GLOBAL EXISTENCE AND DECAY RATES OF THE SOLUTIONS FOR A CHEMOTAXIS SYSTEM WITH LOTKA-VOLTERRA TYPE MODEL FOR CHEMOATTRACTANT AND REPELLENT

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Abstract. We study global existence and asymptotic behavior of the solutions for a chemotaxis system with chemoattractant and repellent in three dimensions. To accomplish this, we use the Fourier transform and energy method. We consider the case when the mass is conserved and we use the Lotka-Volterra type model for chemoattractant and repellent. Also, we establish $L^q$ time-decay for the linear homogeneous system by using a Fourier transform and finding Green’s matrix. Then, we find $L^q$ time-decay for the nonlinear system using solution representation by Duhamel’s principle and time-weighted estimates.

1. Introduction. We consider the initial value problem of the system in $\mathbb{R}^3$

$$
\begin{align*}
\partial_t n + \nabla \cdot (nu) &= 0 \\
\partial_t (nu) + \nabla \cdot (nu \otimes u + p(n)) &= n(\nabla c_1 - \nabla c_2) - \nu nu \\
\partial_t c_1 &= \Delta c_1 + c_1(-a_1 + a_{11}c_1 + a_{12}c_2 + a_{13}n) \\
\partial_t c_2 &= \Delta c_2 + c_2(-a_2 + a_{21}c_1 + a_{22}c_2 + a_{23}n),
\end{align*}
$$

(1.1)

where $a_1 > 0, a_2 > 0$. The signs of the other coefficients are generally given as

- $a_{11} > 0, a_{12} < 0, a_{13} \geq 0,$
- $a_{21} < 0, a_{22} > 0, a_{23} \geq 0.$

The initial data is given by

$$
(n, u, c_1, c_2) \mid_{t=0} = (n_0, u_0, c_{1,0}, c_{2,0})(x), \ x \in \mathbb{R}^3.
$$

(1.2)

The movement of an organism or a cell in response to a chemical stimulus, often in the direction corresponding to a gradient of increasing or decreasing concentration of a particular substance, is referred as chemotaxis. In the above system, the first two equations are the conservation of mass and momentum for the cells. In angiogenesis or vasculogenesis the cells are the endothelial cells constituting blood vessels. The gradients of $c_1$ and $c_2$ cause cells to grow toward and away from the higher density of $c_1$ and $c_2$, respectively. For this reason, they are called attractants and repellents, respectively. In this paper, we use the reaction-diffusion equations

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for them, and for the interactions among them and cells we use a Lotka-Volterra type competitive model. An example of an attractant is the vascular endothelial growth factor (VEGF) and it is a signal protein produced by cancer cells that stimulate the formation of blood vessels. An example of repellent can be anti-VEGF medications that block VEGF. We consider both VEGF and anti-VEGF, and use the reaction-diffusion equations for them. For the interaction between substances and cells, and there are constants \( \lambda \) obtained above satisfies for \( t \geq t_0 \) with \( t_0 > 0 \) a sufficiently large time that:

\[
\|n - n_{\infty}\|_{L^q} \leq C(1 + t)^{\frac{\alpha}{2} + \frac{q}{p}}, \tag{1.6}
\]

\[
\|u\|_{L^q} \leq C(1 + t)^{\frac{\alpha}{2} + \frac{q}{p}}, \tag{1.7}
\]

Subtracting the first equation from the second equation, we will consider the simplified chemotaxis fluid equations taking the following form

\[
\begin{aligned}
\partial_t n + \nabla \cdot (nu) &= 0, \\
\partial_t u + u \cdot \nabla u + \nabla p(n) &= \nabla c_1 - \nabla c_2 - \nu u, \\
\partial_t c_1 &= \Delta c_1 - a_1c_1 + a_{11}c_1^2 + a_{12}c_1c_2 + a_{13}c_1n, \\
\partial_t c_2 &= \Delta c_2 - a_2c_2 + a_{21}c_1c_2 + a_{22}c_2^2 + a_{23}c_2n,
\end{aligned}
\tag{1.3}
\]

with initial data

\[
(n, u, c_1, c_2) \mid_{t=0} = (n_0, u_0, c_{1,0}, c_{2,0})(x), \ x \in \mathbb{R}^3.
\tag{1.4}
\]

(\(n_0, u_0, c_{1,0}, c_{2,0}\))(\(x\)) \to (n_{\infty}, 0, 0, 0) as \( |x| \to \infty \), for some constant \( n_{\infty} > 0 \).

Throughout this paper, we assume the following: \( p(.) \) is the smooth function of \( n \) and \( p'(n) > 0 \). The main goals of this paper are to show the local and global existence of solutions in \( H^N(\mathbb{R}^3) \) and \( L^q \) time-decay rates of solutions for the Cauchy problem for the above system (1.1)-(1.2). The main result of this paper is stated as follows.

**Theorem 1.1.** Let \( N \geq 4 \). There exists a positive number \( \epsilon_0 \) such that if

\[
\|n_0 - n_{\infty}, u_0, c_{1,0}, c_{2,0}\|_{H^N} \leq \epsilon_0,
\]

the Cauchy problem (1.3)-(1.4) has a unique solution \((n, u, c_1, c_2)(t)\) globally in time which satisfies

\[
(n - n_{\infty}, u)(t) \in C([0, \infty); H^N(\mathbb{R}^3)) \cap C^1([0, \infty); H^{N-1}(\mathbb{R}^3)),
\]

\[
(c_1, c_2)(t) \in C([0, \infty); H^N(\mathbb{R}^3)) \cap C^1([0, \infty); H^{N-2}(\mathbb{R}^3))
\]

and there are constants \( \lambda_1 > 0, \lambda_2 > 0, \lambda_3 > 0 \) and \( C_0 > 0 \) such that

\[
\|n - n_{\infty}, u, c_1, c_2\|_{H^N}^2 + \lambda_1 \int_0^t \|\nabla(n - n_{\infty})\|_{H^{N-1}}^2 + \lambda_2 \int_0^t \|\nabla(c_1, c_2)\|_{H^N}^2 + \lambda_3 \int_0^t \|u, c_1, c_2\|_{H^N}^2 \leq C_0\|n_0 - n_{\infty}, u_0, c_{1,0}, c_{2,0}\|_{H^N}^2.
\tag{1.5}
\]

Moreover, if \((n_0 - n_{\infty}, u_0, c_{1,0}, c_{2,0}) \in L^1(\mathbb{R}^3)\), the global solution \([n, u, c_1, c_2]\) obtained above satisfies for \( t \geq t_0 \) with \( t_0 > 0 \) a sufficiently large time that:

\[
\|n - n_{\infty}\|_{L^q} \leq C(1 + t)^{\frac{\alpha}{2} + \frac{q}{p}},
\tag{1.6}
\]

\[
\|u\|_{L^q} \leq C(1 + t)^{\frac{\alpha}{2} + \frac{q}{p}},
\tag{1.7}
\]
\[(c_1, c_2) \|_{L^q} \leq C(1 + t)^{-2}, \quad (1.8)\]

with \(2 \leq q \leq \infty\), where \(C > 0\) is a positive constant independent of time.

The proof of the existence of global solutions in Theorem 1.1 is based on the local existence and an a priori estimate. The local existence can be proved by constructing a sequence of approximation functions based on an iteration by following the methods in Kato [10] and Majda [16]. The a priori estimate can be obtained by the energy method. Moreover, to obtain the time-decay rate in \(L^q\) norm of solutions in Theorem 1.1, our approach is a combined analysis of Green’s function of the linear system and the refined energy estimates with the help of Duhamel’s principle. We obtain Green’s matrix of the linear system by Fourier transform.

We mention some previous related works about chemotaxis models. Such chemotaxis models are based on Keller and Segel [11, 12]. Wang [20] explored the interactions between the nonlinear diffusion and logistic source on the solutions of the attraction-repulsion chemotaxis system in three dimensions. E. Lankeit and J. Lankeit [13] proved the global existence of classical solutions to a chemotaxis system with singular sensitivity. Liu and Wang [14] established the existence of global classical solutions and steady states to an attraction-repulsion chemotaxis model in one dimension based on the method of energy estimates. Luca, Chavez-Ross, Edelstein-Keshet, and Mogilner [15] investigated conditions that lead to aggregation of microglia and developed a model for chemotaxis in response to a combination of chemoattractant and chemorepellent signaling chemicals.

Concerning the chemotaxis models based on fluid dynamics, there are two approaches, incompressible and compressible. For the incompressible case, Chae, Kang and Lee [5], and Duan, Lorz, and Markowich [8] showed the global-in-time existence for the incompressible chemotaxis equations near the constant states, if the initial data is sufficiently small. Rodríguez, Ferreira and Villamizar-Roa [17] showed the global existence for an attraction-repulsion chemotaxis fluid model with a logistic source. Tan and Zhou [19] proved the global existence and time-decay estimate of solutions to the Keller-Segel system in \(R^3\) with small initial data. For the compressible case, Ambrosi, Bussolino, and Preziosi [1] discussed the vasculogenesis using the compressible fluid dynamics for the cells and the diffusion equation for the attractant. Modeling aspects of vasculogenesis are studied in [2, 9, 18].

Many related approaches that use Fourier transform, and we only mention that Duan [6] and Duan, Liu, and Zhu [7] proved the time-decay rate by the combination of energy estimates and spectral analysis.

Lotka-Volterra models for the ordinary differential equations (ODEs) are well-known in Biology and generally written as
\[
\begin{align*}
\frac{dc_1}{dt} &= c_1(-a_1 + a_{11}c_1 + a_{12}c_2) \\
\frac{dc_2}{dt} &= c_2(-a_2 + a_{21}c_1 + a_{22}c_2).
\end{align*}
\]

An example is a predator-prey model. The classifications of the flows of the model are discussed in [3, 4]. One characteristic of the model is that \(c_i (i = 1, 2)\) are factors for the equations for \(c_i\), respectively. This makes the nonnegative region of \(c_i (i = 1, 2)\) invariant, which means the solutions starting in the nonnegative region of \(c_i (i = 1, 2)\) will stay in that region for positive \(t\). By the maximum principle it is not difficult to show that the same observation holds true for the \(c_i (i = 1, 2)\) in (1.1) provided that they are \(C^2\)-solutions.
For later use in this paper, we give some notations. \( C \) denotes some positive constant, \( \lambda_i \), where \( i = 1, 2 \), denotes some positive (generally small) constant, where both \( C \) and \( \lambda_i \) may take different values in different places. For any integer \( m \geq 0 \), we use \( H^m \) to denote the Sobolev space \( H^m(\mathbb{R}^d) \) and \( H_m \) the \( m \)-th order homogeneous Sobolev space. Set \( L^2 = H^0 \). For simplicity, the norm of \( H^m \) is denoted by \( \| \cdot \|_m \) with \( \| \cdot \| = \| \cdot \|_0 \). We set \( \partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3} \) for a multi-index \( \alpha = [\alpha_1, \alpha_2, \alpha_3] \). The length of \( \alpha \) is \( |\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \) and we also set \( \partial_j = \partial_{x_j} \) for \( j = 1, 2, 3 \). For an integrable function \( f : \mathbb{R}^3 \to \mathbb{R} \), its Fourier transform is defined by \( \hat{f} = \int_{\mathbb{R}^3} e^{-ix \cdot \xi} f(x) dx \), \( x \cdot \xi = \sum_{i=0}^3 x_i \xi_i \), \( x, \xi \in \mathbb{R}^3 \), where \( i = \sqrt{-1} \) is the imaginary unit. Let us denote the space
\[
X(0, T) = \{ \{ \rho, u \} \in C([0, T]; H^N(\mathbb{R}^3)) \cap C^1 ([0, T]; H^{N-1}(\mathbb{R}^3)) \}
\]
\[
\quad (c_1, c_2) \in C([0, T]; H^N(\mathbb{R}^3)) \cap C^1 ([0, T]; H^{N-2}(\mathbb{R}^3)).
\]

This paper is organized as follows. In section 2, we reformulate the Cauchy problem under consideration. In section 3, we prove the global existence and uniqueness of solutions. In section 4, we investigate the linearized homogeneous system to obtain the \( L^q \) time-decay property and the explicit representation of solutions. In section 5, we study the \( L^q \) time-decay rates of solutions to the reformulated nonlinear system and finish the proof of Theorem 1.1.

2. Reformulation of the system (1.3). Let \( U(t) = [n, u, c_1, c_2] \) be a smooth solution to the Cauchy problem of the chemotaxis fluid equations (1.3) with initial data \( U_0 = [n_0, u_0, c_{1,0}, c_{2,0}] \). Set
\[
n(x, t) = \rho(x, t) + n_\infty.
\]
Then the Cauchy problem (1.3)-(1.4) are reformulated as
\[
\begin{aligned}
\partial_t \rho + n_\infty \nabla \cdot u &= -\nabla \cdot (\rho u) \\
\partial_t u + u \cdot \nabla u + \nu u + \frac{\rho u n_\infty}{\rho + n_\infty} \nabla \rho &= \nabla c_1 - \nabla c_2 \\
\partial_t c_1 - \Delta c_1 + (a_1 - a_1 n_\infty) c_1 &= a_{11} c_1^2 + a_{12} c_1 c_2 + a_{13} c_1 \rho \\
\partial_t c_2 - \Delta c_2 + (a_2 - a_2 n_\infty) c_2 &= a_{21} c_1 c_2 + a_{22} c_2^2 + a_{23} c_2 \rho
\end{aligned}
\]
(2.2)
with initial data
\[
(\rho, u, c_1, c_2) \big|_{t=0} = (\rho_0, u_0, c_{1,0}, c_{2,0}) \to (0, 0, 0, 0),
\]
(2.3)
as \( |x| \to \infty \), where \( \rho_0 = n_0 - n_\infty \). We assume that \( (a_{12} - a_1 n_\infty a_{11}) > 0 \) and \( (a_{22} - a_2 n_\infty a_{21}) > 0 \). In the following, we set \( N \geq 4 \). Besides, for \( U = [\rho, u, c_1, c_2] \), we use \( \mathcal{E}_N(U(t)) \) to denote the energy functional and \( \mathcal{D}_N(U(t)) \) the dissipation rates. Here,
\[
\mathcal{E}_N(U(t)) \sim \| [\rho, u, c_1, c_2] \|^2_N,
\]
(2.4)
\[
\mathcal{D}_N(U(t)) \sim \| \nabla (c_1, c_2) \|^2_N,
\]
(2.5)
and
\[
\mathcal{D}_N^H(U(t)) \sim \| [\nabla \rho] \|^2_{N-1} + \| [u, c_1, c_2] \|^2_N.
\]
(2.6)
Then, concerning the reformulated Cauchy problem (2.2)-(2.3), one has the following global existence result.

**Proposition 1.** Suppose that \( \| [\rho_0, u_0, c_{1,0}, c_{2,0}] \|_H^N \) is sufficiently small. Then, the Cauchy problem (2.2)-(2.3) has a unique solution \( U(t) = (\rho, u, c_1, c_2)(t) \) globally in time which satisfies \( U(t) \in X(0, \infty) \) and
\[ E_N(U(t)) + \lambda_1 \int_0^t D_N(U(t))ds + \lambda_2 \int_0^t D_N^2(U(t))ds \leq C_0 E_N(U_0), \tag{2.7} \]

for any \( t \geq 0 \).

Moreover, the solutions obtained in Proposition 1 indeed have the decay rates in time under some extra conditions on the initial data. For that, given \( U_0 = [\rho_0, u_0, c_{1,0}, c_{2,0}] \), set \( \epsilon_N(U_0) \) as

\[ \epsilon_N(U_0) = \|U_0\|_N + \|U_0\|_{L^1}, \tag{2.8} \]

for \( N \geq 4 \). Then, we have the following two Propositions:

**Proposition 2.** Let \( U = [\rho, u, c_1, c_2] \) be the solution to the Cauchy problem (2.2) with initial data \( U_0 = (\rho_0, u_0, c_{1,0}, c_{2,0}) \). If \( \epsilon_{N+1}(U_0) > 0 \) is sufficiently small, then the solution \( U = [\rho, u, c_1, c_2] \) satisfies

\[ \|U(t)\|_N \leq \epsilon_{N+1}(U_0)(1 + t)^{-\frac{2}{N}}, \tag{2.9} \]

and

\[ \|\nabla U(t)\|_N \leq \epsilon_{N+1}(U_0)(1 + t)^{-\frac{2}{N}}, \tag{2.10} \]

for any \( t \geq 0 \).

**Proposition 3.** Let \( 2 \leq q \leq \infty \). Suppose that \( U(t) = [\rho, u, c_1, c_2] \) is the solution to the Cauchy problem (2.2)-(2.3) obtained in Proposition 1. Then the solution \( U(t) = [\rho, u, c_1, c_2] \) satisfies the following \( L^q \)-time decay estimates:

\[ \|\rho\|_{L^q} \leq C(1 + t)^{-\frac{2}{q} + \frac{2}{N}}, \tag{2.11} \]

\[ \|u\|_{L^q} \leq C(1 + t)^{-\frac{2}{q} + \frac{2}{N}}, \tag{2.12} \]

\[ \|(c_1, c_2)\|_{L^q} \leq C(1 + t)^{-\frac{2}{q}}, \tag{2.13} \]

for any \( t \geq 0, 2 \leq q \leq \infty \).

The existence of global solutions in Theorem 1.1 is obtained directly from Proposition 1 and the derivation of rates in Theorem 1.1 is based on Proposition 3.

### 3. Global solution of the nonlinear system (2.2)

The goal of this section is to prove the global existence of solutions to the Cauchy problem (2.2) when initial data is a small, smooth perturbation near the steady-state \( (n_\infty, 0, 0, 0) \). The proof is based on some uniform a priori estimates combined with the local existence that will be shown in subsections 3.1 and 3.2.

#### 3.1. Existence of local solutions

The local existence of smooth solutions for symmetrizable hyperbolic equations (2.2) and (2.2)_2 can be proved as in [10, 16]. Since (2.2)_3 and (2.2)_4 are the heat equations, the local solutions exist. We construct a solution sequence \( (\rho^j, w^j, c_{1,j}, c_{2,j})_{j \geq 0} \) by iteratively solving the Cauchy problem on the following system

\[
\begin{aligned}
\rho_{t}^{j+1} + n_{\infty} \nabla \cdot w^j &= -\nabla \cdot (\rho^j w^j), \\
\partial_t w^{j+1} + \nu w^{j+1} + \frac{\rho^j}{n_{\infty}} \nabla \rho^{j+1} &= -w^j \cdot \nabla w^{j+1} + \nabla c_1^j - \nabla c_2^j, \\
-(\rho^j w^j)_{t} + \frac{\rho^j}{n_{\infty}} \nabla \rho^{j+1} &= -\nabla (n_{\infty} \rho^j) - \frac{\rho^j}{n_{\infty}} \nabla \rho^{j+1}, \\
\partial_t c_{1,j}^{j+1} - \Delta c_{1,j}^{j+1} + (a_1 - a_{13} n_{\infty}) c_{1,j}^{j+1} &= a_{11} c_{1,j}^{j+1} + a_{12} c_{2,j}^{j+1} + a_{13} \rho^j c_{1,j}^{j+1}, \\
\partial_t c_{2,j}^{j+1} - \Delta c_{2,j}^{j+1} + (a_2 - a_{23} n_{\infty}) c_{2,j}^{j+1} &= a_{21} c_{1,j}^{j+1} + a_{22} c_{2,j}^{j+1} + a_{23} \rho^j c_{2,j}^{j+1},
\end{aligned}
\tag{3.1}
with initial data
\[(\rho^{j+1}, u^{j+1}, c_1^{j+1}, c_2^{j+1}) |_{t=0} = (\rho_0, u_0, c_{1,0}, c_{2,0}), \quad (3.2)\]
for \(j \geq 0\), where \((\rho^0, u^0, c_1^0, c_2^0) \equiv (0, 0, 0, 0)\) holds. For simplicity, in what follows, we write \(U^j = (\rho^j, u^j, c_1^j, c_2^j)\) and \(U_0 = (\rho_0, u_0, c_{1,0}, c_{2,0})\).

**Lemma 3.1.** There are constants \(T_1 > 0, \epsilon_0 > 0, B > 0\) such that if the initial data \(U_0 \in H^N(\mathbb{R}^3)\) and \(\|U_0\|_N \leq \epsilon_0\), then for each \(j \geq 0\), \(U^j \in C([0, T_1] : H^N(\mathbb{R}^3))\) is well-defined and
\[
\sup_{0 \leq t \leq T_1} \|U^j(t)\|_N \leq B, \quad j \geq 0. \quad (3.3)
\]
Moreover, \((U^j)_{j \geq 0}\) is a Cauchy sequence in Banach space \(C([0, T_1]; H^N(\mathbb{R}^3))\), and the limit function \(U(x, t)\) of \((U^j)_{j \geq 0}\) satisfies
\[
\sup_{0 \leq t \leq T_1} \|U(t)\|_N \leq B, \quad (3.4)
\]
and \(U = (\rho, u, c_1, c_2)\) is a solution over \([0, T_1]\) to the Cauchy problem (2.2)-(2.3).

Finally, the Cauchy problem (2.2)-(2.3) admits at most one solution \(U \in C([0, T_1] : H^N(\mathbb{R}^3))\) satisfying (3.4).

**3.2. A Priori Estimates.** In this subsection, we provide some estimates for the solutions for any \(t > 0\). We establish the uniform-in-time a priori estimates for smooth solutions to the Cauchy problem (2.2)-(2.3) by applying some basic energy estimates.

**Lemma 3.2.** (a priori estimates) Suppose that there exist a solution \(U(t) = (\rho, u, c_1, c_2) \in C([0, T]; H^N(\mathbb{R}^3))\) to the Cauchy problem (2.2)-(2.3), with
\[
\sup_{0 \leq t \leq T} \|\rho, u, c_1, c_2\|(t) \|_N \leq \epsilon \quad (3.5)
\]
for \(0 < \epsilon \leq 1\). Then, there are \(\epsilon_0 > 0, C_0 > 0\) and \(\lambda > 0\) such that for any \(\epsilon \leq \epsilon_0\),
\[
\mathcal{E}_N(U(t)) + \lambda_1 \int_0^t \mathcal{D}_N(U(t)) ds + \lambda_2 \int_0^t \mathcal{D}_N^2(U(t)) ds \leq C_0 \mathcal{E}_N(U_0) \quad (3.6)
\]
holds for any \(t \in [0, T]\).

**Proof.** At first, we find the zero-order estimates. For the estimate of \(\rho\), multiplying \(\rho\) to both sides of the first equation of (2.2) and taking integrations in \(x \in \mathbb{R}^3\), we obtain
\[
\int_{\mathbb{R}^3} \rho \rho_t dx + n_\infty \int_{\mathbb{R}^3} \rho \nabla \cdot u dx = - \int_{\mathbb{R}^3} \rho \nabla \cdot (\rho u) dx.
\]
Using integration by parts and the Cauchy-Schwarz inequality, we have
\[
\frac{1}{2} \int_{\mathbb{R}^3} (\rho^2)_t dx + n_\infty \int_{\mathbb{R}^3} \rho \nabla \cdot u dx \leq C \|\rho\|_2 \int_{\mathbb{R}^3} |u|^2 + |\nabla \rho|^2 dx. \quad (3.7)
\]
For the estimate of \(u\), multiplying \(u\) to both sides of the second equation of (2.2) and taking integrations in \(x \in \mathbb{R}^3\), we obtain
\[
\int_{\mathbb{R}^3} u \cdot u_t dx + \int_{\mathbb{R}^3} u \cdot (u \cdot \nabla u) dx - \nu \int_{\mathbb{R}^3} u^2 dx + \frac{\rho'(n_\infty)}{n_\infty} \int_{\mathbb{R}^3} u \cdot \nabla \rho dx = \int_{\mathbb{R}^3} u \cdot \nabla c_1 dx - \int_{\mathbb{R}^3} u \cdot \nabla c_2 dx - \int_{\mathbb{R}^3} u \cdot \left( \frac{\rho(n_\infty)}{\rho + n_\infty} - \frac{\rho'(n_\infty)}{n_\infty} \right) \nabla \rho dx.
\]
By using integration by parts and the Cauchy-Schwarz inequality, we have
\[
\frac{1}{2} \int_{\mathbb{R}^3} (u^2)_t \, dx + \nu \int_{\mathbb{R}^3} |u|^2 \, dx - \frac{\rho'(n_\infty)}{n_\infty} \int_{\mathbb{R}^3} \rho \nabla \cdot udx \leq \|u\|_3 \int_{\mathbb{R}^3} |u|^2 \, dx \\
+ C \int_{\mathbb{R}^3} |\nabla c_1|^2 + |u|^2 \, dx + C \int_{\mathbb{R}^3} |\nabla c_2|^2 + |u|^2 \, dx \\
+ C\|\rho\|_2 \int_{\mathbb{R}^3} |\nabla \rho|^2 + |u|^2 \, dx.
\]
(3.8)

For the estimates of \(c_1\), we multiply \(c_1\) to both sides of the equation of \(c_1\) and integrate with respect to \(x\), to get
\[
\int_{\mathbb{R}^3} c_1(c_1)_t \, dx - \int_{\mathbb{R}^3} c_1 \Delta c_1 \, dx + (a_1 - a_{13} n_\infty) \int_{\mathbb{R}^3} |c_1|^2 \, dx \leq a_{11} \sup_x |c_1| \int_{\mathbb{R}^3} |c_1|^2 \, dx \\
+ a_{12} \sup_x |c_2| \int_{\mathbb{R}^3} |c_1|^2 \, dx + a_{13} \sup_x |u| \int_{\mathbb{R}^3} |c_1|^2 \, dx.
\]
(3.9)

Similarly, as above, from the equation of \(c_2\), we have
\[
\frac{1}{2} \int_{\mathbb{R}^3} (c_2')_t \, dx + \int_{\mathbb{R}^3} |\nabla c_2|^2 \, dx + (a_2 - a_{23} n_\infty) \int_{\mathbb{R}^3} |c_2|^2 \, dx \leq a_{21} \|c_1\|_2 \int_{\mathbb{R}^3} |c_1|^2 \, dx \\
+ a_{22} \|c_2\|_2 \int_{\mathbb{R}^3} |c_1|^2 \, dx + a_{23} \|\rho\|_2 \int_{\mathbb{R}^3} |c_1|^2 \, dx.
\]
(3.10)

By choosing the constant \(d_1 = \frac{\rho'(n_\infty)}{n_\infty}\) and as long as \(E^T_N(U)\) is small so that
\[
(a_1 - a_{13} n_\infty) > (a_{11} + a_{12} + a_{13}) E^T_N(U), \quad (a_2 - a_{23} n_\infty) > (a_{21} + a_{22} + a_{23}) E^T_N(U)
\]
are satisfied, we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \left( |u|^2 + d_1 |\rho|^2 + |c_1|^2 + |c_2|^2 \right) \, dx + \nu \int_{\mathbb{R}^3} |u|^2 \, dx + \int_{\mathbb{R}^3} |\nabla c_1|^2 \, dx + \int_{\mathbb{R}^3} |\nabla c_2|^2 \, dx \\
+ (a_1 - a_{13} n_\infty) \int_{\mathbb{R}^3} |c_1|^2 \, dx + (a_2 - a_{23} n_\infty) \int_{\mathbb{R}^3} |c_2|^2 \, dx \leq C \|\rho\|_2 \int_{\mathbb{R}^3} |\nabla \rho|^2 \, dx.
\]
(3.11)

Now, we make estimates of the high-order derivatives of \((\rho, u, c_1, c_2)\). Take \(\alpha\) with \(1 \leq |\alpha| \leq N\). Applying \(\partial^{\alpha}\) to the second equation of (2.2), multiplying by \(\partial^{\alpha} u\) and then integrating in \(x\), we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} (\partial^{\alpha} u)^2 \, dx + \nu \int_{\mathbb{R}^3} \partial^{\alpha} u \cdot \partial^{\alpha} u \, dx + \int_{\mathbb{R}^3} \partial^{\alpha} u \cdot \partial^{\alpha} (\rho'(n_\infty)\nabla \rho) \, dx \\
= - \int_{\mathbb{R}^3} \partial^{\alpha} u \cdot \partial^{\alpha} (u \cdot \nabla u) \, dx + \int_{\mathbb{R}^3} \partial^{\alpha} u \cdot \partial^{\alpha} \nabla c_1 \, dx - \int_{\mathbb{R}^3} \partial^{\alpha} u \cdot \partial^{\alpha} \nabla c_2 \, dx.
\]
Thus

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} (\partial^a u)^2 \, dx + \nu \int_{\mathbb{R}^3} |\partial^a u|^2 \, dx + \int_{\mathbb{R}^3} \partial^a u \cdot \left( \frac{p'(\rho + \eta_n)}{\rho + \eta_n} \right) \partial^a \nabla \rho \, dx \\
= \int_{\mathbb{R}^3} \partial^a u \cdot \sum_{\beta=1}^3 C_{\alpha}^{\beta} (p'(\rho + \eta_n)) \partial^a \beta \nabla \rho \, dx - \int_{\mathbb{R}^3} \partial^a u \cdot \sum_{\beta=1}^3 C_{\alpha}^{\beta} (\partial^a \beta u \cdot \nabla \partial^\beta u) \, dx \\
+ \int_{\mathbb{R}^3} \partial^a u \cdot \partial^a \nabla c_1 \, dx - \int_{\mathbb{R}^3} \partial^a u \cdot \partial^a \nabla c_2 \, dx.
\]

(3.12)

We can estimate the third term on the left-hand side of the previous equality by using integration by parts and using the first equation, to give

\[
\int_{\mathbb{R}^3} \partial^a u \cdot \left( \frac{p'(\rho + \eta_n)}{\rho + \eta_n} \right) \partial^a \nabla \rho \, dx \\
= - \int_{\mathbb{R}^3} \partial^a \nabla \cdot u \left( \frac{p'(\rho + \eta_n)}{\rho + \eta_n} \right) \partial^a \rho \, dx - \int_{\mathbb{R}^3} \partial^a u \cdot \nabla \left( \frac{p'(\rho + \eta_n)}{\rho + \eta_n} \right) \partial^a \rho \, dx \\
= \int_{\mathbb{R}^3} \partial^a \left[ \frac{1}{\rho + \eta_n} \rho_t + \frac{\nabla (\rho + \eta_n) u}{\rho + \eta_n} \right] p'(\rho + \eta_n) \partial^a \rho \, dx - \int_{\mathbb{R}^3} \partial^a u \cdot \nabla \left( \frac{p'(\rho + \eta_n)}{\rho + \eta_n} \right) \partial^a \rho \, dx \\
= \frac{1}{2} \int_{\mathbb{R}^3} p'(\rho + \eta_n) \left( \partial^a \rho \right)^2 \, dx + \int_{\mathbb{R}^3} \frac{p'(\rho + \eta_n)}{\rho + \eta_n} \partial^a \left( \frac{1}{\rho + \eta_n} \rho_t \right) \partial^a \rho \, dx \\
+ \int_{\mathbb{R}^3} \frac{1}{\rho + \eta_n} \partial^a \rho \cdot u p'(\rho + \eta_n) \partial^a \rho \, dx + \int_{\mathbb{R}^3} \partial^a \left( \frac{1}{\rho + \eta_n} \nabla \rho \cdot u \right) p'(\rho + \eta_n) \partial^a \rho \, dx \\
- \int_{\mathbb{R}^3} \partial^a u \cdot \nabla \left( \frac{p'(\rho + \eta_n)}{\rho + \eta_n} \right) \partial^a \rho \, dx.
\]

(3.13)

Putting this into (3.12) gives

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} (\partial^a u)^2 \, dx + \nu \int_{\mathbb{R}^3} |\partial^a u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} p'(\rho + \eta_n) \left( \partial^a \rho \right)^2 \, dx \\
= - \int_{\mathbb{R}^3} p'(\rho + \eta_n) \partial^a \left( \frac{1}{\rho + \eta_n} \rho_t \right) \partial^a \rho \, dx - \int_{\mathbb{R}^3} \frac{1}{\rho + \eta_n} \partial^a \left( \nabla \rho \cdot u \right) p'(\rho + \eta_n) \partial^a \rho \, dx \\
- \int_{\mathbb{R}^3} \partial^a \left( \frac{1}{\rho + \eta_n} \nabla \rho \cdot u \right) p'(\rho + \eta_n) \partial^a \rho \, dx + \int_{\mathbb{R}^3} \partial^a u \cdot \nabla \left( \frac{p'(\rho + \eta_n)}{\rho + \eta_n} \right) \partial^a \rho \, dx \\
+ \int_{\mathbb{R}^3} \partial^a u \cdot \sum_{\beta=1}^3 C_{\alpha}^{\beta} (p'(\rho + \eta_n)) \partial^a \beta \nabla \rho \, dx - \int_{\mathbb{R}^3} \partial^a u \cdot \sum_{\beta=1}^3 C_{\alpha}^{\beta} (\partial^a \beta u \cdot \nabla \partial^\beta u) \, dx \\
+ \int_{\mathbb{R}^3} \partial^a u \cdot \partial^a \nabla c_1 \, dx - \int_{\mathbb{R}^3} \partial^a u \cdot \partial^a \nabla c_2 \, dx.
\]

(3.14)

Then, the terms on the right-hand side of (3.12) are bounded by

\[
C\|\rho\|^N \int_{\mathbb{R}^3} (|\partial^a u|^2 + |\partial^a \rho|^2) \, dx + C\|u\|^N \int_{\mathbb{R}^3} |\partial^a u|^2 \, dx \\
+ C \int_{\mathbb{R}^3} (|\partial^a u|^2 + |\partial^a \nabla c_1|^2) \, dx + C \int_{\mathbb{R}^3} (|\partial^a u|^2 + |\partial^a \nabla c_2|^2) \, dx.
\]

(3.15)

Plugging (3.13)-(3.15) into (3.12), integrating with respect to \( t \), and using the Cauchy-Schwarz inequality, we get

\[
\frac{1}{2} \|\partial^a u\|^2 + C_1 \|\partial^a \rho\|^2 + \nu \int_0^t \|\partial^a u\|^2 \, ds
\]
In a similar way as above, we estimate \( c_1 \) and \( c_2 \) as follows:

\[
\frac{1}{2} \| \partial^\alpha c_1 \|^2 + \int_0^t \| \nabla \partial^\alpha c_1 \|^2 ds + (a_{12} - n_\infty a_{11}) \int_0^t \| \partial^\alpha c_1 \|^2 ds \leq C \| \partial^\alpha c_{1,0} \| \\
+ C \| \rho \| \int_0^t \| \partial^\alpha c_1 \|^2 ds + C \| c_1 \| \int_0^t (\| \partial^\alpha c_1 \|^2 + \| \partial^\alpha \rho \|^2) ds \\
+ C \| c_2 \| \int_0^t \| \partial^\alpha c_1 \|^2 ds + C \| c_1 \| \int_0^t (\| \partial^\alpha c_1 \|^2 + \| \partial^\alpha c_2 \|^2) ds, \tag{3.17}
\]

and

\[
\frac{1}{2} \| \partial^\alpha c_2 \|^2 + \int_0^t \| \nabla \partial^\alpha c_2 \|^2 ds + (a_{22} - n_\infty a_{21}) \int_0^t \| \partial^\alpha c_2 \|^2 ds \leq C \| \partial^\alpha c_{2,0} \| \\
+ C \| \rho \| \int_0^t (\| \partial^\alpha c_2 \|^2 ds + C \| c_2 \| \int_0^t (\| \partial^\alpha c_2 \|^2 + \| \partial^\alpha \rho \|^2) ds \\
+ C \| c_1 \| \int_0^t (\| \partial^\alpha c_2 \|^2 \| + \| \partial^\alpha c_1 \|^2) ds. \tag{3.18}
\]

Then, by taking the summation of (3.16)-(3.18) over \(|\alpha| \leq N\), we have

\[
\frac{1}{2} (\| u \|_N^2 + C_1 \| \rho \|_N^2 + \| c_1 \|_N^2 + \| c_2 \|_N^2) + \nu \int_0^t \| u \|_N^2 ds + \int_0^t \| \nabla c_1 \|_N^2 ds + \int_0^t \| \nabla c_2 \|_N^2 ds \\
+ (a_1 - n_\infty a_{13}) \int_0^t \| c_1 \|_N^2 ds + (a_2 - n_\infty a_{23}) \int_0^t \| c_2 \|_N^2 ds \\
\leq \tilde{C} (u_0 \| \rho \|_N \int_0^t (\| u \|_N^2 + \| \rho \|_N^2 + \| c_1 \|_N^2 + \| c_2 \|_N^2) ds + C \| u, c_1, c_2 \| \int_0^t \| \rho \|_N^2 ds \\
+ C \| u \| \int_0^t \| u \|_N^2 ds + C \| c_1 \|_N \int_0^t (\| u \|_N^2 + \| c_1 \|_N^2 + \| c_2 \|_N^2) ds \\
+ C \| c_1 \|_N \int_0^t (\| c_1 \|_N^2 + \| c_2 \|_N^2) ds + C \| c_2 \| \int_0^t (\| c_1 \|_N^2 + \| c_2 \|_N^2) ds. \tag{3.19}
\]

Let \(|\alpha| \leq N - 1\). Applying \( \partial^\alpha \) to (2.2)2, multiplying it by \( \partial^\alpha \nabla \rho \) and taking integrations in \( x \) gives

\[
\int_{\mathbb{R}^3} \partial^\alpha \nabla \rho \cdot \partial^\alpha u dx + \nu \int_{\mathbb{R}^3} \partial^\alpha \nabla \rho \cdot \partial^\alpha u dx + \frac{\rho'(t_{n_\infty})}{n_\infty} \int_{\mathbb{R}^3} \partial^\alpha \nabla \rho \partial^\alpha \nabla \rho dx \\
= -\int_{\mathbb{R}^3} \partial^\alpha \nabla \rho \cdot \partial^\alpha (u \cdot \nabla u) dx + \int_{\mathbb{R}^3} \partial^\alpha \nabla \rho \partial^\alpha \nabla c_1 dx - \int_{\mathbb{R}^3} \partial^\alpha \nabla \rho \partial^\alpha \nabla c_2 dx \\
- \int_{\mathbb{R}^3} \partial^\alpha \nabla \rho \partial^\alpha u \left( \frac{\rho'(t_{n_\infty})}{t_{n_\infty}} - \frac{\rho'(t_{n_\infty})}{n_\infty} \right) \nabla \rho dx.
\]
which further, by replacing $\partial_t \rho$ from the first equation of (2.2) and then using integration by parts, implies
\[
\int_{\mathbb{R}^3} (\partial^\alpha \nabla \cdot \partial^\alpha u) dx + \frac{\nu(n_\infty)}{n_\infty} \int_{\mathbb{R}^3} |\partial^\alpha \nabla \rho|^2 dx \\
= -\nu \int_{\mathbb{R}^3} \partial^\alpha \nabla \rho \partial^\alpha u dx - \int_{\mathbb{R}^3} \partial^\alpha \nabla \rho \cdot \partial^\alpha (u \cdot \nabla u) dx \\
+ \int_{\mathbb{R}^3} \partial^\alpha \nabla \rho \partial^\alpha \nabla c_1 dx - \int_{\mathbb{R}^3} \partial^\alpha \nabla \rho \partial^\alpha \nabla c_2 dx \\
- \int_{\mathbb{R}^3} \partial^\alpha \nabla \rho \partial^\alpha (\frac{\nu'(\rho + n_\infty)}{\rho + n_\infty} - \frac{\nu'(n_\infty)}{n_\infty}) \nabla \rho) dx \\
- \int_{\mathbb{R}^3} \partial^\alpha \nabla \cdot u \partial^\alpha \nabla ((\rho + n_\infty)u) dx.
\]

Applying the Cauchy-Schwarz inequality we obtain
\[
\frac{d}{dt} \int_{\mathbb{R}^3} (\partial^\alpha \nabla \cdot \partial^\alpha u) dx + \lambda_2 \|\partial^\alpha \nabla \rho\|^2 \\
\leq C(\|\nabla \cdot \partial^\alpha u\|^2 + \|\partial^\alpha u\|^2) + C\|\partial^\alpha \nabla [c_1, c_2]\|^2 \\
+ C(\|\rho, u\|_N \|\nabla \cdot \partial^\alpha [\rho, u]\|^2).
\]

Then, after taking summation over $|\alpha| \leq N - 1$ and integrating with respect to $t$, we obtain
\[
\sum_{|\alpha| \leq N - 1} \int_{\mathbb{R}^3} \partial^\alpha \nabla \cdot \partial^\alpha u dx + \lambda_2 \int_0^\infty \|\nabla \rho\|^2_{N-1} ds \
\leq \sum_{|\alpha| \leq N - 1} \int_{\mathbb{R}^3} \partial^\alpha \nabla \rho \cdot \partial^\alpha u dx|_{t=0} \\
+ C \int_0^t \|u\|^2_N ds + C \int_0^t \|\nabla [c_1, c_2]\|^2_{N-1} ds \\
+ C\|\rho, u\|_N \int_0^t \|\nabla \cdot [\rho, u]\|^2_{N-1} ds.
\]

By taking a linear combination (3.11) + (3.19) + k(3.20), we have
\[
\|U\|^2_N + k \sum_{|\alpha| \leq N - 1} \int_{\mathbb{R}^3} \partial^\alpha \nabla \rho \cdot \partial^\alpha u dx \\
+ \lambda_1 \int_0^t \|\nabla c_1, c_2\|^2_N ds + \lambda_2 \int_0^t (\|\nabla \rho\|^2_{N-1} + \|u, c_1, c_2\|^2_N) ds \
\leq C_0 \|U_0\|^2_N
\]

for constant $0 < k \ll 1$. Then
\[
\|U\|^2_N + k \sum_{|\alpha| \leq N - 1} \int_0^t \int_{\mathbb{R}^3} \partial^\alpha \nabla \rho \partial^\alpha u dx ds \sim \|U\|^2_N.
\]

This completes the proof of Lemma 3.2. □

Based on the argument in Lemma 3.1 and Lemma 3.2, now we start to prove Proposition 1.

\textbf{Proof of Proposition 1.} Choose a positive constant $\bar{\epsilon} = \min\{\epsilon_0, \epsilon_1\}$, where $\epsilon_0 > 0$ and $\epsilon_1 > 0$ are given in Lemma 3.1 and Lemma 3.2. Let $U_0 \in H^N(\mathbb{R}^3)$ satisfy
\[
\|U_0\|_{H^N} \leq \frac{\bar{\epsilon}}{2\sqrt{C_0} + 1}.
\]
Now, let us define
\[ T = \{ t \geq 0 : \sup_{0 \leq s \leq t} \| U(s) \|_{H^N} \leq \bar{\epsilon} \}. \]

Note that
\[ \| U_0 \|_{H^N} \leq \frac{\bar{\epsilon}}{2\sqrt{C_0} + 1} \leq \frac{\bar{\epsilon}}{2} < \epsilon_0. \]

Then \( T > 0 \) holds from the local existence result. If \( T \) is finite, from the definition of \( T \), we have
\[ \sup_{0 \leq s \leq t} \| U \|_{H^N} = \bar{\epsilon}. \] (3.22)

On the other hand, by Lemma 3.2 we have
\[ \sup_{0 \leq s \leq t} \| U(s) \|_{H^N} \leq \sqrt{C_0} \| U_0 \|_{H^N} \leq \frac{\bar{\epsilon}\sqrt{C_0}}{2\sqrt{C_0} + 1} \leq \frac{\bar{\epsilon}}{2}, \]

which is a contradiction to 3.22. Then \( T = \infty \) holds true. This implies that local solution \( U(t) \) obtained in Lemma 3.1 can be extended to infinity in time. Thus, we have a global solution \( (\rho, u, c_1, c_2)(t) \in C([0, \infty); H^N) \). This completes the proof of Proposition 1. \( \square \)

4. Linearized homogeneous system. In this section, to study the time-decay property of solutions to the nonlinear system (2.2), we have to consider the following Cauchy problem for the corresponding linearized equations around the constant state \([n_{\infty}, 0, 0, 0]\). Then \( U = [\rho, u, c_1, c_2] \) satisfies
\[
\begin{cases}
\partial_t \rho + n_{\infty} \nabla \cdot u = g_1 \\
\partial_t u + \nu u + \frac{\mu_0(n_{\infty})}{n_{\infty}} \nabla \rho + \nabla c_1 - \nabla c_2 = g_2 \\
\partial_t c_1 - \Delta c_1 + (a_1 + n_{\infty} a_{13}) c_1 = g_3 \\
\partial_t c_2 - \Delta c_2 + (a_2 + n_{\infty} a_{23}) c_2 = g_4,
\end{cases}
\] (4.1)

with initial data
\[ (\rho, u, c_1, c_2) \mid_{t=0} = (\rho_0, u_0, c_{1,0}, c_{2,0}). \] (4.2)

Here the nonlinear source term takes the form
\[
\begin{aligned}
g_1 &= -\nabla \cdot (\rho u) \\
g_2 &= -u \cdot \nabla u - \frac{\mu'(\rho + n_{\infty})}{\rho + n_{\infty}} - \frac{\mu'(n_{\infty})}{n_{\infty}} \nabla \rho \\
g_3 &= a_{11} c_1^2 + a_{12} c_1 c_2 + a_{13} c_1 \rho \\
g_4 &= a_{21} c_1 c_2 + a_{22} c_2^2 + a_{23} c_2 \rho.
\end{aligned}
\] (4.3)

To obtain the time-decay rates of the solution to the system (4.1) in the next section, we are concerned with the following Cauchy problem for the linearized homogenous system corresponding with the system (4.1):
\[
\begin{cases}
\partial_t \rho + n_{\infty} \nabla \cdot u = 0 \\
\partial_t u + \nu u + \frac{\mu_0(n_{\infty})}{n_{\infty}} \nabla \rho - \nabla c_1 + \nabla c_2 = 0 \\
\partial_t c_1 - \Delta c_1 + (a_1 - n_{\infty} a_{13}) c_1 = 0 \\
\partial_t c_2 - \Delta c_2 + (a_2 - n_{\infty} a_{23}) c_2 = 0.
\end{cases}
\] (4.4)

with initial data
\[ (\rho, u, c_1, c_2) \mid_{t=0} = U_0 = (\rho_0, u_0, c_{1,0}, c_{2,0}). \] (4.5)

In this section, we let \( U = [\rho, u, c_1, c_2] \) be the solution to the system (4.4).
4.1. Representation of solutions. In this subsection, we find the explicit representation of the Fourier transform of the solution \( U = e^{B t} U_0 \) to the Cauchy problem (4.4)-(4.5), where \( e^{B t} \) is the linear solution operator.

After taking the Fourier transform in \( x \) for the first equation of (4.4), we have

\[
\hat{\rho}_t + n_\infty i \hat{\rho} = 0,
\]

with initial data \( \hat{\rho} \big|_{t=0} = \hat{\rho}_0 \).

Similarly, by taking the Fourier transform for the second equation of (4.4), we get

\[
\hat{u}_t + \nu \hat{u} + \frac{c'(n_\infty)}{n_\infty} i \hat{\xi} \hat{\rho} - i \hat{\xi} \hat{c}_1 = 0,
\]

with initial data \( \hat{u} \big|_{t=0} = \hat{u}_0 \).

Further, by taking the Fourier transform for the second equation of (4.4), we have

\[
\hat{\xi} \cdot \hat{u}_t + \nu \hat{\xi} \cdot \hat{u} + i \frac{c'(n_\infty)}{n_\infty} \hat{\xi} \cdot \hat{\rho} - i \hat{\xi} \cdot \hat{c}_1 + i \hat{\xi} \cdot \hat{c}_2 = 0.
\]

Here and in the sequel, we set \( \hat{\xi} = \frac{\xi}{|\xi|} \) for \( |\xi| \neq 0 \).

Similarly for \( [c_1, c_2] \), by taking the Fourier transform for the third and fourth equations of (4.4), we get

\[
\partial_t \hat{c}_1 + |\xi|^2 \hat{c}_1 + (a_1 - a_{13} n_\infty) \hat{c}_1 = 0
\]

\[
\partial_t \hat{c}_2 + |\xi|^2 \hat{c}_2 + (a_2 - a_{23} n_\infty) \hat{c}_2 = 0.
\]

Then, we have

\[
\begin{cases}
\hat{\rho}_t + i n_\infty \hat{\xi} \cdot \hat{u} = 0 \\
\hat{\xi} \cdot \hat{u}_t + \nu \hat{\xi} \cdot \hat{u} + i \frac{c'(n_\infty)}{n_\infty} \hat{\xi} \cdot \hat{\rho} - i \hat{\xi} \cdot \hat{c}_1 + i \hat{\xi} \cdot \hat{c}_2 = 0 \\
\partial_t \hat{c}_1 + |\xi|^2 \hat{c}_1 + (a_1 - a_{13} n_\infty) \hat{c}_1 = 0 \\
\partial_t \hat{c}_2 + |\xi|^2 \hat{c}_2 + (a_2 - a_{23} n_\infty) \hat{c}_2 = 0.
\end{cases}
\]

(4.9)

We can rewrite (4.9) as

\[
\partial_t \hat{U} = A(\xi) \hat{U},
\]

with \( \hat{U}(t, \xi) = (\hat{\rho}(t, \xi), \hat{\xi} \cdot \hat{u}(t, \xi), \hat{c}_1(t, \xi), \hat{c}_2(t, \xi))^T \) and

\[
A(\xi) = \begin{bmatrix}
0 & -i n_\infty |\xi| & 0 & 0 \\
-i \frac{c'(n_\infty)}{n_\infty} \frac{\xi^2}{|\xi|^2} & -\nu & i |\xi| & -i |\xi| \\
0 & 0 & -|\xi|^2 - (a_1 - a_{13} n_\infty) & 0 \\
0 & 0 & 0 & -|\xi|^2 - (a_2 - a_{23} n_\infty)
\end{bmatrix},
\]

where \( T \) denotes the transpose of a row vector. Then, the eigenvalues of the system are as follows

\[
\begin{align*}
\lambda_1 &= -\frac{1}{2} \nu + \frac{1}{2} \sqrt{\nu^2 - 4p'(n_\infty)|\xi|^2} \\
\lambda_2 &= -\frac{1}{2} \nu - \frac{1}{2} \sqrt{\nu^2 - 4p'(n_\infty)|\xi|^2} \\
\lambda_3 &= -|\xi|^2 - (a_1 - a_{13} n_\infty) \\
\lambda_4 &= -|\xi|^2 - (a_2 - a_{23} n_\infty).
\end{align*}
\]

Therefore, the eigenvectors corresponding to the eigenvalues \( \lambda \) of \( A(\xi) \) that satisfy \( (A - \lambda I)X = 0 \) are

\[
v_1 = \begin{bmatrix} m_\infty |\xi| \\ -\lambda_1 \\ 0 \\ 0 \end{bmatrix}.
\]
and

\[
v_2 = \begin{bmatrix}
  \frac{n_\infty |\xi|^2}{\lambda_3} \\
  -\lambda_2 \\
  0 \\
  0
  \end{bmatrix}
\]

\[
v_3 = \begin{bmatrix}
  \frac{n_\infty |\xi|^2}{i\xi} \\
  \frac{p'(n_\infty)|\xi|^2}{\lambda_3} + (\nu + \lambda_3) \\
  0 \\
  \frac{p'(n_\infty)|\xi|^2}{\lambda_4} + (\nu + \lambda_4)
  \end{bmatrix}
\]

\[
v_4 = \begin{bmatrix}
  -\frac{n_\infty |\xi|^2}{\lambda_4} \\
  -\frac{i|\xi|}{\lambda_4} \\
  0 \\
  \frac{p'(n_\infty)|\xi|^2}{\lambda_4} + (\nu + \lambda_4)
  \end{bmatrix}
\]

From above, one can define the general solution of (4.9) as

\[
\begin{bmatrix}
\hat{\rho} \\
\hat{\xi} \\
\hat{u} \\
\hat{c}
\end{bmatrix} = \begin{bmatrix}
  \frac{n_\infty |\xi| e^{\lambda_3 t}}{\lambda_3} & \frac{n_\infty |\xi| e^{\lambda_3 t}}{\lambda_3} & -\frac{n_\infty |\xi|^2}{\lambda_3} & -\frac{n_\infty |\xi|^2}{\lambda_4} \\
  -\lambda_1 e^{\lambda_3 t} & -\lambda_2 e^{\lambda_4 t} & \frac{n_\infty |\xi|^2}{i\xi} e^{\lambda_3 t} & \frac{n_\infty |\xi|^2}{i\xi} e^{\lambda_4 t} \\
  0 & 0 & (\frac{p'(n_\infty)|\xi|^2}{\lambda_3} + (\nu + \lambda_3)) e^{\lambda_3 t} & 0 \\
  0 & 0 & (\frac{p'(n_\infty)|\xi|^2}{\lambda_4} + (\nu + \lambda_4)) e^{\lambda_4 t} & (\frac{p'(n_\infty)|\xi|^2}{\lambda_4} + (\nu + \lambda_4)) e^{\lambda_4 t}
\end{bmatrix}
\begin{bmatrix}
d_1 \\
d_2 \\
d_3 \\
d_4
\end{bmatrix},
\]

where \(d_1, d_2, d_3, d_4\) satisfy

\[
\begin{bmatrix}
\hat{\rho} \\
\hat{\xi} \\
\hat{u} \\
\hat{c}
\end{bmatrix} = \begin{bmatrix}
  \frac{n_\infty |\xi|}{i(\lambda_1 - \lambda_2)} \\
  -\lambda_1 & -\lambda_2 & \frac{n_\infty |\xi|^2}{\lambda_3} & -\frac{n_\infty |\xi|^2}{\lambda_4} \\
  0 & 0 & \frac{p'(n_\infty)|\xi|^2}{\lambda_3} + (\nu + \lambda_3) & 0 \\
  0 & 0 & 0 & \frac{p'(n_\infty)|\xi|^2}{\lambda_4} + (\nu + \lambda_4)
\end{bmatrix}
\begin{bmatrix}
d_1 \\
d_2 \\
d_3 \\
d_4
\end{bmatrix}.
\]

From this we deduce that

\[
\begin{bmatrix}
\hat{\rho} \\
\hat{\xi} \\
\hat{u} \\
\hat{c}
\end{bmatrix} = \begin{bmatrix}
  \frac{1}{n_\infty |\xi|}(\lambda_1 - \lambda_2) \\
  -\lambda_2 & -\lambda_1 & \frac{n_\infty |\xi|^2}{\lambda_3} & -\frac{n_\infty |\xi|^2}{\lambda_4} \\
  0 & 0 & \frac{a_3}{n_\infty |\xi|^2(\lambda_3 - \lambda_2)} & \frac{a_3}{n_\infty |\xi|^2(\lambda_4 - \lambda_2)} \\
  0 & 0 & 0 & \frac{a_4}{n_\infty |\xi|(\lambda_1 - \lambda_2)}
\end{bmatrix}
\begin{bmatrix}
\hat{\rho}_0 \\
\hat{\xi}_{1,0} \\
\hat{u}_{1,0} \\
\hat{c}_{1,0}
\end{bmatrix},
\]

where \(a_3 = \frac{p'(n_\infty)|\xi|^2}{\lambda_3} + (\nu + \lambda_3)\) and \(a_4 = \frac{p'(n_\infty)|\xi|^2}{\lambda_4} + (\nu + \lambda_4)\).

It is straightforward to obtain

\[
\hat{\rho} = \frac{\lambda_1 e^{\lambda_3 t} - \lambda_2 e^{\lambda_4 t}}{\lambda_1 - \lambda_2} \hat{\rho}_0 - \frac{n_\infty |\xi|^2(\lambda_3 - \lambda_2)}{\lambda_1 - \lambda_2} \hat{\xi} \cdot \hat{u}_0
\]

\[
+ \frac{n_\infty |\xi|^2(\lambda_3 - \lambda_2) e^{\lambda_4 t} - n_\infty |\xi|^2(\lambda_3 - \lambda_2) e^{\lambda_3 t} + \frac{2n_\infty |\xi|^2}{\lambda_3} (\lambda_1 - \lambda_2) e^{\lambda_3 t}}{a_3(\lambda_1 - \lambda_2)} \hat{c}_{1,0}
\]

\[
+ \frac{n_\infty |\xi|^2(1 - \lambda_3) e^{\lambda_4 t} - n_\infty |\xi|^2(1 - \lambda_3) e^{\lambda_3 t} + \frac{2n_\infty |\xi|^2}{\lambda_4} (\lambda_1 - \lambda_2) e^{\lambda_4 t}}{a_4(\lambda_1 - \lambda_2)} \hat{c}_{2,0}.
\]
For

By solving the initial value problem (4.16) and (4.17), we have

It is straightforward to get

\[ \hat{c}_1 = e^{\lambda_1 t} \hat{c}_{1,0}, \]

\[ \hat{c}_2 = e^{\lambda_2 t} \hat{c}_{2,0}. \]

Now, by taking the curl for the second equation of (4.4), we have

\[ \partial_t (\nabla \times u) + \nu (\nabla \times u) = 0. \]

Taking the Fourier transform for the above equation in \( x \), we have

\[ \partial_t (\hat{\xi} \times \hat{u}) + \nu \hat{\xi} \times \hat{u} = 0. \]

Initial data is given as

\[ (\hat{\xi} \times \hat{u})|_{t=0} = \hat{\xi} \times \hat{u}_0. \]

By solving the initial value problem (4.16) and (4.17), we have

\[ (\hat{\xi} \times \hat{u}) = e^{-\nu t} \hat{\xi} \times \hat{u}_0. \]

For \( t \geq 0 \) and \( \xi \in \mathbb{R}^3 \) with \( |\xi| \neq 0 \), one has the decomposition \( \hat{u} = \hat{\xi} \hat{\xi} \cdot \hat{u} - \hat{\xi} \times (\hat{\xi} \times \hat{u}) \).

It is straightforward to get

\[
\hat{u} = \frac{\lambda_1 \lambda_2 \xi}{in_{\infty} |\xi|^2} \left( e^{\lambda_1 t} - e^{\lambda_2 t} \right) \hat{\rho}_0 + \left( \frac{\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \right) \hat{\xi} \cdot \hat{u}_0 - e^{-\nu t} \hat{\xi} \times (\hat{\xi} \times \hat{u}_0) \\
+ \frac{i \lambda_1 \xi \left( \frac{\lambda_2}{\lambda_3} - 1 \right) e^{\lambda_1 t} - i \lambda_2 \xi \left( \frac{\lambda_1}{\lambda_3} - 1 \right) e^{\lambda_2 t} + i \xi (\lambda_1 - \lambda_2) e^{\lambda_3 t}}{a_3 (\lambda_1 - \lambda_2)} \hat{c}_{1,0} \\
- \frac{i \lambda_1 \xi \left( \frac{\lambda_2}{\lambda_3} - 1 \right) e^{\lambda_1 t} + i \lambda_2 \xi \left( \frac{\lambda_1}{\lambda_3} - 1 \right) e^{\lambda_2 t} + i \xi (\lambda_1 - \lambda_2) e^{\lambda_3 t}}{a_4 (\lambda_1 - \lambda_2)} \hat{c}_{2,0}.
\]

Then

\[ \hat{u} = \frac{\lambda_1 \lambda_2 \xi}{in_{\infty} |\xi|^2} \left( e^{\lambda_1 t} - e^{\lambda_2 t} \right) \hat{\rho}_0 \\
+ \left( \frac{\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \right) \frac{\xi \otimes \xi}{|\xi|^2} \hat{u}_0 + e^{-\nu t} \left( I_3 - \frac{\xi \otimes \xi}{|\xi|^2} \right) \hat{u}_0 \\
+ \frac{i \lambda_1 \xi \left( \frac{\lambda_2}{\lambda_3} - 1 \right) e^{\lambda_1 t} - \lambda_2 \xi \left( \frac{\lambda_1}{\lambda_3} - 1 \right) e^{\lambda_2 t} + i \xi (\lambda_1 - \lambda_2) e^{\lambda_3 t}}{a_3 (\lambda_1 - \lambda_2)} \hat{c}_{1,0} \\
+ \frac{-i \lambda_1 \xi \left( \frac{\lambda_2}{\lambda_3} - 1 \right) e^{\lambda_1 t} + i \lambda_2 \xi \left( \frac{\lambda_1}{\lambda_3} - 1 \right) e^{\lambda_2 t} - i \xi (\lambda_1 - \lambda_2) e^{\lambda_3 t}}{a_4 (\lambda_1 - \lambda_2)} \hat{c}_{2,0}. \]

After summarizing the above computations on the explicit representation of Fourier transform of the solution \( U = [\rho, u, c_1, c_2] \), we have

\[
\begin{bmatrix}
\hat{\rho}(t, \xi) \\
\hat{u}(t, \xi) \\
\hat{c}_1(t, \xi) \\
\hat{c}_2(t, \xi)
\end{bmatrix} = \hat{G}(t, \xi) \begin{bmatrix}
\hat{\rho}(0, \xi) \\
\hat{u}(0, \xi) \\
\hat{c}_1(0, \xi) \\
\hat{c}_2(0, \xi)
\end{bmatrix}, \tag{4.19}
\]
where

\[
\hat{\mathbf{G}}(t, \xi) = \begin{bmatrix}
\hat{G}_{11} & \hat{G}_{12} & \hat{G}_{13} & \hat{G}_{14} \\
\hat{G}_{21} & \hat{G}_{22} & \hat{G}_{23} & \hat{G}_{24} \\
\hat{G}_{31} & \hat{G}_{32} & \hat{G}_{33} & \hat{G}_{34} \\
\hat{G}_{41} & \hat{G}_{42} & \hat{G}_{43} & \hat{G}_{44}
\end{bmatrix}
\]

is the Green’s matrix and it is the Fourier transform of the Green’s function \( G(t, \xi) = e^{tB} \). The elements of Green’s matrix \( \hat{\mathbf{G}}(t, \xi) \) in (4.19) are given by

\[
\begin{align*}
\hat{G}_{11} &= \frac{\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_1 - \lambda_2}, & \hat{G}_{12} &= -i n_\infty \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \xi, \\
\hat{G}_{13} &= \frac{n_\infty |\xi|^2 (\lambda_2^2 - 1)e^{\lambda_1 t} - n_\infty |\xi|^2 (\lambda_1^2 - 1)e^{\lambda_2 t} + \frac{n_\infty |\xi|^2}{\lambda_3^2} (\lambda_1 - \lambda_2)e^{\lambda_3 t}}{\lambda_1 - \lambda_2} & a_3(\lambda_1 - \lambda_2), \\
\hat{G}_{14} &= -i \frac{\lambda_1 \xi (\lambda_2^2 - 1)e^{\lambda_1 t} + i \lambda_2 \xi (\lambda_1^2 - 1)e^{\lambda_2 t} + i \xi (\lambda_1 - \lambda_2)e^{\lambda_3 t}}{a_4(\lambda_1 - \lambda_2)}, \\
\hat{G}_{21} &= \frac{\lambda_1 \lambda_2 \xi}{in_\infty |\xi|^2} \left( e^{\lambda_1 t} - e^{\lambda_2 t} \right), \\
\hat{G}_{22} &= \frac{(\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t})}{\lambda_1 - \lambda_2} \xi \otimes \xi \frac{|\xi|^2}{|\xi|^2} + e^{-\nu t} (I - \xi \otimes \xi), \\
\hat{G}_{23} &= \frac{i \lambda_1 \xi (\lambda_2^2 - 1)e^{\lambda_1 t} - \lambda_2 \xi (\lambda_1^2 - 1)e^{\lambda_2 t} + i \xi (\lambda_1 - \lambda_2)e^{\lambda_3 t}}{a_3(\lambda_1 - \lambda_2)}, \\
\hat{G}_{24} &= \frac{-i \lambda_1 \xi (\lambda_2^2 - 1)e^{\lambda_1 t} + i \lambda_2 \xi (\lambda_1^2 - 1)e^{\lambda_2 t} - i \xi (\lambda_1 - \lambda_2)e^{\lambda_3 t}}{a_4(\lambda_1 - \lambda_2)}, \\
\hat{G}_{31} &= \hat{G}_{32} = \hat{G}_{34} = 0, & \hat{G}_{33} &= e^{\lambda_3 t}, \\
\hat{G}_{41} &= \hat{G}_{42} = \hat{G}_{43} = 0, & \hat{G}_{44} &= e^{\lambda_4 t}.
\end{align*}
\]

4.2. Refined \( L^2 - L^q \) time-decay property. In this subsection, we use (4.19) to obtain some refined \( L^2 - L^q \) time-decay property for \( \hat{U} = [\rho, u, c_1, c_2] \). To do so, we need to find the time-frequency pointwise estimate on \( \hat{U} \) in the following Lemma:

**Lemma 4.1.** Suppose \( U = [\rho, u, c_1, c_2] \) is the solution to the linear homogeneous system (4.4) with the initial data \( U|_{t=0} = U_0 = (\rho_0, u_0, c_{1,0}, c_{2,0}) \). Then, there are constants \( \epsilon > 0 \), \( C > 0 \), \( \lambda > 0 \) such that for all \( t > 0 \), \( |\xi| \leq \epsilon \),

\[
\begin{align*}
|\hat{\rho}(t, \xi)| &\leq C(e^{-\lambda|\xi|^2 t} + |\xi|^2 e^{-\nu \lambda t})|\hat{\rho}_0(\xi)| + C(|\xi|e^{-\lambda|\xi|^2 t} + |\xi|e^{-\nu \lambda t})|\hat{\rho}_0(\xi)| \\
&+ C|\xi|^2(e^{-\lambda|\xi|^2 t} + e^{-\nu \lambda t} + e^{-\lambda t})|\hat{c}_{1,0}(\xi), \hat{c}_{2,0}(\xi)|, \\
|\hat{u}(t, \xi)| &\leq C|\xi|(e^{-\lambda|\xi|^2 t} + e^{-\nu \lambda t})|\hat{\rho}_0(\xi)| + C(|\xi|^2 e^{-\lambda|\xi|^2 t} + e^{-\nu \lambda t})|\hat{u}_0(\xi)| \\
&+ C(|\xi|^3 e^{-\lambda|\xi|^2 t} + C|\xi|e^{-\nu \lambda t} + C|\xi|e^{-\lambda t})|\hat{c}_{1,0}(\xi), \hat{c}_{2,0}(\xi)| \quad (4.20)
\end{align*}
\]

\[
\begin{align*}
|\hat{c}_1| &\leq Ce^{-\lambda t}|\hat{c}_{1,0}(\xi)| \quad (4.21) \\
|\hat{c}_2| &\leq Ce^{-\lambda t}|\hat{c}_{2,0}(\xi)|. \quad (4.22)
\end{align*}
\]

For all \( t > 0 \), \( |\xi| \geq \epsilon \),

\[
\begin{align*}
|\hat{\rho}(t, \xi)| &\leq Ce^{-\lambda t}|\hat{U}_0|, \\
|\hat{u}(t, \xi)| &\leq Ce^{-\lambda t}|\hat{U}_0|, \quad (4.23)
\end{align*}
\]

\[
\begin{align*}
|\hat{c}_1| &\leq Ce^{-\lambda t}|\hat{c}_{1,0}(\xi)|, \\
|\hat{c}_2| &\leq Ce^{-\lambda t}|\hat{c}_{2,0}(\xi)|. \quad (4.24)
\end{align*}
\]
\[ |\hat{c}_1| \leq Ce^{-\lambda t}|\hat{c}_{1,0}|, \quad (4.26) \]

and

\[ |\hat{c}_2| \leq Ce^{-\lambda t}|\hat{c}_{2,0}|. \quad (4.27) \]

**Proof.** To obtain the upper bound of \(\hat{U}(t, \xi)\), we have to estimate the elements of Green’s matrix \(\hat{G}(t, \xi)\) in (4.19). If \(\nu^2 - 4p'(n_\infty)|\xi|^2 \geq 0\), then \(\lambda_{1,2} = -\nu \pm \frac{1}{2} \sqrt{\nu^2 - 4p'(n_\infty)|\xi|^2}\) are real. It is straightforward to obtain

\[ \lambda_1 \sim -O(1)|\xi|^2, \]
\[ \lambda_2 \sim -\nu + O(1)|\xi|^2, \]
\[ \lambda_{3,4} \sim -O(1), \]

as \(|\xi| \to 0\).

On other hand, if \(\nu^2 - 4p'(n_\infty)|\xi|^2 \leq 0\), then \(\lambda_{1,2} = -\nu \pm \frac{1}{2} i \sqrt{4p'(n_\infty)|\xi|^2 - 1}\) are complex conjugates. Moreover, we have

\[ |\lambda_{1,2}| \sim O(1)|\xi|, \]
\[ \lambda_1 - \lambda_2 \sim iO(1)|\xi|, \]
\[ \lambda_{3,4} \sim -O(1)|\xi|^2, \]

as \(|\xi| \to \infty\). Then, there exists \(\epsilon \leq \sqrt{\frac{\nu^2}{4p'(n_\infty)}} \leq R\), with \(0 < \epsilon \ll 1 < R < \infty\) such that we can estimate Green’s matrix \(\hat{G}(t, \xi)\) as follows:

\[ |\hat{G}_{11}| \leq C(e^{-\lambda|\xi|^2 t} + |\xi|^2 e^{-\nu t}) \]
\[ |\hat{G}_{12}| \leq C|\xi|(e^{-\lambda|\xi|^2 t} + e^{-\nu t}) \]
\[ |\hat{G}_{13}| \leq C|\xi|^2(e^{-\lambda|\xi|^2 t} + e^{-\nu t} + e^{-\lambda t}) \]
\[ |\hat{G}_{14}| \leq C|\xi|^2(e^{-\lambda|\xi|^2 t} + e^{-\nu t} + e^{-\lambda t}) \]
\[ |\hat{G}_{21}| \leq C|\xi|(e^{-\lambda|\xi|^2 t} + e^{-\nu t}) \]
\[ |\hat{G}_{22}| \leq Ce^{-\nu t} + C(|\xi|^2 e^{-\lambda|\xi|^2 t} + e^{-\nu t}) \]
\[ \leq C|\xi|^2 e^{-\lambda|\xi|^2 t} + Ce^{-\nu t} \]
\[ |\hat{G}_{23}, \hat{G}_{24}| \leq C(|\xi|^3 e^{-\lambda|\xi|^2 t} + C|\xi|e^{-\nu t} + C|\xi|e^{-\lambda t}) \]
\[ |\hat{G}_{33}| \leq Ce^{-\lambda t} \]
\[ |\hat{G}_{44}| \leq Ce^{-\lambda t} \]

as \(|\xi| \leq \epsilon\), and

\[ |\hat{G}_{ij}| \leq Ce^{-\lambda t}, \quad 1 \leq i, j \leq 4, \]

as \(|\xi| > R\).

When the eigenvalues \(\lambda_1\) and \(\lambda_2\) coalesce, since the real part is negative, we have \(te^{-\frac{\nu}{2} t}\) in the solution, but this decays exponentially. Then, we get \(te^{-\frac{\nu}{2} t} \leq e^{-\lambda t}\).

Now, we can estimate \(\hat{U} = [\hat{\rho}, \hat{u}, \hat{c}_2, \hat{c}_3]\) as follows

\[ |\hat{\rho}(t, \xi)| = |\hat{G}_{11}\hat{\rho}_0 + \hat{G}_{12}\hat{u}_0 + \hat{G}_{13}\hat{c}_{1,0} + \hat{G}_{14}\hat{c}_{2,0}| \]
\[ \leq |\hat{G}_{11}||\hat{\rho}_0| + |\hat{G}_{12}||\hat{u}_0| + |\hat{G}_{13}||\hat{c}_{1,0}| + |\hat{G}_{14}||\hat{c}_{2,0}| \]
for $|\xi| \leq \epsilon$. Finally, (4.24)-(4.27) can be proved in the completely same way as for (4.20)-(4.23). This completes the proof of Lemma (4.1).

**Theorem 4.2.** Let $2 \leq q \leq \infty$, and let $m \geq 0$ be an integer. Assume $U = e^{Bt}U_0$ is the solution to the Cauchy problem (4.4)-(4.5). Then for any $t \geq 0$, $U = [\rho, u, c_1, c_2]$ satisfies:

\[
\begin{align*}
\|\nabla^m \rho(t)\|_{L^q} &\leq C(1 + t)^{-\frac{3}{2}(1 - \frac{1}{q})} \|\rho_0\|_{L^1} + e^{-\lambda t} \|\nabla^{m+3(1 - \frac{1}{q})} + 1\| U_0, \quad (4.28) \\
\|\nabla^m u(t)\|_{L^q} &\leq C(1 + t)^{-\frac{3}{2}(1 - \frac{1}{q})} \|\rho_0\|_{L^1} + e^{-\lambda t} \|\nabla^{m+3(1 - \frac{1}{q})} + 1\| U_0, \quad (4.29) \\
\|\nabla^m c_1(t)\|_{L^q} &\leq Ce^{-\lambda t}(\|c_1(0)\| + \|\nabla^{m+3(1 - \frac{1}{q})} + 1\| c_1(0)), \quad (4.30) \\
\|\nabla^m c_2(t)\|_{L^q} &\leq Ce^{-\lambda t}(\|c_2(0)\| + \|\nabla^{m+3(1 - \frac{1}{q})} + 1\| c_2(0)), \quad (4.31)
\end{align*}
\]

where $C = C(m, q)$ and $\lfloor 3(\frac{1}{2} - \frac{1}{q}) \rfloor$ is defined by

\[
\lfloor 3(\frac{1}{2} - \frac{1}{q}) \rfloor = \begin{cases} 0 & \text{if } q = 2 \\
\lfloor 3(\frac{1}{2} - \frac{1}{q}) \rfloor + 1 & \text{otherwise,}
\end{cases}
\]

where $\lfloor \cdot \rfloor$ denotes the integer part of the argument.

**Proof.** Take $2 \leq q \leq \infty$ and let $m \geq 0$ be an integer. Set $U = e^{Bt}U_0$. Using the Haussdorf-Young inequality and (4.20), we prove (4.28) as follows,

\[
\|\nabla^m \rho(t)\|_{L^q(\mathbb{R}^d)} \leq C\|\xi^m \hat{\rho}(\xi, t)\|_{L^q(\mathbb{R}^d)} \leq C\|\xi^m \hat{\rho}(\xi, t)\|_{L^q(|\xi| \leq \epsilon)} + C\|\xi^m \hat{\rho}(\xi, t)\|_{L^q(|\xi| \geq \epsilon)}, \quad (4.32)
\]

where $\frac{1}{q} + \frac{1}{q'} = 1$.

We estimate the first term of (4.32) as

\[
\begin{align*}
\|\xi^m \hat{\rho}(\xi, t)\|_{L^q(\mathbb{R}^d)} &\leq C \int_{|\xi| \leq \epsilon} \left( |\xi|^m q e^{-q\lambda |\xi|^2 t} + |\xi|^{m+2} q e^{-q\nu t} \right) \hat{\rho}(\xi) d \xi \\
&+ C \int_{|\xi| \leq \epsilon} \left( |\xi|^m q e^{-q\lambda |\xi|^2 t} + |\xi|^{m+2} q e^{-q\nu t} \right) \hat{u}(\xi) d \xi \\
&+ C \int_{|\xi| \leq \epsilon} \left( |\xi|^m q e^{-q\lambda |\xi|^2 (t+1)} + |\xi|^{m+2} q e^{-q\lambda t} \right) d \xi \\
&+ C \sup_{\xi} \hat{\rho}(\xi) \int_{|\xi| \leq \epsilon} \left( |\xi|^m q e^{-q\lambda |\xi|^2 (t+1)} + |\xi|^{m+2} q e^{-q\nu t} \right) d \xi \\
&\leq C \sup_{\xi} \hat{\rho}(\xi) \int_{|\xi| \leq \epsilon} \left( |\xi|^m q e^{-q\lambda |\xi|^2 (t+1)} + |\xi|^{m+2} q e^{-q\nu t} \right) d \xi
\end{align*}
\]
Thus,

\[ + C \sup_{\xi} |\tilde{\psi}_0| e^{q^* \lambda t} \int_{|\xi| \leq \epsilon} (|\xi|^{m+2} a^* |e^{-q^* \lambda t}| + |\xi|^{m+2} a^* e^{-q^* \lambda t}) d\xi \]

\[ + C \sup_{\xi} |\tilde{c}_{1,0}(\xi), \tilde{c}_{2,0}| e^{q^* \lambda t} \int_{|\xi| \leq \epsilon} (|\xi|^{m+2} a^* |e^{-q^* \lambda t}| + |\xi|^{m+2} a^* e^{-q^* \lambda t}) d\xi \]

\[ + \sup_{|\xi| \leq \epsilon} \| \phi \|_{L^1} \left[ \int_{|\xi| < \epsilon} \| e^{-q^* \lambda t} |\hat{\psi}_0| \|_{L^1} \right] \]

\[ \leq C(1 + t) e^{-\frac{3}{2} t} \sup_{|\xi| \leq \epsilon} \| \phi \|_{L^1} + C(1 + t) e^{-\frac{3}{2} t} \| \phi \|_{L^1} + C e^{-q^* \lambda t} \| \psi_0 \|_{L^1} \]

\[ + C(1 + t) e^{-\frac{3}{2} t} \| \phi \|_{L^1} \]

Thus,

\[ \| \xi \|^{m+2} \| e^{-q^* \lambda t} |\hat{\psi}_0| \|_{L^1} \]

\[ \leq C(1 + t) e^{-\frac{3}{2} t} \sup_{|\xi| \leq \epsilon} \| \phi \|_{L^1} + C(1 + t) e^{-\frac{3}{2} t} \| \phi \|_{L^1} + C e^{-q^* \lambda t} \| \psi_0 \|_{L^1} \]

\[ + C(1 + t) e^{-\frac{3}{2} t} \| \phi \|_{L^1} \]

(4.33)

Now, we estimate the second term of (4.32) using the Hölder inequality and fixing \( \epsilon > 0 \) small enough. Then,

\[ \| \xi \|^{m+2} \| e^{-q^* \lambda t} |\hat{\psi}_0| \|_{L^1} \]

\[ \leq C(1 + t) e^{-\frac{3}{2} t} \sup_{|\xi| \leq \epsilon} \| \phi \|_{L^1} + C(1 + t) e^{-\frac{3}{2} t} \| \phi \|_{L^1} + C e^{-q^* \lambda t} \| \psi_0 \|_{L^1} \]

\[ + C(1 + t) e^{-\frac{3}{2} t} \| \phi \|_{L^1} \]

(4.34)

Substituting (4.33) and (4.34) in (4.32), we obtain (4.28).

The proof of (4.29) is similar. Using the Hausdorff-Young inequality

\[ \| \nabla u(t) \|_{L^q(\mathbb{R}^2)} \leq C \| \xi \|^{m} \hat{\psi}(\xi, t) \|_{L^q(\mathbb{R}^2)} \]

\[ \leq C \| \xi \|^{m} \hat{\psi}(\xi, t) \|_{L^q(|\xi| \leq \epsilon)} + C \| \xi \|^{m} \hat{\psi}(\xi, t) \|_{L^q(|\xi| \geq \epsilon)} \]

(4.35)

and (4.21), we estimate the first term of (4.35) as

\[ \| \xi \|^{m} \hat{\psi}(\xi, t) \|_{L^q(|\xi| \leq \epsilon)} \leq C \int_{|\xi| \leq \epsilon} (|\xi|^{m+2} a^* |e^{-q^* \lambda t}| + |\xi|^{m+2} a^* e^{-q^* \lambda t}) |\hat{\psi}_0| d\xi \]

\[ + C \int_{|\xi| \leq \epsilon} (|\xi|^{m+2} a^* |e^{-q^* \lambda t}| + |\xi|^{m+2} a^* e^{-q^* \lambda t}) |\hat{\psi}_0| d\xi \]

(4.36)
Similarly to obtaining (4.34), one has
\[ C \int_{\xi \in \mathcal{R}} (|\xi|^{(m+3)}q e^{-q'\lambda \xi^2(t+1)} + |\xi|^{m q' e^{-q'\nu \lambda t}}
+ |\xi|^{m q' e^{-q'\lambda t}}) \xi_{1,0}(\xi, \xi_{2,0}(\xi)) q' d\xi \]
\[ \leq C (1 + t)^{-\frac{m q' + q' + \lambda t}{2}} \| \rho_0 \|_{L^1} + (1 + t)^{-\frac{m q' + q' + \lambda t}{2}} \| u_0 \|_{L^1}
+ C (1 + t)^{-\frac{m q' + q' + \lambda t}{2}} \| c_{1,0} \|_{L^1} + C e^{-q'\nu \lambda t} \| U_0 \|_{L^1} + C e^{-q'\lambda t} \| [c_{1,0}, c_{2,0}] \|_{L^1}. \]

It follows that
\[ \| \xi \|^m \check{a}(t, \xi, L^\nu((|\xi| \leq \xi)) \leq C (1 + t)^{-\frac{3}{2} - \frac{m q' + q' + \lambda t}{2}} \| \rho_0 \|_{L^1} + (1 + t)^{-\frac{3}{2} - \frac{m q' + q' + \lambda t}{2}} \| u_0 \|_{L^1}
+ (1 + t)^{-\frac{3}{2} - \frac{m q' + q' + \lambda t}{2}} \| c_{1,0} \|_{L^1} + C e^{-q'\nu \lambda t} \| U_0 \|_{L^1} + C e^{-q'\lambda t} \| [c_{1,0}, c_{2,0}] \|_{L^1}, \]
\[ \| \xi \|^m \check{a}(t, \xi, L^\nu((|\xi| \geq \xi)) \leq C e^{-q'\nu \lambda t} \| \nabla^{m+3} \|_{L^1} - \frac{1}{2} - \| U_0 \|. \] (4.37)

Thus, plugging (4.37) and (4.36) into (4.35) implies (4.29). We prove (4.30) and (4.31) in the similar way. This completes the proof of Theorem 4.2.

**Corollary 1.** Assume that \( U = e^{Bt}U_0 \) is the solution to the Cauchy problem (4.4) with initial data \( U_0 = [\rho_0, u_0, c_{1,0}, c_{2,0}] \). Then \( U = [\rho, u, c_{1}, c_{2}] \) satisfies the following:

\[ \| \rho(t) \| \leq C (1 + t)^{-\frac{3}{2}} \| U_0 \|_{L^1} + e^{-\lambda t} \| U_0 \|, \]
\[ \| u(t) \| \leq C (1 + t)^{-\frac{3}{2}} \| U_0 \|_{L^1} + e^{-\lambda t} \| U_0 \|, \]
\[ \| \rho(t) \|_{L^\infty} \leq C (1 + t)^{-\frac{3}{2}} \| U_0 \|_{L^1} + e^{-\lambda t} \| \nabla^2 U_0 \|, \]
\[ \| u(t) \|_{L^\infty} \leq C (1 + t)^{-\frac{3}{2}} \| U_0 \|_{L^1} + e^{-\lambda t} \| \nabla^2 U_0 \|, \]
\[ \| c_{1}(t) \| \leq C e^{-\lambda t} \| c_{1,0} \|, \]
\[ \| c_{2}(t) \| \leq C e^{-\lambda t} \| c_{2,0} \|, \]
\[ \| c_{1}(t) \|_{L^\infty} \leq C e^{-\lambda t} (\| c_{1,0} \| + \| \nabla^2 c_{1,0} \|), \]
\[ \| c_{2}(t) \|_{L^\infty} \leq C e^{-\lambda t} (\| c_{2,0} \| + \| \nabla^2 c_{2,0} \|). \]

**5. Time-Decay Rates for the nonlinear system.** In this section, we will prove Proposition 2 and Proposition 3. The main idea is to combine the energy estimates and spectral analysis. We apply the linear \( L^2 - L^3 \) time-decay property of the homogeneous system (4.4) studied in the previous section to the nonlinear case. We need the mild form of the original nonlinear Cauchy problem (2.2). Throughout this section, we suppose that \( U = [\rho, u, c_{1}, c_{2}] \) is the solution to the Cauchy problem (2.2) with initial data \( U_0 = (\rho_0, u_0, c_{1,0}, c_{2,0}) \) satisfying (2.3).

Then, by Duhamel’s principle, the solution \( U = [\rho, u, c_{1}, c_{2}] \) can be formally written as
\[ U(t) = e^{Bt}U_0 + \int_0^t e^{(t-s)B} [g_1, g_2, g_3, g_4] ds, \]
where \( e^{Bt}, t \geq 0, \) is called the linear solution operator and the nonlinear source term takes the form (4.3).
5.1. Decay rates for the energy functional and high-order energy functional. In this subsection, we will prove the decay rate for the energy functional \(\|U(t)\|_N^2\) and the decay rate for the high-order energy functional \(\|\nabla U(t)\|_N^2\). For that, we investigate the time-decay rates of solutions in Proposition 1 under an extra condition (2.8).

Proof of Proposition 2. Suppose \(\epsilon_{N+1}(U_0)\) is sufficiently small. Then, from Proposition 1 the solution \(U = [\rho, u, c_1, c_2]\) satisfies:

\[
\frac{d}{dt} \mathcal{E}_N(U(t)) + \lambda_1 \mathcal{D}_N(U(t)) + \lambda_2 \mathcal{D}_N^h(U(t)) \leq 0,
\]

for any \(t \geq 0\), where \(\mathcal{E}_N(U(t)) \sim \|\rho, u, c_1, c_2\|_N^2\) denotes the energy functional and \(\mathcal{D}_N(U(t)) \sim \|\nabla(c_1, c_2)\|_N^2\) and \(\mathcal{D}_N^h(U(t)) \sim \|\nabla\rho\|_{N-1}^2 + \|u, c_1, c_2\|_N^2\) the dissipation rates.

Now, we begin with the time-weighted estimate and iteration for inequality (5.2). Let \(l \geq 0\). Multiplying (5.2) by \((1 + t)^l\) and integrating over \([0, t]\) give

\[
(1 + t)^l \mathcal{E}_N(U(t)) + \lambda_1 \int_0^t (1 + s)^l \mathcal{D}_N(U(s)) ds + \lambda_2 \int_0^t (1 + s)^l \mathcal{D}_N^h(U(s)) ds \\
\leq \mathcal{E}_N(U_0) + l \int_0^t (1 + s)^l \mathcal{E}_N(U(s)) ds
\]

\[
\leq \mathcal{E}_N(U_0) + CL \int_0^t (1 + s)^l \mathcal{E}_N(U(s)) ds + \mathcal{D}_N^h(U_0) + \|\rho(s)\|^2 ds,
\]

where we have used

\[
\mathcal{E}_N(U(t)) \leq CD_{N-1}(U(t)) + C\mathcal{D}_N^h(U(t)) + \|\rho(t)\|^2.
\]

Using (5.2) again, we have

\[
\mathcal{E}_{N+1}(U(t)) + \lambda_1 \int_0^t \mathcal{D}_{N+1}(U(t)) + \lambda_2 \int_0^t \mathcal{D}_{N+1}^h(U(t)) \leq \mathcal{E}_{N+1}(U_0),
\]

and

\[
(1 + t)^l \mathcal{E}_{N+1}(U(t)) + \lambda_1 \int_0^t (1 + s)^l \mathcal{D}_{N+1}(U(s)) ds
\]

\[
+ \lambda_2 \int_0^t (1 + s)^l \mathcal{D}_{N+1}^h(U(s)) ds \leq \mathcal{E}_{N+1}(U_0)
\]

\[
+ CL \int_0^t (1 + s)^l \mathcal{E}_{N+1}(U(s)) ds \leq \mathcal{E}_{N+1}(U_0(t))
\]

\[
+ CL \int_0^t (1 + s)^l \mathcal{D}_{N}U(s) + C\mathcal{D}_{N+1}^h(U(s)) + \|\rho(s)\|^2 ds.
\]

Then, for \(1 < l < 2\), it follows by iterating the previous estimates that

\[
(1 + t)^l \mathcal{E}_N(U(t)) + \lambda_1 \int_0^t (1 + s)^l \mathcal{D}_N(U(s)) ds + \lambda_2 \int_0^t (1 + s)^l \mathcal{D}_N^h(U(s)) ds
\]

\[
\leq \mathcal{E}_{N+1}(U_0) + C \int_0^t (1 + s)^l \|\rho(s)\|^2 ds.
\]

On the other hand, to estimate the integral term on the right-hand side of the previous inequality, let us define

\[
\mathcal{E}_{N,\infty}(U(t)) = \sup_{0 \leq s \leq T} (1 + t)^{\frac{3}{2}} \mathcal{E}_N(U(t)).
\]
By applying the linear estimate on $\rho$ in (4.38) to the mild form (5.1), one has
\[
\|\rho(t)\| \leq C(1 + t)^{-\frac{3}{2}} \|U_0\|_{L^1} + C e^{-\lambda t} \|U_0\| + C \int_0^t (1 + s)^{-\frac{3}{2}} \|g_1(t, g_2, g_3, g_4(s))\|_{L^1} ds + C \int_0^t e^{-\lambda(t-s)} \|g_1(t, g_2, g_3, g_4(s))\| ds.
\]
(5.4)

Recall the definitions (4.3) of $g_1$ and $g_2$. It is direct to check that for any $0 \leq s \leq t$,
\[
\|g_1(t, g_2, g_3, g_4(s))\|_{L^1 \cap L^2} \leq C \mathcal{E}_N(U(t)) \leq C (1 + s) \frac{3}{2} \mathcal{E}_{N, \infty}(U(t)).
\]
Putting this into (5.4) gives
\[
\|\rho(t)\| \leq C(1 + t)^{-\frac{3}{2}} \|U_0\|_{L^1 \cap L^2} + \mathcal{E}_{N, \infty}(U(t)).
\]
(5.5)

Next, we prove the uniform-in-time bound of $\mathcal{E}_{N, \infty}(U(t))$ which implies the decay rates of the energy functional $\mathcal{E}_N(U(t))$. In fact, by taking $t = \frac{3}{2} + \epsilon$ in (5.3) where $\epsilon > 0$ is small enough, it follows that
\[
(1 + t)^{\frac{5}{2} + \epsilon} \mathcal{E}_N(U(t)) + \lambda_1 \int_0^t (1 + s)^{\frac{5}{2} + \epsilon} D_N(U(s)) ds + \lambda_2 \int_0^t (1 + s)^{\frac{5}{2} + \epsilon} D_N^h(U(s)) ds
\leq \mathcal{E}_{N+1}(U_0) + C \int_0^t (1 + s)^{\frac{5}{2} + \epsilon} \|\rho(s)\|^2 ds.
\]
Here, using (5.5), we obtain
\[
\int_0^t (1 + s)^{\frac{5}{2} + \epsilon} \|\rho(t)\|^2 ds \leq C(1 + t)^{\epsilon} (\mathcal{E}_{N, \infty}^2(U(t)) + \|U_0\|^2_{L^1 \cap L^2}).
\]
Therefore, it follows that
\[
(1 + t)^{\frac{5}{2} + \epsilon} \mathcal{E}_N(U(t)) + \lambda_1 \int_0^t (1 + s)^{\frac{5}{2} + \epsilon} D_N(U(s)) ds + \lambda_2 \int_0^t (1 + s)^{\frac{5}{2} + \epsilon} D_N^h(U(s)) ds
\leq \mathcal{E}_{N+1}(U_0) + C(1 + t)^{\epsilon} (\mathcal{E}_{N, \infty}^2(U(t)) + \|U_0\|^2_{L^1 \cap L^2}),
\]
which implies
\[
(1 + t)^{\frac{5}{2}} \mathcal{E}_N(U(t)) \leq C(\mathcal{E}_{N+1}(U_0) + \|\rho_0, u_0\|^2_{L^1} + \mathcal{E}_{N, \infty}^2(U(t))),
\]
and thus
\[
\mathcal{E}_{N, \infty}(U(t)) \leq C(\mathcal{E}_{N+1}^2(U_0) + \mathcal{E}_{N, \infty}^2(U(t))).
\]
Since $\mathcal{E}_{N+1}(U_0) > 0$ is sufficiently small, it holds that $\mathcal{E}_{N, \infty}(U(t)) \leq C \mathcal{E}_{N+1}^2(U_0)$ for any $t \geq 0$, which gives $\|U(t)\|_N \leq C(\mathcal{E}_N(U(t)))^{\frac{1}{2}} \leq C \mathcal{E}_{N+1}(U_0)(1 + t)^{-\frac{3}{4}}$. This proves (2.9).

Now, we estimate the high-order energy functional. By comparing the definitions of $\mathcal{E}_N(U(t))$, $D_N(U(t))$, and $D_N^h(U(t))$, it follows from (5.2) that
\[
\frac{d}{dt} \|\nabla U(t)\|_N^2 + \lambda \|\nabla U(t)\|_N^2 \leq C \|\nabla \rho(t)\|^2,
\]
which implies
\[
\|\nabla U(t)\|_N^2 \leq e^{-\lambda t} \|\nabla U_0\|_N^2 + C \int_0^t e^{-\lambda(t-s)} \|\nabla \rho(s)\|^2 ds.
\]
(5.6)

for any $t \geq 0$. To estimate the time integral term on the (r.h.s.) of the above inequality, one can apply the linear estimate (4.28) to the mild form (5.1) of the solution $U(t)$ so that
||\nabla \rho(t)|| \leq C(1 + t)^{-\frac{3q}{2}}||U_0||_{L^1} + C e^{-\lambda t}||\nabla U_0|| \\
+ C \int_0^t (1 + t - s)^{-\frac{3q}{2}} ||g_1, g_2, g_3, g_4(s)||_{L^1} ds + C \int_0^t e^{-\lambda(t-s)} ||\nabla [g_1, g_2, g_3, g_4(s)]|| ds.
\tag{5.7}

Recall the definition (4.3) of \( g_1, g_2, g_3 \) and \( g_4 \). It is straightforward to check that for any \( 0 \leq s \leq t \)
\[ ||[g_1, g_2, g_3, g_4(s)]||_{L^1 \cap H^1} \leq C \mathcal{E}_N U(s) \leq C \epsilon_{N+1}^2 (U_0)(1 + s)^{-\frac{3}{2}}. \]
Putting this into (5.7) gives
\[ ||\nabla \rho(t)|| \leq C \epsilon_{N+1} (U_0)(1 + t)^{-\frac{3}{2}}. \tag{5.8} \]

Then, by using (5.8) in (5.6), we have
\[ ||\nabla U(t)||_{H^N} \leq e^{-\lambda t} ||\nabla U_0||_{H^N} + C \epsilon_{N+1}^2 (U_0)(1 + t)^{-\frac{3}{2}}, \]
which implies (2.10). The proof of Proposition 2 is complete.

5.2. Decay rates in \( L^q \). In this subsection, we will prove Proposition 3 for time-decay rates in \( L^q \) corresponding to (1.6)-(1.8) in Theorem 1.1. For \( N \geq 4 \), Proposition 2 shows that if \( \epsilon_{N+1}(U_0) \) is small enough,
\[ ||U(s)||_{H^2} \leq C \epsilon_{N+1}(U_0)(1 + t)^{-\frac{3}{2}}, \tag{5.9} \]
and
\[ ||\nabla U(t)||_{H^N} \leq C \epsilon_{N+1}(U_0)(1 + t)^{-\frac{3}{2}}. \tag{5.10} \]

Now, let us establish the estimates on \( \rho, u, \) and \([c_1, c_2]\) in the following.

Estimate on \( ||\rho(t)||_{L^q} \). For the \( L^2 \) rate, it is easy to see from (5.5) and (5.9) that
\[ ||\rho(t)||_{L^2} \leq C \epsilon_{N+1}(U_0)(1 + t)^{-\frac{3}{2}} \leq C(1 + t)^{-\frac{3}{2}}. \]

For the \( L^\infty \) rate, by applying the \( L^\infty \) linear estimate on \( \rho \) in (4.40) to the mild form (5.1), we have
\[ ||\rho(t)||_{L^\infty} \leq C(1 + t)^{-\frac{3}{2}} ||U_0||_{L^1} + C e^{-\lambda t} ||\nabla U_0|| + C \int_0^t (1 + t - s)^{-\frac{3}{2}} ||[g_1, g_2, g_3, g_4(s)]||_{L^1} ds \\
+ C \int_0^t e^{-\lambda(t-s)} ||\nabla [g_1, g_2, g_3, g_4(s)]|| ds \\
\leq C(1 + t)^{-\frac{3}{2}} ||U_0||_{L^1 \cap H^2} + C \int_0^t (1 + t - s)^{-\frac{3}{2}} ||[g_1, g_2, g_3, g_4(s)]||_{L^1 \cap H^2} ds. \tag{5.11} \]

Since by (5.9)
\[ ||[g_1, g_2, g_3, g_4(s)]||_{L^1 \cap H^2} \leq C ||\nabla U(t)||_{H^N} ||U(s)||_{H^N} + C ||U(s)||_{H^N} \leq C \epsilon_{N+1}^2 (U_0)(1 + s)^{-\frac{3}{2}}, \]
putting the above inequality into (5.11), gives
\[ ||\rho(t)||_{L^\infty} \leq C \epsilon_{N+1}(U_0)(1 + t)^{-\frac{3}{2}}. \]

Then, by \( L^2 \) \(-\) \( L^\infty \) interpolation,
\[ ||\rho||_{L^q} \leq C \epsilon_{N+1}(U_0)(1 + t)^{-\frac{3}{2}} \tag{5.12} \]
for \( 2 \leq q \leq \infty \).

Estimate on \( ||u(t)||_{L^q} \). For the \( L^2 \) rate, utilizing the \( L^2 \) estimate on \( u \) in (4.39) to (5.1), we have
\[ \|u(t)\| \leq C(1 + t)\frac{\beta^2}{2} \|U_0\|_{L^1} + Ce^{-\lambda t}\|U_0\| + C\int_0^t (1 + t - s)^{-\frac{\beta^2}{2}} \|g_1, g_2, g_3, g_4\|_{L^1} ds \]
\[+ \int_0^t e^{-\lambda(t-s)} \|[g_1, g_2, g_3, g_4](s)\|_{L^1} ds. \] (5.13)

Due to (5.9),
\[ \|[g_1, g_2, g_3, g_4](s)\|_{L^1 \cap L^2} \leq C\|U(s)\|_N^2 \leq C\epsilon_{N+1}(U_0)(1 + t)^{-\frac{\beta^2}{2}}. \]

Then (5.13) implies the slower decay estimate
\[ \|u(t)\| \leq C\epsilon_{N+1}(U_0)(1 + t)^{-\frac{\beta^2}{2}} \leq C(1 + t)^{-\frac{\beta^2}{2}}. \] (5.14)

For the \(L^\infty\) rate, utilizing the \(L^\infty\) estimate on \(u\) in (4.41) to (5.1), we have
\[ \|u(t)\|_{L^\infty} \leq (1 + t)^{-2}\|U_0\|_{L^1 \cap H^2} + C\int_0^t (1 + t - s)^{-2} \|[g_1, g_2, g_3, g_4](s)\|_{L^1 \cap H^2} ds. \] (5.15)

Since by (5.9) and (5.10)
\[ \|[g_1, g_2, g_3, g_4](s)\|_{L^1 \cap H^2} \leq C\|\nabla U(t)\|_N\|U(s)\|_N + C\|U(s)\|_N^3 \leq C\epsilon_{N+1}(U_0)(1 + s)^{-2} + C\epsilon_{N+1}^2(U_0)(1 + s)^{-\frac{\beta^2}{2}} \leq C\epsilon_{N+1}^2(U_0)(1 + s)^{-\frac{\beta^2}{2}}. \]

It follows from (5.15) that
\[ \|u(t)\|_{L^\infty} \leq C\epsilon_{N+1}(U_0)(1 + s)^{-\frac{\beta^2}{2}}. \]

Therefore, by \(L^2 - L^\infty\) interpolation,
\[ \|u(t)\|_{L^q} \leq C\epsilon_{N+1}(U_0)(1 + t)^{-\frac{\beta^2}{2} + \frac{q}{\beta^2}} \] (5.16)
for \(2 \leq q \leq \infty\).

Estimate on \(\|c_1(t)\|_{L^q}\). For the \(L^2\) rate, we utilize the \(L^2\) estimate on \(c_1\) in (4.42) to (5.1), we have
\[ \|c_1\| \leq Ce^{-\lambda t}\|c_1, 0\|_{L^2} + C\int_0^t e^{-\lambda(t-s)}\|g_3\|_{L^2} ds. \] (5.17)

Since
\[ \|g_3(s)\| \leq C\|U(s)\|_N^2 \leq C\epsilon_{N+1}(U_0)(1 + t)^{-\frac{\beta^2}{2}}. \]

Then (5.17) implies the slower decay estimate
\[ \|c_1\| \leq C\epsilon_{N+1}(U_0)(1 + t)^{-\frac{\beta^2}{2}}. \] (5.18)

Similarly, we have
\[ \|c_2\| \leq C\epsilon_{N+1}(U_0)(1 + t)^{-\frac{\beta^2}{2}}. \] (5.19)

For \(L^\infty\) rate, we can utilize the \(L^\infty\) estimate on \(c_1\) in (4.44) to (5.1), we get
\[ \|c_1\|_{L^\infty} \leq Ce^{-\lambda t}\|c_1, 0\|_{L^1 \cap H^2} + C\int_0^t e^{-\lambda(t-s)}\|g_3\|_{L^1 \cap H^2} ds. \] (5.20)

From (5.9), we obtain
\[ \|g_3(s)\|_{L^1 \cap H^2} \leq C\|U(s)\|_N^2 \leq C\epsilon_{N+1}(U_0)(1 + t)^{-\frac{\beta^2}{2}}, \]
and thus
\[ \|c_1\|_{L^\infty} \leq C\epsilon_{N+1}(U_0)(1 + t)^{-\frac{\beta^2}{2}}. \] (5.21)

Similarly, we have
\[ \|c_2\|_{L^\infty} \leq C\epsilon_{N+1}(U_0)(1 + t)^{-\frac{\beta^2}{2}}. \] (5.22)
So, by $L^2 - L^\infty$ interpolation,

$$\| [c_1, c_2] \|_{L^q} \leq C \epsilon N+1 (U_0) (1 + t)^{-u}, \quad (5.23)$$

for $2 \leq q \leq \infty$.

This completes the proof of Proposition 2 and hence Theorem 1.1.

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