$C^{1,\alpha}$-regularity for surfaces with $H \in L^p$

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**Abstract**

In this paper we prove several results on the geometry of surfaces immersed in $\mathbb{R}^3$ with small or bounded $L^2$ norm of $|A|$. For instance, we prove that if the $L^2$ norm of $|A|$ and the $L^p$ norm of $H$, $p > 2$, are sufficiently small, then such a surface is graphical away from its boundary. We also prove that given an embedded disk with bounded $L^2$ norm of $|A|$, not necessarily small, then such a disk is graphical away from its boundary, provided that the $L^p$ norm of $H$ is sufficiently small, $p > 2$. These results are related to previous work of Schoen-Simon [12] and Colding-Minicozzi [4].

1 Introduction

Inspired by the ideas of Schoen-Simon in [12] and Colding-Minicozzi in [4], in this paper we prove several results on the geometry of surfaces with small or bounded $L^2$ norm of $|A|$, where $|A| = \sqrt{k_1^2 + k_2^2}$ denotes the norm of the second fundamental form; $k_1, k_2$ are the principal curvatures.

Throughout this paper, $M$ will be a smooth, compact, oriented surface with boundary, immersed in $\mathbb{R}^3$. Given $x \in M$, we let $B_R(x)$ and $\mathcal{B}_R(x)$ denote the intrinsic and extrinsic open balls of radius $R$ centered at $x$. Often and when the center of these balls is clear from the context we will write $B_R$ and $\mathcal{B}_R$ instead of $B_R(x)$ and $\mathcal{B}_R(x)$. We let $H = k_1 + k_2$ denote the mean curvature.

One of the main theorems of this paper, Theorem 1.1 below, states that if $\mathcal{B}_R$ is an embedded disk with bounded $L^2$ norm of $|A|$, then $\mathcal{B}_R$ is graphical away from its boundary, provided that the $L^p$ norm of $H$ is sufficiently small, $p > 2$. This is related to previous results by Colding-Minicozzi for minimal surfaces [4], see also [2]. These previous results assume stronger conditions on $H$ and deliver point-wise estimates for $|A|$. Clearly, one cannot expect point-wise estimates for $|A|$ with our assumptions.

Let $\nu: M \to S^2$ denote the Gauss map and let

$$g(x) = \sqrt{1 - \nu(x) \cdot e_3} = \frac{1}{\sqrt{2}} |\nu(x) - e_3|, \quad x \in M.$$
Theorem 1.1. Given $K > 0$ and $p > 2$, there exist $\varepsilon = \varepsilon(K, p)$ and $\gamma = \gamma(K)$, such that the following holds. Let $\mathcal{B}_R := \mathcal{B}_R(x_0) \subset M \setminus \partial M$ be an embedded disk such that

$$\int_{\mathcal{B}_R} |A|^2 d\mathcal{H}^2 \leq Kr^4 \quad \text{and} \quad R^{\frac{p-2}{p}} \left( \int_{\mathcal{B}_R} |H|^p d\mathcal{H}^2 \right)^{\frac{1}{p}} \leq \varepsilon r^2$$

for some $r \in [0, 1/4]$. Then, after a rotation,

$$\sup_{\mathcal{B}_R} g \leq \frac{5}{4}r.$$

Remark 1.2. In Section 4 we actually prove a slightly more general version of Theorem 1.1 that does not require $\mathcal{B}_R$ to be a disk, cf. Theorem 4.4.

A key ingredient in proving Theorem 1.1 and indeed an interesting geometric result in its own right, is Theorem 1.3 below; it states that if the $L^2$ norm of $|A|$ and the $L^p$ norm of $H$ are sufficiently small, $p > 2$, then a geodesic ball is graphical away from its boundary. In contrast to Theorem 1.1, in this result we are neither assuming that the geodesic ball is a disk nor that it is embedded. This theorem was motivated by a classical result of Schoen-Simon for surfaces with quasi-conformal Gauss map [12]. It is also related to the Choi-Schoen Curvature Estimate for minimal surfaces [3] and our extension of it to surfaces with “small” mean curvature [2].

Theorem 1.3. There exist constants $c_1 > 0$ and $\beta \in (0, \frac{1}{2})$ such that the following holds. Given $p > 2$ there exists $c_2 = c_2(p)$ such that if $\mathcal{B}_R := \mathcal{B}_R(x_0) \subset M \setminus \partial M$ is such that

$$\int_{\mathcal{B}_R} |A|^2 d\mathcal{H}^2 \leq c_1 r^2 \quad \text{and} \quad \|H\|_{L^p(\mathcal{B}_R)} R^{\frac{p-2}{p}} \leq c_2 r$$

for some $r \in [0, 1]$ then, after a rotation,

$$\sup_{\mathcal{B}_{\beta r}} g \leq r.$$

Note that the assumption on the mean curvature is necessary as the $C^1$ norm of a smooth function over a bounded domain in $\mathbb{R}^2$ is not in general bounded by its $W^{2,2}$ norm.

In Corollary 3.2 we prove a similar result for surfaces with bounded, not necessarily small, $L^p$ norm of $H$. In this case the $L^2$ bound for $|A|$ depends on the $L^p$ bound for $H$.

Using Theorem 1.3, we can immediately prove an analogous result with the intrinsic ball replaced by an extrinsic one.
Corollary 1.4. Let $M$ be an orientable surface containing the origin with $\partial M \subset \partial B_R(0)$,
\[
\int_M |A|^2 d\mathcal{H}^2 \leq c_1 r^2 \quad \text{and} \quad \|H\|_{L^p(M \cap B_R)} R^{\frac{n-2}{p}} \leq c_2 r
\]
for some $r \in [0, \frac{1}{\sqrt{3}}]$, $p > 2$. Then, if $M_R$ is a connected component of $M \cap B_{\beta r}$
containing the origin, after a rotation
\[
\sup_{M_R} g \leq r,
\]
where the constants $c_1$, $c_2$ and $\beta$ are the constants in Theorem 1.3.

Our main theorems deal with intrinsic balls and, as an immediate consequence, we obtain results such as Corollary 1.4, where stronger hypotheses on the extrinsic geometry of $M$ are assumed. Among many other crucial findings, several results related to Corollary 1.4 can be found in [1, 12, 16].

After having proved our main results, the $C^{1,\alpha}$-regularity follows from standard PDE theory, see [7] (cf. Remark 5.3).

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2 Some results on the topology of $B_R$

In order to prove the main theorems, we first need to prove some more general results on the geometry and topology of a geodesic ball with small (or bounded) $L^2$ norm of the second fundamental form. For this we recall the isoperimetric inequality (see Poincare inequality, [15, Theorem 18.6] and [9]):

For any open subset $F$ of $M$, with $\overline{F} \subset M \setminus \partial M$, it is
\[
|F|^{1/2} \leq C \left( |\partial F| + \int_F |H| \, d\mathcal{H}^2 \right),
\]
where $C$ is an absolute constant and $|F|$, $|\partial F|$ denote the area of $F$ and the length of $\partial F$ respectively.

In the rest of the paper we will denote by $|U|$ the $n$-dimensional Hausdorff measure of $U$, whenever $U$ is a set of Hausdorff dimension $n$, as we did above with $|F|$ and $|\partial F|$.

The first lemma is a lower bound for the area of a surface, whose mean curvature has bounded $L^2$ norm.

Lemma 2.1. If $B_\rho := B_\rho(x_0) \subset M \setminus \partial M$ and $C\|H\|_{L^2(B_\rho)} \leq \frac{1}{2}$, where $C$ is the isoperimetric constant given in (1), then
\[
|B_\rho| \geq \frac{1}{16C^2} \rho^2.
\]
Proof. The isoperimetric inequality (1), with $F = B_\rho$ gives

$$|B_\rho|^{\frac{1}{2}} \leq C \left( |\partial B_\rho| + \int_{\partial B_\rho} |H| d\mathcal{H}^2 \right) \leq C \left( |\partial B_\rho| + |B_\rho|^{\frac{1}{2}} \|H\|_{L^2(B_\rho)} \right).$$

Thus, if $C\|H\|_{L^2(B_\rho)} \leq \frac{1}{2}$ we obtain that

$$|B_\rho|^{\frac{1}{2}} \leq 2C|\partial B_\rho|. \quad (2)$$

Therefore for almost every $\rho$

$$|B_\rho|^{\frac{1}{2}} \leq 2C \frac{d}{d\rho} |B_\rho| \implies \frac{d}{d\rho} (|B_\rho|^{\frac{1}{2}}) \geq \frac{1}{4C}. \quad (3)$$

Integrating the equation above finishes the proof of the lemma. $\square$

Recall that

$$H^2 = (k_1 + k_2)^2 = k_1^2 + k_2^2 + 2k_1k_2 \leq 2(k_1^2 + k_2^2) = 2|A|^2.$$

Therefore, if we assume $C\|A\|_{L^2(B_\rho)} \leq \frac{1}{4}$, then the hypothesis on $H$ in Lemma 2.1 and thus the conclusion of the lemma still hold.

It follows from the work in [6, 8, 13, 14] that for all $s > 0$,

$$|\partial B_s| = \int_0^s \int_{\partial B_\rho} k_g d\mathcal{H}^1 d\rho - F(s)$$

where $k_g$ is the geodesic curvature of $\partial B_\rho$ and $F(s)$ is a non-decreasing function with $F(0) = 0$. By using now the Gauss-Bonnet theorem, we have that

$$|\partial B_s| \leq \int_0^s \left( 2\pi \chi(B_\rho) - \int_{\partial B_\rho} K d\mathcal{H}^2 \right) d\rho \quad (4)$$

where $\chi(B_\rho)$ denotes the Euler characteristic of $B_\rho$ (see also [4, 10]). We are now going to use Lemma 2.1 and equation (4) to study the topology of geodesic balls with small total curvature.

Given $B_R \subset M \setminus \partial M$ let

$$T_1 = \{ \rho \in [0, R] : \chi(B_\rho) = 1 \}$$
$$T_0 = \{ \rho \in [0, R] : \chi(B_\rho) \leq 0 \}$$

and for $i = 0, 1$, define $\alpha_i \in [0, 1]$ to be such that $|T_i| = \alpha_i R$. Since

$$\chi(B_\rho) = 2 - \kappa - 2g, \quad (5)$$
where $\kappa$ is the number of components of $\partial B_\rho$ and $g$ is the genus, we have that
\[ \chi(B_\rho) \leq 1 \] for all $\rho \in [0, R]$ and thus $|T_1| + |T_0| = R$ giving $\alpha_1 + \alpha_0 = 1$.

Using Lemma 2.1 and equation (3) in its proof, and equation (4), we obtain that
\[
\frac{R}{4C} \leq |B_R|^\frac{1}{2} \leq 2C|\partial B_R| \leq 4C\pi \int_0^R \chi(B_s)ds - 2C \int_0^R \int_{B_s} K d\mathcal{H}^2 ds,
\]
provided that $\|H\|_{L^2(B_R)} \leq \frac{1}{2C}$. Since the Gauss equation gives
\[ -K = \frac{|A|^2 - |H|^2}{2} \leq \frac{|A|^2}{2}, \]
we obtain
\[
\frac{R}{4C} \leq 4C\pi \alpha_1 R + CR \int_{B_R} |A|^2 d\mathcal{H}^2.
\]
Therefore, if $\int_{B_R} |A|^2 d\mathcal{H}^2 \leq \frac{1}{8C^2}$ (which also implies that $\|H\|_{L^2(B_R)} \leq \frac{1}{2C}$) we get
\[
\frac{1}{32\pi C^2} \leq \alpha_1,
\]
namely
\[
|T_1| \geq \frac{1}{32\pi C^2} R.
\]

Note that, by (5), if $\rho \in T_1$, then $B_\rho$ is homeomorphic to a disk. Thus, we have proven the following lemma on the topology of geodesic balls with small $L^2$ norm of $|A|$.

**Lemma 2.2.** Let $B_R := B_R(x_0) \subset M \setminus \partial M$ be such that $\int_{B_R} |A|^2 d\mathcal{H}^2 \leq \frac{1}{8C^2}$, where $C$ is the isoperimetric constant given in (1), then
\[
|T_1| \geq \frac{1}{32\pi C^2} R,
\]
where $T_1 = \{ \rho \in [0, R] : \chi(B_\rho) = 1 \}$.

The next lemma is an estimate from above for the area of a geodesic ball and the length of its boundary in terms of the $L^2$ norm of $|A|$.

**Lemma 2.3.** If $B_\rho := B_\rho(x_0) \subset M \setminus \partial M$ then
\[
|B_\rho| \leq \pi \rho^2 + \frac{1}{2} \rho^2 \int_{B_\rho} |A|^2 d\mathcal{H}^2
\]
and
\[
|\partial B_\rho| \leq 2\pi \rho + \frac{1}{2} \rho \int_{B_\rho} |A|^2 d\mathcal{H}^2.
\]
Proof. Integrating equation (4) from 0 to $\rho$ we have

$$|B_\rho| \leq 2\pi \int_0^\rho \int_0^{s_0} \chi(B_s) d\rho d\sigma - \int_0^{s_0} \int_{B_s} K dH^2 d\rho d\sigma. \quad (6)$$

Since $-K \leq \frac{|A|^2}{2}$ and $\chi(B_s) \leq 1$ for all $s$ we obtain from (4) with $s = \rho$

$$|\partial B_\rho| \leq 2\pi \rho + \frac{1}{2} \rho \int_{B_\rho} |A|^2 dH^2$$

and from (6)

$$|B_\rho| \leq 2\pi \int_0^\rho s_0 d\rho + \frac{1}{2} \rho^2 \int_{B_\rho} |A|^2 dH^2 \leq \pi \rho^2 + \frac{1}{2} \rho^2 \int_{B_\rho} |A|^2 dH^2.$$

\[ \square \]

3 Small total curvature implies graphical

In this section we prove Theorem 1.3 and its corollaries. Namely, we prove that for any geodesic ball, if the $L^2$ norm of $|A|$ and the $L^p$ norm of $H$, $p > 2$, are sufficiently small, then such ball is graphical away from its boundary.

We begin by proving a lemma stating that given a compact surface $M$ whose boundary satisfies certain geometric conditions, if the $L^2$ norm of $|A|$ and the $L^p$ norm of $H$, $p > 2$, are sufficiently small, $M$ is (locally) a graph over a fixed plane.

Recall the definition in the introduction; let $\nu : M \to S^2$ denote the Gauss map and let

$$g(x) = \sqrt{1 - \nu(x) \cdot e_3} = \frac{1}{\sqrt{2}} |\nu(x) - e_3|, \quad x \in M.$$ 

Note that if $g \leq \frac{1}{4}$ implies that $M$ is locally graphical over the plane $\{x_3 = 0\}$ with gradient bounds (cf. Lemma 5.2). The core of the proof of the lemma follows the ideas in [12].

**Lemma 3.1.** Given $p > 2$ there exists a constant $c_3 > 0$, depending only on $p$, such that the following holds. Let $M$ be a compact orientable surface with boundary such that

$$\int_M |A|^2 dH^2 \leq \frac{\pi}{2} r^2 \quad \text{and} \quad \|H\|_{L^p(M)} |M|^{\frac{p-2}{2p}} \leq c_3 r$$

for some $r \in (0,1]$. If either

$$g < r \quad \text{on} \quad \partial M$$

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or
\[ g > r \text{ on } \partial M \text{ and } \inf_M g < \frac{3}{4} r, \]
then
\[ g < r \text{ on } \partial M \text{ and } \sup_M g \leq \frac{5}{4} r. \]

**Proof.** Let
\[ \tilde{g} = \begin{cases} r - g, & \text{if } g > r \text{ on } \partial M, \\ g - r, & \text{if } g < r \text{ on } \partial M, \end{cases} \]
so that \( \tilde{g} < 0 \text{ on } \partial M. \) We claim that
\[ \tilde{g} \leq \frac{r}{4} \text{ on } M. \] (7)

We first show how the lemma follows easily from (7) above. If (7) were true, then it remains to show that \( \tilde{g} \) must be in fact equal to \( g - r, \) because in that case
\[ \tilde{g} \leq \frac{r}{4} \implies g \leq \frac{5}{4} r. \]
Suppose that instead \( \tilde{g} = r - g, \) i.e. \( g > r \) on \( \partial M. \) Then (7) implies that \( \tilde{g} = r - g \leq \frac{r}{4} \) on \( M \) and thus
\[ g \geq \frac{3}{4} r \text{ on } M. \]
This contradicts the fact that \( \inf_M g < \frac{3}{4} r. \)

We now prove equation (7), i.e. that \( \tilde{g} \leq \frac{r}{4} \) on \( M. \) We begin by defining the following sequence
\[ r_0 = 0, \quad r_1 = \frac{r}{2^2}, \quad \ldots, \quad r_k = \sum_{i=1}^{k} \frac{r}{2^i} = r \frac{2^k - 1}{2^{k+2}}, \quad \ldots \]
for which we note that
\[ 0 = r_0 < r_1 < \cdots < r_k < \cdots < r/4 \text{ and } r_k - r_{k-1} = \frac{r}{2^{k+2}}. \]

Since \( |\nabla \tilde{g}| = |\nabla g| \leq |A| \) (see for instance [11, Proof of Lemma 1]), the Jacobian of \( \tilde{g} \) is bounded by \( |A| \) and thus applying the co-area formula [15, §10] we obtain that
\[ \int_{r_{k-1}}^{r_k} |\Gamma_s| ds \leq \int_{M_k} |A| d\mathcal{H}^2 \] (8)
where, for any \( s \in (r_{k-1}, r_k), \)
\[ \Gamma_s = \{ x \in M : \tilde{g}(x) = s \} \text{ and } M_k = \{ x \in M : r_{k-1} < \tilde{g}(x) < r_k \}. \]
Applying Sard’s Theorem, for each \( k \) we can pick \( s_k \in (r_{k-1}, r_k) \), such that \( \Gamma_{s_k} \) is a collection of smooth Jordan curves and such that
\[
|\Gamma_{s_k}| \leq \frac{2^{k+2}}{r} \int_{M_k} |A| d\mathcal{H}^2. \tag{9}
\]

For each \( k \) let
\[
U_k := \{ x \in M : \tilde{g}(x) > s_k \},
\]
with the \( s_k \)'s as above, and note that \( U_k \subset M \setminus \partial M \), since on \( \partial M \) we have \( \tilde{g} < 0 \).
Furthermore
\[
s_1 < s_2 < \cdots < s_k < \ldots \Rightarrow U_1 \supset U_2 \supset \cdots \supset U_k \supset \ldots
\]
and \( \lim_{k \to \infty} s_k = r/4 \). Let
\[
U_\infty := \{ x \in M : \tilde{g}(x) \geq r/4 \} = \bigcap_{k \in \mathbb{N}} U_k,
\]
then, to prove (7) it suffices to show that
\[
|U_\infty| = \left| \bigcap_{k \in \mathbb{N}} U_k \right| = \lim_{k \to \infty} |U_k| = 0. \tag{10}
\]
That is because if claim (7) does not hold, then there would be a point \( p \in M \) such that \( \tilde{g}(p) > \frac{r}{4} \) implying, since \( \tilde{g} \) is continuous, that \( |U_\infty| > 0 \).

In order to prove equation (10), let \( G : S^2 \to \mathbb{R} \) denote the map
\[
G((\nu_1, \nu_2, \nu_3)) = \sqrt{1 - \nu_3} = \frac{1}{\sqrt{2}} |(\nu_1, \nu_2, \nu_3) - e_3|
\]
and let
\[
\tilde{G} = \begin{cases} \quad r - G, & \text{if } g > r \text{ on } \partial M, \\ G - r, & \text{if } g < r \text{ on } \partial M. \end{cases}
\]
Note that \( g = \tilde{G} \circ \nu \) where \( \nu \) is the Gauss map of \( M \). Then
\[
\nu(\partial U_k) \subset D_k := \left\{ (\nu_1, \nu_2, \nu_3) \in S^2 : \tilde{G}((\nu_1, \nu_2, \nu_3)) = s_k \right\}
\]
and
\[
\nu(U_k) \subset \Delta_k := \left\{ (\nu_1, \nu_2, \nu_3) \in S^2 : \tilde{G}((\nu_1, \nu_2, \nu_3)) > s_k \right\}.
\]
Since \( -K \) is the signed area magnification of the Gauss map, we have
\[
\int_{U_k} (-K) d\mathcal{H}^2 = n|\Delta_k| \tag{11}
\]
where \( n \in \mathbb{Z} \) is the degree of the map \( \nu \). Therefore,

\[
\int_{U_k} |K| \, d\mathcal{H}^2 \geq |n| |\Delta_k|.
\]

We claim that

\[
|\Delta_k| \geq 2\pi \min \left\{ \left( \frac{3}{4} r \right)^2, \frac{7}{16} \right\}.
\]

(13)

In order to prove the claim, we need to discuss two separate cases depending on the definition of \( \tilde{g} \) and thus of \( \tilde{G} \).

Case 1: \( \tilde{g} = r - g \) and \( \tilde{G} = r - G \). In this case

\[
D_k = \{ (\nu_1, \nu_2, \nu_3) \in S^2 : \nu_3 = 1 - (r - s_k)^2 \},
\]

\[
\Delta_k = \{ (\nu_1, \nu_2, \nu_3) \in S^2 : \nu_3 > 1 - (r - s_k)^2 \}.
\]

Since \( s_k < \frac{r}{4} \), this implies that \( \Delta_k \) contains the upper spherical cap that has boundary \( \nu_3 = 1 - \left( \frac{3}{4} r \right)^2 \), whose area is \( 2\pi \left( \frac{3}{4} r \right)^2 \). Therefore,

\[
|\Delta_k| \geq 2\pi \left( \frac{3}{4} r \right)^2.
\]

Case 2: \( \tilde{g} = g - r \) and \( \tilde{G} = G - r \). In this case

\[
D_k = \{ (\nu_1, \nu_2, \nu_3) \in S^2 : \nu_3 = 1 - (r + s_k)^2 \},
\]

\[
\Delta_k = \{ (\nu_1, \nu_2, \nu_3) \in S^2 : \nu_3 < 1 - (r + s_k)^2 \}.
\]

Since \( s_k < \frac{r}{4} \) and \( r \leq 1 \), this implies that \( \Delta_k \) contains the lower spherical cap that has boundary \( \nu_3 = 1 - \left( \frac{5}{4} r \right)^2 \), whose area is \( 2\pi \left( 2 - \left( \frac{5}{4} r \right)^2 \right) \). Therefore,

\[
|\Delta_k| \geq 2\pi \left( 2 - \left( \frac{5}{4} r \right)^2 \right) = 2\pi \frac{7}{16}.
\]

Hence the claim (13) is true, that is

\[
|\Delta_k| \geq 2\pi \min \left\{ \left( \frac{3}{4} r \right)^2, \frac{7}{16} \right\}.
\]

By the inequalities (12) and (13) and recalling the hypothesis of the lemma on \( \int |A|^2 \, d\mathcal{H}^2 \), we have

\[
2\pi \min \left\{ \left( \frac{3}{4} r \right)^2, \frac{7}{16} \right\} |n| \leq |\Delta_k| |n| \leq \int_{U_k} |K| \, d\mathcal{H}^2
\]

\[
\leq \int_{U_k} \frac{|A|^2}{2} \, d\mathcal{H}^2 \leq \frac{1}{2} \int_M |A|^2 \, d\mathcal{H}^2 \leq \frac{\pi r^2}{4},
\]

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which implies that \( n = 0 \), since \( r \leq 1 \).

Now, since \( n = 0 \), equation (11) gives that

\[
\int_{U_k} -K \, d\mathcal{H}^2 = 0.
\]

Hence by the Gauss equation

\[
\int_{U_k} |A|^2 \, d\mathcal{H}^2 = \int_{U_k} H^2 \, d\mathcal{H}^2. \tag{14}
\]

Applying the isoperimetric inequality (1) with \( F = U_k \) we obtain

\[
|U_k|^{1/2} \leq C \left( |\partial U_k| + \int_{U_k} |H| \, d\mathcal{H}^2 \right)
\]

and since \( \partial U_k = \Gamma_{s_k} \), using (9) we get

\[
|U_k|^{1/2} \leq C \left( \frac{2^{k+2}}{r} \int_{M_k} |A| \, d\mathcal{H}^2 + \int_{U_k} |H| \, d\mathcal{H}^2 \right)
\]

\[
\leq C \left( \frac{2^{k+2}}{r} \int_{U_{k-1}} |A| \, d\mathcal{H}^2 + \int_{U_k} |H| \, d\mathcal{H}^2 \right)
\]

\[
\leq C \frac{2^{k+3}}{r} \int_{U_{k-1}} |A| \, d\mathcal{H}^2,
\]

where we have used the facts \( |H| \leq 2|A| \), \( M_k \subset U_{k-1}, \ U_k \subset U_{k-1} \) and \( 2 < \frac{2^{k+2}}{r} \), since \( r \leq 1 \). Using Holder inequality and then squaring both sides of the inequality gives

\[
|U_k| \leq C_1 \frac{2^{2k}}{r^2} |U_{k-1}| \int_{U_{k-1}} |A|^2 \, d\mathcal{H}^2
\]

where \( C_1 = (8C)^2 \) is an absolute constant.

Applying (14) and Holder inequality we have

\[
|U_k| \leq C_1 2^{2k} r^{-2} |U_{k-1}| \int_{U_{k-1}} |H|^2 \, d\mathcal{H}^2 \leq C_1 2^{2k} r^{-2} \| H \|_{L^p(M)}^2 |U_{k-1}|^{\frac{q+1}{q}}, \tag{15}
\]

where \( q \) is such that \( 1/q + 2/p = 1 \). By iterating (15) we obtain:

\[
|U_k|^{\frac{q}{q+p}} \leq \left( C_1 r^{-2} \| H \|_{L^p(M)}^2 \right)^{\frac{q}{q+p}} 4^{k \frac{q}{q+p}} |U_{k-1}|
\]

\[
|U_k|^{\left( \frac{q}{q+p} \right)^2} \leq \left( C_1 r^{-2} \| H \|_{L^p(M)}^2 \right)^{\left( \frac{q}{q+p} \right)^2} 4^{k \left( \frac{q}{q+p} \right)^2} |U_{k-1}|^{\frac{q}{q+p}}
\]

\[
\leq \left( C_1 r^{-2} \| H \|_{L^p(M)}^2 \right)^{\left( \frac{q}{q+p} \right)+\left( \frac{q}{q+p} \right)^2} 4^{(k-1) \frac{q}{q+p}+k(\frac{q}{q+p})^2} |U_{k-2}|
\]
\[
|U_1| \leq C_2 \|H\|_{L^p(M)}^{\frac{q+1}{q}} |M|^{\frac{q}{q+1} r - 2} 
\]
where \(C_2\) is a constant that depends solely on \(p\). Assume that equation (10) is not true, namely assume that \(|U_\infty| > 0\). Then, since \(U_\infty = \bigcap_{k \in \mathbb{N}} U_k \subset U_k\) and \(\frac{q}{q+1} < 1\), we have
\[
\lim_{k \to \infty} |U_k| \left(\frac{q}{q+1}\right)^{k-1} = 1
\]
and using this in (16) we get
\[
1 \leq C_2 \|H\|_{L^p(M)}^{2q+2} |M|^{\frac{q+1}{q}} r^{-2} - 2q.
\]
Therefore if in the hypotheses of the lemma we take \(c_3\) to be
\[
c_3 = \left(\frac{1}{2C_2}\right)^{\frac{1}{q+1}}
\]
then
\[
\|H\|_{L^p(M)} |M|^{\frac{1}{2q}} r^{-1} \leq \left(\frac{1}{2C_2}\right)^{\frac{1}{q+1}}
\]
which in turn gives
\[
1 \leq C_2 \|H\|_{L^p(M)}^{2q+2} |M|^{\frac{q+1}{q}} r^{-2} - 2q \leq \frac{1}{2}.
\]
This contradiction proves that actually \(|U_\infty| = 0\). As we discussed before, this implies claim (7), that is \(\tilde{g} \leq \frac{1}{2}\), which in turn implies the lemma and thus, taking \(c_3\) as given by equation (17) finishes the proof.

We are now ready to prove Theorem 1.3 in the introduction. It says that if the \(L^2\) norm of \(|A|\) and the \(L^p\) norm of \(H\) are sufficiently small, \(p > 2\), then a geodesic ball is graphical away from its boundary. For convenience, we recall its statement.
Theorem 1.3. There exist constants $c_1 > 0$ and $\beta \in (0, \frac{1}{2})$ such that the following holds. Given $p > 2$ there exists $c_2 = c_2(p)$ such that if $B_R := B_R(x_0) \subset M \setminus \partial M$ is such that
\[
\int_{B_R} |A|^2 d\mathcal{H}^2 \leq c_1 r^2 \quad \text{and} \quad \|H\|_{L^p(B_R)} R^{\frac{p-2}{p}} \leq c_2 r
\]
for some $r \in [0, 1]$ then, after a rotation,
\[
\sup_{B_{\beta R}} g \leq r.
\]

Proof. Let $c_2 = \frac{c_3}{4\pi}$, where $c_3 = c_3(p)$ is the constant in Lemma 3.1. To prove this theorem we will show that there exists $s_0 \in [r/2, 4r/5]$ and $\beta \in (0, \frac{1}{2})$, such that if $c_1$ is small enough, then all the hypotheses of Lemma 3.1 are satisfied with $M = B_{\beta R}$ and $r = s_0$.

After rotating the surface, we can assume that $\nu(x_0) = c_3$, i.e. $g(x_0) = 0$, where recall that $x_0$ is the center of the given geodesic ball $B_R := B_R(x_0)$. By Lemma 2.3,
\[
\int_{B_\beta R} |A|^2 d\mathcal{H}^2 \leq c_1 r^2 \leq 2\pi c_1 \implies |B_R| \leq \pi R^2 (1 + c_1).
\]  
(18)

Note also that if $c_1 \leq \pi/8$ then for any $\beta \in (0, 1]$ and any $s \in [r/2, 4r/5]$, we have
\[
\int_{B_{\beta R}} |A|^2 d\mathcal{H}^2 \leq c_1 r^2 \leq \frac{\pi}{8} \left(\frac{r}{2}\right)^2 \leq \frac{\pi}{2} s^2
\]
and using (18) we also have
\[
\|H\|_{L^p(B_{\beta R})} \leq \frac{c_3}{4\pi} R^{\frac{p-2}{p}} |B_R|^{\frac{p-2}{p}} \leq \frac{c_3}{4\pi} R^{\frac{p-2}{p}} (\pi (1 + c_1))^{\frac{p-2}{p}} \leq \frac{c_3}{4\pi} (2\pi)^{\frac{p-2}{p}}
\]

To apply Lemma 3.1 it remains to show that there exists $s_0 \in [r/2, 4r/5]$ and $\beta \in (0, 1]$, such that on $\partial B_{\beta R}$ either $g > s_0$ or $g < s_0$. We obtain this by showing that we can find $\beta$ and $s_0$ such that $\partial B_{\beta R}$ consists of exactly one connected component and $g \neq s_0$ on $\partial B_{\beta R}$. In fact, we will show that $B_{\beta R}$ is homeomorphic to a disk.

Arguing exactly as we did in the proof of Lemma 3.1, equation (8), since $|\nabla g| \leq |A|$, the Jacobian of $g$ is bounded by $|A|$ and thus applying the co-area formula in $B_R$ for the function $g$, we get:
\[
\int_{\Gamma_s} |s| ds \leq \int_{M_r} |A| d\mathcal{H}^2 \leq |M_r|^{1/2} \left(\int_{M_r} |A|^2 d\mathcal{H}^2\right)^{1/2} \leq |B_R|^{1/2} \left(\int_{B_R} |A|^2 d\mathcal{H}^2\right)^{1/2} \leq Rr (\pi c_1 (1 + c_1))^{1/2}
\]
where \( M_r = \{ x \in B_R : \frac{r}{2} < g(x) < \frac{4r}{5} \} \) and for any \( s \in [\frac{r}{2}, \frac{4r}{5}] \), \( \Gamma_s = \{ x \in B_R : g(x) = s \} \) and where we have used the inequality in (18).

By Sard’s theorem, for almost all \( s \in [r/2, 4r/5] \), \( \Gamma_s \) is collection of smooth, simple curves and we can pick \( s_0 \in [r/2, 4r/5] \) such that

\[
|\Gamma_{s_0}| \leq \frac{10}{3r} Rr (\pi c_1 (1 + c_1))^{\frac{1}{2}} < 4R (\pi c_1 (1 + c_1))^{\frac{1}{2}}.
\]

Thus, if we let \( \Delta := \{ \rho \in (0, R) : \Gamma_{s_0} \cap \partial B_\rho \neq \emptyset \} \) then

\[
|\Delta| \geq R - 4R (\pi c_1 (1 + c_1))^{\frac{1}{2}} = R \left(1 - 4 (\pi c_1 (1 + c_1))^{\frac{1}{2}}\right).
\]  

(19)

Note that \( g|_{\partial B_\rho} \neq s_0 \) for any \( \rho \in \Delta \). However, since \( \partial B_\rho \) consists of possibly more than one connected components, this does not imply that \( g - s_0 \) has a sign on \( \partial B_\rho \).

If we can find \( \rho \) in \( \Delta \) for which \( \chi(B_\rho) = 1 \) then, for this \( \rho \), \( B_\rho \) is homeomorphic to a disk, \( \partial B_\rho \) consists of a unique connected component and hence \( g - s_0 \) does have a sign on \( \partial B_\rho \).

Recall that Lemma 2.2 states that

\[
\int_{B_R} |A|^2 dH^2 \leq \frac{1}{8C^2} \Rightarrow |T_1| \geq \frac{1}{32\pi C^2} R,
\]  

(20)

where \( C \) is the isoperimetric constant given in (1), and where

\[ T_1 = \{ \rho \in [0, R] : \chi(B_\rho) = 1 \}. \]

Let \( \beta = \frac{1}{2(32\pi C^2)} \), then (20) becomes

\[ |T_1| \geq 2\beta R. \]

If we take \( c_1 \) sufficiently small such that

\[ 4 (\pi c_1 (1 + c_1))^{\frac{1}{2}} \leq \beta \]

then the hypothesis and thus the implication of (20) holds and (19) becomes

\[ |\Delta| \geq R (1 - \beta). \]

Therefore for some \( \gamma \geq \beta \) we have that \( \gamma R \in \Delta \cap T_1 \), namely \( B_{\gamma R} \) is homeomorphic to a disk, \( \partial B_{\gamma R} \) consists of one connected component and \( g \neq s_0 \) on \( \partial B_{\gamma R} \). Hence, by applying Lemma 3.1 to \( B_{\gamma R} \) we have that

\[
\sup_{B_{\beta R}} g \leq \sup_{B_{\gamma R}} g \leq \frac{5}{4} s_0 \leq r.
\]

This finishes the proof of the theorem.

\[ \square \]
From the above theorem, an extrinsic version of the same theorem follows.

**Corollary 1.4.** Let $M$ be an orientable surface containing the origin with $\partial M \subset \partial B_R(0)$,

\[
\int_M |A|^2 d\mathcal{H}^2 \leq c_1 r^2 \quad \text{and} \quad \|H\|_{L^p(M \cap B_R)} R^{\frac{p-2}{p}} \leq c_2 r
\]

for some $r \in [0, \frac{1}{\sqrt{2}}]$, $p > 2$. Then, if $M_R$ is a connected component of $M \cap B_{\beta R}$ containing the origin, after a rotation

\[
\sup_{M_R} g \leq r,
\]

where the constants $c_1$, $c_2$ and $\beta$ are the constants in Theorem 1.3.

**Proof.** Let $M_R$ be a connected component of $M \cap B_{\beta R}$ containing the origin and let $B_R$ be “the” geodesic ball of radius $R$ centered at the origin. Note that since $M$ is not assumed to be embedded, the pre-image of the origin in $\mathbb{R}^3$ may consist of several points in $M$. Thus, by $B_R$ we indicate a geodesic ball of radius $R$ centered at one of those pre-images related to $M_R$. By the previous theorem, after a rotation,

\[
\sup_{B_{\beta R}} g \leq r
\]

and since $r \leq \frac{1}{\sqrt{2}}$, this gives that $B_{\beta R}$ contains a graph over a domain $\Omega \subset \mathbb{R}^2$ with

\[
\left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq \left( \frac{\beta R}{2} \right)^2 \right\} \subset \Omega
\]

(see Lemma 5.2) and hence $\partial \Omega \cap B_{\beta R} = \emptyset$, which finishes the proof of the corollary. \(\square\)

In the next corollary we prove that if the $L^p$ norm of the mean curvature is bounded, $p > 2$, then, if the $L^2$ norm of $|A|$ is sufficiently small, a geodesic ball is graphical away from its boundary.

**Corollary 3.2.** Given any $p > 2$ and $K > 0$, there exists $\varepsilon = \varepsilon(p, K)$ such that if $B_R := B_R(x_0) \subset M \setminus \partial M$,

\[
\int_{B_R} |A|^2 d\mathcal{H}^2 \leq \varepsilon r^2 \quad \text{and} \quad \|H\|_{L^p(B_R)} R^{\frac{p-2}{p}} \leq Kr
\]

for some $r \in [0, 1]$ then, after a rotation,

\[
\sup_{B_{\beta R}} g \leq r,
\]

where $\beta$ is as in Theorem 1.3.
Proof. Let $c_1$ be as in Theorem 1.3 and let $p' = \frac{p}{2} + 1 > 2$. We will show that there exists $\varepsilon = \varepsilon(p, K)$ such that if the hypotheses of Corollary 3.2 are satisfied, i.e.

$$
\int_{B_R} |A|^2 \, d\mathcal{H}^2 \leq \varepsilon r^2 \quad \text{and} \quad \|H\|_{L^p(B_R)} R^{\frac{p}{p-2}} \leq Kr,
$$

(21)

for some $r \in [0, 1]$, then also the hypotheses of Theorem 1.3 are satisfied, i.e.

$$
\int_{B_R} |A|^2 \, d\mathcal{H}^2 \leq c_1 r^2 \quad \text{and} \quad \|H\|_{L^{p'}(B_R)} R^{\frac{p'-2}{p'}} \leq c_2 r
$$

(22)

with the same $r$. Then by Theorem 1.3 we have that

$$\sup_{B_{\beta R}} g \leq r$$

and thus the Corollary is true.

To show that (21) implies (22), note first that by picking $\varepsilon \leq c_1$, gives that

$$
\int_{B_R} |A|^2 \, d\mathcal{H}^2 \leq \varepsilon r^2. \quad \text{By using that} \quad |H|^2 \leq 2|A|^2 \quad \text{and Holder inequality, we have that}
$$

$$
\|H\|_{L^{p'}(B_R)} R^{\frac{p'-2}{p'}} \leq R^{\frac{p'-2}{p'}} \left( \int_{B_R} |H|^2 \, d\mathcal{H}^2 \right)^{\frac{1}{2p'}} \leq R^{\frac{p'-2}{p'}} \left( \int_{B_R} |A|^2 \, d\mathcal{H}^2 \right)^{\frac{1}{2p'}} \left( \int_{B_R} |H|^2 \, d\mathcal{H}^2 \right) \leq (2\varepsilon r^2)^{\frac{1}{2p'}} (Kr)^{\frac{p}{2p'}} R^{\frac{2-p}{2p'}} \leq (\varepsilon K^p)^{\frac{1}{2p}} r \leq c_2 r
$$

with the last inequality being true provided that

$$\varepsilon \leq c_2^{p+2} K^{-p}.\]

So picking $\varepsilon = \varepsilon(p, K)$ as above the hypotheses (22) are satisfied and this finishes the proof of the Corollary.

\[ \square \]

4 Graph representation in terms of $\|A\|_{L^2}$, when $\|H\|_{L^p}$ is small

In this section we use results from the previous sections to prove that an embedded geodesic disk with bounded $L^2$ norm of $|A|$ and sufficiently small $L^p$ norm of the mean curvature, $p > 2$, is graphical away from its boundary. This is related to previous results by Colding-Minicozzi for minimal surfaces [4], see also [2].
Definition 4.1. Let $M$ be a simply-connected surface embedded in $\mathbb{R}^3$. For any $x \in M$ and $R > 0$ such that $B_R(x) \subset M \setminus \partial M$, $[M \cup \partial M] \setminus B_R(x)$ has a unique connected component $A$ with $\partial M \subset A$. We denote the complement of $A$ in $[M \cup \partial M]$ by $B^*_R(x)$. Elementary topological arguments show that $B^*_R(x)$ is simply-connected, $B_R(x) \subset B^*_R(x)$ and $\partial B_R(x) \subset \partial B^*_R(x)$.

The following lemma shows that given an embedded geodesic ball $B_R$ in a simply-connected surface, if $B_R$ has small $L^2$ norm of $|A|$ away from the origin and also $B^*_R$ has sufficiently small $L^p$ norm of $H$, then this geodesic ball is graphical away from its boundary.

Lemma 4.2. Given $K \geq 0$ and $N \geq 20$ there exists $\varepsilon_1 = \varepsilon_1(K, N) > 0$ such that for any $p > 2$ the following holds. Let $M$ be a simply-connected surface embedded in $\mathbb{R}^3$ containing the origin and let $B_N := B_N(0) \subset M \setminus \partial M$ be such that

$$\int_{B_N} |A|^2 d\mathcal{H}^2 \leq K, \quad \int_{B_N \setminus B_1} |A|^2 d\mathcal{H}^2 \leq c_1(\varepsilon_1 r)^2$$

and

$$(16C^2 |B^*_N|)^{\frac{2}{2-p}} \left( \int_{B^*_N} |H|^p d\mathcal{H}^2 \right)^{\frac{1}{p}} \leq c_2 \varepsilon_1 r,$$

for some $r \in [0, \frac{1}{4}]$, where $c_1, c_2 = c_2(p)$ are as in Theorem 1.3 and where $C$ is the isoperimetric constant as in (1). Then, after a rotation,

(i) $$\sup_{B_{N-1} \setminus B_2} g \leq r,$$

(ii) $$\int_{B_2^*} |A|^2 d\mathcal{H}^2 \leq (c_2 \varepsilon_1 r)^2 + 24\pi \varepsilon_1 r \left( \frac{K + 4\pi + 4(N - 3)}{\beta} + 1 + c_1 \right),$$

where $\beta$ as in Theorem 1.3 and

(iii) $$\sup_{B_2^*} g \leq \frac{5}{4} \sqrt{r}.$$

Proof. Note that

$$\left( \int_{B_N} |H|^2 d\mathcal{H}^2 \right)^{\frac{1}{2}} \leq \left( \int_{B_N} |H|^2 d\mathcal{H}^2 \right)^{\frac{1}{2}} \leq |B^*_N|^{\frac{p-2}{2-p}} \left( \int_{B^*_N} |H|^p d\mathcal{H}^2 \right)^{\frac{1}{p}} \leq \frac{c_2 \varepsilon_1 r}{(16C^2)^{\frac{2}{2-p}}}.$$

Hence, if $\varepsilon_1 \leq \frac{1}{2cC}$ then $\frac{c_2 \varepsilon_1 r}{(16C^2)^{\frac{2}{2-p}}} \leq \frac{1}{2C}$, and we can apply Lemma 2.1, which gives

$$N^2 \leq 16C^2 |B_N| \leq 16C^2 |B^*_N|.$$
Therefore,

\[ N^{\frac{p-2}{p}} \left( \int_{B_N} |H|^p \, dH^2 \right)^{\frac{1}{p}} \leq (16C^2|B_n|)^{\frac{p-2}{2p}} \left( \int_{B_N} |H|^p \, dH^2 \right)^{\frac{1}{p}} \leq c_2\varepsilon_1 r \]

Furthermore, for any \( x \in B_{N-1} \setminus B_2 \), we have that \( B_1(x) \subset B_N \setminus B_1 \) and thus, by the previous discussion and the assumptions on \( |A|^2 \), we note that the hypotheses of Theorem 1.3 are satisfied with \( B_R(x_0) \) replaced by \( B_1(x) \) and with \( r \) replaced by \( \varepsilon_1 r \). Applying Theorem 1.3 gives then that

\[ \frac{1}{\sqrt{2}} |\nu(y) - \nu(x)| \leq \varepsilon_1 r, \quad \forall y \in B_\beta(x), \quad \forall x \in B_{N-1} \setminus B_2 \quad (23) \]

with \( \beta \) as in Theorem 1.3. Since \( B_{N-1} \setminus B_2^* \subset B_{N-1} \setminus B_2 \), by using the triangle inequality and (23), we obtain the following estimate: for any \( p, q \in B_{N-1} \setminus B_2^* \) let \( \gamma \subset B_{N-1} \setminus B_2^* \) be a curve connecting \( p \) and \( q \), then

\[ \frac{1}{\sqrt{2}} |\nu(p) - \nu(q)| \leq \left( \frac{2|\gamma|}{\beta} + 1 \right) \varepsilon_1 r, \quad (24) \]

where recall that \( |\gamma| \) denotes the length of the curve \( \gamma \). To see this, let \( \{p_i\}_{i=0}^m \) be points on \( \gamma \) such that \( p_0 = p, \ p_m = q \) and \( \text{dist}_\Sigma(p_i, p_{i+1}) \leq \beta/2 \). Note that we can do this with \( m = \left[ \frac{2|\gamma|}{\beta} \right] + 1 \) points. Then,

\[ \frac{1}{\sqrt{2}} |\nu(p_i) - \nu(p_{i+1})| \leq \varepsilon_1 r, \quad \forall i = 0, 1, \ldots, m - 1 \implies \]

\[ \frac{1}{\sqrt{2}} |\nu(p) - \nu(q)| \leq m \varepsilon_1 r \leq \left( \frac{2|\gamma|}{\beta} + 1 \right) \varepsilon_1 r \]

Thus, in order to prove (i) of the lemma, it remains to bound the diameter of \( B_{N-1} \setminus B_2^* \). By Lemma 2.3 we have

\[ |\partial B_2^*| \leq |\partial B_2| \leq 4\pi + \int_{B_2} |A|^2 \, dH^2 \leq 4\pi + K. \]

This implies that any two points in \( B_{N-1} \setminus B_2^* \) can be connected by a curve \( \gamma \subset B_{N-1} \setminus B_2^* \) such that

\[ |\gamma| \leq \frac{1}{2} |\partial B_2^*| + 2((N-1) - 2) \leq 2\pi + \frac{K}{2} + 2(N-3). \]

Finally, combining (24) and (25), we have that for any \( p, q \in B_{N-1} \setminus B_2^* \)

\[ \frac{1}{\sqrt{2}} |\nu(p) - \nu(q)| \leq \left( \frac{K + 4\pi + 4(N-3)}{\beta} + 1 \right) \varepsilon_1 r. \]
Taking
\[ \varepsilon_1 \leq \left( \frac{K + 4\pi + 4(N - 3)}{\beta} + 1 \right)^{-1} \]
and applying a rotation finishes the proof of (i) in the lemma. In fact, by letting
\[ \delta = \left( \frac{K + 4\pi + 4(N - 3)}{\beta} + 1 \right) \varepsilon_1 \leq 1, \]
we have that for any \( p, q \in B_{N-1} \setminus B_2^* \)
\[ \frac{1}{\sqrt{2}} |\nu(p) - \nu(q)| \leq \delta r. \] (26)
which implies that, after possibly applying a rotation, \( B_{N-1} \setminus B_2^* \) is locally graphical over the plane \( \{ x_3 = 0 \} \) with the norm of the gradient bounded by \( 3\delta r \) (cf. Lemma 5.2).

Part (ii) of the lemma states that the \( L^2 \) norm of \( |A| \) is small on \( B_2^* \). We intend to show this by using the Gauss-Bonnet theorem together with our bound on the \( L^p \) norm of the mean curvature. To that end, we need to find a curve bounding a disk containing \( B_2^* \) and which has small total geodesic curvature.

We begin by showing that the projection of \( \partial (B_{(N+1)/2} \cup B_2^*) \) on the plane \( \{ x_3 = 0 \} \) is away from the origin. In particular, \( \partial (B_{(N+1)/2} \cup B_2^*) \) is extrinsically distant from the origin.

**Claim 4.3.** Let \( C_\rho := \{ (x_1, x_2, x_3) : x_1^2 + x_2^2 \leq \rho^2 \} \) then
\[ \partial (B_{(N+1)/2} \cup B_2^*) \cap \partial C_{N-11} = \emptyset. \]
In particular \( \partial B_2^* \) lies inside \( C_2 \) and \( \partial (B_{(N+1)/2} \cup B_2^*) \) outside \( C_{N-11} \).

**Proof of Claim 4.3.** Given \( x \in \partial B_{N+1} \setminus B_2^* \), consider the geodesic ball \( B_{N-1}^{{\rho_2}}(x) \). Note that \( \partial B_2^* \) is clearly contained in \( C_2 \) and that by our choice of radii, there exists at least one point \( p \in \partial B_{N-3}^{{\rho_2}}(x) \cap \partial B_2^* \). Since \( B_{N-3}^{{\rho_2}}(x) \subset B_{N-1} \setminus B_2^* \), by (26) we have that \( B_{N-3}^{{\rho_2}}(x) \) is locally graphical over the plane \( \{ x_3 = 0 \} \) with norm of the gradient bounded by \( 3\delta r \). In particular, if we let
\[ \Pi : \mathbb{R}^3 \to \{ x_3 = 0 \} \]
be the projection to the plane \( \{ x_3 = 0 \} \), then \( B_{N-3}^{{\rho_2}}(x) \) contains a graph over the disk in the plane \( \{ x_3 = 0 \} \) centered at \( \Pi(x) \) and of radius \( \frac{N-3}{2\sqrt{1 + (3\delta r)^2}} \) (cf. Lemma 5.2). This implies that for any \( q \in \partial B_{N-3}^{{\rho_2}}(x) \),
\[ |\Pi(x) - \Pi(q)| \geq \frac{N-3}{2\sqrt{1 + (3\delta r)^2}} - \left( \frac{N-3}{2} - \frac{N-3}{2\sqrt{1 + (3\delta r)^2}} \right) \]
\[ = \frac{N-3}{\sqrt{1 + (3\delta r)^2}} - \frac{N-3}{2}. \]
The above inequality holds because if \( \gamma \) is a geodesic connecting \( q \) and \( x \) of length \( N - \frac{3}{2} \), then, by the previous discussion, there exists \( y \in \gamma \) such that

\[
|\Pi(y) - \Pi(x)| = \frac{N - 3}{2\sqrt{1 + (3\delta r)^2}}
\]

and then the intrinsic distance between \( y \) and \( q \) is at most \( \frac{N - 3}{2} - \frac{N - 3}{2\sqrt{1 + (3\delta r)^2}} \).

Finally, since the above inequality holds with \( q \) replaced by any \( p \in \partial \mathcal{B}_{N - \frac{3}{2}}(x) \cap \partial \mathcal{B}_2^* \neq \emptyset \) and because for such a \( p \) the inequality \( |\Pi(p)| \leq 2 \) holds, we have that

\[
|\Pi(x)| \geq \frac{N - 3}{\sqrt{1 + (3\delta r)^2}} - \frac{N - 3}{2} - 2.
\]

Since \( \delta \leq 1 \), \( N \geq 20 \) and \( r \leq 1/4 \),

\[
\frac{N - 3}{\sqrt{1 + (3\delta r)^2}} - \frac{N - 3}{2} - 2 \geq \frac{3}{10}(N - 3) - 2 \geq \frac{N - 11}{4}.
\]

This finishes the proof of the claim. \( \square \)

By the above claim and since \( \mathcal{B}_{(N+1)/2} \setminus \mathcal{B}_2^* \) is embedded and locally a graph over the plane \( \{x_3 = 0\} \), we have that \( \partial C_{N - \frac{11}{4}} \cap (\mathcal{B}_{(N+1)/2} \setminus \mathcal{B}_2^*) \) is the union of simple closed curves that are graphs over

\[
S_{N - \frac{11}{4}} = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = \left( \frac{N - 11}{4} \right)^2 \right\}.
\]

By elementary topological arguments, there exists a component \( \Gamma_0 \) of \( \partial C_{N - \frac{11}{4}} \cap (\mathcal{B}_{(N+1)/2} \setminus \mathcal{B}_2^*) \) such that \( \Gamma_0 \) bounds a disk in \( M \) containing \( \mathcal{B}_2^* \). To see this, note that otherwise it would be possible to connect \( \partial (\mathcal{B}_{(N+1)/2} \cup \mathcal{B}_2^*) \) with \( \partial \mathcal{B}_2^* \) without intersecting \( \partial C_{N - \frac{11}{4}} \), which is clearly a contradiction since the former boundary is outside \( C_{N - \frac{11}{4}} \), while the latter boundary is inside \( C_{N - \frac{11}{4}} \). Note that for each \( x \in \Gamma_0 \), we have that

\[
\mathcal{B}_{N - \frac{11}{4}}(x) \subset \mathcal{B}_{N - 1} \setminus \mathcal{B}_2^*.
\]

Hence, using (26) and applying Lemma 5.2 in each of the geodesic balls \( \mathcal{B}_{N - \frac{11}{4}}(x) \), we conclude that there exists a “thick” neighborhood of \( \Gamma_0 \) in \( M \) that can be written as a graph over the plane \( \{x_3 = 0\} \). Namely there exist \( b > \frac{N - 11}{4} > a > 0 \) such that if \( \Omega \) denotes the annulus \( \{(x_1, x_2) : a^2 \leq x_1^2 + x_2^2 \leq b^2\} \) then there exists a function

\[
u : \Omega \to M
\]
such that the following holds: the curve $\Gamma_0$ is contained in the graph of $u$, $\Gamma_0 \subset \text{graph } u|_\Omega$, the gradient of $u$ satisfies $|Du| \leq 3\delta r$ and for $a, b$ we have that
\[
\begin{align*}
b - a &= \frac{N - 19}{2} \cdot \frac{1}{\sqrt{1 + (3\delta r)^2}}, \\
b &= \frac{N - 11}{4} + \frac{N - 19}{4} \cdot \frac{1}{\sqrt{1 + (3\delta r)^2}}, \quad a > 2. \tag{27}
\end{align*}
\]

Now we note that
\[
\int_{\text{graph } u|_\Omega} |A|^2 d\mathcal{H}^2 \leq \int_{B_N \setminus B_1} |A|^2 d\mathcal{H}^2 \leq c_1(\epsilon_1 r)^2
\]
and thus we can apply Lemma 5.1, with $r$ and $\epsilon$ replaced by $3\delta r$ and $c_1(\epsilon_1 r)^2$ respectively, to conclude that for some $\rho \in (a, b)$
\[
\int_{\text{graph } u|_{S_\rho}} k \, ds \leq 2\pi \left( 1 + 3\sqrt{2} \delta r + \left( \frac{2c_1(\epsilon_1 r)^2 b}{b - a} \right)^\frac{1}{2} \right)
\]
where $k$ is the curvature of $\text{graph } u|_{S_\rho}$. Using now (27), we have that
\[
\frac{b}{b - a} \leq \frac{1}{2} \frac{N - 11}{N - 19} \cdot \sqrt{1 + (3\delta r)^2} + \frac{1}{2} \leq 10,
\]
where we have used that $N \geq 20$, $\delta \leq 1$ and $r \leq \frac{1}{4}$. Thus we get
\[
\int_{\text{graph } u|_{S_\rho}} k \, ds \leq 2\pi \left( 1 + 3\sqrt{2} \delta r + \left( 20c_1(\epsilon_1 r)^2 \right)^\frac{1}{2} \right) \tag{28}
\]

Let $\Gamma = \text{graph } u|_{S_\rho}$. Then, by construction $\Gamma$ bounds a disk $\Delta$ that contains $B_2^*$. Let $k_g$ denote the geodesic curvature of $\Gamma$. Using the Gauss Bonnet theorem we have that
\[
2\pi - \int_\Gamma k_g \, ds = \int_\Delta K_\Sigma \, d\mathcal{H}^2 = \frac{1}{2} \int_\Delta (H^2 - |A|^2) \, d\mathcal{H}^2. \tag{29}
\]
Since
\[
\left( \int_{B_N^*} |H|^2 \, d\mathcal{H}^2 \right)^\frac{1}{2} \leq |B_N^*|^{\frac{p - 2}{2p}} \left( \int_{B_N^*} |H|^p \, d\mathcal{H}^2 \right)^\frac{1}{p} \leq \frac{c_2 \epsilon_1 r}{(16C)^{\frac{p - 2}{2p}}} \leq c_2 \epsilon_1 r,
\]
using equation (29), $|k_g| \leq k$ and (28) gives
\[
\int_{B_2^*} |A|^2 \, d\mathcal{H}^2 \leq \int_\Delta |A|^2 \, d\mathcal{H}^2 \leq -4\pi + \int_\Delta H^2 \, d\mathcal{H}^2 + 2 \int_\Gamma k_g \, ds \leq -4\pi + \int_{B_N^*} |H|^2 \, d\mathcal{H}^2 + 2 \int_\Gamma k \, ds \leq -4\pi + (c_2 \epsilon_1 r)^2 + 4\pi (1 + 6\delta r + 5c_1 \epsilon_1 r). \nonumber
\]
Finally, since $\delta = \varepsilon_1 \left( \frac{K + 4\pi + 4(N-3)}{\beta} + 1 \right)$, we have

$$
\int_{B_2^*} |A|^2 \, d\mathcal{H}^2 \leq \left( c_2 \varepsilon_1 r \right)^2 + 24\pi \varepsilon_1 r \left( \frac{K + 4\pi + 4(N-3)}{\beta} + 1 + c_1 \right).
$$

This finishes the proof of (ii) in the lemma.

The proof of (iii) is a simple consequence of (ii) and Lemma 3.1. Let $\Delta$ be the previously defined disk, see equation (29). The disk $\Delta$ contains $B_2^*$ and

$$
\int_{\Delta} |A|^2 \, d\mathcal{H}^2 \leq \int_{B_2^*} |A|^2 \, d\mathcal{H}^2 + \int_{B_N \setminus B_1} |A|^2 \, d\mathcal{H}^2
\leq (c_2 \varepsilon_1 r)^2 + 24\pi \varepsilon_1 r \left( \frac{K + 4\pi + 4(N-3)}{\beta} + 1 + c_1 \right) + c_1 (\varepsilon_1 r)^2
\leq (\varepsilon_1 r)^2 \left( c_2^2 + c_1 \right) + 24\pi \varepsilon_1 r \left( \frac{K + 4\pi + 4(N-3)}{\beta} + 1 + c_1 \right)
\leq r \varepsilon_1 \left( c_2^2 + c_1 + 24\pi \left( \frac{K + 4\pi + 4(N-3)}{\beta} + 1 + c_1 \right) \right).
$$

Moreover, since $\Delta \subset B_N^*$ we have that

$$
\|H\|_{L^p(\Delta)} \left| \Delta \right|^{\frac{p-2}{p}} \leq \|H\|_{L^p(B_N^*)} \left| B_N^* \right|^{\frac{p-2}{p}} \leq c_2 \varepsilon_1 r
$$

Therefore, if $\varepsilon_1$ is taken sufficiently small, such that

$$
\varepsilon_1 \leq \left( c_2^2 + c_1 + 24\pi \left( \frac{K + 4\pi + 4(N-3)}{\beta} + 1 + c_1 \right) \right)^{-1} \cdot \frac{\pi}{2}
$$

and

$$
\varepsilon_1 \leq c_2^{-1} c_3,
$$

where $c_3$ is as in Lemma 3.1, and since

$$
\partial \Delta \subset \text{graph } u_{\Omega} \subset B_{N-1} \setminus B_2^* \Rightarrow \sup_{\partial \Delta} g \leq r \leq \sqrt{r},
$$

(see (23)) we can apply Lemma 3.1 with $M$ and $r$ replaced by $\Delta$ and $\sqrt{r}$ respectively. This application then gives

$$
\sup_{B_2^*} g \leq \sup_{\Delta} g \leq \frac{5}{4} \sqrt{r},
$$

which finishes the proof of (iii) and of the lemma.

The next theorem shows that given an embedded geodesic ball $B_R(x_0)$ in a simply-connected surface, if $B_R(x_0)$ has bounded $L^2$ norm of $|A|$ and $B_R^*(x_0)$ has sufficiently small $L^p$ norm of $H$ then this geodesic ball is graphical away from its boundary.
Theorem 4.4. Given $K > 0$ and $p > 2$, there exist $\varepsilon = \varepsilon(K, p)$ and $\gamma = \gamma(K)$, such that the following holds. Let $M$ be a simply-connected surface embedded in $\mathbb{R}^3$ and let $\mathcal{B}_R := \mathcal{B}_R(x_0) \subset M \setminus \partial M$ be such that

$$\int_{\mathcal{B}_R} |A|^2 \, d\mathcal{H}^2 \leq Kr^4 \text{ and } |\mathcal{B}_R|^{\frac{n-2}{p}} \left( \int_{\mathcal{B}_R} |H|^p \, d\mathcal{H}^2 \right)^{\frac{1}{p}} \leq \varepsilon r^2$$

for some $r \in [0, 1/4]$. Then, after a rotation $\sup_{\mathcal{B}_R^*} g \leq \frac{5}{4} r$.

Proof. Without loss of generality, let us assume that $x_0 = 0$. Note also that by rescaling it suffices to prove the theorem for $R = 1$, i.e. we assume that $\mathcal{B}_1 := \mathcal{B}_1(0) \subset M \setminus \partial M$,

$$\int_{\mathcal{B}_1} |A|^2 \, d\mathcal{H}^2 \leq Kr^4 \text{ and } |\mathcal{B}_1|^{\frac{n-2}{p}} \left( \int_{\mathcal{B}_1} |H|^p \, d\mathcal{H}^2 \right)^{\frac{1}{p}} \leq \varepsilon r^2.$$ 

We will show that this theorem is a consequence of Lemma 4.2. In order to do this we begin by proving the following claim.

Claim 4.5. Given $\varepsilon_1 > 0$, there exists $s \in [20^{-n_0}, 20^{-1}]$, such that

$$\int_{\mathcal{B}_s} |A|^2 \, d\mathcal{H}^2 \leq c_1(\varepsilon_1 r^2)^2,$$

where $n_0 = \left\lceil \frac{K}{c_1 \varepsilon_1} \right\rceil + 1$ and $c_1$ is as in Theorem 1.3.

Proof of Claim 4.5. Note that $\mathcal{B}_{20^s} \setminus \mathcal{B}_s \subset \mathcal{B}_1$, $\forall s \leq 20^{-1}$ and writing

$$\mathcal{B}_1 = \bigcup_{j=1}^{n_0} (\mathcal{B}_{20^{-j-1}} \setminus \mathcal{B}_{20^{-j}}) \cup \mathcal{B}_{20^{-n_0}}$$

we have that

$$\sum_{j=1}^{n_0} \int_{\mathcal{B}_{20^{-j-1}} \setminus \mathcal{B}_{20^{-j}}} |A|^2 \, d\mathcal{H}^2 \leq \int_{\mathcal{B}_1} |A|^2 \, d\mathcal{H}^2 = Kr^4$$

which implies that for some $j_0 \in \{1, \ldots, n_0\}$, we have that

$$\int_{\mathcal{B}_{20^{-j_0-1}} \setminus \mathcal{B}_{20^{-j_0}}} |A|^2 \, d\mathcal{H}^2 \leq \frac{Kr^4}{n_0} \leq c_1(\varepsilon_1 r^2)^2.$$

Hence the claim is true with $s = 20^{-j_0}$, with $j_0$ being as above. \qed
Let \( \varepsilon_1 = \varepsilon_1(K) \) be as in Lemma 4.2 with \( N = 20 \) and let \( s \in [20^{-n_0}, 20^{-1}] \) be as in the previous claim, i.e. so that
\[
\int_{B_{20} \setminus B_{s}} |A|^2 \, d\mathcal{H}^2 \leq c_1(\varepsilon_1 r^2)^2
\]
where \( c_1 \) is as in Theorem 1.3 and \( n_0 = \left\lceil \frac{K}{c_1 \varepsilon_1} \right\rceil + 1 \).

Let \( \tilde{M} = s^{-1}M \) be the rescaling of \( M \) by \( s^{-1} \) and \( \tilde{A}, \tilde{H} \) the corresponding second fundamental form and mean curvature and let \( \tilde{B} \) denote the geodesic balls of \( \tilde{M} \). Then we have
\[
\int_{\tilde{B}_{20}} |\tilde{A}|^2 \, d\mathcal{H}^2 = \int_{B_{20}s} |A|^2 \, d\mathcal{H}^2 \leq \int_{B_1} |A|^2 \, d\mathcal{H}^2 \leq K r^4 \text{ and }
\]
\[
\int_{\tilde{B}_{20} \setminus \tilde{B}_1} |\tilde{A}|^2 \, d\mathcal{H}^2 = \int_{B_{20} \setminus B_s} |A|^2 \, d\mathcal{H}^2 \leq c_1(\varepsilon_1 r^2)^2.
\]
Furthermore
\[
|\tilde{B}^*_2|^{\frac{\mu-2}{2p}} \left( \int_{\tilde{B}^*_2} |\tilde{H}|^p \, d\mathcal{H}^2 \right)^{\frac{1}{p}} = |B^*_2 s|^{\frac{\mu-2}{2p}} \left( \int_{B^*_2 s} |H|^p \, d\mathcal{H}^2 \right)^{\frac{1}{p}} \leq |B^*_1|^{\frac{\mu-2}{2p}} \left( \int_{B^*_1} |H|^p \, d\mathcal{H}^2 \right)^{\frac{1}{p}} \leq \varepsilon r^2.
\]

Let \( \varepsilon = c_2 \varepsilon_1 (16C^2)^{\frac{2-p}{2p}} \), where \( c_2 \) is as in Theorem 1.3, (and Lemma 4.2) and where \( C \) is the isoperimetric constant as in (1). Then
\[
(16C^2 |\tilde{B}^*_2|)^{\frac{\mu-2}{2p}} \left( \int_{\tilde{B}^*_2} |\tilde{H}|^p \, d\mathcal{H}^2 \right)^{\frac{1}{p}} \leq c_2 \varepsilon_1 r^2
\]
and by applying Lemma 4.2 to \( \tilde{B}_{20} \subset \tilde{M} \), with \( N = 20 \) and with \( r \) replaced by \( r^2 \) we obtain, after possibly a rotation, the following estimate:
\[
\sup_{\tilde{B}^*_2} g \leq \frac{5}{4} r.
\]
Since the quantity \( g \) is scale invariant, we have that \( \sup_{B^*_2 s} g \leq \frac{5}{2} r \). Let \( \gamma = 2 \cdot 20^{-n_0} \), since \( s \geq 20^{-n_0} \) this gives that
\[
\sup_{B^*_2} g \leq \frac{5}{4} r.
\]
This finishes the proof of the theorem. \( \square \)
We finally show that we can derive Theorem 1.1 in the introduction by the above
Theorem 4.4. Theorem 1.1 states that if $B$ is an embedded disk with bounded
$L^2$ norm of $|A|$, then $B$ is graphical away from its boundary, provided that the
$L^p$ norm of $H$ is sufficiently small. For convenience we recall the statement of the
theorem.

**Theorem 1.1.** Given $K > 0$ and $p > 2$, there exists $\varepsilon = \varepsilon(K, p)$ and $\gamma = \gamma(K)$, such that the following holds. Let $B_R := B_R(x_0) \subset M \setminus \partial M$ be an embedded disk such that
\[
\int_{B_R} |A|^2 d\mathcal{H}^2 \leq Kr^4 \quad \text{and} \quad R^\frac{p-2}{p} \left( \int_{B_R} |H|^p d\mathcal{H}^2 \right)^\frac{1}{p} \leq \varepsilon r^2
\]
for some $r \in [0, 1/4]$. Then, after a rotation,
\[
\sup_{B \cap B_R} g \leq \frac{5}{4} r.
\]

**Proof.** Since $B_R$ is a disk, we have that $B_R = B^*_R$ and furthermore by Lemma 2.3 we have that
\[
|B_R| \leq \left( \pi + \frac{Kr^4}{2} \right) R^2.
\]
Therefore
\[
|B^*_R|^{\frac{p-2}{2p}} \left( \int_{B^*_R} |H|^p d\mathcal{H}^2 \right)^\frac{1}{p} \leq \left( \pi + \frac{Kr^4}{2} \right) \frac{p-2}{2p} R^{\frac{p-2}{p}} \left( \int_{B_R} |H|^p d\mathcal{H}^2 \right)^\frac{1}{p}
\]
\[
\leq \left( \pi + \frac{K}{2} \right) \frac{p-2}{2p} \varepsilon r^2
\]
and hence we can directly apply Theorem 4.4.

\[\square\]

5 **Appendix**

For the sake of completeness, in this appendix we prove two results in differential
geometry that are used throughout the paper. In Remark 5.3 we also discuss the
$C^{1,\alpha}$ regularity.

Let
\[
\Omega := \{(x_1, x_2) \in \mathbb{R}^2 | a^2 < x_1^2 + x_2^2 < b^2\}
\]
for certain $b > a > 0$ and let $\mathcal{A}$ denote the graph above $\Omega$ of a smooth function $u$. That is $u \in C^\infty(\Omega)$ and graph $u = \mathcal{A}$.
Lemma 5.1. Assume that

\[ |Du| \leq r \leq 1 \quad \text{and} \quad \int_A |A|^2 \, d\mathcal{H}^2 \leq \varepsilon \]

Then there exists \( \rho \in (a, b) \) for which

\[ \int_{u(S_\rho)} k \, ds \leq 2\pi \left( 1 + r\sqrt{2} + \left( \frac{2\varepsilon b}{b-a} \right)^{\frac{1}{2}} \right), \]

where \( k \) is the curvature of the curve graph \( u|_{S_\rho} \) and \( S_\rho = \{(x_1, x_2) : x_1^2 + x_2^2 = \rho^2\} \).

Proof. Recall that in graphical coordinates

\[ A_{ij}(x, u(x)) = \frac{\partial^2}{\partial x_i \partial x_j} (x_1, x_2, u(x_1, x_2)) \perp \left( 0, 0, D_{ij}u \right) = \frac{D_{ij}u}{\sqrt{1 + |Du|^2}}, \]

where \( A_{ij}, i = 1, 2 \), are the coefficients of the second fundamental form, and also that

\[ |A|^2 = A_{ij}A_{kl}g^{ik}g^{jl}, \]

where \( g \) is the induced metric. We have that

\[ |D^2u|^2 = \sum_{i,j=1}^2 |D_{ij}u|^2 \leq |A|^2(1 + |Du|^2)^3 \]

(see for example [5]). On \( \Omega \) we are assuming that

\[ |Du(x)| \leq r, \quad \forall x \in \Omega \]

and this, together with the area formula, gives

\[ \int_\Omega |D^2u(x)|^2 \, dx = \int_\Omega \frac{|D^2u|^2}{(1 + |Du|^2)^3} (1 + |Du|^2)^3 \, dx \]
\[ \leq (1 + r^2)^{\frac{3}{2}} \int_\Omega \frac{|D^2u|^2}{(1 + |Du|^2)^3} \sqrt{1 + |Du|^2} \, dx \]
\[ = (1 + r^2)^{\frac{3}{2}} \int_A |A|^2 \, d\mathcal{H}^2 \leq 2(1 + r)\varepsilon. \]

By the coarea formula we can pick \( \rho \in (a, b) \), so that

\[ \int_{S_\rho} |D^2u|^2 \, dx \leq \frac{2(1 + r)\varepsilon}{b-a}. \]

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Let \( \Gamma = \text{graph } u|_{S_\rho} \). \( \Gamma \) is a closed curve in \( A \) and we want to compute
\[
\int_\Gamma k \, ds.
\]

Let \( \gamma : [0, 1] \to S_\rho \) be the following parametrization of \( S_\rho \):
\[
\gamma(t) = (\rho \cos 2\pi t, \rho \sin 2\pi t)
\]
and consider the parametrization of \( \Gamma \) given by
\[
f(t) = (\gamma(t), u(\gamma(t))), \quad t \in [0, 1].
\]

Recall that
\[
\int_\Gamma k \, ds = \int_0^1 k(f(t))|f'(t)| \, dt \leq \int_0^1 \frac{|f''|}{|f'|} \, dt,
\]
since
\[
k = \frac{|f' \times f''|}{|f'|^3} \implies k \leq \frac{|f''|}{|f'|^2}.
\]
Furthermore
\[
|f'|^2 = |\gamma'(t)|^2 + \left| \frac{d}{dt} u(\gamma(t)) \right|^2 \implies (2\pi \rho)^2 = |\gamma'(t)|^2 + |Du| |\gamma'(t)|^2 = |f'|^2
\]
and
\[
|f''| = \left| \left( \gamma''(t), \frac{d^2}{dt^2} u(\gamma(t)) \right) \right| \leq |\gamma''(t)| + \left| \frac{d^2}{dt^2} u(\gamma(t)) \right| \leq \rho(2\pi)^2 + \sqrt{2}|D^2 u| |\gamma'(t)|^2 + \sqrt{2}|Du| |\gamma''(t)| \leq \rho(2\pi)^2 + \sqrt{2}|D^2 u|(2\pi \rho)^2 + \sqrt{2}r \rho (2\pi)^2,
\]
where we have used the computation:
\[
\left| \frac{d^2}{dt^2} u(\gamma) \right| = \left| \frac{d}{dt} (Du \cdot \gamma') \right| \leq |\gamma'|^2 \left( \sum_{i,j=1}^2 |D_{ij} u| \right) + |Du| (|\gamma'_1| + |\gamma'_2|)
\]
\[
\leq |\gamma'|^2 \sqrt{2} \left( \sum_{i,j=1}^2 |D_{ij} u|^2 \right)^{\frac{1}{2}} + |Du| \sqrt{2}|\gamma''| = |\gamma'|^2 \sqrt{2}|D^2 u| + |Du| \sqrt{2}|\gamma''|.
\]
Hence
\[
\int_\Gamma k \, ds \leq \int_0^1 \rho(2\pi)^2 + \sqrt{2}|D^2 u|(2\pi \rho)^2 + r \rho \sqrt{2}(2\pi)^2 \frac{dt}{2\pi \rho} = 2\pi(1 + \sqrt{2}r) + 2\sqrt{2}\pi \rho \int_0^1 |D^2 u(\gamma(t))| dt.
\]
Using again the area formula we have
\[
\int_0^1 |D^2u(\gamma(t))|dt = \int_0^1 \frac{|D^2u(\gamma(t))|}{|\gamma'(t)|} |\gamma'(t)|dt = \frac{1}{2\pi \rho} \int_0^1 |D^2u(\gamma(t))||\gamma'(t)|dt
\]
\[
= \frac{1}{2\pi \rho} \int_{S_\rho} |D^2u(x)|dx \leq \frac{1}{2\pi \rho} \left( \int_{S_\rho} |D^2u(x)|^2dx \right)^{\frac{1}{2}} |S_\rho|^{\frac{1}{2}}
\]
\[
\leq \frac{1}{2\pi \rho} \left( \frac{4\pi \rho(1+r)\varepsilon}{b-a} \right)^{\frac{1}{2}} = \left( \frac{(1+r)\varepsilon}{\pi \rho(b-a)} \right)^{\frac{1}{2}}.
\]
Hence
\[
\int_{\Gamma} k ds \leq 2\pi(1+r\sqrt{2}) + 2\sqrt{2}\pi \rho \left( \frac{(1+r)\varepsilon}{\pi \rho(b-a)} \right)^{\frac{1}{2}} = 2\pi \left( 1 + r\sqrt{2} + \left( \frac{2(1+r)\varepsilon \rho}{\pi(b-a)} \right)^{\frac{1}{2}} \right)
\]
\[
\leq 2\pi \left( 1 + r\sqrt{2} + \left( \frac{2\varepsilon b}{b-a} \right)^{\frac{1}{2}} \right).
\]

\[
\square
\]

**Lemma 5.2.** Let $B_R := B_R(x_0) \subset M \setminus \partial M$ and assume that
\[
g(x) = \frac{1}{\sqrt{2}} |\nu(x) - e_3| \leq r, \forall x \in B_R,
\]
for some $r \in \left[0, \frac{\pi \sqrt{2}}{2}\right]$. Then $B_R$ is locally graphical over the plane $\{x_3 = 0\}$ with gradient bounded by $3r$. Moreover, $B_R$ contains a graph of a function $u$ over the disk in the plane $\{x_3 = 0\}$ centered at $\Pi(x_0)$ and of radius $\rho = \frac{R}{\sqrt{1+(3r)^2}}$, where $\Pi$ denotes the projection on the plane $\{x_3 = 0\}$.

**Proof.** Since $g(x) \leq 1$ for all $x \in B_R$, we have that $B_R$ is locally a graph over the plane $\{x_3 = 0\}$ and at each point $x = (x_1, x_2, u(x_1, x_2)) \in B_R$, we have
\[
\nu(x) = \left( -\frac{D_1u}{\sqrt{1+|Du|^2}}, -\frac{D_2u}{\sqrt{1+|Du|^2}}, \frac{1}{\sqrt{1+|Du|^2}} \right)
\]
where $\nu$ is the upward pointing unit normal. We estimate now $|Du|^2 = |D_1u|^2 + |D_2u|^2$.
\[ |D_2u|^2 \] using the estimate for \( g \) as follows: Note first that
\[
g(x) = \frac{1}{\sqrt{2}}|\nu(x) - e_3| = \frac{1}{\sqrt{2}} \left( \frac{|D_1u|^2}{1 + |Du|^2} + \frac{|D_2u|^2}{1 + |Du|^2} + \left( 1 - \frac{1}{\sqrt{1 + |Du|^2}} \right)^2 \right)^{\frac{1}{2}}
\]
\[
= \frac{1}{\sqrt{2}} \left( \frac{|Du|^2}{1 + |Du|^2} + \frac{1}{1 + |Du|^2} + 1 - 2 \frac{1}{\sqrt{1 + |Du|^2}} \right)^{\frac{1}{2}}
\]
\[
= \frac{1}{\sqrt{2}} \left( 2 - 2 \frac{1}{\sqrt{1 + |Du|^2}} \right)^{\frac{1}{2}} = \left( 1 - \frac{1}{\sqrt{1 + |Du|^2}} \right)^{\frac{1}{2}}.
\]

Hence, since \( g(x) \leq r \), we get
\[
g(x) = \left( 1 - \frac{1}{\sqrt{1 + |Du|^2}} \right)^{\frac{1}{2}} \leq r \implies 1 - \frac{1}{\sqrt{1 + |Du|^2}} \leq r^2 \implies \\
\sqrt{1 + |Du|^2} \leq \frac{1}{1 - r^2} \leq 1 + 2r^2,
\]
with the last inequality being true since \( r \leq \frac{1}{\sqrt{2}} \implies r^2 - 2r^4 \geq 0 \). Squaring both sides we obtain
\[
1 + |Du|^2 \leq 1 + 4r^4 + 4r^2 \implies |Du|^2 \leq 9r^2 \implies |Du| \leq 3r.
\]
This finishes the proof of the first part of the lemma.

By the previous discussion, \( B_R \) is a graph of a function \( u \) around the point \( x_0 \). Let \( \rho \) be such that \( u \) is defined on the disk centered at \( \Pi(x_0) \) of radius \( \rho \) in the plane \( \{x_3 = 0\} \). Without loss of generality, let \( x_0 = 0 \). We will prove a lower estimate for the radius \( \rho \) of the disk where the function \( u \) is defined. To do this, let \( \rho \) be the maximum such radius. Then there exists a point \( (x_1, x_2) \in \partial D_\rho(0) \), for which \( (x_1, x_2, u(x_1, x_2)) \in \partial B_R \), else \( u \) maps \( \partial D_\rho(0) \) in the interior of \( B_R \) and since \( B_R \) is locally a graph over the plane \( \{x_3 = 0\} \) we could increase \( \rho \). Let \( \gamma(t) \) be the path in \( B_R \) defined by
\[
\gamma : [0, 1] \to B_R, \quad \gamma(t) = (tx_1, tx_2, u(t(x_1, x_2))).
\]
The path \( \gamma \) joins 0, that is the center of \( B_R \), with \( x \in \partial B_R \), therefore it must have length at least \( R \), from which we get
\[
R \leq \text{Length}(\gamma) = \int_0^1 |\dot{\gamma}|dt \leq \int_0^1 \rho \sqrt{1 + |Du|^2} dt \leq \rho \sqrt{1 + (3r)^2}
\]
which implies that
\[
\rho \geq \frac{R}{\sqrt{1 + (3r)^2}}
\]
and this finishes the proof of the lemma. \( \square \)
Remark 5.3. Standard PDE theory implies that under the hypotheses of Lemma 5.2 and if in addition \( r \leq \frac{1}{\sqrt{3}} \) and \( H \in L^p(B_R) \), \( p > 2 \), we obtain \( C^{1,\alpha} \) estimates in \( B_{\frac{R}{2}} \); namely, there exist constants \( \alpha \in (0, 1) \) and \( C \), depending on \( r, R, \int_{B_R} |H|^p dH^2 \), and \( p \), such that

\[
\frac{|\nu(x) - \nu(y)|}{|x - y|^{\alpha}} \leq R^{-\alpha}C, \quad \forall x, y \in B_{\frac{R}{2}}
\]

To see why the above remark is true, we can assume without loss of generality that \( x_0 = 0 \). Note that by Lemma 5.2, \( B_R \) contains a graph of a function \( u \) over \( \Omega := \{(x_1, x_2) : x_1^2 + x_2^2 \leq \rho^2\} \) with \( |Du| \leq 3r \) and with

\[
\rho = \frac{R}{\sqrt{1 + (3r)^2}} \geq \frac{R}{\sqrt{10}} \geq \frac{R}{2}.
\]

Thus \( B_{\frac{R}{2}} \subset \text{graph } u \). Furthermore \( u \) satisfies the equation

\[
\sum_{i=1}^{2} D_i \left( \frac{Du(x)}{\sqrt{1 + |Du(x)|^2}} \right) = H(x, u(x)).
\]

By differentiating the above equation, we obtain that \( w = D_k u \), for \( k = 1, 2 \), is a solution to the equation

\[
\sum_{i,j=1}^{2} D_i (a^{ij} D_j w) = D_k H
\]

(cf. [7, pages 319-320]) with

\[
a^{ij} = \frac{\delta_{ij}}{\sqrt{1 + |Du|^2}} - \frac{D_i u D_j u}{\sqrt{1 + |Du|^2}}
\]

Note that since \( |Du| \) is bounded we have that \( H \in L^p(B_R) \implies H \in L^p(\Omega) \). We can then apply Theorem 8.22 in [7] to obtain that

\[
\sup_{x,y \in \Omega} \frac{|Du(x) - Du(y)|}{|x - y|^{\alpha}} \leq \rho^{-\alpha}C
\]

for some \( \alpha \in (0, 1) \) and with \( C \) and \( \alpha \) depending on \( \sup_{\Omega} |Du|, \rho^{1-\frac{p}{2}} \|H\|_{L^p(\Omega)} \) and \( p \), namely on \( r, R^{1-\frac{p}{2}} \|H\|_{L^p(B_R)} \) and \( p \), and \( \rho \geq \frac{R}{2} \). Using the formula for \( \nu \) as in (30), we get the required estimate.
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