_w$ are of the form $(h; \sigma) = (h_1, \ldots, h_w; \sigma)$, with $h_1, \ldots, h_w \in \mathbb{Z}_\ell$ and $\sigma \in \mathfrak{S}_w$. For any such element, and for any $k$-cycle $\kappa = (j, j+\ell, \ldots, j+k\ell-1)$ in $\sigma$, we define the cycle product of $(h; \sigma)$ and $\kappa$ by

$$g((h; \sigma), \kappa) = h_jh_{j+k\ell-2}\ldots h_{j+(k-1)\ell} \in \mathbb{Z}_\ell.$$

If $\sigma$ has cycle structure $\pi$ say, then we form $\ell$ partitions $(\pi_1, \ldots, \pi_\ell)$ from $\pi$ as follows: any cycle $\kappa$ in $\pi$ gives a cycle of the same length in $\pi_i$ if $g((h; \sigma), \kappa) = g_i$.

The resulting $\ell$-tuple of partitions of $w$ describes the cycle structure of $(h; \sigma)$, and two elements of $\mathbb{Z}_\ell \wr \mathfrak{S}_w$ are conjugate if and only if they have the same cycle structure. An element of $\mathbb{Z}_\ell \wr \mathfrak{S}_w$ is said to be regular if it has no cycle product equal to 1.

The irreducible characters of $\mathbb{Z}_\ell \wr \mathfrak{S}_w$ are also canonically parametrized by the $\ell$-tuples of partitions of $w$ in the following way. Write $\text{Irr}(\mathbb{Z}_\ell) = \{\alpha_1, \ldots, \alpha_\ell\}$, and take $\beta_\lambda = (\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \vdash w$, with $\lambda^{(i)} \vdash k_i$ as above ($1 \leq i \leq \ell$). The irreducible character $\alpha_{k_1}^{\lambda_1} \otimes \cdots \otimes \alpha_{k_\ell}^{\lambda_\ell}$ of the base group $\mathbb{Z}_\ell^w$ can be extended in a natural way to its inertia subgroup $(\mathbb{Z}_\ell \wr \mathfrak{S}_{k_1}) \times \cdots \times (\mathbb{Z}_\ell \wr \mathfrak{S}_{k_\ell})$, giving
the irreducible character $\prod_{i=1}^\ell \alpha_i^k_i$. The tensor product $\prod_{i=1}^\ell \alpha_i^k_i \otimes \chi_{\lambda(i)}$ is an irreducible character of $(\mathbb{Z}_\ell \wr S_{k_1}) \times \cdots \times (\mathbb{Z}_\ell \wr S_{k_\ell})$ which extends $\prod_{i=1}^\ell \alpha_i^k_i$, and it remains irreducible when induced to $\mathbb{Z}_\ell \wr S_w$. We denote by $\chi_{\lambda(i)}$ this induced character. Furthermore, any irreducible character of $\mathbb{Z}_\ell \wr S_w$ can be obtained in this way.

In [6], the authors show that the map $\chi_{\lambda} \mapsto \chi_{\lambda(i)}$ is a generalized perfect isometry between $B$ and $\text{Irr}(\mathbb{Z}_\ell \wr S_w)$, with respect to $\ell$-regular elements of $S_n$ and regular elements of $\mathbb{Z}_\ell \wr S_w$.

On the other hand, they show that, writing $\text{reg}$ for the set of regular elements of $\mathbb{Z}_\ell \wr S_w$, we have, for all $\chi \in \text{Irr}(\mathbb{Z}_\ell \wr S_w)$,

$$\mathbb{Z} \ni \ell^w w! \langle \chi, 1_{\mathbb{Z}_\ell \wr S_w}\rangle_{\text{reg}} \equiv (-1)^w \text{ (mod } \ell),$$

where $1_{\mathbb{Z}_\ell \wr S_w}$ is the trivial character of $\mathbb{Z}_\ell \wr S_w$. In particular, $\langle \chi, 1_{\mathbb{Z}_\ell \wr S_w}\rangle_{\text{reg}} \neq 0$.

Using the generalized perfect isometry we described above, this implies that there exists a character $\chi_{\lambda} \in B$ such that, for all $\chi_{\mu} \in B$, we have $\langle \chi_{\lambda}, \chi_{\mu}\rangle_C \neq 0$, where $C$ is the set of $\ell$-regular elements of $S_n$. In particular, all the characters in $B$ belong to the same $\ell$-block of $S_n$, which ends the proof of Theorem 2.1.

### 2.2 $\ell$-defect of characters

Using the ingredients in the proof of Theorem 2.1, we can now compute explicitly the $\ell$-defects of the irreducible characters of $S_n$ (that is, their $C$-defect, where $C$ is the set of $\ell$-regular elements of $S_n$).

As we remarked earlier, if $\lambda$ is a partition of $n$ of $\ell$-weight $w$, then, because of the generalized perfect isometry we described above, the $\ell$-defect $d_{\ell}(\chi_{\lambda})$ is the same as the $\text{reg}$-defect $d_{\text{reg}}(\chi_{\beta_{\lambda}})$ of $\chi_{\beta_{\lambda}} \in \text{Irr}(\mathbb{Z}_\ell \wr S_w)$, where $\beta_{\lambda}$ is the $C$-quotient of $\lambda$. It is in fact these $\text{reg}$-defects we will compute.

First note that, if $w = 0$, then $\lambda$ is its own $C$-core, so that $\chi_{\lambda}$ is alone in its $\ell$-block, and $d_{\ell}(\chi_{\lambda}) = 1$. We therefore now fix $w \geq 1$.

We write $\pi$ the set of primes dividing $\ell$. Every positive integer $m$ can be factorized uniquely as $m = m_{\pi} m_{\pi'}$, where every prime factor of $m_{\pi}$ belongs to $\pi$ and no prime factor of $m_{\pi'}$ is contained in $\pi$. We call $m_{\pi}$ the $\pi$-part of $m$.

Using results of Donkin (cf [2]) and equality (1), Külschammer, Olsson and Robinson proved the following

**Theorem 2.2.** ([6], Theorem 6.2) The $\text{reg}$-defect of the trivial character of $\mathbb{Z}_\ell \wr S_w$ is $\ell^w w!_{\pi}$.

In particular, since $1_{\mathbb{Z}_\ell \wr S_w}$ has maximal $\text{reg}$-defect, we see that, for any $\chi \in \text{Irr}(\mathbb{Z}_\ell \wr S_w)$, $d_{\text{reg}}(\chi)$ is a $\pi$-number.

We can now compute the $\text{reg}$-defect of any irreducible character $\chi$ of $\mathbb{Z}_\ell \wr S_w$. It turns out that it is sufficient to know the $\text{reg}$-contribution of $\chi$ with the trivial character, and this is given by (1). We have the following

**Proposition 2.3.** Take any integers $\ell \geq 2$ and $w \geq 1$. Then, for any $\chi \in \text{Irr}(\mathbb{Z}_\ell \wr S_w)$, we have $d_{\text{reg}}(\chi) = \frac{\ell^w w!_{\pi}}{\chi(1)_{\pi}}$. 

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Proof. Take $\chi \in \text{Irr}(\mathbb{Z}_\ell \wr \mathfrak{S}_w)$. Recall that, by (1),

$$
\mathbb{Z} \ni \frac{\ell^w w!}{\chi(1)} \langle \chi, 1 \rangle_{\text{reg}} \equiv (-1)^w \pmod{\ell}.
$$

Now $d_{\text{reg}}(\chi)$ is a $\pi$-number, so that $\langle \chi, 1 \rangle_{\text{reg}}$ is a rational whose (reduced) denominator is a $\pi$-number. This implies that $\frac{\ell^w w!}{\chi(1) \pi} \langle \chi, 1 \rangle_{\text{reg}} \in \mathbb{Z}$.

Furthermore, from (1), we also deduce that, for each $p \in \pi$,

$$
\frac{\ell^w w!}{\chi(1)} \langle \chi, 1 \rangle_{\text{reg}} \not\equiv 0 \pmod{p}.
$$

Thus, for any $p \in \pi$, $\frac{\ell^w w!}{\chi(1)} \langle \chi, 1 \rangle_{\text{reg}} \not\equiv 0 \pmod{p}$. Hence $\frac{\ell^w w!}{\chi(1) \pi}$ is the smallest positive integer $d$ such that $d \langle \chi, 1 \rangle_{\text{reg}} \in \mathbb{Z}$. This implies that $\frac{\ell^w w!}{\chi(1) \pi}$ divides $d_{\text{reg}}(\chi)$ (indeed, by definition, $d_{\text{reg}}(\chi) \langle \chi, 1 \rangle_{\text{reg}} \in \mathbb{Z}$, and $d_{\text{reg}}(\chi)$ is a $\pi$-number).

Now, conversely, if $\psi \in \text{Irr}(\mathbb{Z}_\ell \wr \mathfrak{S}_w)$, then $\langle \chi, \psi \rangle_{\text{reg}} \in \mathbb{Q}$, so also (since $\chi(1)$ divides $|\mathbb{Z}_\ell \wr \mathfrak{S}_w| = \ell^w w!$) $\frac{\ell^w w!}{\chi(1)} \langle \chi, \psi \rangle_{\text{reg}} \in \mathbb{Q}$. However,

$$
\frac{\ell^w w!}{\chi(1)} \langle \chi, \psi \rangle_{\text{reg}} = -\frac{\ell^w w!}{\chi(1) \pi} \sum_{g \in \text{reg}/\sim} K_g \chi(g) \psi(g^{-1})
$$

(where the sum is taken over representatives for the regular classes, and, for $g$ such a representative, $K_g$ is the size of the conjugacy class of $g$). And, for each $g$ in the sum, $\frac{K_g \chi(g)}{\chi(1) \pi}$ and $\psi(g^{-1})$ are both algebraic integers. Hence $\frac{\ell^w w!}{\chi(1)} \langle \chi, \psi \rangle_{\text{reg}}$ is also an algebraic integer, and thus an integer. Thus

$$
\forall \psi \in \text{Irr}(\mathbb{Z}_\ell \wr \mathfrak{S}_w), \quad \frac{\ell^w w!}{\chi(1)} \langle \chi, \psi \rangle_{\text{reg}} \in \mathbb{Z}
$$

and this implies that $d_{\text{reg}}(\chi)$ divides $\frac{\ell^w w!}{\chi(1) \pi}$, and, $d_{\text{reg}}(\chi)$ being a $\pi$-number, $d_{\text{reg}}(\chi)$ divides $\frac{\ell^w w!}{\chi(1) \pi}$. Hence we finally get $d_{\text{reg}}(\chi) = \frac{\ell^w w!}{\chi(1) \pi}$.

We want to express the $\ell$-defect of a character in terms of hook-lengths. For any $\lambda \vdash n$, we write $\mathcal{H}(\lambda)$ for the set of hooks in $\lambda$, and $\mathcal{H}_\ell(\lambda)$ for the set of hooks in $\lambda$ whose length is divisible by $\ell$. Similarly, if $\beta_\lambda = (\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \vdash w$, we define a hook in $\beta_\lambda$ to be a hook in any of the $\lambda^{(i)}$’s, and write $\mathcal{H}(\beta_\lambda)$ for the set of hooks in $\beta_\lambda$. Finally, for any hook $h$ (in a partition or a tuple of partitions), we write $|h|$ for the length of $h$.

We will use the following classical results about hooks (cf for example [5], §2.3 and §2.7)

**Theorem 2.4.** Let $n \geq \ell \geq 2$ be any two integers, and let $\lambda$ be any partition of $n$. Then

(i) (Hook-Length Formula, [5], Theorem 2.3.21) We have $|\mathfrak{S}_n|_{\chi(1)} = \prod_{h \in \mathcal{H}(\lambda)} |h|$

(ii) ([5], 2.7.40) if $\lambda$ has $\ell$-weight $w$, then $|\mathcal{H}_\ell(\lambda)| = w$

(iii) ([5], Lemma 2.7.13 and Theorem 2.7.16) if $\beta_\lambda$ is the $\ell$-quotient of $\lambda$, then $\{ |h|, h \in \mathcal{H}_\ell(\lambda) \} = \{ 0, \ldots, \ell |h'|, h' \in \mathcal{H}(\beta_\lambda) \}$.
We can now establish the following

**Proposition 2.5.** If \( n \geq \ell \geq 2 \) are any two integers, \( \pi \) is the set of primes dividing \( \ell \), and \( \lambda \vdash n \) has \( \ell \)-weight \( w \neq 0 \) and \( \ell \)-quotient \( \beta \), then

\[
\frac{\ell^w(w!)}{\chi_{\beta}(1)} = \prod_{h \in H_{\beta}(\lambda)} |h|_{\pi},
\]

Proof. Write \( \beta = (\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \), where \( \lambda^{(i)} \vdash k_i \) for \( 1 \leq i \leq \ell \). First note that, by construction of \( \chi_{\beta}(1) \), and since the irreducible characters of \( \mathbb{Z}_\ell \) all have degree 1, we have

\[
\chi_{\beta}(1) = \frac{\ell^w w!}{\prod_{h \in H_{\beta}(\lambda)} |h|} \chi_{\lambda^{(1)}(1)} \cdots \chi_{\lambda^{(\ell)}(1)}.
\]

Thus, by the Hook-Length Formula (Theorem 2.4 (i)),

\[
\chi_{\beta}(1) = \frac{w!}{\prod_{h \in H_{\beta}(\lambda)} |h|} \text{ and } \frac{\ell^w |\mathbb{S}_w|}{\chi_{\beta}(1)} = \ell^w \prod_{h \in H_{\beta}(\lambda)} |h|.
\]

We therefore get

\[
\frac{\ell^w(w!)}{\chi_{\beta}(1)} = \prod_{h \in H_{\beta}(\lambda)} |h|_{\pi} = \ell^w \prod_{h \in H_{\beta}(\lambda)} |h|_{\pi},
\]

Now, by Theorem 2.4 (ii) and (iii), we have \( |H_{\beta}(\lambda)| = w \), so that \( \ell^w \prod_{h \in H_{\beta}(\lambda)} |h| = \prod_{h \in H_{\beta}(\lambda)} \ell |h| \), and, by Theorem 2.4 (iii), \( \prod_{h \in H_{\beta}(\lambda)} \ell |h| = \prod_{h \in H_{\beta}(\lambda)} |h| \). Taking \( \pi \)-parts, we obtain \( \ell^w (w!) = \prod_{h \in H_{\beta}(\lambda)} |h|_{\pi} \), as announced. \( \square \)

Combining Propositions 2.3 and 2.5, we finally get

**Theorem 2.6.** Let \( 2 \leq \ell \leq n \) be any two integers, and let \( B \) be an \( \ell \)-block of \( \mathbb{S}_n \) of weight \( w \). Then

(i) If \( w = 0 \), then \( B = \{ \chi_{\lambda} \} \) for some partition \( \lambda \) of \( n \), and \( d_\ell(\chi_{\lambda}) = 1 \).

(ii) If \( w > 0 \), and if \( \chi_{\lambda} \in B \), then \( d_\ell(\chi_{\lambda}) = \prod_{h \in H_{\beta}(\lambda)} |h|_{\pi} \), where \( \pi \) is the set of primes dividing \( \ell \) (that is, \( d_\ell(\chi_{\lambda}) \) is the \( \pi \)-part of the product of the hook-lengths divisible by \( \ell \) in \( \lambda \)).

3 McKay Conjecture

3.1 McKay Conjecture, generalization

In this section, we want to study an \( \ell \)-analogue of the following

**Conjecture 3.1 (McKay).** Let \( G \) be a finite group, \( p \) be a prime, and \( P \) be a Sylow \( p \)-subgroup of \( G \). Then the numbers of irreducible complex characters whose degree is not divisible by \( p \) are the same for \( G \) and \( N_G(P) \).
First note that the McKay Conjecture was proved by J. B. Olsson for the symmetric group (cf [9]). In order to generalize this to an arbitrary integer \( \ell \), we will use the results of [4], which we summarize here. Let \( 2 \leq \ell \leq n \) be integers. Suppose furthermore that \( n < \ell^2 \), and write \( n = \ell w + r \), with \( 0 \leq w, r < \ell \).

We define a Sylow \( \ell \)-subgroup of \( \mathfrak{S}_n \) to be any subgroup of \( \mathfrak{S}_n \) generated by \( w \) disjoint \( \ell \)-cycles. In particular, if \( \ell \) is a prime \( p \), then the Sylow \( \ell \)-subgroups of \( \mathfrak{S}_n \) are just its Sylow \( p \)-subgroups. Then any two Sylow \( \ell \)-subgroups of \( \mathfrak{S}_n \) are conjugate, and they are Abelian. Let \( \mathcal{L} \) be a Sylow \( \ell \)-subgroup of \( \mathfrak{S}_n \). In [4], a notion of \( \ell \)-regular element is given, which coincide with the notion of \( p \)-regular element if \( \ell \) is a prime \( p \). Using this, one can construct the \( \ell \)-blocks of \( N_{\mathfrak{S}_n}(\mathcal{L}) \), and show that they satisfy an analogue of Broué’s Abelian Defect Conjecture (cf [4], Theorem 4.1). We will show that, still in the case where \( n < \ell^2 \), an analogue of the McKay Conjecture also holds. However, if we just replace \( \ell \) by any integer \( \ell \), and consider irreducible characters of degree not divisible by \( \ell \), or even coprime to \( \ell \), then the numbers differ in \( \mathfrak{S}_n \) and \( N_{\mathfrak{S}_n}(\mathcal{L}) \). Instead, we will use the notion of \( \ell \)-defect, and prove that the numbers of irreducible characters of maximal \( \ell \)-defect are the same in \( \mathfrak{S}_n \) and \( N_{\mathfrak{S}_n}(\mathcal{L}) \) (note that, if \( \ell \) is a prime, then both statements coincide).

### 3.2 Defect and weight

In order to study characters of \( \mathfrak{S}_n \) of maximal \( \ell \)-defect, we need the following result, which tells us where to look for them.

**Proposition 3.2.** Let \( \ell \geq 2 \) and \( 0 \leq w, r < \ell \) be any integers, and let \( \lambda \) be a partition of \( n = \ell w + r \). If \( \chi_\lambda \in \text{Irr}(\mathfrak{S}_n) \) has maximal \( \ell \)-defect, then \( \lambda \) has (maximal) \( \ell \)-weight \( w \).

**Proof.** First note that, if \( \ell \) is a prime, then this can be proved in a purely arithmetic way (cf [4]). This doesn’t seem to be the case when \( \ell \) is no longer a prime, and we will use the abacus instead. For a complete description of the abacus, we refer to [4], §2.7 (note however that the abacus we use here is the horizontal mirror image of that described by James and Kerber).

Suppose, for a contradiction, that \( \lambda \) has \( \ell \)-weight \( v < w \). By the previous section, the \( \ell \)-defect of \( \chi_\lambda \) is the \( \pi \)-part of the product of the hook-lengths divisible by \( \ell \) in \( \lambda \). Now these are visible on the \( \ell \)-abacus of \( \lambda \). This has \( \ell \) runners, and a hook of length \( k\ell \) \((k \geq 1)\) corresponds to a bead sitting, on a runner, \( k \) places above an empty spot. In particular, the \((\ell)\)-hooks (i.e. those whose length is divisible by \( \ell \)) in \( \lambda \) are stored on at most \( v \) runners. To establish the result, we will construct a partition \( \mu \) of \( n \) of weight \( w \), and such that \( d_{\ell}(\chi_\mu) > d_{\ell}(\chi_\lambda) \).

Start with the \( \ell \)-abacus of any partition \( \nu \) of \( r \). On the (at most) \( v \) runners used by \( \lambda \), take some beads up to encode the same \((\ell)\)-hooks as for \( \lambda \). Then, on \( w-v \) of the (at least) \( \ell - v > w-v \) remaining runners, take the highest bead one place up. The resulting abacus then corresponds to a partition of \( n = r + \ell w = r + \ell v + \ell(w-v) \), and we see that \( d_{\ell}(\chi_\mu) = t^{w-v}d_{\ell}(\chi_\lambda) \) (indeed, the \((\ell)\)-hooks in \( \mu \) are precisely those in \( \lambda \), together with \( w-v \) hooks of length \( \ell \)). This proves the result.

\[ \square \]
3.3 Generalized Perfect Isometry

We describe here the analogue of Broué’s Abelian Defect Conjecture given in [4] (Theorem 4.1). We take any integers $\ell \geq 2$ and $0 \leq w, r < \ell$, and $G = S_{tw+r}$. We take an Abelian Sylow $\ell$-subgroup $L$ of $G$; that is, $L \cong \mathbb{Z}_\ell^w$ is generated by $w$ disjoint $\ell$-cycles. Then $L$ is a natural subgroup of $S_{tw}$, and we have $N_G(L) \cong N_{S_{tw}}(L) \times S_r$ and $\text{Irr}(N_G(L)) = \text{Irr}(N_{S_{tw}}(L)) \otimes \text{Irr}(S_r)$. Now $N_{S_{tw}}(L) \cong N \wr S_w = N_{S_w}(L) \wr S_w$, where $L = \langle \pi \rangle \cong \mathbb{Z}_\ell$ is a (subgroup of $S_\ell$) generated by a single $\ell$-cycle. As in the sketch of the proof of Theorem 2.4, we see that the conjugacy classes and irreducible characters of $N_{S_{tw}}(L)$ are parametrized by the $s$-tuples of partitions of $w$, where $s$ is the number of conjugacy classes of $N$. Among these, there is a unique conjugacy class of $\ell$-cycles, for which we take representative $\pi$. We take representatives $\{g_1 = \pi, g_2, \ldots, g_s\}$ for the conjugacy classes of $N$. Considering as $\ell$-regular any element of $N$ not conjugate to the $\ell$-cycle $\pi$, we can construct the $\ell$-blocks of $N$, and show that the principal $\ell$-block contains $\ell$ characters, which we label $\psi_1, \ldots, \psi_\ell$, and that each of the remaining $s - \ell$ characters, labeled $\psi_{s+1}, \ldots, \psi_s$, is in its $\ell$-block (cf [4], Section 2). Using the construction presented here, we label the conjugacy classes and irreducible characters of $N \wr S_w$ by the $s$-tuples of partitions of $w$. An element of $N \wr S_w$ of cycle type $(\pi_1, \ldots, \pi_s) \vdash w$ is called $\ell$-regular if $\pi_1 = \emptyset$ (and $\ell$-singular otherwise). Then one shows that the $\ell$-blocks of $N \wr S_w$ are the principal $\ell$-block, $b_0 = \{\chi^{a_1}, \alpha = (\alpha_1, \ldots, \alpha_\ell, \emptyset, \ldots, \emptyset) \vdash w\}$, and blocks of size $1$, $\{\chi^0\}$, whenever $\alpha \vdash w$ is such that $\alpha_k \neq \emptyset$ for some $\ell < k \leq s$ ([4], Theorem 3.7 and Corollary 3.11).

Finally, an element of $N_G(L) \cong N_{S_{tw}}(L) \times S_r$ is said to be $\ell$-regular if its $N_{S_{tw}}(L)$-part is $\ell$-regular in the above sense (so that, if $\ell$ is a prime $p$, then the notions of $\ell$-regular and $p$-regular coincide). Then, we can summarize the results of [4] as follows:

**Theorem 3.3** ([4], Theorem 4.1). Let the notation be as above. Then any $\ell$-block of $N_G(L)$ has size 1 or belongs to $\{b_0 \otimes \{\psi\}, \psi \in \text{Irr}(S_r)\}$. Furthermore, for any $\psi \in \text{Irr}(S_r)$, there is a generalized perfect isometry (with respect to $\ell$-regular elements) between $b_0 \otimes \{\psi\}$ and $B_\psi$, where $B_\psi$ is the $\ell$-block of $S_{tw+r}$ consisting of the irreducible characters labeled by partitions with $\ell$-core $\psi$.

(Note that any partition of $r$ does appear as $\ell$-core of a partition of $\ell w + r$ (for example, if $\gamma \vdash r$, then $\gamma$ is the $\ell$-core of $(\gamma, 1^{\ell w}) \vdash \ell w + r$).)

3.4 Analogues of the McKay Conjecture

We can now give the analogue of the McKay Conjecture we announced. Let, as before, $\ell \geq 2$ and $0 \leq w, r < \ell$ be integers, $n = \ell w + r$, and $L$ be an Abelian Sylow $\ell$-subgroup of $S_n$. By Proposition 3.2, any irreducible character of $S_n$ of maximal $\ell$- defect has (maximal) $\ell$-weight $w$, hence belongs to one of the $B_\psi$’s, $\psi \in \text{Irr}(S_r)$. Since any generalized perfect isometry preserves the defect, Theorem 3.3 provides a bijection between the sets of irreducible characters of maximal $\ell$-defect and $\ell$-weight $w$ of $S_n$ and of characters of maximal $\ell$-defect in $N_{S_n}(L)$. We therefore obtain

**Theorem 3.4.** With the above notations, the numbers of irreducible characters of maximal $\ell$-defect are the same in $S_n$ and $N_{S_n}(L)$. 

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Remark: furthermore, we have an explicit bijection, essentially given by taking \( \ell \)-quotients of partitions.

In fact, Theorem 3.3 gives something a bit stronger, namely

**Theorem 3.5.** For any \( \ell \)-defect \( \delta \neq 1 \), there is a bijection between the set of irreducible characters of \( S_n \) of \( \ell \)-weight \( w \) and \( \ell \)-defect \( \delta \) and the set of irreducible characters of \( N_{S_n}(L) \) of \( \ell \)-defect \( \delta \).

Now, McKay’s Conjecture is stated (and, in the case of symmetric groups, proved) without any hypothesis on the Sylow \( p \)-subgroups. One would therefore want to generalize the above results to the case where \( n \geq \ell^2 \). Examples seem to indicate that such analogues do indeed hold in this case, and that a bijection is given by taking, not only the \( \ell \)-quotient of a partition, but its \( \ell \)-tower (cf [9]).

In order to prove these results, one would first need to generalize Proposition 3.2 showing that, for any \( 2 \leq \ell \leq n \), if \( \chi_{\lambda} \in \text{Irr}(S_n) \) has maximal \( \ell \)-defect, then \( \lambda \) has maximal \( \ell \)-weight, but also maximal \( \ell^2 \)-weight, maximal \( \ell^3 \)-weight, ... . If \( \ell \) is a prime, then this is known to be true (cf [7]). However, it seems hard to prove in general, even when \( n = \ell^2 \). The particular case where \( \ell \) is squarefree is much easier.

Also, one would need to generalize the results of [4], while making sure that, when \( \ell \) is a prime \( p \), the notions of \( \ell \)-regular and \( p \)-regular elements still coincide.

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Jean-Baptiste Gramain
EPFL
IGAT
Bâtiment de Chimie (BCH)
CH-1015 Lausanne
Switzerland

jean-baptiste.gramain@epfl.ch