On the largest eigenvalue of a sparse random subgraph of the hypercube

Alexander Soshnikov
University of California, Davis
Department of Mathematics
Davis, CA 95616, USA
soshniko@math.ucdavis.edu

Abstract

We consider a sparse random subgraph of the $n$-cube where each edge appears independently with small probability $p(n) = O(n^{-1+o(1)})$. In the most interesting regime when $p(n)$ is not exponentially small we prove that the largest eigenvalue is $\Delta(G)^{1/2}(1+o(1)) = \frac{n \log 2}{\log(p-1)} \times (1+o(1))$ almost surely, where $\Delta(G)$ is the maximum degree of $G$. If $p(n)$ is exponentially small but not proportional to $2^{-n/k_n^{-1}}$, $k = 1, 2, \ldots$, then with probability going to one $\lambda_{\text{max}}(G) = \left(\Delta(G)\right)^{1/2} = \left(\frac{n \log 2}{\log(p-1)-\log n}\right)^{1/2}$. If $p(n)$ is proportional to $2^{-n/k_n^{-1}}$, $k = 1, 2, \ldots$, then with probability going to one $\left(\Delta(G)\right)^{1/2} \leq \lambda_{\text{max}}(G) < \left(\Delta(G) + 1\right)^{1/2}$ and $|\Delta(G) - \left[\frac{n \log 2}{\log(p-1)-\log n}\right]| \leq 1$.

1 Introduction and Formulation of Results

Let $Q^n$ be a graph of the $n$-cube consisting of $2^n$ vertices $V = \{x = (x_1, \ldots, x_n); \ x_i \in \{0, 1\}, \ i = 1, \ldots, n\}$ and $n2^{n-1}$ edges $E = \{\{x, y\} : x, y \in V, x \neq y\}$. 

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\[ \sum |x_i - y_i| = 1 \}. \] In this paper we study a random subgraph \( G(Q^n, p(n)) \), where each edge appears independently with probability \( p(n) \). Random subgraphs of the hypercube were studied by Burtin [5], Erdős and Spencer [8], Ajtai, Komlós and Szemerédi [1] and Bollobás [4], among others. In particular it was shown that a giant component emerges shortly after \( p = 1/n \) ([1]) and the graph becomes connects shortly after \( p = 1/2 \) ([5],[8],[4]). Recently the model has become of interest in mathematical biology ([7], [14], [15]). In this paper we are concerned with the behavior of the largest eigenvalues of a sparse random graph \( p(n) \leq n^{-1+o(1)} \).

The adjacency matrix of \( G \) is a \( 2^n \times 2^n \) matrix \( A \) whose entries are either one or zero depending on whether the edge \( (x, y) \) is present in \( G \) or not. \( A \) is a random real symmetric matrix with the eigenvalues denoted by \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{2^n} \). It follows from the Perron-Frobenius theorem that the largest eigenvalue is equal to the spectral norm of \( A \), i.e. \( \lambda_{\text{max}}(G) = \lambda_1 = \|A\| = \max_j |\lambda_j| \).

**Remark 1** It is easy to see that for a subgraph of the hypercube, or in general, for any bipartite graph, \( \lambda_k(G) = \min_{|V|\leq k}(G), \quad k = 1,2,\ldots, \) in particular \( |\lambda_{\text{min}}(G)| = \lambda_{\text{max}}(G) \).

Our main result is concerned with the asymptotic behavior of the largest (smallest) eigenvalues of \( A \).

**Theorem** Let \( G(Q^n, p(n)) \) is a random subgraph of the \( n \)-cube and \( p(n) \leq n^{-1+o(1)} \). Then the following statements hold.

(i) If \( \gamma^{-n} \ll p(n) \leq n^{-1+o(1)} \) for all \( \gamma > 1 \), then \( \lambda_{\text{max}} = (\Delta(G))^{1/2} \times (1 + o(1)) = \left( \frac{n \log 2}{\log(p^{-1})} \right)^{1/2} (1 + o(1)) \) almost surely, where \( \Delta(G) \) is the maximum degree of \( G \). Also for any \( 0 < \alpha < 1 \) there exists some positive constant depending on \( \alpha \) such that with probability at least \( 1 - 2^{2n} \times \exp\left(-\frac{1}{2} (\log(p^{-1}))^{-2}\right) \) there exist at least \( 2^{\alpha n} / (2n^2) \) eigenvalues greater or equal to \( \sqrt{\frac{(1-\alpha)n \log 2 (1-1/\log \log(p^{-1}))}{\log(p^{-1})}} - 2 \).

(ii) If \( 2^{-n} n^{-1} \ll p(n) \leq \gamma^{-n} \) for some \( \gamma \in (1,2] \), and \( p(n) \) is not proportional to \( 2^{-n/k} n^{-1} \), \( k = 2,3,\ldots, \) then with probability going to one \( \lambda_{\text{max}}(G) = \left( \Delta(G) \right)^{1/2} = \left( \frac{n \log 2}{\log(p^{-1}) - \log n} \right)^{1/2} \).

(iii) If \( p(n) \) is proportional to \( 2^{-n/k} n^{-1} \), with some \( k = 2,3,\ldots, \) then with probability going to one \( \left( \Delta(G) \right)^{1/2} \leq \lambda_{\text{max}}(G) < \left( \Delta(G) + 1 \right)^{1/2} \), and \( |\Delta(G) - \left[ \frac{n \log 2}{\log(p^{-1}) - \log n} \right]| \leq 1 \).
(iv) If $p(n)2^n n \to \nu$ as $n \to \infty$, then $\lambda_{\max}(G)$ converges in distribution to $\text{Be}(e^{-\nu})$.

(v) If $p(n) \ll 2^{-n}n^{-1}$, then with probability going to one $G$ is empty and $\lambda_{\max}(G) = 0$.

Remark 2 $\sqrt{\Delta(G)}$ is an obvious lower bound for $\lambda_{\max}(G)$ since $\|A \times f\|^2 \geq \Delta(G)$, where $f$ is a delta-function with the support at the vertex $x$ of the maximum degree $\deg(x) = \Delta(G)$.

Remark 3 The result of the theorem is similar to a recent result of Krivelevich and Sudakov [12] on the largest eigenvalue of a random subgraph $G(n, p(n))$ of a complete graph, who proved that $\lambda_{\max}(G(n, p(n))) = \max(\Delta(G(n, p(n)))^{1/2}, np)(1 + o(1))$.

To some extent our approach has been influenced by [12].

The results claimed to take place almost surely hold with probability one on the product of probability spaces corresponding to $G(Q^n, p(n))$, $n = 1, 2, \ldots$. We use the standard notations $a_n = \Theta(b_n)$, $a_n = O(b_n)$ and $a_n = \Omega(b_n)$ for $a_n > 0, b_n > 0$ as $n \to \infty$ if there exist constants $C_1$ and $C_2$ such that $C_1b_n < a_n < C_2 b_n$, or $a_n > C_1b_n$ correspondingly. The equivalent notations $a_n = o(b_n)$ and $a_n \ll b_n$ mean that $a_n/b_n \to 0$ as $n \to \infty$.

The rest of the paper is organized as follows. Section 2 is devoted to the proof of Theorem. Several results of auxiliary nature are collected in Section 3. The concluding remarks are given in Section 4.

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2 Proof of Theorem

We start with the case $n^{-\Theta(1)} \leq p(n) \leq n^{-1+o(1)}$, i.e. $1 \leq \lim inf \frac{\log(p^{-1})}{\log n} \leq \lim sup \frac{\log(p^{-1})}{\log n} < +\infty$. Let us denote $\max(e^{5np}, \exp(\log n/\log \log n))$ by $r_n$. We decompose the set of all vertices $V$ into the disjoint union $V = V_1 \sqcup V_2 \sqcup V_3$, where $V_1 = \{x \in V : d(x) \leq r_n\}$, $V_2 = \{x \in V : r_n < d(x) \leq \frac{n}{\log(p^{-1})}r_n^{-2}\}$, and $V_3 = \{x \in V : d(x) > \frac{n}{\log(p^{-1})}r_n^{-2}\}$.

We recall that $d(x)$ denotes the degree of the vertex $x$. Let us denote the induced graphs by $G_i = G[V_i]$, $i = 1, 2, 3$. We also denote by $G_4, G_5$...
and $G_6$ the bipartite subgraphs consisting of all edges of $G$ between $V_1$ and $V_2$, $V_1$ and $V_3$, and $V_2$ and $V_3$ correspondingly.

**Lemma 1**

$$\lambda_{\text{max}}(G) \leq \sum_{i=1}^{6} \lambda_{\text{max}}(G_i).$$

**Lemma 2**

$$\lambda_{\text{max}}(G_1) \leq \max\left(\exp\left(\frac{\log n}{\log \log n}\right), e^{5np}\right),$$

$$\lambda_{\text{max}}(G_4) \leq \left(\frac{n}{\log n}\right)^{1/2} \left(\max(\exp\left(\frac{\log n}{\log \log n}\right), e^{5np})\right)^{-1/2}.$$  

The proofs of Lemmas 1 and 2 are rather standard (see e.g. [13]) and will be omitted.

**Lemma 3**

$$\mathbb{E}(\#(x \in V_2 \sqcup V_3 : \sum_{y \in V_2 \sqcup V_3 \setminus \{x\}} (A^2)(x,y) > \frac{n}{\log(p^{-1})} \frac{\log(\log(n))}{\log(n)}) > D_n).$$

$$= O\left(\exp\left(-\frac{1}{4}n \exp\left(\log n / \log \log n\right)\right)\right).$$

**Proof**

Let us denote $\frac{n}{\log(p^{-1})} \frac{\log(\log(n))}{\log(n)}$ by $D_n$. We estimate the mathematical expectation in (1) from above by

$$2^n \sum_{m=r_n+1}^{n} \binom{n}{m} p^m \sum_{s_1+\ldots+s_m > D_n} \prod_{j=1}^{m} \binom{n}{s_j} p^{s_j} \left\{ \binom{n-2}{r_n-2} p^{r_n-2}\right\}^{\frac{1}{2}(s_1+\ldots+s_m)}$$

Indeed, we can choose the vertex $x$ in $2^n$ ways. The probability that the degree of $x$ is $m$, $r_n < m \leq n$, is $\binom{n}{m} p^m (1-p)^n-m \leq \binom{n}{m} p^m$. We shall call the vertices whose distance in $G$ from $x$ is one by the vertices of the first generation. Similarly, we shall call the vertices whose distance from $x$ is two by the vertices of the second generation, etc. If $\text{deg}(x) = m$, then there are exactly $m$ vertices of the first generation. We denote by $s_j$, $j = 1, \ldots, m$, the number of edges that connect the $j$-th vertex of the first generation to $V_2 \sqcup V_3 \setminus \{x\}$. Since $\sum_{j=1}^{m} s_j = \sum_{y \in V_2 \sqcup V_3 \setminus \{x\}} (A^2)(x,y)$, we conclude that $s_1 + \ldots + s_m > D_n$. The probability

$$\Pr\left(\bigcap_{j=1}^{m} \{j\text{th vertex has } s_j \text{ edges connecting it to } V_2 \sqcup V_3 \setminus \{x\}\} \right)$$
is estimated from above by $\prod_{l=1}^{m} \binom{n}{s_j} p^{s_j}$. since the events are independent. The number of the vertices of the second generation in $V_2 \sqcup V_3 \setminus \{x\}$ is at least $\frac{1}{2}(s_1 + \ldots + s_m)$ (each vertex of the second generation is connected to at most two vertices of the first generation). Finally, each of the vertices of the second generation is connected to at least $(r_n - 2)$ vertices of the third generation. Because the edges are independent, the last factor in the bound of the mathematical expectation is

\[
\{ \left( \binom{n-2}{r_n-2} p^{r_n-2} \right)^{\frac{1}{2}(s_1+\ldots+s_m)} \}.
\]

Let us denote the sum $s_1 + \ldots + s_m$ by $B$ and the term

\[
2^n \binom{n}{m} p^n \prod_{j=1}^{m} \binom{n}{s_j} p^{s_j} \left\{ \left( \binom{n-2}{r_n-2} p^{r_n-2} \right)^{\frac{1}{2}(s_1+\ldots+s_m)} \right\}
\]

by $T(s_1, \ldots, s_m)$. We estimate $\log T(s_1, \ldots, s_m)$ from above as

\[
\log T \leq n \log 2 - m \log m + m \log(np) + m + \sum_{j=1}^{m} (-s_j \log s_j + s_j \log(np) + s_j) + \frac{1}{2}(-r_n \log r_n + r_n \log(np) + r_n)(s_1 + \ldots + s_m) \leq n \log 2 - \frac{1}{3}r_n(s_1 + \ldots + s_m).
\]

Then the mathematical expectation in (1) is bounded from above by

\[
\sum_{m=r_n+1}^{n} \sum_{B \geq B_n} \binom{B + m - 1}{m - 1} \exp(n \log 2 - \frac{1}{3}r_n B) = O\left( \exp\left( -\frac{1}{4} n \exp(\log n / \log \log n) \right) \right).
\]

Lemma 3 is proven.

**Lemma 4**

\[
E\left( \#(x \in V_1 : \sum_{y=1}^{2^n} (A^2)(x, y) > \frac{n \log 2}{\log(p-1)}(1 + \frac{4}{\log \log n} + \frac{2 \log r_n}{\log n}) \right) = O\left( \exp\left( -\frac{n}{\log \log n} \right) \right).
\]
Proof
Let us denote \( n \log \frac{2}{\log(p-1)} (1 + \frac{4}{\log \log n} + \frac{2 \log r_n}{\log n}) \) by \( L_n \). Let \( 1 \leq m = \deg(x) \leq r_n \) be the number of the vertices of the first generation. Then there are at least \( L_n - m \) vertices of the second generation and similarly to the proof of Lemma 3 we can estimate the l.h.s. of (2) from above by

\[
2^n \sum_{m=1}^{r_n} \binom{n}{m} p^m \left( \frac{(n-1)m}{L_n - m} \right)^{L_n - m} \leq r_n \max_{1 \leq m \leq r_m} \left( \exp \left( n \log 2 - m \log m + m \log(np) + (L_n - m) \log m - (L_n - m) \log(L_n - m) + (L_n - m) \log(np) + L_n - m \right) \right) \leq n \exp \left( n \log 2 + r_n \log n - L_n (\log L_n - \log(np) - \log r_n - \frac{r_n}{L_n} \log L_n) \right) \leq \exp \left( n \log 2 + (r_n + 1) \log n - \frac{n \log 2}{\log(p-1)} (1 + \frac{4}{\log \log n} + \frac{2 \log r_n}{\log n}) \times (\log(p-1) + O(1) - \log \log(p-1) - \log r_n) \right) \leq O \left( - \exp \left( \frac{n}{\log \log n} \right) \right).
\]

Lemma 4 is proven.

**Lemma 5**

\[
E \left( \#(x \in V_3 : \sum_{y \in V_2 \cup V_3} (A(x, y))^2 > \frac{n \log 2}{\log(p-1) \log n}) \right) = O \left( \exp \left( -n \exp \left( \frac{\log n}{2 \log \log n} \right) \right) \right).
\]

Proof
We estimate the l.h.s. of (3) from above by
\[ 2^n \sum_{m \geq \log(p^{-1}) \log n} \binom{n}{m} p^m \left\{ \binom{n}{r_n} p_n \right\}^m \leq O\left( \exp(n \log 2 + \log n \log(p^{-1}) \log n) \right) \leq O\left( \exp(-n \exp(\frac{\log n}{2 \log \log n})) \right). \]

Lemma 5 is proven.

**Lemma 6**

\[ E\left( \#(x \in V_3 : \deg_{G_3}(x) = \sum_{y \in V_3} (A(x,y))^2 > \left( \frac{n \log 2}{\log(p^{-1}) \log n} \right)^{1/2} \right) = O\left( \exp(-n^{4/3}) \right). \]

**Proof**

Let us denote \( \frac{n \log 2}{\log(p^{-1}) \log n} \) by \( M_n \) and \( \left( \frac{n \log 2}{\log(p^{-1}) \log n} \right)^{1/2} \) by \( N_n \). We estimate the l.h.s. of (4) from above by
\[ 2^n \sum_{m = N_n + 1}^n \binom{n}{m} p^m \left\{ \binom{n}{M_n} p^{M_n} \right\}^m \leq n \exp\left(n \log 2 - (M_n \log M_n - M_n - M_n \log(np))N_n \right) = \exp\left(-n^{3/2} r_n^{-3} \right) = O\left( \exp(-n^{4/3}) \right). \]

Lemma 6 is proven.

**Lemma 7** Let \( G \) be a random subgraph of the \( n \)-cube, \( G = G(Q^n, p(n)) \), where \( p(n) \leq n^{-1 + o(1)} \). Let us define \( \kappa(n) := \max\{k : 2^n \binom{n}{k} p^k(1-p)^{n-k} \geq 1\} \). Then the following statements hold.

(i) If \( p(n) \) is not exponentially small in \( n \), then
\[ \Pr(\Delta(G) < \kappa(n) - j) \leq \exp\left(-\left( \frac{\log 2}{p \log(p^{-1})} \right)^j (1 + o(1)) \right), \]
Pr(Δ(G) > κ(n) + j) ≤ \left(\frac{p \log(p^{-1})}{\log 2}\right)^j (1 + o(1)), \quad (6)

for j = 1, 2, \ldots. In particular, with probability one, there exists sufficiently large (random) n∗ such that for n > n∗ we have |Δ(G) − κ(n)| ≤ 2.

(ii) If p(n) = Θ(2^{−n/k}n^{−1}), then 2^{\binom{n}{k}}p^k = Θ(1), κ(n) = k − 1 or k, and

Pr(Δ(G) = k − 1) = \exp(-2^n \binom{n}{k} p^k)(1 + o(1)),

Pr(Δ(G) = k) = 1 − O(2^{−n/k}). \quad (7)

(iii) If p(n) is exponentially small, but not proportional to 2^{−n/k}n^{−1}, then κ(n) = \left\lfloor \frac{n \log 2}{\log(p − 1) − \log n} \right\rfloor, \quad EX_{κ(n)+1} ≪ 1 ≪ EX_{κ(n)}, \quad (8)

Pr(Δ(G) > κ(n)) = O(EX_{κ(n)+1}),

Pr(Δ(G) < κ(n)) ≤ \exp(-Θ(EX_{κ(n)})). \quad (9)

Proof
Let us denote the number of vertices of G(Q^n, p) with degrees larger than k − 1 by X_k. Then X_k = \sum_{i=1}^{2^n} I_i, where we denoted by the first 2^n positive integers the vertices of Q^n and by I_i the indicator of the event that deg(i) ≥ k. By its definition X_k is monotone (non-decreasing) with respect to k. One can easily calculate the mathematical expectation EX_k = 2^n \sum_{l≥k} \binom{n}{l} p^l(1 − p)^{n−l}. Estimating EX_k from above as

EX_k ≤ 2^n \binom{n}{k} p^k ≤ \exp(n \log 2 − k \log k + k \log(np) + O(k))

we obtain that for k ≥ \frac{n \log 2^{1+1/\log \log(p−1)}}{\log(p−1)}

EX_k = O\left(\exp\left(-\frac{n}{2 \log \log(p−1)}\right)\right). \quad (10)

On the other hand if k ≤ \frac{n \log 2^{1−1/\log \log(p−1)}}{\log(p−1)} then
\[ E_k = \Omega\left(\exp\left(\frac{n}{2 \log \log(p^{-1})}\right)\right). \quad (11) \]

It is clear that \( \kappa(n) \) must satisfy the inequalities

\[ \frac{n \log 2(1 - 1/\log \log(p^{-1}))}{\log(p^{-1})} - 1 \leq \kappa(n) \leq \frac{n \log 2(1 + 1/\log \log(p^{-1}))}{\log(p^{-1})} + 1. \quad (12) \]

We claim that for such \( k \) the probability \( \Pr(\Delta(G) < k) = \Pr(X_k = 0) \) is equal, up to a small error term, to \( \exp(-EX_k) \). More precisely the following inequalities take place

\[ \exp\left( -\frac{EX_k}{1 - 2^{-n}EX_k} \right) \leq \Pr(X_k = 0) \leq \exp\left( -EX_k\left(1 - EX_k\frac{p^{-1}}{2^n} \left(\frac{k^2}{n^2} + p\right) \exp(n2^{-n+1}EX_k)\right)\right). \quad (13) \]

The l.h.s. of (13) follows from the FKG inequality ([3],[9]). Since the events \( \{\text{deg}(i) < k\}_{i=1}^{2^n} \) are monotone with respect to the edge indicators we have

\[ \Pr(X_k = 0) = \Pr\left(\bigcap_{i=1}^{2^n}\{\text{deg}(i) < k\}\right) \geq \prod_{i=1}^{2^n}(1 - EI_i) \leq \prod_{i=1}^{2^n} \exp\left( -\frac{EI_i}{1 - EI_i} \right) = \exp\left( -\frac{\sum_{i=1}^{2^n} EI_i}{1 - \max_{1 \leq i \leq 2^n} EI_i} \right) = \exp\left( -\frac{EX_k}{1 - EI} \right). \quad (14) \]

The l.h.s. of (13) now follows from \( EI = 2^{-n}EX_k \). To prove the r.h.s. of (13) we apply the Suen’s type inequality (see e.g. [9], Theorem 2.22, part (i)) that states that

\[ \Pr(X_k = 0) \leq \exp\left( -EX_k + \epsilon e^{2\delta} \right), \quad (15) \]

where \( \epsilon = \frac{1}{2} \sum_{i \sim j} E(I_iI_j) \), and \( \delta = \max_i \sum_{k \sim i} EI_k \). Here we use the notation \( i \sim j \) if \( i \neq j \) and \( I_i \) and \( I_j \) are dependent random variables.
It is easy to see that in our case \( \delta = n 2^{-n} E X_k \), and
\[
\epsilon \leq 2^{n-1} n \left( p \left[ \left( \frac{n-1}{k-1} \right) p^{k-1} \right]^2 + (1-p) \left[ \left( \frac{n-1}{k} \right) p^k \right]^2 \right) \leq n 2^{n-1} 2^{-2n} \frac{k^2}{n^2 p} + 1) (E X_k)^2 \leq (E X_k)^2 \frac{1}{2^n p} \left( \frac{k^2}{n^2} + p \right).
\]  
(16)

Let us now consider the case (i) in more detail. Taking into account that \( E X_{k+1} = \frac{k \log(p^{-1})}{\log 2} E X_k (1 + o(1)) \) for \( k = \kappa(n)(1 + o(1)) \) we obtain from the definition of \( \kappa(n) \) that for any fixed \( j = 0, 1, 2, \ldots \),
\[
\left( \frac{\log 2}{p \log(p^{-1})} \right)^j (1 + o(1)) \leq E X_{\kappa(n)-j} \leq \left( \frac{\log 2}{p \log(p^{-1})} \right)^{j+1} (1 + o(1)),
\]  
(17)

\[
E X_{\kappa(n)+j} \leq \left( \frac{p \log(p^{-1})}{\log 2} \right)^{j-1} (1 + o(1)),
\]  
(18)

Applying (17) and the r.h.s. of (13) we infer
\[
\Pr(\Delta(G) < \kappa(n) - j) = \Pr(X_{\kappa(n)-j} = 0) \leq \exp(-E X_{\kappa(n)-j}(1 + o(1))) \leq \exp\left(-\left( \frac{\log 2}{p \log(p^{-1})} \right)^j (1 + o(1)) \right) \quad j = 1, 2, ..
\]  
(19)

In a similar manner
\[
\Pr(\Delta(G) > \kappa(n) + j) = 1 - \Pr(X_{\kappa(n)+j+1} = 0) \leq 1 - \exp(-E X_{\kappa(n)+j+1}(1 + o(1))) \leq 1 - \exp\left(-\left( \frac{p \log(p^{-1})}{\log 2} \right)^j (1 + o(1)) \right) \leq \left( \frac{p \log(p^{-1})}{\log 2} \right)^j (1 + o(1)), \quad j = 1, 2, ...
\]  
(20)

Let us now consider the case (ii). Since \( p(n) = \Theta(2^{-n/k} n^{-1}) \) we have \( E X_k = 2^n \binom{n}{k} p^k (1-p)^{n-k} = \Theta(1) \), and \( \kappa(n) = k - 1 \) or \( k \), depending on whether \( E X_k < 1 \) or \( E X_k \geq 1 \). It follows from (13) that
\[
\Pr(\Delta(G) < k) = \exp(-E X_k)(1 + o(1)) = \exp(-2^n \binom{n}{k} p^k)(1 + o(1)).
\]  
(21)
Applying the r.h.s. of (13) and
\[ \mathbb{E} X_{k-1} \frac{p^{-1}}{2^n} \left( \frac{k^2}{n^2} + p \right) \exp(n2^{-n+1} \mathbb{E} X_k) = o(1), \quad k \geq 2, \]
we obtain
\[ \Pr(\Delta(G) < k - 1) = \exp(-\mathbb{E} X_{k-1}(1 + o(1))) = \exp\left(-\Theta\left(\frac{1}{p \log(p^{-1})}\right)\right) = \exp\left(-\Theta\left(2^{-n/k}\right)\right). \]

To estimate \( \Pr(\Delta(G) > k) \) we observe that \( \mathbb{E} X_{k+1} = \Theta(2^{-n/k}) \) which implies
\[ \Pr(\Delta(G) > k) = 1 - \Pr(X_{k+1} = 0) = \Theta(2^{-n/k}). \]

If \( p(n) \) is exponentially small but not proportional to \( 2^{-n/k} n^{-1} \), \( k = 1, 2, \ldots \), then \( \kappa(n) = \left[ \frac{n \log 2}{\log(p^{-1}) - \log n} \right] \) and \( \mathbb{E} X_{\kappa(n)+1} \ll 1 \ll \mathbb{E} X_{\kappa(n)} \). In a similar way to (i), (ii) one has
\[ \Pr(\Delta(G) > \kappa(n)) = O(\mathbb{E} X_{\kappa(n)+1}), \]

To estimate \( \Pr(\Delta(G) < \kappa(n)) \) we consider first the case \( p(n) > 2^{-\frac{n}{2} + 2}/(n \sqrt{\log n}) \). Then
\[ \mathbb{E} X_{\kappa(n)} \frac{p^{-1}}{2^n} \left( \frac{\kappa(n)^2}{n^2} + p \right) \exp(n2^{-n+1} \mathbb{E} X_{\kappa(n)}) = o(1), \]
and
\[ \Pr(\Delta(G) < \kappa(n)) \leq \exp(-\Theta(\mathbb{E} X_{\kappa(n)})). \]

In the case \( 2^{-n} n^{-1} \ll p(n) \leq 2^{-\frac{n}{2} + 2}/(n \sqrt{\log n}) \) one has \( \kappa(n) = 1 \) and \( \Delta(G) < \kappa(n) \) iff the graph is empty. This probability is equal to \( \exp(-\Theta(n2^n p)) \) since it is the probability that \( n2^{n-1} \) independent Bernoulli random variables \( B \epsilon(p) \) all equal zero.

Lemma 7 is proven.

Combining the results of Lemmas 1-7 we are now ready to prove part (i) of the theorem. Indeed, applying Borel-Contelli Lemma we obtain that with probability one there exists sufficiently large (random) \( n_* \) such that for all
the counting numbers from Lemmas 3-6 are all zero. Let $n > n_*$.

It follows from Lemma 3 that

$$\max_{x \in V_2} \sum_{y \in V_2} (A^2)(x, y) \leq \frac{n}{\log(p-1)} \frac{\log \log n}{\log n} + \max_{x \in V_2} (A^2)(x, x) =$$

$$\frac{n}{\log(p-1)} \frac{\log \log n}{\log n} + \max_{x \in V_2} \deg_{G_2}(x) \leq$$

$$\frac{n}{\log(p-1)} \frac{\log \log n}{\log n} + \frac{n \log 2}{\log(p-1)} r^{-2} =$$

$$o\left(\frac{n}{\log(p-1)}\right).$$

(28)

Since

$$(\lambda_{\text{max}}(G_2))^2 \leq \max_{x \in V_2} \sum_{y \in V_2} (A^2)(x, y)$$

we conclude that

$$\lambda_{\text{max}}(G_2) = o\left(\left(\frac{n}{\log(p-1)}\right)^{1/2}\right)$$

(29)

almost surely. Similar estimates hold for $\lambda_{\text{max}}(G_3)$ and $\lambda_{\text{max}}(G_6)$. The estimate

$$\lambda_{\text{max}}(G_3) = o\left(\left(\frac{n}{\log(p-1)}\right)^{1/2}\right)$$

(30)

follows from $\lambda_{\text{max}}(G_3) \leq \max_{x \in V_3} \deg_{G_3}(x)$ and Lemma 6. To prove

$$\lambda_{\text{max}}(G_6) = o\left(\left(\frac{n}{\log(p-1)}\right)^{1/2}\right)$$

(31)

we employ (28), Lemma 3 and Lemma 5 to see that

$$\max_{x \in V_3} \sum_{y \in V_2, z \in V_3} A(x, y)A(y, z) = o\left(\frac{n}{\log(p-1)}\right),$$

$$\max_{x \in V_2} \sum_{y \in V_3, z \in V_2} A(x, y)A(y, z) = o\left(\frac{n}{\log(p-1)}\right).$$

(32)

Finally, we claim that

$$\lambda_{\text{max}}(G_5) = \left(\left(\frac{n \log 2}{\log(p-1)}\right)^{1/2}\right) \text{ (a.s.),}$$

(33)
which follows from Lemmas 3, 4 and 7. Combining Lemmas 1, 2 and (29) – (32) we prove \( \lambda_{\text{max}}(G(Q^n, p)) = (\Delta(G(Q^n, p)))^{1/2}(1 + o(1)) \) almost surely for \( n^{-\Theta(1)} \leq p(n) \leq n^{-1+o(1)} \). To find the the eigenvalues of \( A \) close to the \( \lambda_{\text{max}} \) we use Lemmas 8 and 9 from the next section. Assuming Lemma 8 we can construct \( 2^{[\alpha n]} \) \( \delta \)-functions \( \{f_i\}_{i=1}^{2^{[\alpha n]}} \) supported at the vertices \( \{x_i\}_{i=1}^{2^{[\alpha n]}} \) of degrees greater or equal than \( \kappa(n - [\alpha n]) - 2 \geq \frac{\log(p^{1/2})}{\log(p^{-1})} - 2 \). The constructed vectors \( \{f_i\}_{i=1}^{2^{[\alpha n]}} \) form an orthonormal family such that \( (A^2f_i, f_i) > \frac{(1-\alpha)n \log 2}{\log(p^{-1})} \). Since each vertex has at most \( (n^2 - 1) \) vertices within distance 2 one can select a sub-family of size at least \( \frac{2^{[\alpha n]}}{n^2} \) such that for the vectors from the sub-family \( (A^2f_i, f_j) = 0 \) if \( i \neq j \). Applying Lemma 9 one obtains that there are at least \( \frac{2^{[\alpha n]}}{n^2} \) eigenvalues of \( A^2 \) greater or equal to \( \kappa(n - [\alpha n]) - 2 \). Since the spectrum of \( A \) is central symmetric with respect to the origin this implies that there are at least \( \frac{2^{[\alpha n]}}{2n^2} \) eigenvalues of \( A \) greater or equal to \( \left( \kappa(n - [\alpha n]) - 2 \right)^{1/2} \).

The case \( \log n \ll \log(p^{-1}) \ll n \) is very similar to the previous one. We again represent \( V \) as \( V = V_1 \sqcup V_2 \sqcup V_3 \), where now \( V_1 = \{x \in V : d(x) \leq \tau_n\} \), \( V_2 = \{x \in V : \tau_n < d(x) \leq \frac{n}{\log(p^{-1})} \tau_n^{-2}\} \), and \( V_3 = \{x \in V : d(x) > \frac{n}{\log(p^{-1})} \tau_n^{-2}\} \), with \( \tau_n = \exp(\frac{\log(\Delta_n)}{\log(p^{-1})}) \), and \( \Delta_n = \frac{n \log 2}{\log(p^{-1})} \). We claim that the analogues of Lemmas 3-6 hold, namely:

\[
E\left(\#(x \in V_2 \sqcup V_3 : \sum_{y \in V_2 \cup V_3 \setminus \{x\}} (A^2)(x, y) > \frac{n}{\log(p^{-1})} \tau_n^{1/3}\right) = O\left(\exp\left(-n \tau_n^{1/2}\right)\right),
\]

\[
E\left(\#(x \in V_1 : \sum_{y=1}^{n}(A^2)(x, y) > \frac{n \log 2}{\log(p^{-1})} (1 + 1/ \log \log(p^{-1})))\right) = O\left(\exp\left(-\frac{n}{\log \log(p^{-1})}\right)\right),
\]

\[
E\left(\#(x \in V_3 : \sum_{y \in V_2 \cup V_3} (A(x, y))^2 > \frac{n \log 2}{\log(p^{-1})} \tau_n^{1/2}\right) = O\left(\exp\left(-n \tau_n^{1/2}\right)\right),
\]

(34)

(35)

(36)
and
\[ E\left(\#(x \in V_3 : \deg_{G_3}(x) = \sum_{y \in V_3} (A(x, y))^2 > \left(\frac{n \log 2}{\log(p^{-1}) \tau_n}\right)^{1/2})\right) = O\left(\exp(-n \tau_n)\right). \tag{37} \]

The proofs of (34) – (37) are very similar to the arguments given in Lemmas 3-6 and left to the reader.

Let us now consider the case when \( p(n) \) is exponentially small in \( n \). We denote by \( Y_k \) the number of isolated components with \( k \) edges, \( k = 1, 2, \ldots \). It is easy to see that
\[ EY_k = \Theta(EX_k) = \Theta(2^n n^k p^k). \tag{38} \]

If \( p(n) \) is not proportional to \( 2^{-n/k} n^{-1}, \ k = 1, 2, 3, \ldots \), then it follows from Lemma 7, part (iii) and (38) that with probability going to one the maximum degree of \( G(Q^n, p) \) is \( \kappa(n) = \left\lfloor \frac{n \log 2}{\log(p^{-1}) \log n} \right\rfloor \) and there are no components with more than \( \kappa(n) \) edges. Since the largest eigenvalue of \( G \) is the maximum of the eigenvalues of its connected components and the largest eigenvalue of a component with \( k \) edges is not greater than \( \sqrt{k} \) (and is equal to \( \sqrt{k} \) only if the component is a star on \( k+1 \) vertices), we prove that with probability going to one \( \lambda_{\max}(G) = \sqrt{\kappa(n)} = \sqrt{\Delta(G)} \).

Finally if \( p(n) \) is proportional to \( 2^{-n/k} n^{-1}, \ k = 1, 2, 3, \ldots \), then with probability going to one \( \Delta(G) \in \{k-1, k\} \) and there are no connected components with more than \( k \) edges.

Theorem is proven.

### 3 Auxiliary Results

In this section we present two auxiliary lemmas. Our first result claims that there are many vertices with degrees close to the maximum degree

**Lemma 8** If \( p(n) \) is not exponentially small, then for any fixed \( 0 < \alpha < 1 \) with probability at least \( 1 - 2^{\alpha n} \exp\left(-\left(2p \log(p^{-1})\right)^{-2}\right) \) there exist at least \( 2^{\alpha n}/(2n^2) \) vertices with degrees greater or equal to \( \frac{(1-\alpha) n \log 2 (1-1/\log \log(p^{-1}))}{\log(p^{-1})} - 2 \).
Proof
Consider $Q^n$ as a disjoint union of $2^{[\alpha n]}$ cubes of dimension $n - [\alpha \times n]$, $Q^n = \sqcup_{i=1}^{2^{[\alpha n]}} Q_i$. Consider random subgraphs $G_i = G(Q_i, p(n))$ induced by the edges of $G$. According to Lemma 7 the maximum degree of $Q_i$ is at least $\kappa(n - [\alpha n]) - 2 \geq (1 - \alpha)n \log \left( \frac{2(1 - 1/\log \log(p^{-1}))}{\log(p^{-1})} \right) - 2$ with probability at least $\exp \left( - \left( 2p \log(p^{-1}) \right)^{-2} \right)$. The intersection of these events has probability at least $1 - 2^{\alpha n} \exp \left( - \left( 2p \log(p^{-1}) \right)^{-2} \right)$.

Lemma 8 is proven.

We finish this section with elementary lemma from linear algebra.

Lemma 9 Let $A$ be a Hermitian (or real symmetric) matrix and \{\(f_i\), $i = 1, \ldots, n\}$ be an orthonormal family of vectors such that \((Af_i, f_i) \geq \lambda\) for some $\lambda$, $i = 1, 2, \ldots, n$. Suppose that \((Af_i, f_j) = 0\) if $i \neq j$. Then the number of eigenvalues of $A$ greater or equal to $\lambda$ is at least $n$.

Proof
Let the number of the eigenvalues greater or equal to $\lambda$ be less than $n$. Then there exists a non-zero linear combination $f = \sum_{i=1}^{n} x_i f_i$ orthogonal to all eigenvectors with the eigenvalues greater or equal to $\lambda$, which implies $(Af, f) < \lambda(f, f)$. On the other hand it follows from the conditions of the lemma that

$$(Af, f) = \sum_{i=1}^{n} |x_i|^2 (Af_i, f_i) \leq \lambda \sum_{i=1}^{n} |x_i|^2 = \lambda(f, f).$$

4 Concluding Remarks

It would be also interesting to study the regime $n^{-1+o(1)} \leq p(n) \leq 1$ and to prove the analogue of the Krivelevich- Sudakov theorem there as well.

There are several other important questions that are beyond the reach of presented technique. The most fundamental is perhaps the local statistics of the eigenvalues, in particular the local statistics near the edge of the spectrum. For the results in this direction for other random matrix models we refer the reader to [18],[19],[17]. A recent result of Alon, Krivelevich and Vu [2] states that the deviation of the first, second, etc largest eigenvalue from
its mean is at most of order of $O(1)$. Unfortunately our results give only
the leading term of the mean.

Second, and perhaps even more difficult question is whether the local
behavior of the eigenvalues is not sensitive to the details of the distribution
of the matrix entries of $A$. We refer the reader to [16],[6],[17],[10] for the
results of that nature for unitary invariant and Wigner random matrices.

References

[1] M. Ajtai, J.Komlós and E. Szemerédi, Largest random component
of a k-cube, *Combinatorica*, 2, No. 1, 1-7, (1982).

[2] N.Alon, M.Krivelevich and V.H.Vu, On the concentration of eigen-
values of random symmetric matrices, preprint (2001).

[3] N.Alon, J.H.Spencer, *The Probabilistic Method*, Wiley, New York,
1992.

[4] B.Bollobás, *Random Graphs*, Academic Press, New York, 1985.

[5] Yu. Burtin, The probability of connectedness of a random graph, (in
Russian) *Problem Peredaci Informacii*, 13, No.2, 90-95, (1977).

[6] P. Deift, *Orthogonal Polynomials and Random Matrices: A
Riemann- Hilbert Approach*, Courant Lecture Notes in Mathematics
3, New York, 1999.

[7] S.Gavrillets, J.Gravner, Percolation on the fitness hypercube and the
evolution of reproductive isolation, *J.Theor. Biol.*, 184, No. 1, 51-
64, (1997).

[8] P.Erdös, J.H.Spencer, Evolution of the n-cube, *Comput. Math. Appl.*,
vol. 5, No. 1, 33-39, (1979).

[9] S.Janson, T.Luczak and A.Rucinski, *Random Graphs*,Wiley, New
York, 2000.

[10] K.Johansson, Universality of the local spacing distribution in certain
Hermitian Wigner matrices, *Commun. Math. Phys.*, 215, 683–705,
(2001).
[11] A.Khorunzhy, Sparse random matrices: spectral edge and statistics of rooted trees, Adv. Appl. Probab., 33, (2001).

[12] M.Krivelevich, B.Sudakov, The largest eigenvalue of sparse random graphs, preprint, 2001, available at arXiv:math.CO/0106066.

[13] L. Lovasz, Combinatorial Problems and Exercises, North Holland, Amsterdam, 1993.

[14] E. van Nimwegen, J.P. Crutchfield and M.Nuynen, Neutral evolution of mutational robustness, P. Natl. Acad. Sci. USA, 96, No.17, 9716-9720, (1999).

[15] E. van Nimwegen, J.P. Crutchfield, Metastable evolutionary dynamics: crossing fitness barriers or escaping via neutral path ?, B. Math Biol., 62, No. 5, 799-848, (2000).

[16] L.Pastur, M.Shcherbina, Universality of the local eigenvalue statistics for a class of unitary invariant random matrix ensembles, J.Stat. Phys., 86, 109-147, (1997).

[17] A.Soshnikov, Universality at the edge of the spectrum in Wigner random matrices, Commun. Math. Phys., 207, 697-733, (1999).

[18] C.A.Tracy, H.Widom, Level-spacing distributions and the Airy kernel, Commun. Math. Phys. 159, 151-174, (1994).

[19] C.A.Tracy, H.Widom, On orthogonal and symplectic matrix ensembles, Commun. Math. Phys. 177, 727-754, (1996).