Fission and Fusion Bound States of $p$-brane Solitons

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ABSTRACT

Supersymmetric $p$-branes that carry a single electric or magnetic charge and preserve $1/2$ of the supersymmetry have been interpreted as the constituents from which all supersymmetric $p$-branes can be constructed as bound states, albeit with zero binding energy. Here we extend the discussion to non-supersymmetric $p$-branes, and argue that they also can be interpreted as bound states of the same basic supersymmetric constituents. In general, the binding energy is non-zero, and can be either positive or negative depending on the specific choice of constituents. In particular, we find that the $a = 0$ Reissner-Nordstrøm black hole in $D = 4$ can be built from different sets of constituents such that it has zero, positive or negative binding energy.

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1 Introduction

Isotropic $p$-brane solitons in M-theory or string theory have been extensively studied, and their classification has been discussed from various points of view. One approach is to organise the various solutions using the U duality group of the theory. In particular, it was shown that $p$-brane solutions form representations of the U Weyl group [1]. Another approach is to interpret lower dimensional solutions from the viewpoint of the fundamental dimension of the theory, namely $D = 11$ in the case of M-theory. In other words, the lower-dimensional solutions can be oxidised, by the inverse of the Kaluza-Klein reduction procedure, to solutions in $D = 11$. It has been shown that lower dimensional supersymmetric $p$-branes can be viewed as intersecting M-branes [2-7] or boosted intersecting M-branes [7] in $D = 11$. A third approach, which until now has also been applied only to supersymmetric $p$-branes, is to view those solutions that carry more than one kind of charge as bound states of single-charge solutions [3, 4, 5]. For example, the $a = 1, 1/\sqrt{3}$ and 0 black holes in $D = 4$ can be viewed as bound states of two, three or four $a = \sqrt{3}$ black holes [8]. Another example is provided by the dyonic string [11] in $D = 6$, which can be viewed as a bound state of an electric and a magnetic string [7]. All the above solutions Strictly speaking, the term bound state is a misnomer, since the binding energy is actually zero for these supersymmetric $p$-branes. This zero binding energy is consistent with the fact that the charges in the above multi-charge solutions can be located independently; the bound states can be “pulled apart” into constituents that can sit in static equilibrium at any separation. This type of multi-charge, multi-center solution was first discussed in [10], where a four-charge supersymmetric black hole in $D = 4$ was “split” into two $a = 1$ two-charge black holes (which themselves can be further split into $a = \sqrt{3}$ black holes).

The majority of $p$-brane solutions are in fact not supersymmetric [12]. The purpose of this paper is to extend the previous discussion of bound states to encompass these non-supersymmetric solutions. As we shall see, all these solutions can also be viewed as bound states of the same supersymmetric building blocks, namely the single-charge $p$-branes. The major difference from the supersymmetric bound states is that now the binding energy is non-zero. In some cases, the binding energy is positive, implying that the supersymmetric building blocks will undergo a spontaneous fusion to form the non-supersymmetric $p$-brane. In other cases, the binding energy is negative, and the non-supersymmetric $p$-brane will undergo a spontaneous fission into its supersymmetric constituents.

The binding energy of a $p$-brane can easily be calculated by comparing its mass with the sum of the masses of its individual constituents when their locations are widely separated.
Of course if the binding energy is non-zero, this configuration will not be an exact solution. However, it can be made arbitrarily good by taking the separations to be sufficiently large. We shall discuss the binding energy for single-scalar solutions in section 2. In these solutions, the charges carried by the various participating field strengths arise in fixed ratios, which implies that they are formed as bound states of constituents whose charges have the same ratios. In the case of supersymmetric $p$-branes, more general solutions are known in which the charges can be independently specified \cite{13}, implying that these are bound states of constituents with independent charges. The analogous solutions with independent charges are not in general known for the non-supersymmetric cases, and so the exact discussion is generally restricted to the cases where the constituents have the necessary fixed ratios of charges. In section 3, however, we shall present an explicit exact solution with two independent charges in one particular non-supersymmetric case, namely a black hole dyon in $D = 4$, with one field strength that carries both an electric charge $Q_e$ and a magnetic charge $Q_m$. The mass of the bound state is

$$m = \left(\frac{Q_e^2}{3} + \frac{Q_m^2}{3}\right)^{3/2},$$

while the widely-separated electric and magnetic black hole constituents have total mass $m_\infty = Q_e + Q_m$. Thus the binding energy in this case is negative. If the charges $Q_e$ and $Q_m$ are equal, the solution reduces to an extremal $a = 0$ Reissner-Nordstrøm black hole. This is quite distinct from the usual four-charge $a = 0$ Reissner-Nordstrøm black hole which, being supersymmetric, has zero binding energy. By contrast, this new two-charge Reissner-Nordstrøm black hole is non-supersymmetric, and is like a dyon fission bomb with a yield of about 29\% of the mass of the dyon. In fact we can also construct an eight-charge $a = 0$ Reissner-Nordstrøm black hole with positive binding energy.

We have not found exact solutions for any other non-supersymmetric $p$-branes with independent charges. Instead, in section 4, we present a general perturbative analysis for charges which are close to the fixed ratios of the exact single-scalar solutions. In certain cases, we can use these results to conjecture the analogue of the mass formula (1).

## 2 Binding energy of single-scalar $p$-branes

The Kaluza-Klein reduction of eleven-dimensional supergravity to $D$ dimensions gives rise to various field strengths coming both from the 4-form and from the vielbein in $D = 11$. Denoting the compactified $(11 - D)$ internal indices by $i, j, \cdots$, we have $F_4, F_3^{(i)}$, $F_2^{(ij)}$ and $F_1^{(ijk)}$ from the former, and $F_2^{(i)}$ and $F_1^{(ij)}$ from the latter, where the subscript index denotes
the degree of the form. The detailed expression for the bosonic Lagrangian can be found in \[12\]. It admits a consistent truncation to the Lagrangian
\[
e^{-1}L = R - \frac{1}{2}(\partial\vec{\phi})^2 - \frac{1}{2n!} \sum_{\alpha=1}^{N} e^{-\vec{a}_{\alpha} \cdot \vec{\phi}} F_{\alpha}^2 + L_{FFA},
\]
where $\vec{\phi} = (\phi_1, \cdots, \phi_N)$, $F_{\alpha}$ are a set of $N$ antisymmetric tensor field strengths of rank $n$, and $\vec{a}_{\alpha}$ are the associated constant vectors characterising their couplings to the dilatonic scalars $\vec{\phi}$. The term $L_{FFA}$ represents the dimensional reduction of the $FFA$ term in $D = 11$. This, together with the Chern-Simons modifications to the field strengths, vanishes for the solutions we shall construct, and we shall not consider them further. (This puts constraints on the possible charges that can be used to construct $p$-brane solutions; a full discussion of these constraints can be found in \[12\].) The $n$-rank field strengths can be used to construct elementary $p$-branes with world-volume dimension $d = n - 1$, or solitonic $p$-branes with $d = D - n - 1$. In both cases, the $p$-brane metric takes the form
\[
ds^2 = e^{2A} dx^\mu dx^\nu \eta_{\mu\nu} + e^{2B} dy^m dy^m,
\]
where $x^\mu$ are the coordinates on the $d$-dimensional world volume. The functions $A$ and $B$ depend only on the coordinates $y^m$ of the transverse space. In all the extremal solutions that we shall be considering in this paper, they satisfy the relation $dA + dB = 0$, where $\tilde{d} \equiv D - d - 2$.

A further truncation to the single-scalar Lagrangian
\[
e^{-1}L = R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2n!} e^{-a\phi} F^2
\]
is possible, where $a$, $\phi$ and $F$ are given by \[12\]
\[
a^2 = \left( \sum_{\alpha,\beta} (M^{-1})_{\alpha\beta} \right)^{-1}, \quad \phi = a \sum_{\alpha,\beta} (M^{-1})_{\alpha\beta} \vec{a}_{\alpha} \cdot \vec{\phi}, \\
F_{\alpha}^2 = a^2 \sum_{\beta} (M^{-1})_{\alpha\beta} F_{\beta}^2,
\]
and $M_{\alpha\beta} = \vec{a}_{\alpha} \cdot \vec{a}_{\beta}$. The parameter $a$ can be conveniently re-expressed as
\[
a^2 = \Delta - \frac{2\tilde{d}d}{D - 2},
\]
where $\Delta$ is preserved under dimensional reduction \[14\]. The equations of motion following from \[1\] admit extremal $p$-brane solutions given by
\[
ds^2 = \left(1 + \frac{k}{\eta^d}\right)^{-2\Delta(D-2)/d} dx^\mu dx^\nu \eta_{\mu\nu} + \left(1 + \frac{k}{\eta^d}\right)^{4\Delta(D-2)/d} dy^m dy^m, \\
e^{\phi} = \left(1 + \frac{k}{\eta^d}\right)^{2\Delta},
\]
where \( \epsilon = 1 \) for elementary solutions and \( \epsilon = -1 \) for solitonic solutions. The field strength \( F \) carries electric or magnetic charge \( Q, k = 2\sqrt{\Delta}Q/\tilde{d}, \) and the mass per unit \( p \)-volume is given by

\[
m = \frac{2Q}{\sqrt{\Delta}}.
\]

The mass (8) of a single-scalar \( p \)-brane solution can be compared with the total mass \( m_\infty \) of its widely-separated constituents. If follows from (8) that each field strength \( F_\alpha \) carries a charge

\[
Q_\alpha \equiv Q \, c_\alpha , \quad \text{with} \quad c_\alpha \equiv a \left( \sum _\beta (M^{-1})_{\alpha \beta} \right)^{1/2}.
\]

The constants \( c_\alpha \) describe the fixed fractions of the total normalised charge \( Q \) that are carried by the field strengths \( F_\alpha \). Note that \( \sum _\alpha c_\alpha ^2 = 1 \). The total mass \( m_\infty \) is

\[
m_\infty = \sum _\alpha Q_\alpha = Q \sum _\alpha c_\alpha ,
\]

since each constituent is a \( \Delta = 4 \) solution. The binding energy is then given by

\[
\delta E = Q \left( \sum _\alpha c_\alpha - \frac{2}{\sqrt{\Delta}} \right).
\]

For supersymmetric \( p \)-branes, the dilaton vectors \( \vec{a}_\alpha \) satisfy the dot products

\[
M_{\alpha \beta} = 4\delta_{\alpha \beta} - \frac{2dd}{D-2},
\]

which implies that the \( c_\alpha \) are all equal, and that \( \Delta = 4/N \). Thus from (11) we recover the previously-known result that the supersymmetric \( p \)-branes have zero binding energy.

For non-supersymmetric solutions, the binding energy can be of either sign, depending on the detailed structure of the dot products of the dilaton vectors. The discussion becomes particularly simple for \( p \)-brane solutions using 3-form field strengths (i.e. elementary strings or solitonic \( (D-5) \)-branes). The dot products \( M_{\alpha \beta} \) in this case are given by (12)

\[
M_{\alpha \beta} = 2\delta_{\alpha \beta} - \frac{2(D-6)}{D-2},
\]

implying that the \( c_\alpha \) are again all equal, \( c_\alpha = 1/\sqrt{N} \), and \( \Delta = 2 + 2/N \). Thus the binding energy (11) is

\[
\delta E = Q\sqrt{N} \left( 1 - \sqrt{\frac{2}{N+1}} \right),
\]

which is positive for all \( N \geq 2 \). Thus it is energetically favourable for a set of \( N \) widely-separated \( \Delta = 4 \) constituents carrying the appropriate set of charges to coalesce to form a \( p \)-brane soliton of this kind. The exact solution that would describe this collapse would of
course be non-static and extremely complicated. It would also be non-supersymmetric; however, in the limit where the separation between the constituents goes to infinity, supersymmetry would be asymptotically restored locally in the neighbourhood of each constituent. A similar asymptotic local enhancement of supersymmetry also occurs for the supersymmetric bound states. The difference in that case however is that the supersymmetry is never totally broken even when the constituents coalesce. Furthermore the configuration is static, and hence it is somewhat misleading to refer to the supersymmetric p-branes as bound states, since the constituents will remain in neutral equilibrium at any separation. On the other hand, the non-supersymmetric p-branes are consequences of the natural evolution of widely-separated supersymmetric constituents. Note that in addition to 3-form solutions in M-theory, the above solutions can also be used to describe a $\Delta = 3$ string with vanishing dilaton in type IIB supergravity, which involves both the NS-NS and R-R 3-form field strengths.

In the above supersymmetric and non-supersymmetric examples, we have $c_\alpha$’s that are all equal. It follows from (9) that each individual field strength carries an equal charge. Further non-supersymmetric equal-charge p-branes can also be constructed using 2-form field strengths. For example, the dot products of the dilaton vectors of the 2-form field strengths $F_{\alpha}$ coming from the vielbein are given by

$$M_{\alpha\beta} = 2\delta_{\alpha\beta} + \frac{2}{D - 2}.$$  \hspace{1cm} (15)

it is straightforward to verify that $c_\alpha$ are equal, and that $\Delta = 2 + 2/N$. Thus the binding energy for elementary black holes or solitonic $(D - 4)$-branes carrying equal charges of these kinds is again given by (14).

While all supersymmetric single-scalar solutions carry equal charges, in many non-supersymmetric cases the ratios of the individual charges can be different. This can occur when a solution involves more than two different type of charges. A simple example is provided by an $a = 0$ black hole in $D = 9$. The solution involves all the three 2-form field strengths, namely $F^{(1)}$ and $F^{(2)}$ coming from the $D = 11$ vielbein and $F^{(12)}$ coming from the 4-form in $D = 11$. It has $\Delta = 12/7$ and the fractions $c_\alpha$ of the total normalised

\footnote{It is worth remarking here that all the extremal single-scalar p-branes admit multi-center generalisations, implying a non-force condition, whether or not they are supersymmetric \cite{13}. However, the supersymmetric ones allow a much more general kind of separation of centers, in which charges of different species can be located independently in the transverse space. Obviously, such static solutions with separated charge species are not possible for the non-supersymmetric p-branes, precisely because their binding energy is non-zero.

\footnote{This is the first of a number of examples of non-supersymmetric $a = 0$ black holes in $D < 9$ that lie outside the classification given in \cite{2}, involving $(12 - D)$ 2-form field strengths.}}
charge $Q$ carried by each field strength turn out to be $\sqrt{2/7}$, $\sqrt{2/7}$ and $\sqrt{3/7}$ respectively. Thus it follows from (11) that the 9-dimensional $a = 0$ black hole has a positive binding energy. Another example is an $a = 0$ eight-charge Reissner-Nordstrøm black hole in $D = 4$, which is non-supersymmetric. Indeed it was well known that there exist supersymmetric Reissner-Nordstrøm black holes with zero binding energy, which can carry, for example, three electric charges and one magnetic charges by the 2-form field strengths $F^{(12)}$, $F^{(34)}$, $F^{(56)}$ and $F^{(7)}$ respectively. In the non-supersymmetric case we are discussing here, there are five electric charges carried by the 2-forms $F^{(3)}, \ldots, F^{(7)}$, and three magnetic charges carried by $F^{(1)}$, $F^{(2)}$ and $F^{(12)}$. The solution has $\Delta = 1$, and the fractions $c_\alpha$ are $1/\sqrt{12}$ for each electric charge, and $1/\sqrt{6}, 1/\sqrt{6}$ and $1/2$ for the three magnetic charges. Thus the binding energy is again positive. In the next section, we shall describe a two-charge dyonic black hole with negative binding energy.

Let us, at this point, use $D = 9$ as an example to illustrate the different types of bound-state black holes mentioned above. There are three basic black-hole building blocks in $D = 9$, namely those where $F^{(1)}$, $F^{(2)}$ or $F^{(12)}$ carry the electric charge. We shall call them type A, B and C respectively. We can build three two-charge bound states: AC and BC are both supersymmetric with $\Delta = 2$; in fact they form a doublet under the U Weyl group $S_2$. AB is non-supersymmetric with $\Delta = 3$, and it has positive binding energy. Thus if two constituents A and C, or B and C, are initially separated, they will remain in neutral equilibrium. On the other hand, if the constituents A and B are initially separated, they will tend to attract one another. Similarly, as we mentioned above, there is also an ABC bound state with positive binding energy.

In all the examples that we have discussed explicitly above, the binding energy is non-negative. This seems to be true for the majority of the $p$-brane solutions. However, there are cases where the binding energy can be negative. For example there is a solitonic string in $D = 4$, with charges carried by the 1-form field strengths $F^{(12)}$, $F^{(34)}$ and $F^{(135)}$, in the fractions $c_\alpha = \sqrt{3/10}, \sqrt{3/10}$ and $2/\sqrt{10}$ respectively. The solution has $\Delta = 2/5$, and hence the binding energy $\delta E = Q(2\sqrt{3} - 8)/\sqrt{10}$ is negative.

### 3 A black hole dyon in $D = 4$

In $D = 2n$, $n$-form field strengths can carry both electric and magnetic charges simultaneously, giving rise to dyonic $(n-2)$-branes. As discussed in [12], there are two kinds of dyonic $p$-branes. The first kind involves more than one field strength, with each field strength car-
ryring either electric or magnetic charge but not both. An example is the supersymmetric four-charge Reissner-Nordstrøm black hole, which we discussed in the previous section. The second kind is more genuinely dyonic, in that each field strength carries both electric and magnetic charge. In this paper, we reserve the term “dyon” exclusively for dyonic p-branes of the second kind. An example is the dyonic string in \( D = 6 \) \[11\], which preserves 1/4 of the supersymmetry. Another example is a dyonic black hole using two field strengths, for example \( F^{(12)} \) and \( F^{(3)} \), with electric charges \( Q_1 \) and \( Q_2 \), and magnetic charges \( Q_2 \) and \( Q_1 \) respectively \[12\]. The second example is non-supersymmetric \[12\]. It reduces to a non-supersymmetric Reissner-Nordstrøm black hole in \( D = 4 \), with zero binding energy. This is understandable since it also involves four charges, as in the supersymmetric case.

In this section, we shall construct a new dyonic black hole in \( D = 4 \), with only one field strength, carrying both electric charge \( Q_e \) and magnetic charge \( Q_m \). It is in fact the extremal limit of the dyonic Toda black hole constructed in \[16\]. The equations of motion in the extremal limit are given by \[16\]

\[
\phi'' = 8\sqrt{3}(Q_e^2 e^{\sqrt{3}\phi} - Q_m^2 e^{-\sqrt{3}\phi})e^{2A},
A'' = 4(Q_e^2 e^{\sqrt{3}\phi} + Q_m^2 e^{-\sqrt{3}\phi})e^{2A},
4A'^2 + \phi'^2 = 16(Q_e^2 e^{\sqrt{3}\phi} + Q_m^2 e^{-\sqrt{3}\phi})e^{2A},
\]

(16)

where a prime denotes a derivative with respect to \( \rho \equiv 1/r \). Defining new functions \( q_1 \) and \( q_2 \) by

\[
A = \frac{1}{4}(q_1 + q_2 - 2\log(16Q_eQ_m)) , \quad \phi = \frac{\sqrt{3}}{2}(q_2 - q_1) + \frac{1}{\sqrt{3}} \log \frac{Q_m}{Q_e} ,
\]

(17)

the equations of motion \[16\] become

\[
q_1'' = e^{2q_1-q_2} , \quad q_2'' = e^{2q_2-q_1} ,
H \equiv \frac{1}{3}(p_1^2 + p_2^2 + p_1p_2) - e^{2q_1-q_2} - e^{2q_2-q_1} = 0 ,
\]

(18)

(19)

where \( H(p_1, p_2, q_1, q_2) \) is the Hamiltonian. Thus Hamilton’s equations \( p_i' = \partial H/\partial q_i \) imply that \( p_1 = 2q_1' - q_2' \), and \( p_2 = 2q_2' - q_1' \), while \( p_i' = -\partial H/\partial q_i \) gives precisely the equations of motion \[18\]. Thus the original equations of motion for the dyonic black hole can be cast into the \( SU(3) \) Toda equations \[18\]. The vanishing of the Hamiltonian is a consequence of requiring that the solution be extremal. In \[16\], the general non-extremal case with non-vanishing Hamiltonian was discussed. We can obtain the extremal solution either by taking the limit of the non-extremal solution in \[16\], or by directly solving the equations \[18\].

The required extremal solution can be obtained by making the ansatz \( e^{-q_2} = e^{-q_1} + \text{const} \). With this ansatz, it is easy to verify that \( (e^{-q_1})'' = 1 = (e^{-q_2})'' \). Thus the solution is given
by these two simple equations, subject to the first order constraint (19). Requiring that the function $A$ and the dilaton $\phi$ be zero in the asymptotic limit $\rho = 1/r = 0$, the solution is

$$e^{\phi/\sqrt{3} - 2A} = 1 + 4Q^2/3(Q_e^2 + Q_m^2)^{1/2} + 8Q^2/3Q_m^4/3 \frac{1}{r^2},$$
$$e^{-\phi/\sqrt{3} - 2A} = 1 + 4Q^2/3(Q_e^2 + Q_m^2)^{1/2} + 8Q^2/3Q_m^4/3 \frac{1}{r^2},$$

(20)

together with $B = -A$. There is an horizon at $r = 0$. All physical quantities, including the dilaton, field strength and curvatures, are finite on the horizon. The entropy is

$$S = 8\pi Q_e Q_m,$$

(21)

and the temperature $T$ is zero. The non-extremal generalisation was given in [16]. It is easy to verify that in the near-extremal regime, the entropy and temperature satisfy the relation

$$S = 8\pi Q_e Q_m + 128(Q_e Q_m)^4/3 \sqrt{Q_e^2/3 + Q_m^2/3} T.$$  

(22)

The mass of the dyonic black hole is

$$m = (Q_e^2/3 + Q_m^2/3)^{3/2}.$$  

(23)

On the other hand, the total mass of the purely electric and magnetic constituents, at large separation, is given by $m_\infty = Q_e + Q_m$. Thus the binding energy is negative whenever both charges are non-vanishing.

It is interesting to note that if the two charges $Q_e$ and $Q_m$ are equal, $Q_e = Q_m = Q/\sqrt{2}$, the dilaton $\phi$ decouples, and the solution becomes precisely the extremal $a = 0$ Reissner-Nordstrøm black hole, whose metric is given by

$$ds^2 = -\left(1 + \frac{2Q}{r}\right)^{-2} dt^2 + \left(1 + \frac{2Q}{r}\right)^2 dy^m dy^m,$$

(24)

and its mass is $m = 2Q$. Thus we see that the $a = 0$ Reissner-Nordstrøm black hole can be embedded in $D = 4$ maximal supergravity in four inequivalent ways. One is the usual supersymmetric embedding with four non-zero charges for appropriate field strengths, e.g. $F^{(12)}$, $F^{(34)}$, $F^{(56)}$ and $F^{(7)}$.\footnote{In fact this solution itself has a non-supersymmetric variant [13, 17], achieved by making an alternative sign choice for any one of the charges, which appear quadratically in the bosonic equations of motion. The fermionic equations of motion and the supersymmetry transformations involve the field strength linearly, and thus are sensitive to this sign choice.} In accordance with that fact that the solution is supersymmetric, this embedding has zero binding energy. The other three inequivalent embeddings are all non-supersymmetric, and involve two charges, four charges or eight charges. They
have negative, zero and positive binding energies respectively. The first case is the one
that we have just discussed above. The third case is the $a = 0$ eight-charge black hole
that we discussed in the previous section. The second case, the $a = 0$ non-supersymmetric
four-charge black hole with zero binding energy, can be obtained from the dyonic black hole
with two 2-forms that we discussed in the first paragraph of this section, by setting the
charges $Q_1$ and $Q_2$ equal.

4 Multi-scalar non-supersymmetric $p$-branes

In section 2 we discussed single-scalar $p$-brane solutions, in which the non-vanishing charges
occur in fixed ratios that are determined by the dot products of the associated dilaton
vectors $\vec{a}_\alpha$. If the solution is supersymmetric, it can easily be generalised to a multi-scalar
solution where all the charges become independent. On the other hand, if the solution
is non-supersymmetric, such a generalisation is not known, except for the new two-charge
dyonic black hole which we discussed in the previous section. In this section, we shall
present first-order perturbative solutions for charges which are close to the fixed ratios of
the exact single-scalar solutions. The Lagrangian is given by (2). The equations of motion
are

\begin{equation}
\varphi''_\alpha = \frac{8}{d^2} \epsilon \sum_\beta M_{\alpha\beta} Q_\beta^2 e^{\varphi_\beta + 2dA},
\end{equation}

\begin{equation}
d(D - 2) A'^2 + \frac{1}{2} \tilde{d} \sum_{\alpha,\beta} (M^{-1})_{\alpha\beta} \varphi'_\alpha \varphi'_\beta = \frac{8}{d} \sum_\alpha Q_\alpha^2 e^{\varphi_\alpha + 2dA},
\end{equation}

where a prime again denotes a derivative with respect to $\rho = 1/r, \varphi_\alpha = \vec{a}_\alpha \cdot \vec{\phi}$, and the
function $A$ is given by

\begin{equation}
A = \frac{\epsilon \tilde{d}}{D - 2} \sum_{\alpha,\beta} (M^{-1})_{\alpha\beta} \varphi_\alpha.
\end{equation}

Defining $\Phi_\alpha = \epsilon \sum_\beta (M^{-1})_{\alpha\beta} \varphi_\beta$, it follows from (27) that (25) becomes

\begin{equation}
\Phi''_\alpha = \frac{8Q_\alpha^2}{d^2} \exp \left( \sum_\beta (M_{\alpha\beta} + \frac{2d\tilde{d}}{D - 2}) \Phi_\beta \right).
\end{equation}

In the supersymmetric case, the dot products of dilaton vectors satisfy (12), implying that
these equations are diagonal Liouville equations in $\Phi_\alpha$ and can be easily solved. The mass
of the supersymmetric $p$-brane is given by

\begin{equation}
m = \sum_\alpha Q_\alpha,
\end{equation}
which is exactly the same as the total mass of the constituents when they are widely separated. In general, the equations (28) have the general form of Toda equations, but with precise coefficients that seem to render them non-integrable. However, the exact solution can be obtained if the charges occur in the fixed ratios given by (9) since then the equations reduce to those for the single-scalar solutions. In this case, \( \Phi_\alpha = \Phi_0^\alpha \equiv \epsilon c_\alpha^2 \phi/a \), where \( \phi \) is the remaining dilaton of the single-scalar solution, given by (7).

Let us consider a perturbation in which the charges \( Q_\alpha \) are displaced slightly from their single-scalar values:

\[
Q_\alpha = c_\alpha Q(1 + \varepsilon_\alpha) , \quad \Phi_\alpha = \Phi_0^\alpha + \sum_\beta \varepsilon_\beta f_{\alpha\beta} .
\]  

(30)

Substituting these equations into (28), we obtain the equations of motion for the first-order functions \( f_{\alpha\beta} \):

\[
f''_{\alpha\beta} = \frac{8Q^2}{d^2} e^{\epsilon \Delta \phi/a} \left( 2c_\alpha^2 \delta_{\alpha\beta} + a^2 \sum_{\gamma,\delta} (M^{-1})_{\alpha\delta}(M_{\alpha\gamma} + \frac{2d}{D-2})f_{\gamma\beta} \right) ,
\]

(31)

where the only summations are those indicated explicitly. Defining \( f_\alpha = \sum_\beta f_{\beta\alpha} \), we have

\[
f''_\alpha = \frac{8Q^2}{d^2} e^{\epsilon \Delta \phi/a} \left( 2c_\alpha^2 + \Delta f_\alpha \right) .
\]

(32)

It follows from (31) that we can solve for the functions \( f_\alpha \), obtaining

\[
f_\alpha = \frac{2c_\alpha^2}{\Delta} \left( (1 + kp)^{-1} - 1 \right) .
\]

(33)

Thus we obtain the function \( A \) in the metric, given by

\[
A = \frac{\tilde{d}}{D-2} \sum_\alpha \Phi_\alpha = A^0 + \frac{2\tilde{d}}{D-2} \sum_\alpha f_\alpha \varepsilon_\alpha ,
\]

(34)

where \( A^0 \) is the unperturbed metric function. Thus we have

\[
e^{2A} = \left( 1 + kp \right)^{-\frac{4\tilde{d}}{\Delta(D-2)}} \left( 1 + \frac{4\tilde{d}}{(D-2)\Delta}((1 + kp)^{-1} - 1) \sum_\alpha c_\alpha^2 \varepsilon_\alpha \right) ,
\]

(35)

to first order in \( \varepsilon_\alpha \), where \( \rho = r^{-\tilde{d}} \) and \( k = 2Q\sqrt{\Delta}/\tilde{d} \). The mass per unit \( p \)-volume of the perturbed \( p \)-brane is

\[
m = \frac{2Q}{\sqrt{\Delta}} \left( 1 + \sum_\alpha c_\alpha^2 \varepsilon_\alpha \right) .
\]

(36)

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\[\text{We have chosen the constants of integration so that the perturbations} f_\alpha, \text{like the original functions} \Phi_0^\alpha, \text{vanish at infinity (i.e. at} \rho = 0, \text{and also so that the perturbations remain finite on the horizon} \rho = \infty.\]
It is easy to verify that the small perturbation mass formula (36) is consistent with the general mass formula (29) for supersymmetric solutions, since in this case, $c_\alpha^2 = 1/N$ and $\Delta = 4/N$. It is also consistent with the mass formula (23) for the new dyonic black hole in the previous section. It would be of interest to know the exact mass formulae for the cases that we have been analysing perturbatively in this section. At least for the cases where the $c_\alpha$ are all equal, a natural conjecture for the exact mass formula would be

$$m = \left( \sum_\alpha Q_\alpha^x \right)^{1/x}, \quad (37)$$

where $x$ is a constant that can be determined by requiring consistency with the mass for the single-scalar solution where all the charges are equal. Thus we have $x = \log N / \log(2\sqrt{N/\Delta})$. This relation correctly produces $x = 1$ if we apply it to the supersymmetric cases, and $x = 2/3$ for the new black hole dyon of section 3. It is straightforward to show that (37) is also consistent with the perturbative result (36). If one would be able to generalise the non-supersymmetric equal-charge single-scalar solutions discussed in section 2 to multi-scalar solutions, it would follow from (37) that $x > 1$, and hence all these solutions would have positive binding energy for all values of the charges.

5 Conclusions

In this paper, we have argued that non-supersymmetric $p$-branes can be viewed as bound states of $\Delta = 4$ supersymmetric $p$-branes. In general the binding energy is non-zero, with a sign that depends on the specific choice of the constituents, and in particular on the dot products $M_{\alpha\beta}$ of their dilaton vectors. By contrast, the single-scalar supersymmetric $p$-branes with $\Delta = 4/N = 2, 4/3, \ldots$ are also bound states of appropriate $\Delta = 4$ constituents, but with zero binding energy. In fact it is not clear that the single-scalar supersymmetric solutions can be sensibly interpreted as distinct entities in their own right, since there is nothing (other than initial conditions imposed to infinite precision) to prevent the different charge species from drifting apart to give a multi-scalar multi-center solution. In other words, these supersymmetric single-scalar $p$-branes are nothing but multi-scalar multi-center solutions where, improbably, the centers happen to have become coincident.\footnote{It is curious that the three-charge black hole in $D = 5$ and four-charge black hole in $D = 4$ have non-vanishing entropy, which can be understood microscopically from string theory \cite{19, 20, 21}, and yet an infinitesimally displacement of their constituents, at no cost in energy, would cause the classical entropy to vanish.}
The existence of non-supersymmetric solutions with positive binding energy indicates that these configurations are energetically more favourable than those where the associated constituents remain separated. Thus although the widely-separated constituents with the charge quantum numbers of a supersymmetric $p$-brane will be stable, if the charges are instead those of a non-supersymmetric $p$-brane with positive binding energy, the constituents will be unstable to collapse. Of course, the non-supersymmetric $p$-brane may itself be unstable to quantum corrections. Nonetheless, the existence of the lower-energy classical configuration is an indication of the instability of the widely-separated constituents, each of which is asymptotically supersymmetric locally in its neighbourhood, even at the quantum level. On the other hand, non-supersymmetric $p$-branes with negative binding energy indicate that the associated constituents in such cases will tend to repel one another.

**Note Added**

After submitting this paper, we learned that the dyonic black hole in section 3 was also discussed in [21, 22], and its fission into electric and magnetic black holes was discussed in [22].

**References**

[1] H. Lü, C.N. Pope and K.S. Stelle, *Weyl group invariance and p-brane multiplets*, hep-th/9602140, to appear in Nucl. Phys. B.

[2] G. Papadopoulos and P.K. Townsend, *Intersecting M-branes*, hep-th/9603087.

[3] A.A. Tseytlin, *Harmonic superpositions of M-branes*, hep-th/9604033.

[4] I.R. Klebanov and A.A. Tseytlin, *Intersecting M-branes as four-dimensional black holes*, hep-th/9604166.

[5] J.P. Gaunlett, D.A. Kastor and J. Traschen, *Overlapping branes in M-theory*, hep-th/9604179.

[6] V. Balasubramanian and F. Larsen, *On D-branes and black holes in four dimensions*, hep-th/9604189.

[7] N. Khviengia, Z. Khviengia, H. Lü and C.N. Pope, *Intersecting M-branes and bound states*, hep-th/9605077.
[8] J. Rahmfeld, *Extremal black holes as bound states*, Phys. Lett. **B372** (1996) 198.

[9] J. Rahmfeld and M.J. Duff, *Bound states of black holes and other p-branes*, [hep-th/9605085](http://arxiv.org/abs/hep-th/9605085).

[10] R.E. Kallosh, A. Linde, T. Ortin, A. Peet and A. Van Proeyen, *Supersymmetry as a cosmic censor*, Phys. Rev. **D46** (1992) 5278.

[11] M.J. Duff, S. Ferrara, R.R. Khuri and J. Rahmfeld, *Supersymmetry and dual string solitons*, Phys. Lett. **B356** (1995) 479.

[12] H. Lü and C.N. Pope, *p-brane solitons in maximal supergravities*, [hep-th/9512012](http://arxiv.org/abs/hep-th/9512012), to appear in Nucl. Phys. **B**.

[13] H. Lü and C.N. Pope, *Multi-scalar p-brane solitons*, [hep-th/9512153](http://arxiv.org/abs/hep-th/9512153), to appear in Mod. Phys. Lett. **A**.

[14] H. Lü, C.N. Pope and K.S. Stelle, *Stainless super p-branes*, Nucl. Phys. **B456** (1996) 669.

[15] H. Lü, C.N. Pope and K.S. Stelle, *Vertical versus diagonal dimensional reduction for p-branes*, [hep-th/9605082](http://arxiv.org/abs/hep-th/9605082).

[16] H. Lü, C.N. Pope and K.W. Xu, *Liouville and Toda solitons in M-theory*, [hep-th/9604058](http://arxiv.org/abs/hep-th/9604058).

[17] R.R. Khuri and T. Ortin, *A non-supersymmetric dyonic extremal Reissner-Nordstrøm black hole*, [hep-th/9512178](http://arxiv.org/abs/hep-th/9512178).

[18] A. Strominger and C. Vafa, *Microscopic origin of Bekenstein-Hawking entropy*, [hep-th/9601029](http://arxiv.org/abs/hep-th/9601029).

[19] C.V. Johnson, R.R. Khuri and R.C. Myers, *Entropy of 4-D extremal black holes*, [hep-th/9603061](http://arxiv.org/abs/hep-th/9603061).

[20] A. Strominger and J. Maldacena, *Statistical entropy of four-dimensional extremal black holes*, [hep-th/9603060](http://arxiv.org/abs/hep-th/9603060).

[21] G.W. Gibbons and D.L. Wiltshire, *Black holes in Kaluza-Klein theory*, Ann. of Phys. **167** (1986) 201.
[22] G.W. Gibbons and R.E. Kallosh, Topology, entropy and Witten index of dilaton black holes, Phys. Rev. D51 (1995) 2839.