The Asymptotic Behavior of Bouncing Cosmological Models in $F(G)$ Gravity Theory

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Abstract

We reconstruct $F(G)$ gravity theory with an exponential scale factor to realize the bouncing behavior in the early universe and examine the asymptotic behavior of late-time solutions in this model. We propose an approach to the construction of asymptotic expansions of solutions of the Friedmann equations on the basis of Puiseux series.

1 Introduction

The Big Bang era is one of the least understood periods of the evolution of our Universe, and the physics behind this era is still inconceivable. The classical cosmological approach leads inevitably to an initial singularity, which is a rather "embarrassing" feature of the classical description, because due to this singularity, the closed time-like geodesics which pass from this singularity, have a finite proper length, but no end points to normal space away from the singularity. However, not so long ago there was an alternative description - the matter bounce scenario \cite{1,2,3,4}. In this scenario, in the contraction phase the universe is dominated by matter, and a non-singular bounce occurs. Also, the density perturbations whose spectrum is consistent with the observations can be produced (for a review, see \cite{5}). In addition, after the contracting phase, the so-called BKL instability \cite{6} happens, so that the universe will be anisotropic. The way of avoiding this instability \cite{7} and issues of the bounce \cite{8,9} in the Ekpyrotic scenario \cite{10} has been investigated \cite{11,12,13}. Moreover, the density perturbations in the matter bounce scenario with two scalar fields has recently been examined \cite{14}. On the other hand, various cosmological observations support the current cosmic accelerated expansion. To explain this phenomenon in the homogeneous and isotropic universe, it is necessary to assume the existence of dark energy, which has negative pressure, or propose that gravity is modified on large scales (for recent reviews on issues of dark energy and modified gravity theories, see, e.g., \cite{15,16,17,18,19,20}). Regarding the latter approach, there have been proposed a number of modified gravity theories such as $F(R)$ gravity. Bounces in modified gravity of $F(R)$ type mainly have been studied in \cite{21,22,23,24,25,26}. A relation between the bouncing behavior and the anomalies on the cosmic microwave background radiation also been discussed \cite{27}.
The asymptotic behavior of bouncing models is interesting due to various reasons. First of all, it may give the information about \( \Lambda \)CDM era of bounce cosmology \cite{28, 29, 25, 30, 38}. Second, it maybe important for the understanding of possible presence of weak singularities at the very-early and very-late bounce universe. Indeed, it is known that bounce universe maybe still weakly singular, see examples in \cite{22, 23}. The study of weak singularities in bouncing universe may suggest the way to remove finally all the singularities and construct the totally regular bounce. Finally, the asymptotic behavior of solutions may indicate new, not yet explored possibilities for the universe evolution.

2 \( F(\mathcal{G}) \) theory of gravity

In this paper, we explore bounce cosmology in \( F(\mathcal{G}) \) gravity. This class of modified gravity is based on the use of the Gauss-Bonnet invariant \( \mathcal{G} = R^2 - 4 R_{\mu \nu} R_{\mu \nu} + R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma} \), where \( R_{\mu \nu} \) is the Ricci tensor and \( R_{\mu \nu \rho \sigma} \) is the Riemann tensor.

The action of \( F(\mathcal{G}) \) gravity model is described as \cite{31}

\[
S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left( R + F(\mathcal{G}) \right) + S_{\text{matter}}, \tag{1}
\]

where \( g \) is the determinant of the metric tensor \( g_{\mu \nu} \) and \( S_{\text{matter}} \) is the matter action. We use units of \( k_B = c = \hbar = 1 \), where \( c \) is the speed of light, and denote the gravitational constant \( 8\pi G \) by \( \kappa^2 \equiv 8\pi / M_{Pl}^2 \) with the Planck mass of \( M_{Pl} = G^{-1/2} = 1.2 \times 10^{19} \) GeV.

It follows from this action that the gravitational field equation reads

\[
R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R - \frac{1}{2} g_{\mu \nu} F(\mathcal{G}) +
\begin{align*}
&+ \left( 2 R R_{\mu \nu} - 4 R_{\mu \nu} R_{\rho \sigma} + 2 R_{\mu}^{\rho \sigma \tau} R_{\nu \rho \sigma \tau} - 4 g_{\mu \rho \sigma} g_{\nu \alpha \beta} R_{\rho \sigma \alpha \beta} \right) F'(\mathcal{G}) - \\
&- 2 \left( \nabla_{\mu} \nabla_{\nu} F'(\mathcal{G}) \right) R + 2 g_{\mu \nu} \left( \Box F'(\mathcal{G}) \right) R - 4 \left( \Box F'(\mathcal{G}) \right) R_{\mu \nu} + \\
&+ 4 \left( \nabla_{\rho} \nabla_{\mu} F'(\mathcal{G}) \right) R_{\rho \nu} + 4 \left( \nabla_{\rho} \nabla_{\nu} F'(\mathcal{G}) \right) R_{\mu \rho} - 4 g_{\mu \nu} \left( \nabla_{\rho} \nabla_{\sigma} F'(\mathcal{G}) \right) R_{\rho \sigma} + \\
&+ 4 \left( \Box F'(\mathcal{G}) \right) g_{\mu \nu} g_{\alpha \beta} R_{\mu \rho \nu \sigma} = \kappa^2 T^{(\text{matter})}_{\mu \nu}.
\end{align*}
\]

Here, the prime denotes the derivative with respect to \( \mathcal{G} \), \( \nabla_{\mu} \) is the covariant derivative, \( \Box \equiv g^{\mu \nu} \nabla_{\mu} \nabla_{\nu} \) is the covariant d’Alembertian, and

\[
T_{\mu \nu}^{(\text{matter})} = \text{diag} \left( -\rho_{\text{matter}}, p_{\text{matter}}, p_{\text{matter}}, p_{\text{matter}} \right)
\]

is the energy-momentum tensor of matter, where \( \rho_{\text{matter}} \) and \( p_{\text{matter}} \) are the energy density and pressure of matter, respectively.

We take the flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric, given by

\[
ds^2 = -dt^2 + a^2(t) \sum_{i=1,2,3} \left( dx^i \right)^2,
\]

where \( a \) is the scale factor, \( H = \dot{a} / a \) is the Hubble parameter, and the dot shows the time derivative. In this background, we have \( R = 6H + 12H^2 \) and \( \mathcal{G} = 24H^2 \left( \dot{H} + H^2 \right) \). The gravitational field equations become \cite{32}

\[
6H^2 + F(\mathcal{G}) - \mathcal{G} F'(\mathcal{G}) + 24H^3 \dot{\mathcal{G}} F''(\mathcal{G}) = 2\kappa^2 \rho_{\text{matter}}, \tag{2}
\]
\[ 4\dot{H} + 6H^2 + F(G) - \mathcal{G} F'(G) + 16H\dot{\mathcal{G}} \left( \dot{H} + H^2 \right) F''(G) + 8H^2\dot{\mathcal{G}} F'(G) + 8H^2\mathcal{G}^2 F''(G) = -2\kappa^2 p_{\text{matter}}. \]

In what follows, we investigate only gravity part of the action in Eq. (1) without its matter part.

We examine the following form of the scale factor
\[ a(t) = \exp \left( \alpha t^2 \right), \quad \alpha > 0. \]

(3)

Here \( \alpha \) is a constant with the dimension of mass squared \([\text{Mass}^2]\). From this expression we have
\[ H(t) = 2\alpha t, \quad \mathcal{G}(t) = 192t^2\alpha^3(1 + 2t^2\alpha). \]

(4)

We should note that \( \mathcal{G} \geq 0 \) for any \( t \).

From Eq. (4) we see that a cosmological bounce happens in the early universe at the time \( t = 0 \). On the other hand, when \( \alpha t^2 \gg 1 \), the universe can be considered to be at the dark energy dominated stage, because taking into account Eq. (3), we get
\[ \ddot{a}(t) = 2\alpha (1 + 2\alpha t^2) \exp(\alpha t^2) > 0. \]

This implies the accelerated expansion of the universe happens. Thus we see that in the case when the scale factor is given by Eq. (3) the late-time cosmic acceleration as well as the bouncing behavior in the early universe can be realized in a unified manner.

### 3 Reconstruction method of \( F(G) \) gravity

Next, we reconstruct \( F(G) \) gravity models by using the method \cite{33, 34, 35}. Introducing proper functions \( P(t) \) and \( Q(t) \) of a scalar field \( t \), which is interpreted as the cosmic time, the action in Eq. (1) without matter is described as
\[ S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left( R + P(t)\mathcal{G} + Q(t) \right). \]

(5)

By varying this action with respect to \( t \), we obtain
\[ \frac{dP(t)}{dt}\mathcal{G} + \frac{dQ(t)}{dt} = 0. \]

(6)

Solving this equation in terms of \( t \), we get \( t = t(G) \). The substitution of \( t = t(G) \) into Eq. (3) yields \( F(G) = P(t)\mathcal{G} + Q(t) \). Using this equation and Eq. (2), we find
\[ Q(t) = -6H^2(t) - 24H^3(t)\frac{dP(t)}{dt}. \]

(7)

With this equation and the relation \( F(G) = P(t)\mathcal{G} + Q(t) \), we acquire
\[ 2H^2\frac{d^2P(t)}{dt^2} + 2H \left( 2\dot{H} - H^2 \right) \frac{dP(t)}{dt} + \dot{H} = 0. \]

(8)

Suppose the scale factor is given by Eq. (3), the general solution of Eq. (8) becomes
\[ P(t) = c_1 + c_2 \left( 2\alpha t_1 \right)^{1/2} F_1 \left( \frac{1}{2}; \frac{3}{2}; \alpha t^2 \right) - \frac{1}{t} e^{\alpha t^2} - \frac{1}{12} t^2 F_2 \left( 1, 1; 2 \frac{5}{2}; \alpha t^2 \right) - \frac{1}{8\alpha} \ln(\alpha t^2), \]
where \(c_1\) and \(c_2\) are arbitrary constants, \(\mathbf{1}_F_1(a; b; z)\) and \(\mathbf{2}_F_2(a_1, a_2; b_1, b_2; z)\) are generalized hypergeometric functions. From Eq. (7), we obtain

\[
Q(t) = -24\alpha^2 t^2 - 192c_2\alpha^3te^{\alpha t^2} + 48\alpha^2 t^2e^{\alpha t^2} \mathbf{1}_F_1\left(\frac{1}{2};\frac{3}{2};-\alpha t^2\right).
\]

Plugging this expression with Eq. (6), we have

\[
t = \pm \frac{1}{4\sqrt{3\alpha}} \sqrt{-12\alpha + \sqrt{6\sqrt{G} + 24\alpha^2}}, \quad G \geq 0.
\] (9)

Accordingly, by solving \(F(G) = P(t)G + Q(t)\), we find the most general form of \(F(G)\) as

\[
F(G) = c_1G + c_2\left(2\alpha tG \mathbf{1}_F_1\left(\frac{1}{2};\frac{3}{2};\alpha t^2\right) - (G + 192t^2\alpha^3)\frac{1}{t}e^{\alpha t^2}\right) - 24\alpha^2 t^2 +
\]

\[
+ 48\alpha^2 t^2e^{\alpha t^2} \mathbf{1}_F_1\left(\frac{1}{2};\frac{3}{2};-\alpha t^2\right) - \frac{1}{12}t^2G \mathbf{2}_F_2\left(1, 1;\frac{5}{2};\alpha t^2\right) - \frac{1}{8\alpha}G \ln(\alpha t^2),
\] (10)

where \(t\) determined by the expression (9).

### 4 Asymptotic behaviour of solutions

We explore the exponential form of the scale factor (3) and for this case Friedmann equation (2) has the form

\[
p_2(G)\frac{d^2F(G)}{dG^2} + p_1(G)\frac{dF(G)}{dG} + F(G) = b(G),
\] (11)

with

\[
p_2(G) = -192\alpha^2G - 16\alpha(G + 48\alpha^2)(12\alpha - \sqrt{6\sqrt{G} + 24\alpha^2}),
p_1(G) = -G,
b(G) = \frac{1}{2}\left(12\alpha - \sqrt{6\sqrt{G} + 24\alpha^2}\right).
\]

It is not difficult to see that the differential equation has two singularities. One of them \((G = 0)\) is a regular singularity and another \((G = \infty)\) is an irregular singularity.

One can consider the homogeneous equation corresponding to Eq. (11)

\[
\frac{d^2F(G)}{dG^2} + q_1(G)\frac{dF(G)}{dG} + q_2(G)F(G) = 0.
\] (12)

Here the coefficients \(q_1(G)\) and \(q_2(G)\) have the following form

\[
q_1(G) = \frac{p_1(G)}{p_2(G)}, \quad q_2(G) = \frac{1}{p_2(G)}.
\]

It is obviously that \(F(G) = G\) is a solution of Eq. (12).

We seek a solution of Eq. (11) in the neighborhood of \(G = 0\). First of all, we construct a fundamental system of solutions of the homogeneous equation (12). Since
the coefficients $q_k(\mathcal{G})$ for $k = 1, 2$ has a pole of order not higher than $k$ at $\mathcal{G} = 0$, then we can obtain

$$ q_1(\mathcal{G}) = \frac{\bar{q}_1(\mathcal{G})}{\mathcal{G}}, \quad q_2(\mathcal{G}) = \frac{\bar{q}_2(\mathcal{G})}{\mathcal{G}^2}, $$

where $\bar{q}_1(\mathcal{G})$ and $\bar{q}_2(\mathcal{G})$ are holomorphic functions in a neighborhood of $\mathcal{G} = 0$. We construct a fundamental system of solutions of Eq. (12) in the neighborhood of $\mathcal{G} = 0$. Solutions will be found in the form of a generalized series

$$ F(\mathcal{G}) = \mathcal{G}^{\mu} \sum_{k=0}^{\infty} A_k \mathcal{G}^k. $$

By combining this expression with Eq. (12), we acquire

$$ \mu (\mu - 1) + \bar{q}_1(0) \mu + \bar{q}_2(0) = 0. \quad (13) $$

By solving this equation, we get $\mu_1 = 1/2$ and $\mu_2 = 1$. The value $\mu = 1$ corresponds to solution $F(\mathcal{G}) = \mathcal{G}$. The second solution of Eq. (12) we represent

$$ F(\mathcal{G}) = \sqrt{\mathcal{G}} \varphi(\mathcal{G}), $$

where $\varphi(\mathcal{G})$ is holomorphic function in the neighborhood of $\mathcal{G} = 0$ at that $\varphi(\mathcal{G}) \neq 0$. Substituting Eq. (13) into the homogeneous equation, we obtain a recurrent system from which we consistently find the coefficients $A_0$, $A_1$, $\ldots$. Thus, the fundamental system of the homogeneous equation has the form

$$ F_1(\mathcal{G}) = \mathcal{G}, \quad F_2(\mathcal{G}) = \sqrt{\mathcal{G}} \left(1 - \frac{\mathcal{G}^2}{2^{12}3^3 \alpha^4} + O(\mathcal{G}^3)\right), \quad \mathcal{G} \to 0. $$

By solving the inhomogeneous equation (11) using the method of variation of constants, we obtain an approximate solution at $\mathcal{G} \to 0$

$$ F(\mathcal{G}) = c_1 \mathcal{G} + c_2 \sqrt{\mathcal{G}} \left(1 - \frac{\mathcal{G}^2}{2^{12}3^3 \alpha^4} + O(\mathcal{G}^3)\right) - \frac{1}{8\alpha} \mathcal{G} \ln \mathcal{G} + \frac{\mathcal{G}^2}{2^{8}3^2 \alpha^3} + O(\mathcal{G}^3), $$

where $c_1$ and $c_2$ are arbitrary constants. In addition, it should be mentioned that we could have obtained the same result if we expand Eq. (10) in a fractional power of $\mathcal{G}$.

Next, we will construct an asymptotic expansion of solution of Eq. (11) at $\mathcal{G} \to \infty$. As above, we find a fundamental system of solutions of homogeneous equation (12), where the coefficients of equation are represented as asymptotic series

$$ q_1(\mathcal{G}) = -\frac{1}{16\sqrt{6} \alpha} \frac{1}{\mathcal{G}^{1/2}} - \frac{1}{4\mathcal{G}} - \frac{3\sqrt{6} \alpha}{8} \frac{1}{\mathcal{G}^{3/2}} + \ldots, \quad q_2(\mathcal{G}) = -\frac{1}{\mathcal{G}} q_1(\mathcal{G}). $$

Asymptotic solutions will be found in the form of a Puiseux series

$$ F(\mathcal{G}) = \exp(\lambda \mathcal{G}^{1/2}) \mathcal{G}^{\sigma} \sum_{k=0}^{\infty} A_k \mathcal{G}^{-k/2} $$

By combining this expression with Eq. (12), we acquire

$$ \frac{\lambda}{\lambda - 1} = 0, \quad \sigma = \sqrt{6 - 36\alpha \lambda}.$$
Solutions of this equation are $\lambda_1 = 0$, $\sigma_1 = 1$ and $\lambda_2 = 1/(8\sqrt{6}\alpha)$, $\sigma = -1/4$. The first of them is correspond to solution $F(\mathcal{G}) = \mathcal{G}$. Therefore the asymptotic expansion of the second solution of Eq. (12) can be represented in the form

$$F(\mathcal{G}) = \exp\left(\frac{\mathcal{G}^{1/2}}{8\sqrt{6}\alpha}\right) \varphi(\mathcal{G}), \quad (14)$$

where $\varphi(\mathcal{G})$ is Puiseux asymptotic series at $\mathcal{G} \to \infty$. Substituting Eq. (14) into the homogeneous equation, we obtain a recurrent system from which we consistently find the coefficients $A_0, A_1, \ldots$. Thus, the asymptotic expansion for solution of Eq. (11) has the form

$$F(\mathcal{G}) = \exp\left(\frac{\mathcal{G}^{1/2}}{8\sqrt{6}\alpha}\right) \varphi(\mathcal{G}), \quad (14)$$

Asymptotic expansion for solution of inhomogeneous equation (11) will be found in the form

$$F(\mathcal{G}) = \mathcal{G} \sum_{k=0}^{\infty} A_k \mathcal{G}^{-k/2}. \quad (15)$$

Substituting Eq. (15) into Eq. (11) we find the coefficients $A_k$. As a result, we obtain

$$F(\mathcal{G}) = A_0 \mathcal{G} - \sqrt{6} \mathcal{G}^{1/2} - 18\alpha + O\left(\frac{1}{\mathcal{G}^{1/2}}\right), \quad \mathcal{G} \to \infty,$$

where $A_0$ is arbitrary constant.

Note that there is another asymptotic expansion for the solution of Eq. (11)

$$F(\mathcal{G}) = A_0 \mathcal{G} - \sqrt{6} \mathcal{G}^{1/2} - 18\alpha + O\left(\frac{1}{\mathcal{G}^{1/2}}\right) + B_0 \exp\left(\frac{\mathcal{G}^{1/2}}{8\sqrt{6}\alpha}\right) \varphi(\mathcal{G})^{-1/4} \left(1 + O\left(\frac{1}{\mathcal{G}^{1/2}}\right)\right), \quad \mathcal{G} \to \infty,$$

where $A_0$ and $B_0$ are arbitrary coefficients. Varying these coefficients leads to the asymptotic behaviour change. In particular, it is easy to see that if we take $A_0 = 0$ and $B_0 = 0$, the asymptotic behaviour solutions of Eq. (11) will be determined by $\mathcal{G}^{1/2}$.

In addition, consider bouncing cosmological models with scale factor

$$a(t) = \exp(\alpha t^2) + \exp(\alpha^2 t^4), \quad \alpha > 0.$$
5 Summary

We have reconstructed $F(G)$ gravity model with exponential scale factor and found that in this model the bouncing behavior can happen. Also, we have explored the behavior of solutions of Friedmann equations for this model at the singularities of the differential equation. In particular, the Puiseux series were used to obtain the asymptotic expansion at an irregular singularity. In addition, it has been verified that in a sum of two exponential functions model of the scale factor, asymptotic behavior at late-time cosmic acceleration is similar to that of the model discussed above. Recently there was an paper [38] one can demonstrated that in the context of LQC, it is possible to realize a deformed matter bounce scenario, in which the deformation practically alters the late-time behavior of the model. Would be interesting to apply the proposed mechanism for the constructed in our paper models.

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