A THEORETICAL APPROACH TO UNDERSTANDING RUMOR PROPAGATION DYNAMICS IN A SPATIALLY HETEROGENEOUS ENVIRONMENT

LINHE ZHU*
School of Mathematical Sciences, Jiangsu University
Zhenjiang, 212013, China

WENSHAN LIU
School of Mathematical Sciences, Jiangsu University
Zhenjiang, 212013, China
School of Mathematical Sciences, Nanjing Normal University
Nanjing, 210023, China

ZHENGDI ZHANG
School of Mathematical Sciences, Jiangsu University
Zhenjiang, 212013, China

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Abstract. Most of the previous work on rumor propagation either focus on ordinary differential equations with temporal dimension or partial differential equations (PDE) with only consideration of spatially independent parameters. Little attention has been given to rumor propagation models in a spatiotemporally heterogeneous environment. This paper is dedicated to investigating a SCIR reaction-diffusion rumor propagation model with a general nonlinear incidence rate in both heterogeneous and homogeneous environments. In spatially heterogeneous case, the well-posedness of global solutions is established first. The basic reproduction number \( R_0 \) is introduced, which can be used to reveal the threshold-type dynamics of rumor propagation: if \( R_0 < 1 \), the rumor-free steady state is globally asymptotically stable, while \( R_0 > 1 \), the rumor is uniformly persistent. In spatially homogeneous case, after introducing the time delay, the stability properties have been extensively studied. Finally, numerical simulations are presented to illustrate the validity of the theoretical analysis and the influence of spatial heterogeneity on rumor propagation is further demonstrated.

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* Corresponding author: Linhe Zhu.
1. Introduction. As an important form of social communication and a pervasive social phenomenon, rumor refers to the information that disseminate without officially publicized confirmation. People spread rumor, often driven by inundant moralities or angry emotions. With the advent of the new media era, online social network has become the main distributing center for rumor. Online social network, on the one hand, with its convenience and flexibility provides a rapid way for information communication and information sharing. However, on the other hand, considering the openness and the dispersed information sources of the online social network, the rumor propagation has caused many negative effects on public opinion environment, social order and national security. Therefore, the investigation on the mechanism of rumor propagation is a arduous task which brooks no delay.

Most existing mathematical models of rumor propagation have evolved from classical infectious disease models [43, 12], such as: SIS model and SIR model. The researches on rumor propagation can be traced back to 1960s with the basic DK (Daley-Kendal) model [6] formalized mathematically by Daley and Kendal. Henceforth, the DK model, together with its variants, such as MK (Maki) model [18], were modified and improved for quantitative studies of rumor propagation which brought a breakthrough in rumor propagation research. Currently, an increasing number of researchers have set out to investigate the problem of rumor propagation in online social networks. The IS2R2 rumor propagation model [14] had been studied to show that rumor could diffuse in a multi-lingual atmosphere. Zhu and Liu [44] established a novel delayed SIR rumor propagation model with forced silence function in both homogeneous and heterogeneous networks. By using mean-field equations, Wang et al.[30] established the SIRaRu rumor propagation model in complex networks and analyzed its dynamic characteristics. These studies above provided theoretical support for us to control rumor propagation more effectively [45, 46, 5, 47, 48].

The rumor propagation models we mentioned above, it is worth noting, are based on the assumption of “homogeneous mixing”, with users mixing uniformly, without regard to location. It has been universally accepted that spatial diffusion and environmental heterogeneity are crucial factors that should be considered in the infectious disease transmission [17, 25, 33, 20]. Since the pioneering work of Allen et al. [2] on the influence of the environmental heterogeneity and the movement of individuals on disease transmission, there are many scholars begin to study the reaction-diffusion equations in a spatially heterogeneous environment [3, 4, 15, 41]. Lou and Zhao [16] proposed a reaction-diffusion malaria model with incubation period in the vector population, in which several system parameters were spatially dependent. In Ref.[35], to investigate the effect of the spatial heterogeneity, Wu and Zhao established a reaction-diffusion model of vector-borne disease with periodic delays. Due to the certain similarities between rumor propagation and disease transmission, web users can be affected not only on the basis of mode of transmission, but also on the basis of network resource availability and network interaction, so it is appropriate to consider a user which differs from each other in the cyberspace when rumor spreads.

The above statements related to rumor propagation model seem to get little attention which is also the innovation of our work. Our paper is devoted to using reaction-diffusion equations to explore the influence of spatiotemporal heterogeneities and general incidence function on the dynamics behaviors of a SCIR
rumor propagation model. We firstly establish the well-posedness of global solutions. Inspired by a general approach in Refs. [31, 8, 29], we calculate the basic reproduction number $R_0$ which includes the spatial homogeneity as a special case. Moreover, the threshold-type dynamics are deeper analyzed: the global stability of the rumor-free steady state is proved by using the comparison principle and Lyapunov function, and the uniform persistence of all positive solutions is obtained by the persistence theory. In addition, in order to better reflect the influence of spatial heterogeneity and enrich our work, we also propose the corresponding homogeneous model by incorporating a time delay which is more in line with the reality. The basic reproduction number $R_0$ is obtained by using the existence of the positive equilibrium point. Further, we expound the local and global stabilities of the rumor-free and rumor-prevailing equilibrium point. Finally, some carefully designed numerical simulations are presented to illustrate the theoretical predictions.

The remainder of this paper is arranged as follows. In Section 2, the spatially heterogeneous $SCIR$ rumor propagation model is investigated. Some preliminaries for the reaction-diffusion equations and the well-posedness of global solutions are provided. The basic reproduction number is calculated and the threshold criteria on the global stability of rumor-free steady state are proposed. Moreover, we also prove the uniform persistence of all positive solutions [4, 40, 33]. In Section 3, the spatially homogenous model is established [39, 19, 38]. The basic reproduction number, the existence of equilibrium points as well as the stability properties are illustrated. In addition to theoretical analysis, in Section 4, we design some numerical simulations to support our theoretical results. Finally, a conclusion section completes the paper.

2. Spatially heterogeneous model.

2.1. Model formulation over both temporal and spacial dimensions. Depending on the user’s different states in online social networks, users are classified into four distinct classes: susceptible ($S$), collected ($C$), infective ($I$) and refractory ($R$), where $S$, $C$, $I$ and $R$ denote, respectively, the people who have never heard the rumor and are susceptible to the rumor (Susceptible), those who collect the rumor but does not spread it (Collected), those who are spreading the rumor (Infective), and the ones who have no response to the rumor (Refractory). The propagation rules of the $SCIR$ model can be summarized as follows.

(1) When a susceptible user comes into contact with a infected user, the susceptible user deems the rumor has collecting value so that he/she collects the rumor (such as putting them in their Internet link collections) with probability $\beta$, namely collecting rate.

(2) When rumor spreads in full force, the collector may click again on the link to the rumor he/she collected before, then he/she may have interest in the rumor and become a sharer with probability $\theta$.

(3) Due to the decrease of heat for the rumor and the constant updating of information, infective users may become refractory users with probability $\delta$, namely recovery rate.

(4) Due to the compatibility and openness of online social networks, users can freely enter or exit the online social networks at rate $A$ and $\mu$ respectively.

It is well known that cyberspace is not necessarily one-dimensional, that is to say, rumor does not travel only along time orientation. Thus it is reasonable to consider a PDE rumor propagation model in a general bounded space domain.
In general, it is assumed that the related parameters of rumor propagation are constants, however, the related parameters involving space may be determined by the spatial variable \(x\), instead of constants, that is, with spatial heterogeneity.

Set \(\Omega \subset \mathbb{R}\) be a spatial domain with smooth boundary \(\partial \Omega\). \(\frac{\partial}{\partial n}\) denotes the outward normal derivative on smooth boundary \(\partial \Omega\) where \(n\) is the outward normal vector on \(\partial \Omega\). In the light of the works in Section 1 and the discussion above, a spontaneous consideration of a spatially heterogeneous environment and general incidence function motivate us to investigate the following SCIR diffusive rumor propagation model:

\[
\begin{aligned}
\frac{\partial S(t,x)}{\partial t} &= D \Delta S + A(x) - \beta(x)f(x,I)S(t,x) - \mu(x)S(t,x), \quad t > 0, x \in \Omega, \\
\frac{\partial C(t,x)}{\partial t} &= D \Delta C + \beta(x)f(x,I)S(t,x) - [\theta(x) + \mu(x)]C(t,x), \quad t > 0, x \in \Omega, \\
\frac{\partial I(t,x)}{\partial t} &= D \Delta I + \delta(x)C(t,x) - [\delta(x) + \mu(x)]I(t,x), \quad t > 0, x \in \Omega, \\
\frac{\partial R(t,x)}{\partial t} &= D \Delta R + \delta(x)I(t,x) - \mu(x)R(t,x), \quad t > 0, x \in \Omega,
\end{aligned}
\]

with the homogeneous Neumann boundary conditions

\[
\frac{\partial S(t,x)}{\partial n} = \frac{\partial C(t,x)}{\partial n} = \frac{\partial I(t,x)}{\partial n} = \frac{\partial R(t,x)}{\partial n} = 0, \quad t > 0, x \in \partial \Omega, \tag{2}
\]

and the initial conditions

\[
\begin{aligned}
S(0,x) &= \phi_1(x), C(0,x) = \phi_2(x), \\
I(0,x) &= \phi_3(x), R(0,x) = \phi_4(x),
\end{aligned} \tag{3}
\]

for \(x \in \Omega\). \(S(t,x), C(t,x), I(t,x)\) and \(R(t,x)\) in system (1) stand for the density of susceptible users, collectors, infective users and refractory users at position \(x\) and time \(t\), respectively. The model above is based on the assumption that the self-contained environment is spatially heterogenous, that is, the parameters \(A(x), \mu(x), \beta(x), \delta(x)\) and \(\theta(x)\) are space dependent which account for the rates of natural entry of network users, natural exit of network users, rumor collection, rumor recovery and rumor conversion from collectors to infective users. Moreover, the movement of network users is subject to Fick’s law of diffusion, and the diffusion rate of susceptible users, collectors, infective users and refractory users is the same and is a constant \(D\). The function \(f(x,I)S\) represents the general nonlinear incidence rate. The homogeneous Neumann boundary conditions indicate that no users can move across the boundary \(\partial \Omega\). The symbol \(\Delta = \frac{\partial^2}{\partial x^2}\) denotes the Laplace operator. The initial condition (3) is nonnegative functions.

We work under the following hypotheses similar to Refs.[25, 4, 33]:

(H1) \(A(x), \mu(x), \beta(x), \delta(x)\) and \(\theta(x)\) are Hölder continuous functions which are strictly positive and uniformly bounded on \(\overline{\Omega}\).

(H2) \(f(\cdot, I) \in C^1(\Omega \times \mathbb{R}_+)\) with \(f(x,0) = 0\), \(f(0) > 0\) and \(f(x, I) > 0\) for all \(x \in \Omega, I > 0\).

(H3) \(I/f(\cdot, I)\) is a monotonically increasing function for \((x, I) \in \Omega \times (0, +\infty)\).
Let \( X \) therefore it is sufficient to study the following system in later discussion

\[
\begin{align*}
\frac{\partial S(t,x)}{\partial t} & = D \Delta S + A(x) - \beta(x)f(x, I)S(t,x) - \mu(x)S(t,x), & t > 0, x \in \Omega, \\
\frac{\partial C(t,x)}{\partial t} & = D \Delta C + \beta(x)f(x, I)S(t,x) - [\theta(x) + \mu(x)]C(t,x), & t > 0, x \in \Omega, \\
\frac{\partial I(t,x)}{\partial t} & = D \Delta I + \theta(x)C(t,x) - [\delta(x) + \mu(x)]I(t,x), & t > 0, x \in \Omega,
\end{align*}
\]

with the homogeneous Neumann boundary conditions

\[
\frac{\partial S(t,x)}{\partial \mathbf{n}} = \frac{\partial C(t,x)}{\partial \mathbf{n}} = \frac{\partial I(t,x)}{\partial \mathbf{n}} = 0, \quad t > 0, x \in \partial \Omega,
\]

and the initial conditions

\[
S(0,x) = \phi_1(x) \geq 0, C(0,x) = \phi_2(x) \geq 0, I(0,x) = \phi_3(x) \geq 0.
\]

### 2.2. Preliminaries

In this subsection, we first give some notations and some preliminaries which will be used frequently hereafter through Refs.\[17,4,16,40\]. Normally, denote by \( \mathbb{R}^3_+ \) the positive cone in \( \mathbb{R}^3 \), where

\[
\mathbb{R}^3_+ := \{ \omega = (S,C,I)^T \in \mathbb{R}^3 | S \geq 0, C \geq 0, I \geq 0 \}.
\]

Define the supremum norm of the Banach space \( \mathcal{X} := C(\overline{\Omega}, \mathbb{R}^3) \) as

\[
\|\varpi\|_{\mathcal{X}} := \max \left\{ \sup_{x \in \Omega} |\varpi_1(x)|, \sup_{x \in \Omega} |\varpi_2(x)|, \sup_{x \in \Omega} |\varpi_3(x)| \right\}, \varpi = (\varpi_1, \varpi_2, \varpi_3) \in \mathcal{X}.
\]

Let \( \mathcal{X}_+ := C(\overline{\Omega}, \mathbb{R}^3_+) \), then \( (\mathcal{X}, \mathcal{X}_+) \) constitute a strongly ordered space.

Denote the compact and strongly positive operator by \( T_i(t) : C(\overline{\Omega}, \mathbb{R}) \to C(\overline{\Omega}, \mathbb{R}_+) \) \((i = 1, 2, 3)\) the \( C_0 \)-semigroups associated with the operators \( D \Delta - \rho_i(\cdot) \) subject to Neumann boundary condition by Ref.\[23\], where \( \rho_1(x) = \mu(x), \rho_2(x) = \theta(x) + \mu(x) \) and \( \rho_3(x) = \delta(x) + \mu(x) \) respectively, and

\[
T_i(t) \vartheta(x) = \int_\Omega G_i(t,x,y) \vartheta(y) dy, \quad i = 1, 2, 3,
\]

\( G_i(t,x,y) \) \((i = 1, 2, 3)\) are the Green function associated with \( D \Delta - \rho_i(\cdot) \) subject to Neumann boundary condition and \( \vartheta \in C(\overline{\Omega}, \mathbb{R}) \) by Ref.[22].

Thus, system \((4)-(6)\) can be rewritten as the integral equations below

\[
\begin{align*}
S(t,\cdot,\varphi) & = T_1(t)\phi_1 + \int_0^t T_1(t-s)F_1(u(s,\cdot,\varphi)) ds, \\
C(t,\cdot,\varphi) & = T_2(t)\phi_2 + \int_0^t T_2(t-s)F_2(u(s,\cdot,\varphi)) ds, \\
I(t,\cdot,\varphi) & = T_3(t)\phi_3 + \int_0^t T_3(t-s)F_3(u(s,\cdot,\varphi)) ds,
\end{align*}
\]

where \( u(t,\cdot,\varphi) = (S(t,\cdot,\varphi), C(t,\cdot,\varphi), I(t,\cdot,\varphi))^T \) is the solution of system \((4)\), and

\[
\begin{align}
F_1(\varphi)(x) & = A(x) - \beta(x)f(x, \phi_3(x)) \phi_1(x), \\
F_2(\varphi)(x) & = \beta(x)f(x, \phi_3(x)) \phi_1(x), \\
F_3(\varphi)(x) & = \theta(x)\phi_2(x),
\end{align}
\]

for every initial value function \( \phi(x) = (\phi_1(x), \phi_2(x), \phi_3(x)) \in \mathcal{X}_+ \).
2.3. The well-posedness of the solutions. In this subsection, on the existence and uniqueness of global solutions as well as the ultimate boundedness for system (4)-(6), we can establish the following results. The similar conclusion as those in Refs. [17, 16] implies that the subsequent lemma holds.

**Lemma 1.** For the following scalar reaction-diffusion equations

\[
\begin{aligned}
\frac{\partial y(t,x)}{\partial t} &= D \Delta y(t,x) + A(x) - \mu(x)y(t,x), \quad t > 0, x \in \Omega, \\
\frac{\partial y(t,x)}{\partial n} &= 0, \quad t > 0, x \in \partial \Omega, \\
y(0, x) &= y_0(x) \geq 0, \quad x \in \Omega.
\end{aligned}
\]  

(9)

A standard analysis as in Ref. [16] claims that Eq. (9) admits a unique positive steady state \( y^* \), which is globally asymptotically stable in \( C(\Omega, \mathbb{R}) \). In particular, if \( A(\cdot) \equiv A, \mu(\cdot) \equiv \mu \) are positive constants, then \( y^* = \frac{A}{\mu} \).

In the following, we will study the existence and uniqueness of the global classical solution for system (4)-(6) based on the method in Refs. [34, 13].

**Theorem 1.** If the conditions \((H_{11}) - (H_{13})\) are satisfied, then system (4)-(6) has a unique globally nonnegative and bounded solution \((S(t,x), C(t,x), I(t,x))\) defined on \([0, \infty) \times \Omega\) with initial function \( \phi(x) \in \mathcal{X}_+ \).

**Proof.** According to Ref. [23] and the assumptions \((H_{11}) - (H_{13})\), for any initial function \( \phi(x) \in \mathcal{X}_+ \), system (4)-(6) admits a unique continuously differentiable classical solution \((S(t,x), C(t,x), I(t,x)) \in \mathcal{X}_+ \) on \([0, \tau_\infty) \times \Omega\), and \( S(t,x) \geq 0, C(t,x) \geq 0, I(t,x) \geq 0 \) by Ref. [1]. To show that \((S(t,x), C(t,x), I(t,x))\) is a global solution of system (4)-(6), we now try to prove that \( \tau_\infty = \infty \). For this purpose, we only need to prove the upper bound of \((S(t,x), C(t,x), I(t,x))\) according to Ref. [23].

Denote \( Y(t,x) = S(t,x) + C(t,x) + I(t,x) \), from the equations of \( S, C \) and \( I \) in system (4), it arrives

\[
\begin{aligned}
\frac{\partial Y(t,x)}{\partial t} &= D \Delta Y(t,x) + A(x) - \delta(x)I(t,x) - \mu(x)Y(t,x), \\
&\leq D \Delta Y(t,x) + A(x) - \mu(x)Y(t,x), \\
&\leq D \Delta Y(t,x) + \overline{A} - \mu Y(t,x),
\end{aligned}
\]  

(10)

where \( \overline{A} = \max_{x \in \Omega} A(x), \mu = \min_{x \in \Omega} \mu(x) \). Using Lemma 1 and the comparison theorem [21], we have

\[
\limsup_{t \to +\infty, x \in \Omega} Y(t,x) \leq \frac{\overline{A}}{\mu}.
\]

That is to say, \( Y(t,x) \) is bounded combining the nonnegative of \( S(t,x), C(t,x) \) and \( I(t,x) \). Thus, there exists positive constants \( \kappa_1, \kappa_2 \) and \( \kappa_3 \) such that \( 0 \leq S(t,x) \leq \kappa_1, \quad 0 \leq C(t,x) \leq \kappa_2 \) and \( 0 \leq I(t,x) \leq \kappa_3 \). As a consequence, \( \tau_\infty = \infty \). This finishes the proof.

2.4. Basic reproduction number. In this subsection, we define the basic reproduction number of system (4) and show its properties based on Refs. [32, 7, 28, 26].
In order to get the basic reproduction number, we first need to find the rumor-free steady state. Setting \( I(t, x) = 0 \) in the \( S(t, x) \) equation in (4), we obtain

\[
\begin{aligned}
\frac{\partial S(t, x)}{\partial t} &= D \Delta S(t, x) + A(x) - \mu(x)S(t, x), \quad t > 0, x \in \Omega, \\
\frac{\partial S(t, x)}{\partial n} &= 0, \quad t > 0, x \in \partial \Omega.
\end{aligned}
\] (11)

According to Lemma 1, system (4) has a unique rumor-free steady state \( E_0(x) = (S_0(x), 0, 0) \), where \( S_0(x) \) is a positive solution of

\[
\begin{aligned}
- D \Delta S(t, x) &= A(x) - \mu(x)S(t, x), \quad t > 0, x \in \Omega, \\
\frac{\partial S(t, x)}{\partial n} &= 0, \quad t > 0, x \in \partial \Omega.
\end{aligned}
\] (12)

Let \( F(x, I, S) \) be the input rate of newly infective network users, \( V(x, I, S, C) \) be the transfer rate of network users, then we obtain

\[
F(x, S, I) = \begin{pmatrix}
0 \\
\beta(x)f(x, I)S(t, x)
\end{pmatrix},
\]

and

\[
V(x, S, C, I) = \begin{pmatrix}
\beta(x)f(x, I)S(t, x) + \mu S(t, x) - A(x) \\
[\gamma(x) + \mu(x)]C(t, x) \\
[\delta(x) + \mu(x)]I(t, x) - \theta(x)C(t, x)
\end{pmatrix}.
\]

Under assumption \( H_{12} \), we can calculate the Jacobian matrices of \( F(x) \) and \( V(x) \) at \( E_0(x) \) respectively

\[
DF = \begin{pmatrix}
F(x) & 0 \\
0 & 0
\end{pmatrix}, \quad DV = \begin{pmatrix}
V(x) & 0 \\
J(x) & -M^0(x)
\end{pmatrix},
\]

where

\[
F(x) = \begin{pmatrix}
0 & \beta(x)S_0(x)f_1(0) \\
0 & 0
\end{pmatrix}, \quad V(x) = \begin{pmatrix}
\theta(x) + \mu(x) & 0 \\
-\theta(x) & \delta(x) + \mu(x)
\end{pmatrix}.
\]

According to assumption \( (H_{12}) \), linearizing system (4) at \( E_0(x) \), we obtain the linear system of variables related to rumor transmission as follows

\[
\begin{aligned}
\frac{\partial u_I(t, x)}{\partial t} &= D \Delta u_I + F(x)u_I(t, x) - V(x)u_I(t, x), \quad t > 0, x \in \Omega, \\
\frac{\partial u_I}{\partial n} &= 0, \quad t > 0, x \in \partial \Omega,
\end{aligned}
\] (13)

where \( u_I(t, x) = (C(t, x), I(t, x))^T \), \( D = diag\{D, D\} \). Substituting \( u_I(t, x) = e^{M^0}\phi(x) \) into system (13), it gives rise to the following eigenvalue problem

\[
\begin{aligned}
\lambda \phi(x) &= D \Delta \phi(x) + F(x)\phi(x) - V(x)\phi(x), \quad t > 0, x \in \Omega, \\
\frac{\partial \phi(x)}{\partial n} &= 0, \quad t > 0, x \in \partial \Omega,
\end{aligned}
\] (14)

where \( \phi = (\phi_2, \phi_3) \in C(\overline{\Omega}, \mathbb{R}_+ \times C(\overline{\Omega}, \mathbb{R}_+) \).

It is easy to verify that system (13) is cooperative. Obviously, system (14) has a simple principal real eigenvalue \( \lambda_0 = \lambda_{00}(D, S_0(x)) \) associated with a strictly positive eigenvector \( \phi_0(x) = (\phi_{02}(x), \phi_{03}(x))^T \), and \( Re(\lambda) < \lambda_0 \) for any other eigenvalue of system (14) based on Refs.[16, 31, 23].
Obviously, \( T(t) : C(\bar{\Omega}, \mathbb{R}^2) \to C(\bar{\Omega}, \mathbb{R}^2) \) is the strongly continuous semigroup generated by the following system

\[
\begin{cases}
\frac{\partial u_I(t, x)}{\partial t} = D \Delta u_I - V(x)u_I(t, x), & t > 0, x \in \Omega, \\
\frac{\partial u_I(t, x)}{\partial n} = 0, & t > 0, x \in \partial \Omega,
\end{cases}
\]  

(15)

Define the linear operator

\[
L(\phi)(x) := \int_{0}^{+\infty} F(x)T(t)\phi(x)dt = F(x) \int_{0}^{+\infty} T(t)\phi(x)dt.
\]

According to Refs.[32, 7, 28, 26], the spectral radius of \( L \) can be defined as the basic reproduction number \( R_0 \) of system (4). That is

\[
R_0 := r(L).
\]

In the following proposition, we will make a connection between \( R_0 \) and \( \lambda_0 \) by Refs.[31, 32, 7].

**Proposition 1.** \( R_0 - 1 \) has the same sign as \( \lambda_0 \).

**Remark 1.** Here, we point out that in Ref.[31], the above procedure of defining the basic reproduction number for reaction-diffusion systems has become standardized recently. Note that if the parameters are spatially dependent, it is mathematically difficult to express the basic reproduction number \( R_0 \) in an explicit formula. But when all parameters are location independent (spatially homogeneous), we can derive the explicit formula of \( R_0 \) by Ref.[31] and the similar analysis as in Ref.[34].

**Theorem 2.** Assume that \( \beta(x) \equiv \beta, A(x) \equiv A, \theta(x) \equiv \theta, \mu(x) \equiv \mu \) and \( \delta(x) \equiv \delta \) are all positive constants in system (4), then the basic reproduction number \( R_0 \) is given as

\[
R_0 = \frac{\beta \theta S_0 f_I(0)}{(\delta + \mu)(\theta + \mu)},
\]

where \( S_0 = \frac{A}{\mu} \).

**Proof.** According to \( \beta(x) \equiv \beta, A(x) \equiv A, \theta(x) \equiv \theta, \mu(x) \equiv \mu \) and \( \delta(x) \equiv \delta \), we obtain that \( F(x) \) and \( V(x) \) which we defined before can be degenerated to the constant matrices below

\[
F = \begin{pmatrix} 0 & \beta S_0 f_I(0) \\ 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} \theta + \mu & 0 \\ -\theta & \delta + \mu \end{pmatrix}.
\]

For any \( \varepsilon > 0 \) and \( \forall \phi \in X \), define

\[
F_\varepsilon(\phi) = F(\phi) + \varepsilon \phi, \quad L_\varepsilon(\phi) = F_\varepsilon \left( \int_{0}^{+\infty} [T(t)\phi]dt \right).
\]

Together with the Krein-Rutman theorem and Ref.[34], we have

\[
L_\varepsilon(\alpha) = T_\varepsilon(\alpha), \quad \forall \alpha = (\alpha_1, \alpha_2)^T \in \mathbb{R}^2,
\]

where \( T_\varepsilon = F_\varepsilon V^{-1} = (F + \varepsilon E)V^{-1} \). A direct calculation yields

\[
T_\varepsilon = \begin{pmatrix} \frac{\varepsilon}{\delta + \mu} + \frac{\beta \theta S_0 f_I(0)}{(\delta + \mu)(\theta + \mu)} & \frac{\beta \theta S_0 f_I(0)}{\delta + \mu} \\ \frac{\beta \theta S_0 f_I(0)}{(\delta + \mu)(\theta + \mu)} & \frac{\beta \theta S_0 f_I(0)}{\delta + \mu} \end{pmatrix}.
\]  

(16)
According to the uniqueness of the principal eigenvalue confirmed in our problem, it then follows that $r(L_{ε}) = r(T_{ε})$. Setting $ε \to 0^+$, it is easy to show that

$$R_0 = r(L_{ε}) = r(T_{ε}) = \frac{βθS_0 f_I(0)}{(δ + μ)(θ + μ)}.$$ 

This finishes the proof. □

**Remark 2.** It is worth mentioning that if we let $μ(x) = d_T$, $θ(x) + μ(x) = d_I$, $δ(x) + μ(x) = d_V$ and $f(x, I) = I$, $R_0$ will degenerate into the form of the basic reproduction number in Ref.[13]. It is well known that the basic reproduction number $R_0$ in heterogeneous environments plays an important role in numerical simulation. Next, we would like to make an estimate of the value of $R_0$, which is not mentioned in Ref.[13]. To avoid getting too complicated, suppose that $β(x)$ is spatially dependent and the rest of the parameters are constant. Adopting the similar analysis as in Ref.[50], one further achieves the subsequent theorem.

**Theorem 3.** Define

$$β^m = \min_{x \in Ω} β(x), β^M = \max_{x \in Ω} β(x),$$

we obtain

$$R_0 \in \left[ \frac{θA f_I(0) β^m}{μ(δ + μ)(θ + μ)}, \frac{θA f_I(0) β^M}{μ(δ + μ)(θ + μ)} \right].$$

**(17)**

**Proof.** Consider the following eigenvalue problem

$$\begin{cases}
-D Δ ψ_1 + (θ + μ)ψ_1 = \frac{1}{s} \frac{β(x)}{μ} ψ_2, & t > 0, x \in Ω, \\
-D Δ ψ_2 + (δ + μ)ψ_2 - θψ_1 = 0, & t > 0, x \in Ω, \\
\frac{∂ψ_1}{∂n} = \frac{∂ψ_2}{∂n} = 0, & t > 0, x \in ∂Ω, 
\end{cases}$$

(18)

According to Ref.[15], $R_0 = s_*$, where $s_*$ is a unique principal eigenvalue of problem (18), and $R_0$ increases monotonically with respect to $β(x)$ based on Ref.[50] and the results given in Theorem 2. Therefore, the value of $R_0$ should be between the value of $R_0$ when $β(x)$ is at its minimum and the value of $R_0$ when $β(x)$ is at its maximum. This finishes the proof. □

In next subsection, we are ready to prove the stability of the rumor-free steady state, including the locally and the globally asymptotic stability.

### 2.5. Extinction of rumor.

For cooperative system (4)-(6), from the perspective of the application background for the basic reproduction number, we have the following results about the local stability of the rumor-free steady state $E_0(x)$ by Ref.[31].

**Lemma 2.** Let $R_0$ be defined as in the preceding subsection, then the following statements hold.

(i) If $R_0 < 1$, then the rumor-free steady state $E_0(x)$ is locally asymptotically stable for system (4).

(ii) If $R_0 > 1$, then the rumor-free steady state $E_0(x)$ is unstable for system (4).

We are now in a position to establish the following theorem on the global stability of the rumor-free steady state $E_0$.

**Theorem 4.** If $R_0 < 1$, the rumor-free steady state $E_0(x)$ is globally asymptotically stable.
Proof. It can be calculated directly that
\[ \frac{\partial S}{\partial t} = D\triangle S + A(x) - \beta(x)f(x, I)S(t, x) - \mu(x)S(t, x) \leq D\triangle S(t, x) + A(x) - \mu(x)S(t, x), \]
for \( x \in \Omega, t > 0 \). By applying the comparison principle, it follows that \( \lim_{t \to +\infty} \sup S(t, x) \leq S_0(x) \). That is to say, for \( \forall \eta_0 > 0 \) (It may as well suppose that \( \eta_0 \) is small enough), \( \exists \theta_0 > 0 \) such that \( S(t, x) \leq S_0(x) + \eta_0 \), for all \( t \geq t_0, x \in \overline{\Omega} \).

In addition, according to assumption \((H_{13})\), it is easily seen that \( f(\cdot, I)/I \) monotonically decreasing for \((x, I) \in \Omega \times (0, +\infty)\). This result, together with the definition of the derivative, gives rise to the following inequality
\[ \frac{f(\cdot, I)}{I} \leq \lim_{t \to 0} \frac{f(\cdot, I)}{I} = f(\cdot, 0). \] (19)

Thus,
\[ \begin{cases}
\frac{\partial C(t, x)}{\partial t} \leq D\triangle C + \beta(x)[S_0(x) + \eta_0]f(0)I(t, x) - [\theta(x) + \mu(x)]C(t, x), & t > t_0, x \in \Omega, \\
\frac{\partial I(t, x)}{\partial t} = D\triangle I + \theta(x)C(t, x) - [\delta(x) + \mu(x)]I(t, x), & t > t_0, x \in \Omega, \\
\frac{\partial t}{\partial n} = \frac{\partial (I(t, x))}{\partial n} = 0, & t > t_0, x \in \partial \Omega. 
\end{cases} \] (20)

By the comparison principle, we have
\[ (C(t, x), I(t, x)) \leq \overline{M}[(\phi_{\eta_0}(x), \psi_{\eta_0}(x))e^{\lambda_0(\eta_0)(t-t_0)}], \quad \forall t \geq t_0, \]
where \( \overline{M} \) is a large enough constant, \( \lambda_0(\eta_0) = \lambda_0(D, S_0(x) + \eta_0) \), and \((\phi_{\eta_0}(x), \psi_{\eta_0}(x))\) is the positive eigenvector corresponding to the principal eigenvalue \( \lambda_0(\eta_0) \) based on system \((20)\) when the first equation in system \((20)\) takes the equal sign.

If \( R_0 < 1 \), then combining the continuity of \( \lambda_0 \) and Proposition 1, we have
\[ \lim_{\eta_0 \to 0} \lambda_0(\eta_0) = \lambda_0(D, S_0(x)) < 0. \]

In addition, we have proved \( C(t, x) \geq 0 \) and \( I(t, x) \geq 0 \) before. Thus, we can obtain that \( \lim_{t \to +\infty} (C(t, x), I(t, x)) = (0, 0) \). In the following, we think about the convergence of \( S(t, x) \).

Let \( \overline{S}(t, x) = S(t, x) - S_0(x) \), one has
\[ \begin{cases}
\frac{\partial \overline{S}(t, x)}{\partial t} \leq D\triangle \overline{S} - \mu(x)\overline{S}(t, x), & t > 0, x \in \Omega, \\
\frac{\partial \overline{S}(t, x)}{\partial n} = 0, & t > 0, x \in \partial \Omega, \\
\overline{S}(0, x) = S(0, x) - S_0(x), & x \in \Omega. 
\end{cases} \] (21)

Similar to the proof above. According to the comparison principle, we have \( \overline{S}(t, x) \to 0 \) uniformly for \( x \in \overline{\Omega} \) as \( t \to \infty \). That is, \( \lim_{t \to +\infty, x \in \pi} S(t, x) = S_0(x) \).

Consequently, the rumor-free steady state \( E_0(x) \) is globally asymptotically stable. This finishes the proof.

In the above statement, we prove the global asymptotic stability of the rumor-free steady state \( E_0(x) \) by means of definition. In the following, we will give a new condition for the global stability of the rumor-free steady state \( E_0(x) \) based on the Lyapunov stability theory.
Theorem 5. Denote

\[ P = \max_{x \in \Omega} \left\{ \frac{\kappa_1 f_I(0) \beta(x) \theta(x)}{\theta(x) + \mu(x) \delta(x) + \mu(x)} \right\}. \]

If \( P < 1 \) holds, then the rumor-free steady state \( E_0(x) \) of system (4) is globally asymptotically stable.

Proof. Consider the following Lyapunov function

\[ V_1 = \int_{\Omega} \frac{\theta(x)}{\theta(x) + \mu(x)} C(t,x) dx + \int_{\Omega} I(t,x) dx. \] (22)

Calculating the time derivative of \( V_1 \) along the positive solutions of system (4), one has

\[
\frac{dV_1}{dt} \bigg|_{\Omega} = \int_{\Omega} \left\{ \frac{\theta(x)}{\theta(x) + \mu(x)} \Delta C + \frac{\theta(x)}{\theta(x) + \mu(x)} \beta(x) f(x) S(t,x) - \theta(x) C(t,x) \right\} dx \\
+ \int_{\Omega} \left\{ \Delta I + \theta(x) C(t,x) - [\delta(x) + \mu(x)] I(t,x) \right\} dx \\
= \int_{\Omega} \left\{ \frac{\theta(x)}{\theta(x) + \mu(x)} \Delta C + \Delta I + \frac{\theta(x)}{\theta(x) + \mu(x)} \beta(x) f(x) S(t,x) \right\} dx \\
- \int_{\Omega} \left\{ \delta(x) + \mu(x) \right\} I(t,x) dx.
\]

According to the homogeneous Neumann boundary conditions (2) and by integrating by parts [9], there’s no doubt that the following results hold that

\[
\int_{\Omega} \Delta S(t,x) dx = 0, \\
\int_{\Omega} \Delta C(t,x) dx = 0, \\
\int_{\Omega} \Delta I(t,x) dx = 0.
\]

Thus, we have

\[
\frac{dV_1}{dt} \bigg|_{\Omega} \leq \int_{\Omega} \left\{ \frac{\kappa_1 \theta(x) \beta(x) f_I(0)}{\theta(x) + \mu(x)} I(t,x) - [\delta(x) + \mu(x)] I(t,x) \right\} dx \\
= \int_{\Omega} \left\{ \frac{\kappa_1 \theta(x) \beta(x) f_I(0)}{\theta(x) + \mu(x)} - [\delta(x) + \mu(x)] \right\} I(t,x) dx.
\]

From the condition of the theorem, it follows that \( \frac{dV_1}{dt} \leq 0 \). \( \frac{dV_1}{dt} = 0 \) if and only if \( I(t,x) = 0 \). In addition, when \( I(t,x) = 0 \), we have \( S(t,x) = S_0(x) \), \( C(t,x) = 0 \) by substituting \( I(t,x) = 0 \) into the first and second equation of system (4). According to the LaSalle’s invariance principle [11], the rumor-free steady state \( E_0(x) \) is globally asymptotically stable. This finishes the proof. \( \Box \)

Remark 3. Note that if \( A(x) \equiv A \), \( \theta(x) \equiv \theta \), \( \mu(x) \equiv \mu \) and \( \delta(x) \equiv \delta \) are all positive constants, and \( \beta(x) \) is spatially dependent, the condition for Theorem 5 can be simplified to

\[ P = \max_{x \in \Omega} \left\{ \frac{\theta A f_I(0) \beta(x)}{\mu(\theta + \mu)(\delta + \mu)} \right\} = \frac{\theta A f_I(0) \beta M}{\mu(\delta + \mu)(\theta + \mu)} < 1. \]
Combining Theorem 3, we can obtain the following equivalence relation
\[ R^M_0 < 1 \iff P < 1, \]
where \( R^M_0 = \max_{x \in \Omega} \{ R_0 \} \).

2.6. Persistence of rumor propagation. In this section, we study the results on the uniform persistence of rumor propagation with respect to \( R_0 \) which indicates that \( R_0 \) is also a threshold index for rumor persistence. The main outcome about the uniform persistence of positive solutions of system (4) is given in the following theorem.

**Theorem 6.** If \( R_0 > 1 \), then there exists a constant \( \sigma > 0 \) such that any solution \((S(t, x, \phi), C(t, x, \phi), I(t, x, \phi))\) of system (4) satisfy
\[
\lim_{t \to +\infty, x \in \Omega} \inf S(t, x, \phi) \geq \sigma, \quad \lim_{t \to +\infty, x \in \Omega} \inf C(t, x, \phi) \geq \sigma, \quad \lim_{t \to +\infty, x \in \Omega} \inf I(t, x, \phi) \geq \sigma
\]
for \( \phi = (\phi_1, \phi_2, \phi_3) \in X_+ \) with \( \phi_2 \neq 0 \) and \( \phi_3 \neq 0 \), where \( \sigma \) is defined below.

**Proof.** To obtain our results, we shall use the similar analysis as those in Refs.\[24, 35, 34\]. Modifications to the proof are necessary to ensure the accuracy of our articles. The proof is divided into the following four steps to verify the conditions in Theorem 3 in Ref.\[24\].

**Step 1:** Firstly, we want to verify the condition \((C1)\) of Theorem 3 in Ref.\[24\], which claims that the solution semiflow \( \Phi(t) = u(t, \cdot) : X_+ \to X_+ \) of system (4)-(6) admits a global attractor in \( X_+ \). Due to the boundedness of the nonnegative solutions \( u(t, x) \) for system (4)-(6). It indicates that \( \Phi(t) \) is point dissipative. Therefore, \( \Phi(t) = u(t, \cdot) \) admits a global compact attractor by the methods in Refs.\[10, 36\].

**Step 2:** Secondly, make the following notations. Define
\[ X_0 = \{ \phi = (\phi_1, \phi_2, \phi_3) \in X_+ : \phi_2 \neq 0, \phi_3 \neq 0 \}. \]
Apparently, one has
\[ \partial X_0 := X_+/X_0 = \{ \phi \in X_+ : \phi_2 \equiv 0 \text{ or } \phi_3 \equiv 0 \}. \]
According to the second and third equation of system (4), one obtains
\[
\begin{aligned}
\frac{\partial C(t, x)}{\partial t} &\geq D\triangle C - [\theta(x) + \mu(x)] C(t, x), \quad t > 0, x \in \Omega, \\
\frac{\partial I(t, x)}{\partial t} &\geq D\triangle I - [\delta(x) + \mu(x)] I(t, x), \quad t > 0, x \in \Omega.
\end{aligned}
\]
Since the initial value \( \phi_2(x) \neq 0 \) and \( \phi_3 \neq 0 \), by applying the strong maximum principle \[21\], it follows that
\[
C(t, x) > 0, \quad I(t, x) > 0, \quad \forall t > 0, \quad x \in \Omega. \tag{23}
\]
This implies that \( \Phi(t)X_0 \subseteq X_0 \). In other words, \( X_0 \) is positively invariant for the solution semiflow \( \Phi(t) \). Denote
\[ M_\phi := \{ \phi \in X_+ : \Phi(t)\phi \in \partial X_0, \forall t \geq 0 \}. \]
Let \( \omega(\phi) \) be the omega limit set of the forward orbit of \( \Phi(t) \) through \( \phi \in X_+ \), and set
\[ M_1 := \{(S_0(x), 0, 0)\}. \]
Now, we are ready to prove that $M_1$ is isolated in $\mathbb{X}_+$. For this purpose, we need to show that if $R_0 > 1$, the solution $u(t,x)$ of system (4) meets

$$
\lim_{t \to +\infty} \sup \| u(t, \cdot) - E_0(x) \|_{\mathbb{X}_+} \geq \epsilon_0,
$$

where $\epsilon_0 > 0$ is small enough. Suppose on the contrary that the conclusion above does not hold. That is to say,

$$
\lim_{t \to +\infty} \sup \| u(t, \cdot) - E_0(x) \|_{\mathbb{X}_+} < \epsilon_0.
$$

Then there exists a sufficiently large $T$ such that the following inequalities hold

$$
\begin{cases}
S_0(x) - \epsilon_0 < S(t,x) < S_0(x) + \epsilon_0, & t > T, x \in \bar{\Omega}, \\
0 < C(t,x) < \epsilon_0, & t > T, x \in \bar{\Omega}, \\
0 < I(t,x) < \epsilon_0, & t > T, x \in \bar{\Omega}.
\end{cases}
$$

According to

$$
\frac{f(\cdot, I)}{T} \geq \frac{f(\cdot, \epsilon_0)}{\epsilon_0} \geq f_I(\cdot, \epsilon_0)
$$

we have

$$
\begin{cases}
\frac{\partial C(t,x)}{\partial t} \geq D\Delta C + \beta(x)[S_0(x) - \epsilon_0]f_I(x, \epsilon_0)I(t,x) - [\theta(x) + \mu(x)]C(t,x), & t > T, x \in \Omega, \\
\frac{\partial I(t,x)}{\partial t} = D\Delta I + \theta(x)C(t,x) - [\delta(x) + \mu(x)]I(t,x), & t > T, x \in \Omega.
\end{cases}
$$

If $R_0 > 1$, then $\lambda_0(\epsilon_0) > 0$, where $\lambda_0(\epsilon_0) = \lambda_0(D, S_0(x) - \epsilon_0)$ is the principal eigenvalue based on system (27) when the first equation in system (27) takes the equal sign with the positive eigenvector $(\phi_{\epsilon_0}(x), \psi_{\epsilon_0}(x))$. By using the comparison principle, we have

$$(C(t,x), I(t,x)) \geq m(\phi_{\epsilon_0}(x), \psi_{\epsilon_0}(x))e^{\lambda_0(\epsilon_0)(t-T)}, \quad \forall t > T,$$

where $m$ is a given constant that’s small enough.

Since $\lambda_0(\epsilon_0) > 0$, one has

$$\lim_{t \to +\infty} C(t,x) = \infty, \quad \lim_{t \to +\infty} I(t,x) = \infty,$$

which is a contradiction with the boundedness of $(C(t,x), I(t,x))$. Thus, (24) holds.

**Step 3:** With $\omega(\phi)$ and $M_1$ defined above, we want to show $\cup_{\phi \in M_0} \omega(\phi) = M_1$.

(I) $\cup_{\phi \in M_0} \omega(\phi) \subset M_1$.

**Case 1.** ($I(t, \cdot, \phi) \equiv 0, \forall t \geq 0$). In this case, according to system (4), we have

$$
\begin{cases}
\frac{\partial S(t,x)}{\partial t} = D\Delta S(t,x) + A(x) - \mu(x)S(t,x), & t > 0, x \in \Omega, \\
C(t,x) \equiv 0, & t > 0, x \in \Omega, \\
\frac{\partial S(t,x)}{\partial n} = \frac{\partial C(t,x)}{\partial n} = 0, & t > 0, x \in \partial \Omega.
\end{cases}
$$

Obviously, $\lim_{t \to +\infty} S(t,x) = S_0(x)$. This therefore shows $\omega(\phi) = E_0(x), \forall \phi \in \partial M$. 

Case 2. \(I(\tilde{t}, \cdot, \phi) \neq 0\), for some \(\tilde{t} \geq 0\). In this case, by Ref.[21], one achieves that \(I(t, x) > 0\) for all \(t > \tilde{t}\) and \(x \in \Omega\). In fact, for any given \(\phi \in M_0\), it follows that \(\Phi(t) \phi \in \partial \Omega_0\), \(\forall t \geq 0\). Consequently, \(C(t, \phi) \equiv 0\) for all \(t \geq 0\). Then, we have

\[
\begin{align*}
\frac{\partial S(t, x)}{\partial t} &\leq D \Delta S + A(x) - \mu(x)S(t, x), \quad t > 0, x \in \Omega, \\
\frac{\partial I(t, x)}{\partial t} &= D \Delta I - [\delta(x) + \mu(x)] I(t, x), \quad t > 0, x \in \Omega, \\
\frac{\partial S(t, x)}{\partial n} &= \frac{\partial I(t, x)}{\partial n} = 0, \quad t > 0, x \in \partial \Omega.
\end{align*}
\]

Thus, we obtain that \(\lim_{t \to +\infty, x \in \Omega} S(t, x) = S_0(x)\) and \(\lim_{t \to +\infty, x \in \Omega} I(t, x) = 0\) [27]. That is to say, \(\omega(\phi) = E_0(x)\) for \(\forall \phi \in \partial M\).

Combining Case 1 and Case 2, we obtain that \(\cup_{\phi \in M_0} \omega(\phi) \subset M_1\).

\(\lim_{t \to +\infty, x \in \Omega} S(t, x) = S_0(x)\) for all \(t \geq 0\) by a direct calculation.

**Step 4:** Applying Ref.[24], define a continuous function \(p : X_+ \to \mathbb{R}_+\) by

\[p(\phi) := \min \left\{ \min_{x \in \Omega} \phi_2(x), \min_{x \in \Omega} \phi_3(x) \right\}, \quad \phi = (\phi_1, \phi_2, \phi_3) \in X_+.
\]

Applying, if \(p(\phi) > 0\), one has

\[\min_{x \in \Omega} \phi_2(x) > 0, \min_{x \in \Omega} \phi_3(x) > 0, \quad \phi \in X_+,
\]

this therefore implies \(p^{-1}(0, +\infty) \subset X_0\).

Further, if \(p(\phi) > 0\), according to (23), we obtain that \(p(\Phi(t)(\phi)) > 0, \forall t > 0\). Thus, \(p\) is a generalized distance function for \(\Phi(t) : X_+ \to X_0\) defined in Ref.[24].

Define \(W^S(M_1) = \{ \phi \in X_+ : \lim_{t \to +\infty} \sup \| \Phi(t) \phi - E_0(x) \|_{X_+} = 0 \}\) be the stable set of \(M_1\). According to \(\lim_{t \to +\infty} \sup \| u(t, \cdot, \phi) - E_0(x) \|_{X_0} \geq \epsilon_0\) for \(\forall \phi \in X_0\) in step 2, it can be easily seen that \(W^s(M_1) \cap X_0 = \emptyset\). Note that \(p^{-1}(0, +\infty)\) is a subset of \(X_0\), it follows that \(W^s(M_1) \cap p^{-1}(0, +\infty) = \emptyset\). Now, the last condition of Theorem 3 in Ref.[24] is satisfied.

So far, all the conditions in Theorem 3 in Ref.[24] have been verified. Thus we can find a \(\sigma_1 > 0\) satisfying

\[\lim_{t \to +\infty, x \in \Omega} \inf C(t, x, \phi) \geq \sigma_1, \quad \lim_{t \to +\infty, x \in \Omega} \inf I(t, x, \phi) \geq \sigma_1, \quad \text{for } \forall \phi \in X_0.
\]

Further considering

\[\frac{\partial S(t, x)}{\partial t} = D \Delta S + A(x) - \beta(x)f(x, I)S(t, x) - \mu(x)S(t, x),
\]

\[\geq D \Delta S + \underline{A} - (\| \mu \| + m \| \beta \| )S(t, x),
\]

where \(\underline{A} = \min_{x \in \Omega} A(x), m = \max_{t \in [0, \| S_0(x) \|]} f(I)\), we have

\[\lim_{t \to +\infty, x \in \Omega} \inf S(t, x, \phi) \geq \frac{\underline{A}}{\| \mu \| + m \| \beta \| } \equiv \sigma_2.
\]
By choosing the sufficiently small $\sigma := \min\{\sigma_1, \sigma_2\}$, we have
\[
\lim_{t \to +\infty, x \in \Omega} \inf S(t, x, \phi) \geq \sigma, \quad \lim_{t \to +\infty, x \in \Omega} \inf C(t, x, \phi) \geq \sigma, \quad \lim_{t \to +\infty, x \in \Omega} \inf I(t, x, \phi) \geq \sigma.
\]
This finishes the proof. \qed

By the general results in Ref. [42], the following result about the existence of rumor-prevailing steady state of system (4) is valid.

**Theorem 7.** If $R_0 > 1$, then system (4) has at least one positive rumor-prevailing steady state $E_* (x) = (S_*(x), C_*(x), I_*(x))$.

**Remark 4.** What we need to point out here is that we now have only obtained the existence of rumor-prevailing steady state for system (4). In order to further explore the dynamic characteristics of system (4), we need to study the stability of the rumor-prevailing steady state. However, due to the diversity and complexity of the rumor-prevailing steady state, it is a pity that no effective conclusions are given in the general spatial heterogeneous model. In numerical simulation, the spatiotemporal complex dynamics of rumor-prevailing steady state will be further demonstrated.

### 3. Spatially homogeneous model.

#### 3.1. Model formulation.

In this section, we consider the special form of spatial heterogeneous system: when all system parameters are constant, that is, spatial homogeneous system. We will further discuss the above problems and give more accurate results. Meanwhile, we select a specific expression of the abstract function $f(x, I) = \frac{I(t, x)}{1 + \alpha I(t, x)}$ for theoretical analysis which also satisfies the assumption $(H_{12})$ and $(H_{13})$. In addition, we will consider the influence of time delay. Before the collector gather information, he/she will first judge whether the rumor has collecting value. There will obviously be an identification time, which is expressed by $\tau$. Based on the above statements, the system to be studied later takes the following form

\[
\begin{aligned}
\frac{\partial S(t, x)}{\partial t} &= DS(t, x) + A - \frac{\beta S(t, x)I(t, x)}{1 + \alpha I(t, x)} - \mu S(t, x), & t > 0, x \in \Omega, \\
\frac{\partial C(t, x)}{\partial t} &= DC(t, x) + \frac{\beta S(t - \tau, x)I(t - \tau, x)}{1 + \alpha I(t - \tau, x)}e^{-\rho \tau} - (\theta + \mu)C(t, x), & t > 0, x \in \Omega, \\
\frac{\partial I(t, x)}{\partial t} &= DI(t, x) + \theta C(t, x) - (\delta + \mu)I(t, x), & t > 0, x \in \Omega, \\
\frac{\partial R(t, x)}{\partial t} &= DR(t, x) + \delta I(t, x) - \mu R(t, x), & t > 0, x \in \Omega, \\
\frac{\partial S(t, x)}{\partial n} &= \frac{\partial C(t, x)}{\partial n} = \frac{\partial I(t, x)}{\partial n} = \frac{\partial R(t, x)}{\partial n} = 0, & t > 0, x \in \partial \Omega,
\end{aligned}
\]

(29)

with initial conditions

\[
\begin{aligned}
S(t, x) &= \phi_1(\theta, x) \geq 0 \text{ and } \neq 0, \quad (\theta, x) \in [-\tau, 0] \times \Omega, \\
C(t, x) &= \phi_2(\theta, x) \geq 0 \text{ and } \neq 0, \quad (\theta, x) \in [-\tau, 0] \times \Omega, \\
I(t, x) &= \phi_3(\theta, x) \geq 0 \text{ and } \neq 0, \quad (\theta, x) \in [-\tau, 0] \times \Omega, \\
R(t, x) &= \phi_4(\theta, x) \geq 0 \text{ and } \neq 0, \quad (\theta, x) \in [-\tau, 0] \times \Omega,
\end{aligned}
\]

(30)

where $D$, $A$, $\beta$, $\alpha$, $\mu$, $\theta$, $\delta$ and $\tau$ are all positive constants which have the same meanings to the parameters in system (4). We choose the closed interval $\Omega = [0, \pi]$ as the spatial region mainly for the simplicity of calculating the symbols in the standard forms and for the convenience of the numerical simulation. Without loss
of generality, for any closed interval \([a, b]\), they can be converted into \([0, \pi]\) by performing translation and rescaling.

Since the last equation is decoupled from the other three equations, thus it suffices to discuss the following subsystem

\[
\begin{align*}
\frac{\partial S(t, x)}{\partial t} &= D \Delta S(t, x) + A - \frac{\beta S(t, x)I(t, x)}{1 + \alpha I(t, x)} - \mu S(t, x), \quad t > 0, x \in \Omega, \\
\frac{\partial C(t, x)}{\partial t} &= D \Delta C(t, x) + \frac{\beta S(t - \tau, x)I(t - \tau, x)}{1 + \alpha I(t - \tau, x)} e^{-\mu \tau} - (\theta + \mu)C(t, x), \quad t > 0, x \in \Omega, \\
\frac{\partial I(t, x)}{\partial t} &= D \Delta I(t, x) + \theta C(t, x) - (\delta + \mu)I(t, x), \quad t > 0, x \in \Omega, \\
\frac{\partial S(t, x)}{\partial n} &= \frac{\partial I(t, x)}{\partial n} = 0, \quad t > 0, x \in \partial \Omega,
\end{align*}
\]

(31)

with initial conditions

\[
\begin{align*}
S(t, x) &= \phi_1(\theta, x) \geq 0 \text{ and } \neq 0, \quad (\theta, x) \in [-\tau, 0] \times \Omega, \\
C(t, x) &= \phi_2(\theta, x) \geq 0 \text{ and } \neq 0, \quad (\theta, x) \in [-\tau, 0] \times \Omega, \\
I(t, x) &= \phi_3(\theta, x) \geq 0 \text{ and } \neq 0, \quad (\theta, x) \in [-\tau, 0] \times \Omega,
\end{align*}
\]

(32)

3.2. Well-posedness and existence of the equilibrium points. In this part, we demonstrate the existence, positivity and boundedness of the solutions for system (31). At the same time, the threshold value and the existence of equilibrium points under some necessary assumptions are investigated.

Before going further, we make the following notations:

Denote

\[
\mathcal{C} = C([-\tau, 0] \times [0, \pi], \mathbb{R}^3), \mathbb{N}_0 = \{0, 1, 2, \cdots\},
\]

\[
\mathcal{X} = \{\phi \in C^2([0, \pi], \mathbb{R}^3) : \frac{d\phi(x)}{dx} = 0 \text{ on } \partial \Omega\}.
\]

Define \(\varphi_t \in \mathcal{C}\) by

\[
\varphi_t(s, x) = \varphi(t + s, x), \forall s \in [\tau, 0], x \in \Omega,
\]

where \(\varphi : [-\tau, \tau] \times [0, \pi] \rightarrow \mathbb{R}^3\) for \(t > 0\) is a continuous function from \([0, t]\) to \(\mathcal{C}\).

\[ \textbf{Theorem 8.} \] For any initial function \(\phi(x) = (\phi_1(\theta, x), \phi_2(\theta, x), \phi_3(\theta, x)) \in \mathcal{C}\), system (31)-(32) has a unique nonnegative and ultimately bounded solution \(u(t, x) = (S(t, x), C(t, x), I(t, x))\) defined on \([0, \infty) \times \Omega\). More accurately, one has

\[
\limsup_{t \to +\infty, x \in \Omega} (S(t, x), C(t, x), I(t, x)) \leq \frac{A}{\mu} e^{-\mu t} (e^{\mu t}, 1, 1).
\]

(33)

\[ \textbf{Proof.} \] Define \(F = (F_1, F_2, F_3) : \mathcal{C} \rightarrow \mathbb{R}^3\) with any \(\phi = (\phi_1, \phi_2, \phi_3) \in \mathcal{C}\) by

\[
\begin{align*}
F_1(\phi) &= A - \frac{\beta \phi_1(0, x)\phi_3(0, x)}{1 + \alpha \phi_3(0, x)} - \mu \phi_1(0, x), \\
F_2(\phi) &= \frac{\beta \phi_1(-\tau, x)\phi_3(-\tau, x)}{1 + \alpha \phi_3(-\tau, x)} e^{-\mu \tau} - (\theta + \mu)\phi_2(t, x), \\
F_3(\phi) &= \theta \phi_2(t, x) - (\delta + \mu)\phi_3(t, x).
\end{align*}
\]

(34)

Thus, system (31)-(32) can transformed into the following abstract functional differential equation

\[
\frac{d\varphi(t)}{dt} = D\varphi + F(\varphi_t), \quad \varphi(0) = \phi \in \mathcal{C},
\]

(35)

where \(\varphi = (S, C, I)^T, D = diag\{D, D, D\}\). It is obviously that \(F\) is locally Lipschitz in \(\mathcal{C}\). By the similar argument as those in Refs.\[21, 27\], we firstly infer that system (31)-(32) has a unique local solution \(\varphi_t = (S(t, x), C(t, x), I(t, x))\) defined...
Theorem 9. For system (31), we define the value $T$ similar to the spatial heterogeneous model, then it follows from the standard theory that $C$ correspondingly applying the similar results for $\phi$. By the comparison principle, we have $H(t,x) = e^{-\mu t}S(t,x) + C(t+\tau,x) + I(t+\tau,x)$. (36) Differentiating $H(t,x)$ with respect to $t$ along the solution of system (31), a direct calculation yields

$$
\frac{\partial H(t,x)}{\partial t} = e^{-\mu t} \left( \frac{\partial S(t,x)}{\partial t} + \frac{\partial C(t+\tau,x)}{\partial t} + \frac{\partial I(t+\tau,x)}{\partial t} \right) + e^{-\mu t} \left( D \frac{\partial^2 S(t,x)}{\partial x^2} + A - \frac{\beta S(t,x)I(t,x)}{1+\alpha I(t,x)} - \mu S(t,x) \right) + D \frac{\partial^2 C(t+\tau,x)}{\partial x^2} + \frac{\beta S(t,x)I(t,x)}{1+\alpha I(t,x)} e^{-\mu t} - (\theta + \mu)C(t+\tau,x) + \frac{\partial^2 I(t+\tau,x)}{\partial x^2} + \theta C(t+\tau,x) - (\delta + \mu)I(t+\tau,x) = D \frac{\partial^2 H(t,x)}{\partial x^2} + Ae^{-\mu t} - \mu H(t,x).
$$

By the comparison principle, we have

$$
\limsup_{t \to +\infty, x \in \Omega} H(t,x) \leq \frac{A}{\mu} e^{-\mu t}.
$$

Combined with the fact that $S(t,x), C(t,x)$ and $I(t,x)$ are non-negative, by Eq.(36) we know, for any $t \geq -\tau$, $e^{-\mu t}S(t,x) \leq H(t,x)$, which indicates that

$$
\limsup_{t \to +\infty, x \in \Omega} S(t,x) \leq \limsup_{t \to +\infty, x \in \Omega} e^{\mu t}H(t,x) = \frac{A}{\mu}.
$$

Correspondingly, applying the similar results for $C(t,x)$ and $I(t,x)$, one can obtain that

$$
\limsup_{t \to +\infty, x \in \Omega} C(t,x) \leq \frac{A}{\mu} e^{-\mu t}, \quad \limsup_{t \to +\infty, x \in \Omega} I(t,x) \leq \frac{A}{\mu} e^{-\mu t}.
$$

Similar to the spatial heterogeneous model, then it follows from the standard theory that $T_\infty = +\infty$ which implies that system (31) admits a unique globally solution $(S(t,x), C(t,x), I(t,x))$. 

Theorem 9. For system (31), we define the value

$$
R^0 = \frac{\beta \theta S_0}{(\mu + \theta)(\delta + \mu)} e^{-\mu t}.
$$

Then the following statements hold.

1. System (31) always exists a rumor-free equilibrium point $E^0 = (S_0, 0, 0)$ for any feasible parameters.
2. There exists a unique rumor-prevailing equilibrium point $E^* = (S^*, C^*, I^*)$ when $R^0 > 1$. 
Proof. It can be easily shown that $E^0 = (S_0, 0, 0)$ is always an equilibrium of system (31). In this case, the number of rumor-susceptible individuals reaches the maximum level. In the following, we want to get the conditions for the existence of the rumor-prevaling equilibrium point. To the end, let the right-hand side of system (31) equal to zero, we obtain

$$
\begin{align*}
D\Delta S(t,x) + A - \frac{\beta S(t,x)I(t,x)}{1 + \alpha I(t,x)} - \mu S(t,x) &= 0, \\
D\Delta C(t,x) + \frac{\beta S(t-\tau,x)I(t-\tau,x)}{1 + \alpha I(t-\tau,x)} - e^{-\mu\tau} - (\theta + \mu)C(t,x) &= 0, \\
D\Delta I(t,x) + \theta C(t,x) - (\delta + \mu)I(t,x) &= 0.
\end{align*}
$$

A straightforward calculation yields:

$$
\begin{align*}
S^* &= \frac{(1+\alpha I^*)(\theta + \mu)(\delta + \mu)}{\beta}\mu, \\
C^* &= \frac{\delta + \mu}{\theta} I^*, \\
I^* &= \frac{A\beta e^{-\mu\tau} - \mu(\mu + \theta)(\delta + \mu)}{(\theta + \mu)(\delta + \mu)}.
\end{align*}
$$

It is easily seen that if $I^* > 0$, then $S^* > 0$ and $C^* > 0$. In order to ensure the existence of the positive equilibrium point of system (31), let $I^* > 0$, one has

$$\frac{A\beta e^{-\mu\tau}}{\mu(\mu + \theta)(\delta + \mu)} > 1.$$

Hence, we denote

$$R^0 = \frac{\beta S_0}{(\mu + \theta)(\delta + \mu)} e^{-\mu\tau}.$$

If $R^0 < 1$, this implies that system (31) always has only one rumor-free equilibrium point $E^0$. Otherwise, if $R^0 > 1$, there exists an interior equilibrium $E^* = (S^*, C^*, I^*)$ (namely rumor-prevaling equilibrium) in addition to the rumor-free equilibrium point $E^0$ which reveals the existence of susceptible users, collectors and infective users in online social networks. The proof is completed.

3.3. Stability analysis of equilibrium points. Considering that the stabilities of the equilibrium points for system (31) will directly reflect the dynamic process of rumor propagation in online networks, it is of great significance to investigate the rumor propagation model from the perspective of stability. In this section, we use relevant theories of functional differential equations to study the local and global asymptotic stability of rumor-free equilibrium $E^0$ and rumor-prevaling equilibrium $E^*$. First, we make the following statements.

Denote

$$
\begin{align*}
P(S(t,x), C(t,x), I(t,x)) &= A - \frac{\beta S(t,x)I(t,x)}{1 + \alpha I(t,x)} - \mu S(t,x), \\
Q(S(t,x), C(t,x), I(t,x)) &= \frac{\beta S(t-\tau,x)I(t-\tau,x)}{1 + \alpha I(t-\tau,x)} - e^{-\mu\tau} - (\theta + \mu)C(t,x), \\
Z(S(t,x), C(t,x), I(t,x)) &= \theta C(t,x) - (\delta + \mu)I(t,x).
\end{align*}
$$

Let $E^* = (S^*, C^*, I^*)$ be any nonnegative equilibrium points of system (31). Without loss of generality, making the transformation that $\hat{S}(t,x) = S(t,x) - S^*$, $\hat{C}(t,x) = C(t,x) - C^*$, $\hat{I}(t,x) = I(t,x) - I^*$. To simplify the notation, system
(31) can be written by dropping bars in the following form

\[
\begin{align*}
\frac{\partial S(t,x)}{\partial t} &= D\Delta S(t,x) - (\mu + \frac{\beta I^*}{1 + \alpha I^*}) S(t,x) - \frac{\beta S^*}{(1 + \alpha I^*)^2} I(t,x) + \sum_{i+j\geq 2} \frac{1}{i!j!} P_{ij} S(t,x)^i I(t,x)^j, \\
\frac{\partial C(t,x)}{\partial t} &= D\Delta C(t,x) + \frac{\beta I^*}{1 + \alpha I^*} e^{-\mu \tau} S(t - \tau, x) - (\theta + \mu) C(t,x) + \frac{\beta S^*}{(1 + \alpha I^*)^2} e^{-\mu \tau} I(t - \tau, x) \\
\frac{\partial I(t,x)}{\partial t} &= D\Delta I(t,x) + \theta C(t,x) - (\delta + \mu) I(t,x) + \sum_{i+j\geq 2} \frac{1}{i!j!} Z_{ij} C^i(t,x) I^j(t,x), \\
\end{align*}
\]

where

\[
P_{ij} = \frac{\partial^{i+j} P}{\partial S(t,x)^i \partial I(t,x)^j}, \quad Q_{hij} = \frac{\partial^{h+i+j} Q}{\partial S(t,x)^i \partial C(t,x)^j \partial I(t,x)^k}, \quad Z_{ij} = \frac{\partial^{i+j} Z}{\partial C(t,x)^i \partial I(t,x)^j}.
\]

Apparently, system (40) has a zero equilibrium point \((0,0,0)\), which is corresponding to the nonnegative equilibrium points \((S^*, C^*, I^*)\) of system (31). Therefore, the study on the stability of \(E^0\) and \(E^*\) of system (31) can be transformed into the study on the stability at the zero equilibrium point of system (40).

Denote

\[U(t) = (S(t,\cdot), C(t,\cdot), I(t,\cdot))^T,\]

so in phase space \(\mathcal{C}\), system (40) is equivalent to the following abstract differential equation

\[
d\frac{dU(t)}{dt} = D\Delta U(t) + L(U_t) + w(U_t),
\]

where \(L : \mathcal{C} \to \mathbb{R}^3\) as well as \(w : \mathcal{C} \to \mathbb{R}^3\) are defined, respectively, by

\[
L(\varphi) = \begin{pmatrix}
-\left(\mu + \frac{\beta I^*}{1 + \alpha I^*}\right) \varphi_1(0) - \frac{\beta S^*}{(1 + \alpha I^*)^2} \varphi_2(0) \\
\frac{\beta I^*}{1 + \alpha I^*} e^{-\mu \tau} \varphi_1(-\tau) - (\theta + \mu) \varphi_2(0) + \frac{\beta S^*}{(1 + \alpha I^*)^2} e^{-\mu \tau} \varphi_3(-\tau) \\
\theta \varphi_2(0) - (\delta + \mu) \varphi_3(0)
\end{pmatrix}
\]

and

\[
w(\varphi) = \begin{pmatrix}
\sum_{i+j\geq 2} \frac{1}{i!j!} P_{ij} \varphi_1^{(i)}(0) \varphi_2^{(j)}(0) \\
\sum_{h+i+j\geq 2} \frac{1}{h!i!j!} Q_{hij} \varphi_1^{(h)}(-\tau) \varphi_2^{(i)}(0) \varphi_3^{(j)}(-\tau) \\
\sum_{i+j\geq 2} \frac{1}{i!j!} Z_{ij} \varphi_2^{(i)}(0) \varphi_3^{(j)}(0)
\end{pmatrix}
\]

with \(\varphi = (\varphi_1, \varphi_2, \varphi_3)^T \in \mathcal{C}\). The linearization system of (40) at the equilibrium point \((0,0,0)\) is

\[
\dot{U} = D\Delta U(t) + L(U_t).
\]

Substituting \(U(t) = e^{\lambda t} y\) into system (44), the corresponding characteristic equation can be calculated as follows

\[
\lambda y - D\Delta y - L(e^{\lambda t} y) = 0, y \in \text{dom}(\Delta), y \neq 0, \text{dom}(\Delta) \subset \mathcal{X}.
\]

Adopting the properties of the Laplacian operator defined on \(\Omega\) with homogeneous Neumann boundary conditions, the operator \(\Delta\) has the eigenvalues \(-k^2\), where \(k \in \mathbb{N}_0\) is the wave number, with the relevant eigenfunctions defined by

\[
e_k^1 = \begin{pmatrix}
\gamma_k \\
0
\end{pmatrix}, \quad e_k^2 = \begin{pmatrix}
0 \\
\gamma_k
\end{pmatrix}, \quad e_k^3 = \begin{pmatrix}
0 \\
0
\end{pmatrix}, \quad \gamma_k = \cos(kx).
\]
Note that the sequence \( \{e_1^i, e_2^i, e_3^i\}_{i=1}^{\infty} \) constitute a basis of the space \( \mathcal{X} \). Thus, any element \( y \) in \( \mathcal{X} \) can be expanded into the following Fourier series

\[
y = \sum_{k=0}^{\infty} Y_k^T \begin{pmatrix} e_1^k \\ e_2^k \\ e_3^k \end{pmatrix}, \quad Y_k = \begin{pmatrix} \langle y, e_1^k \rangle \\ \langle y, e_2^k \rangle \\ \langle y, e_3^k \rangle \end{pmatrix},
\]

(47)

where \( \langle y, e_i^k \rangle \) is defined as the inner product of \( y \) and \( e_i^k \) in \( \mathcal{X} \) \( (i = 1, 2, 3) \).

Through the appropriate calculation, it follows that

\[
L(Y_k^T \begin{pmatrix} e_1^k \\ e_2^k \\ e_3^k \end{pmatrix}) = L(Y_k)^T \begin{pmatrix} e_1^k \\ e_2^k \\ e_3^k \end{pmatrix}, \quad \Delta y = -\sum_{k=0}^{\infty} k^2 Y_k^T \begin{pmatrix} e_1^k \\ e_2^k \\ e_3^k \end{pmatrix}.
\]

(48)

Based on the above discussion, and combining (47) and (48), the characteristic equation of the linearized system of (40) at \((0,0,0)\) is offered as follows

\[
\sum_{k=0}^{\infty} Y_k^T \left[ \lambda E_3 + k^2 D - \begin{pmatrix} (-\mu + \frac{\beta I^*}{1 + \alpha I^*}) & 0 & \frac{\beta S^*}{(1 + \alpha I^*)^2} \\ \frac{\beta I^*}{1 + \alpha I^*} e^{-\mu \tau - \lambda \tau} & 0 & \frac{\beta S^*}{(1 + \alpha I^*)^2} e^{-\mu \tau - \lambda \tau} \\ 0 & \theta & \lambda + Dk^2 + \theta + \mu \end{pmatrix} \right] \begin{pmatrix} e_1^k \\ e_2^k \\ e_3^k \end{pmatrix} = 0.
\]

(49)

Consequently, the characteristic equation of the linearized system of (40) at \((0,0,0)\) is offered as follows

\[
\begin{vmatrix}
\lambda + Dk^2 + \mu + \frac{\beta I^*}{1 + \alpha I^*} & 0 & \frac{\beta S^*}{(1 + \alpha I^*)^2} \\
-\frac{\beta I^*}{1 + \alpha I^*} e^{-\mu \tau - \lambda \tau} & \lambda + Dk^2 + \theta + \mu & -\frac{\beta S^*}{(1 + \alpha I^*)^2} e^{-\mu \tau - \lambda \tau} \\
0 & \theta & \lambda + Dk^2 + \delta + \mu
\end{vmatrix} = 0.
\]

(50)

Next, we shall give the local and global asymptotic stability of the rumor-free equilibrium point \( E^0 \) in the following theorems.

**Theorem 10.** For system (31), and \( \mathcal{R}^0 \) is defined as (37), the following statements hold.

(i) If \( \mathcal{R}^0 \leq 1 \), then the rumor-free equilibrium point \( E^0 \) is locally asymptotically stable.

(ii) If \( \mathcal{R}^0 > 1 \), the rumor-free equilibrium point \( E^0 \) is unstable.

**Proof.** Through the appropriate calculation and simplification, the corresponding characteristic equation (50) at \( E^0 \) is given as follows

\[
(\lambda + Dk^2 + \theta + \mu)(\lambda + Dk^2 + \delta + \mu) - \beta \theta S_0 e^{-\mu \tau - \lambda \tau} = 0.
\]

(51)

It is obvious that \( \lambda = -Dk^2 - \mu \) is one eigenvalue of the characteristic equation, and the remaining eigenvalues satisfy the following equation

\[
(\lambda + Dk^2 + \theta + \mu)(\lambda + Dk^2 + \delta + \mu) - \beta \theta S_0 e^{-\mu \tau - \lambda \tau} = 0.
\]

(52)

By calculation, we have

\[
\lambda^2 + (2Dk^2 + 2\mu + \theta + \delta)\lambda + (Dk^2 + \theta + \mu)(Dk^2 + \delta + \mu) - \beta \theta S_0 e^{-\mu \tau - \lambda \tau} = 0.
\]

(53)

Assuming that \( \lambda = r_1 + i\omega_1 (\omega_1 > 0) \) is a complex root of Eq.(53). Substituting \( \lambda \) into Eq.(53), it follows that

\[
\begin{align*}
\lambda - \omega_1^2 + 2r_1\omega_1 i + (2Dk^2 + 2\mu + \theta + \delta)(r_1 + \omega_1 i) + (Dk^2 + \theta + \mu)(Dk^2 + \delta + \mu) \\
= \beta \theta S_0 e^{-\mu \tau - r_1 \tau}(\cos \omega_1 \tau - \sin \omega_1 \tau).
\end{align*}
\]
Separating the real parts and the imaginary parts, we obtain
\[
\begin{align*}
\{ & r_1^2 - \omega_1^2 + (2Dk^2 + 2\mu + \theta + \delta)r_1 + (Dk^2 + \theta + \mu)(Dk^2 + \delta + \mu) = \beta\theta S_0 e^{-\mu \tau - r_1 \tau} \cos \omega_1 \tau, \\
2r_1 \omega_1 + (2Dk^2 + 2\mu + \theta + \delta)\omega_1 = -\beta\theta S_0 e^{-\mu \tau - r_1 \tau} \sin \omega_1 \tau.
\end{align*}
\] (54)

Taking square on both sides of the two equations of (54) and summing them up, one has
\[
\omega_1^4 + a_1 \omega_1^2 = a_2,
\] (55)
where
\[
a_1 = (2Dk^2 + 2\mu + 2r_1 + \theta + \delta)^2 - 2[r_1^2 + (2Dk^2 + 2\mu + \theta + \delta)r_1 + (Dk^2 + \theta + \mu)(Dk^2 + \delta + \mu)]
\]
\[
= 2r_1^2 + 2r_1(2Dk^2 + 2\mu + \theta + \delta) + (Dk^2 + \theta + \mu)^2 + (Dk^2 + \delta + \mu)^2 > 0,
\]
\[
a_2 = \beta^2 \theta^2 S_0^2 e^{-2\mu \tau - 2r_1 \tau} - [r_1^2 + (2Dk^2 + 2\mu + \theta + \delta)r_1 + (Dk^2 + \theta + \mu)(Dk^2 + \delta + \mu)]^2.
\]

In the following, this problem will be discussed on three cases.

**Case 1:** If $R^0 < 1$, $r_1 > 0$, then $\beta^2 \theta^2 S_0^2 e^{-2\mu \tau} < (\mu + \theta)^2 (\delta + \mu)^2$ which leads to
\[
a_2 = \beta^2 \theta^2 S_0^2 e^{-2\mu \tau - 2r_1 \tau} - [r_1^2 + (2Dk^2 + 2\mu + \theta + \delta)r_1 + (Dk^2 + \theta + \mu)(Dk^2 + \delta + \mu)]^2
\]
\[
< \beta^2 \theta^2 S_0^2 e^{-2\mu \tau} - (Dk^2 + \theta + \mu)^2(Dk^2 + \delta + \mu)^2
\]
\[
< (\mu + \theta)^2 (\delta + \mu)^2 - (Dk^2 + \theta + \mu)^2(Dk^2 + \delta + \mu)^2 < 0.
\]

From (55), it follows that the left-hand side is always greater than 0, and the right-hand side is always less than 0, which is impossible.

**Case 2:** If $R^0 < 1$, $r_1 = 0$, then
\[
a_1 = (Dk^2 + \theta + \mu)^2 + (Dk^2 + \delta + \mu)^2 > 0,
\]
\[
a_2 = (Dk^2 + \theta + \mu)^2(Dk^2 + \delta + \mu)^2 - \beta^2 \theta^2 S_0^2 e^{-2\mu \tau} < 0.
\]
This therefore implies Eq. (55) has no positive roots about $\omega_1$.

**Case 3:** If $R^0 > 1$, denote
\[
G(\lambda) = \lambda^2 + (2Dk^2 + 2\mu + \theta + \delta)\lambda + (Dk^2 + \theta + \mu)(Dk^2 + \delta + \mu) - \beta\theta S_0 e^{-\mu \tau - \lambda \tau}.
\]

It is clearly that $G(0) = (Dk^2 + \theta + \mu)(Dk^2 + \delta + \mu) - \beta\theta S_0 e^{-\mu \tau - \lambda \tau}$. If $k = 0$, it follows that $G(0) < 0$. Meanwhile, we have $\lim_{\lambda \to +\infty} G(\lambda) = +\infty$. To sum up, when $k = 0$, $G(\lambda) = 0$ has a positive real root which implies that Eq. (53) has positive real root.

In conclusion, if $R^0 < 1$ holds, all roots of Eq. (53) have negative real parts which leads to the result that the rumor-free equilibrium point $E^0$ is locally asymptotically stable for $\forall \tau \geq 0$. If $R^0 > 1$, the rumor-free equilibrium $E^0$ is unstable. The proof is completed.

In the following, we explore the global asymptotic stability of the rumor-free equilibrium point $E^0$.

**Theorem 11.** If $R^0 < 1$ holds, then the rumor-free equilibrium point $E^0$ of system (31) is globally asymptotically stable.

**Proof.** Constructing the following Lyapunov function
\[
V_2(t) = \int_\Omega \left[ S(t,x) - S_0 - S_0 \ln \frac{S(t,x)}{S_0} + e^{\mu \tau} C(t,x) + k \cdot e^{\mu \tau} I(t,x) + \int_{t-\tau}^t \frac{\beta S(\rho,x) I(\rho,x)}{1 + \alpha I(\rho,x)} d\rho \right] dx,
\] (56)
where \( k_* \) is a positive constant that can be fixed in later calculations. Calculating the time derivative of \( V_2(t) \) along the positive solutions of system (31), one has

\[
\frac{dV_2(t)}{dt} \bigg|_{(31)} = \int_{\Omega} \left\{ \left( \frac{S_0}{S(t,x)} - S_0 \right)^2 + e^{\mu \tau} \left[ k_* \theta - (\theta + \mu) \right] C(t,x) + \left[ \beta S_0 - k_* e^{\mu \tau} (\mu + \delta) \right] I(t,x) \right\} dx.
\]

Choose positive constant \( k_* = \frac{\theta + \mu}{\beta} \) such that

\[
\frac{dV_2(t)}{dt} \bigg|_{(31)} \leq \int_{\Omega} -\mu \left( \frac{S(t,x) - S_0}{S(t,x)} \right)^2 dx + \int_{\Omega} \left\{ \beta S_0 - e^{\mu \tau} \frac{(\theta + \mu) (\delta + \mu)}{\theta} \right\} I(t,x) dx
\]

\[
= -\mu \int_{\Omega} \left( \frac{S(t,x) - S_0}{S(t,x)} \right)^2 dx + \int_{\Omega} \left\{ \beta S_0 \left( 1 - \frac{1}{R_0} \right) \right\} I(t,x) dx.
\]

Since \( R_0 < 1 \), then we can get \( \frac{dV_2(t)}{dt} \bigg|_{(31)} \leq 0 \). The equality \( \frac{dV_2(t)}{dt} \bigg|_{(31)} = 0 \) holds if and only for \( S(t,x) = S_0, I(t,x) = 0 \). Meanwhile, by the second equation of system (31) we can easily obtain \( C(t,x) = 0 \). According to LaSalle’s invariance principle which leads to the conclusion that the rumor-free equilibrium point \( E^0 \) is globally asymptotically stable. This completes the proof.

The result of the local asymptotic stability of the rumor-prevailing equilibrium point \( E^* \) of system (31) reads as follows.

To simplify the notation, define

\[
m = \frac{\beta I^*}{1 + \alpha I^*} > 0, \quad n = \frac{\beta \theta S^*}{(1 + \alpha I^*)^2} > 0,
\]

\[
b_1 = 3\mu + \theta + \delta + m > 0, \quad b_2 = (\mu + m)(2\mu + \theta + \delta) + (\theta + \mu)(\delta + \mu) > 0,
\]

\[
b_3 = (\mu + m)(\theta + \mu)(\delta + \mu) > 0, \quad b_4 = ne^{-\mu \tau} > 0.
\]

A basic calculation shows that

\[
b_1 b_2 - 3b_3 = (3\mu + \theta + \delta + m) \left[ (\mu + m)(2\mu + \theta + \delta) + (\theta + \mu)(\delta + \mu) \right] - 3(\mu + m)(\mu + \theta)(\delta + \mu) > 0. \quad (57)
\]
Clearly $b_1b_2 - b_3 > 0$, and
\[
b_1^2 - 2b_2 = (\mu + \theta + \delta + m)^2 - 2(\mu + m)(\mu + \theta + \delta) - 2(\theta + \mu)(\delta + \mu) = 3\mu^2 + 2m\mu + 2\theta\mu + 2\delta\mu + \theta^2 + \delta^2 + m^2 > 0.
\] (58)

For the sake of later proof, we make the following assumptions:
(H$_{21}$): $b_2^2 - 2b_1b_3 - b_4^2 > 0,$  
(H$_{23}$): $b_2 - b_4 > 0,$  
(H$_{22}$): $b_1 - b_4 > 0,$  
(H$_{24}$): $b_3 - b_4(\mu + 2m) > 0.$

**Theorem 12.** If $R^0 > 1$, and assumptions (H$_{21}$) – (H$_{24}$) are all satisfied, then the rumor-prevailing equilibrium point $E^*$ is locally asymptotically stable.

**Proof.** Through proper simplification of (50), the characteristic equation of the rumor-prevailing equilibrium point $E^*$ can be calculated as follows
\[
\lambda^3 + (3Dk^2 + b_1)\lambda^2 + (3D^2k^4 + 2b_1Dk^2 + b_2)\lambda + D^3k^6 + b_1D^2k^4 + b_2Dk^2 + b_3 - [b_4\lambda + b_4Dk^2 + (\mu + 2m)b_4]e^{-\lambda\tau} = 0.
\] (59)

Making an assumption that $\lambda = r_2 + i\omega$ ($\omega > 0$) is a complex root of Eq.(59). Substituting $\lambda$ into Eq.(59) and separating the real and imaginary parts, it arrives
\[
\begin{cases}
-\omega_2^2 (3Dk^2 + B_1 + 3r_2) + D^3k^6 + (3r_2 + b_1)D^2k^4 + (3r_2^2 + 2r_2b_1 + b_2)Dk^2 + r_2^3 + r_2b_1 + r_2b_2 + b_3 = (Dk^2 + r_2 + \mu + 2m) b_4 e^{-r_2\tau} \cos \omega_2 \tau + b_2 b_4 e^{-r_2\tau} \sin \omega_2 \tau, \\
-\omega_3^2 + \omega_2 [3D^2k^4 + 2(3r_2 + b_1)Dk^2 + 3r_2^2 + 2r_2b_1 + b_2] + b_1^2 + b_2^2 + b_3^2 = 0
\end{cases}
\]
\[
\begin{cases}
\omega_2^2 (3Dk^2 + B_1 + 3r_2) + D^3k^6 + (3r_2 + b_1)D^2k^4 + (3r_2^2 + 2r_2b_1 + b_2)Dk^2 + r_2^3 + r_2b_1 + r_2b_2 + b_3 = (Dk^2 + r_2 + \mu + 2m) b_4 e^{-r_2\tau} \cos \omega_2 \tau - (Dk^2 + r_2 + \mu + 2m) b_4 e^{-r_2\tau} \sin \omega_2 \tau
\end{cases}
\]

**Case 1:** If $R^0 > 1$, $r_2 > 0$, squaring and adding the both sides of the two equations respectively, one has
\[
\begin{align}
\omega_2^2 + \omega_3^2 & \left[3D^2k^4 + 2(3r_2 + b_1)Dk^2 + 3r_2^2 + 2r_2b_1 + b_2^2 - 2b_2\right] + \omega_2^2 \left\{3D^4k^8 + 4(b_1 + 3r_2)D^3k^6 + 2(3r_2 + b_1)D^2k^4 + \left[4r_2^2b_1^2 + 12r_2(r_2 + b_1) + 2(b_1b_2 - 3b_3)\right]Dk^2 + 3r_2^2 + 4b_1^3 + 2b_2^2r_2 + 2r_2(b_1b_2 - b_3) + b_2^2 - 2b_1b_3 - b_4^2\right\} + \left\{D^3k^6 + (3r_2 + b_1)D^2k^4 + (3r_2^2 + 2r_2b_1 + b_2)Dk^2 + r_2^3 + r_2b_1 + b_2r_2 + b_3 + b_4(Dk^2 + r_2 + \mu + 2m)\right\} \\
\{D^3k^6 + (3r_2 + b_1)D^2k^4 + (3r_2^2 + 2r_2b_1 + b_2)Dk^2 + r_2^3 + r_2b_1 + b_2r_2 + b_3 + b_4(Dk^2 + r_2 + \mu + 2m)\} & < 0.
\end{align}
\] (60)

When all the above assumptions are satisfied, no positive $\omega_2$ satisfies the condition, which implies that all the roots of the characteristic equation (59) have non-positive real parts.

**Case 2:** If $R^0 > 1$, $r_2 = 0$, correspondingly, we can obtain
\[
\begin{align}
\omega_2^2 + \omega_3^2 & \left[3D^2k^4 + 2b_1Dk^2 + b_2^2 - 2b_2\right] + \omega_2^2 \left\{3D^4k^8 + 4b_1D^3k^6 + 2b_2^2D^2k^4 + 2(b_1b_2 - 3b_3)Dk^2 + b_2^2 - 2b_1b_3 - b_4^2\right\} + \left\{D^3k^6 + b_1D^2k^4 + (b_2 - b_4)Dk^2 + \left[b_3 - b_4(\mu + 2m)\right]\right\} = 0
\end{align}
\] (61)

Once again, when all the assumptions are true, then Eq.(61) has no positive roots which indicates that Eq.(59) has no pure imaginary roots.

Therefore, the real parts of all the eigenvalues of the characteristic equation (59) are negative based on (H$_{21}$)–(H$_{24}$). Consequently, the rumor-prevailing equilibrium point $E^*$ is locally asymptotically stable. 

\[\square\]
4. Numerical simulations. In this section, we will use Matlab to simulate and analyze the spatial-temporal dynamic characteristics of the proposed rumor propagation model, including the effects of spatial heterogeneity, different diffusion coefficient $D$ and different incidence function $f(x, I)$ on rumor propagation dynamics [49] in spatial heterogeneous case. In spatial homogeneous case, we also verify the influence of key parameters on the basic reproduction number $R_0$, the stability of the rumor-free equilibrium point and the rumor-prevailing equilibrium point. For brevity, we simulate on $\Omega = [0, \pi]$. From the perspective of numerical simulation, we verify the correctness of the previous theoretical analysis by selecting different parameter values in different simulations.

4.1. The spatially heterogeneous case.

4.1.1. Stability of rumor-free steady state $E_0(x)$. In order to better estimate the value of $R_0$, we select the following parameters and function in system (4).

$$
\begin{align*}
A &= 0.3, \mu = 0.3, \theta = 0.5, \delta = 0.4, \\
\beta(x) &= 0.2 + 0.05 \sin(x), D = 0.001, \\
f(x, I) &= I/(1 + \alpha I^2), \alpha = 0.1.
\end{align*}
$$

Resort to Theorem 3, we obtain

$$
R_0 \leq \frac{A \theta \beta^M}{\mu (\mu + \delta)(\mu + \theta)} = \frac{0.3 \times 0.5 \times 0.25}{0.3 \times (0.4 + 0.3) \times (0.5 + 0.3)} = 0.2232 < 1.
$$

It is easily seen that the rumor-free steady state $E_0(x) = (1, 0, 0)$ is globally asymptotically stable according to Theorem 4. Fig.1 shows that the density of rumor-susceptible users, collectors and rumor-infected users finally converges to a constant, and the solution trajectory of system (4) tends to the rumor-free equilibrium point which implies that the rumor is extinct. The numerical results shown in Fig.1 (c)-(f)-(i) implies that the maximum value of the oscillation of the solution of system (4) on the interval decreases gradually, and the curve tends to be stable as time evolves.
4.1.2. The uniform persistence of rumor propagation. Choosing the following parameters and function in system (4).

\[
\begin{align*}
A &= 0.3, \\
\mu &= 0.3, \\
\theta &= 0.5, \\
\delta &= 0.1, \\
\beta(x) &= 0.8 + 0.05 \sin(x), \\
D &= 0.001, \\
f(x, I) &= I/(1 + \alpha I^2), \\
\alpha &= 0.1.
\end{align*}
\]

By means of Theorem 3, it arrives

\[
R_0 \geq \frac{A\theta \beta^m}{\mu(\mu + \delta)(\mu + \theta)} = \frac{0.3 \times 0.5 \times 0.8}{0.3 \times (0.1 + 0.3) \times (0.5 + 0.3)} = 1.2500 > 1.
\]

By Theorem 7, we can deduce that system (4) has at least one positive rumor-prevailing steady state \( E_* \). As shown in Fig.2, the density of rumor-susceptible users, collectors and rumor-infected users are always greater than 0, that is, rumor continues to circulate. This therefore implies that the solution of system (4) is uniformly persistent which is consistent with the result in Theorem 6. Further, we can observe that the effect of the spatial heterogeneity on dynamic behaviors of system (4) are highly sensitive, and the numerical results shown in Fig.2 (c)-(f)-(i) indicate that when \( t \) is relatively small, the solution fluctuates greatly with the change of \( x \). As time evolves, the curves changes gently. This phenomenon also suggests that it is difficult to prove the asymptotic stability of the rumor-prevailing steady state.
4.1.3. The effect of different diffusion coefficient $D$ on rumor propagation dynamics. In order to investigate the influence of different diffusion coefficients $D$ on rumor propagation, we fix the parameters and function as follows.

\[
\begin{aligned}
A &= 0.3, \mu = 0.3, \theta = 0.5, \delta = 0.05, \\
\beta(x) &= 0.8 + 0.05 \sin(x), \\
f(x, I) &= I/(1 + \alpha I^2), \alpha = 5.
\end{aligned}
\]

In Fig. 3, we draw the plan of $C(t, x)$ and $I(t, x)$ in the $tx$-plane. For fixed $t = 0.5$, we also draw a graph of $C(t, x)$ and $I(t, x)$ with respect to $x$ in Fig. 4. As we can see from these pictures that the larger the diffusion coefficient, the smaller the vibration range of $C(t, x)$ and $I(t, x)$.

Figure 2. The uniform persistence of rumor propagation.
4.1.4. The effect of different incidence functions \( f(x, I) \) on rumor propagation dynamics. It is well known that the incidence function plays an important role in rumor propagation modeling. Taking the parameters in system (4)

\[
\begin{align*}
A &= 0.3 + 0.05 \sin(x), \\
\mu &= 0.3 + 0.05 \sin(x), \\
\theta &= 0.15 + 0.01 \sin(x), \\
\delta &= 0.35 + 0.02 \sin(x), \\
\beta(x) &= 0.8 + 0.05 \sin(x), \\
D &= 0.001,
\end{align*}
\]
to observe the distribution of the rumor-infective users changes with the incidence function \( f(x, I) = I/(1 + \alpha I^2) (\alpha = 55) \) and \( f(x, I) = I \). The influence of both functions on \( I(t, x) \) are shown in Fig.5. Observe that the use of non-monotonic incidence function reduces the density of \( I(t, x) \). Actually, the amount of contacts between rumor-infective users and rumor-susceptible users is reduced because the regulatory authorities impose a mandatory silence on rumor-infective users or screen and protect the information browned by rumor-susceptible users [37]. This effect cannot be expressed with bilinear incidence. Thus the results of numerical simulation agree with the reality.

![Figure 5. Two incidence functions.](image)

### 4.2. The spatially homogeneous case.

#### 4.2.1. The relation between the basic reproduction number \( R^0 \), \( \beta \), \( \theta \) and \( A \).

As we all know, the basic reproduction number plays a very important role in the rumor propagation model which reflects the dynamic characteristics of rumor propagation to some extent. Therefore, it is of great significance to investigate the influence of system parameters on the basic reproduction number.

In the inherent parameters of the system, the transmission rate \( \beta \) reflects the ability of rumor propagation, \( \theta \) reflects the conversion ability of collectors to infective users, and \( A \) expresses the number level of new users. Therefore, these three parameters can more effectively represent the characteristics of rumor propagation.

We select \( \beta \), \( \theta \), \( A \) as the variables of the three axes in Fig.6 and the rest of the system variables are set as \( \mu = 0.2 \), \( \delta = 0.2 \), \( \tau = 0.2 \). In Fig.6, we draw three contour surfaces, \( R^0 = 0.1 \), \( R^0 = 1 \) and \( R^0 = 2 \), which are different from the direct relation diagram of basic reproduction number and system parameters. It can be seen from the figure that in order to completely terminate rumor propagation, the values of \( \beta \), \( \theta \), and \( A \) must be above the surface \( R^0 = 1 \), otherwise rumor will continue to circulate.
4.2.2. Stability of rumor-free equilibrium point $E^0$. Considering system (31) with parameters $\mu = 0.3$, $A = 0.4$, $\beta = 0.5$, $\theta = 0.5$, $\delta = 0.1$, $\tau = 5$, $\alpha = 0.2$ and $D = 2$ and the initial condition $S(t, x) = 1.4333, C(t, x) = 0.3, I(t, x) = 0.3$. A straightforward calculation yields: $R^0 = 0.8966 < 1$. Therefore, there only exists a rumor-free equilibrium point $E^0 = (1.3333, 0, 0)$. Moreover, the rumor-free equilibrium point $E^0 = (1.3333, 0, 0)$ is global asymptotically stable from Theorem 11. As Fig.7 (a)-(c) shows that the density of the susceptible users eventually converges to a positive constant, the densities of the rumor collectors and the infective users stabilize to 0.

In order to better reflect the impact of initial densities of susceptible users, rumor collectors and the infective users in rumor propagation, we select the following groups of initial values to plot: $(1.35, 0.11, 0.11)$, $(1.3, 0.2, 0.2)$, $(1.3, 0.3, 0.3)$, $(1.1, 0.3, 0.3)$, $(0.9, 0.2, 0.2)$ and $(0.9, 0.11, 0.11)$. Similarly, by selecting the system parameters as before. It can be clearly observed from Fig.7 (d) that, regardless of the initial value of susceptible users, collectors and infective users, the trajectory solution of system (31) eventually tends to the rumor-free equilibrium point $E^0 = (1.3333, 0, 0)$. That is to say, both the collectors and the infective users no longer exist in the end. In this case, the rumor is extinct.
4.2.3. Stabilities of rumor-prevailing equilibrium point \( E^\star \). Choose parameters \( A = 0.34, \mu = 0.2, \theta = 0.5, \delta = 0.1, \alpha = 55, \beta = 0.6 \) and \( \tau = 0.5 \) in system (31). A straightforward calculation shows that \( R^0 = 2.5852 > 1 \). That is to say, system (31) admits a unique rumor-prevailing equilibrium point \( E^\star = (1.9366, 0.0164, 0.0273) \). In addition, we can also have \( b_1 - b_4 = 1.1227 > 0, b_3 - b_4 = 0.2527 > 0, b_2^2 - 2b_1b_3 - b_3^2 = 0.0016 > 0, b_3 - b_4(\mu + 2m) = 0.0255 > 0 \). Thus, according to Theorem 12, the rumor-prevailing equilibrium point \( E^\star \) is locally asymptotically stable. Consider the initial value \( S(t, x) = 1.4333, C(t, x) = 0.3, I(t, x) = 0.3 \), we have drawn three-dimensional figures of individual densities, time and space respectively. As shown in Fig.8 (a)-(c), the densities of the susceptible users, rumor collectors and the infective users eventually converge to positive constants. Furthermore, in Fig.8 (d), we can easily find that the trajectory solution tends to the rumor-prevailing equilibrium point \( E^\star \) eventually. In other words, the densities of susceptible users, rumor collectors and the infective users are 1.9366, 0.0164 and 0.0273 at the end of rumor propagation.
5. Conclusions. In this paper, we investigate the threshold dynamics of a SCIR reaction-diffusion rumor propagation model with a general nonlinear incidence rate in both heterogeneous and homogeneous environments. The novelty of this model is that it considers not only the diffusion of the environment, but also the spatial heterogeneity. Firstly, we show the existence, uniqueness, nonnegativity and boundedness of solutions of system (4). By appealing to the theory developed in Ref. [31], we obtain a threshold index with biological significance: the basic reproduction number $R_0$. $R_0$ is expressed by the spectral radius of the next infection operator. By applying the comparison principle, Lyapunov function as well as the abstract results in persistence theory, we obtain that the rumor-free steady state $E_0(x)$ is globally asymptotically stable if $R_0 < 1$. While if $R_0 > 1$, all positive solutions of system (4) are uniformly persistent and there exists at least one rumor-prevailing equilibrium point. Further, we introduce time delay on the basis of the original model in homogeneous environment. We also define the basic reproduction number $R_0^\delta$, and extensively discuss the stability properties. In the numerical simulations, some numerical examples are carried out to illustrate the theoretical results.

From the application background, threshold dynamics plays an significant role in the control strategies of the online rumor propagation. It can be obtained that the rumor-free steady state $E_0(x)$ is globally asymptotically stable when $R_0 < 1$, which indicates that the rumor will extinct. While if $R_0 > 1$, the rumor is uniformly persistent which implies that the rumor will exist forever. If one can choose the system parameters such that $R_0 < 1$, we can clear the online rumor finally. From the expression of $R_0$ , it is key to control the online rumor propagation by controlling the rumor transmission rate and the conversion rate from rumor collectors to infective users. In addition, the time delay also has an important impact on rumor propagation. Due to the openness and diversity of the online social networks, it is far enough to eradicate the rumor, which requires us to seek better ways to suppress rumor.

In our model (4), we assume that all users share the same diffusion rate $D$ (independent of $x$), which plays a pivotal role in the proof of the boundedness of
solution of (4). Nevertheless, due to the heterogeneity of the spatial environment, web users may disperse at different rates, so it is more practical to consider a model with different diffusion rates. It has been commonly accepted that age structure is an important factor that should be incorporated in rumor propagation models. So we are naturally inspired to investigate the joint effects of the spatial heterogeneity, age structure, the diffusion of web users on rumor propagation in online social networks. Although we have made a detailed study of the proposed models, it is a pity that we have not established rich results for system (4).

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E-mail address: zlhuua@126.com
E-mail address: liuwenshan6130163.com
E-mail address: dyzhang@ujs.edu.cn