Degenerate operators on the half-line

G. Metafune, L. Negro and C. Spina

Abstract. We study elliptic and parabolic problems governed by the singular elliptic operators

\[ y^\alpha \left( D_{yy} + \frac{c}{y} D_y \right) - V(y), \quad \alpha \in \mathbb{R} \]

in \( \mathbb{R}_+ \), where \( V \) is a potential having nonnegative real part.

1. Introduction

In this paper, we study solvability and regularity of elliptic and parabolic problems associated with the degenerate operators

\[ L = y^\alpha \left( D_{yy} + \frac{c}{y} D_y \right) - V \quad \text{and} \quad D_t - L \]

in the half-line \( \mathbb{R}_+ \).

Here, \( c, \alpha \) are real numbers and \( V \in L^1_{\text{loc}} \left( \mathbb{R}_+, y^{c-\alpha} \right) \) is a potential having nonnegative real part. The operator \( B = D_{yy} + \frac{c}{y} D_y \) is a Bessel operator and satisfies the scaling property

\[ I^{-1}_s B I_s = s^2 B, \quad I_s u(y) = u(sy). \]

We study \( L \) in the weighted spaces \( L^p_m := L^p \left( \mathbb{R}_+, y^m dy \right) \), \( m \in \mathbb{R} \), and we characterize all \( m \) such that \( L \) generates a \( C_0 \)-semigroup. When \( V \geq 0 \), we also prove that the generated semigroup is analytic and we show that it has maximal regularity, which means that both \( D_t v \) and \( L v \) have the same regularity as \( (D_t - L) v \). In the case \( V(y) = y^\alpha \), we finally characterize the domain of \( L \).

We observe that the results already available for \( B \), see [13, Section 3] and also [8–11,15] for the \( N \)-d version of \( B \), imply the corresponding ones for \( y^\alpha B \) in \( L^p_m \) by a change of variables, as described in Sect. 3. The change of variables varies the underlying measure and explains why we need the full scale of \( L^p_m \) spaces.

Mathematics Subject Classification: 35K67, 35B45, 47D07, 35J70, 35J75

Keywords: Degenerate elliptic operators, Boundary degeneracy, Vector-valued harmonic analysis, Maximal regularity.
More effort is needed to add the potential term. We consider first $B - V$ in $L^2(\mathbb{R}_+; y^c \mathrm{d}y)$. We use form methods to construct an analytic semigroup, and we prove kernel bounds for complex times via Davies–Gaffney estimates and provide a core. Then, with the methods of Sect. 3, we deduce similar results for $y^\alpha B - V$ in $L^2(\mathbb{R}_+; y^{c-\alpha} \mathrm{d}y)$. Next we prove that the semigroup can be extended to $L_m^p$ under sharp conditions on $p$ and $m$. Finally, we prove that for every $\epsilon > 0$ the family of operators \[
abla \in \mathbb{R}_+^{\frac{1}{2} - \epsilon}, 0 \leq V \in L^1_{\text{loc}}(\mathbb{R}_+, y^{c-\alpha}) \]
is $\mathcal{R}$-bounded in $L_m^p$, which implies the maximal regularity of the semigroup when $V \geq 0$.

As a motivation for our investigation, we point out that, in the special case $V(y) = y^\alpha$, all the results above play a crucial role in \[14\] in the investigation of the degenerate operators \[
abla = y^\alpha \Delta_x + \frac{c}{y} D_y - \frac{b}{y^2}. \]

Let us suppose, for simplicity, $b = 0$, $\alpha_1 = \alpha_2 := \alpha$. Assuming that $y^\alpha(\Delta_x u + B_y u) = f$ and taking the Fourier transform $\mathcal{F} u$ or $\hat{u}$ with respect to $x$, we obtain $y^\alpha |\xi|^2 \hat{u}(\xi, y) = -y^\alpha |\xi|^2 (y^\alpha |\xi|^2 - y^\alpha B_y)^{-1} \hat{f}(\xi, y)$. Therefore, \[
abla y^\alpha \Delta_x \mathcal{L}^{-1} = \mathcal{F}^{-1} \left( y^\alpha |\xi|^2 (y^\alpha |\xi|^2 - y^\alpha B_y)^{-1} \right) \mathcal{F} \]
and the boundedness of $y^\alpha \Delta_x \mathcal{L}^{-1}$ is equivalent to that of the multiplier \[
\xi \in \mathbb{R}^N \rightarrow y^\alpha |\xi|^2 (y^\alpha |\xi|^2 - y^\alpha B_y)^{-1}. \]

For this reason, we prove in Sect. 8 that certain multipliers associated with $y^\alpha B - V$ satisfy a vector-valued Mikhlin theorem. These results rely on square function estimates which we deduce from kernel bounds and the following equality, which allows to treat $\lambda$ or $|\xi|^2$ as spectral parameters simultaneously \[
\left( \lambda - y^\alpha B + |\xi|^2 y^\alpha \right)^{-1} = \left( |\xi|^2 - B + \frac{\lambda}{y^\alpha} \right)^{-1} \frac{1}{y^\alpha}.
\]

We restrict ourselves to $\alpha < 2$ and consider $y^\alpha B$ with Neumann boundary condition at 0, namely $\lim_{y \to 0} y^c D_y u(y) = 0$. This is equivalent to require $y^{\alpha - 1} D_y u \in L_m^p$, see \[12, Proposition 5.11\]. The restriction $\alpha < 2$ is not really essential since one can deduce from it the case $\alpha > 2$, which requires a boundary condition at $\infty$, using the change of variables described in Sect. 3.

Besides this, our strategy can be easily adapted to different boundary conditions and to more general operators $y^\alpha \left( D_{yy} + \frac{c}{y} D_y - \frac{b}{y^2} \right) - V$. We do this (in much more generality) in \[14, Sections 7, 8\].
The paper is organized as follows. In Sect. 2, we briefly recall the harmonic analysis background needed in the paper, such as square function estimates, $\mathcal{R}$-boundedness and a vector-valued multiplier theorem.

In Sect. 3, we exploit an elementary change of variables, in a functional analytic setting, to reduce our operators to the simpler case where $\alpha = 0$.

Section 4 is devoted to the study of the Bessel operator $y^\alpha B$. In Sects. 5, 6 and 7, we perturb the Bessel operator by adding the potential $V$ and we prove real and complex kernel estimates, generation results and maximal regularity for $y^\alpha B - V$. Finally in Sect. 8, we treat the case $V(y) = y^\alpha$ and characterize the domain of $y^\alpha B - y^\alpha$.

**Notation.** For $m \in \mathbb{R}$, we consider the measure $y^m \, d\, y$ in $\mathbb{R}_+$ and we write $L^p_m$ for $L^p(\mathbb{R}_+, y^m \, d\, y)$. Similarly, $W^{k,p}_m = \{u \in L^p_m : \partial^\alpha u \in L^p_m \mid |\alpha| \leq k\}$. When we write $V \in L^q_{loc}(\mathbb{R}_+, y^m \, d\, y)$, we mean that $V \in L^q([0, b], y^m \, d\, y)$ for every $b < \infty$.

We use $\mathbb{C}^+ = \{\lambda \in \mathbb{C} : \text{Re}\, \lambda > 0\}$, and for $|\theta| \leq \pi$, we denote by $\Sigma_{\theta}$ the open sector $\{\lambda \in \mathbb{C} : \lambda \neq 0, |\text{Arg}(\lambda)| < \theta\}$.

**Statements and declarations.** Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

2. Harmonic analysis and maximal regularity

The study of maximal regularity of parabolic problems of the form $u_t = Au + f, u(0) = 0$, where $A$ is the generator of an analytic semigroup on a Banach space $X$, consists in proving estimates like

$$ \|u_t\|_p + \|Au\|_p \leq \|f\|_p $$

where the $L^p$ norm is that of $L^p([0, T]; X)$. This can be interpreted as closedness of $D_t - A$ on the intersection of the respective domains or, equivalently, boundedness of the operator $A(D_t - A)^{-1}$ in $L^p([0, T]; X)$.

Nowadays this strategy is well established and relies on Mikhlin vector-valued multiplier theorems. Let us state the relevant definitions and main results we need, referring the reader to [5,6,17] or [7].

Let $S$ be a subset of $B(X)$, the space of all bounded linear operators on a Banach space $X$. $S$ is $\mathcal{R}$-bounded if there is a constant $C$ such that

$$ \left\| \sum_i \varepsilon_i S_i x_i \right\|_{L^p(\Omega; X)} \leq C \left\| \sum_i \varepsilon_i x_i \right\|_{L^p(\Omega; X)} $$

for every finite sum as above, where $(x_i) \subset X, (S_i) \subset S$ and $\varepsilon_i : \Omega \to \{-1, 1\}$ are independent and symmetric random variables on a probability space $\Omega$. The smallest constant $C$ for which the above definition holds is the $\mathcal{R}$-bound of $S$, denoted by $\mathcal{R}(S)$. It is well known that this definition does not depend on $1 \leq p < \infty$ (however, the constant $\mathcal{R}(S)$ does) and that $\mathcal{R}$-boundedness is equivalent to boundedness when
Let $S \subset B(L^p(\Sigma))$, $1 < p < \infty$. Then, $S$ is $\mathcal{R}$-bounded if and only if there is a constant $C > 0$ such that for every finite family $(f_i) \in L^p(\Sigma)$, $(S_i) \in S$

$$\left\| \left( \sum_i |S_i f_i|^2 \right)^{1/2} \right\|_{L^p(\Sigma)} \leq C \left\| \left( \sum_i |f_i|^2 \right)^{1/2} \right\|_{L^p(\Sigma)}.$$

The best constant $C$ for which the above square functions estimates hold satisfies $\kappa^{-1} C \leq \mathcal{R}(S) \leq \kappa C$ for a suitable $\kappa > 0$ (depending only on $p$). The proposition above $\mathcal{R}$-boundedness follows from domination.

**Corollary 2.2.** Let $S, T \subset B(L^p(\Sigma))$, $1 < p < \infty$ and assume that $T$ is $\mathcal{R}$ bounded and that for every $S \in S$ there exists $T \in T$ such that $|Sf| \leq |Tf|$ pointwise, for every $f \in L^p(\Sigma)$. Then, $S$ is $\mathcal{R}$-bounded.

Let $(A, D(A))$ be a densely defined, sectorial operator in a Banach space $X$; this means that $\rho(-A) \supset \Sigma_{\pi-\phi}$ for some $\phi < \pi$ and that $\lambda(\lambda + A)^{-1}$ is bounded in $\Sigma_{\pi-\phi}$. The infimum of all such $\phi$ is called the spectral angle of $A$ and denoted by $\phi_A$. Note that $-A$ generates a strongly continuous analytic semigroup if and only if $\phi_A < \pi/2$. The definition of $\mathcal{R}$-sectorial operator is similar, substituting boundedness of $\lambda(\lambda + A)^{-1}$ with $\mathcal{R}$-boundedness in $\Sigma_{\pi-\phi}$. As above one denotes by $\phi_R^A$ the infimum of all $\phi$ for which this happens; since $\mathcal{R}$-boundedness implies boundedness, we have $\phi_A \leq \phi_R^A$.

The $\mathcal{R}$-boundedness of the resolvent characterizes the regularity of the associated inhomogeneous parabolic problem, as we explain now.

An analytic semigroup $(e^{-tA})_{t \geq 0}$ on a Banach space $X$ with generator $-A$ has maximal regularity of type $L^q (1 < q < \infty)$ if for each $f \in L^q([0, T]; X)$ the function $t \mapsto u(t) = \int_0^t e^{-(t-s)A} f(s) \, ds$ belongs to $W^{1,q}([0, T]; X) \cap L^q([0, T]; D(A))$. This means that the mild solution of the evolution equation

$$u'(t) + Au(t) = f(t), \quad t > 0, \quad u(0) = 0,$$

is in fact a strong solution and has the best regularity one can expect. It is known that this property does not depend on $1 < q < \infty$ and $T > 0$. A characterization of maximal regularity is available in UMD Banach spaces, through the $\mathcal{R}$-boundedness of the resolvent in a suitable sector $\omega + \Sigma_{\phi}$, with $\omega \in \mathbb{R}$ and $\phi > \pi/2$ or, equivalently, of the scaled semigroup $e^{-(A+\omega)t}$ in a sector around the positive axis. In the case of $L^p$ spaces, it can be restated in the following form, see [7, Theorem 1.11]

**Theorem 2.3.** Let $(e^{-tA})_{t \geq 0}$ be a bounded analytic semigroup in $L^p(\Sigma)$, $1 < p < \infty$, with generator $-A$. Then, $T(\cdot)$ has maximal regularity of type $L^q$ if and only if the set $\{\lambda(\lambda + A)^{-1}, \lambda \in \Sigma_{\pi/2+\phi}\}$ is $\mathcal{R}$-bounded for some $\phi > 0$. In an equivalent way, if and only if there are constants $0 < \phi < \pi/2$, $C > 0$ such that for every finite sequence $(\lambda_i) \subset \Sigma_{\pi/2+\phi}$, $(f_i) \subset L^p$
proof.

or, equivalently, there are constants \(0 < \phi' < \pi/2\), \(C' > 0\) such that for every finite sequence \((z_i) \subset \Sigma_{\phi'}\), \((f_i) \subset L^p\)

\[
\left\| \left( \sum_i |e^{-z_i A} f_i|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Sigma)} \leq C' \left\| \left( \sum_i |f_i|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Sigma)}
\]

Finally, we state a version of the operator-valued Mikhlin multiplier theorem in the \(N\)-dimensional case, see [5, Theorem 3.25] or [7, Theorem 4.6].

**Theorem 2.4.** Let \(1 < p < \infty\), \(M \in C^N(\mathbb{R}^N \setminus \{0\}; B(L^p(\Sigma)))\) be such that the set

\[
\left\{ |\xi|^{|\alpha|} D_{\xi}^\alpha M(\xi) : \xi \in \mathbb{R}^N \setminus \{0\}, |\alpha| \leq N \right\}
\]

is \(\mathcal{R}\)-bounded. Then, the operator \(T_M = \mathcal{F}^{-1} M \mathcal{F}\) is bounded in \(L^p(\mathbb{R}^N, L^p(\Sigma))\), where \(\mathcal{F}\) denotes the Fourier transform.

We end this section with the following lemma on radially symmetric multipliers.

**Lemma 2.5.** Let \(1 < p < \infty\), \(m \in C^N(\mathbb{R}^+; B(L^p(\Sigma)))\) be such that the set

\[
\left\{ s^k m^{(k)}(s) : s \in \mathbb{R}^+, k \leq N \right\}
\]

is \(\mathcal{R}\)-bounded. For \(a \in \mathbb{R}\), let \(M(\xi) = m(|\xi|^a)\). Then, \(M \in C^N(\mathbb{R}^N \setminus \{0\}; B(L^p(\Sigma)))\) and

\[
\left\{ |\xi|^{|\alpha|} D_{\xi}^\alpha M(\xi) : \xi \in \mathbb{R}^N \setminus \{0\}, |\alpha| \leq N \right\}
\]

is \(\mathcal{R}\)-bounded and

\[
\mathcal{R} \left\{ |\xi|^{|\alpha|} D_{\xi}^\alpha M(\xi) : \xi \in \mathbb{R}^N \setminus \{0\}, |\alpha| \leq N \right\} \leq C(N) \mathcal{R} \left\{ s^k m^{(k)}(s) : s \in \mathbb{R}^+, k \leq N \right\}.
\]

**Proof.** Let us first observe that for any multi-index \(\alpha\) with \(0 < |\alpha| \leq N\) one has

\[
D_{\xi}^\alpha M(\xi) = \sum_{i=1}^{|\alpha|} h_{i,\alpha}(\xi)m^{(i)}(|\xi|^a)
\]

(1)

where \(h_{i,\alpha} \in C^\infty(\mathbb{R}^N \setminus \{0\})\) are homogeneous functions of degree \(ia - |\alpha|\). Obviously, (1) is valid for \(|\alpha| = 1\) since \(\nabla M(\xi) = a m'(|\xi|^a)|\xi|^{a-2}\xi\) and follows by induction, and the derivatives of \(h_{i,\alpha}\) are homogeneous of degree \(ia - |\alpha| - 1\).

The proof of the lemma now follows by Corollary (2.2) since from (1) one has for \(f \in L^p(\Sigma)\)

\[
|\xi|^{|\alpha|} D_{\xi}^\alpha M(\xi) f(\xi) \leq |\xi|^{|\alpha|} \sum_{i=1}^{|\alpha|} h_{i,\alpha}(\xi)|m^{(i)}(|\xi|^a) f(\xi)| \leq C \sum_{i=1}^{|\alpha|} |\xi|^{|\alpha|} |m^{(i)}(|\xi|^a) f(\xi)|
\]

\(\square\)
3. Degenerate operators and similarity transformations

We investigate when the operators
\[ B = D_{yy} + \frac{c}{y} D_y, \quad y^\alpha B = \frac{y^\alpha}{y} \left( D_{yy} + \frac{c}{y} D_y \right) \]
can be transformed one into the other by means of change of variables. Here, \( \alpha, c \) are unrestricted real coefficients.

For \( \beta \in \mathbb{R}, \beta \neq -1 \) let
\[ T_\beta u(y) := |\beta + 1|^\frac{1}{p} u(y^{\beta+1}), \quad y \in \mathbb{R}_+. \quad (2) \]
Observe that
\[ T^{-1}_\beta = T_{-\frac{\beta}{\beta+1}}. \]

**Proposition 3.1.** Let \( 1 \leq p \leq \infty, k, \beta \in \mathbb{R}, \beta \neq -1 \). The following properties hold.

(i) For every \( m \in \mathbb{R}, T_\beta \) maps isometrically \( L^p_m \) onto \( L^p_m \) where
\[ \tilde{m} = m - \frac{\beta}{\beta + 1}. \]

(ii) For every \( u \in W^{2,1}_{\text{loc}} (\mathbb{R}_+) \), one has
\[ 1. \ y^\alpha T_\beta u = T_\beta \left( y^{\frac{\alpha}{\beta + 1}} u \right), \text{ for any } \alpha \in \mathbb{R}; \]
\[ 2. \ D_y T_\beta u = T_\beta \left( (\beta + 1)^\frac{\beta}{\beta + 1} D_y u \right), \]
\[ D_{yy} (T_\beta u) = T_\beta \left( (\beta + 1)^2 y^{\frac{2\beta}{\beta + 1}} D_{yy} u + (\beta + 1)\beta y^{\frac{\beta-1}{\beta + 1}} D_y u \right). \]

**Proof.** The proof of (i) follows after observing the Jacobian of \( y \mapsto y^{\beta+1} \) is \( |1 + \beta| y^\beta \).

Then, we compute
\[ D_y T_\beta u(y) = \left| \beta + 1 \right|^\frac{1}{p} \left( (\beta + 1)^\frac{\beta}{\beta + 1} y^\beta D_y u(y^\beta + 1) \right) = T_\beta \left( (\beta + 1)^\frac{\beta}{\beta + 1} y^\beta D_y u \right) \]
and similarly
\[ D_{yy} T_\beta u(y) = T_\beta \left( (\beta + 1)^2 y^{\frac{2\beta}{\beta + 1}} D_{yy} u + (\beta + 1)\beta y^{\frac{\beta-1}{\beta + 1}} D_y u \right). \]

\[ \square \]

**Proposition 3.2.** Let \( T_\beta \) be the isometry above defined. The following properties hold. For every \( u \in W^{2,1}_{\text{loc}} (\mathbb{R}_+) \), one has
\[ T^{-1}_\beta \left( y^\alpha B \right) T_\beta u = \left( (\beta + 1)^2 y^{\frac{\alpha + 2\beta}{\beta + 1}} B \right) u \]
where $\tilde{B}$ is the operator defined as in (1) with parameter $c$ replaced, respectively, by

$$\tilde{c} = \frac{c + \beta (c + 1 + \beta)}{(\beta + 1)^2}.$$

**Proof.** Using Proposition 3.1, we can compute

$$BT_{\beta} u(y) = T_{\beta} \left[ (\beta + 1)^2 y^{\frac{2\beta}{p+1}} Dy u + (\beta + 1) \beta y^{\frac{\beta+1}{p+1}} D_y u + c(\beta + 1) y^{\frac{\beta-1}{p+1}} D_y u - by^{-\frac{\beta}{p+1}} u \right]$$

$$= T_{\beta} \left[ y^{\frac{2\beta}{p+1}} (\beta + 1)^2 D_{yy} u + (\beta + 1) (\beta + c) y D_y u - b u \right]$$

which implies

$$T_{\beta}^{-1} (y^{\alpha} B) T_{\beta} u = y^{\alpha + \frac{2\beta}{p+1}} \tilde{B} u.$$

4. The Bessel operator $y^{\alpha} B^\alpha$

In this section, we consider for $\alpha < 2$, $c \in \mathbb{R}$ the operator

$$y^{\alpha} B = y^{\alpha} \left( D_{yy} + \frac{c}{y} D_y \right)$$

in the space $L^p_m$ under Neumann boundary conditions.

According to Proposition 3.2, for $0 < (m + 1)/p < c + 1 - \alpha$, we use the isometry

$$T_{-\frac{\alpha}{2}} : L^p_{\tilde{m}} \rightarrow L^p_{m} \quad T_{-\frac{\alpha}{2}} u(y) = \left| 1 - \frac{\alpha}{2} \right|^{\frac{1}{\beta}} u(y^{1-\frac{\alpha}{2}}),$$

$\tilde{m} = \frac{m + \frac{\alpha}{2}}{1-\frac{\alpha}{2}}$, under which $y^{\alpha} B$ becomes isometrically equivalent to $T_{-\frac{\alpha}{2}} (y^{\alpha} B) T_{-\frac{\alpha}{2}} = (1 - \frac{\alpha}{2})^2 \tilde{B}$ where $\tilde{B} = D_{yy} + \frac{\tilde{c}}{y} D_y$, $\tilde{c} = \frac{\alpha - \frac{\alpha}{2}}{1-\frac{\alpha}{2}}$ and $0 < (\tilde{m} + 1)/p < \tilde{c} + 1$.

All the results for $y^{\alpha} B$ in $L^p_m$ are then immediate consequence of those of $\tilde{B}$ in $L^p_{\tilde{m}}$ already proved in [13, Section 3] (see also [9–11,15] for analogous results in the multi-dimensional case).

If $1 < p < \infty$, we define

$$W^{2,p}_{\mathcal{N}}(\alpha, m) = \left\{ u \in W^{2,p}_{loc}(\mathbb{R}_+) : u, \ y^{\alpha} D_{yy} u, \ y^{\frac{\alpha}{2}} D_y u, \ y^{\alpha-1} D_y u \in L^p_m \right\}$$

and refer to [12] where these spaces are studied in detail in $\mathbb{R}^{N+1}_+$. The Neumann boundary condition, denoted by the pedix $\mathcal{N}$, is enclosed in the requirement $y^{\alpha-1} D_y u \in$
4.7], that is for \( \alpha < 2 \) it is a core when \((m+1)/p > 1 - \alpha \) and equivalent to \( D_y u(y) \to 0 \) as \( y \to 0 \), when \((m+1)/p < 1 - \alpha \), see [12, Proposition 4.3].

Consequently, we write \( y^\alpha B^n \) or, more pedantically \( y^\alpha B^n_{m,p} \) if necessary, for the operator \( y^\alpha B \) endowed with the domain \( W^{2,p}_N(\alpha, m) \). This time the suffix \( n \) reminds the Neumann boundary condition at \( y = 0 \).

**Remark 4.1.** The restriction \( \alpha < 2 \) is not really essential since one can deduce from it the case \( \alpha > 2 \), which requires boundary condition at \( \infty \), using the change of variables described in Sect. 3 or directly from the equality \( T_{-\frac{\alpha}{2}}^{-1}(y^\alpha B)T_{-\frac{\alpha}{2}} = (1 - \frac{\alpha}{2})^2 \tilde{B} \) which is valid for any \( \alpha \neq 2 \). However, here and in what follows, we keep to it in order to simplify the exposition.

**Theorem 4.2.** If \( 0 < \frac{m+1}{p} < c + 1 - \alpha \), then \( y^\alpha B^n \) endowed with domain \( W^{2,p}_N(\alpha, m) \) generates a bounded positive analytic semigroup of angle \( \pi/2 \) on \( L^p(\mathbb{R}^+, y^m dy) \).

**Proof.** We use the identity \( T_{-\frac{\alpha}{2}}^{-1}(y^\alpha B^n)T_{-\frac{\alpha}{2}} = (1 - \frac{\alpha}{2})^2 \tilde{B} \) and apply [13, Proposition 3.3] in \( L^p_m \). Note that \( D(y^\alpha B^n_{m,p}) = T_{-\frac{\alpha}{2}} D(\tilde{B}^n_{m,p}) \) which means

\[
  u \in D(y^\alpha B^n_{m,p}) \iff \nu(y) := u(y \frac{2}{\alpha-2}) \in D(\tilde{B}^n_{m,p}).
\]

Under the hypothesis of Theorem 4.2, the domain of \( y^\alpha B^n \) consists of all functions in the maximal domain satisfying a Neumann condition at 0, see [12, Proposition 4.6, 4.7], that is

\[
  D(y^\alpha B^n_{m,p}) = \left\{ u \in W^{2,p}_{loc}(\mathbb{R}^+) : u, y^\alpha Bu \in L^p_m \text{ and } \lim_{y \to 0} y^c D_y u = 0 \right\}.
\]

(The condition \( \lim_{y \to 0} y^c D_y u = 0 \) can be deleted in the range \( 0 < \frac{m+1}{p} \leq c - 1 \).)

When \( c \geq 1 \), the domain can also be described involving a Dirichlet, rather than Neumann, boundary condition

\[
  D(y^\alpha B^n_{m,p}) = \left\{ u \in W^{2,p}_{loc}(\mathbb{R}^+) : u, y^\alpha Bu \in L^p_m \text{ and } \lim_{y \to 0} y^{c-1} u = 0 \right\}, \quad \text{if } c > 1;
\]

\[
  D(y^\alpha B^n_{m,p}) = \left\{ u \in W^{2,p}_{loc}(\mathbb{R}^+) : u, y^\alpha Bu \in L^p_m \text{ and } \lim_{y \to 0} u = 0 \right\}, \quad \text{if } c = 1.
\]

We close this section by describing a core which does not depend on \( \alpha, m, p \) and on the coefficients of the operator.

**Proposition 4.3.** If \( 0 < \frac{m+1}{p} < c + 1 - \alpha \), then a core for \( y^\alpha B^n \) is

\[
  \mathcal{D} = \left\{ u \in C^\infty_c((0, \infty)) : u \text{ constant in a neighborhood of } 0 \right\}.
\]

**Proof.** The proof immediately follows by observing that, by [13, Proposition 5.4], \( \mathcal{D} \) is a core when \( \alpha = 0 \), that is for \( \tilde{B}^n_{m,p} \), and the isometry \( T_{-\frac{\alpha}{2}} \) leaves invariant \( \mathcal{D} \) since \( \alpha < 2 \).
Remark 4.4. We point out that, by the proof of [13, Proposition 5.4] or by [12, Remark 4.14], it follows that if \( u \in D(y^\alpha B^n_{m,p}) \) has support in \([0, b]\), then there exists a sequence \((u_n)_{n \in \mathbb{N}} \in \mathcal{D}\) such that \( \text{supp } u_n \subseteq [0, b] \) and \( u_n \to u \) in \( D(y^\alpha B^n_{m,p}) \).

5. The operator \( B^n - V \)

We start our investigation by adding a potential \( 0 \leq V \in L^1_{loc}(\mathbb{R}^+, y^c \,dy) \) to \( B^n \). Here, we prove kernel bounds and construct a core.

5.1. Kernel bounds

For \( c + 1 > 0 \) and \( 0 \leq V \in L^1_{loc}((\mathbb{R}^+, y^c \,dy) \), we prove upper bounds for the heat kernel of \( B^n - V \), following the method used in [3, Sections 3, 4].

Setting \( H^1_c = \{ u \in L^2_c, u' \in L^2_c \} \), we recall that from [13, Section 2] the operator \( B^n_{c,2} \) is associated with the nonnegative, symmetric and closed form in \( L^2_c \)

\[
\mathbb{a}(u, v) := \int_0^\infty D_y u D_y v y^c \,dy, \quad D(\mathbb{a}) = H^1_c.
\]

We consider the perturbed form \( \mathbb{a}_V \) in \( L^2_c \) defined by

\[
\mathbb{a}_V(u, v) = \mathbb{a}(u, v) + \langle Vu, v \rangle_{L^2_c} = \int_{\mathbb{R}^+} (D_y u D_y \tilde{v} + Vu \tilde{v}) y^c \,dy
\]

\[
D(\mathbb{a}_V) = D(\mathbb{a}) \cap L^2(\mathbb{R}^+, Vy^c \,dy) \tag{3}
\]

and define \( B^n - V \) in \( L^2_c \) as the operator associated with the form \( \mathbb{a}_V \)

\[
D(B^n - V) = \{ u \in D(\mathbb{a}_V) : \exists f \in L^2_c \text{ such that } \mathbb{a}_V(u, v) = \int_0^\infty f \tilde{v} y^c \,dy \text{ for every } v \in D(\mathbb{a}_V) \},
\]

\[
B^n u - Vu = -f.
\]

The positivity of \( V \) implies that the norm induced by the form \( \mathbb{a}_V \) is stronger than the one induced by \( \mathbb{a} \): As an immediate consequence, one deduces that \( \mathbb{a}_V \) is closed. By standard theory on sesquilinear forms, we have the following result.

**Proposition 5.1.** If \( c + 1 > 0 \), \( 0 \leq V \in L^1_{loc}((\mathbb{R}^+, y^c \,dy) \), then \( \mathbb{a}_V \) is a nonnegative, symmetric and closed form in \( L^2_c \). Its associated operator \(-B^n + V\) is nonnegative and self-adjoint, and \( B^n - V \) generates a contractive analytic semigroup \( \{ e^{zt(B^n - V)} : z \in \mathbb{C}_+ \} \) in \( L^2_c \). Moreover:

(i) The semigroup \( \{ e^{t(B^n - V)} \}_{t \geq 0} \) is sub-Markovian (i.e., it is positive and \( L^\infty \)-contractive), and it is dominated by \( e^{tB^n} \), that is

\[
|e^{t(B^n - V)} f| \leq e^{tB^n} |f|, \quad t > 0, \quad f \in L^2_c.
\]
(ii) \( ( e^{t(B^n - V)} )_{t \geq 0} \) is a semigroup of integral operators, and its heat kernel \( p_V \), taken with respect to the measure \( \rho^c \mathrm{d}\rho \), satisfies

\[
0 \leq p_V (t, y, \rho) \leq C t^{-\frac{1}{2}} \rho^{-c} \left( \frac{\rho}{t} \wedge 1 \right)^c \exp \left( -\frac{|y - \rho|^2}{\kappa t} \right).
\]

**Proof.** The first claim follows from the property of \( a_V \). \( e^{t(B^n - V)} \) is sub-Markovian from [16, Corollary 2.17]. The domination property follows from [16, Corollary 2.21]. (ii) is a consequence of [2, Proposition 1.9] since \( e^{t(B^n - V)} \) is dominated by the positive integral operator \( e^{tB^n} \) whose kernel satisfies the stated estimate, see [13, Proposition 2.8], where, however, the kernel is written with respect to the Lebesgue measure. \( \square \)

To extend the above heat kernel estimates to the half-plane \( \mathbb{C}_+ \), we need the following lemma.

**Lemma 5.2.** Let \( c + 1 > 0 \) and for \( y_0, r > 0 \)

\[
Q_c(y_0, r) := \int_{[y_0, y_0 + r]} y^c \mathrm{d}y.
\]

Then one has

\[
Q_c(y_0, r) \simeq r^{c+1} \left( \frac{y_0}{r} \right)^c \left( \frac{y_0}{r} \wedge 1 \right)^{-c}, \quad r, y_0 > 0.
\]

In particular, the function \( Q_c \) satisfies, for some constants \( C \geq 1 \), the doubling condition

\[
\frac{Q_c(y_0, s)}{Q_c(y_0, r)} \leq C \left( \frac{s}{r} \right)^{1/(c+1)}, \quad \forall y_0 > 0, \ 0 < r < s.
\]

**Proof.** A scaling argument immediately yields \( Q_c(y_0, r) = r^{c+1} Q_c \left( \frac{y_0}{r}, 1 \right) \), and we may therefore assume \( r = 1 \). The local integrability of \( y^c \) implies that \( Q_c(y_0, 1) \) is continuous as a function of \( y_0 \) and moreover \( Q_c(y_0, 1) \to \int_{(0, 1)} y^c \mathrm{d}y > 0 \) as \( y_0 \to 0 \). Therefore, if \( y_0 \leq 1 \), then

\[
Q_c(y_0, 1) \simeq 1.
\]

On the other hand, if \( y_0 > 1 \), then \( y \simeq y_0 \) for any \( y \in (y_0, y_0 + 1) \) which implies

\[
Q_c(y_0, 1) = \int_{(y_0, y_0 + 1)} y^c \mathrm{d}y \simeq y_0^c.
\]

The last two inequalities yield \( Q_c(y_0, 1) \simeq (y_0)^c \left( y_0 \wedge 1 \right)^{-c} \). The doubling condition follows from the previous estimates and the fact that for \( 0 < r < s \) one has

\[
\frac{Q_c(y_0, s)}{Q_c(y_0, r)} \leq C \begin{cases} \left( \frac{s}{r} \right)^{c+1}, & \text{if } \frac{y_0}{r} \leq \frac{s}{r} \leq 1; \\ \left( \frac{s}{y_0} \right)^c, & \text{if } \frac{y_0}{r} \leq 1 < \frac{y_0}{s}; \\ \frac{s}{r}, & \text{if } 1 \leq \frac{y_0}{s} \leq \frac{y_0}{r}. \end{cases}
\]

(Note that in the range \( \frac{y_0}{s} \leq 1 < \frac{y_0}{r} \) one has \( \left( \frac{s}{y_0} \right)^c \leq 1 \) if \( c < 0 \) and \( \left( \frac{s}{y_0} \right)^c \leq \left( \frac{s}{r} \right)^c \) if \( c \geq 0 \). \( \square \)
Proposition 5.3. Let \( c + 1 > 0, \ 0 \leq V \in L_{\text{loc}}^1(\mathbb{R}^+, y^c \, dy) \). The semigroup \( \{ e^{z(B^n-x)} : z \in \mathbb{C}_+ \} \) consists of integral operators

\[
e^{z(B^n-x)} f(y) = \int_0^\infty p_V(z, y, \rho) f(\rho) \rho^{c} d\rho, \quad f \in L_{c}^2, \quad y > 0.
\]

Furthermore for every \( \epsilon > 0 \), there exist \( k_\epsilon, C_\epsilon > 0 \) such that, for every \( z \in \Sigma_{\frac{\epsilon}{2}} \) and \( y, \rho > 0 \),

\[
|p_V(z, x, y)| \leq C_\epsilon |z|^{\frac{1}{2}} \rho^{-c} \left( \frac{\rho}{|z|^{\frac{1}{2}}} \wedge 1 \right)^c \exp \left( -\frac{|y - \rho|^2}{\kappa \epsilon |z|} \right).
\]

**Proof.** Using the previous lemma, we rewrite Proposition 5.1 (ii) as

\[ 0 \leq p_V(t, y, \rho) \leq C \frac{1}{Q_c(\rho, \sqrt{t})} \exp \left( -\frac{|y - \rho|^2}{\kappa t} \right). \]

Furthermore by [4, Theorem 3.3], \( e^{t(B^n-x)} \) satisfies the Davies–Gaffney estimates

\[
|\langle e^{t(B^n-x)} f_1, f_2 \rangle| \leq \exp \left( -\frac{r^2}{4t} \right) \| f_1 \|_{L_{c}^2} \| f_2 \|_{L_{c}^2}
\]

for all \( t > 0, U_1, U_2 \) open subsets of \((0, +\infty), r := d(U_1, U_2) = \min \{|x - y| : x \in U_1, y \in U_2\} \) and \( f_i \in L_{c}^2(U_i, y^c \, dy) \). By [4, Corollary 4.4] and Lemma 5.2, we then obtain for \( z \in \Sigma_{\frac{\epsilon}{2}} \) and \( y, \rho > 0 \)

\[
|p_V(z, x, y)| \leq C_\epsilon \frac{1}{Q_c(y, \sqrt{|z|})^{\frac{1}{2}} (Q_c(\rho, \sqrt{|z|})^{\frac{1}{2}})^c} \exp \left( -\frac{|y - \rho|^2}{\kappa \epsilon |z|} \right)
\]

\[
\leq C_\epsilon' |z|^{-\frac{c+1}{2}} \left( \frac{y}{\sqrt{|z|}} \right)^{-\frac{\epsilon}{2}} \left( 1 \wedge \frac{\rho}{\sqrt{|z|}} \right)^\frac{\epsilon}{2} \exp \left( -\frac{|y - \rho|^2}{\kappa \epsilon |z|} \right).
\]

This is an equivalent form (after modifying the constant in the exponential) of the estimate in the statement, by [13, Lemma 10.2] with \( \gamma_1 = \gamma_2 = -\frac{c}{2} \).

**Remark 5.4.** We remark that in [4], the authors work in an abstract metric measure space \((M, d, \mu)\) and assume that the heat kernel \( p \) associated with a semigroup \( e^{-zL} \), where \( L \) is a nonnegative self-adjoint operator on \( L^2(M, d\mu) \), is continuous with respect to the space variables. In such a case, in fact,

\[
\sup_{x \in U_1, y \in U_2} |p(z, x, y)| = \sup_M \int e^{-zL} f_1 f_2 d\mu, \quad \| f_1 \|_{L^1(U_1, d\mu)} = \| f_2 \|_{L^1(U_2, d\mu)} = 1.
\]
In our setting, the continuity assumption on \( p \) can be avoided since the proofs of [4, Theorem 4.1, Corollary 4.4] hold only assuming that for a.e. \( x, y \in M \)

\[
p(z, x, y) = \lim_{s \to 0} \int_M e^{-zL} f_1 \overline{f_2} \, d\mu = \lim_{s \to 0} \frac{1}{\mu(B(x, s)\mu(B(y, s))} \int_{B(x, s) \times B(y, s)} p(z, \bar{x}, \bar{y}) \, d\mu(\bar{x}) \, d\mu(\bar{y}),
\]

where \( f_1 = \frac{\chi_{B(x, s)}}{\mu(B(x, s))}, f_2 = \frac{\chi_{B(y, s)}}{\mu(B(y, s))} \). This holds, outside a set of zero measure, when the measure \( \mu \) is doubling, by the Lebesgue differentiation theorem.

5.2. A core for \( B^n - V \)

We prove that under mild hypotheses the set

\[ D = \{ u \in C^\infty_c((0, \infty)) : u \text{ constant in a neighborhood of } 0 \} \]

is a core for \( B^n - V \) in \( L^2_c \). Note that this is true when \( V = 0 \), by Proposition 4.3.

We need some elementary lemmas. Unless explicitly stated, we only assume that \( 0 \leq V \in L^1_{loc}(\mathbb{R}^+, y^c \, dy) \).

**Lemma 5.5.** Assume that \( 0 \leq V \in L^2_{loc}(\mathbb{R}^+, y^c \, dy) \). Then, \( D(a_V) = H^1_c \cap L^2(\mathbb{R}^+, Vy^c \, dy) \) is dense in \( H^1_c \).

**Proof.** By Proposition 4.3, \( D \) is dense in \( D(B^n) \) with respect to the graph norm. Moreover, since \( V \in L^2_c \) locally, \( D \subset D(a_V) \). The claim follows from the density of \( D(B^n) \) in \( H^1_c \).

**Lemma 5.6.** Let \( u \in H^1_c \) such that \( V u \in L^2_c \). Then, \( u \in D(B^n) \) if and only if \( u \in D(B^n - V) \). Moreover,

\[
(B^n - V)u = Bu - Vu.
\]

**Proof.** Let \( u \in D(B^n) \). Then, \( u \in D(a) \) and there exists \( f \in L^2_c \) such that

\[
a(u, v) = \int_0^\infty D_y u D_y \overline{v} y^c \, dy = \int_0^\infty f \overline{v} y^c \, dy
\]

for every \( v \in H^1_c \). Setting \( g = f + Vu \in L^2_c \), we have

\[
a_V(u, v) = \int_0^\infty (D_y u D_y \overline{v} + Vu \overline{v}) y^c \, dy = \int_0^\infty (f + Vu) \overline{v} y^c \, dy
\]

for every \( v \in H^1_c \) and, in particular, for every \( v \in D(a_V) \subseteq H^1_c \). Therefore \( u \in D(B^n - V) \). Conversely, if \( u \in D(B^n - V) \), then \( u \in D(a_V) \) and there exists \( g \in L^2_c \) such that

\[
a_V(u, v) = \int_0^\infty (D_y u D_y \overline{v} + Vu \overline{v}) y^c \, dy = \int_0^\infty g \overline{v} y^c \, dy
\]
for every \( v \in D(a_V) \). Setting \( f = g - Vu \in L^2_c \), we have that

\[
a(u, v) = \int_0^\infty f \overline{v} y^c \, dy
\]

for every \( v \in D(a_V) \), hence for every \( v \in H^1_c \), by Lemma 5.5.

**Lemma 5.7.** Let \( u \in D(B^n - V) \) and \( \eta \) be a smooth function such that \( \eta = 1 \) for \( 0 \leq y \leq 1 \) and \( \eta = 0 \) for \( y \geq 2 \). Then, \( \eta u \in D(B^n - V) \) and

\[
(B^n - V)(\eta u) = \eta(B^n - V)u + 2D_y \eta D_y u + uD_{yy} \eta + cu \frac{D_y \eta}{y}.
\]

**Proof.** Let \( u \in D(B^n - V) \), then \( \eta u \in D(a_V) \) and, setting \( f = (B^n - V)u \),

\[
a_V(\eta u, v) = \int_0^\infty (D_y (\eta u)D_y \overline{v} + V \eta u \overline{v}) y^c \, dy
\]

\[
= \int_0^\infty (D_y u D_y (\eta \overline{v}) + V u \eta \overline{v} + uD_y \eta D_y \overline{v} - D_y u \eta \overline{v}) y^c \, dy
\]

\[
= -\int_0^\infty \eta f \overline{v} y^c \, dy - \int_0^\infty D_y u D_y \eta \overline{v} y^c \, dy + \int_0^\infty uD_y \eta D_y \overline{v} y^c \, dy
\]

\[
= -\int_0^\infty \eta f \overline{v} y^c \, dy - \int_0^\infty D_y u D_y \eta \overline{v} y^c \, dy - \int_0^\infty \overline{v} D_y (u D_y \eta \overline{v}) \, dy
\]

\[
= -\int_0^\infty \eta f \overline{v} y^c \, dy - 2\int_0^\infty D_y u D_y \eta \overline{v} y^c \, dy
\]

\[
- \int_0^\infty \overline{v} u D_{yy} \eta y^c \, dy - \int_0^\infty \frac{cu}{y} \overline{v} D_y \eta y^c \, dy
\]

for every \( v \in D(a_V) \).

**Lemma 5.8.** Let \( u \in D(B^n - V) \). Then, there exists \( (u_k) \subseteq D(B^n - V) \) with compact support such that \( (u_k) \to u \) in \( D(B^n - V) \).

**Proof.** Let \( \eta \) be a smooth function such that \( \eta = 1 \) for \( 0 \leq y \leq 1 \) and \( \eta = 0 \) for \( y \geq 2 \). Setting \( \eta_k(y) = \eta \left(\frac{y}{k}\right) \), by Lemma 5.7, \( u_k = \eta_k u \in D(B^n - V) \) and

\[
(B^n - V)(\eta_k u) = \eta_k(B^n - V)u + 2D_y \eta_k D_y u + uD_{yy} \eta_k + \frac{cu}{y} D_y \eta_k.
\]

Then, \( u_k \to u \), \( \eta_k(B^n - V)u \to (B^n - V)u \) in \( L^2_c \) by dominated convergence and, since \( D_y \eta_k = 0 \) in \([0, 1]\),

\[
\left| D_y \eta_k D_y u + u D_{yy} \eta_k + \frac{cu}{y} D_y \eta_k \right| \leq C \left( \frac{|u|}{k} + \frac{|u|}{k^2} + \frac{|D_y u|}{k} \right) \chi_{[k, \infty]} \to 0.
\]
Lemma 5.7 shows that functions with compact support are a core for $B^n - V$. To show that $D$ is a core, we need more information on the behavior near $y = 0$ of functions in the domain of $B^n - V$.

We start by recalling some well-known facts about the modified Bessel functions $I_\nu$ and $K_\nu$ which constitute a basis of solutions of the modified Bessel equation

$$z^2 \frac{d^2 v}{dz^2} + z \frac{dv}{dz} - (z^2 + \nu^2)v = 0, \quad Re z > 0.$$ 

We recall that for $Re z > 0$ one has

$$I_\nu(z) = \sum_{m=0}^{\infty} \frac{1}{m! \Gamma(\nu + 1 + m)} \left( \frac{z}{2} \right)^{2m}, \quad K_\nu(z) = \frac{\pi}{2} I_{-\nu}(z) - I_\nu(z) \frac{\sin \pi \nu}{\nu},$$

where limiting values are taken for the definition of $K_\nu$ when $\nu$ is an integer. The basic properties of these functions we need are collected in the following lemma, see, e.g., [1, Sections 9.6 and 9.7].

**Lemma 5.9.** For $\nu > -1$, $I_\nu$ is increasing and $K_\nu$ is decreasing (when restricted to the positive real half-line). Moreover, they satisfy the following properties if $z \in \Sigma_{\pi/2 - \epsilon}$.

(i) $I_\nu(z) \neq 0$ for every $Re z > 0$.

(ii) $I_\nu(z) \approx \frac{1}{\Gamma(\nu + 1)} \left( \frac{z}{2} \right)^\nu$, as $|z| \to 0$, $I_\nu(z) \approx \frac{\epsilon^{\nu}}{\sqrt{\pi z}} (1 + O(|z|^{-1}))$, as $|z| \to \infty$.

(iii) If $\nu \neq 0$, $K_\nu(z) \approx \frac{\nu}{|\nu|} \frac{1}{2} \Gamma(|\nu|) \left( \frac{z}{2} \right)^{-|\nu|}$, $K_0(z) \approx -\log z$, as $|z| \to 0$.

(iv) $I'_\nu(z) = I_{\nu+1}(z) + \frac{\nu}{z} I_\nu(z)$, $K'_\nu(z) = -K_{\nu+1}(z) + \frac{\nu}{z} K_\nu(z)$, for every $Re z > 0$.

Note that

$$|I_\nu(z)| \approx C_{\nu,\epsilon} (1 \wedge |z|)^{\nu+1} e^{Re z} \sqrt{|z|}, \quad z \in \Sigma_{\pi/2 - \epsilon}$$

for suitable constants $C_{\nu,\epsilon} > 0$ which may be different in lower an in the upper estimate.

The following estimates of the resolvent operator of $B^n - V$ are a consequence of the domination property stated in Proposition 5.1.

**Proposition 5.10.** Let $c + 1 > 0$ and $\lambda > 0$. Then, for every $f \in L^2_c$,

$$(\lambda - B^n + V)^{-1} f = \int_0^{\infty} G(\lambda, y, \rho) f(\rho) \rho^c d\rho$$

with

$$0 \leq G(\lambda, y, \rho) \leq G_n(\lambda, y, \rho)$$
where
\[ G^n(\lambda, y, \rho) := \begin{cases} 
  y^{\frac{1-c}{2c}} \rho^{\frac{1-c}{2c}} I_{\frac{|1-c|}{2}}(\sqrt{\lambda} y) K_{\frac{|1-c|}{2}}(\sqrt{\lambda} \rho) & y \leq \rho \\
  [1.5ex] y^{\frac{1-c}{2c}} \rho^{\frac{1-c}{2c}} I_{\frac{|1-c|}{2}}(\sqrt{\lambda} \rho) K_{\frac{|1-c|}{2}}(\sqrt{\lambda} y) & y \geq \rho,
\end{cases} \] (5)
is the integral kernel (taken with respect to the measure \( \rho^c d\rho \)) of the operator \((\lambda - B^n)^{-1}\).

**Proof.** Writing \((\lambda - B^n + V)^{-1} = \int_0^\infty e^{-\lambda t} e^{t(B^n - V)} dt\) and using property (i) of Proposition 5.1, we get that
\[ |(\lambda - B^n + V)^{-1} f| \leq (\lambda - B^n)^{-1} |f|, \quad \lambda > 0, \quad f \in L^2_c. \]
This yields the domination \( G(\lambda, y, \rho) \leq G^n(\lambda, y, \rho) \). (The existence of the kernel follows by [2, Proposition 1.9] as in Proposition 5.1.) Formula (5) is proved in [13, Proposition 2.4]. □

We now prove local pointwise estimates for functions in the domain of \( B^n - V \).

**Proposition 5.11.** Let \( c + 1 > 0 \). Then, there exists \( C > 0 \), independent of \( V \), such that for every \( u \in D(B^n - V) \) and \( 0 < y < 1 \)

(i) if \( -1 < c < 3 \)
\[ |u(y)| \leq C \left( \|u\|_{L^2_c} + \|(B - V)u\|_{L^2_c} \right). \]

(ii) if \( c = 3 \)
\[ |u(y)| \leq C \left( \|u\|_{L^2_c} + \|(B - V)u\|_{L^2_c} \right) |\log y|^{\frac{1}{2}}, \]

(iii) if \( c > 3 \)
\[ |u(y)| \leq C \left( \|u\|_{L^2_c} + \|(B - V)u\|_{L^2_c} \right) y^{\frac{3-c}{2}}. \]

**Proof.** Let \( u \in D(B^n - V) \) and \( f = u - (B^n - V)u \in L^2_c \) so that \( u = (I - B^n + V)^{-1} f \).
Let us distinguish between the following cases and always take \( 0 < y < 1 \).

(i) If \( -1 < c < 1 \), Lemma 5.9 implies that for \( y \leq 1 \)
\[ G(1, y, \rho) \approx \begin{cases} 
  1, & \rho < 1, \\
  \rho^{-\frac{c}{2}} e^{-\rho}, & 1 < \rho.
\end{cases} \]
Then, one has
\[ |u(y)| \leq \int_0^\infty G(1, y, \rho) |f(\rho)| \rho^c d\rho \leq C \left( \int_0^1 |f(\rho)| \rho^c d\rho + \int_1^\infty \rho^{-\frac{c}{2}} e^{-\rho} |f(\rho)| \rho^c d\rho \right) \leq C \left( \|f\|_{L^2_c(0, 1)} + \|\rho^{-\frac{c}{2}} e^{-\rho} \|_{L^2_c((1, \infty))} \|f\|_{L^2_c((1, \infty))} \right) \leq C \|f\|_{L^2_c}. \]
(ii) If $c = 1$, Lemma 5.9 gives for $y \leq 1$

\[
G(1, y, \rho) \simeq \begin{cases} 
|\log y| \leq |\log \rho|, & \rho < y < 1, \\
|\log \rho|, & y < \rho < 1, \\
\rho^{-\frac{1}{2}}e^{-\rho}, & 1 < \rho.
\end{cases}
\]

Then, analogously

\[
|u(y)| \leq C \left( \int_0^1 |\log \rho| |f(\rho)| \rho d\rho + \int_1^\infty \rho^{\frac{1}{2}}e^{-\rho}|f(\rho)| d\rho \right) \\
\leq C \left( \|\log \rho\|_{L^2_c((0,1),1)}\|f\|_{L^2_c((0,1),1)} + \|\rho^{-\frac{1}{2}}e^{-\rho}\|_{L^2_c((1,\infty),1)}\|f\|_{L^2_c((1,\infty),1)} \right) \leq C\|f\|_{L^2_c}.
\]

(iii) Let now $1 < c$. Then, Lemma 5.9 implies that for $y \leq 1$

\[
G(1, y, \rho) \simeq \begin{cases} 
y^{1-c} \leq \rho^{1-c}, & \rho < y < 1, \\
\rho^{1-c}, & y < \rho < 1, \\
\rho^{-\frac{1}{2}}e^{-\rho}, & 1 < \rho.
\end{cases}
\]

If $c < 3$, one has

\[
|u(y)| \leq C \left( \int_0^1 \rho^{1-c}|f(\rho)|\rho^c d\rho + \int_1^\infty \rho^{-\frac{1}{2}}e^{-\rho}|f(\rho)|\rho^c d\rho \right) \\
\leq C \left( \|\rho^{1-c}\|_{L^2_c((0,1),1)}\|f\|_{L^2_c((0,1),1)} + \|\rho^{-\frac{1}{2}}e^{-\rho}\|_{L^2_c((1,\infty),1)}\|f\|_{L^2_c((1,\infty),1)} \right) \leq C\|f\|_{L^2_c}.
\]

If $c = 3$, then we get

\[
|u(y)| \leq C \left( \int_0^1 |f(\rho)|\rho^3 d\rho + \int_1^\infty \rho^{-2}|f(\rho)|\rho^3 d\rho + \int_1^\infty \rho^{-\frac{3}{2}}e^{-\rho}|f(\rho)|\rho^3 d\rho \right) \\
\leq C\|f\|_{L^2_c} \left( \int_0^y |f(\rho)|\rho^3 d\rho + \int_1^\infty \rho^{-\frac{3}{2}}e^{-\rho}\rho^3 d\rho \right) \\
\leq C\|f\|_{L^2_c} \left( 1 + |\log y|^{\frac{1}{2}} \right)
\]

and finally if $c > 3$

\[
|u(y)| \leq C \left( \int_0^1 |f(\rho)|\rho^c d\rho + \int_1^\infty \rho^{1-c}|f(\rho)|\rho^c d\rho + \int_1^\infty \rho^{-\frac{3}{2}}e^{-\rho}|f(\rho)|\rho^c d\rho \right) \\
\leq C\|f\|_{L^2_c} \left( \int_0^y |f(\rho)|\rho^c d\rho + \int_1^\infty \rho^{-\frac{3}{2}}e^{-\rho}\rho^c d\rho \right) \\
\leq C\|f\|_{L^2_c} \left( y^{1-c} + \rho^{-\frac{3}{2}}e^{-\rho}\rho^c d\rho \right) \\
\leq C\|f\|_{L^2_c} y^{\frac{3-c}{2}}.
\]

\[\Box\]

We can now show that, under stronger assumptions, the potential term $V$ can be seen as a perturbation of $B^n$ near 0, that is $Vu \in L^2_c$ for every $u \in D(B_n)$ having compact support. In particular, we prove that $\mathcal{D}$ is a core for $B^n - V$. 

**Proposition 5.12.** Let \( c + 1 > 0 \) and assume that 

(i) \( c < 3 \) and \( V \in L^2_{\text{loc}}(\mathbb{R}^+, y^c \text{d}y) \) or 

(ii) \( c = 3 \) and \( V|\log y|^\frac{3}{2} \in L^2_{\text{loc}}(\mathbb{R}^+, y^c \text{d}y) \) or 

(iii) \( c > 3 \) and \( V y \frac{3-c}{2} \in L^2_{\text{loc}}(\mathbb{R}^+, y^c \text{d}y) \).

If \( C_r := \{ u \in L^2_c : \text{supp } u \subseteq [0, r] \} \), then \( D(B^n - V) \cap C_r = D(B^n) \cap C_r \) with equivalence of norms

\[ \|u\|_{D(B^n-V)} \simeq \|u\|_{D(B^n)}, \quad \forall u \in D(B^n) \cap C_r. \]

Finally,

\[ \mathcal{D} = \{ u \in C^\infty([0, \infty)) : u \text{ constant in a neighborhood of } 0 \} \]

is a core for \( B^n - V \).

**Proof.** Let \( u \in C_r \). Then, the hypotheses on \( V \) and Proposition 5.11 imply that \( Vu \in L^2_c \) and \( \|Vu\|_{L^2_c} \leq C \|u - (B - V)u\|_{L^2_c} \). Then, by Lemma 5.6 \( u \in D(B^n - V) \) if and only if \( u \in D(B^n) \). This shows the equality \( D(B^n - V) \cap C_r = D(B^n) \cap C_r \). Using Proposition 5.11 again, we also have \( \|Vu\|_{L^2_c} \leq C_1 \|u - Bu\|_{L^2_c} \) for any \( u \in D(B^n) \cap C_r \), which proves the equivalence of the graph norms. Finally, let \( u \in D(B^n - V) \). We have to prove that \( u \) can be approximated in the graph norm with functions belonging to \( \mathcal{D} \). Using Lemma 5.8, we may suppose, without any loss of generality, that \( \text{supp } u \subseteq (0, r) \). Then, by Proposition 4.3, there exist \( (u_n) \subset \mathcal{D} \) such that \( u_n \to u \) in the graph norm \( \| \cdot \|_{D(B^n)} \). We may also assume, after multiplying by a suitable cutoff function, that \( \text{supp } u_n \subseteq (0, 2r) \) for every \( n \). Then, the previous point implies that \( Vu_n \to Vu \) in \( L^2_c \), too.

\[ \square \]

6. **The operator \( y^\alpha B^n - V \) in \( L^2_{c-\alpha} \)**

We consider now for \( c \in \mathbb{R}, \alpha < 2, \) and \( 0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^+, y^{c-\alpha} \text{d}y) \) the operator

\[ y^\alpha B^n - V = y^\alpha \left( D_{yy} + \frac{c}{y} D_y \right) - V \]

in the space \( L^2_{c-\alpha} \). As in Sect. 4, we use the isometry \( T_{-\frac{\alpha}{2}} u(y) = \left| 1 - \frac{\alpha}{2} \right|^{\frac{1}{2^*}} u(y^{1-\frac{\alpha}{2}}) \),

\[ T_{-\frac{\alpha}{2}} : L^2_c \to L^2_{c-\alpha}, \quad \tilde{c} = \frac{c - \frac{\alpha}{2}}{1 - \frac{\alpha}{2}} \]

under which \( y^\alpha B - V \) becomes similar to

\[ T_{-\frac{\alpha}{2}}^{-1} \left( y^\alpha B - V \right) T_{-\frac{\alpha}{2}} = \left( 1 - \frac{\alpha}{2} \right)^2 \left( \tilde{B} - \tilde{V} \right) \]

where \( \tilde{B} = D_{yy} + \frac{\tilde{c}}{y} D_y \) and \( \tilde{V}(y) = \left( 1 - \frac{\alpha}{2} \right)^{-2} V \left( y^{\frac{2}{2-\alpha}} \right) \in L^1_{\text{loc}}(\mathbb{R}^+, y^\tilde{c} \text{d}y) \).
Defining
\[ D(y^\alpha B^n - V) := T_{-\frac{\alpha}{2}} \left( D(\tilde{B}^n - \tilde{V}) \right), \]
onel on obtains that when \( c > -1 + \alpha, \) \( y^\alpha B^n - V \) generates a contractive analytic semigroup \( \{ e^{z(y^\alpha B^n - V)} : z \in \mathbb{C}_+ \} \) in \( L^2_{c-\alpha} \) which satisfies
\[ e^{z(y^\alpha B^n - V)} = T_{-\frac{\alpha}{2}} \left( e^{z(1-\frac{\alpha}{2})^2(\tilde{B}-\tilde{V})} \right) T_{-\frac{\alpha}{2}}^{-1}. \]  

We state the properties obtained so far, together with a density result which is a restating of Proposition 5.12 under the isometry \( T_{-\frac{\alpha}{2}} \).

**Proposition 6.1.** Let \( c + 1 - \alpha > 0 \) and \( 0 \leq V \in L^1_{loc}(\mathbb{R}^+, y^{c-\alpha}) \). Then, the operator \( y^\alpha B^n - V \) generates a contractive analytic semigroup in \( L^2_{c-\alpha} \). If, in addition,

(i) \( c < 3 - \alpha \) and \( V \in L^2_{loc}(\mathbb{R}^+, y^{c-\alpha}) \) or

(ii) \( c = 3 - \alpha \) and \( V|\log y|^{\frac{1}{2}} \in L^2_{loc}(\mathbb{R}^+, y^{c-\alpha}) \) or

(iii) \( c > 3 - \alpha \) and \( V y^{\frac{3-c-\alpha}{2}} \in L^2_{loc}(\mathbb{R}^+, y^{c-\alpha}) \),

then
\[ \mathcal{D} = \{ u \in C^\infty_{c}(\mathbb{R}, (0, \infty)) : u \text{ constant in a neighborhood of 0} \} \]

is a core for \( y^\alpha B^n - V \) in \( L^2_{c-\alpha} \).

**Remark 6.2.** If \( V(y) = y^\alpha \), then \( V \) always satisfies (ii) and (iii) when \( c \geq 3 - \alpha \). Instead, if \( c < 3 - \alpha \), we need \( c + 1 - |\alpha| > 0 \).

Let \( a_{\tilde{V}} \) be the form in \( L^2_{\alpha} \), defined in (3), associated with \( \tilde{B}^n - \tilde{V} \). In \( L^2_{c-\alpha} \), we introduce the form \( a_{\alpha, V} \) which is the image of \( a_{\tilde{V}} \) under the isometry \( T_{0, -\frac{\alpha}{2}} \), that is
\[ a_{\alpha, V}(u, v) := a_{\tilde{V}} \left( T_{-\frac{\alpha}{2}}^{-1} u, T_{-\frac{\alpha}{2}}^{-1} v \right) = \int_{\mathbb{R}^+} \left( y^\alpha D_y u D_y \tilde{V} + Vu \tilde{V} \right) y^{c-\alpha} dy, \]
\[ D(a_{\alpha, V}) := T_{-\frac{\alpha}{2}} D(a_{\tilde{V}}) = \left\{ u \in L^2_{c-\alpha} : u' \in L^2_{\alpha} \right\} \cap L^2 (\mathbb{R}^+, V y^{c-\alpha} dy). \]  

To keep consistency of notation, we often write \( a_{\alpha, V} = a_{\tilde{V}} \). By construction, \( y^\alpha B^n - V \) is the operator associated with the form \( a_{\alpha, V} \) in \( L^2_{c-\alpha} \)
\[ D(y^\alpha B^n - V) = \{ u \in D(a_{\alpha, V}) : \exists f \in L^2_{c-\alpha} \text{ such that } a_{\alpha, V}(u, v) = \int_{0}^{\infty} f \tilde{V} y^{c-\alpha} dy \text{ for every } v \in D(a_{\alpha, V}) \}, \]
\[ y^\alpha B^n u - Vu = -f. \]

The next lemma, which follows from the considerations above, will be used later to relate the resolvents of \( y^\alpha B^n - y^\alpha \) and \( B^n - y^{-\alpha} \).
Lemma 6.3. Let $a_{\alpha, y^\alpha}$ and $a_{y^{-\alpha}}$ be the sesquilinear forms associated, respectively, with the operator $y^\alpha B^n - y^\alpha$ in $L^2_{c-\alpha}$ and $B^n - y^{-\alpha}$ in $L^2_c$. Then,

$$a_{\alpha, y^\alpha}(u, v) = \int_{\mathbb{R}^+} \left(D_y u D_y \overline{v} + u \overline{v}\right) y^\epsilon \, dy,$$

$$a_{y^{-\alpha}}(u, v) = \int_{\mathbb{R}^+} \left(D_y u D_y \overline{v} + y^{-\alpha} u \overline{v}\right) y^\epsilon \, dy.$$

on the common form domain

$$D(a_{\alpha, y^\alpha}) = D(a_{y^{-\alpha}}) = \{ u \in L^2_{c-\alpha} \cap L^2_c : u' \in L^2_c \}$$

Note that the above operators act in different Hilbert spaces; in particular, their domains are different. However, the form domains coincide.

7. The operator $y^\alpha B^n - V$ in $L^p_m$

Here, we investigate properties of $y^\alpha B - V$, $\alpha < 2$, in $L^p_m$ when $0 < \frac{m+1}{p} < c + 1 - \alpha$.

We introduce the family of integral operators $(S^\beta(t))_{t \geq 0}$ on $L^p_m$

$$S^\beta(t)f(y) := t^{-\frac{1}{2}} \int_{\mathbb{R}^+} \left(\frac{\rho}{t^{1-\alpha}} \wedge 1\right)^{-\frac{\beta + \frac{\alpha}{2}}{2}} \exp \left(-\frac{|y^{1-\frac{\alpha}{2}} - \rho^{1-\frac{\alpha}{2}}|^2}{\kappa t}\right) f(\rho) \rho^{-\frac{\alpha}{2}} \, d\rho$$

and note that

$$S^\beta(t) = T_{-\alpha} \circ S_0^\beta(t) \circ T_{-\alpha}^{-1}, \quad \tilde{\beta} = \frac{\beta - \frac{\alpha}{2}}{1 - \frac{\alpha}{2}}.$$

As usual $T_{-\alpha} u(y) = |1 - \frac{\alpha}{2}|^\frac{1}{2} u(y^{1-\frac{\alpha}{2}})$ is an isometry from $L^p_m$ onto $L^p_m$, $\tilde{m} = \frac{m+\frac{\alpha}{2}}{1-\alpha}$.

Here, $\kappa$ is a positive constant, but we omit the dependence on it. The following result has been proved for $\alpha = 0$ in [13].

Lemma 7.1. Let $m \in \mathbb{R}$, and let $p \in (1, \infty)$ such that $0 < \frac{m+1}{p} < 1 - \alpha - \beta$. The families $(S^\beta(t))_{t \geq 0}$ and $\{\Gamma(\lambda) = \int_0^\infty \lambda e^{-\lambda t} S^\beta(t) \, dt, \lambda > 0\}$ are $\mathcal{R}$-bounded in $L^p_m$.

Proof. Since the $\mathcal{R}$-boundedness is preserved under isometries, from $S^\beta(t) = T_{-\alpha} \circ S^\beta_0(t) \circ T_{-\alpha}^{-1}$ we may assume that $\alpha = 0$. (Note that $0 < \frac{m+1}{p} < -\beta + 1 - \alpha$ is equivalent to $0 < \frac{\tilde{m}+\frac{\alpha}{2}}{p} < -\tilde{\beta} + 1$.) The first result is then a consequence of [13, Theorem 7.7]. The family

$$\Gamma(\lambda) = \int_0^\infty \lambda e^{-\lambda t} S^\beta_0(t) \, dt, \quad \lambda > 0$$

is $\mathcal{R}$-bounded by [7, Corollary 2.14].
We can now prove our main results for the operator $y^\alpha B - V$.

**Theorem 7.2.** Let $0 \leq V \in L^1_{loc}(\mathbb{R}^+, y^{c-\alpha} \, dy)$. For any $p \in (1, \infty)$ such that $0 < \frac{m+1}{p} < c + 1 - \alpha$, the semigroup $e^{z(y^\alpha B^n - V)}$ initially defined on $L^{2}_{c-\alpha}$ extends to a bounded analytic semigroup on $L^{p}_{m}$ of angle $\pi/2$ which consists of integral operators. Moreover, the generated semigroup has maximal regularity and the following properties hold.

(i) For every $\epsilon > 0$, there exist $C = C(\epsilon, \alpha) > 0$ (independent of $V$) such that
\[
|e^{z(y^\alpha B^n - V)} f| \leq C S^{\alpha}_{\epsilon}(|z|)|f|, \quad f \in L^{p}_{m}, \quad |\text{arg } z| < \frac{\pi}{2} - \epsilon.
\]

(ii) For every $\epsilon > 0$, the families of operators
\[
\{ e^{z(y^\alpha B^n - V)} : z \in \Sigma_{\frac{\pi}{2} - \epsilon}, \ 0 \leq V \in L^1_{loc}(\mathbb{R}^+, y^{c-\alpha}) \},
\]
\[
\{ \lambda (\lambda - y^\alpha B^n + V)^{-1} : \lambda \in \Sigma_{\frac{\pi}{2} - \epsilon} : 0 \leq V \in L^1_{loc}(\mathbb{R}^+, y^{c-\alpha}) \}
\]
are $\mathcal{R}$-bounded in $L^{p}_{m}$.

**Proof.** By Proposition 5.3 and (6), (i) holds for any $f \in L^{2}_{c-\alpha}$. The boundedness of $e^{z(y^\alpha B^n - V)}$ in $L^{p}_{m}$ follows from the previous lemma, and (i) extends to $L^{p}_{m}$. The semigroup law is inherited from $L^{2}_{c-\alpha}$ via a density argument, and we have only to prove the strong continuity at $0$. Using the isometry $T^{z}_{-\alpha}$, we may suppose that $\alpha = 0$. Let $f, g \in C^{\infty}_{c}(0, \infty)$. Then as $z \to 0$, $z \in \Sigma_{\frac{\pi}{2} - \epsilon},$
\[
\int_{0}^{\infty} (e^{(y^\alpha B^n f)} g \ y^{m}) \, dy = \int_{0}^{\infty} (e^{(y^\alpha B^n f)} g \ y^{m-c} \ y^{c}) \, dy
\]
\[
\to \int_{0}^{\infty} f \ g y^{m-c} \ y^{c} \, dy = \int_{0}^{\infty} f \ g \ y^{m} \, dy,
\]
by the strong continuity of $e^{z(B^n - V)}$ in $L^{2}_{c}$. By density and uniform boundedness of the family $(e^{z}(B^n - V))_{z \in \Sigma_{\frac{\pi}{2} - \epsilon}}$, this holds for every $f \in L^{p}_{m}, g \in L^{p'}_{m}$. The semigroup is then weakly continuous, hence strongly continuous.

The $\mathcal{R}$-boundedness of $e^{z(y^\alpha B^n - V)}$ follows then by domination from Lemma 7.1, see Corollary 2.2. To prove the $\mathcal{R}$-boundedness of the resolvent family, for $\lambda \in \Sigma_{\frac{\pi}{2} - \epsilon} \setminus \{0\}$ let $\theta = \frac{|\text{arg } \lambda|}{\frac{\pi}{2} - \frac{\epsilon}{2}}$, so that $\mu := e^{-i\theta} \lambda \in \Sigma_{\frac{\pi}{2} - \frac{\epsilon}{2}}$. Then,
\[
|\lambda (\lambda - y^\alpha B^n + V)^{-1} f| = \left| \mu (\mu - e^{-i\theta} (y^\alpha B^n - V))^{-1} f \right|
\]
\[
= \left| \int_{0}^{\infty} \mu e^{-\mu t} e^{-i\theta t} (y^\alpha B^n - V) f \, dt \right|
\]
\[
\leq C \int_{0}^{\infty} |\mu| e^{-Re \mu t} S^{\alpha}_{\epsilon}(t) |f| \, dt
\]
\[
\leq C \int_{0}^{\infty} |\lambda| e^{-|\lambda| \sin \frac{\pi}{2} t} S^{\alpha}_{\epsilon}(t) |f| \, dt.
\]
The $\mathcal{R}$-boundedness of the second family in (ii) now follows from [7, Corollary 2.14] and the maximal regularity of the semigroup from Theorem 2.3.

In our investigation of degenerate Nd problems, see [14], we need also a weaker version of the result above for potentials having nonnegative real part. We formulate it in the next proposition.

**Proposition 7.3.** Let $V \in L^1_{\text{loc}}(\mathbb{R}^+, y^{c-\alpha} \, dy)$ be a potential having nonnegative real part. Then, for any $1 < p < \infty$ such that $0 < \frac{m+1}{p} < c + 1 - \alpha$, $y^\alpha B^n - V$ generates a $C_0$-semigroup on $L^p_m$. The generated semigroup consists of integral operators, and the following estimates hold

$$\left| e^{t(y^\alpha B^n - V)} f \right| \leq e^{ty^\alpha B^n} |f|, \quad f \in L^p_m, \quad t \geq 0$$

In particular, the families of operators

$$\left\{ e^{t(y^\alpha B^n - V)} : t \geq 0, \ V \in L^1_{\text{loc}}(\mathbb{R}^+, y^{c-\alpha}), \ Re\ V \geq 0 \right\},$$

$$\left\{ \lambda (\lambda - y^\alpha B^n + V)^{-1} : \lambda > 0, \ V \in L^1_{\text{loc}}(\mathbb{R}^+, y^{c-\alpha}), \ Re\ V \geq 0 \right\}$$

are $\mathcal{R}$-bounded in $L^p_m$.

**Proof.** Using the isometry $T_{0,-\frac{\alpha}{2}}$, we may assume that $\alpha = 0$. Let us treat first the symmetric case in $L^2_c$. The generation results can be proved as in Proposition 5.1 (where we assumed $V \geq 0$). If $a$ is the form associated with $B^n$, then $B^n - V$ is associated with $a_V := a(u, v) + \langle Vu, v \rangle_{L^2_c}$ and, by the standard theory on sesquilinear forms, $B^n - V$ generates a $C_0$-semigroup on $L^2_c$.

The domination properties follow from [16, Theorem 2.21]. Let $u, v \in D(\alpha_V) = D(a) \cap L^2_c(\mathbb{R}^+, |V| y^{c-\alpha} \, dy)$ such that $u \bar{v} \geq 0$. Since $e^{tB^n}$ is positive, one has $\text{Re} \ a(u, v) \geq \text{Re} \ a(|u|, |v|)$. Moreover,

$$\text{Re} \ a_V(u, v) = \text{Re} \ a(u, v) + \int_0^\infty \text{Re} \ V \, u \bar{v} \, y^c \, dy \geq \text{Re} \ a(|u|, |v|)$$

which by [16, Theorem 2.21] again implies the stated domination of the generated semigroups. (One easily verifies that $D(\alpha_V)$ is an ideal of $D(a)$ since this last is an ideal in itself, by the positivity of $e^{tB^n}$, see [16, Proposition 2.20].) The extrapolation on $L^p_m$ follows as in Theorem 7.2. The domination of the resolvent is a straightforward consequence of that of the semigroup. The $\mathcal{R}$-boundedness of the semigroup follows by domination from the $\mathcal{R}$-boundedness of $(e^{tB^n})_{t \geq 0}$ proved in Theorem 7.2. The $\mathcal{R}$-boundedness of the resolvent follows as in Theorem 7.2.

**8. The operator $y^\alpha B^n - y^\alpha$**

We end the paper by thoroughly investigating the special case $V(y) = y^\alpha$, keeping $\alpha < 2$. We prove, in particular, that the domain of $y^\alpha B - V$ is $D(y^\alpha B) \cap D(V)$, under slightly more restrictive hypotheses than those of Theorem 7.2.
As explained in Introduction, this case plays a crucial role in [14] in the investigation of the degenerate operators

\[
\mathcal{L} = y^{\alpha_1} \Delta_x + y^{\alpha_2} \left( D_{yy} + \frac{c}{y} D_y - \frac{b}{y^2} \right), \quad \alpha_1, \alpha_2 \in \mathbb{R}
\]

in the spaces \( L^p \left( \mathbb{R}^{N+1}_+, y^m dx dy \right) \). In particular, we prove in Propositions 8.3 and 8.4 that the multipliers

\[
\xi \in \mathbb{R}^N \rightarrow N_\lambda(\xi) = \lambda(\lambda - y^\alpha B y + y^\alpha |\xi|^2)^{-1},
\]

\[
\xi \in \mathbb{R}^N \rightarrow M_\lambda(\xi) = y^\alpha |\xi|^2(\lambda - y^\alpha B y + y^\alpha |\xi|^2)^{-1}
\]
satisfy the hypothesis of Theorem 2.4.

We start with the following lemma.

**Lemma 8.1.** Assume that \( c + 1 > 0 \) and \( c + 1 - \alpha > 0 \); that is, \( B^n \) generates a \( C_0 \)-semigroup in \( L^2_c \) and \( y^\alpha B^n \) generates a \( C_0 \)-semigroup in \( L^{2-\alpha}_c \). If \( \lambda \in \mathbb{C}^+ \) and \( \mu > 0 \), then

\[
(\lambda - y^\alpha B^n + \mu y^\alpha)^{-1} f = \left( \mu - B^n + \frac{\lambda}{y^\alpha} \right)^{-1} \left( \frac{f}{y^\alpha} \right), \quad \forall f \in C_c^\infty((0, \infty)).
\]

**Proof.** Under the assumptions, \( y^\alpha B^n - \mu y^\alpha \) and \( B^n - \lambda y^{-\alpha} \) generate a semigroup on \( L^{2-\alpha}_c \) and \( L^2_c \), respectively, see Theorem 7.2. Since \( \text{Re} \lambda > 0 \), \( \mu > 0 \), both resolvents are well defined but map to different spaces.

Let \( a_{\alpha,\mu y^\alpha} \), \( a_{\lambda y^{-\alpha}} \) be the forms associated with \( y^\alpha B^n - \mu y^\alpha \) in \( L^{2-\alpha}_c \) and \( B^n - \lambda y^{-\alpha} \) in \( L^2_c \)

\[
a_{\alpha,\mu y^\alpha}(u, v) = \int_{\mathbb{R}_+} \left( D_\gamma u D_\gamma \overline{v} + \mu uv \right) y^\epsilon dy,
\]

\[
a_{\lambda y^{-\alpha}}(u, v) = \int_{\mathbb{R}_+} \left( D_\gamma u D_\gamma \overline{v} + \lambda y^{-\alpha} uv \right) y^\epsilon dy.
\]

By Lemma 6.3, they are defined on the common domain

\[
\mathcal{F} := \left\{ u \in L^{2-\alpha}_c \cap L^2_c : u' \in L^2_c \right\}
\]

Given \( f \in C_c^\infty((0, \infty)) \), let \( u := \left( \mu - B^n + \frac{\lambda}{y^\alpha} \right)^{-1} \left( \frac{f}{y^\alpha} \right) \). In order to prove that the equality \( u = (\lambda - y^\alpha B^n + \mu y^\alpha)^{-1} f \) holds, we have to show that \( u \in \mathcal{F} \) and that for every \( v \in \mathcal{F} \), \( u \) satisfies the weak equality

\[
\int_0^\infty f v y^{\epsilon-\alpha} dy = \int_0^\infty \lambda u \overline{v} y^{\epsilon-\alpha} dy + a_{\alpha,\mu y^\alpha}(u, v)
\]

\[
= \int_0^\infty (\lambda y^{-\alpha} u \overline{v} + D_\gamma u D_\gamma \overline{v} + \mu uv) y^\epsilon dy. \quad (8)
\]
By construction, \( u \) is in the domain of \( B^n - \lambda y^{-\alpha} \) which is contained in \( F \) and satisfies

\[
\int_0^\infty \frac{f}{y^\alpha} \bar{v} y^c \, dy = \int_0^\infty \mu u \bar{v} y^c \, dy + a_{\alpha, \lambda y^{-\alpha}} (u, v) = \int_0^\infty (\mu u \bar{v} + D_y u D_y \bar{v} + \lambda y^{-\alpha} u \bar{v}) y^c \, dy,
\]

which is the same as (8).

In the next results, we relate the resolvent of \( y^\alpha B^n - y^\alpha \) with that of \( B^n - 1 \). We shall assume both the conditions \( 0 < \frac{m+1}{p} < c + 1 - \alpha \) and \( -\alpha < \frac{m+1}{p} < c + 1 - \alpha \) (that is \( \alpha^- < \frac{m+1}{p} < c + 1 - \alpha \)). The first guarantees that \( y^\alpha B^n \) is a generator in \( L^p_m \) and the second that \( B^n \) is a generator in \( L^p_{m+\alpha p} \).

**Corollary 8.2.** Assume that \( \alpha^- < \frac{m+1}{p} < c + 1 - \alpha \). If \( \lambda \in \mathbb{C}^+ \) and \( \mu > 0 \), then

(i) for every \( f \in L^p_m \)

\[
(\lambda - y^\alpha B^n + \mu y^\alpha)^{-1} f = \left(\mu - B^n + \frac{\lambda}{y^\alpha}\right)^{-1} \left(\frac{f}{y^\alpha}\right) \in L^p_{m+\alpha p} \cap L^p_m;
\]

(ii) the operator \( y^\alpha (\lambda - y^\alpha B^n + \mu y^\alpha)^{-1} \) is bounded in \( L^p_m \);

(iii) the operator \( \frac{1}{y^\alpha} \left(\mu - B^n + \frac{\lambda}{y^\alpha}\right)^{-1} \) is bounded in \( L^p_{m+\alpha p} \).

**Proof.** Equality (i) is proved in Lemma 8.1 for any \( f \in C^\infty_c((0, \infty)) \). Since \( (\lambda - y^\alpha B^n + \mu y^\alpha)^{-1} \) is bounded form \( L^p_m \) into itself and \( \left(\mu - B^n + \frac{\lambda}{y^\alpha}\right)^{-1} \left(\frac{f}{y^\alpha}\right) \) is bounded from \( L^p_m \) to \( L^p_{m+\alpha p} \), by density, (i) holds for every \( f \in L^p_m \). Parts (ii), (iii) are consequence of (i).
The map \( Tf = f/y^\alpha \) is an isometry of \( L^p_m \) onto \( L^{p+\alpha}_m \) and by Corollary 8.2,
\[
m_\lambda(\mu) = T^{-1}\mu\left(\mu - B^n + \frac{\lambda}{y^\alpha}\right)^{-1} T.
\]

The family
\[
\left\{ \mu^k D^k_{\mu}(\Gamma_\lambda)(\mu) : \mu > 0, \; k \leq N, \; \lambda \in \mathbb{C}^+ \right\}, \quad \Gamma_\lambda(\mu) = \mu\left(\mu - B^n + \frac{\lambda}{y^\alpha}\right)^{-1}
\]
is \( \mathcal{R} \)-bounded in \( L^{p+\alpha}_m \). Indeed,
\[
\Gamma_\lambda(\mu) = \int_0^\infty \mu e^{-\mu t} e^{t(B^n - \frac{\lambda}{y^\alpha})} \, dt
\]
and \( \left\{ e^{t(B^n - \frac{\lambda}{y^\alpha})} : t \geq 0, \; \lambda \in \mathbb{C}^+ \right\} \) is \( \mathcal{R} \)-bounded in \( L^{p+\alpha}_m \), by Theorem 7.3. The \( \mathcal{R} \)-boundedness of the derivatives follows either by the resolvent equation or by differentiating the last equation under the integral and using [7, Corollary 2.14]. In fact, if \( h(\mu, t) = \mu e^{-\mu t} \), then
\[
\mu^k \int_0^\infty |D^k_{\mu} h(\mu, t)| \, dt \leq C_k, \; \mu > 0.
\]

Next we deal with \( N_\lambda \) which is crucial in [14] for the proof that \( \mathcal{L} = y^\alpha(\Delta_x + B_y) \) generates an analytic semigroup.

**Proposition 8.4.** Assume that \( \alpha^- < \frac{m+1}{p} < c + 1 - \alpha \) and let for \( \lambda \in \mathbb{C}^+, \; \xi \neq 0 \)
\[
N_\lambda(\xi) = (\lambda - y^\alpha B^n + |\xi|^2 y^\alpha)^{-1} \in \mathcal{B}(L^p_m).
\]
Then, the family
\[
\left\{ |\xi|^{|\beta|} D^\beta_{\xi}(\lambda N_\lambda)(\xi) : \xi \in \mathbb{R}^N \setminus \{0\}, \; |\beta| \leq N, \; \lambda \in \mathbb{C}^+ \right\}
\]
is \( \mathcal{R} \)-bounded in \( L^p_m \).

**Proof.** For \( \mu > 0 \), let \( n_\lambda(\mu) = (\lambda - y^\alpha B^n + \mu y^\alpha)^{-1} \). Using Lemma 2.5, we have to show that the family
\[
\left\{ \mu^k D^k_{\mu}(n_\lambda)(\mu) : \mu > 0, \; k \leq N, \; \lambda \in \mathbb{C}^+ \right\}
\]
is \( \mathcal{R} \)-bounded in \( L^p_m \).

Theorem 7.2 with \( V(y) = \mu y^\alpha \) and Proposition 8.3 imply that the families
\[
\left\{ \lambda n_\lambda(\mu) : \mu > 0, \; \lambda \in \mathbb{C}^+ \right\}, \quad \left\{ \mu y^\alpha n_\lambda(\mu) : \mu > 0, \; \lambda \in \mathbb{C}^+ \right\}
\]
(10)
are $\mathcal{R}$-bounded in $L^p_m$.
We have that $n_\lambda(\cdot) \in C^1(\mathbb{R}_+, \mathcal{B}(L^p_m))$ and
$$D_\mu(n_\lambda(\mu)) = -n_\lambda(\mu)y^\alpha n_\lambda(\mu). \quad (11)$$
Indeed setting $A = \lambda - y^\alpha B^n_y$, $V = y^\alpha$, we have
$$\frac{n_\lambda(\mu + h) - n_\lambda(\mu)}{h} = \frac{(A + (\mu + h)V)^{-1} - (A + \mu V)^{-1}}{h}$$
$$= \frac{(A + \mu V)^{-1}(A + (\mu + h)V)^{-1} - I}{h}$$
$$= -(A + \mu V)^{-1} V (A + (\mu + h)V)^{-1}$$
which tends to $-n_\lambda(\mu) y^\alpha n_\lambda(\mu)$ as $h \to 0$ in the norm of $\mathcal{B}(L^p_m)$ since, by Corollary 8.2,
$$\mu \mapsto V(A + \mu)V^{-1} = \mu y^\alpha \left( \mu - B^n + \frac{\lambda}{y^\alpha} \right)^{-1} \frac{1}{y^\alpha}$$
is continuous from $(0, \infty)$ to $\mathcal{B}(L^p_m)$. This shows (11) and then $n_\lambda(\cdot) \in C^\infty(\mathbb{R}_+, \mathcal{B}(L^p_m))$ and
$$D^k_\mu(n_\lambda(\mu)) = a_k n_\lambda(\mu) \left( y^\alpha n_\lambda(\mu) \right)^k, \quad a_1 = -1, \quad a_{k+1} = -(k+1)a_k. \quad (12)$$
Formula (12) follows by induction after observing that since $y^\alpha n_\lambda(\mu)$ and its derivative $D_\mu(y^\alpha n_\lambda(\mu)) = -(y^\alpha n_\lambda(\mu))^2$ commute, then
$$D^k_\mu(y^\alpha n_\lambda(\mu)) = k D^k_\mu(y^\alpha n_\lambda(\mu)) \left( y^\alpha n_\lambda(\mu) \right)^{k-1} = -k \left( y^\alpha n_\lambda(\mu) \right)^{k+1}.$$  

The $\mathcal{R}$-boundedness of the family (9) then follows from the $\mathcal{R}$-boundedness of the families (10) since
$$\mu^k D^k_\mu(n_\lambda(\mu)) = a_k \lambda n_\lambda(\mu) \left( y^\alpha n_\lambda(\mu) \right)^{k+1}.$$  

In order to characterize the domain of $y^\alpha B^n - y^\alpha$, we denote by
$$D(y^\alpha) = \{ u \in L^p_m : y^\alpha u \in L^p_m \}$$
the domain of the potential $V = y^\alpha$ in $L^p_m$. Recalling that Theorem 4.2 assures that $D(y^\alpha B^n) = W^{2,p}_N(\alpha, m)$, we consider, for $0 < \frac{m+1}{p} < c + 1 - \alpha$, the Banach space
$$W^{2,p}_N(\alpha, m) \cap D(y^\alpha) = \left\{ u \in W^{2,p}_N(\mathbb{R}_+) : u, y^\alpha u, y^\alpha D_u u, y^\alpha D_y u, y^{\alpha-1}D_y u \in L^p_m \right\}$$
endowed with norm $\|y^\alpha Bu\|_{L^p_m} + \|y^\alpha u\|_{L^p_m} + \|u\|_{L^p_m}$. 

Theorem 8.5. Let $\alpha < 2$, $\mu > 0$, $c \in \mathbb{R}$. Then, for any $1 < p < \infty$ such that \(\alpha^- < \frac{m+1}{p} < c + 1 - \alpha\) the operator \(L = y^\alpha B^n - \mu y^\alpha\) with domain \(W^{2,p}_N(\alpha, m) \cap D(y^\alpha)\) generates a bounded analytic semigroup in \(L^p_m\) which has maximal regularity. Moreover,

\[ \mathcal{D} = \{ u \in C^\infty_c([0, \infty)) : u \text{ constant in a neighborhood of } 0 \} \]

is a core for \(y^\alpha B^n - \mu y^\alpha\).

Proof. The generation properties as well as the maximal regularity follow from Theorem 7.2. Without any loss of generality, we may assume that \(\mu = 1\). We prove preliminarily that \(\mathcal{D}\) is dense in \(W^{2,p}_N(\alpha, m) \cap D(y^\alpha) = D(y^\alpha B^n) \cap D(y^\alpha)\). Let \(u \in W^{2,p}_N(\alpha, m) \cap D(y^\alpha)\); up to using a standard cutoff argument we may suppose that \(\text{supp} \, u \subseteq [0, b]\) for some \(b > 0\). Using Remark 4.4, let \((u_n)_{n \in \mathbb{N}} \subseteq \mathcal{D}\) such that \(\text{supp} \, u_n \subseteq [0, b]\) and \(u_n \to u\) in \(W^{2,p}_N(\alpha, m)\). Then by [12, Proposition 3.2 (ii)]

\[ \| y^\alpha(u_n - u) \|_{L^p_m} \leq C \| y^{\alpha+1}(D_y u_n - D_y u) \|_{L^p_m} \leq C b^2 \| y^{\alpha-1} D_y (u_n - u) \|_{L^p_m} \]

which tends to 0 as \(n \to \infty\). This proves the density of \(\mathcal{D}\).

Let us now characterize the domain. By definition, \(D(y^\alpha B^n - y^\alpha) = (1 - y^\alpha B^n + y^\alpha)^{-1}(L^p_m)\). Let \(u = (1 - y^\alpha B^n + y^\alpha)^{-1} f\) with \(f \in L^p_m\). Using Corollary 8.2 (ii), we obtain

\[ \| y^\alpha u \|_{L^p_m} + \| y^\alpha Bu \|_{L^p_m} \leq C \left( \| (y^\alpha B - y^\alpha) u \|_{L^p_m} + \| u \|_{L^p_m} \right) \]

which proves the inclusion \(D(y^\alpha B^n - y^\alpha) \subseteq D(y^\alpha B^n) \cap D(y^\alpha)\). To prove the reverse property, we observe that since the graph norm of \(y^\alpha B^n - y^\alpha\) is clearly weaker than the norm of \(D(y^\alpha B^n) \cap D(y^\alpha)\), inequality (13) again shows that they are equivalent on \(D(y^\alpha B^n - y^\alpha)\), in particular on \(\mathcal{D}\) which is dense in \(D(y^\alpha B^n) \cap D(y^\alpha)\), by the previous step. Therefore, \(D(y^\alpha B^n - y^\alpha) = D(y^\alpha B^n) \cap D(y^\alpha)\) and in particular \(\mathcal{D}\) is a core.

We remark that Theorem 7.2 assures that \(y^\alpha B^n - y^\alpha\) generates a semigroup on \(L^p_m\) under the milder assumption \(0 < \frac{m+1}{p} < c + 1 - \alpha\) and \(c + 1 > 0\). However, the hypothesis \((m+1)/p + \alpha > 0\) must be added when \(\alpha < 0\) in order that \(\mathcal{D} \subseteq D(y^\alpha)\).

The same method yields the domain of \(B^n - \frac{1}{y^\alpha}\), using Corollary 8.2 (iii) with \(m\) replaced by \(m - \alpha p\).

Corollary 8.6. If \(\alpha^+ < \frac{m+1}{p} < c + 1\), then the domain of \(B^n - \frac{1}{y^\alpha}\) is \(W^{2,p}_N(0, m) \cap D(\frac{1}{y^\alpha})\).

Funding Open access funding provided by Università del Salento within the CRUCARE Agreement.
REFERENCES

[1] Abramowitz, M., and Stegun, I. A. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, tenth printing ed. National Bureau of Standards Applied Mathematics Series 55, Dover, New York City, 1964.

[2] Arendt, W., and Bukhvalov, A. V. Integral representations of resolvents and semigroups. *Forum Mathematicum* 6 (1994), 111–136.

[3] Calvaruso, G., Metafune, G., Negro, L., and Spina, C. Optimal kernel estimates for elliptic operators with second order discontinuous coefficients. *Journal of Mathematical Analysis and Applications* 485, 1 (2020), 123763.

[4] Coulhon, T., and Sikora, A. Gaussian heat kernel upper bounds via the Phragmén-Lindelöf theorem. *Proceedings of the London Mathematical Society* 96, 2 (2008), 507–544.

[5] Denk, R., Hieber, M., and Prüss, J. $\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type, vol. 166 (n.788) of *Memoirs of the American Mathematical Society*. Amer. Math. Soc., 2003.

[6] Hytönen, T., Van Neerven, J., Veraar, M., and Weis, L. *Analysis in Banach Spaces, Vol. II: Probabilistic Methods and Operator Theory*. Springer, 2017.

[7] Kunstmann, P. C., and Weis, L. Maximal $L^p$-regularity for parabolic equations, Fourier multiplier theorems and $H^\infty$-functional calculus. In Iannelli M., Nagel R., Piazzera S. (eds) *Functional Analytic Methods for Evolution Equations*, vol. 1855 of *Lecture Notes in Mathematics*. Springer, Berlin, 2004.

[8] Metafune, G., Negro, L., Sobajima, M., and Spina, C. Rellich inequalities in bounded domains. *Mathematische Annalen* 379, 2 (2021), 765–824.

[9] Metafune, G., Negro, L., and Spina, C. Sharp kernel estimates for elliptic operators with second-order discontinuous coefficients. *Journal of Evolution Equations* 18 (2018), 467–514.

[10] Metafune, G., Negro, L., and Spina, C. Gradient estimates for elliptic operators with second-order discontinuous coefficients. *Mediterranean Journal of Mathematics* 16, 138 (2019).

[11] Metafune, G., Negro, L., and Spina, C. Maximal regularity for elliptic operators with second-order discontinuous coefficients. *Journal of Evolution Equations* 21 (2021), 3613–3637.

[12] Metafune, G., Negro, L., and Spina, C. Anisotropic sobolev spaces with weights. *Submitted* (2022). Online preprint: https://arxiv.org/abs/2112.01791.

[13] Metafune, G., Negro, L., and Spina, C. $L^p$ estimates for the Caffarelli-Silvestre extension operators. *Journal of Differential Equations* 316 (2022), 290–345.

[14] Metafune, G., Negro, L., and Spina, C. A unified approach to degenerate problems in the half-space. *Submitted* (2022). Online preprint: https://arxiv.org/abs/2201.05573.

[15] Negro, L., and Spina, C. Asymptotic behaviour for elliptic operators with second-order discontinuous coefficients. *Forum Mathematicum* 32, 2 (2020), 399–415.

[16] Ouhabaz, E. M. *Analysis of Heat Equations on Domains*. Princeton University Press, 2009.

[17] Prüss, J., and Simonett, G. *Moving Interfaces and Quasilinear Parabolic Evolution Equations*, vol. 105. Springer-Verlag, 2016.
