On the computational tractability of a geographic clustering problem arising in redistricting

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Abstract

Redistricting is the problem of dividing up a state into a given number $k$ of regions (called districts) where the voters in each district are to elect a representative. The three primary criteria are: that each district be connected, that the populations of the districts be equal (or nearly equal), and that the districts are “compact”. There are multiple competing definitions of compactness, usually minimizing some quantity.

One measure that has been recently promoted by Duchin and others (see e.g. \cite{7}) is number of cut edges. In redistricting, one is generally given atomic regions out of which each district must be built (e.g., in the U.S., census blocks). The populations of the atomic regions are given. Consider the graph with one vertex per atomic region and an edge between atomic regions that share a boundary. Define the weight of a vertex to be the population of the corresponding region. A districting plan is a partition of vertices into $k$ pieces so that the parts have nearly equal weights and each part is connected. The districts are considered compact to the extent that the plan minimizes the number of edges crossing between different parts.

There are two natural computational problems: find the most compact districting plan, and sample districting plans (possibly under a compactness constraint) uniformly at random.

Both problems are NP-hard so we consider restricting the input graph to have branchwidth at most $w$. (A planar graph’s branchwidth is bounded, for example, by its diameter.) If both $k$ and $w$ are bounded by constants, the problems are solvable in polynomial time. For simplicity of notation, assume that each vertex has unit weight. We would like algorithms whose running times are of the form $O(f(k, w)n^c)$ for some constant $c$ independent of $k$ and $w$ (in which case the problems are said to be fixed-parameter tractable with respect to those parameters), we show that, under standard complexity-theoretic assumptions, no such algorithms exist. However, we do show that there exist algorithms with running time $O(c^w n^{k+1})$. Thus if the diameter of the graph is moderately small and the number of districts is very small, our algorithm is useable.

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1 Introduction

For an undirected planar graph $G$ with vertex-weights and a positive integer $k$, a connected partition of the vertices of $G$ is a partition into parts each of which induces a connected subgraph. If $G$ is equipped with nonnegative integral vertex weights and $[L,U]$ is an interval we say such a partition has part-weight in $[L,U]$ if the sum of weights of each part lies in the interval. If $G$ is equipped with nonnegative edge costs, we say the cost of such a partition is the sum of costs of edges $uv$ where $u$ and $v$ lie in different parts.

Consider the following computational problems:

- **optimization**: Given a planar graph $G$ with vertex weights and edge costs, a number $k$, and a weight interval $[L,U]$, find the minimum cost of a partition into $k$ connected parts with part-weight in $[L,U]$.

- **sampling**: Given in addition a number $C$, generate uniformly at random a cost-$C$ partition into $k$ connected parts with part-weight in $[L,U]$.

These problem arises in political redistricting. Each vertex represents a small geographical region (such as a census block or census tract or county), and its weight represents the number of
Figure 3: In the planar dual, each atomic region is represented by a node. (There is also a node for the single infinite region outside the state boundary but here we ignore that node.

Figure 4: For each maximal contiguous boundary segment between a pair of atomic regions, the planar dual has an edge between the corresponding pair of nodes. (Again, the dual also has edges corresponding to segments of the boundary of the state but we ignore those here.

Figure 5: This figure shows an example of a districting plan with seven districts. Each district is the union of several atomic regions.
Figure 6: The districting plan superimposed on the planar dual, showing that it corresponds to a partition of the atoms into connected parts; the cost of the solution is the sum of costs of edges of the dual that cross between different parts. In this paper, a districting plan is compact to the extent that the total cost of the solution is small.

Figure 7: Let $G$ be the graph of atomic regions. As stated in Section 2.2, the radial graph of $G$ has a node for every vertex of $G$ and a node for every face of $G$, and an edge between a vertex-node and a face-node if the vertex lies on the face’s boundary. This diagram shows that every face is reachable from the outer face within six hops in the radial graph of the graph $G$ of atomic regions. This implies that the branchwidth of $G$ and of its dual are at most six.
Figure 8: This map shows the twenty-one cantons for the department “Sarthe” of France. The cantons are the atomic regions for the redistricting of Sarthe. The corresponding radial graph has radius six, so there is a branch decomposition of width $w = 6$. For the upcoming redistricting of France, Sarthe must be divided into $k = 3$ districts.
people living in the region. Each part is a district. A larger geographic region (such as a state) must be partitioned into \( k \) districts when the state is to be represented in a legislative body by \( k \) people; each district elects a single representative. The partition is called a *districting plan*.

The rules governing this partitioning vary from place to place, but usually there are (at least) three important goals: *contiguity*, *compactness*, and *population balance*.

- **Contiguity** is often interpreted as connectivity; we represent this by requiring that the set of small regions forming each district is connected via shared boundary edges.
- **Population balance** requires that two different districts have approximately equal numbers of people.
- One measure of *compactness* is the number of pairs of adjacent small regions that lie in distinct districts.

Thus, in the definitions of the *optimization* and *sampling* problems above, the connectivity constraint reflects the contiguity requirement, the part-weight constraint reflects the population balance requirement, and the cost is a measure of compactness.

The *optimization* problem described above arises in computer-assisted redistricting; an algorithm for solving this problem could be used to select a districting plan that is optimally compact subject to contiguity and desired population balance, where compactness is measured as discussed above.

The *sampling* problem arises in evaluating a plan; in court cases [4, 31, 21, 20, 32] expert witnesses argue that a districting plan reflects an intention to gerrymander by comparing it to districting plans randomly sampled from a distribution. The expert witnesses use Markov Chain Monte Carlo (MCMC), albeit unfortunately on Markov chains that have not been shown to be rapidly mixing, which means that the samples are possibly not chosen according to anything even close to a uniform distribution. There have been many papers addressing random sampling of districting plans (e.g. [1, 3, 7, 20, 21]) but, despite the important role of random sampling in court cases, there are no results on provably uniform or nearly uniform sampling from a set of realistic districting plans for a realistic input in a reasonable amount of time.

It is known that even basic versions of these problems are NP-hard. If the vertex weights are allowed to very large integers, expressed in binary, the NP-hardness of SUBSET SUM already implies the NP-completeness of partitioning the vertices into two equal-weight subsets. However, in application to redistricting the integers are not very large. For the purpose of seeking hardness results, it is better to focus on a special case, the *unit-weight* case, in which each vertex has weight one. For this case, Dyer and Frieze [10] showed that, for any fixed \( k \geq 3 \), it is NP-hard to find a weight-balanced partition into \( k \) connected parts of the vertices of a planar graph. Najt, Deford, and Solomon [29] showed that even for \( k = 2 \) and without the constraint on balance, uniform sampling of partitions into two connected parts is NP-hard.

\[1\] These terms are often not formally defined in law.
Following Ito et al. [24, 23] and Najt et al. [29], we therefore consider a further restriction on the input graph: we consider graphs with bounded branchwidth. The branchwidth of a graph is a measure of how treelike the graph is: often even an NP-hard graph problem is quickly solvable when the input is restricted to graphs with low branchwidth. For planar graphs in particular, there are known bounds on branchwidth that are relevant to the application.

**Lemma 1.** A planar graph on $n$ vertices has branchwidth $O(\sqrt{n})$.

**Lemma 2.** A planar graph of diameter $d$ has branchwidth $O(d)$.

There is a stronger bound, which we will review in Section 2.2.

Najt, Deford, and Solomon [29] show that, for any fixed $k$ and fixed $w$, the optimization and sampling problems without the constraint on population balance can be solved in polynomial time on graphs of branchwidth at most $w$. Significantly, the running time is $O(n^c)$ for some constant $c$ independent of $k$ and $w$. Such an algorithm is said to be fixed-parameter tractable with respect to $k$ and $w$, meaning that as long as $k$ and $w$ are fixed, the problem is considered tractable. Fixed-parameter tractability is an important and recognized way of coping with NP-completeness.

However, their result has two disadvantages. First, as the authors point out, the big O hides a constant that is astronomical; for NP-hard problems, it is expected that the dependence on the parameters be exponential but in this case it is a tower of exponentials. As the authors state, the constants in the theorems on which they rely are “too large to be practically useful.”

Second, because their algorithm cannot handle the constraint on population balance, the algorithm would not be applicable to redistricting even if it were tractable. The authors discuss (Remark 5.11 in [29]) the extension of their approach to handle balance: “It is easy to add a relational formula...that restricts our count to only balanced connected $k$-partitions.... From this it should follow that ... [the problems are tractable]. However ... the corresponding meta-theorem appears to be missing from the literature.”

In our first result, we show that in fact what they seek does not exist: under a standard complexity-theoretic assumption, there is no algorithm that is fixed-parameter tractable with respect to both $k$ and $w$.

More precisely, we use the analogue of NP-hardness for fixed-parameter tractability, $W[1]$-hardness.

**Theorem 1.** For unit weights, finding a weight-balanced $k$-partition of a planar graph of width $w$ into connected parts is $W[1]$-hard with respect to $k + w$.

In the theory of fixed-parameter tractability (see e.g. Section 13.4 of [6]) this is strong evidence that no algorithm exists with a running time of the form $O(f(k, w)n^c)$ for fixed $c$ independent of $k$ and $w$.

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$^2$Treewidth and branchwidth are very similar measures; they are always within a small constant factor of each other. Thus a graph has small treewidth if and only if it has small branchwidth.

$^3$They use treewidth but the results are equivalent.
This is bad news but there is a silver lining. The lower bound guides us in seeking good algorithms, and it does not rule out an algorithm that has a running time of the form \( f(k)n^{O(w)} \) or \( f(w)n^{O(k)} \). That is, according to the theory, while there is no algorithm that is fixed-parameter tractable with respect to both \( k \) and \( w \) simultaneously, there could be one that is fixed-parameter tractable with respect to \( k \) alone and one that is fixed-parameter tractable with respect to \( w \) alone.

These turn out to be true. Ito et al. \[24, 23\] show that, even for general (not necessarily planar) graphs there is an algorithm with running time \( O((w + 1)^{2(w+1)}U^{2(w+1)}k^2n) \), where \( U \) is the upper bound on the part weights. Thus for unit weights, the running time is \( O((w + 1)^{2(w+1)}n^{2w+3}) \). We can significantly reduce the exponent \( 2(w + 1) \), and, by restricting to planar graphs, can also reduce the constant.

However, for the application we have in mind this is not the bound to try for. In some real-world redistricting problems, the number \( k \) of districts is very small.\footnote{For example, in the U.S. there are seventeen states for which the number of districts is greater than one but no more than five. In one of the most complex redistricting problems for France, the number of districts is three.} The number of atoms of course tends to be much larger, but the diameter of the graph is in some cases not so large. Thus we need an algorithm that can tolerate a very small number \( k \) of districts and a moderately small branchwidth \( w \).

**Theorem 2.** For the optimization problem and the sampling problem, there are algorithms that run in \( O(c^w u^k sn(\log U + \log S) + n^3) \) time, where \( c \) is a constant, \( k \) is the number of districts, \( u \geq k \) is an upper bound on the branchwidth of the planar graph, \( n \) is the number of vertices of the graph, \( U \) is the upper bound on the weight of a part, and \( S \) is an upper bound on the cost of a desired solution.

**Remarks:**

1. In the unit-cost case (every edge cost is one), \( S \leq n \).
2. In the unit-weight, unit-cost case, the running time is \( O(c^w n^{k+2} \log n) \).
3. For practical use the input weights need not be the populations of the atoms; if approximate population is acceptable, the weight of an atom with population \( p \) can be, e.g., \( \lceil p/100 \rceil \).
4. The \( n^3 \) term in the running time accounts for the time required to find the branchwidth of the input graph; as we will see, that is usually unnecessary in our application to redistricting.

An implementation of the optimization algorithm has been developed; another paper will describe the implementation and report on experiments in which the algorithms are applied to real (but small) redistricting problems.
2 Preliminaries

2.1 Branchwidth

A branch decomposition of a graph $G$ is a rooted binary tree with the following properties:

1. Each node $x$ is labeled with a subset $C(x)$ of the edges of $G$.

2. The leaves correspond to the edges of $G$: for each edge $e$, there is a leaf $x$ such that $C(x) = \{e\}$.

3. For each node $x$ with children $x_1$ and $x_2$, $C(x)$ is the disjoint union of $C(x_1)$ and $C(x_2)$.

We refer to a set $C(x)$ as a branch cluster. A vertex $v$ of $G$ is a boundary vertex of $C(x)$ if $G$ has at least one edge incident to $v$ that is in $C(x)$ and at least one edge incident to $v$ that is not in $C(x)$. The width of a branch cluster is the number of boundary vertices, and the width of a branch decomposition is the maximum cluster width. The branchwidth of a graph is the minimum $w$ such that the graph has a branch decomposition of width $w$.

For many optimization problems in graphs, if the input graph is required to have small branchwidth then there is a fast algorithm, often linear time or nearly linear time, and often this algorithm can be adapted to do uniform random sampling of solutions. Therefore Najt, Deford, and Solomon [29] had good reason to expect that there would be a polynomial-time algorithm to sample from balanced partitions where the degree of the polynomial was independent of $w$ and $k$.

2.2 Radial graph

For a planar embedded graph $G$, the radial graph of $G$ has a node for every vertex of $G$ and a node for every face of $G$, and an edge between a vertex-node and a face-node if the vertex lies on the face’s boundary. Note that the radial graph of $G$ is isomorphic to the radial graph of the dual of $G$. There is a linear-time algorithm that, given a planar embedded graph $G$ and a node $r$ of the radial graph, returns a branch decomposition whose width is at most the number of hops required to reach every node of the radial graph from $r$ (see, e.g., [26]). For example, Figure 7 shows that the number of hops required is at most six, so the linear-time algorithm would return a branch decomposition of width $w$ at most six.

Using this result, some real-world redistricting graphs can be shown to have moderately small branchwidth. For example, Figure 8 shows a department of France, Sarthe, that will need to be divided into $k = 3$ districts. The number of hops required for this example is six, so we would get a branch decomposition of width $w$ at most six.

2.3 Sphere-cut decomposition

The branch decomposition of a planar embedded graph can be assumed to have a special form. The radial graph of $G$ can be drawn on top of the embedding of $G$ so that a face-node is embedded in
the interior of a face of \( G \) and a vertex-node is embedded in the same location as the corresponding vertex. We can assume that the branch decomposition has the property that corresponding to each branch cluster \( C \) is a cycle in the radial graph that encloses exactly the edges belonging to the cluster \( C \), and the vertices on the boundary of this cluster are the vertex-nodes on the cycle. This is called a sphere-cut decomposition \([9]\). If the branch decomposition is derived from the radial graph using the linear-time algorithm mentioned above, the sphere-cut decomposition comes for free. Otherwise, there is an \( O(n^3) \) algorithm to find a given planar graph’s least-width branch decomposition, and if this algorithm is used it again gives a sphere-cut decomposition.

### 3 Related work

There is a vast literature on partitioning graphs, in particular on partitions that are in a sense balanced. In particular, in the area of decomposition of planar graphs, there are algorithms \([33, 30, 34]\) for sparsest cut and quotient cut, in which the goal is essentially to break off a single piece such that the cost of the cut is small compared to the amount of weight on the smaller side. The single piece can be required to be connected. There are approximation algorithms for variants of balanced partition \([16, 14]\) into two pieces. These only address partitioning into \( k = 2 \) pieces, the pieces are not necessarily connected, and the balance constraint is only approximately satisfied;

There are many papers on algorithms relevant to computer-aided redistricting (a few examples are \([19, 22, 28, 15]\)), including papers that have appeared at SIGSPATIAL \([11, 5]\).

Finally, there many papers on \( W[1]\)-hardness and more generally lower bounds on fixed-parameter tractability, as this is a well-studied area of theoretical computer science. Our result is somewhat rare in that most graph problems are fixed-parameter tractable with respect to branchwidth/ treewidth. However, there are by now other \( W[1]\)-hardness results with respect to treewidth \([8, 2, 13, 27, 18, 17]\) and a few results \([2, 12]\) were previously known even under the restriction that the input graph must be planar.

### 4 \( W[1]\)-Hardness

In this section, we show that the problem is \( W[1]\)-hard parameterized by \( k + w \), where \( k \) is the number of districts and \( w \) the treewidth of the graph.

We start with the following lemma that shows that it is enough to prove that a more structure version of the problem (bounded vertex weights, each region must have size greater than 1) is \( W[1]\)-hard.

**Lemma 3.** If the planar vertex-weighted version of the problem is \( W[1]\)-hard parameterized by \( k + w \) when the total weight of each region should be greater than 1, and the smallest weight is 1 and the largest weight is \( |V|^c \) for some constant \( c \), then the planar unweighted version of the problem is \( W[1]\)-hard parameterized by \( k + w \).
Proof. Consider the following transformations of a vertex-weighted instance of the problem. First, rescale all the weights of vertices by a factor $W/w$ where $W$ is the largest vertex weight and $w$ is the largest vertex weight. For each vertex $v$ of weight $w(v)$, create $w(v) - 1$ unit-weight dummy vertices and connect each of them to $v$ with a single edge, then remove the weight of $v$.

This yields a unit-weight graph which satisfies the following properties. First, if the input graph was planar, then the resulting graph is also planar. Second, since the ratio $W/w$ is at most $|V|^c$, the total number of vertices in the new graph is at most $|V|^{c+1}$. Finally, any solution for the problem on the vertex-weighted graph can be associated to a solution for the problem on the unit-weight graph: for each vertex $v$ of the original graph, assign each of the $w(v) - 1$ dummy vertices to the same region as $v$. We have that the associated solution has connected regions of exactly the same weight as the solution in the weighted graph. Moreover, we claim that any solution for the unit-weight graph is associated to a solution of the input weighted graph: this follows from the assumption that the prescribed weights for the regions is greater than 1 and that the regions must be connected. Thus for each vertex $v$, in any solution all the $w(v) - 1$ dummy vertices must belong to the region of $v$.

Therefore, if the planar vertex-weighted version of the problem is $W[1]$-hard parameterized by $k + w$ when the smallest weight is at least 1, the total weight of each region should be greater than 1, and the total weight of the graph is at most $|V|^c$ for some constant $c$, then the planar unit-weight version of the problem is $W[1]$-hard parameterized by $k + w$.

By Lemma 3, we can focus without loss of generality on instances where the vertices have weights between $1$ and $|V|^c$ for some fixed constant $c$. We next show that the problem is $W[1]$-hard on these instances.

We reduce from the Bin Packing problem with polynomial weights. Given a set of integer values $v_1, \ldots, v_n$ and two integers $B$ and $k$, the Bin Packing problem asks to decide whether there exists a partition of $v_1, \ldots, v_n$ into $k$ parts such that for each part of the partition, the sum of the values is at most $B$. The Bin Packing problem with polynomially bounded weights assumes that there exists a constant $c$ such that $B = O(n^c)$. Note that for the case where the weights are polynomially bounded, we can assume w.l.o.g. that the sum of the weights is exactly $kB$ by adding $kB - \sum_{i=1}^n v_i$ elements of value 1. Since the weights are polynomially bounded and that each weight is integer we have that (1) the total number of new elements added is polynomial in $n$, hence the size of the problem is polynomial in $n$, and (2) there is a solution to the original problem if and only if there is a solution to the new problem; the new elements can be added to fill up the bins that are not full in the solution of the original problem.

We will make use of the following theorem of Jansen et al. [25].

Theorem 3 ([25]). The Bin Packing problem with polynomial weights is $W[1]$-hard parameterized by the number of bins $k$. Moreover, there is no $f(k)n^{o(k/\log k)}$ time algorithm assuming the exponential time hypothesis (ETH).

We now proceed to the proof of Theorem 1. From an instance of Bin Packing with polynomially
bounded weights and whose sum of weights is $kB$, create the following instance for the problem. For each $i \in [2k + 1]$, create

$$
\ell_i = \begin{cases} 
k & \text{if } i \text{ is odd} \\
k + 1 & \text{if } i \text{ is even}
\end{cases}
$$

vertices $s_1^i, \ldots, s_{\ell_i}^i$. Let $S_i = \{s_1^i, \ldots, s_{\ell_i}^i\}$. Moreover, for each odd $i < n$, for each $1 \leq j \leq k$, connect $s_j^i$ to $s_{i-1}^j$ and $s_{i+1}^j$, and when $j < k$, also to $s_{i+1}^j$ and $s_{i+1}^{j+1}$. Let $G$ be the resulting graph.

It is easy to see that $G$ is planar. We let $f_\infty$ be the longest face:

$$
\{s_1^1, \ldots, s_{k+1}^1, s_{2n+1}^1, s_{2n+1}^{k-1}, \ldots, s_{2n+1}^1, s_{2n+1}^{k-1}, \ldots, s_{2n+1}^1, s_{2n+1}^1, s_{2n+1}^1\}.
$$

We claim that the treewidth of the graph is at most $7k$. To show this we argue that the face-vertex incidence graph $G$ of $G$ has diameter at most $2k + 4$ and by Lemma 3 this immediately yields that the treewidth of $G$ is at most $10k$. We show that each vertex of $G$ is at hop-distance at most $k + 2$ of the vertex corresponding to $f_\infty$. Indeed, consider a vertex $s_1^1$ (for a face, consider a vertex $s_1^1$ on that face). Recall that for each $i_0, j_0$, we have that $s_{i_0}^0$ is adjacent to $s_{i+1}^0$ and $s_{i+1}^{n+1}$ and so, $s_1^i$ is at hop-distance at most $k + 1$ from either $s_1^i$ or $s_1^i$ in $G$. Moreover both $s_1^i$ and $s_n^i$ are on face $f_\infty$ and so $s_1^i$ is at hop-distance at most $k + 2$ from $f_\infty$ in $G$. Hence the treewidth of $G$ is at most $10k$.

Our next step is to assign weights to the vertices. Then, we set the weight $w(s_1^i)$ of every vertex $s_1^i$ of $\{s_1^1, \ldots, s_k^1\}$ to be $(kB)^2$ and the weight $w(s_1^i)$ of every vertex $s_1^i$ of $\{s_1^1, \ldots, s_k^1\}$ to be $(kB)^4$. For each odd $i \neq 1, 2n + 1$ we set a weight of $1/(2n - 2)$. Finally, we set the weight of each vertex $s_1^i$ where $i$ is even to be $v_i$. Let $T = (kB)^2 + (kB)^4 + 1/2 + kB$, and recall that $kB = \sum_{i=1}^{n} v_i$.

**Fact 1.** Consider a set $S$ of vertices containing exactly one vertex of $S_i$ for each $i$. Then the sum of the weights of the vertices in $S$ is $T$.

We now make the target weight of each region to be $(kB)^2 + (kB)^4 + kB + B = T + B$. We have the following lemma.

**Lemma 4.** In any feasible solution to the problem, there is exactly 1 vertex of $\{s_1^1, \ldots, s_k^1\}$ and exactly 1 vertex of $\{s_1^n, \ldots, s_{\ell(n)}^n\}$ in each region.

**Proof.** Recall that by definition we have that $\sum_{i=1}^{n} v_i = kB$. Moreover, the number of vertices with weight equal to $(kB)^2$ is exactly $k$. Thus, since the target weight of each region is $(kB)^2 + (kB)^4 + B + kB$, each region has to contain exactly 1 vertex from $\{s_1^1, \ldots, s_k^1\}$ and exactly 1 vertex from $\{s_1^n, \ldots, s_{\ell(n)}^n\}$.

We now turn to the proof of completeness and soundness of the reduction. We first show that if there exists a solution to the Bin Packing instance, namely that there is a partition into $k$ parts such that for each part of the partition, the sum of the values is $B$, then there exists a feasible solution to the problem. Indeed, consider a solution to the Bin Packing instance $\{B_1, \ldots, B_k\}$ and construct the following solution to the problem. For each odd $i$, assign vertices $s_1^i, \ldots, s_k^i$ to regions
defines an equivalence relation on \( \Omega \): two elements are equivalent if they are in the same subset. A partition \( \pi \) on simplicity rather than on achieving the best constant possible as the base of \( k \) Algorithm proof is complete.

We then bound the total weight of each region. Let’s partition the vertices of a region \( R_j \) into two: Let \( S_{R_j} \) be a set that contains one vertex from each \( S_i \), and let \( \overline{S}_{R_j} \), be the rest of the elements. The total weight of the vertices in \( S_{R_j} \) is by Fact \( \ref{fact:unique-minimal} \) exactly \( T \). The total weight of the remaining vertices corresponds to the sum of the values \( v_i \) such that \( |R_j \cap S_i| = 2 \) which is \( \sum_{v_i \in B_j} v_i = B \) since it is a solution to the Bin Packing problem. Hence the total weight of the region is \( T + B \), as prescribed by the problem.

We finally prove that if there exists a solution for the problem with the prescribed region weights, then there exists a solution to the Bin Packing problem. Let \( R_1, \ldots, R_k \) be the solution to the problem. By Lemma \( \ref{lemma:unique-minimal} \) each region contains one vertex of \( s_1^1, \ldots, s_1^k \) and one vertex of \( s_2^1, \ldots, s_{2n+1}^k \). Since the regions are required to be connected, there exists a path joining these two vertices and so by the pigeonhole principle for each odd \( i \), each region contains exactly one vertex of \( s_i^1, \ldots, s_i^k \). Moreover for each even \( i \), each region contains at least one vertex of \( s_i^1, \ldots, s_i^{k+1} \) and exactly one region contains two vertices. Let \( \phi(i) \in [k] \) be such that \( |R_{\phi(i)} \cap \{s_i^1, \ldots, s_i^{k+1}\}| = 2 \). We now define the following solution for the Bin Packing problem. Define the \( j \)th bin as \( B_j = \{v_i \mid \phi(i) = j\} \). We claim that for each bin \( B_j \) the sum of the weights of the elements in \( B_j \) is exactly \( B \). Indeed, observe that region \( R_j \) contains exactly one vertex of \( s_1^1, \ldots, s_k^k \) for each odd \( i \) and exactly one vertex of \( s_i^1, \ldots, s_i^{k+1} \) for each even \( i \) except for the sets \( s_i^1, \ldots, s_i^{k+1} \) where \( \phi(i) = j \) for which it contains two vertices. Thus by Fact \( \ref{fact:unique-minimal} \) the total sum of the weights is \( T + \sum_{i|\phi(i)=j} v_i \) and since the target weight is \( T + B \) we have that \( \sum_{i|\phi(i)=j} v_i = B \). Since the weight of \( B_j \) is exactly \( \sum_{i|\phi(i)=j} v_i \) the proof is complete.

5 Algorithm

In this section, we describe the algorithms of Theorem \( \ref{thm:algorithm} \). In describing the algorithm, we will focus on simplicity rather than on achieving the best constant possible as the base of \( k \).

5.1 Partitions

A partition of a finite set \( \Omega \) is a collection of disjoint subsets of \( \Omega \) whose union is \( \Omega \). A partition defines an equivalence relation on \( \Omega \): two elements are equivalent if they are in the same subset.

There is a partial order on partitions of \( \Omega \): \( \pi_1 \prec \pi_2 \) if every part of \( \pi_1 \) is a subset of a part of \( \pi_2 \). This partial order is a lattice. In particular, for any pair \( \pi_1, \pi_2 \) of partitions of \( \Omega \), there is a unique minimal partition \( \pi_3 \) such that \( \pi_1 \prec \pi_3 \) and \( \pi_2 \prec \pi_3 \). (By minimal, we mean that for any partition \( \pi_4 \) such that \( \pi_1 \prec \pi_4 \) and \( \pi_2 \prec \pi_4 \), it is the case that \( \pi_3 \prec \pi_4 \).) This unique minimal
partition is called the *join* of $\pi_1$ and $\pi_2$, and is denoted $\pi_1 \vee \pi_2$.

It is easy to compute $\pi_1 \vee \pi_2$: initialize $\pi := \pi_1$, and then repeatedly merge parts that intersect a common part of $\pi_2$.

In a slight abuse of notation, we define the join of a partition $\pi_1$ of one finite set $\Omega_1$ and a partition $\pi_2$ of another finite set $\Omega_2$. The result, again written $\pi_1 \vee \pi_2$, is a partition of $\Omega_1 \cup \Omega_2$. It can be defined algorithmically: initialize $\pi$ to consist of the parts of $\pi_2$, together with a singleton part $\{\omega\}$ for each $\omega \in \Omega_2 - \Omega_1$. Then repeatedly merge parts of $\pi$ that intersect a common part of $\pi_2$.

### 5.2 Noncrossing partitions

The sphere-cut decomposition is algorithmically useful because it restricts the way a graph-theoretic structure (such as a solution) can interact with each cluster. For a cluster $C$, consider the corresponding cycle in the radial graph, and let $\theta_C$ be the cyclic permutation $(v_1 v_2 \ldots v_m)$ of boundary vertices in the order in which they appear in the radial cycle. (By a slight abuse of notation, we may also interpret $\theta_C$ as the set $\{v_1, \ldots, v_m\}$.)

First consider a partition $\rho^\text{in}$ of the vertices incident to edges belonging to $C$, with the property that each part induces a connected subgraph of $C$. Planarity implies that the partition induced by $\rho^\text{in}$ on the boundary vertices $\{v_1, \ldots, v_m\}$ has a special property.

**Definition 1.** Let $\pi$ be a partition of the set $\{1, \ldots, w\}$. We say $\pi$ is crossing if there are integers $a < b < c < d$ such that one part contains $a$ and $c$ and another part contains $b$ and $d$.

It follows from connectivity that the partition induced by $\rho^\text{in}$ on the boundary vertices $\theta_C$ is a noncrossing partition. Similarly, let $\rho^\text{out}$ be a partition of the vertices incident to edges that do not belong to $C$; then $\rho^\text{out}$ induces a noncrossing partition on the boundary vertices of $C$.

The asymptotics of the Catalan numbers imply the following (see, e.g., [9]).

**Lemma 5.** There is a constant $c_1$ such that the number of noncrossing partitions of $\{1, \ldots, w\}$ is $O(c_1^w)$.

Finally, suppose $\rho$ is a partition of all vertices of $G$ such that each part is connected. Then $\rho = \rho^\text{in} \vee \rho^\text{out}$ where $\rho^\text{in}$ is a partition of the vertices incident to edges in $C$ (in which each part is connected) and $\rho^\text{out}$ is a partition of the vertices incident to edges not in $C$ (in which each part is connected).

Because the only vertices in both $\rho^\text{in}$ and $\rho^\text{out}$ are those in $\theta_C$, the partition $\rho$ induces on $\theta_C$ is $\pi^\text{in} \vee \pi^\text{out}$ where $\pi^\text{in}$ is the partition induced on $\theta_C$ by $\rho^\text{in}$ and $\pi^\text{out}$ is the partition induced on $\theta_C$ by $\rho^\text{out}$.

### 5.3 Algorithm overview

The algorithms for optimization and sampling are closely related.
The algorithms are based on dynamic programming using the sphere-cut decomposition of the planar embedded input graph $G$.

Each algorithm considers every vertex $v$ of the input graph and selects one edge $e$ that is incident to $v$, and designates each branch cluster that contains $e$ as a home cluster for $v$.

We define a topological configuration of a cluster $C$ to be a pair $(\pi^{\text{in}}, \pi^{\text{out}})$ of noncrossing partitions of $\theta_C$ with the following property:

\[ \pi^{\text{in}} \lor \pi^{\text{out}} \text{ has at most } k \text{ parts.} \tag{1} \]

The intended interpretation is that there exist $\rho^{\text{in}}$ and $\rho^{\text{out}}$ as defined in Section 5.2 such that $\phi^{\text{in}}$ is the partition $\rho^{\text{in}}$ induces on $\theta_C$ and $\phi^{\text{out}}$ is the partition $\rho^{\text{out}}$ induces on $\theta_C$.

We can assume that the vertices of the graph are assigned unique integer IDs, and that therefore there is a fixed total ordering of $\theta_C$. Based on this total ordering, for any partition $\pi$ of $\theta_C$, let $p$ be the number of parts of $\pi$, and define representatives($\pi$) to be the $p$-vector $(v_1, v_2, \ldots, v_p)$ obtained as follows:

- $v_1$ is the smallest-ID vertex in $\theta_C$,
- $v_2$ is the smallest-ID vertex in $\theta_C$ that is not in the same part as $v_1$,
- $v_2$ is the smallest-ID vertex in $\theta_C$ that is not in the same part as $v_1$ and is not in the same part as $v_2$,

and so on.

This induces a fixed total ordering of the parts of $\pi^{\text{in}} \lor \pi^{\text{out}}$.

We define a weight configuration of $C$ to be a $k$-vector $w = (w_1, \ldots, w_k)$ where each $w_i$ is a nonnegative integer less than $U$. There are $U^k$ such vectors.

We define a weight/cost configuration of $C$ to be a $k$-vector together with a nonnegative integer $s$ less than $S$. There are $U^k S$ such configurations.

We define a configuration of $C$ to be a pair consisting of a topological configuration and a weight/cost configuration. The number of configurations of $C$ is bounded by $c w U^k S$.

The algorithms use dynamic programming to construct, for each cluster $C$, a table $T_C$ indexed by configurations of $C$. In the case of optimization, the table entry $T_C[\Psi]$ corresponding to a configuration $\Psi$ is true or false. For sampling, $T_C[\Psi]$ is a cardinality.

Let $\Psi = ((\pi^{\text{in}}, \pi^{\text{out}}), ((w_1, \ldots, w_k), s))$ be a configuration of $C$. Let count($\Psi$) be the number of partitions $\rho^{\text{in}}$ of the vertices incident to edges belonging to $C$ with the following properties:

- $\rho^{\text{in}}$ induces $\pi^{\text{in}}$ on $\theta_C$.
- Let $\pi = \pi^{\text{in}} \lor \phi^{\text{out}}$. Let representatives($\pi$) = $(v_1, \ldots, v_p)$. Then for $j = 1, \ldots, p$, $w_j$ is the total weight of vertices $v$ for which $C$ is a home cluster and such that $v$ belongs to the same part of $\rho^{\text{in}} \lor \pi^{\text{out}}$ as $v_j$. 

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For optimization, \( T_C[\Psi] \) is true if \( \text{count}(\Psi) \) is nonzero. For sampling, \( T_C[\Psi] = \text{count}(\Psi) \). We describe in Section 5.5 how to populate these tables. Next we describe how they can be used to solve the problems.

### 5.4 Using the tables

For the root cluster \( \hat{C} \), the cluster that contains all edges of \( G \), \( \theta_{\hat{C}} \) is empty. Therefore there is only one partition of \( \theta_{\hat{C}} \), the trivial partition \( \pi_0 \) consisting of a single part, the empty set.

To determine the optimum cost in the optimization problem, simply find the minimum nonnegative integer \( s \) such that, for some \( w = (w_1, \ldots, w_k) \) such that each \( w_i \) lies in \( [L, U) \), the entry \( T_{\hat{C}}[((\pi_0, \pi_0), (w, s))] \) is true. To find the solution with this cost, the algorithm needs to find a “corresponding” configuration for each leaf cluster \( C(\{uv\}) \); that configuration tells the algorithm whether the two endpoints \( u \) and \( v \) are in the same district. This information is obtained by a recursive algorithm, which we presently describe.

Let \( C_0 \) be a cluster with child clusters \( C_1 \) and \( C_2 \). For \( i = 0, 1, 2 \), let \( (\pi^\text{in}_i, \pi^\text{out}_i) \) be a topological configuration for cluster \( C_i \). Then we say these topological configurations are consistent if the following properties hold:

- For \( i = 1, 2 \), \( \pi^\text{out}_i = \pi^\text{out}_0 \lor \pi^\text{in}_{3-i} \).
- \( \pi^\text{in}_0 = \pi^\text{in}_1 \lor \pi^\text{in}_2 \).

For \( i = 0, 1, 2 \), let \( (w_i, s_i) \) be a weight/cost configuration for \( C_i \). We say they are consistent if \( w_0 = w_1 + w_2 \) and \( s_0 = s_1 + s_2 \).

Finally, for \( i = 0, 1, 2 \), let \( \Psi_i = ((\pi^\text{in}_i, \pi^\text{out}_i), (w_i, s_i)) \) be a configuration for cluster \( C_i \). Then we say \( \Psi_1, \Psi_2, \Psi_3 \) are consistent if the topological configurations are consistent and the weight/cost configurations are consistent.

**Lemma 6.** For a configuration \( \Psi_0 \) of \( C_0 \), \( \text{count}(\Psi_0) = \sum_{\Psi_1, \Psi_2} \text{count}(\Psi_1) \cdot \text{count}(\Psi_2) \) where the sum is over pairs \( (\Psi_1, \Psi_2) \) of configurations of \( C_1, C_2 \) such that \( \Psi_0, \Psi_1, \Psi_2 \) are consistent.

The recursive algorithm, given a configuration \( \Psi \) for a cluster \( C \) such that \( T_C[\Psi] \) is true, finds configurations for all the clusters that are descendants of \( C \) such that, for each nonleaf descendant and its children, the corresponding configurations are consistent; for each descendant cluster \( C' \), the configuration \( \Psi' \) selected for it must have the property that \( T_{C'}[\Psi'] \) is true.

The algorithm is straightforward:

**define DESCEND**\((C_0, \Psi_0)\):

- **precondition**: \( T_{C_0}[\Psi_0] = \text{true} \)
- assigns \( \Psi_0 \) to \( C_0 \)
- if \( C_0 \) is not a leaf config
  - for each config \( \Psi_1 = ((\pi^\text{in}_1, \pi^\text{out}_1), (w_1, s_1)) \) of \( C_0 \)'s left child \( C_1 \),
if $T_{C_1}[\Psi_1]$ is true

for each topological config $(\pi_2^{\text{in}}, \pi_2^{\text{out}})$ of $C_0$’s right child $C_2$

let $(w_2, s_2)$ be the weight/cost config of $C_2$ such that

$\Psi_0, \Psi_1, \Psi_2$ are consistent

where $\Psi_2 = ((\pi_2^{\text{in}}, \pi_2^{\text{out}}), (w_2, s_2))$

if $T_{C_2}[\Psi_2] = \text{true}$

call DESCEND$(C_1, \Psi_1)$ and DESCEND$(C_2, \Psi_2)$

return, exiting out of loops

Lemma 6 shows via induction from root to leaves that the procedure will successfully find configurations for all clusters that are descendants of $C_0$. For the root cluster $\hat{C}$ and a configuration $\hat{\Psi}$ of $\hat{C}$ such that $T_{\hat{C}}[\hat{\Psi}]$ is true, consider the $\Psi_C$ configurations found for each leaf cluster, and let $(\pi_C^{\text{in}}, \pi_C^{\text{out}})$ be the topological configuration of $\Psi_C$. Consider the partition

$$\rho = \bigvee_C \pi_C^{\text{in}}$$

where the join is over all leaf clusters $C$. Because there are no vertices of degree one, for each leaf cluster $C(\{uv\})$, both $u$ and $v$ are boundary vertices, so $\rho$ is a partition of all vertices of the input graph. Induction from leaves to root shows that this partition agrees with the weight/cost part $(\hat{w}, \hat{s})$ of the configuration $\hat{\Psi}$. In particular, the weights of the parts of $\rho$ correspond to the weights of $\hat{w}$, and the cost of the partition equals $\hat{s}$.

In the step of DESCEND that selects $(w_2, s_2)$, there is exactly one weight/cost config that is consistent (it can be obtained by permuting the elements of $w_1$ and then subtracting from $w_0$ and subtracting $s_1$ from $s_0$). By an appropriate choice of an indexing data structure to represent the tables, we can ensure that the running time of DESCEND is within the running time stated in Theorem 2. For optimization, it remains to show how to populate the tables.

define DESCEND$(C_0, \Psi_0, p)$:

precondition: $p \leq T_{C_0}[\Psi_0]$

assign $\Psi_0$ to $C_0$

if $C_0$ is not a leaf config

for each config $\Psi_1 = ((\pi_1^{\text{in}}, \pi_1^{\text{out}}), (w_1, s_1))$ of $C_0$’s left child $C_1$,

for each topological config $(\pi_2^{\text{in}}, \pi_2^{\text{out}})$ of $C_0$’s right child $C_2$

let $(w_2, s_2)$ be the weight/cost config of $C_2$ such that

$\Psi_0, \Psi_1, \Psi_2$ are consistent

where $\Psi_2 = ((\pi_2^{\text{in}}, \pi_2^{\text{out}}), (w_2, s_2))$

$$\Delta := T_{C_1}[\Psi_1] \cdot T_{C_2}[\Psi_2]$$

if $p \leq \Delta$

$q := \lfloor p/T_{C_2}[\Psi_2] \rfloor$

$r := r \mod T_{C_2}[\Psi_2]$
call `DESCEND(C_1, \Psi_1, q)` and `DESCEND(C_2, \Psi_2, r)`
return
else \( p := p - \Delta \) and continue

Induction shows that this procedure, applied to root cluster \( \hat{C} \) and a configuration \( \hat{\Psi} \) and an integer \( p \leq T_{\hat{C}}[\hat{\Psi}] \), selects the \( p^{th} \) solution among those “compatible” with \( \hat{\Psi} \). This can be used for random generation of solutions with given district populations and a given cost. Again, the running time for the procedure is within that stated in Theorem 2.

5.5 Populating the tables

For this section, let us focus on the tables needed for sampling. Populating the table for a leaf cluster is straightforward. Therefore, suppose \( C_0 \) is a cluster with children \( C_1 \) and \( C_2 \). We first observe that, given noncrossing partitions \( \pi_0^{\text{out}} \) of \( \theta_{C_0} \), \( \pi_1^{\text{in}} \) of \( \theta_{C_1} \), and \( \pi_2^{\text{in}} \) of \( \theta_{C_2} \), there are unique partitions \( \pi_1^{\text{in}}, \pi_1^{\text{out}}, \pi_2^{\text{out}} \) such that the topological configurations \( (\pi_0^{\text{in}}, \pi_0^{\text{out}}), (\pi_1^{\text{in}}, \pi_1^{\text{out}}), (\pi_2^{\text{in}}, \pi_2^{\text{out}}) \) are consistent. (The formulas that show this are in the pseucode below.)

For the second observation, consider a configuration \( \Psi_0 = (\kappa_0, (w_0, s_0)) \) of \( C_0 \). Then \( \text{count}(\Psi_0) \) is

\[
\sum_{\kappa_1, \kappa_2} \sum \text{count}((\kappa_1, (w_1, s_1))) \cdot \text{count}((\kappa_2, (w_2, s_2)))
\]

where the first sum is over pairs of topological configurations \( \kappa_1 \) for \( C_1 \) and \( \kappa_2 \) for \( C_2 \), and the second sum is over pairs of weight/cost configurations that are consistent with \( (w_0, s_0) \). Note that because of how weight/cost configuration consistency is defined, the second sum mimics multivariate polynomial multiplication. We use these observations to define the procedure that populates the table for \( C_0 \) from the tables for \( C_1 \) and \( C_2 \).

```python
def Combine(C_0, C_1, C_2):
    initialize each entry of \( T_{C_0} \) to zero
    for each noncrossing partition \( \pi_0^{\text{out}} \) of \( \theta_{C_0} \)
        for each noncrossing partition \( \pi_1^{\text{in}} \) of \( \theta_{C_1} \)
            for each noncrossing partition \( \pi_2^{\text{in}} \) of \( \theta_{C_2} \)
                \( \pi_1^{\text{out}} = \pi_0^{\text{out}} \lor \pi_2^{\text{in}} \)
                \( \pi_2^{\text{out}} = \pi_0^{\text{out}} \lor \pi_1^{\text{in}} \)
                \( \pi_0^{\text{in}} = \pi_1^{\text{in}} \lor \pi_2^{\text{in}} \)
                comment: now we populate entries of \( T_{C_0}[\cdot] \) indexed by configurations of \( C_0 \) with topological configuration \( (\pi_0^{\text{in}}, \pi_0^{\text{out}}) \).
            for \( i = 1, 2 \),
                let \( p_i(x, y) \) be a polynomial over variables \( x_1, \ldots, x_k, y \)
                such that the coefficient of \( x_1^{w_1} \cdots x_k^{w_k} y^a \)
        \end{comment}
```

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is $T_{C_1}([((\pi_0^{\text{in}}, \pi_0^{\text{out}}), ((w_1, \ldots, w_k), s))]$

let $p(x, y)$ be the product of $p_1(x, y)$ and $p_2(x, y)$

for every weight/cost configuration $((w_1, \ldots, w_k), s)$

add to $T[((\pi_0^{\text{in}}, \pi_0^{\text{out}}), ((w_1, \ldots, w_k), s))]$ the

coefficient of $x_1^{w_1} \cdots x_k^{w_k} y^s$ in $p(x, y)$

The three loops involve at most $c^w$ iterations, for some constant $c$. Multivariate polynomial multiplication can be done using multidimensional FFT. The time required is $O(N \log N)$, where $N = U^k S$. (This use of FFT to speed up an algorithm is by now a standard algorithmic technique.) It follows that the running time of the algorithm to populate the tables is as described in Theorem 2.

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