A full parametrization of the $9 \times 9$ active-sterile flavor mixing matrix in the inverse or linear seesaw scenario of massive neutrinos

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Abstract

The inverse and linear seesaw scenarios are two typical extensions of the canonical seesaw mechanism, which contain much more sterile degrees of freedom but can naturally explain the smallness of three active neutrino masses at a sufficiently low energy scale (e.g., the TeV scale). To fully describe the mixing among three active neutrinos, three sterile neutrinos and three extra gauge-singlet neutral fermions in either of these two seesaw paradigms, we present the first full parametrization of the $9 \times 9$ flavor mixing matrix in terms of 36 rotation angles and 36 CP-violating phases. The exact inverse and linear seesaw formulas are derived, respectively; and possible deviations of the $3 \times 3$ active neutrino mixing matrix from its unitary limit are discussed by calculating the effective Jarlskog invariants and unitarity nonagons.

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1 Introduction

The absence of the right-handed neutrino fields in the standard model (SM) makes it impossible to accommodate a gauge-invariant neutrino mass term like the Yukawa-interaction term for the charged leptons \[1\]. Therefore, the simplest way to go beyond the SM and generate finite masses for three active neutrinos (i.e., \(\nu_e, \nu_\mu, \nu_\tau\)) is to introduce three right-handed neutrino fields and allow for the gauge-invariant neutrino Yukawa interactions. But such a Dirac neutrino mass term has no way to explain why the active neutrinos are so light as compared with the respective charged leptons, which participate in the standard weak interactions in the same way. Moreover, the right-handed neutrino fields and their charge-conjugate counterparts (which are left-handed) can form a gauge-invariant but lepton-number-violating Majorana mass term, which should not be discarded in general. Combining this Majorana neutrino mass term with the aforementioned Dirac mass term leads us to the canonical (type-I) seesaw mechanism \[2–6\], in which all six neutrino mass eigenstates have the Majorana nature (i.e., their charge conjugates are equal to themselves; “fulcrum” of such a seesaw is just around the electroweak scale \(\Lambda_{\text{EW}}\)), and the smallness of three active neutrino masses \(m_i\) can naturally be attributed to the largeness of three sterile neutrino masses \(M_i\) (for \(i = 1, 2, 3\)). Note that \(M_i\) act as the cut-off scales in the SM effective field theory with a unique dimension-five Weinberg operator for generating tiny masses of the active neutrinos \[8\], and the “fulcrum” of such a seesaw is just around the electroweak scale \(\Lambda_{\text{EW}} \sim 10^2\) GeV. Unfortunately, the naturalness prerequisite \(M_i \gg \Lambda_{\text{EW}}\) implies that this conventional seesaw picture has essentially lost its testability in any feasible high-energy experiments \[9\].

In this regard a possible way out is to lower the seesaw scale (i.e., the values of \(M_i\)) by introducing more extra degrees of freedom, and the typical examples of this kind include the inverse (or double) seesaw scenario \[10, 11\] and the linear seesaw scenario \[11–14\]. Besides the left-handed neutrino fields \(\nu_{\alpha L}\) and the right-handed neutrino fields \(N_{\alpha R}\) (for \(\alpha = e, \mu, \tau\)), the inverse or linear seesaw mechanism requires the adding of three neutral SM gauge-singlet fermions \(S_{\alpha R}\) (for \(\alpha = e, \mu, \tau\)) and one scalar singlet \(\Phi\). With such new degrees of freedom, a generic gauge-invariant neutrino mass term can be written as follows:

\[
-L_{\text{mass}} = \ell_L Y_e H \bar{N}_R + \frac{1}{2} (N_R)^c M_R N_R + \nu_L Y_\mu \Phi S_R + (N_R)^c Y_S \Phi S_R + \frac{1}{2} (S_R)^c \mu S_R + \text{h.c.},
\]

where \(\nu_L = (\nu_{eL}, \nu_{\mu L}, \nu_{\tau L})^T\), \(N_R = (N_{eR}, N_{\mu R}, N_{\tau R})^T\) and \(S_R = (\nu_{eR}, \nu_{\mu R}, \nu_{\tau R})^T\) stand respectively for the vector columns of left-handed neutrino fields, right-handed neutrino fields and gauge-singlet fermion fields, \(\ell_L\) denotes the lepton doublet of the SM, \(H\) is defined as \(H \equiv i \sigma_2 H^*\) with \(H\) being the Higgs doublet of the SM, and the mass matrices \(M_R\) and \(\mu\) are both symmetric. After spontaneous symmetry breaking, Eq. (1) turns out to be

\[
-L'_{\text{mass}} = \frac{1}{2} [\nu_L (N_R)^c (S_R)^c] \begin{pmatrix}
0 & M_D & \varepsilon \\
M_D^T & M_R & M_S \\
\varepsilon^T & M_S^T & \mu
\end{pmatrix} \begin{pmatrix}
(\nu_L)^c \\
N_R \\
S_R
\end{pmatrix} + \text{h.c.}, \quad (2)
\]

where \(M_D = Y_e \langle H \rangle\), \(\varepsilon = Y_\nu' \langle \Phi \rangle\) and \(M_S = Y_S \langle \Phi \rangle\). Switching off the extra neutral gauge-singlet fermion fields, we are immediately left with the neutrino mass term of the canonical seesaw mech-

\[1\] A natural extension of the inverse seesaw scenario to the multiple seesaw scenario has been proposed in Ref. [15].
anism from Eqs. (1) and (2). To partly reduce the number of new free parameters associated with the overall $9 \times 9$ neutrino mass matrix in Eq. (2), one has considered the following two simplified but phenomenologically interesting cases.

- The *inverse* seesaw scenario with $M_R = 0$ and $\varepsilon = 0$ [10,11], where the mass scale of $\mu$ is naturally small because it is the only lepton-number-violating term, and the mass scale of $M_S$ can be considerably larger than that of $M_D$ if $\langle \Phi \rangle \gg \langle H \rangle$ and $Y_S \sim Y_\nu$ are taken. The effective Majorana neutrino mass matrix for three active neutrinos is therefore given by

$$M_\nu \simeq M_D(M_S^T)^{-1}\mu(M_S)^{-1}M_D^T$$  \hspace{1cm} (3)

in the leading-order approximation. So the tiny mass eigenvalues of $M_\nu$ are mainly attributed to the smallness of $\mu$, and they are further suppressed by the largeness of $M_S$ as compared with $M_D$ even if $M_S \sim \mathcal{O}(1)$ TeV is assumed.

- The *linear* seesaw scenario with $M_R = 0$ and $\mu = 0$ [11–14], where $\varepsilon$ is the only lepton-number-violating term, and thus its mass scale is naturally small. In this case the effective Majorana neutrino mass matrix for three active neutrinos is found to be

$$M_\nu \simeq -\varepsilon M_S^{-1}M_D^T - (\varepsilon M_S^{-1}M_D^T)^T$$  \hspace{1cm} (4)

in the leading-order approximation. As a result, the smallness of $M_\nu$ is mainly ascribed to that of $\varepsilon$ and further suppressed by the ratio of the mass scales of $M_D$ to $M_S$.

In either case the seesaw scale can be successfully lowered to the TeV scale which is experimentally accessible at the Large Hadron Collider (LHC).

Although a lot of work has been done to explore various phenomenological consequences of the inverse or linear seesaw scenario (see, e.g., Refs. [16–30]), a complete description of the $9 \times 9$ active-sterile flavor mixing matrix for such a seesaw picture has been lacking. Following the full Euler-like parametrization of active-sterile flavor mixing in the type-I or type-(I+II) seesaw mechanism with three sterile neutrinos done previously by one of us [31–33], here we are going to present the *first* full parametrization of the $9 \times 9$ flavor mixing matrix in terms of 36 Euler rotation angles and 36 CP-violating phases in the inverse or linear seesaw scenario. The exact inverse or linear seesaw formula will also be derived; and possible deviations of the $3 \times 3$ active neutrino mixing matrix (i.e., the so-called Pontecorvo-Maki-Nakagawa-Sakata lepton flavor mixing matrix [34,36]) from its unitary limit will be discussed by calculating the effective Jarlskog invariants and unitarity nonagons. This study is expected to be useful for generally describing possible interplays between three active neutrino species and up to six species of extra (light or heavy) degrees of freedom no matter whether a specific seesaw scenario is taken into account or not.

The remaining parts of this paper are organized as follows. In section 2 we parameterize the $9 \times 9$ flavor mixing matrix with 36 Euler rotation angles and 36 CP-violating phases in such a way that the primary unitary $3 \times 3$ flavor mixing submatrices of three active neutrinos, three sterile neutrinos and three extra neutral fermions are linked and modified by the intermediate flavor mixing matrices. Section 3 is devoted to deriving the exact inverse or linear seesaw formula.
and reproducing its approximate expression shown in Eq. (3) or Eq. (4). In section 4 we examine possible deviations of the $3 \times 3$ PMNS matrix from its unitary limit by calculating the effective Jarlskog invariants and discussing the unitarity nonagons. A brief summary of our main results, together with some further discussions, is finally made in section 5.

## 2 Parametrization of the $9 \times 9$ flavor mixing matrix

To fully describe the parameter space of flavor mixing between three active neutrinos and their six sterile counterparts, one may write out the $9 \times 9$ active-sterile flavor mixing matrix $U$ in terms of 36 two-dimensional unitary matrices $O_{ij}$ ($1 \leq i < j \leq 9$) as follows:

$$U = \left( \begin{array}{ccc} O_{89} & O_{79} & O_{69} \\ O_{59} & O_{49} & O_{39} \\ O_{29} & O_{19} & 0 \end{array} \right) \left( \begin{array}{ccc} O_{78} & O_{68} & O_{58} \\ O_{48} & O_{38} & O_{28} \\ O_{18} & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} O_{67} & O_{57} & O_{47} \\ O_{37} & O_{27} & O_{17} \end{array} \right),$$

(5)

where only the combination $(O_{23}O_{13}O_{12})$ is purely associated with the flavor mixing sector of three active neutrinos. Following the example of Refs. [31–33], let us adjust the ordering of $O_{ij}$ in $U$ in such a way that

$$U = \left( \begin{array}{ccc} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & S_0 \end{array} \right) L_3 \left( \begin{array}{ccc} I & 0 & 0 \\ 0 & U'_0 & 0 \\ 0 & 0 & I \end{array} \right) L_1 L_2 \left( \begin{array}{ccc} U_0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{array} \right),$$

(6)

where $I$ denotes the $3 \times 3$ identity matrix; the $3 \times 3$ unitary matrices

$$U_0 = O_{23}O_{13}O_{12}, \quad U'_0 = O_{56}O_{46}O_{45}, \quad S_0 = O_{89}O_{79}O_{78},$$

(7)

describe the primary flavor mixing sectors of three active neutrinos, three sterile neutrinos and three extra neutral fermions, respectively; and the $9 \times 9$ matrices

$$L_1 = O_{36}O_{26}O_{16}O_{35}O_{25}O_{15}O_{34}O_{24}O_{14} = \left( \begin{array}{ccc} A_1 & R_1 & 0 \\ S_1 & B_1 & 0 \\ 0 & 0 & I \end{array} \right),$$

$$L_2 = O_{39}O_{29}O_{19}O_{38}O_{28}O_{18}O_{37}O_{27}O_{17} = \left( \begin{array}{ccc} A_2 & 0 & R_2 \\ 0 & I & 0 \\ S_2 & 0 & B_2 \end{array} \right),$$

$$L_3 = O_{69}O_{59}O_{49}O_{68}O_{58}O_{48}O_{67}O_{57}O_{47} = \left( \begin{array}{ccc} I & 0 & 0 \\ 0 & A_3 & R_3 \\ 0 & S_3 & B_3 \end{array} \right),$$

(8)

describe the interplays between any two of the three sectors. There are two obvious advantages associated with the parametrization of $U$ in Eq. (6): on the one hand, the flavor mixing angles of $L_1$ and $L_2$ are naturally small because they measure the strength of interplay between the active sector and two sterile sectors; on the other hand, the three flavor sectors will automatically become decoupled if the off-diagonal elements of $L_1$, $L_2$ and $L_3$ are all switched off.
To be more specific, the three primary $3 \times 3$ unitary flavor mixing matrices are given by

$$
U_0 = \begin{pmatrix}
    c_{12}c_{13} & \hat{s}_{12}c_{13} & \hat{s}_{13}
  \\
    -\hat{s}_{12}c_{23} - c_{12}\hat{s}_{13}\hat{s}_{23} & c_{12}c_{23} - \hat{s}_{12}\hat{s}_{13}\hat{s}_{23} & c_{13}\hat{s}_{23}
  \\
    \hat{s}_{12}\hat{s}_{23} - c_{12}\hat{s}_{13}c_{23} & -c_{12}\hat{s}_{23} - \hat{s}_{12}\hat{s}_{13}c_{23} & c_{13}c_{23}
\end{pmatrix},
$$

$$
U' = \begin{pmatrix}
    c_{45}c_{46} & \hat{s}_{45}c_{46} & \hat{s}_{46}
  \\
    -\hat{s}_{45}c_{56} - c_{45}\hat{s}_{46}\hat{s}_{56} & c_{45}c_{56} - \hat{s}_{45}\hat{s}_{46}\hat{s}_{56} & c_{46}\hat{s}_{56}
  \\
    \hat{s}_{45}\hat{s}_{56} - c_{45}\hat{s}_{46}c_{56} & -c_{45}\hat{s}_{56} - \hat{s}_{45}\hat{s}_{46}c_{56} & c_{46}c_{56}
\end{pmatrix},
$$

$$
S_0 = \begin{pmatrix}
    c_{78}c_{79} & \hat{s}_{78}c_{79} & \hat{s}_{79}
  \\
    -\hat{s}_{78}\hat{s}_{79} - c_{78}\hat{s}_{79}\hat{s}_{89} & c_{78}\hat{s}_{89} - \hat{s}_{78}\hat{s}_{79}\hat{s}_{89} & c_{79}\hat{s}_{89}
  \\
    \hat{s}_{78}\hat{s}_{89} - c_{78}\hat{s}_{79}c_{89} & -c_{78}\hat{s}_{89} - \hat{s}_{78}\hat{s}_{79}c_{89} & c_{79}\hat{s}_{89}
\end{pmatrix},
$$

where $c_{ij} \equiv \cos \theta_{ij}$ and $\hat{s}_{ij} \equiv e^{i\delta_{ij}} \sin \theta_{ij}$ with $\theta_{ij}$ and $\delta_{ij}$ (for $1 \leq i < j \leq 9$) standing respectively for the rotation angles and phase angles. Moreover, the four $3 \times 3$ submatrices of $L_1$ read as

$$
A_1 = \begin{pmatrix}
    c_{14}c_{15}c_{16} & 0 & 0 \\
    -c_{14}c_{15}\hat{s}_{16}\hat{s}_{26} - c_{14}\hat{s}_{15}\hat{s}_{25}\hat{s}_{26} & c_{24}\hat{s}_{25}\hat{s}_{26} & 0 \\
    -\hat{s}_{14}c_{24}\hat{s}_{25}\hat{s}_{26} & -c_{24}\hat{s}_{25}\hat{s}_{26}\hat{s}_{36} - c_{24}\hat{s}_{25}\hat{s}_{35}\hat{s}_{36} & c_{34}\hat{s}_{35}\hat{s}_{36}
\end{pmatrix},
$$

$$
B_1 = \begin{pmatrix}
    -c_{14}\hat{s}_{15}\hat{s}_{25}\hat{s}_{36} - c_{14}\hat{s}_{15}\hat{s}_{26}\hat{s}_{36} & c_{15}\hat{s}_{25}\hat{s}_{26} & 0 \\
    -\hat{s}_{14}\hat{s}_{15}\hat{s}_{26}\hat{s}_{35} & -c_{15}\hat{s}_{25}\hat{s}_{35}\hat{s}_{36} - c_{15}\hat{s}_{25}\hat{s}_{26}\hat{s}_{36} & c_{16}\hat{s}_{26}\hat{s}_{36}
\end{pmatrix},
$$

and

$$
R_1 = \begin{pmatrix}
    \hat{s}_{14}c_{15}c_{16} & \hat{s}_{16} & \hat{s}_{15}c_{16} \\
    -\hat{s}_{14}\hat{s}_{15}\hat{s}_{16}\hat{s}_{26} - \hat{s}_{14}\hat{s}_{15}\hat{s}_{25}\hat{s}_{26} & -\hat{s}_{15}\hat{s}_{16}\hat{s}_{26} + c_{15}\hat{s}_{25}\hat{s}_{26} & c_{16}\hat{s}_{26}
  \\
    +c_{14}\hat{s}_{24}\hat{s}_{25}\hat{s}_{26} & -\hat{s}_{15}\hat{s}_{16}\hat{s}_{26}\hat{s}_{36} - c_{15}\hat{s}_{25}\hat{s}_{26}\hat{s}_{36} & c_{16}\hat{s}_{26}\hat{s}_{36}
\end{pmatrix},
$$

$$
S_1 = \begin{pmatrix}
    \hat{s}_{14}\hat{s}_{24}\hat{s}_{34}\hat{s}_{35} & \hat{s}_{24}\hat{s}_{34}\hat{s}_{35} & \hat{s}_{24}\hat{s}_{34}\hat{s}_{35} \\
    -c_{14}\hat{s}_{15}\hat{s}_{24}\hat{s}_{35} & -c_{15}\hat{s}_{24}\hat{s}_{35} & -c_{16}\hat{s}_{35}
  \\
    +c_{14}\hat{s}_{15}\hat{s}_{24}\hat{s}_{36} & +c_{15}\hat{s}_{24}\hat{s}_{36} & -c_{16}\hat{s}_{36}
\end{pmatrix}.
$$
The exact results in Eqs. (10) and (11) have already been obtained in Refs. [31–33] for the canonical (type-I) seesaw mechanism. All the nine flavor mixing angles associated with $A_1$, $B_1$, $R_1$ and $S_1$ are very small and can at most reach the level of $O(0.1)$ [37–40], as they describe the strength of active-sterile (or light-heavy) neutrino mixing and thus are well constrained by current neutrino oscillation data and precision measurements of various electroweak processes. In fact, the smallness of $\theta_{i4}$, $\theta_{i5}$ and $\theta_{i6}$ (for $i = 1, 2, 3$) implies that $A_1$ and $B_1$ are essentially equal to the identity matrix $I$, and the magnitude of each element of $R_1$ and $S_1$ is at most of $O(0.1)$. Similarly, the four $3 \times 3$ submatrices of $L_2$ can be expressed as

$$A_2 = \begin{pmatrix}
 c_{17}c_{18}c_{19} & 0 & 0 \\
 -c_{17}\hat{s}_{19}\hat{s}_{29} - c_{17}\hat{s}_{18}\hat{s}_{28}c_{29} & c_{27}c_{28}c_{29} & 0 \\
 -\hat{s}_{17}\hat{s}_{27}c_{29} & 0 & \\
 -c_{17}\hat{s}_{18}\hat{s}_{19}c_{29} + c_{17}\hat{s}_{18}\hat{s}_{28}\hat{s}_{29}c_{39} & -c_{27}\hat{s}_{28}\hat{s}_{29}c_{39} & -c_{27}\hat{s}_{28}\hat{s}_{38}c_{39} & c_{37}c_{38}c_{39} \\
 -\hat{s}_{17}\hat{s}_{18}\hat{s}_{28}\hat{s}_{38}c_{39} - \hat{s}_{17}\hat{s}_{18}\hat{s}_{28}\hat{s}_{38}c_{39} & -\hat{s}_{27}\hat{s}_{37}c_{38}c_{39} & \\
 +\hat{s}_{17}\hat{s}_{18}\hat{s}_{28}\hat{s}_{38}c_{39} & 0 & 0 \\
 \end{pmatrix}; \quad (12)$$

and

$$B_2 = \begin{pmatrix}
 c_{17}c_{27}c_{37} & 0 & 0 \\
 -c_{17}\hat{s}_{27}\hat{s}_{38}c_{38} - c_{17}\hat{s}_{27}\hat{s}_{28}c_{38} & c_{18}c_{28}c_{38} & 0 \\
 -\hat{s}_{17}\hat{s}_{27}c_{28}c_{38} & 0 & \\
 -c_{17}\hat{s}_{27}\hat{s}_{37}\hat{s}_{38} + c_{17}\hat{s}_{27}\hat{s}_{28}\hat{s}_{29}\hat{s}_{39} & -c_{18}\hat{s}_{28}\hat{s}_{38}\hat{s}_{29}\hat{s}_{39} - c_{18}\hat{s}_{28}\hat{s}_{29}\hat{s}_{39} & c_{19}c_{29}c_{39} \\
 -c_{17}\hat{s}_{27}\hat{s}_{28}\hat{s}_{29}c_{39} + \hat{s}_{17}\hat{s}_{18}\hat{s}_{28}\hat{s}_{38}\hat{s}_{39} & -\hat{s}_{18}\hat{s}_{18}\hat{s}_{29}c_{39} & \\
 +\hat{s}_{17}\hat{s}_{18}\hat{s}_{28}\hat{s}_{29}c_{39} & 0 & 0 \\
 \end{pmatrix}; \quad (13)$$

One can see that the triangular matrices $A_2$ and $B_2$ have the same forms as $A_1$ and $B_1$, and the former can be directly obtained from the latter with the replacements $\theta_{i4} \rightarrow \theta_{i7}$, $\theta_{i5} \rightarrow \theta_{i8}$, $\theta_{i6} \rightarrow \theta_{i9}$ and $\delta_{i4} \rightarrow \delta_{i7}$, $\delta_{i5} \rightarrow \delta_{i8}$, $\delta_{i6} \rightarrow \delta_{i9}$ (for $i = 1, 2, 3$). The same is true for the forms of $R_2$ and $S_2$ as
compared respectively with those of $R_1$ and $S_1$. It is obvious that the flavor mixing angles $\theta_{ij}$, $\theta_{ij}$ and $\theta_{ij}$ are all small because they describe the interplay between three active neutrinos and three extra neutral fermions. Finally, the four $3 \times 3$ submatrices of $L_3$ are

$$
A_3 = 
\begin{pmatrix}
  c_{47}c_{48}c_{49} & 0 & 0 \\
  -c_{47}s_{48}\hat{s}_{49}s_{59} - c_{47}\hat{s}_{48}s_{58}\hat{s}_{c59} & c_{57}c_{58}c_{59} & 0 \\
  -\hat{s}_{47}\hat{s}_{57}c_{58}c_{59} & 0 & 0
\end{pmatrix},
$$

$$
B_3 = 
\begin{pmatrix}
  c_{47}c_{57}c_{67} & 0 & 0 \\
  -c_{47}s_{57}\hat{s}_{67}\hat{s}_{c58}c_{68} & c_{48}s_{58}c_{66} & 0 \\
  -\hat{s}_{47}\hat{s}_{57}c_{58}c_{68} & 0 & 0
\end{pmatrix};
$$

\hspace{1cm} (14)

and

$$
R_3 = 
\begin{pmatrix}
  \hat{s}_{47}s_{48}s_{49} & \hat{s}_{48}c_{49} & \hat{s}_{49} \\
  -\hat{s}_{47}s_{48}\hat{s}_{49}s_{59} - \hat{s}_{47}\hat{s}_{48}s_{58}\hat{s}_{c59} & -\hat{s}_{48}s_{49}\hat{s}_{59} + c_{48}\hat{s}_{58}\hat{s}_{c59} & c_{49}\hat{s}_{59} \\
  +c_{47}\hat{s}_{57}c_{58}c_{59} & 0 & 0
\end{pmatrix},
$$

$$
S_3 = 
\begin{pmatrix}
  -\hat{s}_{47}\hat{s}_{57}c_{58}c_{59} & 0 & 0 \\
  \hat{s}_{47}\hat{s}_{57}\hat{s}_{67}\hat{s}_{c58}c_{68} & \hat{s}_{57}\hat{s}_{67}\hat{s}_{c58}c_{68} & -c_{67}\hat{s}_{68} \\
  -c_{47}s_{48}c_{58}c_{66} & 0 & 0
\end{pmatrix};
$$

\hspace{1cm} (15)

Different from the eighteen active-sterile flavor mixing angles $\theta_{ij}$, $\theta_{ij}$ and $\theta_{ij}$ (for $j = 4, 5, \cdots, 9$), which are all expected to be strongly suppressed in magnitude, the nine sterile-sterile flavor mixing angles $\theta_{ij}$, $\theta_{ij}$ and $\theta_{ij}$ (for $j = 7, 8, 9$) are completely unconstrained. Given the very fact that the right-handed neutrino fields and the extra neutral fermion fields are both hypothetical, one may naively conjecture that these two sterile sectors might be essentially disconnected and thus their interplay might be very weak. Here we only focus our attention on the issue of active-sterile flavor mixing. Without loss of any generality, we have chosen the basis in which the flavor eigenstates of three charged leptons are identical with their mass eigenstates throughout this work.
After Eqs. (6), (7) and (8) are taken into account, the $9 \times 9$ active-sterile flavor mixing matrix $U$ can be explicitly written as

$$
U = \begin{pmatrix}
A_2 A_1 U_0 & A_2 R_1 & R_2 \\
(R_3 S_2 A_1 + A_3 U_0^T S_1) U_0 & R_3 S_2 R_1 + A_3 U_0^T B_1 & R_3 B_2 \\
S_0 (B_3 S_2 A_1 + S_3 U_0^T S_1) U_0 & S_0 (B_3 S_2 R_1 + S_3 U_0^T B_1) & S_0 B_3 B_2
\end{pmatrix} .
$$

(16)

The unitarity of $U$ (i.e., $UU^\dagger = U^\dagger U = I_{9 \times 9}$ with $I_{9 \times 9}$ being the $9 \times 9$ identity matrix) allows us to obtain a number of constraint equations, as listed in appendix A. In the chosen flavor basis the key role of $U$ is to transform the flavor eigenstates of three active neutrinos, three sterile neutrinos and three extra neutral fermions shown in Eq. (2) into their mass eigenstates; namely,

$${U}^\dagger \begin{pmatrix} 0 & M_D & \varepsilon \\
M_D^T & M_R & M_S \\
\varepsilon^T & M_S^T & \mu \end{pmatrix} U^* = \begin{pmatrix} D_\nu & 0 & 0 \\
0 & D_N & 0 \\
0 & 0 & D_S \end{pmatrix} ,
$$

(17)

where $D_\nu = \{m_1, m_2, m_3\}$, $D_N = \{M_1, M_2, M_3\}$ and $D_S = \{M'_1, M'_2, M'_3\}$ with $m_i$, $M_i$ and $M'_i$ being the respective masses of the active neutrinos $\nu_i$, sterile neutrinos $N_i$ and extra neutral fermions $N'_i$ (for $i = 1, 2, 3$). We are therefore left with

$$
\begin{pmatrix} \nu_e \\
\nu_\mu \\
\nu_\tau \end{pmatrix}_L = A_2 A_1 U_0 \begin{pmatrix} \nu_1 \\
\nu_2 \\
\nu_3 \end{pmatrix}_L + A_2 R_1 \begin{pmatrix} N_1 \\
N_2 \\
N_3 \end{pmatrix}_L + R_2 \begin{pmatrix} N'_1 \\
N'_2 \\
N'_3 \end{pmatrix}_L .
$$

(18)

The weak charged-current interactions of three active neutrinos, three sterile neutrinos and three extra neutral fermions turn out to be

$$
-L_{cc} = \frac{g}{\sqrt{2}} (e \mu \tau)_L \gamma^\mu \left[ U \begin{pmatrix} \nu_1 \\
\nu_2 \\
\nu_3 \end{pmatrix}_L + R \begin{pmatrix} N_1 \\
N_2 \\
N_3 \end{pmatrix}_L + R' \begin{pmatrix} N'_1 \\
N'_2 \\
N'_3 \end{pmatrix}_L \right] W^- + \text{h.c.} ,
$$

(19)

where $U \equiv A_2 A_1 U_0$ represents the effective PMNS matrix, $R \equiv A_2 R_1$ and $R' \equiv R_2$ characterize the contributions of new degrees of freedom to $L_{cc}$. Once the extra neutral fermions are switched off (i.e., $A_2 = I$ and $R_2 = 0$), Eq. (19) will be reduced to the canonical seesaw case [31,33].

Note that the $3 \times 3$ PMNS matrix $U$ is not exactly unitary, and its slight deviation from the unitarity limit can be clearly seen from

$$
UU^\dagger = A_2 A_1 A_1^T A_2^T = I - RR^\dagger - R'R'^\dagger ,
$$

(20)

where the relation in Eq. (A.1) has been used. This issue will be further discussed in section 4.

Note also that the full parametrization of the $9 \times 9$ unitary flavor mixing matrix $U$ obtained in Eq. (16) is a quite general result, and it is actually not subject to the seesaw scenario and the corresponding $9 \times 9$ neutrino mass matrix considered in Eqs. (1) and (2). For example, the texture zeros in the original neutrino mass matrix will help establish some correlations between the mass and flavor mixing parameters, including the exact seesaw formulas. This point will be clearly seen later on in the inverse and linear seesaw mechanisms.
3 Exact and approximate seesaw formulas

3.1 The inverse seesaw scenario

As for the inverse seesaw scenario with $M_R = 0$ and $\varepsilon = 0$, the corresponding neutrino mass matrix takes the well-known Fritzsch texture \[41\]

$$F = \begin{pmatrix} 0 & M_D & 0 \\ M_D^T & 0 & M_S \\ 0 & M_S^T & \mu \end{pmatrix} = U \begin{pmatrix} D_\nu & 0 & 0 \\ 0 & D_N & 0 \\ 0 & 0 & D_S \end{pmatrix} U^T. \quad (21)$$

The texture zero $F_{11} = 0$ allows us to obtain the exact inverse seesaw formula

$$U D_\nu U^T + R D_N R + R' D_S R'^T = 0, \quad (22)$$

where $U \equiv A_2 A_1 U_0$, $R \equiv A_2 R_1$ and $R' \equiv R_2$ have been defined below Eq. (19), and they are correlated with one another through $U U^T + R R^T + R' R'^T = I$ as can be seen from Eq. (20). If the extra neutral fermion sector is switched off (i.e., $A_2 = I$ and $R' = 0$), one will reproduce the exact seesaw formula of the canonical seesaw mechanism \[31-33\]. Moreover, the texture zeros $F_{22} = 0$ and $F_{13} = F_{31} = 0$ lead us to the constraint equations

(a) : \[\begin{array}{c}
R_3 S_2 A_1 U_0 + A_3 U_0^T S_1 U_0 D_\nu \left(U_0^T A_1^T S_2^T R_3^T + U_0^T S_1^T U_0^T A_3^T \right) \\
+ \left(R_3 S_2 R_1 + A_3 U_0 B_1 \right) D_N \left(R_1^T S_2^T R_3^T + B_1^T U_0^T A_3 \right) + R_3 B_2 D_S B_2^T R_3^T = 0,
\end{array}\]

(b) : \[\begin{array}{c}
\left(S_0 B_3 S_2 A_1 U_0 + S_0 S_3 U_0^T S_1 U_0 \right) D_\nu U_0^T A_1^T A_2^T \\
+ \left(S_0 B_3 S_2 R_1 + S_0 S_3 U_0^T B_1 \right) D_N R_1^T A_2^T + S_0 B_3 B_2 D_S R_2^T = 0.
\end{array}\] \quad (23)

These two equations, together with Eq. (22), characterize the correlations between the mass and flavor mixing parameters in this seesaw scenario.

We proceed to derive the approximate but more instructive inverse seesaw formula given in Eq. (3) from Eqs. (22) and (23). To this end, we take into account the fact that the interplay between three active neutrinos and those hypothetical degrees of freedom must be strongly suppressed in magnitude. In other words, the active-sterile flavor mixing angles $\theta_{1j}$, $\theta_{2j}$ and $\theta_{3j}$ (for $j = 4, 5, \cdots, 9$) are all very small, and thus the eight $3 \times 3$ matrices $A_{1,2}$, $B_{1,2}$, $R_{1,2}$ and $S_{1,2}$ in Eqs. (10)—(13) can be reliably simplified to the forms listed in appendix \[B\]. As very good approximations, Eqs. (22) and (23) become

$$U_0 D_\nu U_0^T + R_1 D_N R_1^T + R_2 D_S R_2^T \simeq 0, \quad (24)$$

and

$$A_3 U_0^T D_N U_0^T A_3^T + R_3 D_S R_3^T \simeq 0, \quad S_3 U_0^T D_N R_1^T + B_3 D_S R_2^T \simeq 0. \quad (25)$$

Since $D_\nu$ is strongly suppressed in magnitude as compared with $D_N$ and $D_S$, the sum of the second and third terms on the left-hand side of Eq. (24) should be as small as the first term and possess
the opposite sign. Moreover, \( D_N \simeq -D_S \) is expected to hold due to the smallness of \( \mu \), as one can see when solving the eigenvalue equation of \( \mathcal{F} \). This observation implies that \( R_1 \simeq R_2 \) is a good approximation. Then \( A_3 U_0' \simeq -R_3 \) and \( S_3 U_0' \simeq B_3 \) can be derived from Eq. (25), and their opposite signs will be explained later. In this case the effective 3 × 3 Majorana neutrino mass matrix of three active neutrinos can be defined as

\[
M_\nu \equiv U_0 D_\nu U_0^T \simeq -R_1 D_N R_1^T - R_2 D_S R_2^T ,
\]

where Eq. (24) has been used. To see how \( M_\nu \) is related to \( M_D, M_S \) and \( \mu \), we start from Eq. (21) and obtain the following results:

\[
M_D = R_1 D_N (R_1^T S_2^T R_3^T + U_0'^T A_3^T) + R_2 D_S R_3^T \\
\simeq R_1 D_N U_0'^T A_3^T + R_2 D_S R_3^T \\
\simeq R_1 D_N (U_0'^T A_3^T - R_3) \\
\simeq 2 R_1 D_N U_0'^T A_3^T ,
\]

\[
M_S = (R_3 S_2 R_1 + A_3 U_0') D_N (R_1^T S_2^T B_3^T S_0^T + U_0'^T S_3^T S_0^T) + R_3 D_S B_3^T S_0^T \\
\simeq A_3 U_0'^T D_N U_0'^T S_3^T S_0^T + R_3 D_S B_3^T S_0^T \\
\simeq (A_3 U_0' - R_3) D_N U_0'^T S_3^T S_0^T \\
\simeq 2 A_3 U_0'^T D_N U_0'^T S_3^T S_0^T ,
\]

\[
\mu = (S_0 S_3 R_2 U_0' + S_0 S_3 R_3 U_0' B_1) D_N (R_1^T S_2^T B_3^T S_0^T + B_1 U_0'^T S_3^T S_0^T) + S_0 B_3^T S_0^T \\
\simeq S_0 S_3 U_0'^T D_N U_0'^T S_3^T S_0^T + S_0 B_3^T S_0^T \\
\simeq S_0 B_3^T S_0^T \\
\simeq S_0 B_3^T S_0^T \approx (R_1^{-1} R_2 D_S R_2^T R_1^{-1} - D_N) B_3^T S_0^T .
\]

(27)

It is known that a natural TeV-scale inverse seesaw scenario requires \( M_D \sim \mathcal{O}(1) \) GeV, \( M_S \sim \mathcal{O}(1) \) TeV and \( \mu \sim \mathcal{O}(1) \) keV (see, e.g., Ref. [16]). Hence the two terms in \( M_D \) and \( M_S \) should be of the same sign to avoid a significant cancellation. That is why we have chosen \( A_3 U_0' \simeq -R_3 \) below Eq. (25). On the other hand, the two terms in \( \mu \) should nearly cancel each other, implying that \( S_3 U_0' \simeq B_3 \) is a reasonable choice. Substituting the first equation in Eq. (25) into Eq. (26) and then making use of the approximate expressions obtained in Eq. (27), we simply arrive at

\[
M_\nu \simeq -R_1 D_N R_1^T - R_2 D_S R_2^T \\
\simeq R_1 ( -D_N - R_1^{-1} R_2 D_S R_2^T R_1^{-1} ) R_1^T \\
\simeq R_1 U_0'^{-1} S_3^{-1} B_3 (-D_N - R_1^{-1} R_2 D_S R_2^T R_1^{-1} ) B_3^T S_3^{-1} (U_0'^T)^{-1} R_1^T \\
\simeq \left[ R_1 U_0'^{-1} S_3^{-1} S_0^{-1} \right] \mu \left[ R_1 U_0'^{-1} S_3^{-1} S_0^{-1} \right]^T \\
\simeq 2 R_1 D_N U_0'^T A_3^T (2 S_0 S_3 U_0' D_N U_0'^T A_3^T)^{-1} \mu \left[ (2 A_3 U_0' D_N U_0'^T S_3^T S_0^T)^{-1} (2 R_1 D_N U_0'^T A_3^T)^T \right] \\
\simeq M_D (M_S^T)^{-1} \mu (M_S)^{-1} M_D^T .
\]

(28)

This leading-order result is just the well-known inverse seesaw formula given in Eq. (3). While such an approximate expression may be more instructive in some explicit model-building exercises,
our exact inverse seesaw formula and full parametrization of the $9 \times 9$ active-sterile flavor mixing matrix will be more useful for a generic study of the inverse seesaw picture.

At this point it is worth remarking that we have only paid attention to the leading-order effects when defining the effective Majorana neutrino mass matrix of three active neutrinos in Eq. (26) and reproducing the inverse seesaw formula for it in Eqs. (27) and (28). The next-to-leading-order corrections to the leading-order inverse seesaw formula have been discussed at the tree level [20]. We find that it will be a messy and tangled business to include such next-to-leading-order effects into our approach, because both our generic Euler-like parametrization of the $9 \times 9$ flavor mixing matrix will be more useful for a generic study of the inverse seesaw picture. This is also true of our discussions about the non-unitary effects on flavor mixing and CP violation based on our parametrization in section (4).

3.2 The linear seesaw scenario

Since $M_R = 0$ and $\mu = 0$ are taken for the linear seesaw scenario, one may reconstruct the corresponding traceless neutrino mass matrix in the following way:

$$F' = \begin{pmatrix} 0 & M_D & \varepsilon \\ M_D^T & 0 & M_S \\ \varepsilon^T & M_S^T & 0 \end{pmatrix} = U \begin{pmatrix} D_\nu & 0 & 0 \\ 0 & D_N & 0 \\ 0 & 0 & D_S \end{pmatrix} U^T. \tag{29}$$

In this case the texture zero $F'_{11} = 0$ leads us to the exact linear seesaw formula

$$UD_\nu U^T + RD_N R + R'D_S R'^T = 0, \tag{30}$$

where $U \equiv A_2 A_1 U_0$, $R \equiv A_2 R_1$ and $R' \equiv R_2$ are defined and $UU^\dagger + RR^\dagger + R'R'^\dagger = I$ holds. One can immediately see that Eq. (30) is formally the same as Eq. (22) obtained in the inverse seesaw case, although their physical contexts are somewhat different. This interesting observation tells us that such an exact seesaw formula is actually universal and valid for the more general seesaw scenario described by the mass terms in Eqs. (1) and (2).

Taking account of the texture zeros $F'_{22} = F'_{33} = 0$ in the linear seesaw scenario, we arrive at two further constraint equations

(a) : \( (R_3 S_2 A_1 U_0 + A_3 U'_0 S_1 U_0) D_\nu (U_0^T A_1^T S_2^T R_3^T + U_0^T S_1^T U'_0 A_3^T) + (R_3 S_2 R_1 + A_3 U'_0 B_1) D_N (R_1^T S_2^T R_3^T + B_1 U'_0 A_3^T) + R_3 B_2 D_S B_2^T R_3^T = 0, \)

(b) : \( (S_0 B_3 S_2 A_1 U_0 + S_0 S_3 U'_{0} S_1 U_0) D_\nu (U_0^T A_1^T S_2^T B_3^T S_0^T + U_0^T S_1^T U'_0 S_3^T S_0^T) + (S_0 B_3 S_2 R_1 + S_0 S_3 U'_0 B_1) D_N (B_1^T S_2^T S_0^T R_3^T + R_1^T S_2^T B_3^T S_0^T) + S_0 B_3 D_S B_3^T S_0^T = 0. \) \tag{31}

Once the approximations made in Eqs. (B1)—(B4) are taken into consideration, Eqs. (30) and (31) can be easily simplified to

$$U_0 D_\nu U_0^T + R_1 D_N R_1^T + R_2 D_S R_2^T \simeq 0, \tag{32}$$

and

$$A_3 U'_0 D_N U'_0 A_3^T + R_3 D_S R_3^T \simeq 0,$$
\[ S_3 U'_0 D_N U'^T_0 S_3^T + B_3 D_S B_3^T \simeq 0 . \]  

As in the inverse seesaw scenario, we similarly have \( D_N \simeq -D_S \) and \( R_1 \simeq R_2 \) in the linear seesaw scenario. The relations \( A_3' U'_0 \simeq -R_3 \) and \( S_3' U'_0 \simeq B_3 \) are expected to remain valid. The reason why these two approximate relations have the opposite signs is also that the terms in \( M_D \) and \( M_S \) should avoid significant cancellations while \( \varepsilon \) should be strongly suppressed in magnitude.

Now let us use Eq. (32) to define the effective \( 3 \times 3 \) Majorana neutrino mass matrix of three active neutrinos:

\[
M_\nu \equiv U_0 D_s U'^T_0 \simeq -R_1 D_N R'^T_1 - R_2 D_S R'^T_2 .
\]

Starting from Eq. (29) and taking into account the approximate relations obtained in and below Eq. (33), we have

\[
M_D = R_1 D_N (R'^T_1 S'^T_2 R'^T_3 + U'^T_0 A'^T_3) + R_2 D_S R'^T_3 \\
\simeq 2 R_1 D_N U'^T_0 A'^T_3 ,
\]

\[
M_S = (R_3 S_R R_1 + A_3' U'_0) D_N (R'^T_1 S'^T_2 B'^T_3 S'^T_0 + U'^T_0 S'^T_3 S'^T_0) + R_3 D_S B'^T_3 S'^T_0 \\
\simeq 2 A_3' U'_0 D_N U'^T_0 S'^T_3 S'^T_0 \\
\varepsilon = R_1 D_N (R'^T_1 S'^T_2 B'^T_3 S'^T_0 + U'^T_0 S'^T_3 S'^T_0) + R_2 D_S B'^T_3 S'^T_0 \\
\simeq R_1 D_N U'^T_0 S'^T_3 S'^T_0 + R_2 D_S B'^T_3 S'^T_0 .
\]

It is then easy to arrive at

\[
R_1 D_N R'^T_1 - R_1 D_N R'^T_2 \simeq R_1 U'^T_0 S'^T_3 (S'_3 U'_0 D_N R'^T_1 + B_3 D_S R'^T_2) \\
\simeq (2 R_1 D_N U'^T_0 A'^T_3) (2 S_3' S'_3 U'_0 D_N U'^T_0 A'^T_3)^{-1} (R_1 D_N U'^T_0 S'^T_3 S'^T_0 + R_2 D_S B'^T_3 S'^T_0)^T \\
\simeq M_D (M_S^T)^{-1} \varepsilon^T ,
\]

\[
R_1 D_N R'^T_2 + R_2 D_S R'^T_2 \simeq R_1 D_N (R_2 B'^T_3 S'_3 U'_0)^T + R_2 D_S R'^T_2 \\
\simeq (R_1 D_N U'^T_0 S'^T_3 S'^T_0 + R_2 D_S B'^T_3 S'^T_0) (2 R_3 D_S B'^T_3 S'^T_0)^{-1} (R_2 D_S R'^T_3)^T \\
\simeq \varepsilon (M_S)^{-1} M_D^T .
\]

Combining Eq. (36) with Eq. (34) allows us to obtain the linear seesaw formula

\[
M_\nu \simeq -R_1 D_N R'^T_1 - R_2 D_S R'^T_2 \\
\simeq -M_D (M_S^T)^{-1} \varepsilon^T - \varepsilon (M_S)^{-1} M_D^T .
\]

This result is fully consistent with Eq. (4).

## 4 Jarlskog invariants and unitarity nonagons

### 4.1 The Jarlskog invariants

In the standard three-flavor scheme without any new degrees of freedom, the strength of CP violation in neutrino oscillations is measured by the unique Jarlskog invariant [42]

\[
J_0 = J_{\alpha\beta}^{ij} \equiv \text{Im} \left( U_{\alpha i} U_{\beta j} U_{\alpha j}^* U_{\beta i}^* \right) ,
\]

\[ S_3 U'_0 D_N U'^T_0 S_3^T + B_3 D_S B_3^T \simeq 0 . \]
where the Latin (or Greek) indices $i$ and $j$ (or $\alpha$ and $\beta$) run cyclically over 1, 2 and 3 (or $e$, $\mu$ and $\tau$). Since $U = U_0$ holds in this case, the parametrization of $U_0$ in Eq. (9) leads us to

$$J_0 = c_{12}s_{12}^2s_{13}^2s_{23}s_{23}\sin \delta,$$

where $\delta = \delta_{13} - \delta_{12} - \delta_{23}$ is sometimes referred to as the “Dirac” CP-violating phase. Note that this phase parameter may also manifest itself in the lepton-number-violating neutrino-antineutrino oscillations $^{43}[44]$, simply because it is a nontrivial CP phase of Majorana neutrinos.

In the inverse or linear seesaw scenario, however, the $3 \times 3$ PMNS matrix $U = A_2A_1U_0$ is not exactly unitary. As a consequence, some additional effects of CP violation described by the deviations of $J^{ij}_{\alpha\beta}$ from $J_0$ will manifest themselves in neutrino oscillations. To explicitly calculate the nine Jarlskog invariants $J^{ij}_{\alpha\beta}$, let us rewrite the approximate expressions of $A_1$ in Eq. (B1) and $A_2$ in Eq. (B3) as follows:

$$A_{1(2)} \simeq \begin{pmatrix} 1 & 0 & 0 \\ -X_{1(2)} & 1 & 0 \\ -Y_{1(2)} & -Z_{1(2)} & 1 \end{pmatrix},$$

where only the leading term of each matrix element is kept, and

$$X_1 \equiv \hat{s}_{14}\hat{s}_{24}^* + \hat{s}_{15}\hat{s}_{25}^* + \hat{s}_{16}\hat{s}_{26}^*,$$

$$Y_1 \equiv \hat{s}_{14}\hat{s}_{34}^* + \hat{s}_{15}\hat{s}_{35}^* + \hat{s}_{16}\hat{s}_{36}^*,$$

$$Z_1 \equiv \hat{s}_{24}\hat{s}_{34}^* + \hat{s}_{25}\hat{s}_{35}^* + \hat{s}_{26}\hat{s}_{36}^*, \quad (41)$$

together with

$$X_2 \equiv \hat{s}_{17}\hat{s}_{27}^* + \hat{s}_{18}\hat{s}_{28}^* + \hat{s}_{19}\hat{s}_{29}^*,$$

$$Y_2 \equiv \hat{s}_{17}\hat{s}_{37}^* + \hat{s}_{18}\hat{s}_{38}^* + \hat{s}_{19}\hat{s}_{39}^*,$$

$$Z_2 \equiv \hat{s}_{27}\hat{s}_{37}^* + \hat{s}_{28}\hat{s}_{38}^* + \hat{s}_{29}\hat{s}_{39}^*. \quad (42)$$

Now that $A_1$ and $A_2$ signify the departure of $U$ from $U_0$, the magnitudes of $X_{1(2)}$, $Y_{1(2)}$ and $Z_{1(2)}$ are at most of the percent level. On the other hand, the smallest flavor mixing angle of $U_0$ is $\theta_{13}$, and its size is about 0.16. So it is reasonable to omit the terms of $\mathcal{O}(X_{1(2)}s_{13})$, $\mathcal{O}(Y_{1(2)}s_{13})$, $\mathcal{O}(Z_{1(2)}s_{13})$, $\mathcal{O}(X_{1(2)}^2)$, $\mathcal{O}(Y_{1(2)}^2)$ and $\mathcal{O}(Z_{1(2)}^2)$ in calculating $J^{ij}_{\alpha\beta}$. Our results are

$$J^{12}_{\mu\mu} \simeq J_0 + c_{12}s_{12}c_{23}\text{Im}\left[(X_1 + X_2)e^{-i\delta_{12}}\right],$$

$$J^{12}_{\tau e} \simeq J_0 + c_{12}s_{12}s_{23}\text{Im}\left[(Y_1 + Y_2)e^{-i(\delta_{12} + \delta_{23})}\right],$$

$$J^{12}_{\mu\tau} \simeq J_0 + c_{12}s_{12}c_{23}s_{23}\left\{s_{23}\text{Im}\left[(X_1 + X_2)e^{-i\delta_{12}}\right] + c_{23}\text{Im}\left[(Y_1 + Y_2)e^{-i(\delta_{12} + \delta_{23})}\right]\right\},$$

$$J^{23}_{\mu\tau} \simeq J_0 + c_{12}s_{12}c_{23}s_{23}\left\{s_{23}\text{Im}\left[(X_1 + X_2)e^{-i\delta_{12}}\right] + c_{23}\text{Im}\left[(Y_1 + Y_2)e^{-i(\delta_{12} + \delta_{23})}\right]\right\},$$

$$+ c_{12}\text{Im}\left[(Z_1 + Z_2)e^{-i\delta_{23}}\right],$$

$$J^{31}_{\mu\tau} \simeq J_0 + s_{12}c_{23}s_{23}\left\{c_{12}s_{23}\text{Im}\left[(X_1 + X_2)e^{-i\delta_{12}}\right] + c_{12}c_{23}\text{Im}\left[(Y_1 + Y_2)e^{-i(\delta_{12} + \delta_{23})}\right]\right\}.$$
\(-s_{12} \text{Im} \left[ (Z_1 + Z_2) e^{-i \delta_{23}} \right] \),

(43)

and \(J_{e\mu}^{23} \simeq J_{e\tau}^{31} \simeq J_{\tau e}^{23} \simeq J_{\tau e}^{31} \simeq J_0\). One can see that five of the nine Jarlskog invariants are sensitive to the active-sterile flavor mixing angles and the associated CP-violating phases.

As for the probabilities of neutrino oscillations in vacuum, let us assume that all the sterile particles are heavy enough and hence kinematically forbidden to participate in a realistic long-baseline oscillation process. In this case one does not have to worry about the masses of those hypothetical particles and their differences from the masses of three active neutrinos, and the Jarlskog invariants calculated above determine the CP-violating asymmetry \(A_{\alpha\beta} \equiv P(\nu_\alpha \to \nu_\beta) - P(\bar{\nu}_\alpha \to \bar{\nu}_\beta)\) (for \(\alpha, \beta = e, \mu, \tau\)) as follows [31]:

\[
A_{\alpha\beta} = -4 \left( \frac{4}{(U U^\dagger)_{\alpha \alpha}} \frac{\sum_{i<j} J_{\alpha \beta}^{ij} \sin \frac{\Delta m_{ij}^2 L}{2E}}{(U U^\dagger)_{\beta \beta}} \right),
\]

(44)

where \(\Delta m_{ij}^2 \equiv m_i^2 - m_j^2\) for \(i, j = 1, 2, 3\) are defined, \(E\) and \(L\) stand respectively for the neutrino beam energy and baseline length, and the approximations \(A_1 A_1^\dagger \simeq I\) and \(A_2 A_2^\dagger \simeq I\) have been used. In practice, however, terrestrial matter effects must be taken into account for a long-baseline neutrino oscillation experiment which is sensitive to leptonic CP violation (see, e.g., Refs. [45–51]). That is why a careful analysis of the non-unitary flavor mixing effects and matter effects is needed for a realistic long-baseline neutrino (or antineutrino) oscillation experiment [54], no matter whether CP violation is concerned or not.

4.2 The unitarity nonagons

It is well known that six orthogonality relations of the \(3 \times 3\) unitary PMNS matrix \(U = U_0\) define six triangles in the complex plane, which are referred to as the unitarity triangles [52–53]. Their areas are all equal to \(J_0/2\), thanks to the unitarity of \(U_0\). Three of the six triangles, defined by

\[
\Delta_\tau : U_{e1}^* U_{\mu1} + U_{e2}^* U_{\mu2} + U_{e3}^* U_{\mu3} = 0,
\]

\[
\Delta_e : U_{\mu1}^* U_{\tau1} + U_{\mu2}^* U_{\tau2} + U_{\mu3}^* U_{\tau3} = 0,
\]

\[
\Delta_\mu : U_{\tau1}^* U_{e1} + U_{\tau2}^* U_{e2} + U_{\tau3}^* U_{e3} = 0,
\]

(45)

where \(U = U_0\) is implied, are directly associated with leptonic CP violation in neutrino oscillations.

Given three sterile neutrinos and three extra neutral fermions which slightly mix with the active neutrinos, the corresponding \(9 \times 9\) unitary flavor mixing matrix \(U\) allows us to totally define 36 nonagons in the complex plane. Among them, only three unitarity nonagons defined by the orthogonality conditions \((U U^\dagger)_{\epsilon \mu} = 0\), \((U U^\dagger)_{\epsilon \tau} = 0\) and \((U U^\dagger)_{\tau \epsilon} = 0\) are relevant to the flavor oscillations of three active neutrinos. Since the sides associated with sterile flavors must be very short as compared with the sides \(|U_{\alpha i} U_{\beta i}^*|\) for \((\alpha, \beta) = (e, \mu), (\mu, \tau), (\tau, e)\) and \(i = 1, 2, 3\), these three unitarity nonagons can be regarded as the deformed versions of unitarity triangles \(\Delta_\tau, \Delta_e\) and
Taking account of Eqs. (40)—(42) in this case, the three unitarity nonagons under discussion can therefore be expressed as

\[
\begin{align*}
\Delta'_{\tau} : & \quad U_{e1}U_{\mu1}^* + U_{e2}U_{\mu2}^* + U_{e3}U_{\mu3}^* \simeq -X_1^* - X_2^* , \\
\Delta'_{e} : & \quad U_{\mu1}U_{\tau1}^* + U_{\mu2}U_{\tau2}^* + U_{\mu3}U_{\tau3}^* \simeq -Z_1^* - Z_2^* , \\
\Delta'_{\mu} : & \quad U_{\tau1}U_{e1}^* + U_{\tau2}U_{e2}^* + U_{\tau3}U_{e3}^* \simeq -Y_1 - Y_2 .
\end{align*}
\]

Comparing Eq. (46) with Eq. (45), one can see that the new sides originating from those sterile degrees of freedom are all of \(O(s_{ij}^2)\) with \(i = 1, 2, 3\) and \(j = 4, 5, 6, 7, 8, 9\). That is why we use \(\Delta'_{\alpha}\) to denote the unitarity nonagon defined by \((UU^\dagger)_{\beta\gamma} = 0\) (for \(\alpha, \beta\) and \(\gamma\) running cyclically over \(e, \mu\) and \(\tau\)), simply because it deviates only slightly from \(\Delta_{\alpha}\).

In the lack of definite information about the active-sterile flavor mixing parameters (i.e., \(\theta_{ij}\) and \(\delta_{ij}\) for \(i = 1, 2, 3\) and \(j = 4, 5, 6, 7, 8, 9\)), one may simply treat the new sides of \(\Delta'_{\alpha}\) (for \(\alpha = e, \mu\) or \(\tau\)) on the right-hand side of Eq. (46) as an effective “single” side. In this case the unitarity nonagon is reduced to an effective unitarity quadrangle whose shortest side signifies the existence of sterile neutrinos and extra neutral fermions, as schematically illustrated by Fig. 1 for \(\Delta'_{\tau}\). For simplicity, we have made the choice that the side \(U_{e1}U_{\mu1}^*\) lies in the horizontal direction and forms a sharp angle to the side \(U_{e2}U_{\mu2}^*\) in Fig. 1. Then the side \(U_{e3}U_{\mu3}^*\) may form either a sharp angle or an obtuse angle to the horizontal side, and the shortest side defined by \(-X_1^* - X_2^*\) is likely to link these two longer sides in several different topologies.

To constrain the shortest side of each effective unitarity quadrangle, one may study those lepton-flavor-violating processes such as neutrino oscillations and radiative decays of charged leptons. In the latter case the new degrees of freedom can mediate the one-loop \(\alpha^- \rightarrow \beta^- + \gamma\) transitions for \((\alpha, \beta) = (\mu, e), (\tau, e)\) and \((\tau, \mu)\), and the relevant loop functions depend on the masses of three sterile neutrinos and three extra neutral fermions (i.e., \(M_{\alpha}^i\) and \(M_{\alpha}^{i'}\) for \(i = 1, 2, 3\)) \[26, 55, 56\]. A systematic analysis of such lepton-flavor-violating processes in the inverse and linear seesaw scenarios will be done elsewhere.

### 5 Concluding remarks

The inverse and linear seesaw scenarios are two simple extensions of the canonical (type-I) seesaw mechanism aiming to lower the mass scales of those hypothetical particles and hence enhance the experimental testability. In this regard a high price that one has to pay is the introduction of three species of extra neutral fermions besides three species of sterile neutrinos. How to determine or constrain the flavor structure of such a complicated seesaw scenario and describe the corresponding flavor mixing pattern turns out to be a highly nontrivial issue.

Instead of trying to reduce the number of free parameters by imposing some kind of flavor symmetry or empirical assumptions on the texture of the \(9 \times 9\) mass matrix (or its \(3 \times 3\) submatrices) in either the inverse seesaw scenario or the linear seesaw scenario \[33, 57, 59\], here we have focused on a full description of the \(9 \times 9\) active-sterile flavor mixing matrix in terms of 36 rotation angles and 36 CP-violating phases. Such a generic and model-independent work is certainly new. The most
Figure 1: A schematic illustration of possible shapes of the effective unitarity quadrangle $\triangle^\prime_\tau$ as an example in the complex plane, where the three long sides correspond to $U_{e1}U_{\mu1}^*$, $U_{e2}U_{\mu2}^*$ and $U_{e3}U_{\mu3}^*$, and the fourth (effective) side is measured by $-X_1^* - X_2^*$. 
salient feature of our parametrization that the primary unitary $3 \times 3$ flavor mixing submatrices of three active neutrinos, three sterile neutrinos and three extra neutral fermions, which would look like three isolated islands if the Yukawa-like interactions among them were absent, are linked and modified by the intermediate flavor mixing matrices. This approach proves to be useful for describing possible deviations of the $3 \times 3$ PMNS matrix of three active neutrinos from its unitary limit, as it allows us to calculate the effective Jarlskog invariants and the deformed unitarity triangles in a good approximation and without loss of any generality.

Starting from our generic result, one may easily arrive at a full Euler-like parametrization of the $7 \times 7$ active-sterile flavor mixing matrix in the so-called minimal inverse seesaw model which contains only two species of sterile neutrinos and two species of extra neutral fermions (see, e.g., Refs. [16, 26, 60–64]). A similar simplification can also be applied to the so-called minimal linear seesaw model [65] and the study of its various phenomenological consequences.

Finally, it is also worth pointing out that our exact Euler-like parametrization of the $n \times n$ unitary matrix (for $3 \leq n \leq 9$) is likely to find applications in some other physical systems.

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A The unitary conditions of $\mathcal{U}$

Given the expression of the $9 \times 9$ active-sterile flavor mixing matrix $\mathcal{U}$ in Eq. (16), one may obtain either the six unitary conditions for its $3 \times 3$ submatrices from $\mathcal{U} \mathcal{U}^\dagger = I_{9 \times 9}$:

(1) $A_2 A_1 A_2^\dagger + A_2 R_1 R_1^\dagger A_2^\dagger + R_2 R_2^\dagger = 0$

(2) $(R_3 S_2 A_1 + A_3 U_0' S_1) (A_1^\dagger S_2^\dagger R_3^\dagger + S_1^\dagger U_0'^\dagger A_3^\dagger) + (R_3 S_2 R_1 + A_3 U_0' B_1) (R_1^\dagger S_2^\dagger R_3^\dagger + B_1^\dagger U_0'^\dagger A_3^\dagger) + R_3 B_2 B_2^\dagger R_3^\dagger = 0$

(3) $(B_3 S_2 A_1 + S_3 U_0' S_1) (A_1^\dagger S_2^\dagger B_3^\dagger + S_1^\dagger U_0'^\dagger S_3^\dagger) + (B_3 S_2 R_1 + S_3 U_0' B_1) (B_1^\dagger U_0'^\dagger S_3^\dagger + R_1^\dagger S_2^\dagger B_3^\dagger) + B_3 B_2 B_2^\dagger B_3^\dagger = 0$

(4) $A_2 A_1 (A_1^\dagger S_2^\dagger R_3^\dagger + S_1^\dagger U_0'^\dagger A_3^\dagger) + A_2 R_1 (R_1^\dagger S_2^\dagger R_3^\dagger + B_1^\dagger U_0'^\dagger A_3^\dagger) + R_2 B_2^\dagger R_3^\dagger = 0$

(5) $(R_3 S_2 A_1 + A_3 U_0' S_1) (A_1^\dagger S_2^\dagger B_3^\dagger + S_1^\dagger U_0'^\dagger S_3^\dagger) + (R_3 S_2 R_1 + A_3 U_0' B_1) (B_1^\dagger U_0'^\dagger S_3^\dagger + R_1^\dagger S_2^\dagger B_3^\dagger) + R_3 B_2 B_2^\dagger B_3^\dagger = 0$

(6) $(B_3 S_2 A_1 + S_3 U_0' S_1) A_2 A_2^\dagger + (B_3 S_2 R_1 + S_3 U_0' B_1) R_1^\dagger A_2^\dagger + B_3 B_2 R_2^\dagger = 0$; \hfill (A.1)

or the six unitary conditions for its $3 \times 3$ submatrices from $\mathcal{U}^\dagger \mathcal{U} = I_{9 \times 9}$:

(1) $U_0^\dagger A_2 A_2^\dagger U_0 + U_0^\dagger (A_1^\dagger S_2^\dagger R_3^\dagger + S_1^\dagger U_0'^\dagger A_3^\dagger) (R_3 S_2 A_1 + A_3 U_0' S_1) U_0 + U_0^\dagger (A_1^\dagger S_2^\dagger B_3^\dagger + S_1^\dagger U_0'^\dagger S_3^\dagger) (B_3 S_2 A_1 + S_3 U_0' S_1) U_0 = I$, 

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\[ (2) \quad R_1^t A_2^t A_2 R_1 + (R_1^t S_2^t R_3^t + B_3^t U_0'^t A_3^t) (R_3 S_2 R_1 + A_3 U_0'B_1) \\
+ (B_1^t U_0'^t S_1^t + R_1^t S_2^t B_3^t) (B_3 S_2 R_1 + S_3 U_0'B_1) = I , \\
(3) \quad R_2^t R_2 + B_2^t R_3^t R_2 B_2 + B_2^t B_3^t B_3 B_2 = I , \\
(4) \quad A_1^t A_2^t A_2 R_1 + (A_1^t S_2^t R_3^t + S_1^t U_0'^t A_3^t) (R_3 S_2 R_1 + A_3 U_0'B_1) \\
+ (A_1^t S_2^t B_3^t + S_1^t U_0'^t S_3^t) (B_3 S_2 R_1 + S_3 U_0'B_1) = 0 , \\
(5) \quad R_1^t A_2^t R_2 + (R_1^t S_2^t R_3^t + B_3^t U_0'^t A_3^t) R_3 B_2 + (B_1^t U_0'^t S_1^t + R_1 S_2^t B_3^t) B_3 B_2 = 0 , \\
(6) \quad R_2^t A_2^t A_1 + B_2^t R_3^t (R_3 S_2 A_1 + A_3 U_0'S_1) + B_2^t B_3^t (B_3 S_2 A_1 + S_3 U_0'S_1) = 0 . \quad \text{(A.2)} \]

Switching off the extra neutral fermion sector, for example, we arrive at \( A_2 = I \) and \( R_2 = 0 \). In this case the first relation in Eq. (A.1) can be simplified to \( A_1 A_1^t + R_1 R_1^t = I \), which is valid for the canonical seesaw mechanism. \[ \textbf{31, 33} \].

\section*{B \quad The approximate forms of \( A_{1,2} \), \( B_{1,2} \), \( R_{1,2} \) and \( S_{1,2} \)}

Given the very fact that the eighteen active-sterile flavor mixing angles \( \theta_{1j}, \theta_{2j} \) and \( \theta_{3j} \) (for \( j = 4, 5, \cdots, 9 \)) must be strongly suppressed in magnitude, one may simplify the four \( 3 \times 3 \) matrices \( A_1, B_1, R_1 \) and \( S_1 \) in Eqs. (10) and (11) to the following forms:

\[
A_1 \simeq I - \begin{pmatrix}
\frac{1}{2} (s_{14}^2 + s_{15}^2 + s_{16}^2) & 0 & 0 \\
\hat{s}_{14} \hat{s}_{24} + \hat{s}_{15} \hat{s}_{25} + \hat{s}_{16} \hat{s}_{26} & \frac{1}{2} (s_{24}^2 + s_{25}^2 + s_{26}^2) & 0 \\
\hat{s}_{14} \hat{s}_{34} + \hat{s}_{15} \hat{s}_{35} + \hat{s}_{16} \hat{s}_{36} & \hat{s}_{24} \hat{s}_{34} + \hat{s}_{25} \hat{s}_{35} + \hat{s}_{26} \hat{s}_{36} & \frac{1}{2} (s_{34}^2 + s_{35}^2 + s_{36}^2)
\end{pmatrix}
\]

\[
B_1 \simeq I - \begin{pmatrix}
\frac{1}{2} (s_{14}^2 + s_{24}^2 + s_{34}^2) & 0 & 0 \\
\hat{s}_{14} \hat{s}_{15} + \hat{s}_{24} \hat{s}_{25} + \hat{s}_{34} \hat{s}_{35} & \frac{1}{2} (s_{15}^2 + s_{25}^2 + s_{35}^2) & 0 \\
\hat{s}_{14} \hat{s}_{16} + \hat{s}_{24} \hat{s}_{26} + \hat{s}_{34} \hat{s}_{36} & \hat{s}_{15} \hat{s}_{16} + \hat{s}_{25} \hat{s}_{26} + \hat{s}_{35} \hat{s}_{36} & \frac{1}{2} (s_{16}^2 + s_{26}^2 + s_{36}^2)
\end{pmatrix}
\quad \text{(B.1)}
\]

where the terms of \( \mathcal{O}(s_{ij}^4) \) (for \( i = 1, 2, 3 \) and \( j = 4, 5, 6 \)) have been omitted; and

\[
S_1 \simeq -R_1^t \simeq - \begin{pmatrix}
\hat{s}_{14} & \hat{s}_{24} & \hat{s}_{34} \\
\hat{s}_{15} & \hat{s}_{25} & \hat{s}_{35} \\
\hat{s}_{16} & \hat{s}_{26} & \hat{s}_{36}
\end{pmatrix}
\quad \text{(B.2)}
\]

where the terms of \( \mathcal{O}(s_{ij}^3) \) (for \( i = 1, 2, 3 \) and \( j = 4, 5, 6 \)) have been omitted. Similarly, the four \( 3 \times 3 \) matrices \( A_2, B_2, R_2 \) and \( S_2 \) in Eqs. (12) and (13) can be simplified as follows:

\[
A_2 \simeq I - \begin{pmatrix}
\frac{1}{2} (s_{17}^2 + s_{18}^2 + s_{19}^2) & 0 & 0 \\
\hat{s}_{17} \hat{s}_{27} + \hat{s}_{18} \hat{s}_{28} + \hat{s}_{19} \hat{s}_{29} & \frac{1}{2} (s_{27}^2 + s_{28}^2 + s_{29}^2) & 0 \\
\hat{s}_{17} \hat{s}_{37} + \hat{s}_{18} \hat{s}_{38} + \hat{s}_{19} \hat{s}_{39} & \hat{s}_{27} \hat{s}_{37} + \hat{s}_{28} \hat{s}_{38} + \hat{s}_{29} \hat{s}_{39} & \frac{1}{2} (s_{37}^2 + s_{38}^2 + s_{39}^2)
\end{pmatrix}
\]

\[
B_2 \simeq I - \begin{pmatrix}
\frac{1}{2} (s_{17}^2 + s_{27}^2 + s_{37}^2) & 0 & 0 \\
\hat{s}_{17} \hat{s}_{18} + \hat{s}_{27} \hat{s}_{28} + \hat{s}_{37} \hat{s}_{38} & \frac{1}{2} (s_{18}^2 + s_{28}^2 + s_{38}^2) & 0 \\
\hat{s}_{17} \hat{s}_{19} + \hat{s}_{27} \hat{s}_{29} + \hat{s}_{37} \hat{s}_{39} & \hat{s}_{18} \hat{s}_{19} + \hat{s}_{28} \hat{s}_{29} + \hat{s}_{38} \hat{s}_{39} & \frac{1}{2} (s_{19}^2 + s_{29}^2 + s_{39}^2)
\end{pmatrix}
\quad \text{(B.3)}
\]
where the terms of $\mathcal{O}(s_{ij}^4)$ (for $i = 1, 2, 3$ and $j = 7, 8, 9$) have been omitted; and

$$S_2 \simeq -R_2^\dagger \simeq - \left( \begin{array}{ccc} \hat{s}_{17} & \hat{s}_{27} & \hat{s}_{37} \\
\hat{s}_{18} & \hat{s}_{28} & \hat{s}_{38} \\
\hat{s}_{19} & \hat{s}_{29} & \hat{s}_{39} \end{array} \right),$$  \hspace{1cm} (B.4)

where the terms of $\mathcal{O}(s_{ij}^3)$ (for $i = 1, 2, 3$ and $j = 7, 8, 9$) have been omitted.

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