Holographic Entanglement Entropy, Fractional Quantum Hall Effect and Lifshitz-like Fixed Point

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Abstract. The entanglement entropy has been very important in various subjects such as quantum information theory, condensed matter physics, and quantum gravity. In the first half of this talk, I will give an overview of recent progresses in the holographic calculation of entanglement entropy. Next, I will apply it to string theory duals of anisotropic scale invariant theories (Lifshitz-like points), which are obtained from a string theory realization of fractional quantum Hall effect.

1. Introduction

The idea of holography has been crucially important in string theory. Holography claims that the degrees of freedom in (d+2)-dimensional quantum gravity are much more reduced than we naively think, and will be comparable to those of quantum many body systems in d+1 dimensions [1, 2]. This was essentially found by remembering that the entropy of a black hole is not proportional to its volume, but to its area of the event horizon Σ (the Bekenstein-Hawking formula [3]):

\[ S_{BH} = \frac{\text{Area}(\Sigma)}{4G_N}, \]  

where \( G_N \) is the Newton constant. Owing to the discovery of the AdS/CFT correspondence [4], we know explicit examples where the holography is manifestly realized. The AdS/CFT argues that the quantum gravity on (d+2)-dimensional anti-de Sitter spacetime (AdS\(_{d+2}\)) is equivalent to a certain conformal field theory in d+1 dimensions (CFT\(_{d+1}\)) [4].

Even after quite active researches of AdS/CFT for these ten years, fundamental mechanism of the AdS/CFT correspondence still remains a mystery, in spite of so many of evidences in various examples. In particular, we cannot answer which region of AdS is responsible to particular information in the dual CFT. To make any progresses for this long standing problem, we believe that it is important to understand and formulate the holography in terms of a universal observable, rather than quantities which depend on the details of theories such as specific operators or Wilson loops etc. We only expect that a quantum gravity in some spacetime is dual to (i.e. equivalent to) a certain theory which is governed by the law of quantum mechanics. We would like to propose that an appropriate quantity which can be useful in this universal viewpoint is the entanglement entropy. Indeed, we can always define the entanglement entropy in any quantum mechanical system.
The entanglement entropy $S_A$ in quantum field theories or quantum many body systems is a non-local quantity as opposed to correlation functions. It is defined as the von Neumann entropy $S_A$ of the reduced density matrix when we ‘trace out’ (or smear out) degrees of freedom inside a $d$-dimensional space-like submanifold $B$ in a given $(d+1)$-dimensional QFT, which is a complement of $A$. $S_A$ measures how the subsystems $A$ and $B$ are correlated with each other. Intuitively we can also say that this is the entropy for an observer in $A$ who is not accessible to $B$ as the information is lost by the smearing out in region $B$. This origin of entropy looks analogous to the black hole entropy. Indeed, this was the historical motivation of considering the entanglement entropy in quantum field theories [5, 6]. Interestingly, the leading divergence of $S_A$ is proportional to the area of the subsystem $A$, called the area law [5, 6] (refer also to the review articles [7, 8]).

Now we come back to our original question where in AdS given information in CFT is saved. Since the information included in a subsystem $B$ is evaluated by the entanglement entropy $S_A$, we can formulate this question more concretely as follows: “Which part of AdS space is responsible for the calculation of $S_A$ in the dual gravity side?” We propose that a holographic formula of the entanglement entropy in [9, 10]:

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N^{(d+2)}} ,$$  \hspace{1cm} (2)\

where $\gamma_A$ is the $d$-dimensional minimal surface $\gamma_A$ whose boundary is given by the $(d-1)$-dimensional manifold $\partial \gamma_A = \partial A$ (see Fig. 1); the constant $G_N^{(d+2)}$ is the Newton constant of the general gravity in AdS$_{d+2}$. This formula can be applied equally well to asymptotically AdS static spacetimes.

The purpose of this talk is to review this holographic description and discuss its applications. In particular, the examples we study will include confining gauge theories, fractional quantum Hall systems and Lifshitz-like fixed points, where the entanglement entropy plays the role of quantum order parameters.

2. Holographic Entanglement Entropy
We start with a review of the definition and properties of the entanglement entropy.

2.1. Definition of Entanglement Entropy
Consider a quantum mechanical system with many degrees of freedom such as spin chains. More generally, we can consider arbitrary lattice models or quantum field theories (QFTs). We put such a system at zero temperature and then the total quantum system is described by the pure ground state $|\Psi\rangle$. We assume no degeneracy of the ground state. Then, the density matrix is that of the pure state

$$\rho_{\text{tot}} = |\Psi\rangle\langle\Psi| .$$  \hspace{1cm} (3)\

The von Neumann entropy of the total system is clearly zero $S_{\text{tot}} = -\text{tr} \rho_{\text{tot}} \log \rho_{\text{tot}} = 0$.

Next we divide the total system into two subsystems $A$ and $B$. In the spin chain example, we just artificially cut off the chain at some point and divide the lattice points into two groups. Notice that physically we do not do anything to the system and the cutting procedure is an imaginary process. Accordingly the total Hilbert space can be written as a direct product of two spaces $\mathcal{H}_{\text{tot}} = \mathcal{H}_A \otimes \mathcal{H}_B$ corresponding to those of subsystems $A$ and $B$. The observer who is only accessible to the subsystem $A$ will feel as if the total system is described by the reduced density matrix $\rho_A$

$$\rho_A = \text{tr}_B \rho_{\text{tot}} ,$$  \hspace{1cm} (4)\

where the trace is taken only over the Hilbert space $\mathcal{H}_B$. 

Now we define the entanglement entropy of the subsystem $A$ as the von Neumann entropy of the reduced density matrix $\rho_A$

$$S_A = -\text{tr}_A \rho_A \log \rho_A .$$

This quantity provides us with a convenient way to measure how closely entangled (or how "quantum") a given wave function $|\Psi\rangle$ is.

It is also possible to define the entanglement entropy $S_A(\beta)$ at finite temperature $T = \beta^{-1}$. This can be done just by replacing (3) with the thermal one $\rho_{\text{thermal}} = e^{-\beta H}$, where $H$ is the total Hamiltonian. When $A$ is the total system, $S_A(\beta)$ is clearly the same as the thermal entropy.

2.2. Properties

There are several useful properties which the entanglement entropy enjoys generally. We summarize some of them as follows (the derivations and other properties of the entanglement entropy can be found in e.g. the textbook [11]):

- If the density matrix $\rho_{\text{tot}}$ is pure such as in the zero temperature system, then we find the following relation assuming $B$ is the complement of $A$:

$$S_A = S_B .$$

This manifestly shows that the entanglement entropy is not an extensive quantity. This equality (6) is violated at finite temperature.

- For any three subsystems $A, B$ and $C$ that do not intersect each other, the following inequalities hold:

$$S_{A+B+C} + S_B \leq S_{A+B} + S_{B+C} ,$$

$$S_A + S_C \leq S_{A+B} + S_{B+C} .$$

These inequalities are called the strong subadditivity [12], which is the most powerful inequality obtained so far with respect to the entanglement entropy.

- By setting $B$ empty in (7), we can find the subadditivity relation

$$S_{A+B} \leq S_A + S_B .$$

The subadditivity (9) allows us to define an interesting quantity called mutual information $I(A, B)$ by

$$I(A, B) = S_A + S_B - S_{A+B} \geq 0 .$$

2.3. Entanglement Entropy in QFTs

Consider a QFT on a $(d+1)$-dimensional manifold $\mathbb{R} \times N$, where $\mathbb{R}$ and $N$ denote the time direction and the $d$-dimensional space-like manifold, respectively. We define the subsystem by a $d$-dimensional submanifold $A \subset N$ at fixed time $t = t_0$. We call its complement the submanifold $B$. The boundary of $A$, which is denoted by $\partial A$, divides the manifold $N$ into two submanifolds $A$ and $B$. Then we can define the entanglement entropy $S_A$ by the previous formula (5). Sometimes this kind of entropy is called geometric entropy as it depends on the geometry of the submanifold $A$. Since the entanglement entropy is always divergent in a continuum theory, we introduce an ultraviolet cut off $a$ (or a lattice spacing). Then the coefficient in front of the divergence turns out to be proportional to the area of the boundary $\partial A$ of the subsystem $A$ as first pointed out in [5, 6],

$$S_A = \gamma \cdot \frac{\text{Area}(\partial A)}{a^{d-1}} + \text{subleading terms} ,$$

(11)
where $\gamma$ is a constant which depends on the system. This behavior can be intuitively understood since the entanglement between $A$ and $B$ occurs at the boundary $\partial A$ most strongly. This result (11) was originally found from numerical computations [6, 5] and checked in many later arguments.

The simple area law (11), however, does not always describe the scaling of the entanglement entropy in generic situations. Indeed the entanglement entropy of 2D CFT scales logarithmically with respect to the length $l$ of $A$ [13, 14]. If we assume the total system is infinitely long, it is given by the simple formula [13, 14]

$$S_A = \frac{c}{3} \log \frac{l}{a},$$

(12) where $c$ is the central charge of the CFT. We will later see that the scaling behavior (12) is consistent with the generic structure (22) expected from AdS/CFT.

The result for a finite size system at finite temperature has been obtained in [15] for a free Dirac fermion (i.e. $c = 1$) in two dimensions. In the high temperature expansion, the result becomes (we set $L = 1$)

$$S_A(\beta, l) = \frac{1}{3} \log \left[ \frac{\beta}{\pi a} \sinh \left( \frac{\pi l}{\beta} \right) \right] + \frac{1}{3} \sum_{m=1}^{\infty} \log \left[ \frac{(1 - e^{2\pi^2/3} e^{-2\pi^2/3}) (1 - e^{-2\pi^2/3} e^{-2\pi^2/3})}{(1 - e^{-2\pi^2/3})^2} \right]$$

$$+ 2 \sum_{k=1}^{\infty} (-1)^k \frac{\pi k l}{k} \coth \left( \frac{\pi k l}{\beta} \right) - 1 \sinh \left( \frac{\pi k}{\beta} \right).$$

(13)

Using this expression, we can find the relation between the thermal entropy $S_{\text{thermal}}(\beta)$ and the entanglement entropy

$$S_{\text{thermal}}(\beta) = \lim_{\epsilon \to 0} \left( S(\beta, 1 - \epsilon) - S(\beta, \epsilon) \right).$$

(14)

2.4. Holographic Entanglement Entropy Proposal

Here we would like to explain the holographic calculation of the entanglement entropy. For a detailed review article, refer to [8]. In order to simplify the notations and reduce ambiguities, we consider the setup of the AdS/CFT correspondence, though it will be rather straightforward to extend our results to general holographic setups.

The AdS/CFT correspondence argues that (quantum) gravity in the $(d+2)$-dimensional anti-de-Sitter space AdS$_{d+2}$ is equivalent to a $(d+1)$-dimensional conformal field theory CFT$_{d+1}$ [4]. Below we mainly employ the Poincare metric of AdS$_{d+2}$ with radius $R$:

$$ds^2 = R^2 \left( \frac{dz^2 - dx_0^2}{z^2} + \sum_{i=1}^{d-1} dx_i^2 \right).$$

(15)

The dual CFT$_{d+1}$ is supposed to live on the boundary of AdS$_{d+2}$ which is $R^{1,d}$ at $z \to 0$ spanned by the coordinates $(x^0, x^i)$. The extra coordinate $z$ in AdS$_{d+2}$ is interpreted as the length scale of the dual CFT$_{d+1}$ in the RG sense. Since the metric diverges in the limit $z \to 0$, we put a cut off by imposing $z \geq a$. Then the boundary is situated at $z = a$ and this cut off $a$ is identified with the ultraviolet cut off in the dual CFT. Under this interpretation, a fundamental principle of AdS/CFT, known as the bulk to boundary relation, is simply expressed by the equivalence of the partition functions in both theories

$$Z_{\text{CFT}} = Z_{\text{AdS Gravity}}.$$
Since (non-normalizable) perturbations in the AdS background by exciting fields in the AdS side are dual to the shift of background in the CFT side, we can compute the correlation functions in the CFT by taking the derivatives with respect to the perturbations. In generic parameter regions, the gravity should be treated in string theory to take the quantum corrections into account. Nevertheless, in particular interesting limit, typically strong coupling limit of CFT, the quantum corrections become negligible and we can employ supergravity to describe AdS spaces. Moreover, most of our examples shown in this article are simple enough that we can apply general relativity. In this situation, the right-hand side of (16) is reduced to the exponential $e^{-S_{EH}}$ of the on-shell Einstein-Hilbert action. So far we applied the AdS/CFT to the pure AdS spacetime (15). However, the AdS/CFT can be applied to any asymptotically AdS spacetimes including the AdS black holes.

Now we are in a position to present how to calculate the entanglement entropy in CFT $d + 1$ from the gravity on AdS $d + 2$. This argument here can be straightforwardly generalized to any static backgrounds.

To define the entanglement entropy in the CFT $d + 1$, we divide the (boundary) time slice $N$ into $A$ and $B$ as we explained before (see Fig. 1). In the Poincare coordinate (15), we are setting $N = R^d$ and the CFT $d + 1$ is supposed to live on the boundary $z = a \to 0$ of AdS $d + 2$. To have its dual gravity picture, we need to extend this division $N = A \cup B$ to the time slice $M$ of the bulk spacetime. In the setup (15), $M$ is the $(d + 1)$-dimensional hyperbolic spacetime $H_{d+1}$. Thus we extend $\partial A$ to a surface $\gamma_A$ in the entire $M$ such that $\partial \gamma_A = \partial A$. Notice that this is a surface in the time slice $M$, which is a Euclidean manifold. Of course, there are infinitely many different choices of $\gamma_A$. We claim that we have to choose the minimal area surface among them. This means that we require that the variation of the area functional vanishes; if there are multiple solutions, we choose the one whose area takes the minimum value. This procedure singles out a unique minimal surface and we call this $\gamma_A$ again (see Fig. 1).

In this setup we propose that the entanglement entropy $S_A$ in CFT $d + 1$ can be computed from the following formula [9, 10]

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N^{(d+2)}}.$$  \hspace{1cm} (17)

We stress again that the manifold $\gamma_A$ is the $d$-dimensional minimal area surface in AdS $d + 2$ whose boundary is given by $\partial A$. Its area is denoted by $\text{Area}(\gamma_A)$. Also $G_N^{(d+2)}$ is the $(d + 2)$-dimensional Newton constant of the AdS gravity. We can easily show that the leading divergence $\sim a^{-(d-1)}$ in (17) is proportional to the area of the boundary $\partial A$ and this immediately reproduces the area law property (11).

The appearance of the formula (17) looks very similar to the area law of the Bekenstein-Hawking formula (1) of black hole entropy. Indeed, we can regard our formula (17) as a generalization of (1) because in the presence of event horizon such as the AdS Schwarzschild black hole solutions, the minimal surface tends to wrap the horizon.

2.5. Entanglement Entropy from AdS$_3$/CFT$_2$

Consider AdS$_3$/CFT$_2$ as one of the simplest setups of AdS/CFT. Since the entanglement entropy in two-dimensional CFT can be analytically obtained as we mentioned, we can test our holographic formula explicitly. The central charge of CFT is related to radius of AdS$_3$ via

$$c = \frac{3R}{2G_N^{(3)}}.$$  \hspace{1cm} (18)

We are interested in the entanglement entropy $S_A$ in an infinitely long system when $A$ is an interval of length $l$. To compute this via the holographic formula (17), we need to find a geodesics
Figure 1. The holographic calculation of entanglement entropy via AdS/CFT.

between the two points \((x^1, z) = (-l/2, a)\) and \((x^1, z) = (l/2, a)\) in the Poincare coordinate (15). It is actually given by the half circle

\[(x, z) = \frac{l}{2}(\cos s, \sin s), \quad (\epsilon \leq s \leq \pi - \epsilon), \quad (19)\]

where \(\epsilon = \frac{2a}{l}\). The length of \(\gamma_A\) can be found as

\[\text{Length}(\gamma_A) = 2R \int_\epsilon^{\pi/2} ds \frac{\sin s}{\sin s} = -2R \log(\epsilon/2) = 2R \log \frac{l}{a}. \quad (20)\]

Finally the entropy can be obtained as follows

\[S_A = \frac{\text{Length}(\gamma_A)}{4G_N^{(3)}} = \frac{c}{3} \log \frac{l}{a}. \quad (21)\]

This perfectly agrees with the result (12) in the CFT side.

2.6. General Predictions from Holographic Calculations

For conformal field theories in higher dimensions \((d > 1)\), our holographic method predicts the following general form of \(S_A\) for relativistic quantum field theories, assuming that \(\partial A\) is a smooth and compact manifold

\[S_A = p_1 \frac{l}{a}^{d-1} + p_3 \frac{l}{a}^{d-3} + \cdots\]

\[\cdots + \begin{cases} p_{d-1} \frac{l}{a} + p_d, & d: \text{even} \\ p_{d-2} \left(\frac{l}{a}\right)^2 + \hat{c} \log \frac{l}{a}, & d: \text{odd} \end{cases} \quad (22)\]

where \(l\) is the typical length scale of \(\partial A\). This result includes the known result for \(d = 1\) (12) with \(\hat{c} = c/3\). Also, in the case of \((3+1)\)-dimensional conformal field theories \((d = 3)\), the scaling law (22) has been confirmed by direct field theoretical calculations based on Weyl anomaly [10], where again the coefficient of the logarithmic term \(\hat{c}\) is given in terms of central charges of \((3+1)\) CFTs. For \(d = \text{even}\), Eq. (22) has been the only known analytical result for (interacting) conformal field theories. Even though we assumed conformal field theories in the above, the
same scaling formula (22) should be true for a quantum field theory with a UV fixed point, i.e., at a relativistic quantum critical point.

When the boundary $\partial A$ is not a smooth manifold such as the one with cusp singularities, we will have other terms $(l/a)^{d-2}, (l/a)^{d-4}, \ldots$ which do not obey the scaling law in (22). For example, for a three-dimensional CFT ($d = 2$), if $\partial A$ has a cusp with the angle $\Omega$, then $S_A$ includes a logarithmic term $\sim -f(\Omega) \log l/a$, for a certain function $f$.

Also, if we consider a gapped system in three dimensions ($d = 2$) which is described (at low energies) by a topological field theory (called topologically ordered phase), the scaling of the entanglement entropy is the same as (22) with $d = 2$. The constant $p_d = p_2$ in a topologically ordered phase is, however, invariant under a smooth deformation of the boundary $\partial A$. In this case, $S_{top} = p_2$ is called the topological entanglement entropy [17, 16].

2.7. Holographic Proof of Strong Subadditivity

One of the most important properties of the entanglement entropy is the strong subadditivity [12] given by the inequalities (7) and (8). This represents the concavity of the entropy and is somehow analogous to the second law of thermodynamics. Actually, it is possible to check that our holographic formula (17) satisfies this property in a rather simple argument as shown in [18] (see also [19] for the explicit numerical studies).

Let us start with three regions $A$, $B$ and $C$ on a time slice of a given CFT so that there are no overlaps between them. We extend this boundary setup toward the bulk AdS. Consider the entanglement entropy $S_{A+B}$ and $S_{B+C}$. In the holographic description (17), they are given by the areas of minimal area surfaces $\gamma_{A+B}$ and $\gamma_{B+C}$ which satisfy $\partial \gamma_{A+B} = \partial (A + B)$ and $\partial \gamma_{B+C} = \partial (B + C)$ as before. Then it is easy to see that we can divide these two minimal surfaces into four pieces and recombine into (i) two surfaces $\gamma'_B$ and $\gamma'_{A+B+C}$ or (ii) two surfaces $\gamma'_A$ and $\gamma'_{C}$, corresponding to two different ways of the recombination. Here we again meant $\gamma'_X$ is a surface which satisfies $\partial \gamma'_X = \partial X$. Since in general $\gamma'_X$s are not minimal area surface, we have $\text{Area}(\gamma'_X) \geq \text{Area}(\gamma_X)$. Therefore, as we can easily find that this argument immediately leads to

\[
\text{Area}(\gamma_{A+B}) + \text{Area}(\gamma_{B+C}) = \text{Area}(\gamma'_B) + \text{Area}(\gamma'_{A+B+C}) \geq \text{Area}(\gamma_B) + \text{Area}(\gamma_{A+B+C}) ,
\]

\[
\text{Area}(\gamma_{A+B}) + \text{Area}(\gamma_{B+C}) = \text{Area}(\gamma'_A) + \text{Area}(\gamma'_{C}) \geq \text{Area}(\gamma_A) + \text{Area}(\gamma_{C}) .
\]

In this way, we are able to check the strong subadditivity (7) and (8).

3. Entanglement Entropy and Confinement/deconfinement Transition

In recent discussions in condensed matter physics, the entanglement entropy is expected to play a role of an appropriate order parameter describing quantum phases and phase transitions. For example, it can be particularly useful for a system which realizes a topological order, such as fractional quantum Hall systems. At low energies, such systems can be described by a topological field theory, and the correlation functions are not useful order parameter as they are trivial. However, the entanglement entropy can capture important information of the topological ground state [17, 16].

The main purpose of this section is to apply the entanglement entropy to the confinement/deconfinement transition of gauge theories [20, 21]. It is much easier to employ our holographic calculation as we need to deal with strongly coupled gauge theories.

3.1. Confinement/Deconfinement Transition

One of the most interesting applications of the entanglement entropy is that it can be used as an order parameter for the confinement/deconfinement phase transition in the confining gauge
theory. When we divide one of the spatial direction into a line segment with length \(l\) and its complement, the entanglement entropy between the two regions measures the effective degrees of freedom at the energy scale \(\Lambda \sim 1/l\). Then, in the confining gauge theory, the behavior entanglement entropy should become trivial (i.e. \(S_A\) approaches to a constant) as \(l\) becomes large, i.e. the infrared limit \(\Lambda \to 0\).

Such a transition can be captured by the holographic entanglement entropy if there are confining backgrounds dual to the confining gauge theories [20, 21]. We find that there are two candidates for the minimal surface with the same endpoints at the boundary in the confining background. One is the two disconnected straight lines extending from the endpoints of the line segment to inside the bulk, and the other is the curved line connecting these two points. The connected curve correspond to the deconfinement phase in dual gauge theory because the entanglement entropy depends on the length \(l\). Above which the disconnected lines are favored, while below which the connected curve is favored, as we will see below explicitly.

For example, we consider the AdS soliton solution [22]

\[
ds^2 = R^2 \frac{dr^2}{r^2 f(r)} + \frac{r^2}{R^2}(-dt^2 + f(r)d\theta^2 + dx_1^2 + dx_2^2),
\]

where \(f(r) = 1 - r_0^4/r^4\) and the \(\theta\) direction is compactified with the radius \(L = \pi R^2/r_0\) to avoid the conical singularity at \(r = r_0\). This can be obtained from the double Wick rotation of the AdS Schwarzschild solution. The dual gauge theory is \(N = 4\) super Yang-Mills on \(R^{1,2} \times S^1\), but the supersymmetry is broken due to the anti-periodic boundary condition for fermions along the \(\theta\) direction. Then the scalar fields acquire non-zero masses from radiative corrections, and the theory becomes almost the same as the \((2 + 1)\)-dimensional pure Yang-Mills, which shows the confinement behavior [22].

To define the entanglement entropy, let us divide the boundary region into two parts \(A\) and \(B\): \(A\) is defined by \(-l/2 \leq x_1 \leq l/2\), \(0 \leq x_2 \leq V(\rightarrow -\infty)\) and \(0 \leq \theta \leq L\), and \(B\) is the complement of \(A\). The minimal surface \(\gamma_A\), whose boundary coincides with the endpoint \(\partial A\), can be obtained by minimizing the area

\[
\text{Area} = LV \int^{l/2}_{-l/2} dx_1 \frac{r}{R} \sqrt{\left(\frac{dr}{dx_1}\right)^2 + \frac{r^4 f(r)}{R^4}}.
\]

Regarding \(x_1\) as a time, then the energy conservation leads to

\[
\frac{dr}{dx_1} = \frac{r^2}{R^2} \sqrt{f(r) \left(\frac{r^6 f(r)}{r_*^6 f(r_*)} - 1\right)},
\]

where \(r_*\) is the minimal value of \(r\). When integrating this relation, we should also take the boundary condition into account

\[
\frac{l}{2} = \int_{r_*}^{r_\infty} dr \frac{R^2}{r^2 \sqrt{f(r) \left(\frac{r^6 f(r)}{r_*^6 f(r_*)} - 1\right)}},
\]

which relates \(r_*\) with \(l\). Here we introduced the UV cutoff at \(r = r_\infty\). After eliminating \(l\) in (2) and (4), we find the entanglement entropy as

\[
S_A^{(\text{com})} = \frac{LV}{2RG_N^{(5)}} \int_{r_*}^{r_\infty} \frac{r^4 \sqrt{f(r)}}{\sqrt{r^6 f(r) - r_*^6 f(r_*)}}.
\]
It is important that $l$ is bounded from above due to the relation (4)
\[ l \leq l_{\text{max}} \simeq 0.22L . \tag{6} \]

Then, when $l$ becomes large, there is no minimal surface that connects the two boundaries of $\partial A$. Instead, the disconnected straight lines actually dominate before $l$ becomes greater than $l_{\text{max}}$. The entanglement entropy is easy to be found
\[
S_{A}^{(\text{discon})} = \frac{V L}{2G_{N}^{(5)}} \int_{r_{0}}^{r_{\infty}} \frac{dr}{R} = \frac{V L}{4G_{N}^{(5)} R} (r_{\infty}^{2} - r_{0}^{2}) . \tag{7}
\]

When $\Delta S_{A}$ becomes positive at the critical length $l_{c}(< l_{\text{max}})$, the disconnected surface dominates, i.e., becomes minimal. Then there happens a phase transition at $l = l_{c}$, which corresponds to the confinement/deconfinement phase transition in dual gauge theory [20, 21]. A similar analysis has been done in [21] in the more general backgrounds including the Klebanov-Strassler solution [23]. These results indicate that the entanglement entropy can be a good order parameter for a phase transition. A benefit of the holographic entanglement entropy is that in order to detect the confinement/deconfinement, we do not need finite temperature black brane solutions, which are often difficult to get analytically.

In summary, our holographic analysis predicts the following behavior of the finite part of the entanglement entropy in $(d+1)$-dimensional confining large $N$ gauge theories at vanishing temperature (we subtracted the area law divergence $\sim a^{-(d-1)}$):
\[
S_{A}(l)|_{\text{finite}} = -VF(l) ,
\]
where $F(l) \simeq c_{1}N^{2}l^{-(d-1)} \ (l \to 0)$ ,
\[
F(l) = c_{2}N^{2} \ (l > l_{c}) , \tag{8}
\]
where $V$ is the volume of the non-compact $d-2$ directions transverse to the separation. The numerical coefficients $c_{1}$ and $c_{2}$ depend on each theory.

Remarkably, the numerical computation of the entanglement entropy in the lattice gauge theory has been done in [24], and the non-analytic behavior (8) has been confirmed. These results would also support the validity of the holographic formula (17) of the entanglement entropy in the AdS/CFT correspondence.

4. Fractional Quantum Hall Effect and Entanglement Entropy from String Theory

Another example where the entanglement entropy plays the role of quantum order parameter is the fractional quantum Hall effect (FQHE). Thus we would like to embed FQHE in string theory using D-branes and discuss its entanglement entropy via AdS/CFT.

The simplest series of the fractional quantum Hall states is known as the Laughlin state which has $\nu = \frac{k}{k}$ with $k$ being an odd integer. At low energies, the Laughlin states are described by a (2+1)-dimensional $U(1)$ Chern-Simons theory coupled to an external electromagnetic field $\tilde{A}$ [25]:
\[
S_{\text{QHE}} = \frac{k}{4\pi} \int A \wedge dA + \frac{e}{2\pi} \int \tilde{A} \wedge dA , \tag{9}
\]
where $A$ is the $U(1)$ gauge field that describes the internal degrees of freedom and $e$ is the charge of the electrons. It follows straightforwardly from (9) that the Hall conductivity (defined by $j_{x} = \sigma_{xy}E_{y}$) is fractionally quantized: $\sigma_{xy} = \frac{\nu e^{2}}{h}$. Since the Chern-Simons gauge theory is topological [26] and has no propagating degree of freedom in contrast to the Maxwell theory, the FQHE provides an example for the gapped many-body system whose ground state can be
described by a topological field theory. Therefore we would like to realize this topological theory in a holographic setup.

We start with the $\mathcal{N} = 4$ super Yang-Mills in four dimensions and compactify one of its three spatial directions (denoted by $\theta$). Then we impose the anti-periodic boundary condition on $\theta$, which makes the fermions massive. The quantum corrections give a mass to scalar fields and in the end we obtain a pure Yang-Mills theory in $(2 + 1)$ dimensions in the IR limit. This theory is dual to the $AdS_5$ soliton (or double Wick-rotated $AdS_5$ black brane) in IIB string theory as we explained in (1). Since the cycle $\partial \theta$ shrinks at $r = r_0$, the two-dimensional space spanned by $(r, \theta)$ is topologically a two-dimensional disk $D_2$. The boundary of the disk is at $r = \infty$ and this is the boundary of the AdS. The dual gauge theory lives in $R^{1,2}$ whose coordinate is $(t, x, y)$.

Now we would like to deform this theory so that it includes the Chern-Simons term. Remember the WZ-term of a D3-brane,

$$\frac{1}{4\pi} \int_{D3} \chi F \wedge F = - \frac{1}{4\pi} \int_{D3} d\chi \wedge A \wedge F,$$

where $\chi$ is the axion field in IIB theory. If we assume the background axion field $\chi = \frac{k}{L} \theta$, this leads to the CS coupling

$$\frac{k}{4\pi} \int_{R^{1,2}} A \wedge F.$$

In this way, we get a Maxwell-Chern-Simons theory which flows into the level $k$ pure Chern-Simons theory in the IR. This argument can be easily generalized to $N$ D3-branes and we get the non-abelian Chern-Simons term $\frac{k}{4\pi} \int_{R^{1,2}} \mathrm{Tr} \left( A \wedge dA + \frac{2}{3} A^3 \right)$.

The axion field $\chi = \frac{k}{L} \theta$ can be regarded as $k$ D7-branes located at $r = r_0$. Therefore we find that the $U(N)_k$ Yang-Mills-Chern-Simons theory is holographically dual to this $AdS_5$ soliton background with the $k$ D7-branes. The IR limit of the CFT side is dual to the small $r$ region of the dual background and this is given by the D7-brane theory. Due to the Chern-Simons term $\frac{N}{4\pi} \int_{R^3} A \wedge F$ from the RR 5-form flux, we get $U(k)_N$ pure Chern-Simons theory. Thus this holography in the low energy limit is equivalent to the level-rank duality of the Chern-Simons theory. Indeed, the level-rank duality becomes more direct for $U(N)_k$ than for $SU(N)_k$ as shown in [27]. Indeed, we can prove the following identity for the partition functions on $S^3$ [27]:

$$Z(S^3, U(N)_k) = Z(S^3, U(k)_N), \quad Z(S^3, SU(N)_k) = \sqrt{\frac{k}{N}} Z(S^3, SU(k)_N).$$

Now we would like to examine the gauge theory dual to the gravity background (1) with $k$ D7-branes. To interpret our model as one for the quantum Hall effect, we need to couple it to an external gauge field. To this end, we assume that the background RR 2-from field has the form

$$B_{RR} = A_{RR} \wedge d\tilde{\theta},$$

where $A_{RR}$ is a 1-form in $R^{1,2}$ and will serve as the external gauge field. We define $\tilde{\theta} \equiv \frac{2\pi}{L} \theta$ so that the period of $\tilde{\theta}$ is $2\pi$.

The relevant RR-coupling on D3-branes reads

$$\frac{1}{4\pi^2} \int_{D3} B_{RR} \wedge \text{Tr} F = \frac{1}{2\pi} \int_{R^{1,2}} A_{RR} \wedge \text{Tr} F.$$

Therefore in the IR limit, the QFT side becomes the Chern-Simons theory coupled to an external gauge field

$$S_{D3} = \frac{k}{4\pi} \int \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) + \frac{1}{2\pi} \int A_{RR} \wedge \text{Tr} F,$$
This is a non-abelian version of the standard Chern-Simons description of the quantum Hall effect.

It is also important to identify the anyons in this system. A D5-brane wrapped on $S^5$ is the baryon vertex in $AdS_5 \times S^5$. Due to the string creation, it comes with $N$ F-strings stretching between the D5 and $N$ D3-branes. The F-strings induce a unit charge for each $U(1)$ gauge theory in $U(1)^N \subset U(N)$. Thus when two baryons are interchanged, the full wave-function acquires a phase factor $e^{\frac{2\pi i N}{k}}$, which means that the baryons are anyons for generic values of $N$. We would like to ask the reader to refer to the original paper [28] for the holographic calculations of the Hall conductivity.

The topological entanglement entropy $S_{\text{top}}$ [16, 17] is defined as the finite part of the entanglement entropy

$$S_A = \gamma \frac{l}{a} + S_{\text{top}},$$

(16)

where $a$ is the UV cutoff and $\gamma$ a certain numerical factor proportional to the number of UV degrees of freedom. The entanglement entropy $S_A$ is defined as the von-Neumann entropy when we trace out a subsystem $A$ on a time-slice. We assume $A$ is a two-dimensional disk in a time-slice. $l$ is the length of the boundary $\partial A$ of $A$. When the theory has a mass gap, the quantity $S_{\text{top}}$ can be shown to be invariant under a continuous deformation of the region $A$ [16, 17].

In the holographic calculation [9], we introduce the (negative) deficit angle $\delta = 2\pi (1 - n)$ on the $\partial A$ at $r = \infty$. Then we extend this deficit angle surface toward the bulk AdS, called $\gamma$. Since the entanglement entropy is defined by $S_A = -\frac{\partial}{\partial n} \log Z_n$, where $Z_n$ is the partition function on the manifold with the deficit angle $\delta = 2\pi (1 - n)$, in our setup we obtain

$$S_A = \frac{\text{Area}(\gamma)}{4G_N} + S_{D7},$$

(17)

where the part $S_{D7}$ is the contribution of $k$ D7-branes. The action principle tells us that $\gamma$ is the minimal area surface [9] whose boundary coincides with $\partial A$. It is clear from this holographic expression (17) that the topological entanglement entropy is given by $S_{D7}$ as the Chern-Simons term appears due to the D7-branes. Since we are interested in the $(2+1)$-dimensional Yang-Mills-Chern-Simons theory, the Kaluza-Klein modes need to be negligible. This requires that the size of region $A$ be much larger than the compactified radius $L$. When there is no D7-brane, the surface $\gamma$ ends at the bubble wall $r = r_0$ as shown in [20, 21, 29]. In our case with D7-branes, and with their backreactions neglected, the deficit angle surface $\gamma$ is extended into the $(2+1)$-dimensional theory on the $k$ D7-branes. Since the large $N$ limit corresponds to the large level limit, the D7-brane theory becomes the classical Chern-Simons theory. Then, the AdS/CFT tells us that $S_{\text{top}}$ is given by $S_{D7}$, i.e. the topological entanglement entropy of $U(k)_N$ Chern-Simons theory. Using the level-rank duality, this reproduces the $S_{\text{top}}$ of the original CS theory. In this way, we find how to reproduce the correct topological entanglement entropy via AdS/CFT. Its more direct calculation from the bulk gravity itself will be an important future problem.

5. Lifshitz-like Fixed Points and Entanglement Entropy

5.1. Holographic Dual of Lifshitz-like Fixed Points

It is natural to try to extend the AdS/CFT correspondence to a holography for the following anisotropic spacetime

$$ds^2 = r^{2\varepsilon} \left( -dt^2 + \sum_{i=1}^p dx_i^2 \right) + r^2 \sum_{j=p+1}^d dy_j^2 + \frac{dr^2}{r^2},$$

(18)
where $0 \leq p \leq d - 1$, and the parameter $z(\neq 1)$ measures the degree of Lorentz symmetry violation and anisotropy. Since the metric (18) is invariant under the scaling $(t, x_i, y_j, r) \rightarrow (\lambda^2 t, \lambda^2 x_i, \lambda y_j, \lambda r)$, we expect that on the field theory side it is dual to a fixed point which is invariant under the scaling transformation
\[(t, x_i, y_j) \rightarrow (\lambda^2 t, \lambda^2 x_i, \lambda y_j).\] (19)

In general, fixed points with the anisotropic scaling property (19) are called Lifshitz(-like) fixed points. This generalization of AdS/CFT correspondence to Lifshitz-like fixed points (18) was first proposed and analyzed in [30] in the particular case of $p = 0$. The simplest case with $p = 0$ represents non-relativistic fixed points with dynamical critical exponent $z$, which appear in many examples of quantum criticality in condensed matter physics (see the references in [30]). The other cases where $1 \leq p \leq d - 1$ are not only generalizations of $p = 0$ case but can also be interpreted as space-like anisotropic fixed points. Lifshitz fixed points with space-like anisotropic scale invariance appear in realistic magnets such as MnP and the axial next-nearest-neighbor Ising model.

To understand holographic duals of such gravity backgrounds, it is the best to embed them into string theory, where microscopic interpretations are often possible by using D-branes. One such example can be obtained by considering the near horizon of the backreacted geometry for a D3-D7 system which we studied in the previous section as found in [31]. The metric in Einstein frame is given by
\[ds^2_{E} = \tilde{R}^2 \left[ r^2 (-dt^2 + dx^2 + dy^2) + \frac{dr^2}{r^2} + R^2 ds^2_{\mathbb{S}^5} \right],\] (20)
where $R^2 = \frac{12}{11} \tilde{R}^2$ and $R$ is the standard radius of $AdS_5 \times S^5$. The dilaton now depends on the radial coordinate as $e^\phi = e^{\phi_0} r^{2/3}$. The metric (20) is invariant under the scaling
\[(t, x, y, w, r) \rightarrow \left( \lambda t, \lambda x, \lambda y, \lambda^2 w, \frac{r}{\lambda} \right),\] (21)
and therefore is expected to be holographically dual to Lifshitz-like fixed points with space-like anisotropic scale invariance. Note that the metric corresponds to $z = 3/2$, $p = 2$ and $d = 3$.

We can also straightforwardly generalize them to black brane solutions which have regular event horizons. The metric in the Einstein frame is
\[ds^2_{E} = \tilde{R}^2 \left[ r^2 (-F(r)dt^2 + dx^2 + dy^2) + \frac{dr^2}{r^2 F(r)} + R^2 ds^2_{\mathbb{S}^5} \right],\] (22)
where \[F(r) = 1 - \frac{\mu}{r^{1+\frac{3}{d}}}.\] (23)
The constant $\mu$ represents the mass parameter of the black brane. The dilaton and RR fields remain the same.

Requiring the smoothness of the Euclidean geometry of (22) gives the Hawking temperature
\[T_H = \frac{11}{12 \pi} \mu^{\frac{1}{d+3}}.\] (24)
The Bekenstein-Hawking entropy is then
\[S_{BH} = \gamma \cdot \left( \frac{\pi^3}{\text{Vol}(\mathbb{S}^5)} \right) \cdot N^2 \cdot T_H^8 \cdot V_2 \cdot L,\] (25)
where $\gamma \approx 3.729$, and $V_2$ represents the area in the $(x, y)$ direction. The entropy (25) is proportional to $N^2$ and thus is consistent with the planar limit of a certain gauge theory.
For an anisotropic system, an interesting question is “how does the scaling behavior of the entanglement entropy depend on the direction along which the subsystems are delineated?” Here we will study the entanglement entropy of various subsystems of the \((x, y, w)\) space at the boundary \((r \to \infty)\) of the 5D part of the D3-D7 scaling solution \((20)\). The field theoretical computation of the entanglement entropy is expected to be difficult as the system will be strongly coupled. We will instead compute its holographic dual on the gravity side using the formula \((17)\).

Let’s first consider the subsystem \(A\) cut out along the \(x\)-direction:

\[ x \in [0, \ell_x < L_x], \quad y \in [0, L_y], \quad w \in [0, L_w]. \]  

We finally obtain the holographic entanglement entropy for the subsystem divided out along the \(x\)-direction:

\[ S_{EE-x} = \left( \frac{11}{12} \right)^3 \frac{\pi^2 \text{Vol}(X_5)}{N^2 L_y L_w} \frac{1}{d-1} \left[ \frac{1}{a^{d-1}} - \left( \frac{2}{\ell_x} \right)^{d-1} \left( \frac{\sqrt{\pi} \Gamma \left( \frac{1+d}{2d} \right)}{\Gamma \left( \frac{1}{2d} \right)} \right)^d \right] \]

with \(d = \frac{8}{3}\). \(\gamma_1\) and \(\gamma_2\) are numerical constants.

The next subsystem we can consider is to divide along the \(w\)-direction:

\[ w \in [0, \ell_w < L_w]. \]

This leads to the entanglement entropy of subsystem along the \(w\)-direction

\[ S_{EE-w} = \left( \frac{11}{12} \right)^3 \frac{\pi^2 \text{Vol}(X_5)}{N^2 L_x L_y} \frac{1}{D-1} \left[ \frac{1}{a^{D-1}} - \left( \frac{3}{2} \right)^{d_w-1} \left( \frac{2}{\ell_w} \right)^{d-1} \left( \frac{\sqrt{\pi} \Gamma \left( \frac{1+d_w-1}{2d_w} \right)}{\Gamma \left( \frac{1}{2d_w} \right)} \right)^{d_w} \right] \]

where \(D - 1 \equiv \frac{2(d_w-1)}{3} = 2\). The negative finite part has the same form as the result for the subsystem divided along the \(x\)-direction with \(d = \frac{8}{3}\) replaced by \(d_w = 4\). We can confirm that both results are consistent with the anisotropic scale invariance of our system.

6. Conclusions

In this talk, we reviewed a holographic calculation of entanglement entropy and its application to various systems. In all examples we analyzed we find complete agreement between the holographic analysis and the field theoretic results. Moreover, our holographic method predicts in strongly coupled higher dimensional field theories, where the field theoretic analysis is very complicated. The most important future problem is to rigorously prove our holographic entanglement entropy formula from the first principle of AdS/CFT correspondence. Another interesting problem to understand the holography in de Sitter spacetime as almost nothing has been known on this in spite of many efforts. Since the entanglement entropy is quite general quantity and needs only quantum mechanics for its definition, we can always study this quantity for any kinds of holography.
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