Fermionic cosmologies with Yukawa-type interactions

M. O. Ribas¹(a), F. P. Devecchi²(b) and G. M. Kremer²(c)

1 Departamento de Física, Universidade Tecnológica Federal do Paraná - Curitiba, Brazil
2 Departamento de Física, Universidade Federal do Paraná - Curitiba, Brazil

received 9 December 2010; accepted 20 December 2010
published online 28 January 2011

PACS 98.80.-k – Cosmology
PACS 98.80.Cq – Particle-theory and field-theory models of the early Universe (including cosmic pancakes, cosmic strings, chaotic phenomena, inflationary universe, etc.)
PACS 98.80.Jk – Mathematical and relativistic aspects of cosmology

Abstract – In this work we discuss if fermionic sources could be responsible for accelerated periods in a Friedmann-Robertson-Walker spatially flat universe, including a usual self-interaction potential of the Nambu-Jona-Lasinio type together with a fermion-scalar interaction potential of the Yukawa type. The results show that the combination of these potentials could promote an initially accelerated period, going through a middle decelerated era, with a final eternal accelerated period, where the self-interaction contribution dominates.

The identification of constituents that promoted the inflationary period in the evolution of the universe is a fundamental topic in cosmology. Several candidates have been tested for describing both the inflationary period and the present accelerated era: scalar fields, exotic equations of state and cosmological constant [1]. Besides these, fermionic fields have also been tested as gravitational sources of an expanding universe. In fact, fermionic sources can be responsible for accelerated periods with different regimes emerging from it [2–5]. In some of these models the fermionic field plays the role of the inflaton in the early period of the universe and of dark energy for the old universe, without the need of a cosmological constant term or a scalar field. In an old universe scenario an initially matter-dominated period gradually turns into a dark (fermion) energy period when an accelerated regime starts and remains for the rest of the evolution of the system. These fermionic sources have been investigated using several approaches, with results including numerical solutions, exact solutions, anisotropy-to-isotropy scenarios and cyclic cosmologies (see, for example [2–5]). When considering these models, one important point is the choice of the fermionic potential and in previous works [3] self-interacting potentials were tested to account for accelerated regimes. One complementary/alternative approach would be to consider the interactions between a scalar and a fermionic field through the presence of a potential of the Yukawa type [6]. This potential was proposed originally in particle physics to describe the behavior of strong force interactive fermions and here we want to test which kind of role the Yukawa potential could play in an young accelerating universe. Besides, as in previous works [3,4], a fermionic self-interaction term of the Nambu-Jona-Lasinio type [7] is included. We have used the metric signature (+, −, −, −) and natural units with \( 8\pi G = c = \hbar = 1 \).

Let us start with a brief review of the elements of the tetrad formalism employed to include fermionic fields in a dynamical curved space-time, since, as it is well known, the tetrad formalism permits the inclusion of fermions in gravitational models. Following the general covariance principle, a connection between the tetrad and the metric tensor \( g_{\mu\nu} \) is established through the relation \( g_{\mu\gamma} = e_{\mu}^{a} e_{\gamma}^{b} \eta_{ab} \), \( a = 0, 1, 2, 3 \), where \( e_{\mu}^{a} \) denotes the tetrad or vierbein and \( \eta_{ab} \) is the Minkowski metric tensor (see, e.g., [6,8,9]). Here Latin indices refer to the local inertial frame whereas Greek indices to the general system. Furthermore, the general covariance principle imposes that the usual Dirac-Pauli matrices \( \gamma^{a} \) must be replaced by their generalized counterparts \( \Gamma^{a} = e_{\mu}^{a} \gamma^{\mu} \), where these matrices satisfy the extended Clifford algebra, i.e., \( \{ \Gamma^{a}, \Gamma^{b} \} = 2g^{\mu\nu} \).

The generally covariant Dirac Lagrangian density for the fermionic field is given by

\[
\mathcal{L}_{\text{Dirac}}(\psi) = \frac{i}{2}[\bar{\psi} \Gamma^{\mu} D_{\mu} \psi - (D_{\mu} \bar{\psi}) \Gamma^{\mu} \psi] - V(\psi), \tag{1}
\]

where \( \psi \) and \( \bar{\psi} = \psi^{\dagger} \gamma^{0} \) denote the spinor field and its adjoint, respectively, and \( V(\psi) \) is the self-interaction
potential of the fermionic field which is supposed to be massless. In the above equation the covariant derivatives read
\[ D_\mu \psi = \partial_\mu \psi - \Omega_\mu \psi, \quad D_\mu \bar{\psi} = \partial_\mu \bar{\psi} + \psi \Omega_\mu, \]
where the spin connection \( \Omega_\mu \) is given by
\[ \Omega_\mu = -\frac{1}{4} g_{\rho\sigma} \Gamma^\rho_{\mu\delta} - e^\rho_\mu (\partial_\mu e^\delta_\rho)[\Gamma^\sigma_{\delta\gamma}], \]
with \( \Gamma^\rho_{\lambda\sigma} \) denoting the Christoffel symbols.

The Yukawa interaction of Yukawa type can be written as a massive scalar field that are connected through a constant of the Yukawa potential.

\[ \mathcal{L}_{\text{Yukawa}}(\phi, \psi) = -\lambda \bar{\psi} \psi \phi, \]
where \( \lambda \) is a coupling constant and \( \phi \) is the accompanying Yukawa field.

The total action for a massless fermionic field and a massive scalar field that are connected through an interaction of Yukawa type can be written as
\[ S = \int \sqrt{-g} d^4x \left\{ \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 - \lambda \bar{\psi} \psi \right\}, \]
where \( R \) is the scalar curvature.

We obtain the Dirac equations for the spinor field and its adjoint coupled to the gravitational field from the Euler-Lagrange equations for \( \psi \) and \( \bar{\psi} \), namely
\[ i \Gamma^\mu D_\mu \psi - \psi \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 - \lambda \bar{\psi} \psi = 0, \]
\[ i D_\mu \bar{\psi} \Gamma^\mu - \bar{\psi} \partial^\mu \phi \partial_\mu \phi + \lambda \bar{\psi} \psi = 0. \]

Likewise the Euler-Lagrange equation for \( \phi \) leads to the modified Klein-Gordon equation
\[ \nabla_\mu \nabla^\mu \phi + m^2 \phi + \lambda \bar{\psi} \psi = 0. \]

From the variation of the total action (6) with respect to the tetrad we obtain the Einstein field equations
\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -T_{\mu\nu}, \]
where \( T_{\mu\nu} \) is the total energy-momentum tensor which is a sum of the contributions from the fermionic and scalar fields. Since we are dealing with a fermionic field in a space-time without torsion, the total energy-momentum tensor is symmetric and reads
\[ T^{\mu\nu} = \frac{1}{4} \left[ \bar{\psi} \Gamma^\mu D^\nu \psi + \psi \Gamma^\nu D^\mu \psi - D^\mu \bar{\psi} \Gamma^\nu \psi \right] - D_\mu \bar{\psi} \Gamma^\nu D^\mu \psi + \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 - \lambda \bar{\psi} \psi + \frac{1}{2} \left( \bar{\psi} \Gamma^\lambda D_\lambda \psi + D_\lambda \bar{\psi} \Gamma^\lambda \psi \right) - V(\psi). \]

The Friedmann-Robertson-Walker metric
\[ ds^2 = dt^2 - a(t)^2(dx^2 + dy^2 + dz^2), \]
with \( a(t) \) denoting the cosmic scale factor, mirrors the homogeneity and isotropy properties of a spatially flat universe. For this metric the components of the tetrad, Dirac-Pauli matrices and spin connection read
\[ e^0_0 = \delta^0_0, \quad e^1_0 = \frac{1}{a(t)} \delta^1_0, \quad \Gamma^0 = \gamma^0, \]
\[ \Gamma^i = \frac{1}{a(t)} \gamma^i, \quad \Omega_0 = 0, \quad \Omega_i = \frac{1}{2} \dot{a}(t) \gamma^i \gamma^0, \]
where the dot represents the derivative with respect to time.

Furthermore, from the hypothesis of homogeneity and isotropy it follows that the spinor and scalar fields depend only on time so that the Dirac (7), (8) and Klein-Gordon (9) equations reduce to
\[ \dot{\psi} + \frac{3}{2} a \dot{\phi} + m^2 \phi + \lambda \bar{\psi} \psi = 0, \]
\[ \ddot{\psi} + \frac{3}{2} \dot{a} \dot{\phi} - m^2 \phi - \lambda \bar{\psi} \psi = 0, \]
\[ \dot{\phi} + \frac{3}{2} \dot{a} \phi + m^2 \phi + \lambda \bar{\psi} \psi = 0. \]

From the Einstein field equations (10) it follows the Friedmann and acceleration equations
\[ \left( \frac{\dot{a}}{a} \right)^2 = \frac{1}{3} \rho, \quad \ddot{a} = -\frac{1}{6} (\rho + 3p), \]
respectively, where the total energy density \( \rho \) and total pressure \( p \) are are given by
\[ \rho = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} m^2 \phi^2 + \lambda \bar{\psi} \psi + V(\psi), \]
\[ p = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} m^2 \phi^2 - V(\psi) + \frac{\partial V}{\partial \psi} \frac{\dot{\psi}}{2} + \frac{\psi}{2} \frac{\partial V}{\partial \psi}. \]

We suppose now that the self-interaction potential of the fermionic field is given by \( V = \Delta (\bar{\psi} \psi)^n \), where \( \Delta \) is a coupling constant and \( n \) a constant exponent. If we substitute this potential into the Dirac equations (15) and (16), multiply the first one by \( \psi \), the second one by \( \bar{\psi} \) and add the resulting equations, we obtain a differential equation for the bilinear \( \bar{\psi} \psi \) whose solution is given by \( \bar{\psi} \psi = C/a^3 \), where \( C \) denotes an integration constant.

Once we know the explicit form of the bilinear \( \bar{\psi} \psi = C/a^3 \) the total energy density (19) and total pressure (20) become
\[ \rho = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} m^2 \phi^2 + \frac{C_1}{a^3} \phi + \frac{C_2}{a^{3n}}, \]
\[ p = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} m^2 \phi^2 + (n - 1) \frac{C_2}{a^{3n}}. \]
Fermionic cosmologies with Yukawa-type interactions

where we have introduced the constants $C_1 = C\lambda$ and $C_2 = C\Lambda$.

Therefore, from (17) and (18) we have a coupled system of non-linear differential equations for the determination of $a(t)$ and $\phi(t)$ that include the modified Klein-Gordon equation and the acceleration equation, namely,

$$\ddot{\phi} + \frac{3}{a} \dot{a} \dot{\phi} + m^2 \phi + \frac{C_1}{a^3} = 0,$$

$$\ddot{a} + \frac{1}{6} \left[ 2\dot{\phi}^2 - m^2 \phi^2 + \frac{C_1}{a^3} \phi + (3n - 2) \frac{C_2}{a^{3n}} \right] = 0.$$

The coupled system of the differential equations (23) and (24) was solved numerically with the initial conditions that the cosmic scale factor and the energy density were normalized at $t = 0$, i.e., $a(0) = 1$ and $\rho(0) = 1$, which from the Friedmann equation (18) implies that $\dot{a}(0) = 1/\sqrt{3}$. Furthermore, it was supposed that the scalar field at time $t = 0$ was zero with a very small slope, i.e., $\phi(0) = 0$ and $\dot{\phi}(0) = 10^{-4}$. With these initial conditions, the coupling constant related with the self-interaction potential of the fermionic field is no more arbitrary, since from the expression for the energy density (21) we must have $C_2 = \rho(0) - \phi(0)^2/2$. However, it remains to specify the coupling constant of the Yukawa potential $C_1$, the mass of the scalar field $m$ and the exponent of the self-interaction potential of the fermionic field $n$. We have chosen fixed normalized values for the mass of the scalar field and for the exponent of the self-interaction potential of the scalar field —namely, $m = 10^{-6}$ and $n = 0.5$— and variable values for the coupling constant of the Yukawa potential in order to see the influence of the strength of this potential on the solutions of the coupled system of differential equations.

As a result of the numerical integration, the scale factor behaves according to an ever-expanding universe. The classification of eras can be done in terms of the time evolution of the acceleration field. In fact, in fig. 1 we have plotted the behavior of the acceleration $\ddot{a}$ as a function of time, for three different (normalized) values of the coupling constant of the Yukawa potential: $|C_1| = 2.0$, 2.4 and 2.8. From this figure it is possible to infer that we have three periods which can be interpreted as an initial inflationary period where the acceleration is positive, a period dominated by radiation/matter where the acceleration is negative and a period dominated by the dark energy where the acceleration becomes again positive. Furthermore, we observe that longer periods of deceleration are related with larger values of $|C_1|$. By increasing the value of $|C_1|$ beyond 2.8 the solution of acceleration field presents only an accelerated period followed by a decelerated one, while for values of $|C_1|$ smaller than 2.0 the acceleration field has only one solution corresponding to an accelerated period.

In fig. 2 we show the behavior of the Yukawa potential $\bar{\psi} \phi \bar{\psi}$ as a function of time for different values of the coupling constant $C_1$. The behavior of the curves in this figure corroborates the explanation given above, i.e., by the increase of the coupling constant the amplitude of the Yukawa potential becomes more accentuated, so that it has a direct influence on the decelerated period.
The coupling constant of the Yukawa potential has also an influence in the time decay of the total energy density. This fact can be observed from the analysis of fig. 3, where the total energy density is plotted as a function of time for different values of $C_1$.

To sum up, in this work we have investigated the possibility of a fermionic cosmology to describe the different accelerated regimes of our universe. This is possible due to the inclusion of two potentials, one self-interactive and one of the Yukawa type. When the universe is still young, the combination of these contributions promote a short positive accelerated regime that can be associated to an inflationary period, followed by a decelerated period that would correspond to a matter/radiation-dominated universe. We conclude that the intensity of the Yukawa coupling determines the time interval and the amplitude of the decelerated regime. On the other hand, the constant that controls the intensity of the self-interactive potential is decisive for the existence of a final accelerated period, that could correspond to a universe dominated by dark energy.

***

FPD and GMK acknowledge the support by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq).

REFERENCES

[1] Peebles P. J. E. and Ratra B., Astrophys. J. Lett., 17 (1988) 325; Copeland E., Sami M. and Tsujikawa S., Int. J. Mod. Phys. D, 15 (2006) 1573; Capozziello S., De Laurentis M., Nojiri S. and Odintsov S. D., Phys. Rev. D, 79 (2009) 124007; Capozziello S., De Martino S. and Falanga M., Phys. Lett. A, 299 (2002) 494; Kremer G. M., Phys. Rev. D, 68 (2003) 123507.

[2] Obukhov Y. N., Phys. Lett. A, 182 (1993) 214; Saha B. and Shikin G. N., Gen. Relativ. Gravit., 29 (1997) 1099; Saha B., Phys. Rev. D, 64 (2001) 123501; Saha B. and Boyadjiev T., Phys. Rev. D, 69 (2004) 124010; Saha B., Phys. Rev. D, 74 (2006) 124030.

[3] Ribas M. O., Deveccchi F. P. and Kremer G. M., Phys. Rev. D, 72 (2005) 123502; EPL, 81 (2008) 19001; Chimento L. P., Deveccchi F. P., Forte M. and Kremer G. M., Class. Quantum Grav., 25 (2008) 085007; de Souza R. C. and Kremer G. M., Class. Quantum Grav., 25 (2008) 225006; Samojeden L. L., Deveccchi F. P. and Kremer G. M., Phys. Rev. D, 81 (2010) 027301; Ribas M. O. and Kremer G. M., Gravit. Cosmol., 16 (2010) 173.

[4] Saha B., Phys. Rev. D, 69 (2004) 124006; Gu Y.-Q., Int. J. Mod. Phys. A, 22 (2007) 4667; Cai Y.-F. and Wang J., Class. Quantum Grav., 25 (2008) 165014; Kassandrov V. V., Gravit. Cosmol., 14 (2008) 53; Saha B. and Visinescu M., Int. J. Theor. Phys., 49 (2010) 1411; Cai Y.-F., Sariidakis E. N., Setare M. R. and Xia J.-Q., Phys. Rep., 493 (2010) 1; Wang J., Cui S.-W. and Zhang C.-M., Phys. Lett. B, 683 (2010) 110; Rakhi R., Vidayagovindan G. V., Indulekha K. and Abraham N. P., Int. J. Mod. Phys. A, 25 (2010) 1267.

[5] Armendáriz-Picón C. and Greene P. B., Gen. Relativ. Gravit., 35 (2003) 1637.

[6] Ryder L. H., Quantum Field Theory (Cambridge University Press, Cambridge) 1996; Itzykson C. and Zuber J.-B., Quantum Field Theory (McGraw-Hill, New York) 1980; Bjorken J. D. and Drell S. D., Relativistic Quantum Mechanics (McGraw-Hill, New York) 1984.

[7] Nambu Y. and Jona-Lasinio G., Phys. Rev., 122 (1961) 345.

[8] Wald R. M., General Relativity (The University of Chicago Press, Chicago) 1984.

[9] Weinberg S., Cosmology (Oxford University Press, New York) 2008.