Attempt to Assess the Scope of Applicability of a Hyperelastic Murnaghan’s Material Model in the Case of Elastomers

Stanislaw Jemiolo 1, Aleksander Franus 1, Wlodzimierz Domanski 2

1 Faculty of Civil Engineering, Warsaw University of Technology, Armii Ludowej 16, 00-637 Warsaw, Poland
2 Faculty of Cybernetics, Military University of Technology, Urbanowicza 2, 00-908 Warsaw 49, Poland
s.jemiolo@il.pw.edu.pl

Abstract. In this work, we use the MCMV slightly compressible hyperelastic constitutive model of rubber-like materials that was proposed in the authors’ work and implemented in the MES ABAQUS / Standard program. The stored energy function of the MCMV model is approximated by polynomials relative to the Lagrange strain tensor, attempting to evaluate the scope of applicability of Murnaghan’s models in the boundary-value problem of statics. We analyze in detail the homogeneous deformation problems. The point is to determine the approximate range of deformation, in which we get similar results of the problems mentioned, using the MCMV material model and its approximation.

1. Preliminary remarks
The purpose of this paper is to assess the scope of applicability of a hyperelastic Murnaghan’s material model [1] in the case of isotropic elastomers, often called rubber-like materials. In the Murnaghan model, consistent approximations of the stored energy function (SEF) with respect to the invariants of the Lagrange strain tensor $E$ are considered [1,2-4]. Consequently, models of isotropic materials with 2, 5 and 9 elastic constants respectively of the second, third and fourth order are obtained. The Murnaghan’s model is the basic model of hyperelastic material in acoustic-elasticity and dynamic nonlinear elasticity [5, 6, 7]. The method of implementing this material model with II and III order elasticity constants in the MES ABAQUS program as a part of the UHYPER user procedure, was given, among others, in the works of the authors [3, 8].

In the case of rubber-like materials, the volumetric compressibility module is a few orders of magnitude larger than the shear module $K_0 >> \mu_0$ [3, 9]. Therefore, when interpreting typical experimental results of uniaxial and biaxial stretching and simple shear, universal relationships resulting from the adoption of a hyperelastic model of the incompressible material are used. Taking into account the compressibility of the material consists in using the so-called model of a slightly compressible material in which the function of unitary elastic energy (ES) is the sum of two functions: i.e. the function of the unitary energy of elasticity of the isochoric deformation $W_p$ and the function of the unitary elastic energy of volume deformation $W_v$. 
In this work, we use the MCMV material model that was proposed in the authors’ work [5, 8] and implemented in the MES ABAQUS / Standard program [10]. The SEF of the MCMV model is approximated by polynomials relative to the tensor $\mathbf{E}$, attempting to evaluate the scope of applicability of Murnaghan’s models in the boundary-value problems of statics. We analyze in detail the homogeneous deformation problems. The point is to determine the approximate range of deformation, in which we get similar results of the problems mentioned, using the MCMV material model and its approximation.

2. Constitutive relations of slightly compressible hyperelastic isotropic materials

Hyperelastic models of isotropic slightly compressible materials are generalizations of the model of incompressible materials in which there is no coupling between the stored energy function of the isochoric $W_p$ and volume $W_V$ deformation [2,3,8,18]. To the function $W_p$ of incompressible material models, which depend on two invariants of the isochoric deformation $I_1, I_2$, a sufficiently regular function $W_V(J)$ is added dependent on the invariant describing volume deformation: $J = \det \mathbf{F}$, where $\mathbf{F}$ is the so called deformation gradient tensor [9, 13]. Consequently, we obtain the SEF function of the form:

$$W = W_p(\bar{I}_1, \bar{I}_2) + W_V(J).$$

(1)

In the case of hyperelastic isotropic materials, the constitutive relationship in the description of Euler (in the configuration of the deformed body) can be obtained from the following formula [16]:

$$\mathbf{B} = \frac{2}{J} \left. \frac{\partial W}{\partial \mathbf{B}} \right|_{\mathbf{B} = \mathbf{F}} = \frac{2}{J} \left. \frac{\partial W}{\partial \mathbf{B}} \right|_{\mathbf{B} = \mathbf{B}^T},$$

(2)

where $\mathbf{B}$ is the stress Cauchy tensor, and $\mathbf{B}$ is the left deformation tensor ($\mathbf{B} = \mathbf{F}^T \mathbf{F}$, the symbol “$T$” means a transposition of the tensor). Due to the form of the function (2.1), the constitutive relationship of hyperelastic slightly compressible materials is divided into two independent parts, deviatoric and volumetric:

$$\mathbf{s} = \mathbf{B} - \frac{I}{3}(\mathbf{tr} \mathbf{B}) \mathbf{I} = \frac{2}{J} \left( \frac{\partial W_p}{\partial \bar{I}_1} \mathbf{B} - \frac{\partial W_p}{\partial \bar{I}_2} \mathbf{B}^{-1} \right), \quad \mathbf{tr} \mathbf{B} = \frac{3}{J} \frac{\partial W_p}{\partial J},$$

(3)

In relation (3) there are deviators of the left Cauchy-Green tensor of isochoric deformation [3, 16]:

$$\mathbf{B}_D = \mathbf{B} - \frac{1}{3} \bar{I}_1 \mathbf{I}, \quad \mathbf{B}_D^{-1} = \mathbf{B}^{-1} - \frac{1}{3} \bar{I}_2 \mathbf{I},$$

(4)

We remind that: $\mathbf{B} = J^2 \mathbf{F}$ and $\bar{I}_1 = \mathbf{tr} \mathbf{B}, \bar{I}_2 = \mathbf{tr} \mathbf{B}^{-1}$ and these invariants are the same for deformation tensors $\mathbf{B}$ and $\mathbf{C} = J^2 \mathbf{F}^{-T}$, where $\mathbf{C} = \mathbf{F}^T \mathbf{F}$ is the right Cauchy-Green deformation tensor.

We obtain constitutive relations related to the initial configuration from (2) - (3) and the following relations [4,8]:

$$\mathbf{T} = J \mathbf{F}^{-1} \mathbf{F}^{-T}, \mathbf{S} = \mathbf{F} \mathbf{F}^T,$$

(5)

where $\mathbf{T}$ and $\mathbf{S}$ are respectively the first and the second Pioli-Kirchhoff stress tensors.
3. Model of a slightly compressible material MCMV

We consider the MCMV model of the SE function

\[ W = W_0(T_1, T_2) + K_0 \alpha(J), \]  

where the part of isochoric deformation is identical to that of the MV model [3]:

\[ W_0(T_1, T_2) = \frac{1}{2} a_1 (T_1 - 3) + \frac{1}{2} a_2 (T_1^2 - 9) + \frac{1}{3} a_3 (T_1^3 - 27) + a_4 (T_2 - 3) + a_5 (T_1 T_2 - 9) = C_{10} (T_1 - 3) + C_{20} (T_1 - 3)^2 + C_{30} (T_1 - 3)^3 + C_{40} (T_2 - 3) + C_{11} (T_1 - 3) (T_2 - 3), \]

while

\[ \alpha(J) = \frac{1}{4} (J^2 - 1) - \frac{1}{2} \ln J. \]

Function \( \alpha(J) = K_0 \alpha(J) [14, 18] \) defines SEF volume deformation. The constant should be understood as the initial module of volume compressibility.

In constitutive relations (2.3), we substitute the following material functions:

\[ 2 \frac{\partial W_0}{\partial T_1} = a_1 + a_2 T_1 + a_3 T_1^2 + a_4 T_2, \quad -2 \frac{\partial W_0}{\partial T_2} = -a_4 - a_5 T_1, \quad \frac{\partial \alpha}{\partial J} = K_0 \frac{\partial \alpha}{\partial J} = \frac{K_0}{2J} (J^2 - 1), \]  

In [18] we have published some examples of SEF volume deformation functions in the context of hyperelastic models of small-volume materials used in literature [5]. Function \( w(3.1) \) is a convex function, it fulfills the condition of natural state and

\[ \alpha(J \rightarrow 0) = +\infty, \quad \frac{\partial \alpha(J)}{\partial J} \bigg|_{J=0} = -\infty, \quad \alpha(J \rightarrow +\infty) = +\infty, \quad \frac{\partial \alpha(J)}{\partial J} \bigg|_{J \rightarrow +\infty} = +\infty \]  

The functions considered in [18] are scaled by the volume compressibility module \( K_0 \) (identical to the Hooke’s relationship of the linear elastic theory), meaning that only one experimental test, which can be performed in the range of small strains, is necessary to determine them. Other functions contain additional material parameters.

4. Model of Murnaghan’s material

The elastic potential of the Murnaghan model (M) as a function of the invariants of the strain tensor \( E = (C - I) / 2 \) has the following form, por. [1-4, 8,18]:

\[ W = \frac{1}{2} \lambda_0 (\text{tr}E)^2 + \mu_0 \text{tr}E^2 + \frac{1}{6} \nu_1 (\text{tr}E)^3 + \nu_2 \text{tr}E \text{tr}E^2 + \frac{4}{3} \nu_3 \text{tr}E^3, \]  

where }
where $\lambda_0$ and $\mu_0$ are Lame constants (or traditionally referred to as Murnaghan II order elasticity constants), while $\nu_i$ ($i=1,2,3$) they are Murnaghan III order elasticity constants.

The literature also introduces, instead of constants $\nu_i$ in (11), the parameters $l$, $m$ and $n$

$$l = \frac{1}{2} \nu_1 + \nu_2, \quad m = \nu_2 + 2 \nu_3, \quad n = 4 \nu_3,$$

(12)

Then it is convenient to present the SEF (11) in a form dependent on invariants of the basic tensor, see [2]. Relationships between the basic invariants of $E$ and its traces are as follows:

$$N_i(E) = \text{tr} E, \quad N_2(E) = \frac{1}{2} \left[ (\text{tr} E)^2 - \text{tr} E^2 \right],$$

$$N_3(E) = \text{det} E = \frac{1}{3} \text{tr} E^3 - \frac{1}{2} \text{tr} E^2 \text{tr} E + \frac{1}{6} \text{tr}^3 E.$$

(13)

Then the SE function of the M model (11) is of the form:

$$W = \frac{1}{2} (\lambda_0 + 2 \mu_0) M_1^2 - 2 \mu_0 M_2 + \frac{1}{3} (l + 2m) M_3 + nM_4 - 2mM_1M_2,$$

(14)

where there are basic invariants of deformation tensor $M_j = N_i(E)$, cf. (13).

Potential of the form:

$$W = \frac{1}{2} (\lambda_0 + 2 \mu_0) M_1^2 - 2 \mu_0 M_2 + \frac{1}{3} (l + 2m) M_3 + nM_4 - 2mM_1M_2 +$$

$$+ \nu_1 M_1^4 + \nu_2 M_1^2 M_2 + \nu_3 M_1 M_3 + \nu_4 M_2^2,$$

(15)

is also called the Murnaghan (FO) model with 9 constants of elasticity. In comparison to the function (11), in (15) we have an additional four terms and four fourth order elastic constants $\hat{\nu}_k$ ($k=1,2,3,4$). The exemplary constants of this model for aluminum are given in [5], see also [2,4].

The constitutive relation is as follows:

$$T = (\gamma_1 + M_1 \gamma_2) I - \gamma_2 E + \gamma_3 M_3 E^{-1} = (\gamma_1 + M_1 \gamma_2 - M_1 \gamma_3) I + (\gamma_2 + M_1 \gamma_3) E + \gamma_3 E^2 \equiv$$

$$\equiv \delta_1 I + \delta_2 E + \gamma_3 E^2,$$

(16)

with
5. Approximation of the SEF and constitutive relationships of the MCMV model

In the further part of the work, we use the square and cubic approximations of deformation tensors and their invariants to the tensor $E$. To explain the potential of elasticity and the constitutive relations of the MCMV model, we present below the useful formulas that result from the use of the generalized Taylor formula and the definition of the appropriate tensor quantities and their invariants.

Let us note at the outset that formulas (13) show an unambiguous relationship between the tensor's basic invariants $E$ and $C$, i.e.:

\[ I_1 = N_1(E) = 2M_1 + 3, \quad I_2 = N_2(E) = 3 + 4M_1 + 4M_2, \]

\[ I_3 = N_3(E) = J^2 = 1 + 2M_1 + 4M_2 + 8M_3, \]

where we remind that $M_i = N_i(E)$.

Accordingly, using the approximation formulas with respect to tensor $E$ in the case of the SE function of the MCMV model (6), we obtain the following formulas for material parameters in the SE function of the M model (15):

\[ \lambda_0 = K_0 - \frac{4}{3}(C_{10} + C_{01}), \quad \mu_0 = 2(C_{10} + C_{01}), \]

\[ l = \frac{8}{9}(C_{10} + 2C_{01}), \quad m = -K_0 - 4\left( C_{10} + \frac{4}{3}C_{01} \right), \quad n = -8(C_{10} + 2C_{01}). \]

In addition, in the ES (15) function there are the fourth order Murnaghan's constants of the form:

\[ \hat{\nu}_1 = K_0 + \frac{1}{27}(112C_{10} + 160C_{01} + 48C_{11} + 48C_{20}), \]

\[ \hat{\nu}_2 = -4K_0 - \frac{4}{9}(40C_{10} + 60C_{01} + 24C_{11} + 24C_{20}), \]

\[ \hat{\nu}_3 = 4K_0 + 16(C_{10} + 2C_{01}), \]

\[ \hat{\nu}_4 = 2K_0 + \frac{2}{3}(16C_{10} + 24C_{01} + 24C_{11} + 24C_{20}). \]
Note that there are only three material constants in dependencies (19) $K_0$, $C_{10}$ and $C_{01}$. On the other hand, only the approximation of the fourth order shows all the material parameters of the MCMV model.

We emphasize that in the generalized Mooney-Rivlin model [2,3,8], we have the following form of the SEF function:

$$W = C_{10} (\tilde{T}_1 - 3) + C_{01} (\tilde{T}_2 - 3) + K_0 \mathcal{P}(J),$$

(21)

6. Comparison of the applicability range of the MCMV and Murnaghan models

As mentioned in the introduction of the work, in the case of rubber-like materials, the volume compressibility module is several orders of magnitude greater than the shear modulus, i.e. in the case of a typical rubber we have $\mu_0 \equiv 0.4 \text{[MPa]}$ and $K_0 \equiv 1950 \text{[MPa]}$, cf. [3] and the literature cited there.

| Material   | $C_{10}$ [MPa] | $C_{20}$ [MPa] | $C_{30}$ [MPa] | $C_{01}$ [MPa] | $C_{11}$ [MPa] |
|------------|----------------|----------------|----------------|----------------|----------------|
| neoprene   | 1.865E–01      | –3.140E–03     | 3.567E–05      | 1.329E–02      | –6.573E–05     |
| rubber     | 1.792E–01      | –1.896E–03     | 4.517E–05      | 7.725E–03      | –9.000E–05     |

In table 1 we summarize the material parameters of the MV incompressible material model and the FO model in two variants, i.e. determined on the basis of approximation of the results of the studies of Alexander [12] and Treloar [20].

The parameters of the fourth order IV of the FO model as a function of the initial volume compressibility module are:

a) neoprene

$$\tilde{v}_1 = K_0 + 0.8465, \tilde{v}_2 = -4K_0 - 3.635, \tilde{v}_3 = 4K_0 + 3.409, \tilde{v}_4 = 2K_0 + 2.1502 \text{[MPa]},$$

(22)

b) rubber

$$\tilde{v}_1 = K_0 + 0.7857, \tilde{v}_2 = -4K_0 - 3.3712, \tilde{v}_3 = 4K_0 + 3.115, \tilde{v}_4 = 2K_0 + 2.004 \text{[MPa]}.$$  

(23)

Thus, it can be seen that in the case of rubber-like materials, these parameters are practically multiple of $K_0$, $\tilde{v}_1$, $\tilde{v}_3$, and $\tilde{v}_4$ are positive, while $\tilde{v}_2$ is negative.

The attempt to assess the applicability of the approximation of the MCMV model with the M and FO models was carried out on the basis of the verification of the compatibility of the SE function and the stress-elongation relationship with homogeneous deformations.
Figure 1. Graphs of the SEF in the case of a plane deformation in cross-sections a) $\lambda_1 = \lambda_2$, b) $\lambda_2 = 1$, c) and d) $\lambda_1 = \lambda_1^{-1}$ respectively with $K_0 = 10\mu_0$ and $K_0 = 1000\mu_0$ (neoprene)

Figure 1 presents the graphs of the SEF of the MCMV, M and FO models for three tests: biaxial uniform stretching in a plane deformation state (figure 1a), uniaxial strain (figure 1b) and simple shear, respectively $K_0 = 10\mu_0$ and $K_0 = 1000\mu_0$ (figure 1 c and d). However, figure 2 contains contour diagrams of the energy function in a plane state of strain, depending on the elongation, including the constraints of incompressibility $J = 1$ and constraints $\text{tr} \, E = 0$.

The diagrams contained in figures 1 and 2, cf. [3], indicate unambiguously that the multilayer models postulated on the basis of invariants of the Lagrange tensor constitute an incorrect approximation of a small-volume material with a potential of elasticity (11). This is related formally to the loss of proper consideration of the description of the multiplicative decomposition of a deformation, in particular in the neighborhood of kinematic constraints $J = 1$. The applicability range then depends significantly on the adopted value of the compressibility modulus. As it grows, there is a clear approximation of the FE function to the function describing constraints $\text{tr} \, E = 0$, while in the case of the MCMV model we observe this effect in relation to the constraints of incompressibility $\det C = 1$, see figure 2. At the $K_0 = 1000\mu_0$, similar to rubber-like materials, it can be assumed that the
M model is an acceptable approximation only in the range \( \lambda = (0.98, 1.02) \). The FO model slightly expands this range, i.e. \( \lambda = (0.9, 1.1) \) what in this case of rubber-like materials is not a satisfactory result.

\[ \text{Figure 2. Graphical plots of the ES function in the case of a plane state of deformation (neoprene) a) } K_0 = 10 \mu_0 \text{ and b) } K_0 = 1000 \mu_0 , \text{ red lines - the constraints of incompressibility } J = 1 , \text{ blue lines - the function describing constraints } \text{tr } E = 0 . \]

In the case of volumetric deformation, i.e. \( J = \lambda^3 \) we get the following expressions describing the elastic energy \( W_{vol} \):

\[
\begin{align*}
\text{M: } W_{vol}(J) &= \frac{9}{8} K_0 \left( J^{2/3} - 1 \right)^2, \\
\text{FO: } W_{vol}(J) &= \frac{3}{16} K_0 \left( J^{2/3} - 1 \right) \left( J^{4/3} - 2 J^{2/3} + 7 \right), \\
\text{MCMV: } W_{vol}(J) &= K_0 \bar{\sigma}(J),
\end{align*}
\]

7. Example of a boundary value problem.

The assessment of the applicability of the discussed models is also performed on the basis of a boundary problem with non-uniform deformations. It concerns axial compression of a pipe with characteristic dimensions: height \( h = 20 R_m \), internal radius \( R_w = 0.95 R_m \), external radius \( R_e = 1.05 R_m \), where \( R_m = 1 \text{[m]} \). The computation model consists of 6002 finite elements (3 elements by thickness) of type C3D8H with linear interpolation functions [11]. The task was solved using the M, FO and MCMV models and assuming the values of neoprene material parameters according to table 1 in two variants, i.e. \( K_0 = 10 \mu_0 \) and \( K_0 = 1000 \mu_0 \).
The load is realized by displacement control. For this purpose, at the one end of the pipe (B) boundary conditions \( u_1 = u_2 = u_3 = 0 \) and at the other end (A) \( u_1 = u_2 = 0, \ u_3 = -5R_m \) are given, according to the coordinate system as shown in figure 3.

![FEM mesh of the considered model](image)

**Figure 3.** FEM mesh of the considered model

Figure 4 shows the curves of the stress \( \sigma \) as a function of the displacement \( u_3 / R \) in the axial compression of the cylinder. The values \( \sigma \) were obtained by averaging the stress values \( \sigma_{3i} \) from the Gauss points of the first band of finite elements of the loaded cylinder end, cf. figure 3.

Observations regarding the scope of applicability of the M and FO models under consideration, described in point 6 are clearly reflected in the results obtained. In the case of the M model, the solution was obtained in only a small range of displacement \( u_3 / R \) and in both variants of the values of the compressibility modulus. The FO model for \( K_0 = 10 \mu_0 \) is a good approximation of the MCMV model in the range \( u_3 / R \approx -1.3 \). In the case of \( K_0 = 1000 \mu_0 \) the scope of the solution obtained is significantly smaller, i.e. \( u_3 / R \approx -0.25 \). Using the MCMV model, a solution was obtained in the full range of the given displacement in the two considered variants.

The critical stress values in the three material models used differ slightly at \( K_0 = 10 \mu_0 \). In the case of \( K_0 = 1000 \mu_0 \) a divergence of the algorithm in the problem with the model M appears much below the critical stress value obtained by the FO and MCMV models, cf. figure 4.
8. Conclusions

In this article the approximations of the model of hyperelastic MCMV materials, whose function of the stored energy function $W$ is the sum of part concerning isochoric deformation $W_i$ and the function of volumetric deformation $W_v$, of the models of Murnaghan M and FO materials with elasticity constant II and III (M) and II, III and IV (FO) order, were discussed. The models mentioned have respectively 6 (MCMV), 5 (M) and 9 (FO) material parameters.

The MCMV, M and FO models have identical second order elastic constants. Third order constants depend on three material parameters, $K_0$, $C_{ij0}$ and $C_{ijkl0}$. Two of the third order constants have negative values, and $m \equiv -K_0$. Only the approximation of the fourth order has all the material parameters of the MCMV model.

In the case of slightly compressible materials, we have the following approximate values of the fourth order: $\tilde{V}_1 \equiv K_0, \tilde{V}_2 \equiv -4K_0, \tilde{V}_3 \equiv 4K_0, \tilde{V}_4 \equiv 2K_0$.

It was shown that in a small range of deformations M and FO models, which are formally the models of compressible materials, are proper approximation of the MCMV model (a slightly compressible material model). This is particularly evident in the problems of plane deformation and very low compressibility of the material, i.e. $K_0 \approx 1000\mu_0$, when the plots of stored energy function should be close to the diagram of incompressibility constraints $J = 1$ instead of constraints $\text{tr} \mathbf{E} = 0$, which is the case for M and FO models. It relates to an incorrect consideration in the approximations of the stored energy function with the models M and FO of the multiplicative decomposition of a deformation.

References
[1] F.D. Murnaghan, Finite deformation of an elastic solids, J. Wiley, New York, 1951.
[2] S. Jemioło, and C. Suchocki, Hyperelasticity and its modifications. An outline of the theory, pseudo-hyperelasticity and quasi-linear visco-elasticity [in Polish], OW PW, Warsaw, 2018.
[3] S. Jemioło, Study of hyperelastic properties of isotropic materials. Modeling and numerical implementation, Scientific Works, Civil Engineering 140 [in Polish], OW PW, Warsaw, 2002.
[4] A.I. Lurie, Nonlinear theory of elasticity, North-Holland, Amsterdam-Tokyo 1990.
[5] S. Gopalakrishnan, Wave Propagation in Materials and Structures, CRC Press, Taylor & Francis Group, 2016.
[6] A. Romano, and A. Marasco, Continuum mechanics. Advanced topics and research trends, Brikhäuser, Boston-Basel-Berlin, 2010.
[7] Z. Wesołowski, Dynamical problems of nonlinear elasticity [in Polish], PWN, Warsaw, 1974.
[8] S. Jemioło, and A. Franus, Isotropic constitutive models of hyperelasticity. Determination of material parameters and numerical implementation [in Polish], OW PW, Warsaw, 2019.
[9] R.W. Ogden, Non-linear elastic deformations, Ellis Horwood, 1984.
[10] ABAQUS 2016 Analysis user’s guide. Volume III: Materials, Dassault Systèmes, 2015.
[11] ABAQUS 2016 Theory manual, Simulia, Dassault Systèmes, 2015.
[12] H. Alexander, A constitutive relation for rubber-like materials, Int. J. of Engineering Science, 6, pp. 549–563, 1968.
[13] P.G. Ciarlet, Mathematical elasticity. Vol.1: Three-dimensional elasticity. In: Studies in mathematics and its applications, Vol.20, North-Holland, Amsterdam-Tokyo, 1988.
[14] S. Doll, K. Schweizerhof, On the development of volumetric strain energy function, ASME J. Appl. Mech., 67, pp. 17–21, 2000.
[15] A. Franus, and S. Jemioło, Implementation of generalized MV models of hyperelastic slightly compressible materials in the ABAQUS program, Chapter XI in monograph: Nonlinear elasticity. Constitutive modelling, stability and wave problems [in Polish], S. Jemioło [ed], pp. 155-166, OW PW, Warsaw, 2019.
[16] G.A. Holzapfel, Nonlinear solid mechanics, John Wiley & Sons Ltd., New York 2010.
[17] James A. G., Green A., Simpson G. M.: Strain energy function of rubber. I. Characterization of gum vulcanizates, Journal of applied Polymer Science, 19, 1975, pp. 2033–2058.
[18] S. Jemioło [red.]: Elasticity and hyperelasticity. Modelling and applications (in Polish), OW PW, Warsaw 2012.
[19] R. S. Rivlin, Large elastic deformation of isotropic materials: I. Fundamental concepts. II. Some uniqueness theorem for pure homogeneous deformation, Phylosophical Transactions of Royal Society, A240, 1948, pp. 459–508.
[20] L.R.G. Treloar, Stress-strain data for vulcanized rubber under various types of deformation, Transactions of the Faraday Society, 40, pp. 59–70, 1944.