Analytic solutions of 2-photon and two-mode Rabi models

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Abstract

Applying Bogoliubov transformations and Bargmann-Hilbert spaces, we obtain analytic representations of solutions to the 2-photon and two-mode quantum Rabi models. In each case, a transcendental function is analytically derived whose zeros give the energy spectrum of the model. The zeros can be numerically found by standard root-search techniques. We also present analytic solution to the driven Rabi model with broken $\mathbb{Z}_2$ symmetry.

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1 Introduction

Spin-boson systems describe interactions between spins and harmonic oscillators (boson modes) and have played a prominent role in modeling the ubiquitous matter-light interactions in modern physics. One of the well-known spin-boson systems is the quantum Rabi model, introduced by Rabi over 70 years ago. This model describes the interaction of a two-level atom with a harmonic mode of quantized electromagnetic field. Due to the simplicity of its Hamiltonian, the Rabi model has served as the basis for understanding matter-light interactions, and has a variety of applications ranging from quantum optics [1] to solid state semiconductor systems [2] and molecular physics [3]. It also played a significant role in the novel research field of cavity and circuit quantum electrodynamics [4, 5].

The Hamiltonian of the Rabi model is given by

$$H_R = \omega b^\dagger b + \Delta \sigma_z + g \sigma_x (b^\dagger + b),$$

where $g$ is the spin-boson interaction strength, $\sigma_z, \sigma_x$ are the Pauli matrices describing the two atomic levels separated by energy difference $2\Delta$, and $b^\dagger (b)$ are creation (annihilation)
operators of a boson mode with frequency $\omega$. In the Bargmann space with the monomials $\{ w^n \sqrt{n!} \}$ as basis vectors, the boson creation and annihilation operators can be realized as $b^\dagger = w$, $b = \frac{d}{dw}$. In terms of the two-component wavefunction $\psi(w) = (\psi_+(w), \psi_-(w))^T$, the time-independent Schrödinger equation of the model gives a system of two coupled linear differential equations with rational coefficients. The system depends on a spectral parameter $E$. The wavefunction components are elements of Bargmann-Hilbert space of entire functions of one variable $w \in \mathbb{C}$ [6]. The scalar product of any two elements $f(w), g(w)$ in the Bargmann-Hilbert space is given by

$$ (f, g) = \int \overline{f(w)} g(w) d\mu(w), \quad (2) $$

where $d\mu(w) = \frac{1}{\pi} e^{-|w|^2} dx dy$. The energy $E$ belongs to the spectrum of the problem if and only if for this value of $E$ the system of coupled differential equations (given by the time-independent Schrödinger equation) has entire solution [6].

By means of the Bargmann-Hilbert space approach, Braak [7] recently presented a pair of transcendental functions defined as infinite power series expansions with coefficients satisfying three-term recurrence relations, and argued that the spectrum of the Rabi model is given by the zeros of the transcendental functions. This theoretical progress has renewed the interest in the Rabi and related models [8]-[17].

In this paper we solve the 2-photon and two-mode generalizations of the quantum Rabi model. The 2-photon and two-mode Rabi models are phenomenological models describing a two-level atom interacting with 2 photons and 2 harmonic modes, respectively. They can be experimentally realized in circuit quantum electromagnetic systems [5], and have established applications in many research fields, including Rubidium atoms [18] and quantum dots [19, 20].

The reset of this paper is as follows. Applying the Bogoliubov type transformations and Bargmann-Hilbert spaces, we obtain analytic representations of solutions to the two models in sections 2 and 3, respectively. For each case, a transcendental function is analytically found whose zeros give the energy eigenvalues of the model for arbitrary coupling strength and detuning. We draw conclusions in section 4. In the Appendix, we apply our approach to the driven Rabi model without $\mathbb{Z}_2$ symmetry and derive a continued fraction equation for its energy spectrum.

## 2 2-photon quantum Rabi model

The Hamiltonian of the 2-photon Rabi model reads

$$ H_{2p} = \omega b^\dagger b + \Delta \sigma_z + g \sigma_x [(b^\dagger)^2 + b^2]. \quad (3) $$
Let us make a canonical Bogoliubov transformation from \( b, b^\dagger \) to the squeezed bosons \( a, a^\dagger \)
[21],
\[
\begin{align*}
  b &= \frac{a + \tau a^\dagger}{\sqrt{1 - \tau^2}}, \\
  b^\dagger &= \frac{\tau a + a^\dagger}{\sqrt{1 - \tau^2}},
\end{align*}
\]
where \(|\tau| < 1\) is a real parameter. In terms of the squeezed bosons, the Hamiltonian (3) takes the form
\[
\hat{H}_{2p} = \Delta \sigma_z + \frac{1}{1 - \tau^2} \left[ \left( \omega \tau + g \sigma_x (1 + \tau^2) \right) \left( (a^\dagger)^2 + a^2 \right) \right.
\]
\[
\left. + \left( \omega (1 + \tau^2) + 4 g \tau \sigma_x \right) a^\dagger a + \omega \tau^2 + 2 g \tau \sigma_x \right].
\]
(5)

Introduce the operators \( K_\pm, K_0 \)
\[
K_+ = \frac{1}{2} (a^\dagger)^2, \quad K_- = \frac{1}{2} a^2, \quad K_0 = \frac{1}{2} \left( a^\dagger a + \frac{1}{2} \right).
\]
(6)

Then (5) becomes
\[
\hat{H}_{2p} = \Delta \sigma_z + \frac{1}{1 - \tau^2} \left[ 2 \left( \omega \tau + g \sigma_x (1 + \tau^2) \right) (K_+ + K_-) \right.
\]
\[
\left. + 2 \left( \omega (1 + \tau^2) + 4 g \tau \sigma_x \right) K_0 \right] - \frac{1}{2} \omega.
\]
(7)

The operators \( K_\pm, K_0 \) form the \( su(1,1) \) Lie algebra. Its quadratic Casimir, \( C = K_+ K_- - K_0 (K_0 - 1) \), takes the particular values \( C = \frac{3}{16} \) in the representation (6). This is the well-known infinite-dimensional unitary irreducible representation \( D^+(q) \) of \( su(1,1) \) with \( q = \frac{1}{4}, \frac{3}{4} \). Thus the Fock-Hilbert space decomposes into the direct sum of two subspaces \( \mathcal{H}^q \) labeled by \( q = 1/4, 3/4 \).

Using the representation \( a^\dagger = w \) and \( a = \frac{d}{dw} \), we can show [15] that in the Bargmann space with basis vectors given by the monomials in \( z = w^2 \), \( \left\{ z^n / \sqrt{2(n+q-1/2)!} \right\} \), the operators \( K_\pm, K_0 \) (6) have the single-variable 2nd order differential realization
\[
K_0 = z \frac{d}{dz} + q, \quad K_+ = \frac{z}{2}, \quad K_- = 2z \frac{d^2}{dz^2} + 4q \frac{d}{dz}.
\]
(8)

The Bargmann space is the Hilbert space of entire functions on the complex plane if the inner product
\[
(f, g) = \int \overline{f(w)} g(w) \, d\mu(w)
\]
(9)
is finite for an appropriate measure \( d\mu(w) \). Take \( d\mu(w) = \frac{1}{\pi} |w|^{4(q-1/4)} e^{-|w|^2} \, dx \, dy \). Then we can show
\[
(z^m, z^n) = (w^{2m}, w^{2n}) = [2(n+q-1/4)!] \delta_{mn}.
\]
(10)

Thus the monomials \( \left\{ z^n / \sqrt{2(n+q-1/2)!} \right\} \) form an orthonormal basis of the Bargmann-Hilbert space. It is now not difficult to see that if \( f(z) = \sum_{n=0}^\infty c_n z^n \) then
\[
||f||^2 = \sum_{n=0}^\infty |c_n|^2 [2(n+q-1/4)!]
\]
(11)
and \( f(z) \) is entire if the sum on the right hand side converges.

Using this differential realization, working in a representation defined by \( \sigma_x \) diagonal and choosing \( \tau \) such that \( \omega \tau + g(1 + \tau^2) = 0 \), i.e.†

\[
\tau = -\frac{\omega}{2g}(1 - \Omega), \quad \Omega = \sqrt{1 - \frac{4g^2}{\omega^2}},
\]

where \( \frac{|2g|}{\omega} < 1 \), then the transformed 2-photon Rabi Hamiltonian becomes the matrix differential operator

\[
\tilde{H}_{2p} = \begin{pmatrix}
2\omega\Omega \left( z \frac{d}{dz} + q \right) - \frac{1}{2}\omega & \Delta \\
\Delta & -8g\frac{d^2}{dz^2} + \frac{2}{\Omega} [\omega(2 - \Omega^2)z - 8gq] \frac{d}{dz} - \frac{2g}{\Omega} z + \frac{2\omega(2 - \Omega^2)q}{\Omega} - \frac{1}{2}\omega
\end{pmatrix}.
\]

In terms of two-component wavefunction \( \psi(z) = (\psi_+(z), \psi_-(z))^T \), the time-independent Schrödinger equation, \( \tilde{H}_{2p} \psi(z) = E \psi(z) \), yields a system of coupled differential equations,

\[
\begin{align*}
2\omega\Omega \left( z \frac{d}{dz} + q \right) - \frac{1}{2}\omega - E & \psi_+ + \Delta \psi_- = 0, \\
8g\frac{d^2}{dz^2} + (-2\omega(2 - \Omega^2)z + 16gq) \frac{d}{dz} & + 2g - 2\omega(2 - \Omega^2)q + (\frac{1}{2}\omega + E)\Omega \psi_- - \Omega \Delta \psi_+ = 0.
\end{align*}
\]

This is a system of differential equations of Fuchsian type. Solutions to these equations must be analytic in the whole complex plane if \( E \) belongs to the spectrum of \( \tilde{H}_{2p} \). So we are seeking solutions of the form

\[
\psi_+(z) = \sum_{n=0}^{\infty} K_n^+(E) z^n, \quad \psi_-(z) = \sum_{n=0}^{\infty} K_n^-(E) z^n,
\]

which converge in the entire complex plane, i.e. solutions which are entire.

Substituting (16) into (14), we obtain

\[
K_n^+ = \frac{\Delta}{E + \frac{1}{2}\omega - (2n + 2q)\omega\Omega} K_n^-.
\]

So \( K_n^+ \) is not analytic in \( E \) but has simple poles at

\[
E = -\frac{1}{2}\omega + (2n + 2q)\omega\Omega, \quad n = 0, 1, \cdots.
\]

†The quadratic equation has two roots \( \tau = -\frac{2g}{\omega}(1 \mp \Omega) \). The root given in (12) is real and obeys \(|\tau| < 1\), the requirements of \( \tau \) from the Bogoliubov transformation, provided that \( \frac{|2g|}{\omega} < 1 \). This is seen as follows. Assume \(|\tau| = \left|\frac{2g}{\omega}\right| \left(1 - \sqrt{1 - \left(\frac{2g}{\omega}\right)^2}\right) \geq 1 \), i.e. \( 1 - \left|\frac{2g}{\omega}\right| \geq \sqrt{1 - \left(\frac{2g}{\omega}\right)^2} \). Then we would have \( \sqrt{1 - \left|\frac{2g}{\omega}\right|} \geq \sqrt{1 + \left|\frac{2g}{\omega}\right|} \), which is impossible.
The energies (18) appear for special values of model parameters [21, 15] and correspond to the exceptional solutions of the 2-photon Rabi model. If (18) is satisfied, the infinite series expansions (16) truncate and reduce to polynomials in $z$ but only if the system parameters satisfy certain constraints [15]. Majority part of the spectrum of the 2-photon Rabi model is regular for which (18) is not satisfied. The regular spectrum of the model is given by the zeros of the transcendental function $F(E)$ obtained below. Thus similar to the Rabi case, the spectrum of the 2-photon Rabi model consists of two parts, the regular and the exceptional spectrum.

From (15), we obtain the 3-step recurrence relation for $K_{n-1}$,

$$K_1^+ + A_0 K_0^- = 0,$$
$$K_{n+1}^- + A_n K_n^- + B_n K_{n-1}^- = 0, \quad n \geq 1,$$

where

$$A_n = \frac{1}{8g(n+1)(n+2q)} \left[ -(2n+2q)\omega(2 - \Omega^2) 
+ \left( E + \frac{1}{2}\omega - \frac{\Delta^2}{E + \frac{1}{2}\omega - (2n+2q)\omega\Omega} \right) \Omega \right],$$
$$B_n = \frac{1}{4(n+1)(n+2q)}.$$

(20)

The coefficients $A_n, B_n$ have the behavior as $n \to \infty$

$$A_n \sim a n^\alpha, \quad B_n \sim b n^\beta$$

(21)

with

$$a = -\frac{\omega}{4g}(2 - \Omega^2), \quad \alpha = -1, \quad b = \frac{1}{4}, \quad \beta = -2.$$

(22)

Thus the asymptotic structure of solutions to the 2nd equation of (19) depends on the Newton-Puiseux diagram formed with the points $P_0(0,0), P_1(1,-1), P_2(2,-2)$ [22]. Let $\gamma$ be the slope of $P_0 P_1$ and $\delta$ the slope of $P_1 P_2$ so that $\gamma = \alpha$ and $\delta = \beta - \alpha$. Then we have $\gamma = \delta = \alpha$. The characteristic equation of the $n \geq 1$ part of (19) reads $t^2 + at + b = 0$ with $a, b$ given in (22). It has two roots $t_1 = \frac{\omega}{4g}, t_2 = \frac{a}{\omega}$. Remembering the condition $\frac{2a}{\omega} < 1$, we have $|t_2| < |t_1|$. Applying the Perron-Kreuser theorem (i.e. Theorem 2.3 of [22]), we conclude that the two linearly independent solutions $K_{n,1}^-$ and $K_{n,2}^-$ of the $n \geq 1$ part (i.e. the truly 3-term part) of (19) satisfy

$$\lim_{n \to \infty} \frac{K_{n+1,r}^-}{K_{n,r}^-} \sim t_r n^{-1}, \quad r = 1, 2.$$

(23)

So $K_{n,2}^-$ is a minimal solution and $K_{n,1}^-$ is a dominant one. By (11), we can see that the infinite power series in (16) with expansion coefficients $K_{n,r}^-$ is entire if the sum

$$\sum_{n=0}^{\infty} \left| K_{n,r}^- \right|^2 \left[ 2(n + q - 1/4) \right]!$$

(24)
converges. Using the asymptotic form (23) we get
\[
\lim_{n \to \infty} \frac{|K_{n+1,r}|^2 [2(n + 1 + q - 1/4)]!}{|K_{n,r}|^2 [2(n + q - 1/4)]!} = 4 |t_r|^2
\] (25)
which is less than 1 for \( r = 2 \) and greater than 1 for \( r = 1 \). Thus by the ratio test, the sum (24) converges for the minimal solution \( K_n^{\text{min}} \equiv K_{n,2} \) and diverges for the dominant solution \( K_n^{-1} \). It follows that the infinite power series expansions \( \psi_{\pm}^{\text{min}}(z) \), obtained by substituting \( K_n^{\text{min}} \) for the \( K_n^{-} \)'s in (17) and (16), converge in the whole complex plane, i.e. they are entire.

We now find energy eigenvalues \( E \) corresponding to the minimal solution \( K_n^{\text{min}} \) (and thus to the entire wavefunctions \( \psi_{\pm}^{\text{min}}(z) \)). We follow a procedure presented in [23] that uses the relationship between minimal solutions and infinite continued fractions [22]. Such a procedure was applied in [9] to analyze the Rabi model.

By the Pincherle theorem (i.e. Theorem 1.1 of [22]), the ratio of successive elements of the minimal solution sequence \( K_n^{\text{min}} \) is expressible as continued fractions,
\[
R_n = \frac{K_{n+1}^{\text{min}}}{K_n^{\text{min}}} = -\frac{B_{n+1}}{A_{n+1}} - \frac{B_{n+2}}{A_{n+2}} - \frac{B_{n+3}}{A_{n+3}} - \cdots
\] (26)
which for \( n = 0 \) gives
\[
R_0 = \frac{K_1^{\text{min}}}{K_0^{\text{min}}} = -\frac{B_1}{A_1} - \frac{B_2}{A_2} - \frac{B_3}{A_3} - \cdots
\] (27)
Note that the ratio \( R_0 = \frac{K_1^{\text{min}}}{K_0^{\text{min}}} \) involves \( K_0^{\text{min}} \), although the above continued fraction expression is obtained from the 2nd equation of (19), i.e the recurrence (19) for \( n \geq 1 \). However, for single-ended sequences such as those appearing in the infinite series expansions (16), the ratio \( R_0 = \frac{K_1^{\text{min}}}{K_0^{\text{min}}} \) of the first two terms of a minimal solution is unambiguously fixed by the first equation of the recurrence (19), namely,
\[
R_0 = -A_0 = \frac{1}{16gq} \left[ 2q\omega(2 - \Omega^2) - \left( E + \frac{1}{2} \omega - \frac{\Delta^2}{E + \frac{1}{2} \omega - 2q\omega\Omega} \right) \Omega \right]
\] (28)
In general, the \( R_0 \) computed from the continued fraction (27) can not be the same as that from (28) for arbitrary values of recurrence coefficients \( A_n \) and \( B_n \). As a result, general solutions to the recurrence (19) are dominant and are usually generated by simple forward recursion from a given value of \( K_0^{-} \). Physical meaningful solutions are those that are entire in the Bargmann-Hilbert space [6]. They can be obtained if \( E \) can be adjusted so that equations (27) and (28) are both satisfied. Then the resulting solution sequence \( K_n^{-}(E) \) will be purely minimal and the power series expansion (16) will converge in the whole complex plane.

Therefore, if we define the transcendental function \( F(E) = R_0 + A_0 \) with \( R_0 \) given by the continued fraction in (27), then the zeros of \( F(E) \) correspond to the points in
the parameter space where the condition (28) is satisfied. In other words, \( F(E) = 0 \) is the eigenvalue equation of the 2-photon Rabi model, which may be solved for \( E \) by standard nonlinear root-search techniques (see e.g. [23, 24] and references therein). Only for the denumerable infinite values of \( E \) which are the roots of \( F(E) = 0 \), do we get entire solutions of the differential equations (14) and (15).

3 Two-mode quantum Rabi model

The Hamiltonian of the two-mode Rabi model is

\[
H_{2m} = \omega(b_1^\dagger b_1 + b_2^\dagger b_2) + \Delta \sigma_z + g \sigma_x (b_1^\dagger b_2 + b_2^\dagger b_1),
\]

(29)

where we assume that the boson modes are degenerate with the same frequency \( \omega \). Introduce the two-mode Bogoliubov transformation,

\[
b_1 = \frac{a_1 + \sigma a_2^\dagger}{\sqrt{1 - \sigma^2}}, \quad b_1^\dagger = \frac{\sigma a_2 + a_1^\dagger}{\sqrt{1 - \sigma^2}}, \quad b_2 = \frac{a_2 + \sigma a_1^\dagger}{\sqrt{1 - \sigma^2}}, \quad b_2^\dagger = \frac{\sigma a_1 + a_2^\dagger}{\sqrt{1 - \sigma^2}}.
\]

(30)

Here \( |\sigma| < 1 \) is a real parameter and \( a_1, a_2, a_1^\dagger, a_2^\dagger \) are squeezed bosons satisfying the canonical commutation relations \( [a_i, a_j^\dagger] = 1, \quad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0, \quad i, j = 1, 2. \) In terms of the 2-mode squeezed bosons, the Hamiltonian (29) has the form

\[
\tilde{H}_{2m} = \Delta \sigma_z + \frac{1}{1 - \sigma^2} \left[ (2\omega \sigma + g \sigma_x (1 + \sigma^2)) (a_1^\dagger a_2 + a_2^\dagger a_1) + \left( \omega(1 + \sigma^2) + 2g \sigma_x \right) (a_1^\dagger a_1 + a_2^\dagger a_2) + 2\omega \sigma^2 + 2g \sigma \sigma_x \right] + \Delta \sigma_z.
\]

(31)

Introduce the operators \( K_\pm, K_0 \)

\[
K_+ = a_1^\dagger a_2^\dagger, \quad K_- = a_1 a_2, \quad K_0 = \frac{1}{2} (a_1^\dagger a_1 + a_2^\dagger a_2 + 1).
\]

(32)

Then (31) becomes

\[
\tilde{H}_{2m} = \Delta \sigma_z + \frac{1}{1 - \sigma^2} \left[ (2\omega \sigma + g \sigma_x (1 + \sigma^2)) (K_+ + K_-) + 2 \left( \omega(1 + \sigma^2) + 2g \sigma \sigma_x \right) K_0 \right] - \omega.
\]

(33)

The operators \( K_\pm, K_0 \) form the \( su(1,1) \) Lie algebra. Its quadratic Casimir, \( C = K_+ K_- - K_0 (K_0 - 1) \), takes the particular values \( C = \kappa (1 - \kappa) \) in the representation (32), where \( \kappa = 1/2, 1, 3/2, \cdots \). This is the well-known infinite-dimensional unitary irreducible representation of \( su(1,1) \) known as the positive discrete series \( D^+ (\kappa) \). Thus the Fock-Hilbert space decomposes into the direct sum of infinite subspaces \( \mathcal{H}^\kappa \) labeled by \( \kappa = 1/2, 1, 3/2, \cdots \).
Using the representation $a^+_i = w_i$ and $a_i = \frac{d}{dz}$, $i = 1, 2$, we may show [15] that in the Bargmann space with basis vectors given by the monomials in $z = w_1 w_2$, \( \{ z^n/\sqrt{n!(n+2\kappa-1)!} \} \), the operators $K_\pm, K_0$ (32) have single-variable differential realization

\[
K_0 = z \frac{d}{dz} + \kappa, \quad K_+ = z, \quad K_- = z \left( \frac{d^2}{dz^2} + 2\kappa \frac{d}{dz} \right),
\]

where $\kappa = 1/2, 1, 3/2, \cdots$. The Bargmann space is the Hilbert space of entire functions on $\mathbb{C}^2$ if the inner product

\[
(f, g) = \int \int f(w_1, w_2) g(w_1, w_2) \, d\mu(w_1, w_2)
\]

is finite for an appropriate measure $d\mu(w_1, w_2)$. It can be easily seen that if we choose

\[
d\mu(w_1, w_2) = \frac{1}{\pi^2} |w_1|^{2(2\kappa-1)} e^{-|w_1|^2-|w_2|^2} d^2w_1 d^2w_2,
\]

then

\[
(z^m, z^n) = ((w_1 w_2)^{2m}, (w_1 w_2)^{2n}) = n! (n+2\kappa-1)! \delta_{mn}.
\]

Thus the monomials \( \{ z^n/\sqrt{n!(n+2\kappa-1)!} \} \) form an orthonormal basis of the Bargmann-Hilbert space. It is now not difficult to see that if $f(z) = \sum_{n=0}^\infty c_n z^n$ then

\[
||f||^2 = \sum_{n=0}^\infty |c_n|^2 n! (n+2\kappa-1)!
\]

and $f(z)$ is entire if the sum on the right hand side converges.

Using this differential realization, working in a representation defined by $\sigma_x$ diagonal and choosing $\sigma$ such that $2\omega\sigma + g(1 + \sigma^2) = 0$, i.e.$^\dagger$

\[
\sigma = -\frac{\omega}{g} (1 - \Lambda), \quad \Lambda = \sqrt{1 + \frac{g^2}{\omega^2}},
\]

where $|\sigma| < 1$, then the transformed Hamiltonian (33) becomes a matrix differential operator

\[
\tilde{H}_{2m} = \begin{pmatrix}
2\omega \Lambda \left( z \frac{d}{dz} + \kappa \right) - \omega & \Delta \\
\Delta & -2g z \frac{d^2}{dz^2} + \frac{2}{\lambda} \left[ \omega(2-\Lambda^2)z - 2g\kappa \right] \frac{d}{dz} - \frac{2g}{\lambda} z + \frac{2\omega (2-\Lambda^2)\kappa}{\lambda} - \omega
\end{pmatrix}.
\]

In terms of two-component wavefunction $\phi(z) = (\phi_+(z), \phi_-(z))^T$, the time-independent Schrödinger equation, $\tilde{H}_{2m} \phi(z) = E \phi(z)$, yields a system of coupled differential equations,

\[
\begin{align}
2\omega \Lambda \left( z \frac{d}{dz} + \kappa \right) - \omega - E \phi_+ + \Delta \phi_- &= 0, \\
2gz \frac{d^2}{dz^2} + (-2\omega(2-\Lambda^2)z + 4g\kappa) \frac{d}{dz} + 2gz - 2\omega(2-\Lambda^2)\kappa + (E + \omega) \Lambda \phi_- - \Lambda \Delta \phi_+ &= 0.
\end{align}
\]

$^\dagger$By arguments similar to the 2-photon case we can show that $\sigma$ given by (38) obeys the requirements that it is real and $|\sigma| < 1$, provided that $|\sigma| < 1$. 

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This is a system of differential equations of Fuchsian type. Solutions to these equations must be analytic in the whole complex plane if $E$ belongs to the spectrum of $\tilde{H}_{2m}$. Similar to the 2-photon Rabi case, we seek solutions of the form

$$
\phi_+(z) = \sum_{n=0}^{\infty} Q_n^+(E) z^n, \quad \phi_-(z) = \sum_{n=0}^{\infty} Q_n^-(E) z^n,
$$

which converge in the entire complex plane.

Substituting (42) into (40), we obtain

$$
Q_n^+ = \frac{\Delta}{E + \omega - (2n + 2\kappa)\omega \Lambda} Q_n^-,
$$

(43)

So $Q_n^+$ is not analytic in $E$ but has simple poles at

$$
E = -\omega + (2n + 2\kappa)\omega \Lambda, \quad n = 0, 1, \ldots
$$

(44)

The energies (44) appear for special values of model parameters [15] and correspond to the exceptional solutions of the two-mode Rabi model. If (44) is satisfied, the infinite series expansions (42) truncate and reduce to polynomials in $z$ but only if the model parameters obey certain constraints [14]. Majority part of the spectrum of the two-mode Rabi model is regular spectrum which does not have the form (44). The regular spectrum of the model is given by the zeros of the transcendental function $G(E)$ obtained below. Thus again, the spectrum of the two-mode Rabi model consists of two parts, the regular and the exceptional spectrum.

From (41), we obtain the 3-step recurrence relation for $Q_n^-$,

$$
Q_1^- + C_0 Q_0^- = 0,
Q_{n+1}^- + C_n Q_n^- + D_n Q_{n-1}^- = 0, \quad n \geq 1,
$$

(45)

where

$$
C_n = \frac{1}{2g(n+1)(n+2\kappa)} \left[ -(2n + 2\kappa)\omega (2 - \Lambda^2) \\
+ \left( E + \omega - \frac{\Delta^2}{E + \omega - (2n + 2\kappa)\omega \Lambda} \right) \Lambda \right],
$$

(46)

$$
D_n = \frac{1}{(n+1)(n+2\kappa)}.
$$

The coefficients $C_n, D_n$ have the behavior as $n \to \infty$

$$
C_n \sim c n^\mu, \quad D_n \sim d n^\rho
$$

(47)

with

$$
c = -\frac{\omega}{g} (2 - \Lambda^2), \quad \mu = -1, \quad d = 1, \quad \rho = -2.
$$

(48)
By analysis similar to the 2-photon case, we see that the two linearly independent solutions $Q_{n,1}$ and $Q_{n,2}$ of the $n \geq 1$ part of the recurrence (19) obey

$$\lim_{n \to \infty} \frac{Q_{n+1,r}}{Q_{n,r}} \sim t_r n^{-1}, \quad r = 1, 2,$$

(49)

where $t_1 = \frac{\omega}{g}$, $t_2 = \frac{g}{\omega}$ and $|t_2| < |t_1|$ (from the condition $|\frac{g}{\omega}| < 1$). Thus $Q_{n,2}$ is a minimal solution and $Q_{n,1}$ is a dominant one. Using (37) and by similar analysis to the 2-photon case, we can conclude that the infinite power series expansions $\phi_{\pm}^\text{min}(z)$ generated by substituting the minimal solution $Q_n^\text{min} \equiv Q_{n,2}$ for the $Q_n$'s in (43) and (42), converge in the whole complex plane.

The ratio of successive elements of the minimal solution $Q_n^\text{min}$ can be expressed as continued fractions,

$$S_n = \frac{Q_{n+1}^\text{min}}{Q_n^\text{min}} = -\frac{D_{n+1}}{C_{n+1}} - \frac{D_{n+2}}{C_{n+2}} - \frac{D_{n+3}}{C_{n+3}} - \cdots,$$

(50)

which for $n = 0$ reduces to

$$S_0 = \frac{Q_1^\text{min}}{Q_0^\text{min}} = -\frac{D_1}{C_1} - \frac{D_2}{C_2} - \frac{D_3}{C_3} - \cdots.$$

(51)

The ratio $S_0 = \frac{Q_{n+1}^\text{min}}{Q_n^\text{min}}$ involves $Q_0^\text{min}$, although the above continued fraction expression is obtained from the 2nd equation of (45). On the other hand, the ratio $S_0 = \frac{Q_1^\text{min}}{Q_0^\text{min}}$ of the first two terms of a minimal solution is unambiguously fixed by the $n = 0$ part of the recurrence (45), that is,

$$S_0 = -C_0 = \frac{1}{4g\kappa} \left[ 2\kappa \omega (2 - \Lambda^2) - \left( E + \omega - \frac{\Delta^2}{E + \omega - 2\kappa \omega \Lambda} \right) \Lambda \right].$$

(52)

In general, (51) and (52) can not be both satisfied for arbitrary values of the recurrence coefficients $C_n$ and $D_n$. Thus general solutions to the recurrence (45) are dominant and are usually generated by simple forward recursion from a given value of $Q_0$. Physical meaningful solutions are those that are entire in the Bargmann-Hilbert space [6]. They can be obtained if $E$ can be adjusted so that equations (51) and (52) are both satisfied. Then the resulting solution sequence $Q_n(E)$ will be purely minimal and the power series expansion (42) will converge in the whole complex plane. Equating the right hand sides of (51) and (52) yields an implicit continued fraction equation for the regular spectrum $E$. That is, the regular energies $E$ of the two-mode Rabi model are determined by the zeros of the transcendental function $G(E) = S_0 + C_0$ with $S_0$ and $C_0$ given by (51) and (52), respectively. Only for the denumerable infinite values of $E$ which are the roots of $G(E) = 0$, do we get entire solutions of the differential equations (40) and (41). The transcendental eigenvalue equation $G(E) = 0$ may be solved for $E$ by standard root-search algorithms (see e.g. [23, 24] and references therein).
4 Conclusions

We have presented analytic representations of solutions to the 2-photon and two-mode quantum Rabi models. Our main result is that the respective spectrum of the models consists of two parts, the regular and exceptional spectrum. Exceptional solutions of the models appear for special values of system parameters and were previously found in [21, 15]. Majority of the eigenvalues of the models are regular and correspond to arbitrary coupling strength and detuning. The regular eigenvalues of the models are given by the zeros of the transcendental functions, which have been analytically found by applying the Bogoliubov transformations and Bargmann-Hilbert spaces. The zeros can be determined numerically by standard root-search techniques.

This work should be of general interest due to its focus on the two simplest generalizations of the popular quantum Rabi model, but also be intriguing to those interested in analytic solutions of Fuchsian differential equations in physics. Our approach is quite general and could be applied to a large class of dynamical systems. As an example, in the Appendix, we will solve the driven Rabi model with broken $Z_2$ symmetry [7].

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A Driven Rabi model

In this appendix we revisit the driven quantum Rabi model and derive a transcendental equation for its regular energy spectrum.

The driven Rabi model is a generalization of the Rabi model with broken $Z_2$ symmetry. It has recently been used in [25] to examine quantum thermalization. The Hamiltonian of the model reads

$$H_{dR} = \omega b^\dagger b + \Delta \sigma_z + g \sigma_x (b^\dagger + b) + \delta \sigma_x,$$

where $\delta$ is the drive amplitude. After a canonical Bogoliubov transformation $b = a + \frac{\lambda}{\omega}, \quad b^\dagger = a^\dagger + \frac{\lambda}{\omega}$, where $\lambda$ is a real parameter, the Hamiltonian (53) becomes

$$\tilde{H}_{dR} = \omega a^\dagger a + \Delta \sigma_z + (g \sigma_x + \lambda)(a^\dagger + a) + \delta \sigma_x + \frac{2g\lambda}{\omega} \sigma_x + \frac{\lambda^2}{\omega}. \quad (54)$$

Using the Bargmann realization $a^\dagger = z, \quad a = \frac{d}{dz}$, working in a representation defined by $\sigma_x$ diagonal and choosing $\lambda = -g$, we can turn the transformed driven Rabi Hamiltonian
into the matrix differential operator

\[
\tilde{H}_{dR} = \begin{pmatrix}
\omega z \frac{d}{dz} + \delta - \frac{g^2}{\omega} \\
\omega(z - \frac{2g}{\omega}) \frac{d}{dz} - 2gz - \delta + \frac{3g^2}{\omega}
\end{pmatrix} \Delta.
\] (55)

In this case, the Bargmann-Hilbert space is the space of entire functions \( f(z) \) with inner product (2) and orthonormal basis \( \{ z^n \sqrt{n!} \} \). So if \( f(z) = \sum_{n=0}^{\infty} c_n z^n \), then

\[
||f||^2 = \sum_{n=0}^{\infty} |c_n|^2 n!
\] (56)

and \( f(z) \) is entire iff this sum converges.

In terms of two-component wavefunction \( \varphi(z) = (\varphi_+(z), \varphi_-(z))^T \), the time-independent Schrödinger equation, \( \tilde{H}_{dR} \varphi(z) = E \varphi(z) \), yields a system of couple of differential equations

\[
\begin{align*}
\left( \omega z \frac{d}{dz} + \delta - \frac{g^2}{\omega} - E \right) \varphi_+ + \Delta \varphi_- &= 0, \\
\left[ \omega \left( z - \frac{2g}{\omega} \right) \frac{d}{dz} - 2gz + \frac{3g^2}{\omega} - \delta - E \right] \varphi_- + \Delta \varphi_+ &= 0.
\end{align*}
\] (57) (58)

Solutions to these equations must be analytic in the whole complex plane if \( E \) belongs to the spectrum of \( \tilde{H}_{dR} \). So we are looking for solutions of the form

\[
\varphi_+(z) = \sum_{n=0}^{\infty} R_+^n(E) z^n, \quad \varphi_-(z) = \sum_{n=0}^{\infty} R_-(E) z^n,
\] (59)

which converge in the entire complex plane, i.e. solutions which are entire.

Substituting (59) into (57), we obtain

\[
R_+^n = \frac{\Delta}{E - n\omega - \delta + \frac{g^2}{\omega}} R_-^n.
\] (60)

Thus \( R_+^n \) is not analytic in \( E \) but has simple poles at

\[
E = n\omega + \delta - \frac{g^2}{\omega}, \quad n = 0, 1, \ldots.
\] (61)

The energies (61) appear for special value of model parameters and correspond to the exceptional solutions of the driven Rabi model. If (61) is satisfied, the infinite series expansions (59) truncate and reduce to polynomials in \( z \) but only if the system parameters satisfy certain constraints. This can be easily verified by following the procedure in [15] (and thus the driven Rabi model is quasi-exactly solvable). Majority part of the spectrum of the driven Rabi model is regular for which (61) is not satisfied. The regular spectrum of the model is given by the zeros of the transcendental function \( Q(E) \) obtained below.

From (58), we obtain the 3-term recurrence relation for \( R_-^n \)

\[
\begin{align*}
R_-^1 + X_0 R_-^0 &= 0, \\
R_-^{n+1} + X_n R_-^n + Y_n R_-^{n-1} &= 0, \quad n \geq 1,
\end{align*}
\] (62)
where
\[ X_n = \frac{1}{2g(n+1)} \left[ E - n\omega + \delta - \frac{3g^2}{\omega} + \frac{\Delta^2}{E - n\omega - \delta + \frac{\omega^2}{\omega}} \right], \]
\[ Y_n = \frac{1}{n+1}. \] (63)

The characteristic equation for the \( n \geq 1 \) part of the recurrence relation (62) is \( t^2 - \omega^2 g t = 0 \), which gives two distinct roots \( t_1 = 0 \) and \( t_2 = \frac{\omega}{2g} \). Applying the Poincaré-Perron theorem (i.e. theorems 2.1 and 2.2 of [22]), the two linearly independent solutions \( R_{n,1}^- \) and \( R_{n,2}^- \) of the 2nd equation of (62) satisfy
\[ \lim_{n \to \infty} \frac{R_{n+1,r}^-}{R_{n,r}^-} = t_r, \quad r = 1, 2, \] (64)
and \( R_{n}^{\text{min}} \equiv R_{n,1}^- \) is a minimal solution. It is not difficult to show that the infinite series expansions (59) with coefficients given by the minimal solution \( R_{n}^{\text{min}} \) are entire.

The ratio of successive elements of the minimal solution sequence \( R_{n}^{\text{min}} \) is expressible in terms of continued fractions,
\[ T_n = \frac{R_{n+1}^{\text{min}}}{R_{n}^{\text{min}}} = -\frac{Y_{n+1}^-}{X_{n+1}^-} - \frac{Y_{n+2}^-}{X_{n+2}^-} - \frac{Y_{n+3}^-}{X_{n+3}^-} \cdots, \] (65)
which for \( n = 0 \) reduces to
\[ T_0 = \frac{R_{1}^{\text{min}}}{R_{0}^{\text{min}}} = -\frac{Y_{1}^-}{X_{1}^-} - \frac{Y_{2}^-}{X_{2}^-} - \frac{Y_{3}^-}{X_{3}^-} \cdots. \] (66)

However, for the single-ended sequences appearing in the infinite series expansions (59), the first equation (i.e. the \( n = 0 \) part) of the recurrence (62) requires that
\[ T_0 = -X_0 = -\frac{1}{2g} \left( E + \delta - \frac{3g^2}{\omega} + \frac{\Delta^2}{E - \delta + \frac{\omega^2}{\omega}} \right). \] (67)

In general, (66) and (67) can not be both satisfied for arbitrary values of the recurrence coefficients \( X_n \) and \( Y_n \). Physical meaningful solutions are those that are entire in the Bargmann-Hilbert space [6]. They can be obtained if \( E \) can be adjusted so that equations (66) and (67) are both satisfied. Then the resulting solution sequence \( R_n(E) \) will be purely minimal and the power series expansions (59) will converge in the whole complex plane. Equating the right hand sides of (66) and (67) yields a transcendental equation for the spectrum \( E \). That is, regular energies of the driven Rabi model are given by the zeros of the transcendental function \( Q(E) = T_0 + X_0 \), where \( T_0 \) is the continued fraction in (66) and \( X_0 \) is the right hand side of (67). The transcendental eigenvalue equation \( Q(E) = 0 \) may be solved for \( E \) by standard root-search techniques.
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