ON UNIVERSAL CONTINUOUS ACTIONS ON THE CANTOR SET

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Abstract. Using the notion of proper Cantor colorings we prove the following theorem. For any countably infinite group Γ, there exists a free continuous action ζ : Γ ↷ C on the Cantor set, which is universal in the following sense: for any free Borel action α : Γ ↷ X on the standard Borel space, there exists an injective Borel map Θα : X → C such that Θα ◦ α = ζ ◦ Θα. We extend our theorem for (nonfree) Borel (Γ, Z)-actions, where Z is a uniformly recurrent subgroup.

Keywords. Borel actions, continuous actions, uniformly recurrent subgroups

1. Introduction

In [4] Hjorth and Molberg proved that for any countable group Γ there exists a free and continuous action of Γ on the Cantor set C that admits an invariant probability measure. The first goal of this note is to show that one can say much more.

Theorem 1. For any countably infinite group Γ, there exists a free continuous action ζ : Γ ↷ C on the Cantor set such that for any free Borel action α : Γ ↷ X on the standard Borel space, there exists an injective Borel map Θα : X → C, satisfying Θα ◦ α = ζ ◦ Θα.

Therefore, if the free Borel action α : Γ → X admits an invariant probability measure µ and µΘα is the pushdown of µ onto C, then the probability measure µΘα is invariant with respect to ζ. The theorem above will easily follow from the proposition below.

Proposition 1.1. For any countably infinite group Γ and for any free Borel action α : Γ → X on the standard Borel space, there exists an injective equivariant Borel map Θ′α : X → Free(CΓ) (where Free(CΓ) is the free part of the Bernoulli C-shift space), such that the closure of the set Θ′α(X) is still in Free(CΓ).

Note that Seward and Tucker-Drob [7] (see also [1]) proved the following result: For any free Borel action α : Γ → X there exists a (not necessarily injective) equivariant Borel map Θ : X → Free({0, 1}Γ) such that the closure
of the set $\Theta(X)$ is still in $\text{Free}(\{0,1\})$. It seems however that the technique of [7] cannot easily be extended to nonfree actions.

So, let $\beta : \Gamma \to M$ be a continuous minimal nonfree action of a countable group on a compact metric space. Then we have a natural equivariant map $\text{Stab}_\beta : M \to \text{Sub}(\Gamma)$ (see Section 3 for further details) from our space $M$ to the compact space of subgroups of $\Gamma$, mapping each point $x \in M$ to its stabilizer subgroup. Glasner and Weiss (Proposition 1.2 [3]) proved that the set $M_0$ of the continuity points of the map $\text{Stab}_\beta$ is a dense, invariant $G_\delta$ subset of $M$ and the closure of $\text{Stab}_\beta(M_0)$ in $\text{Sub}(\Gamma)$ is a minimal closed invariant subset of $\text{Sub}(\Gamma)$, that is, a uniformly recurrent subgroup. They asked if for any uniformly recurrent subgroup $Z \subset \text{Sub}(\Gamma)$ there exists a minimal continuous action $\beta : \Gamma \to M$ such that $M_0 = M$. This question has been answered in [2] and [6]. In Section 3 we will prove that one can answer the question of Glasner and Weiss in a uniform way.

**Theorem 2.** Let $\Gamma$ be a countable group and $Z \subset \text{Sub}(\Gamma)$ be uniformly recurrent subgroup. Then there exists a continuous action $\zeta_Z : \Gamma \curvearrowright C$ such that

- $\text{Stab}_{\zeta_Z} : C \to \text{Sub}(\Gamma)$ is continuous everywhere and $\text{Stab}_{\zeta_Z}(C) = Z$.
- For any Borel action $\alpha : \Gamma \curvearrowright X$ such that for any $x \in X$ the group $\text{Stab}_\alpha(x)$ is in $Z$ (we call these actions $(\Gamma,Z)$-actions), there exists an injective Borel map $\Theta_\alpha : X \to C$ such that $\Theta_\alpha \circ \alpha = \zeta_Z \circ \Theta_\alpha$.

It was proved in Section 5 [2] that there exist countable groups $\Gamma$ and uniformly recurrent subgroups $Z \subset \text{Sub}(\Gamma)$ such that no Borel $(\Gamma,Z)$-action admits an invariant probability measure. However, we have the following nonfree analogue of the aforementioned result of Hjorth and Molberg.

**Corollary 1.1.** Let $\Gamma$ be a countable group and let $Z \subset \text{Sub}(\Gamma)$ be a uniformly recurrent subgroup. If there exists a Borel $(\Gamma,Z)$-action $\alpha : \Gamma \curvearrowright X$ that admits an invariant probability measure, then there exists a continuous $(\Gamma,Z)$-action $\beta : \Gamma \curvearrowright C$ on the Cantor set admitting an invariant probability measure such that

- The map $\text{Stab}_\beta : C \to \text{Sub}(\Gamma)$ is continuous everywhere and
- $\text{Stab}_\beta(C) = Z$.

### 2. The proof of Theorem 1

Let $\Gamma$ be a countable infinite group and $\{\sigma_i\}_{i=1}^\infty$ be a generating system of $\Gamma$. Also for $n \geq 1$, let $\Gamma_n$ be the subgroup of $\Gamma$ generated by the elements $\{\sigma_i\}_{i=1}^n$. Let $\alpha : \Gamma \curvearrowright X$ be a Borel action of $\Gamma$ on the standard Borel space $X$. We define a sequence $\{G_n\}_{n=1}^\infty$ of Borel graph structures on $X$ in the following way. If $p, q \in X$, $p \neq q$, then let $(p,q) \in E(G_n)$ if there exists $1 \leq i \leq n$ such that $\alpha(\sigma_i)(p) = q$ or $\alpha(\sigma_i)(q) = p$. A Borel $\mathcal{C}$-coloring of $X$ is a Borel map $\varphi : X \to \mathcal{C}$, where $\mathcal{C} = \{0,1\}^\mathbb{N}$ is the Cantor set. We say that $\varphi$ is a proper
C-coloring with respect to \( \alpha : \Gamma \curvearrowright X \) if for any \( r > 0 \) there exists \( S_r > 0 \) such that for any \( p, q \in X \) \( (\varphi(p))_{S_r} \neq (\varphi(q))_{S_r} \), provided that \( 0 < d_{G_r}(p, q) \leq r \), where

- \( d_{G_r} \) is the shortest path metric on the components of the Borel graph \( G_r \).
- For \( \kappa \in C \) and \( s > 0 \), \( (\kappa)_s \in \{0, 1\}^{|s|} \), denotes the projection of \( \kappa \) onto its first \( s \) coordinates.

The Borel coloring \( \varphi \) is called separating if for any \( p \neq q \in X \), \( \varphi(p) \neq \varphi(q) \).

**Lemma 2.1.** For any Borel action \( \alpha : \Gamma \curvearrowright X \) there exists a separating, proper \( C \)-coloring with respect to \( \alpha \).

**Proof.** First, for any \( r \geq 1 \) we construct a new Borel graph \( H_r \) of bounded vertex degree on \( X \) such that \((p, q) \in E(H_r)\) if \( 0 < d_{G_r}(p, q) \leq r \). By the classical result of KeCHRIS, SolecK and TodorceVic [5], there exists an integer \( m_r > 0 \) and a Borel coloring \( \psi_r : X \rightarrow \{0, 1\}^{[m_r]} \) such that \( \psi_r(p) \neq \psi_r(q) \), whenever \( p \) and \( q \) are adjacent vertices in the Borel graph \( H_r \). Then \( \varphi_1(p) = \{\psi_1(p)\psi_2(p)\ldots\}\in\{0, 1\}^N \), defines a proper \( C \)-coloring of \( X \) with respect to \( \alpha \).

Now we use the usual trick to obtain a separating coloring. Let \( \varphi_2 : X \rightarrow C \) be a Borel isomorphism. Then if \( \varphi_1(p) = \{a_1a_2a_3\ldots\} \) and \( \varphi_2(p) = \{b_1b_2b_3\ldots\} \) let \( \varphi(p) = \{a_1b_1a_2b_2\ldots\} \). Clearly, \( \varphi \) is a separating, proper \( C \)-coloring with respect to \( \alpha \).

Now we prove Proposition 1.1. Let \( \alpha : \Gamma \curvearrowright X \) be a free Borel action and let \( \varphi : X \rightarrow C \) be a separating, proper \( C \)-coloring. Consider the Bernoulli shift \( C^\Gamma \) with the natural left action

\[ L_\delta(\rho)(\gamma) = \rho(\gamma\delta) . \]

The map \( \Theta'_\alpha : X \rightarrow C^\Gamma \) is defined as usual by

\[ \Theta'_\alpha(x)(\gamma) = \varphi(\alpha(\gamma)(x)) . \]

Clearly, \( \Theta'_\alpha \) is Borel and \( \Gamma \)-equivariant and since \( \varphi \) is separating \( \Theta'_\alpha \) is injective as well. Also, \( \Theta'_\alpha(X) \subset \text{Free}(C^\Gamma) \). We need to show that if \( \rho : \Gamma \rightarrow C \) is in the closure of \( \Theta'_\alpha(X) \), then \( L_\gamma(\rho) \neq \rho \), whenever \( \gamma \neq 1_\Gamma \). It is enough to see that if \( \gamma \neq e_\Gamma \), then \( \rho(\gamma) \neq \rho(e_\Gamma) \). Let \( \lim_{n \to \infty} \Theta'_\alpha(x_n) = \rho \in C^\Gamma \). Also, let \( r > 0 \) such that

- \( \gamma \in \Gamma_r \),
- \( d_{G_r}(e_\Gamma, \gamma) \leq r \), where \( d_{G_r} \) is the shortest path metric on the right Cayley graph \( \text{Cay}(\Gamma_r, \{\sigma_i\}_{i=1}^r) \).

Thus, for any \( n \geq 1 \),

\[ d_{G_r}(\alpha(\gamma)(x_n), x_n) \leq r . \]

Hence, for any \( n \geq 1 \),

\[ (\varphi(\alpha(\gamma)(x_n)))_r \neq (\varphi(x_n))_r . \]
Since \( \rho(\gamma) = \lim_{n \to \infty} \varphi(\alpha(\gamma)(x_n)) \) and \( \rho(\epsilon_1) = \lim_{n \to \infty} \varphi(x_n) \), we have that \( \rho(\gamma) \neq \rho(\epsilon_1) \). Hence our proposition follows.

Now we prove Theorem 1. Let \( \iota : \Gamma \curvearrowright \text{Free}(C^\ell) \) be the natural free action and \( \Theta'_1 : \text{Free}(C^\ell) \to \text{Free}(C^\ell) \) be the injective \( \Gamma \)-equivariant Borel map defined in Proposition 1.1. Let \( W \) be the closure of \( \Theta'_1(\text{Free}(C^\ell)) \) in \( C^\ell \). Thus, \( W \subset \text{Free}(C^\ell) \) is a closed invariant subset. Then, \( W \) can be written as the disjoint union of \( W_1 \) and \( W_2 \), where \( W_1 \) is a countable invariant subset and \( W_2 \) is a closed invariant subset homeomorphic to the Cantor set. Let \( \alpha : \Gamma \curvearrowright X \) be a free Borel action and \( \Theta'_2 : X \to \text{Free}(C^\ell) \) be the injective \( \Gamma \)-equivariant Borel map defined in Proposition 1.1. Let \( \Theta = \Theta'_1 \circ \Theta'_2 \). Clearly, \( \Theta : X \to W \) is an injective \( \Gamma \)-equivariant Borel map. Now we define the injective \( \Gamma \)-equivariant Borel map \( \Theta' : X \to W_2 \) by modifying \( \Theta \) on countably many orbits. Since \( W_2 \) is homeomorphic to \( C \) our Theorem follows.

3. Uniformly recurrent subgroups

Let \( \Gamma \) be a countable group and let \( \text{Sub}(\Gamma) \) be the space of subgroups of \( \Gamma \) \cite{[3]}: \( \text{Sub}(\Gamma) \) is a compact, metrizable space and conjugations define a continuous action \( c : \Gamma \curvearrowright \text{Sub}(\Gamma) \). Let \( Z \subset \text{Sub}(\Gamma) \) be a uniformly recurrent subgroup as in the Introduction. We define the Bernoulli shift space \( C^Z \) of \( Z \) in the following way. Let

\[
C^Z = \bigcup_{H \in Z} F(H),
\]

where \( F(H) \) is the set of maps \( \rho : \Gamma/H \to C \) from the right coset space of \( H \) to the Cantor set. The action of \( \Gamma \) on \( C^Z \) is defined as follows.

- If \( \rho \in F(H) \) then \( L_{\delta}\rho \in F(\delta H\delta^{-1}) \) and
- \( L_{\delta}\rho(\delta H\delta^{-1} \gamma) = \rho(H\gamma\delta) \).

**Lemma 3.1.** \( L : \Gamma \to \text{Homeo}(C^Z) \) is a homomorphism.

**Proof.** We need to show that if \( \rho : \Gamma/H \to C \) and \( \delta_1, \delta_2 \in \Gamma \), then

\[
L_{\delta_1}(L_{\delta_2}(\rho)) = L_{\delta_1 \delta_2}(\rho).
\]

Observe that

\[
L_{\delta_1}(L_{\delta_2}(\rho)) \in \mathcal{F}(\delta_1 \delta_2 H \delta_2^{-1} \delta_1^{-1}) \quad \text{and} \quad L_{\delta_1 \delta_2}(\rho) \in \mathcal{F}(\delta_1 \delta_2 H \delta_2^{-1} \delta_1^{-1}).
\]

Now

\[
L_{\delta_1}(L_{\delta_2}(\rho))(\delta_1 \delta_2 H \delta_2^{-1} \delta_1^{-1} \gamma) = L_{\delta_1 \delta_2}(\rho)(\delta_2 H \delta_2^{-1} \gamma \delta_1) = \rho(H\gamma \delta_1 \delta_2) = L_{\delta_1 \delta_2}(\rho)(\delta_1 \delta_2 H \delta_2^{-1} \delta_1^{-1} \gamma),
\]

hence our lemma follows.

We can equip \( C^Z \) with a compact metric structure \( d \) such that \( (C^Z, d) \) is homeomorphic to the Cantor set and the \( \Gamma \)-action above is continuous. Let \( \rho_1 : \Gamma/H_1 \to C, \rho_2 : \Gamma/H_2 \to C \) be elements of \( C^Z \). We say that \( \rho_1 \) and \( \rho_2 \) are \( n \)-equivalent, \( \rho_1 \equiv_n \rho_2 \) if
For any $\gamma \in \Gamma$, $d_{\Gamma_n}(e_\Gamma, \gamma) \leq n$, $\gamma \in H_1$ if and only if $\gamma \in H_2$.

For any $\gamma \in \Gamma$, $d_{\Gamma_n}(e_\Gamma, \gamma) \leq n$,

\[
(\rho_1(H\gamma))_n = (\rho_2(H\gamma))_n.
\]

Then we define $d(\rho_1, \rho_2) := \frac{1}{n}$ whenever $\rho_1 \equiv_n \rho_2$ and $\rho_1 \neq_{n+1} \rho_2$. Let \( \{H_n\}_{n=1}^\infty, H \in \text{Sub}(\Gamma) \), $\rho \in \mathcal{F}(H)$ and for any $n \geq 1$ let $\rho_n \in \mathcal{F}(H_n)$. Observe that $\{\rho_n\}_{n=1}^\infty \to \rho$ in the $d$-metric if and only if

- $H_n \to H$ in the compact space $\text{Sub}(\Gamma)$ and
- for any $\gamma \in \Gamma$, $\rho_n(H\gamma) \to \rho(H\gamma)$.

We can define $\text{Free}(C^Z)$ in the usual way. We have that $\rho : \Gamma / H \to C \in \text{Free}(C^Z)$ if $L_\delta(\rho) \neq \rho$ for any $\delta \notin H$. Clearly, if for any $\delta \notin H$, $\rho(H) \neq \rho(H\delta)$, then $\rho \in \text{Free}(C^Z)$. Now let $\alpha : \Gamma \curvearrowright X$ be a Borel $(\Gamma, Z)$-action and $\varphi : X \to \mathcal{C}$ be a separating, proper $\mathcal{C}$-coloring with respect to $\alpha$. We define $\Theta^C_\alpha : X \to \text{Free}(C^Z)$ as follows.

- $\Theta^C_\alpha(x) \in \mathcal{F}(H)$, where $H = \text{Stab}_\alpha(x)$.
- $\Theta^C_\alpha(x)(H\gamma) = \varphi(\alpha(\gamma)(x))$.

Clearly, $\Theta^C_\alpha : X \to \text{Free}(C^Z)$ is an injective Borel map and $\Theta^C_\alpha(X) \subset \text{Free}(C^Z)$.

**Lemma 3.2.** The map $\Theta^C_\alpha : X \to \text{Free}(C^Z)$ is $\Gamma$-equivariant.

**Proof.** Let $\delta \in \Gamma$. Then

\[
L_\delta(\Theta^C_\alpha(x))(\delta H\delta^{-1}\gamma) = \Theta^C_\alpha(x)(H\gamma\delta) = \varphi(\alpha(\gamma)(x)) .
\]

On the other hand,

\[
\Theta^C_\alpha(\alpha(\delta)(x))(\delta H\delta^{-1}\gamma) = \varphi(\alpha(\delta)\alpha(\gamma)(x)) = \varphi(\alpha(\delta\gamma))(x) .
\]

\qed

Now we prove the nonfree analogue of Proposition 1.1.

**Proposition 3.1.** For any countably infinite group $\Gamma$ and for any free Borel $(\Gamma, Z)$-action $\alpha : \Gamma \curvearrowright X$, there exists an injective equivariant Borel map $\Theta'_\alpha : X \to \text{Free}(C^Z)$, such that the closure of the set $\Theta'_\alpha(X)$ is still in $\text{Free}(C^Z)$.

**Proof.** Let $\varphi$ and $\Theta^C_\alpha$ be as above. Let $\{x_n\}_{n=1}^\infty \subset X$ such that

\[
\lim_{n \to \infty} \Theta^C_\alpha(x_n) = \rho \in \mathcal{F}(H)
\]

and $\delta \notin H$. We need to show that $\rho \in \text{Free}(C^Z)$. Observe that

$\{\text{Stab}_\alpha(x_n)\}_{n=1}^\infty \to H$ in $\text{Sub}(\Gamma)$. Hence, there exists $N > 0$ such that $\delta \notin H_n$ if $n \geq N$. By properness, there exists $m > 0$ such that for all $n \geq N$

\[
(\varphi(\alpha(\delta)(x_n)))_m \neq (\varphi(x_n))_m .
\]

Since, $\lim_{n \to \infty} \varphi(\alpha(\delta)(x_n)) = \rho(H\delta)$ and $\lim_{n \to \infty} \varphi(x_n) = \rho(H)$ we have that $\rho(H) \neq \rho(H\delta)$. Therefore $\rho \in \text{Free}(C^Z)$.

\qed

Now Theorem 2 follows from Proposition 3.1 exactly the same way as Theorem 1 follows from Proposition 1.1. \qed
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