Geometric phase of very slow neutrons

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The geometric phase (GP) acquired by a neutron passing through a uniform magnetic field elucidates a subtle interplay between its spatial and spin degrees of freedom. In the standard setup using thermal neutrons, the kinetic energy is much larger than the typical Zeeman split. This causes the spin to undergo nearly perfect precession around the axis of the magnetic field and the GP becomes a function only of the corresponding cone angle. Here, we perform a plane wave analysis of the GP of very slow neutrons, for which the precession feature breaks down. Purely quantum-mechanical matter wave effects, such as resonance, reflection, and tunneling, become relevant for the behavior of the GP in this low energy scattering regime.

I. INTRODUCTION

When a spin-polarized neutron passes through a uniform magnetic field, it acquires a measurable phase that depends only on the path taken by the spin. This geometric phase (GP) is traditionally calculated under the assumption that the neutron moves at sufficiently large speed so that the effect of the magnetic field can be treated as a small perturbation to the free motion [1]. In this ‘high-speed’ case, the spin performs pure precession around the direction of the magnetic field; for one such precession cycle, the GP becomes [2]

$$\gamma = -\pi(1 - \cos \theta),$$

(1)

where $\theta$ is the angle between the spin and the magnetic field. This result has been confirmed and examined experimentally in the past [3–5].

Here, we ask: how is the GP modified if the neutron moves at very low speed, i.e., when its kinetic energy is comparable to the Zeeman split? In this case, the magnetic field cannot be treated as a small perturbation and the neutron spin no longer performs pure precession. The situation must be treated as a scattering problem, in which the spin components along the direction of the magnetic field are scattered differently, causing a speed dependent spin path.

We perform a plane wave analysis of the spin dependent scattering of the neutron through the magnetic field region. We derive results for the GP in different energy regimes and discuss the feasibility of experimental parameters needed to see measurable deviations from the ideal GP in Eq. (1). We examine the influence of resonance, reflection, and tunneling effects on the GP; purely quantum-mechanical matter wave effects that become relevant in this low energy scattering regime.

II. MODEL

The basic setup is shown in Fig. 1. The linear motion of the neutron is described by the Hamiltonian

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \mu \cdot b.$$  

(2)

where

$$\kappa_{\pm} = \sqrt{\frac{2m(E \mp V_0)}{\hbar^2}},$$

$$k = \frac{\sqrt{2mE}}{\hbar},$$

$$\mathbf{b} = b_0 \mathbf{e}_z,$$

$$V_0 = \mu b_0,$$

FIG. 1: (Color online) Setup for realizing a nontrivial GP. Neutrons with energy $E$ enter a homogeneous magnetic field region of width $L$. The spin up (down) sees a potential barrier (well) due to the Zeeman interaction. This results in the wave vectors $\kappa_{\pm}$ for the two spin components in the magnetic field region. Here, $2V_0 = \mu b_0$ is the Zeeman split, $\mu$ being the magnetic moment of the neutron. The difference $\kappa_+ - \kappa_-$ causes spin precession and a nontrivial GP.

Here, $m = 1.675 \times 10^{-27}$ kg and $\mu = \mu_R \mathbf{S}$ with $\mu = 9.662 \times 10^{-27}$ J/T are the mass and magnetic moment, respectively, of the neutron. $\mathbf{S} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of the standard Pauli operators. We further introduce the spin basis $|\pm\rangle_z$ such that $\sigma_z|\pm\rangle_z = \pm|\pm\rangle_z$. The magnetic field is taken to be

$$\mathbf{b} = \begin{cases} b_0 \mathbf{e}_z, & 0 \leq x \leq L, \\ 0, & \text{otherwise}, \end{cases}$$

(3)

which implies a spin dependent potential $V(x) = V_0 \sigma_z$, $0 \leq x \leq L$, and $V(x) = 0$ otherwise, where $2V_0 = \mu b_0$ is the Zeeman split between the two spin states $|\pm\rangle_z$. The coordinate system is chosen so that $V_0 > 0$. We assume the kinetic en-

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ergy of the incoming neutron is $E$. The behavior of the GP is essentially determined by the ratio

$$\frac{V_0}{E} = \frac{\mu b_0}{mv^2},$$  \hspace{1cm} (4)$$

the ideal case being $\frac{V_0}{E} \ll 1$. For example, a typical triple-Laue experiment utilizes neutrons with speed $v \sim 2000$ m/s and magnetic fields $b_0 \sim 10^{-2}$ T, yielding a ratio $\frac{V_0}{E} \sim 10^{-8}$, which is well within this ideal regime.

The spin evolution can be treated as a one-dimensional scattering problem. Let

$$|\psi\rangle = c_+|+\rangle_z + c_-|\rangle_z = \sum_{l=\pm} c_l|l\rangle_z,$$

be the normalized spin state at $x = 0$, which is where the neutron enters the magnetic field region. While the $|+\rangle_z$ state sees the potential barrier $V_0$ when entering the region, the $|-\rangle_z$ state sees the potential well $-V_0$. We can solve for these two cases in terms of the wave numbers

$$\kappa_{\pm} = \sqrt{\frac{2m(E \mp V_0)}{\hbar^2}} \equiv \sqrt{k^2 \mp k_0^2},$$ \hspace{1cm} (6)$$

$k$ and $k_0$ being determined by the speed of the incoming neutron and the Zeeman split between its two spin states, respectively. We can write the spatial wave function for $x \leq 0$ as $\psi_+(x) = e^{ikx} + \sqrt{r_+}e^{ikx}e^{-ikx}$ with $r_+$ the reflection probability and $\delta_+$ the scattering phase shift acquired upon reflection. We further write $\psi_+(x) = A_+e^{ik_{\pm}x} + B_+e^{-ik_{\pm}x}$ and $\psi_-(x) = C_+e^{ik(x-L)}$ on $0 \leq x \leq L$ and $x \geq L$, respectively. The matching conditions at $x = 0$ and $x = L$ yield

$$\sqrt{r_+}e^{i\delta_+} = \frac{(k^2 - k_{\pm}^2) \sin \kappa_+ L}{(k^2 + k_{\pm}^2) \sin \kappa_+ L + 2i k\kappa_+ \cos \kappa_+ L}$$ \hspace{1cm} (7)$$

and

$$\frac{B_+}{A_+} = e^{2ik_{\pm}L} \left( \kappa_{\pm} - k \right).$$ \hspace{1cm} (8)$$

The neutron spin can be followed through the setup:

(i) $x \leq 0$:

$$|\phi^{(i)}(x)\rangle = \sum_{l=\pm} \left( \frac{e^{ikx} + \sqrt{r_+}e^{ikx}e^{-ikx}}{1 + \sqrt{r_+}e^{ikx}e^{-ikx}} \right) c_l|l\rangle_z,$$ \hspace{1cm} (9)$$

where we have normalized the spin components so as to satisfy $|\phi^{(i)}(0)| = |\psi\rangle$.

(ii) $0 \leq x \leq L$:

$$|\phi^{(ii)}(x)\rangle = \sum_{l=\pm} \left( \frac{k_+ \cos \kappa_+ (x-L) + ik \sin \kappa_+ (x-L)}{k_+ \cos \kappa_+ L - ik \sin \kappa_+ L} \right) c_l|l\rangle_z,$$ \hspace{1cm} (10)$$

again with $|\phi^{(ii)}(0)| = |\psi\rangle$.

(iii) $x \geq L$:

$$|\phi^{(iii)}(x)\rangle = e^{ik(x-L)} \sum_{l=\pm} \frac{k_+}{k_+ \cos \kappa_+ L - ik \sin \kappa_+ L} c_l|l\rangle_z.$$ \hspace{1cm} (11)$$

Since the $x$ dependence is fully confined to the overall phase factor $e^{ik(x-L)}$, the GP vanishes in this region for all choices of Zeeman split and speed of the incoming neutron.

To cover both cyclic and noncyclic evolution of the neutron spin, we use the open path GP [6]. This phase takes the form

$$\gamma = \arg \langle \phi(x_0)|\phi(x_1)\rangle + i \int_{x_0}^{x_1} \left( \frac{d}{dx} \frac{\langle \phi(x)|\phi(x)\rangle}{\langle \phi(x)|\phi(x)\rangle} \right) dx$$

$$= \arg \left[ \sum_{l=\pm} f_l^+(x_0)f_l(x_1) \right] + i \int_{x_0}^{x_1} \left[ \sum_{l=\pm} |f_l(x)|^2 \right]^{-1} \sum_{l=\pm} \left[ f_l^+(x) \frac{d}{dx} f_l(x) - f_l(x) \frac{d}{dx} f_l^+(x) \right] dx,$$ \hspace{1cm} (12)$$

where we have introduced the $x$ dependent complex-valued amplitudes $f_l(x)$ according to

$$|\phi(x)\rangle = \sum_{l=\pm} f_l(x)|l\rangle_z.$$ \hspace{1cm} (13)$$

Equation (12) can be applied to any interval $x_0 \leq x \leq x_1$ by identifying $f_l(x)$ from Eqs. (9)-(11).

\section{III. CYCLIC GP IN RESONANT SCATTERING}

The energy barrier becomes transparent if the length $L$ is an integer multiple of half the effective wave length of the neutron inside the magnetic field region. If this resonance condition is satisfied for both spin state components simultaneously, the spin undergoes loss-free unitary evolution through the magnetic field. If the winding numbers of phase factors of the two spin states are both even or both odd, the spin performs cyclic evolution. This creates a GP of cyclic evolution
enforced by the simultaneous resonance scattering of the two spin states; this low-energy GP can be compared to the ideal GP in Eq. (1).

As is evident from Eq. (10), the probability amplitudes undergo the change
\[ c_i \mapsto \frac{k_i}{\kappa_L \cos \kappa_L L - i k \sin \kappa_L L} c_i \] (14)
during the passage through the magnetic field. Thus, the resonance conditions that enforce cyclic evolution become
\[ \kappa_L = \frac{\pi}{L} n_i, \] (15)
where \( n_i \) are either both even or both odd integers. Note that these conditions can only be satisfied if \( E > V_0 \). Furthermore, since \( \kappa_- > \kappa_+ \) for \( V_0 > 0 \) it follows that \( n_- > n_+ \). The resonance conditions put restrictions on the Zeeman split and the speed of the incoming neutron according to
\[ \kappa_0 = \frac{\pi}{L} \sqrt{\frac{n_-^2 - n_+^2}{2}}, \]
\[ k = \frac{\pi}{L} \sqrt{\frac{n_-^2 + n_+^2}{2}}. \] (16)

As can be seen from Eq. (7), the conditions in Eq. (15) imply \( r_1 = 0 \). Thus, the GP \( \gamma^{(ii)} \) on \( x \leq 0 \) vanishes and a nontrivial GP can only be acquired on \( 0 \leq x \leq L \) in this case. Equation (10) reduces to
\[ |\phi^{(ii)}(s)\rangle = \sum_{l = k}^l \left( \cos n_i s + \frac{i}{\kappa_L} \sin n_i s \right) c_l |l\rangle, \] (17)
where we have defined \( s = (\pi/L)x \) such that \( s \in [0, \pi] \) over the magnetic field region. Since \( n_i \) are either both even or both odd and \( n_- > n_+ \), we can write \( n_- = n_+ + 2\xi \), \( \xi \) being a positive integer. It follows that
\[ |\phi^{(ii)}(\pi)\rangle = (-1)^{n_-} |\phi^{(ii)}(0)\rangle. \] (18)

In the high-speed limit where \( k \gg \kappa_0 \) (11), we have \( \kappa_L \approx k \), which implies
\[ |\phi^{(ii)}(s)\rangle \approx \sum_{l = k}^l e^{i n_i s} c_l |l\rangle. \] (19)
The corresponding GP becomes
\[ \gamma^{(ii)} \approx \arg \langle \phi^{(ii)}(0) | \phi^{(ii)}(\pi) \rangle + i \int_0^\pi \langle \phi^{(ii)}(s) | \frac{d}{ds} \phi^{(ii)}(s) \rangle ds = \pi n_+ - \pi \left( n_- |c_-|^2 + n_+ |c_+|^2 \right) = -\xi 2\pi |c_-|^2. \] (20)

By letting \( |c_-| = \sin \frac{\theta}{2} \), \( \theta \) being the angle between the spin and the magnetic field, we may rewrite Eq. (20) as
\[ \gamma^{(ii)} \approx -\xi \pi (1 - \cos \theta), \] (21)
which coincides with the expected expression in Eq. (1) up to the ‘winding number’ \( \xi \) that counts the number of precessions of the spin when passing through the magnetic field.

Now, by turning to the exact treatment, we use Eq. (12) and find
\[ \gamma^{(ii)} = \pi n_+ - \sqrt{\frac{n_-^2 + n_+^2}{2}} \times \int_0^\pi \left\{ \sum_{n=\pm} \left( \cos^2 n_i s + \frac{n_-^2 + n_+^2}{2n_i^2} \sin^2 n_i s \right) |c_l|^2 \right\}^{-1} ds. \] (22)

This expression is valid for any parameter choice as long as \( k > \kappa_0 \) and the resonance conditions in Eq. (15) are satisfied.

We examine numerically the deviation of the GP per turn \( \gamma^{(ii)} / \pi \) in terms of the ratio \( \frac{n_-}{n_+} \equiv q > 1 \). Figure 2 shows the exact GP per turn for \( q = 1.2, 3, \) and 10, as the solid, dashed, and dotted blue lines, respectively. We find a considerable deviation from the ideal GP (shown as the red line) obtained in the \( q \to 1 \) limit.

By combining Eqs. (4), (9), and (16), we can relate the magnetic field strength \( b_0 \), the neutron speed \( v \), and the parameter \( q \) according to
\[ b_0[T] = \left( \frac{q^2 - 1}{4q^2 + 1} \right)^{0.17} \times \left( v[\text{ms}^{-1}] \right)^2. \] (23)
We see that thermal neutrons, corresponding to \( v \sim 2000 \text{ ms}^{-1} \), would require \( q^2 - 1 = 10^{-7} \) in order to achieve a manageable magnetic field strength, say, about 1 T. For such small \( q \), \( \gamma^{(ii)} \) is practically indistinguishable from the ideal GP. We need instead to consider ultracold neutrons, typically having speed in the order of 1 ms\(^{-1}\), to achieve a measurable difference between the exact and ideal GPs for feasible magnetic field strengths. Indeed, we find \( b_0 = 0.17 \text{ T}, 0.14 \text{ T}, 0.031 \text{ T} \) for \( q = 10, 3, 1.2, \) respectively, when \( v = 1 \text{ m/s} \).
IV. GP IN OFF-RESONANT CASE

In the off-resonant case, the low-energy GP shows new phenomena since the transmission probabilities of the two spin states are no longer unity. This means that the neutron beam is partly backscattered by the magnetic field.

A. GP for $x < 0$

A key novel aspect in the off-resonance case is that the spin state may be $x$ dependent for $x < 0$. This implies that the neutron may pick up a nontrivial GP even before it enters the magnetic field.

![Graph showing GP evolution](image)

FIG. 3: (Color online) GP $\gamma^{(i)}$ for one cycle as a function of $\frac{V_0}{E}$ for the symmetric input state $c_- = c_+ = \frac{1}{\sqrt{2}}$. The blue curve shows the GP for $kL = \sqrt{10}\pi$, which admits resonance scattering at $\frac{V_0}{E} = \frac{3}{2}$, corresponding to $n_- = 4$ and $n_+ = 2$. At this energy ratio, the magnetic field becomes fully transparent for both spin components and $\gamma^{(i)}$ would vanish, as confirmed by the simulation. The red curve shows $kL = 4\pi$, for which no nonzero energy ratio can give rise to $r_x = 0$ simultaneously. A closer inspection shows that $\gamma^{(i)}$ is indeed nonvanishing for all $\frac{V_0}{E} > 0$.

The spin state given by Eq. (9) repeatedly undergoes cyclic evolution with period $\pi/k$ for any starting point $x_0 \leq -\pi/k$. It follows that the GP $\gamma^{(i)}_{\zeta}$ for any number $\zeta$ of cycles is independent of $x_0$ and can be written as $\gamma^{(i)}_{\zeta} = \zeta \gamma^{(i)}$, where $\gamma^{(i)}$ is the GP for one cycle. We find

$$
\gamma^{(i)} = \pi - \sum_{l=\pm} \left( \frac{1 - r_l}{1 + r_l + 2 \sqrt{r_l} \cos \delta_l} \right) |c_l|^2 \\
\times \int_0^{\pi} \left\{ \sum_{l=\pm} \left[ \frac{1 + r_l + 2 \sqrt{r_l} \cos (\delta_l - 2s)}{1 + r_l + 2 \sqrt{r_l} \cos \delta_l} \right] |c_l|^2 \right\}^{-1} ds,
$$

where the integral is expressed in terms of the dimensionless variable $s = ks$ and we have made use of the $x_0$ independence to shift the integration interval to $[0, \pi]$. Thus, the neutron may acquire a nonzero GP before it enters the magnetic field region, due to a nonzero reflection probability of at least one of the two spin states.

The GP $\gamma^{(i)}$ is fully determined by the energy ratio $\frac{V_0}{E}$ and the kinematic parameter $kL$ via the reflection probabilities $r_l$ and phase shifts $\delta_l$. In Fig. 3, we show $\gamma^{(i)}$ as a function of $\frac{V_0}{E}$ for two different $kL$ and the symmetric input state $c_- = c_+ = \frac{1}{\sqrt{2}}$. We have chosen $kL = \sqrt{10}\pi$ (blue line), for which there is an energy ratio that corresponds to resonant scattering and vanishing $\gamma^{(i)}$. Explicitly, we find this ratio to be $\frac{V_0}{E} = \frac{3}{2}$ with $n_- = 4$ and $n_+ = 2$. We confirm in Fig. 3 that $\gamma^{(i)}$ vanishes at this energy ratio. The other choice (red line) is $kL = 4\pi$ with resonance solution $n_- = n_+ = 4$ corresponding to the trivial case $V_0 = 0$. Thus, the barrier never becomes transparent for both spin components simultaneously for this choice, and the GP $\gamma^{(i)}$ is nonvanishing for all $\frac{V_0}{E} > 0$, as is confirmed by the simulation.

The incoming neutron speed corresponding to the two curves in Fig. 3 are very low and have been chosen mainly for illustrative purposes. Indeed, for $kL \sim 10$, one obtains $\nu [\text{m/s}] \sim 10^{-6}/L [\text{m}]$, which would imply a reasonable high speed only for very small $L$. In principle, this could be achieved by using a thin metallic foil to create the magnetic field, for instance by using a setup similar to that of Ref. [8].

![Graph showing spin direction evolution](image)

FIG. 4: (Color online) Evolution of the spin direction when the neutron passes through the magnetic field. The wave vector $\kappa$ of the $|\rangle_{+}$ spin state satisfies $\kappa L = 5\pi$. We have chosen $\frac{V_0}{E} = 1.01$ (blue curve) and $\frac{V_0}{E} = 2$ (red curve), both starting at $(c_+, c_-) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. The evolution is genuinely noncyclic. The decay towards the $|\rangle_{-}$ state at the negative $z$ axis is clearly visible.

B. Tunneling-induced GP

For extremely slow neutrons, at kinetic energies corresponding to $k \ll \kappa_0$, the classical intuition entails that the
magnetic field would act as a filter that allows only the spin state $|−⟩$ to enter the magnetic field region, in case of which $γ^{(ii)}$ would be expected to vanish. However, a portion of the $|+⟩$ state is transmitted via the tunnel effect. This creates a tunneling-induced GP.

While the $|−⟩$ state undergoes an oscillatory evolution, the $|+⟩$ state decays during the passage through the magnetic field when $k < k_0$. In this regime, we may write $κ_+ = ik$ with $κ = \sqrt{k_0^2 - k^2}$, yielding

$$c_+ \mapsto \left( \frac{κ \cosh [κ(x - L)] + ik \sinh [κ(x - L)]}{κ \cosh κL - ik \sinh κL} \right)c_+. \quad (25)$$

Figure 5 shows the evolution of the spin direction for the entire passage of the neutron through the magnetic field with $κL = 5π$ and $(c_+, c_-) = (\frac{\sqrt{5}}{2}, \frac{1}{2})$. We choose $\frac{V}{E} = 1.01$ and $2$ corresponding to, respectively, an energy of the incoming neutron slightly below and half the magnetic-field-induced barrier seen by the $|+⟩$ state. The curves can be understood as a combined decay and oscillatory motion. The spin spirals down to the $|−⟩$ state, pointing in the negative $z$ direction. The evolution is genuinely noncyclic.

FIG. 5: (Color online) Tunneling-induced GP as a function of the angle $θ$ between the directions of the input spin and the magnetic field. The wave vector $κ_+ \equiv \frac{\sqrt{E}}{2}$ of the $|−⟩$ spin state satisfies $κ_+L = π$. The solid, dashed, and dotted lines show $\frac{V}{E} = 1.01$, $2$, and $5$, respectively.

Figure 5 shows the tunneling-induced GP as a function of the angle $θ$ between the incoming spin and the magnetic field. We see that the GP is quenched more rapidly when the kinetic energy tends to zero. It remains large, however, for initial spins close to the positive $z$ axis, since the spin is forced to traverse a long path towards the negative $z$ direction, thereby enclosing a substantial solid angle.

V. CONCLUSIONS

Spin-polarized neutrons passing through a uniform magnetic field have offered a rich field to study subtle aspects of the geometric phase (GP) experimentally. These studies have typically used neutrons moving at sufficiently high speed so that the magnetic field can be treated as a small perturbation to the free motion. In this work, we relax the high energy requirement and consider the case where the kinetic energy of the incoming neutron is comparable to the Zeeman shift. In this regime, the two spin states along the direction of the magnetic field scatter in such a way that the spin evolution and the concomitant GP deviate significantly from the pure precession that occurs at high energies.

We have demonstrated the existence of cyclic spin evolution during the passage through the magnetic field for low energy neutrons and have calculated the corresponding GP. The evolution becomes cyclic if the two spin components simultaneously satisfy certain resonance conditions. When these resonance conditions do not hold, however, the incoming spin state may become nontrivially $x$ dependent and may pick up a nonzero GP even before the neutron enters the magnetic field. At even lower velocities, one eventually enters the tunneling regime, where one of the spin components decays while the other remains in an oscillatory mode. In this case, we have shown the existence of a tunneling-induced GP that can be large, even when the kinetic energy of the incoming neutron is smaller than the Zeeman shift.

The realization of the GP effects discussed in this work requires a combination of low velocity and high magnetic field that put restrictions on the experimental feasibility. For instance, thermal neutrons, moving at speed in the order of 2000 ms$^{-1}$, would require unrealistically large magnetic fields to see deviations from the ideal case. Therefore, to verify experimentally the rich behavior of the GP, ultracold neutrons moving at speed in the order of a few ms$^{-1}$ would be needed.

Another possibility is to replace the neutrons with atoms, for which the magnetic moment is larger, still having a mass of the same order of magnitude. For instance, for atomic hydrogen the magnetic moment would be dominated by that of the single electron, which is essentially the Bohr magneton $μ_B = 9.27 \cdot 10^{-24}$ J/T. For this atom, one would therefore expect the feasible velocity range to be increased by a factor of $(μ_B/μ)^{1/2} \sim 30$ in order to see the low-energy GP effects.

Acknowledgments

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