ON ALGEBRAIC PROPERTIES OF LOW RANK APPROXIMATIONS OF
PRONY SYSTEMS

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Abstract. We consider the reconstruction of spike train signals of the form
\[ F(x) = \sum_{i=1}^{d} a_i \delta(x - x_i), \]
from their moments measurements
\[ m_k(F) = \int x^k F(x) \, dx = \sum_{i=1}^{d} a_i x^k. \]
When some of the nodes \( x_i \) near collide the inversion becomes unstable. Given noisy moments measurements, a typical consequence is that reconstruction algorithms estimate the signal \( F \) with a signal having fewer nodes, \( \tilde{F} \). We derive lower bounds for the moments difference between a signal \( F \) with \( d \) nodes and a signal \( \tilde{F} \) with strictly less nodes, \( l \). Next we consider the geometry of the non generic case of \( d \) nodes signals \( F \), for which there exists an \( l < d \) nodes signal \( \tilde{F} \), with moments
\[ m_0(\tilde{F}) = m_0(F), \ldots, m_p(\tilde{F}) = m_p(F), \quad p > 2l - 1. \]
We give a complete description for the case of a general \( d, l \) and \( p = 2l - 1 \) which can be inferred from earlier work.

1. Introduction

In this paper we consider the classical Prony system of algebraic equations
\[ \sum_{j=1}^{d} a_j x_j^k = m_k, \quad k = 0, 1, \ldots, N, \]
with the unknowns \( a_j, x_j, \ j = 1, \ldots, d \), and known right hand side formed by the moment “measurements” \( m_0, \ldots, m_N \). We will refer to the unknowns \( a = (a_1, \ldots, a_d) \) as amplitudes and to the unknowns \( x = (x_1, \ldots, x_d) \) as nodes.

Prony systems appear in many classical theoretical and applied mathematical problems [2, 4, 8, 18, 10, 9]. In particular, the bibliography in [3] contains more than 50 pages. Explicit solution of Problem (1.1) was given by Prony himself already in [28].

Many of the more recent applications are in Signal Processing. As a very partial sample we mention that in [15] and in many other publications a method, essentially equivalent to solving a Prony system, was used in reconstructing signals with a “finite rate of innovation”. In [24, 26] the applicability of Prony-type systems was extended to some new wide and important classes of signals. In [14, 17] multidimensional Prony systems were investigated via symmetric tensors, in particular, connecting them to the polynomial Waring problem. In [21] Prony system appears in the general

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context of Compressed Sensing. In [6, 11] Prony-like systems were used in reconstructing piecewise-smooth functions from their Fourier data. Finally, in [6] the same reconstruction accuracy as for smooth functions was demonstrated (thus confirming the Eckhoff conjecture).

In what follows we will identify the unknown tuple \((a, x)\) with a “spike-train signal” \(F\),

\[
F(x) = \sum_{j=1}^{d} a_j \delta(x - x_j).
\]

Clearly, the moments \(m_k(F) = \int x^k F(x)dx, \ k = 0, 1, \ldots\), are given by \(m_k(F) = \sum_{j=1}^{d} a_j x_j^k\), so reconstructing \(F\) from its \(N\) initial moments is equivalent to solving (1.1), with \(m_k = m_k(F)\).

In practice it is important to have a stable method of inversion and many research efforts are devoted to this task (see e.g. [5, 12, 20, 25, 27, 31] and references therein). A basic question here is the following. We are given noisy measurements \(\nu = (\nu_0, \ldots, \nu_N)\) with

\[
|\nu_k - m_k| \leq \epsilon, \quad k = 0, 1, \ldots, N,
\]

where \(m_k\) are actual moments for some signal \(F\). The goal is to solve the Prony system (1.1) with right hand side \(\nu\), so as to minimize the worst case reconstruction error.

An important case that poses major mathematical and numerical difficulties is when some of the nodes \(x_j\) of the measured signal nearly collide. In particular, this happens in the context of the “super-resolution problem”, which was investigated in many recent publications. See [1, 7, 12, 16, 19, 22, 29] as a small sample.

We now introduce the moments Hankel matrix which is important in the next calculations, and is used in reconstruction algorithms that are based on Prony method. Given a moments vector \(m = (m_0, \ldots, m_N)\), with \(N = 2d - 1, d \in \mathbb{N}^+\), consider the associated \(d \times d\) Hankel matrix \(H_d(m)\),

\[
H_d(m) = \begin{bmatrix}
m_0 & m_1 & m_2 & \cdots & m_{d-1} \\
m_1 & m_2 & m_3 & \cdots & m_d \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
m_{d-1} & m_d & m_{d+1} & \cdots & m_{2d-2}
\end{bmatrix}.
\]

We say that a signal \(F\) as above, has \(d\) nodes if its amplitudes \(a_i, \ i = 1, \ldots, d\) are non zero and the nodes are distinct. For exact measurements vector \(m = (m_0, \ldots, m_N)\) of a signal \(F\), the rank of the associated Hankel matrix \(H_d(m)\) is equal to number of nodes of \(F\). Given noisy moment measurements \(\nu = (\nu_0, \ldots, \nu_N)\), generalized Prony methods for reconstruction of \(F\) typically estimate the numerical rank \(r\), of the associated Hankel matrix \(H_d(\nu)\). The next step is to recover \(F\) from \(\nu\) with the number of nodes equal to \(r\).

As the nodes collide the rank of \(H_d(\nu)\) drops, effectively causing such methods of reconstruction to estimate the source signal with a signal with less nodes. Typically, each cluster of nodes will be reduced to a single node.

In the present paper we consider two problems related to a low rank approximation of Prony systems.

Denote by \(\mathcal{P} = \mathcal{P}_d\) the parameter space of signals \(F\) with \(d\) nodes,

\[
\mathcal{P}_d = \{(a, x) = (a_1, \ldots, a_d, x_1, \ldots, x_d) \in \mathbb{R}^{2d}, \ x_1 < x_2 < \ldots < x_d, \ a_i \neq 0, \ i = 1, \ldots, d\}.
\]

For the sack of completion we define \(\mathcal{P}_0\) to be the singleton containing the zero signal \(F_0(x) = 0\). Denote by \(\mathcal{P}_d^a\) and by \(\mathcal{P}_d^x\) the parameter spaces of the amplitudes \(a\) and the nodes \(x\), respectively.
Finally denote by \( \mathcal{M} = \mathcal{M}_d \cong \mathbb{R}^{2d} \) the moment space consisting of the \( 2d \)-tuples of the form \((\nu_0, \nu_1, \ldots, \nu_{2d-1})\).

The first main question of this paper, considered in section \( 2 \) is the following:

**Problem 1.1.** Given the triplet \((d, l, p)\) of natural numbers with \( d > l > 0 \), describe the geometry of the set of all signals \( F \in \mathcal{P}_d \) such that there exists a signal \( \tilde{F} \) with at most \( l \) nodes, \( \tilde{F} \in \mathcal{P}_l, i \leq l \), satisfying

\[
m_k(\tilde{F}) = m_k(F), \quad k = 0, \ldots, p.
\]

That is, \( \tilde{F} \) matches the \( p + 1 \) initial moments of \( F \).

For each triplet \((d, l, p)\) as in Problem 1.1, we will denote by \( \Sigma_{d,l,p} \subset \mathcal{P}_d \) the set of all signals satisfying the condition of the Problem for this case. Fixing \( p \) and \( d \) and then varying \( l \) naturally leads to a stratification of \( \mathcal{P}_d \) according to the sets \( \Sigma_{d,l,p} \).

Let \( \hat{\Sigma}_{d,l,p} \subset \mathcal{P}_d \times \mathcal{P}_l \) be defined by the algebraic conditions

\[
m_k(F) = m_k(\hat{F}), \quad k = 0, \ldots, p.
\]

Then for \( \pi : \mathcal{P}_d \times \mathcal{P}_l \rightarrow \mathcal{P}_d \), the projection to the first factor, we have

\[
\Sigma_{d,l,p} = \pi(\hat{\Sigma}_{d,l,p}).
\]

In particular, this implies that \( \Sigma_{d,l,p} \) is a semi-algebraic subset of \( \mathcal{P}_d \). Counting the parameters, we can expect that for \( p > 2l - 1 \) it is generically of codimension \( p - 2l + 1 \). For \( p = 2l - 1 \) and \( F \in \mathcal{P}_d \) the condition \( F \in \Sigma_{d,l,p} \) is equivalent to the solvability of the Prony system \( m_k(\tilde{F}) = m_k(F), \quad k = 0, \ldots, p \), for signals \( \tilde{F} \in \mathcal{P}_l \) with real nodes and amplitudes. These conditions can be given explicitly (see, e.g. [13, 22]).

Our main result with respect to Problem 1.1 is a complete description of the geometry of the set of signals meeting the condition of the Problem for the case \((d, 1, 2)\). See Theorems 2.1 and 2.2.

The second main question, considered in section 3 is to provide lower bounds for the *errors in the moments* which appears as the consequence of approximating a signal \( F \) with \( d \) nodes by signals with at most \( d - 1 \) nodes. We give a bound of this form in terms of the minors of the moment Hankel matrix \((1.4)\) formed by the moments of \( F \). See Theorems 3.1 and Corollary 3.1. Finally, as a special case, we consider a situation where the nodes of \( F \) form a cluster of a size \( h \ll 1 \).

### 2. Exact moment fitting

In this section we consider Problem 1.1 of exact fitting of the moments of a signal \( F \) by a signal with strictly less nodes \( \tilde{F} \).

For each signal \( F = (a, x) \in \mathcal{P}_d \), consider the \( d \times d \) matrix \( D = D(x) \), its \( i,j \) entry is given by \( D_{i,j} = \delta_{i,j}^2, d_{i,j} = x_i - x_j \). The matrix \( D \) is called an Euclidean Distance Matrix which has many important applications (for applications in signal processing see [23]).

In what follows we describe the geometry of the set of signals \( \Sigma_{d,1,2} \). We show in Theorem 2.1 that \( F = (a, x) \in \Sigma_{d,1,2} \) iff the amplitudes vector of \( F \), \( a \), is a zero of the quadratic form induced by the Euclidean Distance Matrix supported by the nodes of \( F \), \( D(x) \). This holds for all signals \( F \) in \( \Sigma_{d,1,2} \) with \( m_0(F) \neq 0 \). For signals \( F \in \Sigma_{d,1,2} \) with \( m_0(F) = 0 \), the description is straightforward and is given below as well.

Next we use the spacial structure of \( D(x) \) to show that for a fixed nodes vector \( x \), the set of amplitudes vectors of signals \( F = (a, x) \in \Sigma_{d,1,2} \) having \( m_0(F) \neq 0 \), is a union of two sets...
For a nodes vector $x = (x_1, x_2, \ldots, x_d) \in \mathcal{P}_d^*$, denote by $V(x) = V_d(x)$ the Vandermonde matrix with infinite row index and $d$ columns and with the nodes $x_1, \ldots, x_d$.

\begin{equation}
V_d(x) = \begin{bmatrix}
1 & x_1 & \cdots & x_{d-1} \\
x_1 & x_2 & \cdots & x_{d} \\
x_1^2 & x_2^2 & \cdots & x_{d}^2 \\
\vdots & \vdots & \ddots & \vdots \\
\end{bmatrix}.
\end{equation}

We denote by $V^{0:k-1}(x) = V_d^{0:k-1}(x)$ the $k \times d$ submatrix of $V(x)$ formed by the first $k$ rows of $V(x)$.

**Theorem 2.1.** For $F = (a, x) \in \mathcal{P}_d$, $F \in \Sigma_{d,1,2}$ iff $a_i \neq 0$, $i = 1, \ldots, d$ and exactly one of the following mutually exclusive conditions is met:

(I) $m_0(F) = 0$ and the amplitudes vector $a$ is in the null space of $V^0:2(x)$.

(II) $m_0(F) \neq 0$ and the amplitudes vector $a$ is a zero of the quadratic form $a^T D(x)a$.

The condition $a_i \neq 0$, $i = 1, \ldots, d$ above is a mere technicality needed to ensure that $F$ has $d$ nodes.

**Proof.** Let $F = (a, x) \in \Sigma_{d,1,2}$, $x = (x_1, \ldots, x_d)$, $a = (a_1, \ldots, a_d)$. If $m_0(F) = 0$, by assumption there exists $\tilde{F} \in \mathcal{P}_1$ or $\tilde{F} \in \mathcal{P}_0$ such that $m_0(\tilde{F}) = m_0(F) = 0$. Then $\tilde{F} = F_0 \in \mathcal{P}_0$, the identically 0 signal. Since $\tilde{F}$ has all its moments equal to 0 we have that $m_0(F) = m_1(F) = m_2(F) = 0$. The last condition is equivalent to the amplitudes vector of $F$, $a$, being in the null space of truncated Vandermonde $V^0:2(x)$.

Else $m_0(F) \neq 0$. Then, $\tilde{F}$ is a single node signal with a non zero amplitude, $\tilde{F}(x) = \tilde{a}_1 \delta(x - \tilde{x}_1)$, $\tilde{F} \in \mathcal{P}_1$.

**Lemma 2.1.** Let $F \in \mathcal{P}_d$, $F(x) = \sum_{i=1}^{d} a_i \delta(x - x_i)$, $d > 1$. Let $\tilde{F}(x) = \tilde{a}_1 \delta(x - \tilde{x}_1)$ be a single node signal such that $m_0(\tilde{F}) = m_0(F) \neq 0$ and $m_1(\tilde{F}) = m_1(F)$. Then,

\begin{equation}
m_2(\tilde{F}) - m_2(F) = \frac{\sum_{i<j} a_i a_j (x_i - x_j)^2}{\sum_{i=1}^{d} a_i} = \frac{a^T D(x)a}{\sum_{i=1}^{d} a_i}.
\end{equation}
Proof. $m_0(\tilde{F}) = m_0(F)$ and $m(\tilde{F})_1 = m_1(F)$ imply that $\tilde{a}_1 = m_0(F)$ and $\tilde{x}_1 = \frac{m_1(F)}{m_0(F)}$. Then $m_2(\tilde{F}) = \tilde{a}_1 \tilde{x}_1^2 = \frac{m_2(F)}{m_0(F)}$ and we have

$$m_2(F) - m_2(\tilde{F}) = \frac{m_0(F)m_2(F) - m_1^2(F)}{m_0(F)}$$

$$= \frac{(\sum_{i=1}^d a_i)(\sum_{i=1}^d a_i x_i^2) - (\sum_{i=1}^d a_i x_i)^2}{\sum_{i=1}^d a_i}$$

$$= \sum_{i=1}^d a_i x_i^2 + \sum_{1 \leq i < j \leq d} (a_i a_j x_i x_j) - \sum_{i=1}^d a_i^2 x_i^2 - 2 \sum_{1 \leq i < j \leq d} a_i a_j x_i x_j$$

$$= \sum_{1 \leq i < j \leq d} a_i a_j (x_i - x_j)^2$$

This conclude the proof of Lemma 2.1 \( \square \)

Case II of Theorem 2.1 now follows from Lemma 2.1 which concludes the proof of the Theorem. \( \square \)

We now consider case II of Theorem 2.1.

Introduce the maps $b_1, b_2 : P_2^d \rightarrow P_2^a$ which are certain continuous parametrizations of amplitudes vectors by the nodes. The exact definition of the maps $b_1, b_2$ will be given within the proof of theorem 2.2.

Denote by $\mathbf{1} \in P_2^a$ the amplitudes vector with all entries equal to 1.

**Definition 2.1.** For each $x \in P_2^d$ the sets $P_1(x), P_2(x) \subset P_2^d$ are defined via the mappings $b_1(x), b_2(x)$ as follows:

$$P_1(x) = \left\{ \lambda \left( \frac{1}{d} + b_1(x) + u_1 \right) \in P_2^d : \lambda \neq 0, u_1 \perp \text{span}(\mathbf{1},b_1(x)) \right\}.$$ 

$$P_2(x) = \left\{ \lambda \left( \frac{1}{d} \mathbf{1} + b_2(x) + u_2 \right) \in P_2^d : \lambda \neq 0, u_2 \perp \text{span}(\mathbf{1},b_2(x)) \right\}.$$ 

**Remark 2.1.** For a given nodes vector $x \in P_2^d$, the set $P_1(x) \subset P_2^a$ (and similarly $P_2(x)$) is a relatively “simple” “punctured” vector space of dimension $d - 1$ given by

$$\text{span}\left( \frac{1}{d} \mathbf{1} + b_1(x), \text{Ker}(\mathbf{1}, b_1(x)) \right) \cap P_2^d$$

minus the subspace $\text{Ker}(\mathbf{1}, b_1(x))$, where $\text{Ker}(\mathbf{1}, b_1(x))$ denotes the subspace of vectors perpendicular to $\mathbf{1}, b_1(x)$.

**Theorem 2.2.** For $F = (a,x) \in P_d$ with $m_0(F) \neq 0$, $F \in \Sigma_{d,1,2}$ iff the amplitudes vector of $F$, $a$, belongs to at least one of the sets $P_1(x), P_2(x) \subset P_2^a$.

**Proof.** Let $F = (a,x) \in \Sigma_{d,1,2}$, $x = (x_1, \ldots, x_d)$, $a = (a_1, \ldots, a_d)$, such that $m_0(F) \neq 0$. By Theorem 2.1 we have that this is equivalent to the amplitudes vector of $F$, $a$, being a zero of the quadratic form $a^T D(x) a$. We now analyse the zeros of $a^T D(x) a$.

Consider the following notation which simplifies the presentation. For a nodes vector $x = (x_1, \ldots, x_d)$:
\[\begin{align*}
\bullet \: \mu = \mu(x) &= \frac{1}{d} \sum_{i=1}^{d} x_i \text{ is the mean of the nodes vector.} \\
\bullet \: \bar{x} = x - \mu(x) \mathbb{1} \text{ is the nodes vector centered to its mean } \mu(x). \\
\text{Finally } || \cdot || \text{ denotes the euclidean norm.}
\end{align*}\]

**Lemma 2.2.** For \(x = (x_1, \ldots, x_d) \in \mathbb{R}^d, a = (a_1, \ldots, a_d) \in \mathbb{R}^d, \) with \(x_i, i = 1, \ldots, d, \) distinct and \(\sum_{i=1}^{d} a_i \neq 0,\) we have that

\[a^T D(x)a = 0 \iff a = (\sum_{i=1}^{d} a_i)(\frac{1}{d} \mathbb{1} + \alpha_k(x)\bar{x} + u), \quad k \in \{0, 1\},\]

where:

\[\begin{align*}
\alpha_{1,2} &= \frac{x^T D\mathbb{1} + c_{1,2}}{d ||\bar{x}||^4}, \\
c_{1,2} &= \pm \sqrt{\left(\frac{x^T D\mathbb{1}}{d}\right)^2 + 4 ||\bar{x}||^4 \frac{1}{d^2} \sum_{1 \leq i<j \leq d} d_{i,j}^2}
\end{align*}\]

and \(u\) is any vector that is orthogonal to \(\mathbb{1}\) and \(\bar{x}.

**Proof.**

\[D(x) = diag(xx^T)\mathbb{1}^T + \mathbb{1} diag(xx^T)^T - 2xx^T,\]

where \(diag(xx^T) = (x_1^2, \ldots, x_d^2)\) taken as a column vector. Consider the orthogonal projection into the subspace \(\{x \mid x^T \cdot \mathbb{1} = 0\}\) given by \(I - \frac{\mathbb{1}\mathbb{1}^T}{d},\) where \(I\) is the \(d \times d\) identity matrix. Using equation (2.3), by direct calculation we have that

\[\frac{1}{2} \left(I - \frac{\mathbb{1}\mathbb{1}^T}{d}\right) D \left(I - \frac{\mathbb{1}\mathbb{1}^T}{d}\right) = -(x - \mu)(x - \mu^T) = -\bar{x}\bar{x}^T.\]

Let \(a \in \mathbb{R}^d\) with \(\sum_{i=1}^{d} a_i \neq 0.\) Then \(a = (\sum_{i=1}^{d} a_i)(\frac{1}{d} \mathbb{1} + \alpha \bar{x} + u),\) for some \(\alpha \in \mathbb{R}\) and for some vector \(u\) which is orthogonal to subspace spanned by the vectors \(\mathbb{1}\) and \(\bar{x}.\) Then

\[\begin{align*}
\frac{1}{2} a^T Da &= \frac{1}{d} (\frac{1}{d} \mathbb{1} + \alpha \bar{x} + u)^T D\left(\frac{1}{d} \mathbb{1} + \alpha \bar{x} + u\right) \\
&= \frac{1}{d^2} (\bar{x}^T D\mathbb{1} + \alpha^2 \bar{x}^T \bar{x} + \alpha \bar{x}^T \mathbb{1}) \\
&= \frac{1}{d^2} \sum_{1 \leq i<j \leq d} d_{i,j}^2 - \alpha^2 ||\bar{x}||^4 + \frac{1}{d} \bar{x}^T \mathbb{1},
\end{align*}\]

where for the penultimate equality we used equation (2.3) to get \(\frac{1}{2} \alpha^2 \bar{x}^T \bar{x} = -\alpha^2 ||\bar{x}||^4.\) Then setting

\[\frac{1}{d^2} \sum_{1 \leq i<j \leq d} d_{i,j}^2 - \alpha^2 ||\bar{x}||^4 + \frac{1}{d} \bar{x}^T \mathbb{1} = 0\]

we get that \(\alpha\) is as declared in (2.3). This concludes the proof of Lemma 2.2. \(\Box\)

Now setting \(b_1(x) = \alpha_1(x)\bar{x}\) and \(b_2(x) = \alpha_2(x)\bar{x}\) concludes the proof of Theorem 2.2. \(\Box\)

3. Lower bounds

In this section we derive lower bounds on the moments difference between a signal with \(d\) nodes, \(F \in \mathcal{P}_d,\) and a signal with strictly less nodes, \(\tilde{F}.\)

We will consider only the first \(2d - 1\) consecutive moments of each signal. Accordingly, we denote by \(\mathcal{M}_d \cong \mathbb{R}^{2d-1}\) the restricted moment space consisting of \(2d - 1\) tuples \((\nu_0, \ldots, \nu_{2d-2}),\) and for any signal \(G,\) we put \(\tilde{m}(G) = (m_0(G), \ldots, m_{2d-2}(G)) \in \mathcal{M}_d.\)
The Hankel matrix introduced in equation (1.4) plays an important part in what follows. Recall that for a moment vector \( m = (m_0, \ldots, m_{2d-1}) \in \mathcal{M}_d \) we defined
\[
(3.1) \quad H_d(m) = \begin{bmatrix}
m_0 & m_1 & m_2 & \cdots & m_{d-1} \\
m_1 & m_2 & m_3 & \cdots & m_d \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
m_{d-1} & m_d & m_{d+1} & \cdots & m_{2d-2}
\end{bmatrix}.
\]

Note that this matrix depends only in the first \( 2d - 2 \) entries of \( m \).

For a signal \( G = (a, x) \in \mathcal{P}_l, \ l \leq d, \ x = (x_1, \ldots, x_{l}) \), \( a = (a_1, \ldots, a_l) \), we denote by \( H_d(G) \) the Hankel matrix as above, formed by the \( 2d - 1 \) initial moments of \( G \). As it was stated in the introduction, the rank of \( H_d(G) \) is equal to the number of nodes in \( G \). This fact can be seen as follows. Let \( D_l(a) = \text{diag}(a_1, \ldots, a_l) \) be the diagonal matrix with the entries \( a_1, \ldots, a_l \). Let \( V_l^{0:d-1}(x) \) be the matrix formed by the first \( d \) rows of the Vandermonde matrix \((2.1)\) with the nodes \( x \). We have the next identity
\[
(3.2) \quad H_d(G) = V_l^{0:d-1}(x) D_l(a) V_l^{0:d-1}(x)\trans.
\]

From (3.2) we conclude that the rank of \( H_d(G) \) is equal to the number of the nodes in \( G \), that is \( l \).

Informally we can expect that the size of the minors of order \( l \) of \( H_d(F) \) measures the distance from \( F \) to the set of the signals \( \hat{F} \) with at most \( l - 1 \) nodes. The following definition, and Theorem 3.1 below make this observation rigorous.

**Definition 3.1.** For \( l = 1, \ldots, d \), define \( \delta_l(F) \) as the maximum of the absolute values of all the \( l \)-minors of the moment Hankel matrix \( H_d(F) \).

To simplify the statement of our results we assume below that both the nodes \( x_j \) and the amplitudes \( a_j \) are bounded in absolute value by 1, and denote by \( \mathcal{P}_d \) the corresponding part of the signal space \( \mathcal{P}_d \). However, for the approximating low-rank signal \( \hat{F} \), no such assumptions are made.

**Theorem 3.1.** Let a signal \( F \in \mathcal{P}_d \) be given. Then for each signal \( \hat{F} \in \mathcal{P}_{l-1}, \ l \leq d \), we have
\[
||\hat{m}(F) - m(\hat{F})|| \geq \min \left(1, \frac{\delta_l(F)}{\sqrt{2l(l+1)d^{-1}}\sqrt{l!}}\right).
\]

**Proof.** By our assumptions we have \( |m_k(F)| \leq d, \ k = 0, 1, \ldots \). In other words, \( \hat{m}(F) \in Q_d \subset \mathcal{M}_d \), where \( Q_d \) is the coordinate cube of radius \( d \) centered at the origin of \( \mathcal{M}_d \). We can assume also that \( \hat{m}(\hat{F}) \in Q_{d+1} \), since otherwise \( ||\hat{m}(F) - \hat{m}(\hat{F})|| \geq 1 \). Therefore we restrict the consideration to the cube \( Q_{d+1} \).

Now we fix \( l \in \{1, \ldots, d\} \), and let \( \hat{H}_l(F) \) be the minor of \( H_d(F) \) for which the determinant \( \Delta_l(F) = \det \hat{H}_l(F) \) attains, in absolute value, the maximum \( \delta_l(F) \). This determinant \( \Delta_l(F) \), considered as a function \( \Delta_l(\nu) \) of the moments entering the minor \( \hat{H}_l(F) \), is a polynomial of degree \( l \) in \( \nu = (\nu_0, \ldots, \nu_{2d-2}) \in \mathcal{M}_d \). On \( Q_{d+1} \), this polynomial is bounded in absolute value by \( l! \) of \( l \)-products of the moments. Applying to \( \Delta_l(\nu) \) the classical Markov inequality (see e.g. [30]) (with an appropriate adaptation to the cubic domain \( Q_{d+1} \)) we have that
\[
\max_{\nu \in Q_{d+1}} \left|\frac{\partial \Delta_l(\nu)}{\partial \nu_i}\right| \leq \frac{l^2}{d+1} \max_{\nu \in Q_{d+1}} |\Delta(\nu)|.
\]

We conclude that
\[
||\text{grad} \Delta_l(\nu)|| \leq \zeta := \sqrt{2l-1} l \sqrt{l!} (d+1)^{l-1}
\]
for each $\nu \in Q_{d+1}$.

Consequently, for any two points $\nu, \nu' \in Q_{d+1}$ we have

$$|\Delta_l(\nu) - \Delta_l(\nu')| \leq \zeta ||\nu - \nu'||,$$

or

$$||\nu - \nu'|| \geq \frac{1}{\zeta} |\Delta_l(\nu) - \Delta_l(\nu')|.$$  \hspace{1cm} (3.3)

Now we notice that for any signal $\tilde{F}$ with at most $l - 1$ nodes we have $\Delta_l(\tilde{m}(\tilde{F})) = 0$. Indeed, this follows immediately from the fact that the rank of $H_d(\tilde{F})$ is at most $l - 1$. Applying (3.3) to the points $\tilde{m}(F), \tilde{m}(\tilde{F})$ we get

$$||\tilde{m}(F) - \tilde{m}(\tilde{F})|| \geq \frac{1}{\zeta} |\Delta_l(\tilde{m}(F)) - \Delta_l(\tilde{m}(\tilde{F}))| = \frac{\delta_l(F)}{\zeta}.$$  

This completes the proof of Theorem 3.1. \hfill $\Box$

**Remark 1.** The inequalities provided by Theorem 3.1 for different $l$ are not completely independent from one another. Indeed, the assumption that $\delta_l(F) > 0$ implies $\delta_{l'}(F) > 0, l' < l$. Via linear algebra one can get explicit lower bound in this direction. We plan to present these results separately.

**Remark 2.** The result of Theorem 3.1 can be improved as follows: the same lower bound on the difference of the moments of $F$ and $\tilde{F}$ remains valid as applied only to those moments which enter the minor $\tilde{H}_l(\tilde{F})$. The proof remains verbally the same.

In order to describe specific classes of signals $F \in \tilde{P}_d$ for which Theorem 3.1 works, we have to make explicit assumptions on the separation of the nodes $x$ of the signal $F$, and on the lower bound of the size of its amplitudes $a$.

By the assumptions, the nodes $x_1, \ldots, x_d$ of a signal $F \in \tilde{P}_d$ belong to the interval $I = [-1, 1]$. Let us assume now that for a certain $\eta$ with $0 < \eta \leq \frac{2}{d-1}$, the distance between the neighboring nodes $x_j, x_{j+1}, j = 1, \ldots, d - 1$, is at least $\eta$. We also assume that for a certain positive $\gamma \leq 1$ the amplitudes $a_1, \ldots, a_d$ satisfy $|a_j| \geq \gamma$, $j = 1, \ldots, d$. We will call signals $F$ satisfying these conditions, $(\eta, \gamma)$-regular.

**Theorem 3.2.** Let a signal $F = (a, x) \in \tilde{P}_d$, $x = (x_1, \ldots, x_d)$, $a = (a_1, \ldots, a_d)$, be $(\eta, \gamma)$-regular. Then

$$\delta_d(F) \geq \prod_{i=1}^{d-1} (d!)^2 \eta^d (d-1) \gamma^d.$$  

**Proof.** We use factorization (3.2) of the moment Hankel matrix $H_d(F)$:

$$H_d(F) = V_d^{0:d-1}(x) D_d(a) [V_d^{0:d-1}(x)]^T.$$  

Taking the determinant, we obtain

$$\delta_d(F) = |\det H_d(F)| = \prod_{i=1}^{d} |a_i| (\det V_d^{0:d-1}(x))^2 \geq \gamma^d (\det V_d^{0:d-1}(x))^2.$$  

For the determinant of the Vandermonde matrix we have

$$|\det V_d^{0:d-1}(x)| = \prod_{i>j} |x_i - x_j| \geq \eta^\frac{d(d-1)}{2} \prod_{i>j} |i-j| = \eta^\frac{d(d-1)}{2} \prod_{i=1}^{d-1} i!,$$
since for the nodes $x_j$ of an $(\eta, m)$-regular signal $F$ we have $|x_i - x_j| \geq \eta|i - j|$. We conclude that
\[ \delta_d(F) \geq \gamma^d \eta^{d(d-1)} \prod_{i=1}^{d-1} (i!)^2. \]

This completes the proof of Theorem 3.2.

Now we can apply Theorem 3.1 with $l = d$, and get a lower bound on the error of any low-rank approximation of the moments of an $(\eta, \gamma)$-regular signal $F \in \mathcal{P}_d$:

**Corollary 3.1.** Let a signal $F \in \mathcal{P}_d$ be $(\eta, \gamma)$-regular. Then for each $\hat{F} \in \mathcal{P}_l$, $l < d$, we have
\[ ||\bar{m}(F) - \bar{m}(\hat{F})|| \geq \theta := \min \left( 1, \frac{\eta^{d(d-1)} \prod_{i=1}^{d-1} (i!)^2}{\sqrt{2d - 1} d! (d + 1)^{d-1}} \right). \]

**Remark 3.** As it was mentioned in Remark 1 above, the lower bound for $\delta_d(F)$ provided by Theorem 3.2 for $(\eta, \gamma)$-regular signals $F$, implies explicit lower bounds for each $\delta_i(F), l \leq d$. We expect these bounds to contain, for smaller $l$, smaller powers of the parameter $\eta$. In the case of “positive” signals $F$ (i.e. for all the amplitudes $a_j$ positive), such improved bounds for the principal minors of $H_d(F)$ can be, presumably, obtained via the Silvester criterion.

An important special case of the low-rank approximation problem is when the nodes of the signal $F$ near collide. (Our assumption of $(\eta, \gamma)$-regularity, essentially, excludes nodes near collisions.) A natural initial step in the study of signals with near-colliding nodes is to assume that the nodes form a cluster of a size $h \ll 1$, but inside the cluster the nodes are positioned in a relatively uniform way.

**Definition 3.2.** A signal $F$ is said to form an $(h, \eta, \gamma)$-regular cluster, if its nodes are obtained by an $h$-downscaling of an $(\eta, \gamma)$-regular signal $G \in \mathcal{P}_d$.

**Corollary 3.2.** Let a signal $F \in \mathcal{P}_d$ form an $(h, \eta, \gamma)$-regular cluster. Then for each $\hat{F} \in \mathcal{P}_l$, $l < d$, we have
\[ ||\bar{m}(F) - \bar{m}(\hat{F})|| \geq \theta_h := \min \left( h^{2d-2}, \frac{\eta^{d(d-1)} h^{2d-2} \prod_{i=1}^{d-1} (i!)^2}{\sqrt{2d - 1} d! (d + 1)^{d-1}} \right). \]

**Proof.** Under a scaling by $h$ the $k$-th moment of $F$ is multiplied by $h^k$. So the difference in the $k$-th coordinate of $\bar{m}(F)$ and $\bar{m}(\hat{F})$ is multiplied by $h^k \geq h^{2d-2}$. □

**Remark 4.** If we could bound from below the difference in the lower-order moments of $F$ and $\hat{F}$, it would provide a better asymptotic behavior in $h \to 0$ in the bound of Corollary 3.2.

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