Weak localization and conductance fluctuations of a chaotic quantum dot with tunable spin-orbit coupling

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In a two-dimensional quantum dot in a GaAs heterostructure, the spin-orbit scattering rate is substantially reduced below the rate in a bulk two-dimensional electron gas [B.I. Halperin et al, Phys. Rev. Lett. 86, 2106 (2001)]. Such a reduction can be undone if the spin-orbit coupling parameters acquire a spatial dependence, which can be achieved, e.g., by a metal gate covering only a part of the quantum dot. We calculate the effect of such spatially non-uniform spin-orbit scattering on the weak localization correction and the universal conductance fluctuations of a chaotic quantum dot coupled to electron reservoirs by ballistic point contacts, in the presence of a magnetic field parallel to the plane of the quantum dot.

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In disordered metals, interference of time-reversed trajectories leads to a small negative correction to the conductivity, known as weak localization. With strong spin-orbit scattering, the effect of such interference is opposite, causing weak antilocalization, a positive correction to the conductivity. Both interference corrections are suppressed when time-reversal symmetry is broken by a magnetic field. The same phenomena are observed in a two-dimensional electron gas, such as is formed in GaAs heterostructures. While spin-orbit scattering in metals is largely due to scattering from the metal ions or from impurities, in a GaAs heterostructure, spin-orbit effects mainly arise from the asymmetry of the potential creating the quantum well (Rashba term), as well as from the lack of inversion symmetry which may occur in the crystal structure of the material forming the heterostructure (Dresselhaus term).

Recently, it has become possible to study spin-orbit scattering in finite size systems, such as metal grains and semiconductor quantum dots. In the universal regime, where all relevant time scales (spin-orbit time $\tau_{so}$, inverse level spacing/broadening) are much larger than the electronic time scales, the spin-orbit scattering term is not symplectic, but unitary. As this “vector potential” has no spatial dependence, and hence no “flux”, a suitable gauge transformation removes the spin-orbit scattering term from the Hamiltonian up to corrections of order $L/\lambda_{x,y}$ which arise due to the non-Abelian nature of the “vector potential” of Eq. (1). As a result, the spin-orbit scattering time is increased by a large factor, $\sim \lambda_{x}\lambda_{y}/L^{2} \sim \tau_{so}/\tau_{erg}$ over its value $\tau_{so} \sim 2\lambda_{x}\lambda_{y}/v_F L$ in a bulk two-dimensional electron gas with Fermi velocity $v_F$ and mean free path $\ell$ equal to that in the dot, or with $\ell \sim L$ for the case of a ballistic dot. Moreover, as was shown in Ref. 12, the symmetry of the transformed spin-orbit scattering term is not symplectic, but unitary.

In this paper we investigate the case where the spin-orbit coupling parameters $\lambda_{x}$ and $\lambda_{y}$ are not constant throughout the quantum dot. Experimentally, such a situation could be created with the help of a metal gate parallel to the two-dimensional electron gas that changes the asymmetry of the quantum well. If the metal gate covers only a part of the quantum dot, as is shown schematically in Fig. 1, the translational invariance of $H_{so}$ is lifted. Hence, the spin-orbit scattering can no longer be gauged away to leading order in $L/\lambda_{x}$, $L/\lambda_{y}$. In other words, the “non-Abelian vector potential” in Eq. (1) now represents a nonzero “flux”. The consequence is a significant increase of the spin-orbit scattering rate and a restoration of the symplectic symmetry of $H_{so}$. Thus, a metal gate that changes the asymmetry of the quantum well in only part of the quantum dot has a fundamentally different effect than a metal gate that changes the quantum well potential uniformly throughout the dot. A gate that covers only part of the dot is clearly the more effective tool to tune the spin-orbit scattering rate.

As an example, let us consider a quantum dot for which the spin-orbit scattering is due to the asymme-
try of the potential well (Rashba term) only, so that \[ |\lambda_x| = |\lambda_y| = \lambda. \] If there is a metal gate over half of the dot as in Fig. 1, the spin-orbit scattering rate may take two different values \( \lambda^{-1} = \lambda^{-1} \pm \frac{1}{2} \lambda^{-1} \) in the two halves of the dot. As discussed in Refs. 11,12, the spatially uniform component of \( H_{so} \) leads to a unitary perturbation of the Hamiltonian, with a characteristic time

\[ \tau_{so}^u \sim \tau_{erg}(\lambda/L)^4 \sim (\tau_{so}^{\infty})^2/\tau_{erg}. \] (2)

The spatially varying component gives rise to a symplectic perturbation of the Hamiltonian, with a characteristic time that can be estimated as the time to accumulate a “flux quantum” from the “vector potential” in Eq. (1),

\[ \tau_{so}^v \sim \tau_{erg}(\lambda_B/L)^2. \] (3)

If \( \lambda_s \sim \lambda, \) \( \tau_{so}^v \) becomes comparable to \( \tau_{so}^{\infty} \), the spin-orbit scattering time in a bulk two-dimensional electron gas.

**FIG. 1.** Schematic drawing of a quantum dot with a metal gate (hatched) over part of the dot. The role of the gate is to change the asymmetry of the potential of the quantum well beneath it, and hence the spin-orbit parameters \( \lambda_x \) and \( \lambda_y. \)

We now present a quantitative calculation of how such a tunable spin-orbit scattering time affects the quantum interference corrections to the conductance: the weak (anti)localization correction \( \langle G \rangle \) and the conductance autocorrelation function \( \text{cov}[G(\vec{B}),G(\vec{B'})] \), where \( \vec{B} \) is a magnetic field. Our work extends previous works of Efetov and Frahm for the magnetic-field dependent quantum interference corrections without spin-orbit scattering or a parallel magnetic field. The effect of the spatially uniform component of \( H_{so} \) on the weak localization correction was calculated in Ref. 12.

We consider a quantum dot coupled to two electron reservoirs (labeled 1 and 2), via ballistic point contacts that have \( N_1 \) and \( N_2 \) channels each. We assume that the electron motion in the quantum dot is chaotic, so that random matrix theory can be used to calculate the conductance distribution in the universal regime \( g \mu_B B, h/\tau_{so} \ll \hbar/\tau_{erg}. \) (Here \( g \) is the electron g-factor and \( \mu_B \) the Bohr magneton.) The quantum dot is described in terms of its scattering matrix \( S \), which, for particles with spin, is a \( N \times N \) unitary matrix of quaternions, \( N = N_1 + N_2 \). Quaternions are \( 2 \times 2 \) matrices with special rules for transposition and complex conjugate. Starting point of the calculation is the Landauer formula for the two-terminal conductance \( G \) of the quantum dot at zero temperature,

\[ G = \frac{2e^2}{h} N_1 N_2 - \frac{e^2}{h} \text{tr} S A S^\dagger A, \] (4)

where the diagonal matrix \( A \) has elements

\[ A_{jj} = \begin{cases} N_j/N, & j = 1, \ldots, N_1, \\ -N_j/N, & j = N_1 + 1, \ldots, N. \end{cases} \]

In order to find the average and variance of the conductance \( G \) it is sufficient to compute the average

\[ \langle S_{kl} u u' \rangle = \frac{\pi}{2} \delta_{kl} \delta(u u'), \]

in the presence of spin-orbit scattering and for arbitrary values of the magnetic field \( \vec{B} \) and Fermi energy \( \varepsilon \). (Roman indices refer to the propagating channels in the leads, greek indices refer to spin.) In a random-matrix approach, the statistical properties of the scattering matrix \( S \) can either be calculated from a hermitian random matrix that represents the Hamiltonian of the quantum dot, or from a random unitary matrix. Here we use the latter approach; equivalence of the two approaches, including the dependence on an external parameter, was shown in Ref. 21. The \( N \times N \) matrix \( S \) is written as

\[ S = PU(1 - Q^2 RQU)^{-1} P^\dagger, \] (5)

where \( U \) is an \( M \times M \) random unitary symmetric matrix taken from Dyson’s circular orthogonal ensemble and \( R \) is a unitary matrix of size \( M - N \). The \( N \times M \) matrix \( P \) and the \( (M - N) \times M \) matrix \( Q \) are projection matrices with \( P_{ij} = \delta_{ij} \) and \( Q_{ij} = \delta_{i+N,j}. \) The quaternion elements of the matrices \( U, P, \) and \( Q \) are all proportional to the \( 2 \times 2 \) unit matrix \( 1 \). The matrix \( R \) is given by

\[ R(\varepsilon, \vec{B}, \tau_{so}) = \exp \left[ i \frac{\pi}{M \Delta} (\varepsilon - H'(\vec{B}, \tau_{so})) \right], \] (6)

where \( \Delta \) is the mean level spacing of the dot and \( H' \) is an \( (M - N) \) dimensional quaternion matrix generating the perturbations to the dot Hamiltonian that correspond to the magnetic field and the spin-orbit scattering,

\[ H' = \frac{\mu_B g}{2} \vec{B} \cdot \vec{\sigma} + i \frac{\pi \Delta}{2 \pi} \frac{\vec{B} \cdot \vec{\sigma}}{\hbar} + i \frac{\hbar}{2 \pi \tau_{so}} (A_1 \sigma_x + A_2 \sigma_y). \] (7)

Here \( A_j (j = 1, 2) \) and \( X \) are real antisymmetric matrices of dimension \( M - N \), with \( \text{tr} A_j A_j^T = M^2 \delta_{ij} \) and \( \text{tr} X X^T = M^2 \). The symmetry of the spin-orbit term in Eq. (7) is chosen in accordance with Eq. (1), taking into account that the spin-orbit Hamiltonian has symplectic symmetry once the coupling parameters \( \lambda_x \) and \( \lambda_y \) depend on position. In Eq. (7) the orbital and Zeeman
effects of the magnetic field have been separated. The first term describes the Zeeman coupling to the spin of the electrons. The second term models the orbital effect, where \( x \) is related to the perpendicular component of the magnetic field,
\[
x^2 = c \varepsilon^2 L^4 B^2 / (\hbar \tau_{\text{erg}} \Delta),
\]
e being a numerical coefficient of order unity. At the end of the calculation, the limit \( M \to \infty \) should be taken.

We now describe our calculation, which was done to leading order in \( 1/N \). Corrections for finite \( N \) are discussed at the end of this paper. To leading order in \( 1/M \) and \( 1/N \), it is sufficient to consider the elements of \( U \) as random Gaussian variables with zero mean and with variance \( \langle U_i U^*_j \rangle = M^{-1}(\delta_{i,k} \delta_{j,l} + \delta_{i,l} \delta_{j,k}) \).

We then expand Eq. (5) in powers of \( U \) and perform the Gaussian averages to leading order in \( 1/N \). We thus find
\[
\langle S_{k\mu\nu}(\varepsilon, \vec{B}) S_{k'\mu'\nu'}(\varepsilon', \vec{B}') \rangle \equiv \delta_{k'k} \delta_{\nu'\nu} D_{\mu\nu} c_{\mu'\nu'} + \delta_{k'k} \delta_{\nu'\nu} (T C T)_{\mu\nu} c_{\mu'\nu'},
\]
and with multiplication rules for the quaternion complex conjugate \( c^\ast \), we then find
\[
\langle S_{k\mu\nu}(\varepsilon, \vec{B}) S_{k'\mu'\nu'}(\varepsilon', \vec{B}') \rangle \equiv \delta_{k'k} \delta_{\nu'\nu} D_{\mu\nu} c_{\mu'\nu'} + \delta_{k'k} \delta_{\nu'\nu} (T C T)_{\mu\nu} c_{\mu'\nu'},
\]
where \( D = (M I \otimes \mathbb{I} - \text{tr } R \otimes R^t)^{-1} \),
\[
C = (M I \otimes \mathbb{I} - \text{tr } R \otimes R^t)^{-1}.
\]
Here \( R^t \ast \) is the quaternion complex conjugate of \( R^t \), \( T = \mathbb{I} \otimes \sigma_2 \), and the tensor multiplication should be understood as “backwards multiplication” for the second matrix, i.e., with the multiplication rules
\[
(\sigma_i \otimes \sigma_j)(\sigma_{i'} \otimes \sigma_{j'}) = (\sigma_{i} \sigma_{i'}) \otimes (\sigma_{j} \sigma_{j'}).
\]

The two contributions \( C \) and \( D \) are the equivalents of cooperon and diffusion in the conventional diagrammatic perturbation theory.

Taking the limit \( M \to \infty \) and defining a dimensionless magnetic field \( b = \pi g \mu_B B / \Delta \) and spin-orbit scattering rate \( \alpha^2 = 2 \pi \hbar / \tau_0 \Delta \), we find
\[
D^{-1} = N_D (I \otimes I) + i \vec{b} \cdot (\sigma \otimes I) - i \vec{b}^\prime \cdot (I \otimes \sigma) + 2a^2 (I \otimes I) - a^2 (\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y),
\]
\[
C^{-1} = N_C (I \otimes I) + i \vec{b} \cdot (\sigma \otimes I) + i \vec{b}^\prime \cdot (I \otimes \sigma) + 2a^2 (I \otimes I) - a^2 (\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y),
\]
where \( N_D \) and \( N_C \) are given by
\[
N_D = N - 2\pi i (\varepsilon - \varepsilon') / \Delta + (1/2)(x - x')^2,
\]
\[
N_C = N - 2\pi i (\varepsilon - \varepsilon') / \Delta + (1/2)(x + x')^2.
\]

We now set \( \vec{b} = b \hat{\varepsilon}, \vec{b}^\prime = b' \hat{\varepsilon}' \), take the inverses in Eqs. (8) and (11), and calculate the average and covariance of the conductance \( G \) from Eq. (5). For \( G \), we find
\[
\langle G(\varepsilon, x, b) \rangle = \frac{e^2}{\hbar} \frac{N_1 N_2}{N_1 + N_2} \left[ 1 - \frac{1}{2} \left( \frac{1}{N_C + 2a^2} + \frac{1}{N_C + 4a^2} - \frac{2a^2}{N_C(N_C + 2a^2) + 4b^2} \right) \right],
\]
where \( N_C = N + 2x^2 \), as follows from Eq. (12) with \( \varepsilon = \varepsilon', \varepsilon = \varepsilon' \). To calculate the zero temperature conductance fluctuations, it is sufficient to know the two-point correlator \( \langle G(\varepsilon, x, b), G(\varepsilon', x', b') \rangle \) to leading order in \( 1/N \). (Contributions from higher-order correlators vanish since they contain a factor \( \text{tr } \Lambda = 0 \).) We then find
\[
\text{cov} \left[ \langle G(\varepsilon, x, b) \rangle, \langle G(\varepsilon', x', b') \rangle \right] = \frac{e^2}{\hbar} \left[ \frac{N_1 N_2}{N_1 + N_2} \right]^2 (F_D + F_C),
\]
where
\[
F_D = \frac{2(b - b')^2 + 2|N_D + a^2|^2 + 2a^4}{|b - b'|^2 + N_D(2a^2 + N_D^2)} - \frac{2a^4 + 2(b + b')^2 + 2|N_D + a^2|^2}{|b + b'|^2 + (N_D + 2a^2)(N_D + 2a^2)|^2},
\]
(14)

\( F_C \) is obtained from \( F_D \) by the substitution \( x' \to -x' \), \( b' \to -b' \), and \( N_D \to N_C \), and \( N_D \) and \( N_C \) are given in Eqs. (13) and (14) above. The conductance fluctuations at finite temperature are obtained from Eq. (14) by multiplication with the derivatives of the Fermi function at energies \( \varepsilon \) and \( \varepsilon' \) and subsequent integration over \( \varepsilon \) and \( \varepsilon' \). Equations (13) and (14) recover the result of Refs. 18,11 for the spinless case \( a = b = 0 \). Further, one has \( \langle DG \rangle = (e^2/\hbar)(N_1 N_2/N^2) \), \( \text{var } G = 2(e^2/\hbar)^2(N_1 N_2/N^2)^2 \) if there is no parallel magnetic field, while spin orbit scattering is strong \( (a^2 \gg N) \), in agreement with known results for the circular symplectic ensemble. For comparison, no positive \( \langle DG \rangle \) is observed for the case of a spatially uniform spin-orbit coupling, see Ref. 12. Equations (13) and (14) are illustrated in Fig. 3 where we show \( \langle DG \rangle \) and \( \text{var } G \) as a function of the perpendicular magnetic field \( x \) for various values of the dimensionless spin-orbit scattering rate \( a \).

In the presence of both strong spin-orbit scattering \( (a^2 \gg N) \) and a large parallel field \( (b^2 \gg N^2, Na_2) \), all terms contributing to the cooperon \( C \) are suppressed. As a result, there is no weak localization correction to the conductance, \( \delta G = 0 \), and the conductance fluctuations are reduced by an additional factor two, \( \text{var } G = (e^2/\hbar)^2(N_1 N_2/N^2)^2 \), see also Ref. 11. Furthermore, the conductance is no longer symmetric under reversal of the
perpendicular magnetic field. This observation is best illustrated by the correlator \((G(x) - G(-x))^2\), which goes to \(2 \text{Var} G\) for perpendicular magnetic field strengths \(x^2 \gg N\), see Eq. (14). For comparison, in the presence of either spin-orbit scattering or a parallel field (but not both), one has \(G(x) = G(-x)\) for all \(x\).

On a phenomenological level, dephasing can be added to the current description via the voltage probe model of Büttiker. In this model, a fictitious voltage probe is attached to the quantum dot. Electrons escape from the dot into the voltage probe at a rate \(1/\tau_\phi\), where \(\tau_\phi\) is the dephasing time, and are then reinjected from the voltage probe without phase memory. The escape into the voltage probe is described by an imaginary term \(i\hbar/2\tau_\phi\) in the Hamiltonian. With the form \((\hbar/2\tau_\phi)^{1/2}\) of the Landau formula, the reinjection of particles from the voltage probe has no effect on the conductance to order \(N^0\). Hence, to leading order in \(1/N\), inclusion of dephasing amounts to the replacement

\[
N_{D,C} \to N_{D,C} + \frac{2\pi\hbar}{\tau_\phi \Delta}
\]

in Eqs. (13) and (14) above.

![Graph](https://via.placeholder.com/150)

**FIG. 2.** Left panel: Weak localization correction \((\delta G)\) as a function of the dimensionless perpendicular magnetic field \(x\), for \(a = 0\) (bottom curve), \(a = 0.3\), \(a = 0.5\), \(a = 1.0\), and \(a \to \infty\) (top curve). Right panel: \(\text{Var} G\) versus \(x\) for the same values of \(a\) (a = 0 is top curve, \(a \to \infty\) is bottom curve). In both cases, there is no parallel component of the magnetic field and \(G\) is measured in units of \(\hbar^2/2\Delta N_1 N_2/N^2\).

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