Reduction theory for a rational function field

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Abstract. Let $G$ be a split reductive group over a finite field $\mathbb{F}_q$. Let $G = \mathbb{F}_q(t)$ and let $A$ denote the adèles of $F$. We show that every double coset in $G(F) \backslash G(A)/K$ has a representative in a maximal split torus of $G$. Here $K$ is the set of integral adelic points of $G$. When $G$ ranges over general linear groups this is equivalent to the assertion that any algebraic vector bundle over the projective line is isomorphic to a direct sum of line bundles.

Keywords. Automorphic form; function field.

1. Introduction

Let $F$ be a global field, $A$ its ring of adèles and $G$ a reductive group defined over $F$. The theory of automorphic forms involves the study of spaces of functions on $G(A)/K$ as representations of $G(A)$. The functions involved are often required to be right invariant under certain large compact subgroups $K$ of $G(A)$ because (among other reasons) the double coset space $G(F) \backslash G(A)/K$ admits nice interpretations. For example, the classical study of the upper half plane modulo the action of arithmetic subgroups of the real special linear group is a special case of the above when $F$ is the field of rational numbers (see e.g., ([13], §1). Another special case, which corresponds to taking $F$ to be a field of rational functions in one variable and $G$ to be $GL(2)$ is discussed by Weil in [15]. When $F$ is a function field, Harder describes a fundamental domain for the action of $G(F)$ on $G(A)$ in ([10], §1) using results from [8] and [9]. This is an analogue of the Siegel domain described by Godement in [6] for $F = \mathbb{Q}$. Proposition 14 in this article is analogous to these results and the proof proceeds along the lines of [6]. Harder’s description of the fundamental domain is a very basic result in the theory of automorphic forms over function fields (see e.g., [12], §9 and Appendix E).

From now on let $G$ be a split reductive group defined over a finite field $\mathbb{F}_q$ with $q$ elements. Fix a Borel subgroup $B$ defined over $\mathbb{F}_q$ with unipotent radical $N$, and a maximal $\mathbb{F}_q$-split torus $T$ contained in $B$. Set $F = \mathbb{F}_q(t)$. For a valuation $v$ of $F$, we denote the corresponding local field by $F_v$ and its ring of integers by $O_v$. For each $v$, fix a uniformizing element $\pi_v \in F \cap O_v$. In particular, fix $\pi_\infty = t^{-1}$ as a uniformizing element at the place $\infty$ whose local field is $\mathbb{F}_q((t^{-1}))$. Let $K$ be the maximal compact subgroup $\prod_v G(O_v)$ of $G(A)$. This article concerns the double coset space

$$G(F) \backslash G(A)/K$$

which may be interpreted as the set of isomorphism classes of principal $G$-bundles on the projective line. In [7], Grothendieck proves that when $G$ is a complex reductive group any
holomorphic $G$-bundle over the complex projective line admits a reduction of structure
group to a maximal torus. (In fact this result has been attributed to Dedekind and Weber
for $G = GL(n)$ by Geyer ([5], §6) who deduces it from a statement in ([3], §22).) In
our ad`elic setting, this should correspond to the assertion that every double coset has a
representative in $T(\A)$. 

Let $X_s(T)$ denote the lattice $\text{Hom}(G_m, T)$ of algebraic co-characters of $T$. Given $\eta \in X_s(T)$, and a valuation $v$ denote by $\pi_v^\eta$ the element $\eta(\pi_v) \in T(F_v) \subset T(\A)$. Recall that
$\eta \in X_s(T)$ is called antidominant if $|\alpha_i \circ \eta(\pi_v)|_v \geq 1$ for each simple root $\alpha_i$ (see §3). 

Precisely stated, the main result of this article is the following:

**Theorem 1.** Every double coset in

$$G(F) \setminus G(\A) \slash K$$

has a unique representative of the form $(t^{-1})^\eta$, where $\eta \in X_s(T)$ is antidominant.

In §6, we will deduce Theorem 1 from the following local result which is proved in §5. Let $F_\bullet$ be the local field $\mathbb{F}_q((\pi))$ of Laurent series in $\pi$ with coefficients in $\mathbb{F}_q$. It contains,
as its ring of integers, the discrete valuation ring $O = \mathbb{F}_q[[\pi]]$, and as a discrete subring,
the polynomial ring $R = \mathbb{F}_q[\pi^{-1}]$. Let $\Gamma = G(R)$.

**Theorem 2.** Every double coset in

$$\Gamma \setminus G(F_\bullet) \slash G(O)$$

has a unique representative of the form $\pi^\eta$, where $\eta \in X_s(T)$ is antidominant.

The main results proved in this article should be known to the experts, but we have not
found them in the literature beyond the case of $GL(2)$, for which Theorem 2 is proved in
([15], §3). The results proved in this paper have played an important role in the author’s
work [14], as well as in the work of other authors on $\mathbb{F}_q(t)$ [4,1,11].

2. Normed local vector spaces

Let $V$ be a vector space defined over $\mathbb{F}_q$. Let $e_1, \ldots, e_n$ be a basis of the free $O$-module $V(O)$ (so that $V(O)$ is isomorphic to the free $O$-module generated by the $e_i$s). Given
a vector $x \in V(F_\bullet)$, we may write $x = x_1e_1 + \cdots + x_ne_n$, uniquely, with $x_i \in F_\bullet$.

Define

$$\|x\| = \sup\{|x_1|, \ldots, |x_n|\}. \tag{1}$$

**Lemma 4.** If $g \in GL(V(O))$, then $\|xg\| = \|x\|$.

**Proof.** Let $(g_{ij})$ be the matrix of $G$ with respect to the basis chosen above. Let $y = xg$. If
$y = y_1e_1 + \cdots + y_ne_n$, then

$$y_j = \sum_{i=1}^n x_ig_{ij}$$
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and

\[ \|y\| = \sup_{1 \leq j \leq n} \left| \sum_{i=1}^{n} x_i g_{ij} \right| \]
\[ \leq \sup_{1 \leq j \leq n} \sup_{1 \leq i \leq n} |x_i g_{ij}| \quad \text{(ultrametric inequality)} \]
\[ \leq \sup_{1 \leq j \leq n} \sup_{1 \leq i \leq n} |x_i| \quad \text{(since } g_{ij} \in \mathbf{O}) \]
\[ = \|x\|. \]

Hence

\[ \|y\| \leq \|x\|. \]

We may apply the same reasoning to \( g^{-1} \) to show that

\[ \|x\| \leq \|y\|. \]

Therefore,

\[ \|y\| = \|x\|. \]

\[ \square \]

**COROLLARY 5**

*The norm \( \| \cdot \| \) is independent of our choice of basis of \( V(\mathbf{O}) \).*

**Proof.** The coordinates of a vector with respect to two different bases differ by a matrix with entries in \( \mathbf{O} \). The argument in the proof of Lemma 4 shows that the norms with respect to two different bases are equal. \( \square \)

**Lemma 6.** The norm \( \| \cdot \| \) satisfies the ultrametric triangle inequality, i.e., for vectors \( x, y \) in \( V(F_\ast) \),

\[ \|x + y\| \leq \sup\{\|x\|, \|y\|\}. \]

**Proof.** Write \( x = x_1 e_1 + \cdots + x_n e_n \) and \( y = y_1 e_1 + \cdots + y_n e_n \). Then

\[ \|x + y\| = \sup\{|x_1 + y_1|, \ldots, |x_n + y_n|\} \]
\[ \leq \sup\{\sup\{|x_1|, |y_1|\}, \ldots, \sup\{|x_n|, |y_n|\}\} \]
\[ = \sup\{|x_1|, |y_1|, \ldots, |x_n|, |y_n|\} \]
\[ = \sup\{\|x\|, \|y\|\}. \]

\[ \square \]

**Lemma 7.** For a scalar \( \lambda \in F_\ast \) and a vector \( x \in V(F_\ast) \),

\[ \|\lambda x\| = |\lambda| \|x\|. \]

**Lemma 8.** If \( g \in GL(V(F_\ast)) \), then there is a constant \( C_g > 0 \), such that for any vector \( x \in V(F_\ast) \),

\[ \|xg\| \leq C_g \|x\|. \]
Proof. Suppose that \( g \) has matrix \( (g_{ij}) \), and \( x \) has coordinates \( (x_1, \ldots, x_n) \) with respect to the basis \( e_1, \ldots, e_n \). Then
\[
\|xg\| = \sup \left\{ \sum_{i=1}^{n} x_i g_{i1}, \ldots, \sum_{i=1}^{n} x_i g_{in} \right\}
\leq \sup_{1 \leq j \leq n} \sup_{1 \leq i \leq n} |g_{ij}| \|x\|.
\]
Therefore, let
\[
C_g = \sup_{1 \leq j \leq n} \sup_{1 \leq i \leq n} |g_{ij}|.
\]

Lemma 9. If \( x \in V(R) \) is a non-zero vector then \( \|x\| \geq 1 \).

Proof. By Corollary 5, we may assume that the elements \( e_i \) of a basis used to define \( \| \cdot \| \) lie in \( V(F_q) \). Then at least one coordinate of \( x \) is non-zero in \( R \). But any non-zero element in \( R \) has norm at least one. Therefore, \( \|x\| \geq 1 \).

PROPOSITION 10

For any non-zero vector \( x \in V(F_q) \) and any \( g \in GL(V(F_q)) \), there is a positive constant \( E \) such that for all \( \gamma \in GL(V(R)) \),
\[
\|x\gamma g\| \geq E.
\]
Consequently, for any subset \( S \) of \( GL(V(R)) \), the set \( \{\|xsg\| : s \in S\} \) has a positive minimal element.

Proof. Applying Lemma 8 to \( g^{-1} \), and Lemma 9 to \( x\gamma \) (which lies in \( V(R) \)), we have
\[
\|x\gamma g\| \geq C_{g^{-1}} \|x\gamma\| \geq C_{g^{-1}} > 0.
\]
The second part of the assertion follows by noting that the values taken by the norm \( \| \cdot \| \) are of the form \( q^j \), where \( j \) is an integer.

3. Fundamental representations

Let \( \alpha_1, \ldots, \alpha_r \) be the simple roots with respect to \( B \) in the root system \( \Phi(G, T) \) of \( G \) with respect to \( T \). Let \( W = N_G(T)/T \) be the Weyl group of \( G \) with respect to \( T \). To each simple root \( \alpha_i \), we associate an element \( s_i \) of order two in \( W \) in the usual way.

Given a subset \( D \) of \( \{1, \ldots, r\} \), let \( W_D \) denote the subgroup of \( W \) generated by \( \{s_j | j \in D\} \), and let \( P_D \) denote the parabolic subgroup \( BW_DB \) of \( G \) containing \( B \). This group has a Levi decomposition
\[
P_D = L_DU_D.
\]
where \( L_D \) is a reductive group of rank \( |D| \) and \( U_D \) is the unipotent radical of \( P_D \). \( L_D \cap B \) is a Borel subgroup for \( L_D \) containing the split torus \( T \). The set of simple roots of \( L_D \) with respect to \( L_D \cap B \) is \( \{\alpha_j | j \in D\} \). Denote by \( P_i \) (resp., \( L_i, U_i \)) the parabolic subgroup (resp., Levi subgroup, unipotent subgroup) corresponding to the set \( \{1, \ldots, i-1, i+1, \ldots, r\} \). These are the maximal proper parabolic subgroups of \( G \) containing \( B \).
Theorem 11 [2]. There exist irreducible finite dimensional representations \((\rho_i, V_i)\) of \(G\), vectors \(v_i \in V_i(F_q)\) that are unique up to scaling, and characters \(\Delta_i : P_i \to G_m\), for \(i = 1, \ldots, r\) all defined over \(F_q\), such that

1. \(P_i\) is the stabilizer of the line generated by \(v_i\) and \(v_i \rho_i(p) = \Delta_i(p)v_i\) for each \(p \in P_i\) for \(i = 1, \ldots, r\).
2. The restrictions \(\mu_i\) to \(T\) of \(\Delta_i\) are antidominant weights of \(T\) with respect to \(B\), which generate \(X^*(T) \otimes \mathbb{Q}\) as a vector space over the rational numbers.

4. Ordering by roots

Lemma 12. Let \(L\) be a Levi subgroup of \(G\) associated to a parabolic subgroup \(P\) containing \(B\). Then there is a canonical surjection

\[
G(F_\bullet)/G(O) \xrightarrow{\phi^G_L} L(F_\bullet)/L(O).
\]

If \(Q = MN\) is a parabolic subgroup of \(G\) containing \(B\) and contained in \(P\), then \(M\) is a Levi subgroup for \(L\) corresponding to the parabolic subgroup \(L \cap Q\) of \(L\), and \(\phi^L_M \circ \phi^G_L = \phi^G_M\).

Proof. Given \(g \in G(F_\bullet)\), we may use the Iwasawa decomposition to write \(g = luk\), where \(l \in L(F_\bullet), u \in U(F_\bullet)\) and \(k \in G(O)\). Moreover, if \(g = l' u' k'\) is another such decomposition, then, setting \(l_0 = l'^{-1} l\) and \(k_0 = k' k^{-1}\),

\[
u'^{-1} l_0 u = k_0 \in G(O).
\]

On the other hand,

\[
k_0 = u'^{-1} l_0 u = l_0'^{-1} u'^{-1} l_0 u.
\]

Since \(L\) normalizes \(U\), \(l_0'^{-1} u'^{-1} l_0 \in U(F_\bullet)\), and hence, setting \(u_0 = l_0^{-1} u'^{-1} l_0 u \in U(F_\bullet)\),

\[
l_0 = k_0 u_0 \in G(O)U(F_\bullet) \cap L(F_\bullet).
\]

Therefore \(l_0 u_0^{-1} = k_0 \in G(O) \cap P(F_\bullet) = P(O)\), so that \(l_0 \in L(O)\). This shows that \(luk \mapsto l\) induces a well defined map \(\Phi^G_L : G(F_\bullet)/G(O) \to L(F_\bullet)/L(O)\). It is clear that this map is surjective. To see that \(\Phi^L_M \circ \Phi^G_L = \Phi^G_M\), note that we may write \(g = muk\) with \(m \in M(F_\bullet), u \in N(F_\bullet)\) and \(k \in G(O)\). But \(N(F_\bullet) = (N(F_\bullet) \cap L(F_\bullet)) U(F_\bullet)\), so we may write \(u = u_1 u_2\), where \(u_1 \in N(F_\bullet) \cap L(F_\bullet)\) and \(u_2 \in U(F_\bullet)\). Therefore, we see that \(mM(O) = \Phi^L_M(mu_1) = \Phi^G_M(g)\).

In the sequel we denote \(\Phi^G_L\) simply by \(\Phi\). Define

\[
\Omega_G := \{g \in G(F_\bullet) : |\alpha_i \circ \Phi(g)| \geq 1 \text{ for } i = 1, \ldots, r\}.
\]

PROPOSITION 14

\[G(F_\bullet) = \Gamma \Omega_G.\]
Proof.

The rank one case (following [15]): Here $G$ has one simple root $\alpha_1$, and one fundamental representation $(\rho_1, V_1)$ and a vector $v_1 \in V_1(F_q)$ such that for any element $g$ in the parabolic subgroup $B = TN$, where $N$ is the unipotent radical of $B$,

$$v_1 \rho_1(b) = \Delta_1(b)v_1,$$

(3)

where the character $\Delta_1 : B \mapsto G_m$ (defined over $F_q$) restricts to an anti-dominant weight $\mu_1$ on the maximal split torus $T$. Let $g \in G(F_\ast)$. We wish to show that $g \in \Gamma \Omega_G$. To this end, by Proposition 10, and by replacing $g$, if necessary by an appropriate element of $\Gamma g$, we may assume that $g$ has the property that

$$\|v_1 \rho_1(g)\| \geq \|v_1 \rho_1(1)\| \quad \text{for all } \gamma \in \Gamma.$$

(4)

Write $g = tnk$, where $t \in T(F_\ast)$, $n \in N(F_\ast)$ and $k \in G(O)$. By Theorem 11 and Lemma 4,

$$\|v_1 \rho(g)\| = |\Delta_1(t)||v_1|||\mu_1(t)|.$$

(5)

Fix an isomorphism $u_{a_1} : G_\ast \to N$ defined over $F_q$, and let $c \in F_\ast$ be such that $n = u_{a_1}(x)$. Choose $\sigma$ in the nontrivial $T(F_q)$-coset of $N_T(F_q)$. Note that if $S \in R$, then $\sigma u_{a_1}(S)$ is isomorphic to $G$. Therefore, using Proposition 10,

$$|\mu_1(t)| = \|v_1 \rho_1(g)\|
\leq \|v_1 \rho_1(\sigma u_{a_1}(S)tu_{a_1}(x))\|
= \|v_1 \rho_1(\sigma t\sigma u_{a_1}(\alpha_1(t)^{-1}S + \alpha_1(t)x))\|
= |\mu_1(t)|^{-1}\|v_1 \rho_1(u_{-a_1}(\alpha(t)^{-1}S + x))\|.
$$

Here $u_{-a_1} = \sigma u_{a_1} \sigma^{-1}$, and its image is the root subgroup for $-\alpha_1$. The element $u_{-a_1}(\alpha(t)^{-1}S + x)$ lies in the derived group of $G$ which is isomorphic to either $SL_2$ or $PGL_2$ in the rank one case. When the derived group of $G$ is isomorphic to $SL_2$, we may take $V_1$ to be the right action of $SL_2$ on the space of $1 \times 2$-matrices by right multiplication. One may take the torus $T$ to consist of diagonal matrices in $SL_2$, $B$ the upper triangular matrices in $SL_2$ and $v_1$ to be the vector $(0, 1)$. Calculating with matrices, one may verify that

$$\|v_1 \rho_1(u_{-a_1}(\alpha(t)^{-1}S + x))\| \leq \sup\{1, |\alpha(t)^{-1}S + x|\}.$$

Therefore,

$$\sup\{1, |\alpha(t)^{-1}S + x|\} \geq |\mu_1(t)|^2.$$

(6)

Choose $S$ in $R$ such that $|S + \alpha(t)x| < 1$. Then $|\alpha(t)^{-1}S + x| < |\alpha(t)|^{-1}$. Suppose that $|\alpha(t)^{-1}S + x| \geq |\mu_1(t)|^2$. Then $|\alpha(t)|^{-1} > |\mu_1(t)|^2$. This is impossible, since $\alpha(t)^{-1} = \mu_1(t)^2$. It follows that $|\alpha(t)^{-1}S + x| < |\mu_1(t)|^2$. Therefore, (6) can hold only if $1 \geq |\mu_1(t)|^2$, which is the same as $|\alpha(t)| \geq 1$. This completes the proof of Proposition 14 when the derived group of $G$ is isomorphic to $SL_2$.

When the derived group of $G$ is isomorphic to $PGL_2$, then $G$ is the product of its centre with $PGL_2$. Therefore, the assertion of Proposition 14 for $G$ follows from that for $PGL_2$. However, the assertion for $PGL_2$ follows easily from that for $GL_2$. The derived group
of \(GL_2\) is \(SL_2\), hence the proposition holds for \(GL_2\) by the argument in the previous paragraph, completing the proof of Proposition 14 in the rank one case.

The general case: Let \(G\) be a group of rank \(r\), and \(g \in G(F_\gamma)\). By modifying \(g\) on the left by an element of \(\Gamma\), we may, for the purposes of this proof, assume, using the second assertion of Proposition 10, that

\[
\|v_1 \rho_1(g)\| \leq \|v_1 \rho_1(\gamma g)\| \quad \text{for all } \gamma \in \Gamma.
\]  

(7)

Note that if \(\gamma \in P_1(F_\gamma) \cap \Gamma\), then \(v_1 \rho_1(\gamma g) = \Delta_1(\gamma) v_1 \rho_1(g)\). Since \(\Delta_1(\gamma) \in F_\gamma[\pi^{-1}]^\times\), \(|\Delta_1(\gamma)| = 1\). Therefore, \(\|v_1 \rho_1(\gamma g)\| = \|\Delta_1(\gamma) v_1 \rho_1(g)\|\). We may use the second assertion of Proposition 10 again, to assume, for the purposes of this proof, that

\[
\|v_2 \rho_2(g)\| \leq \|v_2 \rho_2(\gamma g)\| \quad \text{for all } \gamma \in \Gamma \cap P_1(F_\gamma)
\]  

(8)

while preserving (7). Continuing in this manner, we may assume that

\[
\|v_j \rho_j(g)\| \leq \|v_j \rho_j(\gamma g)\| \quad \text{for all } \gamma \in \Gamma \cap P_1(F) \cap \ldots \cap P_{j-1}(F),
\]  

(9)

for \(j = 1, \ldots, r\). Therefore, it suffices to prove the following:

Lemma 22. If an element \(g \in G(F_\gamma)\) satisfies the inequalities (9) for each integer \(1 \leq j \leq r\), then \(g \in \Omega_G\).

The proof of Proposition 14 in the rank one case shows that Lemma 22 is true when \(G\) is of semisimple rank one. We prove it in general assuming the validity of Theorem 2 in the rank one case.

Suppose that \(g\) satisfies the inequalities (9) for each \(1 \leq j \leq r\). Write \(g = bk\), with \(b \in B(F_\gamma)\) and \(k \in G(O)\). Then \(b\) can be written as \(lu\), where \(l \in L_{[i]}(F_\gamma) \cap B(F_\gamma)\) and \(u \in U_{[i]}(F_\gamma)\). Since \(U_{[i]}\) fixes \(v_i\), the inequalities (9) imply that

\[
\|v_i \rho_i(l)\| \leq \|v_i \rho_i(\gamma l)\| \quad \text{for all } \gamma \in L_{[i]}(R).
\]  

(10)

From the rank one case, \(l = \gamma \pi^0 k\) for some \(\gamma \in L_{[i]}(R), k \in L_{[i]}(O)\) and \(\eta \in X_s(T)\) such that \(|\alpha_i(\pi^0)| \geq 1\). \(\rho_i(\gamma)\) maps \(v_i\) into \(V(R)\). From Lemma 24 it follows that

\[
\|v_i \rho_i(l)\| \geq \|v_i \rho_i(\pi^0)\|.
\]

Equation (10) implies that the above must be an equality. This forces \(\gamma \in L_{[i]}(R) \cap P_i(R)\), and hence also \(k \in L_{[i]}(O) \cap P_i(O)\). Write \(b = tn\) with \(t \in T(F_\gamma)\) and \(n \in N(F_\gamma)\). Then viewing \(\alpha_i\) as a rational character of \(B(F_\gamma)\) that is trivial on \(N(F_\gamma)\), we have

\[
|\alpha_i(t)| = |\alpha_i(l)| = |\alpha_i(\pi^0)| \geq 1.
\]

Repeating this argument for each \(i\) completes the proof of Lemma 22.

5. Local reduction theory

In order to prove the existence part of Theorem 2, it suffices to show that every element \(g\) in \(\Omega_G\) may be written as \(g = \gamma \pi^0 k\), where \(\gamma \in \Gamma\), \(\eta \in X_s(T)\) is antidominant and \(k \in G(O)\). To this end, we may assume (using the Iwasawa decomposition) that we are
given \( g \in \Omega_G \), with \( g = tn \), with \( t \in T(F_\bullet) \) and \( n \in N(F_\bullet) \). Since \( g \), and hence \( t \), is in \( \Omega_G \), \(|a_i(t)| \geq 1\), so that \( a_i(t)^{-1} \in \mathcal{O} \), for \( i = 1, \ldots, r \). For each root \( \alpha \in \Phi(G, T) \), let \( U_\alpha \) denote the corresponding root subgroup. Fix an isomorphism \( u_\alpha : G_a \rightarrow U_\alpha \) defined over \( F_q \). Then for \( x \in F_\bullet \), we have
\[
t u_\alpha(x) = (tu_\alpha(x)t^{-1})t = u_\alpha(\alpha(t)x)t.
\]
Therefore, if we write \( \alpha(t)x = P + h \), where \( P \in R \) and \( h \in \mathcal{O} \), then
\[
t u_\alpha(x) = tu_\alpha(\alpha(t)^{-1}P)u_\alpha(\alpha(t)^{-1}h) = u_\alpha(P)tu_\alpha(\alpha(t)^{-1}h).
\]
Given two positive roots \( \alpha \) and \( \beta \), the commutator \([U_\alpha, U_\beta]\) is contained in the product of root subgroups \( U_{\alpha'} \) where the \( \alpha' \) are roots which can be written as positive linear combinations of \( \alpha \) and \( \beta \) and are distinct from either \( \alpha \) or \( \beta \). Moreover, we may enumerate the positive roots as \( \beta_1, \beta_2, \ldots \) so that if \( j > i \), then \( \beta_i \) cannot be written as a sum of \( \beta_j \) and any other positive roots.

Write \( n \) as \( \prod_i \ u_\beta_i(x_i) \). Then
\[
tn = tu_{\beta_1}(x_1) \prod_{i>1} u_{\beta_i}(x_i).
\]
If we write \( \beta_1(t)x_1 = P_1 + h_1 \), where \( P_1 \in F_q[\pi^{-1}] \) and \( h \in \mathcal{O} \), then
\[
tn = u_{\beta_1}(P_1)tu_{\beta_1}(\beta_1(t)^{-1}h_1) \prod_{i>1} u_{\beta_i}(x_i).
\]
Since \( u_{\beta_1}(P_1) \in \Gamma \), \( \beta_1(t)^{-1} \in \mathcal{O} \), and the image of \( u_{\beta_1} \) normalizes all the subsequent root subgroups whose elements appear in the above expression, we may assume for the purpose of proving Theorem 2, that
\[
tn = t \prod_{i>1} u_{\beta_i}(x_i'),
\]
for \( x_i' \in F_\bullet \). We may continue in this manner to reduce \( tn \) to \( t \). It is then easy to see (using the decomposition \( F_\bullet^* = \pi^k \mathcal{O}^* \)) that \( t \) may be replaced by \( \pi^k \) for some \( k \). Since \(|\alpha_i(\pi^k)| \geq 1\), it follows that \( \pi^k \) is antidominant, proving the existence part of Theorem 2.

We now prove the uniqueness part of Theorem 2. In order to do this, it suffices to show that if \( \eta \) and \( v \) are two dominant co-weights, and \( \pi^v = \gamma \pi^\eta k \) for some \( \gamma \in \Gamma \) and \( k \in G(\mathcal{O}) \), then \( v = \eta \). Since the weights \( \mu_1, \ldots, \mu_r \) corresponding to the fundamental representations in Theorem 11 generate the vector space \( X^*(T) \otimes \mathbb{Q} \), it suffices to show that \( \langle \mu_i, \eta \rangle = \langle \mu_i, v \rangle \) for each \( i \). In order to do this, we need the following:

Lemma 24. For any non-zero vector \( v \in V_i(F_\bullet) \) and any antidominant co-weight \( \mu \in X^*_\ast(T) \),
\[
\frac{\|v, \rho_i(\pi^\mu)\|}{\|v\|} \geq \frac{\|v, \rho_i(\pi^\mu)\|}{\|v\|}.
\]

Proof. Since \( \pi^k \) is \( F_q \)-split and \( \rho_i \) is defined over \( F_q \), \( V \) has a decomposition (over \( F_q \)) into root subspaces
\[
V = \bigoplus \mathcal{V}_\lambda
\]
where $T$ acts on $V_\lambda$ by the character $\lambda : T \to G_m$. It is easy to see that $\mu_i$ is the lowest weight of $T$ occurring in $(\rho_i, V_i)$, so that $(\mu_i, \mu) \geq (\lambda, \mu)$ for any weight $\lambda$ of $T$ occurring in $(\rho_i, V_i)$ and any antidominant co-weight $\mu$. Given any vector $v \in V(F_\ast)$, we may write

$$v = \sum x_j u_j,$$

where $x_j \in F_\ast$ and $u_j \in V_{\lambda_j}(F_q)$ for each $j$ and the $\lambda_j$s are not necessarily distinct. Thus

$$\|v \rho_i(\pi^\mu)\| = \left\| \sum \lambda_j(\pi^\mu)x_j u_j \right\|$$

$$= \sup_j \left\| \lambda_j(\pi^\mu)x_j \right\|$$

$$= q^{(\lambda_j, \mu)} \sup_j \left\| x_j \right\|$$

$$\geq q^{(\mu_i, \mu)} \sup_j \left\| x_j \right\|$$

$$= \|v_i \rho_i(\pi^\mu)\| \|v\|.$$ 

Since $\|v_i\| = 1$, this completes the proof of Lemma 24.

Lemma 24 allows us to compare $(\mu_i, v)$ and $(\mu_i, \eta)$:

$$q^{-(\mu_i, \eta)} = \frac{\|v \rho_i(\pi^\eta)\|}{\|v\|}$$

$$\leq \frac{\|v \rho_i(\gamma \pi^\eta)\|}{\|v \rho_i(\gamma)\|}$$

$$\leq \frac{\|v \rho_i(\gamma \pi^\eta)\|}{\|v\|}$$

$$= \|v \rho_1(\pi^\gamma)\| \|v\|$$

$$= q^{-(\mu_i, \gamma)}.$$ 

The first inequality is Lemma 24 applied to $v = v \rho_i(\gamma)$. The second inequality follows from Lemma 9 with $x = v \rho_i(\gamma)$. Interchanging the roles of $\eta$ and $v$ in the above arguments shows that $(\mu_i, \eta) = (\mu_i, v)$ for each $i$. This completes the proof of the uniqueness part of the assertion of Theorem 2.

6. Global reduction theory

If $g = (g_v)_v$ is an element of $G(A)$ then, since $g_v \in G(O_v)$ for all but finitely many places $v$ of $F$, we may assume, for the purpose of proving Theorem 1 that $g$ is a finite product $g = g_\infty g_{v_1} g_{v_2} \cdots g_{v_k}$, with $g_\infty \in G(F_\infty)$ and $g_{v_j} \in G(F_{v_j})$, $u_j \neq \infty$, for $1 \leq j \leq k$. By Theorem 2, there is a decomposition

$$g_{v_k} = \gamma_k x_{v_k}^{\eta_k} \kappa_k,$$
where $\gamma_k \in G(F_q[\pi_{v_k}^{-1}])$, $\eta_k \in X_\omega(T)$, and $\kappa_k \in G(O_{v_k})$. Now $\gamma_k$ and $\pi_{v_k}^{\eta_k}$ are both contained in $G(F)$ and in $G(O_v)$ for all $v \neq \infty$. Therefore, by multiplying $g$ on the left by $\pi_{v_k}^{\eta_k}\gamma^{-1}$ we get an element of the subset
\[ G(F_\infty) \times \prod_{j=1}^{k-1} G(F_{v_j}) \times \prod_{\text{all other } v} G(O_v) \]
of $G(A)$.

We have now reduced $g$ to an element with non-trivial entries only at most $k - 1$ places and $\infty$. We may continue in this manner until the entries at all places except $\infty$ are trivial. Finally, the use of Theorem 2 to $v = \infty$ gives us a representative each double coset of type asserted by Theorem 1.

The uniqueness part of the theorem follows from the corresponding assertion in the local situation, because two elements $g$ and $h$ of $G(F_\infty)$ lie in the same double coset if and only if $g = \gamma h k$, with $\gamma \in G(F_q[t])$ and $k \in G(O_\infty)$.

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