LIE-ALGEBRAS AND LINEAR OPERATORS WITH INVARIANT SUBSPACES

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Abstract. A general classification of linear differential and finite-difference operators possessing a finite-dimensional invariant subspace with a polynomial basis (the generalized Bochner problem) is given. The main result is that any operator with the above property must have a representation as a polynomial element of the universal enveloping algebra of some algebra of differential (difference) operators in finite-dimensional representation plus an operator annihilating the finite-dimensional invariant subspace. In low dimensions a classification is given by algebras $sl_2(\mathbb{R})$ (for differential operators in $\mathbb{R}$) and $sl_2(\mathbb{R})_q$ (for finite-difference operators in $\mathbb{R}$), $osp(2, 2)$ (operators in one real and one Grassmann variable, or equivalently, $2 \times 2$ matrix operators in $\mathbb{R}$), $sl_3(\mathbb{R})$, $sl_2(\mathbb{R}) \oplus sl_2(\mathbb{R})$ and $gl_2(\mathbb{R}) \ltimes \mathbb{R}^{r+1}$, $r$ a natural number (operators in $\mathbb{R}^2$). A classification of linear operators possessing infinitely many finite-dimensional invariant subspaces with a basis in polynomials is presented. A connection to the recently-discovered quasi-exactly-solvable spectral problems is discussed.

S. Bochner (1929) asked about a classification of differential equations

$$T\varphi = \epsilon \varphi \quad (0)$$

where $T$ is a linear differential operator of $k$th order in one real variable $x \in \mathbb{R}$ and $\epsilon$ is the spectral parameter, having an infinite sequence of orthogonal polynomial solutions [1] (see also [2]).

Definition 0.1. Let us give the name of the generalized Bochner problem to the problem of classification of linear differential (difference) operators, for which the eigenvalue problem (0) has a certain number of eigenfunctions in the form of a finite-order polynomial in some variables.

Following this definition the original Bochner problem is simply a particular case. In [3] a general method has been formulated for generating eigenvalue problems for linear differential operators, linear matrix differential operators and linear finite-difference operators in one and several variables possessing polynomial solutions. The method was based on considering the eigenvalue problem for the representation of a polynomial element of the universal enveloping algebra of the Lie algebra in a finite-dimensional, ‘projectivized’ representation of this Lie algebra [3]. Below it is shown that this method provides both necessary and sufficient conditions for the
existence of polynomial solutions of linear differential equations and a certain class of finite-difference equations.

The generalized Bochner problem can be subdivided into two parts: (i) a classification of linear operators possessing an invariant subspace (subspaces) with a basis in polynomials of finite degree and (ii) a description of the conditions under which such operators are symmetrical. This paper will be devoted to a solution of the first problem; as for the second one, only the first step has been done (see below). The plan of the paper is the following: Section 1 is devoted to the case of differential operators in one real variable; in Section 2 finite-difference operators in $\mathbb{R}$ are treated; operators in one real and one Grassmann variables are considered in Section 3, while the case of operators in $\mathbb{R}^2$ is given in Section 4. The general situation is described in the Conclusion.

1. Ordinary differential equations

Consider the space of all polynomials of order $n$

$$\mathcal{P}_n = \langle 1, x, x^2, \ldots, x^n \rangle,$$

where $n$ is a non-negative integer and $x \in \mathbb{R}$.

**Definition 1.1.** Let us name a linear differential operator of the $k$th order, $T_k$, quasi-exactly-solvable, if it preserves the space $\mathcal{P}_n$. Correspondingly, the operator $E_k$, which preserves the infinite flag $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \cdots \subset \mathcal{P}_n \subset \cdots$ of spaces of all polynomials, is named exactly-solvable.

**Lemma 1.1.** (i) Suppose $n > (k - 1)$. Any quasi-exactly-solvable operator $T_k$, can be represented by a $k$-th degree polynomial of the operators

$$J^+ = x^2 \partial_x - nx,$$

$$J^0 = x \partial_x - \frac{n}{2},$$

$$J^- = \partial_x,$$

(the operators (2) obey the $\mathfrak{sl}_2(\mathbb{R})$ commutation relations $[\mathfrak{sl}_2(\mathbb{R})]$). If $n \leq (k - 1)$, the part of the quasi-exactly-solvable operator $T_k$ containing derivatives up to order $n$ can be represented by an $n$th degree polynomial in the generators (2).

(ii) Conversely, any polynomial in (2) is quasi-exactly solvable.

(iii) Among quasi-exactly-solvable operators there exist exactly-solvable operators $E_k \subset T_k$.

**Comment 1.** If we define the universal enveloping algebra $U_g$ of a Lie algebra $g$ as the algebra of all ordered polynomials in generators, then $T_k$ at $k < n + 1$ is simply an element of the universal enveloping algebra $U_{\mathfrak{sl}_2(\mathbb{R})}$ of the algebra $\mathfrak{sl}_2(\mathbb{R})$ taken in representation (2). If $k \geq n + 1$, then $T_k$ is represented as an element of $U_{\mathfrak{sl}_2(\mathbb{R})}$ plus $B \frac{d^{n+1}}{dx^{n+1}}$, where $B$ is any linear differential operator of order not higher than $(k - n - 1)$.

**Proof.** The essential part of the proof is based on the Burnside theorem (see, e.g., [4]):

1 The representation (2) is one of the ‘projectivized’ representations (see [3]). This realization of $\mathfrak{sl}_2(\mathbb{R})$ has been derived at the first time by Sophus Lie.

2 Burnside theorem is a particular case of more general Jacobson theorem (see [3], Chapter XVII.3). I am grateful to V. Kac for this comment.
Let $A_1, \ldots, A_k$ be linear operators in a linear space $E$ over real numbers, $\dim E < \infty$. Let us assume that there is no linear space $L$ over real numbers, $0 < \dim L < \dim E$, such that $A_i : L \mapsto L$ for all $i = 1, 2, \ldots, k$. Then any linear operator acting in $E$ can be represented as a polynomial in $A_1, \ldots, A_k$.

The operators $J^\pm, 0$ act in $P_n$ irreducibly and therefore the theorem can be applied. Also there exist operators $B$ having $P_n$ as a kernel, $B : P_n \mapsto 0$. Clearly, those operators have a form $B(x, \partial_x) d^{n+1}x^n + dx^n$, where $B(x, \partial_x)$ is any linear differential operator, and they make no contribution in $T_k$ if $k < n + 1$. It completes the proof of part (i). Parts (ii) and (iii) are easy to prove based on part (i).

Since $sl_2(\mathbb{R})$ is a graded algebra, let us introduce the grading of generators (2):

$$\deg(J^+) = +1, \quad \deg(J^0) = 0, \quad \deg(J^-) = -1,$$

hence

$$\deg[(J^+)^n (J^0)^m (J^-)^p] = n_+ - n_-.$$  

The grading allows us to classify the operators $T_k$ in the Lie-algebraic sense.

**Lemma 1.2.** A quasi-exactly-solvable operator $T_k \subset U_{sl_2(\mathbb{R})}$ has no terms of positive grading, if and only if it is an exactly-solvable operator.

**Theorem 1.1.** Let $n$ be a non-negative integer. Take the eigenvalue problem for a linear differential operator of the $k$th order in one variable

$$T_k \varphi = \varepsilon \varphi,$$

where $T_k$ is symmetric. The problem (5) has $(n + 1)$ linearly independent eigenfunctions in the form of a polynomial in variable $x$ of order not higher than $n$, if and only if $T_k$ is quasi-exactly-solvable. The problem (5) has an infinite sequence of polynomial eigenfunctions, if and only if the operator is exactly-solvable.

**Comment 2.** The “if” part of the first statement is obvious. The “only if” part is a direct corollary of Lemma 1.1.

This theorem gives a general classification of differential equations

$$\sum_{j=0}^{k} a_j(x) \varphi^{(j)}(x) = \varepsilon \varphi(x)$$

having at least one polynomial solution in $x$. The coefficient functions $a_j(x)$ must have the form

$$a_j(x) = \sum_{i=0}^{k+j} a_{j,i} x^i$$

The explicit expressions (7) for coefficient function in (6) are obtained by the substitution (2) into a general, $k$th degree polynomial element of the universal enveloping algebra $U_{sl_2(\mathbb{R})}$. Thus the coefficients $a_{j,i}$ can be expressed through the coefficients of the $k$th degree polynomial element of the universal enveloping algebra $U_{sl_2(\mathbb{R})}$.

The number of free parameters of the polynomial solutions is defined by the number of parameters characterizing a general $k$th degree polynomial element of the universal enveloping algebra $U_{sl_2(\mathbb{R})}$.
universal enveloping algebra $U_{sl_2}(\mathbb{R})$. A rather straightforward calculation leads to the following formula

$$\text{par}(T_k) = (k + 1)^2$$

where we denote the number of free parameters of operator $T_k$ by the symbol $\text{par}(T_k)$. For the case of an infinite sequence of polynomial solutions expression (7) simplifies to

$$a_j(x) = \sum_{i=0}^{j} a_{j,i} x^i$$

in agreement with the results of H.L. Krall’s classification theorem [1] (see also [2]).

In this case the number of free parameters is equal to

$$\text{par}(E_k) = \frac{(k + 1)(k + 2)}{2}$$

In the present approach Krall’s theorem is simply a description of differential operators of $k$th order in one variable preserving a finite flag $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \cdots \subset \mathcal{P}_k$ of spaces of polynomials. One can easily show that the preservation of such a set of polynomial spaces implies the preservation of an infinite flag of such spaces.

One may ask a more general question: which non-degenerate linear differential operators have a finite-dimensional invariant sub-space of the form

$$\langle \alpha(z), \alpha(z)\beta(z), \ldots, \alpha(z)\beta(z)^n \rangle,$$

where $\alpha(z), \beta(z)$ are some functions. Such operators are easily obtained from the quasi-exactly-solvable operators (see Lemma 1.1) making the change of variable $x = \beta(z)$ and the “gauge” transformation $\tilde{T} = \alpha(z) T \alpha(z)^{-1}$. Since any one- or two-dimensional invariant sub-space can be presented in the form (11), the important general statement takes place: there are no linear operators possessing one- or two-dimensional invariant sub-space with an explicit basis other than given by Lemma 1.1. Therefore for any linear operator possessing one- or two-dimensional invariant sub-space with an explicit basis the eigenvalue problem (5) can be reduced to the form (16) or (15), respectively (see below).

Let us consider the second-order differential equation (6), which can possess polynomial solutions. From Theorem 1.1 it follows that the corresponding differential operator must be quasi-exactly-solvable and can be represented as

$$T_2 = c_{++} J^+ J^+ + c_{+0} J^+ J^0 + c_{+-} J^+ J^- + c_{0-} J^0 J^- + c_{--} J^- J^- +$$

$$c_+ J^+ + c_0 J^0 + c_- J^- + c,$$

where $c_{\alpha\beta}, c_{\alpha}, c \in \mathbb{R}$. The number of free parameters is $\text{par}(T_2) = 9$. Under the condition $c_{++} = c_{+0} = c_+ = 0$, the operator $T_2$ becomes exactly-solvable (see Lemma 1.2) and the number of free parameters is reduced to $\text{par}(E_2) = 6$.

**Lemma 1.3.** If the operator (12) is such that

$$c_{++} = 0 \quad \text{and} \quad c_+ = (\frac{n}{2} - m)c_{+0}, \text{ at some } m = 0, 1, 2, \ldots$$

then the operator $T_2$ preserves both $\mathcal{P}_n$ and $\mathcal{P}_m$. In this case the number of free parameters is $\text{par}(T_2) = 7.$
In fact, Lemma 1.3 means that $T_2(J^n(\alpha), c_{\alpha\beta}, c_{\alpha})$ can be rewritten as $T_2(J^n(\alpha), c_{\alpha\beta}', c_{\alpha}')$. As a consequence of Lemma 1.3 and Theorem 1.1, in general, among polynomial solutions of (6) there are polynomials of order $n$ and order $m$.

**Remark.** From the Lie-algebraic point of view Lemma 1.3 means the existence of representations of second-degree polynomials in the generators (2) possessing two invariant sub-spaces. In general, if $n$ in (2) is a non-negative integer, then among representations of $k$th degree polynomials in the generators (2), lying in the universal enveloping algebra, there exist representations possessing $1, 2, ..., k$ invariant sub-spaces. Even starting from an infinite-dimensional representation of the original algebra ($n$ in (2) is not a non-negative integer), one can construct the elements of the universal enveloping algebra having finite-dimensional representation (e.g., the parameter $n$ in (13) is non-integer, however $T_2$ has the invariant sub-space of dimension $(m + 1)$). Also this property implies the existence of representations of the polynomial elements of the universal enveloping algebra, which can be obtained starting from different representations of the original algebra.

Substituting (2) into (12) and then into (6), we obtain

$$- P_2(x) \partial_x^2 \varphi(x) + P_1(x) \partial_x \varphi(x) + P_2(x) \varphi(x) = \varepsilon \varphi(x),$$

(14)

where the $P_j(x)$ are polynomials of $j$th order with coefficients related to $c_{\alpha\beta}, c_{\alpha}$ and $n$. In general, problem (14) has $(n + 1)$ polynomial solutions. If $n = 1$, as a consequence of Lemma 1.1, a more general spectral problem than (14) arises

$$- F_3(x) \partial_x^2 \varphi(x) + Q_2(x) \partial_x \varphi(x) + Q_1(x) \varphi(x) = \varepsilon \varphi(x),$$

(15)

where $F_3$ is an arbitrary real function of $x$ and $Q_j(x), j = 1, 2$ are polynomials of order $j$, possessing only two polynomial solutions of the form $(ax + b)$. For the case $n = 0$ (one polynomial solution) the spectral problem (6) becomes

$$- F_2(x) \partial_x^2 \varphi(x) + F_1(x) \partial_x \varphi(x) + Q_0 \varphi(x) = \varepsilon \varphi(x),$$

(16)

where $F_{2,1}(x)$ are arbitrary real functions and $Q_0$ is a real constant. After the transformation

$$\Psi(z) = \varphi(x(z)) e^{- A(z)},$$

(17)

where $x(z)$ is a change of the variable and $A(z)$ is a certain real function, one can reduce (14)–(16) to the Sturm-Liouville-type problem

$$(- \partial_x^2 + V(z)) \Psi(z) = \varepsilon \Psi(z),$$

(18)

with the potential, which is equal to

$$V(z) = (A')^2 - A'' + P_2(x(z)),$$

if

$$A = \int \left( \frac{P_3}{P_4} \right) dx - \log z', \ z = \int \frac{dx}{\sqrt{T_4}}.$$ 

for the case of (14). If the functions (17), obtained after transformation, belong to the $L_2(\mathcal{D})$-space, we reproduce the recently discovered quasi-exactly-solvable problems, where a finite number of eigenstates was found algebraically. For example,

$$T_2 = -4J^0 J^- + 4aJ^+ + 4bJ^0 - 2(n + 1 + 2k)J^- + b(2n + 1 + 2k)$$

(19)

4Depending on the change of variable $x = z(x)$, the space $\mathcal{D}$ can be whole real line, semi-line and a finite interval.
leads to the spectral problem (18) \((x = z^2)\) with the potential
\[
V(z) = a^2 z^6 + 2abz^4 + [b^2 - (4n + 3 + 2k)a]z^2,
\]
for which at \(k = 0(1)\) the first \((n + 1)\) eigenfunctions, even (odd) in \(x\), can be found algebraically. Of course, the number of those ‘algebraized’ eigenfunctions is nothing but the dimension of the irreducible representation (1).

Taking different exactly-solvable operators \(E_2\) for the eigenvalue problem (6) one can reproduce the equations having the Hermite, Laguerre, Legendre and Jacobi polynomials as solutions \([3]\). It is worth noting that for general exactly-solvable operator \(E_2\) the eigenvalues are quadratic in number of eigenstate and can be presented as follows
\[
e_n = c_00n^2 + c_0n + \text{const}
\]
(for details see \([3]\)).

Under special choices of the general element \(E_2(E_0, E_8)\), one can reproduce all known fourth-(sixth-, eighth-)order differential equations giving rise to infinite sequences of orthogonal polynomials (see \([3]\) and other papers in this volume).

Recently, A. González-López, N. Kamran and P. Olver \([9]\) gave the complete description of second-order polynomial elements of \(\mathfrak{sl}_2(\mathbb{R})\) leading to the square-integrable eigenfunctions of the Sturm-Liouville problem (18) after transformation (17). Consequently, for second-order ordinary differential equations (14) the combination of their result and Theorem 1.1 gives a general solution of the Bochner problem as well as the more general problem of classification of equations possessing a finite number of orthogonal polynomial solutions.

2. Finite-difference equations in one variable

Let us introduce the finite-difference analogue of the differential operators (2)
\[
\tilde{J}^+ = x^2D - \{n\}x \\
\tilde{J}^0 = xD - \hat{n} \\
\tilde{J}^- = D,
\]
where \(\hat{n} \equiv \{n\}{n+1\over2n+2}\), \(\{n\} = {1-q^n\over1-q}\) is the quantum symbol (or \(q\)-number), \(q \in \mathbb{R}\) is a number characterizing the deformation, \(Df(x) = f(x) - f(qx)\) is a shift or a finite-difference operator (or the so-called Jackson symbol (see \([11]\))). The operators (21) after multiplication by some factors
\[
\tilde{J}^0 = q^{-n}\{2n+2\}\{n+1\}\tilde{j}^0 \\
\tilde{j}^\pm = q^{-n/2}\tilde{j}^\pm
\]
(see \([10]\)) form a quantum \(\mathfrak{sl}_2(\mathbb{R})_q\) algebra with the following commutation relations
\[
q\tilde{J}^0\tilde{J}^- - \tilde{J}^-\tilde{J}^0 = -\tilde{J}^- \\
q^2\tilde{J}^+\tilde{J}^- - \tilde{J}^-\tilde{J}^+ = -(q + 1)\tilde{J}^0 \\
\tilde{J}^0\tilde{J}^+ - q\tilde{J}^+\tilde{J}^0 = \tilde{J}^+
\]
(22)

\(^5\)For instance, setting the parameter \(a = 0\) in (19), the equation (15) converts to the Hermite equation (after some substitution).
LIE-ALGEBRAS AND LINEAR OPERATORS . . .

7

(this algebra corresponds to the second Witten quantum deformation of \( sl_2 \) in the classification of C. Zachos \([12]\)). If \( q \to 1 \), the commutation relations (22) reduce to the standard \( sl_2(\mathbb{R}) \) ones. A remarkable property of generators (21) is that, if \( n \) is a non-negative integer, they form the finite-dimensional representation. For \( q \) others than root of unity this representation is irreducible.

Similarly as for differential operators one can introduce quasi-exactly-solvable \( \tilde{T}_k \) and exactly-solvable operators \( \tilde{E}_k \).

**Lemma 2.1.** (i) Suppose \( n > (k - 1) \). Any quasi-exactly-solvable operator \( \tilde{T}_k \), can be represented by a \( k \)th degree polynomial of the operators (21). If \( n \leq (k - 1) \), the part of the quasi-exactly-solvable operator \( \tilde{T}_k \) containing derivatives up to order \( n \) can be represented by a \( n \)th degree polynomial in the generators (21).

(ii) Conversely, any polynomial in (21) is quasi-exactly solvable.

(iii) Among quasi-exactly-solvable operators there exist exactly-solvable operators \( \tilde{E}_k \subset \tilde{T}_k \).

**Comment 3.** If we define an analogue of the universal enveloping algebra \( U_q \) for the quantum algebra \( \tilde{g} \) as an algebra of all ordered polynomials in generators, then a quasi-exactly-solvable operator \( \tilde{T}_k \) at \( k < n + 1 \) is simply an element of the ‘universal enveloping algebra’ \( U_{sl_2(\mathbb{R})_q} \) of the algebra \( sl_2(\mathbb{R})_q \) taken in representation (21). If \( k \geq n + 1 \), then \( \tilde{T}_k \) is represented as an element of \( U_{sl_2(\mathbb{R})_q} \) plus \( BD^{n+1} \), where \( B \) is any linear difference operator of order not higher than \( (k - n - 1) \).

Similar to \( sl_2(\mathbb{R}) \), one can introduce the grading of generators (21) of \( sl_2(\mathbb{R})_q \) (see (3)) and, hence, of monomials of the universal enveloping \( U_{sl_2(\mathbb{R})_q} \) (see (4)).

**Lemma 2.2.** A quasi-exactly-solvable operator \( \tilde{T}_k \subset U_{sl_2(\mathbb{R})_q} \) has no terms of positive grading, iff it is an exactly-solvable operator.

**Theorem 2.1.** Let \( n \) be a non-negative integer. Take the eigenvalue problem for a linear difference operator of the \( k \)-th order in one variable

\[
\tilde{T}_k \varphi(x) = \varepsilon \varphi(x),
\]

where \( \tilde{T}_k \) is symmetric. The problem (23) has \( (n + 1) \) linearly independent eigenfunctions in the form of a polynomial in variable \( x \) of order not higher than \( n \), if and only if \( T_k \) is quasi-exactly-solvable. The problem (23) has an infinite sequence of polynomial eigenfunctions, if and only if the operator is exactly-solvable \( \tilde{E}_k \).

**Comment 4.** Saying the operator \( \tilde{T}_k \) is symmetric, we imply that, considering the action of this operator on a space of polynomials of degree not higher than \( n \), one can introduce a positively-defined scalar product, and the operator \( \tilde{T}_k \) is symmetric with respect to it.

This theorem gives a general classification of finite-difference equations

\[
\sum_{j=0}^{k} a_j(x) D^j \varphi(x) = \varepsilon \varphi(x)
\]

having polynomial solutions in \( x \). The coefficient functions must have the form

\[
a_j(x) = \sum_{i=0}^{k+j} \tilde{a}_{j,i} x^i.
\]
In particular, this form occurs after substitution (21) into a general $k$th degree polynomial element of the universal enveloping algebra $U_{sl_2(R)}$. It guarantees the existence of at least a finite number of polynomial solutions. The coefficients $\tilde{a}_{j,i}$ are related to the coefficients of the $k$th degree polynomial element of the universal enveloping algebra $U_{sl_2(R)}$. The number of free parameters of the polynomial solutions is defined by the number of free parameters of a general $k$-th order polynomial element of the universal enveloping algebra $U_{sl_2(R)}$. A rather straightforward calculation leads to the following formula

$$\text{par}(\tilde{T}_k) = (k+1)^2 + 1$$

(for the second-order finite-difference equation $\text{par}(\tilde{T}^2) = 10$). For the case of an infinite sequence of polynomial solutions the formula (25) simplifies to

$$\tilde{a}_j(x) = \sum_{i=0}^{j} \tilde{a}_{j,i} x^i$$

(26)

and the number of free parameters is given by

$$\text{par}(\tilde{E}_k) = \frac{(k+1)(k+2)}{2} + 1$$

(for $k = 2$, $\text{par}(\tilde{E}^2) = 7$). The increase in the number of free parameters compared to ordinary differential equations is due to the presence of the deformation parameter $q$. In $[8]$ one can find a description in the present approach of the $q$-deformed Hermite, Laguerre, Legendre and Jacobi polynomials (for definitions of these polynomials see $[11]$).

Lemma 2.3. If the operator $\tilde{T}_2$ (see (12)) is such that

$$\tilde{c}_{++} = 0 \quad \text{and} \quad \tilde{c}_+ = (\tilde{n} - \{m\})\tilde{c}_{+0}, \text{ at some } m = 0, 1, 2, \ldots$$

(27)

then the operator $\tilde{T}_2$ preserves both $\mathcal{P}_n$ and $\mathcal{P}_m$, and polynomial solutions in $x$ with $8$ free parameters occur.

As usual in quantum algebras, a rather outstanding situation occurs if the deformation parameter $q$ is equal to the root of unity. For instance, the following statement holds.

Lemma 2.4. If a quasi-exactly-solvable operator $\tilde{T}_k$ preserves the space $\mathcal{P}_n$ and the parameter $q$ satisfies to the equation

$$q^n = 1,$$

(28)

then the operator $\tilde{T}_k$ preserves an infinite flag of polynomial spaces $\mathcal{P}_0 \subset \mathcal{P}_n \subset \mathcal{P}_{2n} \subset \cdots \subset \mathcal{P}_{kn} \subset \ldots$.

It is worth emphasizing that, in the limit as $q$ tends to one, Lemmas 2.1,2,3 and Theorem 2.1 coincide with Lemmas 1.1,2,3 and Theorem 1.1, respectively. Thus the case of differential equations in one variable can be treated as particular case.

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$^6$For quantum $sl_2(R)_q$ algebra there are no polynomial Casimir operators (see, e.g., $[14]$). However, in the representation (21) the relationship between generators analogous to the quadratic Casimir operator

$$q\tilde{J}^+ \tilde{J}^- - \tilde{J}_3^2 \tilde{J}_0 + \{(n+1) - 2\tilde{n}\} \tilde{J}_0 = \tilde{n}(\tilde{n} - \{n + 1\})$$

appears. It reduces the number of independent parameters of the second-order polynomial element of $U_{sl_2(R)_q}$. It becomes the standard Casimir operator at $q \to 1$. 
of finite-difference ones. Evidently, one can consider the finite-difference operators, which are a mixture of generators (21) with the same value of $n$ and different $q$’s.

3. Operators in one real and one Grassmann variable

Define the following space of polynomials in $x, \theta$

$$P_{N,M} = \langle x^0, x^1, \ldots, x^n, x^0\theta, x^1\theta, \ldots, x^M\theta \rangle$$

where $N, M$ are non-negative integers, $x \in \mathbb{R}$ and $\theta$ is a Grassmann (anticommuting) variable.

The projectivized representation of the algebra $osp(2,2)$ is given as follows.

The algebra $osp(2,2)$ is characterized by four bosonic generators $T^{\pm,0}, J$ and four fermionic generators $Q_{1,2}, \overline{Q}_{1,2}$ and given by the commutation relations

$$[T^0, T^\pm] = \pm T^\pm, \quad [T^+, T^-] = -2T^0, \quad [J, T^\alpha] = 0, \quad \alpha = +, -, 0$$

$$\frac{1}{2}((\overline{Q}_1, Q_1) + (\overline{Q}_2, Q_2)) = +J, \quad \frac{1}{2}((\overline{Q}_1, Q_1) - (\overline{Q}_2, Q_2)) = T^0,$$

$$[Q_1, Q_1] = [Q_2, Q_2] = [Q_1, Q_2] = 0, \quad [\overline{Q}_1, \overline{Q}_1] = [\overline{Q}_2, \overline{Q}_2] = [\overline{Q}_1, \overline{Q}_2] = 0,$$

$$[Q_1, T^+] = Q_2, \quad [Q_2, T^+] = 0, \quad [Q_1, T^+] = 0, \quad [Q_2, T^-] = -Q_1,$$

$$[\overline{Q}_1, T^+] = 0, \quad [\overline{Q}_2, T^+] = -\overline{Q}_1, \quad [\overline{Q}_1, T^-] = \overline{Q}_2, \quad [\overline{Q}_2, T^-] = 0,$$

$$[Q_{1,2}, T^0] = \mp \frac{1}{2} Q_{1,2}, \quad [\overline{Q}_{1,2}, T^0] = \mp \frac{1}{2} \overline{Q}_{1,2}$$

(30)

This algebra has the algebra $sl_2(\mathbb{R}) \oplus \mathbb{R}$ as sub-algebra.

The algebra (30) possesses the projectivized representation

$$T^+ = x^2\partial_x - nx + x\theta\partial_\theta,$$

$$T^0 = x\partial_x - \frac{n}{2} + \frac{1}{2}\theta\partial_\theta,$$

$$T^- = \partial_x, \quad J = -\frac{n}{2} - \frac{1}{2}\theta\partial_\theta$$

(31)

for bosonic (even) generators and

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} \partial_\theta \\ x\partial_\theta \end{bmatrix}, \quad \bar{Q} = \begin{bmatrix} Q_1 \\ \bar{Q}_2 \end{bmatrix} = \begin{bmatrix} x\theta\partial_x - n\theta \\ -\theta\partial_x \end{bmatrix},$$

(32)

for fermionic (odd) generators, where $x$ is a real variable and $\theta$ is a Grassmann one. Inspection of the generators shows that if $n$ is a non-negative integer, the representation (31), (32) is a finite-dimensional representation of dimension $(2n+1)$.

The polynomial space $P_{n,n-1}$ describes the corresponding invariant sub-space.

**Definition 3.1.** Let us name a linear differential operator of the $k$-th order, $T_k(x, \theta)$, **quasi-exactly-solvable** if it preserves the space $P_{n,n-1}$. Correspondingly, the operator $E_k(x, \theta) \in T_k(x, \theta)$, which preserves the infinite flag $P_{0,0} \subset P_{1,0} \subset P_{2,1} \subset \cdots \subset P_{n,n-1} \subset \cdots$ of spaces of all polynomials, is named **exactly-solvable**.
Lemma 3.1. Consider the space $P_{n,n-1}$.

(i) Suppose $n > (k-1)$. Any quasi-exactly-solvable operator $T_k(x, \theta)$, can be represented by a $k$th degree polynomial of the operators (31), (32). If $n \leq (k-1)$, the part of the quasi-exactly-solvable operator $T_k(x, \theta)$ containing derivatives in $x$ up to order $n$ can be represented by an $n$th degree polynomial in the generators (31), (32).

(ii) Conversely, any polynomial in (31), (32) is a quasi-exactly solvable operator.

(iii) Among quasi-exactly-solvable operators there exist exactly-solvable operators $E_k \subset T_k(x, \theta)$.

Let us introduce the grading of the bosonic generators (31)

$$\deg(T^+) = +1, \quad \deg(J, T^0) = 0, \quad \deg(J^-) = -1$$

and fermionic generators (32)

$$\deg(Q_2, Q_1) = +\frac{1}{2}, \quad \deg(Q_1, Q_2) = -\frac{1}{2}$$

Hence the grading of monomials of the generators (31), (32) is equal to

$$\deg[(T^+)^n(T^0)^m(J)^r(T^-)^sQ_1^{m_1}Q_2^{m_2}Q_1^{\overline{m}_1}Q_2^{\overline{m}_2}] = (n_+ - n_-) - (m_1 - m_2 - \overline{m}_1 + \overline{m}_2)/2$$

The $n$’s can be arbitrary non-negative integers, while the $m$’s are equal to either 0 or 1. The notion of grading allows us to classify the operators $T_k(x, \theta)$ in the Lie-algebraic sense.

Lemma 3.2. A quasi-exactly-solvable operator $T_k(x, \theta) \subset U_{osp(2,2)}$ has no terms of positive grading other than monomials of grading $+1/2$ containing the generator $Q_1$ or $Q_2$, iff it is an exactly-solvable operator.

Theorem 3.1. Let $n$ be a non-negative integer. Take the eigenvalue problem for a linear differential operator in one real and one Grassmann variable

$$T_k(x, \theta)\phi = \varepsilon \phi$$

where $T_k$ is symmetric. In general, the problem (36) has $(2n+1)$ linearly independent eigenfunctions in the form of polynomials in variables $x, \theta$ of order not higher than $n$, if and only if $T_k$ is quasi-exactly-solvable. The problem (36) has an infinite sequence of polynomial eigenfunctions, if and only if the operator is exactly-solvable.

This theorem gives a general classification of differential equations

$$\sum_{i,j=0}^{i=k, j=1} a_{i,j}(x, \theta)\varphi^{(i,j)}_{x,\theta}(x, \theta) = \varepsilon \varphi(x, \theta)$$

where the notation $\varphi^{(i,j)}_{x,\theta}$ means the $i$th order derivative with respect to $x$ and $j$th order derivative with respect to $\theta$, having at least one polynomial solution in $x, \theta$, thus resolving the generalized Bochner problem. Suppose that $k > 0$, then the coefficient functions $a_{i,j}(x, \theta)$ should have the form

$$a_{i,0}(x, \theta) = \sum_{p=0}^{k+i} a_{i,0,p} x^p + \theta \sum_{p=0}^{k+i-1} \overline{a}_{i,0,p} x^p$$
The explicit expressions (38) are obtained by substituting (31), (32) into a general, the $k$th order, polynomial element of the universal enveloping algebra $U_{osp(2,2)}$. Thus the coefficients $a_{i,j,p}$ can be expressed through the coefficients of the $k$-th order polynomial element of universal enveloping algebra $U_{osp(2,2)}$. The number of free parameters of the polynomial solutions is defined by the number of parameters characterizing a general, $k$th order polynomial element of the universal enveloping algebra $U_{osp(2,2)}$. However, in counting parameters some relations between generators should be taken into account, specific for the given representation (31), (32), like

\begin{align*}
2T^+J - \overline{Q}_1 Q_2 &= nT^+, \\
T^+ Q_1 - T^0 Q_2 &= -Q_2, \\
T^+ \overline{Q}_2 + T^0 \overline{Q}_1 &= \frac{(1-n)}{2} \overline{Q}_1, \\
J Q_2 &= \frac{n}{2} Q_2, \\
J \overline{Q}_1 &= \frac{(n+1)}{2} \overline{Q}_1, \\
T^+ T^- - T^0 T^0 - JJ + T^0 &= -\frac{n}{2} (n-1), \\
JJ &= (n + \frac{1}{2})J - \frac{n}{4}(n+1), \\
Q_1 \overline{Q}_1 + Q_2 \overline{Q}_2 - 2nJ &= -n(n+1), \\
2T^0 J + Q_1 \overline{Q}_1 - (n+1)T^0 - nJ &= -\frac{n}{2}(n+1), \\
T^- Q_2 - T^0 Q_1 &= \frac{(n}{2}+1)Q_1, \\
T^- \overline{Q}_1 - T^0 \overline{Q}_2 &= \frac{(n-1)}{2} \overline{Q}_2, \\
J Q_1 &= \frac{n}{2} Q_1, \\
J \overline{Q}_2 &= \frac{n+1}{2} \overline{Q}_2, \\
2JT^- - Q_1 \overline{Q}_2 &= (n+1)T^- 
\end{align*}

between quadratic expressions in generators (and the ideals generated by them). Straightforward analysis leads to the following formula for the number of free parameters

\begin{equation}
\text{par}(T_k(x, \theta)) = 4k(k + 1) + 1,
\end{equation}

For the case of an exactly-solvable operator (an infinite sequence of polynomial solutions of Eq. (37)), the expressions (38) simplify and reduce to

\begin{equation*}
a_{i,0}(x, \theta) = \sum_{p=0}^{i} a_{i,0,p} x^p + \theta \sum_{p=0}^{i-1} \overline{a}_{i,0,p} x^p
\end{equation*}
\[
a_{i,1}(x, \theta) = \sum_{p=0}^{i} a_{i,1,p} x^p + \theta \sum_{p=0}^{i-1} a_{i,1,p+1} x^p
\]  
(41)

Correspondingly, the number of free parameters reduces to
\[
\text{par}(E_k(x, \theta)) = 2k(k+2) + 1 \quad (42)
\]

Hence, Eq. (37) with the coefficient functions (41) gives a general form of eigenvalue problem for the operator \( T_k \), which can lead to an infinite family of orthogonal polynomials as eigenfunctions. If in (41) we put formally all coefficients, \( a_{i,0,0} \) and \( a_{i,1}(x, \theta) \) equal to zero, we reproduce the eigenvalue problem for the differential operators in one real variable, which gives rise to all known families of orthogonal polynomials in one real variable (see \[13\]).

3.1. Second-order differential equations in \( x, \theta \). Now let us consider in more detail the second-order differential equation Eq. (37), which can possess polynomial solutions. As follows from Theorem 3.1, the corresponding differential operator \( T_2(x, \theta) \) should be quasi-exactly-solvable. Hence, this operator can be expressed in terms of \( osp(2,2) \) generators taking into account the relations (39)
\[
T_2 = c_{++} T^+ T^+ + c_{+0} T^+ T^0 + c_{+-} T^+ T^- + c_{0-} T^0 T^- + c_{-} T^- T^- +
\]
\[
c_{+,} T^+ J + c_{0,} T^0 J + c_{-,} T^- J +
\]
\[
c_{+1} T^+ Q_1 + c_{+2} T^+ Q_2 + c_{+1} T^+ Q_1 + c_{+2} T^+ Q_2 + c_{01} T^0 Q_1 +
\]
\[
c_{02} T^0 Q_2 + c_{1} T^- Q_1 + c_{-} T^- Q_2 +
\]
\[
c_{+} T^+ + c_{0} T^0 + c_{-} T^- + c_{J} J + c_{1} Q_1 + c_{2} Q_2 + c_{+1} Q_1 + c_{+2} Q_2 + c
\]  
(43)

where \( c_{\alpha\beta}, c_{\alpha}, c \) are parameters. Following Lemma 3.2, under the conditions
\[
c_{++} = c_{+0} = c_{+T} = c_{+2} = c_{T} = c_{+2} = c_{+J} = c_{+} = 0 \quad (44)
\]

the operator \( T_2(x, \theta) \) becomes exactly-solvable (see (41)).

Now we proceed to the detailed analysis of the quasi-exactly-solvable operator \( T_2(x, \theta) \). Set
\[
c_{++} = 0 \quad (45)
\]
in Eq. (43). The remainder possesses an exceptionally rich structure. The whole situation can be subdivided into three cases
\[
c_{+2} \neq 0, \quad c_{+T} = 0 \quad (\text{case I}) \quad (46)
\]
\[
c_{+2} = 0, \quad c_{+T} \neq 0 \quad (\text{case II}) \quad (47)
\]
\[
c_{+2} = 0, \quad c_{+T} = 0 \quad (\text{case III}) \quad (48)
\]

We emphasize that we keep the parameter \( n \) in the representation (31), (32) as a fixed, non-negative integer.
Case I. The conditions (45) and (46) are fulfilled (see Fig. 3.1).

Case I.1. If

\[ (n + 2)c_{+0} + nc_{+J} + 2c_+ = 0 \, , \]
\[ c_+\mathcal{T} = c_\mathcal{T} = 0 \, , \]
\[ (n + 1)c_0\mathcal{T} + 2c_\mathcal{T} = 0 \, , \]

then \( T_2(x, \theta) \) preserves \( P_{n,n-1} \) and \( P_{n+1,n-1} \).

Case I.1.1. If

\[ (n + 2)c_{+0} + nc_{+J} + 2c_+ = 0 \, , \]
\[ c_+\mathcal{T} = c_\mathcal{T} = 0 \, , \]
\[ (n + 1)c_0\mathcal{T} + 2c_\mathcal{T} = 0 \, , \]

then \( T_2(x, \theta) \) preserves \( P_{n,n-1} \) and \( P_{n+1,n-1} \).

Case I.1.2. If

\[ (n + 4 + 2m)c_{+0} + nc_{+J} + 2c_+ = 0 \, , \]
\[ c_+\mathcal{T} = c_\mathcal{T} = 0 \, , \]
\[ c_0\mathcal{T} = c_\mathcal{T} = c_-\mathcal{T} = 0 \, , \]

at a certain integer \( m \geq 0 \), then \( T_2(x, \theta) \) preserves \( P_{n,n-1} \) and \( P_{n+2+m,n-1} \). If \( m \) is non-integer, then \( T_2(x, \theta) \) preserves \( P_{n,n-1} \) and \( P_{\infty,n-1} \).

Case I.1.3. If

\[ (n + 1)c_{+J} + 2c_+ = 0 \, , \]
\[ c_{+0} = 0 \, , \]
\[ c_{+\mathcal{T}} = c_\mathcal{T} = 0 \, , \]
\[ c_{0\mathcal{T}} = c_\mathcal{T} = c_-\mathcal{T} = 0 \, , \]

then \( T_2(x, \theta) \) preserves the infinite flag of polynomial spaces the \( P_{n+m,n-1} \), \( m = 0, 1, 2, \ldots \).

Case I.2.1. If

\[ (n - 3)c_{+0} + (n + 1)c_{+J} + 2c_+ = 0 \, , \]
\[ (n - 1)c_{+\mathcal{T}} = c_\mathcal{T} \, , \]
\[ (n - 1)c_{0\mathcal{T}} + 2c_\mathcal{T} = 0 \, , \]

then \( T_2(x, \theta) \) preserves \( P_{n,n-1} \) and \( P_{n,n-2} \).

Case I.2.2. If

\[ 3c_{+0} - c_{+J} = 0 \, , \]
\[ (2k + 2n + 4)c_{+0} + 2c_+ = 0 \, , \]
\[ c_{+\mathcal{T}} = c_\mathcal{T} = 0 \, , \]
\[ (2k - n + 3)c_{0\mathcal{T}} + 2c_\mathcal{T} = 0 \, , \]

at a certain integer \( k \geq 0 \), then \( T_2(x, \theta) \) preserves \( P_{n,n-1} \) and \( P_{k+2,k} \).

Case I.2.3. If

\[ c_{+0} = c_{+J} = c_+ = 0 \, , \]
\[ c_{+\mathcal{T}} = c_\mathcal{T} = 0 \, , \]
\[ c_{0\mathcal{T}} = c_\mathcal{T} = 0 \, , \]

then \( T_2(x, \theta) \) preserves \( P_{n,n-1} \) and the infinite flag of the polynomial spaces \( P_{k+2,k} \), \( k = 0, 1, 2, \ldots \). Note in general for this case \( c_-\mathcal{T} \neq 0 \).

Case I.3.1. If

\[ (n - 5 - 2m)c_{+0} + (n + 1)c_{+J} + 2c_+ = 0 \, , \]
\[ c_{+\mathcal{T}} = c_\mathcal{T} = 0 \, , \]
\[ c_0 = c_\tau = c_{-\tau} = 0 \, , \]

at a certain integer \( 0 \leq m \leq (n-3) \), then \( T_2(x, \theta) \) preserves \( P_{n,n-1} \) and \( P_{n,n-3-m} \).

**Case I.3.2.** If

\[
\begin{align*}
\sum_{j=0}^{2} c_j &= 0, \\
(n+1)c_{-j} + 2c_+ &= 0, \\
c_{\tau} &= c_{-\tau} = 0, \\
\end{align*}
\]

then \( T_2(x, \theta) \) preserves \( P_{n,n-1} \) and the sequence of the polynomial spaces \( P_{n,n-3-m} \), \( m = 0, 1, 2, \ldots, (n-3) \).

**Case I.3.3.** If

\[
\begin{align*}
(2k+1-n)c_0 + (n+1)c_{-j} + 2c_+ &= 0, \\
(2m+5)c_{-j} - c_{-\tau} &= 0, \\
c_0 &= c_{-\tau} = c_{-\tau} = 0, \\
\end{align*}
\]

at certain integers \( k, m \geq 0 \), then \( T_2(x, \theta) \) preserves \( P_{n,n-1} \) and \( P_{k+3+m,k} \).

**Case I.3.4.** If

\[
\begin{align*}
c_0 &= c_{-\tau} = c_0 = 0, \\
c_+ &= c_{-\tau} = 0, \\
c_0 &= c_{-\tau} = c_{-\tau} = 0, \\
\end{align*}
\]

then \( T_2(x, \theta) \) preserves \( P_{n,n-1} \) and the infinite flag of polynomial spaces \( P_{k+3+m,k} \), \( k, m = 0, 1, 2, \ldots \) (cf. Cases I.1.3 and I.2.3).

**Case II.** The conditions (45) and (47) are fulfilled (see Fig. 3.II).

**Case II.1.1.** If

\[
\begin{align*}
(n+1)c_0 + (n+1)c_{-j} + 2c_+ &= 0, \\
c_2 &= 0, \\
\end{align*}
\]

then \( T_2(x, \theta) \) preserves \( P_{n,n-1} \) and \( P_{n,n} \).

**Case II.1.2.** If

\[
\begin{align*}
(n+3)c_0 + (n+1)c_{-j} + 2c_+ &= 0, \\
(n+2)c_0 + 2c_1 &= 0, \\
c_1 &= c_2 = 0, \\
\end{align*}
\]

then \( T_2(x, \theta) \) preserves \( P_{n,n-1} \) and \( P_{n,n+1} \).

**Case II.1.3.** If

\[
\begin{align*}
(2k+5+n)c_0 + (n+1)c_{-j} + 2c_+ &= 0, \\
c_0 &= c_1 = 0, \\
c_1 &= c_2 = 0, \\
c_{-1} &= 0, \\
\end{align*}
\]

at a certain integer \( k \geq 0 \), then \( T_2(x, \theta) \) preserves \( P_{n,n-1} \) and \( P_{n,n+2+k} \).
Case II.1.4. If
\[
\begin{align*}
    c_{+0} &= 0, \\
    (n + 1)c_{+J} + 2c_+ &= 0, \\
    c_{01} &= c_1 = 0, \\
    c_{+1} &= c_2 = 0, \\
    c_{-1} &= 0,
\end{align*}
\]
then $T_2(x, \theta)$ preserves $P_{n,n-1}$ and the infinite flag of polynomial spaces $P_{n,n+k}$, $k = 0, 1, 2, \ldots$ (cf. Cases I.1.3, I.2.3 and I.3.4).

Case II.2.1. If
\[
\begin{align*}
    (n - 2)c_{+0} + nc_{+J} + 2c_+ &= 0, \\
    c_{+1} &= c_2 = 0,
\end{align*}
\]
then $T_2(x, \theta)$ preserves $P_{n,n-1}$ and $P_{n-1,n-1}$.

Case II.2.2. If
\[
\begin{align*}
    (n + 1)c_{+0} - c_+ &= 0, \\
    3c_{+0} + c_{+J} &= 0, \\
    nc_{01} + 2c_1 &= 0, \\
    c_{+1} &= c_2 = 0,
\end{align*}
\]
then $T_2(x, \theta)$ preserves $P_{n,n-1}$ and $P_{n-1,n}$.

Case II.2.3. If
\[
\begin{align*}
    (n - 2)c_{+0} + nc_{+J} + 2c_+ &= 0, \\
    (2k + 1)c_{+0} + c_{+J} &= 0, \\
    c_{+1} &= c_2 = 0, \\
    c_{01} &= c_1 = c_{-1} = 0,
\end{align*}
\]
at a certain integer $k \geq 0$, then $T_2(x, \theta)$ preserves $P_{n,n-1}$ and $P_{n-1,n+k+1}$.

Case II.2.4. If
\[
\begin{align*}
    c_{+0} &= c_{+J} = c_+ = 0, \\
    c_{+1} &= c_2 = 0, \\
    c_{01} &= c_1 = c_{-1} = 0,
\end{align*}
\]
then $T_2(x, \theta)$ preserves $P_{n,n-1}$ and the infinite flag of the polynomial spaces $P_{n-1,n+k}$, $k = 0, 1, 2, \ldots$ (cf. Cases I.1.3, I.2.3, I.3.4 and II.1.4).

Case II.3.1. If
\[
\begin{align*}
    (n - 4)c_{+0} + nc_{+J} + 2c_+ &= 0, \\
    (n - 2)c_{01} + 2c_1 &= 0, \\
    c_{+1} &= c_2 = 0,
\end{align*}
\]
then $T_2(x, \theta)$ preserves $P_{n,n-1}$ and $P_{n-2,n-1}$.

Case II.4.1. If
\[
\begin{align*}
    (m - 2n)c_{+0} + c_+ &= 0, \\
    3c_{+0} + c_{+J} &= 0, \\
    (2m + 2 - n)c_{01} + 2c_1 &= 0, \\
    c_{+1} &= c_2 = 0,
\end{align*}
\]
at a certain integer \( m \geq 0 \), then \( T_2(x, \theta) \) preserves \( P_{n,n-1} \) and \( P_{m,m+1} \).

**Case II.4.2.** If

\[
\begin{align*}
(2m - n)c_{+0} + nc_{+J} + 2c_+ &= 0, \\
(2k + 5)c_{+0} + 2c_{+J} &= 0, \\
c_{+1} &= c_2 = 0, \\
c_{01} &= c_1 = c_{-1} = 0
\end{align*}
\]

(69)

at certain integers \( k \geq 0, m \geq 0 \), then \( T_2(x, \theta) \) preserves \( P_{n,n-1} \) and \( P_{m,m+2+k} \).

**Case II.4.3.** If

\[
\begin{align*}
c_{+0} &= c_{+J} = c_+ = 0, \\
c_{+1} &= c_2 = 0, \\
c_{01} &= c_1 = c_{-1} = 0
\end{align*}
\]

(70)

then \( T_2(x, \theta) \) preserves \( P_{n,n-1} \) and the infinite flag of the polynomial spaces \( P_{m,m+1+k}, m,k = 0,1,2,\ldots \) (cf. **Cases I.1.3, I.2.3, I.3.4, II.1.4 and II.2.4**).

**Case III.** The conditions (45) and (48) are fulfilled (see Fig. 3.III).

**Case III.1.1.** If

\[
\begin{align*}
(2m - n)c_{+0} + nc_{+J} + 2c_+ &= 0, \\
c_{+0} + c_{+J} &= 0, \\
(m - n)c_{+1} + c_2 &= 0,
\end{align*}
\]

(71)

at a certain integer \( m \geq 0 \), then \( T_2(x, \theta) \) preserves \( P_{n,n-1} \) and \( P_{m,m} \).
Case III.1.2. If
\[ c_{+0} = c_{+J} = c_{+} = 0, \]
\[ c_{+1} = c_{2} = 0, \]
then \( T_{2}(x, \theta) \) preserves \( P_{n,n-1} \) and the infinite flag of the polynomial spaces \( P_{m,m}, m = 0,1,2,\ldots \) (cf. Cases I.1.3, I.2.3, I.3.4, II.1.4, II.2.4 and II.4.3).

Case III.2.1. If
\[ (2m - n)c_{+0} + nc_{+J} + 2c_{+} = 0, \]
\[ c_{+0} - c_{+J} = 0, \]
\[ mc_{\theta} - \theta = 0, \]
at a certain integer \( m \geq 0 \), then \( T_{2}(x, \theta) \) preserves \( P_{n,n-1} \) and \( P_{m,m-1} \).

Case III.2.2. If
\[ c_{+0} = c_{+J} = c_{+} = 0, \]
\[ c_{+\theta} = \theta = 0 \]
then \( T_{2}(x, \theta) \) preserves \( P_{n,n-1} \) and the infinite flag of polynomial spaces \( P_{m,m-1}, m = 0,1,2,\ldots \) (cf. Cases I.1.3, I.2.3, I.3.4, II.1.4, II.2.4, II.4.3 and III.1.2). This case corresponds to exactly-solvable operators \( E_{k} \).

In \[13\] it has been shown that under a certain condition some quasi-exactly-solvable operators \( T_{2}(x) \) in one real variable can preserve two polynomial spaces of different dimensions \( n \) and \( m \) (see Lemma 1.3). It has been shown that those quasi-exactly-solvable operators \( T_{2}(x) \) can be represented through the generators of \( sl_{2}(\mathbb{R}) \) in a projectivized representation characterized either by the mark \( n \) or by the mark \( m \). The above analysis shows that the quasi-exactly-solvable operators \( T_{2}(x, \theta) \) in two variables (one real and one Grassmann) possess an extremely rich variety of internal properties. They are characterized by different structures of invariant sub-spaces. However, generically the quasi-exactly-solvable operators \( T_{2}(x, \theta) \) can preserve either one, or two, or infinitely many polynomial spaces. For the latter, those operators become 'exactly-solvable' (see Cases I.1.3, I.2.3, I.3.4, II.1.4, II.2.4, II.4.3 and III.1.2) giving rise to eigenvalue problems (36) possessing infinite sequences of polynomial eigenfunctions. In general, for the two latter cases the interpretation of \( T_{2}(x, \theta) \) in terms of \( osp(2,2) \) generators characterized by different marks does not exist, unlike the case of quasi-exactly-solvable operators in one real variable. The only exceptions are given by Case III.2.1 and Case III.2.2.
Newton diagrams describing invariant subspaces $P_{N,M}$ of the second-order polynomials in the generators of $osp(2,2)$. The lower line corresponds to the part of the space of zero degree in $\theta$ and the upper line of first degree in $\theta$. The letters without brackets indicate the maximal degree of the polynomial in $x$. The letters in brackets indicate the maximal (or minimal) possible degree, if the degree can be varied. The thin line displays schematically the length of polynomial in $x$ (the number of monomials). The thick line shows that the length of polynomial can not be more (or less) than that size. The dashed line means that the length of polynomial can take any size on this line. If the dashed line is unbounded, it means that the degree of the polynomial can be arbitrary up to infinity. The numbering of the figures I-III corresponds to the cases, which satisfy the conditions (45), (46) (Case I); (45), (47) (Case II) and (45), (48) (Case III).

Fig. 3.0. Basic subspace

Fig. 3.I.1 . Subspaces for the Case I.1

Fig. 3.I.2 . Subspaces for the Case I.2

Fig. 3.I.3 . Subspaces for the Case I.3
\[ \Theta \]
\[
\begin{array}{ccc}
n & n+1 & (n+2) n + 2 + k \\
n & n & n - 1 \\
\end{array}
\]

(a) (b) (c)

\[ \Theta \]
\[
\begin{array}{ccc}
n - 1 & n & (n + 1) n + 1 + k \\
n - 1 & n - 1 & n - 1 \\
\end{array}
\]

(a) (b) (c)

Fig. 3.II.1 . Subspaces for the Case II.1

Fig. 3.II.2 . Subspaces for the Case II.2

Fig. 3.II.3 . Subspace for the Case II.3

Fig. 3.II.4 . Subspaces for the Case II.4

Fig. 3.III.1 . Subspaces for the Case III.1

Fig. 3.III.2 . Subspaces for the Case III.2
3.2. 2 x 2 matrix differential equations in \( x \). It is well known that anti-commuting variables can be represented by matrices. In our case the matrix representation is as follows: substitute \( \theta \) and \( \partial_\theta \) in the generators (31), (32) by the Pauli matrices \( \sigma^+ \) and \( \sigma^- \), respectively, acting on two-component spinors. In fact, all main notations are preserved such as quasi-exactly-solvable and exactly-solvable operator, grading etc.

In the explicit form the fermionic generators (32) in matrix representation are written as follows:

\[
Q = \begin{bmatrix} \sigma^+ \\ x \sigma^+ \end{bmatrix}, \quad \bar{Q} = \begin{bmatrix} x \sigma^- \partial_x - n \sigma^- \\ -\sigma^- \partial_x \end{bmatrix}.
\]

(75)

The representation (75) assumes that in the spectral problem (36) an eigenfunction \( \varphi(x) \) is treated as a two-component spinor

\[
\varphi(x) = \begin{bmatrix} \varphi_1(x) \\ \varphi_2(x) \end{bmatrix},
\]

(76)

In the matrix formalism, the polynomial space (29) has the form:

\[
P_{N,M} = \{ x^0, x^1, \ldots, x^M \\ x^0, x^1, \ldots, x^N \}.
\]

(77)

where the terms of zero degree in \( \theta \) come in as the lower component and the terms of first degree in \( \theta \) come in as the upper component. The operator \( T_k(x, \theta) \) becomes a \( 2 \times 2 \) matrix differential operator having derivatives in \( x \) up to \( k \)-th order. In order to distinguish the matrix operator in \( x \) from the operator in \( x, \theta \), we will denote the former as \( T_k(x) \). Finally, as a consequence of Theorem 3.1, we arrive at the eigenvalue problem for a \( 2 \times 2 \) matrix quasi-exactly-solvable differential operator \( T_k(x) \), possessing in general \( (2n+1) \) polynomial solutions of the form \( P_{n,n-1} \). This eigenvalue problem can be written in the form (cf. Eq.(37))

\[
\sum_{i=0}^{i=k} a_{k,i}(x) \varphi_x^{(i)}(x) = \varepsilon \varphi(x),
\]

(78)

where the notation \( \varphi_x^{(i)} \) means the \( i \)-th order derivative with respect to \( x \) of each component of the spinor \( \varphi(x) \) (see Eq. (76)). The coefficient functions \( a_{k,i}(x) \) are given by \( 2 \times 2 \) matrices and generically for the \( k \)-th order quasi-exactly-solvable operator their matrix elements are polynomials. Suppose that \( k > 0 \). Then the matrix elements are given by the following expressions

\[
a_{k,i}(x) = \begin{pmatrix} A_{k,i}^{[k+i]} & B_{k,i}^{[k+i-1]} \\ C_{k,i}^{[k+i+1]} & D_{k,i}^{[k+i]} \end{pmatrix}
\]

(79)

at \( k > 0 \), where the superscript in square brackets displays the order of the corresponding polynomial.

It is easy to calculate the number of free parameters of a quasi-exactly-solvable operator \( T_k(x) \)

\[
\text{par}(T_k(x)) = 4(k+1)^2
\]

(80)

(cf. Eq.(40)).
For the case of exactly-solvable problems, the matrix elements (79) of the coefficient functions are modified
\[ a_{k,i}(x) = \begin{pmatrix} A_{k,i}^{[i]} & B_{k,i}^{[i-1]} \\ C_{k,i}^{[i+1]} & D_{k,i}^{[i]} \end{pmatrix} \] (81)
where \( k > 0 \). An infinite family of orthogonal polynomials as eigenfunctions of Eq. (78), if they exist, will occur, if and only if the coefficient functions have the form (81). The number of free parameters of an exactly-solvable operator \( E_k(x) \) and, correspondingly, the maximal number of free parameters of the \( 2 \times 2 \) matrix orthogonal polynomials in one real variable, is equal to
\[ \text{par}(E_k(x)) = 2k(k + 3) + 3 \] (82)
(cf. Eq.(42)).

The increase in the number of free parameters for the \( 2 \times 2 \) matrix operators with respect to the case of the operators in \( x, \theta \) is connected to the occurrence of extra monomials of degree \( (k + 1) \) in generators of \( osp(2,2) \) (see Eqs.(31), (32), (75)), leading to the \( k \)th order differential operators in \( x \).

Thus, the above formulas describe the coefficient functions of matrix differential equations (78), which can possess polynomials in \( x \) as solutions, resolving the analogue of the generalized Bochner problem Eq. (0) for the case of \( 2 \times 2 \) matrix differential equations in one real variable.

Now let us take the quasi-exactly-solvable matrix operator \( T_2(x) \) and try to reduce Eq. (36) to the Schrödinger equation
\[ -\frac{1}{2} \frac{d^2}{dy^2} + V(y)\Psi(y) = E\Psi(y) \] (83)
where \( V(y) \) is a two-by-two hermitian matrix, by making a change of variable \( x \mapsto y \) and “gauge” transformation
\[ \Psi = U\phi \] (84)
where \( U \) is an arbitrary \( 2 \times 2 \) matrix depending on the variable \( y \). In order to get some “reasonable” Schrödinger equation one should fulfill two requirements: (i) the potential \( V(y) \) must be hermitian and (ii) the eigenfunctions \( \Psi(y) \) must belong to a certain Hilbert space.

Unlike the case of quasi-exactly-solvable differential operators in one real variable (see [3]), this problem has no complete solution so far. Therefore it seems instructive to display a particular example [14].

Consider the quasi-exactly-solvable operator
\[ T_2 = -2T^0T^- + 2T^-J - i\beta T^0Q_1 + \alpha T^0 - (2n + 1)T^- - \frac{i\beta}{2}(3n + 1)Q^1 + \frac{i}{2}\alpha\beta Q^2 - i\beta Q_1 , \] (85)
where \( \alpha \) and \( \beta \) are parameters. Upon introducing a new variable \( y = x^2 \) and after straightforward calculations one finds the following expression for the matrix \( U \) in Eq. (84)
\[ U = \exp(-\frac{\alpha y^2}{4} + \frac{i\beta y^2}{4} \sigma_1) \] (86)
and for the potential $\hat{V}$ in Eq. (83)

$$
\hat{V}(y) = \frac{1}{8}(\alpha^2 - \beta^2)y^2 + \sigma_2[-(n + \frac{1}{4})\beta + \frac{\alpha\beta}{4}y^2 - \frac{\alpha}{4}\tan\frac{\beta y^2}{2}] \cos\frac{\beta y^2}{2} + \\
\sigma_3[-(n + \frac{1}{4})\beta + \frac{\alpha\beta}{4}y^2 - \frac{\alpha}{4}\cot\frac{\beta y^2}{2}] \sin\frac{\beta y^2}{2} 
$$

(87)

It is easy to see that the potential $\hat{V}$ is hermitian; $(2n + 1)$ eigenfunctions have the form of polynomials multiplied by the exponential factor $U$ and they are obviously normalizable.

4. Polynomials in two real variables

For further consideration it is convenient to illustrate the space of polynomials of finite degree in several variables through Newton diagrams. In order to do this, let us introduce a $d$-dimensional integer lattice in $\mathbb{R}^d$ and put into correspondence to each node with the coordinates $(k_1, k_2, \ldots, k_d)$ the monomial $x_1^{k_1}x_2^{k_2}\cdots x_d^{k_d}$.

4.1. Polynomials of the first type. Now let us describe the projectivized representation of the algebra $sl_3(\mathbb{R})$ in the differential operators of the first order acting on functions of two real variables. It is easy to show that the generators have the form [14]

$$
J_1 = y^2\partial_y + xy\partial_x - ny, \quad J_2 = x^2\partial_x + xy\partial_y - nx, \\
J_3 = -y\partial_x, \quad J_1' = -\partial_x, \quad J_2' = -\partial_y, \quad J_3' = -x\partial_y, \\
J_d = y\partial_y + 2x\partial_x - n, \quad \tilde{J}_d = 2y\partial_y + x\partial_x - n
$$

(88)

where $x, y$ are the real variables and $n$ is a real number. If $n$ is a non-negative integer, the representation becomes finite-dimensional of dimension $(1+n)(1+n/2)$.

The invariant sub-space has a polynomial basis and is presented as a space of all polynomials of the following type

$$
\mathcal{P}_n^{(I)} = \langle 1; x, y; x^2, xy, y^2; \ldots; x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n \rangle
$$

(89)

or, graphically, the space (89) is given by the Newton diagram of Figure 4.1.

![Newton Diagram](image)

Fig. 4.1. Graphical representation (Newton diagram) of the space $\mathcal{P}_n^{(I)}$ (see (89)).

Definition 4.1. Let us name a linear differential operator of the $k$-th order a quasi-exactly-solvable of the first type, $T_k^{(I)}(x, y)$, if it preserves the space $\mathcal{P}_n^{(I)}$. Correspondingly, the operator $E_k^{(I)}(x, y) \in T_k^{(I)}(x, y)$, which preserves the infinite flag $\mathcal{P}_0^{(I)} \subset \mathcal{P}_1^{(I)} \subset \mathcal{P}_2^{(I)} \subset \cdots \subset \mathcal{P}_n^{(I)} \subset \cdots$ of spaces of all polynomials of the type (89), is named an exactly-solvable of the first type.
Lemma 4.1. Consider the space $\mathcal{P}_n^{(I)}$.

(i) Suppose $n > (k - 1)$. Any quasi-exactly-solvable operator of the first type $T_k^{(I)}(x,y)$, can be represented by a $k$-th degree polynomial of the generators (88). If $n \leq (k - 1)$, the part of the quasi-exactly-solvable operator $T_k^{(I)}(x,y)$ of the first type containing derivatives in $x,y$ up to the order $n$ can be represented by a $n$-th degree polynomial in the generators (88).

(ii) Conversely, any polynomial in (88) is a quasi-exactly solvable operator of the first type.

(iii) Among quasi-exactly-solvable operators of the first type there exist exactly-solvable operators of the first type $E_k^{(I)}(x,y) \subset T_k^{(I)}(x,y)$.

Comment 5. The meaning of the lemma is the following: $T_k^{(I)}(x,y)$ at $k < n+1$ is simply an element of the universal enveloping algebra $U_{sl(3)}(R)$ of the algebra $sl_3(R)$ in realization (88). If $k \geq n+1$, then $T_k(x,y)$ is represented as a polynomial of $n$th degree in (88) plus $B_{\partial x^{n-1} \partial y^m}$, where $m = 0, 1, \ldots (n + 1)$ and $B$ is any linear differential operator of order not higher than $(k - n - 1)$.

Let us introduce the grading of the generators (88) in the following way. The generators are characterized by the two-dimensional grading vectors

\[ \deg(J_3) = (0, +1), \quad \deg(J_2^i) = (+1, 0), \]
\[ \deg(J_3^i) = (-1, +1), \quad \deg(J_2^{i1}) = (+1, -1), \]
\[ \deg(J_d) = (0, 0), \quad \deg(J_d^i) = (0, 0), \]
\[ \deg(J_1^i) = (0, -1), \quad \deg(J_2^i) = (-1, 0). \quad (90) \]

It is apparent that, the grading vector of a monomial in the generators (88) can be defined by the grading vectors of the generators by the rule

\[ \deg(T) = \deg(J_3) = \deg(J_2^i) = \deg(J_3^i) = \deg(J_d) = \deg(J_d^i) = \deg(J_1^i) = \deg(J_2^i) \equiv (\deg_x(T), \deg_y(T)) \]

(91)

Here the $n$’s can be arbitrary non-negative integers.

Definition 4.2. Let us name the grading of a monomial $T$ in generators (75) the number

\[ \deg(T) = \deg_x(T) + \deg_y(T). \]

We will say that a monomial $T$ possesses positive grading if this number is positive. If this number is zero, then a monomial has zero grading. The notion of grading allows one to classify the operators $T_k^{(I)}(x,y)$ in a Lie-algebraic sense.

Lemma 4.2. A quasi-exactly-solvable operator $T_k^{(I)} \subset U_{sl(3)}(R)$ has no terms of positive grading if and only if it is an exactly-solvable operator of the first type.

It is worth noting that among exactly-solvable operators there exists a certain important class of degenerate operators, which preserve an infinite flag of spaces of all homogeneous polynomials

\[ \bar{\mathcal{P}}_m^{(I)} = \{ x^m, x^{m-1}y, \ldots, xy^{m-1}, y^m \} \quad (92) \]

(represented by a horizontal line in Fig. 4.1).
Lemma 4.3. A linear differential operator $T_k(x, y)$ preserves the infinite flag $\hat{\mathcal{P}}_m(1) \subset \hat{\mathcal{P}}_{m+1}(1) \subset \cdots \subset \hat{\mathcal{P}}_n(1) \subset \cdots$ of spaces of all polynomials of the type (89), if and only if it is an exactly-solvable operator having terms of zero grading only. Any operators of such a type can be represented as a polynomial in the generators $J^2_2, J^3_2, J_d, \tilde{J}_d$ (see (88)), which form the algebra $so_3 \oplus R$. If such an operator contains only terms with zero grading vectors, this operator preserves any space of polynomials.

Theorem 4.1. Let $n$ be a non-negative integer. In general, the eigenvalue problem for a linear symmetric differential operator in two real variables $T_k(x, y)$:

$$T_k(x, y) \varphi(x, y) = \varepsilon \varphi(x, y)$$

has $(n + 1)(n/2 + 1)$ eigenfunctions in the form of a polynomial in variables $x, y$ belonging to the space (89), if and only if $T_k(x, y)$ is a symmetric quasi-exactly-solvable operator of the first type. The problem (93) has an infinite sequence of eigenfunctions in the form of polynomials of the type (89), if and only if the operator is a symmetric exactly-solvable operator.

This theorem gives a general classification of differential equations

$$\sum_{m=0}^{k} \sum_{i=0}^{m} a_{i,m-i}^{(m)}(x, y) \frac{\partial^m \varphi(x, y)}{\partial x^i \partial y^{m-i}} = \varepsilon \varphi(x, y)$$

having at least one eigenfunction in the form of a polynomial in $x, y$ of the type (89). In general, the coefficient functions $a_{i,m-i}^{(m)}(x, y)$ have quite cumbersome functional structure and we do not display them here (below we will give their explicit form for $T_k^{(1)}(x, y)$). They are polynomials in $x, y$ of order $(k + m)$ and always contain a general inhomogeneous polynomial of order $m$ as a part. The explicit expressions for those polynomials are obtained by substituting (88) into a general, $k$th order polynomial element of the universal enveloping algebra $U_{sl_3(R)}$ of the algebra $sl_3(R)$. Thus, the coefficients in the polynomials $a_{i,m-i}^{(m)}(x, y)$ can be expressed through the coefficients of the $k$-th order polynomial element of the universal enveloping algebra $U_{sl_3(R)}$. The number of free parameters of the polynomial solutions is defined by the number of parameters characterizing a general, $k$th order polynomial element of the universal enveloping algebra $U_{sl_3(R)}$. In counting free parameters some relations between generators should be taken into account, specifically for the given representation (88)

$$J^2_2 J_d - 2 J^3_2 \tilde{J}_d - 3 J^1_2 J^3_2 = n J^2_2$$

$$J^3_2 \tilde{J}_d - 2 J^1_2 J^3_2 - 3 J^2_2 J^2_2 = n J^3_2$$

$$J^2_2 J_d + J^3_2 \tilde{J}_d - 3 J^1_2 J^3_2 = (n + 3) J^2_2$$

$$J^3_2 J_d + J^2_3 \tilde{J}_d - 3 J^3_2 J^1_2 = (n + 3) J^3_2$$

$$3(J^2_2 J^3_2 + J^3_2 J^1_2 + J^2_3 J^2_2) + J_d J_d + J_d J_d - J_d J_d = 3 J_d + 3 n + n^2$$

$$2 J_d J_d + 2 \tilde{J}_d \tilde{J}_d - 5 J_d \tilde{J}_d + 9 J^2_2 J^2_2 = (n + 6) J_d + (n - 3) \tilde{J}_d + n^2 + 3 n$$

$$J_d J_d - \tilde{J}_d \tilde{J}_d + 3(J^2_2 J^2_2 J^2_2 - J_3^3 J^1_2) = (n + 3)(J_d - \tilde{J}_d)$$

$$J_d J^2_2 - 2 J_d J^1_2 - 3 J^2_2 J^1_2 = n J^2_2$$

$$\tilde{J}_d J^3_2 - 2 J_d J^3_2 - 3 J^2_3 J^1_2 = n J^3_2$$

$$J_d J^3_2 - 2 J_d J^3_2 - 3 J^3_2 J^1_2 = n J^3_2$$

(95)
between quadratic expressions in generators (and the ideals generated by them).[7] For the case of exactly-solvable problems, the coefficient functions $a_{i,m-1}^{(m)}(x,y)$ take the form

$$a_{i,m-1}^{(m)}(x,y) = \sum_{p,q=0}^{p+q \leq m} a_{i,m-1,p,q} x^p y^q$$

with arbitrary coefficients.

Now let us proceed to the case of the second-order differential equations. The second-order polynomial in the generators (88) can be represented as

$$T_2 = c_{\alpha\beta,\gamma\delta} J^\alpha_{\beta} J^\gamma_{\delta} + c_{\alpha\beta} J^\alpha_{\beta} + c$$

where we imply summation over all repeating indices; $\alpha, \beta, \gamma, \delta$ correspond to the indices of operators in (88) and for the Cartan generators we suppose both indices simulate $d$ or $\tilde{d}$, all $c$'s are set to be real numbers. After substitution the expressions (88) into $T_2$, the explicit form of the quasi-exactly-solvable operator is given by

$$T_2^{(I)}(x,y) = \left[ x^2 F_{2,2}^{xx}(x,y) + x F_{2,1}^{xx}(x,y) + \tilde{F}_{2,0}^{xx}(x,y) \right] \frac{\partial^2}{\partial x^2} +$$

$$\left[ xy F_{2,2}^{xy}(x,y) + P_{3,1}^{xy}(x,y) + \tilde{P}_{2,0}^{xy}(x,y) \right] \frac{\partial^2}{\partial x \partial y} +$$

$$\left[ y^2 F_{2,2}^{yy}(x,y) + y P_{2,1}^{yy}(x,y) + \tilde{P}_{2,0}^{yy}(x,y) \right] \frac{\partial^2}{\partial y^2} +$$

$$\left[ x P_{2,2}^{xx}(x,y) + P_{2,1}^{xx}(x,y) + \tilde{P}_{1,0}^{xx}(x,y) \right] \frac{\partial}{\partial x} +$$

$$\left[ y P_{2,2}^{yy}(x,y) + P_{2,1}^{yy}(x,y) + \tilde{P}_{1,0}^{yy}(x,y) \right] \frac{\partial}{\partial y} +$$

$$\left[ P_{0,2}^{00}(x,y) + P_{1,1}^{00}(x,y) + \tilde{P}_{0,0}^{00}(x,y) \right]$$

where $P_{k,m}^{\xi}(x,y)$ and $\tilde{P}_{k,m}^{\xi}(x,y)$ are homogeneous and inhomogeneous polynomials of order $k$, respectively, the index $m$ numerates them, the superscript $\xi$ characterizes the order of derivative, which this coefficient function corresponds to. For the case of the second-order-exactly-solvable operator $E^{(I)}_2(x,y)$, the structure of the coefficient function is similar to (97), except for the fact that all tilde-less polynomials disappear. The number of free parameters is equal to

$$\text{par}(T_2^{(I)}(x,y)) = 36$$

while for the case of an exactly-solvable operator (an infinite sequence of polynomial eigenfunctions in the problem (93))

$$\text{par}(E_2^{(I)}(x,y)) = 25$$

An important particular case is when the quasi-exactly-solvable operator $T_2^{(I)}(x,y)$ possesses two invariant sub-spaces of the type (89). This situation is described by the following lemma:

---

[7] It was shown that there exist 56 cubic relations between the generators (88). However, it turned out that all of them (including standard cubic Casimir operator in given representation (88)) are functionally-dependent on quadratic relations (95) (Personal communication by G. Post).
Lemma 4.4. Suppose $T_i^{(f)}(x,y)$ has no terms of grading 2
\[ c_{12,12} = c_{13,13} = c_{12,13} = 0 \] (98)
If there exist some coefficient $c$’s and a non-negative integer $N$ such that the conditions
\[ c_{12} = (n - N - m)c_{12,d} + (n - 2N + m)c_{12,d} + (N - m)c_{12,32}, \]
\[ c_{13} = (n - N - m)c_{13,d} + (n - 2N + m)c_{13,d} + mc_{13,23}, \] (99)
are fulfilled at all $m = 0, 1, 2, \ldots, N$, then the operator $T_2^{(f)}(x,y)$ preserves both $P_n$ and $P_N$, and $\text{par}(T_2^{(f)}(x,y)) = 31$.

Now let us proceed to the important item: under what conditions on the coefficients in (97) the second-order-quasi-exactly-solvable operators can be reduced to a form of the Schroedinger operator after some gauge transformation (conjugation)
\[ f(x,y)e^{(x,y)}T_2(x,y)^{(f)}e^{-(x,y)} = \Delta_g + V(x,y,n) \] (100)
where $f, t, V$ are some functions in $\mathbb{R}^2$, and $\Delta_g$ is the Laplace-Beltrami operator with some metric tensor $g_{\mu\nu}$; $\mu, \nu = 1, 2$. This is a difficult problem which has no complete solution yet. In [14, 16] a few multi-parametric examples have been constructed.

In general, if one can perform the transformation (100), we arrive at potentials $V(x,y,n)$ containing explicitly a dependence on the parameter $n$ and hence the dimension of the initial invariant subspace (89). Finally, we end up at quasi-exactly-solvable Schroedinger equations with a certain number of eigenfunctions known algebraically. Taking an exactly-solvable operator $E_2^{(f)}(x,y)$ instead of the quasi-exactly-solvable one and performing the transformation (100), a certain exactly-solvable Schroedinger operator emerges. Evidently, they will have no dependence on the parameter $n$ and correspondently an infinite sequence of eigenfunctions of the Schroedinger equation has a form of polynomials.

Also it is worth noting that there exists an important particular class of quasi-exactly-solvable operators, lying in a certain sense in between quasi-exactly-solvable and exactly-solvable operators. The algebra $so_3(\mathbb{R})$ in realization (88) contains the algebra $sl_3(\mathbb{R})$ as a sub-algebra in such a way that
\[ J^1 = (1 + y^2)\partial_y + xy\partial_x - ny, \quad J^2 = (1 + x^2)\partial_x + xy\partial_y - nx, \]
\[ J^3 = x\partial_y - y\partial_x \] (101)
which is not graded anymore. If the parameter $n$ is a non-negative integer (and coincides with that in (88)), the same finite-dimensional invariant subspace $P_n^{(f)}$ (see (89) and Fig. 4.1) as in the original $sl_3(\mathbb{R})$ occurs. For this case a corresponding finite-dimensional representation is reducible and unitary. It has been proven [14, 15], that any symmetric bilinear combination of generators (101), $T_2^{(c)} = c_{\alpha\beta} J^\alpha J^\beta$, where $c_{\alpha\beta} = c_{\beta\alpha}$ are real numbers, can be reduced to a form of the

\[ \text{Our further consideration will be restricted the case } f = 1 \text{ only.} \]
LIE-ALGEBRAS AND LINEAR OPERATORS

27

Laplace-Beltrami operator plus a scalar function making a certain gauge transformation (conjugation). 9 In general, the metric tensor $g_{\mu \nu}$ is not degenerate 10 and the obtained potential $V(x, y)$ is given by a rational function of the coordinates. Moreover, the potential $V(x, y)$ has no dependence on the label $n$! All $n$-dependent terms contain no $x, y$-dependence and can be included into redefinition of the reference point for eigenvalues. At first sight, those quasi-exactly-solvable operators look as if they are exactly-solvable, but is not so. As a consequence of the fact that the quadratic Casimir operator for $so_3(\mathbb{R})$ in the realization (101) is non-trivial and commutes with $T_2(e)$, the functional space of $T_2(e)$ is subdivided into the finite-dimensional blocks of increasing sizes, corresponding to the irreducible representations of $so_3(\mathbb{R})$ (unlike truly exactly-solvable operators, for which the sizes of blocks are fixed and equal to one). We name such operators exactly-solvable of the second type.

Another interesting property appears, if the matrix $c_{\alpha \beta}$ has one vanishing eigenvalue: a certain mysterious relation holds 14 (see also 17, 18)

$$V(x, y, \{c\}) = \frac{3}{16} R(x, y, \{c\}),$$

(102)

where $R(x, y, \{c\})$ is the scalar curvature calculated through the metric tensor attached in the Laplace-Beltrami operator. This can imply that the corresponding Schroedinger operator has the form of a purely geometrical object! The real meaning of this fact is not understood so far.

4.2. Polynomials of the second type. In 13 we studied the quasi-exactly-solvable operators in one real variable. It turned out that the solution to this problem was found using a connection with the projectivized representation of the algebra $sl_2(\mathbb{R})$. As a natural step in developing the original idea, let us consider the projectivized representation of the direct sum of two algebras $sl_2(\mathbb{R})$.

The algebra $sl_2(\mathbb{R}) \oplus sl_2(\mathbb{R})$ taken in projectivized representation acts on functions of two real variables. The generators have the form (see (2))

$$J^+_x = x^2 \partial_x - nx , \quad J^+_y = y^2 \partial_y - my ,$$

$$J^0_x = x \partial_x - \frac{n}{2} , \quad J^0_y = y \partial_y - \frac{m}{2} ,$$

$$J^-_x = \partial_x , \quad J^-_y = \partial_y ,$$

(103)

where $x, y$ are the real variables and $n, m$ are numbers. If $n, m$ are non-negative integers, there exists the finite-dimensional representation of dimension $(n + 1)(m + 1)$. Evidently, the invariant sub-space has a polynomial basis and is presented as a space of all polynomials described by the Newton diagram of Fig. 4.2. We denote this space as $\mathcal{P}_{n,m}$. 9

9 In fact, in 17 has been proven a more general theorem, that any symmetric bilinear combination of generators of unitary representation of the semisimple Lie algebra, realized in the first-order differential operators, can be reduced to a form of the Laplace-Beltrami operator with some metric tensor plus a scalar function making a certain gauge transformation (conjugation).

10 Degeneracy of $g_{\mu \nu}$ occurs, for example, if $c_{\alpha \beta}$ has two vanishing eigenvalues.
Fig. 4.2. Graphical representation (Newton diagram) of the space $\mathcal{P}_{n,m}^{(II)}$. 

Definition 4.3. Let us name a linear differential operator of the $(k,p)$th order, containing derivatives in $x$ and $y$ up to $k$th and $p$th orders, respectively, a quasi-exactly-solvable of the second type, $T_{k,p}^{(II)}(x,y)$, if it preserves the space $\mathcal{P}_{n,m}^{(II)}$. Correspondingly, the operator $E_{k,p}^{(II)}(x,y) \in T_{k,p}^{(II)}(x,y)$, which preserves either the infinite flag $\mathcal{P}_{0,m}^{(II)} \subset \mathcal{P}_{1,m}^{(II)} \subset \mathcal{P}_{2,m}^{(II)} \subset \cdots \subset \mathcal{P}_{n,m}^{(II)} \subset \cdots$, or the infinite flag $\mathcal{P}_{n,0}^{(II)} \subset \mathcal{P}_{n,1}^{(II)} \subset \mathcal{P}_{n,2}^{(II)} \subset \cdots \subset \mathcal{P}_{n,m}^{(II)} \subset \cdots$ of spaces of all polynomials, is named an exactly-solvable of the type $2_x$ or $2_y$, respectively.

Lemma 4.5. Consider the space $\mathcal{P}_{n,m}^{(II)}$.

(i) Suppose $n > (k-1)$ and $m > (p-1)$. Any quasi-exactly-solvable operator of the second type $T_{k,p}^{(II)}(x,y)$, can be represented by a $(k,p)$th degree polynomial of the generators (103). If $n \leq (k-1)$ and/or $m \leq (p-1)$, the part of the quasi-exactly-solvable operator $T_{k,p}^{(II)}(x,y)$ of the second type containing derivatives in $x,y$ up to order $n,m$, respectively, can be represented by an $(n,m)$th degree polynomial in the generators (103).

(ii) Conversely, any polynomial in (103) is a quasi-exactly-solvable operator of the second type.

(iii) Among quasi-exactly-solvable operators of the second type there exist exactly-solvable operators of the second type $E_{k,p}^{(II)}(x,y) \subset T_{k,p}^{(II)}(x,y)$.

Similarly, as it has been done for the algebra $sl_3(\mathbb{R})$, one can introduce the notion of grading:

$$
\deg(J_x^+) = (+1,0), \quad \deg(J_y^+) = (0,+1),
$$

$$
\deg(J_x^0) = (0,0), \quad \deg(J_y^0) = (0,0),
$$

$$
\deg(J_x^-) = (-1,0), \quad \deg(J_y^-) = (0,-1).
$$

(104)

It is apparent that, the grading vector of a monomial in the generators (90) can be defined by the grading vectors (104) of the generators by the rule

$$
\deg(T) \equiv \deg((J_x^+)^{n_+} (J_y^0)^{n_{0+}} (J_x^0)^{n_{0-}} (J_y^+)^{n_+} (J_y^0)^{n_{0+}} (J_y^-)^{n_{-+}}) = 
$$

$$
(n_{x+} - n_{x-}, n_{y+} - n_{y-}) \equiv (\deg_x(T), \deg_y(T)) \quad (105)
$$

Here the $n$’s can be arbitrary non-negative integers.

---

11 So a leading derivative has a form $\frac{\partial^{(k+p)}}{\partial y \partial y}$. Also we will use a notation through this section $T_N(x,y)$ implying that in general all derivatives of the order $N$ are presented.
Definition 4.4. Let us name the $x$-grading, $y$-grading and grading of a monomial $T$ in generators (103) the numbers $\text{deg}_x(T)$, $\text{deg}_y(T)$ and $\text{deg}(T) = \text{deg}_x(T) + \text{deg}_y(T)$, respectively. We say that a monomial $T$ possesses positive $x$-grading ($y$-grading, grading), if the number $\text{deg}_x(T)$ ($\text{deg}_y(T)$, $\text{deg}(T)$) is positive. If $\text{deg}_x(T)(\text{deg}_y(T), \text{deg}(T)) = 0$, then a monomial has zero $x$-grading ($y$-grading, grading).

The notion of grading allows one to classify the operators $T_{k,p}^{(II)}(x,y)$ in the Lie-algebraic sense.

Lemma 4.6. The quasi-exactly-solvable operator $T_{k,p}^{(II)}(x,y)$ preserves the infinite flag $\mathcal{P}_{0,m} \subset \mathcal{P}_{1,m} \subset \mathcal{P}_{2,m} \subset \cdots \subset \mathcal{P}_{n,m} \subset \cdots$ of spaces of all the polynomials, if and only if it is an exactly-solvable operator of type $2_x$ having no terms of positive $x$-grading, $\text{deg}_x(T) > 0$.

The quasi-exactly-solvable operator $T_{k,p}^{(II)}(x,y)$ preserves the infinite flag $\mathcal{P}_{n,0} \subset \mathcal{P}_{n,1} \subset \mathcal{P}_{n,2} \subset \cdots \subset \mathcal{P}_{n,m} \subset \cdots$ of all spaces of the polynomials, if and only if it is an exactly-solvable operator of type $2_y$ having no terms of positive $y$-grading, $\text{deg}_y(T) > 0$.

If a quasi-exactly-solvable operator of the second type contains no terms of
positive grading, this operator preserves the infinite flag $\mathcal{P}_0(1) \subset \mathcal{P}_1(1) \subset \mathcal{P}_2(1) \subset \cdots \subset \mathcal{P}_n(1) \subset \cdots$ of spaces of all the polynomials of the type (76) and is attached to the exactly-solvable operator of the first type.

**Theorem 4.2.** Let $n, m$ be non-negative integers. In general, the eigenvalue problem (93) for a linear symmetric differential operator in two real variables $T_{k,p}(x,y)$ has $(n+1)(m+1)$ eigenfunctions in the form of a polynomial in variables $x,y$ belonging to the space $\mathcal{P}_n,m(11)$, if and only if $T_{k,p}(x,y)$ is a quasi-exactly-solvable symmetric operator of the second type. The problem (93) has an infinite sequence of eigenfunctions in the form of polynomials belonging the space $\mathcal{P}_n,m(11)$ at fixed $m$ ($n$), if and only if the operator is an exactly-solvable symmetric operator of type $2_x$ ($2_y$).

This theorem gives a general classification of differential equations (94), having at least one eigenfunction in the form of a polynomial in $x, y$ of type $\mathcal{P}_n,m(11)$. In general, the coefficient functions $a_{i,m,n}^{(m)}(x,y)$ in (94) have a quite cumbersome functional structure and we do not display them here (below we will give their explicit form for $T_{2,x}(11)(x,y)$). They are polynomials in $x, y$ of the order $(k + m)$. The explicit expressions for those polynomials are obtained by substituting (103) into a general, $k$th order polynomial element of the universal enveloping algebra $U_{sl_2(\mathbb{R}) \oplus sl_2(\mathbb{R})}$ of the algebra $sl_2(\mathbb{R}) \oplus sl_2(\mathbb{R})$. Thus, the coefficients in the polynomials $a_{i,m,n}^{(m)}(x,y)$ can be expressed through the coefficients of the $k$-th order polynomial element of the universal enveloping algebra $U_{sl_2(\mathbb{R}) \oplus sl_2(\mathbb{R})}$. The number of free parameters of the polynomial solutions is defined by the number of parameters characterizing a general, $k$th order polynomial element of the universal enveloping algebra $U_{sl_2(\mathbb{R}) \oplus sl_2(\mathbb{R})}$. In counting free parameters some relations between generators should be taken into account, specifically for the given representation (103)

\[
\begin{align*}
J_x^+ J_x^- - J_x^0 J_x^0 + J_x^0 &= \frac{m}{2} \left( \frac{m}{2} + 1 \right) \\
J_y^+ J_y^- - J_y^0 J_y^0 + J_y^0 &= \frac{m}{2} \left( \frac{m}{2} + 1 \right)
\end{align*}
\]

(106) between quadratic expressions in generators (and the ideals generated by them). [3]

Now let us proceed to the case of the second-order differential equations. The second-order polynomial in the generators (103) can be represented as

\[
T_2 = c_x^{x \alpha \beta} J_x^\alpha J_x^\beta + c_y^{x \alpha \beta} J_x^\alpha J_y^\beta + c_y^{y \alpha \beta} J_y^\alpha J_y^\beta + c_x^{y \alpha \beta} J_x^\alpha J_y^\beta + c_y^{y \alpha \beta} J_y^\alpha + c \ , \tag{107}
\]

where we imply summation over all repeating indices and $\alpha, \beta = \pm, 0$; all $c$'s are set to be real numbers. Taking (106) in account, it is easy to show that $T_2$ is characterized by 26 free parameters. Substituting (103) into (107), we obtain the explicit form of the second-order-quasi-exactly-solvable operator

\[
T_{2,x}^{(11)}(x,y) = \tilde{P}_{4,0,x}(x,y) \frac{\partial^2}{\partial x^2} + [x^2 y^2 \tilde{P}_{0,2}^{xy} + x y \tilde{P}_{2,0}^{xy}(x,y)] \frac{\partial^2}{\partial x \partial y} + \tilde{P}_{4,0,y}(y) \frac{\partial^2}{\partial y^2} + \tilde{P}_{0,2}^{xy}(x,y) \frac{\partial^2}{\partial y^2} + \tilde{P}_{2,0}^{xy}(x,y) \frac{\partial^2}{\partial x \partial y} + \tilde{P}_{4,0,y}^{xy}(y) \frac{\partial^2}{\partial y^2} + \tilde{P}_{2,0}^{xy}(x,y) \frac{\partial^2}{\partial x \partial y} + \tilde{P}_{4,0,y}^{xy}(y) \frac{\partial^2}{\partial y^2} + \tilde{P}_{0,2}^{xy}(x,y) \frac{\partial^2}{\partial y^2} + \tilde{P}_{2,0}^{xy}(x,y) \frac{\partial^2}{\partial x \partial y}
\]

[12] For this case they correspond to the quadratic Casimir operators.
\[ [\hat{P}^{x}_{3,1}(x) + y\hat{P}^{x}_{2,0}(x)] \frac{\partial}{\partial x} + [\hat{P}^{y}_{3,1}(y) + x\hat{P}^{y}_{2,0}(y)] \frac{\partial}{\partial y} + \hat{P}^{0}_{2,0}(x, y) \]  

(108)

where \( P_{k,m}(x, y) \) and \( \hat{P}_{k,m}(x, y) \) are homogeneous and inhomogeneous polynomials of order \( k \), respectively, the index \( m \) marks them, and the superscript \( c \) characterizes the order of the derivative corresponding to this coefficient function. For the case of the second-order-exactly-solvable operator of type 2

\[ E_{2}^{(II)}(x, y) = \]  

\[ \hat{Q}^{x}_{2,0}(x) \frac{\partial^2}{\partial x^2} + [x\hat{Q}^{xy}_{2,1}(y) + \hat{Q}^{xy}_{2,0}(y)] \frac{\partial^2}{\partial x \partial y} + \hat{Q}^{y}_{4,0}(y) \frac{\partial^2}{\partial y^2} + \]  

\[ \hat{Q}^{x}_{1,1}(x) + y\hat{Q}^{y}_{1,0}(x)] \frac{\partial}{\partial x} + \hat{Q}^{y}_{1,1}(y) + x\hat{Q}^{y}_{1,0}(y)] \frac{\partial}{\partial y} + \hat{Q}^{0}_{2,0}(y) \]  

(109)

For the case of the second-order-exactly-solvable operator of type 2, the functional form is similar to (109) with the interchange \( x \leftrightarrow y \).

It is easy to show that the numbers of free parameters are

\[ par(T_{2}^{(II)}(x, y)) = 26. \]  

\[ par(E_{2}^{(II)}(x, y)) = 20. \]

As in the previous cases, there is an important particular case of quasi-exactly-solvable operators of the second order, where they possess two invariant sub-spaces.

**Lemma 4.7.** Suppose in (107) there are no terms of \( x \)-grading 2

\[ c^{x}_{++} = 0. \]  

(110)

If there exist some coefficient \( c \)'s and a non-negative integer \( N \) such that the conditions

\[ c^{y}_{++} = 0, \]  

\[ c^{y}_{+-} = 0, \]  

\[ c^{x}_{+} = (n/2 - N)c^{x}_{+0} + (m/2 - k)c^{y}_{+0}, \]  

(111)

are fulfilled at all \( k = 0, 1, 2, \ldots, m \), then the operator \( T_{2}^{(II)}(x, y) \) preserves both \( P_{n,m}^{II} \) and \( P_{N,m}^{II} \), and \( par(T_{2}^{(II)}(x, y)) = 22 \).

Generically, the question of the reduction of quasi-exactly-solvable operators of the second type \( T_{2}^{(II)}(x, y) \) to the form of the Schroedinger operator is still open. Recently, several multi-parametrical families of those Schroedinger operators were constructed in \[ \[4, 16\]. As well as the case of the first type of quasi-exactly-solvable operators, corresponding Schroedinger operators contain in general the Laplace-Beltrami operator characterizing a non-flat space metric tensor.

The above analysis of linear differential operators preserving the space \( P_{n,m}^{(II)} \) can be naturally extended to the case of linear finite-difference operators defined through the Jackson symbol \( D \) (see Section 2) with the action:

\[ Df(x) = \frac{f(x) - f(qx)}{(1 - q)x} + f(qx)D \]
instead of a continuous derivative, where $f(x)$ is a real function and $q$ is a number. All above-described results hold with replacement of the algebra $sl_2(\mathbb{R}) \oplus sl_2(\mathbb{R})$ by the quantum algebra $sl_2(\mathbb{R})_q \oplus sl_2(\mathbb{R})_q$ in ‘projectivized’ representation (21) (see [3, 13]). Obviously, in the limit $q \to 1$ all results coincide to the results of the present Section.

4.3. Polynomials of the third type. The third case, which we shall discuss here, corresponds to the family of Lie algebras $gl_2(\mathbb{R}) \ltimes \mathbb{R}^{r+1}$ (semidirect sum of $gl_2(\mathbb{R})$ with a $(r+1)$-dimensional abelian ideal; Case 24 in the classification given in [19]). This family of Lie algebras, depending on an integer $r > 0$, can be realized in terms of the first-order differential operators

$$J^1 = \partial_x ,$$

$$J^2 = x\partial_x - \frac{n}{3} , J^3 = y\partial_y - \frac{n}{3r} ,$$

$$J^4 = x^2\partial_x + rxy\partial_y - nx ,$$

$$J^{5+i} = x^i\partial_y , i = 0, 1, \ldots, r ,$$

(112)

where $x, y$ are real variables and $n$ is a real number. If $n$ is a non-negative integer, the representation becomes finite-dimensional. The invariant sub-space has a polynomial basis and is presented as a space of all polynomials of the form

$$P_{r,n}^{(III)} = \sum_{i,j \geq 0} a_{ij} x^i y^j$$

(113)

or, graphically, (113) is given by the Newton diagram of Figure 4.3. The general formula for the dimension of the corresponding finite-dimensional representation (113) is given by

$$dim P_{r,n}^{(III)} = \frac{n^2 + (r+2)n + \alpha_{r,n}}{2r}$$

(114)

where for small $r$

$$\alpha_{1,n} = 2 ,$$

$$\alpha_{2,n} = \begin{cases} 3 & \text{at odd } n \\ 4 & \text{at even } n \end{cases}$$

$$\alpha_{3,n} = \begin{cases} 4 & \text{at } (n+1)=0 \pmod{3} \\ 6 & \text{at other } n \end{cases}$$

$$\alpha_{4,n} = \begin{cases} 5 & \text{at } (n+1)=0 \pmod{4} \\ 8 & \text{at other } n \\ 9 & \text{at } (n+3)=0 \pmod{4} \end{cases}$$

13 with minor modifications
14 It is worth noting that at $r = 1$ the algebra \{gl_2(\mathbb{R}) \ltimes \mathbb{R}^{2}\} \subset sl_3(\mathbb{R})$. Thus, this case is reduced to one about the first type polynomials (see Section 4.1). Hereafter, we include $r = 1$ into consideration just for the sake of completeness.
Definition 4.5. Let us name a linear differential operator of the \( k \)th order a quasi-exactly-solvable of the \( r \)-third type, \( T^{(r,III)}_k(x,y) \), if it preserves the space \( P^{(III)}_{r,n} \). Correspondingly, the operator \( E^{(r,III)}_r(x,y) \in T^{(r,III)}_k(x,y) \), which preserves the infinite flag \( P^{(III)}_{r,0} \subset P^{(III)}_{r,1} \subset \cdots \subset P^{(III)}_{r,n} \subset \cdots \) of spaces of all polynomials of the type (113), is named an exactly-solvable of the \( r \)-third type.

Lemma 4.8. Take the space \( P^{(III)}_{r,n} \).

(i) Suppose \( n > (k - 1) \). Any quasi-exactly-solvable operator \( T^{(r,III)}_k(x,y) \) can be represented by a \( k \)th degree polynomial of the generators (112). If \( n \leq (k - 1) \), the part of the quasi-exactly-solvable operator \( T^{(r,III)}_k(x,y) \) of the \( r \)-third type containing derivatives in \( x, y \) up to order \( n \) can be represented by a \( n \)-th degree polynomial in the generators (112).

(ii) Conversely, any polynomial in (112) is a quasi-exactly solvable operator of the \( r \)-third type.

(iii) Among quasi-exactly-solvable operators of the \( r \)-third type there exist exactly-solvable operators of the \( r \)-third type \( E^{(r,III)}_r(x,y) \subset T^{(r,III)}_k(x,y) \).

One can introduce the grading of the generators (112) in an analogous way as has been done before for the cases of the algebra \( sl_3(\mathbb{R}) \) (see (90)) and \( sl_2(\mathbb{R}) \oplus sl_2(\mathbb{R}) \) (see (104)). All generators are characterized by the two-dimensional grading vectors

\[
\begin{align*}
\text{deg}(J^1) &= (-1, 0), \\
\text{deg}(J^2) &= (0, 0), \quad \text{deg}(J^3) = (0, 0), \\
\text{deg}(J^4) &= (1, 0), \\
\text{deg}(J^5) &= (0, -1), \quad \text{deg}(J^6) = (1, -1), \ldots, \quad \text{deg}(J^{5+r}) = (r, -1)
\end{align*}
\]

(115)

Similarly as before, the grading vector of a monomial in the generators can be defined through the grading vectors of the generators (112) (cf. (90), (105)).

Definition 4.6. Let us name the grading of a monomial \( T \) in generators (112) the number

\[
\text{deg}(T) = \text{deg}_x(T) + r \text{ deg}_y(T)
\]

(cf. the case of \( sl_3(\mathbb{R}) \)). We say that a monomial \( T \) possesses positive grading if \( \text{deg}(T) \) is positive. If it is zero, then a monomial has zero grading.

The notion of grading allows us to classify the operators \( T_k(x,y) \) in a Lie-algebraic sense.
Lemma 4.9. A quasi-exactly-solvable operator $T^{(r,III)}_k \subset U_{gl_2(R) \ltimes R^{r+1}}$ has no terms of positive grading iff it is an exactly-solvable operator.

Theorem 4.3. Let $n$ and $(r - 1)$ be non-negative integers. In general, the eigenvalue problem (93) for a linear symmetric differential operator in two real variables $T_k(x,y)$ has a certain number of eigenfunctions in the form of a polynomial in variables $x, y$ belonging to the space (113), if and only if $T_k(x,y)$ is a quasi-exactly-solvable, symmetric operator of the $r$-third type. The problem (93) has an infinite sequence of eigenfunctions in the form of polynomials belonging to (113), if and only if the operator is an exactly-solvable, symmetric operator of the $r$-third type.

This theorem gives a general classification of differential equations (94), having at least one eigenfunction in the form of a polynomial in $x, y$ of the type $P^{(r,III)}_{m,n}$. In general, the coefficient functions $a_{i,m-i}^{(m)}(x,y)$ in (94) are polynomials in $x, y$ and have a quite cumbersome functional structure and we do not display them here (below we will give their explicit form for $T^{(r,III)}_2(x,y)$). The explicit expressions for those polynomials are obtained by substituting (112) into a general, $k$th order polynomial element of the universal enveloping algebra $U_{gl_2(R) \ltimes R^{r+1}}$ of the algebra $gl_2(R) \ltimes R^{r+1}$. Thus, the coefficients in the polynomials $a_{i,m-i}^{(m)}(x,y)$ can be expressed through the coefficients of the $k$th order polynomial element of the universal enveloping algebra. The number of free parameters of the polynomial solutions is defined by the number of parameters characterizing a general, $k$th order polynomial element of the universal enveloping algebra. In counting free parameters some relations between generators should be taken into account, specifically for a given representation (112)

\[
J^2 J^5 - J^1 J^6 + \frac{n}{3} J^5 = 0,
\]

\[
J^1 J^4 - J^2 J^3 - r J^2 J^3 - J^2 - r(n + 1)J^3 = -\frac{n}{3} (\frac{n}{3} + 1),
\]

(116)

\[
J^2 J^{6+i} + r J^3 J^{6+i} - J^4 J^{5+i} - (\frac{n}{3} + 1) J^{6+i} = 0,
\]

at $i = 0, 1, 2, \ldots, (r - 1),
\]

(117)

\[
J^1 J^{7+i} - J^2 J^{6+i} - (\frac{n}{3} + 1) J^{6+i} = 0,
\]

at $i = 0, 1, 2, \ldots, (r - 2),
\]

(118)

\[
J^5 J^{7+i} = \cdots = J^{5+k} J^{7+i-k},
\]

at $k = 0, 1, 2, \ldots$ and $2k \leq (2 + i)$, and $i = 0, 1, 2, \ldots, (2r - 2)
\]

(119)

between quadratic expressions in generators (and the ideals generated by them).

Now let us proceed to the case of the second-order differential equations. Taking the relations (116)-(119) in account, one can find the number of free parameters

\[
\text{par}(T^{(r,III)}_2(x,y)) = 5(r + 4)
\]

(120)
and obtain the explicit form of the second-order-quasi-exactly-solvable operator

\[ T_{2}^{(r,III)}(x, y) = \]

\[ \hat{P}_{4,0}^{x,x}(x) \frac{\partial^2}{\partial x^2} + \left[ \hat{P}_{r+2,1}^{xy}(x) + y\hat{P}_{3,0}^{xy}(x) \right] \frac{\partial^2}{\partial x \partial y} + \left[ \hat{P}_{2r,0}^{xy}(x) + y\hat{P}_{r+1,0}^{xy}(x) \right] \frac{\partial^2}{\partial y^2} + \]

\[ [\hat{P}_{3,0}^{x}(x)] \frac{\partial}{\partial x} + [\hat{P}_{r+1,1}^{y}(x) + y\hat{P}_{1,0}^{y}(x)] \frac{\partial}{\partial y} + \hat{P}_{2,0}^{y}(x) \]  

(121)

where \( \hat{P}_{k,m}^{c}(x, y) \) are inhomogeneous polynomials of order \( k \), the index \( m \) numerates them, and the superscript \( c \) characterizes the order of the derivative corresponding to this coefficient function. For the case of the second-order-exactly-solvable operator of \( r \)-third type

\[ E_{2}^{(r,III)}(x, y) = \]

\[ \hat{P}_{2,0}^{x,x}(x) \frac{\partial^2}{\partial x^2} + \left[ \hat{P}_{r+1,1}^{xy}(x) + y\hat{P}_{1,0}^{xy}(x) \right] \frac{\partial^2}{\partial x \partial y} + \left[ \hat{P}_{r,0}^{y}(x) + y\hat{P}_{r,0}^{y}(x) \right] \frac{\partial^2}{\partial y^2} + \]

\[ [\hat{P}_{1,0}^{x}(x)] \frac{\partial}{\partial x} + [\hat{P}_{r,1}^{y}(x) + y\hat{P}_{0,0}^{y}(x)] \frac{\partial}{\partial y} + \hat{P}_{0,0}^{y}(x) \]  

(122)

In this case the number of free parameters is equal to

\[ \text{par}(E_{2}^{(r,III)}(x, y)) = 5(r + 3) \]  

(123)

As in the previous cases, there is an important particular case of \( r \)-third type quasi-exactly-solvable operators, where they possess two invariant sub-spaces.

**Lemma 4.10.** Suppose \( E_{2}^{(r,III)}(x, y) \) has no terms of grading 2

\[ c_{44} = 0 . \]  

(124)

If there exist some coefficient \( c \)'s and a non-negative integer \( N \) such that the conditions

\[ c_{4,5+r} = 0 , \]

\[ c_{4} = \left( \frac{N}{3} - m \right)c_{24} + \left( \frac{N}{3} - k \right)c_{34} , \]  

(125)

are fulfilled at all \( m, k = 0, 1, 2, \ldots \) such that \( m + rk = N \), then the operator \( T_{2}^{(r,III)}(x, y) \) preserves both \( P_{r,n}^{11} \) and \( P_{r,N}^{11} \), and \( \text{par}(T_{2}^{(r,III)}(x, y)) = 5r + 17 \).

Generically, the question of the reduction of the quasi-exactly-solvable operator \( T_{2}^{(r,III)}(x, y) \) to the form of the Schrödinger operator is still open. Initially in [10] a few multi-parametrical families of those Schrödinger operators have been constructed. Similar to the case of quasi-exactly-solvable operators of the first and the second types, corresponding Schrödinger operators contain in general the Laplace-Beltrami operator with non-flat space metric tensor.
4.4. Discussion. Through all the above analysis, one very crucial requirement played a role: we considered the problem in general position, assuming the spaces of all polynomials contain polynomials with all possible real (complex) coefficients. If this requirement is not fulfilled, then the above connection of the finite-dimensional space of all polynomials and the finite-dimensional representation of the Lie algebra is lost and degenerate cases occur. This demands a separate investigation.

So, we described three types of finite-dimensional spaces (see Fig. 4.1-3) with a basis in polynomials of two real variables, which can be preserved by linear differential operators. The natural question emerges: is it possible to find linear differential operators having no representation through generators of a Lie algebra, which preserve the finite-dimensional space of polynomials, other than those shown in Fig. 4.1,2,3? The answer is given by the following conjecture [20].

**Conjecture 1.** If a linear differential operator \( T_k(x, y) \) does preserve a finite-dimensional space of all polynomials (presented by a convex Newton diagram) other than \( P_{n}^{(I)}, P_{n,m}^{(II)}, P_{r,n}^{(III)} \), then this operator preserves either one of those spaces as well, or a certain infinite flag of finite-dimensional spaces of all polynomials.

The general question about reducibility of a second-order polynomial in generators of the algebras \( sl_3(\mathbb{R}), sl_2(\mathbb{R}) \oplus sl_2(\mathbb{R}) \) and \( \{gl_2(\mathbb{R}) \times \mathbb{R}^{r+1}\}, r = 2, 3, 4, \ldots \) to the form of the Laplace-Beltrami operator plus a potential is still open. For all cases where this reducibility has been performed, a metric tensor appeared which corresponded to non-flat space. All attempts to find the quasi-exactly-solvable or exactly-solvable problems in \( \mathbb{R}^2 \) corresponding to the flat space did lead to a problem with separability of variables. We believe that the following conjecture holds.

**Conjecture 2.** In \( \mathbb{R}^2 \) there exist no quasi-exactly-solvable or exactly-solvable problems containing the Laplace-Beltrami operator with flat-space metric tensor, which are characterized by non-separable variables.

Also the next question, under what conditions the eigenfunctions obtained belong to some \( L^2 \)-space, is lacking a solution. The analysis analogous to that which has been performed in [14] for one-dimensional quasi-exactly-solvable problems should be done.

5. General considerations

As the main conclusion one can formulate as the following theorem.

**(Main) Theorem.** Consider a Lie algebra \( g \) of the differential operators of the first order, which possesses a finite-dimensional irreducible representation \( \mathcal{P} \). Any linear differential operator acting in \( \mathcal{P} \) (having \( \mathcal{P} \) as the invariant subspace) can be represented by a polynomial in generators of the algebra \( g \) plus an operator annihilating \( \mathcal{P} \).

As a probable extension of this theorem, we assume that the following conjecture holds [20].
Conjecture 3. If a linear differential operator $T$ acting on functions in $\mathbb{R}^k$ does possess a single finite-dimensional invariant subspace with polynomial basis (presented by a convex Newton diagram), this finite-dimensional space coincides with a certain finite-dimensional representation of some Lie algebra, realized by differential operators of the first order.

The interesting question is how to describe finite modules of smooth functions in $\mathbb{R}^k$ which can serve as invariant sub-spaces of linear operators. Evidently, if we could find some realization of a Lie algebra $g$ of differential operators, possessing an irreducible finite module of smooth functions, we can immediately consider the direct sum of several species of $g$ acting on the functions given at the corresponding direct products of spaces, where the original algebra $g$ acts in each of them. As an example, this procedure is presented in Section 4.2: the direct sum of two $sl_2(\mathbb{R})$ generates the linear differential operators acting on the functions at $\mathbb{R}^2$ having a rectangular $\mathcal{P}_{n,m}$ as the invariant sub-space. Taking the direct sum of $k$ species of $sl_2(\mathbb{R})$ acting at $\mathbb{R}^k$, we arrive to the operators possessing a $k$-dimensional parallelopipied as the invariant sub-space. In this case the algebra $sl_2(\mathbb{R})$ plays the role of a “primary” algebra giving rise to a multidimensional convex geometrical figure (Newton diagram) as the invariant sub-space of some linear differential operators. In the space $\mathbb{R}^1$ there is only one such “irreducible” convex Newton diagram - a finite interval, in other words, a space of all polynomials in $x$ of degree not higher than a certain positive integer. Thus, in $\mathbb{R}^2$ the rectangular Newton diagram is “reducible” stemming from the direct product of two intervals as one-dimensional Newton diagrams. In turn, the irreducible convex Newton diagrams in $\mathbb{R}^2$ are likely exhausted by different triangles, connected to the algebras $sl_3(\mathbb{R})$ and $gl_2(\mathbb{R}) \ltimes \mathbb{R}^{r+1}$, $r = 2, 3, 4, \ldots$ (see Figures 4.1, 4.3). Those algebras play a role of primary algebras. At least, we could not find any other convex Newton diagrams in $\mathbb{R}^2$ (see conjecture 1).

Before description of our present knowledge about convex Newton diagrams for the $\mathbb{R}^k$ case, let us recall the regular representation of the Lie algebra $sl_N(\mathbb{R})$ given on the flag manifold which acts on the smooth functions $\mathcal{N}(\mathbb{N} - 1)/2$ real variables $z_{i,q}$, $i = 1, 2, \ldots, (N - 1)$, $q = 1, 2, \ldots, (N - i)$. The explicit formulas for generators in the Chevalley basis are given by (see [21])

$$D(e_i) = -\frac{\partial}{\partial z_{i,i+1}} - \sum_{q=i+2}^{N} z_{i+1,q} \frac{\partial}{\partial z_{i,q}}$$

$$D(f_i) = \sum_{q=1}^{i} z_{q,i+1} z_{i,i+1} \frac{\partial}{\partial z_{q,i+1}} + \sum_{q=1}^{i-1} (z_{q,i+1} - z_{q,i} z_{i,i+1}) \frac{\partial}{\partial z_{q,i}}$$

$$- \sum_{q=i+2}^{N} z_{i,q} \frac{\partial}{\partial z_{i+1,q}} - n_i z_{i,i+1}$$

$$D(h_i) = -\sum_{q=i+1}^{N} z_{i,q} \frac{\partial}{\partial z_{i,q}} + \sum_{q=1}^{i-1} z_{q,i} \frac{\partial}{\partial z_{q,i}}$$

$$\sum_{q=i+2}^{N} z_{i+1,q} \frac{\partial}{\partial z_{i+1,q}} - \sum_{q=1}^{i} z_{q,i+1} \frac{\partial}{\partial z_{q,i+1}} + n_i$$

(126)
where we use a notation $e_i, f_i, h_i, i = 1, 2, \ldots, (N-1)$ for generators of positive and negative roots, and the Cartan generators, respectively. The algebra $sl_N(\mathbb{R})$ is generated by generators $e_i, f_i, h_i$. If $n_i$ are non-negative integers, the finite-dimensional irreducible representation of $sl_N(\mathbb{R})$ will occur in the form of inhomogeneous polynomials in variables $z_{i,i+q}$. The highest weight vector is characterized by the integer numbers $n_i, i = 1, 2, \ldots, (N-1)$. If some integers $n_i$ become zero, the representation (126) is degenerated and acts on a space of dimension lower than $N(N-1)/2$ (see, e.g., (88) as an example of a degenerate representation of $sl_3(\mathbb{R})$).

In connection with the space $\mathbb{R}^3$, at least two irreducible convex Newton diagrams appear: a tetrahedron, related to a degenerate finite-dimensional representation of $sl_4(\mathbb{R})$ (the highest weight vector is characterized by two zeroing and one non-zeroing components), and certain geometrical bodies, corresponding to the regular representation of $sl_3(\mathbb{R})$ on the flag manifold (see (126) at $N = 3$) and a representation of $sp_2(\mathbb{R})$, respectively. We do not know whether those exhaust all irreducible Newton diagrams or not.

For the general case $\mathbb{R}^k$, firstly, we wish to indicate, that there also exist Newton polyhedra related to the almost degenerate representation of $sl_{k+1}(\mathbb{R})$ (in formulas (126) all integers except the one characterizing the highest weight vector become zero). Also there exist some convex geometrical figures related to some finite-dimensional degenerate and, sometimes, regular representations of $sl_{k+1-i}(\mathbb{R}), i > 0$. It is easy to show that the algebra $so_{k+1}(\mathbb{R})$ acts on $\mathbb{R}^k$ in a degenerate representation

$$H_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}, \quad i > j , \quad i, j = 1, 2, \ldots, k$$

$$G_i = \sum_{j=1, i \neq j}^k x_i x_j \frac{\partial}{\partial x_j} + (1 + x_i^2) \frac{\partial}{\partial x_i} - nx_i , \quad i = 1, 2, \ldots, k$$

(127)

where $x_i, i = 1, 2, \ldots, k$ are real variables and $n$ is a number. If $n$ is a non-negative integer, the algebra (127) possesses the finite-dimensional invariant sub-space, which has the form of a polyhedron in $\mathbb{R}^k$ and coincides with a finite-dimensional invariant sub-space of the algebra $sl_{k+1}(\mathbb{R})$ in the almost degenerate representation, acting on $\mathbb{R}^k$. Making a gauge transformation, any symmetric bilinear combination in the generators (127) of the algebra $so_{k+1}(\mathbb{R})$ with real coefficients can be reduced to the form of a Laplace-Beltrami operator with some metric tensor plus a scalar function giving rise to exactly-solvable operators of the second type (for the proof see [17] and the discussion in the end of Section 4.1).

The above procedure can be extended to the case of the Lie super-algebras and also the quantum algebras. For the former case, our experience is very limited. The only statement one can make is that for linear operators acting on functions of one real and one Grassmann variable (or, equivalently, on two-component spinors in on real variable) there can appear finite-dimensional invariant sub-spaces other than finite-dimensional representations of the algebra $osp(2, 2)$ (see Section 3.1 and the

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15In order to decide what algebras act on $\mathbb{R}^k$, it is necessary to perform some analysis of the Young tableau. I am grateful to W. Rühl for an enlightening discussion of this question.

16The corresponding finite-dimensional representation is unitary and reducible (see, e.g., the discussion in the end of Section 4.1 and also in [17]).
discussion in [15]). Nevertheless, those sub-spaces are connected with the representations of polynomial elements of the universal enveloping algebra of the algebra $osp(2, 2)$ and, correspondingly, they can be constructed through finite-dimensional representations of the original super-algebra $osp(2, 2)$. Probably, there are no more irreducible convex Newton diagrams in this space. For the latter (quantum algebras), there exists an irreducible Newton diagram in $\mathbb{R}$ – an interval (see Section 2) – but we could not find any irreducible (non-trivial) Newton diagrams in $\mathbb{R}^k$ for $k > 1$; however, at least, in $\mathbb{R}^2$ they occur, if a quantum plane is introduced, $xy = qyx$, instead of an ordinary one, $xy = yx$, or a quantum hyperplane [30] for the case of more than two variables.

It is worth noting that one can construct quasi-exactly-solvable and exactly-solvable operators considering “mixed” algebras: a direct sum of the Lie algebras, Lie superalgebras and the quantum algebras, realized in the first-order differential and/or difference operators, possessing finite-dimensional invariant subspaces. In this case the Newton diagrams are reducible.

As for the cases considered in Sections 1-4, the problem of reducibility of general quasi-exactly-solvable and exactly-solvable operators to the Schroedinger-type operators is open.

Above we have described the linear operators which possess a finite-dimensional invariant sub-space with a polynomial basis. Certainly, making changes of variables and gauge transformations one can obtain the operators with an invariant sub-space with a non-polynomial basis (but emerging from a polynomial one, see e.g. (11)). Those ‘induced’ linear spaces we consider as the equivalent to the original, polynomial ones. A general problem can be posed: is it possible to describe all linear differential operators possessing a finite-dimensional invariant sub-space with a basis in a certain explicit form, which cannot be reduced to a basis of polynomials using a change of variables and a gauge transformation? Equivalently, this problem can be re-formulated as a problem of a description of finite-dimensional linear spaces of functions, where some linear differential operators can act. Since the original aim of the present investigation was mainly the construction of the quasi-exactly-solvable Schroedinger operators [7, 14, 17, 9, 16, 18], we did not touch on the general problem (except for a short remark on page 4). A solution of this problem plays a crucial role for a complete classification of the Schroedinger operators possessing a finite-dimensional invariant sub-space with an explicit basis in functions. Although it is rather surprising, it is very likely that this general and abstract problem has a constructive solution, at least for the one-dimensional Schroedinger operators. This work is in progress.

Concluding remarks.

(I). A crucial approach to the problem of classification of linear operators possessing a finite-dimensional invariant sub-space(s) with a basis in functions in an explicit form is connected to a theory of Riemann surfaces.

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17I am grateful to O.V. Ogievetsky for an interesting discussion of this subject and valuable comment.

18It is quite difficult to define precisely the meaning of the words “space with a basis in a certain explicit form.” One of the examples is the linear space $< 1, x, f_1(x), f_2(x), \ldots, f_n(x) >$, where $f_i, i = 1, 2, \ldots n$ are any functions on $R^1$. Other examples of such spaces are the spaces (1), (11), (29), (77), (89), (113).

19Recently, a classification of linear differential operators possessing a finite-dimensional invariant sub-space with a basis in monomials was completed.

20see footnote 18
Take a symmetric operator depending on a real parameter polynomially and possessing the infinite discrete set of eigenvalues. Searching of eigenvalues is equivalent to solving the transcendental secular equation. It is rather natural that the roots of this equation (eigenvalues) form an infinite-sheet Riemann surface as a parameter manifold. Generically, this surface is characterized by square-root branch points (see the excellent paper by Bender and Wu [25] and also [26, 27, 28]). Existence of \( N \)-dimensional invariant sub-space with an explicit basis implies, that the secular equation is reduced to two equations: an algebraic one and a transcendental one. It reflects the fact, that the original determinant factors into the product of two determinants. Equivalently, the infinite-sheet Riemann surface is split off into two disjoint Riemann surfaces: \( N \)-sheet one and an infinite-sheet one. If there exist \( k \) finite-dimensional invariant sub-spaces (for instance, which are embedded consequently one into the other, or, another case, any two have no intersection), the original, infinite-sheet Riemann surface is split off into \( k \) finite-sheet Riemann surfaces and an infinite-sheet one. Namely, such a situation occurs for a general quasi-exactly-solvable operator. As for exactly-solvable operators, the corresponding infinite-sheet Riemann surface is split off completely for separate sheet [24]. Following this idea, in [21] a complete classification was done of second-order differential operators in \( \mathbb{R} \) related with \( \mathfrak{sl}_2(\mathbb{R}) \)-hidden algebra. This insight allowed to separate the quasi-exactly-solvable and exactly-solvable operators from generic operators, possessing abstract finite-dimensional invariant sub-spaces. I hope that such an approach can lead to an abstract classification of linear operators.

(II). The above-discussed questions about the existence of convex Newton diagrams (equivalently, the finite-dimensional linear spaces with polynomial basis) as an invariant sub-space of some linear differential operator are related to a classification of finite-dimensional Lie algebras of first-order differential operators possessing an invariant sub-space. Originally, this problem was formulated by Sophus Lie, who solved this problem for the algebras on \( \mathbb{C}^1 \) and gave a complete classification of the algebras of vector fields on \( \mathbb{C}^2 \) (see [22] and references therein). A complete classification of Lie algebras of first-order differential operators on \( \mathbb{C}^2 \) has been presented just recently [14], while the same problem for the algebras on \( \mathbb{R}^2 \) has not been solved so far, although the vector fields on \( \mathbb{R}^2 \) have been described [22]. The same problem for algebras on \( \mathbb{C}^3 \) has also been discussed by S. Lie, who presented 35 algebras of vector fields, emphasizing that this list is not complete. As far as we know at present, similar questions about a classification of the Lie super-algebras, both vector fields and first-order differential operators in real and Grassmann variables, are never discussed in literature. The same situation takes place for the algebras of first-order finite-difference operators on the real line, or a quantum hyperplane [30].

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\footnote{Of course, if the eigenfunctions related with the infinite flag of invariant sub-spaces form a complete set of the eigenfunctions of an original operator. In contrary, an additional Riemann surface is left as well.}
discussions. Also I am very grateful to the Institute of Theoretical Physics, ETH-Honggerberg, Zurich, where this work was mainly done, and the Research Institute for Theoretical Physics, University of Helsinki and the Institut des Hautes Études Scientifiques, Bures-sur-Yvette, where this work was completed, for their kind hospitality extended to me. I am deeply appreciate to N. Kamran and P. Olver for their kind invitation to make a contribution to present volume.

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In connection to the finite-dimensional representations of the algebras of first-order differential (difference) operators some operator identities occur \[24\].

1. The following operator identity holds:

\[
(J^+_n)^{n+1} \equiv (x^2 \partial_x - nx)^{n+1} = x^{2n+2} \partial_x^{n+1}, \quad \partial_x \equiv \frac{d}{dx}, \quad n = 0, 1, 2, \ldots
\]

(A.1)

**Proof.** The proof is straightforward:

(i) the operator \((J^+_n)^{n+1}\) annihilates the space of all polynomials of power not higher than \(n\), 

\[
\mathcal{P}_n(x) = \text{Span}\{x^i : 0 \leq i \leq n\};
\]

(ii) in general, an \((n+1)\)-th order linear differential operator annihilating \(\mathcal{P}_n(x)\) must have the form \(B(x)\partial_x^{n+1}\), where \(B(x)\) is an arbitrary function

(iii) since \((J^+_n)^{n+1}\) is a graded operator, \(\deg(J^+_n) = +1\), \(\deg((J^+_n)^{n+1}) = n + 1\), hence \(B(x) = bx^{2n+2}\), while clearly the constant \(b = 1\).

It is worth noting that taking the power in (A.1) different from \((n+1)\), the l.h.s. in (A.1), generally, will contain all derivative terms from zero up to \((n+1)\)-th order with non-vanishing coefficients.

The identity (A.1) has a Lie-algebraic interpretation. The operator \((J^+_n)^{n+1}\) is the positive-root generator of the algebra \(sl_2\) of first-order differential operators (the other \(sl_2\)-generators are \(J^0_n = x\partial_x - n/2\), \(J^-_n = \partial_x\), see (2)). Correspondingly, the space \(\mathcal{P}_n(x)\) is nothing but the \((n+1)\)-dimensional irreducible representation of \(sl_2\). The identity (A.1) is a consequence of the fact that \((J^+_n)^{n+1} = 0\) in matrix representation \[^{22}\].

Another Lie-algebraic interpretation of (A.1) is connected with occurrence of some relations between elements of the universal enveloping algebra of three-dimensional Heisenberg algebra, \(H_1\): \(\{P, Q, 1\}\). Let \([P, Q] = 1\), then

\[
(Q^2 P - nQ)^{n+1} = Q^{2n+2} P^{n+1}, \quad n = 0, 1, 2, \ldots
\]

(A.2)

and certainly this identity is more general than (A.1). Taking different representations for \(P\) and \(Q\) in terms of differential operators (others than the standard one, \(P = \partial_x\) and \(Q = x\)), one can get various families of operator identities, other than (A.1). Also the formula (A.2) has a meaning of a formula of an ordering: \(Q\)-operators are placed on the left, \(P\)-operators on the right.

One can easily check, that once two operators obey \([P, Q] = 1\), then

\[
J^+ = Q^2 P - nQ,
\]

\[
J^0 = QP - n/2,
\]

\[
J^- = P,
\]

(A.3)

obey \(sl_2\)-algebra commutation relations. The representation (2) is a particular case of this representation. In fact, (A.3) is one of possible embeddings of the algebra \(sl_2\) into the universal enveloping algebra of three-dimensional Heisenberg algebra.

[^{22}]: So \(J^+_n\) maps \(x^k\) to a multiple of \(x^{k+1}\).

[^{23}]: It is a particular case of more general statement: in the algebra \(gl_k\) a positive (negative) root generator \(J_{\pm, \alpha}\) in a finite-dimensional representation of the dimension \(d\), taken in power of \(\alpha\) of the dimension \(d\) is equal to zero, \((J_{\pm, \alpha})^d = 0\).
There exist other algebras of differential or finite-difference operators (in more than one variable), which admit a finite-dimensional representation. This leads to more general and remarkable operator identities. In fact, (A.1) is one representative of an infinite family of identities for differential and finite-difference operators. Also there occur generalizations of the relation (A.2) for a certain polynomial elements of an infinite family of identities for differential and finite-difference operators. Also there exist other algebras of differential or finite-difference operators (in more

2. The Lie-algebraic interpretation presented above allows us to generalize (A.1) for the case of differential operators of several variables, taking appropriate powers of the highest-positive-root generators of (super) Lie algebras of first-order differential operators, possessing a finite-dimensional invariant sub-space. First we consider the case of $sl_3$. There exists a representation of $sl_3(\mathbb{R})$ as differential operators on $\mathbb{R}^2$ (see Section 4.1, eq.(88)). One of the generators is

$$J_2(n) = x^2 \partial_x + xy \partial_y - nx .$$

The space (89), $P_n(x,y) = \text{Span}\{x^i y^j : 0 \leq i + j \leq n\}$ is a finite-dimensional representation for $sl_3$, and due to the fact that $(J_2^1(n))^n + 1 = 0$ on the space $P_n(x,y)$, we have

$$(J_2^1(n))^{n+1} = (x^2 \partial_x + xy \partial_y - nx)^{n+1} = 
\sum_{k=0}^{n+1} \binom{n+1}{k} x^{2n+2-k} y^k \partial_x^{n+1-k} \partial_y^k . \quad (A.4)$$

This identity is valid for $y \in \mathbb{R}$ (as described above), but also if $y$ is a Grassmann variable, i.e., $y^2 = 0$\footnote{In this case just two terms in the l.h.s. of (A.4) survive.}. In the last case, $J_2^1(n)$ is a generator $T^+$ of the algebra $osp(2,2)$ (see (31)).

More generally (using algebra $sl_{k+1}$ instead of $sl_3$), the following operator identity holds:

$$(J_k^{-1}(n))^{n+1} = (x_1(\sum_{m=1}^{k} x_m \partial_{x_m} - n))^{n+1} = 
\sum_{j_1 + j_2 + \ldots + j_k = n+1} C_{j_1,j_2,\ldots,j_k}^{n+1} x_1^{j_1} x_2^{j_2} \ldots x_k^{j_k} \partial_{x_1}^{j_1} \partial_{x_2}^{j_2} \ldots \partial_{x_k}^{j_k} , \quad (A.5)$$

where $C_{j_1,j_2,\ldots,j_k}^{n+1}$ are the generalized binomial (multinomial) coefficients. If $x \in \mathbb{R}^k$, then $J_k^{-1}(n)$ is a generator of the algebra $sl_{k+1}(\mathbb{R})$, generated by $D(f_i)$ (see (126)). While some of the variables $x$’s are Grassmann ones, the operator $J_k^{-1}(n)$ is a generator of a certain super Lie algebra of first-order differential operators. The operator in the l.h.s. of (A.5) annihilates the linear space of polynomials $P_n(x_1, x_2, \ldots x_k) = \text{Span}\{x_1^{j_1} x_2^{j_2} \ldots x_k^{j_k} : 0 \leq j_1 + j_2 + \ldots + j_k \leq n\}$, which is represented by tetrahedron Newton diagram in $\mathbb{R}^k$.

Considering $(2k + 1)$-dimensional Heisenberg algebra:

$$[P_i, Q_j] = \delta_{ij}, \quad [P_i, P_j] = 0, \quad [Q_i, Q_j] = 0, \quad i, j = 1, 2, \ldots k$$
one can rewrite (A.5) in more general form

\[(Q_1(\sum_{m=1}^{k} Q_m P_m - n))^{n+1} = \]

\[Q_1^{n+1} \sum_{j_1+j_2+\ldots+j_k=n+1} C_{j_1,j_2,\ldots,j_k}^{n+1} Q_1^{j_1} Q_2^{j_2} \ldots Q_k^{j_k} P_1^{j_1} P_2^{j_2} \ldots P_k^{j_k}, \quad (A.6)\]

(cf.(A.2)). It is clear, the formulas for \(sl_{k+1}\)-generators at \(k > 1\) (see (126)) in terms of \(P_i, Q_i\), analogous to (A.3), can be easily derived.

3. For the case of the algebra \(gl_2(\mathbb{R}) \ltimes \mathbb{R}^{r+1}\), \(r = 1, 2, \ldots\) (see (112)) the situation becomes rather sophisticated

\[(J_4^{n})^{n+1} \equiv (x^2 \partial_x + rxy \partial_y - nx)^{n+1} = \sum_{k=0}^{n+1} r^k \binom{n+1}{k} x^{2n+2-k} y^{n+1-k} \partial_x^k \partial_y^{n+1-k} = Q_{r,n}, \quad (A.7)\]

where \(Q_{1,n} = 0\) and also

\[Q_{r,n} = \begin{cases} 0 & \text{at } n=0 \\ r(r-1)x^2y \partial_y & \text{at } n=1 \\ r(r-1)x^3y[3ry \partial_y^2 + 3x \partial_x y + (r-2) \partial_y] & \text{at } n=2 \end{cases} \]

(cf. (A.4)). For arbitrary \(n\) and \(r\)

\[Q_{r,n} = \frac{r(r-1)n(n+1)}{2} \left[ x^{2n} y \partial_x^{n-1} \partial_y + r(n-1)x^{2n-1} y^2 \partial_x^{n-2} \partial_y + \frac{r^2}{4}(n-1)(n-2)x^{2n-2} y^3 \partial_x^{n-3} \partial_y^3 + \ldots \right] + [\text{lower order derivatives}]. \]

The r.h.s. of (A.7) still has quite regular structure, which can be analyzed using the following trick: variables \(y\)'s in the l.h.s. of (A.4), (A.7) can be related to each other through the very simple connection

\[y_{(A.7)} = y_{(A.4)}^r, \]

and then \(J_4^n\) (see (A.4)) coincides with \(J_4^n\).

4. The above-described family of operator identities can be generalized for the case of finite-difference operators with the Jackson symbol, \(D_x\) (see, e.g., [12]), obeyed the following Leibnitz rule

\[D_x f(x) = \frac{f(x) - f(q^2 x)}{(1 - q^2)x} + f(q^2 x) D_x, \]

instead of the ordinary, continuous derivative\(^{25}\). Here, \(q\) is an arbitrary complex number. The following operator identity holds:

\[(\tilde{J}_n^+)^{n+1} = (x^2 D_x - \{n\})^{n+1} = q^{2n(n+1)} x^{2n+2} D_x^{n+1}, n = 0, 1, 2, \ldots \quad (A.8)\]

(cf. (A.1)), where \(\{n\} = \frac{1-q^{2n}}{1-q}\) is a so-called \(q\)-number. The operator in the l.h.s. annihilates the space \((1), P_n(x)\). The proof is similar to the proof of identity (A.1).

\(^{25}\)The definition of the Jackson symbol used here is slightly different than that presented in Section 2.
From the algebraic point of view the operator \( \hat{J}_{n}^{+} \) is the generator of a \( q \)-deformed algebra \( sl_{2}(\mathbb{R})_{q} \) of first-order finite-difference operators on the line: 
\[
\hat{J}_{n}^{0} = xD - \hat{n}, \quad \hat{J}_{n}^{-} = D, \quad \text{where} \quad \hat{n} \equiv \frac{[n_{1}|n_{2}+1]}{q^{2n_{2}+2}}. \]
(see \[\text{10}\] and also \[\text{3}\]), obeying the commutation relations (22).

One can show, that once two operators obey \( \hat{P}\hat{Q} - q^{2}\hat{Q}\hat{P} = 1 \), then
\[
(\hat{Q}^{2}\hat{P} - \{n\}\hat{Q})^{n+1} = q^{2n(n+1)}\hat{Q}^{2n+2}\hat{P}^{n+1}, \quad n = 0, 1, 2, \ldots \quad (A.9)
\]
holds and
\[
\begin{align*}
\hat{J}^{+} & = \hat{Q}^{2}\hat{P} - \{n\}\hat{Q} , \\
\hat{J}^{0} & = \hat{Q}\hat{P} - \hat{n} , \\
\hat{J}^{-} & = \hat{P} ,
\end{align*}
\]
(A.10) 
obey \( sl_{2}(\mathbb{R})_{q} \)-algebra commutation relations (22). It is evident, that this representation is more general than the representation (21).

An attempt to generalize (A.4) replacing continuous derivatives by Jackson symbols immediately leads to requirement to introduce the quantum plane and \( q \)-differential calculus \[\text{[30]}\]
\[
\begin{align*}
xy & = qyx , \\
D_{x}x & = 1 + q^{2}xD_{x} + (q^{2} - 1)yD_{y} , \quad D_{x}y = qyD_{x} , \\
D_{y}x & = qxD_{y} , \quad D_{y}y = 1 + q^{2}yD_{y} , \\
D_{x}D_{y} & = q^{-1}D_{y}D_{x} .
\end{align*}
\]
(A.11)
The formulae analogous to (A.4) have the form
\[
(\hat{J}_{2}(n))^{n+1} = (x^{2}D_{x} + xyD_{y} - \{n\}x)^{n+1} = \\
\sum_{k=0}^{n+1} q^{2n^{2}-n(k-2)+k(k-1)} \binom{n+1}{k} q^{2n+2-k}y^{k}D_{x}^{n+1-k}D_{y}^{k} ,
\]
(A.12)
where
\[
\binom{n}{k}_{q} \equiv \frac{\{n\}!}{\{k\}!\{n-k\}!}, \quad \{n\}! = \{1\}\{2\} \ldots \{n\}
\]
are \( q \)-binomial coefficient and \( q \)-factorial, respectively. As in all previous cases, if \( y \in \mathbb{R} \), the operator \( \hat{J}_{2}(n) \) is one of generators of the \( q \)-deformed algebra \( sl_{3}(\mathbb{R})_{q} \) of finite-difference operators, acting on the quantum plane and having the linear space \( \mathcal{P}_{n}(x, y) = \text{Span}\{x^{i}y^{j} : 0 \leq i + j \leq n\} \) as a finite-dimensional representation; the l.h.s. of (A.12) annihilates \( \mathcal{P}_{n}(x, y) \). If \( y \) is a Grassmann variable, \( \hat{J}_{2}(n) \) is a generator of the \( q \)-deformed superalgebra \( osp(2, 2)_{q} \) possessing finite-dimensional representation.

Introducing two couples of the operators \( \hat{P}_{1,2}, \hat{Q}_{1,2} \) obeying the following relations:
\[
\begin{align*}
Q_{1}Q_{2} & = qQ_{2}Q_{1} , \\
P_{1}Q_{1} & = 1 + q^{2}Q_{1}P_{1} + (q^{2} - 1)Q_{2}P_{2} , \quad P_{1}Q_{2} = qQ_{2}P_{1} , \\
P_{2}Q_{1} & = qQ_{1}P_{2} , \quad P_{2}Q_{2} = 1 + q^{2}Q_{2}P_{2} , \\
P_{1}P_{2} & = q^{-1}P_{2}P_{1} .
\end{align*}
\]
(A.13)
It is easy to derive a family of abstract identities similar to (A.6) at \( k = 2 \)
\[
(Q_{2}^{2}P_{1} + Q_{1}Q_{2}P_{2} - \{n\}Q_{1})^{n+1} =
\]
\[
\sum_{k=0}^{n+1} q^{2n^2-n(k-2)+k(k-1)} \binom{n+1}{k} q^{2n+2-k} Q_1^k Q_2^{n+1-k} P_1^k P_2^k,
\]  
(A.14)

while (A.12) is a particular case corresponding a special choice of a representation (A.13) in the form of (A.11).

Introducing a quantum hyperplane [30], one can generalize the whole family of operator identities (A.5) replacing first of all continuous derivatives by finite-difference operators and then introducing the abstract operators like it was done above for the case of (A.11)-(A.12).

The Lie-algebraic interpretation of above operator identities allows to make a general conclusion that an existence of finite-dimensional representations of the Lie algebra $\mathfrak{gl}_k$ of differential operators leads to an appearance of some specific operator identities analogous to those families described above.

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