NEW EXPLICIT CMC CYLINDERS AND SAME–LOBED CMC MULTIBUBBLETONS

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ABSTRACT. We provide explicit parametrisations of all Darboux transforms of Delaunay surfaces. Using the Darboux transformation on a multiple cover, we obtain this way new closed CMC surfaces with dihedral symmetry. These can be used to construct closed same-lobed CMC multibubbletons by applying Bianchi permutability.

1. INTRODUCTION

In this paper, we investigate Darboux transforms of Delaunay surfaces, that is, of constant mean curvature surfaces (CMC) of revolution with non–vanishing mean curvature. Recall that the classical Darboux transformation [13] is defined for isothermic surfaces, those surfaces which allow a conformal curvature line parametrisation. Geometrically, a Darboux pair is given by a pair of isothermic surfaces, both conformally enveloping a sphere congruence. Since a CMC surface $f : M \to \mathbb{R}^3$ is isothermic, one can apply the classical Darboux transformation; however, a classical Darboux transform $\hat{f}$ is in general just isothermic, and has constant mean curvature only if $\hat{f}$ has pointwise constant distance from the parallel constant mean curvature surface of $f$, see [15]. We will denote these Darboux transforms for short by CMC Darboux transforms. CMC Darboux transforms are equivalent to the Bianchi-Bäcklund transforms [3] and simple-factor dressings of the extended frame [6,7,12,21].

We will construct CMC Darboux transforms via the $\mu$–Darboux transformation for CMC surfaces: this transformation is a special case of the (generalised) Darboux transformation for conformal immersions, [5]. We briefly recall the $\mu$–Darboux transformation (see also Section 2): By the Ruh–Vilms Theorem [24] a surface has constant mean curvature if and only if its Gauss map is harmonic. The harmonicity of the Gauss map allows for an introduction of an associated family of flat complex connections $d_\lambda$, $\lambda \in \mathbb{C}_* = \mathbb{C} \setminus \{0\}$. Then the $\mu$–Darboux transforms of a CMC surface are exactly given by the $d_\mu$–parallel sections for some fixed spectral parameter $\mu \in \mathbb{C}_*$. In the case when $\mu \in \mathbb{R}_* = \mathbb{R} \setminus \{0\}$, this construction gives exactly all (classical) CMC Darboux transforms, [9].

In the case of CMC Darboux transforms of Delaunay surfaces, for a generic spectral parameter $\mu \in \mathbb{R}_*$ the only periodic CMC Darboux transforms are trivial, that is

2020 Mathematics Subject Classification. Primary 53A10; Secondary 37K35, 58E20.
Key words and phrases. CMC surfaces, Darboux transformations, Delaunay surfaces.
the Darboux transforms are again reparametrisations, rotations and translations of the original Delaunay surface.

In case of a round cylinder however Sterling and Wente [25] discovered, while using the Bianchi–Bäcklund transform, that there are special spectral parameters, the so–called resonance points, so that there are closed, non–trivial CMC Darboux transforms; these surfaces are called (CMC) bubbletons and are not embedded [17]. For each \( n \in \mathbb{N}, n > 1 \), there is a corresponding resonance point \( \mu_n \), and \( n \) gives the number of lobes on the bubble of the bubbleton in case of a single cover of the surface. This process has been extended [19, 20] to obtain bubbletons from any Delaunay surface (and in all space forms).

From our point of view (Section 3), closed CMC Darboux transforms are characterised by parallel sections with multiplier: going around the circle direction, the parallel section only changes by a complex multiple. Resonance points are then spectral parameter \( \mu \in \mathbb{R} \setminus \{0, \pm1\} \) for which all parallel sections are sections with multiplier. In the case of Delaunay surfaces we first provide an explicit parameterisation of all Darboux transforms in terms of Jacobi elliptic functions and the incomplete elliptic integral of the third kind by computing parallel sections in a quaternionic formalism (see Theorem 3.1). We then investigate the parallel sections with multipliers, and in particular obtain resonance points \( \mu_n \) which depend on the necksize \( r \) of the Delaunay surface. For necksize \( r = \frac{1}{2} \) we have the round cylinder, for \( 0 < r < \frac{1}{2} \) an unduloid and \( r < 0 \) a nodoid. As before \( n \in \mathbb{N}, n > 1 \), gives the number of lobes if the Delaunay surface is embedded. In the case of nodoids, that is, non–embedded Delaunay surfaces, the necksize \( r < 0 \) imposes the condition \( n > 1 - 2r \) on the number of lobes on a bubbleton.

Given two Darboux transforms \( f_1 \) and \( f_2 \) one can algebraically construct a common Darboux transform \( \hat{f} \) via Bianchi permutability, [2, 5]. Applying this construction in our setting, we obtain a doublebubbleton for two resonance points \( \mu_n, \mu_l \in \mathbb{R}_+, n \neq l, n, l \in \mathbb{N} \), and corresponding bubbletons \( f_1 \) and \( f_2 \) with \( n \) and \( l \) lobes respectively, that is, a common closed Darboux transform \( \hat{f} \) of \( f_1 \) and \( f_2 \) which has constant mean curvature. Such a doublebubbleton has two bubbles, one with \( n \) lobes, the other with \( l \) lobes. For this construction the assumption \( n \neq l \) is critical: for \( \mu_n = \mu_l \) the common CMC Darboux transform obtained by Bianchi permutability is the original Delaunay surface. Repeating the procedure we can obtain multibubbletons with an arbitrary number of bubbles but each two bubbles have different numbers of lobes. Put differently, with this approach we cannot obtain closed, same–lobbed CMC multibubbletons.

On the other hand, we showed in [11] by using a Sym–type argument that any closed classical double Darboux transform of a Delaunay surface with the same spectral parameter can only have constant mean curvature if it is the original surface. Therefore, if we restrict to a single cover of the Delaunay surface, no same–lobed closed CMC multibubbletons can occur.

We now consider multiple covers of Delaunay surfaces (Section 4). This is motivated by results on Darboux transforms of isothermic tori [1], circletons [18] and periodic smooth and discrete Darboux transforms [10]: in these cases (additional) resonance points can be found when considering multiple covers of the domain.
In the case of a multiple-cover of the round cylinder already Sterling and Wente [25] observed that additional resonance points $\mu^m_n$ can occur: one obtains again a closed CMC surface with $n$ lobes on the $m$–fold cover of the round cylinder where $n$ must be bigger than $m$. This is a natural generalisation of the situation on a single cover. Indeed, in this paper we show that for nodoids and unduloids, one can also obtain additional resonance points $\mu^m_n$ for $n > m$. Geometrically, these bubbletons close on an $m$–fold cover and have non-embedded ends but look similar to the “normal” bubbletons with $n$ lobes on a single bubble (if $m$ and $n$ are co–prime). As before, in the case of nodoids there is an obstruction on the number of lobes $n > m(1 - 2r)$.

However, the situation is more intriguing for unduloids and nodoids: in this situation the case $n < m$ can occur (Theorem 4.1). Geometrically, now the “bubbles” are not localised anymore but the number $n$ still gives the dihedral symmetry of the surface. In contrast to the case of a single cover, there is a restriction on the order of the symmetry for unduloids $n < m(1 - 2r)$ whereas for any choice of $n < m$ we obtain a bubbleton of a nodoid. In particular, for all nodoids (and unduloids with $0 < r < \frac{m-1}{2m}$) it is possible to obtain a CMC surface with $n = 1$, which just has a reflectional symmetry and one “lobe”.

To obtain closed, same–lobed CMC multibubbletons, we change our point of view (Section 5). Limiting to the case when $n > m$ (to obtain localised bubbles with lobes on our bubbletons), we investigate multibubbletons: if there are two co–prime pairs $(m_1, n)$ and $(m_2, n)$, $m_1 \neq m_2$, so that the corresponding Darboux transforms $f_1$ and $f_2$ have $n$ bubbles on the $m_1$ and $m_2$–fold cover respectively, then the common Darboux transform of $f_1$ and $f_2$ given by Bianchi permutability has two bubbles, each with $n$ lobes — we obtain a CMC same–lobed doublebubbleton which closes on an $L$–fold cover, where $L$ is the least common multiple of $m_1$ and $m_2$.

Finally, we observe that the number $1 \leq l_0 \leq m$ of resonance points given by $n$, that is the number of $n$–lobed bubbles we can put on a Delaunay surface, depends on the necksize of the Delaunay surface and co–prime pairs $(m, n)$: for any $l \leq l_0$ we show that there exists a multibubbleton with $l$ bubbles with $n$ lobes each (Theorem 5.1).

Acknowledgements. The second author would like to thank the members of the Research Institute for Mathematical Sciences at Kyoto University for their hospitality during her stay as Visiting Professor at the Institute. The third author gratefully acknowledges the support from Grants-in-Aid of JSPS Research Fellowships for Young Scientist 21K13799.

2. Background

In this section we will give a short summary of results and methods used in this paper. For details on the quaternionic formalism and CMC surfaces we refer to [7,8,9,14,15].

2.1. CMC surfaces in the quaternionic model. In this paper, we investigate the Darboux transformation on conformal immersions $f : M \to \mathbb{R}^3$ from a Riemann
surface into 3–space with constant mean curvature, where we assume without loss of generality that the mean curvature is $H = 1$. By the Ruh–Vilms theorem [24] the Gauss map $N$ of a CMC surface $f$ is harmonic. This allows to introduce a spectral parameter and to define, for example, the associated family of flat connections, the associated family of CMC surfaces, and the spectral curve of a CMC torus, [4, 16, 23].

Here we use a quaternionic model and identify 3–space with the imaginary quaternions $\mathbb{R}^3 = \text{Im} \mathbb{H}$ where $\mathbb{H} = \text{span}_R \{1, i, j, k\}$ and $i^2 = j^2 = k^2 = ijk = -1$. In the following we will identify $H = \text{Re} H \oplus \text{Im} H = \mathbb{R} \oplus \mathbb{R}^3$.

For imaginary quaternions the product in the quaternions is related to the inner product $\langle \cdot, \cdot \rangle$ and the cross product in $\mathbb{R}^3$ by $ab = -\langle a, b \rangle + a \times b$, $a, b \in \text{Im} \mathbb{H}$.

In particular, we see $S^2 = \{n \in \text{Im} \mathbb{H} | n^2 = -1\}$. Thus, if $f : M \to \mathbb{R}^3$ is an immersion then its Gauss map $N : M \to S^2$ is a complex structure $N^2 = -1$ on $\mathbb{R}^4 = \mathbb{H}$. Moreover, if $(M, J_{TM})$ is a Riemann surface, where $J_{TM}$ is the complex structure on the tangent space, then $f : M \to \mathbb{R}^3$ is conformal if and only if $\ast df = N df = -df N$, where $\ast$ denotes the negative Hodge star operator, that is, $\ast \omega(X) = \omega(J_{TM} X)$ for $X \in TM$, $\omega \in \Omega^1(M)$.

Denoting the $(1, 0)$–part ($(0, 1)$–part, respectively) of a 1–form $\omega$ with respect to the complex structure $N$ by $\omega' = \frac{1}{2}(\omega - N \ast \omega)$ (and $\omega'' = \frac{1}{2}(\omega + N \ast \omega)$, respectively), we have for any conformal immersion $f : M \to \mathbb{R}^3$ that

$$df H = -(dN)' \tag{2.1}$$

In the case when $H = 1$, the decomposition of $dN$ into $(0, 1)$– and $(1, 0)$–part is thus given by $dN = dg - df$ where $dg = (dN)''$ and $df = -(dN)'$. Put differently, $g = f + N$ is the parallel CMC surface of $f$ since $N_g = -N$ and thus $dg = -(dN_g)'$.

In particular, we obtain the following reformulation of the Ruh–Vilms theorem: a conformal immersion $f$ has constant mean curvature $H$ if and only if $N$ is harmonic, that is,

$$d((dN)') = 0.$$  

Harmonicity allows for the introduction of a complex spectral parameter (for details in our setting, see [9]) to obtain the family of connections $d_\lambda$, $\lambda \in \mathbb{C}_* = \mathbb{C} \setminus \{0\}$, given by

$$d_\lambda \alpha = d\alpha - \frac{1}{2}(dN)'(N\alpha(a-1) + \alpha b) \tag{2.2}$$
where \( a = \lambda + \lambda^{-1}, b = i \lambda^{-1} - \lambda \).

The associated family can now be used to characterise CMC surfaces: the connections \( d_\lambda \) of the associated family (2.2) of a map \( N : M \to S^2 \) are flat for all \( \lambda \in \mathbb{C}_* \) if and only if \( N \) is harmonic.

**Remark 2.1.** In case when \( \lambda = a + ib \in S^1 \) we have \( a, b \in \mathbb{R} \) and the family of flat connections is quaternionic. In general, the family of flat connections is defined on the trivial \( \mathbb{C}^2 \)–bundle \( M \times \mathbb{C}^2 = \mathbb{C}^2 \) over \( M \) where we consider \( \mathbb{C}^2 = (\mathbb{H}, I) \) with the complex structure \( I \) given by right multiplication by \( i \). In this case, we have the reality condition

\[
(2.3) \quad d_\lambda(\alpha_j) = (d_{\bar{\lambda}} - 1)\alpha_j.
\]

**2.2. CMC surfaces and Darboux transforms.** We are now in the position to recall the Darboux transformation of CMC surfaces. In [5] a Darboux transformation on conformal immersions \( f : M \to \mathbb{R}^3 \) was defined, generalising the classical Darboux transformation in [13]. In this paper we consider the so–called \( \mu \)–Darboux transformation on CMC surfaces, which gives the \( \mu \)–Darboux transforms by the choice of a spectral parameter \( \mu \in \mathbb{C}_* \) and a \( d_\mu \)–parallel section. The \( \mu \)–Darboux transformation is a special case of the generalised Darboux transformation and preserves the CMC property, up to translation. Additionally, the \( \mu \)–Darboux transforms with \( \mu \in \mathbb{R} \setminus \{0, 1\} \) are exactly the classical Darboux transforms of a CMC surface which are CMC. In this paper, we investigate these (classical) CMC Darboux transforms in the case of Delaunay surfaces. For a detailed study of the link between these different Darboux transformations, we refer to [22].

We recall:

**Theorem 2.2 ([9]).** Let \( f : M \to \mathbb{R}^3 \) be a CMC surface with \( H = 1 \), and let \( d_\lambda \) be its associated family of flat connections. For \( \mu \in \mathbb{R} \setminus \{0, 1\} \), let \( \alpha \in \Gamma(\mathbb{H}) \) be a \( d_\mu \)–parallel section of the trivial \( \mathbb{C}^2 \)–bundle \( \tilde{M} \times \mathbb{C}^2 = \tilde{M} \times \mathbb{H} \) over the universal cover \( \tilde{M} \) of \( M \).

Then \( \hat{f} = f + T \) is a CMC Darboux transform of \( f \) where \( T \) is given algebraically by

\[
T^{-1} = \frac{1}{2}(N(a - 1) + aba^{-1}),
\]

and \( a = \frac{\mu + \mu^{-1}}{2}, b = i\frac{\mu^{-1} - \mu}{2} \). All classical CMC Darboux transforms of \( f \) arise this way.

**Remark 2.3.** Note that \( \mu = a + ib \) and therefore \( \mu \in \mathbb{R} \) if and only if \( a \in \mathbb{R}, b \in i\mathbb{R} \).

In this case, the condition \( a^2 + b^2 = 1 \) shows that \( \mu^{-1} = a - ib \). Therefore, by the reality condition (2.3) \( \alpha \) is \( d_{a + ib} \)–parallel if and only if \( \alpha \) is \( d_{a - ib} \)–parallel.

In particular, in the case of real parameter, the two parameter \( \mu = a + ib \) and \( \mu^{-1} = a - ib \) give rise to the same Darboux transforms.

Moreover, for \( \mu = -1 \), that is, \( a = -1, b = 0 \), the Darboux transform is independent of the parallel section \( \alpha \) and we obtain the parallel CMC surface \( \hat{f} = f + N \) as the only Darboux transform.
We now recall the Bianchi permutability theorem [2, 5] in our setting which gives two–step Darboux transforms algebraically, that is, without further integration, from the parallel sections of the first CMC surface.

**Theorem 2.4 ([22]).** Let \( f : M \to \mathbb{R}^3 \) be a CMC surface and \( \alpha_l \in \Gamma(\mathbb{H}) \) parallel sections with respect to the flat connections \( d_{\mu_l} \) where \( \mu_l \in \mathbb{R} \setminus \{0, 1\}, \ l = 1, 2, \mu_1 \neq \mu_2 \). Let \( f_l = f + \alpha_l \beta_l^{-1} \) be the CMC Darboux transforms given by \( \alpha_l \) and

\[
(2.4) \quad \beta_l = \frac{1}{2}(N\alpha_l(a_l - 1) + \alpha_l b_l)
\]

where \( a_l = \frac{\mu_l + \mu_l^{-1}}{2}, b_l = i\frac{\mu_l^{-1} - \mu_l}{2} \).

Then the common \( \mu \)-Darboux transform \( \hat{f} \) of \( f_1 \) and \( f_2 \) is a CMC surface. Moreover, the common \( \mu \)-Darboux transform is given by \( \hat{f} = f_1 + \alpha_2 \beta_2^{-1} \) where

\[
(2.5) \quad \alpha = \alpha_2 - \alpha_1 \beta_1^{-1} \beta_2, \quad \beta = \beta_2 - \beta_1 \alpha_1^{-1} \alpha_2 \frac{a_2}{a_1} - 1 \frac{1}{a_1}.
\]

Note that one could also allow \( \mu_1 = \mu_2 \) and consider two independent \( d_{\mu_l} \)-parallel sections, but in this case the resulting common CMC Darboux transform is \( \hat{f} = f \).

To investigate closed surfaces we recall:

**Definition 2.5.** Let \( f : M \to \mathbb{R}^3 \) be a CMC surface and \( d_{\lambda} \) its associated family of flat connections. For \( \mu \in \mathbb{C}_* \) a \( d_{\mu} \)-parallel section \( \alpha \) is called a section with multipier if \( \gamma^* \alpha = a_{h, \gamma} \) for all \( \gamma \in \pi_1(M) \). Here \( h, \gamma : \pi_1(M) \to \mathbb{C}_* \) is a group homomorphism, and \( \gamma^* \alpha = \alpha \circ \gamma_\# \) where \( \gamma_\# \) is the associated deck transformation of \( \gamma \in \pi_1(M) \).

A parameter \( \mu \in \mathbb{C}_* \) is called a resonance point if all \( d_{\mu} \)-parallel sections have a multiplier.

With this definition we have in the case of a CMC surface (for the general case of a conformal immersion, see [5]):

**Proposition 2.6.** Let \( f : M \to \mathbb{R}^3 \) be a CMC surface and \( \hat{f} = f + T \) a \( \mu \)-Darboux transform given by a \( d_{\mu} \)-parallel section \( \alpha, \mu \in \mathbb{R} \setminus \{0, 1\} \). Then \( \hat{f} \) is defined on \( M \), rather than the universal cover \( \hat{M} \), if and only if \( \alpha \) is a section with multiplier.

In particular, we can apply this to Bianchi permutability:

**Corollary 2.7.** Let \( f : M \to \mathbb{R}^3 \) be a CMC surface and \( f_l : M \to \mathbb{R}^3 \) be the CMC Darboux transforms given by \( d_{\mu_l} \)-parallel sections \( \alpha_l \) with multipliers where \( \mu_l \in \mathbb{R} \setminus \{0, 1\}, \ l = 1, 2, \mu_1 \neq \mu_2 \). Then the common CMC Darboux transform \( \hat{f} : M \to \mathbb{R}^3 \) of \( f_1 \) and \( f_2 \) is defined on \( M \) as well.

**Proof.** Let \( h_l \) denote the multipliers of the \( d_{\mu_l} \)-parallel sections \( \alpha_l \). By (2.4) we see that \( \gamma^* \beta_l = \beta_l(h_l)_\gamma \). Then (2.5) shows that

\[
\gamma^* \alpha = \gamma^* \alpha_2 - (\gamma^* \alpha_1)(\gamma^* \beta_1)^{-1}(\gamma^* \beta_2) = \alpha_2(h_2)_\gamma - \alpha_1(h_1)_\gamma(h_1)_\gamma^{-1} \beta_1^{-1} \beta_2(h_2)_\gamma
\]

\[
= \alpha(h_2)_\gamma.
\]

Therefore, \( \alpha \) is a section with multiplier, and \( \hat{f} = f_1 + \alpha \beta^{-1} \) is defined on \( M \). \( \square \)
3. Delaunay bubbletons

We will now recall the construction of (multi-) bubbletons which are CMC Darboux transforms at a resonance point of Delaunay surfaces, that is, of CMC surfaces of revolution.

More generally, we will give all closed CMC Darboux transforms of Delaunay surfaces in terms of Jacobi elliptic functions and elliptic integrals. To this end, we need to find all parallel sections of the associated family of flat connections explicitly.

Recall that surfaces of revolution can be conformally parametrised by

\[ f(x, y) = ip(x) + jq(x)e^{-iy}, \quad \text{with} \quad (p')^2 + (q')^2 = q^2. \]

In the cases of Delaunay surfaces, \( p, q \) are given by the Jacobi elliptic functions with parameter \( M = 1 - (1 - \frac{1}{r^2})^2 \) where \( r \in (-\infty, \frac{1}{2}] \) is the necksize and

\[ q(x) = (1 - r) \, \text{dn}((1 - r)x, M), \quad p'(x) = q(x)^2 + r(1 - r). \]

Note that \( r = \frac{1}{2} \) gives a cylinder, \( r \in (0, \frac{1}{2}) \) unduloids and \( r \in (-\infty, 0) \) nodoids.

To find parallel sections \( \alpha \) of a flat connection \( d_\mu \) with \( \mu \in \mathbb{R} \setminus \{0, 1\} \), in the associated family \( d_\lambda \) of the harmonic Gauss map \( N \) of \( f \), we need to solve

\[ \frac{d\alpha}{f} = -\frac{1}{2}d(N\alpha(a - 1) + \alpha b), \]

where \( a = \frac{e^{i\mu}}{2}, b = i\frac{e^{i\mu}}{2} \). Here we used (2.1) to express (2.2) in terms of the CMC surface \( f \). Since \( f \) is conformal we have \( f_x N = -f_y \) and \( f_y N = f_x \) so that we need to solve the PDEs

\begin{align*}
(3.1) & \quad \alpha_x = \frac{1}{2}(f_y\alpha(a - 1) - f_x\alpha b), \\
(3.2) & \quad \alpha_y = -\frac{1}{2}(f_x\alpha(a - 1) + f_y\alpha b).
\end{align*}

We consider the second equation (3.2) first. We decompose \( \alpha = \alpha_0 + j\alpha_1 \) and set \( \alpha_1 = e^{iy}\alpha_1 \) to obtain the linear system

\[
\begin{pmatrix}
\alpha_0 \\
\alpha_1
\end{pmatrix}
= \frac{1}{2}
\begin{pmatrix}
-ip'(a - 1) & iq\bar{b} + q'(a - 1) \\
 iq\bar{b} - q'(a - 1) & i(2 + p'(a - 1))
\end{pmatrix}
\begin{pmatrix}
\alpha_0 \\
\alpha_1
\end{pmatrix}
=: A
\begin{pmatrix}
\alpha_0 \\
\alpha_1
\end{pmatrix}.
\]

The matrix \( A \) has eigenvalues \( \frac{1}{2}(1 \pm t) \) where

\[ t = \sqrt{1 + 2r(1 - r)(a - 1)}. \]

If \( t = 0 \) then \( \alpha_+ = \alpha_- \) gives only one solution. In the case when \( r = \frac{1}{2} \), we obtain \( \mu = -1 \), and the general solutions are given by \( \alpha = e^{\frac{\pi}{2}c}, c \in \mathbb{H} \). By Remark 2.3 the only Darboux transform in this case is the parallel CMC surface. For \( r < \frac{1}{2} \) and \( t = 0 \), the spectral parameter are \( \mu_0 = \frac{r}{r-1} \) and \( \mu_b^{-1} = \frac{r-1}{r} \), and the matrix \( A \) is not diagonalisable. Therefore, we obtain a second independent parallel section \( \hat{\alpha} \) by the generalised eigenvector, but \( \hat{\alpha} \) is not a section with multiplier. This shows that \( \mu_b, \mu_b^{-1} \) are not resonance points. (Indeed they are the branch points of the multiplier spectral curve of the unduloids and nodoids with necksize \( r \neq \frac{1}{2} \).)
Therefore, we can from now on assume \( t \neq 0 \). We obtain all \( d_\mu \)-parallel sections, since the matrix has constant coefficients in \( y \) and \( p' - q^2 = r(1 - r) \), by the following fundamental sections

\[
\alpha_\pm = e^{\mp \frac{it}{2}} (qb - iq'(a - 1) + j (1 + p'(a - 1) \pm t)) e^{\pm \frac{it}{2}} c_\pm,
\]

where \( c_\pm \) are complex valued functions in \( x \).

To solve for \( c_\pm \), we compute \((\alpha_\pm)_x\) in two ways. Since \( c_\pm \) only depend on \( x \), we can evaluate at \( y = 0 \) and obtain

\[
(\alpha_\pm)_x = (q'b - i q''(a - 1) + j p''(a - 1)) c_\pm \\
+ (qb - i q'(a - 1) + j (1 + p'(a - 1) \pm t)) c'_\pm.
\]

By (3.1) we also have at \( y = 0 \)

\[
(\alpha_\pm)_x = \frac{1}{2} (f_\mu \alpha_\pm(a - 1) - f_\beta \alpha_{\mu \beta}) \\
= \frac{1}{2} \left( -i q (1 - 2 p'(a - 1) \pm t) (a - 1) + q' (1 \pm t) b \\
+ j (p''(a - 1) + i p'(1 \pm t) b) \right) c_\pm,
\]

where we used \( a^2 + b^2 = 1 \), \((p')^2 + (q')^2 = q^2 \) and \( p'' = 2qq' \) since \( p' = q^2 + r(1 - r) \).

By comparing the \( \mathbb{C} \) and \( j \mathbb{C} \) parts, respectively, we obtain

\[
(bq - i(a - 1)q') c_\pm' = \frac{1}{2} (-i(a - 1)q(1 \pm t - 2 p'(a - 1)) - bq'(1 \mp t) + 2i(a - 1)q'') c_\pm,
\]

\[
(1 \pm t + (a - 1)p') c_\pm' = \frac{1}{2} (- (a - 1)p'' + i(1 \pm t)bp') c_\pm.
\]

Up to multiplication by the factor \(-\frac{bq + i(a - 1)q'}{1 + t + (a - 1)p'}\) these two equations are the same.

To solve for \( c_\pm \), we note that when \( 1 \pm t + (a - 1)p' \neq 0 \), the second equation gives

\[
\frac{c_\pm'}{c_\pm} = -\frac{1}{2} \left( \log |1 \pm t + (a - 1)p'| \right)' + \frac{ib(1 \pm t)}{2(a - 1)} - \frac{ib(1 \pm t)^2}{2(a - 1)} \frac{1}{1 \pm t + (a - 1)p'}.
\]

Integrating both sides, we obtain

\[
\log |c_\pm| = -\frac{1}{2} \int \log |1 \pm t + (a - 1)p'| + \frac{ib(1 \pm t)}{2(a - 1)} x - \frac{ib(1 \pm t)^2}{2(a - 1)} \int \frac{1}{1 \pm t + (a - 1)p'} dx.
\]

Abbreviating the Jacobi elliptic functions so that, for example, \( \operatorname{dn}((1 - r)x, M) = \operatorname{dn} \), we have \( p' = q^2 + r(1 - r) \) for \( q = (1 - r) \) \( \operatorname{dn} \). Thus, using \( \operatorname{dn}^2 = 1 - M \operatorname{sn}^2 \), one calculates

\[
1 \pm t + (a - 1)p' = (1 \pm t + (a - 1)(1 - r))(1 - N \operatorname{sn}^2)
\]

where \( N = \frac{(a - 1)(1 - r)^2 M}{1 + t + (a - 1)(1 - r)} \). Therefore, it remains to compute \( \int \frac{1}{1 - N \operatorname{sn}^2} dx \), which is well-known to be the incomplete elliptic integral of the third kind

\[
\int \frac{1}{1 - N \operatorname{sn}^2} dx = \frac{1}{1 - r} \Pi(N; \operatorname{am}((1 - r)x, M) | M),
\]

where \( \operatorname{am}((1 - r)x, M) \) is the Jacobi amplitude.

Observing that \( \alpha_\pm \) are parallel sections, our results extend smoothly into the zeros of \( 1 \pm t + (a - 1)p' \). We summarise:
Theorem 3.1. Let \( f(x, y) = ip + jqe^{-iy} \) be a Delaunay surface with necksize \( r \).
For \( \mu \in \mathbb{R} \setminus \{0, \pm 1\} \), let \( a = \frac{\mu + \mu^{-1}}{2} \) and 
\[
t = \sqrt{1 + 2r(1 - r)(a - 1)}.
\]
Then all \( d_\mu \)-parallel sections for \( t \neq 0 \) are explicitly given by 
\[
\alpha = \alpha_+ m_+ + \alpha_- m_-
\]
with constants \( m_\pm \in \mathbb{C} \), where 
\[
\alpha_\pm = e^{\frac{\mu}{2}} (qb - iq'(a - 1) + j(1 + p'(a - 1) \pm t)) e^{\pm \frac{ia}{2} c_\pm}
\]
and 
\[
c_\pm = \exp \left( \frac{\varphi(1 \mp t)}{2(a - 1)} \left( x - \frac{1 \mp t}{1 + t + (a - 1)(1 - r)} \frac{1}{1 - r} \Pi(\text{am}((1 - r)x, M), M) \right) \right) \sqrt{1 \mp t + (a - 1)p'}.
\]
Here \( \Pi \) is the incomplete elliptic integral of the third kind and \( \text{am} \) the Jacobi amplitude.

Since by Theorem 2.2 all CMC Darboux transforms of Delaunay surfaces are given algebraically by \( d_\mu \)-parallel sections, we obtain also:

Theorem 3.2. The CMC Darboux transforms of a Delaunay surfaces can be expressed explicitly in terms of Jacobi elliptic functions and the incomplete elliptic integral of the third kind.

Remark 3.3. The parallel sections obtained here are special cases of the parallel sections of a surface of revolution in \([11, 22]\) (after adjusting the spectral parameter to account for the choice of dual surface): we obtain exactly those classical Darboux transforms which are CMC.

Now, observe that \( \alpha_\pm \) are sections with multipliers 
\[
h_\pm = -e^{\pm \pi i \sqrt{1 + 2r(1 - r)(a - 1)}},
\]
and therefore, since \( c_\pm \) are independent of \( y \), resonance points have to satisfy 
\[
1 + 2r(1 - r)(a - 1) = n^2, \quad n \in \mathbb{N}.
\]
Since \( \mu \neq 1 \) and \( t \neq 0 \) by assumption, the cases \( n = 0 \) and \( n = 1 \) cannot occur. Therefore, the (relevant) resonance points are given by 
\[
\mu_\pm^\pm = \frac{1}{2r(1 - r)} \left( n^2 - 1 + 2r(1 - r) \pm \sqrt{(n^2 - 1)^2 + 4r(1 - r)(n^2 - 1)} \right),
\]
for \( n \in \mathbb{N}, n > 1 \).

We also recall that \( \mu_\pm^\pm \) give rise to the same Darboux transform by Remark 2.3, allowing us to consider only one of the solutions \( \mu_n = \mu_n^+ \).

The (non–trivial) CMC Darboux transforms of a Delaunay surface at resonance points \( \mu_n \) are called bubbletons. Geometrically, the integer \( n \in \mathbb{N} \) gives the number of lobes of the bubble on the bubbleton (see Figures 1 and 2).
Figure 1. Bubbletons of the cylinder, that is, $r = \frac{1}{2}$, at resonance points $\mu_2 = 7 + 4\sqrt{3}, \mu_3 = 17 + 12\sqrt{2}$.

Figure 2. An unduloid with $r = \frac{1}{5}$ and its bubbletons with $n = 2$ and $n = 3$.

Note that we require $\mu_n^\pm \in \mathbb{R}_+$ so that there is an additional condition on $n$ in the case of a nodoid, see also [19]:

$$\frac{1 - n}{2} \leq r < 0.$$ 

The case when $n = 1 - 2r$ gives $\mu = -1$ and the corresponding CMC Darboux transform is the parallel CMC surface, that is, we only obtain bubbletons for $\frac{1 - n}{2} < r$ (see Figure 3). (In the case of the cylinder and unduloids, all $\mu_n^\pm \in \mathbb{R} \setminus \{0, \pm 1\}$ for $n \in \mathbb{N}, n > 1$.)
Finally, we can use Bianchi permutability to obtain multibubbletons. Given two parallel sections at distinct resonance points $\mu_n \neq \mu_l$, that is, $n \neq l$, we obtain two bubbletons with $n$ and $l$ lobes respectively. The common Darboux transform $\hat{f}$, a \textit{doublebubbleton}, of $f_n, f_l$ is a surface with two bubbles on it, where one bubble has $n$ lobes and the other one has $l$ lobes (see Figure 4).

Applying Bianchi permutability one can now add an arbitrary number of bubbles to the Delaunay surface. Since Bianchi permutability needs distinct spectral parameter, we obtain CMC multibubbletons with any number of bubbles but every pair of bubbles has different numbers of lobes.

On the other hand, in a recent paper [11] we showed that the double Darboux transformation of a Delaunay surface which uses the same spectral parameter twice (using a Sym–type argument) does not give new closed CMC surfaces: In particular, we cannot obtain same-lobed CMC multibubbletons this way.

4. New CMC cylinders via the Darboux transformation

In this section we will discuss CMC Darboux transforms of Delaunay surfaces which close on a multiple cover. This is primarily motivated by our aim to find same-lobed CMC multibubbletons but also in itself gives rise to new interesting CMC surfaces.
In the case of a cylinder, Sterling and Wente [25] discussed bubbletons on an \( m \)-fold cover via the Bianchi–Bäcklund transform. In this case, the number of lobes \( n \) must be bigger than \( m \). This naturally extends the result for the single cover cylinder where \( n > 1 \). We will recover this result in this section by investigating the multipliers of parallel sections of a Delaunay surface on a multiple cover. Surprisingly, in the case of unduloids and nodoids, additional cases appear: in addition to the natural generalisations of the single cover conditions \( n > m \), we also obtain solutions for \( n < m \).

To this end, we will first investigate the resonance points of a multiple cover of a Delaunay surface, that is, we will consider the surfaces

\[ f(x, y) = ip(x) + jq(x)e^{-iy} \]

on an \( m \)-fold cover, that is, \( x \in \mathbb{R}, y \in [0, 2m\pi], m \in \mathbb{N} \). Then the parallel sections for \( \mu \in \mathbb{R} \setminus \{0, 1\} \) are still given by Theorem 3.1

\[ \alpha_{\pm}^m = e^{\frac{iu}{2}} (qb - iq'(a - 1) + j (1 + p'(a - 1) \pm t)) e^{\pm \frac{it}{2}c_{\pm}} \]

for \( t = \sqrt{1 + 2r(1 - r)(a - 1)} \neq 0 \).

However, the resonance points and multipliers change since we are now investigating the parallel sections on an \( m \)-fold cover. As before excluding \( t = 0 \), the multipliers of the parallel sections \( \alpha_{\pm}^m \) are

\[ h_{\pm}^m = (-1)^m e^{\pm m\pi it} \]

which shows that the resonance points \( \mu_{\pm}^m \) have to satisfy

\[ 1 + 2r(1 - r)(a - 1) = \frac{n^2}{m^2}, \quad n \in \mathbb{N}_+, \]

that is, the resonance points must be of the form

\[ \mu_{\pm}^m = \frac{1}{2r(1 - r)} \left( \frac{n^2 - m^2}{m^2} + 2r(1 - r) \pm \sqrt{\frac{n^2 - m^2}{m^2} \left( \frac{n^2 - m^2}{m^2} + 4r(1 - r) \right)} \right). \]

Note that \( \alpha_{\pm}^m \) are only sections with multipliers if we consider them as sections on a \( m \)-fold cover. That is, we cannot construct Darboux transforms with period \( 2\pi \tau \) with \( \tau > 0 \) this way if \( \tau \not\in \mathbb{Z} \). Moreover, we can assume without loss of generality that \( (m, n) \) are co–prime since \( \mu_{\pm}^m = \mu_{\pm}^{m'} \) for \( \frac{n}{m} = \frac{n'}{m'} \).

Additionally, \( (m, n) \) must satisfy the condition

\[ \frac{n^2 - m^2}{m^2} \left( \frac{n^2 - m^2}{m^2} + 4r(1 - r) \right) > 0, \]

since we consider only real spectral parameter \( \mu \in \mathbb{R} \setminus \{0, \pm 1\} \).

Note that by Theorem 2.2 the resulting \( \mu \)-Darboux transforms for spectral parameter \( \mu_{\pm}^m \in \mathbb{R} \setminus \{0, \pm 1\} \) are CMC surfaces in 3–space. Thus we have extended the result for round cylinders in [25] (obtained via Bianchi’s Bäcklund transformation) to general Delaunay surfaces:

**Theorem 4.1.** Let \( f(x, y) = ip + jqe^{-iy} \) be a Delaunay surface, \( r \in (-\infty, \frac{1}{2}) \), \( r \neq 0 \), \( M = 1 - \left(1 - \frac{1}{2r} \right)^2 \), \( q(x) = (1 - r) \text{dn}((1 - r)x, M), p'(x) = q^2 + r(1 - r) \).

Let \( (m, n) \) with co–prime \( m, n \in \mathbb{N}_+ \) be an admissible pair, that is,
Then every Darboux transform with parameter $\mu^m_n$ is a closed CMC surface on the $m$–fold cover of the Delaunay surface. Put differently, for each admissible pair $(m,n)$ the spectral parameter

$$\mu^m_n = \frac{1}{2r(1-r)} \left( \frac{n^2-m^2}{m^2} + 2r(1-r) \pm \sqrt{\frac{n^2-m^2}{m^2} \left( \frac{n^2-m^2}{m^2} + 4r(1-r) \right)} \right)$$

is a resonance point and $\mu^m_n \in \mathbb{R} \setminus \{0, \pm 1\}$.

Note that [25] only considered the case of cylinders via the Bianchi–Bäcklund transformation, and thus, only considered the case when $n > m$. However, in the case of unduloids and nodoids, interesting new CMC cylinders appear for $n < m$. We discuss the three cases separately, assuming that $(m,n)$ is an admissible pair:

**Cylinder.** In the case of a cylinder, $r = \frac{1}{2}$, we have $\mu^m_n \in \mathbb{R} \setminus \{0, 1\}$ if and only if $n > m$. The resulting Darboux transforms have a localised bubble with $n$ lobes and close on the $m$–fold cover of the Delaunay surface. These are the bubbletons given in [25] (see Figure 5).

![Figure 5](image)

**Unduloid.** In the case of an unduloid we have $0 < r < \frac{1}{2}$ and $4r(1-r) \in (0,1)$. Thus, for $n > m$ we still have $\mu^m_n \in \mathbb{R}_\ast$. In this case, we obtain Darboux transforms with a localised bubble with $n$ lobes as in the case of cylinders (see Figure 6).

![Figure 6](image)
Figure 6. Darboux transform of an unduloid with necksize $r = \frac{1}{5}$, $n = 3$ on the 2-fold cover, next to a bubbleton with $n = 3$ on the single cover.

However, for unduloids, we obtain additional solutions which could not be observed in [25]: if $n < m$ and $r < \frac{m-n}{2m}$ we also obtain closed CMC surfaces (see Figure 7). Note that in this case, the “bubble” of the surface is less localised, and the number $n$ gives the dihedral symmetry of the surface rather than the number of lobes.

Figure 7. Darboux transform with $n = 2$, $m = 7$, of an unduloid with necksize $r = \frac{1}{5}$.

**Nodoid.** In the case of a nodoid, we have $r < 0$, and as in the case of the single cover, we obtain an obstruction for the number $n$ of lobes on the bubbleton: in this case we have a bubbleton with a localised bubble with $n > m$ lobes on the $m$-fold cover if $\frac{m-n}{2m} > r$ (see Figure 8).

Figure 8. Darboux transform of a nodoid with necksize $r = -\frac{1}{5}$, $n = 3$ and $m = 2$, next to a bubbleton with $n = 3$ on the single cover.
In the case of a nodoid, every choice of $n < m$ gives a bubbleton on the $m$–fold cover. In this case, the number $n$ again gives the dihedral symmetry of the surface.

![Image](image1.png)

**Figure 9.** Darboux transform of a nodoid with necksize $r = -\frac{1}{8}$, $n = 3$ and $m = 4$ with two different initial conditions.

In particular, we can also use $n = 1$ (see Figure 10):

![Image](image2.png)

**Figure 10.** Darboux transform of a nodoid with necksize $r = -\frac{1}{8}$, $n = 1$ and $m = 2$.

We will return to a more detailed analysis of these new examples of closed CMC Darboux transforms of nodoid and unduloids in a future paper.

## 5. Same–lobed CMC multibubbletons

Changing our point of view, we now fix the number of lobes $n$ and consider all possible $m$ such that $\mu_n^m$ is a resonance point. To obtain same–lobed CMC multibubbletons (with localised bubbles) we restrict here to the case $n > m$: for such an admissible pair $(m, n)$ we then obtain a bubbleton with $n$ lobes on an $m$–fold cover, and each $\mu_n^m \in \mathbb{R}^*_+$ is a resonance point.

Recall that Bianchi permutability requires different spectral parameters $\mu_1 \neq \mu_2$ to obtain new surfaces as common Darboux transform of the two surfaces $f_1$ and $f_2$. Using now resonance points $\mu_n^{m_1} \neq \mu_n^{m_2}$ given by two admissible pairs $(m_1, n)$ and $(m_2, n)$ with $m_1 \neq m_2$, we obtain by Bianchi permutability a CMC doublebubbleton $\hat{f}$ which has two bubbles with $n$ lobes each. Note that $\hat{f}$ closes on the $L$–fold cover of the Delaunay surface where $L$ is the least common multiple of $m_1$ and $m_2$. 
Repeating the procedure we obtain further multibubbletons as long as there are distinct admissible pairs \((m, n)\), \(m < n\). Denoting by 
\[
\varphi(n) = \#\{m \in \mathbb{N} \mid m, n \text{ co-prime}, 1 \leq m < n, \frac{m - n}{2m} < r\},
\]
the number of admissible pairs with \(m < n\), (for cylinders and unduloids, this is just Euler’s totient function \(\varphi\) of \(n\)), we obtain our final result (see also Figure 11).

**Theorem 5.1.** Let \(f\) be a Delaunay surface with necksize \(r\) and denote by \(\varphi(n)\) the number of admissible pairs \((m, n)\) with \(m < n\). Then for \(1 \leq l \leq \varphi(n)\) there exists a CMC multibubbleton with \(l\) bubbles, each with \(n\) lobes, which closes on a multiple cover of the Delaunay surface.

**Figure 11.** CMC same-lobed multibubbletons of a cylinder and unduloid given via Bianchi permutability: here \(n = 5\), \(\varphi(5) = 4\) and all available resonance points \(\mu_1^5, \mu_2^5, \mu_3^5\) and \(\mu_4^5\) are used. The multibubbletons close on the 12-fold cover and have 4 bubbles with 5 lobes each.

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