Research Article

Extended Error Expansion of Classical Midpoint Rectangle Rule for Cauchy Principal Value Integrals on an Interval

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Received 21 January 2021; Revised 17 February 2021; Accepted 14 March 2021; Published 24 March 2021

Academic Editor: Zhaoqing Wang

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The classical composite midpoint rectangle rule for computing Cauchy principal value integrals on an interval is studied. By using a piecewise constant interpolant to approximate the density function, an extended error expansion and its corresponding superconvergence results are obtained. The superconvergence phenomenon shows that the convergence rate of the midpoint rectangle rule is higher than that of the general Riemann integral when the singular point coincides with some priori known points. Finally, several numerical examples are presented to demonstrate the accuracy and effectiveness of the theoretical analysis. This research is meaningful to improve the accuracy of the collocation method for singular integrals.

1. Introduction

Singular integrals, especially Cauchy principal value integrals, are usually encountered in the fields of Boundary Element Method (BEM) [1–3], for example, the fluid mechanics, the elasticity and fracture mechanics, the acoustics, and the electromagnetics. In these fields (including their related physical problems), much attention has been paid to the Cauchy principal value integrals [4–8]. We now consider the following integral:

\[ I(f; s) = \int_a^b f(x) \frac{dx}{x - s}, \quad s \in (a, b), \]  

(1)

where \( f(x) \) is Holder continuous on the interval \([a, b]\), \( f(x)/(x - s) \) is the density function, \( c.p \int_a^b \) denotes a Cauchy principal value integral, and \( s \) is the singular point. There are many ways to define equation (1). However, these definitions can be proved to be mathematically equivalent. In this paper, we adopt the following definition:

\[ c.p \int_a^b \frac{f(x)}{x - s} \frac{dx}{x - s} = \lim_{\epsilon \to 0} \left\{ \int_a^{s-\epsilon} \frac{f(x)}{x - s} \frac{dx}{x - s} + \int_{s+\epsilon}^b \frac{f(x)}{x - s} \frac{dx}{x - s} \right\}, \quad s \in (a, b). \]

(2)

For this level of regularity of integrand, some methods based on the Chebyshev series expansion are much effective. They are easy to include the adaptive feature and can be applied to much difficult integrands [9–11]. At the same time, there are numerous works that have been devoted to developing efficient quadrature formulas, such as the Gaussian method [12,13], the Newton–Cotes method [14–16], the spline method [17,18], and some other methods [19–25]. Usually, Gaussian rules have good accuracy if the integrand is smooth, while Newton–Cotes rules are attractive due to their ease of implementation and flexibility of mesh. To improve the accuracy of boundary element analysis, an efficient method called general (composite) Newton–Cotes rule has been studied and used to compute Cauchy principal value integrals and Hadamard finite-part integrals [26, 27]. When the singular point \( s \) coincides with some priori known points, Newton–Cotes rules can reach a high-order convergence rate [28, 29]. This is the so-called pointwise superconvergence phenomenon of the Newton–Cotes rules.

In this paper, we will focus on the superconvergence phenomenon of midpoint rectangle rules for Cauchy principal integrals with the density function \( f(x)/(x - s) \) being replaced by the approximation function...
(f(\(\bar{x}_i\))/(\(\bar{x}_i - s\))), \(i = 0, 1, \ldots, n - 1\), where \(\bar{x}_i\) is the middle point of each subinterval of \([a, b]\). Different from the idea provided by Linz in [30] to calculate the hypersingular integral on an interval, we will present a direct method to compute the Cauchy principal integral. Based on the error estimate [31–33], the error function is determined by a certain special function \(S_0(\tau)\). We will also give the necessary and sufficient conditions to be satisfied by the superconvergence points. In addition, we will not only try to obtain the error estimate of the superconvergence phenomenon but also make some investigation about the superconvergence points.

The rest of this paper is organized as follows. In Section 2, some basic formulas of the midpoint rectangle rule are introduced, and our main results are presented. In Section 3, some lemmas are given and the proof of the main results is completed. In Section 4, several numerical examples are provided to validate our analysis. At last, the concluding remarks are presented.

2. The Superconvergence of the Composite Midpoint Rectangle Rule

Let \(a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b\) be a uniform partition of the interval \([a, b]\) with a mesh size \(h = (b - a)/n\). We first define the piecewise constant interpolant for \(f(x)\):

\[
f_C(x) = f(\bar{x}_i), \quad \bar{x}_i = x_i + \frac{h}{2}, \quad x \in (x_i, x_{i+1}),
\]

and then, we define a linear transformation:

\[
x = \bar{x}_i (\tau) = \frac{(\tau + 1)(x_{i+1} - x_i)}{2} + x_i, \quad \tau \in [-1, 1],
\]

which maps the reference element \([-1, 1]\) onto the subinterval \([x_i, x_{i+1}]\). By replacing \((f(x))/(x - s)\) in equation (1) with \((f_C(x))/(\bar{x}_i - s)\), we can obtain the following composite midpoint rectangle rule:

\[
I_n(f; s) = c \cdot p \int_a^b \frac{f_C(x)}{x - s} \, dx = \sum_{i=0}^{n-1} w_i(s) f(\bar{x}_i)
\]

\[
= c \cdot p \int_a^b \frac{f(x)}{x - s} \, dx - E_n(f; s),
\]

where \(w_i(s)\) denotes the Cotes coefficient given by \(w_i(s) = (h/\bar{x}_i - s)\) and \(E_n(f; s)\) is the error function.

Theorem 1 (see [5]). Assume that \(f(x) \in C^1[a, b], \quad a \in (0, 1)\). For the midpoint rectangle rule \(I_n(f; s)\) defined in equation (5), assume that \(s \in [x_m, x_{m+1}], \quad s \neq \bar{x}_m, \quad m = 0, 1, 2, \ldots, n - 1, \) and there exists a positive constant \(C\) that is independent of \(h\) and \(s\), such that

\[
|E_n(f; s)| \leq C(\ln h + |\ln \gamma(\tau)|)h^a,
\]

where

\[
\gamma(\tau) = \min_{0 \leq |x_\tau| \leq 1} \left| \frac{s - x_\tau}{h} \right| = \frac{1 - |\tau|}{2}.
\]

Before presenting the main results, we first define

\[
\phi_0(x) = \left\{
\begin{array}{ll}
\frac{1}{2\pi} c \cdot p \int_{|x| < 1} \frac{r}{r - x} \, dr, & |x| < 1, \\
\frac{1}{2\pi} c \cdot p \int_{|x| > 1} \frac{r}{r - x} \, dr, & |x| > 1,
\end{array}
\right.
\]

\[
S_0(\tau) = \phi_0(\tau) + \sum_{j=1}^{\infty} \left[ \phi_0(2i + \tau) + \phi_0(-2i + \tau) \right], \quad \tau \in (-1, 1).
\]

Theorem 2. Assume that \(f(x) \in C^l[a, b], \quad l \geq 2\). For the midpoint rectangle rule \(I_n(f; s)\) defined in equation (5), assume that \(s = x_m + (1 + \tau)h/2, \quad s \neq \bar{x}_m, \quad m = 0, 1, 2, \ldots, n - 1, \) and there exists a positive constant \(C\) that is independent of \(h\) and \(s\), such that

\[
E_n(f; s) = -f(s)S_0(\tau) + R_n(s),
\]

where

\[
|R_n(s)| \leq C \left( \ln h + \frac{\gamma^l(\tau)}{h^l} \right) h^l,
\]

where \(\gamma(\tau)\) is defined by equation (7) and

\[
\eta(s) = \max \left\{ \frac{1}{s - a}, \frac{1}{b - s} \right\}.
\]

Remark 1. Under the same assumptions of Theorem 2, when \(\tau^*\) is the zero of \(S_0(\tau^*)\), we have

\[
|E_n(f; s)| \leq C \left( \ln h + \frac{\gamma^l(\tau^*)}{h^l} \right) h^l.
\]

3. The Proof of the Main Results

In this section, we mainly complete the proof of Theorem 2.

3.1. Some Necessary Lemmas

Lemma 1. Under the same assumptions of Theorem 2, there holds that

\[
(x_i - s)f(x) - (x - s)f(\bar{x}_i) = f(s)(\bar{x}_i - x)
\]

\[
+ \sum_{k=1}^{l-1} \frac{f^{(k)}(s)}{k!}(x - s)^k(\bar{x}_i - x)
\]

\[
- \sum_{k=1}^{l-1} \sum_{j=k}^{l-1} \frac{f^{(j)}(s)}{k!(j-k)!}(x - s)^{j-k+1}(\bar{x}_i - x)^k + R_f(x),
\]

where
\( R_j(x) = R^1_j(x) + R^2_j(x) + R^3_j(x) \),

with

\[
R^1_j(x) = \frac{f^{(i)}(\theta)}{i!} (x-s)(\bar{x}_i - x),
\]

\[
R^2_j(x) = \frac{f^{(i)}(\theta)}{i!} (x-s)^i(\bar{x}_i - x),
\]

\[
R^3_j(x) = -\sum_{k=1}^{l-1} \frac{f^{(i)}(\theta)}{k! (l-k)!} (x-s)^{l-k+1}(\bar{x}_i - x),
\]

and \( \theta_1, \theta_2 \in (x_i, x_{i+1}), \theta_3 \in (x, s) \) or \( (s, x) \).

**Proof.** Note that \( f(x) \in C^4[a, b] \). By applying the following Taylor expansion to \( f(\bar{x}_i) \) at the point \( x \), we obtain

\[
f(\bar{x}_i) = f(x) + \sum_{k=1}^{l-1} \frac{f^{(k)}(x)}{k!}(\bar{x}_i - x)^k
\]

\[
+ \frac{f^{(i)}(\theta)}{i!}(\bar{x}_i - x)^i, \quad \theta_1 \in (x_i, x_{i+1}).
\]

Similarly, we have

\[
f(x) = f(s) + \sum_{k=1}^{l-1} \frac{f^{(k)}(s)}{k!}(x-s)^k
\]

\[
+ \frac{f^{(i)}(\theta)}{i!}(x-s)^i, \quad \theta_2 \in (x_i, x_{i+1}).
\]

Thus,

\[
(\bar{x}_i - s)f(x) - (x-s)f(\bar{x}_i) = f(s)(\bar{x}_i - x)
\]

\[
+ \sum_{k=1}^{l-1} \frac{f^{(k)}(s)}{k!}(x-s)^k(\bar{x}_i - x)
\]

\[
- \sum_{k=1}^{l-1} \frac{f^{(k)}(x)}{k!}(x-s)^k(\bar{x}_i - x) - \frac{f^{(i)}(\theta)}{i!}(\bar{x}_i - x)^i(x-s)
\]

\[
+ \frac{f^{(i)}(\theta)}{i!}(x-s)^i(\bar{x}_i - x).
\]

On the contrary, for \( k = 1, 2, \ldots, l-1 \), we have

\[
f^{(k)}(x) = f^{(k)}(s) + f^{(k+1)}(s)(x-s) + \ldots + \frac{f^{(i)}(\theta)}{(i-k)!}(x-s)^{i-k}
\]

\[
= \sum_{j=k}^{l-1} \frac{f^{(j)}(s)}{(j-k)!}(x-s)^{j-k} + \frac{f^{(i)}(\theta)}{(i-k)!}(x-s)^{i-k},
\]

where \( \theta_i \in (x, s) \) or \( (s, x) \). According to equations \( 18 \) and \( 19 \), we can obtain equation \( 13 \). The proof is completed.

Define the error function:

\[
E_m(x) = f(x) - \frac{f(x_m) - f(x)}{(x_m - x)}
\]

\[
- \sum_{k=1}^{l-1} \frac{f^{(k)}(s)(x-s)^k}{k!}(x_m - x)
\]

\[
+ \sum_{k=1}^{l-1} \frac{f^{(i)}(s)(x-s)^{i-k}}{(i-k)!}(x_m - x)^i
\]

**Lemma 2.** Assume that \( s \in (x_i, x_{i+1}) \) for the integer \( m \) and let \( c_i = (2s - x_i)/h - 1 \), \( 0 \leq i \leq n-1 \). Then, we have

\[
\varphi_0(c_i) = \begin{cases} 
-\frac{1}{2}c.p \int_{x_i}^{x_{m+i}} \frac{\tilde{x}_m - x}{(x-s)(\tilde{x}_m - s)} \, dx, & i = m, \\
\frac{1}{2}c.p \int_{x_i}^{x_{m+i}} \frac{\tilde{x}_m - x}{(x-s)(\tilde{x}_m - s)} \, dx, & i \neq m.
\end{cases}
\]

**Proof.** If \( i = m \), by using the definition of equation \( 2 \) and the linear transformation of equation \( 4 \), we have

\[
c.p \int_{x_i}^{x_{m+i}} \frac{\tilde{x}_m - x}{(x-s)(\tilde{x}_m - s)} \, dx
\]

\[
= \lim_{\tau \to 0} \left\{ \int_{x_i}^{x_{m+i}} \frac{\tilde{x}_m - x}{(x-s)(\tilde{x}_m - s)} \, dx + \int_{x_{m+i}}^{x_{m+i}} \frac{\tilde{x}_m - x}{(x-s)(\tilde{x}_m - s)} \, dx \right\}
\]

\[
= c.p \int_{x_i}^{x_{m+i}} \frac{\tilde{x}_m - x}{(x-s)(\tilde{x}_m - s)} \, dx
\]

\[
= -2\varphi_0(c_m).
\]

If \( i \neq m \), it can be proved by applying the same approach to the corresponding Riemann integral. The proof is completed. \( \square \)

**Lemma 3.** Under the same assumptions of Theorem 2, for the function \( E_m(x) \) in equation \( 20 \), there holds that

\[
c.p \int_{x_i}^{x_{m+i}} E_m(x) \, dx \leq C h^i |\ln y(\tau)|,
\]

where \( y(\tau) \) is defined in equation \( 7 \).

**Proof.** Since \( f(x) \in C^4[a, b] \), by using Taylor expansion, we have

\[
\left| E_m^{(i)}(x) \right| \leq C h^{i-1}, \quad i = 0, 1.
\]

From the following defined relationship,

\[
c.p \int_{x_i}^{b} f(x) \frac{dx}{x-s} = \int_{a}^{b} f(x) \frac{dx}{x-s} + f(s) \ln \frac{b-s}{s-a}
\]

\[
\int_{a}^{b} f(x) \frac{dx}{x-s} = \int_{a}^{b} f(x) \frac{dx}{x-s} + f(s) \ln \frac{b-s}{s-a}
\]
From the definition of $E_m(x)$ in equation (20), we have

\[ c.p \int_{x_m}^{x_{m+1}} E_m(x) \, dx = c.p \int_{x_m}^{x_{m+1}} \frac{E_m(x) - E_m(s)}{x - s} \, dx + f(s)c.p \int_{x_m}^{x_{m+1}} \frac{(\tilde{x}_m - x)}{(x - s)(\tilde{x}_m - s)} \, dx \]

\[ + \sum_{k=1}^{l-1} \frac{f^{(k)}(s)}{k!} c.p \int_{x_m}^{x_{m+1}} \frac{(x - s)^{k-1}(\tilde{x}_m - x)}{(\tilde{x}_m - s)^{k+1}} \, dx \]

\[ - \sum_{k=1}^{l-1} \frac{f^{(k)}(s)}{k!} c.p \int_{x_m}^{x_{m+1}} \frac{(x - s)^{k-1}(\tilde{x}_m - x)}{k!(\tilde{x}_m - s)^{k+1}} \, dx. \]

From equations (26)–(28), we can obtain equation (23).

The proof is completed. \(\square\)

**Lemma 4** (see [5]). For $\tau \in (-1, 1)$ and $m \geq 1$, we have

\[ \sum_{l=0}^{\infty} \frac{\varphi_0(2i + \tau)}{i} + \sum_{i=0}^{\infty} \frac{\varphi_0(-2i + \tau)}{i} \leq C \eta(s). \] (29)

### 3.2. Proof of Theorem 2

*Proof.* By Lemma 1, we have

\[ \left| E_m(s) \right| \leq C \eta(s). \]

From equations (26)–(28), we can obtain equation (23). The proof is completed.
Now, we estimate these four terms one by one. For $f V_h$, we obtain

$$
c.P \int_a^b f(x) \, dx = \sum_{i=0}^{n-1} c.P \int_{x_i}^{x_{i+1}} \frac{f(x)}{x_i - x} \, dx
$$

where

$$
R_n(s) = R_n^{(1)}(s) + R_n^{(2)}(s) + R_n^{(3)}(s) + R_n^{(4)}(s),
$$

with

$$
R_n^{(1)}(s) = \sum_{k=1}^{l-1} \frac{f^{(k)}(s)}{k!} \left[ \sum_{i=0, i \neq m}^{n-1} \int_{x_i}^{x_{i+1}} \frac{(x-s)^{k-1}(\bar{x}_i - x)}{(\bar{x}_i - s)} \, dx + c.P \int_{x_m}^{x_{m+1}} \frac{(x-s)^{k-1}(\bar{x}_m - x)}{(\bar{x}_m - s)} \, dx \right],
$$

$$
R_n^{(2)}(s) = -\sum_{i=0, i \neq m}^{n-1} \int_{x_i}^{x_{i+1}} \frac{f^{(l)}(s)}{l!} \frac{(\bar{x}_i - x)^l}{(\bar{x}_i - s)} \, dx + \sum_{i=0, i \neq m}^{n-1} \int_{x_i}^{x_{i+1}} \frac{f^{(l)}(s)}{l!} \frac{(\bar{x}_i - x)^l}{(\bar{x}_i - s)} \, dx,
$$

$$
R_n^{(3)}(s) = c.P \int_{x_m}^{x_{m+1}} \frac{E_m(x)}{x-s} \, dx,
$$

$$
R_n^{(4)}(s) = f(s) \left[ \sum_{i=m}^{m} \varphi_0(2i + \tau) + \sum_{i=m+1}^{\infty} \varphi_0(-2i + \tau) \right].
$$

Now, we estimate these four terms one by one. For $R_n^{(1)}(s)$, by the definition of equation (2) and the linear transformation of equation (4), we have

$$
\left[ \sum_{k=1}^{l-1} \sum_{j=k}^{l-1} \frac{f^{(j)}(s)}{k!}\frac{k!}{j!(j-k)!} \int_{x_i}^{x_{i+1}} \frac{(x-s)^{j-k}(\bar{x}_i - x)^k}{(\bar{x}_i - s)} \, dx + c.P \int_{x_m}^{x_{m+1}} \frac{(x-s)^{j-k}(\bar{x}_m - x)^k}{(\bar{x}_m - s)} \, dx \right]
$$

\begin{align*}
\leq C \left[ \sum_{k=1}^{l-1} \sum_{j=k}^{l-1} \frac{f^{(j)}(s)}{k!}\frac{k!}{j!(j-k)!} \left( \frac{h}{2} \right)^{j-k} \frac{1}{c_{m}^{j-k}} \right] & \leq C h^{l-1}(\tau).
\end{align*}

Then, we obtain

$$
\left| c.P \int_{x_m}^{x_{m+1}} \frac{(x-s)^{j-k}(\bar{x}_m - x)^k}{(\bar{x}_m - s)} \, dx \right| \leq C \frac{1}{c_{m}^{j-k}} \left( \frac{h}{2} \right)^{j-k}.
$$

Similarly, by applying the same approach to the corresponding Riemann integral, we obtain

$$
R_n^{(2)}(s) = \ldots
$$

$$
R_n^{(3)}(s) = \ldots
$$

$$
R_n^{(4)}(s) = \ldots
$$
In the same way, we can obtain

\[
\left| \int_{x_i}^{x_{i+1}} \frac{(x-s)^{j-k} \left( \xi - x \right)^k}{\left( \xi_i - s \right)} \ dx \right| \leq C \left( \frac{1}{e} \right)^{j}. \tag{40}
\]

By substituting equations (39) and (40) into the second part of \( R_n'(s) \), we can obtain

\[
\left| \int_{x_i}^{x_{i+1}} \frac{f^{(j)}(s)}{k!} \left( \sum_{i=0,j \neq m}^{n-1} \frac{x_i^{(j)}}{l!(\xi_i - s)} \ dx \right) + c.P \right| \leq C h^{j-1}(s). \tag{41}
\]

In the same way, we can obtain

\[
\left| \sum_{k=1}^{n} \frac{f^{(j)}(s)}{k!} \right| \leq C h^{j-1}(s), \tag{42}
\]

As for \( R_n''(s) \), we have

\[
\left| \sum_{i=0,j \neq m}^{n-1} \frac{x_i^{(j)}}{l!(\xi_i - s)} \right| \leq C h^{j-1}(s), \tag{43}
\]

From equation (40), we can obtain

\[
\left| \sum_{i=0,j \neq m}^{n-1} \frac{f^{(j)}(s)}{l!(\xi_i - s)} \right| \leq C \left( \frac{1}{e} \right)^{j}, \tag{44}
\]

According to Lemmas 3 and 4, we know

\[
R_n'(s) \leq C h^{j} |\ln(s)|, \quad R_n''(s) \leq C h^{j+k}(s), \tag{45}
\]

From the above estimates, equation (10) can be obtained.

\[ S_0(\varphi_0, \tau) = -(\pi/2) \tan(\pi(\tau + 1)/2). \]

4. Numerical Examples

Example 1. We first consider the Cauchy principal value integral with \( f(x) = x^6 \), \( a = -1 \), and \( b = 1 \). The exact value can be expressed as

\[ 2s^5 + 2/3s^3 + 2/5s + s^2 \log(1 - s/1 + s). \]

We adopt the uniform mesh method to examine the convergence rate of the midpoint rectangle rule \( L_n(x^6, s) \) with the dynamic points \( s = x[n] + (1 + \tau)h/2 \) and \( s = a + (1 + \tau)h/2 \), respectively. For different values of \( n \), the error distributions are shown in Figure 1 and Figure 2, respectively.
When \( s = x_{[n/4]} + (1 + \tau)h/2 \), the convergence rate of the midpoint rectangle rule is \( O(h^2) \) at the superconvergence points, and there is no convergence rate at the non-superconvergence points, as shown in Table 1. When \( s = a + (1 + \tau)h/2 \), because of the influence of \( \eta(s) \), no superconvergence phenomenon occurs at both the superconvergence points and the non-superconvergence points, which coincides with the theoretical analysis, as shown in Table 2. For the case of \( l = 2 \), these numerical results agree quite well with the theoretical results in Theorem 2.

Example 2. In the second example, we further study the accuracy of the midpoint rectangle rule. We next consider the Cauchy principal value integral with \( f(x) = x^2 - 1, a = -1, b = 1 \). The exact value can be expressed as \( 2s + (1 - s^2)\log|1 + s/1 - s| \).

We adopt the uniform mesh method to examine the convergence rate of the midpoint rectangle rule \( I_\tau(x^2 - 1, s) \) with the dynamic points \( s = x_{[n/4]} + (1 + \tau)h/2 \) and \( s = a + (1 + \tau)h/2 \), respectively. For different values of \( n \), the error distributions are shown in Figure 3 and Figure 4, respectively.

For the case of \( s = x_{[n/4]} + (1 + \tau)h/2 \), when the local coordinate of singular point \( \tau = \pm 1 \), the convergence rate of the midpoint rectangle rule is \( O(h^2) \) at the superconvergence points, and there is no convergence rate at the non-superconvergence points, as shown in Table 3. For the case of \( s = a + (1 + \tau)h/2 \) because of the limitation of the boundary condition \( f(a) = f(b) = 0 \), the convergence rate is \( O(h) \) at both the superconvergence points and the non-superconvergence points, as shown in Table 4. These numerical results are consistent with the theoretical results of \( l = 2 \) in Theorem 2.
Figure 2: Error distributions for the midpoint rectangle rule with
$s = a + (1 + \tau)h/2$.

**Table 1:** Errors and convergence for the midpoint rectangle rule with $s = a + (1 + \tau)h/2$.

| $n$  | $\tau = 1/2$ | $\tau = -1$ | $\tau = 1/2$ | $\tau = 2/3$ |
|------|--------------|--------------|--------------|--------------|
| 32   | 6.2074e-004 | 7.2296e-004 | 2.6547e-002 | 1.4010e-002 |
| 64   | 1.6800e-004 | 1.8082e-004 | 3.6631e-002 | 2.0390e-002 |
| 128  | 4.3607e-005 | 4.5210e-005 | 4.2533e-002 | 2.4147e-002 |
| 256  | 1.1026e-005 | 1.1303e-005 | 4.5724e-002 | 2.6186e-002 |
| 512  | 2.8007e-006 | 2.8257e-006 | 4.7384e-002 | 2.7249e-002 |
| 1024 | 7.0330e-007 | 7.0643e-007 | 4.8230e-002 | 2.7791e-002 |
| Convergence ratio | 1.9571 | 1.9998 | — | — |

**Table 2:** Errors and convergence for the midpoint rectangle rule with $s = x_{n/4} + (1 + \tau)h/2$.

| $n$  | $\tau = 1/2$ | $\tau = -1$ | $\tau = 1/2$ | $\tau = 2/3$ |
|------|--------------|--------------|--------------|--------------|
| 32   | 2.2989e-002 | —            | 2.3986e-000 | 1.3505e-000 |
| 64   | 2.9689e-002 | —            | 2.7767e-000 | 1.5907e-000 |
| 128  | 3.3079e-002 | —            | 2.9831e-000 | 1.7230e-000 |
| 256  | 3.4782e-002 | —            | 3.0909e-000 | 1.7924e-000 |
| 512  | 3.5635e-002 | —            | 3.1459e-000 | 1.8280e-000 |
| 1024 | 3.6062e-002 | —            | 3.1738e-000 | 1.8460e-000 |
| Convergence ratio | — | — | — | — |
\( r = 1 \times 10^{-3} \)

\( r = -1 \times 10^{-3} \)

\( r = 1/2 \)

\( r = 2/3 \)

\( \tau = 1 \times 10^{-4} \)

\( \tau = -1 \times 10^{-4} \)

\( \tau = 1 \)

\( \tau = -1 \)

\( \tau = 1/2 \)

\( \tau = 2/3 \)

Figure 3: Error distributions for the midpoint rectangle rule with \( s = x_{[n/4]} + (1 + \tau)h/2 \).

Figure 4: Continued.
5. Conclusions

In this paper, the interpolation method and Taylor expansion method are used to obtain the extended error expansion of classical composite midpoint rectangle rule for the computation of Cauchy principal value integrals. For the case of \( f(x) \in C^k[a, b], \quad k \geq 2 \), the error expansion and its accuracy are analyzed by theoretical proofs and numerical experiments. Based on the expansion of the error function, some superconvergence results are obtained. It shows that the increased rate of convergence occurs at the singular points whose location changes are allowed. This kind of Cauchy principal value integral can be widely used in many engineering areas, and the positions of singular points are fixed in real applications. Moreover, it is very possible to extend the above presented results to improve the accuracy of the collocation method for singular integrals by choosing the superconvergence points to be the collocation points.

Data Availability

The data used to support the results of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The work in this research was supported by the National Natural Science Foundation of China (no. 11301459).

Table 3: Errors and convergence for the midpoint rectangle rule with \( s = x_{[n/4]} + (1 + \tau)h/2 \).

| n   | \( \tau = 1 \)      | \( \tau = -1 \)         | \( \tau = 1/2 \)        | \( \tau = 2/3 \)        |
|-----|---------------------|-------------------------|-------------------------|-------------------------|
| 32  | 3.5138e–004         | 4.3272e–004             | 2.4969e+000             | 1.4503e+000             |
| 64  | 9.7712e–005         | 1.0842e–004             | 2.4282e+000             | 1.4065e+000             |
| 128 | 2.5746e–005         | 2.7122e–005             | 2.3926e+000             | 1.3837e+000             |
| 256 | 6.6071e–006         | 6.7814e–006             | 2.3745e+000             | 1.3721e+000             |
| 512 | 1.6735e–006         | 1.6954e–006             | 2.3654e+000             | 1.3662e+000             |
| 1024| 4.2110e–007         | 4.2385e–007             | 2.3608e+000             | 1.3633e+000             |
|     | Convergence ratio   | 1.9409                  |                        |                         |

Table 4: Errors and convergence for the midpoint rectangle rule with \( s = x_{[n/4]} + (1 + \tau)h/2 \).

| n   | \( \tau = 1 \)      | \( \tau = -1 \)         | \( \tau = 1/2 \)        | \( \tau = 2/3 \)        |
|-----|---------------------|-------------------------|-------------------------|-------------------------|
| 32  | 4.4135e–003         | —                       | 2.9313e–001             | 1.8911e–001             |
| 64  | 2.2443e–003         | —                       | 1.4833e–001             | 9.5823e–002             |
| 128 | 1.3131e–003         | —                       | 7.4603e–002             | 4.8228e–002             |
| 256 | 5.6792e–004         | —                       | 3.7411e–002             | 2.4193e–002             |
| 512 | 2.8452e–004         | —                       | 1.8733e–002             | 1.2116e–002             |
| 1024| 1.4240e–004         | —                       | 9.3735e–003             | 6.0631e–003             |
|     | Convergence ratio   | 0.9908                  | —                       | —                       |

Figure 4: Error distributions for the midpoint rectangle rule with \( s = a + (1 + \tau)h/2 \).
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