Algebraizable Logics and a functorial encoding of its morphisms

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Abstract

The present work presents some results about the categorial relation between logics and its categories of structures. A (propositional, finitary) logic is a pair given by a signature and Tarskian consequence relation on its formula algebra. The logics are the objects in our categories of logics; the morphisms are certain signature morphisms that are translations between logics ($\text{AFLM}_1$, $\text{AFLM}_2$, $\text{AFLM}_3$, $\text{FC}$). Morphisms between (Lindenbaum) algebraizable logics ($\text{BP}$) are translations that preserves algebraizing pairs ($\text{MaMe}$); they can be completely encoded by certain functors defined on the quasi-variety canonically associated to the algebraizable logics. This kind of results can be usefull in the development of a categorial approach to the representation theory of general logics ($\text{MaPi}$)

1 Introduction

The main motivation of categories of logics was the combining logics. In the 1990’s rise many methods of combinations of logics ($\text{CC3}$). They appear in dual aspects: as processes of decomposition or analysis of logics (e.g., the ”Possible Translation Semantics” of W. Carnielli, $\text{[Car]}$) or as a processes of composition or synthesis of logics (e.g., the ”Fibrings” of D. Gabbay, $\text{[Ga]}$). The combining of logics is still a young topic in contemporary logic. Besides the pure philosophical interest of define mixed logic systems in which distinct operators obey logics of different nature, there also exist many pragmantical and methodological reasons for consider combined logics. The major concern in the study of categories of logics (CLE-UNICAMP, IST-Lisboa) is to describe condition for preservation, under the combination method, of meta-logical properties ($\text{CCCSS}$, $\text{ZSS}$). Our complementary approach to this field is study the ”global” aspects of categories of logics ($\text{AFLM}_1$, $\text{AFLM}_2$, $\text{AFLM}_3$, $\text{MaMe}$).

The initial steps on ”global” approach to categories of logics are given in the sequence of papers $\text{AFLM}_1$, $\text{AFLM}_2$, $\text{AFLM}_3$: they present very simple but too strict notions of logics and morphisms, with ”good” categorial properties ($\text{AR}$) but unsatisfactory treatment of the ”identity problem” of logics ($\text{Bez}$). More flexible notions of morphisms between logics are considered in $\text{FC}$, $\text{BCC1}$, $\text{BCC2}$, $\text{CG}$: this alternative notion allows better approach to the identity problem however has many categorical ”defects”. A ”refinement” of those ideas is provided in $\text{MaMe}$: are considered categories of logics satisfying simultaneously certain natural conditions: (i) represent the major part of logical systems; (ii) have good categorical properties; (iii) allow a natural notion of algebraizable logical system ($\text{BP}$, $\text{[?]}$); (iv) allow satisfactory treatment of the ”identity problem” of logics.

Another important category of logics is the category of algebraizable logics. About the ideas of describe a precise connection between Boolean algebra and classic propositional logic presented by Lindenbaum – Tarski, Blok and Pigozzi in 1989 $\text{BP}$ gave the concept of algebraizable logics for the first time as a mathematical definition. This is the notion of algebraizable logics that we use here.

In this work we established results about the categories above mentioned and the relationship between logics and its structure, more precisely, given a morphism of algebraizable logics, there is a contravariant functor between structures such that commutes over set and restricts to quasivariety. About this functor we also present an anti-isomorphism between the class of morphisms of signature and functores of structures. Moreover, we have an anti-isomorphism between morphisms of algebraizable logics and some functores among quasi-varieties.

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2 Preliminaries

The appearance of several process of combining of logics were the main motivations for the systematic study of categories of logics. The category theory is concerned in the relations between different mathematical objects. This is exactly the proposal in that we will apply the theory of categories in logic. Here the objects in this category of logics are signature and consequence operator pairs, the morphisms are translations between logics. In the study of categories of logics appears problems relating the logic properties and categories. In view of this, appear different definitions of categories of logics, more precisely, different definitions of morphisms between logic systems.

Definition 2.1. A signature is a sequence of pairwise disjoint sets $\Sigma = (\Sigma_n)_{n \in \mathbb{N}}$. In what follows, $X = \{x_0, x_1, \ldots, x_n, \ldots\}$ will denote a fixed enumerable set (written in a fixed order). Denote $F(\Sigma)$ (respectively $F(\Sigma)[n]$), the set of $\Sigma$-formulas over $X$ (respect. $\{x_0, \ldots, x_{n-1}\}$).

A Tarskian consequence relation is a relation $\Gamma \vdash \varphi$ on a signature $\Sigma = (\Sigma_n)_{n \in \mathbb{N}}$, such that, for every set of formulas $\Gamma$, $\Delta$ and every formula $\varphi, \psi$ of $F(\Sigma)$, it satisfies the following conditions:

- **Reflexivity:** If $\varphi \in \Gamma$, $\Gamma \vdash \varphi$
- **Cut:** If $\Gamma \vdash \varphi$ and for every $\psi \in \Gamma$, $\Delta \vdash \psi$, then $\Delta \vdash \varphi$
- **Monotonicity:** If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$
- **Finitarity:** If $\Gamma \vdash \varphi$, then there is a finite subset $\Delta$ of $\Gamma$ such that $\Delta \vdash \varphi$.
- **Structurality:** If $\Gamma \vdash \varphi$ and $\sigma$ is a substitution, then $\sigma[\Gamma] \vdash \sigma(\varphi)$

The notion of logic that we consider is:

**Definition 2.2.** A logic of type $\Sigma$, or a $\Sigma$-logic, is a pair $(\Sigma, \vdash)$ where $\Sigma$ is a signature and $\vdash$ is a Tarskian consequence relation. A logic is finitary if it satisfies the finitarity condition. A finitary logic is also called a sentential logic.

The set of all consequence relations on a signature $\Sigma$, denoted by $\text{Cons}_\Sigma$, is endowed with the partial order: $\vdash_0 \leq \vdash_1$ iff for each $\Gamma \subseteq (F(\Sigma))$, $\{\varphi \in F(\Sigma); \Gamma \vdash_0 \varphi\} = \Gamma^0 \subseteq \Gamma = \{\varphi \in F(\Sigma); \Gamma \vdash_1 \varphi\}$.

**Remark 2.3.** For each signature $\Sigma$, the poset $(\text{Cons}_\Sigma, \leq)$ is a complete lattice. It is in fact an algebraic lattice where the compact elements are the "finitely generated logics", i.e., the logics over $\Sigma$ given by a finite set of axioms and a finite set of (finitary) inference rules.

2.1 Categories of signatures and logics with strict morphism

Initially we define the category of signature with "strict" morphism $\mathcal{S}_s$ according to $\text{AFLM1}$, $\text{AFLM2}$ e $\text{AFLM3}$.

**Definition 2.4.** The objects of the category $\mathcal{S}_s$ are signature. If $\Sigma, \Sigma'$ are signature then a morphism $f : \Sigma \rightarrow \Sigma'$ is a sequence of functions $f = (f_n)_{n \in \mathbb{N}}$, where $f_n : \Sigma_n \rightarrow \Sigma'_n$. For each morphism $f : \Sigma \rightarrow \Sigma$ there is only one function $\hat{f} : F(\Sigma) \rightarrow F(\Sigma')$, called the extension of $f$, such that:

- $\hat{f}(x) = x$ if $x \in X$ ($X$ is a enumerable set)
- $\hat{f}(c) = f_0(c)$ if $c \in \Sigma_0$
- $\hat{f}(c_n(\psi_0, \ldots, \psi_{n-1})) = f_n(c_n)(\hat{f}(\psi_0), \ldots, \hat{f}(\psi_{n-1}))$ if $c_n \in \Sigma_n$, $n > 0$

Then, by induction, $\hat{f}(\varphi(\psi_0), ..., \psi_{n-1}) = \hat{f}(\varphi)(\hat{f}(\psi_0), ..., \hat{f}(\psi_{n-1}))$

The categories $\mathcal{S}_s$ and $\text{Set}^\mathbb{N}$ are equivalent, thus we have that $\mathcal{S}_s$ has good categorial properties, namely $\mathcal{S}_s$ is a finitely locally presentable category (fp) and the finitely presentable signatures are the "finite support" signatures.
Remark 2.5. (i) (Sub): For any substitution function \( \sigma : X \to F(\Sigma) \), there is only one extension \( \bar{\sigma} : F(\Sigma) \to F(\Sigma) \) such that \( \bar{\sigma} \) is an homomorphism \( \bar{\sigma}(x) = \sigma(x) \), for all \( x \in X \) and

\[
\bar{\sigma}(c_n(\psi_0, ..., \psi_{n-1})) = c_n(\bar{\sigma}(\psi_0), ..., \bar{\sigma}(\psi_{n-1}))
\]

for all \( c_n \in \Sigma_n, n \in \mathbb{N} \). The identity substitution induces the identity homomorphism on the formula algebra; the composition substitution of the substitutions \( \sigma, \sigma' : X \to F(\Sigma) \) is the substitution \( \sigma'' : X \to F(\Sigma), \sigma'' = \sigma \circ \sigma' := \sigma \circ \sigma \circ \sigma' \).

Let \( f : \Sigma \to \Sigma' \) be a \( S_s \)-morphism. Then for any substitution \( \sigma : X \to F(\Sigma) \) there is another substitution \( \sigma' \) such that \( \sigma' \circ \bar{f} = \bar{\sigma} \).

(ii) Let \( f : \Sigma \to \Sigma' \) and \( \theta \in F(\Sigma) \). If \( \text{var}(\theta) \subseteq \{x_0, ..., x_{n-1}\} \), then

\[
\bar{f}(\theta(\bar{x})[\bar{x}]) = \bar{f}(\theta(\bar{x}))[\bar{f}(\psi)].
\]

Moreover \( \text{var}(\bar{f}(\theta)) = \text{var}(\theta) \) and then \( \bar{f} \) restricts to maps \( \bar{f} \mid_n : F(\Sigma)[n] \to F(\Sigma')[n] \).

Now we give the definition of category of logics with "strict" morphism \( \mathcal{L}_s \).

Definition 2.6. The objects of \( \mathcal{L}_s \) are \( l = (\Sigma, \vdash) \), where \( \Sigma \) is a signature and \( \vdash \) is a tarskian consequence operator. A \( \mathcal{L}_s \)-morphism, \( f : l \to l' \) is a (strict) signature morphism \( f \in S_s(\Sigma, \Sigma') \) such that \( \bar{f} : F(\Sigma) \to F(\Sigma') \) is a \( (\vdash, \vdash') \)-translation: \( \Gamma \vdash \psi \Rightarrow \bar{f}(\Gamma) \vdash' \bar{f}(\psi) \).

\( \mathcal{L}_s \) is a \( \omega \)-locally presentable category and the fp logics are given by a finite set of "axioms" and "inference rules" over a fp signature.

Between the categories \( \mathcal{L}_s \) and \( S_s \) there exist a forgetful functor such that forget the consequence relation.

The categories above mentioned have good categorial properties, but unsatisfactory treatment of the logic problems, e.g., the "identity problem" of logics \( \text{Bez} \). Two presentation of classic propositional logic with signatures \( \{\neg, \to\} \) and \( \{\neg, \lor\} \) not admits strict morphism between them (because any such morphism must takes \( \to \) to \( \lor \) and they does not preserve \( \vdash \)) while it was expected that these presentations would be isomorphic.

2.2 Categories of signatures and logics with flexible morphism

In this moment, it is given a definition of category of logics where the ideas behind was found in [IKE], [FC], [BCC1], [BCC2] and [CG]. This definition give a appropriate treatment for the "identity problem" of logics.

Similarly to the previous case firstly we define the category of signature with "flexible" morphism \( \mathcal{S}_f \). Before to define this category, it is introduced the following notation:

If \( \Sigma = (\Sigma_n)_{n \in \mathbb{N}} \) is a signature, then \( T(\Sigma) := (F(\Sigma)[n])_{n \in \mathbb{N}} \) is a signature too. We have the inverse bijections (just notations): \( h \in S_f(\Sigma, \Sigma') \Leftrightarrow h^\dagger \in S_s(\Sigma, T(\Sigma'))\); \( f \in S_s(\Sigma, T(\Sigma')) \Leftrightarrow f^\ddagger \in S_f(\Sigma, \Sigma')\).

For each signature \( \Sigma \) and \( n \in \mathbb{N} \), let the function:

\[
(f\Sigma)_n : \Sigma_n \to F(\Sigma)[n]
\]

\( c_n \mapsto c_n(x_0, ..., x_{n-1}) \)

For each morphism \( f : \Sigma \to \Sigma' \) in \( \mathcal{S}_f \) there is only one function \( \bar{f} : F(\Sigma) \to F(\sigma') \), called the extension of \( f \), such that:
2.3 Other categories of logics

Definition 2.7. The category $S_f$ is the category of signature and flexible morphism as above. The composition in $S_f$ is given by $(f \bullet f^n) := (\tilde{f} \circ f^n)$. The identity $id_{\Sigma}$ in $S_f$ is given by $(id_{\Sigma})^n := ((\Sigma)_{n})_{n \in \mathbb{N}}$

As well as the category $S_s$, we have that the category $S_f$ satisfies the conditions of $\mathbb{Z}_f$.

Definition 2.8. The category $L_f$ is the category of propositional logics and flexible translations as morphisms. This is a category "built above" the category $L_f$, that is, there is an obvious forgetful functor $U_f : L_f \rightarrow S_f$.

Due to flexible morphism, this category allows better approach to the identity problem of logics. Consider the flexible morphism $f : \Sigma \rightarrow \Sigma'$ in $S_f$ such that "preserves the consequence relation", that is, for all $\Gamma \cup \{ \psi \} \subseteq F(\Sigma)$, if $\Gamma \vdash \psi$ then $\tilde{f}[\Gamma] \vdash \tilde{f}[\psi]$. Composition and identities are similar to $S_f$.

Remark 2.9. It follows easily from the facts above that the forgetful functor $U_f : L_f \rightarrow S_f : ((\Sigma, \lnot \rightarrow) \rightarrow (\Sigma', \lnot \rightarrow')) \rightarrow (\Sigma \rightarrow \Sigma')$ has a left and right adjoint functors: the left adjoint $\perp_f : S_f \rightarrow L_f$ and the right adjoint $\top_f : S_f \rightarrow L_f$ take a signature $\Sigma$ to, respectively, $\perp_f (\Sigma) = (\Sigma, \lnot_{\min})$ (the first element of Cons$_S$) and $\top_f (\Sigma) = (\Sigma, \lnot_{\max})$ (the last element of Cons$_S$). Moreover, $U_f \circ \perp_f = 1_{S_f}$, $U_f \circ \top_f = \perp_f$ and $U_f$ preserves all limits and colimits that exists in $S_f$.

Remark 2.10. It is known that $L_f$ has weak products, coproducts and some pushouts, and in the Remark above we see that $U_f$ preserves limits and colimits. As $U_f$ also "lift" limits and colimits - the constructions in $L_f$ are analogous to in $L_s$ (in $\mathbb{Z}_{fAM}$), just replace $\tilde{f}$ by $f$ - then given a small category $T$, $L_f$ is $T$-complete (respectively, $T$-cocomplete) if and only if $S_f$ is $T$-complete (respectively, $T$-cocomplete). As the category $S_f$ has colimits for any (small) diagram entails that $L_f$ has colimits for any (small) diagram "in $L_s$", in particular, it has all unconstrained fibrings (= coproducts) and the constrained fibrings (= pushouts) "based in $L_s$".

2.3 Other categories of logics

Due to difficult found in the categories of logics mentioned above, are presented others categories of logics that help the overcome these "defects".

- On the category $L_f$ we take the quotient category $QL_f$: $f, g \in L_f(l, l')$, $f \sim g$ if $f(\varphi) \dashv l' \vdash g(\varphi)$. Thus two logics $l, l'$ are equipollent if only if $l$ and $l'$ are $QL_f$-isomorphic $\mathbb{Z}_f$

- Still on the category $L_f$ we have the "congruential" logics $L'_f$. This category is a subcategory of $L_f$ where the logics are congruential, i.e., logics that satisfies:

$$\varphi_0 \vdash \psi_0, ..., \varphi_{n-1} \vdash \psi_{n-1} \Rightarrow c_n(\varphi_0, ..., \varphi_{n-1}) \vdash c_n(\psi_0, ..., \psi_{n-1}).$$

The inclusion functor $L'_f \rightarrow L_f$ has a left adjoint given by congruential closure operator.

- In $\mathbb{Z}_{fAM}$ we found the category $QL'_f$ (or simply $Q'_f$). This category of logics satisfies simultaneously certain natural conditions:

(i) represent the major part of logical systems;

(ii) have a good categorial approach (e.g., they are complete, cocomplete and accessible categories);

(iii) allow a natural notion of algebraizable logical system $\mathbb{Z}_{fAM}$; and

(iv) allow satisfactory treatment of the "identity problem" of logics.

In $\mathbb{Z}_{fAM}$ is shown that the categories $S_s$ and $S_f$ in a way are associated, i.e., there is a pair of adjoint functors between them, namely $(+)_S : S_s \rightarrow S_f$ and $(-)_S : S_f \rightarrow S_s$. Moreover there is a monad or triple $T = (T_S, \mu_S, \eta_S)$ on $S_s$ canonically associated with this adjunction such that $T$ preserve filtered colimits, reflects isomorphisms and,
mainly, that $Kleisli(T) = S_f$ [Mac], where derives some additional informations about the category $S_*$: e.g., this is all coproducts.

This adjunction between $S_*$ and $S_f$ through forgetful functors $U_s$ and $U_f$ give a pair of adjoint functors $(+)_L : \mathcal{L}_s \to \mathcal{L}_f$, $(-)_L : \mathcal{L}_f \to \mathcal{L}_s$, i.e.:

\begin{itemize}
  \item $U_f \circ (+)_L = (+)_S \circ U_s$
  \item $U_s \circ (-)_L = (-)_S \circ U_f$
  \item $U_s \eta_L = \eta_S U_f$
  \item $U_f \varepsilon_L = \varepsilon_S U_f$
\end{itemize}

The signature monad $T_S = (T_S, \mu_S, \eta_S)$ associated to the signature adjunction $(\eta_S, \varepsilon_S)$ (i.e., $\mu_S = (-)_S \varepsilon_S (+(+))$) "lifts" to a logic monad $T_L = (T_L, \mu_L, \eta_L)$ associated to the signature adjunction $(\eta_L, \varepsilon_L)$ (i.e., $\mu_L = (-)_L \varepsilon_L (+(+))$) and is such that $Kleisli(T_L) = \mathcal{L}_f$. Moreover, the functors $(+)_L$ and $(-)_L$ are precisely the canonical functors associated to the adjunction of the Kleisli category of a monad.

### 2.4 The category of algebraizable logics

The idea behind of algebraizing a logic emerged due to the need to connect two independent approach to logic, on the one hand was the equivalence of logic and on the other hand was the assertion and inference. On the two approaches and the ideas of Hilbert, begin the attempts of connect them.

Traditionally algebraic logic has focused on the algebraic investigation of particular classes of algebras of logic, whether or not they could be connected to some known assertional system by means of the Lindenbaum-Tarski method. However, when such a connection could be established, there was interest in investigating the relationship between various meta-logical properties of the logical system and the algebraic properties of the associated class of algebras.

Firstly we will define algebraizing logics.

#### Remark 2.11

Let $\Sigma$ be a signature. $\Sigma - Str$ is the class of all structures (in the sense of universal algebra) on the signature $\Sigma$. Thus $F(\Sigma) \in \Sigma - Str$.

#### Definition 2.12

Given a class of algebras $K$ of algebraic similarity type $L$, the equational consequence associated with $K$ is the relation $|=K$ between a set of equations $\Gamma$ and a single equation $\varphi \approx \psi$ of type $l = (\Sigma, \vdash)$ defined by:

$$\Gamma \vdash_K \varphi \approx \psi \text{ if and only if } \text{ for every } A \in K \text{ and every } h : F(\Sigma) \to A,$$

$$h(\eta) = h(\nu) \text{ for all } \eta \approx \nu \in \Gamma, \text{ then } h(\varphi) = h(\psi).$$

#### Definition 2.13

Let $l = (\Sigma, \vdash)$ be a logic and $K$ be a class of $\Sigma - algebra$. $K$ is a equivalent algebraic semantics to $l$ if $\vdash$ can be interpreted in $|=K$ of the following form:

(1) there is a finite system $\delta_i(p) \approx \epsilon_i(p), i = 1, ..., n$ of equations in a single variable $p$ such that for all $\Gamma \cup \{\varphi\} \subseteq F(\Sigma)$ and for $j < n$ has been:

$$\Gamma \vdash \varphi \Leftrightarrow \{\delta[\gamma/p] \approx \epsilon[\gamma/p] : \gamma \in \Gamma\} \vdash_M \delta[\varphi/p] \approx \epsilon[\varphi/p] \text{ where } \delta(p) \approx \epsilon(p) \text{ abbreviates the equation systems}\n$$

\(\delta_i(p) \approx \epsilon_i(p), i = 1, ..., n\).

(2) there is a finite system $\Delta_j(p, q), j = 1, ..., m$ of two variables formulas (formed by derived binary connectives) such that for all equation $\varphi \approx \psi$,

- $\varphi \approx \psi = |K|= \delta(\varphi \Delta \psi) \approx \epsilon(\varphi \Delta \psi)$

\(\varphi \Delta \psi = \Delta(\varphi, \psi) \text{ where } \Delta(\varphi, \psi) \text{ abbreviates } \Delta_j(\varphi, \psi), j = 1, ..., m\).

In this case we say that a logic $l$ is algebraizable. The set $\langle \Delta(p, q), \langle \epsilon(p), \delta(p) \rangle \rangle$ is called algebraizing pair.

As example of algebraizable logics we have, in addition with CPC (Classic Propositional Calculus) and IPC (Intuitionistic propositional calculus), the modal logics, the Post and Lukasiewicz multi-valued logics, and many of several versions of quantic logic.

In case of CPC and IPC the pair algebraizing $\langle \Delta(p, q), \langle \epsilon(p), \delta(p) \rangle \rangle$ is:
1. \( \Delta(p, q) = \{ p \leftrightarrow q \} \)
2. \( \varepsilon(p) = p \)
3. \( \delta(p) = \top \)

and \( K \) is the class of Boolean algebra and Heyting algebra respectively. Here the signature of CPC and IPC have as binary connective \( \leftrightarrow \).

An other example more common between mathematicians is the logic of groups theory \([Pa]\). The groups theory can be formalized as a deductive system on signature \( \Sigma = \{ \cdot, ^{-1}, e \} \), where \( \cdot \) is a binary connective, \( ^{-1} \) is a unary connective, and \( e \) is a constant.

Axioms of \( G \)

\[\begin{align*}
\text{G}_1 & \quad (p \cdot q) \cdot r \cdot (p \cdot (q \cdot r))^{-1} \\
\text{G}_2 & \quad (p \cdot e) \cdot p^{-1} \\
\text{G}_3 & \quad (e \cdot p) \cdot p^{-1} \\
\text{G}_4 & \quad p \cdot p^{-1} \\
\text{G}_5 & \quad p^{-1} \cdot p
\end{align*}\]

Rules

\[\begin{align*}
\text{R}_1 & \quad p \cdot q^{-1} \vdash q \cdot p^{-1} \\
\text{R}_2 & \quad p \cdot q^{-1} \vdash p^{-1} \cdot q^{-1}^{-1} \\
\text{R}_3 & \quad \{ p \cdot q^{-1}, q \cdot r^{-1} \} \vdash p \cdot r^{-1} \\
\text{R}_4 & \quad \{ p \cdot q^{-1}, r \cdot s^{-1} \} \vdash (p \cdot r) \cdot (q \cdot s)^{-1} \\
\text{R}_5 & \quad p \vdash p \cdot e^{-1} \\
\text{R}_6 & \quad p \cdot e^{-1} \vdash p
\end{align*}\]

The logic of groups theory has as pair algebrizing \( \langle \Delta(p, q), (\varepsilon(p), \delta(p)) \rangle \):

1. \( \Delta(p, q) = p \cdot q^{-1} \)
2. \( \delta(p) = p \)
3. \( \varepsilon(p) = e \)

\( K \), in this case, is the class of groups. Worth pointing out that the logic of groups theory, in some sense, does not admit Deduction Theorem.

**Proposition 2.14.** For each algebrizable logic \( a \), let \( (\Delta_i \equiv \varepsilon_i, \Delta_i) \), an algebrizing pair, and \( K_i \) an equivalent algebraic semantic, for each \( i \in \{0, 1\} \). For any class \( K' \) of \( \Sigma \)-algebras let us denote \( (K')^Q \) the \( \Sigma \)-quasivariety generated by \( K' \). Then some uniqueness conditions holds: on quasivariety semantics: \( (K_0)^Q = (K_1)^Q \); on equivalence formulas: \( \Delta_0 \equiv \Delta_1 \); on defining equations: \( \delta_0 \equiv \varepsilon_0 =_{K} \delta_1 \equiv \varepsilon_1 \) (where \( K \equiv (K_0)^Q = (K_1)^Q \)).

**Proposition 2.15.** Let \( K \) an equivalent algebraic semantic for the algebrizable logic \( a = (\Sigma, \vdash) \) with algebrizing pair \( (\Delta(p, q), (\varepsilon(p), \delta(p))) \), then:

1. For all set of equations \( \Gamma \) and for all equation \( \varphi \equiv \psi \), we have that
   \[ \Gamma \vdash K \varphi \equiv \psi \iff \{ \xi \Delta \eta : \xi \approx \eta \equiv \Gamma \} \vdash \varphi \Delta \psi \]
2. For each \( \vartheta \in F(\Sigma) \) we have that
\[ \vartheta \models \delta(\vartheta) \Delta e(\vartheta). \]

Conversely, if there is a logic \( a = (\Sigma, \vdash) \) and formulas \( \langle \Delta(p, q), \langle \epsilon(p), \delta(p) \rangle \rangle \) such that satisfies the conditions 1. and 2., then \( K \) is an equivalent algebraic semantics for \( a \).

Due to proposition 2.15 we have that the two different algebrazing pair for CPC and IPC given above create a same class of algebra for its respective logics.

An attempt to determine if a given logic is algebraizable, at times found difficulties about the definition given above. Thus we have the following characterization.

**Theorem 2.16.** Let \( a = (\Sigma, \vdash) \) a logic and \( \Delta \subseteq F(\Sigma)[2], \ (\delta \equiv \epsilon) \subseteq F(\Sigma)[1] \times F(\Sigma)[2] \) such that the conditions below are satisfied

(a) \( \vdash \varphi \Delta \varphi, \) for all \( \varphi \in F(\Sigma) \);
(b) \( \varphi \Delta \psi \vdash \psi \Delta \varphi, \) for all \( \varphi, \psi \in F(\Sigma) \);
(c) \( \varphi \Delta \psi, \psi \Delta \varphi \vdash \varphi \Delta \varphi, \) for all \( \varphi, \psi, \vartheta \in F(\Sigma) \);
(d) \( \varphi_0 \Delta \psi_0, ..., \varphi_{n-1} \Delta \psi_{n-1} \vdash c_n(\psi_0, ..., \psi_{n-1}) \Delta c_n(\psi_0, ..., \psi_{n-1}), \) for all \( c_n \in \Sigma_n \) and all \( \varphi_0, \psi_0, ..., \varphi_{n-1}, \psi_{n-1} \in F(\Sigma) \);
(e) \( \vartheta \vdash \delta(\vartheta) \Delta e(\vartheta), \) for all \( \vartheta \in F(\Sigma) \).

Then \( a \) is an algebraizable logic with \( \Delta \) as equivalence formulas and \( (\delta \equiv \epsilon) \) as defining equations.

Conversely if \( a = (\Sigma, \vdash) \) is an algebraizable logics with algebrazing pair \( \langle \Delta(p, q), \langle \epsilon(p), \delta(p) \rangle \rangle \), then the conditions (a) to (e) are satisfied for these formulas.

With the definition of categories of logics given above, it is possible define categories of algebraizable logics. Others categories of algebraizable logics can be found in [JKE, FC].

- \( A_s \) is the category of algebraizable logics with morphism in \( L_s \) such that preserves algebrazing pair. In the sequence of works, [AFLM1], [AFLM2], [AFLM3] is proven that the category \( A_s \) is a relatively complete \( \omega \)-accessible category [AR].
- \( A_f \) is the category of algebraizable logics with morphisms in \( L_f \) such that preserves algebrazing pair. \( A_f \) is a subcategory of \( L_f, A_f \hookrightarrow L_f \).
- On the category \( A_f \) we have the following subcategories: \( A_f, QA_f \) and \( QA_f \).
- The "Lindenbaum algebraizable" logics are logics \( l \in A \) such that given formulas \( \varphi, \psi \in F(\Sigma), \varphi \vdash \psi \iff \varphi \Delta \psi \). The Lindenbaum algebraizable logics lead a subcategory of the category of algebraizable logics (\( j : Lind(A_f) \hookrightarrow A_f \)). \( Lind(A_f) \) has a importance in the representation theory of logics that we present below. The inclusion functor \( Lind(A_f) \hookrightarrow A_f \) has a left adjoint functor \( L : A_f \rightarrow Lind(A_f) \).

The following diagram represent the functors (and its adjoints) between the categories mentioned above:
3 Relation between logics and structures

Below we present some results found about categories of algebraizable logics, in particular, the category of Lindenbaum algebraizable logics and its relationship with structures and associated quasivarieties.

We seen above that given a algebraizable logic $a = (\alpha, \vdash)$, we have a quasivariety $QV(a)$ associated with $a$ such that is a subcategory of $\alpha - \text{Str} I : QV(a) \rightarrow \alpha - \text{Str}$.

**Lemma 3.1.** The inclusion functor has a left adjoint $(L, I) : QV \rightleftarrows \alpha - \text{Str}$: given by $M \mapsto M/\theta_M$ where $\theta_M$ is the least $\Sigma$-congruence in $M$ such that $M/\theta_M \in QV$. Moreover, the unity of the adjunction $(L, I)$ has components $(q_M)_{M \in \Sigma - \text{Str}}$, where $q_M : M \rightarrow M/\theta_M$ is the quotient homomorphism.

**Demonstration:** Consider $\Gamma_M = \{\theta \subseteq |M| \times |M| : \text{is congruence relation and } M/\theta \text{ is } QV\}$. $\Gamma$ is not empty, because $\theta = |M| \times |M|$ is a congruence relation and $M/\theta$ is $QV$. Let $\theta_M = \cap \Gamma_M$. We will show that $\theta \in \Gamma_M$.

Let $c_n \in \Sigma_n$ and $\varphi_0 \theta_M \psi_0, \ldots, \varphi_{n-1} \theta_M \psi_{n-1}$, $\theta_M = \cap \Gamma_M$, then $\varphi_0 \theta_M \psi_0, \ldots, \varphi_{n-1} \theta_M \psi_{n-1}$ for all $\theta \in \Gamma_M$, therefore $c_n(\varphi_0, \ldots, \varphi_{n-1}) \theta_M c_n(\psi_0, \ldots, \psi_{n-1})$ for all $\theta \in \Gamma_M$, follow that $c_n(\varphi_0, \ldots, \varphi_{n-1}) \theta_M c_n(\psi_0, \ldots, \psi_{n-1})$.

To show that $M/\theta_M$ is $QV$, consider $c : M \rightarrow \prod_{\theta \in \Gamma_M} M/\theta; \quad m \mapsto ([m]_{\theta})_{\theta \in \Gamma_M}$, $c$ is a morphism of $\Sigma - \text{Str}$. We will show that $Ker(c) = \theta_M$.

$\quad (m, n) \in Ker(c) \iff ([m]_{\theta})_{\theta \in \Gamma_M} = ([n]_{\theta})_{\theta \in \Gamma_M} \iff [m]_{\theta} = [n]_{\theta} \forall \theta \in \Gamma_M \iff m \theta n \forall \theta \in \Gamma_M \iff m \theta_M n$.

By ”theorem of homomorphism"

$$
\begin{array}{ccc}
M & \xrightarrow{c} & \prod_{\theta \in \Gamma_M} M/\theta \\
\downarrow{q} & & \downarrow{\ell} \\
M/\theta_M & & \\
\end{array}
$$

As $QV$ is closed by product, we have that $\prod_{\theta \in \Gamma_M} M/\theta$ is $QV$. We also have that $QV$ is closed by substructure and isomorphism, then $M/\theta_M$ is $QV$.

We define $L : \Sigma - \text{Str} \rightarrow QV$ such that $L(M) = M/\theta_M$ and given $M \xrightarrow{L} N$, $M/\theta_M \xrightarrow{L(f)} N/\theta_N$ where $L(f)([m]_{\theta_M}) = [f(m)]_{\theta_N}$.

We will proof that $L(f)$ is a homomorphism of $QV$.

Let $m, n \in M$ such that $(f(m), f(n)) \notin \theta_N$. Observe that $f^{-1}[\theta] := \{(m, n) \in |M| \times |M| : f(m) \theta_N f(n)\} \subseteq \Gamma_M$, so $(m, n) \notin \theta_M$, like this $L(f)$ is good defined.

Let $c_k \in \Sigma_k$ and $[m]_{\theta_M} = c_k^{M/\theta_M}([m]_{\theta_M}, \ldots, [m_k]_{\theta_M}) = c_k^M(m_1, \ldots, m_k)_{\theta_M} \in M/\theta_M$. By definition $L(f)([m]_{\theta_M}) = [f(m)]_{\theta_N} = [c_k^N(f(m_1), \ldots, f(m_k))] = c_k^{N/\theta_N}([f(m_1)]_{\theta_N}, \ldots, [f(m_k)]_{\theta_N})$. By induction hypothesis we have that $L(f)([m]_{\theta_M}) = c_k^{N/\theta_N}(L(f)([m_1]_{\theta_M}), \ldots, L(f)([m_k]_{\theta_M}))$.

It is easy to see that respect the composition. Hence $L$ is a functor.

Now we will proof that $L$ is a left adjoint of $I$. It is enough that $M \xrightarrow{q_M} I(M/\theta_M)$ satisfies the universal property.

Let $N$ a $QV$ and $M \xrightarrow{f} I(N)$. Due to $N$ to be a $QV$, is closed by substructure and isomorphism, so we have that $M/Ker(f)$ is $QV$, hence $Ker(f) \in \Gamma_M$.

$$
\begin{array}{ccc}
M & \xrightarrow{q_M} & I(M/\theta_M) \\
\downarrow{q} & & \downarrow{I(f)} \\
I(M/Ker(f)) & \xrightarrow{I(\tilde{f})} & I(N) \\
\end{array}
$$

As $\theta_M = \cap \Gamma_M$, we have $\tilde{q} : M/\theta_M \rightarrow M/Ker(f)$ where $\tilde{q}([m]_{\theta_M}) = [m]_{Ker(f)}$ is good defined. So $I(\tilde{q}) \circ q_M = q$.

By ”theorem of homomorphism” there is $\tilde{f} : M/Ker(f) \rightarrow N$ such that $I(\tilde{f}) \circ q = f$.

We define $\tilde{f} := \tilde{f} \circ \tilde{q}$, this way $I(\tilde{f}) \circ q_M = f$ and is the unique that do it. □
Remark 3.2. The (forgetful) functor \((QV \xrightarrow{\iota} \Sigma \xrightarrow{\iota} \text{Set})\) has the (free) functor \((\text{Set} \xrightarrow{\iota} \Sigma \xrightarrow{\iota} QV)\), \(Y \mapsto F(Y)/\theta_{F(Y)}\), as left adjoint. Moreover, if \(\alpha_Y : Y \mapsto U \circ F(Y)\) is the \(Y\)-component of the unity of the adjunction \((F,U)\), then \((Y \xrightarrow{\iota} \text{U}F(Y) \xrightarrow{\iota} \text{U}F(Y)) = (Y \xrightarrow{\iota} \text{U}F(Y) \xrightarrow{\iota} \text{U}F(Y))\) is the \(Y\)-component of the adjunction \((L \circ F, U \circ I)\).

Theorem 3.3. Let \(h \in \mathcal{A}_f(a_0, a_1)\), then the induced functor \(h^* : \alpha_1 \rightarrow \alpha_0 \rightarrow \alpha_l (M_1 \mapsto (M_1^h)), \) “commutes over Set” (i.e., \(U_0 \circ h^* = U_1\)) and has the following additional properties:

(a) it has restriction \(h^*; QV(a_0) \rightarrow QV(a_0)\) (i.e. \(I_0 \circ h^* = h^* \circ I_1\));

(b) there is a natural epimorphism \(\hat{h} : L_0 \circ h^* \rightarrow h^* \circ L_1\), that restricts to \(L_0 \circ h^* \circ I_1 = h^* \circ L_1 \circ I_1\)

![Diagram](attachment:image.png)

Demonstration:

Given \(h : a_0 \rightarrow a_1\) we define \(h^* : \alpha_1 \rightarrow \alpha_0 \rightarrow \alpha_l\) such that given \(M \in \alpha_0 \rightarrow \alpha_l\) then \(|M| = |M^h| = |h^*(M)|\) and given \(c_k \in \alpha_0^k\) (\(c_k\) is a k-ary connective),

\[
c_k^h : M^h \times \cdots \times M^h \rightarrow M^h; c_k^h(m_0, \ldots , m_{k-1}) = h(c_k)(m_0, \ldots , m_{k-1})
\]

Let \(g \in \alpha_1 \rightarrow \alpha_l(M, N)\), we define \(h^*(g) = g\). Therefore \(h^*\) is a functor.

**item (a)**

\(\bullet\) It follows from the description of a set of quasi-identities that define the unique equivalent quasi-varietiy semantics associated to algebraizable logic in Theorem 2.17 in [BP] it follows that, if \(((\delta, \epsilon), \Delta)\) is an algebraizable pair for \(\iota = (\Sigma, \iota),\) then the set of quasi-identities \(S' = S_0' \sqcup S_1' \sqcup S_2'\) axiomatizes \(QV(\iota)\), where:

\[
S_0' = \{ (\delta(\psi_0) = \epsilon(\psi_0) \wedge \ldots \wedge \delta(\psi_{n-1}) = \epsilon(\psi_{n-1}) ) \rightarrow \delta(\varphi) = \epsilon(\varphi) : \{ \psi_0, \ldots , \psi_{n-1} \} \vdash \varphi \};
\]

\[
S_1' = \{ \delta(x_0 \Delta x_1) = \epsilon(x_0 \Delta x_1) \rightarrow x_0 = x_1 \};
\]

\[
S_2' = \{ \delta(x_0 \Delta x_0) = \epsilon(x_0 \Delta x_0) \}
\]

As \(h \in \mathcal{A}(a_0, a_1)\), if \(((\delta, \epsilon), \Delta)\) is an algebraizable pair for \(a_0\), then \(((\hat{h}(\delta), \hat{h}(\epsilon)), \hat{h}(\Delta))\) is an algebraizable pair for \(a_1\) and denoting \(h\) the extension of \(f\) to first-order formulas (instead \(\hat{f}\) that its the extension for propositional \(\Sigma\)-formulas = first-order terms)

Let \(M \in QV(a_1)\). If \(\{\psi_0, \ldots , \psi_{n-1}\} \vdash_{a_0} \varphi\), then \(\hat{h}\psi_0, \ldots , \hat{h}\psi_{n-1} \vdash_{a_0} \hat{h}\varphi\), denote \(\Omega = (\delta(\psi_0) = \epsilon(\psi_0) \wedge \ldots \wedge \delta(\psi_{n-1}) = \epsilon(\psi_{n-1})) \rightarrow \delta(\varphi) = \epsilon(\varphi) (Omega \in S_0')\), thus \(\hat{h}(\Omega) = \tilde{M}(\hat{h}(\psi_0) \ldots \hat{h}(\psi_{n-1}) \rightarrow \hat{h}(\varphi) = \tilde{M}(\hat{h}(\varphi))\), thus

\[
M \vdash_{a_1} \hat{h}(\Omega), \text{ but (institution) } M \vdash_{a_0} \hat{h}(\Omega) \Leftrightarrow M^h \vdash_{a_0} \hat{h}(\Omega). \text{ For the quasi-identities } \Omega \text{ in } S_0' \cup S_2', \text{ we can conclude in analogous way that } M^h \vdash_{a_0} \hat{h}(\Omega), \text{ thus } M^h \in QV(a_0).
\]

**item (b)**

Given \(\theta \in \Gamma_M\), we have \(h^*[\theta] := \hat{h}(\theta)\) is a congruence relation in \(M^h\). Observe that due to “theorem of homomorphism” \((M/\theta)^h = M^h/\theta^h\). Than \(M^h/\theta^h \in QV(a_0)\).

Define \(\eta : L_0 \circ h^* \Rightarrow h^*_l \circ L_1\) such that \(\eta_M : L_0 \circ h^*(M) = M^h/\theta_{M^h} \rightarrow M^h/\theta_{M^h} = h^*_l \circ L_1\); \(\eta_M((m^h)_{\theta_M}) = ([m]_{\theta_M})^h\)

\(\eta\) is a natural isomorphism.
Proposition 3.4. Let $g_0, g_1 : l \to a \in L_f$, with $a \in \text{Lind}(A_f)$.

(a) $g_0$ is dense $\Rightarrow g^* : QV(a) \to \Sigma - \text{str}$ is full, faithful, injective on objects and satisfies the structurality condition, i.e., given $M \in QV(a)$ and $M'$ substructure of $g^* (M)$ ($T \in QV(a)$) then there is $T$ substructure $M$ such that $g^* (T) = M'$.

(b) $[g_0] = [g_1] \in Q_f \iff g_0^* = g_1^* : QV(a) \to \Sigma - \text{str}$.

Proof:

(a) Full: Let $M_1, M_2 \in QV(a)$ and $f : g_1^*(M_1) \to g_2^*(M_2)$ in $\Sigma - \text{Str}$. As $|g_i^*(M_i)| = |M_i|$, $i \in \{1, 2\}$, we have $U(f) : |M_1| \to |M_2|$ is a function. We will prove that $f : M_1 \to M_2$ is in $QV(a)$.

Let $c_n \in \alpha_n$. As $g$ is dense, there is a $\varphi_n \in F(\Sigma)[u]$ such that $\check{g}(\varphi_n(x_0, ..., x_{n-1})) \models c_n(x_0, ..., x_{n-1})$. $a$ is Lindenbaum algebraizable, hence $\models \check{g}(\varphi_n(x_0, ..., x_{n-1})) \Delta c_n(x_0, ..., x_{n-1})$ and then $|=QV(a) \check{g}(\varphi_n(x_0, ..., x_{n-1})) \equiv c_n(x_0, ..., x_{n-1})$.

Let $h : X \to M_1$ a function. Consider $m_0 = h(x_0), ..., m_{n-1} = h(x_{n-1})$. So $\check{g}(\varphi_n(x_0, ..., x_{n-1}))^{M_1} = (e(x_0, ..., x_{n-1}))^{M_1}$.

Therefore $f$ is a $QV(a)$ morphism.

Faithful: By $g_i^*(f) = f$. We have that given $f_1, f_2 : M \to N$, if $g_1^*(f_1) = g_1^*(f_2)$ then $f_1 = f_2$.

Injective on objects: Let $M_1, M_2 \in QV(a)$ such that $g_1^*(M_1) = g_1^*(M_2)$, so $|M_1| = |M_2|$. Given $c_n \in \alpha_n$ and $m_0, ..., m_{n-1}$ there is $\varphi_n(x_0, ..., x_{n-1}) \in F(\Sigma)[u]$ such that

$\varphi_n(x_0, ..., x_{n-1}) \models c_n(x_0, ..., x_{n-1}) \Rightarrow \varphi_n(x_0, ..., x_{n-1}) \Delta c_n(x_0, ..., x_{n-1}) \models |QV(a) \varphi_n(x_0, ..., x_{n-1}) \equiv c_n(x_0, ..., x_{n-1})$

Hence

$c_n^M(m_0, ..., m_{n-1}) = \varphi_n^{M_1^0}(m_0, ..., m_{n-1})$

$= \varphi_n^{M_2^0}(m_0, ..., m_{n-1})$

$= c_n^{M_2^0}(m_0, ..., m_{n-1})$

Than $M_1 = M_2$

Structurality: Let $M \in QV(a)$ and $M'$ substructure of $g^*(M) = M^g$. Consider $T \subseteq M$ where $|T| = |M'|$. As $g$ is dense, given $c_n \in \alpha_n$ there is $\varphi_n \in F(\Sigma)[u]$ such that

$c_n^*(m_0, ..., m_{n-1}) \models \check{g}(\varphi_n(x_0, ..., x_{n-1})) \iff \models c_n^*(x_0, ..., x_{n-1}) \Delta \check{g}(\varphi_n(x_0, ..., x_{n-1}))$

$\models c_n^*(x_0, ..., x_{n-1}) \equiv \check{g}(\varphi_n(x_0, ..., x_{n-1}))$

Define $c_n^T(m_0, ..., m_{n-1}) = \varphi_n^{M'}(m_0, ..., m_{n-1})$. So $c_n^T(m) = \varphi_n^{M'}(m) = \varphi_n^M(m) = \check{g}(\varphi_n)(m) = c_n^M(m)$

Therefore $T$ is a substructure of $M$. As $M \in QV(a)$, we have $T \in QV(a)$.

Let $c_n \in \Sigma_n$, so $c_n^T = g(c_n)^T = g(c_n)^M = c_n^M$. Hence $g^*(T) = M'$

(b) $\Rightarrow$
Let $M \in QV(a)$ and $c_n \in \Sigma_n$. As $[g_0] = [g_1]$ we have that

$$g_0(c_n)(x_0, ..., x_{n-1}) \vDash g_1(x_0, ..., x_{n-1}) \Rightarrow \vdash g_0(c_n)(x_0, ..., x_{n-1}) \Delta g_1(x_0, ..., x_{n-1}) \Rightarrow \vdash QV(a) g_0(c_n)(x_0, ..., x_{n-1}) \equiv g_1(x_0, ..., x_{n-1})$$

Therefore

$$c_n^{M \circ \theta} = (g_0(c_n))^M = (g_1(c_n))^m = c_n^{M \circ \phi}$$

With this $g^*_0 = g^*_1$

" $\Leftarrow$ "

Suppose that $g^*_0 = g^*_1$. Let $c_n \in \Sigma_n$. Hence $c_n^{M \circ \theta} = c_n^{M \circ \phi}$ for all $M \in QV(a)$. So $\vdash QV(a) g_0(c_n)(x_0, ..., x_{n-1}) \equiv g_1(c_n)(x_0, ..., x_{n-1})$. Due to $a$ to be algebraizable, $\vdash g_0(c_n)(x_0, ..., x_{n-1}) \Delta g_1(c_n)(x_0, ..., x_{n-1}) \Rightarrow g_0(c_n)(x_0, ..., x_{n-1}) \vDash g_1(c_n)(x_0, ..., x_{n-1})$.

By induction we have that $g_1(\varphi) \vDash \hat{g}_0(\varphi)$ for all $\varphi \in F(\Sigma)$. Therefore $[g_0] = [g_1]$

**Proposition 3.5.** Let $a = (\Sigma, \vdash)$ be a Lindenbaum algebraizable then:

(a) $F(\Sigma)/\Delta = F(\Sigma)/(\vdash \vdash)$ is a $\Sigma$-structure.

(b) $F(\Sigma)/\Delta \in QV(a)$.

(c) $F(\Sigma)/\Delta$ is the free $QV(a)$-object over the set $X = \{x_0, ..., x_n, \ldots\}$.

**Proposition 3.6.** Let $a \xrightarrow{\downarrow} a' \in \text{Lind}(A_f)$, then $f^*_1 : QV(a') \to QV(a)$ has a left adjoint $G$. In case that $f$ is dense, let $M' \in QV(a')$, $G(M') = M'/\theta_{M'}$ where $\theta_{M'}$ is the least $\Sigma$-congruence in $M'$ such that $M/\theta_{M'} = h[M']$ and $h : M' \to f^*_1(M)$ for some $M \in QV(a)$.

**Remark 3.7.** Let $a = IPC$ and $a' = CPC$ both Lindenbaum algebraizable logics with the same signature. We have the inclusion morphism $f : IPC \to CPC$. So $f^*_1 : BA \to HA$ has a adjoint functor $G : HA \to BA$. Observe that $f^*_1$ is the inclusion functor. Hence given $H \in HA$, $G(H) = H/F_H$, where $F_H = \{(a \leftrightarrow \neg a) \in H\}$. Its possible to proof that $G(H) = H_{\neg \neg}$, where $H_{\neg \neg}$ denote the poset of regular elements of $H$, that is, those elements $x \in H$ such that $\neg \neg x = x$.

This way we believe that the Gödel translation has a connection with the functor $G$.

The same way that the Gödel translation can be in association with the functor $G$, we could have a type of "Gödel translation generalized" in association with the left adjoint of morphisms between Lindenbaum algebraizable logics.

**Proposition 3.8.** Let $a$ and $a'$ be Lindenbaum algebraizable logics. If $a \xrightarrow{\downarrow} a'$ is a pair of inverse $QLind(A_f)$-isomorphisms (i.e., are $Q^n_f$-isomorphisms that preserve algebraizing pair) then: $QV(a) \xrightarrow{\downarrow} QV(a')$ is an isomorphism of categories.

**Lemma 3.9.** Let $\Sigma, \Sigma' \in \text{Obj}(S_f)$. Consider $H : \Sigma' \to \text{Str}$ a functor that "commutes over Set" (i.e. $U \circ H = U'$ and, for each set $Y$, let $\eta_H(Y) : F(Y) \to H(F'(Y))$ be the unique $\Sigma$-morphism such that $(Y \xrightarrow{\gamma} U'(Y)) UHF'(Y) = (Y \xrightarrow{\gamma} U'(Y'))$). Then:

(a) For each set $Y$ and each $\psi \in F(Y)$, $\text{Var}(\eta_H(Y)(\psi)) \subseteq \text{Var}(\psi)$

(b) $(\eta_H(Y))_{Y \in \text{Set}}$ is a natural transformation $\eta : F \to H \circ F'$

(c) If $H$ is an isomorphism of categories, then $\eta_H(Y)$ "preserves variables" (i.e., $\forall \psi \in F(Y), \text{Var}(\eta_H(Y)(\psi)) = \text{Var}(\psi)$) and $H$ preserves (strictly) products and substructures.

(d) For each $n \in \mathbb{N}$, let $X_n := \{x_0, \ldots, x_{n-1}\} \subseteq X$, if $\eta_H(X_n)$ "preserves variables", then the mapping $c_n \in \Sigma_n \mapsto \eta_H(X_n)(c_n(x_0, \ldots, x_{n-1}))$ determines a $\mathcal{F}$-morphism $m_H : \Sigma \to \Sigma'$

(a) Let $\Sigma, \Sigma' \in \text{Obj}(S_f)$. Let $H : \Sigma' \to \text{Str}$ be a "signature" functor, i.e. a functor satisfying (s1), (s2), (s3):

(s1) $H$ "commutes over Set"

(s2) $\eta_H$ "preserves variables"

(s3) $H$ preserves (strictly) products and substructures.

(b) Denote $S^\dagger_f$ the subcategory of the category of diagrams (i.e., the category whose objects are categories and the arrows are change of base morphisms (i.e., some pairs (functors, natural transformations)), given by all the categories $\Sigma - \text{str}$ and morphisms $(H, \eta_H)$ where $H$ is a signature functor.

(c) Let $a, a' \in \text{Obj}(Lind(A_f))$. Let $H : \Sigma' \to \text{Str}$ be a "Lindenbaum" functor, i.e. a signature functor also satisfying (11), (12), (13):

(11) $H$ has a (unique) restriction to the quasivarieties $H \upharpoonright : QV(a') \to QV(a)$
\( (l) \) \( \hat{m}_H(\Delta) \vdash \Delta' \)
\( (l) \) \( \hat{m}_H(\delta) = \hat{m}_H(\varepsilon) \vdash_{QV(\alpha')} \delta' = \varepsilon' \).

(d) Let \( a, a' \in \text{Obj}(\text{Lind}(A_f)) \) and \( H : \Sigma' - \text{Str} \rightarrow \Sigma - \text{Str} \) be a “Lindenbaum” functor. For each set \( Y \), let \( \tilde{\eta}_H(Y) : LF(Y) \rightarrow H \upharpoonright (L'F(\alpha')) \) be the unique \( QV(\alpha') \)-morphism such that \( (Y \upharpoonright UILF(Y) \uparrow U(\tilde{\eta}_H(Y)) \uparrow UH \upharpoonright L'F'(Y)) = (Y \upharpoonright L'F'(Y)) \). Then \( (\tilde{\eta}_H(Y))_{Y \in \text{Set}} \) is a natural transformation \( \tilde{\eta}_H : L \circ F \rightarrow H \downarrow \circ L' \circ F'. \)

(e) Denote \( \text{Lind}(A_f) \uparrow \) the subcategory of the category of diagrams, given by all the subcategories \( QV(a) \leftarrow \Sigma - \text{str} \) and morphisms \( (H, \tilde{\eta}_H) \) where \( H \) is a Lindenbaum functor.

**Theorem 3.10.** (a) The categories \( S_f \) and \( S_f' \) are anti-isomorphic. More precisely, given \( \Sigma, \Sigma' \in S_f \), the mappings \( h \in S_f(\Sigma, \Sigma') \) \( \mapsto (h^*, \tilde{\eta}_H) \in S_f(\Sigma' - \text{str}, \Sigma - \text{str}) \) and \( (H, \tilde{\eta}_H) \in S_f(\Sigma' - \text{str}, \Sigma - \text{str}) \) \( \mapsto m_H \in S_f(\Sigma, \Sigma') \) are inverse bijections.

(b) The pair of inverse anti-isomorphisms above restricts to a pair of inverse anti-isomorphisms between the categories \( \text{Lind}(A_f) \) and \( \text{Lind}(A_f)' \).

Moreover, the inverse isomorphisms establish a correspondence:

(c) If \( h \in \text{Lind}(A_f)(a, a') \) and \( H \in \text{Lind}(A_f)' \) are in correspondence, then also are in correspondence the equivalence class \( \{h' \in \text{Lind}(A_f)(a, a') : [h] = [h'] \in Q\text{Lind}(A_f)(a, a')\} \) and the equivalence class \( \{H' \in \text{Lind}(A_f)' : H' \mid H, \tilde{\eta}_{H'} = \tilde{\eta}_H\} \).

(d) If \( h \in \text{Lind}(A_f)(a, a') \) and \( H \in \text{Lind}(A_f)' \) are in correspondence, then \( [h] \) is a \( Q\text{Lind}(A_f)-\)isomorphism (i.e., \( h \) is an equivalence of logics) \( \iff (H, \tilde{\eta}_H) \) is an isomorphism of change of bases.

### 4 Future works

As future works we pretend develop a representation theory of logics through the categories theory. Influenced by process of analysis and synthesis of combining of logics we define a notion of Morita equivalence of logics. So we apply the mathematical device above to take information about logics and study meta-logics properties that are preserved by the process of analysis and synthesis of combining of logics we define a notion of Morita equivalence of logics. So we apply our representation theory in others logics, in particular LFIs (Logics of Formal Inconsistency) \( [BCC1, BCC2] \). In \( [EFJ] \) we find a compilation of results about algebraizable logics and meta-logic properties. Then we pretend study local-global principle to meta-logic properties.

Another point that we pretend to investigate if its possible obtain a functorial treatment to ”Gödel translation” in Lindenbaum algebraizable logics as already mentioned in \( [CZ] \).

### References

[**AFLM1**] P. Arndt, R. A. Freire, O. O. Luciano, H. L. Mariano, *Fibring and Sheaves*, Proceedings of IICAI-05, Special Session at the 2nd Indian International Conference on Artificial Intelligence, Pune, India, (2005), 1679-1698.

[**AFLM2**] P. Arndt, R. A. Freire, O. O. Luciano, H. L. Mariano, *On the category of algebraizable logics*, CLE e-Prints 6(1) (2006), 24 pages.

[**AFLM3**] P. Arndt, R. A. Freire, O. O. Luciano, H. L. Mariano, *A global glance on categories in Logic*, Logica Universalis 1 (2007), 3-39.

[**AR**] J. Adámek, J. Rosický, *Locally Presentable and Accessible Categories*, Lecture Notes Series of the LMS 189, Cambridge University Press, Cambridge, Great Britain, 1994.

[**Bez**] J.Y. Béziau, From Consequence Operator to Universal Logic: A Survey of General Abstract Logic in *Logica Universalis: Towards a General Theory of Logic* (J.-Y. Beziau, ed.), Birkhaeuser, Basel, 2005.
[BC] J. Bueno-Soler, W.A. Carnielli, *Possible-translations algebraization for paraconsistent logics*, Bulletin of the Section of Logic 34(2) (2005), 77-92, University of Lodz, Polonia; CLE e-Prints 5(6) (2005), 13 pages.

[BCC1] J. Bueno, M.E. Coniglio, W.A. Carnielli, *Finite algebraizability via possible-translations semantics*, Proceedings of CombLog 04 - Workshop on Combination of Logics: Theory and Applications, (editors: W.A. Carnielli, F.M. Dionisio e P. Mateus), (2004), 79-86.

[BCC2] J. Bueno-Soler, M.E. Coniglio, W.A. Carnielli, *Possible-Translations Algebraizability*, Paraconsistency with no Frontiers (2006) (editors: J.-Y. Beziau, W. Carnielli), North-Holland.

[BP] W. J. Blok, D. Pigozzi, *Algebraizable logics*, Memoirs of the AMS 396, American Mathematical Society, Providence, USA, 1989.

[Car] W. A. Carnielli, *Many-valued logics and plausible reasoning*, Proceedings of the XX International Congress on Many-Valued Logics, IEEE Computer Society, University of Charlotte, USA, (1990), 328-335.

[Con] M. E. Coniglio, *The Meta-Fibring environment: Preservation of meta-properties by fibring*, CLE e-Prints 5(4) (2005), 36 pages.

[Cze1] J. Czelakowski, *Protoalgebraic logic*, Trends in Logic, Studia Logica Library, Kluwer Academic Publishers, 2001.

[Cze2] J. Czelakowski *Logical matrices and the amalgamation property*, Studia Logica 41 (1982), 329-342

[CC1] W. A. Carnielli, M. E. Coniglio, *A categorial approach to the combination of logics*, manuscript 22 (1999), 64-94.

[CC2] W. A. Carnielli, M. E. Coniglio, *Transfers between logics and their applications*, Studia Logica 72 (2002); CLE e-Prints 1(4) (2001), 31 pages.

[CC3] W. A. Carnielli, M. E. Coniglio, *Combining Logics*, Stanford Encyclopedia of Philosophy, [http://plato.stanford.edu/entries/logic-combining/](http://plato.stanford.edu/entries/logic-combining/)

[CCCGS] W. A. Carnielli, M. Coniglio, D. Gabbay, P. Gouveia, C. Sernadas, *Analysis and Synthesis of Logics*, volume 35 of Applied Logic Series, (2008), Springer.

[CCRS] C. Caleiro, W. Carnielli, J. Rasga, C. Sernadas, *Fibring of Logics as a Universal Construction*, Handbook of Philosophical Logic 13 (2005) (editors: D. Gabbay, F. Guenthner), Kluwer Academic Publishers.

[ CCCSS] C. Caleiro, W. Carnielli, M. E. Coniglio, A. Sernadas, C. Sernadas, *Fibring Non-Truth-Functional Logics: Completeness Preservation*, Journal of Logic, Language and Information 12(2) (2003), 183-211; CLE e-Prints 1(1) (2001), 34 pages.

[CG] C. Caleiro, R. Gonçalves, *Equipollent logical systems*, Logica Universalis: Towards a General Theory of Logic (Editor J.-Y. Beziau) (2007), 97-110.

[CR] C. Caleiro, J. Ramos, (2004), *Cryptofibring*, Proceedings of CombLog 04 - Workshop on Combination of Logics: Theory and Applications, Lisboa, Portugal (editors: W. A. Carnielli, F. M. Dionísio, P. Mateus) (2004), 87-92.

[CSS2] M. E. Coniglio, A. Sernadas, C. Sernadas, *Fibring logics with topos semantics*, Journal of Logic and Computation 13(4) (2003), 595-624.

[FC] V. L. Fernández, M. E. Coniglio, *Fibring algebraizable consequence systems*, Proceedings of CombLog 04 - Workshop on Combination of Logics: Theory and Applications, (editors: W.A. Carnielli, F.M. Dionisio and P. Mateus), (2004), 93-98.

[FJP] J. M. Font, R. Jansana, D. Pigozzi, *A Survey of Abstract Algebraic Logic*, Studia Logica 74 (2003)

[Ga] D. Gabbay, *Fibred Semantics and the weaving of logics: part 1*, Journal of Symbolic Logic 61(4) (1996), 1057-1120.

[Ho] E. Hoogland, *Algebraic characterizations of various Beth definability properties*, Studia Logica 65 (2000), 91-112
A. Jánossy, Á. Kurucz, Á. Eiben, *Combining Algebrizable Logics*, Notre Dame Journal of Formal Logic 37(2), (1996), 366-381, Springer.

MacLane, S., *Categories for the Working Mathematician*, Graduate texts in Mathematics 5, Springer-Verlag, Berlin, 1971.

H. L. Mariano, C. A. Mendes, *Towards a good notion of categories of logics*, arXiv preprint, [http://arxiv.org/abs/1404.3780](http://arxiv.org/abs/1404.3780) 2014.

H. Mariano and D. Pinto. Representation theory of logics: a categorial approach. arXiv preprint, [http://arxiv.org/abs/1405.2429](http://arxiv.org/abs/1405.2429) 2014

M. Makkai, R. Paré, *Accessible categories: The Foundations of Categorical Model Theory*, Contemporary Mathematics 104, American Mathematical Society, Providence, USA, 1989.

J. Patrícia, *Sistemas dedutivos não algebizáveis*, [http://www2.mat.ua.pt/martins/documentos/didaticos/SistDednAlg.pdf](http://www2.mat.ua.pt/martins/documentos/didaticos/SistDednAlg.pdf), 2012.

C. Sernadas, J. Rasga, W. A. Carnielli, *Modulated fibring and the collapsing problem*, The Journal of Symbolic Logic 67 (2002), 1541-1569; CLE e-Prints 1(2) (2001), 34 pages.

A. Sernadas, C. Sernadas, C. Caleiro, *Fibring of logics as a categorial construction*, Journal of logic and computation, 9(2) (1999), 149-179.

A. Zanardo, A. Sernadas, C. Sernadas, *Fibring: Completeness preservation*, The Journal of Symbolic Logic 66(1) (2001), 414-439.