Antimatroids Induced by Matchings

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Abstract

An antimatroid is a combinatorial structure abstracting the convexity in geometry. In this paper, we explore novel connections between antimatroids and matchings in a bipartite graph. In particular, we prove that a combinatorial structure induced by stable matchings or maximum-weight matchings is an antimatroid. Moreover, we demonstrate that every antimatroid admits such a representation by stable matchings and maximum-weight matchings.

1 Introduction

In this paper, we explore a novel connection between antimatroids and matchings in bipartite graphs. In particular, we prove that a combinatorial structure induced by stable matchings or maximum-weight matchings is an antimatroid. Moreover, we demonstrate that every antimatroid admits such a representation by stable matchings and maximum-weight matchings.

An antimatroid is a combinatorial abstraction of the convexity in geometry, which is represented by a nonempty set system \((E, F)\) satisfying (i) accessibility: every nonempty \(X \in F\) has an element \(e \in X\) such that \(X - e \in F\) and (ii) union-closedness: \(X \in F\) and \(Y \in F\) imply \(X \cup Y \in F\). An antimatroid is known to be equivalent to a convex geometry by complementation (i.e., \(\{E \setminus X \mid X \in F\}\) against an antimatroid \((E, F)\)), which is also equivalent to a path-independent choice function by a suitable construction. For more details, see [2, 8].

In a stable matching instance, we are given a bipartite graph in which each vertex has a strict order on the set of its neighbors (or, equivalently, of its incident edges). Since the seminal paper by Gale and Shapley [3], the stable matching and its generalizations have been widely studied in mathematics, economics, and computer science. In particular, Conway [7] pointed out that the set of stable matchings forms a distributive lattice under a natural dominance relation. Conversely, Blair [1] proved that every finite distributive lattice equals to the set of stable matchings in some instance. See [5, 9, 10] for more details.

In a weighted matching instance, we are given a bipartite graph with edge weights, and required to find a matching with the maximum total weight. This problem is one of the most fundamental combinatorial optimization problems on graphs. Through the researches of this problem and its generalizations, a variety of concepts and techniques in combinatorial optimization have been developed, e.g., good characterization, augmenting-path-type algorithms, and polyhedral approaches. See [12] for the details.

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2 Preliminaries

We consider matchings in a bipartite graph \( G = (U, V; E) \), where \( U \) and \( V \) are the disjoint vertex sets and \( E \subseteq U \times V \) is the set of edges. For a vertex \( r \in U \cup V \), we denote by \( \delta_G(r) \) the set of edges incident to \( r \), i.e., \( \delta_G(r) = \{ (u, v) \in E \mid u = r \text{ or } v = r \} \), and denote by \( N_G(r) \) the set of neighbors of \( r \), i.e., \( N_G(r) = \{ t \in U \cup V \mid (t, r) \in E \text{ or } (r, t) \in E \} \). For a vertex subset \( X \subseteq U \cup V \), define \( E[X] := \{ (u, v) \in E \mid u \in X \text{ and } v \in X \} \) and \( G[X] := (U \cap X, V \cap X; E[X]) \).

An edge subset \( M \subseteq E \) is called a matching in \( G \) if no two edges in \( M \) have a common vertex, i.e., \( |M \cap \delta_G(r)| \leq 1 \) for every \( r \in U \cup V \). We write \( M_G \) for the set of all matchings in \( G \). For a matching \( M \) and an edge \((u, v), \in M \), let \( M(u) := v \) and \( M(v) := u \).

Let \( F \) be a map from \( 2^U \) to \( 2^V \) that is induced by stable matchings or maximum-weight matchings as we will see below. Our purpose is to study the structure of the codomain, i.e., \( \{ F(U') \mid U' \subseteq U \} \).

2.1 Stable Matchings

Let us consider a bipartite graph \( G = (U, V; E) \) with preferences (strict orders) \( \succ \) on \( N_G(r) \) for all \( r \in U \cup V \). We denote the profile \( \{ \succ_r \}_{r \in U \cup V} \) of preferences simply by \( \succ \), and refer to a pair \((G, \succ)\) as a stable matching instance. For a vertex subset \( X \subseteq U \cup V \), we mean by \((G, \succ)_X\) the stable matching instance \((G[X], \{ \succ_r \}_{r \in X})\) obtained by restricting \((G, \succ)\) to \( X \).

Let \( M \subseteq E \) be a matching in \( G \). An edge \((u, v) \in E \) is called a blocking pair against \( M \) in \( G \) if \( [\delta_G(u) \cap M = \emptyset \text{ or } v \succ_u M(u)] \) and \( [\delta_G(v) \cap M = \emptyset \text{ or } u \succ_v M(v)] \). A matching \( M \in M_G \) is called a stable matching if there exists no blocking pair against \( M \) in \( G \). It is well-known that, for any stable matching instance, there exists at least one stable matching, and moreover all stable matchings consist of the same set of vertices \([4]\). Hence, for each subset \( U' \subseteq U \), the set of vertices in \( V \) who are matched in a stable matching in \((G, \succ)_{U' \cup V}\) is uniquely determined, and this fact naturally defines a map from \( 2^U \) to \( 2^V \). In what follows, we define this map algorithmically.

A stable matching can be obtained by a simple algorithm, so-called the deferred acceptance algorithm \([3][10]\) (see Algorithm \([1]\)). In each iteration, an unmatched left vertex \( u \) proposes to the most-preferred right vertex \( v \) in \( u \)'s preference list to whom it hasn’t yet proposed. Then \( v \) accepts the proposal if \( v \) is unmatched or prefers \( u \) to the current partner \( u' \) (in this case, \( u' \) becomes unmatched). Otherwise, i.e., if \( v \) prefers the current partner \( u' \) to \( u \), the proposal is rejected. The process is repeated until every left vertex is matched or rejected by all its neighbors.

A significant feature of this algorithm is that the output does not depend on the order of proposals. For a stable matching instance \((G = (U, V; E), \succ)\) and a subset \( U' \subseteq U \), we denote by \( \text{SM}(G, \succ; U') \) the output of the deferred acceptance algorithm for the restricted instance \((G, \succ)_{U' \cup V}\).

Definition 2.1. The map \( F : 2^U \rightarrow 2^V \) induced by a stable matching instance \((G = (U, V; E), \succ)\) is defined by

\[
F(U') := \{ v \mid (u, v) \in \text{SM}(G, \succ; U') \} \quad (U' \subseteq U).
\]

Note that, since all stable matchings consist of the same set of vertices in a stable matching instance, one can replace \( \text{SM}(G, \succ; U') \) in the above definition with an arbitrary stable matching in the restricted instance \((G, \succ)_{U' \cup V}\).

Let us mention two important properties of \( F \), which can be derived from properties of the deferred acceptance algorithm.
Algorithm 1: Deferred Acceptance Algorithm

Input: A bipartite graph $G = (U, V; E)$ and a preference profile $\succ = \{r_f\}_{f \in U \cup V}$

Output: A stable matching $\text{SM}(G, \succ) \subseteq E$

1. let $T := U$ and $M := \emptyset$;
2. foreach $u \in U$ do $R_u := N_G(u)$;
3. while $T \neq \emptyset$ do
   4. pick $u \in T$ arbitrarily;
   5. if $R_u = \emptyset$ then $T := T - u$;
   6. else take $v \in R_u$ so that $v \succ_u v'$ for all $v' \in R_u$;
   7. if $\delta_G(v) \cap M = \emptyset$ then $M := M + (u, v)$, $T := T - u$;
   8. else let $u' := M(v)$;
   9. if $u' \succ_v u$ then $R_u := R_u - v$;
10. else $M := M + (u, v) - (u', v)$, $R_{u'} := R_{u'} - v$, and $T := T + u - u'$;
11. return $M$;

Lemma 2.2. The map $F: 2^U \rightarrow 2^V$ induced by a stable matching instance satisfies the followings.

(a) If $U_2 \subseteq U_1 \subseteq U$, then $F(U_2) \subseteq F(U_1)$.

(b) If $U_2 \subseteq U_1 \subseteq U$ and $|F(U_1)| = |U_1|$, then $|F(U_2)| = |U_2|$.

We give an example of the map induced by a stable matching instance.

Example 2.3. Suppose that $U = \{u_1, u_2, u_3\}$, $V = \{v_1, v_2, v_3\}$, and

$$E = \{(u_1, v_1), (u_1, v_2), (u_2, v_1), (u_2, v_3), (u_3, v_1), (u_3, v_2)\}.$$ 

Consider an instance $(G = (U, V; E), \succ)$, where

- $u_1: v_1 \succ_{u_1} v_2$
- $u_2: v_1 \succ_{u_2} v_3$
- $u_3: v_2 \succ_{u_3} v_1$
- $v_1: u_3 \succ_{v_1} u_2 \succ_{v_1} u_1$
- $v_2: u_1 \succ_{v_2} u_3$
- $v_3: u_2$.

Then, for example, $F(U) = \{v_1, v_2, v_3\}$ because $\text{SM}(G, \succ; U) = \{(u_1, v_2), (u_2, v_3), (u_3, v_1)\}$. By similar calculations, we obtain that the codomain of $F$ is

$$\{F(U') \mid U' \subseteq U\} = \{\emptyset, \{v_1\}, \{v_2\}, \{v_1, v_2\}, \{v_1, v_2, v_3\}\},$$

which forms an antimatroid on $V$.

We remark that the choice function $\text{Ch}: 2^U \rightarrow 2^U$ defined by $\text{Ch}(U') := \{u \mid (u, v) \in \text{SM}(G, \succ; U')\}$ ($U' \subseteq U$) is path-independent (and size-monotone), i.e., $\text{Ch}(\text{Ch}(U_1) \cup U_2) = \text{Ch}(U_1 \cup U_2)$ for all $U_1, U_2 \subseteq U$ (and $|\text{Ch}(U_1)| \leq |\text{Ch}(U_2)|$ for all $U_1 \subseteq U_2 \subseteq U$).
2.2 Maximum-Weight Matchings

Given a bipartite graph $G = (U, V; E)$ with weights $w: E \to \mathbb{R}$, the weight of a matching $M$, denoted by $w(M)$, is defined to be the sum $\sum_{e \in M} w(e)$ of the weights of the edges in $M$. We refer to a pair $(G, w)$ as a weighted matching instance. A maximum-weight matching in $G$ is a matching in $\mathcal{M}_G$ with weight $\max_{M \in \mathcal{M}_G} w(M)$. If there exist multiple maximum-weight matchings, we pick the lexicographically smallest (with respect to a fixed order on the edges) one among them. We can obtain such a matching as the unique maximum-weight matching by applying a small perturbation to the weights. Throughout the paper, we assume that each matching in $\mathcal{M}_G$ has a distinct weight. For a subset $U' \subseteq U$, let $\text{MM}(G, w; U')$ denote the unique maximum-weight matching in $G[U' \cup V]$.

**Definition 2.4.** The map $F: 2^U \to 2^V$ induced by a weighted matching instance $(G = (U, V; E), w)$ is defined by

$$F(U') := \{ v \mid (u, v) \in \text{MM}(G, w; U') \} \quad (U' \subseteq U).$$

The map $F$ induced by a weighted matching instance has the same properties as in Lemma 2.2.

**Lemma 2.5.** The map $F: 2^U \to 2^V$ induced by a weighted matching instance $(G = (U, V; E), w)$ satisfies the followings.

(a) If $U_2 \subseteq U_1 \subseteq U$, then $F(U_2) \subseteq F(U_1)$.

(b) If $U_2 \subseteq U_1 \subseteq U$ and $|F(U_1)| = |U_1|$, then $|F(U_2)| = |U_2|$.

**Proof.** (a): To prove by contradiction, suppose that $F(U_2) \nsubseteq F(U_1)$ and let $u^* \in F(U_2) \setminus F(U_1)$. Define $M_i := \text{MM}(G, w; U_i)$ for $i = 1, 2$. Then, the symmetric difference $M_1 \triangle M_2 = (M_1 \setminus M_2) \cup (M_2 \setminus M_1)$ forms disjoint cycles and paths, because at most two edges in $M_1 \triangle M_2$ is incident to each vertex. Since $u^* \in F(U_2) \setminus F(U_1)$, the connected component of $G[M_1 \triangle M_2]$ containing $u^*$ is a path of length at least 1, and let $P \subseteq M_1 \triangle M_2$ be the set of edges in the path. Then, $M_i \triangle P \in \mathcal{M}_{G[U_i \cup V]}$ $(i = 1, 2)$, and $w(M_1) + w(M_2) = w(M_1 \triangle P) + w(M_2 \triangle P)$. On the other hand, for $i = 1, 2$, since $M_i$ is a unique maximum-weight matching in $G[U_i \cup V]$, we have $w(M_i) > w(M_i \triangle P)$, a contradiction.

(b): It is sufficient to prove the case when $U_1 = U_2 + q$ for some $q \in U \setminus U_2$. Define $M_i := \text{MM}(G, w; U_i)$ for $i = 1, 2$. Then, the symmetric difference $M_1 \triangle M_2$ forms a path from $q$ (or the empty set, which can be regarded as a path of length 0), since otherwise (i.e., if it contains a cycle or a path disjoint from $q$ that is of length at least 1) we can improve at least one of $M_1$ and $M_2$. Therefore, $|U_2| + 1 = |U_1| = |F(U_1)| = |M_1| \leq |M_2| + 1 = |F(U_2)| + 1 \leq |U_2| + 1$, in which the equalities must hold throughout.

Let us see an example of the map induced by a weighted matching instance.

**Example 2.6.** Suppose that $U = \{u_1, u_2, u_3\}$, $V = \{v_1, v_2\}$, and

$$E = \{(u_1, v_1), (u_1, v_2), (u_2, v_1), (u_3, v_1)\}.$$

Consider an instance $(G = (U, V; E), w)$, where

$$w((u_1, v_1)) = 20, \ w((u_1, v_2)) = 8, \ w((u_2, v_1)) = 9, \text{ and } w((u_3, v_1)) = 15.$$

Then, for example, $F(\{u_1, u_2\}) = \{v_1\}$ because $\text{MM}(G, w; \{u_1, u_2\}) = \{(u_1, v_1)\}$. By similar calculations, we obtain that the codomain of $F$ is

$$\{ F(U') \mid U' \subseteq U \} = \{\emptyset, \{v_1\}, \{v_1, v_2\}\},$$

which forms an antimatroid on $V$.
3 Antimatroids Induced by Matchings

In this section, we prove that any stable matching or weighted matching instance induces an antimatroid.

3.1 Antimatroids Induced by Stable Matchings

**Theorem 3.1.** Let \( F : 2^U \to 2^V \) be the map induced by a stable matching instance \((G = (U, V; E), \succ)\), and \( F := \{ F(U') \mid U' \subseteq U \} \). Then the set system \((V, F)\) is an antimatroid.

**Proof.** We have \( \emptyset \in F \) since \( F(\emptyset) = \emptyset \). To see the accessibility, let us fix \( V' \in F \setminus \{ \emptyset \} \) and let \( U' \) be a subset of \( U \) such that \( F(U') = V' \). Define \( M := SM(G, \succ; U') \) and \( U'' = \{ u \mid (u, v) \in M \} \). Then \( M \) is also a stable matching in \( G[U'' \cup V] \) because \( U'' \subseteq U' \) and \( M \in \mathcal{M}_G[U'' \cup V] \). Thus we have \( F(U'') = V' \) and \( |U''| = |V'| \geq 1 \). Let us fix \( u' \in U'' \). Then \( F(U'' - u') \subseteq V' \) and \( F(U'' - u') = |V'| - 1 \) by Lemma 2.2. Therefore, there exists \( v \in V' \) such that \( V' - v = F(U'' - u') \in F \).

In what follows, we show that \( F \) is union-closed. Fix any two subsets \( U_1, U_2 \subseteq U \), and let \( V^* := F(U_1) \cup F(U_2) \). We shall show that there exists \( U^* \subseteq U_1 \cup U_2 \) such that \( V^* = F(U^*) \). Note that, due to (a) in Lemma 2.2 we have \( F(U_1) \subseteq F(U_1 \cup U_2) \) and \( F(U_2) \subseteq F(U_1 \cup U_2) \), and hence \( V^* \subseteq F(U_1 \cup U_2) \).

Let \( M_i := SM(G, \succ; U_i) \) for \( i = 1, 2 \), \( \hat{E} := \{ (u, v) \mid (u, v') \in M_1 \cup M_2, v \succ u \} \), and \( \hat{G} := (U, V; \hat{E}) \). Note that \( \{ v \mid (u, v) \in \hat{E} \} \subseteq V^* \), because if there exists \( (u, v) \in \hat{E} \) such that \( v \notin V^* \), then it is a blocking pair against \( M_1 \) or \( M_2 \). For each \( r \in U \cup V \), let \( \succ_r \) denote the restriction of \( \succ \) to \( N_{\hat{G}}(r) \), i.e., \( \succ_r \) is a strict order on \( N_{\hat{G}}(r) \) such that, for every \( x, y \in N_{\hat{G}}(r) \), \( x \succ_r y \) if and only if \( x \succ y \). We then have

\[
SM(\hat{G}, \succ; U_i) = M_i = SM(G, \succ; U_i) \quad (i = 1, 2).
\]

Let \( \hat{F} : 2^U \to 2^V \) be the map induced by \((\hat{G}, \succ)\). Then, by (a) in Lemma 2.2,

\[
F(U_i) = \{ v \mid (u, v) \in M_i \} = \hat{F}(U_i) \subseteq \hat{F}(U_1 \cup U_2) \quad (i = 1, 2),
\]

and hence \( V^* = F(U_1) \cup F(U_2) \subseteq \hat{F}(U_1 \cup U_2) \). In addition, since \( \{ v \mid (u, v) \in \hat{E} \} \subseteq V^* \), we have \( \hat{F}(U_1 \cup U_2) \subseteq V^* \). Thus, we obtain \( \hat{F}(U_1 \cup U_2) = V^* \).

Let \( \hat{M} := SM(\hat{G}, \succ_r; U_1 \cup U_2) \) and \( U^* := \{ u \mid (u, v) \in \hat{M} \} \subseteq U_1 \cup U_2 \). We define

\[
M^* := SM(\hat{G}, \succ_r; U^*).
\]

Note that \( \hat{F}(U_1 \cup U_2) = \hat{F}(U^*) \) because \( \hat{M} \) is a stable matching in \( \hat{G}[U^* \cup V] \). In addition, every vertex in \( U^* \) is matched in \( M^* \) because \( |M^*| = |\hat{F}(U^*)| = |\hat{F}(U_1 \cup U_2)| = |\hat{M}| = |U^*| \).

The proof is completed by showing that \( M^* \) is a stable matching also in \( G[U^* \cup V] \) (with respect to \( \succ \)) because this implies \( F(U^*) = \{ v \mid (u, v) \in M^* \} = \hat{F}(U^*) = \hat{F}(U_1 \cup U_2) = V^* \). To obtain a contradiction, suppose that there exists a blocking pair \((u^*, v^*) \in E[U^* \cup V] \) against \( M^* \). Since \( M^* \) is a stable matching in \( \hat{G}[U^* \cup V] \), we can assume that \((u^*, v^*) \notin \hat{E} \). Then, by the definition of \( \hat{E} \), we have \( v \succ u^* \) \( v^* \) for every \( v \in N_{\hat{G}}(u^*) \). As \((u^*, M^*(u^*)) \in \hat{E} \) implies \( M^*(u^*) \succ u^* \) \( v^* \), we get that \((u^*, v^*) \) cannot be a blocking pair against \( M^* \) in \( G[U^* \cup V] \). This contradicts our assumption. \qed
3.2 Antimatroids Induced by Maximum-Weight Matchings

**Theorem 3.2.** Let $F : 2^U \to 2^V$ be the map induced by a weighted matching instance $(G = (U, V; E), w)$, and $\mathcal{F} := \{ F(U') \mid U' \subseteq U \}$. Then the set system $(V, \mathcal{F})$ is an antimatroid.

**Proof.** We have $\emptyset \in \mathcal{F}$ since $F(\emptyset) = \emptyset$. Also, we can derive the accessibility from Lemma [2,5] similarly to the proof of Theorem [5,1].

It remains to prove that $\mathcal{F}$ is union-closed. Fix any two subsets $U_1, U_2 \subseteq U$, and let $V^* := F(U_1) \cup F(U_2)$. Let $M_i := MM(G, w; U_i)$ for $i = 1, 2$. For each vertex $v \in V \setminus V^*$, we create $\{U\}$ new vertices $h_{u,j}$ ($j = 1, \ldots, |U|$), and let $H_v$ denote the set of those vertices. We define a new weighted matching instance $(\tilde{G} = (U, \tilde{V}; \tilde{E}), \tilde{w})$ as follows:\footnote{The weighted matching instance $(\tilde{G} = (U, \tilde{V}; \tilde{E}), \tilde{w})$ does not satisfy the condition that each matching in $\mathcal{M}_{\tilde{G}}$ has a distinct weight. Hence, we define $MM(G, \tilde{w}; U^*)$, for $U^* \subseteq U$, to be the lexicographically smallest matching in $\arg\max\{w(M) \mid M \in \mathcal{M}_{\tilde{G}[(U \setminus \tilde{V})]} \}$, with respect to a fixed order on the edges $\tilde{E}$. Note that this definition is not essential in the proof.}

\begin{align*}
\tilde{V} &:= V^* \cup \bigcup_{v \in V \setminus V^*} H_v, \\
\tilde{E} &:= \{ (u, v) \mid (u, v) \in E, v \in V^* \} \cup \{ (u, h_{u,j}) \mid (u, v) \in E, v \notin V^*, j \in \{1, \ldots, |U|\} \},
\end{align*}

and for each $(u, v) \in \tilde{E}$,

\[ \tilde{w}((u, v)) := \begin{cases} w((u, v)) & \text{if } v \in V^*, \\
                  w((u, v')) & \text{if } v \in H_v'. \end{cases} \]

Let $\tilde{M}$ be the maximum-weight matching $MM(\tilde{G}, \tilde{w}; U)$, $M^* := \tilde{M} \cap E$, and $U^* := \{ u \mid (u, v) \in M^* \}$. In addition, let $\tilde{F} : 2^U \to 2^V$ be the map induced by $(\tilde{G}, \tilde{w})$.

We first claim that $\tilde{F}(U^*) = V^*$. Note that $MM(G, w; U_i) = MM(\tilde{G}, \tilde{w}; U_i) = M_i$ for $i = 1, 2$ by the definition of $(\tilde{G}, \tilde{w})$. As $F(U_i) = \{ v \mid (u, v) \in M_i \} = \tilde{F}(U_i) \subseteq \tilde{F}(U_1 \cup U_2)$ by Lemma [2,5] we have $V^* = F(U_1) \cup F(U_2) \subseteq \tilde{F}(U_1 \cup U_2)$. Thus we get $\tilde{F}(U^*) = \tilde{F}(U_1 \cup U_2) \cap V^* = V^*$.

Next, we observe that $M^* = MM(G, w; U^*)$. Suppose that $M^* \triangle MM(G, w; U^*) \neq \emptyset$ and let $X \subseteq M^* \triangle MM(G, w; U^*)$ be one of its connected components. Then we have $w(M^* \triangle X) > w(M^*)$ or $w(MM(G, w; U^*) \triangle X) > w(M^*)$, a contradiction.

Consequently, we obtain $V^* = \tilde{F}(U^*) = \{ v \mid (u, v) \in M^* \} = F(U^*) \in \mathcal{F}$. $\blacksquare$

4 Matching Representations of Antimatroids

In this section, we provide a representation of an antimatroid as a matching instance.

Let $(S, \mathcal{F})$ be an antimatroid. Let $d : \mathcal{F} \setminus \{\emptyset\} \to S$ be a function such that $d(X) \in X$ and $X - d(X) \in \mathcal{F}$ for every $X \in \mathcal{F} \setminus \{\emptyset\}$. There exists such a function $d$ since $\mathcal{F}$ satisfies accessibility.

Let $>^*$ be a total order on $\mathcal{F} \setminus \{\emptyset\}$ such that $X >^* Y$ whenever $X \subseteq Y$. Namely, $X >^* Y$ implies $Y \nsubseteq X$. Also, let $>^X$ be the order on each $X = \{a_1, \ldots, a_k\} \in \mathcal{F} \setminus \{\emptyset\}$ such that $a_1 >^X \ldots >^X a_k$, where $a_i = d(X \setminus \{a_{i+1}, \ldots, a_k\})$ ($i = 1, 2, \ldots, k$). Note that $\{a_1, \ldots, a_i\} \in \mathcal{F}$ ($i = 0, 1, \ldots, k$) by the definition of $d$.\footnote{The weighted matching instance $(\tilde{G} = (U, \tilde{V}; \tilde{E}), \tilde{w})$ does not satisfy the condition that each matching in $\mathcal{M}_{\tilde{G}}$ has a distinct weight. Hence, we define $MM(G, \tilde{w}; U^*)$, for $U^* \subseteq U$, to be the lexicographically smallest matching in $\arg\max\{w(M) \mid M \in \mathcal{M}_{\tilde{G}[(U \setminus \tilde{V})]} \}$, with respect to a fixed order on the edges $\tilde{E}$. Note that this definition is not essential in the proof.}
4.1 Representation by Stable Matchings

We construct a stable matching instance \( (G = (U, V; E), \succ) \) as follows:

- \( U := \mathcal{F} \setminus \{\emptyset\} \) and \( V := \mathcal{S} \);
- \( E := \{ (u, v) \mid u \in U, v \in u \} \);
- \( \succ_u := \succ^u \) for each \( u \in U \);
- let \( \succ_v \) be the restriction of \( \succ^v \) to \( \{ u \mid v \in u \} \) for each \( v \in V \).

We prove that this stable matching instance derives the desired antimatroid.

**Theorem 4.1.** Let \( (S, \mathcal{F}) \) be an antimatroid, and \( F : 2^{\mathcal{F} \setminus \{\emptyset\}} \to 2^S \) the map induced by the stable matching instance \( (G, \succ) \) defined as above. Then the codomain of \( F \) coincides with \( \mathcal{F} \).

**Proof.** Let \( \mathcal{F}' = \{ F(U') \mid U' \subseteq U \} \). We claim that \( \mathcal{F}' = \mathcal{F} \).

We first see \( \mathcal{F} \subseteq \mathcal{F}' \). Let \( X = \{ u_1, \ldots, u_k \} \in \mathcal{F} \) such that \( v_1 \succ_X \cdots \succ_X v_k \), i.e., \( v_i = d(X \setminus \{ v_{i+1}, \ldots, v_k \}) \) \( (i = 1, 2, \ldots, k) \). We define \( u_i := \{ v_1, \ldots, v_i \} \in \mathcal{F} \setminus \{\emptyset\} = U \) for \( i = 1, \ldots, k \). Then, \( X = F(\{ u_1, \ldots, u_k \}) \in \mathcal{F}' \) because \( SM(G, \succ; \{ u_1, \ldots, u_k \}) = \{(u_1, v_1), \ldots, (u_k, v_k)\} \) by \( u_1 \succ^* \cdots \succ^* u_k = X \) and \( v_1 \succ_{u_i} \cdots \succ_{u_i} v_i \) \( (i = 1, \ldots, k) \).

Next, we prove the opposite direction, i.e., \( \mathcal{F}' \subseteq \mathcal{F} \). Let \( U' \subseteq U \) and \( M := SM(G, \succ^*; U') \). Since each \( u \in U' \) matched with someone in \( M \) proposes only to the neighbors \( v' \in N_{G[U' \cup V]}(u) = u \) such that \( v' \succeq_u M(u) \) throughout the deferred acceptance algorithm (recall Algorithm 1), we have

\[
F(U') = \{ v \mid (u, v) \in M \} = \bigcup_{(u, v) \in M} \{ v' \mid v' \in u, v' \succeq_u v \} \subseteq \mathcal{F},
\]

where the last membership follows from the facts that \( \{ v' \mid v' \in u, v' \succeq_u v \} \subseteq \mathcal{F} \) for all \( (u, v) \in E \) (recall the definitions of \( \succ^* = \succ^u \) and \( d \)) and that \( \mathcal{F} \) is union-closed. Therefore, we get \( \mathcal{F}' \subseteq \mathcal{F} \). □

4.2 Representation by Maximum-Weight Matchings

We define \( b \) to be a unique bijection from \( \mathcal{F} \setminus \{\emptyset\} \) to \( \{1, 2, \ldots, |\mathcal{F} \setminus \{\emptyset\}|\} \) consistent with \( \succ^* \), i.e., for any distinct \( X, Y \in \mathcal{F} \setminus \{\emptyset\} \), we have \( b(X) > b(Y) \) if and only if \( X \succ^* Y \). We build a weighted matching instance \( (G = (U, V; E), w) \) as follows:

- \( U := \mathcal{F} \setminus \{\emptyset\} \) and \( V := \mathcal{S} \);
- \( E := \{ (u, v) \mid u \in U, v \in u \} \);
- \( w((u, v_i)) := 2^{|V| - b(u) + i} \) for \( u = \{ v_1, \ldots, v_k \} \in \mathcal{F} \setminus \{\emptyset\} \) and \( v_i \in u \) such that \( v_1 \succ^u \cdots \succ^u v_k \).

We show that this weighted matching instance also derives the desired antimatroid. Note that the maximum-weight matching is lexicographically maximum (with respect to the order of edge weights) because the edge weights are distinct power-of-two values. With a similar proof to that of Theorem 4.1, we can prove the following theorem.

**Theorem 4.2.** Let \( (S, \mathcal{F}) \) be an antimatroid, and \( F : 2^{\mathcal{F} \setminus \{\emptyset\}} \to 2^S \) the map induced by the weighted matching instance \( (G, w) \) defined as above. Then the codomain of \( F \) coincides with \( \mathcal{F} \). □
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