An alternative way to characterize a q-Gaussian distribution by a robust heavy tail measurement

E.L de Santa Helena, C. M. Nascimento and G. J. L. Gerhardt

Departamento de Física, Universidade Federal do Sergipe, Aracaju, Brazil.
Departamento de Física e Química, Universidade de Caxias do Sul, Brazil.

Abstract – The identification of a q-Gaussian distribution from empirical data is done through a measure of heavy tail using robust statistic. Numerical methods are used to generate artificial data, to figure out the tail weight and the curve fitting between the tail weight measurement and $q$ value. It is shown that this measure of tail weight is not changed when applied to a distribution originated from long memory processes with any Hurst exponent. A routine is created to calculate the $q$ value and its standard deviation from empirical data. We found that the calculation of tail weight done through robust statistic provides a value of $q$ with the same accuracy as compared to values usually found in the literature, using only 10% of sample. We hope that this method will open new opportunities for identifying physical phenomena that belongs to nonextensive frameworks.

Introduction. – It is sometimes a puzzling question to unveil the statistical description related to empirical data coming from physical systems. In general, to know the probability distribution function (PDF) related to the system, it is not sufficient to describe it precisely, and the informations about their correlation relationship are so important as the PDF itself. In this sense, in the last decades, many attempts to describe phenomena in several scientific fields were not done either by their PDF or by exact correlations, but in terms of their PDFs asymptotic behaviors. In most cases the PDFs related to the variables used to describe the physical systems obey a power law distribution. Power law distributions are present in many scientific fields such as physical [1]; science [2]; Geophysical [4]; social science [5,6], ecology [8] and economics [9]. In many other classes of systems, it is possible to associate an appropriate “scale parameter” to a PDF and a power law asymptotic behaviour can not be observed (see discussion in ref. [10]). Although in both cases the PDFs associated to the dynamics variable are stable, in the first case the second moment is not well defined [11], and this is a necessary condition for the observation of the power law. See refs. [10,12] for a great discussion concerning this kind of approach and its observation in an extraordinarily diverse range of phenomena. Still in this line of approach, the description of several phenomena have been done by q-Gaussians distributions [13]. These distributions are also a class of stable distribution but in these case the usual requirement of independent dynamical variables is not necessary [14].

The q-Gaussian probability density function [13], usually named qPDF, with $q$-mean $\mu_q$ and $q$-variance $\sigma_q$ is:

$$\rho(x; \mu_q, \sigma_q) = \alpha \sqrt{\frac{\beta}{\pi}} \left[ 1 + \left( \frac{x - \mu_q}{\sigma_q} \right)^2 \right]^{1/(1-q)}$$  \hspace{1cm} (1)

where $\beta = [(3-q)\sigma_q^2]^{-1}$ and

$$\alpha = \begin{cases} 
\sqrt{1-q}I_0^2(\frac{3-2q}{2-q})/\Gamma(\frac{2-q}{2}) & \text{if } q < 1, \\
1 & \text{if } q = 1, \\
\sqrt{q}I_0^2(\frac{1}{q-1})/\Gamma(\frac{3-2q}{2}) & \text{if } 1 < q < 3.
\end{cases}$$ \hspace{1cm} (2)

Let us make the following convenient replacement of variable, $Z = (q-1)/(3-q)$ and assume $\mu_q = 0, \sigma_q = 1$. 

PACS 02.50.-r – Probability theory, stochastic processes, and statistics
PACS 89.75.-k – Complex Systems
PACS 02.60.-x – Numerical approximation and analysis
to write down the standard q-Gaussian like

$$\rho(x; 0, 1) = \alpha^\prime \sqrt{\frac{2}{\pi}} [1 + Zx^2]^{-(q + 1)/2}$$

where $\alpha^\prime = \alpha / \sqrt{(q - 1)}$. In the limit of $q \rightarrow 1$ a qPDF tends to a standard Gaussian distribution, but for $q < 1$ the qPDF has compact support, and on the other hand, in case of $1 < q < 3$ the qPDF has a heavy tail, i.e., the power law asymptotic behavior is also well described by this class of distribution.

A weakness in this approach (as in any other statistical approach), is the difficulty in associating a q value (or several q values) to the system. In most cases, this limitation arises due to the poorness of statistical data (limitation on size of the random realizations of the dynamical variables). In such cases, the q value is achieved only via adjustment (fitting). In this paper, we propose an alternative way of estimating the q value with a precision at least equal to that obtained using the standard statistical method, medcouple. We show that medcouple works very precisely even in cases where the number of data is limited, and the traditional method fail.

In this work we present a way to obtain a q value from medcouple, a measurement of tail weights. In the theoretical background we presente (i) the medcouple and (ii) the long memory process definition. In the following section we describe how the numerical calculations are carried on.

**Theoretical Background.**

**Medcouple.** The kurtosis is a classical measurement of tail weight of a distribution that is very sensitive to outlying values. Outliers occur in the data set due to measurement error or contamination and may become more apparent when the sample size is small. The robust statistics [15] seeks to estimate moments and derived quantities to nicely fit the bulk of the data when the data contain, or not, outliers. The median is an example of a robust estimator of the data middle and a measurement using this estimate should help to identify a heavy tail distribution. Since we are seeking a method to identify a q-Gaussian from the experimental data, we will use the medcouple originally introduced by Brys et al. [16] to quantify skewness as follows. Given a sorted sample $\{x_1 < \ldots < x_n\}$ from a univariate distribution, the kernel function is defined as:

$$h(x_i, x_j) = \frac{(x_i - \tilde{x}) - (x - x_i)}{(x_j - x_i)}$$

that measures the distances from $x_i$ and $x_j$ to the median $\tilde{x}$. Remembering the median definition $\tilde{x}$ applied to this set:

$$med(x) = \begin{cases} x_{(n+1)/2} & \text{if } n \text{ is odd} \\ (x_{n/2} + x_{(n/2)+1})/2 & \text{if } n \text{ is even} \end{cases}$$

the medcouple is defined as:

$$MC = med_{i \leq j \leq x} h(x_i, x_j)$$

applied to all pairs that satisfied condition $x_i \leq \tilde{x} \leq x_j$. The main feature of this measure is that it is invariant under scale and location changes. For more details, see [16].

The application of medcouple in each side of the distribution leads to two measures which allow quantification of the tail weight [17].

$$LMC = -MC(\{x_1 < \ldots < \tilde{x}\})$$

and

$$RMC = MC(\{\tilde{x} < \ldots < x_n\})$$

the left and right medcouple. In the case of symmetric distributions, both measures are equivalent.

**Long Memory process and Self-Similarity.** A stochastic process characterized by a probability density function (PDF) $Y_t$ (t is the time parameter) is called self-similar when the rescaled PDF $c^{-H}Y_{ct}$ (time scale ct and c > 0) presents the PDF of the original process [18].

We will consider only $Y_t$ self-similar with stationary increments, $X_i = Y_i - Y_{i-1} = 0$ for $i = 1, 2, \ldots$, where, for time lags ($n > 0$), the covariance between $X_i$ and $X_{i+n}$ can be shown to be

$$\gamma(n) = \frac{1}{2} \sigma^2 [n + 1]^{2H} - 2 |n|^{2H} + |n - 1|^{2H}$$

where $H$ is known as the Hurst exponent, and $\sigma^2$ is the variance of the increment process $X_i = Y_i - Y_{i-1}$. $\gamma(n)$ is non-negative only when $0 < H < 1$ and can be seen in this case as a legitimate covariance [19]. A long memory process with covariance given by eq. [19] is called fractional Gaussian noise and the corresponding self-similar process $Y_t$ is called fractional Brownian motion.

The correlation $\rho(n) = \gamma(n)/\sigma^2$ has the asymptotic behavior $\rho(n) = H(2H - 1)n^{2H-2}$ from which we conclude that $\lim_{n \rightarrow \infty} \rho(n) = 0$ when $H < 1$. When $H = 1/2$, the process $X_t$ is uncorrelated because $\rho(n) = 0$ for any lag $n \neq 0$. For $1/2 < H < 1$ the process has long range memory since $\sum \rho(n) = \infty$ for all $n$, and for $0 < H < 1/2$ the process has short range dependence because $\sum \rho(n) = 0$.

Samorodnitsky [19] draws attention to the important issue that a long memory increment process gradually seems to stop showing the stationary behavior when the correlation, measured by $H$, is far from a half. This caveat will be used in a careful choice of a long memory process.
Given the covariance matrix Σ,

\[
\Sigma = \begin{pmatrix}
\gamma_0 & \gamma_1 & \cdots & \gamma_{N-1} \\
\gamma_1 & \gamma_0 & \cdots & \gamma_{N-2} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{N-1} & \gamma_{N-2} & \cdots & \gamma_0
\end{pmatrix}
\]

(10)

where the elements of the matrix γₙ are obtained from eq. 9, we obtain L by Choleski factorization \(\Sigma = LL'\) where L is lower triangular.

For artificially generating a long memory process [21], \(\tilde{X}_c\), we multiply a vector of an uncorrelated stationary process \(\tilde{X}\) by a transformation matrix L.

\[
\tilde{X}_c = L\tilde{X}
\]

(11)

To our knowledge, there is no fast algorithm to generate long-memory q-Gaussian noise.

**Numerical Calculus.** — The box-muler algorithm was implemented as described in [20] using the Mersenne-Twister algorithm as a random number generator in R [22] to generate the q-Gaussian probability density function (qPDF), eq 6. For any distinct q value, this procedure allowed us to create \(K = 2^n\) artificial time series \(X_k(q)\) (replications) in three length scales \(N = \{2^{13}, 2^{14}, 2^{15}\}\).

The Robustbase [23] is a robust statistical package that implements the calculation of medcouple as described in [17], was implemented in a straightforward way (see Appendix). As we are only dealing with symmetric distributions, the RMC choice was made through a coin toss. Time series with different lengths \(N\) were used to estimate the RMC standard deviation, while only the series with \(N = 2^{14}\) were used in curve fittings by non-linear least-squares. We chose a representative set of q values in the range \(-1 < Z(q) < 6\) to build up \(m(q)\) PDFs of random variable \(m_k(q) = RMC(X_k(q)), k = 1, 2, \ldots, K\). This is the first time it is shown that the medcouple may be used to characterize distributions of compact support.

The Choleski factorization is used to generate stationary long memory process. Since this method is computationally heavy, we need to keep the length of the vector in an acceptable computational size. Furthermore, to figure out what the covariance matrix elements \(\gamma\) are, we use a \(H\) value slightly larger than a half to avoid numerical problems and to ensure a generation of a stationary long memory increment process.

**Results.** — We verified through a Shapiro Test that the \(m(q)\) PDFs \(\{m_1(q), \ldots, m_K(q)\}\) have a gaussian shape, and this ensures its mean, \(m\), to be an accurate measurement of more probable value. On the other hand, a median is more accurate when \(m(q)\) PDFs has a poisson shape, which occurs at \(-q < O(10)\).

In fig. 1 we can see \(m\) versus \(Z\) that gives us an idea of how to make a good curve fitting. In fig. 2 we can see numerical values of \(dm/dZ\) versus \(Z\), which allows us to infer the existence of an inflection point around \(Z = 0.5(q = 5/3)\). We judge convenient to split the curve fitting in two parts, \(-\infty < q < 5/3\) and \(5/3 < q < 3\) without any difficulty. This procedure aims at providing that the ansatz

\[
m(Z) = \tanh[a + bZ + cZ^2],
\]

(12)

ensure a smooth curve fit that was done with the parameters set \(a, b, c\) listed in table 1.

![Fig. 1: The mean value \(m\) as a function of \(Z = (q-1)/(3-q)\). At \(Z = 0.5\) \((q = 5/3)\) there is an inflection point that separates the two parts of the curve fitting eq. (12) according to the values of table 1.](image)

| \(q < 5/3\) | \(5/3 < q < 3\) |
|-----------------|-----------------|
| \(a\) | \(0.20177505633664\) | \(0.170714501818342\) |
| \(b\) | \(0.282139173677106\) | \(0.387670970311634\) |
| \(c\) | \(0.083140836156118\) | \(-0.008371647346576\) |

Table 1: Adjust parameters values

Theses curve fitting allows us to calculate an adjusted \(q\) value:

\[
q(m) = 3 - \frac{2}{1 + Z(m)}
\]

(13)

with \(Z(m)\) obtained from eq. (12).

It was observed that the behavior of the standard deviation of \(m(q = 1.33)\) as a function of increasing \(K\) (replications) decreases quadratically to a value around \(K \approx 2^8\) and beyond this point it decreases linearly. Therefore, it is enough to have \(K = 2^9\) to obtain a good estimate of the \(m\) value standard deviation, \(\delta m\) and to assume it as an asymptotic value \((K \to \infty)\).

For each one of the \(m(q)\) PDFs, generated from the time series with different lengths \(\{2^{13}, 2^{14}, 2^{15}\}\), the standard
deviation $\delta m$ was calculated and a graph was drawn as shown in fig. 3. Starting from the different scales we can collapse the data and adjust a relationship between $\delta m$ and $(N, q)$ as follows:

$$\delta m \approx \frac{e^{0.5}}{\sqrt{N}} \times \begin{cases} 1 & \text{if } q \leq 5/3 \\ 0.5(q-5/3) & \text{if } q > 5/3 \end{cases} \quad (14)$$

The $q$ value standard deviation $\delta q$ was estimated by the usual process of error propagation

$$\delta q = \left| \frac{dq}{dm} \right| \delta m. \quad (15)$$

Since $\delta m$ is fairly constant for $q < 5/3$ as can be seen in fig. 3, the $\delta q$ is governed by the factor $\left| \frac{dq}{dm} \right|$ that tends to increase, despite the fact that $\left| \frac{dZ}{dm} \right| \to 0$ as $q \to -\infty$ ($Z \to -1$). This behavior may be inferred from eq. (13) and fig. 2. However, in practice, we do not need to worry about this because experimental data always have some degree of contamination with outliers.

For symmetric PDFs, we can enhance the $q$ value estimate, including the LMC measurements in calculating the mean and standard deviation of $m$. In this case, the accuracy is increased since it seems as if the size sample, $N$, is doubled. In the appendix we present a $R$ routine that implements the $q$ value estimate and standard deviation.

Many empirical data exhibit long-memory such as financial assets like stocks market returns and currency pairs series. Furthermore, they exhibit $q$-Gaussian behavior. Therefore, it is useful to verify the behavior of $\delta m$ and $(N, q)$ as a function of $(N, q)$. The points of the upper curve correspond to $N = 2^{13}$ and those of the lower curve to the values of $N = 2^{15}$. $\delta m$ is constant only until around $q = 5/3$ and its variability is inversely proportional to $N$ size.

Table 2: Mean and (standard deviation) of $m$ value calculated from uncorrelated and correlated time series

| $q$   | H   | 0    | 1.04 | 1.69 |
|-------|-----|------|------|------|
| 0.5   | .115(.015) | .206(.012) | .358(.011) |
| 0.567 | .117(.012) | .205(.012) | .349(.012) |

Discussion and Conclusions. – Usually to obtain a reliable fitting of a $q$-Gaussian distribution to the empirical data, a large amount of data is needed. In geophysics, for example, the author [24] used 400,000 earthquakes to obtain $q = 1.75 \pm 0.15$ from PDFs of the energy differences and used $10^9$ avalanches to obtain a PDF of the avalanche size differences characterized by $q = 1.75 \pm 0.15$. From 2.5 $\times$ $10^6$ values of temperature fluctuation obtained from WMAP [27], it was possible to adjust a nonextensive distribution with $q = 1.04 \pm 0.01$. Liu et al. [25] measured the distribution of position of $10^7$ particles immersed in a plasma and have identified an anomalous diffusion process. After using a low and high energy laser to heat the samples, they obtained distributions of posi-
tions fitted with $q = 1.08 \pm 0.01$ and $q = 1.05 \pm 0.014$, respectively. In economics, the authors [13] discuss how q-Gaussian distributions fit very well empirical distributions of returns SP500 stocks index. For the empirical returns stocks volumes from NASDAQ and NYSE they found $q = 1.41, 1.44, 1.43$ for fitting $10^6$ data points with $\Delta t = 1, 2, 3$ min. In these cases, it is reasonable to assume that the uncertainty is in the range of $\pm 0.01$.

It is worth calling attention that the q-Gaussian behavior could arise from a normalization process applied to the empirical data [30]. This spurious behavior is not observed because medcouple does not need a normalized data set.

Furthermore, non-Gaussianity can arise as a finite-size effect in a data analysis [29]. Our method is somewhat less influenced by finite-size effects compared to the usual method of curve fitting with $10^6$ points, since it is possible to obtain values of $q$ with accuracy of $\pm 0.01$ using samples with only $10^5$ points. Moreover, the proposed method is not affected if the data have long memory. Taking all this into account, our method opens new perspectives for identifying phenomena within nonextensive frameworks. The goal is not to replace the classical distribution fitting but rather to supplement it since this loses computational efficiency for samples with more than $10^6$ data points.

Appendix. – The function \texttt{q GMC} to calculate the value of $q$ and its standard deviation is shown below:

```r
library(robustbase)
qgmc=function(x){
  N=length(x)
  yy=cut(x, c(min(x)-1,median(x), max(x)),label=c(0,1))
  mm=data.frame(x,yy)
  vmcl=abs(by(mm[,1],factor(mm[,2]),mc)[2])
  vmcr=abs(by(mm[,1],factor(mm[,2]),mc)[1])
  vmax=(vmcl+vmcr)/2
  if (vmct > 0.348) j=c(0.1797145,.38767097,-.00837164)
  else dm=exp(.5)/sqrt(N)*.5
  dqdm=cosh(j[1]+j[2]*Z+j[3]*Z^2)/((j[2]+2*j[3]*Z+2*j[3]^2+1+Z)^2)
  attr(dq, 'names' )= c('Estimate', 'Std. Error')
  return(c(qv,dq))
}
```

REFERENCES

[1] Sornette D., Critical Phenomena in Natural Sciences: Chaos, Fractals, Selforganization and Disorder: Concepts and Tools, edited by SPRINGER 2000
[2] Lotka A. J., J. Wash. Acad. Sci., 16 (1926) 317.
[3] de S. Price D. J., Science, 149 (1965) 510.
[4] Gutenberg B. and Richter R. F., Bull. Seismol. Soc. Am., 34 (1944) 185.
[5] Zipf G. K., Human Behaviour and the Principle of Least Effort, edited by ADDISON-WESLEY 1949
[6] Estoup J. B., Gammes Stenographiques, edited by INSTITUT STENOGRAPHIQUE DE FRANCE 1916
[7] Levy J. S., War in the Modern Great Power System 1495-1975, edited by UNIVERSITY OF KENTUCKY PRESS 1983.
[8] Viswanathan G. M., Afanasyev V., Buldyrev S. V., Murphy E. J., Prince P. A., and Stanley, H. E., Nature, 381 (1996) 413.
[9] Mantegna R. N. and Stanley, H. E., Nature, 376 (2002) 46.
[10] Newman M. E. J., Contemporary Physics, 46 (2005) 323.
[11] Mantegna R. N., Stanley H. E., Introduction to Econophysics: Correlations and Complexity in Finance, edited by CAMBRIDGE UNIVERSITY PRESS 2000
[12] Buchanan M., Ubiquity: The Science of History Or Why the World is Simpler than we Think, edited by WEIDENFELD & NICOLSON 2000
[13] Tsallis C., Introduction to Nonextensive Statistical Mechanics, edited by SPRINGER 2009
[14] Umarov, S., Tsallis, C. and Steinberg, S., Milan Journal of Mathematics., 76 (2008) 307.
[15] Maronna R.A., Martin R.D. and Yohai V.J., Robust Statistics: Theory and Methods, edited by SPRINGER 2006
[16] Brys G., Hubert M. and Struyf A., J. Comput. Graphical Statist., 13(4) (2004) 1.
[17] Brys G., Hubert M. and Struyf A., Comput. Statist. Data Anal., 50 (2006) 733.
[18] Beran J., Statistics for Long-Memory Processes, Monographs on Statistics and Applied Probability, edited by CHAPMAN AND HALL 1994
[19] Samorodnitsky G., Annales de la Faculte des Sciences de Toulouse, 15 (2006) 107.
[20] Thistleton W., Marsh J.A., Nelson K. and Tsallis C., IEEE Transactions on Information Theory, 53(12) (2007) 4805.
[21] Diebold F.X. and Rudebusch G.D., Economics Letters, 35 (1991) 155.
[22] R Development Core Team, R: A language and environment for statistical computing (R Foundation for Statistical Computing, Vienna, Austria ) 2003, p. ISBN 3-900051-00-3.
[23] Rousseuw P., Croux C., Todorov V., Ruckstuhl A., Salibian-Barrera M., Verbeke T., Koller M., Maechler M., (robustbase): Basic Robust Statistics 2014 (http://CRAN.R-project.org/package=robustbase).
[24] Caruso F., Pluchino A., Latora V., Vinciguerra S. and Rapisarda A., Phys. Rev. E, 75 (2007) 055101(R).
[25] Liu B. and Goree J., Phys. Rev. Lett, 100 (2008) 055003.
[26] Cortines A.A.G. and Riera R., Physica A, 377 (2007) 181.
[27] Bernui A., Tsallis C. and Villela T., Europhys. Lett., 78 (2007) 19001.
[28] Osorio R., Borland L. and Tsallis C., Nonextensive Entropy: Interdisciplinary Applications, edited by C. Tsallis and M. Gell-Mann (Santa Fe Institute Studies in the Science of Complexity, Oxford) 2004, p. 321.
[29] Milotti E., Phys Rev. E, 83 (2011) 042103.
[30] Vignat C. and Plastino A., Physica A, 388 (2009) 601.