FDR CONTROL FOR MULTIPLE HYPOTHESIS TESTING ON COMPOSITE NULLS

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Multiple hypothesis testing often involves composite nulls, i.e., nulls that are associated with two or more distributions. In many cases, it is reasonable to assume that there is a prior distribution on the distributions despite it is unknown. When the number of distributions under true nulls is finite, we show that under the above assumption, the false discover rate (FDR) can be controlled using \( p \)-values computed under constraints imposed by the empirical distribution of the observations. Comparing to FDR control using \( p \)-values defined as maximum significance level over all null distributions, the proposed FDR control can have substantially more power.

1. Introduction. In hypothesis testing, a relatively simple case is where the data associated with true nulls and those with false nulls each follow a common distribution (“simple versus simple”) [4, 6]. On the other hand, in many cases, either the data associated with true nulls follow different distributions (“composite nulls”) or those associated with false nulls follow different distributions (“composite alternatives”). In the current literature on multiple testing, once appropriate test statistics such as \( p \)-values are computed, testing procedures based on the statistics usually do not distinguish between the simple and composite cases [11, 10, 16, 7, 14]. At the time when a procedure is applied, it only has the test statistics available. For this reason, how the test statistics are defined plays an important role in the overall performance of the procedure.

For composite nulls, \( p \)-values are usually defined as maximum probabilities over all null distributions [10]. Following the random-effects extension for composite alternatives [6], a Bayesian approach to calculating \( p \)-values can be used. Specifically, one assumes that there is a known prior distribution on the null distributions. Since the overall distribution of the data associated with true nulls can now be determined by an integral of the null distributions weighted by the prior, the composite case is essentially reduced to the simple one.

The focus of the article lies between the above two approaches. The underlying premise is that there is a prior distribution on the null distributions, however, the prior is unknown. The basic observation is that, in the presence of a large number of nulls, the empirical distribution of the data provides

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useful information on the prior. More specifically, the mixture of the null distributions, if multiplied by the population fraction of true nulls, is dominated by the empirical distribution of the data plus a small margin. This constrains the set of possible priors. We shall explore the observation for the case where there are only a finite number of null distributions. On the one hand, the $p$-values will be calculated as maximum probabilities. On the other, the maximization is over a range of linear combinations of the null distributions, with the coefficients being constrained. As a result, the $p$-values can be computed by linear programming.

The article does not consider the case of composite alternatives. The position here is that, since oftentimes no information on the distributions under false nulls is available, it is sensible to regard data associated with false nulls as being sampled from a single overall distribution.

Although our focus is the evaluation of $p$-values under constraints, we start with Section 2 on FDR control using maximum probabilities without constraints. That the BH procedure can control the FDR in this case is known [3]. The purpose of the section is to set up suitable framework for following sections, by making a more general description of the BH procedure and indicating where constrained maximization may be introduced.

Section 3 considers two ways to compute $p$-values. The first one is sequential, such that the $p$-value of each observation is obtained under linear constraints imposed by observations whose $p$-values have already been computed. In the second one, in principle, the $p$-values can be computed for the observations simultaneously under the linear constraints imposed by the entire data. Both types of $p$-values are then processed by the BH procedure. Analytically, it is easier to establish FDR control based on the first type of $p$-values because the sequential computation allows one to use a stopping time argument [15]. On the other hand, since there are more constraints imposed on the second type of $p$-values, presumably they may lead to more improvement in multiple testing. However, the simulation study reported in Section 4 indicates that the two types of $p$-values lead to similar performance of multiple testing. Some possible explanations for this will be given at the end of Section 4. The study shows that, the BH procedure is substantially more powerful when using the two types of $p$-values than using $p$-values computed by the usual unconstrained maximization. In addition to power, we will also compare the FDR and positive FDR (pFDR) realized by the $p$-values.

The results in Section 4 indicate that in general, for the case of composite nulls, the prior on the null distributions cannot be estimated consistently. Basically, this is because the constraints imposed by the data cannot yield exact details of the prior and also because the above two ways to evaluate $p$-values usually select different linear combinations of the null distributions for different observations. This is in contrast to the simple case, where the fraction of true nulls can be estimated consistently [2, 8, 15]. Conceptually it is of interest to ask whether there are conditions that allow the prior of
the null distributions to be estimated consistently. In Section 5, for the case where there are only a finite number of null distributions, a necessary and sufficient condition will be given for the consistent estimation of the prior using maximum likelihood estimation (MLE). Note that, in the MLE, the distribution under false nulls is unknown, and the data are treated as though all are sampled from true nulls. An example will be given to show that for any finite set of linearly independent null distributions, one can construct a large class of distributions that satisfy the condition.

Section 6 contains a brief discussion. Most technical details are collected in the Appendix.

1.1. Assumptions and notation. Let \( \{F_{\theta}, \theta \in \Theta\} \) be a family of distributions on \( \mathbb{R}^d \). Given random observations \( X_1, \ldots, X_n \in \mathbb{R}^d \), the composite nulls to be tested are

\[ H_i : X_i \sim F_{\theta} \text{ for some } \theta \in \Theta. \]

Each \( F_{\theta} \) is a null distribution.

Our discussion will be under the following random mixture model. The distribution under false nulls is \( G \notin \{F_{\theta}, \theta \in \Theta\} \) and the fraction of false nulls among all nulls is \( a \in (0, 1) \). There is a prior probability measure \( \nu \) on \( \Theta \). The data are sampled as follows. Define probability measure \( \mu \) on \( \Theta \cup \{\ast\} \), where \( \ast \) is any element not in \( \Theta \), such that \( \mu(\{\ast\}) = a \) and \( \mu(A) = (1 - a)\nu(A) \) for \( A \subset \Theta \). Sample \( \eta_1, \ldots, \eta_n \) iid \( \sim \mu \). If \( \eta_i = \ast \), then sample \( X_i \sim G \); otherwise, sample \( X_i \sim F_{\eta_i} \). Thus \( \eta_i \) can be thought of as the identity of \( X_i \), indexing the distribution \( X_i \) is sampled from.

Throughout we will make two assumptions. First, \( \nu \) is unknown. Indeed, if \( \nu \) is known, then under true \( H_i \), \( X_i \sim F = \int F_{\theta} \nu(d\theta) \) and thus the composite null can be reduced to a simple null. Second, \( G \) is unknown. This assumption is especially intended for the case where \( \Theta \) is finite. Indeed, if \( G \) is known, then for \( n \gg 1 \), both \( a \) and \( \nu \) can be estimated accurately by the MLE, which reduces the testing problem into one only involving simple nulls.

Recall that for a multiple testing procedure, if \( R \) is the number of rejected nulls, and \( V \) that of rejected true nulls, then

\[
\text{FDR} = E \left[ \frac{V}{R \vee 1} \right], \quad \text{pFDR} = E \left[ \frac{V}{R} \middle| R > 0 \right].
\]

Furthermore, if there are \( n \) nulls and \( N \) of them are true, then

\[
\text{power} = E \left[ \frac{R - V}{(n - N) \vee 1} \right].
\]

2. Testing based on maximum probabilities. Usually, a description of multiple testing procedure starts with \( p \)-values, treating them as already available. For our discussion later, it is useful to start with how \( p \)-values are computed. The \( p \)-values are absent in the continuous version of our description, but explicit in the discrete version.
Let \( \{ D_t : t \in \mathcal{I} \} \) be a family of Borel sets in \( \mathbb{R}^d \) satisfying the following conditions, where \( \mathcal{I} \neq \emptyset \) is an open interval in \( \mathbb{R} \).

D1. The family is increasing and right-continuous, i.e. \( D_t = \bigcap_{s > t, s \in \mathcal{I}} D_s \), for \( t \in \mathcal{I} \).
D2. \( \bigcup_{t \in \mathcal{I}} D_t = \mathbb{R}^d \).
D3. \( G(\bigcap_{t \in \mathcal{I}} D_t) = F_\theta(\bigcap_{t \in \mathcal{I}} D_t) = 0, \theta \in \Theta. \)

For each \( \theta \in \Theta \), define
\[
\phi_\theta(t) = \begin{cases} 
F_\theta(D_t) & \text{if } t \in \mathcal{I}, \\
0 & \text{if } t \leq \inf \mathcal{I}, \\
1 & \text{if } t \geq \sup \mathcal{I},
\end{cases}
\]
(2.1)
i.e., \( \phi_\theta(t) \) is the significance level of the region \( D_t \) under \( F_\theta \). By D2 and D3, \( \phi_\theta \) is nondecreasing and continuous at \( \inf \mathcal{I} \) and \( \sup \mathcal{I} \). Denote
\[
M(t) = \sup_{\theta} \phi_\theta(t),
\]
i.e., \( M(t) \) is the significance level of \( D_t \) associated with \( \{ F_\theta, \theta \in \Theta \} \). It is nondecreasing with \( M(t) = 0 \) for \( t \leq \inf \mathcal{I} \) and \( M(t) = 1 \) for \( t \geq \sup \mathcal{I} \).

We can regard \( M(t) \) as \( \sup_\mu \int \phi_\theta(t) \, d\mu(\theta) \), where the supremum is taken over all possible probability measures \( \mu \) on \( \Theta \). By our assumption, there is a prior \( \nu \) on \( \Theta \). If there is no information on the value of \( \nu \), then the supremum is justified. If, on the other hand, it is known that \( \nu \) satisfies certain conditions, then it makes sense to use the conditions to constrain the supremum, even though the conditions may not uniquely determine \( \nu \). This may yield a \( M(t) \) closer to \( \int \phi_\theta(t) \, d\nu(\theta) \) that improves the performance of multiple testing.

Once \( M(t) \) are in place, the BH procedure can be applied. The procedure can be described in two ways. The continuous version features a stopping time that may simplify the analysis of FDR control (cf. [15]), while the discrete one is easier to implement. For \( t \in \mathcal{I} \), denote
\[
R_n(t) = \sum_{i=1}^{n} 1 \{ X_i \in D_t \}, \quad V_n(t) = \sum_{i=1}^{n} 1 \{ X_i \in D_t, \eta_i \in \Theta \}.
\]

**Procedure 2.1 (Continuous version).** Given control parameter \( \alpha \in (0, 1) \), let
\[
\mathcal{I}_R = \left\{ t \in \mathcal{I} : \frac{M(t)}{\alpha} \leq \frac{R_n(t) \vee 1}{n} \right\}.
\]
If \( \mathcal{I}_R \neq \emptyset \), set \( \tau = \inf \mathcal{I}_R \) and reject \( H_i \) if and only if \( X_i \in D_\tau \). Otherwise, set \( \tau = \inf \mathcal{I} \) and accept all \( H_i \).

To describe the discrete version of Procedure 2.1, define
\[
s(x) = \inf \{ t \in \mathcal{I} : x \in D_t \}, \quad s_i = s(X_i), \quad i = 1, \ldots, n.
\]
(2.3)
By D2, the set in (2.3) is nonempty, so \( s(x) \) is well-defined and \( s(x) < \sup \mathcal{I} \).
Proposition 2.1. Under D1-3, the following statements hold.

1) \( s_i \in \mathcal{I} \) almost surely.
2) For any \( t \in \mathcal{I} \), \( s_i \leq t \iff X_i \in D_t \) and hence \( R_n(t) = \sum 1 \{ s_i \leq t \} \).
3) Given \( \theta \), if \( X_i \sim F_\theta \), then \( s_i \sim \phi_\theta \).
4) For \( i = 1, \ldots, n \), the distribution function of \( s_i \) is
   \[
   Q(t) = (1 - a) \int \phi_\theta(t) \, d\theta + aG(D_t).
   \]
5) If \( \phi_\theta \in C(\mathbb{R}) \) for all \( \theta \), then \( M(t) \) is left-continuous.

By Proposition 2.1, \( \phi_\theta(s_i) \) is the \( p \)-value of \( X_i \) under \( F_\theta \). Therefore, \( M(s_i) \) can be used as a \( p \)-value under the composite null \( H_1 \) [10].

Procedure 2.2 (Discrete version). Let \( s_{(1)} \leq \ldots \leq s_{(n)} \) be the order statistics of \( s_i \) and \( s_{(0)} = \inf \mathcal{I} \). Reject \( H_i \) if and only \( s_i \leq s_{(R)} \), where
   \[
   R = \max \left\{ i \geq 0 : \frac{M(s_{(i)})}{\alpha} \leq \frac{i}{n} \right\}.
   \]

Proposition 2.2. Suppose \( \phi_\theta \in C(\mathbb{R}) \) for all \( \theta \). Then Procedures 2.1 and 2.2 are the same, and both have FDR \( \leq (1 - a)\alpha \).

In single hypothesis tests, nested rejection regions are usually indexed by significance level. For FDR control, other indices can be used. This allows one to think about the rejection regions in more natural terms and also avoids problems when different regions have the same significance levels.

Example 2.1. Suppose \( X_i \in \mathbb{R} \). To use lower-tail probabilities as \( p \)-values, set \( D_t = (-\infty, t] \), \( t \in \mathcal{I} = \mathbb{R} \). Then \( s_i = X_i \) and \( \phi_\theta(s_i) = F_\theta(X_i) \).

To use upper-tail probabilities as \( p \)-values, set \( D_t = (-t, \infty) \), \( t \in \mathcal{I} = \mathbb{R} \).

Then \( s_i = -X_i \) and \( \phi_\theta(s_i) = F_\theta([-s_i, \infty)) = F_\theta([-X_i, \infty)) \). Suppose each \( F_\theta \) is continuous at 0. If we use \( D_t = [-t, t], \ t \in \mathcal{I} = [0, \infty) \), then \( s_i = |X_i| \) and \( \phi_\theta(s_i) = F([-|X_i|, |X_i|]) \).

3. Testing based on constrained maximum probabilities.

3.1. Outlines. Testing using maximum probabilities can be very conservative. Our goal is to find alternative methods when \( \Theta \) is a finite set \( \{ \theta_k, \ k = 1, \ldots, L \} \). The probability measure \( \nu \) on \( \Theta \) can now be specified by \( \nu = (\nu_1, \ldots, \nu_L)^\top \) with \( \nu_k = \nu(\{ \theta_k \}) \). Henceforth, a letter in boldface will stand for an \( L \)-dimensional vector. Denote \( \phi_k(t) = \phi_{\theta_k}(t) \). In this section, we assume that all \( F_k \) and hence all \( \phi_k(t) \) are continuous. Denote
   \[
   F_n(t) = R_n(t)/n,
   \]
i.e. the empirical distribution based on \( s_1, \ldots, s_n \) defined in (2.3).
Instead of \( M(t) = \max_k \phi_k(t) \) as in Procedure 2.1, for finite \( \Theta \), the proposed functions to use have the general form

\[
M_n(t) = \sup\{ c^\top \phi(t) : c \in C, \ c^\top \phi \in \mathcal{A}_{n,t} \},
\]

where \( C \) is a suitable subset of

\[
\Delta = \{ c \in [0,1]^L : c_1 + \cdots + c_L \leq 1 \}
\]

and for \( n \geq 1 \) and \( t \in \mathcal{I} \), \( \mathcal{A}_{n,t} \) is a family of functions on \( \mathcal{I} \). In general, \( C \) is constructed based on deterministic knowledge on \( \nu \) and \( a \). On the other hand, \( \mathcal{A}_{n,t} \) is constructed based on the data and hence both \( M_n(t) \) and \( \mathcal{A}_{n,t} \) may be random. If \( C = \Delta \) and \( \mathcal{A}_{n,t} \) is the entire family of functions on \( \mathcal{I} \), then \( M_n(t) \) is constructed based on deterministic knowledge on \( \nu \) and \( a \). If \( C \) is random, this may result in higher power. In particular, if \( C = \{ \nu \} \), then \( M_n(t) = \nu^\top \phi(t) \), which reduces the testing problem to the one for simple nulls.

Oftentimes, there is no direct knowledge on \( \nu \) or \( a \) so one has to set \( C = \Delta \); constraints on \( c \) are indirectly imposed through the condition \( c^\top \phi \in \mathcal{A}_{n,t} \). Then \( M_n(t) \) takes the form

\[
M_n(t) = \sup\{ c^\top \phi(t) : c \in \Delta, \ c^\top \phi \in \mathcal{A}_{n,t} \}.
\]

In Section 4, we will consider the case where \( C \) can be chosen smaller than \( \Delta \), and in Section 5, a case where substantial knowledge on \( \nu \) can be attained by estimation will be considered.

Recall that \( (1 - a)\nu^\top \phi(t) \) is the population fraction of true nulls with \( X_i \in D_t \). In order for \( M_n(t) \) not to underestimate the fraction, a basic requirement is \( M_n(t) \geq (1 - a)\nu^\top \phi(t) \). In general, since \( \mathcal{A}_{n,t} \) is random, this requires that \( \mathcal{A}_{n,t} \) have the property that as long as \( n \) is large enough, with probability close to 1, \( (1 - a)\nu^\top \phi \in \mathcal{A}_{n,t} \) for all \( t \in \mathcal{I} \).

A basic fact to use in order to satisfy the condition is that, almost surely, as \( n \to \infty \),

\[
\sup_{t} |F_n(t) - Q(t)| \to 0,
\]

where \( Q(t) \) is the distribution function of \( s_i = s(X_i) \) defined in (2.3), i.e.

\[
Q(t) = (1 - a)\nu^\top \phi(t) + aG(D_t).
\]

Then, with probability close to 1, \( (1 - a)\nu^\top \phi \) is less than \( F_n(t) \) plus a small margin. Moreover, \( Q(t) - (1 - a)\nu^\top \phi(t) = aG(D_t) \) is increasing in \( t \). Then for \( n \gg 1 \), with probability close to 1,

\[
F_n(u) - (1 - a)\nu^\top \phi(u) > F_n(v) - (1 - a)\nu^\top \phi(v) - \epsilon_n, \text{ for all } u > v.
\]

Therefore, in calculating \( M_n(t) \), the maximization can be constrained to those \( c \) such that, when they replace \( (1 - a)\nu \), the inequalities still hold.
3.2. Construction using data sequentially. Given the relative ease to establish FDR control by using a stopping time as the random cut-off for rejection, we first consider a construction of $A_{n,t}$ that allows a stopping time to be defined.

Incorporating the facts discussed just now, a basic form of $A_{n,t}$ is

$$(3.2) \quad A_{n,t} = \begin{cases} h \in C(\mathcal{I}) : h(s_i) \leq F_n(s_i) + \epsilon_n \text{ for } s_i \geq t \\ \mathbb{F}_n(t_2) - \mathbb{F}_n(t_1) \geq h(t_2) - h(t_1) - \epsilon_n \text{ for } t_1, t_2 \in T_n \text{ with } t \leq t_1 < t_2 \end{cases},$$

where $T_n \subset \mathcal{I}$ is a finite set of points. Although $T_n$ can contain any number of points, to reduce computation, the number of points in $T_n$ needs to be relatively small.

It is easy to see $M_n(t) = 0$ if $t \leq \inf \mathcal{I}$. Some other useful properties of $M_n(t)$ are as follows.

**Lemma 3.1.** $M_n$ is always nondecreasing. Furthermore, if $\phi_i \in C(\mathbb{R})$ for all $i$, then almost surely, 1) $M_n$ is continuous at every $t$ other than $s_1, \ldots, s_n$ and 2) it is left-continuous and has a right-hand limit at each $s_i$.

The continuous and discrete versions of the BH procedure using $M_n(t)$ are described below. Similar to Procedure 2.2, the two versions are equivalent. As in Procedure 2.1, the random variable $\tau$ in the continuous version is a stopping time.

**Procedure 3.1.** Given control parameter $\alpha \in (0, 1)$, let

$$\mathcal{I}_R = \left\{ t \in \mathcal{I} : \frac{M_n(t)}{\alpha} \leq \frac{R_n(t) \vee 1}{n} \right\}.$$ 

If $\mathcal{I}_R \neq \emptyset$, set $\tau = \sup \mathcal{I}_R$ and reject $H_i$ if and only if $s_i \leq \tau$. Otherwise, set $\tau = \inf \mathcal{I}$ and accept all $H_i$.

Equivalently, sort $s_i$ into $s(1) \leq \ldots \leq s(n)$ and set $s(0) = \inf \mathcal{I}$. Reject $H_i$ if and only if $s_i \leq s(R)$, where

$$R = \max \left\{ i \geq 0 : \frac{M_n(s(i))}{\alpha} \leq \frac{R_n(s(i)) \vee 1}{n} \right\}. \quad \square$$

For each $i$, $M_n(s(i))$ is the maximum of $c^T \phi(s(i))$, with $c_k$ satisfying

1) $c_k \geq 0$, $\sum c_k \leq 1$;
2) $c^T \phi(s(j)) \leq \mathbb{F}_n(s(j)) + \epsilon_n$ for $j \geq i$;
3) $\mathbb{F}_n(t_2) - \mathbb{F}_n(t_1) \geq \sum_{k=1}^L c_k [\phi_k(t_2) - \phi_k(t_1)] + \epsilon_n$ for $t_1, t_2 \in T_n$ with $s(i) \leq t_1 < t_2$.

All the constraints are linear. As a result, $M_n(s(i))$ can be computed by linear programming. The computation is termed sequential because each
$M_n(s_{(i)})$ is computed based on the data greater than $s_{(i)}$. Therefore, if we imagine that $s_{(i)}$ are input one by one, starting with the largest one, then $M_n(s_{(i)})$ can be computed only after all $s_{(j)}$, $j \geq i$, have been input.

The FDR control of Procedure 3.1 is given in the next result. The main tool for the proof is martingale stopping time and the Dvoretzky-Kiefer-Wolfowitz (DKW) inequality [12].

**Theorem 3.1.** Suppose 1) $\phi_i \in C(\mathbb{R})$, 2) $\nu^T \phi(t) > 0$ for all $t \in \mathcal{I}$ and 3) $G(D_i)$ in continuous in $t$. Then for $n \geq 1$, provided $\exp(-2n\epsilon_n^2) \leq 1/2$, Procedure 3.1 satisfies

$$FDR \leq \alpha + 2(1 + |T_n|)\exp(-2n\epsilon_n^2) + E \left[ \frac{1\{R > 0\}}{R \lor 1} \right].$$

The bound contains terms in addition to $\alpha$. For appropriate $\epsilon_n$ and $T_n$, the term $2(1 + |T_n|)\exp(-2n\epsilon_n^2)$ is $o(1)$ as $n \to \infty$. Under certain conditions, $R$ is of the same order as $n$ and hence the bound shows FDR can be asymptotically controlled at $\alpha$. However, the simulation study in Section 4 indicates that usually the realized FDR is substantially lower than $\alpha$, which is reasonable because $M_n(t)$ is an overestimation of $(1 - a)\nu^T \phi(t)$.

### 3.3. Construction using entire data.

In place of $A_n$, which depends on $t$, we can use a single family of functions $A_n$. In order to impose maximum amount of linear constraints, $A_n$ should incorporate all $X_i$. Based on the same considerations underlying (3.2), we define

$$A_n = \left\{ h \in \mathcal{C}(\mathcal{I}) : h(s_i) \leq \mathbb{F}_n(s_i) + \epsilon_n \text{ for all } s_i \right\}.$$

Corresponding to (3.1), for $t \in \mathcal{I}$, define

$$M_n(t) = \sup \left\{ c^T \phi(t) : c \in \Delta, c^T \phi \in A_n \right\}.$$

It is easy to see that $M_n$ is nondecreasing. Therefore, corresponding to Procedure 3.1, the following BH procedure obtains.

**Procedure 3.2.** Given control parameter $\alpha \in (0, 1)$, let

$$\mathcal{I}_R = \sup \left\{ t \in \mathcal{I} : \frac{M_n(t)}{\alpha} \leq \frac{R_n(t) \lor 1}{n} \right\}.$$

If $\mathcal{I}_R \neq \emptyset$, set $\tau = \sup \mathcal{I}_R$ and reject $H_i$ if and only if $s_i \leq \tau$. Otherwise, set $\tau = \inf \mathcal{I}$ accept all $H_i$.

Equivalently, sort $s_i$ into $s_{(1)} \leq \ldots \leq s_{(n)}$ and set $s_{(0)} = \inf \mathcal{I}$. Reject $H_i$ if and only if $s_i \leq s_{(R)}$, where

$$R = \max \left\{ i \geq 0 : \frac{M_n(s_{(i)})}{\alpha} \leq \frac{R_n(s_{(i)}) \lor 1}{n} \right\}.$$
Like Procedure 3.1, \( M_n(s_{(i)}) \) can be computed using linear programming. For comparison, we list the constraints for the maximization. For each \( i \), \( M_n(s_{(i)}) \) is the maximum of \( c^T \phi(s_{(i)}) \), with \( c_k \) satisfying

1) \( c_k \geq 0, \sum c_k \leq 1; \)
2) \( c^T \phi(s_{(j)}) \leq F_n(s_{(j)}) + \epsilon_n \) for all \( j = 1, \ldots, n. \)
3) \( F_n(t_2) - F_n(t_1) \geq \sum_{k=1}^L c_k[\phi_k(t_2) - \phi_k(t_1)] + \epsilon_n \) for all \( t_1, t_2 \in T_n \) with \( t_1 < t_2 \).

It is worth pointing out that although the set of constraints on \( c \) is the same for all \( s_i \), for different \( i \), because \( \phi(s_i) \) are different, the value of \( c \) that yields \( M_n(s_i) \) will be different.

Unlike Procedures 2.1 and 3.1, since \( \tau \) in Procedure 3.2 is determined by the entire \( s_1, \ldots, s_n \), it is not a stopping time. Because the martingale stopping time argument cannot be used to establish FDR control for finite \( n \), we will work out an asymptotic statement instead.

For \( s \in \mathbb{R} \) and \( S \subset \mathbb{R} \), denote the distance from \( s \) to \( S \) by \( d(s, S) = \inf \{ |s - t| : t \in S \} \). Define \( \delta(S, T) = \sup \{ d(s, S) : s \in T \} \) for \( S, T \subset \mathbb{R} \). A sequence \( S_n \) of finite sets is said to be increasingly dense in \( T \) if for any \( r > 0, \delta(S_n, T \cap [-r, r]) \to 0 \) as \( n \to \infty \).

**Theorem 3.2.** Suppose 1) all \( \phi_i \) are continuous and \( c^T \phi \) is strictly increasing in \( I \) 2) \( G(D_t) \) is continuous in \( t \), and 3) as \( n \to \infty, \epsilon_n \to 0, nc_n^2 \to \infty \) and \( T_n \) is increasingly dense in \( I \). Then, under Assumption A given below, for Procedure 3.2, \( \lim_{n \to \infty} \text{FDR} \leq \alpha \).

Furthermore, asymptotically the procedure is equivalent to the one that reject \( H_i \) if and only if \( s_i \leq t_\ast \), where \( t_\ast \) is defined in (3.6) below.

Intuitively, as \( n \to \infty \), in certain sense \( A_n \) should tend to \( A = \{ h \in C(I) : Q - h \geq 0 \} \) is nondecreasing. Consequently, \( M_n(t) \) should tend to

\[
(3.5) \quad m(t) = \sup \{ c^T \phi(t) : c \in \Delta, c^T \phi \in A \}.
\]

If this is true, then, as in [6], the asymptotic of FDR as \( n \to \infty \) may be characterize by a fixed point derived from \( m(t) \) and \( Q(t) \). Let

\[
(3.6) \quad t_\ast = \sup \{ t \in I : m(t) \leq \alpha Q(t) \}.
\]

**Assumption A.** \( t_\ast \in I \) and there is \( t_0 < t_\ast \), such that \( m(t) < \alpha Q(t) \) on \( (t_0, t_\ast) \).

4. Numerical study.

4.1. Setup. Because the properties of \( M_n(t) \) in (3.1) and (3.4) are hard to keep track of, it is difficult to analyze the power and pFDR of Procedures 3.1 and 3.2. We resort to numerical simulations to get a handle to these two quantities. For comparison, Procedure 2.1 and the BH procedure with the prior probabilities \( \nu_1, \ldots, \nu_L \) being known are also included.
We only consider univariate observations. To use lower-tail $p$-values, we set $D_t = (-\infty, t]$. By (2.3), if an observation $X$ takes value $x$, then $s(X) = x$ and hence $\phi_k(s(X)) = F_k(x)$, the left-tail $p$-value of $X$ under $F_k$. Also, given observations $X_1, \ldots, X_n$, from $R_n(t) = \sum_{i=1}^{n} 1 \{X_i \in D_t\}$, $R_n(X_i)$ is the rank of $X_i$.

In each simulation, we draw iid samples $X_1, \ldots, X_n$ from a mixture distribution $(1-a)\sum_{k=1}^{p} \nu_k F_k(x) + aG(x)$, where $G \not= F_1, \ldots, F_L$. To test nulls

$$H_i : X_i \sim F_k \text{ for some } k, \quad i = 1, \ldots, n,$$

we compute four types of $p$-values:

1) $p_i,\text{seq} = M_n(X_i)$ defined by (3.1) and (3.2), where “seq” in the subscript stands for “sequential”, indicating that as the calculation of $M_n(X_i)$ precedes to smaller $X_i$, linear constraints are added sequentially;

2) $p_i,\text{glb} = M_n(X_i)$ defined by (3.3) and (3.4), where “glb” in the subscript stands for “global”, indicating that $M_n(X_i)$ are calculated under linear constraints imposed by all $X_1, \ldots, X_n$;

3) $p_i,\text{max} = \max_k F_k(X_i)$;

4) $p_i,\text{mix} = \sum_k \nu_k F_k(X_i)$, i.e., the $p$-value of $X_i$ when the values of $\nu_1, \ldots, \nu_L$ are known.

The computation of $p_i,\text{seq}$ and $p_i,\text{glb}$ is done by linear programming. By (3.1) and (3.4), both are maxima of $c^T F(X_i) = c_1 F_1(X_i) + \cdots + c_L F_L(X_i)$. In the simulations, the constraints are a little different from those basic ones given in (3.2) and (3.3). However, the analysis is the same.

Denote by $\bar{\Gamma}^*(z; \alpha, \beta)$ the $z$-th upper-tail quantile of the Gamma distribution with shape parameter $\alpha$ and scale parameter $\beta$. For $i = 1, \ldots, n$, to compute $p_i,\text{seq}$, the constraints on $c_1, \ldots, c_L$ are

1) $c_k \geq 0, \sum c_k \leq 1$;

2) $c^T F(X_j) \leq u(X_j)$ for $X_j \geq X_i$, where

$$u(X_j) = \begin{cases} \frac{1}{n} \bar{\Gamma}^* \left( \frac{1}{n}; R_n(X_i), \frac{1}{0.95} \right), & \text{if } R_n(X_j) \leq n^{0.2}, \\ \frac{1}{n} \bar{\Gamma}^* \left( \frac{1}{n}; R_n(X_i), \frac{1}{0.95} \right) + \epsilon_n, & \text{otherwise}; \end{cases}$$

3) $\bar{F}_n(t_2) - \bar{F}_n(t_1) \geq c^T [F(t_2) - F(t_1)] + \epsilon_n$ for $t_1, t_2 \in T_n$ with $X_i \leq t_1 < t_2$, where $\bar{F}_n(t) = R_n(t)/n$.

In all the simulations, $\epsilon_n = \sqrt{\ln n/n}$ and $T_n$ consists of $[(\ln n)^2]$ equally spaced points with the first and last ones being $\min X_i$ and $\max X_i$.

The only difference between the above constraints and those in (3.2) is the modified upper bound $u(X_j)$ when $R_n(X_j) \leq n^{0.2}$. This aims to impose stronger constraint on $c_k$. In the definition of $u(X_j)$, $n^{0.2}$ can be changed to any $a_n = o(n)$ and the scale parameter $1/0.95$ to any $1/\beta$ with $\beta \in (0, 1)$. As Appendix A.4 shows, at control parameter $\alpha$, Procedure 3.1 using $p_i,\text{seq}$
computed under the above constraints obtains

\[
(4.1) \quad \text{FDR} \leq \alpha + r_n + E \left[ \frac{1\{R > 0\}}{R \vee 1} \right].
\]

with \( r_n \to 0 \) as \( n \to \infty \).

With similar modifications to (3.3), for \( i = 1, \ldots, n \), to compute \( p_{i,\text{glb}} \), the constrains on \( c_1, \ldots, c_L \) are

1. \( c_k \geq 0, \sum c_k \leq 1 \);
2. \( c^\top F(X_j) \leq u(X_j) \) for all \( X_j \geq X_i \); and
3. \( F_n(t_2) - F_n(t_1) \geq c^\top [F(t_2) - F(t_1)] + \epsilon_n \) for all \( t_1, t_2 \in T_n \) with \( t_1 < t_2 \).

We then apply the BH procedure to the above \( p \)-values, specifically, Procedure 3.1 to \( p_{i,\text{seq}} \), Procedure 3.2 to \( p_{i,\text{glb}} \), Procedure 2.2 to \( p_{i,\text{max}} \), and the BH procedure to \( p_{i,\text{mix}} \). For each set of \( F_1, \ldots, F_L \) and \( G \), we draw 1000 iid samples of \( X_1, \ldots, X_n \) with \( n = 5000 \). In this case, \( r_n \leq 9.7 \times 10^{-3} \) in (4.1); see Appendix A.4. The power, FDR and pFDR of each procedure are calculated by averaging over the samples. Throughout, \( a = 0.05 \).

All the simulations are conducted in R language [13]; \( p_{i,\text{seq}} \) and \( p_{i,\text{glb}} \) are computed by the R linear programming package \texttt{glpk}.

4.2. Results. We conduct 5 groups of simulations. The parameters of the simulations are shown in Table 1.

\[
\begin{array}{|c|c|c|c|}
\hline
F_1, \ldots, F_L & \nu_1, \ldots, \nu_L & G \\
\hline
1 & N(0,1), N(-1,1), N(-2,1) & .75, .15, .1 & N(-4,1) \\
2 & t_{20, -1}, t_{20,-2} & .75, .15, .1 & t_{20,-4} \\
3 & N(0,1), N(-1,1), N(-2,1) & .6, .25, .15 & N(-4,1) \\
4 & N(0,1), N(-1,1.5), N(-2,1.5) & .75, .15, .1 & N(-4,1) \\
5 & N(\mu,1), \mu = 0, -1, -2, -3, -4 & .65, .15, .1, .05, .05 & N(-5,1) \\
\hline
\end{array}
\]

Table 1

Parameters for the simulations. \( F_k \) are null distributions, \( \nu_k \) their prior probabilities, and \( G \) the distribution under false nulls. In each simulation, \( a = 0.05 \). \( t_{n,c} \) denotes the noncentral \( t \) distribution with \( n \) df and noncentrality \( c \).

The results of the simulations are summarized in Table 2. In all the simulations, the control parameter \( \alpha \) is equal to 0.25. As expected, because \( p_{i,\text{mix}} \) incorporates \( \nu_1, \ldots, \nu_L \), which is information not accessible by the other types of \( p \)-values, they yield the highest power with substantial margin. On the other hand, \( p_{i,\text{seq}} \) and \( p_{i,\text{glb}} \) yield substantially higher power than \( p_{i,\text{max}} \). This shows that even when \( \nu_1, \ldots, \nu_L \) are unknown, by utilizing properties of empirical processes to reduce overestimation of \( p \)-values, the power of the BH procedure can still be significantly improved.

In agreement with known results [1, 15], the FDR attained by using \( p_{i,\text{mix}} \) or \( p_{i,\text{max}} \) is close to or lower than \((1 - a)\alpha = 0.2375\). However, the large gap between the FDR by using \( p_{i,\text{max}} \) and \((1 - a)\alpha \) indicates that testing based on \( p_{i,\text{max}} \) can be very conservative. On the other hand, in all the
simulations, the FDR attained by using $p_{i, \text{seq}}$ or $p_{i, \text{glb}}$ lies between the above two, substantially lower than the first one but substantially higher than the second. Together with the simulation result on power, this shows that multiple testing based on $p_{i, \text{seq}}$ and $p_{i, \text{glb}}$ is more conservative than based on $p_{i, \text{mix}}$, but can be much less conservative than based on $p_{i, \text{max}}$.

The conservativeness of multiple testing based on the $p$-values other than $p_{i, \text{mix}}$ does not necessarily help the control of pFDR. In simulations 1 and 3, for each type of $p$-value, the power is relatively high, implying $P(R \geq 1) \approx 1$. As a result, the pFDR is almost identical to the FDR. In simulations 2, 4 and 5, the power yielded by $p_{i, \text{max}}$ is low ($\leq .05$), and, consistent with this, the pFDR is substantially higher than the FDR. In contrast, in simulations 2 and 5, by using $p_{i, \text{seq}}$ or $p_{i, \text{glb}}$, the pFDR and FDR are similar to each other. The worst case is simulation 4, where the pFDR is almost twice as high as the control parameter $\alpha = .25$ when $p_{i, \text{seq}}$ or $p_{i, \text{glb}}$ are used. Observe that in simulation 4, negative observations with large absolute values are more likely to be associated with true nulls than with false nulls. This explains the poor control of the pFDR by the BH procedure using $p_{i, \text{seq}}$ or $p_{i, \text{glb}}$.

To see in more detail why $p_{i, \text{seq}}$ and $p_{i, \text{glb}}$ in general yield better multiple testing results than $p_{i, \text{max}}$, we compare the plots of the $p$-values. Because all the procedures in the study are variants of the BH procedure, it is more informative to compare the plots of $p(i)/(i/n) = np(i)/i$ than to compare those of $p(i)$, $i = 1, \ldots, n$, where $p(i)$ is the $i$th smallest $p$-value of a given

| simul. | 1 | 2 |
|--------|---|---|
|        | power FDR pFDR | power FDR pFDR |
| $p_{i, \text{seq}}$ | .495 $8.61 \times 10^{-2}$ | .236 $8.57 \times 10^{-2}$ |
| $p_{i, \text{glb}}$ | .494 $8.60 \times 10^{-2}$ | .235 $8.57 \times 10^{-2}$ |
| $p_{i, \text{max}}$ | .223 $2.55 \times 10^{-2}$ | .035 $2.87 \times 10^{-2}$ |
| $p_{i, \text{mix}}$ | .770 .238 | .634 .238 |

| simul. | 3 | 4 |
|--------|---|---|
|        | power FDR pFDR | power FDR pFDR |
| $p_{i, \text{seq}}$ | .449 .103 | 4.82 $10^{-4}$ .465 |
| $p_{i, \text{glb}}$ | .449 .102 | 4.82 $10^{-4}$ .465 |
| $p_{i, \text{max}}$ | .229 $3.77 \times 10^{-2}$ | 8.42 $10^{-2}$ .523 |
| $p_{i, \text{mix}}$ | .685 .236 | .144 .226 |

| simul. | 5 |
|--------|---|
|        | power FDR pFDR |
| $p_{i, \text{seq}}$ | 4.53 $10^{-2}$ 6.42 $10^{-2}$ 6.85 $10^{-2}$ |
| $p_{i, \text{glb}}$ | 4.62 $10^{-2}$ 6.51 $10^{-2}$ 6.94 $10^{-2}$ |
| $p_{i, \text{max}}$ | 3.22 $10^{-3}$ 1.68 $10^{-2}$ 4.00 $10^{-2}$ |
| $p_{i, \text{mix}}$ | .448 .239 .239 |

**Table 2**

Performance of the BH procedure applied to different types of $p$-values in simulations 1–5. In each simulation, the control parameter is set at $\alpha = 0.25$. 
type. Figures 1 display the plots of $np(i)/i$ versus $i/n$ in the simulations, where $\bar{p}(i)$ is the average over the repetitions. The figure clearly shows that for small $i/n$, $np(i)/i$ and $np(i)_{\text{seq}}/i$ are similar to each other, both are substantially lower than $np(i)_{\text{max}}/i$, and both increase more rapidly than $np(i)_{\text{mix}}/i$. This is consistent with the observation that multiple testing using $p_{i,\text{seq}}$ and that using $p_{i,\text{glob}}$ perform similarly in terms of power, FDR and pFDR, and in general both have higher power than multiple testing using $p_{i,\text{max}}$ at the same value of $\alpha$.

We next look at how $p_{i,\text{seq}}$ and $p_{i,\text{glob}}$ are computed by linear programming. For each $p_{i,\text{seq}}$ or $p_{i,\text{glob}}$, denote by $c_1, \ldots, c_{L,i}$ the values of coefficients that yield $p$-values under the corresponding constraints. After the $p$-values are sorted, let $c_{k,(i)}$ be the values corresponding to $p_{(i),\text{seq}}$ or $p_{(i),\text{glob}}$. We plot $c_{k,(i)}$ versus $i/n$ for $k = 1, \ldots, L$. Figure 2 shows the plots for simulations 1 and 5. The plots for the other simulations are qualitatively similar. As can be seen, although $p_{i,\text{seq}}$ and $p_{i,\text{glob}}$ in the simulations are similar, this is not the case for the corresponding coefficients $c_{k,i}$. For each $k$, when $i/n$ is small, $c_{k,i}$ for the two types of $p$-values are similar. However, as $i/n$ increases, to compute $p_{i,\text{seq}}$, essentially only one $c_k$ stays nonzero. In all the simulations, this unique $c_k$ is associated with the last null distribution of the null, i.e., $F_L$, which also has the smallest sup-norm distance from $G$ among all $F_k$. In contrast, to compute $p_{i,\text{glob}}$, more complicated combinations of $c_1, \ldots, c_L$ are picked. This difference between the coefficients for $p_{i,\text{seq}}$ and $p_{i,\text{glob}}$ may be partially attributed to how linear programming is implemented by the package used. However, it also indicates linear programming may not yield consistent estimation of $c_1, \ldots, c_L$.

Note that in Figure 2, for small $i/n$, the sum of $c_{k,(i)}$ is quite smaller than 0.4. Since $a = 1 - \sum c_k$, this would imply the fraction of false nulls could be as high as 0.6, which is improbable in many cases. This raises the possibility that, by imposing some constraint on the sum of $c_k$, the power may be improved. Recall that $a = 0.05$ in the simulation study. We simulate the scenario where it is known that $a \leq 0.1$. For both $p_{i,\text{seq}}$ and $p_{i,\text{glob}}$, the first constraint on $c_1, \ldots, c_L$ is expanded to become

$$1^\circ c_k \geq 0, \ 0.9 \leq \sum c_k \leq 1.$$  

Denote the $p$-values computed with the expanded linear constraints by $p'_{i,\text{seq}}$ and $p'_{i,\text{glob}}$, and those computed previously still by $p_{i,\text{seq}}$ and $p_{i,\text{glob}}$. In Table 3, the power and pFDR of the BH procedures when applied to the $p$-values are compared. In all the cases, the FDR is substantially lower than $(1-a)\alpha = 0.2375$ and hence not shown. In place of FDR, the standard deviation of $\frac{R-V}{n-N}$ over 1000 repetitions is reported. Recall $R$ is the number of rejections, $V$ that of false rejections, $n = 5000$ is the total number of nulls, and $N$ is number of true nulls. In simulations 1–3, there is a small but significant increase in power by using $p'_{i,\text{seq}}$ and $p'_{i,\text{glob}}$. This is not the case in simulations 4 and 5, where the power is very low for all the 4 types of $p$-values.
In Figure 3, we compare the plots of \( np_{(i)}/i \) for the \( p \)-values. Since all rejections occur when \( i \ll n \), we only compare the plots with \( i/n \leq 0.05 \). It is seen that for small \( i/n \), the plots for \( p_{i, \text{seq}} \) and \( p_{i, \text{glb}} \) are very close to each other, explaining why the performances of the BH procedure based on the two types of \( p \)-value are similar. Likewise, the plots for \( p'_{i, \text{seq}} \) and \( p'_{i, \text{glb}} \) are very close to each other, and in simulations 1–3, both are significantly lower than the plots of \( p_{i, \text{seq}} \) and \( p_{i, \text{glb}} \), which explains the improved power yielded by \( p'_{i, \text{seq}} \) and \( p'_{i, \text{glb}} \). Finally, comparing Figures 2 and 4, we can see that the extra constraint \( c_1 + \cdots + c_{\ell} \geq 0.9 \) substantially changes the plots of the coefficients. In particular, for \( p'_{i, \text{seq}} \) with \( i \ll n \), the linear programming sets two coefficients nonzero, as opposed to only one for \( p_{i, \text{seq}} \).

From the above results, it is seen that the performances of the BH procedure based on \( p_{i, \text{seq}} \) and \( p_{i, \text{glb}} \) are close to each other, even though the latter one are subject to more constraints. The reason seems to lie in how \( p_{(i), \text{seq}} \) are computed. The evaluation of \( p_{(i), \text{seq}} \) incorporates the constraints imposed by \( s_{(j)} \) with \( j \geq i \). For small \( i \), the set of constraints is only different by a small fraction from those that are imposed by the entire set of \( s_{(j)} \). Under regular conditions, constraints imposed by \( s_{(j)} \) with \( j < i \) will not change the maximization substantially. This implies that for small \( i \), \( p_{(i), \text{seq}} \) and \( p_{(i), \text{glb}} \) are close to each other, as can be seen from Figure 3. Since the

| simul. | power | SD \( \frac{\hat{R} - V}{n - N} \) | pFDR | 2 | power | SD \( \frac{\hat{R} - V}{n - N} \) | pFDR |
|-------|-------|----------------|-------|---|-------|----------------|-------|
| \( p_{i, \text{seq}} \) | .495  | 5.20 \times 10^{-2} | 8.61 \times 10^{-2} | \( p_{i, \text{glb}} \) | .494  | 5.20 \times 10^{-2} | 8.60 \times 10^{-2} |
| \( p'_{i, \text{seq}} \) | .541  | 5.25 \times 10^{-2} | .103  | \( p'_{i, \text{glb}} \) | .541  | 5.25 \times 10^{-2} | .103  |
| simul. | 3 |  | | 4 |  |  |
| \( p_{i, \text{seq}} \) | .449  | 5.35 \times 10^{-2} | .103  | \( p_{i, \text{glb}} \) | .449  | 5.35 \times 10^{-2} | .102  |
| \( p'_{i, \text{seq}} \) | .473  | 5.38 \times 10^{-2} | .113  | \( p'_{i, \text{glb}} \) | .473  | 5.38 \times 10^{-2} | .113  |
| simul. | 5 |  | |  |  |  |
| \( p_{i, \text{seq}} \) | 4.53 \times 10^{-2} | 3.38 \times 10^{-2} | 6.85 \times 10^{-2} | \( p_{i, \text{glb}} \) | 4.62 \times 10^{-2} | 4.53 \times 10^{-2} | 6.94 \times 10^{-2} |
| \( p'_{i, \text{seq}} \) | 4.65 \times 10^{-2} | 3.32 \times 10^{-2} | 6.89 \times 10^{-2} | \( p'_{i, \text{glb}} \) | 4.65 \times 10^{-2} | 3.32 \times 10^{-2} | 6.89 \times 10^{-2} |

Table 3 Performance of the BH procedure applied to \( p \)-values computed under different linear constraints: \( p_{i, \text{seq}} \) and \( p_{i, \text{glb}} \) are the same as in Table 2, \( p'_{i, \text{seq}} \) and \( p'_{i, \text{glb}} \) are computed with the additional constraint \( c_1 + \cdots + c_{\ell} \geq 0.9 \). For each simulation, \( \frac{R - V}{n - N} \) is the fraction of rejected false nulls among all false nulls in a repetition. The SD is obtained over 1000 repetitions.
BH procedure only reject nulls with small \( p \)-values, its performance based on either type of \( p \)-values will be similar.

5. MLE for prior probabilities of nulls. Let \( \Theta = \{ \theta_1, \ldots, \theta_L \} \). As indicated in Section 4, for composite nulls, in general the prior \( \nu \) may not be estimated consistently. In this section, we consider under what conditions \( \nu \) can be estimated consistently. Under the setup in Section 1, suppose each \( F_k \) has a density \( f_k \) and \( G \) is absolutely continuous with respect to the distribution under true nulls. By the Radon-Nikodym theorem, \( G \) has a density \( \rho(x) \nu^\top f(x) \) with \( \rho(x) \geq 0 \). Then the data \( X_1, \ldots, X_n \) are iid with density

\[
q(x) = [1 - a + a \rho(x)] \nu^\top f(x).
\]

Pretending all the nulls are true, the MLE for \( \nu \) is

\[
\hat{\nu}_n = \arg\sup_{\nu \in S} \sum_{i=1}^n \ln[c^\top f(X_i)],
\]

where \( S \) is a suitable set. Usually, one would choose \( S = \{ c \in [0,1]^L : \sum c_k = 1 \} \) because by the definition of prior probabilities, \( \nu_k \geq 0 \) and \( \sum \nu_k = 1 \). For the reason described below, we shall make the setting a little more general. Still suppose that the distribution under true nulls is a linear combination of \( F_k \). However, now \( \nu_k \) are allowed to be negative. In this setting, it had better merely regard \( f_k \) as a basis for a set of densities. Then set

\[
(5.1) \quad S = \{ c : c_1 + \cdots + c_L = 1, \, c^\top f \geq 0 \}.
\]

A reason for this choice of \( S \) can be seen when density functions under true nulls are linearly dependent. In this case, it is desirable to pick a basis from them, say \( f_1, \ldots, f_L \), and represent the others as \( g_j = \sum \lambda_{jk} f_k \). By linear dependence, \( \lambda_{jk} \) can be negative. Let the mixture density under true nulls be \( a^\top f + b^\top g \), with \( \sum a_k + \sum b_j = 1 \) and \( a_k, b_j \geq 0 \). By representing it as \( \nu^\top f \), we get \( \nu_k = a_k + \sum b_j \lambda_{jk} \), which can be negative. On the other hand, \( \sum \nu_k = 1 \) and \( \nu^\top f \geq 0 \). Therefore, \( S \) in (5.1) contains \( \nu \).

Recall that if \( A \subset \mathbb{R}^d \), its interior is \( A^o = \{ x : B(x,r) \subset A \text{ for some } r > 0 \} \), where \( B(x,r) = \{ z : |z_k - x_k| < r, k = 1, \ldots, d \} \). By this definition, \( S^o = \emptyset \). However, regarding \( S \) as a subset in \( \{ c : \sum c_k = 1 \} \), we have \( S^o = \{ c : \text{for some } r > 0, c + v \in S \forall v \in B(0,r) \text{ with } \sum v_k = 0 \} \). Both \( S \) and \( S^o \) are convex. Since \( S \) contains all \( c \) with \( c_k \geq 0 \) and \( \sum c_k = 1 \), \( S^o \neq \emptyset \).

**Proposition 5.1.** Suppose \( \int q | \ln f_k | < \infty \) and \( f_1, \ldots, f_L \) are linearly independent. Let \( a \in (0,1) \). If \( \nu \in S^o \), then

\[
\hat{\nu}_n \overset{P}{\to} \nu \iff \int p f_k = 1 \text{ for all } k.
\]
Apparently, if \( \rho = 1 \), then \( \int \rho f_k = 1 \). A question is whether nontrivial \( \rho \geq 0 \) satisfying the condition exists. Since \( \int \rho (f_k - f_1) = 0 \), provided \( f_k \in L^2 \), one might search for \( \rho \) among functions in \( L^2 \) that are orthogonal to \( f_k - f_1 \). However, such functions are not always nonnegative. Moreover, oftentimes \( f_k \not\in L^2 \). The construction below avoids these potential problems and seems to be general.

**Example 5.1.** We only consider how to construct \( \rho \geq 0 \) that are unbounded on \( E = \{ x : \nu^T f(x) > 0 \} \). The general case follows the same idea. The main step is to find bounded \( \psi_1, \ldots, \psi_L \in C(\mathbb{R}^d) \), such that the \( L \times L \) matrix \( M = (M_{ik}) \) is nonsingular, where \( M_{ik} = \int \psi_i f_k \). Once this is done, to construct \( \rho \), fix \( \phi \geq 0 \) continuous such that \( \int \phi f_k < \infty \) and \( \sup_{x \in E} \phi(x) = \infty \). Such \( \phi \) always exist. By \( \det M \neq 0 \), there are unique \( a_1, \ldots, a_L \in \mathbb{R} \), such that \( \sum a_i M_{ik} = 1 - \int \phi f_k \) for each \( k \). Then \( \int h f_k = 1 \), where \( h = \phi + \sum a_i \psi_i \). It is easy to see \( h \in C(\mathbb{R}^d) \) is lower bounded and \( \sup_{x \in E} h(x) = \infty \). Then for \( c > 0 \) small enough, \( \rho = 1 - c + ch \in C(\mathbb{R}^d) \) is nonnegative with \( \sup_{x \in E} \rho(x) = \infty \) and \( \int \rho f_k = 1 - c + c \int h f_k = 1 \).

To see that \( \psi_1, \ldots, \psi_L \) as above exist, recall

\[
\det M = \sum_{\sigma} \text{sgn}(\sigma) \prod_k \int f_{\sigma(k)} \psi_k \\
= \int \sum_{\sigma} \text{sgn}(\sigma) \prod_k f_{\sigma(k)}(x_k) \psi_k(x_k) \, dx \\
= \int \prod_k \psi_k(x_k) \det[f_i(x_k)] \, dx.
\]

where the sum is over all permutations \( \sigma \) of \( 1, \ldots, L \) and \( \text{sgn}(\sigma) \) is the sign of \( \sigma \). Denote \( D(x) = \det[f_i(x_k)] \). Since \( |D(x)| \leq \sum_{\sigma} \prod_k f_{\sigma(k)}(x_k) \), \( D \in L^1 \).

Because \( f_1, \ldots, f_L \) are linearly independent, we claim

\[
(5.2) \quad \ell(x : D(x) = 0) \neq 0,
\]

where \( \ell \) is the Lebesgue measure. If (5.2) holds, then the characteristic function of \( D \) is nonzero. Therefore, there are \( t_1, \ldots, t_L \neq 0 \), such that \( \int e^{i(t_1 x_1 + \cdots + t_L x_L)} D(x) \, dx \neq 0 \). It follows that there are \( \psi_k(x) \) of the form \( \sin(t_k x) \) or \( \cos(t_k x) \), such that \( \det M \neq 0 \).

We use induction to prove (5.2). For \( L = 2 \), if \( D(x) = 0 \) a.e., then \( f_1(x_1) f_2(x_2) = f_1(x_2) f_2(x_1) \) a.e. Integrating over \( x_2 \) yields \( f_1(x_1) = f_2(x_1) \) a.e., contradicting the assumption that \( f_1 \) and \( f_2 \) are linearly independent.

For \( L > 2 \), suppose (5.2) holds for \( L - 1 \) linearly independent \( f_i \). Now

\[
D(x) = \sum_{i=1}^{L} (-1)^{L+i} f_i(x_L) M_i(x_1, \ldots, x_{L-1}),
\]

where \( M_i(x_1, \ldots, x_{L-1}) \) is the determinant of the \((L - 1) \times (L - 1)\) matrix consisting of \( f_l(x_k) \), \( l \neq i \), \( k = 1, \ldots, L - 1 \). Given \( x_1, \ldots, x_{L-1} \), \( D(x) \) is a
linear combination of $f_i(x_L)$. Therefore, if $D(x) = 0$ a.e., then, by the linear independence of $f_i(x)$, $M_i(x_1, \ldots, x_{L-1}) = 0$ a.e. for each $i = 1, \ldots, L$. However, this contradicts the induction hypothesis.

6. Discussion. In the article, we have focused on the case of finitely composite nulls, where true nulls are only associated with a finite number of distributions. Formally, it is straightforward to generalize the constrained maximization to the case of infinitely composite nulls. However, usually the maximization will involve infinitely many degrees of freedom and it becomes unclear how to accommodate this with a finite number of observations. A more direct approach might be to partition the set of null distributions into a finite number of subsets and use the envelopes of the subsets to compute $p$-values. To be more specific, given a partition $\Theta_1, \ldots, \Theta_L$ of $\Theta$, let $u_k(t) = \sup_{\theta \in \Theta_k} \phi_\theta(t)$ and $l_k(t) = \inf_{\theta \in \Theta_k} \phi_\theta(t)$. Then, define, for example, $M_n(t) = \sup \{ c^\top u(t) : c \in \Delta, c^\top l(t) \}$ is dominated by $F_n(t)$ up to a small margin. Unfortunately, some of the constraints available to the finitely composite case can no longer be used. Another issue is how to select the partition. Too coarse partition will only yield loose constraints on $c_k$ and too fine partition will result in many degrees of freedom. Either way, the obtained $M_n(t)$ may not be much different from the unconstrained maximum probability.

As is known, FDR control can be realized by the local FDRs [5]. For the simple case, the local FDR at $x$ is $(1-a)f_0(x)/h(x)$, where $a$ may be replaced with $0$, $f_0$ is the density under true nulls, and $h$ is the overall density of the data $X_1, \ldots, X_n$ or an estimate of the density. For the finitely composite case where the null distributions have densities $f_1, \ldots, f_L$, we may derive a conservative estimate of the local FDR by $\rho(x)/h(x)$, where

$$\rho(x) = \sup \{ c^\top f(x) : c \in \Delta \text{ and } c^\top f \leq h \}.$$  

Alternatively, if the dimension of $X_i$ is high, then we may work on $s_i = s(X_i)$, with the local FDR defined as $\rho(s_i)/f(s_i)$, where $h$ is now the overall density of $s_1, \ldots, s_n$ or an estimate, while

$$\rho(t) = \sup \{ c^\top \phi(t) : c \in \Delta \text{ and } c^\top \phi \leq h \}.$$  

It is worth pointing out that, unlike the simple case, the BH procedure based on $M_n(t)$ and the FDR control based on $\rho(x)/h(x)$ are no longer equivalent. The reason is that $M_n(t)$ is of the form $\max_{c} c^\top \phi$. The density of $M_n(t)$, if existent, in general is different from $\max_{c} c^\top \phi$ that is associated with the local FDR. It remains to be seen how much difference the two approaches may have.

Appendix. In this section, we give proofs of the theoretical statements of the article. The Lebesgue measure on $\mathbb{R}^d$ will be denoted by $\ell$. For any nondecreasing function $f$ defined on $\mathbb{R}$ and $x \in \mathbb{R}$, if $A := \sup \{ t : f(t) \leq x \} \neq \emptyset$, define $f^*(x) = \sup A$, otherwise, define $f^*(x) = -\infty$. By this definition, if $f$ is left-continuous and $x \in f(I)$, then $f(f^*(x)) = x$.  


A.1. Proofs for Section 2.

Proof of Proposition 2.1. Since $s_i = -\infty \iff X_i \in D_t$ for all $t$, by D3 and the random mixture model, the probability of the event is 0, hence proving 1). By the right-continuity of $D_t$,

$$s_i \leq t \iff X_i \in D_s \text{ for all } s > t \iff X_i \in D_t,$$

yielding 2). By $P(s_i \leq t) = P(X_i \in D_t) = \phi_\theta(t)$, 3) holds and 4) follows from 3) and the random mixture model. To get 5), given $t$, for any $\epsilon > 0$, there is $\theta \in \Theta$ such that $M(t) \leq \phi_\theta(t) + \epsilon$. By D3, $M(s) \geq \phi_\theta(s) \rightarrow \phi_\theta(t)$ as $s \uparrow t$, giving $M(t-)+\epsilon \geq M(t)$. Since $M$ is nondecreasing and $\epsilon$ is arbitrary, this implies $M(t-) = M(t)$.

Proof of Proposition 2.2. To see that Procedures 2.1 and 2.2 are the same, by Proposition 2.1,

Procedure 2.1 accepts $H_i \iff s_i > \tau \iff M(t) \frac{\alpha}{\tau} > \frac{R_n(t)}{n} \forall t \geq s_i$.

Because $M(t)$ is nondecreasing and $R_n(t)$ is a nondecreasing step function that has jumps only at $s_i$,

Procedure 2.1 accepts $H_i \iff \frac{M(s_j)}{\alpha} > \frac{R_n(s_j)}{n} \forall s_j \geq s_i$.

Taking into account the possibility of ties, it is not hard to see that the condition on the right hand side is equivalent to $s_i > s(\tau)$, which implies Procedures 2.1 and 2.2 always reject the same set of nulls.

By the random mixture model, for $X_i$ under true nulls, the distribution of $F_{\eta_i}(D_{s_i})$ is a mixture of those of $\phi_\theta(s(X))$ under $F_\theta$, $\theta \in \Theta$. By Proposition 2.1, under $F_\theta$, $\phi_\theta(s(X)) \sim \text{Unif}(0,1)$. Therefore, for $X_i$ under true nulls, $s_i$ are iid $\sim \text{Unif}(0,1)$.

Procedure 2.2 is the BH procedure applied to $M(s_i)$. Since $M(s_i) \geq F_{\eta_i}(D_{s_i})$, under true nulls, $P(M(s_i) \leq x) \leq P(F_{\eta_i}(D_{s_i}) \leq x) = x$. The proof then follows from Theorem 5.1 and the comment that follows in [3].

A.2. Proofs for Section 3. First, note that for Procedures 3.1 and 3.2, the number of rejections and that of false rejections are $R = R_n(\tau)$ and $V = V_n(\tau)$, respectively.

Proof of Lemma 3.1. Let $s < t$. Then $A_{n,s} \subset A_{n,t}$ and $c^T\phi(s) \leq c^T\phi(t)$ for any $c \in \Delta$, giving $M_n(s) \leq M_n(t)$. Thus $M_n$ is nondecreasing. Next suppose $\phi_i \in C(\mathbb{R})$ for all $i$.

1) Given $t$, as $0 < t - u \ll 1$, $[u,t)$ has no point in $\tau_n$ and, almost surely, no $s_i$. Thus $A_{n,u} = A_{n,t}$. Let $K = \{c \in \Delta : c^T\phi \in A_{n,t}\}$. It is seen that $K$ is compact and $c^T\phi(s)$ is a uniformly continuous function in $(c,s) \in K \times \mathcal{I}$. 
Then $\sup_{c \in K} c^\top \phi(s)$ is continuous in $s$, yielding $M_n(u) \to M_n(t)$ as $u \uparrow t$. Thus $M_n$ is left-continuous.

2) Since $M_n$ is nondecreasing, $M_n$ has a right-hand limit at every $t$. It only remains to be shown that at every $t \notin \{s_1, \ldots, s_n\}$, $M_n$ is right-continuous. Now, as $0 < u - t \ll 1$, $[t, u)$ contains no point in $T_n$ and no $s_i$, yielding $A_{n,u} = A_{n,t}$. Then the right-continuity follows from the same argument for the left-continuity.

In addition to Lemma 3.1, we need a few lemmas to prove Theorem 3.1. For $t \in \mathcal{I}$, define $\sigma$-field

$$\mathcal{F}_t = \mathcal{F}(R_n(t-), V_n(t-), R_n(s), V_n(s) : s \geq t).$$

Then $\{\mathcal{F}_t, t \in \mathcal{I}\}$ is a backward filtration, i.e., $\mathcal{F}_t \subset \mathcal{F}_s$ for $t > s$.

**Lemma A.2.1.** Suppose $\phi_i \in C(\mathbb{R})$ for all $i$. Then for $t \in \mathbb{R}$, $M_n(t)$ is $\mathcal{F}_t$-measurable.

**Proof.** It suffices to show that given $a \geq 0$, $\{M_n(t) \leq a\} \in \mathcal{F}_t$ for $t \in \mathcal{I}$. For $c \in \Delta$, $c^\top \phi \in C(\mathbb{R})$ and $\{c^\top \phi \in A_{n,t}\} = E_1 \cap E_2$, where

$$E_1 = \left\{ c^\top \phi(s_i) \leq \mathbb{F}_n(s_i) + \epsilon_n \text{ for } s_i \geq t \right\},$$

and

$$E_2 = \left\{ \mathbb{F}_n(t_2) - \mathbb{F}_n(t_1) \geq c^\top \phi(t_2) - \phi(t_1) - \epsilon_n \text{ for } t_i \in T_n \cap [t, \tau_2] \text{ with } t_1 < t_2 \right\}.$$  

Note $E_1 = \{c^\top \phi(s) \leq R_n(s)/n + \epsilon_n \forall s \geq t \text{ with } R_n(s) > R_n(s-))$. Since $R_n(s- \in \mathcal{F}_t$ for $s \geq t$, it can be seen that $E_1 \in \mathcal{F}_t$. On the other hand, $E_2 \in \mathcal{F}_t$. Therefore, $\{c^\top \phi \in A_{n,t}\} \in \mathcal{F}_t$.

Since $c^\top \phi \in A_{n,t}$ implies $r^\top \phi \in A_{n,t}$ for any $r \in \mathbb{Q}^t \cap \Delta$ with $r_i \leq c_i$, where $\mathbb{Q}$ is the set of rational numbers, $M_n(t) = \sup\{r^\top \phi(t) : r \in \mathbb{Q}^t \cap \Delta, \ r^\top \phi \in A_{n,t}\}$. Notice that $r^\top \phi(t)$ is nonrandom. Then

$$\{M_n(t) \leq a\} = \bigcap_{r \in \mathbb{Q}^t \cap \Delta \text{ s.t. } r^\top \phi(t) > a} \{r^\top \phi \not\in A_{n,t}\} \in \mathcal{F}_t. \quad \Box$$

The next goal is to show $\tau$ is a stopping time of the backward filtration $\mathcal{F}_t$. If $\sup \mathcal{I} = \infty$, then $\tau$ has to start at $\infty$. To get around this problem, we use truncations. Let $\mathcal{I}_R$ be as in Procedure 3.1. Given $c < \sup \mathcal{I}$, define

$$\mathcal{I}_c = \mathcal{I}_R \cap (-\infty, c], \quad \tau_c = \begin{cases} \sup \mathcal{I}_c & \text{if } \mathcal{I}_c \neq \emptyset \\ \inf \mathcal{I} & \text{otherwise} \end{cases}$$

**Lemma A.2.2.** As $c \uparrow \sup \mathcal{I}$, $\tau_c \uparrow \tau$ a.s.

**Proof.** It suffices to show $\tau < \sup \mathcal{I}$ a.s. By definition, $\tau \leq \sup \mathcal{I}$. The event $\{\tau = \sup \mathcal{I}\}$ implies there are $t_k \uparrow \sup \mathcal{I}$, such that $M_n(t_k) \leq \alpha[R_n(t_k) \vee 1] / n$. By Lemma 3.1, $M_n(t_k) \to M_n(\sup \mathcal{I}) = 1$ a.s. On the other hand, $[R_n(t_k) \vee 1] / n \leq 1$. Therefore, $P(\tau = \sup \mathcal{I}) = 0$. \[\Box\]
Lemma A.2.3. Suppose $\phi_k \in C(\mathbb{R})$. Then 1) there is $t_0 > \inf \mathcal{I}$ such that for any $c \in \mathcal{I}$, $\tau_c \geq t_0$, 2) for $c \in \mathcal{I}$, $\tau_c$ is a stopping time of the backward filtration $\{\mathcal{F}_t, t \in (\inf \mathcal{I}, c]\}$.

Proof. 1) Let $u(t) := \phi_1(t) + \cdots + \phi_L(t)$. Then $M_n(t) \leq u(t)$ and

$$\tau_c \geq \sup \left\{ t \in (\inf \mathcal{I}, c) : \frac{u(t)}{\alpha} \leq \frac{R_n(t) \vee 1}{n} \right\} \geq t_0 := \sup \left\{ t \in (\inf \mathcal{I}, c) : \frac{u(t)}{\alpha} \leq \frac{1}{n} \right\}.$$  

Since $\phi_k \in C(\mathbb{R})$ and $\phi_k(t) \to 0$ as $t \to \inf \mathcal{I}$, the set on the right hand side is nonempty, yielding $t_0 > \inf \mathcal{I}$.

2) By definition, $\tau_c$ is a stopping time of the backward filtration $\mathcal{F}_t$ if $\{\tau_c \geq t\} \in \mathcal{F}_t$ for every $t \in (\inf \mathcal{I}, c]$. Denote $E = \{\tau_c \geq t\}$. We first show

$$E = \left\{ \exists s \in [t, c] \text{ such that } \frac{M_n(s)}{\alpha} \leq \frac{R_n(s) \vee 1}{n} \right\}. \tag{A.1}$$

The right hand side of (A.1) equals $\{\mathcal{I}_c \cap [t, c] \neq \emptyset\}$, which is a subset of $E$. On the other hand, the difference between the two events is

$$\{\tau_c \geq t, \mathcal{I}_c \cap [t, c] = \emptyset\} = \{\mathcal{I}_c \neq \emptyset, \mathcal{I}_c \cap [t, c] = \emptyset, \sup \mathcal{I}_c \geq t\} \subset \left\{ \frac{M_n(t)}{\alpha} > \frac{R_n(t) \vee 1}{n}, \exists t_k \uparrow t \text{ with } \frac{M_n(t_k)}{\alpha} \leq \frac{R_n(t_k) \vee 1}{n} \right\}.$$  

Since by Lemma 3.1 $M_n$ is left-continuous, $M_n(t_k) \to M_n(t)$. On the other hand, $R_n(t_k) \to R_n(t -) \leq R_n(t)$. Thus, the last event is empty and (A.1) holds. Note that by similar argument,

$$M_n(\tau_c)/\alpha \leq [R_n(\tau_c) \vee 1]/n. \tag{A.2}$$

Let $A = \{M_n(t)/\alpha \leq [R_n(t) \vee 1]/n\}$. Then $A \subset E$ and $A \in \mathcal{F}_t$. We next show $E = A \cup \Gamma$, where $\Gamma = \bigcap_{k=1}^{\infty} \bigcup_{r \in \mathbb{Q} \cap (t, c]} \Gamma_{r,k}$, with

$$\Gamma_{r,k} = \left\{ \frac{M_n(r)}{\alpha} \leq \frac{R_n(r + 1/k) \vee 1}{n} \right\}.$$  

Once this is done, by $M_n(r) \in \mathcal{F}_r$ (cf. Lemma A.2.1) and $R_n(r + 1/k) \in \mathcal{F}_r$, $\Gamma_{r,k} \in \mathcal{F}_r \subset \mathcal{F}_t$ for any $r > t$. Then $E \in \mathcal{F}_t$.

Note $E - A$ implies $\tau_c > t$, which in turn implies there are $r_k \in \mathbb{Q}$ with $t < r_k < \tau_c < r_k + 1/k$. By

$$\frac{M_n(r_k)}{\alpha} \leq \frac{M_n(\tau_c)}{\alpha} \leq \frac{R_n(\tau_c) \vee 1}{n} \leq \frac{R_n(r_k + 1/k) \vee 1}{n},$$

$\Gamma_{r_k,k}$ holds for all $k$. Thus $E - A \subset \Gamma$. 

It remains to show that $\Gamma \subset E$. Suppose there are $r_k \in Q \cap (t, c]$ with $M_n(r_k)/\alpha \leq [R_n(r_k + 1/k) \vee 1]/n$. Then $r_k$ has a subsequence, say, itself, that converges to some $s \in [t, c]$. Since $M_n$ is nondecreasing and left-continuous, while $R_n$ is nondecreasing and right-continuous,

$$\frac{M_n(s)}{\alpha} \leq \lim_{k} \frac{M_n(r_k)}{\alpha} \leq \lim_{k} \frac{R_n(r_k + 1/k) \vee 1}{n} \leq \frac{R_n(s) \vee 1}{n}.$$

Therefore $s \in T_c$ and $T_c \cap [t, c] \neq \emptyset$. Thus $\Gamma \subset E$. \qed

**Lemma A.2.4.** For $n \geq 1$, denote

$$(A.3) \quad \Gamma_n = \{(1-a)\nu^\top \phi \in A_{n,t}, \forall t \in T\}.$$ 

Suppose $Q$ is continuous. Then, provided $\exp(-2n\epsilon_n^2) \leq 1/2$,

$$P(\Gamma_n) \geq 1 - (1 + |T_n|) \exp\left\{-2n\epsilon_n^2\right\}.$$ 

**Proof.** Since $Q$ is continuous, by the DKW inequality [12], for $\lambda > 0$ and $n \geq 1$, as long as $\exp(-2n\lambda^2) \leq 1/2$,

$$P\{\sup(Q - F_n) \geq \lambda\} \leq \exp(-2n\lambda^2).$$

By $Q(t) = (1-a)\nu^\top \phi(t) + aG(D_t)$,

$$P\left\{(1-a)\nu^\top \phi(t) \geq F_n(t) + \lambda \text{ for some } t\right\} \leq \exp(-2n\lambda^2).$$

DKW inequality also implies that, given $x \in \mathbb{R}$,

$$(A.4) \quad P\left\{\sup_{t \geq x} \{(Q(t) - Q(x)) - [F_n(t) - F_n(x)]\} \geq \lambda\right\} \leq \exp(-2n\lambda^2).$$

Assuming (A.4) is true for now, it follows that

$$P\left\{Q(t) - Q(t_i) \geq F_n(t) - F_n(t_i) + \lambda \right\} \leq |T_n| \exp(-2n\lambda^2).$$

Since $Q(t) - Q(t_i) \geq (1-a)\nu^\top [\phi(t) - \phi(t_i)]$ for $t > t_i$, by letting $\lambda = \epsilon_n$, the Lemma then follows.

Finally, to get (A.4), let $y = Q(x)$. By quantile transformation,

$$\sup_{t \geq x} \{(Q(t) - Q(x)) - [F_n(t) - F_n(x)]\} \sim \xi = \sup_{s \geq y} \{s - y - [G_n(s) - G_n(y)]\},$$

where $G_n$ is the empirical distribution of $U_i = Q(X_i)$. Since $U_i$ are iid $\sim\text{Unif}(0, 1)$, $V_i = U_i - y + 1 \{U_i \leq y\}$ are iid $\sim\text{Unif}(0, 1)$ as well and $\xi = \sup_{0 \leq s \leq 1-y} [s - G_n'(s)]$, where $G_n'$ is the empirical distribution of $V_i$. Applying DKW inequality to $\xi$, it is seen that (A.4) follows. \qed
PROOF OF THEOREM 3.1. By Proposition 2.1, under true $H_i, s_i \sim \nu^\top \phi$, which is continuous and positive on $\mathcal{I}$. As a result, $\{V(t-) / \nu^\top \phi(t), \mathcal{F}_t, t \in \mathcal{I}\}$ is a left-continuous backward martingale.

Fix $c \in \mathcal{I}$. By Lemma A.2.3, $\tau_c$ is a stopping time of $\{\mathcal{F}_t, t \in (\inf \mathcal{I}, c]\}$ with $\tau_c \geq t_0 > \inf \mathcal{I}$. Then $\nu^\top \phi(\tau_c) > 0$ and $V_n(\tau_c-)/[\nu^\top \phi(\tau_c)]$ is well-defined. By the optional sampling theorem (cf. [9], Ch. 1, Thm 3.22),

$$E \left[ \frac{V_n(\tau_c-)}{\nu^\top \phi(\tau_c)} \right] = E \left[ \frac{V_n(c-)}{\nu^\top \phi(c)} \right] = (1 - a)n.$$

Let $c \uparrow \sup \mathcal{I}$. By Lemma A.2.2, $\tau_c \uparrow \tau$. Because $V_n(\tau_c-) \uparrow V_n(\tau-) \leq n$, $\phi_k(\tau_c) \uparrow \phi_k(\tau)$ and $\nu^\top \phi(\tau_c) \geq \nu^\top \phi(t_0) > 0$, by dominated convergence,

(A.5) $$E \left[ \frac{V_n(\tau-)}{\nu^\top \phi(\tau)} \right] = (1 - a)n.$$

On the other hand, because $Q$ is continuous, by Lemma A.2.4, with $\Gamma_n$ defined as in (A.3),

$$E \left[ \frac{V_n(\tau)}{R_n(\tau) \vee 1} \right] = E \left[ \frac{V_n(\tau-)}{R_n(\tau) \vee 1} \right] + E \left[ \frac{V_n(\tau) - V_n(\tau-)}{R_n(\tau) \vee 1} \right] \leq E \left[ \frac{V_n(\tau-)}{R_n(\tau) \vee 1} \right] \Gamma_n \, P(\Gamma_n) + P(\Gamma_n^c) + E \left[ \frac{V_n(\tau) - V_n(\tau-)}{R_n(\tau) \vee 1} \right] \Gamma_n \, P(\Gamma_n) + P(\Gamma_n^c).

From (A.2), $M_n(\tau)/\alpha \leq [R_n(\tau) \vee 1]/n$. On the other hand, conditional on $\Gamma_n$, $M_n(\tau) \geq (1 - a)\nu^\top \phi(\tau)$. Thus, by (A.5)

$$E \left[ \frac{V_n(\tau-)}{R_n(\tau) \vee 1} \right] \Gamma_n \, P(\Gamma_n) \leq E \left[ \frac{\alpha V_n(\tau-)/n}{(1 - a)\nu^\top \phi(\tau)} \right] \Gamma_n \, P(\Gamma_n) \leq E \left[ \frac{\alpha V_n(\tau-)/n}{(1 - a)\nu^\top \phi(\tau)} \right] = \alpha.$$

By Lemma A.2.4, $P(\Gamma_n^c) \leq (1 + |\mathcal{T}|) \exp(-2n \epsilon_n^2)$. Finally, note that $R_n(\tau) = 0$ implies $V_n(\tau) - V_n(\tau-) = 0$ while $V_n(\tau) - V_n(\tau-) \geq 2$ implies at least two true nulls have the same value of $s_i$. Since $s_i$ under true nulls are iid with a density, the probability of the latter event is 0. Therefore, $V_n(\tau) - V_n(\tau-) \leq 1 \{R > 0\}$ a.s. This then finishes the proof. \(\square\)

We next proof Theorem 3.2. For $n \geq 1$, define

$$\Gamma_n = \{ c \in \Delta : c^\top \phi \in \mathcal{A}_n \}.$$  

For each $r > 0$, corresponding to (3.3), define

$$\Gamma_r = \left\{ c \in \Delta : c^\top \phi(t) \leq Q(t) + r, \quad Q(t_2) - Q(t_1) \geq c^\top [\phi(t_2) - \phi(t_1)] - r, \quad t_1 \leq t_2 \right\}.$$
Both $\Gamma_n$ and $\Gamma_r$ are nonempty since they contain 0. It is not hard to see that $\Gamma_n$ and $\Gamma_r$ are convex and closed, with $\Gamma_r$ being increasing and $\Gamma_0 = \bigcap_{r > 0} \Gamma_r$. Also, whereas $\Gamma_n$ are random, $\Gamma_r$ are nonrandom.

Observe that each $t \in \mathcal{I}$,

(A.6) \[ M_n(t) = \sup \{ c^\top \phi(t) : c \in \Gamma_n \}, \quad m(t) = \sup \{ c^\top \phi(t) : c \in \Gamma_0 \}. \]

Because $\Gamma_n$ is compact, there is a random $c(t) \in \Gamma_n$, such that

(A.7) \[ M_n(t) = c(t)^\top \phi(t). \]

As commented after Theorem 3.2, we need to get $M_n \rightarrow m$. One way to do this is to first get $\Gamma_n \rightarrow \Gamma_0$, which is formalized below.

**Lemma A.2.5.** Let $r > 0$. Then under the conditions of Theorem 3.2, $P(\Gamma_0 \subset \Gamma_n \subset \Gamma_r) \rightarrow 1$.

**Proof.** By the assumptions, $Q(t)$ is continuous. Let

\[ E_n = \left\{ \sup_t |F_n(t) - Q(t)| \leq \epsilon_n/2 \right\}. \]

Then, as in the proof of Lemma A.2.4, for $n \geq 1$, as long as $\exp(-n\epsilon_n^2/2) \leq 1/2$, $P(E_n^c) \leq 2 \exp \left\{ -n\epsilon_n^2/2 \right\}$. It is not hard to see that $E_n$ implies $\Gamma_0 \subset \Gamma_n$. As $n\epsilon_n^2 \rightarrow \infty$, $P(\Gamma_0 \subset \Gamma_n) \geq P(E_n) \rightarrow 1$.

Since $c^\top \phi$ is supported by and strictly increasing in $\mathcal{I}$, almost surely, as $n \rightarrow \infty$, the set of $s_i$ under true nulls is increasingly dense in $\mathcal{I}$, and thus so is $S_n = \{s_1, \ldots, s_n\}$. Because $\phi_k$ and $Q$ are continuous distribution functions, they are equicontinuous. Given $r > 0$, fix $C > 0$ and $\delta > 0$, such that

\[ \max_k [\phi_k(-C) + 1 - \phi_k(C)] + Q(-C) + 1 - Q(C) < r, \]

\[ \max_k [\phi_k(s) - \phi_k(t)] + |Q(s) - Q(t)| < r, \quad \text{if } |s - t| < \delta. \]

Let $E_n' = \{\delta(S_n, [-C, C]) + \delta(\mathcal{T}_n, [-C, C]) < \delta\}$. Conditional on $E_n \cap E_n'$, if $t \in [-C, C]$, then $|Q(t) - F_n(t)| \leq \epsilon_n$ and there is $s_i$ with $|t - s_i| < \delta$. Let $c \in \Gamma_n$. By $c_k \geq 0$, $c_1 + \cdots + c_L \leq 1$ and $c^\top \phi(s_i) \leq F_n(s_i) + \epsilon_n$,

\[ c^\top \phi(t) \leq c^\top \phi(s_i) + \max_k [\phi_k(t) - \phi_k(s_i)] \]

\[ \leq F_n(s_i) + \epsilon_n + r \]

\[ \leq Q(s_i) + 2\epsilon_n + r \]

\[ < Q(t) + 2\epsilon_n + 2r. \]

If $t \leq -C$, then $c^\top \phi(t) \leq \max_k \phi_k(-C) \leq r \leq Q(t) + r$. If $t \geq C$, then $c^\top \phi(t) \leq 1 \leq Q(t) + r$. In any case, $c^\top \phi(t) \leq Q(t) + 2\epsilon_n + 2r$.

Similarly, for $t_1 < t_2$, it can be shown that $c^\top [\phi(t_2) - \phi(t_1)] < Q(t_2) - Q(t_1) + 3\epsilon_n + 4r$. As a result, $c \in \Gamma_\sigma$, with $\sigma = 3\epsilon_n + 4r$. Then $E_n \cap E_n' \subset \{ \Gamma_n \subset \Gamma_\sigma \}$. Because $\epsilon_n \rightarrow 0$, $P(E_n \cap E_n') \rightarrow 1$ and $r$ is arbitrary, the proof is complete. \qed
Lemma A.2.6. Suppose \( a < 1 \). Then, under the conditions of Theorem 3.2, as \( n \to \infty \), \( P(M_n \in C(\mathbb{R})) \to 1 \) and \( \sup |M_n - m| \overset{P}{\to} 0 \). Also, \( m \in C(\mathbb{R}) \).

Proof. Because each \( \phi_k \) is bounded, nondecreasing and continuous, \( \phi \) is uniformly continuous on \( \mathbb{R} \). Since \( \Gamma_n \) is compact, \( c^T \phi(t) \), \( c \in \Gamma_n \) as a family of functions in \( t \) tends to 1, implying \( P|\sup_{t \in \Gamma_n} \phi(t)| \leq \sqrt{L} \sigma \)

\[
M_n(t) \leq c_0(t)^T \phi(t) + \sqrt{L} \sigma \leq m(t) + \sqrt{L} \sigma.
\]

Thus, \( \{\Gamma_0 \subset \Gamma_n \subset \Gamma_r\} \subset \{0 \leq M_n(t) - m(t) \leq \sqrt{L} \sigma \text{ all } t\} \). Because \( \sigma \) is arbitrary, by Lemma A.2.5, \( \sup |M_n - m| \overset{P}{\to} 0 \).

Proof of Theorem 3.2. The proof follows closely the one in [6]. By Assumption A and the continuity of \( m \) and \( Q \), for any \( 0 < \epsilon \ll t_s - t_0 \),

\[
\delta = \min \left\{ \inf_{t \in (t_0 + \epsilon, t_0 - \epsilon)} [\alpha Q(t) - m(t)], \inf_{t > t_s + \epsilon} [m(t) - \alpha Q(t)] \right\} > 0.
\]

Let \( Q_n(t) = [R_n(t) \vee 1]/n \). As \( n \to \infty \), because \( \sup |Q_n - Q| \overset{P}{\to} 0 \) and \( \sup |M_n - m| \overset{P}{\to} 0 \), the probability that

\[
\min \left\{ \inf_{t \in (t_0 + \epsilon, t_0 - \epsilon)} [\alpha Q_n(t) - M_n(t)], \inf_{t > t_s + \epsilon} [M_n(t) - \alpha Q_n(t)] \right\} \geq \delta/2
\]

tends to 1, implying \( P(|\tau - t_s| \leq \epsilon) \to 1 \). Therefore, \( \tau \overset{P}{\to} t_s \), which leads to the last claim of the theorem. Since \( t_s > \inf I \) and \( c^T \phi(t) \) is strictly increasing, \( Q(t_s) \geq (1-a)\nu^T \phi(t_s) > 0 \). By the Week Law of Large Numbers and dominated convergence,

\[
\text{FDR} = E \left[ \frac{V_n(\tau)/n}{Q_n(\tau)} \right] \to \frac{(1-a)\nu^T \phi(t_s)}{Q(t_s)} \leq \frac{m(t_s)}{Q(t_s)} = \alpha,
\]

where the last equality is due to the continuity of \( m \) and \( Q \) at \( t_s \). \( \square \)

A.3. Proofs for Section 5. We need two lemmas for the proof of Proposition 5.1.

Lemma A.3.1. Suppose \( f_1, \ldots, f_L \) are linearly independent. Then \( S \) in (5.1) is a convex compact set.
PROOF. It is easy to see that $S$ is convex and closed, so it suffices to show $S$ is bounded. Suppose there are $c_l \in S$ with $|c_l| \to \infty$ as $l \to \infty$. Since $c_{l1} + \cdots + c_{lt} = 1$, this implies $\max_k c_{lk} \to \infty$ and $\min_k c_{lk} \to -\infty$. There is a subsequence of $c_l$ and a partition of $\Theta$ into $\{\theta_{i1}, \ldots, \theta_{ir}\}$ and $\{\theta_{j1}, \ldots, \theta_{js}\}$, with $r > 0$, $t > 0$ and $r + t = L$, such that $c_{lii} \geq 0$ and $c_{lj,i} < 0$ for each $c_l$ in the subsequence. Without loss of generality, assume $c_{li} \geq 0$ for $i = 1, \ldots, r$ and $c_{li} < 0$ for $i = r + 1, \ldots, L$. Denote $d_{li} = -c_{i,r+i}$ for $i = 1, \ldots, t$. Then for every $x$,

$$\sum_{k=1}^r c_{lk} f_k(x) \geq \sum_{k=1}^t d_{lk} f_{r+k}(x), \quad \sum_{k=1}^r c_{lk} = 1 + M_l, \quad \text{with } M_l = \sum_{k=1}^t d_{lk}. $$

Divide both sides of the inequality by $M_l$ and let $l \to \infty$. Since $M_l \to \infty$, there is a sequence of $l$ along which $(c_{l1}, \ldots, c_{lr})^\top/M_l$ and $(d_{l1}, \ldots, d_{lt})^\top/M_l$ have limits, say $(u_1, \ldots, u_r)^\top$ and $(v_1, \ldots, v_t)^\top$. Then

$$\sum_{k=1}^r u_k f_k(x) \geq \sum_{k=1}^t v_k f_{r+k}(x), \quad \text{all } x. $$

It is easy to see that $u_k \geq 0$, $v_k \geq 0$ and $\sum u_k = \sum v_k = 1$. Because the integrals of both sides are equal to 1, equality must hold. As a result, $f_1, \ldots, f_L$ are linearly dependent, which is a contradiction. $\square$

**Lemma A.3.2.** Suppose $\int q |\ln f_k| < \infty$ for all $k$.

1) For $c \in S^o$ and $r > 0$, if $c + v \in S$ and $\forall v \in B(0, r)$ with $\sum v_k = 0$, then

(A.1) \quad $c^\top f(x) \geq r [M_f(x) - m_f(x)]$, \quad all $x$, where $M_f(x) = \max_k f_k(x)$ and $m_f(x) = \min_k f_k(x)$.

2) For any $c \in S^o$, $\ln(c^\top f) \in L^1(Q)$.

3) Let $\ell(c) := \int q \ln(c^\top f)$. Then $\ell \in C(S^o)$.

4) For any $c \in S^o$ and $x$, $c^\top f(x) = 0 \iff f(x) = 0$.

5) If $f_1, \ldots, f_L$ are linearly independent, then $\ell$ is strictly concave in $S^o$.

**Proof.** 1) For any $v \in B(0, r)$ with $\sum v_k = 0$, by $(c + v)^\top f(x) \geq 0$, $c^\top f(x) \geq - \sum v_k f_k(x)$. Let $v_k = -r$ if $k = \min \{i : f_i(x) = M_f(x)\}$, $v_k = r$ if $k = \min \{i : f_i(x) = m_f(x)\}$, and $v_k = 0$ otherwise. Then (A.1) follows.

2) Let $t^+ = t \vee 0$ and $t^- = (-t) \wedge 0$. By Lemma A.3.1, $\sum c_k^-$ and $\sum c_k^+$ are bounded on $S$. Fix $\lambda \in (0, 1)$ such that $(1 - \lambda) \sum c_k^- \leq \lambda/2$ on $S$. If $M_f(x) > m_f(x)/\lambda$, then by (A.1), $c^\top f(x) \geq r(\lambda^{-1} - 1)m_f(x)$.
$M_f(x) \leq m_f(x)/\lambda$, then

$$c^\top f(x) = \sum c_k^+ f_k(x) - \sum c_k^- f_k(x) \geq \sum c_k^+ m_f - \sum c_k^- M_f \geq \left(\sum c_k^+ - \frac{1}{\lambda} \sum c_k^-\right) m_f = \left[1 - \left(\frac{1}{\lambda} - 1\right) \sum c_k\right] m_f \geq m_f/2.$$ 

Thus, there is a constant $\kappa > 0$ such that $c^\top f(x) \geq \kappa (r^1 m_f)$.

On the other hand, $c^\top f(x) \leq \sum c_k^+ M_f(x) \leq \kappa' M_f(x)$, where $\kappa' < \infty$ is another constant. As a result, as $n \to \infty$.

\[\ln|c^\top f(x)| \leq \max\{\ln[\kappa'M_f(x)], \ln[\kappa(r^1 m_f(x))]\}\]

Then by $\ln f_k \in L^1(Q)$, $\ln(c^\top f) \in L^1(Q)$.

3) Follows from (A.2) and dominated convergence.

4) If $c^\top f(x) = 0$, then by (A.1), $M_f(x) = m_f(x)$ and hence $f_k(x)$ are all equal. As a result, $f_k(x) = c^\top f(x) = 0$.

5) For $c_1, c_2 \in S^o$ and $\theta \in (0, 1)$, since $S^o$ is convex, $c := (1 - \theta)c_1 + \theta c_2 \in S^o$. Because $\ln z$ is strictly concave on $(0, \infty)$, $(1 - \theta)\ell(c_1) + \theta\ell(c_2) \leq \ell(c)$, with “$=$” $\iff c_1^\top f(x) = c_2^\top f(x)$ for $x$ with $q(x) > 0$. On the other hand, if $q(x) = 0$, then $c_1^\top f(x) = 0$ and by 4), $f(x) = 0$. Therefore, “$=$” implies $c_1^\top f(x) = c_2^\top f(x)$ for all $x$. Since $f_k$ are linearly independent, it follows that “$=$” $\iff c_1 = c_2$. Therefore, $\ell$ is strictly concave.

**Proof of Proposition 5.1.** By Lemma A.3.2, for $c \in S^o$ and $X \sim Q$, $\ln|c^\top f(X)| \in L^1$, so by the Weak Law of Large Numbers, as $n \to \infty$, $\frac{1}{n} \sum_{i=1}^n \ln|c^\top f(X_i)| \to_P \ell(c)$. Since $S$ is compact and $\ell$ is continuous and strictly concave on $S^o$, by standard argument, if $\ell$ has a maximum point in $S^o$, then the point is unique and $\nu_0$ converges in probability to it. Thus, to finish the proof, it suffices to show that $\nu$ is the maximum point of $\ell(c)$ if and only if $\int \rho f_k = 1$.

Let $\pi$ be the map $c \to (c_1, \ldots, c_{L-1})^\top$ and $d_k(x) = f_k(x) - f_L(x)$, $k < L$. Since $c_1 + \cdots + c_L = 1$ for $c \in S$, then

$$c^\top f(x) = f_L(x) + \sum_{k=1}^{L-1} c_k[f_k(x) - f_L(x)] = f_L(x) + \pi(c)^\top d(x).$$

Denote $h(u, x) = f_L(x) + u^\top d(x)$ and $H(u) = \int q(x) \ln h(u, x) \, dx$. Then $\ell(c) = H(\pi(c))$. Since $\ell$ is strictly concave in $S^o$, so is $H$ on $\Gamma^o$, with $\Gamma = \pi(S) = \{u : f_L + u^\top d \geq 0\}$. Note that $\pi : S \to \Gamma$ is bijective with $\pi^{-1}(u) = (u, 1 - \sum u_k)$ and $\pi(S^o) = \Gamma^o$. It remains to be seen that $H$ is differentiable in $\Gamma^o$, with

$$\frac{\partial H(u)}{\partial u_k} = \int \frac{q(x)d_k(x)}{h(u, x)} \, dx, \quad k = 1, \ldots, L - 1.$$
Once this obtains, by the strict concavity of $H$ and $(\nu_1, \ldots, \nu_{L-1})^\top \in \Gamma^0$,

$\nu$ is the maximum point of $\ell$

$\iff (\nu_1, \ldots, \nu_{L-1})$ is the maximum point of $H$

$\iff \int \frac{q(x)k(x)}{\nu^\top f(x)} \, dx = 0$

$\iff \int [1 - a + \rho(x)][k(x) - f_L(x)] \, dx = 0$

$\iff (a) \int \rho k_x = \int \rho f_L \iff \int \rho f_k = 1 \text{ all } k,$

where $(a)$ is due to $a > 0$ and $\int f_k = 1$ and $(b)$ is due to the fact that $\int \rho f_k$

being all equal implies each being equal to $\int \rho f^\top f = 1$. Given $u \in \Gamma^0$, fix $r > 0$ such that $B(u, 2r) \subset \Gamma^0$. It is not hard to see that there is $\sigma > 0$, such that for any $v \in B(u, r)$, $\pi^{-1}(v) + w \in S^0$,

$\forall w \in B(0, 2\sigma)$ with $\sum w_k = 0$. Then by (A.1),

(A.3) \quad h(v, x) \geq \sigma[M_f(x) - m_f(x)], \text{ all } v \in B(u, r) \text{ and } x.$

For $x$ with $q(x) > 0$ by Lemma A.3.2, $h(u + v, x) > 0$, $\forall v \in B(0, r)$.

Therefore, $\ln[h(u + v, x)/h(u, x)]$ is well-defined and by Taylor’s expansion,

$$\ln \frac{h(u + v, x)}{h(u, x)} = \sum_{k=1}^{L-1} \left[ \frac{d_k(x)u_k}{h(u, x)} - \frac{d_k(x)^2 u_k^2}{2h(u + zv, x)^2} \right]$$

for some $z = z(v, x) \in [0, 1]$. As $u + zv \in B(u, r)$, by Lemma A.3.2, $h(u + zv, x) > 0$ and by (A.3), $h(u + zv, x) \geq \sigma[M_f(x) - m_f(x)]$. On the other hand, $|d_k(x)| \leq M_f(x) - m_f(x)$. Thus $|d_k(x)/h(u + zv, x)| \leq 1/\sigma$. Likewise, $|d_k(x)/h(u)| \leq 1/\sigma$. As a result,

$$H(u + v) - H(u) = \int q(x) \ln \frac{h(u + v, x)}{h(u, x)} \, dx$$

$$= \sum_{k=1}^{L-1} v_k \int q(x)\frac{d_k(x)}{h(u, x)} \, dx + O(|v|^2),$$

which finishes the proof.

**A.4. Proofs for Section 4.**

**Proof of (4.1).** Recall that the overall distribution under true nulls is

$$\sum_{k=1}^{L} \nu_k F_k$$

and the distribution of $X_1, \ldots, X_n$ is $Q = (1 - a)\nu^\top F + aG$. Then $(1 - a)\nu^\top F(X_i) \leq Q(X_i)$. By the assumption, $Q$ is continuous, which implies that $Q(X_i)$ are iid $\sim \text{Unif}(0, 1)$. Then for the rank statistics $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)},$

$$\left( Q(X_{(k)}), 1 \leq k \leq n \right) \sim \left( \frac{\xi_1 + \cdots + \xi_k}{\xi_1 + \cdots + \xi_{n+1}}, 1 \leq k \leq n \right),$$
where \( \xi_1, \ldots, \xi_{n+1} \) are iid with density \( e^{-x} 1 \{ x \geq 0 \} \). By exponential inequality, for \( \beta \in (0, 1) \), \( P(\xi_1 + \cdots + \xi_{n+1} < \beta(n + 1)) \leq (\beta e^{1-\beta})^{n+1} \). Therefore, for each \( k \leq a_n \),

\[
P \left( Q(X(i)) \leq \Gamma^*(1/n; i, 1/\beta) \right)
= P \left( \frac{\sum_{i=1}^{k} \xi_i}{\sum_{i=1}^{n+1} \xi_i} \leq (1/n) g^*(1/n; i, 1/\beta) \right)
\geq P \left( \sum_{i=1}^{k} \xi_i \leq \beta g^*(1/n; i, 1/\beta), \sum_{i=1}^{n+1} \xi_i \geq \beta n \right)
\geq P \left( \frac{1}{\beta} \sum_{i=1}^{k} \xi_i \leq g^*(1/n; i, 1/\beta) \right) - P \left( \sum_{i=1}^{n+1} \xi_i < \beta n \right)
\]

Because \( \beta^{-1} \sum_{i=1}^{k} \xi_i \) follows the Gamma distribution with shape parameter \( k \) and scale parameter \( \beta \), by above inequalities yield

\[
P \left( Q(X(i)) \leq \Gamma^*(1/n; i, 1/\beta) \right) \geq 1 - \frac{1}{n} - (\beta e^{1-\beta})^{n+1}.
\]

As a result,

\[
P \left( (1 - a) \nu^\top F(X(i)) \leq \Gamma^*(1/n; i, 1/\beta), \text{ all } i \leq a_n \right)
\geq P \left( Q(X(i)) \leq \Gamma^*(1/n; i, 1/\beta), \text{ all } i \leq a_n \right)
\geq 1 - a_n \left[ \frac{1}{n} + (\beta e^{1-\beta})^{n+1} \right].
\]

Following the proof of Theorem 3.1,

\[
\text{FDR} \leq \alpha + E \left[ \frac{1 \{ R > 0 \}}{R \vee 1} \right] + 2(1 + |T_n|) \exp(-2n\epsilon_n^2) + a_n \left[ \frac{1}{n} + (\beta e^{1-\beta})^{n+1} \right].
\]

Note \( \beta e^{1-\beta} < 1 \). With \( \epsilon_n = \sqrt{\ln n/n} \) and \( |T_n| = [(\ln n)^2] \), it is easy to see \( r_n \to 0 \) as \( n \to \infty \). Furthermore, for \( a_n = n^{0.2}, \beta = 0.95, \) and \( n = 5000 \), \( r_n \approx 9.64 \times 10^{-3} \).

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Fig 1. Plots of $n\bar{p}(i)/i$ versus $i/n$ in simulations 1–5 for different types of $p$-values: $p_{i,\text{seq}}$ ("lp-sequential"), $p_{i,\text{glb}}$ ("lp-global"), $p_{i,\text{max}}$ ("max"), and $p_{i,\text{mix}}$ ("mix").
Fig 2. Plots of $c_{k,(i)}$ versus $i/n$, $k = 1, \ldots, n$ in simulations 1 and 5, where $c_{1,(i)}, \ldots, c_{L,(i)}$ are the coefficients to attain $P_{(i),\text{seq}}$ (left) or $P_{(i),\text{glb}}$ (right).
Fig 3. Plots of $n\bar{p}_i / i$ versus $i/n$ in simulations 1–5, with $i/n \leq 0.05$. The plots with open symbols are those of $p_{i, \text{seq}}$ and $p_{i, \text{glb}}$ as in Figure 1. The plots with closed symbols are those of $p_{i, \text{seq}}$ and $p_{i, \text{glb}}$ computed with the extra constraint $c_1 + \cdots + c_L \geq 0.9$. 
Fig 4. Plots of $c_{k,i}$ versus $i/n$, $k = 1, \ldots, n$ in simulations 1 and 5, where $c_{k,i}, \ldots, c_{L,i}$ are the coefficients to attain $p'_{(i),\text{seq}}$ (left) or $p'_{(i),\text{glob}}$ (right), under the constraint $c_1 + \cdots + c_L \geq 0.9$ in addition to those for $p_{(i),\text{seq}}$ and $p_{(i),\text{glob}}$ in Figure 2.