Gravitational scattering at the seventh order in $G$: nonlocal contribution at the sixth post-Newtonian accuracy

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A recently introduced approach to the classical gravitational dynamics of binary systems involves intricate integrals (linked to a combination of nonlocal-in-time interactions with iterated 1/potential scattering) which have so far resisted attempts at their analytical evaluation. By using computing techniques developed for the evaluation of multi-loop Feynman integrals (notably Harmonic Polylogarithms and Mellin transform) we show how to analytically compute all the integrals entering the nonlocal-in-time contribution to the classical scattering angle at the sixth post-Newtonian accuracy, and at the seventh order in Newton’s constant, $G$ (corresponding to six-loop graphs in the diagrammatic representation of the classical scattering angle).

I. INTRODUCTION

The detection of the gravitational wave signals emitted by compact binary systems [1] has opened a new path for investigating the structure of the Universe, and offers a novel tool for studying the gravitational interaction. The full exploitation of this new observational tool poses, however, the theoretical challenge to model with improved accuracy the gravitational wave signals emitted during the last orbits of coalescing black-hole binaries.

The latter theoretical challenge has recently motivated the construction of a new approach [2] to the analytical description of the classical conservative dynamics of binary systems. The latter approach is based on a novel way of combining results from several theoretical formalisms, developed for studying the gravitational potential within classical General Relativity (GR): post-Newtonian (PN) expansion, post-Minkowskian (PM) expansion, multipolar-post-Minkowskian expansion, effective-field-theory, gravitational self-force approach, and effective one-body method. Another feature of the approach of Ref. [2] is to combine knowledge from gauge-invariant characterizations of the 6PN expansion, multipolar characterizations of the first index of the conservative dynamics (c.m.) energy, $E = \sqrt{\mathcal{E}}$, and of the total c.m. angular momentum, $J$. Both quantities are given as double expansions in powers of the gravitational constant $G$ (PM expansion), and of the inverse velocity of light $1/c$ (PN expansion), each term of these expansions being a polynomial in the symmetric mass-ratio $\nu = m_1 m_2 / (m_1 + m_2)^2$. Most of the $O(200)$ coefficients entering the latter gauge-invariant characteristics of the 6PN dynamics have been analytically obtained within the TF method except for six coefficients entering the local-in-time Hamiltonian. In addition, the explicit implementation of the TF method requires the evaluation of a certain number of “scattering integrals,” $A_{mnk}$, arising in the computation of the nonlocal-in-time contribution to the scattering angle. Previous work [5] only succeeded in analytically computing a fraction of the latter scattering integrals: namely the $A_{mnk}$’s for $m = 0, 1$ and for $(mnk) = (200), (221)$. Some other scattering integrals (namely $A_{2nk}$ for $(nk) = (20), (40), (41), (42)$) were only numerically evaluated (with a modest, 8-digit accuracy).

Many computing techniques [6–22] have been developed for the evaluation of multi-loop Feynman integrals. We show here how the use of some of these techniques, notably involving the use of Mellin transforms [8], Harmonic Polylogarithms (HPL) [9], and expansion of hypergeometric functions about half-integer parameters, allows one to derive the analytical values of all the scattering coefficients $A_{mnk}$’s entering the nonlocal-in-time contribution at the seventh order in $G$ and at the 6PN accuracy (the $O(G^6)$ order corresponds to the value $m = 3$ of the first index $m$ of the scattering integrals $A_{mnk}$). In particular, the present work will determine the exact, analytical values of the $O(G^6)$ scattering integrals $A_{2nk}$ that were left undetermined in Ref. [5], and which enter the full determination of the 6PN local-in-time dynamics, via the combination $D$, defined as (see Eq. (6.29) of [5])

$$D = \frac{1}{\pi} \left( \frac{5}{2} A_{221} + \frac{15}{8} A_{200} + A_{242} \right).$$

The present work is an extension of Ref. [23] which derived the analytical expressions of the scattering coefficients $A_{2nk}$ entering the nonlocal-in-time contribution at the sixth order in $G$. "
II. SETUP ON THE GR SIDE

The TF method extracts information from various classical GR observables. In particular, one of the crucial gauge-invariant observables used in this approach is the conservative\(^1\) classical scattering angle $\chi$ during a gravitational encounter, considered as a function of the total c.m. energy, $E = \sqrt{s}$, the total c.m. angular momentum, $J$, and the symmetric mass ratio $\nu$. We use the notation

$$M \equiv m_1 + m_2; \mu \equiv \frac{m_1 m_2}{m_1 + m_2}; \nu \equiv \frac{\mu}{M} = \frac{m_1 m_2}{(m_1 + m_2)^2}. \quad (2.1)$$

The TF approach decomposes $\chi(E, J; \nu)$ into three separate contributions:

$$\chi(E, J; \nu) = \chi^{\text{loc}, f} + \chi^{\text{nonloc}, h} + \chi^{\text{\text{-}h}}, \quad (2.2)$$

corresponding to an analogous decomposition of the total Hamiltonian: $H(t) = H^{\text{loc}, f}(t) + H^{\text{nonloc}, h}(t) + \Delta^{\text{\text{-}h}}H(t)$. Here $\chi^{\text{loc}, f}$ is the scattering angle that would be induced by the (f-route) local-in-time piece of the Hamiltonian, $H^{\text{loc}, f}(t)$. By contrast, $\chi^{\text{nonloc}, h}$ is induced by the (h-route) nonlocal-in-time piece of the Hamiltonian, $H^{\text{nonloc}, h}(t)$, while the last contribution, $\chi^{\text{\text{-}h}}$, is induced by the complementary (f-route) term $\Delta^{\text{\text{-}h}}H(t)$, which is algorithmically derived \(^3\) from the $\nu$-structure of $\chi^{\text{nonloc}, h}$. The present work will focus on $\chi^{\text{nonloc}, h}$, which is perturbatively determined as a double expansion in powers of the gravitational constant $G$ (PM expansion), and of the inverse velocity of light $1/c$ (PN expansion). It is convenient to express the combined PM+PN expansion of $\chi^{\text{nonloc}, h}$ in terms of the dimensionless variables

$$p_\infty \equiv \sqrt{\gamma^2 - 1}, \quad \text{and} \quad j \equiv \frac{cJ}{G m_1 m_2}, \quad (2.3)$$

where the dimensionless energy parameter $\gamma$ is defined in terms of the total c.m. energy $E = \sqrt{s}$ by

$$\gamma \equiv \frac{E^2 - m_1^2 c^4 - m_2^2 c^4}{2 m_1 m_2 c^4}. \quad (2.4)$$

The variable $\gamma$ is equal both to the Lorentz factor between the two incoming worldlines, and to the $\mu c^2$-rescaled effective energy $E_{\text{eff}}$ entering the effective-one-body description \(^2\) of the binary dynamics.

As $j < \frac{c}{\gamma}$, the PM expansion of $\chi^{\text{nonloc}, h}$ is equivalent to an expansion in inverse powers of $j$, and reads

$$\frac{1}{2} \chi^{\text{nonloc}, h}(\gamma; j; \nu) \equiv + j p_\infty^4 \left( \frac{A_0^{\text{\text{\text{-}h}}}(p_\infty, \nu)}{j^4} + \frac{A_1^{\text{\text{\text{-}h}}}(p_\infty, \nu)}{j^3} \right) + A_2^0(p_\infty, \nu) + A_2^1(p_\infty, \nu) + p_\infty j^5 + O\left(\frac{1}{j^8}\right). \quad (2.5)$$

The last-written contribution $\propto A_2^1(p_\infty, \nu)/(p_\infty j^7)$ belongs to the 7PM approximation, $O(G^7)$. The dimensionless coefficients $A_m(p_\infty, \nu)$, $m = 0, 1, 2, 3, \ldots$, then admit a PN expansion, i.e., an expansion in powers of $p_\infty = O\left(\frac{1}{j}\right)$, modulo logarithms of $p_\infty$, say

$$A_m^0(p_\infty, \nu) = \sum_{n \geq 0} A_m^n(\nu) \ln\left(\frac{p_\infty}{2}\right)^n. \quad (2.6)$$

The coefficient $A_m^n(\nu)$ is a polynomial in $\nu$ of order $n$ and parametrizes a term of order $m^4 + m^3 + \ldots + m\nu^2$ (with $m \geq 0, n \geq 0$) in the combined PM+PN expansion of the nonlocal scattering angle. The leading-order contribution to the nonlocal dynamics is at the combined 4PM and 4PN level, i.e., $O(G^4/c^8)$ \(^23\). The corresponding nonlocal scattering coefficient, coming from $m = 0$ and $n = 0$, is $A_0^0(p_\infty, \nu) = \frac{\pi}{2} \frac{\ln\left(\frac{p_\infty}{2}\right)}{\sqrt{\gamma^2 - 1}} + O(p_\infty^2) \quad (2.7)$

The higher-order logarithmic coefficients $A_m^n(\nu)$ were analytically determined \(^3\) so that we shall henceforth focus on the non-logarithmic coefficients $A_m^n(\nu)$. Finally, the numerical scattering coefficient $\chi_{\text{num}, k}$ is defined as the coefficient of the $k$th power of the symmetric mass ratio $\nu$ in $A_m^n(\nu)$:

$$A_m^n(\nu) \equiv \sum_{k=0}^{n} A_{m}^{nk} \nu^k, \quad (2.7)$$

with $k = 0, 1, 2, \ldots$.

III. CLASSICAL PERTURBATIVE EXPANSION OF THE NONLOCAL-IN-TIME SCATTERING ANGLE

Ref. \(^{23}\) has derived a general link (valid to first order in tail effects, i.e., up to $O(G^4/c^8)^2 = O(G^8/c^{16})$) between the nonlocal-in-time contribution $\chi^{\text{nonloc}, h}$ to the scattering angle and the integrated nonlocal action. Namely,

$$\chi^{\text{nonloc}, h}(E, J; \nu) = \frac{\partial W^{\text{nonloc}, h}(E, J; \nu)}{\partial J}, \quad (3.1)$$

where

$$W^{\text{nonloc}, h}(E, J; \nu) \equiv \int_{-\infty}^{+\infty} \mathrm{d}t \ H^{\text{nonloc}, h}(t), \quad (3.2)$$

is the integrated (h-route) nonlocal action. The TF method expresses the latter quantity by the following explicit (regularized) two-fold integral (to be evaluated along an hyperbolic-motion solution of the local-in-time Hamiltonian $H^{\text{loc}, f}(t)$),

$$W^{\text{nonloc}, h} = \alpha \mathrm{Pf}_{\Delta t} \int_{-\infty}^{+\infty} \frac{\mathrm{d}tt}{|t|} \xi_{\text{split}} \mathrm{d}W_{G \text{\text{\text{-}s}}}(t, t') + O(\alpha^2). \quad (3.3)$$

\(^1\) See Refs. \(^{24,27}\) for discussions including the radiation-reaction contribution to the scattering angle.
Here: $\alpha \equiv GE/c^5 = G\sqrt{\gamma}/c^5$; $\text{PF}_h$ denotes the partie-
finie regularization of the logarithmically divergent time-derivative. 
The second-order tail contribution is at the 6.5PN level, 
the PN accuracy needed for knowing 
the 1PN fractional accuracy [34, 51]. Their explicit 
expressions (in the center-of-mass harmonic coordinate frame) 
have been recalled in Eq. (3.3) and in Table I of Ref. [3].

Introducing the shorthand notation
$$\langle F \rangle_\infty \equiv \int_{-\infty}^{\infty} dt F(t),$$
and expressing the partie-finie operation $\text{PF}_{2/c}$ entering 
Eq. (3.3) in terms of a partie-finie operation $\text{PF}_{2/c}$ involving 
an intermediate length scale $s$, we decompose the 
nonlocal integrated action $W_{\text{nonloc},h}$ into two contributions
$$W_{\text{nonloc},h}(E, j) = W_{1\text{tail},h}(E, j) + W_{2\text{tail},h}(E, j) + O(\alpha^2),$$
where
$$W_{1\text{tail},h}(E, j) \equiv -\alpha \left\langle \text{PF}_{2/c} \int_{-\infty}^{\infty} \frac{dt'}{t-t'} \mathcal{F}_{\text{split}}(t, t') \right\rangle,$$
and
$$W_{2\text{tail},h}(E, j) \equiv 2\alpha \left\langle \mathcal{F}_{\text{split}}(t) \ln \left( \frac{\eta}{s} \right) \right\rangle.$$
The integrated nonlocal action $W_{\text{nonloc},h}(E, j)$, and 
therefore each partial contribution, Eqs. (3.8), (3.9), has 
been evaluated along a 2PN-accurate hyperbolic motion.

IV. QUASI-KEPLERIAN PARAMETRIZATION 
OF THE HYPERBOLIC MOTION, AND ITS 
LARGE-ECCENTRICITY EXPANSION

In view of Eq. (3.1), the PM expansion (2.5) of 
$\chi_{\text{nonloc},h}$ is equivalent to the following expansion of the 
inverted nonlocal action $W_{\text{nonloc},h}(E, j)$ in inverse powers of $j$,
$$c W_{\text{nonloc},h}(\gamma; j; \nu) = -\nu^4 \left( A_0^h(p_\infty, \nu) + \frac{A_1^h(p_\infty, \nu)}{4p_\infty j^4} \right).$$
Remembering the proportionality between $j = cJ/(GM_1M_2)$ and the impact parameter $b$ (via $J = bP_\text{c.m.}$, where $P_\text{c.m.}$ is the c.m. linear momentum of each body), we see that the computation of the scattering coefficients $A_{i,j}^h(p_\infty, \nu)$ amounts to expanding the 
inverted nonlocal action in inverse powers of $b$. 
An explicit way to compute the large-impact-parameter expansion of $W_{\text{nonloc},h}$ is to use the quasi-Keplerian 
parametrization of the 2PN-accurate hyperbolic-motion solution 
[59] of the 2PN dynamics of a binary system in harmonic coordinates [40, 41].
The hyperbolic quasi-Keplerian parametrization involves a semi-major-axis-like quantity \( a_r \), together with several eccentricity-like quantities \( e_r, \epsilon_r, e_\phi \). The variable parametrizing the time development is an eccentric-anomaly-like (hyperbolic) angle \( v \) varying from \(-\infty\) to \(+\infty\):

\[
\begin{align*}
r &= \bar{a}_r(e_r, \cos v - 1), \\
\ell &= \bar{n}(t - t_P) = e_t \sinh v - v + f_t V(v) + g_t \sinh v, \\
\bar{\phi} &= \phi - \phi_P = V(v) + f_\phi \sin 2V(v) + g_\phi \sin 3V(v).
\end{align*}
\]

(4.2)

Here, we use adimensionalized variables (and \( c = 1 \)), notably \( r = r_{\text{phys}}/(GM) \), \( t = t_{\text{phys}}/(GM) \), while \( V(v) \) is given by

\[ V(v) = 2 \arctan \left[ \Omega_{e_\phi} \tanh \frac{v}{2} \right], \]

where

\[ \Omega_{e_\phi} \equiv \left( \frac{\epsilon_\phi + 1}{\epsilon_\phi - 1} \right). \]

(4.4)

The expressions (as functions of the specific binding energy \( E \equiv (E_{\text{tot}} - M c^2)/\mu m^2 \)) and of the dimensionless angular momentum \( j = c j/(GM\mu) \) of the orbital parameters \( \bar{n} \) (hyperbolic mean motion) and \( K \) (hyperbolic periastron precession), as well as \( \bar{a}_r, e_r, \epsilon_r, e_\phi, f_t, g_t, f_\phi, g_\phi, \) can be found in Appendix A of Ref. [8]. Let us only recall here the expressions of \( \bar{a}_r, \) and \( e_r \) in terms of \( E \) and \( j \):

\[
\begin{align*}
\bar{a}_r &= \frac{1}{2E} \left[ 1 - \frac{1}{2} E j^2 (-7 + \nu) \right. \\
&\quad + \frac{1}{4} E^2 \eta^4 \left( 1 + \nu^2 - 8(\frac{4}{2} + 7\nu) \right) \right], \\
e_r^2 &= 1 + 2E j^2 + E[5E j^2 (\nu - 3) + 2\nu - 12] \eta^2 \\
&\quad + \frac{E^2}{3} (4\nu^2 + 85\nu - 45) \bar{E} j^2 \times \\
&\quad + (\nu^2 + 34\nu + 30) \bar{E} j^2 + 56\nu - 32 \eta^2. \quad (4.5)
\end{align*}
\]

When using this quasi-Keplerian parametrization, the combined PM+PN expansion of \( W_{\text{nnloc.h}}(\gamma, j; \nu) \) can be constructed from the combined large-\( e_r \)-large-\( a_r \) expansion of the function \( W_{\text{nnloc.h}}(e_r, a_r) \). On the one hand, as the tail action starts at the 4PN level, we need to work to the next-to-next-to-leading-order (NNLO) in \( \frac{1}{a_r} \sim \frac{\bar{E}}{\bar{a}_r} \) in order to reach the 6PN accuracy. On the other hand, as the tail action starts at the 4PM level (O(\( G^4 \))), we need to work to the next-to-next-to-next-to-leading-order (N3LO) in \( \frac{1}{a_r} \) in order to reach the 7PM, \( O(G^5) \), accuracy (seventh order in \( \frac{1}{\bar{a}_r} \)).

Without presenting too many technical details, let us illustrate the origin of some of the structures entering the scattering integrals \( A_{mnk} \) by explaining how one can compute the large-eccentricity expansion of the crucial nonlocal integral

\[
\iint dt dt' \frac{dt dt'}{|t - t'|} W_{\text{split}}^{\text{GW}}(t, t') \quad (4.6)
\]

entering \( W_{\text{nnloc.h}} \). The first step is to introduce the auxiliary time variable \( T \in [-1, 1] \):

\[ T \equiv \tanh \frac{v}{2}. \]

(4.7)

In terms of this variable, the 2PN-accurate functional relation between the original (rescaled) time variable \( t \equiv \frac{t_{\text{phys}}}{GM} \) and the hyperbolic eccentric anomaly \( v \) reads

\[
t = \frac{2}{\bar{n}} \left[ e_r \left( \frac{T}{1 - T^2} \right) - \arctanh(T) \right. \\
&\left. + f_t \arctan \left( \frac{\Omega_{e_\phi} T}{1 + \Omega_{e_\phi}^2 T^2} \right) + g_t \right], \quad (4.8)
\]

with a corresponding expression for \( t' \) vs \( T' \), whose 2PN-accurate large-eccentricity expansion reads

\[
|t - t'| = |T - T'| \left[ 1 + \frac{T T'}{(1 - T^2)(1 - T'^2)} \right]^3 a_r^2 e_r \\
\times \left[ 2 - (1 + 2\nu) \frac{\eta^2}{a_r} + \frac{1}{4} - 8\eta^2 - 8\nu - 1 \frac{\eta^4}{a_r^2} \right] \\
\times \left[ 1 + \frac{1}{e_r} P_1 + \frac{1}{e_r^2} P_2 + \frac{1}{e_r^3} P_3 + O \left( \frac{1}{e_r^4} \right) \right]. \quad (4.9)
\]

with coefficients \( P_1, P_2 \) and \( P_3 \) of the form

\[
\begin{align*}
P_1 &= P_{10}(T, T') \eta^2 a_r - P_{14}(T, T') \frac{\eta^4}{a_r} , \\
P_2 &= P_{24}(T, T') \frac{\eta^4}{a_r^2} , \\
P_3 &= P_{34}(T, T') \frac{\eta^4}{a_r^3} .
\end{align*}
\]

(4.10)

Let us illustrate the structure of the coefficients \( P_{nm}(T, T') \) entering the \( P_n \)’s by citing the expressions of the first few of them. Introducing the shorthand notation

\[ At(T, T') \equiv \arctan(T) - \arctan(T'), \]
\[ Ath(T, T') \equiv \arctanh(T) - \arctanh(T'), \]

we have
\[ P_{10}(T, T') = \frac{(1 - T'^2)(1 - T^2)}{(T' + 1)(T - T')} \text{Arth}(T, T'), \]
\[ P_{12}(T, T') = \frac{1}{2} (-8 + 3\nu) P_{10}(T, T'), \]
\[ P_{14}(T, T') = \frac{1}{8} (-29 + 3\nu) P_{10}(T, T') + \frac{1}{8} (15 + \nu) \frac{(TT' - 1)(1 - T'^2)(1 - T^2)}{(1 + T')(1 + T^2)(T' + 1)} \bigg( \frac{1}{2} \nu - 15 \bigg) \frac{TT'(1 - T'^2)(1 - T^2)}{(1 + T^2)^2(1 + T'^2)^2}, \]
\[ P_{24}(T, T') = -\frac{3}{2} (-5 + 2\nu) \frac{(1 - T'^2)(1 - T^2)}{(T' + 1)(T - T')} \text{At}(T, T') + \frac{1}{2} \nu \bigg( \nu - 15 \bigg) \frac{TT'(1 - T'^2)(1 - T^2)}{(1 + T^2)^2(1 + T'^2)^2} \bigg( \frac{1}{2} \nu - 15 \bigg) \frac{TT'(1 - T'^2)(1 - T^2)}{(1 + T^2)^2(1 + T'^2)^2}, \]
\[ P_{34}(T, T') = \frac{3}{2} (-4 + 7\nu) P_{10}(T, T') - \frac{1}{2} \nu \bigg( \nu - 15 \bigg) \frac{TT'(T - T') - 1(1 - T'^2)(1 - T^2)}{(T' + 1)(1 + T^2)^2(1 + T'^2)^2} \bigg( \frac{1}{2} \nu - 15 \bigg) \frac{TT'(1 - T'^2)(1 - T^2)}{(1 + T^2)^2(1 + T'^2)^2} \bigg( \frac{1}{2} \nu - 15 \bigg) \frac{TT'(1 - T'^2)(1 - T^2)}{(1 + T^2)^2(1 + T'^2)^2}, \]

Using the above relations one can compute the large-

eccentricity expansion of the measure

\[ \frac{dt dt'}{|t - \nu'|} = \frac{1}{|T - \nu(T')|} \frac{dt}{dT} \frac{dt'}{dT} dT dT' \equiv dM_{(T,T')} \]  

Its schematic 2PN-accurate structure reads

\[ dM_{(T,T')} = 2\ep_a a^3/2 \left[ 1 - \frac{1 + 2\nu}{2\ep_a} \eta^2 - \frac{1 + 8\nu - 8\eta^2}{8\ep_a^2} \eta^4 \right] \]
\[ \times \frac{(1 + T'^2)(1 + T^2)dTdT'}{(1 - T'^2)(1 - T^2)(1 + T^2)(1 + T'^2)(T + T')}, \]
\[ \times \left( \frac{M_1}{\ep_r} + \frac{M_2}{\ep^2} + \frac{M_3}{\ep^3} + O\left( \frac{1}{\ep^4} \right) \right), \]

(4.14)

where we have explicitly shown only the LO contribution

in the large-eccentricity expansion. The NLO, NNLO and \( N^3 \)LO contributions (respectively described by the coefficients \( M_1(T, T'; \nu, \eta) \), \( M_2(T, T'; \nu, \eta) \) and \( M_3(T, T'; \nu, \eta) \)) have long expressions that we do not explicitly display here. Let us simply note that (recalling the definitions Eq. (4.11)) \( M_1(T, T'; \nu, \eta) \) involves the function \( \text{At}(T, T') \) linearly, \( M_2(T, T'; \nu, \eta) \) involves \( \text{At}(T, T'), \text{At}(T, T') \) and \( \text{At}(T, T') \), while \( M_3(T, T'; \nu, \eta) \) involves \( \text{At}(T, T'), \text{At}(T, T') \) and \( \text{At}(T, T') \) as well as \( \text{At}(T, T') \) and \( \text{At}(T, T') \).

As illustrated here, apart from rational functions of \( T \) and \( T' \), the large-eccentricity expansion has a polynomial dependence on the transcendental functions \( \text{arctan}(T) \), \( \text{arctan}(T') \), \( \text{arctanh}(T) \) and \( \text{arctanh}(T') \). Using these expansions (as well as corresponding expansions of the various multipole moments), one finally gets explicit integral expressions for the scattering coefficients \( A_{mnk} \) of the form

\[ A_{mnk} = \int_{-1}^{+1} \int_{-1}^{+1} \frac{dT dT'}{|T - T'|} a_{mnk}(T, T'), \]

(4.15)

with integrands \( a_{mnk}(T, T') \) of the form

\[ a_{mnk}(T, T') = \sum_{p,q \geq 0} R_{pq}^{mnk}(T, T') \text{At}(T, T')^p \text{At}(T, T')^q, \]

(4.16)

where \( R_{pq}^{mnk}(T, T') \) are rational functions of \( T \) and \( T' \), and where we used the shorthands (4.11). The highest power of \( \text{At}(T, T') \equiv \text{arctanh}(T) - \text{arctanh}(T') \) in this expression is directly equal to the order of expansion in \( 1/\ep_r \) (and therefore in \( G \), recalling the leading-order expression \( e_r = \sqrt{1 + 2E/\ep_r^2} \)) of the relativistic hyperbolic motion.

Ref. succeeded in analytically computing (up to the 6PN accuracy) the numerical coefficients \( A_{mnk} \) when \( m = 0 (G^4 \text{ level}) \) and \( m = 1 (G^5 \text{ level}) \). By contrast, the integrands of Eq. (4.15) become so involved when \( m = 2 \) and \( m = 3 (G^6 \text{ and } G^7 \text{ levels}) \) that most of them resisted analytical integration by standard integration methods.

V. MULTIPLE POLYLOGARITHMS AND HARMONIC POLYLOGARITHMS

To determine the analytic expressions of the scattering integrals \( A_{2nk} \) we follow one of the strategies used in the realm of multi-loop Feynman calculus, namely the reduction to iterated integrals [11]. Given a sequence of univariate functions \( g_a(x), g_{a_2}(x), \ldots, g_{a_n}(x), \ldots \),
assumed (say) to be regular at \( x = 0 \), iterated integrals are recursively defined by 
\[ G(a_1, a_2, \ldots, a_n; x) = \int_0^x dt_1 g_{a_1}(t_1)G(a_2, \ldots, a_n; t_1), \]
with the starting value \( G(0; x) = 1 \). The simplest class of iterated integrals are the multiple polylogarithms defined by considering a sequence of inverse-linear functions: 
\[ g_a(x) = (x - a)^{-1}. \]
These were introduced by Poincaré [42], and have been the topic of many mathematical studies, e.g., [13, 17]. They also came up as important tools for expressing certain multi-loop Feynman integrals [17, 48]. On the other hand, from the practical point of view, a subclass of the multiple logarithms, the harmonic polylogarithms (HPL) [2], has turned out to be sufficient, and very useful, to express many Feynman integrals. They are defined by restricting the singular points \( a_i \) entering \( G(a_1, a_2, \ldots, a_n; x) \) to taking one of the three values \( +1, -1, \) or \( 0 \), and by normalizing the inverse-linear factors in a slightly different way. Specifically, the HPLs are defined as the recursive integrals,

\[ H_{i_1, i_2, \ldots, i_n}(x) = \int_0^x dt_1 f_{i_1}(t_1)H_{i_2, \ldots, i_n}(t_1), \quad (5.1) \]

with \( f_{\pm 1}(x) = (1 \mp x)^{-1}, \) \( f_0(x) = 1/x, \) and a regularization at \( x = 0 \) such that \( H_{0, 0, \ldots, 0}(x) \equiv \ln^n(x)/n! \).

A crucial feature of the multiple polylogarithms, and therefore of the HPLs, is that they enjoy special algebraic properties, going under the names of: shuffle algebra, stuffle algebra, scaling invariance, shuffle-antipode relations, Hölder convolution, integration-by-parts identities, etc. In addition, all these special algebraic properties respect a filtration by the weight, i.e. by the number \( n \) of singular values, \( a_1, a_2, \ldots, a_n \), or the number \( n \) of indices on \( H_{i_1, i_2, \ldots, i_n}(x) \). The weight corresponds to the number of iterations appearing in the nested integral representation. For instance, at weight 1 a multiple polylogarithm is a simple logarithm, while at weight 2, it is a linear combination of a dilogarithm and a squared logarithm. The remarkable algebraic properties of multiple polylogarithms (and HPLs) allow one to express them algebraically, at any given weight \( n \), in terms of a minimal subset of them, having weights \( n' \leq n \). For instance, at weights \( n = 2, 3, \) and 4 the minimal subsets are formed by 3, 8, and 18 elements, respectively. In addition, their evaluation for special values of their arguments \( a_1, a_2, \ldots, a_n; x \) can often be reduced to a relatively small number of transcendental constants. This is particularly the case if, besides 0, the arguments \( a_1, a_2, \ldots, a_n; x \) are roots of unity. For introductions to the vast literature on the properties, and evaluation, of multiple polylogarithms and HPLs (including computer-program implementations) see, e.g., [10, 12, 14, 17, 19, 21, 22, 43, 44, 49, 50].

VI. ANALYTIC EVALUATION OF THE O(G^6) SCATTERING INTEGRALS VIA HARMONIC POLYLOGARITHMS

Let us now sketch how we could analytically compute the \( O(G^6) \) scattering integrals, i.e. Eq. (1.15), with \( m = 2 \), by reducing these two-fold definite integrals to the evaluation of HPLs, of weight \( \leq 4 \), for the values \( x = 1, i \) of the HPL variable.

First, using symmetry properties of the integrands \( a_{mnk}(T, T') \) entering Eq. (1.10), it is possible to reduce the double integration to the triangle \( 0 < T < 1 < T' < T \). Let us start by discussing the integration over \( T' \) on the interval \( 0 < T' < T \). The crucial information needed for discussing this first integration concerns the structure of the integrands \( a_{mnk}(T, T') \), and particularly of the denominators entering the rational coefficients \( R_{pq}^{mnk}(T, T') \) in Eq. (1.10), when \( m = 2 \). To be concrete, let us discuss the case (\( mnk \) = (242)) and exhibit one representative part of the integrand \( e_{242}(T, T') \). It reads

\[ \frac{-16(1 - T^2)^3(1 - T'^2)^3P_2(T, T')}{315(1 + T^2)^6(1 + T'^2)^4(1 + T T')^3(T - T')^3} \times \left\{ \frac{\{\text{arctanh}(T) - \text{arctanh}(T')\}^2}{2(1 - T^2)} \right\}, \quad (6.1) \]

where \( P_2(T, T') \) is a (symmetric) polynomial in \( T \) and \( T' \), of order 14 in both variables. By partial fractioning \( (6.1) \) with respect to \( T' \) (keeping \( T \) fixed) one is reduced to evaluating integrals of the type

\[ \int dT' \text{arctanh}^p(T'), \quad (T' - a)^q, \quad (6.2) \]

where \( p = 0, 1, 2, 1 \leq q \leq 8 \) and \( a = \pm i, -\frac{1}{\sqrt{2}}, T \) or \( 0 \). Integrating by parts (with respect to \( T' \)), one can reduce the power \( q \) down to \( q = 1 \). At this stage, remembering that \( \text{arctanh}(T) = \frac{1}{2} \ln((1 + T)/(1 - T)) \) (and \( \text{arctan}(T) = \text{arctanh}(iT)/i \) for other denominators) are (as explained above) of weight 1, we see that the highest-weight term in the numerator, \( \propto \ln^3((1 + T)/(1 - T')) \), is of weight 2, so that its integration over \( T' \) with the additional kernel \( (T' - a)^{-1} \) will generate terms of weight 3. The explicit computation of the needed integration over \( T' \in [0, T] \), with the values of \( a \) listed above, is found to involve at most the trilogarithm \( Li_3(z) \) at the rational arguments

\[ z = -\frac{1 + i}{1 - T}, \quad \text{or} \quad z = -\frac{1 + i}{1 + T}. \]

Having so obtained an explicit weight-3 expression for the result of the integration over \( T' \), we need to perform the final integration over \( T \in [0, 1] \). This is done in three steps. The first step is the same that was used for the \( T' \) integration. There are now polynomial denominators involving powers of \( T^2 + 1 \), powers of \( T \pm 1 \), and also powers of \( T \), \( T^2 + 1 \), \( T \pm 1 \), and also powers of \( T \). Partial-fractioning, and integrating by parts, one can reduce these powers to the first power. Second, we
use the definition of HPLs to express the integrals containing $T^{-1}$ and $(T \pm 1)^{-1}$ in terms of HPLs. Third, we consider the integrals containing $(T \pm i)^{-1}$: these cannot be directly cast in HPL format (which admits poles only at $T = 0, \pm 1$). Therefore, we modify the integrands by the insertion of a parameter $x$, to be later replaced by a suitable value, so as to obtain the original integral back. Following a technique introduced many years ago to analytically evaluate multi-loop Feynman integrals \[10,11\], the integral, now function of $x$, is reduced to iterated integrals of the type $\int_0^1 dx_1 \left(x_1 - a_1 \right)^{-1} \int_0^1 dx_2 \left(x_2 - a_2 \right)^{-1} \cdots$, by combining repeated differentiations with respect to $x$ with partial-fractions, and integrations by parts, followed by quadratures to get back the original integral.

Let us show an example of this technique: all the $A_{2nk}$ integrals contain, after the $T'$ integration, the same combination of integrals of weight $w = 4$,

$$J = \int_0^1 dT \frac{-2 \ln^3 \left(\frac{1}{1+T} \right) - 3 \text{Li}_3 \left(-\left(\frac{1}{1+T} \right)^2 \right)}{1 + T^2}.$$

We modify the integral \[6,3\], to let it acquire a dependence on the new parameter $x$, i.e. $J \rightarrow J(x)$, in the following way:

$$J(x) \equiv \int_0^1 dT \left(1 - x^2 \right) \times -2 \ln^3 \left(\frac{1}{1+T} \right) - 3 \text{Li}_3 \left(\frac{(1-T)(1-x)}{1+T(1+x)} \right)^2 \times \frac{2x(T+x)(T+1/x)}{2x(T+x)(T+1/x)}.$$

It is easily seen that the original integral is recovered at the value $x = i$, that is $J = J(i)$, and that $J(1) = 0$. By differentiating and reintegrating three times over $x$, on the model of $J(x) = \int_0^1 dx (dJ(x)/dx)$, $J(x)$ can be expressed in terms of HPLs of weight $w \leq 4$, namely:

$$i J(x) = \frac{23}{240} \pi^4 - 21 \ln(2) \zeta(3) + \pi^2 \ln^2(2) - \ln^4(2) - 24a_4$$

$$+ 21 H_{-1}(x) \zeta(3) - \frac{3}{2} H_0(x) \zeta(3) + \frac{21}{2} H_1(x) \zeta(3)$$

$$+ \frac{1}{2} \pi^2 H_{0,-1}(x) + \frac{1}{2} \pi^2 H_{0,1}(x) - \frac{3}{2} \pi^2 H_{-1,-1}(x)$$

$$- \frac{3}{2} \pi^2 H_{-1,1}(x) - \frac{3}{2} \pi^2 H_{1,-1}(x) - \frac{3}{2} \pi^2 H_{1,1}(x)$$

$$+ 12 H_{0,1,-1}(x) \ln(2) + 12 H_{0,1,1}(x) \ln(2)$$

$$- 12 H_{0,-1,1}(x) + 6 H_{0,-1,0}(x) - 12 H_{0,-1,-1}(x)$$

$$+ 6 H_{0,-1,0}(x) - 12 H_{0,-1,1}(x) + 6 H_{0,1,-1}(x)$$

$$- 6 H_{1,1,-1,0}(x) - 6 H_{1,1,-1,1}(x) - 6 H_{1,1,1,0}(x)$$

$$- 6 H_{1,1,1,1}(x) + 12 H_{0,-1,1}(x) \ln(2) + 12 H_{0,1,-1}(x) \ln(2).$$

This result expresses $\tilde{J} = J(i)$ in terms of the values at the fourth root of unity, $i$, of HPLs of weight $w \leq 4$.

The fourth root of unity, allowed by quadratures to get back the original integral, now function of $x$, is reduced to iterated integrals of weight $w = 4$, by combining repeated differentiations with respect to $x$ with partial-fractions, and integrations by parts, followed by quadratures to get back the original integral.

Further details about our integration procedures, and our intermediate results, are provided in the Supplemental Material \[53\].

### Table 1: Analytical results for the $O(G^6)$ scattering coefficients $A_{2nk}$

| coefficient | value |
|------------|-------|
| $\pi^{-1} A_{200}$ | $-\frac{22}{407} \zeta(3)$ |
| $\pi^{-1} A_{220}$ | $-\frac{1}{127} \zeta(3)$ |
| $\pi^{-1} A_{221}$ | $\frac{1937}{407} + 407 \zeta(3)$ |
| $\pi^{-1} A_{240}$ | $\frac{103549}{407} + 407 \ln(2) - \frac{40711}{128} \zeta(3)$ |
| $\pi^{-1} A_{241}$ | $\frac{583751}{407} + 21 \ln(2) - \frac{256}{128} \zeta(3)$ |
| $\pi^{-1} A_{242}$ | $-\frac{864}{64} + 6 \zeta(3)$ |

(Together with $a_4 = \text{Li}_4(1/2)$, and lower-weight quantities such as $\pi^2$ and $\zeta(3)$). Using \[12\], we expressed the needed values of the HPLs at $x = i$ in terms of a small subset of irreducible constants of weight $w \leq 4$, namely: $K = \text{ImLi}_2(i) = \sum_{n=1}^{\infty} (-1)^n/(2n + 1)^2$ (Catalan's constant), $Q_1 = \text{ImLi}_{0.1,1}(i)$, $Q_4 = \text{ImLi}_{0.1,1,1}(i)$, $a_4 = \text{Li}_4(1/2)$ and $\beta(4) = \text{ImLi}_4(i)$. The irreducible weight-4 constants are found to cancel when evaluating $\tilde{J} = J(i)$ by means of Eq. \[6.6\] to yield

$$\tilde{J} = J(i) = -\frac{1}{2} \pi^2 K + \frac{9}{2} \pi \zeta(3).$$

Applying our technique to all the scattering integrals $A_{2nk}$, we found that they could all be expressed in terms of the values of HPLs of weight $w \leq 4$ at the arguments $x = 1$ or $x = i$. Similarly to what happens for $\tilde{J} = J(i)$, the irreducible weight-4 constants are found to cancel in the evaluation of all the scattering integrals $A_{2nk}$. Actually, the final results for the $A_{2nk}$'s are found to factorize as the product of $\pi$ with constants of weight $\leq 3$. For instance, we found

$$A_{242} = -\pi \left( \frac{583751}{864} + \frac{100935}{64} \right) \zeta(3).$$

Our complete analytical results for the $A_{2nk}$'s are listed in Table \[1\]. We give below the relations between such coefficients and those used in Ref. \[2\] to parametrize the (non-logarithmic) part of the scattering angle (see Eq. \[4.15\] there)

$$\pi^{-1} A_{200} = d_{00},$$

$$\pi^{-1} A_{220} = d_{20} + 3d_{00},$$

$$\pi^{-1} A_{221} = d_{21} + 2d_{00},$$

$$\pi^{-1} A_{240} = d_{20} + d_{00} + \frac{3}{2} d_{00},$$

$$\pi^{-1} A_{241} = d_{21} - \frac{11}{2} d_{00} + d_{21} - 2d_{00},$$

$$\pi^{-1} A_{242} = d_{22} - 2d_{21} + 3d_{00}.$$
At the $O(G^7)$ level, i.e., for the integrals $A_{mk}$ with index $m = 3$, the structure of the integrands $\alpha_{mk}(T, T')$ becomes more complex. The rational functions $R_{pq}^{mk}(T, T')$ entering as coefficients in Eq. (4.10) involve higher-order polynomials in their numerators, but their most important feature, namely the location of the poles in the denominators, stays the same as at the $O(G^6)$ level. Again the poles are located at $T = T'$, $T = -1/T'$, $T = \pm i$, $T' = \pm i$, and $T' = \pm 1$. However, an important change concerns the powers $p$ and $q$ with which the functions $A(T, T') \equiv \arctan(T) - \arctan(T')$ and $A_{th}(T, T') \equiv \arctanh(T) - \arctanh(T')$ enter the numerator of $\alpha_{mk}(T, T')$. At the $O(G^7)$ level, we have the values $(p, q) = (0, 0), (1, 0), (2, 0), (3, 0), (0, 1), (1, 1)$. In particular, the highest value of $p + q$ is 3, and is reached via the presence of a term proportional to $A_{th}^3(T, T')$. We already noticed that both $A_{th}(T, T')$ and $A(T, T')$ are of weight 1. The integrand $R_{pq}^{mk}(T, T') A_{th}^3(T, T')$ is therefore of weight 3. Its double integral over $T$ and $T'$ can therefore be a priori expected to be of weight 5.

We succeeded in finding the analytic expressions of the $O(G^7)$ integrals entering the 6PN nonlocal scattering angle by using several methods. As a preliminary method, we combined very-high-precision (200 digits) numerical computation of the integrals (using a double-exponential change of variables) with the PSLQ algorithm [53] and a basis of transcendental constants indicated by the structure of the integrands. Let us recall that such an experimental mathematics strategy is often used in the realm of multi-loop Feynman calculus, when a direct analytic integration seems prohibitive, see e.g., Refs. [48, 54, 55]. Previous uses of experimental mathematics and high-precision arithmetics within studies of binary systems include Refs. [46, 55]. We note in passing that one of the integrals (in momentum space) contributing to the 4PN-static term of the two-body potential, used in [55] and originally obtained by analytic recognition [60], was later analytically confirmed by direct integration (in position space) [61].

The application of experimental mathematics to the $A_{3nk}$ integrals has shown that, similarly to what happened for the $A_{2nk}$ integrals, the final results were simpler than what was a priori expected. In particular, we found that the final results do not go beyond weight 4, and that the only weight-4 quantity entering (some of) the results is simply $\zeta(4) \propto \pi^4$.

Having obtained such simple semi-analytic expressions for the $A_{3nk}$ integrals, we embarked on confirming them by means of a purely analytical derivation. We found that an efficient method for doing so was to reformulate the time-domain integral defining the integrated action Eq. (3.3) in frequency space. Actually, when decomposing $W_{\text{nonloc}, h}(E, j)$ in the two contributions entering Eq. (3.7), the most difficult one to evaluate is

$$W_{1}^{\text{tail}, h}(E, j) \equiv -\alpha \left\langle \text{Pr}_{2s/c} \int_{-\infty}^{\infty} \frac{dt'}{t - t'} F_{\text{split}}^{2PN}(t, t') \right\rangle_{\infty}.$$

In order to express $W_{1}^{\text{tail}, h}(E, j)$ in the frequency domain, the first step is to Fourier transform the multipole moments. For example,

$$I_{ab}(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega t} I_{ab}(\omega),$$

where

$$I_{ab}(\omega) = \int_{-\infty}^{+\infty} dte^{i\omega t} I_{ab}(t),$$

with the associated PN expansion

$$I_{ab}(\omega) = \mathcal{I}_{h}^{N}(\omega) + \eta^{2} \mathcal{I}_{h}^{1PN}(\omega) + \eta^{4} \mathcal{I}_{h}^{2PN}(\omega) + O(\eta^{6}).$$

Inserting these Fourier representations into Eq. (7.5) then yields (see Section V of Ref. [25] for details)

$$W_{1}^{\text{tail}, h}(E, j) = \frac{2G^{2} H_{\text{tot}}}{\pi c^{3}} \int_{0}^{\infty} d\omega \mathcal{K}(\omega) \ln \left( \frac{2\omega}{c} \right),$$

where

$$\mathcal{K}(\omega) = \frac{1}{5} \omega^{6} |\mathcal{I}_{ab}(\omega)|^{2} + \eta^{2} \left[ \frac{\omega^{8}}{189} |\mathcal{I}_{abc}(\omega)|^{2} + \frac{16}{45} \omega^{6} |\mathcal{J}_{ab}(\omega)|^{2} \right] + \eta^{4} \left[ \frac{\omega^{10}}{9072} |\mathcal{I}_{abcd}(\omega)|^{2} + \frac{\omega^{8}}{84} |\mathcal{J}_{abc}(\omega)|^{2} \right].$$

and we have used the result

$$\text{Pr}_{T} \int_{0}^{\infty} dt \frac{\cos \omega t}{t} = - \ln(|\omega| T e^{\gamma})$$

with $\gamma = 0.577215 \ldots$. Note the close link between the expression (7.5) for $W_{1}^{\text{tail}, h}(E, j)$ and the frequency-domain expression of the total energy flux emitted during the scattering process, namely

$$\Delta E_{GW} = \frac{G}{\pi c^{2}} \int_{0}^{\infty} d\omega \mathcal{K}(\omega).$$

The difference between the two expressions is embodied in the logarithmic factor $\ln \left( \frac{\omega}{c} e^{\gamma} \right)$, which is characteristic of the tail in the frequency domain [61].

\footnote{In the following, we use $GM = 1$, i.e., we work with $GM$-rescaled time and frequency variables.}
The relation between $\Delta E_{GW}$ and $W_{1}^{\text{tail},h}(E, j)$ is clarified by stating them in the framework of the Mellin transform. Let us first note that it is convenient to replace the frequency $\omega$ by the variable $u$, using

$$
\omega \equiv \frac{u}{e_r a_r^{3/2}},
$$

(7.9)

so that Eq. (7.5) becomes

$$
W_{1}^{\text{tail},h}(E, j) = 2 \ln \alpha_s G \hbar \Delta E_{GW} + \frac{2}{\pi e^5} \frac{1}{e_r a_r^{3/2}} \int_0^\infty du \mathcal{K}(u) \ln u,
$$

(7.10)

where

$$
\Delta E_{GW} = \frac{G}{\pi e^5} \frac{1}{e_r a_r^{3/2}} \int_0^\infty du \mathcal{K}(u),
$$

(7.11)

with

$$
\mathcal{K}(u) = \mathcal{K}(\omega)|_{\omega = u/(e_r a_r^{3/2})},
$$

(7.12)

and

$$
\alpha_s = \frac{2s}{ce_r a_r^{3/2}}e^\gamma.
$$

(7.13)

We recall that the Mellin transform of a function $f(u)$ (with $u \in [0, +\infty]$) is defined as

$$
g(s) \equiv \mathcal{M}\{f(u); s\} = \int_0^\infty u^{s-1}f(u)du.
$$

(7.14)

It is then easily seen that the first two terms of the Taylor expansion of $g(s)$ around $s = 1$ are respectively given by

$$
g(1) = \int_0^\infty f(u)du,
$$

(7.15)

and

$$
\left.\frac{dg(s)}{ds}\right|_{s=1} = \int_0^\infty f(u)\ln udu.
$$

(7.16)

This shows the possible usefulness of the Mellin transform in connecting $W_{1}^{\text{tail},h}$ to $\Delta E_{GW}$.

We have indeed been able to use the Mellin transform to analytically compute all the scattering integrals at $O(G^7)$, i.e. the values of the integrals appearing in the double PN+PM (or $\eta^2 e_r^{-1}$) expansion of

$$
\int_0^\infty du \ln u [\mathcal{K}(u)]^{\text{PN+PM}},
$$

(7.17)

where

$$
[\mathcal{K}(u)]^{\text{PN+PM}} = \mathcal{K}_N^{\text{LO}}(u) + \eta^2 \mathcal{K}_N^{\text{LO}}(u) + \eta^4 \mathcal{K}_N^{\text{LO}}(u)
$$

$$
+ \frac{1}{e_r} \mathcal{K}_N^{\text{LO}}(u) + \frac{\eta^2}{e_r^2} \mathcal{K}_N^{\text{LO}}(u) + \frac{\eta^4}{e_r^4} \mathcal{K}_N^{\text{LO}}(u)
$$

$$
+ \frac{1}{e_r^2} \mathcal{K}_N^{\text{LO}}(u) + \frac{\eta^2}{e_r^4} \mathcal{K}_N^{\text{LO}}(u) + \frac{\eta^4}{e_r^6} \mathcal{K}_N^{\text{LO}}(u),
$$

(7.18)

as well as their simpler analogs appearing in the double $\eta^2 e_r^{-1}$ expansion of

$$
\int_0^\infty du [\mathcal{K}(u)]^{\text{PN+PM}}.
$$

(7.19)

The starting point of this approach rests on the simple value of the Fourier transform of the multipole moments at the lowest PN order, i.e. at the Newtonian order $(O(\eta^3))$, but at all orders in $\frac{1}{e_r}$

$$
[\mathcal{K}(u)]_N = \mathcal{K}_N^{\text{LO}}(u) + \frac{1}{e_r} \mathcal{K}_N^{\text{LO}}(u) + \frac{\eta^2}{e_r^2} \mathcal{K}_N^{\text{LO}}(u) + \cdots
$$

(7.20)

In the elliptic-motion case, it is well-known that the (discrete) Fourier expansion of the Newtonian multipole moments involve ordinary Bessel functions, namely $J_{p+k}(pe_r)$, where $p$ and $k$ are integers. In the hyperbolic-motion case the (continuous) Fourier transform of the Newtonian-level multipole moments involve integrals of the form

$$
\int_{-\infty}^{\infty} e^{q \sinh u-(p+k)u}du = 2e^{-\frac{q^2}{4}(p+k)} K_{p+k}(u),
$$

(7.21)

involving the modified Bessel function $K_{p+k}(u)$ of real argument $u$, Eq. (7.5), but of order $p + k$, where $k = 0, \pm 1, \cdots$ is an integer, while $p$ defined as

$$
p \equiv \frac{q}{e_r}, \quad q \equiv i u,
$$

(7.22)

is purely imaginary, and $u$-dependent. The Newtonian-level energy integrand $[\mathcal{K}(u)]_N$ is quadratic in time-derivatives of the Newtonian multipole moments. Remembering the fact that the variable $u$ is proportional to the frequency, $[\mathcal{K}(u)]_N$ therefore involves functions of the type

$$
u^{k_1} K_{p+k_2}(u) K_{p+k_3}(u),
$$

(7.23)

with some integers $k_1, k_2, k_3$.

There are several technical features which allow one to compute integrals involving bilinear quantities in Bessel K functions of the type (7.23). First, the Mellin transform $g_{KK}(s; \mu, \nu)$ of the function $f_{KK}(u; \mu, \nu) \equiv$
$K_\mu(u) K_\nu(u)$ has a simple explicit expression, namely
\[
g_{kk}(s; \mu, \nu) = \frac{2^{-s-3}}{\Gamma(s)} \Gamma \left( \frac{s + \mu + \nu}{2} \right) \Gamma \left( \frac{s - \mu + \nu}{2} \right) \times \Gamma \left( \frac{s + \mu - \nu}{2} \right) \Gamma \left( \frac{s - \mu - \nu}{2} \right). \tag{7.24}
\]

Differentiating the result \((7.24)\) with respect to the Mellin parameter $s$ then allows one to compute the $\ln u$-weighted integral of integrands of the form \((7.23)\).

The situation becomes more involved when going beyond the Newtonian level. Indeed, the post-Newtonian-level Fourier-domain integrands $K^{LO}_{\nu}(u)$, $K^{NLO}_{\nu}(u)$, etc can no longer be explicitly computed. For instance, the 1PN-level, $\frac{1}{\epsilon_v}$ NLO term $K^{NLO}_{1\text{PN}}(u)$ reads
\[
\mathcal{K}^{NLO}_{1\text{PN}}(u) = \frac{16}{21} u^{-3} \left[ \left( u^4 - 46u^2 - \frac{111}{5} \right) K_0(u)^2 + \frac{122}{5} u \left( u^2 - \frac{653}{122} \right) K_0(u) K_1(u) \right] + \frac{48}{5^7} \int_{-\infty}^{\infty} dv \arctan \left( \frac{\ln u}{2} \right) \left[ \frac{175}{3} \ln(2) \left( \mathcal{K}_0(u) + 2uK_1(u) \right) \cos(\ln u) \right] - \frac{64}{21} u^3 \left[ \left( u^4 - \frac{21u^2}{2} - \frac{3}{4} \right) K_0(u)^2 - \frac{6}{5} u \left( u^2 + \frac{95}{24} \right) K_0(u) K_1(u) \right]
\]
\[
\left. + \left( u^4 - \frac{23u^2}{2} - \frac{21}{20} \right) K_1(u)^2 \right] \nu. \tag{7.25}
\]

TABLE II: Analytical results for the $O(G^7)$ scattering coefficients $A_{3nk}$.

| coefficient | value |
|-------------|-------|
| $A_{330}$   | $2.072 \times 10^{-4} - 1.85 \times 10^{-3} \ln(2) - 6.45 \times 10^{-6} \left( \frac{\zeta(3)}{\pi^3} \right)$ |
| $A_{340}$   | $-2.43 \times 10^{-4} - 1.85 \times 10^{-3} \ln(2) + 6.45 \times 10^{-6} \left( \frac{\zeta(3)}{\pi^3} \right)$ |
| $A_{341}$   | $5.73 \times 10^{-4} + 2.17 \times 10^{-3} \ln(2) + 1.35 \times 10^{-4} \left( \frac{\zeta(3)}{\pi^3} \right)$ |
| $A_{342}$   | $-2.43 \times 10^{-4} - 1.85 \times 10^{-3} \ln(2) - 6.45 \times 10^{-6} \left( \frac{\zeta(3)}{\pi^3} \right)$ |

Parameters, see Eq. \((A26)\), were able to derive analytic expressions for all the scattering coefficients $A_{3nk}$ (which confirmed the results previously obtained by experimental mathematics techniques). More technical details on our analytical derivations are given in Appendix A. The final results for the $N^3$LO scattering coefficients $A_{3nk}$ appearing at the 6PN level are listed in Table II.

VIII. FINAL RESULTS FOR THE NONLOCAL CONTRIBUTIONS TO THE SCATTERING ANGLE AT $O(G^7)$

As briefly recalled in Sec. 11 there are three types of contributions to the scattering angle, as displayed in Eq. \((2.2)\): the $f$-route local contribution $\chi^{\text{loc},f}$, the $h$-route nonlocal contribution $\chi^{\text{nonloc},h}$, and the additional...
The h-route nonlocal contribution \( \chi_{\text{non loc}, h} \), in Eq. (2.2), is directly linked (via Eq. (3.1)) to the integrated action \( W_{\text{non loc}, h} \). The work done in the sections above has allowed us to derive results \([2–5]\) on contributions to nonlocal effects. Previous contributions are related to nonlocal effects. The two remaining contributions to the function \( \chi_{\text{non loc}, h} \), as per Eq. (8.5), are explained in Refs. \([3–5]\). We gather the final results for the function \( \chi_{\text{non loc}, h} \) in the following subsection.

The last contribution, \( \chi^{f-h} \), in Eq. (2.2) to the scattering angle is indirectly related to nonlocal effects. As discussed in Refs. \([3–5]\), the flexibility factor is determined, modulo some gauge freedom, by the few contributions to the function \( \chi_{\text{non loc}, h} \) that violate the simple \( \nu \)-dependence rules \([62]\) satisfied by the total scattering angle \( \chi^{\text{tot}} \). The resulting value of \( \chi^{f-h} \) will be discussed in the second subsection below.

Before listing our results for the various contributions to the scattering angle, let us recall our conventional definition of the expansion coefficients in the large-\( j \) limit (which include a factor \( \frac{1}{2} \)):

\[
\frac{1}{2} \chi^{\text{tot}}(\gamma; j; \nu) = \sum_{n \geq 1} \frac{\chi_n(\gamma; \nu)}{j^n},
\]  

with

\[
\chi_n(\gamma; \nu) = \chi_n^{\text{loc}, f}(\gamma; \nu) + \chi_n^{\text{non loc}, f}(\gamma; \nu).
\]

The various pieces of the nonlocal part

\[
\chi_n^{\text{non loc}, f}(\gamma; \nu) = \chi_n^{\text{non loc}, h}(\gamma; \nu) + \chi_n^{f-h}(\gamma; \nu),
\]

with

\[
\chi_n^{\text{non loc}, h} = \chi_n^{h, \alpha} + O(\alpha^3),
\]

will be shown as a 6PN-accurate expansion (keyed by the powers of \( p_{\infty} \equiv \sqrt{\gamma^2 - 1} \)) of the type

\[
\chi_n^{h, \alpha} = \chi_n^{h, \alpha, 4\text{PN}} + \chi_n^{h, \alpha, 5\text{PN}} + \chi_n^{h, \alpha, 6\text{PN}},
\]

and

\[
\chi_n^{h, \alpha^2} = \chi_n^{h, \alpha^2, 5\text{PN}}.
\]

Note that the third-order-tail contribution starts at the 7PN level, which is beyond the PN accuracy sought for in the present work.

A. The h-route first-order-tail contribution to the scattering angle

The \( \frac{1}{2} \)-expansion coefficients of the \( 4 + 5 + 6\text{PN} \) contribution to the first-order-tail part of the scattering angle are given by

\[
\begin{align*}
\pi^{-1} \chi_4^{h, \alpha, 4\text{PN}} &= \left[ \frac{57}{5} \ln \left( \frac{2}{\nu} \right) - \frac{63}{4} \right] p_{\infty}^4, \\
\pi^{-1} \chi_4^{h, \alpha, 5\text{PN}} &= \left[ \frac{1357}{280} + \frac{111}{10} \nu \ln \left( \frac{2}{\nu} \right) - \frac{2753}{120} + \frac{1071}{40} \nu \right] p_{\infty}^6, \\
\pi^{-1} \chi_4^{h, \alpha, 6\text{PN}} &= \left[ \frac{27953}{360} + \frac{2517}{560} \nu - \frac{111}{8} \nu^2 \ln \left( \frac{2}{\nu} \right) - \frac{155473}{8960} + \frac{109559}{40320} - \frac{186317}{5040} \nu^2 \right] p_{\infty}^8.
\end{align*}
\]
the Newtonian approximation to the Fourier transform

At our present level of accuracy, it is enough to use $B$ where we have used the result $H_{\gamma} = \frac{1}{2}\pi^2 \ln \frac{p_{\infty}}{2} - \frac{99}{4} - \frac{2079}{8} \zeta(3) \nu p_{\infty}^2$.

Working in the Fourier domain we find $H_{\gamma} = \frac{1}{2}\pi^2 \ln \frac{p_{\infty}}{2} - \frac{99}{4} - \frac{2079}{8} \zeta(3) \nu p_{\infty}^2$.

\( \pi^{-1} \chi^h_{\alpha,2PN} = \left[ \frac{9344}{15} \ln \left( \frac{p_{\infty}}{2} \right) + \frac{7934}{2} - \frac{88576}{75} \zeta(3) \right] v p_{\infty}^6, \)
\( \pi^{-1} \chi^h_{\alpha,5PN} = \left[ \frac{284224}{15} + \frac{48256}{15} \nu \ln \left( \frac{p_{\infty}}{2} \right) - \frac{621}{20} \pi^4 + \frac{2349}{15} + \frac{642}{525} \zeta(3) + \frac{9016}{175} \zeta(3) - \frac{384}{175} \right] v p_{\infty}^3, \)
\( \pi^{-1} \chi^h_{\alpha,6PN} = \left[ \frac{118912}{2} + \frac{1156416}{1575} + \frac{587984}{567} \ln \left( \frac{p_{\infty}}{2} \right) + \frac{238612}{17325} \zeta(3) + \frac{5759748}{51975} - \frac{1684}{25} \pi^2 + \frac{31772}{448} \pi^4 \right] v p_{\infty}^5. \)

\( \chi^h_{\alpha,4PN} = \left[ \frac{46597}{1208} - \frac{785}{168} \nu^2 + \frac{75505}{225} \nu \ln \left( \frac{p_{\infty}}{2} \right) - \frac{40711}{128} \zeta(3) \right] v + \nu \left[ \frac{55504}{2205} - \frac{123771328}{11025} \zeta(3) \right] v^2 \)

\( \chi^h_{\alpha,5PN} = \left[ \frac{2672}{45} - \frac{13952}{1125} \nu \ln \left( \frac{p_{\infty}}{2} \right) + \frac{11436}{1125} + \frac{221504}{525} \nu \right] v p_{\infty}^5, \)
\( \chi^h_{\alpha,6PN} = \left[ \frac{881392}{11025} - \frac{268224}{1575} \nu \ln \left( \frac{p_{\infty}}{2} \right) + \frac{21632}{2315} \nu \right] v p_{\infty}^7. \)

\( \chi^h_{\alpha,7PN} = \left[ \frac{6656}{45} - \frac{6272}{45} \ln \left( \frac{p_{\infty}}{2} \right) \right] v p_{\infty}^3, \)

\( W^{nonloc}_{5.5PN} = \alpha^2 \frac{B}{2} \left[ \int_{-\infty}^{\infty} \frac{d\tau}{\tau} H^{split}(t, \tau) \right] \),

where $B = \frac{107}{105}$ and

\( H^{split}(t, \tau) = \frac{G}{5c^5} \left[ I_{ij}^{(3)}(t) I_{ij}^{(4)}(t + \tau) - I_{ij}^{(3)}(t) I_{ij}^{(4)}(t - \tau) \right]. \)

Working in the Fourier domain we find

\[ W^{nonloc}_{5.5PN} = -\alpha^2 B \frac{G}{5c^5} \int_0^\infty d\omega \sin^2 \frac{\omega \tau}{\tau} = \pi. \]

At our present level of accuracy, it is enough to use the Newtonian approximation to the Fourier transform $\hat{I}_{ij}(\omega)$ of the quadrupole moment. Using the relations given in the previous section we have then

\[ W^{nonloc}_{5.5PN} = \alpha^2 \frac{107}{105} \frac{G}{c^5} \int_0^\infty du u K_N(u). \]

Using the results of Section 5 in \[8\], extending the large-eccentricity expansion to the NNLO order and using the frequency-domain integrals presented in Appendix \[A\], one finds

\[ \chi^h_{\alpha,2.5PN} = -\frac{47936}{675} \nu p_{\infty}^6, \]
\[ \chi^h_{\alpha,2.5PN} = -\frac{10593}{560} \pi^2 \nu p_{\infty}^5, \]
\[ \chi^h_{\alpha,2.5PN} = \left( \frac{499904}{1575} + \frac{4738816}{23625} \pi^2 \right) \nu p_{\infty}^4. \]

C. The f-h additional contribution to the scattering angle

The flexibility factor $f(t)$ has been determined in terms of the 6PN-accurate, $O(G^6)$ h-route nonlocal scattering angle in Section VII of \[8\]. Our new results at the $O(G^7)$
level do not change the determination of the flexibility factor.] The corresponding additional contribution

$$\Delta f^{\text{-h}} H = \frac{2 G H_{\text{tot}}}{c^5} \mathcal{F}^{\text{split}}_{2\text{PN}}(t, t) \ln(f(t)), \quad (8.19)$$

to the f-route nonlocal Hamiltonian has been determined in \[5\] (Eq. (7.29) there) to be equal, modulo an irrelevant canonical transformation, to

$$\Delta f^{\text{-h}} H'_{5+6\text{PN}} = \Delta f^{\text{-h}} H'^{\text{min}}_{5+6\text{PN}} + \Delta f^{\text{-h}} H'^{CD}_{5+6\text{PN}}. \quad (8.20)$$

Here, \(\Delta f^{\text{-h}} H'^{\text{min}}_{5+6\text{PN}}\) denotes the minimal part of the canonically-transformed \(\Delta f^{\text{-h}} H\) (built with the minimal solution, Eq. (7.28) there), while \(\Delta f^{\text{-h}} H'^{CD}_{5+6\text{PN}}\) denotes the part that involves six arbitrary flexibility parameters, namely: \(C_2, C_3, D_2^1, D_3^1,\) and \(D_4 = D_3^1 + \nu D_4^1.\) Explicitly, the latter contribution reads

$$\begin{align*}
\frac{\Delta f^{\text{-h}} H'^{CD}_{5+6\text{PN}}}{M} & = C_2 \frac{\nu^3 \rho^2}{r^5} + C_3 \frac{\nu^3}{r^6} \\
& + \left(D_2^0 + \frac{14}{3} \nu C_2\right) \frac{\nu^3 \rho^2}{r^5} \\
& + \left[D_3^0 + \nu \left(-\frac{3}{2} C_2 + 6 C_3\right)\right] \frac{\nu^3 \rho^2}{r^6} + (D_4^0 + \nu D_4^1) \frac{\nu^3}{r^7}. \\
& \quad (8.21)
\end{align*}$$

On the other hand, the fully determined minimal Hamiltonian \(\Delta f^{\text{-h}} H'^{\text{min}}_{5+6\text{PN}}\) given in Eq. (7.30) of \[5\] involves the coefficient

$$D = \frac{1}{r} \left(\frac{5}{2} A_{221} + \frac{15}{8} A_{200} + A_{242}\right), \quad (8.22)$$

which could not be analytically determined in \[5\]. Our new results, presented above, allow one to determine the exact analytical expression of the coefficient \(D.\) Though the individual scattering coefficients \(A_{2nk}\) entering \(D\) involve \(\zeta(3),\) it is remarkably found that \(D\) turns out to be equal to the rational number

$$D = -\frac{12607}{108}, \quad (8.23)$$

which is compatible with the previous numerical estimate of \[5\], namely \(D^{\text{num}} = \frac{-116.7348147(1)}{}\). The value of \(D\) then determines the minimal value of the flexibility coefficient \(D'^{\text{min}}_3\) (see Eq. (7.28) in \[5\]), namely

$$D'^{\text{min}}_3 = \frac{-68108}{945} \nu, \quad (8.24)$$

as well as the \(f\)-related, 6PN-level contribution to the periastron precession (see Eq. (8.30) in \[5\]):

$$K^{f^{\text{-h}}, \text{circ}, \min}(j) = \frac{68108}{945} \nu^3. \quad (8.25)$$

Inserting the analytical value of \(D\) in Eq. (7.30) of \[5\] also determines the analytical value of \(\Delta f^{\text{-h}} H'^{\text{min}}_{5+6\text{PN}},\) namely

$$\begin{align*}
\frac{\Delta f^{\text{-h}} H'^{\text{min}}_{5+6\text{PN}}}{M} & = \nu^3 \frac{168 \rho^2}{r^4} + \nu^3 \left(\frac{271066}{4725} + \frac{21736}{189} \nu^{3/2}\right) \frac{\rho^2}{r^4} \\
& - \nu^4 \frac{39712 \rho^2}{189} - \nu^4 \frac{68108 \rho^2}{945} \frac{1}{r^6}. \\
& \quad (8.26)
\end{align*}$$

Using the (canonically transformed) additional Hamiltonian \(8.20\), it is a straightforward matter to compute the large-eccentricity expansion of the corresponding integrated action

$$W^{f^{\text{-h}}} = +\frac{2 G H_{\text{tot}}}{c^5} \int dt \mathcal{F}^{\text{split}}_{2\text{PN}}(t, t) \ln(f(t))$$

$$= \int dt \Delta f^{\text{-h}} H'_{5+6\text{PN}}, \quad (8.27)$$

and the corresponding (halved) scattering angle contribution

$$\frac{1}{2} \chi^{f^{\text{-h}}} = \frac{1}{2 M^2 \nu} \frac{\partial W^{f^{\text{-h}}}(\gamma, j; \nu)}{\partial j} \quad (8.28)$$

We find
\[ \pi^{-1} \chi_4^{f-h} = -\frac{3}{32} C_1 \nu^2 p_\infty^8 + \left( \frac{27}{64} C_1 \nu - \frac{3}{64} C_1 - \frac{15}{256} D_1 \right) \nu^2 p_\infty^8, \]
\[ \chi_5^{f-h} = \left( \frac{8}{5} C_1 - \frac{8}{15} C_2 \right) \nu^2 p_\infty^6 + \left[ \frac{276}{35} C_1 + \frac{32}{15} C_2 \right] \nu - \frac{172}{35} C_1 - \frac{4}{15} C_2 - \frac{8}{7} D_1 - \frac{8}{35} D_2 \right] \nu^2 p_\infty^7, \]
\[ \pi^{-1} \chi_6^{f-h} = \left( \frac{45}{32} C_1 - \frac{15}{16} C_2 - \frac{15}{16} C_3 \right) \nu^2 p_\infty^4 + \left[ \frac{495}{64} C_1 + \frac{275}{64} C_2 + \frac{105}{32} C_3 \right] \nu - \frac{615}{64} C_1 - \frac{95}{32} C_2 - \frac{15}{32} C_3 - \frac{75}{64} D_1 - \frac{15}{32} D_2 - \frac{5}{32} D_3 \right] \nu^2 p_\infty^6, \]
\[ \chi_7^{f-h} = \left( -8C_1 - 8C_2 - 16C_3 \right) \nu^2 p_\infty^3 + \left[ \frac{252}{5} C_1 + \frac{216}{5} C_2 + \frac{344}{5} C_3 \right] \nu - 100C_1 - \frac{292}{5} C_2 - \frac{264}{5} C_3 - 8D_1 - \frac{24}{5} D_2 - \frac{16}{5} D_3 - \frac{16}{5} D_4 \right] \nu^2 p_\infty^5, \]

with minimal values (for vanishing values of \( C_2, C_3, D_2^0, D_3^0 \), and \( D_4 = D_4^0 + \nu D_4^1 \), and the minimal values \( C_1^{\min}, D_1^{\min}, D_2^{\min}, D_4^{\min} \) given in Eq. (7.28) in [3], with Eq. (8.24) above)

\[ \pi^{-1} \chi_4^{f-h} = -\frac{63}{20} \nu^2 p_\infty^6 + \left( \frac{199037}{40320} + \frac{27331}{10080} \right) \nu^2 p_\infty^8, \]
\[ \chi_5^{f-h}_{\min} = \frac{1344}{25} \nu^2 p_\infty^5 + \left( \frac{7629872}{33075} + \frac{2448608}{33075} \right) \nu^2 p_\infty^7, \]
\[ \pi^{-1} \chi_6^{f-h} = -\frac{189}{5} \nu^2 p_\infty^4 + \left( \frac{786449}{2016} + \frac{12607}{108} \right) \nu^2 p_\infty^6, \]
\[ \chi_7^{f-h}_{\min} = \frac{1344}{5} \nu^2 p_\infty^3 + \left( \frac{18044528}{4725} + \frac{90464}{75} \right) \nu^2 p_\infty^5. \] (8.29) (8.30)

**IX. CONCLUSIONS**

By using computing techniques developed for the evaluation of multi-loop Feynman integrals, we have advanced the analytical knowledge of classical gravitational scattering at the seventh order in \( G \), and at the sixth post-Newtonian accuracy, by fully determining the nonlocal-in-time contribution to the scattering angle. The present work has given a new instance of a fruitful synergy between classical GR and QFT techniques leading to an improved theoretical description of gravitationally interacting binary systems.

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**Appendix A: Details on the frequency domain computation**

The first step is to Fourier transform\(^5\) the multipolar moments (see, e.g., Eq. (6.3a)). At the Newtonian level the computation is done by using the integral representation of the Hankel functions of the first kind of order \( p \) and argument \( q \)

\[ H_p^{(1)}(q) = \frac{1}{i \pi} \int_{-\infty}^{\infty} e^{i \nu \sinh v - \nu \nu} dv. \] (A1)

As the argument \( q = \nu u \) of the Hankel function is purely imaginary, the Hankel function becomes converted into a Bessel \( K \) function, according to the relation

\[ H_p^{(1)}(\nu u) = \frac{2}{\pi} e^{-\nu^{(p+1)}} K_p(u). \] (A2)

Note that the order \( p = \nu u/c \) of the Bessel functions is purely imaginary, and proportional to the (frequency-dependent) argument \( u = \omega r a_\nu^{3/2} \). However, the order

\(^5\) In the following, we use \( GM = 1 \), i.e., we work with \( GM \)-rescaled time and frequency variables.
p tends to zero when $\epsilon_r \to \infty$, which allows most integrals to be explicitly computed when performing a large-eccentricity expansion. A typical term at the Newtonian level ($O(\eta^0)$) is of the kind $e^{g \sinh v - (p+k)v}$, the Fourier transform of which is

$$e^{g \sinh v - (p+k)v} \to 2e^{-i\frac{\pi}{2}(p+k)} K_{p+k}(u), \quad (A3)$$

involving Bessel functions having the same argument $u$, but various orders differing by integers. However, standard identities valid for Bessel functions allow one to reduce the orders $p+k$ to either $p$ or $p+1$. When taking the large-eccentricity expansion, one expands with respect to derivatives of $K_v$ involving $d$.

Mellin transform, we then have that

$$\int e^{\eta u} f(u) K_v(u) du = \frac{\Gamma(s+\nu)}{\Gamma(s+\nu+1)} \left( \frac{s+\nu+1}{2} \right) \left( \frac{s+\nu+1}{2} \cosh^2 v \right), \quad (A12)$$

One then generally has terms of the form $e^{g \sinh v - (p+k)v} f_j(v)$, involving non-trivial functions $f_j(v)$, which cannot be integrated analytically. However, in most cases one can overcome this difficulty by integrating over $u$, before integrating over $v$.

1. Integrating over the frequency spectrum and Mellin transform

The integrated nonlocal action $W^{\text{tail, h}}_{\nu}$ (Eq. (6.10)) and the GW energy $\Delta E_{\text{GW}}$ (Eq. (7.11)) are connected by the Mellin transform (Eq. (7.13)) of the function $K(u)$, being defined in terms of the integrals

$$I_{W_i} = \frac{d^2(s)}{ds^2} \bigg|_{s=1}, \quad \text{Mellin transforms are well implemented in standard symbolic algebra manipulators.}$$

At the Newtonian level, the function $K(u)$ is expressed in terms of modified Bessel functions of the second kind. The typical term has the form

$$u^k K_\nu(u) K_\nu(u), \quad (A8)$$

so that it is enough to compute the Mellin transform $g_{KK}(s; \mu, \nu)$ of the function

$$f_{KK}(u; \mu, \nu) = K_\mu(u) K_\nu(u), \quad (A9)$$

(see Eq. (7.24)) and its first derivative with respect to $s$, further using the property $\Re(x^s f(x); s) = g(s + k)$.

At higher PN orders also appear terms like

$$u^k K_\nu(u) \cos(\mu \sinh v), \quad u^k K_\nu(u) \sin(\mu \sinh v), \quad (A10)$$

to be integrated both over $u$ and $v$. Hence we also need the Mellin transform $g_{K\cos}(s; \nu, \nu)$ and $g_{K\sin}(s; \nu, \nu)$ of the functions

$$f_{K\cos}(u; \nu, \nu) = K_\nu(u) \cos(\mu \sinh v), \quad f_{K\sin}(u; \nu, \nu) = K_\nu(u) \sin(\mu \sinh v), \quad (A11)$$

and their first derivatives with respect to $s$. Their Mellin transforms are given by

$$g_{K\cos}(s; \nu, \nu) = \frac{2^{s-1}}{\cosh^{2-\nu} v} \left( \frac{s+\nu+1}{2} \cosh^2 v \right), \quad (A12)$$

For each of them ($i = \text{KK}, \text{Kcos}, \text{Ksin}$) we need then

$$g_{ks} = \frac{\partial}{\partial s} g_i, \quad g_{ivos} = \frac{\partial^2}{\partial s \partial \nu} g_i, \quad g_{isov} = \frac{\partial^3}{\partial s \partial \nu^2} g_i, \quad (A13)$$

(for example, $g_{KK} = g_{KK}(s; \mu, \nu)$, etc.) and higher derivatives with respect to the order $\nu$ for increasing PN accuracy as well as level of expansion in the eccentricity parameter. Explicit expressions can be obtained which generally involve HPLs, coming from the derivatives of the hypergeometric functions with respect to their parameters ($s, \nu$) (see below).

2. Results

The function $K(u)$ can be decomposed as in Eq. (7.18) (here in a conveniently rescaled form)

$$K(u) = K_N(u) + \frac{\eta^2}{a_\nu} K_{1PN}(u) + \frac{\eta^4}{a_\nu^4} K_{2PN}(u), \quad (A14)$$
Using the same decomposition as above for \( K_{nPN}(u) \) with

\[
K_{nPN}(u) = \frac{\nu^2}{\epsilon_r^2 u^2} \left[ K^\text{LO}_{nPN}(u) + \frac{\pi}{\epsilon_r^2} K^\text{NLO}_{nPN}(u) \right] + \frac{1}{\epsilon_r^2} K^\text{NNLO}_{nPN}(u) + \frac{\pi}{\epsilon_r^2} K^\text{NNLO}_{1nPN}(u),
\]

up to the 2PN order and to the N^3LO order in the large eccentricity.

\[
K^\text{LO}_{N}(u) = \frac{32}{5} u^2 \left( \frac{1}{3} + u^2 \right) K_0^2(u) + 3u K_0(u) K_1(u) + (1 + u^2) K_2^2(u),
\]

\[
K^\text{NLO}_{N}(u) = u K^\text{LO}_{N}(u),
\]

\[
K^\text{NNLO}_{N}(u) = \frac{\pi^2}{2} u^2 K^\text{LO}_{N}(u) - \frac{32}{5} u^2 \left\{ (1 + 3u^2) K_0^2(u) + 7u K_0(u) K_1(u) + (1 + 2u^2) K_2^2(u) \right. \\
+ u^2 \left[ \left( u^2 + \frac{1}{3} \right) K_0(u) + \frac{3}{2} u K_1(u) \right] \left. \frac{\partial^2 K_{\nu\nu}(u)}{\partial \nu^2} \right|_{\nu=0} + u^2 \left[ \frac{3}{2} u K_0(u) + (u^2 + 1) K_1(u) \right] \frac{\partial^2 K_{\nu\nu}(u)}{\partial \nu^2} \right|_{\nu=1},
\]

\[
K^\text{NNLO}_{N}(u) = u K^\text{NNLO}_{N}(u) - \frac{\pi^2}{3} u^4 K^\text{LO}_{N}(u).
\]

Using the same decomposition as above for \( I_{\Delta E,N} \) we find

\[
I^\text{LO}_{\Delta E,N} = \frac{32}{15} \left[ 3g_{KK}(5;1,1) + 3g_{KK}(3;1,1) + 9g_{KK}(4;0,1) + 3g_{KK}(5;0,0) + g_{KK}(3;0,0) \right]
\]

\[
= \frac{37}{15} \pi^2,
\]

\[
I^\text{NLO}_{\Delta E,N} = \frac{32}{15} \left[ 3g_{KK}(6;1,1) + 3g_{KK}(4;1,1) + 9g_{KK}(5;0,1) + 3g_{KK}(6;0,0) + g_{KK}(4;0,0) \right]
\]

\[
= \frac{1568}{45},
\]

\[
I^\text{NNLO}_{\Delta E,N} = \frac{16}{15} \left[ -9g_{KK\nu\nu}(6;1,0) - 6g_{KK\nu\nu}(7;0,0) - 2g_{KK\nu\nu}(5;0,0) \right]
\]

\[
+ \frac{16}{15} \left[ -9g_{KK\nu\nu}(6;0,1) - 6g_{KK\nu\nu}(7;1,1) - 6g_{KK\nu\nu}(5;1,1) \right]
\]

\[
+ \frac{16}{15} \left[ 3\pi^2 \left( g_{KK}(7;0,0) + g_{KK}(7;1,1) + g_{KK}(5;1,1) + 3g_{KK}(6;0,1) + \frac{1}{3} g_{KK}(5;0,0) \right) \right.
\]

\[
- 12g_{KK}(5;1,1) - 6g_{KK}(3;1,1) - 42g_{KK}(4;0,1) - 18g_{KK}(5;0,0) - 6g_{KK}(3;0,0) \right]
\]

\[
= \frac{281}{10} \pi^2,
\]

\[
I^\text{N^3LO}_{\Delta E,N} = \frac{16}{45} \left[ -27g_{KK\nu\nu}(7;1,0) - 18g_{KK\nu\nu}(8;0,0) - 6g_{KK\nu\nu}(6;0,0) \right]
\]

\[
+ \frac{16}{45} \left[ -27g_{KK\nu\nu}(7;0,1) - 18g_{KK\nu\nu}(8;1,1) - 18g_{KK\nu\nu}(6;1,1) \right]
\]

\[
+ \frac{16}{45} \left[ 3\pi^2 \left( g_{KK}(8;1,1) + g_{KK}(6;1,1) + 3g_{KK}(7;0,1) + \frac{1}{3} g_{KK}(6;0,0) \right) \right.
\]

\[
- 18g_{KK}(4;1,1) - 36g_{KK}(6;1,1) - 126g_{KK}(5;0,1) - 54g_{KK}(6;0,0) - 18g_{KK}(4;0,0) \right]
\]

\[
= \frac{7808}{45},
\]

(A17)

where the values of the various Mellin transforms are listed in Table III. The corresponding result for \( I_{W_{1,N}} \) is obtained
simply by replacing each of them by its derivative with respect to the Mellin parameter, leading to

\[
I_{W_1, N}^{\text{LO}} = \left( \frac{40}{3} - \frac{74}{5} \ln(2) - \frac{74}{15} \pi^2 \right),
\]

\[
I_{W_1, N}^{\text{NLO}} = \frac{4448}{135} + \frac{3136}{45} \ln(2) - \frac{3136}{45} \pi^2,
\]

\[
I_{W_1, N}^{\text{NNLO}} = \frac{2479}{30} - \frac{843}{5} \ln(2) - \frac{281}{5} \pi^2 + \frac{2079}{20} \zeta(3),
\]

\[
I_{W_1, N}^{\text{NNLO}} = - \frac{23936}{675} + \frac{15616}{45} \ln(2) - \frac{15616}{45} \pi^2 + \frac{88576}{225} \zeta(3).
\]

Starting from the 1PN level, the Fourier transform of the multipolar moments can be explicitly done only partly, so that the resulting function \( K(u) \) is not fully determined in closed form. Consider, for instance, the NLO term

\[
K_{\text{IPN}}^{\text{NLO}}(u) = \frac{16}{21} u^3 \left[ \left( u^4 - 46u^2 - \frac{141}{5} \right) K_0(u)^2 + \frac{122}{5} \left( u^2 - \frac{653}{122} \right) K_0(u)K_1(u) + \left( u^4 - \frac{333u^2}{10} - \frac{39}{5} \right) K_1(u)^2 \right]
\]

\[
- \frac{48}{5\pi} u^4 \int_\infty^{-\infty} dv \arctan \left[ \tanh \frac{v}{2} \right] \sinh 2v (K_0(u) + 2uK_1(u)) \cos(u \sinh v)
\]

\[
+ \frac{1}{2} \cosh 3v \left( K_1(u) \right) \sin(u \sinh v)
\]

\[
- \frac{64}{21} \left[ \left( u^4 - \frac{21u^2}{20} - \frac{3}{4} \right) K_0(u)^2 - \frac{6}{5} u \left( u^2 + \frac{95}{24} \right) K_0(u)K_1(u) + \left( u^4 - \frac{23u^2}{20} - \frac{21}{20} \right) K_1(u)^2 \right] \nu.
\]

It is convenient taking the Mellin transform first (i.e., integrating over \( u \)), and then integrating over \( v \). We find

\[
I_{\Delta E, \text{IPN}}^{\text{NLO}} = - \frac{888}{35} g_{KK}(6; 1, 1) - \frac{208}{35} g_{KKK}(6; 1, 1) + \frac{16}{21} g_{KKK}(8; 1, 1) + \frac{1952}{105} g_{KKK}(7; 0, 1) - \frac{10448}{105} g_{KK}(5; 0, 1)
\]

\[
+ \frac{1}{2} \int dv \arctan \left[ \tanh \frac{v}{2} \right] \times
\]

\[
- \left[ \frac{48}{5} \sinh(2v)(2g_{KC}(6; 1, 1) + g_{KC}(5; 0, 0)) + \frac{96}{5} \cosh(v)(\cosh(v)^2 - 2)(g_{K\sin}(5; 1, 1) + g_{K\sin}(6; 0, 0)) \right]
\]

\[
+ \left[ \frac{64}{21} g_{KK}(8; 1, 1) + \frac{368}{105} g_{KK}(6; 1, 1) + \frac{16}{8} g_{KK}(4; 1, 1) + \frac{128}{35} g_{KK}(7; 0, 1) + \frac{304}{21} g_{KK}(5; 0, 1) \right]
\]

\[
- \frac{64}{21} g_{KK}(8; 0, 0) + \frac{16}{5} g_{KK}(6; 0, 0) + \frac{16}{7} g_{KK}(4; 0, 0) \right] \nu
\]

\[
= - \frac{25616}{315} \cosh(v) \sinh(v) \left( - \frac{4032}{5} + \frac{2448}{\cosh(v)} \right) \frac{1136}{45} \nu.
\]

where we have used

\[
g_{KC}(5; 0, v) = - \frac{3\pi}{2 \cosh^9 v} (8 \cosh^4 v - 40 \cosh^2 v + 35),
\]

\[
g_{KC}(6; 1, v) = \frac{45\pi}{2 \cosh^{11} v} (8 \cosh^4 v - 28 \cosh^2 v + 21),
\]

\[
g_{K\sin}(5; 1, v) = - \frac{15\pi \sinh v}{2 \cosh^9 v} (4 \cosh^2 v - 7),
\]

\[
g_{K\sin}(6; 0, v) = \frac{15\pi \sinh v}{2 \cosh^{11} v} (8 \cosh^4 v - 56 \cosh^2 v + 63).
\]
| $1 + k$ | $\nu$ | $\nu' G_{KK}$ | $\nu' G_{KK}$ | $\nu G_{KK}$ | $\nu G_{KK}$ |
|---------|------|----------------|----------------|----------------|----------------|
| 3       | 0    | $\frac{1}{4}$ | $\pi^2 (6 \ln(2) - 5 + 2 \gamma)$ | $\frac{9}{4 \pi^2}$ | $\pi^2 (\pi^2 - 8)$ |
| 3       | 0    | $\frac{1}{7}$ | $\frac{1}{2} (2 \ln 2 - 2 \gamma - 1)$ | $\frac{1}{2} (3 + \gamma)$ | $\frac{1}{12}$ |
| 3       | 1    | $3 \frac{2}{4}$ | $\pi^2 (18 \ln(2) - 11 + 6 \gamma)$ | $\frac{43}{64}$ | $\pi^2 (3 \pi^2 - 8)$ |
| 4       | 0    | $\frac{1}{6}$ | $\frac{1}{2} \ln(2) + \frac{1}{2} \gamma - \frac{1}{2} \gamma$ | $\frac{3}{32} \pi^2 (9 \pi^2 - 68)$ | $\frac{1}{2}$ |
| 4       | 0    | $\frac{1}{4}$ | $\pi^2 (18 \ln(2) - 17 + 6 \gamma)$ | $\frac{1}{2} \ln(2) + \frac{1}{2} \gamma$ | $\frac{1}{6}$ |
| 4       | 1    | $\frac{2}{3}$ | $\ln(2) - \frac{1}{2} \gamma - \frac{2}{3} \gamma$ | $\frac{2}{3} \ln(2)$ | $\frac{2}{3}$ |
| 5       | 0    | $\frac{26}{12}$ | $\pi^2 (12 \ln(2) - 13 + 4 \gamma)$ | $\frac{3}{8} \pi^2 (12 \ln(2) - 13 + 4 \gamma)^2$ | $\frac{3}{16} \pi^2 (12 \ln(2) - 13 + 4 \gamma)^2$ |
| 5       | 0    | $\frac{9}{5}$ | $\frac{1}{2} \ln(2) + \frac{1}{2} \gamma - \frac{1}{2} \gamma$ | $\frac{1}{5} \ln(2)$ | $\frac{1}{5} \ln(2)$ |
| 5       | 1    | $\frac{1}{5}$ | $\frac{1}{2} \ln(2) + \frac{1}{2} \gamma - \frac{1}{2} \gamma$ | $\frac{1}{5} \ln(2)$ | $\frac{1}{5} \ln(2)$ |
| 6       | 0    | $\frac{18}{15}$ | $\ln(2) + \frac{172}{225} - \frac{15}{15} \gamma$ | $\frac{4}{3} + \frac{5}{3} \delta \gamma$ | $\frac{1}{10} \ln(2)$ |
| 6       | 1    | $\frac{145}{2024}$ | $\ln(2) + \frac{432}{4076} - \frac{36}{36} \gamma$ | $\frac{143}{2024} \ln(2)$ | $\frac{143}{2024} \ln(2)$ |
| 6       | 1    | $\frac{135}{2024}$ | $\ln(2) + \frac{432}{4076} - \frac{36}{36} \gamma$ | $\frac{143}{2024} \ln(2)$ | $\frac{143}{2024} \ln(2)$ |
| 6       | 1    | $\frac{8}{5}$ | $\ln(2) + \frac{16}{225} + \frac{8}{8} \gamma$ | $\frac{5}{3} + \frac{4}{3} \delta \gamma$ | $\frac{1}{10} \ln(2)$ |
| 7       | 0    | $\frac{125}{4060}$ | $\ln(2) + \frac{1635}{16384} + \frac{1635}{16384} \gamma$ | $\frac{5}{3} \ln(2)$ | $\frac{5}{3} \ln(2)$ |
| 7       | 0    | $\frac{1}{4}$ | $\ln(2) + \frac{432}{4076} - \frac{36}{36} \gamma$ | $\frac{143}{2024} \ln(2)$ | $\frac{143}{2024} \ln(2)$ |
| 7       | 1    | $\frac{16}{5}$ | $\frac{1}{2} \ln(2) + \frac{122}{125} - \frac{16}{16} \gamma$ | $\frac{172}{45} + \frac{14}{3} \delta \gamma$ | $\frac{1}{10} \ln(2)$ |
| 7       | 1    | $\frac{1575}{4096}$ | $\ln(2) + \frac{1635}{16384} + \frac{1635}{16384} \gamma$ | $\frac{5}{3} \ln(2)$ | $\frac{5}{3} \ln(2)$ |
| 8       | 0    | $\frac{24}{15}$ | $\ln(2) + \frac{1562}{1225} - \frac{238}{238} \gamma$ | $\frac{128}{3} + \frac{128}{3} \delta \gamma$ | $\frac{1}{10} \ln(2)$ |
| 8       | 1    | $\sqrt{7755} \pi^2$ | $\ln(2) + \frac{1562}{1225} - \frac{238}{238} \gamma$ | $\frac{128}{3} + \frac{128}{3} \delta \gamma$ | $\frac{1}{10} \ln(2)$ |
| 8       | 1    | $\frac{120525}{8192}$ | $\ln(2) + \frac{1562}{1225} - \frac{238}{238} \gamma$ | $\frac{128}{3} + \frac{128}{3} \delta \gamma$ | $\frac{1}{10} \ln(2)$ |
The corresponding result for \( I_{W_1,1PN}^{\text{NLO}} \) is

\[
I_{W_1,1PN}^{\text{NLO}} = -\frac{3536}{135} - \frac{51232}{315} \ln(2) + \frac{51232}{315} \gamma \\
+ \int dv \arctan \left( \tanh \left( \frac{v}{2} \right) \right) \frac{\sinh v}{\cosh^2 v} \left[ -\frac{576}{5} + \left( \frac{8064}{5} \ln(2) + \frac{8064}{5} \gamma + \frac{16128}{5} \ln(\cosh v) - \frac{34464}{5} \right) \frac{1}{\cosh^2 v} \\
+ \left( -9792 \ln(\cosh v) + \frac{86592}{5} - 4896 \ln(2) - 4896 \gamma \right) \frac{1}{16 \cosh^2 v} \\
+ \left( \frac{77744}{945} - \frac{2272}{45} \ln(2) + \frac{2272}{45} \gamma \right) v \right] \\
= -\frac{56144}{3375} + \frac{1888}{1575} \ln(2) + \frac{1888}{1575} \gamma + \left( \frac{77744}{945} - \frac{2272}{45} \ln(2) + \frac{2272}{45} \gamma \right) v, \tag{A22}
\]

where we have used

\[
g_{\text{Kcos}}(5; 0, v) = -\frac{12 \pi}{\cosh v} \left[ \left( 2 \cosh^4 v - 10 \cosh^2 v + \frac{35}{4} \right) \ln(\cosh v) + \left( \gamma - \frac{25}{6} + \ln(2) \right) \cosh^4 v \\
+ \left( \frac{107}{6} - 5 \ln(2) - 5 \gamma \right) \cosh^2 v + \frac{35}{8} \ln(2) - \frac{44}{3} + \frac{35}{8} \gamma \right],
\]

\[
g_{\text{Kcos}}(6; 1, v) = -\frac{180 \pi}{\cosh^3 v} \left[ \left( 2 \cosh^4 v - 7 \cosh^2 v + \frac{21}{4} \right) \ln(\cosh v) + \frac{1}{15} \cosh^6 v + \left( \gamma + \ln(2) - \frac{127}{30} \right) \cosh^4 v \\
+ \left( \frac{7}{2} \ln(2) + \frac{1583}{120} - \frac{7}{2} \right) \cosh^2 v + \frac{21}{8} \ln(2) + \frac{21}{8} \gamma - \frac{563}{60} \right],
\]

\[
g_{\text{Kins}}(5; 1, v) = \frac{\pi \sinh v}{2 \cosh^4 v} \left[ (120 \cosh^2 v - 210) \ln(\cosh v) + 6 \cosh^4 v \\
+ (60 \gamma - 229 + 60 \ln(2)) \cosh^2 v - 105 \gamma + 352 - 105 \ln(2) \right],
\]

\[
g_{\text{Kins}}(6; 0, v) = -\frac{60 \pi \sinh v}{\cosh^4 v} \left[ \left( \frac{63}{4} + 2 \cosh^4 v - 14 \cosh^2 v \right) \ln(\cosh v) + \left( \gamma - \frac{137}{30} + \ln(2) \right) \cosh^4 v \\
+ \left( \frac{809}{30} - 7 \gamma - 7 \ln(2) \right) \cosh^2 v + \frac{63}{8} \ln(2) - \frac{563}{20} + \frac{63}{8} \gamma \right]. \tag{A23}
\]

At the NNLO the derivatives of the hypergeometric functions entering the Mellin transforms \([A12]\) also generates HPLs of weight 2. Consider, for instance, the Mellin transform \(g_{\text{Kcos}}(6; 0, v)\) and its derivative \(g_{\text{Kcos}}(6; 0, v)\). We find

\[
g_{\text{Kcos}}(6; 0, v) = \left( -\frac{120}{\cosh v} + \frac{840}{\cosh^2 v} - \frac{945}{\cosh^4 v} \right) v \sinh v + \frac{274}{\cosh^6 v} - \frac{1155}{\cosh^8 v} + \frac{945}{\cosh^{10} v}, \tag{A24}
\]

and

\[
g_{\text{Kcos}}(6; 0, v) = -g_{\text{Kcos}}(6; 0, v) \left( \ln(\cosh v) - \ln(2) + \gamma - \frac{3}{2} \right) \\
+ \frac{64}{\cosh^6 v} \frac{\partial}{\partial s} \left[ T_1 \left( s, \frac{1 - s}{2}, \frac{1}{2} \right) \tanh(v)^2 \right] \bigg|_{s=6} \tag{A25}
\]

respectively. The latter term can be computed, e.g., by using the tool \texttt{HypExp2} \[13\], which allows for Taylor-expanding hypergeometric functions around their parameters. It reads

\[
\frac{\partial}{\partial s} \left[ T_1 \left( s, \frac{1 - s}{2}, \frac{1}{2} \right) \tanh(v)^2 \right] \bigg|_{s=6} = \left[ \left( \frac{15}{8 \cosh v} + \frac{105}{8 \cosh^3 v} - \frac{945}{64 \cosh^5 v} \right) v \sinh v \\
+ \frac{945}{1155} - \frac{137}{32} \ln(\cosh v) \right] v \sinh v \\
+ \left( \frac{128 \cosh^8 v}{64 \cosh^6 v} - \frac{16 \cosh^6 v}{16 \cosh^8 v} + \frac{15}{16 \cosh^2 v} \right) |\sinh v| H_{\gamma + \left( \frac{1}{2} \tanh v \right)} \\
+ \left( \frac{247}{141} - \frac{3921}{39} + \frac{23}{219} \right) v \sinh v \\
+ \left( \frac{128 \cosh^8 v}{64} - \frac{128 \cosh^5 v}{128 \cosh^2 v} \right), \tag{A26}
\]
where
\[ H_{\ldots}(|\tanh v|) = 2 \ln(2 \cosh v)|v| + \text{Li}_2 \left( \frac{1}{2} \frac{|\sinh v|}{2 \cosh v} \right) \]
\[ - \text{Li}_2 \left( \frac{1}{2} \frac{|\sinh v|}{2 \cosh v} \right), \]  
(A27)
is an HPL with weights ±, which can be in turn converted into HPLs with integer weights according to the rule
\[ H_{\ldots}(x) = -H_{-1,-1}(x) - H_{-1,1}(x) + H_{1,-1}(x) + H_{1,1}(x). \]  
(A28)

Going to the 2PN level we get more involved expressions, but with the same structure (furtherv including terms containing derivatives of the Bessel functions with respect to the order up to the fourth at N^3LO as well as HPLs of increasing weight).

Appendix B: Summary of final results for the integrated nonlocal action

We recap below our final results for the integrated nonlocal action \( W^{\text{tail},h} \) up to the N^3LO order in the large eccentricity expansion, showing also equivalent forms corresponding to different choices of orbital parameters used as independent variables, i.e., either \((\bar{a}_r, e_r)\) or \((\bar{E}, j)\), which are related by Eq. (4.5).

1. First-order-tail part

The 2PN-accurate values of the two contributions to the first-order tail \( W^{\text{tail},h} = W_1^{\text{tail},h} + W_2^{\text{tail},h} \) i.e.,
\[ W_1^{\text{tail},h} = W_1^{\text{tail},h,\text{LO}} + W_1^{\text{tail},h,\text{NLO}} + W_1^{\text{tail},h,\text{NNLO}} + W_1^{\text{tail},h,\text{N^3LO}} + O(e_r^{-7}), \]
(B1)

are listed in Tables [IV] and [V]. It is easily seen that the intermediate scale \( s \) cancels between the two contributions.

Re-expressing \( \bar{a}_r \) and \( e_r \) in terms of \( \bar{E} \) and \( j \) we get
\[ W^{\text{tail},h} = \frac{W_3}{j^3} + \frac{W_4}{j^4} + \frac{W_5}{j^5} + \frac{W_6}{j^6} + O(j^{-7}). \]
(B2)

with coefficients \( W_k \), \( k = 3, 4, 5, 6 \) listed in Table [VI].

The \( f \)-induced additional contribution reads
\[ W^{\text{f-h}} = \frac{M \nu^3}{\bar{a}_r^{9/2}} H_{\text{tot}} \eta^2 \left[ \frac{\pi}{e_r^2} \left( \frac{21}{\nu} \frac{e_r}{25} + \frac{15120}{15120} \nu + \frac{580661}{60480} \nu^2 \right) + \frac{1}{e_r^6} \left( \frac{672}{25} + \left( \frac{1668832}{33075} \nu + \frac{4703992}{33075} \right) \nu^2 \right) \right] + \frac{\pi}{e_r^2} \left[ \frac{441}{20} \frac{1732117}{30240} \nu + \frac{594173}{4480} \nu^2 \frac{1}{\bar{a}_r} \right], \]
(B3)

with the various \( W^{\text{f-h},\text{NLO}} \) listed in Table [VII].

Using the minimal solution of the 5+6PN constraints
\[ C_1^{\text{min}} = \frac{168}{5}, \quad C_2^{\text{min}} = 0, \quad C_3^{\text{min}} = 0; \]
\[ D_1^{\text{min}} = \frac{271066}{189}, \quad D_2^{\text{min}} = \frac{21736}{189}, \quad D_3^{\text{min}} = \frac{39712}{189}, \]
(B4)

the previous expression becomes
\[ W^{\text{f-h}} = \frac{W_3}{j^3} + \frac{W_4}{j^4} + \frac{W_5}{j^5} + \frac{W_6}{j^6} + O(j^{-7}), \]
(B6)

with the coefficients \( W_n^{\text{f-h}} \) listed in Table [VIII] below.
TABLE IV: Expressions for the various coefficients $W_{1}^{\text{tail,h,LO}}$ of the large-$c_r$ expansion [141] of the first-order-tail $W_{1}^{\text{tail,h}}$.

| Coefficient | Expression |
|-------------|------------|
| $W_{1}^{\text{tail,h,LO}}$ | $\frac{2}{15} \frac{\alpha \pi^2}{\alpha_t^2} H_{\text{tot}} \left\{ 100 + 37 \ln \left( \frac{10}{\pi} \right) + 685 \frac{1}{2} + 1017 \frac{14}{14} \nu + \frac{3429}{96} + \frac{37}{2} \nu \ln \left( \frac{10}{\pi} \right) \right\} \frac{2^2}{\pi} $ |
| $W_{1}^{\text{tail,h,NLO}}$ | $\frac{2}{15} \frac{\alpha \pi^2}{\alpha_t^2} H_{\text{tot}} \left\{ \frac{224}{9} + 156 \ln \left( \frac{10}{\pi} \right) + \left[ \frac{29072}{225} - \frac{39872}{63} \right] + \left[ \frac{344}{105} - \frac{1136}{3} \right] \nu \ln \left( \frac{10}{\pi} \right) \right\} \frac{2^2}{\pi} $ |
| $W_{1}^{\text{tail,h,NNLO}}$ | $\left( \frac{2}{15} \frac{\alpha \pi^2}{\alpha_t^2} H_{\text{tot}} \left\{ \frac{2479}{4} + 637 \ln \left( \frac{10}{\pi} \right) + 2 \nu \ln \left( \frac{10}{\pi} \right) \right\} \frac{2^2}{\pi} \right. $ |
| $W_{1}^{\text{tail,h,N3LO}}$ | $\left( \frac{2}{15} \frac{\alpha \pi^2}{\alpha_t^2} H_{\text{tot}} \left\{ \frac{11608}{9} + \frac{4622}{19} \ln \left( \frac{10}{\pi} \right) \right\} \frac{2^2}{\pi} \right. $ |

TABLE V: Expressions for the various coefficients $W_{2}^{\text{tail,h,LO}}$ of the large-$c_r$ expansion [141] of the first-order-tail $W_{2}^{\text{tail,h}}$.

| Coefficient | Expression |
|-------------|------------|
| $W_{2}^{\text{tail,h,LO}}$ | $\frac{2}{15} \frac{\alpha \pi^2}{\alpha_t^2} H_{\text{tot}} \left\{ \frac{85}{9} - 37 \ln \left( \frac{10}{\pi} \right) + \frac{9679}{224} + \frac{981}{86} \nu + \left( \frac{3429}{96} + \frac{37}{2} \nu \ln \left( \frac{10}{\pi} \right) \right\} \frac{2^2}{\pi} $ |
| $W_{2}^{\text{tail,h,NLO}}$ | $\frac{2}{15} \frac{\alpha \pi^2}{\alpha_t^2} H_{\text{tot}} \left\{ \frac{278}{9} - 156 \ln \left( \frac{10}{\pi} \right) + \frac{2204}{225} - \frac{599}{45} = \frac{545}{3} \right\} \frac{2^2}{\pi} $ |
| $W_{2}^{\text{tail,h,NNLO}}$ | $\frac{2}{15} \frac{\alpha \pi^2}{\alpha_t^2} H_{\text{tot}} \left\{ \frac{414199}{181} + \frac{42872}{86} \ln \left( \frac{10}{\pi} \right) \right\} \frac{2^2}{\pi} $ |
| $W_{2}^{\text{tail,h,N3LO}}$ | $\frac{2}{15} \frac{\alpha \pi^2}{\alpha_t^2} H_{\text{tot}} \left\{ \frac{1728}{9} - 809 \ln \left( \frac{10}{\pi} \right) \right\} \frac{2^2}{\pi} $ |

Using the minimal value solutions of the $C_1$ and $D_1$ we find

$$ W_{\text{tail,h,LO}}^{\text{min}} = \left( \frac{2 \tilde{E}}{\tilde{E}} \right)^3 \frac{M^2 \nu^2 \eta^2}{\bar{g}^2} \left\{ \begin{array}{l} \frac{21}{10} + \frac{294293}{60480} - \frac{10229}{3024} \left( \frac{2 \tilde{E}}{\tilde{E}} \right)^2 \end{array} \right\} \left( \frac{2 \tilde{E}}{\tilde{E}} \right)^2 $$

$$ + \frac{2 \tilde{E}}{\tilde{E}} \left\{ \begin{array}{l} \frac{672}{25} + \frac{4370596}{33075} - \frac{2446756}{33075} \left( \frac{2 \tilde{E}}{\tilde{E}} \right)^3 \end{array} \right\} \left( \frac{2 \tilde{E}}{\tilde{E}} \right)^3 $$

$$ + \pi \left( \frac{2 \tilde{E}}{\tilde{E}} \right)^{-1} \left\{ \begin{array}{l} \frac{189}{10} + \frac{834707}{5040} - \frac{22813}{270} \left( \frac{2 \tilde{E}}{\tilde{E}} \right)^2 \end{array} \right\} \left( \frac{2 \tilde{E}}{\tilde{E}} \right)^2 $$

$$ + \frac{\pi \left( 2 \tilde{E} \right)^{-3}}{\tilde{E}^2} \left\{ \begin{array}{l} \frac{448}{5} + \frac{18520808}{14175} - \frac{143384}{225} \left( \frac{2 \tilde{E}}{\tilde{E}} \right)^2 \end{array} \right\} \left( \frac{2 \tilde{E}}{\tilde{E}} \right)^2 \right\}. \quad \text{(B7)}
TABLE VI: Coefficients $W_n$ entering the large-$j$ expansion \([B9]\) of the first-order tail $W^\text{tail,h} = W_1^\text{tail,h} + W_2^\text{tail,h}$.

| Coefficient | Expression |
|-------------|-------------|
| $W_2$      | $\pi(2E)^3M^2\nu^2\left\{\frac{215}{56}C_1 + \frac{371}{112}\nu + \frac{2293}{80} \ln\left(\frac{E}{2}\right)\right\}(2E)^2\eta^2$ + $\left[\frac{1899347}{806} - \frac{29231\nu}{2016} + \frac{29231\nu^2}{4032} + \left(\frac{7415}{384}\right)\nu + \frac{49965}{8192}\nu^2\right]\ln\left(\frac{E}{2}\right)(2E)^2\eta^2$ |
| $W_3$      | $\pi(2E)^3M^2\nu^2\left\{\frac{69}{32}\nu + \frac{1565}{1536}\ln(8E) + \left[\frac{1565}{1536} - \frac{27\ln(5\nu)}{6\nu} + \frac{2548}{3\nu}\right]\ln(8E)\right\}(2E)^2\eta^2$ + $\left[\frac{26516567}{931120} + \frac{7800556\nu^2}{232940} + \frac{1004327\nu}{232940} - \frac{67099}{70}\nu + \frac{4353}{4}\nu^2\right]\ln(8E)(2E)^2\eta^4$ |
| $W_4$      | $\pi(2E)^3M^2\nu^2\left\{\frac{215}{56}C_1 + \frac{371}{112}\nu + \frac{2293}{80} \ln\left(\frac{E}{2}\right)\right\}(2E)^2\eta^2$ + $\left[\frac{1899347}{806} - \frac{29231\nu}{2016} + \frac{29231\nu^2}{4032} + \left(\frac{7415}{384}\right)\nu + \frac{49965}{8192}\nu^2\right]\ln\left(\frac{E}{2}\right)(2E)^2\eta^2$ |
| $W_5$      | $\pi(2E)^3M^2\nu^2\left\{\frac{69}{32}\nu + \frac{1565}{1536}\ln(8E) + \left[\frac{1565}{1536} - \frac{27\ln(5\nu)}{6\nu} + \frac{2548}{3\nu}\right]\ln(8E)\right\}(2E)^2\eta^2$ + $\left[\frac{26516567}{931120} + \frac{7800556\nu^2}{232940} + \frac{1004327\nu}{232940} - \frac{67099}{70}\nu + \frac{4353}{4}\nu^2\right]\ln(8E)(2E)^2\eta^4$ |

TABLE VII: Coefficients $W^{f-h,\text{LO}}_n$ entering the large-$e_0$ expansion \([B3]\) of $f$-$h$ contribution $W^f$ to the first-order tail.

| Coefficient | Expression |
|-------------|-------------|
| $W^{f-h,\text{LO}}$ | $\frac{1}{16}C_1 + \left(-\frac{3}{64}\nu^2C_1 + \frac{5}{32}C_1 + \frac{5}{128}D_1\right)\frac{E}{2\pi}$ |
| $W^{f-h,\text{NLO}}$ | $\frac{2}{3}C_1 + \frac{1}{6}C_2 + \left(-\frac{2}{9}C_2 - \frac{5}{36}C_1\right)\nu + \frac{1}{3}D_1 + \frac{2}{9}C_2 + \frac{5}{36}D_2 + \frac{10}{27}C_1\frac{E}{\pi}$ |
| $W^{f-h,\text{NNLO}}$ | $\frac{4}{5}C_1 + \frac{1}{6}C_2 + \frac{2}{3}C_2$ + \left(-\frac{141}{147}C_1 - \frac{41}{36}C_2 - \frac{25}{108}C_3\right)\nu + \frac{145}{36}C_2 + \frac{41}{36}D_3 - \frac{13}{10}D_2 - \frac{13}{36}C_3 + \frac{13}{10}D_1\frac{E}{10}$ |
| $W^{f-h,\text{N^3LO}}$ | $\frac{4}{15}C_1 + \frac{1}{6}C_2 + \frac{2}{3}C_2$ + \left(-\frac{141}{147}C_1 - \frac{41}{36}C_2 - \frac{25}{108}C_3\right)\nu + \frac{145}{36}C_2 + \frac{41}{36}D_3 - \frac{13}{10}D_2 - \frac{13}{36}C_3 + \frac{13}{10}D_1\frac{E}{10}$ |

2. Second-order-tail part

Finally, the second-order-tail contribution turns out to be

$$W^\text{tail,h,5PN} = \frac{M^2\nu^2}{e^2a_r^2}\left[\frac{23968}{675} + \frac{10593\pi^3}{1400}\right]\frac{1}{e_r} + \left(\frac{835456}{4725} + \frac{4738816}{70875}\right)\frac{1}{e_r^2} + \left(\frac{499904}{4725} + \frac{4738816}{70875}\right)\frac{1}{e_r^3},$$

or equivalently

$$W^\text{tail,h,5PN} = \frac{M^2\nu^2(2E)^2}{j^2}\left[\frac{23968}{675}(2E) + \frac{10593\pi^3(2E)^{1/2}}{j}\right]\frac{1}{e_r^2} + \left(\frac{835456}{4725} + \frac{4738816}{70875}\right)\frac{1}{e_r^4}.$$
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