Mixmaster Hořava-Witten Cosmology

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We discuss various superstring effective actions and, in particular, their common sector which leads to the so-called pre-big-bang cosmology (cosmology in a weak coupling limit of heterotic superstring). Then, we review the main ideas of the Hořava-Witten theory which is a strong coupling limit of heterotic superstring theory. Using the conformal relationship between these two theories we present Kasner asymptotic solutions of Bianchi type IX geometries within these theories and make predictions about possible emergence of chaos. Finally, we present a possible method of generating Hořava-Witten cosmological solutions out of the well-known general relativistic or pre-big-bang solutions.

1. Introduction

There are various superstring effective theories (type I, type IIA, type IIB and heterotic $E_8 \times E_8$ or $SO(32)$) which can be represented by effective actions with suitable field equations which generalize general relativity. In view of duality symmetry the superstring effective actions are not necessarily the right description of physics at strong coupling, where string coupling parameter $g_s = \exp(\phi/2) \to \infty$. It appears that at strong coupling regime the physics is 11-dimensional and can be described by M-theory with its low-energy limit – 11-dimensional supergravity. One of the proposals for M-theory is Hořava-Witten theory. The main issue of this talk is to discuss the question about the implications of superstring/M-theory onto the evolution of the universe.

2. Cosmology of the common sector (pre-big-bang)

Let us start with the presentation of superstring effective actions. The first action under consideration is type IIA superstring effective action. It has $N = 2$ supersymmetries of opposite chirality and reads

$$S_{\text{IIA}} = \frac{1}{2\lambda_s^8} \left\{ \int d^10 x \sqrt{-g_{10}} \left[ e^{-\phi} \left( R_{10} + (\nabla\phi)^2 \right) - \frac{1}{12} (H_{10}^{(1)})^2 \right] - \frac{1}{2} (\nabla\phi)^2 \right\}. \quad (2)$$

Here the bosonic massless excitations arising in the NS–NS sector are the dilaton, $\phi$, the metric, $g_{10}$, the antisymmetric 2–form potentials, $B_2$, and the dilaton field, $\phi$. The RR sector contains antisymmetric $p$–form potentials, $F_p$, where $p$ is odd. The NS–NS sector couples directly to the dilaton, but the RR fields do not.

The bosonic type IIB superstring effective theory has $N = 2$ supersymmetries of the same chirality and its action reads

$$S_{\text{IIB}} = \frac{1}{2\lambda_s^8} \left\{ \int d^10 x \sqrt{-g_{10}} \left[ e^{-\phi} \left( R_{10} + (\nabla\phi)^2 \right) - \frac{1}{12} (H_{10}^{(1)})^2 \right] - \frac{1}{2} (\nabla\phi)^2 - \frac{1}{12} (H_3^{(2)})^2 + \chi H_3^{(1)} \right\}. \quad (1)$$

where $R_{10}$ is the Ricci scalar curvature of the spacetime with metric $g_{10}$ and $g_{10} \equiv \det g_{\mu\nu}, \lambda_s = \alpha'$ is the fundamental string length. Strings sweep out geodesic surfaces with respect to the metric $g_{10}$. The antisymmetric tensor field strengths are defined by $H_3 = dB_2, F_2 = dA_1, F_3 = dA_3$ and $F_4 = F_4 + A_1 \wedge H_3$, where $A_p, B_p$ denote antisymmetric $p$–form potentials and $d$ is the exterior derivative. The last term in \[1\] is a Chern–Simons term and is a necessary consequence of supersymmetry \[3\]. The NS–NS sector of the action contains the graviton, $g_{\mu\nu}$, the antisymmetric 2–form potential, $B_2$, and the dilaton field, $\phi$. The RR sector contains antisymmetric $p$–form potentials, where $p$ is odd. The NS–NS sector couples directly to the dilaton, but the RR fields do not.
$g_{\mu\nu}$, and the antisymmetric, 2–form potential, denoted here by $B_{\mu\nu}^{(1)}$. The RR sector contains a scalar axion field, $\chi$, a 2–form potential, $B_{\mu\nu}^{(2)}$ (we dropped a 4–form potential, $D_{\mu\nu\rho\sigma}$ here) and the RR field strengths are defined by $H_3^{(2)} = dB_2^{(2)}$ and $F_5 = dA_4 + B_2^{(2)} \wedge H_3^{(1)}$.

In heterotic superstring theories supersymmetry is imposed only in the right-moving sector so these theories are $N = 1$ supersymmetric. Quantization of the left-moving sector requires the gauge groups to be either $SO(32)$ or $E_8 \times E_8$ and the choice of the group depends on the boundary conditions. The heterotic superstring effective action reads

$$S_H = \frac{1}{2\lambda_s^3} \int d^{10}x \sqrt{g_{10}} e^{-\phi} \left[ R_{10} + \left( \nabla \phi \right)^2 - \frac{1}{12} H_3^2 - \frac{1}{4} F_2^2 \right]$$

(3)

where $F_2^2$ is the field strength corresponding to the gauge groups $SO(32)$ or $E_8 \times E_8$ and $H_3 = dB_2$ is the field strength of a 2–form potential, $B_2$.

The last of the superstring theories is an open string theory or type I theory. Because of the obvious reason (left and right moving sectors must be the same) it has $N = 1$ supersymmetry and the effective action reads

$$S_I = \frac{1}{2\lambda_s^3} \int d^{10}x \sqrt{g_{10}} \left[ e^{-\phi} \left( R_{10} + \left( \nabla \phi \right)^2 \right) - \frac{1}{12} H_3^2 - \frac{1}{4} e^{-\phi/2} F_2^2 \right]$$

(4)

where $F_2^2$ is the Yang–Mills field strength taking values in the gauge group $G = SO(32)$ and $H_3 = dB_2$ is the field strength of a 2–form potential, $B_2$. We note that this field strength is not coupled to the dilaton field in this frame and since both actions (3) and (4) have the same particle content this is the only difference between the two theories.

It is not difficult to notice that all the above theories (3), (4) in the string frame have the common sector which (except different coupling of $H_3$ in type I) is

$$S = \frac{1}{2\lambda_s^3} \int d^{10}x \sqrt{g_{10}} e^{-\phi} \left[ R_{10} + \left( \nabla \phi \right)^2 - \frac{1}{12} H_3^2 \right].$$

(5)

In particular, (3) represents a weak coupling limit ($g_s \to 0$) of heterotic $E_8 \times E_8$ theory. It is also the zeroth–order expansion in both the string coupling $g_s$ and the inverse string tension $\alpha'$. The common sector (5), after a suitable dimensional reduction to 4 dimensions on a constant Calabi-Yau manifold, gives the following elementary cosmological solutions for flat isotropic Friedmann geometry (with no axion $H_3 = 0$)

$$a(t) = \left| t \right|^{\frac{1}{\sqrt{3}}},$$

$$e^{\phi(t)} = \left| t \right|^{\pm \sqrt{3} - 1},$$

(6)

where $a$ is the scale factor and $\phi$ the dilaton. These solutions led to the cosmological scenario which is called pre-big-bang cosmology (3–5). It is easy to notice that the solutions (6) admit a phase of expansion for negative times as well as for positive times (with the singularity formally located at $t = 0$). It is both a curvature singularity (Ricci scalar diverges) and a string coupling singularity $g_s \to \infty$.

From (6) we realize that there are four possible types of the evolution for the scale factor together with four corresponding types of evolution for the dilaton. These with ‘−’ sign in (6) will be numbered as 1 and 2 while those with ‘+’ sign in (6) will be numbered as 3 and 4. All of them are commonly called branches. Branches 1 and 3 apply for negative times ($t < 0$) while branches 2 and 4 apply for positive times ($t > 0$). The four types of evolution are connected by the underlying symmetry of string theory namely $T$–duality (also called $O(d,d)$ symmetry [6]) which in the context of isotropic cosmology is called scale factor duality (SFD). Its mathematical realization is given by the relation which interchange the scale factor and the dilaton, leaving field equations unchanged, i.e.,

$$a(t) \iff \frac{1}{a(t)},$$

$$\phi(t) \iff \phi(t) - \ln a(t).$$

(7)

(8)

SFD relates 1 and 3 or 2 and 4 whose domains are either for $t < 0$ or $t > 0$. However, there is also time-reflection symmetry

$$t \iff -t,$
which together with SFD gives relation between \(1\) and \(4\) as follows

\[
a_1(t) = (-t)^{-\frac{1}{4}} \iff t^{-\frac{1}{3}} = \frac{1}{a_4(-t)}. \quad (10)
\]

It is easy to show that for branch \(1\)

\[
\frac{\ddot{a}_1}{a_1} > 0,
\]

which means that it describes inflation which undergoes without a violation of the energy conditions and it is called superinflation. This comes form the fact that there is only kinetic term for the dilaton in the action (5) and there is no potential energy at all. Let us remind that standard inflation is potential-energy-driven inflation. It is easy to notice that the branch \(4\) is deflationary, i.e.,

\[
\frac{\ddot{a}_4}{a_4} < 0,
\]

and it describes standard radiation-dominated evolution. Branches \(1\) and \(4\) are duality-related, though they are divided by the singularity of curvature and strong coupling. The solutions with axion are qualitatively the same – still there is a curvature singularity and strong coupling singularity though a bounce of the scale factor \(a(t)\) appears.

There are some problems with pre-big-bang scenario. One of the most interesting is that in the conformally related Einstein frame the action (6) is the same as the Einstein relativity minimally coupled to a scalar field (or stiff fluid pressure = energy density) and the solutions for negative times \(t < 0\) are just collapsing solutions and no superinflation is present. Another is the so-called "graceful-exit" problem which is a possible physical mechanism for a transition from superinflation to a radiation-dominated universe. An interesting issue appears if one considers less symmetric geometries like homogeneous Bianchi or Kantowski-Sachs type models. In that case a choice of the antisymmetric tensor potential is unclear since one is possible to make it timely \((B_{\mu \nu} = B_{\mu \nu}(t))\) or spatially \((B_{\mu \nu} = B_{\mu \nu}(x))\) dependent. The first case (though makes energy momentum tensor time-dependent only) cannot be admitted to some geometries and even if it can, it may prevent isotropization of the universe for late times which is observationally unfavourable.

3. M-theory and Hořava-Witten theory

M-theory is defined as strong coupling limit of superstring theories. It is 11-dimensional and it has got as a weak coupling limit 11-dimensional supergravity with the action

\[
S_{SUGRA} = \frac{1}{16\pi G_{11}} \left( \int d^{11}x \sqrt{-g_{11}} \left[ R_{11} - \frac{1}{48} F_{4}^2 \right] \right.
\]

\[
+ \frac{1}{6} \int A_3 \wedge F_4 \wedge F_4, \quad (13)
\]

where \(G_{11}\) is 11-dimensional Newton constant and the Chern–Simons term arises as a direct consequence of the \((N=1)\) supersymmetry.

It has been shown that the compactification of \(N = 1, D = 11\) supergravity on a circle, \(S^1\), results in the type IIA supergravity theory which was interpreted as the strongly coupled limit of the type IIA superstring (with \(N = 2\) supersymmetries) in terms of an 11-dimensional theory. This correspondence gave Hořava and Witten the idea that one can also compactify eleven-dimensional supergravity on a \(S^1/Z_2\) orbifold (which is a unit interval \(I\)) in order to get a heterotic theory with only \(N = 1\) supersymmetry. Other words, they proved that the 10-dimensional \(E_8 \times E_8\) theory results from an 11-dimensional theory compactified on the orbifold \(R^{10} \times S^1/Z_2\) in the same way as the type IIA theory results from an 11-dimensional theory compactified on \(R^{10} \times S^1\). This identifies strongly coupled limit of heterotic \(E_8 \times E_8\) theory as 11-dimensional supergravity compactified on an orbifold. The action for such a theory reads as

\[
S = S_{SUGRA} + S_{YM}, \quad (14)
\]

where

\[
S_{YM} = -\frac{1}{8\pi \kappa^2_{11}} \left( \frac{e_{11}}{4\pi} \right)^\frac{1}{2} \left\{ \int_{M_{10}^{(1)}} \sqrt{-g_{10}} \left[ tr \left( F^{(1)} \right)^2 \right.ight.
\]

\[
- \frac{1}{2} tr R^2 \bigg] + \int_{M_{10}^{(2)}} \sqrt{-g_{10}} \left[ tr \left( F^{(2)} \right)^2 - \frac{1}{2} tr R^2 \right] \bigg\}, \quad (15)
\]
The action (15) is composed of the two $E_8$ Yang-Mills theories on 10-dimensional orbifold fixed planes – manifolds $M^{(i)}_{10}$ ($i = 1, 2$) and $F^{(i)}$ are the two gauge field strengths.

4. Hořava-Witten cosmological solutions

For further cosmological investigations we can compactify Hořava-Witten models on a Calabi-Yau deformed manifold $X$ with orbifold coordinate to make decomposition $M^{(1)} = M^4 \times X \times S^1/Z_2$. It is important that the size of the orbifold is much bigger than the radius of the Calabi-Yau space and we can discuss 5-dimensional effective theory with the action [3]

$$S = \int_{M^5} \sqrt{-g_5} \left(\frac{1}{2} R - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{6} \alpha_5 e^{-2\sqrt{2} \phi} \right)$$

$$+ \sqrt{2} \sum_{i=1}^{2} \int_{M^4(i)} \sqrt{-g_4} \alpha_0 e^{-\sqrt{2} \phi}, \quad (16)$$

where $M^{(1)}_4$, $M^{(1)}_5$ are orbifold fixed planes, $\phi = 1/\sqrt{2} \ln V$ is a scalar field (dilaton) which parametrizes the radius of Calabi-Yau space and $g_{ij}, i, j = 0, 1, 2, 3$ is the pull-back of 5-dimensional metric onto $M^{(1)}_4$ and $M^{(1)}_5$. In the action (15) we dropped other important fields like p-form fields, gravitini, RR scalar and fermions.

The effective field equations for the action (16) are $(i, j = 0, 1, 2, 3, \mu, \nu = 0, 1, 2, 3, 5)$

$$R^\nu_\mu = \nabla_\mu \nabla^\nu \phi + \frac{\alpha_0^2}{9} g^{\nu} g^{\mu} e^{-2\sqrt{2} \phi} + \sqrt{2} \alpha_0 e^{-\sqrt{2} \phi} \left[ g^{\mu \nu} - \frac{1}{3} g^{\mu \nu} g_{\sigma \tau} g^{\sigma \tau} \right] [\delta(y) - \delta(y - \pi \lambda)] \quad (17)$$

$$\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} \partial^\mu \phi \right) = -\frac{\sqrt{2}}{3} \alpha_0^2 e^{-2\sqrt{2} \phi} \left[ \delta(y) - \delta(y - \pi \lambda) \right]. \quad (18)$$

In (18) $y \equiv x^5 \in [-\pi \lambda, \pi \lambda]$ is a coordinate in the orbifold direction and the orbifold fixed planes are at $y = 0, \pi \lambda$. $Z_2$ acts on $S^1$ by $y \rightarrow -y$. The terms involving delta functions arise from the stress energy on the boundary planes.

Before going further we present the form of 11-dimensional metric for our cosmological solutions which is

$$ds^2_{11} = e^{-\frac{2\sqrt{2} \phi}{3}} g_{\mu \nu} dx^\mu dx^\nu + e^{\frac{2\sqrt{2} \phi}{3}} \Omega_{mn} dy^m dy^n. \quad (19)$$

and $m, n = 6, \ldots, 11$ so that the last term is simply Calabi-Yau metric. The 5-dimensional metric is given by

$$ds^2_5 = g_{\mu \nu} dx^\mu dx^\nu = -N^2(\tau, y) d\tau^2 + ds^2_3 + d^2(\tau, y) dy^2. \quad (20)$$

and our main task in this paper is to consider the most general form of the 3-metric of homogeneous type Bianchi IX (or Mixmaster)

$$ds^2_3 = a^2(\tau, y)(\sigma^1)^2 + b^2(\tau, y)(\sigma^2)^2 + c^2(\tau, y)(\sigma^3)^2. \quad (21)$$

where the orthonormal forms $\sigma^1, \sigma^2, \sigma^3$ are given by

$$\sigma^1 = \cos \psi d\theta + \sin \psi \sin \theta d\varphi, \quad (22)$$

$$\sigma^2 = \sin \psi d\theta - \cos \psi \sin \theta d\varphi, \quad (23)$$

$$\sigma^3 = d\psi + \cos \theta d\varphi, \quad (24)$$

and the angular coordinates $\psi, \theta, \varphi$ span the following ranges,

$$0 \leq \psi \leq 4\pi, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi \leq 2\pi. \quad (25)$$

One should notice that under a choice $\sigma^1 = dx^1, \sigma^2 = dx^2, \sigma^3 = dx^3$ one gets Bianchi type I flat geometry.

Similarly as in [10] we will look for separable solutions of the form

$$N(\tau, y) = n(\tau) \tilde{\alpha}(y), \quad a(\tau, y) = \alpha(\tau) \tilde{a}(y), \quad b(\tau, y) = \beta(\tau) \tilde{a}(y), \quad c(\tau, y) = \gamma(\tau) \tilde{a}(y), \quad (26)$$

$$d(\tau, y) = \delta(\tau) \tilde{d}(y), \quad V(\tau, y) = e^{\sqrt{2} \phi(\tau, y)} = \varepsilon(\tau) \tilde{V}(y).$$

In fact $\alpha(\tau), \beta(\tau), \gamma(\tau)$ are worldvolume scale factors and $\delta(\tau)$ is an orbifold scale factor. It appears that the suitable equations of motion are fully separable into the orbifold-dependent part and spacetime-dependent part provided [11]

$$n(\tau) = 1, \quad \delta(\tau) = \varepsilon(\tau). \quad (27)$$
where the first condition is simply the choice of the lapse function, while the second tells us that Calabi-Yau space is tracking the orbifold. One can show that orbifold-dependent part can be solved by
\begin{align*}
\tilde{a} &= a_0 H^{1/2}(y), \\
\tilde{d} &= d_0 H^2(y), \\
\tilde{V} &= d_0 H^3(y),
\end{align*}
where
\begin{align*}
H(y) &= \frac{\sqrt{2}}{3} \alpha_o |y| + h_0, \\
H''(y) &= \frac{2\sqrt{2}}{3} \alpha_o \left[ \delta(y) - \delta(y - \pi \lambda) \right],
\end{align*}
and we have applied
\begin{align*}
|y'|^2 = \epsilon(y) - \epsilon(y - \pi \lambda) - 1,
\end{align*}
so that
\begin{align*}
|y'|^2 = 2\delta(y) - 2\delta(y - \pi \lambda),
\end{align*}
(factor 2 comes from the fact that y is periodic) and
\begin{align*}
\epsilon(y) = 1 & \quad if \quad y \geq 0, \\
\epsilon(y) = -1 & \quad if \quad y < 0.
\end{align*}
After all these substitutions one can write down 5-dimensional metric in the form
\begin{align*}
d^2 = -a_0^2 H(y)d\tau^2 + a_0^2 H(y) [\alpha^2(\tau)(\sigma^1)^2 \\
+ \beta^2(\tau)(\sigma^2)^2 + \gamma^2(\tau, y)(\sigma^3)^2] + d^2 H^4(y)d^2 y^2.
\end{align*}
Elementary solutions of Hořava-Witten theory for Friedmann flat geometry analogous to [3] in pre-big-bang cosmology are given by taking one scale factor \(\tilde{a} = \alpha = \beta = \gamma\) in [3] and read
\begin{align*}
\tilde{a}(\tau) &= |\tau|^{p_\tau}, \quad p_\tau = 3/11 \mp 4/11\sqrt{3}, \\
\delta(\tau) &= |\tau|^{q_\tau}, \quad q_\pm = 2/11 \pm 4\sqrt{3}/11,
\end{align*}
for the worldvolume and orbifold respectively. From [3] one can easily deduce that there are four types of evolution of the worldvolume \(M^4\) and the orbifold, namely: both worldvolume and orbifold contracts, both worldvolume and orbifold expand, worldvolume contracts while orbifold expands (superinflationary) and worldvolume expands while orbifold contracts. The former case corresponds to superinflation while the latter to standard radiation-dominated evolution in pre-big-bang scenario.

5. Kasner asymptotics of Mixmaster Hořava-Witten and pre-big-bang solutions

Using the relationship between weakly (pre-big-bang [3]) and strongly (Hořava-Witten [5]) coupled heterotic string theories [12] we study the problem of the emergence of chaotic oscillations in Mixmaster cosmologies based on these theories. In particular, we discuss Kasner asymptotic states (anisotropic solutions of zero curvature to the field equations) of homogeneous Bianchi type IX geometries in these string cosmologies. In order to present the time-dependent part of the Hořava-Witten field equations we use a new time coordinate [3]
\begin{align*}
d\eta = \frac{d\tau}{\alpha^3 \beta \gamma \delta}.
\end{align*}
From now on we will use the notation \((...)_\eta = d/d\eta\). To further simplify the equations we additionally define
\begin{align*}
\hat{\alpha} = \ln \alpha, \quad \hat{\beta} = \ln \beta, \quad \hat{\gamma} = \ln \gamma, \quad \hat{\delta} = \ln \delta,
\end{align*}
so that we get
\begin{align*}
\left( \hat{\alpha} + \hat{\beta} + \hat{\gamma} + \hat{\delta} \right)_{,\eta} + \frac{1}{2} \hat{\beta}^2_{,\eta} &= 2 \left( \hat{\alpha}_{,\eta} \hat{\beta}_{,\eta} + \hat{\alpha}_{,\eta} \hat{\gamma}_{,\eta} \right) \\
+ \hat{\beta}_{,\eta} \hat{\gamma}_{,\eta} + 2 \left( \hat{\alpha}_{,\eta} + \hat{\beta}_{,\eta} + \hat{\gamma}_{,\eta} \right) \hat{\delta}_{,\eta},
\end{align*}
\begin{align*}
2\hat{\alpha}_{,\eta} &= \left( (\beta^2 - \gamma^2)^2 - \alpha^4 \right) \delta^2, \\
2\hat{\beta}_{,\eta} &= \left( (\alpha^2 - \gamma^2)^2 - \beta^4 \right) \delta^2, \\
2\hat{\gamma}_{,\eta} &= \left( (\beta^2 - \gamma^2)^2 - \alpha^4 \right) \delta^2, \\
\hat{\delta}_{,\eta} &= 0.
\end{align*}
The important point is that pre-big-bang field equations are the same except the constraint which now takes the form \((\delta = -\phi)\)
\begin{align*}
\left( \hat{\alpha} + \hat{\beta} + \hat{\gamma} \right)_{,\eta} - \delta^2_{,\eta} &= 2 \left( \hat{\alpha}_{,\eta} \hat{\beta}_{,\eta} + \hat{\alpha}_{,\eta} \hat{\gamma}_{,\eta} + \hat{\beta}_{,\eta} \hat{\gamma}_{,\eta} \right) \\
- 2 \left( \hat{\alpha}_{,\eta} + \hat{\beta}_{,\eta} + \hat{\gamma}_{,\eta} \right) \phi_{,\eta}.
\end{align*}
The Kasner asymptotic solutions are of the type
\begin{align*}
\alpha(\tau) = \alpha_0 \tau^{p_\alpha}, \beta(\tau) = \beta_0 \tau^{p_\beta}, \gamma = \gamma_0 \tau^{p_\gamma}, \delta = e^\phi = \delta_0 \tau^{p_\delta}.
\end{align*}
where $\alpha, \beta, \gamma$ are scale factors, $\phi$ is the scalar field, $\alpha_0, \beta_0, \gamma_0, \delta_0$ constants and $p_1, p_2, p_3, p_4$ the so-called Kasner indices (it is obvious that the ‘fourth’ Kasner index $p_4$ refers to the scalar field).

It is well-known that the approach to a singularity in Bianchi type IX models of the universe happens through a sequence of the Kasner-to-Kasner transitions (Mixmaster oscillations). These transitions can be described as the replacements of the Kasner indices of the type
$$p_i' \equiv p_i(p_i).$$

The main problem is to establish whether the number of these replacements will be finite (no chaos) or infinite (chaos). It is easy to notice from the field equations \[\text{[33]}\] that these transitions are possible only if the terms of the type
$$\alpha^4 \sigma^2 \propto \tau^{(2p_1+p_4)} = \tau^{(1+p_1-p_2-p_3)},$$
$$\beta^4 \sigma^2 \propto \tau^{(2p_2+p_4)} = \tau^{(1+p_2-p_3-p_4)},$$
$$\gamma^4 \delta^2 \propto \tau^{(2p_3+p_4)} = \tau^{(1+p_3-p_4-p_2)},$$

increase in the limit $\eta \to \infty$ ($\tau \to 0$). The increase is possible in the regions which are marked in the Figures 1 and 2. Fig. 1 corresponds to pre-big-bang while Fig. 2 corresponds to Hořava-Witten. However, this is not enough to answer the question about the infinite continuation of Kasner transitions since the increase may, in general, be replaced by the decrease of these indices in the marked regions either. What is important is whether the increase (or the decrease) of the indices refers to all of them and whether they all may have the same sign. If they do have the same sign, the expansion becomes monotonic - a singularity is reached and there is no chaos.

To establish that in both limits of the heterotic $E_8 \times E_8$ theory we obtain suitable conditions for Kasner indices ("Kasner sphere") which are \[\text{[14]}\]:
$$p_1 + p_2 + p_3 + p_4 = 1, \quad p_1^2 + p_2^2 + p_3^2 = 1 \quad (48)$$
for weak coupling limit (pre-big-bang) and \[\text{[11]}\]
$$p_1 + p_2 + p_3 + p_4 = 1, \quad p_1^2 + p_2^2 + p_3^2 + \frac{3}{2} p_4^2 = 1 \quad (49)$$
for strong coupling limit. For further discussion it is important that both Kasner asymptotics contain the isotropic solutions of Friedmann type as special cases in their domains.

For \[\text{[18]}\] these points are defined by $p_1 = p_2 = p_3 = \pm 1/\sqrt{3}, p_4 = \mp \sqrt{3} + 1$ while for \[\text{[49]}\] these points are defined by $p_1 = p_2 = p_3 = p_4 = 3/11 \mp 4/11 \sqrt{3}, p_4 = q_4 = 2/11 \pm 4 \sqrt{3}/11$. This means there is a nonzero region in the parameter space where all indices can have the same sign (e.g. all positive). Once it happens the chaotic scatterings of the anisotropic Bianchi IX model stop which means that it is impossible to approach singularity in a chaotic way.

6. Weak coupling versus strong coupling – how to generate Hořava-Witten solutions

In fact, there is a relationships between the pre-big-bang \[\text{[18]}\] and Hořava-Witten \[\text{[19]}\] solutions and the result of our previous section is an example of generation of solutions from those known in one theory, into the other - in particular, into solutions of Hořava-Witten theory \[\text{[12]}\]. This happens due to conformal relation between the theoreis:
$$g_{\mu \nu}^E = e^{-\phi} g_{\mu \nu}^S = e^{-\phi} g_{\mu \nu}^{HW}.$$  
(50)

where $E, S, HW$ refer to Einstein frame, string frame and Hořava-Witten, respectively, and the
relation is true, provided 3-branes are given by the separable ansatz equations \[ (17) \]. In particular, lots of exact solutions are available for the Einstein frame (general relativity + stiff fluid matter) - having them, one is able to generate Hořava-Witten type solutions and study their properties.

Our conclusions are as follows. Hořava-Witten theory admits Mixmaster type cosmology with Kasner type asymptotic solutions. Due to conformal relations one can generate Hořava-Witten cosmological solutions (with separable 3-brane) of many types and study their properties. Hořava-Witten Mixmaster cosmology (for truncated spectrum of particles), similarly as pre-big-bang (truncated) cosmology does not admit chaos.

Finally, one should mention some open issues in the topic. Firstly, one should study non-separable ansätze for Hořava-Witten cosmology and investigate how they relate to pre-big-bang. Secondly, one should study Mixmaster behaviour of cosmological models which involve non-truncated spectrum of particles and inhomogeneities \[ (17, 18) \].

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