LIST COLORING AND $n$-MONOPHILIC GRAPHS

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Abstract. In 1990, Kostochka and Sidorenko proposed studying the smallest number of list-colorings of a graph $G$ among all assignments of lists of a given size $n$ to its vertices. We say a graph $G$ is $n$-monophilic if this number is minimized when identical $n$-color lists are assigned to all vertices of $G$. Kostochka and Sidorenko observed that all chordal graphs are $n$-monophilic for all $n$. Donner (1992) showed that every graph is $n$-monophilic for all sufficiently large $n$. We prove that all cycles are $n$-monophilic for all $n$; we give a complete characterization of 2-monophilic graphs (which turns out to be similar to the characterization of 2-choosable graphs given by Erdős, Rubin, and Taylor in 1980); and for every $n$ we construct a graph that is $n$-choosable but not $n$-monophilic.

1. Introduction

Suppose for each vertex $v$ of a graph $G$ we choose a list $L(v)$ of a fixed number $n$ of colors, and then to each $v$ we assign a color chosen randomly from its color list $L(v)$. If our goal is to maximize the probability of getting the same color for at least two adjacent vertices, then it seems intuitively plausible that we should give every vertex of $G$ the same list. But this turns out to be false for some graphs! Graphs which do satisfy this property are called “$n$-monophilic” (defined more precisely below). It is natural to ask: Which graphs are $n$-monophilic for a given $n$? This question has been open at least since 1990.

We work with finite, simple graphs, and use the notation and terminology of Diestel [1]. Given a graph $G = (V, E)$, a list assignment (resp. $n$-list assignment, $n \in \mathbb{N}$) for $G$ is a function that assigns a subset (resp. $n$-subset) of $\mathbb{N}$ to each vertex $v \in V$, denoted $L(v)$. Given a list assignment $L$ for $G$, a (proper) coloring of $G$ from $L$ is a function $\gamma : G \to \mathbb{N}$ such that for each vertex $v \in V$, $\gamma(v) \in L(v)$, and for any pair of adjacent vertices $v$ and $w$, $\gamma(v) \neq \gamma(w)$. We denote the number of distinct colorings of $G$ from $L$ by $\text{col}(G, L)$. In the special case where $L(v) = [n] = \{1, \ldots, n\}$ for every $v \in V$, we also write $\text{col}(G, n)$ for $\text{col}(G, L)$. We say $G$ is $n$-monophilic if $\text{col}(G, n) \leq \text{col}(G, L)$ for every $n$-list assignment $L$ for $G$. Clearly a graph is $n$-monophilic iff each connected component of it is $n$-monophilic. So we restrict attention to connected graphs only.

In 1990, Kostochka and Sidorenko [7] proposed studying the minimum value $f(n)$ attained by $\text{col}(G, L)$ over all $n$-list assignments $L$ for a given graph $G$. They observed that for chordal graphs $f(n)$ equals the chromatic polynomial of $G$ evaluated at $n$; i.e., chordal graphs are $n$-monophilic for all $n$. In 1992 Donner [4] showed that for any fixed graph $G$, $f(n)$ equals the chromatic polynomial of $G$ for
there appears to be no further literature on this subject since then. A graph $G$ is said to be $n$-colorable if $\text{col}(G, n) \geq 1$; and $G$ is said to be $n$-choosable (or $n$-list colorable) if $\text{col}(G, L) \geq 1$ for every $n$-list assignment $L$ for $G$. The chromatic number of $G$, denoted $\chi(G)$, is the smallest $n$ such that $G$ is $n$-colorable. The list chromatic number of $G$ (also called the choice number of $G$), denoted $\chi_l(G)$ (or ch($G$)), is the smallest $n$ such that $G$ is $n$-choosable. Since $\chi$ and $\chi_l$ are well-known and have been studied extensively, it is interesting to compare the concept of $n$-monophilic graphs to them. The following are clear from definitions. For every graph $G$,

1. $\chi(G) \leq \chi_l(G)$;
2. if $n < \chi(G)$, then $G$ is $n$-monophilic;
3. if $\chi(G) \leq n < \chi_l(G)$, then $G$ is not $n$-monophilic.

The interesting region is $\chi_l(G) \leq n$, which contains $n$-monophilic graphs (e.g., all cycles and all chordal graphs), as well as non-$n$-monophilic graphs (Section 5).

Deciding whether a graph is $n$-choosable turns out to be difficult. Even deciding whether a given planar graph is 3-choosable is NP-hard [6]. Thus one might expect the decision problem for $n$-monophilic graphs to be NP-hard as well; so a “nice characterization” (i.e., one that would lead to a polynomial time decision algorithm) of $n$-monophilic graphs might not exist. In this paper we prove that all cycles are $n$-monophilic for all $n$, and $G$ is not 2-monophilic iff all its cycles are even and it contains at least two cycles whose union is not $K_{2,3}$. This characterization of 2-monophilic graphs is fairly similar to that given by Erdős, Rubin, and Taylor [4]. But, as we show in Section 5 for every $n \geq 2$ there is a graph that is $n$-choosable but not $n$-monophilic.

2. Chordal graphs are $n$-monophilic

A graph is chordal if every cycle in it of length greater than 3 has a chord. Kostochka and Sidorenko [7] observed that all chordal graphs are $n$-monophilic for all $n$. Because the proof is short, we include it below. Observe that if $H$ is a subgraph of $G$, then $L$ restricts in a natural way to give a list assignment for $H$, and $\text{col}(H, L)$ denotes the number of colorings of $H$ from this restricted list assignment.

**Lemma 1.** Let $G$ be an $n$-monophilic graph, and suppose $v_1, \ldots, v_k$ induce a complete subgraph of $G$. Let $G'$ be the graph obtained from $G$ by adding a new vertex and connecting it to $v_1, \ldots, v_k$. Then $G'$ is $n$-monophilic.

**Proof.** Let $L$ be an $n$-list assignment for $G'$. If $n \leq k$, then $\text{col}(G', n) = 0$ and we are done. So assume $n > k$. Then each coloring of $G$ from $L$ extends to at least $n - k$ distinct colorings of $G'$ from $L$. Hence $\text{col}(G', L) \geq (n - k)\text{col}(G, L) \geq (n - k)\text{col}(G, n) = \text{col}(G', n)$. 

A graph has a simplicial elimination ordering if its vertices can be ordered as $v_1, \ldots, v_k$ such that for each $v_i$ the subgraph induced by $\{v_i\} \cup N(v_i) \cap \{v_1, \ldots, v_{i-1}\}$, where $N(v_i)$ denotes the set of neighboring vertices of $v_i$, is a complete graph.

**Theorem.** (Dirac [2]) A graph is chordal iff it has a simplicial elimination ordering.

The above lemma and Dirac’s Theorem give us:
Corollary. (Kostochka and Sidorenko [7]) Every chordal graph is \( n \)-monophilic for every \( n \).

Note that trees and complete graphs are chordal and hence are \( n \)-monophilic for every \( n \).

3. Cycles are \( n \)-monophilic

In this section we show that every \( m \)-cycle is \( n \)-monophilic for all \( m, n \). We first need some definitions. Let \( L \) be a list assignment for a graph \( G \). For \( i = 1, \cdots, k \), let \( v_i \) be a vertex of \( G \), and \( c_i \) a color in \( L(v_i) \). Then \( \text{col}(G, L, v_1, c_1, \cdots, v_k, c_k) \) denotes the number of colorings of \( G \) from \( L \) which assign color \( c_i \) to \( v_i \), \( i = 1, \cdots, k \). We say \( L \) is minimizing for \( G \) if \( \text{col}(G, L) \leq \text{col}(G, L') \) for every list assignment \( L' \) where \( |L'(v)| = |L(v)| \) for every \( v \).

Lemma 2. Let \( G_1 \) and \( G_2 \) be disjoint subgraphs of a graph \( G \), with \( v_i \) a vertex of \( G_i \), such that \( G = G_1 \cup G_2 + v_1 v_2 \). Let \( L \) be a list assignment for \( G \). Then there exists a list assignment \( L' \) such that \( |L'(v)| = |L(v)| \) for every \( v \), \( L'(v_1) \subseteq L'(v_2) \) or \( L'(v_2) \subseteq L'(v_1) \), and \( \text{col}(G, L') \leq \text{col}(G, L) \). Moreover, the inequality is strict provided there exist \( c_1 \in L(v_1) \setminus L(v_2) \) and \( c_2 \in L(v_2) \setminus L(v_1) \) with \( \text{col}(G_1, L, v_1, c_1) \neq 0 \) and \( \text{col}(G_2, L, v_2, c_2) \neq 0 \).

Proof. If \( L(v_1) \subseteq L(v_2) \) or \( L(v_2) \subseteq L(v_1) \), then there is nothing to show. So we can assume there exist colors \( c_1 \in L(v_1) \setminus L(v_2) \) and \( c_2 \in L(v_2) \setminus L(v_1) \).

Let \( L' \) be the list assignment that is identical to \( L \) except that in the lists assigned to the vertices of \( G_2 \) every \( c_1 \) is replaced with \( c_2 \) and every \( c_2 \) with \( c_1 \). Then, for each \( c \neq c_1 \) in \( L(v_1) \), \( \text{col}(G, L', v_1, c) = \text{col}(G, L, v_1, c) \) (since \( c \neq c_2 \), as \( c_2 \notin L(v_1) \)). Furthermore,

\[
\text{col}(G, L', v_1, c_1) = \text{col}(G, L, v_1, c_1) - \text{col}(G, L, v_1, c_2) = \text{col}(G_1, L, v_1, c_1) \cdot \text{col}(G_2, L, v_2, c_2)
\]

Hence, \( \text{col}(G, L', v_1, c_1) > \text{col}(G, L, v_1, c_1) \) if \( \text{col}(G_1, L, v_1, c_1) \) and \( \text{col}(G_2, L, v_2, c_2) \) are both nonzero.

Now, by renaming \( L' \) as \( L \) and then repeating this process as long as \( L(v_1) \nsubseteq L(v_2) \) and \( L(v_2) \nsubseteq L(v_1) \), we eventually obtain the desired \( L' \). \( \square \)

The length of a path is the number of edges it contains. For \( n \geq 2 \), an \((n, n-1)\)-list assignment for a path of length at least one is a function that assigns \( n \)-color lists to the path’s interior vertices, if any, and \((n-1)\)-color lists to its two terminal vertices. Suppose the interior vertices of the path have identical lists, each of which contains as a subset the \((n-1)\)-color list of each of the two terminal vertices. If, in addition, these two \((n-1)\)-color lists are identical, we say \( L \) is type A, and denote \( \text{col}(P, L) \) by \( A_k \); otherwise we say \( L \) is type B, and denote \( \text{col}(P, L) \) by \( B_k \). Note that, up to renaming colors, all type A \((n, n-1)\)-list assignments for a given path are equivalent, and similarly for type B.

Lemma 3. Let \( n \geq 2 \), and let \( L \) be an \((n, n-1)\)-list assignment for a path \( P \) of length \( k \geq 2 \). Then: (a) \( A_k - B_k = (-1)^k \), and \( A_k = \frac{n-1}{n}((-n-1)^{k+1} + (-n)^k) \); (b) \( \text{col}(P, L) \geq \min(A_k, B_k) \); and (c) for \( n \geq 3 \), \( L \) is minimizing only if \( k \) is odd and \( L \) is type A or \( k \) is even and \( L \) is type B.
Proof. Part (a): Let \( v \) be a terminal vertex of \( P \), and let \( w \) be the vertex adjacent to \( v \). Suppose \( L \) is type A. Then, for each color that we choose to assign to \( v \), there remains an \((n - 1)\)-color list of choices for \( w \), and this list is not the same as the \((n - 1)\)-color list of the other terminal vertex of \( P \). Thus we get
\[
A_k = (n - 1)B_{k-1}
\] (2)
By a similar (but slightly longer) reasoning, we see that
\[
B_k = A_{k-1} + (n - 2)B_{k-1}
\] (3)
Subtracting (3) from (2) gives
\[
A_k - B_k = (-1)(A_{k-1} - B_{k-1})
\] (4)
Now, by direct calculation, \( A_1 = (n - 1)(n - 2) \), and \( B_1 = (n - 2)^2 + (n - 1) \). It follows that \( A_1 - B_1 = -1 \), which together with (4) inductively yield
\[
A_k - B_k = (-1)^k
\] (5)
Finally, combining (2) with (5) gives \( A_k = (n - 1)(A_{k-1} + (-1)^k) \). It follows by induction from the base case \( A_1 = (n - 1)(n - 2) \) that \( A_k = \frac{n-1}{n}((n-1)^{k+1} + (-1)^k) \).

Part (b): This follows immediately from Lemma 2 and the definition of \( A_k \) and \( B_k \).

Part (c): Assume \( k \geq 2 \), since the case \( k = 1 \) is trivial. Suppose, toward contradiction, that \( v_1, v_2 \) are adjacent vertices of \( P \) such that \( L(v_1) \not\subseteq L(v_2) \) and \( L(v_2) \not\subseteq L(v_1) \). Then, in the proof of Lemma 2, the term \( \text{col}(G_1, L, v_1, c_1) \cdot \text{col}(G_2, L, v_2, c_2) \) being subtracted in equation (1) is positive since each \( G_i \) is now a path and \( n \geq 3 \). This would imply that \( L \) is not minimizing. Hence, if \( L \) is minimizing, it must be type A or type B. The result now follows from equation (5).

Remark: For \( n = 2 \) a minimizing list need not be type A or type B; examples are easy to construct.

Lemma 4. Let \( L \) and \( L' \) be distinct list assignments for a path \( P \) such that for every vertex \( v \) in \( P \) we have \( L(v) \subseteq L'(v) \) and \( |L'(v)| \geq 2 \). Then \( \text{col}(P, L) < \text{col}(P, L') \).

Proof. Since \( L(v) \subseteq L'(v) \) for every \( v \), every coloring of \( P \) from \( L \) is also a coloring of \( P \) from \( L' \). And since \( L \) and \( L' \) are distinct, for some vertex \( w \) there is a color \( c \in L'(w) \setminus L(w) \). By hypothesis, \( |L'(v)| \geq 2 \) for every \( v \); hence \( \text{col}(P, L', w, c) \geq 1 \), i.e., there is at least one coloring of \( P \) from \( L' \) that is not a coloring of \( P \) from \( L \). The result follows.

Let \( L \) be a list assignment for a graph \( G \), and let \( v_1, \ldots, v_k \) be vertices in \( G \), where \( k < |G| \). Let \( c_i \in L(v_i) \). We define the list assignment \( L_{c_1, \ldots, c_k} \text{ induced by} \ L, \ c_1, \ldots, c_k \) on the graph \( H = G - \{v_1, \ldots, v_k\} \) by: for every vertex \( v \in H \), \( L_{c_1, \ldots, c_k}(v) = L(v) \setminus \{c_i : v_i \in N(v)\} \) where \( N(v) \) denotes the set of vertices in \( G \) adjacent to \( v \). Then clearly \( \text{col}(G, L, v_1, c_1, \ldots, v_k, c_k) = \text{col}(H, L_{c_1, \ldots, c_k}) \).

Theorem 1. Every cycle is \( n \)-monophilic for all \( n \geq 2 \).

Proof. Let \( C \) be a cycle of length \( k \geq 3 \). Suppose we assign the color list \([n]\) to every vertex of \( C \). Then, by Lemma 3 for each \( c \in [n] \), \( \text{col}(C, n, v, c) = A_{k-2} \). Therefore \( \text{col}(C, n) = nA_{k-2} \). Let \( L \) be an \( n \)-list assignment that does not assign identical lists to all vertices of \( C \). We will show \( \text{col}(C, L) \geq nA_{k-2} \). Since \( L \) does
not assign identical lists to all vertices of \( C \), there are adjacent vertices \( v \) and \( w \) such that \( L(v) \neq L(w) \). Let \( P \) be the path \( C - v \). We have two cases.

Case 1: \( k \) is odd. For each \( c \in L(v) \) we have \( \text{col}(C, L, v, c) \geq A_{k-2} \), then we are done. So assume for some \( c_0 \in L(v) \) we have \( \text{col}(C, L, v, c_0) < A_{k-2} \). Then \( \text{col}(P, L_{c_0}) < A_{k-2} \) since \( \text{col}(P, L_{c_0}) = \text{col}(C, L, v, c_0) \). As \( P \) has even length \( k - 2 \), it follows from Lemma 3 that \( L_{c_0} \) must be a type B \((n, n-1)\)-list assignment for \( P \) and \( \text{col}(P, L_{c_0}) = B_{k-2} - 1 \). Let \( u \) be the vertex adjacent to \( v \) in \( C - w \). Then \( c_0 \) is in both \( L_u \) and \( L_w \), and not in \( L_{c_0}(u) \cup L_{c_0}(w) = L_{c_0}(x) = L(x) \), where \( x \) is any vertex in \( P - \{u, w\} \). Therefore, for each \( c \neq c_0 \in L(v) \), the induced list assignment \( L_c \) on \( P \) is not type B because \( c_0 \) is in \( L_c(u) \) and \( L_c(w) \) but not in \( L_c(x) = L(x) \). Hence \( \text{col}(C, L, v, c) > B_{k-2} \), i.e., \( \text{col}(C, L, v, c) \geq A_{k-2} \).

Now, as follows that \( \text{col}(C, L, v, d) \geq A_{k-2} + 1 \). Note that \( \text{col}(L_d) = \text{col}(L) \) contains \( n \) colors. Let \( L' \) be an \((n, n-1)\)-list assignment for \( P \) obtained from \( L_d \) by removing one element other than \( c_0 \) from \( L_d(w) \), and also one element from \( L_d(u) \) if \( |L_d(u)| = n \). Since \( c_0 \in L'_d(w) \), \( L'_d \) is not a type B list assignment for \( P \). So, by Lemma 3 \( \text{col}(P, L'_d) \geq B_{k-2} + 1 = A_{k-2} \). Hence, by Lemma 4 \( \text{col}(P, L_d) = \text{col}(P, L'_d) \geq A_{k-2} \), as desired.

Thus we get

\[
\text{col}(C, L) = \text{col}(C, L, v, c_0) + \text{col}(C, L, v, d) + \sum_{c \in L(v) \setminus \{c_0, d\}} \text{col}(C, L, v, c) \\
\geq B_{k-2} + (A_{k-2} + 1) + (n - 2)A_{k-2} \\
= nA_{k-2}
\]

as desired.

Now suppose \( n \geq 3 \). Clearly \( \text{col}(C, 2) = 2 \). Let \( Q \) be the path obtained by removing the edge \( vw \) (but not its vertices) from \( C \). We will show there are at least two colorings of \( Q \) from \( L \) that extend to colorings of \( C \). First, we need a definition. Let \( x \) and \( y \) be any two vertices in a graph \( G \) with a given list assignment \( M \). We say that \( c \in M(x) \) forces \( d \in M(y) \) if \( \text{col}(G, M, x, c, y, d) \geq 1 \) and for every \( d' \neq d \in M(y) \), \( \text{col}(G, M, x, c, y, d') = 0 \).

Denote the vertices of \( Q \) by \( v_0, \ldots, v_k \), where \( v_0 = v \), \( v_k = w \), and \( v_i \) is adjacent to \( v_{i+1} \) for \( i = 0, \ldots, k-1 \). Suppose, toward contradiction, that each color in \( L(v) \) forces a color in \( L(w) \). Then each color in \( L(v_i) \) must be in \( L(v_{i+1}) \). Hence \( L(v_1) = L(v_{i+1}) \). But \( L(v) \neq L(w) \). So at least one of the colors in \( L(v) \) forces no color in \( L(w) \). Let \( L(v) = \{\alpha, \beta\} \) and \( L(w) = \{\gamma, \delta\} \). Then, without loss of generality, \( \alpha \) forces neither \( \gamma \) nor \( \delta \). Therefore \( \text{col}(Q, L, v, \alpha, w, \gamma) \) and \( \text{col}(Q, L, v, \alpha, w, \delta) \) are both nonzero, since \( \text{col}(Q, L, v, \alpha) \geq 1 \). Now, if \( \alpha \) is different from both \( \gamma \) and \( \delta \), then any coloring of \( Q \) that assigns \( \alpha \) to \( v \) extends to a coloring of \( C \), and we're done. On the other hand, suppose \( \alpha \) is not different from both \( \gamma \) and \( \delta \). Then, without loss of generality, \( \alpha = \gamma \). So any coloring of \( Q \) with \( \alpha \) assigned to \( v \) and \( \delta \) to \( w \) extends to a coloring of \( C \). Also, \( \beta \neq \gamma \) since \( \beta \neq \alpha \). And \( \beta \neq \delta \) since
\(\{\alpha, \beta\} \neq \{\gamma, \delta\}\). So any coloring of \(Q\) with \(\beta\) assigned to \(v\) also extends to a coloring of \(C\). As \(\text{col}(Q, L, v, \beta) \geq 1\), we are done again.

\[\square\]

Note that although cycles are 2-monophilic, every even cycle has a minimizing 2-list assignment that does not assign the same list to every vertex: assign the list \(\{1, 2\}\) to two adjacent vertices, and the list \(\{2, 3\}\) to all the remaining vertices.

4. A characterization of 2-monophilic graphs

The \textbf{core} of a connected graph \(G\) is the subgraph of \(G\) obtained by repeatedly deleting vertices of degree 1 until every remaining vertex has degree at least 2.

**Lemma 5.** A connected graph is \(n\)-monophilic iff its core is \(n\)-monophilic.

**Proof.** This is proved easily using Lemma 2 and induction on the number of vertices in the graph.

\[\square\]

Let \(\theta_{a,b,c}\) denote the graph consisting of two vertices connected by three paths of lengths \(a, b, c\) with mutually disjoint interiors. In particular, \(\theta_{2,2,2}\) is the complete bipartite graph \(K_{2,3}\). In the paper by Erdős, Rubin, and Taylor [4], we find the following result by Rubin:

**Theorem.** (A. L. Rubin) A connected graph is 2-choosable iff its core is a single vertex, an even cycle, or \(\theta_{2,2,2m}\) for some \(m \geq 1\).

We use this to prove that a connected graph is 2-monophilic iff its core is a single vertex, is an even cycle, is \(K_{2,3}\), or contains an odd cycle.

**Lemma 6.** \(K_{2,3}\) is 2-monophilic.

**Proof.** In Figure 1 the five vertices of \(K_{2,3}\) have been labeled as \(u, v, w, x, y\). Since \(K_{2,3}\) has no odd cycles, \(\text{col}(K_{2,3}, 2) = 2\). Let \(L\) be a 2-list assignment for \(K_{2,3}\). We will show that \(\text{col}(K_{2,3}, L) \geq 2\). We consider three cases, depending on the number of colors that \(L(x)\) and \(L(y)\) share.

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{k23.png}
\caption{\(K_{2,3}\)}
\end{figure}

\textbf{Case 1.} \(|L(x) \cap L(y)| = 2\). Then there are two ways to assign the same color to \(x\) and \(y\); and for each way, there is at least one way to color each of \(u, v,\) and \(w\). Hence \(\text{col}(K_{2,3}, L) \geq 2\).

\textbf{Case 2.} \(|L(x) \cap L(y)| = 1\). Without loss of generality, \(L(x) = \{1, 2\}\) and \(L(y) = \{1, 3\}\). If at least one of the vertices \(u, v, w,\) does not contain color 1 in its list, then there are at least two distinct colorings of \(K_{2,3}\) with color 1 assigned to both \(x\) and \(y\). On the other hand, if all three vertices \(u, v, w\) contain color 1
their lists, then we can obtain one coloring by assigning color 2 to \( x \), 3 to \( y \), and 1 to \( u, v, w \), and another coloring by assigning color 1 to both \( x \) and \( y \), and using the second color in each of the lists for \( u, v, w \).

**Case 3.** \( |L(x) \cap L(y)| = 0 \). Without loss of generality, assume \( L(x) = \{1, 2\} \), \( L(y) = \{3, 4\} \). Then there are four ways to color the pair \( x, y \). If at least two of these extend to a coloring of \( K_{2,3} \), we are done. Otherwise, without loss of generality, \( L(u) = \{1, 3\} \), \( L(v) = \{1, 4\} \), and \( L(w) = \{2, 3\} \). Then \((u, v, w, x, y) = (1, 1, 3, 2, 4)\) and \((u, v, w, x, y) = (3, 1, 3, 2, 4)\) are two distinct colorings of \( K_{2,3} \).

\[\square\]

**Theorem 2.** A connected graph is 2-monophilic iff its core is a single vertex, is a cycle, is \( K_{2,3} \), or contains an odd cycle.

Equivalently: A graph is not 2-monophilic iff all its cycles are even and it contains at least two cycles whose union is not \( K_{2,3} \).

**Proof.** Clearly a single vertex and a graph that contains an odd cycle are both 2-monophilic. Also, by Theorem 1 and Lemma 6, all cycles and \( K_{2,3} \) are 2-monophilic. Using Lemma 5, this gives us one direction of the theorem.

To prove the converse, let \( G \) be a 2-monophilic graph. If \( G \) is not 2-colorable, then it must contain an odd cycle, and we are done. So assume \( \chi(G) \) is 1 or 2. Then \( G \) is also 2-choosable since it is 2-monophilic. So, by Rubin’s theorem above, it is enough to show that for \( m \geq 2 \), \( \theta_{2,2,2m} \) is not 2-monophilic.

Figure 2 shows a 2-list assignment \( L \) for the case when \( m = 2 \). When \( m > 2 \), we add an even number of vertices to the interior of the edge \( uv \) in Figure 2 and assign to each new vertex the list \( \{1, 2\} \). It is then easy to check that for \( m \geq 2 \), \( \text{col}(\theta_{2,2,2m}, L) = 1 < 2 = \text{col}(\theta_{2,2,2m}, 2) \), as desired.

5. Examples of \( n \)-choosable, non-\( n \)-monophilic graphs

Given the close similarity between Theorem 2 and Rubin’s theorem above, it is natural to wonder how similar or different the notions of \( n \)-choosable and \( n \)-monophilic are. In this section, for each \( n \geq 2 \) we construct a graph \( H_n \) that is \( n \)-choosable but not \( n \)-monophilic. To make the notation simpler, we work with \( H_{n+1} \) with \( n \geq 1 \) instead of \( H_n \) with \( n \geq 2 \).

First, consider the complete bipartite graph \( K_{n,n^n} \). Fix \( n \geq 1 \), and denote the vertices of \( K_{n,n^n} \) by \( a_1, \ldots, a_n, b_1, \ldots, b_n^n \). Let \( L_0 \) be an \( n \)-list assignment for
for every vertex \( v \in j \) vertices, colors are renamed so that colors 1, \( \cdots \), \( n \) smallest integer such that \( a \in n \)\smallest integer such that \( a \in n \) \( \) and each \( L_0(a_i) \) shares exactly one element with each \( L_0(b_k) \). Then there are \( n^n \) distinct ways to assign a color to each of \( a_1, \cdots, a_n \), and each of them will preclude assigning a color to \( b_k \) for some \( k \). It follows that \( \text{col}(K_{n,n^n}, L_0) = 0 \).

Let \( L'_0 \) be an \( n \)-list assignment for \( K_{n,n^n} \) that is the same as \( L_0 \) except that its colors are renamed so that colors 1, \( \cdots \), \( n \) do not appear in any of its lists. For each \( j \in [n] \), let \( L_j \) be the \((n+1)\)-list assignment for \( K_{n,n^n} \) given by \( L_j(v) = L'_0(v) \cup \{j\} \) for every vertex \( v \in K_{n,n^n} \). Let \( x = \text{col}(K_{n,n^n}, L_j) \); clearly \( x \) is nonzero and independent of \( j \).

Let \( \{G_{i,j} : i, j \in [n]\} \) be a set of \( n^2 \) disjoint copies of \( K_{n,n^n} \). Let \( p \) be the smallest integer such that \( n^p > x^{n^2} \). Let \( K_{n,p} \) be a complete bipartite graph with vertices \( v_1, \cdots, v_n, w_1, \cdots, w_p \). We connect each \( v_i \) to all vertices of \( G_{i,1}, \cdots, G_{i,n} \).

This describes the graph \( H_{n+1} \).

**Lemma 7.** For all \( n \geq 1 \), the graph \( H_{n+1} \) is not \((n+1)\)-monoplicial.

**Proof.** Define an \((n+1)\)-list assignment \( L \) for \( H_{n+1} \) as follows. For all \( k \in [p] \), \( L(w_k) = \{n+1, n+2, \cdots, 2n+1\} \); for each \( i \in [n] \), \( L(v_i) = [n] \cup \{n+i\} \); and on each \( G_{i,j} \), \( L = L_j \).

Let \( \gamma \) be a coloring of \( H_{n+1} \) from \( L \). Since \( \text{col}(K_{n,n^n}, L_0) = 0 \), for each \( i, j \in [n] \), \( \gamma \) must assign color \( j \) to at least one vertex of \( G_{i,j} \). Hence for all \( i \in [n] \), \( \gamma(v_i) = n+i \); and for all \( k \in [p] \), \( \gamma(w_k) = 2n+1 \). It follows that \( \text{col}(H_{n+1}, L) = x^{n^2} \).

On the other hand, \( \text{col}(H_{n+1}, n+1) \geq n^p \): there are \( n^p \) ways to color \( w_1, \cdots, w_p \) from just \([n]\); then assign color \( n+1 \) to every \( v_i \); and finally color every \( G_{i,j} \) using colors 1 and 2. Hence \( \text{col}(H_{n+1}, L) < \text{col}(H_{n+1}, n+1) \), as desired.

So it remains to show that \( H_{n+1} \) is \((n+1)\)-choosable. We do this in the three following lemmas. We say two list assignments \( L \) and \( L' \) for a graph \( G \) are **equivalent** if one can be obtained from the other by renaming colors and vertices, i.e., there is a bijection \( f : \mathbb{N} \to \mathbb{N} \) and an automorphism \( \phi : G \to G \) such that for every vertex \( v \in G \), \( L'(v) = f(L(\phi(v))) \).

**Lemma 8.** Let \( L \) be a list assignment for \( K_{n,n^n} \) such that for every vertex \( v \in K_{n,n^n} \), \( |L(v)| \geq n \). If \( \text{col}(K_{n,n^n}, L) = 0 \), then \( L \) is equivalent to \( L_0 \).

**Proof.** Denote the two vertex-partitions of \( K_{n,n^n} \) by \( A \) and \( B \), with \( |A| = n \) and \( |B| = n^n \). Suppose for some \( a_1 \neq a_2 \) in \( A \), \( L(a_1) \cap L(a_2) \neq \emptyset \). If we assign the same color to \( a_1 \) and \( a_2 \), and to each \( a' \neq a_1, a_2 \) we assign a color to from \( L(a') \), then for every \( b \in B \), \( L(b) \) contains at least one color that was not assigned to any \( a \in A \). Hence \( \text{col}(K_{n,n^n}, L) > 0 \), which is a contradiction. So for all \( a_1 \neq a_2 \) in \( A \), \( L(a_1) \cap L(a_2) = \emptyset \).

Now, suppose we have colored all \( a \in A \). Since \( K_{n,n^n} \) is not colorable from \( L \), there must exist some \( b \in B \) whose color list is exactly the \( n \) colors we have chosen for the vertices in \( A \). Since there are only \( n^n \) vertices in \( B \), if there were more than \( n^n \) ways to color \( A \), then \( \text{col}(K_{n,n^n}, L) \) would not be zero. So each \( L(a) \) must contain exactly \( n \) colors, and there are exactly \( n^n \) ways to color \( A \). It follows that every \( L(b) \) must contain exactly one color from \( L(a) \) for each \( a \in A \), and no other colors; Furthermore, distinct vertices in \( B \) must have distinct lists. This proves \( L \) is equivalent to \( L_0 \).
Lemma 9. Let \( v \) denote the vertex in the one-element partition of the complete tripartite graph \( K_{n,n+1} \). Let \( L \) be an \( (n+1) \)-list assignment for \( K_{n,n+1} \) such that \( L(v) = [n+1] \). Suppose for some \( j \in [n] \), \( \text{col}(K_{n,n+1}, L, v, j) = 0 \). Then \( L \) is equivalent to \( L_j \); and for all \( i \neq j \), \( \text{col}(K_{n,n+1}, L, v, i) > 0 \).

Proof. Let \( L' \) be the list assignment for \( K_{n,n+1} \) obtained by deleting color \( j \) from every list in the restriction of \( L \) to \( K_{n,n+1} \). Then \( \text{col}(K_{n,n+1}, L') = 0 \). So, by Lemma 8, \( L' \) is equivalent to \( L_0 \), and hence \( L \) is equivalent to \( L_j \). If for some \( i \neq j \), \( \text{col}(K_{n,n+1}, L, v, i) = 0 \), then it would follow that both \( i \) and \( j \) are in every list of \( L \), which contradicts the fact that \( L' \) is equivalent to \( L_0 \). \( \square \)

Lemma 10. For all \( n \geq 1 \), \( H_{n+1} \) is \( (n+1) \)-choosable.

Proof. Let \( L \) be an \( (n+1) \)-list assignment for \( H_{n+1} \). For each \( i, j \in [n] \), let \( G_{i,j} \) be the subgraph of \( H_{n+1} \) induced by \( G_{i,j} \) and \( v_i \). By Lemma 9, there is at most one color \( c \) in \( L(v_i) \) such that \( \text{col}(G_{i,j}, L, v_i, c) = 0 \). Since \( |L(v_i)| = n+1 \), there exists \( c_i \in L(v_i) \) such that for every \( j \in [n] \), \( \text{col}(G_{i,j}, L, v_i, c_i) \neq 0 \). Furthermore, since each \( w_k \) has only \( n \) neighbors, \( L(w_k) \{ c_1, \ldots, c_n \} \) is non-empty. Hence \( \text{col}(H_{n+1}, L) \neq 0 \). \( \square \)

6. Questions

In this section we offer (and try to motivate) two questions. The Dinitz Conjecture, proved by Galvin [5], states that the line graph of the complete bipartite \( K_{n,n} \) is \( n \)-choosable. The List Coloring Conjecture (which is open as of this writing), generalizes the Dinitz Conjecture: for every graph \( G \), \( \chi_l(L(G)) = \chi(L(G)) \), where \( L(G) \) denotes the line graph of \( G \).

Note that the line graph of \( K_{n,n} \) is isomorphic to the product \( K_n \times K_n \), where the product \( G \times H \) is defined by \( V(G \times H) = V(G) \times V(H) \), with two vertices \( (g, h) \) and \( (g', h') \) in \( G \times H \) declared to be adjacent if \( g = g' \) and \( h \) is adjacent to \( h' \) or if \( h = h' \) and \( g \) is adjacent to \( g' \). Thus, another way to generalize the Dinitz Conjecture is:

Question 1. Is the product of two \( n \)-monophilic graphs \( n \)-monophilic?

For \( n = 2 \) the answer to this question is No; Letting \( P_n \) denote the path of length \( i \), it follows from Theorem 2 that \( P_3 \times P_3 \) is not 2-monophilic, while by Theorem 3 every \( P_i \) is 2-monophilic. However, it is possible that the \( n = 2 \) case is special and for \( n \geq 3 \) the answer is Yes.

Donner's result [3] allows us to define the monophilic number of \( G \), denoted \( \chi_m(G) \), in two possible natural ways:

1. the smallest \( n \) for which \( G \) is \( n \)-colorable and \( n \)-monophilic, or
2. the smallest \( n \) such that \( G \) is \( n' \)-monophilic for all \( n' \geq n \).

We do not know whether or not these two definitions are equivalent; it depends on the answer to the following:

Question 2. If a graph is \( n \)-colorable and \( n \)-monophilic, is it necessarily \( (n+1) \)-monophilic?

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