Generalized $k$-regular sequences III: Arithmetical properties of generalized $k$-regular series

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Abstract

Let $F(z)$ be a $k$-regular series in $\mathbb{Z}[z]$ and $b$ be an integer with $b \geq 2$. Bell, Bugeaud and Coons [BelBC] proved that $F(\frac{1}{b})$ is either rational or transcendental. In [Mi], we introduce a generalized $k$-regular sequence as a unification of several kinds of important sequences including $k$-regular, $k$-additive and $k$-multiplicative sequences. In this paper, we give a generalization of the result of Bell, Bugeaud and Coons for certain generalized $k$-regular series. Especially, we show that the values of irrational generating functions of certain sum of $k$-additive sequences and certain $k$-multiplicative sequences are either rational or transcendental. Moreover, we also give a partly generalization of a result obtained by Tachiya [Ta]. Especially, we show that the values of irrational generating functions of certain $k$-additive sequences and certain $k$-multiplicative sequences give transcendental numbers.

1 Introduction

Let $a := (a(n))_{n \geq 0}$ be a given sequence. For any non-negative integer $e$, set

$$S_e(a) := \{(a(k^e n + j))_{n \geq 0} | 0 \leq j \leq k^e - 1\}.$$ 

Allouche and Shallit [AlS] introduced the notion of $k$-regular sequence as follows. A sequence $(a(n))_{n \geq 0}$ is defined to be $k$-regular if the set $S$ is contained in a finitely generated $\mathbb{Q}$-module of sequences. Allouche-Shallit also proved that the set of generating functions of $k$-regular sequence (called a $k$-regular series) forms a ring under the usual addition and the canonical convolution. Later Becker [Bec] and Nishioka [Ni], which characterizes $k$-regular sequences by using the $k$-regular series. (See Theorem 5.1.2 in [Ni].)

**Theorem 1.1** [Bec, Ni] A sequence $(a(n))_{n \geq 0}$ is $k$-regular if and only if there exist a positive integer $d$, $d$ power series $f_1(z) \cdots f_d(z) \in \mathbb{Q}[z]$ with $f_1(z) = f(z)$ given in and a $d \times d$ matrix $A(z)$ whose entries are polynomials in $z$ of degrees less than $k$ with coefficients in $\mathbb{Q}$ such that

$$
\begin{pmatrix}
  f_1(z) \\
  f_2(z) \\
  \vdots \\
  f_d(z)
\end{pmatrix}
= A(z)
\begin{pmatrix}
  f_1(z^k) \\
  f_2(z^k) \\
  \vdots \\
  f_d(z^k)
\end{pmatrix}.
$$

(1.1)

By Theorem 1.1 the $k$-regular series can be regard as Mahler function. (See chapter 5 in [Ni]). Therefore, the arithmetical properties of the special value of $k$-regular series was investigated in Mahler function theory. Recently, Bell, Bugeaud and Coons [BelBC] proved the following theorem. (See Theorem 8.1 in [BelBC] or Theorem 2.5.1 in [CoS].)
Theorem 1.2 [BelBC] Let $F(z)$ be a $k$-regular series in $\mathbb{Z}[[z]]$ and $b$ be an integer with $b \geq 2$. Then $F\left(\frac{1}{b}\right)$ is either rational or transcendental.

The proof of Theorem 1.2 relies on $p$-adic Schmidt subspace theorem.

On the other hand, Gel’fond [Gel] introduced the two functions related with the base $k$-representation as follows. A sequence $(a(n))_{n \geq 0}$ is $k$-additive if and only if, for any non-negative integers $e$, $n$ and $j$ with $0 \leq j \leq k^e - 1$, $(a(n))_{n \geq 0}$ satisfies the following additive relation

$$a(k^e n + j) = a(k^e n) + a(j) \quad (1.2)$$

and $a(0) = 0$. A sequence $(a(n))_{n \geq 0}$ is $k$-multiplicative if and only if, for any non-negative integers $e$, $n$ and $j$ with $0 \leq j \leq k^e - 1$, $(a(n))_{n \geq 0}$ satisfies the following multiplicative relation

$$a(k^e n + j) = a(k^e n)a(j) \quad (1.3)$$

and $a(0) = 1$.

Recently, we introduce a generalized $k$-regular sequence as a unification of several kinds of important sequences including $k$-regular, $k$-additive and $k$-multiplicative sequences in [Mi].

Definition 1.1 [Mi] A sequence $(a(n))_{n \geq 0}$ is generalized $k$-regular if and only if, there exist an integer $d$, for any non-negative integer $e$, the dimension of the $\mathbb{Q}$-module of a sequence generated by $S_e$ is at most $d$. The generating series of generalized $k$-regular sequences is called a generalized $k$-regular series. To show the role of $d$ more precisely, a generalized $k$-regular sequence or series is also called a generalized $(k,d)$-regular sequence or series, respectively.

In [Mi], we give the following generalization of Theorem 1.1 for generalized $k$-regular sequences as follows.

Theorem 1.3 [Mi] A sequence $(a(n))_{n \geq 0}$ is generalized $(k,d)$-regular if and only if, for any non-negative integer $e$, there exist a positive integer $d$ power series $f_{e,1}(z), f_{e,2}(z), \ldots, f_{e,d}(z) \in \mathbb{Q}[[z]]$ with $f_{0,1}(z)$ being $f(z)$ given in and a $d \times d$ matrix $A_e(z)$ whose entries are polynomials in $z$ of degrees less than $k$ with coefficients in $\mathbb{Q}$ such that

$$\begin{pmatrix}
  f_{e,1}(z) \\
  f_{e,2}(z) \\
  \vdots \\
  f_{e,d}(z)
\end{pmatrix} = A_e(z) \begin{pmatrix}
  f_{e+1,1}(z^k) \\
  f_{e+1,2}(z^k) \\
  \vdots \\
  f_{e+1,d}(z^k)
\end{pmatrix} \quad (e \geq 0). \quad (1.4)$$

Now we give the natural three examples of Theorem 1.3. Let $g_0(z)$ be the generating function of a $k$-multiplicative sequence $(a(n))_{n \geq 0}$ (generalized $(k,1)$-regular series). For any non-negative integer $e$, we define $g_e(z)$ as $g_e(z) := \sum_{n=0}^{\infty} a(k^e n) z^n$. The series $g_0(z)$ has the following infinite chains equations

$$g_e(z) = \left(\sum_{j=0}^{k-1} a(j k^e) z^j\right) g_{e+1}(z^k). \quad (1.5)$$

The arithmetical properties of infinite product (1.5) was investigated in Mahler function theory. (See [AmV1, AmV2, Ta].) Let $h_0(z)$ be the generating function of $k$-additive sequence $(b(n))_{n \geq 0}$ (generalized $(k,2)$-regular series). For any non-negative integer $e$, we
define \( h_e(z) \) as \( h_e(z) := \sum_{n=0}^{\infty} b(k^n) n^z \). The series \( h_0(z) \) has the following infinite chains matrix equations
\[
\begin{pmatrix}
\frac{h_e(z)}{1-z} \\
\frac{h_{e+1}(z)}{1-z}
\end{pmatrix} = \begin{pmatrix}
\sum_{j=0}^{k-1} \frac{z^j}{j!} & \sum_{j=0}^{k-1} \frac{b(jk^e) z^j}{j!} \\
\sum_{j=0}^{k-1} \frac{b(jk^e) z^j}{j!} & \sum_{j=0}^{k-1} \frac{b(jk^e) z^j}{j!}
\end{pmatrix} \begin{pmatrix}
\frac{h_e(z)}{1-z} \\
\frac{h_{e+1}(z)}{1-z}
\end{pmatrix}.
\]
(1.6)

The set of generalized \( k \)-regular sequences forms a ring under the usual addition and the canonical convolution. (See Theorem 2.2 in [Mi] ). Therefore, the power series \( g_0(z) + h_0(z) \) is also a generalized \( k \)-regular series. The power series \( g_0(z) + h_0(z) \) has the following infinite chains matrix equations
\[
\begin{pmatrix}
g_e(z) + h_e(z) \\
h_e(z)
\end{pmatrix} = \begin{pmatrix}
m_{0,j=0} \frac{(a(jk^e) + 1) z^j}{j!} - \sum_{j=0}^{k-1} \frac{z^j}{j!} & - \sum_{j=0}^{k-1} \frac{a(jk^e) z^j}{j!} \\
0 & \sum_{j=0}^{k-1} \frac{z^j}{j!}
\end{pmatrix} \begin{pmatrix}
g_e(z) + h_e(z) \\
h_e(z)
\end{pmatrix} + \begin{pmatrix}
g_{e+1}(z) + h_{e+1}(z) \\
h_{e+1}(z)
\end{pmatrix}.
\]
(1.7)

The purpose of this paper investigates the arithmetical properties of certain generalized \( k \)-regular series as follows. We denote a \((i,j)\)-componet of \( A_e(z) \) by \( \sum_{n=0}^{k-1} a_{e,s,i,j} n^z \). We assume that, for any non-negative integers \( e \) and \( j \) with \( 1 \leq j \leq d \), there exists a positive constant \( C \)
\[
|f_{e,j}(0)| \leq C
\]
(1.8)
and, for any \( \epsilon > 0 \), there exists an integer \( N(\epsilon) \) such that, for any \( e \geq N(\epsilon) \), \( i,j \) with \( 1 \leq i,j \leq d \) and \( s \) with \( 0 \leq s \leq k-1 \),
\[
|a_{e,s,i,j}| \leq e^{\epsilon k^e}.
\]
(1.9)
Moreover, we assume that, for any non-negative integer \( e \),
\[
A_e(z) \in \mathbb{Z}[z]^{d \times d}.
\]
(1.10)

**Theorem 1.4** Let \( b \) be an integer with \( b \geq 2 \) and \( f(z) = f_{0,1}(z) \) be satisfies the equations (1.4) with (1.5), (1.6) and (1.8). Then \( f(\frac{1}{b}) \) is either rational or transcendental.

We prove Theorem 1.4 by modifying the method of proof of Theorem 1.2. (See the proof of Theorem 2.5.1 in [CoS]). By (1.7), we get the following corollary of Theorem 1.4.

**Corollary 1.1** Let \( g_0(z) \) and \( h_0(z) \) be defined by the above and \( b \) be an integer with \( b \geq 2 \). Assume that, for any non-negative integers \( e \) and \( j \) with \( 0 \leq j \leq k-1 \), \( a(k^e) \) and \( b(k^e) \) are integers, \( a(jk^e) \) and \( b(jk^e) \) satisfy (1.9). Then \( g_0(\frac{1}{b}) + h_0(\frac{1}{b}) \) is either rational or transcendental.

By Theorem 1.4 and the most classical Mahler method (See 20p in [Ma]), we also prove the following theorem.

**Theorem 1.5** Let \( b \) be an integer with \( b \geq 2 \) and irrational powers series \( f(z) = f_{0,1}(z) \) be satisfies the equations (1.4) with (1.5), (1.6) and (1.8). Assume that, for any non-negative integer \( e \), let \( A_e(\frac{1}{b}) \neq 0 \). Then at least one among the numbers \( f(\frac{1}{b}) = f_{0,1}(\frac{1}{b}), f_{0,2}(\frac{1}{b}), \ldots, f_{0,d}(\frac{1}{b}) \) is transcendental.

Theorem 1.5 gives a partly generalization of a result obtained by Theorem 1 in [Ta]. By (1.5), we get the following corollary of Theorem 1.5.
Corollary 1.2 Let \( g_0(z) \) be defined by the above with irrational and \( b \) be an integer with \( b \geq 2 \). Assume that, for any non-negative integers \( e \) and \( j \) with \( 0 \leq j \leq k - 1 \), \( a(jk^e) \) is an integer, \( a(jk^e) \) satisfies \( \sum_{j=0}^{k-1} a(jk^e) \frac{1}{b^j} \neq 0 \). Then \( g_0(\frac{1}{b}) \) is transcendental.

This corollary is covered by Theorem 1 in [Ta]. By (1.6), we get the following corollary of Theorem 1.5.

Corollary 1.3 Let \( h_0(z) \) be defined by the above with irrational and \( b \) be an integer with \( b \geq 2 \). Assume that, for any non-negative integers \( e \) and \( j \) with \( 0 \leq j \leq k - 1 \), \( b(jk^e) \) is an integer and \( b(jk^e) \) satisfies \( \sum_{j=0}^{k-1} b(jk^e) \frac{1}{b^j} \neq 0 \). Then \( h_0(\frac{1}{b}) \) is transcendental.

This corollary is new.

This paper is organized as follows. In section 2, we gather lemmas for the proof of the theorems and the proposition. In section 3, we give a proof of Theorem 1.4. In section 4, we give a proof of Theorem 1.5. In section 5, we give the other examples of Theorem 1.5 by related with the certain digital pattern sequences.

2 Preliminaries

In this section, we gather lemmas for the proof of the theorems. The following lemma is need for the proof of Theorem 1.4. The following lemma is known as Siegel’s lemma. (See Lemma 1.4.2 in [Ni].)

Lemma 2.1 (Siegel’s lemma) Consider the \( m \) equations in \( n \) unknowns

\[
a_{k1}x_1 + \cdots + a_{kn}x_n = 0 \quad k = 1, 2, \ldots, m
\]

with rational integral coefficients \( a_{ij} \), and with \( 0 < m < n \). Let \( A \) be a positive integer such that \( A \geq |a_{ij}| \), for all \( i \) and \( j \). Then there is a nontrivial solution \( x_1, x_2, \ldots, x_n \) in rational integers of equations \((2.1)\) such that

\[
|x_j| < 1 + (nA)^{n/(n-m)} \quad j = 1, 2, \ldots, n.
\]

The following lemma is need for the proof of Theorem 1.4. The following lemma is known as \( p \)-adic Schmidt subspace theorem. (See Theorem E.10 in [Bu] or Theorem 2.5.4 in [CoS].)

Lemma 2.2 (\( p \)-adic Schmidt subspace theorem) Let \( n \geq 2, \delta > 0 \), and let \( p_1, \ldots, p_s \) be distinct prime numbers. Further, let \( L_{1\infty}, \ldots, L_{n\infty} \) be linearly independent linear forms in \( X_1, \ldots, X_n \) with algebraic coefficients in \( \mathbb{C} \), and for \( j = 1, \ldots, s \), \( L_{1j}, \ldots, L_{nj} \) be linearly independent linear forms in \( X_1, \ldots, X_n \) with algebraic coefficients in \( \mathbb{Q}_p \). Consider the inequality

\[
|L_{1\infty}(x) \cdots L_{n\infty}(x)| \prod_{j=1}^{s} |L_{1j}(x) \cdots L_{nj}(x)|_p < \max\{x_1, \ldots, x_m\}^{-\delta}
\]

with \( x := (x_1, \ldots, x_m) \) in \( \mathbb{Z}^n \). There are a finite number of proper linear subspaces \( T_1, \ldots, T_t \) of \( \mathbb{Q}^n \) such that all solutions of \((2.3)\) lie in \( T_1 \cup \cdots \cup T_t \).

The following notion is need for the construct of the examples of Theorem 1.5. (See Definition 1 in [AmV1].)

Definition 2.1 [AmV1] Let \( f(z) \in K[[z]] \). We define the irrationality measure \( \mu(f) \) to be the infimum of \( \mu \) such that;

\[
\text{ord}(A(z)f(z) - B(z)) \leq \mu M
\]

for all nonzero \( A(z), B(z) \in K[z] \) with \( \max(\text{deg}A(z), \text{deg}B(z)) \leq M \) (for \( M \geq M_0 \), some \( M_0 \) depend only on \( f(z) \)). If there does not exist such a \( \mu \), \( \mu(f) := +\infty \).
The following lemma is need for the construct of the examples of Theorem 1.5 (See Theorem 5 in [DuN].)

**Lemma 2.3** [DuN] Let $K$ be a commutative field and $c_1$, $c_2$, $c_3$ be real numbers with $0 < c_1 < c_2$, $c_3 \geq 1$. Let $(m(n))_{n \geq 0}$ be an increasing sequence of nonnegative integers satisfying $m(n+1) - m(n) \leq c_3$. Let $k \geq 2$ be an integer and $f(z) \in K[[z]]$. Suppose that for large positive integer $n$ there exists a sequence $(P_n(z), Q_n(z))_{n=0}^{\infty}$ in $K[z]^2$ satisfying

\begin{align*}
P_n(z)Q_{n+1}(z) - P_{n+1}(z)Q_n(z) & \neq 0, \\
\deg Q_n(z), \deg P_n(z) & \leq c_1k^{m(n)}, \\
\ord(Q_n(z)f(z) - P_n(z)) & \geq c_2k^{m(n)}.
\end{align*}

Then $\mu(f) < +\infty$.

### 3 Proof of Theorem 1.4

In this section, we prove Theorem 1.4. Let $p$ be a positive integer parameter with $p > d + 5$. We shall denote by $c_1, c_2, \cdots$ positive constants independent of $e, p, e$. For any non-negative integers $e$ and $j$ with $1 \leq j \leq d$, we define the $(a_{e,j}(n))_{n \geq 0}$ by $f_{e,j}(z) = \sum_{n=0}^{\infty} a_{e,j}(n)z^n$.

**Lemma 3.1** Notation is the same as for section 1. Then, for any $e \geq N(e)$, $j$ with $1 \leq j \leq d$ and $n \geq 0$,

\begin{equation}
|a_{e,j}(n)| \leq e^{e^k(1+n)}.
\end{equation}

**Proof.** By 1.8 and 1.9, one can show analogously to the proof of Lemma 3 in [Ta].

**Lemma 3.2** Notation is the same as for section 1. For any $e \geq N(e)$, $j$ with $1 \leq j \leq d$ and $n \geq 0$ there exist auxiliary functions for any $e \geq N(e)$, $j$ with $1 \leq j \leq d$ and $n \geq 0$, we have

\begin{equation}
Q_e(z)f_{e,j}(z) - P_{e,j}(z) = z^{dp+p+1}G_{e,j}(z).
\end{equation}

with polynomials $Q_e(z) = \sum_{i=0}^{pd} q_{e,i}z^i, P_{e,j}(z) = \sum_{i=0}^{dp} p_{e,j,i}z^i, G_{e,j}(z) = \sum_{n=0}^{\infty} g_{e,j}(n)z^n$, such that

\begin{align*}
|q_{e,i}| & \leq 1 + ((dp + 1)e^{e^k(1+dp)})^{dp+1} e^{e^1p^2k^e}, \\
|p_{e,j,i}| & \leq (dp + 1)(1 + ((dp + 1)e^{e^k(1+dp)})^{dp}e^{e^k(1+dp)}) e^{e^2p^2k^e}, \\
|g_{e,j}(n)| & \leq (dp + 1)(1 + ((dp + 1)e^{e^k(1+dp)})^{dp}e^{e^k(1+n)}) e^{e^1c^2p^2+n}e^ek^e.
\end{align*}

**Proof.** By Lemma 2.1 and 3.1, one can show analogously to the proof of Lemma 5 in [Am2].

**Lemma 3.3** Notation is the same as for section Lemma 3.2 Then, for any $e \geq N(e)$, $j$ with $1 \leq j \leq d$,

\begin{equation}
\ord Q_e(z) \leq \ord P_{e,j}(z).
\end{equation}

**Proof.** We denote $\ord Q_e(z)$ by $J_e$ and $Q_e'(z) \in \mathbb{Z}[z]$ by $Q_e(z) = z^{J_e}Q'_e(z)$.

\begin{equation}
\frac{Q_e'(z)f_{e,j}(z) - P_{e,j}(z)}{z^{J_e}} = z^{dp+p+1-J_e}G_{e,j}(z).
\end{equation}
By the definition of $J_e$,
\[ dp + p + 1 - J_e \geq 0. \]  \hspace{1cm} (3.8)

By \((3.8)\) and right hand side of \((3.7)\), $P_e(z)z^{-e} \in \mathbb{Z}[z]$. Therefore, we get \((3.6)\).

There exists an integer $J$ with $J \leq dp$ such that
\[ \# \{ e \mid \text{ord } Q_e(z) = J \} = \infty. \] \hspace{1cm} (3.9)

We denote the set $\{ e \mid \text{ord } Q_e(z) = J \}$ by $B$. By \((3.2)\), for any integer $e$ in $B$ and $j$ with $1 \leq j \leq d$, we have
\[ Q_e(z) f_e(j) - \frac{P_{e,j}(z)}{z^{e+j}} = z^{dp+p+1-J} G_{e,j}(z). \] \hspace{1cm} (3.10)

We replace $Q_e(z)$ and $P_{e,j}(z)$ by $Q_e(z)$ and $P_{e,j}(z)$. By \((3.4)\) and \((3.10)\), there exist polynomials $a_{e,j,0}(z)$ ($1 \leq j \leq d$) with degrees at most $k^e$ such that
\[ Q_e(z^{k^e}) f(z) - \sum_{j=1}^{d} a_{e,j,0}(z) P_{e,j}(z^{k^e}) = z^{(dp+p+1-J)k^e} \sum_{j=1}^{d} a_{e,j,0}(z) G_{e,j}(z^{k^e}). \] \hspace{1cm} (3.11)

Lemma 3.4 If $|z| \leq \frac{2}{3e^{e+1}k^e}$, then, for sufficiently large integer $e$,
\[ |Q_e(z^{k^e})| \geq 1/2. \] \hspace{1cm} (3.12)

Proof. By the definition of $Q_e(z)$ and \((3.3)\), we have
\[ |Q_e(z^{k^e})| \geq 1 - \frac{(2e)^{e+2k^e}}{(3e)^{e+2k^e}} - \cdots - \frac{(2e)^{e+2k^e}}{(3e)^{e+2k^e}} \geq 1/2. \] \hspace{1cm} (3.13)

Lemma 3.5 Let $b$ be an integer with $b \geq 2$. If
\[ 3e^{2e+dp^2} < b \] \hspace{1cm} (3.14)

then, for sufficiently large integer $e$,
\[ \left| f \left( \frac{1}{b} \right) - \sum_{j=1}^{d} a_{e,j,0}(\frac{1}{b}) P_{e,j}(\frac{1}{b}) \right| \leq \frac{1}{b^{dp+p+1-J}k^e} 2C_1(e) \frac{b}{b-1} e^{e+2k^e} \frac{b^{k^e}}{b^{k^e} - e^{k^e}}. \] \hspace{1cm} (3.15)

where $C_1(e)$ is a positive constant independent of $e$. In particular,
\[ \lim_{e \to \infty} \sum_{j=1}^{d} a_{e,j,0}(\frac{1}{b}) P_{e,j}(\frac{1}{b}) = \lim_{e \to \infty} \sum_{j=1}^{d} a_{e,j,0}(\frac{1}{b}) P_{e,j}(\frac{1}{b}) = f \left( \frac{1}{b} \right). \] \hspace{1cm} (3.16)

Proof. By $a_{e,0} \neq 0$, \((3.11)\) and \((3.3)\), for any sufficiently large $e$, we have
\[ |Q_e(\frac{1}{b^{k^e}}) f(\frac{1}{b}) - \sum_{j=1}^{d} a_{e,j,0}(\frac{1}{b}) P_{e,j}(\frac{1}{b^{k^e}}) | \leq \frac{1}{b^{dp+p+1-J}k^e} \sum_{j=1}^{d} a_{e,j,0}(\frac{1}{b}) \sum_{n=0}^{\infty} e^{(c(3p^2+n)k^e)} \] \[ \leq \frac{1}{b^{dp+p+1-J}k^e} C_1(e) d^e e^{k^e} b \leq \frac{1}{b^{dp+p+1-J}k^e} C_1(e) d^e e^{k^e} b \leq \frac{1}{b^{dp+p+1-J}k^e} C_1(e) d^e e^{k^e} b \leq \frac{1}{b^{dp+p+1-J}k^e} C_1(e) d^e e^{k^e} b. \] \hspace{1cm} (3.17)
By Lemma 3.12, 3.14 and 3.17, we get
\[ |f\left(\frac{1}{b^{k^e}}\right) - \frac{1}{Q_c\left(\frac{1}{b^{k^e}}\right)} \| e^{\frac{d}{p+2}}C_1(\epsilon)d^e e^{k^e}b \| b-1 \| e^{c_3p^2}b^{k^e} - e^{k^e}. \]  
(3.18)

Moreover, by (3.3), we have
\[ |Q_c\left(\frac{1}{b^{k^e}}\right) - q_{e,0}| \leq dp\left(\frac{e^{c_3p^2}}{b}\right)^{k^e}. \]  
(3.19)

By (3.14), (3.18) and (3.19), we get (3.16).

Assume that (3.14). For any non-negative integer \( e \), we define the integer tuples \( (D_{e,0}, \ldots, D_{e,dp-J}D_{e,dp-J+1}) \) as follows
\[ (D_{e,0}, \ldots, D_{e,dp-J}, D_{e,dp-J+1}) := (b^{(dp+1-J)k^e}q_{e,0}, \ldots, b^{k^e}q_{e,dp-J}, b^{(dp+1-J)k^e}d_{e,0,0})^{\sum_{j=1}^d a_{e,j,0}(\epsilon)P_{e,j}(\epsilon)\left(\frac{1}{b^{k^e}}\right)}. \]  
(3.20)

From (3.3) and (3.4), we have
\[ \max\{D_{e,0}, \ldots, D_{e,dp-J}, D_{e,dp-J+1}\} \leq b^{(dp+1-J)k^e}e^{c_2p^2k^e}2C_1(\epsilon)d^e e^{k^e}b \| b-1 \| b^{(dp+3)k^e}. \]  
(3.21)

Moreover, there exist the integer sets \( T := \{s_1, s_2, \ldots, s_l\} \) with \( 0 \leq s_1 < s_2 < \cdots < s_l \leq dp - J + 1 \) such that
\[ \#\{e \mid D_{e,i} \neq 0 \text{ for } i \in T \text{ and } D_{e,i} = 0 \text{ for } i \in \{0, \ldots, dp - J + 1\}/T\} = \infty \]  
(3.22)

We put the set \( E := \{e \mid D_{e,i} \neq 0 \text{ for } i \in T \text{ and } D_{e,i} = 0 \text{ for } i \in \{0, \ldots, dp - J + 1\}/T\}. \)

We assume that \( f\left(\frac{1}{b}\right) \neq 0 \) is an algebraic number. By \( f\left(\frac{1}{b}\right) \neq 0, q_{e,0} \neq 0 \) and (3.16), we have
\[ s_1 = 0, s_l = dp - J + 1. \]  
(3.23)

Let \( S \) be the set of prime factor of \( b \). We define the linear form
\[ L_{i,\infty} = x_i \quad (1 \leq i \leq l-1) \]  
(3.24)

and
\[ L_{i,\infty} = f\left(\frac{1}{b}\right)\sum_{i=1}^{l-1} x_i + x_l. \]  
(3.25)

Moreover, for any prime \( p \) in \( S \), we define the linear form
\[ L_{i,p} = x_i \quad (1 \leq i \leq l). \]  
(3.26)

For any sufficiently large integer \( e \) in \( E \), we define \( (x_1, \ldots, x_l) := (D_{e,0}, D_{e,s_2}, \ldots, D_{e,dp-J+1}). \)

\[ |L_{1,\infty}(x) \cdots L_{l,\infty}(x)| \prod_{p \leq S_{l=1}^l |L_{1j}(x) \cdots L_{lj}(x)|_p} \leq \frac{1}{b^{(p+2)k^e}C_1(\epsilon)d^e e^{k^e}b \| b-1 \| e^{c_3p^2}e^{c_3dp^2}k^e \| b^{k^e} - e^{k^e}} \]  
\[ \leq \frac{1}{b^{(p+3)k^e}} \leq \left(\frac{1}{b^{(dp+3)k^e}}\right)^{\frac{p-3}{dp+3}} \leq \left(\frac{1}{\max\{D_{e,0}, \ldots, D_{e,dp-J}, D_{e,dp-J+1}\}}\right)^{\frac{p-3}{dp+3}}. \]  
(3.27)
By (3.27) and Lemma 2.2, \((x_1, \ldots, x_l) := (D_{e,0}, D_{e,s_2}, \ldots, D_{e,dp-J+1})\) with in \(E\) lie in finitely many proper linear subspaces of \(\mathbb{Q}^l\). There exist an infinite set of distinct positive integers \(E' \subset E\) and a nonzero integer triple \((z_1, \ldots, z_l)\) such that

\[
z_1 b^{(dp+1-J)k_e} q_{e,0} + z_2 D_{e,s_2} + \cdots + z_l b^{(dp+1-J)k_e} = z_1 D_{e,0} + z_2 D_{e,s_2} + \cdots + z_l D_{e,dp-J+1} = 0, \quad \text{for any } e \in E'.
\] (3.28)

We define the integer \(m\) as \(m := \min\{i \mid z_i \neq 0\}\). If \(1 < m\), we have

\[
z_m b^{(dp+1-J-s_m)k_e} q_{e,s_m} + \cdots + z_{l-1} b^{(dp+1-J)k_e} q_{e,s_{l-1}} = z_l b^{(dp+1-J)k_e} \sum_{j=1}^d a_{e,j,0}(\frac{1}{b^{k_e}}) P_{e,j}(\frac{1}{b^{k_e}}), \quad \text{for any } e \in E'.
\] (3.29)

By (3.30) \(|q_{e,0}| \geq 1\) and (3.16), for sufficiently large \(e\), we have

\[
|\sum_{j=1}^d a_{e,j,0}(\frac{1}{b^{k_e}}) P_{e,j}(\frac{1}{b^{k_e}})| \geq \frac{|f(\frac{1}{b^{k_e}})|}{2} \neq 0.
\] (3.30)

By (3.29) and (3.30), \(e\) tend to infinity, we get

\[
z_l = 0.
\] (3.31)

By (3.29) and (3.31)

\[
z_m = 0.
\] (3.32)

This contradicts the definition of \(m\). Therefore, \(m = 1\). Dividing (3.28) by \(b^{(dp+1-J)k_e} q_{e,0}\) and (3.16), \(e\) tend to infinity, we get

\[
z_1 + z_l f(\frac{1}{b}) = 0.
\] (3.33)

By (3.33), \(f(\frac{1}{b})\) is a rational number. This completes the proof of Theorem 1.4. \(\square\)

### 4 Proof of Theorem 1.5

Now we prove Theorem 1.5. For any non-negative integer \(e\), we define the matrix \(B_e(z)\) as \(B_e(z) = A_0(z) \cdots A_{e-1}(z^{k_e-1})\).

**Lemma 4.1** For any sufficiently large integer \(e\), there exists an non-zero integer \(D_e\) such that \(D_e B_e^{-1}(\frac{1}{b}) \in \mathbb{Z}^{d \times d}\) and

\[
D_e \leq C_2(\epsilon) d! p b^{d k_e} e^{d k_e}
\] (4.1)

where \(C_2(\epsilon)\) is a positive constant independent of \(e\).

**Proof.** By the computation of numerator of det \(A_i(z)\) with (1.8) and (1.9).

We assume that

\[
b > e^{e+3p^2}
\] (4.2)
and \( f\left(\frac{1}{n}\right) = f_{0,1}\left(\frac{1}{n}\right), f_{0,2}\left(\frac{1}{n}\right), \ldots, f_{0,d}\left(\frac{1}{n}\right) \) are rational. For an integer \( j \) with \( 1 \leq j \leq d \), we define integers \( p_j, q_j \neq 0 \) by \( f_{0,j}(1/b) = \frac{p_j}{q_j} \). By the irrationality of \( f(z) \) and (3.11), for any non-negative integer \( e \), we have

\[
Q_e\left(z^{k^e}\right) f(z) - \sum_{j=1}^{d} a_{e,j,0}(z) P_{e,j}(z^{k^e}) = z^{(dp+1-j)k^e} \sum_{j=1}^{d} a_{e,j,0}(z) G_{e,j}(z^{k^e}) \neq 0. \tag{4.3}
\]

From (4.3), there exist integers \( i_e \) with \( 1 \leq i_e \leq d \) and \( L_e \) such that

\[
\text{ord } G_{e,i_e}(z^{k^e}) = L_e k^e. \tag{4.4}
\]

By (3.5), (4.1) and (4.2), for any sufficiently large integer \( e \), we get

\[
0 < \frac{1}{bL_e k^e} (1 - e^{e \epsilon^2 k^e}) \leq |G_{e,i_e}(\frac{1}{b k^e})| \leq \frac{e^{e \epsilon^2 k^e} b k^e}{b k^e - e k^e}. \tag{4.5}
\]

By (3.10), we get

\[
Q_e(z^{k^e}) \begin{pmatrix} f_{0,1}(z) \\
    f_{0,2}(z) \\
    \vdots \\
    f_{0,d}(z) \end{pmatrix} - B_e(z) \begin{pmatrix} P_{e,1}(z^{k^e}) \\
    P_{e,2}(z^{k^e}) \\
    \vdots \\
    P_{e,d}(z^{k^e}) \end{pmatrix} = B_e(z) z^{(dp+p-1-j)k^e} \begin{pmatrix} G_{e,1}(z^{k^e}) \\
    G_{e,2}(z^{k^e}) \\
    \vdots \\
    G_{e,d}(z^{k^e}) \end{pmatrix}. \tag{4.6}
\]

We denote the \((i,j)\)-component of \( B_e^{-1}(\frac{1}{b}) \) by \( b_{e,i,j} \). By (4.6), we have

\[
(b_{e,i_e,1}, \ldots, b_{e,i_e,d}) Q_e(\frac{1}{b k^e}) \begin{pmatrix} f_{0,1}(\frac{1}{b}) \\
    f_{0,2}(\frac{1}{b}) \\
    \vdots \\
    f_{0,d}(\frac{1}{b}) \end{pmatrix} - P_{e,i_e}(\frac{1}{b k^e}) = \frac{1}{b(dp+p-1+j)k^e} G_{e,i_e}(\frac{1}{b k^e}). \tag{4.7}
\]

We define a positive integer \( C_{1,e} \) by \( C_{1,e} := \min\{D \mid D(b_{e,i_e,1}, \ldots, b_{e,i_e,d}) \in \mathbb{Z}^d\} \). For any non-negative integer \( e \), we define an integer \( I_e \) as follows

\[
I_e := \prod_{i=1}^{d} q_i C_{1,e} b^{(dp+p-1-j)k^e}(b_{e,i_e,1}, \ldots, b_{e,i_e,d}) Q_e(\frac{1}{b k^e}) \begin{pmatrix} f_{0,1}(\frac{1}{b}) \\
    f_{0,2}(\frac{1}{b}) \\
    \vdots \\
    f_{0,d}(\frac{1}{b}) \end{pmatrix} - \prod_{i=1}^{d} q_i C_{1,e} b^{(dp+p-1-j)k^e} P_{e,i}(\frac{1}{b k^e}). \tag{4.8}
\]

By (4.5), (4.7) and Lemma 4.1, we have

\[
0 < |I_e| \leq \frac{C_{1,e} \prod_{i=1}^{d} q_i e^{e \epsilon^2 k^e} b k^e}{b k^e - e k^e} \leq \frac{C_{2}(\epsilon) d! c b^{d+1} e^{d} k^e}{b k^e} \prod_{i=1}^{d} q_i e^{e \epsilon^2 k^e} b k^e. \tag{4.9}
\]

By (4.9), Lemma 4.1 and \( p > d + 5 \), for any sufficiently large integer \( e \), we have

\[
0 < |I_e| < 1. \tag{4.10}
\]

This contradicts that \( I_e \) is an integer. This completes the proof of Theorem 1.5. \( \square \)
5 The other examples of Theorem 1.5 by related the digital pattern sequences.

In this section, we give the examples of Theorem 1.5 by related the certain digital pattern sequences as follows. Let \( A \) be a set and \( A^* \) be the free monoid generated by \( A \). Let \( A \) and \( B \) be two finite words on \( \mathbb{Z}^* \), and let \( AB \) be the concatenation of \( A \) and \( B \). Let \( m \) be a positive integer. Let \( b_i \in \mathbb{Z} \) with \( 1 \leq i \leq m \), and \( A := b_1 b_2 \cdots b_m \in \{ b_1, b_2, \cdots, b_m \}^* \). Then, for any element \( f \in \mathbb{Z} \), we define a word \( f(A) \) by,

\[
f(A) := f \cdot b_1 f \cdot b_2 \cdots f \cdot b_m \in \{ f \cdot b_1, f \cdot b_2, \cdots, f \cdot b_m \}^*,
\]

where, for any \( b \in \mathbb{Z} \), \( f(b) := f \cdot b \in \mathbb{Z} \) with denoting the multiplication on \( \mathbb{Z} \) by the dot symbol. Note that the length of \( f(A) \) as well as that of \( A \) is \( m \). The element \( f \) can be regarded as the coding on \( \mathbb{Z}^* \). (See the definition of coding 9p in [AlS2]) By Theorem 3.2 in [Mi] (See also example 3.4 in [Mi]), the following sequences \( (a(n))_{n \geq 0} \) and \( (b(n))_{n \geq 0} \) are generalized \( k \)-regular. Let \( A_0 = a, B_0 = b \) with \( a, b \in \mathbb{Z} \). We define the words \( A_{n+1} \) and \( B_{n+1} \) of length \( 2^n + 1 \) recursively as

\[
A_{n+1} := A_n B_n, \tag{5.1}
\]
\[
B_{n+1} := A_n f_n(B_n),
\]

where \( f_n \in \mathbb{Z} \). For \( a = b = 1 \), we define the \( (a(n))_{n \geq 0} \) by \( (a(n))_{n \geq 0} = \lim_{n \to \infty} A_n \). If \( f_n = -1 \) \( (n \geq 0) \), then the sequence \( (a(n))_{n \geq 0} \) is known as the Rudin-Shapiro sequence. (See 126p in [Fo].)

Remark 5.1 By th definition of the sequence \( (a(n))_{n \geq 0} \), the sequence \( (a(n))_{n \geq 0} \) has the following digital pattern definition. We define the counting functions \( d_1(n; 2^y + 2^{y+1}) \) as

\[
d_1(n; 2^y + 2^{y+1}) := \begin{cases} f_y & \text{if } 2^y + 2^{y+1} \text{ is appeared in the base-2 representation of } n \\ 1 & \text{otherwise.} \end{cases}
\]

The sequence \( (a(n))_{n \geq 0} \) has the following definition

\[
a(n) = \prod_{y=0}^{\infty} d_1(n; 2^y + 2^{y+1}). \tag{5.2}
\]

Moreover, for \( a = 1, b = -1 \), we also define the \( (b(n))_{n \geq 0} \) by \( (b(n))_{n \geq 0} = \lim_{n \to \infty} A_n \). Let \( f(z) := \sum_{n=0}^{\infty} a(n)z^n \) and \( g(z) := \sum_{n=0}^{\infty} b(n)z^n \). From the proof of Theorem 3.2 in [Mi], \( f(z) \) and \( g(z) \) has the following representation

\[
\begin{pmatrix} f(z) \\ g(z) \end{pmatrix} = \prod_{e=0}^{\infty} \begin{pmatrix} 1 + \frac{1+e}{2} z^{2^e} & \frac{1-e}{2} z^{2^e} \\ 1 + \frac{1-e}{2} z^{2^e} & \frac{1+e}{2} z^{2^e} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lim_{e \to \infty} \left( 1 + \frac{1+e}{2} z \right) \left( 1 + \frac{1-e}{2} z \right) \cdots \left( 1 + \frac{1+e}{2} z \right) \left( 1 + \frac{1-e}{2} z \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{5.3}
\]

This representation show that \( f(z) \) and \( g(z) \) satisfy the equations with \( A_e(z) = \left( 1 + \frac{1+e}{2} z, \frac{1+e}{2} z \right) \). Now we give the examples of Theorem 1.5 as follows.
Proposition 5.1 Notation is same as above. Let \( b \) be an integer with \( b \geq 2 \). Assume that, for any non-negative integer \( e \), the integer \( f_e \neq 1 \) is odd. Moreover, assume that, for any \( \epsilon > 0 \), there exists an integer \( N(\epsilon) \) such that, for any \( e \geq N(\epsilon) \),

\[
|f_e| \leq e^{\epsilon e}.
\]

(5.4)

Then at least one among the numbers \( f(\frac{1}{b}) \), \( g(\frac{1}{b}) \) is transcendental.

Proof. By (5.1), we have

\[
(a(n))_{n \geq 0} := A_n B_n A_n f_n(B_n) \cdots .
\]

(5.5)

For any non-negative integer \( n \), we define the polynomial \( P_n(z) \) as the generating function of \( A_n B_n \). We also define the polynomial \( Q_n(z) \) by \( Q_n(z) = (1 - z^{2^{n+1}}) \). By the definitions of \( P_n(z) \), \( Q_n(z) \), we have

\[
\text{deg} P_n(z), \text{deg} Q_n(z) \leq 2^{n+1}.
\]

(5.6)

From the definitions of \( P_n(z) \), \( Q_n(z) \) and (5.5), we have

\[
\text{ord}(Q_n(z)f(z) - P_n(z)) \geq 2^{n+1} + 2^n.
\]

(5.7)

By the \( f_n \neq 1 \) and the definitions of \( P_n(z) \), we have

\[
z^{2^{n+1}}(P_{n+1}(z) - z^{2^{n+1}}P_n(z)) \neq P_{n+1}(z) - P_n(z).
\]

(5.8)

From (5.8), we have

\[
P_n(z)Q_{n+1}(z) - Q_n(z)P_{n+1}(z) \neq 0.
\]

(5.9)

By (5.1), (5.7), (5.9) and Lemma 2.3 we have

\[
\mu(f(z)) < \infty.
\]

(5.10)

From (5.10), \( f(z) \) is irrational. Therefore, by Theorem 1.5 at least one among the numbers \( f(\frac{1}{b}) \), \( g(\frac{1}{b}) \) is transcendental.

By the similar way of Proposition 5.1 one can construct the other examples of Theorem 1.5 in \( k \)-recursive sequences. (See Definition 3.1 in [Mi]).

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