Terms of Lucas sequences having a large smooth divisor

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Abstract. We show that the $Kn$–smooth part of $a^n - 1$ for an integer $a > 1$ is $a^{o(n)}$ for most positive integers $n.$

1 Introduction

It is known that if for every $n$, the sequence $(\binom{2n}{n})$ can be computed in $O(\log^k n)$ arithmetic operations for a fixed constant $k$, then integers can be factored efficiently [3, 5]. We ask if there exist linearly recurrent sequences which contain many small factors like $(\binom{2n}{n})$. If such sequences exist, they can be used instead of $(\binom{2n}{n})$ to factor integers. This is because the $n$th term of any linearly recurrent sequence can be computed in $O(\log n)$ arithmetic operations using repeated squaring of the companion matrix [1]. We first set up some notation to formally state our question.

Let $P(n)$ be the largest prime factor of $n$ and $s_y(n)$ be the largest divisor $d$ of $n$ with $P(d) \leq y.$ Thus, $s_y(n)$ is the $y$-smooth part of $n.$ Given a sequence $u = (u_n)_{n \geq 0}$ of positive integers we ask whether we can find $c > 1$ and $K$ such that

$$\mathcal{A}_{K,c,u} = \{ n : s_{Kn}(u_n) > c^n \}$$

contains many elements. For example, if $u_n = (\binom{2n}{n})$ is the sequence of middle binomial coefficients, then $\mathcal{A}_{2,2,u}$ contains all the positive integers. The main question we tackle in this paper can be formally stated as follows.

Question 1.1 Does there exist a linearly recurrent sequence $u$ such that $\mathcal{A}_{K,c,u}$ is infinite?

Here, we address the problem in the simplest case namely $u_n = a^n - 1$ for some positive integer $a$. Our results are easily extendable to all Lucas sequences, in particular, the sequence of Fibonacci numbers.

To start we recall the famous ABC-conjecture. Put

$$\text{rad}(n) = \prod_{p | n} p$$

for the algebraic radical of $n.$
**Conjecture**  For all $\varepsilon > 0$ there exists a constant $K_\varepsilon$ such that whenever $A, B, C$ are coprime nonzero integers with $A + B = C$, then

$$\max\{|A|, |B|, |C|\} \leq K_\varepsilon \text{rad}(ABC)^{1+\varepsilon}.$$ 

Throughout this paper, $a > 1$ is an integer and $u_n = a^n - 1$.

We have the following result.

**Theorem 1.1**  Assume the ABC conjecture. Then for any $K > 0$, $c > 1$, the set $\mathcal{A}_{K,c,u}$ is finite.

One can ask what can one prove unconditionally. Maybe we cannot prove that $\mathcal{A}_{K,c,u}$ is finite but maybe we can prove that it is thin, that is that it does not contain too many integers. This is the content of the next theorem.

**Theorem 1.2**  We have

$$\#(\mathcal{A}_{K,c,u} \cap [1, N]) \ll N \exp\left(-\frac{\log N}{156 \log \log N}\right).$$

In particular, if one wants to find for all large $N$ an interval starting at $N$ of length $k$, that is $[N + 1, \ldots, N + k]$ which has nonempty intersection with $\mathcal{A}_{K,c,u}$ then infinitely often one should take $k > \exp(\log N/(157 \log \log N))$. But if the ABC conjecture is true, one will no longer find elements of $\mathcal{A}_{K,c,u}$ in the above interval for large $N$ no matter how large $k$ is. Regarding Theorem 1.2, see [6] for a more general result which applies to any linearly recurrent sequence but which gives a slightly weaker bound when specialised to our sequence $u$.

2 Proofs

2.1 The proof of Theorem 1.1

We apply the ABC conjecture to the equation

$$a^n - 1 = st, \quad s := s_{Kn}(u_n), \quad t = (a^n - 1)/s$$

for $n \in \mathcal{A}_{K,c,u}$ with the obvious choices. Note that

$$\text{rad}(s) = \prod_{\substack{p \leq Kn \\text{p|a^n-1}}} p \quad \text{and} \quad t < (a/c)^n.$$ 

We then have

$$a^n \ll_\varepsilon (a \cdot \text{rad}(s)t)^{1+\varepsilon} \ll \left(\prod_{\substack{p \leq Kn \\text{p|a^n-1}}} p\right)^{1+\varepsilon} (a/c)^{n(1+\varepsilon)}.$$
We may of course assume that \(1 < c < a\). Then
\[
\sum_{p \leq Kn, p \mid a^n - 1} \log p \geq \frac{n}{1 + \epsilon} \left( \log a - (1 + \epsilon) \log(a/c) \right) + O(1).
\]
We choose \(\epsilon > 0\) small enough so that \(\log a - (1 + \epsilon) \log(a/c) > 0\). Then, we get
\[
(2.1) \quad S_{a,K}(n) := \sum_{p \leq Kn, p \mid a^n - 1} \log p \gg \epsilon n.
\]
The next lemma shows that the left–hand side above is \(\leq n^{2/3 + o(1)}\) as \(n \to \infty\). This is unconditional and finishes the proof of Theorem 1.1.

**Lemma 2.1** We have
\[
S_{K,a}(n) \leq K^{1/2} n^{1/2 + o(1)}
\]
as \(n \to \infty\).

**Proof** Let \(\ell_p\) be the order of \(a\) modulo \(p\); that is the smallest positive integer \(k\) such that \(a^k \equiv 1 \pmod{p}\). Since primes \(p\) participating in \(S_{K,a}(n)\) have \(p \mid a^n - 1\), it follows that \(\ell_p \mid n\). Since also such primes are \(O(n)\), it follows that
\[
S_{K,a} \ll \#P_{K,a} \log n,
\]
where \(P_{K,a}(n) := \{p \leq Kn : \ell_p \mid n\}\). To estimate \(P_{K,a}(n)\) we fix a divisor \(d\) of \(n\) and look at primes \(p \leq Kn\) such that \(\ell_p = d\). Such primes \(p\) have the property that \(p \equiv 1 \pmod{d}\) by Fermat’s Little Theorem. In particular, the number of such (without using results on primes in progressions) is at most
\[
\left\lfloor \frac{Kn}{d} \right\rfloor \leq \frac{Kn}{d}.
\]
However, since these primes divide \(a^d - 1\), the number of them is \(O(d)\). Thus, for a fixed \(d\) the number of such primes is
\[
\ll \min \left\{ \frac{Kn}{d}, d \right\} \ll (Kn)^{1/2}.
\]
Summing this up over all divisors \(d\) of \(n\) we get that
\[
\#P_{K,a}(n) \ll d(n) (Kn)^{1/2} \leq K^{1/2} n^{1/2 + o(1)}
\]
as \(n \to \infty\), where we used \(d(n)\) for the number of divisors of \(n\) and the well-known estimate \(d(n) = n^{o(1)}\) as \(n \to \infty\) (see Theorem 315 in [2]). Hence,
\[
S_{K,a}(n) \ll \#P_{K,a}(n) \log n \leq K^{1/2} n^{1/2 + o(1)}
\]
as \(n \to \infty\), which is what we wanted. \(\blacksquare\)

**Remark 2.2** The current Lemma 2.1 was supplied by the referee. Our initial statement was weaker. The combination between Lemma 2.1 and estimate (2.1) shows that
we can even take $K$ growing with $n$ such as $K = n^{1 - \epsilon}$ in the hypothesis of Theorem 1.1 and retain its conclusion. This has been also noticed in [4].

2.2 The proof of Theorem 1.2

It is enough to prove an upper bound comparable to the upper bound from the right-hand side of (1.1) for $\#(A_{K,c,u} \cap (N/2, N])$ as then we can replace $N$ by $N/2$, then $N/4$, etc. and sum up the resulting inequalities. So, assume that $n \in (N/2, N]$. We estimate

$$Q_N := \prod_{n \in (N/2, N]} s_{K_n}(u_n).$$

On the one hand, since $s_{K_n}(u_n) \geq s_{K_n}(u_n) \geq c^n \geq c^{N/2}$ for all $n \in A_{K,c,u}$, we get that

$$\log Q_N \gg N(\#(A_{K,c,u} \cap (N/2, N]).$$

Next, writing $v_p(m)$ for the exponent of $p$ in the factorisation of $m$, we have

$$\log Q_N = \sum_{n \in (N/2, N]} \sum_{p \leq KN} v_p(u_n) \log p \leq \sum_{p \leq KN} \log p \sum_{n \in (N/2, N]} v_p(u_n).$$

Let $o_p := v_p(u_{\ell_p})$. It is well-known that if $p$ is odd then

$$v_p(u_n) = \begin{cases} o_p + v_p(n), & \text{if } \ell_p | n; \\ 0, & \text{otherwise} \end{cases}$$

(see, for example, (66) in [7]). In particular, if $p \mid u_n$, then $p^{o_p} \mid u_n$. Furthermore, for each $k \geq 0$, the exact power of $p$ in $u_n$ is $o_p + k$ if and only if $\ell_p p^k$ divides $n$ and $\ell_p p^{k+1}$ does not divide $n$. When $p = 2$, we may assume that $a$ is odd (otherwise $v_2(u_n) = 0$ for all $n \geq 1$), and the right-hand side of the above formula needs to be amended to

$$v_2(u_n) = \begin{cases} o_2, & \text{if } 2 \mid n; \\ o_2 + v_2(a + 1) + v_2(n/2), & \text{if } 2 \not| n. \end{cases}$$

Thus, for odd $p$,

$$\sum_{n \in (N/2, N]} v_p(u_n) = o(p)\#\{N/2 < n \leq N : \ell_p | n\} + \sum_{k \geq 1} \#\{N/2 < n \leq N : \ell_p p^k | n\}.$$

A similar formula holds for $p = 2$. In particular, for $p = 2$, we have

$$\sum_{n \in (N/2, N]} v_2(u_n) = O(N).$$

Thus, the prime $p = 2$ contributes a summand of size $O(N)$ to the right-hand side of (2.2). From now on, we assume that $p$ is odd. The first cardinality in the right-hand side of formula (2.3) above is

$$\#\{N/2 < n \leq N : \ell_p | n\} \leq \left\lfloor \frac{N}{2\ell_p} \right\rfloor + 1 \ll \frac{N}{\ell_p}. \]
The remaining cardinalities on the right-above can be bounded as

\[ \# \{ N/2 < n \leq N : \ell_p p^k \mid n \} \leq \left\lfloor \frac{N}{2\ell_p p^k} \right\rfloor + 1 \ll \frac{N}{\ell_p p^k}. \]

Thus,

\[ \sum_{n \in (N/2, N]} v_p(u_n) \ll \frac{\nu_p(u_n)}{\ell_p} + \sum_{k \geq 1} \frac{N}{\ell_p p^k} \ll \frac{\nu_p(u_n)}{\ell_p} + \frac{N}{\ell_p}. \]

We thus get

\[ \log Q_N \ll N \sum_{p \leq Kt} \frac{a_p \log p}{\ell_p} + N \sum_{p \leq Kt} \frac{\log p}{\ell_p p} \ll N \sum_{p \leq Kt} \frac{a_p \log p}{\ell_p} := S. \]

It remains to bound $S$. Since $p^{a_p} \mid a^{t_p} - 1$, we get that $p^{a_p} < a^{t_p}$ so $a_p \log p \ll \ell_p$. Hence,

\[ S = N \sum_{p \leq Kt} \frac{a_p \log p}{\ell_p} \ll N \pi(Kt) \ll K \frac{N^2}{\log N}. \]

We get the first nontrivial upper bound on $\#(A_{K,c,u} \cap (N/2, N])$, namely

\[ N\#(A_{K,c,u} \cap (N/2, N]) \ll \log Q_N \ll S \ll \frac{N^2}{\log N} + N \log \log N \ll K \frac{N^2}{\log N}, \]

so

\[ \#(A_{K,c,u} \cap (N/2, N]) \ll K \frac{N}{\log N}. \]

To do better, we need to look more closely at $a_p \log p/\ell_p$ for primes $p \leq Kt$. We split the sum $S$ over primes $p \leq Kt$ in two subsums. The first is over the primes in the set $Q_1$ consisting of $p$ such that $a_p \log p/\ell_p < 1/y_N$, where $y_N$ is some function of $N$ which we will determine later. We let $Q_2$ be the complement of $Q_1$ in the set of primes $p \leq Kt$. The sum over primes $p \in Q_1$ is

\[ S_1 = N \sum_{p \in Q_1} \frac{a_p \log p}{\ell_p} \leq N \pi(Kt) \ll K \frac{N^2}{y_N \log N}. \]

For $Q_2$, we use the trivial estimate

\[ S_2 = N \sum_{p \in Q_2} \frac{a_p \log p}{\ell_p} \ll N \#Q_2, \]

and it remains to estimate the cardinality of $Q_2$. Note that $Q_2$ consists of primes $p$ such that $a_p > \ell_p/(y_N \log p) \gg \ell_p/(y_N \log N)$. We put $\ell_p$ in dyadic intervals. That is $\ell_p \in (2^i, 2^{i+1}]$ for some $i \geq 0$. Then primes $p \leq Kt$ in $Q_2$ with such $\ell_p$ have the property that $a_p \gg 2^{i/(y_N \log N)}$. Hence,
\[
\frac{2^i \#(Q_2 \cap (2^i, 2^{i+1}])}{y_N \log N} \ll \sum_{p \in Q_2 \cap (2^i, 2^{i+1})} \nu_p(a^\ell - 1) \log p \leq \sum_{\ell \in (2^i, 2^{i+1})} \log(a^\ell - 1)
\]

\[
\ll \sum_{\ell \in (2^i, 2^{i+1})} \ell \ll 2^{2i},
\]

which gives

\[
\#(Q_2 \cap (2^i, 2^{i+1})] \ll 2^i y_N \log N.
\]

Summing up over all the \(i\), we get

\[
\#Q_2 \leq 2^I y_N \log N,
\]

where \(I\) is maximal such that \((2^I, 2^{I+1})\) contains an element \(p\) of \(Q_2\). By a result of Stewart (see Lemma 4.3 in [7]),

\[
2^I < \ell_p < \alpha_p y_N \log N < p \exp\left(-\frac{\log p}{51.9 \log \log p}\right) y_N \log N \log \ell_p
\]

\[
\ll KN \exp\left(-\frac{\log(KN)}{51.9 \log \log(KN)}\right) y_N \log(KN)^2
\]

\[
\ll K N \exp\left(-\frac{\log N}{51.95 \log \log N}\right) y_N (\log N)^2.
\]

Thus,

\[
\#Q_2 \ll 2^I y_N \log N \ll K N \exp\left(-\frac{\log N}{51.95 \log \log N}\right) y_N^2 (\log N)^3
\]

\[
\ll N \exp\left(-\frac{\log N}{52 \log \log N}\right) y_N^2.
\]

Choosing \(y_N := \exp\left(c \frac{\log N}{\log \log N}\right)\) with a positive constant \(c\) to be determined later, we get

\[
N \#(A_{K,c,u} \cap (N/2, N]) \ll N \#Q_2 + \frac{N}{y_N \log N}
\]

\[
\ll K N \left(\exp\left(\left(2c - \frac{1}{52}\right) \frac{\log N}{\log \log N}\right) + \exp\left(-\frac{c \log N}{\log \log N}\right)\right).
\]

Choosing \(c := 1/156\), we get

\[
\#(A_{K,c,u} \cap (N/2, N]) \ll N \log N \exp\left(-\frac{\log N}{156 \log \log N}\right),
\]

which is what we wanted.

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