A Note on Overshoot Estimation in Pole Placements

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Abstract

In this note we show that for a given controllable pair $(A, B)$ and any $\lambda > 0$, a gain matrix $K$ can be chosen so that the transition matrix $e^{(A+BK)t}$ of the system $\dot{x} = (A+BK)x$ decays at the exponential rate $e^{-\lambda t}$ and the overshoot of the transition matrix can be bounded by $M\lambda^L$ for some constants $M$ and $L$ that are independent of $\lambda$. As a consequence, for any $h > 0$, a gain matrix $K$ can be chosen so that the magnitude of the transition matrix $e^{(A+BK)t}$ can be reduced by $\frac{1}{h}$ (or by any given portion) over $[0, h]$. An interesting application of the result is in the stabilization of switched linear systems with any given switching rate (see [1]).

Key words: Linear system, transition matrix, Squashing Lemma.

1 Introduction

Consider a linear system

$$\dot{x} = Ax + Bu,$$  \hspace{1cm} (1)

where $x(\cdot)$ takes values in $\mathbb{R}^n$, $u(\cdot)$ takes values in $\mathbb{R}^m$, and where $A$ and $B$ are matrices of appropriate dimensions. Suppose $(A, B)$ is a controllable pair. It is a well known fact that for any $\lambda > 0$, a gain matrix $K$ can be chosen so that the transition matrix of the system $\dot{x} = (A+BK)x$ decays exponentially at the rate of $e^{-\lambda t}$, that is, for some $R > 0$,

$$\|e^{(A+BK)t}\| \leq Re^{-\lambda t},$$

where and hereafter $\|\cdot\|$ denotes the operator norm induced by the Euclidean norm on $\mathbb{R}^n$. To get a faster decay rate, it is natural to consider a “higher gain” matrix $K_1$. However, such a gain matrix in general results in a bigger overshoot for the transition matrix $e^{(A+BK_1)t}$. In this note, we show that in the pole placement practice, a gain matrix $K$ can be chosen so that the overshoot of the transition matrix $e^{(A+BK)t}$ can be bounded by $M\lambda^L$ for some constants $M$ and $L$ independent of $\lambda$. As a consequence, one sees that for any $h > 0$, a gain matrix $K$ can be chosen so that the magnitude of the transition matrix $e^{(A+BK)t}$ can be reduced by $\frac{1}{h}$ (or by any given portion) over $[0, h]$. Note that this is a stronger requirement than merely requiring $e^{(A+BK)t}$ to decay at an exponential rate. An interesting application of the result is in the stabilization of switched linear systems with a given switching rate (see [1]).

The estimate of the overshoots of transition matrices in the practice of pole assignments has been studied widely (see e.g. [5], [9] and [7]). Our main result in this note can be considered an enhancement of the Squashing Lemma (see [7], [6] and [4]) which says the following: for any $\tau_0 > 0$, $\delta > 0$, any $\lambda > 0$, it is possible to find $K$ such that

$$\|e^{(A+BK)t}\| \leq \delta e^{\lambda(t-\tau_0)}.$$  \hspace{1cm} (2)

In the current note, we show that $K$ can be chosen so that the estimate in (2) can be strengthened to

$$\|e^{(A+BK)t}\| \leq M\lambda^L e^{-\lambda t}$$

for some constants $M$ and $L$ which are independent of $\lambda$. Our proof is constructive that shows explicitly how $M$ and $L$ are chosen.
2 Main Result

In this section we present our main result.

Proposition 2.1 Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ be two matrices such that the pair $(A, B)$ is controllable. Then for any $\lambda > 0$, there exists a matrix $K \in \mathbb{R}^{m \times n}$ such that

$$\left\| e^{(A+BK)t} \right\| \leq M \lambda^t e^{-\lambda t}, \quad \forall t \geq 0,$$

where $L = (n-1)(n+2)/2$ and $M > 0$ is a constant, which is independent of $\lambda$ and can be estimated precisely in terms of $A$, $B$ and $n$.

Comparing with the Squashing Lemma obtained in [7], Proposition 2.1 has two improvements: (i) In (2), the estimate on the transient overshoot is exponentially proportional to the decay rate $\lambda$, which resulted in an estimation of the transition matrix in terms of $e^{-\lambda(t-\tau_0)}$ instead of $e^{-\lambda t}$. In (3), the estimate on the transient overshoot is proportional to $\lambda^t$ instead of $e^{\lambda t}$ as in (2). This distinction between the two types of estimations may be significant for some possible extensions of our results to systems with external inputs. (ii). The value of the constant $M$ in estimate (3) can be precisely calculated by using our constructive proof (see equation (10) in the sequel). This is certainly a very desirable feature for practical purposes. See Example 3.1 for some illustrations.

Proposition 2.1 was primarily presented and applied to a stabilization problem of switched linear systems in [2]. It was found later that a recent paper [3] also provides a similar result with similar proofs. The difference is that [3] only considered the single input case and the upper bound $M \lambda^t$ in (3) was found to be a polynomial $p(\lambda)$ in [3] without an explicit expression. Hence, our result has obvious merits in control design.

Proof of Proposition 2.1. First we consider a linear system $(A, b)$ of a single input. Without loss of generality, we assume that $(A, b)$ is in the Brunovsky canonical form:

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_n \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

Let $\lambda_1, \ldots, \lambda_n$ be $n$ distinct, negative real numbers. There exists some $k \in \mathbb{R}^{1 \times n}$ such that the characteristic equation of the closed-loop system $A + bk$ is $p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$. Note that the closed-loop system is given by

$$\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_3, \\
\vdots \\
\dot{x}_{n-1} &= x_n, \\
\dot{x}_n &= \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n
\end{align*}$$

for some $\beta_1, \beta_2, \ldots, \beta_n \in \mathbb{R}$. Hence, $x_1$ satisfies the equation

$$x_1^{(n)} = \beta_1 x_1 + \beta_2 x_1 + \cdots + \beta_n x_1^{(n-1)},$$

whose characteristic equation is the same as $p(\lambda)$. Hence, the general solution of (4) is

$$x_1(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \cdots + c_n e^{\lambda_n t},$$

where $c_1, c_2, \ldots, c_n$ are constants. From the equations $x_2 = \dot{x}_1, x_3 = \dot{x}_2, \ldots, x_n = \dot{x}_{n-1}$, we have $x(t) = \Lambda_0 e^{Dt} c$, where

$$\Lambda_0 = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix},$$

and where $c = \begin{pmatrix} c_1 \ c_2 \ \cdots \ c_n \end{pmatrix}^T$. Now, observe that $x(0) = \Lambda_0 c$, that is, $c = \Lambda_0^{-1} x(0)$ (note that $\Lambda_0$ is an invertible Vandermonde matrix). Comparing this with the transition matrix of the system, one sees that

$$e^{(A+bk)t} = \Lambda_0 e^{Dt} \Lambda_0^{-1}.$$  \hspace{1cm} (5)

Let $\lambda_{\text{max}} = \max\{|\lambda_1|, \ldots, |\lambda_n|\}$. Without loss of generality, assume that $\lambda_{\text{max}} \geq 1$. To get an estimate on $\|\Lambda_0\|$ and $\|\Lambda_0^{-1}\|$, we need the following simple fact: for an $n \times n$ matrix $C$, let $c_{\text{max}} = \max_{1 \leq i,j \leq n} |c_{ij}|$. It is not hard to see that

$$\|C\| \leq n c_{\text{max}}.$$  \hspace{1cm} (6)

Hence, we have

$$\|\Lambda_0\| \leq n \lambda_{\text{max}}^{n-1}.$$  \hspace{1cm} (6)

To get an estimate on $\Lambda_0^{-1}$, first note that

$$\Lambda_0^{-1} = \frac{1}{\det \Lambda_0} \text{adj} \Lambda_0,$$  \hspace{1cm} (7)

where $\text{adj} \Lambda_0$ denotes the adjoint matrix of $\Lambda_0$, and that

$$\det \Lambda_0 = \prod_{j>1} (\lambda_j - \lambda_i).$$

Hence, if we choose $\lambda_1, \ldots, \lambda_n$ in such a way that $\lambda_{i+1} \leq \lambda_i - 1$ with $\lambda_1 < 0$, we get $|\det \Lambda_0| \geq 1$. \hspace{1cm} (7)
Taking the structure of $\text{adj}\Lambda_0$ into account, it is easy to see that for $C = \text{adj} \Lambda_0$,

$$c_{\max} \leq (n-1)!\lambda_{\max}^{1+2+\cdots+(n-1)} = (n-1)!\lambda_{\max}^{\frac{n(n-1)}{2}}.$$  

Hence, by (7), we have

$$\|\Lambda_0^{-1}\| \leq \|\text{adj} \Lambda_0\| \leq n(n-1)!\lambda_{\max}^{\frac{n(n-1)}{2}}. \quad (8)$$

Consequently, (6) and (8) yield that

$$\|\Lambda_0 e^{Dt} \Lambda_0^{-1}\| \leq n(n-1)!\lambda_{\max}^{\frac{n(n-1)}{2}} e^{-\lambda_{\min} t},$$

where $\lambda_{\min} = \min\{|\lambda_1|, \ldots, |\lambda_n|\}$.

Suppose for some $\rho > 1$, $\lambda_{\max} \leq \rho \lambda_{\min}$. Then, it follows that

$$\|\Lambda_0 e^{Dt} \Lambda_0^{-1}\| \leq M \lambda_{\min}^{\frac{n(n-1)}{2}} e^{-\lambda_{\min} t}, \quad (9)$$

where

$$M = n(n-1)!\rho^{\frac{n(n-1)}{2}} \rho \lambda_{\min}^{\frac{n(n-1)}{2}}.$$  

In summary, we need the following conditions on the $\lambda_i$’s:

- $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct, real, and negative;
- $\lambda_{i+1} \leq -\lambda_i$ for $1 \leq i \leq n-1$, and hence, $\lambda_{\max} = |\lambda_n|$, $\lambda_{\min} = |\lambda_1|$;
- $|\lambda_n| \leq \rho |\lambda_1|$, for some constant $\rho > 1$.

Obviously, for any given $\lambda > 0$, it is easy to choose $\lambda_1, \ldots, \lambda_n$ to satisfy all the above conditions together with the condition that $\lambda_1 \leq -\lambda$. For example, one can choose $\lambda_1 < \min\{-1, -\lambda\}$, and let $\lambda_{i+1} = \lambda_i - 1$ for $1 \leq i \leq n-1$. Since $|\lambda_n| = |\lambda_1 - (n-1)| \leq n |\lambda_1|$, we see that $\rho$ can be set as $\rho = n$.

With such choices of $\lambda_1, \lambda_2, \ldots, \lambda_n$, we see from (5) and (9) that the desired result hold.

Now we consider the case when $(A, b)$ is not in the Brunovsky canonical form. In this case, find an invertible $T \in \mathbb{R}^{n \times n}$ such that $(T^{-1} A T, T^{-1} b)$ is in the Brunovsky canonical form.

For any given $\lambda > 0$, the above proof has shown that for $A_1 = T^{-1} A T$, $b_1 = T^{-1} b$, one can find $k_0 \in \mathbb{R}^{1 \times n}$ such that

$$e^{(A_1 + b_1 k_0) t} \leq M L e^{-\lambda t},$$

where $M$ is given by (10) for some chosen $\rho$, and $L = (n-1)(n+1)/2$. Clearly, with $k = k_0 T^{-1}$, one has

$$e^{(A + bk) t} = T e^{(A_1 + b_1 k_0) t} T^{-1} \leq M_1 L e^{-\lambda t}, \quad (11)$$

where $M_1 = M \|T\| \|T^{-1}\|$.

Finally, we consider the multi-input system

$$\dot{x} = Ax + Bu,$$  

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$. Suppose that the system is controllable. By Heymann’s Lemma (c.f., e.g., page 187 of [8]), one sees that for any $v \in \mathbb{R}^m$ such that $b := B v \neq 0$, there exists some $K_0 \in \mathbb{R}^{m \times n}$ such that $(A + B K_0, b)$ is itself controllable. Hence, the conclusion of single-input case that has just been proved above is applicable to the controllable pair $(A + B K_0, b)$, and one then sees that there exists some $k \in \mathbb{R}^{1 \times n}$ such that $\|e^{(A + B K_0 + bk) t}\| \leq M L e^{-\lambda t}$ for all $t \geq 0$. Hence, with $K = K_0 + v k$, it holds that

$$\|e^{(A + B K) t}\| \leq M L e^{-\lambda t} \quad \forall t \geq 0. \quad (13)$$

This completes the proof. \qed

Remark 2.2 In the above proof, we have used the fact that for a single input system $(A, b)$ which is controllable, when it is not in the Brunovsky canonical form, one can find an invertible matrix $T$ such that $(T^{-1} A T, T^{-1} b)$ is in the canonical form. To be more precise, the matrix $T$ can be chosen as (see e.g., [8]):

$$T = \begin{pmatrix} a_{n-1} & \cdots & a_1 & 1 \vline & 1 & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots \vline & 1 & 0 \\ a_1 & \cdots & 1 & \vline & 0 & 0 \\ 1 & 0 & \cdots & \vline & 0 & 0 \end{pmatrix},$$

where $a_1, \ldots, a_{n-1}$ are as in the characteristic polynomial of $A$ given by

$$\det(s I - A) = s^n + a_1 s^{n-1} + \cdots + a_{n-2} s + a_{n-1}.$$  

From this one can find an estimate of $\|T\|$ and $\|T^{-1}\|$, which in turn will lead to an estimate of $M_1$ in (11).

3 An Example

The design technique is demonstrated in the following example.

Example 3.1 Consider the following controllable linear system:

$$A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 2 & 1 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

With the help of MATLAB, we first calculate the transfer matrix

$$T_1 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$  

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With the transfer matrix $T_1$, one has

$$T_1^{-1} A_1 T_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 2 \end{pmatrix}, \quad T_1^{-1} B_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Calculation shows that $\|T_1\| = 1.80193754431757$ and $\|T_1^{-1}\| = 2.24697960199992$. Taking $\rho = n(= 3)$, we have

$$L = \frac{(n-1)(n+2)}{2} = 5,$$ (14)

$$M = \|T_1\|\|T_1^{-1}\|nn!n^{(n-1)(n-2)/2} \approx 218.642.$$ (15)

Suppose for some design purpose, a decay constant $\lambda = 49.894$ is given. Choosing $\lambda_1 = -\lambda$, $\lambda_2 = \lambda_1 - 1$, $\lambda_3 = \lambda_2 - 1$, the feedback $K_1$ can be easily calculated (under the normal form) as

$$\hat{K}_1 \approx \begin{pmatrix} -151.681 & -776.9474 & -131773.562 \end{pmatrix}.$$  

Back to the original coordinate frame, we have

$$K_1 = \hat{K}_1 T_1^{-1} \approx \begin{pmatrix} -124155.769 & 776.9474 & -761793.793 \end{pmatrix}.$$  

With such a choice of $K_1$, we get the desired decay estimate

$$\|e^{(A+B)K}t\| \leq M\lambda^L e^{-\lambda t} \quad \forall t \geq 0,$$

for the given decay constant $\lambda = 49.894$ with $L$ and $M$ given as in (14)–(15). \hfill \Box 

### 4 Conclusion

In this note we show that if $(A, B)$ is controllable, then for any $\lambda > 0$, a gain matrix $K$ can be chosen such that the transition matrix $e^{(A+B)K}t$ decays at the exponential rate $e^{-\lambda t}$ and the overshoot of $e^{(A+B)K}t$ can be bounded by $M\lambda^L$ for some constants $M$ and $L$ that are independent of the decay constant $\lambda$. The result provides a convenient tool for control design, particularly for switched systems, see [1].

### References

[1] D. Cheng, L. Guo, Y. Lin, Y. Wang, Stabilization of switched linear systems, *IEEE Trans. Autom. Contr.*, (accepted).

[2] L. Guo, Y. Wang, D. Cheng, Y. Lin, State feedback stabilization of switched linear systems, *Proc. 21st Chinese Control Conference*, Hangzhou, pp. 429–434, 2002.

[3] Y. Fang, K.A. Loparo, Stabilization of continuous-time jump linear systems, *IEEE Trans. Aut. Contr.*, Vol. 47, No. 10, pp. 1590-1603, 2002.

[4] Hespanha, J.P and A.S Morse, Stability of switched systems with average dwell-time, *Proc. of 38th CDC*, Phoenix, Arizona, pp. 2655-2660, 1999.

[5] Loan, C.V., The sensitivity of the matrix exponential, *SIAM J. Numer. Anal.*, Vol. 14, No. 6, pp. 971-981, 1977.

[6] Morse, A.S., Supervisory control of families of linear set-point controllers–part 1: exact matchings, *IEEE Trans. Automatic Control*, Vol. 41, pp. 1413–1431, October, 1996.

[7] Pait, F.M., A.S. Morse, A cyclic switching strategy for parameter-adaptive control, *IEEE Trans. Automatic Control*, Vol. 39, No. 6, pp. 1172-1183, 1994.

[8] Sontag, E.D., *Mathematical Control Theory: Deterministic Finite Dimensional Systems*, Springer-Verlag, New York, 2nd ed., 1998.

[9] Valcarce, R.L., S. Dasgupta, One property of the matrix exponential, *IEEE Trans. Circ. Sys. – Analog and Digital Signal*, Vol. 48, No. 2, pp. 213-215, 2001.

### Erratum

There is a mild flaw in the statement of Proposition 2.1 in the above paper (cf. [1]). We restate it as follows.

**Proposition** Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ be two matrices such that the pair $(A, B)$ is controllable. Then for any $\lambda \geq 1$, there exists a matrix $K \in \mathbb{R}^{m \times n}$ such that

$$\|e^{(A+B)K}t\| \leq M\lambda^L e^{-\lambda t}, \quad \forall t \geq 0,$$ (16)

where $L = (n-1)(n+2)/2$ and $M > 0$ is a constant, which is independent of $\lambda$ and can be estimated precisely in terms of $A, B$ and $n$.

The proof of Proposition 2.1 in [1] is only valid for the case when $\lambda \geq 1$ (instead of the original version of $\lambda > 0$), because the eigenvalues $\lambda_1, \ldots, \lambda_n$ were chosen to satisfy $\lambda_1 \leq -1$, and $\lambda_k \leq \lambda_1$ for $k \geq 1$. For more details, we refer the reader to the discussions that followed formula (10) in [1].

A main motivation of the work in [1] was for us to develop the results in [2]. As in most applications of overshoot estimation for pole placements, the parameter $\lambda$ in [2] was chosen as a number of large value. Hence, the correction does not affect our results in [2].

**Acknowledgment.** The authors would like to thank Prof. Elena De Santis for pointing out the error.