Effectively attractive Bose-Einstein condensates in a rotating toroidal trap

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We examine an effectively attractive quasi-one-dimensional Bose-Einstein condensate of atoms confined in a rotating toroidal trap, as the magnitude of the coupling constant and the rotational frequency are varied. Using both a variational mean-field approach, as well as a diagonalization technique, we identify the phase diagram between a uniform and a localized state and we describe the system in the two phases.

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Effectively one-dimensional clouds of trapped atoms at very low temperatures are interesting systems because of a number of reasons. First of all, highly anisotropic traps allow us to realize this situation experimentally [1]. For example, in elongated traps, when the oscillator energy transversely to the long axis of the trap is much larger than the strength of the interaction, the degrees of freedom along this direction are frozen out and the system is effectively one-dimensional. In addition, such systems are interesting from a theoretical point of view since they allow us to check one-dimensional models which have been developed over the years.

Recently Strecker et al. [2] and Khaykovich et al. [3] have managed to create and observe bright solitons – localized blobs of atoms – in quasi-one-dimensional clouds of atoms after switching the effective coupling constant from positive (corresponding to an effective repulsive interaction between the atoms) to negative (corresponding to an effective attractive interaction) with use of the so-called Feshbach resonances [4]. References [5] have examined these experiments theoretically.

Inspired by these experiments, Refs. [6] and [7] have studied the behavior of a cloud of atoms confined in a toroidal trap. Their basic conclusion is that while the density of such a system is homogeneous for an effective repulsion between the atoms, when the interaction becomes attractive, there is a critical value of the coupling constant below which the atoms form a localized blob. An easy way of understanding this effect is that the kinetic energy that results from the Heisenberg uncertainty principle favors a homogeneous density, whereas an attractive interaction between the atoms favors the formation of a localized density.

Extending their study of Ref. [6], in a more recent paper Kanamoto et al. have examined the same problem when the toroidal trap is rotated [8]. While in a non-rotating torus there is only one parameter to vary, i.e., the ratio between the interaction and the kinetic energy (which we call \( \gamma \), in the present problem there is an extra degree of freedom, namely the rotational frequency of the torus \( \Omega \). Using the mean-field approximation, as well as numerical diagonalization of the Hamiltonian, Kanamoto et al. have identified the phase diagram that separates the phase of uniform density from the phase of localized density in the \( \Omega - \gamma \) plane.

In our study we attack the same problem using different methods, and our results are in agreement with those of Ref. [8]. In the first method we use the mean-field approximation and calculate the order parameter variationally. In the second method we truncate appropriately the Hamiltonian of the system and diagonalize it analytically using a Bogoliubov transformation. Our results are also relevant with those of Lundh et al. [9], where the rotational properties of an attractive Bose gas that is confined in an anharmonic potential have been investigated. As shown in this study, the effective potential felt by the atoms is toroidal and the angular momentum is carried either by exciting the center of mass, or by creating vortex states. One may argue that the localized phase of our problem corresponds to the center of mass excitation, while the uniform phase corresponds to the vortex phase [10].

Let us therefore consider a Bose-Einstein condensate of \( N \) atoms confined in a toroidal trap which has a radius \( R \) and a cross section \( S = \pi r^2 \), where we assume that \( r \ll R \). Measuring the length in units of \( R \) and the energy in units of \( \hbar^2/2MR^2 \), with \( M \) being the atom mass, the Hamiltonian \( \hat{H} \) in the rotating frame of reference is

\[
\hat{H} = \int_0^{2\pi} d\theta \left[ \hat{\psi}^\dagger(\theta)(\hat{\mathcal{L}} - \Omega^2/2)\hat{\psi}(\theta) + \frac{1}{2} U_0\hat{\psi}^\dagger(\theta)\hat{\psi}^\dagger(\theta)\hat{\psi}(\theta)\hat{\psi}(\theta) \right].
\] (1)

Here \( \hat{\psi}(\theta) \) is the field operator, \( \hat{\mathcal{L}} = -i\partial/\partial\theta \) is the operator of the angular momentum, with \( \theta \) being the azimuthal angle, \( 2\Omega \) is the angular frequency of rotation, and \( U_0 = 8\pi aR/S \), with \( a \) being the scattering length for elastic atom-atom collisions. In the above expression the energy has been shifted by the term \( \Omega^2/2 \) for convenience.

Let us start with the mean-field approximation. Within this scheme the system is described by the order parameter \( \psi(\theta) \). Expanding \( \psi(\theta) \) in the basis of plane-wave states \( \phi_l(\theta) = e^{i\theta}/\sqrt{2\pi} \), we write

\[
\psi(\theta) = \sum_l c_l \phi_l,
\] (2)
where we treat the coefficients $c_l$ as variational parameters. As shown in Refs. [11], the physical quantities of a system described by the Hamiltonian of Eq. (1) are given by the phase winding number $J$,

$$ J = [\Omega + 1/2], $$

(3)

where $[x]$ is the largest integer that does not exceed $x$, and by the angular frequency relative to $J$, i.e., $\Omega - J$. Therefore the dominant term in the expansion of Eq. (2) is the one with $l = J$, where $J$ is given by Eq. (3) for a given value of $\Omega$. When the density of the gas is uniform, the occupancy of all other states is zero. However when the density is localized more states contribute to $\psi$. Including the $l = J \pm 1$ states we write,

$$ \psi(\theta) = c_{J-1}\phi_{J-1} + c_{J}\phi_{J} + c_{J+1}\phi_{J+1}, $$

(4)

where in order for $\psi$ to be normalized, $|c_{J-1}|^2 + |c_J|^2 + |c_{J+1}|^2 = 1$. Close to the phase boundary the dominant terms in the order parameter are the ones with $l = J, J \pm 1$, since other states (with $l = J \pm 2$, etc.) have relatively higher kinetic energy and for this reason we can neglect them. As a variational method, our approach is more reliable close to the phase boundary, while further away from it, one needs to include more basis states.

We proceed by expressing the energy per particle $\epsilon_J$ in the state of Eq. (4) in terms of the coefficients $c_l$, which are then determined by minimizing $\epsilon_J$ with respect to them [7,12]. We get that

$$ \epsilon_J = (J - \Omega)^2|c_J|^2 $$

$$ + |c_{J-1}|^2 + (J + 1 - \Omega)^2|c_{J+1}|^2 $$

$$ + (\gamma/2)^{2} |c_{J-1}|^{4} + |c_{J}|^{4} + |c_{J+1}|^{4} $$

$$ + 4|c_{J-1}|^{2}|c_{J}|^{2} + 4(\gamma/2)|c_{J-1}|^{2}|c_{J+1}|^{2} $$

$$ + 2\gamma^{2}|c_{J-1}|^{2}|c_{J+1}|^{2} + 2(\gamma/2)^{2}|c_{J-1}|^{2}|c_{J+1}|^{2}, $$

(5)

where $\gamma = NU_0/(2\pi)$. If we write $c_l = |c_l|e^{i\theta_l}$, there are three phases involved in the problem. One phase is arbitrary, the other one is free (reflecting the rotational invariance of the problem) and we choose it here so that the maximum of the density is at $\theta = 0$, and the third one is chosen so that the energy is minimum. To minimize $\epsilon_J$ we choose $\theta_{J-1} + \theta_{J+1} = 2\theta_J = 0$ in order to make the last two terms in Eq. (5) as large as possible [7]. Using the normalization condition we eliminate $c_J$ thus getting

$$ \epsilon_J - \gamma/2 - (\Omega - J)^2 = |c_{J-1}|^{2}[1 - 2(J - \Omega)] $$

$$ + |c_{J+1}|^{2}[1 + 2(J - \Omega)] $$

$$ + |c_{J-1}|^{4} - |c_{J+1}|^{4} + |c_{J-1}|^{2} + |c_{J+1}|^{2} $$

$$ - 2|c_{J-1}|^{2}|c_{J+1}|^{2} + 2|c_{J-1}||c_{J+1}| $$

$$ - 2|c_{J-1}||c_{J+1}| - 2|c_{J-1}||c_{J+1}|, $$

(6)

When the density of the system is uniform, $|c_{J\pm 1}| = 0$, and the energy per particle is $\epsilon_J = \gamma/2 + (\Omega - J)^2$, in agreement with the exact solution of the mean-field approximation [8].

From Eq. (6) we now determine the phase boundary and the order parameter. Minimizing Eq. (6) we get two coupled equations,

$$ 1 + 2(J - \Omega) + \gamma((\lambda^{\pm 1} + 1)(1 - |c_{J\pm 1}|^{2}) - 2|c_{J+1}|^{2} $$

$$ - 3|c_{J+1}||c_{J-1}| = 0, $$

(7)

where $\lambda = |c_{J+1}|/|c_{J-1}|$. Close to the phase boundary $|c_{J\pm 1}|$ is either zero, or $< 1$. This observation allows us to write Eq. (7) as

$$ 1 - 2(J - \Omega) + \gamma(1 + \lambda) = 0 $$

$$ 1 + 2(J - \Omega) + \gamma(1 + 1/\lambda) = 0, $$

(8)

and in order for a solution to exist, $\gamma = -1/2 + 2(J - \Omega)^2$, which is the equation that gives the phase boundary. Figure 1 shows this curve and is periodic in $\Omega$, with a period of $\Omega = 1$. For $\gamma$ larger than this critical value the state is uniform, while for smaller values it is localized. Reference [8] has derived the same result by examining the stability of the exact mean-field solution. Furthermore the transition between the two phases is of second order, since the order parameter is continuous across the phase boundary. Expanding we find that for $\gamma \lesssim -1/2$ and $\Omega - J \approx 0$,

$$ |c_l|^2 = \frac{1 + 2\gamma}{7\gamma} \left[ 1 + \alpha_l(\Omega - J) \right], $$

(9)

where $\alpha_{J-1} = -18/7$ and $\alpha_{J+1} = 38/7$, while $|c_J|^2 = 1 - |c_{J-1}|^2 - |c_{J+1}|^2$.

Going even further, we have solved the two coupled Eqs. (7). Figure 2 shows the density of the gas $|\psi|^2$ for $\gamma = -0.55$ and $-0.45$ in each graph, and for $|\Omega - J| = 0, 1/4, 1/2$. For the smaller value of $\gamma$ the density is more peaked, as one expects intuitively. On the other hand the difference between the two states gets smaller as one gets further away from the phase boundary. Comparing our variational solution with the exact one (within the mean-field approximation) for $\gamma = -0.55$ of Ref. [8], we see that in general the difference is small and it increases as $|\Omega - J|$ increases. This is expected, however, since for higher values of $|\Omega - J|$ one is further away from the phase boundary and more states have to be included in the expansion of the order parameter. Furthermore, when $|\Omega - J| = 0$ the solution coincides with the one of a non-rotating cloud [6,7], apart from the phase factor $e^{i\theta}$, while for $|\Omega - J| = 1/2$ the order parameter has a node, since at this point the phase jumps by $\pi$ and $J$ increases by one unit [8].

Let us now turn to the second approach we have developed, namely the diagonalization of the Hamiltonian $H$. Following the same steps as before, we expand the field operator $\psi$, and for the same reasons as before,

$$ \hat{\psi}(\theta) = \hat{c}_{J-1}\phi_{J-1} + \hat{c}_{J}\phi_{J} + \hat{c}_{J+1}\phi_{J+1}, $$

(10)

where in this case $\hat{c}_l$ is the annihilation operator of an atom with angular momentum $l$. When the density of the
gas is uniform, the states \( l = J \pm 1 \) are equally occupied and their occupancy goes to zero in the \( N \to \infty \) limit. In addition the occupancy of other states \( (l = J \pm 2 \), etc.) goes more rapidly to zero (as \( N \to \infty \), since these states have relatively higher kinetic energy and for this reason we neglect them. Writing the Hamiltonian (1) in second-quantized form,

\[
\hat{H} = \sum_l (l - \Omega)^2 \hat{c}_l^\dagger \hat{c}_l + \frac{U_0}{4\pi} \sum_{klmn} \hat{c}_{l+\frac{1}{2}}^\dagger \hat{c}_m^\dagger \hat{c}_n \delta_{m+n-k-l},
\]

we notice that it can be diagonalized with a Bogoliubov transformation as discussed before [12]. We work with the basis vectors \( |m\rangle = |N_{J-1}, N_J, N_{J+1}\rangle \equiv |(J - 1)^m, J^{N-2m}, (J + 1)^m\rangle \),

\[
(12)
\]

where \( N_l \) is the occupancy of the state with angular momentum \( l \). The states \( |m\rangle \) have been constructed to have \( N \) atoms, \( \sum_l N_l = N, N J \) units of angular momentum, \( \sum_l l N_l = N J \), and an equal occupancy of the \( l = J \pm 1 \) states (i.e., \( m \) atoms in each of them). The diagonal matrix elements are,

\[
\langle m | \hat{H} | m \rangle = m [(J - 1 - \Omega)^2 + (J + 1 - \Omega)^2] + (N - 2m)(J - \Omega)^2 + \frac{U_0}{4\pi} [(N - 2m)(N - 2m - 1) + 2m(m - 1) + 8m(N - 2m) + 4m^2].
\]

(13)

When the system is on the side of the “uniform state” (in the limit \( N \to \infty \)) then \( m \) is of order \( N^2 \), which allows us to neglect the terms of order \( m^2 \) compared to those of order \( N^2 \) and \( Nm \) (on the contrary, \( m \) becomes of order \( N \) when a localized state forms),

\[
\langle m | \hat{H} | m \rangle \approx N (J - \Omega)^2 + \gamma (N - 1)/2 + 2m(1 + \gamma).
\]

(14)

We also get for the off-diagonal matrix elements,

\[
\langle m | \hat{H} | m + 1 \rangle = \frac{U_0}{4\pi} 2 \sqrt{(N - 2m)(N - 2m - 1)(m + 1)^2} \approx \gamma (m + 1)
\]

(15)

in the limit \( m \ll N \). From Eqs. (14) and (15) \( \hat{H} \) can be written as

\[
\hat{H} - \gamma (N - 1)/2 - N (J - \Omega)^2 = (1 + \gamma) (\hat{c}_{J-1}^\dagger \hat{c}_{J-1} + \hat{c}_{J+1}^\dagger \hat{c}_{J+1}) + \gamma (\hat{c}_{J-1}^\dagger \hat{c}_{J+1} + \hat{c}_{J-1} \hat{c}_{J+1}).
\]

(16)

The above Hamiltonian can be diagonalized with use of a Bogoliubov transformation [13], i.e., introducing the operators

\[
\hat{b} = \lambda_1 \hat{c}_{J-1}^\dagger + \lambda_2 \hat{c}_{J+1} \quad \text{and} \quad \hat{d} = \lambda_2 \hat{c}_{J-1} + \lambda_1 \hat{c}_{J+1}.
\]

(17)

Following the usual tricks, we get that the eigenvalues of \( \hat{H} \) are given by

\[
\varepsilon_{n_b, n_d} - \gamma (N - 1)/2 - N (J - \Omega)^2 = -\gamma + 1 + \sqrt{1 + 2\gamma (1 + n_b + n_d)},
\]

(18)

where \( n_b, n_d \) are the eigenvalues of the number operators \( \hat{b}^\dagger \hat{b} \) and \( \hat{d}^\dagger \hat{d} \) respectively. To get the above solution, \( \gamma \) has to be larger than \(-1/2\). For \( N \to \infty \), one can identify the term \( \varepsilon_{0,0}/N = \gamma/2 + (J - \Omega)^2 \) as the total energy of the condensate with uniform density, in agreement with our variational approach and with the exact solution of the mean-field theory, Ref. [8].

Furthermore, one can see that for the states \( |m\rangle \), \( \langle \hat{b}^\dagger \hat{b} - \hat{d}^\dagger \hat{d} |m\rangle = 0 \), which implies that \( n_b = n_d = 0, 1, 2, \ldots \). Therefore, the first excited state of the system is (measured from the lowest energy) \( 2\sqrt{1 + 2\gamma} \) as Eq. (18) implies, if \( n_b = n_d = 1 \). This result is in agreement with Ref. [8].

Up to now the only condition we have found in order for a solution to exist is that \( \gamma > -1/2 \). Turning to the stability of our solution against small perturbations, we consider two states where one atom with angular momentum \( l = J - 1 \) has been transferred to the state with \( l = J + 1 \), or vice versa,

\[
|m^\pm\rangle = |(J - 1)^{m-1}, J^{N-2m}, (J + 1)^{m+1}\rangle.
\]

(19)

The states \( |m^\pm\rangle \) have \( N \) atoms, and they carry an angular momentum that differs by two units compared to that of the state \( |m\rangle \). As \( \sum_l |N_l| = NJ \pm 2 \), one can follow the same steps as before and get that the energy is

\[
\varepsilon_{n_b, n_d}^\pm - \gamma (N - 1)/2 - N (J - \Omega)^2 \mp 4(J - \Omega) = -\gamma + 1 + \sqrt{1 + 2\gamma (1 + n_b + n_d)}.
\]

(20)

In this case \( \langle \hat{b}^\dagger \hat{b} - \hat{d}^\dagger \hat{d} |m^\pm\rangle = \pm 2|m^\pm\rangle \) and thus the lowest energy of the system is that with \( n_b = 2, n_d = 0 \), or vice versa. In order for our solution to be stable, the lowest energy of \( \varepsilon_{n_b, n_d}^\pm \) has to be higher than that of \( \varepsilon_{n_b, n_d} \), or

\[
\sqrt{1 + 2\gamma} \geq 2|\Omega - J|,
\]

(21)

which can also be written as \( \gamma > -1/2 + 2(J - \Omega)^2 \), in agreement with the expression we derived earlier within our variational scheme. If Eq. (21) is not satisfied, even an infinitesimally small anisotropy in the trapping potential suffices to convert the density of the system from uniform to localized.

Finally we calculate the depletion of the condensate \( \Delta N \),

\[
\Delta N = |\langle \hat{c}_{J-1}^\dagger \hat{c}_{J-1} \rangle + \langle \hat{c}_{J+1}^\dagger \hat{c}_{J+1} \rangle | = \gamma + 1 \sqrt{1 + 2\gamma} - 1,
\]

(22)

which is singular only when \( J = \Omega \). More precisely, close to the phase boundary and for small values of \( |J - \Omega| \), \( \Delta N \approx 1/(4|J - \Omega|) \), but it is finite otherwise. These results are also in agreement with those of the mean-field approach [8].
To summarize, we examined a one-dimensional Bose-Einstein condensate of atoms confined in a rotating torus, using two independent methods, namely a variational mean-field approach, as well as an exact diagonalization of the Hamiltonian. Based on these models, we examined the two phases which appear of uniform and localized density, and we identified the phase boundary which separates them, as the strength of the interaction and the frequency of rotation vary. Our results are in agreement with those of the exact mean-field approach and with numerical diagonalization of the Hamiltonian of the system [8].

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[1] This limit was first reached experimentally in a Bose-Einstein condensate of $^{23}$Na atoms; see: A. Görlitz, J. M. Vogels, A. E. Leanhardt, C. Raman, T. L. Gustavson, J. R. Abo-Shaeer, A. P. Chikkatur, S. Gupta, S. Inouye, T. P. Rosenband, D. E. Pritchard, and W. Ketterle, Phys. Rev. Lett. 87, 130402 (2001).
[2] K. E. Strecker, G. B. Partridge, A. G. Truscott, and R. G. Hulet, Nature (London) 417, 150 (2002). [3] L. Khaykovich, F. Schreck, G. Ferrari, T. Bourdel, J. Cubizolles, L. D. Carr, Y. Castin, and C. Salomon, Science 296, 1290 (2002).
[4] S. Inouye, M. R. Andrews, J. Stenger, H. J. Miesner, D. M. Stamper-Kurn, and W. Ketterle, Nature (London) 392, 151 (1998). [5] L. D. Carr and Y. Castin, Phys. Rev. A 66, 063602 (2002); U. Al Khawaja, H. T. C. Stoof, R. G. Hulet, K. E. Strecker, and G. B. Partridge, Phys. Rev. Lett. 89, 200404 (2002); L. Salasnich, A. Parola, and L. Reatto, Phys. Rev. A 66, 043603 (2002).
[6] R. Kanamoto, H. Saito, and M. Ueda, Phys. Rev. A 67, 013608 (2003). [7] G. M. Kavoulakis, Phys. Rev. A 67, 011601(R) (2003). [8] R. Kanamoto, H. Saito, and M. Ueda, e-print cond-mat/0305307. [9] E. Lundh, A. Collin and K.-A. Suominen, e-print cond-mat/0304186. [10] E. Lundh, private communication. [11] A. J. Leggett, Low Temperature Physics, edited by M. J. R. Hoch and R. H. Lemmer (Springer Verlag, Berlin); Physica Fennica 8, 125 (1973). [12] A similar problem has been attacked in the context of weakly-interacting Bose-Einstein condensates under rotation in: G. M. Kavoulakis, B. Mottelson, and C. J. Pethick, Phys. Rev. A 62, 063605 (2000).
[13] N. N. Bogoliubov, J. Phys. (Moscow) 11, 23 (1947).