FUNDAMENTAL GROUPS OF COMPLEMENTS OF BRANCH CURVES AS SOLVABLE GROUPS

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Abstract. In this paper we show that fundamental groups of complements of curves are "small" in the sense that they are "almost solvable". Thus we can start to compute \( \pi_2 \) as a module over \( \pi_1 \) in order to produce new invariants of surfaces that might distinguish different components of a moduli space.

0. Applications of the calculations of fundamental groups to algebraic surfaces.

Our study of fundamental groups of branch curves is aimed towards understanding algebraic surfaces.

Algebraic surfaces are classified by discrete and continuous invariants. Fixing the discrete invariants (of homotopy type), one gets a family of algebraic surfaces parametrized by an algebraic variety which is called the moduli space (the word "moduli" stands for "continuous invariants"). Very little is known about the structure of moduli space of surfaces of general type.

We study algebraic surfaces in order to understand the structure of their moduli spaces. We intend to construct new invariants that will distinguish between different connected components of moduli space. We study invariants that come from pluricanonical embeddings, the corresponding generic projections to \( \mathbb{CP}^2 \) and branch curves. More specifically, we investigate the fundamental groups of the complements in \( \mathbb{CP}^2 \) of those branch curves.

Let \( V \) be an algebraic surface of general type pluricanonically embedded in \( \mathbb{CP}^N \), \( \pi: V \to \mathbb{CP}^2 \) be a generic projection, \( S_V (\subset \mathbb{CP}^2) \) be the corresponding branch curve. The topological invariants of \( \mathbb{CP}^2 - S_V \) do not change when the complex structure of \( V \) is changes continuously (for more details see the introduction of [MoTe6]). Thus, one can use such invariants to distinguish the connected components of the corresponding moduli space.

Artin's braid groups \( B_n \) are our most important tool. In recent years these groups are becoming more and more popular in different branches of mathematics, such as the theory of Jones polynomials (which is a current interest of theoretical physicists). Our former study shows that some nontrivial properties of braid groups are intimately connected with the existence of nontrivial geometrical objects of complex geometry and algebraic surfaces. Thus, our study of algebraic surfaces and their branch curves gives new insight in the field of braid groups and vice versa.

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The fact that algebraic surfaces are nontrivial geometric objects was remarkably confirmed by S. Donaldson who showed that among algebraic surfaces one can find homeomorphic non-diffeomorphic (simply-connected) 4-manifolds. In particular, he produced the first counterexamples to the h-cobordism conjecture in dimension four. Donaldson’s theory was also used to construct the first examples of homeomorphic non-diffeomorphic (simply-connected) algebraic surfaces of general type ([FMoM], [Mo4]).

We expect that the connected components of moduli spaces of algebraic surfaces (of general type) correspond to the principal diffeomorphism classes of corresponding topological 4-manifolds. Thus, it is possible that Donaldson’s polynomial invariants will distinguish these connected components. However, the definition of Donaldson’s invariants seems to be too “transcendental” for direct computation. We believe that a more direct geometrical approach must be applied.

An algebraic surface can be considered as a “Riemann surface” of an algebraic function of two variables, that is, as a finite ramified covering of $\mathbb{CP}^2$. For classification problems we can restrict ourselves to the so-called stable case, that is, to the case when the corresponding branch curve $S$ in $\mathbb{CP}^2$ is cuspidal (having only nodes and cusps as singularities). In such an approach, the most important thing is to study the topology of the complement $\mathbb{CP}^2 - S$, in particular $\pi_1(\mathbb{CP}^2 - S, *)$ and $\pi_2(\mathbb{CP}^2 - S)$ as $\pi_1$-module, etc.

The first results on such $\pi_1(\mathbb{CP}^2 - S, *)$ were obtained by O. Zariski in the thirties. Let $X_n$ be a non-singular hypersurface in $\mathbb{CP}^3$ of deg $n$, $X_{ab}$ be a projective embedding of $\mathbb{CP}^1 \times \mathbb{CP}^1$ corresponding to $a\ell_1 + b\ell_2$, where $a, b$ are positive integers ($\ell_1 = pt \times \mathbb{CP}^1$, $\ell_2 = pt \times \mathbb{CP}^1$). Let $S_n$ (resp. $S_{ab}$) be the branch curve in $\mathbb{CP}^2$ corresponding to a generic projection of $X_n$ (resp. $X_{ab}$) to $\mathbb{CP}^2$. Zariski proved that $\pi_1(\mathbb{CP}^2 - S_3, *) \cong \mathbb{Z}/2 \ast \mathbb{Z}/3$, and $\pi_1(\mathbb{CP}^2 - S_{16}) \cong$ braid group of $S^2$ with $2b$ strings. In 1981 B. Moishezon generalized Zariski’s result to $X_n$, proving that $\pi_1(\mathbb{CP}^2 - S_n, *) \cong B_n/\text{center}$.

These results were almost the only ones known about $\pi_1(\mathbb{CP}^2 - S)$’s (in the stable case) till the middle of the eighties. They gave the impression that these groups are very big (in particular, contain large free groups). Such an impression was partly responsible for the general belief in the following conjecture: The Galois coverings corresponding to generic projections of algebraic surfaces to $\mathbb{CP}^2$ have (as a rule) infinite fundamental groups. This was a partial case of Bogomolov’s conjecture which stated that algebraic surfaces of general type with positive index have infinite fundamental groups. This is equivalent to the following: for any simply-connected algebraic surface of general type the Chern numbers satisfy the inequality $C_1^2 < 2C_2$.

In 1984 we disproved this conjecture ([MoTe1]), showing that Galois coverings $\bar{X}_{ab}$ corresponding to $X_{ab} \to \mathbb{CP}^2$ (generic projection) have finite fundamental groups. One can check that $\pi_1(\bar{X}_{ab}) \cong \ker(\pi_1(\mathbb{CP}^2 - S_{ab})/\langle \Gamma^2 \rangle) \xrightarrow{\psi} S_k$ ($\langle \Gamma^2 \rangle$ denotes the normal subgroup generated by squares of “geometric generators”), $\psi$ the standard monodromy homomorphism to the symmetric group $S_k$, $k = \deg X_{ab}$). We proved that $\ker \psi$ is a finite abelian group. In 1985 we computed $\ker \Psi$ explicitly as an $S_k$-module. These results gave the first sign that, in general, the fundamental groups $\pi_1(\mathbb{CP}^2 - S, *)$ are not as big and complicated as the earlier theorems of Zariski et al. (see above) indicated.

With all the above in mind, one can start to attack the general problem of explic-
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Itly computing \( \pi_1(\mathbb{C}P^2 - S_V, \ast) \) for many classes of algebraic surfaces. Our belief is that in the case of simply-connected surfaces, the kernels
\[
\ker(\pi_1(\mathbb{C}P^2 - S_V, \ast) \xrightarrow{\psi} \text{symmetric group}) \quad \Psi \quad \text{the standard monodromy homomorphism}
\]
are very often solvable groups.

We start by considering here the branch curve of a Veronese surface. Completing the computation of \( \pi_1 \) will allow us to start the investigation of \( \pi_2(\mathbb{C}P^2 - S_V) \) (as \( \pi_1 \)-module) which is our next objective.

For \( V \) be a projective algebraic surface, \( S_V \) the branch curve corresponding to a generic projection of \( V \) to \( \mathbb{C}P^2 \). \( S_V \) is a cuspidal curve. Our method of studying \( \mathbb{C}P^2 - S_V \) is based on the explicit formulas for braid monodromy. In our papers [MoTe5], [MoTe6] we developed algorithms and methods to obtain such formulas for many interesting geometrical cases. These formulas contain nontrivial information which connects some properties of braid groups to the existence of cuspidal branch curves and the corresponding algebraic surfaces. The classical Van Kampen theorem gives an algorithm for deducing a finite presentation of \( \pi_1(\mathbb{C}P^2 - S_V, \ast) \) from the explicit knowledge of the braid monodromy. At first glance, this presentation looks hopeless. We manage to work with such presentations, since we have discovered certain symmetry properties in the braid monodromies (we call them “invariance properties”). These properties sharply distinguish our subject from its real analog – classical knot theory, where finite presentations of a knot group usually cannot be simplified.

We shall also use certain new quotients of braid groups and related algebraic objects described in [MoTe9] which are important ingredients in the final description of fundamental groups \( \pi_1(\mathbb{C}^2 - S_V, \ast) \), as in [MoTe9] and of \( \pi_4(\mathbb{C}P^2 - S_V, \ast) \) in this paper. To get the results in [MoTe9] we also used degenerations of \( V \) into a union of simple rational surfaces which give degenerations of the corresponding branch curves into configurations of simple curves (lines and conics), (see [MoTe7]). Such degenerations make it possible to move from the local analysis of singularities to the global analysis, which is the most difficult part of the subject, and to compute the braid monodromy related to \( S \). Using the braid monodromy, we computed \( \pi_1(\mathbb{C}^2 - S_V, \ast) \) in [MoTe9].

Our conjecture is that for many classes of simply-connected algebraic surfaces the fundamental groups \( \pi_1(\mathbb{C}P^2 - S_V) \) will be extensions of symmetric groups by solvable groups. At the present stage of our knowledge it is difficult to predict the precise answers. One possibility is that certain general theorems will be proved about the structure of the solvable groups in question. Our knowledge of \( \pi_2(\mathbb{C}P^2 - S_V) \)'s is practically non-existent at the present time. That means that it is impossible to even formulate conjectures about them before our computations are completed. Still it is possible that certain general rules define their structure (as \( \pi_1 \)-modules). The possibilities described above, if realized, will demand a study of branch curves of more complicated algebraic surfaces, for example, the non-simply-connected ones. It is known that many abstract groups (in particular, all finite ones) could be fundamental groups of algebraic surfaces.

This paper follows our series of papers, Braid Group Techniques, I – V (referred to also as BGTI – BGTV, [MoTe5], [MoTe6], [MoTe7], [MoTe8], [MoTe9]). In BGTI we gave an algorithm to compute the braid monodromy of different line arrangements in \( \mathbb{C}P^2 \). In BGTII we considered nodal curves and cuspidal branch curves. In BGTIII we started to treat the Veronese surface separately. We constructed
special degenerations of the Veronese surface to the union of planes with which we computed in BGTIV the braid monodromy of the branch curve of a generic projection of the Veronese surface to $\mathbb{CP}^2$. Using the results in BGTIV we computed in BGTV the fundamental group $\pi_1(\mathbb{C}^2 - S_V)$.

In Section 1 of this paper we discuss the Van Kampen Theorem which relates the fundamental group of the complement of a curve to the braid monodromy of the curve. In Section 2 we prove our main result: a solvability type theorem on $\pi_1(\mathbb{C}^2 - S_V)$, and in Section 3 we present the projective version of the affine results.

There exists an epimorphism

\[ \pi_1(\mathbb{C}^2 - S,*) \to \pi_1(\mathbb{C}^2 - S,0), \]

so a set of generators for $\pi_1(\mathbb{C}^2 - S,0)$ determines a set of generators for $\pi_1(\mathbb{C}^2 - S,*)$.

There is a classical theorem of Van Kampen from the 30’s that all relations in $\pi_1(\mathbb{C}^2 - S,*)$ is “almost” a solvable group in the sense that it has a solvable subgroup of finite index. Such groups are “small enough” so that the computation of $\pi_2$ as a model over $\pi_1$ makes sense.

1. The Braid Group and the Van Kampen Theorem.

Let $S$ be a cuspidal curve in $\mathbb{C}^2$, $p = \deg S$, $\mathcal{C}_u = \{(u,y)\}$.

The group $\pi_1(\mathbb{C} - N, u)$ is a free group.

There exists an epimorphism $\pi_1(\mathbb{C}_u - S, u_0) \to \pi_1(\mathbb{C}^2 - S, u_0)$, so a set of generators for $\pi_1(\mathbb{C}_u - S, u_0)$ determines a set of generators for $\pi_1(\mathbb{C}^2 - S, u_0)$.

There is a classical theorem of Van Kampen thefrom the 30’s that all relations in $\pi_1(\mathbb{C}^2 - S, u_0)$ come from the braid group $B_p$ via the braid monodromy $\varphi_u$ of $S$. We shall formulate the theorem precisely in 1.2.

We start with the definition of braid group and a half-twist.

**Definition.** Braid group $B_n[D,K]$: Let $D$ be a closed disc in $\mathbb{R}^2$, $K \subset D$, $K$ finite. Let $B$ be the group of all diffeomorphisms $\beta$ of $D$ such that $\beta(K) = K$, $\beta|_\partial D = \text{Id}|_\partial D$. For $\beta_1, \beta_2 \in B$, we say that $\beta_1$ is equivalent to $\beta_2$ if $\beta_1$ and $\beta_2$ induce the same automorphism of $\pi_1(D - K, u)$. The quotient of $B$ by this equivalence relation is called the braid group $B_n[D,K]$ ($n = \#K$). We sometimes denote by $\beta$ the braid represented by $\beta$. The elements of $B_n[D,K]$ are called braids.

**Definition.** $H(\sigma)$, half-twist defined by $\sigma$: Let $D, K$ be as above. Let $a, b \in K$, $K_{a,b} = K - a - b$ and $\sigma$ be a simple path in $D - \partial D$ connecting $a$ with $b$ s.t. $\sigma \cap K = \{a,b\}$. Choose a small regular neighborhood $U$ of $\sigma$ and an orientation preserving diffeomorphism $f : \mathbb{R}^2 \to \mathbb{C}^1$ ($\mathbb{C}^1$ is taken with the usual “complex” orientation) such that $f(\sigma) = [-1,1]$, $f(U) = \{z \in \mathbb{C} : |z| < 2\}$. Let $a(r), r \geq 0$, be a real smooth monotone function such that $a(0) = 1$ for $r \in [0,\frac{1}{2}]$ and $a(r) = 0$ for $r \geq 2$. Let $H(re^{i\theta}) = re^{i(\theta + a(r))}$, and let $H(\sigma) : D \to D$ be defined by $fHf^{-1}$.

The following lemma is a technical lemma to be used in the simplified Van Kampen theorem.

**Lemma 1.1.** Let $V$ be a half-twist in $B_p[D,K], u_0 \notin K$. Then there exists $A_V, B_V \in \pi_1(D - K, u_0)$ s.t. $A_V, B_V$ can be extended to a g-base of $\pi_1(D - K, u_0)$ and $(A_V)V = B_V$.

**Proof.** [MoTe10], XIII.1.1 of [MoTe5].

We use the existence of $A_V, B_V$ in the formulation of the Van Kampen theorem. In [MoTe10] and [MoTe5] we also introduced an algorithm for expressing $A_V, B_V$ in terms of $\{\Gamma_j\}$.

To formulate the Van Kampen Theorem we consider the following situations and use the following notations:

Let $S$ be a cuspidal curve in $\mathbb{CP}^2$, $p = \deg S$. Let $L$ be a line at infinity,
$\mathbb{C}^2 = \mathbb{C} \mathbb{P}^2 - L$.

Choose coordinates $x, y$ on $\mathbb{C}^2$.

$\pi : \mathbb{C}^2 \to \mathbb{C}$ projection on the first coordinate, $x$-coordinate, $C_x = \pi^{-1}(x)$.

$K(x) = \pi^{-1}(x) \cap S$ (Finitely presented). $N = \{ x \mid \#K(x) \leq p \}$.

Let us choose $u \in \mathbb{C}$, $u$ real s.t. $x \ll u \, \forall \, x \in N$.

Let $B_p = B[C_u, C_u \cap S]$.

Let $\varphi_u : \pi_1(\mathbb{C} - N, u) \to B_p$ the braid monodromy of $S$ with respect to $\pi, u$.

The group $\pi_1(\mathbb{C} - S, u_0)$ is a free group.

$M' = \{ x \in S \mid \pi_1(x) \not\text{ is etale at } x \}$ (Finitely presented).

Assume $\#C_x \cap M' = 1 \, \forall \, x \in N$. For $x \in N$, let $x' = C_x \cap M'$. The point $x'$ is either a branch point, a node, or a cusp.

Let $\{ \delta_i \}$ be a free geometric base ($g$-base) for $\pi_1(\mathbb{C} - N, u)$.

By Theorem VI.3.3 of [MoTe10] (see also [MoTe9]), for every $\delta_i$ there exist $V_i$ and $\nu_i$ where $V_i$ is a half-twist and $\nu_i$ is a number s.t. $\varphi_u(\delta_i) = V_i^{\nu_i}$. Moreover, $\nu_i = 1, 2, 3$ if $c_i$ = a branch point, node or a cusp, respectively.

Let $u_0 \in C_u, u_0 \not\in S, u_0$ below real lines far enough s.t. $B_p$ does not move $u_0$.

$[A, B] = ABA^{-1}B^{-1}$.

$\langle A, B \rangle = ABAB^{-1}A^{-1}B^{-1}$.

**Theorem 1.2 (Van Kampen Theorem).**

Let $S$ be cuspidal curve in $\mathbb{C}^2$.

Let $u, u_0, \varphi_u, A_V, B_V$ be as above:

Let $\{ \delta_i \}$ be a $g$-base of $\pi_1(\mathbb{C} - N, u)$.

Let $\varphi_u(\delta_i) = V_i^{(\nu_i)} V_i$ a half-twist. $\nu_i = 1, 2, 3$ (as above).

Let $\{ \Gamma_j \}_{j=1}^p$ be a $g$-base for $\pi_1(\mathbb{C} - S, u_0)$. Then $\pi_1 = \pi_1(\mathbb{C}^2 - S, u_0)$ is generated by the images of $\Gamma_j$ in $\pi_1$ and we get a complete set of relations from those induced from $\varphi_u(\delta_i) = V_i^{\nu_i}$, as follows: $A_{V_i} = B_{V_i}$ when $\nu_i = 1$, $[A_{V_i}, B_{V_i}] = 1$ when $\nu_i = 2$, $\langle A_{V_i}, B_{V_i} \rangle = 1$ when $\nu_i = 3$, and $A_{V_i}, B_{V_i}$ are expressed in terms of $\{ \Gamma_j \}$.

**Proof.** (See [VK1], [Z1]).

We shall also quote here an equivalent form of this theorem.

Let $z_{12} = \text{the \"shortest\" path connecting } q_1 \text{ and } q_2$. Let $Z = H(z_{12})$. Since every half-twist of $B_p$ is conjugate to $Z$ (see Proposition VI.4.4, [MoTe10] or [MoTe5]), we get $\rho_u(\delta_i) = Q_i^{-1}Z^{\nu_i}Q_i$ for some $Q_i \in B_p$.

**Theorem 1.3.** (The original version of the Van Kampen Theorem).

Let $S, u, u_0, \{ \delta_i \}, Q_i$ be as above.

Let $\{ \Gamma_j \}_{j=1}^p$ be a $g$-system of generators for $F = \pi_1(\mathbb{C} - S, u_0)$. Then $\pi_1 = \pi_1(\mathbb{C}^2 - S, u_0)$ is finitely presented by the images of $\Gamma_j$ in $\pi_1$ and the relations which are images of: $Q_i(\Gamma_1) = Q_i(\Gamma_2)$ when $\nu_i = 1$, $Q_i(\Gamma_1)Q_i(\Gamma_2) = Q_i(\Gamma_2)Q_i(\Gamma_1)$ when $\nu_i = 2$, and $Q_i(\Gamma_1)Q_i(\Gamma_2)Q_i(\Gamma_1) = Q_i(\Gamma_2)Q_i(\Gamma_1)Q_i(\Gamma_2)$ when $\nu_i = 3$, for each $c_i \in N$.

It is easy to see that $((\Gamma_1)H(z_{12}) = \Gamma_2$ (thus for $Z = H(z_{12})$, we have $\Gamma_1 = A_Z, \, \Gamma_2 = B_Z$). Therefore, $\forall \, i \, (\Gamma_1)Q_i \cdot Q_i^{-1} \cdot ZQ_i = \Gamma_2Q_i$. Thus $((\Gamma_1)Q_i)V_i = (\Gamma_2)Q_i$. So $\Gamma_1Q_i = A_{V_i}, (\Gamma_2)Q_i = B_{V_i}$. This establishes the equivalence of the different formulations of the theorem.
Theorem 1.4 (Projective Van Kampen Theorem). Let \( \overline{S} \) be a cuspidal curve in \( \mathbb{CP}^2 \), transversal to the line in infinity. Let \( S = \overline{S} \cap \mathbb{C}^2 \). Let \( \Gamma_1, \ldots, \Gamma_p \) be a \( g \)-base for \( \pi_1(\mathbb{C}^2 - S, u_0) \). Then \( \pi_1(\mathbb{CP}^2 - \overline{S}, *) \simeq \pi_1(\mathbb{C}^2 - S, *) \left/ \left( \prod_{j=1}^p \Gamma_j \right) \right. \).

In \([\text{MoTe}5]\) we proved that for a \( g \)-base \( \{\delta_i\} \) of \( \pi_1(\mathbb{C}^2 - S, u_0) \) we have \( \Delta_p^2 = \prod \varphi_u(\delta_i) \) where \( \varphi_u \) is the braid monodromy w.r.t. \( S, \pi, u \) and \( \Delta_p^2 \) is the generator of \( B_p[\mathbb{C}_u, \mathbb{C}_u \cap S] \). Moreover, we proved there that the set of such product-forms of \( \Delta_p^2 \) form a complete class under Hurwitz equivalence of factorized expressions. Moreover, from all these equivalent factorizations we try to choose the product form which will be most useful for fundamental group calculations. So to calculate the braid monodromy is equivalent to finding certain factorizations of \( \Delta_p^2 \).

Invariance properties are results in which we prove that the braid monodromy factorization of \( \Delta_p^2 \) is invariant under certain elements of \( B_{2p} \). We look for elements of \( B_{2p} \) that will give us equivalent factorizations of \( \Delta_p^2 \). Establishing invariance properties is essential in order to simplify the calculations which follow from the Van Kampen Theorem.

For defining invariance properties we need the following definitions:

Definition. Hurwitz move

Let \( g_1 \cdots g_k = h_1 \cdots h_k \) be two factorized expressions of the same element in a group \( G \). We say that \( g_1 \cdots g_k \) is obtained from \( h_1 \cdots h_k \) by a Hurwitz move if \( \exists 1 \leq p \leq k - 1 \) s.t. \( g_i = h_i, \ i \neq p, p + 1 \), \( g_p = h_p h_{p+1} h_p^{-1} \) and \( g_{p+1} = h_p \) or \( g_p = h_{p+1} \) and \( g_{p+1} = h_p h_p h_{p+1} \).

Definition. Hurwitz equivalence of factorized expressions

Let \( g_1 \cdots g_k = h_1 \cdots h_k \) be two factorized expressions of the same element in a group \( G \). We say that \( g_1 \cdots g_k \) is a Hurwitz equivalent to \( h_1 \cdots h_k \) if \( h_1 \cdots h_k \) is obtained from \( g_1 \cdots g_k \) by a finite number of Hurwitz moves. We denote it by \( g_1 \cdots g_k \simeq h_1 \cdots h_k \).

Definition. A factorized expression invariant under \( h \)

Let \( g_1 \cdots g_k \) be a factorized expression in \( G, h \in G \). We say that \( g = g_1 \cdots g_k \) is invariant under \( h \) if \( g_1 \cdots g_k \) is Hurwitz equivalent to \( (g_1)_h \cdots (g_k)_h \), where \( (g_i)_n = h^{-1} g_i h \).

Invariance properties are important in view of the following lemma (\([\text{MoTe}5]\)):

Lemma 1.5. If a braid monodromy factorization \( \Delta_p^2 = \prod Z_i \) is invariant under \( h \) then the equivalent factorization \( \Delta_p^2 = \prod (Z_i)_h \) is also a braid monodromy factorization.

Since every factor of a braid monodromy factorization induces a relation on \( \pi_1(\mathbb{C}^2 - S) \) by proving invariant properties we get more information on \( \pi_1(\mathbb{C}^2 - S) \). (each \( (Z_i)_h \) induces a new relation) and thus it is an essential addition to the Van Kampen theorem. In the next theorem we shall explain how to get new relations from invariance properties.
Theorem 1.6. Let \( S, u, \rho, B \) be as above. Let \( \Delta_p = \prod Z_i \) be a braid monodromy factorization of \( \Delta^2 \) w.r.t. \( \varphi \). If a subfactorization \( \prod_{i=s}^r Z_i \) is invariant under \( h \), and \( \prod Z_i \) induces a relation \( \Gamma_1 \cdot \ldots \cdot \Gamma_t \) on \( G \) via the Van Kampen method then \( (\Gamma_1) \cdot \ldots \cdot (\Gamma_t) \) is also a relation. If \( \prod Z_i \) is invariant under \( h \), and if \( R = (\Gamma_1) \cdot \ldots \cdot (\Gamma_t) \) is a relation on \( G = \pi_1(\mathbb{C}^2 - S, u_0) \), then \( (\Gamma_1)h \cdot \ldots \cdot (\Gamma_t)h \) is also a relation.

Proof.

Any relation on \( \pi_1(\mathbb{C}^2 - S, u_0) \) is a product of the relations induced by \( \prod Z_i \), via the Van Kampen method. Thus, it is enough to consider relations induced from the braid monodromy. Assume that \( V^\nu \) is a factor in \( \prod_{i=2}^r Z_i \). Assume that the relation induced by it is:

\[
A_V = B_V \\
[A_V, B_V] = 1 \\
\langle A_V, B_V \rangle = 1
\]

Since \( \prod_{i=s}^r Z_i \) is invariant under \( h \), \( \prod_{i=s}^r (Z_i)_h \) is also part of a braid monodromy factorization. In \( \prod_{i=s}^r (Z_i)_h \) we have a factor of the form \((V^\nu)_h\). Since \((V^\nu)_h = (V_h)^\nu\), \((V^\nu)_h\) induces the following relation on \( \pi_1(\mathbb{C}^2 - S, u) \):

\[
A_{V_h} = B_{V_h} \\
[A_{V_h}, B_{V_h}] = 1 \\
\langle A_{V_h}, B_{V_h} \rangle = 1
\]

Since \( A_{V_h} = (A_V)_h \), \( B_{V_h} = (B_V)_h \), \((V^\nu)_h\) induces the following relation on \( \pi_1(\mathbb{C}^2 - S, u) \):

\[
(A_V)_h = (B_V)_h \\
[(A_V)_h, (B_V)_h] = 1 \\
\langle (A_V)_h, (B_V)_h \rangle = 1
\]

Thus, if \( V^\nu \) induces the relation \( \Gamma_1 \cdot \ldots \cdot \Gamma_t = 1 \) then \((V^\nu)_h\) induces the relation \((\Gamma_1)_h \cdot \ldots \cdot (\Gamma_t)_h = 1 \). \( \square \)

2. The topology of affine complements of Veronese branch curves.

Let \( S_{V_3} \) be the branch curve of a generic projection \( V_3 \to \mathbb{P}^2 \) defined in Section 2. For short we shall denote \( S_{V_3} \) by \( S \) in the sequel.

In [MoTe7] we proved a result concerning \( G = \pi_1(\mathbb{C}^2 - S, *) \). In this section, we shall quote this result and prove further results concerning \( G \), presenting it as an “almost” solvable group.

Let \( B_n \) be the braid group, as in Section 1. In order to formulate these results we need a few definitions.
Definition. **Transversal half-twists**

The half-twists $H(\sigma_1)$ and $H(\sigma_2)$ will be called *transversal* if $C_1$ and $C_2$ intersect transversally in one point which is not an end point of either of the $\sigma_i$’s.

Definition. $\tilde{B}_n, \tilde{X}_i$

Let $T_n$ be the subgroup of $B_n$ normally generated by $[X,Y]$ for $X,Y$ transversal half-twists. $\tilde{B}_n$ is the quotient of $B_n$ modulo $T_n$. We choose a frame $X_i$ of $B_n$. We denote their images in $\tilde{B}_n$ by $\tilde{X}_i$.

We shall use a slightly different presentation for $\tilde{B}_9$ than the one induced from the standard Artin presentation:

$$\tilde{B}_9 = \langle \tilde{T}_1, \ldots, \tilde{T}_9, \ i \neq 4 \rangle$$

with the following complete list of relations:

$$[\tilde{T}_i, \tilde{T}_j] = 1 \quad t_i, t_j \text{ disjoint}$$
$$\langle \tilde{T}_i, \tilde{T}_j \rangle = 1 \quad \text{otherwise}$$
$$[\tilde{T}_1, \tilde{T}_2 \cdot \tilde{T}_3 \tilde{T}_2] = 1$$
$$[\tilde{T}_5, \tilde{T}_8 \tilde{T}_9 \tilde{T}_8] = 1$$

where $T_i$ is a half-twist corresponding to a path $t_i$ and the $t_i$ are arranged in the following configuration:

**Proposition-Definition 2.0.** $G_0(n), \tau, u_1$

Let $A_{n-1}$ be the free abelian group on $w_1, \ldots, w_{n-1}$. Let us define a $\mathbb{Z}/2$ skew-symmetric form on $A_{n-1}$ as follows:

$$w_i \cdot w_j = \begin{cases} 1 & |i - j| = 1 \\ 0 & \text{otherwise}. \end{cases}$$

There exists a unique central extension $G_0(n)$, of $\mathbb{Z}/2$ by $A_{n-1}$, with generators $u_1, \ldots, u_{n-1}$ that satisfies

$$1 \to \mathbb{Z}/2 \xrightarrow{b} G_0(n) \xrightarrow{a} A_{n-1} \to 1$$
$$a(u_i) = w_i$$
$$[u_i, u_j] = b(w_i \cdot w_j) = \begin{cases} \tau & |i - j| = 1 \\ 0 & \text{otherwise}. \end{cases}$$
We always consider $G_0(n)$ with the standard $\tilde{B}_n$-action as follows:

\[
(u_i)\tilde{X}_k = \begin{cases} 
   u_i^{-1}\tau & k = i \\
   u_k u_i & |i-k| = 1 \\
   u_i & |i-k| \geq 2
\end{cases}
\]

**Claim 2.1.** $\text{Ab}(G_0(n)) = A_{n-1}$ (free abelian group on $n-1$ generators), $G_0(n)' = \{\tau, 1\}$ ($\simeq \mathbb{Z}/2$).

**Proof.** Claim III.6.4 of [MoTe9]. □

Consider the semidirect product, $\tilde{B}_9 \ltimes G_0(9)$, with respect to the standard $\tilde{B}_9$ action on $G_0(9)$.

We will work with a more concrete presentation of $G_0(9)$ that will be compatible with the chosen presentation of $\tilde{B}_9$:

Let $G_0(9)$ be the group generated by $g_i$, $i = 1, \ldots, 9$, $i \neq 4$, with the following list of relations.

\[
\begin{align*}
[g_1, g_2]^2 &= 1 \\
[g_1, g_2] &\in \text{Center}(G_0(9)) \\
g_i, g_j &= \begin{cases} 
   1 & t_i, t_j \text{ are disjoint} \\
   [g_i, g_j] & \text{otherwise.}
\end{cases}
\end{align*}
\]

Denote $\nu = [g_1, g_2]$. Let us reformulate the relations of $G_0(9)$ as follows:

\[
G_0(9) = \langle g_1, \ldots, \tilde{g}_4, \ldots, g_9 \mid [g_i, g_j] = \begin{cases} 
   1 & T_i, T_j \text{ are disjoint} \\
   \tau^2 = 1 & \tau g_i = \tau \end{cases} \rangle
\]

**Definition.** $\nu_1, N_9, G_9, \tilde{\psi}_9 : G_9 \to S_9$

$\nu_1 = (\tilde{X}_2 X_1 X_2^{-1})^2 X_1^{-2}$ for a frame $X_1, \ldots, X_8$ of $B_9$.

$N_9 = \text{The subgroup normally generated by } c\tau^{-1}, (u_1\nu_1^{-1})^3$ (an element of $G_0(9)$, see above) and $c$ an element of $\tilde{B}_9$ (see above).

$G_9 = \tilde{B}_9 \ltimes G_0(9)/N_9$

$\tilde{\psi}_9 : G_9 \to S_9$, defined by $\tilde{\psi}_9(\alpha, \beta) = \tilde{\psi}_9(\alpha)$, where $\tilde{\psi}_9 : \tilde{B}_9 \to S_9$ is the homomorphism to the symmetric group, induced from the standard homomorphism $B_9 \to S_9$.

**Proposition 2.2.** Let $V_3$ be the Veronese surface of order 3. Let $S_3$ be the branch curve of a generic projection $V_3 \to \mathbb{CP}^2$. Let $\mathbb{CP}^2$ be a big “affine piece” of $\mathbb{CP}^2$. Let $S = S_3 \cap \mathbb{CP}^2$. Let $G = \pi_1(\mathbb{C}^2 - S)$. Then $G \cong G_9$ s.t. $\psi : G \to S_9$ is compatible with $\tilde{\psi}_9 : G_9 \to S_9$.

**Proof.** Theorem 6.1 of [MoTe9].

Let $\psi_n$ be the standard homomorphisms $B_n \to S_n (= \text{symmetric group})$. Let $\text{Ab}$ be the standard homomorphism $B_n \to \mathbb{Z}$.

$\psi_n([X, Y]) = 1$, and $\text{Ab}([X, Y]) = 1$, $\psi_n$ and $\text{Ab}$ induce homomorphisms on $\tilde{B}_n$.

**Definition.** $\tilde{\psi}_n, \tilde{\Psi}_n, \tilde{P}_n, c$

$\tilde{\psi}_n : \tilde{B}_n \to S_n$, the induced homomorphism from $\psi_n$.

$\tilde{\text{Ab}} : \tilde{B}_n \to \mathbb{Z}$, the induced homomorphism from $B_n \to \mathbb{Z}$. 
\[ \tilde{P}_n = \ker \tilde{\psi}_n. \]
\[ \tilde{P}_{n,0} = \ker \tilde{\psi}_n \cap \ker \tilde{A}b = \ker \tilde{P}_n \rightarrow Ab(\tilde{B}_n) = \mathbb{Z}. \]
\[ c = [X_1^2, X_2^2] \] for 2 consecutive half-twists.

For the proof of the main result (Proposition 2.4) we need Proposition 2.3 concerning quotients of the braid group. For a subgroup \( H \) denote by \( H' \) the commutator subgroup of \( H \).

**Proposition 2.3.** Let \( \hat{X}_i \) be a frame in \( B_n \). Let \( c = [\hat{X}_1^2, \hat{X}_2^2] \). Then
\[ c = [\hat{X}_1^2, \hat{X}_2^2] = [\hat{X}_1^2, \hat{X}_2^2] \cdots = [\hat{X}_n^2, \hat{X}_{n-1}^2]. \]
Moreover, \( (\tilde{P}_n)' = (\tilde{P}_{n,0})' = \{1, c\} \cong \mathbb{Z}_2. \)
\[ Ab(\tilde{P}_n) = \text{free abelian group on } n \text{ generators}; \xi_1, \ldots, \xi_{n-1}, \hat{X}_1^2, \text{ where } \xi_1 = (\hat{X}_2\hat{X}_1\hat{X}_2^{-1})^2\hat{X}_2^{-2} \text{ and } \xi_i \text{ is conjugate to } \xi_1, \xi_i \in \hat{P}21_{n,0}. \]
\[ \hat{B}_n \text{ acts on } \hat{P}_{n,0} \text{ by conjugation.} \]
\[ \tilde{P}_{n,0} \text{ with this action is isomorphic to } G_0(n) \text{ with the standard } \hat{B}_n\text{-action as defined previously.} \]

There exists a series: \( 1 \subset (\tilde{P}_{n,0})' \subset \tilde{P}_{n,0} \subset \tilde{P}_n \subset \hat{B}_n \text{ s.t. } \hat{B}_n/\hat{P}_n = S_n, \)
\[ \hat{P}_n/\hat{P}_{n,0} \cong \mathbb{Z}, \tilde{P}_{n,0}/(\tilde{P}_{n,0})' \cong A_{n-1} \cong Ab(G_0(n)), (\tilde{P}_{n,0})' \cong \mathbb{Z}_2. \]

**Proof.** Theorem III.6.4 of [MoTe9]. See [MoTe5], Chapters 4, 5 for more information about \( \tilde{P}_n \) and \( \tilde{P}_{n,0}. \) \( \square \)

The following theorem is the main result of this paper. It is a structure theorem for \( G = G_9 \), which states that \( G_9 \) is an almost solvable group. Using this result, one can start to compute \( \pi_2 \) as a model over \( \pi_1. \)

**Definition.** \( Ab_9, H_9, H_{9,0} \)
\[ Ab_9 = \text{Abelization map of } G_9. \]
\[ H_9 = \ker \tilde{\psi}_9 \]
\[ H_{9,0} = \ker \tilde{\psi}_9 \cap \ker Ab_9 \]

**Proposition 2.4.** There exists a series
\[ 1 \subset H_{9,0}' \subset H_{9,0} \subset H_9 \subset G_9, \text{ where } \]
\[ G_9/H_9 \cong S_9, H_9/H_9' \cong \mathbb{Z}, H_{9,0}/H_{9,0}' \cong (\mathbb{Z} + \mathbb{Z}/3)^8, \]
\[ H_{9,0}' = H'_9 = \{1, c\} \cong \mathbb{Z}/2, \text{ where } c \in \text{Center}(G_9). \text{ Moreover, } Ab(G_9) = \mathbb{Z}. \]

**Proof.** Let \( \tilde{T}_i \ i = 1 \ldots 9 \ i \neq 4 \) be the base of \( \hat{B}_9 \) as above. Let \( g_i, i = 1, \ldots, 9, \ i \neq 4, \) be the chosen base for \( G_0(9). \)

Let
\[ c = [\tilde{T}_1^2, \tilde{T}_2^2] \]
\[ \tau = [g_1, g_2]. \]
\[ \hat{B}_9 \text{ acts on } G_0(9) \text{ as follows:} \]
\[ (g_i)_{\tau_k} = \begin{cases} g_i & \text{i, k disjoint} \\ g_i^{-1} \tau & \text{i = k} \\ g_ig_k^{-1} & \text{or } g_kg_i \text{ otherwise.} \end{cases} \]

It is easy to check that we have the following relations in \( \hat{B}_9 \times G_0(9) \)
\[ [\tilde{T}_i^2, \tilde{T}_j^2] = \begin{cases} 1 & \text{i, j disjoint or } i = j \\ c & \text{otherwise} \end{cases} \]
\[ [g_i, g_j] = \begin{cases} 1 & i, j \text{ disjoint or } i = j \\ \tau & \text{otherwise} \end{cases} \]

\[ c_j \tau \in \text{Center} \tilde{B}_9 \rtimes G_0(9) \]

\[ G_0(9)' = \{1, \tau\} \]

\[ c^2 = \tau^2 = 1 \]

\[ c \neq \tau. \]

We recall from the proof of Proposition 2.3 (which appears in [MoTe9]) that \( \tilde{P}_9,0 \) is generated by \( \xi_1, \ldots, \xi_9 \) \( i \neq 4 \), \( \tilde{P}_9 \) is generated by \( \tilde{P}_9,0 \) and \( \tilde{T}_2^2 \), \( \tilde{P}'_9 = \tilde{P}'_9,0 = \{1, c\} \), where

\[ [\xi_i, \xi_j] = \begin{cases} 1 & i, j \text{ disjoint, } i = j \\ c & \text{otherwise} \end{cases} \]

(\( \xi_1 = (\tilde{T}_2 \tilde{T}_1 \tilde{T}_2^{-1})^2 \tilde{T}_2^{-2} \) and \( \xi_i \) is conjugate to \( \xi_1 \)).

Let \( \xi_i = g_i \xi_i^{-1} \).

Since \( \tilde{T}_1, \tilde{T}_2 \) can be extended to a frame of \( \tilde{B}_9 \), \( \xi_1 \) can be considered as \( v_1 \) in the above definition. From the above commutators one can see that \( \xi_i^3 \) is conjugated to \( \xi_1^3 \), up to multiplication by \( c\tau \). Thus \( N_9 \) can be represented as

\[ N_9 = \langle c\tau, \xi_i^3 \mid i = 1 \ldots 9 \text{, } i \neq 4 \rangle. \]

To prove the theorem we consider first another quotient of \( \tilde{B}_9 \rtimes G_0(9) \):

Let

\[ C_9 = \langle c\tau \rangle \]

\[ \hat{G}_9 = \tilde{B}_9 \rtimes G_0(9)/C_9. \]

There exist natural epimorphisms \( \hat{I}, \hat{J} \).

\[ \tilde{B}_9 \rtimes G_0(9) \xrightarrow{\hat{I}} \hat{G}_9, \quad \hat{G} \xrightarrow{\hat{J}} G_9, \]

s.t. \( \hat{I}, \hat{J} \) in the natural epimorphism \( \tilde{B}_9 \rtimes G_0(9) \rightarrow \tilde{B}_9 \rtimes G_0(9)/N_9 = G_9 \). Let

\[ M_9 = \langle \xi_i^3 \mid i = 1, \ldots, 9 \text{, } i \neq 4 \rangle. \]

Let \( \hat{M}_9 \) be the image under \( \hat{I} \) of \( M_9 \) in \( \hat{G}_9 \).

Remark. By abuse of notation we use the same notation for elements of \( \tilde{B}_9, G_0(9) \) and their images in \( \tilde{B}_9 \rtimes G_0(9) \) and its different quotients.

We divide the proof into 11 claims and corollaries.

Claim 1.

(a) \( \ker \hat{I} = C_9 \cong \mathbb{Z}_2 \).

(b) \( \hat{I}(N_9) = \hat{M}_9 \), \( N_9/C_9 \cong \hat{M}_9 \).

(c) \( \hat{M}_9 \) is a normal commutative subgroup of \( \hat{G}_9 \).

(d) The induced homomorphism \( G_9 = \tilde{B}_9 \rtimes G_0(9)/N_9 \xrightarrow{\hat{I}} \hat{G}_9/\hat{M}_9 \) is an isomorphism.
(e) $\hat{M}_9 = \ker(\hat{G}_9 \rightarrow G_9)$, $\hat{G}_9/\hat{M}_9 = G_9$.
(f) There exists a short exact sequence

$$1 \rightarrow \hat{M}_9 \rightarrow \hat{G}_9 \rightarrow G_9 \rightarrow 1.$$ 

**Proof.**

(a) Clearly $C_9 = \ker \hat{I}$. Since $c, \tau \in \text{Center } \tilde{B}_9 \ltimes G_0(9)$ and $c^2 = \tau^2 = 1$ then $(c\tau)^2 = 1$ since $c \neq \tau$, $C_9 = \mathbb{Z}_2$.

(b) $N_9 = (M_9, C_9) \Rightarrow I(N_9) = (\tilde{I}(N_9), \tilde{I}(C_9)) = (\hat{M}_9, 1) = \hat{M}_9$. Since $\ker \tilde{I} = C_9 \subseteq N_9$, $\ker \left( \tilde{I} \mid N_9 \right) = C_9$. Thus $N_9/C_9 = \hat{M}_9$.

(c) Consider $\tilde{B}_9 \ltimes G_0(9)$. By Lemma IV.6.1 of [MoTe9]:

$$[\tilde{T}_j^{\pm 2}, g_i^{\pm 1}] = \begin{cases} 1 & t_i, t_j \text{ disjoint, } i = j \\ c & \text{otherwise} \end{cases}$$

By Claim II.4(e) of [MoTe9]

$$[T_j^{-2} T_k^2, g_i^{\pm 1}] = \begin{cases} 1 & t_i, t_k \text{ disjoint} \\ c & \text{otherwise} \end{cases}$$

By Corollary IV.4.2 for every $\xi_\ell$ and every $i$, s.t. $t_i \cap t_\ell \neq \emptyset$ there exists $k$ s.t. $t_k \cap t_i \neq \emptyset$ and $\xi_\ell = T_j^{\pm 2} T_k^{\pm 2}$. Thus

$$[\xi_\ell^{\pm 1}, g_i^{\pm 1}] = \begin{cases} 1 & i, \ell \text{ disjoint} \\ c & \text{otherwise} \end{cases}$$

On the other hand,

$$[g_\ell^{\pm 1}, g_i^{\pm 1}] = \begin{cases} 1 & t_i, t_\ell \text{ disjoint} \\ \tau & \text{otherwise} \end{cases}$$

Since $c, \tau$ are central elements and $\zeta_\ell = g_\ell \xi_\ell^{-1}$, the last 2 equations imply:

$$[\zeta_\ell^{\pm 1}, g_i^{\pm 1}] = \begin{cases} 1 & t_i, t_\ell \text{ disjoint} \\ c\tau & \text{otherwise} \end{cases}$$

So

$$[\zeta_\ell^{\pm 3}, g_i^{\pm 1}] = \begin{cases} 1 & t_i, t_\ell \text{ disjoint} \\ c\tau & \text{otherwise} \end{cases}$$

Therefore, $\zeta_\ell^3$ and $g_i$ commute in $\tilde{B}_9 \ltimes G_0(9)/C_9$. The same arguments imply that $\xi_\ell^3$ and $g_i$ commute. Thus, $\zeta_\ell^3$ and therefore with $\zeta_\ell^3$. Thus, $\hat{M}_9$ is commutative.

(d) $\ker(\tilde{B}_1 \ltimes G_0(9) \rightarrow \hat{G}_9) = C_9$. Thus $\ker \left( \tilde{B}_9 \ltimes G_0(9) \rightarrow \hat{G}_9/\hat{M}_9 \right) = (C_9, M_9) = N_9$. By the isomorphism theorem we get (d).

(e) $\ker(\tilde{B}_9 \ltimes G_0(9) \rightarrow \tilde{B}_9 \ltimes G_0(9)/N_9) = N_9$. Thus $\ker(\tilde{B}_9 \ltimes G_0(9)/C_9 \rightarrow \tilde{B}_9 \ltimes G_0(9)/N_9) = N_9/C_9$ which equals $\hat{M}_9$.

(f) From (e). □
Thus we have

\[ \tilde{B}_9 \times G_0(9) \xrightarrow{I} \tilde{G}_9 \xrightarrow{J} \hat{G}_9 \]
\[ \cup \hspace{1cm} \cup \]
\[ N_9 \hspace{1cm} \rightarrow M_9 \hspace{1cm} \rightarrow 1 \]
\[ \tilde{B} \times G_0(9)/N_9 \xrightarrow{I} \tilde{G}_9/\tilde{M}_9 \xrightarrow{J} \hat{G}_9 \]

Claim 2.

\[ Ab(G_9) = Ab(\hat{G}_9) = Ab(\tilde{B}_9 \times G_0(9)) = Ab(\tilde{B}_9) = Ab(B_9) = \mathbb{Z}. \]

Proof. Recall that conjugate elements are equal under abelization (\( Ab(G) = G'/G' \)). Since \( B_n \) is generated by half-twists and they are all conjugate to each other, \( Ab(B_n) \) is generated by one element, whose infinite order remains valid under abelization. So \( Ab(B_9) = \mathbb{Z} \). Since \( \tilde{B}_9 = B_9/\text{subgroup of } B'_9 \), \( Ab(\tilde{B}_9) \) is also \( \mathbb{Z} \). From the action of \( \tilde{B}_9 \) on \( G_0(9) \) one can see that for every \( i, j \leq 9, i \neq j \), \( \exists k, \ i \neq k, \) \(, \) s.t. \( (g_i)_{\tilde{f}_k} = g_i g_k^{\pm 1} \). Under abelization this relation becomes \( \tilde{f}_i = g_i g_i^{\pm 1} \). Therefore, \( g_k^{\pm 1} = 1 \). So in the abelianization of \( \tilde{B}_9 \times G_0(9) \), the elements of \( G_0(9) \) are 1. Thus \( Ab(\tilde{B}_9 \times G_0(9)) = \mathbb{Z} \). Since \( C_9 \subseteq (\tilde{B}_9 \times G_0(9))^r \), \( Ab(\tilde{B}_9 \times G_0(9)/C_9) \) is also \( \mathbb{Z} \). So \( Ab(\hat{G}_9) = \mathbb{Z} \). Since \( M_9 \) consists of degree 0 elements, \( Ab(G_9/M_9) = \mathbb{Z} \). Thus \( Ab(G_9) = \mathbb{Z} \).

\[ \square \]

Let \( \tilde{H}_9 \) be \( \ker \tilde{G}_9 \mapsto S_9 \) (= the symmetric group on 9 elements).

Claim 3.

(a) \( \hat{H}_9 \simeq \tilde{H}_9 \times G_0(9)/C_9 \).

(b) There exists an epimorphism \( \hat{H}_9 \rightarrow G_0(18) \) which is compatible with \( Ab(\hat{H}_9) \rightarrow A_{17} \) \( (A_{17} = Ab G_0(18)) \).

(c) \( Ab(\hat{H}_9) \) is freely generated by \( \xi_1 \ldots \xi_9 i \neq 4, \ g_1 \ldots g_9 i \neq 4, \overline{T}_2 \).

(d) \( M_9 \subseteq \text{Center } \hat{H}_9 \).

(e) \( H_9 \simeq \tilde{F}_9 \times G_0(9)/N_9 \).

(f) \( \hat{H}_9 \) maps onto \( H_9 \) under \( \tilde{G}_9 \xrightarrow{J} G_9 \).

(g) \( H_9/M_9 \simeq H_9 \).

Proof.

(a) \( \ker(\tilde{B}_9 \rightarrow S_9) = \tilde{P}_9, \) and \( \tilde{B}_9 \times G_0(9) \rightarrow S_9 \) factors through \( \tilde{B}_9 \). So \( \ker(\tilde{B}_9 \times G_0(9) \rightarrow S_9) = \tilde{P}_9 \times G_0(9). \) So \( \tilde{H}_9 = \ker(\tilde{B}_9 \times G_0(9)/C_9 \rightarrow S_9) = \tilde{P}_9 \times G_0(9)/C_9 \).

(b) (c) We shall skip the precise proof of (b) and (c). The idea is to derive the presentation of \( \hat{H}_9 \) from a presentation of \( \tilde{P}_9 \). (Proposition 2.3) and Proposition-Definition 2.0.

(d) The relevant relations are to be found in the proof of Claim 3(c).

(e) Similar to (a).

(f) (g) \( \tilde{G}_9 \xrightarrow{\psi_9} S_9 \) factors through \( G_9 \xrightarrow{\psi_9} S_9, \) i.e., \( \tilde{G}_9 \rightarrow G_9 \xrightarrow{\psi_9} S_9 \). So \( \ker \psi_9 \) is mapped into \( \ker \psi_9 \). We can see it differently using (a) and (e): Since \( C_9 \subset N_9 \), there exists a natural epimorphism (restriction of \( J \)) \( \tilde{P}_9 \times G_0(9)/C_9 \rightarrow \tilde{P}_9 \times G_0(9)/N_9 \).
By (a) and (c) this map is actually
\[ \hat{H}_9 \rightarrow H_9. \]

The kernel of this map is \( N_9/C_9 \), which is \( \hat{M}_9 \) by Claim 1(b).

Claim 4.
(a) \( \hat{M}_9 \) is isomorphic to its image in \( \text{Ab}(\hat{H}_9) \).
(b) \( \hat{M}_9 \cap \hat{H}_9' = \{ 1 \} \).

Proof. By Claim 3 (c), \( \text{Ab}(\hat{H}_9) \) is freely generated by 17 elements \( \xi_1, \ldots, \xi_9, i \neq 4, g_1, \ldots, g_9, i \neq 4, \) \( \tilde{T}_1^2 \), regarded as elements of the abelization.

\( M_9 \) is generated by \( \zeta_1, \ldots, \zeta_9, i \neq 4 \) and \( \zeta_i = \xi_i g_i^{-1} \). The image of \( \hat{M}_9 \) in the abelization is generated by \( \{ g_i \xi_i^{-1} \}_{i=1}^{9} \) which are 8 products of different elements of a free base. Thus \( M \simeq \text{Image of } M \) in \( \text{Ab}(\hat{H}_9) \). Thus \( \text{ker} \text{Ab}(\hat{H}_9) \mid \hat{M}_9 = 1 \). But this kernel is \( \hat{M}_9 \cap \hat{H}_9' \).

Claim 5. \( \hat{H}_9' \simeq H_9' \).

Proof. Consider \( \hat{G}_9 \rightarrow G_9 \) with ker \( \hat{M}_9 \). Under this map \( \hat{H}_9 \rightarrow H_9 \) (Claim 3(f)) and \( \hat{H}_9' \rightarrow H_9' \). Evidently, ker \( \hat{H}_9' \rightarrow H_9' \) is \( H_9' \cap \hat{M}_9 \) which is \( \{ 1 \} \) by the previous claim.

Claim 6. \( (\hat{H}_9') \simeq \mathbb{Z}_2 \).

Proof. By Claim 3(a), \( \hat{H}_9 = \tilde{P}_9 \times G_9(9)/C_9 \). \( \tilde{P}_9' = \{ 1, c \} \), \( (G_9(9))' = \{ 1, \tau \} \). Thus \( (\tilde{P}_9 \times G_9(9))' = \{ 1, c, \tau \} \). When dividing by \( c = \tau \) \( (C_9 = (c\tau)) \) we get \( (\hat{H}_9)' = \{ 1, \tau \} \subseteq \mathbb{Z}_2 \). We have to prove that it does not collapse completely. We have:

\[ 1 \rightarrow (\hat{H}_9)' \rightarrow \hat{H}_9 \rightarrow \text{Ab}(\hat{H}_9) \rightarrow 1. \]

Consider the short exact sequence from Proposition-Definition 2.0:

\[ 1 \rightarrow \mathbb{Z}_2 \rightarrow G_9(18) \rightarrow A_{17} \rightarrow 1. \]

By Claim 3(c) we have \( \text{Ab}(\hat{H}_9) \simeq A_{17} \). By Claim 3(b) there exists \( \hat{H}_9 \rightarrow G_9(18) \). Thus there exists \( (\hat{H}_9)' \rightarrow \mathbb{Z}_2 \). So \( \hat{H}_9' \simeq \mathbb{Z}_2 \).

Corollary 7. \( H_9' = \mathbb{Z}_2 \).

Similar to Claims 3, 5, 6, we have:

Claim 8.
(a) \( \text{Ab}(\hat{H}_{9,0}) \) is freely generated by \( g_1, \ldots, g_9, i \neq 4, \xi_1, \ldots, \xi_9, i \neq 4 \).
(b) \( \hat{H}_{9,0}/M_9 \simeq H_{9,0} \).
(c) \( (\hat{H}_{9,0})' = H_{9,0} \).
(d) \( (\hat{H}_{9,0})' = \mathbb{Z}_2 \).
Corollary 9.
\[ \tilde{H}_{9,0} = H_9 \cong \mathbb{Z}_2. \]

Claim 10.
\[ H_{9,0}/\tilde{H}_{9,0} = (\mathbb{Z}_3 \oplus \mathbb{Z})^8. \]

Proof. By Claim 8(c), \( \tilde{H}_{9,0}/\tilde{M}_9 \cong H_{9,0} \). Thus \( H_{9,0}/\tilde{H}_{9,0} = Ab(H_{9,0}) = Ab(\tilde{H}_{9,0}/\tilde{M}_9) \). Since \( \tilde{M}_9 \) is isomorphic to its image in \( Ab(\tilde{H}_9) \) (Claim 4(a)), we get \( Ab(H_{9,0})/\tilde{M}_9 \). We can take \( \{ g_i, \zeta_i \}_{i \neq 4} \) as free generators for \( Ab(H_{9,0}) \). Thus
\[ Ab(\tilde{H}_{9,0})/\tilde{M}_9 = \left( \sum_{i=1}^{9} (g_i + \sum_{i=1}^{9} (\zeta_i)) \right) / \sum_{i=1}^{9} (\zeta_i^3) = \sum_{i=1}^{9} (g_i) \oplus \sum_{i=1}^{9} (\zeta_i/\zeta_i^3) \]
\[ \cong (\mathbb{Z} \oplus \mathbb{Z}_3)^8. \]

Corollary 11.
\[ \tilde{H}/H_{9,0} = \mathbb{Z}. \]

Proof. \( H_{9,0} = \ker(H_9 \to Ab(\tilde{G}_9)) \). Since \( Ab \tilde{G}_9 \cong Ab H_9 = \mathbb{Z} \), we have \( H_{9,0} = \ker(H_9 \to \mathbb{Z}). \]

The proof of the different statements of the theorem are Claims and Corollaries 11, 10, 9, 2, and the definition of \( H_9 \) for \( G_9/H_9 \cong S_9 \). □ for Theorem 2.4

3. The fundamental group of complements of a Veronese branch curve in \( \mathbb{CP}^2 \).

Theorem 3.1. Let \( \overline{S} \) be the Veronese surface of order 3. Let \( \overline{S} \) be the branch curve of a generic projection \( V_3 \to \mathbb{CP}^2 \). Let \( \overline{G} = \pi_1(\mathbb{CP}^2 - \overline{S}) \). Then there exist \( w_0 \in H_{9,0} \) s.t. \( \overline{G} \cong G_9 = G_9/(X_1^{18}w_0) \).

Theorem 3.2. Let \( \overline{H}_9 \) and \( \overline{H}_{9,0} \) be the images of \( H_{9,0} \) and \( H_q \) in \( \overline{G}_9 \). Then \( \overline{H}_{9,0} = \overline{H}_q \) and
\[ 1 \subseteq \overline{H}_{9,0} \subseteq \overline{H}_q \subseteq \overline{G}_9 \]
where
\[ \overline{G}_9/\overline{H}_9 \cong S_9, \quad \overline{H}_9/\overline{H}_{9,0} \cong \mathbb{Z}_3, \quad \overline{H}_{9,0}/\overline{H}_{9,0} \cong (\mathbb{Z} + \mathbb{Z}/3)^8, \quad \overline{H}_{9,0} = \overline{H}_9 \cong \mathbb{Z}/2. \]

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