LIE ALGEBRAS AND THE FOUR COLOR THEOREM

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Abstract. We present a “reasonable” statement about Lie algebras that is equivalent to the Four Color Theorem.

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1. Introduction

Let us start by recalling a well-known construction that associates to any finite dimensional metrized Lie algebra \(L\) a numerical-valued functional \(W_L\) defined on the set of all oriented trivalent graphs \(G\) (that is, trivalent graphs in which every vertex is endowed with a cyclic ordering of the edges emanating from it). This construction underlies the gauge-group dependence of gauge theories in general and of the Chern-Simons topological field theory in particular (see e.g. [B-N1, AS1, AS2]) and plays a prominent role in the theory of finite type (Vassiliev) invariants of knots ([B-N2, B-N3, B-N4]) and most likely also in the theory of finite type invariants of 3-manifolds ([O, GO, R]).

Fix a finite dimensional metrized Lie algebra \(L\) (that is, a finite dimensional Lie algebra with an \(ad\)-invariant symmetric non-degenerate bilinear form), choose some basis \(\{L_a\}_{a=1}^{\dim L}\) of \(L\), let \(t_{ab} = \langle L_a, L_b \rangle\) be the metric tensor, let \(t^{ab}\) be the inverse matrix of \(t_{ab}\), and let \(f_{abc}\) be the structure constants of \(L\) relative to \(\{L_a\}\):

\[f_{abc} = \langle L_a, [L_b, L_c] \rangle.\]

Let \(G\) be some oriented trivalent graph. To define \(W_L\), label all half-edges of \(G\) by symbols from the list \(a, b, c, \ldots, a_1, b_1, \ldots\), and sum over \(a, b, \ldots, a_1, \ldots \in \{1, \ldots, \dim L\}\) the product over the vertices of \(G\) of the structure constants “seen” around each vertex times the product over the edges of the \(t\)'s seen on each edge. This definition is much better explained by an example, as in figure 1.

By introducing an explicit change-of-basis matrix as in [B-N2] or by re-interpreting \(W_L(G)\) in terms of abstract tensor calculus as in [B-N3], one can verify that \(W_L(G)\) does not depend on the choice of the basis \(\{L_a\}\). Typically one chooses a “nice” orthonormal (or almost...
reducing the number of summands in the definition of $W$ metric to be the matrix trace in the defining representation: 

$$\langle N \rangle$$

is a polynomial in $N$ (proof in section 2). If a connected (orthonormal) basis $\{L_a\}$, so that most of the constants $t^{ab}$ and $f_{abc}$ vanish, thus greatly reducing the number of summands in the definition of $W_L(G)$.

Unless otherwise stated, whenever dealing with a Lie algebra of matrices, we will take the metric to be the matrix trace in the defining representation: $\langle L_a, L_b \rangle = \text{tr}(L_a L_b)$.

**Lemma-Definition 1.1.** (proof in section 2) If a connected $G$ has $v$ vertices, then $W_{sl(N)}(G)$ is a polynomial in $N$ of degree at most $\frac{v}{2} + 2$ in $N$. Thus we can set $W_{sl(N)}^{top}(G)$ to be the coefficient of $N^{\frac{v}{2}+2}$ in $W_{sl(N)}(G)$.

The following statement sounds rather reasonable; it just says that if $G$ is “$sl(2)$-trivial”, then it is at least “$sl(N)$-degenerate”. For us who grew up thinking that all that there is to learn about $sl(N)$ is already in $sl(2)$, this is not a big surprise:

**Statement 1.** For a connected oriented trivalent graph $G$, $W_{sl(2)}(G) = 0$ implies $W_{sl(N)}^{top}(G) = 0$.

Lie-theoretically, there is much to say about $sl(2)$ and $sl(N)$. There are representations of $sl(2)$ into $sl(N)$, there is an “almost decomposition” of $sl(N)$ into a product of $sl(2)$’s, and there are many other similarities. A-priori, the above statement sounds within reach. The purpose of this note is to explain why statement [1] is equivalent to the Four Color Theorem [2].

This equivalence follows from the following two propositions, proven in sections 2 and 3, respectively:

**Proposition 1.2.** Let $G$ be a connected oriented trivalent graph. If $G$ is 2-connected, $|W_{sl(N)}^{top}(G)|$ is is equal to the number of embeddings of $G$ in an oriented sphere. Otherwise, $W_{sl(N)}^{top}(G) = 0$.

**Proposition 1.3.** (Penrose [3]. See also [Ka1, Ka2, KS].) If $G$ is planar with $v$ vertices and $G^c$ is the map defined by its complement, than $|W_{sl(2)}(G)|$ is $2^{\frac{v}{2}-2}$ times the total number of colorings of $G^c$ with four colors so that adjacent states are colored with different colors.

1 See [B-NG] for a similar context in which the different $sl(2)$’s “decouple”.

2 The Four Color Theorem was conjectured by Francis Guthrie in 1852 and proven by K. I. Appel and W. Haken [AH] in 1976. See also [SK].
Indeed, statement 1 is clearly equivalent to
\[|W_{\text{top}}^{\text{sl}(N)}(G)| \neq 0 \implies |W_{\text{sl}(2)}(G)| \neq 0,\]
which by propositions 1.2 and 1.3 is the same as saying
\[
\left(G \text{ has a planar embedding} \quad \text{with } G^c \text{ a map}\right) \implies (G^c \text{ has a 4-coloring}).
\]
Notice that if \(G\) is connected, \(G^c\) is a map (does not have states that border themselves) iff \(G\) is 2-connected.

**Remark 1.4.** We’ve chosen the formulation of statement 1 that we felt was the most appealing. With no change to the end result, one can replace \(\text{sl}(N) = A_{N-1}\) by \(B_N, C_N, D_N,\) or \(\text{gl}(N)\) and \(\text{sl}(2)\) by \(\text{so}(3)\) in the formulation of statement 1. In fact, in the proofs we actually work with \(\text{gl}(N)\) and \(\text{so}(3)\) rather than with \(\text{sl}(N)\) and \(\text{sl}(2)\).

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2. **Understanding \(W_{\text{sl}(N)}\)**

As Lie algebras, \(\text{gl}(N)\) is just \(\text{sl}(N)\) plus an Abelian factor. As Abelian Lie algebras have vanishing structure constants, \(W_{\text{sl}(N)}(G) = W_{\text{gl}(N)}(G)\) for any oriented trivalent graph \(G\). So let us concentrate on computing \(W_{\text{gl}(N)}(G)\) for such \(G\).

For the basis of \(\text{gl}(N)\), we pick the matrices \(\{L_a\}_{a=1}^{N^2}\) where \(L_{ij}\) is the matrix with 1 in the \(ij\) entry and 0 everywhere else. As the basis is indexed by a double index rather than by a single index, it is convenient to label every half-edge of \(G\) by two symbols from the list \(i, j, \ldots, i_1, \ldots\) and double all the edges:

\[
\begin{align*}
\begin{array}{c}
  a \\
  b
\end{array} & \quad \rightarrow \quad 
\begin{array}{c}
  j \\
  k
\end{array} 
\end{align*}
\]

The metric \(t_{ab} = t^{(ij)(kl)}\) of \(\text{gl}(N)\) is given by \(t^{(ij)(kl)} = \text{tr} L_{ij} L_{kl} = \delta_{jk} \delta_{il}\), and its inverse is given by the same formula:

\[
t^{(ij)(kl)} = \delta_{jk} \delta_{il}.
\]

This formula means that in the summation defining \(W_{\text{gl}(N)}(G)\) we can assume the equalities \(j = k\) and \(l = i\) along each edge as in (4). In other words, it is enough to label every doubled edge with just one pair of indices, getting an overall picture like

\[
\sum_{i,j,...,u=1}^{N} f^{(ij)(kl)(mn)} f^{(ji)(rs)(ut)} f^{(lk)(tu)(qp)} f^{(nm)(pq)(sr)},
\]

where \(f^{(ij)(kl)(mn)}\) are the structure constants in our basis:

\[
\begin{align*}
\begin{array}{c}
  i \\
  j
\end{array} & \quad = \quad 
\begin{array}{c}
  k \\
  l
\end{array} 
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
  m \\
  n
\end{array} & \quad = \quad 
\begin{array}{c}
  i \\
  j
\end{array} = \delta_{jk} \delta_{lm} - \delta_{nk} \delta_{il} \delta_{jm} = \delta_{jk} \delta_{lm} - \delta_{nk} \delta_{il} \delta_{jm}.
\end{align*}
\]
In the last equation, indices connected by a line can be assumed to be equal in the summation defining $W_{gl(N)}(G)$. Once the edges and vertices of $G$ are “thickened” as in (1) and (2), the summation over $i, j, \ldots$ becomes the counting of the number of solutions of the equalities determined by the connected components of the thickened $G$. This number is simply $N$ raised to the number of connected components:

\[
\sum_{i,j,\ldots,u=1}^{N} \delta_{il} \delta_{lp} \cdots = \# \left\{ 1 \leq i,j,\ldots \leq N : \begin{array}{c}
i = l = p = m = j = t = k = n = r \\
u = q = s
\end{array} \right\} = N^2.
\]

Summarizing, we find the formula\(^3\)

\[
W_{gl(N)}(G) = \sum_{\text{markings } M \text{ of } G} \text{sign}(M) N^{b(T_M)},
\]

where:

- A markings $M$ of $G$ is a marking of each vertex of $G$ by a sign in $\{+, -\}$, and $\text{sign}(M)$ is the product of these signs.
- The thickening $T_M$ corresponding to a marking $M$ is the oriented surface with boundary obtained from $G$ as follows:
  - Replace the vertices marked by a “+” with “joints” and the vertices marked by a “−” with “twisted joints” as in (2).
  - Orient these surface pieces using the following black($b$)/white($w$) convention for the thickening of vertices:
    - “+”-vertices are embedded in the thickening of $G$ so that they are seen as oriented counterclockwise from the white side of the thickening, while “−”-vertices are seen as oriented clockwise from the white side of the thickening.
  - Finally, connect the joints together along the edges of $G$ by bands, in the only way consistent with the orientations of the joints.
- $b(T_M)$ is the number of boundary components of $T_M$.

If $M$ is a marking of $G$, let $S_M$ be the closed oriented surface obtained by gluing a disk into each boundary component of the thickening $T_M$. With $\chi$ denoting Euler characteristic and $g$ denoting genus, we have

\[
2 - 2g(S_M) = \chi(S_M) = \chi(T_M) + b(T_M) = \chi(G) + b(T_M).
\]

Remembering that $G$ is trivalent and thus $\chi(G) = -\frac{v}{2}$, we get

\[
b(T_M) = -\chi(G) + 2 - 2g(S_M) = \frac{v}{2} + 2 - 2g(S_M).
\]

\(^3\)Compare with [B-N3, equation (36)]; for similar formulas in the cases of $so(N)$ and $sp(N)$, see [B-N3, equation (33)] and [E-N3, exercise 6.37].
Thus $b(T_M)$ is maximal when $g(S_M) = 0$ and in that case $b(T_M) = \frac{v}{2} + 2$. With (8), this proves lemma [11]. Furthermore, calling a marking $M$ spherical when $S_M$ is a sphere, we get the formula

$$W_{\text{top}}^{sl(N)}(G) = W_{\text{top}}^{gl(N)}(G) = \sum_{\text{spherical markings } M \text{ of } G} \text{sign}(M).$$

**Proof of proposition 1.2.** Let $G$ be 2-connected, and consider the map

$$\Theta : \{\text{spherical markings } M \text{ of } G\} \longrightarrow \{\text{embeddings of } G \text{ in an oriented sphere}\}$$

defined by mapping $M$ to the natural embedding of $G$ in $S_M$. It is clear that $\Theta$ is a bijection. Indeed, if an embedding of $G$ in an oriented sphere $S^2$ is given, one can reconstruct $M$ by marking the vertices that are oriented counterclockwise within $S_M$ (as seen from the white side of $S_M$) by a “+” sign, and marking all other vertices by a “−”.

To conclude the proof, it is enough to show that for spherical markings, $\text{sign}(M)$ is independent of the marking. Clearly, $\text{sign}(M)$ is also equal to $(-1)^{v_-}$, where $v_-$ is the number of vertices of $G$ that are embedded clockwise in $S_M$ by $\Theta(M)$ (with $S_M$ viewed from its white side). By a theorem of H. Whitney [W1, W2],4 one can get from any spherical embedding of a trivalent 2-connected graph $G$ to any other such embedding by a sequence of flips as in figure 2. Such flips do not change the parity of $v_-$, since the number of vertices that is flipped is even.

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**Figure 2.** A flip takes a part of a graph that connects to the rest via only two edges, and flips it over.

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3. UNDERSTANDING $W_{sl(2)}$

Proposition 1.3 is due to Penrose [P] (see also [Ka1, Ka2, KS]). For completeness, we reproduce its proof in this section.

As Lie algebras, $sl(2)$ is isomorphic to $so(3)$, so let us work with $so(3)$ instead. The standard basis of $so(3)$ is given by the matrices

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let us pick the scalar product $\langle \cdot, \cdot \rangle$ on $so(3)$ to be the one that makes this basis orthonormal, and let us denote the corresponding functional on graphs by $\overset{\sim}{W}_{so(3)}$, with the “~” on top of the $so(3)$ to remind us that we are not using the standard matrix-trace scalar product.

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4Check [L, corollary 6] for a version closer to what we need, and remember that our graph is 2-connected and hence some of the moves in [L] are irrelevant for us.
One can easily verify that $\frac{1}{2}\langle \cdot, \cdot \rangle$ is the scalar product induced on $so(3)$ from matrix-trace in $sl(2)$. Thus, remembering that in the construction of $W_L$ vertices scale with the scalar product and edges scale with its inverse, we find that

$$W_{sl(2)}(G) = \left(\frac{1}{2}\right)^{v-e} W_{so(3)}(G) = 2\pi W_{so(3)}(G),$$

where $G$ is an oriented trivalent graph with $v$ vertices and $e$ edges. Let us fix such a $G$ once and for all, and let us assume that it is planar and that all the vertices of $G$ are oriented counterclockwise in the plane. Flipping the orientation of any given vertex just reverses the sign of $W_{so(3)}(G)$, and so the latter assumption does not limit the generality of our arguments.

Proof of proposition 1.3. With (1) in mind, proposition 1.3 clearly follows from the following two lemmas.

Lemma 3.1. (Penrose [P]. See also [Ka1, Ka2, KS].) For a planar $G$ as above, $|W_{so(3)}(G)|$ is the number of colorings of the edges of $G$ with three colors $\{1, 2, 3\}$, so that the edges emanating from any single vertex are of different colors.

Lemma 3.2. (Tait’s theorem [T]) Edge-3-colorings as in the previous lemma are in a bijective correspondence with 4-colorings of the map $G^c$ that fix the color of the “state at infinity”.

Proof of lemma 3.1. In the basis $\{L_a\}$, the structure constants of $so(3)$ are given by

$$f_{abc} = \epsilon_{abc} = \begin{cases} \text{sign}(abc) & \text{if } abc \text{ is a permutation}, \\ 0 & \text{otherwise}. \end{cases}$$

Remembering also that $\{L_a\}$ is orthonormal by choice, the computation of $W_{so(3)}(G)$ is given (on a simple example) by:

$$W_{so(3)}(G) = \prod_{\text{edge-3-colorings of } G} \prod_{\text{a sign per vertex}},$$

where the sign at each vertex is the sign of the permutation of $\{1, 2, 3\}$ induced by an edge-3-coloring, as read counterclockwise around the vertex.

The $\epsilon$ symbols force the indices coming into each vertex to be different, and hence clearly

$$W_{so(3)}(G) = \sum_{\text{edge-3-colorings of } G} \prod_{\text{a sign per vertex}},$$

where the sign at each vertex is the sign of the permutation of $\{1, 2, 3\}$ induced by an edge-3-coloring, as read counterclockwise around the vertex.

The only thing left to show is that the product of signs in (5) is independent of the edge-3-coloring. A clever way to do that, discovered by L. H. Kauffman, is to replace every edge colored by a “3” by a pair of edges colored “1” and “2” (in symbols, $1 + 2 = 3$). This defines two families of circles in the plane, labeled by “1” and by “2” (see figure 3). By lumping together the signs on each end of a “3” edge and taking the product over all of those edges, one sees that the overall sign depends only on the parity of the number of “3” edges (always
\( \mathbb{Z}/2\mathbb{Z} \) intersection number of the “1” family of circles with the “2” family of circles. By the Jordan curve theorem, the latter is always 0.

\[ \xrightarrow{\text{Figure 3. The two families of circles obtained by splitting every “3” edge.}} \]

**Proof of lemma [3.3].** This is a well known result (see e.g. [BM, theorem 9.12]), so let us only sketch the proof. Consider the group \( H = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \). Given any 4-coloring of \( G^c \) by elements of \( H \), one may associate to it an edge-3-coloring of \( G \) by the non-zero elements of \( H \), by coloring every edge by the difference of the colors in the two faces adjacent to it. One then verifies that this edge 3-coloring is well defined and that we get a bijection between the set of 4-colorings of \( G^c \) that color the state at infinity with 0 and the set of edge-3-coloring of \( G \).

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