ON ARTINIAN GORENSTEIN ALGEBRAS ASSOCIATED TO THE FACE POSETS OF REGULAR POLYHEDRA

AKIKO YAZAWA

Abstract. We introduce Artinian Gorenstein algebras defined by the face posets of regular polyhedra. We consider the strong Lefschetz property and Hodge–Riemann relation for the algebras. We show the strong Lefschetz property of the algebras for all Platonic solids. On the other hand, for some Platonic solids, we show that the algebras do not satisfy the Hodge–Riemann relation with respect to some strong Lefschetz elements.

1. Introduction

A matroid is a simplicial complex with the independence augmentation property. Facets of a matroid are called bases for the matroid. They satisfy a property of “symmetricity”, so called the basis exchange property. For a matroid \( M \), define the polynomial

\[ F_M = \sum_{F} \prod_{i \in F} x_i \]

where \( F \) runs the collection of facets of \( M \). The polynomials are studied from various viewpoints. In \cite{1, 2, 3, 4, 5}, it was shown that the polynomials are log-concave on the positive orthant. For a polynomial \( F \), to study log-concavity and to study the Hessian matrix are equivalent. More precisely, \( F \) is log-concave on the positive orthant if and only if the Hessian matrix \( H_F(a) \) of \( F \) has exactly one positive eigenvalue for \( a \) in the positive orthant. In \cite{9}, it was shown that the polynomials are strictly log-concave on the positive orthant, equivalently, the Hessian matrices have exactly one positive eigenvalue and are not degenerate. For an Artinian Gorenstein algebra associated to the polynomial, it was also shown that the strong Lefschetz property and Hodge–Riemann relation at degree one.

We introduce the homogeneous polynomial \( F_P \) for a regular polyhedron \( P \) as an analogue of above polynomials. For a regular polyhedron \( P \), we define a homogeneous polynomial by

\[ F_P = \sum_{F \in F(P)} \prod_{v \in F} x_v, \]

where \( F(P) \) is the collection of the facets, and \( v \) is a 0-dimensional face. Then we consider the Artinian Gorenstein algebra

\[ A_P = \mathbb{R}[\partial_v | v \in V(P)] / \text{Ann}(F_P), \]

where \( \partial_v \) is the partial derivative operator of \( x_v \), and \( V(P) \) is the collection of 0-dimensional faces. We discuss the strong Lefschetz property of \( A_P \) and Hodge–Riemann relation with respect to the Poincaré duality

\[ P^k_{F_P} : A_k \times A_{s-k} \to \mathbb{R}, \quad (f, g) \mapsto fgF_P. \]
In this paper, we consider the Platonic solids. And, we show the following, and study the strong Lefschetz elements.

**Theorem 1.1** (cf. Theorems 3.3, 3.5, 3.8, 3.11 and 3.15). The following hold for the Platonic solids:

- The Artinian Gorenstein algebras have the strong Lefschetz property.
- For the regular tetrahedron and octahedron, the algebra satisfy the Hodge–Riemann relation on the positive orthant. For the others, the algebras do not satisfy the Hodge–Riemann relation with respect to some strong Lefschetz elements.

This paper is organized as follows: In Section 2, we recall the strong Lefschetz property and Hodge–Riemann relation. Then we see the relation between the strong Lefschetz property and the Hessian matrices, and between the Hodge–Riemann relation and the Hessian matrices. In Section 3, we discuss the strong Lefschetz property and Hodge–Riemann relation for the Artinian Gorenstein algebras defined by the Platonic solids. In Section 4, we calculate the Hessian matrices of the homogeneous polynomials of the Platonic solids.

**Acknowledgment.** This work was supported by the Sasakawa Scientific Research Grant from The Japan Science Society.

2. **Strong Lefschetz property, Hodge–Riemann relation and Hessian matrices**

We recall some properties for a graded algebra, the strong Lefschetz property and Hodge–Riemann relation.

Let $A = \bigoplus_{k=0}^{s} A_k$, $A_s \neq 0$, be a graded Artinian algebra with a symmetric bilinear map $P^k$ from $A_k \times A_{s-k}$ to $\mathbb{R}$. For a graded algebra $A = \bigoplus_{k=0}^{s} A_k$, define $h_k$ by the dimension of $k$-th homogeneous component of $A$. We call the sequence $(h_0, h_1, \ldots, h_s)$ the **Hilbert series** of $A$.

We say that $A$ has the **strong Lefschetz property** if there exists an element $\ell \in A_1$ such that the linear map

\[ x \mapsto x^{s-2k} : A_k \to A_{s-k}, \quad f \mapsto \ell^{s-2k} f \]

is bijective for each nonnegative integer $k \leq \frac{s}{2}$. We call $\ell \in A_1$ with this property a **strong Lefschetz element**. If $A$ has the strong Lefschetz property, then the Hilbert series of $A$ is palindromic.

We say that $A$ satisfies the **Hodge–Riemann relation** with respect to $\ell \in A_1$ if the symmetric bilinear form

\[ Q^\ell_k : A_k \times A_k \to \mathbb{R}, \quad (f, g) \mapsto (-1)^k P^k(f, \ell^{s-2k} g) \]

is positive definite on the kernel $\times x^{s-2k+1} : A_k \to A_{s-k+1}$ for each nonnegative integer $k \leq \frac{s}{2}$.

The strong Lefschetz property and Hodge–Riemann relation are defined on a general graded algebra with a symmetric bilinear form, but we consider those properties on a graded Artinian Gorenstein algebra with the Poincaré duality in this paper.

Let $\partial_i = \frac{\partial}{\partial x_i}$ be the partial derivative operator of $x_i$. The polynomial ring $\mathbb{R}[\partial_1, \partial_2, \ldots, \partial_n]$ acts on $\mathbb{R}[x_1, x_2, \ldots, x_n]$ in the usual manner. For a homogeneous polynomial $F \in \mathbb{R}[x_1, x_2, \ldots, x_n]$, we define the annihilator $\text{Ann}(F)$ by

\[ \text{Ann}(F) = \{ f \in \mathbb{R}[\partial_1, \ldots, \partial_n] \mid f F = 0 \} . \]
Then $\text{Ann}(F)$ is a homogeneous ideal of $\mathbb{R}[\partial_1, \ldots, \partial_n]$. Let $A = \mathbb{R}[\partial_1, \ldots, \partial_n]/\text{Ann}(F)$. Since $\text{Ann}(F)$ is homogeneous, the algebra $A$ is graded. Furthermore $A$ is an Artinian Gorenstein algebra. Conversely, a graded Artinian Gorenstein algebra $A$ has the presentation

$$A = \mathbb{R}[\partial_1, \ldots, \partial_n]/\text{Ann}(F)$$

for some homogeneous polynomial $F \in \mathbb{R}[x_1, x_2, \ldots, x_n]$. The socle degree $s$ of $A$ is the degree of $F$. Thus the maps

$$P^k_F : A_k \times A_{s-k} \rightarrow \mathbb{R}, \quad (f, g) \mapsto fgF$$

are bilinear maps. We call $P_F = \bigoplus_k P^k_F$ the Poincaré duality of $A$. The Hilbert series is palindromic for every graded Artinian Gorenstein algebra.

Let $A = \mathbb{R}[\partial_1, \ldots, \partial_n]/\text{Ann}(F) = \bigoplus_{k=0}^s A_k$ be a graded Artinian Gorenstein algebra, and $\Lambda_k$ the basis for $A_k$. We define the matrix $H^k_F$ by

$$H^k_F = (e_ie_jF)_{e_i, e_j \in \Lambda_k}.$$ 

The matrix $H^k_F$ is called the $k$-th Hessian matrix and $\det H^k_F$ is called the $k$th Hessian of $F$ with respect to the basis $\Lambda_k$. We define the 0th Hessian of $F$ to be $F$. If $\Lambda_1 = \{ \partial_1, \partial_2, \ldots, \partial_n \}$, then $H^1_F$ coincides with the usual Hessian matrix of $F$.

**Theorem 2.1** (Watanabe [12], Maeno–Watanabe [8]). Let $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$, and $\ell_a = a_1\partial_1 + a_2\partial_2 + \cdots + a_n\partial_n$. The multiplication map $\times \ell_a^{s-k} : A_k \rightarrow A_{s-k}$ is bijective if and only if $\det H^k_F(a) \neq 0$.

**Remark 2.2.** By Theorem 2.1 a strong Lefschetz element comes from an open dense space where the determinants do not vanish. Thus, if the $k$-th Hessian does not vanish as a polynomial for all $k$, then the Artinian Gorenstein algebra $A$ has the strong Lefschetz property.

**Proposition 2.3.** Assume that $F(a) > 0$ for $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$. Let $\ell_a = a_1\partial_1 + a_2\partial_2 + \cdots + a_n\partial_n$. The algebra $A$ satisfies the condition (2) when $k = 1$ with respect to $\ell_a$ if and only if the first Hessian matrix $H^1_F(a)$ has $n - 1$ negative eigenvalues and one positive eigenvalue.

**Proof.** Since the map

$$\times \ell_a^s : A_0 \xrightarrow{\times \ell_a} A_1 \xrightarrow{\times \ell_a^{-1}} A_s$$

is bijective, we have $A_1 = \mathbb{R}\ell_a \oplus \text{Ker}(\times \ell_a^{-1})$. And we have

$$Q^1_{\ell_a}(\ell_a, \ell_a) = -P^1_F(\ell_a, \ell_a^{s-2}\ell_a)$$

$$= -P^1_F(\ell_a, \ell_a^{s-1})$$

$$= -\ell_a^sF$$

$$= -s!F(a) < 0.$$ 

Hence, the algebra $A$ satisfies the condition (2) when $k = 1$ with respect to $\ell_a$ if and only if $Q^1_{\ell_a}$ has $n - 1$ positive eigenvalues and one negative eigenvalue.
Furthermore, the representing matrix associated to $Q^1_{\ell_a}$ with respect to a basis $\Lambda_1$ for $A_1$ is given by the first Hessian matrix $-H^1_F(a)$. In fact, for $e_i, e_j \in \Lambda_1$,

$$Q^1_{\ell_a}(e_i, e_j) = -P^1_F(e_i, \ell^{a-2}_ae_j) = -\ell^{a-2}_ae_ie_jF$$

$$= -(a_1\partial_1 + a_2\partial_2 + \cdots + a_n\partial_n)s^{-2}e_ie_jF$$

$$= -(s-2)!(e_ie_jF)(a)$$

$$= -(s-2)! \left( H^1_F(a) \right)_{i,j}.$$

\[ \square \]

3. Main results

In this section, we discuss the strong Lefschetz property and Hodge–Riemann relation for the Artinian Gorenstein algebras defined by the regular polyhedra.

For a polyhedron $P$, we call a 0-dimensional face a vertex of $P$, an 1-dimensional face an edge of $P$, and a 2-dimensional face, simply, a face of $P$. Let $V(P)$, $E(P)$ and $F(P)$ denote the collection of vertices, edges and faces of $P$, respectively. We focus a combinatorial data, the face poset of $P$. The face poset of $P$ is the set $V(P) \cup E(P) \cup F(P)$ ordered by inclusion. We regard edges and faces of $P$ as subsets of $V(P)$.

In this paper, we are only interested in the regular polyhedra. For a regular polyhedron $P$, we define a homogeneous polynomial by

$$F_P = \sum_{F \in F(P)} \prod_{v \in F} x_v.$$  

The number of vertices appears as the number of variables, and the number of faces appears as the numbers of terms. The shape of faces appears as the degree. For example, if faces of $P$ are $d$-gons, then $F_P$ is a homogeneous of degree $d$.

For a polyhedron $P$, $A_P = \bigoplus_{k=0} F^k_{F_P}$ denotes the Artinian Gorenstein algebra defined by the homogeneous polynomial $F_P$. We call $A_P$ the algebra associated to $P$. For the algebra $A_P$, we consider the Poincaré duality $P_{F_P} = \bigoplus_k P^k_{F_P}$, where

$$P^k_{F_P} : A_k \times A_{s-k} \to \mathbb{R}, \quad (f, g) \mapsto fgF_P.$$  

We see the strong Lefschetz property of $A_P$ and Hodge–Riemann relation with respect to the Poincaré duality $P_{F_P} = \bigoplus_k P^k_{F_P}$ of $A_P$.

Remark 3.1. As mentioned in Section 2, if $\{ \partial_i \}_{i \in V(P)}$ is a basis for $A_1$, then the first Hessian matrix $H^1_{F_P}$ coincides with the usual Hessian matrix of $F_P$. Unless noted, we calculate $H^1_{F_P}$ as the Hessian matrix of $F_P$. In other words, we calculate $H^1_{F_P}$ with respect to $\{ \partial_i \}_{i \in V(P)} \subset A_1$.

3.1. Regular tetrahedron. Let us consider the regular tetrahedron $P$. The regular tetrahedron has 4 vertices, 6 edges and 4 faces. We assign the number 1, 2, 3, 4 to the vertices. Let

$$F(P) = \{ \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\} \}.$$  

See Figure [4] Then
The homogeneous polynomial \( F_P \) is equal to the elementary symmetric polynomial \( e_3(x_1, x_2, x_3, x_4) \) of degree 3 in 4 variables. In [7], Maeno and Numata show that the algebra defined by the annihilator of the elementary symmetric polynomial \( e_d(x_1, x_2, \ldots, x_n) \) of degree \( d \) in \( n \) variables satisfies the strong Lefschetz property for every \( d \) and \( n \).

Furthermore, \( F_P \) is equal to the Kirchhoff polynomial of the cycle graph \( C_4 \) with 4 vertices. In [10], Nagaoka and Yazawa show that the algebra defined by a Kirchhoff polynomial, a homogeneous polynomial defined by a graph, satisfies the strong Lefschetz property and Hodge–Riemann relation “at degree one” (it will be explained in Remark 3.2) on the positive orthant. More generally, in [9], Murai, Nagaoka and Yazawa show that the algebra defined by the basis generating polynomial, a generalization of a Kirchhoff polynomial, satisfies the strong Lefschetz property and Hodge–Riemann relation at degree one on the positive orthant.

**Remark 3.2.** Let \( A = \bigoplus_{k=0}^r A_k \) be a graded algebra with a symmetric bilinear map \( F^k \) from \( A_k \times A_{r-k} \) to \( R \). We say that \( A \) satisfies the strong Lefschetz property at degree one if the condition [1] holds when \( k = 1 \), and that \( A \) satisfies the Hodge–Riemann relation with respect to \( \ell \in A_1 \) if the condition [2] holds when \( k = 1 \).

To summarize of this section, we obtain the following.

**Theorem 3.3.** The algebra associated to the regular tetrahedron satisfies the strong Lefschetz property and Hodge–Riemann relation on the positive orthant.

### 3.2. Regular hexahedron.

Let us consider the regular hexahedron \( P \). The regular hexahedron has 8 vertices, 12 edges and 6 faces. We assign the number 1, 2, 3, 4 to the vertices. Let

\[
F(P) = \{ \{ 1, 2, 3, 4 \}, \{ 2, 3, 6, 7 \}, \{ 3, 4, 5, 8 \}, \{ 1, 4, 5, 8 \}, \{ 1, 2, 5, 6 \}, \{ 5, 6, 7, 8 \} \}.
\]

See Figure 2 Then

\[
F_P = x_1x_2x_3x_4 + x_2x_3x_6x_7 + x_3x_4x_7x_8 + x_1x_4x_5x_8 + x_1x_2x_5x_6 + x_5x_6x_7x_8.
\]
Proposition 3.4. The eigenvalues of the matrix $H_{P_{\mathcal{P}}}(1,1,...,1)$ are 9, 1, −3 with the dimensions of the eigenspaces 1, 3, 4, respectively.

The matrix $H_{P_{\mathcal{P}}}(1,1,...,1)$ is in Section 4. Also the proof of Proposition 3.4 is in Section 4.

From Proposition 3.4, Theorem 2.1 and Proposition 2.3, we obtain the following.

Theorem 3.5. The algebra associated to the regular hexahedron satisfies the strong Lefschetz property with a strong Lefschetz element $\sum_{i=1}^{8} \partial_{i}$, but does not satisfy the Hodge–Riemann relation with respect to $\sum_{i=1}^{8} \partial_{i}$.

Remark 3.6. If $\{ v, v' \}$ does not a subset of a face of $\mathcal{P}$, then $\partial_{v} \partial_{v'}$ is in Ann($F_{P}$). In this case,

$$\partial_{1} \partial_{7}, \partial_{2} \partial_{8}, \partial_{3} \partial_{5}, \partial_{4} \partial_{6} \in \text{Ann}(F_{P}).$$

Further, monomials of degree 2 of $A_{\mathcal{P}}$ have the following relations:

$$\partial_{1} \partial_{7} = \partial_{2} \partial_{8}, \quad \partial_{1} \partial_{4} = \partial_{6} \partial_{7}, \quad \partial_{1} \partial_{5} = \partial_{3} \partial_{4}, \quad \partial_{2} \partial_{3} = \partial_{5} \partial_{8}, \quad \partial_{2} \partial_{6} = \partial_{4} \partial_{8}, \quad \partial_{3} \partial_{4} = \partial_{5} \partial_{6}.$$ (3)

The other monic monomials of degree two and monomials in left hand sides of the equations (3) generate $A_{2}$. Furthermore, since the second Hessian with respect to them does not vanish, they form basis for $A_{2}$. Thus the Hilbert series of the algebra associated to $\mathcal{P}$ is $(1,8,18,8,1)$. The second Hessian matrix $H_{P_{\mathcal{P}}}^{2}$ is in Section 4.

3.3. **Regular octahedron.** Let us consider the regular octahedron $\mathcal{P}$. The regular octahedron has 6 vertices, 12 edges and 8 faces. We assign the number 1, 2, ..., 6 to the vertices. Let

$$F(\mathcal{P}) = \{ \{ 1, 2, 3 \}, \{ 1, 3, 4 \}, \{ 1, 4, 5 \}, \{ 1, 2, 5 \}, \{ 2, 3, 6 \}, \{ 3, 4, 6 \}, \{ 4, 5, 6 \}, \{ 2, 5, 6 \} \}.$$ 

See Figure 3. Then

$$F_{P} = x_{1}x_{2}x_{3} + x_{1}x_{3}x_{4} + x_{1}x_{4}x_{5} + x_{1}x_{2}x_{5} + x_{2}x_{3}x_{6} + x_{3}x_{4}x_{6} + x_{4}x_{5}x_{6} + x_{2}x_{3}x_{6}.$$ 

The algebra $A_{\mathcal{P}}$ has 6 variables.

We can reduce the number of variables.
Proposition 3.7. For the regular octahedron $P$, we have

$$A_P \cong \mathbb{R}[t_1, t_2, t_3]/(t_1^2, t_2^2, t_3^2).$$

Proof. Let

$$\Phi : \mathbb{R}[\partial_1, \partial_2, \ldots, \partial_6] \to \mathbb{R}[t_1, t_2, t_3]/(t_1^2, t_2^2, t_3^2)$$

$$\partial_1, \partial_6 \mapsto t_1,$$

$$\partial_2, \partial_4 \mapsto t_2,$$

$$\partial_3, \partial_5 \mapsto t_3.$$

We calculate Ker($\Phi$). In our situation, we have a way to calculate the kernel: For sets $X = \{ x_1, x_2, \ldots, x_n \}$, $Y = \{ y_1, y_2, \ldots, y_m \}$ of variables, and a subset $D$ of the polynomial ring $\mathbb{R}[Y]$, consider

$$\varphi : \mathbb{R}[X] \to \mathbb{R}[Y]/\langle D \rangle,$$

$$G = \{ x_i - \varphi(x_i) \mid i \in \{1, 2, \ldots, n\} \} \cup D.$$

Then we have Ker($\varphi$) = $\langle G \rangle_{\mathbb{R}[X,Y]} \cap \mathbb{R}[X]$. For $\Phi$, the map defined above, we consider $D = \{ t_1^2, t_2^2, t_3^2 \}$. Then we have

$$G = \{ \partial_1 - t_1, \partial_6 - t_1, \partial_2 - t_1, \partial_4 - t_2, \partial_3 - t_3, \partial_5 - t_3, t_1^2, t_2^2, t_3^2 \}.$$

A Gröbner basis for $\langle G \rangle_{\mathbb{R}[X,Y]}$ with respect to the lexicographical order with $t_4 > t_2 > t_1 > \partial_6 > \partial_5 > \partial_4 > \partial_3 > \partial_2 > \partial_1$ is

$$\{ \partial_1 - t_1, \partial_6 - t_1, \partial_2 - t_2, \partial_3 - t_3, \partial_4 - \partial_6, \partial_2 - \partial_4, \partial_3 - \partial_5, \partial_1^2, \partial_2^2, \partial_3^2 \}.$$

Hence, Ker($\Phi$) = $\langle \partial_1 - \partial_6, \partial_2 - \partial_4, \partial_3 - \partial_5, \partial_1^2, \partial_2^2, \partial_3^2 \rangle_{\mathbb{R}[X]}$.

The Hilbert series of $A_P = \mathbb{R}[\partial_1, \partial_2, \ldots, \partial_6]/\text{Ann}(F_P)$ is $(1, 3, 3, 1)$ since the degree of $F_P$ is 3, and the first Hessian $H_{F_P}^1$ with respect to the $\{ \partial_1, \partial_2, \partial_3 \}$ does not vanish.
It is obvious that $\text{Ker}(\varphi) \subset \text{Ann}(F)$, and the Hilbert series of $\mathbb{R}[\partial_1, \partial_2, \ldots, \partial_n]/\text{Ker}(\varphi)$ is $(1, 3, 3, 1)$, so we obtain

$$A_P \cong \mathbb{R}[\partial_1, \partial_2, \ldots, \partial_n]/\text{Ker}(\varphi) \cong \mathbb{R}[t_1, t_2, t_3]/(t_1^2, t_2^2, t_3^2).$$

In general, the algebra $\mathbb{R}[\partial_1, \partial_2, \ldots, \partial_n]/(\partial_1^{a_1+1}, \partial_2^{a_2+1}, \ldots, \partial_n^{a_n+1})$, where $a_i \geq 0$ for all $i$ satisfies the strong Lefschetz property with a strong Lefschetz element $\sum_{i=1}^n \partial_i$ (Theorem 3.8). If $F = x_1^{a_1}x_2^{a_2} \cdots x_n^{a_n}$, then we have $A_F = \mathbb{R}[\partial_1, \partial_2, \ldots, \partial_n]/(\partial_1^{a_1+1}, \partial_2^{a_2+1}, \ldots, \partial_n^{a_n+1})$. In [10] and [9], when $a_1 = a_2 = \cdots = a_n = 1$, namely $F = x_1x_2 \cdots x_n$, they show that the algebra $\mathbb{R}[\partial_1, \partial_2, \ldots, \partial_n]/(\partial_1^2, \partial_2^2, \ldots, \partial_n^2)$ satisfies the strong Lefschetz property and Hodge–Riemann relation at degree one on the positive orthant. In fact, the homogeneous polynomial $F = x_1x_2 \cdots x_n$ coincides with the Kirchhoff polynomial a tree graph with $n + 1$ vertices.

To summarize of this section, we obtain the following.

**Theorem 3.8.** The algebra associated to the regular octahedron satisfies the strong Lefschetz property and Hodge–Riemann relation on the positive orthant.

### 3.4. Regular dodecahedron

Let us consider the regular dodecahedron $\mathcal{P}$. The regular dodecahedron has 20 vertices, 30 edges and 12 faces. We assign the number 1, 2, 3, ... to the vertices. Let

$$F(\mathcal{P}) = \{ 1, 2, 3, 4, 5 \}, \{ 1, 2, 6, 7, 19 \}, \{ 2, 3, 7, 8, 20 \}, \{ 3, 4, 8, 9, 16 \}, \{ 4, 5, 9, 10, 17 \}, \{ 1, 5, 6, 10, 18 \}, \{ 11, 12, 13, 14, 15 \}, \{ 11, 12, 16, 17, 9 \}, \{ 12, 13, 17, 18, 10 \}, \{ 13, 14, 18, 19, 6 \}, \{ 14, 15, 19, 20, 7 \}, \{ 11, 15, 16, 20, 8 \}.$$  

See Figure 4. Then

$$F_\mathcal{P} = x_1x_2x_3x_4x_5 + x_1x_2x_6x_7x_9 + x_2x_3x_7x_8x_9 + x_3x_4x_8x_9x_{16}$$

$$+ x_4x_5x_9x_{10}x_{17} + x_1x_5x_6x_{10}x_{18} + x_{11}x_{12}x_{13}x_{14}x_{15} + x_{11}x_{12}x_{16}x_{17}x_{9}$$

$$+ x_{12}x_{13}x_{17}x_{18}x_{10} + x_{13}x_{14}x_{18}x_{19}x_9 + x_{14}x_{15}x_{19}x_{20}x_7 + x_{11}x_{15}x_{16}x_{20}x_8.$$

The socle degree of $A_P$ is 5, in other words, $A_P = \bigoplus_{k=0}^5 A_k$. We calculate the first and second Hessian matrices of $F_\mathcal{P}$.
Proposition 3.9. Let $a = (a_i) \in \mathbb{R}^{20}$ with $a_i = 1$ for $i \notin \{6, 7, 8, 9, 10\}$ and $a_i = 0$ for $i \in \{6, 7, 8, 9, 10\}$. The matrix $H^1_{F_{\mathcal{P}}}(a)$ is non-degenerate. Moreover, the number of the positive eigenvalues is more than two.

Proposition 3.10. Let $a = (a_i) \in \mathbb{R}^{20}$ with $a_1 = 0$ and $a_i = 1$ for $i \neq 1$. The matrix $H^2_{F_{\mathcal{P}}}(a)$ with some basis for $A_2$ is not degenerate.

The matrices $H^1_{F_{\mathcal{P}}}(a)$ and $H^2_{F_{\mathcal{P}}}(a)$ are in Section 4. Also the proof of Propositions 3.9 and 3.10 are in Section 4. To show Proposition 3.10 we use CoCalc III.

To summarize of this section, we obtain the following.

Theorem 3.11. The algebra associated to the regular dodecahedron satisfies the strong Lefschetz property, but does not satisfy the Hodge–Riemann relation with respect to $\sum_{i \neq 6, 7, 8, 9, 10} \partial_i$.

Remark 3.12. Let $a = (a_i) \in \mathbb{R}^{20}$ with $a_i = 1$ for all $i$; $b = (b_i) \in \mathbb{R}^{20}$ with $b_i = 1$ for $i \notin \{6, 7, 8, 9, 10\}$ and $b_i = 0$ for $i \in \{6, 7, 8, 9, 10\}$; and $c = (c_i) \in \mathbb{R}^{20}$ with $c_1 = 0$ and $c_i = 1$ for $i \neq 1$. We have

- $\det H^1_{F_{\mathcal{P}}}(a) = 0$ and $\det H^2_{F_{\mathcal{P}}}(a) = 0$,
- $\det H^1_{F_{\mathcal{P}}}(b) \neq 0$ and $\det H^2_{F_{\mathcal{P}}}(b) = 0$,
- $\det H^1_{F_{\mathcal{P}}}(c) \neq 0$ and $\det H^2_{F_{\mathcal{P}}}(c) \neq 0$.

Hence, $\ell_a = \sum_{i=1}^{20} a_i \partial_i$ and $\ell_b = \sum_{i=1}^{20} b_i \partial_i$ are not strong Lefschetz elements. The linear element $\ell_c = \sum_{i=1}^{20} c_i \partial_i$ is a strong Lefschetz element. We use CoCalc for calculation the determinant $\det H^1_{F_{\mathcal{P}}}(c)$.

Remark 3.13. The Hilbert series of $A_{\mathcal{P}}$ is $(1, 20, 90, 20, 1)$.

3.5. Regular icosahedron. Let us consider the regular icosahedron $\mathcal{P}$. The regular icosahedron has 12 vertices, 30 edges and 20 faces. We assign the number 1, 2, \ldots, 12 to the vertices. Let

$$F(\mathcal{P}) = \left\{ 1, 2, 3 \right\}, \left\{ 1, 3, 4 \right\}, \left\{ 1, 4, 5 \right\}, \left\{ 1, 5, 6 \right\}, \left\{ 1, 2, 6 \right\}, \left\{ 2, 3, 8 \right\}, \left\{ 3, 8, 9 \right\}, \left\{ 2, 6, 7 \right\}, \left\{ 2, 7, 8 \right\}, \left\{ 10, 11, 12 \right\}, \left\{ 4, 5, 10 \right\}, \left\{ 4, 9, 10 \right\}, \left\{ 5, 6, 11 \right\}, \left\{ 5, 10, 11 \right\}, \left\{ 9, 10, 12 \right\}, \left\{ 8, 9, 12 \right\}, \left\{ 6, 7, 11 \right\}, \left\{ 7, 11, 12 \right\}, \left\{ 7, 8, 12 \right\}.$$

See Figure 5 Then
\[ F_p = x_1x_2x_3 + x_1x_3x_4 + x_1x_4x_5 + x_3x_4x_9 + x_1x_5x_6 + x_1x_2x_6 + x_2x_3x_8 + x_3x_5x_9 + x_2x_6x_7 + x_2x_7x_8 + x_{10}x_{11}x_{12} + x_4x_5x_{10} + x_4x_9x_{10} + x_5x_6x_{11} + x_5x_{10}x_{11} + x_9x_{10}x_{12} + x_6x_7x_{11} + x_7x_{11}x_{12} + x_7x_8x_{12}. \]

**Proposition 3.14.** The eigenvalues of the matrix \( H^1_{F_p}(1,1,\ldots,1) \) of the homogeneous polynomial \( F_p \) are \( 10, -2, 2\sqrt{5}, -2\sqrt{5} \) with the dimensions of the eigenspaces 1,5,3,3, respectively.

The matrix \( H^1_{F_p}(1,1,\ldots,1) \) is in Section 4. Also the proof of Proposition 3.14 is in Section 4.

To summarize of this section, we obtain the following.

**Theorem 3.15.** The algebra associated to the regular icosahedron satisfies the strong Lefschetz property with a strong Lefschetz element \( \sum_{i=1}^{20} \partial_i \), but does not satisfy the Hodge–Riemann relation with respect to \( \sum_{i=1}^{20} \partial_i \).

### 4. Hessian matrices

Here, we calculate the Hessian matrices of the homogeneous polynomials of the regular polyhedra.

**4.1. Regular tetrahedron.** Let \( P \) be the regular tetrahedron. The matrix \( H^1_{F_p}(1,1,1,1) \) is

\[
\begin{pmatrix}
0 & 2 & 2 & 2 \\
2 & 0 & 2 & 2 \\
2 & 2 & 0 & 2 \\
2 & 2 & 2 & 0
\end{pmatrix}.
\]

The eigenvalues of the matrix are 6, \(-2, -2, -2\).

**4.2. Regular hexahedron.** Let \( P \) be the regular hexahedron. The matrix \( H^1_{F_p}(1,1,\ldots,1) \) is the following, and has the following block decomposition:

\[
\begin{pmatrix}
0 & 2 & 1 & 2 & 2 & 1 & 0 & 1 \\
2 & 0 & 2 & 1 & 1 & 2 & 1 & 0 \\
1 & 2 & 0 & 2 & 0 & 1 & 2 & 1 \\
2 & 1 & 2 & 0 & 1 & 0 & 1 & 2 \\
2 & 1 & 0 & 1 & 0 & 2 & 1 & 2 \\
1 & 2 & 1 & 0 & 2 & 0 & 2 & 1 \\
0 & 1 & 2 & 1 & 1 & 2 & 0 & 2 \\
1 & 0 & 1 & 2 & 2 & 1 & 2 & 0
\end{pmatrix}.
\]

By Section 2, the eigenvalues of the matrix come from the eigenvalues of the following four matrices \( M_0, M_1, M_2, M_3 \):

\[
M_0 = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}, \quad
M_1 = M_3 = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}, \quad
M_2 = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}.
\]

The eigenvalues of \( M_0 \) are 9, 1. The eigenvalues of \( M_1 = M_3 \) are 1, -3. The eigenvalues of \( M_2 \) are -3, -3. Hence we have Proposition 3.4.
By Section 3, we know that

\[ \Lambda_2 = \{ \partial_1 \partial_3, \partial_2 \partial_4, \partial_1 \partial_5, \partial_2 \partial_6, \partial_1 \partial_6, \partial_2 \partial_7, \partial_3 \partial_8, \partial_6 \partial_8, \partial_5 \partial_7, \partial_1 \partial_4, \partial_4 \partial_2, \partial_4 \partial_5, \partial_2 \partial_3, \partial_3 \partial_4 \} \]

generates \( A_2 \). Let us calculate the second Hessian matrix with respect to \( \Lambda_2 \) is

\[
\begin{pmatrix}
A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & A & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & A & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & A & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & A & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A
\end{pmatrix},
\]

where

\[ A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \]

The eigenvalues are 1 and \(-1\). The multiplicity of both are 9. Since the second Hessian with respect to \( \Lambda_2 \) does not vanish, \( \Lambda_2 \) is basis for \( A_2 \).

4.3. Regular octahedron. Let \( \mathcal{P} \) be the regular octahedron. By Proposition 3.7, the algebra \( A_\mathcal{P} \) is isomorphic to the graded Artinian Gorenstein algebra \( R[\partial_1, \partial_2, \partial_3]/(\partial_1^2, \partial_2^2, \partial_3^2) \). If \( F = x_1x_2x_3 \), then \( R[\partial_1, \partial_2, \partial_3]/(\partial_1^2, \partial_2^2, \partial_3^2) = R[\partial_1, \partial_2, \partial_3]/\text{Ann}(F) \).

The first Hessian matrix of \( F \) evaluated 1 is the following.

\[
\begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix}
\]

The eigenvalues of the matrix are 2, \(-1\), \(-1\).

4.4. Regular dodecahedron. Let \( \mathcal{P} \) be the regular dodecahedron. The socle degree of \( A_\mathcal{P} \) is 5, in other words, \( A_\mathcal{P} = \bigoplus_{k=0}^5 A_k \). We calculate the first and second Hessian matrices of \( F_\mathcal{P} \).

At first, we consider the first Hessian. The matrix \( H^1_{F_\mathcal{P}}(1,1,\ldots,1) \) is degenerate. In fact, if we set

\[
A = \begin{pmatrix}
0 & 2 & 1 & 1 & 2 \\
2 & 0 & 2 & 1 & 1 \\
1 & 2 & 0 & 2 & 1 \\
1 & 1 & 2 & 0 & 2 \\
2 & 1 & 1 & 2 & 0
\end{pmatrix}, \quad B = \begin{pmatrix}
2 & 1 & 0 & 0 & 1 \\
1 & 2 & 1 & 0 & 0 \\
0 & 1 & 2 & 1 & 0 \\
0 & 0 & 1 & 2 & 1 \\
1 & 0 & 0 & 1 & 2
\end{pmatrix}, \quad C = \begin{pmatrix}
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0
\end{pmatrix},
\]

\[
D = \begin{pmatrix}
0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0
\end{pmatrix}, \quad 0 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad A' = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 0
\end{pmatrix}.
\]
then the matrix $H_{F_P}^1(1,1,\ldots,1)$ is

$$
\begin{pmatrix}
A & B & 0 & C \\
B & D & C & 2C \\
0 & C & A & B \\
C & 2C & B & D
\end{pmatrix}.
$$

One can check that the matrix $H_{F_P}^1(1,1,\ldots,1)$ is degenerate. By Theorem 2.1, the linear element $\sum_{i=1}^{20} \partial_i$ is not a strong Lefschetz element of $A_P$. Instead of the matrix, we consider another matrix $H'$ evaluated the following:

$$
x_i = 1 \text{ for } i \not\in \{6,7,8,9,10\}, \\
x_i = 0 \text{ for } i \in \{6,7,8,9,10\}.
$$

The matrix $H'$ is the following, and has the following block decomposition

$$
\begin{pmatrix}
A' & 0 & 0 & 0 \\
0 & D & C & C \\
0 & C & A' & 0 \\
0 & C & 0 & 0
\end{pmatrix}.
$$

The eigenvalues of $A'$ is 4, $-1$. The multiplicities are 1 and 3, respectively. The matrix

$$
\begin{pmatrix}
D & C & C \\
C & A' & 0 \\
C & 0 & 0
\end{pmatrix}
$$

is non-degenerate since $A'$ and $C$ are non-degenerate. Moreover, since the trace is zero, the matrix has at least one positive eigenvalue. Therefore, $H'$ has at least two positive eigenvalues.

Next, we consider the second Hessian. The second Hessian matrix of $F_P$ is too large to calculate by hands, so we use CoCalc [11] to calculation.

```python
xx=[var("x%d" % i) for i in range(20)]
Facet=[
[0, 1, 2, 3, 4],
[0, 1, 5, 6, 18],
[1, 2, 6, 7, 19],
[2, 3, 7, 8, 15],
[3, 4, 8, 9, 16],
[0, 4, 5, 9, 17],
[10, 11, 12, 13, 14],
[8, 10, 11, 15, 16],
[9, 11, 12, 16, 17],
[5, 12, 13, 17, 18],
[6, 13, 14, 18, 19],
[7, 10, 14, 15, 19]
]
F=sum(prod(xx[i] for i in f) for f in Facet)
```

A2=Combinations(xx,2).list()
Basis=[]
for i in A2:
    if (F.derivative(i[0])).derivative(i[1])==0:
        pass
    else:
        Basis=Basis+[i]

ss={}
for xi in xx:
    ss[xi]=1
    ss[x0]=0

h1=[F.derivative(i[0]).derivative(i[1]) for i in Basis]
h=[[hi.derivative(j[0]).derivative(j[1]) for j in Basis] for hi in h1]
H=Matrix([[ZZ(hij.substitute(ss)) for hij in hi] for hi in h])
det(H)

By this program, you obtain \( \det H_{F_{P}}^{2}(\alpha) = 342456532992 \), where \( \alpha = (a_{i}) \in \mathbb{R}^{20} \) with \( a_{1} = 0 \) and \( a_{i} = 1 \) for \( i \neq 1 \).

4.5. Regular icosahedron. The first Hessian matrix \( H_{F_{P}}^{1}(1,1,...,1) \) of the homogeneous polynomial of the regular icosahedron is the following, and has the following block decomposition:

\[
\begin{pmatrix}
0 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 0 \\
2 & 0 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 \\
0 & 2 & 0 & 2 & 0 & 0 & 2 & 2 & 0 & 2 \\
0 & 0 & 2 & 0 & 2 & 0 & 0 & 2 & 2 & 2 \\
2 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 2 & 0 \\
2 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 \\
0 & 2 & 2 & 0 & 0 & 2 & 0 & 2 & 0 & 0 \\
0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 2 \\
0 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 \\
2 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 2 & 2 \\
\end{pmatrix}
\]

By [13] Section 2], the eigenvalues of the matrix are come from the eigenvalues of the following five matrices \( M_{0}, M_{1}, M_{2}, M_{3}, M_{4} \): Let \( \zeta_{5} \) be the primitive 5th root of unity and \( k \in \{1, 2, 3, 4 \} \). Then we define the matrices \( M_{0}, M_{1}, M_{2}, M_{3}, M_{4} \) by

\[
M_{0} = \begin{pmatrix}
4 & 4 & 2 & 0 \\
4 & 4 & 0 & 2 \\
10 & 0 & 0 & 0 \\
0 & 10 & 0 & 0 \\
\end{pmatrix}
\]

\[
M_{k} = \begin{pmatrix}
2(\zeta_{5}^{k} + \zeta_{5}^{-k}) & 2(1 + \zeta_{5}^{-k}) & 2 & 0 \\
2(1 + \zeta_{5}^{k}) & 2(\zeta_{5}^{k} + \zeta_{5}^{-k}) & 0 & 2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
Note that $M_1 = M_4$ and $M_2 = M_3$. Moreover, the eigenvalues of $M_0$ come from the eigenvalues of the following two matrices $M'_0, M''_0$, and the eigenvalues of $M_k$ come from some eigenvalues of the following two matrices $M'_k, M''_k$:

$$
M'_0 = \begin{pmatrix} 8 & 2 \\ 10 & 0 \end{pmatrix}, \quad M''_0 = \begin{pmatrix} 0 & 2 \\ 10 & 0 \end{pmatrix},
$$

$$
M'_k = \begin{pmatrix} 2(\zeta^k_5 + \zeta^{-k}_5) & 2(1 + \zeta^{-k}_5) \\ 2(1 + \zeta^k_5) & 2(\zeta^k_5 + \zeta^{-k}_5) \end{pmatrix}, \quad M''_k = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
$$

For $M'_0$, the vectors

$$
\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -5 \end{pmatrix}
$$

are the eigenvectors associated to the eigenvalues $10$ and $-2$, respectively. For $M''_0$, the vectors

$$
\begin{pmatrix} 1 \\ \sqrt{5} \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -\sqrt{5} \end{pmatrix}
$$

are the eigenvectors associated to the eigenvalues $2\sqrt{5}$ and $-2\sqrt{5}$, respectively. For $M'_k$, the vectors

$$
\begin{pmatrix} 2 + \zeta^k_5 + \zeta^{-k}_5 \\ -(1 + \zeta^k_5) \end{pmatrix}, \quad \begin{pmatrix} 1 + \zeta^{-k}_5 \\ -(\zeta^2_5 + \zeta^{-2}_5) \end{pmatrix}
$$

are eigenvectors associated to the eigenvalues $-2$ and $2\left((\zeta^k_5 + \zeta^{-k}_5) - (\zeta^{2k}_5 + \zeta^{-2k}_5)\right)$, respectively. For $k \in \{1, 2, 3, 4\}$, we have

$$
((\zeta^k_5 + \zeta^{-k}_5) - (\zeta^{2k}_5 + \zeta^{-2k}_5))^2 = 5.
$$

If $k = 1, 4$, then

$$
(\zeta^k_5 + \zeta^{-k}_5) - (\zeta^{2k}_5 + \zeta^{-2k}_5) > 0.
$$

If $k = 2, 3$, then

$$
(\zeta^k_5 + \zeta^{-k}_5) - (\zeta^{2k}_5 + \zeta^{-2k}_5) < 0.
$$

Hence the eigenvalues of $M_1 = M_4$ are $-2, 2\sqrt{5}$, and the eigenvalues of $M_2 = M_3$ are $-2, -2\sqrt{5}$. For $k \in \{0, 1, 2, 3, 4\}$, define

$$
z_k = \begin{pmatrix} 1 \\ \zeta^k_5 \\ \zeta^{2k}_5 \\ \zeta^{3k}_5 \\ \zeta^{4k}_5 \end{pmatrix}.
$$

Then, by [13 Lemma 2.1], the vector

$$
\begin{pmatrix} z_0 \\ z_0 \\ 1 \\ 1 \end{pmatrix}
$$
is a eigenvector of $H^1_{P'}(1,1,\ldots,1)$ associated to the eigenvalue 10. The vectors
\[
\begin{pmatrix}
  z_0 \\
  -z_0 \\
  5
\end{pmatrix}, \quad
\begin{pmatrix}
  (2 + \zeta_5 + \zeta_5^{-1}) z_1 \\
  -(1 + \zeta_5) z_1 \\
  0
\end{pmatrix}, \quad
\begin{pmatrix}
  (2 + \zeta_5^2 + \zeta_5^{-2}) z_2 \\
  -(1 + \zeta_5^2) z_2 \\
  0
\end{pmatrix},
\]
are the eigenvectors of $H^1_{P'}(1,1,\ldots,1)$ associated to the eigenvalue $-2$. The vectors
\[
\begin{pmatrix}
  z_0 \\
  -z_0 \\
  \sqrt{5}
\end{pmatrix}, \quad
\begin{pmatrix}
  (1 + \zeta_5) z_1 \\
  -(\zeta_5^2 + \zeta_5^{-2}) z_1 \\
  0
\end{pmatrix}, \quad
\begin{pmatrix}
  (1 + \zeta_5) z_4 \\
  -(\zeta_5^2 + \zeta_5^{-2}) z_4 \\
  0
\end{pmatrix},
\]
are the eigenvectors of $H^1_{P'}(1,1,\ldots,1)$ associated to the eigenvalue $2\sqrt{5}$. The vectors
\[
\begin{pmatrix}
  z_0 \\
  -z_0 \\
  \sqrt{5}
\end{pmatrix}, \quad
\begin{pmatrix}
  (1 + \zeta_5^2) z_2 \\
  -(\zeta_5^{-1} + \zeta_5) z_2 \\
  0
\end{pmatrix}, \quad
\begin{pmatrix}
  (1 + \zeta_5^2) z_3 \\
  -(\zeta_5 + \zeta_5^{-1}) z_3 \\
  0
\end{pmatrix},
\]
are the eigenvectors of $H^1_{P'}(1,1,\ldots,1)$ associated to the eigenvalues $-2\sqrt{5}$.

Finally, we obtain Proposition 3.14.

REFERENCES

[1] Nima Anari, Kuikui Liu, Shayan Oveis Gharan, and Cynthia Vinzant, Log-concave polynomials III: Mason’s Ultra-Log-Concavity Conjecture for Independent Sets of Matroids, arXiv:1811.01600, URL https://arxiv.org/abs/1811.01600
[2] ______, Log-concave polynomials II: High-dimensional walks and an FPRAS for counting bases of a matroid, STOC’19—Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing, ACM, New York, 2019, pp. 1–12, URL https://doi.org/10.1145/3313276.3316385
[3] Nima Anari, Shayan Oveis Gharan, and Cynthia Vinzant, Log-concave polynomials, entropy, and a deterministic approximation algorithm for counting bases of matroids, 59th Annual IEEE Symposium on Foundations of Computer Science—FOCS 2018, IEEE Computer Soc., Los Alamitos, CA, 2018, pp. 35–46, URL https://doi.org/10.1109/FOCS.2018.00013
[4] Petter Brändén and June Huh, Hodge-Riemann relations for Potts model partition functions, arXiv:1811.01699, URL https://arxiv.org/abs/1811.01699
[5] ______, Lorentzian polynomials, Ann. of Math. (2) 192 (2020), no. 3, 821–891, URL https://doi.org/10.4007/annals.2020.192.3.4
[6] Tadahito Harima, Toshiaki Maeno, Hideaki Morita, Yasuhide Numata, Akihito Wachi, and Junzo Watanabe, The Lefschetz properties, Lecture Notes in Mathematics, vol. 2080, Springer, Heidelberg, 2013, URL https://doi.org/10.1007/978-3-642-38206-2
[7] Toshiaki Maeno and Yasuhide Numata, Sperner property and finite-dimensional Gorenstein algebras associated to matroids, J. Commut. Algebra 8 (2016), no. 4, 549–570, URL https://doi.org/10.1216/JCA-2016-8-4-549
[8] Toshiaki Maeno and Junzo Watanabe, *Lefschetz elements of Artinian Gorenstein algebras and Hessians of homogeneous polynomials*, Illinois J. Math. **53** (2009), no. 2, 591–603, URL [http://projecteuclid.org/euclid.ijm/1266934795](http://projecteuclid.org/euclid.ijm/1266934795). MR 2594646

[9] Satoshi Murai, Takahiro Nagaoka, and Akiko Yazawa, *Strictness of the log-concavity of generating polynomials of matroids*, arXiv:2003.09568, URL [https://arxiv.org/abs/2003.09568](https://arxiv.org/abs/2003.09568).

[10] Takahiro Nagaoka and Akiko Yazawa, *Strict log-concavity of the Kirchhoff polynomial and its applications to the strong Lefschetz property*, arXiv:1904.01800, URL [https://arxiv.org/abs/1904.01800](https://arxiv.org/abs/1904.01800).

[11] Sagemath, Inc., *CoCalc – Collaborative Calculation and Data Science*, 2020, [https://cocalc.com](https://cocalc.com).

[12] Junzo Watanabe, *A remark on the Hessian of homogeneous polynomials*, The curves seminar at Queen’s, vol. XIII, Queen’s Papers in Pure and Appl. Math., vol. 119, Queen’s Univ., Kingston, ON, 2000, pp. 171–178.

[13] Akiko Yazawa, *The hessian of the complete and complete bipartite graphs and its application to the strong lefschetz property*, arXiv:1812.07199, URL [https://arxiv.org/abs/1812.07199](https://arxiv.org/abs/1812.07199).

(Akiko Yazawa) DEPARTMENT OF SCIENCE AND TECHNOLOGY, GRADUATE SCHOOL OF MEDICINE, SCIENCE AND TECHNOLOGY, SHINSHU UNIVERSITY, MATSUMOTO, NAGANO, 390-8621, JAPAN

*Email address*: yazawa@math.shinshu-u.ac.jp