Simple metric for a magnetized, spinning, deformed mass

V. S. Manko† and E. Ruiz‡

†Departamento de Física, Centro de Investigación y de Estudios Avanzados del IPN,
A.P. 14-740, 07000 Ciudad de México, Mexico
‡Instituto Universitario de Física Fundamental y Matemáticas,
Universidad de Salamanca, 37008 Salamanca, Spain

Abstract

We present and discuss a 4-parameter stationary axisymmetric solution of the Einstein-Maxwell equations able to describe the exterior field of a rotating magnetized deformed mass. The solution arises as a system of two overlapping corotating magnetized non-equal black holes or hyperextreme disks and we write it in a concise explicit form very suitable for concrete astrophysical applications. An interesting peculiar feature of this solution is that its first four electric multipole moments are zeros; it also has a non-trivial extreme limit which we elaborate completely in terms of four polynomial factors. We speculate that the formation of the binary configurations of this type, which is accompanied by a drastic change of the system’s total angular momentum due to strong dragging effects, might be one of the mechanisms giving birth to relativistic jets in the galactic nuclei.

PACS numbers: 04.20.Jb, 04.70.Bw, 97.60.Lf
I. INTRODUCTION

During the last decade there has been considerable interest in the observational confirmation of the nature of the known black hole (BH) candidates as yet another possible test of general relativity [1, 2]. It is clear that for being able to recognize a Kerr BH [3] by analyzing the data obtained in a real astronomical observation, it is also necessary to know well the properties of the non-Kerr spacetimes which could only slightly differ from those of a BH. Moreover, proposals for the study of the properties of black holes with electromagnetic radiation [4, 5] make desirable the knowledge of simple models for compact magnetized objects permitting a clear physical interpretation. Such models may arise in particular from the extended 2-soliton electrovac solution with equatorial symmetry [6] determining a large 6-parameter family of the two-body configurations and permitting to analytically approximate the exterior fields of compact astrophysical objects. Note that in order to form a configuration with reflection symmetry, a two-body system must be composed either of two separated identical corotating constituents, or by two nonequal overlapping constituents with a common center and with a corotation or a counterrotation. While the former type of two-body systems is more habitual for the analysis and studies, the systems of the latter type as a rule are given no attention at all because their interpretation may look clumsy from the black hole point of view. At the same time, the use of overlapping black holes for modeling compact objects of astrophysical interest seems to us very natural, and for instance a pair of superposed Schwarzschild black holes with a common center could be a good model of a static deformed mass. Similarly, the double-Reissner-Nordström solution [7] in which the separation distance $R$ is set equal to zero would describe a simple model of a charged static deformed mass; besides, as can be easily checked, in the $R = 0$ limit this solution simplifies considerably and becomes a static specialization of the solution [6]. In the present paper we will explore further the second type of the equatorially symmetric configurations and consider a 4-parameter electrovac metric for a magnetized rotating deformed mass which also could be regarded as representing two overlapping black-hole constituents and which was identified with the help of a recent paper on the charged rotating masses [8]. The solution will be shown to have a simple form, with various distinctive features and interesting limits, which makes it potentially suitable for the use in astrophysical applications.

Our paper is organized as follows. In the next section we present the 4-parameter elec-
trovac solution, the corresponding complete metric and expression of the magnetic potential. Here we also analyze the sub- and hyperextreme cases of the solution and its multipole structure. In Sec. III we consider interesting specializations of the solution and work out the solution’s extreme limit. Concluding remarks are given in Sec. IV.

II. THE 4-PARAMETER ELECTROVAC SOLUTION AND METRIC FUNCTIONS

Let us begin this section with reminding that the general equatorially symmetric 2-soliton electrovac solution [6] (henceforth referred to as the MMR solution), obtained with the aid of Sibgatullin’s integral method [9, 10], is defined by the axis data

\[ e(z) = \frac{(z - m - ia)(z + ib) + k}{(z + m - ia)(z + ib) + k}, \quad f(z) = \frac{qz + ic}{(z + m - ia)(z + ib) + k}, \]

where \( e(z) \) and \( f(z) \) are the expressions of the Ernst complex potentials \( E \) and \( \Phi \) [11] on the upper part of the symmetry \( z \)-axis; the six arbitrary real parameters entering (1) are \( m, a, b, k, q \) and \( c \).

The 4-parameter specialization of the MMR solution which we are going to report in this paper is determined by the following simple choice of the parameters in (1):

\[ b = a, \quad k = m_1m_2 - \mu^2, \quad q = 0, \quad c = m\mu, \]

the mass parameters \( m_1 \) and \( m_2 \) being such that \( m_1 + m_2 = m \); the charge parameter \( q \) is set equal to zero because of its irrelevance for astrophysical applications, and hence the electromagnetic field in this solution is defined solely by the magnetic dipole parameter \( \mu \). Then the axis data determining this particular case take the form

\[ e(z) = \frac{(z - m_1)(z - m_2) - ia(m_1 + m_2) + a^2 - \mu^2}{(z + m_1)(z + m_2) + ia(m_1 + m_2) + a^2 - \mu^2}, \]
\[ f(z) = \frac{i\mu(m_1 + m_2)}{(z + m_1)(z + m_2) + ia(m_1 + m_2) + a^2 - \mu^2}, \]

where the four arbitrary real parameters are \( m_1, m_2, a \) and \( \mu \). The particular parameter choice (2) occurred to us when we noticed that the metric for two unequal counterrotating charged masses [8] becomes, in the limit \( R = 0 \), a member of the MMR solution.

An attractive feature of the data (3) is that the algebraic equation

\[ e(z) + \bar{e}(z) + 2f(z)\bar{f}(z) = 0 \]

(4)
(the bars over symbols denote complex conjugation), which plays an important role in Sibgatullin’s method, yields for this case four very simple roots \( \alpha_i \), namely,

\[
\alpha_1 = -\alpha_2 = \sqrt{m_1^2 - a^2 + \mu^2} \equiv \sigma_1, \quad \alpha_3 = -\alpha_4 = \sqrt{m_2^2 - a^2 + \mu^2} \equiv \sigma_2, \quad (5)
\]

which in turn define the following four functions of the Weyl-Papapetrou coordinates \((\rho, z)\):

\[
R_\pm = \sqrt{\rho^2 + (z \pm \sigma_1)^2}, \quad r_\pm = \sqrt{\rho^2 + (z \pm \sigma_2)^2}. \quad (6)
\]

Formulas (3), (5) and (6) permit one to construct, by purely algebraic manipulations, the Ernst potentials \( E \) and \( \Phi \), as well as the corresponding metric functions \( f \), \( \gamma \) and \( \omega \) of the line element

\[
ds^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f (dt - \omega d\varphi)^2, \quad (7)
\]

using only the determinantal formulas of the paper [12]. In our concrete case the desired expressions can be also worked out from the respective formulas of the MMR solution.\(^1\) As a result, the potentials \( E \) and \( \Phi \) of the 4-parameter solution can be shown to have the form

\[
E = \frac{A - B}{A + B}, \quad \Phi = \frac{C}{A + B},
\]

\[
A = \sigma_1 \sigma_2 [(m_1^2 + m_2^2)(R_+ + R_-)(r_+ + r_-) - 4m_1m_2(R_+R_- + r_+r_-)] \nonumber \\
- (m_2^2 \sigma_1^2 + m_1^2 \sigma_2^2)(R_+ - R_-)(r_+ - r_-) + i\alpha (m_1^2 - m_2^2) \nonumber \\
\times [\sigma_1(R_+ + R_-)(r_+ - r_-) - \sigma_2(R_+ - R_-)(r_+ + r_-)], \nonumber \\
B = -2(m_1^2 - m_2^2)\{\sigma_1 \sigma_2[m_2(R_+ + R_-) - m_1(r_+ + r_-)] + i\alpha [m_2 \sigma_2(R_+ - R_-) 
- m_1 \sigma_1(r_+ - r_-)]\}, \nonumber \\
C = 2i\mu (m_1^2 - m_2^2)[m_1 \sigma_2(R_+ - R_-) - m_2 \sigma_1(r_+ - r_-)], \quad (8)
\]

while for the metrical fields \( f \), \( \gamma \) and \( \omega \) one gets the following expressions:

\[
f = \frac{A\bar{A} - B\bar{B} + CC}{(A + B)(A + B)} e^{2\gamma} = \frac{A\bar{A} - B\bar{B} + CC}{K_0 K_0 R_+ R_- r_+ r_-}, \quad \omega = -\frac{\text{Im}[G(A + B) + C\bar{I}]}{AA - BB + CC},
\]

\[
G = -2zB + (m_1^2 - m_2^2)\{\sigma_1(2m_2^2 + \mu^2)(R_+ + R_-)(r_+ + r_-) 
- \sigma_2(2m_1^2 + \mu^2)(R_+ - R_-)(r_+ - r_-) - 2i\alpha (m_1^2 - m_2^2)(R_+ - R_-)(r_+ - r_-) 
- 2\sigma_2[2m_2 \sigma_1^2 - \mu^2(m_1 + m_2)](R_+ - R_-) + 2\sigma_1[2m_1 \sigma_2^2 - \mu^2(m_1 + m_2)](r_+ - r_-)
\]

\(^1\) Unfortunately, the general formulas of the paper [8] are not helpful for elaborating the \( R = 0 \) limit because of the misprints crept into that paper, so that Eqs. [5] and [6] have been worked out with the aid of our computer codes developed for the MMR solution.
aid of the Hoenselaers-Perjés procedure [15] rectified by Sotiriou and Apostolatos [16] yields of the asymmetric dihole spacetime considered in [13].

\[ -4ia\sigma_1\sigma_2[m_2(R_+ + R_-) - m_1(r_+ + r_-)], \]
\[ I = -i\mu(m_1 + m_2)\left(\sigma_1\sigma_2[(m_1 + m_2)(R_+ + R_-)(r_+ + r_-) - 4m_1R_+R_- - 4m_2r_+r_-] \right. \]
\[ \left. -(m_2\sigma_1^2 + m_1\sigma_2^2)(R_+ - R_-)(r_+ - r_-) + ia(m_1 - m_2)[\sigma_1(R_+ + R_-)(r_+ - r_-) \right. \]
\[ \left. -\sigma_2(R_+ - R_-)(r_+ + r_-)] - 2(m_1 - m_2)\{\sigma_1\sigma_2[(2m_1 + m_2)(R_+ + R_-) \right. \]
\[ \left. -(m_1 + 2m_2)(r_+ + r_-) + 2m_1^2 - 2m_2^2] + ia(m_1 + m_2)[\sigma_2(R_+ - R_-) - \sigma_1(r_+ - r_-)]\} \right) \]
\[ K_0 = 4\sigma_1\sigma_2(m_1 - m_2)^2. \] (9)

It should be also noted that the electric \( A_t \) and magnetic \( A_\varphi \) components of the electromagnetic 4-potential defined by the solution \( \text{(8)} \) have the form
\[ A_t = -\text{Re}\left(\frac{C}{A + B}\right), \quad A_\varphi = \text{Im}\left(\frac{I - zC}{A + B}\right), \] (10)
and these formulas complement the description of our particular 4-parameter electrovac spacetime.

Turning now to the discussion of the properties of the solution \( \text{(8)} \), we first mention that the form of \( \sigma_1 \) and \( \sigma_2 \) defined in \( \text{(5)} \) clearly shows which type of constituents may form the two-body configurations described by this solution. In the subextreme case, the quantities \( \sigma_1 \) and \( \sigma_2 \) take pure imaginary values, which means that both \( m_1 \) and \( m_2 \) must fulfill the inequalities \( m_1^2 > a^2 - \mu^2 \) and \( m_2^2 > a^2 - \mu^2 \). Similarly, in the hyperextreme case both \( \sigma_1 \) and \( \sigma_2 \) take pure imaginary values, which means that \( m_1^2 < a^2 - \mu^2 \) and \( m_2^2 < a^2 - \mu^2 \). Since \( \sigma_1 \neq \sigma_2 \) generically, we can suppose, say, that \( m_1 > m_2 \); then the mixed subextreme-hyperextreme case arises when \( m_2^2 < a^2 - \mu^2 < m_1^2 \). These three basic types of the two-body configurations described by the solution \( \text{(8)} \) are depicted in Fig. 1. The fact that the constituents are overlapping can be most easily seen by setting the rotational parameter \( a \) equal to zero in the above formulas and observing that in this case the solution reduces to the \( R = 0 \) limit of the asymmetric dihole spacetime considered in [13].

The calculation of the first five Beig-Simon relativistic multipole moments [14] with the aid of the Hoenselaers-Perjés procedure [15] rectified by Sotiriou and Apostolatos [16] yields for the solution \( \text{(8)} \) the expressions
\[ P_0 = m_1 + m_2, \quad P_1 = ia(m_1 + m_2), \quad P_2 = -(m_1 + m_2)(m_1m_2 + a^2 - \mu^2), \]
\[ P_3 = -ia(m_1 + m_2)(m_1m_2 + a^2 - \mu^2), \]
\[ P_4 = (m_1 + m_2)(m_1m_2 + a^2 - \mu^2)^2 + \frac{1}{70}(m_1 + m_2)^3(10m_1m_2 - 7\mu^2), \]
whence it follows that the parameters $m_1$ and $m_2$ can be associated with the individual masses of the first and second constituent, respectively. Moreover, the expression of the total angular momentum $P_1$ indicates that the constituents are corotating with the same angular momentum per unit mass ratio: $j_1/m_1 = j_2/m_2 = a$, $j_1$ and $j_2$ being angular momenta of the first and second constituent, respectively. A surprising feature of the electromagnetic moments $Q_n$ is that the first four electric multipoles (these are represented by the real parts of the respective $Q_n$) are zeros, the first nonzero electric moment being the hexadecapole one.

\[ Q_0 = 0, \quad Q_1 = i\mu(m_1 + m_2), \quad Q_2 = 0, \quad Q_3 = -i\mu(m_1 + m_2)(m_1m_2 + a^2 - \mu^2), \]
\[ Q_4 = -\frac{1}{10}a\mu(m_1 + m_2)^3, \quad (11) \]

III. PHYSICALLY INTERESTING LIMITS OF THE SOLUTION

The 4-parameter solution (8) has various physically interesting limits which we will briefly consider below.

A. The static limit

When $a = 0$, the solution describes a static deformed mass endowed with magnetic dipole moment, and it coincides, as was already mentioned earlier, with the $R = 0$ specialization of the asymmetric dihole solution considered in [13]. In this case $\sigma_1 = \sqrt{m_1^2 + \mu^2}$, $\sigma_2 = \sqrt{m_2^2 + \mu^2}$, which means that only the subextreme type of overlapping constituents is possible (see Fig. 1(a)). An interesting particular case of this magnetostatic solution is the magnetized Schwarzschild metric which arises by further setting to zero one of the parameters $m_1$ or $m_2$, and its explicit form is the following ($|C|^2 \equiv C\bar{C}$):

\[ \mathcal{E} = \frac{A - B}{A + B}, \quad \Phi = \frac{C}{A + B}, \quad A_\varphi = -\mu + \frac{2\mu I}{A + B}, \]
\[ f = \frac{A^2 - B^2 + |C|^2}{(A + B)^2}, \quad e^{2\gamma} = \frac{A^2 - B^2 + |C|^2}{16\sigma^2 R_+ R_- r_+ r_-}, \]
\[ A = \sigma(R_+ + R_-)(r_+ + r_-) - \mu(R_+ - R_-)(r_+ - r_-), \]
\[ B = 2m\sigma(r_+ + r_-), \quad C = 2im\mu(R_+ - R_-), \]
\[ I = 2\sigma(R_+ + m)(R_- + m) - mz(R_+ - R_-), \]
\[ R_\pm = \sqrt{\rho^2 + (z \pm \sigma)^2}, \quad r_\pm = \sqrt{\rho^2 + (z \pm \mu)^2}, \quad \sigma = \sqrt{m^2 + \mu^2}. \quad (12) \]
By putting $\mu = 0$ in (12), one gets the Schwarzschild solution.

**B. The pure vacuum limit**

In the absence of the magnetic dipole parameter $\mu$, the solution (8) reduces to the $R = 0$ special case of the metric for two unequal counterrotating black holes [17]. Our alternative derivation of the solution [17] permitted us to work out a simple representation for the $R = 0$ case which we give below:

$$f = \frac{A \bar{A} - B \bar{B}}{(A + B)(A - B)}, \quad e^{2\gamma} = \frac{AA - BB}{K_0 K_0 R_+ R_- r_+ r_-}, \quad \omega = -\frac{2 \text{Im}[G(\bar{A} + \bar{B})]}{AA - BB},$$

$$A = (\sigma_1 + \sigma_2)^2(R_+ - R_-)(r_+ - r_-) - 4\sigma_1 \sigma_2 (R_+ - r_-)(R_- - r_+),$$

$$B = 2(m_1^2 - m_2^2)[\sigma_2(R_+ - R_-) + \sigma_1(r_+ - r_-)],$$

$$G = -zB + (m_1^2 - m_2^2)[\sigma_1(R_+ + R_-)(r_+ - r_-) - \sigma_2(R_+ - R_-)(r_+ + r_-) - 2\sigma_1 \sigma_2 (R_+ + R_- - r_+ - r_-)],$$

$$R_\pm = \frac{m_1 \mp \sigma_1 - ia}{m_1 \pm \sigma_1 + ia} \sqrt{\rho^2 + (z \pm \sigma_1)^2}, \quad r_\pm = \frac{m_2 \mp \sigma_2 - ia}{m_2 \pm \sigma_2 + ia} \sqrt{\rho^2 + (z \pm \sigma_2)^2},$$

$$\sigma_1 = \sqrt{m_1^2 - a^2}, \quad \sigma_2 = \sqrt{m_2^2 - a^2}, \quad K_0 = 4 \sigma_1 \sigma_2 (m_1 - m_2)^2 / (m_1 m_2), \quad (13)$$

and please note that the functions $R_\pm$ and $r_\pm$ are defined here in a slightly different way than in (6). Of course, the above formulas (13) are fully equivalent to the $\mu = 0$ limit of the solution [8]-[9].

We would like to remark that it was precisely the work [17] that motivated us to write this paper after we incidentally discovered a misprint in the formula (38) of [17] and then took notice of a rather unusual property of that formula whose correct form is

$$J_2 = -\frac{J_1 M_2}{M_1} \left( \frac{R + M_1 - M_2}{R - M_1 + M_2} \right), \quad (14)$$

where $M_i$ and $J_i$ are, respectively, the Komar [18] masses and angular momenta of the black-hole constituents, while $R$ is the separation distance. Indeed, as it follows from (14), for all $R > |M_1 - M_2|$ the two Kerr black holes are counterrotating; nevertheless, for $0 \leq R < |M_1 - M_2|$ the black holes become corotating, as the expression in parentheses on the right-hand side of (14) then takes negative values. Obviously, the authors of [17] were only interested in the configurations with $R > M_1 + M_2$, when the counterrotating black holes are separated by a massless strut, so that they discarded other possibilities as unphysical or
uninteresting. However, in our opinion, it is the \( R = 0 \) case that is probably most interesting from the physical point of view because this is the only case of unequal constituents with equatorial symmetry, and also because it might represent a legitimate final state of two merging Kerr black holes. It is worth pointing out that the change from counterrotation to corotation does not occur in the case of equal black holes \((M_1 = M_2)\) in \([13]\), so that the intrinsic inequality of black holes is necessary for the formation of the final configuration of corotating Kerr black holes described by \([13]\). The change of the total angular momentum of the system between its final \((R = 0)\) state and the initial state of infinitely separated sources \((R = \infty)\) is given by the simple formula

\[
\Delta J = 2J_1M_2/M_1, \tag{15}
\]

and we believe that this change of the total angular momentum, which should certainly be attributed to the extremely strong frame-dragging effects inside a larger black hole, could be related to the production of relativistic jets in the centers of galaxies.

C. Magnetized Kerr solution

By choosing \( m_1 = m, m_2 = 0 \) in \([8]\), we get a 3-parameter variant of the magnetized Kerr spacetime of the form

\[
\begin{align*}
\mathcal{E} &= \frac{A - B}{A + B}, \quad \Phi = \frac{C}{A + B}, \quad A_\phi = \text{Im} \left( \frac{I - zC}{A + B} \right), \\
f &= \frac{A\bar{A} - B\bar{B} + C\bar{C}}{(A + B)(A + B)}, \quad e^{2\gamma} = \frac{A\bar{A} - B\bar{B} + C\bar{C}}{K_0K_0R_+R_-r_+r_-}, \\
A &= \sigma_1\sigma_2(R_+ + R_-)(r_+ + r_-) - \sigma_2^2(R_+ - R_-)(r_+ - r_-) \\
&\quad + ia[\sigma_1(R_+ + R_-)(r_+ + r_-) - \sigma_2(R_+ - R_-)(r_+ + r_-)], \\
B &= 2m\sigma_1[\sigma_2(r_+ + r_-) + ia(r_+ - r_-)], \\
C &= 2im\sigma_2(R_+ - R_-), \\
G &= -2zB + \sigma_1\mu^2(R_+ + R_-)(r_+ - r_-) - \sigma_2(2m^2 + \mu^2)(R_+ - R_-)(r_+ + r_-) \\
&\quad - 2im^2a(R_+ - R_-)(r_+ - r_-) + 2\sigma_2m\mu^2(R_+ - R_-) \\
&\quad + 2m\sigma_1(\mu^2 - 2a^2)(r_+ - r_-) + 4ima\sigma_1\sigma_2(r_+ - r_-), \\
I &= -i\mu[\sigma_1\sigma_2(R_+ + R_-)(r_+ + r_-) - \sigma_2^2(R_+ - R_-)(r_+ + r_-) + ia[\sigma_1(R_+ + R_-)(r_+ - r_-) \\
&\quad - \sigma_2(R_+ - R_-)(r_+ + r_-)] - 2\sigma_1\sigma_2[2(R_+ + m)(R_- + m) - m(r_+ + r_-)]
\end{align*}
\]
\[-2ima[\sigma_2(R_+ - R_-) - \sigma_1(r_+ - r_-)]\},
\]
\[
R_{\pm} = \sqrt{\rho^2 + (z \pm \sigma_1)^2}, \quad r_{\pm} = \sqrt{\rho^2 + (z \pm \sigma_2)^2},
\]
\[
\sigma_1 = \sqrt{m^2 - a^2 + \mu^2}, \quad \sigma_2 = \sqrt{\mu^2 - a^2}, \quad K_0 = 4\sigma_1\sigma_2.
\] (16)

This solution is different from the known generalization of the Kerr solution obtained by one of us more than two decades ago [19]. The difference is clearly seen if one considers the axis data of the solution (16), namely,
\[
e(z) = \frac{(z - m - ia)(z + ia) - \mu^2}{(z + m - ia)(z + ia) - \mu^2}, \quad f(z) = \frac{im\mu}{(z + m - ia)(z + ia) - \mu^2},
\] (17)
where the magnetic dipole parameter \( \mu \) enters the expressions of both \( e(z) \) and \( f(z) \), unlike the solution [19] whose \( e(z) \) coincides with the axis data of the Kerr metric [3] and hence is free of the magnetic parameter. However, the presence of \( \mu \) on the right-hand side of \( e(z) \) in (17) is quite acceptable as it is well known that magnetic field is able to distort the stars [20], thus affecting the structure of their gravitational multipoles.

D. The extreme limit

The extreme limit of the solution (16) corresponds to the case of equal overlapping constituents, when \( M_1 = M_2 \) and \( \sigma_1 = \sigma_2 \), and the application of the L’Hôpital rule to formulas (8) and (9) is then required. By introducing the spheroidal coordinates \( x \) and \( y \) via the relations
\[
x = \frac{1}{2\sigma}(r_+ + r_-), \quad y = \frac{1}{2\sigma}(r_+ - r_-), \quad r_{\pm} = \sqrt{\rho^2 + (z \pm \sigma)^2},
\]
\[
\sigma = \sqrt{m^2 - a^2 + \mu^2},
\] (18)
it is possible to write down the resulting expressions in terms of four polynomials \( \lambda \), \( \nu \), \( \kappa \) and \( \chi \). Thus, for the potentials \( E \), \( \Phi \) and \( A_\varphi \) we get
\[
E = \frac{A - B}{A + B}, \quad \Phi = \frac{C}{A + B}, \quad A_\varphi = \text{Im} \left( \frac{I}{A + B} \right),
\]
\[
A = \lambda^2 + 2m^2[\lambda + 2ia\sigma xy(1 - y^2)],
\]
\[
B = 2m[(\sigma x + iay)\lambda + 2im^2ay(1 - y^2)],
\]
\[
C = 2i\mu m y \lambda,
\]
\[
I = \frac{i}{2}\mu(1 - y^2)[\kappa + 4m^2(\sigma x + m - iay)^2].
\] (19)
while the metrical fields $f$, $\gamma$ and $\omega$ are defined by the expressions

\begin{align*}
  f &= \frac{N}{D}, \quad e^{2\gamma} = \frac{N}{\sigma^2(x^2 - y^2)^2}, \quad \omega = \frac{(y^2 - 1)W}{N}, \\
  N &= \lambda^4 - \sigma^2(x^2 - 1)(1 - y^2)\nu^2, \\
  D &= N + \lambda^2\kappa + (1 - y^2)\nu\chi, \\
  W &= \sigma^2(x^2 - 1)\nu\kappa + \lambda^2\chi, \\
  \lambda &= \sigma^2(x^2 - y^2) - m^2(1 - y^2), \\
  \nu &= 4m^2\sigma^2, \\
  \kappa &= 4m[\sigma^2(\sigma x + 2m)(x^2 - y^2) + m^2\sigma x(y^2 + 1) + m(2m^2 + \mu^2)\nu^2], \\
  \chi &= 4ma[\sigma^2(\sigma x + 2m)(x^2 - y^2) + m^2\sigma x(1 - y^2)].
\end{align*}

Note that in the literature on exact solutions the polynomials $\lambda$, $\nu$, $\kappa$ and $\chi$ have been previously used exclusively in application to the metric functions $f$, $\gamma$ and $\omega$, so that our paper actually pioneers the use of these polynomials for getting a concise form of the Ernst potentials $\mathcal{E}$ and $\Phi$ too. We also note that the magnetic potential $A_\varphi$ can be written alternatively in the form

\begin{align*}
  A_\varphi &= \frac{2\mu(y^2 - 1)F}{D}, \\
  F &= \lambda \left[ \frac{1}{4}\kappa + m^2(\sigma x + m)^2 - m^2a^2\nu^2 \right] \left[ \lambda + 2m(\sigma x + m) \right] \\
  &\quad - 4m^3a^2\nu^2(\sigma x + m) \left[ \lambda + 2m(\sigma x + m)(1 - y^2) \right].
\end{align*}

Remarkably, the vacuum ($\mu = 0$) limit of the solution (19) differs from the well-known Tomimatsu-Sato [23] and Kinnersley-Chitre [24] solutions. This was a real surprise to us since our initial intention was only to see how this limit is contained in the Kinnersley-Chitre 5-parameter family of solutions. In view of the potential interest the new vacuum solution might represent, below we write it out explicitly:

\begin{align*}
  \xi &= \frac{1 - \mathcal{E}}{1 + \mathcal{E}} = \frac{2m[(\sigma x + iay)\lambda + 2im^2ay(1 - y^2)]}{\lambda^2 + 2m^2[\lambda + 2i\sigma xy(1 - y^2)]}, \\
  \lambda &= m^2(x^2 - 1) - a^2(x^2 - y^2), \quad \sigma = \sqrt{m^2 - a^2}.
\end{align*}

The corresponding metric functions are easily obtainable from (20).
IV. CONCLUDING REMARKS

We believe that the 4-parameter electrovacuum solution considered in the present paper, as well as some of its particular limits, provide interesting new opportunities for modeling the exterior gravitational and electromagnetic fields of rotating bodies and enlarge our knowledge about possible final states of two interacting black holes. The solution has a clear physical interpretation since it arises within a legitimate binary system of counterrotating nonequal black holes, and the corotation of its constituents, though unexpected at first glance but still natural, should be attributed to strong dragging effects that involve a smaller black hole in corotation with the larger one. We have shown that the overlapping constituents in the case of two Kerr black holes have a larger total angular momentum than at infinite separation, the increase being defined by formula (15), and the latter formula also holds in the presence of the electromagnetic field. It looks to us plausible to suppose that the aforementioned change of the angular momentum could be related to the mechanisms that are responsible for the production of relativistic jets at the galactic nuclei [25, 26]; indirectly, this also suggests a recent article [27] in which the analysis of binary static systems of black holes with magnetic charges has been considered as a right step towards a better understanding of the jets phenomenon. We hope to be able to shed more light on the details of that relation in the future.

As a final remark we would like to mention that the 4-parameter solution considered in this paper can be trivially generalized to include an additional parameter of electric dipole moment $\varepsilon$ by the substitution $i\mu \rightarrow \varepsilon + i\mu$, $\mu^2 \rightarrow \varepsilon^2 + \mu^2$, but we have not found for ourselves any physical justification to do it here.

Acknowledgments

This work was partially supported by CONACYT of Mexico, and by Project FIS2015-65140-P (MINECO/FEDER) of Spain.

[1] C. Bambi, Astron. Rev. 8, 4 (2013).

[2] E. Berti, E. Barausse, V. Cardoso et al., Class. Quantum Grav. 32, 243001 (2015).
[3] R. P. Kerr, Phys. Rev. Lett. 11, 237 (1963).
[4] T. Johannsen, Class. Quantum Grav. 33, 124001 (2016).
[5] C. Bambi, Rev. Mod. Phys. 89, 025001 (2017).
[6] V. S. Manko, J. Martín, and E. Ruiz, J. Math. Phys. 36, 3063 (1995).
[7] V. S. Manko, Phys. Rev. D 76, 124032 (2007).
[8] I. Cabrera-Munguia, Phys. Rev. D 91, 044005 (2015).
[9] N. R. Sibgatullin, Oscillations and Waves in Strong Gravitational and Electromagnetic Fields (Berlin: Springer, 1991).
[10] V. S. Manko and N. R. Sibgatullin, Class. Quantum Grav. 10, 1383 (1993).
[11] F. J. Ernst, Phys. Rev. 168, 1415 (1968).
[12] E. Ruiz, V. S. Manko, and J. Martín, Phys. Rev. D 51, 4192 (1995).
[13] V. S. Manko, E. Ruiz, and J. Sánchez-Mondragón, Phys. Rev. D 79, 084024 (2009).
[14] W. Simon, J. Math. Phys. 25, 1035 (1984).
[15] C. Hoenselaers and Z. Perjés, Class. Quantum Grav. 7, 1819 (1990).
[16] T. P. Sotiriou and T. A. Apostolatos, Class. Quantum Grav. 21, 5727 (2004).
[17] I. Cabrera-Munguia, C. Lämmerzahl, and A. Macías, Class. Quantum Grav. 30, 175020 (2013).
[18] A. Komar, Phys. Rev. 113, 934 (1959).
[19] V. S. Manko, Class. Quantum Grav. 10, L239 (1993).
[20] S. Bonazzola and F. Gourgoulhon, Astron. Astrophys. 312, 675 (1996).
[21] F. J. Ernst, J. Math. Phys. 17, 1091 (1976).
[22] Z. Perjés, J. Math. Phys. 30, 2197 (1989).
[23] A. Tomimatsu and H. Sato, Phys. Rev. Lett. 29, 1344 (1972).
[24] W. Kinnersley and D. M. Chitre, J. Math. Phys. 19, 2037 (1978).
[25] R. D. Blandford and R. L. Znajek, Mon. Not. R. Astron. Soc. 179, 433 (1977).
[26] S. L. Shapiro, Phys. Rev. D 95, 101303(R) (2017).
[27] M. J. Rodrigues, Binary black hole in a double magnetic monopole field, arXiv:1706.06212
FIG. 1: Three different types of systems with overlapping constituents: (a) subextreme configuration, (b) subextreme-extreme configuration, (c) hyperextreme configuration.