ON NEARLY PARALLEL $G_2$-MANIFOLDS

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Abstract. In contrast to simply connected compact nearly Kähler manifolds, which are always formal, the simply connected compact Sasaki–Einstein 7-manifold $Q(1,1,1) = (SU(2) \times SU(2) \times SU(2))/(U(1) \times U(1))$ is non-formal. In this paper we study the formality of nearly parallel $G_2$-manifolds, showing that the Berger space and the Aloff–Wallach spaces are both formal. Moreover, we obtain new examples of non-formal simply connected compact Sasaki–Einstein 7-manifolds. We construct minimal associative 3-folds in the Aloff–Wallach spaces and in any regular Sasaki–Einstein 7-manifold; in particular, in the space $Q(1,1,1)$ with the natural $S^1$-family of nearly parallel $G_2$-structures induced by the Sasaki–Einstein structure. We find also examples of formal manifolds and non-formal manifolds, which are locally conformal parallel Spin(7)-manifolds.

1. Introduction

A nearly parallel $G_2$-structure on a 7-dimensional manifold $P$ is given by a positive 3-form $\varphi$ satisfying the differential equation $d\varphi = \tau_0 \ast \varphi$, for some non-zero constant $\tau_0$, where $\ast$ is the Hodge star operator determined by the induced metric by $\varphi$. Nearly parallel $G_2$-manifolds were introduced as manifolds with weak holonomy $G_2$ by Gray in [33].

In [9], it was shown that there is a 1-1 correspondence between nearly parallel $G_2$-manifolds and Killing spinors. As a consequence, the metric $g$ induced by the 3-form $\varphi$ has to be Einstein with positive scalar curvature $\text{scal}(g) = \frac{21}{8} \tau_0^2$. Thus, if $(P, g)$ is complete, then $P$ is a compact manifold with finite fundamental group due to Myers’ theorem.

Another equivalent description of nearly parallel $G_2$-structures is in terms of the metric cone $(\tilde{P} = P \times \mathbb{R}^+, g_c = r^2 g + dr^2)$, which has to have holonomy contained in Spin(7), viewed as subgroup of SO(8). If $(P, g)$ is simply connected and not isometric to the standard sphere, then there are three possible cases: the holonomy of $(\tilde{P}, g_c)$ is $\text{Sp}(2)$ or, equivalently, $(P, g)$ is a 3-Sasakian manifold; the holonomy of $(\tilde{P}, g_c)$ can be $\text{SU}(4)$ if and only if $(P, g)$ is a Sasaki–Einstein manifold; or the holonomy of...
(\(\widehat{P}, g^c\)) coincides with Spin(7), in which case the \(G_2\)-structure is called proper. We recall that these three cases correspond to the existence of a space of Killing spinors of dimension 3, 2 or 1, respectively (see [29]). In particular, by [11] a 7-dimensional Sasaki–Einstein manifold has a canonical \(S^1\)-family of nearly parallel \(G_2\)-structures inducing the Sasaki–Einstein metric. Moreover, by [30] a 7-dimensional 3-Sasakian manifold carries a second Einstein metric induced by a proper nearly parallel \(G_2\)-structure (see Section 2 for details).

The torsion-free Spin(7)-structure on the metric cone \((\widehat{P}, g^c)\) gives rise to a canonical closed 4-form which defines a calibration. A 3-dimensional submanifold \(Y\) in \((P, \varphi)\) is called associative if \(\varphi|_X = \vol_X\), or equivalently if its cone \(\widehat{Y}\) in \(\widehat{P}\) is calibrated by the 4-form (see [39] Lemma 2.10). Associative submanifolds in nearly parallel \(G_2\)-manifolds \(P\) are minimal and their infinitesimal deformations were considered by Kawai in [39]. Associative 3-folds have been studied by Lotay [43] when \(P\) is the standard 7-sphere, by Kawai [38] when \(P\) is the squashed 7-sphere, and by Ball and Madnick when \(P\) is the Berger space [6], the squashed 7-sphere and the squashed exceptional Aloff–Wallach space \(W_{1,1}\) [7]. However, nothing seems to be known for other nearly-parallel \(G_2\)-manifolds.

Nearly parallel \(G_2\)-manifolds are in various ways rather similar to nearly Kähler manifolds, which in dimension 6 are also Einstein manifolds admitting a Killing spinor. Moreover, the metric cone of a nearly Kähler 6-manifold has holonomy contained in \(G_2\). By [3], simply connected compact nearly Kähler manifolds are formal. In contrast to the nearly Kähler case, an example of a non-formal simply connected compact nearly parallel \(G_2\)-manifold is given by the simply connected compact Sasaki–Einstein manifold \(Q(1, 1, 1) = (SU(2) \times SU(2) \times SU(2))/(U(1) \times U(1))\) (see [12], also Theorem 4.3).

Formality of simply connected compact 3-Sasakian 7-manifolds was previously considered in [25]. In this paper we study the formality of nearly-parallel \(G_2\)-manifolds, proving that the Berger space and the Aloff–Wallach spaces are both formal. Moreover, we obtain new examples of non-formal simply connected compact Sasaki–Einstein 7-manifolds, constructed as total spaces of circle bundles over the manifold \(X_k = P_k \times S^2\), where 3 \(\leq k \leq 8\) and \(P_k = \mathbb{C}P^2 \# k\bar{\mathbb{C}P}^2\) is a del Pezzo surface.

The compact Sasaki–Einstein homogeneous space \(Q(1, 1, 1)\) is the total space of a principal \(S^1\)-bundle over \(S^2 \times S^2 \times S^2\) with Euler class the Kähler form on \(S^2 \times S^2 \times S^2\). We first make explicit the canonical \(S^1\)-family \(\varphi_t\) of nearly parallel \(G_2\)-structures associated to the Sasaki–Einstein structure on \(Q(1, 1, 1)\). Then, considering the Calabi–Yau cone over \(Q(1, 1, 1)\), we construct minimal associative 3-folds in \(Q(1, 1, 1)\), and we determine a family of non-trivial associative deformations. We also study minimal associative 3-folds in any 7-dimensional regular Sasaki–Einstein manifold as well as in the Aloff–Wallach spaces.
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Given a nearly parallel G\(_2\)-manifold \(P\), the product manifold \(P \times S^1\) or, more generally, the mapping torus of \(P\) by a diffeomorphism preserving the nearly parallel G\(_2\)-structure, carries a natural locally conformal parallel Spin(7)-structure. Applying the mapping torus construction to the nearly-parallel G\(_2\)-manifolds \((Q(1,1,1), \varphi_t)\), the Berger space and the Aloff–Wallach spaces we construct examples of formal and non-formal locally conformal parallel Spin(7)-manifolds.

The paper is organized as follows. In Section 2, we review some definitions and results about nearly parallel G\(_2\)-structures and associative 3-folds. In particular, we recall the general construction of the canonical \(S^1\)-family of nearly parallel G\(_2\)-structures associated to a Sasaki–Einstein structure on a 7-manifold. In Section 3, we review concepts about minimal models and formality. Then, in Section 4 and Section 5 we examine the formality of nearly parallel G\(_2\)-manifolds. In Section 6, we apply the construction of the canonical \(S^1\)-family of nearly parallel G\(_2\)-structures to the Sasaki–Einstein manifold \(Q(1,1,1)\). Section 7 is devoted to the construction of minimal associative 3-folds in 7-dimensional regular Sasaki–Einstein manifolds and, in particular, in \(Q(1,1,1)\). Deformations of minimal associative 3-folds in any 7-dimensional regular Sasaki–Einstein manifold are also considered in Section 7. In Section 8, we study the associative 3-dimensional submanifolds of the Aloff–Wallach spaces. Finally, Section 9 contains the construction of locally conformal parallel Spin(7)-manifolds as mapping tori of each of the following spaces: \((Q(1,1,1), \varphi_t)\), the Berger space and the Aloff–Wallach spaces.

2. NEARLY PARALLEL G\(_2\)-STRUCTURES

In this section, we review some definitions and results about associative 3-folds and coassociative 4-folds in a nearly parallel G\(_2\)-manifold. Moreover, we recall the canonical \(S^1\)-family of nearly parallel G\(_2\)-structures that exists on any 7-dimensional Sasaki–Einstein manifold \([1, 14, 28]\), and the proper nearly parallel G\(_2\)-structure on any 7-dimensional 3-Sasakian manifold \([29]\).

A 7-dimensional manifold \(P\) has a G\(_2\)-structure if there is a reduction of the structure group of its frame bundle from GL(7, \(\mathbb{R}\)) to the exceptional Lie group G\(_2\). Gray in \([32]\) proved that a smooth 7-manifold \(P\) carries G\(_2\)-structures if and only if \(P\) is orientable and spin.

The presence of a G\(_2\)-structure is equivalent to the existence of a differential 3-form \(\varphi\) (the G\(_2\)-form) on \(P\), which can be described locally as

\[
\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245},
\]

with respect to some basis \(\{e^1, \ldots, e^7\}\) of the (local) 1-forms on \(P\). Here, \(e^{127}\) stands for \(e^1 \wedge e^2 \wedge e^7\), and so on.
Since $G_2 \subseteq \text{SO}(7)$, a $G_2$-structure $\varphi$ on $P$ determines a Riemannian metric $g = g_\varphi$ and an orientation on $P$ such that
\[ g(U, V) \text{ vol} = \frac{1}{6} (U \lrcorner \varphi) \wedge (V \lrcorner \varphi) \wedge \varphi, \tag{2.1} \]
for any vector fields $U, V$ on $P$, where vol is the volume form on $P$.

Denote by $\Omega^*(P)$ the algebra of the differential forms on $P$. The first-order local invariants of $\varphi$ are completely encoded in the four differential forms $\tau_0 \in \Omega^0(P)$, $\tau_1 \in \Omega^1(P)$, $\tau_2 \in \Omega^2(P)$ and $\tau_3 \in \Omega^3(P)$, which are called the torsion forms of the $G_2$-structure $\varphi$, and they are defined by the equations
\[
\begin{cases}
    d\varphi = \tau_0 \star \varphi + 3 \tau_1 \wedge \varphi + \star \tau_3, \\
    d \star \varphi = 4 \tau_1 \wedge \star \varphi - \star \tau_2,
\end{cases}
\]
for any vector fields $U, V$ on $P$, where $\star$ is the Hodge star operator determined by the metric $g_\varphi$ and the volume form on $P$ induced by $\varphi$.

**Definition 2.1.** A $G_2$-structure $\varphi$ on a 7-manifold $P$ is said to be nearly parallel if there exists a non-zero real number $\tau_0$ such that
\[ d\varphi = \tau_0 \star \varphi. \tag{2.2} \]

A 7-manifold $P$ with a nearly parallel $G_2$-structure is called a nearly parallel $G_2$-manifold.

Clearly, if $\varphi$ is a nearly parallel $G_2$-structure on a 7-manifold $P$, then $\varphi$ is coclosed, i.e. $d \star \varphi = 0$, and the $G_2$-structure can be defined, in terms of the torsion forms, by the conditions $0 = \tau_1 = \tau_2 = \tau_3$.

### 2.1. Associative and coassociative submanifolds

From the view point of $G_2$-geometry, the most natural class of submanifolds in a nearly-parallel $G_2$-manifold are the associative submanifolds. Let $P$ be a 7-manifold equipped with a $G_2$-structure $\varphi \in \Omega^3(P)$. An oriented 3-dimensional submanifold $X \subset P$ is called an associative 3-fold if
\[ \varphi|_X = \text{vol}_X. \]

There are two cases that we must highlight. The first case is when $d\varphi = 0$, then $\varphi$ is a calibration [35], and hence associative 3-folds in $P$ are area-minimizing. The second case occurs if $\varphi$ is nearly parallel, that is $d\varphi = \tau_0 \star \varphi$, then associative 3-folds in $P$ are the links of Cayley cones in the metric cone over $P$, and hence are also minimal submanifolds of $P$. Even more, Ball and Madnick in [5, Theorem 2.18] prove that the largest torsion class of $G_2$-structures for which every associative 3-fold is minimal is the class for which the nearly parallel condition (2.2) holds.
If $\varphi \in \Omega^3(P)$ is a $G_2$-structure on a 7-manifold $P$, then an oriented 4-dimensional submanifold $Y \subset P$ is called coassociative 4-fold if

$$\star \varphi|_Y = \text{vol}_Y.$$  \hfill (2.3)

**Proposition 2.2** ([43, Lemma 3.2]). A nearly parallel $G_2$-manifold does not admit any coassociative 4-folds.

The argument in [43] uses the property that (2.3) implies $\varphi|_Y = 0$ which leads to a contradiction $\text{vol}_Y = \tau_0^{-1} d\varphi|_Y = 0$.

**Remark 2.3.** For closed submanifolds, the result of Proposition 2.2 is an instance of a more general fact: if a calibration on a manifold $M$ is given by an exact form $d\phi_M$, then there is no closed submanifold $N \subset M$ calibrated by $d\phi_M$. This follows at once by an application of Stokes’ theorem $\int_N \text{vol}_N = \int_N d\phi_M = 0$.

### 2.2. Sasaki–Einstein and 3-Sasakian manifolds.

Let us recall that an odd-dimensional Riemannian manifold $(S, g)$ is called Sasakian if its cone $(S \times \mathbb{R}^+, g^c = r^2 g + dr^2)$ is Kähler, that is the cone metric $g^c = r^2 g + dr^2$ admits a compatible integrable almost complex structure $J$ so that $(S \times \mathbb{R}^+, g^c = r^2 g + dr^2, J)$ is a Kähler manifold. In this case the Reeb vector field $\xi = J\partial_r$ on $S$ is a Killing vector field of unit length. The corresponding 1-form $\eta$ defined by $\eta(U) = g(\xi, U)$, for any vector field $U$ on $S$, is a contact form, meaning $\eta \wedge (d\eta)^n \neq 0$ at every point of $S$, where $\dim S = 2n + 1$.

The Kähler form on the cone can be expressed as

$$\omega^c = \frac{1}{2} d(r^2 \eta),$$  \hfill (2.4)

where we extended $\eta$ (and $\xi$) to $S \times \mathbb{R}^+$ by $\eta(U) = \frac{1}{r} g^c(J(r\partial_r), U)$. Let $\nabla$ be the Levi–Civita connection of $g$. The endomorphism $\Phi : TS \to TS$ of the tangent bundle $TS$ of $S$, given by $\Phi U = \nabla_U \xi$, satisfies the following identities

$$\Phi^2 = -\text{Id} + \eta \otimes \xi, \quad g(\Phi U, \Phi V) = g(U, V) - \eta(U)\eta(V), \quad d\eta(U, V) = 2g(\Phi U, V),$$  \hfill (2.5)

for any vector fields $U, V$ on $S$. If the integral curves of the Reeb vector field $\xi$ are closed, hence circles, then $\xi$ integrates to a locally free isometric action of $S^1$ on $(S, g)$. A Sasakian manifold is called regular when this latter $S^1$-action is free.

If $(S, g)$ is a Sasakian manifold, of dimension $2n + 1$, then $(S, g)$ is said to be Sasaki–Einstein if the cone metric $g^c = r^2 g + dr^2$ on $S \times \mathbb{R}^+$ is Kähler and Ricci-flat or, equivalently, the restricted holonomy group of $g^c$ is contained in $\text{SU}(n + 1)$. Equivalently, the Sasakian metric $g$ is Einstein with Einstein constant $2n$. If $(S, g)$ is complete, then it follows from Myers’ theorem that $S$ is compact with finite fundamental group.

Compact regular Sasakian manifolds, of dimension $2n + 1$, are principal circle bundles over compact Kähler manifolds whose Kähler form defines a cohomology class proportional to an integral cohomology class [15, 28, 40]. In the case of complete
regular Sasaki–Einstein manifolds, the base manifold is Kähler–Einstein with positive scalar curvature (therefore a simply connected projective algebraic variety, more precisely a Fano variety); the converse also holds. The following result is stated in [15] and attributed to Kobayashi and Hatakeyama.

**Theorem 2.4** ([15, cf. Theorem 2.8]). Let \((X, g_X)\) be a compact Kähler manifold whose Kähler form \(\omega_X\) defines an integral cohomology class. Let \(S\) be the total space of a principal circle bundle 
\[ S^1 \hookrightarrow S \xrightarrow{\pi} X \]
with Euler class \([\omega_X] \in H^2(X, \mathbb{Z})\) and let \(\eta\) be a connection 1-form on \(S\) with curvature \(d\eta = 2\pi^*\omega_X\). Then \(S\) with metric \(g = \pi^*g_X + \eta \otimes \eta\) is a (regular) Sasakian manifold, with contact form \(\eta\). If also the metric \(g_X\) is Einstein with positive scalar curvature, then \(g\) is Sasaki–Einstein.

Let \((S, g)\) be a Sasaki–Einstein manifold of dimension 7 with contact form \(\eta\). Then, according to [1, pp. 723–724], \(S\) has an \(S^1\)-family of nearly parallel \(G_2\)-structures \(\varphi_t\), which are given by
\[ \varphi_t = \Omega \wedge \eta + \cos t \Psi_+ + \sin t \Psi_- \]  
(2.6)
Here \(\Omega\) is the horizontal Kähler form related to the Ricci-flat Kähler form \(\omega^c\) on the cone \(S \times \mathbb{R}^+\) via \(\Omega \wedge \eta = \frac{1}{2} \partial_r \omega^c|_{r=1}\), equivalently \(\Omega = \pi^*\omega_X\) in the case when \((S, g)\) is regular. Further, \(\Psi = \Psi_+ + i\Psi_- = \partial_r \hat{\Psi}|_{r=1}\) is a horizontal complex volume form, where \(\hat{\Psi}\) denotes a holomorphic 4-form of unit length on \(S \times \mathbb{R}^+\). Now, a straightforward computation using (2.1) shows that every \(\varphi_t\) induces the Sasaki–Einstein metric \(g\) on \(S\).

**Remark 2.5.** Note that the expression for \(\varphi_t\) given in [1] (where the authors write \(\sigma_t\) instead of \(\varphi_t\)) is
\[ \varphi_t = -\Omega \wedge \eta + \cos t \Psi_+ + \sin t \Psi_- \]
with \(\eta\) the contact form of the Sasakian structure on \(S\). The change that we made of the first term on the right-hand side of (2.6) is due to the following. By (2.5), we have \(d\eta(U, V) = 2g(\Phi U, V)\), while in [1] the authors consider \(d\eta(U, V) = 2g(U, \Phi V)\), for all vector fields \(U, V\) on \(S\).

There is an important subclass of Sasaki–Einstein manifolds called 3-Sasakian manifolds. Let us recall that a 3-Sasakian structure is a collection of three Sasakian structures \((\varphi_i, \xi_i, \eta_i, g)\) on a \((4n + 3)\)-dimensional Riemannian manifold satisfying quaternionic-like identities. More precisely, a Riemannian manifold \((S, g)\) of dimension \(4n + 3\) is called 3-Sasakian if its cone \((S \times \mathbb{R}^+, g^c = r^2g + dr^2)\) is hyperkähler, that is the metric \(g^c = r^2g + dr^2\) admits three compatible integrable almost complex structure \(J_i, (i = 1, 2, 3)\), satisfying the quaternionic relations, i.e., \(J_1J_2 = -J_2J_1 = J_3\), such that \((S \times \mathbb{R}^+, g^c = t^2g + dt^2, J_1, J_2, J_3)\) is a hyperkähler manifold. Equivalently, the holonomy group of the cone metric \(g^c\) is a subgroup of \(\text{Sp}(n + 1)\). In this case the
Reeb vector fields $\xi_i = J_i \partial_t$ ($i = 1, 2, 3$) are Killing vector fields. The three Reeb vector fields $\xi_i$, the three 1-forms $\eta_i$ and the three $(1,1)$-tensor fields $\phi_i$, where $i = 1, 2, 3$, satisfy the relations

\[
\eta_i(\xi_j) = g(\xi_i, \xi_j) = \delta_{ij},
\]

\[
\phi_i \xi_j = -\phi_j \xi_i = \xi_k,
\]

\[
\eta_i \circ \phi_j = -\eta_j \circ \phi_i = \eta_k,
\]

\[
\phi_i \circ \phi_j - \eta_j \otimes \xi_i = -\phi_j \circ \phi_i + \eta_i \otimes \xi_j = \phi_k,
\]

for any cyclic permutation $(i, j, k)$ of $(1, 2, 3)$. The three Sasakian structures $(\eta_i, \xi_i, \phi_i, g)$, where $i \in \{1, 2, 3\}$, are called the 3-Sasakian structure on $S$.

Note that in [30] (see also [29] and [45]) it is shown that if $(S, g)$ is a 7-dimensional 3-Sasakian manifold, then $S$ carries a second nearly parallel $G_2$-structure whose underlying Einstein metric is such that its cone metric has holonomy equal to Spin(7). This second Einstein metric $\tilde{g}$ on $S$ is given by

\[
\tilde{g} = \frac{1}{\sqrt{5}} g|_H + g|_V,
\]

(2.7)

where $V$ is the 3-dimensional distribution $V = \text{span}\{\xi_1, \xi_2, \xi_3\}$, and $H = V^\perp$. The explicit expression of the second nearly parallel $G_2$-structure inducing the metric $\tilde{g}$ is given in [30, Prop. 2.4] (see also [45]).

Let $(S, g)$ be a 3-Sasakian manifold of dimension $4n + 3$. Since the Reeb vector fields $\xi_i$ satisfy the following relations $g(\xi_i, \xi_j) = \delta_{ij}$ and $[\xi_i, \xi_j] = 2\xi_k$, they span an integrable 3-dimensional distribution on a 3-Sasakian manifold. Denote by $\mathcal{F}$ the 3-dimensional foliation generated by the Reeb vector fields $\{\xi_1, \xi_2, \xi_3\}$.

If $(S, g)$ is a compact 3-Sasakian manifold, then the Reeb vector fields $\xi_i$ are complete and the leaves of the foliation $\mathcal{F}$ are compact. Hence, $\mathcal{F}$ is quasi-regular so that the space of leaves is a $4n$-dimensional orbifold. A 3-Sasakian structure $(\eta_i, \xi_i, \phi_i, g)$, $i \in \{1, 2, 3\}$, on a manifold $S$ is said to be regular if the 3-dimensional foliation $\mathcal{F}$ is regular, that is, the space of leaves $S/\mathcal{F}$ is a Riemannian manifold, with Riemannian metric $\tilde{g}$, and the natural projection $\pi: (S, g) \to (S/\mathcal{F}, \tilde{g})$ is a Riemannian submersion.

On the structure of a 3-Sasakian manifold, in [13] the following property is proved. Let $(S, g)$ be a 3-Sasakian manifold of dimension $4n + 3$ such that the Reeb vector fields $\{\xi_1, \xi_2, \xi_3\}$ are complete. Then the space of leaves $S/\mathcal{F}$ has the structure of a quaternionic Kähler orbifold $(\mathcal{O}, g_0)$ of dimension $4n$ such that the natural projection $\pi: S \to \mathcal{O}$ is a principal V-bundle with group SU(2) or SO(3), and $\pi$ is a Riemannian orbifold submersion such that the scalar curvature of $g_0$ is $16n(n + 2)$.

Some topological properties of compact 3-Sasakian manifolds are the following.
Proposition 2.6 ([30]). For any compact 3-Sasakian manifold \((S, g)\) of dimension \(4n + 3\), the odd Betti numbers \(b_{2i+1}(S)\) vanish, for \(0 \leq i \leq n\).

Proposition 2.7 ([16, Proposition 13.5.6 and Theorem 13.5.7]). Let \((S, g)\) be a compact regular 3-Sasakian manifold of dimension \(4n + 3\). Then \(\pi_1(S) = 0\) unless \(S = \mathbb{RP}^{4n+3}\). Always \(b_2(S) \leq 1\).

Note that there are compact 3-Sasakian manifolds with \(b_2 \geq 2\) [18]. They are not regular.

Let \((S, g)\) be a 3-Sasakian manifold. Then the isometry group \(\text{Iso}(S, g)\) of \((S, g)\) is non-trivial, and it has dimension \(\geq 3\), since each Sasakian structure has an isometry group of dimension \(\geq 1\). Denote by \(\text{Aut}(S, g) \subset \text{Iso}(S, g)\) the subgroup of the isometry group which preserves the 3-Sasakian structure \((g, \xi_i, \eta_i, \phi_i), i = 1, 2, 3\) on \(S\). A 3-Sasakian manifold \((S, g)\) is said to be a 3-Sasakian homogeneous space if the group \(\text{Aut}(S, g)\) acts transitively on \(S\). Any compact 3-Sasakian homogeneous space is a regular 3-Sasakian manifold. Moreover, if \((S, g)\) is a regular 3-Sasakian manifold of dimension less than or equal to 15, then \((S, g)\) is a 3-Sasakian homogeneous space.

In dimension 7, any regular 3-Sasakian manifold is homogeneous [28] and the only compact homogeneous 3-Sasakian manifolds are the sphere \(S^7\), the projective space \(\mathbb{RP}^7 \cong \text{Sp}(2)/\text{Sp}(1) \times \mathbb{Z}_2\), with their standard metrics, and the Aloff–Wallach space \(W_{1,1} = \text{SU}(3)/\text{S}(\text{U}(1) \times \text{U}(1))\) (see [17]). The space \(W_{1,1}\) can equivalently be defined as the total space of the principal \(\text{SO}(3)\)-bundle over \(\mathbb{CP}^2\) with Pontryagin class \(-3x\), where \(x \in H^4(\mathbb{CP}^2, \mathbb{Z})\) is the positive generator (see [22], Subsection 5.2 and Section 8). The 3-Sasakian metric then comes from this fibre bundle structure, using the Fubini–Study metric on \(\mathbb{CP}^2\). In fact, \(W_{1,1}\) arises as an instance of a more general construction of 3-Sasakian structures on \(\text{SO}(3)\)-bundles over homogeneous positive quaternionic Kähler manifolds \(\text{Gr}_2(\mathbb{C}^n)\) [27, Prop. 4.2]. We shall revisit the space \(W_{1,1}\) in Subsection 5.2 and Section 8.

### 3. Minimal models and formality

In order to analyze the property of formality of nearly parallel \(\text{G}_2\)-manifolds (Section 4 and Section 5) and of 8-manifolds with a locally conformal parallel \(\text{Spin}(7)\)-structure (see Section 9), we review here concepts about minimal models and formality (see [21], [23], [26] for more details).

Simply connected compact manifolds of dimension \(\leq 6\) are always formal [26], so dimension 7 is the lowest dimension in which formality is an issue.

A differential graded algebra (or DGA) over the real numbers \(\mathbb{R}\), is a pair \((A, d)\) consisting of a graded commutative algebra \(A = \oplus_{k \geq 0} A^k\) over \(\mathbb{R}\), and a differential \(d\) satisfying the Leibniz rule \(d(a \cdot b) = (da) \cdot b + (-1)^{|a|} a \cdot (db)\), where \(|a|\) is the degree of \(a\). Given a differential graded commutative algebra \((A, d)\), we denote its cohomology
by $H^*(A)$. The cohomology of a differential graded algebra $H^*(A)$ is naturally a DGA with the product inherited from that on $A$ and with the differential being identically zero. The DGA $(A, d)$ is connected if $H^0(A) = \mathbb{R}$, and $A$ is 1-connected if, in addition, $H^1(A) = 0$. Henceforth we shall assume that all our DGAs are connected.

In our context, the main example of DGA is the de Rham complex $(\Omega^*(M), d)$ of a connected differentiable manifold $M$, where $d$ is the exterior differential.

Morphisms between DGAs are required to preserve the degree and to commute with the differential. A morphism $f : (A, d) \to (B, d)$ is a quasi-isomorphism if the map induced in cohomology $f^* : H^*(A, d) \to H^*(B, d)$ is an isomorphism.

A DGA $(M, d)$ said to be minimal if

1. $M$ is free as an algebra, that is, $M$ is the free algebra $\bigwedge V$ over a graded vector space $V = \bigoplus V^i$, and
2. there is a collection of generators $\{x_\tau\}_{\tau \in I}$ indexed by some well ordered set $I$, such that $|x_\mu| \leq |x_\tau|$ if $\mu < \tau$, and each $dx_\tau$ is expressed in terms of preceding $x_\mu$, $\mu < \tau$.

We say that $(\bigwedge V, d)$ is a minimal model of the differential graded commutative algebra $(A, d)$ if $(\bigwedge V, d)$ is minimal and there exists a quasi-isomorphism $f : (\bigwedge V, d) \to (A, d)$. A connected DGA $(A, d)$ has a minimal model unique up to isomorphism. For 1-connected DGAs, this is proved in [21].

A minimal model of a connected differentiable manifold $M$ is a minimal model $(\bigwedge V, d)$ for the de Rham complex $(\Omega^*(M), d)$ of differential forms on $M$. If $M$ is a simply connected manifold, then the dual of the real homotopy vector space $\pi_i(M) \otimes \mathbb{R}$ is isomorphic to $V^i$ for any $i$ (see [21]).

We say that a DGA $(A, d)$ is a model of a manifold $M$ if $(A, d)$ and $M$ have the same minimal model. Thus, if $(\bigwedge V, d)$ is the minimal model of $M$, we have

$$(A, d) \xleftarrow{g} (\bigwedge V, d) \xrightarrow{f} (\Omega^*(M), d),$$

where $f$ and $g$ are quasi-isomorphisms.

A minimal algebra $(\bigwedge V, d)$ is called formal if there exists a morphism of differential algebras $f : (\bigwedge V, d) \to (H^*(\bigwedge V), 0)$ inducing the identity map on cohomology. Also a differentiable manifold $M$ is called formal if its minimal model is formal.

Many examples of formal manifolds are known: all compact symmetric spaces (e.g., spheres, projective spaces, compact Lie groups, flag manifolds), compact Kähler manifolds and simply connected manifolds of dimension 6 or less.

The formality property of a minimal algebra is characterized as follows.

**Proposition 3.1** ([21]). A minimal algebra $(\bigwedge V, d)$ is formal if and only if the space $V$ can be decomposed into a direct sum $V = C \oplus N$ with $d(C) = 0$ and $d$ injective on $N$, such that every closed element in the ideal $I(N)$ in $\bigwedge V$ generated by $N$ is exact.
This characterization of formality can be weakened using the concept of $s$-formality introduced in \cite{26}.

**Definition 3.2.** A minimal algebra $(\bigwedge V, d)$ is $s$-formal ($s > 0$) if for each $i \leq s$ the space $V^i$ of generators of degree $i$ decomposes as a direct sum $V^i = C^i \oplus N^i$, where the spaces $C^i$ and $N^i$ satisfy the three following conditions:

1. $d(C^i) = 0$,
2. the differential map $d: N^i \to \bigwedge V$ is injective, and
3. any closed element in the ideal $I_s = I(\bigoplus_{i \leq s} N^i)$, generated by the space $\bigoplus_{i \leq s} N^i$ in the free algebra $\bigwedge (\bigoplus_{i \leq s} V^i)$, is exact in $\bigwedge V$.

A differentiable manifold $M$ is $s$-formal if its minimal model is $s$-formal. Clearly, if $M$ is formal then $M$ is $s$-formal, for any $s > 0$. The main result of \cite{26} shows that sometimes the weaker condition of $s$-formality implies formality.

**Theorem 3.3** (\cite{26}). Let $M$ be a connected and orientable compact manifold of dimension $2n$ or $(2n - 1)$. Then $M$ is formal if and only if it is $(n - 1)$-formal.

One can check that any simply connected compact manifold is 2-formal. Therefore, Theorem 3.3 implies that any simply connected compact manifold of dimension at most 6 is formal. (This result was proved earlier in \cite{46}.)

**Lemma 3.4.** Let $M$ be a 7-dimensional compact manifold with $b_1(M) = 0$ and $b_2(M) \leq 1$. Then, $M$ is 3-formal and so formal.

**Proof.** The proof is exactly the same as the one given in \cite{25} for 7-dimensional simply connected compact manifolds with $b_2 \leq 1$. \hfill $\square$

**Lemma 3.5** (\cite{26}). Let $M_1$ and $M_2$ be differentiable manifolds. For any $s \geq 0$, the product manifold $M_1 \times M_2$ is $s$-formal if and only if $M_1$ and $M_2$ are $s$-formal. In particular, $M_1 \times M_2$ is formal if and only if $M_1$ and $M_2$ are both formal.

In order to detect non-formality, instead of computing the minimal model, which is usually a lengthy process, one can use Massey products, which are obstructions to formality. The simplest type of Massey product is the triple (also known as ordinary) Massey product. This will be defined next.

Let $(A, d)$ be a DGA (in particular, it can be the de Rham complex of differential forms on a smooth manifold). Suppose that there are cohomology classes $[a_i] \in H^{p_i}(A)$, $p_i > 0$, $1 \leq i \leq 3$, such that $a_1 \cdot a_2$ and $a_2 \cdot a_3$ are exact. Write $a_1 \cdot a_2 = da_{1,2}$ and $a_2 \cdot a_3 = da_{2,3}$. The (triple) Massey product of the classes $[a_i]$ is defined as

$$\langle [a_1], [a_2], [a_3] \rangle = [a_1 \cdot a_{2,3} + (-1)^{p_1+1} a_{1,2} \cdot a_3] \in H^{p_1+p_2+p_3-1}(A)$$

$$\frac{H^{p_1+p_2+p_3-1}(A)}{[a_1] \cdot H^{p_2+p_3-1}(A) + [a_3] \cdot H^{p_1+p_2-1}(A)}.$$
Note that a Massey product \([\langle a_1, a_2, a_3 \rangle]\) on \((A, d_A)\) is zero (or trivial) if and only if there exist \(x, y \in A\) such that \(a_1 \cdot a_2 = d_A x\), \(a_2 \cdot a_3 = d_A y\) and
\[
0 = [a_1 \cdot y + (-1)^{p_1+1} x \cdot a_3] \in H^{p_1+p_2+p_3-1}(A).
\]
Massey products are related to formality by the following well-known result.

**Theorem 3.6** ([21]). A DGA which has a non-zero Massey product is not formal.

We will also use the following properties.

**Lemma 3.7** ([10]). Let \(M\) be a connected smooth manifold. Then, Massey products on \(M\) can be calculated by using any model of \(M\).

**Models of fibrations.** Let \(F \to E \to B\) be a fibration of simply connected spaces. Let \((A_B, d_B)\) be a model (not necessarily minimal) of the base \(B\), and let \((\bigwedge V_F, d_F)\) be a minimal model of the fiber \(F\). By [23, section 15], a model of \(E\) is the KS-extension \((A_B \otimes \bigwedge V_F, D)\), where \(D\) is defined as \(Db = d_B b\), for \(b \in A_B\), and \(Dx = d_F x + \Theta(x)\), for \(x \in V_F\), and where
\[
\Theta : V_F \to A_B
\]
is called the *transgression map*. This is also true in the case that \(F, E\) and \(B\) are nilpotent spaces and the fibration is nilpotent, that is \(\pi_1(B)\) acts nilpotently in the homotopy groups \(\pi_j(F)\) of the fiber.

We will need two cases. In the case that \(E\) and \(B\) are simply connected and \(F = SU(2) = S^3\) or \(F = S^3/\mathbb{Z}_r\), with \(r > 0\), the fibration is nilpotent. The minimal model of \(S^3\) is \((\bigwedge a, d)\), with \(|a| = 3\) and \(da = 0\). Both spaces \(S^3\) and \(S^3/\mathbb{Z}_r\) are rationally homotopy equivalent, so a fibration \(S^3/\mathbb{Z}_r \to E \to B\) is a rational \(S^3\)-fibration (that is, after rationalization of the spaces, it becomes a fibration). The transgression map \(\Theta\) is such that \(\Theta(a) \in A_B^1\) is a closed element of degree 4 defining the (rational) Euler class \(e(E)\) of the fibration.

The second case that we will need is for fibrations with \(F = S^1 \times S^2\). Then the minimal model of \(F\) is \((\bigwedge (b, c, x), d)\), where \(|b| = 1, |c| = 2, |x| = 3\), and the differential map is defined by \(db = 0, dc = 0\) and \(dx = c^2\). If \(B\) is simply connected then the fibration is nilpotent, and a model of \(E\) is given as \((A_B \otimes \bigwedge (b, c, x), D)\), where the transgression map \(\Theta\) satisfies that \(\Theta(b) \in A_B^2\) is a closed element of degree 2 defining the Chern class of the \(S^1\)-bundle induced from \(E\) via the fiberwise projection \(S^1 \times S^2 \to S^1\); and \(\Theta(c) = 0\), and \(\Theta(x) \in A_B^4\) is a closed element of degree 4, defining the Pontryagin class of the \(S^2\)-bundle induced from \(E\) via the fiberwise projection \(S^1 \times S^2 \to S^2\).

4. **On the formality of nearly parallel \(G_2\)-manifolds**

Let \(P\) be a simply connected compact 7-manifold with a nearly parallel \(G_2\)-structure \(\varphi\) inducing a metric \(g\) such that \((P, g)\) is not isometric to the standard sphere \(S^7\). Let
$(\tilde{P} = P \times \mathbb{R}^+, g^c = r^2 g + dr^2)$ be the metric cone on $(P, g)$. As already mentioned in the introduction, $(P, g)$ belongs to one of the following three classes: $(P, g)$ is a proper nearly parallel $G_2$-manifold if the holonomy of $(\tilde{P}, g^c)$ is $\text{Spin}(7)$; $(P, g)$ is a Sasaki–Einstein manifold if the holonomy of $(\tilde{P}, g^c)$ coincides with $\text{SU}(4)$; and $(P, g)$ is a 3-Sasakian manifold if the holonomy of $(\tilde{P}, g^c)$ is $\text{Sp}(2)$.

Formality of simply connected compact 3-Sasakian manifolds, of dimension 7, was studied in [25]. The following result is proved there.

**Theorem 4.1** ([25]). Let $(S, g)$ be a simply connected compact 3-Sasakian manifold, of dimension 7. Then, $S$ is formal if and only if its second Betti number $b_2(S) \leq 1$.

**Remark 4.2.** Note that Theorem 4.1 is also true if $S$ is not simply connected but $b_1(S) = 0$ which, by Proposition 2.5, always happens if $(S, g)$ is a compact 3-Sasakian manifold. The proof of this case is exactly the same as that given in [25] for Theorem 4.1.

In dimension 7, the only simply connected regular 3-Sasakian manifolds are the sphere $S^7$ and the space $W_{1,1} = \text{SU}(3)/S^1_{1,1}$. The sphere $S^m$ is formal, for any $m$. For the space $W_{1,1}$ we know that $b_1(W_{1,1}) = 0$ by Proposition 2.6 and $b_2(W_{1,1}) \leq 1$ by Proposition 2.7. Hence, $W_{1,1}$ is formal by Lemma 3.4. (Note that the formality of $W_{1,1}$ as a 3-Sasakian homogeneous space is proved in [27].) In [18] it is shown that there exist many 3-Sasakian manifolds, of dimension 7, with arbitrary second Betti number.

Regarding the formality of Sasaki–Einstein manifolds, the following result is proved in [12] Theorem 3.2].

**Theorem 4.3.** [12] Let $\omega = \omega_1 + \omega_2 + \omega_3$ be the Kähler form on $S^2 \times S^2 \times S^2$, where $[\omega_1]$, $[\omega_2]$ and $[\omega_3]$ are the generators of the integral cohomology group of each of the $S^2$-factors on $S^2 \times S^2 \times S^2$. Then, the total space of the principal $S^1$-bundle

$$S^1 \hookrightarrow Q(1,1,1) \longrightarrow S^2 \times S^2 \times S^2,$$

with Euler class $[\omega] \in H^2(S^2 \times S^2 \times S^2, \mathbb{Z})$ is non-formal.

The above is a consequence of the following more general result.

**Proposition 4.4.** Consider the principal $S^1$-bundle

$$S^1 \hookrightarrow Q \longrightarrow S^2 \times S^2 \times S^2,$$

with Euler class $e_1a_1 + e_2a_2 + e_3a_3$, where $e_1, e_2, e_3 \in \mathbb{Z}$, and $a_i$ is the generator of $H^2(S^2, \mathbb{Z})$ for the $i$-th copy of $S^2$, $i = 1, 2, 3$. Then, $Q$ is formal if and only if $e_1e_2e_3 = 0$.

**Proof.** We will determine a model of the 7-manifold $Q$. A minimal model of $S^2 \times S^2 \times S^2$ is the differential algebra $(\wedge(a_1, a_2, a_3, x_1, x_2, x_3), d)$, where $|a_i| = 2$ while $|x_i| = 3$.
with $1 \leq i \leq 3$, and the differential $d$ is defined by $da_i = 0$ and $dx_i = a_i^2$. Therefore, a model of the total space of a fiber bundle

$$S^1 \hookrightarrow Q \twoheadrightarrow S^2 \times S^2$$

is the differential algebra over the vector space $V$ generated by the elements $y$ of degree 1, $a_1, a_2, a_3$ of degree 2, and $x_1, x_2, x_3$ of degree 3, and the differential $d$ is given by

$$da_i = 0, \quad dx_i = a_i^2, \quad 1 \leq i \leq 3, \quad dy = e_1a_1 + e_2a_2 + e_3a_3,$$

where $e_1a_1 + e_2a_2 + e_3a_3 \in H^2(S^2 \times S^2 \times S^2, \mathbb{Z})$ is the Euler class of the $S^1$-bundle.

If all $e_1 = e_2 = e_3 = 0$, then $(\bigwedge V, d)$ is a minimal DGA, and so it is the minimal model of $Q$. But $(\bigwedge V, d)$ is also the minimal model of $S^1 \times S^2 \times S^2 \times S^2$, which is formal being the product of formal manifolds. Thus $Q$ is formal because its minimal model is so.

If not all $e_i$ are zero, we can assume $e_1 \neq 0$ (up to reordering). Then $e_1a_1 = dy - e_2a_2 - e_3a_3$, so

$$e_1^2dx_1 = e_1^2a_1^2 = (dy - e_2a_2 - e_3a_3)^2 = d(y \cdot dy) - 2 \sum_{i=2}^{3} e_i d(y \cdot a_i) + \sum_{i=2}^{3} e_i^2dx_i + 2e_2e_3a_2 \cdot a_3.$$

Thus, letting

$$\tilde{x}_1 = x_1 - e_1^{-2} \left( y \cdot dy + 2 \sum_{i=2}^{3} e_i y \cdot a_i - \sum_{i=2}^{3} e_i^2 x_i \right),$$

we have

$$d\tilde{x}_1 = 2e_1^{-2}e_2e_3a_2 \cdot a_3.$$

Then, the differential algebra $(\bigwedge (a_2, a_3, \tilde{x}_1, x_2, x_3), d)$ is a model of $Q$. In fact, the map $f: (\bigwedge (a_2, a_3, \tilde{x}_1, x_2, x_3), d) \to (\bigwedge (a_1, a_2, a_3, x_1, x_2, x_3, y), d)$ defined by $f(a_i) = a_i$, $f(x_i) = x_i$ ($i = 2, 3$), and $f(\tilde{x}_1) = x_1 - e_1^{-2} (y \cdot dy + 2 \sum_{i=2}^{3} e_i y \cdot a_i - \sum_{i=2}^{3} e_i^2 x_i)$ is a quasi-isomorphism.

Let us assume that $e_2e_3 = 0$. In this case, $d\tilde{x}_1 = 0$, and $(\bigwedge (a_2, a_3, x_2, x_3, \tilde{x}_1), d)$ is a minimal differential graded algebra. So $(\bigwedge (a_2, a_3, x_2, x_3, \tilde{x}_1), d)$ is the minimal model of $Q$. Hence $Q$ is formal since $(\bigwedge (a_2, a_3, x_2, x_3, \tilde{x}_1), d)$ is also the minimal model of $S^2 \times S^2 \times S^3$, which is formal being the product of formal manifolds.

Finally, if $e_2e_3 \neq 0$, then

$$a_2 \cdot a_3 = \frac{e_1^2}{2e_2e_3}d\tilde{x}_1.$$

We are going to show that $Q$ is non-formal because there exists a non-zero Massey product on $Q$. By Lemma 3.7 we know that Massey products on a manifold can be computed by using any model for the manifold. Since $(\bigwedge (a_2, a_3, x_2, x_3, \tilde{x}_1), d)$
is a model of $Q$, we have $H^*(Q) \cong H^*(\bigwedge(a_2, a_3, x_2, x_3, \bar{x}_1), d)$, so that $H^4(Q) = 0 = H^4(Q)$, and by Poincaré duality for the 7-manifold $Q$, $H^4(Q) = 0$ and $H^5(Q)$ has dimension 2. Moreover, the Massey product $\langle[a_2], [a_2], [a_3]\rangle$ is defined and

$$\langle[a_2], [a_2], [a_3]\rangle = \left[\frac{e_1^3}{2e_2 e_3} a_2 \cdot \bar{x}_1 - x_2 \cdot a_3\right].$$

This element in $H^5(Q)$ cannot be exact since there is no non-zero element $x \in \bigwedge^4(a_2, a_3, x_2, x_3, \bar{x}_1)$ such that $dx = \frac{e_1^3}{2e_2 e_3} a_2 \cdot \bar{x}_1 - x_2 \cdot a_3$. Moreover, the indeterminacy of the Massey product is zero because $H^3(Q) = 0$. So $\langle[a_2], [a_2], [a_3]\rangle \neq 0$, and hence $Q$ is non-formal.

In order to exhibit further examples of non-formal simply connected Sasaki–Einstein manifolds, we consider a del Pezzo surface $P_k$, for $3 \leq k \leq 8$, that is the blow-up of the complex projective space $\mathbb{CP}^2$ at $k$ points,

$$P_k = \mathbb{CP}^2 \# k\mathbb{CP}^2 = \mathbb{CP}^2 \# \mathbb{CP}^2 \# \cdots \# \mathbb{CP}^2, \quad 3 \leq k \leq 8,$$

where $\mathbb{CP}^2$ is $\mathbb{CP}^2$ with the opposite of the standard orientation. Then the de Rham cohomology of $P_k$ is

- $H^0(P_k) = \langle 1 \rangle$,
- $H^1(P_k) = 0$,
- $H^2(P_k) = \langle a, a_1, \ldots, a_k \rangle$,
- $H^3(P_k) = 0$,
- $H^4(P_k) = \langle \nu \rangle$,

where $\nu = a^2$ is the volume form, and $a$ is the integral cohomology class defined by the Kähler form on $\mathbb{CP}^2$. Among these cohomology classes, the following relations are satisfied

$$a^2 = -a_i^2 = \nu, \quad \text{for } 1 \leq i \leq k, \quad a \cdot a_i = 0 = a_i \cdot a_j, \quad \text{for } 1 \leq i, j \leq k \text{ and } i \neq j.$$

Now consider the 6-manifold

$$X_k = P_k \times S^2, \quad 3 \leq k \leq 8.$$

**Theorem 4.5.** Let $S_k$ be the total space of the circle bundle $S^1 \rightarrow S_k \rightarrow X_k = P_k \times S^2$, with Euler class $N(a - \sum_{i=1}^k \epsilon_i a_i + b)$, for some $\epsilon_i > 0$ small, and where $b$ is the generator of $H^2(S^2, \mathbb{Z})$, and $N$ is a large integer satisfying that $N \epsilon_i \in \mathbb{Z}$ for all $i$. Then, for $3 \leq k \leq 8$, $S_k$ is a simply connected compact Sasaki–Einstein manifold, with second Betti number $b_2 = k + 1$, which is non-formal.

**Proof.** By the assumption, $N(a - \sum_{i=1}^k \epsilon_i a_i + b)$ is an integral cohomology class defined by the Kähler form on the complex manifold $X_k = P_k \times S^2$. (Certainly it is Kähler
for $\epsilon_i > 0$ small, and integral if $N\epsilon_i \in \mathbb{Z}$.) Therefore, there is a circle bundle $S_k \rightarrow X_k = P_k \times S^2$ with Euler class equal to $N(a - \sum_{i=1}^{k} \epsilon_i a_i + b)$, where $b$ is the generator of $H^2(S^2, \mathbb{Z})$.

Clearly $S_k$ is a 7-dimensional simply connected, compact manifold, with second Betti number $b_2 = k + 1$. Moreover, $S_k$ is Sasaki–Einstein. Indeed, Tian and Yau in [52], proved that there are Kähler–Einstein structures with $c_1 > 0$ on any manifold $P_k = \mathbb{CP}^2 \# k\mathbb{CP}^2$, for $3 \leq k \leq 8$. Then, there exists a Kähler–Einstein metric on any manifold $X_k = P_k \times S^2$, for $3 \leq k \leq 8$ [29]. Thus, according to Section 2, $S_k$ is a Sasaki–Einstein manifold.

By Theorem 3.7, if $S_k$ has a non-zero Massey product, then $S_k$ is non-formal. By Lemma 3.7, we know that Massey products on a manifold can be computed using any model for the manifold. Since $X_k$ is a compact Kähler manifold, $X_k$ is formal. Thus, a model of $X_k$ is $(H^*(X_k), 0)$, where $H^*(X_k)$ is the de Rham cohomology algebra of $X_k$, that is

\[
\begin{align*}
H^0(X_k) &= \langle 1 \rangle, \\
H^1(X_k) &= H^3(X_k) = H^5(X_k) = 0, \\
H^2(X_k) &= \langle a, a_1, \ldots, a_k, b \rangle, \\
H^4(X_k) &= \langle a^2, a \cdot b, a_1 \cdot b, \ldots, a_k \cdot b \rangle, \\
H^6(X_k) &= \langle a^2 \cdot b \rangle.
\end{align*}
\]

Then, a model of $S_k$ is the differential graded algebra $(\mathcal{A}, d)$, where $\mathcal{A} = H^*(X_k) \otimes \wedge(y)$, with $|y| = 1$, $d(H^*(X_k)) = 0$ and $dy = N(a - \sum_{i=1}^{k} \epsilon_i a_i + b)$. Write $\tilde{y} = \frac{1}{N} y$.

Then, $H^1(\mathcal{A}, d) = H^3(\mathcal{A}, d) = 0$, $H^2(\mathcal{A}, d) = \langle [a], [a_1], \ldots, [a_k] \rangle$.

Using this model, we compute the Massey product $\langle [a], [a], [a_1] \rangle$. In this model $a \cdot a_i = 0$ ($1 \leq i \leq k$), and

\[
(a - \sum_{i=1}^{k} \epsilon_i a_i + b) \cdot (a - \sum_{i=1}^{k} \epsilon_i a_i - b) = (1 - \sum_{i=1}^{k} \epsilon_i^2) a^2,
\]

since $b^2 = 0$. Then, $a \cdot a = (1 - \sum_{i=1}^{k} \epsilon_i^2)^{-1} d((a - \sum_{i=1}^{k} \epsilon_i a_i - b) \cdot \tilde{y})$. So the Massey product $\langle [a], [a], [a_1] \rangle$ is defined, and

\[
\langle [a], [a], [a_1] \rangle = \left[ - (1 - \sum_{i=1}^{k} \epsilon_i^2)^{-1} \left( (a - \sum_{i=1}^{k} \epsilon_i a_i - b) \cdot \tilde{y} \right) \cdot a_1 \right] = \left[ - (1 - \sum_{i=1}^{k} \epsilon_i^2)^{-1} (\epsilon_1 \nu - b \cdot a_1) \cdot \tilde{y} \right],
\]

where $\nu$ is the standard 3-form on $[3, 3]$.
which is non-zero in $H^5(S_k)$. Therefore, $S_k$ is non-formal. Note that there is no indeterminacy of this Massey product, since it lives in $[a] \cdot H^3(S_k) + [a_1] \cdot H^3(S_k)$, and we know that $H^3(S_k) \cong H^3(A, d) = 0$.

\[\square\]

Note that in [25] there is an example of a 7-dimensional regular simply connected Sasaki–Einstein manifold, with second Betti number $b_2 \geq 2$, which is formal, and so it does not admit any 3-Sasakian structure by Theorem 4.1. Such a manifold is the total space of an $S^1$-bundle over the blow up of the complex projective space $\mathbb{CP}^3$ at four points. Other examples of simply connected formal Sasaki–Einstein manifolds, of dimension 7, are the total space of a circle bundle over the Kähler–Einstein manifold $\mathbb{CP}^2 \times S^2$ [10], and the space $W_{1,1}$ mentioned above which, as a Sasaki–Einstein manifold, is the total space of a circle bundle over the flag manifold $F(1, 2)$ [29].

**Remark 4.6.** It is well-known that compact Kähler manifolds are formal [21]. However, it is an open problem to know whether all the simply connected compact manifolds with special holonomy ($G_2$ or Spin(7)) are formal. Nearly parallel $G_2$-manifolds are one way to generalize manifolds with holonomy $G_2$, and they can be considered as analogous to nearly Kähler manifolds of dimension 6. By [3] simply connected compact nearly Kähler manifolds are formal. In contrast to the nearly Kähler case, we have that the manifolds $Q(1, 1, 1)$ and $S_k$ are non-formal by Theorem 4.3 and Theorem 4.5, respectively.

5. **Formality of proper nearly parallel $G_2$-manifolds**

According to [29], the only examples of proper nearly parallel $G_2$-manifolds, whose underlying metric is homogeneous, are the squashed 7-sphere $S^7_{sq}$, the Berger space $\mathcal{B} = SO(5)/SO(3)$ and the Aloff–Wallach spaces $W_{k,l} = SU(3)/S^1_{k,l}$. The only compact non-homogeneous examples of proper nearly parallel $G_2$-manifolds, known in the literature, are 7-dimensional compact non-homogeneous 3-Sasakian manifolds $(S, g)$ with the canonical variation metric $\tilde{g}$ of $g$ given by (2.7). (Examples of such 3-Sasakian manifolds are given in [18].)

5.1. **The Berger space.** Consider the usual action of $SO(3)$ on $\mathbb{R}^3 = \text{span}\{x, y, z\}$. This action extends to an action of $SO(3)$ on the polynomial ring $\mathbb{R}[x, y, z]$. Let $V_n \subset \mathbb{R}[x, y, z]$ be the $SO(3)$-submodule of homogeneous polynomials of degree $n$, and let $\mathcal{H}_n \subset V_n$ denote the $SO(3)$-submodule of harmonic polynomials of degree $n$, an irreducible $SO(3)$-module of dimension $2n + 1$. Every finite dimensional irreducible $SO(3)$-module is isomorphic to $\mathcal{H}_n$, for some $n$. The irreducible representation $\mathcal{H}_2$ of $SO(3)$ has dimension 5, and so defines a non-standard embedding $SO(3) \subset SO(5)$. The Berger space is the compact homogeneous space

$$\mathcal{B} = SO(5)/SO(3),$$
given by the quotient of SO(5) by the copy of SO(3) embedded in SO(5) via the irreducible representation $\mathcal{H}_2$ of SO(3). The space $\mathcal{B}$ has a metric such that the holonomy of its cone metric is $\text{Spin}(7)$ \cite{19}. The proper nearly parallel $G_2$-structure on $\mathcal{B}$ is given explicitly in \cite[Subsection 2.4.1]{6}.

Berger \cite{11} proved that $\mathcal{B}$ is a rational homology sphere with $H^4(\mathcal{B}, \mathbb{Z}) = \mathbb{Z}_{10}$, and it has the cohomology ring of an $S^3$-bundle over $S^4$. Therefore, $\mathcal{B}$ and the sphere $S^7$ have the same minimal model. In the following theorem, we determine the minimal model of the total space of a principal $S^3$-bundle over $S^4$, and we show that such a space is formal. We apply this to conclude that the space $\mathcal{B}$ and $S^7$ have the same minimal model, and hence $\mathcal{B}$ is formal.

**Theorem 5.1.** Consider an $S^3$-bundle $S^3 \to P \to S^4$. Then $P$ is formal. In particular, the Berger space $\mathcal{B} = \text{SO}(5)/\text{SO}(3)$ is formal.

*Proof.* Let $e[\omega]$ be the Euler class of the bundle, where $e \in \mathbb{Z}$, and $[\omega]$ is the generator of the integral cohomology group $H^4(S^4, \mathbb{Z})$. The minimal model of $S^4$ is the differential graded algebra $(\bigwedge (a, u, d), d)$, with $|a| = 4$, $|u| = 7$, $da = 0$ and $du = a^2$. The minimal model of $S^3$ is $(\bigwedge b, d)$, where $|b| = 3$ and $db = 0$. Then, according to Section 3 a model of $P$ is $(\bigwedge (a, u, b), d)$, with

$$da = 0, \quad du = a^2, \quad db = e\cdot a.$$ 

If $e = 0$, then the DGA $(\bigwedge (a, u, b), d)$ is minimal, and so it is the minimal model of $P$. But $(\bigwedge (a, u, b), d)$ is the minimal model of $S^3 \times S^4$, which is formal being the product of two formal manifolds. Therefore, $P$ is formal.

Suppose now that $e \neq 0$. In this case we have $du = a^2 = e^{-1}d(b \cdot a)$. So the element $\tilde{u} = u - e^{-1}b \cdot a$ has degree 7 and $d\tilde{u} = 0$. Then the DGA $(\bigwedge (a, \tilde{u}, b), d)$ is also a model of $P$, because it is quasi-isomorphic to $(\bigwedge (a, u, b), d)$. Moreover, $(\bigwedge \tilde{u}, 0)$ is the minimal model of $P$. In fact, $(\bigwedge \tilde{u}, 0)$ is a minimal DGA, and a model of $P$ because, taking into account that $db = e\cdot a$, the map $f : (\bigwedge \tilde{u}, 0) \to (\bigwedge (a, \tilde{u}, b), d)$ given by $f(\tilde{u}) = \tilde{u}$ is a quasi-isomorphism. Therefore, $P$ is formal since $(\bigwedge \tilde{u}, 0)$ is the minimal model of the sphere $S^7$, which is formal.

In \cite{31} it is proved that the Berger space is diffeomorphic to the $S^3$-bundle over $S^4$ with Euler class $-10[\omega]$. Thus $\mathcal{B}$ and $S^7$ have the same minimal model, and consequently $\mathcal{B}$ is formal. \qed
5.2. The Aloff–Wallach spaces. Let $k, l \in \mathbb{Z}$ be non-zero, co-prime integers, and $S^1_{k,l}$ be a circle subgroup of $SU(3)$ consisting of elements of the form
\[
\begin{pmatrix}
    e^{ik\theta} & 0 & 0 \\
    0 & e^{il\theta} & 0 \\
    0 & 0 & e^{im\theta}
\end{pmatrix},
\]
where $k + l + m = 0$. The Aloff–Wallach space
\[W_{k,l} = SU(3)/S^1_{k,l}\]
is the quotient of $SU(3)$ by this circle subgroup \[2\]. Note that there are examples of different pairs $(k, l)$ such that the corresponding Aloff–Wallach spaces are homeomorphic but not diffeomorphic \[41\]. The spaces $W_{k,l}$ are called generic if \[
\{k, l, -(k + l)\} \neq \{1, 1, -2\} \text{ or } \{1, -1, 0\}.
\]
We will denote by $W_{1,1}$ and $W_{1,-1}$ the two exceptional Aloff–Wallach spaces.

By \[48\] (see also \[55\]) all the spaces $W_{k,l}$ admit two homogeneous Einstein metrics. If $(k, l) = (1, 1)$ one of those metrics is the 3-Sasakian structure on the space $W_{1,1}$ mentioned in Section \[4\] and the other is induced by a proper homogeneous nearly parallel $G_2$-structure. If $(k, l) = (1, -1)$, the space $W_{1,-1}$ admits only one proper homogeneous nearly parallel $G_2$-structure, up to homotheties \[48\]. On the generic Aloff-Wallach spaces the two metrics are induced by proper homogeneous nearly parallel $G_2$-structures \[9\], which by \[48, 49\] are only two, up to homotheties. The expressions of those two $G_2$-structures are given in \[8, 20\].

The manifold $W_{k,l}$ is simply connected with \[H^2(W_{k,l}, \mathbb{Z}) \cong \mathbb{Z} \text{ and } H^3(W_{k,l}, \mathbb{Z}) = 0\] (see \[41\]). Thus, $b_1(W_{k,l}) = b_3(W_{k,l}) = 0$ and $b_2(W_{k,l}) = 1$. Hence, $W_{k,l}$ is formal by Lemma \[3.4\].

In \[34\] (see also \[8\]) it is shown that there is a canonical fibration
\[\pi : W_{k,l} \to \mathbb{CP}^2,\]
whose fibers are the lens spaces $S^3/\mathbb{Z}_{k+l}$ if $k + l \neq 0$, or $S^1 \times S^2$ if $k + l = 0$. In the two following theorems, we determine the minimal model of the total space of a $F$-fiber bundle over $\mathbb{CP}^2$, where $F = S^1 \times S^2$, or $F = S^3/\mathbb{Z}_p$ with $p > 0$, and we prove that such a space is formal. In particular, we show that $W_{k,l}$ and $S^5 \times S^2$ have the same minimal model.

**Theorem 5.2.** Let $P$ be the total space of an $S^3/\mathbb{Z}_p$-bundle $S^3/\mathbb{Z}_p \to P \to \mathbb{CP}^2$ with $p > 0$. Then $P$ is formal. In particular, if $k + l \neq 0$, the Aloff–Wallach space $W_{k,l}$ is formal, and $W_{k,l}$ and the product manifold $S^2 \times S^5$ have the same minimal model.

**Proof.** We will determine the minimal model of $P$ and show that it is formal. According to Section \[3\] the fibre bundle $S^3/\mathbb{Z}_p \to P \to \mathbb{CP}^2$ is a rational $S^3$-fibration. Let $e a^2$ be its (rational) Euler class, where $e \in \mathbb{Q}$ and $a = [\omega]$ is the generator of the
integral cohomology group $H^2(\mathbb{CP}^2, \mathbb{Z})$. By [23], if $(A, d_A)$ is a model of $\mathbb{CP}^2$, we have that $(A \otimes \wedge u, d)$, with $|u| = 3, d|_A = d_A$ and $du = e a^2$, is a model of $P$.

The minimal model of $\mathbb{CP}^2$ is the differential graded algebra $(\wedge(a, x), d)$, where $|a| = 2, |x| = 5, da = 0$ and $dx = a^3$. Therefore, the KS-model of $P$ is $(\wedge(a, x, u), d)$, with $du = e a^2$.

If $e = 0$ then the differential graded algebra $(\wedge(a, x, u), d)$ is minimal, and so it is the minimal model of $P$. Moreover, $(\wedge(a, x, u), d)$ is the minimal model of $S^3 \times \mathbb{CP}^2$, which is formal being the product of two formal manifolds. Thus, $P$ is formal.

Suppose now that $e \neq 0$. In this case we have $a^2 = e^{-1} du$, and so the element of degree five $\bar{x} = x - e^{-1} a \cdot u$ is such that $d\bar{x} = 0$. Clearly $(\wedge(a, \bar{x}, u), d)$ is a minimal DGA, and a model of $P$ because the map $f: (\wedge(a, \bar{x}, u), d) \to (\wedge(a, x, u), d)$ given by $f(a) = a$, $f(\bar{x}) = x - e^{-1} a \cdot u$ and $f(u) = u$ is a quasi-isomorphism. Therefore, $(\wedge(a, \bar{x}, u), d)$ is the minimal model of $P$. Thus, $P$ is formal since $(\wedge(a, \bar{x}, u), d) = (\wedge \bar{x} \otimes \wedge(a, u), d)$ is the minimal model of $S^5 \times S^2$, which is formal.

All this shows not only that $P$ is formal but also the minimal model of $P$. Indeed, the minimal model of $P$ is either the minimal model of $S^3 \times \mathbb{CP}^2$ or the minimal model of $S^5 \times S^2$. Thus, if the third Betti number of $P$ is $b_3(P) = 1$, then $P$ and $S^3 \times \mathbb{CP}^2$ have the same minimal model, while if $b_3(P) = 0$, then the minimal model of $P$ is the minimal model of $S^5 \times S^2$. Therefore, for $k + l \neq 0$, the space $W_{k,l}$ and $S^5 \times S^2$ have the same minimal model since $b_3(W_{k,l}) = 0$. □

**Theorem 5.3.** The total space of an $S^1 \times S^2$-bundle $S^1 \times S^2 \to P \to \mathbb{CP}^2$ is formal. In particular, for $k \neq 0$, the Aloff–Wallach space $W_{k,-k}$ is formal, and $W_{k,-k}$ and the product manifold $S^2 \times S^5$ have the same minimal model.

**Proof.** We know that the minimal model of $\mathbb{CP}^2$ is the differential graded algebra $(\wedge(a, x), d)$, where $|a| = 2, |x| = 5, da = 0$ and $dx = a^3$. Moreover, the minimal model of $S^1 \times S^2$ is $(\wedge(b, c, y), d)$, where $|b| = 1, |c| = 2, |y| = 3$, and the differential map is defined by $db = 0, dc = 0$ and $dy = c^2$. Hence the KS-model of $P$ is of the form

$$(\wedge(a, x, b, c, y), d), \quad da = 0, \quad dx = a^3, \quad db = e a, \quad dc = 0, \quad dy = c^2 + f a^2,$$

for some $e, f \in \mathbb{Z}$.

If $e = f = 0$ then $(\wedge(a, x, b, c, y), d)$ is the minimal model of $S^1 \times S^2 \times \mathbb{CP}^2$, which is formal being the product of three formal manifolds. Hence $P$ is formal.

Suppose now that $e \neq 0$ and $f = 0$. In this case, $a = e^{-1} db$. Then the element $\bar{x}$ of degree 5 given by $\bar{x} = x - e^{-1} a \cdot u$ is such that $d\bar{x} = 0$. Thus, $(\wedge(a, \bar{x}, b, c, y), d)$ is a model of $P$ because this DGA is quasi-isomorphic to $(\wedge(a, x, b, c, y), d)$. Moreover, $(\wedge(\bar{x}, c, y), d)$ is a minimal DGA, and a model of $P$. In fact, taking into account that $a = e^{-1} db$, one can check that the map $f: (\wedge(\bar{x}, c, y), d) \to (\wedge(a, \bar{x}, b, c, y), d)$ given
by \( f(\tilde{x}) = \tilde{x}, \ f(c) = c \) and \( f(y) = y \) is a quasi-isomorphism. Therefore, \( (\wedge(\tilde{x}, c, y), d) \)

is the minimal model of \( P \). Thus, \( P \) is formal since \( (\wedge(\tilde{x}, c, y), d) \) is the minimal model of \( S^5 \times S^2 \), which is formal.

If \( e \neq 0 \) and \( f \neq 0 \), as before we determine the minimal model of \( P \). Take the elements \( \tilde{x} = x - e^{-1}b \cdot a^2 \) and \( \tilde{y} = y - f e^{-1}a \cdot b \) of degree 5 and 3, respectively. Then, we get the model \( (\wedge(a, \tilde{x}, b, c, \tilde{y}), d) \) of \( P \) with \( da = 0, \ dx = 0, \ db = e \cdot a, \ dc = 0 \) and \( d\tilde{y} = c^2 \). Consider the differential graded algebra \( (\wedge(\tilde{x}, c, \tilde{y}), d) \), which is a minimal DGA, and a model of \( P \). In fact, the map \( f : (\wedge(\tilde{x}, c, \tilde{y}), d) \to (\wedge(a, \tilde{x}, b, c, \tilde{y}), d) \) given by \( f(\tilde{x}) = \tilde{x}, \ f(c) = c \) and \( f(\tilde{y}) = \tilde{y} \) is a quasi-isomorphism. Therefore, \( (\wedge(\tilde{x}, c, \tilde{y}), d) \) is the minimal model of \( P \). Thus, \( P \) is formal since \( (\wedge(\tilde{x}, c, \tilde{y}), d) \) is the minimal model of \( S^5 \times S^2 \), which is formal.

Finally, suppose that \( e = 0 \) and \( f \neq 0 \). Then the model \( (\wedge(a, x, b, c, y), d) \) of \( P \) is such that \( da = 0, \ dx = a^3, \ db = 0, \ dc = 0 \) and \( dy = c^2 + f a^2 \). This implies that the differential graded algebra \( (\wedge(a, x, b, c, y), d) \) is minimal, and so it is the minimal model of \( P \). To show that \( P \) is formal we proceed as follows. First note that \( \wedge(a, x, b, c, y) = \wedge b \otimes \wedge(a, x, c, y) \). Clearly, \( (\wedge b, 0) \) is the minimal model of \( S^1 \), which is formal, and \( (\wedge(a, x, c, y), d) \) is the minimal model of the total space \( X \) of an \( S^2 \)-bundle

\[
S^2 \to X \to \mathbb{C}P^2.
\]

The de Rham cohomology of \( X \) is \( H^1(X) = H^3(X) = H^5(X) = 0 \), \( H^2(X) = \langle [a], [c] \rangle \), \( H^4(X) = \langle [a]^2, [a] \cdot [c] \rangle \) and \( H^6(X) = \langle [a]^2 \cdot [c] \rangle \). Thus, the 6-manifold \( X \) is formal since \( b_1(X) = 0 \). In fact, denote by \( V \) the graded vector space generated by the elements \( a, x, c \) and \( y \). Because \( V \) is a graded vector space, we can consider the space \( V^i \) of generators of degree \( i \), which decomposes as a direct sum \( V^i = C^i \oplus N^i \), with \( C^1 = N^1 = 0 \), \( C^2 = \langle a, c \rangle \) and \( N^2 = 0 \). Thus, according to Definition 3.2, \( X \) is 2-formal, and by Theorem 3.3, \( X \) is formal.

Therefore, the minimal model \( (\wedge(a, x, b, c, y), d) \) of \( P \) is the minimal model of the product manifold \( S^1 \times X \), which is formal being the product of two formal manifolds. Hence \( P \) is formal.

Thus, if \( P \) is the total space of an \( S^1 \times S^2 \)-bundle over \( \mathbb{C}P^2 \), then \( P \) and \( S^5 \times S^2 \)

have the same minimal model, or the minimal model of \( P \) is the minimal model of the manifold \( M \), where \( M = S^1 \times S^2 \times \mathbb{C}P^2 \), or \( M = S^1 \times X \). So, for \( k \neq 0 \), \( W_{k,-k} \) and \( S^5 \times S^2 \)

have the same minimal model since \( b_1(W_{k,-k}) = 0 \).

**Remark 5.4.** In [22] it is proved that, for \( k, l \) co-prime integers, the space \( W_{k,l} = SU(3)/S^1_{k,l} \) can be described as follows. Let \( X \) be the total space of the \( S^2 \)-bundle \( S^2 \to X \to \mathbb{C}P^2 \) with Pontryagin class \( p_1 = -3 \) and Stiefel-Whitney class \( w_2 \neq 0 \). Then, \( W_{k,l} \) is the total space of the circle bundle

\[
S^1 \to W_{k,l} \xrightarrow{\pi} X
\]

(5.1)
determined by the Euler class $ka + lb$, where $a, b$ are the generators of $H^2(X, \mathbb{Z}) \cong \mathbb{Z}^2$, and the base space $X = SU(3)/T^2$.

Note that the space $W_{k,l}$ thus defined can be also considered as the aforementioned $F$-fiber bundle $F \rightarrow W_{k,l} \rightarrow \mathbb{CP}^2$, where $F = S^1 \times S^2$, or $F = S^3/Z_p$ with $p \neq 0$. Indeed, for each $z$, consider $X(z) = \pi^{-1}(z)$ and $W_{k,l}(z) = \text{pr}^{-1}(z)$, where $\text{pr} = \pi \circ \varphi$. If we restrict to each $X(z)$, this circle bundle $S^1 \rightarrow W_{k,l}(z) \rightarrow X(z)$ must be $W_{k,l}(z) = S^1 \times S^2$ if the Euler class is $e = 0$, or a lens space $S^3/Z_e$ if the Euler class is $e \neq 0$. Varying over all $z \in \mathbb{CP}^2$, we have a fiber bundle $F \rightarrow W_{k,l} \rightarrow \mathbb{CP}^2$, where $F = S^1 \times S^2$ or $F = S^3/Z_e$.

6. The nearly parallel $G_2$-manifold $Q(1, 1, 1)$

In this section, we consider the regular Sasaki–Einstein 7-manifold $Q(1, 1, 1)$ and we work out explicitly, in coordinates, the $S^1$-family of nearly parallel $G_2$-structures on it, which we will use in Section 7 and Section 9.

We start from the Kähler manifold $X = S^2 \times S^2 \times S^2$ with Kähler form

$$\omega = \omega_1 + \omega_2 + \omega_3, \quad (6.1)$$

where $\omega_1$, $\omega_2$ and $\omega_3$ are the generators of the integral cohomology group for each of the $S^2$-factors on $S^2 \times S^2 \times S^2$. Let $M$ be the total space of the principal $S^1$-bundle

$$S^1 \hookrightarrow M \xrightarrow{\pi} X = S^2 \times S^2 \times S^2, \quad (6.2)$$

with Euler class $[\omega] \in H^2(X, \mathbb{Z})$. Then, by application of Theorem 2.4, $M$ is a simply connected compact Sasaki–Einstein manifold, with contact form $\eta$ such that $d\eta = 2\pi^*_M(\omega)$. From [9 §4.2] (see also [28, 29]), we know that $M$ is the homogeneous space

$$M = Q(1, 1, 1) = \left( SU(2) \times SU(2) \times SU(2) \right) / \left( U(1) \times U(1) \right). \quad (6.3)$$

Throughout this section, the notation $M$ and $X$ will always mean the manifolds defined in (6.2) and (6.3).

We apply the Kobayashi construction [40] to determine the contact form for the Sasaki–Einstein metric of the principal circle bundle $M$. Let $\theta_j \in (0, \pi)$ and $\phi_j \in (0, 2\pi)$, $j = 1, 2, 3$, be the standard spherical coordinates on each of the $S^2$-factors in $X$ and $z_j = \cot(\theta_j/2)e^{i\phi_j} \in \mathbb{C}$ a complex coordinate defined via the stereographic projection. The Fubini–Study metric of volume 1 on the $j$-th $S^2$ factor has the Kähler form $\omega_j = \frac{1}{4\pi} dd^c \log(1 + z_j \bar{z}_j) = \alpha_{2j-1} \wedge \alpha_{2j}$, where we used an orthonormal co-frame field

$$\alpha_{2j-1} = \frac{-1}{2\sqrt{\pi}} d\theta_j, \quad \alpha_{2j} = \frac{1}{2\sqrt{\pi}} \sin \theta_j d\phi_j, \quad j = 1, 2, 3, \quad (6.4)$$
on the open dense region \( U_X \subset X \) defined by \( z_j \neq 0 \) for all \( j \). The Kähler form \((6.1)\) on \( X \) restricts to an exact form on \( U_X \)

\[
\omega = \alpha_1 \wedge \alpha_2 + \alpha_3 \wedge \alpha_4 + \alpha_5 \wedge \alpha_6 = \frac{1}{4\pi} \sum_{j=1}^{3} d(\cos \theta_j d\phi_j).
\]

(6.5)

The principal \( S^1 \)-bundle \( M \) trivializes over \( U_X \), \( \pi_M^{-1}(U_X) \cong U_X \times S^1 \). Let \( s \in (0, 4\pi) \) denote a coordinate on the fibre \( S^1 \) and consider on \( \pi_M^{-1}(U_X) \) a 1-form

\[
\eta = \frac{1}{2\pi} (-ds + \sum_{j=1}^{3} \cos \theta_j d\phi_j).
\]

(6.6)

Noting that \( \omega \in H^2(X, \mathbb{Z}) \) is an integral cohomology class, it can be checked by following the general construction in \([40, \S 2]\) that \( \eta \) extends smoothly from \( \pi_M^{-1}(U_X) \) to a well-defined connection 1-form on all of \( M \) (still denoted by \( \eta \)). Since

\[
d\eta = 2\pi^* M(\omega),
\]

(6.7)

it also follows that \( \eta \) is a well-defined contact form on \( M \) (cf. Section 2.2).

We shall slightly abuse the notation by writing \( \alpha_i \) also for the lifting to \( M \) via \( \pi_M^* \) of the local 1-forms \((6.4)\). Then \( \alpha_1 + i\alpha_2, \alpha_3 + i\alpha_4, \alpha_5 + i\alpha_6 \) can be considered as point-wise orthonormal \((1, 0)\)-forms on the complex cone \( \pi^{-1}(U_X) \times \mathbb{R}^+ \) as \( \pi_M \) is a Riemannian submersion and the associated bundle projection \( M \times \mathbb{R}^+ \to X \) is holomorphic.

The Riemannian cone \( M \times \mathbb{R}^+ \) is simply-connected and Ricci-flat Kähler. Therefore, \( M \times \mathbb{R}^+ \) has a non-vanishing holomorphic \((4, 0)\)-form \( \hat{\Psi} \), which is parallel with respect to the Kähler metric. In local coordinates, the Kähler form on \( M \times \mathbb{R}^+ \) is given by \( r dr \wedge \eta + r^2 \sum_{j=1}^{3} \alpha_{2j-1} \wedge \alpha_{2j} \) (cf. (2.4)) and \( \hat{\Psi} \) can be written as

\[
\hat{\Psi} = r^3 e^{i\mu} \left( dr + \frac{i\pi}{2} r\eta \right) \wedge (\alpha_1 + i\alpha_2) \wedge (\alpha_3 + i\alpha_4) \wedge (\alpha_5 + i\alpha_6),
\]

(6.8)

where a smooth real function \( \mu \) on \( M \) is determined, up to adding a constant, by the condition \( d\hat{\Psi} = 0 \). Setting

\[
\Psi = e^{-i\mu}(\alpha_1 + i\alpha_2) \wedge (\alpha_3 + i\alpha_4) \wedge (\alpha_5 + i\alpha_6),
\]

(6.9)

it is straightforward to check that

\[
d\Psi = -2\pi i \Psi \wedge \eta,
\]

(6.10)

thus, noting also that \( \Psi \wedge d\eta = 0 \), one can take \( \mu = -s \) in \((6.8)\). The holomorphic 4-form \( \hat{\Psi} \) extends from \( \pi^{-1}(U_X) \times \mathbb{R}^+ \) to all of \( M \times \mathbb{R}^+ \) (e.g. by patching with a different choice of spherical coordinate neighbourhood in \((6.4)\)). Respectively, \( \Psi \) extends to a well-defined horizontal complex volume form on \( M \).
We can now write a coordinate expression for the $S^1$-family of nearly parallel $G_2$-structures $\varphi_t$ on $M$ induced by the Sasaki–Einstein structure, cf. (2.6),

$$\varphi_t = \frac{\pi}{2} \omega \wedge \eta + \cos(s + t) \Psi_+ + \sin(s + t) \Psi_- = \frac{\pi}{2} \omega \wedge \eta + \text{Re}(e^{-it}\Psi)$$  \hspace{1cm} (6.11)

where the real 3-forms $\Psi_+$, $\Psi_-$ in (6.11) are determined by $\Psi = e^{-is}(\Psi_+ + i\Psi_-)$ and can be written in coordinates using (6.9). The respective $G_2$ 4-forms are

$$\ast_{\varphi_t} \varphi_t = \frac{1}{2} \omega \wedge \omega + (\cos(s + t) \Psi_- - \sin(s + t) \Psi_+) \wedge \eta = \frac{1}{2} \omega \wedge \omega + \text{Im}(e^{-it}\Psi) \wedge \eta,$$

and from (6.7) and (6.10) it follows that

$$d\varphi_t = 2\pi \ast \varphi_t,$$  \hspace{1cm} (6.12)

for each $t$. The induced metric $g = g_{\varphi_t}$ on $M$ does not depend on $t$ and is given by

$$g = \frac{1}{16} \left( ds - \sum_{j=1}^{3} \cos \theta_j \, d\phi_j \right)^2 + \frac{1}{4\pi} \sum_{j=1}^{3} \left( d\theta_j^2 + \sin^2 \theta_j \, d\phi_j^2 \right),$$

so the local coframe of 1-forms $\{\alpha_1, \ldots, \alpha_6, \alpha_7 = \eta\}$ is orthonormal.

### 7. Associative 3-folds in Sasaki–Einstein 7-manifolds

We now turn to consider minimal associative 3-folds in nearly parallel $G_2$-manifolds. In this section, we deal with the case when the nearly parallel $G_2$-structure arises from a Sasaki–Einstein structure on a 7-manifold, with particular attention to the regular Sasaki–Einstein manifolds $Q(1,1,1)$ and $S_k$ discussed in Section 4 and Section 6.

Let $(S, g_S)$ be a Sasaki–Einstein 7-manifold and $\varphi_t$ the corresponding $S^1$-family of nearly parallel $G_2$-structures on $S$ defined in (2.6). Assume that $S$ is simply-connected, so the metric cone $S \times \mathbb{R}^+$ has holonomy in SU(4), and denote by $\omega^c$ the Kähler form and by $\hat{\Psi}$ a non-vanishing holomorphic 4-form on $S \times \mathbb{R}^+$. Then $S \times \mathbb{R}^+$ has a 1-parameter family of torsion-free $\text{Spin}(7)$-structures induced by the closed 4-forms

$$\hat{\Phi}_t = \frac{1}{2} \omega^c \wedge \omega^c + \text{Re}(e^{-it}\hat{\Psi}), \quad t \in \mathbb{R}. \hspace{1cm} (7.1)$$

cf. [37, Prop. 13.1.4]. Recall that we identify the 7-manifold $S$ as a submanifold $S \times \{1\}$ of the cone. The nearly parallel $G_2$-structures $\varphi_t$ on $S$ are then related to the $\text{Spin}(7)$-structures (7.1) by

$$\varphi_t = \partial_t \hat{\Phi}_t |_{r=1}, \quad \text{for each } t,$$

cf. [11 pp. 723–724]. If a 4-dimensional submanifold $Z$ of $S \times \mathbb{R}^+$ is calibrated by $\hat{\Phi}$ (i.e. $Z$ is a Cayley submanifold) and $Z = Y \times \mathbb{R}^+$ for some submanifold $Y \subset S$, then $(\varphi_t|_Y) \wedge dr$ is the volume form of the conical metric $r^2 g_S + dr^2$ on $Y \times \mathbb{R}^+$ as $\partial_t$ defines a unit normal vector field along each $Y \times \{r\}$. It follows that $Y$ is an associative
3-fold in $S$. Conversely, if $Y \subset S$ is an associative 3-fold then $Z = Y \times \mathbb{R}^+ \subset S \times \mathbb{R}^+$ is Cayley.

Examples of Cayley submanifolds, in the case when the Spin(7)-structure is of the form $(\mathbb{C}P^1)$ induced by an SU(4)-structure, include complex surfaces and special Lagrangian submanifolds. We shall consider these two possibilities in order.

If $Z = Y \times \mathbb{R}^+$ is a complex surface in $S \times \mathbb{R}^+$, so the tangent spaces of $Z$ are preserved by the (integrable) almost complex structure $J$ on $S \times \mathbb{R}^+$, then $Y$ is an ‘invariant’ submanifold for the contact structure on $S$ as defined in [13, §8.1]. More explicitly, the endomorphism $\Phi$ discussed in Section 2.2 maps each tangent space of $Y$ into itself and the Reeb vector field $\xi$ is tangent to $Y$.

**Proposition 7.1.** Let $S$ be a regular Sasaki–Einstein 7-manifold with contact form $\eta$ arising from a principal $S^1$-bundle $\pi : S \to X$ with Euler class $c_1 = [\omega]$, where $X$ is a projective complex 3-fold with Kähler form $\omega$ and $d\eta = \pi^*(\omega)$. Let $\varphi_t$ be the corresponding 1-parameter family of nearly parallel $G_2$-forms defined in (2.6).

Then, for each complex curve $\Sigma$ in $X$, the preimage $Y_\Sigma = \pi^{-1}(\Sigma) \subset S$ is an invariant submanifold for the contact structure $\eta$ and a (minimal) associative 3-fold with respect to $\varphi_t$ for each $t$.

In particular, if $S$ is $Q(1,1,1)$ or $S_k$, so $X = \mathbb{C}P^1 \times P$, with $P = \mathbb{C}P^1 \times \mathbb{C}P^1$ or $P = P_k$ ($3 \leq k \leq 8$) a del Pezzo surface, and $\omega = \omega_1 + \omega_\rho$, where $[\omega_1]$ is the (positive) generator of $H^2(\mathbb{C}P^1, \mathbb{Z})$, $[\omega_\rho] \in H^2(P, \mathbb{Z})$ and $\Sigma = \mathbb{C}P^1 \times \{p\}$, $p \in P$, then the minimal associative $Y_\Sigma$ is diffeomorphic to the sphere $S^3$.

**Proof.** Recall from (2.4) that the Kähler form on the cone $S \times \mathbb{R}^+$ is

$$\omega^c = r^2d\eta + 2r\,dr \wedge \eta = r^2\pi^*(\omega) + 2r\,dr \wedge \eta,$$

(7.2)

where $r \in \mathbb{R}^+$ and we extended $\pi$ to a projection $\pi : S \times \mathbb{R}^+ \to X$ independent of the $\mathbb{R}^+$ factor. Pulling back to submanifolds $Y_\Sigma \times \mathbb{R}^+$ and $\Sigma \subset Y_\Sigma \times \{1\}$ we find that

$$\frac{1}{2}(\omega^c \wedge \omega^c)|_{Y_\Sigma \times \mathbb{R}^+} = \pi^*(\omega|_{\Sigma \times \{1\}}) \wedge 2\eta \wedge dr = \text{vol}_{Y_\Sigma \times \mathbb{R}^+},$$

noting also that $\pi$ is a Riemannian submersion and $2\eta$ has unit length in the metric induced by $\varphi_t$. Thus, $Y_\Sigma \times \mathbb{R}^+$ is calibrated by $\frac{1}{2}\omega^c \wedge \omega^c$ and must be a complex surface in $S \times \mathbb{R}^+$ by the Wirtinger inequality. The link $Y_\Sigma$ is therefore associative.

For the last part, we can consider $S \times \mathbb{R}^+$ as the total space of a holomorphic line bundle over $X$. The restriction of this bundle to a projective line $\Sigma$ is isomorphic to the hyperplane bundle $\mathcal{O}(1)$ over $\mathbb{C}P^1$. The total space $Y_\Sigma \times \mathbb{R}$ is then biholomorphic to $\mathbb{C}P^2 \setminus \{(0:0:1)\}$. We obtain that the associative 3-fold $Y_\Sigma$ is diffeomorphic to the sphere $S^3$, noting that the fibres of $Y_\Sigma \times \mathbb{R}$ correspond to projective lines through $(0:0:1)$, so the zero section $\Sigma$ can be identified as a projective line in $\mathbb{C}P^2$ avoiding $(0:0:1)$.
Remarks.

(i) It is not difficult to see that every minimal associative $Y_\Sigma$ in Proposition 7.1 is invariant under the (isometric) $S^1$-action on the principal bundle $S$ and every deformation of the holomorphic curve $\Sigma$ in $X$ induces an associative deformation of $Y_\Sigma$.

(ii) We obtain from (2.6) that $\varphi_t|_{Y_\Sigma} = 2\eta \wedge \Omega_\Sigma$, where $\Omega_\Sigma = \pi^*(\omega|_\Sigma)$. As $\pi$ is a Riemannian submersion, the form $\Omega_\Sigma$ at each point in $Y_\Sigma$ can be written as a wedge product of two unit-length covectors which are orthogonal to $\eta$ in the metric induced by $\varphi_t$. (If $S = Q(1,1,1)$ and $\Sigma = \mathbb{CP}^1 \times \{(p_1,p_2)\}$, then $\Omega_\Sigma = \alpha_1 \wedge \alpha_2$ in the notation of (6.4).)

(iii) In the last part of Proposition 7.1, one can more generally take $\Sigma$ to be the graph of a holomorphic embedding $\mathbb{CP}^1 \rightarrow P$. For $S = Q(1,1,1)$, the ambiguity of taking such $\Sigma$ corresponds to a generic choice of two rational functions of one complex variable.

Concerning deformations of minimal associative 3-folds for nearly parallel $G_2$-structures of this type, we more generally have:

**Proposition 7.2.** Let $S$ be a regular Sasaki–Einstein 7-manifold with contact form $\eta$ arising from a principal $S^1$-bundle $\pi : S \rightarrow X$ and let $\varphi_t$ be the induced $S^1$-family (2.6) of nearly parallel $G_2$-structures on $S$.

If $Y \subset S$ is an associative 3-fold with respect to $\varphi_{t_0}$ for some fixed $t_0$ and $Y \neq \pi^{-1}(\Sigma)$ for any real 2-dimensional submanifold $\Sigma \subset X$, then the free $S^1$-action on the principal bundle $S$ induces non-trivial $\varphi_{t_0}$-associative deformations of $Y$ (in particular, $Y$ is not rigid).

**Proof.** The free $S^1$-action on $S$ preserves the family $\varphi_t$ and all $\varphi_t$ induce the same (Sasaki–Einstein) metric on $S$. Therefore, the $S^1$-orbit of $Y$ consists of volume minimizing submanifolds of $S$ which are in general distinct from $Y$ by the hypothesis.

The result now follows by application of [35, Theorem II.4.2]. □

We now turn to consider associative 3-folds arising as links of Cayley submanifolds $Z = Y \times \mathbb{R}^+$ which are a special Lagrangian in $S \times \mathbb{R}^+$, so $\omega_c|_Z = 0$ and $\text{Re}(e^{-it} \hat{\Psi})|_Z = 0$ for some fixed $t$. Then (and only then) the cross-section $Y$ is, by definition, a special Legendrian submanifold of the Sasaki–Einstein manifold $S$. Equivalently, $\eta|_Y = 0$ and $\text{Re} \Psi_t|_Y = 0$, where $\Psi_t = \partial_r(e^{-it} \hat{\Psi})|_{r=1}$ is the horizontal volume form, cf. [44, Prop. 3.3]. (As before, $\eta$ is the contact form on $S$.) Since $Y$ is associative with respect to the $G_2$-structure $\varphi_t$, $Y$ is a minimal submanifold for the Sasaki–Einstein metric on $S$. Conversely, it is known that an oriented Legendrian submanifold $Y \subset S$ is minimal if and only if $Y$ is special Legendrian [44, Prop. 3.2].

The following result will be useful for producing examples of associative 3-folds.
Proposition 7.3 ([11] Prop. 3.4]). Let $S$ be a Sasaki–Einstein $(2n+1)$-manifold and let $\tau_S : S \times \mathbb{R}^* \to S \times \mathbb{R}^*$ be an anti-holomorphic involution for the complex structure of the Kähler–Einstein cone $S \times \mathbb{R}^*$ fixing the coordinate $r \in \mathbb{R}^+$. If the fixed point set $C_{\tau}$ of $\tau$ is not empty, then the link $C_{\tau} \cap (S \times \{1\})$ is a special Legendrian submanifold of $S$.

In the case of a regular Sasaki–Einstein manifold $S$ we have the following.

**Theorem 7.4.** Let $S$ be a regular Sasaki–Einstein 7-manifold with contact form $\eta$ arising from a principal $S^1$-bundle $\pi : S \to X$ with Euler class $c_1 = [\omega]$, where $X$ is a Kähler–Einstein Fano 3-fold with Kähler form $\omega$ and $d\eta = \pi^*(\omega)$. Let $\varphi_\eta$ be the corresponding 1-parameter family of nearly parallel $G_2$-forms defined in $(2.6)$.

Then for each compact special Legendrian submanifold $Y \subset S$, the restriction $\pi|_Y : Y \to Y_X$ is a finite covering of a Lagrangian submanifold $Y_X \subset X$.

If $Y_X \subset X$ is a compact simply-connected Lagrangian submanifold, thus a Lagrangian 3-sphere, then $Y_X$ lifts to an $S^1$-family of Legendrian submanifolds $Y_s \subset S$ such that $\pi(Y_s) = Y_X$ for each $s \in S^1$.

Assume that $\tau : X \to X$ is an isometric anti-holomorphic involution. If the fixed point set $Y_X \subset X$ of $\tau$ is non-empty, then $Y_X$ is Lagrangian and diffeomorphically lifts to a special Legendrian (hence associative) submanifold of $S$.

**Proof.** If a submanifold $Y$ is Legendrian, i.e. $\eta|_Y = 0$, then $Y$ cannot be tangent to any fibre in the principal circle bundle $S$ and $\pi$ maps $Y$ locally diffeomorphically onto the image $\pi(Y) = Y_X$ which is a submanifold of $X$. Since $Y$, hence also $Y_X$, are compact we obtain that $\pi|_Y$ is a finite cover. Considering $(7.2)$ we find that $Y_X$ is Lagrangian submanifold of $X$, $\omega|_{Y_X} = 0$.

For the second claim, since $c_1 = [\omega]$ vanishes on $Y_X$ and $d\eta = \pi^*(\omega)$, the connection $\eta$ on the $S^1$-bundle $S$ restricts to a flat connection over $Y_X$. Furthermore, $\eta|_{Y_X}$ must be a trivial product connection as $Y_X$ is simply connected. So $\pi^{-1}(Y_X) \cong Y_X \times S^1$ with $\eta|_{Y_X}$ corresponding to $ds$, $s \in S^1$, and $Y_s = Y_X \times \{s\}$ for each $s$, is a Legendrian submanifold of $S$.

The principal bundle $S$ is associated to the anticanonical bundle $K_X^{-1}$ and $S \times \mathbb{R}^*$ is biholomorphic to the complement of the zero section. The hypotheses on $\tau$ imply that $\tau^*\omega_X = -\omega_X$ and $\tau^*K_X^{-1}$ is a holomorphic line bundle isomorphic to $K_X^{-1}$. We find that $\tau^*$ defines a lift of $\tau$ to an antiholomorphic involution on $K_X^{-1}$ and (by restriction) on $S \times \mathbb{R}^+$ with $\tau^*\eta = -\eta$ and $\tau \circ r = r$. Thus $\tau^*$ preserves the Kähler-Einstein metric on $S \times \mathbb{R}^+$ and the fixed point set of $\tau^*$ is an oriented (hence trivial) real line bundle over $Y_X$. The desired special Legendrian is obtained as the intersection with $S = S \times \{1\}$ by application of Proposition 7.3. □
If the Fano 3-fold $X$ also appears as a smooth fibre in a Lefschetz fibration $\lambda : E \to \Delta$ over a disc with $0 \in \Delta$ the only critical value, then the vanishing cycles in $X$ (the cycles that degenerate to a point as $X$ is deformed to a fibre over $0$) are represented by Lagrangian 3-spheres (see [4] or [50, § 4]).

In the case when the nearly parallel $G_2$-manifold $S$ is $Q(1,1,1)$ or $S_3$ (see Theorem 4.5) we can say more.

**Proposition 7.5.** Let $\pi_M : M = Q(1,1,1) \to X = S^2 \times S^2 \times S^2$ be the principal $S^1$-bundle (6.2) and $\varphi_t$ the 1-dimensional family of nearly parallel $G_2$-structures (6.11) on $M$. Let $L \subset X$ be a 3-torus defined by $\theta_j = \pi/2$, $j = 1,2,3$, in the spherical coordinates $\phi_j, \theta_j$ on $X$ (see Section 6).

Then $L$ lifts via $\pi_M$ to a family of minimal Legendrian 3-tori $L_s \subset M$, $s \in \mathbb{R}/2\pi\mathbb{Z}$. For each $s$, the 3-torus $L_s$ is associative with respect to $\varphi_t$ for all $t$.

As $X$ and hence $Q(1,1,1)$ are toric manifolds, the existence of a compact special Legendrian submanifold in $Q(1,1,1)$ follows from the main result in [44]. However, the explicit examples of special Legendrians in $Q(1,1,1)$ are not considered in [44].

**Proof of Proposition 7.5.** It follows at once from the expression (6.5) for the Kähler form that the 3-torus $L$ is Lagrangian in $X$. We know from (6.7) that $\omega$ is proportional to the curvature of the principal $S^1$-bundle $M$, thus the restriction of this bundle to $L$ has first Chern class zero and is a trivial bundle. Furthermore, with $\theta_j = \pi/2$ the 1-form $\eta$ in (6.6) defines a trivial product connection on $\pi_M^{-1}(L)$. Thus $L$ lifts via $\pi$ to a family of horizontal 3-tori $L_s \subset M$, parametrised by $s \in S^1$.

When $s + t = \pi/2$, from (6.11) and (6.9) we obtain that $\varphi_t|_{L_s} = \Psi_-|_{L_s} = -\alpha_2 \wedge \alpha_4 \wedge \alpha_6$, whence $L_s$ (with appropriate orientation) is $\varphi_t$-associative and minimal Legendrian. Then $L_s$ is $\varphi_t$-associative, for all $t$, by application of Proposition 7.2. □

The Fano 3-fold $X_3 = P_3 \times \mathbb{CP}^1$ is toric because $P_3$ is so. Recall that the del Pezzo surface is the blow-up $P_3 = \mathbb{CP}^2 \# 3 \bar{\mathbb{CP}}^2$ and is, up to isomorphism, independent of the choice of three non-colinear points in $\mathbb{CP}^2$. Also, $P_3$ admits a Kähler–Einstein metric (51 or 52) which can be taken to be invariant under the torus action. (The induced nearly parallel $G_2$-structure $\varphi_t$ of (2.6) on $S_3$ therefore is also torus-invariant for all $t$.)

We construct a special Legendrian associative 3-fold in $S_3$ as a fixed point set of an anti-holomorphic involution. The existence of this special Legendrian is of course again a special case of the main result in [44], but is rather simpler than in a general case.

The complex surface $P_3$ can be identified as the simultaneous solution of two complex bilinear equations

$$\{(z, w) \in \mathbb{CP}^2 \times \mathbb{CP}^2 : z_0 w_0 = z_1 w_1 = z_2 w_2\} \quad (7.3)$$
and the blow up $P_3 \to \mathbb{CP}^2$ is the restriction of the first projection. The effective torus action on $P_3$ is given by $(z_0, z_1, z_2, w_0, w_1, w_2) \mapsto (\xi_0 z_0, \xi_1 z_1, \xi_2 z_2, \xi_0^{-1} w_0, \xi_1^{-1} w_1, \xi_2^{-1} w_2)$, where $\xi_i \in \mathbb{C}^\ast$, $\xi_0 \xi_1 \xi_2 = 1$, and this can be interpreted as an embedding of $(\mathbb{C}^\ast)^2$ in $P_3$ as an open dense subset. We deduce that in the notation of (7.3) the map $\tau_3(z_i, w_i) = (\bar{w}_i, \bar{z}_i)$ induces an antiholomorphic isometry of the Kähler–Einstein $P_3$. The fixed point set of this involution is diffeomorphic to the 2-torus $\mathbb{T}^2$.

Extending to an antiholomorphic involution of the Fano 3-fold $X_3$ with the complex conjugation on the $\mathbb{CP}^1$-factor, we obtain, by application of Theorem 7.4.

**Proposition 7.6.** There exists an associative 3-torus in the nearly parallel $G_2$-manifold $(S_3, \varphi_1)$

8. Associative 3-folds in the Aloff–Wallach spaces

We now revisit the Aloff–Wallach spaces $W_{k,l}$, discussed in Subsection 5.2 and find examples of associative submanifolds in $W_{k,l}$. Recall that $k, l$ are non-zero, co-prime integers. We begin by briefly recalling from [8, 20] the construction of proper nearly parallel $G_2$-structures on $W_{k,l}$.

Let $\{e_i\}_{i=0, \ldots, 7}$ be a basis of left-invariant vector fields on $SU(3)$ such that $e_0$ is everywhere tangent to the orbits of the $S^1_{k,l}$ action. We shall interchangeably think of $e_i$ as elements of the Lie algebra $su(3)$ with $e_0 \in su_{k,l}$. Choose $\{e_i\}$ so that writing $\{e^i\}$ for the dual basis of $su(3)^\ast$ (or the left-invariant 1-forms) the Maurer–Cartan form $\Omega = \sum e_i e^i$ on $SU(3)$ with values in $su(3)$ is expressed as

$$\Omega = \frac{1}{\sqrt{2}} \begin{pmatrix} i\left(\frac{k}{\sqrt{3}}s e^0 + \frac{l-m}{3s}e^4\right) & e^1 + ie^5 & -e^3 + ie^7 \\ -e^1 + ie^5 & i\left(\frac{k}{\sqrt{3}}s e^0 + \frac{m-k}{3s}e^4\right) & e^2 + ie^6 \\ e^3 + ie^7 & -e^2 + ie^6 & i\left(\frac{m}{\sqrt{3}}s e^0 + \frac{k-l}{3s}e^4\right) \end{pmatrix},$$

(8.1)

where $m = -k - l$ and $s = \sqrt{(k^2 + l^2 + m^2)/6}$.

Then for any choice of non-zero constants $A, B, C, D$ the 3-form

$$\varphi_W = ABC(e^{123} - e^{167} + e^{257} - e^{356}) - D(A^2 e^{15} + B^2 e^{26} + C^2 e^{37}) \wedge e^4$$

(8.2)

descends to a well-defined positive 3-form giving a coclosed $G_2$-structure on $W_{k,l}$. The induced orientation corresponds to a non-vanishing 7-form $D e^{1234567}$ and the coframe $A e^1, B e^2, C e^3, D e^4, A e^5, B e^6, C e^7$ of 1-forms is orthonormal in the induced metric $g_{\varphi_W}$. Note that we may assume without loss of generality that $A > 0$ and $B > 0$. From the results in [20] it follows that if $\{k, l, m\}$ are not $\{1, 1, -2\}$ or $\{1, -1, 0\}$, then every homogeneous, coclosed $G_2$-structure on $W_{k,l}$ is of the form (8.2). Furthermore, by using the Maurer–Cartan equation $d\Omega = -\frac{1}{2} [\Omega, \Omega]$, it was proved that exactly two such $G_2$-structures are nearly parallel, more precisely, these are proper nearly parallel.
Proposition 8.1. Let $\varphi_W$ be a nearly parallel $G_2$-structure of the form (8.2) on an Aloff–Wallach space $W_{k,l}$, with $k, l$ non-zero and co-prime integers such that $\{k, l, m\}$ are not $\{1, 1, -2\}$ or $\{1, -1, 0\}$. Then $W_{k,l}$ is a smooth fibre bundle $\pi_{k,l} : W_{k,l} \to \mathbb{CP}^2$ with typical fibre the spherical space form $S^3/\mathbb{Z}_{|k+l|}$. Furthermore, the fibres of $\pi_{k,l}$ are embedded minimal associative 3-folds with respect to $\varphi_W$.

When the $G_2$-structure $\varphi_W$ in Proposition 8.1 is not nearly parallel, the fibres of $\pi_{k,l}$ are associative 3-folds but need not be minimal.

Proof. The fibre bundle $\pi : W_{k,l} \to \mathbb{CP}^2$, noted in Subsection 5.2 above, can be obtained by considering the embedding of $U(2)$ as a Lie subgroup of $SU(3)$ consisting of the block-diagonal matrices with blocks $A e^{i\theta}$ and $1/\det(A e^{i\theta}) = e^{-2i\theta}$, for all $A \in SU(2)$ and $\theta \in \mathbb{R}$. This induces a fibre bundle

$$\pi_{k,l} : W_{k,l} \to SU(3)/U(2) \cong \mathbb{CP}^2$$

with fibre $U(2)/S^1_{k,l} \cong S^3/\mathbb{Z}_{|k+l|}$.

Comparing the tangent spaces to the cosets of $U(2)$ in $SU(3)$ with (8.1), we find that the vertical spaces of the principal $U(2)$-bundle $SU(3) \to \mathbb{CP}^2$ are defined by the vanishing of $e^2, e^3, e^6, e^7$. Thus $\varphi_W$ restricts on each fibre of $\pi_{k,l}$ to the volume form $A^2 De^{145}$ of the metric induced by $g_{\varphi_W}$. □

Remark 8.2. The exceptional Aloff–Wallach space $W_{1,-1}$ has, up to homotheties, only one homogeneous nearly parallel $G_2$-structure, which is of the form (8.2). The argument and result of Proposition 8.1 applies to this latter $G_2$-structure noting that the fibres of $\pi_{1,-1}$ are now $S^2 \times S^1$ as $S^1_{1,-1}$ is a subgroup of $SU(2)$.

Note that a fibration of $W_{k,l}$ over $\mathbb{CP}^2$ by spherical space forms is not unique. The Weyl group of $SU(3)$ contains an element of order 3 which induces a diffeomorphism $W_{k,l} \to W_{l,m}$. A composition with this diffeomorphism defines a different fibration, generally by different spherical space forms. We next show that the fibres of this latter map are also minimal associative for the same $G_2$-structure $\varphi_W$ as in Proposition 8.1.

To this end, we further note that $\pi_{k,l}$ factors through the flag manifold $F(1, 2) \cong SU(3)/T^2$ (where $T^2 \subset SU(3)$ is the subgroup of diagonal matrices),

$$W_{k,l} \to F(1, 2) \to \mathbb{CP}^2.$$  

Here the first map is a principal $S^1$-bundle and the second map is a $\mathbb{CP}^1$-bundle associated to the principal $U(2)$-bundle $SU(3) \to \mathbb{CP}^2$ discussed above. In particular, the vertical space of the $S^1$-bundle is spanned by $e_4$.

It is well-known that $F(1, 2)$ is a complex manifold with a Kähler structure defined by the Kirillov–Kostant–Souriau symplectic form. We can consider a different choice of the $\mathbb{CP}^1$-bundle $h_3 : F(1, 2) \to \mathbb{CP}^2$ corresponding to a different embedding of $U(2)$ as a Lie subgroup of $SU(3)$, with the tangent space at the identity now spanned
by $e_0, e_3, e_4, e_7 \in \text{su}(3)$ (rather than $e_0, e_1, e_4, e_5$ chosen above). Let $Y$ be a fibre of the map $W_{k,l} \to F(1, 2) \xrightarrow{h_3} \mathbb{CP}^2$. Then the forms $e^1, e^5, e^2, e^6$ vanish on $Y$ and so $\varphi_W|_Y = C^2 D e^{347}$. We thus have the following.

**Proposition 8.3.** Assume the hypotheses of Proposition 8.1. Then $Y$ is an embedded minimal associative 3-fold with respect to a $G_2$-structure $\varphi_W$ of the form (8.2) on $W_{k,l}$.

We similarly obtain a further family of associative 3-folds by replacing the $\mathbb{CP}^1$-bundle $h_3$ with $h_2 : F(1, 2) \to \mathbb{CP}^2$ where the vertical spaces are now spanned by $e_0, e_2, e_4, e_6$. Thus, for suitably ‘generic’ $k,l$, the Aloff–Wallach space $W_{k,l}$ has three 4-dimensional deformation families of minimal associative spherical space forms.

**Remark 8.4.** For the exceptional Aloff–Wallach space $W_{1,1}$ there are still exactly two homogeneous Einstein metrics [47]. However, one of these metrics is induced by a proper nearly parallel $G_2$-structure on $W_{1,1}$ which is not of the form (8.2), whereas the other metric is induced by a 3-Sasakian structure on $W_{1,1}$. For this latter 3-Sasakian manifold, the above construction of minimal associative 3-folds based on (8.3) remains valid. On the other hand, very recently Ball and Madnick constructed in $W_{1,1}$ examples of associative 3-folds diffeomorphic to an $S^1$-bundle over a genus $g$ surface for all $g \geq 0$ [7, Theorem 5.9].

9. Locally conformal parallel Spin(7)-structures

In this section we give examples of formal compact 8-manifolds, with a locally conformal parallel Spin(7)-structure.

An 8-dimensional manifold $N$ has a Spin(7)-structure if there is a reduction of the structure group of its frame bundle from $\text{GL}(8, \mathbb{R})$ to the exceptional Lie group Spin(7). In opposite to the existence of $G_2$-structures on any orientable and spin manifold of dimension 7, it happens that not every 8-dimensional spin manifold $N$ admits a Spin(7)-structure. In fact, in [42] it is proved that $N$ has a Spin(7)-structure if and only if

$$p_1^2(N) - 4 p_2(N) + 8 \chi(N) = 0,$$

for an appropriate choice of the orientation, where $p_1(N), p_2(N)$ and $\chi(N)$ are the first Pontryagin class, the second Pontryagin class and the Euler characteristic of $N$, respectively.

The presence of a Spin(7)-structure is equivalent to the existence of a nowhere vanishing global differential 4-form $\Omega$ on the 8-manifold $N$, which can be written locally as

$$\Omega = \left(e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}\right) \wedge e^8$$
$$+ e^{1234} + e^{1256} + e^{1367} + e^{1457} + e^{2357} + e^{2467} + e^{3456},$$
with respect to some (local) basis \(\{e^1, \ldots, e^8\}\) of the (local) 1-forms on \(N\).

Since \(\text{Spin}(7) \subset \text{SO}(8)\), a \(\text{Spin}(7)\)-structure \(\Omega\) on \(N\) determines a Riemannian metric \(g_{\Omega}\) and an orientation on \(N\) such that

\[
g_{\Omega}(U, V) \text{ vol} = \frac{1}{7} (U \lrcorner \Omega) \wedge \ast_8 (V \lrcorner \Omega),
\]

for any vector fields \(U, V\) on \(N\), where \(\text{vol}\) is the volume form on \(N\), and \(\ast_8\) is the Hodge star operator determined by \(g_{\Omega}\). Note that if \(\Omega\) is a \(\text{Spin}(7)\)-structure on \(N\), then \(\Omega\) is a self-dual 4-form, i.e. \(\ast_8 \Omega = \Omega\).

Let us recall that a \(\text{Spin}(7)\)-structure \(\Omega\) on a 8-manifold \(N\) is said to be parallel if the induced metric by \(\Omega\) has holonomy contained in \(\text{Spin}(7)\). This is equivalent to say that the 4-form \(\Omega\) is parallel with respect to the Levi–Civita connection of the metric \(g_{\Omega}\), which happens if and only if \(d\Omega = 0\) [24].

**Definition 9.1.** A \(\text{Spin}(7)\)-structure \(\Omega\) on a 8-manifold \(N\) is said to be locally conformal parallel if

\[
d\Omega = \Omega \wedge \Theta,
\]

for a closed non-vanishing 1-form \(\Theta\), which is known as the Lee form of the \(\text{Spin}(7)\)-structure. A manifold endowed with such a structure is called locally conformal parallel \(\text{Spin}(7)\)-manifold.

Let \(\varphi\) be a nearly parallel \(G_2\)-structure on a 7-manifold \(P\), so that \(d\varphi = \tau_0 \ast \varphi\) by (2.2). Then, the product manifold \(P \times S^1\) carries a natural locally conformal parallel \(\text{Spin}(7)\)-structure \(\Omega\) defined by

\[
\Omega = \varphi \wedge \theta \ast \varphi,
\]

where \(\theta\) is the volume form on \(S^1\), and \(\ast\) is the Hodge star operator determined by the induced metric by \(\varphi\) on \(P\). Note that the Lee form \(\Theta\) is given by \(\Theta = \tau_0 \theta\). In fact, using (9.2), we have \(d\Omega = \tau_0 \ast \varphi \wedge \theta = \tau_0 \Omega \wedge \theta\), that is \(d\Omega = \Omega \wedge \Theta\).

By Theorem [13] we know that \(Q(1,1,1)\) is non-formal. Hence the product manifold \(Q(1,1,1) \times S^1\) is non-formal by Lemma [3.5]. Then, if we consider the \(S^1\)-family of nearly parallel \(G_2\)-structures \(\varphi_t\) defined in (6.11), we have:

**Proposition 9.2.** The product manifold \(Q(1,1,1) \times S^1\) is non-formal and has an \(S^1\)-family of locally conformal parallel \(\text{Spin}(7)\)-structures \(\Omega_t = \varphi_t \wedge \theta \ast \varphi_t\), with Lee form \(\Theta = 4 \theta\).

Let \(P\) be a differentiable manifold and let \(\rho : P \to P\) be a diffeomorphism. The mapping torus \(P_{\rho}\) of \(\rho\) is the manifold obtained from \(P \times [0,1]\) by identifying the ends with \(\rho\), that is

\[
P_{\rho} = \frac{P \times [0,1]}{(x,0) \sim (\rho(x),1)}.
\]
It is a differentiable manifold, because it is the quotient of \( P \times \mathbb{R} \) by the infinite cyclic group generated by \( (x,t) \rightarrow (\rho(x), t+1) \). The natural map \( \pi: P_\rho \rightarrow S^1 \) defined by \( \pi(x,t) = e^{2\pi i t} \) is the projection of a locally trivial fiber bundle (here we think of \( S^1 \) as the interval \([0,1]\) with identified end points). Thus, any \( \rho \)-invariant form \( \beta \) on \( P \) defines a form on \( P_\rho \), since the pullback of \( \beta \) to \( P \times \mathbb{R} \) is invariant under the diffeomorphism \( (x,t) \mapsto (\rho(x),t+1) \). For the same reason, the 1-form \( dt \) on \( \mathbb{R} \), where \( t \) is the coordinate on \( \mathbb{R} \), induces a closed 1-form \( \nu \) on \( P_\rho \).

A theorem of Tischler [53] asserts that a compact manifold is a mapping torus if and only if it admits a non-vanishing closed 1-form. This result was extended to locally conformal parallel Spin(7) manifolds by Ivanov, Parton and Piccinni [36] as follows.

**Theorem 9.3 ([36, Theorem B]).** A compact Riemannian 8-manifold \( N \) admits a locally conformal parallel Spin(7)-structure if and only if there exists a fibre bundle \( N \rightarrow S^1 \) with abstract fibre \( P/\Gamma \), where \( P \) is a compact simply connected nearly parallel \( G_2 \)-manifold and \( \Gamma \) is a finite subgroup of isometries of \( P \) acting freely. Moreover, the cone of \( P/\Gamma \) covers \( N \) with cyclic infinite covering transformation group.

Notice that if \( P \) is a 7-dimensional compact manifold endowed with a nearly parallel \( G_2 \)-structure \( \varphi \), and \( \rho: P \rightarrow P \) is a diffeomorphism such that \( \rho^*\varphi = \varphi \), then \( \rho \) preserves the orientation and the metric on \( P \) induced by the \( G_2 \)-structure \( \varphi \). So, \( \rho^*(\ast \varphi) = \ast \varphi \), and the mapping torus \( N = P_\rho \) has a locally conformal parallel Spin(7)-structure \( \Omega \) given by

\[
\Omega = \varphi \wedge \nu + \ast \varphi.
\]

Let us recall that if \( P \) is a differentiable manifold and \( \rho: P \rightarrow P \) is a diffeomorphism, then the cohomology of the mapping torus \( N = P_\rho \) of \( \rho \) sits in an exact sequence

\[
0 \rightarrow C^r \rightarrow H^r(N) \rightarrow K^r \rightarrow 0,
\]

where \( K^r \) is the kernel of \( \varphi^* - \text{Id}: H^r(P) \rightarrow H^r(P) \), and \( C^r \) is its cokernel. Thus,

\[
H^r(P_\rho) \cong \ker (\varphi^* - \text{Id}: H^r(P) \rightarrow H^r(P)) \oplus [\nu] \wedge \frac{H^{r-1}(P)}{\text{im} (\varphi^* - \text{Id}: H^{r-1}(P) \rightarrow H^{r-1}(P))}.
\]

Now we consider the manifold \( M = Q(1,1,1) \) with the family of nearly parallel \( G_2 \)-structures \( \varphi_t \) defined by (6.11). Let \( \rho: Q(1,1,1) \rightarrow Q(1,1,1) \) be the diffeomorphism of \( Q(1,1,1) \) given by

\[
\rho(\theta_1, \phi_1, \theta_2, \phi_2, \theta_3, \phi_3, s) = (\theta_2, \phi_2, -\theta_1, \phi_1, \theta_3, \phi_3, s).
\]

Then, the diffeomorphism \( \rho \) on the 1-forms \( \{\alpha_1, \ldots, \alpha_6, \alpha_7 = \zeta\} \) on \( Q(1,1,1) \) is given by

\[
\rho^*\alpha_1 = \alpha_3, \quad \rho^*\alpha_2 = \alpha_4, \quad \rho^*\alpha_3 = -\alpha_1, \quad \rho^*\alpha_4 = -\alpha_2, \quad \rho^*\alpha_5 = \alpha_5, \quad \rho^*\alpha_6 = \alpha_6, \quad \rho^*\alpha_7 = \alpha_7.
\]
Proposition 9.4. The mapping torus \( M_\rho = Q(1, 1, 1)_\rho \) is formal and it has an \( S^1 \) family of locally conformal parallel \( \text{Spin}(7) \)-structures.

Proof. Clearly the diffeomorphism \( \rho \) preserves the nearly parallel \( G_2 \)-structures \( \varphi_t \) defined in (6.11) and, taking into account (6.12), \( \rho \) preserves also \( *\varphi_t \). Then, by Theorem 9.3, we know that \( M_\rho = Q(1, 1, 1)_\rho \) carries a \( S^1 \)-family of locally conformal parallel \( \text{Spin}(7) \)-structures.

In order to prove that \( M_\rho = Q(1, 1, 1)_\rho \) is formal we proceed as follows. As in Section 3 we write with the same symbol the lifting to \( M \) defined in (6.11) and, taking into account (6.12), \( \rho \omega \). Clearly the diffeomorphism \( \rho \) preserves the \( 1 \)-forms \( \alpha_i \), \( 1 \leq i \leq 6 \), and of the Kähler forms \( \omega_j \), \( 1 \leq j \leq 3 \), where \([\omega_1], [\omega_2] \) and \([\omega_3] \) are the generators of the integral cohomology group of each of the \( S^2 \)-factors on \( S^1 \times S^2 \times S^2 \). Now, for simplicity of notation we write \( a_1 = [\omega_1], a_2 = [\omega_2] \) and \( a_3 = [\omega_3] \). Since \( M = Q(1, 1, 1) \) is the principal \( S^1 \)-bundle

\[
S^1 \longrightarrow M = Q(1, 1, 1) \longrightarrow S^2 \times S^2 \times S^2
\]

with first Chern class equal to \( a_1 + a_2 + a_3 \), the Gysin sequence gives that

\[
H^0(M, \mathbb{Z}) = H^7(M, \mathbb{Z}) = \mathbb{Z},
H^1(M, \mathbb{Z}) = H^3(M, \mathbb{Z}) = H^6(M, \mathbb{Z}) = 0,
H^2(M, \mathbb{Z}) = H^5(M, \mathbb{Z}) = \mathbb{Z}^2,
H^4(M, \mathbb{Z}) = \mathbb{Z}(a_1a_2, a_1a_3, a_2a_3)/\langle a_1a_2 + a_1a_3, a_2a_1 + a_2a_3, a_3a_1 + a_3a_2 \rangle = \mathbb{Z}_2.
\]

So, the de Rham cohomology groups of \( M = Q(1, 1, 1) \) up to the degree 4 are

\[
H^0(M) = \langle 1 \rangle, \quad H^1(M) = H^3(M) = H^6(M) = 0, \quad H^2(M) = \langle a_1, a_2 \rangle.
\]

Since \( \rho^*\omega_1 = \omega_2 \) and \( \rho^*\omega_2 = \omega_1 \), the de Rham cohomology groups of the mapping torus \( M_\rho = Q(1, 1, 1)_\rho \) up to the degree 4 are

\[
H^0(M_\rho) = \langle 1 \rangle, \quad H^1(M_\rho) = \langle [\nu] \rangle, \quad H^2(M_\rho) = \langle a_1 + a_2 \rangle,
H^3(M_\rho) = \langle (a_1 + a_2) \wedge [\nu] \rangle, \quad H^4(M_\rho) = 0.
\]

Therefore, the minimal model of \( M_\rho \) must be a differential graded algebra \( (\bigwedge V, d) \), being \( \bigwedge V \) the free algebra of the form \( \bigwedge V = \bigwedge (a, b, x) \otimes \bigwedge V^{\geq 4} \), where \( |a| = 1, |b| = 2, |x| = 3 \), and \( d \) is defined by \( da = 0 = db, dx = b^2 \). According to Definition 3.2, we get \( N^j = 0 \) for \( j = 1, 2 \), thus \( M_\rho \) is 2-formal. Moreover, \( M_\rho \) is 3-formal. In fact, take \( z \in I(N^{\leq 3}) \) a closed element in \( \bigwedge V \). As \( H^*(\bigwedge V) = H^*(M_\rho) \) has non-zero cohomology in degrees 0, 1, 2, 3, 5, 6, 7, 8, it must be \( \deg z = 5, 6, 7, 8 \). If \( \deg z = 5 \) then \( z = b \cdot x \) which is not closed because \( d(b \cdot x) = b^3 \neq 0 \). If \( \deg z = 6 \) then \( z = a \cdot b \cdot x \) which is not closed because \( d(a \cdot b \cdot x) = -a \cdot b^3 \neq 0 \). If \( \deg z = 7 \) then \( z = b^2 \cdot x \) which is not closed, and if \( \deg z = 8 \) then \( z = a \cdot b^2 \cdot x \) which is not closed either. Thus, according to Definition 3.2 \( M_\rho \) is 3-formal and, by Theorem 3.3 \( M_\rho \) is formal. \( \square \)
Next, for $M = B$ or $M = W_{k,l}$, where $B = \text{SO}(5)/\text{SO}(3)$ is the Berger space and $W_{k,l} = \text{SU}(3)/S^1_{k,l}$ are the Aloff–Wallach spaces, we study the formality of the mapping torus of a diffeomorphism of $M$ preserving a nearly-parallel $G_2$-structure on $M$ (see subsections 5.1 and 5.2).

**Proposition 9.5.** Let $\rho : B \to B$ be a diffeomorphism preserving a nearly-parallel $G_2$-structure on $B$. Then, the mapping torus $B_\rho$ is formal and it has a locally conformal parallel $\text{Spin}(7)$-structure.

**Proof.** We consider a nearly parallel $G_2$-structure on $B$, which is preserved by the diffeomorphism $\rho : B \to B$. (As we already mentioned in subsection 5.1, the only known nearly parallel $G_2$-structure on $B$ is proper and is explicitly given in [6].) Theorem 9.3 implies that $B_\rho$ has a locally conformal parallel $\text{Spin}(7)$-structure.

According to the proof of Theorem 5.1, we know that $B$ and the 7-sphere $S^7$ have the same minimal model. Hence, the de Rham cohomology groups of $B$ up to the degree 6 are

$$H^0(B) = \langle 1 \rangle, \quad H^k(B) = 0, \quad 1 \leq k \leq 6.$$

So, the de Rham cohomology groups of the mapping torus $B_\rho$ up to the degree 6 are

$$H^0(B_\rho) = \langle 1 \rangle, \quad H^1(B_\rho) = \langle \nu \rangle, \quad H^k(B_\rho) = 0, \quad 2 \leq k \leq 6.$$

Therefore, the minimal model of $B_\rho$ must be a differential graded algebra $(\wedge V, d)$, being $\wedge V$ the free algebra of the form $\wedge V = \wedge \langle a \rangle \otimes \wedge V^{\geq 7}$, with $|a| = 1$, and $d$ is given by $da = 0$. Now, according to Definition 3.2 we have $C^i = \langle a \rangle$, $C^i = 0$ for $2 \leq i \leq 4$, and $N^j = \langle 0 \rangle$ for $1 \leq j \leq 6$. Thus, Definition 3.2 implies that $B_\rho$ is 3-formal and, by Theorem 3.3, $B_\rho$ is formal.

**Proposition 9.6.** Let $\rho : W_{k,l} \to W_{k,l}$ be a diffeomorphism preserving a nearly parallel $G_2$-structure (not necessarily proper) on $W_{k,l}$. Then, the mapping torus $(W_{k,l})_\rho$ is formal and it has a locally conformal parallel $\text{Spin}(7)$-structure.

**Proof.** If $\rho : W_{k,l} \to W_{k,l}$ is a diffeomorphism preserving a nearly parallel $G_2$-structure on $W_{k,l}$, the mapping torus $(W_{k,l})_\rho$ has a locally conformal parallel $\text{Spin}(7)$-structure by Theorem 9.3.

In order to prove that $(W_{k,l})_\rho$ is formal we proceed as follows. By Theorem 5.2 and Theorem 5.3, we know that $W_{k,l}$ and the product manifold $S^2 \times S^5$ have the same minimal model, and hence the same de Rham cohomology. So, the de Rham cohomology groups of $W_{k,l}$ are:

$$H^0(W_{k,l}) = \langle 1 \rangle, \quad H^1(W_{k,l}) = H^3(W_{k,l}) = H^4(W_{k,l}) = H^6(W_{k,l}) = 0,$$

$$H^2(W_{k,l}) = \langle \xi \rangle, \quad H^5(W_{k,l}) = \langle \tau \rangle, \quad H^7(W_{k,l}) = \langle \xi \wedge \tau \rangle.$$

Let us consider the map $\rho^* : H^2(W_{k,l}) \to H^2(W_{k,l})$. It is clear that either $\rho^*(\xi) \neq \xi$ or $\rho^*(\xi) = \xi$. 

We deal first with the possibility that $\rho^*(\xi) \neq \xi$. Then, the cohomology of $(W_{k,l})_\rho$ up to the degree 4 is

$$H^0((W_{k,l})_\rho) = \langle 1 \rangle, \quad H^1((W_{k,l})_\rho) = \langle [\nu] \rangle, \quad H^i((W_{k,l})_\rho) = 0, \quad 2 \leq i \leq 4.$$  

Therefore, the minimal model of $(W_{k,l})_\rho$ must be a differential graded algebra $(\Lambda V, d)$, being $\Lambda V$ the free algebra of the form $\Lambda V = \Lambda(a) \otimes \Lambda V^{\geq 5}$, where $|a| = 1$, and $d$ is defined by $da = 0$. According to Definition 3.2 we get $C^1 = \langle a \rangle$, $C^i = 0$ for $2 \leq i \leq 4$, and $N^j = 0$ for $1 \leq j \leq 4$. Hence, $(W_{k,l})_\rho$ is 3-formal and, by Theorem 3.3, $(W_{k,l})_\rho$ is formal.

Suppose now that $\rho^*(\xi) = \xi$. In this case, the cohomology of $(W_{k,l})_\rho$ up to the degree 4 is

$$H^0((W_{k,l})_\rho) = \langle 1 \rangle, \quad H^1((W_{k,l})_\rho) = \langle [\nu] \rangle, \quad H^2((W_{k,l})_\rho) = \langle \xi \rangle, \quad H^4((W_{k,l})_\rho) = 0.$$  

Thus, the minimal model of $(W_{k,l})_\rho$ must be a differential graded algebra $(\Lambda V, d)$, being $\Lambda V$ the free algebra of the form $\Lambda V = \Lambda(a, b, x) \otimes \Lambda V^{\geq 4}$, where $|a| = 1$, $|b| = 2$, $|x| = 3$, and $d$ is defined by $da = db = 0$ and $dx = b^2$. According to Definition 3.2 we get $N^j = 0$ for $j \leq 2$, thus $(W_{k,l})_\rho$ is 2-formal. Moreover, $(W_{k,l})_\rho$ is 3-formal.

In fact, take $\alpha \in I(N^{\leq 3})$ a closed element in $\Lambda V$. As $H^*(\Lambda V) = H^*((W_{k,l})_\rho)$ has cohomology in all the degrees except in degree 4, and since $|\alpha| \geq 4$, it must be $|\alpha| = 5, 6, 7, 8$. If $|\alpha| = 5$, then $\alpha = b \cdot x$, which is not closed. If $|\alpha| = 6$, then $\alpha = a \cdot b \cdot x$, which is not closed either. If $|\alpha| = 7$, then $\alpha = b^2 \cdot x$, and if $|\alpha| = 8$, then $\alpha = a \cdot b^2 \cdot x$, but $\alpha$ is not closed in either case. So, according to Definition 3.2 $(W_{k,l})_\rho$ is 3-formal, and by Theorem 3.3 $(W_{k,l})_\rho$ is formal.

Acknowledgements. The first author was partially supported by the Basque Government Grant IT1094-16 and by the Grant PGC2018-098409-B-100 of the Ministerio de Ciencia, Innovación y Universidades of Spain. The second author is supported by the project PRIN “Real and complex manifolds: Topology, Geometry and Holomorphic Dynamics” and by GNSAGA of INdAM. The fourth author was partially supported by Project MINECO (Spain) PID2020-118452GB-I00.

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