An atomic polyadic algebra of infinite dimension is completely representable if and only if is completely additive

Tarek Sayed Ahmed
Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt.

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Abstract. Answering a question posed by Hodkinson, we show that for infinite ordinals $\alpha$, every atomic polyadic algebra of dimension $\alpha$ ($\text{PA}_\alpha$) is completely representable if and only if it is completely additive. We readily infer that the class of completely representable $\text{PA}_\alpha$s is elementary. This is in sharp contrast to the cylindric case. However, we do not settle the question as to whether there are atomic polyadic algebras that are not completely additive, hence not completely representable. In this connection, we could only show that the proper reduct of polyadic algebras, obtained by discarding all non bijective transformations, with the exception of replacements, is not completely additive. Our proof of the equivalence in the title uses a neat embedding theorem together with a simple topological argument, namely, that principal ultrafilters in the Stone space of a Boolean algebra lie outside nowhere dense sets, and if the algebra is atomic they form a dense subset of the Stone topology. An analogous result is proved for many modifications of polyadic algebras. In all cases the signature contains all substitutions, so that the cylindrifier free reduct of such algebras, can be viewed as a transformation system. Finally, we give a metalogical interpretation to our algebraic result, which is a Vaught’s theorem for Keisler’s logic. Certain atomic theories have atomic models. Based on work of Ferenczi, we discuss various metalogical property for reducts of Keisler’s logic endowed with equality. In particular, for such logic, we show that any atomic theory has an atomic model. Further generalizations are discussed.

Polyadic algebras were introduced by Halmos [18] to provide an algebraic reflection of the study of first order logic without equality. Later the algebras were enriched by diagonal elements to permit the discussion of equality. That the notion is indeed an adequate reflection of first order logic was demonstrated

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by Halmos’ representation theorem for locally finite polyadic algebras (with and without equality). Daigneault and Monk proved a strong extension of Halmos’ theorem, namely that, every polyadic algebra of infinite dimension (without equality) is representable [16].

There are several types of representations in algebraic logic. Ordinary representations are just isomorphisms from boolean algebras with operators to a more concrete structure (having the same similarity type) whose elements are sets endowed with set-theoretic operations like intersection and complementation. Complete representations, on the other hand, are representations that preserve arbitrary conjunctions whenever defined. The notion of complete representations has turned out to be very interesting for cylindric algebras, where it is proved in [21] that the class of completely representable algebras is not elementary.

Lately, it has become fashionable to study representations that preserve infinitary meets and joins. This phenomena is extensively discussed in [13], where it is shown that it has affinity with the algebraic notion of complete representations for cylindric like algebras and atom-canonicity in varieties of Boolean algebras with operators, a prominent persistence property studied in modal logic.

The typical question is: given an algebra and a set of meets, is there a representation that carries this set of meets to set theoretic intersections? (assuming that our semantics is specified by set algebras, with the concrete Boolean operation of intersection among its basic operations.) When the algebra in question is countable, and we have countably many meets; this is an algebraic version of an omitting types theorem; the representation omits the given set meets or non-principal types. When it is only one meet consisting of co-atoms, in an atomic algebra, this representation is a complete one.

The correlation of atomicity to complete representations has caused a lot of confusion in the past. It was mistakenly thought for a long time, among algebraic logicians, that atomic representable relation and cylindric algebras are completely representable, an error attributed to Lyndon and now referred to as Lyndon’s error.

For boolean algebras, however this is true. The class of completely representable algebras is simply the class of atomic ones. An analogous result holds for certain countable reducts of polyadic algebras [5]. The notion of complete representations has been linked to Martin’s axiom, omitting types theorems and existence of atomic models in model theory [6], [4], [12]. Such connections will be worked out below in the context of Keisler’s logic the infinitary logic corresponding to $\mathcal{PA}_\alpha$. This logic allows formulas of infinite length and infinite quantification.

In this paper we show that an atomic polyadic algebra of infinite dimension is also completely representable, if and only if it is completely additive.
From this we conclude that the class of completely representable algebras is elementary by just spelling out first order formulas, one for each substitution stipulating that it is completely additive. This gives continuum many formulas, however, they share one schema.

Our result is in sharp and, indeed, interesting contrast to the cylindric and quasi-polyadic equality cases [21], [9] (these are completely additive varieties). This result also adds to the long list of results existing in the literature, further emphasizing, the commonly accepted view that cylindric algebras and polyadic algebras belong to different paradigms often exhibiting conflicting behaviour. It also answers a question raised by Ian Hodkinson, see p. 260 in [20] and Remark 6.4, p. 283 in op.cit, and a question in [2], though admittedly the latter is ours [13].

1 Main result

Our notation is in conformity with [2] which is based on the notation [19]. However, we chose to deviate from [19], when we felt that it was compelling to reverse the order. We write $f \upharpoonright A$ instead of $A \upharpoonright f$, for the restriction of a function $f$ to a set $A$, which is the more usual standard notation. On the other hand, following [19], for given sets $A, B$ we write $A \sim B$ for the set \{ $x \in A : x \notin B$ \}. Gothic letters are used for algebras, and the corresponding Roman letter will denote their universes. Also for an algebra $\mathfrak{A}$ and $X \subseteq A$, $\mathfrak{S}_g^A X$, or simply $\mathfrak{S}_g X$ when $\mathfrak{A}$ is clear from context, denotes the subalgebra of $\mathfrak{A}$ generated by $X$. $Id$ denotes the identity function and when we write $Id$ we will be tacitly assuming that its domain is clear from context. We now recall the definition of polyadic algebras as formulated in [19]. Unlike Halmos’ formulation, the dimension of algebras is specified by ordinals as opposed to arbitrary sets.

**Definition 1.1.** Let $\alpha$ be an ordinal. By a *polyadic algebra* of dimension $\alpha$, or a $\text{PA}_\alpha$, for short, we understand an algebra of the form

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_\Gamma, s_\tau \rangle_{\Gamma \subseteq \alpha, \tau \in ^{\alpha} \alpha}$$

where $c_\Gamma$ ($\Gamma \subseteq \alpha$) and $s_\tau$ ($\tau \in ^{\alpha} \alpha$) are unary operations on $A$, such that postulates below hold for $x, y \in A, \tau, \sigma \in ^{\alpha} \alpha$ and $\Gamma, \Delta \subseteq \alpha$

1. $\langle A, +, \cdot, -, 0, 1 \rangle$ is a boolean algebra
2. $c_\Gamma 0 = 0$
3. $x \leq c_\Gamma x$
4. $c_\Gamma(x \cdot c_\Gamma y) = c_\Gamma(x \cdot c_\Gamma y)$
5. $c_{(\Gamma)}c_{(\Delta)}x = c_{(\Gamma \cup \Delta)}x$

6. $s_\tau$ is a boolean endomorphism

7. $s_{id}x = x$

8. $s_{\sigma \circ \tau} = s_\sigma \circ s_\tau$

9. If $\sigma \upharpoonright (\alpha \sim \Gamma) = \tau \upharpoonright (\alpha \sim \Gamma)$, then $s_\sigma c_{(\Gamma)}x = s_\tau c_{(\Gamma)}x$

10. If $\tau^{-1}\Gamma = \Delta$ and $\tau \upharpoonright \Delta$ is one to one, then $c_{(\Gamma)}s_\tau x = s_\tau c_{(\Delta)}x$.

We will sometimes add superscripts to cylindrifiers and substitutions indicating the algebra they are evaluated in. The class of representable algebras is defined via set-theoretic operations on sets of $\alpha$-ary sequences. Let $U$ be a set. For $\Gamma \subseteq \alpha$ and $\tau \in {}^\alpha \alpha$, we set

$$c_{(\Gamma)}X = \{ s \in {}^\alpha U : \exists t \in X, \forall j \not\in \Gamma, t(j) = s(j) \}$$

and

$$s_\tau X = \{ s \in {}^\alpha U : s \circ \tau \in X \}.$$

For a set $X$, let $\mathcal{B}(X)$ be the boolean set algebra $(\wp(X), \cup, \cap, \sim)$. The class of representable polyadic algebras, or $\text{RPA}_\alpha$ for short, is defined by

$$\text{SP}\{ (\mathcal{B}(^\alpha U), c_{(\Gamma)}, s_\tau)_{\Gamma \subseteq \alpha, \tau \in {}^\alpha \alpha} : U \text{ a set} \}.$$

Here $\text{SP}$ denotes the operation of forming subdirect products. It is straightforward to show that $\text{RPA}_\alpha \subseteq \text{PA}_\alpha$. Daigneault and Monk [16] proved that for $\alpha \geq \omega$ the converse inclusion also holds, that is $\text{RPA}_\alpha = \text{PA}_\alpha$. This is a completeness theorem for certain infinitary extensions of first order logic without equality [22].

In this paper we are concerned with the following question: If $\mathfrak{A}$ is a polyadic algebra, is there a representation of $\mathfrak{A}$ that preserves infinite meets and joins, whenever they exist? (A representation of a given abstract algebra is basically a non-trival homomorphism from this algebra into a set algebra). To make the problem more tangible we need to prepare some more. In what follows $\prod$ and $\sum$ denote infimum and supremum, respectively. We will encounter situations where we need to evaluate a supremum of a given set in more than one algebra, in which case we will add a superscript to the supremum indicating the algebra we want. For set algebras, we identify notationally the algebra with its universe, since the operations are uniquely defined given the unit of the algebra.

Let $\mathfrak{A}$ be a polyadic algebra and $f : \mathfrak{A} \rightarrow \wp(^\alpha U)$ be a representation of $\mathfrak{A}$. If $s \in X$, we let

$$f^{-1}(s) = \{ a \in \mathfrak{A} : s \in f(a) \}.$$
An atomic representation \( f : \mathfrak{A} \to \wp^\alpha U \) is a representation such that for each \( s \in V \), the ultrafilter \( f^{-1}(s) \) is principal. A complete representation of \( \mathfrak{A} \) is a representation \( f \) satisfying

\[
f(\prod X) = \bigcap f[X]
\]

whenever \( X \subseteq \mathfrak{A} \) and \( \prod X \) is defined.

A completely additive boolean algebra with operators is one for which all extra non-boolean operations preserve arbitrary joins.

Lemma 1.2. Let \( \mathfrak{A} \in \mathsf{PA}_\alpha \). A representation \( f \) of \( \mathfrak{A} \) is atomic if and only if it is complete. If \( \mathfrak{A} \) has a complete representation, then it is atomic and is completely additive.

Proof. The first part is like [21]. For the second part, we note that \( \mathsf{PA}_\alpha \) is a discriminator variety with discriminator term \( c(\alpha) \). And so because all algebras in \( \mathsf{PA}_\alpha \) are semi-simple, it suffices to show that if \( \mathfrak{A} \) is simple, \( X \subseteq A \), is such that \( \bigvee X = 1 \), and there exists an injection \( f : \mathfrak{A} \to \wp(\alpha) \), such that \( \bigcup_{x \in X} f(x) = V \), then for any \( \tau \in \alpha \), we have \( \bigvee s_\tau X = 1 \). So assume that this does not happen for some \( \tau \in \alpha \). Then there is a \( y \in \mathfrak{A} \), \( y < 1 \), and \( s_\tau x \leq y \) for all \( x \in X \). Now

\[
1 = s_\tau \left( \bigcup_{x \in X} f(x) \right) = \bigcup_{x \in X} s_\tau f(x) = \bigcup_{x \in X} f(s_\tau x).
\]

(Here we are using that \( s_\tau \) distributes over union.) Let \( z \in X \), then \( s_\tau z \leq y < 1 \), and so \( f(s_\tau z) \leq f(y) < 1 \), since \( f \) is injective, it cannot be the case that \( f(y) = 1 \). Hence, we have

\[
1 = \bigcup_{x \in X} f(s_\tau x) \leq f(y) < 1
\]

which is a contradiction, and we are done.

By Lemma 1.2, a necessary condition for the existence of complete representations is the condition of atomicity and complete additivity. We now prove a converse to this result, namely, that when \( \mathfrak{A} \) is atomic and completely additive, then \( \mathfrak{A} \) is completely representable.

We need to recall from [19, definition 5.4.16], the notion of neat reducts of polyadic algebras, which will play a key role in our proof of the main theorem.

Definition 1.3. Let \( J \subseteq \beta \) and \( \mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c(\tau), s_\tau \rangle_{\tau \in \beta^J} \in \mathsf{PA}_\beta \). Let \( Nr_J \mathfrak{B} = \{ a \in A : c(\beta^J) a = a \} \). Then

\[
\mathfrak{A} = \langle Nr_J \mathfrak{B}, +, \cdot, -, c(\tau), s_\tau \rangle_{\tau \in J, \tau \in \alpha^J}
\]
where \( s' = s\tau. \) Here \( \tau = \tau \cup Id_{\beta \sim \alpha}. \) The structure \( \mathfrak{N}_J \mathfrak{B} \) is an algebra, called the \( J \) compression of \( \mathfrak{B}. \) When \( J = \alpha, \alpha \) an ordinal, then \( \mathfrak{N}_\alpha \mathfrak{B} \in \mathcal{PA}_\alpha \) and is called the neat \( \alpha \) reduct of \( \mathfrak{B} \) and its elements are called \( \alpha \)-dimensional.

The notion of neat reducts is also extensively studied for cylindric algebras [10]. We also need, [16] theorem 2.1 and top of p.161 in op.cit:

**Definition 1.4.** Let \( \mathfrak{A} \in \mathcal{PA}_\alpha. \)

(i) If \( J \subseteq \alpha, \) an element \( a \in A \) is independent of \( J \) if \( c(J)a = a; \) \( J \) supports \( a \) if \( a \) is independent of \( \alpha \sim J. \)

(ii) The effective degree of \( \mathfrak{A} \) is the smallest cardinal \( e \) such that each element of \( \mathfrak{A} \) admits a support whose cardinality does not exceed \( e. \)

(iii) The local degree of \( \mathfrak{A} \) is the smallest cardinal \( m \) such that each element of \( \mathfrak{A} \) has support of cardinality < \( m. \)

(iv) The effective cardinality of \( \mathfrak{A} \) is \( c = |Nr_J A| \) where \( |J| = e. \) (This is independent of \( J). \)

We chose to highlight he following simple basic known facts about boolean algebras and topological spaces.

(1) Let \( \mathfrak{B} \) be a boolean algebra, and let \( S \) be its Stone space whose underlying set consists of all ultrafilters of \( \mathfrak{B}. \) The topological space \( S \) has a clopen base of sets of the form \( N_b = \{ F \in S : b \in F \} \) for \( b \in B. \) Assume that \( X \subseteq B \) and \( c \in B \) are such that \( \sum X = c. \) Then the set \( N_c \sim \bigcup_{x \in X} N_x \) is nowhere dense in the Stone topology. In particular, if \( c \) is the top element, then it follows that \( S \sim \bigcup_{x \in X} N_x \) is nowhere dense. (A nowhere dense set is one whose closure has empty interior).

(2) Let \( X = (X, \tau) \) be a topological space. Let \( x \in X \) be an isolated point in the sense that there is an open set \( G \in \tau \) containing \( x, \) such that \( G \cap X = \{x\}. \) Then \( x \) cannot belong to any nowhere dense subset of \( X. \)

The proofs of these very elementary facts are entirely straightforward. They follow from the basic definitions. Now we formulate and prove the main result. The proof is basically a Henkin construction together with a simple topological argument. The proof also has affinity with the proofs of the main theorems in [16] and [8], endowed with a topological twist.

The idea is simple. Start with an atomic completely additive \( \mathfrak{A} \in \mathcal{PA}_\alpha. \) Then \( \mathfrak{A} \) neatly embeds into an algebra \( \mathfrak{B} \) having enough spare dimensions, called a dilation of \( \mathfrak{A}, \) that is \( \mathfrak{A} = \mathfrak{N}_\alpha \mathfrak{B}. \) As it turns out, \( \mathfrak{B} \) is also atomic, and by complete additivity the sum of all all substituted versions of the set of atoms is the top element in \( \mathfrak{B}. \) The desired representation is built from any
principal ultrafilter that preserves this set of infinitary joins as well as some infinitary joins that have to do with eliminating cylindrifiers. A principal ultrafilter preserving these sets of joins can always be found because, on the one hand, the set of principal ultrafilters are dense in the Stone space of the Boolean reduct of \( \mathfrak{B} \) since the latter is atomic, and on the other hand, finding an ultrafilter preserving the given set joints amounts to finding a a principal ultrafilter outside a nowhere dense set corresponding to the infinitary joins. Any such ultrafilter can be used to build the desired representation. But first a definition:

**Definition 1.5.** A transformation system is a quadruple of the form \((\mathfrak{A}, I, G, S)\) where \(\mathfrak{A}\) is an algebra of any similarity type, \(I\) is a non-empty set (we will only be concerned with infinite sets), \(G\) is a subsemigroup of \((I, \circ)\) (the operation \(\circ\) denotes composition of maps) and \(S\) is a homomorphism from \(G\) to the semigroup of endomorphisms of \(\mathfrak{A}\). Elements of \(G\) are called transformations.

The cylindrifier free reducts of polyadic algebras can be viewed as transformation systems where \(I\) is the dimension and \(G = I^{I}\). Now we prove our main result:

**Theorem 1.6.** Let \(\alpha\) be an infinite ordinal. Let \(\mathfrak{A} \in \text{PA}_\alpha\) be atomic and completely additive. Then \(\mathfrak{A}\) has a complete representation.

**Proof.** Let \(c \in A\) be non-zero. We will find a set \(U\) and a homomorphism from \(\mathfrak{A}\) into the set algebra with universe \(\wp(\alpha U)\) that preserves arbitrary suprema whenever they exist and also satisfies that \(f(c) \neq 0\). \(U\) is called the base of the set algebra. Let \(m\) be the local degree of \(\mathfrak{A}\), \(c\) its effective cardinality and \(n\) be any cardinal such that \(n \geq c\) and \(\sum_{s < m} n^s = n\). The cardinal \(n\) will be the base of our desired representation.

Substitutions in \(\mathfrak{A}\), induce a homomorphism of semigroups \(S : \alpha \alpha \to \text{End}(\mathfrak{A})\), via \(\tau \mapsto s_\tau\). The operation on both semigroups is composition of maps; the latter is the semigroup of endomorphisms on \(\mathfrak{A}\). For any set \(X\), let \(F(\alpha X, \mathfrak{A})\) be the set of all functions from \(\alpha X\) to \(\mathfrak{A}\) endowed with boolean operations defined pointwise and for \(\tau \in \alpha \alpha\) and \(f \in F(\alpha X, \mathfrak{A})\), put \(s_\tau f(x) = f(x \circ \tau)\). This turns \(F(\alpha X, \mathfrak{A})\) to a transformation system as well. The map \(H : \mathfrak{A} \to F(\alpha \alpha, \mathfrak{A})\) defined by \(H(p)(x) = s_\tau p\) is easily checked to be an isomorphism. Assume that \(\beta \supseteq \alpha\). Then \(K : F(\alpha \alpha, \mathfrak{A}) \to F(\beta \alpha, \mathfrak{A})\) defined by \(K(f)x = f(x \upharpoonright \alpha)\) is an isomorphism. These facts are straightforward to establish, cf. theorem 3.1, 3.2 in [16]. Call \(F(\beta \alpha, \mathfrak{A})\) a minimal functional dilation of \(F(\alpha \alpha, \mathfrak{A})\). Elements of the big algebra, or the (cylindrifier free) functional dilation, are of form \(s_\sigma p, p \in F(\beta \alpha, \mathfrak{A})\) where \(\sigma\) is one to one on \(\alpha\), cf. [16] theorem 4.3-4.4.

We can assume that \(|\alpha| < n\). Let \(\mathfrak{B}\) be the algebra obtained from \(\mathfrak{A}\), by discarding its cylindrifiers, then taking a minimal functional dilation and then
re-defining cylindrifiers in the bigger algebra, by setting for each \( \Gamma \subseteq n \):

\[
c(\Gamma) p = \sum_{\tau \in \gamma} c_\rho(\Gamma) \rho^{-1} c_\rho(\{\gamma\}) p.
\]

Here \( \rho \) is a any permutation such that \( \rho \circ \sigma(\alpha) \subseteq \sigma(\alpha) \). The definition is sound, that is, it is independent of \( \rho, \sigma, p \); furthermore, it agrees with the old cylindrifiers in \( A \). Identifying algebras with their transformation systems we have \( A \cong Nr_\alpha B \), via \( H \) defined for \( f \in A \) and \( x \in \beta \alpha \) by, \( H(f)x = f(y) \) where \( y \in \alpha \alpha \) and \( x \, | \, \alpha = y \), cf. [16] theorem 3.10.

The local degree of \( B \) is the same as that of \( A \), in particular, each \( x \in B \) admits a support of cardinality \( < m \). Furthermore, \( | n \sim \alpha | = | n | \) and for all \( Y \subseteq A \), we have \( Sg^\beta Y = Nr_\alpha Sg^\beta Y \). All this can be found in [16], see the proof of theorem 1.6.1 therein; in such a proof, \( B \) is called a minimal dilation of \( A \), due to the fact that \( B \) is unique up to isomorphisms that fix \( A \) pointwise.

We show that, like \( A \), \( F = F(m_\alpha, A) \), hence the boolean reduct of \( B \), is atomic. Let \( a \) be a non-zero element in \( F \). Then \( a : m_\alpha \rightarrow A \), such that \( a(t) \neq 0 \) for some \( t \in m_\alpha \). Choose an atom \( b(t) \) in \( A \) below \( a(t) \) and define \( b : m_\alpha \rightarrow A \) via \( t \mapsto b(t) \), and otherwise \( b(t) = 0 \). Then clearly \( b \) is an atom in \( F \) below \( a \), and so \( B \) is atomic. (Note that the fact that \( A \) is the full neat reduct of \( B \) and that \( A \) generates \( B \) is not enough to show that atomicity of \( A \), implies that of \( B \) (see example [1] below). But in the context of polyadic algebras we are lucky; the dilations of atomic algebras are constructed in such a way that they are atomic, as well.)

Let \( \Gamma \subseteq \alpha \) and \( p \in A \). Then in \( B \) we have, see [16] the proof of theorem 1.6.1,

\[
c(\Gamma) p = \sum \{ s_\tau p : \tau \in \alpha, \, \tau \, | \, \alpha = \Gamma = Id \}. \tag{1}
\]

Here, and elsewhere throughout the paper, for a transformation \( \tau \) with domain \( \alpha \) and range included in \( n \), \( \bar{\tau} = \tau \cup Id_{n \sim \alpha} \). Let \( X \) be the set of atoms of \( A \). Since \( A \) is atomic, then \( \sum X = 1 \). By \( A = Nr_\alpha B \), we also have \( \sum X = 1 \). By complete additivity we have for all \( \tau \in \alpha \), we have,

\[
\sum X = 1. \tag{2}
\]

Let \( S \) be the Stone space of \( B \), whose underlying set consists of all boolean ultrafilters of \( B \). Let \( X^* \) be the set of principal ultrafilters of \( B \) (those generated by the atoms). These are isolated points in the Stone topology, and they form a dense set in the Stone topology since \( B \) is atomic. So we have \( X^* \cap T = \emptyset \) for every nowhere dense set \( T \) (since principal ultrafilters, which are isolated points in the Stone topology, lie outside nowhere dense sets). For \( a \in B \), let \( N_a \) denote the set of all boolean ultrafilters containing \( a \). Now for all \( \Gamma \subseteq \alpha \), \( p \in B \) and \( \tau \in \alpha \), we have, by the suprema, evaluated in (1) and
(2):
\[ G_{\Gamma,p} = N_{\xi(\Gamma)}p \sim \bigcup_{\tau \in ^{\alpha}\mathfrak{n}} N_{s_{\tau}p} \]  
(3)
and
\[ G_{X,\tau} = S \sim \bigcup_{x \in X} N_{s_{\tau}x}. \]  
(4)

are nowhere dense. Let \( F \) be a principal ultrafilter of \( S \) containing \( c \). This is possible since \( \mathfrak{B} \) is atomic, so there is an atom \( x \) below \( c \); just take the ultrafilter generated by \( x \). Then \( F \in X^{*} \), so \( F \notin G_{\Gamma,p}, F \notin G_{X,\tau} \), for every \( \Gamma \subseteq \alpha, p \in A \) and \( \tau \in ^{\alpha}\mathfrak{n} \). Now define for \( a \in A \)
\[
f(a) = \{ \tau \in ^{\alpha}\mathfrak{n} : s_{\tau}^{a}a \in F \}.\]

Then \( f \) is a homomorphism from \( \mathfrak{A} \) to the full set algebra with unit \( ^{\alpha}\mathfrak{n} \), such that \( f(c) \neq 0 \). We have \( f(c) \neq 0 \) because \( c \in F \), so \( Id \in f(c) \). The rest can be proved exactly as in [8]; the preservation of the boolean operations and substitutions is fairly straightforward. Preservation of cylindrifications is guaranteed by the condition that \( F \notin G_{\Gamma,p} \) for all \( \Gamma \subseteq \alpha \) and all \( p \in A \). (Basically an elimination of cylindrifications, this condition is also used in [10] to prove the main representation result for polyadic algebras.) Moreover \( f \) is an atomic representation since \( F \notin G_{X,\tau} \) for every \( \tau \in ^{\alpha}\mathfrak{n} \), which means that for every \( \tau \in ^{\alpha}\mathfrak{n} \), there exists \( x \in X \), such that \( s_{\tau}^{a}x \in F \), and so \( \bigcup_{x \in X} f(x) = ^{\alpha}\mathfrak{n} \).

We conclude that \( f \) is a complete representation by Lemma 1.2.

Let CPA\( _{\alpha} \) denote the class of polyadic algebras of dimension \( \alpha \). Contrary to cylindric algebras, we have:

**Corollary 1.7.** The class CPA\( _{\alpha} \) is elementary, and it is axiomatizable by a finite schema in first order logic.

**Proof.** Atomicity can be expressed by a first order sentence, and complete additivity can be captured by the following continuum many formulas, that form a single schema. Let At\( (x) \) be the first order formula expressing that \( x \) is an atom. That is At\( (x) \) is the formula \( x \neq 0 \land (\forall y)(y \leq x \rightarrow y = 0 \lor y = x) \).

For \( \tau \in ^{\alpha}\mathfrak{n} \), let \( \psi_{\tau} \) be the formula:
\[
y \neq 0 \rightarrow \exists x(At(x) \land s_{\tau}x \neq 0 \land s_{\tau}x \leq y).\]

Let \( \Sigma \) be the set of first order formulas obtained by adding all formulas \( \psi_{\tau} \) (\( \tau \in ^{\alpha}\mathfrak{n} \)) to the polyadic schema. We show that CPA\( _{\alpha} = \text{Mod}(\Sigma) \). Let \( \mathfrak{A} \in \text{CPA}_{\alpha} \). Then, by theorem [1][2] we have \( \sum_{x \in X} s_{\tau}x = 1 \) for all \( \tau \in ^{\alpha}\mathfrak{n} \). Let \( \tau \in ^{\alpha}\mathfrak{n} \). Let \( a \) be non-zero, then \( a \cdot \sum_{x \in X} s_{\tau}x = a \neq 0 \), hence there exists...
$x \in X$, such that $a \cdot \text{s}_x x \neq 0$, and so $\mathfrak{A} \models \psi$. Conversely, let $\mathfrak{A} \models \Sigma$. Then for all $\tau \in \alpha$, $\sum_{x \in X} \text{s}_x x = 1$. Indeed, assume that for some $\tau$, $\sum_{x \in X} \text{s}_x x \neq 1$. Let $a = 1 - \sum_{x \in X} \text{s}_x x$. Then $a \neq 0$. But then there exists $x \in X$, such that $\text{s}_x x \cdot \sum_{x \in X} \text{s}_x x = 0$ which is impossible.

We do not need all the $\psi_{\tau}$ for if $\tau$ is a bijection then $\text{s}_x$ is self-conjugate, hence completely additive.

The question that imposes itself now, is whether there are atomic polyadic algebras that are not completely additive. We do not know. But here we give an example taken from [3], and slightly modified, to show that if we restrict substitutions to only injective ones, together with all replacements and ones whose kernels induce finite equivalence classes, then the resulting variety is not completely additive.

However, the algebra that witnesses the non-complete additivity of the substitution operator corresponding to one replacement is not atomic. Notice that here the resulting semigroup, generated by the available substitutions, has the same cardinality as $|\alpha|$, so that we are discarding quite a few substitutions, and this is a significant change.

But we believe that the non complete additivity of replacements (even in this restricted context), does give an indication that there are (full) polyadic algebras that are not completely additive; but nevertheless atomicity could well be a prohibiting factor to non-complete additivity of substitutions corresponding to non-bijective maps.

However, we also show here that atomic polyadic algebras of dimension 2 may not be completely representable (The class of completely representable algebras, in this case, is also elementary. This can be proved exactly as 1.2 above. For higher finite dimensions the class of completely representable algebras is not even elementary [9]. This is a non-trivial result and the proof in [9] uses a rainbow construction.)

Example 1.8. (1) Let $\mathfrak{B}$ be an atomless Boolean algebra that has a Stone representation with unit $U$ such that for any distinct $u, v \in U$, there is $X \in B$ such that $u \in X$ and $v \in X$. Let $\alpha$ be an infinite ordinal. Let $R = \{\prod X_i : i \in \alpha, X_i \in \mathfrak{B}\} \subseteq \alpha U$ and $A = \{\bigcup S : S \subseteq R : |S| < \omega\}$. Then $A$ is the base of a subalgebra of $\wp(\alpha U)$, call this algebra $\mathfrak{A}$. This can be proved exactly as in [3] by noting that substitutions corresponding to injections just permute components, and that for $\Gamma \subseteq \alpha$, $C(\Gamma)R$ is the element in $A$ that agrees with $R$ off of $\Gamma$, that is it is the same as $R$ in all components, except for $i \in \Gamma$; here the $i$th component is blown up to $U$.

Let $S = \{X \times \sim X \times U \times U \times \ldots : X \in B\}$. Then, like in [3], we have $S^0_0(\sum S) = \alpha U$ and $\sum\{S^0_0(Z) : Z \in S\} = \emptyset$.

If $\tau \in \alpha$ is not bijective, then we face a problem; the algebra $\mathfrak{A}$ defined
We show that atomic polyadic algebras of dimension 2 may not be complete representation entails that the algebra is atomic which is not the case. So though arbitrary meets exist in the algebra (by completeness) they are not necessarily reflected by infinite intersections in the Stone representation. In other words, given a partition of their equivalence classes, each such class renders only a finite intersection. 

And even if we take a complete atomless Boolean algebra, which exists, then we know that this algebra cannot be completely representable, for a complete representation entails that the algebra is atomic which is not the case. So though arbitrary meets exist in the algebra (by completeness) they are not necessarily reflected by infinite intersections in the Stone representation. In other words, given \( X_i : i \in I, I \) an infinite set, and \( X_i \in \mathcal{B} \), there is nothing to guarantee that \( \bigcap_{i \in I} X_i \) is in \( \mathcal{B} \). We find that this example is a near miss, and we conjecture that it can be appropriately modified to give a polyadic algebra of infinite dimension that is not completely additive. But this reasoning also tells us that we can count in those not necessarily injective maps whose kernels give finite equivalence classes, each such class renders only a finite intersection.

(2) We show that atomic polyadic algebras of dimension 2 may not be completely representable. For every infinite cardinal \( \kappa \) we construct such an algebra with cardinality \( \kappa \). By our theorem \( \mathbb{1.2} \) it suffices to show that one of the operations is not completely additive. The example is also from \( \mathbb{3} \). We give the outline. Let \( |U| = \mu \) be an infinite set and \( |I| = \kappa \) be a cardinal such that \( Q_n, n \in \kappa \), is a family of relations that form a partition of \( U \times U \). Let \( i \in I \), and let \( J = I \sim \{i\} \). Then of course \( |I| = |J| \). Assume that \( Q_i = D_{01} = \{s \in V : s_0 = s_1\} \), and that each \( Q_n \) is symmetric; that is for any \( S_{[0,1]} Q_n = Q_n \) and furthermore, that \( \text{dom} Q_n = \text{range} Q_n = U \) for every \( n \in \kappa \). It is straightforward to show that such partitions exist.

Now fix \( F \) a non-principal ultrafilter on \( J \), that is \( F \subseteq \mathcal{P}(J) \). For each \( X \subseteq J \), define

\[
R_X = \begin{cases} 
\bigcup \{Q_k : k \in X\} & \text{if } X \notin F, \\
\bigcup \{Q_k : k \in X \cup \{i\}\} & \text{if } X \in F 
\end{cases}
\]

Let

\[
\mathfrak{A} = \{R_X : X \subseteq I \sim \{i\}\}.
\]

Notice that \( |\mathfrak{A}| \geq \kappa \). Also \( \mathfrak{A} \) is an atomic set algebra with unit \( R_J \), and its atoms are \( R_{\{k\}} = Q_k \) for \( k \in J \). (Since \( F \) is non-principal, so \( \{k\} \notin F \).
for every $k$). This can be proved exactly like in [3]. The subalgebra $\mathcal{B}$ generated by the atoms is as required. We should also mention that this example shows that Pinters algebra, which are cylindrifier free algebras of polyadic algebras for all dimensions may not be completely representable answering an implicit question of Hodkinson’s [20] top of p. 260. (In the absence of cylindrifiers the construction lifts to arbitrary dimensions, because we do not require that $domQ_n = rangeQ_n = U$.) For infinite dimensions weak set algebras can be used. A weak set algebra is one whose unit is the set of sequences agreeing co-finitely with a given one.

**Example 1.9.** For cylindric algebras minimal dilations of atomic algebras may not be atomic. This is quite easy to show. Let $\mathfrak{A} \in \mathbb{RCA}_n$ such that $\mathfrak{A}$ is atomic. Let $\mathfrak{B} \in \mathbb{CA}_\omega$ such that $\mathfrak{A} = \mathbb{Mr}_n\mathfrak{B}$. Obviously such algebras exist. Let $\mathfrak{B}' = \mathbb{Eq}^\mathfrak{A}\mathfrak{A}$, then $\mathfrak{B}'$ is locally finite, and $\mathfrak{A} = \mathbb{Mr}_n\mathfrak{B}'$. Locally finite algebras are not atomic. Another even easier example is that if one takes a simple locally finite non-atomic algebra $\mathfrak{A}$, then $\mathbb{Nr}_0\mathfrak{A} = \{0, 1\}$. In fact, for cylindric algebras minimal dilations may not be unique, so that unlike polyadic algebras, we cannot speak about the minimal dilation of an even representable algebra. This property, is strongly linked to the amalgamation property for the class of representable cylindric algebras, or rather, the lack of [14].

**Question 1.10.** Are there atomic polyadic algebras that are not completely additive, hence not completely representable?

Dedekind or MacNeille completions for $\mathbb{PA}$, which we refer to as minimal completions, are also problematic, and they have to be re-defined to adapt the possibility of non-complete additivity.

A definition of completions of not necessarily completely additive varieties of Boolean algebras with operators is given in [3], but to our mind it is not satisfactory, for it gives, for example, that the completion of the atomic algebra $\mathfrak{A}$ in the second item of the previous example, it itself, for this algebra is complete. And so the completon is not completely additive, the completion of $\mathfrak{A}$ is not $\text{CmAt}\mathfrak{A}$.

However, if $\mathfrak{A} \in \mathbb{PA}_\alpha$ happens to be completely additive, then it has a minimal completion, because the equations axiomatizing $\mathbb{PA}_\alpha$ are Sahlqvist, and these are preserved under taking minimal completions of a completely additive algebra... We do not know whether $\mathbb{PA}_\alpha$ is atom-canonical, nor even single-persistent. That is, if $\mathfrak{A}$ is atomic and not completely additive, is the complex algebra of its atom structure a polyadic algebra, in symbols, $\text{CmAt}\mathfrak{A} \in \mathbb{PA}_\alpha$? Is the algebra generated by the singletions of $\mathfrak{A}\mathfrak{A}$ a $\mathbb{PA}_\alpha$?, not that an affirmative answer to the first question implies an affirmative answer to the second, but the converse is not true.
Note that in this case, since $\text{CmAt}\mathfrak{A}$ is completely additive (complex algebras are completely additive) and $\mathfrak{A}$ is not, so that $\mathfrak{A}$ does not embed into $\text{CmAt}\mathfrak{A}$ via $a \mapsto \{x \in \text{At}\mathfrak{A} : x \leq a\}$. If $\mathfrak{A}$ is completely additive and atomic, then the complex algebra of its atom structure, namely, $\text{CmAt}\mathfrak{A}$ is just its ordinary completion, with the above embedding.

**Question 1.11.** Is $\text{PA}_\alpha$ atom canonical?

On the other hand, $\text{PA}_\alpha$ is a canonical variety because again it is axiomatized by Sahlqvist equations (in fact, positive ones). Furthermore, if $\mathfrak{A} \in \text{PA}_\alpha$, then its canonical extension is $\mathfrak{N}_\alpha\mathfrak{B}^+$ where $\mathfrak{B}^+$ is the complex algebra of the minimal dilation $\mathfrak{B}$ of $\mathfrak{A}$ in $\geq \omega$ many dimensions ($\mathfrak{B}^+$ is unique, it does not depend on the number of extra dimensions).

### 2 A metalogical interpretation in Keisler’s logic

Polyadic algebras of infinite dimension correspond to a certain infinitary logic studied by Keisler, and referred to in the literature as Keisler’s logic. Keisler’s logic allows formulas of infinite length and quantification on infinitely many variables, but is does not allow infinite conjunctions, with semantics defined as expected. While Keisler [22], and independently Monk and Daigneault [16], proved a completeness theorem for such logics, our result implies a ‘Vaught’s theorem’ for such logics, namely, that certain atomic theories, namely those whose Tarski Lindenbaum algebra is completely additive, have atomic models, in a sense to be made precise.

Let $\mathcal{L}$ denote Keisler’s logic with $\alpha$ many variables ($\alpha$ an infinite ordinal). For a structure $\mathfrak{M}$, a formula $\phi$, and an assignment $s \in ^\alpha\mathfrak{M}$, we write $\mathfrak{M} \models \phi[s]$ if $s$ satisfies $\phi$ in $\mathfrak{M}$. We write $\phi^\mathfrak{M}$ for the set of all assignments satisfying $\phi$.

**Definition 2.1.** Let $T$ be a given $\mathcal{L}$ theory.

1. A formula $\phi$ is said to be complete in $T$ iff for every formula $\psi$ exactly one of

   \[ T \models \phi \rightarrow \psi, T \models \phi \rightarrow \neg\psi \]

   holds.

2. A formula $\theta$ is completable in $T$ iff there there is a complete formula $\phi$ with $T \models \phi \rightarrow \theta$.

3. $T$ is atomic iff if every formula consistent with $T$ is completable in $T$.

4. A model $\mathfrak{M}$ of $T$ is atomic if for every $s \in ^\alpha\mathfrak{M}$, there is a complete formula $\phi$ such that $\mathfrak{M} \models \phi[s]$.
We denote the set of formulas in a given language by $\mathfrak{Fm}$ and for a set of formula $\Sigma$ we write $\mathfrak{Fm}_\Sigma$ for the Tarski-Lindenbaum quotient (polyadic) algebra.

**Theorem 2.2.** Let $T$ be an atomic theory in $\mathcal{L}$ and assume that $\phi$ is consistent with $T$. Assume further that $\mathfrak{Fm}_T$ is completely additive. Then $T$ has an atomic model in which $\phi$ is satisfiable.

**Proof.** Assume that $T$ and $\phi$ are given. Form the Lindenbaum Tarski algebra $\mathfrak{A} = \mathfrak{Fm}_T$ and let $a = \phi/T$. Then $\mathfrak{A}$ is an atomic completely additive polyadic algebra, since $T$ is atomic, and $a$ is non-zero, because $\phi$ is consistent with $T$. Let $\mathfrak{B}$ be a set algebra with base $M$, and $f: \mathfrak{A} \rightarrow \mathfrak{B}$ be a complete representation such that $f(a) \neq 0$. We extract a model $\mathfrak{M}$ of $T$, with base $M$, from $\mathfrak{B}$ as follows. For a relation symbol $R$ and $s \in \alpha M$, $s$ satisfies $R$ if $s \in h(R(x_0, x_1, \ldots))/T$. Here the variables occur in their natural order. Then one can prove by a straightforward induction that $\phi^\mathfrak{M} = h(\phi/T)$. Clearly $\phi$ is satisfiable in $\mathfrak{M}$. Moreover, since the representation is complete it readily follows that $\bigcup \{ \phi^\mathfrak{M} : \phi \text{ is complete} \} = \alpha M$, and we are done.

If we add infinite conjunctions to our logic and stipulate that for any theory $T \vdash s, \bigwedge \phi_T \equiv \bigwedge s_\tau \phi_T$, then we get a proper extension of Keisler’s logic such that Tarski Lindenbaum algebras are completely additive, and so in this case atomic theories with no extra conditions will have atomic models. One way to do this is to stipulate the axiom $\vdash s, \bigwedge \phi \equiv \bigwedge s_\tau \phi$. Then for such expansion of Keisler’s logic with infinite conjunction, every atomic theory with no further conditions has an atomic model.

Results in algebraic logic are more interesting when they have immediate impact on the logic side be it first order logic or extensions thereof. For ordinary first order logic atomic theories in countable languages have atomic models, as indeed Vaught proved, but in the first order context countability is essentially needed. Furthermore, in the context of first order logic, atomic countable models for atomic theories are unique (up to isomorphism).

Our theorem can also be regarded as an omitting types theorem, for possibly uncountable languages, for the representation constructed in our theorem omits the set (or infinitary type) of co-atoms in the sense that the representation $f$, defined in our main theorem, satisfies $\bigcap_{x \in X^{-}} f(x) = \emptyset$.

A standard omitting types theorem for Keisler’s logic, addressing the omission of a family of types, not just one, is highly problematic since, even in the countable case, i.e. when the base of the algebra considered is countable, since we have uncountably many operations. Nevertheless, a natural omitting types theorem can be formulated as follows.

Let $\mathcal{L}$ denote Keisler’s logic, and let $T$ be an $\mathcal{L}$ theory. A set $\Gamma \subseteq \mathfrak{Fm}$ is principal, if there exist a formula $\phi$ consistent with $T$, such that $T \vdash \phi \rightarrow \psi$ for all $\psi \in \Gamma$. Otherwise $\Gamma$ is non-principal. A model $\mathfrak{M}$ of $T$ omits $\Gamma$, if
\[ \bigcap_{\phi \in \Gamma} \phi^\mathcal{L} = \emptyset, \] where \( \phi^\mathcal{L} \) is the set of assignments satisfying \( \phi \) in \( \mathcal{M} \). Then the omitting types theorem in this context says: If \( \Gamma \) is non-principal, then there is a model \( \mathcal{M} \) of \( T \) that omits \( \Gamma \). Algebraically:

\textbf{OTT} . Let \( \mathfrak{A} \in \mathcal{PA}_\alpha \) be completely additive and \( a \in A \) be non-zero. Assume that \( X \subseteq A \), is such that \( \prod X = 0 \). Then there exists a set algebra \( \mathfrak{B} \) and a homomorphism \( f : \mathfrak{A} \to \mathfrak{B} \) such that \( f(a) \neq 0 \) and \( \bigcap_{x \in X} f(x) = \emptyset \).

Unlike omitting types theorems for countable languages, the proof cannot resort to the Baire Category theorem, for the simple reason that the Baire category theorem applies only to the countable case. Nevertheless, our proof of theorem 1.6 shows how to omit the non principal type consisting solely of co-atoms, basically because the set principal ultrafilters is dense in the Stone topology, and a principal ultrafilter lie outside nowhere dense sets. It is not at all clear how to omit arbitrary non-principal types, when the algebra in question is not atomic. If we take only the set of infinitary joins corresponding to quantifier elimination, then an ultrafilter preserving them can be found giving an ordinary representation \[16\], but there is no topology involved here, at least in the proof of Diagneault and Monk; the ultrafilter is built in a step by step fashion. In case our algebra is completely additive, then we would have a second infinitary meet that we want to omit, and the corresponding set (which is now an infinite intersection) in the Stone space is also nowhere dense.

But in this case, when this meet is not the set of co-atoms, example when the algebra is atomless, we do not guarantee that such a set consisting of the ultrafilters (models) not omitting the non principal type do not exhaust the set consisting of those ultrafilters preserving only cylindrifier elimination, these are the ultrafilters from which we obtain representations. (This cannot happen in the countable case where the Baire Category theorems entails that the complement of such a set or even a complement of \( < \text{cov}K \) union of such sets is dense. Here \( \text{cov}K \) is the least cardinal such that the Baire Category theorem for compact Hausdorff spaces fail and also the largest cardinal for which Martin’s axiom on countable Boolean algebra holds, and it is the best estimate for number of non isolated types omitted.)

But in this form of generality the omitting types theorem fails as the next easy example show:

\textbf{Example 2.3.} Let \( T \) be a an uncountable complete first order theory \textit{without} equality, in an uncountable languages having a sequence of variables of order type \( \omega \). Assume that there exists a non principal type \( \Gamma \) of this theory that cannot be omitted. Easy examples are known. Then take the \textit{locally finite} polyadic Tarski Lindenbaum algebra \( \mathfrak{A} \in \mathcal{PA}_\omega \) based on this theory, and let \( X = \{ \phi_T : \phi \in \Gamma \} \). Then \( \prod X = 0 \) and there is no (locally finite) polyadic set algebra omitting this meet.
It is worthy of note that locally finite algebras, except in trivial cases, are atomless.

Nevertheless, there is yet another interesting connection between complete representations and omitting types. More can be said here. A classical theorem of Vaught for first order logic says that countable atomic theories have countable atomic models, such models are necessarily prime, and a prime model omits all non principal types. We have an analogous situation here, and the proof is very simple, assuming simplicity (in the universal algebraic sense) of our algebra, that is, assuming that the corresponding theory in Keisler’s logic is complete. The general case is not much harder, we just work with disjoint unions of square units.

**Theorem 2.4.** Let \( f : \mathfrak{A} \to \wp(\alpha U) \) be a complete representation of \( \mathfrak{A} \in \text{PA}_\alpha \). Then for any set \( I \), for any given family \((Y_i : i \in I)\) of subsets of \( \mathfrak{A} \), if \( \prod Y_i = 0 \) for all \( i \in I \), then we have \( \bigcap_{y \in Y_i} f(y) = \emptyset \) for all \( i \in I \).

**Proof.** Let \( i \in I \). Let \( Z_i = \{-y : y \in Y_i\} \). Then \( \sum Z_i = 1 \). \( \mathfrak{A} \) is completely representable, hence \( \mathfrak{A} \) is atomic, and so for any atom \( x \), we have \( x.\sum Z_i = x \neq 0 \). Hence there exists \( z \in Z_i \), such that \( x.z \neq 0 \). But \( x \) is an atom, hence \( x.z = x \) and so \( x \leq z \). We have shown that for every atom \( x \), there exists \( z \in Z_i \) such that \( x \leq z \). It follows immediately, since a complete representation is an atomic one, that \( \alpha U = \bigcup_{x \in \text{At} \mathfrak{A}} f(x) \leq \bigcup_{z \in Z_i} f(z) \), and so, \( \bigcap_{y \in Y_i} f(y) = \emptyset \), and we are done. \( \square \)

Every principal ultrafilter in a completely additive polyadic algebra gives rise to a complete representation (an atomic model). Are these representations, like first order logic, isomorphic, that is, are the generalized set algebras (obtained by taking the disjoint union of the bases over non-zero elements of the algebra) base isomorphic?

This is not case because the base of our algebras can be any of any cardinality \( \geq n \) with \( n \) as in the proof of theorem [1.6] so that set algebras constructed cannot be base isomorphic. It is true that the dilation is unique in a fixed dimension, but if we take a larger dimension, we get another also unique dilation but in this larger dimension. Evidently the two dilations cannot be isomorphic for the very simple reason that they have different similarity types. This is a significant deviation from first order logic.

Let us formulate this last paragraph in a theorem. Let \( \mathfrak{A} \in \text{PA}_\alpha \) be simple, infinite and hereditary atomic (every subalgebra is atomic). Assume that \( |\mathfrak{A}| = |\alpha| \). Then the number of principal ultrafilters \( \leq |\alpha| \), but if it not hereditary atomic then this number is \( \leq |\alpha|^2 \). However, regardless of the number of ultrafilters in the Stone space, we have:

**Theorem 2.5.** Let \( T \) be a complete atomic theory in Keisler’s logic such that \( \mathfrak{A} = \varphi(T) \) is completely additive. Let \( m \) be the local degree of \( \mathfrak{A} \), \( c \) its effective
cardinality and \( n \) be any cardinal such that \( n \geq c \) and \( \sum_{s<m} n^s = n \). Then \( T \) has an atomic model of size \( n \).

**Proof.** See the proof of theorem 1.6. Here the representing function is injective, due to simplicity of \( \mathfrak{S}m_T \) inducing an isomorphism.

So here each principal ultrafilter gives rise to infinitely many atomic representations. and we are infront of an Upward Skolem Theorem addressing atomic models, for every atomic model there is one with larger cardinality. A natural question here is that if \( n < m \), is there perhaps a subbase isomorphism between the representation corresponding to adding \( n \) witnesses, into that corresponding to adding \( m \) witnesses, which is an algebraic reflection of an elementary embedding?

We should mention that an analogous result holds for several reducts of polyadic algebras (without equality), namely, complete additivity and atomicity is equivalent to complete representability. For example if we take the reduct by restricting cylindrifiers on only those subsets \( \Gamma \) of \( \alpha \) such that \( |\Gamma| < \kappa \leq |\alpha| \), where \( \kappa \) is an infinite cardinal, then we get the same result.

The point is, as long as we have all substitutions, then we have rich transformation systems, hence functional dilations, from which we can get dilations of the algebra in question by discarding cylindrifiers, forming the functional dilation using substitutions, then re-defining cylindrifiers (to agree with the old ones in the neat reduct) as we did in our proof of 1.6. Furthermore, in all cases if the original algebra is atomic, then so will be the dilation (because it is basically a product of the atomic boolean algebra), hence the proof survives verbatim.

### 2.1 Modified Keisler’s logic with equality

The case of polyadic algebras of infinite dimension with equality is much more involved. In this case the class of representable algebras is not a variety; it is not closed under ultraproducts, although every algebra has the neat embedding property (can be embedded into the neat reduct of algebras in every higher dimension). In particular, we do not guarantee that atomic algebras are even representable, let alone admit a complete representation. Still we can ask whether atomic representable algebras are completey representable. The question seems to be a hard one, because we cannot resort to a neat embedding theorem as we did here for the equality free case.

In fact, finding neat embedding theorems for polyadic equality algebras, that enforce even relativized representations is a very active topic, that is gaining a lot of momentum [2]. One can find well motivated appropriate notions of relativized semantics by first locating them while giving up classical semantical prejudices. It is hard to give a precise mathematical underpinning

to such intuitions. What really counts at the end of the day is a completeness theorem stating a natural fit between chosen intuitive concrete-enough, but perhaps not excessively concrete, semantics and well behaved axiomatizations. The move of altering semantics has radical philosophical repercussions, taking us away from the conventional Tarskian semantics captured by Fregean-Godel-like axiomatization; the latter completeness proof is effective but highly undecidable; in modal logic and finite variable fragments of first order logic, which have a modal formalism, this is highly undesirable.

Now we show that such algebras, when atomic, admit complete relativized representations.

Ferenczi has a lot of work in this direction [17]. In the latter article, he defines an abstract equational class $\text{CPEA}_\alpha$, $\alpha$ an infinite ordinal, which is like cylindric algebras in that it has only finite cylindrifiers and like polyadic algebras, in that its cylindrifier free reduct, forms a transformation system; all substitutions are available. Also, and this is the most important part this class has diagonal elements. The presence of diagonal elements makes the variety completely additive, which is an acet, in our context seeking complete representations. However, full fledged commutativity of cylindrifiers do not hold here.

Ferenczi proves a strong completeness theorem in analogy the Diagneault Monk representation theorem for polyadic algebras, namely, $\text{CPEA}_\alpha = \text{Gp}_\alpha$, where $\text{Gp}_\alpha$ is a class of set algebras whose units are relativized. (It is a union of weak set algebras that are not necessarily disjoint). The technique is a Henkin construction implemented via a neat embedding theorem. This is definitely an achievement, because for classical polyadic equality algebras, when cylindrifiers commute, neat embeddability into infinitely many extra dimensions does not enforce representability.

But the choice of representing ultrafilters (which Ferenczi calls perfect) here is more delicate, and the representation is somewhat more intricate than the classical case. This is due to the fact that cylindrifiers and substitutions do not commute, in cases where it is consistent that they do as witnessed by classical representations. Sacrificing commutativity of cylindrifiers and for that matter commutativity of cylindrifiers and substitutions make relativized representability possible. Also the representant class of relativized algebras turns out to be a variety; this is not the case with classical representations for polyadic equality algebras having square Tarskian semantics, even if we restrict the similarity type to only finite cylindrifiers. In the latter case the presence of diagonal elements together with infinitary substitutions make this class not closed under ultraproducts.

But such modifications will survive our proof. And indeed we can show also using a neat embedding theorem together with our previous topological argument, baring in mind that we can omit the condition of complete additivity
since it holds anyway, we have: (However, we give the general idea and some of the details will be omitted, but can be recovered easily from [17].)

**Theorem 2.6.** Every atomic CPEA\(\alpha\) is completely representable

**Proof.** Let \(c \in A\) be non-zero. We will find a \(B \in Gp_\alpha\) and a homomorphism from \(f : A \to B\) that preserves arbitrary suprema whenever they exist and also satisfies that \(f(c) \neq 0\). Now there exists \(B \in CPEA_n, n\) a regular cardinal, such that \(A \subseteq \Nr_\alpha B\) and \(A\) generates \(B\). Note that \(|n \sim \alpha| = |n|\) and for all \(Y \subseteq A\), we have \(\Sr g^A Y = \Nr_\alpha \Sr g^B Y\). This dilation also has Boolean reduct isomorphic to \(F(\alpha, A)\), in particular, it is atomic because \(A\) is atomic. Also cylindrifiers are defined on this minimal functional dilation exactly like above by restricting to singletons. Let \(adm\) be the set of admissible substitutions. \(\tau \in B\) is admissible if \(Do\psi \subseteq \alpha\) and \(Rg\psi \cap \alpha = \emptyset\). Then we have for all \(i < n\) and \(\sigma \in adm\),

\[s_\sigma c_i p = \sum s_\sigma s_j^i p\]  

(5)

This uses that \(c_k = \sum s_j^k x\), which is proved like the cylindric case; the proof depends on diagonal elements. Let \(X\) be the set of atoms of \(A\). Since \(A\) is atomic, then \(\sum^B X = 1\). By \(A = \Nr_\alpha B\), we also have \(\sum^B X = 1\). Because substitutions are completely additive we have for all \(\tau \in \alpha\)

\[\sum s_\tau^B X = 1\]  

(6)

Let \(S\) be the Stone space of \(B\), whose underlying set consists of all boolean ultrafilters of \(B\), and let \(F\) be a principal ultrafilter chosen as before. Let \(B'\) be the minimal completion of \(B\). Exists by complety additivity. Take the filter \(G\) in \(B'\) generated by the generator of \(F\) and let \(\text{let } F = G \cap B\). Then \(F\) is a perfect ultrafilter. Because our algebras have diagonal algebras, we have to factor our base by a congruence relation that reflects equality. Define an equivalence relation on \(\Gamma = \{i \in \beta : \exists j \in \alpha : c_i d_{ij} \in F\}\), via \(m \sim n\) iff \(d_{mn} \in F\). Then \(\Gamma \subseteq \alpha\) and the desired representation is defined on a \(Gp_\alpha\) with base \(\Gamma/\sim\). We omit the details.

The metalogical interpretation of the above theorem is also interesting. It gives Vaught’s theorem for a variant of Keisler’s logic with equality, which we call modified Keisler’s logic. This variant is defined by taking only finite cylindrifiers, and weakening the axioms (concerning commutativity of the various non boolean operations). This calculus is complete, but with respect to relativized semantics. This is proved by Ferenczi. Here our theorem says that any atomic theory as defined above, for Keisler’s logic, adapted to the present context, has an atomic model. We do not need to add the algebraic condition of additivity of the formula algebra, it is completely additive. This is a more
elegant formulation; it does not refer to an algebraic property that has only a vague logical counterpart. In fact the modified Keisler’s logics share quite a few properties with first order logic, as we proceed to show next. Let $\mathcal{L}^=\alpha$ be the modified Keislers logic. Then

**Theorem 2.7.**

1. $\mathcal{L}^=\alpha$ is strongly complete with respect to relativized semantics. That is, for any set of formulas $\Gamma \cup \{\phi\}$, if $\Gamma \models \phi$, then $\Gamma \vdash \phi$.

2. $\mathcal{L}^=\alpha$ has the interpolation property, hence Beth definability

3. If $T$ be a complete atomic theory in the modified Keisler’s logic with equality. Let $m$ be a regular cardinal such that $|\alpha| < m$. Then $T$ has an atomic model of size $m$.

4. If $T$ is an atomic complete theory, then $T$ omits all non-principal types

**Proof.** The first part is due to Ferenczi. The third part follows from the fact that principal ultrafilters respecting the given set of infinitary joins (corresponding to substituted versions of co atoms and elimination of cylindrifiers) always exist in the Stone space of the atomic dilation in $m$ extra dimensions, when the latter is a regular cardinal $> |\alpha|$. The fourth part follows from the argument used above for Keisler’s logic.

For the second part we give only a sketch. The full proof will be submitted elsewhere. Let $\beta$ be a cardinal, and $\mathfrak{A} = \exists \tau_{\beta}\text{CPEA}_\alpha$ be the free algebra on $\beta$ generators. Let $X_1, X_2 \subseteq \beta$, $a \in \mathfrak{Sg}^\beta X_1$ and $c \in \mathfrak{Sg}^\beta X_2$ be such that $a \leq c$. We show that there exists $b \in \mathfrak{Sg}^\beta(X_1 \cap X_2)$ such that $a \leq b \leq c$. This is the algebraic version of the Craig interpolation property. Assume that $\mu$ is a regular cardinal $> \max(|\alpha|, |A|)$. Let $\mathfrak{B} \in \text{CPEA}_\mu$, such that $\mathfrak{A} = \text{Nr}_\alpha \mathfrak{B}$, and $A$ generates $\mathfrak{B}$. Such dilations exist. Ultrafilters in dilations used to represent algebras in CPEA are as before defined via the admitted substitutions, which we denote by $adm$. Recall that very admitted substitution has a domain $\text{Dor} \tau$ which is subsets of $\alpha$ and a range, $\text{Rg} \tau$ such that $\text{Rg} \tau \cap \alpha = \emptyset$. One defines special filters in the dilations $\mathfrak{Sg}^\beta X_1$ and in $\mathfrak{Sg}^\beta X_2$ like but they have to be compatible on the common subalgebra. This needs some work. Assume that no interpolant exists in $\mathfrak{Sg}^\beta(X_1 \cap X_2)$. Then no interpolant exists in $\mathfrak{Sg}^\beta(X_1 \cap X_2)$. We eventually arrive at a contradiction. Arrange $adm \times \mu \times \mathfrak{Sg}^\beta(X_1)$ and $adm \times \mu \times \mathfrak{Sg}^\beta(X_2)$ into $\mu$-termed sequences:

$$\langle (\tau_i, k_i, x_i) : i \in \mu \rangle$$

and

$$\langle (\sigma_i, l_i, y_i) : i \in \mu \rangle$$

is as desired. Thus we can define by recursion (or step-by-step) $\mu$-termed sequences of witnesses:

$$\langle u_i : i \in \mu \rangle$$

and

$$\langle v_i : i \in \mu \rangle$$
such that for all \( i \in \mu \) we have:

\[
u_i \in \mu \setminus (\Delta a \cup \Delta c) \cup \bigcup_{j \leq i}(\Delta x_j \cup \Delta y_j \cup Do\tau_j \cup Rg\tau_j \cup Do\sigma_j \cup Rg\sigma_j) \cup \{u_j : j < i\} \cup \{v_j : j < i\}\]

and

\[
v_i \in \mu \setminus (\Delta a \cup \Delta c) \cup \bigcup_{j \leq i}(\Delta x_j \cup \Delta y_j \cup Do\tau_j \cup Rg\tau_j \cup Do\sigma_j \cup Rg\sigma_j) \cup \{u_j : j \leq i\} \cup \{v_j : j < i\}\]

For an algebra \( D \) we write \( BlD \) to denote its boolean reduct. For \( i, j < \mu \), \( i \neq j \), \( s_i^j x = c_i(d_j \cdot x) \) and \( s_i^j x = x \). \( s_i^j \) is a unary operation that abstracts the operation of substituting the variable \( v_i \) for the variable \( v_j \) such that the substitution is free. For a boolean algebra \( C \) and \( Y \subseteq C \), we write \( fl^B \) to denote the boolean filter generated by \( Y \) in \( C \). Now let

\[
Y_1 = \{a\} \cup \{-s_{r_i}c_i, x_i + s_{r_i}s_{i}'c_i, x_i : i \in \mu\},
\]

\[
Y_2 = \{-c\} \cup \{-s_{r_i}c_i, y_i + s_{r_i}s_{i}'c_i, y_i : i \in \mu\},
\]

\[
H_1 = fl^{BlG}(x_1), \ H_2 = fl^{BlG}(x_2),
\]

and

\[
H = fl^{BlG}(X_1 \cap X_2) \left[(H_1 \cap GgB(X_1 \cap X_2) \cup (H_2 \cap GgB(X_1 \cap X_2)) \right].
\]

Then \( H \) is a proper filter of \( GgB(X_1 \cap X_2) \). This can be proved by a tedious induction, with the base provided by the fact that no interpolant exists in the dilation. Proving that \( H \) is a proper filter of \( GgB(X_1 \cap X_2) \), let \( H^* \) be a (proper boolean) ultrafilter of \( GgB(X_1 \cap X_2) \) containing \( H \). We obtain ultrafilters \( F_1 \) and \( F_2 \) of \( GgB(X_1) \) and \( GgB(X_2) \), respectively, such that

\[
H^* \subseteq F_1, \ H^* \subseteq F_2
\]

and (**)

\[
F_1 \cap GgB(X_1 \cap X_2) = H^* = F_2 \cap GgB(X_1 \cap X_2).
\]

Now for all \( x \in GgB(X_1 \cap X_2) \) we have

\[
x \in F_1 \text{ if and only if } x \in F_2.
\]

Also from how we defined our ultrafilters, \( F_i \) for \( i \in \{1, 2\} \) are perfect, a term introduced by Ferenczi. Then one defines homomorphisms, one on each subalgebra, like in \([14]\) p. 128-129, using the perfect ultrafilters to define a congruence relation on \( \beta \) so that the defined homomorphisms respect diagonal elements. Then freeness will enable paste these homomorphisms, to a single one defined to the set of free generators, which we can assume to be, without any loss, to be \( X_1 \cap X_2 \) and it will satisfy \( h(a. -c) \neq 0 \) which is a contradiction.
The notion of relativized representations constitute a huge topic in both algebraic and modal logic, see the introduction of [2], [17], [15]. Historically, in [19] square units got all the attention and relativization was treated as a side issue. However, extending original classes of models for logics to manipulate their properties is common. This is no mere tactical opportunism, general models just do the right thing.

The famous move from standard models to generalized models is Henkin’s turning round second order logic into an axiomatizable two sorted first order logic. Such moves are most attractive when they get an independent motivation.

The idea is that we want to find a semantics that gives just the right action while additional effects of square set theoretic representations are separated out as negotiable decisions of formulation that can threaten completeness, decidability, and interpolation. (This comes across very much in cylindric algebras, especially in finite variable fragments of first order logic, and classical polyadic equality algebras, in the context of Keisler’s logic with equality [23].)

Using relativized representations Ferenczi [17], proved that if we weaken commutativity of cylindrifiers and allow relativized representations, then we get a finitely axiomatizable variety of representable quasi-polyadic equality algebras (analogous to the Andréka-Resek Thompson CA version, cf. [14] and [17], for a discussion of the Andréka-Resek Thompson breakthrough for cylindric-like algebras); even more this can be done without the merry go round identities. This is in sharp contrast with the long list of complexity results proved for the commutative case [17]. Ferenczi’s results can be seen as establishing a hitherto fruitful contact between neat embedding theorems and relativized representations, with enriching repercussions and insights for both notions.

Now coming back to the classical case, where we have full fledged commutativity of cylindrifiers and classical Tarskian square semantics, if we restrict the signature of \( \text{PA}_\alpha \) to only these substitutions whose support has cardinality less than \( \kappa \), where \( \kappa \leq |\alpha| \), then we conjecture that in this new signature atomic completely additive algebras, in the classical sense, may not be completely representable, not only that, but we further conjecture that the class of completely representable algebras may turn out non-elementary. Here the support of a map \( \tau \) on \( \alpha \) is \( \{ i \in \alpha : i \neq \tau(i) \} \). As a matter of fact, if we have a single diagonal element then indeed this will be the case, as shown below using a cardinality argument of Hirsch and Hodkinson [21]. Call this class \( \mathbb{K}_\alpha \).

**Theorem 2.8.** The class of completely representable algebras in \( \mathbb{K}_\alpha \) is not elementary

Let \( \mathcal{C} \in \mathbb{K}_\alpha \) such that \( \mathcal{C} \models d_{01} < 1 \). Such algebras exist, for example one can take \( \mathcal{C} \) to be \( \wp(\alpha 2) \). Assume that \( f : \mathcal{C} \rightarrow \wp(\alpha X) \) is a complete representation.
Since \( C | d_{01} < 1 \), there is \( s \in h(-d_{01}) \) so that if \( x = s_0 \) and \( y = s_1 \), we have \( x \neq y \). For any \( S \subseteq \alpha \) such that \( 0 \in S \), set \( a_S \) to be the sequence with \( i \)th coordinate is \( x \), if \( i \in S \) and \( y \) if \( i \in \alpha \sim S \). By complete representability every \( a_S \) is in \( h(1) \) and so in \( h(\mu) \) for some unique atom \( \mu \).

Let \( S, S' \subseteq \alpha \) be distinct and assume that each contains 0. Then there exists \( i < \alpha \) such that \( i \in S \), and \( i \notin S' \). So \( a_S \in h(d_{01}) \) and \( a'_S \in h(-d_{01}) \). Therefore atoms corresponding to different \( a_S \)’s are distinct. Hence the number of atoms is equal to the number of subsets of \( \alpha \) that contain 0, so it is at least \( |\alpha|^2 \). Now using the downward Lowenheim Skolem Tarski theorem, take an elementary substructure \( B \) of \( C \) with \( |B| \leq |\alpha| \). This is possible since the scope of cylindrifiers and the support of substitutions are \( < \kappa \). Then in \( B \) we have \( B \models d_{01} < 1 \). But \( B \) has at most \( |\alpha| \) atoms, and so \( B \) cannot be completely representable.

This does not hold for Ferenczi’s relativized algebras, precisely because they are relativized. The above argument depended essentially on the cardinality of the square of \( \alpha^2 \).

### 2.2 Possible extensions

Finally, we mention that Ferenczi’s ideas can be transferred to the countable paradigm, by using countable transformation systems. In more detail given a countable ordinal \( \alpha \) one defines a strongly rich semigroup \( G \) as in [7], with \( G \subseteq ^{\alpha} \alpha \). The signature is now like polyadic equality algebras except that substitutions are restricted to \( G \), and cylindrifiers are finite. Such semigroups are adequate to define dilations. Postulating Ferenczi’s axioms for this new signature, we get all the results obtained in this paper, using relativized semantics.

An important addition in this new context is an omitting types theorem since now we can apply the Baire Category theorem. Indeed in such a context one can easily prove an exact analogue of the Orey-Henkin omitting types, omitting even \( < \text{cov}K \) many non principal types, where the latter, as mentioned earlier is the least cardinal such the Baire category theorem for compact second countable Hausdoff spaces fail, and it is also the largest for which Martin’s axiom for countable Boolean algebras holds, hence the best estimate for the number of non principal types that can be omitted in countable theories. We think that this could be a reasonable solution to the so called finitizability problem in algebraic logic, which has been open for ages for the equality case. The solution for logics without equality is provided by Sain.

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