Electron-phonon coupling close to a metal-insulator transition in one dimension

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Abstract

We consider a one-dimensional system of electrons interacting via a short-range repulsion and coupled to phonons close to the metal-insulator transition at half filling. We argue that the metal-insulator transition can be described as a standard one dimensional incommensurate to commensurate transition, even if the electronic system is coupled to the lattice distortion. By making use of known results for this transition, we prove that low-momentum phonons do not play any relevant role close to half-filling, unless their coupling to the electrons is large in comparison with the other energy scales present in the problem. In other words the effective strength of the low-momentum transferred electron-phonon coupling does not increase close to the metal-insulator transition, even though the effective velocity of the mobile carriers is strongly diminished.

I. INTRODUCTION

The behavior of one-dimensional (1D) electronic systems coupled to phonons is a well studied problem especially in the context of the physics of quasi-1D compounds (see e.g.
Apart from their immediate applications to these materials, the 1D models might be helpful to understand how phonons may affect higher dimensional strongly-correlated electronic systems like, for instance, systems close to a metal-insulator transition (MIT). A well known example would be a model close to an incommensurate to commensurate transition (e.g. the Hubbard model close to half filling). A main question in this case is how the vanishing velocity of the mobile carriers close to the MIT influences the electron-phonon coupling. If the renormalization of the electron-phonon coupling induced by the proximity to the MIT were negligible, then one would conclude that the closer is the MIT, i.e. the smaller is the quasi-particle velocity, the stronger is the effect of the electron-phonon coupling. This in turn might lead to anomalous but appealing scenarios where for instance superconductivity is favored by approaching the MIT. The other possibility is that the MIT induces a strong suppression of the phonon-mediated electron-electron interaction so that no additional features are introduced by the electron-phonon coupling if its strength is not particularly large. At least in one dimension it is possible to give a definite answer to the above question, which is what we are going to do in this paper.

Most of the theoretical works on electron-phonon coupling in 1D deal with large momentum phonons ($q \sim 2k_F$), since, due to the nesting properties of the 1D Fermi surface, these phonons get strongly coupled to the $2k_F$ charge density wave fluctuations. Recently Martin and Loss have considered the different case of low-momentum acoustic phonons coupled to electrons described by a 1D Hubbard Hamiltonian. By adding the electron-phonon coupling to the Luttinger-liquid model which describes at low energy the Hubbard model, they argue that, close to half filling, the system is always unstable to a whatever small phonon-induced attractive interaction. Near this instability, spin density wave (SDW) and charge density wave (CDW) fluctuations are strongly depressed and the system is pushed through a metallic phase (which was originally found in Ref. 5) towards a superconducting phase and finally into phase separation (the so-called Wentzel-Bardeen singularity). What they find is therefore a point in favor of the scenario where the phonon-mediated interaction is not strongly renormalized close to the MIT and this in turn implies that the electron-phonon coupling
constant is always strong if compared to the effective bandwidth of the mobile carriers. This situation could possibly occur even in higher dimensions where the bandwidth reduction due to correlations also appears.

This result is however puzzling firstly because a similar behavior is not shown by any physical system (e.g. doped polyacetylene) and secondly because the tendency towards superconductivity which they find close to half filling is difficult to reconcile with the experimental and theoretical evidence of a Peierls phase at half filling (although one may invoke for this purpose the region of phase separation). At first sight, a possible explanation of this questionable result could be the neglect of the $2k_F$-phonons, which are known to be essential to describe the behavior of these systems. However one easily realizes that this is not the solution to the puzzle, since the $2k_F$-phonons would not change the result of Ref. 3. In fact, let us consider for instance the extreme situation of strong electron-phonon coupling. In this case it is known that the lattice distortion (i.e. the $2k_F$-phonon field) moves with the $2k_F$-CDW fluctuations. The system is then described by the same type of Luttinger-liquid Hamiltonian as in the absence of electron-phonon coupling, although with renormalized parameters. On the other hand, charge modes described by a Luttinger-liquid model is also the starting point of Martin and Loss’s calculation, which therefore can be redone even in this limiting situation of electrons strongly coupled with $2k_F$-phonons. As a consequence their main conclusions remain valid in so far as the Hubbard model plus the $2k_F$-phonons has a metal-insulator transition at half filling. That their results do not depend on the neglect of large momentum phonons is even more transparent if the $2k_F$-phonon frequency is comparable to the electron bandwidth. Then the electron-electron interaction induced by the $2k_F$-phonons can be taken as unretarded and simply renormalizes the on-site repulsion (to higher value since it is effectively repulsive at large transferred momentum).

In this paper we re-analyze this problem in a more systematic way. We find, contrary to Martin and Loss, that the re-normalization of the low-momentum transferred phonon-mediated interaction is very important close to the MIT. The effect is to diminish the value of the coupling constant to such an extent that the ratio of the bare electron-phonon
coupling to the *bare* bandwidth still determines the effective strength of the electron-phonon interaction, irrespective of the filling. When applied to the Hubbard model, our result implies that, unless the coupling constant between the electrons and the low-momentum phonons is sufficiently strong in comparison with the *bare* hopping, nothing changes in the phase diagram and, as the system approaches half filling, a transition occurs from a metallic phase with dominant density-wave fluctuations to an insulating one.

II. THE WENTZEL-BARDEEN SINGULARITY

Let us consider a 1D system of interacting electrons close to the metal-insulator transition, e.g. the Hubbard model close to half filling, in the presence of electron-phonon interaction.

We start summarizing some known results on the low energy behavior of the Hubbard model. The main important feature is that there are no coherent single-particle excitations (even in the metallic phase away from half filling), but only collective charge and spin sound modes, which are dynamically independent. These collective fluctuations can be imagined as sound waves in an elastic string. In a different language, they both can be described in terms of the Luttinger liquid phenomenology.

A Luttinger liquid has the following Hamiltonian:

\[
\hat{H} = \frac{v_\lambda}{2} \int dx \left\{ K_\lambda \Pi_\lambda^2(x) + \frac{1}{K_\lambda} (\nabla \phi_\lambda(x))^2 \right\},
\]

where \( \Pi_\lambda \) is the momentum conjugate to the bosonic field \( \phi_\lambda \), and the latter are related to the charge (\( \lambda = \rho \)) or spin (\( \lambda = \sigma \)) densities by the relations

\[
\rho_\uparrow(x) + \rho_\downarrow(x) = \sqrt{\frac{2}{\pi}} \nabla \phi_\rho(x) + \frac{2}{\pi \alpha} \sin \left( 2k_F x + \sqrt{2\pi} \phi_\rho(x) \right) \cos \left( \sqrt{2\pi} \phi_\sigma(x) \right),
\]

\[
\rho_\uparrow(x) - \rho_\downarrow(x) = \sqrt{\frac{2}{\pi}} \nabla \phi_\sigma(x) + \frac{2}{\pi \alpha} \cos \left( 2k_F x + \sqrt{2\pi} \phi_\rho(x) \right) \sin \left( \sqrt{2\pi} \phi_\sigma(x) \right),
\]

being \( \rho_\uparrow(x) \) and \( \rho_\downarrow(x) \) the densities of spin-up and spin-down electrons respectively, and \( \alpha \) an ultraviolet cut-off of the order of the lattice spacing. \( \nu_\lambda \) are the velocities of the
charge and spin sound waves, while the positive parameters $K_\lambda$ are related to the exponents which characterize the asymptotic power law decay of all the correlation functions (for non-interacting fermions $K_\lambda = 1$, and for spin isotropic interactions $K_\sigma = 1$).

For the case of the Hubbard model, it is known that approaching the metal-insulator transition, i.e. for filling $\nu \to 1/2$, $v_\rho \to 0$ while $K_\rho \to 1/2$. As a consequence the charge compressibility $\chi \propto K_\rho/v_\rho$ diverges. On the contrary, the spin sector is almost unaffected by the proximity to half-filling. Exactly at $\nu = 1/2$, the charge sector acquires a gap and the correlation functions behaves as if the effective $K_\rho = 0$. In other words $K_\rho$ jumps discontinuously from 1/2 to 0.

Now let us add to the on-site repulsion $U$ the electron-phonon coupling $g(x - y)$ via the term

$$\int dx g(x - y)\rho(x)u(y), \quad (2)$$

where $u(y)$ is the phonon displacement field. By integrating out the phonons, (2) generates in the action $S$ a retarded electron-electron interaction via phonon exchange, given by:

$$\delta S = -\frac{1}{2} \int dx dt dx' dt' V(x - x', t - t')\rho(x, t)\rho(x', t'), \quad (3)$$

where $V(x, t)$ is the Fourier transform of:

$$V(q, \omega) = \frac{|g(q)|^2}{\zeta} \frac{1}{\omega^2 - \omega_q^2 + i\eta}, \quad (4)$$

being $\zeta$ the ionic mass density and $\omega_q$ the phonon frequency at momentum $q$. We can distinguish two different situations. In the so called molecular crystal model (MC), the phonons are optical and $g(q \to 0) \neq 0$. On the contrary, for acoustic phonons $\omega_q = cq$ and $g(q) = g\omega_q$. This is not the most general way of introducing phonons. Another possibility would be to couple phonons with electrons via a modulation of the hopping matrix element as in the Su, Schrieffer and Heeger model (SSH) for acoustic phonons. If we were interested in the low energy behavior, we could write also in this case an expression like (3) with $g(q) = ig \sin(q)$ in (4).
A possible approach to analyze the effects of (3) close to half filling is that one followed by Martin and Loss (3) which we shall now discuss in order to point out the basic limits and approximations. Following them, for the time being, we limit our analysis to $V(q, \omega)$ with $q \sim 0$, and disregard the $q \sim 2k_F$ component. This is in principle an unrealistic approximation since the $2k_F$-phonons are strongly coupled to the electrons. However, as we have said in the Introduction, even if $2k_F$-phonons are taken into account, the charge modes can still be described by a Hamiltonian like (1), with renormalized, in particular smaller, parameters $v_\rho$ and $K_\rho$. A more relevant point is that the phonon-induced $q \sim 0$ interaction (3) introduces in the model new energy scales which is related to the frequency dependence of (3), and can be taken as a characteristic phonons frequency. If we could assume that this energy scale is much smaller than the energy scale $\omega_L$ below which the Hubbard model (or better the Hubbard model plus the $2k_F$ lattice distortion) can be described in terms of a Luttinger liquid, then it would be justified to proceed along the lines of Ref. 3 (this energy scale inequality is the crucial approximation which underlies Martin and Loss’s approach).

We therefore first represent the Hubbard model as an effective Luttinger liquid as in Eq. (1), with the appropriate parameters $v_\rho$ and $K_\rho$. This step implies that we are interested in energies $\ll \omega_L$. Secondly, to this low energy model we add the electron-phonon coupling of (2) limited only to the low momentum component. This implies that

$$\rho_\uparrow + \rho_\downarrow \approx \sqrt{\frac{2}{\pi}} \nabla_\rho \phi_\rho(x)$$

in (2). If the previous assumption remains valid close to half-filling, then the effective model can be diagonalized exactly (see Ref. 3), and describes low momentum phonons coupled to charge sound waves with vanishing velocity $v_\rho$, which in turn means that $V$ is always in the strong coupling regime. A simple consequence of this is that the charge compressibility should diverge before half-filling: in fact $\chi^{-1} - v_\rho/K_\rho \propto \lim_{q \to 0} V(q, 0)$ and therefore for any weak attraction $V < 0$ there is a filling $\nu$ such that $v_\rho$ is so small that $\chi^{-1}$ vanishes. The divergent compressibility identifies the Wentzel-Bardeen singularity, as discussed by Martin and Loss, or, equivalently, the instability to phase separation. Notice that this conclusion
is more or less independent of the particularly chosen phonons (i.e. optical or acoustic) as well as of the electron-phonon coupling (MC or SSH).

We believe that the above approach to the problem is not correct. The simple reason is that $\omega_L$ is a function of the density and vanishes approaching half filling. In fact the opening of the charge gap at $\nu = 1/2$ is a consequence of the so-called Umklapp scattering. This process is singular only at half filling. Away from it, the singularity is cut-off at energies of order

$$\omega_{Um} \simeq v_F(1 - 2\nu)/a \simeq t(1 - 2\nu),$$

(5)
a being the lattice spacing and $t$ the nearest neighbor hopping. $\omega_{Um}$ depends in general also on $U$ – Eq. (5) is its weak coupling limit. The consequence is twofold:

(1) only at energies lower than $\omega_{Um}$ the gapless Luttinger liquid behavior shows up – consequently $\omega_L$ should be identified with $\omega_{Um}$;

(2) since at energies larger than $\omega_{Um}$ the Umklapp processes are important, the closer the system is to half filling, the larger is the influence of the Umklapp scattering to the parameters of the effective Luttinger liquid Hamiltonian.

Point (1) implies that close to half filling the energy scales introduced by the electron-phonon interaction become necessarily comparable to the energy scale $\omega_L$. Therefore it is more correct to analyze the effect of low momentum phonons on the full Hubbard model (plus eventually the $2k_F$-phonons) than on the effective Luttinger liquid model describing its low energy behavior without $q \sim 0$ phonons. Notice that analogous considerations apply near $\nu = 0$ since $\omega_L$ vanishes approaching zero filling as well, $\omega_L \simeq v_F/a \simeq tv$. On the other hand, point (2) implies that, in order to describe correctly the physics of the model close to half filling, one has to take into account the effects of the Umklapp scattering processes, both those generated by the on-site repulsion $U$ and those induced by the electron-phonon coupling. Umklapp processes will renormalize the phonon-exchanged interaction as well as they renormalize the direct el-el interactions. This makes it meaningless to add the bare
phonon-exchanged interaction (3) to the fully renormalized low-energy Hamiltonian (1) while approaching half filling (or zero filling).

We will now re-analyze the electron-phonon interaction moving along the guidelines of the previous discussion.

III. THE HUBBARD MODEL NEAR HALF FILLING

We have seen that at the metal-insulator transition, i.e. for $\nu \to 1/2$, $v_\rho \to 0$ while the charge exponent $K_\rho$ jumps discontinuously from the asymptotic value $1/2$ to $0$. A way to understand these results without invoking the exact Bethe ansatz solution would be to use the Renormalization Group (RG) and analyze the metal-insulator transition within the context of commensurate-incommensurate transitions. The RG scaling equations for the Hubbard model correctly predict that, in the absence of Umklapp scattering, the charge sector scales to a Luttinger liquid, while, when present, the Umklapp scattering scales to strong coupling, which is interpreted as the opening of a gap. Close to half filling a better approach would be to use a two cut-off scaling theory. For energy higher than $\omega_{U,m}$ [see (3)] the correct RG equations are those in the presence of Umklapp terms, while below $\omega_{U,m}$ these terms are taken to be equal to zero, and the RG equations correspond to those of a Tomonaga–Luttinger model. This weak coupling approach has the merit of reproducing the decrease of $K_\rho$ as $\nu \to 1/2$, but fails in reproducing other important features. In fact

1. it predicts $v_\rho$ to be an increasing function of the density, while in the reality $v_\rho \to 0$ for $\nu \to 1/2$;

2. it predicts $v_\rho$ to be an increasing function of the interaction, this is not true close to half filling;

3. it predicts $K_\rho$ to vanish continuously as $\nu \to 1/2$.

Therefore one has to resort to more refined methods than weak coupling RG which can not reproduce too detailed features when flowing to strong coupling.
The first aspect which merits to be clarified is why the point $K_\rho = 1/2$ is so important. First of all, this value corresponds to the limit $U = \infty$ of the Hubbard model. In this limit the on-site interaction is the strongest possible, so it clearly constitutes a lower bound to $K_\rho$, at least approaching half filling (longer range repulsion would allow lower values).

Secondly, $K_\rho = 1/2$ is exactly the value at the Luther–Emery line, where the charge Hamiltonian in the presence of Umklapp scattering can be rewritten in terms of free spinless fermions with a mass term $14$. At half filling the chemical potential lies in the middle of the gap, while away from half filling it crosses the conduction (or valence) band, so that the particle (hole) occupation of the conduction (valence) band is proportional to the deviation from half filling. The standard approach (see e.g. Ref. 16) is then to describe the gapless low-energy effective charge modes by linearizing the spinless fermion spectrum around the new Fermi points. Clearly when the density is very close to half filling, the chemical potential lies close to the bottom (top) of the conduction (valence) band, so that the velocity of the gapless modes is very small $v \sim (1 - 2\nu)$, correctly reproducing the result of the exact solution. Moreover, if the starting $K_\rho \neq 1/2$, one can still work at the Luther–Emery line, the difference $(K_\rho - 1/2)$ being accounted for by an effective interaction between the resulting spinless fermions. However a (weak) interaction among spinless fermions is not effective at low density; rather its strength vanishes when $\nu \to 1/2$. Hence for $\nu \to 1/2$ any generic electronic model with a relevant Umklapp tends asymptotically to the Luther-Emery line.15

Going back to the Hubbard model, this in turn implies that close to $\nu = 1/2$ this model behaves as if $U$ were infinite. This equivalence holds in the sense that it correctly predicts the asymptotic behavior of the correlation functions, but it obviously fails in determining $U$-dependent pre-factors.

These results are obtained for unretarded Umklapp scattering. We argue that they are still valid in the presence of the retarded interaction induced by the phonons. Specifically we claim that even if the $2k_F$-phonons are taken into account, the metal-insulator transition at half-filling can still be described in terms of a model close to the Luther-Emery line.
\( (K_\rho \rightarrow 1/2) \).

The immediate consequence is that any additional interaction (even an attractive interaction induced by low-momentum acoustic phonons) is relevant only if its bare value is comparable to the bare bandwidth, and not to the renormalized charge velocity. This, we believe, more rigorously explains why Martin and Loss’s results for the onset of the Wentzel-Bardeen singularity are incorrect close to half filling.

The above conclusions are simple to demonstrate for phonon frequencies larger than (or comparable to) the electron bandwidth. In this case, in fact, one can easily map the original electron-phonon coupled model into a model of electrons only with unretarded interactions. This amounts to approximate \( V(q, \omega) \) of (4) with its static limit
\[
V(q, \omega) \approx V(q, 0) = -\frac{|g(q)|^2}{\zeta \omega_q^2}.
\]
The strength of this effective attraction has to be compared with that of the on-site repulsion \( U \). If \( U + 2V(0, 0) - V(2k_F, 0) > 0 \) then \( K_\rho < 1 \) and the system still shows dominant density-wave fluctuations. As a consequence, one recovers at half filling a standard metal-to-insulator transition. If, on the contrary, that inequality does not hold, \( K_\rho > 1 \) and superconducting fluctuations dominate. In this case there is no commensurate transition at half filling. As to the spin sector, if \( U + V(2k_F, 0) > 0 \) no spin gap opens and \( K_\sigma = 1 \) (CDW and SDW coexist). In the opposite case the spin modes acquire a gap.

On the other hand, the reduction to unretarded interactions can be generalized to include the case when phonon frequencies are smaller (but not too smaller) than the electron bandwidth and will be discussed in the following Sections for weak electron-phonon coupling.

The opposite case of strong electron-phonon coupling and/or low \( 2k_F \)-phonon frequency corresponds to the situation where the \( 2k_F \)-lattice distortion can be described in terms of amplitude and phase fluctuations (as if the amplitude of the lattice distortion order parameter acquires a finite average value). Due to the finite restoring force acting against the amplitude fluctuations, these latter can be neglected at zero temperature. The phase fluctuations of the order parameter are on the contrary massless and they get strongly coupled to
the charge density waves. Practically the two move together. These collective electronic and lattice gapless modes are described again by a Luttinger liquid type of Hamiltonian with velocity $v_\rho$ and exponent $K_\rho$ renormalized to lower values by the electron-phonon coupling (see for instance Fukuyama in Ref. 1). The metal-insulator transition which takes place at half-filling is driven by an Umklapp term of the same form as in the pure Hubbard model. Therefore even in this limiting situation the approach to half filling can be still described as in the usual incommensurate-commensurate transition; the decrease of $K_\rho$ induced by the $2k_F$ processes will indeed increase the relevance of Umklapp scattering.

IV. INFINITE U LIMIT

To support the picture given in the previous Section and strongly simplify the analysis of retardation we consider, with some caution, the $U \to \infty$ limit, since, as we said, it falls in the same universality class of the model for arbitrary $U$ close to half-filling as far as charge degrees of freedom are concerned. The caution is necessary since at $U \to \infty$ only $4k_F$-CDW fluctuations survive while $2k_F$-ones are suppressed. This implies that, as the system approaches half-filling with large on-site repulsion, the $2k_F$-lattice distortion vanishes.

Keeping in mind that taking $U \to \infty$ we disregard interesting properties which appear only at finite $U$, let us consider how the system approaches half-filling in this limit. In this case it is known that the model is equivalent to a model of free spinless fermions (whose position on the chain correspond to the position of the electrons, independently of the spin label) plus an Heisenberg model on the squeezed chain. The Fermi momentum of the spinless fermions, which we will denote in the following as $k^{sf}_F$, is twice larger than the spinning fermion one, i.e. $k^{sf}_F = 2k_F$. This for instance means that half filling for the electrons means filling one for the spinless fermions. Close to this filling, the spinless fermions almost occupy all the levels of the band: the Fermi energy is close to the top of the band and the Fermi velocity is very small (notice the analogy with the Luther-Emery line). The electron-phonon interaction translates into an interaction between spinless fermions and
phonons. If the phonons are coupled to the electron density, the spin sector is unaffected by
the phonons and still describes an Heisenberg model – no spin gap opens close to half filling
at large $U$. On the contrary, in the SSH coupling the modulation of the hopping induces
a modulation of the exchange in the effective Heisenberg model and in turns leads to a
spin-Peierls transition with the consequent opening of a spin gap (which anyway vanishes as
$U \to \infty$). As to the charge sector, we are left with the problem of spinless fermions (whose
electronic energy scale is the bare bandwidth of the spinning electrons) in the presence
of phonon-induced interaction. We start by just considering the zero frequency limit of
this interaction, thus ignoring its detailed frequency dependence. The important scattering
processes are those at transferred momentum $q \sim 0$ and $q$ of the order of twice the spinless-
fermion Fermi momentum, which implies four times the spinning-fermion Fermi momentum
$q \sim 2k_F^{sf} = 4k_F$. The equivalent g-ological model has the effective $g_2$ scattering process
given by:

$$g_2 = V(0) - V(4k_F).$$

Close to filling one for the spinless fermions, i.e. half filling for the original electrons,
$k_F^{sf} = 2k_F \to \pi$ therefore $[V(0) - V(4k_F)] \to [V(0) - V(2\pi)] = 0$. Once again it comes out
that any small-$q$ interaction close to half filling is not very efficient. The same conclusion
is obtained in the zero filling limit. Notice the crucial importance of formally keeping both
the low and high transferred-momentum electron-phonon coupling. In the opposite case, in
fact, we would have reached the wrong conclusion that the system is pushed towards the
Wentzel-Bardeen singularity.

Similarly to what happens for the Heisenberg model with negative $J_z$ at finite
magnetization, one still expect an instability to phase-separation but only for strong
enough attraction $|V| \simeq t$, and not for $|V| \simeq v_F/a$.

The physical meaning of this result is indeed simple and predictable. It states that long
wavelength phonons are not relevant close to a commensurate transition.

One may still wonder about retardation effects, which we are going to discuss in the next
V. RETARDATION EFFECTS

We now analyze the consequences of retardation for the effective spinless fermion model which describes the $U \to \infty$ limit of the Hubbard model discussed in the previous Section 21.

We start by considering free spinless fermions coupled to optical phonons with a momentum independent coupling constant and arbitrary filling. Later on we will discuss the case of acoustic phonons. The frequency dependent effective $g_{eff}^2$ is:

$$g_{eff}^2(\epsilon_R, \epsilon_L, \omega) = V(\omega) - V(\epsilon_L - \omega - \epsilon_R),$$

where

$$V(\omega) = \frac{g^2}{\zeta} \frac{1}{\omega^2 - \omega_0^2},$$

$\omega_0$ being the phonon frequency [see Eq. (4)]. $\epsilon_R$ and $\epsilon_L$ are the frequencies of the incoming right and left fermions respectively, and $\epsilon_R + \omega$ and $\epsilon_L - \omega$ their outgoing frequencies. The first term on the right hand side is a pure $g_2$ term (low momentum transferred). The second is instead a $g_1$ interaction (momentum transferred $\sim 2k_F$), which for spinless fermions can be cast in the form of a $g_2$ scattering process. If all the fermionic lines are at zero frequency, $g_{eff}^2 = 0$. However, already at second order a finite and repulsive $g_{eff}^2$ is generated

$$g_{eff}^2 = \frac{1}{2\pi v_F} \left( \frac{g^2}{\omega_0^2 \zeta} \right)^2 \ln \left( \frac{\omega_c}{\omega_0} \right).$$

Eq. (6) has been obtained for a linearized band with a bandwidth cut-off $\omega_c$. At higher orders three kinds of terms are found. The first kind include all the logarithmic divergent terms which would be there if a constant $g_{eff}^2$ given by (6) were assumed from the beginning. The second kind consists of all the non divergent diagrams proportional to powers of $\ln(\omega_c/\omega_0)$. These terms can be interpreted as the higher order (finite) corrections to (6). The last class of terms include the divergent diagrams generated by the previous introduced higher order correction to $g_{eff}^2$. If
\[
\frac{1}{2\pi v_F} \left( \frac{g_2^2}{\omega_0^2 \zeta} \right) \ln \left( \frac{\omega_c}{\omega_0} \right) \ll 1
\]  

(7)

then one is allowed to neglect all the higher order corrections to \( g_2^{\text{eff}} \). It corresponds to a situation where not even the amplitude of the lattice distortion order parameter develops. In this case the problem reduces to a simple \( g \)-ological model with the repulsive \( g_2^{\text{eff}} \) Eq. (6). Consequently, the system of spinless fermions has dominant CDW fluctuations in the asymptotically low energy regime. For smaller phonon-frequency one has to use better approximations. A quite useful approach is the renormalization group, which can be implemented in various equivalent ways to take into account the retardation in both spinning and spinless systems (see e.g. Refs. [22–24]). All these analyses show that retardation favors CDW, confirming the simple calculation we have just described. Applied to the original \( U \to \infty \) Hubbard model it implies that \( K_\rho \) gets smaller than 1/2 when a local interaction with optical phonons is added.

This analysis can be pushed towards filling one (or zero) for the spinless model (which means half filling for the \( U \to \infty \) Hubbard model). If we calculate in this limit the effective \( g_2 \), as we have done to obtain Eq. (3) for intermediate fillings, we find a \( g_2^{\text{eff}} \) vanishing as filling zero or one is approached. This in turns implies that \( K_\rho \to 1/2 \) (from below) as \( \nu \to 1/2 \). The reason of the vanishing \( g_2^{\text{eff}} \) is simply that the effective bandwidth (i.e. the interval around the Fermi energy where the energy dispersion relation can be approximated by a linear dispersion) vanishes approaching filling zero or one. This implies that the electron-phonon interaction is practically un-retarded (\( \omega_0 \gg \omega_c \)), and being local in space, it is zero for Pauli’s principle. In terms of the starting spinning-model, we find once again that any additional interaction close to half filling is unimportant unless its strength is sufficiently large.

What would it change for acoustic phonons? This case was studied by Chen et al. taking into account only the small transferred-momentum electron-phonon coupling, which can be diagonalized exactly. Their results in the spinless case are analogous to Martin and Loss’s, which are derived for spinning fermions, so that they suffer from the same problems.
In fact if we apply the method of Ref. 5 to a spinless fermion model close to filling one (or zero), where the Fermi velocity is very small (eventually smaller than the phonon velocity), then we arrive to the conclusions 1) that superconductivity always dominates; 2) that just before filling one (or zero) phase separation occurs which onset is characterized by a Wentzel-Bardeen singularity.

In order to correctly describe the approach to filling one, it turns out important to include also the $2k_F^sF$-scattering processes, since in this limit $2k_F^sF \rightarrow 0$ and one is not allowed to neglect these processes while keeping the $q \sim 0$ ones. A simple way, which neglects the second order correction to these processes previously discussed and therefore underestimates their effects, is to approximate the $2k_F^sF$-scattering by its zero frequency limit, approximation valid for weak electron-phonon coupling and/or $2k_F^sF$-phonon frequency comparable to the bandwidth (this is surely the case when the density of the spinless fermions is close to one). This approximation amounts to write

$$\lim_{q \rightarrow 2k_F^sF} V(q, \omega) = \lim_{q \rightarrow 2k_F^sF} \frac{g^2}{\zeta} \frac{\omega_q^2}{\omega^2 - \omega_q^2} = -\frac{g^2}{\zeta}.$$ 

It is important to realize that these $2k_F^sF$-scattering processes generate both an effective $g_2$ and a so-called $g_4$ interaction, so that the effective phonon induced interaction can be rewritten as

$$g_2(q, \omega) = g_4(q, \omega) = V(q, \omega) - V(2k_F^sF, 0) = \frac{g^2}{\zeta} \frac{\omega^2}{\omega^2 - \omega_q^2}. \quad (8)$$

The Hamiltonian can be still diagonalized even in the presence of the $2k_F^sF$-scattering processes, which change profoundly the behavior as compared to that one predicted by using the approximation of Ref. 5. For instance we do not encounter any Wentzel-Bardeen singularity for vanishing spinless fermion Fermi velocity. If $v_F$ is the bare Fermi velocity and $c$ the phonon velocity, we find, for $v_F \ll c$, a renormalized charge velocity given by

$$v_{F}^{\text{eff}} = v_F \sqrt{\frac{2\pi \zeta c^2}{2\pi \zeta c^2 + g^2 v_F}},$$

which is diminished by the coupling with the phonons but yet remains positive for any $v_F > 0$. In fact, analogously to the case of optical phonons, we again predict dominant
CDW-fluctuations. In particular, in terms of the original parameter, we still find $K_\rho \to 1/2$ from below. Rigorously speaking, one can not apply this method really close to filling one when $2k_F^{\text{eff}} \to 0$, since it is not anymore correct to approximate the fermionic band with a linearized dispersion relation. However we do not expect any singularity in this limit and therefore we conclude that even in the case of acoustic phonons the metal-insulator transition is not preceded by any phase separation.

VI. CONCLUSIONS

In the preceding Sections, we have discussed the approach to half filling of a 1D system of repulsively interacting electrons in the presence of electron-phonon coupling. We have argued that the metal-insulator transition which occurs in a pure system at half-filling can be described, even in the presence of a strong coupling to the $2k_F$-lattice deformation, as a standard 1D incommensurate-commensurate transition\textsuperscript{15,16}. By making use of known results for this transition, we have shown that the coupling constant between the electrons and the low-momentum phonons (both optical and acoustic) is strongly renormalized downwards close to half filling. In particular we find that the renormalized coupling constant vanishes at least like the charge velocity as the density $\nu$ goes to $1/2$. Our result thus disagrees with that one found by Martin and Loss\textsuperscript{3}, according to whom this coupling constant remains finite when $\nu \to 1/2$. As to the phase diagram of the Hubbard model, they claim that the insulating phase at half filling is preceded (in density, keeping fixed all the other parameters of the Hamiltonian) by a region of phase separation, which one could imagine as a region of coexistence of the insulating phase with the superconductive one which they find for smaller densities. On the contrary we think that, if an insulating phase does exist at half-filling, the MIT is still preceded by a region where charge density wave fluctuations dominate, as in the absence of phonons. Rather, even if we neglect the coupling with $2k_F$-phonons, by taking for instance the $U \to \infty$ limit, we find that low-momentum ($q \sim 0$ modulus a reciprocal lattice vector) phonons only provide an effective repulsive interaction close to half-filling which
effect is to diminish the value of the charge exponent \( K_\rho \) from its \( U \to \infty \) limit \( K_\rho = 1/2 \) (although, as the MIT is approached, \( K_\rho \to 1/2 \) anyway).

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\textsuperscript{21} The $U \rightarrow \infty$ limit considerably simplifies the analysis firstly because the spin degrees of freedom are explicitly decoupled from the charge ones, and secondly because one can disregard the charge scattering at $2k_F$, and consequently the coupling to the $2k_F$-phonons, and only consider the $4k_F = 2k_F$\textsuperscript{sf} scattering-processes.

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\textsuperscript{25} In particular the RG analysis for spinning fermions show that the coupling to the $2k_F$ phonons can always be absorbed into an effective unretarded repulsion in the weak coupling limit. This in turns justifies the claim we made that the system of interacting fermions close to half filling is also close to the Luther-Emery line, even if $2k_F$ phonons are taken into account.

\textsuperscript{26} In fact the $2k_F$-scattering induces a Fermi velocity renormalization which can be interpreted in terms of so-called $g_4$ processes. This can be simply seen when we consider optical phonons with frequency much larger then the electron bandwidth. In this case the phonon induced interaction is practically unretarded. Since it is local in space, it is zero for Pauli’s principle ($n(x)^2 = n(x)$ for spinless fermions). This implies that the interaction induced
by the $q \sim 0$ phonons must cancel exactly that one induced by the $q \sim 2k_F$ phonons. Since the former contains a $g_4$ process, the latter should contain it too.