H$^\alpha$-flow of mean convex, complete graphical hypersurfaces

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We consider the evolution of hypersurfaces in $\mathbb{R}^{n+1}$ with normal velocity given by a positive power of the mean curvature. The hypersurfaces under consideration are assumed to be strictly mean convex (positive mean curvature), complete, and given as the graph of a function. Long-time existence of the $H^\alpha$-flow is established by means of approximation by bounded problems.

1 Introduction

Let $\alpha > 0$. The flow by the $\alpha$th power of the mean curvature, or short $H^\alpha$-flow, is the evolution of a hypersurface $M_t$, such that at each point the normal velocity equals $H^\alpha$, the $\alpha$th power of the mean curvature $H$. The hypersurface is assumed to be strictly mean convex ($H > 0$) at all times. If $X(\cdot, t): M^n \to \mathbb{R}^{n+1}$ are (local) embeddings of the time-dependent hypersurface $M_t$, then $H^\alpha$-flow is described by the equation ($\nu$ is the normal vector)

$$\langle \dot{X}, \nu \rangle = H^\alpha.$$  \hspace{1cm} (1)

In the case of $\alpha = 1$, one obtains the mean curvature flow. The mean curvature flow has been studied extensively and continues to be an active area of research. Among all curvature functions one can impose for the normal speed the mean curvature certainly is outstanding in its importance and comparable simplicity. A very influential paper was \cite{Huisken} where Huisken has shown that mean curvature flow shrinks convex hypersurfaces to “round points.” Similar results have been proven for various other normal speeds of homogeneity one before Schulze investigated the $H^\alpha$-flow, which has homogeneity $\alpha$, in \cite{Schulze} (also cf. \cite{15, 3, 2}). The $H^\alpha$-flow is a step away from homogeneity one but still has a fairly simple structure.

In this paper we are concerned with graphical hypersurfaces. For the mean curvature flow of entire graphs long-time existence has been established in \cite{Franzen}. Franzen

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generalized this to $H^\alpha$-flow of entire graphs in [8] under some technical assumptions. Sáez and Schnürer generalized the mean curvature flow of entire graphs in a different direction: In [13] they considered complete graphs. In contrast to entire graphs these need not be defined over all of $\mathbb{R}^n$ and are instead defined over an open subset. To represent a complete hypersurface, the graph representation must tend to infinity at the boundary of this subset. It turns out that in this case, too, no singularities occur on a finite level, where they would be visible. Instead, singularities form at infinity in some sense.

In this paper we bring together these two trends of generalization and prove long-time existence for the $H^\alpha$-flow of complete graphs. The following is the main result.

**Theorem 1.** Let $\alpha > 0$. Let $\Omega_0 \subset \mathbb{R}^n$ be open. Let $u_0: \Omega_0 \to \mathbb{R}$ be smooth and such that graph $u_0$ is of positive mean curvature. Furthermore, we suppose that the sets $\{x \in \mathbb{R}^n : u_0(x) \leq a\}$ are compact for any $a \in \mathbb{R}$.

Then there exists an $H^\alpha$-flow $(u, \Omega)$ of complete graphs with initial value $(u_0, \Omega_0)$ (cf. Theorem 1 for a precise formulation).

At this point it is mandatory to say more about the existing literature and on how this result fits in. Concerning entire graphs, we have already mentioned [8], where the $H^\alpha$-flow is considered, too, and the results of which we could improve by dropping a certain condition called “$\nu$-condition” as well as expanding the result to complete graphs. Still concerning entire graphs, the author wants to mention [9], where flows with speed $S^1/k$ with $S_k$ an elementary symmetric polynomial of the principal curvatures are considered, and [1], where general curvature flows of homogeneity one are investigated. Inspired by the mean curvature flow of complete graphs there is a number of papers that are concerned with the flow of complete graphs by various normal speeds. In [17], Xiao provides the needed a priori estimates for general curvature flows of homogeneity one. In two papers [4, 5] the existence of the flow of complete graphs by powers of the Gaussian curvature and the $Q_k$-flow of convex, complete graphs is established. In [12], the flow by powers of general curvature functions is considered for complete graphs that are convex. While the $H^\alpha$-flow that is considered in the present paper is fully non-linear and not of homogeneity one, it is still subsumed in the latter paper [12]. However, in the present paper we do not demand that the hypersurfaces be convex; we only assume mean convexity which is the natural assumption in this case. This does not only make the a priori estimates more difficult but also significantly changes the approximation scheme. In the convex case one usually cuts off at some height and reflects at that height to get a convex closed hypersurface. These usually behave nicely under geometric flows. In particular the two symmetric parts stay graphical. This idea does not work anymore without convexity.

Having already touched upon the proof, let us briefly summarize the main ideas. To established long-time existence for the $H^\alpha$-flow of complete graphs, we approximate the non-compact problem by compact ones. To this end, we cut off the initial complete graph at increasing heights. These will then serve as initial data for initial boundary value problems with Dirichlet boundary condition. These approximation problems can then be solved using parabolic PDE-theory in Hölder-spaces. Instead of $H^\alpha$-flow
the author found it easier to consider $H^{\alpha(x)}$-flow for these problems. The reason is that estimates at the boundary are difficult for the fully nonlinear flow and changing the equation towards the boundary to the quasilinear mean curvature flow ($\alpha = 1$) facilitates the problem. For being able to extract a limit from the approximation problems, it is crucial to have local a priori estimates. These are gained through the maximum principle applied to a suitable test function. Here it suffices to have bounds that are local in height and one may utilize the height function as a localization function. With these it is possible to pass to a limit using a variation on the Arzelà-Ascoli theorem from [13].

The paper is organized as follows. After some preliminaries about our sign-conventions, different parametrizations for the $H^{\alpha}$-flow, and evolution equations, we introduce and solve the approximation problems in Section 3. Section 4 is devoted to the needed a priori estimates that are local in height. Finally, in Section 5 the main theorem is proven.

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2 Preliminaries

2.1 Sign convention

Let $X(\cdot, t): M^n \to \mathbb{R}^{n+1}$ be a family of immersions with parameter $t$ which we interpret as time. Our convention for the normal $\nu$ is that it points in the same direction as the mean curvature vector $\vec{H} = \Delta X$ ($\Delta$ denotes the Laplace-Beltrami operator with respect to the induced metric). This is a well-defined choice of normal because the hypersurfaces are assumed to be strictly mean convex. Accordingly, the second fundamental form is defined by $h_{ij} = \langle X_{ij}, \nu \rangle$ and the mean curvature is given by $H = g^{ij} h_{ij}$, where $(g_{ij})$ is the inverse of the metric $(g^{ij})$ induced by $X(\cdot, t)$. For example, if $M_t$ bounds a convex domain, then $(h_{ij})$ is non-negative and $\nu$ points into the interior.

2.2 Parametrizations

Equation (1) is invariant under time-dependent reparametrizations. So there is still a certain (large) degree of freedom in that equation. This can be fixed by fully prescribing $\dot{X}$ instead of only its normal component. Of course there are many different ways to do this, which we refer to as different parametrizations. We will use two different ones, the parametrization along the normal and the graphical parametrization. In a parametrization with $\dot{X}$ pointing along the normal direction equation (1) becomes

$$\dot{X} = H^{\alpha} \nu.$$  \hspace{1cm} (2)

If the evolving hypersurface is described by graphs of functions $u(\cdot, t)$ with the assumption $\langle \nu, e_{n+1} \rangle > 0$ (with $e_{n+1} = (0, \ldots, 0, 1)$) and if we use a graphical parametrization,
i.e., $X(x,t) = (x, u(x,t))$, equation (1) is equivalent to

$$\frac{\partial_t u}{\sqrt{1 + |\nabla u|^2}} = \left( \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) \right)^\alpha \equiv H^\alpha,$$

where we have used

$$w := \langle \nu, e_{n+1} \rangle \equiv \frac{1}{\sqrt{1 + |\nabla u|^2}}.$$

The parametrization along the normal (2) is geometrically appealing and yields comprehensible evolution equations for geometric quantities. We shall use it for deriving the needed estimates. The graphical parametrization (3) is preferable from a PDE point of view. It is a scalar parabolic differential equation and we can harness the power of the theory of those equations. Higher estimates and existence of solutions are our takings.

### 2.3 Evolution equations

For the estimates we rely on the maximum principle. It is applied in conjunction with the linear parabolic operator

$$\mathcal{L} := \frac{\partial_t}{\alpha(X) H^\alpha(X)} - \Delta_M.$$

This operator is defined on a hypersurface that evolves by $H^\alpha(X)$-flow with parametrization along the normal, i.e., the immersions $X(\cdot,t)$ satisfy (2). The position-dependence of the exponent is needed in the approximation problems and therefore incorporated.

For the a priori estimates of Section 4 we do not need the dependence on the position. We will apply the operator $\mathcal{L}$ to different geometric quantities. The outcome is summarized in the following lemma.

**Lemma 2.** The evolution equations for $X$, $\nu$, $H^\alpha(X)$, and $|A|^2$ are given by (we suppress the dependence of $\alpha$ on $X$ in the notation)

1. $\mathcal{L} X = \left( \frac{1}{\alpha} - 1 \right) H \nu,$
2. $\mathcal{L} \nu = |A|^2 \nu - \frac{1}{\alpha} H \log(H) X^\beta \alpha_{\beta} g^{ij} X_j,$
3. $\mathcal{L} H^\alpha = \left( |A|^2 + \frac{1}{\alpha} H \log(H) \partial_\alpha H \right) H^\alpha,$
4. $\mathcal{L} |A|^2 = -2 |\nabla|A|^2 + 2 (\alpha - 1) H^{-1} h^{ij} \nabla_i H \nabla_j H + 2 |A|^4 + 2 \frac{1 - \alpha}{\alpha} H h^i h_j h^k,$
5. $$+ 4 \left( \log(H) + \frac{1}{\alpha} \right) \alpha_{\beta} X^\beta h^{ij} \nabla_j H$$

$$+ 2 \frac{1}{\alpha} H \log(H) \left( (\alpha_{\beta} + \log(H) \alpha_{\beta} \alpha_{\gamma}) X^\beta X^\gamma h^{ij} + \partial_\alpha |A|^2 \right).$$
Proof. The first follows directly from $\dot{X} = H^\alpha \nu$ and $\Delta X = H \nu$.

For the normal $\nu$, we notice that the condition $\langle \nu, \nu \rangle = 1$ implies $\langle \dot{\nu}, \nu \rangle = 0$ and $\langle \nu, \nu_i \rangle = 0$. Applying a time derivative to $\langle \nu, X_i \rangle = 0$ we obtain

$$
\langle \dot{\nu}, X_i \rangle = - \langle \nu, \dot{X}_i \rangle = - \langle \nu, \left( H^\alpha(X) \nu \right)_i \rangle = - \left( H^\alpha(X) \right)_i = -\alpha H^{\alpha-1} H_i - \log(H) H^\alpha \alpha_\beta X_i^\beta.
$$

Therefore, we conclude

$$
\dot{\nu} = - \left( \alpha H^{\alpha-1} H_i + \log(H) H^\alpha \alpha_\beta X_i^\beta \right) g^{ij} X_j.
$$

By Weingarten’s equation and Codazzi’s equation there holds

$$
\Delta \nu = g^{ij} \nu_{ij} = -g^{ij} (h^k_i X_k)_j = - (\nabla^k H) X_k - h^k_j h_k^i \nu.
$$

Combing the last two yields the assertion.

Before we continue with the evolution equation for $H^\alpha(X)$, we consider

$$
\frac{d}{dt} g_{ij} = \frac{d}{dt} \langle X_i, X_j \rangle = \left\langle \left( H^\alpha(X) \nu \right)_i, X_j \right\rangle + \left\langle X_i, \left( H^\alpha(X) \nu \right)_j \right\rangle = 0 - H^\alpha h_{ij} + 0 - H^\alpha h_{ij} = -2H^\alpha h_{ij},
$$

$$
\frac{d}{dt} g^{ij} = -g^{ik} \left( \frac{d}{dt} g_{kl} \right) g^{lj} = 2H^\alpha h^{ij},
$$

$$
\frac{d}{dt} h_{ij} = -\frac{d}{dt} \langle \nu_j, X_i \rangle = -\langle \dot{\nu}_j, X_i \rangle - \langle \nu_j, \dot{X}_i \rangle = -\left\langle \left( \nabla^k \left( H^\alpha(X) \right) X_k \right)_j, X_i \right\rangle + \left\langle h^k_j X_k, \left( H^\alpha(X) \nu \right)_i \right\rangle = \nabla_j \nabla_i H^\alpha(X) + 0 + 0 - H^\alpha h^k_j h_{ik}. \tag{8}
$$

Now we are in position:

$$
\frac{d}{dt} H^\alpha(X) = \alpha H^{\alpha-1} \left( g^{ij} \left( \frac{d}{dt} h_{ij} \right) + \left( \frac{d}{dt} g^{ij} \right) h_{ij} \right) + H^\alpha \log(H) \alpha_\beta \frac{d}{dt} X_i^\beta
$$

$$
= \alpha H^{\alpha-1} \left( \Delta H^\alpha(X) - H^\alpha |A|^2 + 2H^\alpha |A|^2 \right) + H^\alpha \log(H) \alpha_\beta H^\alpha \nu^\beta
$$

$$
= \alpha H^{\alpha-1} \left( \Delta H^\alpha(X) + H^\alpha |A|^2 + \frac{1}{\alpha} H^{\alpha+1} \log(H) \partial_\alpha \right).
$$
Next we compute the evolution equation of \( h_{ij} \). To this end we compute the term

\[
\nabla_j \nabla_i H^\alpha(X) = \nabla_j \left( \alpha(X) H^{\alpha(X)-1} \nabla_i H + H^{\alpha(X)} \log H \alpha_\beta(X) X^\beta_i \right)
\]

\[
= \alpha H^{\alpha-1} \nabla_j \nabla_i H + \alpha_\beta X^\beta_j H^{\alpha-1} \nabla_i H
\]

\[
+ \alpha (\alpha - 1) H^{\alpha-2} \nabla_j H \nabla_i H + \alpha H^{\alpha-1} \log(H) \alpha_\beta X^\beta_j \nabla_i H
\]

\[
+ \alpha H^{\alpha-1} \nabla_j H \log(H) \alpha_\beta X^\beta_i + H^\alpha (\log H)^2 \alpha_\gamma X^\gamma_j \alpha_\beta X^\beta_i
\]

\[
+ H^\alpha \nabla_j H \alpha_\beta X^\beta_j + H^\alpha \log(H) \alpha_\beta_\gamma X^\beta_i X^\gamma_j
\]

\[
+ H^\alpha \log(H) \alpha_\beta \beta^\beta h_{ij}.
\]

From (8) we infer

\[
\mathcal{L} h_{ij} = -g^{kl} \nabla_k \nabla_l h_{ij} + \nabla_j \nabla_i H + (\alpha - 1) H^{-1} \nabla_i H \nabla_j H
\]

\[
+ \left( \log(H) + \frac{1}{\alpha} \right) \left( \alpha_\beta X^\beta_j \nabla_i H + \alpha_\beta X^\beta_i \nabla_j H \right)
\]

\[
+ \frac{1}{\alpha} H \log(H) \left( (\alpha_\beta_\gamma + \log(H) \alpha_\beta \alpha_\gamma) X^\beta_j X^\gamma_i + \partial_\alpha \alpha h_{ij} \right)
\]

(9)

To cancel the second derivatives of \( h_{ij} \) on the right hand side, we will need the following identity:

\[
\nabla_j \nabla_i h_{kl} = \nabla_j \nabla_i h_{ik} + R_{jli}^a h_{ak} + R_{jik}^a h_{ia}
\]

\[
= \nabla_i \nabla_j h_{ik} + (h_{ij} b_{ik}^a - h_{ik} b_{ij}^a) h_{ak} + (h_{jk} b_{ik}^a - h_{ik} b_{jk}^a) h_{ia}
\]

\[
= \nabla_i \nabla_k h_{ij} + h_{ij} h_{ik} h_{al} - h_{il} h_{ij} h_{ak} + h_{jk} h_{ik} h_{al} - h_{ik} h_{al} h_{aj}.
\]

We apply this in the following way

\[
\nabla_j \nabla_i H = \nabla_j \nabla_i (g^{kl} h_{kl}) = g^{kl} \nabla_j \nabla_k h_{ij} = g^{kl} \nabla_i \nabla_k h_{ij} + |A|^2 h_{ij} - H h_{ij}^a h_{aj}.
\]

Inserting this into (9) yields

\[
\mathcal{L} h_{ij} = |A|^2 h_{ij} - \left( 1 + \frac{1}{\alpha} \right) H h_{ij}^a h_{aj} + (\alpha - 1) H^{-1} \nabla_i H \nabla_j H
\]

\[
+ \left( \log(H) + \frac{1}{\alpha} \right) \left( \alpha_\beta X^\beta_j \nabla_i H + \alpha_\beta X^\beta_i \nabla_j H \right)
\]

\[
+ \frac{1}{\alpha} H \log(H) \left( (\alpha_\beta_\gamma + \log(H) \alpha_\beta \alpha_\gamma) X^\beta_j X^\gamma_i + \partial_\alpha \alpha h_{ij} \right).
\]
Finally, we turn to the evolution equation for $|A|^2 = g^{ik} g^{jl} h_{ij} h_{kl}$.

\[ \mathcal{L} |A|^2 = \frac{1}{\alpha} H (2 h^{ik} g^{jl} h_{ij} h_{kl} + 2 g^{ik} h^{jl} h_{ij} h_{kl}) + (\mathcal{L} h_{ij}) h^{ij} + h^{kl} \mathcal{L} h_{kl} - 2 |\nabla A|^2 \]

\[ = 2 h^{ij} (\mathcal{L} h_{ij}) + 4 \frac{1}{\alpha} H h^i_j h^k_l h^l_i - 2 |\nabla A|^2 \]

\[ = -2 |\nabla A|^2 + 2 (\alpha - 1) H^{-1} h^{ij} \nabla_i H \nabla_j H + 2 |A|^4 + 2 \frac{1}{\alpha} H h^i_j h^k_l h^l_i \]

\[ + 2 h^{ij} \left( \log(H) + \frac{1}{\alpha} \right) (\alpha \beta X^\beta_i \nabla_i H + \alpha \beta X^\beta_i \nabla_j H) \]

\[ + 2 h^{ij} \frac{1}{\alpha} H \log(H) \left( (\alpha \beta \gamma + \log(H) \alpha \beta \alpha \gamma) X^\beta_i X^\gamma_j + \partial_i \alpha h_{ij} \right). \]

3 Auxiliary problems

The aim of this section is to provide the solutions of the approximation problems. In a first attempt, one may wish to solve the initial boundary value problem for the $H^\alpha$-flow ((3))

\[
\begin{aligned}
\partial_t u &= \frac{1}{\alpha} H^\alpha = \sqrt{1 + |\nabla u|^2} \left( \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) \right)^\alpha, \\
& \quad \text{on } Q \times (0, T), \\
& \quad \text{for } x \in \partial Q, t \in [0, T], \\
& \quad (10)
\end{aligned}
\]

for $Q \subset \mathbb{R}^n$ open and bounded, $T > 0$, and for (smooth) initial data $u_0$ with $u_0|_{\partial Q} \equiv 0$ and such that the mean curvature of graph $u_0$ is positive. However, this initial boundary value problem is problematic for a number of reasons. Firstly, the right hand side of the equation must vanish at the boundary. So the mean curvature vanishes at the boundary. This destroys the uniform parabolicity of the equation (unless $\alpha = 1$). Secondly, compatibility conditions are only satisfied for initial data $u_0$ with specific behavior near the boundary. On the more technical side, we will need to prove $C^2$-estimates at the boundary for this fully nonlinear equation. These are hard to work out for geometric flows, especially if the homogeneity is different from one.

To bypass these problems we consider a different auxiliary problem. Eventually, all we need of our auxiliary problem is that it has a given $u_0$ as initial condition and that, below a certain given height, the equation describes $H^\alpha$-flow. This can be accomplished by a number of different initial boundary value problems and we do not need to insist on (10). The route from (10) to a new initial boundary value problem that is easier to solve is described now. To ensure the parabolicity of the equation at the boundary, we introduce time dependent boundary values $u(x, t) = ct$ for $(x, t) \in \partial Q \times [0, T]$. Moreover, we add a term to the right hand side of the equation that depends on $u_0$ and will ensure compatibility conditions of any order. Lastly, we make the equation quasilinear near the boundary by introducing a position dependence for the exponent $\alpha$ such that $\alpha \equiv 1$ near the boundary.
We consider the initial boundary value problem
\[
\begin{cases}
\dot{u} = G(\nabla^2 u, \nabla u, x) - \xi(x)(G_0(x) - c) & \text{on } Q \times (0, T), \\
\dot{u}(x, 0) = ct & \text{for } x \in \partial Q, t \in [0, T], \\
u(\cdot, 0) = u_0.
\end{cases}
\]

We suppose:
\begin{itemize}
\item $T > 0$, and $Q \subset \mathbb{R}^n$ is a smooth, bounded, and open domain.
\item The operator is given by
\[
G(\nabla^2 u, \nabla u, x) = \frac{1}{w(\nabla u)}H(\nabla^2 u, \nabla u)^{\alpha(x)}
\]
\[
= \sqrt{1 + |\nabla u|^2} \left( \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) \right)^{\alpha(x)}.
\]
So the equation $\dot{u} = G(\nabla^2 u, \nabla u, x)$ describes $H^{\alpha(x)}$-flow for graph $u(\cdot, t)$.
\item $\alpha(x)$ is a smooth function on $Q$ with values in $(0, \infty)$ and which is constant $\alpha \equiv 1$ in a neighborhood of the boundary $\partial Q$.
\item The initial datum $u_0$ is smooth on $\overline{Q}$ and satisfies $u_0 \leq 0$, $u_0|_{\partial Q} \equiv 0$, and its graph has positive mean curvature $H(\nabla^2 u_0, \nabla u_0) > 0$ on $\overline{Q}$.
\item The constant $c > 0$ is chosen in such a way that $G_0(x) := G(\nabla^2 u_0, \nabla u_0, x) \geq c$ holds.
\item The function $\xi(x)$ is a smooth cut-off function with values in $[0, 1]$. We demand $\xi \equiv 1$ in a neighborhood of the boundary $\partial Q$ and that $\alpha \equiv 1$ on a neighborhood of the support of $\xi$.
\end{itemize}

**Lemma 3.** The initial boundary value problem (11) has a unique smooth solution, even for $T = \infty$.

**Proof.** The problem satisfies compatibility conditions of any order because if one substitutes $u_0$ for $u$, then the right hand side of the equation becomes equal to the constant $c$ in a neighborhood of the boundary. This is perfectly compatible with the boundary condition $u(x, t) = ct$ for any order of differentiation. For the sake of clarity, we explicitly check the first compatibility conditions. For the zeroth compatibility condition we fix $x \in \partial Q$ and we put $t = 0$ in the second line of (11), which yields $u(x, 0) = 0$. The third line in (11) yields $u(x, 0) = u_0(x) = 0$ because $u_0 \equiv 0$ on $\partial Q$. So the two lines are compatible. For the first compatibility condition one has to check the equation (first line) where the time derivative is computed from the second line and the spatial derivatives are derived from the third line. The second line gives $\dot{u}(x, 0) = c$. The third line ($u(\cdot, 0) = u_0$) yields
\[
G(\nabla^2 u, \nabla u, x) - \xi(x)(G_0(x) - c) = (1 - \xi(x))G_0(x) + \xi(x)c = c
\]
at \( t = 0 \) and for \( x \) in the neighborhood of the boundary where \( \xi \equiv 1 \). In particular the first compatibility condition is satisfied. For the second compatibility condition one takes a time derivative in the equation. This gives
\[
\ddot{\bar{u}} = G_{r_{ij}}(\nabla^2 u, \nabla u, x) \dot{u}_{ij} + G_{p_i}(\nabla^2 u, \nabla u, x) \dot{u}_i .
\]
But we have already seen that \( \dot{u}(\cdot, 0) \equiv c \) where \( \xi \equiv 1 \) when computed from \( u_0 \) using the equation. Therefore, the spatial derivatives of \( \dot{u} \) on the right-hand side vanish. As \( \ddot{u} \) also vanishes when computed from \( u(x, t) = ct \), the second compatibility condition holds true too. The higher order conditions are similar.

To investigate the parabolicity properties of the equation, we differentiate the right-hand side with respect to \( \nabla^2 u \):
\[
\frac{\partial}{\partial r_{ij}} \left( G(r, p, x) - \xi(x)(G_0(x) - c) \right) = \frac{1}{w(p)} \frac{\partial}{\partial r_{ij}} (H(r, p))^{\alpha(x)}
\]
\[
= \frac{\alpha H(r, p)^{\alpha(x) - 1}}{w(p)} \frac{\partial}{\partial r_{ij}} H(r, p)
\]
\[
= \frac{\alpha H(r, p)^{\alpha(x) - 1}}{w(p)} \left( \delta_{ij} - \frac{p^i p^j}{1 + |p|^2} \right)
\]
This positive definite matrix has both-sided bounds on its eigenvalues if \( p = \nabla u \) is bounded and there is a both-sided bound \( C^{-1} \leq H \leq C \) for the mean curvature.

We are clearly going to need some a priori estimates. Let \( u \) be a solution of (11) with some \( T > 0 \) which is not necessarily that of the assertion of the lemma. Our goal now is to work through the following program.

1. \( c \leq \dot{u} \leq \sup G_0 \)
2. \( u_0(x) \leq u(x,t) - ct \leq 0 \)
3. \( \nabla u \) is uniformly bounded at the boundary \( \partial Q \)
4. upper bound on \( H^{\alpha(x)} \)
5. interior gradient bound
6. positive lower bound on \( H^{\alpha(x)} \)
7. bound for the \( C^{2,1} \)-norm in a neighborhood of \( \partial Q \)
8. estimation of \( |A|^2 \) corresponding to graph \( u \).

(11) By differentiating the equation (11), we obtain
\[
\frac{\partial}{\partial t} \ddot{u} = G_{r_{ij}} \dot{u}_{ij} + G_{p_i} \dot{u}_i ,
\]
a linear parabolic equation for \( \dot{u} \). On the lateral boundary \( \partial Q \times [0, T] \), the boundary condition yields \( \dot{u} \equiv c \). On the bottom part \( Q \times \{0\} \), we have \( \dot{u} = (1 - \xi) G_0 + \xi c \). So \( \dot{u} \) attains values in the range \( [c, \sup G_0] \) on the parabolic boundary. According to the parabolic maximum principle, \( \dot{u} \) attains its minimum and maximum on the parabolic boundary. Hence, we have \( c \leq \dot{u} \leq \sup G_0 \) on all of \( \overline{Q} \times [0, T] \).
Because of \( u_0|_{\partial Q} = 0 \), immediately induces the boundary gradient estimate
\[
|\nabla u(x,t)| = |\nabla (u(x,t) - ct)| \leq |\nabla u_0(x)|
\]
for \((x,t) \in \partial Q \times [0,T]\).

An upper bound for \( H^{\alpha(x)} \) follows from (11):
\[
H^{\alpha(x)} = w \cdot G(x) = w \left( \hat{u} + \xi (G_0 - c) \right) \leq 1 \cdot (\sup G_0 + 1 \cdot (G_0 - 0))
\]
\[
\leq 2 \sup G_0.
\]

Taking a spatial derivative in (11) yields
\[
\dot{u}_k = G_{r,i} u_{kij} + G_{p,j} u_{ki} + G_{x} - \xi_k (G_0 - c) - \xi \partial_x G_0.
\]
(14)
We shall show that \( G_{x} \) is bounded in a controlled way. It holds
\[
G_{x} = \frac{\partial}{\partial x} \frac{1}{w} H^{\alpha(x)} = \frac{1}{w} \log(H) H^{\alpha(x)} \partial_x \alpha.
\]
Because \( a^\alpha a \to 0 \) as \( a \to 0 \) for any \( \alpha > 0 \), an upper bound for \( |G_{x}| \) follows from the upper bound (11) for \( H^{\alpha(x)} \) (of course the (least) upper bound for \( \alpha \) enters too).

Let \( C > 0 \) be a constant depending only on the data such that \( |G_{x} - \xi (G_0 - c) - \xi \partial_x G_0| < C \). The maximum principle applied to equation (12) implies that \( u_k - C t \) attains its maximum on the parabolic boundary of \( Q \times [0,T] \). In the same way \( u_k + C t \) attains its minimum on the parabolic boundary. There, on the parabolic boundary, we have \( |u_k| \leq |\nabla u_0| \) by (3). So we obtain a \((T\text{-dependent})\) bound for \( |\nabla u| \) on all of \( Q \times [0,T] \).

From (5) we have a controlled positive lower bound for \( w > 0 \). On the other hand, we have the lower bound \( \dot{u} \geq c \) from (14). Combined, these give a lower bound for \( H^{\alpha(x)} \):
\[
H^{\alpha(x)} = w \left( \dot{u} + \xi (G_0 - c) \right) \geq w \dot{u} \geq c \inf w > 0.
\]

In a neighborhood of \( \partial Q \), there holds \( \alpha \equiv 1 \). There, the equation is quasilinear as the right hand side of (12) does not depend on \( r \) for \( \alpha \equiv 1 \). From general regularity theory of parabolic equations [11, Chapter 6], we obtain from \( C^{1,1} \)-estimates (11, (2), (3)) and estimates on the parabolicity constants, which follow from (11, 3, 4), local higher order estimates: in a first step \( C^{1+\beta,1+\beta} \)-estimates and then all higher order estimates from the theory of Schauder. This establishes local a priori estimates for all derivatives of \( u \) in some neighborhood of \( \partial Q \times [0,T] \).
Now we turn to the estimation of $|A|^2$. We do calculations on the time-dependent hypersurface which is given by the graph of $u(\cdot,t)$ and we choose a parametrization $X$ such that $\partial_t X$ points in normal direction (cf. (23)). Slightly abusing notation, we extend $\alpha$ to a function on $\mathbb{R}^{n+1}$ by setting $\alpha(x^1, \ldots, x^n, x^{n+1}) = \alpha(x^1, \ldots, x^n)$.

We consider the quantity $f := \log |A|^2 - p \log (H^\alpha(X) - b)$ for a constant $b > 0$ such that $0 \leq \frac{1}{2} H^\alpha \leq b^{-1}$ and a constant $p > 0$ to be chosen later. Finding a controlled upper bound for $|A|^2$ is equivalent to finding a controlled upper bound for $f$. The reason being that we have already established both-sided bounds for $H^\alpha(X)$, i.e., we have control on $b$, and consequently we have both-sided control on the second term in the expression for $f$. So let $(p_0, t_0)$ be a maximal point of $f$. Our goal is to prove that $f$ is controllably bounded at $(p_0, t_0)$.

If $(x^1, \ldots, x^n)(p_0, t_0)$ is in the support of $\xi$ then $f(p_0, t_0)$ is bounded by virtue of (7). Furthermore, $f(p_0, t_0)$ is controlled by $u_0$ if $t_0 = 0$. Therefore, we may assume that $(X^1, \ldots, X^n)(p_0, t_0), (Q \times (0, T))) \setminus (\text{supp } \xi \times [0, T))$ holds.

Because $p_0$ is an interior point, the maximality condition implies that the first derivative vanishes. It follows

$$\frac{\nabla |A|^2}{|A|^2} = p \frac{\nabla H^\alpha(X)}{H^\alpha(X) - b} \quad \text{at } (p_0, t_0).$$

We consider the differential operator

$$\mathcal{L} := \frac{\partial}{\partial t} - \Delta.$$  

Here, $\Delta$ denotes the Laplace-Beltrami-Operator of the hypersurface given by the graph of $u(\cdot,t)$.

Outside of the support of $\xi$, $u$ solves the graphical $H^\alpha(X)$-flow. In particular, we may apply Lemma 23. Because the point $(p_0, t_0)$ is a parabolically interior point and because of its maximality condition for $f = \log |A|^2 - p \log (H^\alpha - b)$, at this point holds

$$0 \leq \mathcal{L} (\log |A|^2 - p \log (H^\alpha - b)) = \mathcal{L} \frac{\nabla |A|^2}{|A|^2} - p \frac{\nabla (H^\alpha - b)}{H^\alpha - b} - p \frac{\nabla H^\alpha}{H^\alpha - b} = \mathcal{L} \frac{\nabla |A|^2}{|A|^2} - p \frac{\nabla (H^\alpha - b)}{H^\alpha - b} + (p^2 - p) \frac{\nabla H^\alpha}{H^\alpha - b}$$

due to (15). From (9), (7), and (17) we obtain

$$0 \leq -2 \frac{\nabla |A|^2}{|A|^2} + 2|\alpha - 1| \frac{\nabla H}{|A| H}$$

$$+ 4 \frac{1}{\alpha} (1 + \alpha |\log H|) \frac{1}{|A|} \text{log}(H \partial_\alpha |\nabla H| + (p^2 - p) \frac{\nabla H^\alpha}{H^\alpha - b}$$

$$+ \left(2 - p \frac{H^\alpha}{H^\alpha - b}\right) \left(|A|^2 + \frac{1}{\alpha} H \text{log}(H) \partial_\alpha \alpha\right) + 2 \frac{1 - \alpha}{\alpha} H |A|$$

$$+ \left(2 \frac{H}{\alpha |A|} \text{log}(H) \left(|D^2 \alpha| + |\log H| |\alpha|^2\right)$$.

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We have already established upper and lower bounds for \( H \). The exponent-function
\[ \alpha : \mathbb{R}^{n+1} \to \mathbb{R} \] is fixed and it is, together with its derivatives, \( D_0 \) and \( D^2 \alpha \), to be seen as
to controlled. Moreover, we may assume for an arbitrary, but fixed, \( \delta > 0 \) that
\( H \leq \delta |A| \)
Otherwise, \( |A| \) is controlled by \( H \) (and \( \delta \)) and the desired curvature estimate would
follow. Therefore, we conclude from (13)
for a universal constant \( C > 0 \)
\[
0 \leq -2 \frac{|\nabla A|^2}{|A|^2} + 2|\alpha - 1|\delta \frac{|\nabla H|^2}{H^2}
+ \frac{1}{\alpha} (1 + \alpha |\log H|) \frac{1}{|A|} |D\alpha| |\nabla H| + (p^2 - p) \frac{|\nabla H^\alpha|}{H^\alpha - b}
+ \left( 2 - p \frac{H^\alpha}{H^\alpha - b} + 2 \frac{|1 - \alpha|}{\alpha} \delta \right) |A|^2 + C .
\]
Thereon we apply the following results:
\[
|\nabla |A|^2|^2 = 2 h^{ij} \nabla h_{ij}|^2 \leq 4 |A|^2 |\nabla A|^2
\]
\[
\Rightarrow -2 \frac{|\nabla A|^2}{|A|^2} \leq \frac{1}{2} \frac{|\nabla A|^2}{|A|^2} = \frac{p^2}{2} \frac{|\nabla H^\alpha|}{H^\alpha - b} ,
\]
\[
\frac{\nabla H^\alpha}{H^\alpha} = \nabla \log H^\alpha = \alpha \nabla \log H + \log(H) \nabla \alpha = \alpha \frac{\nabla H}{H} + \log(H) \nabla \alpha
\]
\[
\Rightarrow \frac{|\nabla H|^2}{H^2} \leq C |\nabla \alpha|^2 + C \leq C |\nabla H|^2 \leq C + \delta^2 C |\nabla H^\alpha| \frac{|\nabla H^\alpha|}{H^\alpha - b} .
\]
With these three inequalities we obtain from (19)
\[
0 \leq -2 \frac{|\nabla A|^2}{|A|^2} + 2|\alpha - 1|\delta \frac{|\nabla H|^2}{H^2}
+ \left( 2 - p \frac{H^\alpha}{H^\alpha - b} + 2 \frac{|1 - \alpha|}{\alpha} \delta \right) |A|^2 + C .
\]
Now we set \( p = 2 - \frac{b^2}{2} \). Then, using \( b \leq \frac{1}{\delta} H^\alpha \leq b^{-1} \),
\[
2 - p \frac{H^\alpha}{H^\alpha - b} = \frac{(2 - p)H^\alpha - 2b}{H^\alpha - b} = \frac{\frac{b^2}{2} H^\alpha - 2b}{H^\alpha - b}
\leq \frac{b^2}{2} H^\alpha \leq \frac{b^2}{2} b^{-1} = - \frac{b^2}{2} .
\]
Because of \( p < 2, -p + \frac{1}{\delta} p^2 < 0 \) holds. If we choose \( \delta > 0 \) sufficiently small, but still
in a controlled fashion, we can conclude from (20)
\[
0 \leq 0 \cdot \frac{|\nabla H|^2}{H^\alpha - b} - \frac{b^2}{4} |A|^2 + C .
\]
This demonstrates that \( |A|^2 \leq C \) holds at the point \((p_0, t_0)\). Of course, this implies
that \( f(p_0, t_0) \) is bounded (with control). To show this was our goal.
Conclusion of the proof of Lemma 3. The estimates (3), (4), and (6) ensure that the equation is uniformly parabolic along any solution with a priori bounds on the parabolicity constants.

Following the short time existence proof in [6] yields a smooth solution to (11) for some time. The same argument shows that for the maximal time interval $[0,T)$ the estimates for $u(\cdot,t)$ degenerate as $t \to T$. To prove long time existence ($T = \infty$), it remains to exclude this behavior.

We have already proven $C^{2,1}$-estimates for $u$. Using these, we can infer a $C^\beta$-estimate for $u$ from (13) through Krylov-Safonov estimates: We can apply in our case the interior estimate of [16]. (Steps 1 and 2 of their proof are sufficient for our purpose. These consist of applying the linear Krylov-Safonov estimate to equation (13) and then using theory for, in our case, quasilinear elliptic equations for spatial $C^{2,\beta}$-estimates on $u(\cdot,t)$. In a second step, classical elliptic Schauder theory, applied to $(t_1 - t_0)^{-\beta/4} (u(\cdot,t_1) - u(\cdot,t_0))$, yields H"older estimates in time for $\nabla^2 u(\cdot,t)$. Cf. [16] for more details.) In a neighborhood of the boundary, estimates follow from [11] because the equation is quasilinear there, as we have discussed above in (7).

With the $C^{2+\beta,\frac{2}{\beta+2}}$ estimates for $u$, we can start a bootstrapping argument using Schauder estimates and conclude estimates for any derivatives of $u$. Thus, we have achieved long time existence.

Uniqueness of the solution follows from the maximum principle.

4 A priori estimates

We consider strictly mean convex, graphical hypersurfaces $M_t = \text{graph } u(\cdot,t)$ in this section. We assume that $u \geq 0$ so that $M_t$ lies in the upper half-space $\{x \in \mathbb{R}^{n+1}: x_{n+1}^+ \geq 0\}$. For a fixed chosen height $a \in \mathbb{R}$, we suppose that $\{x: u(x,t) \leq a\}$ is compact for all times $t$ and that $u$ solves (3) on $\{(x,t): u(x,t) < a + 1\}$, i.e., the part of $M_t$ below height $a + 1$ moves by the $H^\alpha$-flow. How $M_t$ behaves above that height is irrelevant because we will use a cut-off function which vanishes above height $a$.

Let $X(\cdot,t)$ be a parametrization of $M_t$ with a suitable (time dependent) parameter space such that (2) holds. We will use this parametrization throughout this section.

We prove the estimates via a maximum principle type argument. The general strategy pursued here is to multiply the quantity to be estimated with a localization function. From there, a test function is constructed. If one applies a parabolic operator to a function, then the result has a certain sign at parabolically interior, minimal or maximal points of the function. By using a suitable parabolic operator this can be exploited to bound the test function if it has been chosen adequately. In a last step one recovers from the bound on the test function a local estimate for the quantity in question. A suitable linear parabolic differential operator is given in our case by

$$L := \frac{1}{\alpha H^{\alpha-1}} \partial_t - \Delta M_t.$$ (21)

We will again use

$$w := \langle \nu, e_{n+1} \rangle.$$ (22)
Lemma 4. On \( \{ (p, t) : X^{n+1}(p, t) \leq a \} \) hold
\[
\mathcal{L} X^{n+1} = \frac{1 - \alpha}{\alpha} H w, \tag{23}
\]
\[
\mathcal{L} w = |A|^2 w, \tag{24}
\]
\[
\mathcal{L} H^\alpha = |A|^2 H^\alpha, \tag{25}
\]
\[
\mathcal{L} |A|^2 = -2 |\nabla A|^2 + 2 (\alpha - 1) H^{-1} h^{kt} \nabla_k H \nabla_l H + 2 |A|^4 + 2 \left( 1 - \frac{1}{\alpha} \right) H h^{ij} h^{kl} h^k. \tag{26}
\]

Proof. Follows directly from Lemma 4. \( \blacksquare \)

For the computations the following identity for functions \( \varphi > 0 \) is helpful
\[
\mathcal{L} \log \varphi = \frac{1}{\alpha H^{\alpha-1}} \frac{\partial \varphi}{\varphi} g^{ij} \nabla_i \varphi \nabla_j \varphi = \frac{1}{\alpha H^{\alpha-1}} \frac{\partial \varphi}{\varphi} \left( \frac{\nabla_i \nabla_j \varphi}{\varphi} - \frac{\nabla_i \varphi \nabla_j \varphi}{\varphi^2} \right)
= \mathcal{L} \varphi + \left| \nabla \varphi \right|^2. \tag{27}
\]

Before we start estimating, we still have to define the cut-off function. For a parameter \( b \geq 0 \) and \( a \) from above it is of the form
\[
\psi := (a - b t - X^{n+1})^+. \tag{28}
\]

By assumption, \( \{ p : \psi(p, t) > 0 \} \) is compact for any \( t \geq 0 \). Where \( \psi > 0 \),
\[
\mathcal{L} \log \psi = \mathcal{L} \frac{\psi}{\psi} + \left| \nabla \frac{\psi}{\psi} \right|^2 = -\frac{1 - \alpha}{\alpha} H w \psi^{-1} - \frac{b}{\alpha H^{\alpha-1}} \psi^{-1} + \left| \nabla \frac{\psi}{\psi} \right|^2 \tag{29}
\]
holds.

The usage of the height function \( X^{n+1} \) is the most frequently applied localization method in the study of curvature flows without singularities. The idea to add the time dependent term \( b t \) has been adopted from \([4, 5]\).

**Gradient bound.** Because of \( w \equiv (1 + |\nabla u|^2)^{-1/2} \), an upper bound for \( |\nabla u| \) is equivalent to a lower positive bound for \( w \). We have
\[
\psi^{-1} w \geq \begin{cases} 
\inf_{t=0} \psi^{-1} w & \alpha \leq 1, \\
\min \left\{ \inf_{t=0} \psi^{-1} w, \frac{b}{a (\alpha - 1)}, \frac{b}{\alpha (\alpha - 1)} \right\} & \alpha > 1.
\end{cases} \tag{30}
\]

Proof. We work on the set \( \{ \psi > 0 \} \). By assumption, \( \{ p : X^{n+1}(p, t) \leq a - b t \} \) is compact for any \( t \). Hence, \( f := \log \psi^{-1} w = \log w - \log \psi \) attains a minimum on any compact time interval \([0, T] \). It suffices to prove (30) for times in \([0, T] \) if \( T \) is
kept arbitrary. Let the minimum be attained at \((p_0, t_0)\). Note that \((p_0, t_0)\) is also a minimum point of \(\psi^{-1} w\). Because of this, it suffices to prove (30) at this point \((p_0, t_0)\).

If \(t_0 = 0\), the assertion (30) follows.

If \(t_0 > 0\) and taking into account that \(f \to \infty\) as \(p \to \partial\{p: X^{n+1}(p, t) \leq a - b t\}\), and hence that \(p_0\) is in the interior, \((p_0, t_0)\) is a parabolically interior minimum point. We infer \(\nabla f = 0\) as well as the differential inequality \(\mathcal{L} f \leq 0\) at \((p_0, t_0)\). The condition \(\nabla f = 0\) gives

\[
\frac{\nabla w}{w} = \frac{\nabla \psi}{\psi} \quad \text{at } (p_0, t_0).
\]  

(31)

The other condition yields by virtue of (24), (27), (29), and (31) at \((p_0, t_0)\)

\[
0 \geq \mathcal{L} f = \mathcal{L} \log w - \mathcal{L} \log \psi
= |A|^2 + \left| \frac{\nabla w}{w} \right|^2 + \frac{1 - \alpha}{\alpha} H w \psi^{-1} + \frac{b}{\alpha H^{\alpha - 1}} \psi^{-1} - \left| \frac{\nabla \psi}{\psi} \right|^2
= |A|^2 + \frac{1 - \alpha}{\alpha} H w \psi^{-1} + \frac{b}{\alpha H^{\alpha - 1}} \psi^{-1}.
\]

In the case \(\alpha \leq 1\), this is impossible, and hence we must have \(t_0 = 0\), and (30) follows for this case.

In the case \(\alpha > 1\), we use \(|A|^2 \geq \frac{1}{n} H^2\) and \(\psi \leq a\) to obtain at \((p_0, t_0)\)

\[
\psi^{-1} w \geq \frac{\alpha}{n (\alpha - 1)} H + \frac{b}{a (\alpha - 1)} H^{-\alpha} \geq \begin{cases} \frac{\alpha}{a (\alpha - 1)} & H \geq 1, \\ \frac{b}{a (\alpha - 1)} & H \leq 1 \end{cases}
\]

This establishes (30) at \((p_0, t_0)\), which we noted to be sufficient.

**Lower bound on** \(H^\alpha\).

\[
\psi^{-1} H^\alpha \geq \begin{cases} \inf_{t=0} \psi^{-1} H^\alpha & \alpha \leq 1, \\ \min \left\{ \inf_{t=0} \psi^{-1} H^\alpha, \frac{b}{a (\alpha - 1)} \right\} & \alpha > 1 \end{cases}
\]  

(32)

**Proof.** Again, it suffices to prove the estimate (32) on time intervals \([0, T]\) where the end point \(T\) is kept arbitrary. Let \((p_0, t_0)\) be the minimal point of \(\psi^{-1} H^\alpha\) over those times. If \(t_0 = 0\), (32) follows. So let us assume \(t_0 > 0\). Because \(\psi\) is a localization function (in particular it vanishes towards the boundary of the set \(\{\psi > 0\}\), which we consider), we can conclude from \(t_0 > 0\) that \((p_0, t_0)\) is a parabolically interior point. The point \((p_0, t_0)\) also is a minimal point for \(\log H^\alpha - \log \psi\). Hence, it holds at \((p_0, t_0)\)

\[
\frac{\nabla H^\alpha}{H^\alpha} = \frac{\nabla \psi}{\psi}
\]
\[
0 \geq \mathcal{L}(\log H^\alpha - \log \psi) = \frac{\mathcal{L} H^\alpha}{H^\alpha} - \frac{\mathcal{L} \psi}{\psi} + 0 \\
= |A|^2 + \frac{1 - \alpha}{\alpha} H w \psi^{-1} + \frac{b}{\alpha H^{\alpha-1}} \psi^{-1} \\
\geq \frac{1 - \alpha}{\alpha} H w \psi^{-1} + \frac{b}{\alpha H^{\alpha-1}} \psi^{-1}.
\]

For \( \alpha \leq 1 \) this is impossible, and (32) follows in this case. If \( \alpha > 1 \), we use \( w \leq 1 \) and rearrange to

\[
\psi^{-1} H^\alpha \geq \frac{b}{\alpha - 1} \psi^{-1} \geq \frac{b}{\alpha (\alpha - 1)},
\]

and (32) holds in this case, too.

**Upper bound on \( H \).** From now on, \( b = 0 \) is possible and we will henceforth choose it that way.

We assume a gradient bound: \( w \geq 2w \) on the set \( \{(p, t): \psi(p, t) > 0\} \) where \( w > 0 \) is a constant. In what follows, constants may depend on \( w \) and consequently depend on the gradient bound.

The following localized bound for \( H \) holds:

\[
\psi H \leq C(n, \alpha, a, w) \max \left\{ \sup_{t=0} \psi H, 1 \right\}.
\]

**Proof.** Without loss of generality, we may only consider the finite time interval \([0, T]\) for an arbitrary, but fixed \( T > 0 \).

We define the test function

\[
f := \log H^\alpha + \alpha \log \psi - \log(\psi H^\alpha).
\]

Let \((p_0, t_0)\) be a maximal point of \( f \) over this time interval. If \( t_0 = 0 \), then \( f \leq \sup_{t=0} f \).

By exponentiation, this translates to

\[
\psi^\alpha H^\alpha \leq (w - w) \sup_{t=0} \frac{\psi^\alpha H^\alpha}{w - w} \leq \frac{1}{w} \sup_{t=0} \psi^\alpha H^\alpha,
\]

where we have used \( w \leq w - w \leq 1 \). This shows (34) in the case \( t_0 = 0 \).

Suppose \( t_0 > 0 \). Then \((p_0, t_0)\) is a parabolically interior point because \( \psi \) is a localization function. At this maximal point \((p_0, t_0)\), there hold \( \nabla f = 0 \) and \( \mathcal{L} f \geq 0 \), which respectively amount to

\[
\frac{\nabla H^\alpha}{H^\alpha} = -\alpha \frac{\nabla \psi}{\psi} + \frac{\nabla w}{w - w}.
\]

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and

\[
0 \leq \frac{\mathcal{L} H^\alpha}{H^\alpha} + \left| \frac{\nabla H^\alpha}{H^\alpha} \right|^2 + \alpha \, \mathcal{L} \log \psi - \frac{\mathcal{L} w}{w-w} - \left| \frac{\nabla w}{w-w} \right|^2 \\
= |A|^2 + \left( \frac{\nabla H^\alpha}{H^\alpha} \right)^2 + (\alpha - 1) \, H w \psi^{-1} + \alpha \left| \frac{\nabla \psi}{\psi} \right|^2 - \frac{w}{w-w} |A|^2 - \left| \frac{\nabla w}{w-w} \right|^2 
\]

(36)

\[
= - \frac{w}{w-w} |A|^2 + (\alpha - 1) \, H w \psi^{-1} + \alpha \left| \frac{\nabla H^\alpha}{H^\alpha} \right|^2 + \alpha \left| \frac{\nabla \psi}{\psi} \right|^2 - \left| \frac{\nabla w}{w-w} \right|^2 .
\]

From (35) we can deduce for \( \varepsilon > 0 \)

\[
\left| \frac{\nabla H^\alpha}{H^\alpha} \right|^2 \leq (1 + \varepsilon^{-1}) \alpha^2 \left| \frac{\nabla \psi}{\psi} \right|^2 + (1 + \varepsilon) \left| \frac{\nabla w}{w-w} \right|^2 .
\]

(37)

Furthermore, the following hold

\[
|\nabla \psi| = |\nabla X^{n+1}|^2 \leq 1 , \tag{38}
\]

\[
\nabla_i w = -\nabla_i X^{n+1} = -h_i^t \nabla_j X^{n+1} , \tag{39}
\]

\[
|\nabla w|^2 \leq |A|^2 |\nabla X^{n+1}|^2 \leq |A|^2 . \tag{40}
\]

By (35), (37), (40), (38), and \( w-w \geq w \).

\[
0 \leq - \frac{w}{w-w} |A|^2 + (\alpha - 1) \, H w \psi^{-1} + (\alpha + (1 + \varepsilon^{-1}) \alpha^2) \left| \frac{\nabla \psi}{\psi} \right|^2 + \varepsilon \left| \frac{\nabla w}{w-w} \right|^2 \\
\leq - \frac{w - \varepsilon/w}{w-w} |A|^2 + (\alpha - 1) \, H w \psi^{-1} + (\alpha + (1 + \varepsilon^{-1}) \alpha^2) \psi^{-2} .
\]

If we set \( \varepsilon := \frac{1}{2} w^2 \) and use \( |A|^2 \geq \frac{1}{n} H^2 , \ w-w \leq 1 , \) and \( w \leq 1 \), we obtain

\[
0 \leq - \frac{1}{2} \frac{H^2}{n} + C H \psi^{-1} + C \psi^{-2} ,
\]

where the constant \( C > 0 \) depends on \( \alpha \) and on \( w \). Rearranging and adjusting the constant, which may now also depend on \( n \), yields

\[
(\psi H)^2 \leq C (\psi H + 1) .
\]

It is now easy to see that \( \psi H \) is bounded in a controlled fashion at \( (p_0 , t_0) \), the point at which our calculations take place, and which is the maximal point of our test function \( f \). Put together with the previous case of \( t_0 = 0 \), we conclude (by considering \( \exp f \))

\[
\psi^\alpha H^\alpha (w-w)^{-1} \leq \max \left\{ \sup_{t=0} \psi^\alpha H^\alpha (w-w)^{-1} , C \right\} .
\]

Because of \( w \leq w-w \leq 1 \), the assertion (34) easily follows from here. \( \Box \)
Curvature bound. We assume the following both-sided bound on $H^\alpha$:

$$0 < c \leq \frac{1}{2} H^\alpha \leq c^{-1}. \quad (41)$$

Then, depending on this control for $H^\alpha$ and the control on the gradient, the following localized curvature bound holds:

$$\psi |A| \leq C(c, w, a, \alpha) \max \left\{ \sup_{t=0} \psi |A|, 1 \right\}. \quad (42)$$

**Proof.** Our strategy is the same as with the other estimates before. This time, we consider the test function

$$f := \log |A|^2 + 2 \log \psi - \beta \log (H^\alpha - c),$$

where we are going to choose $\beta > 0$ later.

First of all, we record an inequality we are going to use:

$$|\nabla |A|^2|^2 = |2 h_{ij} \nabla h_{ij}|^2 \leq 4 |A|^2 |\nabla A|^2$$

$$= -2 |\nabla A|^2 |A|^2 \leq -\frac{1}{2} |\nabla |A|^2|^2. \quad (43)$$

By (25), (27), and (43),

$$L \log |A|^2 = \frac{\mathcal{L} |A|^2}{|A|^2} + \frac{|\nabla |A|^2|^2}{|A|^2}$$

$$= -2 \frac{|\nabla A|^2}{|A|^2} + 2(\alpha - 1) \frac{1}{\alpha} H^{-1} h^{ij} \nabla \nabla H \nabla H |A|^{-2}$$

$$+ 2 |A|^2 + 2 \frac{1 - \alpha}{\alpha} H h^i_j h^k_i |A|^{-2} + \frac{|\nabla |A|^2|^2}{|A|^2}$$

$$\leq \frac{1}{2} |\nabla |A|^2|^2 + 2 |\alpha - 1| (H |A|)^{-1} |\nabla H||^2 + 2 |A|^2 + 2 \frac{|\alpha - 1|}{\alpha} H |A|. \quad (44)$$

In a parabolically interior, first maximal point $(p_0, t_0)$ of $f$, $L f \geq 0$ holds:

$$0 \leq \mathcal{L} f = \mathcal{L} \log |A|^2 + 2 \mathcal{L} \log \psi - \beta \mathcal{L} \log (H^\alpha - c)$$

$$\leq \frac{1}{2} \frac{|\nabla |A|^2|^2}{|A|^2} + 2 |\alpha - 1| (H |A|)^{-1} |\nabla H||^2 + 2 |A|^2 + 2 \frac{|\alpha - 1|}{\alpha} H |A|$$

$$+ 2 \frac{|\alpha - 1|}{\alpha} H w \psi^{-1} + 2 |\nabla \psi|^2 - \beta \frac{H^\alpha}{H^\alpha - c} |A|^2 - \beta \frac{|\nabla H^\alpha|^2}{H^\alpha - c}. \quad (44)$$

The vanishing of $\nabla f$ at $(p_0, t_0)$ yields

$$\frac{|\nabla |A|^2|^2}{|A|^2} = \beta \frac{\nabla H^\alpha}{H^\alpha - c} - \frac{2 |\nabla \psi|^2}{\psi}$$

$$\leq (1 + \varepsilon) \beta^2 \frac{\nabla H^\alpha}{H^\alpha - c} + (1 + \varepsilon^{-1}) \frac{|\nabla \psi|^2}{\psi}. \quad (45)$$
for an arbitrary \( \varepsilon > 0 \) to be chosen. We substitute (45) into (44). Let us for the moment assume \( H \leq \delta |A| \) at \((p_0, t_0)\) for a (small) constant \( \delta > 0 \) to be chosen. We obtain

\[
0 \leq \left( (1 + \varepsilon) \frac{\beta^2}{2} - \beta \right) \left| \frac{\nabla H^\alpha}{H^\alpha - c} \right|^2 + 2 |\alpha - 1| \delta \left| \frac{\nabla H}{H} \right|^2 \\
+ \left( 2 - \beta \frac{H^\alpha}{H^\alpha - c} + 2 \frac{|\alpha - 1|}{\alpha} \delta \right) |A|^2 \\
+ 2 \frac{|\alpha - 1|}{\alpha} H w \psi^{-1} + (2 + (1 + \varepsilon^{-1})) \left| \frac{\nabla \psi}{\psi} \right|^2. 
\]

(46)

We use

\[
\left| \frac{\nabla H}{H} \right|^2 = |\nabla \log H|^2 = |\alpha^{-1} \nabla \log H^\alpha|^2 = \frac{1}{\alpha^2} \left| \frac{\nabla H^\alpha}{H^\alpha - c} \right|^2 \leq \frac{1}{\alpha^2} \left| \frac{\nabla H^\alpha}{H^\alpha - c} \right|^2
\]

and (38) to obtain from (46) with certain controlled constants

\[
0 \leq \left( (1 + \varepsilon) \frac{\beta^2}{2} - \beta + 2 \frac{\alpha - 1}{\alpha^2} \delta \right) \left| \frac{\nabla H^\alpha}{H^\alpha - c} \right|^2 \\
+ \left( 2 - \beta \frac{H^\alpha}{H^\alpha - c} + 2 \frac{|\alpha - 1|}{\alpha} \delta \right) |A|^2 \\
+ C \psi^{-1} + C(\varepsilon) \psi^{-2}. 
\]

(47)

We choose \( \beta = 2 - \frac{c^2}{2} \). For sufficiently small (but controlled) \( \varepsilon, \delta > 0 \), then hold

\[
(1 + \varepsilon) \frac{\beta^2}{2} - \beta + 2 \frac{\alpha - 1}{\alpha^2} \delta = \frac{1}{2} \left( 4 - 2c^2 + \frac{c^4}{4} \right) - \left( 2 - \frac{c^2}{2} \right) + \varepsilon \frac{\beta^2}{2} + 2 \frac{|\alpha - 1|}{\alpha^2} \delta \\
= -\frac{c^2}{2} + \frac{c^4}{4} + \varepsilon \frac{\beta^2}{2} + 2 \frac{|\alpha - 1|}{\alpha^2} \delta \\
\leq 0 \quad \text{(note that } c \leq 1\text{)}
\]

(48)

and

\[
2 - \beta \frac{H^\alpha}{H^\alpha - c} + 2 \frac{\alpha - 1}{\alpha} \delta = \frac{2 (H^\alpha - c) - (2 - \frac{c^2}{2}) H^\alpha}{H^\alpha - c} + 2 \frac{|\alpha - 1|}{\alpha} \delta \\
= -\frac{c^2}{2} + \frac{c^4}{4} + \frac{\alpha - 1}{\alpha} \delta \\
\leq -\frac{c^2}{2} + \frac{\alpha - 1}{\alpha} \delta. 
\]

(49)
We substitute (48) and (49) into (47) and multiply by $\psi^2$:

$$0 \leq -\frac{c^2}{4} |A|^2 \psi^2 + C \psi + C(\varepsilon).$$

The localization function $\psi$ is bounded by $a$ and $\varepsilon > 0$ is a controlled quantity. Thus, there is a controlled constant such that

$$\psi |A| \leq C \text{ at } (p_0, t_0).$$

We have assumed $H \leq \delta |A|$ at $(p_0, t_0)$. Now we rectify this. If $|A| < \frac{H}{\delta}$, then we still find $\psi |A| \leq C$ by the control on $H$, $\delta$, and $\psi$. Because of (11) it thus holds

$$f(p_0, t_0) = (\log |A|^2 + 2 \log \psi - \beta \log(H^\alpha - c))|_{(p_0, t_0)} \leq C.$$

At any point $(p, t)$ the function $f$ is bounded by this constant or by its initial values. Through consideration of $\exp \frac{f}{2}$, we obtain

$$\psi |A| \leq (H^\alpha - c)^{\beta/2} \max \left\{ \sup_{t=0} \frac{\psi |A|}{(H^\alpha - c)^{\beta/2}}, C \right\} \leq C \max \left\{ \sup_{t=0} \psi |A|, 1 \right\},$$

which is the asserted inequality (12).

\[\square\]

5 Complete hypersurfaces

Theorem 1. Let $\alpha > 0$. Let $\Omega_0 \subset \mathbb{R}^n$ be open. Let $u_0: \Omega_0 \to \mathbb{R}$ be smooth and such that graph $u_0$ is of positive mean curvature $H[u_0] > 0$. Furthermore, we suppose that the sets $\{x : u_0(x) \leq a\}$ are compact for any $a \in \mathbb{R}$.

Then, there exists a relatively open set $\Omega \subset \mathbb{R}^n \times [0, \infty)$ compatible with the $\Omega_0$ from above ($\Omega \cap (\mathbb{R}^n \times \{0\}) = \Omega_0$), and there exists a continuous function $u: \Omega \to \mathbb{R}$ which is smooth on $\Omega \setminus (\Omega_0 \times \{0\})$, and such that $u(\cdot, 0) = u_0$, and that $u$ is a solution of the graphical $H^\alpha$-flow (3). Moreover, $u$ fulfills the following maximality condition: There is a continuous function $\overline{u}: \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ such that $\{(x, t) : \overline{u}(x, t) \in \mathbb{R}\} = \Omega$ and $\overline{u}|\Omega = u$.

Remark 5. The maximality condition implies the completeness at every fixed time. On the other hand it implies that the flow is maximal in the sense that it is not arbitrarily stopped or runs into any singularities.

Proof. Let $(a_k) \subset \mathbb{R}$ be a sequence of heights such that $a_k \to \infty$ and such that $Q_k := \{u_0 < a_k\}$ are smooth, open, and bounded domains. (The existence of such a sequence is guaranteed by Sard’s Theorem.) For fixed $k \in \mathbb{N}$, we consider the initial boundary value problem (14) with domain $Q_k$ and initial datum $u_0|Q_k$. (We observe that $u_0|\partial Q_k \equiv a_k$ because if $x \in \partial Q_k$ and $Q_k \ni x \to x \in \partial Q_k$, then, by the compactness of $\{y : u_0(y) \leq a_k\}$, the boundary point is in this set: $x \in \{y : u_0(y) \leq a_k\} \subset \Omega_0$. Therefore, $u_0(x)$ is well-defined and $u_0(x) = a_k$ must hold by the continuity of $u_0.$) The exponent function will be denoted $\tilde{a}$ here. It is chosen such that $\tilde{a} \equiv a$ on
\{a_0 \leq a_k - 1\) and such that \(\tilde{\sigma}(x)\) is always between \(\alpha\) and 1 (including those values).

The cut-off function \(\xi\) in (11) is chosen according to the assumptions on it, as is the constant \(c\). Let \(v_k : Q_k \times [0, \infty) \to \mathbb{R}\) be the solution from Lemma 4. We extend \(v_k\) to \(\mathbb{R}^n \times [0, \infty)\) by setting \(\varpi_k(x, t) = v_k(x, t)\) if \((x, t) \in Q_k \times [0, \infty)\) and \(\varpi_k(x, t) = a_k + ct\) if \((x, t) \notin Q_k \times [0, \infty)\). Then \(\varpi_k\) is a continuous function on \(\mathbb{R}^n \times [0, \infty)\).

Because the solutions of (11) are growing in time, the \(t\)-dependent set \(\{u(\cdot, t) \leq a_k - 1\}\) is shrinking with \(t\). For this reason, \((\text{graph } \varpi_k(\cdot, t))\) moves by the \(H^\alpha\)-flow below the height \(a_k - 1\). This makes the a priori bounds from section 4 applicable. These yield estimates on \(\nabla \varpi_k\) and \(\nabla^2 \varpi_k\) at points \((x, t)\) where \(\varpi_k(x, t) < a\) and \(t < t_*\) for arbitrary \(a < a_k - 2\) and \(t_* > 0\). The estimates depend on \(a\) and \(t_*\) but not on \(k\). Estimates on \(\partial_t \varpi_k\) of the same kind are obtained from the equation of graphical \(H^\alpha\)-flow once we have the estimates on \(\nabla \varpi_k\) and \(\nabla^2 \varpi_k\). This is enough to apply the Arzelà-Ascoli argument of [13]. It gives a subsequence of \(\varpi_k\) and a continuous function \(\varpi : \mathbb{R}^n \times [0, T) \to \mathbb{R}\) such that \(\varpi_k \to \varpi\) pointwise and locally uniformly on \(\Omega := \{\varpi \in \mathbb{R}\}\).

In addition to the local \(C^{2,1}\)-estimates mentioned above, section 4 also supplies us with the estimates \((8.2)\) and \((8.3)\) which give both-sided control on \(H^\alpha\). This lets us locally control the parabolicity constants. Here, “local” means on sets \(\{(x, t) : \varpi_k(x, t) < a, \ t < t_*\}\). As in the proof of Lemma 4, local estimates on all derivatives of the \(\varpi_k\) uniform in \(k\) now follow from general theory. These estimates are local in time also for \(t \to 0\): Uniform estimates are only obtained for \(t > \varepsilon\) and the constants then depend on \(\varepsilon\). These estimates are still sufficient to show that the locally uniform convergence \(\varpi_k \to \varpi\) on \(\Omega\) is a locally smooth convergence on \(\Omega \setminus (\Omega_0 \times \{0\})\).

We set \(u := \varpi|_{\Omega}\). It is quite clear that \(u(x, 0) = u_0(x)\) for \(x \in \Omega_0\) and that \(\varpi(x, 0) = \infty\) for \(x \notin \Omega_0\). That \(u\) is a solution of the graphical \(H^\alpha\)-flow is fairly easy to see too: Because \((\text{graph } \varpi_k(\cdot, t))\) moves by \(H^\alpha\)-flow below the height \(a_k - 1\) and \(\varpi_k\) converges locally smoothly to \(u\) on \(\Omega \setminus (\Omega_0 \times \{0\})\), \(u\) must solve the graphical \(H^\alpha\)-flow on \(\Omega \setminus (\Omega_0 \times \{0\})\). \(\square\)

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