When is a Riesz distribution a complex measure?

Alan D. Sokal*
Department of Physics
New York University
4 Washington Place
New York, NY 10003 USA
sokal@nyu.edu

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Abstract

Let \( R_\alpha \) be the Riesz distribution on a simple Euclidean Jordan algebra, parametrized by \( \alpha \in \mathbb{C} \). I give an elementary proof of the necessary and sufficient condition for \( R_\alpha \) to be a locally finite complex measure (= complex Radon measure).

Soit \( R_\alpha \) la distribution de Riesz sur une algèbre de Jordan euclidienne simple, paramétrisée par \( \alpha \in \mathbb{C} \). Je donne une démonstration élémentaire de la condition nécessaire et suffisante pour que \( R_\alpha \) soit une mesure complexe localement finie (= mesure de Radon complexe).

Key Words: Riesz distribution, Jordan algebra, symmetric cone, Gindikin’s theorem, Wallach set, tempered distribution, positive measure, Radon measure, relatively invariant measure, Laplace transform.

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*Also at Department of Mathematics, University College London, London WC1E 6BT, England.
1 Introduction

In the theory of harmonic analysis on Euclidean Jordan algebras (or equivalently on symmetric cones) [12], a central role is played by the Riesz distributions $R_\alpha$, which are tempered distributions that depend analytically on a parameter $\alpha \in \mathbb{C}$. One important fact about the Riesz distributions is the necessary and sufficient condition for positivity, due to Gindikin [13]:

**Theorem 1.1 [12, Theorem VII.3.1]** Let $V$ be a simple Euclidean Jordan algebra of dimension $n$ and rank $r$, with $n = r + \frac{d}{2}r(r - 1)$. Then the Riesz distribution $R_\alpha$ on $V$ is a positive measure if and only if $\alpha = 0, \frac{d}{2}, \ldots, (r - 1)\frac{d}{2}$ or $\alpha > (r - 1)\frac{d}{2}$.

The “if” part is fairly easy, but the “only if” part is reputed to be deep [12, 13, 20].

The purpose of this note is to give a completely elementary proof of the “only if” part of Theorem 1.1, and indeed of the following strengthening:

**Theorem 1.2** Let $V$ be a simple Euclidean Jordan algebra of dimension $n$ and rank $r$, with $n = r + \frac{d}{2}r(r - 1)$. Then the Riesz distribution $R_\alpha$ on $V$ is a locally finite complex measure (= complex Radon measure) if and only if $\alpha = 0, \frac{d}{2}, \ldots, (r - 1)\frac{d}{2}$ or $\text{Re} \alpha > (r - 1)\frac{d}{2}$.

This latter result is also essentially known [18, Lemma 3.3], but the proof given there requires some nontrivial group theory.

The idea of the proof of Theorem 1.2 is very simple: A distribution defined on an open subset $\Omega \subset \mathbb{R}^n$ by a function $f \in L^1_{\text{loc}}(\Omega)$ can be extended to all of $\mathbb{R}^n$ as a locally finite complex measure only if the function $f$ is locally integrable also at the boundary of $\Omega$ (Lemma 2.1); furthermore, this fact survives analytic continuation in a parameter (Proposition 2.3). In the case of the Riesz distribution $R_\alpha$, a simple computation using its Laplace transform (Lemma 3.4) plus a bit of extra work (Lemma 3.5) allows us to determine the allowed set of $\alpha$, thereby proving Theorem 1.2.

Theorem 1.2 thus states a necessary and sufficient condition for $R_\alpha$ to be a distribution of order 0. It would be interesting, more generally, to determine the order of the Riesz distribution $R_\alpha$ for each $\alpha \in \mathbb{C}$.

It would also be interesting to know whether this approach is powerful enough to handle the multiparameter Riesz distributions $R_\alpha$ with $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{C}^r$ [12, Theorem VII.3.2] and/or the Riesz distributions on homogeneous cones that are not symmetric (i.e. not self-dual) and hence do not arise from a Euclidean Jordan algebra [13, 20].

In an Appendix I comment on a beautiful but little-known elementary proof of Theorem 1.1 — which does not extend, however, to Theorem 1.2 — due to Shanbhag [27] and Casalis and Letac [9].

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1 The set of values of $\alpha$ described in Theorem 1.1 is the so-called Wallach set [10–12, 21, 25, 29].
2 A general theorem on distributions

We assume a basic familiarity with the theory of distributions [19, 26] and recall some key notations and facts.

For each open set $\Omega \subseteq \mathbb{R}^n$, we define the space $\mathcal{D}(\Omega)$ of $C^\infty$ functions having compact support in $\Omega$, the corresponding space $\mathcal{D}'(\Omega)$ of distributions, and the space $\mathcal{D}^k(\Omega)$ of distributions of order $\leq k$. In particular, the space $\mathcal{D}^0(\Omega)$ consists of the distributions that are given locally (i.e. on every compact subset of $\Omega$) by a finite complex measure.

Let $f : \Omega \rightarrow \mathbb{C}$ be a measurable function, and extend it to all of $\mathbb{R}^n$ by setting $f \equiv 0$ outside $\Omega$. We say that $f \in L^1_{\text{loc}}(\Omega)$ if, for every $x \in \Omega$, $f$ is (absolutely) integrable on some neighborhood of $x$. Any $f \in L^1_{\text{loc}}(\Omega)$ defines a distribution $T_f \in \mathcal{D}'^0(\Omega)$ by

$$T_f(\varphi) = \int \varphi(x) f(x) \, dx$$

for all $\varphi \in \mathcal{D}(\Omega)$. \hspace{1cm} (2.1)

We are interested in knowing under what circumstances the distribution $T_f \in \mathcal{D}'^0(\Omega)$ can be extended to a distribution $\widetilde{T}_f \in \mathcal{D}'^0(\mathbb{R}^n)$, i.e. one that is locally everywhere on $\mathbb{R}^n$ a finite complex measure.

Lemma 2.1 Let $f : \Omega \rightarrow \mathbb{C}$ be in $L^1_{\text{loc}}(\Omega)$, and let $T_f \in \mathcal{D}'^0(\Omega)$ be the corresponding distribution. Then the following are equivalent:

(a) $f \in L^1_{\text{loc}}(\Omega)$, i.e. for every $x \in \overline{\Omega}$, $f$ is integrable on some neighborhood of $x$.\footnote{Since this has already been assumed for $x \in \Omega$, the content of hypothesis (a) is that it should hold also for $x \in \partial \Omega$.}

(b) There exists a distribution $\widetilde{T}_f \in \mathcal{D}'^0(\mathbb{R}^n)$ that extends $T_f$ and is supported on $\overline{\Omega}$.

(c) There exists a distribution $\widetilde{T}_f \in \mathcal{D}'^0(\mathbb{R}^n)$ that extends $T_f$.

Proof. (a) $\implies$ (b): It suffices to define $\widetilde{T}_f(\varphi) = \int_{\Omega} \varphi(x) f(x) \, dx$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

(b) $\implies$ (c) is trivial.

(c) $\implies$ (a): By hypothesis, for every $x \in \partial \Omega$ and every compact neighborhood $K \ni x$, there exists a finite complex measure $\mu_K$ supported on $K$ such that $\widetilde{T}_f(\varphi) = \int K \varphi \, d\mu_K$ for every $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with support in $K$. But since $\widetilde{T}_f$ extends $T_f$, the restriction of $\mu_K$ to every compact subset of $K \cap \Omega$ must coincide with the measure $f(x) \, dx$. Since $K \cap \Omega$ is $\sigma$-compact, this implies that $\int_{K \cap \Omega} |f(x)| \, dx = |\mu_K|(K \cap \Omega) < \infty$, so that $f$ is integrable in a neighborhood of $x$. \hfill $\square$

We now extend this idea to allow for analytic dependence on a parameter. Let $\Omega$ be an open set in $\mathbb{R}^n$, let $D$ be a connected open set in $\mathbb{C}^m$, and let $F : \Omega \times D \rightarrow \mathbb{C}$
be a continuous function such that \( F(x, \cdot) \) is analytic on \( D \) for each \( x \in \Omega \). Then, for each \( \lambda \in D \), define

\[
T_\lambda(\varphi) = \int \varphi(x) F(x, \lambda) \, dx \quad \text{for all } \varphi \in \mathcal{D}(\Omega) .
\]

(2.2)

**Lemma 2.2** With \( F \) as above, the map \( \lambda \mapsto T_\lambda \) is analytic from \( D \) into \( \mathcal{D}'(\Omega) \) in the sense that \( \lambda \mapsto T_\lambda(\varphi) \) is analytic for all \( \varphi \in \mathcal{D}(\Omega) \).

**Proof.** This is an immediate consequence of the hypotheses on \( F \) together with standard facts about scalar-valued analytic functions in \( \mathbb{C} \) (either Morera’s theorem or the Cauchy integral formula) and \( \mathbb{C}^m \) (e.g. the weak form of Hartogs’ theorem). \( \Box \)

**Remark.** Weak analyticity in the sense used here is actually equivalent to strong analyticity: see e.g. [15, pp. 37–39, Théorème 1 and Remarque 1] [5, Theorems 3.1 and 3.2] [14, Theorem 1]. Indeed, our hypothesis on \( F \) is equivalent to the even stronger statement that the map \( \lambda \mapsto F(\cdot, \lambda) \) is analytic from \( D \) into the space \( C^0(\Omega) \) of continuous functions on \( \Omega \), equipped with the topology of uniform convergence on compact subsets [15, p. 41, example (a)]. But we do not need any of these facts; weak analyticity is enough for our purposes. \( \Box \)

Putting together these two lemmas, we obtain:

**Proposition 2.3** Let \( F \) be as above, let \( D_0 \subseteq D \) be a nonempty open set, and let \( \lambda \mapsto \tilde{T}_\lambda \) be a (weakly) analytic map of \( D \) into \( \mathcal{D}'(\mathbb{R}^n) \) such that \( \tilde{T}_\lambda \) extends \( T_\lambda \) for each \( \lambda \in D_0 \). Then, for each \( \lambda \in D \), we have:

(a) \( \tilde{T}_\lambda \) extends \( T_\lambda \).

(b) If \( \tilde{T}_\lambda \in \mathcal{D}'(\mathbb{R}^n) \), then \( F(\cdot, \lambda) \in L^1_{\text{loc}}(\overline{\Omega}) \).

**Proof.** (a) This is immediate by analytic continuation: for each \( \varphi \in \mathcal{D}(\Omega) \), both \( \tilde{T}_\lambda(\varphi) \) and \( T_\lambda(\varphi) \) are (by hypothesis and Lemma 2.2 respectively) analytic functions of \( \lambda \) on \( D \) that coincide on \( D_0 \), therefore they must coincide on all of \( D \).

(b) This is immediate from (a) together with Lemma 2.1 \( \Box \)

We shall apply this setup with \( F(x, \lambda) = f(x)^\lambda \) where \( f: \Omega \to (0, \infty) \) is a continuous function; in fact, we shall take \( f \) to be a polynomial.

**Remark.** Let \( P \) be a polynomial that is strictly positive on \( \Omega \) and vanishes on \( \partial \Omega \), and define for \( \text{Re} \lambda > 0 \) a tempered distribution \( \mathcal{P}_{\Omega}^\lambda \in \mathcal{S}'(\mathbb{R}^n) \) by the formula

\[
\mathcal{P}_{\Omega}^\lambda(\varphi) = \int_{\Omega} P(x)^\lambda \varphi(x) \, dx \quad \text{for } \varphi \in \mathcal{S}(\mathbb{R}^n) .
\]

(2.3)
Then $P_\Omega^\lambda$ is a tempered-distribution-valued analytic function of $\lambda$ on the right half-plane, and it is a deep result of Atiyah, Bernstein and S.I. Gelfand [1–4] that $P_\Omega^\lambda$ can be analytically continued to the whole complex plane as a meromorphic function of $\lambda$ with poles on a finite number of arithmetic progressions. It is important to note that our Proposition 2.3 does not rely on this deep result; rather, it says that whenever such an analytic continuation exists (however it may be constructed), the analytically-continued distribution $P_\Omega^\lambda$ can be a complex measure only if $P_\Omega^\lambda \in L^1_{\text{loc}}(\Omega)$.

\[ \square \]

3 Application to Riesz distributions

We refer to the book of Faraut and Korányi [12] for basic facts about symmetric cones and Jordan algebras. Let $V$ be a simple Euclidean (real) Jordan algebra of dimension $n$ and rank $r$, with Peirce subspaces $V_{ij}$ of dimension $d$; recall that $n = r + \frac{d}{2}r(r-1)$\(^3\). We denote by $(x|y) = \text{tr}(xy)$ the inner product on $V$, where $\text{tr}$ is the Jordan trace and $xy$ is the Jordan product. Let $\Omega \subset V$ be the positive cone (i.e. the interior of the set of squares in $V$, or equivalently the set of invertible squares in $V$); it is self-dual, i.e. $\Omega^* = \Omega$. We denote by $\Delta(x) = \det(x)$ the Jordan determinant on $V$: it is a homogeneous polynomial of degree $r$ on $V$, which is strictly positive on $\Omega$ and vanishes on $\partial \Omega$, and which satisfies [12, Proposition III.4.3]

\[ \Delta(gx) = \text{Det}(g)^{r/n} \Delta(x) \quad \text{for } g \in G, x \in V, \tag{3.1} \]

where $G$ denotes the identity component of the linear automorphism group of $\Omega$ [it is a subgroup of $\text{GL}(V)$] and $\text{Det}$ denotes the determinant of an endomorphism. We then have the following fundamental Laplace-transform formula:

**Theorem 3.1 [12, Corollary VII.1.3]** For $y \in \Omega$ and $\text{Re } \alpha > (r-1)\frac{d}{2} = \frac{n}{2} - 1$,

\[ \int_{\Omega} e^{-(x|y)} \Delta(x)^{\alpha - \frac{d}{2}} \, dx = \Gamma_\Omega(\alpha) \Delta(y)^{-\alpha} \tag{3.2} \]

\[^3\] See [12, Chapter V] for the classification of simple Euclidean Jordan algebras. There are five cases [12, p. 97]:

(a) $V = \text{Sym}(m, \mathbb{R})$, the space of $m \times m$ real symmetric matrices ($d = 1$, $r = m$);
(b) $V = \text{Herm}(m, \mathbb{C})$, the space of $m \times m$ complex hermitian matrices ($d = 2$, $r = m$);
(c) $V = \text{Herm}(m, \mathbb{H})$, the space of $m \times m$ quaternionic hermitian matrices ($d = 4$, $r = m$);
(d) $V = \text{Herm}(3, \mathbb{O})$, the space of $3 \times 3$ octonionic hermitian matrices ($d = 8$, $r = 3$); and
(e) $V = \mathbb{R} \times \mathbb{R}^{n-1}$ ($d = n - 2$, $r = 2$).

In cases (a)–(d) the positive cone $\Omega$ is the cone of positive-definite matrices; in case (e) it is the Lorentz cone $\{(x_0, x): x_0 > \sqrt{x^2}\}$.
where
\[ \Gamma_\Omega(\alpha) = (2\pi)^{(n-r)/2} \prod_{j=0}^{r-1} \Gamma\left(\alpha - j \frac{d}{2}\right). \] (3.3)

Thus, for \( \Re \alpha > (r - 1)\frac{d}{2} \), the function \( \Delta(x)^{\alpha - \frac{n}{r}} / \Gamma_\Omega(\alpha) \) is locally integrable on \( \Omega \) and polynomially bounded, and so defines a tempered distribution \( \mathcal{R}_\alpha \) on \( V \) by the usual formula
\[ \mathcal{R}_\alpha(\varphi) = \frac{1}{\Gamma_\Omega(\alpha)} \int_{\Omega} \varphi(x) \Delta(x)^{\alpha - \frac{n}{r}} \, dx \quad \text{for} \ \varphi \in \mathcal{S}(V). \] (3.4)

Using (3.2), a beautiful argument — which is a special case of Bernstein’s general method for analytically continuing distributions of the form \( \mathcal{P}_\Omega^\lambda [2,4] \) — shows that the Riesz distributions \( \mathcal{R}_\alpha \) can be analytically continued to the whole complex \( \alpha \)-plane:

**Theorem 3.2 [12, Theorem VII.2.2 et seq.]** The distributions \( \mathcal{R}_\alpha \) can be analytically continued to the whole complex \( \alpha \)-plane as a tempered-distribution-valued entire function of \( \alpha \). Furthermore, the distributions \( \mathcal{R}_\alpha \) have the following properties:

\[ \mathcal{R}_0 = \delta \] (3.5a)
\[ \mathcal{R}_\alpha * \mathcal{R}_\beta = \mathcal{R}_{\alpha + \beta} \] (3.5b)
\[ \Delta(\partial / \partial x) \mathcal{R}_\alpha = \mathcal{R}_{\alpha - 1} \] (3.5c)
\[ \Delta(x) \mathcal{R}_\alpha = \frac{1}{\prod_{j=0}^{r-1} (\alpha - j \frac{d}{2})} \mathcal{R}_{\alpha + 1} \] (3.5d)

(here \( \delta \) denotes the Dirac measure at 0) and
\[ \mathcal{R}_\alpha(\varphi \circ g^{-1}) = \text{Det}(g)^{\alpha r/n} \mathcal{R}_\alpha(\varphi) \quad \text{for} \ g \in G, \ \varphi \in \mathcal{S}(V) \] (3.6)

(in particular, \( \mathcal{R}_\alpha \) is homogeneous of degree \( \alpha r - n \)). Finally, the Laplace transform of \( \mathcal{R}_\alpha \) is
\[ (\mathcal{L}\mathcal{R}_\alpha)(y) = \Delta(y)^{-\alpha} \] (3.7)

for \( y \) in the complex tube \( \Omega + iV \).

The property (3.5d) is not explicitly stated in [12], but for \( \Re \alpha > (r - 1)\frac{d}{2} \) it is an immediate consequence of (3.3)/(3.4), and then for other values of \( \alpha \) it follows by analytic continuation (see also [18, Proposition 3.1(iii) and Remark 3.2]). Likewise, the property (3.6) is not explicitly stated in [12], but for \( \Re \alpha > (r - 1)\frac{d}{2} \) it is an immediate consequence of (3.1)/(3.4), and then for other values of \( \alpha \) it follows by analytic continuation (see also [18, Proposition 3.1(i)])). It follows from (3.5a,b) that the distributions \( \mathcal{R}_\alpha \) are all nonzero; and it follows from this and (3.6) that \( \mathcal{R}_\alpha \not= \mathcal{R}_\beta \) whenever \( \alpha \not= \beta \).

It is fairly easy to find a sufficient condition for the Riesz distributions to be a positive measure:
Proposition 3.3 [12, Proposition VII.2.3] (see also [18, Section 3.2] [6, 21])

(a) For \( \alpha = k \frac{d}{2} \) with \( k = 0, 1, \ldots, r - 1 \), the Riesz distribution \( \mathcal{R}_\alpha \) is a positive measure that is supported on the set of elements of \( \overline{\Omega} \) of rank exactly \( k \) (which is a subset of \( \partial \Omega \)).

(b) For \( \alpha > (r - 1) \frac{d}{2} \), the Riesz distribution \( \mathcal{R}_\alpha \) is a positive measure that is supported on \( \Omega \) and given there by a density (with respect to Lebesgue measure) that lies in \( L^1_{\text{loc}}(\Omega) \).

The interesting and nontrivial fact (Theorem 1.1 above) is that the converse of Proposition 3.3 is also true: the foregoing values of \( \alpha \) are the only ones for which \( \mathcal{R}_\alpha \) is a positive measure. Here I shall use Proposition 2.3 together with the Laplace-transform formula (3.2)/(3.7) to provide an alternate and extremely elementary proof of the stronger converse result stated in Theorem 1.2.

Lemma 3.4 \( \Delta^\lambda \in L^1_{\text{loc}}(\Omega) \) if and only if \( \text{Re}\lambda > -1 \); or in other words, \( \Delta^{\alpha - \frac{n}{r}} \in L^1_{\text{loc}}(\Omega) \) if and only if \( \text{Re}\alpha > (r - 1) \frac{d}{2} = \frac{n}{r} - 1 \).

**Proof.** Since \( |\Delta(x)|^\lambda = \Delta(x)^{\text{Re}\lambda} \), it suffices to consider real values of \( \lambda \).

For \( \lambda > -1 \) [i.e. \( \alpha > (r - 1) \frac{d}{2} \)], fix any \( y \in \Omega \): the fact that the integral (3.2) is convergent, together with the fact that \( x \mapsto e^{+(x|y)} \) is locally bounded, implies that \( \Delta^\lambda \in L^1_{\text{loc}}(\Omega) \).

Now consider \( \lambda = -1 \): again fix any \( y \in \Omega \), and let \( \mu = \inf_{x \in C} (x|y) > 0 \) where \( \| \cdot \| \) is any norm on \( V \). Choose \( R > 0 \) such that \( |\Delta(x)| \leq 1 \) whenever \( \|x\| \leq R \). Then

\[
\int_{x \in \Omega \atop \|x\| \leq R} e^{-(x|y)} \Delta(x)^{-1} dx = \lim_{\lambda \downarrow -1} \int_{x \in \Omega \atop \|x\| \leq R} e^{-(x|y)} \Delta(x)^\lambda dx \tag{3.8}
\]

by the monotone convergence theorem. We now proceed to obtain a lower bound on

\[
M_\lambda := \int_{x \in \Omega \atop \|x\| \leq R} e^{-(x|y)} \Delta(x)^\lambda dx. \tag{3.9}
\]
For any $\beta \geq 1$, we have
\[\int_{x \in \Omega} e^{-\langle x \mid y \rangle} \Delta(x)^\lambda \, dx = \beta^{n+r}\lambda \int_{x \in \Omega} e^{-\beta\langle x \mid y \rangle} \Delta(x)^\lambda \, dx \leq \|x\| \leq \beta R\]
(3.10a)
\[\leq \beta^{n+r}\lambda e^{-\beta\langle x \mid y \rangle} \Delta(x)^\lambda \int_{x \in \Omega} e^{-\langle x \mid y \rangle} \Delta(x)^\lambda \, dx \leq R \leq \beta R\]
(3.10b)
\[\leq \beta^{n+r}\lambda e^{-\beta\langle x \mid y \rangle} \Delta(x)^\lambda \int_{x \in \Omega} e^{-\langle x \mid y \rangle} \Delta(x)^\lambda \, dx \leq R \leq \beta R\]
(3.10c)
where the first equality used the homogeneity of $\Delta$. Now sum this over $\beta = 2^k$ ($k = 1, 2, 3, \ldots$); the sum is convergent, and we conclude that
\[\int_{x \in \Omega} e^{-\langle x \mid y \rangle} \Delta(x)^\lambda \, dx \leq CM_\lambda\]
(3.11)
for a universal constant $C < \infty$ that is independent of $\lambda$ for $-1 < \lambda \leq 0$. Since (3.2) tells us that
\[\lim_{\lambda \downarrow -1} \int_{x \in \Omega} e^{-\langle x \mid y \rangle} \Delta(x)^\lambda \, dx = +\infty\]
(3.12)
due to the pole of the gamma function at $\alpha = (r - 1)\frac{d}{2}$, we conclude that $\lim_{\lambda \downarrow -1} M_\lambda = +\infty$ as well. Therefore
\[\int_{x \in \Omega} e^{-\langle x \mid y \rangle} \Delta(x)^{-1} \, dx = +\infty,\]
(3.13)
which proves that $\Delta^{-1} \notin L^1_{\text{loc}}(\Omega)$.

Since $\Delta$ is locally bounded, it also follows that $\Delta^\lambda \notin L^1_{\text{loc}}(\Omega)$ for $\lambda < -1$. □

We shall also need a uniqueness result related to Proposition 3.3(a). If $\mu$ is a locally finite complex measure on $V$, we say that $\mu$ is $G$-relatively invariant with exponent $\kappa$ in case
\[\mu(gA) = \text{Det}(g)^\kappa \mu(A) \quad \text{for } g \in G, \ A \text{ compact } \subseteq V.\]
(3.14)
In particular, every such $\mu$ is $G \cap SL(V)$-invariant, i.e.
\[\mu(gA) = \mu(A) \quad \text{for } g \in G \cap SL(V), \ A \text{ compact } \subseteq V.\]
(3.15)
Now define $\Omega_k = \{x \in \Omega: \text{rank}(x) = k\}$, so that $\partial\Omega = \bigcup_{k=0}^{r-1} \Omega_k$ and $\Omega = \Omega_r$. We then have the following result, which seems to be of some interest in its own right:
Lemma 3.5

(a) The group $G \cap SL(V)$ acts transitively on each set $\Omega_k$ $(0 \leq k \leq r - 1)$.

(b) Let $\mu$ be a locally finite complex measure that is supported on $\Omega_k$ $(0 \leq k \leq r - 1)$ and is $G \cap SL(V)$-invariant. Then $\mu$ is a multiple of $\mathcal{R}_{kd/2}$.

(c) Let $\mu$ be a locally finite complex measure that is supported on $\partial \Omega$ and is $G$-relatively invariant with some exponent $\kappa$. Then there exists $k \in \{0, 1, \ldots, r-1\}$ such that $\mu$ is a multiple of $\mathcal{R}_{kd/2}$ (and hence $\kappa = kdr/2n$ if $\mu \neq 0$).

Proof. (a) Fix a Jordan frame $c_1, \ldots, c_r$, and let $V = \bigoplus_{1 \leq i < j \leq r} V_{ij}$ be the corresponding orthogonal Peirce decomposition [12, Theorem IV.2.1]. Then, for $\lambda > 0$, let $M_\lambda = P(c_1 + \ldots + c_{r-1} + \lambda c_r) \in GL(V)$, where $P$ is the quadratic representation [12, p. 32]. From [12, p. 32 and Theorem IV.2.1(ii)] we see that $M_\lambda$ acts as multiplication by $\lambda^2$ on the space $V_{rr}$, as multiplication by $\lambda$ on the spaces $V_{ir}$ with $1 \leq i \leq r - 1$, and as the identity on the other subspaces. We have $M_\lambda \in G$ [12, Proposition III.2.2] and $\text{Det}(M_\lambda) = \lambda^{(r-1)d+2} = \lambda^{2n}/r$.

Now write $e_k = c_1 + \ldots + c_k$. By construction we have $M_\lambda e_k = e_k$ for $0 \leq k \leq r - 1$. Now, we know [12, Proposition IV.3.1] that $\Omega_k = Ge_k$, so that for any $x \in \Omega_k$ there exists $g \in G$ such that $x = ge_k$. Therefore, if we set $\lambda = \text{Det}(g)^{-r/2n}$, we have $x = gM_\lambda e_k$ with $gM_\lambda \in G \cap SL(V)$.

(b) follows from (a) and Proposition 3.3(a) together with a standard result about the uniqueness of invariant measures: see e.g. [7, Chapitre 7, sec. 2.6, Théorème 3], [24, p. 138, Theorem 1] or [30, Theorem 7.4.1 and Corollary 7.4.2].

(c) is now an easy consequence, as we can write (uniquely) $\mu = \sum_{k=0}^{r-1} \mu_k$ with $\mu_k$ supported on $\Omega_k$, and each $\mu_k$ is $G$-relatively invariant with exponent $\kappa$ [since each set $\Omega_k$ is a separate $G$-orbit]. But in at most one case can $\kappa$ take the correct value $kd/2n$; so all but one of the measures $\mu_k$ must be zero. \hfill \Box

Remarks. 1. Assertions (a) and (b) are false when $k = r$: the determinant $\Delta(x)$ is invariant under the action of $G \cap SL(V)$ [cf. 3.11], so $G \cap SL(V)$ cannot act transitively on $\Omega_r$; and all the measures $\mathcal{R}_\alpha$ with $\text{Re} \alpha > (r - 1)\frac{d}{2}$ are $G$-relatively invariant [hence $G \cap SL(V)$-invariant] and supported on $\Omega_r$.

2. A slight weakening of Lemma 3.5(b) — in which “$G \cap SL(V)$-invariant” is replaced by “$G$-relatively invariant with some exponent $\kappa$” — is asserted in [21, p. 391, Remarque 3], but the proof given there is insufficient (if it were valid, it would apply also to $k = r$). However, Michel Lassalle has kindly communicated to me a simple alternative proof of this result, based on [21, Théorème 3 and Proposition 11(b)].

3. Further information on the Riesz measures $\mathcal{R}_{kd/2}$ for $0 \leq k \leq r - 1$ can be found in [6,21]. \hfill \Box

More generally, we see that $P(\sum \lambda_i c_i)$ acts as multiplication by $\lambda_i \lambda_j$ on $V_{ij}$.
Proof of Theorem 1.2. We already know from Proposition 3.3(b) that $\mathcal{R}_\alpha$ is a locally finite complex measure for $\text{Re}\alpha > (r - 1)\frac{d}{2}$. On the other hand, by applying Proposition 2.3 to $F(x, \alpha) = \Delta(x)^{\alpha - \frac{d}{2}}/\Gamma_\Omega(\alpha)$ and using Lemma 3.4, we deduce that $\mathcal{R}_\alpha$ is not a locally finite complex measure whenever $\text{Re}\alpha \leq (r - 1)\frac{d}{2}$ and $\Gamma_\Omega(\alpha) \neq \infty$.

So it remains only to study the values of $\alpha$ for which $\Gamma_\Omega(\alpha) = \infty$, namely $\alpha \in\{0, \frac{d}{2}, \ldots, (r-1)\frac{d}{2}\} - \mathbb{N}$. For $\alpha \in \{0, \frac{d}{2}, \ldots, (r-1)\frac{d}{2}\}$ we know from Proposition 3.3(a) that $\mathcal{R}_\alpha$ is a positive measure. For $\alpha \in \{0, \frac{d}{2}, \ldots, (r-1)\frac{d}{2}\} - \mathbb{N}$, we know from Proposition 3.3(a) and (3.5c) that $\mathcal{R}_\alpha$ is a distribution supported on $\partial \Omega$; and by (3.6) and Lemma 3.5(b) we conclude that it cannot be a locally finite complex measure (here we use the fact that $\mathcal{R}_\alpha \neq \mathcal{R}_\beta$ when $\alpha \neq \beta$).

Remark. For $\text{Re}\alpha < 0$, an alternate proof that $\mathcal{R}_\alpha$ is not a complex measure can be based on the following fact, which is a special case of the $N = 0$ case of [19, Theorem 7.4.3] (compare [19, Theorem 7.3.1]) but can also easily be proven by direct computation:

**Lemma 3.6** Let $\Omega$ be a proper open convex cone in a real vector space $V$, and let $\Omega^* \subset V^*$ be the open dual cone. Let $T \in S'(V) \cap \mathcal{D}^0(V)$ be a tempered distribution of order 0 (i.e. a polynomially bounded complex measure) that is supported in $\overline{\Omega}$. Then the Laplace transform $LT$ is analytic in the complex tube $\Omega^* + iV^*$ and is bounded in every set $K + \Omega^* + iV^*$ where $K$ is a compact subset of $\Omega^*$.

It then follows from (3.7) that $\mathcal{R}_\alpha$ cannot be a locally finite complex measure when $\text{Re}\alpha < 0$, because $\Delta(y)^{-\alpha}$ is unbounded at infinity. This argument handles (without the need for Lemma 3.5) the cases $d = 1$ (real symmetric matrices) and $d = 2$ (complex hermitian matrices) in Theorem 1.2.

Remark on an elementary proof of Theorem 1.1

Casalis and Letac [9, Proposition 5.1] have given an elementary proof of Theorem 1.1 that deserves to be more widely known than it apparently is. They employ a method due to Shanbhag [27, p. 279, Remark 3] — who proved Theorem 1.1 for the cases of real symmetric and complex hermitian matrices — which they abstract as a general “Shanbhag principle” [9, Proposition 3.1]. Here I would like to abstract their method even further, with the aim of revealing its utter simplicity and beauty.

Let $V$ be a finite-dimensional real vector space, and let $V^*$ be its dual space. We then make the following trivial observations:

---

5 Science Citation Index shows only ten publications citing [9], and six of these have an author in common with [9].
(a) If \( \mu \) is a positive (i.e. nonnegative) measure on \( V \), then its Laplace transform
\[
L(\mu)(y) = \int e^{-\langle y,x \rangle} \, d\mu(x)
\] (A.1)
is nonnegative on any subset of \( V^* \) where it is well-defined (i.e. where the integral is convergent).

(b) If \( \mu \) is a positive measure on \( V \), then so is \( f \mu \) for every continuous (or even bounded measurable) function \( f \) on \( V \) that is nonnegative on \( \text{supp} \, \mu \).

(c) If \( \mu \) is a (positive or signed) measure on \( V \) whose Laplace transform is well-defined (and finite) on a nonempty open set \( \Theta \subseteq V^* \), then the same is true for \( P\mu \), where \( P \) is any polynomial on \( V \); furthermore, \( L(P\mu) = P(-\partial)L(\mu) \).

Putting together these observations, we conclude:

**Proposition A.1 (Shanbhag–Casalis–Letac principle)** If \( \mu \) is a positive measure on \( V \) whose Laplace transform is well-defined (and finite) on a nonempty open set \( \Theta \subseteq V^* \), and \( P \) is a polynomial on \( V \) that is nonnegative on \( \text{supp} \, \mu \), then \( P(-\partial)L(\mu) \geq 0 \) everywhere on \( \Theta \).

**Remark.** Proposition A.1 also has a strong converse, which we shall state and prove at the end of this appendix. \( \square \)

Using Proposition A.1, we can give the following slightly simplified version of the Shanbhag–Casalis–Letac argument:

**Proof of Theorem 1.1, based on [9, Proposition 5.1].** In view of Proposition 3.3, it suffices to prove the converse statement. So let \( \alpha \in \mathbb{R} \) and suppose that \( \mathcal{R}_\alpha \) is a positive measure. Using Proposition A.1 with \( P = \Delta \) together with the Laplace-transform formula (3.7), we conclude that
\[
\Delta(-\partial/\partial y) \Delta(y)^{-\alpha} \geq 0 \quad \text{for all } y \in \Omega. \tag{A.2}
\]
But the “Cayley” identity [12, Proposition VII.1.4] tells us that
\[
\Delta(\partial/\partial y) \Delta(y)^\lambda = \Delta(y)^{\lambda-1} \prod_{j=0}^{r-1} \left( \lambda + j \frac{d}{2} \right), \tag{A.3}
\]
hence (since \( \Delta \) is homogeneous of degree \( r \))
\[
\Delta(-\partial/\partial y) \Delta(y)^{-\alpha} = \Delta(y)^{-\alpha-1} \prod_{j=0}^{r-1} \left( \alpha - j \frac{d}{2} \right). \tag{A.4}
\]

\(^6\) Indeed, the same holds if the measure \( \mu \) is replaced by a distribution \( T \in \mathcal{D}'(V) \). See [26, Chapitre VIII] or [19, Section 7.4] for the theory of the Laplace transform on \( \mathcal{D}'(V) \).
It follows from (A.2) and (A.4) that $R_{\alpha}$ is \textit{not} a positive measure when $(r-2)\frac{d}{2} < \alpha < (r-1)\frac{d}{2}$. But using the convolution equation (3.5b) with $\beta = d/2$ together with the fact that $R_{d/2}$ is a positive measure [Proposition 3.3(a)], we conclude successively that $R_{\alpha}$ is not a positive measure when $(k-1)\frac{d}{2} < \alpha < k\frac{d}{2}$ for any integer $k \leq r-1$. This leaves only negative multiples of $d/2$; and the argument given after Lemma 3.6 shows that $R_{\alpha}$ is not a positive measure whenever $\alpha < 0$. \hfill $\square$

\textbf{Remark.} This method has been used recently by Letac and Massam [22, proof of Proposition 2.3] to determine the set of acceptable powers $p$ for the noncentral Wishart distribution, generalizing the earlier proof of Shanbhag [27] and Casalis and Letac [9] for the ordinary Wishart distribution (which is essentially Theorem 1.1). \hfill $\square$

But this is not yet the end of the story; the proof can be further simplified. The use of the Laplace transform in the foregoing proof is in reality a red herring, as it is used \textit{twice} in opposite directions: once in the proof of Proposition A.1, and once again in the proof of (A.3). We can therefore give a direct proof that makes almost no reference to the Laplace transform:

\textbf{SECOND PROOF OF THEOREM 1.1} Consider first $(r-2)\frac{d}{2} < \alpha < (r-1)\frac{d}{2}$. If $R_{\alpha}$ is a positive measure, then so is $\Delta(x)R_{\alpha}$, which by (3.5d) equals $C_{\alpha}R_{\alpha+1}$, where

$$C_{\alpha} = \prod_{j=0}^{r-1}(\alpha - j\frac{d}{2}) < 0. \quad (A.5)$$

It follows that $R_{\alpha+1}$ must be a negative (i.e. nonpositive) measure. But this is surely not the case, as the Laplace-transform formula (3.1) immediately implies that \textit{no} $R_{\beta}$ can be a negative measure. This shows that $R_{\alpha}$ is not a positive measure when $(r-2)\frac{d}{2} < \alpha < (r-1)\frac{d}{2}$. The proof is then completed as before. \hfill $\square$

\textbf{Alternate argument:} For $k = 1, 2, 3, \ldots$ we know from Proposition 3.3(a,b) and 3.6 that $R_{kd/2}$ is a positive measure that is not supported on a single point. If $R_{-kd/2}$ were a positive measure (recall that we know it is nonzero), then $R_{kd/2}*R_{-kd/2}$ could not be supported on a single point, contrary to the fact that $R_{kd/2}*R_{-kd/2} = \delta$ [cf. (3.5a,b)].

The simplest proof of (A.3) is probably the one given in [12, Proposition VII.1.4], using Laplace transforms. However, direct combinatorial proofs are also possible: see [8] for a detailed discussion in the cases of real symmetric and complex hermitian matrices.

It would be interesting to know whether this residual use of the Laplace transform can be avoided. For $d \leq 2$ it can definitely be avoided, as $\alpha + 1 > (r-1)\frac{d}{2}$, so that $R_{\alpha+1}$ is a nonzero positive measure by Proposition 3.3(b); but for $d > 2$ I do not know.

The argument given after Lemma 3.6 explicitly uses the Laplace transform. But the alternate argument given in footnote $\square$ does not.
It would be interesting to know whether this approach is powerful enough to handle the multiparameter Riesz distributions [12, Theorem VII.3.2] and/or the Riesz distributions on homogeneous cones that are not symmetric and hence do not arise from a Euclidean Jordan algebra [13, 20].

To conclude, let us give the promised strong converse to Proposition A.1:

**Proposition A.2** Let \( T \in \mathcal{D}'(V) \) be a distribution whose Laplace transform is well-defined on a nonempty open set \( \Theta \subseteq V^* \). Let \( S \subseteq V \) be a closed set, and suppose that there exists \( y_0 \in \Theta \) such that \( [P(-\partial)L(T)](y_0) \geq 0 \) for all polynomials \( P \) on \( V \) that are nonnegative on \( S \). Then \( T \) is in fact a positive measure that is supported on \( S \).

**Proof.** By replacing \( T(x) \) by \( e^{-\langle y_0, x \rangle}T(x) \), we can assume without loss of generality that \( y_0 = 0 \). Then the derivatives of \( L(T) \) at the origin give us the moments of \( T \); and the hypothesis \( [P(-\partial)L(T)](y_0) \geq 0 \) implies, by Haviland’s theorem [16, 17] [23, Theorem 3.1.2], that there exists a positive measure \( \mu \) supported on \( S \) that has these moments. Furthermore, the analyticity of \( L(T) \) in the open set \( \Theta + iV^* \) implies that these moments satisfy a bound of the form \( |c_n| \leq AB^n|n|! \), so that \( \int e^{\epsilon|x|} d\mu(x) < \infty \) for some \( \epsilon > 0 \). It follows that the Laplace transform \( L(\mu) \) is well-defined and analytic in a neighborhood of the origin; and since its derivatives at the origin agree with those of \( L(T) \), we must have \( L(\mu) = L(T) \). But by the injectivity of the distributional Laplace transform [26, p. 306, Proposition 6], it follows that \( \mu = T \). \( \square \)

In Proposition A.2 it is essential that the Laplace transform of \( T \) be well-defined on a nonempty open set \( \Theta \ni y_0 \), or in other words (when \( y_0 = 0 \)) that \( T \) have some exponential decay at infinity [in the sense that \( \cosh(\epsilon|x|)T \in S'(V) \) for some \( \epsilon > 0 \)]. It is not sufficient for \( T \) to have finite moments of all orders satisfying \( T(P) \geq 0 \) for all polynomials \( P \) on \( V \) that are nonnegative on \( S \). Indeed, Stieltjes’ [28] famous example

\[
f(x) = \begin{cases} 
  e^{-\log^2 x} \sin(2\pi \log x) & \text{for } x > 0 \\
  0 & \text{for } x \leq 0
\end{cases}
\]

(A.6)

belongs to \( S(\mathbb{R}) \) and has zero moments of all orders [i.e. \( T(P) = 0 \) for all polynomials \( P \)] but is not nonnegative.

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