PURELY ADDITIVE REDUCTION OF ABELIAN VARIETIES WITH TORSION

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Abstract. Let $O_K$ be a discrete valuation ring with fraction field $K$ of characteristic 0 and algebraically closed residue field $k$ of characteristic $p > 0$. Let $A/K$ be an abelian variety of dimension $g$ with a $K$-rational point of order $p$. In this article, we are interested in the reduction properties that $A/K$ can have. After discussing the general case, we specialize to $g = 1$, and we study the possible Kodaira types that can occur.

Keywords: Abelian variety, Purely additive reduction, Néron model, Torsion point, Kodaira type.

1. Introduction

Let $O_K$ be a discrete valuation ring with valuation $v_K$, fraction field $K$ of characteristic 0 and algebraically closed residue field $k$ of characteristic $p > 0$. Let $A/K$ be an abelian variety of dimension $g$. In this paper we investigate if the existence of a $K$-rational point of order $p$ on $A/K$ imposes restrictions on the reduction properties of $A/K$. We prove Theorem 1.1 below, using ideas from [5].

Theorem 1.1. Let $O_K$ be a discrete valuation ring with valuation $v_K$, fraction field $K$ of characteristic 0 and residue field $k$ which is assumed to be algebraically closed of characteristic $p > 0$. Let $A/K$ be an abelian variety with a $K$-rational point of order $p$. Let $L/K$ be an extension of minimal degree over which $A/K$ acquires semi-stable reduction. If $v_K(p) < p - 1 \frac{[L:K]}{6}$, then $A/K$ cannot have purely additive reduction.

We then turn our attention to the case where $g = 1$ and $\text{char}(k) \geq 5$. Suppose that $E/K$ is an elliptic curve and assume that $E/K$ has a $K$-rational point of order $N \geq 5$. We study the possible reduction types that $E/K$ can have. The situation where $p \nmid N$ is well-known. In this case, $E/K$ can only have semi-stable reduction (see Corollary 6.5 of [33] or Theorem 2 of [9]). When $p \mid N$, additive reduction for $E/K$ can also occur. The first to take up the study of elliptic curves with additive reduction and a $K$-rational point of order $p$ were Lenstra and Oort in [17]. We study in this article what are the possible Kodaira types of reduction that can occur in this situation. More precisely, we prove the following theorems.

Theorem 1.2. Assume that $\text{char}(k) = p \geq 5$ and let $E/K$ be an elliptic curve. Suppose also that $E/K$ has a $K$-rational point of order $p^n$ for some $n \geq 1$.

(i) If $v_K(p) < \frac{p^{n-1}(p-1)}{6}$, then $E/K$ can only have semi-stable reduction.

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Assume now that $E/K$ does not have semi-stable reduction.

- (ii) If $v_K(p) \leq \frac{p^{n-1}(p-1)}{6}$, then $E/K$ can only have reduction of type II.
- (iii) If $v_K(p) \leq \frac{p^{n-1}(p-1)}{3}$, then $E/K$ can only have reduction of type II or III.
- (iv) If $v_K(p) \leq \frac{p^{n-1}(p-1)}{2}$, then $E/K$ can only have reduction of type II, III, or IV.

**Theorem 1.3.** Assume that $\text{char}(k) = p \geq 5$ and let $E/K$ be an elliptic curve. Suppose also that $E/K$ has a $K$-rational point of order $p^n$ for some $n \geq 1$.

- (i) If $v_K(p) < \frac{5p^{n-1}(p-1)}{6}$, then $E/K$ cannot have reduction of type $II^*$.
- (ii) If $v_K(p) < \frac{3p^{n-1}(p-1)}{4}$, then $E/K$ cannot have reduction of type $II^* \text{ or III}^*$.
- (iii) If $v_K(p) < \frac{2p^{n-1}(p-1)}{3}$, then $E/K$ cannot have reduction of type $II^*, III^*, \text{ or IV}^*$.

This paper is organized as follows. Theorems 1.1, 1.2, and 1.3 are the main results of Section 2. In Section 3, we present some examples which show that the bounds of Theorems 1.1 and 1.3 are sharp. Section 4 is devoted to an alternative way of proving Theorems 1.2 and 1.3 for small primes using explicit equations for modular curves. Finally, in the last section we explain a difference, for dimension 1, between the equicharacteristic case, i.e., $\text{char}(K) = p$, which is studied in [18], and our case.

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## 2. Proofs of Theorems 1.1, 1.2, and 1.3

In this section, we prove Theorems 1.1, 1.2, and 1.3. Before we begin the proof we need to set up some notation as well as recall some basic facts concerning reduction of abelian varieties. Let $\mathcal{O}_K$ be a discrete valuation ring with valuation $v_K$, fraction field $K$ of characteristic 0 and algebraically closed residue field $k$ of characteristic $p > 0$. If $L/K$ is a finite field extension and $\mathcal{O}_K$ is complete, then we will denote by $\mathcal{O}_L$ the integral closure of $\mathcal{O}_K$ in $L$ and by $v_L$ the associated normalized discrete valuation of $L$. This notation will be fixed throughout this section.

**2.1.** Let $A/K$ be an abelian variety of dimension $g$. We denote by $A/\mathcal{O}_K$ the Néron model of $A/K$ (see [2] for the definition as well as the basic properties of Néron models). The special fiber $A_k/k$ of $A/\mathcal{O}_K$ is a smooth commutative group scheme. We denote by $A^0_k/k$ the connected component of the identity of $A_k/k$. The finite étale group scheme defined by $\Phi_k := A_k/A^0_k$ is called the component group of $A/\mathcal{O}_K$. By a theorem of Chevalley (see Theorem 1.1 of [6]) we have a short exact sequence

$$0 \rightarrow T \times U \rightarrow A^0_k \rightarrow B \rightarrow 0,$$

where $T/k$ is a torus, $U/k$ is a unipotent group, and $B/k$ is an abelian variety. The number $\dim(U)$ (resp. $\dim(T), \dim(B)$) is called the unipotent (resp. toric, abelian) rank of $A/K$. By
construction, \( g = \dim(U) + \dim(T) + \dim(B) \). We say that \( A/K \) has purely additive reduction if \( g = \dim(U) \), or equivalently, if \( \dim(T) = \dim(B) = 0 \).

Let \( \mathcal{O}_K \subset \mathcal{O}_{K'} \) be a local extension of discrete valuation rings with fraction fields \( K \) and \( K' \). Assume that \([K' : K] \) is finite and let \( e(K'/K) \) be the ramification index of \( \mathcal{O}_{K'}/\mathcal{O}_K \). We will say that the extension \( K'/K \) is tame if \( e(K'/K) \) is coprime to the characteristic of \( k \) (note that we assume that \( k \) is algebraically closed).

We are now ready to proceed to the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Assume that \( A/K \) has purely additive reduction and that \( v_K(p) < \frac{1}{[L : K]} \), and we will find a contradiction. Let \( \mathcal{O}_{K'} \) be the completion of \( \mathcal{O}_K \) and let \( K' \) be the field of fractions of \( \mathcal{O}_{K'} \). Since \( \mathcal{O}_{K'} \) has ramification index 1 over \( \mathcal{O}_K \), the base change \( A_{K'}/K' \) of \( A/K \) has the same reduction as \( A/K \). Therefore, we can assume that \( \mathcal{O}_K \) is complete.

Recall that \( L/K \) is an extension of minimal degree such that the base change \( A_L/L \) has semi-stable reduction and set \( m := [L : K] \). If \( L/K \) is not tame, then \( p \leq m \) so \( \frac{1}{m} < 1 \). Therefore, we can assume that \( L/K \) is tame. We denote by \( \mathcal{O}_L \) the integral closure of \( \mathcal{O}_K \) in \( L \), which is again a discrete valuation ring because \( \mathcal{O}_K \) is complete. We denote by \( v_L \) the associated (normalized) valuation of \( \mathcal{O}_L \). Since \( \mathcal{O}_L \) has ramification index \( m \) over \( \mathcal{O}_K \), the restriction \( v_L|_K \) of \( v_L \) to \( K \) satisfies \( v_L|_K = mv_K \). Let \( A/\mathcal{O}_K \) be the Néron model of \( A/K \) and let \( \mathcal{A}'/\mathcal{O}_L \) be the Néron model of \( A_L/L \). If \( A_L/\mathcal{O}_L \) is the base change of \( A/\mathcal{O}_K \) to \( \mathcal{O}_L \), we obtain, using the universal property of \( \mathcal{A}'/\mathcal{O}_L \), an \( \mathcal{O}_L \)-morphism \( \phi: \mathcal{A}_L \to \mathcal{A}' \) which gives rise to a \( k \)-morphism \( \psi: (A^0_L)_k \to (A^0)'_k \) between the connected components of the special fibers. Since \( A/K \) has purely additive reduction, \( (A^0_L)_k \) is a unipotent group and since \( A_L/L \) has semi-stable reduction, \( (A^0)'_k \) is an extension of a torus by an abelian variety. Since there are no nonconstant morphisms from a unipotent group to an abelian variety (see Lemma 2.3 of \[6\]), and there are no nonconstant morphisms from a unipotent group to a torus (see Corollary 14.18 of \[28\]), we obtain that \( \psi \) is constant.

Recall that we assume that \( A/K \) has purely additive reduction and, hence, the toric rank of \( A/K \) is zero. Proposition 1.8 of \[21\] (see also \[20\] for a complement to this Proposition 1.8) tells us that the component group of \( A/K \) is killed by \( [L : K]^2 \) and since \( L/K \) is tame, we obtain that \( p \) cannot divide the order of the component group of \( A/K \). Therefore, the image of \( P \) under the reduction map belongs to the connected component of the identity.

Let \( A_L^{1k}(L) \) denote the kernel of the reduction map \( A_L(L) \to A_L^{1k}(k) \). Since \( v_L(p) < m(\frac{1}{m}) = p - 1 \), the reduction map of the base change is injective on torsion points (see the Appendix of \[14\]) and, hence, \( P \notin A_L^{1k}(L) \). Therefore, the image of \( P \) under the reduction map belongs to the connected component of the identity and has order \( p \). However, this is a contradiction since \( \psi \) is constant.

**Corollary 2.2.** Let \( \mathcal{O}_K \) be a discrete valuation ring with valuation \( v_K \), fraction field \( K \) of characteristic 0 and residue field \( k \) which is assumed to be algebraically closed of characteristic \( p > 0 \). Let \( A/K \) be an abelian surface with a \( K \)-rational point of order \( p \). Assume that \( v_K(p) = 1 \).

(i) If \( p > 13 \), then \( A/K \) cannot have purely additive reduction.

(ii) If \( p = 11 \) or \( 13 \), and \( A/K \) has purely additive reduction, then \( A/K \) has potentially good reduction.

**Proof.** We can without loss of generality assume that \( \mathcal{O}_K \) is complete. Let \( L/K \) be the minimal Galois extension of \( K \) such that the base change \( A_L/L \) of \( A/K \) to \( L \) has semi-stable reduction.
If \( x = p_1^{a_1} \ldots p_n^{a_n} \) is a positive integer with \( p_i \) distinct prime numbers, then we let
\[
L(x) := \begin{cases} 
\sum_{i=1}^{n} p_i^{a_i-1}(p_i - 1) & \text{if } \ord_2(x) \neq 1, \\
L\left(\frac{x}{2}\right) & \text{if } \ord_2(x) = 1, \\
0 & \text{if } x = 0,1.
\end{cases}
\]

**Proof of (i):** Assume that \( A/K \) has purely additive reduction and we will arrive at a contradiction. Let \( \epsilon \) be the exponent of the group \( \Gal(L/K) \) and write \( \epsilon = p^w \cdot \epsilon(p) \), for some \( w \geq 0 \) with \( p \nmid \epsilon(p) \). If \( t_L \) be the toric rank of \( A_L/L \), then Proposition 3.1 of [23] tells us that \( \max(L(p^w), L(\epsilon(p))) \leq 2 \cdot 2^0 + t_L \leq 4 \). If \( w > 0 \), then \( L(p^w) > 4 \) because \( p > 13 \). Therefore, \( w = 0 \) and, hence, \( p \nmid \epsilon(p) \). Since the exponent and the order of a finite group have the same prime divisors, we obtain that \( L/K \) is tame. Since \( L/K \) is also totally ramified, we have that \( L/K \) is cyclic and, hence, \( \epsilon = [L : K] \). Moreover, since \( L(\epsilon) \leq 4 \), we obtain that \( \epsilon \leq 12 \) by the definition of \( L(\epsilon) \). Finally, since \( p > 13 \), we find that \( v_K(p) = 1 < \frac{p-1}{\epsilon} \) and, therefore, Theorem [1.1] gives that \( A/K \) cannot have purely additive reduction.

**Proof of (ii):** Assume that \( p = 11 \) or 13, that \( A/K \) has purely additive reduction, and \( A_L/L \) does not have potentially good reduction, and we will find a contradiction. Let \( t_L \geq 1 \) be the toric rank of \( A_L/L \) and let \( m = [L : K] \). Proceeding in exactly the same way as in (i) we can show that \( L/K \) is tame and that \( L(m) \leq 2 \cdot 2^0 + t_L = 4 - t_L < 4 \). Therefore, \( m \leq 8 \) by the definition of \( L(m) \). Since \( p = 11 \) or 13, we find that \( v_K(p) = 1 < \frac{p-1}{m} \). Therefore, Theorem [1.1] implies that \( A/K \) cannot have purely additive reduction, which is a contradiction. This concludes the proof. \( \square \)

**Remark 2.3.** Keeping the same hypotheses as in Corollary [2.2], Penniston has proved (see Theorem 6.2 of [31]) that if \( p > 5 \) and \( A/K \) is the Jacobian of a smooth, projective, and geometrically connected curve such that \( A(K) \) contains a point of order \( p^2 \), then \( A/K \) cannot have purely additive reduction. Examples of abelian surfaces satisfying the hypotheses of Part (ii) of Corollary [2.2] can be found in Remark [3.13].

We now turn our attention to \( g = 1 \). The key ingredient for the proof of Theorem [1.2] is Proposition [2.4] below. This proposition is well known to the experts, and we include a proof here for completion.

**Proposition 2.4.** Let \( \mathcal{O}_K \) be a complete discrete valuation ring with valuation \( v_K \), fraction field \( K \) and algebraically closed residue field \( k \) of characteristic \( p > 5 \). Let \( E/K \) be an elliptic curve. Then there is an extension \( L/K \) of minimal degree over which the base change \( E_L/L \) of \( E/K \) has semi-stable reduction. The degree \( [L : K] \) is determined by the Kodaira type of \( E/K \) as follows:
\[
\begin{align*}
[L : K] = 2 & \quad \text{if } E/K \text{ has reduction of type } I_n^*, \text{for some } n \geq 0, \\
[L : K] = 3 & \quad \text{if } E/K \text{ has reduction of type } IV \text{ or } IV^*, \\
[L : K] = 4 & \quad \text{if } E/K \text{ has reduction of type } III \text{ or } III^*, \\
[L : K] = 6 & \quad \text{if } E/K \text{ has reduction of type } II \text{ or } II^*.
\end{align*}
\]

**Proof.** If the reduction of \( E/K \) is of type \( I_n^* \) for some \( n \geq 1 \), then there is an extension \( L/K \) over which the base change \( E_L/L \) of \( E/K \) has multiplicative reduction. Moreover, \( [L : K] = 2 \) by Part (d) of Theorem 14.1 in Appendix C.14 of [35].

Assume now that \( E/K \) has potentially good reduction. Let \( v_K(\Delta) \) be the minimal discriminant valuation of \( E/K \). Then \( L \) is obtained by adjoining to \( K \) the \( d \)-th power of a uniformizer, where \( d \) is the minimal positive integer such that \( dv_K(\Delta) \) is divisible by 12, as we now explain.
Let $O_L$ be the integral closure of $O_K$ in $L$. We also denote by $v_L$ the associated (normalized) valuation of $O_L$. Then $v_L(\Delta) = d v_K(\Delta)$. We know that the valuation of minimal discriminant for the base $E_L/L$ must be between 0 and 12 by Remark VII.1.1 of [35]. Moreover, when we perform a change of Weierstrass equation for an elliptic curve the valuation of the discriminant changes by a multiple of 12. Therefore, since $12 | v_L(\Delta)$, $E_L/L$ has good reduction. The corresponding values for $d$ depending on each Kodaira type, which can be found using the table on page 46 of [30], give the degree of $L/K$. This concludes the proof of our proposition. \hfill \square

**Proof of Theorem 1.2.** By an argument similar to the start of the proof of Theorem 1.1 we can assume that $O_K$ is complete.

Let $E_{ns}(k)$ denote the set of non-singular $k$-rational points of $E/k$, $E^1(K)$ denote the kernel of the reduction map, and $E^0(K)$ denote the set of points whose image under the reduction map is a non-singular.

We first prove that if $v_K(p) < \frac{p^{n-1}(p-1)}{2}$, then it is not possible for $E/K$ to have reduction of type $I_m^*$ for any $m \geq 0$. Assume that $E/K$ has reduction of type $I_m^*$ for some $m \geq 0$ and we will find a contradiction. Let $P \in E(K)[p^n]$ be the $K$-rational point of order $p^n$. Since $E/K$ has additive reduction, we obtain that $|E(K)/E^0(K)| \leq 4$. Therefore, since $p \geq 5$ and $P$ has order $p^n$, we get that $P \in E^0(K)$.

Assume first that $m > 0$. Let $L/K$ be an extension of $K$ over which $E_L/L$ achieves semistable reduction. Assume also that the degree $[L : K]$ is minimal. By Proposition 2.4 we obtain that $[L : K] = 2$. Let $O_L$ be the integral closure of $O_K$ in $L$, which is again a discrete valuation ring because $O_K$ is complete. We also denote by $v_L$ the associated (normalized) valuation of $O_L$. Since $O_L$ has ramification index 2 over $O_K$, we have that the restriction $v_L|_K$ of $v_L$ to $K$ satisfies $v_L|_K = 2v_K$. Using Theorem 5.9.4 of [16], since $v_L(p) = 2v_K(p) < 2(\frac{p^{n-1}(p-1)}{2}) = p^{n-1}(p-1)$, we obtain that $P \notin E_L^1(L)$. Let $E/O_K$ be the Néron model of $E/K$ and let $E'/O_L$ be Néron model of $E_L/L$. By looking at the base change $E_L/O_L$ of $E/O_K$ over $O_L$ we obtain, using the universal property of $E'/O_L$, that there is an $O_L$-morphism $\phi : E_L \to E'$ which gives rise to a $k$-morphism $(E_0^0)_k \to (E_0^0)_k$. Since $E/K$ has additive reduction, we get that $(E_0^0)_k \cong G_{a,k}$ and on the other hand since $E_L/L$ has multiplicative reduction, we obtain that $(E_0^0)_k \cong G_{m,k}$. However, because of the fact that there are no non-constant $k$-morphisms from $G_{a,k}$ to $G_{m,k}$ (see Corollary 14.18 of [28]) we get that the image of $(E_0^0)_k$ is the identity of $(E_0^0)_k$. This is a contradiction since this means that $P \in E_L^1(L)$ and this contradicts the fact that $P \notin E_L^1(L)$ showed above. This proves that if $v_K(p) < \frac{p^{n-1}(p-1)}{2}$, then it is not possible for $E/K$ to have reduction of type $I_m^*$ for any $m > 0$. The case where $m = 0$ follows directly from Theorem 1.1 using Proposition 2.4.

Using similar arguments as the proof above combined with Proposition 2.4 we can prove the following three statements. If $v_K(p) < \frac{p^{n-1}(p-1)}{3}$, then it is not possible for $E/K$ to have reduction of type IV or IV*. If $v_K(p) < \frac{p^{n-1}(p-1)}{4}$, then it is not possible for $E/K$ to have reduction of type III or III*. If $v_K(p) < \frac{p^{n-1}(p-1)}{6}$, then it is not possible for $E/K$ to have reduction of type II or II*.

Finally, using Theorem 1.3 we can rule out the cases II*, III*, and IV* in each part of the theorem. Putting now everything together, we have proved Theorem 1.2. \hfill \square

**Proof of Theorem 1.3.** By an argument similar to the start of the proof of Theorem 1.1 we can assume that $O_K$ is complete. Let $E/K$ be an elliptic curve with additive and potentially good reduction. Let $L/K$ be an extension of $K$ of minimal degree over which the base change $E_L/L$
of $E/K$ to $L$ acquires good reduction. Let $O_L$ be the integral closure of $O_K$ in $L$, which is again a discrete valuation ring because $O_K$ is complete. We also denote by $v_L$ the associated (normalized) valuation of $O_L$. Note that the restriction $v_L|_K$ of $v_L$ to $K$ satisfies $v_L|_K = mv_K$, where $m := [L : K]$.

Since we assume that $p \geq 5$, we can find short minimal Weierstrass equations

$$W_{\text{min}} : \ y^2 = x^3 + ax + b \quad \text{and} \quad W_{\text{min}}^L : \ y^2 = x^3 + a'x + b'$$

for $E/K$ and $E_L/L$ respectively. We will denote by $\Delta_{\text{min}}$ the discriminant of $W_{\text{min}}$.

**Lemma 2.5.** Let $P \in E(K)$ such that $P \notin E^1(K)$ but $P \in E_L^1(L)$. If $P$ corresponds to $(x_0, y_0)$ on $W_{\text{min}}$, then $v_L(x_0) = 0 = v_L(y_0)$. Moreover, the corresponding point $(x'_0, y'_0)$ on $W_{\text{min}}^L$ satisfies

$$v_L(x'_0) = \frac{-v_L(\Delta_{\text{min}})}{6} \quad \text{and} \quad v_L(y'_0) = \frac{-v_L(\Delta_{\text{min}})}{4}.$$ 

**Proof.** There exists an isomorphism $W_{\text{min}} \longrightarrow W_{\text{min}}^L$ defined over $L$ which is of the form $(x, y) \mapsto (u^{-2}x, u^{-3}y)$, for some $u \in L^*$. Since this isomorphism alters the discriminant by $u^{-12}$ and $E_L/L$ has good reduction by assumption, we find that $v_L(u) = \frac{v_L(\Delta_{\text{min}})}{12}$.

Since $(x'_0, y'_0) := (u^{-2}x_0, u^{-3}y_0) \in E_L^1(L)$ and, hence, $(x'_0, y'_0)$ reduces to the point at infinity we have

$$v_L(u^{-2}x_0) < 0, \quad v_L(u^{-3}y_0) < 0 \quad \text{and} \quad 3v_L(u^{-2}x_0) = 2v_L(u^{-3}y_0).$$

Therefore, $v_L(x_0) < 2v_L(u) = \frac{v_L(\Delta_{\text{min}})}{6}$ and $v_L(x_0) < 3v_L(u) = \frac{v_L(\Delta_{\text{min}})}{6}$.

On the other hand, since $P \notin E^1(K)$ we obtain that $v_K(x_0) \geq 0$ and $v_K(y_0) \geq 0$. Moreover, $3v_L(u^{-2}x_0) = 2v_L(u^{-3}y_0)$ implies that $3v_L(x_0) = 2v_L(y_0)$ and, hence, $3v_K(x_0) = 2v_K(y_0)$. Write $v_K(x_0) = 2r$, for some $r \geq 0$. Since $E/K$ has potentially good reduction, we have that $v_K(\Delta_{\text{min}}) < 12$ (see Table 4.1 of [34]). This implies that $v_K(x_0) < 2$ and since $v_K(x_0)$ is even we obtain that $v_K(x_0) = 0$. Therefore, $v_L(x_0) = 0$ and also $v_L(y_0) = 0$ because $3v_L(x_0) = 2v_L(y_0)$.

Finally, since $v_L(u) = \frac{v_L(\Delta_{\text{min}})}{12}$ and $(x'_0, y'_0) = (u^{-2}x_0, u^{-3}y_0)$, we obtain that $v_L(x'_0) = \frac{-v_L(\Delta_{\text{min}})}{6}$ and $v_L(y'_0) = \frac{-v_L(\Delta_{\text{min}})}{4}$, as needed. This proves the lemma.

□

**Corollary 2.6.** Suppose that $v_L(p) < \frac{(p^n - p^{n-1})v_L(\Delta_{\text{min}})}{12}$ for some $n \geq 1$ and that $P \in E(K)$ has order $p^n$. Then $P \notin E_L^1(L)$.

**Proof.** Suppose that $P \in E_L^1(L)$ and we will find a contradiction. If $P$ corresponds to $(x'_0, y'_0)$ on $W_{\text{min}}^L$, then Lemma 2.5 gives that $v_L(-x'_0/y'_0) = \frac{v_L(\Delta_{\text{min}})}{12}$. On the other hand, Theorem IV.6.1 of [34] gives that $v_L(-x'_0/y'_0) \leq \frac{v_L(p)}{p^n - p^{n-1}} < \frac{v_L(\Delta_{\text{min}})}{12}$. This is a contradiction and we have proved our corollary.

□

We are now ready to conclude our proof.

**Proof of (i):** Assume that $v_K(p) < \frac{5p^{n-1}(p-1)}{6}$ and that $E/K$ has reduction of type $\Pi^*$ and we will arrive at a contradiction. Let $P$ a point of order $p^n$. Proceeding in a similar way as in the proof of Theorem 1.1 we will find a contradiction provided that we can show that $P \notin E_L^1(L)$. Since $E/K$ has reduction of type $\Pi^*$, Proposition 2.4 gives that $[L : K] = 6$. Moreover, from Table 4.1 of [35] we obtain that $v_K(\Delta_{\text{min}}) = 10$. Therefore, $v_L(\Delta_{\text{min}}) = 60$ and $v_L(p) = 6v_K(p)$. This implies that if $v_K(p) < \frac{5p^{n-1}(p-1)}{6}$, then $v_L(p) < \frac{(p^n - p^{n-1})v_L(\Delta_{\text{min}})}{12}$ and, hence, Corollary 2.6 gives that $P \notin E_L^1(L)$, as needed. This proves (i).
The proofs of parts (ii) and (iii) are similar using Proposition 2.4 and Table 4.1 of [34].

Our next proposition shows that the statements of Theorem 1.3 are the best that one can hope for.

**Proposition 2.7.** Let \( \mathcal{O}_K \) be a complete discrete valuation ring with valuation \( v_K \), fraction field \( K \) of characteristic 0 and residue field \( k \) which is assumed to be algebraically closed of characteristic \( p \geq 5 \). Let \( E/K \) be an elliptic curve with a rational point of order \( p \geq 5 \) and assume that \( E/K \) has additive reduction.

(i) If \( E/K \) has reduction of type II and \( L/K \) is a finite extension with \( v_L(p) = 5v_K(p) \), then \( E_L/L \) has reduction of type II*.  
(ii) If \( E/K \) has reduction of type III and \( L/K \) is a finite extension with \( v_L(p) = 3v_K(p) \), then \( E_L/L \) has reduction of type III*.  
(iii) If \( E/K \) has reduction of type IV and \( L/K \) is a finite extension with \( v_L(p) = 2v_K(p) \), then \( E_L/L \) has reduction of type IV*.

**Proof.** The proof is a consequence of the table on page 46 of [36]. Assume \( E/K \) has reduction of type II (resp. III, IV) and let \( \Delta \) be the minimal discriminant of \( E/K \). By the table on page 46 of [36] \( v_K(\Delta) = 2 \) (resp. \( v_K(\Delta) = 3, v_K(\Delta) = 4 \)). Let \( L/K \) be a field extension with \( v_L(p) = 5v_K(p) \) (resp. \( v_L(p) = 3v_K(p), v_L(p) = 2v_K(p) \)). Then \( v_L(\Delta) = 10 \) (resp. \( v_L(\Delta) = 9, v_L(\Delta) = 8 \)) so by the table on page 46 of [36] we get that the base change of \( E/K \) to \( L \), \( E_L/L \), has reduction type of type II* (resp. III*, IV*). This proves the proposition.

**Remark 2.8.** When \( p = 3 \) and \( v_K(p) = 1 \), Kozuma (see Proposition 3.5 and Lemma 3.6 of [15]) has described the possible reduction types of elliptic curves \( E/K \) that have a \( K \)-rational point of order 3 using explicit Weierstrass equations. We note that in this case the reduction types II* and III* cannot occur.

### 3. Some Examples

In this section, we present examples of abelian varieties with torsion points and purely additive reduction. Our examples show that the ramification bounds in Theorems 1.1 and 1.2 are sharp.

**Example 3.1.** Let \( p \) be an odd prime and \( s \) be an integer with \( 1 \leq s \leq p - 2 \). Consider the smooth projective curve \( C_{p,s}/\mathbb{Q} \) birational to

\[
y^p = x^s(1 - x).
\]

The curve \( C_{p,s}/\mathbb{Q} \) has genus \( \frac{p-1}{2} \). The Jacobian \( J_{p,s}/\mathbb{Q} \) of \( C_{p,s}/\mathbb{Q} \) has a \( \mathbb{Q} \)-rational point of order \( p \) (see Theorem 1.1 of [10]). Each \( C_{p,s}/\mathbb{Q} \) is a quotient of the Fermat curve \( F_p/\mathbb{Q} \) (see Example 5.1 of [24] for more details). It turns out that each \( J_{p,s}/\mathbb{Q} \) has good reduction away from \( p \) and potentially good reduction modulo \( p \). When \( C_{p,s}/\mathbb{Q} \) is tame (see Example 5.1 of [24] for the definition) \( J_{p,s}/\mathbb{Q} \) has purely additive reduction modulo \( p \) and achieves good reduction after a totally ramified extension of degree \( 2(p - 1) \) (see Anmerkung in page 339 of [26] for the last statement).

Let \( K \) be as in the previous section. Let \( C/K \) be a smooth, proper, and geometrically connected curve of genus 2 with Jacobian \( J/K \). Assume that \( J(K) \) contains a point of prime order \( p \). When \( p = 7, 11, 13 \) and \( v_K(p) = 1 \), purely additive reduction for \( J/K \) is still allowed
by Corollary 2.2. The examples below when $p = 7$ show that the special fiber of the minimal regular model of the curve $C/K$ is surprisingly simple.

**Example 3.2.** Consider the curve $C/Q$ given by the following equation

$$y^2 + (x^3 + 1)y = 22x^6 - 10x^5 - 24x^4 + 5x^3 + 11x^2 - x - 2.$$ 

This is a genus 2 curve with LMFD [22] label 40817.a.40817.1. The Jacobian $\text{Jac}(C)/Q$ of $C/Q$ has a $Q$-rational point of order 7.

Using the command `genus2reduction()` in SAGE [37] we see that $C/Q$ has reduction type $[\text{VIII}-1]$ ([29], page 156) modulo 7 and, hence, potentially good reduction modulo 7. The special fiber of the minimal regular model is irreducible with multiplicity 1. Let $C_{\mathbb{Q}_7}^{unr}/\mathbb{Q}_7^{unr}$ be the base change of $C/Q$ to the maximal unramified extension of $\mathbb{Q}_7$. Since the minimal strict normal crossings model of $C_{\mathbb{Q}_7}^{unr}/\mathbb{Q}_7^{unr}$ contains no irreducible components of genus greater than 0, by Theorem 6.1 of [25] the abelian rank of $\text{Jac}(C)/Q$ modulo 7 is zero. Therefore, since $C/Q$ has potentially good reduction modulo 7, $\text{Jac}(C)/Q$ has purely additive reduction modulo 7.

One way of trying to find a curve of genus 2 with a Jacobian having a $K$-rational point of order $p$ is to find a degree 2 cover of an elliptic curve having a $K$-rational point of order $p$. This is our method for producing Examples 3.3 and 3.4 below.

**Example 3.3.** Consider the curve $C/Q$ defined by the equation

$$y^2 = x^6 - 182763x^2 + 31201254.$$ 

This is a curve of genus 2. Using the command `TorsionSubgroup()` in MAGMA [3], we find that the Jacobian $\text{Jac}(C)/Q$ of $C/Q$ has a $Q$-rational point of order 7.

Using the command `genus2reduction()` in SAGE [37] we see that $C/Q$ has reduction type $[\text{V}]$ ([29], page 156) modulo 7, and, hence, potentially good reduction. The special fiber of the minimal regular model consists of two irreducible components of multiplicity 1. Using a similar argument as in Example 3.2 we find that $\text{Jac}(C)/Q$ has purely additive reduction modulo 7. Note that, by Table 1 of [10], the minimal degree of an extension over which the base change $\text{Jac}(C)_{\mathbb{Q}_7}^{unr}/\mathbb{Q}_7^{unr}$ acquires good reduction is 6. This shows that Theorem 1.1 is sharp in the sense that in our example $\frac{1}{m} = 1$ and $v_K(p) = 1$, but $\text{Jac}(C)_{\mathbb{Q}_7}^{unr}/\mathbb{Q}_7^{unr}$ has purely additive reduction.

**Example 3.4.** Consider the curve $C/Q$ defined by the equation

$$y^2 = x^6 + \frac{1}{4}x^4 - 141x^2 + 657.$$ 

This is a curve of genus 2. Using the command `TorsionSubgroup()` in MAGMA [3], we find that the Jacobian $\text{Jac}(C)/Q$ of $C/Q$ has a $Q$-rational point of order 7.

Using the command `genus2reduction()` in SAGE [37] we see that $C/Q$ has reduction type $[\text{II}-\text{II}-0]$ ([29], page 163) modulo 7, and, hence, potentially good reduction. The special fiber of the minimal regular model is irreducible of multiplicity 1. Therefore, $\text{Jac}(C)/Q$ has potentially good reduction modulo 7.

**Remark 3.5.** When $g = 2$, $p = 11$, and $v_K(p) = 1$, purely additive reduction for Jacobians of dimension $g$ with a $K$-rational point of order $p$ is still allowed. In order for such an example to exist, the minimal degree of an extension over which $J/K$ acquires good reduction must be 10 or 12. Families of curves of genus 2 defined over $\mathbb{Q}$ whose Jacobian has a $\mathbb{Q}$-rational point of order 11 have been constructed in [8], in [7], and some sporadic examples can be found in [1].
We performed a search through those families. For all the examples that we found there, the reduction of the Jacobian modulo \( p = 11 \) had either a positive toric rank or good reduction.

**Example 3.6.** There exists a field \( K \) and a prime number \( p \) as in Theorem 1.2, Part (iv), such that \( v_K(p) = \frac{p-1}{2} \) and such that there exists an elliptic curve \( E_K/K \) with a \( K \)-rational point of order \( p \) and Kodaira type \( I_0^* \). Indeed, consider the curve \( E/\mathbb{Q}(\sqrt{5}) \) given by

\[
y^2 + \left( \frac{1 - \sqrt{5}}{2} + 1 \right) xy + \left( \frac{1 - \sqrt{5}}{2} + 1 \right) y = x^3 + x^2 + \left( \frac{-314(1 - \sqrt{5})}{2} - 1031 \right) x + \frac{5958(1 - \sqrt{5})}{2} + 12717.
\]

This is the curve with label 2.2.5.1-1100.1-i in the LMFDB database \[22\]. Let \( M = \mathbb{Q}(\sqrt{5}) \) and \( \mathfrak{p} \) be the prime above \( p := 5 \) in \( M \). If \( K := M_{\mathfrak{p}}^{unr} \) is the maximal unramified extension of the completion of \( K \) with respect to \( \mathfrak{p} \), then the base change \( E_K/K \) of \( E/\mathbb{Q}(\sqrt{5}) \) has a \( K \)-rational torsion point of order 10, because \( E/\mathbb{Q}(\sqrt{5}) \) does. Moreover, \( E_K/K \) has additive reduction of type \( I_0^* \). The proof of Theorem 1.2 does not apply to this example because, using the same notation as in the proof, looking at the elliptic curve over the minimal extension of degree 2 over \( \mathbb{Q}(\sqrt{5}) \) in which \( E \) achieves semi-stable reduction, say \( L \), we get that \( 2P \in (E_L)_1(L) \).

**Example 3.7.** There exists a field \( K \) and a prime number \( p \) as in Theorem 1.2, Part (iv), such that \( v_K(p) = \frac{p-1}{2} \) and such that there exists an elliptic curve \( E_K/K \) with a \( K \)-rational point of order \( p \) and Kodaira type \( I_0^* \). In the tables of Van Hoeij \[11\] there is an elliptic curve \( E/M \) (with \( N = 58 \), \( \text{deg}v = 28 \), \( \text{deg}j = 1 \), \( j = -3375 \) in Van Hoeij’s notation) which is defined over a number field \( M \) with \([M : \mathbb{Q}] = 28 \) and with a \( M \)-rational point of order 29 (in fact it has a point of order 58). This elliptic curve is a CM elliptic curve and there is a prime ideal \( \mathfrak{p} \) of \( M \) above \( p := 29 \) such that \( e(\mathfrak{p}|29) = \frac{29-1}{2} = 14 \). If \( K := M_{\mathfrak{p}}^{unr} \) is the maximal unramified extension of the completion of \( K \) with respect to \( \mathfrak{p} \), then the base change \( E_K/K \) of \( E/M \) has a \( K \)-rational torsion point of order 29, because \( E/M \) does. Moreover, \( E_K/K \) has additive reduction of type \( I_0^* \).

**Example 3.8.** There exists a field \( K \) and a prime number \( p \) as in Theorem 1.2, Part (iii), such that \( v_K(p) = \frac{p-1}{3} \) and such that there exists an elliptic curve \( E_K/K \) with a \( K \)-rational point of order \( p \) and Kodaira type IV. Let

\[
M = \mathbb{Q}[c]/(e^{12} - 4e^{11} + 9e^{10} - 6e^9 - 2e^8 - 3e^7 + 11e^6 - 12e^4 + 7e^3 + 2e^2 - 3e + 1)
\]

and consider the elliptic curve \( E/M \) given by

\[
E(b, c) : y^2 + (1 - c)xy - by = x^3 - bx^2
\]

where \( b, c \) are given as follows. Let \( g = \frac{u}{10142} \) where

\[
u = -(530e^{11} - 1314e^{10} + 2235e^9 + 1428e^8 - 646e^7 - 5201e^6 - 658e^5 + 4866e^4 + 808e^3 - 574e^2 - 1834e - 99)
\]

and let

\[
r = \frac{e^2g - eg + g - 1}{e^2g - e}, \quad s = \frac{eg - g + 1}{eg}.
\]

Finally, put

\[
b = rs(r - 1) \quad \text{and} \quad c = s(r - 1).
\]

We have that \( E/M \) is an elliptic curve with an \( M \)-rational point of order \( p := 19 \) (in fact it has a point of order 57) and \([M : \mathbb{Q}] = 12 \). This is a CM elliptic curve that appears in the tables of Van Hoeij \[11\]. Let \( \mathfrak{p} \) be the prime ideal \((19, e + 1) \) of \( M \). We have that \( e(\mathfrak{p}|19) = \frac{19-1}{3} = 6 \). If \( K := M_{\mathfrak{p}}^{unr} \) is the maximal unramified extension of the completion of \( K \) with respect to \( \mathfrak{p} \),
then the base change $E_K/K$ of $E/M$ has a $K$-rational torsion point of order 19, because $E/M$ does. Moreover, $E_K/K$ has additive reduction of type IV.

**Example 3.9.** There exists a field $K$ and a prime number $p$ as in Theorem 1.2 Part (ii), such that $v_K(p) = \frac{p-1}{2}$ and such that there exists an elliptic curve $E_K/K$ with a $K$-rational point of order $p$ and Kodaira type III. Let

$$M = \mathbb{Q}[e]/(e^8 + 4e^7 + 7e^6 + 8e^5 + 8e^4 + 6e^3 + 4e^2 + 2e + 1)$$

and consider the elliptic curve $E/M$ given by

$$E(b, c) : y^2 + (1 - c)xy - by = x^3 - bx^2$$

where

$$b = \frac{-11e^7 - 34e^6 - 38e^5 - 16e^4 - 11e^3 - 11e^2 - e + 3}{17}$$

and

$$c = \frac{9e^7 + 31e^6 + 59e^5 + 60e^4 + 50e^3 + 30e^2 + 25e + 6}{17}.$$

This elliptic curve appears in the table of page 18 of [1] and is a CM elliptic curve. We have that $[M : \mathbb{Q}] = 8$ and that $E/M$ has a $M$-rational point of order $p := 17$ (in fact it even has a $M$-rational point of order 34). Let $p$ be the prime ideal $(17, e - 7)$ of $M$. Then using SAGE we can check that $e(p|17) = \frac{17 - 1}{4} = 4$. If $K := M_{p^{ur}}$ is the maximal unramified extension of the completion of $K$ with respect to $p$, then the base change $E_K/K$ of $E/M$ has additive reduction of type III.

**Remark 3.10.** There exists a field $K$ and a prime number $p$ as in Theorem 1.2 Part (i), such that $v_K(p) = \frac{p-1}{6}$ and such that there exists an elliptic curve $E_K/K$ with a $K$-rational point of order $p$ and Kodaira type II. In fact, using SAGE [37] we found many examples of elliptic curves that are defined over number fields, have a point of order $p := 13$, and have reduction type II modulo some prime $p$ of the ring of integers $\mathcal{O}_K$ that lies above $p$ and such that the ramification index of $p$ over $p$ is equal to $\frac{p-1}{6}$. We now explain the method we used. If $K$ is any number field and $E/K$ is an elliptic curve with a $K$-rational point of order 13, then there exist $s, t \in K$ which are solutions to the modular curve $X_1(13) : s^2 = t^6 - 2t^5 + t^4 - 2t^3 + 6t^2 - 4t + 1$, and such that $E/K$ can be given by the following equation

$$y^2 + axy + by = x^3 + bx^2,$$

where

$$a = \frac{(t - 1)^2(t^2 + t - 1)s - t^7 + 2t^6 + 3t^5 - 2t^4 - 5t^3 + 9t^2 - 5t + 1}{2t^5},$$

$$b = \frac{(t - 1)^2((t^5 + 2t^4 - 5t^2 + 4t - 1)s - t^8 - t^7 + 4t^6 + 2t^5 + t^4 - 13t^3 + 14t^2 - 6t + 1)}{2t^9}$$

(see also [32] page 14). Consider $t_n := 13n - 3$, for $n \in \mathbb{N}$ and let $(s_n, t_n)$ be a point of $X_1(13)(\mathbb{Q}(s_n))$. Curves corresponding to the points $(s_n, t_n)$, for $n = 1, 2, ..., 3000$, have reduction type II modulo the prime above 13. It seems likely that curves corresponding to the points $(s_n, t_n)$ have reduction type II modulo the prime above 13 for all $n \in \mathbb{N}$.

**Remark 3.11.** There exist elliptic curves $E/K$ defined over quadratic number fields $K/\mathbb{Q}$ ramified at $p := 11$ with a $K$-rational point of order $p$ and with reduction type II modulo the prime above $p$. We now explain how to find such examples. If $K$ is any number field and $E/K$
is an elliptic curve with a $K$-rational point of order 11, then there exist $s, t \in K$ which are solutions to $X_1(11) : s^2 - s = t^3 - t^2$, and such that $E/K$ can be given by the following equation

$$y^2 + (st + t - s^3)xy + s(s - 1)(s - t)t^2y = x^3 + s(s - 1)(s - t)tx^2$$

(see [32] page 10). Consider $t_n := 11n - 3$, for $n \in \mathbb{N}$ and let $(s_n, t_n)$ be a point of $X_1(11)(\mathbb{Q}(s_n))$. Curves corresponding to the points $(s_n, t_n)$, for $n = 1, 2, ..., 3000$, have reduction type II modulo the prime above 11. It seems likely that curves corresponding to the points $(s_n, t_n)$ have reduction type II modulo the prime above 11 for all $n \in \mathbb{N}$.

**Remark 3.12.** When $p := 7$ and $v_K(7) = 1$, then there exist elliptic curves with a $K$-rational point of order 7 and reduction type II. For instance, the elliptic curve with Cremona label 294b2 has a $\mathbb{Q}$-rational point of order 7 and reduction type II modulo 7.

When $p := 7$ and $v_K(7) = 2$, then there exist elliptic curves with a $K$-rational point of order 7 and reduction types III and IV. Indeed, if an elliptic curve $E/\mathbb{Q}$ has reduction type II modulo 7, then after a quadratic extension $K/\mathbb{Q}$ ramified at 7 the base change $E_K/K$ will have reduction type IV modulo the prime of $K$ above 7. Moreover, using the LMFDB database one can find examples of elliptic curves defined over a quadratic extension $K/\mathbb{Q}$ that have $K$-rational point of order 7 and reduction type III modulo the prime of $K$ above 7. One such example is the elliptic curve $E/K$ with LMFDB label 2.2.21.1-980.1-p1 defined over $\mathbb{Q}(\sqrt{21})$.

**Remark 3.13.** Let $p := 11$ or 13 and let $K/\mathbb{Q}$ be a quadratic number field ramified at $p$. Let $E/K$ be an elliptic curve with a $K$-rational point of order $p$. Assume that $E/K$ has reduction type II modulo the prime $\mathfrak{p}$ above $p$ (see Remarks 3.10 and 3.11 for such examples). Then the Weil restriction $\text{Res}_{K/\mathbb{Q}}(E)/\mathbb{Q}$ of $E/K$ from $K$ to $\mathbb{Q}$ has a $\mathbb{Q}$-rational point of order $p$ and purely additive reduction modulo $\mathfrak{p}$, as we now explain. The base change $\text{Res}_{K/\mathbb{Q}}(E)_K/K$ of $\text{Res}_{K/\mathbb{Q}}(E)/\mathbb{Q}$ to $K$ is isomorphic over $K$ to $E \times E^*$, where $E^*/E$ is the base change of $E/K$ along the map $\sigma : K \to K$. Moreover, $E/K$ has additive reduction modulo $\mathfrak{p}$, and since $\mathfrak{p}$ is the unique prime above $p$, it follows that $E^*/K$ has additive reduction modulo $\mathfrak{p}$. Therefore, the toric and abelian ranks of $\text{Res}_{K/\mathbb{Q}}(E)_K/K$ are equal to zero. Since the toric and abelian ranks cannot decrease upon base change, it follows that $\text{Res}_{K/\mathbb{Q}}(E)/\mathbb{Q}$ has purely additive reduction modulo $p$.

4. **An alternative way to prove Theorems 1.2 and 1.3 for small primes**

Let $O_K$ be a discrete valuation ring with valuation $v_K$, fraction field $K$ of characteristic 0 and algebraically closed residue field $k$ of characteristic $p > 5$. We now present an alternative method to prove some partial results towards Theorems 1.2 and 1.3 using explicit equations for modular curves. The idea is that using explicit equations for modular curves we can write down the general equation of an elliptic curve with a torsion point of order $p$ and then we can use Tate’s algorithm [36] to analyze the reduction of the curve based on the equation that we have. We illustrate the method for $p = 5$ below and we refer to the author’s Ph.D. thesis [27] for similar results for $p \leq 23$.

**Proposition 4.1.** Assume $\text{char}(k) = 5$ and let $E/K$ be an elliptic curve. Suppose also that $E/K$ has a $K$-rational torsion point of order 5 and that $E/K$ has additive reduction. Then

(i) If $v_K(5) = 1$, then $E/K$ can only have reduction type II or III.

(ii) If $v_K(5) = 2$, then $E/K$ can only have reduction type II, III, IV or $I_n$ for $n \geq 0$.

(iii) If $v_K(5) = 3$, then $E/K$ cannot have reduction type $I^r$. 

(iv) If \( v_K(5) = 1 \) and \( E/K \) has a \( K \)-rational torsion point of order 10, then \( E/K \) can only have reduction type III.

**Proof.** By completing \( K \) with respect to the valuation \( v_K \) we can assume, without loss of generality, that \( K \) is complete. Let \( E/K \) be an elliptic curve with a \( K \)-rational point of order 5. Then \( E/K \) can be given by a Weierstrass equation of the form

\[
y^2 + (1 - c)xy - by = x^3 - bx^2
\]

(see [12] Section 4.4), with

\[
b = \lambda, \quad c = \lambda, \quad \text{and} \quad \lambda \in K.
\]

Let \( \lambda = \frac{4}{t} \) with \( s, t \in \mathcal{O}_K \) and such that \( v_K(s) = 0 \) or \( v_K(t) = 0 \). Using the transformation \( x \to x/t^2 \) and \( y \to y/t^3 \), we obtain a new Weierstrass equation of the form

\[
y^2 + (t - s)xy - st^2y = x^3 - stx^2.
\]

The discriminant of this Weierstrass equation is

\[
\Delta = s^5t^5(s^2 - 11st - t^2),
\]

with invariants

\[
c_4(s, t) = 24st^2(-s + t) + (s^2 - 6st + t^2)^2,
\]

and

\[
c_6(s, t) = -(s^2 + t^2)(s^4 - 18s^3t + 74s^2t^2 + 18st^3 + t^4).
\]

Note that if \( v_K(s) > 0 \) or \( v_K(t) > 0 \), then \( E/K \) has split multiplicative reduction so in what follows we must have that \( v_K(s) = 0 \) and \( v_K(t) = 0 \) since we assume that \( E/K \) has additive reduction.

**Claim 4.2.** There exist polynomials \( A(s, t), B(s, t) \in \mathbb{Z}[s, t] \) such that

\[
A(s, t)c_4(s, t) + B(s, t)c_6(s, t) = 2^{12}3^65^9.
\]

Consider first the polynomials

\[
f(x) = c_4(x, 1) = x^4 - 12x^3 + 14x^2 + 12x + 1
\]

and

\[
g(x) = c_6(x, 1) = -(x^2 + 1)(x^4 - 18x^3 + 74x^2 + 18x + 1).
\]

Using the WOLFRAM [13] command PolynomialExtendedGCD\([f(x), g(x)]\) we obtain that \( f(x) \) and \( g(x) \) are coprime in \( \mathbb{Q}[x] \) and moreover if

\[
a(x) := \frac{1}{2^{12}3^65}(6471756x^5 - 1171065600x^4 + 4965235200x^3 - 472780800x^2 + 490020480x + 698035968)
\]

and

\[
b(x) := \frac{1}{2^{12}3^65}(64717056x^3 - 782763264x^2 + 980543232x + 683106048),
\]

then

\[
a(x)f(x) + b(x)g(x) = 1.
\]

Setting \( x = \frac{4}{t} \) and multiplying the last expression by \( 2^{12}3^65^9 \) we obtain that if

\[
A(s, t) := a\left(\frac{s}{t}\right)2^{12}3^65^5 \quad \text{and} \quad B(s, t) := b\left(\frac{s}{t}\right)2^{12}3^65^3,
\]

then \( A(s, t) \) and \( B(s, t) \) are in \( \mathbb{Z}[s, t] \), and

\[
A(s, t)c_4(s, t) + B(s, t)c_6(s, t) = 2^{12}3^65^9.
\]
This proves the claim.

Proof of (i): Recall that we assume that \( v_K(s) = v_K(t) = 0 \). If \( v_K(c_4(s, t)) = 0 \) or \( v_K(c_6(s, t)) = 0 \), then by Tableau 1 of [30] we find that \( E/K \) has semi-stable reduction. If \( v_K(c_4(s, t)) > 0 \) and \( v_K(c_6(s, t)) > 0 \), then since \( v_K(t) = 0 \) and \( v_K(5) = 1 \), by Claim 4.2 we obtain that \( \min\{v_K(c_4), v_K(c_6)\} \leq 1 \). As a result, Tableau 1 of [30] implies that if \( E/K \) has additive reduction, then the reduction type can only be II or III.

Proof of (ii): Assume that \( v_K(5) = 2 \). Since \( v_K(t) = 0 \), Claim 4.2 implies that \( \min\{v_K(c_4), v_K(c_6)\} \leq 2 \). Therefore, by Tableau 1 of [30] we find that if \( E/K \) has additive reduction, then the reduction type can only be II, III, IV or \( I^*_n \) for some \( n \geq 0 \).

Proof of (iii): Assume now that \( v_K(5) = 3 \). By Claim 4.2, since \( v_K(t) = 0 \) and \( v_K(5) = 3 \), we obtain \( \min\{v_K(c_4), v_K(c_6)\} \leq 3 \) so by Tableau 1 of [30] we find that reduction type II* cannot occur.

Proof of (iv): Assume that \( E/K \) has additive reduction and let \( P \in E(K)[10] \). Since \( 2P \) has order 5, Part (i) implies that \( E/K \) has reduction type II or III. We will show that reduction type II cannot occur. Indeed, consider the point \( Q = 5P \) which is a point of order 2. By Theorem 5.9.4 of [16] we get that \( Q \) is not in the kernel of the reduction. This implies that \( Q \) reduces to the singular point. Indeed, if \( Q \) reduces to a non-singular point, then the reduction of \( Q \) has order 2, which is impossible since \( \tilde{E}_a(k) \cong \mathbb{G}_a(k) \) and the latter does not have points of order 2. Therefore, \( Q \) reduces to the singular point. As a result, 2 divides the component group of \( E/Q \) at 5. Finally, by the table on page 46 of [36] we find that reduction type II cannot occur.

\[ \square \]

Remark 4.3. We note that if \( v_K(5) = 1 \), then both of the allowed reduction types in Part (i) of Proposition 4.1 occur. For instance, the elliptic curves \( E/Q \) with Cremona labels 50b1 and 50b2 have a \( Q \)-rational point of order 5 and reduction of type II modulo 5. The elliptic curve with Cremona label 150a4 has a \( Q \)-rational point of order 10 (and hence of order 5) and reduction of type III modulo 5.

Concerning Part (ii) of Proposition 4.1, we first note that if an elliptic curve \( E/Q \) has reduction type II (resp. III) modulo 5, then after a quadratic extension ramified above \( 5 \) the base change will have reduction type IV (resp. \( I_2^* \)) modulo the prime above 5. Moreover, using the LMFDB database we found examples of elliptic curves defined over \( \mathbb{Q}(\sqrt{5}) \) that have reduction \( I_0^* \) modulo the prime above 5. For instance, the elliptic curve with LMFDB label 2.2.5.1-2525.2-g1 is one such example. Thus, the types IV, \( I_0^* \), \( I_2^* \) in Part (ii) of Proposition 4.1 occur.

5. DIFFERENCE BETWEEN THE MIXED AND THE EQUICHARACTERISTIC CASES

Let \( E/K \) be an elliptic curve with a \( K \)-rational point of order \( p \). The reduction types that can occur in the case where \( K \) is the fraction field of a discrete valuation ring of characteristic \( p \geq 5 \) have already been studied by Liedtke and Schröer in [18]. More precisely, among other results in [18], they prove the following theorem (see Theorem 4.3 of [18] and the Theorem in page 2158 of [18]).

**Theorem 5.1.** Let \( R \) be a henselian discrete valuation ring of characteristic \( p \geq 5 \), with fraction field \( K \) and residue field \( k \) which is assumed to be algebraically closed. Let \( E/K \) be elliptic curve with additive reduction and containing a \( K \)-rational point of order \( p \). Then \( E/K \) has potentially supersingular reduction. Moreover,

(i) If \( p \equiv 1 \pmod{12} \), then \( E/K \) can only have reduction type \( I_0^* \).
(ii) If \( p \equiv 5 \pmod{12} \), then \( E/K \) can only have reduction type II, IV, \( I_0^* \), IV* or IP*.

(iii) If \( p \equiv 7 \pmod{12} \), then \( E/K \) can only have reduction type III, \( I_0^* \) or III*.

(iv) If \( p \equiv 11 \pmod{12} \), then \( E/K \) can only have reduction type II, III, IV, \( I_0^* \), IV*, III* or IP*.

We note that it is not true in the mixed characteristic case that an elliptic curve \( E/K \) with additive reduction and integral \( j \)-invariant has potentially supersingular reduction when it has a \( K \)-rational point of prime order \( p \geq 5 \). There are examples of elliptic curves defined over \( \mathbb{Q} \) with a 7-torsion point that have additive reduction of Kodaira type II (see Remark 3.12). The following proposition shows that for \( p = 7 \) the reduction of such elliptic curves cannot be potentially supersingular.

**Proposition 5.2.** Let \( R \) be a discrete valuation ring with fraction field \( K \), where \( K \) is of any characteristic, and with residue field characteristic \( p \geq 5 \). Let \( E \) be an elliptic curve over \( K \). Then we have the following:

(i) If \( E/K \) has reduction of type II, II*, IV or IV*, then the reduction is potentially supersingular if and only if \( p \equiv 2 \pmod{3} \).

(ii) If \( E/K \) has reduction of type III or III*, then the reduction is potentially supersingular if and only if \( p \equiv 3 \pmod{4} \).

**Proof.** Assume that \( E/K \) has reduction type II, II*, IV or IV*. By the table on page 46 of [36] the \( j \)-invariant of \( E/K \) reduces to zero in the residue field. However, 0 is a supersingular \( j \)-invariant if and only if \( p \equiv 2 \pmod{3} \) by [35] Example V.4.4. This proves Part (i).

Assume that \( E/K \) has reduction type III or III*. By the table on page 46 of [36] the \( j \)-invariant of \( E/K \) reduces to 1728 in the residue field. Since 1728 is a supersingular \( j \)-invariant if and only if \( p \equiv 3 \pmod{4} \) by [35] Example V.4.4, we have that reduction is potentially supersingular if and only if \( p \equiv 3 \pmod{4} \). This proves Part (ii). □

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