Genus expansion of HOMFLY polynomials

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ABSTRACT

In the planar limit of the 't Hooft expansion, the Wilson-loop average in 3d Chern-Simons theory (i.e. the HOMFLY polynomial) depends in a very simple way on representation (the Young diagram): \( H_R(A|q = 1) = \left( \sigma_{[1]}(A) \right)^{|R|} \) so that the (knot-dependent) Ooguri-Vafa (OV) partition function \( \sum_R H_R \chi_R \{ \bar{p}_k \} \) becomes a trivial KP \( \tau \)-function. We study higher genus corrections to this formula for \( H_R \) in the form of expansion in powers of \( z = q - q^{-1} \). Expansion coefficients are expressed through the eigenvalues of the cut-and-join operators, i.e. symmetric group characters. Moreover, the \( z \)-expansion is naturally exponentiated. Representation through cut-and-join operators makes contact with Hurwitz theory and its sophisticated integrability properties. Our formulas describe the shape of genus expansion for the HOMFLY polynomials, which for their matrix model counterparts is usually controlled by Virasoro like constraints and AMM/EO topological recursion. The genus expansion differs from the better studied weak coupling expansion at finite number of colors \( N \), which is described in terms of the Vassiliev invariants and Kontsevich integral.

1 Introduction

Today the knot polynomials are among the most interesting new special functions, the closest relatives of conformal blocks, Nekrasov functions and \( \tau \)-functions of KP/Toda integrable hierarchies. The central object in this world is HOMFLY polynomial \( \mathcal{H}^{K \in \mathcal{M}}_R(A, q) \) which depends on the knot (or link) \( K \) embedded into the 3d space \( \mathcal{M} \), on representation (Young diagram) \( R \) and on two variables \( A \) and \( q \). The value of this polynomial at \( A = q^N \) can be interpreted as the Wilson line average along \( K \) in the 3d Chern-Simons theory on \( \mathcal{M} \) [2], with the coupling constant \( \kappa \) and \( q = \exp \frac{2\pi i}{N} \). Further generalizations of the HOMFLY polynomial are: (i) superpolynomial depending on one extra parameter \( t \) [3] which is related to the Khovanov-Rozansky categorification [4] and has much to do with the MacDonald deformation of the Schur symmetric functions [5 6],

\[
P_R^{K \in \mathcal{M}}(A, q, t) \xrightarrow{t=q} \mathcal{H}^{K \in \mathcal{M}}_R(A, q)
\]

and (ii) extended knot polynomial [7 8], depending on infinitely many time-variable \( p_k \),

\[
\mathcal{H}^{K \in \mathcal{M}}_R(p_k | q) \xrightarrow{p_k=p^*_k} \mathcal{H}^{K \in \mathcal{M}}_R(A, q),
\]

\[
p^*_k = \frac{A^k - A^{-k}}{q^k - q^{-k}}
\]

Of main interest at the current stage is the search for a system of interrelations between the knot polynomials for \( \mathcal{M} = S^3 \) (when they are indeed Laurent polynomials in \( A \) and \( q \) variables), especially the study of one-parametric families of these polynomials, considered as functions of a single variable. Examples of such interrelations are skein relations [9], difference equations [10] (also known as "quantum \( A \)-polynomials") and [11 12], "evolutions" within torus, twist and similar families [6 13] and, closest to the subject of the present paper, representation dependence of the special polynomials [6 14]:

\[
\sigma_R(A) = \sigma_{[1]}(A)^{|R|}.
\]
The special polynomials are obtained from the reduced HOMFLY polynomials $H^K_{R}(A, q)$ in the limit $q \to 1$:

$$\sigma^K_{R}(A) = \lim_{q \to 1} H^K_{R}(A, q) = \lim_{q \to 1} \frac{H^K_{R}(A, q)}{H^K_{R}(A, q)}, \quad K \in S^3$$

(4)

A similar limit for the superpolynomial is also of great interest, but the factorization properties of the special superpolynomials are more obscure: they depend on the kind of representation and, probably, even on the complexity of knots [15, 16].

The limit in (4) is taken at $A = \text{const}$, and it is different from the limit $q \to 1$ at $N = \text{const}$, described by the Kontsevich integral [17], where the Vassiliev invariants [18] and chord diagrams arise. It is actually a ’t Hooft large $N$ limit, where the ’t Hooft coupling $\log A = N \log q$ is kept constant. Relation (5) is then the usual factorization property of multitrace operators in the planar limit, in this case it means that the link polynomials in the ’t Hooft limit decompose into products of constituent knot polynomials (links are unlinked into the individual knots), then (3) immediately follows from the cabling approach [19]. Really interesting in this limit is the genus expansion (the one controlled by the Virasoro constraints in the well-studied case of matrix models).

Thus the task of the present paper is to investigate, what happens when the $q$-dependence is “perturbatively” restored: as the expansion into powers of $z = q - q^{-1}$ or $\hbar = \log q$ around the point $q = 1$, where the entire representation dependence is fully described by (3). As we demonstrate, deviations have a very interesting structure expressed through the action of ”cut-and-join” $W$-operators from [20], the generators of ”closed string” commutative Ivanov-Kerov algebra, which have all the linear group $\text{SL}(\infty)$ characters $S_R\{p_k\}$ (the Schur functions) as their common eigenvectors and all the symmetric group $S(\infty)$ characters $\varphi_R(\Delta)$ as the corresponding eigenvalues:

$$\hat{W}_\Delta\{p\} S_R\{p\} = \varphi_R(\Delta) S_R\{p\}$$

(5)

To be more precise, $\varphi_R(\Delta)$ is proportional to the standard symmetric group $S(|R|)$ character $\chi_R(\Delta)$ at $|\Delta| = |R|$

$$\varphi_R(\Delta) = \frac{\chi_R(\Delta)}{d_R z_\Delta}$$

(6)

while at $|R| = |\Delta| + k$ it is equal to

$$\varphi_R(\Delta) = \frac{(r_\Delta + k)!}{k! r_\Delta!} \varphi_R(\Delta, 1, \ldots, 1)$$

(7)

where $r_\Delta$ is the number of lines of the unit length in $\Delta$, $d_R = S_R\{p\}\big|_{p_k=\delta_{k,1}}$ and $z_\Delta$ is the order of automorphism of the Young diagram $\Delta$ [21].

To make the study possible it is most convenient to use the generating function

$$Z^K(\vec{p}|A, q) = \sum_R \mathcal{H}^K_R(A, q) S_R(\vec{p})$$

(8)

also considered in [22]. Relation of the expansion in this paper to the Ooguri-Vafa expansions with appropriately introduced integer-valued coefficients is non-trivial and deserves further investigation. The simplest of cut-and-join operators, $\hat{W}_{[2]}$ already appeared in OV-related considerations [23].

Our main result is that the $R$-dependence of the genus expansion of HOMFLY polynomials is controlled by the symmetric group characters $\varphi_R(\Delta)$, usually studied in Hurwitz theory, which hence acquires a straightforward relation to Chern-Simons and knot theory. We give an explicit form of eq.(44) from [12],

$$H^K_R(q|A) = \left(\sigma^K_{R}(A)\right)^{|R|} \exp \left(\sum_j (q - q^{-1})^j \sum_{|Q|\leq j+1} \varphi_R(Q) \frac{s^K_{\Delta}(A)}{(\sigma^K_{\Delta}(A))^{2j}}\right)$$

(9)

including concrete examples of the ”higher” special polynomials $s^K_{\Delta}(A)$ for particular knots. The set of these polynomials is labeled by the integer and the Young diagram. Of course, for a particular knot $K$ higher special polynomials are not all independent (it is enough to say that the entire exponential (9) is going to be a Laurent polynomial in $q$ and $A$). However, it is unclear if there are universal relations between them on the entire space of all knots.
2 Perturbative expansions of HOMFLY polynomials

Perturbative expansions of the HOMFLY polynomials for knots and links are actively studied since late 1980’s. We briefly consider here three of them. Every expansion leads to its own family of nontrivial invariants, reveals particular underlying structures and provides peculiar relations to other topics.

2.1 $\hbar$–expansion

One of the first expansions to study was the perturbative series for the vacuum expectation value of the Wilson loop in Chern-Simons gauge theory, which leads to the Kontsevich integral for the Vassiliev invariants. It provides a relation to quantum invariants and invariants of finite type (Vassiliev).

This expansion goes in powers of the variable $\hbar$, which is related to the Chern-Simons coupling constant $\kappa$ as follows:

$$\hbar = \frac{2\pi i}{\kappa}. \quad (10)$$

Then, the HOMFLY polynomial is represented as

$$H^K_R(A = e^{N\hbar/2}|q = e^{\frac{\hbar}{2}}) = \sum_{i=0}^{\infty} \hbar^i \sum_{j=1}^{N_i} r^{(R)}_{i,j} v^{K}_{i,j}, \quad (11)$$

where $v_{i,j}$ are the Vassiliev invariants, $r_{i,j}$ is a basis in the vector space of trivalent diagrams, $N_i$ is a dimension of the vector space at order $i$. This expansion is given for the HOMFLY polynomials which are associated with the Chern-Simons theory with the gauge group $SU(N)$, however, similar expansions can be given for any semi-simple group $G$. There is a vast literature on the subject, see, e.g., [24, 25, 17, 26].

Thus, this perturbative expansion is described as

$$\hbar \to 0 \quad |R| \text{ fixed} \quad N \text{ fixed} \quad (12)$$

2.2 "Volume" expansion

The second notorious example of the perturbative expansion is that related to the so-called "volume" conjecture [27], which states that for the knot $K$

$$\lim_{|R| \to \infty} \lim_{q \to 1} \frac{\log H^K_R(A = q^2|q)}{|R|} = \frac{1}{2\pi} \text{Vol}(S^3 \setminus K), \quad (13)$$

where $\text{Vol}(S^3 \setminus K)$ is the simplicial volume of the knot complement $S^3 \setminus K$. This time the variable $q$ is parameterized as follows:

$$q = e^{\frac{2\pi i \cdot u}{|R|}}. \quad (14)$$

In terms of the coupling constant $\hbar$ from (10)

$$u = 2\pi i \left(\frac{|R|}{\kappa} - 1\right) \quad \text{or} \quad (15)$$

$$u = \hbar |R| - 2\pi i. \quad (16)$$
The volume expansion can be done either in powers of $\hbar$ or $|R|^{-1}$ equally well, since in this expansion

\begin{equation}
\begin{aligned}
\hbar &\to 0 \\
|R| &\to \infty \\
N &= \text{fixed} \\
\hbar|R| &= \text{fixed}
\end{aligned}
\end{equation}

(17)

\subsection*{2.3 Genus expansion}

The third known expansion is the genus expansion or $\frac{1}{N}$ expansion. It is very well-known in QFT and matrix models. Our paper is devoted exactly to this expansion, its detailed analysis is presented in the next section. It is this expansion that reveals the relations with integrable KP hierarchy, Hurwitz theory and especially with the cut-and-join operators. The genus expansion of the HOMFLY polynomials has been studied earlier in [22, 28, 29], this led to a number of interesting results, including the Ooguri-Vafa invariants, relations to the Gromov-Witten invariants and peculiar matrix models. There is certain evidence [30, 29], that the AMM/EO topological recursion [31] is nicely applicable to this expansion, though the underlying counterpart of Virasoro constraints and especially their dependence on the choice of the knot still remains to be explicitly formulated in the generic case.

The genus expansion is described as

\begin{equation}
\begin{aligned}
\hbar &\to 0 \\
|R| &= \text{fixed} \\
N &\to \infty \\
\hbar N &= \text{fixed}
\end{aligned}
\end{equation}

(18)

\section{Structure of genus expansion}

\subsection*{3.1 Special polynomials and separation of knot and representation dependencies}

The HOMFLY polynomials, if they are obtained as vacuum expectation values of the Wilson loops in the 3d Chern-Simons theory, are un reduced. For this reason they are singular when $q \to 1$: they diverge as $(q - q^{-1})|R|$. However this singularity does not depend on the knot: the HOMFLY polynomial for the unknot behaves exactly in the same way. Hence, one can consider the reduced HOMFLY polynomial

\begin{equation}
\frac{H_R^K(A, q)}{H_R^{\text{unknot}}(A, q)}.
\end{equation}

(19)

which is well-defined in the limit $q \to 1$.

The $H_R^{\text{unknot}}$ in the denominator is actually the Schur polynomial $S_R(p^*)$ [22], evaluated at the topological locus, i.e. at $p_k = p_k^* = \frac{A^k - A^{-k}}{q^k - q^{-k}}$. For $A = q^N$ this $S_R(p^*)$ is nothing but the quantum dimension of representation $R$ of the algebra $SU(N)$.

At genus zero, one considers the reduced HOMFLY polynomial at $q \to 1$ and finite $A = q^N$, i.e. this is the planar limit with the coupling constant $\hbar \sim \kappa^{-1} \to 0$ and $N \to \infty$. What one gets is the "special" polynomial defined as

\begin{equation}
\sigma^K_R(A) = \lim_{q \to 1} \frac{H_R^K(A, q)}{S_R(A, q)}.
\end{equation}

(20)
In some respects it is "dual" to the Alexander polynomials
\[ N^K_R := \lim_{A \to 1} \frac{H^K_R(A, q)}{S_R(A, q)} \]
while in other respects the properties of special polynomials are somewhat simpler.

In particular, since in the planar limit the averages of multi-trace operators decompose into products of averages, the cabling techniques immediately implies that the special polynomial has a very simple dependence on \( R \) [6]:
\[ \sigma^K_R(A) = \left( \sigma^{K_1}_R(A) \right)^{|R|} \]
This property will be a starting point in our brief consideration of integrable properties of the HOMFLY polynomials in sect.4.2 below.

As already mentioned, the genus expansion goes actually in powers of \( h \) at fixed \( A \). However, if literally \( h \) is used, some properties can get obscure: in particular, the HOMFLY polynomial is a Laurent polynomial in \( q \), while it is a series in \( h \). Hence, it makes sense to use a somewhat different parameter \( z = q - q^{-1} = h + O(h^3) \).

As a next step, we "perturbatively" restore the reduced HOMFLY polynomial by studying the \( z \)-corrections to the special polynomial, and focus on the deformation of the factorization property (22):
\[ H^K_R(A, q) = o \sigma^K_R(A) + z \sigma^K_R(A) \cdot z + 3 \sigma^K_R(A) \cdot z^2 + 4 \sigma^K_R(A) \cdot z^3 + \ldots \]
All \( o \sigma^K_R(A) \), \( i = 0, 1, 2, \ldots \) are polynomials in \( A \) depending on \( K \). Sometimes we omit the labels \( A \) and \( K \) to simplify formulas. Also we identify \( o \sigma^K_R(A) \equiv \sigma^K_R(A) \). Of main interest for us is the \( R \)-dependence. It turns out that the dependence of the perturbative \( z \)-corrections on the representation is spanned by the symmetric group characters with finitely many terms at each order of perturbation. To understand what it looks like, we write down first few terms in (23):

\[
\begin{align*}
0 \sigma^K_R &= \left( \sigma^K_{[1]} \right)^{|R|} \\
1 \sigma^K_R &= \left( \sigma^K_{[1]} \right)^{|R| - 2} \sigma^K_{[2]} \varphi_R([2]) \\
2 \sigma^K_R &= \left( \sigma^K_{[1]} \right)^{|R| - 4} \left( 2 \sigma^K_{[1]} \varphi_R([1]) + 3 \sigma^K_{[1]} \varphi_R([11]) + 3 \sigma^K_{[1]} \varphi_R([3]) + 3 \sigma^K_{[1]} \varphi_R([22]) \right) \\
3 \sigma^K_R &= \left( \sigma^K_{[1]} \right)^{|R| - 6} \left( 3 \sigma^K_{[2]} \varphi_R([2]) + 3 \sigma^K_{[1]} \varphi_R([21]) + 4 \sigma^K_{[1]} \varphi_R([4]) + 4 \sigma^K_{[1]} \varphi_R([211]) + 3 \sigma^K_{[3]} \varphi_R([32]) + 3 \sigma^K_{[2]} \varphi_R([22]) \right) \\
4 \sigma^K_R &= \left( \sigma^K_{[1]} \right)^{|R| - 8} \left( 4 \sigma^K_{[1]} \varphi_R([3]) + 6 \sigma^K_{[1]} \varphi_R([11]) + 6 \sigma^K_{[1]} \varphi_R([3]) + 6 \sigma^K_{[1]} \varphi_R([111]) + 4 \sigma^K_{[2]} \varphi_R([22]) + 4 \sigma^K_{[2]} \varphi_R([21]) + 4 \sigma^K_{[2]} \varphi_R([33]) + 4 \sigma^K_{[2]} \varphi_R([21]) + 4 \sigma^K_{[2]} \varphi_R([33]) \right)
\end{align*}
\]
We call \( o \sigma^K_A(A) \) at the r.h.s. higher special polynomials, these are coefficients in front of \( \varphi_R(\Delta) \). Let us emphasize that they are polynomials in \( A \), they depend on the knot \( K \), but no longer depend on the representation \( R \). The dependence on \( R \) is fully concentrated in the simple factor \( \left( \sigma^K_{[1]} \right)^{|R|} \) and in \( \varphi_R(\Delta) \), which are the eigenvalues of cut-and-join operators:
\[ \tilde{W}_\Delta S_R[p] = \varphi_R(\Delta) S_R[p], \]
where \( R \) and \( \Delta \) are the Young diagrams, \( \tilde{W} \) is a cut-and-join operator. For definitions, various representations and properties of the cut-and-join operators we refer to papers [20, 33, 34]. As explained there, \( \varphi_R(\Delta) \) are actually proportional to the symmetric group characters (generated by the command \( \text{Chi}(R, \Delta) \) in Maple in the package combinat).

Thus, formulas (24) provide expressions for the coefficients \( o \sigma^K_R(A) \) of the Taylor series (23) as linear combinations of the eigenvalues \( \varphi_R(\Delta) \) of the cut-and-join operators \( \tilde{W}_\Delta \), the higher special polynomials \( o \sigma^K_R(A) \) being coefficients in these linear combinations. In this way one gets a complete separation of the \( K \) and \( R \)-dependencies: the former one is quite complicated and is encoded in the set of higher special polynomials, the latter one is relatively simple and is encoded in the symmetric group characters.
3.2 Knot-independent relations between special polynomials and exponentiation of genus expansion

However the story does not end here. It turns out that there are nonlinear relations on higher special polynomials, which are presumably universal, i.e. do not depend on the knot. These relations can be obtained from explicit computations for the torus knots and for the figure eight knot:

\[ \sigma_{22} = (\sigma_2)^2 \quad ("22 = 2 \cdot 2") \]  
\[ \sigma_{211} = (\sigma_2)^2 \sigma_{11} \quad ("211 = 2 \cdot 11") \]  
\[ \sigma_{32} = (\sigma_3)^2 \sigma_2 \quad ("32 = 3 \cdot 2") \]  
\[ \sigma_{222} = (\sigma_2)^3 \quad ("222 = 2 \cdot 2 \cdot 2") \]

\[ \sigma_{111} = 3 \sigma_{11}^2 \quad ("1111 = 11 \cdot 11") \]  
\[ \sigma_{311} = 2 \sigma_{11} \sigma_3 \quad ("311 = 3 \cdot 11") \]  
\[ \sigma_{42} = (\sigma_2)^2 \sigma_4 \quad ("42 = 4 \cdot 2") \]  
\[ \sigma_{33} = 3 \sigma_3^2 \quad ("33 = 3 \cdot 3") \]  
\[ \sigma_{2211} = (\sigma_2)^2 \sigma_{11} \quad ("2211 = 2 \cdot 2 \cdot 11") \]  
\[ \sigma_{322} = (\sigma_3)^2 \sigma_2 \quad ("322 = 3 \cdot 2 \cdot 2") \]  
\[ \sigma_{2222} = (\sigma_2)^4 \quad ("2222 = 2 \cdot 2 \cdot 2 \cdot 2") \]

One can see how this works in explicit examples in tables [128 129] below.

Let us concentrate on the first issue and represent the expansion (23) as an exponential by taking into account relations (26-36):

\[ H_{R}(A, q) = \left( \sigma_1 \right)^{\left\lfloor R \right\rfloor} \exp \left\{ \frac{z}{\sigma_1^2} \sigma_2 \varphi_R([2]) + \left( \frac{z}{\sigma_1^2} \right)^2 \left( \sigma_1 \varphi_R([1]) + 2 \sigma_{11} \varphi_R([11]) + 2 \sigma_3 \varphi_R([3]) + 1 \sigma_2^2 \left( \varphi_R([2]) - \frac{1}{2} \varphi_R^2([2]) \right) \right) \right. \\
+ \left( \frac{z}{\sigma_1^2} \right)^3 \left( \sigma_2 \varphi_R([2]) + 3 \sigma_{21} \varphi_R([21]) + 3 \sigma_4 \varphi_R([4]) + 1 \sigma_2^2 \sigma_{11} \left( \varphi_R([211]) - \varphi_R([2]) \varphi_R([11]) \right) - 1 \sigma_2^2 \varphi_R([2]) \varphi_R([1]) + \right. \\
left. + 1 \sigma_2^2 \sigma_3 \left( \varphi_R([32]) - \varphi_R([2]) \varphi_R([3]) \right) + 1 \sigma_3^2 \left( \varphi_R([222]) - \varphi_R([2]) \varphi_R([22]) + \frac{1}{3} \varphi_R^3([2]) \right) \right\} \quad (37) \]

Note that the terms proportional to

\[ \varphi_R([11]) \]
\[ \varphi_R([22]) - \frac{1}{2} \varphi_R^2([2]) \]
\[ \varphi_R([211]) - \varphi_R([2]) \varphi_R([11]) \]
\[ \varphi_R([32]) - \varphi_R([2]) \varphi_R([3]) \]
\[ \varphi_R([222]) - \varphi_R([2]) \varphi_R([22]) + \frac{1}{3} \varphi_R^3([2]) \]

... 

can be spanned by the (multiplicative) basis of \( \varphi_{\rho}([p]) \) with single line Young diagrams \([p]\) so that (37) can be further simplified. To this end, one suffices to note that the cut-and-join operators form a commutative associative algebra [20]

\[ \bar{W}_{\Delta_1} \bar{W}_{\Delta_2} = \sum_{\Delta} C_{\Delta_1, \Delta_2} \bar{W}_{\Delta}. \quad (39) \]
and so do their eigenvalues \( \varphi_R(\Delta) \). This imposes the relations which can be used to express (38) through \( \varphi_R([p]) \). For some particular examples of \( C^{\Delta_1,\Delta_2} \) see s.B1 in the Appendix, borrowed from [20]. Then the exponential (37) reduces to

\[
H_R(A, q) = \left( \sigma_1 \right)^{|R|} \cdot \exp \left\{ \frac{z}{\sigma_1^2} \cdot \bar{s}_2 \varphi_R([2]) \right\} \cdot \left( \frac{z}{\sigma_1^2} \right)^2 \cdot \left( \frac{\bar{s}_1 \varphi_R([1]) + \bar{s}_1, \varphi_R([1])^2 + \bar{s}_3 \varphi_R([3])}{\sigma_1^2} \right) + \\
+ \left( \frac{z}{\sigma_1^2} \right)^3 \cdot \left( \bar{s}_2 \varphi_R([2]) + 3 \bar{s}_{1,2} \varphi_R([1]) \varphi_R([2]) + 3 \bar{s}_4 \varphi_R([4]) \right) + \\
+ \left( \frac{z}{\sigma_1^2} \right)^4 \cdot \left( \bar{s}_1 \varphi_R([1]) + 4 \bar{s}_{1,1} \varphi_R([1])^2 + \\
+ 4 \bar{s}_{1,1,2} \varphi_R([1]) \varphi_R([2]) + 4 \bar{s}_{2,2} \varphi_R([2])^2 + 4 \bar{s}_{1,3} \varphi_R([1]) \varphi_R([3]) + 4 \bar{s}_3 \varphi_R([3]) + 4 \bar{s}_4 \varphi_R([5]) \right) + \ldots
\] (40)

Note that the label \( \delta = \delta_1, \delta_2, \ldots \) of \( s_\Delta \) is not a Young diagram, but a set of single line Young diagrams so that the polynomial \( s_{s_1, s_2, \ldots, s_m} \) is multiplied by the product \( \varphi_R([\delta_1]) \cdot \varphi_R([\delta_2]) \cdot \ldots \cdot \varphi_R([\delta_m]) \). The polynomials \( \bar{s}_\Delta \) form a new linear combinations, which we denote through \( \bar{\sigma}_\Delta \). Then one can write the HOMFLY polynomial (10) as

\[
H^K_R(A, q) = \left( \frac{\sigma^K_{|1|} (A)}{\sigma^K (A)} \right)^{|R|} \cdot \exp \left\{ \sum_{j=1}^{j+1} \sum_{i=1}^{j+1} \frac{\sigma^K_{|1|} (A) \varphi_R(\Delta)}{\sigma^K (A)} \right\}
\] (41)

3.3 Genus expansion in the additive basis

Now one can use relations (39) in the opposite direction and change (37) to the additive basis, i.e. that with the characters \( \varphi_R(\Delta) \) entering only linearly. Then, one definitely needs the complete set of the characters, and the polynomials \( \bar{\sigma}_\Delta \) form new linear combinations, which we denote through \( \bar{\sigma}_\Delta \). Then one can write the HOMFLY polynomial (10) as

3.4 What is seen in the (anti)symmetric representations

It deserves making an immediate important comment about the expansion (41). Note that the first order of expansion \( \bar{\sigma}^K_{|1|} \) and the second order \( \bar{\sigma}^K_{|2|} \) are completely defined by the only (anti)symmetric representations, i.e. the knowledge of the HOMFLY polynomials in these representations fixes the first two corrections to the special polynomial in any representation.

The same is the case for the third order \( \bar{\sigma}^K_{|3|} \) in spite of the term \( \sigma^K_{|2|} \varphi_R(\Delta) \). Namely, this term can be determined by symmetric representation [3] because \( \varphi_{[3]} (21) \neq 0 \).

However in the fourth order \( \bar{\sigma}^K_{|4|} \) it is no longer the case. The point is that \( \varphi_R([3]) \) and \( \varphi_R([111]) \) are not linearly independent for (anti)symmetric representations, while both are present in the fourth order. Appearance of non-symmetric representation is related with the well known fact that HOMFLY polynomials in (anti)symmetric representations can not distinguish mutant knots, however they do so in non-symmetric representations. For instance, the (mirrored) Conway knot \( K11n34 \) and the (mirrored) Kinoshita-Terasaka knot \( K11n42 \) are a mutant pair of knots with 11 intersections, they are notoriously difficult to tell apart and are distinguished only by the HOMFLY polynomials starting from representation [21] (or by the framed Vassiliev invariant of type 11), [35].

3.5 Restoration of the HOMFLY polynomial

Let us now see how the HOMFLY polynomial is restored from the expansion (28). The fact that it is a polynomial imposes a severe constraints (relations) on higher special polynomials. We consider two basic examples: that of the trefoil and of the figure eight knot, both in the fundamental representation.
Figure eight knot. In this case from table (123) in Appendix B one gets that
\[ H_{\Delta}^{(1)}(A, q) = \sigma_0^{(1)}(A) + \sigma_1^{(1)}(A) \cdot z + \sigma_2^{(1)}(A) \cdot z^2 + \sigma_3^{(1)}(A) \cdot z^3 + \sigma_4^{(1)}(A) \cdot z^4 + \ldots = \]
\[ = \frac{2 - A^2}{A} + z \cdot 0 + z^2 \left( \frac{A}{A^2 - 2A^2} \right)^3 \cdot \frac{(2 - A^2)^3}{A^4} = \frac{2 - A^2}{A} + (q - q^{-1})^2 \cdot \frac{1}{A} = \]
\[ = \frac{2}{A} - A + \frac{q^2}{A} - \frac{2}{A} + \frac{1}{Aq^2} = \frac{-Aq^2 + q^4 + 1}{Aq^2} \quad \text{(42)} \]

Figure eight knot. In this case from table (123) in Appendix B one gets that
\[ H_{\Delta}^{(1)}(A, q) = \sigma_0^{(1)}(A) + \sigma_1^{(1)}(A) \cdot z + \sigma_2^{(1)}(A) \cdot z^2 + \sigma_3^{(1)}(A) \cdot z^3 + \sigma_4^{(1)}(A) \cdot z^4 + \ldots = \]
\[ = A^4 - A^2 + 1 \cdot A^2 + z \cdot 0 + z^2 \left( \frac{A^2}{A^4 - A^2 + 1} \right)^3 \cdot \frac{(A^4 - A^2 + 1)^3}{A^6} = \frac{A^4 - A^2 + 1}{A^2} - (q - q^{-1})^2 = \]
\[ = A^2 - 1 + \frac{1}{A^2} - q^2 + 2 - \frac{1}{q^2} = A^2 + \frac{1}{A^2} - q^2 + 1 - \frac{1}{q^2} = \frac{A^4q^2 - A^2q^4 + A^2q^2 - A^2 + q^2}{A^2q^2} \quad \text{(43)} \]

3.6 New special polynomials

Higher special polynomials \( \sigma^K \) in formulas (24) can be evaluated from the coefficients \( \sigma^K_r \) of expansion by decomposition into the sum over representations \( \Delta \), though this is not very straightforward. In fact, there is a much easier way to calculate them which leads us to consider new polynomials which are generating functions of the higher special polynomials.

Let us suppose that we already know that the coefficients \( \sigma^K_r \) in the expansion (23) are the sums over Young diagrams of the higher special polynomials, multiplied by the eigenvalues of the cut-and-join operators, precisely like in (24):
\[ i\sigma^K_r = \left( \sigma^K_{(1)} \right)^{|R| - 2i} \sum_{\Delta} i\sigma^K_{\Delta} \varphi_R(\Delta) \quad \text{(44)} \]

Now let us consider these eigenvalues given by formula (25) as a matrix \( \varphi_R(\Delta) \) in the pair of indices \( R \) and \( \Delta \). Then one can define the inverse matrix \( \psi_R(\Delta) \) as follows
\[ \sum_R \psi_R(\Delta) \varphi_R(\Delta') = \delta_{\Delta\Delta'} \quad \text{(45)} \]

to obtain
\[ i\sigma^K_{\Delta}(A) = \sum_R i\sigma^K_r(A) \cdot \psi_R(\Delta) \cdot \left( \sigma^K_{(1)}(A) \right)^{2i - |R|} \quad \text{(46)} \]

Now using the matrix \( \psi_R(\Delta) \) it is easy to decompose \( i\sigma^K_r(A) \) into a sum over representations like (24). Using the matrix \( \psi_R(\Delta) \) one can also define new polynomials as
\[ \Sigma^K_{\Delta} := \sum_R H^K_R \psi_R(\Delta) \cdot \left( \sigma^K_{(1)}(A) \right)^{2i - |R|} \quad \text{(47)} \]

In fact, they are nothing but the generating functions of the higher special polynomials
\[ \Sigma^K_{\Delta}(A, q) = \sum_i \sigma^K_r(A) \cdot z^i \quad \text{(48)} \]

Our actual calculations are made by using this technical (perhaps, not just a technical) tool.
4 Comments

4.1 OV partition function

In order to reveal various properties of the HOMFLY polynomials such as integrability, it is often more convenient
to work with generating functions. We define the generating function \[^{[22]}\] of the HOMFLY polynomials as
\[
Z^K(p, A, q) = \sum_R \mathcal{H}_R(A, q) S_R(p)
\]
and use some results from sect.3 First, we need formula \[^{[19]}\]:
\[
Z^K(p, A, q) = \sum_R \frac{\mathcal{H}_R(A, q)}{S_R(p^*)} S_R(p^*) S_R(p) = \sum_R H_R(A, q) S_R(p^*) S_R(p)
\]
Second with the help of expansion \[^{[28]}\] one gets
\[
Z^K(p, A, q) = \sum_R (\sigma^K_\alpha(A) + \sigma^K_\beta(A) \cdot z + \sigma^K_\gamma(A) \cdot z^2 + \ldots) S_R(p^*) S_R(p)
\]
Finally, due to formula \[^{[41]}\] one obtains
\[
Z^K(p, A, q) = \sum_R \left( \sigma^K_{\alpha, \beta}(A) \right)^{|R|} \cdot \exp \left\{ \sum_{j=1}^{j+1} \left( \sum_{i=1}^{j} s^K_\delta(A) \varphi_{\delta}(\Delta) \right) \right\} S_R(p^*) S_R(p) = \sum_R \exp \left\{ \sum_{j=1}^{j+1} \left( \sum_{i=1}^{j} s^K_\delta(A) \varphi_{\delta}(\Delta) \right) \right\} \cdot S_R(p^*) S_R(p) \cdot \left( \sigma^K_{\alpha, \beta}(A) \right)^{|R|} \Rightarrow
\]
\[
Z^K(p, A, q) = \exp \left\{ \sum_{j=1}^{j+1} \left( \sum_{i=1}^{j} s^K_\delta(A) \varphi_{\delta}(\Delta) \right) \right\} \cdot \sum_R \exp \left\{ \sum_{k=1}^{k+1} \left( \sigma^K_{\alpha, \beta}(A)^k \right) \right\}
\]

4.2 Integrability properties

Integrability of the OV partition function \[^{[19]}\] means that it is a tau-function of the KP hierarchy. The linear
combination of characters, \( \sum_R \xi_R S_R(p) \) is a tau-function iff the coefficients \( \xi_R \) satisfy the infinite set of the
quadratic Plücker relations:
\[
\xi_{[22]} \xi_{[0]} - \xi_{[21]} \xi_{[1]} + \xi_{[2]} \xi_{[11]} = 0
\]
\[
\xi_{[32]} \xi_{[0]} - \xi_{[31]} \xi_{[1]} + \xi_{[3]} \xi_{[11]} = 0
\]
\[
\xi_{[22]} \xi_{[0]} - \xi_{[21]} \xi_{[1]} + \xi_{[2]} \xi_{[1]} = 0
\]
\[
\xi_{[32]} \xi_{[0]} - \xi_{[31]} \xi_{[1]} + \xi_{[3]} \xi_{[11]} = 0
\]
\[
\xi_{[42]} \xi_{[0]} - \xi_{[41]} \xi_{[1]} + \xi_{[4]} \xi_{[11]} = 0
\]
\[
\xi_{[33]} \xi_{[0]} - \xi_{[31]} \xi_{[2]} + \xi_{[3]} \xi_{[21]} = 0
\]
\[
\xi_{[32]} \xi_{[0]} - \xi_{[31]} \xi_{[1]} + \xi_{[3]} \xi_{[11]} = 0
\]
\[
\xi_{[22]} \xi_{[0]} - \xi_{[21]} \xi_{[1]} + \xi_{[2]} \xi_{[11]} = 0
\]
\[
\xi_{[22]} \xi_{[0]} - \xi_{[21]} \xi_{[1]} + \xi_{[2]} \xi_{[11]} = 0
\]
\[
\xi_{[22]} \xi_{[0]} - \xi_{[21]} \xi_{[1]} + \xi_{[2]} \xi_{[11]} = 0
\]
\[
\xi_{[22]} \xi_{[0]} - \xi_{[21]} \xi_{[1]} + \xi_{[2]} \xi_{[11]} = 0
\]
\[
\xi_{[22]} \xi_{[0]} - \xi_{[21]} \xi_{[1]} + \xi_{[2]} \xi_{[11]} = 0
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\[
\xi_{[22]} \xi_{[0]} - \xi_{[21]} \xi_{[1]} + \xi_{[2]} \xi_{[11]} = 0
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\[
\xi_{[22]} \xi_{[0]} - \xi_{[21]} \xi_{[1]} + \xi_{[2]} \xi_{[11]} = 0
\]
\[
\xi_{[22]} \xi_{[0]} - \xi_{[21]} \xi_{[1]} + \xi_{[2]} \xi_{[11]} = 0
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\xi_{[22]} \xi_{[0]} - \xi_{[21]} \xi_{[1]} + \xi_{[2]} \xi_{[11]} = 0
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\xi_{[22]} \xi_{[0]} - \xi_{[21]} \xi_{[1]} + \xi_{[2]} \xi_{[11]} = 0
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\xi_{[22]} \xi_{[0]} - \xi_{[21]} \xi_{[1]} + \xi_{[2]} \xi_{[11]} = 0
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\xi_{[22]} \xi_{[0]} - \xi_{[21]} \xi_{[1]} + \xi_{[2]} \xi_{[11]} = 0
\]
\[
\xi_{[22]} \xi_{[0]} - \xi_{[21]} \xi_{[1]} + \xi_{[2]} \xi_{[11]} = 0
\]
\[
\xi_{[22]} \xi_{[0]} - \xi_{[21]} \xi_{[1]} + \xi_{[2]} \xi_{[11]} = 0
\]
\[
\xi_{[22]} \xi_{[0]} - \xi_{[21]} \xi_{[1]} + \xi_{[2]} \xi_{[11]} = 0
\]
Let us look at examples.
Unknot. The HOMFLY polynomial $H_R(A,q)$ for unknot is equal to 1. Then the partition function reduces to

$$Z^{\text{unknot}}(\tilde{p}|A,q) = \exp \left\{ \sum_k \frac{1}{k} \tilde{p}_k p^*_k \right\}, \quad (54)$$

which is the simplest KP tau-function, and the characters $S_R\{p\}$ certainly do satisfy the Plücker relations for any $\{p\}$.

Weak coupling limit. Consider the OV partition function $(51)$ in the weak coupling limit:

$$Z^\mathcal{K}(\tilde{p}|A,q)|_{q=1} = \sum_R \left( \sigma_{[1]}^R(A) \right)^{|R|} S_R\{p^*\} S_R\{\tilde{p}\}. \quad (55)$$

Since the Plücker relations are homogeneous in $R$, and $S_R\{p^*\}$ satisfy the Plücker relations, $(\sigma_{[1]}^R(A))^{\sum \Delta} S_R\{\tilde{p}\}$ also satisfy the Plücker relations. This means that, for every knot, the OV partition function in the planar limit is a KP tau-function $[6]$.

Unfortunately, the HOMFLY polynomials do not satisfy the Plücker relations, they do this only in the weak coupling limit. For this reason one may associate the Plücker relations $(53)$ only with the classical groups $(q \to 1)$, while the case of generic $q \neq 1$ which involves quantum groups requires some deformation of the Plücker relations. Hence, this is the key point in the story of HOMFLY integrability: to construct a ”quantum” version of the Plücker relations which suits the HOMFLY polynomials.

4.3 Connection to the two-point Hurwitz partition function

Note a similarity of $(52)$ with the two-point Hurwitz partition function given by the following formula $[20]$:

$$Z(p,\bar{p}|\beta) = \exp \left( \sum_{\Delta} \beta_{\Delta} W_{\Delta} \right) \cdot \exp \left( \sum_k \frac{1}{k} p_k \bar{p}_k \right). \quad (56)$$

A connection with the OV partition function $(52)$ is provided for a particular choice of $\beta_{\Delta}$’s and getting off the topological locus $p_k = p^*_k$. The Hurwitz partition function is the $\tau$-function of the KP hierarchy both in $p$ and in $\bar{p}$ when $\sum_{\Delta} \beta_{\Delta} W_{\Delta}$ is any (linear) combination of the Casimir operators $[36, 33]$, in particular, when $\beta_{[2]} \neq 0$, $\beta_{\Delta} = 0 \forall \Delta \neq [2]$. The OV partition function is not this case, i.e. it is generically not a tau-function. However, for the torus knots there is another representation due to M.Rosso and V.F.R.Jones $[37, 29]$:

$$Z^{T[m,n]}(p,\tilde{p}) = q^{-\frac{1}{2}W_{[2]}} \cdot \exp \left( \sum_k \frac{1}{k} p_k \tilde{p}_k \right). \quad (57)$$

From this representation it is immediate to check that the OV partition function for the torus knots (and links) is a $\tau$-function of the KP hierarchy in $p$ (but not in $\tilde{p}$) $[8]$. However it is not clear how to see this fact directly from representation $(52)$.

4.4 AMM/EO topological recursion

It is an extremely interesting question if the genus expansion $(41)$ satisfies the topological recursion. There is certain evidence in favour of this $[30]$, but still a lot should done to better understand this important issue. The situation can be simpler when a matrix model representation is also known for the HOMFLY polynomials which is so far the case only for the torus knots.

4.5 Genus expansion in matrix-model representation

In the case of torus knots, one can work out a representation of vacuum expectation values of the Wilson loops in terms of matrix integrals $[38, 29]$. More precisely, this is an integral over the Cartan algebra of the corresponding group (SU(N) in our case):

$$W^{T[m,n]}_R = \frac{1}{Z_{m,n}} \int du e^{-u^2/2m \hbar} \prod_{\alpha > 0} 4\sinh \frac{u_\alpha}{2n} \sinh \frac{u_\alpha}{2m} S_R(e^u) = \langle S_R \rangle \langle 1 \rangle, \quad (58)$$

where $S_R$ is the character of the representation $R$ of the group.
where

\[ Z_{m,n} = \int du e^{-u^2/2mn} \prod_{\alpha > 0} \sinh \frac{u\alpha}{2n} \sinh \frac{u\alpha}{2m} = \langle 1 \rangle, \]  

(59)

\( u \) is an element of \( \Lambda_w \otimes \mathbb{R} \), \( \alpha > 0 \) are the positive roots.

For the \( SU(N) \) case, (58) leads to the Rosso-Jones formula [29]. In other words, formula (58) gives us the unreduced HOMFLY polynomials:

\[ \frac{\langle S_R \rangle}{\langle 1 \rangle} = H_R(A = q^N|q)^*_{SR} \]  

(60)

On the other hand, the integral (58) in the \( SU(N) \) case can be treated as an eigenvalue integral of the Hermitean matrix model. In order to deal with the special polynomial, i.e. to work in the limit of \( q \to 1 \), one has to consider the large \( N \) (planar) limit of this matrix model. Since \( S_R \) is a graded polynomial in \( \{ p_k \} \) of gradation \( |R| \), and since in the large \( N \) limit all \( p_k \sim N \), one gets that

\[ S_R \sim \frac{d_R}{|R|} \]  

and

\[ \frac{\langle p_1^{|R|} \rangle}{\langle 1 \rangle} = \frac{d_R}{\langle 1 \rangle} \frac{\langle p_1^{|R|} \rangle}{\langle 1 \rangle} = \frac{\langle p_1^{|R|} \rangle}{\langle 1 \rangle} \]  

(61)

Since in the planar limit the correlators factorize,

\[ \frac{\langle p_1^{|R|} \rangle}{\langle 1 \rangle} = \frac{\langle p_1 \rangle}{\langle 1 \rangle} \]  

and

\[ \frac{\langle p_1 | R \rangle}{\langle 1 \rangle} = \frac{\langle p_1 \rangle}{\langle 1 \rangle} \frac{d_R}{\langle 1 \rangle} \langle |R| \rangle \]  

(62)

Then, using formulas (61) and (62), one finally obtains in the planar limit

\[ \sigma_R = \frac{\langle S_R \rangle}{\langle 1 \rangle} S_R^* \frac{d_R}{\langle 1 \rangle} \frac{\langle p_1 \rangle}{\langle 1 \rangle} = \left( \frac{\langle p_1 \rangle}{\langle 1 \rangle} \right)^{|R|} \]  

(63)

Now with the help of matrix model technique one can evaluate the special polynomial \( \sigma_1 \) following [29]. The spectral curve describing the planar limit is given by the equation

\[ y^n(y - 1)^m = A^{-m-n}x^{-mn}(yA^2 - 1)^m, \]  

(64)

where \( x \) and \( y \) are lying on the curve. Define the resolvent of \( u\alpha \)

\[ G(x) = \langle \sum_{\alpha} \frac{x}{x - u\alpha} \rangle = \sum_{k=0}^{\infty} x^{-k} \langle \sum_{\alpha} e^{ku\alpha/m} \rangle, \]  

(65)

In the planar limit one can check that the resolvent of \( (u\alpha)^n \) is equal to \( hN - \ln y \) [29], i.e.

\[ \sum_{k=0}^{\infty} x^{-kn} \langle \sum_{\alpha} e^{ku\alpha/m} \rangle = hN - \ln y \]  

(66)

Since \( \langle \text{Tr } e^{u\alpha} \rangle = \sigma_1(A) \), equation (66) along with the spectral curve (64) determines the special polynomial from the term with \( k = m \), the answer being given by (115).

The next terms are determined by the AMM/EO topological recursion [31] on the matrix model side and by our higher special polynomials on the knot theory side. It deserves further careful studies to establish explicit relations for higher order terms like it is done for the planar limit.

### 4.6 Relation to Ooguri-Vafa representation

There is another natural way to construct the \( z \)-expansion which is inspired by the theory of topological string. That is, in the paper [22], the authors conjectured a connection of the Chern-Simons theory with topological string on the resolution of the conifold. In fact, they proposed that the OV partition function [49] is associated
with the topological string partition function $Z_{str}$. The topological nature of this object implies that the "connected" correlators $f_R(q, A)$ defined by the expansion

$$\log Z^K(p|A,q) = \sum_{n=0,R} \frac{1}{n} f_R(q^n, A^n) S_R(p^{(n)})$$

(67)

with the set of variables $p_k^{(n)} \equiv p_{nk}$, has the generic structure

$$f_R(q, A) = \sum_{n,k} \tilde{N}_{R,n,k} \frac{A^n q^k}{q - q^{-1}}$$

(68)

Therefore, $f_R(q, A)$ has only singularity $1/z$, while the corresponding HOMFLY polynomial behaves as $1/z^{|R|}$, i.e. the leading terms of the HOMFLY z-expansion are canceled, and $f_R(q, A)$ is related with higher orders of the z-expansion. $\tilde{N}_{R,n,k}$ are integer and the parity of $n$ in the sum coincides with the parity of $|R|$ while the parity of $k$ is inverse. These numbers are related to the Gopakumar-Vafa integers $n_{\Delta, n, k}$ [39] by the relation

$$n_{\Delta, n, k} = \sum_R d_R z_{\Delta} \varphi_R(\Delta) \tilde{N}_{R,n,k}$$

where $z_{\Delta}$ is the standard symmetric factor of the Young diagram (order of the automorphism) [21]. The integers $\tilde{N}_{R,n,k}$ are more refined, since their integrality implies that $n_{\Delta, n, k}$ are integer but not vise versa. In fact, one can consider even more refined integers [28]

$$f_R(q, A) = \sum_{n,k,R_1,R_2} C_{RR_1,R_2} \Xi_{R_1}(q) N_{R_2,n,k} A^n z^{2k-1}$$

(69)

where

$$C_{RR_1,R_2} = \sum_{\Delta} z^2 d^2_R \varphi_R(\Delta) \varphi_{R_1}(\Delta) \varphi_{R_2}(\Delta)$$

(70)

are the Clebsh-Gordon coefficients of the symmetric group and $\Xi_R(q)$ is a monomial non-zero only for the corner Young diagrams $R = [l - d, 1^d]$ and is equal to

$$\Xi_R(q) = (-1)^d q^{d-l+1}$$

(71)

First few terms for $f_R$ and $N_{R,n,k}$ are

$$f_{[1]}(q, A) = H_{[1]}(q, A)$$

(72)

$$f_{[2]}(q, A) = H_{[2]}(q, A) - \frac{1}{2} \left( H_{[1]}(q, A)^2 + H_{[2]}(q^2, A^2) \right)$$

(73)

$$f_{[1,1]}(q, A) = H_{[1,1]}(q, A) - \frac{1}{2} \left( H_{[1]}(q, A)^2 - H_{[2]}(q^2, A^2) \right)$$

(74)

$$\ldots$$

(75)

and

$$f_{[1]}(q, A) = \sum_{n,k} N_{[1], n, k} z^{2k-1} A^n$$

(76)

$$f_{[2]}(q, A) = \sum_{n,k} \left( q^{-1} N_{[2], n, k} - q N_{[1,1], n, k} \right) z^{2k-1} A^n$$

(77)

$$f_{[1,1]}(q, A) = \sum_{n,k} \left( -q N_{[2], n, k} + q^{-1} N_{[1,1], n, k} \right) z^{2k-1} A^n$$

(78)

$$\ldots$$

(79)

The expressions connecting $f_R$ and the HOMFLY polynomials are highly non-linear and the relation of $\sum_n N_{R,n,k} A^n$ with our $\hat{o}^K_R(A)$ is quite non-trivial for exception of the fundamental representation, when they coincide. Still by now the OV conjecture is explicitly confirmed in a number of non-trivial examples [40, 41, 42] (see more detailed discussion in [29]).
4.7 Relation to Vassiliev invariants

Coefficients of the higher special polynomials are made of the Vassiliev invariants. In the Vassiliev approach [18], the HOMFLY polynomial can be written as [43, 26]

\[ H^K_R(A = e^{\frac{\hbar}{2}}|q = e^{\frac{c}{2}}}) = \sum_{i=0}^{\infty} \hbar^i \sum_{j=1}^{N_i} r_{i,j}^{(R)} v_{i,j}^K \]

(80)

where \( r_{i,j}^{(R)} \) are the polynomials of degree \(|i|\) in \( N \) corresponding to the trivalent diagrams [24, 26], and \( N_i \) is the dimension of the vector space formed by the trivalent diagrams. Here \( v_{i,j}^K \) are invariants of finite-type or Vassiliev invariants of the knot \( K \). Thus what stands in (80) is the double series in powers of \( \hbar \) and \( N \), such that the degree of \( \hbar \) exceeds or is equal to the degree of \( N \). Such series can be rewritten as a double series in non-negative powers of \( \alpha = \hbar N \) and \( \hbar \):

\[ H^K_R(A = e^{\frac{\hbar}{2}}|q = e^{\frac{c}{2}}}) = \sum_{i=1}^{\infty} c_i^{ij} \alpha^i \hbar^j \]

(81)

Thus, the zeroth powers of \( \hbar \) are controlled by the special polynomial:

\[ \sigma^K_R(A = e^{\frac{\hbar}{2}}) = \sum_{i=1}^{\infty} c_i^{0,i} \alpha^i. \]

(82)

To specify \( c_i^{ij} \) one need to determine trivalent diagrams as polynomials in \( N \). We consider them only up to 4 order:

\[ r_{2,1}^{(R)} = \frac{1}{4} \left( -|R| \cdot N^2 - 2 \varphi_n([2]) \cdot N + |R|^2 \right) \]

(83)

\[ r_{3,1}^{(R)} = \frac{1}{8} \left( N \left( |R| \cdot N^2 + 2 \varphi_n([2]) \cdot N - |R|^2 \right) \right) \]

(84)

\[ r_{4,1}^{(R)} = \frac{1}{16} \left( |R|^2 \cdot N^4 + 4 |R| \varphi_n([2]) \cdot N^3 + 2 (2 \varphi_n([2]) - |R|^3) \cdot N^2 - 4 \varphi_n([2]) |R|^2 \cdot N + |R|^4 \right) \]

(85)

\[ r_{4,2}^{(R)} = \frac{1}{16} \left( N^2 \left( -|R| \cdot N^2 - 2 \varphi_n([2]) \cdot N + |R|^2 \right) \right) \]

(86)

\[ r_{4,3}^{(R)} = \frac{1}{16} \left( |R| \cdot N^4 + 6 \varphi_n([2]) \cdot N^3 + C_1 \cdot N^2 - C_2 \cdot N + 2 |R|^3 \right) \]

(87)

where \( C_1 \) and \( C_2 \) are some coefficients.

Taking into account only leading coefficients we get the special polynomial:

\[ \sigma^K_R(A = e^{\frac{\hbar}{2}}) = 1 - \frac{1}{4} |R| v_{2,1} \alpha^2 + \frac{1}{8} |R| v_{3,1} \alpha^3 + \frac{1}{16} \left( |R|^2 v_{4,1} - |R| v_{4,2} + |R| v_{4,3} \right) \alpha^4 + \ldots \]

(88)

The property \( \sigma_R(A) = \sigma^{(R)}_{[1]}(A) \) implies the very well-known relations on the Vassiliev invariants \( v_{i,j}^K \):

\[ v_{4,1} = \frac{1}{2} v_{2,1} \]

(89)

\[ v_{5,1} = v_{2,1} v_{3,1} \]

(90)

\[ v_{6,1} = \frac{1}{6} v_{2,1} \]

(91)

First correction to special polynomial controls first powers of \( \hbar \). From trivalent diagrams point of view it corresponds to subleading terms, that is we get the following:

\[ \tilde{\sigma}^K_R(A = e^{\frac{\hbar}{2}}) = \sum_{i=1}^{\infty} c_i^{1,i} \alpha^i, \]

(92)

\[ \left( \sigma^{(R)}_{[1]} \right)^{|R|-2} \tilde{\sigma}^K_R([2]) = -\frac{v_{2,1}}{2} \varphi_n([2]) \alpha + \frac{v_{3,1}}{4} \varphi_n([2]) \alpha^2 + \left( \frac{v_{4,1}}{4} |R| - \frac{v_{4,2}}{8} + \frac{3}{8} v_{4,3} \right) \varphi_n([2]) \alpha^3 + \ldots \]

(93)
Let us represent \( \sigma^K_2 \) in the following way:

\[
\sigma^K_2 = \sum_{i=1}^{\infty} \gamma_i \alpha^i
\]

(94)

The fact that this expansion is starting from linear term means that \( \sigma^K_2 \sim (A^2 - 1) \). Then taking into account \[88\] we can get \( \gamma_i \):

\[
\gamma_1 = -\frac{1}{2} v_{2,1}
\]

(95)

\[
\gamma_2 = -\frac{1}{4} v_{3,1}
\]

(96)

\[
\gamma_3 = \frac{1}{8} (4v_{4,1} - v_{4,2} + 3v_{4,3})
\]

(97)

Note that these formulas imply

\[
\left. \frac{\sigma^K_2}{A^2 - 1} \right|_{A=1} = -\frac{1}{2} v_{2,1}
\]

(98)

Now let us consider the examples of trefoil and figure eight knots in the fundamental representation.

**Trefoil.** Vassiliev invariants in the first four orders for the trefoil are

\[
\begin{array}{cccccccc}
v_{0,1} & v_{1,1} & v_{2,1} & v_{3,1} & v_{4,1} & v_{4,2} & v_{4,3} & v_{4,4} \\
1 & 1 & 4 & -8 & 8 & 2 & 3 & 1
\end{array}
\]

(99)

From formula \[88\] one gets

\[
H^T_{\{1\}}(A = e^{\frac{N\hbar}{4}} | q = e^{\frac{\pi}{4}}) = \sum_{i=0}^{\infty} \hbar^i \sum_{j=1}^{N_i} r_{i,j} v_{i,j} = 1 + \hbar^2 \frac{- (N^2 - 1)}{4} \cdot 4 + \hbar^3 \frac{N(N^2 - 1)}{8} \cdot (-8) + \hbar^4 \left( \frac{- (N^2 - 1)}{4} \right)^2 \cdot 8 + \frac{- N^2(N^2 - 1)}{16} \cdot \frac{62}{3} + \frac{(N^2 - 1)(N^2 + 2)}{16} \cdot \frac{10}{3} \right) + o(\hbar^5)
\]

(100)

Now it is easy to get the genus expansion introducing the variable \( \alpha = \hbar N \):

\[
\sigma_{\{1\}}(A = e^{\frac{\pi}{4}}) = 1 + \frac{-\alpha^2}{4} - 4 + \frac{\alpha^3}{8} (-8) + \left( \frac{\alpha^4}{16} 8 + \frac{-\alpha^4}{16} \cdot \frac{62}{3} + \frac{\alpha^4}{16} \cdot \frac{10}{3} \right) + O(\alpha^5) = 1 - \alpha^2 - \alpha^3 - \frac{7}{12} \alpha^4 + o(\alpha^5)
\]

(101)

**Figure eight knot.** The Vassiliev invariants in the first four orders for the figure eight knot are

\[
\begin{array}{cccccccc}
v_{0,1} & v_{1,1} & v_{2,1} & v_{3,1} & v_{4,1} & v_{4,2} & v_{4,3} & v_{4,4} \\
1 & 1 & 4 & -8 & 8 & 3 & 3 & 3
\end{array}
\]

(102)

Actually, taking into account that the figure eight knot is fully symmetric (in particular, under the transformation \( q \leftrightarrow q^{-1} \) which corresponds to the mirror reflection) it is obvious that the Vassiliev invariants for all odd orders vanish.

From formula \[88\] one gets

\[
H^F_{\{1\}}(A = e^{\frac{N\hbar}{4}} | q = e^{\frac{\pi}{4}}) = \sum_{i=0}^{\infty} \hbar^i \sum_{j=1}^{N_i} r_{i,j} v_{i,j} = 1 + \hbar^2 \frac{- (N^2 - 1)}{4} \cdot (-4) + \hbar^3 \frac{N(N^2 - 1)}{8} \cdot 0 + \hbar^4 \left( \frac{- (N^2 - 1)}{4} \right)^2 \cdot 8 + \frac{- N^2(N^2 - 1)}{16} \cdot \frac{34}{3} + \frac{(N^2 - 1)(N^2 + 2)}{16} \cdot \frac{14}{3} \right) + o(\hbar^5)
\]

(103)

Now it is easy to get the genus expansion introducing variable \( \alpha = \hbar N \):

\[
\sigma_{\{1\}}(A = e^{\frac{\pi}{4}}) = 1 + \frac{-\alpha^2}{4} (-4) + \left( \frac{\alpha^4}{16} 8 + \frac{-\alpha^4}{16} \cdot \frac{34}{3} + \frac{\alpha^4}{16} \cdot \frac{14}{3} \right) + O(\alpha^5) = 1 + \alpha^2 + \frac{1}{12} \alpha^4 + o(\alpha^6) = 1 + 2 \sum_{j=1}^{\infty} \frac{\alpha^{2j}}{(2j)!}
\]

(104)

Higher special polynomials appeal for further careful investigations, for more details see \[44\].
4.8 Alexander polynomial

At $A = 1$ the HOMFLY polynomial turns into the Alexander polynomial $\mathcal{R}_R^K(q)$; thus, [24], [47] and [11] provide a genus expansion of these polynomials [45]. However, many terms in these expansions disappear, i.e. many special polynomials $s_\Delta$ vanish at $A = 1$ for arbitrary knots $K$. Among other reasons, this is a necessary condition for the "dual factorization" property [41, 14] to hold

$$\mathcal{R}_R(q) = \mathcal{R}_R^\sqcap(q^{|R|}) \quad \text{for hook diagrams } R$$

Consider in this particular case the limit of large size of the diagram $|R| \to \infty$ such that $q = e^{u/|R|}$ and $A = q^N = e^{uN/|R|}$ with $N$ fixed and finite. Then, the special polynomial expansion [47] would imply that the solution behaves like

$$\mathcal{H}_R^K \sim \sigma^K_{[1]}(A)^{|R|} = \exp \left( |R| \cdot \log \sigma^K_{[1]}(A) \right),$$

i.e. grows exponentially with $|R| = r$, in accordance with the volume conjecture [27]. However, things are not so simple and strongly depends on the range of values of $u$.

First of all, corrections to this formula could also contribute to the exponential growth. Indeed, the next correction is

$$\exp \left( z \phi_R([2]) \cdot \sigma^K_{[1]}(A) \right)$$

Naively, $z\phi_R([2]) \cdot \sigma^K_{[1]}(A)$ grows linearly at large $r$. However, as we just saw in the previous paragraph, since $\sigma^K_{[2]}(A) = 0$ at $A = 1$, in practice, there is no linear growing. Similarly, all the terms that could grow linearly with $r$ do not do this because of the argument of the previous paragraph. This is in accordance with the well known fact that for small enough $u$ there is no exponential growth in $J_r(q = e^{u/r})$. Instead (this formula was realized for the figure eight in [46] and for generic knots in [47] basing on the Melvin-Morton-Rozansky conjecture [48])

$$J_r(q = e^{u/r}) = \exp \left( \sum_{k \geq 0} \binom{u}{r}^{2k} f_k(u) \right) = \frac{1}{\text{Alexander}(q = e^u)} + \sum_{k \geq 1} r^{-k} \frac{w_k(q = e^u)}{\text{Alexander}(q = e^u)^k}$$

with some polynomials $w_k$. In particular,

$$f_0(u) = - \log \left( \text{Alexander}(q = e^u) \right)$$

and this is in a nice accordance with the special polynomial expansion.

Only when $e^u$ exceeds the smallest root of the Alexander polynomial, another solution appears, with $f_{-1}(u) \neq 0$.

At the same time, if one chooses $u = 2\pi i$, there is the exponential behaviour [107] [49] and the resulting coefficient in front of $|R|$, for the Jones polynomial $A = q^2$, is equal to the hyperbolic volume of the knot [27] (volume conjecture).
5 Conclusion

In this paper we study knot polynomial expansions around the point $q = 1$ at fixed $A = q^N$. This is actually a genus expansion, i.e. a weak-coupling and large $N$ expansion at fixed value of the 't Hooft coupling constant $\log A = N \log q$. In the matrix model representation, where the knot polynomials are averages of characters,

$$\mathcal{H}_R(A, q)^K = \langle S_R(U) \rangle^K, \quad Z^K(\hat{p}|A, q) = \sum_R \mathcal{H}_R(A, q) S_R(\hat{p}) = \left\langle \exp \left( \sum_k \frac{1}{k} \hat{p}_k \text{Tr} U^k \right) \right\rangle^K$$

(111)

this expansion should be controlled by a version of the topological recursion [31], and the basic question is if it is the standard Virasoro-based recursion or something essentially different. At the moment it is unclear how the Virasoro-like symmetry can be seen in the knot polynomials, and there are still problems in formulating related integrability properties [7], though there is already a non-trivial evidence in favour of relevance of the standard topological recursion [30, 50].

Note that so far the matrix model representation is known only for the HOMFLY polynomials of torus knots [38, 29]. However, as we discussed in s.5.3, the factorization property of the special polynomials from the point of view of the matrix model is just the factorization of correlators in the planar limit. Since the factorization of the special polynomials is a generic property not restricted to the torus knots [11, 14], it makes a hint that there exists a matrix model representation for an arbitrary, non-torus knot.

In this paper we approached the problem of expansion around the $q = 1$ point directly, by simply expanding the known HOMFLY polynomials into powers of $z = q - q^{-1} = 2 \sinh \hbar$. Our main emphasize was on the representation dependence, because it is closely related to the still unclear integrability properties of the OV partition function $Z^K(\hat{p})$. In the spherical limit, when $q$ is strictly unity, the factorization property [3] implies the KP/Toda-integrability [6, 7], but higher genus corrections spoil it: the averages of characters are not quite characters (see, however, [51]) and the question is what they are, what is the algebraic self-nature of the HOMFLY and other knot polynomials.

What we (expectedly) discovered is that the $R$-dependence of higher genus corrections remains pure algebraic: it is controlled by the symmetric group characters $\varphi_R(\Delta)$, which appear in the study of Hurwitz partition functions. [20] However, this time the coefficients are functions of the 't Hooft coupling constant $A$, which depend on the knot and which we call higher special polynomials, at least, temporally, before their true meaning is revealed. They of course can be expressed through the Ooguri-Vafa numbers, but in a non-trivial and still unclear way. They are also related to the Vassiliev invariants, arising in the ordinary weak coupling expansion (at finite $N$), and the very fact that these functions are polynomials in $A$ encodes an infinite number of non-linear relations between the Vassiliev invariants, some of them universal, some other depending on the particular knot.

Furthermore, we calculated some of these functions explicitly for some knots and observed that there are universal (knot-independent) relations between the higher special polynomials, allowing exponentiation of the genus expansion and thus reducing the number of independent special polynomials in a universal and clever way. It still remains an open question what is the proper labeling of these independent special polynomials, they depend on two representations (Young diagrams) $R$ and $\Delta$ like symmetric group characters $\varphi_R(\Delta)$, but also on something else and also on something less: there are non-trivial multiplicities for given $R$ and $\Delta$, but quite often these multiplicities are zero.

Certain light on the properties of the higher special polynomials can be shed by studying the specialization $A = 1$ of the HOMFLY polynomials to the Alexander polynomials, when the special polynomials reduce just to numbers. Many of them simply vanish, but many remain. The "dual factorization" property [11, 14]

$$R_R(q) = R_{\Delta}(q^{[R]})$$

(112)

imposes additional relations on these numbers. The first attempt on the genus expansion of the Alexander polynomials is made in [15].

An extremely interesting development would be a lifting of the genus expansion from the HOMFLY polynomials to the superpolynomials. Predictably the $W$-operators of [20], which had Schur polynomials as their common eigenvectors and the symmetric group characters $\varphi_R(\Delta)$ as their eigenvalues, are lifted to their MacDonald counterparts, which are known [6] to control the superpolynomial evolution, at least, along the families of the torus knots. Some generic properties of this genus expansion and, specifically, of the first order corrections are discussed in [15, 16], but they are still surrounded by some controversy.

The real problem with this direct approach to the genus expansion is almost a complete lack of knowledge about the knot polynomials in non-trivial (non-symmetric and non-antisymmetric) representations. So far only
some limited results are known about the torus and twist knots. As we demonstrated in this paper, non-trivial representations start to contribute from the third order correction in $z$ in case of the HOMFLY polynomials, and they seem essential already in the first order in the case of superpolynomials. Thus, further development of the knot polynomial calculus along the lines of [30, 50], or in any other way remains extremely important. Not less important are studies of the topological recursion for the knot polynomials, originated in [30, 50] and [29], but they are also extremely tedious and even less straightforward then the direct approach.

In any case, in this paper we demonstrated that the genus expansion of knot polynomial is definitely interesting, explicitly reveals non-trivial connections to other branches of science, and deserve all possible attention, along with other directions of research in Chern-Simons theory, our next basic step after $2d$ conformal theory in the study of fundamental properties of quantum field and string theory.

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A Examples of special polynomials

In this section we give explicit examples of the special polynomials for torus knots, for twist knots (an example of non-torus knots) and for torus links.

A.1 Special polynomial for torus knots

For the torus knots $\sigma_{\[m,n\]}(A) = \frac{A^{-(m-1)n}}{m} \sum_{i=0}^{m-1} (-1)^i \frac{A^{m-1-2i}}{i!(m-1-i)!} \prod_{j=1}^{m-1-i} (n+j) \prod_{j=1}^{i}(n-j)$ (113)

$m$ being the number of strands. In particular,

$m = 2 : \quad \frac{1}{2} A^m (n+1)(n-1)A^{-1}$,

$m = 3 : \quad \frac{1}{6 A^m} (n+1)(n+2)A^2 - 2(n+1)(n-1) + (n-1)(n-2)A^{-2}$,

$m = 4 : \quad \frac{1}{24 A^m} (n+1)(n+2)(n+3)A^3 - 3(n+1)(n+2)(n-1)A +$

$\quad + 3(n+1)(n-1)(n-2)A^{-1} - (n-1)(n-2)(n-3)A^{-3}$,

$\ldots$ (114)

This formula can be considered as a deformation of the naive $\frac{A^{m-n}}{m} (A - A^{-1})^{m-1}$, with restored symmetry $m \leftrightarrow n$. It can be made explicit if the formula is rewritten as follows [52, 7]:

$$\sigma_{\[m,n\]}(A) = \frac{A^{-mn}}{mn} \sum_{i=1}^{\min(m,n)} (-1)^{i-1} A^{m+n-i} \frac{(n+m-i)!}{(i-1)!(m-i)!(n-i)!}$$ (115)

A.2 Higher special polynomial $\sigma_2$ for some torus knots

Here we list the polynomials $\sigma_2$ for some particular series of torus knots. Other particular examples of the higher special polynomials can be found in Appendix [3].

**T[2,2k+1]**

$$\sigma_2 = \frac{k(k+1)(A-1)(A+1)(A^2 + 8A^2 k - 8k - 7)}{6A^2}$$

**T[3,3k+1]**

$$\sigma_2 = \frac{k(2+3k)(A-1)(A+1)}{40A^4} \left( (126k^3 - 69k^2 + 6k + 1)A^6 + (-378k^3 - 183k^2 + 82k + 7)A^4 + 
\quad + (378k^3 + 573k^2 + 178k - 17)A^2 - 321k^2 - 126k^3 - 71 - 266k \right)$$

**T[3,3k+2]**

$$\sigma_2 = \frac{(k+1)(3k+1)(A-1)(A+1)}{40A^4} \left( (126k^3 + 57k^2 + 2k)A^6 + (-378k^3 - 561k^2 - 166k)A^4 + 
\quad + (378k^3 + 951k^2 + 686k + 120)A^2 - 200 - 126k^3 - 447k^2 - 522k \right)$$
To \( T \), it is convenient in this case to introduce the function \( Tw \) knots

\[ T[4,4k+1] \]

\[
\sigma_2 = \frac{k(1+2k)(A-1)(A+1)}{630A^6} \left( (19456k^5 - 26976k^4 + 12928k^3 - 2334k^2 + 61k + 15) A^{10} + (-97280k^5 + 32288k^4 + 28992k^3 - 12326k^2 + 773k + 93) A^6 + (194560k^5 + 140608k^4 - 49024k^3 - 27148k^2 + 5454k + 360) A^6 + (-194560k^5 - 345792k^4 - 156160k^3 + 14100k^2 + 13670k - 318) A^4 + (+97280k^5 + 275488k^4 + 278784k^3 + 114218k^2 + 11701k - 1941) A^2 - 19456k^5 - 75616k^4 - 86510k^3 - 31659k - 4509 - 115520k^3 \right)
\]

\[ T[4,4k+3] \]

\[
\sigma_2 = \frac{(k+1)(1+2k)(A-1)(A+1)}{630A^6} \left( (19456k^5 + 21664k^4 + 7616k^3 + 914k^2 + 15k) A^{10} + (-97280k^5 - 210912k^4 - 1496632k^3 - 420062k^2 - 4065k) A^6 + (194560k^5 + 627008k^4 + 718592k^3 + 353428k^2 + 72642k + 5040) A^6 + (-194560k^5 - 832192k^4 - 1334144k^3 - 982028k^2 - 323046k - 37170) A^4 + (+97280k^5 + 518688k^4 + 1107290k^3 + 1067226k^2 + 503151k + 87570) A^2 - 19456k^5 - 1245256k^4 - 3975336k^3 - 248697k - 61740 - 315392k^3 \right)
\]

A.3 Higher special polynomials for some non-torus knots

Twist knots Twist knot \( Tw \) is made out of an counter-strand braid by twisting its ends. Then, according to [3], it is convenient in this case to introduce the function

\[ F^{(k)} := -\frac{A(A^{2k} - 1)}{A} \]

and the first two special polynomials are

\[
\sigma_1^{(1)} = \frac{(A^2 + A^4F - 2A^2F + F)}{A^2}
\]

\[
\sigma_2^{(2)} = \frac{2(A-1)(A+1)}{A^4} \left( F^2A^6 - 4A^6Fk + A^6F + 2F^2A^6k + 2A^6k - 2F^2A^4 - 6F^2A^4k - 2A^4k + 8A^4Fk + A^4F + 6F^2A^2k - 4FA^2k + F^2A^2 - 2F^2k \right)
\]

Knot 4

\[
\sigma_1^{(1)} = \frac{A^4 - A^2 + 1}{A^2}
\]

\[
\sigma_2^{(2)} = \frac{(A - 1)(A + 1)(A^2 + 1)(2A^4 - 3A^2 + 2)}{A^4}
\]

Knot 5

\[
\sigma_1^{(1)} = \frac{-A^4 + A^2 + 1}{A^2}
\]

\[
\sigma_2^{(2)} = \frac{(A - 1)(A + 1)(5A^6 - 4A^4 - 3A^2 - 2)}{A^4}
\]

Knot 6

\[
\sigma_1^{(1)} = \frac{-(A^2 + A - 1)(-1 - A + A^2)}{A^2}
\]

\[
\sigma_2^{(2)} = \frac{(A - 1)(1 + A)(1 + A^2)(A^2 + A - 1)(-1 - A + A^2)}{A^4}
\]

Knot 8

\[
\sigma_1^{(1)} = \frac{3 + A^4 - 3A^2}{A^2}
\]

\[
\sigma_2^{(2)} = \frac{(4A^4 - 9A^2 + 7)(A - 1)^2(1 + A)^2}{A^4}
\]
A.4 Special polynomial expansion for torus links

We begin with two examples.

Torus link $T[l, lk]$ contains $l$ components, which are nothing but the unknots $T[1, k]$. If one assigns representations $R_1, ..., R_l$ with components of the link $T[l, lk]$, then the special polynomial is equal to

$$\sigma_{R_1, ..., R_l}^T[l, lk] = \prod_{i=1}^l (\sigma_{[1]}^T)^{|R_i|}.$$  \hfill (116)

Torus link $T[4, 4k + 2]$ contains two components, which are nothing but the 2-strand knots $T[2, 2k + 1]$. If one assigns representations $R_1, R_2$ with link $T[4, 4k + 2]$, then the special polynomial is equal to

$$\sigma_{R_1, R_2}^T[4, 4k + 2] = (\sigma_{[1]}^{T[2, 2k + 1]})^{|R_1|} \cdot (\sigma_{[1]}^{T[2, 2k + 1]})^{|R_2|}.$$ \hfill (117)

These two examples demonstrate that the special polynomials for links behave like the HOMFLY polynomials for the composite knots, i.e.

$$H_{R_1, R_2}^T[4, 4k + 2](A, q) \Big|_{q=1} = H_{R_1}^T[2, 2k + 1](A, q) \cdot H_{R_2}^T[2, 2k + 1](A, q) \Big|_{q=1}.$$ \hfill (118)

Consider the deviation of the HOMFLY polynomial for link $T[4, 4k + 2]$ from the HOMFLY polynomial for the composite knot $T[2, 2k + 1]#T[2, 2k + 1]$. We expand this deviation in $z$ variable:

$$H_{R_1, R_2}^T[4, 4k + 2](A, q) - H_{R_1}^T[2, 2k + 1](A, q) \cdot H_{R_2}^T[2, 2k + 1](A, q) = 0$$ \hfill (119)

$$H_{R_1, R_2}^T[4, 4k + 2](A, q) - H_{R_1}^T[2, 2k + 1](A, q) \cdot H_{R_2}^T[2, 2k + 1](A, q) = 0$$ \hfill (120)

$$H_{R_1, R_2}^T[4, 4k + 2](A, q) - H_{R_1}^T[2, 2k + 1](A, q) \cdot H_{R_2}^T[2, 2k + 1](A, q) = -z^2 \cdot (2k + 1)|R_1| |R_2| \left( \sigma_{[2]}^{T[2, 2k + 1]} \right)^{|R_1| + |R_2| - 1} L_1^{T[4, 4k + 2]}(A).$$ \hfill (121)

Here the polynomial $\sigma_{[2]}^{T[2, 2k + 1]}$ is the same as that in formulas (121) and $L_1^{T[4, 4k + 2]}(A)$ is a rational function, which depends on the variable $A$ and the link $T[4, 4k + 2]$ and does not depend on representation.

If $R_1 = [1]$ and $R_2$ is arbitrary then

$$H_{[1], R_2}^T[4, 4k + 2](A, q) - H_{[1]}^T[2, 2k + 1](A, q) \cdot H_{R_2}^T[2, 2k + 1](A, q) = z^3 \cdot \varphi_{R_2}([2]) \cdot \left( \sigma_{[2]}^{T[2, 2k + 1]} \right)^{|R_2| - 2} L_{[R_2]}^{T[4, 4k + 2]}(A).$$ \hfill (122)
B Examples of higher special polynomials

B.1 Examples of structure constants

Here we list some explicit examples of the structure constants in (39): a multiplication table restricted to the case when $|\Delta| \leq 4$.

\[
\hat{W}_1[1]\hat{W}_1[1] = \hat{W}_1[1] + 2\hat{W}_1[1,1],
\]

\[
\hat{W}_1[1]\hat{W}_1[2] = 2\hat{W}_1[2] + \hat{W}_1[2,1],
\]

\[
\hat{W}_1[1]\hat{W}_1[1,1] = 2\hat{W}_1[1,1] + 3\hat{W}_1[1,1,1],
\]

\[
\hat{W}_1[1]\hat{W}_3[3] = 3\hat{W}_3[3] + \hat{W}_3[3,1],
\]

\[
\hat{W}_1[1]\hat{W}_3[2,1] = 3\hat{W}_3[2,1] + 2\hat{W}_3[2,1,1],
\]

\[
\hat{W}_1[1]\hat{W}_1[1,1,1] = 3\hat{W}_1[1,1,1] + 4\hat{W}_1[1,1,1,1],
\]

\[
\hat{W}_1[1]\hat{W}_4[4] = 4\hat{W}_4[4] + \hat{W}_4[4,1],
\]

\[
\hat{W}_1[1]\hat{W}_3[3,1] = 4\hat{W}_3[3,1] + 2\hat{W}_3[3,1,1],
\]

\[
\hat{W}_1[1]\hat{W}_2[2,2] = 4\hat{W}_2[2,2] + \hat{W}_2[2,2,1],
\]

\[
\hat{W}_1[1]\hat{W}_2[2,1,1] = 4\hat{W}_2[2,1,1] + 3\hat{W}_2[2,1,1,1],
\]

\[
\hat{W}_1[1]\hat{W}_1[1,1,1,1] = 4\hat{W}_1[1,1,1,1] + 5\hat{W}_1[1,1,1,1,1],
\]

\[
\hat{W}_1[1,1]\hat{W}_2[2] = \hat{W}_2[2] + 2\hat{W}_2[2,1] + \hat{W}_2[2,1,1],
\]

\[
\hat{W}_1[1,1]\hat{W}_1[1,1] = \hat{W}_1[1,1] + 6\hat{W}_1[1,1,1] + 6\hat{W}_1[1,1,1,1],
\]

\[
\hat{W}_2[2]\hat{W}_2[2] = \hat{W}_1[1,1] + 3\hat{W}_3[3] + 2\hat{W}_2[2,2],
\]

\[
\hat{W}_1[1,1]\hat{W}_3[3] = 3\hat{W}_3[3] + 3\hat{W}_3[3,1] + \hat{W}_3[3,1,1],
\]

\[
\hat{W}_1[1,1]\hat{W}_2[2,1] = 3\hat{W}_2[2,1] + 6\hat{W}_2[2,1,1] + \hat{W}_2[2,1,1,1],
\]

\[
\hat{W}_1[1,1]\hat{W}_1[1,1,1] = 3\hat{W}_1[1,1,1] + 12\hat{W}_1[1,1,1,1] + 10\hat{W}_1[1,1,1,1,1],
\]

\[
\hat{W}_2[2]\hat{W}_3[3] = \hat{W}_3[3,2] + 4\hat{W}_4[4] + 2\hat{W}_2[2,1]
\]

\[
\hat{W}_2[2]\hat{W}_2[2,1] = 2\hat{W}_2[2,2,1] + 3\hat{W}_3[3,1] + 4\hat{W}_2[2,2] + 3\hat{W}_3[3] + 3\hat{W}_1[1,1,1]
\]

\[
\hat{W}_2[2]\hat{W}_1[1,1,1] = \hat{W}_2[2,1] + 2\hat{W}_2[2,1,1] + \hat{W}_2[2,1,1,1],
\]

\[
\ldots
\]

The next five sections of this Appendix contain the tables of first $\sigma_\Delta$ for a few knots, and for the series of twist knots.
B.2 Trefoil

\[ \sigma_{[1]}(A) = \frac{2-A^2}{A^4} \]

\[ \sigma_{[2]}(A) = \frac{(A-1)(A+1)(3A^2-5)}{A^4} \]

| \( \Delta \backslash i \) | 2 | 3 |
|-----------------|---|---|
| [1]             | \( \frac{(2-A^2)^3}{A^4} \) | 0 |
| [2]             | 0 | \( \frac{2}{9} \frac{(A^2-2)^4(A-1)(A+1)(7A^2-17)}{A^6} \) |
| [11]            | \( \frac{1}{2} \frac{(2-A^2)^2(-22A^2+17)+9A^4}{A^4} \) | 0 |
| [3]             | \( \frac{(A^2-2)(27A^2-44)(A-1)^2(A+1)^2}{2A^4} \) | 0 |
| [21]            | 0 | \( \frac{4}{3} \frac{(A^2-2)^3(A-1)(A+1)(27A^4-65A^2+43)}{A^6} \) |
| [11]            | 0 | 0 |
| [4]             | 0 | \( \frac{(A^2-2)^2(A-1)(A+1)(432A^6-1568A^4+1829A^2-683)}{6A^6} \) |
| [31]            | 0 | 0 |
| [22]            | \( \frac{(A-1)^2(A+1)^2(3A^2-5)^2}{A^4} \) | 0 |
| [211]           | 0 | \( \frac{1}{2} \frac{(A^2-2)^2(A-1)(A+1)(3A^2-5)(9A^4+22A^2+17)}{A^6} \) |
| [1111]          | 0 | 0 |
| [5]             | 0 | 0 |
| [41]            | 0 | 0 |
| [32]            | 0 | \( \frac{1}{2} \frac{(A^2-2)^3(A-1)^3(27A^2-44)(3A^2-5)}{A^6} \) |
| [311]           | 0 | 0 |
| [221]           | 0 | 0 |
| [2111]          | 0 | 0 |
| [11111]         | 0 | 0 |
| [222]           | 0 | \( \frac{(A-1)^3(A+1)^3(3A^2-5)^3}{A^6} \) |
| $\Delta \setminus i$ | 4 |
|----------------|---|
| [1]            | 0 |
| [2]            | 0 |
| [11]           | $\frac{(A^2 - 2)^4 (A^4 - 13 A^2 + 11)}{A^8}$ |
| [3]            | $\frac{(A^2 - 2)^5 (470 A^2 - 391)(A - 1)^2 (A + 1)^2}{3 A^8}$ |
| [21]           | 0 |
| [111]          | $\frac{(A^2 - 2)^6 (243 A^6 - 781 A^4 + 905 A^2 - 375)}{3 A^8}$ |
| [4]            | 0 |
| [31]           | $\frac{(A^2 - 2)^4 (-11390 A^2 + 15643 A^4 - 9540 A^6 + 2187 A^8 + 3060)}{8 A^8}$ |
| [22]           | $\frac{(A^2 - 2)^4 (6089 - 21872 A^2 + 30112 A^4 - 18412 A^6 + 4203 A^8)}{12 A^8}$ |
| [211]          | 0 |
| [1111]         | $\frac{3 (A^2 - 2)^4 (17 - 22 A^2 + 9 A^4)^2}{A^8}$ |
| [5]            | $\frac{(A^2 - 2)^3 (10125 A^6 - 36872 A^4 + 42814 A^2 - 15576)(A - 1)^2 (A + 1)^2}{24 A^8}$ |
| [41]           | 0 |
| [32]           | 0 |
| [311]          | $\frac{(A^2 - 2)^3 (27 A^2 - 44)(17 - 22 A^2 + 9 A^4)(A - 1)^2 (A + 1)^2}{4 A^8}$ |
| [221]          | $\frac{(A^2 - 2)^3 (3 A^2 - 5)(216 A^4 - 511 A^2 + 329)(A - 1)^2 (A + 1)^2}{3 A^8}$ |
| [2111]         | 0 |
| [11111]        | 0 |
| [42]           | $\frac{(A^2 - 2)^2 (3 A^2 - 5)(432 A^6 - 1568 A^4 + 1829 A^2 - 683)(A - 1)^2 (A + 1)^2}{6 A^8}$ |
| [33]           | $\frac{(A^2 - 2)^2 (27 A^2 - 44)(A - 1)^4 (A + 1)^4}{4 A^8}$ |
| [2211]         | $\frac{(A^2 - 2)^2 (17 - 22 A^2 + 9 A^4)(A - 1)^2 (A + 1)^2 (3 A^2 - 5)^2}{2 A^8}$ |
| [322]          | $\frac{(A^2 - 2)(27 A^2 - 44)(3 A^2 - 5)^2 (A - 1)^4 (A + 1)^4}{2 A^8}$ |
| [2222]         | $\frac{(A - 1)^4 (A + 1)^4 (3 A^2 - 5)^4}{A^8}$ |
### B.3 Knot $T[2, 5]$ 

\[
\sigma_1(A) = \frac{3 - 2A^2}{A} \quad \sigma_2(A) = \frac{(A - 1)(A + 1)(17A^2 - 23)}{A^2}
\]

| $\Delta \setminus i$ | 2 | 3 |
|----------------------|---|---|
| 1                    | \(\frac{(2A^2 - 3)^3(A - 2)(A + 2)}{A^4}\) | 0 |
| 2                    | 0 | \(235(2A^2 - 3)^4(A - 1)(A + 1)(3A^2 - 5)\) |
| 11                   | \(\frac{(2A^2 - 3)^2(83A^4 - 194A^2 + 123)}{2A^4}\) | 0 |
| 3                    | \(\frac{(2A^2 - 3)(440A^4 - 587)(A - 1)^2(A + 1)^2}{2A^4}\) | 0 |
| 21                   | 0 | \(\frac{2(2A^2 - 3)^3(A - 1)(A + 1)(1427A^4 - 3257A^2 + 1926)}{3A^6}\) |
| 111                  | 0 | 0 |
| 4                    | 0 | \(\frac{(2A^2 - 3)^2(A - 1)(A + 1)(20303A^6 - 67501A^4 + 74088A^2 - 26712)}{6A^6}\) |
| 31                   | 0 | 0 |
| 22                   | \(\frac{(A - 1)^2(A + 1)^2(17A^2 - 23)^2}{A^4}\) | 0 |
| 211                  | 0 | \(\frac{(2A^2 - 3)^2(A - 1)(A + 1)(17A^2 - 23)(83A^4 - 194A^2 + 123)}{2A^6}\) |
| 1111                 | 0 | 0 |
| 5                    | 0 | 0 |
| 41                   | 0 | 0 |
| 32                   | 0 | \(\frac{(2A^2 - 3)(A - 1)^3(A + 1)^3(440A^2 - 587)(17A^2 - 23)}{2A^6}\) |
| 311                  | 0 | 0 |
| 221                  | 0 | 0 |
| 2111                 | 0 | 0 |
| 11111                | 0 | 0 |
| 222                  | 0 | \(\frac{(A - 1)^2(A + 1)^2(17A^2 - 23)^3}{A^6}\) |
| Δ | \( (-3+2A^2)^b \) |
|---|---|
| 1 | \( (-3+2A^2)^7 \) |
| 2 | 0 |
| 11 | \( (-3+2A^2)^6(263-379A^2+133A^4) \) |
| 3 | \( 2(-3+2A^2)^5(6320A^2-8063)(A-1)^2(A+1)^2 \) |
| 21 | 0 |
| 111 | \( (-3+2A^2)^5(-14191+37290A^2-33663A^4+10429A^6) \) |
| 4 | 0 |
| 31 | \( (-3+2A^2)^4(206277-781148A^2+1106241A^4-696058A^6+164176A^8) \) |
| 22 | \( (-3+2A^2)^4(-1532420A^2+2165076A^4-1390752A^6+320633A^8+408399) \) |
| 211 | 0 |
| 1111 | \( \frac{3}{4}(-3+2A^2)^4(123-194A^2+83A^4)^2 \) |
| 5 | \( (-3+2A^2)^3(137376A^4-4574446A^4+4997116A^4-1785093)(A-1)^2(A+1)^2 \) |
| 41 | 0 |
| 32 | 0 |
| 311 | \( (-3+2A^2)^3(440A^2-587)(123-194A^2+83A^4)(A-1)^2(A+1)^2 \) |
| 221 | \( (-3+2A^2)^3(17A^2-23)(5657A^4-12752A^2+7428)(A-1)^2(A+1)^2 \) |
| 2111 | 0 |
| 11111 | 0 |
| 42 | \( (-3+2A^2)^2(17A^2-23)(20303A^6-67601A^4+74088A^2-26712)(A-1)^2(A+1)^2 \) |
| 33 | \( \frac{(-3+2A^2)^2(440A^2-587)^2(A-1)^4(A+1)^4}{4A^8} \) |
| 2211 | \( \frac{(-3+2A^2)^2(123-194A^2+83A^4)(A-1)^2(A+1)^2(17A^2-23)^2}{2A^8} \) |
| 322 | \( \frac{(-3+2A^2)(440A^2-587)(17A^2-23)^2(A-1)^4(A+1)^8}{2A^8} \) |
| 2222 | \( \frac{(A-1)^4(A+1)^4(17A^2-23)^3}{A^8} \) |
B.4 Knot $T[3,4]$

$$\sigma_2(A) = \frac{A^4 - 5A^2 + 5}{A^2}$$

For brevity we omit factor $(A^4 - 5A^2 + 5)^2(A - 1)(A + 1)$ for $\Delta = 4$ and $i = 3$.

| $\Delta \backslash i$ | 2 | 3 |
|----------------------|---|---|
| 1                    | $-\frac{5(A^4 - 5A^2 + 5)^3(A^2 - 2)}{A^2}$ | 0 |
| 2                    | 0 | $(A^4 - 5A^2 + 5)^3(A - 1)(A + 1)(680A^6 - 4579A^4 + 11027A^2 - 8498)$ |
| 22                   | $\frac{(A - 1)^2(A + 1)^2(8A^6 - 59A^4 + 139A^2 - 98)^2}{2A^2}$ | 0 |
| 211                  | 0 | $(A^4 - 5A^2 + 5)^2(A - 1)(A + 1)(8A^6 - 59A^4 + 139A^2 - 98)(-477A^6 + 1312A^4 - 1529A^2 + 650 + 64A^8)$ |
| 1111                 | 0 | 0 |
| 5                    | 0 | 0 |
| 41                   | 0 | 0 |
| 32                   | 0 | $(A^4 - 5A^2 + 5)(A - 1)^3(8A^6 - 59A^4 + 139A^2 - 98)(192A^8 - 1907A^6 + 6989A^4 - 11049A^2 + 6160)$ |
| 311                  | 0 | 0 |
| 11111                | 0 | 0 |
| 5                   | 0 | 0 |
| 221                  | 0 | 0 |
| 2111                 | 0 | 0 |
| 11111                | 0 | 0 |
| 222                  | 0 | $(A - 1)^3(A + 1)^3(8A^6 - 59A^4 + 139A^2 - 98)^3$ |
| \( \Delta \setminus \gamma \) | 4 |
|---|---|
| 1 | \(-\frac{(A^4 - 5A^2 + 5)^2(4^2 - 6)}{A^4} \) |
| 2 | 0 |
| 3 | \( (A^4 - 5A^2 + 5)^5(27600A^4 - 236483A^4 + 772316A^4 - 1129785A^2 + 601045)(A - 1)^2(A + 1)^2 \) |
| 21 | 0 |
| 111 | \( (A^4 - 5A^2 + 5)^5(-252572A^{10} + 1073187A^8 - 2398926A^6 + 2954196A^4 - 1899956A^2 + 500665 + 24576A^{12}) \) |
| 4 | 0 |
| 31 | \( (A^4 - 5A^2 + 5)^6(7056904 + 82299068A^4 - 36038869A^2 + 764274A^8 - 101809624A^6 + 10172458A^4 - 657248A^2 + 10752A^4 - 1616975A^2 + 1464^6) \) |
| 22 | \( (A^4 - 5A^2 + 5)^5(1937464 + 16129145A^4 - 72632994A^2 + 14949118A^8 - 198933448A^6 + 15738681A^4 - 69349804A^2 + 212928A^4 - 3126850A^2 + 1464^6) \) |
| 211 | 0 |
| 11111 | \( \frac{3}{4} (A^4 - 5A^2 + 5)^6(64A^8 - 477A^4 + 1312A^4 - 152A^2 + 650)^2 \) |
| 5 | \( (A^4 - 5A^2 + 5)^3(512000A^{10} - 876875A^{14} + 64624025A^{12} - 266794450A^{10} + 671213335A^8 - 1053157344A^6 + 987632937A^4 - 520542457A^2 + 114311124)(A - 1)^2(A + 1)^2 \) |
| 41 | 0 |
| 32 | 0 |
| 311 | \( (A^4 - 5A^2 + 5)^3(192A^8 - 1907A^4 + 6989A^4 - 11049A^2 + 6160)(64A^8 - 477A^4 + 1312A^4 - 152A^2 + 650)(A - 1)^2(A + 1)^2 \) |
| 221 | \( (A^4 - 5A^2 + 5)^3(8A^6 - 59A^4 + 139A^2 - 98)(4996A^{10} - 41106A^8 + 162021A^6 - 310867A^4 + 286290A^2 - 101450)(A - 1)^2(A + 1)^2 \) |
| 2111 | 0 |
| 111111 | 0 |
| 42 | \( (A^4 - 5A^2 + 5)^2(8A^6 - 59A^4 + 139A^2 - 98)(8192A^{14} - 118975A^{12} + 72338A^{10} - 2374550A^8 + 4517500A^6 - 4952762A^4 + 2889917A^2 - 692540)(A - 1)^2(A + 1)^2 \) |
| 33 | \( (A^4 - 5A^2 + 5)^2(192A^8 - 1907A^4 + 6989A^4 - 11049A^2 + 6160)(A - 1)^2(A + 1)^4 \) |
| 2211 | \( (A^4 - 5A^2 + 5)^2(64A^8 - 477A^4 + 1312A^4 - 152A^2 + 650)(A - 1)^2(A + 1)^2(8A^6 - 59A^4 + 139A^2 - 98)^2 \) |
| 322 | \( (A - 1)^2(A + 1)^2(A^4 - 5A^2 + 5)(192A^8 - 1907A^4 + 6989A^4 - 11049A^2 + 6160)(8A^6 - 59A^4 + 139A^2 - 98)^2 \) |
| 2222 | \( (A - 1)^4(A + 1)^4(8A^6 - 59A^4 + 139A^2 - 98)^4 \) |
B.5 Eight-figure knot

\[ \sigma_{[3]}(A) = \frac{A^4 - A^2 + 1}{A^6} \]

\[ \sigma_2(A) = \frac{A^4 - A^2 + 1}{A^4 - 3A^2 + 2} \]

| \( \Delta \backslash \iota \) | 2 | 3 |
|----------------|---|---|
| 1              | \(-\frac{(A^4 - A^2 + 1)^3}{A^6}\) | 0 |
| 2              | 0 | \(\frac{(A^4 - A^2 + 1)^3(A^4 - 1)(10A^4 - 43A^2 + 10)}{8A^{12}}\) |
| 11             | \(\frac{(A^4 - A^2 + 1)^2(-9A^6 + 6A^4 - 9A^2 + 4A^8 + 4)}{2A^8}\) | 0 |
| 3              | \(\frac{(A^4 - A^2 + 1)(12A^8 - A^6 - 3A^4 - A^2 + 12)(A^4 - 1)^2(A^2 - 1)^2}{2A^8}\) | 0 |
| 21             | 0 | \(\frac{(A^4 - A^2 + 1)^3(A^4 - 1)(32A^6 - 83A^4 + 94A^2 - 83A^2 + 32)}{3A^{12}}\) |
| 111            | 0 | 0 |

| \( \Delta \backslash \iota \) | 4 |
|----------------|---|
| 1              | 0 |
| 2              | 0 |
| 11             | \(\frac{(A^4 - A^2 + 1)^6(-7A^6 + 10A^4 - 7A^2 + A^8 + 1)}{2A^{18}}\) |
| 3              | \(\frac{(A^4 - A^2 + 1)^5(105A^6 - 109A^4 - 123A^4 - 109A^2 + 105)(A^4 - 1)^2(A^2 - 1)^2}{6A^{18}}\) |
| 21             | 0 |
| 111            | \(\frac{(A^4 - A^2 + 1)^5(-283A^{10} + 308A^8 - 260A^6 + 96A^{12} + 308A^4 - 283A^2 + 96)}{6A^{18}}\) |
### B.6 Twist knots

\[ F = F_1^{(k)}(A) = -\frac{\Delta(A^{2k-1})}{\Delta(A)} \]

\[ \sigma_1 = \frac{A^2 + A^4 F - 2A^2 F + F}{A^2} \]

\[ \sigma_2 = \frac{2(A - 1)(A + 1)}{A^2} \left( F^2 A^6 - 4A^6 F k + A^6 F + 2F^2 A^6 k + 2A^6 k - 2A^4 F k + 8A^4 F k + A^4 F + 6F^2 A^2 k + 4F A^2 k + F^2 A^2 - 2F^2 k \right) \]

---

| \( \Delta \) \( n \) | 2 
|-------------------|-----------------|
| 1                 | \(-\frac{4F(A^2 + A^4 F - 2F^2 A^2 + F)}{A^2}\) |
| 2                 | 0 |
| 11                | \(\frac{2(A^2 + A^4 F - 2F^2 A^2 + F)^2}{A^2}\) |
| 3                 | \(-48F^2 A^8 k + 72F^3 A^6 k - 48F^3 A^4 k + 12F^3 A^2 k + 12F^3 A^{10} k + 12F^3 A^{10} k^2 - 10F^3 A^8 + 12F^3 A^8 - 6F^3 A^4 + F^3 A^4 + F^3 A^4 0 - 12A^6 k^2 + 30F^2 A^4 k + 144F^2 A^2 k - 36A^4 F k^2 - 36F^2 A^2 k^2 - 78F^2 A^8 k - 24A^6 F k - 16F^2 A^6 k^2 + 108A^6 F k^2 + 66F^2 A^8 k - 24A^6 F k - 144F^2 A^8 k^2 - 108A^6 F k^2 + 6A^8 k + 24A^8 k^2 + 9A^8 F - 12F^2 A^8 - 18F^2 A^{10} k - 36F^2 A^{10} k^2 + 36A^{10} F k^2 + 6A^{10} k + 24A^{10} F + 7F^2 A^{10} - 12A^{10} k^2 + 60F^3 A^2 k^2 - 60F^3 A^2 k^2 + 120F^3 A^6 k^2 - 120F^3 A^4 k^2 - 12F^3 k^2 + 2F^2 A^4 + A^6 F + 3A^6 F^2 \) |

| 21                | 0 |

| 111               | 0 |

---

### B.6.1 Examples

\[ \sigma_{[1]} = 1 + F_1^{(k)}(A) \{|A|^2 = 1 + F_1^{(k)}(A) \left( A^2 - 2 + A^{-2} \right) \} \]  \hspace{1cm} (123)

\[ \ldots \]  \hspace{1cm} (124)

\[ \begin{align*}
&k = 4 & 9_2 & (-A^8 + A^6 + 1)A^2 \\
&k = 3 & 7_2 & -(A^6 - A^4 - 1)A^2 \\
&k = 2 & 5_2 & -(A^4 - A^2 - 1)A^2 \\
&k = 1 & 3_1 & -(A^2 - 2)A^2 \\
&k = 0 & \text{unknot} & 1 \\
&k = -1 & 4_1 & (-A^2 + A^4 + 1)/A^2 \\
&k = -2 & 6_1 & (A^6 - A^2 + 1)/A^4 \\
&k = -3 & 8_1 & (A^8 - A^2 + 1)/A^6 \\
&k = -4 & 10_1 & (A^{10} - A^2 + 1)/A^8 \\
&\ldots & & \end{align*} \]  \hspace{1cm} (125)
\[
\frac{\sigma_2(A)}{2A^{\sigma_1(A)}}:
\]

\[
\begin{align*}
\sigma_2(A) & : \\
\sigma_1(A) & : \\
\end{align*}
\]

\[
\begin{array}{c|c|c}
 k & A^k & \text{Terms} \\
\hline 
4 & 9_2 & 9A^{14} - 8A^{12} - 3A^6 - 2A^4 - 2A^2 - 2 \\
3 & 7_2 & 7A^{10} - 6A^8 - 3A^4 - 2A^2 - 2 \\
2 & 5_2 & 5A^6 - 4A^4 - 3A^2 - 2 \\
1 & 3_1 & 3A^2 - 5 \\
0 & \text{unknot} & 0 \\
-1 & 4_1 & (1 + A^2)(2A^4 - 3A^2 + 2)/A^8 \\
-2 & 6_1 & (2A^{10} + 2A^8 - A^6 - 3A^2 + 4)/A^{12} \\
-3 & 8_1 & (2A^{14} + 2A^{12} + 2A^{10} - A^8 - 5A^2 + 6)/A^{16} \\
-4 & 10_1 & (2A^{18} + 2A^{16} + 2A^{14} + 2A^{12} - A^{10} - 7A^2 + 8)/A^{20} \\
\end{array}
\]

\[
\begin{array}{c|c|c}
 k & A^k & \text{Terms} \\
\hline 
4 & 9_2 & 9A^{16} - 149A^{14} + 64A^{12} - 5A^8 + 9A^6 + 4A^4 + 4A^2 + 8 \\
3 & 7_2 & 49A^{12} - 89A^{10} + 36A^8 - 5A^6 + 9A^4 + 4A^2 + 8 \\
2 & 5_2 & 25A^8 - 45A^6 + 11A^4 + 9A^2 + 8 \\
1 & 3_1 & 17 + 9A^4 - 22A^2 \\
0 & \text{unknot} & 0 \\
-1 & 4_1 & (4A^8 - 9A^6 + 6A^4 - 9A^2 + 4)/A^8 \\
-2 & 6_1 & (4A^{12} - 4A^{10} - 9A^8 + 5A^6 + 9A^4 - 29A^2 + 16)/A^{12} \\
-3 & 8_1 & (4A^{16} - 4A^{14} - 4A^{12} - 9A^{10} + 5A^8 + 25A^4 - 65A^2 + 36)/A^{16} \\
-4 & 10_1 & (4A^{20} - 4A^{18} - 4A^{16} - 4A^{14} - 9A^{12} + 5A^{10} + 49A^4 - 117A^2 + 64)/A^{20} \\
\end{array}
\]
\[
\frac{\sigma_3(A)}{2A^8\{A\}^2\sigma_1(A)}:
\]

\[
\ldots
\]

\[
k = 4 \quad 9_2 \quad -243A^{20} + 190A^{18} - 2A^{16} - 2A^{14} + 86A^{12} + 61A^{10} + 61A^8 + 36A^4 + 24A^2 + 12
\]

\[
k = 3 \quad 7_2 \quad -147A^{14} + 106A^{12} - 2A^{10} + 68A^8 + 49A^6 + 49A^4 + 24A^2 + 12
\]

\[
k = 2 \quad 5_2 \quad -75A^8 + 46A^6 + 50A^4 + 37A^2 + 12
\]

\[
k = 1 \quad 3_1 \quad -27A^2 + 44
\]

\[
k = 0 \quad \text{unknot} \quad 0
\]

\[
k = -1 \quad 4_1 \quad (12A^8 - A^6 - 3A^4 - A^2 + 12)/A^{12}
\]

\[
k = -2 \quad 6_1 \quad (12A^{14} + 24A^{12} + 11A^{10} + 11A^8 - 9A^6 + 2A^4 - 25A^2 + 48)/A^{18}
\]

\[
k = -3 \quad 8_1 \quad (12A^{20} + 24A^{18} + 36A^{16} + 23A^{14} + 23A^{12} + 23A^{10} - 15A^8 + 2A^6 + 2A^4 - 73A^2 + 108)/A^{24}
\]

\[
k = -4 \quad 10_1 \quad (12A^{26} + 24A^{24} + 36A^{22} + 48A^{20} + 35A^{18} + 35A^{16} + 35A^{14} + 35A^{12} + 21A^{10} + \\
+ 2A^8 + 2A^6 + 2A^4 - 145A^2 + 192)/A^{30}
\]

\[
\ldots
\]
