Anomaly Inflow for Gauge Defects

Julie Blum and Jeffrey A. Harvey

Enrico Fermi Institute, University of Chicago
5640 Ellis Avenue, Chicago, Illinois 60637
Internet: julie@yukawa.uchicago.edu
Internet: harvey@poincare.uchicago.edu

Abstract

Topological defects constructed out of scalar fields and possessing chiral fermion zero modes are known to exhibit an anomaly inflow mechanism which cancels the anomaly in the effective theory of the zero modes through an inflow of current from the space in which the defect is embedded. We investigate the analog of this mechanism for defects constructed out of gauge fields in higher dimensions. In particular we analyze this mechanism for string (one-brane) defects in six dimensions and for fivebranes in ten dimensions.
1. Introduction

In earlier work the gauge and gravitational anomalies of chiral fermion zero modes bound to scalar defects such as axion strings or domain walls were investigated [1]. It was shown that these anomalies are physically sensible when the embedding of the defect in a higher dimensional space and the possibility of charge inflow from this higher-dimensional space is included. Although fermions outside the defect are massive, they nonetheless induce a vacuum current outside the defect in the presence of external gauge fields. This current is conserved away from the defect and has a divergence on the defect which precisely cancels the anomaly present in the low-energy effective action of the chiral zero modes bound to the defect. In addition, the whole process occurs in a way which is consistent with overall gauge covariance [2]. This mechanism provides a simple physical model of the mathematical relation between chiral anomalies in $2n + 2$ dimensions and non-abelian (or gravitational) anomalies in $2n$ dimensions. There have also been recent discussions of the application of this effect both in condensed matter physics [3] and in theories of chiral lattice fermions [4].

This paper generalizes the analysis of [1] and [2] to gauge defects (constructed from Yang-Mills instantons) in $4k + 2$ dimensions. For scalar defects the starting point was a consistent anomaly free theory with a chiral coupling of the fermion fields to the scalar fields responsible for the defect. The current inflow could then be deduced from the presence of couplings of the defect to gauge fields which could be viewed as the result of integrating out the massive fermion degrees of freedom. In trying to extend this picture to gauge defects there is one main new complication. In order to obtain chiral fermion zero modes it is necessary for the gauge fields to have chiral couplings to the fermions. But, in dimensions greater than four, such chiral gauge theories are generically inconsistent because of gauge anomalies. A similar problem arises if we wish to consider energy-momentum inflow and chiral gravitational couplings. However in certain cases of physical interest it is known that the gauge and/or gravitational anomalies can be cancelled through the Green-Schwarz mechanism [5]. This involves introducing a two form field $B$ which has non-trivial transformation properties under gauge and local Lorentz transformations and certain higher dimensional couplings of $B$ to the gauge and gravitational fields. These anomaly cancelling terms, at least in the context of string theory, can be viewed as terms which arise in the low-energy effective Lagrangian as a result of integrating out the massive modes of the string. Thus these theories with anomaly cancellation are consistent gauge
invariant theories which can have gauge defect solutions with chiral fermion zero modes. By analogy with the earlier studies of scalar defects one would expect the apparent anomaly in the effective theory of the chiral zero modes to be canceled by an inflow from outside the defect. One might further expect the anomaly canceling terms to play an important role in understanding how this comes about. As we will show, these expectations are correct.

The outline of the paper is as follows. In the second section we review the inflow mechanism of [1,2] for axion strings in four dimensions. In the third section we introduce a simple model in six dimensions which has a string solution constructed using Yang-Mills instantons and in which the fermion gauge anomaly is cancelled via the Green-Schwarz mechanism. In this model we then show how the gauge anomalies in the low-energy effective action for the chiral zero modes are cancelled by inflow from the outside world. In the fourth section we extend this analysis to fivebrane solutions of the low-energy limit of heterotic string theory which are also constructed using Yang-Mills instantons. We analyze both the gauge and gravitational anomalies on the fivebrane and show how they are cancelled. The final section briefly discusses some of the implications of these results for fundamental and solitonic strings and fivebranes. While this work was in progress we received a paper by Izquierdo and Townsend [3] which addresses some of the same issues as this paper. We will comment on their work in the final section.

2. Anomaly inflow for axion strings

In this section we will briefly review the inflow mechanism of [1,2] as it applies to axion strings in four dimensions. We consider a complex scalar field \( \Phi = \Phi_1 + i\Phi_2 \) with a non-zero vacuum expectation value \( v \). An axion string configuration with winding number \( n \) is given by

\[
\Phi = f(\rho)e^{i\theta}
\]  

(2.1)

with \( \theta = n\phi \) where \((\rho, \phi)\) are polar coordinates in the plane transverse to the string and where \( f \) goes to zero at the origin and to \( v \) at infinity.

We have in addition Dirac fermion fields \( \psi \) in some representation \( r \) of a gauge group \( G \) which interact both with \( \Phi \) and with the gauge field \( A_\mu \)

\[
\mathcal{L} = \bar{\psi}iD\psi - \bar{\psi}(\Phi_1 + i\gamma^5\Phi_2)\psi.
\]  

(2.2)
Since the gauge coupling in (2.2) is vectorial there is no anomaly in the gauge current. Equivalently, the effective action resulting from integrating out the fermion fields,

\[ iS_{\text{eff}} = \log \text{Det}(i\mathcal{D} - f(\rho)e^{i\theta\gamma_5}) \]  

(2.3)
is invariant under infinitesimal gauge transformations, \( \delta A = D\Lambda \) (from here on we will use the language of differential forms with \( A \) a one-form which is also an anti-hermitian element of the Lie algebra of \( G \), \( A = A^A T^A dx^\mu \), and \( D \) the covariant exterior derivative).

The basic puzzle arises because the Dirac equation in the axion string background possesses \(|n|\) chiral two-dimensional zero modes bound to the string with the chirality determined by the sign of \( n \). It would thus seem that the coupling of these chiral fermion zero modes to the background gauge fields must be anomalous and lead to a lack of current conservation. However the full theory is clearly not anomalous. The resolution is that the effective action (2.3) has two physically distinct contributions, each one of which would separately lead to a violation of current conservation, but which combine to give a theory with a conserved and properly covariant current.

The first contribution arises from the \(|n|\) chiral fermion zero modes and is localized on the string. The current derived from the two-dimensional action for the zero modes in the presence of a two-dimensional background gauge field \( A^{(2)} \),

\[ j^{\text{cons}} = \frac{\delta S_2}{\delta A^{(2)}}, \]

(2.4)
has a covariant divergence given by the consistent anomaly,

\[ D^{(2)} j^{\text{cons}} = -\frac{n}{4\pi} d^{(2)} A^{(2)}. \]

(2.5)
(we are using conventions where the current in \( d \) dimensions is a \( d-1 \) form whose dual \( *j \) has components equal to the usual current \( j_{\mu} \).

The second contribution arises from vacuum currents induced by the non-zero fermion modes in the presence of the background gauge field and the topologically non-trivial phase of the axion field. The induced current can be calculated directly at the one-loop level or it can be inferred by removing the coupling of the phase of \( \Phi \) to the fermions by performing a chiral rotation and using the chiral anomaly \([1,2]\). If we work in the “thin string” approximation where we freeze \( f(\rho) \) to its vacuum expectation value \( v \) the second approach leads to a coupling

\[ S_\theta = -\frac{1}{8\pi^2} \int d\theta \omega_3 \]

(2.6)
with \( \omega_3 = \text{Tr}(AF - A^3/3) \). In the thin string approximation \( \theta \) is not well defined at the origin, but this can be accounted for by noting that
\[
\int_{D^2} d^2 \theta = \int_{S^1} d\theta = 2\pi n
\]  
(2.7)
where the first integral is over a disk in the two dimensions transverse to the string and the second is over the \( S^1 \) boundary of the disk. Thus we may treat \( d^2 \theta \) as being \( 2\pi n \) times a delta function in the two transverse dimensions. If we now vary \( S_\theta \) with respect to \( A \) we find a current
\[
J_\theta = \frac{1}{8\pi^2}(2Fd\theta - Ad^2\theta) \equiv J_\infty + \Delta J.
\]  
(2.8)

To see that the two contributions combine to give a covariant and conserved current we integrate the covariant divergence of the four-dimensional current \( J_\theta \) over the two dimensions transverse to the string and show that this when added to the covariant divergence of the two-dimensional current gives zero. We again consider a background gauge field \( A = A^{(2)} \) which is tangent to the string world sheet. We can then write the covariant derivative as \( D = D^{(2)} + d_T \) with \( d_T \) the exterior derivative in the transverse dimensions. Noting that \( D^{(2)}J_\infty = d_T \Delta J = 0 \) we have
\[
\int_{D^2} DJ_\theta = \int_{S^1} J_\infty + D^{(2)} \int_{D^2} \Delta J
\]
\[
= \frac{n}{2\pi} F^{(2)} - D^{(2)} \Delta j
\]  
(2.9)
where \( \Delta j = nA^{(2)}/(4\pi) \). Therefore
\[
\int_{D^2} DJ_\theta + D^{(2)} j^{\text{cons}} = \frac{n}{2\pi} F^{(2)} - \frac{n}{4\pi}(D^{(2)} A^{(2)} + d^{(2)} A^{(2)}) = 0
\]  
(2.10)
and we verify that the total current is conserved. Physically what is going on is that the coupling of the massive fermions to the gauge field and the axion induce a radial current \( J_\infty \) which flows onto the string from infinity. This current is covariant, but has a divergence on the string. This divergence is matched by the divergence of the current flowing along the string. The current along the string has two contributions. The first comes from the fermion zero modes and is necessarily not covariant since it comes from variation of a two-dimensional action \([7]\). The second contribution, \( \Delta j \), while it is localized on the string, is not derivable from a two-dimensional action but only from the four-dimensional action and is precisely the contribution to the current needed to convert the consistent current into the covariant current \([8]\).
An alternative treatment which is closely related to the theories we will discuss in
the next two sections involves writing the theory in a dual formulation with a three-formield strength \( H = v^* d\theta \). The dual formulation is more natural from a geometric point
of view because the two-form \( B \) has a local coupling to the string world-sheet while the
zero form \( \theta \) does not. This formulation has been discussed in \([2]\) and more recently in
\([9]\) so we will be brief. The dual action is written in terms of the two-form \( B \) and its
field strength \( H = dB - \alpha' \omega^3 \) where \( \omega^3 = \text{Tr} (AF - (1/3) A^3) \) is the Chern-Simons three-
form and \( \alpha' = 1/(16\pi^2 v) \). \( H \) is gauge invariant provided that \( B \) transforms under gauge
transformations as \( \delta B = \alpha' \omega^2_1 \equiv \alpha' \text{Tr}(\Lambda dA) \). The action for \( B \) is then
\[
S_B = \int H^* H + 4\pi n v \int B^* V
\]
where \( V \) is the volume two-form on the string world-sheet. Varying \( S \) with respect to \( A \)
we find a current given by
\[
J = 2\alpha' (2F^* H - Ad^* H) = 2\alpha' (2F^* H - 4\pi n v A^* V)
\]
where we have also used the \( B \) equation of motion \( d^* H = 4\pi n v^* V \). The first term in
\( (2.12) \) agrees with \( J_\infty \) while the second term gives the correction to the current \( \Delta J \).

3. Six-dimensional Instanton String

We now turn to a theory which has chiral fermions coupled to gauge fields in six space-
time dimensions. We can use the usual four-dimensional Euclidean Yang-Mills instanton
to construct a one-brane (string) topological defect in this theory.

Our starting point is the action for a gauge field with gauge group \( G \), Weyl fermions
in the representation \( r \) of \( G \), and an antisymmetric tensor field \( B_{\mu\nu} \) which will be necessary
for anomaly cancellation.

\[
S_0 = \int d^6 x \left( -\frac{1}{4g^2} (F_{\mu\nu} F^{\mu\nu}) + \frac{1}{2} \bar{\lambda} (\not{D} \lambda) - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right)
\]
The gauge field strength two-form and antisymmetric tensor field strength three-form are
\[
F = dA + A^2
\]
\[
H = dB - \alpha' \omega^3.
\]
With our conventions the gauge coupling $g$ has dimensions of length and $\alpha'$ is an independent dimensionless coupling. Traces will always be taken in the fermion representation $\mathbf{r}$. The angular brackets denote the Cartan inner product in the Lie algebra of $G$

$$\langle T^A T^B \rangle = -C \operatorname{Tr}(T^A T^B) = \delta^{AB}. \quad (3.3)$$

The normalization constant $C$ will be specified later when we consider specific examples. The bosonic part of the action can be written as

$$S_B = \frac{1}{2} \int \left( \frac{1}{g^2} (F^* F) + H^* H \right) \quad (3.4)$$

and is invariant under the infinitesimal gauge transformation

$$\delta_A A = DA$$
$$\delta_A B = \alpha' \omega_2^1$$ \quad (3.5)

where $\omega_2^1 = \operatorname{Tr}(A dA)$ obeys $\delta_A \omega_3 = d\omega_2^1$.

While this theory is classically gauge invariant, at the one-loop level it has a gauge anomaly which can be determined using the descent equations starting from the chiral anomaly in 8 dimensions:

$$I_8(F) = \frac{1}{(2\pi)^4 (4!)} \operatorname{Tr}(F^4). \quad (3.6)$$

In order for the anomaly cancellation mechanism of [5] to work it is necessary that the quartic trace in (3.6) factorize in the form

$$\operatorname{Tr} F^4 = c (\operatorname{Tr}(F^2))^2 \quad (3.7)$$

for some constant $c$. The anomaly derived from (3.6) by descending to six dimensions is then

$$G_6(\Lambda) = - \int \operatorname{Tr}(A D J_f)$$
$$c' \int w_2^1 \operatorname{Tr}(F^2), \quad (3.8)$$

where $c' = c/(4!(2\pi)^3)$ and $J_f$ is the fermion gauge current.

Given the factorization (3.7) we can add a term to the action which cancels the anomaly by utilizing the non-trivial gauge variation of $B$

$$\Delta S = -\frac{c'}{\alpha'} \int B \operatorname{Tr}(F^2). \quad (3.9)$$
The action \( S = S_0 + \Delta S \) then defines a gauge invariant theory (in this section we will ignore potential gravitational anomalies).

For the axion string it was clear that the overall gauge current must be conserved since there were no gauge anomalies. What is the situation here? If we integrate out the fermions, then the resulting effective action, as a function of \( A \) and \( B \), is gauge invariant by the above construction. This implies that

\[
\delta_{\Lambda} S_{\text{eff}} = \int \text{Tr}(D\Lambda \frac{\delta S_{\text{eff}}}{\delta A}) + \alpha' \omega_2 \frac{\delta S_{\text{eff}}}{\delta B} = 0.
\]

Integrating by parts on the first term we see that there is a conserved current \( J = \delta S_{\text{eff}}/\delta A \) for backgrounds obeying the \( B \) equation of motion \( \delta S_{\text{eff}}/\delta B = 0 \).

The equations of motion which follow from variation of \( S \) are

\[
d^* H = -c' \frac{\alpha'}{\alpha'} \text{Tr}(FF)
\]

\[
-\frac{1}{g^2} D^* F = J_f + J_b
\]

(3.11)

where \( J_f \) is the fermionic contribution to the current and

\[
J_b = -2\alpha'F^* H + \alpha' Ad^* H - \frac{2c'}{\alpha'} FdB
\]

(3.12)

is the bosonic contribution to the current. Using (3.8), the first of (3.11), and (3.12) we check that the total current is covariantly conserved, \( D(J_b + J_f) = 0 \).

We now want to analyze current conservation in a background consisting of an instanton string. This is a configuration with topology \( M^2 \times R^4 \) with \( M^2 \) being two-dimensional Minkowski space. We can split the exterior derivative and gauge one-form into two- and four-dimensional parts as

\[
d = d^{(2)} + d^{(4)}; \quad A = A^{(2)} + A^{(4)}.
\]

(3.13)

For the gauge fields we take \( A^{(4)} \) to be a Yang-Mills instanton, embedded in a \( SU(2) \) subgroup of the gauge group \( G \). In order to study current inflow we then turn on an additional gauge field \( A^{(2)} \) which lives in a subgroup of \( G \) orthogonal to the instanton \( SU(2) \). We will call this subgroup \( G' \). We assume for simplicity that \( A^{(4)} \) and \( A^{(2)} \) depend only on the coordinates on \( R^4 \) and \( M^2 \) respectively.

As in the axion string case, the fermion contribution to the current has two parts, one localized on the string and the other flowing in from infinity. The new feature is
that there is also a bosonic contribution to the current coming from the couplings of the antisymmetric tensor field.

We will first analyze the fermionic contribution. In order to keep the group theory to a minimum we will first do the analysis for a simple choice of the gauge group. We take $G = SU(3)$ with the fermions in the 3 of $SU(3)$ and set $T^A = -i \lambda^A / 2$ with the $\lambda^A$ the usual Gell-Mann matrices. We then have $\mathcal{C} = 2$ and $c = 1/2$. We imbed the instanton in the minimal $SU(2)$ subgroup $SU(3) \supset SU(2) \times U(1)$ with

$$3 \rightarrow 2(\frac{1}{2\sqrt{3}}) + 1(\frac{1}{\sqrt{3}}).$$

An instanton of topological charge

$$p = \frac{1}{8\pi^2} \int \text{Tr} F(4) F(4).$$

will give rise to $p$ chiral fermion zero modes on the string, each carrying charge $q = 1/(2\sqrt{3})$ under the unbroken gauge group $G' \equiv U(1)$.

There are two fermionic contributions to the current, but unlike the axion string model where their divergence added to zero, here the total covariant divergence must equal the six-dimensional consistent anomaly. The covariant divergence of the $U(1)$ component of the six-dimensional consistent current, integrated over the space transverse to the string is

$$\int_{R^4} \text{Tr}(T^8 D j^\text{cons}_f) = -c' \int_{R^4} \text{Tr}(T^8 d(2) A^{(2)}) \text{Tr} F(4) F(4)$$

$$= -\frac{c}{24\pi} \text{Tr}(T^8 d(2) A^{(2)})p = -\frac{p}{96\pi} d(2) A_8^{(2)}.$$ (3.16)

Here $T_8 = -i \text{diag}(1,1,-2)/2\sqrt{3}$ is the $U(1)$ generator and $A_8 = -2 \text{Tr} T_8 A$ is the $U(1)$ component of the gauge field. The two fermion contributions to the current must sum to give (3.16).

The contribution from the chiral zero modes is given by the two-dimensional consistent anomaly and is

$$D^{(2)} j_8^\text{cons} = \frac{q^2}{4\pi} d(2) A_8^{(2)} p = \frac{p}{48\pi} d(2) A_8^{(2)}.$$ (3.17)

We thus infer that there must be a contribution from the non-zero mode fermions, $j_f^{nz}$ which is not purely two-dimensional and which satisfies

$$\int_{R^4} \text{Tr}(T^8 D j_f^{nz}) = -\frac{p}{96\pi} d(2) A_8^{(2)}.$$ (3.18)
We now turn to the bosonic contribution to the current. Since \( D(J_b + J^\text{cons}_b) = 0 \) it is clear that the total current is conserved, i.e. that

\[
D^{(2)} j^\text{cons}_8 + \int_{R^4} \text{Tr} T^8 D(J_b + J^\text{nz}_b) = 0
\]

(3.19)

but it is interesting to note that \( J_b \) has physically distinct contributions. The first term in the bosonic current (3.12) falls off as \( 1/r^3 \) in an instanton background and gives rise to a radial current inflow at infinity. The integral from this first term is

\[
\int_{R^4} \text{Tr} T^8 D(-2\alpha' F^{(2)*} H) = -\frac{p}{48\pi} F^{(2)}_8.
\]

(3.20)

The second term falls off much faster with \( r \) and plays the role of the term \( \Delta J \) in the axion string discussion which was needed to convert the consistent current to the covariant current. In the axion string example it was a delta function on the string because we were working in the thin string approximation. Here it is smeared over the whole core of the instanton. The role of this term will be clearer when we consider a non-abelian \( G' \) since for \( G' = U(1) \) the divergence of the consistent current is “accidentally” covariant. The integral of this second term is

\[
\int_{R^4} \text{Tr} T^8 D(\alpha' A^{(2)} d^* H) = \frac{p}{96\pi} D^{(2)} A^{(2)}_8.
\]

(3.21)

The third term in \( J_b \) is closed and does not contribute to the current inflow.

As a second example which will bring out the role of the bosonic current terms and also illustrate some of the group theoretical subtleties we take the gauge group to be \( G = E_6 \supset SU(6) \times SU(2) \) with fermions in the adjoint 78 and the embedding given by

\[
78 \rightarrow (35, 1) + (20, 2) + (1, 3).
\]

(3.22)

If we normalize the \( E_6 \) generators so that they obey the previous normalization for an \( SU(2) \) subgroup we find \( \mathcal{C} = 1/12 \). Since \( E_6 \) has no independent quartic Casimir (3.7) is satisfied and one can show that \( c = 1/32 \) using the \( SU(2) \) embedding (3.22). An \( SU(2) \) instanton with topological charge \( p \) gives rise to \(|n| \) zero modes with

\[
n = \frac{1}{8\pi^2} \int_{R^4} \text{Tr} F^{(4)} F^{(4)} = 24p.
\]

(3.23)

From the decomposition (3.22) we see that \( 4p \) of these zero modes are singlets under \( G' = SU(6) \) while the rest transform as \( p \) 20’s of \( SU(6) \).
Evaluating the total fermion contribution to the anomaly for a particular \( SU(6) \) current with generator \( T^A \) as in (3.16) we now find

\[
\int_{R^4} \text{Tr}(T^A D J^{\text{cons}}_f) = -\frac{c}{24\pi} \text{Tr}(T^A d^{(2)} A^{(2)}) = -\frac{p}{32\pi} \text{Tr}(T^A d^{(2)} A^{(2)}). \tag{3.24}
\]

The contribution from the chiral zero modes is again given by the consistent anomaly as

\[
\text{Tr} T^A D^{(2)} j^{\text{cons}} = -\frac{p}{4\pi} \text{Tr}_{20} T^A d^{(2)} A^{(2)}. \tag{3.25}
\]

It is crucial to note that in (3.25) the trace on the right hand side is in the representation of the chiral zero modes, that is the \( 20 \) of \( SU(6) \). The traces in (3.24) and (3.25) can be related by noting that for \( SU(6) \), \( \text{Tr}_{35}(T^A T^B) = 2 \text{Tr}_{20}(T^A T^B) \). Thus the two-dimensional contribution can be written as

\[
\text{Tr} T^A D^{(2)} j^{\text{cons}} = -\frac{p}{16\pi} \text{Tr}(T^A d^{(2)} A^{(2)}). \tag{3.26}
\]

and as before is twice the contribution (3.24). The fact that these factors work out in this way may at first sight seem miraculous. The total fermion contribution (3.24) depends on the constant \( c \) appearing in the factorization (3.7) while the two-dimensional contribution depends on the embedding of \( G' \) in \( G \) and the relative normalization of the two traces. Of course these two factors are not independent as can be seen by calculating \( c \) by choosing the generators to be in \( G' \) and using the embedding of \( G' \) in \( G \). As before we deduce that there must be an additional contribution from the non-zero mode fermions which when added to (3.26) gives the total fermion contribution (3.24):

\[
\int_{R^4} \text{Tr} T^A D j^{\mu}_f = \frac{p}{32\pi} \text{Tr} T^A d^{(2)} A^{(2)}. \tag{3.27}
\]

Turning now to the bosonic current there are again two terms that contribute. The first is

\[
-2\alpha' \int_{R^4} \text{Tr} T^A D(F^* H) = \frac{2c}{(2\pi)^4} \text{Tr}(T^A F^{(2)}) \int_{R^4} \text{Tr} F^{(4)} F^{(4)} = \frac{p}{16\pi} \text{Tr}(T^A F^{(2)}). \tag{3.28}
\]

while the second is

\[
\alpha' \int_{R^4} \text{Tr}(T^A D(Ad^* H)) = -\frac{p}{32\pi} \text{Tr} T^A d^{(2)} A^{(2)}. \tag{3.29}
\]
Adding together the contributions (3.28), (3.29), (3.27), and (3.26) we find

\[ \frac{p}{16\pi} \text{Tr} T^A \left( -F^{(2)} + \frac{1}{2} D^{(2)} A^{(2)} - \frac{1}{2} d^{(2)} A^{(2)} + d^{(2)} A^{(2)} \right) = 0. \] (3.30)

Note that the first contribution to the bosonic current is covariant and gives the inflow from infinity while the second term is of the from \( D^{(2)} \Delta J \) found in the axion string analysis. We would also expect the contribution from the non-zero mode fermions to have two physically distinct contributions.

It would be nice to have a better understanding of the non-zero-mode contribution to the fermion current. A current \( J_{nz}^f \) which satisfies (3.27) or (3.18) can be obtained by varying the local action

\[ S_{nz} = c' \int \omega^G_G \omega^G SU(2) \] (3.31)

where the superscripts indicate that the Chern-Simons terms are to be evaluated only in the given subgroup. Varying (3.31) gives

\[ J_{nz}^f = c' (2F^G_G \omega^SU(2) SU - A^G_G d\omega^SU(2)) \] (3.32)

which as mentioned above contains a covariant part which falls off as \( 1/r^3 \) and a non-covariant term localized on the string. However the action (3.31) cannot be obtained from a local \( G \) invariant action since \( \omega^G_G \omega^G = 0 \). We can also understand the action (3.31) as follows. At large distances from the string the \( SU(2) \) gauge field approaches a pure gauge configuration. We can thus remove the coupling of the \( SU(2) \) gauge field to the fermions by a gauge transformation. However, because of the anomaly in the fermion current given by (3.8), this induces an effective coupling between the \( SU(2) \) and \( G' \) gauge fields given by (3.31). This argument only gives the action (3.31) far from the string.

4. Fivebranes in Ten Dimensions

We now turn to an analysis of fivebrane solutions in ten dimensions. Our analysis will parallel that in the previous section except that now we will include both the gauge, gravitational, and mixed anomalies in our analysis.

A number of fivebrane solutions to string theory are known [10]. We will consider the “gauge” solution originally discovered by Strominger [11]. The bosonic action we start
with is the bosonic sector of \( N = 1 \) ten-dimensional supergravity coupled to \( N = 1 \) super Yang-Mills with gauge group \( G = SO(32) \) or \( E_8 \otimes E_8 \). This action is

\[
S = \int d^{10}x \sqrt{g} e^{-2\phi} \left( R + 4 \partial_\mu \phi \partial^\mu \phi - \frac{1}{3} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{\alpha'}{30} \text{Tr} F_{\mu\nu} F^{\mu\nu} \right). \tag{4.1}
\]

where

\[
H = dB - \frac{\alpha'}{30} \omega_3^Y + \alpha' \omega_3^L
\]

and \( \omega_3^Y (\omega_3^L) \) is the Chern-Simons three-form for the gauge field (spin connection).

The inclusion of fermion fields gives an anomalous theory which allows for anomaly cancellation as discovered by Green and Schwarz. The anomaly in this theory is determined by a twelve-form \( I_{12} \) which factorizes in the form

\[
I_{12} = \frac{1}{(2\pi)^6 96} X_4 X_8 \tag{4.3}
\]

with

\[
X_4 = \text{Tr} R^2 - \frac{1}{30} \text{Tr} F^2 \tag{4.4}
\]

and

\[
X_8 = \frac{1}{24} \text{Tr} F^4 - \frac{1}{7200} (\text{Tr} F^2)^2 - \frac{1}{240} \text{Tr} F^2 \text{Tr} R^2 + \frac{1}{8} \text{Tr} R^4 + \frac{1}{32} (\text{Tr} R^2)^2. \tag{4.5}
\]

Traces for the curvature are in the fundamental representation of \( SO(9,1) \). For the forms appearing in the above equations we define other forms by descent as

\[
X_{4n-2}^1(\Lambda) = \text{Tr}(\Lambda X_{4n-2}^A)
\]

\[
dX_{4n-2}^1 = \delta_\Lambda X_{4n-1}
\]

\[
dX_{4n-1} = X_{4n}
\]

The gauge anomaly derived from \( I_{12} \) then has the form

\[
G_{10}(\Lambda) = - \int \text{Tr}(\Lambda D J_{fA})
\]

\[
= c' \int \left( \frac{2}{3} X_6^1(\Lambda) X_4 + \frac{1}{3} X_2^1(\Lambda) X_8 \right), \tag{4.7}
\]

and includes the contribution of Majorana-Weyl fermions of the gauge group where

\[
c' = \frac{1}{96(2\pi)^5} \tag{4.8}
\]
There is also a gravitational anomaly which includes a spin $3/2$ gravitino piece as well as
the gauge group fermionic contribution

\[
G_{10}(\Theta) = \int d^{10}x \, e^\mu \nabla^\nu T_f \mu \nu
\]

\[
= c' \int \left( \frac{2}{3} \, X_6^1(\Theta) \, X_4 + \frac{1}{3} \, X_2^1(\Theta) \, X_8 \right)
\]

and results from a lack of invariance under coordinate transformations

\[
x^\mu \rightarrow x^\mu + \epsilon^\mu
\]

with $\Theta_{\mu \nu} = \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu$. If we consider local Lorentz transformations, this anomaly has
the same form, but the non-conserved current is $J_{f R}$ instead of the energy-momentum

tensor.

Using the variation of $B$, $\delta B = -\alpha' X_2^1$, these anomalies are canceled by adding a
counterterm to the action given by

\[
\Delta S = c' \int \left( \frac{1}{\alpha'} (BX_8) - \frac{2}{3} X_3 X_7 \right).
\]

In the analysis that follows the dilaton field and its equation of motion do not play any
role. We therefore set $\phi = 0$ for simplicity.

In the general case($E_8 \otimes E_8$ or $SO(32)$) we obtain the following equations of motion
for $B$ and $A$

\[
d^* H = \frac{c'}{4 \alpha'} X_8
\]

\[
\frac{4 \alpha'}{30} D^* F = J_{fA} + J_{BA}
\]

where

\[
J_{BA} = \frac{4}{15} \alpha' F^* H - \frac{2}{15} \alpha' Ad^* H
\]

\[
- \frac{2c'}{45} F X_7 + \frac{c'}{45} A X_8 - \frac{2c'}{3} Y_{A9}.
\]

We will not need the precise form of the nine-form $Y_{A9}$ but only the fact that $DY_{A9} = X_6^A X_4$. We then can check that

\[
D J_{BA} = \frac{c'}{90} dAX_8 - \frac{2c'}{3} X_6^A X_4 = - DJ_{fA}.
\]

Turning to gravitational anomalies, they can be handled similarly to gauge anomalies
by focusing on the special case of local Lorentz transformations of the spin connection
\( \omega \). We can define a current \( J_R \) associated with these transformations and demand its conservation. The curvature two-form is

\[
R = d\omega + \omega^2. \tag{4.15}
\]

The above action should, thus, be invariant under gauge transformations of \( \omega \)

\[
\delta_\Theta \omega = D\Theta. \tag{4.16}
\]

Varying the action with respect to \( \omega \) generates the current

\[
J_R = J_{fR} + J_{BR}
\]

\[
J_{BR} = -8\alpha' R^*H + 4\alpha' \omega d^*H
\]

\[
+ \frac{4}{3} c' RX_7 - \frac{2}{3} c' \omega X_8
\]

\[
- \frac{2}{3} c' Y_{R9} + dB R Y_{R4}
\]

where \( DY^{9R} = X^{R6}_0 X_4 \) and \( dY_{R4} = 0 \). Using (4.8), (4.12), and (4.17), we see that \( DJ_R = 0 \).

Since the ten dimensional theory is anomaly free, we can proceed as in section three to examine current inflow for an instanton fivebrane background \((M^6 \times R^4)\). Fields and derivatives are split between \( M^6 \) and \( R^4 \) as discussed previously. The instanton is embedded in an \( SU(2) \) subgroup of the first \( E_8 \) such that \( E_8 \supset SU(2) \times E_7 \) with the decomposition \( 496 \rightarrow (3,1) + (1,133) + (2,56) \). The fivebrane spin connection is an \( SO(5,1) \) gauge field, and the Lorentz group decomposition is \( SO(9,1) \supset SO(5,1) \times SO(4) \) where \( 10 \rightarrow (6,1) + (1,4) \). The number of zero modes contributing to the fivebrane anomaly is \( |n| \) where

\[
n = \frac{1}{8\pi^2} \int_{R^4} \text{Tr} F^{(4)} F^{(4)} = 60p \tag{4.18}
\]

with \( p \) the \( SU(2) \) topological charge. All zero modes contribute to the gravitational anomaly but only the \( p \) 56’s of \( E_7 \) are involved in the gauge anomaly. Since the zero modes are all singlets under the second \( E_8 \) factor, we ignore it in what follows.

Substituting the instanton solution gives the following total fermionic covariant divergence on the fivebrane for the \( E_7 \) and Lorentz currents

\[
\int_{R^4} \text{Tr}(T^A_{E_7} D_{fR}^{cons}) = \frac{p}{48(2\pi)^3} \text{Tr}(T^A_{E_7} d^{(6)} A^{(6)}) \left( \frac{-1}{900} \text{Tr}(F^{(6)})^2 + \frac{1}{60} \text{Tr}(R^{(6)})^2 \right)
\]

\[
\int_{R^4} \text{Tr}(T^{SO(5,1)}_{fR} D_{fR}^{cons}) = \frac{p}{48(2\pi)^3} \left[ \text{Tr}(T^{SO(5,1)} d^{(6)} \omega^{(6)}) \left( \frac{1}{60} \text{Tr}(F^{(6)})^2 - \frac{5}{24} \text{Tr}(R^{(6)})^2 - \frac{1}{6} Y_{R6} \right) \right]. \tag{4.19}
\]
The six-forms $Y_{R6}(Y_{G′6})$ are determined by descent from $\text{Tr} R^4 \ (\text{Tr}_{G′} F^4)$.

The zero mode consistent anomaly on the fivebrane is determined from
\begin{equation}
I_8 = -\frac{1}{(2\pi)^4} \left[ \rho \left( \frac{1}{24} \text{Tr} F^4 - \frac{1}{96} \text{Tr} F^2 \text{Tr} R^2 \right) + \frac{n}{128} \left( \frac{1}{45} \text{Tr} R^4 + \frac{1}{36} (\text{Tr} R^2)^2 \right) \right] \tag{4.20}
\end{equation}
where $n = 60p$ here and the gauge traces are in the 56 of $E_7$. Using the following relations
\begin{equation}
\text{Tr}_{56} F^4 = \frac{1}{24} (\text{Tr}_{56} F^2)^2 \tag{4.21}
\end{equation}
\begin{equation}
\text{Tr}_{133} F^2 = 3 \text{Tr}_{56} F^2 \tag{4.22}
\end{equation}
and multiplying $I_8$ by one-half to account for a Majorana condition on the zero modes, one calculates that
\begin{align}
\text{Tr}(T^A_{E_7} D^{(6)} J^\text{cons}_E) &= \frac{3}{2} \int_{R^4} \text{Tr}(T^A_{E_7} D J^\text{cons}_f) \\
\text{Tr}(T^A_{SO(5,1)} D^{(6)} J^\text{cons}_R) &= \frac{3}{2} \int_{R^4} \text{Tr}(T^A_{SO(5,1)} D J^\text{cons}_f_R) \tag{4.23}
\end{align}

We again deduce that there is a non-zero mode fermion contribution
\begin{align}
J^n_{fE} &= c' \frac{F X_7}{90} - c' \frac{AX_8 - Y_9}{180} \\
J^n_{fR} &= -c' \frac{2RX_7 - \omega X_8 + Y_R}{6} \tag{4.24}
\end{align}
so that
\begin{align}
\int_{R^4} DJ^n_{fE} &= -\frac{1}{3} D^{(6)} J^\text{cons}_{E_7} \\
\int_{R^4} DJ^n_{fR} &= -\frac{1}{3} D^{(6)} J^\text{cons}_{fR} \tag{4.25}
\end{align}
As in the six-dimensional case there is also a bosonic contribution to the inflow which consists of a covariant current at infinity and a term localized on the fivebrane which converts the six-dimensional consistent current into a covariant current which correctly matches the inflow from infinity.

The $SO(32)$ theory can be analyzed in a similar fashion by embedding the instanton in the first $SU(2)$ of $SO(32) \supset SU(2) \times SU(2) \times SO(28)$ where $496 \to (3, 1, 1) + (1, 3, 1) + (1, 1, 378) + (2, 2, 28)$. Note that the embedding $SO(32) \supset SU(2) \times SO(29)$ used in [6] is not the minimal embedding and corresponds to a superposition of two minimal instantons [12]. The number of zero modes is again $|n|$ with $n = 60p$ so that there are $p \ 2 \times 28$’s of $SU(2) \times SO(28)$ and four singlets. In either the $E_8 \times E_8$ or $SO(32)$ model we infer a non-zero mode anomaly that is $-1/3$ times the consistent anomaly. We can derive, for either choice of the group $G$, the non-zero mode contribution from the following local action:
\begin{equation}
S_{nz} = \frac{c'}{3} \int Y^G_{7'} \omega_3^{SU(2)} \tag{4.26}
\end{equation}
where $dY^G_{7'} = (2\pi)^3 24 I_8$, and $G'$ is the fivebrane gauge group.
5. Discussion

Topological defects which are solutions to gauge invariant theories but which have chiral fermion zero modes which lead to apparent anomalies in the low-energy theory on the defect must have a mechanism to cancel this anomaly by inflow from the outside world. For scalar defects this inflow can be determined fairly directly from the coupling of fermion fields to the defect and the background gauge fields. For gauge defects we have seen that the situation is a bit more subtle and involves contributions both from the fermion fields and from the additional interactions of bosonic fields which are required for anomaly cancellation in the underlying theory. A weak point in our analysis is the contribution from the non-zero mode fermions. It would be nice to have a more direct derivation and understanding of the actions (3.31) and (4.25) and the corresponding contributions to the currents.

There is a close relation between anomaly cancellation in spacetime, and the cancellation of sigma-model anomalies in the world-brane theory of defects as was pointed out for the heterotic string in [13]. In [6] this relation is explored in some detail and it is argued that it is possible to redefine the current so that there is no inflow from infinity. However, as also pointed out in [6], this redefinition is only sensible for closed defects which can be enclosed in a sphere of finite radius. Since such configurations are topologically trivial, non-static, and do not possess fermion zero modes (at least for smooth configurations) the implications of the analysis in [6] are not completely clear to us. The situation for singular (fundamental) solutions seems different as these have fermion zero modes even when topologically trivial. In any event, for infinite defects with non-trivial topology there is definitely a current inflow from infinity which is needed to make physical sense of the anomaly in the low-energy effective theory on the defect.

It is also possible to turn the inflow argument around and to deduce the presence of fermion zero modes on a defect given the behavior of the fields around the defect at large distances. This applies to both solitonic and fundamental objects. As an example consider a macroscopic fundamental string [14,15,16]. In the ground state there are no gauge fields excited outside the string and an $H$ field satisfying

$$d^*H = g^*V$$

(5.1)
with $g$ a coupling constant. If we now turn on spacetime gauge and/or gravitational fields
tangent to the string worldsheet there will be an inflow given the first terms in the currents
$J_{BA}$ and $J_{BR}$:

$$J_{BA} = \frac{1}{30} (8\alpha' F^* H - 4\alpha' Ad^* H) \quad J_{BR} = (-8\alpha' R^* H + 4\alpha' \omega d^* H). \quad (5.2)$$

As discussed in [14] and [6], with the correct normalization of (5.1) this inflow precisely
matches the sigma-model anomaly of the heterotic string which is derived by descent from
the four-form

$$X_4 = -\frac{1}{16\pi^2} \left( \frac{1}{30} \text{Tr} F^2 - \text{Tr} R^2 \right) \quad (5.3)$$

In [17] it was pointed out that $X_4$ agrees with the result for two-dimensional Yang-Mills and
gravitational anomalies for a string with 32 left-moving fermions and 8 right-moving
fermions

$$I_4 = -\frac{1}{16\pi^2} \left( \text{tr} F^2 - \frac{r}{24} \text{Tr} R^2 \right) \quad (5.4)$$

with $r = 32 - 8 = 24$ and the gauge trace in the vector representation (for $E_8$ the trace
is in the vector representation of the $SO(16)$ subgroup of $E_8$). This is apparently the field
content of the heterotic string. It was further argued that it should be possible to deduce
fivebrane anomalies starting from the form $X_8$ appearing in the factorization (4.3). This
argument was criticized (correctly we believe) in [6]. The problem is that there are no
two-dimensional gravitational anomalies in the heterotic string [18] and the 32 left-moving
fermions do not couple to the space-time spin connection and thus do not contribute at all
to the sigma-model anomaly. Thus the equivalence between $X_4$ and $I_4$ is in the words of
[6] a “curious fact” which is not relevant to the anomaly cancellation.

The above brings out an apparent difference between fundamental and solitonic strings
or defects. The fermion zero modes of solitonic defects, such as those discussed in earlier
sections, necessarily couple to the spacetime spin connection because they inherit this
coupling from the spacetime fields from which they are constructed. This is not the case
for fundamental strings in the usual formulation or presumably for fundamental fivebranes
if such objects exist.

A possible although rather speculative explanation for the curious equivalence between
$I_4$ and $X_4$ would be provided by the existence of a smooth soliton string solution with the
same zero mode structure and long-range fields as the heterotic string but with zero-modes
coupling universally to the spacetime spin-connection. The solution proposed in [19] does
not satisfy this criterion because it has long-range gauge fields and a different zero-mode structure.

One could also speculate about such an equivalence for solitonic and fundamental fivebranes [17]. Here the situation is even murkier because the soliton solution has long-range gauge fields and the fermion zero modes only transform under a subgroup of the full gauge group (a $(28,2)$ of $SO(28) \times SU(2)$ in the case of the minimal charge fivebrane with $SO(32)$ gauge group). On the other hand one might expect the ground state of a fundamental fivebrane to have no long-range gauge fields and to act only as a source for the six-form potential which appears in the dual formulation of $d = 10$ supergravity. Furthermore the zero mode content conjectured in [17] for a fundamental fivebrane consists of a 32 of $SO(32)$ and does not agree with the zero mode content of the known soliton fivebrane when restricted to a $SO(28) \times SU(2)$ subgroup. Thus while the inflow argument gives definite information about how the zero modes of a fundamental fivebrane would have to couple to the spacetime spin and gauge connection, the relationship between these zero modes and those of the known soliton solutions remains obscure.

Acknowledgements

J. H. would like to thank C. Callan, S. Naculich and A. Strominger for discussions. This work was supported in part by the by NSF Grant No. PHY90-00386. J.H. also acknowledges the support of NSF PYI Grant No. PHY-9196117.
References

[1] C. Callan and J. A. Harvey, Nucl. Phys. B250 (1985) 427.
[2] S. Naculich, Nucl. Phys. B296 (1988) 837.
[3] D. Boyanovsky, E. Dagotto and E. Fradkin, Nucl. Phys. B285 (1987) 340; M. Stone and F. Gaitan, Ann. Phys. 178 (1987) 89; G. E. Volovik, “Vortex motion in fermi superfluids and Callan-Harvey effect”, preprint 1993.
[4] D. Kaplan, Phys. Lett. 288B (1992) 342.
[5] M. Green and J. Schwarz, Phys. Lett. 149B (1984) 117.
[6] J. M. Izquierdo and P. K. Townsend, preprint DAMTP/R-93/18.
[7] J. Wess and B. Zumino, Phys. Lett. B37 (1971) 95.
[8] W. A. Bardeen and B. Zumino, Nucl. Phys. B244 (1984) 421.
[9] K. Lee, “The Dual Formulation of Cosmic Strings and Vortices”, preprint CU-TP-588 (1993).
[10] see C. Callan, J. A. Harvey, and A. Strominger, “Supersymmetric String Solitons” in String Theory and Quantum Gravity ’91, Ed. J. Harvey, R. Iengo, K. Narain, S. Randjbar-Daemi and H. Verlinde, World Scientific (1992) for a review.
[11] A. Strominger, Nucl. Phys. B343 (1990) 167.
[12] C. W. Bernard, N. H. Christ, A. H. Guth and E. Weinberg, Phys. Rev. D16 (1977) 2967.
[13] C. M. Hull and E. Witten, Phys. Lett. 160B (1985) 398.
[14] E. Witten, Phys. Lett. 153B (1985) 243.
[15] A. Dabholkar and J. A. Harvey, Phys. Rev. Lett. 63 (1989) 719.
[16] A. Dabholkar, G. Gibbons, J. A. Harvey and F. Ruiz Ruiz, Nucl. Phys. B340 (1990) 33.
[17] J. A. Dixon, M. J. Duff and J. C Plefka, Phys. Rev. Lett. 69 (1992) 3009.
[18] D. J. Gross, J. A. Harvey, E. Martinec and R. Rohm, Nucl. Phys. B256 (1985) 253.
[19] M. J. Duff and J. X. Lu, Phys. Rev. Lett. 66 (1991) 1402.