PIERI AND MURNAGHAN–NAKAYAMA TYPE RULES FOR CHERN CLASSES OF SCHUBERT CELLS

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ABSTRACT. We develop Pieri type as well as Murnaghan–Nakayama type formulas for equivariant Chern–Schwartz–MacPherson classes of Schubert cells in the classical flag variety. These formulas include as special cases many previously known multiplication formulas for Chern–Schwartz–MacPherson classes or Schubert classes. We apply the equivariant Murnaghan–Nakayama formula to the enumeration of rim hook tableaux.

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1. Introduction

The Chern–Schwartz–MacPherson (CSM) classes, constructed explicitly by MacPherson [36] in order to resolve a conjecture of Deligne and Grothendieck [62], are one way to extend the Chern classes of complex manifolds to complex varieties (possibly singular or noncompact). Their equivariant setting was developed by Ohmoto [48]. The structure of (equivariant) CSM classes for Schubert cells in flag varieties has received much attention in recent years, and evolved into a rich area of research, see for example [3–5, 20, 22, 27, 30, 38, 60, 61]. Such CSM classes also draw special interests from geometric representation theory due to their intimate connections with characteristic cycles of Verma \( \mathcal{D} \)-modules over flag varieties, as well as to stable envelopes for the cotangent bundles of flag manifolds introduced by Maulik and Okounkov [38], see [5, 51, 59].

In this work, we shall focus on the torus equivariant CSM classes \( c_{\text{SM}}^T(Y(w)) \) for Schubert cells \( Y(w) \) in the classical flag variety \( \mathcal{F} \ell(n) \), where \( w \) varies over permutations in the symmetric group \( \mathfrak{S}_n \). A notable feature is that the lowest degree component of \( c_{\text{SM}}^T(Y(w)) \) recovers the equivariant Schubert class \( [Y(w)]_T \). Our goal is to establish Pieri type formulas as well as Murnaghan–Nakayama (MN) type formulas for \( c_{\text{SM}}^T(Y(w)) \).
The Chevalley formula for $c_{SM}^{T}(Y(w)\circ)$ multiplied by divisors was derived by combining the work of Aluffi, Mihalcea, Schürmann and Su [5], and Su [59]. When the divisor corresponds to the first Chern class of the tautological bundle over $\mathcal{F}\ell(n)$, the formula may be described in terms of certain one step walks in the Bruhat graph of $\mathfrak{S}_n$, see Mihalcea, Naruse and Su [40, Theorem 4.2]. A natural and desirable extension of the Chevalley formula is then to develop Pieri or MN type formulas for $c_{SM}^{T}(Y(w)\circ)$. Towards this direction, we prove

- a Pieri formula for multiplying $c_{SM}^{T}(Y(w)\circ)$ by the $r$-th Chern class or Segre class of the tautological bundle over $\mathcal{F}\ell(n)$ (more generally by a Schur polynomial of hook shape), see Theorem A;
- a Pieri formula for multiplying $c_{SM}^{T}(Y(w)\circ)$ by an equivariant Schubert class of a Grassmannian permutation associated to a one row/column partition (more generally to a hook shape), see Theorem B;
- a MN formula for multiplying $c_{SM}^{T}(Y(w)\circ)$ by the Chern character of the tautological bundle over $\mathcal{F}\ell(n)$ (equivalently, by a power sum symmetric polynomial), see Theorem C.

Although one could compute the above multiplications according to the splitting principle by iterating the Chevalley formula, there would be no explicit formulas given for the structure constants because of the occurring of cancellations. Our Pieri formulas are formulated in terms of increasing and decreasing paths in the Bruhat graph of $\mathfrak{S}_n$ with respect to a specific labeling on the edges, which manifestly imply the positivity of the structure constants. We propose general positivity conjectures in Section 9 and discuss their relations to Kumar’s recent conjectures [27]. In particular, we may apply our Pieri formula to show that the CSM class of the Richardson cell is monomial-positive, proving a weaker form of Kumar’s conjecture [27, Conjecture B] in type $A$, see Theorem 9.5.

Each of the above formulas vastly generalizes the Chevalley formula for $c_{SM}^{T}(Y(w)\circ)$. Both Pieri formulas for $c_{SM}^{T}(Y(w)\circ)$ specialize to the Pieri formula for nonequivariant Schubert classes $[Y(w)]$ by Sottile [55], while the Pieri formula in Theorem B may reduce to the Pieri formula for equivariant Schubert classes $[Y(w)]_{T}$ by Robinson [52] (see also Li, Ravikumar, Sottile and Yang [32]). The MN formula in Theorem C is a remarkable generalization from the MN rule for nonequivariant Schubert classes $[Y(w)]$ due to Morrison and Sottile [46] to equivariant CSM classes $c_{SM}^{T}(Y(w)\circ)$. This in particular leads to a MN rule for equivariant Schubert classes, see Corollary 5.14.

The proofs of Theorems A, B and C are mainly combinatorial. In the proof of Theorem A or Theorem C one of the key ingredients we developed is the Rigidity Theorem (see Theorem 4.11, Theorem 6.11), which means that the structure constants in Theorem A or Theorem C can be controlled by the structure constants in their nonequivariant situations. So, to accomplish Theorem A or Theorem C our strategy is to first establish their nonequivariant versions and then employ the Rigidity Theorem. This philosophy has appeared for example in the study of equivariant quantum cohomology by Braverman, Maulik and Okounkov [12], and $K$-theoretic stable envelopes by Okounkov [50, §2.4].

Starting from the work of Ikeda and Naruse [21], it was gradually realized that equivariant Chevalley type formulas could be utilized to deduce hook formulas, see Naruse [47] and the recent work of Morales, Pak and Panova [45] and Mihalcea, Naruse and Su.
We generalize this idea to higher degrees by viewing the equivariant MN formula as a higher degree analogue of the equivariant Chevalley formula. More concretely, we employ our equivariant MN formula to realize the number of standard rim hook tableaux as a coefficient of the Laurent expansion relating to the localization of equivariant Schubert classes, see Theorem 8.1. In this framework, we reveal the enumeration formulas for standard rim hook tableaux due to Alexandersson, Pfannerer, Rubey and Uhlin [2] and Fomin and Lulov [16] in a relatively uniform manner.

This paper is arranged as follows. Section 2 contains descriptions of the main theorems. In Section 3, we give an overview of the background of CSM classes. In Section 4, we prove the Rigidity Theorem for Theorem A. The parallel idea is used in Section 6 to establish the Rigidity Theorem for Theorem C. In Section 5, we finish the proofs of Theorems A and C. Section 8 is devoted to a proof of Theorem C. We investigate the Pieri and MN formulas over Grassmannians in Section 7. In Section 8, we illustrate how the MN formula over Grassmannians can be applied to the enumeration of rim hook tableaux. Some positivity conjectures are discussed in Section 9.

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2. Main Results

In this section, we give detailed descriptions of our main results: two Pieri formulas and a MN formula for equivariant CSM classes of Schubert cells in the classical flag variety. Throughout this paper, let \( n \) be a fixed positive integer, and \( k \) be an integer belonging to the set \([n] := \{1, 2, \ldots, n\}\).

Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) be a partition, namely, \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0 \). We do not distinguish \( \lambda \) with its Young diagram, a left-justified array with \( \lambda_i \) boxes in row \( i \). Let \( s_\lambda(x_1, \ldots, x_k) \) be the Schur polynomial associated to \( \lambda \), see Subsection 3.3 for definition. When \( \lambda \) has exactly one column (resp., one row) with \( r \) boxes, \( s_\lambda(x_1, \ldots, x_k) \) is the elementary symmetric polynomial \( e_r(x_1, \ldots, x_k) \) (resp., the complete homogeneous symmetric polynomial \( h_r(x_1, \ldots, x_k) \)). Note that \( e_r \) and \( h_r \) may serve as, up to a sign, representatives of the \( r \)-th Chern class and Segre class of the \( k \)-th tautological bundle over \( \mathbb{F}\ell(n) \), respectively, see Fulton [18, §14.6]. On the other hand, they are representatives of Schubert classes associated to the Grassmannian permutations corresponding respectively to the one column and one row permutation. These two ways of understanding lead to two different generalizations of the classical Pieri rule to equivariant CSM classes, which are dealt with in Theorem A and Theorem B respectively. As mentioned in Introduction, our formulas are valid for the multiplication by a Schur polynomial or an equivariant Schubert class associated to a partition of hook shape.

Theorem A and Theorem B are closely related to paths in the Bruhat graph on \( S_n \). As usual, write \( t_{ab} \) \((1 \leq a < b \leq n)\) for the transpositions in \( S_n \), and \( \ell(w) \) for the length of \( w \in S_n \). Note that \( \ell(w) \) equals the number of inversion pairs (namely, \((w(a), w(b))\) with \( 1 \leq a < b \leq n \) and \( w(a) > w(b) \)) of \( w \). The Bruhat graph on \( S_n \) is a directed graph whose vertices are permutations in \( S_n \) such that there is a directed edge from \( u \) to \( w \),
denoted \( u \to w \), if \( w = ut_{ab} \) for some \( t_{ab} \) and \( \ell(w) \geq \ell(u) + 1 \) (equivalently, \( u(a) < u(b) \)) \cite{11,13}. For our purpose, we often label the edges in the Bruhat graph by writing \( u \to_{\tau} w \), if \( u \to w \), \( w = ut_{ab} \), and \( \tau = u(a) \).

An edge \( u \to_{\tau} w \) with \( w = ut_{ab} \) is called a \( k \)-edge if \( a \leq k < b \). The \( k \)-Bruhat graph on \( S_n \) is the subgraph induced by all \( k \)-edges. Figure 2 illustrates the 1-Bruhat graph and the 2-Bruhat graph on \( S_3 \), where we use dashed arrows to emphasize the edges \( u \to w \) with \( \ell(w) > \ell(u) + 1 \).

![Figure 1. The 1-Bruhat graph and the 2-Bruhat graph on \( S_3 \)](image)

A path \( \gamma \) of length \( m \) from \( u \) to \( w \) means a sequence of edges
\[
\gamma: \quad u \to_{\tau_1} w_1 \to_{\tau_2} w_2 \to_{\tau_3} \cdots \to_{\tau_m} w_m = w.
\]
We say that \( \gamma \) is increasing (resp., decreasing) if \( \tau_1 < \tau_2 < \cdots < \tau_m \) (resp., \( \tau_1 > \tau_2 > \cdots > \tau_m \)), and is peakless if there exists \( 1 \leq i \leq m \) such that \( \tau_1 > \cdots > \tau_i < \cdots < \tau_m \).

We use \( \text{in}(\gamma) = m - i \) (resp., \( \text{de}(\gamma) = i - 1 \)) to denote one less than the length of the increasing (resp., decreasing) segment of \( \gamma \). Clearly, a peakless path is increasing (resp., decreasing) if \( \text{de}(\gamma) = 0 \) (resp., \( \text{in}(\gamma) = 0 \)).

Define the extended \( k \)-Bruhat order, denoted \( \leq_k \), on \( S_n \) as the order generated by all \( k \)-edges, that is, \( u \leq_k w \) whenever there is a path in the \( k \)-Bruhat graph from \( u \) to \( w \). If restricting the \( k \)-edges to the edges in the Hasse diagram of the Bruhat order (namely, edges \( u \to w \) with \( \ell(w) = \ell(u) + 1 \), then the extended \( k \)-Bruhat order reduces to the well-studied ordinary \( k \)-Bruhat order \cite{6, 31, 55}.

For \( u, w \in S_n \) and a subset \( A \subseteq [n] \), we adopt the following notation
\[
u A := \{u(i): i \in A\},
\]
\[
\Delta_A(u, w) := \{u(i): i \in A\} \setminus \{u(i): u(i) \neq w(i)\}, \quad (2.1)
\]
\[
\Sigma_A(u, w) := \{u(i): i \in A\} \cup \{u(i): u(i) \neq w(i)\}. \quad (2.2)
\]

When \( A = [k] \), we simply use \( \Sigma_k(u, w) \) and \( \Delta_k(u, w) \) to represent \( \Sigma_{[k]}(u, w) \) and \( \Delta_{[k]}(u, w) \), respectively. Moreover, assuming that \( A = \{a_1 < a_2 < \cdots < a_m\} \) and \( f(x_1, \ldots, x_m) \) is any given polynomial, we denote by \( f(x_A) \) the polynomial obtained from \( f(x_1, \ldots, x_m) \) by substituting \( x_i \) with \( x_{a_i} \) for \( 1 \leq i \leq m \). With this notation, the Schur polynomial \( s_{\lambda}(x_1, \ldots, x_k) \) can be briefly written as \( s_{\lambda}(x_{[k]}) \).

A hook shape partition with arm length \( \alpha \) and leg length \( \beta \) will be denoted \( \Gamma = (\alpha + 1, 1^\beta) \), that is, \( \Gamma \) has one row with \( \alpha + 1 \) boxes and one column with \( \beta + 1 \) boxes.
**Theorem A** (see Theorem 5.8). Let \( u \in S_n \), and \( \Gamma = (1 + \alpha, 1^\beta) \) be a hook shape. Then we have

\[
e^\Gamma_{\text{SM}}(Y(u)^\circ) \cdot s^\Gamma(x[k]) = s^\Gamma(t_{u[k]}) \cdot e^\Gamma_{\text{SM}}(Y(u)^\circ) + \sum_{u < k} c_{u,1}^w(t) \cdot e^\Gamma_{\text{SM}}(Y(w)^\circ),
\]

where

\[
c_{u,1}^w(t) = \sum_{\gamma} h_{\alpha - \text{in}(\gamma)}(t_{\Sigma_k(u,w)}) \cdot e_{\beta - \text{de}(\gamma)}(t_{\Delta_k(u,w)})
\]

with the sum taken over all peakless paths from \( u \) to \( w \) in the extended \( k \)-Bruhat order.

Letting \( \alpha = 0 \) or \( \beta = 0 \), Theorem A reduces to a Pieri formula for equivariant CSM classes, see Corollary 5.9. Particularly, if \( \alpha \) and \( \beta \) are both 0, then Theorem A specifies to the Chevalley formula [40, Theorem 4.2]. On the other hand, if taking the lowest degree component of \( e^\Gamma_{\text{SM}}(Y(u)^\circ) \), then Theorem A becomes an expansion for equivariant Schubert classes, see Corollary 5.10. Theorem B replaces the Schur polynomial of a hook shape in Theorem A by an equivariant Schubert class of a Grassmannian permutation of hook shape. For a partition \( \lambda \) of \( \beta \), see Corollary 5.10.

**Theorem B** (see Theorem 5.13). Let \( u \in S_n \), and let \( \Gamma = (1 + \alpha, 1^\beta) \) be a hook shape inside the \( k \times (n-k) \) rectangle. Then we have

\[
e^\Gamma_{\text{SM}}(Y(u)^\circ) \cdot [Y(w)]_T = [Y(w)]_T |_u \cdot e^\Gamma_{\text{SM}}(Y(u)^\circ) + \sum_{u < k} c_{u,1}^w(t) \cdot e^\Gamma_{\text{SM}}(Y(w)^\circ),
\]

where

\[
c_{u,1}^w(t) = \sum_{\gamma} \sum_{\alpha_1 + \alpha_2 = \alpha - \text{in}(\gamma), \beta_1 + \beta_2 = \beta - \text{de}(\gamma)} (-1)^{\alpha_2 + \beta_2} \cdot h_{\alpha_1}(t_{\Sigma_k(u,w)}) \cdot e_{\beta_1}(t_{\Delta_k(u,w)}) \cdot e_{\alpha_2}(t_{[k+\alpha]} \cdot e_{\beta_2}(t_{[k-\beta]})
\]

with the first sum over all peakless paths from \( u \) to \( w \) in the extended \( k \)-Bruhat order. Here, \([Y(w)]_T |_u\) means the localization of \([Y(w)]_T\) at \( u \), as will be defined in Section 5.

Setting \( \alpha = 0 \) or \( \beta = 0 \), we obtain our second Pieri formula for equivariant CSM classes, and in this case the expression for \( c_{u,1}^w(t) \) could be dramatically simplified to the localization of Schubert classes, see Theorem 5.10. Taking the lowest degree part in Theorem B recovers the Pieri formula for equivariant Schubert classes obtained in [52] using an algebraic approach (see [32] for a geometric proof), see Remark 5.17.

The classical MN formula computes the product of a Schur polynomial by a power sum symmetric polynomial, which arises naturally in the computation of irreducible characters of \( S_n \), see Sagan [53] or Stanley [57, Chapter 7]. Its nonequivariant Schubert generalization was found by Morrison and Sottile [46]. Geometrically, a power sum symmetric polynomial can be viewed as a component of Chern characters of tautological bundles up to a scalar, see Subsection 3.3.

We vastly lift the MN formula of Morrison and Sottile from nonequivariant Schubert classes to equivariant CSM classes.
Theorem C. (see Theorem 6.12) Let \( u \in \mathfrak{S}_n \). For \( r \geq 1 \),
\[
c^T_{SM}(Y(u)^\circ) \cdot p_r(x[k]) = p_r(t_{u[k]}) \cdot c^T_{SM}(Y(u)^\circ) + \sum_{\eta \in \mathfrak{S}_n} d^{u, \eta}_{r, t}(t) \cdot c^T_{SM}(Y(u\eta)^\circ),
\]
where the sum runs over \((r' + 1)\) cycles \( \eta \in \mathfrak{S}_n \) with \( 1 \leq r' \leq r \) such that \( u \leq_k u\eta \) in the extended \( k \)-Bruhat order, and
\[
d^{u, \eta}_{r, t}(t) = (-1)^{\text{ht}_k(\eta)} \cdot h_{r-r'}(t_{u M(\eta)}).
\]
Here, \( M(\eta) = \{ 1 \leq i \leq n : \eta(i) \neq i \} \) is the set of non-fixed points of \( \eta \), and \( \text{ht}_k(\eta) = \#\{ i \leq k : i \in M(\eta) \} - 1 \).

Taking the lowest degree component in Theorem C gives a MN formula for equivariant Schubert classes, see Corollary 5.14, which will be applied in Section 8 to the enumeration of standard rim hook tableaux. If further specializing all \( t_i = 0 \), then we are led to the MN formula for Schubert classes by Morrison and Sottile [46].

3. Chern–Schwartz–MacPherson classes

In this section, we shall briefly review the geometric background of Chern–Schwartz–MacPherson classes of Schubert cells in flag varieties. Some properties required in this paper are also included.

3.1. Chern–Schwartz–MacPherson Classes. For a complex variety \( X \), denote by \( \text{Fun}(X) \) the space of constructible functions over \( X \) with coefficients in \( \mathbb{Q} \), namely, the space of functions \( f : X \to \mathbb{Q} \) which can be written as a finite sum
\[
\sum c_W 1_W,
\]
where \( c_W \in \mathbb{Q} \) and \( 1_W \) is the characteristic function of a constructible subset \( W \) of \( X \). It is well known that any proper morphism \( f : Y \to X \) induces a functorial pushforward \( f_* : \text{Fun}(Y) \to \text{Fun}(X) \) given by
\[
f_* \varphi(x) = \sum_{a \in \mathbb{Q}} \chi_c(f^{-1}(x) \cap \varphi^{-1}(a)),
\]
where \( \chi_c \) is the Euler characteristic with compact supports. It was conjectured by Deligne and Grothendieck [62] and proved by MacPherson [36] that there exists a linear map
\[
c_{SM} : \text{Fun}(X) \to H_*(X)
\]
from \( \text{Fun}(X) \) to the Borel–Moore homology of \( X \) which is functorial in \( X \) (with respect to proper pushforward). That is, for any proper morphism \( f : Y \to X \), the following diagram commutes
\[
\begin{array}{ccc}
\text{Fun}(Y) & \xrightarrow{c_{SM}} & H_*(Y) \\
\downarrow f_* & & \downarrow f_* \\
\text{Fun}(X) & \xrightarrow{c_{SM}} & H_*(X)
\end{array}
\]
where the right \( f_* \) is the usual proper pushforward of Borel–Moore homology. For any constructible subset \( W \subseteq X \), define the Chern–Schwartz–MacPherson class (CSM class for abbreviation) to be
\[
c_{SM}(W) = c_{SM}(1_W).
\]
Actually, the CSM class is the unique natural transformation characterized by the following property. For \( X \) smooth,

\[
c_{\text{SM}}(X) = c_1(\mathcal{T}_X) \cup [X] \in H_*(X),
\]

where \( \mathcal{T}_X \) is the tangent bundle of \( X \), \( c_1(\mathcal{T}_X) \) denotes the first Chern class of \( \mathcal{T}_X \), the symbol \( \cup \) stands for the cap product, and \([X]\) is the fundamental class of \( X \).

The equivariant setting of CSM classes was developed by Ohmoto \([48]\). Let \( T \) be a torus, and \( X \) be a \( T \)-variety. For any \( T \)-invariant constructible subset \( W \subseteq X \), denote by \( c_{\text{SM}}(W) \) its corresponding equivariant CSM class.

In this paper, we are only concerned with the case when \( X \) is smooth so that the (equivariant) cohomology can be naturally identified with (equivariant) Borel–Moore homology of \( X \) under Poincaré duality. Hence one may think of \( c_{\text{SM}}(W) \) and \( c_{\text{SM}}^T(W) \) as cohomology classes in \( H^*_T(X) \).

### 3.2. Flag Varieties.

The flag variety \( \mathcal{F}_\ell(n) \) is the variety of (complete) flags of \( \mathbb{C}^n \), namely, chains of subspaces

\[
0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = \mathbb{C}^n
\]

with \( \dim V_i = i \) for \( 0 \leq i \leq n \). For \( 1 \leq i \leq n \), define the \( i \)-th tautological bundle \( \mathcal{V}_i \) over \( \mathcal{F}_\ell(n) \) to be the vector bundle whose fiber at a flag \( (V_i) \in \mathcal{F}_\ell(n) \) is \( V_i \).

Let \( T \) be the subgroup of diagonal matrices of \( GL_n \). There is a natural action of \( T \) on \( \mathcal{F}_\ell(n) \). For \( 1 \leq i \leq n \), denote by \( pt = \text{Spec} \mathbb{C} \) a point with trivial \( T \)-action. Let \( t_i = c_1(C_{-t_i}) \in H^2_T(pt) \) be the first Chern class of the equivariant bundle \( C_{-t_i} \), the one-dimensional representation corresponding to the character sending \( \text{diag}(t_1, \ldots, t_n) \in T \) to \( t_i^{-1} \). By Borel \([9]\), there is an isomorphism

\[
H^*_T(pt) \cong \mathbb{Q}[t_1, \ldots, t_n].
\]

Note that \( H^*_T(\mathcal{F}_\ell(n)) \) is a module over \( H^*_T(pt) \). For \( 1 \leq i \leq n \), let

\[
x_i = c_1^T((\mathcal{V}_i/\mathcal{V}_{i-1})^\vee) \in H^2_T(\mathcal{F}_\ell(n))
\]

be the first equivariant Chern class of the dual bundle of the quotient bundle \( \mathcal{V}_i/\mathcal{V}_{i-1} \). The following seminal result is due to Borel \([9]\), see also Anderson and Fulton \([11] \S 10.6\).

**Theorem 3.1** (Borel \([9]\)). We have

\[
H^*_T(\mathcal{F}_\ell(n)) \cong \frac{\mathbb{Q}[x_1, \ldots, x_n, t_1, \ldots, t_n]}{\langle f(x) - f(t) : f \in \Lambda_n \rangle}
\]

where \( \Lambda_n \) is the ring of symmetric polynomials in \( n \) variables. Moreover,

\[
H^*(\mathcal{F}_\ell(n)) \cong \frac{\mathbb{Q}[x_1, \ldots, x_n]}{\langle f(x) - f(0) : f \in \Lambda_n \rangle}
\]

with the forgetful map \( \epsilon : H^*_T(\mathcal{F}_\ell(n)) \to H^*(\mathcal{F}_\ell(n)) \) given by

\[
\epsilon(f(x, t)) = f(x, 0).
\]
The $T$-fixed points of $\mathcal{F}^T(n)$ are in bijection with the symmetric group $\mathfrak{S}_n$. Precisely, for $w \in \mathfrak{S}_n$, the flag
\[
\phi_w = ((\phi_w)_1, \ldots, (\phi_w)_n)
\]
is fixed by $T$, where $(\phi_w)_i = \text{span}(e_{w(1)}, \ldots, e_{w(i)})$ and $e_i$'s are the standard basis of $\mathbb{C}^n$. Denote the localization map at $w$ by
\[
-\vert_w : H^*_T(\mathcal{F}^T(n)) \rightarrow H^*_T(\phi_w) \cong \mathbb{Q}[t_1, \ldots, t_n].
\]
Under the isomorphism in Theorem 3.1 it is easy to see that
\[
f(x, t)\vert_w = f(wt, t),
\]
where $wt$ means $(t_{w(1)}, t_{w(2)}, \ldots, t_{w(n)})$ (so $f(wt, t)$ is obtained from $f(x, t)$ by replacing $x_i$ with $t_{w(i)}$).

For $w \in \mathfrak{S}_n$, define the Schubert cell $Y(w)^o$ to be the constructible subset of flags $V_\bullet$ such that for $0 \leq i, j \leq n$,
\[
\dim(V_i \cap V_j^0) = \#\{(a, b) : a \leq i, b \leq j, w(a) + b = n + 1\},
\]
where $V_j^0$ is the reversed standard flag, i.e., for $1 \leq i \leq n$,
\[
V_j^0 = \text{span}(e_n, \ldots, e_{n-i+1}).
\]
Notice that $Y(w)^o$ is $T$-invariant and $\phi_w$ is the unique $T$-fixed point over it. In this paper, we shall concentrate on the (equivariant) CSM classes of Schubert cells
\[
c_{SM}^T(Y(w)^o) \in H^*_T(\mathcal{F}^T(n)), \quad c_{SM}(Y(w)^o) = \epsilon(c_{SM}^T(Y(w)^o)) \in H^*(\mathcal{F}^T(n)),
\]
and simply call them (equivariant) CSM classes. The Schubert variety $Y(w)$ is the closure of $Y(w)^o$, and is a closed $T$-subvariety of $\mathcal{F}^T(n)$. So one can define the (equivariant) Schubert class to be its fundamental class
\[
[Y(w)]_T \in H^*_T(\mathcal{F}^T(n)), \quad [Y(w)] = \epsilon([Y(w)]_T) \in H^*(\mathcal{F}^T(n)).
\]

3.3. **Characteristic Classes.** For any $T$-equivariant vector bundle $V$ of rank $k$ over a $T$-variety $X$, there is a homomorphism from the ring $\Lambda_k$ of symmetric polynomials in $k$ variables to $H^*_T(X)$, defined by sending the $r$-th elementary symmetric polynomial
\[
e_r(x[k]) = \sum_{1 \leq i_1 < i_2 \cdots < i_r \leq k} x_{i_1}x_{i_2} \cdots x_{i_r},
\]
to the $r$-th $T$-equivariant Chern class $c^T_r(V)$. Denote by $V^\vee$ the dual of $V$. For $r \geq 1$, the $r$-th $T$-equivariant Segre class of $V^\vee$ is the image of the $r$-th complete homogeneous symmetric polynomial
\[
h_r(x[k]) = \sum_{1 \leq i_1 \leq i_2 \cdots \leq i_r \leq k} x_{i_1}x_{i_2} \cdots x_{i_r},
\]
[1][2]. After suitable completion, we have a well-defined $T$-equivariant Chern character
\[
\text{ch}^T(V) = k + \frac{1}{1!}p_1 + \frac{1}{2!}p_2 + \frac{1}{3!}p_3 + \cdots,
\]
where $p_r$ is the image of the $r$-th power sum symmetric polynomial
\[
p_r(x[k]) = x_1^r + x_2^r + \cdots + x_k^r.
\]
sections of Schur polynomials can be interpreted as the classes of degenerate loci of generic rational 
−n column with s boxes, there hold the following relations

\[ s_{\lambda}(x) = \det \left( x_{i+j}^{\lambda_{i+j}} \right)_{1 \leq i,j \leq k} \]

Schur polynomials can be interpreted as the classes of degenerate loci of generic rational sections of \( V \), see [23, 37] and [18, Chapter 14]. When \( \lambda \) has exactly one row or one column with \( r \) boxes, there hold the following relations

\[ h_{r}(x) = s_{(r)}(x) \quad \text{and} \quad e_{r}(x) = s_{(1^{r})}(x). \]

3.4. Schubert Classes and Schubert Polynomials. The (equivariant) Schubert classes admit a remarkable choice of polynomial representatives called (double) Schubert polynomials as introduced by Lascoux and Schützenberger [29], see also [33, 37]. For \( 1 \leq i \leq n - 1 \), the BGG Demazure operator \( \partial_{i} \) acts on \( \mathbb{Q}[x] = \mathbb{Q}[x_{1}, \ldots, x_{n}] \) by letting

\[ \partial_{i}f = \frac{f - s_{i}f}{x_{i} - x_{i+1}}, \]

where \( s_{i} = t_{i,i+1} \) is the simple transposition, and \( s_{i}f \) is obtained from \( f \) by swapping \( x_{i} \) and \( x_{i+1} \). The double Schubert polynomials \( S_{w}(x, t) \) for \( w \in \mathfrak{S}_{n} \) can be defined recursively by

\[ S_{w_{0}}(x, t) = \prod_{i+j \leq n} (x_{i} - t_{j}), \quad \partial_{i}S_{w}(x, t) = \begin{cases} S_{w_{s_{i}}}(x, t), & \ell(w_{s_{i}}) < \ell(w), \\ 0, & \ell(w_{s_{i}}) > \ell(w), \end{cases} \]

where \( w_{0} = n \cdots 21 \) refers to the longest permutation of \( \mathfrak{S}_{n} \). Setting all \( t_{i} = 0 \) in \( S_{w}(x, t) \) gives the single Schubert polynomial \( S_{w}(x) = S_{w}(x, 0) \). For combinatorial models of (double) Schubert polynomials, see for example [17, 24, 28, 42]. As the original motivation, the equivariant Schubert class \( [Y(w)]_{T} \) is represented by \( S_{w}(x, t) \) under the Borel isomorphism in Theorem 3.1, see for example [11, Theorem 6.4].

We collect some properties concerning Schubert polynomials, which will be used in the proof of Theorem 3.1 in Section 5.

**Proposition 3.2.** (i) For a Grassmannian permutation \( w_{\lambda} \) with descent at \( k \), the single Schubert polynomial \( S_{w_{\lambda}}(x) \) coincides with the Schur polynomial \( s_{\lambda}(x) \).

(ii) There holds the following Giambelli formula

\[ S_{w}(x, t) = \sum_{u \in \mathfrak{S}_{n}} S_{u}(x)S_{v}(-t). \]

(iii) For \( u, v \in \mathfrak{S}_{n} \) with \( M(u) \subseteq [k] \) and \( M(v) \cap [k] = \emptyset \),

\[ S_{uv}(x) = S_{u}(x)S_{v}(x). \]

Here, \( M(w) \) denotes the set of non-fixed points of \( w \in \mathfrak{S}_{n} \) as defined in (4.4).
3.5. Properties of CSM Classes. The computation of (equivariant) CSM classes has received attention in a series of work, see for example [3–5]. For $1 \leq i \leq n - 1$, the (nonhomogeneous) Demazure–Lusztig type operator is defined as

$$T_i = -s_i + \partial_i.$$ 

It can be directly checked that

$$T_i^2 = \text{id},$$

$$T_i T_j = T_j T_i, \quad |i - j| \geq 2,$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}.$$ 

So, for $w \in S_n$, one may define unambiguously that

$$T_w = T_{i_1} T_{i_2} \cdots T_{i_\ell},$$

where $s_{i_1} s_{i_2} \cdots s_{i_\ell}$ is any decomposition (not necessarily reduced) of $w$. The operators $T_i$’s as well as $x_i$’s generate the degenerate affine Hecke algebra introduced by Lusztig [34] in his study of the representation theory of affine Hecke algebras.

Under the Borel isomorphism, both $\partial_i$ and $T_i$ are well defined over $H^*_{T}(\mathcal{F}(\ell(n)))$ by operating on the variables $x_1, \ldots, x_n$. A crucial property we will use is the following.

**Theorem 3.3** (4,5). Let $w \in S_n$. For any $1 \leq i \leq n - 1$,

$$T_i(c^T_{SM}(Y(w))) = c^T_{SM}(Y(ws_i)).$$

Therefore, for $u \in S_n$, we have

$$T_u(c^T_{SM}(Y(w))) = c^T_{SM}(Y(u^{-1}w)).$$

Taking advantage of the above recurrence relation, a number of properties of CSM classes have been established (4,5). Here we list some that we need in this paper.

**Proposition 3.4** (4,5). (i) The CSM classes $c^T_{SM}(Y(w))$ form an $\mathbb{F}$-basis of $H^*_{T}(\mathcal{F}(\ell(n)))_\mathbb{F}$, where $\mathbb{F} = \mathbb{Q}(t_1, \ldots, t_n)$ is the fraction field of $H^*_{T}(pt) = \mathbb{Q}[t_1, \ldots, t_n]$.

(ii) The Schubert class $[Y(w)]_T$ is the lowest degree term of $c^T_{SM}(Y(w))$.

(iii) We have

$$c^T_{SM}(Y(w)|_{id}) = \begin{cases} \prod_{i<j}(1 + t_i - t_j), & w = \text{id}, \\ 0, & \text{otherwise.} \end{cases}$$

**Remark 3.5.** Schubert polynomials are stable in the sense that they are independent of $n$, and can be used to compute the structure constants of Schubert classes, see [1] §10.10.2 and §10.10.4. However, there have been no known polynomial representatives for CSM classes with the stability property. Stability is a necessary condition to lift the structure constants in the multiplication of CSM classes to the level of polynomials. Liu [33] introduced the notion of twisted Schubert polynomials, which represent CSM classes up to a sign, and showed that this family of polynomials enjoy many interesting combinatorial
properties. However, as pointed out in \[33\], twisted Schubert polynomials depend on \(n\), and so are not stable.

### 4. Rigidity Theorem

In this section, we establish the Rigidity Theorem for the proof of Theorem [A], which bridges the gap between the Pieri formulas for nonequivariant and equivariant CSM classes. The analogous idea will be used in Section 6 to establish the Rigidity Theorem (see Theorem 6.11) required in the proof of Theorem [C].

Let \(u ∈ S_n\) and \(Γ = (α + 1, 1^β)\) be a hook shape. For a subset \(A \subseteq [n]\), suppose that

\[
c_{SM}(Y(u)^{w}) \cdot s_{Γ}(x_{A}) = \sum_{w ∈ S_n} c_{w,Γ} \cdot c_{SM}(Y(w)^{w}).
\]

Although the coefficients \(c_{w,Γ}\) depend on \(A\), we simplify the notation since there will cause no confusion from the context.

**Theorem 4.1** (Rigidity Theorem). Let \(u ∈ S_n\), \(Γ = (α + 1, 1^β)\) and \(A \subseteq [n]\). Suppose that

\[
c_{SM}^{T}(Y(u)^{w}) \cdot s_{Γ}(x_{A}) = \sum_{w ∈ S_n} c_{w,Γ}(t) \cdot c_{SM}^{T}(Y(w)^{w}).
\]

Then we have

\[
c_{u,Γ}(t) = \begin{cases} s_{Γ}(t_{u,A}), & w = u, \\ \sum_{α' ≤ α, β' ≤ β} c_{u,Γ'} \cdot h_{α-α'}(t_{Σ_A(u,w)}) \cdot e_{β-β'}(t_{Δ_A(u,w)}), & w ≠ u. \end{cases}
\]

(See (2.1) and (2.2) for the definitions of \(Δ_A(u, w)\) and \(Σ_A(u, w)\).)

If we set all \(t_i = 0\) in (4.1), then the summands on the right-hand side are nonzero only in the case \(α' = α\) and \(β' = β\), and hence we are led to the expected relationship

\[
c_{u,Γ}(0) = c_{u,Γ}.
\]

A combinatorial description of \(c_{u,Γ}\) will be given in Theorem 5.3, which combined with Theorem 4.1 allows us to reach a proof of Theorem A (see Theorem 5.8).

The remaining of this section is devoted to a proof of Theorem 4.1. We first investigate the action of the Demazure operators on a specific family of polynomial quotients related to the generating function of Schur polynomials of hook shapes.

#### 4.1. Demazure Operators and Polynomials \(Q\) and \(Z\).

For a subset \(A \subseteq [n]\), consider

\[
Q(x_A) = \prod_{a ∈ A} (1 + qx_a) \quad \text{and} \quad Z(x_A) = \prod_{a ∈ A} (1 - zx_a),
\]

which are regarded as elements in the ring of formal power series in \(q, z\) over \(\mathbb{Q}[x]\). Here, when \(A = ∅\), we set \(Q(x_A) = Z(x_A) = 1\). Clearly,

\[
Q(x_A) = \sum_{r ≥ 0} q^r e_r(x_A) \quad \text{and} \quad \frac{1}{Z(x_A)} = \sum_{r ≥ 0} z^r h_r(x_A).
\]

Note that \(e_0(x_A) = h_0(x_A) = 1\) for any \(A \subseteq [n]\), and \(e_r(x_∅) = h_r(x_∅) = 0\) for \(r ≥ 1\).
Lemma 4.2. For $A \subseteq [n]$, let
\[
E(q, z, x_A) = \frac{1}{q + z} \left( \frac{Q(x_A)}{Z(x_A)} - 1 \right).
\]

Then we have
\[
E(q, z, x_A) = \sum_{\alpha, \beta \geq 0} z^\alpha q^\beta s_{\alpha(1+\alpha,1^\beta)}(x_A).
\]

Proof. Notice that
\[
E(q, z, x_A) = \frac{1}{q + z} \left( \left( \sum_{r=0}^{\infty} z^r h_r(x_A) \right) \left( \sum_{s=0}^{\infty} q^s e_s(x_A) \right) - 1 \right).
\]

Applying the classical Pieri rule to the multiplication of $h_r(x_A) = s_{(r)}(x_A)$ by $e_s(x_A)$ (see [57, §7.15]), the right-hand side becomes
\[
\frac{1}{q + z} \sum_{r,s \geq 0} z^r q^s \left( s_{(1+r,1^{r-1})}(x_A) + s_{(r,1^s)}(x_A) \right) = \sum_{\alpha, \beta \geq 0} z^\alpha q^\beta s_{\alpha(1+\alpha,1^\beta)}(x_A),
\]

where we used the assumption that $s_{(1+r,1^{r-1})} = s_{(0,1^r)} = 0$ for $r, s \geq 0$. \hfill \Box

Another advantage we consider $E(q, z, x_A)$ is that as $q$ tends to $-z$, $E(q, z, x_A)$ becomes the generating function of power sum symmetric polynomials. This allows us to invoke the results established in this section to prove the Rigidity Theorem for power sum symmetric polynomials as given in Theorem 6.11.

We extend the Demazure operators defined on polynomials in Section 3 to rational functions. For two distinct integers $a, b \in [n]$ and a rational function $f$, let
\[
\partial_{ab}f = \frac{f - t_{ab}f}{x_a - x_b}, \quad (4.2)
\]

where $t_{ab}f$ is obtained from $f$ by interchanging $x_a$ and $x_b$. When $f$ is symmetric in $x_a$ and $x_b$, it is easily seen that $\partial_{ab}(f) = 0$, and for any polynomial $g$,
\[
\partial_{ab}(fg) = f \partial_{ab}g.
\]

Lemma 4.3. For distinct $a, b \in [n]$ and two subsets $A \subseteq B$ of $[n]$, we have
\[
\partial_{ab} \frac{Q(x_A)}{Z(x_B)} = \left( \delta_{a \in A} q - \delta_{b \in A} q - \delta_{a \notin B} z + \delta_{b \notin B} z \right) \frac{Q(x_{A \setminus \{a,b\}})}{Z(x_{B \setminus \{a,b\}})}, \quad (4.3)
\]

where $\delta$ is the Kronecker delta, that is, $\delta_\diamond$ is 1 if the condition $\diamond$ is satisfied and 0 otherwise.

Proof. If any one of the following conditions is satisfied:
\[
a, b \in A, \quad a, b \in B \setminus A, \quad \text{or} \quad a, b \in [n] \setminus B,
\]

then $\frac{Q(x_A)}{Z(x_B)}$ is symmetric in $x_a$ and $x_b$, and in these cases both sides of (4.3) are zero. Noticing that $\partial_{ab} = -\partial_{ba}$, it remains to verify the following three cases.

Case 1. $a \in A$ and $b \in B \setminus A$. In this case,
\[
\partial_{ab} \frac{Q(x_A)}{Z(x_B)} = \frac{Q(x_{A \setminus \{a\}})}{Z(x_{B \setminus \{a,b\}})} \partial_{ab} \frac{1 + qx_a}{(1 - zx_a)(1 - zx_b)}
\]
On the other hand, by (4.5), we have
\[\frac{q}{Z(x_{B\setminus\{a\}})} (1 - z x_a)(1 - z x_b),\]
which, along with the fact \(A \setminus \{a\} = A \setminus \{a, b\}\), agrees with (4.3).

Case 2. \(a \in A\) and \(b \in [n] \setminus B\). In this case,
\[\partial_{ab} \frac{Q(x_A)}{Z(x_B)} = \frac{Q(x_{A\setminus\{a\}})}{Z(x_{B\setminus\{a\}})} \partial_{ab} \frac{1 + q x_a}{(1 - z x_a)} = \frac{Q(x_{A\setminus\{a\}})}{Z(x_{B\setminus\{a\}})} \frac{q + z}{(1 - z x_a)(1 - z x_b)},\]
which is the same as (4.3).

Case 3. \(a \in B \setminus A\) and \(b \in [n] \setminus B\). In this case,
\[\partial_{ab} \frac{Q(x_A)}{Z(x_B)} = \frac{Q(x_A)}{Z(x_{B\setminus\{a\}})} \partial_{ab} \frac{1}{(1 - z x_a)} = \frac{Q(x_A)}{Z(x_{B\setminus\{a\}})} \frac{z}{(1 - z x_a)(1 - z x_b)},\]
which also coincides with (4.3). \(\square\)

For \(w \in S_n\), let
\[M(w) = \{1 \leq i \leq n: w(i) \neq i\}\] (4.4)
represent the set of non-fixed points of \(w\).

**Lemma 4.4.** For \(m \geq 0\), assume that \(w = t_{a_1 b_1} \cdots t_{a_m b_m} \in S_n\). If
\[\partial_{a_1 b_1} \cdots \partial_{a_m b_m} \frac{Q(x_A)}{Z(x_B)} \neq 0\] (4.5)
for some subsets \(A \subseteq B\) of \([n]\), then
\[M(w) = \{a_1, b_1, \ldots, a_m, b_m\}\] (4.6)

**Proof.** It is obvious that \(M(w) \subseteq \{a_1, b_1, \ldots, a_m, b_m\}\). We next prove the reverse inclusion by induction on \(m\). When \(m = 0\), \(w\) is the identity permutation, both sides of (4.6) are empty, and we are done. Now consider the case \(m > 0\). Suppose to the contrary that \(w(x) = x\) for some \(x \in \{a_1, b_1, \ldots, a_m, b_m\}\). For \(0 \leq j \leq m - 1\), let \(w_j = t_{a_{j+1} b_{j+1}} \cdots t_{a_m b_m}\). Let \(i\) be the smallest \(j\) such that \(w_j(x) \neq x\). Clearly, we have \(i > 0\). Since \(w_{i-1}(x) = t_{a_i b_i} w_i(x) = x\) and \(w_i(x) \neq x\), we obtain that
\[\{a_i, b_i\} = \{x, w_i(x)\}\] (4.7)

On the other hand, by (4.5), we have
\[\partial_{a_{i+1} b_{i+1}} \cdots \partial_{a_m b_m} \frac{Q(x_A)}{Z(x_B)} \neq 0,\]
and so it follows by induction that
\[M(w_i) = \{a_{i+1}, b_{i+1}, \ldots, a_m, b_m\},\]
which along with Lemma (4.3) implies that
\[\partial_{a_{i+1} b_{i+1}} \cdots \partial_{a_m b_m} \frac{Q(x_A)}{Z(x_B)} \in \mathbb{Z}[q, z] \frac{Q(x_{A \setminus M(w_i)})}{Z(x_{B \cup M(w_i)})}.\] (4.8)
Since \( w_i(x) \neq x \), we see that \( w_i(w_i(x)) \neq w_i(x) \). Thus both \( x \) and \( w_i(x) \) belong to \( M(w_i) \). In view of (4.7), both \( a_i \) and \( b_i \) belong to \( M(w_i) \). Consequently,

\[
\partial_{a_i} \partial_{b_i} \frac{Q(x_A \setminus M(w_i))}{Z(x_B \setminus M(w_i))} = \frac{Q(x_A \setminus M(a_i))}{Z(x_B \setminus M(a_i))} \partial_{a_i} \frac{1}{(1-zx_{a_i})(1-zx_{b_i})} = 0,
\]

which together with (4.8) would yield

\[
\partial_{a_i} \partial_{b_i+1} \cdots \partial_{a_m b_m} \frac{Q(x_A)}{Z(x_B)} = 0,
\]

crude to the assumption in (4.5). This completes the proof. \( \square \)

4.2. Demazure–Lusztig Operators and Proof of Theorem 4.1. For \( w \in S_n \), fix a reduced word of \( w \):

\[
w = s_{i_1} \cdots s_{i_r}.
\]

For any subset \( J \subseteq [\ell] \), define

\[
w_J = \prod_{j \in J} s_{i_j} \quad \text{and} \quad \Delta_J = \prod_{j=1}^{\ell} \left\{ \begin{array}{ll} s_{i_j}, & j \in J, \\ \partial_{i_j}, & j \notin J, \end{array} \right. \quad (4.9)
\]

where the factors in the product are multiplied from left to right as \( j \) increases. By direct calculation, \( T_i = -s_i + \partial_i \) satisfies the Leibniz rule:

\[
T_i(fg) = T_i f \cdot s_i g + f \cdot \partial_i g. \quad (4.10)
\]

It should be noticed that the twisted operator \( T_i = s_i + \partial_i \), which has been used by Liu [33] to define twisted Schubert polynomials, possesses analogous properties to \( T_i \) [33 Proposition 3.3]. Iterating (4.10), it is not hard to check that

\[
T_w(fg) = \sum_{J \subseteq [\ell]} T_w f \cdot \Delta_J g = \sum_{v \in S_n} T_v f \cdot \sum_{J \subseteq [\ell]} \Delta_J g. \quad (4.11)
\]

For \( v, w \in S_n \), define the skew operator \( T_{w/v} \) as

\[
T_{w/v} = \sum_{J \subseteq [\ell]} \Delta_J. \quad (4.12)
\]

Note that since \( v \) appears as a subword in the reduced word of \( w \), \( T_{w/v} \) is zero unless \( v \leq w \) in the Bruhat order. With the notation in (4.12), (4.11) can be rewritten as

\[
T_w(fg) = \sum_{w \in S_n} T_v f \cdot T_{w/v} g. \quad (4.13)
\]

Let \( \alpha \) be a class in \( H^*_T(F\ell(n)) \). By Proposition 3.4 one has the following expansion

\[
\alpha = \sum_{w \in S_n} c_w^\alpha(t) \cdot c_{SM}(Y(w))^\circ,
\]

where the coefficients \( c_w^\alpha(t) \) are rational functions in \( t \).

Lemma 4.5. We have

\[
c_w^\alpha(t) = \frac{1}{\prod_{1 \leq i < j \leq n} (1 + t_i - t_j)} T_w(\alpha) \big|_{\text{id}}. \quad (4.14)
\]
Proof. By Theorem 3.3 and Proposition 3.4, \( c^w(\alpha, t) \) could be computed by applying the operator \( T_w \) to \( \alpha \) and then invoking the localization map \( |\text{id}| \).

This combined with (4.13) leads to the following important observation.

**Lemma 4.6.** Suppose that for \( u \in \mathcal{S}_n \) and a class \( \alpha \in H^\bullet_T(\mathcal{F}_\ell(n)) \),

\[
c^T_{SM}(Y(u) \circ \alpha) = \sum_{w \in \mathcal{S}_n} c^w_{u, \alpha}(t) \cdot c^T_{SM}(Y(w) \circ \alpha).
\]

Then we have

\[
c^w_{u, \alpha}(t) = T_{w/u}(\alpha)|_{\text{id}}.
\]

**Proof.** Replacing \( \alpha \) by \( c^T_{SM}(Y(u) \circ \alpha) \) in (4.14) gives

\[
c^w_{u, \alpha}(t) = \frac{1}{\prod_{1 \leq i < j \leq n} (1 + t_i - t_j)} \sum_{v \in \mathcal{S}_n} T_v(c^T_{SM}(Y(u) \circ \alpha)|_{\text{id}} \cdot T_{w/v}(\alpha)|_{\text{id}}).
\]

Plugging (4.13) into (4.16) and using Theorem 3.3 and Proposition 3.4, we deduce that

\[
c^w_{u, \alpha}(t) = \frac{1}{\prod_{1 \leq i < j \leq n} (1 + t_i - t_j)} \sum_{v \in \mathcal{S}_n} c^T_{SM}(Y(uv^{-1}) \circ \alpha)|_{\text{id}} \cdot T_{w/v}(\alpha)|_{\text{id}}
\]

\[
= \frac{1}{\prod_{1 \leq i < j \leq n} (1 + t_i - t_j)} c^T_{SM}(Y(\alpha)|_{\text{id}}) \cdot T_{w/u}(\alpha)|_{\text{id}}
\]

as required. □

The last lemma concerns the action of skew operators on the generating function \( E(q, z, x_A) \) of Schur polynomials of hook shapes as defined in Lemma 4.2.

**Lemma 4.7.** For \( u, w \in \mathcal{S}_n \) with \( u \neq w \), we have

\[
T_{w/u}E(q, z, x_A) \in \frac{1}{q + z} Z[q, z] \frac{Q(x_{\Delta A(u, w)})}{Z(x_{\Sigma A(u, w)})}.
\]

**Proof.** For any \( v \in \mathcal{S}_n \), it is easy to check that

\[ v \partial_{ab} = \partial_{v(a)v(b)} v. \]

Hence, for \( J \subseteq [\ell] \) such that \( w_J = u \), we can interchange the Demazure operators appearing in \( \nabla_J \) defined in (4.9) one by one to the rightmost side of \( w_J \), and so we may assume that \( \nabla_J \) takes the form

\[ \nabla_J = u \partial_{a_1 b_1} \cdots \partial_{a_r b_r}. \]

On the other hand, since \( vt_{ab} = t_{v(a)v(b)} v \) for any \( v \in \mathcal{S}_n \), we can use exactly the same procedure with \( \nabla_J \) to deduce that

\[ w = ut_{a_1 b_1} \cdots t_{a_r b_r}, \]
or equivalently,

\[ u^{-1}w = t_{a_1b_1} \cdots t_{a_rb_r}. \]

Combining Lemma 4.3 and Lemma 4.4, we obtain that

\[ \partial_{a_1b_1} \cdots \partial_{a_rb_r} \frac{Q(x_A)}{Z(x_A)} = \frac{Q(x_A \setminus M(u^{-1}w))}{Z(x_A \cup M(u^{-1}w))} \cdot \frac{Q(x_A)}{Z(x_A)}. \] (4.18)

Since \( u \neq w \), we have \( r \geq 1 \) and so

\[ \partial_{a_1b_1} \cdots \partial_{a_rb_r} \frac{1}{q + z} = 0, \]

which together with (4.18) leads to

\[ \partial_{a_1b_1} \cdots \partial_{a_rb_r} E(q, z, x_A) = \partial_{a_1b_1} \cdots \partial_{a_rb_r} \frac{1}{q + z} \frac{Q(x_A)}{Z(x_A)} - \partial_{a_1b_1} \cdots \partial_{a_rb_r} \frac{1}{q + z} \]

\[ \in \frac{1}{q + z} \frac{Q(x_A \setminus M(u^{-1}w))}{Z(x_A \cup M(u^{-1}w))}. \]

Therefore,

\[ \nabla J E(q, z, x_A) = u \partial_{a_1b_1} \cdots \partial_{a_rb_r} E(q, z, x_A) \]

\[ \in \frac{1}{q + z} \frac{Q(x_A \setminus M(u^{-1}w))}{Z(x_A \cup M(u^{-1}w))} \]

\[ = \frac{1}{q + z} \frac{Q(x_{\Delta A(u,w)})}{Z(x_{\Sigma A(u,w)})}, \]

yielding (4.17). \( \square \)

We are finally in a position to complete the proof of Theorem 4.1.

**Proof of Theorem 4.1** Recall that

\[ c_{SM}(Y(u)^\circ) \cdot s_T(x_A) = \sum_{w \in S_n} c^w_{u,1}(t) \cdot c_{SM}(Y(w)^\circ). \]

Let

\[ c^w_u(q, z, t) = \sum_{\alpha, \beta \geq 0} z^\alpha q^\beta \cdot c^w_{u,1}(t). \] (4.19)

Then by Lemma 4.2

\[ c^T_{SM}(Y(u)^\circ) \cdot E(q, z, x_A) = c^T_{SM}(Y(u)^\circ) \cdot \sum_{\alpha, \beta \geq 0} z^\alpha q^\beta \cdot s_{(1+\alpha,1+\beta)}(x_A) \]

\[ = \sum_{w \in S_n} c^w_u(q, z, t) \cdot c^T_{SM}(Y(w)^\circ). \]

By Lemma 4.6, we obtain that

\[ c^w_u(q, z, t) = T_{w/u}[E(q, z, x_A)]_{x_i = t_i}. \] (4.20)
If \( w = u \), then the skew operator \( \mathcal{T}_{u/u} \) is nothing but \( u \) itself, and so
\[
c_u^u(q, z, t) = E(q, z, t_0) = \sum_{\alpha, \beta \geq 0} z^\alpha q^\beta \cdot s_{1+\alpha, 1+\beta}(t_0),
\]
which gives \( c_{u,\Gamma}(t) = s_{1+\alpha, 1+\beta}(t_0) \). If \( w \neq u \), then, by (4.20) and Lemma 4.7, we see that
\[
c_u^w(q, z, t) = f(q, z) \frac{Q(t_{\Delta_A(u,w)})}{Z(t_{\Sigma_A(u,w))})},
\]
where \( f(q, z) \in \frac{1}{q+z} \mathbb{Z}[q, z] \). Setting all \( t_i = 0 \) on both sides, we obtain that \( c_u^w(q, z, 0) = f(q, z) \), and hence,
\[
c_u^w(q, z, t) = c_u^w(q, z, 0) \cdot \sum_{r, s \geq 0} z^r q^s \cdot h_r(t_{\Sigma_A(u,w)}) \cdot e_s(t_{\Delta_A(u,w)})
\]
\[
= \left( \sum_{\alpha', \beta' \geq 0} z^\alpha q^\beta \cdot c_u^w(0) \right) \cdot \sum_{r, s \geq 0} z^r q^s \cdot h_r(t_{\Sigma_A(u,w)}) \cdot e_s(t_{\Delta_A(u,w)}).  \tag{4.21}
\]
Comparing the coefficients of \( z^\alpha q^\beta \) in (4.19) and (4.21), we are led to (4.1). \( \square \)

5. Pieri Type Rules

In this section, we will prove Theorems A and B. To this end, we first establish the expansion formula for multiplying a nonequivariant CSM class by a Schubert polynomial of hook shape, see Theorem 5.5. Combining Theorem 5.5 with the Rigidity Theorem 4.1 gives a proof of Theorem A. Based on Theorem A and properties of double Schubert polynomials, we finish the proof of Theorem B.

5.1. Nonequivariant Case. We begin by proving a Pieri formula for nonequivariant CSM classes. The proof relies on the property of the action of the skew operators \( \mathcal{T}_{w/u} \) on \( e_r(x[k]) \) or \( h_r(x[k]) \), which has been investigated by Liu [33].

**Theorem 5.1 (CSM Pieri formula).** Let \( u \in \mathfrak{S}_n \). We have the following identities in \( H^*(\mathcal{F}(n)) \):

(i) For \( r \geq 0 \),
\[
c_{CSM}(Y(u)^\circ) \cdot e_r(x[k]) = \sum_{w \in \mathfrak{S}_n} c_{CSM}(Y(w)^\circ), \tag{5.1}
\]
where the sum ranges over all \( w \in \mathfrak{S}_n \) such that there is path
\[
u \rightarrow u t_{a_1 b_1} \rightarrow u t_{a_1 b_1} t_{a_2 b_2} \rightarrow \cdots \rightarrow w = u t_{a_1 b_1} \cdots t_{a_r b_r}
\]
from \( u \) to \( w \) in the extended \( k \)-Bruhat order and \( a_1, a_2, \ldots, a_r \) are distinct.

(ii) For \( r \geq 0 \),
\[
c_{CSM}(Y(u)^\circ) \cdot h_r(x[k]) = \sum_{w \in \mathfrak{S}_n} c_{CSM}(Y(w)^\circ), \tag{5.2}
\]
where the sum ranges over all \( w \) as in (i), except that now the integers \( b_1, \ldots, b_r \) are distinct.
Proof. We only give a proof of (5.1), and the arguments for (5.2) are similar. By Lemma 4.6 we need to show that for \( w \in \mathfrak{S}_n \),
\[
\mathcal{T}_{w/u}(e_r(x_{[k]})) \big|_{x_i=0} \neq 0
\]
if and only if \( w \) satisfies the conditions in (i), and in this case, the nonzero value in (5.3) is exactly equal to 1.

Set \( \tilde{\partial}_{w/u} = u^{-1} \mathcal{T}_{w/u} \). We say that \( w \) is legal if \( w \) satisfies the conditions in (i), except that we replace \( r \) by any nonnegative integer \( m \). If \( w \) is not legal, then it follows from [33, Theorem 3.10] that \( \tilde{\partial}_{w/u}(e_r(x_{[k]})) = 0 \). If \( w \) is legal, then [33, Theorem 3.10] gives
\[
\mathcal{T}_{w/u}(e_r(x_{[k]})) = e_r - |A|(x_{u([k]\setminus A)}),
\]
where \( A = \{a_1, \ldots, a_m\} \). So,
\[
T_{w/u}(e_r(x_{[k]})) = u \tilde{\partial}_{w/u} = e_r - |A|(x_{u([k]\setminus A)}),
\]
implying that the value in (5.3) vanishes if \( m \neq r \), and equals 1 if \( m = r \). This completes the proof. \( \square \)

The permutations appearing in Theorem 5.1 can be alternatively characterized in terms of increasing or decreasing paths in the extended \( k \)-Bruhat order.

**Lemma 5.2.** We have the following statements.

1. A permutation \( w \in \mathfrak{S}_n \) satisfies the conditions in (i) of Theorem 5.1 if and only if there exists a decreasing path of length \( r \) from \( u \) to \( w \) in the extended \( k \)-Bruhat order. Moreover, the decreasing path is unique.

2. A permutation \( w \in \mathfrak{S}_n \) satisfies the conditions in (ii) of Theorem 5.1 if and only if there exists an increasing path of length \( r \) from \( u \) to \( w \) in the extended \( k \)-Bruhat order. Moreover, the increasing path is unique.

**Proof.** We only give a proof of the statement in (1), and a similar analysis applies to the statement in (2). Let us first verify the necessity. Suppose that \( w \) satisfies the conditions in (i) of Theorem 5.1. The existence of a decreasing path is implied by the following easily checked claim.

**Claim.** Assume that there is a path
\[
v \xrightarrow{\tau} vt_{a_1}b_1 \xrightarrow{\sigma} vt_{a_1}b_1t_{a_2}b_2 = v'
\]
from \( v \) to \( v' \) in the extended \( k \)-Bruhat order with \( a_1 \neq a_2 \) and \( \tau < \sigma \). Then there is an alternative path from \( v \) to \( v' \):
\[
v \xrightarrow{\sigma} vt_{a_2}b_2 \xrightarrow{\tau} vt_{a_2}b_2t_{a_1}b_1 = v'.
\]

In fact, since
\[
\tau = v(a_1) = vt_{a_1}b_1(b_1) < \sigma = vt_{a_1}b_1(a_2) < vt_{a_1}b_1(b_2),
\]
we have \( b_1 \neq b_2 \). So, \( t_{a_1}b_1 \) and \( t_{a_2}b_2 \) commute, and the claim follows. Given \( u \) and \( w \) as in (ii) of Theorem 5.1 we have a path
\[
u \xrightarrow{\sigma_1} ut_{a_1}b_1 \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_r} ut_{a_1}b_1 \cdots t_{a_i}b_i = w
\]

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from \( u \) to \( w \) in the extended \( k \)-Bruhat order. If this path is not decreasing, we can iterate the above Claim and eventually obtain a decreasing path of length \( r \) from \( u \) to \( w \) in the extended \( k \)-Bruhat order.

We next prove the sufficiency. Let

\[
\gamma': \ u = w'_0 \xrightarrow{\tau'_1} w'_1 \xrightarrow{\tau'_2} \cdots \xrightarrow{\tau'_r} w'_r = w
\]

be a decreasing path of length \( r \) from \( u \) to \( w \) in the extended \( k \)-Bruhat order. Assume that for \( 1 \leq i \leq r \),

\[
w'_i = ut_a t'_i \cdots t'_{a_i}.
\]

To conclude the sufficiency, we show that the integers \( a'_1, a'_2, \ldots, a'_r \) are distinct. We first explain that \( a'_1 \neq a'_2 \). Notice that \( \tau'_1 = u(a'_1) < w'_1(a'_1) \) and \( w'_1(j) = u(j) \) for \( j \in [k] \setminus \{ a'_1 \} \).

Since \( w'_1(a'_2) = \tau'_2 < \tau'_1 \), we have \( w'_1(a'_2) < w'_1(a'_1) \). This implies that \( a'_2 \neq a'_1 \).

Moreover, we see that

\[
w'_2(a'_1) = w'_1(a'_1) > u(a'_1), \quad w'_2(a'_2) = w'_1(b'_2) > w'_1(a'_2) = u(a'_2),
\]

and \( w'_2(j) = u(j) \) for \( j \in [k] \setminus \{ a'_1, a'_2 \} \).

We proceed to check that \( a'_2 \) is different from \( a'_1 \) and \( a'_2 \). Since \( w'_2(a'_1) > u(a'_1) = \tau'_1 \), \( w'_2(a'_2) > u(a'_2) = \tau'_2 \) and \( w'_2(a'_3) = \tau'_3 \), \( w'_2(a'_2) < \tau'_2 < \tau'_1 \), we obtain that \( w'_2(a'_3) \) is smaller than \( w'_2(a'_1) \) and \( w'_2(a'_3) \), and so \( a'_2 \neq a'_1, a'_2 \).

Moreover, we have

\[
w'_3(a'_1) = w'_2(a'_1) > u(a'_1), \quad w'_3(a'_2) = w'_2(a'_2) = w'_1(b'_2) > u(a'_2), \quad w'_3(a'_3) > u(a'_3),
\]

and \( w'_3(j) = u(j) \) for \( j \in [k] \setminus \{ a'_1, a'_2, a'_3 \} \). Using a similar analysis, we can deduce that for \( i = 4, \ldots, r \), \( a'_i \) is different from \( a'_1, \ldots, a'_{i-1} \), and this verifies the sufficiency.

Finally, we show that the decreasing path \( \gamma' \) is unique. Since \( a'_1, \ldots, a'_r \) are distinct, we know that for \( 1 \leq i \leq r \), \( \tau_i = u(a'_i) \), or equivalently, \( a'_i = u^{-1}(\tau_i) \). On the other hand, since \( u(b'_i) = w'_1(a'_1) = w(a'_1) \), we see that \( b'_1 \) is uniquely located. Similarly, since \( w'_1(b'_2) = w'_2(a'_2) = w(a'_2) \) and \( w'_1 = ut_a t'_i \), we see that \( b'_2 \) is uniquely located. Iterating the same process, we can eventually obtain that \( b'_1, b'_2, \ldots, b'_r \) are uniquely determined. This verifies the uniqueness.

\[\square\]

**Remark 5.3.** Let \( u, w \in \mathfrak{S}_n \). Suppose that there is a (unique) decreasing (resp., increasing) path \( \gamma \) from \( u \) to \( w \) in the extended \( k \)-Bruhat order. By the proof of Lemma 5.2, the \( a_i \)'s (resp., \( b_i \)'s) are distinct, thus the length of \( \gamma \) is equal to

\[
\# [k] \cap M(u^{-1}w) \quad (\text{resp., } \# ([n] \setminus [k]) \cap M(u^{-1}w)).
\]

For a path \( \gamma \) in the extended \( k \)-Bruhat order, we use \( \text{end}(\gamma) \) to denote the endpoint permutation of \( \gamma \). By Lemma 5.2, we can reformulate Theorem 5.1 as follows.

**Corollary 5.4 (CSM Pieri Formula).** Let \( u \in \mathfrak{S}_n \). We have the following identities in \( H^*(\mathcal{F}(n)) \):

(i) For \( r \geq 0 \),

\[
\langle c_{SM}(Y(u)^\circ) \cdot e_r(x|k) \rangle = \sum_{\gamma} c_{SM}(Y(\text{end}(\gamma))^\circ), \tag{5.4}
\]

where the sum ranges over all decreasing paths of length \( r \) starting at \( u \) in the extended \( k \)-Bruhat order.
(ii) For \( r \geq 0 \),
\[
c_{SM}(Y(u)^\circ) \cdot h_r(x_{[k]}) = \sum_{\gamma} c_{SM}(Y(\text{end}(\gamma))^\circ),
\]
where the sum ranges over all increasing paths of length \( r \) starting at \( u \) in the extended \( k \)-Bruhat order.

We can now exhibit the expansion formula for the multiplication of a CSM class by a Schur polynomial of hook shape.

**Theorem 5.5.** For a hook shape partition \( \Gamma = (\alpha+1, 1^\beta) \), we have the following identity in \( H^*(\mathcal{F}\ell(n)) :\)
\[
c_{SM}(Y(u)^\circ) \cdot s_{\Gamma}(x_{[k]}) = \sum_{\gamma} c_{SM}(Y(\text{end}(\gamma))^\circ),
\]
where the sum runs over all peakless paths \( \gamma \) starting at \( u \) in the extended \( k \)-Bruhat order with \( \text{in}(\gamma) = \alpha \) and \( \text{de}(\gamma) = \beta \).

**Proof.** We make induction on \( \beta \). The case \( \beta = 0 \) is nothing but (ii) of Corollary 5.4. Now we assume that \( \beta \geq 1 \). By Corollary 5.4, we see that
\[
c_{SM}(Y(u)^\circ) \cdot e_{\alpha+1}(x_{[k]}) = \sum_{\gamma} c_{SM}(Y(w)^\circ),
\]
where the sum is taken over all paths
\[
\gamma' : u = w_0 \xrightarrow{\tau_1} w_1 \rightarrow \cdots \xrightarrow{\tau_{\alpha+\beta+1}} w_{\alpha+\beta+1} = w
\]
in the extended \( k \)-Bruhat order such that
\[
\tau_1 > \cdots > \tau_{\beta} \quad \text{and} \quad \tau_{\beta+1} < \cdots < \tau_{\alpha+\beta+1}.
\]
Since \( \tau_{\beta} = w_{\beta}(i) \) for some \( i > k \) and \( \tau_{\beta+1} = w_{\beta}(j) \) for some \( j \leq k \), we have \( \tau_{\beta} \neq \tau_{\beta+1} \). Thus either \( \tau_{\beta} > \tau_{\beta+1} \) or \( \tau_{\beta} < \tau_{\beta+1} \).

If \( \tau_{\beta} < \tau_{\beta+1} \), then \( \gamma' \) is exactly a peakless path with \( \text{in}(\gamma') = \alpha + 1 \) and \( \text{de}(\gamma') = \beta - 1 \). On the other hand, by induction,
\[
c_{SM}(Y(u)^\circ) \cdot s_{(2+\alpha, 1^{\beta-1})}(x_{[k]}) = \sum_{\gamma''} c_{SM}(Y(\text{end}(\gamma''))^\circ),
\]
where the sum is taken over all peakless paths \( \gamma'' \) with \( \text{in}(\gamma'') = \alpha + 1 \) and \( \text{de}(\gamma'') = \beta - 1 \). Therefore,
\[
c_{SM}(Y(u)^\circ) \cdot h_{\alpha+1}(x_{[k]}) \cdot e_{\beta}(x_{[k]}) - c_{SM}(Y(u)^\circ) \cdot s_{(2+\alpha, 1^{\beta-1})}(x_{[k]})
= c_{SM}(Y(u)^\circ) \cdot (h_{\alpha+1}(x_{[k]}) \cdot e_{\beta}(x_{[k]}) - s_{(2+\alpha, 1^{\beta-1})}(x_{[k]}))
= \sum_{\gamma} c_{SM}(Y(\text{end}(\gamma))^\circ),
\]
where the sum runs over all peakless paths \( \gamma \) starting at \( u \) in the extended \( k \)-Bruhat order with \( \text{in}(\gamma) = \alpha \) and \( \text{de}(\gamma) = \beta \). We arrive at (5.6) by noticing the identity
\[
h_{\alpha+1}(x_{[k]}) \cdot e_{\beta}(x_{[k]}) = s_{(1+\alpha, 1^{\beta})}(x_{[k]}) + s_{(2+\alpha, 1^{\beta-1})}(x_{[k]}),
\]
which follows from the classical Pieri rule for Schur polynomials. \( \square \)
We may alternatively use \( h_{\alpha+1}(x_{[k]}) \cdot e_{\beta}(x_{[k]}) \) instead of \( e_{\beta}(x_{[k]}) \cdot h_{\alpha+1}(x_{[k]}) \) in the proof of Theorem 5.5, which leads to a dual version of Theorem 5.5. We say that a path in the extended \( k \)-Bruhat order is **unimodal** if \( \tau_1 < \cdots < \tau_i > \cdots > \tau_m \) for some \( 1 \leq i \leq m \). Analogously, we use \( \text{in}(\gamma) = i - 1 \) (resp., \( \text{de}(\gamma) = m - i \)) to denote one less than the length of the increasing (resp., decreasing) segment of \( \gamma \).

**Theorem 5.6.** For a hook shape partition \( \Gamma = (1+\alpha, 1^{\beta}) \), we have the following identity in \( H^*(\mathcal{F}_C(n)) \):

\[
c_{\text{SM}}(Y(u)^\circ) \cdot s_T(x_{[k]}) = \sum_{\gamma} c_{\text{SM}}(Y(\text{end}(\gamma))^\circ),
\]

where the sum runs over all unimodal paths \( \gamma \) starting at \( u \) in the extended \( k \)-Bruhat order with \( \text{in}(\gamma) = \alpha \) and \( \text{de}(\gamma) = \beta \).

If we take the lowest degree component in Theorems 5.5 or 5.6, then we recover the formula for the product \( [Y(u)] \cdot s_T(x_{[k]}) \) due to Sottile [55, Theorem 8], where in this case the sum ranges over peakless or unimodal paths in the ordinary \( k \)-Bruhat order.

Theorems 5.5 and 5.6 together lead to the following equidistribution.

**Corollary 5.7.** For permutations \( u, w \in S_n \) and nonnegative integers \( \alpha \) and \( \beta \), the number of peakless paths from \( u \) to \( w \) in the extended \( k \)-Bruhat order with \( \text{in}(\gamma) = \alpha \) and \( \text{de}(\gamma) = \beta \) is equal to the number of unimodal paths from \( u \) to \( w \) in the extended \( k \)-Bruhat order with \( \text{in}(\gamma) = \alpha \) and \( \text{de}(\gamma) = \beta \).

### 5.2. Equivariant Case I: Theorem A

**Theorem 5.8 (=Theorem A).** For \( u \in S_n \) and a hook shape \( \Gamma = (1+\alpha, 1^{\beta}) \), we have the following identity in \( H^*_C(\mathcal{F}_C(n)) \):

\[
c_T(Y(u)^\circ) \cdot s_T(x_{[k]}) = s_T(t_{u_{[k]}}) \cdot c_{\text{SM}}(Y(u)^\circ) + \sum_{u \neq w \in S_n} c_{u, T}(t) \cdot c_{\text{SM}}(Y(w)^\circ),
\]

where

\[
c_{u,T}(t) = \sum_{\gamma} h_{\alpha-\text{in}(\gamma)}(t_{\Delta_k(u,w)}) \cdot e_{\beta-\text{de}(\gamma)}(t_{\Delta_k(u,w)})
\]

with the sum taken over all peakless paths from \( u \) to \( w \) in the extended \( k \)-Bruhat order.

**Proof.** This immediately follows from Theorem 5.5 and the Rigidity Theorem 4.1. \( \square \)

Restricting \( \alpha = 0 \) or \( \beta = 0 \) in Theorem 5.8 we obtain the following Pieri formula for equivariant CSM classes.

**Corollary 5.9** (Equivariant CSM Pieri Formula I). Let \( u \in S_n \). We have the following identities in \( H^*(\mathcal{F}_C(n)) \):

1. For \( r \geq 1 \),

\[
c_T(Y(u)^\circ) \cdot e_{r}(x_{[k]}) = \sum_{\gamma} e_{r-\ell(\gamma)}(t_{\Delta_k(u,w)}) \cdot c_{\text{SM}}(Y(\text{end}(\gamma))^\circ),
\]

where the sum is over all decreasing paths from \( u \) to \( w \) in the extended \( k \)-Bruhat order. Here, \( \ell(\gamma) \) denotes the length of \( \gamma \).
(2) For \( r \geq 1 \),
\[
c^T_{\text{SM}}(Y(u)^\circ) \cdot h_r(x[k]) = \sum_{\gamma} h_{r - \ell(\gamma)}(t_{\Sigma_k(u,w)}) \cdot c^T_{\text{SM}}(Y(\text{end}(\gamma))^\circ),
\]
where the sum is over all increasing paths from \( u \) to \( w \) in the extended \( k \)-Bruhat order.

Taking the lowest degree component of \( c^T_{\text{SM}}(Y(u)^\circ) \), we obtain a formula for multiplying an equivariant Schubert class by a Schur polynomial of hook shape.

**Corollary 5.10.** For a hook shape partition \( \Gamma = (1 + \alpha, 1^\beta) \), we have the following identity in \( H^*_F(\mathcal{F}\ell(n)) \):
\[
[Y(u)]_T \cdot s_{\Gamma}(x[k]) = s_{\Gamma}(t_{u[k]}) \cdot [Y(u)]_T + \sum_{u \neq w \in \mathcal{G}_n} \overline{c}^{w,\Gamma}_u(t) \cdot [Y(w)]_T,
\]
(5.8)

where \( \overline{c}^{w,\Gamma}_u(t) \) has the same expression as \( c^{w,\Gamma}_u(t) \) in (5.7), except that now the peakless paths are restricted to be in the ordinary \( k \)-Bruhat order.

We give an example to illustrate Theorem 5.8 and Corollary 5.10.

**Example 5.11.** Let \( u = 23154 \) and \( k = 2 \). Choose \( \alpha = 1 \) and \( \beta = 1 \), i.e., \( \Gamma = (2,1) \).

For simplicity, denote
\[
\zeta_w = c^T_{\text{SM}}(Y(w)^\circ) \quad \text{and} \quad \sigma_w = [Y(w)]_T.
\]

Table 1 lists all the peakless paths \( \gamma \) in the \( k \)-Bruhat graph of \( \mathcal{G}_5 \) starting from \( u = 23154 \) with \( \text{in}(\gamma) \leq 1 \) and \( \text{de}(\gamma) \leq 1 \), where we use dashed arrows to distinguish the \( k \)-edges \( u \xrightarrow{\tau} w \) with \( \ell(w) > \ell(u) + 1 \). See also Figure 2 for an illustration.

By Theorem 5.8, we have
\[
\zeta_{23154} \cdot s_{(2,1)}(x_1, x_2) = s_{(2,1)}(t_2, t_3) \cdot \zeta_{23154} + (t_2t_3 + t_3^2 + t_3t_5) \cdot \zeta_{53124}
\]
\[
+ (t_2t_3 + t_3^2 + t_3t_4) \cdot \zeta_{43152} + (t_2^2 + t_2t_3 + t_2t_5) \cdot \zeta_{25134}
\]
\[
+ (t_2^2 + t_2t_3 + t_2t_4) \cdot \zeta_{24153} + t_3 \cdot \zeta_{53142} + (t_2 + t_3 + t_5) \cdot \zeta_{35124}
\]
\[
+ (t_2 + t_3 + t_4 + t_5) \cdot \zeta_{45132} + (t_2 + t_3 + t_4 + t_5) \cdot \zeta_{54123}
\]
\[
+ (t_2 + t_3 + t_4) \cdot \zeta_{34152} + t_2 \cdot \zeta_{25143} + \zeta_{45132} + \zeta_{54132} + \zeta_{35142}.
\]

By Corollary 5.10, all the paths in Table 1 with solid edges are what we need in the expansion of \( \sigma_{23154} \cdot s_{(2,1)}(x_1, x_2) \), to wit,
\[
\sigma_{23154} \cdot s_{(2,1)}(x_1, x_2) = s_{(2,1)}(t_2, t_3) \cdot \sigma_{23154} + (t_2 + t_2t_3 + t_2t_5) \cdot \sigma_{25134}
\]
\[
+ (t_2^2 + t_2t_3 + t_2t_4) \cdot \sigma_{24153} + (t_2 + t_3 + t_5) \cdot \sigma_{35124}
\]
\[
+ (t_2 + t_3 + t_4) \cdot \sigma_{34152} + t_2 \cdot \sigma_{25143} + \sigma_{45132} + \sigma_{35142}.
\]

### 5.3. **Equivariant Case II: Theorem**

Recall the Giambelli formula for Schubert polynomials in Proposition 3.2. In the case when the permutation is Grassmannian corresponding to a hook shape, the Giambelli formula has an explicit expression.
Lemma 5.12. For a hook shape $\Gamma = (1 + \alpha, 1^\beta)$, assume that the Grassmannian permutation $w_\Gamma$ belongs to $\mathfrak{S}_n$ and has descent at position $k$. We have the following identity:

$$[Y(w_\Gamma)]_T = \mathfrak{S}_{w_\Gamma^{-1}}(-t) + \sum_{\Gamma' = (1 + \alpha', 1^{\beta'}) \atop \alpha' \leq \alpha, \beta' \leq \beta} s_{\Gamma'}(x_{[k]}) \cdot e_{\alpha - \alpha'}(-t_{[k+\alpha]}) \cdot h_{\beta - \beta'}(-t_{[k-\beta]}).$$

(5.9)

| peakless paths | coefficients |
|----------------|--------------|
| 23154 $\rightarrow$ 53124 | $e_1(t_{(3)})h_1(t_{(1,3,5)}) = t_2t_3 + t_3^2 + t_3t_5$ |
| 23154 $\rightarrow$ 43152 | $e_1(t_{(3)})h_1(t_{(2,3,4)}) = t_2t_3 + t_3^2 + t_3t_4$ |
| 23154 $\rightarrow$ 25134 | $e_1(t_{(2)})h_1(t_{(2,3,5)}) = t_2^2 + t_2t_3 + t_2t_5$ |
| 23154 $\rightarrow$ 24153 | $e_1(t_{(2)})h_1(t_{(2,3,4)}) = t_2^2 + t_2t_3 + t_2t_4$ |
| 23154 $\rightarrow$ 53124 $\rightarrow$ 35124 $\rightarrow$ 54123 | $e_0(t_{(\emptyset)})h_1(t_{(2,3,5)}) = t_2 + t_3 + t_5$ |
| 23154 $\rightarrow$ 43152 $\rightarrow$ 35124 $\rightarrow$ 45132 | $e_0(t_{(\emptyset)})h_1(t_{(2,3,5,4)}) = t_2 + t_3 + t_4 + t_5$ |
| 23154 $\rightarrow$ 45132 $\rightarrow$ 35124 $\rightarrow$ 45132 | $e_0(t_{(\emptyset)})h_1(t_{(2,3,5,4)}) = t_2 + t_3 + t_4 + t_5$ |
| 23154 $\rightarrow$ 34152 $\rightarrow$ 45132 | $e_0(t_{(\emptyset)})h_0(t_{(2,3,5,4)}) = 1$ |
| 23154 $\rightarrow$ 34152 $\rightarrow$ 45132 | $e_0(t_{(\emptyset)})h_0(t_{(2,3,5,4)}) = 1$ |
| 23154 $\rightarrow$ 34152 $\rightarrow$ 45132 | $e_0(t_{(\emptyset)})h_0(t_{(2,3,5,4)}) = 1$ |

Table 1. Computing $\zeta_{23154} \cdot s_{(2,1)}(x_1, x_2)$ and $\sigma_{23154} \cdot s_{(2,1)}(x_1, x_2)$

Figure 2. An illustration of Example 5.11
Proof. By the Giambelli formula in Proposition 3.2,
\[ [Y(w_T)]_T = \mathcal{G}_{w_T}(x, t) = \sum_{\ell(w_T) = \ell(u) + \ell(v)} \mathcal{G}_u(x)\mathcal{G}_v(-t). \]
Since \( w_T \) has descent at \( k \), it is easily verified that
\[ w_T = s_k^{-\beta} \cdots s_{k_1}^{-\beta} s_{k+\alpha}^{-1} \cdots s_{k+1}s_k \] (5.10)
is a reduced word of \( w_T \). If \( u = \text{id} \), then \( v^{-1} = w_T \), and in this case \( \mathcal{G}_u(x)\mathcal{G}_v(-t) \) contributes the term \( \mathcal{G}_{w^{-1}}(-t) \).

We now consider the case when \( u \neq \text{id} \). By analyzing the reduced word in (5.10), it is not hard to check that \( u \) is the product of a latter half (possibly empty) of \( s_k^{-\beta} \cdots s_{k_1}^{-\beta} \) and a latter half (cannot be empty) of \( s_{k+\alpha}^{-1} \cdots s_{k+1}s_k \). That is, there exist \( 0 \leq \alpha' \leq \alpha \) and \( 0 \leq \beta' \leq \beta \) such that
\[ u = s_k^{-\beta} \cdots s_{k_1}^{-\beta} s_{k+\alpha}^{-1} \cdots s_{k+1}s_k \] (5.11)
and
\[ v^{-1} = s_k^{-\beta'} \cdots s_{k_1}^{-\beta'} s_{k+\alpha} \cdots s_{k+1}s_k \] (5.12)
By (5.11), we see that \( u = w_T \), where \( \Gamma' = (1 + \alpha', 1^\beta) \) is hook shape with \( 0 \leq \alpha' \leq \alpha \) and \( 0 \leq \beta' \leq \beta \). By (i) of Proposition 3.2,
\[ \mathcal{G}_u(x) = s_{\Gamma'}(x[k]). \]

We still need to evaluate \( \mathcal{G}_v(-t) \). From (5.12), it follows that \( v \) admits a factorization \( v = v_1v_2 \) with
\[ v_1 = s_{k+\alpha} \cdots s_{k_1}^{-1} \quad \text{and} \quad v_2 = s_k^{-\beta} \cdots s_{k_1}^{-\beta}, \]
which clearly satisfy the condition in (iii) of Theorem 3.2. Note that \( v_1 \) (resp., \( v_2 \)) is a Grassmannian permutation with descent at \( k + \alpha \) (resp., \( k - \beta \)) corresponding to the one column partition \( (1^{\alpha-\alpha'}) \) (resp., the one row partition \( \beta - \beta' \)), whose Schubert polynomial is \( e_{\alpha'-\alpha'}(x[k+\alpha]) \) (resp., \( h_{\beta'-\beta'}(x[k-\beta]) \)). So we have
\[ \mathcal{G}_u(-t) = \mathcal{G}_{v_1}(-t) \cdot \mathcal{G}_{v_2}(-t) = e_{\alpha'-\alpha'}(-t_{k+\alpha}) \cdot h_{\beta'-\beta'}(-t_{k-\beta}). \]

Combining the above gives the desired identity in (5.9). \( \square \)

We are now ready to complete the proof of Theorem B

**Theorem 5.13** (=Theorem B). Let \( u \in \mathfrak{S}_n \), and \( \Gamma = (1 + \alpha, 1^\beta) \) be a hook shape to which the corresponding permutation \( w_T \in \mathfrak{S}_n \) has descent at position \( k \). Then we have the following identity in \( H^*_T(\mathcal{F}(\ell(n))) \):
\[ c^T_\text{SM}(Y(u)\circ) \cdot [Y(w_T)]_T = [Y(w_T)]_T|_u \cdot c^T_\text{SM}(Y(u)\circ) + \sum_{u \neq w \in \mathfrak{S}_n} c^w_{u, T}(t) \cdot c^T_\text{SM}(Y(w)\circ), \]
where
\[ c^w_{u, T}(t) = \sum_{\gamma} h_{\alpha_1}(t_{\Delta(u,w)}) \cdot e_{\alpha_2}(t_{\Delta(u,w)}) \cdot e_{\alpha_2}(-t_{k+\alpha}) \cdot h_{\beta_1}(-t_{k-\beta}) \] (5.13)
with the sum over all peakless paths from \( u \) to \( w \) in the extended \( k \)-Bruhat order.
Proof. Replacing \([Y(w_T)]_T \) by the right-hand side of (5.9), we have
\[
c^T_{SM}(Y(u)^0) \cdot [Y(w_T)]_T \\
= c^T_{SM}(Y(u)^0) \cdot \mathcal{G}_{w_T^{-1}}(-t) \\
+ \sum_{\Gamma'=(1+\alpha',1\beta')} c^T_{SM}(Y(u)^0) \cdot s_{\Gamma'}(x_{[k]}) \cdot e_{\alpha-\alpha'}(-t_{[k+\alpha]}) \cdot h_{\beta-\beta'}(-t_{[k-\beta]}).
\]
By Theorem 5.8, the coefficient of \(c^T_{SM}(Y(u)^0)\) is
\[
\mathcal{G}_{w_T^{-1}}(-t) + \sum_{\Gamma'=(1+\alpha',1\beta')} s_{\Gamma'}(t_{u[k]}) \cdot e_{\alpha-\alpha'}(-t_{[k+\alpha]}) \cdot h_{\beta-\beta'}(-t_{[k-\beta]}),
\]
which, according to Lemma 5.12 and the definition of localization map, is precisely \([Y(w_T)]_T|_u\).

For \(w \neq u\), using Theorem 5.8 again, we obtain that
\[
\partial^w_{u,T}(t) = \sum_{\Gamma'=(1+\alpha',1\beta')} c^{\partial^w_{u,T}}(t) \cdot e_{\alpha-\alpha'}(-t_{[k+\alpha]}) \cdot h_{\beta-\beta'}(-t_{[k-\beta]}) \\
= \sum_{\Gamma'=(1+\alpha',1\beta')} \left( \sum_{\gamma} h_{\alpha'-\in(\gamma)}(t_{\Sigma_k(u,w)}) \cdot e_{\beta'-\de(\gamma)}(t_{\Delta_k(u,w)}) \right) \\
\times e_{\alpha-\alpha'}(-t_{[k+\alpha]}) \cdot h_{\beta-\beta'}(-t_{[k-\beta]}),
\]
which, after exchanging indices, coincides with (5.13). \(\square\)

Taking the lowest degree part of CSM classes leads to a formula for the multiplication of an equivariant Schubert class by an equivariant Schubert class of hook shape.

Corollary 5.14. Adopting the notation in Theorem 5.13, we have
\[
[Y(u)]_T \cdot [Y(w_T)]_T = [Y(w_T)]_T|_u \cdot [Y(u)]_T + \sum_w \partial^w_{u,T}(t) \cdot [Y(w)]_T, \quad (5.14)
\]
where \(\partial^w_{u,T}(t)\) has the same expression as \(c^{\partial^w_{u,T}}(t)\) in (5.13), except that now we assume the peakless paths are restricted to be in the ordinary k-Bruhat order.

When \(\alpha = 0\) or \(\beta = 0\) in Theorem 5.13, we obtain our second Pieri formula for equivariant CSM classes. In this case, the expression in (5.13) for the structure constants could be further simplified. For \(r \geq 0\), let
\[
c[k,r] = s_{k-r+1} \cdots s_k \quad \text{and} \quad c'[k,r] = s_{k+r-1} \cdots s_k.
\]
Note that \(c[k,r]\) (resp., \(c'[k,r]\)) is the Grassmannian permutation corresponding to the one column partition \((1^r)\) (resp., the one row partition \((r)\)) with descent at \(k\).

Lemma 5.15. For \(r \geq 0\), we have
\[
[Y(c[k,r])]_T = \sum_{i+j=r} e_i(x_{[k]}) \cdot h_j(-t_{[k-r+1]}) = \sum_{1 \leq i_1 < \cdots < i_r \leq k} \prod_{i=1}^r (x_{i j} - t_{i j - j + 1})
\]
25
and

\[ [Y(c'[k, r])]_T = \sum_{i+j=r} h_i(x[k]) \cdot e_j(-t_{[k+r-1]}) = \sum_{1 \leq i_1 \leq \cdots \leq i_r \leq k} \prod_{j=1}^r (x_{i_j} - t_{i_j + j - 1}). \]

**Proof.** The first equality in each expression is the special case of \( \alpha = 0 \) or \( \beta = 0 \) in Lemma 5.12. The second equality can be found in \([43, \text{Equations (1.2) and (1.3)}]\). \( \square \)

It turns out that the structure constants in our second Pieri formula may be characterized by the localization of Schubert classes.

**Theorem 5.16 (Equivariant CSM Pieri Formula II).** Let \( u \in S_n \), and let \( c[k, r] \) and \( c'[k, r] \) be permutations in \( S_n \) with descent at position \( k \). We have the following identities in \( H^*_T(\mathcal{F}(n)) \):

(i) For \( r \geq 0 \),

\[ c^T_{SM}(Y(u)^o) \cdot [Y(c[k, r])]_T = \sum_{w \in S_n} c^w_{u,c[k,r]}(t) \cdot c^T_{SM}(Y(w)^o), \]

where the sum ranges over all \( w \in S_n \) such that there exists a decreasing path of length \( r' \leq r \) from \( u \) to \( w \) in the extended \( k \)-Bruhat order, and

\[ c^w_{u,c[k,r]}(t) = [Y(c[k - r', r - r'])]|_{\delta(u,w)}. \]  

(5.15)

Here, \( \delta(u, w) \) can be taken as any permutation in \( S_n \) such that the image set of \( [k - r'] \) is exactly \( \Delta_k(u, w) = \{ u(i) : i \in [k] \} \setminus \{ u(i) : u(i) \neq w(i) \} \).

(ii) For \( r \geq 0 \),

\[ c^T_{SM}(Y(u)^o) \cdot [Y(c'[k, r])]_T = \sum_{w \in S_n} c^w_{u,c'[k,r]}(t) \cdot c^T_{SM}(Y(w)^o), \]

where the sum ranges over all \( w \in S_n \) such that there exists an increasing path of length \( r' \leq r \) from \( u \) to \( w \) in the extended \( k \)-Bruhat order, and

\[ c^w_{u,c'[k,r]}(t) = [Y(c'[k + r', r - r'])]|_{\sigma(u,w)}. \]

(5.16)

Here, \( \sigma(u, w) \) can be taken as any permutation in \( S_n \) such that the image set of \( [k + r'] \) is exactly \( \Sigma_k(u, w) = \{ u(i) : i \in [k] \} \cup \{ u(i) : u(i) \neq w(i) \} \).

**Proof.** We give a proof of (i), and the arguments of (ii) can be carried out similarly. Let \( w \in S_n \) be such that there is a decreasing path from \( u \) to \( w \) of length \( r' \leq r \) in the extended \( k \)-Bruhat order. Notice that \( c[k, r] \) corresponds to the one column hook shape \((1 + \alpha, 1^\beta)\) with \( \alpha = 0 \) and \( \beta = r - 1 \). Hence, by Theorem 5.13, if \( w = u \), then \( r' = 0 \) and the coefficient of \( c^T_{SM}(Y(u)^o) \) is

\[ c^u_{u,c[k,r]}(t) = [Y(c[k, r])]_T|_u, \]  

(5.17)

and if \( w \neq u \), then \( r' > 0 \) and the coefficient of \( c^T_{SM}(Y(w)^o) \) is

\[ c^w_{u,c[k,r]}(t) = \sum_{\beta_1 + \beta_2 = r - r'} e_{\beta_1}(t_{\Delta_k(u,w)}) \cdot h_{\beta_2}(-t_{[k-r+1]}). \]

(5.18)

On the other hand, by the first equality in Lemma 5.15, we have

\[ [Y(c[k - r', r - r'])]|_{T} = \sum_{i+j=r-r'} e_i(x[k-r']) \cdot h_j(-t_{[k-r+1]}). \]
By Remark 5.3, we know that \( \#\Delta_k(u, w) = k - r' \). Moreover, it follows from Lemma 5.15 that \( [Y(c[k - r', r - r])]^T \) is symmetric in \( x_1, \ldots, x_{k-r'} \). Therefore, for any chosen permutation \( \delta(u, w) \in \mathfrak{S}_n \) such that its image set of \( [k - r'] \) is \( \Delta_k(u, w) \), the localization
\[
[Y(c[k - r', r - r'])]^T_{\delta(u, w)}
\]
equals (5.17) when \( w = u \), and equals (5.18) when \( w \neq u \). This concludes the proof. \( \square \)

**Remark 5.17.** In the above proof, we only used the first equality in Lemma 5.15. Applying the second equality, we can get an explicit formula for \( \mathfrak{c}_{u,c[k,r]}(t) \). Precisely, suppose that
\[
\Delta_k(u, w) = \{ \delta_1 < \delta_2 < \cdots < \delta_{k-r'} \}.
\]
Take \( \delta(u, w) \) as the permutation in \( \mathfrak{S}_n \) sending \( i \) to \( \delta_i \) for \( 1 \leq i \leq k - r' \), and fixing all the remaining elements in \( [n] \setminus [k - r'] \). By means of the second equality in Lemma 5.15 the coefficient \( \mathfrak{c}_{u,c[k,r]}(t) \) in (5.15) becomes
\[
\mathfrak{c}_{u,c[k,r]}(t) = \sum_{1 \leq i_1 < \cdots < i_{r-r'} \leq k-r'} \left( \prod_{j=1}^{r-r'} (t_{i_{j+1}} - t_{i_{j+1}}) \right).
\]
Similarly, if we assume that
\[
\Sigma_k(u, w) = \{ \sigma_1 < \sigma_2 < \cdots < \sigma_{k+r'} \},
\]
then the coefficient \( \mathfrak{c}_{u,c'[k,r]}(t) \) in (5.16) can be read as
\[
\mathfrak{c}_{u,c'[k,r]}(t) = \sum_{1 \leq i_1 \leq \cdots \leq i_{r-r'} \leq k+r'} \left( \prod_{j=1}^{r-r'} (t_{\sigma_{j+1}} - t_{\sigma_{j+1}}) \right).
\]
If restricting the decreasing/increasing path from \( u \) to \( w \) to be in the ordinary \( k \)-Bruhat order in Theorem 6.1, we recover the Pieri formula for equivariant Schubert classes, as established by Robinson [32], see also Li, Ravikumar, Sottile and Yang [32].

6. Murnaghan–Nakayama Type Rules

Our goal in this section is to establish the MN formula for equivariant CSM classes, as described in Theorem C. To do this, we begin by deducing a MN formula for nonequivariant CSM classes in Theorem 6.1. Then we derive the Rigidity Theorem for power sum symmetric functions in Theorem 6.11 which together with Theorem 6.1 completes the proof of Theorem C.

6.1. Nonequivariant CSM MN Formula. For a permutation \( w \in \mathfrak{S}_n \), define its \( k \)-height to be one less than the number of non-fixed points of \( w \) at the first \( k \) positions, namely,
\[
ht_k(w) = \# \{ i \leq k : w(i) \neq i \} - 1.
\]

Our MN formula for nonequivariant CSM classes can be stated as follows.

**Theorem 6.1 (CSM MN Formula).** Let \( u \in \mathfrak{S}_n \). For \( r \geq 1 \), we have the following identity in \( H^*(\mathcal{F} \ell(n)) \):
\[
c_{\text{SM}}(Y(u)^\circ) \cdot p_r(x_{[k]}) = \sum_{\eta \in \mathfrak{S}_n} (-1)^{ht_k(\eta)} \cdot c_{\text{SM}}(Y(\eta)^\circ),
\]
where the sum ranges over all \((r + 1)\)-cycles \(\eta \in \mathfrak{S}_n\) such that \(u \leq_k u\eta\) in the extended \(k\)-Bruhat order.

Theorem 6.1 is a direct consequence of Theorem 6.2 and Theorem 6.3.

**Theorem 6.2.** Let \(u \in \mathfrak{S}_n\). For \(r \geq 1\), suppose that

\[
c_{\text{SM}}(Y(u)\circ) \cdot p_r(x_{[k]}) = \sum_{w \in \mathfrak{S}_n} \left[ \sum_{\eta \in \mathfrak{S}_n} d_{u,r}^w \cdot c_{\text{SM}}(Y(w)\circ) \right].
\]

Then,

\[
d_{u,r}^w = \sum_{\gamma} (-1)^{\text{de}(\gamma)},
\]

where the sum runs over all unimodal paths \(\gamma\) of length \(r\) from \(u\) to \(w\) in the extended \(k\)-Bruhat order. Moreover, if \(d_{u,r}^w \neq 0\), then \(w = u\eta\) for some \((m + 1)\)-cycle \(\eta\), where \(m \geq 1\).

**Proof.** To prove (6.3), we need the following expansion

\[
p_r(x_{[k]}) = \sum_{\alpha + \beta + 1 = r} (-1)^\beta s_{(1+\alpha,1^{\beta})}(x_{[k]}),
\]

which is a special case of [57, Theorem 7.17.3]. Applying the Pieri formula in Theorem 5.6 to the right-hand side of (6.4), we obtain that

\[
c_{\text{SM}}(Y(u)\circ) \cdot p_r(x_{[k]}) = \sum_{\gamma} (-1)^{\text{de}(\gamma)} c_{\text{SM}}(Y(\text{end}(\gamma))\circ)
\]

with \(\gamma\) running over all unimodal paths of length \(r\) starting at \(u\) in the extended \(k\)-Bruhat order, which leads to (6.3).

Let \(w \in \mathfrak{S}_n\) be such that \(d_{u,r}^w \neq 0\). In this case, we see from (6.3) that \(\ell(w) > \ell(u)\). By Lemma 4.6, \(d_{u,r}^w \neq 0\) is equivalent to

\[
T_{w/u}(p_r(x_{[k]}))\big|_{x_i=0} \neq 0.
\]

We aim to show that \(w = u\eta\) for some \((m + 1)\)-cycle \(\eta\).

For a subset \(A\) of \([n]\), write

\[
p(x_A) = \sum_{a \in A} \frac{x_a}{1 - zx_a} = p_1(x_A) + zp_2(x_A) + \cdots.
\]

Recall that

\[
T_{w/u} = \sum_{\substack{J \subseteq [n] \\mid \forall j \neq u \exists \, J \subseteq [n] \\mid w_{J \setminus u}}} \nabla_J.
\]

From the proof of Lemma 4.7, it can be seen that

\[
\nabla_J = u \partial_{a_1 b_1} \cdots \partial_{a_m b_m},
\]

and moreover,

\[
w = u t_{a_1 b_1} \cdots t_{a_m b_m}.
\]

Since \(\ell(w) > \ell(u)\), we have \(m \geq 1\).
By direct computation, it is routine to check that
\[ \partial_{ab} p(x_A) = (\delta_{a \in A} \delta_b \neq A - \delta_{b \in A} \delta_a \neq A) \cdot \frac{1}{(1 - zx_a)(1 - zx_b)}. \] (6.7)

We have the following claim:

**Claim.** For \( m \geq 1 \), if
\[ \partial_{a_1b_1} \cdots \partial_{a_mb_m} p(x_A) \neq 0, \] (6.8)
then the set \( \{a_1, b_1, \ldots, a_m, b_m\} \) has cardinality \( m+1 \), and \( t_{a_1b_1} \cdots t_{a_mb_m} \) forms an \((m+1)\)-cycle on \( \{a_1, b_1, \ldots, a_m, b_m\} \).

We verify the above Claim by induction on \( m \). The case \( m = 1 \) is obvious. Assume now that \( m \geq 2 \). By the assumption that \( \partial_{a_1b_1} \cdots \partial_{a_mb_m} p(x_A) \neq 0 \), we have \( \partial_{a_2b_2} \cdots \partial_{a_mb_m} p(x_A) \neq 0 \), and thus by induction, the set \( B = \{a_2, b_2, \ldots, a_m, b_m\} \) has cardinality \( m \), and \( t_{a_2b_2} \cdots t_{a_mb_m} \) forms an \( m \)-cycle on \( B \). By (6.7) and Lemma 4.3
\[ \partial_{a_2b_2} \cdots \partial_{a_mb_m} p(x_A) \in \mathbb{Q}[z] \left( \partial_{a_2b_2} \cdots \partial_{a_{m-1}b_{m-1}} \frac{1}{Z(x_{\{a_r, b_r\}})} \right) \subseteq \mathbb{Q}[z] \frac{1}{Z(x_B)}. \]

To ensure that \( \partial_{a_1b_1} (\partial_{a_2b_2} \cdots \partial_{a_mb_m} p(x_A)) \neq 0 \), we necessarily have \( \partial_{a_1b_1} \frac{1}{Z(x_B)} \neq 0 \), which requires that \( \# \{a_1, b_1\} \cap B = 1 \) by Lemma 4.3. So \( \{a_1, b_1, \ldots, a_m, b_m\} \) contains \( m+1 \) elements. Since \( t_{a_2b_2} \cdots t_{a_mb_m} \) forms an \( m \)-cycle on \( B \), it is easily checked that \( t_{a_1b_1} \cdots t_{a_mb_m} \) forms an \((m+1)\)-cycle on \( \{a_1, b_1, \ldots, a_m, b_m\} \).

Recall that if \( d_{w,r} \neq 0 \), then \( T_{w/u}(p(x_A)) \neq 0 \), implying that \( \partial_{a_1b_1} \cdots \partial_{a_mb_m} p(x_A) \neq 0 \) for some \( m \geq 1 \). Together with the above Claim and (6.6), there must exist some \( m \geq 1 \) such that \( w = u\eta \) with \( \eta = t_{a_1b_1} \cdots t_{a_mb_m} \) an \((m+1)\)-cycle. \( \square \)

**Theorem 6.3.** Let \( u \in \mathcal{S}_n \), and \( \eta \in \mathcal{S}_n \) be an \((m+1)\)-cycle \( (m \geq 1) \) such that \( u \preceq_{k} \eta \) in the extended \( k \)-Bruhat order. Then there exists a unique unimodal path \( \gamma \) from \( u \) to \( \eta \) in the extended \( k \)-Bruhat order. Moreover, \( \gamma \) is a path of length \( m \) and with \( \text{de}(\gamma) = \text{ht}_k(\eta) \).

We remark that an analogous statement to Theorem 6.3 for the Grassmannian Bruhat order appeared in [7, §6.2]. The proof of Theorem 6.3 is quite technical, and will occupy the whole bulk of Subsection 6.2.

6.2. **Proof of Theorem 6.3** To prove Theorem 6.3 we need several lemmas. The first lemma is a nonrecursive criterion for the extended \( k \)-Bruhat order, which is quite useful in the comparison of two permutations in the extended \( k \)-Bruhat order. An analogous criterion for the ordinary \( k \)-Bruhat order has appeared in [6, Theorem A].

**Lemma 6.4.** For two permutations \( u, w \in \mathcal{S}_n \), \( u \preceq_{k} w \) in the extended \( k \)-Bruhat order if and only if for any \( a \leq k < b \), \( u(a) \leq w(a) \) and \( u(b) \geq w(b) \).

**Proof.** The proof is along a similar line to that of [6, Theorem A]. Denote by \( \preceq_{k} \) the relation defined by the condition: for \( a \leq k < b \), \( u(a) \leq w(a) \) and \( u(b) \geq w(b) \). Clearly, \( \preceq_{k} \) is a partial order on \( \mathcal{S}_n \). We need to verify that \( \preceq_{k} \) and \( \preceq_{k}^{'} \) are the same partial order on \( \mathcal{S}_n \).

Assume that \( u \rightarrow w \), that is, \( w = ut_{ab} \) with \( a \leq k < b \), and \( u(a) < u(b) \). It is obvious that \( u \preceq_{k}^{'} w \). So, if \( u \rightarrow u_1 \rightarrow \cdots \rightarrow w \), then \( u \preceq_{k} u_1 \preceq_{k} \cdots \preceq_{k} w \). This implies that if \( u \preceq_{k} w \), then \( u \preceq_{k} w \).
Conversely, assume that \( u \leq^t_w \). We use induction on \( \ell(w) \) to prove \( u \leq_k w \). This is clear in the case \( \ell(w) = 0 \) since both \( u \) and \( w \) are the identity permutation. We now consider \( \ell(w) > 0 \) and \( u \neq w \). To apply induction, we show that there exist \( a \leq k < b \) such that \( wt_{ab} \to w \) and \( u \leq^t_k wt_{ab} \).

Choose the index \( a \leq k \) such that \( u(a) \) is minimal subject to \( u(a) < w(a) \). Such a choice exists since otherwise \( u(i) \geq w(i) \) for all \( i \leq k \) would lead to \( u = w \). Once \( a \) is chosen, locate any index \( b > k \) such that

\[
\w(b) < w(a) \leq u(b).
\]

We explain that the above choice of \( b \) exists. Suppose to the contrary that there does not exist such an index \( b \). We claim clearly holds since \( u(i) \leq w(i) \) for all \( i \leq k \) and \( u(i) \geq w(i) \) would imply the existence of an index \( b \). If \( i \leq k \) and \( w(i) < w(a) \), then the claim clearly holds since \( u(i) \leq w(i) < w(a) \). By this claim, we see that

\[
\{u(i): i \in [n], \ w(i) < w(a)\} = [w(a) - 1] = \{1, 2, \ldots, w(a) - 1\},
\]

which contradicts \( u(a) < w(a) \) since \( a \notin \{i \in [n]: w(i) < w(a)\} \). So the assumption is false, that is, such an index \( b \) always exists.

It is obvious that \( wt_{ab} \to w \) since \( \w(b) < w(a) \). We proceed to show that \( u \leq^t_k wt_{ab} \).

By the choices of \( a \) and \( b \), it suffices to check that

\[
u(a) \leq w(b).
\]

Still, we use contradiction. Suppose otherwise that \( \w(b) < u(a) \). We shall construct an infinite sequence \( b_1, b_2, \ldots \) such that

\[
u(a) > u(b_1) > u(b_2) > \cdots,
\]

which is absurd.

Let \( b_1 \) be the position such that \( u(b_1) = \w(b) \). Then \( u(a) > \w(b) = u(b_1) \). Let \( b_2 \) be such that \( u(b_2) = \w(b_1) \). Since \( u(a) > \w(b) \) and \( u(b) > \w(b) \), it follows that \( b_1 \neq a, b \), and so \( u(b_1) = \w(b) \neq u(b_1) \). By the minimality of \( u(a) \), it follows that \( b_1 > k \) and \( u(b_1) > \w(b_1) = u(b_2) \). Choose the index \( b_3 \) by letting \( u(b_3) = \w(b_2) \). Using the same analysis, we may deduce that \( b_2 > k \) and \( u(b_2) > \w(b_2) = u(b_3) \). Continuing this procedure, we are led to \( u(a) > u(b_1) > u(b_2) > \cdots, \) a contradiction.

Applying induction, we get \( u \leq^t_k wt_{ab} \), which along with \( wt_{ab} \leq_k w \) gives \( u \leq_k w \). □

**Corollary 6.5.** For \( u, w \in \mathfrak{S}_n \), assume that

\[
\gamma: u = w_0 \to w_1 \to \cdots \to w_m = w
\]

is a path in the extended \( k \)-Bruhat order with \( w_i = w_{i-1}t_{a_{i-1}, b_{i-1}} \) for \( 1 \leq i \leq m \). Then

\[
\{a_1, b_1, \ldots, a_m, b_m\} = M(u^{-1}w) = \{i \in [n]: u(i) \neq w(i)\}.
\]

(6.9)

Precisely, by Lemma 6.4

\[
\{a_1, \ldots, a_m\} = \{a \in [n]: u(a) < w(a)\}
\]

and

\[
\{b_1, \ldots, b_m\} = \{b \in [n]: u(b) > w(b)\}.
\]
Proof. It is clear that \( M(u^{-1}w) \subseteq \{a_1, b_1, \ldots, a_m, b_m\} \). It remains to check the reverse inclusion. For \( 1 \leq i \leq m \), we have \( u \leq_k w_{i-1} \rightarrow w_i \leq_k w \). Combining this with Lemma \ref{lem:bruhat_order} gives

\[
\begin{align*}
   u(a_i) &\leq w_{i-1}(a_i) < w_i(a_i) \leq w(a_i), \\
   u(b_i) &\geq w_{i-1}(b_i) > w_i(b_i) \geq w(b_i).
\end{align*}
\]

This implies that both \( a_i \) and \( b_i \) are not fixed points of \( u^{-1}w \), and thus they belong to \( M(u^{-1}w) \), as required. \( \square \)

**Lemma 6.6.** For \( u \neq w \in S_n \) and any path

\[ \gamma: u = w_0 \xrightarrow{\tau_1} w_1 \xrightarrow{\tau_2} \cdots \xrightarrow{\tau_m} w_m = w \]

from \( u \) to \( w \) in the extended \( k \)-Bruhat order, we have

\[ \min\{u(i) : u(i) \neq w(i)\} = \min\{\tau_1, \ldots, \tau_m\}. \quad \tag{6.10} \]

Moreover, letting \( a \in M(u^{-1}w) \) be the index exactly attaining the minimum value in \( \tag{6.10} \), we have \( a \leq k \).

Proof. Let \( a \) be the index attaining the minimum value on the left-hand side of \( \tag{6.10} \). We first verify that \( a \leq k \). Suppose to the contrary that \( a > k \). By Lemma \ref{lem:bruhat_order}

\[ u(a) \geq w(a) = u(u^{-1}w(a)). \quad \tag{6.11} \]

Since \( a \) is not a fixed point of \( u^{-1}w \), \( u^{-1}w(a) \) is also not a fixed point of \( u^{-1}w \). So we have \( u(a) \leq u(u^{-1}w(a)) \), which together with \( \tag{6.11} \) implies that \( a = u^{-1}w(a) \), contrary to the fact that \( u(a) \neq w(a) \). This concludes that \( a \leq k \).

It remains to show that \( u(a) = \min\{\tau_1, \ldots, \tau_m\} \). Let \( i \) be the minimum index such that \( u(a) \neq w_i(a) \). Assume that

\[ w_{i-1} \xrightarrow{\tau_i} w_i = w_{i-1}t_{a_i}b_i \]

for some \( a_i \leq k < b_i \). Since \( i \) is minimum and \( a \leq k \), we have \( a_i = a \) and \( \tau_i = w_{i-1}(a) = u(a) \), meaning that \( u(a) \) appears in the labels \( \tau_1, \ldots, \tau_m \). If there were some \( j \) such that \( \tau_j < u(a) \), say

\[ w_{j-1} \xrightarrow{\tau_j} w_j = w_{j-1}t_{a_j}b_j \]

for some \( a_j \leq k < b_j \), we would deduce from Lemma \ref{lem:bruhat_order} that \( u(a_j) \leq w_{j-1}(a_j) = \tau_j < u(a) \). However, it follows from Corollary \ref{cor:bruhat_order} that \( a_j \in M(u^{-1}w) \), which is contrary to the choice of \( a \). This completes the proof. \( \square \)

To give a proof of Theorem \ref{thm:bruhat_order}, we shall construct a unimodal path from \( u \) to \( u\eta \) in the extended \( k \)-Bruhat order. To do this, we first determine the edge with the minimum label (namely, the first or the last edge) in the unimodal path, as will be done in Lemmas \ref{lem:Bruhat_order} and \ref{lem:Bruhat_order2}.

**Lemma 6.7.** Let \( u \in S_n \), and \( \eta \in S_n \) be an \((m+1)\)-cycle with \( m \geq 2 \) such that \( u \leq_k u\eta \). Assume that \( a \in M(\eta) \) is the index attaining the minimum value \( \min\{u(i) : i \in M(\eta)\} \).

We have the following equivalent statements:

\[ u(\eta^{-1}(a)) < u(\eta(a)) \iff u \rightarrow_{31} u t_{a \eta^{-1}(a)} \leq_k u\eta, \quad \tag{6.12} \]
where $a \leq k < \eta^{-1}(a)$. Moreover, if we write $ut_{a\eta^{-1}(a)} = u'$ and $u\eta = u'\eta'$, then $\eta'$ is an $m$-cycle with $ht_k(\eta') = ht_k(\eta)$, and $u'M(\eta') = uM(\eta) \setminus \{u(a)\}$.

Proof. Since $m \geq 2$, we have $\eta(a) \neq \eta^{-1}(a)$, and both $\eta(a)$ and $\eta^{-1}(a)$ belong to $M(\eta)$. Let us proceed to check that $a \leq k < \eta^{-1}(a)$. Since $u \leq k u\eta$, by Lemma 6.6 we have $a \leq k$. Suppose otherwise that $\eta^{-1}(a) < k$. By Lemma 6.4, we deduce that

$$u(a) = u\eta(\eta^{-1}(a)) \geq u(\eta^{-1}(a)),$$

which, along with the choice of $a$, yields the contradiction that $a = \eta^{-1}(a)$. This verifies $\eta^{-1}(a) > k$.

We next prove the equivalence in (6.12). By Lemma 6.4 the right-hand side of (6.12) is equivalently saying that

$$u(a) < u(\eta^{-1}(a)) \quad (\Longleftrightarrow u \xrightarrow{a} ut_{a\eta^{-1}(a)})$$

and

$$u(\eta^{-1}(a)) \leq u\eta(a) \quad \text{and} \quad u(a) \geq u\eta(\eta^{-1}(a)) = u(a) \quad (\Longleftrightarrow ut_{a\eta^{-1}(a)} \leq u\eta).$$

(6.14)
The choice of $a$ directly implies (6.13). Since $\eta^{-1}(a) \neq \eta(a)$, condition (6.14) is the same as the left-hand side of (6.12). This concludes (6.12).

The fact that $\eta' = t_{a\eta^{-1}(a)}\eta$ is an $m$-cycle follows from direct computation. Actually, $\eta'$ is obtained from $\eta$ by removing the value $\eta^{-1}(a)$. Since $\eta^{-1}(a) > k$, recalling the definition in (6.11), we obtain that $ht_k(\eta') = ht_k(\eta)$. Moreover, it is easily checked that

$$u'M(\eta') = u'(M(\eta) \setminus \{\eta^{-1}(a)\}) = u'M(\eta) \setminus \{u(a)\} = uM(\eta) \setminus \{u(a)\}.$$

This completes the proof. □

A dual statement to Lemma 6.7 can be derived by similar arguments.

**Lemma 6.8.** Let $u$, $\eta$, and $a$ be as given in Lemma 6.7. We have the following equivalent statements:

$$u(\eta^{-1}(a)) > u(\eta(a)) \iff u \leq_k u\eta t_{a\eta^{-1}(a)} \xrightarrow{u(a)} u\eta,$$

(6.15)

where $a \leq k < \eta^{-1}(a)$. Moreover, if we write $u = u'$ and $u\eta t_{a\eta^{-1}(a)} = u'\eta'$, then $ht_k(\eta') = ht_k(\eta) - 1$, and $u'M(\eta') = uM(\eta) \setminus \{u(a)\}$.

The last two lemmas will be used to prove the uniqueness of the unimodal path from $u$ to $u\eta$ in the extended $k$-Bruhat order.

**Lemma 6.9.** Let $u$, $\eta$, and $a$ be as given in Lemma 6.7. Assume that $u(a)$ appears as the first label of a path

$$\gamma: u \xrightarrow{u(a)} ut_{ab} \leq_k u\eta$$

from $u$ to $u\eta$ in the extended $k$-Bruhat order. Then we have $b = \eta^{-1}(a)$. Moreover, we have $u(\eta^{-1}(a)) < u(\eta(a))$ by (6.12).

Proof. By Lemma 6.4 we have

$$u(a) = ut_{ab}(b) \geq u\eta(b) = u(\eta(b)).$$

(6.16)

On the other hand, by Corollary 6.5 we have $b \in M(\eta)$, namely, $\eta(b) \neq b$, implying $\eta(b) \in M(\eta)$. Together with (6.16) and the choice of $a$, we are given $\eta(b) = a$. □

Using similar analysis to Lemma 6.9 we obtain the following dual assertion.
Lemma 6.10. Let $u$, $\eta$, and $a$ be as given in Lemma 6.7. Assume that $u(a)$ appears as the last label of a path

$$\gamma: u \leq_k u\eta a^o b \rightarrow u\eta$$

from $u$ to $u\eta$ in the extended $k$-Bruhat order. Then we have $a' = a$ and $b = \eta^{-1}(a)$. Moreover, we have $u(\eta^{-1}(a)) > u(\eta(a))$ by (6.15).

Proof. Since $u(a)$ appears on the last edge, we have $u\eta a^o b(a') = u\eta(b) = u(a)$, and so $b = \eta^{-1}(a)$. We next verify $a' = a$. By Lemma 6.4,

$$u(a') \leq u\eta a^o b(a') = u\eta(b) = u(a).$$

(6.17)

By Corollary 6.5, we see that $a' \in M(\eta)$, which, together with (6.17) and the choice of $a$, forces $a' = a$, as desired. □

We are finally ready to present a proof of Theorem 6.3.

Proof of Theorem 6.3. In the case when $m = 1$, $\eta$ is a transposition, say, $\eta = t_{ab}$ with $a \leq k < b$. We explain that $u \rightarrow ut_{ab}$ is the unique path from $u$ to $ut_{ab}$ in the extended $k$-Bruhat order. Suppose that $\gamma$ is a path from $u$ to $ut_{ab}$ in the extended $k$-Bruhat order. Since $M(t_{ab}) = \{a,b\}$, by Corollary 6.5, $t_{ab}$ is the only transposition appearing in the construction of $\gamma$, and so $\gamma$ is exactly the path $u \rightarrow ut_{ab}$. Clearly, the path $u \rightarrow ut_{ab}$ is of length one and with $\text{de}(\gamma) = \text{ht}_k(t_{ab}) = 0$.

We next consider the case when $m \geq 2$. Let $a \in M(\eta)$ be the index reaching the minimum value $\min\{u(i): i \in M(\eta)\}$. Since $m \geq 2$, we have $\eta(a) \neq \eta^{-1}(a)$. The discussion is divided into two cases.

Case 1. $u(\eta^{-1}(a)) < u(\eta(a))$. By Lemma 6.7, we see that $u \xrightarrow{u(a)} ut_{a\eta^{-1}(a)} \leq_k u\eta$, where $a \leq k < \eta^{-1}(a)$. Let $u' = ut_{a\eta^{-1}(a)}$ and $u'\eta' = u\eta$ be as defined in Lemma 6.7. By induction on $m$, we assume that there is a unique unimodal path $\gamma'$ from $u'$ to $u'\eta'$ of length $m - 1$, which satisfies $\text{de}(\gamma') = \text{ht}_k(\eta')$. Consider the path

$$\gamma: u \xrightarrow{u(a)} u' \rightarrow \cdots \rightarrow u'\eta'.$$

By Lemma 6.6, the minimum label among the edges in $\gamma'$ is $\min\{u'(i): i \in M(\eta')\} = \min\{u'M(\eta')\}$. By Lemma 6.7, $u'M(\eta') = uM(\eta) \setminus \{u(a)\}$, and by the minimality of $u(a)$, we have

$$\min\{u'(i): i \in M(\eta')\} > u(a).$$

Thus $\gamma$ is a unimodal path from $u$ to $u\eta$ of length $m$ and with $\text{de}(\gamma) = \text{de}(\gamma') = \text{ht}_k(\eta')$. By Lemma 6.7, we have $\text{ht}_k(\eta) = \text{ht}_k(\eta')$, and so we obtain that $\text{de}(\gamma) = \text{ht}_k(\eta)$.

It remains to show that $\gamma$ is unique. Let $\overline{\gamma}$ be any unimodal path from $u$ to $u\eta$ in the extended $k$-Bruhat order. By Lemma 6.6, the minimum label in $\overline{\gamma}$ is $u(a)$. Since $\gamma$ is unimodal, $u(a)$ appears on the first or the last edge. Because $u(\eta^{-1}(a)) < u(\eta(a))$, it follows from Lemma 6.9 that the label $u(a)$ must appear on the first edge, and the transposition corresponding to this edge is $t_{a\eta^{-1}(a)}$. Since the subpath $\overline{\gamma}$ from $u' = ut_{a\eta^{-1}(a)}$ to $u'\eta' = u\eta$ is still unimodal, by induction on $m$, we can assume that $\overline{\gamma}$ is unique. This enables us to conclude that $\overline{\gamma}$ coincides with $\gamma$.

Case 2. $u(\eta^{-1}(a)) > u(\eta(a))$. The arguments rely on Lemmas 6.6, 6.8, and 6.10, which are nearly the same as Case 1, and so are omitted. □
6.3. **Equivariant CSM MN Formula: Theorem** To attain a proof of Theorem \[\text{C}\], we establish the following Rigidity Theorem for multiplying a CSM class by a power sum symmetric function.

**Theorem 6.11** (Rigidity Theorem). Let \( u \in S_n \) and \( A \subseteq [n] \). For \( r \geq 1 \), suppose that
\[
c_{SM}(Y(u)^o) \cdot p_r(x_A) = \sum_{w \in S_n} d_{u,r}^w \cdot c_{SM}(Y(w)^o).
\]
(6.18)

Then, we have
\[
c_{SM}^T(Y(u)^o) \cdot p_r(x_A) = \sum_{w \in S_n} d_{u,r}^w (t) \cdot c_{SM}^T(Y(w)^o),
\]
where
\[
d_{u,r}^w (t) = \begin{cases} p_r(t_{uA}), & w = u, \\ \sum_{1 \leq t', r \leq r} d_{u,r'}^w \cdot h_{r-r'}(t_{uM(u^{-1}w)}), & w \neq u. \end{cases}
\]
(6.20)

Here, recall that
\[
u M(u^{-1}w) = \{ u(i) : w(i) \neq u(i) \}.
\]

**Proof.** The proof is similar to that of Theorem \[\text{4.11}\] and is outlined below. Recall that
\[
p(x_A) = \sum_{a \in A} x_a \frac{x_a}{1 - z x_a} = p_1(x_A) + p_2(x_A)z + \cdots.
\]

Denote
\[
d_{u}^w(z, t) = \sum_{r \geq 1} d_{u,r}^w(t) z^{r-1}.
\]

Then we have
\[
c_{SM}^T(Y(u)^o) \cdot p(x_A) = \sum_{w \in S_n} d_{u}^w(z, t) \cdot c_{SM}^T(Y(w)^o).
\]

By Lemma \[\text{4.6}\] we have
\[
d_{u}^w(z, t) = T_{w/u} (p(x_A))\big|_{x_i = t_i}.
\]

If \( u = w \), then \( T_{w/u} \) is just \( u \), so \( d_{u}^w(z, t) = p(t_{uA}) \), which implies \( d_{u,r}^w(t) = p_r(t_{uA}) \). Now we consider the case \( u \neq w \). Recall the notation \( E(q, z, x_A) \) defined in Lemma \[\text{4.2}\]
\[
E(q, z, x_A) = \frac{1}{q + z} \left( \frac{Q(x_A)}{Z(x_A)} - 1 \right).
\]

By evaluating \( \lim_{q \to -z} E(q, z, x_A) \) using the L’Hospital rule, we obtain the relation
\[
E(-z, z, x_A) = \sum_{a \in A} x_a \frac{x_a}{1 - z x_a} = p(x_A).
\]

In Lemma \[\text{4.17}\] we showed that for \( A \subseteq [n] \), there exists \( f(q, z) \in \mathbb{Z}[q, z] \) such that
\[
T_{w/u} E(q, z, x_A) = \frac{1}{q + z} f(q, z) \frac{Q(x_{\Delta_A(u,w)})}{Z(x_{\Sigma_A(u,w)})}.
\]
(6.21)

Notice the following observation: for \( A \subseteq B \),
\[
\left. \frac{Q(x_A)}{Z(x_B)} \right|_{q = -z} = \frac{1}{Z(x_B \setminus A)} = \sum_{j=0}^{\infty} h_j(x_B \setminus A) z^j.
\]
Moreover, notice that
\[ \Sigma_A(u, w) \backslash \Delta_A(u, w) = uM(u^{-1}w). \]
Taking limit \( q \to -z \) on both sides of (6.21), we get
\[ T_{w/u}(p(x_A)) = \left( \lim_{q \to -z} \frac{f(q, z)}{q + z} \right) \frac{1}{Z(x_{uM(u^{-1}w)})}, \]
and so we have
\[ d^w_u(z, t) = \left( \lim_{q \to -z} \frac{f(q, z)}{q + z} \right) \frac{1}{Z(t_{uM(u^{-1}w)})}. \]
Letting all \( t_i = 0 \) on both sides, we see that
\[ \lim_{q \to -z} \frac{f(q, z)}{q + z} = d^w_u(z, 0), \]
and hence
\[ d^w_u(z, t) = d^w_u(z, 0) \cdot \sum_{j=0}^{\infty} h_j(t_{uM(u^{-1}w)})z^j = \sum_{i=0}^{\infty} d^w_{u,i+1}z^j \cdot \sum_{j=0}^{\infty} h_j(t_{uM(u^{-1}w)})z^j. \]
Equating the coefficient of \( z^{r-1} \) gives (6.20), as desired. \( \square \)

Using Theorems 6.11 and 6.11, we reach a proof of Theorem C:

**Theorem 6.12 (Equivariant CSM MN Formula).** Let \( u \in S_n \). For \( r \geq 1 \), we have the following identity in \( H^*_T(F\ell(n)) \):
\[
c^T_{SM}(Y(u)^\circ) \cdot p_r(x_{[k]}) = p_r(t_{u[k]}) \cdot c^T_{SM}(Y(u)^\circ) + \sum_{\eta \in S_n} d^w_{u,\eta}(t) \cdot c^T_{SM}(Y(u\eta)^\circ),
\]
where the sum runs over \((r'+1)\)-cycles \( \eta \in S_n \) with \( 1 \leq r' \leq r \) such that \( u \leq_k u\eta \) in the extended \( k \)-Bruhat order, and
\[
d^w_{u,\eta}(t) = (-1)^{ht_k(\eta)} \cdot h_{r-r'}(t_{uM(\eta)}). \quad (6.22)
\]

Taking the lowest degree part in Theorem 6.12 leads to the following MN formula for equivariant Schubert classes.

**Corollary 6.13 (Equivariant Schubert MN Formula).** Let \( u \in S_n \). For \( r \geq 1 \), we have the following identity in \( H^*_T(F\ell(n)) \):
\[
[Y(u)]_T \cdot p_r(x_{[k]}) = p_r(t_{u[k]}) \cdot [Y(u)]_T + \sum_{\eta \in S_n} \mathcal{T}^w_{u,\eta}(t) \cdot [Y(w)]_T,
\]
where the sum runs over \((r'+1)\)-cycles \( \eta \in S_n \) with \( 1 \leq r' \leq r \) such that \( \ell(u\eta) = \ell(u)+r' \) and \( u \leq_k u\eta \) in the extended \( k \)-Bruhat order, and \( \mathcal{T}^w_{u,\eta}(t) \) has the same expression as \( d^w_{u,r}(t) \) in (6.22).
We remark that in Corollary 6.13 the condition $\ell(u\eta) = \ell(u) + r'$, together with Theorem 6.3 implies that $u \leq_k u\eta$ is actually in the ordinary $k$-Bruhat order.

If further taking all $t_i = 0$ in Corollary 6.13 we arrive at the MN rule for nonequivariant Schubert classes, as deduced by Morrison and Sottile [46, Theorem 1].

We end this section with an example to illustrate Theorem 6.12 and Corollary 6.13.

**Example 6.14.** With the same setting as in Example 5.11, let us compute Schubert classes, as deduced by Morrison and Sottile [46, Theorem 1].

In the first column in Table 2, we list all

$$\zeta_{23154} \cdot p_3(x_1, x_2) \quad \text{and} \quad \sigma_{23154} \cdot p_3(x_1, x_2).$$

By Theorem 6.12 we have

$$\zeta_{23154} \cdot p_3(x_1, x_2) = p_3(t_2, t_3) \cdot \zeta_{23154}$$

$$\quad + (t_2^2 + t_2 t_5 + t_5^2) \cdot \zeta_{35124} + (t_2^2 + t_2 t_4 + t_4^2) \cdot \zeta_{43152}$$

$$\quad + (t_3^2 + t_3 t_5 + t_5^2) \cdot \zeta_{25134} + (t_3^2 + t_3 t_4 + t_4^2) \cdot \zeta_{24153}$$

$$\quad + (t_3 + t_4 + t_5) \cdot \tau_{34152} - (t_2 + t_3 + t_5) \cdot \zeta_{35124}$$

$$\quad - (t_2 + t_3 + t_4) \cdot \zeta_{34152} + (t_3 + t_4 + t_5) \cdot \zeta_{25143}$$

$$\quad - \tau_{54132} - \zeta_{35142} - \zeta_{45123}.$$

To compute $\sigma_{23154} \cdot p_3(x_1, x_2)$, by Corollary 6.13 we need to pick out the permutations in the first column satisfying that $\ell(w) = \ell(u) + r'$. Such $\ell(w)$ are underlined in the last column in Table 2, and so we obtain that

$$\sigma_{23154} \cdot p_3(x_1, x_2) = p_3(t_2, t_3) \cdot \sigma_{23154}$$

$$\quad + (t_3^2 + t_3 t_5 + t_5^2) \cdot \sigma_{25134} + (t_3^2 + t_3 t_4 + t_4^2) \cdot \sigma_{24153}$$

$$\quad - (t_2 + t_3 + t_5) \cdot \tau_{34152} - (t_2 + t_3 + t_4) \cdot \zeta_{34521}$$

$$\quad + (t_3 + t_4 + t_5) \cdot \tau_{25143} - \tau_{35142} - \zeta_{45123}.$$

| $w$  | $\eta$ | $h_{t_2} + 1$ | $d_{u,3}(t)$ | $\ell(w)$ |
|------|-------|--------------|--------------|---------|
| 23154 | -     | 0            | $p_3(t_2, t_3)$ | 3       |
| 53124 | (14)  | 1            | $h_2(t_{(5,2)}) = t_2^2 + t_2 t_5 + t_5^2$ | 6       |
| 43152 | (15)  | 1            | $h_2(t_{(4,2)}) = t_2^2 + t_2 t_4 + t_4^2$ | 6       |
| 25124 | (24)  | 1            | $h_2(t_{(5,3)}) = t_3^2 + t_3 t_5 + t_5^2$ | 4       |
| 24153 | (25)  | 1            | $h_2(t_{(4,3)}) = t_3 + t_3 t_4 + t_4^2$ | 4       |
| 53142 | (145) | 1            | $h_1(t_{(5,4,2)}) = t_2 + t_4 + t_5$ | 7       |
| 35124 | (124) | 2            | $-h_1(t_{(3,5,2)}) = -t_2 - t_3 - t_5$ | 5       |
| 34152 | (125) | 2            | $-h_1(t_{(3,4,2)}) = -t_2 - t_3 - t_4$ | 5       |
| 25143 | (245) | 1            | $h_1(t_{(5,4,3)}) = t_3 + t_4 + t_5$ | 5       |
| 54132 | (1425)| 2            | $-h_0(t_{(5,4,3,2)}) = -1$ | 8       |
| 35142 | (1245)| 2            | $-h_0(t_{(3,5,4,2)}) = -1$ | 6       |
| 45123 | (1524)| 2            | $-h_0(t_{(4,5,2,3)}) = -1$ | 6       |

Table 2. Computing $\zeta_{23154} \cdot p_3(x_1, x_2)$ and $\sigma_{23154} \cdot p_3(x_1, x_2)$
7. Grassmannian Cases

This section concerns the multiplication formulas for CSM classes and Schubert classes over Grassmannians. As will be seen, all formulas established before have parabolic analogues. We illustrate this by two concrete examples: parabolic versions of Theorem 5.5 and Corollary 6.13 both of which can be described in terms of the combinatorics of partitions. The parabolic version of Corollary 6.13 will be used in Section 8 to investigate the enumeration formulas of rim hook tableaux.

7.1. Geometry of Grassmannians. Denote by \( G_r(k, n) \) the Grassmannian of \( k \)-planes in \( \mathbb{C}^n \). There is a proper \( T \)-equivariant morphism

\[
\pi: \mathcal{F} \ell(n) \longrightarrow G_r(k, n)
\]

by sending a flag \((V_\bullet)\) to its \( k \)-plane \( V_k \). This induces two morphisms: the pullback \( \pi^* \) and the pushforward \( \pi_* \):

\[
\pi^*: H^*_T(G_r(k, n)) \rightarrow H^*_T(\mathcal{F} \ell(n)) \quad \text{and} \quad H^*(G_r(k, n)) \rightarrow H^*(\mathcal{F} \ell(n)),
\]

\[
\pi_*: H^*_T(\mathcal{F} \ell(n)) \rightarrow H^*_T(G_r(k, n)) \quad \text{and} \quad H^*(\mathcal{F} \ell(n)) \rightarrow H^*(G_r(k, n)).
\]

Recall from Section 3 that \( S_n \) acts on \( H^*_T(\mathcal{F} \ell(n)) \) (resp., \( H^*(\mathcal{F} \ell(n)) \)) by permuting the \( x_i \)'s. It is known that \( H^*_T(G_r(k, n)) \) (resp., \( H^*(G_r(k, n)) \)) is isomorphic to the \( (S_k \times S_{n-k}) \)-invariant algebra of \( H^*_T(\mathcal{F} \ell(n)) \) (resp., \( H^*(\mathcal{F} \ell(n)) \)), and \( \pi^* \) coincides with the inclusion map \([8]\).

For any \( w \in S_n \), there is a unique decomposition

\[
w = w_\lambda v,
\]

where \( v \in S_k \times S_{n-k} \), and \( w_\lambda \) is a Grassmannian permutation associated to a partition \( \lambda = (\lambda_1, \ldots, \lambda_k) \). Precisely, \( w_\lambda \) is obtained from \( w \) by rearranging the first \( k \) elements and the last \( n-k \) elements in increasing order, respectively. Formally, \( S_k \times S_{n-k} \) is a maximal parabolic subgroup of \( S_n \), and \( w_\lambda \) is the minimal coset representative of the left coset of \( S_k \times S_{n-k} \) with respect to \( w \). Notice that \( \lambda \) is inside the \( k \times (n-k) \) rectangle. We shall use \( Gr(w) \) to denote the partition \( \lambda \). For example, for \( k = 4 \) and \( w = 516342 \), we have \( w_\lambda = 135624 \) and \( Gr(w) = \lambda = (2, 2, 1, 0) \).

The Schubert cell \( Y(\lambda)_\circ \) refers to the set-theoretic image of \( Y(w_\lambda)_\circ \) under \( \pi \), namely,

\[
Y(\lambda)_\circ = \pi(Y(w_\lambda)_\circ).
\]

The Schubert variety \( Y(\lambda) \) is the closure of \( Y(\lambda)_\circ \). By [4] Proposition 3.5, if \( Gr(w) = \lambda \),

\[
\pi_*\left(c^T_{SM}(Y(w)_\circ)\right) = c^T_{SM}(Y(\lambda)_\circ) \quad \text{and} \quad \pi_*\left(c_{SM}(Y(w)_\circ)\right) = c_{SM}(Y(\lambda)_\circ),
\]

see also [41] Theorem 4.3. This allows the multiplication formulas for CSM classes established in Sections 5 and 6 to be converted into formulas involving the associated partitions.

For \( u \in S_n \) with \( Gr(u) = \lambda \) and a class \( \alpha \in H^*_T(G_r(k, n)) \), suppose that

\[
c^T_{SM}(Y(u)_\circ) \cdot \alpha = \sum_{w \in S_n} c^w_{u, \alpha}(t) \cdot c^T_{SM}(Y(w)_\circ),
\]

where \( c^w_{u, \alpha}(t) \) are the \( c \)-numbers that arise when expanding the \( t \)-coordinate of \( c^T_{SM}(Y(u)_\circ) \cdot \alpha \).
where, via the pullback $\pi^*$, $\alpha$ is also identified with a class in $H^*_T(\mathcal{F}_\ell(n))$ symmetric under $\mathfrak{S}_k \times \mathfrak{S}_{n-k}$. Applying the pushforward map $\pi_*$ to the left-hand side of (7.3) and using the projection formula yields

$$
\pi_*\left(c_{\text{SM}}(Y(u)^\circ) \cdot \alpha \right) = \pi_*\left(c_{\text{SM}}(Y(u)^\circ) \cdot \pi^*(\alpha) \right) = \pi_*\left(c_{\text{SM}}(Y(u)^\circ) \right) \cdot \alpha = c_{\text{SM}}(Y(\lambda)^\circ) \cdot \alpha.
$$

Applying $\pi_*$ to the right-hand side of (7.3) and combining the above, we obtain that

$$
c^T_{\text{SM}}(Y(\lambda)^\circ) \cdot \alpha = \sum_{\mu} \left( \sum_{w \in \mathfrak{S}_n \atop \Gr(w) = \mu} e^w_{u,\alpha}(t) \right) c^T_{\text{SM}}(Y(\mu)^\circ). \tag{7.4}
$$

The same derivation applies to nonequivariant CSM classes.

We turn to Schubert classes. It was shown in [8] that

$$
\pi^*([Y(\lambda)]_T) = [Y(w_\lambda)]_T \quad \text{and} \quad \pi^*([Y(\lambda)]) = [Y(w_\lambda)]. \tag{7.5}
$$

We refer to [11, §10.8] for more information. As a result, multiplication formulas concerning Schubert classes, for example, Corollaries 5.14 and 6.13 can be naturally restricted to Grassmannians in the following sense. For $u \in \mathfrak{S}_n$ and a class $\alpha \in H^*_T(\mathcal{G}_r(k,n))$, suppose that

$$
[Y(u)]_T \cdot \alpha = \sum_{w \in \mathfrak{S}_n} \overline{\tau}^w_{u,\alpha}(t) \cdot [Y(w)]_T, \tag{7.6}
$$

where $\alpha$ is also regarded as a class in $H^*_T(\mathcal{F}_\ell(n))$. We conclude that

$$
[Y(\lambda)]_T \cdot \alpha = \sum_{\mu} \overline{\tau}^\mu_{w_\lambda,\alpha}(t) \cdot [Y(\mu)]_T. \tag{7.7}
$$

Actually, by (7.5), one can identify $[Y(\lambda)]_T$ with $[Y(w_\lambda)]_T$ through $\pi^*$. Thus (7.7) is nothing but the case when $u = w_\lambda$.

In the rest of this section, we shall pay attention to the parabolic versions of Theorem 5.5 and Corollary 6.13, both of which can be described explicitly in terms of operations on partitions. In particular, the parabolic treatment of Corollary 6.13 will be applied in the next section to the enumeration of rim hook tableaux.

### 7.2. Parabolic Version of Theorem 5.5

For two distinct partitions $\lambda$ and $\mu$ inside the $k \times (n-k)$ rectangle, we write $\lambda \rightarrow \mu$ if there exist $u, w \in \mathfrak{S}_n$ with $\Gr(u) = \lambda$ and $\Gr(w) = \mu$ such that $u \rightarrow w$ is a $k$-edge. The $k$-edge $u \rightarrow w$ is not necessarily unique. In fact, it is easily checked that the collection of $k$-edges generating $\lambda \rightarrow \mu$ are of the form $uw \rightarrow uv$, where $v \in \mathfrak{S}_k \times \mathfrak{S}_{n-k}$. Such $k$-edges enjoy the same label, say $\overline{\tau}^u_{w,\lambda,\alpha}(t)$, and $\lambda \rightarrow \mu$ inherits this label, so we may write

$$
\lambda \xrightarrow{\tau} \mu.
$$

The partitions inside the $k \times (n-k)$ rectangle, along with the (labeled) edges $\lambda \xrightarrow{\tau} \mu$, constitute a directed graph, which is called the (labeled) $k$-Bruhat graph on partitions inside the $k \times (n-k)$ rectangle. We also call the edges in this graph $k$-edges since no confusion would arise from the context. We remark that if one ignores the edge
labels, the \( k \)-Bruhat graph on partitions inside the \( k \times (n - k) \) rectangle is exactly the \( \Lambda \)-Bruhat graph on \( \mathcal{S}_n \), \( \mathcal{S}_k \times \mathcal{S}_{n-k} \) [40 Definition 8.4], or the singular Bruhat graph on \( \mathcal{S}_n / \mathcal{S}_k \times \mathcal{S}_{n-k} \) [4 Definition 6.9].

It was pointed out in [40, §10.1] that there exists a \( k \)-edge \( \lambda \rightarrow \mu \) if and only if \( \mu \) can be obtained from \( \lambda \) by adding a rim hook (also called border strip), that is, \( \mu / \lambda \) is a connected skew shape with no \( 2 \times 2 \) square. Figure 3 displays the \( k \)-Bruhat graphs on partitions respectively inside the \( 2 \times 2 \), \( 3 \times 1 \), and \( 1 \times 3 \) rectangles, where dashed edges signify \( \lambda \rightarrow \mu \) with \( \vert \mu / \lambda \vert > 1 \).

**Figure 3.** The \( k \)-Bruhat graphs on partitions inside \( 2 \times 2 \), \( 3 \times 1 \), and \( 1 \times 3 \).

**Lemma 7.1.** Let \( u \in \mathcal{S}_n \) be such that \( \text{Gr}(u) = \lambda \). Then any path

\[
\lambda = \lambda^{(0)} \rightarrow \lambda^{(1)} \rightarrow \cdots \rightarrow \lambda^{(m)}
\]

in the \( k \)-Bruhat graph on partitions inside \( k \times (n - k) \) admits a unique lifting

\[
u = w_0 \rightarrow w_1 \rightarrow \cdots \rightarrow w_m
\]

in the \( k \)-Bruhat graph on \( \mathcal{S}_n \) starting from \( u \), where \( \text{Gr}(w_i) = \lambda^{(i)} \) for \( 0 \leq i \leq m \).

**Proof.** As aforementioned, the collection of \( k \)-edges generating \( \lambda^{(0)} \rightarrow \lambda^{(1)} \) are of the form \( w_0 v \rightarrow w_1 v \), where \( v \in \mathcal{S}_k \times \mathcal{S}_{n-k} \). Since \( w_0 = u \) is fixed, the permutation \( w_1 \) is also uniquely determined. For the same reason, the permutations \( w_i \) for \( i = 2, \ldots, m \) are accordingly uniquely determined. \( \square \)

There is a simple combinatorial rule to determine the label \( \tau \) of a \( k \)-edge \( \lambda \rightarrow \mu \). Draw the Young diagram of \( \lambda \), and label the southeast boundary from bottom left to top right by \( 1, 2, \ldots, n \) in increasing order. Note that the \( k \) labels received by the \( k \) rows are exactly the first \( k \) elements of the corresponding Grassmannian permutation \( w_\lambda \). For example, for \( n = 9 \), \( k = 4 \) and \( \lambda = (4, 2, 2, 0) \), the labeling of \( \lambda \) is illustrated in the left diagram of Figure 3 from which one can read off \( w_\lambda = 145823679 \).

To locate \( \tau \), label the southeast boundary of \( \mu \). Let \( L(\mu / \lambda) \) denote the set of labels received by the southeast boundary of the rim hook \( \mu / \lambda \). For \( \lambda = (4, 2, 2, 0) \) and \( \mu = (4, 4, 3, 0) \), we see from the right diagram of Figure 3 that \( L(\mu / \lambda) = \{4, 5, 6, 7\} \). We have the following observation

\[
\tau = \min L(\mu / \lambda).
\]
This can be understood as follows. Assume that $u \xrightarrow{\tau} w$ is a $k$-edge with $\text{Gr}(u) = \lambda$ and $\text{Gr}(w) = \mu$. Then the set of labels received by the rows of $\lambda$ (resp., $\mu$) is $\{u(1), \ldots, u(k)\}$ (resp., $\{w(1), \ldots, w(k)\}$). Notice that $\tau \in \{u(1), \ldots, u(k)\} \setminus \{w(1), \ldots, w(k)\}$, which, by the definition of $L(\mu/\lambda)$, is exactly the value $\min L(\mu/\lambda)$. With the above $\lambda$ and $\mu$, we see that $\tau = 4$.

The leftmost box in the bottom row of the rim hook $\mu/\lambda$ is referred to as the tail of $\mu/\lambda$. Clearly, the bottom edge of the tail box is endowed with the label $\tau = \min L(\mu/\lambda)$.

As depicted in Figure 4, the tail of $\mu/\lambda$ is marked with a star. With this notion, we have the following characterizations of increasing/decreasing paths in the $k$-Bruhat graph of partitions.

**Lemma 7.2.** Assume that $\lambda$ and $\mu$ are two partitions inside the $k \times (n - k)$-rectangle. Then a path

$$\lambda = \lambda^{(0)} \xrightarrow{\tau_1} \lambda^{(1)} \xrightarrow{\tau_2} \cdots \xrightarrow{\tau_m} \lambda^{(m)} = \mu$$

from $\lambda$ to $\mu$ is increasing (resp., decreasing) if and only if for $1 \leq i \leq m - 1$, the tail of $\lambda^{(i+1)}/\lambda^{(i)}$ is strictly to the right (resp., strictly below) of the tail of $\lambda^{(i)}/\lambda^{(i-1)}$.

Now, Theorem 5.5, formula (7.4), and Lemmas 7.1 and 7.2 together lead to the following multiplication formula over $H^*(\text{Gr}(k, n))$.

**Theorem 7.3.** Let $\lambda$ be a partition inside the $k \times (n - k)$ rectangle. For a hook shape partition $\Gamma = (1 + \alpha, 1^\beta)$, we have the following identity in $H^*(\text{Gr}(k, n))$:

$$c_{\lambda, \Gamma} \cdot s_{\Gamma}(x_{[k]}) = \sum_{\mu} c_{\lambda, \Gamma}^\mu : c_{\mu}(Y(\mu)^\circ),$$

where the sum ranges over partitions inside the $k \times (n - k)$ rectangle, and the coefficient $c_{\lambda, \Gamma}^\mu$ is equal to the number of paths

$$\lambda = \lambda^{(0)} \xrightarrow{\tau_1} \lambda^{(1)} \xrightarrow{\tau_2} \cdots \xrightarrow{\tau_m} \lambda^{(\alpha + \beta + 1)} = \mu$$

such that for $1 \leq i \leq \beta$, the tail of $\lambda^{(i+1)}/\lambda^{(i)}$ is strictly below the tail of $\lambda^{(i)}/\lambda^{(i-1)}$, and for $\beta + 1 \leq j \leq \alpha + \beta$, the tail of $\lambda^{(j+1)}/\lambda^{(j)}$ is strictly to the right of the tail of $\lambda^{(j)}/\lambda^{(j-1)}$.

**Example 7.4.** Write $\zeta_{\lambda} = c_{\lambda, \Gamma}(Y(\lambda)^\circ)$. Let $k = 3, n = 7$, and $\lambda = (3, 2, 0)$. Consider the cases when $\alpha + \beta + 1 = 3$. We use the colors ■, ■, ■ to indicate the first, the second and the third added rim hooks, respectively. For $\alpha = 0$ and $\beta = 2$, we have

![Figure 4. The construction of $w_\lambda$ and $L(\mu/\lambda)$](image-url)
\( s_T(x_1, x_2, x_3) = e_3(x_1, x_2, x_3) \). The decreasing paths inside the \( 3 \times 4 \) rectangle starting from \( \lambda \) are

\[
\begin{array}{cccc}
\text{ decreasing paths } & & & \\
\end{array}
\]

So we have

\[
\zeta_{(3,2,0)} \cdot e_3(x_1, x_2, x_3) = \zeta_{(4,3,1)} + \zeta_{(4,3,2)} + \zeta_{(4,3,3)} + \zeta_{(4,4,1)} + \zeta_{(4,4,2)} + \zeta_{(4,4,3)} + 2\zeta_{(4,4,4)}.
\]

For \( \alpha = 2 \) and \( \beta = 0 \), we have \( s_T(x_1, x_2, x_3) = h_3(x_1, x_2, x_3) \), and the increasing paths inside the \( 3 \times 4 \) rectangle starting from \( \lambda \) are

\[
\begin{array}{cccc}
\text{ increasing paths } & & & \\
\end{array}
\]

This gives

\[
\zeta_{(3,2,0)} \cdot h_3(x_1, x_2, x_3) = \zeta_{(3,3,2)} + \zeta_{(3,3,3)} + \zeta_{(4,2,2)} + \zeta_{(4,3,1)} + \zeta_{(4,3,2)} + 2\zeta_{(4,3,3)} + \zeta_{(4,4,1)} + 2\zeta_{(4,4,2)} + 3\zeta_{(4,4,3)} + 4\zeta_{(4,4,4)}.
\]

For \( \alpha = 1 \) and \( \beta = 1 \), we have \( s_T(x_1, x_2, x_3) = s_{(2,1)}(x_1, x_2, x_3) \). In this case,

\[
\zeta_{(3,2,0)} \cdot s_{(2,1)}(x_1, x_2, x_3) = \zeta_{(3,3,2)} + 2\zeta_{(3,3,3)} + \zeta_{(4,2,2)} + 2\zeta_{(4,3,1)} + 2\zeta_{(4,3,2)} + 3\zeta_{(4,3,3)} + \zeta_{(4,4,0)} + 2\zeta_{(4,4,1)} + 3\zeta_{(4,4,2)} + 6\zeta_{(4,4,3)} + 8\zeta_{(4,4,4)},
\]

with the corresponding peakless paths

\[
\begin{array}{cccc}
\text{ peakless paths } & & & \\
\end{array}
\]

7.3. **Parabolic Version of Corollary 6.13.** Adopting the notion in [57, §7.17], define the height \( \text{ht}(\mu/\lambda) \) of a rim hook \( \mu/\lambda \) to be one less than its number of rows.

**Lemma 7.5.** Let \( \lambda \) and \( \mu \) be two partitions inside the \( k \times (n-k) \) rectangle such that \( \mu/\lambda \) is a rim hook. Then

\[
\text{ht}(\mu/\lambda) = \text{ht}_k(w_\lambda^{-1}w_\mu) \quad \text{and} \quad L(\mu/\lambda) = w_\lambda M(w_\lambda^{-1}w_\mu).
\]

**Proof.** By (6.1), \( \text{ht}_k(w_\lambda^{-1}w_\mu) = \# \{ i \leq k : w_\mu(i) \neq w_\lambda(i) \} - 1 \). Recall that for \( i \leq k \), \( w_\lambda(i) \) equals the label of row \( k + 1 - i \) in \( \lambda \), and for \( i > k \), \( w_\lambda(i) \) equals the label of column \( i - k \) in \( \lambda \). It is easily seen that \( \{ i \leq k : w_\mu(i) \neq w_\lambda(i) \} \) consists exactly of the labels of rows in the rim hook \( \mu/\lambda \), and so we have \( \text{ht}(\mu/\lambda) = \text{ht}_k(w_\mu w_\lambda^{-1}) \). Moreover, \( w_\lambda M(w_\lambda^{-1}w_\mu) = \{ w_\lambda(i) : w_\lambda(i) \neq w_\mu(i) \} \), which is exactly the set \( L(\mu/\lambda) \). \( \square \)
Comparing the classical MN rule for Schur polynomials with the MN rule for Schubert polynomials \[46\], \(\mu/\lambda\) is a rim hook of size \(r\) if and only if there exists an \((r + 1)\)-cycle \(\eta\) such that \(w_\mu = w_\lambda \eta \) and \(w_\lambda \leq_k w_\mu\) in the ordinary \(k\)-Bruhat order. Hence, combining Corollary \[6.13\] (7.7) and Lemma \[7.5\] we arrive at the following equivariant MN rule over \(H^*(G_r(k, n))\).

**Theorem 7.6.** Let \(\lambda\) be a partition inside the \(k \times (n - k)\) rectangle. For \(r \geq 1\), we have the following identity in \(H^*_r(G_r(k, n))\):

\[
[Y(\lambda)]_T \cdot p_r(x[k]) = p_r(t_{w_\lambda[k]}) \cdot [Y(\lambda)]_T + \sum_{\mu} (-1)^{ht(\mu/\lambda)} \cdot h_{r-r'}(t_{L(\mu/\lambda)}) \cdot [Y(\lambda)]_T, \quad (7.8)
\]

where \(\mu\) ranges over partitions inside the \(k \times (n - k)\) rectangle such that \(\mu/\lambda\) is a rim hook of size \(r'\) with \(1 \leq r' \leq r\).

**Example 7.7.** Let \(k = 4\), \(n = 9\) and \(\lambda = (4,2,2,0)\). Denote \(\sigma_\lambda = [Y(\lambda)]_T\). For \(r = 3\), by Theorem \[7.6\], we have

\[
\sigma_{(4,2,2,0)} \cdot p_3(x[4]) = p_3(t_1,t_4,t_5,t_8) \cdot \sigma_{(4,2,2,0)} + (t_8^2 + t_8 t_9 + t_9^2) \cdot \sigma_{(5,2,2,0)} + (t_5^2 + t_5 t_6 + t_6^2) \cdot \sigma_{(4,3,2,0)} + (t_1^2 + t_1 t_2 + t_2^2) \cdot \sigma_{(4,2,2,1)} + (t_5 + t_6 + t_7) \cdot \sigma_{(4,4,2,0)} - (t_4 + t_5 + t_6) \cdot \sigma_{(4,3,3,0)} + (t_1 + t_2 + t_3) \cdot \sigma_{(4,2,2,2)} - \sigma_{(4,4,3,0)}.
\]

The corresponding \(\mu\)'s appearing in \(7.8\) are illustrated below:

8. **Localization and Rim Hook Tableaux**

The purpose of this section is to apply Theorem \[7.6\] to establish a relationship connecting the localization of Schubert classes and the number of standard rim hook tableaux. We discuss how to utilize this connection to deduce formulas for the number of standard rim hook tableaux, including the formulas obtained by Alexandersson, Pfannerer, Rubey and Uhlin \[2\] and Fomin and Lulov \[16\].

Throughout this section, assume that \(r\) and \(d\) are fixed positive integers. Let \(\Lambda/\lambda\) be a skew shape of size \(rd\). A **standard \(r\)-rim hook tableau** of shape \(\Lambda/\lambda\) may be thought of as a sequence

\[
\lambda = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(d)} = \Lambda
\]

of partitions such that for \(1 \leq i \leq d\), \(\lambda^{(i)}/\lambda^{(i-1)}\) is a rim hook of size \(r\). A rim hook of size \(r\) is also called an \(r\)-rim hook. Usually, one assigns each box in the \(r\)-rim hook \(\lambda^{(i)}/\lambda^{(i-1)}\) with the integer \(i\), so that a standard \(r\)-rim hook tableau may be intuitively viewed as a filling of \(\Lambda/\lambda\) with integers \(1, 2, \ldots, d\). For \(r = 2\), \(\lambda = (1)\) and \(\Lambda = (4,4,1)\), there are 4 standard 2-rim hook tableaux (also called domino tableaux) of shape \(\Lambda/\lambda\), as listed below:

\[
\begin{array}{ccc}
1 & 1 & 4 \\
2 & 3 & 3 \\
\end{array} \quad \begin{array}{ccc}
2 & 2 & 4 \\
1 & 3 & 3 \\
\end{array} \quad \begin{array}{ccc}
2 & 3 & 3 \\
1 & 2 & 4 \\
\end{array} \quad \begin{array}{ccc}
2 & 3 & 4 \\
1 & 2 & 3 \\
\end{array}
\]
In the case \( r = 1 \), a standard \( r \)-rim hook tableau of shape \( \Lambda / \lambda \) becomes a standard Young tableau of shape \( \Lambda / \lambda \). Let \( \text{RHT}^r(\Lambda / \lambda) \) stand for the set of standard \( r \)-rim hook tableaux of shape \( \Lambda / \lambda \).

As before, for a partition \( \Lambda \) inside the \( k \times (n - k) \)-rectangle, denote by \( w_\Lambda \in \mathfrak{S}_n \) the Grassmannian permutation associated to \( \Lambda \) with descent at \( k \). Let \( \phi_\Lambda \in \mathcal{G}r(k, n) \) represent the image of the fixed point \( \phi_w \) under the natural projection \( \mathcal{F} \ell(n) \to \mathcal{G}r(k, n) \). Consider the localization morphism

\[
-|_\Lambda : H^*_T(\mathcal{G}r(k, n)) \to H^*_T(\phi_\Lambda) \cong \mathbb{Q}[t_1, \ldots, t_n].
\]

It is easy to see that for a class \( f(x, t) \) in \( H^*_T(\mathcal{G}r(k, n)) \),

\[
f(x, t)|_\Lambda = f(w_\Lambda, t).
\]

We shall pay attention to the localization

\[
[Y(\lambda)]_T|_\Lambda = [Y(w_\Lambda)]_T|_{w_\Lambda}.
\]

A combinatorial formula for \([Y(\lambda)]_T|_\Lambda \) in terms of excited diagrams was given by Ikeda and Naruse \[21\] and Kreiman \[26\], see also Morales, Pak and Panova \[44\]. For general permutations \( u, w \in \mathfrak{S}_n \), the localization \([Y(u)]_T|_w \) can be computed by Billey’s formula \[10\].

8.1. Localization and Rim Hook Tableaux. Denote by \( o(1) \) the space of rational functions in \( \mathbb{Q}(z) \) vanishing at one (thus all) of the primitive \( r \)-th roots of unity. Following the convention of analysis, instead of writing \( f(z) \in g(z) + o(1) \), we will use the notation \( f(z) = g(z) + o(1) \). For example, \( z^r = 1 + o(1) \).

The localization \([Y(\lambda)]_T|_\Lambda \) is a polynomial in \( t_1, \ldots, t_n \). After the specialization \( t_i = z^i \), \([Y(\lambda)]_T|_\Lambda \) becomes a polynomial in \( z \), which is denoted

\[
Y_{\lambda, \Lambda}(z) = ([Y(\lambda)]_T|_\Lambda)_{t_i = z^i}.
\]

For a rim hook tableau \( T \in \text{RHT}^r(\Lambda / \lambda) \), let \( \text{ht}(T) \) denote the total sum of heights of \( r \)-rim hooks appearing in \( T \) (Recall that the height of a rim hook is one less than its number of rows). It is known that for distinct \( T, T' \in \text{RHT}^r(\Lambda / \lambda) \), \( \text{ht}(T) \) and \( \text{ht}(T') \) have the same parity, see for example \[2\] Lemmas 28 and 29. So one may define

\[
\text{sgn}(\Lambda / \lambda) = (-1)^{\text{ht}(T)},
\]

where \( T \) is any rim hook tableau in \( \text{RHT}^r(\Lambda / \lambda) \). In the case that \( \Lambda / \lambda \) does not admit any \( r \)-rim hook tableau, \( \text{sgn}(\Lambda / \lambda) \) is understood as zero.

Our main theorem is the following Laurent expansion.

**Theorem 8.1.** For a skew shape \( \Lambda / \lambda \) of size \( rd \), we have

\[
\frac{Y_{\lambda, \Lambda}(z)}{Y_{\lambda, \Lambda}(1)} = \frac{1}{(z^r - 1)^d} \left( \text{sgn}(\Lambda / \lambda) \frac{\# \text{RHT}^r(\Lambda / \lambda)}{r^d d!} + o(1) \right).
\]

The rest of this subsection is devoted to a proof of Theorem 8.1.

**Lemma 8.2.** Let \( 1 \leq \ell \leq r \), and let \( \zeta \) be a primitive \( r \)-th root of unity. For \( r - \ell < i < r \) and distinct integers \( a_1, \ldots, a_\ell \in \{0, 1, \ldots, r - 1\} \), we have \( h_\ell(\zeta^{a_1}, \ldots, \zeta^{a_\ell}) = 0 \), namely, \( h_\ell(z^{a_1}, \ldots, z^{a_\ell}) = o(1) \).
Proof. Let
\[ f(z) = \frac{1}{1 - z \zeta_1} \cdots \frac{1}{1 - z \zeta_r} = \sum_{j \geq 0} h_j(\zeta_1, \ldots, \zeta_r) z^j \]

denote the generating function of \( h_j(\zeta_1, \ldots, \zeta_r) \). Since \( \zeta \) is a primitive \( r \)-th root of unity, we see that
\[ f(z) - z^r f(z) = \frac{1 - z^r}{(1 - z \zeta_1) \cdots (1 - z \zeta_r)} = \prod_{0 \leq j \leq r-1, j \neq a_1, \ldots, a_r} (1 - z \zeta^j). \]

Extract the coefficient of \( z^i \) for \( r - \ell < i < r \) on both sides. Clearly, the left-hand side contributes \( h_i(\zeta_1, \ldots, \zeta_r) \). Since the right-hand side is a polynomial of degree \( r - \ell \), the coefficient of \( z^i \) is zero. This verifies the lemma.

\[ \square \]

**Theorem 8.3.** Let \( \Lambda/\lambda \) be a nontrivial skew shape. Then
\[ Y_{\lambda, \Lambda}(z) = \frac{1}{z^r - 1} \left( \sum_{|\mu/\lambda| = r} (-1)^{ht(\mu/\lambda)} |\Lambda/\lambda| Y_{\mu, \Lambda}(z) + \sum_{1 \leq |\mu'/\lambda| < r} o(1) \cdot Y_{\mu', \Lambda}(z) \right), \]

where the first sum is over \( \mu \subseteq \Lambda \) such that \( \mu/\lambda \) is a rim hook of size \( r \), and the second sum is over \( \mu' \subseteq \Lambda \) such that \( \mu'/\lambda \) is a rim hook of size \( r' \) with \( 1 \leq r' < r \).

**Proof.** Localizing both sides of (7.8) at \( \Lambda \), we get
\[ [Y(\lambda)]_{|\Lambda|} \cdot (p_r(t_{w_\lambda[k]}) - p_r(t_{w_\lambda[k]})) \]
\[ = \sum_{|\mu/\lambda| = r} (-1)^{ht(\mu/\lambda)} \cdot [Y(\mu)]_{|\Lambda|} + \sum_{1 \leq |\mu'/\lambda| < r} (-1)^{ht(\mu'/\lambda)} \cdot h_{r-r'}(t_{L(\mu'/\lambda)}) \cdot [Y(\mu')]_{|\Lambda|}. \]

To evaluate both sides by setting \( t_i = z^i \), we need the following two claims.

**Claim A.** For \( \mu \) such that \( \mu/\lambda \) is a rim hook of size \( r' \) with \( 1 \leq r' < r \), we have
\[ h_{r-r'}(t_{L(\mu/\lambda)})|_{t_i = z^i} = o(1). \]

Notice that \( L(\mu/\lambda) \) consists of \( r' + 1 \) consecutive integers, say, \( m, \ldots, m + r' \), which are distinct in \( \mathbb{Z}/r\mathbb{Z} \) since \( r' < r \). By Lemma 8.2 since \( r - (r' + 1) < r - r' < r \), we conclude that
\[ h_{r-r'}(t_{L(\mu/\lambda)})|_{t_i = z^i} = h_{r-r'}(z^m, \ldots, z^{m+r'}) = o(1). \]

**Claim B.** We have
\[ (p_r(t_{w_\lambda[k]}) - p_r(t_{w_\lambda[k]}))|_{t_i = z^i} = (z^r - 1)(|\Lambda/\lambda| + o(1)). \]

This claim can be proved by induction on \( |\Lambda/\lambda| \). Let us first check that case when \( |\Lambda/\lambda| = 1 \). Assume that the single box \( \Lambda/\lambda \) is in row \( k + 1 - i \) and column \( j \). Then we see that
\[ (p_r(t_{w_\lambda[k]}) - p_r(t_{w_\lambda[k]}))|_{t_i = z^i} = (z^{n-i+j})^r - (z^{n-i+j-1})^r = (z^r - 1)z^{r(n-i+j-1)}. \]

Clearly,
\[ z^{r(n-i+j-1)} = 1 + o(1), \quad 44 \]
and so the claim follows. We next consider the case $|\Lambda/\lambda| > 1$. Take $\lambda'$ such that $\lambda \subseteq \lambda' \subsetneq \Lambda$. By induction, we have

$$
(p_r(t_{w_\Lambda[k]}) - p_r(t_{w_\lambda[k]}))|_{t_i = z^i} = (z^r - 1)(|\lambda'/\lambda| + o(1))
$$

and

$$
(p_r(t_{w_\Lambda[k]}) - p_r(t_{w_\lambda[k]}))|_{t_i = z^i} = (z^r - 1)(|\Lambda/\lambda'| + o(1)).
$$

Adding them together completes the proof of Claim B.

By Claim A and Claim B, we obtain that

$$
(1 - 1) \cdot (|\Lambda/\lambda| + o(1)) \cdot Y_{\lambda,\Lambda}(z)
$$

$$
= \sum_{|\mu'/\lambda| = r} (-1)^{ht(\mu/\lambda)} \cdot Y_{\mu,\Lambda}(z) + \sum_{1 \leq |\mu'/\lambda| < r} o(1) \cdot Y_{\mu',\Lambda}(z) \cdot Y_{\Lambda,\Lambda}(z).
$$

Since $|\Lambda/\lambda| \neq 0$, we have

$$
\frac{1}{|\Lambda/\lambda| + o(1)} = \frac{1}{|\Lambda/\lambda| + o(1)} + o(1).
$$

So, dividing both sides of (8.3) by $(z^r - 1) \cdot (|\Lambda/\lambda| + o(1))$ yields that

$$
Y_{\lambda,\Lambda}(z) = \sum_{|\mu'/\lambda| = r} \left( (-1)^{ht(\mu'/\Lambda)} \cdot Y_{\mu,\Lambda}(z) \cdot Y_{\Lambda,\Lambda}(z) + \sum_{1 \leq |\mu'/\lambda| < r} \frac{o(1)}{z^r - 1} \cdot Y_{\mu',\Lambda}(z) \right),
$$

which is the same as (8.2).\qed

**Corollary 8.4.** For any skew shape $\Lambda/\lambda$, we have the following estimation

$$
\frac{Y_{\Lambda,\Lambda}(z)}{Y_{\lambda,\Lambda}(z)} = \frac{o(1)}{(z^r - 1)^m},
$$

where $m$ is the minimum nonnegative integer such that $(m + 1)r > |\Lambda/\lambda|$. 

**Proof.** The proof is by induction on the size $|\Lambda/\lambda|$. If $|\Lambda/\lambda| = 0$, then $m = 0$ and the estimation trivially holds. Now assume $|\Lambda/\lambda| > 0$. Dividing both sides of (8.2) by $Y_{\Lambda,\Lambda}(z)$, we obtain that

$$
\frac{Y_{\lambda,\Lambda}(z)}{Y_{\Lambda,\Lambda}(z)} = \frac{1}{z^r - 1} \left( \sum_{|\mu'/\lambda| = r} \frac{(-1)^{ht(\mu'/\Lambda)} \cdot Y_{\mu,\Lambda}(z)}{Y_{\Lambda,\Lambda}(z)} + \sum_{1 \leq |\mu'/\lambda| < r} o(1) \cdot \frac{Y_{\mu',\Lambda}(z)}{Y_{\Lambda,\Lambda}(z)} \right) \cdot Y_{\Lambda,\Lambda}(z).
$$

By induction,

$$
\frac{Y_{\mu,\Lambda}(z)}{Y_{\Lambda,\Lambda}(z)} = \frac{o(1)}{(z^r - 1)^m}
$$

and

$$
\frac{o(1) \cdot Y_{\mu',\Lambda}(z)}{Y_{\Lambda,\Lambda}(z)} = \frac{o(1)}{(z^r - 1)^m} = \frac{o(1)}{(z^r - 1)^{m+1}}.
$$
Putting the above into (8.4), we get
\[ \frac{Y_{\Lambda, \Lambda}(z)}{Y_{\Lambda, \Lambda}(z)} = \frac{1}{z^r - 1} \frac{o(1)}{(z^r - 1)^m} = \frac{o(1)}{(z^r - 1)^m + 1}. \]

Now we can give a proof of Theorem 8.1.

**Proof of Theorem 8.1.**
Since \( r \) divides \( |\Lambda/\lambda| \), for any \( \lambda \subsetneq \mu \subseteq \Lambda \), we have \( dr > |\Lambda/\mu| \). By Corollary 8.4, the second summation of (8.4) in the bracket belongs to \( o(1) \cdot \frac{o(1)}{(z^r - 1)^d} = \frac{o(1)}{(z^r - 1)^d} \). By induction, the first summation of (8.4) in the bracket contributes
\[ \sum_{|\mu/\lambda|=r} \frac{(-1)^{ht(\mu/\lambda)}}{rd} \frac{1}{(z^r - 1)^{d-1}} \left( \sum_{T \in RHT^r(\Lambda/\mu)} \frac{(-1)^{ht(T)}}{rd!} + o(1) \right) \]

\[ = \frac{1}{(z^r - 1)^{d-1}} \left( \sum_{T \in RHT^r(\Lambda/\lambda)} \frac{(-1)^{ht(T)}}{rd!} + o(1) \right) \]

\[ = \frac{1}{(z^r - 1)^{d-1}} \left( \text{sgn}(\Lambda/\lambda) \frac{\#RHT^r(\Lambda/\lambda)}{rd!} + o(1) \right). \]

Plugging the above into (8.4) leads to (8.1), as required. \[ \square \]

### 8.2. Connections with Previous Work.
Still, let \( \zeta \) be a primitive \( r \)-th root of unity. As a direct consequence of Theorem 8.1, we obtain that

**Corollary 8.5.** For a skew shape \( \Lambda/\lambda \) of size \( rd \),
\[ \#RHT^r(\Lambda/\lambda) = \text{sgn}(\Lambda/\lambda) rd! \cdot \lim_{z \to \zeta} \frac{Y_{\Lambda, \Lambda}(z)}{Y_{\Lambda, \Lambda}(z)(z^r - 1)^d}. \]  

(8.5)

We reformulate the right-hand side of (8.5) based on the following lemma.

**Lemma 8.6.** We have
\[ \lim_{z \to \zeta} \frac{1}{(z^r - 1)^d} \prod_{i=1}^{dr} (z^i - 1) = (-1)^{rd-d_r d!}. \]

**Proof.** We evaluate the left-hand side by grouping the factors. Observe that for \( 0 \leq a \leq d - 1 \),
\[ \lim_{z \to \zeta} \frac{1}{z^r - 1} \prod_{i=1}^{r} (z^{ar+i} - 1) = (\zeta - 1) \cdots (\zeta^{r-1} - 1) \cdot \lim_{z \to \zeta} \frac{z^{(a+1)r} - 1}{z^r - 1}. \]

Notice that \((\zeta - 1) \cdots (\zeta^{r-1} - 1) = (-1)^{r-1} r\), which can be deduced by substituting \( z = 1 \) in the following identity
\[ \prod_{i=1}^{r-1} (z - \zeta^i) = \frac{z^r - 1}{z - 1} = 1 + \cdots + z^{r-1}. \]

Moreover, applying L’Hospital’s rule gives
\[ \lim_{z \to \zeta} \frac{z^{(a+1)r} - 1}{z^r - 1} = \left. \frac{(a+1)r \cdot z^{(a+1)r-1}}{r \cdot z^{r-1}} \right|_{z=\zeta} = a + 1. \]
Hence, the limit on the left-hand side in the lemma can be expressed as

\[
\prod_{a=0}^{d-1} \lim_{z \to \zeta} \frac{1}{z^r - 1} \prod_{i=1}^{r} (z^{ar+i} - 1) = (-1)^{rd-d} (r \cdot 1) \cdots (r \cdot d) = (-1)^{rd-d} r!d!.
\]

By Lemma 8.6, we may reformulate (8.5) as

\[
\# \text{RHT}^r(\Lambda/\lambda) = \text{sgn}(\Lambda/\lambda) (-1)^{rd-d} \lim_{z \to \zeta} \frac{Y_{\Lambda,\Lambda}(z)}{Y_{\Lambda,\Lambda}(1)} (z - 1) \cdots (z^{|\Lambda/\lambda|} - 1). \tag{8.6}
\]

It was deduced in [44, Equation (4.10)] that for a skew shape \(\Lambda/\lambda\),

\[
Y_{\Lambda,\Lambda}(z) = (-1)^{|\Lambda/\lambda|} z^{-g(\Lambda)+g(\lambda)} \sum_{T \in \text{SYT}(\Lambda/\lambda)} z^{\text{maj}(T)}, \tag{8.7}
\]

where, for a partition \(\mu = (\mu_1, \ldots, \mu_k)\) inside the \(k \times (n-k)\) rectangle,

\[
g(\mu) = \sum_{i=1}^{k} \left( \mu_i + d + 1 - i \right)/2.
\]

Moreover, by Stanley [57, Proposition 7.19.11],

\[
s_{\Lambda/\lambda}(1, z, z^2, \ldots) = \frac{1}{(1-z) \cdots (1-z^{|\Lambda/\lambda|})} \sum_{T \in \text{SYT}(\Lambda/\lambda)} z^{\text{maj}(T)},
\]

where \(\text{SYT}(\Lambda/\lambda)\) denotes the set of standard Young tableaux of shape \(\Lambda/\lambda\), and the major index \(\text{maj}(T)\) of \(T\) is defined to be the sum of \(i\) such that \(i+1\) appears in a lower row of \(T\) than \(i\).

Combining the above and noting that \(|\Lambda/\lambda| = rd\), (8.6) can be rewritten as

\[
\# \text{RHT}^r(\Lambda/\lambda) = \text{sgn}(\Lambda/\lambda) (-1)^{rd-d} \lim_{z \to \zeta} z^{-g(\Lambda)+g(\lambda)} \sum_{T \in \text{SYT}(\Lambda/\lambda)} z^{\text{maj}(T)} = \text{sgn}(\Lambda/\lambda) (-1)^{rd-d} \frac{1}{\zeta^{g(\lambda)-g(\lambda)}} \sum_{T \in \text{SYT}(\Lambda/\lambda)} \zeta^{\text{maj}(T)}. \tag{8.8}
\]

By [44, Proposition 4.7],

\[
g(\Lambda) - g(\lambda) = k \cdot |\Lambda/\lambda| + \sum_{\Box \in \Lambda/\lambda} c(\Box),
\]

where \(c(\Box) = j - i\) for a box \(\Box\) in row \(i\) and column \(j\). Since \(r\) divides \(|\Lambda/\lambda|\), we get \(\zeta^{k \cdot |\Lambda/\lambda|} = 1\). We next evaluate

\[
\zeta^{\sum_{\Box \in \Lambda/\lambda} c(\Box)} \tag{8.9}
\]

Notice that the integers \(c(\Box)\), where \(\Box\) runs over boxes of any \(r\)-rim hook in \(\Lambda/\lambda\), are distinct in \(\mathbb{Z}/r\mathbb{Z}\). So each \(r\)-rim hook in \(\Lambda/\lambda\) contributes to \(\zeta^{1+2+\cdots+r}\) a value \(\zeta^{r+1} = \zeta^{1+2+\cdots+r}\), which is easily checked to be \((-1)^{r-1}\). Hence the value in \(8.9\) equals \((-1)^{|\Lambda/\lambda|-d} = (-1)^{rd-d}\), and so \(\zeta^{g(\Lambda)-g(\lambda)} = (-1)^{rd-d}\). Putting this into \(8.8\), we recover the following formula deduced by Alexandersson, Pfannerer, Rubey and Uhlin [2, Corollary 30] based on the character theory.
Theorem 8.7 ([2, Corollary 30]). Let $\Lambda/\lambda$ be a skew shape of size $rd$, and $\zeta$ be a primitive $r$-th root of unity. Then

$$\# \text{RHT}^r(\Lambda/\lambda) = \text{sgn}(\Lambda/\lambda) \sum_{T \in \text{SYT}(\Lambda/\lambda)} \zeta^{\text{maj}(T)}.$$ 

We now restrict to straight shapes, that is, $\lambda = \emptyset$ and $\Lambda$ has size $rd$. In this case, $Y_{\lambda,\Lambda}(z) = 1$. By [44, §4.2],

$$Y_{\lambda,\Lambda}(z) = \prod_{\square \in \Lambda} (z^{h(\square)} - 1) \text{ up to a power of } z,$$

which also follows from (8.7) and [57, Corollary 7.21.3]. In this formula, $h(\square)$ is the hook length of a box $\square \in \Lambda$, which is the number of boxes directly to its right or directly below it, including the box itself. So, by (8.5), we see that

$$\# \text{RHT}^r(\Lambda) = r^d d! \cdot \lim_{z \to \zeta} \left| (z^r - 1) \prod_{\square \in \Lambda} \frac{1}{z^{h(\square)} - 1} \right|,$$

where $| \cdot |$ denotes the modulus of complex numbers. Notice that if $r \mid h(\square)$, by L’Hospital’s rule,

$$\lim_{z \to \zeta} \left| \frac{z^r - 1}{z^{h(\square)} - 1} \right| = \frac{r}{h(\square)},$$

and if $r \nmid h(\square)$,

$$\lim_{z \to \zeta} \left| z^{h(\square)} - 1 \right| \neq 0.$$

This observation immediately reveals the well-known criterion that $\# \text{RHT}^r(\Lambda) \neq 0$ if and only if

$$\# \{ \square \in \Lambda : r \mid h(\square) \} = d,$$

see for example Fomin and Lulov [16, Corollary 2.7]. Now, when $\text{RHT}^r(\Lambda)$ is nonempty, (8.10) can be expressed as

$$\# \text{RHT}^r(\Lambda) = \prod_{\square \in \Lambda} \frac{1}{h(\square)} \cdot \frac{r^d d!}{|N|},$$

where

$$N = \prod_{\square \in \Lambda} \begin{cases} 1 - \zeta^{h(\square)}, & \text{if } r \nmid h(\square), \\ 1/r, & \text{if } r \mid h(\square). \end{cases}$$

Lemma 8.8. We have $|N| = 1$.

If $r = 1$, then it is trivially true that $|N| = 1$. If $r = 2$, then $\zeta = -1$ and so $1 - \zeta^{h(\square)} = 2$ for $h(\square)$ odd, which implies that $|N| = 1$ since exactly half of the hook lengths $h(\square)$ are odd. However, we do not have an elementary proof of Lemma 8.8 for general $r$.

Proof of Lemma 8.8. By (8.12), $|N|$ does not depend on the choice of $\zeta$, thus the Galois group $G = \text{Gal}(\mathbb{Q}[\zeta]/\mathbb{Q})$ acts on $|N|$ trivially. That is,

$$|N|^{[G]} = \prod_{\sigma \in G} |\sigma N| = |\text{Nm}(N)|,$$
where \( \text{Nm}: \mathbb{Q}[\zeta] \rightarrow \mathbb{Q} \) is the norm of the field extension \( \mathbb{Q}[\zeta]/\mathbb{Q} \), see [54, §VI.5]. To conclude \(|N| = 1\), it suffices to show that the norm of \( N \) is 1.

For any integer \( h \), the power \( \zeta^h \) is a primitive \( \frac{r}{\gcd(h,r)} \)-th root of unity, and so the norm of \( 1 - \zeta^h \) for the field extension \( \mathbb{Q}[\zeta]/\mathbb{Q} \) depends only on \( \gcd(h,r) \), see [54, Theorem VI.5.1 and Ex VI.19]. This, together with the following claim, will lead to a proof that the norm of \( N \) is 1.

Claim. For \( m | r \),
\[
\# \{ \square \in \Lambda: \gcd(h(\square), r) = m \} = \# \{ i \in [rd]: \gcd(i, r) = m \}. \tag{8.14}
\]

Notice that for any integer \( i \), \( r = \gcd(i, r) \) if and only if \( r | i \), and \( r > \gcd(i, r) \) if and only if \( r \nmid i \). By the above claim, we see that
\[
\text{Nm}(N) = \prod_{\square \in \Lambda} \begin{cases} 
\text{Nm}(1 - \zeta^{h(\square)}), & \text{if } r \nmid h(\square), \\
\text{Nm}(1/r), & \text{if } r | h(\square), 
\end{cases}
\]
\[
= \prod_{i \in [rd]} \begin{cases} 
\text{Nm}(1 - \zeta^i), & \text{if } r \nmid i, \\
\text{Nm}(1/r), & \text{if } r | i, 
\end{cases}
\]
\[
= \text{Nm} \left( \prod_{i \in [rd]} \begin{cases} 
1 - \zeta^i, & \text{if } r \nmid i, \\
1/r, & \text{if } r | i, 
\end{cases} \right)
\]
\[
= \text{Nm} \left( \frac{1}{rd} \left( (1 - \zeta) \cdots (1 - \zeta^{r-1}) \right)^d \right).
\]

Setting \( d = 1 \) in Lemma 8.6 we get \((\zeta - 1) \cdots (\zeta^{r-1} - 1) = (-1)^{r-1}r \), and so \( \text{Nm}(N) = 1 \).

It remains to prove the claim in (8.14). For \( m | r \), by [58, Lemma 7.3], if \( \text{RHT}^r(\Lambda) \) is nonempty, then \( \text{RHT}^m(\Lambda) \) is nonempty. So it follows from (8.11) that
\[
\# \{ \square \in \Lambda: m | h(\square) \} = \frac{r}{m}d = \# \{ i \in [rd]: m | i \}. \tag{8.15}
\]

We explain that the claim in (8.14) follows from (8.15) by Möbius inversion. For any finite sequence of integers \( a_1, \ldots, a_\ell \) and any \( m | r \), denote
\[
f_m = \# \{ i \in [\ell]: m | a_i \}, \quad g_m = \# \{ i \in [\ell]: \gcd(r, a_i) = m \}.
\]
Since \( m | a_i \) if and only if \( m | \gcd(r, a_i) \), it is not hard to check that \( f_{r/m} = \sum_{\delta | m} g_{r/\delta} \).

Therefore,
\[
g_{r/m} = \sum_{\delta | m} f_{r/\delta} \cdot \mu \left( \frac{m}{\delta} \right),
\]
where \( \mu \) is the classical number-theoretic Möbius function. So, \( f_m \) and \( g_m \) are determined by each other, and in particular, (8.15) implies (8.14).  \( \square \)

The above lemma along with (8.12) allows us to reach the following hook formula due to Fomin and Lulov [16, Corollary 2.2].
Theorem 8.9 ([16, Corollary 2.2]). For a partition $\Lambda$ of size $rd$ with $\text{RHT}^r(\Lambda)$ nonempty, we have

$$\#\text{RHT}^r(\Lambda) = \frac{r^d d!}{\prod_{\square \in \Lambda} h(\square)}.$$  

Notice that in the case $r = 1$, Theorem 8.9 specifies to the classical hook formula for standard Young tableaux.

9. Conjectures

In this section, we discuss some positivity conjectures about CSM classes. We mainly adhere to the notation from [5]. Let $G/B$ be the flag variety for a reductive group $G$ over $\mathbb{C}$ with $B$ a fixed Borel subgroup, and $W$ be the associated Weyl group. For an element $w$ in $W$, let $X(w)^\circ = BwB/B$ denote the Schubert cell, and $X(w) = X(w)^\circ$ denote the Schubert variety. We also let $Y(w)^\circ = B^{-1}wB/B$ be the opposite Schubert cell, and $Y(w) = Y(w)^\circ$ be the opposite Schubert variety, where $B^{-1} = w_0Bw_0$ is the opposite Borel subgroup ($w_0$ is the longest element of $W$). Here, we remark that in the above sections of this paper, we are only concerned with opposite Schubert cells/varieties of type $A$, and so we abbreviate “opposite” and simply call them Schubert cells/varieties.

The following is our main conjecture. As will be explained, its geometric version is recently proposed independently by Kumar [27, Conjecture B].

Conjecture 9.1. For $u, v \in W$,

$$c_{\text{SM}}(Y(u)^\circ) \cdot [Y(v)] \in \sum_{w \in W} \mathbb{Z}_{\geq 0} \cdot c_{\text{SM}}(Y(w)^\circ),$$  

(9.1)

where $\mathbb{Z}_{\geq 0}$ is the set of nonnegative integers.

Theorem 5.5 confirms Conjecture 9.1 in type $A$ for $v$ being a Grassmannian permutation of hook shape. This conjecture has been checked for groups of types $A_{\leq 7}$, $B_{\leq 4}$, $C_{\leq 4}$, $D_{\leq 4}$, $G_2$.

Taking the lowest degree component of (9.1) leads to the famous positivity for the structure constants of Schubert classes:

$$[Y(u)] \cdot [Y(v)] \in \sum_{w \in W} \mathbb{Z}_{\geq 0} \cdot [Y(w)].$$

Another implication of (9.1) is

$$c_{\text{SM}}(Y(u)^\circ) \in \sum_{w \in W} \mathbb{Z}_{\geq 0} \cdot [Y(w)],$$  

(9.2)

which was conjectured in [4] and proved in [5] using characteristic cycles of $\mathcal{D}$-modules. Actually, (9.2) is a direct consequence of Kumar’s conjecture [27, Conjecture B], which turns out to be equivalent to Conjecture 9.1.

Conjecture 9.2 ([27, Conj. B] $\iff$ Conj. 9.1). For $u, v \in W$, the CSM class $c_{\text{SM}}(X(u)\cap Y(v)^\circ)$ of the Richardson cell $X(u)\cap Y(v)^\circ$ is effective, namely,

$$c_{\text{SM}}(X(u)^\circ \cap Y(v)^\circ) \in \sum_{w \in W} \mathbb{Z}_{\geq 0} \cdot [Y(w)].$$  

(9.3)
In particular, for $u \in W$,
\[ c_{\text{SM}}(Y(u)^\circ) = \sum_{w \in W} c_{\text{SM}}(X(w)^\circ \cap Y(u)^\circ) \]

is effective assuming Conjecture 9.1.

We explain the equivalence between Conjecture 9.1 and Conjecture 9.2. For a constructible subset $Z$ of a smooth variety $X$, its Segre–Schwartz–MacPherson (SSM) class is defined to be
\[ s_{\text{SM}}(Z) = c_{\text{SM}}(Z) / c(T_X), \]
where $c(T_X)$ is the total Chern class of the tangent bundle of $X$. It was shown in [5, Theorem 7.1] that under Poincaré pairing, CSM classes and SSM classes of Schubert cells are dual. Specifically, for $u, w \in W$, we have
\[ \int_{G/B} s_{\text{SM}}(X(u)^\circ) \cdot c_{\text{SM}}(Y(v)^\circ) \cdot [Y(w)] \geq 0. \]

So (9.1) reads
\[ \int_{G/B} s_{\text{SM}}(X(u)^\circ) \cdot c_{\text{SM}}(Y(v)^\circ) \cdot [Y(w)] = \begin{cases} 1, & u = w, \\ 0, & u \neq w. \end{cases} \]

In other words, $s_{\text{SM}}(X(u)^\circ) \cdot c_{\text{SM}}(Y(v)^\circ)$ is effective. Since $X(u)$ and $Y(v)$ are stratified transverse in the sense of [56], one can conclude that (see also [5, Theorem 3.6])
\[ s_{\text{SM}}(X(u)^\circ) \cdot c_{\text{SM}}(Y(v)^\circ) = c_{\text{SM}}(X(u)^\circ \cap Y(v)^\circ), \]
which is exactly what we require.

Conjecture 9.1 also implies a conjecture by Mihalcea [39, Page 6] that the structure constants of SSM classes are alternately nonnegative.

**Conjecture 9.3 ([39]).** For $u, v \in W$,
\[ s_{\text{SM}}(Y(u)^\circ) \cdot s_{\text{SM}}(Y(v)^\circ) \in \sum_{w \in W} (-1)^{\ell(u) + \ell(v) - \ell(w)} \mathbb{Z}_{\geq 0} \cdot s_{\text{SM}}(Y(w)^\circ). \]

Since the Schubert expansion in (9.2) of CSM classes is nonnegative, Conjecture 9.1 implies
\[ c_{\text{SM}}(Y(u)^\circ) \cdot c_{\text{SM}}(Y(v)^\circ) \in \sum_{w \in W} \mathbb{Z}_{\geq 0} \cdot c_{\text{SM}}(Y(w)^\circ). \tag{9.3} \]

By [5, Theorem 7.5] (see also [61, Equation (2)]), after specialization to the nonequivariant case,
\[ c_{\text{SM}}(Y(u)^\circ) = (-1)^{\ell(u)} s_{\text{SM}}(Y(u)^\circ), \]
where $\mathcal{C} = \sum (-1)^i c_i$ for $c = \sum c_i$ with $c_i \in H^2(G/B)$. Applying the involution $\mathcal{C}$ to both sides of (9.3), we get the assertion in Conjecture 9.3.

Notice that a formula for the coefficients in Conjecture 9.3 was derived in [61], but is not manifestly nonnegative.

Just like the proof of [25, Theorem 3], using [56, Theorem 1.2], Conjecture 9.3 is equivalent to the following geometric interpretation about intersections of Schubert varieties [27, Conjecture D].
Conjecture 9.4 ([27] Conj. D) $\Leftrightarrow$ Conj. [23]. For generic $g, g', g'' \in G$, the Euler characteristic
\[ \chi_c\left( gY(u) \cap g'Y(v) \cap g''Y(w) \right) \in (-1)^\rho \cdot \mathbb{Z}_{\geq 0} \]
where $\rho = \dim(gY(u) \cap g'Y(v) \cap g''Y(w))$.

The Grassmannian (more generally, partial flag varieties of step $\leq 3$) analogue of the above conjecture was confirmed by Knutson and Zinn-Justin [25].

In the case of type A (i.e., the case of $F\ell(n)$), we are able to use the Pieri formula in Theorem 5.1 to prove a weaker form of Kumar’s Conjecture 9.2. Write $\delta = (n-1, n-2, \ldots, 1, 0)$. Since $F\ell(n)$ can be constructed as an iterated projective bundle, we have
\[ H^\bullet(F\ell(n); \mathbb{Q}) = \bigoplus_{a \leq \delta} \mathbb{Q} \cdot x^a, \]
where $a = (a_1, \ldots, a_n) \leq \rho$ means $0 \leq a_i \leq n - i$. This also follows from the following property of Schubert polynomials [35]:
\[ [Y(w)] = \mathcal{S}_w(x) \in \sum_{a \leq \delta} \mathbb{Z}_{\geq 0} \cdot x^a. \]

We say that a class $\alpha \in H^\bullet(F\ell(n))$ is monomial-positive if the coefficient of $x^a$ in $\alpha$ is nonnegative for any $a \leq \delta$. Clearly, an effective class is monomial-positive.

Theorem 9.5. For $u, v \in \mathcal{S}_n$, the CSM class $c_{SM}(X(u) \cap Y(v))$ of the Richardson cell $X(u) \cap Y(v)$ is monomial-positive.

Proof. Write
\[ \mathcal{S}_w(x) = \sum_{a \leq \delta} K_{w,a} x^a. \]

As noticed by Postnikov and Stanley [49 §17],
\[ e_{w_0(s-a)}(x) = \sum_{w \in \mathcal{S}_n} K_{w,a} \mathcal{S}_{w_0w}(x), \]

where
\[ e_a(x) = e_{a_2}(x_{[1]})e_{a_3}(x_{[2]}) \cdots e_{a_n}(x_{[n-1]}). \]

Let $k^a_{u,v}$ be the coefficient of $x^a$ in $c_{SM}(X(u) \cap Y(v))$. Then we have
\begin{align*}
 k^a_{u,v} &= \sum_{w} K_{w,a} \int_{F\ell(n)} c_{SM}(X(u) \cap Y(v)) \cdot [Y(w_0w)] \\
 &= \sum_{w} K_{w,a} \int_{F\ell(n)} c_{SM}(Y(v)) \cdot [Y(w_0w)] \cdot s_{SM}(X(u)).
\end{align*}

(9.4)

We need an involution $\Theta$ over $F\ell(n)$, which sends a flag
\[ 0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = \mathbb{C}^n \]
to its annihilator
\[ 0 = V_n^\perp \subseteq V_{n-1}^\perp \subseteq \cdots \subseteq V_0^\perp = \mathbb{C}^n \]
under any identification $(\mathbb{C}^n)^* = \mathbb{C}^n$. By definition, we have
\[ \Theta(Y(w)) = Y(w_0wv_0)^*, \quad \Theta(X(w)) = X(w_0wv_0)^*. \]
Thus $\Theta$ sends the Schubert class (resp., CSM class) of $w$ to the Schubert class (resp., CSM class) of $w_0 w w_0$. Applying the involution $\Theta$ to (9.4), we obtain that

$$k_{u,v}^a = \sum_w K_{w,a} \int_{F(\ell(n))} c_{SM}(Y(w_0 v w_0)) \cdot [Y(w_0)] \cdot s_{SM}(X(w_0 u w_0))$$

$$= \int_{F(\ell(n))} c_{SM}(Y(w_0 v w_0)) \cdot \sum_w K_{w,a} S_{w_0}(x) \cdot s_{SM}(X(w_0 u w_0))$$

This implies that $k_{u,v}^a$ is the coefficient of $c_{SM}(X(w_0 u w_0))$ in

$$c_{SM}(Y(w_0 v w_0)) \cdot e_{w_0(\delta-a)}(x).$$

By repeatedly using Theorem 5.1, we see that this coefficient is nonnegative. $\Box$

It is worth mentioning that in the case of type A, Conjecture 9.1 can be implied by a conjecture of Fomin and Kirillov. The Fomin–Kirillov algebra $E_n$ is generated by $x_{ab}$ for $1 \leq a < b \leq n$, subject to the following relations

- $x_{ab}^2 = 0$,
- $x_{ab} x_{bc} = x_{ac} x_{ab} + x_{bc} x_{ac}$,
- $x_{bc} x_{ab} = x_{ab} x_{ac} + x_{ac} x_{bc}$,
- $x_{ab} x_{cd} = x_{cd} x_{ab}$ for distinct $a, b, c, d$.

By [15, Lemma 5.1], the Dunkl elements, which are defined as

$$\theta_i = -\sum_{a<i} x_{ai} + \sum_{i<b} x_{ib},$$

are pairwisely commutative. As pointed out in [30, Remark 3.2], $E_n$ acts on $H^*(F(\ell(n)))$ (on the right) by

$$c_{SM}(Y(w)) * x_{ab} = \begin{cases} c_{SM}(Y(w_{t_{ab}})), & \ell(w_{t_{ab}}) \geq \ell(w) + 1, \\ 0, & \text{otherwise}. \end{cases} \quad (9.5)$$

By the CSM Chevalley formula [5], for $w \in S_n$ and $i \in [n]$,

$$c_{SM}(Y(w)) \cdot x_i = c_{SM}(Y(w)) * \theta_i.$$

Hence,

$$c_{SM}(Y(w)) \cdot [Y(u)] = c_{SM}(Y(w)) \cdot S_u(x_1, \ldots, x_n)$$

$$= c_{SM}(Y(w)) * S_u(\theta_1, \ldots, \theta_n),$$

where $S_u$ is the Schubert polynomial of $u$.

**Conjecture 9.6** ([15, Conj. 8.1]). For $w \in S_n$,

$${S}_w(\theta_1, \ldots, \theta_n) \in E_n^+,$$

where $E_n^+$ is the cone of all nonnegative integer linear combinations of (noncommutative) monomials in the generators $x_{ab}$ for $a < b$.
In view of (9.5), the above conjecture implies that the CSM expansion of $c_{SM}(Y(w)\circ) \cdot [Y(u)]$ is nonnegative, as stated in Conjecture 9.1.

Lastly, we investigate the equivariant setting of Conjecture 9.1. Let $T$ be the maximal torus of the Borel subgroup $B$. Denote by $\{\alpha_i\}_{i \in I}$ the set of simple roots. Let $\alpha \in H^*_T(pt)$ be the first Chern class of $C_{\alpha}$. For example, in type $A$, $\alpha_i = -t_i + t_{i+1}$ due to our convention in Subsection 3.2. See also the remarks before Theorem 6.4 in Chapter 10 of [1] for the convention about signs.

Let $Z_{\geq 0}[\alpha_i]_{i \in I}$ be the set of polynomials in $\alpha_i$’s with nonnegative integer coefficients. The equivariant analogue of Conjecture 9.1 can be stated as follows.

**Conjecture 9.7.** For $u, v \in W$,

$$c_{SM}^T(Y(u)\circ) \cdot [Y(v)]_T \in \sum_{w \in W} Z_{\geq 0}[\alpha_i]_{i \in I} \cdot c_{SM}^T(Y(w)\circ).$$

Theorem 5.16 confirms this conjecture in the case of type $A$ when $v$ is a Grassmannian permutation corresponding to a one row/column partition. Actually, thanks to Billey’s formula [10], the coefficients are localizations of Schubert classes which automatically lie in $Z_{\geq 0}[\alpha_i]_{i \in I}$.

Notice also that Conjecture 9.7 generalizes Graham’s positivity theorem [19, Corollary 4.1]:

$$[Y(u)]_T \cdot [Y(v)]_T \in \sum_{w \in W} Z_{\geq 0}[\alpha_i]_{i \in I} \cdot [Y(w)]_T.$$

Along the same line as the nonequivariant case, Conjecture 9.7 has an equivalent geometric description for the CSM classes of the Richardson cells.

**Conjecture 9.8 ($\Leftrightarrow$ Conj. 9.7).** For $u, v \in W$,

$$c_{SM}^T(X(u)\circ \cap Y(v)\circ) \in \sum_{w \in W} Z_{\geq 0}[\alpha_i]_{i \in I} \cdot [X(w)]_T.$$

Conjecture 9.8 implies the following conjecture of Aluffi and Mihalcea [4, Conj. 2]:

$$c_{SM}^T(X(u)\circ) \in \sum_w Z_{\geq 0}[\alpha_i]_{i \in I} \cdot [X(v)]_T.$$
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