CONTROLLABILITY OF THE STRONGLY DAMPED IMPULSIVE SEMILINEAR WAVE EQUATION WITH MEMORY AND DELAY

CRISTI GUEVARA\textsuperscript{1} AND HUGO LEIVA\textsuperscript{2}

ABSTRACT. This article is devoted to study the interior approximated controllability of the strongly damped semilinear wave equation with memory, impulses and delay terms. The problem is challenging since the state equation contains memory and impulsive terms yielding to potential unbounded control sequences steering the system to a neighborhood of the final state, thus fixed point theorems cannot be used directly. As alternative, the A.E Bashirov and et al. techniques are applied and together with the delay allow the control solution to be directed to fixed curve in a short time interval and achieve our result.

1. Introduction

The study of control systems has been of considerable interest for researchers motivated not only by engineering practices but also by biological process. Aiming to improve manufacturing processes, efficiency of energy use, biomedical experimentation, diagnosis, robotics, biological control, systems among others. And surprisingly, it has become of great interest in the social, political and economic spheres for understanding of the dynamics of business, social, and political systems.

Generally speaking, control theory tackles how the behavior of a systems can be modified by some feedback, specifically, how an arbitrary initial state can be directed, either exactly or approximated close, to a given final state using a set of admissible controls. In addition, in practical control systems, abrupt changes, delays and dependance on prior behavior are inherent phenomena and they would modify the controllability of the system. Thus, the conjecture is that controllability of a system won’t change due to perturbations such as delays, impulses or some type of memories.

In this paper, we are concern with the approximated controllability of strongly damped semilinear wave equation (1.1) with memory, impulses and delay terms in \( \Gamma = (0, \tau) \times \Omega \), with \( \Omega \subseteq \mathbb{R}^N \) \((N \geq 1)\) a bounded domain and \( \tau \in \mathbb{R}^+ \).
Setting J. S. Selvi, M. M. Arjunan [26] studied the exact controllability for impulsive differential systems.

In general, the existence of solutions for impulsive evolution equations with delays has been studied by N. Abada, M. Benchohra and H. Hammouche [1] and R. S. Jain and M. B. Dhakne in [16]. Moreover, the controllability of impulsive evolution equations is pretty well understood, just to mention some of the works, D. N. Chalishajar [8] studied the exact controllability of impulsive partial neutral functional differential equations with infinite delay and S. Selvi, M. M. Arjunan [26] studied the exact controllability for impulsive differential systems.

The following estimate holds

\[
\parallel w(t, x) \parallel \leq a_0 \sqrt{|w|^2 + |v|^2} + b_0.
\]

Equations of the form (1.1) appear in a number of different contexts. One of them is in the study of motion of viscoelastic materials. For instance, in one-dimensional case, they model longitudinal vibration of a uniform, homogeneous bar with non-linear stress law. In two-dimensional and three-dimensional cases they describe antiplane shear motions of viscoelastic solids (see for details [13, 17, 25]). Furthermore, the strongly damped wave equation with memory has been used to model the deviation from the equilibrium configuration of a (homogeneous and isotropic) linearly viscoelastic solid [13]. Note that including the impulses on the system (1.1), correspond to a pure mathematical interest and can physically interpret as unexpected changes of state whose duration is negligible in comparison with the duration of the process.

In general, the existence of solutions for impulsive evolution equations with delays has been studied by N. Abada, M. Benchohra and H. Hammouche [11] and R. S. Jain and M.B. Dhakne in [16]. Moreover, the controllability of impulsive evolution equations is pretty well understood, just to mention some of the works, D. N. Chalishajar [8] studied the exact controllability of impulsive partial neutral functional differential equations with infinite delay and S. Selvi, M. M. Arjunan [26] studied the exact controllability for impulsive differential systems.

Along with the initial-boundary conditions and impulses

\[
\left\{ \begin{array}{l}
  w(t, x) = 0, \\
  w(s, x) = \phi_1(s, x), \\
  w(t, x) = \phi_2(s, x), \\
  w(t_k^+, x) = w(t_k^-, x) + I_k(t_k, w(t_k, x), w(t_k, x), u(t_k, x)), \\
  \end{array} \right.
\]

in the space \( \mathcal{Z}^{1/2} = D ((-\Delta)^{1/2}) \times L^2(\Omega) \) as well as the distributed control \( u \in L^2([0, \tau], L^2(\Omega)) \) and \( \Phi = (\phi_1, \phi_2) \in C([-\tau, 0]; \mathcal{Z}^{1/2}) \). Here, \( \eta, \gamma, r \) are positive numbers, \( r \) represents delay, \( 1_\omega \) denotes the characteristic function on \( \omega \) an open nonempty subset of \( \Omega \).

Assuming that the memory \( M \in L^\infty((0, \tau) \times \Omega) \), the impulses \( I_k : [0, \tau] \times \mathbb{R}^3 \rightarrow \mathbb{R} \) for \( k = 1, 2, \ldots, p \), the nonlinear functions \( g : \mathbb{R} \rightarrow \mathbb{R} \), and \( f : [0, \tau] \times \mathbb{R}^3 \rightarrow \mathbb{R} \) are smooth enough so that the problem (1.1) admits only one mild solution on \([-\tau, \tau]\). Moreover, for \( u, w, v \in \mathbb{R} \) the following estimate holds

\[
\parallel f(t, w, v, u) \parallel \leq a_0 \sqrt{|w|^2 + |v|^2} + b_0.
\]

Equations of the form \( \text{(1.1)} \) appear in a number of different contexts. One of them is in the study of motion of viscoelastic materials. For instance, in one-dimensional case, they model longitudinal vibration of a uniform, homogeneous bar with non-linear stress law. In two-dimensional and three-dimensional cases they describe antiplane shear motions of viscoelastic solids (see for details [13, 17, 25]). Furthermore, the strongly damped wave equation with memory has been used to model the deviation from the equilibrium configuration of a (homogeneous and isotropic) linearly viscoelastic solid [13]. Note that including the impulses on the system \( \text{(1.1)} \), correspond to a pure mathematical interest and can physically interpret as unexpected changes of state whose duration is negligible in comparison with the duration of the process.

In general, the existence of solutions for impulsive evolution equations with delays has been studied by N. Abada, M. Benchohra and H. Hammouche [11] and R. S. Jain and M.B. Dhakne in [16]. Moreover, the controllability of impulsive evolution equations is pretty well understood, just to mention some of the works, D. N. Chalishajar [8] studied the exact controllability of impulsive partial neutral functional differential equations with infinite delay and S. Selvi, M. M. Arjunan [26] studied the exact controllability for impulsive differential systems.
with finite delay. L. Chen and G. Li [9] studied the approximate controllability of impulsive differential equations with nonlocal conditions, using measure of noncompactness and Monch fixed point theorem, and assuming that the nonlinear term \( f(t,z) \) does not depend on the control variable. Additionally, in a series of papers the approximate controllability of semilinear evolution equations with impulses is using the Rothe’s Fixed Point Theorem [7, 23, 21], and using A.E. Bashirov and et al. technique which avoid fixed point theorems [22].

Motivated by the work of H. Larez, H. Leiva, J. Rebaza and A. Rios [18, 19] on the approximate controllability of the semilinear strongly damped wave equation with and without impulses and our recent work on for the impulsive semilinear heat equation with memory and delay [15], we prove the interior approximate controllability of the strongly damped semilinear wave (1.1) with memory, impulses and delay terms achieved by applying A.E. Bashirov, N. Ghahramanlou, N. Mahmudov, N. Semi and H. Etikan technique [2, 3, 6] avoiding fixed point theorems.

The structure of this paper is as follow: In section 2, we present the abstract formulation of the strongly damped equation (1.1). Section 3 deals with the controllability of the linear problem. In section 4, the approximated controllability of the strongly damped equation with memory, delay and impulses is proved.

**Definition 1.1. (Approximate Controllability)** The system (1.1) is said to be approximately controllable on \( J \), if for every \( \Phi \in C([-r,0]; Z^{1/2}) \), \( z_1 \in Z^{1/2} \), and \( \epsilon > 0 \), there exists \( u \in L^2(0,\tau; L^2(\Omega)) \) such that the solution \( z(t) \) of (1.1) corresponding to \( u \) verifies:

\[
    z(0) = \Phi(0) \quad \text{and} \quad \| z(\tau) - z_1 \|_{Z^{1/2}} < \epsilon.
\]

The interior approximate controllability of the linear strongly damped wave equation

\[
    w_{tt} + \eta (-\Delta)^{1/2} w_t + \gamma (-\Delta) w = 1_{\omega} u(t,x),
\]

in \( \Gamma \), for all \( \tau > 0 \), with initial-boundary conditions

\[
    w(t,x) = 0, \quad w(t,x) = 0, \quad w_t(0,x) = w_1(x), \quad \text{in } \partial \Gamma,
\]

\[
    w_0(x) = w_0(x), \quad w_1(0,x) = w_1(x), \quad \text{in } \Omega,
\]

was proven in [18].

2. Formulation of the Problem

Let \( \mathcal{X} = L^2(\Omega) = L^2(\Omega,\mathbb{R}) \), consider the linear unbounded operator

\[
    \mathcal{A} : D(\mathcal{A}) \subset \mathcal{X} \rightarrow \mathcal{X},
\]

\[
    \phi \mapsto \mathcal{A}\phi = -\Delta \phi,
\]

where \( D(\mathcal{A}) = H^2(\Omega,\mathbb{R}) \cap H^1_0(\Omega,\mathbb{R}) \). Note that for \( \alpha \geq 0 \), the fractional powered spaces \( \mathcal{X}^\alpha \) are given by

\[
    \mathcal{X}^\alpha = D(\mathcal{A}^\alpha) = \left\{ x \in \mathcal{X} : \sum_{n=1}^{\infty} \lambda_n^{2\alpha} \| E_n x \|^2 < \infty \right\}
\]
endowed with the norm
\[ \|x\|_\alpha^2 = \|A_\alpha x\|^2 = \sum_{n=1}^{\infty} \lambda_n^{2\alpha} \|E_n x\|^2, \]
where \( \{E_j\} \) is a family of complete orthogonal projections in \( \mathcal{X} \); and for the Hilbert space \( Z^\alpha = \mathcal{X}^\alpha \times \mathcal{X} \) the corresponding norm is
\[ \left\| \begin{pmatrix} w \\ v \end{pmatrix} \right\|_{Z^\alpha}^2 = \|w\|^2 + \|v\|^2. \]

**Proposition 2.1.** Given \( j \geq 1 \), the operator \( P_j : Z^\alpha \to Z^\alpha \) defined by
\[ P_j = \begin{bmatrix} E_j & 0 \\ 0 & E_j \end{bmatrix} \quad \text{(2.4)} \]
is a continuous (bounded) orthogonal projections in the Hilbert space \( Z^\alpha \).

Details and proofs can be found in [18].

Now, the system (1.1)-(1.2) can be written as an abstract second order ordinary differential equation in \( \mathcal{X} \)
\[
\begin{cases}
  w'' + \eta A^{1/2} w' + \gamma A w = b_\varpi u + \int_0^t M(t, s) g^e(w(s-r)) dr, & t \in (0, \tau] \text{ and } t \neq t_k, \\
  w(s, \cdot) = \phi_1(s, \cdot), & w'(s, \cdot) = \phi_2(s, \cdot), & s \in [-r, 0], \\
  w'(t_k^+) = w'(t_k^-) + I^e_k(t_k, w(t_k), w'(t_k), u(t_k)), & k = 1, 2, \ldots, p,
\end{cases}
\]
where \( \mathcal{U} = \mathcal{X} = L^2(\Omega) \),
\[
I^e_k : [0, \tau] \times Z^{1/2} \times \mathcal{U} \longrightarrow \mathcal{X} \quad \text{ and } \\
f^e : [0, \tau] \times C(-r, 0; Z^{1/2}) \times \mathcal{U} \longrightarrow \mathcal{X} \quad \text{ and } \\
b_\varpi : \mathcal{U} \longrightarrow \mathcal{U} \quad \text{ and } \\
g^e : C(-r, 0; Z^{1/2}) \longrightarrow Z^{1/2} \quad \text{and } \\
\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \longrightarrow g(\phi_1(\cdot - r)).
\]

By changing variables \( v = w' \), the second order equation (2.5) is written as a first order system of ordinary differential equations with impulses and delay in the space \( Z^{1/2} = \mathcal{X}^{1/2} \times \mathcal{X} \) as
follows:
\[
\begin{cases}
z' = A z + B \varpi u + \int_0^t M_g(t, s, z(s - r)) ds + F(t, z(t - r), u(t)), & z \in Z^{1/2}, \\
\varphi(s) = \Phi(s), & s \in [-r, 0], \\
\varphi_{\kappa}(t, k) = \varphi(t, k) + I_{\kappa}(t, k, u(t)), & k = 1, 2, \ldots, p,
\end{cases}
\]
(2.6)
where
\[A = \left( \begin{array}{cc}
0 & I_x \\
-\gamma A & -\eta A^{1/2}
\end{array} \right)\]
is a unbounded linear operator with domain \(D(A) = D(A^0) \times D(A^{1/2})\),
\(I_x\) is the identity in \(X\), \(z = (w \ v)\), \(u \in C([0, \tau]; U)\), \(\Phi = (\phi_1 \ \phi_2) \in C(-r, 0; Z^{1/2})\),
\[B \varpi : U \to Z^{1/2} \quad u \mapsto (0 \ b \varpi u),\]
\[I_{\kappa} : [0, \tau] \times Z^{1/2} \times U \to Z^{1/2} \quad (t, z, u) \mapsto \left(0 \ I_{\kappa}(t, w, v, u)\right),\]
\[F : [0, \tau] \times C(-r, 0; Z^{1/2}) \times U \to Z^{1/2} \quad (t, \Phi, u) \mapsto \left(0 \ f^e(t, \phi_1(-r), \phi_2(-r), u)\right),\]
and
\[M_g : [0, \tau] \times [0, \tau] \times C(-r, 0; Z^{1/2}) \to Z^{1/2} \quad (t, s, \Phi) \mapsto \left(0 \ M(t, s) g^e(\Phi)\right).\]

The continuous inclusion \(X^{1/2} \subset X\), together with the condition (1.3), yields the following proposition

**Proposition 2.2.** The function \(F\) as defined by (2.7) satisfies
\[\|F(t, \Phi, u)\|_{Z^{1/2}} \leq \tilde{a} \|\Phi(\cdot - r)\|_{Z^{1/2}} + \tilde{b},\]
for all \((t, \Phi, u) \in [0, \tau] \times C(-r, 0; Z^{1/2}) \times U\) and \(\tilde{a}, \tilde{b} > 0\).

It has been proved in [10, 9] that the operator \(A\) generates a strongly continuous and analytic semigroup \(\{T(t)\}_{t \geq 0}\) in the space \(Z^{1/2} = X^{1/2} \times X\). Furthermore, Lemma 2.1 in [20] yields

**Proposition 2.3.** The semigroup \(\{T(t)\}_{t \geq 0}\), generated by the operator \(A\) is compact and represented by
\[T(t)z = \sum_{j=1}^{\infty} e^{A_j t} P_j z,\]
(2.9)
with $z \in Z^{1/2}$, $t \geq 0$, here $\{P_j\}_{j \geq 1}$ is a complete family of orthogonal projections \cite{2.4} in the Hilbert space $Z^{1/2}$ and $A_j = R_jP_j$ for $j = 1, 2, \cdots$, where

$$R_j = \begin{pmatrix} 0 & 1 \\ -\gamma \lambda_j & -\eta \lambda_j^{1/2} \end{pmatrix},$$

with eigenvalues $\lambda = -\lambda_j^{1/2} \left( \frac{\eta + \sqrt{\eta^2 - 4\gamma}}{2} \right)$. Moreover, $e^{A_j t} = e^{R_j t} P_j$, and $A_j^* = R_j^* P_j$

3. CONTROLLABILITY OF THE LINEAR SYSTEM

In this section we present some characterization of the interior approximate controllability of the linear strongly damped wave equations.

The initial value problem

$$\begin{cases} z' = A z + B \varpi u, \\ z(t_0) = z_0, \end{cases}$$

with $z, z_0 \in Z^{1/2}$ and $u \in L^2(0, \tau; U)$, admits only one mild solution for $t \in [t_0, \tau]$, given by

$$z(t) = T(t - t_0)z_0 + \int_{t_0}^{t} T(t - s) B \varpi u(s) ds. \quad (3.11)$$

**Definition 3.1.** For system \eqref{3.10} and $\tau > 0$, the controllability map is given by

$$G_{\tau \delta} : L^2(\tau - \delta, \tau; U) \longrightarrow Z^{1/2}$$

$$u \longmapsto \int_{\tau - \delta}^{\tau} T(\tau - s) B \varpi u(s) ds, \quad (3.12)$$

and its adjoint by

$$G_{\tau \delta}^* : Z^{1/2} \longrightarrow L^2(\tau - \delta, \tau; U)$$

$$z \longmapsto B^* \varpi T^* (\tau - \cdot) z.$$

The Gramian controllability operator is

$$Q_{\tau \delta} = G_{\tau \delta} G_{\tau \delta}^* = \int_{\tau - \delta}^{\tau} T(\tau - t) B \varpi B^* T^* (\tau - t) dt. \quad (3.13)$$

**Lemma 3.1.** The following statements are equivalent to the approximate controllability of the linear system \eqref{3.10} on $[\tau - \delta, \tau]$.

(a) $\text{Rang}(G_{\tau \delta}) = Z^{1/2}$.
(b) $\text{Ker}(G_{\tau \delta}^*) = \{0\}$.
(c) If $0 \neq z \in Z^{1/2}$, then $\langle Q_{\tau \delta} z, z \rangle > 0$.
(d) $\lim_{\alpha \to 0^+} \alpha (\alpha I + Q_{\tau \delta})^{-1} z = 0$. 
(e) For all \( z \in \mathcal{Z} \), \( 0 < \alpha \leq 1 \) and \( u_\alpha = G^*_\tau(\alpha I + Q_{\tau\delta})^{-1}z \) we have that
\[
G_{\tau\delta}u_\alpha = z - \alpha(\alpha I + Q_{\tau\delta})^{-1}z
\]
Hence,
\[
\lim_{\alpha \to 0} G_{\tau\delta}u_\alpha = z
\]
and error \( E_{\tau\delta}z \) of this approximation is
\[
E_{\tau\delta}z = \alpha(\alpha I + Q_{\tau\delta})^{-1}z.
\]
In addition, for each \( v \in L^2(\tau - \delta, \tau; \mathcal{U}) \), the sequence of controls
\[
u_\alpha = G^*_\tau(\alpha I + Q_{\tau\delta})^{-1}z + (v - G^*_\tau(\alpha I + Q_{\tau\delta})^{-1}G_{\tau\delta}v),
\]
satisfies
\[
G_{\tau\delta}u_\alpha = z - \alpha(\alpha I + Q_{\tau\delta})^{-1}(z + G_{\tau\delta}v)
\]
and
\[
\lim_{\alpha \to 0} G_{\tau\delta}u_\alpha = z,
\]
with error \( E_{\tau\delta}z = \alpha(\alpha I + Q_{\tau\delta})^{-1}(z + G_{\tau\delta}v) \).

In general, for a pair \( \mathcal{W} \) and \( \mathcal{Z} \) of Hilbert spaces, a linear bounded operator \( G : \mathcal{W} \to \mathcal{Z} \)
with dense range Lemma 3.1 also holds and the proof can be found in [4, 5, 11, 12, 24]. This Lemma implies that for \( 0 < \alpha \leq 1 \), the family of linear operators \( \Gamma_{\alpha\tau\delta} : Z^{1/2} \to L^2(\tau - \delta, \tau; \mathcal{U}) \), defined by
\[
\Gamma_{\alpha\tau\delta}z = G_{\tau\delta}^*\alpha I + Q_{\tau\delta})^{-1}z,
\]
is an approximated right inverse operator of \( G_{\tau\delta} \), in the sense that
\[
\lim_{\alpha \to 0} G_{\tau\delta}\Gamma_{\alpha\tau\delta} = I.
\]
in the strong topology.

**Lemma 3.2.** \( Q_{\tau\delta} > 0 \), if and only if, the linear system (3.10) is controllable on \([\tau - \delta, \tau]\). Moreover, given an initial state \( y_0 \) and a final state \( z^1 \) we can find a sequence of controls \( \{u_\alpha^\delta\}_{0 < \alpha \leq 1} \subset L^2(\tau - \delta, \tau; \mathcal{U}) \)
\[
u_\alpha = u_\alpha^\delta = G^*_\tau(\alpha I + G_{\tau\delta}G^*_\tau)^{-1}(z^1 - T(\tau)y_0),
\]
such that the solutions \( y(t) = y(t, \tau - \delta, y_0, u_\alpha) \) of the initial value problem
\[
\begin{cases}
y' = Ay + Bu_\alpha(t), & y \in \mathcal{Z}, \quad t > 0, \\
y(\tau - \delta) = y_0,
\end{cases}
\]
satisfies
\[
\lim_{\alpha \to 0^+} y(\tau, \tau - \delta, y_0, u_\alpha) = z^1.
\]
that is,
\[
\lim_{\alpha \to 0^+} y(\tau) = \lim_{\alpha \to 0^+} \left\{ T(\delta)y_0 + \int_{\tau - \delta}^{\tau} T(\tau - s)Bu_\alpha(s)ds \right\} = z^1.
\]
4. CONTROLLABILITY OF THE SEMILINEAR SYSTEM

In this section we prove the interior approximate controllability of the semilinear strongly damped wave equation with memory, impulses and delay, Theorem 4.1. The idea behind of the proof is that for a given final state $z_1$, a sequence of controls is constructed, so that the initial condition $\Phi$ is steering to a small ball around $z_1$. All of this is achieved thanks to the delay, which allows to pullback the family of controls solutions to a fixed trajectory in short time interval, as illustrated below:

**Theorem 4.1.** The wave equation (1.1) under (1.2) and (1.3) is approximately controllable on $[0, \tau]$ with $\tau > 0$.

**Proof.** Provided $\epsilon > 0$, $\Phi \in C$ and a final state $z_1$, we are aiming to find a control $u_\alpha^\delta \in L^2([0, \tau]; U)$ such that it evolves the system (1.1) from $\Phi(0)$ to an $\epsilon$– ball around $z_1$ on time $\tau$. That is, considering the the abstract formulation for (1.1) discussed in section, recall that for all $\Phi \in C(-r, 0; Z^{1/2})$ and $u \in C(0, \tau; U)$ the system (2.6) given by

\[
\begin{aligned}
\frac{dz}{dt} &= Az + B \varpi u + \int_0^t M_g(t,s,z(s-r))ds + F(t,z(t-r),u(t)), \quad z \in Z^{1/2}, \\
z(s) &= \Phi(s), \\
z(t^+_k) &= z(t^-_k) + I_k(t_k, z(t_k), u(t_k)), \
\end{aligned}
\]

We want to prove that, for $\alpha > 0$ and $0 < \delta < \min\{\tau - t_p, r\}$, there exists control $u_\alpha^\delta \in L^2([0, \tau]; U)$ such that corresponding of solutions $z^{\delta \alpha}$ of (2.6) satisfies:

\[
\| z^{\delta \alpha}(\tau) - z_1 \| \leq \epsilon,
\]

and

\[
z^{\delta \alpha}(t) = T(t)\Phi(0) + \int_0^t T(t-s) \left[ B \varpi u(s) + \int_0^s M_g(t,s,z(l-r))dl \right] ds + \\
+ \int_0^t T(t-s) F(s,z(s-r),u(s))ds + \sum_{0 < t_k < t} T(t-t_k)I_k(t_k, z(t_k), u(t_k)),
\]

(4.17)
Since the corresponding solution $y(t) = z(t, 0, \Phi, u)$ of the initial value problem (2.6) and define the controls $u^\alpha(t) \in L^2([0, \tau]; \mathcal{U})$ for $\alpha \in (0, 1]$, such that

$$u^\alpha(t) = \begin{cases} u(t), & 0 \leq t \leq \tau - \delta, \\ u_\alpha(t) = \mathbb{B}^* T^* (\tau - t)(\alpha I + G_{\tau \delta} G^*_{\tau \delta})^{-1} (z_1 - T(\delta) z(\tau - \delta)), & \tau - \delta \leq t \leq \tau, \end{cases}$$

hence,

$$z^{\delta,\alpha}(\tau) = T(\tau) \Phi(0) + \int_0^\tau T(\tau - s) \left[ \mathbb{B}_\omega u^\alpha(s) + \int_0^s M_g(z^{\delta,\alpha}(l - r)) dl \right] ds +$$

$$+ \int_0^\tau T(\tau - s) F(s, z^{\delta,\alpha}(s - r), u^\alpha(s)) ds + \sum_{0 < t_k < \tau} T(t - t_k) \{ k(t_k, z^{\delta,\alpha}(t_k), u^\alpha(t_k)) \}$$

$$= T(\delta) \left\{ T(\tau - \delta) \Phi(0) + \int_0^{\tau - \delta} T(\tau - \delta - s) \left( \mathbb{B}_\omega u^\alpha(s) + F(s, z^{\delta,\alpha}(s - r), u^\alpha(s)) \right) ds + \int_0^\tau T(\tau - s) \int_0^s M_g(s, l, z^{\delta,\alpha}(l - r)) dl ds$$

$$+ \sum_{0 < t_k < \tau - \delta} T(t - \tau - t_k) \{ k(t_k, z^{\delta,\alpha}(t_k), u^\alpha(t_k)) \} \right\} +$$

$$+ \int_0^\tau T(\tau - s) \left( \mathbb{B} u^\alpha(s) + F(s, z^{\delta,\alpha}(s - r), u^\alpha(s)) + \int_0^s M_g(s, l, z^{\delta,\alpha}(l - r)) dl \right) ds,$$

hence,

$$z^{\delta,\alpha}(\tau) = T(\delta) z(\tau - \delta) + \int_0^\tau T(\tau - s) \left( \mathbb{B} u^\alpha(s) + F(s, z^{\delta,\alpha}(s - r), u^\alpha(s)) \right) ds$$

$$+ \int_0^\tau T(\tau - s) \int_0^s M_g(s, l, z^{\delta,\alpha}(l - r)) dl ds.$$
As consequence, the following estimate yields
\[
\|z^{\delta,\alpha}(\tau) - y^{\delta,\alpha}(\tau)\| \leq \int_{\tau-\delta}^{\tau} \|T(\tau-s)\| \left( \tilde{a} \|\Phi(s-r)\| + \tilde{b} \right) ds \\
+ \int_{\tau-\delta}^{\tau} \|T(\tau-s)\| \int_{0}^{s} \|M_g(s,l,z^{\delta,\alpha}(l-r))\| dlds.
\]

Notice that \(0 < \delta < r\) and \(\tau - \delta \leq s \leq \tau\), thus
\[l - r \leq s - r \leq \tau - r < \tau - \delta,
\]

hence
\[z^{\delta,\alpha}(l-r) = z(l-r) \quad \text{and} \quad z^{\delta,\alpha}(s-r) = z(s-r),
\]

which implies that there exists \(\delta > 0\) such that
\[
\|z^{\delta,\alpha}(\tau) - y^{\delta,\alpha}(\tau)\| \leq \int_{\tau-\delta}^{\tau} \|T(\tau-s)\| \left( \tilde{a} \|z(s-r)\| + \tilde{b} \right) ds \\
+ \int_{\tau-\delta}^{\tau} \|T(\tau-s)\| \int_{0}^{s} \|M_g(s,l,z(l-r))\| dlds \\
< \frac{\epsilon}{2}.
\]

Additionally, by Lemma 3.2 we can chose \(\alpha > 0\) such that
\[
\|z^{\delta,\alpha}(\tau) - z_1\| \leq \|z^{\delta,\alpha}(\tau) - y^{\delta,\alpha}(\tau)\| + \|y^{\delta,\alpha}(\tau) - z_1\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\]

which completes our proof.

\[
5. \text{ Final Remarks}
\]

The technique employed here can be applied for the impulsive semilinear beam equation with memory and delay in a bounded domain \(\Omega \subseteq \mathbb{R}^N \ (N \geq 1)\):

\[
\begin{align*}
\frac{\partial^2 y(t,x)}{\partial t^2} &= 2\beta \Delta \frac{\partial y(t,x)}{\partial t} - \Delta^2 y(t,x) + \int_{0}^{t} M(t-s)g(y(s-r,x))ds \\
&\quad + u(t,x) + f(t,y(t-r),y_t(t-r),u), \text{ in } \Omega_\tau, \ t \neq t_k, \\
y(t,x) &= \Delta y(t,x) = 0, \text{ on } \Sigma_\tau, \\
y(s,x) &= \phi(s,x), \ y_t(s,x) = \psi(s,x), \text{ in } \Omega_{-r}, \\
y_t(t_k^+,x) &= y_t(t_k^-,x) + I_k(t_k,y(t_k,x),y_t(t_k,x),u(t_k,x)), \quad k = 1, \ldots, p,
\end{align*}
\]

where \(\Omega_\tau = (0,\tau] \times \Omega, \Sigma_\tau = (0,\tau) \times \partial \Omega, \Omega_{-r} = [-r,0] \times \Omega\).

Furthermore, we believe that the same technique can be applied for controlling diffusion processes systems involving compact semigroups. In particular, our result can be formulated in
a more general setting for the semilinear evolution equation with impulses, delay and memory in a Hilbert space $\mathcal{Z}$

\[
\begin{aligned}
&z' = A z + \mathbb{B}_\infty u + \int_0^t M_g(t, s, z(s - r))ds + F(t, z(t - r), u(t)), \quad z \in \mathcal{Z} \\
&z(s) = \Phi(s), \\
&z(t_k^+) = z(t_k^-) + I_k(t_k, z(t_k), u(t_k)),
\end{aligned}
\]

where $u \in C([0, \tau]; \mathcal{U})$, $\mathcal{U}$ is another Hilbert space, $B : \mathcal{U} \rightarrow \mathcal{Z}$ is a bounded linear operator, $J_k$, $F : [0, \tau] \times C(-r, 0; \mathcal{Z}) \times \mathcal{U} \rightarrow \mathcal{Z}$, $A : D(A) \subset \mathcal{Z} \rightarrow \mathcal{Z}$ is an unbounded linear operator in $\mathcal{Z}$ that generates a strongly continuous semigroup according to Lemma 2.1 from [20]:

\[
T(t)z = \sum_{nj=1}^\infty e^{Aj}P_jz, \quad z \in \mathcal{Z}, \quad t \geq 0,
\]

where $\{P_j\}_{j \geq 0}$ is a complete family of orthogonal projections in the Hilbert space $\mathcal{Z}$ and

\[
\|F(t, \Phi, u)\|_{\mathcal{Z}} \leq \tilde{a}\|\Phi(-r)\|_{\mathcal{Z}^{1/2}} + \tilde{b}, \quad \forall (t, \Phi, u) \in [0, \tau] \times C(-r, 0; \mathcal{Z}^{1/2}) \times \mathcal{U}.
\]

**References**

[1] N. Abada, M. Benchohra and H. Hammouch, *Existence Results for Semilinear Differential Evolution Equations with Impulses and Delay*, CUBO A Mathematical Journal, Vol. 02, (1-17), June 2010.

[2] A. E. Bashirov and N. Ghahramanlou, *On Partial Approximate Controllability of Semilinear Systems*, Cogent Engineering, 1(1):965947, 2014.

[3] A.E. Bashirov and N. Ghahramanlou, *On Partial Complete Controllability of Semilinear Systems*. Abstract and Applied Analysis, Vol. 2013, Article ID 52105, 8 pages.

[4] A.E. Bashirov and K.R. Kerimov, *On Controllability Conception for Stochastic Systems*. SIAM Journal on Control and Optimization, 35 (1997), n. 2, pp.384-398.

[5] A.E. Bashirov and N. Mahmudov, *On Controllability of Deterministic and Stochastic Systems*. SIAM Journal on Control and Optimization, 37 (1999), n. 6, pp.1808-1821.

[6] A.E. Bashirov, N. Mahmudov, N. Semi and H. Etikan, *No Partial Controllability Concepts*. International Journal of Control, Vol. 80, N. 1, January 2007, 1-7.

[7] A. Carrasco, H. Leiva, J.L. Sanchez and A. Tineo M, *Approximate Controllability of the Semilinear Impulsive Beam Equation*. Transaction on IoT and Cloud Computing 2(3) 70-88, 2014.

[8] D. N. Chalishajar, *Controllability of Impulsive Partial Neutral Functional Differential Equation with Infinite Delay*. Int. Journal of Math. Analysis, Vol. 5, 2011, N. 8, 369-380.

[9] L. Chen and G. Li, *Approximate Controllability of Impulsive Differential Equations with Nonlocal Conditions*. International Journal of Nonlinear Science, Vol.10(2010), N. 4, pp. 438-446.

[10] S. Chen and R. Triggiani, *Proof of Extensions of two Conjectures on Structural Damping for Elastic Systems*, Pacific Journal of Mathematics, Vol. 136, N01, 1989.

[11] R. F. Curtain, and A. J. Pritchard, *Infinite Dimensional Linear Systems*. Lecture Notes in Control and Information Sciences, 8, Springer Verlag, Berlin (1978).

[12] R. F. Curtain and H. J. Zwart, *An Introduction to Infinite Dimensional Linear Systems Theory*. Text in Applied Mathematics, 21, Springer Verlag, New York (1995).

[13] G. Duvaut and J-L. Lions, *Inequalities in Mechanics and Physics*, Springer, Berlin, 1976.

[14] W.N. Findley, J.S. Lai, and K.O. Onaran, *Creep and Relaxation of Nonlinear Viscoelastic Materials*. North-Holland, Amsterdam, New York (1976)

[15] C. Guevara and H. Leiva, *Controllability of the Impulsive Semilinear Heat Equation with Memory and Delay* H. J Dyn Control Syst (2016). doi:10.1007/s10883-016-9352-5

[16] R.S. Jain and M.B. Dhakne, *On Mild Solutions of Nonlocal Semilinear Impulsive Functional Integro-Differential Equations*, Applied Mathematics E-Notes, 13(2014), 109-119.
[17] J.K. Knowles, One finite antiplane shear for incompressible elastic material J. Aust. Math. Soc. Ser. B, 19 (1975/1976), pp. 400-415
[18] H. Larez, H. Leiva and J. Rebaza, Approximate Controllability of a Damped Wave Equation, Canadian Applied Math. Quarterly, Vol. 20, No. 3, Fall 2012.
[19] H. Larez, H. Leiva, J. Rebaza and A. Rios, Approximate Controllability of Semilinear Impulsive Strongly Damped Wave Equation, J. Appl. Anal. 21(1) June 2015, DOI: 10.1515/jaa-2015-0005.
[20] H. Leiva, A Lemma on $C_0$-Semigroups and Applications PDEs Systems Quaestiones Mathematicae, Vol. 26, pp. 247-265 (2003).
[21] H. Leiva, Approximate Controllability of Semilinear Impulsive Evolution Equations, Abstract and Applied Analysis, Vol. 2015, Article ID 797439, 7 pages
[22] H. Leiva, Controllability of the Semilinear Heat Equation with Impulses and Delay on the State, Nonautonomous Dynamical Systems, 2015: 2:52-62.
[23] H. Leiva and N. Merentes, Approximate Controllability of the Impulsive Semilinear Heat Equation. Journal of Mathematics and Applications, N. 38, pp 85-104 (2015)
[24] H. Leiva, N. Merentes and J. Sanchez, A Characterization of Semilinear Dense Range Operators and Applications, Abstract and Applied Analysis, Vol. 2013, 1–11.
[25] V.P. Maslov, P.P. Mosolov Nonlinear Wave Equations Perturbed by Viscous Terms Walter de Gruyter, Berlin, New York (2000)
[26] S. Selvi and M.M. Arjunan, Controllability Results for Impulsive Differential Systems with Finite Delay J. Nonlinear Sci. Appl. 5 (2012), 206-219.

1 Louisiana State University, College of Science, Department of Mathematics, Baton Rouge, LA 70803-USA

E-mail address: cguevara@lsu.edu, cristi.guevara@asu.edu

2 Yachaytech University, School of Mathematical Science and Information Technology, Hacienda San José, San Miguel de Urcuquí, Ecuador

E-mail address: hleiva@yachaytech.edu.ec, hleiva@ula.ve