Axisymmetric Stationary Solutions as Harmonic Maps

by

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Abstract: We present a method for generating exact solutions of Einstein equations in vacuum using harmonic maps, when the spacetime possesses two commuting Killing vectors. This method consists in writing the axisymmetric stationary Einstein equations in vacuum as a harmonic map which belongs to the group $SL(2, \mathbb{R})$, and decomposing it in its harmonic “submaps”. This method provides a natural classification of the solutions in classes (Weyl’s class, Lewis’ class etc.).

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Stationary Einstein equations in terms of harmonic maps (HM) [1] are well-known. Sánchez [2] found that the stationary Einstein equations in electrovacuum are in fact HM, but that the corresponding differential equations are non-linear. Neugebauer and Kramer [3] showed that this HM has a Lax pairs-representation (equivalent linear problem) in terms of a HM which is an element of the group SU(2,1), i.e. axisymmetric stationary Einstein-Maxwell equations in potential space (Ernst potential space) are HM $G : M^4 \rightarrow V_p \otimes SU(2,1)$, being $V_p$ a p-dimensional Riemannian space. On the other hand, Belinsky and Zakharov [4] have also given the empty Einstein equations in terms of HM, but in a $SL(2,\mathbb{R})$ representation of spacetime. Recently, Whitman and Stoeger gave a HM representation of these equations [5]. It is also possible to give the empty Einstein equation in potential space in terms of HM if we reduce the group SU(2,1) to SU(1,1). The isomorphism $SU(1,1) \simeq SL(2,\mathbb{R})$ allows us to give a representation of these equations in terms of the group $SL(2,\mathbb{R})$ [6]. Thus we can write the Einstein equations in vacuum as a $SL(2,\mathbb{R})$-HM in the spacetime, and in the potential space.

In this work, we will start from the HM representation of the axisymmetric stationary Einstein equations in vacuum, and develop a method for generating exact solutions. It consists in separating the group $SL(2,\mathbb{R})$ into its one-parametric subgroups and itself giving a representation of the HM, $G : M^4 \rightarrow V_p \otimes SL(2,\mathbb{R})$ in its tangent space (see section 3 or ref. [7]). The result is that each subgroup corresponds to a well-known class of solutions, explaining why do these classes exist.

One could also extend this analysis to the Einstein-Maxwell equations. It may be done in potential space [8] where the HM belongs to the group SU(2,1), but there is no HM representation in spacetime. Nevertheless, five-dimensional gravity admits a HM representation for the stationary equations [9]. Both the spacetime and the potential space have a $SL(3,\mathbb{R})$ HM representation of the field equations [10] (see also ref. [9]). Thus an analysis with harmonic “submaps” can be done [11],[12].

This work pretends to be self-contained. In sections 1 and 2 we give the Einstein’s spacetime field equations in vacuum when the metric depends only on two coordinates. In section 3 we briefly outline the method of subgroups (harmonic “submaps”) which will be applied in section 5.

1. Generalities

We consider a $V_4$ manifold with two commuting Killing-vectors and local coordinates $\{X^\mu\} = (X^r, y^A)$, with $X^r = (X^1, X^2)$ being the “active” (and arbitrary) coordinates and $y^A = (y^1, y^2)$ playing the role of the “passive” coordinates (i.e., with $\partial/\partial y^A$ being the two commuting Killing vectors). The metric can be given in the quasi-diagonalizable form

$$V_4 : g_4 = g_{rs}dx^r \otimes dx^s + K_{AB}dy^A \otimes dy^B.\quad (1.1)$$

The essential “Killingian” part of the metric is thus described by the three real objects

$$K_{AB} = K_{AB}(X^1, X^2) = K_{(AB)}.\quad (1.2)$$

We use for this part of the metric the spinorial notation, because evidently the constant

$$GL(2,\mathbb{R})\quad (1.3)$$

transformations of the passive variables $y^A$ do not affect the structure considered in any manner.

We assume the rules of manipulating the spinorial indices as follows:

$$\psi_A = \epsilon_{AB}\psi^B \iff \psi^A = \psi_B\epsilon^{BA}\quad (1.4)$$

We assume the “Killingian sector” to be of signature (+,−). To ensure this, it is necessary and sufficient to assume that

$$K := det(K_{AB}) = \frac{1}{2}K_{AB}K^{AB} < 0.\quad (1.5)$$
The 2-dimensional metric

\[ V_2 : \; g_2 := g_{rs} dX^r \otimes dX^s \]  

with \( g_{rs} = g_{rs}(X^1, X^2) = g_{(rs)} \) has then the signature \((+,+)\). The condition for this is

\[ \det(g_{rs}) > 0, \]  

assuming, of course, that the coordinates \( X^r \) are real. The \( V_2 \) structure is thus just a proper 2-dimensional Riemannian space. The covariant derivative with respect to \( g_2 \) shall be denoted by “\( | \)’Brien”. The indices \( r,s,... \) are to be manipulated by \( g_{rs} \) and its inverse, the contravariant metric \( g^{rs} \) \((g^{rs}g_{st} = \delta^r_{s})\).

Our problem consists now in studying the Einstein empty spacetime equations working with the chart \( \{X^\mu\} = \{X^r, y^A\} \),

\[ G_{\mu \nu} = 0, \]  

which we can evidently also take in the form

\[ R_{\mu \nu} = 0, \]  

with \( R_{\mu \nu} = R^\sigma_{\mu \nu \sigma} \) being the Ricci tensor.

The Ricci tensor of metric (1.1) can be easily evaluated in the covariant form with respect to the chart used in the description of the metric \( g_2 \), i.e., the coordinates \( \{X^r\} = \{X^1, X^2\} \).

Indeed, one finds that all “mixed” components of \( R_{\mu \nu} \) identically vanish

\[ R_{Ar} = 0 = R_{rA}, \]  

(1.9)

For the “Killingian” components of \( R_{\mu \nu} \) one finds the \( V_2 \)-covariant basic expression

\[ R_{AB} = \frac{1}{2} K_{AB|r} + \frac{1}{4} K^{-1} K^{r |r} K_{AB|r} - \frac{1}{2} K^{-1} K^{PQ} K_{PA|r} K_{QB|r}, \]  

(1.10)

with \( K \) defined by (1.5) and all spinorial indices manipulated according to the rules (1.4).

Now, the curvature of \( V_2 \) is of course characterized entirely by its scalar curvature. If \( R_{stu}^0 \) is the curvature tensor of \( V_2 \), we have

\[ R_{stu}^0 = \frac{1}{2} \delta_{tu}^r R_{r}, \]  

(1.11)

with \( R \) being the scalar curvature of the non-Killingian sector \((g_2)\) of the metric.

Knowing this, one works out the “non-Killingian” components of \( R_{\mu \nu} \), i.e. \( R_{rs} \), in the form

\[ R_{rs} = \frac{1}{2} g_{rs} R + \frac{1}{2} (K^{-1} K_{r|r})_s + \frac{1}{4} K^{-2} K^{PU} K_{QU|r} K^{QV} K_{PV,s}. \]  

(1.12)

The next thing to do is to work out the components of \( R_{AB} \) and \( R_{rs} \) in an appropriate form for our further purposes.

For \( R_{AB} \) we notice first that we have the spinorial identity

\[ K^{PQ} K_{PA} = K^Q_A \delta^P_A. \]  

(1.13)

Employing it, we find that

\[ K^{PQ} K_{PA|r} K_{QB} |^r \]

\[ = (K_{rA}^Q - K^{PQ} K_{PA}) K_{QB} |^r \]

\[ = K_{AB} |^r K_{AP} K^{QP} K_{QB} |^r \]  

(1.14)
but $K^Q_{PB}K^{|r}_{QB} = -K^Q_{B|r}K^{|r}_{QBP}$ implies that

$$\frac{1}{2} \epsilon_{PB} \epsilon^{P'}K^{|r}_{P'Q}|r_{PB}=\frac{1}{2} \epsilon_{PB}K^{MN|r}K_{MN|r}.$$  

Equality (1.14) says then that

$$K^{PQ}K^{|r}_{PA|r}K^{|r}_{QB} = K^{|r}_{PA|r} + K^P_A K^Q_P K^{|r}_{QB}$$

$$= K^{|r}_{PA|r} + \frac{1}{2} K_A^P \epsilon_{PB}K^{MN|r}K_{MN|r}.$$  

(1.15)

We arrive in this way at the identity

$$K^{PQ}K^{|r}_{PA|r}K^{|r}_{QB} = K^{|r}_{PA|r} - \frac{1}{2} K_{AB} K^{MN|r} K_{MN|r}.$$  

(1.16)

Using this in (1.10) we have

$$R_{AB} = \frac{1}{2} K_{AB}^{|r} + \frac{1}{4} K^{-1}K_{AB}^{|r}$$

$$- \frac{1}{2} K^{-1}K^{|r}_{AB} + \frac{1}{4} K^{-1}K_{AB} K^{MN|r} K_{MN|r}$$

$$= \frac{1}{2} K_{AB}^{|r} - \frac{1}{4} K^{-1}K^{|r}_{AB} + \frac{1}{4} K^{-1}K_{AB} K^{MN|r} K_{MN|r}.$$  

(1.17)

We bring this last expression to the slightly simpler form

$$R_{AB} = \frac{1}{2} \sqrt{-K}( \frac{1}{\sqrt{-K}}K_{AB}^{|r})^{|r} + \frac{1}{4} K^{-1}K^{MN|r} K_{MN|r}.$$  

The expression above is already satisfactory for many purposes. We can obtain a more satisfactory equivalent expression by executing the contraction $K^{AP}R_{AB}$, remembering that because of $det(K_{AB}) < 0$, the matrix $K_{AB}$ is invertible. Indeed, employing (1.13) in the form $K^{AP}K_{AB} = K^B_P$, we have

$$K^{AP}R_{AB} = \frac{1}{2} \sqrt{-K}K^{AP}( \frac{1}{\sqrt{-K}}K_{AB}^{|r})^{|r} + \frac{1}{4} \delta^P_BK^{MN|r} K_{MN|r}$$

$$= \frac{1}{2} \sqrt{-K}( \frac{1}{\sqrt{-K}}K^{AP}K_{AB}^{|r})^{|r} - K^{AP}^{|r} \frac{1}{\sqrt{-K}}K_{AB}^{|r}$$

$$+ \frac{1}{4} \delta^P_BK^{MN|r} K_{MN|r}$$

$$= \frac{1}{2} \sqrt{-K}( \frac{1}{\sqrt{-K}}K^{AP}K_{AB}^{|r})^{|r} +$$

$$\frac{1}{4} \delta^P_BK^{MN|r} K_{MN|r} - \frac{1}{2} K^{AP}^{|r} K_{AB}^{|r}.$$  

(1.19)

The last term in this expression, with the index $P$ lowered, $-\frac{1}{2} K^A_{PB}K^{|r}_{AB} = \frac{1}{2} K^A_{B|r}K^{|r}_{AP}$ is antisymmetric in $PB$ and hence equal to

$$-\frac{1}{2} K^A_{PB}K^{|r}_{AB} = -\frac{1}{4} \epsilon_{PB} \epsilon^{P'}K^A_{P'Q}K^{|r}_{AB}$$

$$= -\frac{1}{4} \epsilon_{PB}K^{MN|r} K_{MN|r},$$  

(1.20)

so that, rising again the index $P$

$$-\frac{1}{2} K^{AP}^{|r} K_{AB}^{|r} = -\frac{1}{4} \delta^P_BK^{MN|r} K_{MN|r}.$$  

(1.21)
Consequently, the terms in the last line of (1.19) simply cancel out, and we arrive at

\[ E_{AB} := K_A^R R_{RB} = \frac{1}{2} \sqrt{-K} \left( \frac{1}{\sqrt{-K}} K_A^R K_{RB}^{|r}\right)^r. \] (1.22)

This is perhaps the most condensed form concerning the analytic form of the "Killingian part" of the Ricci tensor. We would like to observe at this point that the expression \( E_{AB} \) defined above has the trace

\[ E_{AA} := K_{AB} R_{AB} \equiv \frac{1}{2} \sqrt{-K} \left( \frac{1}{\sqrt{-K}} \right) K_{AB} \left| K_{AB}^{|r}\right|^r = \frac{1}{2} \sqrt{-K} \left( \frac{1}{\sqrt{-K}} \right) K_{AB}^{|r}\right|^r = -\frac{1}{2} \sqrt{-K} \left( \frac{1}{\sqrt{-K}} \right) K_{AB}^{|r}\right|^r. \] (1.23)

Moreover

\[ K_{AB} E_{AB} = K_{AB} K_A^R R_{RB} = K_{AB}^R R_{AB} \equiv 0, \] (1.24)

so that the objects \( E_{AB} \) defined by (1.22) are linearly dependent.

In the next part of this section, we would like to study the structure of the "non-Killingian" part of the Ricci tensor, (1.2). Because of (1.13), we have

\[ K^{UP} K_{UQ}^{|r} = -K_{UQ} K^{UP} + K_{V}^{|r} \delta_{Q}^{P} \] (1.25)

and therefore the last term in (1.12) transforms to

\[ \frac{1}{4} K^{-2} K^{UP} K_{UQ}^{|r} K^{QV} K_{PV}^{|s} \]

\[ = \frac{1}{4} K^{-2} \left( K_{V}^{|r} \delta_{Q}^{P} - K_{UQ} K^{UP} \right) K^{QV} K_{PV}^{|s} \]

\[ = \frac{1}{4} K^{-2} K_{V}^{|r} \delta_{Q}^{P} K_{PV}^{|s} - \frac{1}{4} K^{-2} K^{QV} K_{QU} K^{UP} K_{VP}^{|s} \]

\[ = \frac{1}{4} K^{-2} K_{V}^{|r} \delta_{Q}^{P} K_{VP}^{|s} - \frac{1}{4} K^{-2} K_{QU} K^{UP} K_{VP}^{|s} \]

\[ = \frac{1}{4} K^{-2} K_{V}^{|r} \delta_{Q}^{P} K_{VP}^{|s} \] (1.26)

With this identity, we have

\[ R_{rs} = \frac{1}{2} g_{rs} \delta_{0}^{P} + \frac{1}{2} \left( K^{-1} K_{s}^{|r}\right)_{s} + \frac{1}{4} K^{-2} K_{V}^{|r} \delta_{Q}^{P} K_{VP}^{|s} - \frac{1}{4} K^{-1} K_{MN}^{|r} K_{MN}^{|s}. \] (1.27)

This can be still slightly simplified, observing that

\[ \frac{1}{2} \left( K^{-1} K_{s}^{|r}\right)_{s} + \frac{1}{4} K^{-2} K_{V}^{|r} \delta_{Q}^{P} K_{VP}^{|s} \equiv \frac{1}{\sqrt{-K}} \left( \sqrt{-K} \right)_{rs}. \] (1.28)

Hence, we have for \( R_{rs} \)

\[ R_{rs} = \frac{1}{2} g_{rs} \delta_{0}^{P} + \frac{1}{\sqrt{-K}} \left( \sqrt{-K} \right)_{rs} - \frac{1}{4} K^{-1} K_{MN}^{|r} K_{MN}^{|s}. \] (1.29)
We can now conveniently evaluate the scalar curvature of the \( g_4 \) metric. Indeed, it is obvious from (1.1) that
\[
\|g_{\mu\nu}\| = \begin{pmatrix} g_{rs} & 0 \\ 0 & K_{AB} \end{pmatrix}, \quad \|g^{\mu\nu}\| = \begin{pmatrix} g^{rs} & 0 \\ 0 & K^{AB} \end{pmatrix},
\]
and consequently
\[
R = g^{rs} R_{rs} + \frac{1}{K} K^{AB} R_{AB},
\]
remembering (1.9). From (1.29) we evaluate now
\[
g^{rs} R_{rs} = \frac{0}{\sqrt{-K}} \frac{1}{(\sqrt{-K})_{|r}} - \frac{1}{4} K^{-1} K^{MN} |r K_{MN}|_r
\]
and from (1.23) we have
\[
\frac{1}{K} K^{AB} R_{AB} = -\frac{\sqrt{-K}}{K} (\sqrt{-K})_{|r} = \frac{1}{\sqrt{-K}} (\sqrt{-K})_{|r}
\]
so that
\[
R = 0 + \frac{2}{\sqrt{-K}} (\sqrt{-K})_{|r} - \frac{1}{4} K^{-1} K^{MN} |r K_{MN}|_r.
\]

Knowing \( R \), we can now evaluate the components of the Einstein tensor
\[
G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R.
\]
In particular, the \((rs)\) components (i.e., the non-Killingian part) of this object are
\[
G_{rs} = R_{rs} - \frac{1}{2} g_{rs} R
= \frac{1}{2} g_{rs} 0 + \frac{1}{\sqrt{-K}} (\sqrt{-K})_{|rs} - \frac{1}{4} K^{-1} K^{MN} |r K_{MN}|_s
\]
\[
- \frac{1}{2} g_{rs} \{ R + \frac{2}{\sqrt{-K}} (\sqrt{-K})_{|t} - \frac{1}{4} K^{-1} K^{MN} |t K_{MN}|_t \}
\]
\[
= \frac{1}{\sqrt{-K}} (\sqrt{-K})_{|rs} - \frac{1}{4} K^{-1} K^{MN} |r K_{MN}|_s
\]
\[
- g_{rs} \{ \frac{1}{\sqrt{-K}} (\sqrt{-K})_{|t} - \frac{1}{8} K^{-1} K^{MN} |t K_{MN}|_t \}.
\]

The cancellation of the term which involves \( \frac{0}{\sqrt{-K}} \) in this expression is of basic importance, the second derivatives of the \( g_2 \) metric \((g_{rs,tt})\) do not enter in the structure of \( G_{rs} \).

Summarizing, we can now state the basic equations of the problem \( R_{\mu\nu} = 0 \) written covariantly with respect to the chart \( \{ X^r \} = \{ X^1, X^2 \} \) in terms of which the metric \( g_2 \) is described as the following hierarchy of differential equations. First we have the “K-sector” Einstein equations
\[
\{ \det(K_{AB}) \neq 0 \} \Rightarrow \{ R_{AB} = 0 \Leftrightarrow E_{AB} := K_A^R R_{RB} = 0 \},
\]
where

\[ E_{AB} \equiv \frac{1}{2} \sqrt{-K} \left( \frac{1}{\sqrt{-K}} K^A R K_{RB|r} \right)^{|r} = 0. \] (1.37)

These equations imply

\[ (\sqrt{-K})^{|r} = 0, \] (1.38)

so they are not independent because of the identity

\[ K^{AB} E_{AB} \equiv 0. \] (1.39)

Then we have the equations

\[ G_{rs} = \frac{1}{\sqrt{-K}} (-K)^{|rs} - \frac{1}{4} K^{-1} K^{MN|s} K_{MN|r}, \]

\[ -g_{rs} \left( \frac{1}{\sqrt{-K}} (\sqrt{-K})^{|t} - \frac{1}{8} K^{-1} K^{MN|s} K_{MN|r} \right) = 0, \] (1.40)

and finally

\[ R \equiv \frac{0}{\sqrt{K}} (\sqrt{-K})^{|r} - \frac{1}{4} K^{-1} K^{MN|r} K_{MN|r} = 0, \] (1.41)

where \( R \) is the scalar curvature of \( g_{rs} \) defined by (1.11).

2. The Equations \( R_{\mu
u} = 0 \) in Weyl’s Coordinates.

The metric \( g_2 \) can be of course always expressed in the conformally flat form

\[ g_2 = \phi^{-2} \{ dX^1 \otimes dX^1 + dX^2 \otimes dX^2 \}, \] (2.1)

or, introducing the complex coordinates

\[ \xi := \frac{1}{\sqrt{2}} (X^1 + iX^2) \]

\[ \bar{\xi} := \frac{1}{\sqrt{2}} (X^1 - iX^2) \] (2.2)

in the simple form

\[ g_2 = 2\phi^{-2} d\xi \otimes d\bar{\xi}. \] (2.3)

We shall call \( \xi \) and \( \bar{\xi} \) the Weyl coordinates. They are a sort of “null variables” and they are arbitrary up to the transformations

\[ \xi = f(\xi'), \] (2.4)

with \( f(z) \) being an arbitrary analytic function such that \( f'(z) \neq 0 \). Under the transformation \( \xi \to \xi' \) the real structural function \( \phi \) transforms according to

\[ \xi \to \xi', \phi \to \phi' = \phi |f'(\xi)|^{-1}. \] (2.5)

Experience with the theory of the exact solutions, e.g., the cases of the D-type solutions or the Tomimatsu-Sato solutions, indicates that the Weyl variables are certainly not the best variables in practice, i.e., in the description of the physically pertinent solutions. On the other hand, these variables are theoretically important in decoding the essential structure of the differential problem stated at the end of
the previous section.

When we employ the Weyl coordinates, a simple manner of proceeding (and taking advantage of the

covariant formulation of the problem outlined at the end of the previous section) is this: we simply take the

metric \( g_2 \) as given in the form

\[
g_2 = 2\phi^{-2}dX^1 \otimes dX^2
\]  

understanding \( X^1 \equiv \xi \) and \( X^2 \equiv \bar{\xi} \). With these complex coordinates all the apparatus of the classical
differential geometry in two dimensions, formally implicates the condition on \( g_2 \) to be positive definite. This
implies that the determinant of the metric

\[
\|g_{rs}\| = \phi^{-2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

must be negative,

\[
det(g_{rs}) = -\phi^{-4},
\]

so that

\[
\sqrt{-det(g_{rs})} = \phi^{-2}.
\]

For the inverse metric we have then

\[
\|g^{rs}\| = \phi^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

so that the tensor density

\[
G_{tu} := \sqrt{-det(g_{rs})}g^{tu}
\]

is just a numeric matrix

\[
\|G^{rs}\| = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

The Weyl coordinates are thus harmonic in the sense that

\[
G^{rs,s} = 0.
\]

The Christoffel symbols \( \Gamma_{rs}^t \) computed from the metric (2.7) and its inverse (2.10) vanish, excepting
the two components

\[
\Gamma^3_{11} = -2(\ln \phi)_{,1} \equiv -2(\ln \phi)_{,\xi},
\]

\[
\Gamma^2_{22} = -2(\ln \phi)_{,2} \equiv -2(\ln \phi)_{,\bar{\xi}}.
\]

This permits us to determine easily the scalar curvature of \( g_2 \), the object \( ^0 R \). We arrive at

\[
^0 R = g^{rs} R^t_{rs} = -4\phi^2(\ln \phi)_{,\xi} \bar{\xi},
\]

Now, equations of the type \( (AB)_{,r}^{\tau} = 0 \) in the present coordinates, being equivalent (when \( A \) and \( B \)
are \( g_2 \) scalars) to \( (G^{rs}AB)_{,s} = 0 \), simply transform to \( (AB)_{,\xi} + (AB)_{,\bar{\xi}} = 0 \). Therefore, the fundamental
equations (1.37) take the form

\[
R_{AB} = 0 \iff \left( \frac{1}{\sqrt{-K}}K_A R_{K_{RB,\xi}}\right)_{,\xi} + \left( -\frac{1}{\sqrt{-K}}K_A R_{K_{RB,\bar{\xi}}}\right)_{,\bar{\xi}} = 0.
\]
The essential fact emerges here since these equations do not involve in any manner the structural function \( \phi \), being them the autonomous "K-sector" equations. The necessary implication of these equations (1.38), can be now simply stated in the form

\[(\sqrt{-K})_{\xi\xi} = 0.\]  \hspace{1cm} (2.17)

Consider now equations (1.40), their (11) component transforms to

\[G_{11} = \frac{1}{\sqrt{-K}}(\sqrt{-K})_{11} - \frac{1}{4}K^{-1}K^{MN}_{,1}K_{MN,1} = 0,\]  \hspace{1cm} (2.18)

or explicitly

\[\sqrt{-K}G_{11} = (\sqrt{-K})_{\xi\xi} + 2(\ln\phi)_{,\xi}(\sqrt{-K})_{,\xi} + \frac{1}{4}\frac{1}{\sqrt{-K}}K^{MN}_{,\xi}K_{MN,\xi} = 0.\]  \hspace{1cm} (2.19)

The complex conjugate of this equation transforms of course to

\[\sqrt{-K}G_{22} = (\sqrt{-K})_{\bar{\xi}\bar{\xi}} + 2(\ln\phi)_{,\bar{\xi}}(\sqrt{-K})_{,\bar{\xi}} + \frac{1}{4}\frac{1}{\sqrt{-K}}K^{MN}_{,\bar{\xi}}K_{MN,\bar{\xi}} = 0.\]  \hspace{1cm} (2.20)

The (12) component of equation (1.40), reads

\[G_{12} = \frac{1}{\sqrt{-K}}(\sqrt{-K})_{12} - \frac{1}{4}K^{-1}K^{MN}_{,1}K_{MN,2},\]  \hspace{1cm} (2.21)

\[-\phi^{-2}\frac{1}{\sqrt{-K}}(\sqrt{-K})_{\xi r} - \frac{1}{8}K^{-1}2\phi^2K^{MN}_{,1}K_{MN,2}.\]  \hspace{1cm} (2.21)

But because of \( \Gamma_{12}^r = 0 \) and \( (\sqrt{-K})_{12} = (\sqrt{-K})_{\xi\xi} \), cancelling the terms with first derivatives, we have

\[G_{12} = \frac{1}{\sqrt{-K}}(\sqrt{-K})_{\xi\xi} - \frac{2}{\sqrt{-K}}(\sqrt{-K})_{,\xi\xi}\]

\[= -\frac{1}{\sqrt{-K}}(\sqrt{-K})_{\xi\xi}.\]  \hspace{1cm} (2.22)

Including that \( R_{AB} = 0 \) implies \( (\sqrt{-K})_{\xi\bar{\xi}} = 0 \), it follows that

\[R_{AB} = 0 \Rightarrow G_{12} = 0,\]  \hspace{1cm} (2.23)

so that the conditions \( G_{12} = 0 \) are automatically fulfilled on the "K-sector" equations.

The last of the field equations which must be described in terms of the Weyl coordinates is (1.41). Assuming that as a consequence of \( R_{AB} = 0 \) we know already that \( \sqrt{-K}_{,r} = 0 \Rightarrow (\sqrt{-K})_{\xi\xi} = 0 \), and using (2.15), this equation transforms to

\[R = -4\phi^2(\ln\phi)_{,\xi\xi} - \frac{1}{2}K^{-1}\phi^2K^{AB}_{,\xi}K_{AB,\xi} = 0,\]  \hspace{1cm} (2.24)

or simply to demand that

\[8(\ln\phi)_{,\xi\xi} + K^{-1}K^{AB}_{,\xi}K_{AB,\xi} = 0\]  \hspace{1cm} (2.25)

Summarizing, we conclude that working in the Weyl coordinates \( (\xi, \bar{\xi}) \), in order to fulfill the field equations \( R_{\mu\nu} = 0 \), we must demand that the real structural functions \( K_{AB} = K_{AB}(\xi, \bar{\xi}) = K(AB), \phi = \phi(\xi, \bar{\xi}) \) should be subject to the sequence of differential conditions:

(a)

\[\left(\frac{1}{\sqrt{-K}}K^{R}_{,\xi}K_{RB,\xi}\right)_{,\xi} + \left(\frac{1}{\sqrt{-K}}K^{R}_{,\bar{\xi}}K_{RB,\bar{\xi}}\right)_{,\bar{\xi}} = 0\]  \hspace{1cm} (2.26)
These equations can be derived from the Lagrangian where the potential $E$ 

\[
\begin{align*}
\frac{1}{2} \sqrt{-K} \xi + 2(\ln \phi) \xi (\sqrt{-K}) \xi + \frac{1}{4} \sqrt{-K} K^{AB} \xi K_{AB} \xi = 0 \\
\frac{1}{2} \sqrt{-K} \xi + 2(\ln \phi) \xi (\sqrt{-K}) \xi + \frac{1}{4} \sqrt{-K} K^{AB} \xi K_{AB} \xi = 0
\end{align*}
\]

The complete set of these conditions, complemented by the basic requirement $-K = -\det(K_{AB}) > 0$, is sufficient to assure $R_{\mu\nu} = 0$, but we do not claim that all of these conditions are (independently) necessary.

To end this section we rewrite equations (2.26) in matrix notation. Let $(\gamma)_{AB} = K_{AB}$ be the 2x2 symmetric matrix corresponding to the Killingian part of the metric (1.1). Observe that $K^B = \gamma \epsilon$ and $K^{AB} = -\rho^2 \gamma^{-1}$ with $\epsilon = \|\epsilon_{AB}\|$, then equations (2.26) can be cast into the form

\[
\begin{align*}
\text{(a)} & \quad (\rho \gamma^{-1} \gamma, \xi) + (\rho \gamma^{-1} \gamma, \xi) = 0 \\
\text{(b)} & \quad \begin{cases}
(\ln \phi^{-2})_{,\xi} = \frac{(\ln \rho)_{,\xi}}{\ln \rho} + \frac{1}{4(\ln \rho)} tr(\gamma, \xi) \gamma^{-1,2} \\
(\ln \phi^{-2})_{,\xi} = \frac{(\ln \rho)_{,\xi}}{\ln \rho} + \frac{1}{4(\ln \rho)} tr(\gamma, \xi) \gamma^{-1,2}
\end{cases}
\text{(c)} & \quad (\ln \phi^{-2})_{,\xi} = (\ln \rho)_{,\xi} (\ln \rho)_{,\xi} + \frac{1}{4} tr(\gamma, \xi) \gamma^{-1,2}
\end{align*}
\]

where we have set $K = -\rho^2$. Equation (2.27c) is a consistency equation of (2.27b)(see also ref. [9]).

3. The Potential Space.

The Ernst potential [13] $E$ has proved to be a very useful tool for finding exact solutions [14]. To define the potential $E$, we need the existence of a time-like Killing vector $Z_{\mu}$. Then one finds that the Einstein equations in vacuum

\[
R_{\mu\nu} Z^\nu = 0
\]

implies that

\[
Q^\mu_{\alpha} = \tilde{Z}^\mu_{\alpha} = 0 \Leftrightarrow Z^\alpha Q_{[\alpha\mu\nu]} = 0
\]

where $Q_{\mu\nu} = \tilde{Z}_{\mu\nu} = Z_{\mu\nu} + \frac{1}{2} \gamma_{\mu\nu,\alpha} Z^{\alpha,\beta}$. Furthermore one finds that the Lie derivative of $Q_{\mu\nu}$ with respect to $Z$ vanishes. This allows us to define the Ernst potential $E_{\mu} = Z^\alpha Q_{\alpha\mu}$, whose integrability conditions are just (3.2) and the vanishing of the Lie derivative of $Q_{\mu\nu}$, i.e.

\[
2E_{,\mu\nu} = Z^\alpha Q_{[\alpha\mu\nu]} - L_Z Q_{\mu\nu} = 0
\]

The Einstein field equations in vacuum in terms of the Ernst potential read [3], [14]

\[
(\mathcal{E} + \bar{\mathcal{E}}) \rho^{-1} (\rho \mathcal{E}_{,\mu})^\mu = 2 \mathcal{E}_{,\nu} \mathcal{E}^\mu.
\]

and $\mathcal{E} \rightarrow \bar{\mathcal{E}}$

These equations can be derived from the Lagrangian

\[
\mathcal{L} = \rho \left( \frac{1}{2 f^2} \mathcal{E}_{,\mu} \mathcal{E}^\mu \right) \Rightarrow f = Re \mathcal{E}
\]

Generation techniques consist in finding invariant transformations of the Lagrangian (3.5). This is equivalent to finding the isometry group of the metric

\[
dS^2 = \frac{1}{2 f^2} d\mathcal{E} d\bar{\mathcal{E}}.
\]
This isometry group is $SU(1,1)$ [14] which is isomorphic to $SL(2,\mathbb{R})$. A straightforward calculation shows that the metric (3.6) of the potential space (defined by this metric) can be cast into the form [3]

$$dS^2 = \frac{1}{4} \text{tr}(dGdG^{-1})$$ (3.7)

where the 2x2 matrix $G$ is given by

$$G = \frac{-1}{\epsilon + \bar{\epsilon}} \begin{pmatrix} 1 + \epsilon \bar{\epsilon} & 1 - \epsilon \bar{\epsilon} + \epsilon - \bar{\epsilon} \\ 1 - \epsilon \bar{\epsilon} + \epsilon - \bar{\epsilon} & -1 - \epsilon \bar{\epsilon} \end{pmatrix}.$$ (3.8)

The matrix (3.8) is an element of the group $SU(1,1)$ restricted to $G^2 = 1$. An other parametrization of (3.7) belongs to the group $SL(2,\mathbb{R})$, it reads [6],[10]

$$G = \frac{1}{f} \begin{pmatrix} f^2 + \epsilon^2 & -\epsilon \\ -\epsilon & 1 \end{pmatrix}$$ (3.9)

where $\epsilon = f + i\epsilon$. Observe that $detG = 1$ in both cases, but $G$ in (3.9) is symmetric. Now we return to the field equations in the potential space. It is clear that the field equation (3.4) can be derived from the Lagrangian

$$\mathcal{L} = \frac{1}{4} \rho \text{tr}(G,MG^{-1},M).$$ (3.10)

We are interested on fields depending on two coordinates $X^1$ and $X^2$. Let $z = X^1 + iX^2$ be the complex variable which $G$ depends on. Equation (3.4) transforms into [6],[10]

$$(\rho G,zG^{-1}), \bar{z} + (\rho G,zG^{-1}), z = 0$$ (3.11)

derived from Lagrangian (3.10). They are the chiral equations with $G \in SL(2,\mathbb{R})$ and $G = G^T$.

4. The $SL(2,\mathbb{R})$ – Chiral Equations

It is now clear why we are interested in developing a technique for solving chiral equations with the group $SL(2,\mathbb{R})$. Solitonic methods for doing so are given in ref.[15]. We want to give another method using an ansatz due to Neugebauer and Kramer [8]. Let us outline it [7].

Suppose $G$ is an element of a Lie group $H$ which depends on a set of parameters $\lambda^a; \ a = 1,\ldots,p$ with $\lambda^a = \lambda^a(z, \bar{z})$ such that these parameters form minimal surfaces on a Riemannian space $V_p$ ($HM, \lambda^a : M^4 \rightarrow V_p$):

$$(\rho \lambda^a),z + (\rho \lambda^a), \bar{z} + 2\rho \gamma^a_{bc} \lambda^b_z \lambda^c_{\bar{z}} = 0$$ (4.1)

$$a, b, c = 1,\ldots,p,$$

where $\gamma^a_{bc}$ are the Christoffel symbols of $V_p$. In terms of these parameters the chiral equation (3.11) reads a)

$$A_{i;j} + A_{j;i} = 0$$

b)

$$A_{i;j} = -[A_i, A_j]$$ (4.2)

where $A_i = (\partial_i, G)G^{-1}$, $A_i$ is the Maurer-Cartan form of the group $H$ and therefor it belongs to the corresponding Lie algebra $\mathfrak{g}$ of $H$. Equation (4.2a) is the Killing equation on $V_p$ for each component of matrix $A_i$. Thus we write $A_i$ in spinor-like notation

$$A_i = \xi_i^j \sigma_j,$$ (4.3)
where $\{\xi_i\}$ is a linearly independent set of Killing vectors on $V_p$ and $\{\sigma_j\}$ is a base of the vector space $G$. Let $L_o$ be the left action of $H_c$ on $H$, being $H_c$ the group of matrices on $H$ which does not depend on $z$ and $\bar{z}$, i.e. $C \xi z = C \bar{z} = 0$. Then the equivalence relation: $A_i^0 \sim A_i$ if there exist $C \xi H_c$ such that $A_i^0 = A_i o L_o$, separates the set $\{A_i\}$ into equivalence classes. Let $TB = \{A_i\}/\sim$ be a set of representatives of each class. Map each representative into the group building the set $B := \{G | G = \exp A_i, A_i \in TB\}$. If we know the set $B$, we can obtain the whole set $H$ by left action of $H_c$ on $H$. We apply it to the group $SL(2,\mathbb{R})$.

Let first $p = 1$ be the one dimensional Riemannian space $V_1$. All one dimensional Riemannian spaces are flat and the geodesic equation (4.1) reduces to

$$ (\rho \lambda, z) + (\rho \lambda, \bar{z}) = 0, $$

which is the Laplace equation in $z, \bar{z}$ coordinates. The Killing equation (4.2a) now reads

$$ A_{1,\lambda} = 0, \quad A_1 = (\partial_\lambda G) G^{-1}. $$

That means that $A_1$ is a constant matrix of $G = sl(2,\mathbb{R})$. We need then the matrices $G$ to be symmetric. Hence it is convenient to choose the left action

$$ L_c(G) = CGCT $$

in order to have $L_c(G)$ also symmetric. With this left action the equivalence relation reads

$$ A_i^0 = A_1 o L_o = CA_1 C^{-1}. $$

The representatives of the classes can be the 2x2, traceless and real Jordan matrices. There is only one representative, i.e.

$$ TB = \{A_i\}/\sim = \{ \begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix} \}. $$

Now we map $A_1$ into the group $H$, using its definition $A_1 = (\partial_\lambda G) G^{-1}$, i.e., solving this matrix differential equation. We have

$$ G_{\lambda} = A_1 G, $$

and get

$$ G_{11,\lambda} = G_{12}, \quad G_{12,\lambda} = G_{22}, $$

$$ G_{12,\lambda} = \alpha G_{11}, \quad G_{22,\lambda} = \alpha G_{12} $$

which imply the differential equations

$$ G_{11,\lambda} - \alpha G_{11} = 0, \quad G_{12,\lambda} - \alpha G_{12} = 0. $$

We separate these differential equations in three cases, $\alpha = a^2 > 0$, $\alpha = -a^2 < 0$ and $\alpha = 0$. The results are given in table 1.

We suppose now $p = 2$ and $V_2$ a 2-dimensional Riemannian space. But all $V_2$ Riemannian spaces are conformally flat and therefore have a metric which may be written in the form

$$ dS^2 = \frac{d\lambda d\tau}{(1 + k\lambda \tau)^2}. $$

In reference [2] and [8] it was shown that the $V_p$ spaces must be symmetric (i.e. all covariant derivatives of the Riemannian tensor vanish), this implies that $k$ must be a constant. Now we choose a base of the Killing
vector space on $V_2$. Let this base be

$$\xi_1 = \frac{1}{2V^2} (kr^2 + 1, k\lambda^2 + 1)$$

$$\xi_2 = \frac{1}{V^2} (-\tau, \lambda) \quad V = (1 + k\lambda \tau)$$

$$\xi_3 = \frac{1}{2V^2} (kr^2 - 1, 1 - k\lambda^2). \quad \text{(4.11)}$$

With this set of Killing vectors the commutation relations (4.2b) read

$$[\sigma_1, \sigma_2] = -4k\sigma_3$$

$$[\sigma_2, \sigma_3] = 4k\sigma_1$$

$$[\sigma_3, \sigma_1] = -4\sigma_2. \quad \text{(4.12)}$$

We have to put $k = -1$ in order to have the commutation relations of $sl(2, \mathbb{R})$. A representation of $sl(2, \mathbb{R})$ is

$$\sigma_1 = 2 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_2 = 2 \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix}, \sigma_3 = 2 \begin{pmatrix} 0 & -b \\ a & 0 \end{pmatrix}, \quad ab = 1,$$

$$\text{(4.13a)}$$

which is a base of the vector space $\mathcal{G} = sl(2, \mathbb{R})$. Now we map the algebra into the group. From the definitions $A_\lambda = (\partial_\lambda G)G^{-1} = \xi_1^2 \sigma_2$ and $A_\tau = (\partial_\tau G)G^{-1} = \xi_2^2 \sigma_2$ we obtain [9]

$$G = \frac{1}{1 - \lambda\tau} \begin{pmatrix} c(1 - \lambda)(1 - \tau) & e(\tau - \lambda) \\ e(\tau - \lambda) & d(1 + \lambda)(1 + \tau) \end{pmatrix}$$

$$-e^2 = cd, \quad a = \frac{c}{e}, \quad b = -\frac{e}{d}. \quad \text{(4.13b)}$$

where $\lambda$ and $\tau$ must fulfill the geodesic equation (4.1). With the metric (4.10) we get

$$(\rho\lambda_{,z})_{,z} + (\rho\lambda_{,\bar{z}})_{,\bar{z}} + \frac{4\tau}{1 - \lambda\tau}\rho\lambda_{,z}\lambda_{,\bar{z}} = 0, \quad \text{(4.14)}$$

and the other one by changing $\lambda$ for $\tau$ and viceversa. The other possibility is to put $k = 0$ in (4.12), but this case corresponds to the one-dimensional subalgebras of $sl(2, \mathbb{R})$ studied before with $\lambda \rightarrow \lambda + \tau$. The corresponding equation (4.1) will be the Laplace equation for $\lambda$ and $\tau$ separately. Observe that $\det G = 1$ if we put $d = -1, c = 1$ in (4.13) and $\det G = -1$ if we put $d = 1 = c$.

Let us introduce a real potential $\alpha$ by

$$\alpha_{,z} = -\frac{1}{2\rho} tr(G_{,z}G^{-1}). \quad \text{(4.15)}$$

and a similar equation for $\alpha_{,\bar{z}}$, with $\bar{z}$ in place of $z$. Since the chiral equations imply $\alpha_{,z\bar{z}} = \alpha_{,z\bar{z}}$, the integrability condition of $\alpha$ follows from the chiral equation (3.11) [8]. This potential can be calculated separately for each case. For the one dimensional subspaces we have

$$\alpha_{,z} = \frac{1}{2\rho} tr(G_{,z}G^{-1})^2$$

$$= \frac{1}{2\rho} tr(G_{,z}G^{-1})^2(\lambda_{,z})^2, \quad \text{(4.16)}$$

$$= \frac{1}{2\rho} trA^2(\lambda_{,z})^2.$$
and for the two-dimensional subspaces, a straightforward calculation gives

$$\alpha_z = -2\rho \frac{(\lambda - \tau)^2}{(1 - \rho \tau)^4} \lambda_z \tau_z. \quad (4.17)$$

These potentials will be important in the following section.

5. The Unified Point of View

In this section we apply the technique developed in section 4 to the results of sections 2 and 3.

Equations (2.27a) and (3.11) are chiral equations, but there is a difference between them. In the first case $\det \gamma = -\rho^2$, in the second one $\det G = 1$. We can transform equation (2.27a) in order to have a (3.11)-like equation. First observe that $\sqrt{-K} = \frac{1}{2}(\xi + \bar{\xi})$. is a solution of equation (2.17). Then we can write $\xi = X^1 + iX^2 = \rho + i\zeta$. Now define a matrix $G$ such that $\gamma = \rho G$ with $\det G = -1$. It is easy to see that $G$ fulfills just the chiral equation (3.11). Let us parametrize the matrix $G$ in Papapetrou form

$$G = -\frac{f}{\rho} \begin{pmatrix} -\frac{1}{4}\rho \omega^2 + \omega^2 & \omega \\ \omega & 1 \end{pmatrix} = \frac{1}{F} \begin{pmatrix} -F^2 + \omega^2 & \omega \\ \omega & 1 \end{pmatrix}, \quad f = e^{-2u}, \quad (5.1)$$

where $F = -\rho/f$. Compare (5.1) with (3.9). That means the following: if we have a solution of the chiral equation (5.2) and $\xi = X^1 + iX^2 = \rho + i\zeta$, then which in the Papapetrou form reads

$$2u = a\lambda + \ln \rho, \quad 4ba^2C_4 \rho e^{\alpha \lambda} = -1, \omega = \text{constant}. \quad (5.6)$$

An other example on this subspace is the solution of the Laplace equation $\lambda = \frac{2}{9} \ln \rho$. If we substitute
this $\lambda$ in (5.2) then we find a cylindrically symmetric solution of the Einstein’s equations with one restriction (equation (20.7) in ref. [14] with $B_2a_1a_2 = -1/4$. See also ref. [9]). Now we apply the transformation (4.6) to get

$$CGC^T = A\left(\begin{array}{c} C_2^+ C_+d_+ \\ C_+d_+ \end{array}\right) e^{a\lambda} + B\left(\begin{array}{c} C_2^- C_-d_- \\ C_-d_- \end{array}\right) e^{-a\lambda}$$

which is the general cylindrically symmetric solution of the Einstein’s equations in vacuum, (equation (20.7) in ref. [14] with $c = C_−d_−, a_1 = bd_+, a_2 = −\frac{1}{2b_0}d_2^2$).

The second case in table 1 corresponds to the solution

$$\gamma = \begin{pmatrix} -\frac{2}{\rho}\cos(a\lambda + \psi) & \rho\sin(a\lambda + \psi) \\ \rho\sin(a\lambda + \psi) & \rho\cos(a\lambda + \psi) \end{pmatrix}$$

where $b^2a^2 = 1$, in order to have $detG = -1$. The function $\phi$ is then a solution of the differential equation

$$(ln\rho^{1/2}\phi^{-2}),z = -a^2\rho(\lambda,z)^2.$$ (5.9)

It is easy to see that

$$\rho^2e^{-4u} = \omega^2 + 1,$$ (5.10)

which corresponds to the Lewis’ class (see eq. (18.22) in ref. [14]). The third case of table 1 corresponds to the degenerated class $f = 0$ [8].

There is only one case in the two-dimensional subsapce. In the spacetime one gets the solution

$$\gamma = \frac{1}{(1-\lambda\tau)}\begin{pmatrix} \rho(1-\lambda)(1-\tau) & \rho(\tau-\lambda) \\ \rho(\tau-\lambda) & -\rho(1+\lambda)(1+\tau) \end{pmatrix}.$$ (5.11)

Comparing (2.27b) with (4.17) we can obtain the function $\phi$ of (2.1) in terms of the parameters $\lambda$ and $\tau$, we arrive at

$$(ln\rho\phi^{-2}),z = -2\rho\frac{(\lambda-\tau)^2}{(1-\lambda\tau)^2}\lambda,\tau,z.$$ (5.12)

where $\lambda$ and $\tau$ are solutions of equation (4.14).

Now we deal with the subsapces in the potential space. Here we need that $detG = 1$, but this can only happen in the first case in table 1. We obtain

$$G = \begin{pmatrix} be^{a\lambda} + \frac{1}{4b_0}e^{-a\lambda} & ba e^{a\lambda} - \frac{1}{4b_0} e^{-a\lambda} \\ ba e^{a\lambda} - \frac{1}{4b_0} e^{-a\lambda} & ba^2e^{a\lambda} + \frac{1}{4b} e^{-a\lambda} \end{pmatrix}.$$ (5.13)

If we compare (5.13) with (3.9), we find that

$$f = \frac{1}{ba^2e^{a\lambda} + \frac{1}{4b} e^{-a\lambda}}, \epsilon = \frac{ba e^{a\lambda} - \frac{1}{4b_0} e^{-a\lambda}}{ba^2e^{a\lambda} + \frac{1}{4b} e^{-a\lambda}}.$$ (5.14)

If we set $b = 1/2, a = e^{\psi_0}$, the Ernst potential $E = f + ic$ can be written as

$$E = \frac{1}{a}[sech(a\lambda + \psi_0) - i tanh(a\lambda + \psi_0)]$$ (5.15)
(compare it with eq. (17.32) of ref. [14]). The function \( k \) in the Papapetrou metric (5.1) can be obtained by integrating the equation [16]

\[
k_z = \sqrt{-2\rho \mathcal{E},z \bar{\mathcal{E}},z}.
\]

(5.16)

We find that \( k = \frac{\sqrt{2}}{4} \alpha \) being \( \alpha \) potential (4.16). It is easy to see that

\[
f^2 = -\epsilon^2 + 1/a^2,
\]

(5.17)

which means that those solutions belong to the Papapetrou’s class (see eq. (18.16) of ref. [14]).

Finally for the two-dimensional subspaces (4.13b) in the potential space we must have \( \det G = 1 \). This can be done if we write \( \zeta = \lambda = \bar{\tau} \), i.e.

\[
G = \frac{1}{1 - \zeta\bar{\zeta}} \begin{pmatrix}
(1 - \zeta)(1 - \bar{\zeta}) & -i(\zeta - \bar{\zeta}) \\
-i(\bar{\zeta} - \zeta) & (1 + \zeta)(1 + \bar{\zeta})
\end{pmatrix}.
\]

(5.18)

A direct comparation of (5.18) with (3.9) shows that \( \mathcal{E} = (1 - \zeta)/(1 + \zeta) \). The geodesic equation (4.14) reads

\[
(\zeta \bar{\zeta} - 1)\frac{1}{\rho}[(\rho \zeta,\bar{z}) + (\rho \bar{\zeta},z)] = 4\bar{\zeta}\zeta,\bar{z}
\]

(5.19)

(compare this equation with eq. (18.28) of ref. [14]). These solutions belong to the Tomimatsu-Sato class. The Kerr solution in terms of the \( \zeta \) potential is \( \zeta = px - iqy \) \( (p^2 + q^2 = 1) \) in prolate coordinates \( x, y \) [14], [19]. The function \( k \) in front of the Papapetrou metric can be calculated. From (4.15) we again arrive at \( k = \frac{\sqrt{2}}{4} \alpha \), but now \( \alpha \) is the potential (4.17). All the results are shown in table 2.

6. Conclusions

We have shown that the solutions of the Einstein’s equations with two commuting Killing vectors can be separated naturally in equivalent classes given by the subgroups of \( SL(2, \mathbb{R}) \). Those classes coincide with the classes of solutions, the one parametric subgroups of \( SL(2, \mathbb{R}) \) give the Weyl’s class, van Stockum class, the general cylindrically symmetric solutions, the Lewis’ and the degenerated classes in the spacetime, and the Papapetrou’s class in the potential space. The group \( SL(2, \mathbb{R}) \) itself gives the Tomimatsu-Sato class in the potential space and a corresponding class in the space time. The left action of the group \( SL(2, \mathbb{R})_C \) over \( SL(2, \mathbb{R}) \) is just the invariance transformations of Neugebauer and Kramer [16], i.e. they are the Ehlers and the Buchdahl transformations to obtain new solutions from a seed one. Then we have shown that a classification of these solutions can be done by the classification of the subgroups of the invariance group.

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References

[1] Matzner, R. A., and Misner, C.W. Phys. Rev.154 (1967), 1229.
Miser, C.W. Phys. Rev. D18 (1978), 4510.
[2] Sánchez, N. Phys. Rev. D26 (1982), 2589.
[3] Neugebauer, G. and Kramer D. In “ Galaxies, axisymmetric systems and relativity” Ed. M.A. H. MacCallum (1985).
[4] Belinsky, V.A. and Zakharov, V.E. Zh. Eksp. Teor. Fis. 75 (1978), 1953.
[5] Whitman, A.P. and Stoeger, W.R. Gen. Rel. Grav. 24 (1992), 641
[6] See for example Matos, T. Rev. Mex.Fis. 36 (1990), 340 or Díaz, C.M.
[7] Matos, T., Rodríguez, G. and Becerril, R. J. Math. Phys. 33, (1992), 3521.
[8] Neugebauer, G. and Kramer, D. "Stationary Axisymmetric Einstein Maxwell Fields Generated by Bäcklund Transformations". Jena Preprint 1989.
[9] Matos, T. Rev. Mex. Fis. 35, (1989), 208.
[10] Matos, T. Phys. Lett. A131, (1989), 423.
[11] Clement, G. Gen. Rel. Grav. 18 (1987), 5.
[12] Matos, T. Ann. Phys. (Leipzig) 46, (1989), 462.
[13] Ernst, F.J.V. Phys. Rev. 167 (1968), 1175.
   Neugebauer, G. and Kramer, D. Ann. Phys. (Leipzig) 24 (1969), 62.
   Ernst, F. and Plebański, J. Ann. Phys. 107 (1977), 266
[14] Kramer, D., Stephani, H., MacCallum, M. and Herlt, E.
   "Exact Solutions of Einstein's Field Equations". VEB-DVW, 1980.
[15] Kramer, D., Neugebauer, G. and Matos, T. J. Math. Phys. 32 (1991), 2727.
   Mendoza, A. and Restuccia, A. J. Math. Phys. 32 (1991), 480.
[16] Neugebauer, G. and Kramer, D. in ref. [13].
[17] Nakamura, Y. J. Math. Phys. 24 (1983), 606.
[18] van Stockum, W.J. Proc. Roy. Soc. Edinburgh 457 (1937), 135.
[19] Tomimatsu, A. and Sato, H. Prog. Theor. Phys. 50, (1973), 95.
1. \( \begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix} \) \( a^2 \) \( G = b \begin{pmatrix} 1 & a \\ a & a^2 \end{pmatrix} e^{a \lambda} + c \begin{pmatrix} 1 & -a \\ -a & a^2 \end{pmatrix} e^{-a \lambda} \)

\[ \det G \] \[ \text{tr}A^2 \]

2. \( -a^2 \) \( G = b \begin{pmatrix} \cos(a \lambda + \psi_0) & -a \sin(a \lambda + \psi_0) \\ -a \sin(a \lambda + \psi_0) & -a^2 \cos(a \lambda + \psi_0) \end{pmatrix} \)

\[ \det G \] \[ \text{tr}A^2 \]

3. 0 \( G = \begin{pmatrix} b \lambda + c & b \\ b & 0 \end{pmatrix} \)

\[ \det G \] \[ \text{tr}A^2 \]

Table 1. One-dimensional subspaces of SL(2,\( \mathbb{R} \)). \( a, b, c, \psi_0 \), constants.

Table 2. Classification of the solutions