Nonseparability and simultaneous readability

Riuji Mochizuki*
Laboratory of Physics, Tokyo Dental College,
2-9-7 Kandasurugadai, Chiyoda-ku, Tokyo 101-0062, Japan

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Abstract

In this study, we investigate quantum nonseparability between an observed system and a measuring apparatus, or multiple measuring apparatuses. We show that the physical meaning of the outcome of the measuring apparatus obtained by weak measurement with a post-selection differs critically from that without any post-selection. In this study, the nonseparability plays the essential role, which is shown to be the same in a simultaneous conventional von Neumann-type measurement of multiple observables. From this viewpoint, we suggest a new concept, known as simultaneous readability, which is the possibility that multiple measuring apparatuses will give the proper information of the observed system simultaneously. Next, we show that different components of the spin of an electron are not simultaneously measurable even if it has EPR correlation with another electron.

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*E-mail: rjmochi@tdc.ac.jp
1 Introduction

Nonseparability is one of the central concepts of quantum mechanics. The unitary evolution of a quantum system obeys the Schrödinger equation but its nonunitary change, i.e., reduction of the wavepacket or its replacement, is also unavoidable in the measurement process. Nonseparability is produced in unitary evolution and appears in the nonunitary change; thus we need careful consideration even after the interaction between the observed system and the measuring apparatus has ended. In this paper, we employ d’Espagnat’s argument\[1\] which was used to clarify the nonseparability of EPR-correlated pairs\[2\][3] to study simultaneous measurability from the viewpoint of the quantum measurement theory. We consider the density matrix of the unified system of the observed system and the measuring apparatuses to clarify their nonseparability.

In the second section, we discuss weak measurement and weak values in view of the nonseparability. Since their concept was developed\[4\][5], many papers concerning them have been written. Applications of the weak measurement technique have been studied in some of them. For example, Lundeen et al.\[6\] have suggested the direct measurement of wavefunctions, which are experimentally verified\[7\][8]. On the other hand, more papers have been written based on the interpretation that the weak values should be conditional expectation values or others of a like nature. In this interpretation, the weak value

\[
\langle \hat{A} \rangle_{FI}^w = \frac{\langle F|\hat{A}|I \rangle}{\langle F|I \rangle}
\]

is regarded as the expectation value of $\hat{A}$ for both an initial state $|I\rangle$ and a final state $|F\rangle$, which would be obtained by post-selection following the weak measurement of $\hat{A}$ for $|I\rangle$. Because the outcomes obtained by the weak measurement without any post-selection agree with the ordinary measured values, it may be expected that the weak values would be interpreted as the expectation values even with the post-selection. This interpretation seems to be supported by many authors\[9\][10][11][12][13] with some differences in their interpretation details. In our previous paper\[14\], though we could not decide against this conjecture, we have noted that some mistakes and insufficiency exist in some important papers on weak measurement and weak values. We have also shown that the real part of the weak value provides the back-action of the weak measurement to the post-selection. In addition, the imaginary part of the weak values has been shown to be interpreted similarly by Dressel et al.\[15\].

What we read in the measurement process are not the observables of the observed system but the outcomes of the measuring apparatuses. These correspond to those in some cases but not in all cases. The expectation value of the pointer’s position, obtained by the weak measurement with the post-selection, has been verified to correspond to the real part of the weak value. The weak value obtained without any post-selection has been verified to agree with the expectation value of the observable of the observed system. However, we have not authenticated whether the readout obtained by the weak measurement with
the post-selection corresponds to the expectation value of the observable of the observed system. In this paper, we show that the post-selection causes the fatal effect on the weak measurement due to the nonseparability between the observed system and the measuring apparatus. The weak values with the post-selection should not be interpreted as expectation values in general, which differ from the weakly measured values without any post-selection. This conclusion is consistent with the discussion in our previous paper[14].

With a similar argument, we consider the simultaneous von Neumann-type measurement[16]. Though two observables belonging to the Hilbert spaces of different measuring apparatuses commute, it does not guarantee that they give the information of the observed system simultaneously. We cannot, in general, regard both of the readouts of the measuring apparatuses as the expectation values of the corresponding observables of the observed system, though either of them can be regard so. Thus, we suggest a new concept called simultaneous readability of the observables of the multiple measuring apparatuses, which is the possibility that they will give the proper information of the observed system simultaneously. We show that the simultaneous readability with a certain interaction Hamiltonian is the necessary and sufficient condition of the simultaneous measurability of the corresponding observables of the observed system. The density matrix of the unified system plays the essential role in our discussion. It reflects the fact that the nonunitary change of quantum states and the nonseparability are indispensable in the quantum measurement theory[1].

Then, we try to solve two troublesome problems of the simultaneous measurability in the latter sections. As mentioned in[17], it has not been obvious whether we can know the expectation values of two noncommuting observables simultaneously if we prepare some eigenstate. For example, it seems possible that we can know the different components of an electron’s spin simultaneously if the $x$-component of the spin is measured for the eigenstate of its $z$-component. Moreover, we can measure the different components of the spins of the EPR-correlated electrons[2][3], which seemingly enables us to know the different components of the spin of one electron. Keeping these discussions in view, Ozawa[18][19] discussed the simultaneous measurability in a state-dependent formulation. We demonstrate by extending the discussion in the preceding sections that the different components of a single electron’s spin cannot be observed simultaneously even in such cases.

2 Weak measurement and weak value

First, we quickly check that the real part of the weak value agrees with the expectation value $\overline{\mathbf{r}}$ of the pointer’s position, obtained by the corresponding weak measurement[4][6]. The interaction Hamiltonian $\hat{H}_I$ between an observable $\hat{A}$ of the observed system and the momentum $\hat{\pi}$ of the pointer is

$$\hat{H}_I \equiv g \hat{A} \hat{\pi}, \quad (1)$$
where \( g \) is the coupling constant. \( \hat{H}_I \) is assumed to be constant and roughly equivalent to the total Hamiltonian over some interaction time \( t \). The initial wavefunction \( \phi(x) \) of the measuring apparatus is assumed to be

\[
\phi(x) = \langle x | \phi \rangle = \left( \frac{1}{\sqrt{2\pi\sigma}} \right)^{1/2} \exp \left( -\frac{(x - x_0)^2}{4\sigma^2} \right).
\]  

The initial state \( |\Phi(0)\rangle = |I\rangle|\phi\rangle \), where \( |I\rangle \) is the initial state of the observed system, of the unified system of the observed system and the measuring apparatus evolves unitarily obeying the Schrödinger equation:

\[
i\hbar \frac{d}{dt} |\Phi(t)\rangle = \hat{H}|\Phi(t)\rangle \sim \hat{H}_I|\Phi(t)\rangle,
\]

and becomes

\[
|\Phi(t)\rangle = \exp \left( -\frac{i\hat{H}_I t}{\hbar} \right) |\Phi(0)\rangle.
\]

Up to the first order of \( g \),

\[
|\Phi_1(t)\rangle = |I\rangle|\phi\rangle - i\frac{gt}{\hbar} \hat{A}|I\rangle|\hat{\pi}|\phi\rangle.
\]

Instead, we can equally describe the unified system by means of the density matrix

\[
\rho(t) = |\Phi_1(t)\rangle \langle \Phi_1(t)|.
\]

Without any post-selection, the expectation value \( \bar{x} \) of the pointer’s position \( \hat{x} \) for this state is

\[
\bar{x} \equiv \text{Tr} \left[ \rho(t) \hat{x} \right] = x_0 + gt \langle I | \hat{A} | I \rangle.
\]

\( \bar{x} \) can be written also in the form

\[
\bar{x} = \text{Tr} \left[ \rho^{(m)}(t) \hat{x} \right],
\]

where \( \rho^{(m)}(t) \) is the partial density matrix only of the measuring apparatus defined as

\[
\rho^{(m)}_1(t) \equiv \text{Tr}^{(s)}[\rho_1(t)] = \langle \phi | \phi \rangle - \frac{i gt}{\hbar} \langle I | \hat{A} | I \rangle \left( \hat{\pi} \langle \phi | \phi \rangle - \langle \phi | \phi \rangle \hat{\pi} \right).
\]

and \( \text{Tr}^{(s)} \) means that the operation of taking the trace is carried over only in the Hilbert space of the observed system.

On the other hand, with the post-selection \( |F\rangle \), the expectation value \( \bar{x}_F \) of the pointer’s position is

\[
\bar{x}_F = \text{Tr} \left[ \rho^{(m)}_F(t) \hat{x} \right] = |\langle F | I \rangle|^2 \left[ x_0 + gt \Re \langle A_F \rangle \right],
\]
where
\[ \rho_{1F}^{(m)}(t) \equiv \text{Tr}^{(s)}[\rho_1(t)|F\rangle\langle F|]. \]  

(11)

These calculations have shown the following. Without any post-selection, the readout of the pointer’s position corresponds to the expectation value of the observable for the initial state belonging to the observed system. With the post-selection, the readout of the pointer’s position corresponds to the real part of the weak value. Thus, it remains to be authenticated whether the readout of the pointer’s position obtained by the weak measurement with the post-selection reflects the information about the observable for the initial and the final states of the observed system. This is written in another form: Can we read both the expectation values of \( \hat{A} \) and \( \hat{F} \equiv |F\rangle\langle F| \) for the state \( |5\rangle \) after the interaction between the observed system and the measuring apparatus has ended?

To answer this question, we assume that the ensemble \( E \) of the observed system and the ensemble \( M \) of the measuring apparatus after their interaction are both separately obtained by combining all the elements of subensembles, each of which is described by its own ket. Then, each element of \( E \) belongs to one of the subensembles \( E_i, \ i = 1, 2, \cdots \) described by \( |s_i\rangle \) and each element of \( M \) belongs to one of the subensembles \( E_\alpha, \ \alpha = 1, 2, \cdots \) described by \( |m_\alpha\rangle \), such that the subensemble \( \varepsilon_{i,\alpha} \) of the unified system, whose elements belong to both \( E_i \) and \( E_\alpha \), is described by the density matrix
\[ \rho_{i,\alpha} = |s_i\rangle|m_\alpha\rangle\langle m_\alpha|\langle s_i|. \]  

(12)

Because the unified system’s ensemble \( \varepsilon \), which is described by \( |5\rangle \), is the union of all the \( \varepsilon_{i,\alpha} \), the density matrix \( \rho' \) describing \( \varepsilon \) should be written as the weighted sum of all the \( \rho_{i,\alpha} \):
\[ \rho' = \sum_{i,\alpha} P_{i,\alpha} \rho_{i,\alpha}, \]  

(13)

where \( P_{i,\alpha} \) are suitable factors. However, \( \varepsilon \) is defined to be described by \( |5\rangle \), such that it should be described by the density matrix \( |5\rangle \). \( \rho_1(t) \) and \( \rho' \) are necessarily different, except in the case that \( |\Phi_1(t)\rangle \) is a product of a vector \( |S\rangle \) in the Hilbert space of the observed system and a vector \( |M\rangle \) in the Hilbert space of the measuring apparatus, i.e.,
\[ |\Phi_1(t)\rangle = |S\rangle|M\rangle. \]  

(14)

\( |5\rangle \) does not have this form. Thus, the previous assumption has been shown to be false.

We must say for the above reason that both the observed system and the measuring apparatus do not have complete sets of their own, though they have no unitary interaction after \( t \) and
\[ \text{Tr}[\rho_1(t)\hat{F}\hat{x}] = \text{Tr}[\rho_1(t)\hat{x}\hat{F}]. \]  

(15)

This is the manifestation of the quantum nonseparability\( ^{[1]} \), which implies that reading the pointer’s position of the measuring apparatus and the post-selection
are mutually dependent even after the interaction between the observed system and the measuring apparatus has ended. In other words, the mixed state of the measuring apparatus, which is described by (11), does not reflect properly its state after the unitary evolution by the Hamiltonian (1). Hence, we cannot expect that the weak value \( \langle \hat{A} \rangle_W^{\psi_F} \) gives the expectation value of \( \hat{A} \) for the initial state \( |I\rangle \) in any situation.

We can expect that \( \overline{x}_F \) gives the expectation value of \( \hat{A} \) for a different state if and only if \( [\hat{A}, \hat{F}] = 0 \). In this case,

\[
\rho_1^{(m)}(t) = \text{Tr}^{(s)}[(\hat{1} - \frac{i g t}{\hbar} \hat{A} \hat{\pi}) \hat{F} |I\rangle \langle \phi| \hat{F} (\hat{1} + \frac{i g t}{\hbar} \hat{A} \hat{\pi})]
\]

so that \( \overline{x}_F \) becomes

\[
\overline{x}_F = x_0 + gt \langle |I\rangle \hat{F} \hat{A} \hat{F} |I\rangle.
\]

Thus, \( \overline{x}_F \) corresponds to the expectation value of \( \hat{A} \) for the state obtained after the projective measurement of \( \hat{F} \) for \( |I\rangle \).

3 Simultaneous measurability and simultaneous readability

We extend the discussion in the previous section to the von Neumann-type measurement [16] and study the simultaneous measurability of two observables \( \hat{A} \) and \( \hat{B} \) of an observed system from the viewpoint of the quantum measurement theory. Let \( |I\rangle \) be the initial state of the observed system and \( |\phi\rangle \) and \( |\psi\rangle \) be the initial states of the measuring apparatuses of \( \hat{A} \) and \( \hat{B} \), respectively. Then, the initial state of the unified system \( |\Psi(0)\rangle \) is

\[
|\Psi(0)\rangle = |I\rangle |\phi\rangle |\psi\rangle.
\]

The initial wavefunctions of the measuring apparatuses are assumed to be

\[
\phi(x_A) = \langle x_A |\phi\rangle = \left( \frac{1}{\sqrt{2\pi} \sigma_A} \right)^{1/2} \exp \left( - \frac{(x_A - (x_A)_0)^2}{4 \sigma_A^2} \right),
\]

\[
\psi(x_B) = \langle x_B |\psi\rangle = \left( \frac{1}{\sqrt{2\pi} \sigma_B} \right)^{1/2} \exp \left( - \frac{(x_B - (x_B)_0)^2}{4 \sigma_B^2} \right).
\]

We define the interaction Hamiltonian [20] as

\[
\hat{H}_I = g_A \hat{A} \hat{\pi}_A + g_B \hat{B} \hat{\pi}_B,
\]

where \( g_A \) and \( g_B \) are coupling constants and \( \hat{\pi}_A \) and \( \hat{\pi}_B \) are the momenta of the pointers. In the same way as the previous section, we assume \( \hat{H}_I \) to be constant and dominant over some interaction time \( t \). Then, the state after the interaction is

\[
|\Psi(t)\rangle = \exp \left( - \frac{i \hat{H}_I t}{\hbar} \right) |\Psi(0)\rangle.
\]
If we read the position of only one pointer of these measuring apparatuses, its readout can be interpreted as the expectation value of the corresponding observable:

$$\bar{x}_A = \text{Tr}\left[ \rho(t) \hat{x}_A \right] = (x_A)_0 + g_A t \langle I | \hat{A} | I \rangle,$$

$$\bar{x}_B = \text{Tr}\left[ \rho(t) \hat{x}_B \right] = (x_B)_0 + g_B t \langle I | \hat{B} | I \rangle,$$

where

$$\rho(t) = |\Psi(t)\rangle \langle \Psi(t)|.$$  \hfill (25)

However, both of the readouts do not, in general, give the information of the observables of the observed system simultaneously. The following discussion is almost the same as that in the previous section. Let $\varepsilon$ be the ensemble described by the partial density matrix $\rho^{(m)}(t)$ only of the measuring apparatuses defined as

$$\rho^{(m)}(t) \equiv \text{Tr}^{(s)}\left[ \rho(t) \right].$$  \hfill (26)

We assume that the ensembles $M_A$ of the measuring apparatuses of $\hat{A}$ and $M_B$ of $\hat{B}$ are both separately obtained by combining all the elements of the subensembles, each of which is described by its own ket. Then, each element of $M_A$ belongs to one of the subensembles $E_\alpha$, $\alpha = 1, 2, \cdots$ described by $|a_\alpha\rangle$ and each element of $M_B$ belongs to one of the subensembles $E_\beta$, $\beta = 1, 2, \cdots$ described by $|b_\beta\rangle$, such that the subensemble $\varepsilon_{\alpha,\beta}$ of the combined measuring apparatus, whose elements belong to both $E_\alpha$ and $E_\beta$, is described by the density matrix

$$\rho_{\alpha,\beta} = |b_\beta\rangle \langle a_\alpha| \langle a_\alpha| \langle b_\beta|. $$

Thus, the ensemble $\varepsilon$ is described as the weighted sum of all the $\rho_{\alpha,\beta}$:

$$\rho' = \sum_{\alpha,\beta} P_{\alpha,\beta} \rho_{\alpha,\beta},$$  \hfill (27)

where $P_{\alpha,\beta}$ are suitable factors. Nevertheless, (27) does not agree with (26), which describes $\varepsilon$ by definition, in general. Thus, the previous assumption has been shown to be false.

If and only if $[\hat{A}, \hat{B}] = 0$, (22) becomes

$$|\Psi(t)\rangle = \exp\left( -\frac{i g_A t}{\hbar} \hat{A} \hat{\pi}_A \right) |\phi\rangle \exp\left( -\frac{i g_B t}{\hbar} \hat{B} \hat{\pi}_B \right) |\psi\rangle |I\rangle,$$  \hfill (28)

and the partial density matrix $\text{Tr}^{(s)}[|\Psi(t)\rangle \langle \Psi(t)|]$ is in the form of (27). In this case, we can simultaneously regard both the readouts of the measuring apparatuses as the expectation values of $\hat{A}$ and $\hat{B}$, respectively.

We then conclude that the multiple measuring apparatuses that have interacted with the same observed system are nonseparable if their corresponding
observables are noncommuting. That is, reading their outcomes mutually affect even after the unitary part of the measuring process has ended. Although we can simultaneously read the values \( x_A \) and \( x_B \) on the measuring apparatuses, we cannot regard them as the expectation values of the noncommuting observables \( \hat{A} \) and \( \hat{B} \), respectively.

On the other hand, \( \hat{x}_A \hat{x}_B \) is an observable of the combined measuring apparatus because \( [\hat{x}_A, \hat{x}_B] = 0 \). We can calculate the expectation value of \( \hat{x}_A \hat{x}_B \):

\[
(\hat{x}_A \hat{x}_B) = \text{Tr}[\rho(t) \hat{x}_A \hat{x}_B] = (x_A)_0(x_B)_0 + g_A t(x_B)_0 \langle I | \hat{A} | I \rangle + g_B t(x_A)_0 \langle I | \hat{B} | I \rangle + \frac{1}{2} g_A g_B t^2 \langle I | (\hat{A} \hat{B} + \hat{B} \hat{A}) | I \rangle.
\]

This is the expectation value of the observable \((1/2)(\hat{A} \hat{B} + \hat{B} \hat{A})\) for the initial state \( |I\rangle \), when \( (x_A)_0 = (x_B)_0 = 0 \). However, we should pay attention to the following fact:

\( (\hat{x}_A \hat{x}_B) \neq \tau_A \tau_B \),

such that \( \tau_A \) and \( \tau_B \) cannot be separately extracted from \( 29 \). If we identify \( \hat{B} \) with the projection operator \( \hat{F} \) of the post-selection, we can reproduce the conclusion in the second section. When \( [\hat{A}, \hat{F}] \neq 0 \), \( (\hat{x}_A \hat{x}_F) \) corresponds to the expectation value of the observable in the Hilbert space of the observed system; however, both \( \tau_A \) and \( \tau_F \) do not. If we select a final state in the weak measurement, we cannot receive the information of the observed system from the readout of the measurement apparatus.

Here, we define the \textit{simultaneous readability}. Let \( |\Psi(t)\rangle \) be the entangled state after the unitary interaction between the observed system and the measuring apparatuses and \( \rho^{(m)}(t) \) be the partial density matrix only of the measuring apparatuses. We call the observables \( \hat{x}_A \) and \( \hat{x}_B \) of the different measuring apparatuses simultaneously readable if and only if \( \rho^{(m)}(t) \) is written in the form of \( 27 \), that is

\[
\rho^{(m)}(t) = \sum_{\alpha, \beta} P_{\alpha, \beta} \rho_{\alpha, \beta}.
\]

Thus, we can regard each of their outcomes as the expectation value of its corresponding observable of the observed system. If not, only one of three observables \( \hat{x}_A \), \( \hat{x}_B \) and \( \hat{x}_A \hat{x}_B \) can be read to receive the information of the observed system. If the interaction between the observed system and the measuring apparatuses is described by the Hamiltonian \( 21 \), the simultaneous readability of \( \hat{x}_A \) and \( \hat{x}_B \) is the necessary and sufficient condition for the simultaneous measurability of the corresponding observables \( \hat{A} \) and \( \hat{B} \).
4 Weak measurement without post-selection

To study the simultaneous measurability by the weak measurement without post-selection, we expand (22) to the first order of $g_A$ and $g_B$:

$$\langle \Psi(t) \rangle = |I\rangle\langle\phi|\psi\rangle - \frac{ig_A}{\hbar}\hat{A}|I\rangle\tilde{\pi}_A|\phi\rangle\langle\psi| - \frac{ig_B}{\hbar}\hat{B}|I\rangle\tilde{\pi}_B|\phi\rangle\langle\psi|. \quad (31)$$

Then, its partial density matrix $\rho^{(m)}(t)$ satisfies (30) up to the first order of the couplings. Nevertheless, we do not think that this fact shows the simultaneous measurability of $\hat{A}$ and $\hat{B}$. Suppose that two observables are measured in turns and their expectation values are obtained after the series of measurements. Then, is it appropriate to regard them as simultaneously measurable? It shows only a similar situation. The commutator of $\hat{A}$ and $\hat{B}$ appears in the second order of the couplings; thus, we cannot discuss the simultaneous measurability in the first order of the couplings.

Up to the second order of the couplings, $|\Psi(t)\rangle$ becomes

$$|\Psi(t)\rangle = (1 - i\hat{H}_I t - \frac{1}{2}H_I^2 t^2)|I\rangle, \quad (32)$$

whose partial density matrix $\rho^{(m)}(t)$ does not satisfy (30) if $[\hat{A}, \hat{B}] \neq 0$. Thus, $\hat{x}_A$ and $\hat{x}_B$ are not simultaneously readable or $\hat{A}$ and $\hat{B}$ are not simultaneously measurable. On the other hand, with $(x_A)_0 = (x_B)_0 = 0$,

$$\langle x_A x_B \rangle = \frac{g_A g_B t^2}{2} \langle I|{\hat{A}}{\hat{B}} + {\hat{B}}{\hat{A}}|I\rangle, \quad (33)$$

which agrees with (29).

5 Measuring a different observable for eigenstates

We consider the spin of an electron with the appropriate normalization. $x$-component $\hat{\sigma}_x$ and $z$-component $\hat{\sigma}_z$ of the spin are not simultaneously measurable or their corresponding observables of the measuring apparatuses are not simultaneously readable. It is not obvious, however, whether the following question has the same answer [17][18][19]: An electron is prepared in the eigenstate of $\hat{\sigma}_z$ at time 0 and its $\hat{\sigma}_x$ will be measured for this state. Then, can we know both the expectation values of $\hat{\sigma}_z$ and $\hat{\sigma}_x$ at time 0?

Let $|\phi\rangle$ be the initial state of the measuring apparatus of $\hat{\sigma}_x$ and $|I\rangle$ be the initial state of the observed system. The state of the unified system after the interaction is

$$|\Phi(t)\rangle = \exp \left(-\frac{igt}{\hbar}\hat{\sigma}_x\pi\right)|\Phi(0)\rangle \quad (34)$$

and its density matrix is

$$\rho(t) = |\Phi(t)\rangle\langle\Phi(t)|, \quad (35)$$
where
\[ |\Phi(0)\rangle = |I\rangle |\phi\rangle. \] (36)

We prepare \(| \uparrow \rangle\), which is the eigenstate of \( \hat{\sigma}_z \) with an eigenvalue +1, as the initial state of the observed system.

In the second section, we considered the weak measurement, and almost the same discussion can be applied to the von Neumann-type measurement. If we take the trace only in the Hilbert space of the measuring apparatus, \((35)\) becomes the proper density matrix of the observed system, and vice versa. Nevertheless, the observed system and the measuring apparatus are nonseparable, so we must lose the information of the observed system at time \( t \) if we read the value on the measuring apparatus. On the other hand, because the evolution from time 0 to \( t \) is unitary, we can calculate the state at time \( t \) if we know that at time 0. Thus, we must conclude that we have no information for the observed state even at time 0 after we read the outcome of the measuring apparatus.

However, we prepared the eigenstate \(| \uparrow \rangle\) as the initial state. We need to reconsider the eigenstate to convince ourselves that there is no contradiction between these statements. We will necessarily obtain +1 if we measure only \( \hat{\sigma}_z \) for \(| \uparrow \rangle\). We should not express this fact by the statement “The \( z \)-component of the spin of \(| \uparrow \rangle\) is +1”. The eigenstate should be understood contextually. We have nothing to say about the \( z \)-component of the spin of the initial state if we read its \( x \)-component. Thus, our answer to the previous question is NO. We cannot know the different components of the spin simultaneously even if the eigenstate of one component of the spin is prepared as the initial state. The above discussion may make us remember the delayed-choice experiment\[^{21}\]\[^{22}\], which seems very strange but is the proper consequence of quantum mechanics.

6 Simultaneous measurement of an EPR-correlated electron

We consider a pair of electrons that have the EPR correlation\[^2\]\[^3\] and their total spin of 0. The operator \( \hat{C}_z \), which measures the correlation, is defined as
\[ \hat{C}_z = (\hat{\sigma}_z)_1(\hat{\sigma}_z)_2, \] (37)
where \((\hat{\sigma}_z)_1\) and \((\hat{\sigma}_z)_2\) are the \( z \)-components of the spins of the electrons 1 and 2, respectively. We prepare \(| -1 \rangle\), which is the eigenstate of \( \hat{C}_z \) with an eigenvalue \(-1\), as the initial state \(| I \rangle\) of the observed system, i.e. the combined system of these electrons. The initial state of the unified system of the observed system and the measuring apparatuses is
\[ |\Psi(0)\rangle = |I\rangle |\phi\rangle |\psi\rangle, \] (38)
where \(|\phi\rangle\) and \(|\psi\rangle\) are the initial states of the measuring apparatuses of the electrons 1 and 2, respectively. Their wavefunctions are given in \[^{19}\] and \[^{20}\].

The interaction Hamiltonian is
\[ \hat{H}_I = g_A (\hat{\sigma}_x)_1 \hat{\pi}_A + g_B (\hat{\sigma}_z)_2 \hat{\pi}_B. \] (39)
Then, the state after the unitary evolution is

$$|\Psi(t)\rangle = \exp\left(-\frac{i\hat{H}_1 t}{\hbar}\right)|\Psi(0)\rangle = \exp\left(-\frac{i\hat{H}_{11} t}{\hbar}\right)|\phi\rangle \exp\left(-\frac{i\hat{H}_{12} t}{\hbar}\right)|\psi\rangle |I\rangle,$$

where

$$\hat{H}_{11} = g_A(\hat{\sigma}_x)\hat{\pi}_A,$$
$$\hat{H}_{12} = g_B(\hat{\sigma}_z)\hat{\pi}_B.$$

Because the partial density matrix $\rho^{(m)}(t)$, which is only for the measuring apparatuses made from (40), has the form of (30), $\hat{x}_A$ and $\hat{x}_B$ are simultaneously readable. Thus, we can expect that both the measuring apparatuses give the proper outcomes simultaneously.

With the help of

$$|I\rangle = \frac{1 - \hat{C}_z}{2} |I\rangle,$$

(40) is rewritten as

$$|\Psi(t)\rangle = \exp\left(-\frac{i\hat{H}_{11} t}{\hbar}\right)|\phi\rangle \exp\left(-\frac{i\hat{H}_{12} t}{\hbar}\right)|\psi\rangle \frac{1 - \hat{C}_z}{2} |I\rangle = \exp\left(-\frac{i\hat{H}_{11} t}{\hbar}\right)|\phi\rangle \exp\left(+\frac{i\hat{H}_{12} t}{\hbar}\right)|\psi\rangle |I\rangle,$$

where

$$\hat{H}_{12} = g_B(\hat{\sigma}_z)\hat{\pi}_B.$$

The partial density matrix $\rho^{(m)}(t)$ made from (42) once again has the form of (30), $\hat{x}_A$ and $\hat{x}_B$ are simultaneously readable. Does (42) thus show that both $(\hat{\sigma}_x)_1$ and $(\hat{\sigma}_z)_1$ are simultaneously measurable?

It is now obvious that its answer is negative as in the previous section. It is shown in (40) that the electron pair and each of the measuring apparatuses are nonseparable. Though we have prepared $|-1\rangle$ as the initial state, this only means that we will necessarily obtain $-1$ if we only measure $\hat{C}_z$ for $|-1\rangle$. Conversely, if we read the outcome of the measuring apparatus of $(\hat{\sigma}_x)_1$, we will receive no information about the correlation of our electron pair at time $t$ and, by extension, at time 0. (41) is not a definition or an identity but an equation that holds only contextually. This equation cannot be used in the context in this section. We can measure any component of the spin of each electron of the EPR-correlated pair but cannot regard it as the measurement of a pair of components of the spin of a single electron.

### 7 Concluding remarks

In our discussion, the density matrix of the unified system of the observed system and the measuring apparatuses has played an essential role. This finding
reflects that the nonseparability and the nonunitary change, i.e., the reduction of the wavepacket or its replacement, are indispensable in the quantum measurement theory. We have not discussed the problem of the von Neumann chain\cite{23} about the extent where the quantum measurement theory treats the reading of the measuring apparatus; thus our word *read* contains some ambiguity in its meaning. Nevertheless, we have shown that reading the outcome of the measuring apparatus, which may be a series of actions, brings the inevitable effect on the unified system, though it is after the unitary interaction between the observed system and the measuring apparatuses. By this observation, we have suggested the simultaneous readability as the necessary and sufficient condition of the simultaneous measurability with the interaction Hamiltonian (21). To understand the simultaneous measurability, it is not sufficient to study the unitary evolution of the unified system. From the viewpoint of the quantum measurement theory, it is in the nonunitary change that the essence of the simultaneous measurability exists.

In this context, the discussion in the second section is valid no matter how weak the measurement is. Starting with (4), we obtain

\[
\text{Tr}[\rho(t)\hat{F}] = \text{Tr} \left[ \exp \left( -\frac{i\hat{H}_1 t}{\hbar} \right) |\Phi(0)\rangle \langle \Phi(0)| \exp \left( +\frac{i\hat{H}_1 t}{\hbar} \right) \hat{F} \right] = \langle I | \hat{F} | I \rangle \tag{43}
\]

if $[\hat{A}, \hat{F}]$ is a c-number. This equation shows that we can receive information about the initial state of the observed system even after its unitary interaction with the measuring apparatus, which is not necessarily weak, has ended. It is the nonunitary change accompanied with reading the outcome of the measuring apparatus that hides the information of the initial state. We well convince this fact to ourselves if we reinterpret (28) as the state after a series of interactions between the observed system and the measuring apparatuses, that is, the interactions of $\hat{B}$ between 0 and $t$ and $\hat{A}$ between $t$ and $2t$. Then, even if $[\hat{A}, \hat{B}] \neq 0$, the unified state at time $2t$ is

\[
|\Psi(2t)\rangle = \exp \left( -\frac{ig_A t}{\hbar} \hat{A} \hat{\pi}_A \right) |\phi\rangle \exp \left( -\frac{ig_B t}{\hbar} \hat{B} \hat{\pi}_B \right) |\psi\rangle |I\rangle. \tag{44}
\]

If we read only $\hat{A}$, we obtain the same result as (23). The unitary part of the measurement of $\hat{B}$ has no effect on this result.

Moreover, it is worth noting that $\hat{x}_A$ and $\hat{x}_B$ are simultaneously readable if the unified system is expressed by (44). We can know the expectation values of $\hat{B}$ at time 0 and $\hat{A}$ at $t$ from those of $\hat{x}_B$ and $\hat{x}_A$, respectively. However, the expectation value of $\hat{A}$ at time $t$ with the reading of the outcome of $\hat{B}$ does not agree with (23) in general. We will obtain the proper expectation value of $\hat{A}$ according to the readout of the measuring apparatus of $\hat{B}$; thus we cannot know the expectation values of $\hat{A}$ and $\hat{B}$ at time 0 simultaneously. On the other hand, as discussed in the fifth section, we cannot know the expectation values of $\hat{A}$ and $\hat{B}$ at time $t$ simultaneously because the observed system and the measuring apparatus of $\hat{A}$ are nonseparable. Thus, we conclude that the noncommuting
observables $\hat{A}$ and $\hat{B}$ are not simultaneously measurable, though $\hat{x}_A$ and $\hat{x}_B$ are simultaneously readable in this case.

As noted in the above paragraph, we have shown in the fifth section that eigenstates should be understood contextually. Thus, Bell’s inequality[24] is not concluded even though a quantum state is assumed to be a simultaneous eigenstate of multiple noncommuting observables. Nevertheless, it does not mean that we can obtain the expectation values of the noncommuting observables from such a state simultaneously. Thus, we do not need to describe the state as the simultaneous eigenstate of noncommuting observables; the quantum mechanics still stay complete for describing our possible knowledge about the quantum state. In addition, it would not support any local hidden variable theories. Bell-nonlocality has not vanished but has shifted between the observed system and the measuring apparatus or between the multiple measuring apparatuses.

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