A local pointwise inequality for a biharmonic equation with negative exponents

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Abstract
In this paper, we are inspired by Ngô, Nguyen and Phan’s (2018 Nonlinearity \textbf{31} 5484–99) study of the pointwise inequality for positive $C^4$-solutions of biharmonic equations with negative exponent by using the growth condition of solutions. They propose an open question of whether the growth condition is necessary to obtain the pointwise inequality. We give a positive answer to this open question. We establish the following local pointwise inequality

$$-\frac{\Delta u}{u} + \alpha \frac{\nabla u}{|u|^2} + \beta u^{-\frac{n+1}{2}} \leq C \frac{1}{R^2}$$

for positive $C^4$-solutions of the biharmonic equations with negative exponent

$$-\Delta^2 u = u^{-q} \text{ in } B_R$$

where $B_R$ denotes the ball centered at $x_0$ with radius $R$, $n \geq 3$, $q > 1$, and some constants $\alpha \geq 0$, $\beta \geq 0$, $C > 0$.

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1. Introduction and main results

In this paper, we consider a pointwise inequality for positive solutions of biharmonic equations with negative exponents

$$-\Delta^2 u = u^{-q} \text{ in } B_R,$$

(1.1)

where $B_R$ denotes the ball centered at $x_0$ with radius $R$, $n \geq 3$, $q > 1$.

Let us briefly describe the geometric background of this equation. Let $g = (g_{ij})$ be the standard Euclidean metric on $\mathbb{R}^n$, where $g_{ij} = 1$ for $i = j$ and $g_{ij} = 0$ for $i \neq j$. Let $\varphi = u^{\frac{4}{n}} g$, $n \neq 4$, be a second metric derived from $g$ by the positive conformal factor $u : \mathbb{R}^n \to \mathbb{R}$. Then $u$ satisfies

$$\Delta^2 u = \frac{n-4}{2} Q_\varphi u^{\frac{n+4}{n}},$$

where $Q_\varphi$ is the $Q$-curvature of $\varphi$, see [3, 12]. We know that if $n = 3$, and $Q_\varphi > 0$ is constant, then the equation (1.1) can be obtained by scaling for $q = 7$. Recently, Lai and Ye [13] showed that if $0 < q \leq 1$, then equation (1.1) admits no entire smooth solution. For any $q > 1$, there exist radial solutions to (1.1), see [11, 14].

We know that the pointwise differential inequality has exerted a great influence on the theory of elliptic partial differential equations. It can be used to solve some famous conjectures and open problems, and also can be applied to the existence theory of solutions of nonlinear partial differential equations. The following pointwise differential inequality was proved by Modica [15], which is one of the main techniques to solve De Giorgi’s [4] conjecture for the Allen–Cahn equation.

**Theorem 1.1. ([15])** Let $F \in C^2(\mathbb{R})$ be a nonnegative function and $u$ be a bounded entire solution of

$$\Delta u = F'(u) \text{ in } \mathbb{R}^n.$$

(1.2)

Then

$$|\nabla u|^2 \leq 2F(u).$$

(1.3)

When $F'(u) = u(u^2 - 1)$, the equation (1.2) is known as the Allen–Cahn equation. The so-called De Giorgi’s conjecture is a monotonic classification of the entire solution of the Allen–Cahn equation in one direction. To be more precise, assume that $u \in C^2(\mathbb{R}^n)$ be a solution to the Allen–Cahn equation satisfying $\partial_\nu u \geq 0$. Then the level sets $\{u = \lambda\}$ must be hyperplanes, for $n \leq 8$. In the last twenty years, great advances in De Giorgi’s conjecture have been achieved, having been fully established in dimensions $n = 2$ by Ghoussoub and Gui [8] and for $n = 3$ by Ambrosio and Cabré [2]. A celebrated result by Savin [17] established its validity for $4 \leq n \leq 8$ under an extra assumption that

$$\lim_{x_n \to \pm \infty} u(x', x_n) = \pm 1.$$

On the other hand, Del Pino, Kowalczyk, and Wei [5] constructed a counterexample for $n \geq 9$.

As far as we know, a similar inequality as (1.3) corresponding to the fourth-order equation is not proved. But some similar results were proved by Aghajani, Cowan and Rădulescu [1] in
a bounded domain. There are many other well known pointwise inequalities for fourth-order Lane–Emden equations with positive exponents, i.e.,

$$\Delta^2 u = u^\theta \quad \text{in} \quad \mathbb{R}^n.$$  \hfill (1.4)

We refer interested readers to [7, 18, 19] and references therein.

There is a big difference between the fourth-order elliptic equations with positive exponents and negative exponents. It is well known that the negative exponent of the nonlinear term is more challenging than the positive exponent, since there exist solutions that grow linearly or superlinearly at infinity, see [3, 6, 9, 14] and references therein.

Recently, Guo and Wei [10] established the following pointwise inequality for the positive $C^4$-solution of equation (1.1),

$$\Delta u \geq \sqrt{\frac{2}{q-1}} u^{\frac{q+1}{2}}.$$  \hfill (1.5)

Inspired by the above nice result, Ngô, Nguyen and Phan [16] proved the following pointwise differential inequality.

**Theorem 1.2.** ([16]) Let $u > 0$ be a $C^4$-solution to (1.1) in $\mathbb{R}^n$ with $n \geq 3$. Assume that two positive constants $\alpha, \beta$ satisfy

$$\begin{align*}
\alpha &\leq \frac{1}{2}, \\
\beta &\leq \sqrt{\frac{2}{q-1-\frac{4\alpha}{n}}}, \\
q &\geq 3\alpha + \sqrt{9\alpha^2 + (1 - 2\alpha)(1 + 16\alpha/n)}.
\end{align*}$$  \hfill (1.6)

Then the following pointwise inequality

$$\Delta u \geq \alpha \frac{\|\nabla u\|^2}{u} + \beta u^{\frac{q+1}{2}}$$  \hfill (1.7)

holds in $\mathbb{R}^n$ for any solution $u$ satisfying the growth condition

$$u(x) = o\left(\frac{1}{|x|^\gamma}\right) \quad \text{as} \quad |x| \to \infty,$$  \hfill (1.8)

where $\gamma$ is arbitrary in $[0, 1)$ such that

$$\begin{align*}
\alpha + \frac{4\alpha(1 - 2\alpha)}{n} - 3\alpha\gamma - \gamma^2 + \gamma > 0, \\
q - \frac{8\alpha}{n} - 2\gamma > 0.
\end{align*}$$  \hfill (1.9)

In particular, the pointwise inequality (1.7) always holds under the assumption (1.6) and

$$u(x) = O(|x|^2), \quad \text{as} \quad |x| \to \infty.$$  \hfill (1.10)

The growth conditions of (1.8) and (1.10) seem not too strict. The evidence supporting this observation is as follows. On one hand, it is proved by Lai and Ye [13] that the radial solutions of the equation (1.1) in $\mathbb{R}^n$ grow at most quadratically at infinity. On the other hand, the inequality (1.5) was proved by Guo and Wei without any growth condition of the solutions.
This observation leads Ngô, Nguyen and Phan to propose an open question of whether the growth condition is necessary to obtain the pointwise inequality (1.7) when \( \alpha > 0 \).

In this paper, we give a positive answer to this open question. Our method of the proof was mainly motivated by the above Ngô, Nguyen and Phan’s [16] result. Our main contribution is to find a new auxiliary function which is different from the one in [16]. When using the maximum principle to this auxiliary function, we do not need any growth condition of the solutions. Here is our main theorem:

**Theorem 1.3.** Let \( u > 0 \) be a \( C^{4} \)-solution to (1.1) on a ball \( B_{R} \) centered at \( x_{0} \) with radius \( R \) in \( \mathbb{R}^{n}(n \geq 3) \). Assume that two nonnegative constants \( \alpha, \beta \) satisfy

\[
\begin{align*}
\alpha &< -(n^{2} + 4n - 4) + \sqrt{(n^{2} + 4n - 4)^{2} + 64n} & (n \geq 4), & \alpha \leq \frac{-7 + 2\sqrt{13}}{4} & (n = 3), \\
\beta &\leq \frac{2}{q - 1 - 4\alpha/n}, & q &\geq 8\alpha/n + 3.
\end{align*}
\]

(1.11)

Then there exist positive constant \( C = C(\alpha, n) \) such that

\[
-\frac{\Delta u}{u} + \alpha \frac{|\nabla u|^{2}}{u^{2}} + \beta u^{-\frac{n+1}{2}} \leq \frac{C}{R^{2}}.
\]

(1.12)

As a direct consequence of theorem 1.3, by taking \( \alpha = 0 \) and letting \( R \to \infty \), we obtain the following corollary:

**Corollary 1.1.** Let \( u > 0 \) be a \( C^{4} \)-solution to (1.1) in \( \mathbb{R}^{n} \) with \( n \geq 3 \). Then

\[
\Delta u \geq \sqrt{\frac{2}{q - 1 - \frac{\alpha}{n}}} u^{\frac{n+1}{2}}.
\]

**Remark 1.1.** This pointwise inequality was concluded by Guo and Wei in [10].

As another direct consequence of theorem 1.3, by letting \( R \to \infty \), we obtain the following corollary:

**Corollary 1.2.** Under the assumptions of theorem 1.3. Then

\[
\Delta u \geq \alpha \frac{|\nabla u|^{2}}{u} + \beta u^{-\frac{n+1}{2}}.
\]

**Remark 1.2.** Now let us compare the conditions in theorem 1.3 with the ones in theorem 1.2. The first assumption of (1.11) can be compared with the first one of (1.9). The constant \( \alpha \) in the first one of (1.9) will be controlled by the constants \( n \) and \( \gamma \). It seems that the constant \( \alpha \) in this paper is more limited than the reference [16]. The second condition of (1.11) is the same as the second one of (1.6). The third condition of (1.11) is similar to the second one of (1.9) for \( \gamma = 1 \). However we do not need any growth conditions of the solution to (1.1). This answers the question in [16].
2. Proof of theorem 1.3

For simplicity, setting \( v = \ln u, \) \( k > \frac{q+1}{2} \), we obtain the following lemma:

**Lemma 2.1.** Let \( u \) be a smooth positive solution of the equation (1.1) on a ball \( B_R^q \) centered at \( x_0 \) with radius \( R \) in \( \mathbb{R}^n \). Then

\[
-\Delta^2 v = 2|\nabla v|^2 \Delta v + 4 \nabla \Delta v \cdot \nabla v + (\Delta v)^2 \\
+ |\nabla v|^4 + 4 \Delta^2 v(\nabla v, \nabla v) + 2 \|\nabla^2 v\|^2 + e^{-2kv},
\]  

(2.1)

where \( \nabla^2 v \) denotes the Hessian matrix of \( v \) and \( \|\nabla^2 v\| \) denotes the Hilbert–Schmidt norm on matrices defined to be

\[
\|\nabla^2 v\| = \left( \sum_{i,j} |v_{ij}|^2 \right)^{\frac{1}{2}}.
\]

**Proof of Lemma 2.1.** Since \( u = e^v \), we deduce that

\[
\Delta^2 u = e^v (2|\nabla v|^2 \Delta v + 2 \nabla \Delta v \cdot \nabla v + (\Delta v)^2 + \Delta^2 v + |\nabla v|^4) \\
+ e^v (4 \nabla^2 v(\nabla v, \nabla v) + \Delta |\nabla v|^2).
\]  

(2.2)

From Bochner formula

\[
\Delta |\nabla v|^2 = 2 \|\nabla^2 v\|^2 + 2 \nabla \Delta v \cdot \nabla v.
\]  

(2.3)

Substituting (1.1) and (2.3) into (2.2), it follows that

\[
-\Delta^2 v = 2|\nabla v|^2 \Delta v + 4 \nabla \Delta v \cdot \nabla v + (\Delta v)^2 \\
+ |\nabla v|^4 + 4 \Delta^2 v(\nabla v, \nabla v) + 2 \|\nabla^2 v\|^2 + e^{-(q+1)v}.
\]  

(2.4)

These complete the proof of this lemma.

We define an auxiliary function \( \omega \) by

\[
\omega = -\Delta v + (\alpha - 1)|\nabla v|^2 + \beta e^{-kv},
\]  

(2.5)

where \( \alpha > 0, \beta > 0 \) are constants to be chosen later.

Now we divide the proof of theorem 1.3 into two cases.

Case I. When \( n \geqslant 4 \). We obtain the following lemma:

**Lemma 2.2.** Let \( u \) be a smooth positive solution of the equation (1.1) on a ball \( B_R^q \) centered at \( x_0 \) with radius \( R \) in \( \mathbb{R}^n (n \geqslant 4) \). Assume that two nonnegative constants \( \alpha, \beta \) satisfy (1.11). Then

\[
\Delta \omega \geqslant -2(\alpha + 1) \nabla v \cdot \nabla \omega + \left( 1 + \frac{2\alpha}{n} \right) \omega^2 + \frac{2}{n}(2 - 4\alpha - n)\alpha \omega |\nabla v|^2 \\
+ \frac{2}{n}(1 - 2\alpha)\alpha |\nabla v|^4.
\]  

(2.6)

**Proof of Lemma 2.2.** We write the Laplacian of \( \omega \)

\[
\Delta \omega = -\Delta^2 v + (\alpha - 1)\Delta |\nabla v|^2 + \beta \Delta e^{-kv}.
\]  

(2.7)
Since
\[ \Delta e^{-kv} = -ke^{-kv} \Delta v + k^2 e^{-kv} |\nabla v|^2. \] (2.8)
Combining (2.3), (2.7) and (2.8) and lemma 2.1, we obtain that
\[ \Delta \omega = 2|\nabla v|^2 \Delta v + 2(\alpha + 1) \nabla \Delta v \cdot \nabla v + (\Delta v)^2 + |\nabla v|^4 + 4\nabla^2 v(\nabla v, \nabla v) + 2k^2 e^{-kv} + 2(\alpha + 1) \beta e^{-kv} |\nabla v|^2 - k\beta e^{-kv} \Delta v. \] (2.9)
From (2.5), we rewrite the first term of the right side of (2.9) by
\[ 2|\nabla v|^2 \Delta v = -2\omega |\nabla v|^2 + 2(\alpha - 1) |\nabla v|^2 + 2\beta e^{-kv} |\nabla v|^2. \] (2.10)
the second term by
\[ 2(\alpha + 1) \nabla \Delta v \cdot \nabla v = -2(\alpha + 1) \nabla v \cdot \nabla \omega + 4(\alpha^2 - 1) \nabla^2 v(\nabla v, \nabla v) - 2(\alpha + 1)k\beta e^{-kv} |\nabla v|^2, \] (2.11)
the third term by
\[ (\Delta v)^2 = \omega^2 + (\alpha - 1)^2 |\nabla v|^4 + \beta^2 e^{-2kv} - 2(\alpha - 1) \omega |\nabla v|^2 - 2\beta \omega e^{-kv} + 2(\alpha - 1) \beta e^{-kv} |\nabla v|^2, \] (2.12)
the last term by
\[ -\beta ke^{-kv} \Delta v = \beta k\omega e^{-kv} - (\alpha - 1) \beta ke^{-kv} |\nabla v|^2 - \beta^2 ke^{-2kv}. \] (2.13)
Substituting (2.10)–(2.13) into (2.9), we get that
\[ \Delta \omega = -2(\alpha + 1) \nabla v \cdot \nabla \omega + \omega^2 - 2\alpha \omega |\nabla v|^2 + (k - 2) \beta \omega e^{-kv} + (k^2 - 3\alpha + 1)k + 2\alpha) \beta e^{-kv} |\nabla v|^2 + (1 + \beta^2 - \beta^2 k)e^{-2kv} + 2\alpha |\nabla^2 v| + 4\alpha^2 \nabla^2 v(\nabla v, \nabla v) + \alpha^2 |\nabla v|^4. \] (2.14)
It follows that
\[ \Delta \omega = -2(\alpha + 1) \nabla v \cdot \nabla \omega + \omega^2 - 2\alpha \omega |\nabla v|^2 + (k - 2) \beta \omega e^{-kv} + (k^2 - 3\alpha + 1)k + 2\alpha) \beta e^{-kv} |\nabla v|^2 + (1 + \beta^2 - \beta^2 k)e^{-2kv} + 2\alpha |\nabla^2 v + \alpha \nabla v \otimes \nabla v||^2 + \alpha^2 (1 - 2\alpha) |\nabla v|^4. \] (2.15)
From the Cauchy–Schwarz inequality, we have that
\[ 2\alpha |\nabla^2 v + \alpha \nabla v \otimes \nabla v||^2 \geq \frac{2\alpha}{n}(\Delta v + \alpha |\nabla v|^2)^2 \]
\[ = \frac{2\alpha}{n} \omega^2 - \frac{4\alpha(2\alpha - 1)\omega |\nabla v|^2}{n} + \frac{4\alpha}{n} \beta \omega e^{-kv} \]
\[ + \frac{2\alpha}{n}(2\alpha - 1)^2 |\nabla v|^4 + \frac{4\alpha(2\alpha - 1)}{n} |\beta e^{-kv} |\nabla v|^2 \]
\[ + \frac{2\alpha \beta^2}{n} e^{-2kv}. \] (2.16)
Combining (2.15) and (2.16), we deduce that

\[
\Delta \omega \geq -2(\alpha + 1) \nabla v \cdot \nabla \omega + \left( 1 + \frac{2\alpha}{n} \right) \omega^2 + \left( \frac{4(1 - 2\alpha)}{n} - 2 \right) \omega |\nabla v|^2 \\
+ \left( k - 2 - \frac{4\alpha}{n} \right) \beta \omega e^{-kv} + \left( \frac{2}{n}(1 - 2\alpha)^2 - 2\alpha^2 + \alpha \right) \alpha |\nabla v|^4 \\
+ \left( 1 + \beta^2 - \beta^2 k + \frac{2\alpha \beta^2}{n} \right) e^{-2kv} \\
+ \left( k^2 - (3\alpha + 1)k + 2\alpha + \frac{4\alpha}{n}(2\alpha - 1) \right) \beta e^{-kv} |\nabla v|^2.
\tag{2.17}
\]

That is

\[
\Delta \omega \geq -2(\alpha + 1) \nabla v \cdot \nabla \omega + \left( 1 + \frac{2\alpha}{n} \right) \omega^2 + Q_1 \omega |\nabla v|^2 + Q_2 \beta \omega e^{-kv} \\
+ I_1 |\nabla v|^4 + I_2 e^{-2kv} + I_3 |\nabla v|^2 e^{-kv},
\tag{2.18}
\]

where

\[
I_1 = \frac{2}{n}(1 - 2\alpha)^2 - 2\alpha^2 + \alpha,
I_2 = 1 + \beta^2 - \beta^2 k + \frac{2\alpha \beta^2}{n} \\
= 1 + \frac{2}{n} \alpha \beta^2 - \frac{q - 1}{2} \beta^2,
I_3 = k^2 - (3\alpha + 1)k + 2\alpha + \frac{4\alpha}{n}(2\alpha - 1) \\
= \frac{q - 1}{2} \left( \frac{q + 1}{2} - \alpha \right) - \alpha \left( q - \frac{8\alpha}{n} + \frac{4}{n} \right),
Q_1 = \frac{4(1 - 2\alpha)}{n} - 2,
Q_2 = k - 2 - \frac{4\alpha}{n} = q - \frac{3}{2} - \frac{4\alpha}{n}.
\tag{2.19}
\]

It is noted that the second inequality and the third inequality of (1.11) guarantee respectively \( I_2 \geq 0 \) and \( Q_2 \geq 0 \). We define a constant \( q_0(n) := 3\alpha + \sqrt{9\alpha^2 + (1 - 2\alpha)(1 + 16\alpha/n)}. \) On one hand, by a simple calculation, we show that \( 8\alpha/n + 3 > q_0(n) \) for \( 0 \leq \alpha < 1/2 \). On the other hand, we find that \( q \geq q_0(n) \) if and only if \( I_3 \geq 0 \). Then the third inequality of (1.11) implies \( I_3 > 0 \).

Let \( 0 \leq \alpha < \frac{1}{2} \), we have that

\[
I_1 = \frac{2}{n}(1 - 2\alpha)^2 - 2\alpha^2 + \alpha \\
= \left( \frac{8}{n} - 2 \right) \alpha^2 + \left( 1 - \frac{8}{n} \right) \alpha + \frac{2}{n} \geq \frac{4 - n}{n} \alpha + \left( 1 - \frac{8}{n} \right) \alpha + \frac{2}{n} \\
= -\frac{4}{n} \alpha + \frac{2}{n}.
\tag{2.20}
\]
Therefore combined with (2.18) and (2.20), these complete the proof of lemma 2.2.

Proof of theorem 1.3 in the case I.

We choose a \( C^2 \) cut-off function \( 0 \leq \eta = \eta(t) \leq 1 \) on \([0, +\infty)\), which is defined as follows:

\[
\eta(t) = \begin{cases} 
1, & t \in [0, 1], \\
> 0, & t \in (1, 2), \\
0, & t \in [2, +\infty).
\end{cases}
\]  

(2.21)
such that it satisfies that for some positive constant \( C \),

\[
\left( \frac{\eta'}{\eta^{1/2}} \right)^2 \leq C, \quad |\eta''| \leq C.
\]  

(2.22)

Let \( \rho(x) \) be the distance function from \( x_0 \), for \( R > 1 \), we define

\[
\varphi = \eta \left( \frac{\rho(x)}{R} \right),
\]  

(2.23)

and

\[
\omega_R = \varphi \omega.
\]  

(2.24)

Then we have

\[
\frac{|\nabla \varphi|^2}{\varphi} = \frac{|\nabla \eta|^2}{R^2 \eta} \leq \frac{C}{R^2}, \quad |\Delta \varphi| \leq \frac{C}{R^2}.
\]  

(2.25)

If \( \omega(x) \leq 0 \) for \( \forall x \in \mathbb{R}^n \), then it is nothing to prove. Suppose that

\[
M = \sup_{B_R} \omega > 0.
\]

Noting that \( \omega_R = 0 \) if \( \rho(x) > 2R \), then there exists \( x_R \in B_{2R} \) such that

\[
M_R = \max_{B_{2R}} \omega_R = \omega_R(x_R).
\]  

(2.26)

Since \( M > 0 \), then \( M_R > 0 \) for \( R > 1 \). According to the necessary conditions of the local maximum, we see that

\[
\begin{align*}
\nabla \omega_R &= 0, \\
\Delta \omega_R &\leq 0, \quad \text{at } x = x_R.
\end{align*}
\]  

(2.27)

This implies that

\[
\nabla \omega = -\frac{\nabla \varphi}{\varphi} \omega,
\]  

(2.28)

and

\[
0 \geq \varphi \Delta \omega + 2\nabla \varphi \cdot \nabla \omega + \omega \Delta \varphi = \varphi \Delta \omega - 2 \frac{|\nabla \varphi|^2}{\varphi} \omega + \omega \Delta \varphi,
\]  

(2.29)
at $x = x_R$. Substituting (2.25) into (2.29), we obtain that
\[ \frac{C}{R^2} \omega \geq \varphi \Delta \omega, \]  
(2.30)
at $x = x_R$. Using lemma 2.2, we see that, at $x = x_R$
\[ \frac{C}{R^2} \omega \geq -2(\alpha + 1)\varphi \nabla v \cdot \nabla \omega + \left(1 + \frac{2\alpha}{n}\right)\varphi \omega^2 \]
\[ + \frac{2}{n}(2 - 4\alpha - n)\alpha \varphi \omega |\nabla v|^2 + \frac{2}{n}(1 - 2\alpha)\alpha \varphi |\nabla v|^4. \]  
(2.31)
Combining (2.25) and (2.28), Cauchy–Schwarz inequality and Young’s inequality, we deduce that, at $x = x_R$
\[ -2(\alpha + 1)\varphi \nabla v \cdot \nabla \omega = 2(\alpha + 1)\omega \nabla \varphi \cdot \nabla v \]
\[ \geq -2(\alpha + 1)\omega |\nabla \varphi||\nabla v| \]
\[ \geq -\frac{1}{\varepsilon_1}(\alpha + 1)^2\omega \frac{|\nabla \varphi|^2}{\varphi} - \varepsilon_1 \varphi \omega |\nabla v|^2 \]
\[ \geq -\frac{C}{R^2\varepsilon_1}(\alpha + 1)^2\omega - \varepsilon_1 \varphi \omega |\nabla v|^2, \]  
(2.32)
here $\varepsilon_1 > 0$ is some number to be determined later. Substituting (2.32) into (2.31) and take $\varepsilon_1 = \frac{4\alpha(1 - 2\alpha)}{n}$, we see that
\[ \frac{C}{R^2} \omega \geq \left(1 + \frac{2\alpha}{n}\right)\varphi \omega^2 - 2\alpha \varphi \omega |\nabla v|^2 + \left(\frac{2}{n} - \frac{4\alpha}{n}\right)\alpha \varphi |\nabla v|^4, \]  
(2.33)
at $x = x_R$. Applying Young’s inequality, we have that
\[-2\alpha \varphi \omega |\nabla v|^2 \geq -\frac{\alpha}{\varepsilon_2} \varphi \omega^2 - \varepsilon_2 \alpha \varphi |\nabla v|^4. \]  
(2.34)
here $\varepsilon_2 > 0$ is some number to be determined later. Substituting (2.34) into (2.33) and take $\varepsilon_2 = \frac{2(1 - 2\alpha)}{n}$, we obtain that, at $x = x_R$
\[ \frac{C}{R^2} \omega \geq \left(1 + \frac{2\alpha}{n} - \frac{\alpha n}{2(1 - 2\alpha)}\right)\varphi \omega^2. \]  
(2.35)
In order to guarantee $1 + \frac{2\alpha}{n} > \frac{\alpha n}{2(1 - 2\alpha)}$, we need that
\[ 8\alpha^2 + (n^2 + 4n - 4)\alpha - 2n < 0. \]  
(2.36)
We can obtain the two real roots of the equation $8\alpha^2 + (n^2 + 4n - 4)\alpha - 2n = 0$ with
\[ \alpha_1 = -\frac{(n^2 + 4n - 4) - \sqrt{(n^2 + 4n - 4)^2 + 64n}}{16} < 0, \]  
(2.37)
and
\[ 0 < \alpha_2 = -\frac{(n^2 + 4n - 4) + \sqrt{(n^2 + 4n - 4)^2 + 64n}}{16} < \frac{1}{2}. \]  
(2.38)
Choosing $0 \leq \alpha < \alpha_2$, from (2.35), we obtain that
\[
\omega = \varphi \omega \leq \max_{B_R} \varphi \omega \leq \frac{C}{R^2}, \quad \text{in } B_R.
\]  
(2.39)
where $C = C(\alpha, n)$ is a positive constant. Combining (2.5) and (2.39), we know that
\[
-\frac{\Delta u}{u} + \alpha \frac{|\nabla u|^2}{u^2} + \beta u^{-\frac{\gamma+1}{\gamma}} \leq \frac{C}{R^2}, \quad \text{in } B_R.
\]  
(2.40)
These complete the proof of the case I of theorem 1.3.

Case II. When $n = 3$. From (2.18), the following lemma it is obvious to obtain.

Lemma 2.3. Let $u$ be a smooth positive solution of the equation (1.1) on a ball $B_R$ centered at $x_0$ with radius $R$ in $\mathbb{R}^3$. Assume that two nonnegative constants $\alpha$, $\beta$ satisfy (1.11). Then
\[
\Delta \omega \geq -2(\alpha + 1)\varphi \nabla v \cdot \nabla \omega + \left(1 + \frac{2\alpha}{3}\right)\omega^2 - \frac{2}{3}(1 + 4\alpha)\alpha \varphi \omega |\nabla v|^2
\]
\[+ \frac{1}{3}(2\alpha^2 - 5\alpha + 2)\alpha |\nabla v|^4.
\]  
(2.41)
Proof of theorem 1.3 in the case II. From (2.30) and lemma 2.3, we see that
\[
\frac{C}{R^2} \omega \geq -2(\alpha + 1)\varphi \nabla v \cdot \nabla \omega + \left(1 + \frac{2\alpha}{3}\right)\omega^2 - \frac{2}{3}(1 + 4\alpha)\alpha \varphi \omega |\nabla v|^2
\]
\[+ \frac{1}{3}(2\alpha^2 - 5\alpha + 2)\alpha \varphi |\nabla v|^4,
\]  
(2.42)
at $x = x_R$. Substituting (2.32) into (2.42) and take $\epsilon_3 = \frac{2\alpha(1+4\alpha)}{3}$, we have that, at $x = x_R$
\[
\frac{C}{R^2} \omega \geq \left(1 + \frac{2\alpha}{3}\right)\omega^2 - \frac{4}{3}(1 + 4\alpha)\alpha \varphi \omega |\nabla v|^2 + \frac{1}{3}(2\alpha^2 - 5\alpha + 2)\alpha \varphi |\nabla v|^4.
\]  
(2.43)
Applying Young’s inequality, we see that, at $x = x_R$
\[
-\frac{4}{3}(1 + 4\alpha)\alpha \varphi \omega |\nabla v|^2 \geq -\frac{4\alpha(1 + 4\alpha)^2}{9\epsilon_3} \omega^2 - \epsilon_3 \alpha \varphi |\nabla v|^4,
\]  
(2.44)
here $\epsilon_3 > 0$ is some number to be determined later. Choose $\epsilon_3 = \frac{2\alpha^2 - 5\alpha + 2}{3}$, substituting (2.44) into (2.43), we obtain that, at $x = x_R$
\[
\frac{C}{R^2} \omega \geq \left(1 + \frac{2\alpha}{3} - \frac{4\alpha(1 + 4\alpha)^2}{3(2\alpha^2 - 5\alpha + 2)}\right)\omega^2.
\]  
(2.45)
We notice that
\[
\frac{4\alpha(1 + 4\alpha)^2}{3(2\alpha^2 - 5\alpha + 2)} = \frac{4\alpha(1 + 4\alpha)^2}{3(1 - 2\alpha)(2 - \alpha)} < \frac{4\alpha(1 + 2)^2}{3(1 - 2\alpha)(2 - \frac{1}{2})} = \frac{8\alpha}{1 - 2\alpha}.
\]  
(2.46)
In order to guarantee \( 1 + \frac{2\alpha}{3} > \frac{4\alpha(1+2\alpha)}{3(2\alpha^2 - 5\alpha + 2)} \), we need that
\[
1 + \frac{2\alpha}{3} \geq \frac{8\alpha}{1 - 2\alpha}.
\] (2.47)
That is
\[
4\alpha^2 + 28\alpha - 3 \leq 0.
\] (2.48)
We can obtain the two real roots of the equation \( 4\alpha^2 + 28\alpha - 3 = 0 \) with
\[
\alpha_3 = \frac{-7 - 2\sqrt{13}}{4} < 0,
\] (2.49)
and
\[
0 < \alpha_4 = \frac{-7 + 2\sqrt{13}}{4} < \frac{1}{2}.
\] (2.50)
Choosing \( 0 \leq \alpha \leq \alpha_4 \), from (2.45), we obtain that
\[
\omega \leq \frac{C}{R^2}, \quad \text{in} \quad B_R,
\] (2.51)
where \( C = C(\alpha) \) is a positive constant. These complete the proof of the case II of theorem 1.3. These complete the proof of theorem 1.3.

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