On Koszulity in homology of moduli spaces of stable n-pointed curves of genus zero

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Abstract
We prove Koszulity of the homology of the of moduli spaces of stable n-pointed curves of genus zero $\overline{M}_{0,n}$ for $n = 5$, using its presentation due to Keel and the Priddy criterion of koszulity. For $n = 6$ we show that it is not Koszul by calculating the Hilbert series of the Koszul dual.

1 Introduction

In the paper [1], due to Keel, dedicated to the intersection theory of the moduli space of n-pointed stable curves of genus zero, it was shown that the canonical map from the Chow ring to the homology is an isomorphism, and the presentation of this ring by generators and relations have been derived.

The variables $D^S$ in this presentation are parameterized by subsets $S \subset \{1, \cdots, n\}$ consisting of two or more elements, variables corresponding to complimentary sets are coincide: $D^S = D^{S^C}$. The relations then looks as follows:

(1). For any four distinct elements $i, j, k, l \in \{1, \cdots, n\}$:

$$\sum_{i,j \in S, k,l \notin S} D^S = \sum_{i,k \in S, j,l \notin S} D^S = \sum_{i,l \in S, j,k \notin S} D^S$$

(2). $D^S D^T$ unless one of the following holds:

$S \subset T, T \subset S, S \subset T^C, T^C \subset S.$

We will use this presentation to examine Koszulity of the ring $A = H_*(\overline{M}_{0,n})$.

For $n = 5$ we suggest the following choice of linearly independent variables: out of all generators

$$\{\delta_{i,j}\} = \{\delta_{12}, \delta_{13}, \delta_{14}, \delta_{15}, \delta_{23}, \delta_{24}, \delta_{25}, \delta_{34}, \delta_{35}, \delta_{45}\}$$

we take those, sitting on the sides of the pentagon:

$$x_i = \delta_{i,i+1}, \ i \in \mathbb{Z}/5\mathbb{Z},$$

then we rewrite all other variables and relations via them. This presentation wakes up some associations with cluster algebras. Using the obtained relations we compute a presentation for the Koszul dual (noncommutative) algebra $A^!$. It turns out that it can be presented by one relation of the form:

$$\sum_{a \neq b, a \neq b+1, a \neq b-1} x_a x_b - \sum_{k=1,\ldots,5} x_k^2 = 0.$$

We can find a change of variables, such that the new relations does not have a square term $x^2$. Thus we have an algebra $A^!$ presented by a quadratic Gröbner basis. Hence due to the Priddy criterion [3,2], we can conclude that $A^!$ is Koszul, and hence $A$ itself as well.
Then we consider $A = M_{0,6}$. Here we demonstrate one of possible calculations, showing that $H_A(t) = 1 + 16t + 240t^2 + 3584t^3 + \cdots$. This means that the equality $H_A(t)H_A(-t) = 1$, which should be true for Koszul algebra, fails in the third term (it should have been 3585, if the algebra is Koszul). By this we prove that $A = H_\bullet(M_{0,6})$ is not Koszul.

2 Koszulity of $M_{0,5}$

We consider first the presentation of $A = M_{0,n}$ due to Keel [1] with generators

$$\{\delta_{i,j}\} = \{\delta_{12}, \delta_{13}, \delta_{14}, \delta_{23}, \delta_{24}, \delta_{34}, \delta_{35}, \delta_{45}\}$$

and corresponding linear and monomial quadratic relations. Let $V = \text{span}_k\{\delta_{i,j}\}$. We suggest to choose as a linear basis in $V$ the following set: $\{x_i = \delta_{i,i+1}, i \in \mathbb{Z}/5\mathbb{Z}\}$. Via this basis we are going to express other elements of $\{\delta_{i,j}\}$, then we rewrite quadratic relations in terms of variables $x_i, i \in \mathbb{Z}/5\mathbb{Z}$. This will allow us later on to pass to the Koszul dual algebra $A^!$, which could be shown to be Koszul. The presentation for the $A^!$ which we obtain, after suitable change of variables, will have a quadratic Gröbner bases. This due to the Priddy’s PBW criterion [3, 2] will imply Koszulity of $A^!$ and hence of $A$.

STEP 1
Expression of $\{\delta_{i,j}\}$ via $x_i = \delta_{i,i+1}, i \in \mathbb{Z}/5\mathbb{Z}$.

Linear relation which we have for quadruple $\{a, b, c, d\} \in \mathbb{Z}/5\mathbb{Z}$ of pairwise distinct elements from 1 to 5, are

$$\delta_{ab} + \delta_{cd} = \delta_{ac} + \delta_{bd} = \delta_{ad} + \delta_{bc}.$$ In particular, $\delta_{ad} = \delta_{ab} + \delta_{cd} - \delta_{bc}$, and for, say $a = i, b = i + 1, c = i + 2, d = i + 3$ we get

$$\delta_{i,i+3} = \delta_{i,i+1} + \delta_{i+2,i+3} - \delta_{i+1,i+2},$$

that is

$$\delta_{i,i+3} = x_i + x_{i+2} - x_{i+1}.$$ The set of elements $\{\delta_{i,j}\}$, which are our original generators can be presented 'geometrically' as a sides and diagonals of the pentagon (analogously to the presentation of cluster variables). Sides correspond to variables $\delta_{i,i+1}$ and diagonals to $\delta_{i,i+3}$, for appropriate $i$. We will call these two types of generators side type and diagonal type generators respectively. Hence, by expressing the diagonal type variables $\delta_{i,i+3}$, via those $\delta_{i,j}$ with nearby indexes, e.i. via the side type variables, we express all generators via the new basis $\{x_i, i \in \mathbb{Z}/5\mathbb{Z}\}$.

STEP 2
Calculation of quadratic relations in terms of $x_i = \delta_{i,i+1}, i \in \mathbb{Z}/5\mathbb{Z}$.

Our initial quadratic relations have a shape

$$\delta_{i,j}\delta_{j,k}, i \neq k$$

This means, for those variables which are side type, we will have

$$(1) \quad \delta_{i,i+1}\delta_{i+1,i+2} = x_ix_{i+1} = 0.$$ If a relation contains both side and diagonal type generators, it have a shape:
\[ \delta_{i,i+3}x_k, k \neq i+1, k \in \mathbb{Z}/5\mathbb{Z}. \]

In the pentagon picture it could be interpreted as a fact, that only the side \( x_{i+1} \) does not intersect the diagonal \( \delta_{i,i+3} \).

So in our new variables these relations look like:

\[
\begin{align*}
(2) \quad (x_i + x_{i+2} - x_{i+1})x_i &= 0(k = i) \\
(x_i + x_{i+2} - x_{i+1})x_{i-1} &= 0(k = i - 1) \\
(x_i + x_{i+2} - x_{i+1})x_{i+2} &= 0(k = i + 2) \\
(x_i + x_{i+2} - x_{i+1})x_{i+3} &= 0(k = i + 3).
\end{align*}
\]

In the case when relations contain two diagonal variables, we also will have:

\[
\delta_{i,i+3}\delta_{j,j+3} = 0,
\]

which means

\[
\delta_{i,i+3}\delta_{i-3,i} = 0.
\]

That is,

\[
(3) \quad (x_i + x_{i+2} - x_{i+1})(x_{i-3} + x_{i-1} - x_{i-2}) = 0.
\]

Since it is obviously the same as

\[
(x_i + x_{i+2} - x_{i+1})(x_{i+2} + x_{i-1} - x_{i+3}) = 0,
\]

we can see that (3) follows from (2) and hence the relations of \( A \) on variables \( x_i \) are formed by (1) and (2).

This system of relations is equivalent to the following one:

\[
x_ix_{i+1} = x_{i+1}x_i = 0
\]

and for any other pair \( a, b, a', b' \in \mathbb{Z}/5\mathbb{Z} \)

\[
x_ax_b = x_{a'}x_{b'} = -x_a^2.
\]

The Hilbert series here is clearly \( H_A = 1 + 5t + t^2 \). This also implies that codimension of the second graded component of the ideal \( I_2 \) is 1. Now using this system of relations, and additional information that the dimension of the space orthogonal to \( I_2 \) is one, we can write down generating relation for the Koszul dual algebra \( A! \):

\[
\sum_{a \neq b, b \neq b+1, a \neq b-1} x_ax_b - \sum_{k=1,...,5} x_k^2 = 0.
\]

This generating system of \( A! \) is not yet form a Gröbner basis of the ideal. But we can make the change of variables which will make this relation free from square of one of variables (say, \( x_2 \)). This will make a presentation of algebra combinatorially free, that is new relations is a Gröbner basis. So the Gröbner basis is quadratic Due to the Priddy PBW criterion this implies that \( A! \) is Koszul. So we have proved
Theorem 2.1. The algebra $A = H_\bullet(M_{0,5})$ is Koszul.

3 Non-Koszulity in the case of $M_{0,6}$

Here we will present an example of computations, showing that

Theorem 3.1. The algebra $A = H_\bullet(M_{0,6})$ is not Koszul.

We will use the system for computations in noncommutative graded algebras GRAAL [4].

Proof. We choose as generators of $A = H_\bullet(M_{0,6})$ all variables

$a = \delta_{13}, b = \delta_{24}, c = \delta_{35}, d = \delta_{46}, e = \delta_{15}, f = \delta_{26};$

$g = \delta_{12}, h = \delta_{23}, i = \delta_{34}, j = \delta_{45}, k = \delta_{56}, l = \delta_{16}, m = \delta_{14}, n = \delta_{25}, o = \delta_{36};$

$p = \delta_{124}, q = \delta_{235}, r = \delta_{346}, s = \delta_{456}, t = \delta_{256}, u = \delta_{136}, v = \delta_{123}, w = \delta_{234}, x = \delta_{345}, y = \delta_{135},$

apart from the following subset: $\{g, h, i, j, k, l, m, n, o\}$. These variables we will express via the rest using linear relations.

The linear Keel relations, rewritten in the normal form by GRAAL w.r.t an ordering

$g > h > i > j > k > l > m > n > o > a > b > c > d > e > f > p > q > r > s > t > u > v > w > x > y$

look as follows:

$n + o - c - f + r + u - x - y$
$m + n - b - e + q + t - w - y$
$-m - o + a + d - p - s + v + y$
$-l + o - c + e - q + r + s - x$
$-l - n + e + f - q + s - u + y$
$-l + m - b + f + s + t - u - w$
$k - o + a - e - r - s + t + v$
$k - n + b - d + p - q - r + w$
$k + m - d - e + p - r + t - y$
$j - m + a - c - p - q + u + v$
$-j - o + c + d + q - s - u + y$
$-j + n + d - f + q + r - s - x$
$-j - l + d + e + r - u - x + y$
$i + k - c - d - q + t + w - y$
$-i + o + b - f + p - t + u - x$
$-i - n + b + c + p - r - t + y$
$-i + m + c - e + p - q - r - w$
$-i - l + m + o + p + s - w - x$
$h - o - b + d - p + q - u + v$
$h + k - n - o - r - u + v + w$
$h + j - b - c - p + s + v - y$
$-h + n + a - e - s + t + u - w$
$-h - m + a + b - q - s + u + y$
$-h - l + a + f - q + t - w + y$
$g + o - a - f + p + r - t - y$
$g - n - a + c + p - t - u + x$
$g - m + d - f + r - s - t + v$
\[ g + k - e - f + p - s + v - y \]
\[ g + j - m - n - q - t + v + x \]
\[ g + i - a - b + r - u + x - y \]

Using the same program we can do gaussian elimination in this linear system and express the set of variables \{g, h, i, j, k, l, m, n, o\} via the remaining ones.

Now we should care about the quadratic relations. The monomial relations consists of three groups. First, all possible products between variables of the group from \( p \) to \( y \), corresponding to three-element subsets will be there. Second, some products of variables from the excluded group of variables \( a, b, c, d, e, f \): \( ac, ae, bd, bf, ce, df \). Third, some products between our main variables (corresponding to two and three elements subsets):

\[
\begin{align*}
ap, & \quad aq, \quad ar, \quad as, \quad aw, \quad ax, 
bq, & \quad br, \quad bs, \quad bt, \quad bv, \quad bx, 
cr, & \quad cs, \quad ct, \quad cu, \quad cv, \quad cw, 
dp, & \quad ds, \quad dt, \quad du, \quad dw, \quad dx, 
ep, & \quad eq, \quad et, \quad eu, \quad ev, \quad ex, 
fq, & \quad fr, \quad fu, \quad fv, \quad fw, 
\end{align*}
\]

We also will have commutativity relations for the following pairs of variables:

\[
\begin{align*}
ab, & \quad ad, \quad af, \quad bc, \quad be, \quad cd, \quad ce, \quad de, \quad ef, \quad at, \quad au, \quad av, \quad bp, \quad bu, \quad bw, 
\end{align*}
\]

And finally, we substitute to the monomial relations, which include the generators from the subset from \( g \) to \( o \), which we expressed via others. This is done also with the help of the normal form calculations in GRAAL, to avoid mistakes. The resulting, last group of quadratic relations, is the following.

\[
\begin{align*}
es + fs + s2 \\
2dv - 2ew + 2fx - 2fy + v^2 - w^2 + x^2 - y^2 \\
dr + er + r2 \\
dc - fc + qc - xc \\
dc - da - fa - ta + va \\
cx + fx + x^2 \\
cq + dq + q2 \\
cf - ef - sf + xf \\
\end{align*}
\]
shows \( H + ab \) algebras: which are not a zero monomials in the monomials (polynomials), which should be present in the relations of the dual, namely those, which are not a zero monomials in \( A \) itself. Actually, since the initial algebra was commutative, the dual will have relations \( ab + ba \), for each pair of variables, so it is polynomials of this type, who will constitute indeterminates of our linear system.

The next task is to calculate the Koszul dual for \( A \). For that we write a system of linear equations on the monomials (polynomials), which should be present in the relations of the dual, namely those, who does not hold, otherwise the third term should have been 3585, since

\[
\frac{1}{H_A(-t)}
\]

Using this presentation of \( A^! \) we can calculate the Hilbert series of \( A^! \) by GRAAL. The calculation shows \( H_{A^!} = 1 + 16t + 240t^2 + 3584t^3 \). This means the equality which should be true for Koszul algebras:

\[
H_A^! = \frac{1}{H_A(-t)}
\]

does not hold, otherwise the third term should have been 3585, since \( H_A(t) = 1 + 16t + 16t^2 + t^3 \).

**Remark.** Similar calculations have been performed with various other choices of generators, and using another systems of symbolic computations in noncommutative algebras (e.g. Bergman).

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