EXACTLY SOLVED MODELS WITH QUANTUM ALGEBRA SYMMETRY*

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ABSTRACT

We have constructed and solved various one-dimensional quantum mechanical models which have quantum algebra symmetry. Here we summarize this work, and also present new results on graded models, and on the so-called string solutions of the Bethe Ansatz equations for the $A_2^{(2)}$ model.

1. INTRODUCTION

The concept of symmetry is fundamental for the description of physical systems. In many cases, such symmetry is codified by a Lie (super) algebra. A generalization of this structure, the so-called quantum Lie (super) algebra, has recently emerged. Our ultimate goal is to understand how quantum algebra symmetry is implemented in physical systems, and to explore the consequences of this symmetry. Such symmetry may eventually prove to be useful for field theory in 4 spacetime dimensions and for string theory.

To date, quantum algebras have been identified as the common mathematical structure linking three types of physical systems: topological (Chern-Simons) field theory in 3 spacetime dimensions, integrable lattice models, and rational conformal field theories and their integrable perturbations. Over the past two years, we have studied primarily the connection between integrable lattice models and quantum algebras. Among the three connections of quantum algebras to physical systems noted above, this is the most direct. Furthermore, it is within this context that quantum algebras were first discovered.

In the course of our investigations, we have constructed and solved various one-dimensional quantum mechanical models which have quantum algebra symmetry. Here we summarize this work, and also present new results on graded models, and on the string solutions of the Bethe Ansatz equations for the $A_2^{(2)}$ model. The construction of these models requires two main ingredients: the $R$ matrix, which can be interpreted as a two-particle scattering amplitude, and the $K$ matrix, which can be interpreted as the amplitude for a particle to reflect elastically from a wall. The integrability of these models comes from demanding that the scattering be consistent with factorization. In Section 2, we introduce $R$ matrices via the Zamolod-
chikov algebra, and summarize some of their important properties. In Section 3, we introduce $K$ matrices through an extension of the Zamolodchikov algebra. In particular, we describe the graded case, which we illustrate with an example connected to the superalgebra $su(2|1)$. In Section 4, we construct open chains of $N$ "spins" (generators of a quantum algebra) with certain nearest-neighbor interactions, which are integrable and which have quantum algebra symmetry. For these models, the transfer matrix (i.e., not just the Hamiltonian) commutes with the generators of a quantum algebra. We also comment on the solution – namely, the eigenvalues of the transfer matrix and the Bethe Ansatz equations – of these models. In order to calculate quantities of physical interest, one must first solve the Bethe Ansatz equations in the $N \to \infty$ limit. For the $A_1^{(1)}$ case, these so-called string solutions are well known. In Section 5, which is a result of a collaboration with A.M. Tsvelik, we investigate string solutions for the $A_2^{(2)}$ model of Izergin and Korepin. We find new types of string solutions, but we are not able to formulate a general string hypothesis. We summarize our results in Section 6.

A more detailed account for the simplest case of $A_1^{(1)}$ can be found in Ref. !!!!.

2. $R$ MATRICES

**Yang-Baxter equation**

We briefly review here how the (graded) Yang-Baxter equation follows from the associativity of the (graded) Zamolodchikov algebra. This algebra is abstracted from studies of scattering in massive relativistic quantum field theories in 1+1 dimensions with an infinite number of conservation laws !!!!.

The Zamolodchikov algebra has generators $A_{\alpha}(u)$, where $u$ is the so-called spectral parameter, and $\alpha = 1, \ldots, n$. These generators obey the relations

$$A_{\alpha}(u) A_{\beta}(v) = \alpha_\beta R_{\alpha' \beta'}(u-v) A_{\beta'}(v) A_{\alpha'}(u). \quad (2.1)$$

The matrix $\alpha_\beta R_{\alpha' \beta'}(u-v)$, which may be interpreted as a two-particle scattering amplitude, is called the $R$ matrix.

By setting $u = v$ in the above relation, and by assuming linear independence of monomials of second degree, we learn that the $R$ matrix is regular,

$$R(0) = \mathcal{P}, \quad (2.2)$$

where $\mathcal{P}$ is the permutation matrix,

$$\alpha_\beta \mathcal{P} = \delta_{\alpha' \beta'} \delta_{\beta \alpha'}. \quad (2.3)$$

Moreover, by interchanging $A_{\alpha}(u) A_{\beta}(v)$ twice using (2.1), we obtain the unitarity relation

$$R(u) \mathcal{P} R(-u) \mathcal{P} = 1. \quad (2.4)$$
Consider now the monomial of third degree $A_\alpha(u) A_\beta(v) A_\gamma(0)$. Associativity of the Zamolodchikov algebra, as well as the assumption of linear independence of monomials of third degree, imply that

$$\alpha\beta R_{\alpha''\beta''}(u-v) \alpha''\gamma R_{\alpha'\gamma'}(u) \beta''\gamma'' R_{\beta'\gamma'}(v) = \beta\gamma R_{\beta''\gamma''}(u) \alpha''\gamma'' R_{\alpha'\gamma'}(u) \alpha''\beta'' R_{\alpha'\beta'}(u-v).$$ (2.5)

This relation is the well-known Yang-Baxter (or factorization) equation. Introducing the notation $R_{12} = R \otimes 1$, so that

$$\alpha\beta\gamma (R_{12})_{\alpha'\beta'\gamma'} = \alpha\beta R_{\alpha'\beta'} \delta_{\gamma'\gamma},$$

and similarly defining $R_{13}$ and $R_{23}$, the Yang-Baxter equation can be rewritten in the compact form

$$R_{12}(u-v) R_{13}(u) R_{23}(v) = R_{23}(v) R_{13}(u) R_{12}(u-v).$$ (2.6)

We remark that the Yang-Baxter equation transforms covariantly under “gauge” (or “symmetry-breaking”) transformations of the Zamolodchikov algebra, as well as the assumption of linear independence of monomials of third degree, imply that $\alpha\beta R_{\alpha''\beta''}(u-v) \alpha'' R_{\alpha'\gamma'}(u) \beta'' R_{\beta'\gamma'}(v) = \beta\gamma R_{\beta''\gamma''}(u) \alpha'' R_{\alpha'\gamma'}(u) \alpha''\beta'' R_{\alpha'\beta'}(u-v).$ (2.5)

We assume that $\alpha\beta R_{\alpha'\beta'}$ are commuting numbers, and that if $\alpha\beta R_{\alpha'\beta'} \neq 0$, then $p(\alpha) + p(\beta) + p(\alpha') + p(\beta') = 0 \mod 2$. Associativity of this graded Zamolodchikov algebra leads to the graded Yang-Baxter equation

$$(-)^{p(\beta'')} [p(\gamma'') - p(\gamma)] \alpha\beta R_{\alpha''\beta''}(u-v) \alpha''\gamma R_{\alpha'\gamma'}(u) \beta''\gamma'' R_{\beta'\gamma'}(v) = (-)^{p(\beta'')} [p(\gamma'') - p(\gamma)] \beta\gamma R_{\beta''\gamma''}(u) \alpha''\gamma'' R_{\alpha'\gamma'}(u) \alpha''\beta'' R_{\alpha'\beta'}(u-v).$$ (2.12)
Solutions of the Yang-Baxter equation

An $R$ matrix is said to be quasi-classical if it depends on an additional parameter $\eta$ which plays the role of Planck’s constant, so that

$$R(u, \eta)\big|_{\eta=0} = \text{const} \ 1. \quad (2.13)$$

There are three known classes of regular quasi-classical solutions of the Yang-Baxter equation: elliptic, trigonometric, and rational (corresponding to the three types of functions of $u$ that appear in $R(u)$).

Being interested in quantum algebras, we focus on the trigonometric solutions. Such solutions are associated with affine Lie algebras $g(k)$, where $g$ is a simple Lie algebra ($A_n = su(n+1)$, $B_n = o(2n+1)$, $C_n = sp(2n)$, $D_n = o(2n)$, etc.) and $k(=1,2,3)$ is the order of a diagram automorphism $\sigma$ of $g$. That is, $\sigma^k = 1$. The cases $k=1$ and $k>1$ are often referred to as “untwisted” and “twisted”, respectively. For instance, in the case of $A_2^{(2)}$ in the fundamental representation, the diagram automorphism is given by the complex conjugation map $\sigma : \lambda^A \rightarrow -\lambda^A^*$, where $\lambda^A$ are the eight Gell-Mann matrices.

We shall later make use of the fact that the automorphism $\sigma$ leaves invariant a subalgebra $g_0$ of $g$. (This subalgebra $g_0$ is in fact the maximal finite-dimensional subalgebra of the affine algebra $g(k)$.) In the $A_2^{(2)}$ example, it is clear that $\sigma$ leaves invariant the purely imaginary matrices $\lambda^5$, $\lambda^6$, $\lambda^7$, which generate an $su(2)$ subalgebra of $su(3)$. A table listing every simple Lie algebra $g$ which has a nontrivial diagram automorphism, along with the corresponding subalgebra $g_0$ which is left invariant by this automorphism, is given in Ref. !!!!, and is reproduced in Ref. !!!!.

The simplest example of an untwisted $R$ matrix is the spin 1/2 $A_1^{(1)}$ matrix

$$R^{(\frac{1}{2}, \frac{1}{2})}(u) = \frac{1}{\sqrt{|sh(u + \eta)sh(-u + \eta)|}} \begin{pmatrix} sh(u + \eta) & sh u & sh \eta \\ sh u & sh u & sh\eta \\ sh \eta & sh \eta & sh(u + \eta) \end{pmatrix} \quad (2.14)$$

In this gauge, the $R$ matrix is “symmetric”; i.e., it is both $P$ invariant ($\mathcal{P}_{12} R_{12} \mathcal{P}_{12} = R_{12}$) and $T$ invariant ($R_{12}^{t_1 t_2} = R_{12}$). The gauge transformation (2.7) with $B(u) = diag(e^{u/2}, e^{-u/2})$ yields the symmetry-broken $R$ matrix

$$R^{(\frac{1}{2}, \frac{1}{2})}(u) = \frac{1}{\sqrt{|sh(u + \eta)sh(-u + \eta)|}} \begin{pmatrix} sh(u + \eta) & sh u & e^u sh \eta \\ e^{-u} sh \eta & sh u & \eta \end{pmatrix} \quad (2.15)$$

which is only $PT$ invariant,

$$\mathcal{P}_{12} R_{12}(u) \mathcal{P}_{12} = R_{12}^{t_1 t_2}(u). \quad (2.16)$$

The transposition $t_i$ refers to the $i^{th}$ space.
The $R$ matrices associated with the nonexceptional affine Lie algebras in the fundamental representation have been given by Bazhanov and Jimbo. (For the graded case, see Ref.) Although in general these $R$ matrices do not have either $P$ or $T$ symmetry, they do have $PT$ symmetry (2.16). Except for $A^{(1)}_n$ ($n > 1$), these $R$ matrices have crossing symmetry,

$$R_{12}^{t_2}(u) \frac{1}{M} R_{12}^{t_2}(-u - 2\rho) \frac{1}{M} = 1,$$

(2.17)

where $M$ is a symmetric matrix ($M^t = M$) which can be deduced from Ref. Moreover, except for $D^{(2)}_n$, these $R$ matrices (in the so-called homogeneous gauge used by Jimbo) satisfy

$$[\tilde{R}(u), \tilde{R}(v)] = 0,$$

(2.18)

where

$$\tilde{R}(u) \equiv \mathcal{P}R(u).$$

(2.19)

Connection with Quantum Algebras

The prototype quantum algebra is $U_q[\mathfrak{su}(2)]$, with generators $\vec{S} = \{S^+, S^-, S^z\}$ which obey

$$q^{S^z + \frac{1}{2}} S^\pm = S^\pm q^{S^z}, \quad [S^+, S^-] = \frac{q^{2S^z} - q^{-2S^z}}{q - q^{-1}},$$

(2.20)

where $q$ is a complex parameter. Given two sets of generators $\vec{S}_1, \vec{S}_2$ of this algebra (with $[\vec{S}_1, \vec{S}_2] = 0$), the generators $\vec{S}$ in the tensor product space are given by

$$q^{S^z} = q^{S^z_1} \otimes q^{S^z_2}, \quad S^\pm = S^\pm_1 \otimes q^{-S^z_2} + q^{S^z_1} \otimes S^\pm_2.$$

(2.21)

The generalization to $U_q[\mathfrak{g}]$ for any simple Lie algebra $\mathfrak{g}$ is discussed in Refs. & Faddeev, et al. emphasize an $R$-matrix formulation of quantum algebras. Taking $U_q[\mathfrak{su}(2)]$ again as an example, define

$$R_\pm = \lim_{u \to \pm \infty} R(u),$$

(2.22)

where $R(u)$ is the spin 1/2 $A^{(1)}_1$ $R$ matrix in the nonsymmetric gauge (2.15); and define the upper, lower triangular matrices

$$T_+ = \left( q^{S^z + \frac{1}{2}} (q - q^{-1}) S^- \begin{array}{c} q^{-S^z - \frac{1}{2}} \end{array} \right), \quad T_- = \left( q^{S^z - \frac{1}{2}} \begin{array}{c} q^{-S^z - \frac{1}{2}} \end{array} \right),$$

(2.23)

with $q = e^\eta$. The relations

$$R_\pm \frac{1}{T_\pm} \frac{2}{\epsilon} = \hat{T}_\epsilon \frac{1}{T_\pm} R_\pm \quad \text{with} \quad \epsilon = \{+, -, \}$$

(2.24)
hold if and only if the operators $\vec{S}$ obey the algebra (2.20). Moreover, consider two sets of such matrices $T_{1\pm}, T_{2\pm}$ constructed from $\vec{S}_1, \vec{S}_2$ respectively. The coproduct matrices $T_{\pm}$ are given by

$$T_{\pm} = T_{1\pm} \otimes T_{2\pm}$$

where the symbol $\otimes$ indicates the tensor product of the algebras and the usual product of the matrices. They are expressed in the form (2.23) in terms of the operators $\vec{S}$ given precisely by the comultiplication rule (2.21).

An important identity is

$$[\hat{R}(u), U_q[su(2)]] = 0,$$

where here by $U_q[su(2)]$ we mean coproducts of the generators. For the general case of an $R$ matrix of the type $g^{(k)}$, the corresponding result is

$$[\hat{R}(u), U_q[g_0]] = 0,$$

where $g_0$ is the subalgebra of $g$ which is left invariant under the diagram automorphism of order $k$. In particular, for both $A_1^{(1)}$ and $A_2^{(2)}$, the matrices $\hat{R}(u)$ commute with $U_q[su(2)]$.

3. \textbf{K MATRICES}

\textbf{Reflection-factorization equation}

We now extend the Zamolodchikov algebra (2.1), by introducing the additional relation

$$A_\alpha(u) = \alpha K_\alpha^\beta(u) A_\beta(-u).$$

The $K$ matrix $\alpha K_\alpha^\beta(u)$ can be interpreted as the amplitude for a particle to reflect elastically from a wall.

By setting $u = 0$, we see that

$$K(0) = 1.$$

Furthermore, using the relation (3.1) twice, we obtain the unitarity relation

$$K(u) K(-u) = 1.$$

Consider now the monomial of second degree $A_\alpha(u) A_\beta(v)$. There are two different ways by which one can apply each of the Zamolodchikov relations (2.1), (3.1) twice to obtain an expression proportional to $A_\alpha^\prime(-u) A_\beta^\prime(-v)$. Using again the assumption of linear independence, we obtain the relation

$$R_{12}(u-v) \frac{1}{K(u)} \mathcal{P}_{12} R_{12}(u+v) \mathcal{P}_{12} \frac{2}{K(v)} = \frac{2}{K(v)} R_{12}(u+v) \frac{1}{K(u)} \mathcal{P}_{12} R_{12}(u-v) \mathcal{P}_{12},$$

(3.4)
to which we shall refer as the reflection-factorization equation. This equation transforms covariantly under the gauge transformation (2.7), provided that the $K$ matrix transforms as follows,

$$K(u) \rightarrow B(u) K(u) B(u).$$

(3.5)

By repeating the above calculation using instead graded Zamolodchikov generators (which obey the relation (2.11)), we obtain the graded reflection-factorization equation,

$$(-)^{p(\beta'')} p(\alpha''') \alpha\beta R_{\alpha''\beta''}(u-v) \alpha''K_{\alpha'''}(u) \beta''\alpha'''R_{\beta''\alpha'}(u+v) \beta''K_{\beta'}(v) = (-)^{p(\beta''') p(\alpha''')} \beta K_{\beta'}(v) \alpha\beta R_{\alpha''\beta''}(u+v) \alpha''K_{\alpha'''}(u) \beta''\alpha'''R_{\beta'\alpha'}(u-v).$$

(3.6)

Here we have assumed that $\alpha K_{\alpha'}$ are commuting numbers, and that if $\alpha K_{\alpha'} \neq 0$, then $p(\alpha) + p(\alpha') = 0$ mod 2.

**Solutions of the reflection-factorization equation**

Given a solution $R(u)$ of the (graded) Yang-Baxter equation, one can solve the (graded) reflection-factorization relation for the corresponding $K(u)$.

**spin 1/2 $A_1^{(1)}$**:

For the spin 1/2 $A_1^{(1)}$ $R$ matrix (2.14), there is a one-parameter family of diagonal $K$ matrices given by

$$K^{(1/2)}(u, \xi) = \frac{1}{\sqrt{|\sinh(u+\xi)\sinh(-u+\xi)|}} \begin{pmatrix} \sinh(u+\xi) & \cosh(u+\xi) \\ -\sinh(u-\xi) & -\cosh(u-\xi) \end{pmatrix},$$

(3.7)

where $\xi$ is an arbitrary parameter.

**spin 1 $A_1^{(1)}$**:

For the spin 1 $A_1^{(1)}$ matrix $R^{(1,1)}$ given in Refs. !!!!, we find

$$K^{(1)}(u, \xi) = \rho(u, \xi) \begin{pmatrix} \sinh(u+\xi) \cosh(u-\eta+\xi) & -\sinh(u+\xi) \cosh(u-\eta+\xi) \\ \sinh(u-\xi) \cosh(u-\eta-\xi) & \sinh(u-\xi) \cosh(u+\eta-\xi) \end{pmatrix},$$

(3.8)

where

$$\rho(u, \xi) = \frac{[\sinh(u+\xi) \cosh(-u+\xi) \sinh(u-\eta+\xi) \cosh(u-\eta+\xi)]^{-\frac{1}{2}}}{\cosh(u+\eta-\xi).}$$

Just as there is a fusion procedure by which $R^{(1,1)}$ may be obtained from $K^{(1/2)}$, there is a similar fusion procedure by which $K^{(1)}$ may be obtained from $K^{(1/2)}$ and $R^{(1/2,1/2)}$. 

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\( A_2^{(2)} \) in fundamental representation:

The \( R \) matrix corresponding to \( A_2^{(2)} \) in the fundamental representation is the simplest example of a “twisted” \( R \) matrix. For this case, we obtain

\[
K(u, \pm) = \rho(u, \pm) \begin{pmatrix}
\text{ch}(\frac{u}{2} + 3\eta) \mp i \text{sh} \frac{u}{2} \\
\text{e}^{-u} \left[ \text{ch}(\frac{u}{2} - 3\eta) \pm i \text{sh} \frac{u}{2} \right] \\
\text{e}^{u} \left[ \text{ch}(\frac{u}{2} - 3\eta) \pm i \text{sh} \frac{u}{2} \right]
\end{pmatrix},
\]

(3.9)

where

\[
\rho(u, \pm) = \left( \left[ \text{ch}(\frac{u}{2} + 3\eta) \mp i \text{sh} \frac{u}{2} \right] \left[ \text{ch}(\frac{u}{2} - 3\eta) \pm i \text{sh} \frac{u}{2} \right] \right)^{-\frac{1}{2}}.
\]

In contrast with the \( A_1^{(1)} \) case, for which there is a one-parameter family of “non-trivial” (i.e., \( K \neq 1 \)) diagonal solutions, here we find only two nontrivial solutions.

\( sl(2|1)^{(2)} \) in fundamental representation:

For the solution of the graded Yang-Baxter equation corresponding to \( sl(2|1)^{(2)} \) in the fundamental representation, we find the following one-parameter family of solutions of the graded reflection-factorization equation:

\[
K(u, \xi) = \rho(u, \xi) \begin{pmatrix}
\text{sh}(u + \xi) \text{ch}(u - \eta + \xi) \\
- \text{sh}(u - \xi) \text{ch}(u + \eta - \xi) \\
- \text{sh}(u - \xi) \text{ch}(u - \eta + \xi)
\end{pmatrix},
\]

(3.10)

where

\[
\rho(u, \xi) = \left[ \text{sh}(u + \xi) \text{sh}(-u + \xi) \text{ch}(u - \eta + \xi) \text{ch}(-u - \eta + \xi) \right]^{-\frac{1}{2}}.
\]

For other trigonometric \( R \) matrices (and in particular, for those enumerated by Bazhanov and Jimbo), finding the general solution of the corresponding reflection-factorization equation remains an interesting open problem. Nevertheless, because of the relation (2.18), the “trivial” \( K \) matrix

\[
K(u) = 1
\]

(3.11)

is a particular solution of the reflection-factorization equation for all the Bazhanov-Jimbo \( R \) matrices except \( D_n^{(2)} \), in the homogeneous gauge. (The \( A_1^{(1)} \) \( K \) matrices (3.7), (3.8) are gauge-equivalent to the identity matrix, in the limits \( \xi \to \pm\infty \).) As we shall see in the next section, this solution is important for constructing models with quantum algebra symmetry.
4. INTEGRABLE MODELS WITH QUANTUM ALGEBRA SYMMETRY

Having established a generalization of the Zamolodchikov algebra corresponding to scattering with walls, we turn to the construction of integrable open quantum spin chains. We shall find a large class of such models which has quantum algebra symmetry.

As it is well known, given an arbitrary solution \( R(u) \) of the Yang-Baxter equation, one can construct an integrable closed chain, with Hamiltonian

\[
H = \sum_{k=1}^{N-1} H_{k,k+1} + H_{N,1}, \tag{4.1}
\]

where

\[
H_{k,k+1} = \frac{d}{du} \tilde{R}_{k,k+1}(u) \bigg|_{u=0}. \tag{4.2}
\]

This Hamiltonian, whose state space is \( \prod_{k=1}^{N} \otimes C^n \), contains only nearest-neighbor interactions. The basic algebraic structure behind the integrability of this chain is the intertwining relation

\[
R_{12}(u-v) \frac{1}{T(u)} \frac{2}{T(v)} = \frac{2}{T(v)} \frac{1}{T(u)} R_{12}(u-v), \tag{4.3}
\]

where the monodromy matrix \( T(u) \) is given by

\[
T(u) = R_{0N}(u) R_{0N-1}(u) \cdots R_{01}(u). \tag{4.4}
\]

The transfer matrix

\[
t(u) = \text{tr} T(u), \tag{4.5}
\]

from which the Hamiltonian is constructed, plays the role of Cartan generators for the above algebraic structure. (The trace in (4.5) is over the auxiliary space, which is denoted by 0 in (4.4).)

In order to construct open integrable spin chains, a generalization of the above structure is necessary. The clue to the introduction of the new transfer matrix is that, while closed chains are related to scattering on an infinite line (∼ circle), open chains are related to scattering on an interval. One finds that the class of \( R \) matrices to be used must now be restricted to those satisfying \( PT \)-invariance (2.16), unitarity (2.4), and crossing symmetry (2.17). The Sklyanin transfer matrix is

\[
t(u) = \text{tr} K_+(u) T(u) K_-(u) T^{-1}(-u), \tag{4.6}
\]

where

\[
K_-(u) = K(u, \xi_-), \quad K_+(u) = K^t(-u - \rho, \xi_+) M, \tag{4.7}
\]

and \( K(u, \xi) \) is a one-parameter (\( \xi \)) family of solutions of the the reflection-factorization equation. Indeed, one can show that

\[
[t(u), t(v)] = 0 \text{ for all } u, v. \tag{4.8}
\]
Also, \( t(u) \) is invariant under gauge transformations (2.7), (3.5). From this transfer matrix, one can construct the local Hamiltonian
\[
H = \sum_{k=1}^{N-1} H_{k,k+1} + \frac{1}{2} \frac{d}{du} K_-(u) \bigg|_{u=0} + \frac{\text{tr}_0 K_+(0) H_{N,0}}{\text{tr} K_+(0)},
\]
where \( H_{k,k+1} \) is given by (4.2).

The novel algebraic structure is generated by
\[
T_-(u) = T(u) K_-(u) T^{-1}(-u),
\]
which, like \( K_-(u) \), obeys the reflection-factorization equation (3.4). It is not known whether the reflection-factorization equation admits solutions other than those of the form (4.10).

Our interest in open quantum spin chains lies in their connection with quantum algebras. Indeed, let us consider the \( R \) matrices of type \( g^{(k)} \) which are listed by Bazhanov and Jimbo. As we have already noted, all of these except for \( A_n^{(1)} (n > 1) \) fulfill the criteria of \( PT \) symmetry, unitarity, and crossing symmetry. Moreover, except for \( D_n^{(2)} \), the reflection-factorization equation has the trivial solution (3.11), which implies
\[
K_-(u) = 1, \quad K_+(u) = M.
\]
For these choices of \( K_\pm \), the Hamiltonian (4.9) reduces to
\[
H = \sum_{k=1}^{N-1} \frac{d}{du} \tilde{R}_{k,k+1}(u) \bigg|_{u=0}.
\]
In showing that the last term in (4.9) gives only a \( c \)-number contribution, one can use the identity
\[
\text{tr}_0 M \tilde{R}_{N,0}(u) = f(u) \frac{N}{1},
\]
(where \( f(u) \) is a scalar function of \( u \)) which follows from the degeneration of (2.18) at \( u = -\rho \) and crossing symmetry. From the identity (2.27), we conclude that the Hamiltonian commutes with the generators of the quantum algebra \( U_q[g_0] \),
\[
[H, U_q[g_0]] = 0.
\]

It is useful to formulate the quantum algebra invariance of these chains in the \( R \) matrix approach. The key is to establish a connection between the quantum algebra generators and the monodromy matrix \( T(u) \). To this end, we define \( R_\pm \) as before (2.22), and similarly, we set
\[
T_\pm = \lim_{u \to \pm \infty} T(u).
\]
By taking the limits \( u \to \pm \infty \) and then \( v \to \pm \infty \) in the fundamental relation (4.3), we obtain the relations
\[
R_\pm \frac{1}{T_\pm} \frac{2}{T(v)} = \frac{T(v)}{T_\pm} R_\pm,
\]
and
\[ R_\pm \tilde{T}_\pm \tilde{T}_\epsilon = \tilde{T}_\epsilon \tilde{T}_\pm R_\pm, \quad \text{with} \quad \epsilon = \{+, -\}. \quad (4.17) \]

We recognize the latter relation as the definition of the quantum algebra \( U_q[\mathfrak{g}_0] \).

Indeed, the entries of \( T_\pm \) can be expressed in terms of quantum algebra generators, as in the \( A_1^{(1)} \) example (2.23). (For the \( A_2^{(2)} \) case, see Ref. !!!!.) The relation (4.16) can be interpreted as defining the tensor character of the operator \( T(v) \) with respect to the quantum algebra. (See also Ref. !!!!.) We remark that the identity
\[ \left[ \tilde{R}_{12}(w), \frac{1}{2} \tilde{T}_\pm \tilde{T}_\pm \right] = 0 \quad (4.18) \]
can also be obtained from the fundamental relation (4.3), if one instead takes the limits \( u, v \to \pm \infty \) with \( u - v = w = \text{finite} \).

Evidently, in the \( R \) matrix approach, quantum algebra symmetry is expressed through commutators with \( T_\pm \). For the open chain transfer matrix
\[ t(u) = \text{tr} M \ T(u) \ T^{-1}(-u) \quad (4.19) \]
corresponding to the \( K \) matrices (4.11), one can establish the result!!!!
\[ [t(u), T_\pm] = 0. \quad (4.20) \]

This directly implies that
\[ [t(u), U_q[\mathfrak{g}_0]] = 0. \quad (4.21) \]

That is, not only the Hamiltonian, but also the transfer matrix commutes with the quantum algebra generators.

Although we have argued that the transfer matrix (4.19) is a quantum algebra invariant, this invariance is not manifest. Clearly, it would be desirable to find a “tensor calculus” formulation of the differential geometry on quantum groups, using which the expression for the transfer matrix would be manifestly invariant. Perhaps this may be accomplished within the noncommutative geometry of Connes recently pursued by Wess and Zumino !!!!. Such a formulation would undoubtedly lead to the construction of additional invariants.

Having constructed a large class of integrable models with quantum algebra symmetry, let us briefly comment on their solutions. In the case of \( A_1^{(1)} \), the eigenvalues of the transfer matrix and the Bethe Ansatz (BA) equations have been determined by the algebraic BA, for both spin 1/2 !!!! and spin 1 !!!!. (The spin 1/2 model was first solved by the coordinate BA in Ref. !!!!.) Actually, these papers contain the solution of more general open spin chains, with a two-parameter \( (\xi_-, \xi_+) \) class of boundary terms, corresponding to the more general solutions (3.7), (3.8) of the reflection-factorization equation. At least a large class of the general \( g^{(k)} \) models with \( U_q[\mathfrak{g}_0] \) symmetry may be solved !!!! by the analytic BA. The BA equations for these open chains are “doubled” with respect to the BA equations for the corresponding closed chains.
5. STRING SOLUTIONS FOR $A_2^{(2)}$ (with A.M. Tsvelik)

Determining the Bethe Ansatz (BA) equations and finding the eigenvalues of the transfer matrix constitute only the first step in calculating quantities of physical interest for the $g^{(k)}$ models. In order to investigate the low-temperature thermodynamics of such models, one must first solve the corresponding BA equations in the limit that the number of spins $N$ tends to infinity. (See, e.g., Ref. !!!!.) Since in the critical regime ($|q| = 1$) these models have non-Hermitian Hamiltonians, one must then make suitable projections on the space of states.

For the spin 1/2 $A_1^{(1)}$ model, the BA equations in the $N \to \infty$ limit have the well-known string solutions of Takahashi-Suzuki !!!!. Moreover, for this model, only the boundary terms are non-Hermitian, as is the case in the Feigin-Fuchs-Dotsenko-Fateev !!!! construction. While details of the projections remain obscure, there is evidence that one obtains !!!!, !!!! the $c < 1$ unitary rational conformal field theories for $q$ a primitive root of unity. For spin $s \geq 1/2$, the corresponding conformal field theories are presumably the $SU(2) \otimes SU(2)/SU(2)$ coset models !!!!. !!!!!

In the generic case, the picture is less clear. In general, the string solutions of the BA equations are not known. Also, the bulk terms of the Hamiltonian (i.e., not just the boundary terms) are non-Hermitian. Such complications appear already for the case $A_2^{(2)}$.

In an effort to begin to understand these issues, we look here for string solutions for the $A_2^{(2)}$ model. We expect that the string solutions for this $U_q[su(2)]$-invariant open chain are a subset of those for the corresponding closed chain. We therefore focus on the BA equations of the closed chain !!!!, !!!!!, which are easier to study:

$$
\left[ \frac{\sinh \eta (\lambda_k + i/2)}{\sinh \eta (\lambda_k - i/2)} \right]^N = \prod_{j \neq k} \frac{\sinh \eta (\lambda_k - \lambda_j + i) \cosh \eta (\lambda_k - \lambda_j - i/2)}{\sinh \eta (\lambda_k - \lambda_j - i) \cosh \eta (\lambda_k - \lambda_j + i/2)},
$$

$$
k = 1, \ldots, M, \quad M = 1, 2, \ldots.
$$

We look for complex solutions

$$
\lambda_k = x_k + iy_k,
$$

with $x_k$, $y_k$ real. We first consider the critical regime $\eta = \text{real}$, with $0 < \eta < \pi$. (The precise normalization of $\eta$ is not important for our exploratory discussion.) Clearly, $y_k$ is determined modulo $\pi/\eta$. Our general strategy is to work with the modulus of these equations. In particular, we proceed in three steps:

1. We fix the value of $M$. Taking the modulus of (5.1), we obtain

$$
\left[ 1 + \frac{\sin \eta \sin 2\eta y_k}{\sinh^2 \eta x_k + \sin^2 \eta (y_k - 1/2)} \right]^N = \prod_{j \neq k} \frac{\sin^2 \eta (x_k - x_j) + \sin^2 \eta (y_k - y_j + 1)}{\sin^2 \eta (x_k - x_j) + \sin^2 \eta (y_k - y_j - 1)} \frac{\sin^2 \eta (x_k - x_j) + \cos^2 \eta (y_k - y_j - 1/2)}{\sin^2 \eta (x_k - x_j) + \cos^2 \eta (y_k - y_j + 1/2)},
$$

$$
k = 1, \ldots, M.
$$

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Consider the limit $N \to \infty$. Provided $x_k, y_k$ have a finite limit, the left-hand-side can have one of three possible values:

\[
\begin{cases}
0 \text{ when } \sin \eta \sin 2\eta y_k < 0 , \\
1 \text{ when } \sin \eta \sin 2\eta y_k = 0 , \\
\infty \text{ when } \sin \eta \sin 2\eta y_k > 0 .
\end{cases}
\]

Correspondingly, for finite $M$, the right-hand-side must have a zero, be equal to one, or have a pole. The only way that the right-hand-side can have a zero or pole for finite values of $x_k, y_k$ is for one of the factors in the numerator or denominator to vanish. This implies certain relations among $\{x_k, y_k\}$. These relations suggest that we look for solutions of the string type – that is, sets of solutions $\{\lambda_1, \cdots, \lambda_m\}$ with a common real part (the “center”) $x_0$,

\[
\lambda_k = x_0 + iy_k , \quad k = 1, \cdots, m .
\]

We shall assume that the center $x_0$ can vary continuously in the $N \to \infty$ limit. The task now is to determine $\{y_k\}$.

(2). Multiplying together all $M$ BA equations (5.1) together, we obtain

\[
\prod_{k=1}^{M} \left[ \frac{\text{sh} \eta (\lambda_k + i/2)}{\text{sh} \eta (\lambda_k - i/2)} \right]^N = 1 ,
\]

which implies

\[
\prod_{k=1}^{M} \left| \frac{\text{sh} \eta (\lambda_k + i/2)}{\text{sh} \eta (\lambda_k - i/2)} \right|^2 = 1 .
\]

For a single string of length $M$, it follows that

\[
\prod_{k=1}^{M} \frac{\text{sh}^2 \eta x_0 + \sin^2 \eta (y_k + 1/2)}{\text{sh}^2 \eta x_0 + \sin^2 \eta (y_k - 1/2)} = 1 .
\]

From the assumption that $x_0$ is arbitrary, we obtain the set of $M$ relations

\[
\sum_{k=1}^{M} (\alpha_k)^n = \sum_{k=1}^{M} (\beta_k)^n , \quad n = 1, \cdots, M ,
\]

where

\[
\alpha_k = \sin^2 \eta (y_k + 1/2) , \quad \beta_k = \sin^2 \eta (y_k - 1/2) .
\]

The only solutions of the above set of relations are

\[(\alpha_1, \alpha_2, \cdots, \alpha_M) = \text{permutation} (\beta_1, \beta_2, \cdots, \beta_M) .\]
In particular, for $\alpha_k = \beta_j$, it follows that
\[ y_k + y_j = \frac{m\pi}{\eta} \quad \text{or} \quad y_k - y_j + 1 = \frac{m'\pi}{\eta}, \tag{5.10} \]
with $m$, $m'$ integers.

(3). Having restricted the possible values of $\{y_k\}$, we return to the full set of equations (5.3), which – at least for small values of $M$ – completely determine $\{y_k\}$ according to the condition of their being poles or zeroes of the right-hand-side.

In this way, we have searched for strings of length $M$ for low values of $M$. Our results are as follows:

$M = 1$: There are two strings of length 1, given by $\lambda = x_0 + iy$ with
\[ y = 0 \quad \text{and} \quad y = \frac{\pi}{2\eta}, \tag{5.11} \]
respectively. These are the so-called strings of positive and negative parity of Takahashi-Suzuki.

$M = 2$: There are two strings of length 2, given by $\lambda_k = x_0 + iy_k$ with
\[ y_1 = \frac{i}{2}, \quad y_2 = -\frac{i}{2}, \tag{5.12} \]
and
\[ y_1 = i \left(-\frac{1}{4} + \frac{\pi}{4\eta}\right), \quad y_2 = i \left(\frac{1}{4} - \frac{\pi}{4\eta}\right), \tag{5.13} \]
respectively. The first string (5.12) is the positive-parity 2-string of Takahashi-Suzuki. The second string, which does not appear for $A_1^{(1)}$, has been studied numerically in Ref. !!!!.

$M = 3$: We find the 3-strings of Takahashi-Suzuki,
\[ y_1 = 1, \quad y_2 = 0, \quad y_3 = -1 \quad (0 < \eta < \frac{\pi}{2}), \tag{5.14} \]
and
\[ y_1 = 1 + \frac{\pi}{2\eta}, \quad y_2 = \frac{\pi}{2\eta}, \quad y_3 = -1 + \frac{\pi}{2\eta} \quad \left(\frac{\pi}{2} < \eta < \pi\right), \tag{5.15} \]
of positive and negative parity, respectively. We also find the solution
\[ y_1 = -\frac{1}{2} + \frac{\pi}{2\eta}, \quad y_2 = 0, \quad y_3 = \frac{1}{2} - \frac{\pi}{2\eta}, \tag{5.16} \]
which can be interpreted as a combination of a negative-parity 2-string and a positive-parity 1-string.

$M = 4$: We find the positive-parity 4-string of Takahashi-Suzuki,
\[ \{y_k\} = \left\{\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{3}{2}\right\} \quad (0 < \eta < \frac{\pi}{3} \text{ and } \frac{2\pi}{3} < \eta < \pi). \tag{5.17} \]
In addition, we find four new candidate 4-strings:

\[ \{ y_k \} = \left\{ \frac{5}{4} - \frac{\pi}{4\eta}, -\frac{1}{4} + \frac{\pi}{4\eta}, -\frac{1}{4} + \frac{\pi}{4\eta}, -\frac{5}{4} + \frac{\pi}{4\eta} \right\} \quad (\frac{\pi}{5} < \eta < \frac{\pi}{3} \text{ and } \frac{\pi}{2} < \eta < \frac{3\pi}{5}), \]

(5.18)

\[ \{ y_k \} = \left\{ 1 - \frac{\pi}{2\eta}, -\frac{1}{2} + \frac{\pi}{2\eta} \right\} \quad \left( \frac{\pi}{3} < \eta < \frac{\pi}{2} \right), \]

(5.19)

\[ \{ y_k \} = \left\{ \frac{3}{4} + \frac{\pi}{4\eta}, \frac{1}{4} - \frac{\pi}{4\eta}, -\frac{1}{4} + \frac{\pi}{4\eta}, -\frac{3}{4} + \frac{\pi}{4\eta} \right\}, \]

(5.20)

\[ \{ y_k \} = \left\{ \frac{3}{4} - \frac{\pi}{4\eta}, \frac{1}{4} + \frac{\pi}{4\eta}, -\frac{1}{4} - \frac{\pi}{4\eta}, -\frac{3}{4} + \frac{\pi}{4\eta} \right\}. \]

(5.21)

For (5.20), (5.21), the analysis of the BA equations is quite intricate, and we have not been able to confirm that these string configurations are in fact solutions.

The group-theoretic significance of these new strings has so far eluded us. The fact that in string configurations there occur steps of both 1 and $\frac{1}{2}$, accompanied by necessary factors of $\frac{\pi}{4\eta}$, makes it difficult to formulate a general string hypothesis. This impedes further progress in computing the thermodynamic properties of this model.

On the other hand, in the noncritical regime $\eta = \text{pure imaginary}$, the situation is much simpler. Let us make the replacement $\eta \to i\eta$ (with $\eta$ real) in the BA equations (5.1). Evidently, $x_k$ is determined modulo $\pi/\eta$. Repeating the steps (1) - (3) in the above analysis, we find only the positive-parity $M$-strings of Takahashi-Suzuki; i.e., $\lambda_k = x_0 + iy_k$ with

\[ \{ y_k \} = \left\{ \frac{M - 1}{2}, \frac{M - 3}{2}, \cdots \frac{3 - M}{2}, \frac{1 - M}{2} \right\}. \]

(5.22)

We do not expect significant difficulties in calculating thermodynamic properties in this regime.

6. CONCLUSIONS

We have obtained a number of results concerning integrable spin chains in connection with quantum algebras. We have presented a generalization of the Zamolodchikov algebra which accommodates reflecting walls, and which reproduces the algebraic relations that are obeyed by the $K$ matrices. By either directly solving these relations or implementing a fusion procedure, we have obtained new $K$ matrices corresponding to the trigonometric $R$ matrices for certain (graded) Lie algebras. We have extended Sklyanin’s approach for constructing integrable open quantum spin chains to $PT$-invariant $R$ matrices, and we have used this formalism to construct and investigate a large class of models with quantum algebra symmetry. These models may be solved by the analytic Bethe Ansatz. Finally, we have exhibited new types of string solutions for the $A_{2}^{(2)}$ model of Izergin and Korepin.
We are frustrated by the difficulty of solving the Bethe Ansatz equations, even in the $N \to \infty$ limit. These equations have a “group theoretical” origin, as they implement the construction of irreducible representations of a certain algebraic structure. Therefore, there should be a straightforward algorithm for obtaining their solutions. This, in turn, should enable one to standardize the calculations of thermodynamic properties, such as specific heat and magnetic susceptibility, of the corresponding models.

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APPENDIX

The solution of the graded Yang-Baxter equation corresponding to $sl(2|1)^{(2)}$ in the fundamental representation is given by

$$R(u) = \begin{pmatrix} a & b & r & y \\ c & x & y \\ r & b & a \\ x & c & d \\ y & y & x & c \end{pmatrix}, \quad (A.1)$$

where

$$a = 1, \quad b = \frac{\text{sh} u \ \text{ch}(u - \eta)}{\text{sh}(u + 2\eta) \ \text{ch}(u + \eta)}, \quad c = \frac{\text{sh} u}{\text{sh}(u + 2\eta)},$$

$$d = \frac{1}{\text{sh}(u + 2\eta)} \left[ \text{sh} u - \frac{\text{ch} \eta \ \text{sh} 2\eta}{\text{ch}(u + \eta)} \right], \quad r = \frac{\text{ch} \eta \ \text{sh} 2\eta}{\text{sh}(u + 2\eta) \ \text{ch}(u + \eta)},$$

$$x = \frac{\text{sh} 2\eta}{\text{sh}(u + 2\eta)}, \quad y = \frac{\text{sh} u \ \text{sh} 2\eta}{\text{sh}(u + 2\eta) \ \text{ch}(u + \eta)} \quad (A.2)$$

The parity assignments are given by $p(1) = p(2) = 0, p(3) = 1.$
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