WILLMORE INEQUALITY ON HYPERSURFACES IN HYPERBOLIC SPACE

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Abstract. In this article, we prove a geometric inequality for star-shaped and mean-convex hypersurfaces in hyperbolic space by inverse mean curvature flow. This inequality can be considered as a generalization of Willmore inequality for closed surface in hyperbolic 3-space.

1. Introduction

The classical isoperimetric inequality and its generalization, the Alexandrov-Fenchel inequalities play an important role in different branches of geometry. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain with boundary $\Sigma$, then the classical isoperimetric inequality is

$$|\Sigma| \geq n \frac{\omega_n}{\omega_{n-1}} \frac{\omega_n}{\omega_{n-1}} |\Omega| \frac{\omega_n}{\omega_{n-1}},$$

and equality in (1) holds if and only if $\Omega$ is a geodesic ball.

For $k \in \{1, \cdots, n-1\}$, we denote by $p_k$ the normalized $k$-th order mean curvature of $\Sigma$, and set $p_0 = 1$ by convention. The celebrated Alexandrov-Fenchel inequalities \cite{1, 2, 15} for convex hypersurface $\Sigma^{n-1} \subset \mathbb{R}^n$ are

$$\frac{1}{\omega_{n-1}} \int_{\Sigma} p_k d\mu \geq \left( \int_{\Sigma} p_j d\mu \right)^{\frac{n-1-k}{n-1}}, \quad 0 \leq j < k \leq n-1,$$

and equality in (2) holds if and only if $\Omega$ is a geodesic ball.

Observe that the isoperimetric inequality holds for non-convex domains, it is natural to extend the original Alexandrov-Fenchel inequality to non-convex domains, see \cite{20, 31, 21, 7, 8, 9, 28}. We should also mention that the Willmore inequality, which is a weaker form of Alexandrov-Fenchel inequality, has been established for closed surfaces in $\mathbb{R}^3$. More precisely, for any closed surface $\Sigma \subset \mathbb{R}^3$, the Willmore inequality \cite{10, 26, 30} is

$$\int_{\Sigma} p_1^2 d\mu \geq \omega_2,$$

and equality in (3) holds if and only if $\Sigma$ is a geodesic sphere.

It is interesting to establish the Alexandrov-Fenchel inequalities for hypersurfaces in hyperbolic space, see \cite{4, 16}. Recently, the following hyperbolic Alexandrov-Fenchel inequalities were obtained.
Theorem A ([17, 18, 32]). Let \( k \in \{1, \cdots, n-1\} \). Any horospherical convex hypersurface \( \Sigma \subset \mathbb{H}^n \) satisfies

\[
\int_{\Sigma} p_k \, d\mu \geq \omega_{n-1} \left[ \left( \frac{\vert \Sigma \vert}{\omega_{n-1}} \right)^{\frac{k}{2}} + \left( \frac{\vert \Sigma \vert}{\omega_{n-1}} \right)^{\frac{2(n-1-k)}{n-1}} \right]^{\frac{k}{2}}.
\]

Equality in (4) holds if and only if \( \Sigma \) is a geodesic sphere.

Inequality (4) was proved in [17] for \( k = 4 \) and in [18] for general even \( k \). For \( k = 1 \), (4) was proved in [18] with a help of a result of Cheng and Zhou [12]. For general integer \( k \), (4) was proved in [32].

For \( k = 2 \), inequality (4) was proved by Li-Wei-Xiong [25] under a weaker condition that \( \Sigma \) is star-shaped and two-convex. More precisely,

Theorem B ([25]). Any star-shaped and 2-convex hypersurface \( \Sigma \subset \mathbb{H}^n \) satisfies

\[
\int_{\Sigma} p_2 \, d\mu \geq \omega_{n-1} \left\{ \frac{n}{n-1} \frac{\vert \Sigma \vert}{\omega_{n-1}} + \vert \Sigma \vert \right\}.
\]

Equality in (5) holds if and only if \( \Sigma \) is a geodesic sphere.

We now give the outline of the proof of Theorem 1. Motivated by [6, 13, 25], we adopt the inverse mean curvature flow (IMCF) in our proof. This flow has been used by Huisken and Ilmanen [22, 23] to prove the Riemannian Penrose inequality in general relativity. We start from a given star-shaped and mean-convex hypersurface \( \Sigma \), and evolve it by IMCF. By the convergence results of Gerhardt [19] (see also [6]), the IMCF exists for all time, and the evolving hypersurface \( \Sigma_t \) with \( \Sigma_0 = \Sigma \) remains star-shaped and mean-convex for all \( t \geq 0 \).
We next consider the quantity:

\[ Q(t) := |\Sigma_t|^{-\frac{n-3}{n-1}} \int_{\Sigma_t} (p_1^2 - 1) d\mu. \]

We study the limit of \( Q(t) \) as \( t \to \infty \). Notice that the roundness estimate for \( \Sigma_t \) is not strong enough to calculate the limit of \( Q(t) \). However, similar to \([6, 25]\), we are able to give a positive lower bound for the limit of \( Q(t) \), which will be used to establish the monotonicity of \( Q(t) \). Finally, we prove that \( Q(t) \) is monotone non-increasing under IMCF. From this, Theorem 1 follows immediately.

2. Preliminaries

In this article, we consider the hyperbolic space \( \mathbb{H}^n = \mathbb{R}^+ \times S^{n-1} \) equipped with the metric

\[ g = dr^2 + \sinh^2 r g_{S^{n-1}}, \]

where \( g_{S^{n-1}} \) is the standard round metric on the unit sphere \( S^{n-1} \). Let \( \Sigma \subset \mathbb{H}^n \) be a closed hypersurface with its unit outward normal vector \( \nu \). The second fundamental form \( h \) of \( \Sigma \) is defined by

\[ h(X, Y) = \langle \nabla_X \nu, Y \rangle \]

for any \( X, Y \in T\Sigma \). The principal curvature \( \kappa = (\kappa_1, \cdots, \kappa_n) \) are the eigenvalues of \( h \) with respect to the induced metric \( g \) on \( \Sigma \). For \( k \in \{1, \cdots, n-1\} \), the normalized \( k \)-th elementary symmetric polynomial of \( \kappa \) is defined as

\[ p_k(\kappa) := \frac{1}{n!}\sum_{i_1 < i_2 < \cdots < i_k} \kappa_{i_1} \cdots \kappa_{i_k}, \]

which can also be viewed as a function of the second fundamental form \( h^j_i = g^{jk} h_{ki} \).

For abbreviation, we write \( p_k \) for \( p_k(\kappa) \).

We now consider the inverse mean curvature flow (IMCF)

\[ \partial_t X = \frac{1}{(n-1)p_1} \nu. \]

where \( \Sigma_t = X(t, \cdot) \) is a family of hypersurfaces in \( \mathbb{H}^n \), \( \nu \) is the unit outward normal to \( \Sigma_t = X(t, \cdot) \). Let \( d\mu_t \) be its area element on \( \Sigma_t \). We list the following evolution equations.

**Lemma 2.** Under IMCF (8), we have:

\[ \partial_t p_1 = - \frac{1}{(n-1)^2} \Delta \left( \frac{1}{p_1} \right) - \frac{1}{(n-1)^2 p_1^2} (|A|^2 + n-1). \]

\[ \partial_t d\mu = d\mu. \]

In [19], Gerhardt investigated the inverse curvature flow of star-shaped hypersurfaces in hyperbolic space.

**Theorem 3 ([19]).** If the initial hypersurface is star-shaped and mean-convex, then the solution for IMCF (8) exists for all time \( t \) and preserves the condition of star-shapedness and mean-convexity. Moreover, the hypersurfaces become strictly convex exponentially fast and more and more totally umbilical in the sense of

\[ |h^j_i - \delta^j_i| \leq C e^{-\frac{t}{n-1}}, \quad t > 0, \]

where \( \delta^j_i \) are the Kronecker delta.
i.e., the principal curvatures are uniformly bounded and converge exponentially fast to one.

3. The asymptotic behavior of monotone quantity

We define the quantity

\[ Q(t) := |\Sigma_t|^{-\frac{n-3}{n-1}} \int_{\Sigma_t} (p_1^2 - 1) d\mu, \]

where |\Sigma_t| is the area of \( \Sigma_t \). In this section, we estimate the lower bound of the limit of \( Q(t) \). First of all, we recall the following sharp Sobolev inequality on \( S^{n-1} \) due to Beckner [3]. This Sobolev inequality is crucial in analyzing the asymptotic behavior of the monotone quantity, see [6, 25], etc.

Lemma 4. For every positive function \( f \) on \( S^{n-1} \), we have

\[ \int_{S^{n-1}} f^{n-3} d\text{vol}_{S^{n-1}} + \frac{n-3}{n-1} \int_{S^{n-1}} f^{n-5} |\nabla f|^2 d\text{vol}_{S^{n-1}} \]

\[ \geq \omega_{n-1}^{\frac{2}{n-1}} \left( \int_{S^{n-1}} f^{n-1} d\text{vol}_{S^{n-1}} \right)^{\frac{n-3}{n-1}}. \]

Moreover, equality in (11) holds if and only if \( f \) is a constant.

Proof. From Theorem 4 in [3], for any positive smooth function \( w \) on \( S^{n-1} \), we have the following inequality

\[ \frac{4}{(n-1)(n-3)} \int_{S^{n-1}} |\nabla w|^2 d\text{vol}_{S^{n-1}} + \int_{S^{n-1}} w^2 d\text{vol}_{S^{n-1}} \]

\[ \geq \omega_{n-1}^{\frac{2}{n-1}} \left( \int_{S^{n-1}} w^{\frac{2(n-1)}{n-3}} d\text{vol}_{S^{n-1}} \right)^{\frac{n-3}{n-1}}. \]

Moreover, the equality holds if and only if \( w \) is a constant. For any positive function \( f \) on \( S^{n-1} \), by letting \( w = f^{\frac{n-3}{n-1}} \), we obtain the desired estimate. \[ \square \]

Proposition 5. Under IMCF (8), we have

\[ \liminf_{t \to \infty} Q(t) \geq \omega_{n-1}^{\frac{1}{n-1}}. \]

Proof. Recall that star-shaped hypersurfaces can be written as graphs of function \( r = r(t, \theta) \), \( \theta \in S^{n-1} \). Denote \( \lambda(r) = \sinh(r) \), then \( \lambda'(r) = \cosh(r) \). We next define a function \( \varphi(\theta) = \Phi(r(\theta)) \), where \( \Phi(r) \) is a positive function satisfying \( \Phi' = \frac{1}{r} \).

Let \( \theta = \{ \theta^j \}, j = 1, \cdots, n-1 \) be a coordinate system on \( S^{n-1} \) and \( \varphi_i, \varphi_{ij} \) be the covariant derivatives of \( \varphi \) with respect to the metric \( g_{S^{n-1}} \). Define

\[ v = \sqrt{1 + |\nabla \varphi|^2_{S^{n-1}}}. \]

From [19], we know that

\[ \lambda = O(e^{-\frac{r}{t}}), \quad |\nabla \varphi|_{S^{n-1}} + |\nabla^2 \varphi|_{S^{n-1}} = O(e^{-\frac{r}{t}}). \]

Since \( \lambda' = \sqrt{1 + \lambda^2} \), we have

\[ \lambda' = \lambda \left( 1 + \frac{1}{2} \lambda^{-2} + O(e^{-\frac{r}{t}}) \right). \]
Moreover,

\[(17)\]

follows,

In terms of \(\varphi\), we can express the metric and the second fundamental form of \(\Sigma\) as follows,

\[g_{ij} = \lambda^2 (\sigma_{ij} + \varphi_i \varphi_j),\]

\[h_{ij} = \frac{\lambda'}{v \lambda} g_{ij} - \frac{\lambda}{v} \varphi_{ij},\]

where \(\sigma_{ij} = g_{\Sigma^{-1}} (\partial_i, \partial_j)\). Denote \(a_i = \sum_k \sigma_{ik} \varphi_k\) and note that \(\sum_i a_i = \Delta_{\Sigma^{-1}} \varphi\).

By \((14)\), the principal curvatures of \(\Sigma_t\) has the following form

\[\kappa_i = \frac{\lambda'}{v \lambda} - a_i + O(e^{-\frac{w}{n-1}}), \quad i = 1, \cdots, n - 1.\]

Then we have

\[p_1 = \frac{\lambda'}{v \lambda} - \frac{\Delta_{\Sigma^{-1}} \varphi}{(n-1)v \lambda} + O(e^{-\frac{w}{n-1}}).\]

By using \((15)\) and \((16)\), we get

\[p_1 = 1 + \frac{1}{2 \lambda^2} - \frac{\|\varphi\|^2_{\Sigma^{-1}}}{2} - \frac{\Delta_{\Sigma^{-1}} \varphi}{(n-1)v \lambda} + O(e^{-\frac{w}{n-1}}).\]

and

\[p_1^2 - 1 = \frac{1}{\lambda^2} - \|\varphi\|^2_{\Sigma^{-1}} - \frac{2}{n-1} \frac{\Delta_{\Sigma^{-1}} \varphi}{\lambda} + O(e^{-\frac{w}{n-1}}).\]

On the other hand,

\[\sqrt{\det g} = \left[\lambda^{n-1} + O(e^{\frac{(n-3)w}{n-1}})\right] \sqrt{\det g_{\Sigma^{-1}}}.\]

So we have

\[\int_{\Sigma_t} (p_1^2 - 1) d\mu = \int_{\Sigma_{n-1}} \lambda^{n-1} (p_1^2 - 1) d\text{vol}_{\Sigma_{n-1}} + O(e^{\frac{(n-5)w}{n-1}})\]

\[= \int_{\Sigma_{n-1}} \left(\lambda^{n-3} - \lambda^{n-1} \|\varphi\|^2_{\Sigma_{n-1}}\right) d\text{vol}_{\Sigma_{n-1}}\]

\[- \frac{2}{n-1} \int_{\Sigma} \lambda^{n-2} \Delta_{\Sigma_{n-1}} \varphi d\text{vol}_{\Sigma_{n-1}} + O(e^{\frac{(n-5)w}{n-1}})\]

\[= \int_{\Sigma_{n-1}} \left(\lambda^{n-3} - \lambda^{n-1} \|\varphi\|^2_{\Sigma_{n-1}}\right) d\text{vol}_{\Sigma_{n-1}}\]

\[+ \frac{2(n-2)}{n-1} \int_{\Sigma} \lambda^{n-3} (\langle \nabla \lambda, \nabla \varphi\rangle_{\Sigma_{n-1}} d\text{vol}_{\Sigma_{n-1}} + O(e^{\frac{(n-5)w}{n-1}})).\]

Since \(\nabla \lambda = \lambda \lambda' \nabla \varphi\), it follows that \(\|\nabla \lambda - \lambda^2 \nabla \varphi\|_{\Sigma_{n-1}} \leq O(e^{-\frac{w}{n-1}})\), we deduce that

\[(17)\]

\[\int_{\Sigma_t} (p_1^2 - 1) d\mu = \int_{\Sigma_{n-1}} \left(\lambda^{n-3} \frac{n-3}{n-1} \lambda^{n-5} \|\nabla \lambda\|^2\right) d\text{vol}_{\Sigma_{n-1}} + O(e^{\frac{(n-5)w}{n-1}}).\]

Moreover,

\[|\Sigma_t|^{\frac{n-3}{n-1}} = \left(\int_{\Sigma_{n-1}} \lambda^{n-1} d\text{vol}_{\Sigma_{n-1}}\right)^{\frac{n-3}{n-1}} + O(e^{\frac{w}{n-1}}).\]
Using Lemma 4, we achieve
\[
\liminf_{t \to \infty} |\Sigma_t|^{-\frac{n-3}{n-1}} \int_{\Sigma_t} (p_1^2 - 1) d\mu \geq \omega_{n-1}^2.
\]

\[\square\]

4. Monotonicity

In this section, we show that \(Q(t)\) is monotone non-increasing under IMCF (8).

**Proposition 6.** Under IMCF (8), the quantity \(Q(t)\) is monotone non-increasing. Moreover, \(\frac{d}{dt} Q(t) = 0\) at some time \(t\) if and only if \(\Sigma_t\) is totally umbilical.

**Proof.** Under IMCF (8), we have
\[
\frac{d}{dt} \int_{\Sigma_t} (p_1^2 - 1) d\mu
= \int_{\Sigma_t} 2p_1 \partial_t p_1 d\mu + \int_{\Sigma_t} (p_1^2 - 1) d\mu
= -\frac{2}{(n-1)^2} \int_{\Sigma_t} p_1 \left[ \Delta \left( |p_1| \right) + \frac{1}{p_1} (|A|^2 - (n-1)) \right] d\mu + \int_{\Sigma_t} (p_1^2 - 1) d\mu
= -\frac{2}{(n-1)^2} \int_{\Sigma_t} \left[ \frac{1}{p_1^2} \| \nabla p_1 \|^2 + (|A|^2 - (n-1)) \right] d\mu + \int_{\Sigma_t} (p_1^2 - 1) d\mu
\leq \frac{n-3}{n-1} \int_{\Sigma_t} (p_1^2 - 1) d\mu,
\]
where the last inequality follows from the trace inequality \(|A|^2 \geq (n-1)p_1^2\). Combining with Proposition 5, we know that the quantity
\[
\int_{\Sigma_t} (p_1^2 - 1) d\mu
\]
is positive under IMCF (8). Therefore, combining with (10) we get
\[
\frac{d}{dt} Q(t) \leq 0.
\]
If the equality holds, then the trace inequality implies that \(\Sigma_t\) is totally umbilical. \(\square\)

Now we finish the proof of Theorem 1.

**Proof of Theorem 1.** Since \(Q(t)\) is monotone non-increasing, we have
\[
Q(0) \geq \liminf_{t \to \infty} Q(t) \geq \omega_{n-1}^2.
\]
This implies that \(\Sigma_0 = \Sigma\) satisfies
\[
\int_{\Sigma} (p_1^2 - 1) d\mu \geq \omega_{n-1}^2 |\Sigma|^{\frac{n-3}{n-1}},
\]
which is equivalent to
\[
\int_{\Sigma} p_1^2 d\mu \geq \omega_{n-1}^2 |\Sigma|^{\frac{2-n}{n-1}} + |\Sigma|.
\]
Now if we assume that equality in (7) is attained, then \(Q(t)\) is a constant. Then Proposition 6 indicates that \(\Sigma_t\) is totally umbilical and therefore a geodesic sphere.
If $\Sigma$ is a geodesic sphere of radius $r$, then $|\Sigma| = \omega_{n-1} \sinh^{n-1} r$ and $p_1 = \coth r$. Hence, we have

$$\int \Sigma p_1^2 \, d\mu = \omega_{n-1} \sinh^{n-1} r \coth^2 r = \omega_{n-1} |\Sigma| \frac{1}{r^2} + |\Sigma|.$$ 

Therefore, equality in (7) holds on a geodesic sphere. This finishes the proof of Theorem 1. \hfill $\Box$

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