A technical note on the calculation of
GJMS (Rac and Di) operator determinants

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GJMS operator determinants in odd dimensions are quickly computed for scalar and spinor fields in both sub– and super–critical cases as a sum of Dirichlet eta functions with polynomials in the (integer) operator order as coefficients.

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1. Introduction

Field theories on spheres for higher derivative propagation occur in an essential way in connection with those AdS/CFT correspondences which have higher spins in the bulk. Particular cases are the GJMS conformal scalars and their Dirac analogues. Formulas for the effective action (‘free energy’) and conformal anomaly have been given for free fields by a purely spherical spectral method in [1,2] and evaluated by various means in [3] and, later, by Brust and Hinterbichler, [4].

If the order of the propagation operator exceeds a certain ‘critical’ value, which depends on the dimension, the operator ceases to exist, except in odd dimensions where a continuation can be made. In this case the effective action acquires an imaginary part, [1], essentially because of the existence of negative modes. Some values are given in [4] for particular dimensions.

More recently, Basile et al, [5], have given the explicit result (for dimension equal to three) of a continuation of the free energy for any integral derivative order, including above critical, in the cases of the higher order Rac and Di representations (equivalent to GJMS scalars and spinors). In the present short, technical note, using a quite different method, I extend the evaluation so as to apply easily to any given, odd dimension.

In the next section I treat GJMS scalars (Rac_k) and then pass on to spinors (Di_l).

2. Scalar determinants

I denote the scalar GJMS propagation operator of order 2k by $P_{2k}^{Rac}$. On the $d$–sphere it is given by the Branson product,

$$P_{2k}^{Rac} = \prod_{j=1}^{k} (B_d^2 - (j - 1/2)^2)$$

$$= \frac{\Gamma(B_d + 1/2 + k)}{\Gamma(B_d + 1/2 - k)} .$$

(1)

For scalars, $B_d = \sqrt{Y_d + 1/4}$ where $Y_d$ is the conformally covariant Penrose–Yamabe Laplacian. $k$ is, initially, an integer but the second expression allows it to be extended to the reals. I will not consider this here.

To save time, I quote the expression derived in [1] for the scalar effective action on the basis of the spectral data for odd spheres. It is
\[
\log \det P_{2k}^{Rac} = \frac{2(-1)^d}{d!} \int_0^k dz \frac{\pi z \tan \pi z}{\prod_{j=1}^{(d-1)/2} (z^2 - (j - 1/2)^2)} \\
\equiv \frac{2(-1)^d}{d!} \int_0^k dz \, P(d, z) \pi \tan \pi z .
\] (2)

The situation as \(k\) increases is described in [1] section 8. I recapitulate a little here. The polynomial \(P\) cancels the poles in \(\tan \pi z\) up to a certain value of \(z\). The first pole in the integrand appears at \(z = d/2\) and so, if \(k > d/2\), the integral is undefined (infinite). However it can be rescued by extending \(z\) into the complex plane and running the integration along the real axis up to \(k\), avoiding any poles. Since, it turns out, the residues at the poles are integers, it doesn’t matter how the poles are skirted. The determinant remains unchanged.

The integral thus splits into a real part, given by its principal value, and an imaginary part coming from the poles. I concentrate first on the real part as the harder to find.

I take \(k\) to be integral. If it wasn’t, then a further, numerical quadrature would be required to make up the difference.

The integral can be broken up into unit pieces and a convenient change of variables then yields,

\[
\int_0^k dz \, P(d, z) \pi \tan \pi z = \sum_{i=0}^{k-1} \int_{-1/2}^{1/2} dx \, P(d, x + i + 1/2) \pi \cot \pi x .
\] (3)

Because of the oddness of the cot, the finite part of the integral is obtained by retaining just the odd powers of \(x\) in \(P\), say,

\[
P(d, x + i + 1/2) = \sum_{p=0}^{(d-1)/2} K_d^p(i) x^{2p+1} + \text{even powers},
\]

where the constants, \(K_d^p(i)\) are polynomials in \(i\) so allowing the sum over \(k\) to be done to give polynomials in \(k\).

Hence the calculation is reduced to finding the integral, \((p \geq 0)\),

\[
C(p, x) \equiv \pi \int_0^x dx \, x^{2p+1} \cot(\pi x) \\
= x^{2p+1} \log \sin(\pi x) - (2p + 1) \int_0^x dx \, x^{2p} \log \sin(\pi x) ,
\] (4)

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evaluated at $x = 1/2$, and I spend a little time on this evaluation. It can, of course, be found in each case from a CAS but it is more satisfying to have a derivation from first principles and a general formula.\(^2\)

In exactly the case when $x = 1/2$, an explicit expression is given by Crandall and Buhler, [6], obtained by expanding the power of $x$ in terms of Clausen functions, and using the rather particular relation (a consequence of trigonometry)

$$\zeta(s) = 2 \int_0^{1/2} dx \cot \pi x S(s, x), \quad (5)$$

where $S$ is the Clausen function

$$S(s, x) = \sum_{n=1}^{\infty} \frac{\sin(2\pi n x)}{n^s},$$

which, for $s$ an odd integer, is a Bernoulli polynomial in $x$,

$$S(s, x) = (-1)^{(s-1)/2} \frac{2^{s-1} \pi^s B_s(x)}{s!}.$$ 

Since the details in [6] are somewhat sketchy, I give my own version here.

The first step is to introduce the twisted (or fermionic or alternating) Clausen function,

$$\tilde{S}(s, x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sin(2\pi n x)}{n^s},$$

which is obtained, up to a sign, from $S(s, x)$ by making the central translation $x \to x + 1/2$,

$$\tilde{S}(s, x) = -S(s, x + 1/2).$$

This enables me without more ado to move directly to the Dirichlet eta function because one now has,

$$\eta(s) = 2 \int_0^{1/2} dx \cot \pi x \tilde{S}(s, x). \quad (6)$$

The polynomial form of the twisted Clausen function is, for $s$ odd,

$$\tilde{S}(s, x) = -(-1)^{(s-1)/2} 2^{s-1} \frac{\pi^s}{s!} B_s(x + 1/2)$$

$$\equiv (-1)^{(s+1)/2} \frac{\pi^s}{2s!} D_s(x).$$

\(^2\) There is some interest in computing the integral (4) for any $x$, being related to polylogarithms (higher Spence functions). An approach which depends on repeated integrations of $\cot \pi x$, or, more conveniently, on $2p$ integrations of $\log \sin \pi x$, i.e. of the Clausen integral will be presented at another time.
Next, the monomial \( x^{\nu} \) in (4) is expanded in the polynomials, [5]. I do this in the following accelerated way.

The Nörlund polynomials, \( D_{\nu}(x) \), can also be expressed in terms of the central derivative,

\[
D_{\nu}(x) = \frac{D}{2} \cosh \frac{D}{2} \cdot (2x)^{\nu}.
\]

which shows that \( D_{\nu}(x) \) is an even polynomial if \( \nu \) is even and an odd one if \( \nu \) is odd. The inversion,

\[
(2x)^{\nu} = \frac{\sinh D/2}{D/2} D_{\nu}(x),
\]

provides the required expansions using the basic relation

\[
D D_{\nu}(x) = 2\nu D_{\nu-1}(x).
\]

The result is, for both odd and even \( \nu \),

\[
\left( \frac{2^{-\nu}}{\nu + 1} \sum_{\mu=1,3,5\ldots}^{\nu,\nu+1} \binom{\nu + 1}{\mu} D_{\nu-\mu+1}(x), \right) \tag{8}
\]

the upper limit being odd. For my purposes, I need only odd \( \nu = 2p + 1 \), and (8), combined with (6) and (7) gives a quick derivation of Crandall and Buhler’s formula for (4) at \( x = 1/2 \),

\[
C(p, 1/2) = \frac{(2p + 1)!}{2^{2p+1}} \sum_{\mu=1,3,\ldots}^{2p+1} \frac{(-1)^{(\mu-1)/2} \eta(\mu)}{(2p + 2 - \mu)! \pi^{\mu-1}}. \tag{9}
\]

I now return to the expression, (3) needed for calculating the (real part of) the effective action,

\[
\sum_{i=0}^{k-1} \sum_{p=0}^{(d-1)/2} \int_{-1/2}^{1/2} dx K_p^d(i) x^{2p+1} \cot \pi x \left( \sum_{p=0}^{(d-1)/2} L^p_d(k) C(p, 1/2) \right), \tag{10}
\]

where,

\[
L^p_d(k) = 2 \sum_{i=0}^{k-1} K^p_d(i),
\]

is a polynomial in \( k \). I will not spend time producing a formula for these polynomials in Bernoullian terms and, since all quantities are now easily computable, I just list
some particular expressions of log det $P_{2k}$, for $d = 3, 5, 7$ and 9 (remembering that $\eta(1) = \log 2$),

$$
k(4k^2 - 1^2) \log 2 - \frac{3k \zeta(3)}{8 \pi^2},
$$

$$
k \left(4k^2 - 3^2\right) \left(4k^2 - 1^2\right) \log 2 - \frac{k(4k^2 - 3) \zeta(3)}{16 \pi^2} + \frac{15k \zeta(5)}{128 \pi^4},
$$

$$
\frac{k(4k^2 - 5^2)(4k^2 - 3^2)(4k^2 - 1^2) \log 2 - k(48k^4 - 200k^2 + 111) \zeta(3)}{161280} + \frac{5k(4k^2 - 5) \zeta(5)}{1024 \pi^4} - \frac{63k \zeta(7)}{2048 \pi^6},
$$

$$
\frac{k \left(4k^2 - 7^2\right) \left(4k^2 - 5^2\right) \left(4k^2 - 3^2\right) \left(4k^2 - 1^2\right) \log 2 - k(6k^6 - 105k^4 + 45k^2 - 410) \zeta(3)}{161280} + \frac{k(6k^4 - 50k^2 + 65) \zeta(5)}{12288 \pi^4} - \frac{21k(4k^2 - 7) \zeta(7)}{16384 \pi^6} + \frac{255k \zeta(9)}{32768 \pi^8}.
$$

The general log 2 term is

$$
\frac{1}{2d!} k^{(d-1)/2} \prod_{j=1}^{d} \left( k^2 - (j - 1/2)^2 \right).
$$

Although eta functions appear more immediately, I have converted them to Riemann zetas as this is how such formulae are usually presented.

The $d = 3$ formula was obtained by Basile et al [5].

The above expressions are valid strictly only for $k$ integral. Evaluation at specific integers produces agreement with the results derived, in a longer way, in [3] and, later, in [4]. Brunt and Hinterblicher, [4], also give super–critical values but they are not expressed as above.
3. The imaginary part

As stated, in the super–critical case (i.e. $k > d/2$) the integral acquires an imaginary part coming from the active poles. In the approach here the exact value depends on the choice of $z$–contour. Two basic contours are one ($C_1$) just below (or above) the real axis and one ($C_2$) that runs alternately above and below the poles. $C_1$ yields the sum of the residues and $C_2$ an alternating sum.

Each residue equals the number of negative modes associated with each of the $k$ factors in the product, (1). These can be labeled by the variable $j$, which runs from 0 to $k - 1$. I denote the residues by $\rho_j$. Actually there are no negative modes, or a pole, for the first factor, $j = 0$, corresponding to the poles being $k - 1$ in number. It is convenient to distinguish ‘Dirichlet’ and ‘Neumann’ modes i.e. those modes which are, respectively, odd and even across the equator of the $d$–sphere, and write the number of negative modes per factor as the $D$ and $N$ sum,

$$\rho_j = \rho_j^N + \rho_j^D$$

$$= \rho_j^N + \rho_{j-1}^N,$$

where the second equality is occasioned by the specific $D, N$ mode structures.

Either by mode counting in the way described in [1], leading to Ehrhart polynomials, or from the explicit Plancherel form of the integrand, it follows that

$$\rho_j^N = \binom{(d - 1)/2 + j}{d}.$$  \hspace{1cm} (11)

The contour $C_1$ then leads to the sum of the residues,

$$N_T(k) \equiv \sum_{j=1}^{k-1} \left[ \binom{(d - 1)/2 + j}{d} + \binom{(d - 3)/2 + j}{d} \right]$$

$$= \binom{(d - 1)/2 + k}{d + 1} + \binom{(d + 1)/2 + k}{d + 1}$$

by the hockey stick identity.\(^3\) This result can be written

$$N_T(k) = \frac{2}{(d+1)!} \prod_{n=0}^{(d-1)/2} (k^2 - n^2).$$

\(^3\) This is typical in that summing over the individual factors to give a GJMS quantity increases the dimension by one in sort of holographic way. See [7] Appendix B.
The imaginary part of the effective action is $\pi N_T$, which agrees with [5], (4.36).

How the contour circulates the poles can be thought of as corresponding to different choices of logarithmic branch for the individual factors in the operator product. There is no obligation to choose the same branch for each and the alternating contour, $C_2$, is another distinguished option. An alternating sum of the residues, taking the relation (11) into account, collapses by telescopage to the first term,

$$N_A(k) = \rho_{k-1} - \rho_{k-2} + \ldots \pm \rho_0 = \rho_k^N.$$  

This quantity is the number of negative eigenvalues of the GJMS operator $P_{2k}^{Rac}$ as follows from the gamma function form of $P_{2k}^{Rac}$, [1]. The number $N_T(k)$ does not give this measure because of sign cancellations between factors in the eigenvalues (which are products).

The alternating sum of the Dirichlet negative eigenvalues equals the number of Dirichlet negative eigenvalues of the product GJMS operator. The same holds for the Neumann case.

4. The Dirac field

The GJMS–like Dirac operator takes the form,

$$P_{2k}^{Di} \equiv B \prod_{h=1}^l (B^2 - h^2) = \prod_{h=-l}^l (B + h),$$

$$= \frac{\Gamma(B + 1/2 + k)}{\Gamma(B + 1/2 - k)}.$$  

(13)

For Dirac spinors $B = (\nabla^2)^{1/2} = |\nabla|$ with an overall sign factor of $\nabla/|\nabla|$ being understood, [8]. The parameter $k$ is a half–integer $k = l + 1/2$, $l = 0, 1, 2, \ldots$. The value $l = 0$ gives the ordinary Dirac case.

It is shown in [2] that the logdet of this operator is,

$$\log \det P_{2k}^{Di} = \frac{S}{d!} \int_0^k dz \, \pi z \, \cot \pi z \prod_{j=1}^{(d-1)/2} (z^2 - j^2)$$

$$\equiv \frac{S}{d!} \int_0^k dz \, P(d, z) \, \pi \, \cot \pi z.$$  

(14)

The factor $S$ (a power of 2) relates mostly to spin degeneracy. I henceforth drop it.
As before, the integral can be split into unit sized pieces, plus, this time, an initial bit,

\[ \int_0^{1/2} \, dz \, P(d, z) \, \pi \cot \pi z + \sum_{j=0}^{l-1} \int_{-1/2}^{1/2} \, dz \, P(d, z + j + 1) \, \pi \cot \pi z, \]

which again involves the integrals \( C(p, 1/2), (4) \). Computation yields the explicit expressions for \( d = 3, 5, 7 \) and 9, as shorter examples,

\[ \frac{k(k^2 - 1^2)}{6} \log 2 - \frac{3k \, \zeta(3)}{16 \, \pi^2}, \]

\[ \frac{k(k^2 - 2^2)(k^2 - 1^2)}{120} \log 2 - \frac{k(2k^2 - 3)}{64} \frac{\zeta(3)}{\pi^2} + \frac{15k \, \zeta(5)}{256 \, \pi^4}, \]

\[ \frac{(k^2 - 3^2)(k^2 - 2^2)(k^2 - 1^2)k}{5040} \log 2 - \frac{k(3k^4 - 20k^2 + 21)}{1920} \frac{\zeta(3)}{\pi^2} + \frac{5k(k^2 - 2) \, \zeta(5)}{512} - \frac{7k \, \zeta(7)}{640 \, \pi^6}, \]

\[ \frac{(k^2 - 4^2)(k^2 - 3^2)(k^2 - 2^2)(k^2 - 1^2)k}{362880} \log 2 - \frac{k(6k^6 - 105k^4 + 455k^2 - 410)}{161280} \frac{\zeta(3)}{\pi^2} + \frac{k(6k^4 - 50k^2 + 65) \, \zeta(5)}{12288} - \frac{21k(2k^2 - 5) \, \zeta(7)}{16384} + \frac{255k \, \zeta(9)}{65536 \, \pi^8}. \]

The general \( \log 2 \) term is

\[ \frac{1}{d!} \prod_{j=1}^{(d-1)/2} (k^2 - j^2) \log 2. \]

The imaginary part follows in the same way as the scalar case. The Dirac mode structure shows that \( h \) plays the same role as \( j \) did before, labelling the factors and the poles \((h = 0, 1, \ldots l)\). Also, the inequality for a negative mode is identical to the Dirichlet scalar one with \( j \to h \). The residues coming from the relevant factors in \( P_{2k}^{Di} \) are then

\[ \rho_h = \left( \frac{(d - 1)/2 + h}{d} \right), \]

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and the total number of negative modes from the factors is,

\[ N^D_{T}(l) = \sum_{h=0}^{l} \rho_{h} = \left( \frac{(d+1)/2 + l}{d+1} \right) \]

\[ = \frac{l + (d+1)/2}{(d+1)!} \prod_{j=1}^{(d-1)/2} (l^2 - j^2) \]

yielding the \( D_{l} \) imaginary part \( \pi N^D_{T}(l) \) as in [5].

I note that there are no negative modes for the first two brackets \( h = 0 \) and \( h = 1 \).

5. From \( d \) to \( d + 1 \)

Equation (2) for the scalar determinant is the algebraic consequence of performing the GJMS sum of \( k \) determinants of second order operators in \( d \) dimensions. By inspection, the integrand is proportional to the Plancherel measure on the \((d + 1)\) dimensional hyperbolic space, \( H^{d+1} \).

Another way of seeing this is to observe that identifying (2) with an alternative (equivalent) construction of the GJMS log det yields the identity, [1],

\[ \log \frac{\det \left[ (Y_{d+1} + 1/4)^{1/2} - k \right]}{\det \left[ (Y_{d+1} + 1/4)^{1/2} + k \right]} = \int_{0}^{k} dz P(d, z) \pi \tan \pi z, \quad (15) \]

and, by differentiating with respect to \( k \), it is seen that the residues of the poles of the integrand at \( d/2 + j \) \((j = 0, \ldots, k - 1)\) are the degeneracies of the eigenlevels, \((d/2 + j)^2\), of the \((d + 1)\) operator, \( Y_{d+1} + 1/4 \), on \( S^{d+1} \). This is confirmed by (12) with (11).

\( S^{d+1} \) is the Cartan dual of \( H^{d+1} \) and the residue statement is in accordance with general theorems on Plancherel measures, e.g. [9].

Equation (15) can be regarded as a \((d + 1)\) dimensional relation, the ratio corresponding to the two boundary conditions in the double trace computation, in AdS/CFT language, e.g. [10]. The derivation here is a purely boundary one.

All these statements can be transcribed into the Dirac case.
6. Conclusion

Another, rather particular, technique has been presented of rapidly computing the determinants of GJMS operators on spheres for any specified odd dimension as a sum of Dirichlet eta functions. Given the Plancherel form of the effective action, it is basically an algebraic method using only mild properties of special functions. Neither Lerch transcendent nor derivatives of the Hurwitz $\zeta$–function appear. These occur, in some numbers, in Basile et al [5], see also Bae [11], who start from a different integral representation of the free energy (effective action), one which is similar to those in [12] where original references can be found.\footnote{The basic ingredient is a Bessel transform the use of which dates back to the very earliest discussions of $\zeta$–functions on spheres and used, on and off, since. It yields a representation of the $\zeta$–function like the heat–kernel one but involving the wave–kernel (obtained from the square root of the propagating operator). This is, more or less, the degeneracy generating function, or, group theoretically, the character. In the hyperbolic case the transform is applied slightly differently but ultimately leads to similar expressions for the determinants after regularisation.}
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