Target Mass Corrections to
QCD Bjorken Sum Rule for
Nucleon Spin Structure Functions

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Abstract

We discuss the possible target mass corrections in the QCD analysis
of nucleon’s spin-dependent structure functions measured in the polarized
deep-inelastic leptoproduction. The target mass correction for the QCD
Bjorken sum rule is obtained from the Nachtmann moment and its magni-
tude is estimated employing positivity bound as well as the experimental
data for the asymmetry parameters. We also study the uncertainty due
to target mass effects in determining the QCD effective coupling constant
$\alpha_s(Q^2)$ from the Bjorken sum rule. The target mass effect for the Ellis-Jaffe
sum rule is also briefly discussed.

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1. Introduction

I would like to talk about possible target mass corrections in the QCD analysis of spin-dependent structure functions which can be measured by the deep inelastic scattering of polarized leptons on polarized nucleon targets [1, 2]. We especially investigate those for the QCD Bjorken sum rule.

As discussed in the previous talks by Dr. Kodaira and Dr. Sloan, the Bjorken sum rule with QCD corrections is given by

$$\int_0^1 dx [g_1^p(x, Q^2) - g_1^n(x, Q^2)] = \frac{1}{6} \frac{G_A}{G_V} [1 - \frac{\alpha_s(Q^2)}{\pi} + O(\alpha_s^2)],$$

where $g_1^p(x, Q^2)$ and $g_1^n(x, Q^2)$ are the spin structure function $g_1$ of proton and neutron, respectively, with $x$ and $Q^2$ being the Bjorken variable and the virtual photon mass squared. On the right-hand side, $G_A/G_V \equiv g_A$ is the ratio of the axial-vector to vector coupling constants. The QCD correction of the order of $\alpha_s(Q^2)$ was first obtained some years ago based on operator product expansion (OPE) and renormalization group (RG) method in ref. [3, 4]. The higher order corrections were calculated in refs. [5, 6, 7, 8].

Now recent experiments on the spin structure function $g_1(x, Q^2)$ for the deuteron, $^3$He and the proton target at CERN and SLAC [9, 10, 11, 12] together with EMC data have provided us with the data for testing the Bjorken sum rule [13] as well as Ellis-Jaffe sum rule [14]. In order to confront the QCD prediction with the experimental data at low $Q^2$ where the QCD corrections are significant, we have to take into account the corrections due to the mass of the target nucleon, which we denote by $M$. In this $Q^2$ region, we cannot neglect the order $M^2/Q^2$ term, which consists of higher-twist effects as well as target mass effects. Here we shall not discuss the higher-twist effect which was mentioned by Dr. Kodaira and will be discussed by Dr. Mueller this afternoon. Here we confine ourselves to the target mass effects. Here we observe that 1) The target mass effects are calculable without any ambiguity; 2) The infinite power series in $M^2/Q^2$ can actually be summed up into a closed analytic form.

In the framework of OPE, the target mass effects of structure functions can be evaluated by taking account of trace terms of composite operators to have a definite spin projection. This amounts to replace the ordinary moments of the structure functions by the Nachtmann moments [15]. The Nachtmann moments for spin structure functions
were obtained in refs. [16, 17]. In ref. [16] the Nachtmann moments were given as an infinite power series, while in ref. [17] they were obtained as a closed analytic form.

2. OPE and Target Mass Effects

The anti-symmetric part of the virtual Compton amplitude can be written in the OPE [18]:

\[
T_{\mu\nu}(p, q, s)^{[A]} \simeq -i\varepsilon_{\mu\nu\lambda\sigma}q^\lambda \sum_{n=1,3,\ldots} \left( \frac{2}{Q^2} \right)^n q_{\mu_1} \cdots q_{\mu_{n-1}} E_1^n(Q^2, g) \langle p, s | R_1^{\sigma_{\mu_1} \cdots \mu_{n-1}} | p, s \rangle \\
- i(\varepsilon_{\mu\nu\lambda\sigma}q^\rho - \varepsilon_{\nu\mu\lambda\sigma}q_\mu q^\rho - q^2\varepsilon_{\mu\nu\lambda\sigma}) \sum_{n=3,5,\ldots} \frac{n-1}{n} \left( \frac{2}{Q^2} \right)^n q_{\mu_1} \cdots q_{\mu_{n-2}} E_2^n(Q^2, g) \times \langle p, s | R_2^{\lambda_{\sigma\mu_1} \cdots \mu_{n-2}} | p, s \rangle,
\]

where the nucleon matrix element of the twist-2 operators \( R_1^n \) is given by

\[
\langle p, s | R_1^{\sigma_{\mu_1} \cdots \mu_{n-1}} | p, s \rangle = -a_n \left\{ s^\sigma p^{\mu_1} \cdots p^{\mu_{n-1}} \right\} - \text{trace terms}
\]

\( \equiv -a_n \left\{ s^\sigma p^{\mu_1} \cdots p^{\mu_{n-1}} \right\}_n \),

(3)

which is totally symmetric in Lorentz indices \( \sigma, \mu_1, \cdots, \mu_{n-1} \) and traceless. (\{ \} denotes symmetrization.) While for the twist-3 operator \( R_2^n \), the matrix element is given by

\[
\langle p, s | R_2^{\lambda_{\sigma\mu_1} \cdots \mu_{n-2}} | p, s \rangle = -d_n \left[ \frac{1}{2} (s^\lambda p^\sigma - s^\sigma p^\lambda) p^{\mu_1} \cdots p^{\mu_{n-2}} - \text{trace terms} \right]
\]

\( \equiv -d_n \left\{ s^\lambda p^{\mu_1} \cdots p^{\mu_{n-2}} \right\} M_n, \)

(4)

which is symmetric in \( \sigma, \mu_1 \cdots \mu_{n-2} \) and anti-symmetric in \( \lambda \sigma \) and also traceless in its Lorentz indices.

Taking account of trace terms we can project out the contribution from a definite spin as follows. By taking a contraction of the tensor appearing in eq.(3) with \( q_{\mu_1} \cdots q_{\mu_{n-1}} \) we get

\[
q_{\mu_1} \cdots q_{\mu_{n-1}} \left\{ s^\sigma p^{\mu_1} \cdots p^{\mu_{n-1}} \right\} - \text{trace terms}
\]

\[= \frac{1}{n^2} \left[ s^\sigma a^{n-1} C_n^{(2)}(\eta) + q^\sigma q \cdot s a^{n-1} \times 4C_n^{(3)}(\eta) + p^\sigma q \cdot s a^{n-2} \times 2C_n^{(3)}(\eta) \right],
\]

where \( C_n^{(m)}(\eta) \) is a Gegenbauer polynomial with

\[\eta = i\nu/Q, \quad \nu = p \cdot q/M, \quad a = -\frac{1}{2} iMQ.\]

(6)
Using the orthogonality property of Gegenbauer polynomials, one can project out the contribution from operators with a definite spin. The closed analytic forms for the Nachtmann moments are given as [17]:

\[
M_n^1(Q^2) = \int_0^1 \frac{dx}{x^2} \xi^{n+1} \left[ \left\{ \frac{x}{\xi} - \frac{n^2}{(n+2)^2} \frac{Mx}{Q} \frac{M\xi}{Q} \right\} g_1(x, Q^2) - \frac{4n}{n+2} \frac{M^2x^2}{Q^2} g_2(x, Q^2) \right],
\]

\[(n = 1, 3, \cdots) \quad (7)\]

\[
M_n^2(Q^2) = \int_0^1 \frac{dx}{x^2} \xi^{n+1} \left[ \frac{x}{\xi} g_1(x, Q^2) + \left\{ \frac{n}{n-1} \frac{x^2}{\xi^2} - \frac{n}{n+1} \frac{M^2x^2}{Q^2} \right\} g_2(x, Q^2) \right],
\]

\[(n = 3, 5, \cdots) \quad (8)\]

where \(\xi\) is a variable given by [19]:

\[
\xi \equiv \frac{2x}{1 + \sqrt{1 + 4M^2x^2/Q^2}}.
\]

Taking the difference between the first moment for the proton target and that for the neutron in eq.(7), we can arrive at the QCD Bjorken sum rule with target mass correction:

\[
\frac{1}{9} \int_0^1 \frac{dx}{x^2} \xi^2 \left[ 5 + 4 \sqrt{1 + \frac{4M^2x^2}{Q^2}} \right] \left[ g_1^p(x, Q^2) - g_1^n(x, Q^2) \right] - \frac{4}{3} \int_0^1 \frac{dx}{x^2} \xi^2 \frac{M^2x^2}{Q^2} \left[ g_2^p(x, Q^2) - g_2^n(x, Q^2) \right] = \frac{1}{6} \frac{G_A}{G_V} 1 - \frac{\alpha_s(Q^2)}{\pi} + O(\alpha_s^2). \quad (10)
\]

Note that in the presence of target mass correction, the other spin structure function \(g_2^{b,n}(x, Q^2)\) also comes into play in the Bjorken sum rule. Here we emphasize that the target mass correction treated through the above procedure is not mere a power correction but given as a closed analytic form. It should also be noted that target mass corrections considered as the expansion in powers of \(M^2/Q^2\) is not valid when \(M^2/Q^2\) is of order unity [20, 21].

Our result can be compared with the target mass correction as a power correction discussed in the literatures. Expanding our Nachtmann moment in powers of \(M^2/Q^2\) we get:

\[
\int_0^1 dx g_1^{p-n}(x, Q^2) = \frac{1}{6} \frac{G_A}{G_V} \left[ 1 - \frac{\alpha_s}{\pi} + \cdots \right]
\]

\[+ \frac{10}{9} \frac{M^2}{Q^2} \int_0^1 dx \frac{x^2}{Q^2} g_1^{p-n}(x, Q^2) + \frac{12}{9} \frac{M^2}{Q^2} \int_0^1 dx \frac{x^2}{Q^2} g_2^{p-n}(x, Q^2). \quad (11) \]
which coincides with the result given by Balitsky-Braun-Kolesnichenko in ref. [21] up to the contribution from the twist-4 operator to the order of $1/Q^2$.

Now, the difference between the left-hand side of (10) and that of (1) leads to the target mass correction $\Delta \Gamma$:

$$\Delta \Gamma = \int_0^1 dx \left\{ \frac{5}{9} \frac{\xi^2}{x^2} + \frac{4}{9} \frac{\xi^2}{x^2} \sqrt{1 + \frac{4M^2x^2}{Q^2} - 1} \right\} \times \left[ g_1^p(x, Q^2) - g_1^n(x, Q^2) \right]$$

$$- \frac{4}{3} \int_0^1 dx \frac{\xi^2}{x^2} \frac{M^2x^2}{Q^2} \left[ g_2^p(x, Q^2) - g_2^n(x, Q^2) \right].$$

(12)

3. Estimation of the Target Mass Effects

Let us now study the size of the target mass correction $\Delta \Gamma$ to the Bjorken sum rule. First we note that the spin structure functions $g_1$ and $g_2$ are written in terms of virtual photon asymmetry parameters $A_1$ and $A_2$, which are measured at the experiments, together with the unpolarized structure function, $F_2(x, Q^2)$, and the ratio of the longitudinal to transverse virtual photon cross sections, $R = \sigma_L/\sigma_T$ [1, 2]. We shall estimate the upper bound for the target mass correction of $\Delta \Gamma$, which we denote by $\Delta \Gamma_{u,b.}$ (i.e. $|\Delta \Gamma| \leq \Delta \Gamma_{u,b.}$) in a variety of methods.

For the first analysis (Analysis I), we apply the positivity bound for the asymmetry parameters [23]:

$$|A_1| \leq 1, \quad |A_2| \leq \sqrt{R}.$$  \hspace{1cm} (13)

We use the parametrization for $R$ taken from the global fit of the SLAC data [24] and the NMC parametrization for $F_2(x, Q^2)$ [25]. In Fig.1 we have plotted the upper bound, $\Delta \Gamma_{u,b.}$, as a function of $Q^2$ for Analysis I by a solid line. Here the error of the upper bounds of $\Delta \Gamma$ due to the parametrizations $R$ and $F_2$ is typically around 10%.

In our second analysis (Analysis II), we employ the experimental data on spin asymmetry $A_1$ and positivity bound for $A_2$ to improve the upper bound. We take the data on $A_1^p$ from SMC data [11] together with EMC data [2] and those for $A_1^n$ from SMC group [9] to extract $A_1^n$, for which we can also use the E142 data [10]. For this case, the upper bound is shown in Fig.1 by the short-dashed line, which is located slightly lower than $\Delta \Gamma_{u,b.}$ for Analysis I. When we decompose the $\Delta \Gamma_{u,b.}$ into two parts, $\Delta \Gamma_1$ and $\Delta \Gamma_2$, which are the contributions from $A_1$ and $A_2$, respectively, it turns out that $\Delta \Gamma_2$ is much larger than $\Delta \Gamma_1$. The value of $\Delta \Gamma_1$ turns out be less than 10% of $\Delta \Gamma_2$. 


The third analysis (Analysis III) uses the recently measured $A_2^p$ by the SMC group [27] in addition to the same data for $A_i^{n,n}$ together with the positivity bound for $A_2^p$ as in Analysis II. We have also plotted the upper bound for Analysis III in Fig.1 by the long-dashed line. Here we took the data on $A_2^p$ obtained by SMC group at the first measurement of transverse asymmetries [27], where the number of data points are still four and the relative error bars are not so small. The $A_2^p$ measured is much smaller than the positivity bound. If the $A_2^n$ for the neutron is also small as mentioned in ref.[10], the $\Delta \Gamma_{u.b.}$ becomes very small.

Finally we briefly comment on uncertainty due to target mass effects in determining the QCD coupling constant from Bjorken sum rule which has recently been discussed by Ellis and Karliner [28]. From the QCD corrections up to $\mathcal{O}(\alpha_s^4)$ [3, 7, 8] they obtained the value $\alpha_s(Q^2 = 2.5\text{GeV}^2) = 0.375^{+0.062}_{-0.081}$ [28], by using the known $g_A = G_A/G_V$ ratio and taking the value $\Gamma(Q^2 = 2.5\text{GeV}^2) = 0.161\pm0.007\pm0.015$, in their analysis of E142 and E143 data [28]. The $Q^2 = 2.5\text{GeV}^2$ is the averaged value of the mean $Q^2$ of the E142 data ($<Q^2> \approx 2\text{GeV}^2$) and the E143 data ($<Q^2> \approx 3\text{GeV}^2$). Here we shall not take into account the higher-twist effects which are considered to be rather small as claimed in refs. [28].

The uncertainty in $\Gamma$ due to target mass effects gives rise to that for the QCD coupling constant $\alpha_s(Q^2 = 2.5\text{GeV}^2)$. Namely, $\Delta \Gamma_{u.b.}(Q^2 = 2.5\text{GeV}^2) = 0.029$, 0.027 and 0.011 for Analyses I, II and III, respectively, we get the ambiguities for $\alpha_s$

$$0.213 \leq \alpha_s(Q^2 = 2.5\text{GeV}^2) \leq 0.474 \quad \text{(Analysis I)},$$

$$0.228 \leq \alpha_s(Q^2 = 2.5\text{GeV}^2) \leq 0.469 \quad \text{(Analysis II)},$$

$$0.315 \leq \alpha_s(Q^2 = 2.5\text{GeV}^2) \leq 0.424 \quad \text{(Analysis III)}. \quad (14)$$

4. Conclusion

In this talk we have examined the possible target mass corrections to the Bjorken sum rule using positivity bound and experimental data on asymmetry parameters. We have found that at relatively small $Q^2$ where the QCD effect is significant, the target mass effects are also non-negligible. We found that to test the target mass correction precisely, we need accurate data for $A_2(x,Q^2)$. In determining the QCD coupling
constant $\alpha_s$ from the Bjorken sum rule, there appears uncertainty due to target mass
effects. This uncertainty can also be removed by the experimental data on $A_2(x, Q^2)$.

Although in this paper we have confined ourselves to the target mass effects in the
Bjorken sum rule, the similar analysis can be carried out for the Ellis-Jaffe sum rule.
For the proton target, $\Delta \Gamma_{u,b}(Q^2 = 2.5 \text{GeV}^2) = 0.017, 0.016$ and $0.0046$ with typical
errors of 10% for Analyses I, II and III, respectively. Those for the neutron turn out
to be $0.012$ and $0.011$ for Analyses I and II.

Finally, we note that there exists the Burkhardt-Cottingham sum rule for $g_2(x, Q^2)$
\begin{equation}
\int_0^1 dx g_2(x, Q^2) = 0, \tag{15}
\end{equation}
which is not only protected from QCD radiative corrections \cite{4, 29, 30} but also free
from target mass effects \cite{17}.

We hope that future experiments at CERN, SLAC and DESY will provide us with
data on $A_1$ possessing higher statistics as well as the data on $A_2$ with high accuracy
which will enable us to study $g_2$ structure functions and also target mass effects more
in detail.

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Fig.1 The upper bound for the target mass correction $\Delta \Gamma$, $\Delta \Gamma_{\text{u.b.}}$, as a function of $Q^2$. The solid, short-dashed and long-dashed lines show the upper bounds for the analyses I, II and III, respectively.