$K_{r+1}$-saturated graphs with small spectral radius

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Abstract

For a graph $H$, a graph $G$ is $H$-saturated if $G$ does not contain $H$ as a subgraph but for any $e \in E(G)$, $G + e$ contains $H$. In this note, we prove a sharp lower bound for the number of paths and walks on length 2 in $n$-vertex $K_{r+1}$-saturated graphs. We then use this bound to give a lower bound on the spectral radii of such graphs which is asymptotically tight for each fixed $r$ and $n \to \infty$.

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1 Introduction

1.1 Notation and preliminaries

In this note we deal with finite undirected graphs with no loops or multiple edges. For a graph $H$, a graph $G$ is $H$-saturated if $H$ is not a subgraph of $G$ but after adding to $G$ any edge results in a graph containing $H$. For a positive integer $n$ and a graph $H$, the extremal number $ex(n, H)$ is the maximum number of edges in an $n$-vertex graph not containing $H$. Clearly, an extremal $n$-vertex graph $G$ not containing $H$ with $|E(G)| = ex(n, H)$ is $H$-saturated. Thus, one can also say that $ex(n, H)$ is the maximum number of edges in an

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n-vertex $H$-saturated graph. On the other hand, the saturation number of $H$, sat$(n, H)$, is the least number of edges in an $H$-saturated graph with $n$ vertices.

Initiating the study of extremal graph theory, Turán [9] determined the extremal number ex$(n, K_{r+1})$. He also proved that there is the unique extremal graph, $T_{n,r}$, the $n$-vertex complete $r$-partite graph whose partite sets differ in size at most 1. The first result on saturation numbers is due to Erdős, Hajnal and Moon [4]:

**Theorem A [4].** If $2 \leq r < n$, then sat$(n, K_{r+1}) = (r - 1)(n - r + 1) + \binom{r-1}{2}$. The only $n$-vertex $K_{r+1}$-saturated graph with sat$(n, K_{r+1})$ edges is the graph $S_{n,r}$ obtained from a copy of $K_{r-1}$ with vertex set $S$ by adding $n - r + 1$ vertices, each of which has neighborhood $S$.

Graph $S_{n,r}$ has clique number $r$ and no $r$-connected subgraphs; in particular, $S_{n,2}$ is a star. For an excellent survey on saturation numbers, we refer the reader to Faudree, Faudree, and Schmitt [5].

Recently, there was a series of publications on eigenvalues of $H$-free graphs. For a graph $G$, let $A(G)$ be its adjacency matrix, and we index the eigenvalues of $A(G)$ in nonincreasing order, $\lambda_1(G) \geq \cdots \geq \lambda_n(G)$. The value $\lambda_1(G)$ is also called the spectral radius of $G$, and denoted by $\rho(G)$.

Studying properties of quasi-random graphs, Chung, Graham, and Wilson [3] proved a theorem implying that, if $n$ is sufficiently large, $0 < c < \frac{1}{2}$ and $G$ is an $n$-vertex $K_c$-free graph with $\lceil cn^2 \rceil$ edges, then either $\lambda_n(G) < -c'n$ or $\lambda_2(G) > c'n$, where $c' = c'(r, c)$ is a positive constant. However, the methods in [3] fail to indicate which of the two inequalities actually holds. Bollobás and Nikiforov [1] observed that if $G$ is a dense $K_r$-free graph, then $\lambda_n(G) < -cn$ for some $c > 0$ independent of $n$. Nikiforov [7] gave a more precise statement that if $G$ is a $K_{r+1}$-free graph with $n$ vertices and $m$ edges, then $\lambda_n(G) < -\frac{2^{r+1}m}{rn^2}$.

Nikiforov [8] also proved that if $G$ is a $K_{r+1}$-free graph with $n$ vertices, then $\rho(G) \leq \rho(T_{n,r})$. Since each $K_{r+1}$-saturated graph is $K_{r+1}$-free, his theorem implies the following.

**Theorem B [7].** If $G$ is a $K_{r+1}$-saturated graph with $n$ vertices, then

$$
\rho(G) \leq \rho(T_{n,r}).
$$

In this note, we give a new lower bound for the spectral radius of an $n$-vertex $K_{r+1}$-saturated graph. This bound is asymptotically tight when $r$ is fixed or grows as $o(n)$. For this, we give a tight lower bound on the sum of the squares of the vertex degrees in an $n$-vertex $K_{r+1}$-saturated graph.

### 1.2 Results

Our main tool will be the following.

**Theorem 1.1.** If $n \geq r + 1$ and $G$ is a $K_{r+1}$-saturated graph with $n$ vertices, then

$$
\sum_{v \in V(G)} d^2(v) \geq (n - 1)^2(r - 1) + (r - 1)^2(n - r + 1). \quad (1)
$$
For $r = 2$, equality in the bound holds only when $G$ is $S_{n,2}$ or a Moore graph with diameter 2. For $r \geq 3$, equality in the bound holds only when $G$ is $S_{n,r}$.

The reason why it is helpful is the following simple observation.

**Lemma 1.2.** For every $n$-vertex graph $G$ with adjacency matrix $A$,

$$\rho^2(A) \geq \frac{1}{n} \sum_{v \in V(G)} d^2(v). \quad (2)$$

Theorem 1.1 together with this observation immediately yield

**Theorem 1.3.** If $2 \leq r < n$ and $G$ is a $K_{r+1}$-saturated graph with $n$ vertices, then

$$\rho(G) \geq \sqrt{\frac{(n-1)^2(r-1) + (r-1)^2(n-r+1)}{n}}. \quad (3)$$

This bound asymptotically is tight because the spectral radius of $S_{n,r}$ is close to $f(n, r)$, where $f(n, r)$ is the lower bound for $\rho(G)$ in Theorem 1.3. More specifically, note that $\rho(S_{n,2}) = f(2, n)$ and for $r \geq 3$, we have $\rho(S_{n,r}) = f(r, n) + \frac{r-2}{2} + O(\frac{1}{\sqrt{n}})$.

**Proposition 1.4.** For integers $2 \leq r < n$,

$$\rho(S_{n,r}) = \frac{r - 2 + \sqrt{(r - 2)^2 + 4(r - 1)(n - r + 1)}}{2}.$$ 

In the next section we prove Theorem 1.1 (in a somewhat stronger form) and in the last section we present proofs for Lemma 1.2 and Proposition 1.4.

For undefined terms, see Brouwer and Haemers [2], Godsil and Royle [6], or West [10].

## 2 Proof of Theorem 1.1

We will derive Theorem 1.1 from the following slightly stronger statement.

**Theorem 2.1.** If $n \geq r + 1$ and $G$ is a $K_{r+1}$-saturated graph with $n$ vertices, then

$$\sum_{v \in V(G)} (d(v) + 1)(d(v) + 1 - r) \geq (r - 1)n(n - r). \quad (4)$$

**Proof.** Let $m = |E(G)|$ and $\overline{m} = |E(G^c)| = \binom{n}{2} - m$. For $v \in V(G)$, let $f(v)$ be the number of pairs of non-adjacent vertices $x$ and $y$ in $N(v)$ such that $G[N(x) \cap N(y) \cap N(v)]$ contains $K_{r-2}$ as a subgraph. Note that if $G[N(v)]$ is a copy of $K_{r-1}$, then $f(v) = 0$.

**Claim 1.** $\overline{m} \leq \frac{1}{r-1} \sum_{v \in V(G)} f(v)$.
We construct an auxiliary bipartite graph $H$ with parts $A$ and $B$ as follows. Let $A = E(G)$ and $B = V(G)$. The graph $H$ has an edge between $xy \in A$ and $v \in B$ iff $x, y \in N(v)$ and $G[N(x) \cap N(y) \cap N(v)]$ contains $K_{r-2}$ as a subgraph. Then for each $v \in B$, we have $|N_H(v)| = f(v)$. Also, since $G$ is $K_{r+1}$-saturated, for each $xy \in A$, $G + xy$ contains $K_{r+1}$ as a subgraph. Thus there exist at least $r - 1$ vertices $v$ such that $x, y \in N(v)$ and $G[N(x) \cap N(y) \cap N(v)]$ contains $K_{r-2}$ as a subgraph, which implies
\[ |N_H(xy)| \geq r - 1. \] (5)

By (5),
\[ (r - 1)m \leq \sum_{xy \in A} d_H(xy) = |E(H)| = \sum_{v \in V(G)} f(v). \] (6)

This proves Claim 1.

Claim 2. For each $v \in V(G)$, we have $f(v) \leq \binom{d(v) - r + 2}{2}.$

Let $H_v = G[N(v)]$, and let $d(v) = p$. Since $G$ contains no $K_{r+1}$, the graph $H_v$ has no $K_r$. Partition the pairs of vertices in $N(v)$ into the sets $E_1, E_2$ and $E_3$ as follows:
(i) $E_1 = E(H_v)$,
(ii) $E_2$ is the set of the edges $xy \in E(G_{H_v})$ such that $H_v + xy$ does not contain $K_r$,
(iii) $E_3$ is the set of the edges $xy \in E(G_{H_v})$ such that $H_v + xy$ contains $K_r$.

Let $m_i = |E_i|$ for $1 \leq i \leq 3$. By definition, $m_3 = f(v)$ and $m_1 + m_2 + m_3 = \binom{p}{2}$. As any $K_r$-free graph is a subgraph of $K_r$-saturated graph on the same vertex set, there exists a $K_r$-saturated graph $H'$ with vertex set $N(v)$ containing $H_v$. Then $E(H') \supseteq E_1$. Furthermore, since $H'$ is $K_r$-free and contains $E_1$, $E(H') \cap E_3 = \emptyset$. By Theorem A, $|E(H')| \geq (r - 2)(p - r + 2) + \binom{r - 2}{2}$. Hence
\[ m_3 \leq \binom{p}{2} - |E(H')| \leq \binom{p}{2} - (r - 2)(p - r + 2) - \binom{r - 2}{2} = \binom{p - r + 2}{2}. \]

This proves Claim 2.

Now we are ready to prove the theorem. By Claims 1 and 2,
\[ \binom{n}{2} = m + m \leq m + \frac{1}{r-1} \sum_{v \in V(G)} f(v) \leq \sum_{v \in V(G)} \left[ \frac{d(v)}{2} + \frac{1}{r-1} (d(v) - r + 2)(d(v) - r + 1) \right]. \]

Multiplying both sides by $2(r - 1)$, we get
\[ (r - 1)n(n - 1) \leq \sum_{v \in V(G)} [(r - 1)d(v) + (d(v) + 1)(d(v) - r + 1) - (r - 1)(d(v) - r + 1)]. \]

This yields
\[ \sum_{v \in V(G)} (d(v) + 1)(d(v) + 1 - r) \geq (r - 1)n(n - 1) - (r - 1)^2n = (r - 1)n(n - r), \]
and Theorem 2.1 is proved.

To obtain Theorem 1.1 observe that (4) implies
\[
\sum_{v \in V(G)} d^2(v) \geq (r - 1)n(n - r) + (r - 1)n + (r - 2)2m.
\]
So, by Theorem A,
\[
\sum_{v \in V(G)} d^2(v) \geq (r - 1)n(n - r) + (r - 1)n + 2(r - 2)(n^2 - (n - r + 1)^2).
\]
(7)

This proves the first part of Theorem 1.1. Furthermore, for \( r \geq 3 \), equality in the bound requires equality in Theorem A. Thus equality holds only for \( S_{n,r} \).

Suppose now \( r = 2 \) and \( G \) is an \( n \)-vertex \( K_3 \)-saturated graph for which (1) holds with equality. As \( G \) is \( K_3 \)-saturated, \( G \) has diameter 2. Equality in the bound requires equality in (6), and hence equality in (5) for every \( xy \in E(G) \). This means \( G \) has no \( C_4 \), which implies that \( G \) has girth at least 5. If \( G \) has no cycles, then \( G \) is a copy of \( S_{n,2} \). Otherwise, \( G \) is a Moore graph with diameter 2.

Recall that there are at most four Moore graphs with diameter 2: \( C_5 \), the Petersen graph, the Hoffman-Singleton graph, and possibly one 57-regular graph of girth 5 with 3250 vertices.

3 Spectral radius

We will use the following standard tool.

**Theorem 3.1** (Rayleigh Quotient Theorem). For a real matrix \( A \)
\[
\rho(A) = \max_{x \in \mathbb{R}^n \setminus \{0\}} \frac{x^T Ax}{x^T x}.
\]
(8)

First, we present a proof of Lemma 1.2. By (8),
\[
\rho^2(A) = \rho(A^2) = \max_{x \in \mathbb{R}^n \setminus \{0\}} \frac{x^T A^2 x}{x^T x} = \max_{x \in \mathbb{R}^n \setminus \{0\}} \frac{(x^T A^T)(Ax)}{x^T x} \geq \frac{(1^T A^T)(A1)}{1^T 1} = \frac{1}{n} \sum_{v \in V(G)} d^2(v).
\]

Thus, (2) holds. Together with Theorem 1.3 this implies Theorem 1.4.

To show that Theorem 1.3 is asymptotically tight, we will determine the spectral radius of \( S_{n,r} \), i.e. prove Proposition 1.4. We will need a new notion. Consider a partition \( V(G) = V_1 \cup \cdots \cup V_s \) of the vertex set of a graph \( G \) into \( s \) non-empty subsets. For \( 1 \leq i, j \leq s \), let \( q_{i,j} \) denote the average number of neighbors in \( V_j \) of the vertices in \( V_i \). The quotient matrix
Q of this partition is the $s \times s$ matrix whose $(i,j)$-th entry equals $q_{i,j}$. The eigenvalues of the quotient matrix interlace the eigenvalues of $G$. This partition is equitable if for each $1 \leq i, j \leq s$, each vertex $v \in V_i$ has exactly $q_{i,j}$ neighbors in $V_j$. In this case, the eigenvalues of the quotient matrix are eigenvalues of $G$ and the spectral radius of the quotient matrix equals the spectral radius of $G$ (see [2], [6] for more details).

[Proof of Proposition 1.4]
Partition $V(S_{n,r})$ into sets $A$ and $B$ such that $S_{n,r}[A]$ is a copy of $K_{r-1}$ and $S_{n,r}[B]$ is an independent set with $n - r + 1$ vertices. Each vertex in $A$ is adjacent to all vertices in $B$. The quotient matrix of the partitions $A$ and $B$ is

\[
\begin{pmatrix}
  r - 2 & n - r + 1 \\
  r - 1 & 0
\end{pmatrix}.
\]

The characteristic polynomial of the matrix is $x^2 - (r - 2)x - (r - 1)(n - r + 1) = 0$. Since the partition $V(S_{n,r}) = A \cup B$ is equitable,

\[
\rho(S_{n,r}) = \frac{r - 2 + \sqrt{(r - 2)^2 + 4(r - 1)(n - r + 1)}}{2}.
\]

This completes the proof of Proposition 1.4.

Note that $\rho(S_{n,2}) = \sqrt{n - 1}$. Thus for $r = 2$, equality in Theorem 1.3 holds if and only if $G$ is $S_{n,2}$ or a Moore graph. For $r \geq 3$, the bound in Theorem 1.3 may be improved, and we guess that the spectral radius of $S_{n,r}$ is the minimum of $\rho(H)$ among all $n$-vertex $K_{r+1}$-saturated graphs $H$.

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