Tunneling Problems between Bose-Einstein Condensates

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Abstract. We investigate transmission and reflection of Bose-Einstein condensate excitations in the low-energy limit across a potential barrier separating two condensates with different densities. Bogoliubov excitation in the low-energy limit has an incident angle where perfect transmission occurs. This condition corresponds to the Brewster’s law for the electromagnetic wave. There also exists the total internal reflection of the Bogoliubov excitation in the low-energy limit. In the case of the normal incidence, our result in the low-energy limit is consistent with a result for weakly interacting one-dimensional Bose gases treated as Tomonaga-Luttinger liquids.

1. Introduction

There are earlier studies of a so-called anomalous tunneling [1], which is described as follows: Bogoliubov excitations experience perfect transmission through a potential barrier in the low-energy limit. In the usual quantum mechanics, on the contrary, a single particle is reflected perfectly by the backscattering barrier in the low-energy limit. This is why the perfect transmission of Bogoliubov excitation is called the anomalous tunneling. However, the perfect transmission has also been discussed in the Tomonaga-Luttinger (TL) liquids, in context of quantum wire [2].

We study transmission and reflection of collective excitations in the condensed Bose system. The aim of this paper is two-fold: (i) exposing novel phenomena of transmission and reflection of the Bogoliubov phonon, (ii) comparing the result of a tunneling problem of the Bogoliubov phonon with that of TL liquids. To achieve these aims, we investigate the tunneling problem of Bogoliubov phonon through a potential barrier separating two Bose-Einstein condensates with different densities. Through this study, we obtain more physical implications for the anomalous tunneling.

2. Transmission and Reflection Coefficients

Our formulation is based on the stationary Gross-Pitaevskii equation [3, 4] and the Bogoliubov equation [5] written in dimensionless forms. The stationary Gross-Pitaevskii equation is given by \( \mathcal{H}\overline{\Psi}(\mathbf{r}) = 0 \), where \( \mathcal{H} \equiv -(1/2)\nabla^2 + \overline{V}(\mathbf{r}) - 1 + |\overline{\Psi}(\mathbf{r})|^2 \), and the Bogoliubov equation is given by

\[
\begin{pmatrix}
\hat{h}' & -\overline{\Psi}(\mathbf{r})^2 \\
\overline{\Psi}'(\mathbf{r})^2 & -\hat{h}'
\end{pmatrix}
\begin{pmatrix}
u(\mathbf{r}; \varepsilon) \\
u'(\mathbf{r}; \varepsilon)
\end{pmatrix}
= \varepsilon
\begin{pmatrix}
u(\mathbf{r}; \varepsilon) \\
u'(\mathbf{r}; \varepsilon)
\end{pmatrix},
\]
where \( \hat{h}' \equiv \hat{V} + |\nabla(r)|^2 \). We have introduced the following notations: \( \nabla \equiv r/\xi, \nabla \equiv \xi \nabla \), \( \nabla(r) \equiv \sqrt{g/\mu} \Psi(r), \nabla(r) \equiv V(r)/\mu \), and \( \varepsilon \equiv \varepsilon/\mu \), where \( g(>0) \) is a coupling constant of two-body short-range interaction, \( \mu \) is a chemical potential, and \( \xi \) is a healing length defined by \( \xi \equiv h/\sqrt{mp} \) with \( m \) being a mass. Henceforward, we omit the bar for simplicity. We take the potential \( V(r) \) to be a function of \( x \). \( V(x) \) is assumed to be the superposition of short-range potential near \( x = 0 \) and the potential step with asymptotic form: \( V_L \), for \( x \ll -1 \), and \( V_R \), for \( x \gg 1 \). Both \( V_L \) and \( V_R \) are smaller than unity.

We consider the situation where the Bogoliubov excitation runs against the potential wall at an angle \( \phi_L \) with respect to the \( x \)-direction. The Bogoliubov phonon is split into a transmitted wave with a refraction at an angle \( \phi_R \) and a reflected wave at \( \phi_L \). From the translational invariance in \( y, z \) directions, Snell’s law \( \sin \phi_L/\sin \phi_R = c_L/c_R \) follows in the low-energy limit. We note that there is no retroreflection in the condensed Bose system. When the Bogoliubov excitation runs toward the wall with an oblique angle, the reflected excitation does not go back the way one has come.

The transmission coefficient \( T(\varepsilon) \) and the reflection coefficient \( R(\varepsilon) \) are, respectively, given by ratios of incident energy flux to transmitted energy flux and to reflected energy flux. In the low-energy limit, \( T(\varepsilon) \) and \( R(\varepsilon) \) are, respectively, given by

\[
\lim_{\varepsilon \to 0} T(\varepsilon) = \frac{4c_Lc_R \cos \phi_L \cos \phi_R}{(c_L \cos \phi_L + c_R \cos \phi_R)^2}, \quad \lim_{\varepsilon \to 0} R(\varepsilon) = \frac{(c_L \cos \phi_L - c_R \cos \phi_R)^2}{(c_L \cos \phi_L + c_R \cos \phi_R)^2}. \tag{2}
\]

These satisfy \( T(\varepsilon) + R(\varepsilon) = 1 \). \( c_L \) and \( c_R \) are sound speeds of Bogoliubov phonon given by \( c_L \equiv \sqrt{1 - V_L} \) and \( c_R \equiv \sqrt{1 - V_R} \). In this proceedings, the outline of the derivation is given and the details will shown in a separate paper [6]. First, we construct equations for functions \( u + v \) and \( u - v \), which are proportional to a phase fluctuation and density fluctuation respectively [7]. Expanding functions \( u + v \) and \( u - v \) with respect to excitation energy, we obtain wavefunctions in the low-energy. In the low-energy limit, one finds that the phase fluctuation \( u + v \) coincides with the condensate wavefunctions, and the density fluctuation \( u - v \) vanishes. Using these properties and also the constancy of the energy flux, transmission and reflection coefficients (2) follows. These expressions obtained by the systematic low-energy expansion are valid at the low-energy limit. In Fig. 1, the transmission coefficients are plotted against the energy. We note that the transmission coefficient in Eq. (2) corresponds to low-energy limit of the numerical results.

We note that \( T(\varepsilon \to 0) \) and \( R(\varepsilon \to 0) \) in Eq. (2) depend on \( V(x) \) only through the asymptotic values \( V_L \) and \( V_R \). As a result, perfect transmission \( T(\varepsilon \to 0) = 1 \) follows from \( c_L = c_R \).
or equivalently \( V_L = V_R \). This result reproduces earlier results of the anomalous tunneling [1, 8, 9, 10]. In the presence of potential step, we find that partial transmission occurs at the normal incidence (\( \phi_L = 0 \)). This partial transmission in the present study and the anomalous tunneling in earlier studies [1, 8, 9, 10] can be understood consistently in the following way: in the low-energy limit, the wavefunction of the excitation coincides with the condensate wavefunction, and hence the property of an elementary excitation in the low-energy limit is similar to that of the condensate [10]. In the absence of the potential step (i.e. in the tunneling problem of excitations between identical Bose-Einstein condensates), the condensate can flow through the potential wall without reflection [9, 12]. From this fact, perfect transmission of low-energy excitations can be understood [10, 11]. In the presence of the potential step (or equivalently tunneling problem of excitations between condensates with different densities), on the other hand, there is a partial reflection of the condensate flow as seen in Ref. [13]. This tunneling property is reflected in partial transmission of low-energy excitations.

When the incident angle \( \phi_L \) of Bogoliubov phonon satisfies \( \phi_L = \arctan(c_L/c_R) \equiv \phi_{L,B} \), we obtain the perfect transmission between condensates with different densities (i.e. in the presence of potential step). This incident angle \( \phi_{L,B} \) corresponds to the Brewster’s angle \( (\pi/2 - \phi_R) \) for the electromagnetic wave [14]. We also find that the Bogoliubov excitation experiences the total internal reflection when the incident angle satisfies the condition \( \phi_L \geq \phi_{L,c} \). The critical angle \( \phi_{L,c} \) is defined by \( \phi_{L,c} = \arcsin(c_L/c_R) \), where \( c_L < c_R \).

### 3. Comparison with Results on Tomonaga-Luttinger Liquids

It is worth relating tunneling properties in condensed-Bose system to those in one-dimensional Bose systems treated as a TL liquid. Kane and Fisher [2] showed that in a TL liquid with Luttinger parameter \( K \) larger than unity (spinless fermions with attractive interactions), a single impurity is irrelevant in the low-energy limit and the perfect transmission occurs. Safi and Schulz studied transport property of inhomogeneous TL liquids [16]. From the results in [16], one can find that a potential barrier separating two TL liquids is irrelevant in the low-energy limit when the Luttinger parameters \( K_L \) and \( K_R \) of the two TL liquids are larger than unity. As a result, the amplitude transmission and reflection coefficients \( \tau \) and \( \gamma \) of excitations injected from the left side to the potential barrier depend only on \( K_L \) and \( K_R \) as \( \tau = 2K_R/(K_L + K_R) \), and \( \gamma = (K_L - K_R)/(K_L + K_R) \), respectively. Here \( \tau \) and \( \gamma \) are defined as ratio of incident current to transmitted and reflected current.

In the repulsive boson case, the Luttinger parameter corresponds to \( K > 1 \), and \( K = 1 \) represents the Tonks-Girardeau gas [15]. In weakly interacting boson systems with short-range interaction, the Luttinger parameter is given by \( K = \pi \hbar \sqrt{\rho_0/m}g \), where \( \rho_0 \) is a ground state density, \( m \) is a mass, and \( g \) is coupling constant of the short-range interaction [17]. Assuming that the difference of Luttinger parameters in the two TL liquids is caused by a potential step, only the ground state densities \( \rho_{0L}, \rho_{0R} \) in the two TL liquids are different. The sound speeds are given by \( c_{R(L)} = \sqrt{g\rho_{0,R(L)}/m} \), and hence \( \tau \) and \( \gamma \) reduce to \( \tau = 2c_R/(c_L + c_R) \), and \( \gamma = (c_L - c_R)/(c_L + c_R) \), respectively.

For comparison, let us be back to the Bogoliubov equations and consider the ratio \( \gamma_c \) of amplitudes of the incident current to the transmitted current of Bogoliubov excitation. We also consider the ratio \( \gamma_c \) of amplitudes of the incident current to the reflected current. Hence, we have \( \tau_c = (\Psi_R/\Psi_L)t \) and \( \gamma_c = \tau \) in the low-energy limit. \( \Psi_L \) and \( \Psi_R \) are the condensate wave functions in the asymptotic regime, and \( t \) and \( \tau \) are amplitude transmission and reflection coefficients in the low-energy limit. As a result, we have expression \( \tau_c = 2c_R/(c_L + c_R) \), and \( \gamma_c = (c_L - c_R)/(c_L + c_R) \). We note that this result is consistent with the result given by TL liquids.

Results on TL liquids suggest to us that perfect transmission occurs in one-dimensional
system, even if the interaction is strong. From this suggestion, we can raise an important issue whether the anomalous tunneling of the excitation can be observed not only in the ultracold Bose gas but also in the superfluid He-4, which is a strongly interacting Bose system.

In connection with TL liquids, we comment on negative density reflection and the Andreev reflection. The dynamics of one-dimensional Bose liquids has been investigated in several papers, where negative density reflection has been found to occur [18, 19]. In these papers, this reflection is called the Andreev-like reflection, in the sense that negative density reflection is analogous to reflection of hole-like excitations at the interface between super and normal conductors. In the condensed Bose system, however, there exists no retroreflection because of the law of reflection, as discussed in Sec. 2. This observation comes from the present study of the tunneling problem between different densities and with arbitrary incident angles. As a result, the Andreev-like reflection in Bose system is different from the Andreev reflection.

4. Conclusion
We summarize results as follows: (i) The Bogoliubov phonon has an incident angle where perfect transmission occurs between condensates with different densities. This condition corresponds to the Brewster’s law. There also exists the total internal reflection of the Bogoliubov excitation in the low-energy limit. (ii) In the case of the normal incidence, our result is consistent with that for weakly interacting one-dimensional Bose gases treated as TL liquids. The negative density reflection given by the theory of the Tomonaga-Luttinger liquids cannot be considered to be the Andreev reflection in Bose system.

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