BIJECTIVE ARITHMETIC CODINGS OF
HYPERBOLIC AUTOMORPHISMS OF THE
2-TORUS, AND BINARY QUADRATIC FORMS

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Abstract. We study the arithmetic codings of hyperbolic automorphisms of the 2-torus, i.e. the continuous mappings acting from a certain symbolic space of sequences with a finite alphabet endowed with an appropriate structure of additive group onto the torus which preserve this structure and turn the two-sided shift into a given automorphism of the torus. This group is uniquely defined by an automorphism, and such an arithmetic coding is a homomorphism of that group onto $\mathbb{T}^2$. The necessary and sufficient condition of the existence of a bijective arithmetic coding is obtained; it is formulated in terms of a certain binary quadratic form constructed by means of a given automorphism. Furthermore, we describe all bijective arithmetic codings in terms the Dirichlet group of the corresponding quadratic field. The minimum of that quadratic form over the nonzero elements of the lattice coincides with the minimal possible order of the kernel of a homomorphism described above.

0. Introduction

In this work we continue studying the symbolic dynamics of ergodic automorphisms of the 2-torus. The dynamics of automorphisms of the torus is related more to number theory than to the general theory of dynamical systems. This is why their coding should be considered as a number-theoretic problem. This was the main idea of [Ver2] and subsequent papers (see [Ver1] and references therein); recently it was developed in [KenVer] and later in the dissertation [Leb]. Recall that to the Markov coding of hyperbolic automorphisms of the torus and more general hyperbolic dynamical systems a number of classical works have been devoted, see, e.g., [AdWe], [Sin], [Bow], [GuSi]. These papers are accented on the structure of Markov partitions, but without special interest to the arithmetic structure. For more details and the history of the problem see the recent survey [Ad].

The quadratic case is studied in detail below, and one sees that the relationship with the theory of quadratic extensions and binary integral quadratic forms becomes even deeper than before. We set certain natural requirements on a symbolic realization of a hyperbolic automorphism of the 2-torus (more precisely, on the maximal commutative subgroup of $GL(2,\mathbb{Z})$ containing this automorphism), see Problem 1 in Section 1. Furthermore, we give the necessary and sufficient condition on the existence of a mapping from a symbolic compactum onto the 2-torus

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which we call an arithmetic coding. Namely, arithmetic coding is a mapping acting from the symbolic compactum provided with “almost group” structure onto the torus as “almost homomorphism” of this structure to the torus as an additive group (see Section 1 for the precise definitions and axiomatics). It is proved that each arithmetic coding is naturally parametrized by a homoclinic point of a given automorphism. In our considerations we use two-sided decompositions of the points of \( T^2 \) whose one-sided restriction coincide with the well-known \( \beta \)-expansions (see [Pa]); however, the two-sided version proves to lead to new effects and problems.

The symbolic compactum in question is either Markov, if the determinant of the matrix specifying an automorphism equals \(-1\), or sofic otherwise. It is proved that in both cases the compactum, after a certain factorization of sequences of zero measure, turns into a group in addition (Proposition 1.4).

An arithmetic coding is a specific mapping from the fixed symbolic compactum \( \mathcal{X} \) onto the torus. This mapping can be considered as expansions of the points of \( T^2 \) into the two-sided convergent series with respect to the orbit of an arbitrary homoclinic point. It has the following form:

\[
\varphi_t(\varepsilon) = \lim_{N \to +\infty} \left( \sum_{-N}^{N} \varepsilon_n T^{-n}t \mod \mathbb{Z}^2 \right),
\]

where \( \varepsilon \) is a sequence from the compactum \( \mathcal{X} \) and \( t \) is a homoclinic point for \( T \) written in coordinates of \( \mathbb{R}^2 \) (see Theorem 1.2 for more details). Such expansions initially appeared in [Ver2], [Ver3].

We also give a criterion of the existence of a bijective arithmetic coding (see Theorem below). In the case, where for a given automorphism there is no bijective arithmetic coding, we present a precise description of some minimal finite-sheeted covering of the torus. A close connection with number theory that we mentioned above is corroborated by the type of existence condition.

For the automorphism \( T \) given by a matrix \( M_T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) we define a very important quadratic form associated with \( T \) by the formula

\[
f_T(x, y) := bx^2 - (a - d)xy - cy^2.
\]

Let \( \lambda \) be an eigenvalue of \( M \), and \( D \) be its discriminant. We recall that the Dirichlet group \( \mathcal{U}_D \) of the quadratic field \( \mathbb{Q}(\sqrt{D}) \) is, by definition, the group of its units (= units of its maximal order), and by the classical theorem of number theory, in our case the Dirichlet group is \( \{ (x + y\sqrt{D})/2 \} \), where \( (x, y) \) is a solution of the Pell equation

\[
x^2 - Dy^2 = \pm 4
\]

(see, e.g., [BorSh] and [Lev, vol. II, Chap. 2]). It is easy to deduce from the cited theorem that if a matrix \( M \) is primitive, i.e. there is no matrix \( K \in GL(2, \mathbb{Z}) \) such that \( M = K^n, \ n \geq 2 \), then \( \mathcal{U}_D = \{ \pm \lambda^n \mid n \in \mathbb{Z} \} \cong \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z}) \). Now we are ready to quote the essential part of the main result, see Theorems 2.5 and 2.6 which concern the existence and properties of the bijective arithmetic codings of \( T \). Item IV is taken from Theorem A.7 (see Appendix).
**Theorem.**

I. The ergodic automorphism $T$ given by a matrix $M_T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ admits bijective arithmetic coding if and only if its matrix $M_T$ is algebraically conjugate in $GL(2, \mathbb{Z})$ to the companion matrix $C_{r, \sigma} = \begin{pmatrix} r & 1 \\ -\sigma & 0 \end{pmatrix}$ with $r = \text{Tr} M_T$ and $\sigma = \det M_T$.

II. A matrix $M_T$ is algebraically conjugate to the corresponding companion matrix if and only if the Diophantine equation

$$f_T(x, y) = \pm 1$$

is solvable.

III. There exists a natural one-to-one correspondence between the set of bijective arithmetic codings of $T$ and the Dirichlet group of the quadratic field $\mathbb{Q}(\lambda)$ where $\lambda$ is an eigenvalue of $M_T$.

IV. The existence of a point $(x, y) \in \mathbb{Z}^2$ such that the linear span for the orbit of $M_T$ of $(x, y)$ is equal to $\mathbb{Z}^2$, is equivalent to the algebraic conjugacy of $M_T$ and the companion matrix $C_{r, \sigma}$. In particular, necessarily $f_T(y, -x) = \pm 1$.

More generally, a *minimal* arithmetic coding, i.e. a coding with the minimal number of preimages, is naturally given by a solution of the Diophantine equation $f_T(x, y) = \pm m$ with the minimal possible positive $m$ (Theorem 3.5).

The precise axiomatic conditions on a symbolic realization of an automorphism of the torus are as follows: the corresponding mapping from the symbolic set of all sequences of nonnegative integers onto the 2-torus should be a continuous homomorphism of semigroups turning the shift into a given automorphism. *A priori* it is not even clear, why so rigid conditions can be satisfied. However, the fact that they really can, yields a purely arithmetic interpretation of a coding, namely, as two-sided convergent power series in powers of the eigenvalue with a specifically chosen collection of digits and Markov or sofic restrictions to their succession. This is nothing but a two-sided generalization of the so-called $\beta$-expansions but with essential sharpenings connected with the requirement of continuity (= convergence).

A good deal of what was said above, might be extended to the general case of a hyperbolic automorphism of $\mathbb{T}^n$, $n \geq 3$ whose principal eigenvalue is a PV number. Let us emphasize that in higher dimensions in general it is not enough to consider natural numbers as coefficients in the symbolic compactum; moreover, in [KenVer] it was shown that these coefficients could be algebraic numbers. The condition of bijectivity is unknown for those cases.

Note also that for constructing examples which corroborate some sharp estimates, we will use the facts from the theory of indefinite quadratic forms contained, e.g., in the monograph [Cas1], see Appendix. The relationship of this kind of problems of dynamical systems theory with the geometry of numbers and the theory of algebraic numbers becomes very important. This link might be used in both directions.
1. Basic notions and the main problem

1.1. Basic notions and constructions. Let $\mathbb{T}^2$ denote the 2-torus considered as the factor $\mathbb{R}^2/\mathbb{Z}^2$. Let $T$ be an arbitrary group automorphism of $\mathbb{T}^2$ given by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z})$ which we will denote by $M_T$. Let $r$ denote the trace of $M_T$, $\sigma$ stand for its determinant. Suppose $T$ is hyperbolic, which in the two-dimensional case is equivalent to the fact that none of the roots of 1 belongs to the spectrum of $M_T$, i.e.

1. if $\sigma = -1$, then $r \neq 0$;
2. if $\sigma = +1$, then $|r| \geq 3$.

The characteristic polynomial of $M_T$ is $x^2 - rx + \sigma$, and its discriminant is $D = r^2 - 4\sigma$. The eigenvalues of $M_T$ are $\frac{1}{2}(r \pm \sqrt{D})$.

Suppose $r$ to be positive; below we will prove that for our purposes the study of the case of a negative trace will be immediate, namely, we will consider the matrix $-M$ and easily reformulate all the claims for it, see the end of Section 3. Let $\lambda = \frac{r + \sqrt{D}}{2} > 1$, and and let $\overline{\lambda}$ denote the algebraic conjugate of $\lambda$, i.e. $\overline{\lambda} = \sigma \lambda^{-1} = r - \lambda$.

We wish to consider symbolic codings as appropriate expansions of the points of the torus in the sense of some generalized “number system” with natural coefficients. Note that for multidimensional hyperbolic automorphisms the coefficients are not necessarily naturals, but always elements of a certain algebraic field, see [KenVer].

As a primary symbolic set of coefficients for further codings we choose $\tilde{\mathbb{X}}$ defined as the set of all two-sided sequences with the coefficients $\{\varepsilon_n\}_{n=\infty}^{\infty} \in \prod_{n=\infty}^{\infty} \mathbb{Z}_+$ such that the series $\sum_{n=1}^{\infty} \varepsilon_n \lambda^{-n}$ and $\sum_{n=1}^{\infty} \varepsilon_n \lambda^{-n}$ converge. We endow $\tilde{\mathbb{X}}$ with the natural (weak) topology and with coordinate-wise addition. It is obvious that $\tilde{\mathbb{X}}$ is a semigroup.

We call a sequence finite, if it contains only a finite number of nonzero coordinates. Let $\tau$ denote the two-sided shift on $\tilde{\mathbb{X}}$, i.e. $\tau\{\varepsilon_n\} = \{\varepsilon'_n\}$, where $\varepsilon'_n = \varepsilon_{n+1}$.

We set up the main problem of arithmetic coding axiomatically.

**Definition.** A one-sided sequence $(\varepsilon_1, \varepsilon_2, \ldots)$ is said to be lexicographically less than a sequence $(\varepsilon'_1, \varepsilon'_2, \ldots)$, if $\varepsilon_n < \varepsilon'_n$ for the least $n \geq 1$ such that $\varepsilon_n \neq \varepsilon'_n$.

Notation: $(\varepsilon_1, \varepsilon_2, \ldots) \prec_{\text{lex}} (\varepsilon'_1, \varepsilon'_2, \ldots)$.

**Problem 1 (description of arithmetic codings).** To describe all continuous semigroup homomorphisms $\varphi : \tilde{\mathbb{X}} \to \mathbb{T}^2$ which turn the shift $\tau$ into $T : \varphi \tau = T \varphi$. For a given $\varphi$ to find a closed, shift-invariant subset $\mathbb{X}$ of $\tilde{\mathbb{X}}$ such that:

1. it is total, i.e. together with a sequence $\{\varepsilon_n\}_{n=\infty}^{\infty}$ it contains each sequence $\{\varepsilon'_n\}_{n=\infty}^{\infty}$ such that $(\varepsilon'_N, \varepsilon'_{N+k+1}, \ldots) \prec_{\text{lex}} (\varepsilon_N, \varepsilon_{N+1}, \ldots)$ for some fixed $N \in \mathbb{Z}$ and any $k \geq 0$;
2. $\varphi$ restricted to $\mathbb{X}$ is surjective and one-to-one on the set of finite sequences of $\tilde{\mathbb{X}}$ belonging to $\mathbb{X}$ (the section over the finite sequences).

**Definition.** For a hyperbolic automorphism $T$ of the 2-torus a pair $(\varphi, \mathbb{X})$ defined in Problem 1, will be called an arithmetic coding of $T$.

We will see that such a coding exists for all hyperbolic automorphisms, the compactum $\mathbb{X}$ depending on the spectrum of $M_T$ (not on $\varphi$). So, sometimes by a coding we will imply a mapping $\varphi$ only. Furthermore, we will show that after small glueings $\mathbb{X}$ acquires the structure of a group and in fact $\varphi$ restricted to $\mathbb{X}$ is an isomorphism.
a group homomorphism. An arithmetic coding is not necessarily bijective almost everywhere, moreover, sometimes there is no bijective arithmetic coding for a given T at all.

**Problem 2 (bijective and minimal arithmetic codings).** To give the necessary and sufficient condition of the existence of a bijective a.e. (with respect to the Lebesgue measure on \( T^2 \)) arithmetic coding for a given automorphism T and to describe all bijective arithmetic codings (BAC). If a BAC does not exist, to find a minimal arithmetic coding (MAC) defined as a coding with the minimal possible number of preimages and to describe all such codings.

We are going to solve Problem 1 in this section and to devote two subsequent ones to Problem 2.

Let \( X_r \) denote the stationary Markov compactum with the state space \( 0, 1, \ldots, r \) and the pairwise restrictions \( \{ \varepsilon_n = r \Rightarrow \varepsilon_{n+1} = 0, \ n \in \mathbb{Z} \} \), and the sofic compactum \( Y_r = \{ \{ \varepsilon_n \}_{n=-\infty}^{\infty} : 0 \leq \varepsilon_n \leq r-1, (\varepsilon_n \ldots \varepsilon_{n+s}) \neq (r-1)(r-2)^{s-2}(r-1) \text{ for any } n \in \mathbb{Z} \text{ and any } s \geq 2 \} \). Each of these compacta is the \( \beta \)-compactum for \( \beta = \lambda \). Let us give the corresponding definition (see [Pa]).

**Definition.** Let \( \beta > 1 \), and \( 1 = d_1 \beta^{-1} + d_2 \beta^{-2} + \ldots \), where \( d_1(\beta) = [\beta], d_2(\beta) = [\beta(\beta)], \ldots \). Then by definition, \( X_\beta = \{ \{ \varepsilon_n \}_{n=-\infty}^{\infty} : (\varepsilon_n, \varepsilon_{n+1}, \ldots) \prec_{\text{lex}} (d_1, d_2, \ldots), n \in \mathbb{Z} \} \). The set \( X_\beta \) endowed with the weak topology is called the \( \beta \)-compactum.

We need to recall one more classical definition.

**Definition.** Let \( T \) be a hyperbolic automorphism of the torus. A point \( x \) is called homoclinic (to zero), if \( T^n(x) \to 0 \) as \( n \to \pm \infty \).

The equivalent definition is that \( x \) belongs to the intersection of the leaves of the stable and unstable foliations for \( T \) going through 0.

A suitable way of obtaining all homoclinic points for a given automorphism was proposed in [Ver3]. Let \( T \) be a hyperbolic automorphism of \( \mathbb{T}^k \) (not necessarily two-dimensional). Consider the linear subspace of \( \mathbb{R}^k \) containing the leaf of the unstable foliation going through 0. Then the projection of a point of the lattice \( \mathbb{Z}^k \) to this subspace along the direction of the stable foliation taken modulo \( \mathbb{Z}^k \) is always a homoclinic point for \( T \), and any homoclinic point can be obtained in such a way (see [Ver3] for more detail). For the two-dimensional case these considerations yield the following complete description of the homoclinic points.

**Lemma 1.1.** For the hyperbolic automorphism \( T \) given by a matrix \( M_T \) its any homoclinic point \( t \) is parametrized by a pair \((u, v) \in \mathbb{Z}^2\) as follows: \( t = (\xi, \eta) \), where

\[
(\xi, \eta) = \left( \frac{v + n \lambda}{\sqrt{D}}, \frac{u + k \lambda}{\sqrt{D}} \right) \mod \mathbb{Z}^2, \tag{1.1}
\]

and

\[
\begin{pmatrix}
  n \\
  k
\end{pmatrix} = - \det M_T \cdot M_T^* \begin{pmatrix}
  v \\
  u
\end{pmatrix}. \tag{1.2}
\]

**Proof.** Let \( M_T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). From the general approach decribed above it follows that to obtain any homoclinic point, one needs to consider the projection \((\xi_0, \eta_0)\) of a certain point \((x, y) \in \mathbb{Z}^2\) onto the eigenline \( y = \lambda^\alpha x \) along the eigenline \( y = \lambda^{-\alpha} x \).
then this homoclinic point is \((\xi_0, \eta_0)\) modulo \(\mathbb{Z}^2\). Solving the equation \(\frac{\xi_0 - n}{b} = \frac{\eta_0 - k}{\lambda - a}\) together with \(b\eta_0 = (\lambda - a)\xi_0\), we get

\[
\xi_0 = \frac{-dn + bk + n\lambda}{\sqrt{D}}, \quad \eta_0 = \frac{cn - ak + k\lambda}{\sqrt{D}}
\]

(in view of the relations \(\lambda + \bar{\lambda} = r, \lambda - \bar{\lambda} = \sqrt{D}\)). Setting \(v := -dn + bk, u := cn - ak\), we complete the proof, because \((v, u) = -\det M_T \cdot M_T^{-1}\). □

**Remark 1.** As we see, there is a natural one-to-one correspondence between the homoclinic points of \(T\) and the projections of the integral points onto the eigenline of \(M_T\) being the leaf of the unstable foliation going through \(0\) along its another eigenline. This fact gives us an occasion to use below coordinates in \(\mathbb{R}^2\) for the homoclinic points of \(T\), which looks more natural.

**Remark 2.** The purpose of such a choice of parameters in Lemma 1.1 will become clear below, see Theorem 3.1.

In the two-dimensional case that we are dealing with, this approach leads to the fact that the group of homoclinic points for \(T\) is isomorphic to \(\mathbb{Z}[\lambda]\), this is why it will be convenient to treat norm in \(\mathbb{Q}(\lambda)\) by means of homoclinic points.

**Definition.** Let \(\|x\| := \min \{|x - n| : n \in \mathbb{Z}\}\). A two-sided series of reals \(\sum_{n=-\infty}^{\infty} w_n\) is said to converge to \(w \in [0,1)\) modulo 1, if \(\|\sum_{n=-k}^{l} w_n - w\| \to 0\) as \(k, l \to +\infty\). The convergence of a pair of series modulo \(\mathbb{Z}^2\) to a point of the torus means the convergence of each coordinate modulo 1. Besides, we will use the following notation: \(\left(\sum_{n=-\infty}^{\infty} w_n\right) \left(\begin{array}{c} \xi \\ \eta \end{array}\right) \mod \mathbb{Z}^2 := \lim_{N \to +\infty} \left(\sum_{n=-N}^{N} \xi w_n, \sum_{n=-N}^{N} \eta w_n\right) \mod \mathbb{Z}^2\). Besides, by multiplication of a homoclinic point by some integer we imply the operation of multiplication in the planar coordinates with (if necessary) further return to the toral coordinates.

**Theorem 1.2.**

(I) The set of arithmetic codings \((\varphi, \mathcal{X})\) of a hyperbolic automorphism \(T\) is in a one-to-one correspondence with the points homoclinic to zero. This correspondence is given by the formula for \(\varphi = \varphi_t\):

\[
\varphi_t(\varepsilon) = \lim_{N \to +\infty} \left(\left(\sum_{n=-N}^{N} \varepsilon_n T^{-n} t\right) \mod \mathbb{Z}^2\right) \mod \mathbb{Z}^2
\]

where \(t\) is a homoclinic point for \(T\) (in coordinates of \(\mathbb{R}^2\)). Conversely, if \(t\) is a homoclinic point for \(T\), mapping (1.3) is a convergent series and specifies an arithmetic coding for \(T\). Furthermore, if \(t \neq 0\), then \(\varphi_t\) is surjective.

(II) For each coding, a shift-invariant subset \(\mathcal{X}\) satisfying the second condition of Problem 1, i.e. the surjectivity of the mapping \(\varphi_t|_{\mathcal{X}}\) and its bijectivity on the set of finite sequences belonging to \(\mathcal{X}\), is the stationary Markov compactum \(\mathcal{X}_r\) for \(\det M_T = -1\), or the sofic compactum \(\mathcal{Y}_r\) for \(\det M_T = +1\).
Proof. (I) Let \( \varphi \) satisfy the conditions of Problem 1. We denote \( u_k = \tau^k(u_0) \), i.e. the sequence having 1 at the \((-k)\)th place and zeroes at all other places. We set \( \mathbf{t} = (\xi, \eta) := \varphi(u_0) \). By virtue of the continuity of the mapping \( \varphi \) and the fact that \( u_k \to 0 \), \( k \to \pm \infty \), we have \( T^k \mathbf{t} \to \mathbf{0} \), \( k \to \pm \infty \), whence by definition, \( \mathbf{t} \) must be a homoclinic point. Hence \( \varphi(u_k) = \lambda^k \mathbf{t} \mod \mathbb{Z}^2 \) for all \( k \in \mathbb{Z} \).

Consider now an arbitrary finite sequence \( \varepsilon = \{\varepsilon_n\} \in \tilde{\mathcal{X}} \). By the additivity of \( \varphi \), we have \( \varphi(\varepsilon) = \sum_n \varepsilon_n \varphi(u_{-n}) \mod \mathbb{Z}^2 \), whence

\[
\varphi(\varepsilon) = \left( \sum_{n=-\infty}^{\infty} \varepsilon_n \lambda^{-n} \right) \left( \begin{array}{c} \xi \\ \eta \end{array} \right) \mod \mathbb{Z}^2.
\]

We can now extend the mapping \( \varphi \) by continuity to all sequences \( \varepsilon \in \tilde{\mathcal{X}} \), because since \( \mathbf{t} \) is a homoclinic point, \( \lambda^N \mathbf{t} \to \mathbf{0} \) as \( N \to \pm \infty \) with exponential rate of convergence, whence \( (\sum_{|n|>N} \varepsilon_n \lambda^{-n}) \mathbf{t} \mod \mathbb{Z}^2 \to \mathbf{0} \) as \( N \to +\infty \) for any sequence \( \{\varepsilon_n\}_{n=1}^{\infty} \in \tilde{\mathcal{X}} \). Thus, if a mapping \( \varphi \) is an arithmetic coding, it must have form (1.3).

Conversely, let a mapping \( \varphi_t \) from \( \tilde{\mathcal{X}} \) onto the 2-torus be specified by formula (1.3) with \( \mathbf{t} = (\xi, \eta) \) being a homoclinic point written in coordinates of \( \mathbb{R}^2 \). The convergence of the series involved follows from the definition of \( \tilde{\mathcal{X}} \). We need to check that \( \varphi_t \) is additive, continuous, and turns the shift into \( T \). The additivity of \( \varphi_t \) on \( \tilde{\mathcal{X}} \) is a consequence of its obvious coordinate-wise additivity. To prove its \textit{continuity}, consider two sequences \( \varepsilon \) and \( \varepsilon' \) such that \( \varepsilon_n = \varepsilon'_n \) for \( |n| \leq N \). Then \( \varphi_t(\varepsilon') - \varphi_t(\varepsilon) = \left( \sum_{|n|>N} (\varepsilon'_n - \varepsilon_n) \right) \mathbf{t} \to \mathbf{0} \) as \( N \to +\infty \). As \( \varphi_t \) is continuous, it suffices to verify the relation \( \varphi_t \tau = T \varphi_t \) on the set of finite sequences. Let \( \varepsilon \in \tilde{\mathcal{X}} \) be finite; then \( \varphi_t \tau(\varepsilon) = \lambda \mathbf{t} \mod \mathbb{Z}^2 \), and \( T \varphi_t(\varepsilon) = T \mathbf{t} = \lambda \mathbf{t} \mod \mathbb{Z}^2 \), because \( \mathbf{t} \) is homoclinic. Finally, let \( \varepsilon \neq \mathbf{0} \). To prove the \textit{surjectivity} of the mapping \( \varphi_t \), we rewrite formula (1.3) in the form

\[
\varphi_t(\varepsilon) = \lim_{N \to +\infty} \left( \sum_{n=-N}^{\infty} \varepsilon_n \lambda^{-n} \right) \mathbf{t} \mod \mathbb{Z}^2.
\]

Thus, the image \( \varphi_t(\tilde{\mathcal{X}}) \) is the closure of the leaf of the unstable foliation going through \( \mathbf{0} \), whence this image is \( \mathbb{T}^2 \), because the leaf has irrational slope and thus is dense.

(II) Suppose now \( \varphi_t|_{\mathcal{X}} \) (we will keep the same notation \( \varphi_t \) for this restriction) to be bijective on the set of finite sequences for some shift-invariant subset \( \mathcal{X} \) of \( \tilde{\mathcal{X}} \). Our goal consists in showing that \( \mathcal{X} = \mathcal{X}_r \) in the case \( \sigma = -1 \) or \( \mathcal{Y}_r \) otherwise. Let for simplicity \( \sigma = -1 \). We first prove the inclusion \( \mathcal{X} \subseteq \mathcal{X}_r \). Let, on the contrary, \( \mathcal{X} \) contain a sequence \( \varepsilon \) not lying in the Markov compactum; then there exists \( k \in \mathbb{Z} \) such that either \( \varepsilon_k = r, \varepsilon_{k+1} \geq 1 \) or \( \varepsilon_k > r \). Recall that by our assumption, \( \mathcal{X} \) is total, and thus, the second case implies the first one. Therefore, \( \mathcal{X} \) contains the sequence \( \ldots, 0, 0, 0, 0, 0, 1, 0, 0, 0, \ldots \). The existence of such a sequence contradicts the assumption that \( \varphi_t \) is bijective on the finite sequences, because \( ru_n + u_{n+1} = u_{n-1} \).

To prove the inverse inclusion, suppose \( \mathcal{X} \subseteq \mathcal{X}_r \). By the stationarity and closeness of \( \mathcal{X} \), this means that there exists a cylinder \( \{\varepsilon_0 = i_0, \ldots, \varepsilon_n = i_n\} \) belonging to \( \mathcal{X} \). Together with all its shifts, \( \mathcal{X} \) will be shown that there exists an ergodic
measure \( \mu \) on \( \mathfrak{X}_r \) such that \( \varphi_t(\mu) \) is the two-dimensional Lebesgue measure. Hence by the ergodic theorem, \( \mu(\mathfrak{X}) = 0 \), and by the fact that any mapping \( \varphi_t : \mathfrak{X}_r \to \mathbb{T}^2 \) of the form (1.3) is bounded-to-one (see Proposition 1.4 below), the Lebesgue measure of the image \( \varphi_t(\mathfrak{X}) \) would be equal to 0, which contradicts the surjectivity of \( \varphi \). \( \square \)

1.2. Group interpretation of an arithmetic coding. The Markov (or sofic) compactum defined above does not form a subsemigroup of the semigroup \( \tilde{\mathfrak{X}} \). Nevertheless, we can introduce a group structure after certain small glueings of some sequences. Let \( \mathfrak{X}_r^{(0)} \) and \( \mathfrak{Y}_r^{(0)} \) denote the subsets of \( \mathfrak{X}_r \) and \( \mathfrak{Y}_r \) respectively consisting of all finite sequences. We have shown above that \( \mathfrak{X}_r^{(0)} \) is in fact the factor of the stationary recurrence relations \( \{u_{n-1} = ru_n + u_{n+1}, n \in \mathbb{Z}\} \). Similarly, \( \mathfrak{Y}_r^{(0)} \) is the factor of \( \mathfrak{X}^{(0)} \) with respect to the relations \( \{u_{n-1} = ru_n - u_{n+1}, n \in \mathbb{Z}\} \).

It is well-known that in both cases in question the finite sequences themselves form an additive semigroup (see, e.g., [FrSa]). Our goal now consists in assigning the structure of an additive group to the whole symbolic compacta. To do this, we first give the well-known definition of normalization (see [Fr]).

**Definition.** Let \( x \in \prod_1^\infty \mathbb{Z}_+ \), \( x = (x_k)_{k=1}^\infty \); we define \( c(x) = \sum_{k=1}^\infty x_k \lambda^{-k} = \sum_{k=k_0}^\infty \varepsilon_k \lambda^{-k} \), where \( \{\varepsilon_k\} \) is the \( \beta \)-expansion of \( c(x) \), i.e. the expansion whose digits are given by the greedy algorithm. Thus, \( (\varepsilon_{-k_0}, \varepsilon_{-k_0+1}, \ldots) \) belongs to the symbolic compactum \( \mathfrak{X}_r \) or \( \mathfrak{Y}_r \) respectively. We define

\[
\mathbf{n}(x) := \{\varepsilon_k\}_{k=k_0}^\infty.
\]

The operation \( \mathbf{n} \) is called the *normalization* of a sequence.

With the help of normalization we can now define addition and subtraction on the symbolic compacta. Let the elements of a sequence \( x \) from the definition of normalization are uniformly bounded, for instance, \( 0 \leq x_k \leq 2r \). Then for the cases in question (i.e. for the quadratic units) it is known that similarly to addition, the normalization of a finite sequence is also finite and the carry to both sides is uniformly bounded, see [FrSo]. Thus, it is easy to define the *two-sided normalization* of almost every sequence with respect to any shift-invariant measure \( \mu \) being positive on each cylinder. Namely, by the result of Frougny and Solomyak cited above, there exists \( L = L(\lambda) \in \mathbb{N} \) such that the one-sided normalization of any sequence with coefficients less than or equal to 2\( r \) which has infinitely many blocks \((0\ldots0)\) \((L \text{ times})\) is blockwise. Thus, one can define the two-sided normalization for any sequence containing this block infinitely many times to both sides from the zero place. Note that the existence of such a block is not necessary but sufficient. For more details see [SidVer], where the precise procedure was described in the case \( \lambda = \sqrt{3}/2 \).

The below theorem-definition is based on the following consideration. We need to define subtraction on \( \mathfrak{X} \), specifically, the operation \( i : \varepsilon \mapsto -\varepsilon \). To do it, we are going to find for each of the compacta involved a sequence which is naturally identified with the zero sequence in the sense of the arithmetic. For this goal we consider different representations in \( \mathfrak{X} \) of the elements \( u_n = \tau^n(u_0) \) and easily see that for the Markov compactum \( \mathfrak{X}_r \),
and for the sofic compactum \( \mathcal{Y}_r \),

\[ u_n = (r - 1)u_{n-1} + (r - 2)u_{n-2} + (r - 2)u_{n-3} + \ldots, \]

whence for \( \mathcal{X}_r \) the sequences \( (\ldots, r, 0, r, 0, \ldots) \) are by continuity identified with the zero sequence, the same is true for the sofic case with the sequence \( (\ldots, r - 2, r - 2, r - 2, \ldots) \). Our idea is to define the operation \( i(\varepsilon) \) for the Markov compactum as the normalization of the sequence defined as \( \varepsilon'_n = r - \varepsilon_n \), similarly, as \( \varepsilon'_n = 2(r - 2) - \varepsilon_n \) for the sofic compactum, i.e. to define \( -\varepsilon \), we subtract \( \varepsilon \) from the sequence whose normalization is the zero sequence. Here is the precise claim.

**Theorem-Definition.** (concerning the group structure on \( \mathcal{X} \)). Let \( \mathcal{X} \) denote one of the compacta \( \mathcal{X}_r \) or \( \mathcal{Y}_r \). We define the operations of summation and turning to the inverse element in addition in \( \mathcal{X} \) as follows: let \( \varepsilon \) and \( \varepsilon' \) belong to the compactum \( \mathcal{X} \); the sequence \( x = \{x_n\}_{-\infty}^{\infty} \) is defined as \( x_k = \varepsilon_k + \varepsilon'_k \). Then the sum of \( \varepsilon \) and \( \varepsilon' \) is by definition the two-sided normalization of \( x \). To define \( -\varepsilon \), consider the sequence \( y = \{y_n\} \) with \( y_k = r - \varepsilon_k \) for the Markov case and \( y_k = 2(r - 2) - \varepsilon_k \) for the sofic case. By definition, \( -\varepsilon \) is the two-sided normalization of \( y \). Both operations are well defined for a.e. sequence (or pair of sequences for summation) with respect to any Borel measure which is positive on each cylinder in \( \mathcal{X} \) (respectively with respect to the square of such a measure for addition).

**Proof.** By the above, the sum of two sequences is well defined for any pair \( (\varepsilon, \varepsilon') \) such that both contain the block \( (0 \ldots 0) \) \((L \text{ times}) \) at the same place infinitely many times to both sides from the zero coordinate.

The operation \( i : \varepsilon \mapsto -\varepsilon \) in the sofic compactum \( \mathcal{Y}_r \) is well defined for \( \varepsilon \) which has the block \( (r - 1, r - 3, r - 2, r - 2, \ldots, r - 2, r - 2, r - 3, r - 1) \) of length \( L + 2 \) infinitely many times to both sides. Indeed, the operation \( \varepsilon_k \mapsto 2(r - 2) - \varepsilon_k \) turns this block into the block \( (r - 3, r - 1, r - 2, r - 2, \ldots, r - 2, r - 2, r - 1, r - 3) \) whose normalization is \( (r - 2, 0, 0, \ldots, 0, 0, r - 2) \) with \( L \) zeroes. Since a.e. sequence \( \{\varepsilon_k\} \) has such a block infinitely many times to both sides, the normalization of \( \{2(r - 2) - \varepsilon_k\} \) is blockwise, and therefore is well defined.

Finally, in the Markov case with \( r \geq 2 \) (the case \( r = 1 \) was considered in [SidVer]) the operation \( i \) is well defined, for instance, for the sequences having the cylinder \( \{\varepsilon_k = r - 1, \varepsilon_{k+1} = r\} \) infinitely many times to both sides. Indeed, the two-sided normalization acts by changing any triple \( (l, r, k) \mapsto (l + 1, 0, k - 1) \) for \( l \leq r - 1, k \geq 1 \), whence, as is easy to see, for the sequence \( \{\varepsilon'_n\} \) with \( \varepsilon'_n = r - \varepsilon_n \), the two-sided normalization is independent for the pieces \( (\ldots, \varepsilon'_{k-1}, \varepsilon'_k) \) and \( (\varepsilon'_{k+1}, \varepsilon'_{k+2}, \ldots) \). Thus, we split a.e. sequence \( \varepsilon \) into such pieces, so that the normalization of \( \varepsilon' \) is blockwise. \( \square \)

Now we are going to make the above claim more precise. We describe all identifications in \( \mathcal{X} \) which turn it into a group in addition.

**Proposition 1.3.** Let \( \mathcal{X}'_r \) and \( \mathcal{Y}'_r \) denote the factor sets \( \mathcal{X}_r / \mathcal{R}_1 \) and \( \mathcal{Y}_r / \mathcal{R}_2 \), where

1. \( \mathcal{R}_1 \) is the identification of the pairs of sequences \( *(k, r, 0, r, \ldots) \sim *(k + 1, 0, 0, 0, \ldots) \), and \( (\ldots_0, r, 0, r, 0, k*) \sim (\ldots_0, r, 0, r, 0, r - 1, k + 1*) \), where \( * \) denotes one and the same arbitrary admissible tail, and \( 0 \leq k \leq r - 1 \).
2. \( \mathcal{R}_2 \) is the identification of the pairs of sequences \( *(k, r - 1, r - 2, r - 2, \ldots) \sim \).
\((s+1,0,0,0,\ldots),\) and \((\ldots 0, r-2, r-2, r-2, r-1, k) \sim (\ldots 0, 0, 0, k+1), 0 \leq k \leq r-2.\) Then the factor sets \(X'_r\) and \(Y'_r\) are groups in addition.

**Proof.** The calculations based on the relations \(u_{n-1} = ru_n + u_{n+1}, n \in \mathbb{Z}\) for the Markov case and \(u_{n-1} + u_{N+1} = (r-1)u_n + (r-2)u_{n+1} + \cdots + (r-2)u_{N-1} + (r-1)u_N, n \in \mathbb{Z}, N \geq n,\) lead exactly to the identifications mentioned in the claim of the proposition. We omit technical computations. For more details see [Ver1], [Ver2] for the Markov case with \(r = 1\) (more general cases are similar in techniques). \(\square\)

**Remark 1.** It is easy to see that according to the rule of gluing given in Proposition 1.3, there are some sequences which are identified with two or three other ones. For instance, by continuity the zero sequence is identified with the \(0\)th sequence \(\varepsilon + 0\). For \(\varepsilon + 0\) on the Markov case with the Markov compactum, and with the sequence \(\ldots, r-2, r-2, r-2, \ldots\) in the sofic compactum.

**Remark 2.** The group \(X'_r\) (resp. \(Y'_r\)) is a compact Abelian group, hence it possesses the Haar measure \(\mu\), which by definition is Borel and positive on each cylinder, i.e. satisfies the conditions of Theorem-Definition. The natural projection \(X_r \mapsto X'_r\) (resp. \(Y_r \mapsto Y'_r\)) as a map of measure spaces is an isomorphism \((\text{mod } 0)\), which follows from the nature of identifications.

**Remark 3.** Let, as above, \(u_k\) denote the sequence having all zeroes except one unity at the \(k\)'th place. The operation \(\varepsilon \mapsto \varepsilon + u_k\) in the sense of group structure defined above, is the two-sided version of adic transformation (see [Ver2]). It turns out that the ordinary adic transformation generates the action of \(\mathbb{Z}\) on the one-sided \(\beta\)-compactum, while the case in question the addition of finite sequences generate the action of \(\mathbb{Z}^2\).

Below we will need the following claim.

**Proposition 1.4.** Any mapping \(\varphi\) from the definition of arithmetic coding is well defined on the factor sets and is a group homomorphism of the groups \(X'_r\) (resp. \(Y'_r\)) and \(\mathbb{T}^2\). Any arithmetic coding as a mapping from \(X_r\) (resp. \(Y_r\)) onto \(\mathbb{T}^2\) is always \(K\)-to-1 a.e. with respect to the measure \(\mu\) for some natural \(K\).

**Proof.** The factor map \(\varphi : X'_r(\mathbb{Z}) \to \mathbb{T}^2\) is well defined, because by the definition of identifications (see Proposition 1.3), \(\varphi(\varepsilon) = \varphi(\varepsilon')\), if \(\varepsilon\) is identified with \(\varepsilon'\). Furthermore, by the nature of the arithmetic in \(X\), we have \(\varphi'(\varepsilon \pm \varepsilon') = \varphi'(\varepsilon) \pm \varphi'(\varepsilon')\). The second claim follows from the theorem on the homomorphic image of a group, from which \((\varphi')^{-1}(x) = (\varphi')^{-1}(0) + \varepsilon\), where \(\varepsilon\) is a sequence in the preimage of \(x \in \mathbb{T}^2\). Thus, \#((\varphi')^{-1}(x)) \equiv \text{const.} \ \square\)

**Remark.** The precise value of the function \(K = K(u, v)\) will be computed in Section 3.

2. **Bijective arithmetic codings of automorphisms and the associated binary quadratic form**

Below we will see that sometimes there are no bijective arithmetic codings of a given automorphism; however, even if they do exist for a certain homoclinic point \(t\), it can happen that for another homoclinic point the mapping \(\varphi\) is not bijective a.e. Here is the simplest example.
Example. Consider the Fibonacci automorphism Φ given by the matrix \(\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\). The corresponding Markov compactum is \(\mathcal{X}_\Phi = \{\{\varepsilon_n\} : \varepsilon_n \in \{0, 1\}, \varepsilon_n \varepsilon_{n+1} = 0, n \in \mathbb{Z}\}\), and \(\lambda = \frac{\sqrt{5}+1}{2}\). By Theorem 1.2, an arithmetic coding of Φ is a mapping \(\varphi_\xi\) from \(\mathcal{X}_\Phi\) onto \(\mathbb{T}^2\) of the form

\[
\varphi_\xi(\{\varepsilon_n\}) = \left(\sum_{n=-\infty}^{\infty} \xi \varepsilon_n \lambda^{-n}\right) \left(\frac{1}{\lambda-1}\right) \mod \mathbb{Z}^2.
\]

Usually, the coefficients \(\xi \varepsilon_n\) assume the values 0 and 1 (see, e.g., [Ber]). However, this mapping (i.e. \(\varphi_1\)) from \(\mathcal{X}_\Phi\) onto the torus proves to be not bijective, but actually 5-to-1 a.e. The kernel of the group homomorphism \(\varphi'_1 : \mathcal{X}'_\Phi \to \mathbb{T}^2\) is the group \(\mathcal{K} = \{0^\infty, (1.000)^\infty, (0.100)^\infty, (0.010)^\infty, (0.001)^\infty\}\), where point denotes the border between negative and nonnegative coordinates of a sequence. Thus, the preimage of a.e. point of the torus consists of five sequences, the difference of any two of them being equal to one of the sequences in \(\mathcal{K}\), and the compactum \(\mathcal{X}_\Phi\) is splitted into five parts \(X_1 \cup \cdots \cup X_5\) such that \(\varphi_1|_{X_k}\) is bijective a.e. for \(1 \leq k \leq 5\).

At the same time, as will be shown below, for the automorphism Φ given by the companion matrix \(M_\Phi = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\) a bijective arithmetic coding does exist, and the proper choice of coefficients is \(\xi \varepsilon_n \in \left\{0, \frac{1}{\sqrt{5}}\right\}\), i.e the mapping \(\varphi_{1/\sqrt{5}}\). In Figure 1 we depict the images of the sets \(\{X_k\}_{k=1}^5\) under the mapping \(\varphi_{1/\sqrt{5}}\). Each of these images is the square with the side \(\frac{1}{\sqrt{5}}\).

![Fig. 1. Splitting the 2-torus into 5 squares](image)

The group \(\{O(0; 0), A(1/5; 2/5), B(3/5; 1/5), C(4/5; 3/5), D(2/5; 4/5)\}\) isomorphic to \(\mathbb{Z}/5\mathbb{Z}\) is the image of the set \(\mathcal{K}\) under the mapping \(\varphi_{1/\sqrt{5}}\). Note that the Fibonacci automorphism cyclically moves the points of this group as follows: \(\Phi : A \to B \to C \to D \to A\).
For a detailed study of the Fibonacci case and the proofs see [SidVer, sec. 1, item 1.6].

We will show that the condition on a homoclinic point \( t \) for the bijectivity a.e. of a mapping \( \varphi_t \) given by formula (1.3), can be interpreted in terms of the area of some fundamental domain. We begin with a class of matrices with the simplest fundamental domain, namely with the case of companion matrices. In this case \( t = (\xi, \pm \lambda^{-1} \xi) \) in coordinates of \( \mathbb{R}^2 \), and the condition of bijectivity will be given in terms of the algebraic norm of \( \xi \). Later it will be shown that the main result depends on the conjugacy class in \( GL(2, \mathbb{Z}) \) and not on a matrix itself.

2.1. Case of the companion matrix. We are going to show that for the automorphism \( T_{r,\sigma} \) given by the companion matrix \( C_{r,\sigma} := \left( \begin{array}{cc} r & 1 \\ -\sigma & 0 \end{array} \right) \) with \( \sigma = \pm 1 \) and \( r \in \mathbb{N} \) for \( \sigma = -1 \) and \( r \geq 3 \) for \( \sigma = +1 \), a BAC always exists and that any such a coding is naturally parametrized by a unit of the field \( \mathbb{Q}(\lambda) \). Note first that the vector \( \left( \frac{\lambda}{-\sigma} \right) \) is an eigenvector of the matrix \( C_{r,\sigma} \). Hence mapping (1.3) in this case is given as follows:

\[
L_\xi(\{\varepsilon_n\}) = \left( \sum_{n=-\infty}^{\infty} \varepsilon_n \lambda^{-n} \right) \left( \begin{array}{c} \xi \\ -\sigma \xi \lambda^{-1} \end{array} \right) \mod \mathbb{Z}^2. \tag{2.1}
\]

To proceed, we need the precise description of possible values of \( \xi \). Recall that by the above, \( (\xi, -\sigma \lambda^{-1} \xi) \) should be a homoclinic point for \( T_{r,\sigma} \). The following claim is a consequence of Lemma 1.1.

**Lemma 2.1.** The set of homoclinic points for the automorphism \( T_{r,\sigma} \) written in coordinates of \( \mathbb{R}^2 \), is

\[
\left\{ \left( \frac{m+n\lambda}{\sqrt{D}}, -\sigma \lambda^{-1} \frac{m+n\lambda}{\sqrt{D}} \right) : (m, n) \in \mathbb{Z}^2 \right\}.
\]

Thus, in formula (2.1),

\[
\xi = \xi(m, n) = \frac{m+n\lambda}{\sqrt{D}} \tag{2.2}
\]

with \((m, n) \in \mathbb{Z}^2\). \( \square \)

Now our goal is to find among all \( \xi \) of the form (2.2) such that \( L_\xi \) is one-to-one a.e. We will see that actually these \( \xi \) have the minimal possible algebraic norm \( N(\xi) := \overline{\xi} \xi \) in modulus, where \( \overline{\xi} \) is the algebraic conjugate of a quadratic irrational \( \xi \).

**Theorem 2.2.** The automorphism of the 2-torus \( T_{r,\sigma} \) given by the companion matrix \( C_{r,\sigma} := \left( \begin{array}{cc} r & 1 \\ -\sigma & 0 \end{array} \right) \), \( \sigma = \pm 1 \), admits a bijective arithmetic coding. If \( \sigma = -1 \) or \( \sigma = +1 \), \( r \geq 4 \), its BAC is always of the form

\[
L_\xi(\{\varepsilon_n\}) = \left( \sum_{n=-\infty}^{\infty} \varepsilon_n \lambda^{-n} \right) \left( \begin{array}{c} \xi \\ -\sigma \xi \lambda^{-1} \end{array} \right) \mod \mathbb{Z}^2,
\]
where \( \xi = \frac{\pm \lambda^k}{\sqrt{D}} \), \( k \in \mathbb{Z} \).

The case \( M = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix} \) is specific. Here \( \lambda = \frac{3 + \sqrt{5}}{2} \), and any BAC is of the form

\[
L_\xi(\{\varepsilon_n\}) = \left( \sum_{n=-\infty}^{\infty} \varepsilon_n \lambda^{-n} \right) \left( \begin{array}{c} \xi \\ -\xi \lambda^{-1} \end{array} \right) \mod \mathbb{Z}^2
\]

with \( \xi = \frac{\pm \theta^k}{\sqrt{D}} \), \( k \in \mathbb{Z} \), \( \theta = \sqrt{\lambda} = \frac{1+\sqrt{5}}{2} \).

**Proof.** Let an arithmetic coding \( L_\xi \) of \( T_{r,\sigma} \) be written in the form (2.1) with \( \xi \) as in formula (2.2). Suppose first \( \sigma = -1 \). Consider an arbitrary sequence \( \{\varepsilon_n\}_n^{\infty} \in \mathcal{X} \). We split it into two pieces \( \{\varepsilon_n\}_0^{\infty} \) and \( \{\varepsilon_n\}_1^{\infty} \) and define \( x_1(\{\varepsilon_n\}) := \sum_{k=1}^{\infty} \varepsilon_k \lambda^{-k}, \) \( x_2 = \sum_{k=0}^{\infty} \varepsilon_k (-\lambda)^{-k} \). It is a direct inspection that \( x_1 \in [0, 1], x_2 \in [-1, 1] \).

Using the relation \( \{\lambda^n\} = \{(-1)^{n+1} \lambda^{-n}\} \), \( n \geq 0 \), we make sure that \( \sum_{n=0}^{\infty} \varepsilon_n \xi_\lambda^{-n} = \xi_1 - \xi x_2 \mod 1 \) and similarly, \( \sum_{n=0}^{\infty} \varepsilon_n \xi_\lambda^{-n-1} = \xi \lambda^{-1} x_1 + \xi_\lambda x_2 \mod 1 \), where, as above, \( \xi_\lambda \) denotes the algebraic conjugate of a quadratic irrational \( \xi \).

Thus, we have the sequence of mappings

\[
\mathcal{X}_r \xrightarrow{F} \mathbb{R}^2 \xrightarrow{b_\xi} \mathbb{R}^2 \xrightarrow{\pi} \mathbb{T}^2,
\]

where \( F(\{\varepsilon_n\}) = (x_1, x_2) \), and \( b_\xi(x_1, x_2) = (\xi x_1 - \xi_\lambda x_2, \xi \lambda^{-1} x_1 + \xi_\lambda x_2) \), i.e. the transfer to the eigenvector coordinates, and finally, \( \pi \) is the projection modulo the lattice \( \mathbb{Z}^2 \). Thus, the mapping \( L_\xi \) is a factor map, i.e.

\[
L_\xi(\{\varepsilon_n\}) = (\pi b_\xi F)(\{\varepsilon_n\}).
\]

By definition, the mapping \( b_\xi F \) is always a bijection onto the image. Note that since \( (\varepsilon_0, \varepsilon_1) \neq (r, k) \) with \( k \neq 0 \), \( F(\mathcal{X}_r) = \Pi = ([0, 1] \times [-1, 1]) \setminus ([\lambda^{-1}, 1] \times [-\lambda^{-1}, \lambda]) \), i.e. the difference of rectangles (see Figure 2 below for the case of the Fibonacci automorphism). The area of \( \Pi \) is \( (\lambda + 1) \lambda^{-1} + (1 - \lambda^{-1}) = \sqrt{D} \), and the linear transformation \( b_\xi = \left( \begin{array}{c} \xi \\ -\xi \lambda^{-1} \end{array} \right) \) from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \) has determinant \( \sqrt{D} N(\xi) \), where \( N(\xi) = \xi \overline{\xi} \) is the algebraic norm of \( \xi \). Thus, the fundamental domain \( \Omega_\xi := (b_\xi F)(\mathcal{X}_r) = b_\xi(\Pi) \) on the plane has area \( S = |DN(\xi)| \).

Recall that \( \#L_\xi^{-1}(x) \) is one and the same for a.e. \( x \in \mathbb{T}^2 \), see Proposition 1.4. Thus, this capacity is necessarily equal to \( S \), and \( L_\xi \) is a bijection a.e. if and only if the area \( S \) of the fundamental domain \( \Omega_\xi \) equals 1, or equivalently, if

\[
N(\xi) = \pm \frac{1}{D}. \tag{2.3}
\]

By Lemma 2.1, \( \xi = \frac{m+n\lambda}{\sqrt{D}} \), and the equation (2.3) is equivalent to the Diophantine equation

\[
N(m + n\lambda) = \pm 1.
\]

Therefore, as is well-known, \( m+n\lambda \) is a unit of the ring \( \mathbb{Z}[\lambda] \) and thus, \( m+n\lambda = \pm \lambda^k \) for some \( k \in \mathbb{Z} \) by virtue of the facts that \( \mathbb{Z}[\lambda] \) is the maximal order of the field \( \mathbb{Q}(\lambda) \) and that \( \lambda \) is its main unit (see, e.g., [BerSh]). Let us recall that the above
equation is in fact the condition on a homoclinic point being the parameter of a coding.

The case \( \sigma = +1 \) is studied in the same way. Since here \( \lambda = \lambda^{-1} \), we have \( x_1 = \sum_{k=1}^{\infty} \varepsilon_k \lambda^{-k} \), \( x_2 = \sum_{k=0}^{\infty} \varepsilon_k \lambda^{-k} \). The set \( \Pi \) here is the difference of the rectangles \((0, 1) \times [0, \lambda) \setminus ((1 - \lambda^{-1}, 1) \times (\lambda - 1, \lambda))\). The rest of the proof is the same, and we come to equation (2.3). Again, \( m + n\lambda \) must be a unit of the ring \( \mathbb{Z}[\lambda] \), whence \( m + n\lambda = \pm \lambda^k \) if \( r \geq 4 \), and \( m + n\lambda = \pm \theta^k \) for \( r = 3 \) with \( \theta \) equal to the golden ratio. \( \square \)

Below we depict the fundamental domain \( \Omega_\xi \) with \( \xi = \frac{1}{\sqrt{5}} \) for the case of Fibonacci automorphism \( \Phi \) (see Example above).

![Figure 2](image.png)

**Fig. 2. Fundamental domain of area 1**

It is visible from the figure that the fundamental domain is projected modulo \( \mathbb{Z}^2 \) onto the unit square. Indeed, consider the square \( O'OR'N = [0, 1] \times [-1, 0] \). The polygon \( OKNPQK' \) lies inside the square, and we project: triangle \( ORK' \) onto \( O'OK \), the triangle \( OLR \) onto \( NPR' \), and finally, \( MNK \) onto \( QR'K' \).

**Corollary 2.3.** Any mapping \( L_\xi \) from the symbolic compactum onto the torus of the form (2.1) is K-to-1 with \( K = |DN(\xi)| \).

**Remark.** Let us give a geometric interpretation of the bijectivity. We know that each parameter of an arithmetic coding of \( T_{r, \sigma} \) is \( \xi = \xi(m, n) = \frac{m + n\lambda}{\sqrt{D}} \), see formula (2.2). Direct computations show that those which yield a bijective arithmetic coding, form the orbit (a kind of “integral hyperbola”)

\[
\left\{(m, n) \in \mathbb{Z}^2 : \left(\begin{array}{c} m \\ n \end{array}\right) = \pm C_k \left(\begin{array}{c} 1 \\ 0 \end{array}\right) \text{ for some } k \in \mathbb{Z}\right\},
\]

where \( C_k \) is the integral cyclic permutation of \( (1, 0, 0, \ldots, 0) \) with period \( k \).
with the exception of the case $\sigma = +1, r = 3$, when these integral points form the orbit \[ \{ \pm \left( \begin{array}{c} 1 \\ 1 \\ 0 \end{array} \right)^k, k \in \mathbb{Z} \}. \]

Recall that the homoclinic equivalence relation on the torus is given as follows: two points $x$ and $y$ are said to be equivalent if $T^n(x - y) \to (0, 0)$ as $|n| \to \infty$ (see, e.g., [Gor]). Let us define the homoclinic equivalence relation on the symbolic compactum.

**Definition.** A sequence in compactum $X_r$ is called homoclinic (to zero) if its right and left tails are either of the form $(0, 0, 0, \ldots)$ or of the form $(r, 0, 0, \ldots)$. Similarly, a sequence in $Y_r$ is called homoclinic if its right and left tails are of the form $(0, 0, 0, \ldots)$ or of the form $(r - 2, r - 2, r - 2, \ldots)$. Similarly to the “toral” definition, we will say that two sequences belong to the same homoclinic class if their difference (which, as we know, is well defined for a.e. pair, see Section 1) is a sequence homoclinic to zero.

**Proposition 2.4.** The image of the homoclinic class of a sequence under a bijective arithmetic coding is the homoclinic class of its image.

**Proof.** By the above, after some identifications touching sequences from one and the same homoclinic class, a BAC becomes a complete bijection. Now the claim follows from the fact that a BAC turns the sequences homoclinic to zero to the points homoclinic to zero. \(\square\)

**Remark 1.** For the case in question it seems more natural to consider the following mapping which naturally generalizes the one sided $\beta$-expansions to the two-sided (= two-dimensional) case:

\[ l(\{\varepsilon_n\}) = \left( \sum_{n=-\infty}^{\infty} \varepsilon_n \lambda^{-n} \right) \left( \frac{-\sigma}{\lambda - 1} \right) \mod \mathbb{Z}^2. \]

It is a particular case of the more general mapping introduced and studied in [Ber]. Obviously, $l$ semiconjugates the shift $\tau$ and the automorphism with the companion matrix $C_{r,\sigma}$; however, from Corollary 2.3 it follows that the mapping $l$ is only $D$-to-$1$ a.e.

**Remark 2.** For the case $\sigma = -1$ the mapping $L = L_{1/\sqrt{D}}$ in the form

\[ L(\varepsilon) = \sum_{k=1}^{\infty} \varepsilon_k \lambda^{-k} \cdot \pi_u(u_0) - \sum_{k=0}^{\infty} \varepsilon_{-k} (-\lambda)^{-k} \cdot \pi_s(u_0) \mod \mathbb{Z}^2 \]

with $u_0 = \left( \begin{array}{c} 0 \\ 1 \end{array} \right)$ and $\pi_u, \pi_s$ being the projections on the leaves of the unstable and stable foliations respectively, was under consideration in the recent Ph. D. dissertation [Leb]. In particular, the author proved its bijectivity a.e. but did not consider in detail its arithmetic properties.

**2.2. General case.** Return now to the general case of ergodic automorphism $T$ given by a matrix $M_T = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in GL(2, \mathbb{Z})$. We begin with two necessary definitions.
**Definition.** Two matrices $M_1$ and $M_2$ will be called *algebraically conjugate*, if there exists a matrix $B \in GL(2, \mathbb{Z})$ such that $BM_1B^{-1} = M_2$. We will write in this case $M_1 \sim M_2$.

**Definition.** The binary quadratic form $f_T(x, y) = bx^2 - (a - d)xy - cy^2$ will be called the *form associated with an automorphism* $T$.

**Remark.** Obviously, a binary integral quadratic form $f(x, y) = \alpha x^2 + \beta xy + \gamma y^2$ is the form associated with some automorphism if and only if $D(f) = r^2 \pm 4$ for some $r$, where $D(f) = \beta^2 - 4\alpha\gamma$ is the discriminant of the form $f$. Since $D(f_T) = D$, we are dealing in fact with all forms with the discriminant of the form $r^2 \pm 4 > 0$. The mapping $\theta : T \mapsto f_T$ will be studied in detail in Appendix.

**Theorem 2.5.**

I. An ergodic automorphism $T$ admits bijective arithmetic coding if and only its matrix $M_T$ is algebraically conjugate to the companion matrix $C_{r, \sigma} = \begin{pmatrix} r & 1 \\ \sigma & 0 \end{pmatrix}$ with $r = \text{Tr}M_T$ and $\sigma = \det M_T$.

II. A matrix $M_T$ is algebraically conjugate to the corresponding companion matrix if and only if one of the equations

$$f_T(x, y) = \pm 1 \quad (2.4)$$

is solvable in $\mathbb{Z}$. Any matrix $B = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in GL(2, \mathbb{Z})$ such that $BM_TB^{-1} = C_{r, \sigma}$ is naturally parametrized by a solution of Diophantine equation (2.4), namely, $(x, y)$ is always a solution of (2.4), and

$$(z, t) = -\det M_T \cdot (x, y)M_T^{-1}.$$

**Proof.** I. Suppose a matrix $B \in GL(2, \mathbb{Z})$ such that $BM_TB^{-1} = C_{r, \sigma}$ exists, and let $Q$ be the toral automorphism given by $B$. Let $L$ be a bijective a.e. mapping from relation (2.1), say, for $\xi = 1/\sqrt{D}$. Recall that the compactum $\mathcal{X}$ is determined only by the spectrum of the matrix specifying an automorphism, whence it is one and the same for $T$ and the automorphism given by $C_{r, \sigma}$. Consider the mapping $\varphi := Q^{-1}L : \mathcal{X} \to \mathbb{T}^2$. We have $\varphi_T = Q^{-1}L_T = Q^{-1}T_T = TQ^{-1}L = T\varphi$, and since $\varphi$ is bijective a.e., it is the desired BAC for $T$.

Conversely, let $T$ admit BAC, and $\varphi$ be the corresponding mapping from the symbolic compactum onto the torus. Consider $Q := L\varphi^{-1} : \mathbb{T}^2 \to \mathbb{T}^2$. It is well defined, because if two sequences $\varepsilon, \varepsilon'$ belong to $L^{-1}(x)$ for some $x \in \mathbb{T}^2$, by the above, $\varphi(\varepsilon) = \varphi(\varepsilon')$. Thus, we make sure that by definition of BAC, $Q$ is a group automorphism of the 2-torus, hence, it is given by some matrix $B \in GL(2, \mathbb{Z})$. Since $QTQ^{-1} = L\varphi^{-1}T\varphi L^{-1} = L\tau L^{-1} = T_{r, \sigma}$, we have $C_{r, \sigma} = BM_TB^{-1}$.

II. It suffices to show that the solvability of one of the Diophantine equations (2.4) is equivalent to the fact that $M_T \sim C_{r, \sigma}$. Let $B = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in GL(2, \mathbb{Z})$ exist, and $BM_T = C_{r, \sigma}B$. Suppose $\sigma = 1$. We have thus the linear system

$$
\begin{align*}
z &= -dx + cy \\
t &= bx - ay \\
x &= az + ct \\
y &= bx + dt
\end{align*}
$$

(2.5)
the last two equations being a consequence of the first two ones. Hence this system together with the condition \( \det B = \pm 1 \) yields the desired condition. For \( \sigma = +1 \) the first two equations in formula (2.5) are the same as for the previous case, so, the argument is also the same.

Conversely, if the equation (2.4) is solvable, then we take some \( x, y \) being its solutions and construct the matrix \( B \) by the equations for \( z, t \) from formula (2.5).

Recall that the Dirichlet theorem claims that given an automorphism \( T \), the group \( D(T) \) defined as the set of all automorphisms of the torus which commute with \( T \), has the form \( \{ \pm S^n, \ n \in \mathbb{Z} \} \) for some primitive automorphism \( S \). The following theorem shows that in the Dirichlet group of \( T \) only four primitive elements \( \pm S, \pm S^{-1} \) can admit BAC (with the unique exclusion, when they are eight).

**Theorem 2.6.**

1. The automorphism \( T \) admits BAC only if its matrix \( M_T \) is primitive, with the exception of the case \( M_T = K^2 \), where \( K \) is algebraically conjugate to \( \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \).

2. Let equation (2.4) be solvable for a given automorphism \( T \). If \( M_T \) is primitive, then there exists a homoclinic point whose coordinates in \( \mathbb{R}^2 \) are \( (\xi_0, \eta_0) \) such that any bijective arithmetic coding of \( T \) is of the form

\[
\varphi^\pm_k(\varepsilon) = \left( \sum_{n=-\infty}^{\infty} \varepsilon_n \lambda^{-n} \right) \left( \pm \xi_0 \lambda^k, \pm \eta_0 \lambda^k \right) \mod \mathbb{Z}^2.
\]

(2.6)

If \( M \) is not primitive, we have \( \lambda = \frac{3+\sqrt{5}}{2} \), and any BAC is of the form

\[
\varphi^\pm_k(\varepsilon) = \left( \sum_{n=-\infty}^{\infty} \varepsilon_n \lambda^{-n} \right) \left( \pm \xi_0 \theta^k, \pm \eta_0 \theta^k \right) \mod \mathbb{Z}^2,
\]

(2.7)

where \( \theta = \sqrt{\lambda} = \frac{1+\sqrt{5}}{2} \).

**Proof.** (1) It is easy to compute that \( f_{T^2} = rf_T, \ f_{T^3} = (r^2 + 1)f_T \) and, more generally, \( f_{T^n} = q_n(r)f_T \), where \( q_n \) is a polynomial of degree \( n \) with nonnegative coefficients, odd for \( n \) odd and even for \( n \) even, namely, \( q_n(r) = \frac{1}{\sqrt{D}}(\lambda^n - \overline{\lambda^n}) \). So, the form \( f_{T^n} \) is not primitive unless \( n = 2, r = 1 \), i.e. its coefficients are not relatively prime, hence Diophantine equations (2.4) have no solutions. Thus, the unique companion matrix which is not primitive, is \( C_{3,1} \sim \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 \). Now if \( T \) admits bijective arithmetic coding, then by Theorem 2.5, \( M_T \sim C_{r,\sigma} \), and the first claim of the theorem follows from the fact that the primitivity is an invariant of algebraic conjugacy.

(2) Since equation (2.4) is solvable in \( \mathbb{Z} \), there exists an infinite number of different BAC’s for \( T \). Fix the notation \( \varphi_0 \) for one of them; let \( \varphi \) be be an arbitrary BAC for \( T \). Consider the mapping \( A := \varphi \varphi_0^{-1} : \mathbb{T}^2 \to \mathbb{T}^2 \). It is well defined by the same arguments as in the proof of Theorem 2.5. Obviously, \( A \) is an automorphism of the 2-torus, and \( A \) commutes with \( T \). By the Dirichlet theorem cited above and the primitivity of \( T \), we have \( \varphi = \pm T^k \varphi_0 \) for some \( k \in \mathbb{Z} \), whence if \( \varphi_0 \) in formula (1.3) is given by a homoclinic point \( (\xi_0, \eta_0) \), the mapping \( \varphi \) is given by \( (\pm)^k \xi_0, (\pm)^k \eta_0 \).
Conversely, if a bijection a.e. \( \varphi \) is given by formula (1.3) with some \((\xi, \eta)\), the mapping \( \varphi' \) defined by the same formula with \((\pm \lambda \xi, \pm \lambda \eta)\) is also a bijection, as \( \varphi(-\varepsilon) = -\varphi(\varepsilon) \), \( \varphi(\tau \varepsilon) = T^{\pm 1} \varphi(\varepsilon) \). The argument for the exclusive case is the same with the exception that here \( \sqrt{M} \) is also a matrix in \( GL(2, \mathbb{Z}) \) and also commutes with \( M \). □

**Remark.** In Appendix we will give a simple example of a matrix which is not conjugate to the corresponding companion matrix, see “Counterexamples”.

Thus, the bijective arithmetic codings in fact are naturally parametrized by elements of the Dirichlet group of the field \( \mathbb{Q}(\sqrt{D}) \).

We finish the section by giving simple algebraic criteria for the existence of a bijective arithmetic coding of a given automorphism of the 2-torus.

**Corollary 2.7.** If two ergodic automorphisms \( T_1 \) and \( T_2 \) whose matrices have one and the same trace and discriminant, both admit bijective arithmetic coding, then their matrices are algebraically conjugate. Conversely, if \( T_1 \) admits BAC and \( M_{T_1} \sim M_{T_2} \), then so does \( T_2 \).

**Proof.** It suffices to recall that both matrices should be algebraically conjugate to the corresponding companion matrix which is one and the same for both ones. □

**Corollary 2.8.** If \(|b| = 1\) or \(|c| = 1\), an automorphism \( T \) with the matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) admits BAC.

**Proof.** One of the equations (2.4) has the trivial solution \( x = 1, y = 0 \) or \( x = 0, y = 1 \) if \( b = \pm 1 \) or \( c = \pm 1 \) respectively. □

### 3. Minimal arithmetic codings

We have already seen that sometimes an ergodic automorphism of the 2-torus does not admit BAC, so, it is meaningful to deal with the notion of minimal arithmetic coding (MAC) introduced in Section 1. Recall that a minimal arithmetic coding of an automorphism is, by definition, a coding \( \varphi \) having the minimal possible number of preimages.

Recall that by formulas (1.1) and (1.2), any arithmetic coding is parametrized by a pair \((u, v) \in \mathbb{Z}^2\), and from Proposition 1.4 it follows that for any coding \((\varphi, \mathcal{X})\) a mapping \( \varphi : \mathcal{X} \to \mathbb{T}^2 \) is \( K \)-to-1 a.e. The following theorem answers the question on the form of the function \( K = K(u, v) \).

**Theorem 3.1.** Let \( T \) be the hyperbolic automorphism of the 2-torus given by a matrix \( M_T \). Then any arithmetic coding \( \varphi_t \) of \( T \) of the form (1.3) with \( t = (\xi, \eta) \) being a homoclinic point defined by formulas (1.1) and (1.2), is \( K \)-to-1 a.e. with

\[
K = K(u, v) = |f_T(u, v)|.
\]

**Proof.** Using the same arguments as in Theorem 2.2, we make sure that \( K \) equals the area of the fundamental domain and that this domain has area given by the formula

\[
S = \sqrt{D} \left| \det \begin{pmatrix} \xi & -\overline{\xi} \\ \eta & -\overline{\eta} \end{pmatrix} \right|.
\]
Furthermore, from direct computations in formula (3.1) which we omit (in view of relation (1.2)), it follows that

\[ K = S = |f_T(u, v)|. \]  

Thus, we proved the following theorem which describes explicitly, in what way an arbitrary arithmetic coding is parametrized by a homoclinic point. □

Let \( m(T) \) denote the minimal possible number of preimages for an arithmetic coding of \( T \).

**Corollary 3.2.** The quantity \( m(T) \) equals the integral minimum of the associated form \( f_T \). Any minimal arithmetic coding of a given automorphism \( T \) is naturally parametrized by a solution of the equation

\[ f_T(u, v) = \pm m, \]  

where \( m = m(T) \).

We are ready now to describe all possible minimal arithmetic codings for a given automorphism more explicitly.

Let below \( M' \) denote the transpose of \( M \), and \( f_T \) stand also for the symmetric matrix of this quadratic form, i.e

\[
 f_T = \begin{pmatrix}
  b & \frac{1}{2}(d - a) \\
  \frac{1}{2}(d - a) & -c
 \end{pmatrix}.
\]

**Lemma 3.3.** We have

\[ M_T f_T M'_T = \det M_T \cdot f_T, \]

i.e. the change of variables given by the matrix \( M'_T \) turns the form \( f_T \) into itself if \( \det M_T = +1 \) and into \( -f_T \) otherwise.

**Proof.** Let, as above, \( \sigma = \det M_T \). Then

\[
 M_T f_T M'_T = \begin{pmatrix}
  a & b \\
  c & d
 \end{pmatrix} \begin{pmatrix}
  b & \frac{1}{2}(d - a) \\
  \frac{1}{2}(d - a) & -c
 \end{pmatrix} \begin{pmatrix}
  a & c \\
  b & d
 \end{pmatrix}
 = \begin{pmatrix}
  \sigma b & \frac{1}{2}\sigma(d - a) \\
  \frac{1}{2}\sigma(d - a) & -\sigma c
 \end{pmatrix}.
\]

**Definition.** An integral change of variables which leaves a binary integral quadratic form unchanged is called its automorph.

Thus, if \( \det M_T = +1 \), then the transformation \( M'_T \) is an automorph of the form \( f_T \).

Suppose from here on \( M_T \) to be primitive. The following proposition answers the question about the structure of the set of solutions of equation (3.3).
Proposition 3.4. Let \( m \) denote the integral minimum of the form \( f_T \). The solutions of the equation (3.3) are described as follows. The congruence
\[
n^2 \equiv D \pmod{4m},
\]
is always solvable, and let \( n \) be its minimum root, i.e. \( 0 \leq n < 2m \), and \( l := \frac{n^2 - D}{4m} \).

Let \( s \) stand for the number of distinct forms \([m,n,l]\) equivalent to \( f_T \). Then there exists a finite collection of solutions of equation (3.3) \((x^{(j)}, y^{(j)})\), \(1 \leq j \leq s\) such that any solution \((x,y)\) of (3.3) is of the form \((x,y) = \pm(x^{(j)}, y^{(j)}) \cdot M_T^n\) for some \( n \in \mathbb{Z} \) and \( 1 \leq j \leq s \). Furthermore, \((x^{(j)}, y^{(j)}) \neq \pm(x^{(i)}, y^{(i)}) \cdot M_T^n\) for \( i \neq j \) and any integer \( n \).

Proof. We use the classical result on the structure of solutions of a quadratic Diophantine equation (see [Lev, vol. II, Theorem 1-12]), by which if \((x,y)\) is a solution of the equation (3.3), say, with \(+m\), then it leads to the series of solutions \( \{V(x,y)\} \), where \( V \) is an automorph of \( f_T \). Besides, any solution of (3.3) is given by such a series with a finite number of basis solutions. This number is given exactly as in the claim. Furthermore, dealing with \( \pm m \), we see that for our purposes we need to consider also the \textit{anti-automorphs}, i.e. the transformations turning \( f_T \) into \(-f_T\). Now it suffices to apply Theorem 1-8 from the same volume and Lemma 3.3 and to recall that \( M_T \) is primitive. Then any automorph or anti-automorph of \( f_T \) is of the form \( V = \pm(M_T^f)^n, n \in \mathbb{Z} \), which completes the proof. \( \square \)

Remark. On the other hand, to prove Proposition 3.4, we may use Proposition A.4 (see Appendix).

We are going to prove an analog of Theorem 2.6.

Theorem 3.5. Each minimal arithmetic coding of the automorphism \( T \) with a primitive matrix \( M_T \) is of the form
\[
\varphi_{k,j}^\pm(\varepsilon) = \left( \sum_{n=-\infty}^{\infty} \varepsilon_n \lambda^{-n} \right) \begin{pmatrix} \pm \xi_j \lambda^k \\ \pm \eta_j \lambda^k \end{pmatrix} \pmod{\mathbb{Z}^2} \tag{3.4}
\]
for some \( n \in \mathbb{Z} \), \( j \in \{0, 1, \ldots, s\} \). Here \((\xi_j, \eta_j)\) is the homoclinic point given by the solution of equation (3.3) \((x^{(j)}, y^{(j)})\) as follows:
\[
\xi_j = \frac{y^{(j)} + n^{(j)} \lambda}{\sqrt{D}}, \quad \eta_j = \frac{x^{(j)} + k^{(j)} \lambda}{\sqrt{D}},
\]
and
\[
\begin{pmatrix} n^{(j)} \\ k^{(j)} \end{pmatrix} = -\det M_T \cdot M_T \begin{pmatrix} y^{(j)} \\ x^{(j)} \end{pmatrix}.
\]

Proof. We use practically the same argument as in the proof of the second part of Theorem 2.6. Let \( \varphi_0 \) and \( \varphi \) be two minimal arithmetic codings for \( T \). Recall that the corresponding factor maps \( \varphi_0' \) and \( \varphi' \) are group homomorphisms of the groups \( X_r \) (or \( Y_r \)) and \( T^2 \). Suppose first \( \text{Ker} \varphi' = \text{Ker} \varphi_0' \). Then \( A := \varphi_0^{-1} : T^2 \to T^2 \) is well defined, and by definition, \( A \) is an automorphism of \( T^2 \) commuting with \( T \). Again, by the Dirichlet theorem and the primitivity of \( T \), we have \( A = \pm T^k \), whence \( \varepsilon = \pm T^k \cdot \varepsilon, k \in \mathbb{Z} \). Thus, for two MAC’s with one and the same kernel, the corresponding factor maps are equal.
claim is proved. Since any minimal arithmetic coding is naturally parametrized by a solution of equation (3.3), it suffices to apply Lemma 1.1 and Proposition 3.4. □

If \( M_T \) is not primitive, this case can be processed in the same spirit; the corresponding formula for \( \varphi_{k,j} \) is similar both to formulas (2.7) and (3.4).

Following the framework of the previous section (cf. the second part of Theorem 2.5), we are going to relate minimal arithmetic codings to the problem of the semiconjuncy of matrices.

**Proposition 3.6.** Any matrix \( B \in GL(2, \mathbb{Q}) \cap M_2\mathbb{Z} \) such that \( BM_T = C_{r,\sigma}B, \det B = \pm m(T) \)

has the form

\[
B = \pm \left( \begin{array}{cc}
x^{(j)} & y^{(j)} \\
-\det M_T \cdot (x^{(j)}, y^{(j)})M_T^{-1} & \\
\end{array} \right) \cdot M_T^n, \quad n \in \mathbb{Z}.
\]

Besides, \( \text{Ker} \ B = \text{Ker} \left( \begin{array}{cc}
x^{(j)} & y^{(j)} \\
(x^{(j)}, y^{(j)})M_T^{-1} & \\
\end{array} \right) \), i.e. there is a finite number of possible kernels for \( B \).

**Proof.** A solution \( B \) of the matrix equation \( BM_T = C_{r,\sigma}B \) together with the condition \( \det B = \pm m(T) \) is in fact a matrix \( B = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \), where \( (x, y) \) is a solution of the equation (3.3), and

\[
(z, t) = -\det M_T \cdot (x, y)M_T^{-1}
\]

(see Theorem 2.5). Now the claim is a direct consequence of Proposition 3.4. □

Thus, we related the problem of description of the kernels of MAC’s for \( T \) to the purely algebraic problem of describing the kernels of the endomorphisms of \( \mathbb{T}^2 \) given by the matrices semiconjugating \( M \) and \( C_{r,\sigma} \). Furthermore, both problems are reduced to finding the basis solutions of the equation (3.3). The following example shows that the situation with distinct series of solutions can take place, which leads to different series of kernels.

**Example.** Let \( M = \begin{pmatrix} 80 & 9 \\ 9 & 1 \end{pmatrix} \). Then \( f_T(x, y) = 9x^2 - 79xy - 9y^2 \), and it is a direct inspection that the equations \( f_T(x, y) = \pm k \) have no solutions for \( 1 \leq k \leq 8 \). Thus, the integral minimum of \(|f_T|\) equals 9. We consider the equation \( f_T(x, y) = \pm 9 \) and choose the pairs of solutions: \( (x_1 = 1, y_1 = -9) \) and \( (x_2 = 9, y_2 = 1) \). Constructing now the matrices \( B_1 = \begin{pmatrix} x_1 & y_1 \\ z_1 & t_1 \end{pmatrix} \) and \( B_2 = \begin{pmatrix} x_2 & y_2 \\ z_2 & t_2 \end{pmatrix} \) by formula (2.5), we obtain thus two matrices from \( GL(2, \mathbb{Q}) \cap M_2\mathbb{Z} \) semiconjugating \( M \) and the companion matrix \( \begin{pmatrix} 81 & 1 \\ 1 & 0 \end{pmatrix} \). However, the matrix \( B_1B_2^{-1} \) is not integral, whence the endomorphisms given by the matrices \( B_1 \) and \( B_2 \) have distinct kernels, so do the corresponding mappings \( \varphi_1 \) and \( \varphi_2 \). Note also that the groups \( \text{Ker} \ B_1 \) and \( \text{Ker} \ B_2 \) being isomorphic as abstract groups, are not isomorphic with respect to \( T \) in the sense that there is no automorphism commuting with \( T \) and turning \( \text{Ker} \ B_1 \) into \( \text{Ker} \ B_2 \).

Thus, the kernel of the minimal arithmetic coding is not an invariant for the integral conjugacy in \( GL(2, \mathbb{Z}) \), as it does not apply even for a single matrix.

The idea of this example is based on the fact that \( m(T) \) is not a prime. It can be shown that for \( m(T) \) prime such a situation cannot take place.
Remark on the case $r < 0$. Finally, we keep our promise and show how to reduce the case $r < 0$ to $r > 0$. Briefly, given an automorphism $T$ whose matrix $M_T$ has the negative trace, we consider the automorphism with the matrix $-M_T$, and make sure that it has the same collection of homoclinic points and the same series with the terms $\varepsilon_n\lambda^{-n}$ but with $0 < \lambda < 1$ and inverted (in the Markov case) restrictions on the digits.

More precisely, let $r < 0$ and let $X_r^-$ be the stationary Markov compactum \( \{ \{\varepsilon_n\}^\infty_{-\infty} : 0 \leq \varepsilon_n \leq |r|, \varepsilon_n = |r| \Rightarrow \varepsilon_{n-1} = 0, n \in \mathbb{Z}\} \). Then any arithmetic coding of $T$ is given by the mapping

\[
\psi_t(\varepsilon) = \left( \sum_{n=-\infty}^{\infty} \varepsilon_n\lambda^{-n} \right) \left( \xi \atop \eta \right) \mod \mathbb{Z}^2,
\]

which formally coincides with the mapping $\varphi_t$ given by formula (1.3), but acting from $X_r^-$ if $\sigma = -1$ and $\mathcal{Y}_r$ otherwise with $\lambda = \frac{r+\sqrt{r^2+4}}{2} \in (0, 1)$ and $\left( \xi, \eta \right)$ being a homoclinic point for $-T$. By formulas (1.1) and (1.2), the set of homoclinic points for $T$ and $-T$ is one and the same. Thus, all claims of the paper for the case $r < 0$ remain valid for $r > 0$.

Note also that the composition mapping $S : X_r \to X_r^-$ (resp. $\mathcal{Y}_r \to \mathcal{Y}_r$) specified by the formula $S = \psi_t^{-1}\varphi_t$ is well defined, does not depend on $t$, and $S(\{\varepsilon_n\}) = \{\varepsilon_{-n}\}$.

**Appendix. Related algebraic questions**

In this appendix we collect all algebraic and number-theoretic claims which are closely related to the main theorems of the paper, but at the same time being practically separate. The authors consider them as known to the specialists or following from certain known facts. However, some of them prove to be important, namely, Theorem A.2 which relates the algebraic conjugacy of the matrices to the equivalence of the binary quadratic forms, Theorem A.7 which answers the question about the number of orbits of a matrix covering $\mathbb{Z}^2$, and finally, Proposition A.9 describing the Pisot group for a given quadratic PV unit; we could not find these claims in the classical sources.

**A.1. Unimodular matrices and quadratic forms.** We are going to prove an assertion which relates our theory to the theory of binary integral quadratic forms. Recall that two binary integral quadratic forms $f$ and $f'$ are called equivalent if $f'(x, y) = f(\alpha x + \beta y, \gamma x + \delta y)$ with $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL(2, \mathbb{Z})$. If $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{Z})$, then they are called properly equivalent. For indefinite quadratic forms the problem of equivalence is rather difficult (see, e.g., [Ven]); note only that for discriminants appearing in our kind of problems the number of equivalence classes is large.

Within the appendix we will denote the quadratic form associated with a matrix $M \in GL(2, \mathbb{Z})$, by $f_M$ instead of $f_T$, which looks more natural here. Our goal is to prove a claim that relates the problem of the conjugacy of matrices $M_1$ and $M_2$ in $GL(2, \mathbb{Z})$ to the equivalence of the forms associated with them. Let $\mathcal{F}$ denote the set of binary integral quadratic forms with discriminant $r^2 \pm 4 > 0$ for some $r \in \mathbb{N}$. We begin with a lemma which studies the mapping $\theta : GL(2, \mathbb{Z}) \to \mathcal{F}$ such that $M \mapsto f_M$. 
Lemma A.1. For a binary quadratic form $f \in \mathfrak{F}$, the preimage $\theta^{-1}(f)$ consists exactly of two matrices. If we denote one of these matrices by $M$, another is $-\det M \cdot M^{-1}$.

Proof. Let $f(x, y) = \alpha x^2 + \beta xy + \gamma y^2$, and the matrix sought is $\begin{pmatrix} a & b \\ c & d \end{pmatrix} =: M$. Then one needs to solve the equations $\alpha = b, \beta = d - a, \gamma = -c$ for the variables $a, b, c, d$ together with the condition $ad - bc = \pm 1$. Solving them, we see that if $M_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a solution, then another solution is $M_2 = \begin{pmatrix} -d & b \\ c & -a \end{pmatrix}$. □

Thus, the mapping $\theta$ is two-to-one, and it is easy to distinguish the two preimages of a given form, as they have different traces, though one and the same determinant.

Theorem A.2.

1. Let matrices $M_1$ and $M_2$ belonging to $GL(2, \mathbb{Z})$ be algebraically conjugate in $GL(2, \mathbb{Z})$, i.e. $BM_1B^{-1} = M_2$ for some $B \in GL(2, \mathbb{Z})$. Then if $\det B = +1$, then the associated forms are properly equivalent. More precisely, we have $Bf_{M_1}B' = f_{M_2}$. If $\det B = -1$, then $Bf_{M_1}B' = -f_{M_2}$.

2. If two binary integral quadratic forms $f_1 \in \mathfrak{F}$ and $f_2 \in \mathfrak{F}$ with one and the same discriminant are properly equivalent, then matrices $M_1 \in \theta^{-1}(f_1)$ and $M_2 \in \theta^{-1}(f_2)$ with equal traces are algebraically conjugate. If the equivalence of forms is not proper, then $M_2 \sim \det M_1 \cdot M_1^{-1}$.

Proof. (1) Let $M_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $M_2 = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ with $a + d = a' + d', \ ad - bc = a'd' - b'c'$. Then it is a direct inspection that the relation

$$B \begin{pmatrix} a & b \\ c & d \end{pmatrix} B^{-1} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

with $B \in SL(2, \mathbb{Z})$ is equivalent to the relation

$$B \begin{pmatrix} b & \frac{1}{2}(d - a) \\ \frac{1}{2}(d - a) & -c \end{pmatrix} B' = \begin{pmatrix} b' & \frac{1}{2}(d' - a') \\ \frac{1}{2}(d' - a') & -c' \end{pmatrix}.$$ 

On the contrary, if $B = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in GL(2, \mathbb{Z}) \setminus SL(2, \mathbb{Z})$, then the relations $BM_1B^{-1} = M_2$ and $Bf_{M_1}B' = -f_{M_2}$ in fact lead to the following one and the same collection of relations:

$$f_M(x, y) = -b',$$

$$f_M(z, t) = c',$$

$$bxz + (d - a)yz - c yt = a - d'.$$

(2) If $B \in SL(2, \mathbb{Z})$, then the claim is already proved in the previous item. If $B \in GL(2, \mathbb{Z}) \setminus SL(2, \mathbb{Z})$, then, similarly to the first item, we make sure that the relations $Bf_{M_1}B' = f_{M_2}$ and $BM_1B^{-1} = \det M_2 \cdot M_2^{-1}$ also yield one and the same collection of relations. □

The rest of item A.1 is devoted to diverse applications of this theorem. The following corollary is straightforward.
Corollary A.3. (1) Let two matrices $M_1$ and $M_2$ from $GL(2, \mathbb{Z})$ have one and the same trace and discriminant. Then they are algebraically conjugate if and only if
- either the forms $f_{M_1}$ and $f_{M_2}$ are properly equivalent
- or $f_{M_1}$ is equivalent to $-f_{M_2}$, and the corresponding change of variables has determinant $-1$.
(2) If $\det M = -1$, then $f_M$ is equivalent to $-f_M$. Hence in this case $M_1 \sim M_2$ if and only if $f_{M_1}$ is equivalent to $f_{M_2}$.

Proof. The first item follows from Theorem A.2. The second one is a consequence of Lemma 3.3 (recall that $M f_M M' = -f_M$, if $\det M = -1$). □

The following example shows that sometimes $f_M$ is not equivalent to $-f_M$ if $\det M = +1$.

Example. Let $M_1 = M = \left( \begin{array}{cc} 3 & 5 \\ 1 & 2 \end{array} \right)$, and $M_2 = M^{-1} = \left( \begin{array}{cc} 2 & -5 \\ -1 & 3 \end{array} \right)$. Then $M_1 \not\sim M_2$, because the form $f_M(x, y) = 5x^2 - xy - y^2$ assumes the value 1, but does not assume the value $-1$. Indeed, the equation $5x^2 - xy - y^2 = -1$ is solvable $(x = 0, y = 1)$, while the equation $5x^2 - xy - y^2 = 1$ has no solutions, as this equation can be rewritten as $(10x - y)^2 = 2y^2 + 20$, whence $\pm 2$ should be a quadratic residue modulo 10, what is wrong. Therefore, $f_M$ is not equivalent to $-f_M$.

The first application of Theorem A.2 is the link between the Dirichlet theorem for $GL(2, \mathbb{Z})$ and the classical theorem on the general form of a proper automorph of an indefinite binary quadratic form. We recall that a change of variables with a primitive, except the exclusive case $\sigma = 1$, $\tau = -1$ (this link between the terms “primitive matrix” and “primitive form” for completely different notions partially explains our choice of terminology for the matrices; see also Remark 3 below).

Then by Theorem A.2, the relation $B f_M B' = f_M$ implies $B M B^{-1} = M$, whence by the Dirichlet theorem, $B = \pm M^n$, and it suffices to use the fact that $\det B = +1$. The exclusive case is studied in the same way. □

Remark 1. This claim can be obtained by using standard number-theoretic arguments as a consequence of the general theorem on the proper automorphs of an indefinite binary quadratic form, see [Lev, vol. II, Th. 1-8]. To this end, one needs to find the minimal positive solution of the Pell equation (0.1) with $+4$. This way is more computational, while the goal of our proof was to establish a link with the classical Dirichlet theorem which is applicable to a priori completely different class of objects.
Remark 2. Note that in the nonexclusive case any transformation of coordinates being either automorph or anti-automorph, is of the form \(\pm(M')^n\), \(n \in \mathbb{Z}\). This is a consequence of the Dirichlet theorem in its complete form.

Remark 3. Generally speaking, it is wrong that the primitivity of a matrix \(M\) implies the primitivity of the associated form \(f_M\). Here is the counterexample: \(M = \begin{pmatrix} 7 & 6 \\ 6 & 5 \end{pmatrix}\).

Now we return to equations (3.3) (or (2.4)) in order to find out if the solvability of one of the equations (3.3) implies the solvability of another.

Lemma A.5. If \(\det M = -1\), then the solvability of one of the equations (3.3) (say, with \(+m\)) implies the solvability of another. On the contrary, for \(\det M = +1\) it is, generally speaking, wrong.

Proof. If \(\det M = -1\), then the claim follows from the equivalence of the forms \(f_M\) and \(-f_M\) (see Lemma 3.3) and the fact that equivalent forms assume one and the same collection of values. As a counterexample for \(\det M = +1\) we can again consider the matrix \(M = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}\) (see Example above). □

Returning now to the problems of Section 2, we will show that for “small” discriminants a primitive matrix is always conjugate to the corresponding companion matrix.

Proposition A.6. (I) Any matrix \(M\) with \(D = r^2 - 4\sigma < 20\) is algebraically conjugate in \(\text{GL}(2, \mathbb{Z})\) to the companion matrix \(C_{r,\sigma}\).

(II) Let \(20 \leq D < 40\) for the matrix \(M\) of an automorphism \(T\). Then either \(M\) is algebraically conjugate to the corresponding companion matrix \(C_{r,\sigma}\) or \(M\) is not primitive. More precisely, there is the following alternative.

1. If \(D = 21\) or \(D = 29\), then \(M \sim C_{r,\sigma}\).

2. If \(D = 20\), then either \(M\) is primitive and \(M \sim \begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix}\) or \(M \sim \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} = \left( \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right)^3\).

3. If \(D = 32\), then either \(M \sim C_{6,1}\) or \(M \sim \left( \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \right)^2\).

Proof. (I) By [Cas1, Ch. II, §4, Theorem VI], if positive integers \(a, b\) are such that for an indefinite binary quadratic form \(f(x, y)\) with the discriminant \(D\) there are no integral \(x, y\) such that \(-a < f(x, y) < b\), then \(D \geq 4ab + \max(a^2, b^2)\). Inverting this assertion and taking \(a = b = 2\), we come to the claim of the proposition, because the integral minimum of the form \(f_M\) in this case equals 1, which is equivalent to the desired claim.

(II) The central point here is the following sharp estimate for the integral minimum of an indefinite binary quadratic form: for a form \(f(x, y) = ax^2 + \beta xy + \gamma y^2\) with discriminant \(D > 0\),

\[
\min \{|f(x, y)| : (x, y) \in \mathbb{Z}^2 \setminus \{(0, 0)\}\} \leq \sqrt{\frac{25D}{221}}
\]

unless \(f\) is equivalent to one of the forms \(l(x^2 - xy - y^2)\) or \((x^2 - 2y^2)\) with \(l \in \mathbb{Z}\) (see [Cas1, Chap. II, §4, Th. 6]). This estimate applied to \(f_M\) for \(D \leq 35\) (which actually means that \(D \leq 40\)) yields the minimum = 1, which is equivalent to the desired claim.
solvability of equation (2.4). Considering the possibilities for the exclusions in the cited claim, we make sure that they could appear only for \( D = 20 \) or \( D = 32 \). In the first case this leads to the equivalence of the forms \( f_M(x, y) \) and \( 2(x^2 - xy - y^2) \), whence by Theorem A.2, \( M \sim \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \), because \( \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix} \). The case \( D = 32 \) is studied in the same way. \( \square \)

Remark. Practically the first claim of Proposition A.6 means that if \( r = 1, 2, 3 \) for \( \sigma = -1 \) and \( r = 3, 4 \) for \( \sigma = +1 \), then \( M \) is algebraically conjugate to the corresponding companion matrix.

We are going to give some “counterexamples” showing that the constants in Proposition A.6 are precise.

“Counterexamples”. 1. The condition \( D < 20 \) cannot be improved. Indeed, it suffices to consider the matrix \( \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \) which is obviously not conjugate to the companion matrix. However, this matrix is not primitive, namely, the cube of \( \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \).

2. For \( D = 40 \) there exists a primitive matrix with this discriminant not algebraically conjugate to the companion matrix, namely, \( M = \begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix} \). Here equation (2.4) is \( 3x^2 - 4xy - 2y^2 = \pm 1 \) and has no integer solutions, as it can be rewritten as \( (3x - 2y)^2 - 10y^2 = \pm 3 \), whence it would follow that \( \pm 3 \) is a quadratic residue modulo 10.

3. Although each matrix with \( D = 5 \) is algebraically conjugate by the above to the companion matrix, this is, generally speaking, wrong for an arbitrary matrix whose spectrum is in the ring \( \mathbb{Z}[\lambda] \) with \( \lambda = \frac{1 + \sqrt{5}}{2} \). Here is an example. Consider \( M = \begin{pmatrix} 2 & 7 \\ 5 & 2 \end{pmatrix} \) whose spectrum is \( \{\lambda^7, -\lambda^{-7}\} \). A detailed analysis shows that the integral minimum of the absolute value of the associated form \( f_M(x, y) = 11x^2 - 25xy - 5y^2 \) equals 5, whence, equation (2.4) for this case has no integral solutions, though \( M \) is primitive. Thus, it is impossible to reformulate Proposition A.6 in purely “ring” terms.

Remark. Another approach to the problem of conjugacy of two matrices in \( GL(2, \mathbb{Z}) \) was proposed in [CamTr]. It is based on the presentation of the group \( PSL(2, \mathbb{Z}) \) as a free product of cyclic groups. The authors express their gratitude to B. Weiss for indicating this reference.

A.2. The number of orbits of a unimodular matrix.

**Theorem A.7.** Let \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z}) \). Let \( Orb_M(x, y) := \{ M^n \begin{pmatrix} x \\ y \end{pmatrix}, n \in \mathbb{Z} \} \) denote the orbit of \( (x, y) \in \mathbb{Z}^2 \). Then the linear span of this orbit \( \langle Orb_M(x, y) \rangle \) is equal to \( \mathbb{Z}^2 \) if and only if \( f_M(y, -x) = \pm 1 \).

More generally, for a given \( M \),

\[
\min \left\{ k : \exists \{(x_j, y_j)\}_{j=1}^k \left\{ \bigcup_{j=1}^k \langle Orb_M(x_j, y_j) \rangle = \mathbb{Z}^2 \right\} \right\} \geq \min_{(x, y) \neq (0, 0)} |f_M(x, y)|. \tag{A.1}
\]

**Proof.** Similarly to Proposition 3.6, for any pair \( (x, y) \in \mathbb{Z}^2 \) there exists a matrix \( B = B(x, y) = \begin{pmatrix} x & y \\ x & y \end{pmatrix} \in GL(2, \mathbb{Z}) \cap M_0\mathbb{Z} \) such that \( BC_{x,y} = MB \), where, as before, \( C_{x,y} := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \) and \( M_0 \) is the subset of matrices whose entries are divisible by 2.
above, \( C_{r, \sigma} = \begin{pmatrix} r & 1 \\ \sigma & 0 \end{pmatrix} \) is the companion matrix. Namely, \( \binom{x}{y} = M^{-1} \binom{x}{y} \). Hence \( BC_{r, \sigma}^n = M^n B \), and
\[
BC_{r, \sigma}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = M^n \begin{pmatrix} x \\ y \end{pmatrix}.
\] (A.2)

Note that by trivial reasons, \( \langle C_{r, \sigma}^n \binom{1}{0}, n \in \mathbb{Z} \rangle = \mathbb{Z}^2 \), as this is equivalent to the fact that the powers of \( \lambda \) form a basis of the module \( \mathbb{Z}[\lambda] \).

Thus, by relation (A.2), we have
\[
\langle \text{Orb}_M(x, y) \rangle = B\mathbb{Z}^2,
\]
whence \( \langle \text{Orb}_M(x, y) \rangle \) coincides with \( \mathbb{Z}^2 \) if and only if \( \det B = \pm 1 \), which is equivalent to \( \pm 1 = xy - yz = \pm (x(-cx + ay) - y(dx - by)) = \pm f_M(y, -x) \).

To prove the second claim of the theorem, we observe that from formula (A.2) follows the fact that \( \min \{ |\det B(x, y)| \mid (x, y) \in \mathbb{Z}^2 \setminus \{(0,0)\} \} = \min \{ |f_M(x, y)| \mid (x, y) \in \mathbb{Z}^2 \setminus \{(0,0)\} \} =: m \), whence one needs at least \( m \) orbits to cover \( \mathbb{Z}^2 \). \( \square \)

**Remark.** Thus, as before, to enumerate all matrices \( B = B(x, y) \) with the minimal possible determinant in modulus, we need to find all solutions of the Diophantine equation
\[
f_M(y, -x) = \pm m.
\] (A.3)

Within one and the same “series of solutions” of equation (A.3) (see Proposition 3.4 for the definitions) \( \langle \text{Orb}_M(x, y) \rangle \equiv \text{const} \), whence it is easy to construct an example with the rigid inequality in formula (A.1). It suffices to consider any matrix with \( m = 2 \), for instance, our “universal” counterexample \( M = \begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix} \). Here all solutions of equation (A.3) form a single orbit, hence the left minimum in the inequality (A.1) is greater than or equal to 3.

**Corollary A.8.** If the linear span for the powers of \( M \) of some vector equals \( \mathbb{Z}^2 \), then \( M \) is algebraically conjugate to the companion matrix. Conversely, if \( M \sim C_{r, \sigma} \), then there exists a vector \( (x, y) \in \mathbb{Z}^2 \) such that \( \langle M^n \binom{x}{y}, n \in \mathbb{Z} \rangle = \mathbb{Z}^2 \).

**A.3. Application to PV numbers.** At the end of the appendix we will relate our results to the classical algebraic theory of Pisot-Vijayaraghavan (PV) numbers.

**Definition.** Let \( \theta \) be an algebraic integer \( \geq 1 \) such that all its Galois conjugates lie inside the unit disc on the complex plane. Then \( \theta \) is called a Pisot-Vijayaraghavan (PV) number.

Thus, in our case \( \lambda \) is a quadratic PV unit. We define
\[
P_\lambda := \{ \xi \in \mathbb{R} : \|\xi \lambda^n\| \to 0, \quad n \to \infty \}.
\]

Obviously, \( P_\lambda \) is a group in addition, and from the definition of a PV number it is clear that if \( \lambda \) is unitary, then \( \mathbb{Z}[\lambda] \subset P_\lambda \). We will call \( P_\lambda \) the Pisot group with a parameter \( \lambda \). The implicit description of the Pisot group is yielded by the classical Pisot-Vijayaraghavan theorem claiming that \( \xi \) belongs to \( P_\lambda \) if and only if \( \xi \in \mathbb{Q}(\lambda) \), and \( \text{Tr}(\xi) \in \mathbb{Z} \), \( \text{Tr}(\lambda \xi) \in \mathbb{Z} \), where \( \text{Tr}(\xi) = \xi + \overline{\xi} \) is the trace of a quadratic irrational (see, e.g., [Cas2, Chap. VIII]). It is not hard to obtain the precise description of the Pisot group from these conditions directly, however, our methods yield its structure almost immediately and relate this theory to the theory of hyperbolic systems.
Proposition A.9. Let $\lambda > 1$ be the PV number which satisfies the equation $\lambda^2 = r\lambda - \sigma$ with $r \geq 1$ for $\sigma = -1$ and $r \geq 3$ for $\sigma = +1$, and let $D = r^2 - 4\sigma$. Then

$$P_\lambda = \frac{Z + \lambda Z}{\sqrt{D}}.$$ 

Proof. Let for simplicity of notation, $\sigma = -1$, and $\xi \in P_\lambda$. Consider the point $x = (\{\xi\}, \{\lambda^{-1}\xi\})$. Obviously, $T_{r,\sigma}^n(x) = (\{\lambda^n\xi\}, \{\lambda^{n-1}\xi\})$, whence by definition of the Pisot group, $T_{r,\sigma}^n(x) \to (0,0)$, $n \to \pm\infty$, and thus, $x$ is a homoclinic point for the automorphism $T_{r,\sigma}$, and by Lemma 2.1, its first coordinate has the form $\frac{m+n\lambda}{\sqrt{D}}$ mod 1 for some $m, n$ integers.

Conversely, let $\xi = \frac{m+n\lambda}{\sqrt{D}}$. Then by Lemma 2.1, the point $x$ is homoclinic for $T_{r,\sigma}$, whence $\lambda^n\xi \to 0$ mod 1, $n \to \infty$. □

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