Localization-delocalization transition in the quasi-one-dimensional ladder chain with correlated disorder

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The generalization of the dimer model on a two-leg ladder is defined and investigated both, analytically and numerically. For the closed system we calculate the Landauer resistance analytically and found the presence of the point of delocalization at the band center which is confirmed by the numerical calculations of the Lyapunov exponent. We calculate also analytically the localization length index and present the numerical investigations of the density of states (DOS). For the open counterpart of this model the distribution of the Wigner delay times is calculated numerically. It is shown how the localization-delocalization transition manifest itself in the behavior of the distribution.

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I. INTRODUCTION

One of the widely accepted results of the disordered systems is that in one-dimensional spaces all the electronic states in independent site-energy random (diagonal disorder) Anderson model with non-random hopping amplitudes are exponentially localized. The diffusion or long-range transport of the initially localized particle in such systems is absent regardless the strength of the disorder. The same result is proved rigorously for quasi-one-dimensional systems.

On the other hand, it is well established that the type of disorder can have a strong effect on localization properties of disordered systems. For instance, in one-dimensional systems with off-diagonal disorder (random hopping models) delocalization takes place at the band center. In quasi-one-dimensional systems with off-diagonal disorder the picture is more complicated. It appeared that in a weakly disordered N-leg quasi-one-dimensional disordered tight-binding hopping model there is a delocalization transition at the critical energy $E_{\text{crit}} = 0$ if and only if $N$ is odd (including $N = 1$ case). The wave-functions remain localized for even $N$. The localization properties of two- and three-channel tight-binding Anderson model with nearest-neighbor interchain hopping was studied in.

Another possibility to have localization-delocalization transition in one dimension is the correlated disorder, when the random variables are correlated at short or long distances. There is intensive current interest both theoretical (see and Refs. therein) and experimental concerning analyze of conditions, under which delocalized states can appear. This kind of models with correlated disorder give explanation of high conductivity of polymers such as doped poly-aniline and also describe the transport properties of random semiconductor super-lattices. Some transport properties of many-mode waveguides with rough surfaces were studied in and the possibility of perfect transmission due to some specific long-range correlations in the surface profiles was investigated. The scaling properties of the models with correlated disorder were studied in. In it was shown, that universality of one parameter scaling breaks down in presence of long range correlated disorder, for instance in the periodic-on-average systems. It is necessary to point out, that the correlated disordered systems do not belong to Zirnbauer’s and Caselle’s classification of fully disordered systems on the basis of Cartan’s classification of symmetric spaces. That happens because, due to correlations, the Brownian motion of the effective degrees of freedom carries out on the restricted part of the coset space. This restriction is general characteristic of all correlated disordered systems.

So far the discussion of correlated disorder was applied only to strictly one-dimensional systems. In order to analyze the existence of localization-delocalization transition in higher dimensional lattices with correlated disorder, it is natural to make the first step in this direction — to study the quasi-one-dimensional systems. To investigate the existence of critical point and address the problem of delocalization transition in this case analytically is an important task. The exact calculation of Landauer localization length and critical indices will allow us to understand whether numerical simulation of Lyapunov exponent shows truly critical point (where the localization length is infinite) or an extended state (where the localization length is finite, but exceeds the system size).

In this paper we will consider the extension of the random dimer model (RDM) and Refs. therein) and experimental concerning analyze of conditions, under which delocalized states can appear. This kind of models with correlated disorder give explanation of high conductivity of polymers such as doped poly-aniline and also describe the transport properties of random semiconductor super-lattices. Some transport properties of many-mode waveguides with rough surfaces were studied in and the possibility of perfect transmission due to some specific long-range correlations in the surface profiles was investigated. The scaling properties of the models with correlated disorder were studied in. In it was shown, that universality of one parameter scaling breaks down in presence of long range correlated disorder, for instance in the periodic-on-average systems. It is necessary to point out, that the correlated disordered systems do not belong to Zirnbauer’s and Caselle’s classification of fully disordered systems on the basis of Cartan’s classification of symmetric spaces. That happens because, due to correlations, the Brownian motion of the effective degrees of freedom carries out on the restricted part of the coset space. This restriction is general characteristic of all correlated disordered systems.

In the next sections we will focus on a particular case, when the same site potentials...
The Hamiltonian of the RLDM is given by
\[
H = \sum_{n=1; i=1, 2}^{N} \left\{ t_{h}(c_{n,i}^+ c_{n+1,i}^+ + c_{n+1,i}^- c_{n,i}^-) + t_{v} c_{n,i}^+ c_{n,i+1} + \epsilon_{n,i} c_{n,i}^+ c_{n,i}^- \right\}.
\]
(1)

Here \( t_{v} \) and \( t_{h} \) are constant vertical and horizontal hopping parameters correspondingly, \( \epsilon_{n,i} \) is the random external potential at site \( n \) of the \( i \)-th chain. \( N \) denotes the number of atoms in each chain. We impose a constraint on the randomness by demanding that four \( \epsilon_{n,1} \) \( (n = 2k - 1, 2k; \ i = 1, 2) \) around the plaquette (see Fig.1) always have the same, but randomly chosen value from the set \( \{ \pm m \} \)
\[
\epsilon_{2k-1, 1} = \epsilon_{2k-1, 2} = \epsilon_{2k, 1} = \epsilon_{2k, 2} \in \{ \pm m \},
\]
\[
k = 1, \ldots, \frac{N}{2}.
\]
(2)

In section III we study the closed system described by Hamiltonian (1). We use the technique introduced in [10, 17, 18] and further applied to the RDM in [19] in order to investigate the RLDM. We calculate analytically the dimensionless Landauer exponent of the model and compare it with numerical simulations of the Lyapunov exponent (Fig.2 and Fig.3), calculated iteratively in standard way [24, 31]. The exact calculation of the Landauer resistance shows the existence of one real critical point with critical energy \( E_{\text{crit}} = 0 \) in the case of \( m = t_{v}/t_{h} \). Otherwise all states are localized.

In the case of \( m = t_{v}/t_{h} < 1 \) we analytically analyze the divergence of the inverse of Landauer exponent and find, that there is delocalization with the critical index \( \nu = 2 \). On Fig.2 and Fig.3 we see very good coincidence of the half of Landauer exponent with the Lyapunov one around the critical point.

The critical behavior is essentially different when \( m = t_{v}/t_{h} = 1 \). In this case also the localization length diverges at the point \( E_{\text{crit}} = 0 \), but with index \( \nu = 1/2 \). The comparison of Landauer and Lyapunov exponents for this case is presented on Fig.4.

The numerical analyze of the density of states (DOS) shows anomalous behavior at the band center in the case, when \( t_{v} = t_{h} \) (see Fig.5) and regular behavior for \( t_{v} < t_{h} \) (see Fig.6).

Section III is devoted to the open counterpart of the system described by (1). There we show that far from the delocalization point the distributions of the Wigner delay times are very similar to those found for the random models with uncorrelated disorder (Fig.7 and Fig.8). At the critical point the behavior of the distribution depends on \( m \). Namely, for \( m = 1 \) the distribution of the re-scaled Wigner delay times tends to the \( \delta \)-function (Fig.9), corresponding to the deterministic, ballistic propagation. While for \( m < 1 \) this distribution is not deterministic, but still bounded (Fig.10), indicating the ballistic propagation with velocities distributed in some interval.

## II. LANDAUER RESISTANCE, CRITICAL EXONENTS AND DENSITY OF STATES

For one particle eigenstates \( \psi_{n}(E) \) of the eigenenergy \( E \) the Schrödinger equation of the Hamiltonian (1) has the following recursion form:
\[
t_{h}(\psi_{n+1,i} + \psi_{n-1,i}) + t_{v}\psi_{n,i+1} = (E - \epsilon_{n,i})\psi_{n,i}.
\]
(3)

Now we can rewrite (3) in the matrix form
\[
\begin{pmatrix}
\psi_{n+1,1} \\
\psi_{n+1,2} \\
\psi_{n+1,3} \\
\psi_{n+1,4}
\end{pmatrix} = T_{n}
\begin{pmatrix}
\psi_{n,1} \\
\psi_{n,2} \\
\psi_{n,3} \\
\psi_{n,4}
\end{pmatrix},
\]
(4)

by introducing \( 4 \times 4 \) transfer matrix \( T_{n} \) as
\[
T_{n} = \begin{pmatrix}
E - \epsilon_{n,1} & -t & -1 & 0 \\
-t & E - \epsilon_{n,2} & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}.
\]
(5)

Here we re-scaled the energies and the vertical hopping parameter \( t_{v} \) (introducing \( t \) instead) by the horizontal hopping parameter \( t_{h} \). One can easily find out, that the \( 4 \times 4 \) transfer matrix \( T_{n} \) possesses the condition \( T_{n}^{+}JT_{n} = J \) with
\[
J = \begin{pmatrix}
0 & 0 & -i & 0 \\
0 & 0 & 0 & -i \\
i & 0 & 0 & 0 \\
0 & i & 0 & 0
\end{pmatrix}
\]
(6)

which means that \( T_{n} \) is an element of the group \( SU(2, 2) \). If we introduce a new matrix \( M_{n} = \prod_{i=1}^{n} T_{i} \), then we can relate the wave functions \( (\psi_{n+1,1}, \psi_{n+1,2}, \psi_{n,1}, \psi_{n,2}) \) with ones at sites 0 and 1 of both chains
\[
\begin{pmatrix}
\psi_{n+1,1} \\
\psi_{n+1,2} \\
\psi_{n,1} \\
\psi_{n,2}
\end{pmatrix} = M_{n}
\begin{pmatrix}
\psi_{1,1} \\
\psi_{1,2} \\
\psi_{0,1} \\
\psi_{0,2}
\end{pmatrix},
\]
(7)

and \( M_{N} \) will denote the total transfer matrix of the system. From the definition of \( M_{N} \) it is obvious, that it
has the same property $M_N^* J M_N = J$. This condition is equivalent to the flux conservation in the theory (i.e., Hermiticity of $H$).

In a physics of disorder particular interests has the limit of eigenvalues of the matrix $\ln(M_N M_N^*) \rightarrow \infty$ when $N \rightarrow \infty$. According to Oseledec theorem this limit exists and the maximal eigenvalue is called Lyapunov exponent. Due to self averaging property it is determined by the maximal eigenvalue of the matrix $M_N$

$$\gamma_{\text{Lyapunov}} = \lim_{N \rightarrow \infty} (\ln \| M_N \|_1)^{1/N},$$

where $\| M_N \|^2 = \sum_{\alpha, \beta} (M_N^*)^\beta_{\alpha} (M_N)^\alpha_{\beta} = \text{Tr}(M_N M_N^*)$ and the average is taken over random distribution of the external potential taking into account the correlations (if they are present in the model under consideration).

It is hard to calculate the Lyapunov exponent exactly and establish the presence of delocalization point. Another variable, which shares the essential characteristics of the localization-delocalization transition but can be calculated exactly is the dimensionless Landauer resistance, which is defined as the ratio of squares of the modules of the reflection and transmission amplitudes. According to arguments of Anderson et al. in one-dimensional systems the Landauer exponent (defined in Eq. (9)) is simply the double of Lyapunov one. The proportionality of these two exponents was confirmed around of the critical point in the paper. Considering in most general terms one dimensional models with transfer matrices belonging to $SU(1,1)$ authors have shown, that in the lowest order of perturbation expansion with some parameter measuring the distance from the critical point, the Landauer exponent is twice the Lyapunov one, just as predicted by [32]. This means, that critical indices of both exponents are equal. In quasi-one dimensional models, which have transfer matrix from $SU(n,n)$, again one can expect the coincidence of critical indices.

The exponential increase of Landauer resistance with distance is determined by the Landauer exponent, which can be defined as

$$\gamma(E) = \lim_{N \rightarrow \infty} \ln(\| M_N \|_1)^{1/N}.$$  

In the articles [16]-[18] it was shown, that the Landauer resistance (and corresponding exponent) can be calculated exactly just by reducing the direct product $M_N \otimes M_N^*$ of fundamental representations of transfer matrices $M$ of $SU(1,1)$ group to the adjoint one. We apply this technique here for the group $SU(2,2)$. In order to calculate this direct product exactly, we use the known representation of the permutation operator via generators $\tau^\mu$ ($\mu = 1, \ldots, 15$) of the $sl(4)$ algebra as $P = \frac{1}{4} (\mathbb{1} + \tau^\mu \otimes \tau^\mu)$. In the matrix elements this formula looks like

$$\delta^\beta_{\alpha_1} \delta^\beta_{\alpha_2} \tau^\mu_{\alpha_1} \tau^\mu_{\alpha_2} = \frac{1}{4} [\delta^\beta_{\alpha_1} \delta^\beta_{\alpha_2} + (\tau^\mu)^{\beta_2}_{\alpha_1} (\tau^\mu)^{\alpha_2}_{\beta_2}],$$

where we suppose summation by the repeating indices $\mu$. Among of generators $\tau^\mu$ there is one, which coincides with the metric $J$ defined in [30]. We denote the corresponding index $\mu$ as $J$, namely $\tau^J = J$.

By simple multiplication of [10] by $T_j$ and $T_j^+$ from the left and right hand sides correspondingly one can express the direct product of $T_j$ and $T_j^+$ via their adjoint representation:

$$(T_j)^\alpha_{\alpha'} (T_j^+)_{\beta'}^\beta = \frac{1}{4} (J)^\alpha_{\alpha'} (J)^\beta_{\beta'} + \frac{1}{4} (\tau^\mu)^{\beta'}_{\alpha} \Lambda^\mu_{\alpha} (J^\tau_{\mu}^\nu)^\alpha_{\beta}.$$  

Here the adjoint representation $\Lambda_n$ of $T_n$ is defined by

$$\Lambda^\mu_{\alpha} = \frac{1}{4} \text{Tr}(T_n \tau^\mu T_n^+ \tau^n)$$

and is an $15 \times 15$ matrix depending on the parameters of the model at the site $n$ of both chains.

Now, in order to calculate the average of Landauer exponent, we use [14] and decompose the direct product of two transfer matrices corresponding to the whole system of $2N$ sites. In the model under consideration we...
attach the same site potential \( \epsilon \) to the quartet of sites (see Fig.1) and calculate the average over random distribution of values \( \epsilon_a \) and \( \epsilon_b \) with probabilities \( p \) and \( 1-p \) respectively. Therefore we should consider the \( \Lambda^2 \) as a constituent transfer matrix and average it in a product

\[
\langle M_N M_N^+ \rangle = \frac{1}{4} J \otimes J + \frac{1}{4} (\tau^\mu J) \otimes (J \tau^\nu) \left( \prod_{j=1}^{N/2} (\Lambda_j^2) \right)^{\mu\nu}. \tag{13}
\]

It is clear, that

\[
(\Lambda_j^2)^{\mu\nu} = \frac{1}{2} \mathrm{Tr} \left( T_j^{2,\mu} [T_j^{2,\nu}]^{-1} \right) \tag{14}
\]

and the average is

\[
\Lambda = \langle \Lambda_n^2 \rangle = p \Lambda_+^2(\epsilon_a) + (1-p) \Lambda_-^2(\epsilon_b), \tag{15}
\]

where \( \Lambda_j(\epsilon_a) \) and \( \Lambda_j(\epsilon_b) \) are calculated for the site potentials \( \epsilon_a \) and \( \epsilon_b \) respectively. The matrix elements of \( \Lambda \) can be found easily from (14) and (15). Hereafter we take \( \epsilon_a = -m \), \( \epsilon_b = +m \) without lose of generality and \( p = 1/2 \) for simplicity.

From the formula (14) one can easily obtain

\[
\langle ||M||^2 \rangle = 4(\Lambda^{N/2})^{JJ} \approx \lambda_{\text{max}}^{N/2}, \tag{16}
\]

where the biggest eigenvalue \( \lambda_{\text{max}} \) defines the asymptotic behavior of Landauer resistance with the distance. In superscript \( JJ \) means \( JJ \) element of the matrix \( \Lambda^{N/2} \) (see notion after formula (10)).

According to (9), the Landauer exponent \( \gamma(E) \) connected with the maximal eigenvalue as

\[
\gamma(E) = \frac{1}{4} \log[\lambda_{\text{max}}] \tag{17}
\]

and defines the inverse of Landauer localization length \( \xi(E) = 1/\gamma(E) \).

The delocalized states correspond to the critical points where \( \xi(E) = \infty \), or \( \lambda_{\text{max}} = 1 \), and can be found by the condition \( \det | I - \Lambda | = 0 \). In our case this characteristic equation is given as

\[
m^4[(E-t)^2-m^2][(E+t)^2-m^2][m^4-E^2(-1+m^2)]t^2 = 0. \tag{18}
\]

From this equation one can find following candidates for critical points

\[
E_{1c} = t \pm m, \tag{19}
\]

\[
E_{2c} = -t \pm m, \tag{20}
\]

\[
E_{3c} = m^4/(m^2-1). \tag{21}
\]

But since the Landauer exponent defined by the maximum eigenvalue of \( \Lambda \) one need to find out whether this points belong to maximum eigenvalue or not. The full characteristic equation \( \det | \lambda^2 - \Lambda | = 0 \) for eigenvalues \( \lambda \) factorizes into the product of three polynomials

\[
\begin{align*}
U_1(\lambda, E, t, m) &= (1 - 2m^2)^2 + [(2m^2 - 1)(3 - 2m^2 + (2E + E^2 - m^2 - 2(1 + E)t + t^2)(E^2 - m^2 - 2E(1 + t) + t(2 + t)))] \lambda + [3 + E^4 + m^4 - 4E^3t - 4t^4 + t^4 - 4Et(3m^2 + t^2 - 2) + m^2(6t^2 - 4) + E^2(6m^2 + 6t^2 - 4)]^2 - \lambda^3 = 0, \tag{22}
\end{align*}
\]

\[
\begin{align*}
U_2(\lambda, E, t, m) &= (1 - 2m^2)^2 + [(2m^2 - 1)(3 - 2m^2 + (2E + E^2 - m^2 - 2(1 + E)t + t^2)(E^2 - m^2 - 2E(1 - t) - t(2 - t))] \lambda + [3 + E^4 + m^4 + 4E^3t - 4t^4 + t^4 + 4Et(3m^2 + t^2 - 2) + m^2(6t^2 - 4) + E^2(6m^2 + 6t^2 - 4)] \lambda^2 - \lambda^3 = 0, \tag{23}
\end{align*}
\]

\[
\begin{align*}
U_3(\lambda, E, t, m) &= -(1 - 2m^2)^2 - (2m^2 - 1) \left[ E^4 + m^4 + (t^2 - 2)^2 - 2E^2(2 + m^2 + t^2) + m^2(6t^2 - 4) \right] \lambda + \ldots
\end{align*}
\]
Eigenvale of this product after statistical averaging just defines the Lyapunov exponent $\gamma$ of the model. If there is a particular value of $E$, where this two transfer matrices commute (this means all matrices in the product commute) then the eigenvalues of the product are given by the product of the corresponding eigenvalues of the constituent random matrices. If, at the same time, the eigenvalues of both transfer matrices at this point are laying on the unit circle (modules of eigenvalues are equal to one), we have a point of delocalization. In order to see that, let us diagonalize the transfer matrix $M_N$ by the matrix $S$, which is also diagonalizing matrix of the individual $T_i$’s along a chain. One obtains $\gamma_{\text{Lyapunov}} = \lim_{N \to \infty} \frac{1}{N} \ln \text{Tr}(SM_dS^{-1}(S^{-1})^tM_d^tS^+) = 0$, where the eigenvalues in $M_d$, as a product of individual eigenvalues of $T_i$, also have module equal to one. Therefore, the argument of logarithm is bounded from above by some constant independent of $N$ and the limit $N \to \infty$ is zero. It remains now to check, that indeed the squares of transfer matrices $E_{\pm m}$ for $\epsilon_n = \pm m$ are commuting at $E = 0$ and for $t = m$. For $t \leq 1$ they have two eigenvalues equal to $-1$ and two complex conjugate eigenvalues on the unit circle ($e^{\pm i\theta}$). This prove the presence of the delocalization point.

In Fig.2, we present the graphic of the half of Landauer exponent $\gamma(E)$ versus $E$ (curve A), defined by the maximum solution of cubic equations for the generic value $m = t_c/t_h = 1/2$. We present here also the Lyapunov exponent $\bar{\gamma}$ (curve B) obtained by the numerical simulations of the chains of the length $N = 40000$ at 60 values of energies in the whole energy interval $[-2.5, 2.5]$. The same exponents in the vicinity of critical point ($E \in [-1.1, 1.1]$) are presented on the Fig.3 in order to demonstrate the coincidence of critical behaviors, as it expected after the articles [32] and [33]. The critical index $\nu = 2$ in this case.

In Fig.4 we present the Landauer and the Lyapunov exponents versus $E$ for the boundary case $m = t_c/t_h = 1$. We present results for close vicinity of $E = 0$ and one can see exactly the same critical behavior of the both exponents. It is necessary to mention, that in contrast to the generic case $m \neq 1$ here the multiplication factor $1/2$ is absent.

For $m > 1$ the delocalization point $E_c = 0$ disappears. The critical index $\nu$ of the Landauer exponent one can obtain by expanding the solutions of the equations near the critical point $E_c = 0$

$$\gamma(E) = \begin{cases} \frac{1}{5}E^2 + \frac{1}{3}E^3 + O(E^4), & \text{if } m = t = \frac{1}{2} \\ \sqrt{\frac{1}{2} - \frac{1}{3}E} + O(E^{3/2}), & \text{if } m = t = 1. \end{cases}$$

As we see the critical indices are different for generic $m = 1/2$, $\nu = 2$ and boundary $m = 1$, $\nu = 1/2$ cases. Fig.3 and Fig.4 demonstrate that difference. The similar situation was in the one-channel RDM [10]. The index $\nu = 1/2$ is equal to the corresponding index in one-channel RDM in the same boundary case $m = 1$,.
when one is approaching to the critical points $E_c = \pm 1$ from the outside of the region $[-1,1]$ (see 11). After the discussion presented above this fact is natural to expect.

In order to fill up the exact results from above, we have analyzed the density of states in the model by the numerical diagonalization of the Hamiltonian 11. In Fig.5 the DOS versus energy for the boundary case $m = \epsilon_a - \epsilon_b = 2$ is presented. Here we calculated the DOS for a system with $N = 50$ sites in each chain. The result comprise the average over 4000 realizations of the disorder. Here one can clearly see the anomaly of DOS at band center $E = 0$ (in accordance with 33), corresponding to delocalization transition. A similar anomaly is present in RDM at the both critical energies $\epsilon_a, \epsilon_b$ in the same case when $\epsilon_a - \epsilon_b = 2$.

In the Fig.6 we present the normalized DOS for the generic case $m = 0.5$, calculated numerically for the ensemble of 2000 coupled chains of length $N = 50$ each. This case corresponds to a different type of critical behavior with respect to the previous one and the DOS is finite in whole energy interval.

III. WIGNER DELAY TIME

In this section we consider the properties of the open counterpart of the model studied in section III. Namely we investigate the statistical distributions of the Wigner delay times for various regimes. The Wigner delay time is defined as the derivative of the total phase of the scattering matrix with respect to the energy $\tau = -i \frac{\partial \Phi}{\partial E}$. This quantity was intensively studied in recent years: for chaotic 35, localized 36, 37, 38, diffusive 38, 39 systems as well as for systems at criticality 38, 40.

In order to study an open system we attach a perfect lead to the semi-infinite disordered sample. Thus $\epsilon_{n,i} = 0$, $t = 1$ for $n \leq 0$ and they take the same values as in the previous section for $n > 0$. Let us first assume more general situation of $M$ channels and derive the expression for the Wigner delay time.

The wavefunction in the lead is a superposition of the plane waves in the longitudinal direction and the standing waves in the transverse direction:

$$\psi_{n,i} = \sum_{j=1}^{M} \left\{ \frac{A_j}{\sqrt{\sin k_j}} e^{ik_jn} \phi_i^j + \frac{B_j}{\sqrt{\sin k_j}} e^{-ik_jn} \phi_i^j \right\},$$

(26)

where $\phi_i^j = \frac{1}{\sqrt{M}} \sin \left( \frac{\pi i}{M+1} j \right)$ is the transverse eigenfunction and $k_j$ is the wave-vectors of $j$th mode determined by the equation $E = 2t_h \cos k_j + 2t_v \cos \left( \frac{\pi j}{M+1} \right)$. In order to decouple different modes it is useful to introduce new variables:

$$\tilde{W}_n = U \tilde{\psi}_n,$$

(27)

where $\tilde{W}_n = (W_{n,1}, \ldots, W_{n,M})^T$, $\tilde{\psi}_n = (\psi_{n,1}, \ldots, \psi_{n,M})^T$ and $U$ is $M \times M$ matrix with elements $U_{i,j} = \phi_i^j$. In new variables each component of the wavefunction in the lead contains contribution of one particular mode only:

$$W_{n,t} = \frac{A_t}{\sqrt{\sin k_t}} e^{ik_tn} + \frac{B_t}{\sqrt{\sin k_t}} e^{-ik_tn}$$

(28)

In order to rewrite 21 in terms of the new variables we notice that the generalization of the transfer matrix 5 for $M$ channel case is $2M \times 2M$ block-matrix:

$$T_n = \begin{pmatrix} C_n & -I \\ I & 0 \end{pmatrix},$$

(29)

where $C_n$ is a three-diagonal matrix generalizing the left upper block of the transfer matrix 5. Now the analog of 21 in new variables takes the similar form:

$$\begin{pmatrix} \tilde{W}_{n+1} \\ \tilde{W}_n \end{pmatrix} = \tilde{T}_n \begin{pmatrix} \tilde{W}_{n+1} \\ \tilde{W}_{n-1} \end{pmatrix},$$

(30)

where the new transfer matrix $\tilde{T}_n$ is giving by

$$\tilde{T}_n = \begin{pmatrix} \tilde{C}_n & -I \\ 1 & 0 \end{pmatrix}, \quad \tilde{C}_n = UC_n U^{-1}$$

(31)

Let us now consider the finite sample of length $N$ and impose Dirichlet boundary condition at $n = N+1$. Then the analog of equation 7 reads

$$\begin{pmatrix} \tilde{O} \\ W_N \end{pmatrix} = \begin{pmatrix} P_{N1}^{11} & P_{N1}^{12} \\ P_{N2}^{11} & P_{N2}^{12} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\sin k}} e^{-iK} A_N + \tilde{B}_N \\ \frac{1}{\sqrt{\sin k}} e^{iK} A_N + \tilde{B}_N \end{pmatrix}$$

(32)

where $P_N$ is the total transfer matrix $P_N = \prod_{n=1}^N \tilde{T}_n$ and matrix $K$ is diagonal $K = \text{diag}(k_1, \ldots, k_M)$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.png}
\caption{Distribution of the Wigner delay times $\tau$ for $E = 0.3$, $t = 1$, $m = 0.1$ and $N = 10^3$ (circles). The line represents the best fit by formula 45 with the fit parameter $a$.}
\end{figure}
The scattering matrix for our geometry consists of only one non-trivial block — the reflection matrix $R_N$, which relates the incoming wave amplitudes $\vec{A}_N$ to the outgoing wave amplitudes $\vec{B}_N$:

$$\vec{B}_N = R_N \vec{A}_N$$  \hspace{1cm} (33)

Using this definition and relation (32) one can easily calculate the reflection matrix:

$$R_N = -\left[ I + \sqrt{\sin K} F_N \left( \sqrt{\sin K} \right)^{-1} e^{ik} \right]^{-1} \times$$

$$\left[ I + \sqrt{\sin K} F_N \left( \sqrt{\sin K} \right)^{-1} e^{-ik} \right]$$

$$F_N \equiv (P_N^1)^{-1} P_N^2$$  \hspace{1cm} (34)

Taking the derivative with respect to energy from this expression one can find the formula for the Wigner delay time:

$$\tau_N = 2 \text{Im} \text{Tr} \left[ I + F_N e^{-ik} \right]^{-1} \left[ \frac{\partial F_N}{\partial E} e^{-ik} - i F_N e^{-iK} \frac{\partial K}{\partial E} \right]$$  \hspace{1cm} (35)

The matrix $F_N$ and its derivative with respect to energy can be calculated recursively. To this end we increase the length of the sample by one and see how the total transfer matrix changes. Then from the definition of $P_N$ and Eq. (31) one can derive the following recursion relations:

$$F_{N+1} = -\left( \tilde{C}_N + F_N \right)^{-1}$$

$$\frac{\partial F_{N+1}}{\partial E} = F_{N+1} \left[ I + \frac{\partial F_N}{\partial E} \right] F_{N+1}$$  \hspace{1cm} (36)

Starting from the initial values $F_1 = -(\tilde{C}_1)^{-1}$, $\partial F_1/\partial E = F_2^2$ and iterating relations (36) one can calculate $F_N$ and $\partial F_N/\partial E$ for any system size $N$. Inserting these values into Eq. (34) we obtain the Wigner delay time $\tau_N$.

Below we present the results of numerical calculations of the distribution of $\tau_N$ for the random ladder dimer model. In these particular case the matrix $\tilde{C}_n$ becomes diagonal:

$$\tilde{C}_n = \begin{pmatrix} E - t - \epsilon_n & 0 \\ 0 & E + t - \epsilon_n \end{pmatrix}$$  \hspace{1cm} (37)

therefore the recursion relations (36) are reduced to two decoupled equations. That makes the numerical simulations especially efficient: in all calculations we used $10^5$ realizations of the random potentials and the system size up to $N = 10^4$.

Let us now using the knowledge gained in the previous section consider different regimes. We start with the localized regime. In Fig. 8 the distribution of the Wigner delay times is presented for $t = 1$, $m = 0.1$ and $E = 0.3$. The distribution is described very nice by the formula:

$$P(\tau) = \frac{a}{\tau^2} e^{-a/\tau}$$  \hspace{1cm} (38)

which is derived for the distribution of $\tau$ for a one-dimensional Anderson model in the weak disorder limit [36, 57]. This indicates that correlations in disordered potential though renormalize the localization length, but does not change the form of the distribution in the weak disorder limit. The increasing the strength of the disorder modifies the short time behavior of $P(\tau)$ (Fig. 8). However the typical $1/\tau^2$ tail of the distribution remains unchanged. This result is again very similar to one obtained for a one-dimensional Anderson model [36].

Next we concentrate on the critical regime. The most surprising behavior of $P(\tau)$ is found for $t = m = 1$ and $E = 0$. As it follows from Fig. 6 the distribution of

![Figure 8: Distribution of the Wigner delay times $\tau$ for $E = 0.3$, $m = t = 1$, and $N = 10^4$ (circles). The dashed line represents $1/\tau^2$ behavior.](image)

![Figure 9: Distribution of the rescaled Wigner delay times $\tilde{\tau} = \tau/N$ for $E = 0$, $m = t = 1$, and various system sizes: (a) $N = 100$, (b) $N = 1000$, (c) $N = 10000$.](image)
the rescaled Wigner delay times $\tilde{\tau} \equiv \tau/N$ goes to the $\delta$-function in the thermodynamic limit. This means that with the probability one the particle propagates through the sample ballistically with a fixed velocity. The ballistic propagation is compatible with the delocalization taking place at this point. The deterministic character of the distribution is however not trivial and can not be deduced from the delocalization only. This becomes very clear when we choose $t = m < 1$. The distribution of the rescaled times (Fig. 10) is bounded (absence of long tails), but not deterministic anymore. In contrast it has a complicated oscillating structure in the middle and high peaks at the edges. The difference in the behavior of the distributions of the Wigner delay times for $t = m = 1$ and $t = m < 1$ is another manifestation of the difference between the critical indices and DOS found in the previous section.

### IV. CONCLUSION

We have formulated and investigated the generalization of random dimer model to the case of ladder chain. As in one-channel case we found delocalization point at $E_c = 0$ with two type of critical behavior when $\epsilon_a - \epsilon_b < 2$ and $\epsilon_a - \epsilon_b = 2$. The comparison of Landauer and Lyapunov exponents shows that they have the same critical behavior in full accordance with the articles [32] and [33].

We studied also the distribution of the Wigner delay times for this model. Our results shows that its behavior ranges from $1/\tau^2$ decay typical for the localized system to the deterministic one at the critical point for $\epsilon_a - \epsilon_b = 2$.

Our analytical and numerical results can be relevant for photoluminescence experiments on electronic properties of GaAs-AlGaAs superlattices [24] and for experiment on a microwave waveguide with inserted correlated scatterers [25].

At the end we would like to make a following remark. The dimer model with the correlated disorder defined here does not belong to the classification of fully disordered systems on the basis of symmetric spaces [27, 28]. This is because the geometry of integration space over random variables, due to the restriction imposed by correlations, does not coincide with geometry of symmetric spaces. The classification of universality classes of the correlated disordered systems is an open and interesting problem.

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