ON THE INITIAL COEFFICIENTS FOR CERTAIN CLASS OF
FUNCTIONS ANALYTIC IN THE UNIT DISC

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Abstract. Let function $f$ be analytic in the unit disk $D$ and be normalized
so that $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$. In this paper we give sharp bounds of
the modulus of its second, third and fourth coefficient, if $f$ satisfies

$$\left| \arg \left( \frac{z}{f(z)} \right)^{1+\alpha} f'(z) \right| < \gamma \frac{\pi}{2} \quad (z \in \mathbb{D}),$$

for $0 < \alpha < 1$ and $0 < \gamma \leq 1$.

1. Introduction and preliminaries

Let $\mathcal{A}$ denote the family of all analytic functions in the unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$ and satisfying the normalization $f(0) = 0 = f'(0) - 1$.

A function $f \in \mathcal{A}$ is said to be strongly starlike of order $\beta, 0 < \beta \leq 1$ if and only if

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \beta \frac{\pi}{2} \quad (z \in \mathbb{D}).$$

We denote this class by $S_\beta^*$. If $\beta = 1$, then $S_1^* \equiv S^*$ is the well-known class of starlike functions.

In [1] the author introduced the class $\mathcal{U}(\alpha, \lambda) \ (0 < \alpha \text{ and } \lambda < 1)$ consisting of functions $f \in \mathcal{A}$ for which we have

$$\left| \left( \frac{z}{f(z)} \right)^{1+\alpha} f'(z) - 1 \right| < \lambda \quad (z \in \mathbb{D}).$$

In the same paper it is shown that $\mathcal{U}(\alpha, \lambda) \subset S^*$ if

$$0 < \lambda \leq \frac{1 - \alpha}{\sqrt{(1 - \alpha)^2 + \alpha^2}}.$$

The most valuable up to date results about this class can be found in Chapter 12 from [4].

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In the paper [2] the author considered univalence of the class of functions \( f \in A \) satisfying the condition
\[
\left| \arg \left( \frac{z}{f(z)} \right)^{1+\alpha} f'(z) \right| < \frac{\gamma \pi}{2} \quad (z \in \mathbb{D})
\]
for \( 0 < \alpha < 1 \) and \( 0 < \gamma \leq 1 \), and proved the following theorem.

**Theorem A.** Let \( f \in A \), \( 0 < \alpha < 2/\pi \) and let
\[
\left| \arg \left( \frac{z}{f(z)} \right)^{1+\alpha} f'(z) \right| < \gamma_*(\alpha) \frac{\pi}{2} \quad (z \in \mathbb{D}),
\]
where
\[
\gamma_*(\alpha) = \frac{2}{\pi} \arctan \left( \sqrt{\frac{2}{\pi \alpha} - 1} \right) - \alpha \sqrt{\frac{2}{\pi \alpha} - 1}.
\]
Then \( f \in S_\beta \), where
\[
\beta = \frac{2}{\pi} \arctan \sqrt{\frac{2}{\pi \alpha} - 1}.
\]

## 2. Main result

In this paper we will give the sharp estimates for initial coefficients of functions \( f \in A \) which satisfied the condition \((1)\). Namely, we have

**Theorem 1.** Let \( f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \) belongs to the class \( A \) and satisfy the condition \((1)\) for \( 0 < \alpha < 1 \) and \( 0 < \gamma \leq 1 \). Then we have the next sharp estimations:

\[ (a) \ |a_2| \leq \frac{2\gamma}{1-\alpha}; \]
\[ (b) \ |a_3| \leq \begin{cases} \frac{2\gamma}{2-\alpha}, & 0 < \gamma \leq \frac{(1-\alpha)^2}{3-\alpha} ; \\ \frac{2(3-\alpha)\gamma^2}{(1-\alpha)^2(2-\alpha)}, & \frac{1}{3-\alpha} \leq \gamma \leq 1 \end{cases} ; \]
\[ (c) \ |a_4| \leq \begin{cases} \frac{2\gamma}{3-\alpha}, & 0 < \gamma \leq \gamma_\nu; \\ \frac{2\gamma}{3-\alpha} \left( 1 + \frac{2(\alpha^2 - 6\alpha + 17)\gamma^2}{(1-\alpha)^2(2-\alpha)} \right), & \gamma_\nu \leq \gamma \leq 1 \end{cases} ; \]

where
\[
\gamma_\nu = \sqrt{\frac{(1-\alpha)^2(2-\alpha)}{\alpha^2 - 6\alpha + 17}}.
\]

**Proof.** We can write the condition \((1)\) in the form
\[
(2) \quad \left( \frac{f(z)}{z} \right)^{(1+\alpha)} f'(z) = \left( \frac{1 + \omega(z)}{1 - \omega(z)} \right)^\gamma \left( 1 + 2\omega(z) + 2\omega^2(z) + \cdots \right),
\]
where $\omega$ is analytic in $D$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, $z \in D$. If we denote by $L$ and $R$ left and right hand side of equality (2), then we have

$$L = \left[ 1 - (1 + \alpha)(a_2z + \cdots) + \left(\frac{-(1 + \alpha)}{2}\right)(a_2z + \cdots)^2 
+ \left(\frac{-(1 + \alpha)}{3}\right)(a_2z + \cdots)^3 + \cdots \right] \cdot (1 + 2a_2z + 3a_2z^2 + 4a_2z^3 + \cdots)$$

and if we put $\omega(z) = c_1z + c_2z^2 + \cdots$:

$$R = 1 + \gamma \left[ 2(c_1z + c_2z^2 + \cdots) + 2(c_1z + c_2z^2 + \cdots)^2 + \cdots \right] 
+ \left(\frac{\gamma}{2}\right) \left[ 2(c_1z + c_2z^2 + \cdots) + 2(c_1z + c_2z^2 + \cdots)^2 + \cdots \right]^2 
+ \left(\frac{\gamma}{3}\right) \left[ 2(c_1z + c_2z^2 + \cdots) + 2(c_1z + c_2z^2 + \cdots)^2 + \cdots \right]^3 + \cdots .$$

If we compare the coefficients on $z$, $z^2$, $z^3$ in $L$ and $R$, then, after some calculations, we obtain

$$a_2 = \frac{2\gamma}{1 - \alpha}c_1, \tag{3}$$

$$a_3 = \frac{2\gamma}{2 - \alpha}c_2 + \frac{2(3 - \alpha)\gamma^2}{(1 - \alpha)^2(2 - \alpha)}c_1^2,$$

$$a_4 = \frac{2\gamma}{3 - \alpha}(c_3 + \mu c_1 c_2 + \nu c_1^3),$$

where

$$\mu = \mu(\alpha, \gamma) = \frac{2(5 - \alpha)\gamma}{(1 - \alpha)(2 - \alpha)} \quad \text{and} \quad \nu = \nu(\alpha, \gamma) = \frac{1}{3} + \frac{2}{3} \frac{\alpha^2 - 6\alpha + 17}{(1 - \alpha)^2(2 - \alpha)} \gamma^2. \tag{4}$$

Since $|c_1| \leq 1$, then by using (3) we easily obtain the result (a) from this theorem. Also, by using $|c_1| \leq 1$ and $|c_2| \leq 1 - |c_1|^2$, from (3) we have

$$|a_3| \leq \frac{2\gamma}{2 - \alpha}|c_2| + \frac{2(3 - \alpha)\gamma^2}{(1 - \alpha)^2(2 - \alpha)}|c_1|^2$$

$$\leq \frac{2\gamma}{2 - \alpha} (1 - |c_1|^2) + \frac{2(3 - \alpha)\gamma^2}{(1 - \alpha)^2(2 - \alpha)}|c_1|^2$$

$$= \frac{2\gamma}{2 - \alpha} + \frac{2\gamma}{2 - \alpha} \left[ \frac{(3 - \alpha)\gamma}{(1 - \alpha)^2} - 1 \right] |c_1|^2$$

and the result depends on the sign of the factor in the last bracket.

The main tool of our proof for the coefficient $a_4$ will be the results of Lemma 2 in the paper [3]. Namely, in that paper the authors considered the sharp estimate of the functional

$$\Psi(\omega) = |c_3 + \mu c_1 c_2 + \nu c_1^3|$$

within the class of all holomorphic functions of the form

$$\omega(z) = c_1 z + c_2 z^2 + \cdots$$
and satisfying the condition \(|\omega(z)| < 1, z \in \mathbb{D}\). In the same paper in Lemma 2, for \(\omega\) of previous type and for any real numbers \(\mu\) and \(\nu\) they give the sharp estimates \(\Psi(\omega) \leq \Phi(\mu, \nu)\), where \(\Phi(\mu, \nu)\) is given in general form in Lemma 2, and here we will use

\[
\Phi(\mu, \nu) = \begin{cases} 
1, & (\mu, \nu) \in D_1 \cup D_2 \cup \{(2, 1)\} \\
|\nu|, & (\mu, \nu) \in \bigcup_{k=3}^{7} D_k
\end{cases},
\]

where

\[
D_1 = \{(\mu, \nu) : |\mu| \leq \frac{1}{2}, -1 \leq \nu \leq 1\},
\]

\[
D_2 = \{(\mu, \nu) : \frac{1}{2} \leq |\mu| \leq 2, \frac{1}{4}(|\mu| + 1)^3 - (|\mu| + 1) \leq \nu \leq 1\},
\]

\[
D_3 = \{(\mu, \nu) : |\mu| \leq \frac{1}{2}, \nu \leq -1\},
\]

\[
D_4 = \{(\mu, \nu) : |\mu| \geq \frac{1}{2}, \nu \leq -\frac{2}{3}(|\mu| + 1)\},
\]

\[
D_5 = \{(\mu, \nu) : |\mu| \leq 2, \nu \geq 1\},
\]

\[
D_6 = \{(\mu, \nu) : 2 \leq |\mu| \leq 4, \nu \geq \frac{1}{12}(\mu^2 + 8)\},
\]

\[
D_7 = \{(\mu, \nu) : |\mu| \geq 4, \nu \geq \frac{2}{3}(|\mu| - 1)\}.
\]

In that sense, we need the values \(\alpha\) and \(\gamma\) such that \(0 < \mu \leq \frac{1}{2}, \mu \leq 2, \mu \leq 4, \nu \leq 1\). So, by using (4), we easily get the next equivalence

\[
0 < \mu \leq \frac{1}{2} \iff \gamma \leq \frac{(1 - \alpha)(2 - \alpha)}{4(5 - \alpha)} := \gamma_{1/2};
\]

\[
\mu \leq 2 \iff \gamma \leq \frac{(1 - \alpha)(2 - \alpha)}{5 - \alpha} := \gamma_2;
\]

\[
\mu \leq 4 \iff \gamma \leq \frac{2(1 - \alpha)(2 - \alpha)}{5 - \alpha} := \gamma_4;
\]

\[
\nu \leq 1 \iff \gamma \leq \sqrt{\frac{(1 - \alpha)^3(2 - \alpha)}{(\alpha^2 - 6\alpha + 17)}} := \gamma_\nu.
\]

It is easily to obtain that all values \(\gamma_{1/2}, \gamma_2, \gamma_4, \gamma_\nu\) are decreasing functions of \(\alpha\), \(0 < \alpha < 1\) and that

\[
0 < \gamma_{1/2} < \frac{1}{10}, \quad 0 < \gamma_2 < \frac{2}{5}, \quad 0 < \gamma_4 < \frac{4}{5}, \quad 0 < \gamma_\nu < \sqrt{\frac{2}{17}} = 0.342997\ldots.
\]

Also, it is clear that

\[
0 < \gamma_{1/2} < \gamma_2 < \gamma_4
\]

and it is easily to obtain that

\[
\gamma_{1/2} \leq \gamma_\nu \quad \text{for} \quad \alpha \in (0, \alpha_\nu],
\]

where \(\alpha_\nu = 0.951226\ldots\) is the root of the equation \(5\alpha^3 - 56\alpha^2 + 177\alpha - 122 = 0\) (of course \(\gamma_\nu \leq \gamma_{1/2}\) for \(\alpha \in [\alpha_\nu, 1]\)).
Further, the next relation is valid:

\[ 0 < \gamma \nu < \gamma_2 < \gamma_4. \]

**Case 1.** \((0 < \gamma \leq \gamma_\nu)\). First, it means that \(\nu \leq 1\). If \(0 < \gamma \leq \gamma_{1/2}\), then \(0 < \mu \leq \frac{1}{2}\) and \(0 < \nu \leq 1\), which by Lemma 2 \([3]\) gives \(\Phi(\mu, \nu) = 1\). If \(\gamma_{1/2} \leq \gamma \leq \gamma_\nu\), \(\alpha \in (0, \alpha_\nu)\), then \(\frac{1}{2} \leq \mu < 2\), \(0 < \nu \leq 1\) and if we prove that

\[ \frac{4}{27}(\mu + 1)^3 - (\mu + 1) \leq \nu, \]

then also by Lemma 2 \([3]\) we have \(\Phi(\mu, \nu) = 1\). In that sense, let denote

\[ L_1 = \frac{4}{27}(\mu + 1)^3 - (\mu + 1) \quad \text{and} \quad R_1 = \nu. \]

Since \(L_1\) is an increasing function of \(\mu\) for \(\mu \geq \frac{1}{2}\) and since \(\gamma \leq \gamma_\nu\), then

\[ \mu \leq \frac{2(3 - \alpha)\gamma_\nu}{(1 - \alpha)(5 - \alpha)} = 2\sqrt{\frac{(1 - \alpha)(5 - \alpha)^2}{(2 - \alpha)(\alpha^2 - 6\alpha + 17)}} < 2\sqrt{\frac{25}{34} = \frac{10}{\sqrt{34}}} \]

(because the function under the square root is decreasing) and so

\[ L_1 < \frac{4}{27}(\frac{10}{\sqrt{34}} + 1)^3 - (\frac{10}{\sqrt{34}} + 1) = 0.249838\ldots, \]

while

\[ R_1 = \nu = \frac{1}{3} + \frac{2(\alpha^2 - 6\alpha + 17)\gamma^2}{3(1 - \alpha^3(2 - \alpha))} > \frac{1}{3} = 0.33\ldots. \]

This implies the desired inequality.

**Case 2.** \((\gamma_\nu \leq \gamma \leq 1)\) In this case we have that \(\nu \geq 1\). If \(\gamma_\nu \leq \gamma \leq \gamma_2\), \(\alpha \in (0, 1)\), then \(0 < \mu \leq 2\), \(\nu \geq 1\), which by Lemma 2 \([3]\) implies \(\Phi(\mu, \nu) = \nu\). If \(\gamma_2 \leq \gamma \leq \gamma_4\), \(\alpha \in (0, 1)\), then \(2 \leq \mu \leq 4\). Also, after some calculations, the inequality \(\nu \geq \frac{1}{15}(\mu^2 + 8)\) is equivalent to

\[ \frac{43 - 23\alpha + 5\alpha^2 - \alpha^3}{(1 - \alpha^3(2 - \alpha)^2}\gamma^2 \geq 1. \]

Since \(\gamma_2 \leq \gamma\), then the previous inequality is satisfied if

\[ \frac{43 - 23\alpha + 5\alpha^2 - \alpha^3}{(1 - \alpha^3(2 - \alpha)^2}\gamma^2 \geq 1. \]

But, the last inequality is equivalent to the inequality \(\alpha^2 - 2\alpha - 3 \leq 0\), which is really true for \(\alpha \in (0, 1)\). By Lemma 2 \([3]\) we also have \(\Phi(\mu, \nu) = \nu\). Finally, if \(\gamma \geq \gamma_4\), then \(\mu \geq 4\) and if \(\nu \geq \frac{2}{3}(\mu - 1)\) we have (by using the same lemma) \(\Phi(\mu, \nu) = \nu\). Really, the inequality \(\nu \geq \frac{2}{3}(\mu - 1)\) is equivalent with

\[ 2(\alpha^2 - 6\alpha + 17)\gamma^2 - 4(1 - \alpha^2(5 - \alpha)\gamma + 3(1 - \alpha^3(2 - \alpha) \geq 0. \]
Since the discriminant of previous trinomial is
\[ D = 8(1 - \alpha)^3(\alpha^3 - 2\alpha^2 + 17\alpha - 52) < 0 \]
for \( \alpha \in (0, 1) \), then the previous inequality is valid. By using (3) we have that
\[ |a_4| \leq \frac{2}{3-\alpha} (\text{Case 1}) \], or \( |a_4| \leq \frac{2}{3-\alpha} \nu (\text{Case 2}) \), and from there the statement of the theorem.

All results of Theorem 1 are the best possible as the functions \( f_i, i = 1, 2, 3 \), defined with
\[ \left( \frac{z}{f_i(z)} \right)^{1+\alpha} f_i'(z) = \left( \frac{1 + z}{1 - z} \right)^\gamma, \]
where \( 0 < \alpha < 1 \), \( 0 < \gamma \leq 1 \). We have that
\[ c_i = 1 \quad \text{and} \quad c_j = 0 \quad \text{when} \quad j \neq i. \]

\[ \square \]

**Remark 1.** By using Theorem A we can conclude that it is sufficient to be \( \gamma \leq \gamma_*(\alpha) \) and \( 0 < \alpha < 2/\pi \) for starlikeness of functions \( f \in A \) which satisfied the condition (1).

Also, these conditions imply that the modulus of the coefficients \( a_2, a_3, a_4 \) is bounded with some constants. Namely, from the estimates given in Theorem 1 we have, for example,
\[ |a_2| \leq \frac{2\gamma}{1-\alpha} \leq \frac{2\gamma_*(\alpha)}{1-\alpha}, \quad |a_3| \leq \frac{2(3-\alpha)\gamma^2_*(\alpha)}{(1-\alpha)^2(2-\alpha)}, \]

etc.

We note that \( \gamma_*(\alpha) < 1 - \alpha \) for \( 0 < \alpha < 2/\pi \). Namely, if we put
\[ \phi(\alpha) =: \gamma_*(\alpha) - (1 - \alpha), \]
then \( \phi'(\alpha) = 1 - \sqrt{\frac{2}{\pi\alpha}} - 1 \). It is easily to see that \( \phi \) attains its minimum \( \phi(1/\pi) = -1/2 \) and since \( \phi(0+) = 0, \phi(2/\pi -) = 2/\pi - 1 < 0 \), we have the desired inequality.

When \( \alpha \to 0 \), then \( \gamma_*(0+) = 1 \), and from Theorem 1 we have the next estimates for \( 0 < \gamma \leq 1 \):
\[ |a_2| \leq 2\gamma \leq 2, \quad |a_3| \leq \begin{cases} \gamma, & 0 < \gamma \leq 1/3 \\ 3\gamma^2, & 1/3 \leq \gamma \leq 1 \end{cases} \]
and
\[ |a_4| \leq \begin{cases} 2\gamma/3, & 0 < \gamma \leq \sqrt{2/17} \\ 2\gamma(1 + 17\gamma^2)/9 \leq 4, & \sqrt{2/17} \leq \gamma \leq 1 \end{cases}. \]

This is the case when we have strongly starlike functions of order \( \gamma \).
For $\gamma = 1$ in Theorem [1], i.e. if $\text{Re} \left[ \left( \frac{z}{f(z)} \right)^{1+\alpha} f'(z) \right] > 0$, $z \in \mathbb{D}$, we have

$$|a_2| \leq \frac{2}{1-\alpha}, \quad |a_3| \leq \frac{2(3-\alpha)}{(1-\alpha)^2(2-\alpha)}$$

and

$$|a_4| \leq \frac{2}{3-\alpha} \left[ \frac{1}{3} + \frac{2}{3} \left( \frac{\alpha^2 - 6\alpha + 17}{(1-\alpha)^2(2-\alpha)} \right) \right].$$

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