Multi-agent coordination using nearest-neighbor rules: revisiting the Vicsek model

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Abstract

Recently, Jadbabaie, Lin, and Morse (2003) offered a mathematical analysis of the discrete time model of groups of mobile autonomous agents raised by Vicsek et al. in 1995. In their paper, Jadbabaie et al. showed that all agents shall move in the same heading, provided that these agents are periodically linked together. This paper sharpens this result by showing that coordination will be reached under a very weak condition that requires all agents are finally linked together. This condition is also strictly weaker than the one Jadbabaie et al. desired.

Index Terms—Decentralized control, multi-agent coordination, switched systems.

1 Introduction

Coordination of groups of mobile autonomous agents (particles [9], or boids [5]) has attracted researchers in a surprisingly wide variety of disciplines ranging from physics [9, 7, 8], to the biological sciences [10, 11], to computer science and engineering [5, 3, 6, 4].

This paper is mainly concerned with one particular discrete time model of groups of mobile autonomous agents, viz., the one proposed by Vicsek et al. [9]

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in 1995. In this model, a group of autonomous agents is moving in the plane with all agents moving at the same speed but with different headings. Each agent’s motion is updated using a local rule based on the average of its own heading and the heading of its “neighbors.” This is known as the nearest-neighbor rule in [3]. Agent \( i \)'s neighbors at time \( t \) are those agents that are either in or on a circle of pre-specified radius \( r \) centered at agent \( i \)'s current position. Known as the Vicsek model, this can be viewed as a special version of a model proposed by Reynolds [5] for simulating animal aggregation for the computer animation industry. Although the Vicsek model is very simple, simulation results in [9] show that, using the local update rule, all agents shall eventually move in the same direction despite the absence of centralized coordination, and that neighborhood of each agent will change.

Recently, Jadbabaie, Lin, and Morse [3] offered a mathematical analysis of this model and provided a theoretical explanation for the observed behavior. They adopt a more conservative approach, which ignores how the neighbor-graphs depend on the agent positions in the plane. Note that under this assumption, the Vicsek model is a graphic example of a switched linear system. Their goal in that paper was to determine for a certain large class of switching signals and for any initial set of agent headings that the headings of all agents will converge into the same steady heading.

Jadbabaie et al. [3] established sufficient conditions given in terms of neighbor-graphs for coordination of agents. One main result of [3] shows that all agents shall eventually move in the same heading if these graph are periodically jointly connected, i.e., the union of any \( T \) sequential graphs is connected for some fixed \( T \). This is a nice result, but as Jadbabaie et al. put it [3, p990, below Theorem 2], what one would prefer instead is to show that coordination would be reached eventually for every switching signal for which there is an infinite sequence of bounded, non-overlapping (but not necessarily contiguous) intervals across which the agents are linked together.

This paper will show above condition desired by Jadbabaie et al. is indeed a sufficient condition for asymptotic convergence. This, however, follows from a more general observation: our main result in this paper shows that convergence will be attained if these neighbor-graphs are finally jointly connected, i.e., the union of all graphs started from any time is connected.

The structure of this paper is as follows. In Section II we give a formal description of the Vicsek model in terms of switching signals. Section III provides the major results, for both leaderless coordination and leader-following coordination. Conclusions and future work are given in the last section.

2 The Vicsek model and the nearest-neighbor rule

In this section, we review some basic definitions concerning the Vicsek model.
The system studied by Vicsek et al. consists of \( n \) autonomous agents, e.g., particles, robots, etc., labeled 1 through \( n \). All agents move in the plane with the same speed but with different headings. The system operates at discrete time \( t = 0, 1, 2, \cdots \). Let \( r > 0 \) and \( v > 0 \) be given numbers associated with the system. The dynamics of agent \( i \) is described by the sequence \( \{x_i(t), y_i(t), \theta_i(t)\} \), where \( x_i(t), y_i(t) \in \mathbb{R} \) are the coordinates of the agent in the plane, and \( \theta_i(t) \) is its heading taking value from \([0, 2\pi)\). At any time \( t = 0, 1, 2, \cdots \), each agent’s heading is updated using a simple rule based on the average of its own heading plus the headings of its neighbors. For any two agents \( i, j \), we say \( j \) is a neighbor of \( i \) at time \( t \), written \( j \in \mathcal{N}_i(t) \), if \( d(i, j) \leq r \),

\[
d(i, j) = \sqrt{(x_i(t) - x_j(t))^2 + (y_i(t) - y_j(t))^2}.
\]

Then, agent \( i \)’s next heading is defined as

\[
\theta_i(t+1) = \frac{\theta_i(t) + \sum_{j \in \mathcal{N}_i(t)} \theta_j(t)}{1 + n_i(t)}
\]

where \( n_i(t) \) is the number of agent \( i \)’s neighbors at time \( t \). Agent \( i \)’s next coordinates are defined as

\[
x_i(t+1) = x(t) + v_i(t) \cos(\theta_i(t))
\]
\[
y_i(t+1) = y(t) + v_i(t) \sin(\theta_i(t)).
\]

For any time \( t \geq 0 \), we define the neighbor-graph of the system described above as the simple undirected graph \( \mathcal{G}(t) \) over \( V = \{1, 2, \cdots, n\} \) where the vertex \( i \) corresponds to agent \( i \) and two vertexes, \( i, j \), are connected by an edge in the graph if they are neighbors at time \( t \), i.e., if \( j \in \mathcal{N}_i(t) \). Since the neighbor relation can change over time, so can the graph that describes them. In the sequel, we write \( \mathcal{P} \) for the collection of simple undirected graphs over \( V \). A switching signal is a function \( \sigma : \mathbb{N} \to \mathcal{P} \) that assigns to each time \( t \) a neighbor-graph that specifies the neighbor relation between agents. Clearly, for a Vicsek model, the function that assigns to each time \( t \) the neighbor-graph \( \mathcal{G}(t) \) is a switching signal.

Note that for the Vicsek model, the neighbor-graph is determined by the initial positions and headings of all agents as well as the pre-specified \( r > 0 \) and \( v > 0 \). A complete description of the model would have to explain explicitly how \( \sigma \) changes over time. As it is difficult to take this into account in a convergence analysis, Jadbabaie et al. adopt a more conservative approach, “which ignores how \( \sigma \) depends on the agent positions in the plane and assumes instead that \( \sigma \) might be any switching signal in some suitably defined set of interests.”

We in this paper follow this basic assumption and formalize the Vicsek model as follows:

\footnote{In certain situation, choosing open neighborhood would give rise to more desirable results, see \cite{ref}.}
Definition 2.1 (Vicsek model). Given \( n \) agents, labeled \( 1, 2, \ldots, n \), moving in the plane at discrete time \( t \in \mathbb{N} \), let \( \mathcal{P} \) be all simple undirected graphs over \( V = \{1, 2, \ldots, n\} \). A Vicsek model is a pair \((V, \sigma)\), where \( \sigma : \mathbb{N} \to \mathcal{P} \) is a switching signal.

For each agent \( i \), define \( i \)'s \( \sigma \)-neighborhood at time \( t \), written \( N_i(t) \), to be the set of agents that is connected to \( i \) by an edge in graph \( \sigma(t) \). That is, agent \( j \) is a neighbor of agent \( i \) if and only if \((i, j)\) is an edge in graph \( \sigma(t) \).

Given an initial heading \( \theta(0) = \langle \theta_i(0) \rangle_{i=1}^n \), agent \( i \)'s heading \( \theta_i(t) \) evolves in discrete time according to Eq. 2. Namely, agent \( i \)'s heading at time \( t + 1 \) is the average of the headings of agent \( i \) and its neighbors at time \( t \).

Remark 2.1. This definition of a multi-agent coordination model is very general and more flexible. Several dimensions of extension/completion could be incorporated in this model: 1) We can choose either closed/open disk, a triangle-like zone or any subset of \( V \) as the neighborhood; 2) The velocity could also change in discrete time; 3) We could consider other state variables of agents besides their headings; and 4) The neighbor-graph could also be directed. This flexibility would be helpful in practical applications.

The goal of this paper is to show for a large class of Vicsek models (or switching signals) and for any initial set of agent headings that the headings of all \( n \) agents will converge into the same heading. Compared with the results obtained in [3], ours are more general.

3 A sufficient condition for multi-agent coordination

3.1 Notations and preliminaries

Suppose \((V, \sigma)\) is a Vicsek model. Following Savkin [6], we define a graph \( \sigma(\infty) \) over \( V = \{1, 2, \ldots, n\} \) as follows: for any two nodes \( i, j \), \((i, j)\) is an edge in \( \sigma(\infty) \) if and only if for any \( K > 0 \), there exists some \( k \geq K \) such that \((i, j)\) is an edge in graph \( \sigma(k) \). For convenience, given a collection of graphs \( \{G_x : x \in X\} \), we write \( \bigcup_{x \in X} G_x \) for the union of these graphs, i.e., any pair \((i, j)\) is an edge in \( \bigcup_{x \in X} G_x \) if and only if it is an edge in some \( G_x \). Then it is easy to show that there exists some \( K > 0 \) such that \( \sigma(\infty) = \bigcup_{k \geq K} \sigma(t) \) holds for all \( k \geq K \). In this paper we will show that all agents shall eventually move in the same heading provided that \( \sigma(\infty) \) is connected. This condition is more general than the one given in [3], where the authors require that the \( \sigma(t) \)'s are periodically jointly connected. In what follows, a switching signal \( \sigma \) is called finally jointly connected if \( \sigma(\infty) \) is connected. Clearly this is equivalent to saying that \( \bigcup_{k \geq K} \sigma(t) \) is connected for any \( k \in \mathbb{N} \).

For a sequence \( \{f(k)\} \) and a number \( u \) in \( \mathbb{R} \), we say \( u \) is an accumulation point of \( \{f(k)\} \) if there is a subsequence of \( \{f(k)\} \) that converges to \( u \). We write
Accu\{f(k)\} for the set of accumulation points of \{f(k)\}.

Given a Vicsek model \((V, \sigma)\) and an initial headings \(\theta(0) = \{\theta_i(0)\}_{i=1}^n\), we now fix some notations concerning the model.

For \(i = 1, 2, \cdots, n\), define
\[
\Theta_i = \{\theta_i(t) : t \in \mathbb{N}\} \quad (5)
\]
\[
m_i = \min\text{Accu}\Theta_i \quad (6)
\]
\[
M_i = \max\text{Accu}\Theta_i \quad (7)
\]
\[
m = \min_{i=1}^n m_i \quad (8)
\]
\[
M = \max_{i=1}^n M_i. \quad (9)
\]

Note that \(\Theta_i\) is a bounded set and, therefore, has minimum and maximum elements.

For any \(t \in \mathbb{N}\), define
\[
\underline{\theta}(t) = \min_{i=1}^n \theta_i(t) \quad (10)
\]
\[
\overline{\theta}(t) = \max_{i=1}^n \theta_i(t). \quad (11)
\]

The following lemma shows \(\underline{\theta}(t) \leq m \leq M \leq \overline{\theta}(t)\).

**Lemma 3.1.** For any \(t \in \mathbb{N}\), we have \(\underline{\theta}(t) \leq \theta(t+1) \leq m \leq \overline{\theta}(t+1) \leq \overline{\theta}(t)\). Consequently, we have \(\lim_{t \to \infty} \underline{\theta}(t) = m\) and \(\lim_{t \to \infty} \overline{\theta}(t) = M\).

**Proof.** For any non-negative \(t\), note that by Vicsek’s nearest-neighbor rule (Eq. 2), we have \(\underline{\theta}(t) \leq \underline{\theta}(t+1) \leq \overline{\theta}(t)\). In particular, we have \(\underline{\theta}(t) \leq \theta(t+1) \leq \overline{\theta}(t+1) \leq \overline{\theta}(t)\). Now since \(\{\underline{\theta}(t)\} \ (\{\overline{\theta}(t)\})\) is a bounded ascending (descending, resp.) sequence, it has a limit. We now show its limit is \(m\) (\(M\), resp.). Take \(\{\underline{\theta}(t)\}\) as an example. Since it is convergent, any subsequence of \(\{\underline{\theta}(t)\}\) also converges to its limit. Suppose \(\{f(k)\}\) is a sequence such that \(\lim_{k \to \infty} \theta_i(f(k)) = m\) for some agent \(i\). Note that \(\underline{\theta}(f(k)) \leq \theta_i(f(k))\) for any \(k\), we have \(\lim_{t \to \infty} \underline{\theta}(t) = \lim_{k \to \infty} \underline{\theta}(f(k)) \leq \lim_{k \to \infty} \theta_i(f(k)) = m\). On the other hand, since there exists some agent \(i\) such that \(\{t : \theta_i(t) = \underline{\theta}(t)\}\) is infinite, we have a sequence \(\{g(k)\}\) such that \(\theta_i(g(k)) = \underline{\theta}(g(k))\). This shows that \(\lim_{t \to \infty} \underline{\theta}(t) = \lim_{k \to \infty} \underline{\theta}(g(k)) = \lim_{k \to \infty} \theta_i(g(k)) \geq m\) since \(m\) is the minimum accumulation point. As a result, we have \(\lim_{t \to \infty} \underline{\theta}(t) = m\). Similarly, we can show \(\lim_{t \to \infty} \overline{\theta}(t) = M\). So we have \(\underline{\theta}(t) \leq m \leq M \leq \overline{\theta}(t)\) for any \(t\).

Note that as shown in the proof of the above lemma, we have a sequence, say \(\{f(k)\}\), such that \(\theta_i(f(k)) = \underline{\theta}(f(k))\) and \(\lim_{k \to \infty} \theta_i(f(k)) = m\) for some agent \(i\). Similarly, we have a sequence, say \(g(k)\), such that \(\lim_{k \to \infty} \theta_j(g(k)) = M\) for some \(j\).
3.2 Leaderless coordination

Theorem 3.1. Given a Vicsek model \((V, \sigma)\), suppose \(\sigma(\infty)\) is connected. Then for any \(\theta(0) = \langle \theta_i(0) \rangle_{i=1}^n\), we have

\[
\lim_{t \to \infty} \theta_i(t) = \theta_{ss} \quad (i = 1, 2, \cdots, n)
\]

where \(\theta_{ss}\) is a number depending only on \(\theta(0)\) and \(\sigma\).

To prove this theorem, we need several lemmas.

Recall \(V = \{1, 2, \cdots, n\}\). For a graph \(G\) over \(V\) and any two disjoint subsets \(A, B\) of \(V\), we say \(A\) and \(B\) are connected if there exist \(a \in A, b \in B\) such that \((a, b)\) is an edge in \(G\). If \(A\) happens to be a singleton \(\{a\}\), we also say node \(a\) is connected to \(B\). In this case, we say alternatively \(a\) has a neighbor in \(B\).

The following lemma suggests that, if the agents are divided into two parts such that the maximum heading of the first part is sufficiently smaller than the minimum of the second part, then, after updating the headings using Eq. 2, the agents will also form two parts such that one part is still sufficiently smaller than the rest.

For \(a < b\) in \(\mathbb{R}\) and any natural number \(t\), we write \(V_t(a, b) = \{i \in V : a < \theta_i(t) < b\}\).

Lemma 3.2. Given \(\alpha < \beta < \gamma\) and set \(\delta = \beta - \alpha, \epsilon = \delta/n^n\), suppose \(V_t(\alpha - \epsilon, \alpha + \epsilon)\) and \(V_t(\beta - \epsilon, \gamma + \epsilon)\) are two nonempty disjoint subsets of \(V\) such that their union is \(V\). We have

1. If \(V_t(\alpha - \epsilon, \alpha + \epsilon)\) and \(V_t(\beta - \epsilon, \gamma + \epsilon)\) are disconnected at time \(t\), then \(V_{t+1}(\alpha - \epsilon, \alpha + \epsilon) = V_t(\alpha - \epsilon, \alpha + \epsilon)\) and \(V_{t+1}(\beta - \epsilon, \gamma + \epsilon) = V_t(\beta - \epsilon, \gamma + \epsilon)\).

2. If \(V_t(\alpha - \epsilon, \alpha + \epsilon)\) and \(V_t(\beta - \epsilon, \gamma + \epsilon)\) are connected at time \(t\), then \(V_{t+1}(\alpha - \epsilon, \alpha + \epsilon) = V_t(\alpha - \epsilon, \alpha + \epsilon) - \{i \in V : i\) has a neighbor in \(V_t(\beta - \epsilon, \gamma + \epsilon)\) at time \(t\}\)

\(\)and \(V - V_{t+1}(\alpha - \epsilon, \alpha + \epsilon) = V_{t+1}(\alpha + \delta/n^n - \epsilon, \gamma + \epsilon)\).

Proof. If \(V_t(\alpha - \epsilon, \alpha + \epsilon)\) and \(V_t(\beta - \epsilon, \gamma + \epsilon)\) are disconnected at time \(t\), then for any \(i \in V_t(\alpha - \epsilon, \alpha + \epsilon)\), its neighbors are all in \(V_t(\alpha - \epsilon, \alpha + \epsilon)\). By Eq. 2 we have \(\alpha - \epsilon < \theta_i(t + 1) < \alpha + \epsilon\). Similarly, for any \(j \in V_t(\beta - \epsilon, \gamma + \epsilon)\), we have \(\beta - \epsilon < \theta_j(t + 1) < \gamma + \epsilon\).

On the other hand, suppose \(V_t(\alpha - \epsilon, \alpha + \epsilon)\) and \(V_t(\beta - \epsilon, \gamma + \epsilon)\) are connected at time \(t\). For \(i \in V_t(\alpha - \epsilon, \alpha + \epsilon)\), if all its neighbors are in \(V_t(\alpha - \epsilon, \alpha + \epsilon)\), then
\( i \in V_{t+1}(\alpha - \epsilon, \alpha + \epsilon) \); if \( i \) has a neighbor, say \( j_0 \), in \( V_t(\beta - \epsilon, \gamma + \epsilon) \), then we have

\[
\theta_i(t + 1) = \frac{\theta_i(t) + \sum_{j \in \mathcal{N}_i(t)} \theta_j(t)}{1 + n_i(t)} = \frac{\theta_i(t) + \theta_{j_0}(t) + \sum_{j \in \mathcal{N}_i(t), j \neq j_0} \theta_j(t)}{1 + n_i(t)} > \frac{(\alpha - \epsilon) + (\beta - \epsilon) + (n_i(t) - 1) \times (\alpha - \epsilon)}{1 + n_i(t)} = \frac{\beta - \alpha + (1 + n_i(t)) \times (\alpha - \epsilon)}{1 + n_i(t)}
\]

\[\geq \frac{\alpha - \epsilon + \delta/n}{1 + n_i(t)} \]

Note that \( \theta_i(t+1) < \gamma + \epsilon \) holds for any \( i \in V \). This shows that, if \( i \in V_t(\alpha - \epsilon, \alpha + \epsilon) \) has a neighbor in \( V_t(\beta - \epsilon, \gamma + \epsilon) \), then \( i \in V_{t+1}(\alpha + \delta/n - \epsilon, \gamma + \epsilon) \). Similarly, for any \( j \in V_t(\beta - \epsilon, \gamma + \epsilon) \), we can show \( \theta_j(t + 1) > \alpha + \delta/n - \epsilon \). In summary, we have \( i \in V_{t+1}(\alpha - \epsilon, \alpha + \epsilon) \) if and only if \( i \in V_t(\alpha - \epsilon, \alpha + \epsilon) \) and it has a neighbor in \( V_t(\beta - \epsilon, \gamma + \epsilon) \) at time \( t \). As for any other agent \( j \), we have \( j \in V_{t+1}(\alpha + \delta/n - \epsilon, \gamma + \epsilon) \). \( \square \)

**Lemma 3.3.** Suppose \( \sigma(\infty) \) is connected and \( \{f(k)\} \) is a sequence. Then we have a subsequence \( \{g(k)\} \) of \( \{f(k)\} \) such that all \( \{\theta_i(g(k))\} \) are convergent for \( i \in V \).

**Proof.** This follows from the compactness of \([0, 2\pi]\) and that \( \theta_i(t) \in [0, 2\pi] \) for any \( i, t \). \( \square \)

**Lemma 3.4.** Suppose \( \{g(k)\} \) is a sequence such that \( \{\theta_i(g(k))\} \) converges to \( l_i \) for \( i = 1, 2, \ldots, n \). Then \( m = \min_{i=1}^n l_i \) and \( M = \max_{i=1}^n l_i \), where \( m = \lim_{t \to \infty} \theta(t) \), \( M = \lim_{t \to \infty} \tilde{\theta}(t) \) and \( \tilde{\theta}(t) = \min_{i=1}^n \theta_i(t) \), \( \theta(t) = \max_{i=1}^n \theta_i(t) \).

**Proof.** Take \( m = \min_{i=1}^n l_i \) as an example. Note that there exists some \( i \) such that \( \{k : \theta_i(g(k)) = \theta(g(k))\} \) is infinite. We have a subsequence \( \{h(k)\} \) of \( \{g(k)\} \) such that \( \theta_i(h(k)) = \tilde{\theta}(h(k)) \) and \( l_i = \lim_{k \to \infty} \theta_i(h(k)) = \lim_{k \to \infty} \tilde{\theta}(h(k)) = m \). That \( M = \max_{i=1}^n l_i \) is similar. \( \square \)

**Proof of Theorem** Suppose \( m < M \) and \( \{g(k)\} \) is a sequence such that \( \{\theta_i(g(k))\} \) converges to \( l_i \) for \( i = 1, 2, \ldots, n \). Recall by Lemma 3.4 that \( m = \min_{i=1}^n l_i \) and \( M = \max_{i=1}^n l_i \). Set \( l = \min\{l_i : l_i > m\} \) and take \( \delta = l - m, \epsilon = \delta/n^n \). Then there exists \( K > 0 \) such that \( \theta_i(g(k)) \in (l_i - \epsilon, l_i + \epsilon) \) for \( k \geq K \) and \( i = 1, 2, \ldots, n \). Moreover, we have \( V_{g(k)}(m - \epsilon, m + \epsilon) = \{i \in V : l_i = m\} \) and \( V_{g(k)}(l - \epsilon, M + \epsilon) = \{i \in V : l_i \geq l\} \). Clearly \( V_{g(k)}(m - \epsilon, m + \epsilon) \) and \( V_{g(k)}(l - \epsilon, M + \epsilon) \) satisfy the condition of Lemma 3.2.
Now since $\bigcup_{t \geq p} \sigma(t)$ is connected for any $p \in \mathbb{N}$, we have some $k \geq K$ such that $V_{g(k)}(m - \epsilon, m + \epsilon)$ and $V_{g(k)}(l - \epsilon, M + \epsilon)$ are connected at time $g(k) + w$ for some $0 \leq w < g(k + 1) - g(k)$. Fix one such $k$ and suppose $g(k + 1) - g(k) = W$. For each $w = 0, 1, \cdots, W$, we define $A_w = V_{g(k) + w}(m - \epsilon, m + \epsilon)$ and $B_w = V - A_w$.

Note that because $A_W = V_{g(k+1)}(m - \epsilon, m + \epsilon) = V_{g(k)}(m - \epsilon, m + \epsilon) = A_0$, it is also true that $m - \epsilon < \theta(g(k)) \leq \theta(g(k + 1)) < m + \epsilon$. Recall that because $\theta(t)$ is an ascending chain (see Lemma 3.2), we also have $\theta(g(k) + w) \in (m - \epsilon, m + \epsilon)$ for any $w = 1, 2, \cdots, W - 1$.

Set $C = \{w \in [0, W) : A_w$ and $B_w$ are connected at time $g(k) + w\}$. Clearly $C$ is not empty since there exists some $w$ such that $A_0$ is connected to $B_0$ at time $g(k) + w$. Suppose $C = \{w_1, w_2, \cdots, w_q\}$ and $0 \leq w_1 < w_2 < \cdots < w_q < W$. We claim

$$A_0 \supseteq A_{w_1 + 1} \supseteq A_{w_2 + 1} \supseteq \cdots \supseteq A_{w_{q-1} + 1} \supseteq A_{w_q + 1} \quad (13)$$

$$B_{w_1 + 1} = V_{g(k) + w_1 + 1}(m + \frac{\delta}{n^s} - \epsilon, M + \epsilon) \quad (s = 1, 2, \cdots, q) \quad (14)$$

As the induction basis, note that $A_0 = V_{g(k)}(m - \epsilon, m + \epsilon) = \{i : l_i = m\}$ and $B_0 = \{i : l_i \geq l\} = V_{g(k)}(l - \epsilon, M + \epsilon) = V_{g(k)}(m + \delta - \epsilon, M + \epsilon)$.

Note that $w_1$ is the first index $w$ such that $A_w$ is connected to $B_w$; by Lemma 3.2 we have $A_0 = A_w$ and $B_w = V_{g(k) + w}(m + \delta - \epsilon, M + \epsilon)$ for any $w \leq w_1$. Moreover, since $A_{w_1} = A_0$ is connected to $B_{w_1} = B_0$ at time $g(k) + w_1$, by Lemma 3.2 we have

$$A_{w_1 + 1} = A_{w_1} - \{i \in A_{w_1} : i \text{ has a neighbor in } B_{w_1} \text{ at time } g(k) + w_1\} \subset A_0 \quad (15)$$

$$B_{w_1 + 1} = V - A_{w_1 + 1} = V_{g(k) + w_1 + 1}(m + \delta/n - \epsilon, M + \epsilon) \quad (16)$$

Recall that $A_{w_1 + 1} \neq \emptyset$ since $\theta(g(k) + w_1 + 1) \in (m - \epsilon, m + \epsilon)$.

Suppose for $s < q$ we have

$$A_0 \supseteq A_{w_1 + 1} \supseteq A_{w_2 + 1} \supseteq \cdots \supseteq A_{w_{s-1} + 1} \supseteq A_{w_s + 1} \quad (17)$$

$$B_{w_s + 1} = V_{g(k) + w_s + 1}(m + \frac{\delta}{n^j} - \epsilon, M + \epsilon) \quad (j = 1, 2, \cdots, s) \quad (18)$$

Note that $s < n - 1$ must hold since $A_0$ contains at most $n - 1$ agents and $A_0 \supseteq A_{w_1 + 1} \supseteq \cdots \supseteq A_{w_s + 1} \neq \emptyset$.

We now show $A_{w_{s-1} + 1} \supseteq A_{w_{s+1} + 1}$ and $B_{w_{s+1} + 1} = V_{g(k) + w_{s+1} + 1}(m + \frac{\delta}{n^{s+1}} - \epsilon, M + \epsilon)$.

Note that $w_s + 1$ is the first index $w > w_s$ such that $A_w$ is connected to $B_w$. By Lemma 3.2 we have $A_w = A_{w_s + 1}$ and $B_w = V_{g(k) + w}(m + \frac{\delta}{n^s} - \epsilon, M + \epsilon) = B_{w_s + 1}$ for any $w \in (w_s, w_{s+1}]$. Moreover, since $A_{w_{s+1}} = A_{w_{s+1} + 1}$ is connected to $B_{w_{s+1}} = B_{w_{s+1}}$ at time $g(k) + w_{s+1}$, by Lemma 3.2 we have

$$A_{w_{s+1} + 1} = A_{w_{s+1}} - \{i \in A_{w_{s+1}} : i \text{ has a neighbor in } B_{w_{s+1}} \text{ at time } g(k) + w_{s+1}\} \supseteq A_{w_{s+1}} \quad (19)$$

$$B_{w_{s+1} + 1} = V - A_{w_{s+1} + 1} = V_{g(k) + w_{s+1} + 1}(m + \frac{\delta}{n^{s+1}} - \epsilon, M + \epsilon) \quad (20)$$
In summary, we have obtained that \( A_0 = A_{w_1} \supseteq A_{w_q+1} \).

Note that if \( w_q < W - 1 \), then \( A_w \) and \( B_w \) are disconnected for any \( w \in (w_q, W) \). By Lemma 3.2 again, we know \( A_w = A_W \) for \( w \in (w_q, W] \). In particular, we have \( A_{w_q+1} = A_W \). On the other hand, if \( w_q = W - 1 \), we also have \( A_{w_q+1} = A_W \).

This suggests that if \( m < M \), then \( A_0 \neq A_W \). This is a contradiction. So our assumption that \( m < M \) cannot hold. This ends the proof of this theorem. \( \square \)

\textbf{Remark 3.1.} Note that if \( \sigma : \mathbb{N} \rightarrow \mathcal{P} \) is a switching signal for which there exists an infinite sequence of bounded, non-overlapping (but not necessarily contiguous) intervals across which the \( n \) agents are linked together, then \( \sigma(\infty) \) is connected. By the above theorem, we know all agents would eventually move in the same heading for this \( \sigma \). Consequently, this theorem shows the desired condition given in [3, p990, below Theorem 2] is a sufficient condition for asymptotic convergence.

The hypothesis of Theorem 3.1 however, is still not necessary. For example, if some \( \sigma(t) \) is the complete graph over \( V \), then a coordination could be achieved at time \( t + 1 \). But if it is not connected, \( \sigma(\infty) \) will have \( 1 < p \leq n \) connected components, say \( G_1, G_2, \cdots, G_p \). Similar to the argument given above for Theorem 3.1, we can show for any \( h = 1, 2, \cdots, p \), there exists a heading \( \hat{\theta}_h \) such that \( \lim_{t \to \infty} \theta_i(t) = \hat{\theta}_h \) for any \( i \in G_h \).

### 3.3 Leader-following coordination

In [3], Jadbabaie et al. also consider a modified version of Vicsek’s discrete-time system, which consists of the same group of \( n \) agents as before except that one leader agent, labeled 0, is added. Agent 0 moves at the same constant speed as its \( n \) followers but with a fixed heading \( \theta_0 \). Agent \( i \) then updates its heading using the average of its own heading plus the headings of its neighbors. Note that this time the leader may be in its neighborhood.

Our abstract Vicsek model with a leader now can be formulated as follows:

\textbf{Definition 3.1.} Suppose \( V^+ = \{0, 1, \cdots, n\} \) and \( \mathcal{P}^+ \) is the collection of simple undirected graphs over \( V^+ \). A leader-following Vicsek model is just a pair \((V^+, \sigma)\), where \( \sigma : \mathbb{N} \rightarrow \mathcal{P}^+ \) is a switching signal.

For each agent \( i > 0 \), define \( i \)'s \( \sigma \)-neighborhood at time \( t \), written \( \mathcal{N}_i(t) \), to be the set of agents that are connected to \( i \) by an edge in the graph \( \sigma(t) \). That is, agent \( j \) is a neighbor of agent \( i \) if and only if \((i, j)\) is an edge in the graph \( \sigma(t) \).

Given an initial heading \( \theta(0) = (\theta_i(0))_{i=1}^n \) and a fixed heading \( \theta_0 \) in which agent 0 moves at all times, for \( i > 0 \), agent \( i \)'s heading evolves in discrete time according to the following equation:

\[
\theta_i(t + 1) = \frac{\theta_i(t) + \sum_{j \in \mathcal{N}_i(t)} \theta_j(t)}{1 + n_i(t)}
\]

(21)

where \( n_i(t) \) is the number of agents in \( \mathcal{N}_i(t) \).
For a leader-following Vicsek model, we have the following correspondence of Theorem 3.1.

**Theorem 3.2.** Given a leader-following Vicsek model \((V^+, \sigma)\), suppose \(\sigma(\infty)\) is connected. Then for any \(\theta(0) = (\theta_i(0))_{i=1}^n\) and \(\theta_0\), we have \(\lim_{t \to \infty} \theta_i(t) = \theta_0\) for all \(i = 1, 2, \ldots, n\).

**Proof.** Note that Lemma 3.2 and Lemma 3.3 still hold for the leader-following case; this theorem follows from a similar argument as given for Theorem 3.1. \(\square\)

### 4 Conclusions and further work

In [3], Jadbabaie *et al.* show that all agents shall move in the same heading if the neighbor-graphs are *periodically jointly connected*, i.e., the union of any \(T\) contiguous neighbor-graphs is connected for some fixed \(T\). In the same paper, they also ask whether this still holds when there exists a sequence of uniformly bounded intervals over the discrete time such that the union of neighbor-graphs across each interval is connected?

This paper has shown that all agents shall move in the same heading under a very weak condition that requires the neighbor-graphs to be *finally jointly connected*, i.e., the union of all graphs started from any time is connected. This result gives an affirmative answer to the question raised in [3].

It should be emphasized that results obtained in this paper are valid for many versions of the Vicsek model (or coordination multi-agent models that use nearest-neighbor rule to update their state) (see Remark 2.1). As for some specific versions of the Vicsek model, there have been some impressive results. Recently, Jadbabaie [2] has shown that, if we choose the neighborhood region to be open, then a necessary and sufficient condition for all headings to converge to the same heading is that the neighbor-graph does not change after finite steps and is connected.\(^2\) Jadbabaie also notes that [2, p.8, last paragraph] the problem would be more complicated if a closed neighborhood region were chosen. It seems this method cannot be directly applied to other kinds of neighborhoods. As a matter of fact, there are often situations when agents do not have disk-like visibility but rather might have a cone-like field of view.\(^3\)

Note that our results are based on the assumption that the switching signal is pre-specified. In our future work, we shall plan to develop a model that can explain how the neighbor-graphs evolve over discrete time, and determine sufficient conditions for coordination of multi-agents in terms of these agents’ initial states. Another thing that should be stressed is that coordination results obtained in ours and in [3] are all asymptotic. It will be interesting to devise other local updating rules such that, using these rules, coordination will be reached quickly and without centralized control. This work is currently being undertaken.

\(^2\)A similar result was obtained by Savkin [6] for a simple version of Vicsek model where the headings \(\theta_i(t)\)'s take their values from a certain finite set.

\(^3\)Such a situation has been investigated (though for continuous time) by Lin *et al.* [4].
References

[1] G. Flierl, D. Grunbaum, S. Levin, and D. Olson. From individuals to aggregations: The interplay between behavior and physics. *Journal of Theoretical Biology*, 196(4):397–454, 1999.

[2] A. Jadababaie. On distributed coordination of mobile agents with changing nearest neighbors. Technical Report, University of Pennsylvania, Philadelphia, PA, 2003.

[3] A. Jadababaie, J. Lin, and A.S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, 48(6):988–1001, 2003.

[4] Z. Lin, M. Broucke, and B. Francis. Local control strategies for groups of mobile autonomous agents. *IEEE Transactions on Automatic Control*, 49(4):622–628, 2004.

[5] C.W. Reynolds. Flocks, birds, and schools: A distributed behavioral model. *Computer Graphics*, 21:25–34, 1987.

[6] A. Savkin. Coordinated collective motion of groups of autonomous mobile robots: Analysis of vicsek’s model. *IEEE Transactions on Automatic Control*, 49(6):981–983, 2004.

[7] J. Toner and Y.H. Tu. Long range order in a two dimensional $xy$ model: How birds fly together? *Physical Review Letters*, 75:4326–4329, 1995.

[8] J. Toner and Y.H. Tu. Flocks, herds, and schools: A quantitative theory of flocking. *Physical Review E*, 58(4):4828–4858, 1998.

[9] T. Vicsek, A. Czirok, E. Ben Jacob, I. Cohen, and O. Schochet. Novel type of phaze transitions in a system of self-driven particles. *Physical Review Letters*, 75:1226–1229, 1995.

[10] K. Warburton and J. Lazarus. Tendency distance models of social cohesion in animal groups. *Journal of Theoretical Biology*, 150(4):473–488, 1991.