Exploring an Alternative Channel of Evolution Towards SNa Ia Explosion

E. Chiosi\textsuperscript{1}, C. Chiosi\textsuperscript{1,2}, P. Trevisan\textsuperscript{2}, L. Piovan\textsuperscript{2}, & M. Orio\textsuperscript{1}

\textsuperscript{1} Astronomical Observatory of Padova, INAF, Vicolo dell’Osservatorio 5, 35112 Padova, Italy
\textsuperscript{2} Department of Physics and Astronomy, University of Padova, Vicolo dell’Osservatorio 2, 35122 Padova, Italy
E-mail: emanuela.chiosi@oapd.inaf.it (EC); cesare.chiosi@unipd.it (CC); patrizia.trevisan@unipd.it (PT); E-mail: lorenzo.piovan@gmail.com (LP); marina.orio@oapd.inaf.it (MO)

Submitted: September 2014; Accepted: ****

ABSTRACT

In this paper we explore the possibility that isolated CO-WDs with mass smaller than the Chandrasekhar limit may undergo nuclear runaway and SNa explosion, triggered by the energy produced by under-barrier pycnoc-nuclear reactions between carbon and light elements. Such reactions would be due to left over impurities of the light elements, which would remain inactive until the WDs transit from the liquid to the solid state. We devise a simple formulation for the coulombian potential and the local density in a ionic lattice affected by impurities and introduce it in the known rates of pycno-nuclear reactions for multi-component plasmas. Our semi-analytical results indicate that the energy generated by these pycno-nuclear reactions exceed the WD luminosity at much younger ages than the age of the Universe, even for WDs with masses as low as 0.8 $M_\odot$. A thermonuclear runaway may thus be triggered in isolated WDs. The explosion would occur sooner in WDs of higher masses, e.g. a few hundred million years in a WD of 1.2 $M_\odot$.

Key words: Stars – structure, evolution; White Dwarfs – pycnoc-nuclear reactions – Supernovae Ia

1 INTRODUCTION

Carbon-Oxygen White Dwarfs (CO-WDs) originate from low and intermediate mass progenitors in the mass interval 0.7 – 0.8 $M_\odot$ to 6 $M_\odot$ (Weidemann 1967, 1973, 1993; Chiosi et al. 1992; Weidemann 2000), which after the central H- and He-burning phases develop a highly electron degenerate CO core (electrons are fully degenerate and nearly relativistic). After the thermally pulsing AGB phase, these stars eject the whole envelope, baring the CO core. A WD is thus formed by a CO core surrounded by thin (if any) layers of lighter elements.

Given any reasonable initial mass function, about 97% of the stars of any generation with lifetime shorter than the age of the Universe (13.7 ± 0.2 Gyr according to Spergel et al. (2003)) end as WDs. Kepler et al. (2007) examined a sample of 7755 DA-WDs from the SDSS catalog and concluded that the vast majority of the 616 DBs and DOs listed in Eisenstein et al. (2006), fall in the mass interval 0.5 – 0.7 $M_\odot$. The peak value is 0.6 $M_\odot$ and the range of masses extends from about 0.1 $M_\odot$ toward the low mass end to 1.2 – 1.3 $M_\odot$ on the opposite side. According to theoretical models, there also is a clear relationship between the progenitor and WD masses, $M_i$ vs $M_{WD}$. (Weidemann 1967).

The most important facts of the WD theory are: (i) Except for a thin surface layer, the equation of state (EOS) can be approximated as fully degenerate electrons ($P \equiv P_e$) with kinematic conditions changing from non relativistic ($P_e = K_{5/3}\rho^{5/3}$) to fully relativistic ($P_e = K_{1/3}\rho^{1/3}$) with increasing central and mean density. (ii) The structure of a WD of given chemical composition (mean molecular weight of ions and/or electrons, $\mu_i$ and $\mu_e$ respectively) is fully determined by its central density. (iii) Assuming Newtonian hydrostatic equilibrium, the WD mass has a maximum value, called the Chandrasekhar mass $M_{Ch}$, for which the central density is infinite so that the EOS is fully relativistic. In this case, $M_{Ch} = 5.85/\mu_e$ with $\mu_e$ the molecular weight of electrons. For a typical CO-WD, $\mu_e \approx 2$ so $M_{Ch} \approx 1.46 M_\odot$. However, the radius of the Chandrasekhar mass is zero. Clearly such non physical situation means that no WD can be born with the Chandrasekhar mass, which therefore is a mere ideal value. (iv) Along the sequence toward the Chandrasekhar value at increasing central density, two important physical
processes can intervene, overall instability driven by General Relativity (GR) effects, and pycno-nuclear burning. WDs more massive than about $1.3 M_\odot$, i.e. denser than about $3 \times 10^9 \text{g cm}^{-3}$, become dynamically unstable because of GR (Shapiro & Teukolsky 1983); C-O WDs denser than about $6 \times 10^9 \text{g cm}^{-3}$, may start pycno-nuclear burning during the liquid-solid regimes (Shapiro & Teukolsky 1983). The onset of these phenomena may lead to a thermo-nuclear runaway. Therefore, all stable WDs we observe must have formed with masses lower than $1.2 - 1.3 M_\odot$.

The current theory of type Ia SNe assumes that the Chandrasekhar mass can be reached by accretion or merging and although the details are not fully known, carbon is ignited via the pycno-nuclear channel. This is followed by C-detonation or C-deflagration (the latter is more likely) and by a thermal runaway, because the gravothermal specific heat of the star is positive. The liberated nuclear energy exceeds the gravitational binding energy and the star is thorn apart (see the recent review by Nomoto et al. 2013 and references therein). Binary models for type Ia SNe are classified as double-degenerate, i.e. the merger of two gravitationally bounded WDs, and single-degenerate, i.e. the evolution to the explosive phases is due to the accreting material from a companion star (see e.g., Napiwotzki et al. 2003; Trimble & Asch万象en 2004; Ori 2013).

Owing to the important role of the pycno-nuclear reactions in destabilizing a WD close to the Chandrasekhar mass and triggering type Ia SNe explosions, it is worth examining in some detail the condition under which pycno-nuclear reactions can occur. The pycno-nuclear regime starts in very dense and cool environments, i.e. in the liquid/solid state. While both the central and mean densities are determined by the WD mass and remain nearly constant if the mass does not change, the temperature decreases because the WD is radiating energy from the surface. Because there are no nuclear sources and the electron are highly degenerate, while the WD radiates, the ions must cool down and transit from gaseous to liquid, and eventually solid (crystallised) conditions, i.e. carbon and oxygen ions form a lattice which is pervaded by a gas of electrons. As the temperature decreases, the ions eventually reach the fundamental energy state under a coulombian potential that approximately has the form of a harmonic oscillator. In quantum physics, the energy of the fundamental state of an harmonic oscillator is $1/2 \hbar \omega$ ($\omega$ is the plasma frequency). This means that even at extremely low temperatures there is a finite probability for the C and O ions to penetrate the repulsive coulombian barrier. In this scheme, the rates of pycno-nuclear reactions between C and/or O ions have been first calculated by Salpeter & van Horri 1969 and then refined over the years.

In their pioneering study, Salpeter & van Horri 1969 first noticed that impurities may enhance the pycno-nuclear reactions among the nuclei of the lattice. The enhancement is due to local over-densities in the sites of the impurities. Furthermore, even the contaminant nuclei themselves may react with the lattice nuclei, thus contributing to the total energy production. If this happens, WDs that are less dense and/or less massive than the limits we discussed above become unstable to the ignition of pycno-nuclear reactions triggered by impurities. Since at the distance scales corresponding to the densities of old WDs, electric forces dominate the scene, impurities due to light elements, such as hydrogen or helium, may induce higher local over-densities so that light elements can more easily cross the coulombian barriers.

In order to explore this idea, first we evaluate the change induced by the light element contamination on the typical inter-ion distance $R_o$. Second, we present the reaction rates in the pycno-nuclear channel for reactions like H+C, He+C, etc. Finally, we explore the possibility that even isolated WDs with mass significantly smaller than the Chandrasekhar limit and relatively low density (i.e. in the ranges $1 - 1.2 M_\odot, 3 \times 10^9 - 10^8 \text{g cm}^{-3}$, respectively) may undergo nuclear runaway, because of the energy produced by under-barrier nuclear reactions by contaminant elements, when the WDs reach the liquid/solid state.

The paper is organized as follows. In Section 2 we first shortly review the history of a WD progenitor and the cooling and crystallization processes of ions in the WD, and introduce the relationships between the progenitor mass and the WD mass and between the WD mass and its central density. In Section 3 we describe the fundamentals of nuclear burnings in WDs, summarize two current sources for the reaction rates in both the thermal and pycno-nuclear regimes for single and multi-component fluids, and present a preliminary comparison of the reaction rates. In Section 4 we evaluate the changes in the rates due to impurities, first in the transmission probability of the coulombian potential barrier penetration and then in the local density. Since impurities by light elements are more efficient than those by heavy elements, in Section 5 we estimate the abundances of hydrogen and helium left over by previous evolutionary phases. After an initial phase at the beginning of the cooling sequence in which part of the energy may be of nuclear origin, for a long time the only source of energy is the thermal energy of the ions and all nuclear sources are turned off. Therefore the light elements remain inactive for a long time until the WD reaches the conditions for the activation of the pycno-nuclear regime. In Section 6 we estimate and compare the energy production by reactions like $^1\text{H} + ^{12}\text{C}$ and $^4\text{He} + ^{12}\text{C}$ showing that they can produce the typical luminosity of an old WD in the pycno-nuclear stage. In Section 7 we follow the evolution of WDs of different mass along their cooling sequences towards the pycno-nuclear regime and explore the effect of different abundances of light elements ($\text{H}$ in particular) on the energy release by nuclear reactions. We estimate the abundance of contaminant elements and the epoch at which the nuclear energy generation equals the WD luminosity. For plausible values of $\text{H}$ and He abundances, WDs with masses as low as $0.8 - 0.9 M_\odot$ reach the critical condition for rekindling the nuclear energy production. Finally, in Section 8 we draw some conclusions about the possible implications of these results for the progenitors of type Ia SNe.
We define low mass stars those that develop an electron degenerate helium core, shortly after leaving the main sequence toward the red giant branch (RGB). When the mass of the He-core has grown to a critical value ($0.45-0.50 \, M_{\odot}$), the precise value depends on the composition, star mass, and input physics), a He-burning runaway (called He flash) starts in the core and continues until electron degeneracy is removed. Then nuclear burning proceeds quietly. The maximum initial mass of the star for this to occur is $M_{\text{HeF}}$. Stars more massive than $M_{\text{HeF}}$ are classified as intermediate-mass or massive depending on the physics of carbon ignition in the core. After core He-exhaustion, intermediate mass stars develop a highly degenerate CO core, and undergo helium shell flashes or thermal pulses as asymptotic giant branch (AGB) stars. The AGB phase is terminated by envelope ejection and formation of a CO-WD (with initial mass $M_1$ in the range $M_{\text{HeF}} \lesssim M_1 \lesssim M_w$) or by carbon ignition and deflagration in a highly degenerate core, once it has grown to the Chandrasekhar limit of 1.4 $M_{\odot}$.

The limit mass $M_w$ is regulated by the efficiency of mass loss by stellar wind during the RGB and AGB phases. The minimum mass of the CO core, below which carbon ignition in non degenerate condition fails and the above scheme holds, is 1.06 $M_{\odot}$. The initial mass to reach a core of 1.06 $M_{\odot}$ is called $M_{\text{up}}$. The exact value of $M_{\text{HeF}}, M_{\text{up}}$, and $M_w$ depend on many details of stellar physics. $M_w$ is mainly controlled by mass loss during the AGB phase and is about 6 $M_{\odot}$. The values of $M_{\text{HeF}}$ and $M_{\text{up}}$ are approximately 1.8 - 2.2 $M_{\odot}$ and $\approx 7 - 9 M_{\odot}$, respectively, in absence of convective overshooting. These ranges become $M_{\text{HeF}} \approx 1.7 - 1.8 M_{\odot}$ and $M_{\text{up}} \approx 6 M_{\odot}$ when convective overshooting is included (Chiosi et al. 1992). Since the ignition mass for a fully degenerate fully relativistic CO core is 1.46 $M_{\odot}$, the possibility that C-ignition may occur in single CO-WDs is definitely ruled out.

### 2.2 Physical state of WD interiors

**Generalities.** The interiors of CO-WDs are made of ions of C, O, traces of other elements, and free electrons. Ions are fully ionized and electrons form a uniform background. In other words, there is a multi-component mixture (commonly named multi-component plasma, MCP) of ion species $i = 1, 2, ...$ with mass numbers $A_i$, atomic number $Z_i$, and number densities $n_i$. The total number density is $n = \sum n_i$. For one component plasma (OCP) the suffix $i$ is omitted. A two components medium is defined a binary ionic medium or BIM.

The number density is related to the mass density $\rho$ of the matter by

$$n_i = \frac{X_i \rho}{A_i m_u} \quad (1)$$

where $X_i$ is the mass fraction or abundance of ionic $i$, and $m_u$ the atomic mass unit ($m_u = 1.660 \times 10^{-24} g$). If the density is not very high, the total mass fraction contained in the nuclei is $X_N = \sum_i X_i \approx 1$, whereas for a density higher than $\approx 4 \times 10^{11} \text{ g cm}^{-3}$ above which neutrons may be free, $X_N < 1$. Introducing the fractional number $x_i = n_i/n$, with $\sum_i x_i = 1$, we use two groups of useful relationships:

$$\langle Z \rangle = \sum_i x_i Z_i, \quad \langle A \rangle = \sum_i x_i A_i \quad (2)$$

where $\langle Z \rangle \langle A \rangle$ are the mean atomic and mass number of ions and

$$n_e = n(Z), \quad \rho = \frac{m_u n(A)}{X_N}, \quad x_i = \frac{X_i/A_i}{\sum_j X_j/A_j}. \quad (3)$$

The physical state is best described by the Coulomb coupling parameter $\Gamma_i$ for ions $i$:

$$\Gamma_i = \frac{(Z_i e)^2}{a_i k_B T} = \frac{Z_i^2 e^{5/3}}{a_i k_B T} \quad (4)$$

$$a_i = \left[ \frac{3}{4 \pi n_e} \right]^{1/3}, \quad a_i = Z_i^{1/3} a_e, \quad a_i = \left[ \frac{3}{4 \pi n_i} \right]^{1/3} \quad (5)$$

where $T$ is the temperature, $k_B$ the Boltzmann constant, $a_e$ the electron-sphere radius, and $a_i$ the ion -sphere radius (a radius of a sphere around a given ion, where the electron and ion charge balance each other). The coupling parameter $\Gamma_i$ is the ratio between the Coulomb energy, $E_C = (Z_i e)^2/\langle a_i \rangle$, to the thermal energy $E_{th} = k_B T$ of the ions. If $\Gamma_i \approx 1$ the ions constitute an almost ideal Boltzmann gas. If $\Gamma_i \gg 1$ the ions are strongly coupled by the Coulomb forces and constitute a Coulomb liquid or a solid. The transition gas to liquid occurs smoothly at $\Gamma_i \approx 1$ with no phase transition. According to Lindemann (1916) and recent accurate Monte Carlo simulations by Dewitt et al. (2001), a classical OCP of ions solidifies at $\Gamma_i \approx 175$ via a weak second-order phase transition.

In most cases, BIMs or MCPs are supposed to represent the composition of WDs, therefore it may be useful to introduce the mean ion Coulombian parameter $\langle \Gamma \rangle = \sum_j x_j \Gamma_j$. Strongly coupling occurs if $\langle \Gamma \rangle \gg 1$, causing a transition from plasma to liquid. The temperature $T_L$ at which this occurs is given by

$$k_B T_L = \sum_j \left[ \frac{Z_j^2 e^2}{a_j} \right] x_j \equiv k_B T(\langle \Gamma \rangle). \quad (6)$$

With decreasing temperature, the ion motions can no longer be considered as classical, but must be quantized. The nuclei form a Debye plasma with temperature $T^P$ associated with the ion plasma frequency $\omega^P$:

$$T^P = \frac{\hbar \omega^P}{k_B}, \quad \omega^P = \sqrt{\sum_j 4 \pi Z_j^2 e^2 n_j / A_j m_u} \quad (7)$$

This is the critical stage at which the specific heat of the material is determined by nuclei oscillations of frequency $\omega^P$ rather than by free thermal motions.

Increasing $\langle \Gamma \rangle$ further, by either lowering the temperature and increasing the density or both, the matter crystallizes into a rigid Coulomb lattice. The solidification (or melting) temperature $T^M$ is given by

$$T^M = \frac{1}{k_B} \sum_j \left[ \frac{Z_j^2 e^2}{a_j} \right] x_j = \frac{T_L}{\langle \Gamma \rangle M} \quad (8)$$
where \( \langle \Gamma \rangle_M = 175 \) \((\text{Dewitt et al. 2001})\). At such high densities, even the small zero point oscillation allow the neighbouring nuclear wave functions to overlap, inducing nuclear reactions that depend on density and not on temperature. This is the pycono-nuclear regime.

**WD cooling.** After a short lived initial phase, during which the energy supply is sustained by some nuclear burning in the two progressively extinguishing surface shells and the large energy losses by neutrinos emission, the evolutionary rate of the WD is driven only by the internal energy of the ions and electrons. Owing to the very different specific heat at constant volume of ions and electrons, the ions are the main contributors to the luminosity of a WD, and their contribution to the cooling rate of the WD can be neglected with respect to that of ions. Therefore the luminosity of a WD is given by

\[
L = -\int_o^M C_v \dot{T} dM,
\]

where \(C_v\) and \(\dot{T}\) are function of the position and time.

As the interior of a WD cools down, the ion specific heat \(C_v^{\text{ion}}\) per gram gradually changes from

\[
C_v^{\text{ion}} \simeq \frac{3}{2} \frac{k_B}{A_{\text{ion}}} \quad \text{to} \quad C_v^{\text{ion}} \simeq \frac{3}{2} \frac{k_B}{A_{\text{nuc}}}
\]

(with the usual meaning of all the symbols) whereby the first relation is for the hot gaseous phase and the second one is for the phase in which the temperature has decreased but it is still far from the Debye value (whereby quantum effects become important). The increase by a factor of 2 is due to increasing correlations of the ion positions driven by the growing importance of the Coulomb interaction energies as compared to the thermal energies of the ions. In other words, the spatial scale of the coulombian interactions is comparable to the inter-ion spacing determined by the density. When the temperature is close to the Debye value, \(C_v^{\text{ion}}\) decreases dramatically and becomes proportional to \(T^3\).

As far as electrons are concerned, the specific heat of the electrons can be neglected with respect to that of ions and their contribution to the cooling rate of the WD can be ignored \((\text{Kippenhahn & Weigert 1990})\). To conclude, as the ions are the main contributors to the luminosity of a WD, the above relationships are used to calculate the cooling sequence of a WD (see Section 5.5 below).

**Liquefaction and crystallization.** The liquefaction and crystallization theory \((\text{van Hori 1968})\) predicts that in WDs ions start to liquefy and freeze in an ordered crystalline lattice from the center to the outer layers when the temperature falls below \(T^L\) and \(T^M\). The phase transition from an isotropic Coulomb liquid to a crystalline solid implies a discontinuity in the distribution of the plasma ions. Because symmetry cannot be achieved instantaneously, the transition is a first-order phase change. Therefore, for \(\langle \Gamma \rangle = \langle \Gamma \rangle_m = 172 - 175 \) \((\text{Kitamura 2000})\) latent heat is released \((\text{van Hori 1968})\).

| \(\rho_c\) | \(M_G\) (in units of \(M_\odot\)) | \(R\) (in units of \(R_\odot\)) | \(\rho_c\) (in \(\text{g cm}^{-3}\)) | \(M_G\) | \(R\) |
|-----|-----|-----|-----|-----|-----|
| 1.93e5 | 2.01e-1 | 2.10e-2 | 1.93e8 | 1.24e0 | 5.33e-3 |
| 3.73e5 | 2.68e-1 | 1.87e-2 | 3.73e8 | 1.30e0 | 4.54e-3 |
| 5.18e5 | 3.08e-1 | 1.77e-2 | 5.18e8 | 1.32e0 | 4.19e-3 |
| 7.19e5 | 3.52e-1 | 1.67e-2 | 7.19e8 | 1.34e0 | 3.85e-3 |
| 1.00e6 | 4.01e-1 | 1.57e-2 | 1.00e9 | 1.39e0 | 3.54e-3 |
| 1.93e6 | 5.08e-1 | 1.39e-2 | 1.93e9 | 1.38e0 | 2.98e-3 |
| 3.73e6 | 6.28e-1 | 1.23e-2 | 3.73e9 | 1.40e0 | 2.49e-3 |
| 5.18e6 | 6.90e-1 | 1.16e-2 | 5.18e9 | 1.41e0 | 2.27e-3 |
| 7.19e6 | 7.52e-1 | 1.08e-2 | 7.19e9 | 1.41e0 | 2.07e-3 |
| 1.00e7 | 8.14e-1 | 1.02e-2 | 1.00e10 | 1.42e0 | 1.88e-3 |

| \(\rho_c\) | \(M_G\) (in units of \(M_\odot\)) | \(R\) (in units of \(R_\odot\)) | \(\rho_c\) (in \(\text{g cm}^{-3}\)) | \(M_G\) | \(R\) |
|-----|-----|-----|-----|-----|-----|
| 1.93e7 | 9.34e-1 | 8.89e-3 | 1.93e10 | 1.42e0 | 1.55e-3 |
| 3.73e7 | 1.04e+0 | 7.33e-3 | 3.73e10 | 1.42e0 | 1.28e-3 |
| 5.18e7 | 1.09e+0 | 7.20e-3 | 5.18e10 | 1.42e0 | 1.15e-3 |
| 7.19e7 | 1.14e+0 | 6.69e-3 | 7.19e10 | 1.42e0 | 1.04e-3 |
| 1.00e8 | 1.18e+0 | 6.21e-3 | 1.00e11 | 1.42e0 | 9.44e-4 |

For an ionic mixture with more than one species of ions, chemical separation may occur, either at solidification or in the fluid phase. The mixture behaviour, in this case, derives from the peculiar shape of the state diagram. Because a phase separation, with the companion stratification of elements, is a source of gravitational energy (caused by sinking of the heavier ions), able to deeply modify the WD cooling time, it is of fundamental importance to obtain detailed phase diagrams for the BM or MCP of interest. Moreover, for an accreting WD, chemical separation may produce chemical stratification, thus affecting the electron capture, opacity and fusion rates.

The most difficult problem with a strongly coupled MCP at low temperatures is understanding its actual state. MonteCarlo simulations of the freezing of classical OCP by \((\text{DeWitt et al. 1992})\) indicate that it freezes into imperfect body-centered cubic (BCC) or face-centered cubic (FCC) micro-crystals. Unfortunately there are no reliable simulations of freezing for MCPs. Cold MCPs are much more complex than OCPs; they can be regular lattices with impurities or an amorphous uniformly mixed structure or a lattice of one phase with admixtures of other ions; or even a mosaic of phase separated regions. Fortunately, these extreme conditions are seldom reached. In a typical CO-WD (with \(X_C = X_O = 0.5\)), the temperature and density plane is confined in the ranges \(7.5 \leq \rho \leq 10.5\) and \(7.0 \leq T \leq 9.5\). At \(\rho > 4 \times 10^{10}\) \text{g cm}^{-3}\), the carbon nuclei cannot survive in dense matter because of beta captures. At \(\rho > 2 \times 10^{10}\) \text{g cm}^{-3}\), the oxygen nuclei are also destroyed by beta captures. In this plane, the loci of \(T^L\), \(T^P\) and \(T^M\) are straight lines whose terminal points \([\log \rho, \log T]\) are: \(T^L[8.4, 9.5]\), \(T^P[10.5, 8.8]\), and \(T^M[10.5, 7.9]\). There is some marginal effect that depends on the fractional abundances \(x_j\) (see \((\text{Yakovlev et al. 2006})\), for more details).
and the WD phase begins. We neglect here the effects of the initial chemical composition on $M_{CO}$, $M_{WD}$ and ages, and focus on the case [$Y=0.26$, $Z=0.017$] typical of a young population. If the chemical composition is taken into account, because of the different total lifetimes of stars of the same mass but different chemical composition and other details of stellar structure, at the low mass end of the WD mass distribution not all combinations of $M_i$ and $M_{WD}$ correspond to realistic cases. The lower mass limit is determined by the age of the Universe, i.e. $13.7 \pm 0.2$ Gyr according to WMAP data
(Spergel et al. 2003).

Finally, we would like to remind that, owing to the different mass size of the external envelope surrounding the $M_{CO}$ core of TP-AGB stars, the core increases little during this phase in low mass stars (up to about $3 M_\odot$) while it increases significantly in stars with mass in the range $3 M_\odot$ to $6 M_\odot$. Furthermore, the upper mass limit $M_{up}$ for AGB phase to occur (and to WDs to be generated) depends on the initial chemical composition; the detailed stellar models by Bertelli et al. (2008, 2009) show that it can vary from $5 M_\odot$ to $6 M_\odot$. Therefore the $M_{WD}$ vs $M_i$ relationship of Table 2 has to be considered as only indicative of the overall trend.

### 3 NUCLEAR BURNING IN WDs

#### 3.1 Generalities

In the first stages of the WD life, nuclear burning can occur in two shells, close to the surface (see Renedo et al. 2010 and references therein). hydrogen may still be burning via the CNO-cycle, because a small amount of it (mass abundances in the range $10^{-4} < X_H < 10^{-5}$) is left on the surface, and the temperature is still sufficiently high to sustain nuclear burning. Similarly, somewhat deeper inside there may be also residual He-burning in a shell surrounding the inert CO-core.

As the temperature decreases, all thermal nuclear burnings are turned off for billions of years until the WD has cooled down to very low temperature, so that the pycno-nuclear regime is reached. This is possible only when the internal energy of ions and the WD luminosity are very low, and the baryonic matter (mostly C and O ions) is crystallized, usually when the WDs are quite old (several Gyr). However, once the pycno-nuclear reactions start, the energy released can easily exceed the luminosity of the WD, possibly causing a thermonuclear runaway, subsequent explosive nuclear burning, and disruption of the whole star (see Section 7).

The pycno-nuclear reactions were studied long ago by Salpeter & van Horn (1969). These authors modeled the pycno-nuclear potential with an harmonic oscillator and discussed the possible influence of electron-screening and other effects. They also suggested that impurities may significantly increase the nuclear burning rates (see Section 4). This suggestion is the starting point of this study.

Other recent formulations of nuclear reactions in the pycno-nuclear regime are by Schramm & Koonin (1990), Ogata et al. (1991), Ichimaru et al. (1992), Kitamura & Ichimaru (1993), Brown & Sawyer (1997), Ichimaru & Kitamura (1999a), and Kitamura (2000).
In WDs the central density ranges from $10^{3.5}$ to $10^{6}$ g cm$^{-3}$ depending on the initial mass of the star, and the density decreases from the center to the surface, so we must consider a wide range of densities. The temperature also spans a wide range, from $10^9K$ at the beginning of the cooling sequence to typical values of about $10^7K$ after a few Gyrs. In the plane $T-\rho$, the WD undergoes two phase transitions: from gas to liquid from liquid to solid (crystallization). In parallel to this, there are five burning regimes (see Salpeter & van Horn 1969, Yakovlev et al. 2006, Beard 2010): (i) the classical thermo-nuclear; (ii) the thermonuclear with strong electron screening; (iii) the thermo-pycno-nuclear; (iv) the thermally enhanced pycno-nuclear; (v) and eventually the zero temperature pycno-nuclear. For a complete description of the regimes see Beard (2010). The five regimes have the following characteristics:

i) The classical thermo-nuclear takes place when $\Gamma_{ij} << 1$. The mean inter-nuclear distance is much greater than the typical scale at which the particles feel electrostatic interaction: the nuclei are fully stripped and there is a small screening effect from the background electron gas. The bulk matter behaves like an ideal gas.

ii) The second regime is bounded by the temperatures $T^b_{ij}$ and $T^c_{ij}$ with $T^c_{ij} \leq T \leq T^b_{ij}$. For temperatures lower than $T^b_{ij}$ the ions are in the liquid phase, and for temperatures close to $T^c_{ij}$, the ions cannot be considered a free gas but part of a lattice with vibrations $\omega_{ij}^P$. The thermoneutrional burning associated to very strong electron screening operates in this temperature range.

iii) The third regime corresponds to the temperature range $T^M_{ij} \leq T \leq T^b_{ij}$ and $\Gamma_{ij} \geq 1$. Nuclei are bound to the lattice sites, so that the reaction occur between highly thermally exited bound nuclei, which oscillate with frequencies higher than the plasma frequency and have energy greater than the zero point energy of the plasma. The nuclei are also embedded in a highly degenerate electron gas, so that the reaction rates are enhanced by the charge screening electron background. For $\Gamma_{ij} \geq 175$, and temperatures lower than $T^M_{ij}$, the nuclei form a solid lattice.

iv) In the thermally enhanced pycno-nuclear regime, $\Gamma_{ij} >> 1$ but the melting temperature is not reached yet (this requires $\Gamma_{ij} > 175$). Most of the nuclei are bound to the lattice, but some nuclei are highly exited states, and reactions may occur between neighboring pairs of these nuclei. Electron screening is always very strong.

v) The last domain is the one of pure pycno-nuclear behaviour ($T \simeq 0$). Almost all the ions are in the fundamental state of the lattice (now crystallized to a solid) but their energies are larger than zero because of the Heisenberg indetermination principle. Therefore, there is still a finite possibility of penetrating the Coulomb barrier (tunneling effect). Reactions are possible only between closest pairs. Because there is no longer a temperature dependence, these reactions are referred to as $T = 0$ pycno-nuclear rates.

### 3.2 Formalism for reaction rates

We are interested in nuclear fusion reactions

$$(A_i, Z_i) + (A_j, Z_j) \rightarrow (A_{ij}, Z_{ij})$$

where $A_c = A_i + A_j$, $Z_c = Z_i + Z_j$ refer to the compound nucleus $c$. To study these reactions we must extend the formalism presented in Sect. 2.2 and introduce the ion-sphere quantities

$$a_{ij} = \frac{a_i + a_j}{2}, \quad \mu_{ij} = m_u \frac{A_i A_j}{A_c}, \quad \Gamma_{ij} = \frac{Z_i Z_j e^2}{a_{ij} k_B T} \quad (11)$$

$$T^h_{ij} = \frac{Z_i Z_j e^2}{a_{ij} k_B}$$

$$T^P_{ij} = \frac{\hbar}{k_B} \omega_{ij} \quad \omega_{ij}^P = \sqrt{\frac{4\pi Z_i Z_j e^2 n_{ij}}{2\mu_{ij}}} \quad (12)$$

where $\mu_{ij}$ is the reduced mass, $a_{ij}$ characterizes the equilibrium distance between neighbouring nuclei (the corresponding number density is $n_{ij} = 3/4\pi a_{ij}^3$), $\Gamma_{ij}$ describes their coulomb coupling, $T^h_{ij}$ is the temperature of the onset of strong coupling (or liquefaction temperature), $T^P_{ij}$ is the Debye temperature for the oscillations of ions $i$ and $j$, and $T^M_{ij}$ is the melting or solidification temperature. Finally, we need the generalized Bohr radius

$$r_{B_{ij}} = \frac{\hbar^2}{2\mu_{ij} Z_i Z_j e^2} \quad (13)$$

which becomes the ion Bohr radius for equal ions $i = j$, and the parameter $\lambda_{ij}$, corresponding to the parameter $\lambda$ introduced by Salpeter & van Horn (1969)

$$\lambda_{ij} = r_{B_{ij}} \left( \frac{n_{ij}}{2} \right)^{1/3} = \frac{2r_{B_{ij}}}{(Z_i^{1/3} + Z_j^{1/3})} \left( \frac{\rho X_N(Z)}{2(A) m_u} \right)^{1/3} \quad (14)$$

### 3.3 Five regimes of nuclear burning

In WDs the central density ranges from $10^9 g cm^{-3}$ to $10^{10} g cm^{-3}$ depending on the initial mass of the star, and the density decreases from the center to the surface, so we must consider a wide range of densities. The temperature also spans a wide range, from $10^9K$ at the beginning of the cooling sequence to typical values of about $10^7K$ after a few Gyrs. In the plane $T-\rho$, the WD undergoes two phase transitions: from gas to liquid from liquid to solid (crystallization). In parallel to this, there are five burning regimes (see Salpeter & van Horn 1969, Yakovlev et al. 2006, Beard 2010): (i) the classical thermo-nuclear; (ii) the thermonuclear with strong electron screening; (iii) the thermo-pycno-nuclear; (iv) the thermally enhanced pycno-nuclear; (v) and eventually the zero temperature pycno-nuclear. For a complete description of the regimes see Beard (2010). The five regimes have the following characteristics:

1. The mean inter-nuclear distance is much greater than the typical scale at which the particles feel electrostatic interaction: the nuclei are fully stripped and there is a small screening effect from the background electron gas. The bulk matter behaves like an ideal gas.

2. The second regime is bounded by the temperatures $T^b_{ij}$ and $T^c_{ij}$ with $T^c_{ij} \leq T \leq T^b_{ij}$. For temperatures lower than $T^b_{ij}$ the ions are in the liquid phase, and for temperatures close to $T^c_{ij}$, the ions cannot be considered a free gas but part of a lattice with vibrations $\omega_{ij}^P$. The thermoneutrional burning associated to very strong electron screening operates in this temperature range.

3. The third regime corresponds to the temperature range $T^M_{ij} \leq T \leq T^b_{ij}$ and $\Gamma_{ij} \geq 1$. Nuclei are bound to the lattice sites, so that the reaction occur between highly thermally exited bound nuclei, which oscillate with frequencies higher than the plasma frequency and have energy greater than the zero point energy of the plasma. The nuclei are also embedded in a highly degenerate electron gas, so that the reaction rates are enhanced by the charge screening electron background. For $\Gamma_{ij} \geq 175$, and temperatures lower than $T^M_{ij}$, the nuclei form a solid lattice.

4. In the thermally enhanced pycno-nuclear regime, $\Gamma_{ij} >> 1$ but the melting temperature is not reached yet (this requires $\Gamma_{ij} > 175$). Most of the nuclei are bound to the lattice, but some nuclei are highly exited states, and reactions may occur between neighboring pairs of these nuclei. Electron screening is always very strong.

5. The last domain is the one of pure pycno-nuclear behaviour ($T \simeq 0$). Almost all the ions are in the fundamental state of the lattice (now crystallized to a solid) but their energies are larger than zero because of the Heisenberg indetermination principle. Therefore, there is still a finite possibility of penetrating the Coulomb barrier (tunneling effect). Reactions are possible only between closest pairs. Because there is no longer a temperature dependence, these reactions are referred to as $T = 0$ pycno-nuclear rates.

### 3.4 Thermal vs pycno-nuclear burnings

There are some important differences in the hypotheses underlying thermo-nuclear and pycno-nuclear reactions that are worth highlighting. In the thermo-nuclear reactions each particle can interact with all the others, therefore the rate contains the product of the two densities of the two interacting species. The reaction rate is

$$R_{ij} = N_i N_j < \sigma v >$$

(15)
The probability of interaction is proportional to the product of two factors: the Maxwell-Boltzmann statistical factor and the probability of tunneling. Their product gives origin to the so-called Gamow window. The first factor is proportional to

$$e^{-E/(KT)}$$  \hspace{1cm} (16)$$

whereas the second one is proportional to

$$e^{-\sqrt{EG/E}}$$  \hspace{1cm} (17)$$

where $E_G$ is the Gamow energy. If we define $E_{ij}^{pk} = \frac{1}{2}\sqrt{E_G k_B T}$ we can write the thermo-nuclear reaction rate as

$$R_{ij}^{th} = \frac{n_i n_j}{1 + \delta_{ij}} \left(\frac{2E_{ij}^{pk}}{3\mu_{ij}}\right)^{1/2} S(E_{ij}^{pk}) e^{-\tau_{ij}}$$  \hspace{1cm} (18)$$

with $\tau_{ij} = \frac{3E_{ij}^{pk}}{k_B T}$.

The rate for the pycno-nuclear reactions is quite different. Each nucleus in the lattice can interact only with the nearest neighbours. Therefore, the rate is

$$R_{ij}^{pyc} = \frac{n_i n_j}{2} < n_i \nu_i >$$  \hspace{1cm} (19)$$

where $p_i$ is the reaction probability between a pair of ions and $\nu_i$ is the number of nearest neighbours. According to Salpeter & van Horn (1959), the reaction rate probability can be written as

$$p_i = D_{pp} \frac{\lambda_i^{-3} C_{pl} S(E_{ij}^{pk})}{h \gamma_i} exp\left(-\frac{C_{exp}}{\lambda_i^{1/2}}\right)$$  \hspace{1cm} (20)$$

where $\lambda_i$ is the ratio of the Bohr radius to the lattice spacing. For more details on the meaning of the various symbols see Salpeter & van Horn (1960); Beard (2010) and Shapiro & Teukolsky (1983).

### 3.5 Rates for pycno-nuclear reactions in mono- and multi-component fluids

In the literature on pycno-nuclear reactions, there are basically three different formulations or models: (i) the classical one, based on the simple harmonic oscillator at zero temperature proposed long ago by Salpeter & van Horn (1960), that we will shortly revisit to explore the effect of impurities (Sect. 3); (ii) the model by Kitamura (2000 and references) and finally (iii) the models by Gasques et al. (2005) and Yakovlev et al. (2006). The last two models include also the effect of temperature, and they are mutually consistent although they make use of different analytical expressions.

#### 3.5.1 Summary of the Kitamura (2000) reaction rates

Kitamura (2000, see also references therein) considerably improved the simple description based on the harmonic oscillator. He derive analytical expressions for the reaction rates, taking into account the dielectric functions of relativistic and non-relativistic electrons, the screening potentials based on the Monte Carlo simulations, and the interaction free energies in dense electron screened BIMs. He found that under-barrier reaction rates can be expressed as the product of three terms

$$R = R_G A_i^{(0)} A_j^{(c)}$$  \hspace{1cm} (21)$$

where $R_G$ is the so-called Gamow channel, representing the basic binary interaction between any two particles, and is expressed as the Gamow rate. It is dominant in tenuous plasmas in which the effect of surrounding particles is negligible, so the nuclei interact via the bare Coulomb potential and the rate is expected to be strongly dependent on temperature. The other two terms take the effects of the surrounding particles into account. The so-called few-particle interactions are expressed by $A_i^{(0)}$ and occur independently of the aggregation state of nuclei. The shielding effect stems from local variations of particle density with respect to the background (also referred to as polarization). The net effect is to reduce the Coulomb potential barrier and strongly enhance the rate. Once the temperature is below a certain value, the rate is expected to increase as the temperature decreases. Finally the third term $A_j^{(c)}$ is due to the many-particles processes that may occur when the electrons can be considered as a uniform background. This is expected to produce a small effect at the typical temperatures and densities of liquid-solid WDs, because the so-called screening temperature $T_s$, at which this effect is important, is much lower. Therefore, the third term is expected to be small and to become dominant only at very low temperatures, unlikely to be reached in WDs.

The most important pycno-nuclear reactions are the few-particle ones rather than the many-particles interactions. Quoting Ichimaru & Kitamura (1999b, page 2694) “Since the enhancement factors generally increase steeply with lowering of the temperature, a maximum pycno-nuclear rate may be attained in a liquid-metallic substance near the conditions of freezing transitions; reaction rates in a solid, depending on the amplitudes of atomic vibrations, increase with the temperature.”

Kitamura (2000)’s rates are particularly useful to understand the effects of impurities because they include both BIMs and OCPs, and provide a better description of the physics during the solid transition, which implies a smoother discontinuity in the reaction rates. We will apply the Kitamura (2000) rates over the whole range of temperatures and densities, for any kind of reacting pairs.

In a BIM of two elements “i” and “j”, with charges $Z_i$ and $Z_j$, mass number $A_i$ and $A_j$, mass fraction $\rho_m$ and temperature $T$, the reaction rates are

$$R_{ij}(\text{reactions cm}^{-3} \text{s}^{-1}) = \frac{1}{1 + \delta_{ij}} \frac{X_i X_j (A_i + A_j)}{Z_i Z_j (A_i A_j)^2} \cdot \left[\rho_m (\text{g cm}^{-3})\right]^2 \cdot [S_{ij}(E_{cm}) \text{(MeV barns})] R_0$$  \hspace{1cm} (22)$$

where $\delta_{ij}$ is the Kronecker delta function ($\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise), $S_{ij}$ is the nuclear cross-section factor for the analyzed reaction and $R_0$ depend the aggregation state of matter.
For the liquid phase, if \( T_s \) is the critical screening temperature and \( T \gg T_s \), we have

\[
R_0^{\text{fluid}} = 2.613 \times 10^{32} \frac{D_{ij}}{r_{ij}} \left[ \frac{1 + \left( \frac{T}{T_s} \right)^8}{1 + \left( \frac{D_{ij} \phi}{r_{ij}} \right)^{3/2}} \right]^{1/2} \exp \left( -\alpha_{ij} \pi \sqrt{\frac{D_{ij}}{r_{ij}}} + \xi_{ij} \right) \tag{23a}
\]

if instead \( T < T_s \)

\[
R_0^{\text{fluid}} = 1.600 \times 10^{33} \left( \frac{D_{ij}}{r_{ij}} \right)^{1/2} \left\{ 1 + \left[ 0.543 \left( \frac{D_{ij} \phi}{r_{ij}} \right)^{1/2} - 1 \right] \left( \frac{T}{T_s} \right)^3 \right\} \exp \left( -\alpha_{ij} \pi \sqrt{\frac{D_{ij}}{r_{ij}}} + \xi_{ij} \right) \tag{23b}
\]

where \( T_s \) is the critical screening temperature at which the Gamow peak energy equals the electrostatic screening energy. The two expressions coincide for \( T = T_s \).

The nuclear reaction rates for the solid state are obtained by substituting in eqn.\( 22 \), the following expression for \( R_0 \):

\[
R_0^{\text{solid}} = 4.83 \times 10^{32} \left( R_s^0 \right)^{1.809} \cdot \exp \left\{ \left[ -3.506 + 0.142 \frac{\alpha_{ij}}{D_{ij}} \right] \left( R_s^0 \right)^{1/2} + 0.144 \left( \frac{\alpha_{ij}}{D_{ij}} \right)^2 + F(Y_s^0) \right\} \tag{24}
\]

The definitions of all terms and meaning of all symbols used in eqns.\( 23 \) and\( 24 \) above can be found in [Kitamura (2000)] to whom the reader should refer. We note that the first term in the exponential factor of eqn.\( 23 \) corresponds to the Gamow thermo-nuclear channel, while the second term corresponds to the under-barrier reaction channel. In this paper, we examine the reactions listed in Table 3. The experimental value of the cross-section factor and Q-value from [Fowler et al. (1973)] are also in Table 3.

3.5.2 Summary of the Gasques - Yakovlev reaction rates

Gasques et al. (2003) elaborated a model for OCPs to calculate the reaction rates in several regimes, from thermo-nuclear to pycno-nuclear. Subsequently Yakovlev et al. (2006) and Beard (2010) extended the model to MCPs. These three papers are based on the Sao Paulo potential, that takes into account the effect of Fermi statistic on the nucleons of the interacting nuclei. The purposes of these studies are: (a) to evaluate the rate for the pycno-nuclear part \( R_{ij}^{\text{pyc}} \) and (b) to apply a phenomenological expression for the temperature and density dependent part \( \Delta R_{ij}(\rho, T) \). All the auxiliary quantities contained in the expressions below have already been introduced in Sect. 2.2 or are given in Table 3.

The pycno-nuclear component is

\[
R_{ij}^{\text{pyc}} = 10^{-6} C_{ij}^{\text{pyc}} \frac{8pX_Nx_Ia_n^cA^cZ^2_iZ^2_j}{(1 - \delta_{ij})A^2_iS(E_{ij}^{\text{pk}})} \times \lambda_{ij}^{3-G_i} \exp(-\frac{C_{exp}}{(\lambda_{ij})^{1/2}} cm^{-3} s^{-1}). \tag{25}
\]

whereas the phenomenological expression for the temperature and density dependent part of the rate that combines all the burning regimes is

\[
R_{ij}(\rho, T) = R_{ij}^{\text{pyc}}(\rho) + \Delta R_{ij}(\rho, T), \tag{26}
\]

\[
\Delta R_{ij}(\rho, T) = \frac{n_{ij} \tau_{ij} S(E_{ij}^{\text{pk}})}{1 + \delta_{ij}} r_{ij}^2 PF, \tag{27}
\]

\[
F = \exp(-\tau_{ij} + C_{ij} \Gamma_{ij} (\phi - \Lambda T_{ij}^{\text{pk}})/T - \Lambda T_{ij}^{\text{pk}}), \tag{28}
\]

\[
\tau_{ij} = \frac{3}{2} \left( \frac{E_{ij}^{\text{pk}}}{k_B T} \right)^{2/3}, \tag{29}
\]

\[
\Gamma_{ij} = \frac{Z_i Z_j e^2}{a_{ij} k_B T}, \tag{30}
\]

\[
\gamma = (T^2 \gamma_1 + (T_{ij}^{\text{pk}})^2 \gamma_2)/((T^2 + (T_{ij}^{\text{pk}})^2)), \tag{31}
\]

\[
E_{ij}^{pk} = \hbar \omega_{ij} + (Z_i Z_j e^2/3 + k_B T \tau_{ij}) \exp(-\Lambda T_{ij}^{\text{pk}}/T). \tag{32}
\]

where \( \gamma_1 = 2/3 \) and \( \gamma_2 = (2/3)(C_{pl} + 0.5) \).

3.5.3 Comparing the two sets of reaction rates

The rates predicted by Kitamura (2000, dashed lines) and Gasques et al. (2003, solid and dotted lines) for the \( ^{12}\text{C}+^{12}\text{C} \) reaction are shown as a function of density in Fig. 1. For the Gasques et al. (2003) rate we adopt the numerical coefficients for MCPs that are listed in Table 3 and show the cases of maximum (solid lines) and minimum (dotted lines) rates. For each case we show separately the sole pycno-nuclear component (red lines) and the pycno-nuclear plus thermal channel (blue lines). The rates by Kitamura (2000) fall always in between those by Gasques et al. (2003) for all regimes. In general, the three cases have a very similar trend, the only difference being the density at which the transition from thermal+pycno to pycno-nuclear regimes occurs and the absolute value of the rates. However, we note that there is a large difference between the maximum and minimum efficiency for the Gasques et al. (2003) rates, amounting to about ten orders of magnitude, and that the Kitamura (2000) rates run closer to the Gasques et al. (2003) case with maximum efficiency. There are large differences in the rates at all densities, causing a large difference in the final results.

4 IMPURITIES IN PYCNO-NUCLEAR REACTIONS RATES

When matter becomes solid, the C and O ions are in a fixed configuration in a crystal whose structure is known from solid state physics. We cannot exclude the possibility that some impurities of elements of any type are present in the lattice. In fact, as long as the WD is liquid, convection is active, mixing all the elements in the star and bringing heavier elements to the surface and lighter elements to the core.
Table 3. Coefficients in the interpolation expressions for a reaction rate for the optimal model of nuclear burning and for the models which maximize and minimize the rate. The parameters $C_T$, $\alpha_{\lambda ij}$, $\alpha_{\omega ij}$ are different for multi-component plasma (MCP) and one component plasma (OCP) (the values for OCP are given in brackets). For a MCP, the models assume a uniformly mixed state (see Yakovlev et al. 2000) for details.

| Model         | $C_{exp}$ | $C_{pyc}$ | $C_{ppl}$ | $C_T$ | $\alpha_{\lambda ij}$ | $\alpha_{\omega ij}$ | $A$ |
|---------------|-----------|-----------|-----------|-------|------------------------|------------------------|-----|
| Optimal       | 2.638     | 3.90      | 1.25      | 0.724 | 1 (1)                  | 1 (1)                  | 0.5 |
| Maximum rate  | 2.450     | 50        | 1.25      | 0.768 | 1.05 (1)               | 0.95 (1)               | 0.35 |
| Minimum rate  | 2.650     | 0.5       | 1.25      | 0.711 | 0.95 (1)               | 1.05 (1)               | 0.65 |

Figure 1. Comparison of the Gasques et al. (2005) and Kitamura (2000) rates for the $^{12}\text{C}+^{12}\text{C}$ reaction. The coefficients of the Gasques et al. (2005) rates are those listed in Table 3. The thermal branch and pycno-nuclear channel are calculated with temperatures of $10^8$ K and $10^9$ K, respectively. The mass abundance of carbon is equal to $X_C = 1$.

Table 4. Parameters for the reaction rates of most common reactions in CO-WDs.

| Reaction       | Products | $Q$-value [MeV] | $S(0)$ [MeV barn] |
|----------------|----------|----------------|-------------------|
| $^{12}\text{C}+^{12}\text{C}$ | Mg$^{24}$ | 13.931 | 8.83 - $10^{16}$ |
| $^{12}\text{C}+^{16}\text{O}$ | Si$^{28}$ | 16.754 | 1.15 - $10^{21}$ |
| $^{16}\text{O}+^{16}\text{O}$ | S$^{32}$ | 16.541 | 2.31 - $10^{27}$ |

Since in an old WD, there are no nuclear reactions involving thermal channels, the chemical composition of is frozen until pycno-nuclear burning begins. The dominant element with the lowest atomic number is carbon ($Z_C = 6$), followed by oxygen ($Z_O = 8$), so impurities can be grouped according to their atomic number: i) heavy elements like magnesium, etc., with $Z > Z_C$; ii) light elements like hydrogen and helium with $Z < Z_C$.

We are particularly interested here in impurities of light elements, like hydrogen and helium. In the next section, we will see that the pycno-nuclear rates for these elements may be particularly high, so the amount of light elements necessary for relevant effects is extremely low. For instance, in the case of hydrogen, abundances in the range $10^{-16} \leq X_H \leq 10^{-25}$ are sufficient to produce nuclear energy in amounts comparable to the luminosity of typical WDs at suitable ages. This should not be confused with the surface content of hydrogen that can be as high as $X_H \approx 10^{-5}$ without triggering thermonuclear burning (see Fujimoto 1982b,a; Renedo et al. 2010, for all details). Higher concentrations are not allowed otherwise the associated thermonuclear energy generation via the CNO cycle would exceed the luminosity of a WD causing a thermonuclear runaway. During the cooling phase, convection/or diffusion may bring some hydrogen and helium to the interior; moreover some hydrogen or helium may be left over from previous evolutionary phases.

In the rest of this section we describe, with a simple formulation, how the coulombian potential and local density in a ionic lattice are modified by impurities. We adopt the expression of Salpeter & van Horn (1969) and Shapiro & Teukolsky (1983) for the rate of pycno-nuclear reactions. The original theory was developed for reactions among the same nuclei (e.g. carbon). Since we consider at least two elements (e.g. reactions among hydrogen and carbon), the coulombian potential must be suitably modified. We are going to present a template, to which more sophisticated formulations must be compared.

4.1 The modified Salpeter-Van Horn model

Using the same formalism of Shapiro & Teukolsky (1983) we consider an OCP made of a certain type of ions, e.g. either pure carbon or pure oxygen. In this framework, first we consider the difference in mass density in a ionic lattice are modified by impurities. We adopt the expression of Salpeter & van Horn (1969) and Shapiro & Teukolsky (1983) for the rate of pycno-nuclear reactions. The original theory was developed for reactions among the same nuclei (e.g. carbon). Since we consider at least two elements (e.g. reactions among hydrogen and carbon), the coulombian potential must be suitably modified. We are going to present a template, to which more sophisticated formulations must be compared.

Using the same formalism of Shapiro & Teukolsky (1983) for the rate of pycno-nuclear burning (see Fujimoto 1982b,a; Renedo et al. 2010, for all details). Higher concentrations are not allowed otherwise the associated thermonuclear energy generation via the CNO cycle would exceed the luminosity of a WD causing a thermonuclear runaway. During the cooling phase, convection/or diffusion may bring some hydrogen and helium to the interior; moreover some hydrogen or helium may be left over from previous evolutionary phases.

In the rest of this section we describe, with a simple formulation, how the coulombian potential and local density in a ionic lattice are modified by impurities. We adopt the expression of Salpeter & van Horn (1969) and Shapiro & Teukolsky (1983) for the rate of pycno-nuclear reactions. The original theory was developed for reactions among the same nuclei (e.g. carbon). Since we consider at least two elements (e.g. reactions among hydrogen and carbon), the coulombian potential must be suitably modified. We are going to present a template, to which more sophisticated formulations must be compared.

4.1 The modified Salpeter-Van Horn model

Using the same formalism of Shapiro & Teukolsky (1983) we consider an OCP made of a certain type of ions, e.g. either pure carbon or pure oxygen. In this framework, first we consider the one-dimensional potential of an array of ions and extend it to 3D. Suppose that the lattice is composed of ions of charge $Z_i$. In one cell we substitute the charge $Z_i$ with a charge $Z_J$ (for instance hydrogen or helium). If $x$ is the displacement from the equilibrium position (see Fig.1), the coulombian potential becomes

$$V(x) = \frac{Z_i Z_J e^2}{R_0 - x} + \frac{Z_J Z_K e^2}{R_0 + x} - \frac{2 Z_i^2 e^2}{R_0}$$

(33)

where $R_0$ is the inter-ionic distance. In eqn. (33) we assume that the energy of the lattice is zero and estimate the energy variation. Removing of a charge $Z_i$ produces the negative term (interaction of the nearest particles with the hole) and the addition of a charge $Z_J$ at position $x$ introduces the positive terms. We suppose that $x < < R_0$ and use the Taylor expansions to derive

$$V(x) = 2 \frac{Z_i Z_J e^2}{R_0} - 2 \frac{Z_i^2 e^2}{R_0} + 2 \frac{Z_i Z_J e^2 x^2}{R_0^3}.$$  

(34)
The constant term shifts all the energy levels. The variable term
\[ + \frac{2Z_1Z_2e^2x^2}{R_0^3} \]
is the dominant one and behaves like an harmonic oscillator with
\[ \omega = \left( \frac{4Z_1Z_2e^2}{R_0^3\hbar} \right)^{1/2}. \]
Using the equation \( E_0 = V(r_0) = V_0 + 1/2Kr_0^2 = V_0 + 1/2\hbar\omega \), the turning point \( r_0 \) is
\[ r_0 = \left( \frac{\hbar R_0^{3/2}}{2(2Z_1Z_2\hbar^{1/2})} \right)^{1/2}. \]
The solutions of the Schrödinger equation for an harmonic oscillator are known. In three dimensions they become
\[ |\psi_{\text{SHO}}|^2 = \frac{\pi^{3/2}}{\Gamma(3/2)} e^{-x^2/2} \]
where \( \tau = 1/r_0 \). Assuming that the exponential value is one,
\[ |\psi_{\text{inc}}|^2 \approx |\psi_{\text{SHO}}|^2 \approx \frac{1}{r_0^{3/2}}. \]

The transmission coefficient for an incident ion with energy \( E_0 \) in the WKB approximation is
\[ T = \exp\left[ -2 \int_a \left( \frac{2\hbar}{\mu}[E_0 - V_0 - 1/2Kx^2] \right)dx \right] \]
where \( E_0 = V_0 + 1/2\hbar\omega \). Following the discussion by Shapiro & Teukolsky (1988), we obtain the reaction rate per ion pair
\[ W = v|\psi_{\text{inc}}|^2 \frac{T}{S(E)} \]
where in the limit \( r_0/R_0 << 1 \)
\[ T = \frac{R_0}{r_0} \exp\left( -2 \frac{R_0^2}{r_0^2} \right). \]

This transmission coefficient is the leading term in the reaction rate and it strongly depends on the charge \( Z_2 \), in fact the exponent is proportional to \( (Z_1Z_2)^{1/2} \), which means that the transmission coefficient is much higher for \( Z_2 < Z_1 \).

We refer to our revision of the Shapiro & Teukolsky (1983) rate as the “modified harmonic oscillator” (MHO). We will later adopt the formulations of Kitamura (2000) and Yakovlev et al. (2006), but the MHO approximation is an efficient way to explore reactions between light and heavier elements when the WD reaches the pycno-nuclear regime. We note that the same formalism can be applied to the case of MCPs because what matters are the pairs of interacting ions, e.g. \(^1\rm{H}+^{12}\rm{C}\) or \(^1\rm{H}+^{16}\rm{O}\). The lower coulombian barrier in the case of \(^{12}\rm{C}\) and \(^1\rm{H}\) definitely favours this ion pair, so most of the discussion will be limited to these two elements.

### 4.2 Local densities around impurities

In order to evaluate the change in the local density and consequently in the rates of energy production by pycno-nuclear reactions caused by impurities, OCPs and MCPs must be treated separately. We can consider three cases:

1. **OCPs, neglecting electrons in the electrostatic force.** Let us consider an array of nuclei of charge \( Z_1 \) (e.g. carbon) as shown in the top panel of Fig. 3. If in one site we substitute the nucleus of charge \( Z_1 \) with a nucleus of charge \( Z_2 \), e.g. hydrogen (the impurity), the carbon nuclei neighbouring the hydrogen nucleus are shifted toward it because of the new equilibrium of the forces as indicated by the arrows (the situation is clearly symmetric). The equilibrium of the forces is expressed by
\[ \frac{Z_1Z_2}{(R_0 - x)^2} = \frac{Z_1^2}{(R_0 + x)^2}. \]

After developing the squares we obtain
\[ (Z_1Z_2 - Z_1^2)x^2 + 2R_0(Z_1Z_2 + Z_1^2)x + (Z_1Z_2 - Z_1^2)R_0^2 = 0. \]

This is a second order equation with two solutions
\[ x_{1,2} = R_0 \left[ \frac{(Z_1Z_2 + Z_1^2)}{(Z_1Z_2 - Z_1^2)} \pm \sqrt{\left(\frac{(Z_1Z_2 + Z_1^2)}{(Z_1Z_2 - Z_1^2)}\right)^2 - 1} \right]. \]

We evaluate the displacement for \( Z_1 = 6 \) (carbon) and \( Z_2 = 1 \) (hydrogen) and keep only the solution \( x < 1 \) that has physical meaning. We obtain \( R_0 = R_0 - x = 0.58 R_0 \), i.e. the new local inter-ion distance \( R_0 \) is smaller than in the unperturbed case. For the case \( Z_1 = 6 \) (carbon) and \( Z_2 = 2 \) (helium) obtaining \( R_0 - x = 0.73 R_0 \). As expected, the effect is decreasing with increasing \( Z_2 \). Hereinafter \( R_0' \) is renamed \( R_0 \).

2. **OCPs with electrons in the electrostatic force.** In the above expression we neglected the contribution by electrons to the balance of electrostatic forces among ions. Taking electrons into account, we assume that in the space between the ions \( Z_1 \) and \( Z_2 \) electrons distribute uniformly and evaluate the field due to electrons at any distance \( x \) in between the ions \( Z_1 \) and \( Z_2 \). In this case eqn. (45) becomes...
the new distance between two neighbouring ions is \( R \) from the thermal one, the functions (2006) rates, in which the pycno-nuclear term is separate (1969) are enhanced by a large factor.

and (40), the pycno-nuclear reaction of Salpeter & van Horn intrinsic in the formulation and cannot be singled out easily.

cated to treat and because the temperature dependence is limited however to a mixture of C + O (the BCC configuration) and only one real with physical meaning, i.e. \( x = 0.4402 \) \( R_0 \). Therefore, in the case of carbon and hydrogen the new distance between two neighbouring ions is \( R_0 = 0.56 \) \( R_0 \).

(3) MCPs with electrons in the electrostatic force. It is worth of interest to consider the case of MCPs, limited however to a mixture of C + O (the BCC configuration) (ichimaru1982) and including the effects of electrons in the force balance. We consider an array of ions in the sequence O (\( Z_1 \) - C (\( Z_2 \)) - H (\( Z_3 \)) - C(\( Z_1 \)) - O(\( Z_2 \)) as shown in the bottom panel of Fig. 5 and evaluate the displacement of the C nuclei bracketing the hydrogen nucleus by imposing the balance of electrostatic forces. In this case eqn. (42) becomes

\[
- \frac{Z_1 Z_2}{(R_0 - x)^2} + \frac{Z_2 Z_3}{(R_0 + x)^2} + \frac{1}{2} \frac{(Z_1 - Z_2) Z_1 x}{R_0^3} + \frac{1}{2} \frac{(Z_1 - Z_3) Z_1 (R_0 - x)}{R_0^3} = 0
\]

the associated fifth degree equation is

\[
- 10x^5 + 5R_0 x^4 + 4R_0^2 x^3 - \frac{38}{5} R_0^4 x + 3R_0^5 = 0
\]

to be solved numerically. The solution has four complex roots and only one real with physical meaning, i.e. \( x = 0.4402 \) \( R_0 \). Therefore, in the case of carbon and hydrogen the new distance between two neighbouring ions is \( R_0 = 0.56 \) \( R_0 \).

Implementing the correction into the reaction rates. The effect of this correction on the interionic distance in the pycno-nuclear rates depend on the physical model for the rates. If we insert the new value of \( R_0 \) into eqns. (40), the pycno-nuclear reaction of Salpeter & van Horn (1969) are enhanced by a large factor.

Using the Gasques et al. (2003), and Yakovlev et al. (2006) rates, in which the pycno-nuclear term is separate from the thermal one, the functions \( a_{ij} \) must be redefined. Finally, the reaction rate Kitamura (2001) is more complicated to treat and because the temperature dependence is intrinsic in the formulation and cannot be singled out easily.

(i) Gasques and Yakovlev rates. There are two approximations depending on the composition of the plasma. If the plasma is made of pure \( ^{12} \text{C} \) with \( ^{3} \text{H} \) impurities, the new \( a_{ij} \) are

\[
a_{ij} = \frac{1}{2} \left( \frac{3Z_i}{4\pi n_e} \right)^{1/3} \quad \text{(50)}
\]

\[
a_{ij} = \frac{1}{2} \left( \frac{3Z_i}{4\pi n_e} \right)^{1/3} \quad \text{(51)}
\]

In such a case the effect is less evident and smaller than before.

In the case of a MCP (for instance \( ^{12} \text{C} + ^{16} \text{O} \)), the new \( a_{ij} \) coefficients are

\[
a_{ij} = 0.5178 a_{CO} = 0.5178 \left( \frac{6^{1/3}}{2} + \frac{8^{1/3}}{2} \right) \left( \frac{3m_n}{4\pi 0.5\rho} \right)^{1/3}
\]

where \( a_{CO} \) is derived from eqn. (51) above.

(ii) Kitamura Rate. In this case, at low densities the \( a_{ij} \) are those of the classical formulation given by eqn. (51), whereas at high density they must coincide with those of...
Figure 4. $^{12}\text{C} + ^{12}\text{C}$ pycno-nuclear reaction rates for the simple MHO model (blue continuous line) compared with the Salpeter & van Horn (1969) model (red dashed line). They are roughly similar.

Figure 5. Top Panel: OCP made of carbon in which a nucleus of carbon ($Z_1$) is replaced by a nucleus of hydrogen ($Z_2$). We evaluate the forces acting on the nuclei $Z_1$ adjacent to the impurity (charge $Z_2$) and the corresponding displacement $x$ of $Z_1$ given by the new condition of force equilibrium. Bottom Panel: the same but for a MCP made of carbon and oxygen nuclei in BCC configuration in which a nucleus of oxygen is replaced by a nucleus of hydrogen.

eqn. [4.3]. In order to transit smoothly from the thermal to the pycno-nuclear regime and to take the correction of local densities due to impurities into account, we assume that the $a_{ij}$ coefficients can be linearly interpolated in $\log \rho$ according to

$$a_{ij} = a_{ij}^{ppc} + (a_{ij}^{th} - a_{ij}^{ppc}) \frac{10 - \log \rho}{10 - 6} \quad (53)$$

where $a_{ij}^{ppc}$ is defined according to eqn. [4.2] at $\log \rho = 10$, and $a_{ij}^{th}$ is defined according to eqn. [51] at $\log \rho = 6$.

4.3 Results from the Modified Harmonic Oscillator

In the MHO formalism derived above, the total number of reactions per second per cm$^3$ is given by

$$P_0 = n_{el} W \quad (54)$$

where $n_{el}$ is the number density of the reacting elements. This is related to the abundance by number $f_{32}$ by $n_{el} = f_{32} N/V$ where $N$ is the total number of ions in the star and $V$ is the volume. In the simple case of reactions among identical ions, converting number densities to mass densities, mass abundances, and number abundances is straightforward, whereas in the case of reactions among different elements the procedure is more complicated. If $N_j$, $X_j$ and $A_j$ are the number of ions of species $j$ with mass abundance $X_j$ and mass number $A_j$,

$$X_j = \frac{N_j A_j m_u}{\sum N_i A_i m_u} \quad (55)$$

where $m_u$ is the mass unit. If a WD, made of pure carbon, is polluted by traces of hydrogen and helium ($N_H << N$ and $N_{He} << N$) we obtain

$$X_H = \frac{f_{12}}{12} \quad \text{and} \quad X_{He} = \frac{f_{12}}{12} \frac{4 f_{He}}{3} \quad (56)$$

For a WD of carbon and oxygen, whose mass abundances are related as $X_C/X_O = \alpha$, we derive

$$X_H = \frac{f_{12} [\alpha + (3/4)]}{\alpha + 1} \quad X_{He} = \frac{f_{12} [\alpha + (3/4)]}{\alpha + 1} \quad (57)$$

for typical values of $\alpha$, $X_H$ is about a factor of 10 lower than $f_{12}$, and $X_{He}$ is a factor of 3 lower than $f_{He}$. Since the mass and structure of a WD depend on its central density, it is useful to express the reaction rates as a function of this parameter. In Fig. 3 we show the energy generation rates for the $^1\text{H} + ^{12}\text{C}$, $^4\text{He} + ^{12}\text{C}$ and $^{12}\text{C} + ^{12}\text{C}$ reactions. These are calculated with the MHO rates with and without the impurities and associated density enhancement. We also show the case of the $^{12}\text{C} + ^{12}\text{C}$ reaction calculated with the MHO. The astrophysical factors $S(0)$ and $Q$-values of the reactions $^1\text{H} + ^{12}\text{C}$, $^4\text{He} + ^{12}\text{C}$ are given in Table 5 those for the $^{12}\text{C} + ^{12}\text{C}$ are in Table 4. The abundances by number and by mass of the elements are $f_{12} = 10^{-20}$, $f_{He} = 10^{-6}$, and $f_{C} = 1$ (or equivalently $X_{12} = 10^{-21}$, $X_{He} = 3 \times 10^{-7}$, and $X_{C} = 1$). For the sake of comparison we show also the mean luminosity of a typical CO-WD, 10$^{34}$ erg s$^{-1}$, which is exceeded by the WD luminosity divided by the WD volume estimated assuming a mean radius of 5000 km.

The results of Fig. 3 are of paramount importance: they clearly show that in the case of hydrogen impurities (with $f_{12} = 10^{-20}$) the energy produced by the pycno-nuclear reactions at zero temperature may exceed the typical luminosity of a WD at a density $\rho \approx 3 \times 10^7$ g/cm$^3$, corresponding to a mass of about 1.05 $M_{\odot}$, significantly smaller than
5 CAN TRACES OF H AND HE BE PRESENT IN THE INTERIOR OF A WD?

The existence of traces of light elements in the core of CO-WDs is at the basis of our investigation. These traces remain inactive in the interior of a WD until the pyccno-nuclear regime is reached. Light elements are the best candidates to consider, because the very high coulombian barriers of high $Z$ elements would quench the penetration probabilities to zero.

Undoubtedly, hydrogen and helium on the WD surfaces in small but significant abundances are predicted theoretically ($10^{-9} \lesssim X_{\text{H}} \lesssim 10^{-5}$) according to Miller Bertolami et al. (2013) and Renedo et al. (2010) and are observed (Bergeron et al. 1990). How much hydrogen or helium can be present in the interior?

Following Kippenhahn & Weigert (1990), the rather high internal temperatures (from $10^8$ to $10^9$ K) of an aging WD set a limit to the possible hydrogen content in the interior. If hydrogen were present with a mass concentration $X_{\text{H}}$, we would expect H-burning via the pp-chain. For average values $T = 5 \times 10^8$ K, $\rho = 10^6$ g cm$^{-3}$, the energy released by the pp-chain $\epsilon_{\text{pp}} \simeq 5 \times 10^6 X_{\text{H}}^2$ erg g$^{-1}$ s$^{-1}$ to which for a $M = M_\odot$ the luminosity would be $L_{\text{H}} \approx M/M_\odot \times \epsilon_{\text{pp}} \approx 2.5 \times 10^4 X_{\text{H}}^2$. The mean observed luminosity of WD $L_{\text{H}} / M_{\odot} \approx 10^{-3}$ allows $X_{\text{H}} \lesssim 2 \times 10^{-4}$, the upper limit for $X_{\text{H}}$ in WD interiors. Stability considerations indeed rule out that the luminosity of normal WD is generated by thermonuclear reactions, which was first pointed out by Mestel (1952a). Nuclear burning could only be expected in nearly cold configurations near $T = 0$ by pyccno-nuclear reactions.

According to Kawaler (1988) sedimentation and diffusion of elements will bring hydrogen ad helium from the interior to surface, although this is a slow process, the initial content of hydrogen can be as high as $10^{-6}$ $M_\odot$.

We argue and try to demonstrate here that traces of light elements (well below the upper limits given above) left over by core nuclear burnings during the previous phases can still be present when the WD is cooling. This would be the analog of the situation in neutron stars, traces of electrons, protons and even heavy nuclei are expected to exist in the medium of free neutrons (Shapiro & Teukolsky 1983). We show that hydrogen abundances in the range $10^{-15} \lesssim X_{\text{H}} \lesssim 10^{-16}$ may be present and cause ignition of pyccno-nuclear reactions. Analogous considerations can be made for helium.

5.1 Content of hydrogen and helium at the end of the AGB phase

We try here a simple numerical experiment to estimate the initial abundances of light elements in the WD interiors. We examine the $3M_\odot$ evolutionary sequence with solar-like composition [X=0.723, Y=0.260, Z=0.017] from the Padova library of stellar models by Bertelli et al. (2009). For this sequence we know all relevant physical quantities as a function of time both in the centre (temperature, density, abundances of elements, size of the convective and H-exhausted core, etc.) and at the surface (total and partial luminosities, effective temperature, etc.). For this star, central H- and He-burning (CNO and 3$\alpha$, respectively) are point-like sources at the centre, so that using simple energy conservation arguments we can write

$$\epsilon_i (X_i, T, \rho) = \frac{L_i}{M}$$  \hspace{1cm} (58)

the index $i$ stands for H- or He-burning, and $L_i$ are the partial luminosities (the total luminosity is $L = L_H + L_{\text{He}} + L_G$, where $L_G$ is the contribution from gravitational contraction, usually negligible). This equation simplifies the complexity of energy production (nuclear burning and gravitational release in convective/radiative conditions, the latter depending on the stellar mass) to an ideal situation in which a gram of matter produces the amount of energy radiated per unit mass and unit time. Using the above equation we implicitly neglect the contribution of the gravitational contraction/expansion to the luminosity. For most of the stellar lifetime this is a reasonable approximation.

For the sake of simplicity, we consider only the nuclear burning occurring in the core and we ignore the nuclear burning in the shell (this implies slightly overestimating the abundances). For a $3M_\odot$ star hydrogen burns via the CNO-cycle with a small contribution from the pp-chain, and helium burns via the 3$\alpha$ process. Finally, in first approximation we consider nuclear burning only in radiative conditions, neglecting continuous refueling by convection which may occur in a $3M_\odot$ star.

For the energy release by H- and He-burning we adopt analytical expressions from classical textbooks (Kippenhahn & Weigert 1990, Weiss et al. 2004):

(i) pp-chain,

$$\epsilon_{\text{pp}} = 2.06 \times 10^6 f_{\text{pp}} g_{\text{pp}} R_X T_6^{-0.66} e^{-0.33 s T_6^{0.33}}$$  \hspace{1cm} (59)

$$g_{\text{pp}} = 1 + 0.0012 T_6^{0.33} + 0.0078 T_6^{-0.66} + 0.0006 T_6$$

$$f_{\text{pp}} = 1 + 0.25 \rho^{0.5} T_6^{-1.5}$$
Figure 6. **Left Panel:** energy rates per unit volume produced by the reaction $^{12}\text{C}(^{1}\text{H}, \gamma)^{13}\text{N}$ with hydrogen abundance $X_{\text{H}} = 10^{-20}$ as a function of density and four different temperatures of the thermal branch. The nuclear rates are from Kitamura (2000). The dotted lines represent the energy without local density correction while the continuous lines represent the model that considers local density correction. **Right panel:** the same as in the left panel but for reaction rates according to Yakovlev et al. (2006). The dotted lines represent the energy without local density correction while the continuous lines represent the model that considers local density correction.

(ii) CNO-cycle

$$\epsilon_{\text{CNO}} = 8.7 \times 10^{27} g_{14.1} X_{\text{CNO}} X_{\text{H}} \rho T_{6}^{-2/3} e^{-152.28/T_{6}^{1/3}}$$

(60)

$$g_{14.1} = 1 + 0.003 T_{6}^{1/3} - 0.0078 T_{6}^{2/3} - 0.0015 T_{6}$$

(iii) and $3\alpha$

$$\epsilon_{3\alpha} = 5.09 \times 10^{11} f_{3\alpha} X_{\text{He}}^{3} \rho^{2} T_{8}^{-3} e^{-44.027/T_{8}}$$

(61)

$$f_{3\alpha} = e/((2.4 \times 10^{-3}) \rho^{1/2} / T_{8}^{3/2})$$

where $T_{6}$ and $T_{8}$ are the temperatures in units of $10^{6}$ and $10^{8}$ K, respectively, the functions $g_{14.1}$ and $g_{3\alpha}$ are the screening factors, $X_{\text{H}}, X_{\text{He}}, X_{\text{CNO}}$ are the abundances by mass of hydrogen, helium and CNO group, respectively. Finally $\epsilon$, and $\rho$ are in cgs units, erg s$^{-1}$ and g cm$^{-3}$, respectively.

In principle the pp-chain and CNO-cycle can occur simultaneously, however with significantly different efficiency. Isolating the dependence on $X_{\text{H}}$ in eqns. (60) and (61), we can write

$$\epsilon_{\text{pp}} + \epsilon_{\text{cno}} \approx \epsilon_{\text{pp}}^{o} X_{\text{H}}^{2} + \epsilon_{\text{cno}}^{o} X_{\text{H}} = L_{\text{H}} / M$$

(62)

where the quantities with superscript $o$ are of obvious definition and are immediately known from eqns. (60) and (61), and $L_{\text{H}}$ is contribution to the total luminosity by H-burning. This quantity is known from the Padova library of stellar models. Eqn. (62) is a second order algebraic equation to be solved for $X_{\text{H}}$ as a function time and the temperature of the mass element (typically at the center). The following recursive relation can be adopted

$$X_{\text{H}}^{\text{new}} = \mathfrak{S}[L/M, \epsilon_{\text{pp}}(X_{\text{H}}^{\text{old}}), \epsilon_{\text{cno}}(X_{\text{H}}^{\text{old}})]$$

(63)

where $X_{\text{H}}^{\text{old}}$ is the hydrogen abundance in the previous time step, and $\mathfrak{S}$ stands for a suitable function that can be easily calculated. Similar reasoning can be followed to derive the helium abundance. We recast eqn. (62) as

$$\epsilon_{3\alpha} = \epsilon_{3\alpha}^{o} X_{\text{He}} = L_{\text{He}} / M$$

(64)

together with

$$X_{\text{He}}^{\text{new}} = \mathfrak{S}[L_{\text{He}} / M, \epsilon_{3\alpha}(X_{\text{He}}^{\text{old}})]$$

(65)

where $X_{\text{He}}^{\text{old}}$ is the helium abundance in the previous time step. All other symbols have the same meaning as in previous case. With this procedure we estimate that the abundances

Figure 7. We compare the rates of the pycno-nuclear reactions $^{12}\text{C}(^{1}\text{H}, \gamma)^{13}\text{N}$ (red lines) and $^{12}\text{C}(^{12}\text{C}, \gamma)^{24}\text{Mg}$ (blue lines) according to the rates by Yakovlev et al. (2006) and different values of $X_{\text{H}}$ as indicated.
of hydrogen and helium at the end of the AGB phase are $X_\text{H} = 10^{-16}$ and $X_\text{He} = 10^{-6}$.

5.2 Decrease of $X_\text{H}$ and $X_\text{He}$ during the WD cooling sequence

In order to follow the hydrogen and helium consumption by nuclear burnings during the early stages of WD cooling, we used the WD evolutionary sequences of Renedo et al. (2010). At the beginning of the WD cooling sequence the luminosity is sustained by CNO burning on the surface. We want to assess whether part of the luminosity may still be due to minor nuclear burning in the interior (if it can occur at all), thus possibly extinguish the inner content of hydrogen. However, using the same recursive procedure above, we estimate that the hydrogen and helium abundances in our test model of $M_\odot$ only decrease to about $X_\text{H} = 10^{-20}$ and $X_\text{He} = 10^{-9}$.

To conclude, this simple experiment suggests that traces of hydrogen and helium can be left over in the interiors of a WD at the end of the whole nuclear history via the thermal channels. However, because our estimates are very approximate, we consider $X_\text{H}$ and $X_\text{He}$ free parameters within the ranges we derived above.

6 RESULTS FOR THE KITAMURA AND YAKOVLEV REACTION RATES

The Kitamura (2000), Gasques et al. (2005) and Yakovlev et al. (2006) formalisms (see also Beard 2010) were originally tailored for reactions like $^{12}\text{C} + ^{12}\text{C}$, $^{12}\text{C} + ^{16}\text{O}$, $^{16}\text{O} + ^{16}\text{O}$. However, they can be extended to reactions involving impurities of light elements, and to incorporate the enhancement in the local density. We have used the reaction rates of Kitamura (2000) to study the reactions $^1\text{H} + ^{12}\text{C}$ and/or $^3\text{He} + ^{12}\text{C}$. We will not examine here the similar reactions occurring with oxygen because of the higher atomic number $Z$. The relative $S$ factors and $Q$ values of the two reactions are listed in Table 5. The abundance of carbon in these test calculations is $X_\text{C} \approx 1$. The results are more general than those of the Salpeter & van Horn (1969) rates because they take also into account the effects of the temperature, extending from the thermal to the pycno-nuclear regime. However, our results match those obtained with the Salpeter & van Horn (1969) rates in the low temperatures and high densities regime. Results for the $^{12}\text{C}(...)$ reaction with $X_\text{H} = 10^{-20}$ are presented in left panel of Fig. 6 in which the generated energy is plotted as a function of central density. The same procedure has been applied to the Yakovlev et al. (2006) formalism. The results for the same reaction and hydrogen abundance are shown in the right panel of Fig. 6. In both cases, four different temperatures have been considered.

In both panels the thermal branches are visible only for the $10^7$ and $10^8$ K temperatures. All curves merge together beyond a certain value of the density that depends on the temperature and source of the rates. The two groups of rates significantly differ both in the thermal and lower in the pycno-nuclear regime compared to those Yakovlev et al. (2006). The rates are enhanced by the increase in the local density. This dependence is shown by the results in Fig. 7 which displays the $^1\text{H} + ^{12}\text{C}$ and $^{12}\text{C} + ^{12}\text{C}$ reaction rates, limited to the case of Yakovlev et al. (2006).
The Coulomb coupling parameter \( \Gamma_{ij} = \frac{e^2 Z_1 Z_2}{a_{ik} k_B T} \) as function of age for the two extreme values of the WD masses under consideration. The underlying nuclear reaction is \(^1\text{H} + ^{12}\text{C}\). The enhancement in the local density induced by the contaminant hydrogen is included (see eqn. 48 and the solution \( R'_0 = 0.5178 R_0 \)). The blue lines show the 0.6\( M_\odot \) WD with \( X_\text{H} = 10^{-15} \), whereas the red lines are the same but for the 1.2\( M_\odot \) WD with \( X_\text{H} = 10^{-24} \). All other combinations of mass and \( X_\text{H} \) abundances fall in between. The corresponding Coulomb coupling parameter with no enhancement in the local density are obtained by scaling the values on display by the factor \( 1/0.5178 \).

The pycno-nuclear energy rates based on the Yakovlev et al. (2006) model for an abundance concentration of hydrogen \( X_\text{H} = 10^{-24} \) (blue continuous line) compared to the luminosity of the WD (red dotted line) according to the models of Althaus et al. (2005) for a 1.2\( M_\odot \) WD. The magenta line represents the energy of the ions. The luminosity is in the same units as the energy generation rates (\( \text{erg} \text{cm}^{-3} \text{s}^{-1} \)).

WDs of given masses follow cooling sequences along which the luminosity varies with time. Therefore we relax now the simplifying assumption of a constant mean luminosity by using the evolutionary sequences of WDs of different mass and compare them with the energy released by nuclear burning. The cooling sequence in the luminosity vs effective temperature plane is determined by the central density and mean molecular weight of the electrons (i.e. the chemical composition of the WD). Therefore the really meaningful comparison is between the energy production of the WD governed by its temperature and density (the central values) and its current luminosity.

For the sake of illustration, let us consider the evolutionary sequence of a CO-WD of 1.2\( M_\odot \) (the border value above which the mean density is high enough to enter the General Relativity domain) calculated by Althaus et al. (2005). All necessary information about central temperature, central density and luminosity, etc is available. The composition of the WD is \( X_\text{C} \approx 0.5 \) and \( X_\text{O} \approx 0.5 \). We are particularly interested in determining when the energy rates given by Kitamura (2000) or by Yakovlev et al. (2006), modified for the effects of contaminants on the local densities, exceed the luminosity. Fig. 8 shows the variation of the WD luminosity (the thick red curve) as a function of time and the energy generated by the reaction \(^{12}\text{C}(^1\text{H},\gamma)^{13}\text{N}\) for differ-
exploring an alternative channel of evolution towards SNa Ia Explosion

7.1 Adding up all sources of energy: critical X_H abundances and ages

We now include all energy sources, i.e. the generation by any of the five possible regimes for the pycno-nuclear reactions (including also the so-called thermally enhanced terms in the early stages of WD cooling sequences) and the thermal energy of the ions.

The proto-type WD of 1.2M☉. Following Yakovlev et al. (2006) we consider only the 12C(H,γ)13N reaction, activated by the traces of hydrogen left over from the previous phases and still present in the WD. This is one of the three starting reactions that ignite the CNO cycle, which however cannot be completed, owing to the extremely low abundances of the intermediate elements. This reaction has a Q-value is 1.94 Mev compared to the 25.02 Mev when the cycle is completed, releasing a factor of 12.5 less energy. The nuclear energy release, the ion internal energy, and the comparison luminosity are all expressed per unit volume of the WD. The volume is calculated from the luminosity and effective temperature of the WD along its cooling sequence. All quantities are provided by Althaus et al. (2005) and Renedo et al. (2010).

The temporal evolution of the WD luminosity (red dotted line), nuclear energy release (blue solid line) and thermal energy of the ions (magenta dashed line) are shown in Fig. Since the ion internal energy is derived here from simple expressions, neglecting the effect of degenerate electrons and the variation of the degrees of freedom in the crystal lattice at very low temperatures, we do not extend it to very high densities or low temperatures, characteristic of very old ages. First of all, we notice that the abundance X_H of the residual hydrogen cannot exceed 10^-21, otherwise the nuclear energy release by the sole 12C(H,γ)13N reaction during the initial stages of WD cooling is comparable to or even exceeds the total luminosity, thus bringing the WD to a risky regime at which a nuclear runaway may start. See also (Renedo et al. 2010) for the energy production by the surface CNO during the same phase. We will see below that the upper limit to X_H to avoid early nuclear runaway changes with the WD mass, it increases at decreasing mass. Incidentally, this constraint may be a way of estimating and calibrating the maximum content of residual hydrogen in a CO-WD.

As cooling proceeds, the energy generation by nuclear reactions in the thermally-enhanced regime decreases and the light contaminants do not have any effect until the pure pycno-nuclear channel begins. In the case of the 1.2M☉ star with the hydrogen abundance of X_H = 10^-24 , the nuclear energy production exceeds the luminosity at the age of about 2.5 × 10^9 as shown in Fig. X_H greater than the above value would lead the WD to intersect the luminosity curve during the thermally enhanced phase, i.e. soon after (or shortly later) the formation of the WD itself (a possibility that cannot be firmly excluded and is temporarily left aside, see below).

Exploring the solution space. Since the abundances X_H cannot be assessed a priori, it is safe to consider them as free parameters whose values fall within a plausible range.
Figure 12. Energy rates per unit volume produced by hydrogen impurities with different abundances of $X_H$ as indicated for WD of different mass. The reaction on display is the $^{12}\text{C}(^1\text{H},\gamma)^{13}\text{N}$ according to Yakovlev et al. (2006). The luminosity is in the same units as the energy generation.

Table 6. Logarithm of the age in yr at which the energy generation by the pycno-nuclear reaction $^{12}\text{C}(^1\text{H},\gamma)^{13}\text{N}$ in erg cm$^{-3}$ s$^{-1}$ matches the WD luminosity in erg cm$^{-3}$ s$^{-1}$ likely starting the nuclear runaway in the star for different $M_{WD}$ in $M_\odot$ and hydrogen abundances $X_H$. The luminosities and radii (volumes) of the WDs are from the cooling sequences of Althaus et al. (2005).

| $M_{WD}$ | $\log X_H$ |
|---------|------------|
| 0.6     | >10.0      |
| 0.8     | 6.5 9.9 10.1 |
| 0.9     | 6.5 9.6 9.8 10.0 |
| 1.0     | 6.5 8.8 9.4 9.8 9.9 |
| 1.1     | 7.7 8.5 9.2 9.6 9.8 |
| 1.2     | 6.6 8.0 8.8 9.2 9.5 |

As expected, at given $M_{WD}$ the intersection occurs later with decreasing $X_H$. For values lower than those indicated, the intersection occurs at lower and lower abundances and younger ages with increasing WD mass. In fact, the luminosity per unit volume (evaluated at the typical age of $10^9$ yr) increases by a factor ten from a 0.8 to a 1.2 $M_\odot$ WD. Also the central and mean density of a WD increase with the mass, and so does the efficiency of the pycno-nuclear reactions. Consequently, the range of abundance for which the intersection may occur within the age of the Universe systematically decrease with increasing WD mass: $X_H \simeq 10^{-17}$ for the 0.8$M_\odot$ star $X_H \simeq 10^{-23}$ for the 1.2$M_\odot$.

Finally, we note that the systematic decrease of the "permitted" values of $X_H$ at increasing WD mass is compatible with the past thermal history of the progenitor star. In fact, the mean temperature in a low mass progenitor is lower than the mean temperature in a more massive one, clearly having effects on nuclear burning. For this reason it is tempting to suggest that $X_H$ decreases with the WD mass.

Concluding this section, we remind the reader that the rates of energy generation as function of age have been calculated at constant $X_H$ and $X_C$. In reality, $X_H$ (and $X_C$) should decrease with time. Therefore a real WD should move along a path in the panel corresponding to its mass gradually shifting to lines of decreasing $X_H$. Consequently, the age at which the liberated energy would equal the luminosity cannot be exactly determined without the aid of real models of WD stars including the energy generated by the reactions between contaminants and the carbon (oxygen) nuclei. The ages reported in Table 6 are merely indicative of the expected trend.

7.2 To be, or not to be: that is the question

(William Shakespeare, Hamlet 3/1)

Adapting Hamlet’s famous soliloquy to our contest, the question is whether or not WDs containing traces of light elements (hydrogen in the case we have considered) during their cooling sequence may produce enough nuclear energy via the $^1\text{H}+^{12}\text{C}$ to balance the luminosity, thus likely initiating a nuclear runaway followed by explosion. Furthermore, is this critical stage reached only during the pure pycno-
nuclear regime or does it occur also in earlier stages? Is there any trend with the mass and the abundance of contaminant elements?

To summarize, the main results of our analysis are:

(i) The above reaction always occurs in physical conditions with $\Gamma_{ij} \geq 1$, transiting from the third (thermally-enhanced) to the fifth regime (pure pycno) of burning (see Section 5.3).

(ii) In low mass WDs, the third regime is initially more efficient than the fifth regime and the latter overwhelms the former only past a certain age, whereas in high mass WDs the fifth regime always prevails. This trend is due to the central and mean density increasing with the WD mass.

(iii) Depending on $X_H$ and $M_{WD}$, the sum of nuclear energy and ion internal energy can equal the luminosity at any age along the cooling sequence.

(iv) Massive WDs are more sensitive to the instability threshold that can be reached even with very small hydrogen abundances. Therefore the reaction $^1H + ^{12}C$ is easily ignited and a nuclear runaway can be started as soon as massive WDs are born. For a typical value of $X_H \approx 10^{-20}$, the above situation should occur for $M_{WD} \geq 1.0$. Although WDs of this mass are rare, the possibility cannot be discarded and it may be possible to identify such progenitors.

(v) WDs of lower mass are likely more stable and should encounter the threshold condition only after a certain amount of time has elapsed since their formation.

(vi) A possible correlation between the WD mass (and its progenitor) and $X_H$ (or $X_{He}$) would translate into a correlation be established between the WD mass and the time of explosion. However, the slope of this relationship which cannot be easily assessed at the present time. We prefer to consider this issue open to discussion and future investigation.

(vii) There are implications also for the final fate of WDs with mass and abundances of light contaminants $X_H$ too low for a prompt explosion in the thermally-enhanced regime, but sufficient to survive the whole cooling period and finally reach the pure pycno-nuclear regime (a time scale of several Gyr, for low masses nearly the Hubble time). It is likely that with even very small values of $X_H$, isolated WDs with mass much smaller than the Chandrasekhar limit may ignite the $^1H + ^{12}C$ reaction. If this occurs, we expect the explosive disruption of the vast majority of WDs before they cool down to the very low temperatures of the so-called undetectability stage.

(viii) Finally, the onset of nuclear runaway should be a consequence of the positive gravothermal specific heat of a mixture of nuclei and electrons whose equation of state is basically driven by the highly degenerate electrons, see the discussion by Rippenhahn & Weigert (1990) and Mestel (1952b, c).

To conclude, there is an ample range of possibilities for WDs contaminated by the presence of traces of light elements in their interiors to reach the critical situation at which nuclear burning is activated, likely followed by nuclear runaway and consequent disruption of the star.

8 CONCLUSIONS

In this paper we investigated the possible effects of impurities on the pycno-nuclear reaction rates in CO-WDs. We reviewed the present-day pycno-nuclear reaction rates, highlighting the peculiarities of each model and the use that has been done until now. We introduced two important modifications in the existing expressions:

i) We extended the Salpeter & van Horn (1969) and Shapiro & Teukolsky (1983) reaction rate, to calculate the variation of the coulombian potential induced by the presence of a lighter ion of hydrogen or helium in an arbitrary node of the C-O ion lattice. Thus we extended the original formulation for pycno-nuclear reaction rates conceived for OCPs to BIMs, including the reactions $^3H + ^{12}C$ and/or $^4He + ^{12}C$.

ii) We evaluated the displacement of nearby nuclei produced by the presence of impurities of different charge and estimated the change in local density, and applied this to the MHO, Kitamura (2004), and Yakovlev et al. (2006) expressions.

Using the above revision of the the theoretical rates, we made a preliminary analysis using the MHO formalism to study the pycno-nuclear reactions between light nuclei (hydrogen or helium) present in extremely low abundances and heavier nuclei like carbon and oxygen, the basic constituents of WDs.

Encouraged by the results obtained with the MHO rates, we used the more sophisticated descriptions of nuclear reactions in high density environments by Kitamura (2004) and Yakovlev et al. (2006) to explore the effects of impurities of light elements in triggering the above reactions.

The main results of the present study suggest that:

- The presence of hydrogen even in extremely low concentrations (from $10^{-16}$ to $10^{-25}$) can raise the pycno-nuclear reaction rates in density intervals from $10^7$ to $10^8$ g cm$^{-3}$. The same is true for helium at somewhat higher threshold densities.

- In the case of hydrogen, the above density interval corresponds to WD masses from 0.8 to 1.2 $M_{\odot}$, well below the known limit of the Chandrasekhar mass.

- In WDs in this mass range, the energy released by pycno-nuclear reactions may exceed the stellar luminosity at ages from 1 to 6 Gyr after their formation, depending on the WD mass.

- Even WDs with masses as low as 0.8 $M_{\odot}$ may experience pycno-nuclear runaway.

Our results could in principle radically change not only the current understanding of the structure and evolution of WDs but also imply that single WDs may be progenitors of type Ia SNe. We may have discovered an alternative channel to SNa Ia explosions. In this way we may be able to explain the star formation rate dependence of the SNa Ia rate (e.g., Mannucci et al. 2006) and also provide some clues to interpreting the observational data on the ejected mass distribution of type Ia SNe showing a significant rate of non-Chandrasekhar-mass progenitors of mass as low as 0.8 $M_{\odot}$ (Scalzo et al. 2014). Therefore, before proceeding further it is mandatory to remind the reader of the limitations of the present approach.

- Even if our computations rely on state-of-the-art WD
models, the correct approach would be to calculate and follow in time stellar models responding to the new source of energy in the course of evolution. The present models, although acceptable for very low nuclear rates, fail to represent the real physical structure of a WD in presence of large energy production. The WD structure may be deeply altered and follow a different evolutionary history that is not easy to foresee at present.

- Our calculations do not take into account yet the additional energy release due to elements stratification, solid state transition (latent heat) and gravitational contraction.

- Only complete, self-consistent models would allow us to correctly determine the amount of energy generated by nuclear reactions, and to deal with the energy transport problem, rigorously comparing production versus transport of energy.

Future work should be the computation of a self-consistent model able to respond to changing physical conditions, indicating the exact age at which the under-barrier reactions become important and the structural response a WD to the novel energy input. After the WD cools down to the temperature for the activation of the under-barrier channel in the liquid/solid phase, we foresee three possible scenarios: (a) reheating with consequent increase of the cooling lifetime; (b) rejuvenation to another type of object; (c) explosion as type Ia SNe (or a different type?).

We urge reconsidering the whole subject of nuclear reactions in these extreme conditions, in particular in presence of impurities that could deeply change our current understanding of the energy sources in WDs.

If the results of our exploratory project are confirmed by further investigation, important implications for the currently accepted scenario for type Ia SNe will follow. The binary origin of type Ia SNe explosion would be no longer strictly necessary. Isolated WDs with masses well below the Chandrasekhar limit may reach the threshold for pycnonuclear burning and consequent SNe explosion due to the survival of traces of light elements. These impurities may remain inactive for long periods of time and be activated only when the WDs reach the liquid-solid regime. Owing to the large range of WD masses that could be affected by the presence of impurities and undergo thermal runaway and consequent SNe explosion, the nature of standard candle so far attributed to type Ia SNe may not be true. Because of the far reaching implications, the whole subject deserves careful future investigation.

ACKNOWLEDGEMENTS

We would like to deeply thank Dr. Maurizio Salaris for his friendship, for the many illuminating discussions, and for providing us his cooling sequences and WD models. We like to thank Drs. S. Ichimaru and H. Kitamura for patiently replying to the numberless emails sent by PT and the many very useful explanations and comments.

REFERENCES

Althaus L. G., Benvenuto O. G., 1997, ApJ, 477, 313
Althaus L. G., Benvenuto O. G., 1998, MNRAS, 296, 206
Althaus L. G., García-Berro E., Isern J., Córsico A. H., Miller Bertolami M. M., 2012, A&A, 537, A33
Althaus L. G., Miller Bertolami M. M., Córsico A. H., 2013, A&A, 557, A19
Althaus L. G., Panei J. A., Miller Bertolami M. M., García-Berro E., Córsico A. H., Romero A. D., Kepler S. O., Rohrmann R. D., 2009, ApJ, 704, 1605
Althaus L. G., Serenelli A. M., Panei J. A., Córsico A. H., García-Berro E., Scóccola C. G., 2005, A&A, 435, 631
Beard M. L., 2010, PhD thesis, University of Notre Dame
Bergeron P., Wesenael F., Fontaine G., Liebert J., 1990, ApJL, 351, L21
Bertelli G., Girardi L., Marigo P., Nasi E., 2008, A&A, 484, 815
Bertelli G., Nasi E., Girardi L., Marigo P., 2009, A&A, 508, 355
Brown L. S., Sawyer R. F., 1997, Reviews of Modern Physics, 69, 411
Catalán S., Isern J., García-Berro E., Ribas I., 2008, MNRAS, 387, 1693
Chandrasekhar S., 1939, An introduction to the study of stellar structure
Chiosi C., Bertelli G., Bressan A., 1992, ARA&A, 30, 235
Dewitt H., Slattery W., Baiko D., Yakovlev D., 2001, Contributions to Plasma Physics, 41, 251
DeWitt H. E., Slattery W. L., Yang J., 1992, in International Conference on the Physics of Strongly Coupled Plasmas, Rochester, NY, 17-21 Aug. 1992 Monte Carlo simulation of the OCP freezing transition, pp 17–21
Eisenstein D. J., Liebert J., Harris H. C., Kleinman S. J., Nitta A., Silvestri N., Anderson S. A., Barentine J. C., Breizhilton J. H., Brinkmann J., Harvanek M., Krzesinski J., Neilsen Jr. E. H., Long D., Schneider D. P., Snedden S. A., 2006, ApJS, 167, 40
Fowler W. A., Caughlan G. R., Zimmerman B. A., 1975, ARAA, 13, 69
Fujimoto M. Y., 1982a, ApJ, 257, 767
Fujimoto M. Y., 1982b, ApJ, 257, 752
Gasques L. R., Afanasjev A. V., Aguilera E. F., Beard M., Chamon L. C., Ring P., Wiescher M., Yakovlev D. G., 2005, PhRvC, 72, 025806
Iben Jr. I., Renzini A., 1983, ARA&A, 21, 271
Ichimaru S., 1982, Reviews of Modern Physics, 54, 1017
Ichimaru S., Kitamura H., 1999a, Physics of Plasmas, 6, 2649
Ichimaru S., Kitamura H., 1999b, Physics of Plasmas, 6, 2649
Ichimaru S., Ogata S., van Horn H. M., 1992, ApJL, 401, L35
Kawaler S. D., 1988, ApJ, 334, 220
Kepler S. O., Kleinman S. J., Nitta A., Koester D., Castanheira B. G., Giovannini O., Althaus L., 2007, in Napowotski R., Burleigh M. R., eds, 15th European Workshop on White Dwarfs Vol. 372 of Astronomical Society of the Pacific Conference Series, The White Dwarf Mass Distribution. p. 35
Kippenhahn R., Weigert A., 1990, Stellar Structure and Evolution
Kitamura H., 2000, ApJ, 539, 888
Kitamura H., Ichimaru S., 1995, ApJ, 438, 300
Lindemann F. A., 1910, Z. Phys., 11, 609
Mannucci F., Della Valle M., Panagia N., 2006, MNRAS,
Exploring an Alternative Channel of Evolution Towards SNe Ia Explosion

Marigo P., 2001, A&A, 370, 194
Mestel L., 1952a, MNRAS, 112, 583
Mestel L., 1952b, MNRAS, 112, 598
Miller Bertolami M. M., Althaus L. G., García-Berro E., 2013, ApJL, 775, L22
Napiwotzki R., Christlieb N., Drechsel H., Hagen H.-J., Heber U., Homeier D., Karl C., Koester D., Leibundgut B., Marsh T. R., Moehler S., Nelemans G., Pauli E.-M., Reimers D., Renzini A., Yungelson L., 2003, The Messenger, 112, 25
Nomoto K., Kobayashi C., Tominaga N., 2013, ARA&A, 51, 457
Ogata S., Iyetomi H., Ichimaru S., 1991, ApJ, 372, 259
Orio M., 2013, The Astronomical Review, 8, 010000
Panei J. A., Althaus L. G., Chen X., Han Z., 2007, MNRAS, 382, 779
Renedo I., Althaus L. G., Miller-Bertolami M. M., Romero A. D., Córscico A. H., Rohrmann R. D., García-Berro E., 2010, ApJ, 717, 183
Salaris M., Althaus L. G., García-Berro E., 2013, A&A, 555, A96
Salaris M., Cassisi S., Pietrinferni A., Kowalski P. M., Isern J., 2010, ApJ, 716, 1241
Salpeter E. E., van Horn H. M., 1969, ApJ, 155, 183
Scalzo R. A., Ruiter A. J., Sim S. A., 2014, ArXiv e-prints
Schramm S., Koonin S. E., 1990, ApJ, 365, 296
Shapiro S. L., Teukolsky S. A., 1983, Black holes, white dwarfs, and neutron stars: The physics of compact objects
Spergel D. N., Verde L., Peiris H. V., Komatsu E., Nolta M. R., Bennett C. L., Halpern M., Hinshaw G., Jarosik N., Kogut A., Limon M., Meyer S. S., Page L., Tucker G. S., Weiland J. L., Wollack E., Wright E. L., 2003, ApJS, 148, 175
Trimble V., Aschwanden M. J., 2004, PASP, 116, 187
van Horn H. M., 1968, ApJ, 151, 227
Weidemann V., 1967, Z. Astrophys., 67, 286
Weidemann V., 1977, A&A, 59, 411
Weidemann V., 1990, ARA&A, 28, 103
Weidemann V., 2000, A&A, 363, 647
Weiss A., Hillebrandt W., Thomas H.-C., Ritter H., 2004, Cox and Giuli's Principles of Stellar Structure
Yakovlev D. G., Gasques L. R., Afanasjev A. V., Beard M., Wiescher M., 2006, PhRvC, 74, 035803