HOMOLOGY OF AFFINE SPRINGER FIBERS IN THE UNRAMIFIED CASE

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Abstract. Assuming a certain “purity” conjecture, we derive a formula for the (complex) cohomology groups of the affine Springer fiber corresponding to any unramified regular semi-simple element. We use this calculation to present a complex analog of the fundamental lemma for function fields. We show that the “kappa” orbital integral which arises in the fundamental lemma is equal to the Lefschetz trace of Frobenius acting on the étale cohomology of a related variety.

1. Introduction

Let $F = \mathbb{C}((\varepsilon))$ be the field of formal Laurent series and let $\mathfrak{o} = \mathbb{C}[[\varepsilon]]$ be its integer ring of formal power series. A fundamental object of study in modern representation theory is the affine Grassmannian $X = G(F)/G(\mathfrak{o})$ associated to a complex reductive algebraic group $G$. Let $\mathfrak{g} = \text{Lie}(G)$. Each regular semisimple element $\gamma \in \mathfrak{g}(\mathfrak{o})$ determines a “vector field” on $X$ whose fixed point set

$$X_\gamma = \{ xG(\mathfrak{o}) \in X : \text{Ad}(x^{-1})(\gamma) \in \mathfrak{g}(\mathfrak{o}) \}$$

is a finite dimensional increasing union of complex projective varieties, which is known as the affine Springer fiber corresponding to $\gamma$. The study of affine Springer fibers was initiated in [KLS88], where many of the basic properties of these varieties were described.

In this paper we suppose $T \subset G$ is a maximal torus defined over $\mathbb{C}$ and we consider the affine Springer fiber $X_\gamma$ associated to regular semisimple elements $\gamma \in \mathfrak{t}(\mathfrak{o})$, where $\mathfrak{t} = \text{Lie}(T)$. (Not all affine Springer fibers are of this type: since $T$ is split over $F$ we refer to this as the unramified case.) The complex torus $T(\mathbb{C})$ acts on $X_\gamma$. In Theorem 9.2, the first principal result of this paper, we give an explicit description of the $T(\mathbb{C})$–equivariant homology of $X_\gamma$ under the assumption that the ordinary homology of $X_\gamma$ is pure, in the sense of mixed Hodge theory. (We conjecture this always holds, cf. §5.3.) The description in §9.2 is made possible by using [GKM98] §6.3 (the lemma of Chang and Skjelbred) which describes the equivariant homology of a $T(\mathbb{C})$–variety in terms of the 0 and 1 dimensional orbits of $T(\mathbb{C})$.

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The computation is reduced to the case of the equivariant homology of affine Springer fibers for the group $SL(2)$. Among the various ways to compute the equivariant homology in this special case of $SL(2)$, we have chosen to describe one which is conceptually simple, but which is computationally rather cumbersome. It uses the existence of a slightly larger torus $\tilde{T}(C)$ which acts on this Springer fiber. To facilitate the exposition we have extracted the combinatorial details and placed them in §12.

Suppose $G$ is adjoint, $s \in \tilde{T}$ is an element of the dual torus and $(H, s)$ is endoscopic data for $G$. Then $H$ and $G$ share the same torus $T$ so $\gamma$ corresponds to a regular semisimple element $\gamma_H \in H$, and there is an associated affine Springer fiber $X^{H}_{\gamma_H}$. Motivated by the fundamental lemma ([La83] §III.1), one predicts that there should be a close relation between the homology $H^*_\ast(X_{\gamma}; \mathbb{C})$ and $H^*_\ast(X^{H}_{\gamma_H}; \mathbb{C})$. However from a geometrical point of view, any relationship between these homology groups comes as a surprise, since the group $H$ may not be a subgroup of $G$. In fact, the varieties $X_{\gamma}$ and $X^{H}_{\gamma_H}$ have very little to do with each other: there does not appear to be any map or correspondence between them, and even their dimensions are different, in general. Moreover, the relation between these homology groups turns out to be rather subtle: they are not simply isomorphic, but rather, they become isomorphic only after a degree shift and a certain localization.

The element $s$ gives rise to a homomorphism $\chi_s(T) \to \mathbb{C}^\times$ (defined on the cocharacter group of $T$) so that the set of co-roots for $H$ is given by

$$\Phi^\vee(H, T) = \{\alpha^\vee \in \Phi^\vee(G, T) : s(\alpha^\vee) = 1\}.$$ 

The same cocharacter group $\chi_s(T)$ acts freely on $X_{\gamma}$. Let $J$ be the multiplicative subset of the group algebra $\mathbb{C}[\chi_s(T)]$ which is generated by the elements $(1 - \alpha^\vee)$ where $\alpha^\vee \in \Phi^\vee(G, T)$ and $s(\alpha^\vee) \neq 1$. For any $\mathbb{C}[\chi_s(T)]$ module $V$, let $V_J = J^{-1}V$ be its localization with respect to $J$. Let $r = r(\gamma)$ be the non-negative integer defined in §10.1. The second principal result in this paper is the following complex analog to the fundamental lemma in the unramified case (a more precise version of which will be stated in Theorem 10.2).

1.1. Theorem. Suppose the homology of $X_{\gamma}$ is pure and the homology of $X^{H}_{\gamma_H}$ is pure. Then for each $i$ there is a homomorphism

$$H^T(C)_i(X_{\gamma}, \mathbb{C}) \to H^T(C)_{i-2r}(X^{H}_{\gamma_H}, \mathbb{C}) \quad (1.1.1)$$

which becomes an isomorphism

$$H^T(C)_i(X_{\gamma}, \mathbb{C})_J \cong H^T(C)_{i-2r}(X^{H}_{\gamma_H}, \mathbb{C})_J \quad (1.1.2)$$

after localizing with respect to $J$.

The homomorphism (1.1.1) is compatible with a number of algebras which act on these homology groups. First, the equivariant homology is a module over the equivariant cohomology $\mathcal{D} = H^T(C)(pt)$ of a point (§4). Second, the group algebra $\mathbb{C}[\chi_s(T)]$ acts. Finally, a
certain group $\widetilde{W}_G^G.H$ of automorphisms of the situation also acts (cf. §9.4 and §10.1), and the homomorphism (1.1.1) transforms by a certain character $\eta$ under $\widetilde{W}_G^G.H$.

Using the action of $D$, it is possible to recover the ordinary homology of $X_\gamma$ from its equivariant homology, so (1.1.1) gives a homomorphism

$$H_i(X_\gamma; \mathbb{C}) \to H_{i-2r}(X_{\gamma_H}; \mathbb{C})$$

of the ordinary homology, which also becomes an isomorphism after localizing with respect to $J$. In §10 the action of the group algebra $\mathbb{C}[\chi_*(T)]$ is used in order to (partially) describe the homology of the quotient $\chi_*(T) \backslash X_\gamma$. Finally, the action of the group $\widetilde{W}_G^G.H$ provides a complex analog for the action of the Frobenius automorphism.

These extra ingredients are related to the fundamental lemma for function fields. In Theorem 15.8, the $\kappa$-orbital integral which occurs in the fundamental lemma is shown to equal the trace of the Frobenius automorphism on the étale cohomology of the quotient $\chi_*(T) \backslash X_\gamma(k)$; where now $X_\gamma(k)$ is the finite characteristic analog of the complex algebraic variety $X_\gamma$. Next, the analog of Theorems 9.2 and 10.2 need to be established in étale homology, a task which we have not fully carried out here. In §15.12 we indicate how these results are related to the fundamental lemma in the unramified case.

We would also like to draw attention to the recent preprint [L02a] in which G. Laumon uses the methods and techniques of the present paper together with a deformation argument from [L02b] to prove (under the same purity assumption) the “geometric fundamental lemma” (that is, the étale cohomology analog of Proposition 11.2) for unitary groups (and for arbitrary regular semi-simple elements $\gamma$).

In §13 and §14 we list the changes which are needed in order to establish similar results for Springer fibers $Y_\gamma$ in the affine flag manifold $Y$. The homology of a Springer fiber in the affine flag manifold carries the additional structure of a (right) action of the affine Weyl group, the Springer action. It was constructed by Lusztig [Lu96] and (using a statement from [KL80] whose proof does not appear in the literature) by Sage [Sa97]. In §14.4 this action is explicitly described using our formula (Theorem 14.3) for the homology of the Springer fiber $Y_\gamma$ (under the assumption that this homology is pure). In Theorem 14.6 we state the main consequence: If $(H, s)$ is endoscopic data for $G$, if $H_*(Y_\gamma; \mathbb{C})$ is pure and $H_*(Y_{\gamma_H}; \mathbb{C})$ is pure, then for each $i$ we obtain a homomorphism

$$H^T_i(Y_\gamma; \mathbb{C}) \to H^T_{i-2r}(Y_{\gamma_H}; \mathbb{C}) \otimes_{\mathbb{C}[\widetilde{W}]} \mathbb{C}[\widetilde{W}]$$

of $D$ modules, and a homomorphism on ordinary homology,

$$H_i(Y_\gamma; \mathbb{C}) \to H_{i-2r}(Y_{\gamma_H}; \mathbb{C}) \otimes_{\mathbb{C}[\widetilde{W}]} \mathbb{C}[\widetilde{W}],$$

each of which is equivariant with respect to the right action of $\widetilde{W}$ and transforms by the character $\eta$ under the action of $\widetilde{W}_G^G.H$. Each of these homomorphisms becomes an isomorphism after localizing with respect to $J$. 

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2. Purity for \textit{ind}-varieties

2.1. Let $X$ be a complex projective algebraic variety with (increasing) weight filtration $W$ of the cohomology $H^* = H^*(X; \mathbb{C})$ (see [D75]). The cohomology of $X$ is pure if

$$\text{Gr}^m W(H^i) = W^m(H^i)/W^{m-1}(H^i) = 0 \text{ for } i \neq m.$$ 

We obtain an increasing weight filtration on the homology $H_* = H_*(X; \mathbb{C})$:

$$W^m(H_i) = (H^i/W^{1-m}(H^i))^*$$

by duality. If $f : X \to Y$ is a morphism of projective varieties, the induced mapping $f_* : H_*(X) \to H_*(Y)$ is strictly compatible with the weight filtration.

Let $X$ be a complex projective \textit{ind}-variety ([Ku96], [Sh82]), that is, a set with a filtration $X_0 \subset X_1 \subset \ldots$ by complex projective varieties such that $X = \bigcup_{n=1}^{\infty} X_n$ and such that each $X_n \to X_{n+1}$ is a closed immersion. A set $Y \subset X$ is closed in the limit topology for $X$ if $Y \cap X_n$ is closed (in the classical topology) for each $n$. Then $H_*(X) = \lim H_*(X_n)$ inherits a limit weight filtration $W'$ from the weight filtrations on $H_*(X_n)$. If $X = \bigcup_{n=1}^{\infty} Y_n$ is an equivalent \textit{ind}-variety structure on $X$ (meaning that the identity mapping $X \to X$ is a morphism of \textit{ind}-varieties) then the resulting weight filtration $W'$ on $H_*(X)$ agrees with $W$. We say the homology of $X$ is pure if $\text{Gr}^{-m} W(H_i(X)) = 0$ for $m \neq i$.

Let $A = (\mathbb{C}^*)^m$ be a complex torus which acts on a projective \textit{ind}-variety $X$ such that each $X_n$ is invariant under $A$ and so that the action of $A$ on $X_n$ is algebraic and is compatible with the immersion $X_n \to X_{n+1}$. In this case we say that $A$ acts algebraically on $X$. The classifying space $BA$ may be taken to be the \textit{ind}-variety $(\mathbb{P}^1)^m \subset (\mathbb{P}^2)^m \ldots$, which has pure homology. The Cartan-Leray spectral sequence for the $A$-equivariant homology of $X_n$ is the spectral sequence for the fibration $X_n \times_A EA \to BA$ with $E^2 \cong H_*(BA) \otimes H_*(X_n)$. It inherits a weight filtration such that the differentials are strictly compatible with the weight. It follows that the Cartan-Leray spectral sequence for the $A$-equivariant homology of $X$ inherits a weight filtration such that the differentials are strictly compatible with the weight. The following fact will be needed for Corollary 10.3.

2.2. Lemma. Suppose $X$ is a complex projective \textit{ind}-variety with an action of an algebraic torus $A$. Suppose the homology of $X$ is pure. Then the homology of $X$ is equivariantly formal [GKM98] §1, that is, the Cartan-Leray spectral sequence for the $A$-equivariant homology of $X$ collapses at $E^2$ and gives rise to an isomorphism

$$H_*^A(X; \mathbb{C}) \cong H_*(X; \mathbb{C}) \otimes_{\mathbb{C}} H_*^A(pt; \mathbb{C}).$$
2.3. Purity of $B\mathbb{Z}^n$. Define an action of the additive group $\mathbb{Z}$ on the contractible ind-variety

$$EZ = (\mathbb{Z} \times \mathbb{C})/(m,0) \sim (m + 1,1)$$

(which is a countable union of affine lines joined each to the next at a single point) by $m \cdot (r,t) = (r + 2m,t)$ (where $m,r \in \mathbb{Z}$ and $t \in \mathbb{C}$). The quotient $B\mathbb{Z} = EZ/\mathbb{Z}$ consists of two copies $V_0, V_1$ of $\mathbb{C}$, joined at two points: $0 \in V_0$ is identified with $1 \in V_1$ and vice versa. The inclusion of the unit circle $T \to B\mathbb{Z}$ given by

$$e^{i\theta} \mapsto \begin{cases} \frac{\theta}{\pi} \in V_0 & \text{if } 0 \leq \theta \leq \pi \\ \frac{\theta}{\pi} - 1 \in V_1 & \text{if } \pi \leq \theta \leq 2\pi \end{cases}$$

is a homotopy equivalence. Let $\Lambda = \mathbb{Z}^n$, and set $E\Lambda = (EZ)^n$ and $B\Lambda = (B\mathbb{Z})^n$. We obtain a homotopy equivalence $T^n = (T^1)^n \to B\Lambda$.

Suppose $X$ is an ind-variety on which $\Lambda$ acts freely by morphisms. Let $E(X) = E\Lambda \times_{\Lambda} X$ be the Borel construction, together with its projections

$$\Lambda \setminus X \xleftarrow{p} E(X) \xrightarrow{\pi} B\Lambda.$$

Let $s : \Lambda \to GL(1,\mathbb{C})$ be a 1-dimensional representation of finite order, and write $C_s$ for the representation space. Then $s$ determines local systems $L_s = C_s \times_{\Lambda} X$ on $\Lambda \setminus X$ and $L_s^B = E\Lambda \times_{\Lambda} C_s$ on $B\Lambda$, and there is a canonical isomorphism of local systems, $\pi^*L_s^B \cong p^*L_s$. Denote this local system on $E(X)$ by $L_s^E$. The homology Leray spectral sequence, with coefficients in $L_s^E$, for the fibration $\pi : E(X) \to B\Lambda$ has

$$E^2_{pq} = H_p(\Lambda; H_q(X; C_s)) = H_p(B\Lambda; L_s^B \otimes H_q) \Rightarrow H_{p+q}(\Lambda \setminus X; L_s) \quad (2.3.1)$$

where $H_q$ denotes the local system on $B\Lambda$ corresponding to $H_q(X; \mathbb{C})$. The following fact will be needed for Proposition 11.2.

2.4. Lemma. Suppose the homology of $X$ is pure. Then the spectral sequence (2.3.1) collapses at $E^2$. Hence there is an isomorphism

$$H_m(\Lambda \setminus X; L_s) \cong \bigoplus_{p+q=m} H_p(\Lambda; H_q(X; C_s)). \quad (2.4.1)$$

2.5. Proof. First consider the case when $n = 1$ and when $s$ is the trivial representation. The homology $H_s(B\mathbb{Z}; H_q)$ may be computed from the Mayer Vietoris sequence for the covering of $B\mathbb{Z}$ by the two (Zariski) closed sets $V_0$ and $V_1$. It is easy to see that this sequence gives a short exact sequence (of mixed Hodge structures)

$$0 \to H_1(B\mathbb{Z}; H_q) \to H_q \xrightarrow{\mu^{-1}} H_q \to H_0(B\mathbb{Z}; H_q) \to 0$$

where $H_q = H_q(X; \mathbb{C})$, and where $\mu$ is the homomorphism induced on homology from the action of $1 \in \mathbb{Z}$. 


Next consider the case of general $n$ but still with $s$ the trivial representation. It follows by induction that the homology $H_*(B\mathbb{Z}^n;H_q)$ is the homology of the Koszul complex $K_*=\wedge^*(\mathbb{C}^n) \otimes H_q$ with differential

$$\partial(e_{i_1} \wedge \ldots \wedge e_{i_r} \otimes h) = \sum_{j=1}^r (-1)^{j+1} e_{i_1} \wedge \ldots \wedge \hat{e}_{i_j} \wedge \ldots \wedge e_{i_r} \otimes (\mu_{ij}(h) - h)$$

where $\mu_1, \ldots, \mu_n$ are the homomorphisms on $H_q$ determined by the action of the standard $n$ basis elements $e_1, \ldots, e_n$. Each $\wedge^k(\mathbb{C}^n) \otimes H_q$ is pure of weight $-q$ since it arises as a direct sum of copies of $H_q$. So each homology group $H_p(B;H_q)$ is pure of weight $-q$. Since the differentials in the spectral sequence (2.3.1) are strictly compatible with the weight filtration, this implies the spectral sequence collapses at $E^2$.

Now consider the case of a nontrivial representation $s$ of finite order. Such a representation factors through the (finite) quotient $\Lambda/N\Lambda$ for some positive integer $N$. Let $\phi : \tilde{B} = E\Lambda/N\Lambda \to B\Lambda$ be the resulting (finite) cover, and let $\tilde{E}$ be the fiber product in the following diagram,

$$\begin{array}{ccc}
\tilde{E} & \xrightarrow{\phi} & E\Lambda \times_\Lambda X \\
\phi \downarrow & & \downarrow \pi \\
\tilde{B} & \xrightarrow{\phi} & B\Lambda
\end{array}$$

The preceding argument may be used to see that the Leray spectral sequence for the mapping $\tilde{\pi}$ (with constant coefficients) collapses at $E^2$.

The covering $\phi$ is a principal homogeneous space for the group $\Lambda/N\Lambda$, which acts by deck transformations on $\tilde{B}$. So it acts on the local system $\phi_*(\mathbb{C})$ as the regular representation, and determines decompositions

$$\phi_*(\mathbb{C}) \cong \bigoplus_{\theta} \mathcal{L}^\mathbb{B}_\Theta \quad \text{and} \quad \Phi_*(\mathbb{C}) \cong \bigoplus_{\theta} \mathcal{L}^E_\Theta$$

(2.5.1)

into one dimensional local systems corresponding to the distinct characters $\theta : \Lambda/N\Lambda \to \mathbb{C}^\times$. It follows (see below) that there is a natural isomorphism

$$H_*{(\tilde{B};H^\mathbb{B}_q)} \cong \bigoplus_{\theta} H_*(B\Lambda;H_q \otimes \mathcal{L}_\Theta)$$

(2.5.2)

where $H^\mathbb{B}_q$ (resp. $H_q$) is the local system on $\tilde{B}$ (resp. on $B\Lambda$) corresponding to $H_q(X;\mathbb{C})$. In fact the whole spectral sequence for $\tilde{\pi}$ decomposes under $\Lambda/N\Lambda$ into a direct sum (over distinct characters $\theta$ of $\Lambda/N\Lambda$) of spectral sequences for $\pi$ with coefficients in $\mathcal{L}_\theta$. It follows that each of these constituent spectral sequences collapses, one of which corresponds to the character $s$. 

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There are several ways to verify equation (2.5.2). One way is to use the homotopy equivalence \( T^n \to B\Lambda \), Poincaré duality, and (2.5.1). However it may also be verified directly. Let \( D_{B\Lambda} \) (resp. \( D_B \)) be the dualizing complex on \( B\Lambda \) (resp. on \( B \)). Since \( \phi \) is a finite covering space, \( \phi_* \cong R\phi_* \cong R\phi! \) and \( \phi^*(D_{B\Lambda}) \cong D_B \). Since the diagram above is a fiber square,

\[
R\phi_*(H^B_q) \cong R\phi_*(\phi^*H_q \otimes \mathbb{C}) \cong H_q \otimes \left( \bigoplus_{\theta} L^B_{\theta} \right).
\]

The homology \( H_*(\tilde{B}; H^B_q) \) is the cohomology (with compact supports) of the sheaf

\[
D_B \otimes H^B_q \cong \phi^*D_B \otimes H^B_q.
\]

Pushing this sheaf forward under \( \phi \) gives

\[
D_B \otimes R\phi_*(H^B_q) \cong D_B \otimes H^B_q \otimes \left( \bigoplus_{\theta} L^B_{\theta} \right)
\]

whose compactly supported cohomology is \( \bigoplus_{\theta} H_*(B\Lambda; H_q \otimes L^B_{\theta}) \) as claimed.  

2.6. **Remarks.** Presumably the theory of weights can be extended to include (co)homology groups such as \( H_k(B\Lambda; H_q \otimes L_s) \), in which case equation (2.5.2) implies that this group is pure of weight \(-q\). We remark that if the homology of \( X \) is both pure and Tate (meaning that it lives only in even degrees and that \( Gr^{-2q}H_*(X) \) is all of Hodge type \((q, q)\)), then the decomposition (2.4.1) is canonical because in this case, the Hodge filtration and the weight filtration are opposed ([D71] Prop. 1.2.5). It seems likely that Lemma 2.4 remains valid even when the local system \( s \) fails to have finite order.

3. **Symmetric algebras**

3.1. Consider the polynomial algebra

\[
S = \mathbb{Q}[x_1, \ldots, x_n] = \bigoplus_{k=0}^{\infty} S_k
\]

in \( n \) variables, graded by degree (with finite dimensional graded pieces); and its dual algebra of differential operators

\[
D = \mathbb{Q}[\partial_1, \ldots, \partial_n] = \bigoplus_{k=0}^{\infty} D_k
\]

with \( \partial_i x_j = \delta_{ij} \). Then \( D_k S_j \subset S_{j-k} \). The natural pairing

\[
D \otimes S \xrightarrow{\langle \cdot, \cdot \rangle} \mathbb{Q}
\]

given by \( \langle \partial, P \rangle = (\partial P)(0) \) satisfies

\[
\langle \partial \partial', P \rangle = \langle \partial, \partial' P \rangle.
\]

It restricts to a nondegenerate (finite dimensional) bilinear form \( D_k \otimes S_k \to \mathbb{Q} \), which vanishes on \( D_k \otimes S_j \) for \( j \neq k \). If \( I = \bigoplus_{k=0}^{\infty} I_k \subset D \) is a homogeneous ideal, define

\[
S\{I\} = \{ s \in S : \partial s = 0 \text{ for all } \partial \in I \}
\]
to be the subspace annihilated by $I$. It follows from (3.1.2) that
\[ S\{I\} = I^\perp = \bigoplus_{k=0}^\infty I^\perp_k \] (3.1.3)
where
\[ I^\perp_k = \{ s \in S_k : \langle \partial, s \rangle = 0 \text{ for all } \partial \in I_k \}. \]
So the pairing (3.1.1) passes to a nondegenerate pairing \((\mathcal{D}_k/I_k) \otimes I^\perp_k \rightarrow \mathbb{Q}\). If $\partial \in \mathcal{D}$ set \(S\{\partial\} = S\{(\partial)\} \).

### 3.2. Lemma
Suppose $\partial, \partial' \in \mathcal{D}$ are relatively prime homogeneous elements of $\mathcal{D}$. Then
\[ S\{\partial\partial'\} = S\{\partial\} + S\{\partial'\} \] (3.2.1)
and
\[ S\{\partial\} \xrightarrow{\partial'} S\{\partial\} \] (3.2.2)
is surjective and lowers degree by $\deg(\partial')$.

### 3.3. Proof
If $I, J \subset \mathcal{D}$ are homogeneous ideals, then it follows from (3.1.3) that
\[ S\{I + J\} = S\{I\} \cap S\{J\} \]
\[ S\{I \cap J\} = S\{I\} + S\{J\}. \]
Taking $I = (\partial)$ and $J = (\partial')$ proves (3.2.1). Since $\partial$ and $\partial'$ are relatively prime, the mapping
\[ \mathcal{D}/(\partial) \xrightarrow{\partial'} \mathcal{D}/(\partial) \]
is an injective homomorphism of graded algebras, which raises degree by $\deg(\partial')$. It follows by duality (3.1.1) that (3.2.2) is surjective. \qed

### 4. EQUIVARIANT HOMOLOGY

#### 4.1.
Let $A = A(\mathbb{C})$ be a complex torus with Lie algebra $\mathfrak{a}$. We will often use without mention the canonical isomorphism $\chi^*(A) \otimes \mathbb{C} \rightarrow \mathfrak{a}^*$. Let
\[ \mathcal{D}(\mathfrak{a}) = \text{Sym}(\mathfrak{a}^*) = \bigoplus_{d=0}^\infty \text{Sym}^d(\mathfrak{a}^*) \]
be the symmetric algebra of linear differential operators (with constant complex coefficients) on $\mathfrak{a}^*$, graded by degree. Let
\[ S(\mathfrak{a}) = \text{Sym}(\mathfrak{a}) = \bigoplus_{d=0}^\infty \text{Sym}^d(\mathfrak{a}^*)^* \]
be the dual symmetric algebra of complex valued polynomial functions on $\mathfrak{a}^*$.

Throughout this paper, cohomology and homology will be taken with complex coefficients (unless otherwise stated), however it is possible to use rational coefficients, (by replacing $\mathfrak{a}^*$ with $\chi^*(A) \otimes \mathbb{Q}$, for example). Let $BA$ be the classifying space for $A$. If $\varphi : A \rightarrow \mathbb{C}^\times$ is
a character let $c_1(\varphi)$ be the first Chern class of the resulting line bundle on $BA$. Then $c_1$ extends to an isomorphism

$$D(\mathfrak{a}) \to H^*(BA) = H^*_A(pt)$$

(the Chern-Weil isomorphism) of graded algebras, which doubles degrees.

Multiplication $A \times A \to A$ determines an H-space structure $m : BA \times BA \to BA$ which gives rise to an algebra structure on $H_*(BA) = H^*_A(pt)$. The homology $H_*(BA)$ is also a module over $H^*(BA)$ under the cap product. If $x \in H^2(BA)$ then the Hopf formula $m^*(x) = 1 \otimes x + x \otimes 1$ implies that $x$ acts on $H_*(BA)$ as a derivation, so $H^*(BA)$ acts on $H_*(BA)$ by differential operators. It follows that the mapping $S(\mathfrak{a}) \to H_*(BA)$ (dual to the Chern-Weil isomorphism) is an algebra isomorphism and is compatible with the cap product, in the sense that the following diagram commutes:

$$
\begin{array}{ccc}
D_k \otimes S_j & \longrightarrow & S_{j-k} \\
\downarrow \cong & & \downarrow \cong \\
H^{2k}(BA) \otimes H_{2j}(BA) & \longrightarrow & H_{2j-2k}(BA)
\end{array}
$$

Suppose $A$ acts on a complex projective algebraic variety $X$ whose cohomology is pure. Then the cohomology of $X$ is equivariantly formal and the Cartan-Leray spectral sequence for the equivariant cohomology of $X$ collapses at $E_2$. Let $T \subset A$ be a (possibly trivial) subtorus and let $I = \ker(\pi^*(BA) \to H^*(BT))$ be the resulting homogeneous ideal. Then the $T$-equivariant cohomology of $X$ may be recovered from the $A$-equivariant cohomology by

$$H^*_T(X) = H^*_A(X) \otimes_{H^*_A(pt)} H^*_T(pt).$$

The equivariant homology $H^*_A(X)$ is a module over $D = H^*_A(pt)$ and by duality, the $T$-equivariant homology of $X$,

$$H^*_T(X) = H^*_A(X) \{I\}$$

is the submodule of $H^*_A(X)$ consisting of those elements which are annihilated by the homogeneous ideal $I$.

4.2. Let $\varphi : A \to \mathbb{C}^\times$ be a nontrivial character with corresponding differential operator $\partial_\varphi \in D(\mathfrak{a})$. Let $M = \ker(\varphi)$ with Lie algebra $\mathfrak{m} = \ker(\partial_\varphi) \subset \mathfrak{a}^\times$. Let $\jmath^*: S(\mathfrak{m}) \to S(\mathfrak{a})$ be the homomorphism induced by inclusion $\mathfrak{m} \subset \mathfrak{a}$. Then the image of $\jmath^*$ consists of the polynomial functions $S(\mathfrak{a}) \{\partial_\varphi\}$ on $\mathfrak{a}^\times$ which are annihilated by $\partial_\varphi$.

Such a nontrivial character $\varphi$ extends to an action of $A$ on $\mathbb{C}P^1 = \mathbb{C}^\times \cup \{0\} \cup \{\infty\}$. As in [GKM98] §7.1, 7.2, the long exact sequence for the pair $(\mathbb{C}P^1, \{0\} \cup \{\infty\})$ breaks into short
exact sequences

$$0 \longrightarrow H^*_A(\mathbb{CP}^1, \{0\} \cup \{\infty\}) \xrightarrow{\partial} H^*_A(\{0\}) \oplus H^T_*(\{\infty\}) \longrightarrow H^*_A(\mathbb{CP}^1) \longrightarrow 0$$

$$\text{S}(m) \xrightarrow{\beta} \text{S}(a) \oplus \text{S}(a)$$

where $\beta(f) = (j^*(f), -j^*(f))$.

4.3. Now suppose that a complex algebraic torus $A$ acts algebraically on a complex projective variety $Y$ whose cohomology is pure. Let $Y_0 \subset Y$ be the set of fixed points and let $Y_1 \subset Y$ be the union of the 0 and 1 dimensional orbits of $A$ in $Y$. The lemma of Chang and Skjelbred [CS74] (cf. [GKM98] §6.3), when translated into homology, says that the following sequence is exact:

$$H^*_A(Y_1, Y_0) \xrightarrow{\beta} H^*_A(Y_0) \longrightarrow H^*_A(Y) \longrightarrow 0. \quad (4.3.1)$$

Suppose, moreover, that there are finitely many fixed points $Y_0 = \{y_1, y_2, \ldots, y_r\}$ and finitely many 1-dimensional orbits $\{E_1, E_2, \ldots, E_d\}$. The closure $E_i$ of each 1-dimensional orbit is obtained by adding two fixed points $\partial E_i = \{y_{i_a}\} \cup \{y_{i_b}\}$. Let $m_i \subset a$ be the Lie algebra of the stabilizer of any point in $E_i$ and let $j_i^* : \text{S}(m_i) \rightarrow \text{S}(a)$ be the resulting homomorphism. Using §4.2, the sequence (4.3.1) becomes

$$\bigoplus_{i=1}^d \text{S}(m_i) \xrightarrow{\beta} \bigoplus_{k=1}^r \text{S}(a) \longrightarrow H^*_A(Y) \longrightarrow 0 \quad (4.3.2)$$

where $\beta = \sum_{i=1}^d \beta_i$ with

$$\beta_i(f_i) = (0, \ldots, j_i^*(f_i), \ldots, -j_i^*(f_i), \ldots, 0).$$

for any $f_i \in \text{S}(m_i)$. (Here, the two nonzero entries occur in the positions corresponding to $y_{i_a}$ and $y_{i_b}$ respectively.)

5. Affine Springer fibers

5.1. **Notation.** Let $\mathbb{G}_m$ denote the multiplicative group. If $k$ is a field, if $\overline{k}$ is an algebraic closure, and if $T$ is an algebraic torus defined over $k$, let $\chi^*(T) = \text{Hom}(\mathbb{G}_m, T) \cong \text{Hom}(\overline{k}^*, T(\overline{k}))$ denote the group of cocharacters of $T$ and let $\chi^*(T) = \text{Hom}(T, \mathbb{G}_m) \cong \text{Hom}(T(\overline{k}), \overline{k}^*)$ denote the group of characters of $T$. The dual torus is $\hat{T} = \text{Hom}(\chi^*(T), \mathbb{G}_m)$.

If $G$ is a connected reductive complex algebraic group, if $T \subset G$ is a maximal torus in $G$ and if $(\chi^*(T), \Phi, \chi^*(T), \Phi^\vee)$ is the resulting root datum for $G$ (where $\Phi$ and $\Phi^\vee$ are the roots and co-roots of $T$ in $G$ respectively), then we denote by $\hat{G}$ the “dual” connected reductive complex algebraic group corresponding to the root datum $(\chi^*(T), \Phi^\vee, \chi^*(T), \Phi)$.

Throughout this paper (except in §15) we let $F = \mathbb{C}((\varepsilon))$ be the field of formal Laurent series over $\mathbb{C}$ and denote by $\mathfrak{o} = \mathbb{C}[[\varepsilon]]$ its ring of integers, the formal power series over $\mathbb{C}$.
The valuation \( \text{val} : F \to \mathbb{Z} \) takes integer values. Let \( T \) be an algebraic torus defined over \( \mathbb{C} \). The following sequence is exact:

\[
1 \longrightarrow T(\mathfrak{o}) \longrightarrow T(F) \xrightarrow{\text{val}} \chi_*(T) \longrightarrow 1.
\]

Here, \( \text{val} \) is defined by the property that \( \alpha(\text{val}(\ell)) = \text{val}(\alpha(\ell)) \) for all \( \alpha \in \chi^*(T) \) and all \( \ell \in T(F) \). (On the left side of this equality, \( \alpha \) is viewed as a homomorphism \( \chi^*(T) \to \mathbb{Z} \).)

The choice of uniformizing parameter \( \varepsilon \) determines a splitting \( \chi_*(T) \to T(F) \) by \( \beta \mapsto \beta(\varepsilon) \), whose image \( \Lambda \) will be called the lattice of translations. It is a free abelian group of rank equal to the dimension of \( T \).

5.2. In this section we recall some definitions and results from [KL88]. Let \( G \) be a connected reductive algebraic group over \( \mathbb{C} \), with Lie algebra \( \mathfrak{g} \). Set \( \mathfrak{g}(F) = \mathfrak{g} \otimes \mathbb{C} F \) and \( \mathfrak{g}(\mathfrak{o}) = \mathfrak{g} \otimes \mathfrak{o} \). Denote by \( \text{Ad} \) the adjoint action of \( G_F \) on \( \mathfrak{g}_F \).

We will often write \( K \) for the group \( G(\mathfrak{o}) \) of \( \mathfrak{o} \)-points of \( G \).

The loop Grassmannian (or affine Grassmannian) is the quotient \( X = G(F)/K \). The affine Grassmannian \( X \) is an \( ind \)-algebraic variety: it is an increasing union \( [KL88] \)

\[
X_0 \subset X_1 \subset X_2 \subset \ldots \subset X
\]

of finite dimensional complex projective algebraic varieties. If \( H \subset G \) is a connected reductive algebraic subgroup then the inclusion of \( H \) into \( G \) induces an injection \( H(K)/H(\mathfrak{o}) \hookrightarrow X \) of the loop Grassmannian for \( H \) into the loop Grassmannian for \( G \).

Each element \( \gamma \in \mathfrak{g}(F) \) gives rise to a “vector field” on \( X \) whose fixed point set

\[
X_\gamma = \{ xK \in G(F)/K : \text{Ad}(x^{-1})(\gamma) \in \mathfrak{g}(\mathfrak{o}) \}
\]

is called an affine Springer fiber. We say the element \( \gamma \) is compact if \( X_\gamma \neq \emptyset \). In [KL88] it is proven that a compact element \( \gamma \in \mathfrak{g}(F) \) is regular and semisimple iff \( X_\gamma \) is finite dimensional, in which case \( X_\gamma \) is an \( ind \)-subvariety of \( X \). It is a union of (possibly) infinitely many irreducible components, each of which is a complex projective variety.

5.3. **Conjecture.** If \( \gamma \in \mathfrak{g}(F) \) is compact, regular, and semisimple then for all \( i \) the homology group \( H_i(X_\gamma; \mathbb{C}) \) is pure of weight \( i \).

In [GKMx] we prove this conjecture for elements \( \gamma \) which have “equal valuation”.

5.4. **Bruhat decomposition.** Let \( T \subset G \) be a maximal torus defined over \( \mathbb{C} \) (hence split over \( F \)), with its lattice of translations \( \Lambda \subset T(F) \subset G(F) \). The identification \( \Lambda \cong T(F)/T(\mathfrak{o}) \) induces an embedding \( \phi : \Lambda \to G(F)/G(\mathfrak{o}) = X \) of the loop Grassmannian for \( T \) into the loop Grassmannian for \( G \). Fix a Borel subgroup \( B \supset T \) and let \( I \subset G(F) \) be the corresponding Iwahori subgroup. Then \( G(F) = I \Lambda K \) so \( X \) decomposes into \( I \) orbits or Bruhat cells,

\[
X = \bigsqcup_{\ell \in \Lambda} I \ell K.
\]

We denote by \( C_\ell = I \ell x_0 \subset X \) the orbit (cell) corresponding to \( \ell \in \Lambda \).
5.5. **Turning torus.** Each \( \lambda \in \mathbb{C}^\times \) determines an automorphism \( \sigma_\lambda \) of the field \( F = \mathbb{C}((\varepsilon)) \) by \( \sigma_\lambda(\varepsilon^m) = \lambda^m \varepsilon^m \). We say that \( \lambda \) acts on \( F \) by “turning the loop” and we refer to the image of \( \mathbb{C}^\times \rightarrow \text{Aut}(F) \) as the “turning torus”. The turning torus preserves \( \mathfrak{o} = \mathbb{C}[\mathfrak{c}] \) and the fixed point set of its action is
\[
\mathfrak{o}^{\mathbb{C}^\times} = F^{\mathbb{C}^\times} = \mathbb{C} \cdot \varepsilon^0.
\]
The resulting action of \( \mathbb{C}^\times \) on \( G(F) \) preserves \( K = G(\mathfrak{o}) \) and induces an action on \( X = G(F)/K \) which commutes with the action of \( T(\mathbb{C}) \subset I \). On each Bruhat cell \( C_\ell \) the action of the extended torus
\[
\widetilde{T}(\mathbb{C}) = T(\mathbb{C}) \times \mathbb{C}^\times
\]
may be described by
\[
(t, \lambda) \cdot g\ell K = \lambda(tg^{-1})\ell K.
\]

5.6. **Affine roots.** Let \( \Phi = \Phi(G, T) \) be the root system for \( G \), with positive roots \( \Phi^+ \) determined by the choice of \( B \), and with root space decomposition
\[
\mathfrak{g}(\mathbb{C}) = \mathfrak{t}(\mathbb{C}) \oplus \bigoplus_{\alpha \in \Phi} \mathbb{C}Y_\alpha
\]
where \( Y_\alpha \in \mathfrak{g}(\mathbb{C})_\alpha \) are root vectors. Let \( \widetilde{\Phi} = \{ (\alpha, k) : \alpha \in \Phi, k \in \mathbb{Z} \} \) be the collection of affine roots. Each affine root \( (\alpha, k) \) may be considered to be a character of the extended torus \( \widetilde{T}(\mathbb{C}) \) (5.5.1) with \( (\alpha, k)(t, \lambda) = \alpha(t)\lambda^k \). Then \( \widetilde{T}(\mathbb{C}) \) acts on the affine root space \( \mathbb{C}\varepsilon^kY_\alpha \) through this character, that is,
\[
(t, \lambda) \cdot \varepsilon^kY_\alpha = \alpha(t)\lambda^k \varepsilon^kY_\alpha.
\]

Let \( C_0 \) be the fundamental alcove determined by \( I \) and let \( \widetilde{\Delta} \) be the set of simple affine roots, so
\[
C_0 = \left\{ a \in \chi_*(T) \otimes_{\mathbb{Z}} \mathbb{R} : \alpha(a) + k > 0 \text{ for all } (\alpha, k) \in \widetilde{\Delta} \right\}.
\]
There is a split short exact sequence
\[
1 \rightarrow I_+ \rightarrow I \rightarrow T(\mathbb{C}) \rightarrow 1
\]
where \( I_+ \) denotes the unipotent radical of \( I \). Its Lie algebra is given by
\[
\mathfrak{N}(I) = \text{Lie}(I_+) = \prod_{n \in \mathbb{Z}} \varepsilon^nT(\mathbb{C}) \oplus \bigoplus_{(\alpha, k)} \mathbb{C}\varepsilon^kY_\alpha
\]
where the product is taken over those affine roots \( (\alpha, k) \in \widetilde{\Phi} \) such that \( \alpha(a) + k > 0 \) for all \( a \in C_0 \). To fix notation we recall the following standard description of the Bruhat cells.
5.7. Lemma. Let \( x_0 = K \in X \) denote the basepoint. Fix \( \ell \in \Lambda \). The exponential map determines a \( \tilde{T}(\mathbb{C}) \)-equivariant isomorphism between the Bruhat cell \( C_\ell = I\ell x_0 \) and the vector space

\[
D_\ell = \bigoplus_{(\alpha, k)} \mathbb{C} \varepsilon^k Y_\alpha
\]  

(5.7.1)

where the sum is over those (finitely many) affine roots \( (\alpha, k) \in \tilde{\Phi} \) such that

\[
\text{val}(\alpha(\ell)) + k < 0 \quad \text{and} \quad \alpha(a) + k > 0 \quad \text{for all} \quad a \in C_0
\]

and where \( \tilde{T}(\mathbb{C}) \) acts on this vector space according to (5.6.1).

5.8. Proof. The subgroup \( I_+ \) acts transitively on \( C_\ell \). The stabilizer \( S_\ell \) of the point \( \ell x_0 \) is \( I_+ \cap \ell G(o)\ell^{-1} \) whose Lie algebra \( \mathfrak{s}_\ell \) is the sum of the affine root spaces \( \mathbb{C} \varepsilon^k Y_\alpha \) such that \( \alpha(a) + k > 0 \) (for \( a \in A \)) and \( \text{Ad}(\ell)(\varepsilon^k Y_\alpha) \in \mathfrak{g}(o) \). The second condition is:

\[\varepsilon^{k + \text{val}(\alpha(\ell))} Y_\alpha \in \mathfrak{g}(o)\]

or equivalently, \( \text{val}(\alpha(\ell)) + k \geq 0 \). But (5.7.1) is a \( \tilde{T}(\mathbb{C}) \)-invariant complement to \( \mathfrak{s}_\ell \) in \( \mathfrak{g}(I) \) so the exponential mapping takes it isomorphically to the Bruhat cell \( I\ell x_0 \).  

5.9. Lemma. The fixed point set of the turning torus on \( X \) is:

\[X^{C^\times} = G(\mathbb{C})\Lambda x_0.\]

The fixed point set of the torus \( T(\mathbb{C}) \) on \( X \) is:

\[X^{T(\mathbb{C})} = \Lambda x_0.\]

5.10. Proof. It is easy to see that \( G(\mathbb{C})\Lambda x_0 \subset X^{C^\times} \). To verify the reverse inclusion it suffices to show that

\[X^{C^\times} \cap C_\ell \subset G(\mathbb{C})\Lambda x_0\]

for each Bruhat cell \( C_\ell \). But the fixed point set of the turning torus on the vector space \( D_\ell \) consists of those factors in the sum (5.7.1) with \( k = 0 \). So their exponentials are contained in \( G(\mathbb{C}) \cap I \). Similarly, the action of \( T(\mathbb{C}) \) on \( D_\ell \) has a single fixed point at the origin so \( T(\mathbb{C}) \) acts on \( C_\ell \) with a single fixed point \( \ell x_0 \).  

5.11. One dimensional orbits. Let \( X_1 \subset X \) denote the union of the 0 and 1-dimensional orbits under the action of \( T(\mathbb{C}) \) on \( X \). If \( \alpha \in \Phi^+ \) let \( U_\alpha \subset G \) be the unique connected \( T \)-stable 1-dimensional unipotent subgroup whose Lie algebra contains the root space \( \mathfrak{g}_\alpha \). Then each \( \alpha \in \Phi^+ \) determines a unique reductive connected algebraic subgroup \( H_\alpha \subset G \) of semi-simple rank 1 which is generated by \( T, U_\alpha, \) and \( U_{-\alpha} \). Let \( X^\alpha = H_\alpha(F)/H_\alpha(o) \) be the loop Grassmannian for this subgroup. The inclusion of \( H_\alpha \) into \( G \) induces an injection \( X^\alpha \hookrightarrow X \).
5.12. Lemma. The union of the 0 and 1-dimensional orbits of $T(\mathbb{C})$ on $X$ is

$$X_1 = \bigcup_{\alpha \in \Phi^+} X^\alpha.$$ 

If $\alpha, \beta \in \Phi^+$ and $\alpha \neq \beta$ then $X^\alpha \cap X^\beta = \Lambda.$

5.13. Proof. If $\dim(T) = 1$ then the result is obvious since both sides coincide with $X,$ so we may assume that $\dim(T) \geq 2.$ The adjoint action of the torus $T(\mathbb{C})$ on $H_\alpha$ factors through an action of the 1-dimensional quotient $T(\mathbb{C})/\ker(\alpha).$ Hence $X^\alpha \subset X_1.$ To show that $X_1 \subseteq \bigcup_{\alpha \in \Phi^+} X^\alpha$ it suffices to show that

$$X_1 \cap C_\ell \subset \bigcup_{\alpha \in \Phi^+} X^\alpha \cap C_\ell$$

for each Bruhat cell $C_\ell.$ By Lemma 5.7 this amounts to determining the 1-dimensional orbits of the action of $T(\mathbb{C})$ on the vector space $C_\ell.$ It follows from (5.6.1) that if $\dim(T) \geq 2$ then these 1-dimensional orbits are precisely the coordinate axes $\mathbb{C}^k Y_\alpha.$ Therefore we may classify the 1-dimensional orbits of $T(\mathbb{C})$ on $C_\ell$ by the positive roots, with those orbits corresponding to a fixed $\alpha \in \Phi^+$ being contained in the subspace

$$D_{\ell,\alpha} = \bigoplus_k \mathbb{C}^k Y_\alpha = \bigoplus_k \mathbb{C}^k \text{Lie}(U_\alpha)(\mathbb{C}).$$

Here, the sum is taken over those $k$ such that $\alpha(a) + k > 0$ (for all $a \in C_0$) and $\text{val}(\alpha(\ell)) + k < 0.$ Let $C_{\ell,\alpha} \subset C_\ell$ be the corresponding subset of the Bruhat cell. Then $C_{\ell,\alpha} = X^\alpha \cap C_\ell,$ which implies (5.13.1). Finally, if $\alpha \neq \beta$ then $D_{\ell,\alpha} \cap D_{\ell,\beta} = \{0\}$ which proves that $X^\alpha \cap X^\beta = \Lambda.$

6. Affine Springer fibers for $\text{SL}(2)$

6.1. In this section we fix $G(\mathbb{C}) = \text{SL}(2, \mathbb{C})$ and set $X^{\text{SL}(2)} = G(F)/K$ with basepoint $x_0 = K = G(\mathfrak{o}).$ Let $T$ be the torus of diagonal matrices and let $\alpha$ be the positive root, $\alpha \left( \begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix} \right) = a^2.$ Let $\alpha^\vee : F^\times \to T(F)$ be the corresponding co-root,

$$\alpha^\vee(b) = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}.$$ 

Then $\Lambda^{\text{SL}(2)} = \langle \alpha^\vee(\varepsilon) \rangle$ is the cyclic group spanned by $\alpha^\vee(\varepsilon),$ whose elements we denote by $\ell_m = \alpha^\vee(\varepsilon^m).$ For any integer $n \leq -1$ set

$$x_n = \begin{pmatrix} 1 & \varepsilon^n \\ 0 & 1 \end{pmatrix} K \in X^{\text{SL}(2)}.$$ 

Let $\alpha' : \mathfrak{t}(F) \to F$ be the differential of $\alpha : T(F) \to F.$ The following fact was observed by D. Nadler.
6.2. Lemma. The affine Grassmannian $X_{\text{SL}(2)}$ is the disjoint union of countably many $T(F)$ orbits,

$$X_{\text{SL}(2)} = \coprod_{n \leq 0} T(F) \cdot x_n.$$

The orbit of the point $x_n$ has (complex) dimension $|n|$. If $\gamma \in t(\mathfrak{o})$ and $v = \text{val}(\alpha'(\gamma))$ then the affine Springer fiber $X_{\gamma}^{\text{SL}(2)}$ is the union of the $T(F)$ orbits

$$X_{\gamma}^{\text{SL}(2)} = \bigcup_{n=-v}^{0} T(F) \cdot x_n$$

(6.2.1)

and it is preserved by the turning torus.

It follows that we may unambiguously denote the affine Springer fiber $X_{\gamma}^{\text{SL}(2)}$ in (6.2.1) by $X_{\leq v}^{\text{SL}(2)}$.

6.3. Proof. First observe that every point $x \in X_{\text{SL}(2)}$ has an expression of the form $gK$ with

$$g = \begin{pmatrix} \varepsilon^m & b_0 \varepsilon^n \\ 0 & \varepsilon^{-m} \end{pmatrix}$$

where either (1) $b_0 = 0$ or (2) $b_0 \in \mathfrak{o}^\times$ and $n - m < 0$. In fact, the standard Borel subgroup acts transitively on $X$, so $x = \begin{pmatrix} a & 0 \\ 0 & a' \end{pmatrix} x_0$ for some $a, a' \in F$. Set $a = a_0 \varepsilon^m$ with $a_0 \in \mathfrak{o}^\times$ and right multiply by $\alpha^\vee(a_0^{-1}) \in K$ to obtain $x = gx_0$ with $g = \begin{pmatrix} \varepsilon^m & b \\ 0 & \varepsilon^{-m} \end{pmatrix}$. If $b = 0$ or if $\text{val}(b) < m$ then we are done. Otherwise set $\tilde{b} = b_0 \varepsilon^n$ with $n \geq m$ and $b_0 \in \mathfrak{o}^\times$. Right multiply by $k = \begin{pmatrix} 1 & -b_0 \varepsilon^n \\ 0 & 1 \end{pmatrix} \in K$ to see that $x = gkK \in \Lambda$.

Now let $x = gK$ with $g = \begin{pmatrix} \varepsilon^m & b_0 \varepsilon^n \\ 0 & \varepsilon^{-m} \end{pmatrix}$. If $b_0 = 0$ then $x = \alpha^\vee(\varepsilon^m)x_0 \in T(F) \cdot x_0$. If $b_0 \in \mathfrak{o}^\times$ and $m > n$ let $a \in \mathfrak{o}^\times$ be a square root of $b_0$. Set $t = \alpha^\vee(a \varepsilon^n) \in T(F)$ and $k = \alpha^\vee(a^{-1}) \in K$. Then $tx_{n-m}k = g$ which proves the first statement. The dimension statement is straightforward. If $\gamma \in t(\mathfrak{o})$ then $X_{\gamma}^{\text{SL}(2)}$ is preserved by $T(F)$ so it is a union of $T(F)$ orbits. To verify (6.2.1), suppose $\gamma = \begin{pmatrix} 0 & a \\ 0 & -a \end{pmatrix} \in t(\mathfrak{o})$. Then $\text{Ad}(x_0)\gamma = \begin{pmatrix} a & -2a^2 \\ 0 & -a \end{pmatrix}$ which is in $\mathfrak{g}(\mathfrak{o})$ if $\text{val}(a) + n \geq 0$.

Each 1-dimensional orbit $\mathcal{O}$ of $\tilde{T}(\mathbb{C})$ necessarily contains exactly two fixed points $\ell_s, \ell_t \in \Lambda$ in its closure, in which case we say that $\mathcal{O}$ "connects" the points $\ell_s$ and $\ell_t$.

6.4. Lemma. For any two lattice points $\ell_s, \ell_t \in \Lambda$ there exists a unique 1-dimensional orbit $\mathcal{O}_{st}$ of $\tilde{T}(\mathbb{C})$ in $X_{\text{SL}(2)}$ which connects them, and this accounts for all the 1-dimensional orbits of $\tilde{T}(\mathbb{C})$ in $X_{\text{SL}(2)}$. The stabilizer of any point in $\mathcal{O}_{st}$ is the kernel of the affine root $(\alpha, s+t)$. The orbit $\mathcal{O}_{st}$ is contained in $X_{\gamma}^{\text{SL}(2)}$ iff $\text{val}(\alpha'(\gamma)) \geq |s-t|$. 

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6.5. **Proof.** By Lemma 6.2 for any $T(\mathfrak{g})$ orbit $\mathcal{T}$ there exists a unique $s \in \mathbb{Z}$ and a unique $n \leq 0$ such that $\ell_s x_n \in \mathcal{T}$. Set $t = s + n$. We will show that

1. the $\tilde{T}(\mathbb{C})$ orbit $\mathcal{O}$ of $\ell_s x_n$ is 1-dimensional,
2. this is the only 1-dimensional orbit of $\tilde{T}(\mathbb{C})$ in $\mathcal{T}$,
3. $\mathcal{O}$ connects $\ell_s$ and $\ell_t = \ell_s + n\alpha^\vee(\varepsilon)$, and
4. $\tilde{T}(\mathbb{C})$ acts on $\mathcal{O}$ through the affine root $(\alpha, 2s + n)$.

First compute the action of $\tilde{T}(\mathbb{C})$ on the $T(\mathfrak{g})$ orbit of the point $\ell x_n$. Let $\alpha^\vee(b) \in T(\mathfrak{g})$ with $b = \sum_{i \geq 0} b_i \varepsilon^i \in \mathfrak{g}^\times$. Let $a \in \mathbb{C}^\times$. Since

$$\ell_s x_n = \begin{pmatrix} \varepsilon^s & \varepsilon^{s+n} \\ 0 & \varepsilon^{-s} \end{pmatrix}$$

the action of $(\alpha^\vee(a), \lambda) \in \tilde{T}(\mathbb{C})$ on the point $\alpha^\vee(b) \ell_s x_n$ is the point

$$y = \begin{pmatrix} (a(\lambda \cdot b)\lambda^s \varepsilon^s & a(\lambda \cdot b)\lambda^{s+n} \varepsilon^{s+n} \\ 0 & a^{-1}(\lambda \cdot b^{-1})\lambda^{-s} \varepsilon^{-s} \end{pmatrix} K.$$ 

Right multiply by $\alpha^\vee(a^{-1}(\lambda \cdot b)^{-1}\lambda^{-s}) \in K$ to find that

$$y = \begin{pmatrix} \varepsilon^s & 2(\lambda \cdot b)^2 \lambda^{2s+n} \varepsilon^{s+n} \\ 0 & \varepsilon^{-s} \end{pmatrix} K. \quad (6.5.1)$$

As $a, \lambda \in \mathbb{C}^\times$ vary, this is a 2-dimensional orbit unless $b = b_0 \in \mathbb{C}^\times$, in which case $\lambda \cdot b = b$, hence $\lim_{a \to 0} y = \ell_s$. To find $\lim_{a \to 0} y$, choose $\lambda = b = 1$ and right multiply by

$$\begin{pmatrix} a^{-3} & a^{-1} \varepsilon^{-n} - a^2 \\ a^{-2} \end{pmatrix} \in K$$

to find

$$y = \begin{pmatrix} a^{-3} \varepsilon^s + \varepsilon^{s+n} & a^{-1} \varepsilon^{s-n} \\ a^{-2} \varepsilon^{-s} & \varepsilon^{-s-n} \end{pmatrix} K.$$ 

So $y \to \ell_s + n\alpha^\vee(\varepsilon)$ as $a \to \infty$. It follows from (6.5.1) that $\tilde{T}(\mathbb{C})$ acts on this 1-dimensional orbit through the character $(\alpha, 2s + n) = (\alpha, s + t)$, cf. (5.6.1). This verifies (1)–(4). Finally, by (6.2.1), the orbit $\mathcal{O} \subset X_\gamma$ iff $|n| \leq \text{val} \alpha^\vee(\gamma)$. \qed

7. **Equivariant homology of $SL(2)$ Springer fibers**

7.1. As in §6 let $G = SL(2)$, let $X = X^{SL(2)} = G(K)/G(\mathfrak{g})$, let $\Lambda = \Lambda^{SL(2)}$. (We use multiplicative notation for the group law in $\Lambda$.) Let $T \subset G$ be the diagonal matrices, with simple root and co-root $\alpha$ and $\alpha^\vee$. Then $\alpha^\vee(\varepsilon) \in \Lambda$ is a generator (which, by abuse of notation, we denote by $\alpha^\vee$) so it determines an isomorphism $\mathbb{Z} \cong \Lambda$, written $s \mapsto \ell_s$. So $\alpha^\vee = \ell_1$ and $\alpha^\vee \ell_s = \ell_{s+1}$. Fix $\gamma \in \mathfrak{t}(\mathfrak{g})$ with corresponding Springer fiber $X_\gamma$.

To simplify the notation, for the remainder of §7 we will write $\tilde{T}$ for $\tilde{T}(\mathbb{C})$, $\mathfrak{t}$ for $\mathfrak{t}(\mathbb{C})$, and so on. Let $\mathcal{D} = \mathcal{D}(\mathfrak{t})$ be the algebra of differential operators on $\mathfrak{t}^*$, which is identified with
$H^*_T(\text{pt})$ by the Chern-Weil homomorphism. Let $S(t)$ be the symmetric algebra of complex valued polynomial functions on $t^*$. The character $\alpha$ corresponds to a differential operator $\partial_\alpha \in D(t)$. Denote by $S(t) \{\partial_d^d\}$ the kernel of $\partial_d^d$. The Chern-Weil homomorphism determines an isomorphism of $D$ modules

$$H^*_T(\Lambda) \cong \mathbb{C}[\Lambda] \otimes \mathbb{C} S(t).$$

7.2. Proposition. Let $v = \text{val}(\alpha'(\gamma))$. The inclusion $\Lambda \subset X$ induces exact sequences,

$$H^*_T(X_\gamma, \Lambda) \xrightarrow{\beta} H^*_T(\Lambda) \longrightarrow H^*_T(X_\gamma) \longrightarrow 0$$

and the image of $\beta$ is the $D$-submodule

$$\sum_{d=1}^v (1 - \alpha^\vee) C[\Lambda] \otimes S(t) \{\partial_d^d\} \subset \mathbb{C}[\Lambda] \otimes S(t).$$

7.3. Proof. The Springer fiber $X_\gamma$ has a paving by affines ([GKMX]): it is an increasing union of complex projective algebraic varieties $\phi \subset (X_\gamma)_0 \subset (X_\gamma)_1 \subset \ldots$ such that each $(X_\gamma)_j - (X_\gamma)_{j-1}$ is isomorphic to a single affine space. In fact the intersection of $X_\gamma$ with a single Bruhat cell is either empty, or else it is a single affine cell in this paving. It follows that the homology of $(X_\gamma)_j$ vanishes in odd degrees. The extended torus $\widetilde{T} = T \times \mathbb{C}^\times$ (§5.5) preserves each $(X_\gamma)_j$ and it acts with finitely many fixed points and finitely many 1-dimensional orbits. So the results of §4.1 and §4.3 may be applied to this action. Since homology commutes with direct limits, we obtain the exact sequences (7.2.1) for both the $\widetilde{T}$-equivariant homology and the $T$-equivariant homology.

The co-root $\alpha^\vee$ determines canonical isomorphisms

$$S(\mathfrak{t}) \cong \mathbb{C}[x, t] \quad \text{and} \quad D(\mathfrak{t}) \cong \mathbb{C}[\partial_x, \partial_t].$$

According to Lemma 6.4, for each unordered pair of distinct integers $a, b \in \mathbb{Z}$ such that $|b - a| \leq v = \text{val}(\alpha'(\gamma))$, there is a unique 1-dimensional orbit $E_{ab} \subset X_\gamma$ which connects $\ell_a$ and $\ell_b$, on which the torus $\widetilde{T}$ acts through the character $\Phi_{ab}$ whose differential $\phi_{ab} : \mathfrak{t} \to \mathbb{C}$ corresponds to the differential operator

$$\partial_{ab} = \partial_\alpha + (a + b)\partial_t = 2\partial_x + (a + b)\partial_t.$$ 

Let $m_{ab} = \ker(\phi_{ab})$ denote the Lie algebra of the stabilizer of any point in this orbit, and let $j^*_ab : S(m_{ab}) \to S(\mathfrak{t})$ be the resulting inclusion. Then by (4.1.1) the image of $j^*_ab$ consists of all polynomial functions on $\mathfrak{t}$ which are annihilated by the differential operator $\partial_{ab}$. Hence

$$j^*_ab S(m_{ab}) = \{g((a + b)x - 2t) : g \in \mathbb{C}[z]\}$$
consists of polynomial functions of \( z = (a + b)x - 2t \). It follows that the image of \( \tilde{\beta} \) is the sum \( \sum_{a,b} M_{ab} \) of submodules \( M_{ab} \) spanned by elements

\[
(\ell_b - \ell_a) \otimes g_{ab}((a + b)x - 2t)
\]

with \(|b - a| \leq v\). This is the module \( P_v \) of §8.2 (with \( 2t \) replaced by \( t \)).

The \( T \)-equivariant homology of \( X_\gamma \) may be recovered (4.1.1) from the \( \tilde{T} \)-equivariant homology of \( X_\gamma \) as the kernel of the operator \( \partial_t \). So the image of \( \beta \) is

\[
\text{Im}(\beta) = P_v \cap \ker(\partial_t).
\]

Then Proposition 12.7 identifies \( P_v \cap \ker(\partial_t) \) with the submodule (7.2.2). \( \square \)

8. Groups of semisimple rank one

For lack of an adequate reference we include a proof of the following well-known fact.

8.1. Lemma. Let \( H \) be a connected reductive complex linear algebraic group of rank \( n \) and of semisimple rank 1. Then \( H \) is isomorphic to either

\[
(1) \ (\mathbb{C}^\times)^{n-1} \times \text{SL}(2, \mathbb{C})
\]

\[
(2) \ (\mathbb{C}^\times)^{n-1} \times \text{PGL}(2, \mathbb{C})
\]

\[
(3) \ (\mathbb{C}^\times)^{n-2} \times \text{GL}(2, \mathbb{C}).
\]

8.2. Proof. Let \((L, \{\alpha\}, L^\vee, \{\alpha^\vee\})\) be the (based) root datum for \( H \) where \( \langle \cdot, \cdot \rangle : L^\vee \times L \to \mathbb{Z} \) is a dual pairing of rank \( n \) lattices, \( \alpha \in L, \alpha^\vee \in L^\vee \), and \( \langle \alpha^\vee, \alpha \rangle = 2 \). If

\[
\langle \alpha^\vee, \cdot \rangle : L \to \mathbb{Z}
\]

is even-valued, then \( \frac{1}{2}\alpha^\vee \in L^\vee \) so

\[
L = (\alpha^\vee)^\perp \oplus \langle \alpha \rangle \text{ while } L^\vee = \alpha^\perp \oplus \langle \frac{1}{2}\alpha^\vee \rangle
\]

which corresponds to case (1). (Here, \( \langle \alpha \rangle \) denotes the cyclic group generated by \( \alpha \in L \), and \( (\alpha^\vee)^\perp \) denotes the kernel of (8.2.1).) Similarly if \( \langle \cdot, \alpha \rangle : L^\vee \to \mathbb{Z} \) is even-valued then \( \frac{1}{2}\alpha \in L \) so

\[
L = (\alpha^\vee)^\perp \oplus \langle \frac{1}{2}\alpha \rangle \text{ while } L^\vee = \alpha^\perp \oplus \langle \alpha^\vee \rangle
\]

which corresponds to case (2).

Now suppose that neither of these homomorphisms is even-valued. Let \( \{e_1, e_2, \ldots, e_{n-1}\} \) be a basis of \( L_0 = (\alpha^\vee)^\perp \). Let \( L_1 = \{x \in L : \langle \alpha^\vee, x \rangle = 1\} \). Then \( L_1 - \frac{1}{2}\alpha \) is a translate of \( L_0 \) so there exists \( a_i \in \{0, \frac{1}{2}\} \) such that \( L_1 - \frac{1}{2}\alpha = L_0 + \sum_{i=1}^{n-1} a_ie_i \) and not all the \( a_i \) are 0. By relabeling the basis we may assume that \( a_1 = a_2 = \ldots = a_r = \frac{1}{2} \) and the remaining coefficients are 0. Let \( v = e_1 + e_2 + \ldots + e_r \). Then \( \{v, e_2, \ldots, e_r\} \) is also a basis for \( L_0 \), and \( \frac{1}{2}\alpha + \frac{1}{2}v \in L \). It follows that

\[
L = Y \oplus \langle e_2, e_3, \ldots, e_{n-1}\rangle
\]

where

\[
Y = \{m\alpha + m'v : m, m' \in \mathbb{Z} \text{ or } m, m' \in \mathbb{Z} + \frac{1}{2}\}.\]
Then this corresponds to case (3), with \( L^\vee = \langle e_2, e_3, \ldots, e_{n-1} \rangle^\perp \oplus Y^\perp. \) \qed

8.3. Let \( H \) be a connected reductive complex linear algebraic group of semisimple rank one. Fix a Borel pair \( B \subset T \subset H \) with resulting positive root \( \alpha \). Let \( X^H \) be the affine Grassmannian for \( H \) and let \( \Lambda^H \subset X^H \) be the lattice of translations for \( T \). Consider the abstract \( SL(2, \mathbb{C}) \) which corresponds to the simple root \( \alpha \). The canonical mapping \( SL(2) \to H \) induces an inclusion \( \phi_\alpha : X^{SL(2)} \subset X^H \) with \( \phi_\alpha (\Lambda^{SL(2)}) = \langle \alpha^\vee (\varepsilon) \rangle \subset \Lambda^H \) (the infinite cyclic subgroup generated by \( \alpha^\vee (\varepsilon) \)). To simplify the notation, identify the co-root \( \alpha^\vee \) with its image \( \alpha^\vee (\varepsilon) \) in \( \Lambda^H \).

8.4. **Lemma.** The following statements hold.

1. The affine Grassmannian for \( H \) is the disjoint union,
   \[
   X^H = \bigsqcup_{\ell \in \frac{\Lambda^H}{\langle \alpha^\vee \rangle}} \ell \phi_\alpha (X^{SL(2)}).
   \]

2. The \( T^H (\mathfrak{o}) \) orbits on \( X^H \) coincide with the \( T^{SL(2)} (\mathfrak{o}) \) orbits on \( X^H \) (which are in turn translates, by elements of \( \Lambda^H \), of \( T^{SL(2)} (\mathfrak{o}) \) orbits on \( X^{SL(2)} \)).

3. Let \( \gamma \in t^H (\mathfrak{o}) \) be a regular element. Then the affine Springer fiber \( X^H_\gamma \) is the disjoint union
   \[
   X^H_\gamma = \bigsqcup_{\ell \in \frac{\Lambda^H}{\langle \alpha^\vee \rangle}} \ell \phi_\alpha (X^{SL(2)}_{\leq v}) \tag{8.4.1}
   \]
   where \( v = val(\alpha^\vee (\gamma)) \), cf. equation (6.2.1).

8.5. **Proof.** For part (1) use Lemma 8.1 to reduce to the case \( H = GL(2) \) or \( PGL(2) \). If \( H = GL(2) \) then the exact sequences
   \[
   I \longrightarrow SL(2, F) \longrightarrow GL(2, F) \stackrel{det}{\longrightarrow} F^\times \longrightarrow 1
   \]
   \[
   I \longrightarrow SL(2, \mathfrak{o}) \longrightarrow GL(2, \mathfrak{o}) \longrightarrow \mathfrak{o}^\times \longrightarrow 1
   \]
   together with the isomorphism \( val : F^\times / \mathfrak{o}^\times \cong \mathbb{Z} \) give rise to a diagram
   \[
   X^{SL(2)} \longrightarrow X^{GL(2)} \stackrel{val \cdot det}{\longrightarrow} \mathbb{Z}
   \]
   \[
   0 \longrightarrow \Lambda^{SL(2)} \longrightarrow \Lambda^{GL(2)} \longrightarrow \mathbb{Z} \longrightarrow 0
   \]
   from which the result follows. Now suppose \( H = PGL(2) \). If \( g \in H(F) \) let \( \tau(g) = val \cdot det g \mod 2 \). If \( \tau(g) = 0 \) then \( det g \) has a square root in \( F \) and \( g/\sqrt{det g} \in SL(2, F) \). So the following exact sequences
   \[
   \{ \pm I \} \longrightarrow SL(2, F) \longrightarrow PGL(2, F) \stackrel{\tau}{\longrightarrow} \mathbb{Z}/(2)
   \]
   \[
   \{ \pm I \} \longrightarrow SL(2, \mathfrak{o}) \longrightarrow PGL(2, \mathfrak{o}) \longrightarrow 1
   \]
give rise to a diagram

\[ \begin{array}{cccccc}
X^{SL(2)} & \longrightarrow & X^{PGL(2)} & \longrightarrow & \mathbb{Z}/(2) \\
\uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & \Lambda^{SL(2)} & \longrightarrow & \Lambda^{PGL(2)} & \longrightarrow & \mathbb{Z}/(2) & \longrightarrow & 0
\end{array} \]

from which the result follows.

For part (2) we may assume \( H = GL(2) \) or \( PGL(2) \). Let \( gK \in X^H \) and let \( t = (\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}) \in T^H(\mathfrak{o}) \). Then \( ad^{-1} \) has a square root \( \lambda \in \mathfrak{o} \). Set \( t' = \alpha^\vee(\lambda) \in T^{SL(2)}(\mathfrak{o}) \). By matrix multiplication, \( tgt^{-1} = t'g(t')^{-1} \) hence \( tgK' = t'gK' \).

Now consider part (3). It is easy to verify for \( H = GL(2) \), \( H = PGL(2) \) and for \( n \leq -1 \) that \( x_n \in X^H_\gamma \) iff \( |n| \leq v = \text{val}(\alpha'_{\lambda}) \), where \( x_n = (\begin{smallmatrix} 1 & \varepsilon^n \\ 0 & 1 \end{smallmatrix})K \). Since \( X^H_\gamma \) is a union of \( T^{SL(2)}(\mathfrak{o}) \) orbits, it follows that

\[ X^H_\gamma \cap \phi_\alpha(X^{SL(2)}) = \phi_\alpha(X^{SL(2)}_{\leq v}). \tag{8.5.1} \]

By Lemma 8.1, equation (8.5.1) holds for any \( H \) of semi-simple rank one. It follows that the union (8.4.1) is contained in \( X^H_\gamma \). Conversely, if \( \mathcal{O} \) is a \( T^H(\mathfrak{o}) \) orbit in \( X^H_\gamma \) then there exists \( \ell \in \Lambda^H \) so that \( \ell \phi_\alpha(x_n) \in \mathcal{O} \), which holds iff \( \phi_\alpha(x_n) \in X^H_\gamma \) or \( x_n \in X^{SL(2)}_{\leq v} \). So

\[ X^H_\gamma \subset \bigcup_{\ell \in \Lambda^H} \ell \phi_\alpha(X^{SL(2)}_{\leq v}) \]

The normalizer of \( \phi_\alpha(X^{SL(2)}_{\leq v}) \) in \( \Lambda^H \) is the sub-lattice \( \langle \alpha^\vee \rangle = \phi_\alpha(\Lambda^{SL(2)}) \), so the union (8.4.1) is disjoint. \( \square \)

9. Equivariant homology of affine Springer fibers

9.1. As in §5 we consider a connected reductive linear algebraic group \( G \) defined over \( \mathbb{C} \) and a Borel pair \( T \subset B \subset G \) with resulting system of positive roots \( \Phi^+ \). Fix \( \gamma \in t(\mathfrak{o}) \) and let \( X_\gamma \) be the resulting affine Springer fiber. It contains the lattice \( \Lambda = \Lambda^G \). We use multiplicative notation for the group operation in \( \Lambda \). For notational simplicity in this section we denote by \( t, t^* \), etc., the complex vector spaces \( t(\mathbb{C}) \), \( t^*(\mathbb{C}) \), etc. Let \( \alpha \in \Phi^+ \) and let \( \alpha^\vee : F^* \rightarrow T(F) \) be the corresponding co-root; it determines a lattice element which we also denote by \( \alpha^\vee \in \Lambda \). Moreover, \( \alpha^\vee \in t = t^\ast \) corresponds to a (degree 1) monomial

\[ x_{\alpha^\vee} \in S(t) \]

while \( \alpha \in t^* \) corresponds to a (degree 1) differential operator

\[ \partial_\alpha \in D(t) \]

such that \( \partial_\alpha(x_{\alpha^\vee}) = 2 \). Let \( S(t) \{ \partial_\alpha^d \} \) be the submodule of polynomial functions which are annihilated by the differential operator \( \partial_\alpha^d \). Define the following submodule of \( H^*_T(\mathbb{C})(\Lambda) = \)
\[ C[\Lambda] \otimes S(t) : \]

\[ L_{\alpha, \gamma} = \sum_{d=1}^{\text{val}(\alpha'(\gamma))} (1 - \alpha')^d C[\Lambda] \otimes S(t) \{ \partial_{\alpha}^d \}. \]

9.2. **Theorem.** Suppose the homology \( H_*(X_\gamma; \mathbb{C}) \) is pure. Then the inclusion \( \Lambda \subset X_\gamma \) induces an exact sequence

\[ 0 \longrightarrow \sum_{\alpha \in \Phi^+} L_{\alpha, \gamma} \longrightarrow C[\Lambda] \otimes S(t) \longrightarrow H_*^{T(\mathbb{C})}(X_\gamma) \longrightarrow 0. \]

9.3. **Proof.** Since the homology of \( X_\gamma \) is equivariantly formal, the exact sequence (4.3.1) becomes

\[ H_*^{T(\mathbb{C})}((X_\gamma)_1, \Lambda) \stackrel{\beta}{\longrightarrow} H_*^{T(\mathbb{C})}(\Lambda) \longrightarrow H_*^{T(\mathbb{C})}(X_\gamma) \longrightarrow 0 \]

(9.3.1)

where \((X_\gamma)_1 \) denotes the closure of the set of 1-dimensional \( T(\mathbb{C}) \) orbits in \( X_\gamma \). By Lemma 5.12, \( X_\gamma^\alpha \cap X_\gamma^\beta = \Lambda \) if \( \alpha \neq \beta \), and

\[ (X_\gamma)_1 = \bigcup_{\alpha \in \Phi^+} X_\gamma^\alpha \]

where \( X_\gamma^\alpha \) is the affine Springer fiber corresponding to \( \gamma \) in the loop Grassmannian \( X^\alpha \) for the group \( H^\alpha \) of semi-simple rank one which is determined by \( \alpha \). So the image of \( \beta \) is the sum over positive roots of the image of the corresponding mapping in the semisimple rank one case. Thus, it suffices to consider the case in which the group \( G \) has semisimple rank one, which we now assume.

Let \( A = \ker(\alpha)^0 \subset T \) and let \( T_1 \subset T \) be the 1-dimensional sub-torus corresponding to the co-root \( \alpha' \). The canonical decomposition \( t = t_1 \oplus a \) determines an isomorphism \( H_*^{T(\mathbb{C})}(pt) \cong H_*^{T_1(\mathbb{C})}(pt) \otimes H_*^{A(\mathbb{C})}(pt) \), that is, \( S(t) \cong S(t_1) \otimes S(a) \) and

\[ S(t) \{ \partial_{\alpha}^d \} \cong S(t_1) \{ \partial_{\alpha}^d \} \otimes S(a). \]

So the exact sequence (9.3.1) for \( T(\mathbb{C}) \)-equivariant homology is obtained from the same exact sequence for \( T_1(\mathbb{C}) \)-equivariant homology

\[ H_*^{T_1(\mathbb{C})}(X_\gamma, \Lambda) \stackrel{\beta}{\longrightarrow} H_*^{T_1(\mathbb{C})}(\Lambda) \longrightarrow H_*^{T_1(\mathbb{C})}(X_\gamma) \longrightarrow 0 \]

(9.3.2)

by tensoring with \( S(a) \). By Lemma 8.4 the homomorphism \( \beta_1 \) decomposes as a direct sum of homomorphisms

\[ \bigoplus_{\ell \in \Lambda^H/\langle \alpha' \rangle} \ell_* \phi_{\alpha *} H_*^{T_1(\mathbb{C})}(X_{\leq \ell}^{\text{SL}(2)}, \Lambda^{\text{SL}(2)}) \longrightarrow \bigoplus_{\ell \in \Lambda^H/\langle \alpha' \rangle} \ell_* \phi_{\alpha *} H_*^{T_1(\mathbb{C})}(\Lambda^{\text{SL}(2)}) \]

(where \( \phi_{\alpha *} \) and \( \ell_* \) denote the homomorphisms on homology which are induced by \( \phi_{\alpha} \) and by translation by \( \ell \in \Lambda^H \), respectively). So by Proposition 7.2 the image of \( \beta_1 \) is the sum

\[ \sum_{\ell \in \Lambda^H/\text{SL}(2)} \ell_* \phi_{\alpha *} \sum_{d=1}^{\nu} (\alpha' - 1)^d C[\Lambda]^{\text{SL}(2)} \otimes S(t_1) \{ \partial_{\alpha}^d \} = \sum_{d=1}^{\nu} (\alpha' - 1)^d C[\Lambda] \otimes S(t_1) \{ \partial_{\alpha}^d \}. \]
Since the image of $\beta$ is obtained by tensoring with $S(a)$, we obtain

$$\text{Im}(\beta) = \sum_{d=1}^{\infty} (\alpha^\vee - 1)^d C[\Lambda] \otimes S(t) \{ \partial_\alpha^d \}. \quad \square$$

In fact, in the semisimple rank one case, as a module over $S(a)$ the vector space of relations $L_{\alpha, \gamma}$ has a basis consisting of the collection of elements

$$f_{\ell, d, \alpha} = (1 - \alpha^\vee)^d \ell \otimes x_{\alpha}^{d-1} \in C[\Lambda] \otimes S(t)$$

with $\ell \in \Lambda$ and $1 \leq d \leq \text{val}(\alpha'((\gamma)))$.

9.4. Let $\text{Aut}$ denote the automorphism group of the based root datum for $G$. Let $W = W(G, T)$ be the Weyl group and let $\tilde{W} = \Lambda \rtimes W$ be the extended affine Weyl group. Then the group

$$\tilde{W} \rtimes \text{Aut}$$

acts on $T$ and on $\Lambda$. It acts through $W \rtimes \text{Aut}$ on $T(\mathbb{C})$ and on the root system $\Phi(G, T)$. Consider the diagonal action (from the left) of the group (9.4.1) on the equivariant homology

$$H_*^{T(\mathbb{C})}(\Lambda) \cong C[\Lambda] \otimes_C S(t).$$

It preserves the kernel

$$L_\infty = \sum_{\alpha \in \Phi^+} \sum_{d=1}^{\infty} (1 - \alpha^\vee)^d C[\Lambda] \otimes S(t) \{ \partial_\alpha^d \}$$

defined in (4.1.1) of the surjection $H_*^{T(\mathbb{C})}(\Lambda) \to H_*^{T(\mathbb{C})}(X)$ so it passes to an action on the equivariant homology of $X$, which we will refer to as the “left” action. If $\gamma \in t(0)$ then the subgroup

$$\left(\tilde{W} \rtimes \text{Aut}\right)_\gamma = \left\{ \tau \in \tilde{W} \rtimes \text{Aut} : \text{val}(\tau\alpha(\gamma)) = \text{val}(\alpha(\gamma)) \text{ for all } \alpha \in \Phi^+ \right\}$$

preserves the module of relations

$$\sum_{\alpha \in \Phi^+} L_{\alpha, \gamma} \subset H_*^{T(\mathbb{C})}(\Lambda).$$

Consequently, if the homology $H_*(X_\gamma)$ is pure, the group $(\tilde{W} \rtimes \text{Aut})_\gamma$ acts (from the left) on the equivariant homology $H_*^{T(\mathbb{C})}(X_\gamma)$ by $D(\xi)$ module homomorphisms. By (4.1.1) the ordinary homology is the submodule $H_*^{T(\mathbb{C})}(X_\gamma) = H_*^{T(\mathbb{C})}(X_\gamma)\{\mathcal{I}\}$ which is killed by the augmentation ideal $\mathcal{I}$, so we obtain an action of $(\tilde{W} \rtimes \text{Aut})_\gamma$ on the ordinary homology of the affine Springer fiber.

We remark that in many cases (including $G = SL(n)$, $Sp(n)$, or $O(n)$) it is possible to construct a (continuous) action of the group (9.4.1) on $X$ which is $T(\mathbb{C})$ covariant, meaning that $\tau(t \cdot x) = \tau(t) \cdot \tau(x)$ for $t \in T(\mathbb{C})$, $x \in X$, and $\tau \in \tilde{W} \rtimes \text{Aut}$. The induced action
on $H_*^{T(\mathbb{C})}(X)$ then agrees with the left action. However, even in these cases, the topological action of the subgroup (9.4.2) does not necessarily preserve the Springer fiber $X_\gamma$.

10. ENDOSCOPIC GROUPS

10.1. In this section we assume, for simplicity only, that $G$ is adjoint. Fix endoscopic data $(H, s)$ for $G$. This means that $H$ is a connected reductive complex algebraic group, that $s \in \hat{T} \subset \hat{G}$, and $\hat{H} = \hat{G}_s$ is the centralizer of $s$ in $\hat{G}$. Then $\hat{H}$ is connected since $\hat{G}$ is simply connected ([Hum95] §2.11). We assume moreover that Borel pairs $T \subset B \subset G$ and $T_H \subset B_H \subset H$ (defined over $\mathbb{C}$) have been chosen, giving rise to a canonical isomorphism

$$T_H \cong T.$$  

Then $\hat{T} = T_H$ is also a maximal torus in $\hat{H}$. The set of co-roots of $T$ in $H$ is

$$\Phi^\vee(H, T_H) = \{ \alpha^\vee \in \Phi^\vee(G, T) : s(\alpha^\vee) = 1 \}.$$  

Let $D = D(t) \cong S(t^*)$ be the graded algebra of differential operators on $t^*$. Each $\alpha \in \Phi(G, T)$ determines a differential operator $\partial_\alpha \in D$ of degree 1. Define the homological transfer factor

$$\Delta = \prod_{\alpha \in \Phi^+(G) - \Phi^+(H)} \partial_\alpha^{\text{val}\alpha^\vee(\gamma)} \in D = H^*_T(\mathbb{C})(pt).$$  

(10.1.2)

It is a homogeneous element of some degree, $r = \sum_{\alpha \in \Phi^+(G) - \Phi^+(H)} \text{val}\alpha^\vee(\gamma)$. The action of $\Delta$ is a surjection of graded $D$ modules

$$S(t) \xrightarrow{\Delta} S(t)[-r] \longrightarrow 0$$

(where $A[-r]$ denotes the shift in grading, $A[-r]_k = A_{k-r}$ for any graded vector space $A = \bigoplus_k A_k$).

Fix $\gamma \in t(\mathfrak{o})$ regular and semisimple and let $\gamma_H \in t_H = \text{Lie}(T_H)$ be the element which corresponds to $\gamma$ under the isomorphism (10.1.1). Then $\gamma_H$ acts as a “vector field” on the affine Grassmannian $X^H = H(F)/H(\mathfrak{o})$ and we denote its fixed point set by $X^H_{\gamma_H}$.

Let $\text{Aut}^{G, H}$ be the subgroup of the automorphism group of the based root datum for $G$ which preserves the roots of $H$. Let $W^H$ be the Weyl group for $H$, set $W^H = \Lambda \times W^H$ and $
abla_{\gamma}^{G, H} = \nabla^H \rtimes \text{Aut}^{G, H}$, and define a subgroup of $\nabla_{\gamma}^{G, H}$ by

$$\nabla_{\gamma}^{G, H} = \left\{ \tau \in \nabla^{G, H} : \text{val}(\tau \alpha(\gamma)) = \text{val}(\alpha(\gamma)) \text{ for all } \alpha \in \Phi^+(G, T) \right\}.$$  

(10.1.3)

Note that the square of $\Delta$ is invariant under $\nabla_{\gamma}^{G, H}$ and hence that there exists a sign character

$$\eta : \nabla_{\gamma}^{G, H} \rightarrow \{\pm 1\}$$

such that $\eta(\Delta) = \eta(\tau) \cdot \Delta$ for all $\tau \in \nabla_{\gamma}^{G, H}$.

Let $J \subset \mathbb{C}[\Lambda]$ be the multiplicative subset which is generated by the elements $1 - \alpha^\vee$ for $\alpha^\vee \in \Phi^\vee(G, T) - \Phi^\vee(H, T_H)$. For any $\mathbb{C}[\Lambda]$ module $M$, let $M_J = J^{-1}M$ denote the $\mathbb{C}[\Lambda]_J = \mathbb{C}[\Lambda][J^{-1}]$ module.
$J^{-1} \mathbb{C}[\Lambda]$ module which is obtained by inverting the elements of $J$. The localization of the module $\mathbb{C}[\Lambda] \otimes \mathbb{C} S(t)$ is $\mathbb{C}[\Lambda]_J \otimes \mathbb{C} S(t)$.

10.2. Theorem. Suppose the homology of $X_\gamma$ and of $X_{\gamma H}$ is pure. Then the surjection

$$1 \otimes \Delta : \mathbb{C}[\Lambda] \otimes \mathbb{C} S(t) \to \mathbb{C}[\Lambda] \otimes \mathbb{C} S(t)[-r] \quad (10.2.1)$$

induces a homomorphism of $\mathcal{D}$ modules

$$\Psi : H^T_{*}(X_\gamma) \to H^T_{*}(X_{\gamma H})[-2r] \quad (10.2.2)$$

which becomes an isomorphism

$$\Psi_J : H^T_{*}(X_\gamma)_J \to H^T_{*}(X_{\gamma H})_J[-2r]$$

after localizing with respect to $J$. The homomorphisms $\Psi$ and $\Psi_J$ are not quite $\tilde{W}^{G,H}_{\gamma}$ equivariant; rather they transform under $\tilde{W}^{G,H}_{\gamma}$ by the sign character $\eta$ defined above.

Assuming the homology of $X_\gamma$ is pure, by Lemma 2.2 and equation (4.1.1) the ordinary homology of $X_\gamma$ is given by the $\mathcal{D}$-submodule of the equivariant homology

$$H_*(X_\gamma; \mathbb{C}) \cong H^T_{*}(X_\gamma; \mathbb{C}) \{I\}$$

which is annihilated by the augmentation ideal $I \subset \mathcal{D}$. We conclude:

10.3. Corollary. The action of $1 \otimes \Delta$ on the equivariant homology induces a homomorphism $H_*(X_\gamma) \to H_*(X_{\gamma H})[-2r]$ and an isomorphism

$$H_*(X_\gamma; \mathbb{C})_J \cong H_*(X_{\gamma H}; \mathbb{C})_J[-2r]. \quad (10.3.1)$$

These maps again transform under $\tilde{W}^{G,H}_{\gamma}$ by the sign character $\eta$.

Although the homomorphism (10.2.2) may be described relatively easily in equivariant homology, the resulting isomorphism (10.3.1) in ordinary homology is much more complicated.

10.4. Proof of Theorem 10.2. The mapping $\Psi$ is well defined because the mapping (10.2.1) kills the submodule

$$L_{\alpha,\gamma} = \sum_{d=1}^{\text{val}(\alpha'(\gamma))} (1 - \alpha^\vee)^d \mathbb{C}[\Lambda] \otimes \mathbb{C} S(t) \{\partial^d\}$$

whenever $\alpha \in \Phi^+(G) - \Phi^+(H)$. If $w \in \tilde{W} \rtimes \text{Aut}$ and if $f \in S(t)$ then $w(\Delta f) = w(\Delta) w(f)$. If $w$ lies in the subgroup $\tilde{W}^{G,H}_{\gamma}$ then $w(\Delta) = \eta(w) \cdot \Delta$. Therefore the maps $1 \otimes \Delta$ and $\Psi$ transform under $\tilde{W}^{G,H}_{\gamma}$ by the sign character $\eta$.

Now let us check that $\Psi_J$ is an isomorphism. The surjection

$$(1 \otimes \Delta)_J : \mathbb{C}[\Lambda]_J \otimes \mathbb{C} S(t) \to \mathbb{C}[\Lambda]_J \otimes \mathbb{C} S(t)[-r]$$
has kernel
\[ \ker(1 \otimes \Delta)_J = \mathbb{C}[\Lambda]_J \otimes S(t) \{ \Delta \} \]
and it restricts to a mapping \( h_J \) in the following diagram.

\[ \begin{array}{cccccc}
0 & \rightarrow & \sum_{\alpha \in \Phi^+(G)} J^{-1}L_{\alpha,\gamma} & \rightarrow & \mathbb{C}[\Lambda]_J \otimes S(t) & \rightarrow & H^T_c(X_\gamma)_J & \rightarrow & 0 \\
& & \downarrow h_J & & \downarrow (1 \otimes \Delta)_J & & \downarrow & & \\
0 & \rightarrow & \sum_{\alpha \in \Phi^+(H)} J^{-1}L_{\alpha,\gamma}[-r] & \rightarrow & \mathbb{C}[\Lambda]_J \otimes S(t)[-r] & \rightarrow & H^T_c(X_{\gamma_H})_J[-2r] & \rightarrow & 0
\end{array} \]

It suffices (by the snake lemma) to show that \( h_J \) is surjective, and that

\[ \ker(1 \otimes \Delta)_J \subseteq \sum_{\alpha \in \Phi^+(G)} J^{-1}L_{\alpha,\gamma}. \] (10.4.1)

If \( \alpha \in \Phi^+(G) - \Phi^+(H) \) (that is, if \( s(\alpha^\vee) \neq 1 \)) then \( 1 - \alpha^\vee \) becomes invertible in \( \mathbb{C}[\Lambda]_J \). If \( a \leq b \) then \( S(t) \{ \partial^a_\alpha \} \subseteq S(t) \{ \partial^b_\alpha \} \). Therefore the localization of the submodule \( L_{\alpha,\gamma} \) is

\[ J^{-1}L_{\alpha,\gamma} = \mathbb{C}[\Lambda]_J \otimes S(t) \left\{ \partial^{\text{val}(\alpha'(\gamma))}_\alpha \right\}. \]

By Lemma 3.2, the sum

\[ \sum_{\alpha \in \Phi^+(G) \atop s(\alpha^\vee) \neq 1} S(t) \left\{ \partial^{\text{val}(\alpha'(\gamma))}_\alpha \right\} = S(t) \{ \Delta \} \]

is killed by \( \Delta \). Therefore

\[ \ker(1 \otimes \Delta)_J = \sum_{\alpha \in \Phi^+(G) \atop s(\alpha^\vee) \neq 1} J^{-1}L_{\alpha,\gamma} \]

which proves (10.4.1). On the other hand, if \( \alpha \in \Phi^+(G) \) and \( s(\alpha^\vee) = 1 \) then

\[ J^{-1}L_{\alpha,\gamma} = \sum_{d=1}^{\text{val}(\alpha'(\gamma))} (1 - \alpha^\vee)^d \mathbb{C}[\Lambda]_J \otimes S(t) \left\{ \partial^d_\alpha \right\}. \]

In this case (by Lemma 3.2) the action of \( \Delta \) is a surjection

\[ S(t) \{ \partial^d_\alpha \} \rightarrow S(t) \{ \partial^d_\alpha \} [-r] \]

which implies that \( h_J \) is surjective. \( \square \)

10.5. Remark. We did not use the full strength of the endoscopic hypothesis on \( H \), and in fact Theorem 10.2 remains valid whenever \( H \) is a connected reductive group for which there exists a Borel pair \( T_H \subset B_H \) and an isomorphism \( T \cong T_H \) such that the set of positive roots \( \Phi^+(H, T_H) \) is a subset of the positive roots \( \Phi^+(G, T) \).
11. The Quotient Under $\Lambda$

11.1. As in §10, fix a Borel pair $T \subset B \subset G$, with resulting positive roots $\Phi^+$. Fix regular $\gamma \in \mathfrak{t}(\mathfrak{o})$ with affine Springer fiber $X_\gamma$ and its lattice of translations $\Lambda$. The quotient $\Lambda \backslash X_\gamma$ admits the structure of a complex projective algebraic variety [KL88]. Let $(H, s)$ be endoscopic data, with $s \in \widehat{T}$, and assume that $s$ has finite order. View $s$ as a character $s : \Lambda \to \text{GL}_1(\mathbb{C})$ and let $C_s$ be the 1-dimensional $C[\Lambda]$ module defined by $s$. Then $s$ determines 1-dimensional local systems

$$L_s = X_\gamma \times_\Lambda C_s \quad \text{and} \quad M_s = X_{\gamma H} \times_\Lambda C_s$$

on $\Lambda \backslash X_\gamma$ and $\Lambda \backslash X_{\gamma H}$ respectively. Let $r = \text{deg}(\Delta)$ as in (10.1.2).

11.2. Proposition. Suppose that $H_*(X_\gamma; \mathbb{C})$ and $H_*(X_{\gamma H}; \mathbb{C})$ are pure. Then there is an isomorphism

$$H_m(\Lambda \backslash X_\gamma; L_s) \cong H_{m-2r}(\Lambda \backslash X_{\gamma H}; M_s). \quad (11.2.1)$$

If the homology of $X_\gamma$ is both pure and Tate, and if the same is true of $X_{\gamma H}$, then this isomorphism may be chosen in a canonical manner.

11.3. Proof. The $E^2$ term of the Cartan-Leray spectral sequence for the above $\Lambda$-equivariant homology group is

$$E^2_{pq}(X_\gamma; \mathbb{C}) = H_p(\Lambda; H_q(X_\gamma; \mathbb{C})) = \text{Tor}^C[\Lambda](H_q(X_\gamma), \mathbb{C}_s) \implies H_{p+q}(\Lambda \backslash X_\gamma; L_s).$$

By Lemma 2.4 the spectral sequence collapses and gives rise to an isomorphism

$$H_m(\Lambda \backslash X_\gamma; L_s) \cong \bigoplus_{p+q=m} \text{Tor}^C[\Lambda](H_q(X_\gamma; \mathbb{C}), \mathbb{C}_s) \quad (11.3.1)$$

which is canonical if $H_*(X_\gamma; \mathbb{C})$ is also Tate.

Let $J \subset C[\Lambda]$ be the multiplicative subset generated by the collection of elements $(1 - \alpha^\vee)$ with $\alpha \in \Phi^+(G)$ and $s(\alpha^\vee) \neq 1$. The homomorphism $C[\Lambda] \to C_s (\ell \mapsto s(\ell))$ which gives rise to the local system $L_s$ may be factored as the composition of ring homomorphisms

$$C[\Lambda] \to C[\Lambda]_J \to C_s \quad (11.3.2)$$

in which the second map is defined by

$$(1 - \alpha^\vee)^{-1} \ell \mapsto (1 - s(\alpha^\vee))^{-1} s(\ell)$$

for any $\alpha^\vee \in \Phi^+(G, T) - \Phi^+(H, T_H)$ and any $\ell \in \Lambda$. It is well defined because $s(\alpha^\vee) \neq 1$. For any $C[\Lambda]$ module $V$, the composition (11.3.2) induces an isomorphism

$$V \otimes_{C[\Lambda]} C_s \cong V \otimes_{C[\Lambda]} C[\Lambda]_J \otimes_{C[\Lambda]} C_s \cong V_J \otimes_{C[\Lambda]} C_s.$$

Taking derived functors gives an isomorphism $\text{Tor}^C[\Lambda](V, C_s) \cong \text{Tor}^C[\Lambda]^J(V_J, C_s)$. Taking $V = H_q(X_\gamma; \mathbb{C})$ gives an isomorphism

$$H_m(\Lambda \backslash X_\gamma; L_s) \cong \bigoplus_{p+q=m} \text{Tor}^C[\Lambda]^J(H_q(X_\gamma; \mathbb{C})_J, C_s).$$
So Proposition 11.2 follows from Corollary 10.3.

11.4. Although we have not constructed the isomorphism (11.2.1) in a canonical way, the
isomorphism induced by $\Delta$,

$$E^2_{pq}(X; \mathbb{C}) \cong E^2_{p,q-2r}(X; \mathbb{C})$$

is canonical, and it transforms under $\tilde{W}_{G,H}^{G,H}$ by the sign character $\eta$. It follows that the
Lefschetz number of any $w \in \tilde{W}_{G,H}^{G,H}$ on $E^2(X; \mathbb{C})$ is equal to that on $E^2_{*}(X; \mathbb{C})$ times
the sign $\eta(w)$.

12. COMBINATORIAL LEMMAS

12.1. Let $\Lambda$ be a free abelian group of rank one, and let $Q[\Lambda]$ be its group ring. (The
group law in $\Lambda$ will be written multiplicatively.) A choice of generator $\alpha \in \Lambda$ determines
an isomorphism $\Lambda \cong \mathbb{Z}$. Write $\ell_a \in \Lambda$ for the element corresponding to $a \in \mathbb{Z}$. Then
multiplication by $\ell_1 = \alpha$ acts as a “shift operator”: $\alpha \ell_a = \ell_{a+1}$.

The algebra $Q[\partial_x, \partial_t]$ of differential operators in two variables acts on the algebra $Q[x, t]$ of polynomials in two variables, and hence also on the algebra $Q[\Lambda] \otimes Q[x, t]$ with

$$\ker(\partial_t) = Q[\Lambda] \otimes Q[x].$$

Fix $d, m \in \mathbb{Z}$ with $d \geq 1$. Define

$$f_{m,d} = \sum_{m \leq a < b \leq m+d} C_{ab}(\ell_b - \ell_a) \otimes ((a+b) x - t)^{d-1} \in Q[\Lambda] \otimes Q[x, t]$$

where $C_{ab} \in \mathbb{Z}$ is the integer

$$C_{ab} = (-1)^{a-b} (a-b) \binom{d}{a-m} \binom{d}{b-m}.$$

Define $J_v \subset Q[\Lambda] \otimes Q[x, t]$ to be the span

$$J_v = \sum_{d=1}^v \sum_{m \in \mathbb{Z}} Qf_{m,d}.$$

12.2. Lemma. Fix $d, m \in \mathbb{Z}$ with $d \geq 1$. Then

$$f_{m,d} = (-1)^d d! (1 - \alpha^\vee)^d \ell_m \otimes x^{d-1} \in \ker(\partial_t).$$

If $v \geq 1$ then

$$J_v = \sum_{d=1}^v (1 - \alpha^\vee)^d Q[\Lambda] \otimes Q[x] \{\partial_x^d\}$$

where $Q[x] \{\partial_x^d\}$ denotes the polynomials which are annihilated by $\partial_x^d$, that is, the polynomials
of degree $\leq d-1$. 27
12.3. Proof. We use the fact [H86] §1.1 that for any polynomial \( p \),
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} p(k) = \begin{cases} 
0 & \text{if } \deg p \leq n - 1 \\
(-1)^n n! & \text{if } p(k) = k^n.
\end{cases}
\] (12.3.1)

Since \( C_{ab} = -C_{ba} \) we may express \( f_{m,d} \) as:
\[
\sum_{a=m}^{m+d} \sum_{b=m}^{m+d} C_{ab} \ell_b \otimes ((a+b)x - t)^{d-1}
= \sum_{a=m}^{m+d} \sum_{b=m}^{m+d} \left( \ell_b \otimes C_{ab} \sum_{j=0}^{d-1} \binom{d-1}{j} (a+b)^j x^j (-1)^{d-1-j} t^{d-1-j} \right)
= \sum_{b=m}^{m+d} \sum_{j=0}^{d-1} \frac{(-1)^b}{j!} \binom{d}{b-m} \binom{d-1}{j} t^d \sum_{a=m}^{m+d} \left( \frac{d}{a-m} (a-b)(a+b)x^j \right)
\]
where \( b' = b + d - 1 - j - m \) and \( d' = d - 1 - j \). By (12.3.1), the innermost sum vanishes unless \( j = d - 1 \), leaving
\[
f_{m,d} = (-1)^d d! \sum_{b=m}^{m+d} \left( \frac{d}{b-m} \ell_b \right) \otimes x^{d-1}
\]
which is (12.2.1). To verify the reverse inclusion in equation (12.2.2) it suffices to show that \( (1 - \alpha)^d \ell_m \otimes x^{e-1} \) is in \( J_v \) for every \( d, e \) with \( 1 \leq e \leq d \leq v \). But
\[
(1 - \alpha)^d \ell_m \otimes x^{e-1} = (1 - \alpha)^e (1 - \alpha)^{d-e} \ell_m \otimes x^{e-1}
= (1 - \alpha)^e \sum_{j=0}^{d-e} (-1)^{d-e-j} \binom{d-e}{j} \ell_{m+j} \otimes x^{e-1}
= \frac{(-1)^{d-e}}{e!} \sum_{j=0}^{d-e} \binom{d-e}{j} f_{m+j,e}
\]
by (12.2.1). Since \( e \leq v \), this quantity lies in \( J_v \). This completes the proof. \( \square \)

12.4. Lemma. Fix \( d, h, v \geq 1 \) with \( v \leq h \). Fix \( m \in \mathbb{Z} \). Let
\[
g = \sum_{\substack{m \leq a < b \leq m+h \\ b-a \leq v}} (\ell_b - \ell_a) \otimes G_{ab} ((a+b)x - t)^{d-1}
\] (12.4.1)
where \( G_{ab} \in \mathbb{Q} \) (with \( b > a \)). If \( \partial_t g = 0 \) and \( d > v \) then \( g = 0 \).
12.5. **Proof.** Write \( g_{ab} = G_{ab} ((a + b)x - t)^{d-1} \). The sum in (12.4.1) may be written either as

\[
\sum_{a=m}^{m+h-1} \min(m+h,a+v) \sum_{b=a+1}^{b-1} \text{ or } \sum_{b=m+1}^{m+h} \min(m+h,a+v) \sum_{a=m}^{a=\max(m,b-v)}
\]

Therefore

\[
g = \sum_{b=m+1}^{m+h} \min(m+h,a+v) \sum_{a=m}^{a=\max(m,b-v)} \ell_b \otimes g_{ab} - \sum_{a=m}^{a=\max(m,b-v)} \sum_{b=a+1}^{b=\max(m,a-v)} \ell_a \otimes g_{ab}
\]

(but if \( a = m \) then the first sum is empty, while if \( a = m + h \) the second sum is empty). Fix \( a \) with \( m \leq a \leq m + h \). Expanding \( ((a + b)x - t)^{d-1} \), the equation \( \partial_t g = 0 \) gives the following system of linear equations:

\[
\sum_{b=\max(m,a-v)}^{a-1} G_{ba}(a + b)^j - \sum_{b=a+1}^{b=\max(m,a-v)} G_{ab}(a + b)^j = 0 \quad \text{(12.5.1)}
\]

for \( 0 \leq j \leq d - 2 \). This is a system of \( d - 1 \) homogeneous equations of van der Monde type, in \( \min(m + h, a + v) - \max(m, a - v) \) variables. If the number of equations equals or exceeds the number of variables, then only the zero solution exists, since the corresponding van der Monde determinant is nonzero. Suppose \( d > v \). First consider the system (12.5.1) corresponding to \( a = m \). Then the first sum in (12.5.1) is empty, leaving the second sum, which consists of \( d - 1 \) (homogeneous) equations (of van der Monde type) in \( \leq v \) unknowns \( G_{mb} \) (with \( a + b \leq b \leq a + v \)). So these coefficients vanish. Suppose by induction that the coefficients \( G_{ab} \) have been shown to vanish for all pairs \((a, b)\) with \( a < t \) and \( a < b \leq \min(m + h, a + v) \). Consider the system (12.5.1) in the case \( a = t \). The terms in the first sum, \( \sum_{b=\max(m,t-v)}^{t-1} G_{xt} \) vanish by the induction hypothesis, leaving only the second sum. This is a system of \( d - 1 \) homogeneous equations in \( \leq v \) variables so only the zero solution \( G_{th} = 0 \) exists. This completes the induction, so we conclude that \( g = 0 \). \( \square \)

We remark that the same (inductive) argument shows that if \( d = v = h \) then there is at most a one dimensional space of solutions, so \( g \) is a multiple of \( f_{m,d} \).

12.6. Define \( P_v \subset \mathbb{Q}[\Lambda] \otimes_{\mathbb{Q}} \mathbb{Q}[x, t] \) to be the vector space spanned by elements

\[
(\ell_b - \ell_a) \otimes g_{ab}((a+b)x-t) \quad \text{(12.6.1)}
\]

where \( g_{ab} \) are polynomials, and where \(|b-a| \leq v\). Let \( P_v \{ \partial_t \} = P_v \cap \ker(\partial_t) \).
12.7. **Proposition.** Fix $v \geq 1$. Then $P_v \{ \partial_t \} = J_v$, that is,

$\ker(\partial_t) \cap \sum_{|b-a| \leq v} \mathbb{Q}(\ell_b - \ell_a) \otimes \mathbb{Q}[(a+b)x-t] = \sum_{d=1}^{v} (1-\alpha^d)^{d} \mathbb{Q}[\Lambda] \otimes \mathbb{Q}[x] \{ \partial^d_x \}$.

12.8. **Proof.** The inclusion $J_v \subset P_v \{ \partial_t \}$ follows from Lemma 12.2, so we only need to verify the reverse inclusion. Let $P_{v,h}$ be the vector subspace of $P_v$ spanned by elements (12.6.1) such that each $g_{ab}(z) = G_{ab}z^h$ is homogeneous of degree $h$ (with $G_{ab} \in \mathbb{Q}$). Set $P_{v,h} \{ \partial_t \} = P_{v,h} \cap \ker(\partial_t)$. Then

$$P_v \{ \partial_t \} = \sum_{h \geq 0} P_{v,h} \{ \partial_t \};$$

and Lemma 12.4 says that $P_{v,h} \{ \partial_t \} = 0$ whenever $h \geq v$. So we need to show that $P_{v,d-1} \{ \partial_t \} \subset J_v$ whenever $d \leq v$. Since both $P_v \{ \partial_t \}$ and $J_v$ are modules over $\mathbb{Q}[\partial_x]$ it suffices to consider the case $d = v$, that is, we must show that $P_{v,v-1} \{ \partial_t \} \subset J_v$. Let

$$g = \sum_{m \leq a < b \leq m+N \atop b-a \leq v} (\ell_b - \ell_a) \otimes G_{ab}((a+b)x-t)^{v-1} \in P_{v,v-1} \{ \partial_t \} \quad (12.8.1)$$

(for some $m, N \in \mathbb{Z}$) where $G_{ab} \in \mathbb{Q}$. The expression (12.1.1) for $f_{m,v}$ contains a unique term with $b-a = v$. However the sum (12.8.1) for $g$ contains only terms with $b-a \leq v$, so by subtracting appropriate multiples of $f_{m,v} \in J_v \cap P_{v,v-1} \{ \partial_t \}$ we can eliminate all terms with $b-a = v$. In other words, there exists $h \in J_v \cap P_{v,v-1} \{ \partial_t \}$ such that $g-h \in P_{v,v-1} \{ \partial_t \} = 0$. Therefore $g \in J_v$. \qed

12.9. **Remarks.** The lemmas in this section refer to the equivariant homology $H^*_T(\mathbb{C}) (X_{\leq v}^{SL(2)})$ of the affine Springer fiber. Each 1-dimensional orbit $E_{ab} \subset X_{\leq v}^{SL(2)}$ of the extended torus $T(\mathbb{C})$ determines a relation in $\mathbb{C}[\Lambda] \otimes \mathbb{C}[x,t] = H^*_T(\mathbb{C}) (\Lambda)$. For $1 \leq d \leq v$ the sum (12.1.1) corresponds to a “constellation” of (finitely many) 1-dimensional orbits, such that the resulting relation $f_{m,d}$ lies in the subalgebra $\mathbb{C}[\Lambda] \otimes \mathbb{C}[x] = H^*_T(\mathbb{C}) (\Lambda)$. This constellation consists of all the 1-dimensional orbits which are contained in a single irreducible component of $X_{\leq d}^{SL(2)}$. One may think of this constellation of 1-dimensional orbits as being attached to the single orbit (the “longest one” in the constellation) which joins the lattice points $\ell_m$ and $\ell_{m+d}$. So $H^*_T(\mathbb{C}) (X_{\leq v})$ is the quotient of $\mathbb{C}[\Lambda] \otimes \mathbb{C}[x]$ by the relations (12.2.1), one for each 1-dimensional $\tilde{T}(\mathbb{C})$-orbit in $X_{\leq v}$.

13. **Affine flag manifold for SL(2)**

The results in this section are parallel to those of §6, §7, §8, and §12 so they will be presented without detailed proofs.
13.1. Throughout this section we take $G = \text{SL}(2)$. Let $B = (\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix})$ be the standard Borel subgroup of $\text{SL}(2)$ with corresponding Iwahori subgroup $I \subset G(\mathfrak{o})$ and let $Y = Y^{\text{SL}(2)} = G(F)/I$ be the affine flag manifold with basepoint $x_0 = I$. Let $T \subset G$ be the torus of diagonal matrices, with its lattice of translations $\Lambda = \{ \alpha^\vee(\epsilon^m) : m \in \mathbb{Z} \}$ where $\alpha^\vee \in \Phi^\vee$ is the simple co-root determined by $T \subset B$ (cf. §5). The torus $T(\mathbb{C})$ acts on $Y$ with fixed points

$$
\ell_m = \begin{pmatrix} \epsilon^m & 0 \\ 0 & \epsilon^{-m} \end{pmatrix} I \quad \text{and} \quad r_m = \begin{pmatrix} 0 & \epsilon^m \\ -\epsilon^{-m} & 0 \end{pmatrix} I
$$

(13.1.1) for $m \in \mathbb{Z}$. Let $W = W(G,T) = \{1, w_0\}$ be the Weyl group of $G$. Since the extended affine Weyl group $\tilde{W} = \Lambda \rtimes W$ acts simply transitively on these fixed points, the choice of basepoint $x_0 = \ell_0 \in Y$ determines an identification of $\tilde{W}$ with this set of fixed points, which identifies $\alpha^\vee(\epsilon^m) \in \Lambda$ with $\ell_m \in Y$, and identifies the reflection $w_0$ with $r_0 \in Y$. Then $\ell_a\ell_b = \ell_{a+b}$, $\ell_ar_b = r_{a+b}$, $r_ar_b = r_{a-b}$, and $r_ar_b = \ell_{a-b}$.

For $m \leq 0$ define

$$x_m = \begin{pmatrix} 1 & \epsilon^m \\ 0 & 1 \end{pmatrix} I \quad \text{and} \quad y_m = \begin{pmatrix} \epsilon^m & -1 \\ 1 & 0 \end{pmatrix} I.
$$

(13.1.2)

Then $x_0 = \ell_0$ is still the basepoint. For notational brevity, put $y_1 = r_0$. Let $\tilde{T}(\mathbb{C})$ denote the extended torus (5.5.1). If $m \leq -1$ and $s \in \mathbb{Z}$, the $\tilde{T}(\mathbb{C})$ orbit of $\ell_s x_m$ is $1$-dimensional, and it connects the fixed points $\ell_s$ and $r_{m+s}$. If $m \leq 0$, the $\tilde{T}(\mathbb{C})$ orbit of $\ell_s y_m$ is $1$-dimensional and it connects the fixed points $r_s$ and $\ell_{m+s}$. Let $\pi : Y^{\text{SL}(2)} \to X^{\text{SL}(2)}$ be the natural projection from the affine flag manifold to the affine Grassmannian. It is a $G(F)$-equivariant fibration with fiber isomorphic to $G(\mathbb{C})/B(\mathbb{C}) \cong \mathbb{C}\mathbb{P}^1$. It satisfies $\pi(\ell_m) = \pi(r_m)$ for all $m$, and $\pi(x_m) = \pi(y_m)$ for all $m \leq 0$.

For $m \leq 0$ the $T(F)$-orbit of $x_m \in Y$ is $(-m)$ dimensional and it projects isomorphically under $\pi$ to the $T(F)$-orbit of the point $\pi(x_m)$ (which was denoted $x_m$ in §6). For $m \leq 1$ the $T(F)$ orbit of $y_m \in Y$ is $1 - m$ dimensional. If $m \leq 0$, it fibers over the $T(F)$ orbit of $\pi(y_m) = \pi(x_m)$ with fiber a $1$-dimensional affine space.

Fix a regular element $\gamma \in \mathfrak{t}(\mathfrak{o})$ and let

$$Y_\gamma = \{ xI \in Y : \text{Ad}(x^{-1})(\gamma) \in \text{Lie}(I) \}$$

be the affine Springer fiber corresponding to $\gamma$ in the affine flag manifold $Y$. It is preserved by the extended torus $\tilde{T}(\mathbb{C})$, and the mapping $\pi : Y_\gamma \to X_\gamma$ is surjective (however it may fail to be a fibration). The homology of $Y_\gamma$ is pure ([GKMx]). The paving of $Y$ by affine cells gives rise to a paving of $Y_\gamma$ by affine cells. In fact, the intersection of $Y_\gamma$ with a single Bruhat cell in $G(F)/I$ is either empty, or else it is a single affine cell.

13.2. Proposition. The affine flag manifold $Y$ is the disjoint union of countably many $T(F)$ orbits,

$$Y = \prod_{n \leq 0} T(F) \cdot \{ x_n \cup y_{n+1} \}.$$  

(13.2.1)
The affine Springer fiber $Y_\gamma$ is the union

$$Y_\gamma = Y_\leq v = \bigcup_{n=-v}^{0} T(F) \cdot \{x_n \cup y_{n+1}\}$$

where $v = \text{val}(\alpha'(\gamma))$. For each ordered pair of integers $s, t \in \mathbb{Z}$ there is a unique one dimensional orbit $E_{st} \subset Y$ of the extended torus $\tilde{T}(\mathbb{C})$ which connects the fixed points $\ell_s$ and $r_t$. This accounts for all the one dimensional orbits of $\tilde{T}(\mathbb{C})$ in $Y$. The torus $\tilde{T}(\mathbb{C})$ acts on the orbit $E_{st}$ through the affine root $(\alpha, s + t)$. The orbit $E_{st}$ is contained in $Y_\gamma$ iff $-v \leq t - s \leq v - 1$.

13.3. **Proof.** If $s - t \geq 1$ then $E_{st}$ is the orbit of the point $\ell_s x_{t-s}$. It is contained in $Y_\gamma$ iff $s - t \leq v$. If $s - t \leq 0$ then $E_{st}$ is the orbit of the point $\ell_t y_{s-t}$. It is contained in $Y_\gamma$ iff $t - s \leq v - 1$. \qed

13.4. As in §7 the co-root $\alpha^\vee$ determines isomorphisms $S(\tilde{t}) \cong \mathbb{C}[x, t]$ and $D(\tilde{t}) \cong \mathbb{C}[\partial_x, \partial_t]$. Fix $\gamma \in t(\mathfrak{o})$ and let $v = \text{val}(\alpha'(\gamma))$. Let $M_{ab}$ be the vector subspace of $\mathbb{C}[\tilde{W}] \otimes_{\mathbb{C}} S(\tilde{t})$ which is spanned by elements

$$(\ell_a - r_b) \otimes g_{ab}((a + b)x - 2t)$$

where $g_{ab}$ are polynomials. Then $M_{ab}$ is a $D(\tilde{t})$ module. Define $Q_v = \sum_{a,b} M_{ab}$ to be the sum of those subspaces $M_{ab}$ such that $-v \leq b - a \leq v - 1$. Then the inclusion $\tilde{W} \subset Y_\gamma$ induces a short exact sequence on $T(\mathbb{C})$-equivariant homology,

$$0 \longrightarrow Q_v \{ \partial_t \} \longrightarrow \mathbb{C}[\tilde{W}] \otimes_{\mathbb{C}} S(\tilde{t}) \longrightarrow H_\ast^{T(\mathbb{C})}(Y_\gamma) \longrightarrow 0 \quad (13.4.1)$$

where the module of relations $Q_v \{ \partial_t \}$ denotes the submodule of $Q_v$ which is annihilated by the differential operator $\partial_t$.

13.5. **Proposition.** Fix $v \geq 1$. The module $Q_v \{ \partial_t \}$ of relations is spanned by

$$\sum_{d=1}^{v} (1 - \alpha^\vee)^d \mathbb{C}[\tilde{W}] \otimes_{\mathbb{C}} S(\tilde{t}) \{ \partial_t^d \} \quad (13.5.1)$$

and

$$\sum_{d=1}^{v} (1 - \alpha^\vee)^d - 1 \mathbb{C}[\tilde{W}] \otimes_{\mathbb{C}} S(\tilde{t}) \{ \partial_t^d \} \cdot (1 - w_a) \quad (13.5.2)$$

13.6. **Proof.** This section is parallel to §12. Fix $a, b \in \mathbb{Z}$ and fix $d \geq 1$. Define

$$f_{a,b,d} \in \mathbb{C}[\tilde{W}] \otimes_{\mathbb{C}} \mathbb{C}[x, t]$$

to be the element

$$\sum_{u = a}^{a + d - 1} \sum_{v = b}^{b + d - 1} (-1)^{u-v} (d - 1) \binom{d - 1}{u - a} \binom{d - 1}{v - b} (\ell_u - r_v) \otimes \((u + v)x - 2t)^{d - 1}.$$
As in §12 we find that
\[ f_{a,b,d} = (-1)^{a+b}(d-1)!(1-\alpha^\vee)^{d-1}(\ell_a - r_b) \otimes x^{d-1} \in \ker(\partial_x). \] (13.6.1)

Fix \( v \geq 1 \). If \( 1 \leq d \leq v \) then \( f_{m,m;d} \in Q_v \) and \( f_{m+1,m;d} \in Q_v \). (These are the relations coming from the one dimensional orbits in each of the two types of irreducible components of \( Y_{\leq d}^{SL(2)} \), cf. Remark 12.9). Define \( K_v \subset \mathbb{C}[\widehat{W}] \otimes \mathbb{C}[x,t] \) to be the span
\[ K_v = \sum_{m \in \mathbb{Z}} \sum_{d=1}^v \mathbb{C}f_{m,m;d} + \mathbb{C}f_{m+1,m;d}. \]

Then \( K_v \) is a \( \mathcal{D}(t) = \mathbb{C}[\partial_x] \) module, and equation (13.6.1) implies that \( K_v \subset Q_v \{ \partial_x \} \). An argument similar to that of §12.6 may be used to show that in fact \( K_v = Q_v \{ \partial_x \} \). Since
\begin{align*}
&f_{m,m;d} + f_{m+1,m,d} = (d-1)!(1-\alpha^\vee)^d \ell_m \otimes x^{d-1} \\
&f_{m+1,m+1;d} + f_{m+1,m,d} = (d-1)!(1-\alpha^\vee)^d r_m \otimes x^{d-1}
\end{align*}
we see that \( K_v \) is the sum
\[ \sum_{d=1}^v (1-\alpha^\vee)^d \mathbb{C}[\widehat{W}] \otimes \mathcal{S}(t) \{ \partial_x^d \} + \sum_{d=1}^v \mathbb{C}f_{0,0;d}. \]

The second set of relations may be replaced by the less efficient but more symmetric collection (13.5.2) by noting that for any \( m \geq 0 \),
\begin{align*}
&\sum_{j=-m}^{m-1} (f_{j,j;d} + f_{j+1,j;d}) + f_{m,m;d} = (d-1)!(1-\alpha^\vee)^{d-1}(\ell_m - r_m) \otimes x^{d-1} \\
&\sum_{j=-m}^{m-1} (f_{j+1,j+1;d} + f_{j+1,j;d}) - f_{m,m;d} = (d-1)!(1-\alpha^\vee)^{d-1}(r_m - \ell_m) \otimes x^{d-1}.
\end{align*}

Since \( K_v \) is a \( \mathbb{C}[\partial_x] \) module, we conclude that for any \( 1 \leq d \leq v \),
\[ f_{0,0;d} \in (1-\alpha^\vee)^{d-1}(1-w_\alpha)\mathbb{C}[\widehat{W}] \otimes \mathcal{S}(t) \{ \partial_x^d \} \subset K_v. \] \( \square \)

13.7. The semisimple rank one case. As in §8.3 let \( H \) be a connected reductive complex algebraic group of semisimple rank one, let \( T \subset B \subset H \) be a Borel pair, let \( I \subset H(F) \) be the corresponding Iwahori subgroup, and let \( Y^H = H(F)/I \) be the affine flag manifold for \( H \). Let \( \alpha, \alpha^\vee \) be the simple positive root and co-root. These determine a reflection \( w_\alpha \in W \) in the (finite) Weyl group for \( H \), a 1-dimensional sublattice \( \langle \alpha^\vee \rangle \) in the lattice \( \Lambda^H \) of translations of \( T \), and a differential operator \( \partial_\alpha \in \mathcal{D}(t) \) of degree 1. Let \( \widehat{W} = \Lambda \rtimes W \) be the extended affine Weyl group. The mapping \( SL(2) \to H \) determined by \( \alpha \) induces an
inclusion $\phi_\alpha: Y^{\text{SL}(2)} \to Y^H$. Let $\gamma \in \mathfrak{t}(\mathfrak{o})$ be a regular element and set $v = \text{val}\alpha'(\gamma)$. Then the affine flag manifold for $H$ is the disjoint union

$$Y^H = \coprod_{\ell \in \Lambda^H/(\alpha^\vee)} \ell \phi_\alpha(Y^{\text{SL}(2)})$$

and the affine Springer fiber for $\gamma$ is the disjoint union

$$Y^H_\gamma = \coprod_{\ell \in \Lambda^H/(\alpha^\vee)} \ell \phi_\alpha(Y^{\text{SL}(2)}_{\leq v})$$

which we may unambiguously denote by $Y^H_{\leq v}$. It follows, as in §9.3, that the $T(\mathbb{C})$ equivariant homology of $Y^H_\gamma$ is again given by the following:

13.8. **Proposition.** Suppose $H$ has semisimple rank one. Fix $T \subset B \subset H$ and fix $v \geq 1$. Then the inclusion of the $T(\mathbb{C})$ fixed points in the affine Springer fiber $Y^H_{\leq v}$ induces a short exact sequence

$$0 \longrightarrow Q_v \{ \partial_t \} \longrightarrow \mathbb{C}[\widehat{W}] \otimes_\mathbb{C} S(t) \longrightarrow H^*_T(\mathbb{C}) (Y^H_{\leq v}) \longrightarrow 0 \quad (13.8.1)$$

where the module of relations $Q_v \{ \partial_t \}$ is the span of (13.5.1) and (13.5.2). \qed

14. **Homology of affine Springer fibers in the affine flag manifold**

The results in this section are parallel to §9 and §10 so they will be presented without detailed proofs.

14.1. As in §5 let $F = \mathbb{C}((\varepsilon))$, let $G$ be a connected reductive complex algebraic group, let $T \subset B \subset G$ be a Borel pair and let $I \subset G(F)$ be the corresponding Iwahori subgroup. The affine flag manifold is $Y = Y^G = G(F)/I$. Let $\Lambda \subset T(F)$ denote the lattice of translations. Identify each co-root $\alpha^\vee \in \Phi^\vee(G, T)$ with its image $\alpha^\vee(\varepsilon) \in \Lambda$. Set $\widetilde{W} = \Lambda \rtimes W$ where $W = W(G(\mathbb{C}), T(\mathbb{C}))$ is the Weyl group. Each cell in the Bruhat decomposition

$$Y = \coprod_{w \in \tilde{W}} IwI/I$$

contains a unique $T(\mathbb{C})$ fixed point. Hence the choice of basepoint determines a one to one correspondence

$$Y^{T(\mathbb{C})} \cong \widetilde{W}$$

between the set of fixed points and the extended affine Weyl group, which is compatible with the action of the lattice $\Lambda$ of translations.

For each root $\alpha \in \Phi^+$ let $w_\alpha \in W$ be its corresponding reflection, let $W_\alpha = \{1, w_\alpha\}$, and let $H^\alpha$ be the connected reductive group of semisimple rank one which contains $T$ and the
root subgroup $U_{\alpha} \subset G$. Let $Y^\alpha$ be the affine flag manifold for $H^\alpha$. If $u \in Y$ is a $T(\mathbb{C})$ fixed point then it determines an isomorphism

$$\phi_u : Y^\alpha \rightarrow H^\alpha \cdot u \subset Y$$

between $Y^\alpha$ and the orbit $H^\alpha \cdot u$. It restricts to an isomorphism $Y_{\leq v}^\alpha \cong Y_\gamma \cap H^\alpha \cdot u$ of affine Springer fibers, for any regular element $\gamma \in \mathfrak{t}(\mathfrak{o})$ with $v = \text{val} \alpha' (\gamma)$. (The action of $w_\alpha$ on $Y^{T(\mathbb{C})}$ preserves the fixed points in the orbit $H^\alpha \cdot u$ but it interchanges those of “type $\ell$” and “type $r$”.) Define

$$Z^\alpha = \bigoplus_{u \in W} H^\alpha \cdot u = \bigoplus_{u \in W} \phi_u (Y^\alpha).$$

The proof of the following lemma is parallel to that of Lemma 5.12.

14.2. **Lemma.** Let $\gamma \in \mathfrak{t}(\mathfrak{o})$ be a regular element. Then the union of the 0 and 1 dimensional orbits in the affine Springer fiber is

$$(Y_\gamma)_1 = \bigcup_{\alpha \in \Phi^+} Z^\alpha_\gamma$$

where $Z^\alpha_\gamma$ is the intersection

$$Z^\alpha_\gamma = Y_\gamma \cap Z^\alpha = \bigoplus_{u \in W} \phi_u (Y_{\leq v}^\alpha).$$

If $\alpha \neq \beta$ then $Z^\alpha_\gamma \cap Z^\beta_\gamma = \tilde{W}$. $\square$

Each $\alpha \in \Phi^+$ corresponds to a degree one differential operator $\partial_\alpha \in \mathcal{D}(\mathfrak{t})$, to an element $\alpha^\vee \in \Lambda$, and to a reflection $w_\alpha \in W$. For $\alpha \in \Phi^+$ define the $\mathcal{D}(\mathfrak{t})$ submodule $M_{\alpha, \gamma}$ to be the sum

$$\sum_{d=1}^{\text{val}(\alpha'(\gamma))} (1 - \alpha^\vee)^d \mathbb{C}[\tilde{W}] \otimes_{\mathbb{C}} \mathcal{S}(\mathfrak{t}) \{ \partial_\alpha^d \} + \sum_{d=1}^{\text{val}(\alpha'(\gamma))} (1 - \alpha^\vee)^{d-1}(1 - w_\alpha) \mathbb{C}[\tilde{W}] \otimes_{\mathbb{C}} \mathcal{S}(\mathfrak{t}) \{ \partial_\alpha^d \}.$$ 

Using Proposition 13.8, the argument of §9.3 gives the following.

14.3. **Theorem.** Let $\gamma \in \mathfrak{t}(\mathfrak{o})$ be a regular element. Suppose the homology $H_*(Y_\gamma; \mathbb{C})$ is pure. Then the inclusion $\tilde{W} \subset Y_\gamma$ induces an exact sequence of $\mathcal{D}(\mathfrak{t})$ modules,

$$0 \longrightarrow \sum_{\alpha \in \Phi^+} M_{\alpha, \gamma} \longrightarrow \mathbb{C}[\tilde{W}] \otimes_{\mathbb{C}} \mathcal{S}(\mathfrak{t}) \longrightarrow H^{T(\mathbb{C})}_*(Y_\gamma) \longrightarrow 0.$$

The group $(\tilde{W} \rtimes \text{Aut})_\gamma$ of (9.4.2) acts on this equivariant homology group and it restricts to an action on the ordinary homology,

$$H_*(Y_\gamma) = H^{T(\mathbb{C})}_*(Y_\gamma) \{ \mathcal{I} \},$$

that is, the subgroup of the equivariant homology which is annihilated by the augmentation ideal $\mathcal{I} \subset \mathcal{D}$. 35
14.4. **Springer action.** Consider the (regular\(\otimes\)trivial) action of \(\tilde{W}\) from the right on

\[ \mathbb{C}[\tilde{W}] \otimes S(t). \]

This action preserves each relation \(M_{\alpha,\gamma}\) (for \(\alpha \in \Phi^+\)). Assume the homology \(H_*(Y_\gamma; \mathbb{C})\) is pure. Then the right action of \(\mathbb{C}[\tilde{W}]\) passes to an action on \(H_*^{T(C)}(Y_\gamma)\) which is compatible with the \(\mathcal{D}(t)\) module structure, and which commutes with the \((\tilde{W} \rtimes \text{Aut})_\gamma\) action. So it restricts to an action (from the right) of \(\tilde{W}\) on the ordinary homology \(H_*(Y_\gamma)\). It can be shown that this action coincides with the Springer action defined by Lusztig [Lu96] and (using a statement from [KL80] whose proof does not appear in the literature) by Sage [Sa97]. So Theorem 14.3 gives a “formula” for the Springer action.

14.5. **Endoscopic groups.** There are parallels to \(\S 10\) and \(\S 11\) for the affine flag manifold also. As in \(\S 10\) suppose \(G\) is adjoint, and let \((H, s)\) be endoscopic data for \(G\). Fix compatible Borel pairs \(T \subset B \subset G\) and \(T_H \subset B_H \subset H\) with resulting identification \(T_H \cong T\). The group \(H\) has root system \(\Phi(H, T_H) \subset \Phi(G, T)\) and (extended) affine Weyl group \(\tilde{W}^H = \Lambda \times W^H \subset \tilde{W}\). Let \(\gamma \in \mathfrak{t}(\mathfrak{o})\) be a regular element with corresponding Springer fibers \(Y_\gamma\) and \(Y_{\gamma_H}^H\). Suppose the homology groups \(H_*(Y_\gamma; \mathbb{C})\) and \(H_*(Y_{\gamma_H}^H; \mathbb{C})\) are pure. Let \(\Delta\) be the homological transfer factor of \((10.1.2)\), let \(r = \deg(\Delta)\), and (as before) let \(\eta\) be the sign character that gives the action of \(\tilde{W}^{G;H}\) on \(\Delta\).

We are going to compare the induced module \(H_*^{T(C)}(Y_{\gamma_H}^H)[-2r] \otimes_{\mathbb{C}[\tilde{W}^H]} \mathbb{C}[\tilde{W}]\) to the module \(H_*^{T(C)}(Y_\gamma)\). One sees immediately that this induced module is equal to the quotient of \(\mathbb{C}[\tilde{W}] \otimes_{\mathbb{C}} S(t)\) by \(\sum_{\alpha \in \Phi^+(H)} M_{\alpha,\gamma}\). Note that the surjection

\[ 1 \otimes \Delta : \mathbb{C}[\tilde{W}] \otimes_{\mathbb{C}} S(t) \to \mathbb{C}[\tilde{W}] \otimes_{\mathbb{C}} S(t) \]

kills \(M_{\alpha,\gamma}\) whenever \(\alpha \in \Phi^+(G) - \Phi^+(H)\).

Let \(J \subset \mathbb{C}[\Lambda]\) be the multiplicative subset generated by the elements \(1 - \alpha^\vee\) for \(\alpha^\vee \in \Phi^\vee(G) - \Phi^\vee(H)\). We use the action of \(\Lambda\) on the Springer fibers to regard their homology groups as modules over the group algebra \(\mathbb{C}[\Lambda]\), allowing us to localize the homology groups using \(J\). Then we have the following result.

14.6. **Theorem.** Suppose \(H_*(Y_\gamma)\) and \(H_*(Y_{\gamma_H}^H)\) are pure. Then the mapping \(1 \otimes \Delta \) induces a homomorphism of \(\mathcal{D}(t)\) modules,

\[ H_*^{T(C)}(Y_\gamma) \to H_*^{T(C)}(Y_{\gamma_H}^H)[-2r] \otimes_{\mathbb{C}[\tilde{W}^H]} \mathbb{C}[\tilde{W}] \]

and a homomorphism on ordinary homology,

\[ H_*(Y_\gamma; \mathbb{C}) \to H_*(Y_{\gamma_H}^H; \mathbb{C})[-2r] \otimes_{\mathbb{C}[\tilde{W}^H]} \mathbb{C}[\tilde{W}]. \]
Both homomorphisms transform under \(\eta\) with respect to the left action of \(\tilde{W}\) and are equivariant with respect to the right action of \(\tilde{W}\), and both homomorphisms become isomorphisms

\[
H^T(C)(Y,\gamma)_J \cong H^T(C)(Y^{\gamma_H}_J)[-2\text{r}] \otimes_{C[\tilde{W}]} \mathbb{C}[\tilde{W}]
\]

\[
H^*(Y,\gamma;\mathbb{C})_J \cong H^*(Y^{\gamma_H};\mathbb{C})_J[-2\text{r}] \otimes_{C[\tilde{W}]} \mathbb{C}[\tilde{W}]
\]

after localizing with respect to \(J\).

\[
\Box
\]

\[
\text{15. An orbital integral}
\]

15.1. \textbf{Set-up}. Let \(k\) be a finite field and \(\bar{k}\) an algebraic closure of \(k\). Let \(F = k((\varepsilon))\) and \(L = \bar{k}((\varepsilon))\). Let \(\mathfrak{o}\) denote the valuation ring \(k[[\varepsilon]]\) of \(F\). We write \(\sigma\) for the Frobenius automorphism of both \(\bar{k}/k\) and \(L/F\), and we write \(\langle \sigma \rangle\) for the infinite cyclic group generated by \(\sigma\). We also choose an algebraic closure \(\bar{L}\) of \(L\) and write \(\sigma\) for \(\text{Gal}(\bar{L}/L)\). We write \(\bar{F}\) for the algebraic closure of \(F\) in \(\bar{\mathbb{T}}\) and \(\Gamma\) for the Galois group \(\text{Gal}(\bar{F}/F)\).

Let \(G\) be a connected reductive group over \(F\), and assume that \(G\) splits over \(L\). Let \(K\) be a parahoric subgroup of \(G(F)\), let \(K_L\) be the corresponding \(\sigma\)-stable parahoric subgroup of \(G(L)\), and write \(\mathfrak{f}\) and \(\mathfrak{f}_L\) for the corresponding parahoric subalgebras (of \(g(F)\) and \(g(L)\) respectively). Let \(X\) be the \(k\)-ind-scheme associated to \(G(L)/K_L\). Then \(X(\bar{k}) = G(L)/K_L\) and \(X(k) = G(F)/K\).

Let \(T\) be a maximal \(F\)-torus in \(G\). Let \(S\) denote the maximal unramified subtorus of \(T\). Thus the cocharacter group \(X_*(S)\) can be identified with \(X_*(T)^I\).

Let \(\alpha\) be a regular element in \(\text{Lie}(T)(F)\) and assume that \(u\) is \textit{integral}, in the sense that \(\alpha(u)\) lies in the valuation ring \(k[[\varepsilon]]\) of \(F\) for every root \(\alpha\) of \(T\) in \(G\). We write \(X_u\) for the affine Springer fiber \(\{x \in X : x^{-1}ux \in \mathfrak{f}_L\}\) studied by Kazhdan-Lusztig in \([KL88]\). (We usually write \(x^{-1}ux\) rather than \(\text{Ad}(x)^{-1}u\).)

For \(\mu \in X_*(S)\) we write \(\varepsilon^\mu\) for \(\mu(\varepsilon)\in S(L)\). The map \(\mu \mapsto \varepsilon^\mu\) is \(\langle \sigma \rangle\)-equivariant and identifies \(X_*(S)\) with a subgroup of \(S(L)\) (and of \(T(L)\)) that we will denote by \(\Lambda\). The group \(T(L)\) centralizes \(u\) and therefore acts by left translations on \(X_u\). The quotient \(\Lambda \backslash X_u\) of \(X_u\) by the subgroup \(\Lambda\) of \(T(L)\) is a non-empty projective scheme of finite type over \(k\) (see \([KL88]\)).

15.2. \textbf{Normalized Haar measure on} \(T(F)\). Recall from \([Kot97, 7.6]\) the exact sequence

\[
1 \to T(L)_1 \to T(L) \xrightarrow{\text{wr}} X_*(T)_I \to 0.
\]

(15.2.1)

Taking invariants under \(\langle \sigma \rangle\), we get another exact sequence

\[
1 \to T(F)_1 \to T(F) \to (X_*(T)_I)^{\langle \sigma \rangle} \to 0,
\]

(15.2.2)

where \(T(F)_1 := T(F) \cap T(L)_1\). Surjectivity at the right end of this last exact sequence is proved in \([Kot97, 7.6]\).
Let $dt$ be the Haar measure on $T(F)$ that gives $T(F)_1$ measure 1. The group $\Lambda^{(\sigma)}$ of $T(F)$ is discrete and cocompact, so the volume of the quotient $\Lambda^{(\sigma)} \backslash T(F)$ is finite. We need to compute this volume (with respect to $dt$).

15.3. Volume computation. We claim that

$$\text{vol}_{dt}(\Lambda^{(\sigma)} \backslash T(F)) = \frac{|\text{cok}[X_*(S) \Gamma \to X_*(T) \Gamma]|}{|\text{ker}[X_*(S) \Gamma \to X_*(T) \Gamma]|}.$$  \hfill (15.3.1)

To prove this claim we begin by noting that the canonical map

$$X_*(S) = X_*(T)^I \to X_*(T)_I$$

has finite kernel and cokernel. Thus this map is in fact injective, since $X_*(S)$ is torsion-free. We let $M$ denote its cokernel, so that we get a short exact sequence

$$0 \to X_*(S) \to X_*(T)_I \to M \to 0$$

doing a short exact sequence

$$0 \to (X_*(S))^{(\sigma)} \to (X_*(T)_I)^{(\sigma)} \to M^{(\sigma)}$$

in which we have identified $X_*(S)_{(\sigma)}$ with $X_*(S)_\Gamma$ and $(X_*(T)_I)_{(\sigma)}$ with $X_*(T)_\Gamma$.

Since $X_*(S)^{(\sigma)} \to (X_*(T)_I)^{(\sigma)}$ is injective, the subgroups $X_*(S)^{(\sigma)}$ and $T(F)_1$ of $T(F)$ have trivial intersection, and therefore

$$\text{vol}_{dt}(\Lambda^{(\sigma)} \backslash T(F)) = \frac{|(X_*(T)_I)^{(\sigma)}|}{X_*(S)^{(\sigma)}}.$$  \hfill (15.3.2)

It follows from (15.3.2) that

$$\frac{|(X_*(T)_I)^{(\sigma)}|}{X_*(S)^{(\sigma)}} = \frac{|M^{(\sigma)}|}{|\text{ker}[X_*(S)_\Gamma \to X_*(T)_\Gamma]|}.$$  \hfill (15.3.3)

Since all groups in the exact sequence

$$0 \to M^{(\sigma)} \to M \overset{\sigma^{-1}}{\to} M \to M_{(\sigma)} \to 0$$

are finite, we have

$$|M^{(\sigma)}| = |M_{(\sigma)}|.$$  \hfill (15.3.4)

Finally, it follows from (15.3.2) that

$$|M_{(\sigma)}| = \text{cok}[X_*(S)_\Gamma \to X_*(T)_\Gamma].$$  \hfill (15.3.5)

Combining the last four equations, we get the claim.
15.4. **Normalized orbital integrals.** For any compactly supported locally constant function $f$ on $g(F)$ we put

$$O_u(f) = \int_{T(F)\backslash G(F)} f(g^{-1}ug) \, dt \, dg,$$

where $dt$ is the normalized Haar measure on $T(F)$ defined above, and $dg$ is the Haar measure on $G(F)$ that gives our chosen parahoric subgroup $K$ measure 1. We then have

$$O_u(f) = \text{vol}_T(\Lambda^{(\sigma)} T(F))^{-1} \int_{\Lambda^{(\sigma)} G(F)} f(g^{-1}ug) \, dg. \quad (15.4.1)$$

In the special case that our function $f$ is $1_t$, the characteristic function of $t$, we have

$$\int_{\Lambda^{(\sigma)} G(F)} 1_t(g^{-1}ug) \, dg = |\{ x \in \Lambda^{(\sigma)} G(F)/K : x^{-1}ux \in t \}|$$

$$= |\Lambda^{(\sigma)} \langle X_u(k) \rangle|. \quad (15.4.2)$$

Combining (15.3.1), (15.4.1), (15.4.2), we find that

$$O_u(1_t) = |\ker[X_*(S)_T \to X_*(T)_T]| \cdot |\Lambda^{(\sigma)} \langle X_u(k) \rangle|. \quad (15.4.3)$$

15.5. **$\kappa$-orbital integrals of $1_t$.** Consider an element $u' \in g(F)$ that is stably conjugate to $u$ (that is, conjugate under $G(T)$). Since the group $H^1(L, T)$ vanishes (see [Ser68, X.7]), there exists $g \in G(L)$ such that $u' = gug^{-1}$. Apply $\sigma$ to this equality to see that $g^{-1}\sigma(g) \in T(L)$. The class of $g^{-1}\sigma(g)$ in $B(T)$ (in other words, the $\sigma$-conjugacy class of $g^{-1}\sigma(g)$ in $T(L)$) is independent of the choice of $g$. Here, as usual, for any linear algebraic group $G$ over $F$ we write $B(G)$ for the set of $\sigma$-conjugacy classes in $G(L)$.

Turning this around, given $t \in T(L)$ whose class in $B(T)$ lies in

$$D(T/F) := \ker[B(T) \to B(G)],$$

we choose $g \in G(L)$ such that $t = g^{-1}\sigma(g)$ and put $u(t) = gug^{-1}$, an element of $g(F)$ that is stably conjugate to $u$ and whose $G(F)$-conjugacy class depends only on the class of $t$ in $B(T)$. The construction $t \mapsto u(t)$ sets up a bijection from $D(T/F)$ to the set of $G(F)$-conjugacy classes in the stable conjugacy class of $u$.

Recall from [Kot97, 7.6] that the canonical homomorphism

$$w_T : T(L) \to X_*(T)_T$$

induces an isomorphism (take coinvariants under $\langle \sigma \rangle$)

$$B(T) \simeq X_*(T)_\Gamma.$$

At this point we fix a prime number $l$ that is non-zero in the field $k$. Put

$$\hat{T} := \text{Hom}(X_*(T), \mathbb{Q}_l^\times).$$
Let $\kappa \in \text{Hom}(X_*(T), \overline{\mathbb{Q}}_l^\times) = \hat{T}^\Gamma$. For $t \in T(L)$ we write $\langle t, \kappa \rangle \in \overline{\mathbb{Q}}_l^\times$ for the value of the character $\kappa$ on the element of $X_*(T)_\Gamma$ obtained as the image of $t$ under 

$$T(L) \to B(T) \simeq X_*(T)_\Gamma.$$ 

For $\overline{\mathbb{Q}}_l$-valued locally constant compactly supported functions $f$ on $g(F)$ we define the $\kappa$-orbital integral $O^\kappa_u(f)$ by

$$O^\kappa_u(f) = \sum_{t \in \mathcal{D}(T/F)} \langle t, \kappa \rangle \cdot O_{u(t)}(f).$$

Note that if $\kappa$ lies in the subgroup $Z(\hat{G})^\Gamma$ of $\hat{T}^\Gamma$ (where we form the Langlands dual group $\hat{G}$ using $\mathbb{Q}_l$, and where $Z(\hat{G})$ denotes the center of $\hat{G}$), then $O^\kappa_u$ is a stable orbital integral.

In the special case that $f$ is $1_r$, we find that

$$O^\kappa_u(1_r) = \frac{|\ker[B(S) \to B(T)]|}{|\text{cok}[B(S) \to B(T)]|} \cdot \sum_{t \in \mathcal{D}(T/F)} \langle t, \kappa \rangle \cdot |\Lambda^{(\sigma)} \setminus X_u^{t\sigma}|,$$ (15.5.1)

where we have written $X_u^{t\sigma}$ for the fixed point set of $t\sigma$ on $X_u$, which appears in our formula since $x \mapsto gx$ induces a bijection from $X_u^{t\sigma}$ to $X_{u(t)}(k)$, where $g \in G(L)$ is chosen so that $t = g^{-1}\sigma(g)$, as before.

15.6. **Local systems $\mathcal{L}_\eta$ on $\Lambda \backslash X_u$.** Any finite dimensional (continuous) $l$-adic representation of $\Lambda \times \text{Gal}(\overline{k}/k)$ (with the topology on $\Lambda$ given by all subgroups of finite index) gives rise to a local system on $\Lambda \backslash X_u$ over $k$ (which becomes constant when pulled back to $X_u$ over $\overline{k}$). Put $\hat{S} := \text{Hom}(X_*(S), \overline{\mathbb{Q}}_l^\times)$. Let

$$\eta \in \text{Hom}(X_*(S)_\Gamma, \overline{\mathbb{Q}}_l^\times) = \hat{S}^\Gamma = \hat{S}^{(\sigma)},$$

and assume that $\eta$ has finite order. Using the isomorphism $\Lambda \simeq X_*(S)$, we also view $\eta$ as a character on $\Lambda$, and since $\eta$ is fixed by $\sigma$, we may extend $\eta$ to a character on $\Lambda \times \text{Gal}(\overline{k}/k)$ by making it trivial on $\text{Gal}(\overline{k}/k)$. We use the resulting 1-dimensional representation of $\Lambda \times \text{Gal}(\overline{k}/k)$ to form a rank 1 local system $\mathcal{L}_\eta$ on $\Lambda \backslash X_u$ over $k$.

Since the action of $T(L)$ on $X_u$ commutes with that of $\Lambda$, it induces an action (over $\overline{k}$) of $T(L)$ on $\Lambda \backslash X_u$ and $\mathcal{L}_\eta$, and therefore $T(L)$ acts on the cohomology groups $H^i(\Lambda \backslash X_u; \mathcal{L}_\eta)$. It follows from the definitions that the subgroup $\Lambda$ of $T(L)$ acts on these cohomology groups through the character $\eta^{-1}$. Moreover the “identity component” $T(L)_1$ of $T(L)$ acts trivially on them. Then since $\Lambda \cdot T(L)_1$ has finite index in $T(L)$ and acts by a character on cohomology, each cohomology group is a semisimple $T(L)$-module. Therefore (using (15.2.1)) we can decompose the cohomology groups as

$$H^i(\Lambda \backslash X_u; \mathcal{L}_\eta) = \bigoplus_{\kappa} H^i(\Lambda \backslash X_u; \mathcal{L}_\eta)_\kappa.$$
where \( \kappa \) runs through the finite set of characters \( \kappa \in \text{Hom}(X_s(T)_\Gamma, \overline{Q}_\kappa^\vee) = \hat{T}^I \) whose image under \( \hat{T}^I \to \hat{S} \) is \( \eta^{-1} \), and where \( H^i(\Lambda \backslash X_u, \mathcal{L}_\eta)_\kappa \) denotes the \( \kappa \)-isotypic subspace of \( H^i(\Lambda \backslash X_u, \mathcal{L}_\eta) \) (viewing \( \kappa \) as a character on \( T(L) \), as before).

15.7. Main result. Now we can state the main result of this section, which gives a cohomological interpretation of the \( \kappa \)-orbital integrals of \( 1_t \).

15.8. Theorem. Let \( \kappa \) be an element of finite order in \( \hat{T}^\Gamma \). Then the \( \kappa \)-orbital integral \( Q_u^\kappa(1_t) \) is given by

\[
Q_u^\kappa(1_t) = \text{Tr}(\sigma^{-1}; H^\bullet(\Lambda \backslash X_u, \mathcal{L}_\eta)_\kappa),
\]

where \( \eta \) is the image of \( \kappa^{-1} \) under \( \hat{T}^\Gamma \to \hat{S}^\Gamma \).

Here we have written \( \text{Tr}(\sigma^{-1}; H^\bullet(\Lambda \backslash X_u, \mathcal{L}_\eta)_\kappa) \) as an abbreviation for

\[
\sum_{i=0}^{2 \dim \ X_u} (-1)^i \text{Tr}(\sigma^{-1}; H^i(\Lambda \backslash X_u, \mathcal{L}_\eta)_\kappa).
\]

Proof. The first step is to apply the Grothendieck-Lefschetz trace formula. We claim that for any \( t \in T(L) \) we have

\[
\text{Tr}((t\sigma)^{-1}; H^\bullet(\Lambda \backslash X_u, \mathcal{L}_\eta)) = \sum_{x \in (\Lambda \backslash X_u)^{t\sigma}} \langle \lambda_x, \eta \rangle,
\]

with notation as follows. We are writing \( (\Lambda \backslash X_u)^{t\sigma} \) for the fixed points of \( t\sigma \) on \( \Lambda \backslash X_u \), and for such a fixed point \( \bar{x} \) we choose a representative \( x \in X_u \) and define \( \lambda_x \in \Lambda \) by the equality \( t\sigma x = \lambda_x x \). The image of \( \lambda_x \) in \( \Lambda_{(\sigma)} = X_s(S)_\Gamma \) is independent of the choice of representative \( x \). The equality (15.8.1) follows from the Grothendieck-Lefschetz trace formula applied to the Frobenius map for a twisted \( k \)-form of \( \Lambda \backslash X_u, \mathcal{L}_\eta \) in which the twisted action of \( \sigma \) is given by \( t\sigma \).

The formula (15.8.1) can be rewritten as

\[
\sum_{\kappa'} \text{Tr}((t\sigma)^{-1}; H^\bullet(\Lambda \backslash X_u, \mathcal{L}_\eta)_{\kappa'}) = \sum_{\lambda \in X_u(\bar{s})_\Gamma} \langle \lambda, \eta \rangle^{-1} \cdot |(\Lambda \backslash X_u)^{\kappa'^\cdot \lambda\sigma}|,
\]

where the sum on the left is taken over \( \kappa' \in \hat{T}^\Gamma \) whose image under \( \hat{T}^\Gamma \to \hat{S}^\Gamma \) is \( \eta^{-1} \). Indeed, since \( (t\sigma)^{-1} \) maps \( H^i(\Lambda \backslash X_u, \mathcal{L}_\eta)_{\kappa'} \) to \( H^i(\Lambda \backslash X_u, \mathcal{L}_\eta)_{\sigma^{-1}(\kappa')} \), only those \( \kappa' \in \hat{T}^I \) that are fixed by \( \sigma \) contribute to the trace. As for the sum on the right, we simply collected like terms in the sum on the right in (15.8.1).

Multiply both sides of (15.8.2) by \( \langle t, \kappa \rangle \), sum over

\[
t \in \text{cok}[B(S) \to B(T)] = \text{cok}[X_s(S)_\Gamma \to X_s(T)_\Gamma],
\]

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and divide by the order \(|\text{cok}|\) of this cokernel. We obtain
\[
\text{Tr}(\sigma^{-1}; H^\bullet(\Lambda \backslash X_u, \mathcal{L}_\eta)_\kappa) = \frac{|\ker|}{|\text{cok}|} \sum_{t \in B(T)} \langle t, \kappa \rangle \cdot |\Lambda^{(\sigma)}(X_u^t)\rangle.
\]
Comparing this equation with equation (15.5.1), we see that in order to finish the proof of the theorem, it suffices to show that if \(X_u^t\) is non-empty, then \(t\) lies in \(\ker[B(T) \to B(G)]\).

Indeed, suppose that \(t\sigma(x) = x\) for some \(x \in G(L)/K_L\). Then \(x^{-1}t\sigma(x) \in K_L\). By Lang's theorem (applied to the finite dimensional quotients of the connected proalgebraic group \(K_L\) over \(k\)) and the completeness of \(L\), every element in \(K_L\) is trivial in \(B(G)\). Therefore \(t\) is trivial in \(B(G)\). □

15.9. **A variant of the main result.** The theorem above can be restated in a slightly different way, since the classical relationship between the cohomology of \(X_u\) and that of \(\Lambda \backslash X_u\) presumably has an \(l\)-adic version
\[
H^i(\Lambda \backslash X_u, \mathcal{L}_\eta)_\kappa = \text{Ext}^{i}_{X_u(T)_l}((\mathbb{Q}_l)_\kappa, R\Gamma(X_u, \mathbb{Q}_l)).
\]
Here \((\mathbb{Q}_l)_\kappa\) denotes the 1-dimensional \(X_u(T)_l\)-module obtained by letting \(X_u(T)_l\) act on \(\mathbb{Q}_l\) via the surjection \(X_u(T)_l \to X_u(T)_\Gamma\) and the character \(\kappa\) of \(X_u(T)_\Gamma\), and the complex \(R\Gamma(X_u, \mathbb{Q}_l)\) is being regarded as an object in a suitable derived category of \(X_u(T)_l\)-modules. Note that \(\sigma\) acts on \(X_u(T)_l\) as well as \(R\Gamma(X_u, \mathbb{Q}_l)\); it then acts on
\[
\text{Ext}^{i}_{X_u(T)_l}((\mathbb{Q}_l)_\kappa, R\Gamma(X_u, \mathbb{Q}_l))
\]
as well since \(\kappa\) is fixed by \(\sigma\). Therefore the theorem above presumably yields the equality
\[
O_u^\kappa(1) = \text{Tr}(\sigma^{-1}; R\text{Hom}^\bullet_{X_u(T)_l}((\mathbb{Q}_l)_\kappa, R\Gamma(X_u, \mathbb{Q}_l))). \tag{15.9.1}
\]

15.10. **A geometric reformulation of the fundamental lemma.** We now assume that \(G\) comes from a (necessarily quasi-split) connected reductive group over \(k\), which we still denote by \(G\). Thus \(G\) is unramified over \(F\) and the parahoric subgroup \(K := G(\mathfrak{o})\) is a hyperspecial maximal compact subgroup of \(G(F)\); we denote the corresponding parahoric subalgebra by \(\mathfrak{k} := \mathfrak{g}(\mathfrak{o})\).

As above we form all Langlands dual groups using \(\mathfrak{Q}_\mathfrak{l}\). Let \(H\) be an endoscopic group for \(G\) and let \(s\) be the usual element in \(Z(\hat{H})^\Gamma\), where \(Z(\hat{H})\) denotes the center of \(\hat{H}\). We write \(\mathfrak{h}\) for the Lie algebra of \(H\). We assume that \(H\) is also unramified, so that it comes from a (quasi-split) group over \(k\), which we still denote by \(H\). Thus we also have \(K_H := H(\mathfrak{o})\) and \(\mathfrak{k}_H := \mathfrak{h}(\mathfrak{o})\).

We use a regular nilpotent element in \(\mathfrak{g}(\mathfrak{o})\) whose image in \(\mathfrak{g}(k)\) is also regular nilpotent in order to form Kostant's section, obtaining as in [Kot99] transfer factors \(\Delta(u_H, u)\) which for \(u\) lying in Kostant’s section take the simple value
\[
D_G(u)D_H(u_H)^{-1},
\]
in other words, \( \Delta_{IV}(u_H, u) \) in the notation of Langlands-Shelstad [LS87]). Here \( D_G \) denotes the usual discriminant function on the Lie algebra.

Let \( u_H \) be an integral \( G \)-regular semisimple element of \( \mathfrak{h}(F) \) and let \( u \) be an image of \( u_H \) (in the sense of Langlands-Shelstad [LS87]) that is \( G(F) \)-conjugate to an element in Kostant’s section. (This uniquely determines the \( G(F) \)-conjugacy class of \( u \).) We then have affine Springer fibers \( X_u, X_{u_H}^H \) for \( u, u_H \) respectively. Using the dimension formula for affine Springer fibers (conjectured by Kazhdan-Lusztig [KL88] and proved by Bezrukavnikov [B96]) we can rewrite the transfer factor as

\[
\Delta(u_H, u) = q^{-r}
\]

where \( r := \text{dim } X_u - \text{dim } X_{u_H}^H \) and \( q \) denotes the cardinality of \( k \).

We let \( T \) denote the centralizer of \( u \) in \( G \), and we identify \( T \) with the centralizer \( T_H \) of \( u_H \) in \( H \) via the unique admissible isomorphism whose differential maps \( u_H \) to \( u \). Via the canonical injection \( \mathcal{Z}(\hat{H}) \hookrightarrow \hat{T} \) the element \( s \in \mathcal{Z}(\hat{H})^{\Gamma} \) determines an element \( \kappa \in \hat{T}^{\Gamma} \).

The conjectural fundamental lemma predicts the equality

\[
O_{u_H}^\kappa(1_{t_H}) = q^{-r} \cdot O_u(1_t),
\]

which in view of (15.9.1) can be rewritten as the equality of

\[
q^r \cdot \text{Tr}(\sigma^{-1}; RHom_{X, (T)_{\mathfrak{t}}}(\mathcal{Q}_{\mathfrak{t}}^\kappa, R\Gamma(X_{u_H}^H, \mathcal{Q}_{\mathfrak{t}})))
\]

and

\[
\text{Tr}(\sigma^{-1}; RHom_{X, (T)_{\mathfrak{t}}}(\mathcal{Q}_{\mathfrak{t}}^\kappa, R\Gamma(X_u, \mathcal{Q}_{\mathfrak{t}})))
\]

We conjecture the stronger statement that

\[
RHom_{X, (T)_{\mathfrak{t}}}(\mathcal{Q}_{\mathfrak{t}}^\kappa, R\Gamma(X_{u_H}^H, \mathcal{Q}_{\mathfrak{t}}[-2r](-r)))
\]

is isomorphic to

\[
RHom_{X, (T)_{\mathfrak{t}}}(\mathcal{Q}_{\mathfrak{t}}^\kappa, R\Gamma(X_u, \mathcal{Q}_{\mathfrak{t}})).
\]

15.11. Transfer factors for unramified tori. Earlier in this paper, when we calculated the homology of Springer fibers, the elements of \( g(F) \) we considered did not necessarily lie in Kostant’s section. Therefore we need a further discussion of transfer factors.

We continue the discussion in the subsection above, making only the following changes. We no longer require that the elements in \( g(F) \) that we consider be \( G(F) \)-conjugate to an element in Kostant’s section. Instead we consider a maximal torus \( T \) of \( G \) over \( k \) (not just over \( F \)), so that in particular \( T \) is unramified over \( F \), and we assume that \( T \) comes from a maximal torus \( T_H \) in \( H \). We identify \( T \) and \( T_H \) using some admissible isomorphism, and for any \( G \)-regular \( \gamma \in \mathfrak{t}(F) \) we denote by \( \gamma_H \) the corresponding element of \( \mathfrak{t}_H(F) \). It follows immediately from the definition of transfer factors on the Lie algebra (see [W97] and perhaps
also [Kot99]) that there is a constant $c_T$ (depending only on the torus $T$ and our choice of admissible isomorphism) such that for any $G$-regular element $\gamma \in \mathfrak{t}(F)$

$$\Delta(\gamma_H, \gamma) = c_T \cdot \Delta_{IV}(\gamma_H, \gamma) \cdot \prod_{\alpha} (-1)^{\text{val}(\alpha(\gamma))},$$

where $\alpha$ runs over a set of representatives for the symmetric orbits of Frobenius on the set of roots of $T$ in $G$ that do not come from $H$. Here $\text{val}$ is the valuation on $\overline{F}$ that takes the value 1 on uniformizing elements for $F$. In fact the constant $c_T$ is equal to 1, since any element $\gamma \in \mathfrak{t}(\mathfrak{o})$ whose image in $\mathfrak{t}(k)$ is $G$-regular (which implies that $\text{val}(\alpha(\gamma)) = 0$ for all roots $\alpha$) is $G(F)$-conjugate to an element in our choice of Kostant’s section, and therefore for such $\gamma$ we have (by [Kot99]) that $\Delta(\gamma_H, \gamma) = \Delta_{IV}(\gamma_H, \gamma)$.

Now recall the homological transfer factor $\Delta = \prod_{\alpha \in \Phi^+(G) - \Phi^+(H)} \partial_{\text{val}(\alpha)}^{\alpha(\gamma)}$. Frobenius acts on $X^*(T)$ by some element $\tau \in \tilde{W}_G^H$ and we have $\tau(\Delta) = \eta(\tau) \cdot \Delta$ (see §10 for the definitions of the group $\tilde{W}_G^H$ and the sign character $\eta$ on it). It is a simple exercise to check that

$$\eta(\tau) = \prod_{\alpha} (-1)^{\text{val}(\alpha(\gamma))},$$

with $\alpha$ again running over a set of representatives for the symmetric orbits of Frobenius on the set of roots of $T$ in $G$ that do not come from $H$. Therefore we have the equality

$$\Delta(\gamma_H, \gamma) = \eta(\tau) \cdot \Delta_{IV}(\gamma_H, \gamma),$$

which for integral $\gamma$ can be rewritten as

$$\Delta(\gamma_H, \gamma) = \eta(\tau) \cdot q^{-r} \quad (15.11.1)$$

with $r$ again defined as $\dim(X_\gamma) - \dim(X_H^{\gamma_H})$.

15.12. **Remarks on the fundamental lemma for unramified tori.** We continue the discussion of the previous subsection. Assume that the étale cohomology groups $H^*(X_\gamma, \overline{Q}_\ell)$ and $H^*(X_H^{\gamma_H}, \overline{Q}_\ell)$ are pure. By [GKMx] this is true in the equal valuation case, in other words, when $\text{val}(\alpha(\gamma))$ is independent of $\alpha$, since in that case there is a paving by affines defined over $k$.

We expect Theorems 9.2 and 10.2 to have corresponding statements in étale cohomology. Assume this is so. Then Theorem 9.2 will give a formula for the $T$-equivariant étale homology of $X_\gamma$ as a quotient

$$0 \longrightarrow \sum_{\alpha \in \Phi^+_G} L_{\alpha, \gamma} \longrightarrow \overline{Q}_\ell[\Lambda] \otimes S(\mathfrak{t}(\overline{Q}_\ell)) \longrightarrow H^*_T(X_\gamma, \overline{Q}_\ell) \longrightarrow 0.$$  

The surjection $\overline{Q}_\ell[\Lambda] \otimes S(\mathfrak{t}(\overline{Q}_\ell)) \rightarrow H^*_T(X_\gamma, \overline{Q}_\ell)$ is Frobenius equivariant, and the action of Frobenius on $\overline{Q}_\ell[\Lambda] \otimes S(\mathfrak{t}(\overline{Q}_\ell))$ is given by $\tau \otimes Q$, where $\tau \in \tilde{W}_G^{H}$ is the element through which the Frobenius element $\sigma$ acts on $X_\gamma(T)$ and $Q$ denotes the endomorphism of the
symmetric algebra induced by multiplication by \( q \) on the vector space \( \mathfrak{t}(\overline{\mathbb{Q}}_\ell) \). It follows that the action of the Frobenius element \( \sigma \) on \( H_2(X_\gamma; \overline{\mathbb{Q}}_\ell) \) is given by \( \sigma = q^i \cdot \tau \).

Therefore Corollary 10.3 gives an isomorphism
\[
H_*(X_\gamma; \overline{\mathbb{Q}}_\ell)_J \cong H_*(X_{\gamma_H}; \overline{\mathbb{Q}}_\ell)_J[-2r](r)
\]
that transforms by the sign \( \eta(\tau) \) under the action of Frobenius. As in the proof of Proposition 11.2, this in turn gives an isomorphism
\[
\text{Tor}^p_A((\overline{\mathbb{Q}}_\ell)_\kappa, H^q(X_\gamma; \overline{\mathbb{Q}}_\ell)) \cong \text{Tor}^p_A((\overline{\mathbb{Q}}_\ell)_\kappa, H^{q-2r}(X_{\gamma_H}; \overline{\mathbb{Q}}_\ell)(r)).
\]
Taking the vector space dual of both sides of this equation, we obtain an isomorphism
\[
\text{Ext}^p_A((\overline{\mathbb{Q}}_\ell)_\kappa, H^q(X_\gamma; \overline{\mathbb{Q}}_\ell)) \cong \text{Ext}^p_A((\overline{\mathbb{Q}}_\ell)_\kappa, H^{q-2r}(X_{\gamma_H}; \overline{\mathbb{Q}}_\ell)(-r)),
\]
that transforms by the sign \( \eta(\tau) \) under the action of Frobenius. It follows that
\[
q^{-r}\eta(\tau)\text{Tr}(\sigma^{-1}; RH\text{Hom}_A^*(((\overline{\mathbb{Q}}_\ell)_\kappa, R\Gamma(X_\gamma; \overline{\mathbb{Q}}_\ell)))) = \text{Tr}(\sigma^{-1}; RH\text{Hom}_A^*(((\overline{\mathbb{Q}}_\ell)_\kappa, R\Gamma(X_{\gamma_H}; \overline{\mathbb{Q}}_\ell)))).
\]
Thus, under the assumptions we have made, the fundamental lemma for unramified tori follows from this last equation together with formulas (15.9.1) and (15.11.1).

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