EXTENSION OF THE $\nu$-METRIC: THE $H^\infty$ CASE

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Abstract. An abstract $\nu$-metric was introduced by Ball and Sasane, with a view towards extending the classical $\nu$-metric of Vinnicombe from the case of rational transfer functions to more general nonrational transfer function classes of infinite-dimensional linear control systems. In this short note, we give an additional concrete special instance of the abstract $\nu$-metric, by verifying all the assumptions demanded in the abstract set-up. This example links the abstract $\nu$-metric with the one proposed by Vinnicombe as a candidate for the $\nu$-metric for nonrational plants.

1. Introduction

We recall the general stabilization problem in control theory. Suppose that $R$ is a commutative integral domain with identity (thought of as the class of stable transfer functions) and let $\mathbb{F}(R)$ denote the field of fractions of $R$. The stabilization problem is:

Given $P \in (\mathbb{F}(R))^{p \times m}$ (an unstable plant transfer function),
find $C \in (\mathbb{F}(R))^{m \times p}$ (a stabilizing controller transfer function),
such that (the closed loop transfer function)

$$H(P, C) := \begin{bmatrix} P \\ I \end{bmatrix} (I - CP)^{-1} \begin{bmatrix} -C & I \end{bmatrix}$$

belongs to $R^{(p+m) \times (p+m)}$ (is stable).

In the robust stabilization problem, one goes a step further. One knows that the plant is just an approximation of reality, and so one would really like the controller $C$ to not only stabilize the nominal plant $P_0$, but also all sufficiently close plants $P$ to $P_0$. The question of what one means by “closeness” of plants thus arises naturally.

So one needs a function $d$ defined on pairs of stabilizable plants such that

1. $d$ is a metric on the set of all stabilizable plants,
2. $d$ is amenable to computation, and
3. stabilizability is a robust property of the plant with respect to this metric.

Such a desirable metric, was introduced by Glenn Vinnicombe in [7] and is called the $\nu$-metric. In that paper, essentially $R$ was taken to be the rational functions without poles in the closed unit disk or, more generally, the disk algebra, and the most important results were that the $\nu$-metric is indeed a metric on the set of stabilizable plants, and moreover, one has the inequality that if $P_0, P \in S(R, p, m)$, then

$$\mu_{P, C} \geq \mu_{P_0, C} - d_{\nu}(P_0, P),$$

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where $\mu_{P,C}$ denotes the stability margin of the pair $(P, C)$, defined by
\[
\mu_{P,C} := \|H(P, C)\|^{-1}_{\infty}.
\]
This implies in particular that stabilizability is a robust property of the plant $P$.

The problem of what happens when $R$ is some other ring of stable transfer functions of infinite-dimensional systems was left open in [7]. This problem of extending the $\nu$-metric from the rational case to transfer function classes of infinite-dimensional systems was addressed in [1]. There the starting point in the approach was abstract. It was assumed that $R$ is any commutative integral domain with identity which is a subset of a Banach algebra $S$ satisfying certain assumptions, labelled (A1)-(A4), which are recalled in Section 2. Then an “abstract” $\nu$-metric was defined in this setup, and it was shown in [1] that it does define a metric on the class of all stabilizable plants. It was also shown there that stabilizability is a robust property of the plant.

In [7], it was suggested that the $\nu$-metric in the case when $R = H^\infty$ might be defined as follows. Let $P_1, P_2$ be unstable plants with the normalized left/right coprime factorizations
\[
P_1 = N_1 D_1^{-1} = \tilde{D}_1^{-1} \tilde{N}_1,
\]
\[
P_2 = N_2 D_2^{-1} = \tilde{D}_2^{-1} \tilde{N}_2,
\]
where $N_1, D_1, N_2, D_2, \tilde{N}_1, \tilde{D}_1, \tilde{N}_2, \tilde{D}_2$ are matrices with $H^\infty$ entries. Then
\[
d_{\nu}(P_1, P_2) = \begin{cases} \|\tilde{G}_2 G_1\|_{\infty} & \text{if } T_{\tilde{G}_1} G_2 \text{ is Fredholm with Fredholm index 0}, \\ 0 & \text{otherwise} \end{cases}
\]
Here $\cdot^*$ has the usual meaning, namely: $G_1^*(\zeta)$ is the transpose of the matrix whose entries are complex conjugates of the entries of the matrix $G_1(\zeta)$, for $\zeta \in \mathbb{T}$. Also in the above, for a matrix $M \in (L^\infty)^{p \times m}$, $T_M : (H^2)^m \to (H^2)^p$ denotes the Toeplitz operator given by
\[
T_M \varphi = P_{(H^2)^p}(M \varphi) \quad (\varphi \in (H^2)^m)
\]
where $M \varphi$ is considered as an element of $(L^2)^p$ and $P_{(H^2)^p}$ denotes the canonical orthogonal projection from $(L^2)^p$ onto $(H^2)^p$.

Although we are unable to verify whether there is a metric $d_{\nu}$ such that the above holds in the case of $H^\infty$, we show that the above does work for the somewhat smaller case when $R$ is the class QA of quasicontinuous functions analytic in the unit disk.

We prove this by showing that this case is just a special instance of the abstract $\nu$-metric introduced in [1].

The paper is organized as follows:

(1) In Section 2 we recall the general setup and assumptions and the abstract metric $d_{\nu}$ from [1].

(2) In Section 3 we specialize $R$ to a concrete ring of stable transfer functions, and show that our abstract assumptions hold in this particular case.

2. Recap of the abstract $\nu$-metric

We recall the setup from [1]:

(A1) $R$ is commutative integral domain with identity.

(A2) $S$ is a unital commutative complex semisimple Banach algebra with an involution $\cdot^*$, such that $R \subset S$. We use $\text{inv } S$ to denote the invertible elements of $S$. 
Given a matrix \( F \), a factorization \( P \) is referred to as a normalized right coprime factorization of \( P \) if there exist matrices \( X,Y \) with entries from \( R \) such that \( XN + YD = I_m \). If moreover it holds that \( N^*N + D^*D = I_m \), then the right coprime factorization is referred to as a normalized right coprime factorization of \( P \).

**Left coprime/normalized coprime factorization:** A factorization \( P = \overline{D}^{-1}\overline{N} \), where \( \overline{N}, \overline{D} \) are matrices with entries from \( R \), is called a left coprime factorization of \( P \) if there exist matrices \( \overline{X}, \overline{Y} \) with entries from \( R \) such that \( \overline{N}\overline{X} + \overline{D}\overline{Y} = I_p \). If moreover it holds that \( \overline{N}\overline{N}^* + \overline{D}\overline{D}^* = I_p \), then the left coprime factorization is referred to as a normalized left coprime factorization of \( P \).

**The notation \( G, \overline{G}, K, \overline{K} \):** Given \( P \in (\mathbb{F}(R))^{p \times m} \) with normalized right and left factorizations \( P = ND^{-1} \) and \( P = \overline{D}^{-1}\overline{N} \), respectively, we introduce the following matrices with entries from \( R \):

\[
G = \begin{bmatrix} N \\ D \end{bmatrix} \quad \text{and} \quad \overline{G} = \begin{bmatrix} -\overline{D} \\ \overline{N} \end{bmatrix}.
\]

Similarly, given \( C \in (\mathbb{F}(R))^{m \times p} \) with normalized right and left factorizations \( C = NCD_C^{-1} \) and \( C = \overline{D}_C^{-1}\overline{N}_C \), respectively, we introduce the following matrices with entries from \( R \):

\[
K = \begin{bmatrix} D_C \\ N_C \end{bmatrix} \quad \text{and} \quad \overline{K} = \begin{bmatrix} -\overline{N}_C \\ \overline{D}_C \end{bmatrix}.
\]

**The notation \( S(R, p, m) \):** We denote by \( S(R, p, m) \) the set of all elements \( P \in (\mathbb{F}(R))^{p \times m} \) that possess a normalized right coprime factorization and a normalized left coprime factorization.

We now define the metric \( d_\nu \) on \( S(R, p, m) \). But first we specify the norm we use for matrices with entries from \( S \).

**Definition 2.1 \( (\|\cdot\|) \).** Let \( \mathfrak{M} \) denote the maximal ideal space of the Banach algebra \( S \). For a matrix \( M \in S^{p \times m} \), we set

\[
\|M\| = \max_{\varphi \in \mathfrak{M}} |M(\varphi)|.
\]
Here $M$ denotes the entry-wise Gelfand transform of $M$, and $\| \cdot \|$ denotes the induced operator norm from $\mathbb{C}^m$ to $\mathbb{C}^p$. For the sake of concreteness, we fix the standard Euclidean norms on the vector spaces $\mathbb{C}^m$ to $\mathbb{C}^p$.

The maximum in (2.1) exists since $\mathcal{M}$ is a compact space when it is equipped with Gelfand topology, that is, the weak-$*$ topology induced from $L(S; \mathbb{C})$. Since we have assumed $S$ to be semisimple, the Gelfand transform $\hat{\cdot} : S \rightarrow \hat{S} (\subset C(\mathcal{M}, \mathbb{C}))$ is an isomorphism. If $M \in S^{1 \times 1} = S$, then we note that there are two norms available for $M$: the one as we have defined above, namely $\|M\|$, and the norm $\| \cdot \|_S$ of $M$ as an element of the Banach algebra $S$. But throughout this article, we will use the norm given by (2.1).

**Definition 2.2** (Abstract $\nu$-metric $d_\nu$). For $P_1, P_2 \in S(R,p,m)$, with the normalized left/right coprime factorizations

\[
P_1 = N_1 D_1 = \tilde{D}_1 \tilde{N}_1,
\]

\[
P_2 = N_2 D_2 = \tilde{D}_2 \tilde{N}_2,
\]

we define

\[
d_\nu(P_1, P_2) := \begin{cases} 
\|\tilde{G}_2 G_1\| & \text{if } \det(G_1^* G_2) \in \text{inv } S \text{ and } \iota(\det(G_1^* G_2)) = \circ, \\
1 & \text{otherwise.}
\end{cases}
\]

The following was proved in [1]:

**Theorem 2.3.** $d_\nu$ given by (2.2) is a metric on $S(R,p,m)$.

**Definition 2.4.** Given $P \in (\mathbb{F}(R))^{p \times m}$ and $C \in (\mathbb{F}(R))^{m \times p}$, the **stability margin** of the pair $(P,C)$ is defined by

\[
\mu_{P,C} = \begin{cases} 
\|H(P,C)\|_{\infty} & \text{if } P \text{ is stabilized by } C, \\
0 & \text{otherwise.}
\end{cases}
\]

The number $\mu_{P,C}$ can be interpreted as a measure of the performance of the closed loop system comprising $P$ and $C$: larger values of $\mu_{P,C}$ correspond to better performance, with $\mu_{P,C} > 0$ if $C$ stabilizes $P$.

The following was proved in [1]:

**Theorem 2.5.** If $P_0, P \in S(R,p,m)$ and $C \in S(R,m,p)$, then

\[
\mu_{P,C} \geq \mu_{P_0,C} - d_\nu(P_0, P).
\]

The above result says that stabilizability is a robust property of the plant, since if $C$ stabilizes $P_0$ with a stability margin $\mu_{P_0,C} > m$, and $P$ is another plant which is close to $P_0$ in the sense that $d_\nu(P, P_0) \leq m$, then $C$ is also guaranteed to stabilize $P$.

3. The $\nu$-metric when $R = QA$

Let $H^\infty$ be the Hardy algebra, consisting of all bounded and holomorphic functions defined on the open unit disk $\mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \}$.

As was observed in the Introduction, it was suggested in [7] to use (1.1) to define a metric on the quotient ring of $H^\infty$. It is tempting to try to do this by using the general setup of [1] with $R = H^\infty$, $S = L^\infty$ and with $\iota$ equal to the Fredholm index of the associated Toeplitz operator. However at this level of generality there is no
guarantee that \( \varphi \) invertible in \( L^\infty \) implies that \( T_\varphi \) is Fredholm (and hence \( \iota \) equal to the Fredholm index of the associated Toeplitz operator is not well-defined on \( \text{inv} \ S \) (condition (A3)). However a perusal of the extensive literature on Fredholm theory of Toeplitz operators from the 1970s leads to the choices \( R \) equal to the class \( QA \) of quasianalytic and \( S \) equal to the class \( QC \) of quasicontinuous functions as conceivably the most general subalgebras of \( H^\infty \) and \( L^\infty \) which fit the setup of [1], as we now explain.

The notation \( QC \) is used for the \( C^* \)-subalgebra of \( L^\infty(\mathbb{T}) \) of quasicontinuous functions:

\[
QC := (H^\infty + C(\mathbb{T})) \cap [H^\infty + C(\mathbb{T})].
\]

An alternative characterization of \( QC \) is the following:

\[
QC = L^\infty \cap \text{VMO},
\]

where \( \text{VMO} \) is the class of functions of vanishing mean oscillation [4, Theorem 2.3, p.368].

The Banach algebra \( QA \) of analytic quasicontinuous functions is

\[
QA := H^\infty \cap QC.
\]

We have the following.

In order to verify (A4), we will also use the result given below; see [2, Theorem 7.36].

**Proposition 3.1.** If \( f \in H^\infty(\mathbb{D}) + C(\mathbb{T}) \), then \( T_f \) is Fredholm if and only if there exist \( \delta, \epsilon > 0 \) such that

\[
|F(re^{it})| \geq \epsilon \text{ for } 1 - \delta < r < 1,
\]

where \( F \) is the harmonic extension of \( f \) to \( \mathbb{D} \). Moreover, in this case the index of \( T_f \) is the negative of the winding number with respect to the origin of the curve \( F(re^{it}) \) for \( 1 - \delta < r < 1 \).

**Theorem 3.2.** Let

\[
R := QA, \\
S := QC, \\
G := \mathbb{Z}, \\
\iota := \Big( \varphi (\in \text{inv} \ QC) \mapsto \text{Fredholm index of } T_\varphi (\in \mathbb{Z}) \Big).
\]

Then (A1)-(A4) are satisfied.

**Proof.** Since \( QA \) is a commutative integral domain with identity, (A1) holds.

The set \( QC \) is a unital \( (1 \in C(\mathbb{T}) \subset QC) \), commutative, complex, semisimple Banach algebra with the involution

\[
 f^*(\zeta) = \overline{f(\zeta)} \quad (\zeta \in \mathbb{T}).
\]

In fact, \( QC \) is a \( C^* \)-subalgebra of \( L^\infty(\mathbb{T}) \). So (A2) holds as well.

[2] Corollary 139, p.354 says that if \( \varphi \in \text{inv} \ QC \), then \( T_\varphi \) is a Fredholm operator. Thus it follows that the map \( \iota : \text{inv} \ QC \to \mathbb{Z} \) given by

\[
\iota(\varphi) := \text{Fredholm index of } T_\varphi \quad (\varphi \in \text{inv} \ QC)
\]
is well-defined. If \( \varphi, \psi \in \text{inv } QC \), then in particular they are elements of \( H^\infty + C(\mathbb{T}) \), and so the semicommutator

\[
T_{\varphi \psi} - T_{\psi} T_{\varphi}
\]

is compact [5 Lemma 133, p.350]. Since the Fredholm index is invariant under compact perturbations (see e.g. [5 Part B, 2.5.2(h)]), it follows that the Fredholm index of \( T_{\varphi \psi} \) is the same as that of \( T_{\psi} T_{\varphi} \). Consequently (A3)(I1) holds.

Also, if \( \varphi \in \text{inv } QC \), then we have that

\[
\iota(\varphi^*) = \iota(\overline{\varphi}) = \text{Fredholm index of } T_{\overline{\varphi}} = \text{Fredholm index of } (T_{\varphi})^* = -(\text{Fredholm index of } T_{\varphi}) = -\iota(\varphi).
\]

Hence (A3)(I2) holds.

The map sending the a Fredholm operator on a Hilbert space to its Fredholm index of compact perturbations (see e.g. [5, Part B, 2.5.2(h)]), it follows that the Fredholm index \( \iota(\varphi) \) is equal to 0. By Proposition 3.1, it follows that there exist a \( \iota, \epsilon > 0 \) such that \( \| T_{\varphi} \| \leq \| \varphi \| \), and so the map \( \varphi \mapsto T_{\varphi} : \text{inv } QC \to \text{Fred}(H^2) \) is continuous. Consequently the map \( \iota \) is continuous from \( \text{inv } QC \) to \( Z \) (where \( Z \) has the discrete topology). Thus (A3)(I3) holds.

Finally, we will show that (A4) holds as well. Let \( \varphi \in H^\infty \cap (\text{inv } QC) \) be invertible as an element of \( H^\infty \). Then clearly \( T_{\varphi} \) is invertible, and so has Fredholm index \( \text{ind } T_{\varphi} \) equal to 0. Hence \( \iota(\varphi) = 0 \). This finishes the proof of the “only if” part in (A4).

Now suppose that \( \varphi \in H^\infty \cap (\text{inv } QC) \) and that \( \iota(\varphi) = 0 \). In particular, \( \varphi \) is invertible as an element of \( H^\infty + C(\mathbb{T}) \) and the Fredholm index \( \text{ind } T_{\varphi} \) of \( T_{\varphi} \) is equal to 0. By Proposition 3.1 it follows that there exist \( \delta, \epsilon > 0 \) such that \( |\Phi(r e^{i\theta})| \geq \epsilon \) for \( 1 - \delta < r < 1 \), where \( \Phi \) is the harmonic extension of \( \varphi \) to \( \mathbb{D} \). But since \( \varphi \in H^\infty \), its harmonic extension \( \Phi \) is equal to \( \varphi \). So \( |\varphi(r e^{i\theta})| \geq \epsilon \) for \( 1 - \delta < r < 1 \). Also since \( \iota(\varphi) = 0 \), the winding number with respect to the origin of the curve \( \varphi(r e^{i\theta}) \) for \( 1 - \delta < r < 1 \) is equal to 0. By the Argument principle, it follows that \( \int \varphi \) cannot have any zeros inside \( r \mathbb{T} \) for \( 1 - \delta < r < 1 \). In light of the above, we can now conclude that there is an \( \epsilon' > 0 \) such that \( |\varphi(z)| > \epsilon' \) for all \( z \in \mathbb{D} \). Thus \( 1/\varphi \) is in \( H^\infty \) with \( H^\infty \)-norm at most \( 1/\epsilon' \) and we conclude that \( \varphi \) is invertible as an element of \( H^\infty \). Consequently (A4) holds.

In the definition of the \( \nu \)-metric given in Definition 2.2 corresponding to Lemma 3.2 the \( \| \cdot \|_\infty \) now means the usual \( L^\infty(\mathbb{T}) \) norm.

**Lemma 3.3.** Let \( A \in QC^{p \times m} \). Then

\[
\| A \| = \| A \|_\infty := \text{ess.sup}_{\zeta \in \mathbb{T}} |A(\zeta)|.
\]

**Proof.** We have that

\[
\| A \|_\infty = \text{ess.sup}_{\zeta \in \mathbb{T}} |A(\zeta)| = \text{ess.sup}_{\zeta \in \mathbb{T}} \sigma_{\text{max}}(A(\zeta)) = \max_{\varphi \in M(L^\infty(\mathbb{T}))} \sigma_{\text{max}}(A(\varphi)) = \max_{\varphi \in M(QC)} \sigma_{\text{max}}(A(\varphi)) = \max_{\varphi \in M(QC)} \| \hat{A}(\varphi) \| = \| A \|.
\]
In the above, the notation \( \sigma_{\text{max}}(X) \), for a complex matrix \( X \in \mathbb{C}^{p \times m} \), means its largest singular value, that is, the square root of the largest eigenvalue of \( X^*X \) (or \( XX^* \)). We have also used the fact that for an \( f \in QC \subset L^\infty(T) \), we have that

\[
\max_{\varphi \in M(L^\infty(T))} \hat{f}(\varphi) = \|f\|_{L^\infty(T)} = \max_{\varphi \in M(QC)} \hat{f}(\varphi),
\]

Also, we have used the fact that if \( \mu \in L^\infty(T) \) is such that

\[
\det(\mu^2 I - A^*A) = 0,
\]

then upon taking Gelfand transforms, we obtain

\[
\det((\hat{\mu}(\varphi))^2 I - (\hat{A}(\varphi))^* \hat{A}(\varphi)) = 0 \quad (\varphi \in M(L^\infty(T))),
\]

to see that \( \sigma_{\text{max}}(A)(\varphi) = \sigma_{\text{max}}(\hat{A}(\varphi)), \varphi \in M(L^\infty(T)) \). \( \square \)

Finally, our scalar winding number condition

\[
\det(G_1^*G_2) \in \text{inv QC} \text{ and Fredholm index of } T_{\det(G_1^*G_2)} = 0
\]
is exactly the same as the condition

\[
T_{G_1^*G_2} \text{ is Fredholm with Fredholm index 0}
\]

in \([1,1]\). This is an immediate consequence of the following result due to Douglas [8, p.13, Theorem 6].

**Proposition 3.4.** The matrix Toeplitz operator \( T_\Phi \) with the matrix symbol \( \Phi = [\varphi_{ij}] \in (H^\infty + C(T))^{n \times n} \) is Fredholm if and only if

\[
\inf_{\zeta \in T} |\det(\varphi(\zeta))| > 0,
\]

and moreover the Fredholm index of \( T_\Phi \) is the negative of the Fredholm index of \( \det \Phi \).

Thus our abstract metric reduces to the same metric given in \([1,1]\), that is, for plants \( P_1, P_2 \in S(QA,p,m) \), with the normalized left/right coprime factorizations

\[
\begin{align*}
P_1 &= N_1 D_1^{-1} = \tilde{D}_1^{-1} \tilde{N}_1, \\
P_2 &= N_2 D_2^{-1} = \tilde{D}_2^{-1} \tilde{N}_2,
\end{align*}
\]

define

\[
d_\nu(P_1, P_2) := \begin{cases}
\|\tilde{G}_2 G_1\|_{\infty} & \text{if } \det(G_1^*G_2) \in \text{inv QC} \text{ and Fredholm index of } T_{\det(G_1^*G_2)} = 0, \\
1 & \text{otherwise}.
\end{cases}
\]

(3.1)

Summarizing, our main result is the following.

**Corollary 3.5.** \( d_\nu \) given by \((3.1)\) is a metric on \( S(QA,p,m) \). Moreover, if \( P_0, P \in S(QA,p,m) \) and \( C \in S(QA,m,p) \), then

\[
\mu_{P,C} \geq \mu_{P_0,C} - d_\nu(P_0,P).
\]
References

[1] J.A. Ball and A.J. Sasane. Extension of the $\nu$-metric. \textit{Complex Analysis and Operator Theory}, to appear.

[2] R.G. Douglas. \textit{Banach algebra techniques in operator theory}. Second edition. Graduate Texts in Mathematics, 179, Springer-Verlag, New York, 1998.

[3] R.G. Douglas. \textit{Banach algebra techniques in the theory of Toeplitz operators}. Expository Lectures from the CBMS Regional Conference held at the University of Georgia, Athens, Ga., June 12–16, 1972. Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 15. American Mathematical Society, Providence, R.I., 1973.

[4] J.B. Garnett. \textit{Bounded analytic functions}. Revised first edition. Graduate Texts in Mathematics, 236. Springer, New York, 2007.

[5] N.K. Nikolski. \textit{Treatise on the shift operator. Spectral function theory. With an appendix by S.V. Khrushchev and V.V. Peller}. Translated from the Russian by Jaak Peetre. Grundlehren der Mathematischen Wissenschaften, 273. Springer-Verlag, Berlin, 1986.

[6] N.K. Nikolski. \textit{Operators, functions, and systems: an easy reading. Vol. 1. Hardy, Hankel, and Toeplitz}. Translated from the French by Andreas Hartmann. Mathematical Surveys and Monographs, 92. American Mathematical Society, Providence, RI, 2002.

[7] G. Vinnicombe. Frequency domain uncertainty and the graph topology. \textit{IEEE Transactions on Automatic Control}, no. 9, 38:1371-1383, 1993.

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