Bose-like few-fermion systems

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Dealing with a few-fermion system in the canonical ensemble, rather than in the grand canonical ensemble, shows that a few-fermion system with odd number fermions behaves differently from a few-fermion system with even number fermions. An even-number-fermion system behaves like a Bose system rather than a Fermi system.

Introduction

A system consisting of finite number particles, strictly speaking, should be dealt with in the canonical ensemble. In quantum statistics, however, few-body systems are always dealt with in the grand canonical ensemble. Considering few-body systems in the grand canonical ensemble is indeed an approximation. In the grand canonical ensemble, the accurate particle number is approximated by the mean particle number. In this paper, by dealing with few-fermion systems in the canonical ensemble rather than in the grand canonical ensemble, we find the difference between odd-particle-number and even-particle-number few-fermion systems: an even-particle-number few-fermion system behaves like a Bose system.

The reason why approximately using the grand canonical ensemble instead of the canonical ensemble to deal with few-particle systems is that in the canonical ensemble the constraint on the particle number makes the calculation complicated. As an expedient treatment, one turns to the grand canonical ensemble which has no constraint on the particle number. Nevertheless, in the grand canonical ensemble one cannot distinguish the difference between odd- and even-particle-number Fermi systems, since in the grand canonical ensemble there is no restriction on the particle number.

The problem of few-body systems draws much interest. Few-body systems exhibit obvious quantum effects at low temperatures. The technology today makes it possible to implement and operate a cold few-body system \cite{14}. Cold and ultracold few-body systems are discussed \cite{5 7}, and the cold and ultracold few-fermion systems \cite{8 9} are studied intensely. Few-fermion systems in one-dimensional \cite{10 11}, the effective theory for trapped few-fermion systems \cite{12}, and strongly interacting few-fermion systems \cite{13 14} are discussed. There are experiments on cold and ultracold few-fermion systems \cite{15 20}. The experiments in one and two dimensions \cite{17 21 23} and the thermal quantities of cold and ultracold few-fermion systems \cite{24 25} are reported. There are studies on two-body correlations for few-body systems \cite{29} and Cooper-like Fermi-Fermi mixtures \cite{30}, Few-fermion thermodynamics is also studied \cite{31 33}.
Ideal Fermi gases in the canonical ensemble. In the canonical ensemble, the canonical partition function of a $\nu$-dimensional ideal Fermi gas with $N$ fermions is

$$Z_{FD}(\beta, N) = \frac{Z^N(\beta)}{N!} \det M_N$$

(1)

with

$$M_N = \begin{pmatrix}
1 & \frac{1}{Z(\beta)} & \cdots & 0 \\
\frac{1}{Z(\beta)} & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{(N-1)^{\nu/2}} & \frac{1}{(N-1)^{\nu/2}} & \cdots & 1 \\
\end{pmatrix},$$

(2)

where $Z(\beta) = \sum e^{-\beta \epsilon_s} = g V$ is the single-particle partition function of an ideal classical gas, $V$ is the volume, $g$ is the number of internal degrees of freedom, and $\lambda = h/\sqrt{2\pi mkT}$ is the thermal wave length. Various thermodynamic quantities in the canonical ensemble can be obtained from the partition function (1), e.g., the specific heat

$$C_V = \frac{Nk}{\nu^2} + \frac{1}{N} \left( 2T \frac{\partial}{\partial T} \ln \det M_N + T^2 \frac{\partial^2}{\partial T^2} \ln \det M_N \right).$$

(3)

The first term in the specific heat, $\frac{Nk}{\nu^2}$, is the specific heat of the corresponding classical gas, and the rest part is the quantum effect coming from the exchange interaction.

Ideal Fermi gases in the grand canonical ensemble. As a comparison, consider the above few-fermion system in the grand canonical ensemble. In the grand canonical ensemble, instead of the canonical partition function (1), one works with the grand partition function $\Xi$, given by

$$\ln \Xi = \sum e^{-\beta \epsilon_s} = g V f_{\nu/2+1}(z)$$

with $f_{\sigma}(z)$ the Fermi-Dirac integral. The calculation of the grand partition function is simple, because there is no limitation on the particle number. In doing so, one loses the information of the particle number. For a finite-size system, however, one need to take the particle number into account. For this purpose, one requires that the mean particle number $\langle N \rangle$ equals the exact particle number $N$. The restriction $\langle N \rangle = N$ gives the relation between the chemical potential and the temperature. As a result, in the grand canonical ensemble, one cannot distinguish the difference caused by the oddevity of particle numbers.

In the grand canonical ensemble, the specific heat reads

$$C_V = \frac{Nk}{\nu^2} + \frac{1}{N} \left( \frac{\nu + 2 f_{\nu/2+1}(z)}{2} f_{\nu/2}(z) - \frac{\nu f_{\nu/2}(z)}{2} f_{\nu/2-1}(z) \right),$$

(4)

where the fugacity is given by

$$\ln z \simeq \frac{\epsilon_F}{kT} \left[ 1 - \frac{\pi^2}{12} \left( \frac{kT}{\epsilon_F} \right)^2 \right]$$

with $\epsilon_F = \frac{h^2}{2m} \left( \frac{3n}{4\pi^2} \right)^{2/3}$ the Fermi energy.

The lower limit on the temperature of a finite size system. Few-particle systems must be of a finite size, and the wavelength of particles must be less than the system size. Therefore, for finite-size systems, there exists a lowest temperature, depending on the system size.

Consider a particle in a container of size $L$. The one-half wavelength should be less than $L$, i.e., $\lambda/2 \lesssim L$. This gives a constraint on the temperature, $T \gtrsim \frac{h^2}{2\pi mk L^2}$, or, equivalently, $T \gtrsim \frac{h^2}{2\pi mk n^{2/3}}$, where $n \sim 1/L^3$ is the particle number density.
Figure 1. The heat capacity of a 3D Fermi gas. In three dimensions, the odd-particle-number few-fermion system and the even-particle-number one behave differently.

The $^3$He gas: an example. The $^3$He atom, a fermion with spin $1/2$ and mass $5.019 \times 10^{-27}$ kg, plays an important role in cold atom experiments [37]. There are many experiments on the the cold and ultracold $^3$He gases [19, 20]. Consider a $^3$He gas with the density $n \simeq 1 \times 10^{23} m^{-3}$ [25, 19]. The lower limit on the temperature, by the above estimation, is about $0.25 \mu K$.

Consider the specific heat of a $^3$He gas with the particle number $N = 2, 3, 4, 5$ in the canonical ensemble for roughly illustrating the problem. As a comparison, we also provide the specific heat of the $^3$He gas in the grand canonical ensemble.

Figures (1) to (3) show that the few-fermion gas of odd-number atoms and the few-fermion gas of even-number atoms behave differently at low temperatures: as the temperature decreases, the heat capacity of the odd-number-fermion system increases while the heat capacity of the even-number-fermion system decreases.

Nevertheless, in the grand canonical ensemble, the behaviors of odd- and even-number-fermion systems are the same. The reason, as mentioned above, is that in the grand canonical ensemble the information of the oddity of the particle number is lost.

Finally compare few-fermion systems with few-boson systems. In order to facilitate comparison, the fictitious Bose system is chosen as a Bose system with the same parameters as the $^3$He gas. The result shows that the specific heat of few-boson systems behaves like the specific heat of even-fermion systems, see Figure (4). It should be emphasized that a three-dimensional ideal Bose gas may perform the BEC phase transition. This is the reason why there is a sharp peak in the specific heat of the three-dimensional Bose gas calculated in the grand canonical ensemble. In contrast, a finite-particle-number system cannot perform phase transitions [35], so the specific heat calculated in canonical ensemble has no sharp peak but only has a maximum value.

A word on spatial dimensions. Figures (1) to (3) show that the specific heat in one, two, and three dimensions behave differently. This is because exchange interactions in different dimensional space are different. The exchange interaction in higher dimensions is more obvious than that in lower dimensions. The difference between the quantum gas and the classical gas is caused by the exchange interaction. Therefore, the difference between odd- and even-particle-number systems in higher dimensions is more obvious than the difference in lower dimensions.

Figure 2. The heat capacity of a 2D Fermi gas. In two dimensions, the odd-particle-number few-fermion system and the even-particle-number one behave differently.

Figure 3. The heat capacity of a 1D Fermi gas. In one dimension, the odd-particle-number few-fermion system and the even-particle-number one behave similarly.
Figure 4. The heat capacity of a 3D Bose gas.

Summary At low temperatures, odd-particle-number few-fermion systems behave differently from even-particle-number few-fermion systems. The specific heat of an even-particle-number few-fermion system behaves like a Bose system rather than a Fermi system. This inspire us to simulate a few-boson system by a few-fermion system. One can switch a Bose-like system to a Fermi system and vice versa by changing the number of fermions.

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