On Pitts’ Relational Properties of Domains

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Abstract

Andrew Pitts’ framework of relational properties of domains is a powerful method for defining predicates or relations on domains, with applications ranging from reasoning principles for program equivalence to proofs of adequacy connecting denotational and operational semantics. Its main appeal is handling recursive definitions that are not obviously well-founded: as long as the corresponding domain is also defined recursively, and its recursion pattern lines up appropriately with the definition of the relations, the framework can guarantee their existence.

Pitts’ original development used the Knaster-Tarski fixed-point theorem as a key ingredient. In these notes, I show how his construction can be seen as an instance of other key fixed-point theorems: the inverse limit construction, the Banach fixed-point theorem and the Kleene fixed-point theorem. The connection underscores how Pitts’ construction is intimately tied to the methods for constructing the base recursive domains themselves, and also to techniques based on guarded recursion, or step-indexing, that have become popular in the last two decades.

1 The Original Result

When reasoning about programs, it is common to compare their behaviors. We might ask if two programs behave equivalently, if their public outputs are equal, or if one program terminates more often than the other, among other questions. Many of these issues can be phrased naturally using recursive relations. For example, to argue that two functions are equivalent, we might want to check if they produce equivalent outputs when applied to equivalent inputs, for some notion of equivalence. However, when reasoning about higher-order or stateful programs, equivalence for inputs and outputs is defined in terms of equivalence for arbitrary programs. Thus, we end up with a circular definition of equivalence, which requires care to justify formally without running into paradoxes.

Andrew Pitts’ framework of relational properties of domains [Pit96] is a powerful tool for constructing such relations. We can summarize the idea as follows.

**Theorem 1.1** ([Pit96]). Let $D$ be an object of a pointed CPO-category $\mathcal{C}$. Suppose that $D$ is equipped with an isomorphism $i : F(D,D) \cong D$ that satisfies the minimal
invariant property; where \( F : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C} \) is a CPO-functor. Suppose moreover that \( \mathcal{C} \) is equipped with an admissible relational structure \( \mathcal{R} \), and that \( F \) acts on \( \mathcal{R} \). Then there exists \( R_D \in \mathcal{R}_D \) such that \( R_D = (i^{-1})^* F(R_D, R_D) \).

Here is how we can read this result intuitively, before diving into formal definitions. The object \( D \) is a universe where we model the behavior of the programs. In Pitts’ original result, \( D \) was assumed to be a complete partial order, or CPO, a domain-theoretic notion for modeling general recursion and nontermination. Here, instead, we assume that \( D \) lives in some pointed CPO-category \( \mathcal{C} \), a generalization allows us to carry the core of Pitts’ arguments while accounting for variations that have been explored in the literature, such as families of CPOs [AFJ20], diagrams of CPOs [Lev02], or CPOs equipped with a metric [Aze+17].

We assume that \( D \) is defined recursively as \( F(D, D) \cong D \). The equation is stated using a functor \( F \), where each recursive occurrence of \( D \) is either contravariant or covariant; being a CPO-functor simply means that \( F \) interacts well with the structure of \( \mathcal{C} \). In principle, there could be many solutions to such equations, but Theorem 1.1 only applies to those that satisfy the minimal invariant property, which roughly means that \( D \) is completely characterized by repeatedly unfolding its definition.

The conclusion of the theorem says that we can construct some “relation” \( R_D \) on \( D \). In the applications we sketched above, \( R_D \) could be a binary relation on a CPO, but the result applies to other settings as well, such as relations of different arities or families of relations. The relational structure \( \mathcal{R} \) formalizes which properties are required of the notion of “relation” for the construction to apply. The definition of \( R_D \) is given by a recursive equation \( R_D = (i^{-1})^* F(R_D, R_D) \), which is derived from an action of \( F \) on \( \mathcal{R} \). Different actions and relational structures yield different definitions, and it is our job to choose them appropriately depending on the application at hand.

Let us now spell out how this works in detail. A complete partial order (CPO) is a poset \((X, \sqsubseteq)\) such that every increasing chain \( x : \mathbb{N} \xrightarrow{\text{mono}} X \) has a limit, or least upper bound, denoted \( \operatorname{lim}_n x(n) \). A CPO \( X \) is pointed if it has a least element \( \bot \in X \).

A function \( f : X \to Y \) between CPOs is continuous, denoted \( f : X \xrightarrow{\text{cont}} Y \), if it is monotone and preserves limits. CPOs and continuous functions between them form a category CPO. This category is cartesian closed; the exponential \( Y^X \) is given by the set of continuous functions of type \( X \xrightarrow{\text{cont}} Y \) ordered pointwise.

A CPO-category is a category \( \mathcal{C} \) where the sets of morphisms \( \mathcal{C}(X, Y) \) are CPOs, and such that composition \( (-) \circ (-) : \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \to \mathcal{C}(X, Z) \) is continuous. (Product CPOs are ordered component-wise.) The most basic example of CPO-category is CPO itself, for the order relation on continuous functions defined above. Another example is given by functor categories of the form \( \mathcal{C}^I \), where \( \mathcal{C} \) is a CPO-category and \( I \) is a small category. A morphism in \( \mathcal{C}^I \) is a family of arrows \( (X_i \to Y_i)_{i \in I} \), and we obtain a CPO-category by ordering such families pointwise. Combined with the previous example, this shows that families or diagrams of CPOs also form CPO-categories. If \( \mathcal{C} \) is a CPO-category, then so is \( \mathcal{C}^{op} \), by inheriting the
structure on $\mathcal{C}$. A CPO-functor $F : \mathcal{C} \to \mathcal{D}$ is a functor whose action on morphisms is a continuous function.

Given a CPO-category, we say that $Y \in \mathcal{C}$ is pointed if $\mathcal{C}(X, Y)$ is pointed for every $X$, and if $\bot \circ f = \bot$ for every $f$. When $\mathcal{C} = \text{CPO}$, this definition of pointedness coincides with the one given above. Every terminal object 1 is pointed: if $X$ is an object, the unique arrow of type $X \to 1$ is the least element. We say that $\mathcal{C}$ itself is pointed if every object is pointed and it has a terminal object 1. In this case, any $\bot : X \to Y$ in $\mathcal{C}$ factors through 1. For example, CPO is not a pointed CPO-category (because not every CPO is pointed according to our definition), but we do obtain a pointed CPO-category by restricting ourselves to pointed CPOs. (More generally, any CPO-category $\mathcal{C}$ with a terminal object has a pointed counterpart $\mathcal{C}_1$ obtained by restricting $\mathcal{C}$ to pointed objects.)

Given an isomorphism $i : F(D, D) \cong D$ in $\mathcal{C}$, where $D$ is pointed, we say that $D$ has the minimal invariant property if the following condition holds. First, given $\pi : D \xrightarrow{\text{cont}} D$, we define $\Phi(\pi) : D \xrightarrow{\text{cont}} D$ as $\Phi(\pi) \equiv i \circ F(\pi, \pi) \circ i^{-1}$, and pose $\pi_i \equiv \Phi^i(\bot)$. Intuitively, each $\pi_i$ is a projection function that truncates $D$ to allow for at most $i$ unfoldings of its definition; hence, $\lim_i \pi_i$ is a projection function that allows for an arbitrary number of unfoldings. Intuitively, we would expect $\lim_i \pi_i$ to be the identity on $D$, because leaving the number of unfoldings unbounded should be tantamount to not truncating $D$ at all. However, this is not necessarily true; the best we can show in general is $\lim_i \pi_i \subseteq 1_D$. The minimal invariant property says precisely that we can strengthen this inequality to $\lim_i \pi_i = 1_D$.

Given a category $\mathcal{C}$, a relational structure on $\mathcal{C}$ is simply a functor $\mathcal{R} : \mathcal{C}^{op} \to \text{CLat}_\lambda$, where $\text{CLat}_\lambda$ is the category of complete lattices and functions that preserve greatest lower bounds. We denote the value of $\mathcal{R}$ at some object $X \in \mathcal{C}$ as $\mathcal{R}_X$, and we use the variables $R, S$ and $T$ to range over the elements of $\mathcal{R}_X$. If $f : X \to Y$ is a morphism in $\mathcal{C}$, we write $\mathcal{R}(f)$ as $f^*$ when $\mathcal{R}$ can be understood from the context.

**Example 1.2.** Our motivating example of relational structure is the one obtained by choosing $\mathcal{C} = \text{CPO}$, and posing $\mathcal{R}_X$ to be the set of binary relations between the elements of $X$ ordered by inclusion. The greatest lower bound of a family of relations is simply their intersection. And the action of a continuous function $f : X \xrightarrow{\text{cont}} Y$ on $\mathcal{R}_Y$ takes the inverse image of a relation by $f$. For intuition, we’ll keep this vocabulary when discussing other relational structures as well.

To define admissible relational structures, it is convenient to shift our perspective a bit. Given a relational structure $\mathcal{R}$ on $\mathcal{C}$, we can build a category, also denoted $\mathcal{R}$, as follows. The objects of $\mathcal{R}$ are pairs $(X, R)$, where $X \in \mathcal{C}$ and $R \in \mathcal{R}_X$. (By abuse of notation, I’ll often use $R$ to represent the object $(X, R) \in \mathcal{R}$.) A morphism $f : (X, R) \to (Y, S)$ is a morphism $f : X \to Y$ in $\mathcal{C}$ such that $R \subseteq f^*S$. In terms of Example 1.2, this simply means that the function $f$ takes elements related by $R$ to elements related by $S$. We can check that identities and composition in $\mathcal{C}$ can be lifted to the morphisms of $\mathcal{R}$. We have a canonical functor $p : \mathcal{R} \to \mathcal{C}$ that maps the object $(X, R)$ to $X$ and acts as the identity on morphisms.

By unfolding definitions, we can restate some of the properties of the functor $\mathcal{R}$ in terms of the above construction:
Lemma 1.3. Let $X$, $Y$ and $Z$ be arbitrary objects of $C$.

1. $1_X : R \to S \iff R \leq S$, for all $R, S \in \mathcal{R}_X$.

2. $f : f^* S \to S$ for any $S \in \mathcal{R}_Y$ and $f : X \to Y$.

3. $gf : R \to S \iff f : R \to g^* S$, for all $f : X \to Y$, $g : Y \to Z$, $S \in \mathcal{R}_Z$ and $R \in \mathcal{R}_X$.

4. $f : R \to \bigcap \{ S_i \}_{i \in I} \iff \forall i \in I, f : R \to S_i$, for any index set $I$, $f : X \to Y$, $R \in \mathcal{R}_X$ and $S \in \mathcal{R}_Y$.

If $\mathcal{R}$ is a relational structure over a pointed CPO-category $C$, we say that a relation $S \in \mathcal{R}_Y$ is admissible if the following conditions hold for all $R \in \mathcal{R}_X$. First, $\bot : R \to S$; second, $\lim f_i : R \to S$ whenever $(f_i : R \to S)_{i \in \mathbb{N}}$ is an increasing sequence of morphisms. We say that $\mathcal{R}$ itself is admissible if every relation is admissible. Intuitively, being admissible means that a relation always holds of diverging programs and is compatible with recursive program definitions, which are constructed using limits via Kleene’s fixed point theorem.

Example 1.4. We can adapt Example 1.2 to obtain an admissible relational structure as follows. First, instead of considering arbitrary CPOs, we just consider pointed ones; that is, we take $C = \text{CPO}_p$. Second, instead of considering arbitrary relations, we consider only those that contain $\bot$ and are closed under taking limits of chains.

The missing piece in the statement of Theorem 1.1 is what it means for a CPO-functor $F : \mathcal{C}^p \times \mathcal{C} \to \mathcal{C}$ to act on $\mathcal{R}$. For each $R \in \mathcal{R}_X$ and $S \in \mathcal{R}_Y$, we assume that there is some $F(R, S) \in \mathcal{R}_{F(X,Y)}$; moreover, if $f : R' \to R$ and $g : S \to S'$ are morphisms in $\mathcal{R}$, then $F(f, g)$ should be a morphism of type $F(R, S) \to F(R', S')$ in $\mathcal{R}$. (Note the contravariance on first argument).

We can now sketch the main idea of Pitts’ original construction. For the rest of the paper, we fix some pointed CPO-category $C$ equipped with an admissible relational structure $\mathcal{R}$, a CPO-functor $F : \mathcal{C}^p \times \mathcal{C} \to \mathcal{C}$ with an action on $\mathcal{R}$, and an object $D$ that satisfies the minimal invariant property for an isomorphism $i : F(D, D) \cong D$.

Proof of Theorem 1.1. The proof relies on the Knaster-Tarski fixed point theorem: every monotone function on a complete lattice has a least fixed point. Since the mapping $R \mapsto (i^{-1})^* F(R, R)$ is not monotone, we need to modify its definition a bit. Pitts’ employed the trick of separating covariant and contravariant arguments: if we pose $L \triangleq \mathcal{R}^p_D \times \mathcal{R}_D$, then the function $\Psi : L \to L$ $\Psi(R^-, R^+) \triangleq ((i^{-1})^* F(R^+, R^-), (i^{-1})^* F(R^-, R^+))$ is monotone, and we can construct a least fixed point $(R^+_D, R^-_D)$. Note that $(R^+_D, R^-_D)$ is also a fixed point, so $(R^-_D, R^+_D) \leq (R^+_D, R^-_D)$ in $L$, and thus $R^+_D \leq R^-_D$. To finish the proof, we just need to show the reverse inequality. This is where the properties of
minimal invariant and the relational structures come into play. We can show that \( \pi_i : R_D^i \rightarrow R_D^+ \) by induction on \( i \), which implies, by admissibility, that \( 1 = \lim_i \pi_i : R_D^i \rightarrow R_D^+ \). But this is equivalent to \( R_D \leq R_D^+ \) by Lemma 1.3, from which the result follows.

**Remark 1.5 (Uniformity)**. This proof shows that a stronger result holds: for all \( i \in \mathbb{N} \),

\[
\pi_i : R_D \rightarrow R_D.
\]

Intuitively, this means that the constructed relation \( R_D \) still holds after we truncate an element of \( D \) after \( i \) unfoldings. This property, known as **uniformity**, will play an important role in Section 3, when constructing \( R_D \) by the Banach fixed-point theorem.

**Remark 1.6**. Pitts’ presentation differs from mine in a few respects [Pit96]. What I call a relational structure here corresponds to what he calls a relational structure with inverse images and intersections. More importantly, his notion of action on a relational structure is different: rather than requiring \( R \) to be admissible, he requires \( F(R, S) \) to be admissible whenever \( S \) is. This is a strengthening of the above notion of action, since it must be defined even for relations that are not admissible. It allows us to formulate more useful coinduction principles associated with the relation \( R_D \), but it does not change the construction of \( R_D \) itself, which is why we do not consider it here.

## 2 Inverse Limit Construction

In practice, minimal invariants such as \( D \) are often obtained with Scott’s *inverse limit construction*. The method can be seen as an adaptation of Kleene’s fixed-point theorem that accounts for mixed-variance functors, and can be carried out for many CPO-categories [Wan79; SP82]. After reviewing the idea, we will see that Pitts’ result, Theorem 1.1, is just an instance of it!

We say that two morphisms \( f^e : X \rightarrow Y \) and \( f^p : Y \rightarrow X \) in \( C \) form an **embedding-projection pair** if \( f^p f^e = 1_X \) and \( f^e f^p \subseteq 1_Y \). We can show that each half of the pair uniquely determines the other. Embeddings and projections compose, so we can form a subcategory \( C^e \) consisting of all embeddings, and \( C^p \) consisting of all projections. With embeddings and projections, we can make mixed-variance functors more symmetric. Since \( F \) is a CPO-functor, its action on morphisms is monotone, and we can show that \( F(f^p, f^e) \) is an embedding, with \( F(f^e, f^p) \) being the corresponding projection. Thus, \( F \) determines a functor \( F^e : C^e \rightarrow C^e \) by posing \( F^e(X) \cong F(X, X) \) on objects, and \( F^e(f^e) \cong F^e(f^p, f^e) \) on morphisms.

Much like Kleene’s fixed-point theorem, we’ll see that we can build \( D \) by considering a chain of finite iterations of \( F^e \) and taking its colimit—which, in the context of Kleene’s construction, would just correspond to a limit in a CPO. Since we are dealing with embeddings, colimits behave particularly symmetrically, a phenomenon known in the literature as the *limit-colimit coincidence*: 

5
**Theorem 2.1** ([SP82]). Let \( X^e : (\mathbb{N}, \leq) \to \mathcal{C}^e \) be a diagram of embeddings, which uniquely corresponds to a diagram \( X^p : (\mathbb{N}, \geq) \to \mathcal{C}^p \) of projections. Let \( A \in \mathcal{C} \). The following conditions are equivalent.

- \( A \) is a colimit of \( X^e \) in \( \mathcal{C} \).
- \( A \) is a limit of \( X^p \) in \( \mathcal{C} \).
- There is a cocone of embeddings \( f^e : X^e \to \Delta A \) such that \( \lim f^e \circ f^p = 1_A \).
- There is a cone of projections \( f^p : \Delta A \to X^p \) such that \( \lim f^e \circ f^p = 1_A \).

In this situation, \( f^e : X^e \to \Delta A \) is a colimiting cocone, and \( f^p : \Delta A \to X^p \) is a limiting cone. We call the pair \((A, f)\) the bilimit of \( X \).

Because of this result, we can show that \( F^e \) preserves bilimits of chains of embeddings in \( \mathcal{C} \). Then, constructing the fixed point of \( F \) becomes simply a matter of adapting the proof of Kleene’s fixed-point theorem.

**Theorem 2.2** ([SP82]). Suppose that \( \mathcal{C} \) has bilimits of chains of embeddings. Then \( F \) has a minimal invariant \( i : F(D, D) \cong D \).

**Proof.** Let \( X_i = (F^e)^i(1) \). Since \( \mathcal{C} \) is pointed, there is an embedding \( f^e_0 = \perp : 1 \to X_1 \). By iterating \( F^e \) on \( f^e_0 \), we can construct a sequence of embeddings \( X_i \to X_{i+1} \).

By hypothesis, this chain has a bilimit, which we call \( g^e : X \to \Delta D \). Since \( F^e \) preserves bilimits of embeddings, we know that \( F^e(g) : F^e(X) = (X_i)_{i \geq 1} \to \Delta F^e(D) \) is a bilimit. Note that \( 1 \) is an initial object of \( \mathcal{C}^e \), so we can extend this cocone to \( h^e : X \to \Delta F^e(D) \) by posing

\[
\begin{align*}
    h^e_0 & : 1 \to F^e(D) \\
    h^e_0 & \equiv \perp \\
    h^e_{i+1} & : F^e(X_i) \to F^e(D) \\
    h^e_{i+1} & \equiv F^e(g_i).
\end{align*}
\]

Since both \( F(D, D) \) and \( D \) satisfy the same universal property, we get an isomorphism \( i : F(D, D) \cong D \). The construction of this isomorphism implies, for every \( j \in \mathbb{N} \),

\[
i_0 h^e_{j+1} = i_0 F(g^p_j, g^e_j) = g^p_{j+1}.
\]

Taking projections on both sides, we obtain

\[
F(g^e_j, g^p_j) i_0^{-1} = g^p_{j+1}.
\]

Combining the two equations, we find

\[
g^e_{j+1} g^p_{j+1} = i_0 F(g^p_j g^e_j, g^e_j g^p_j) i_0^{-1}.
\]

Since \( g^e_0 = \perp \) and \( g^p_0 = \perp \), this implies that \( \pi_j \equiv g^e_j g^p_j \) satisfies exactly the same equations as the projection functions used in the definition of the minimal invariant property. By Theorem 2.1, the limit of this sequence is the identity on \( D \), so \( i : F(D, D) \cong D \) indeed satisfies the minimal invariant property.  \( \square \)
To see how this relates to Pitts’ construction, note that $\mathcal{R}$ can also be seen as a pointed $\mathcal{CPO}$-category, and the projection $p : \mathcal{R} \to \mathcal{E}$ preserves this structure. Indeed, admissibility means that the morphisms of $\mathcal{R}$ have the structure of a pointed $\mathcal{CPO}$ inherited from the morphisms of $\mathcal{E}$. The terminal object of $\mathcal{R}$ is just the terminal object of $\mathcal{E}$ equipped with the greatest relation on $\mathcal{R}_1$, which exists because $\mathcal{R}_1$ is a complete lattice. Moreover, if we see each relation $F(R,S)$ as an object of $\mathcal{R}$, then the action of $F$ on $\mathcal{R}$ can be described equivalently as a $\mathcal{CPO}$-functor of type $\mathcal{R}^{op} \times \mathcal{R} \to \mathcal{R}$ making the following diagram commute:

$$
\begin{array}{ccc}
\mathcal{R}^{op} \times \mathcal{R} & \longrightarrow & \mathcal{R} \\
\downarrow & & \downarrow \\
\mathcal{E}^{op} \times \mathcal{E} & \longrightarrow & \mathcal{E}.
\end{array}
$$

To apply the inverse limit construction to this lifted functor, we just need to show that $\mathcal{R}$ has bilimits of chains of embeddings.

**Lemma 2.3.** Admissible relational structures create bilimits. That is, if $R$ is a chain of embeddings in $\mathcal{R}$, and $f : pX \to \Delta L$ is a bilimit in $\mathcal{E}$, then $f : X \to \Delta R$ is a bilimit in $\mathcal{R}$, where $R \in \mathcal{R}_L$ is defined as

$$
R \triangleq \bigcap_n \left( (f_n^R)^* X_n \right).
$$

**Proof.** By Theorem 2.1, it suffices to show that the corresponding cone of projections is a limiting cone in $\mathcal{R}$. We’ll show that there is a bijective correspondence between morphisms of type $T \to R$ and cones of type $\Delta T \to X$ in $\mathcal{R}$ that is natural in $T$.

In one direction, suppose that $g : T \to \bigcap_n (f_n^R)^* X_n$ is a morphism in $\mathcal{R}$. This means that $f_n^R g : T \to X_n$ is a morphism for every $n \in \mathbb{N}$, and we can check that they form a cone $\Delta T \to X$. Conversely, suppose that we are given a cone $g : \Delta T \to X$. By projecting this cone onto $\mathcal{E}$, we obtain another cone $pg : \Delta pT \to pX$. Since $f^R : \Delta L \to pX$ is limiting, there is a unique mediating morphism $g' : pT \to L$. Moreover, for every $n \in \mathbb{N}$, we have $f_n^R g' = g_n$. Since $g_n : T \to X_n$ by hypothesis, this means that $g' : T \to (f_n^R)^* X_n$ for every $n \in \mathbb{N}$. Thus, $g' : T \to \bigcap_n (f_n^R)^* X_n = R$, and the mediating morphism can be lifted as expected. After checking that this is natural in $T$, we conclude.

**Remark 2.4** (A dual characterization). Since each inverse image function $f^* : \mathcal{R}_Y \to \mathcal{R}_X$ preserves intersections and relations form complete lattices, we can build a corresponding left adjoint $f_! : \mathcal{R}_X \to \mathcal{R}_Y$, called the **direct image by** $f$. This allows us to find an alternative characterization of the bilimit $R$ above, by dualizing the proof.

$$
R = \bigcup_n (f_n^R)_! X_n.
$$

Here, $\bigcup$ refers to the supremum of a family of relations, which exists by completeness.
**Corollary 2.5.** If $\mathcal{C}$ satisfies the hypotheses of Theorem 2.2, then so does $\mathcal{R}$.

This leads to an alternative strategy for constructing recursive relations.

*Proof of Theorem 1.1; inverse limit construction.* Thanks to Corollary 2.5, we can apply Theorem 2.2 to the lifting of $F$ in $\mathcal{R}$, and build a minimal invariant object $i_R : F(R, R) \cong R$ in $\mathcal{R}$. We can check that $pR$ is just $D$ up to isomorphism, since both are built as bilimits and those are preserved by $F$. Thus, we might as well assume that $D = pR$ and $i_R = i$. The fact that $i$ is an isomorphism implies that $R \leq (i^{-1})^* F(R, R)$. To conclude, we just need to show the reverse inequality. We know that $i^{-1} : (i^{-1})^* F(R, R) \to F(R, R)$. Since $i : F(R, R) \to R$, we find by composition that $1 : (i^{-1})^* F(R, R) \to R$, and we conclude that $R = (i^{-1})^* F(R, R)$. □

### 3 Banach Fixed Point

In the last two decades, guarded recursion has emerged as a popular method for defining recursive relations. While originally developed for reasoning about denotational semantics [Nak00], it was shortly after adapted to the operational setting, where it proved to be a convenient interface to step-indexed reasoning [AM01; App+07].

The basic idea is to work with a family of relations $(R_n)_{n \in \mathbb{N}}$. In the case of step indexing, $R_n$ represents a property that holds of terms of a language within at most $n$ steps of computation, such as “if the term terminates in at most $n$ steps, then it is a value of type $\mathbb{N}$”. It is always possible to define such a family recursively if each $R_n$ depends only on the values of $R_m$, for each $m < n$. Manipulating such indices directly quickly becomes cumbersome, so guarded recursion encapsulates this process in a modality $\triangleright$, usually known as “later”. Then, any recursive definition becomes valid, as long as recursive occurrences of the relation appear under $\triangleright$.

After reviewing the basics of guarded recursion, we will see how it leads to an alternative proof of Theorem 1.1. First, we need a general, abstract setting where guarded definitions can be formulated.

**Definition 3.1.** An ordered family of equivalences (OFE) is a tuple $(X, (\equiv_n)_{n \in \mathbb{N}})$, where $X$ is a set, $\equiv_0 \supseteq \equiv_1 \supseteq \cdots$ is a decreasing sequence of equivalence relations on $X$, and $\equiv_0$ is the total relation on $X$. The family should converge to the identity on $X$: $\bigcap_n \equiv_n = \equiv_0$ or, equivalently, $(\forall n. x \equiv_n y) \Rightarrow x = y$ for any $x, y \in X$.

We will soon see examples of OFEs connected to the denotational models we have been studying so far. Before we get there, however, let us go back to the example sketched above: an indexed family of relations on terms $t \in T$. The set of such indexed families forms an OFE: we say that $R \equiv_n S$ if and only if $R_m = S_m$ for any $m < n$. Intuitively, this means that the two relations are equivalent if we restrict ourselves to strictly less than $n$ steps of computation. To define fixed points in such an abstract setting, we require slightly more structure of OFEs.
**Definition 3.2.** Let $X$ be an OFE. A Cauchy sequence on $X$ is a sequence of elements $x : \mathbb{N} \to X$ such that, for every $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that, for any $i, j \geq m$, we have $x_i = x_j$. We say that $X$ is a complete OFE (COFE) if, for every Cauchy sequence $x$, there exists some (necessarily unique) $\lim x \in X$ such that, for every $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $x_i = \lim x$ for every $i \geq m$.

We can show that the OFE of relations sketched above is complete. Intuitively, if we look at the $n$th level of the terms of a Cauchy sequence, they will eventually stabilize at some $R_n$, and we can take the family of such $R_n$ to be the limit of the sequence.

**Theorem 3.3 (Banach Fixed Point).** Suppose that a function $f : X \to X$ on a COFE is contractive; that is, if $x = y$, then $f(x) = f(y)$. Suppose, moreover, that there exists some $x_0 \in X$. The sequence $x_i = f^i(x_0)$ is a Cauchy sequence, and its limit is the unique fixed point of $f$: $f(\lim x) = x$.

**Remark 3.4 (Connection to metric spaces).** Each OFE $X$ gives rise to a metric space as follows: $d(x, y) = 2^{-n}$, where $n$ is the greatest $n$ such that $x = y$ holds (if there is no such $n$, then $x = y$, and we set $d(x, y) = 0$). If $X$ is complete, then the resulting metric space is also complete. If $f : X \to X$ is contractive, in the sense of Theorem 3.3, then $d(f(x), f(y)) \leq \frac{1}{2}d(x, y)$, implying that $f$ is contractive in the traditional sense of metric spaces. This requirements guarantee that the usual metric formulation of Banach’s fixed point theorem applies.

**Remark 3.5 (Defining later).** Given a family of relations $(R_n)_{n \in \mathbb{N}}$ as above, we can define another family $\triangleright R$ by shifting $R$ by 1: $(\triangleright R)_0 = T \times T$, and $(\triangleright R)_{n+1} = R_n$. To define a contractive function on families of relations, it suffices to consider functions of the form $f(R) = g(\triangleright R)$, where $g$ is non-expansive, which means that it preserves each relation $R$. In this case, we say that the definition of $f$ is guarded, which explains the connection to guarded recursion alluded to above. Similar definitions of $\triangleright$ can be stated for other types of relations. Though it will not play a major role in what follows, guardedness is often a convenient way of checking that a definition is contractive (e.g. in a type theory).

To apply Theorem 3.3 to construct recursive relations, we need to show that relations on a minimal invariant $D$ form a COFE. To this end, we restrict ourselves to uniform relations, which are the $R \in \mathcal{R}_D$ such that $\pi_i : R \to R$ for every $i$. As noted in Remark 1.5, the relation that we aim to build is known to be uniform, so there is no harm in restricting our search space to require uniformity from the start. We let $\mathcal{U} \subseteq \mathcal{R}_D$ denote the set of uniform relations.

The reason for focusing on uniform relations is that they are entirely determined by their inverse images by each of the $\pi_n$. Indeed, if $\pi_n R = \pi_n S$, then $R \subseteq \pi_n S$, where the first inequality holds by uniformity. If this holds for every $n$, by admissibility, $1 = (\lim_n \pi_n) : R \to S$ and $R \subseteq S$. An analogous reasoning shows that $S \subseteq R$, and we conclude that $R = S$. This property, in turn, helps us define a COFE structure over $\mathcal{U}$.
Lemma 3.6. If $R$ and $S$ are uniform, then the following conditions are equivalent for every $n \in \mathbb{N}$:

- $\pi^n R = \pi^n S$
- $\pi_n : R \to S$ and $\pi_n : S \to R$.

If one of these conditions holds, we say that $R \equiv^n S$. This assignment endows the set $\mathcal{U}$ with the structure of a COFE.

Proof. The equivalence between the two notions follows from the previous discussion. To show that this indeed defines a COFE, note that we have already seen that $R = S$ when $R \equiv^n S$ for every $n$, so we have a well defined OFE. Thus, we just need to prove completeness. Let $(R_i)$ be a Cauchy sequence on $\mathcal{U}$. For every $i \in \mathbb{N}$, there exists some $m_i \in \mathbb{N}$ such that $\pi^n_i(R_n) = \pi^n_i(R_{m_i})$ for any $n \geq m_i$. Without loss of generality, we can assume that $(m_i)$ is increasing. We pose

$$\lim R \equiv \bigcap_i S_i.$$  

We can show that uniform relations are closed under inverse images by $\pi_i$ and intersections, hence each $S_i$ and $\lim R$ are indeed uniform. Moreover, because $S$ is a subsequence of $R$, it must be a Cauchy sequence, and it must have the same limit as $R$, if one of them does have a limit.

To conclude, we just need to show that $\lim R$ is indeed the limit of $S$. First, note that $(S_i)$ is decreasing. Indeed, given $i \leq j$, we have

$$S_i = \pi^n_i(R_{m_i}) = \pi^n_i(R_{m_j}) = (\pi_j \circ \pi_i)^n(R_{m_j}) = \pi^n_i(\pi^n_j(R_{m_j})) = \pi^n_i(S_j).$$

Thus, $S_j \leq S_i = \pi^n_i(S_j)$ is equivalent to $\pi_i : S_j \to S_i$, which follows from the uniformity of $S_j$.

On the other hand, given $i \in \mathbb{N}$, we have $\lim R \equiv S_i$. Indeed,

$$\pi^n_i(\lim R) = \pi^n_i\left( \bigcap_j S_j \right)$$

$$= \pi^n_i\left( \bigcap_{j \geq i} S_j \right)$$

$S$ is decreasing

$$= \bigcap_{j \geq i} \pi^n_j(S_j)$$

intersections commute with inverse images

$$= \bigcap_{j \geq i} \pi^n_j(S_i)$$

$$= \pi^n_i(S_i),$$

which shows that $S$ does converge to $\lim R$. \qed
Now that we have a COFE, we just need a contractive operator on \( U \).

**Lemma 3.7.** The following defines a contractive operator on \( U \):

\[
\Psi(R) \triangleq (i^{-1})^*(F(R, R)).
\]

**Proof.** We begin with the following auxiliary result. If \( R = S \), for \( R, S \in U \), then

\[
\pi_{n+1} = iF(\pi_n, \pi_n)i^{-1} : \Psi(R) \to \Psi(S).
\]

Indeed, by unfolding definitions, we have \( i^{-1} : (i^{-1})^*F(R, R) \to F(R, R) \) and \( i : F(S, S) \to (i^{-1})^*F(S, S) \). By unfolding \( \Psi \), and by composition, we can prove this statement by showing

\[
F(\pi_m, \pi_m) : F(R, R) \to F(S, S).
\]

This follows from \( R = S \) by Lemma 3.6.

Let us proceed with the main proof. First, note that \( \Psi(R) \) is indeed uniform, so \( \Psi : U \to U \). Indeed, we need to show that \( \pi_n : \Psi(R) \to \Psi(R) \) for any \( n \). If \( n \neq 0 \), we apply the auxiliary result above. If \( n = 0 \), it suffices to show that \( i^{-1} \bot : \Psi(R) \to F(R, R) \). But \( i^{-1} \bot = i^{-1} \bot \bot \leq i^{-1} \bot = \bot \), so \( i^{-1} \bot = \bot \), and we conclude because \( \mathcal{R} \ni F(R, R) \) is pointed. Second, we need to show that \( R = S \) implies \( \Psi(R) = \Psi(S) \). This follows by applying the auxiliary result in both directions, and by using Lemma 3.6. \( \square \)

Combining all these ingredients, we obtain yet another strategy for building \( R_D \).

**Proof of Theorem 1.1; Banach fixed point.** It suffices to apply Theorem 3.3 to the operator \( \Psi : U \to U \) of Lemma 3.7. We just need to find an initial uniform relation to construct the fixed point. Note that \( \mathcal{R}_D \) has an element \( \top \), defined as the intersection of the empty family of relations. Moreover, for any \( f : X \to D \) in \( \mathcal{C} \) and \( R \in \mathcal{R}_X \), we have \( f : R \to \top \) by Lemma 1.3. In particular, \( \pi_i : \top \to \top \) for any \( i \), so \( \top \in U \) and we conclude. \( \square \)

## 4 Kleene Fixed Point

As observed earlier, the inverse limit construction can be seen as a generalization of Kleene’s fixed point theorem:

**Theorem 4.1** (Kleene). Let \( X \) be a pointed CPO and \( f : X \to X \) be a continuous function. Then \( f \) has a least fixed point \( x = f(x) \), given by the limit of the chain \( \bot \subseteq f(\bot) \subseteq f^2(\bot) \subseteq \cdots \).

As a minor variation on Section 2, let us sketch how we can restate those results using Kleene’s theorem, by viewing domains and relations as an ordered structure rather than a category. There are two issues that we need to address. First, \( \mathcal{R} \) contains potentially many morphisms between a pair of objects, whereas a CPO \( X \) seen
as a category has at most one. Second, \( \mathcal{R} \) is not a \textit{skeletal category}: there are objects that are isomorphic, but not equal. By contrast, a CPO seen as a category is skeletal because its order is antisymmetric.

To solve the first issue, consider the slice category \( \mathcal{C}^e/D \). Objects of \( \mathcal{C}^e/D \) are embeddings of type \( X \to D \), and arrows from \( X \to D \) to \( Y \to D \) commuting triangles of embeddings:

\[
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
D & & D
\end{array}
\]

Since embeddings are monomorphisms, if there are two arrows of type \( X \to Y \) in \( \mathcal{C}^e/D \), they must be equal. We can apply a similar idea to \( \mathcal{R} \) by considering \( \mathcal{R}(\mathcal{C}^e/D) \), which is defined as the following pullback:

\[
\begin{array}{ccc}
\mathcal{R}(\mathcal{C}^e/D) & \longrightarrow & \mathcal{R}^e \\
\downarrow & & \downarrow \\
\mathcal{C}^e/D & \longrightarrow & \mathcal{C}^e
\end{array}
\]

Explicitly, objects of \( \mathcal{R}(\mathcal{C}^e/D) \) are triples \( X = ([X], e_X : [X] \to D, R_X : R_X) \), where \( e_X \) is an embedding. An arrow \( f : X \to Y \) is an embedding \( f^e : [X] \to [Y] \) such that \( e_Y f^e = e_X \) and such that, in \( \mathcal{R} \), we have \( f^e : R_X \to R_Y \) and \( f^p : R_Y \to R_X \). Once again, there is at most one arrow of any given type in this category.

To solve the second issue, note that, in many cases of interest, we can replace \( \mathcal{C}^e/D \) (and \( \mathcal{R}(\mathcal{C}^e/D) \)) with equivalent skeletal subcategories, by choosing canonical representatives for their objects. For instance, if \( \mathcal{C} \) is \( \text{CPO}_L \), we can replace an embedding \( e_X : X \to D \) with its image in \( D \), which is isomorphic to \( X \). Two objects in \( \mathcal{C}^e/D \) are isomorphic if and only if their images in \( D \) are equal. In what follows, I’ll assume that such canonical representatives exist, and that \( D \to D \) is its own representative. By abuse of notation, I’ll identify the above categories with their skeletal equivalents.

Both \( \mathcal{C}^e/D \) and \( \mathcal{R}(\mathcal{C}^e/D) \) are CPOs: to compute the least upper bound of a chain, we simply project the chain onto \( \mathcal{C} \) (or \( \mathcal{R} \)), compute its bilimit, and use its canonical representative in \( \mathcal{C}^e/D \) (or \( \mathcal{R}(\mathcal{C}^e/D) \)). Moreover, these CPOs are pointed: their least elements are \( (1, \bot : 1 \to D) \) and \( (1, \bot : 1 \to D, \top) \). This leads to the following alternative proof.

\textit{Proof of Theorem 1.1; Kleene fixed point.} Let \( F : \mathcal{C}^\text{op} \times \mathcal{C} \to \mathcal{C} \) be a CPO-functor. As we have seen in Section 2, we can view the admissible action of \( F \) on \( \mathcal{R} \) as a lifting \( F_{\mathcal{R}} : \mathcal{R}^\text{op} \times \mathcal{R} \rightarrow \mathcal{R} \). These functors give rise to functors \( F^e : \mathcal{C}^e \rightarrow \mathcal{C}^e \) and \( F_{\mathcal{R}}^e : \mathcal{R}^e \rightarrow \mathcal{R}^e \) that preserve colimits of chains. We have the following commutative diagram:
By working with canonical representatives, we can view these functors as continuous functions $f : \mathcal{C}/D \to \mathcal{C}/D$ and $f_R : \mathcal{R}(\mathcal{C}/D) \to \mathcal{R}(\mathcal{C}/D)$, and we can take their fixed points by Theorem 4.1. By construction, the fixed point of $f$ is just $D$, and the above diagram implies that $p(\text{fix}(f_R)) = \text{fix}(f) = D$, where $p : \mathcal{R}(\mathcal{C}/D) \to \mathcal{C}/D$ is the canonical projection. This means, after some unfolding, that the relation component of $\text{fix}(f_R)$ is a relation on $D$ that satisfies the recursive equation we are seeking.

\[ \mathcal{C}/D \xrightarrow{f} \mathcal{C}/D \]
\[ \downarrow \quad \downarrow \]
\[ \mathcal{C}/D \xrightarrow{f_R} \mathcal{C}/D. \]

Remark 4.2. Most categories used in domain theory have canonical representatives of embeddings—we can take the image of an embedding, as we have done above, or we can choose representatives using the axiom of choice. But if images are not available, there is another option that does not rely on the axiom of choice: to work with $\mathcal{C}$, the Karoubi envelope of $\mathcal{C}$. This category extends $\mathcal{C}$ by freely splitting all idempotent arrows in $\mathcal{C}$ (that is, arrows $p : X \to X$ such that $pp = p$). Roughly, this means that $\mathcal{C}$ contains canonical image objects of all idempotents in $\mathcal{C}$. In particular, we can compute the image of the idempotent $f^e f p$ determined by an embedding $f^e : Y \to X$, which yields a choice of representatives for embeddings. Moreover, $\mathcal{C}$ and $\mathcal{R}$ inherit the properties of the original categories that we relied on to carry the above constructions, so our results still apply.

5 Conclusion

We have just reviewed Pitts’ framework of relational properties of domains [Pit96] and seen how it relates to other important fixed-point theorems: the inverse limit construction [SP82], Banach’s fixed-point theorem, and Kleene’s fixed point theorem. These connections are implicit in some of the existing literature, and probably already known by experts. For example, the work of Hermida and Jacobs [HJ98] presents a different method for constructing relations on recursive data types that requires lifting limits and colimits along a fibration; likewise, the proof of Pitts’ method with the inverse limit construction uses Lemma 2.3, which lifts bilimits to a relational structure. As for Banach’s fixed-point theorem, several works for reasoning about denotational models [BST09b; BST09a; Bir+11] employ similar constructions while sometimes noting that Pitts’ framework could have been used instead [BST09b]. Here, we have seen how this connection goes beyond the construction of a particular set of logical relations, and lies at the heart of Pitts’ method. It is worth noting that the connections between these fixed-point theorems go beyond the setting of relational reasoning—e.g. Thamsborg [Tha10] discusses how we can view Banach’s fixed-point theorem as an instance of Kleene’s.

Traditionally, step-indexing uses the steps in some operational semantics to define recursive relations [AM01]. In light of the connections explained above, Pitts’
construction—as well as other applications of Banach’s fixed-point theorem for denotational models [Bir+11; BST09a; BST09b; MPS86]—use a similar trick to ensure that the recursion is well-founded, but count the number of unfoldings of a recursive type instead. In this sense, guarded recursion is more general, since the notion of counting can be tied to anything that can be tracked in the execution of a program, not just the number of unfoldings of the domain equation. On the other hand, relations constructed with Pitts’ method are often cleaner than their guarded counterparts, because they do not have to mention step indices or guards explicitly.

One question that I have not explored is how this connects to variants of Pitts’ construction used for operational semantics, as developed by Birkedal and Harper [BH99] or Crary and Harper [CH07]. Such works note that the projections $\pi_i$ can often be defined as regular programs in a language, and leverage this fact to adapt Pitts’ ideas to establish powerful reasoning principles for program equivalence. Like Pitts’ original construction, these works employ the Knaster-Tarski fixed-point theorem, but I believe that it might be possible to adapt their constructions to leverage other results as well. One possible connection lies in the proof of metric preservation for the Fuzz language [RP10]. Its argument employed step-indexed logical relations, but the indices of the relations tracked the number of recursive unfoldings reduced during execution rather than the number of transitions in a small-step semantics. This idea is similar to constructions by guarded recursion performed in denotational settings [BST09a; BST09b; Bir+11; MPS86], suggesting that it might be possible to obtain an alternative, operational proof of metric preservation for Fuzz along the lines of Birkedal and Harper [BH99] and Crary and Harper [CH07].

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