NEW JACOBI-LIKE IDENTITIES FOR $Z_K$ PARAFERMION CHARACTERS

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ABSTRACT

We state and prove various new identities involving the $Z_K$ parafermion characters (or level-$K$ string functions) $c^\ell_n$ for the cases $K = 4$, $K = 8$, and $K = 16$. These identities fall into three classes: identities in the first class are generalizations of the famous Jacobi $\vartheta$-function identity (which is the $K = 2$ special case), identities in another class relate the level $K > 2$ characters to the Dedekind $\eta$-function, and identities in a third class relate the $K > 2$ characters to the Jacobi $\vartheta$-functions. These identities play a crucial role in the interpretation of fractional superstring spectra by indicating spacetime supersymmetry and aiding in the identification of the spacetime spin and statistics of fractional superstring states.

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1. INTRODUCTION

$Z_K$ parafermion theories$^{[1]}$ have recently found a new application as the basic worldsheet building blocks of fractional superstrings.$^{[2]}$ Fractional superstrings are generalizations of the traditional superstring and heterotic string, and are constructed essentially by replacing the worldsheet supersymmetry of the superstring with a *fractional* supersymmetry (parametrized by an integer $K$) which relates worldsheet bosons not to fermions but to $Z_K$ parafermions. It is found that the critical spacetime dimensions of such string theories are less than ten, and are in fact given by the simple formula

$$D_c = 2 + \frac{16}{K}, \quad K \geq 2.$$  \hspace{1cm} (1.1)

The special case $K = 2$ reproduces the usual superstring and heterotic string with critical dimension $D_c = 10$, and the cases with $K = 4$, $K = 8$, and $K = 16$ yield new fractional superstring theories with critical spacetime dimensions $D_c = 6$, $D_c = 4$, and $D_c = 3$ respectively.

The worldsheet field content of these string theories consists in general of bosons and $Z_K$ parafermions; the special-case $Z_2$ parafermions are equivalent to ordinary Majorana-Weyl fermions. The partition functions of fractional superstrings therefore involve the $Z_K$ parafermion characters for $K > 2$, just as the partition functions of the superstring and heterotic string involve the ordinary “$Z_2$” characters of the Majorana-Weyl fermions. Similarly, just as the usual fermion characters can be written in terms of the classical Jacobi theta-functions $\vartheta_i$, the more general $Z_K$ parafermion characters can be written in terms of the so-called “string functions”...
$c_n^\ell$ originally introduced in the study of infinite-dimensional Lie algebras.\cite{3} For $K = 2$ these string functions are equivalent to the Jacobi $\vartheta$-functions, but such is of course not the case for $K > 2$.

As is well-known in the $K = 2$ superstring theory, the spacetime properties of the string spectrum are reflected at the partition-function level in the properties of these fermion characters. For example, any superstring or heterotic string spectrum exhibiting a spacetime supersymmetry gives rise to a partition function proportional to the factor

$$J \equiv \vartheta_3^4 - \vartheta_2^4 - \vartheta_4^4,$$

and the well-known Jacobi identity $J = 0$ is therefore responsible for the vanishing of such partition functions at all mass levels of the theory. This reflects the exact cancellation of spacetime bosonic states (which arise from the Neveu-Schwarz sector of the theory and yield the terms $\vartheta_3^4 - \vartheta_4^4$ within $J$) and spacetime fermionic states (which arise from the Ramond sector and yield the term $\vartheta_2^4$).

A similar situation exists for the $K > 2$ fractional superstrings, where once again we expect the spacetime properties of the fractional superstring spectrum to be reflected in the partition function through identities satisfied by the $\mathbb{Z}_K$ parafermion characters (i.e., by the string functions $c_n^\ell$). Indeed, since the consistency of fractional superstrings remains to be verified, the existence and use of such identities provides an important step towards that goal. In this paper we obtain and prove these new string-function identities, and discuss as well their relevance to fractional superstrings.
In particular, we state and prove three series of identities. First is a series of
identities for the $Z_K$ parafermion characters which are analogous to the $K = 2$
Jacobi identity and which can be considered to be its $K > 2$ generalizations.
We will see that these new identities arise naturally in the partition functions
of fractional superstrings, and their presence can therefore be interpreted as a
signature of spacetime supersymmetry in fractional superstring spectra. For each
value of $K$, we will find that the corresponding identity in this series involves a
vanishing combination of only the $Z_K$ parafermion characters; we will therefore, for
reasons to become clear, refer to this series of identities as the $[K, 0]$ series. Second,
just as the individual terms within the $K = 2$ Jacobi factor $J$ can be recognized
as arising from either spacetime bosonic or fermionic sectors, we will see that a
similar self-consistent grouping of terms is possible within each of our new Jacobi-
like identities for $K > 2$. This will result in a second series of new string-function
identities, each of which (as we will see) relates the $K > 2$ characters to the $K = 2$
characters. We will therefore refer to this series of identities as the $[K, 2]$ series.
Finally, we find a third series of identities which generalize another well-known
ϑ-function identity $\vartheta_2\vartheta_3\vartheta_4 = 2\eta^3$ (where $\eta$ is the Dedekind $\eta$-function, the inverse
of the boson character). Just as this identity relates the boson character $\eta^{-1}$ to
the fermion characters $\vartheta_i$, our third series of identities relates the boson character
$\eta^{-1}$ to the $Z_K$ parafermion characters $c^{\ell}_n$. We will therefore refer to this series of
identities as the $[K, 1]$ series, since (as we will see) the $K = 1$ character is equal
to the boson character $\eta^{-1}$. As our notation suggests, these are undoubtedly only
three of many such $[K_1, K_2]$-type series of identities relating the characters of $Z_{K_1}$,
and $\mathbb{Z}_{K_2}$ parafermions to each other, and in this paper we also discuss how such general $[K_1, K_2]$ identities may be obtained and proven.

Clearly, many of these new identities may have interpretations in the theory of $\mathbb{Z}_K$ parafermions which are independent of fractional superstrings. As a striking example, our $[16, 2]$ identities indicate that the simple algebraic differences of many of the $\mathbb{Z}_{16}$ characters are nothing but the $\mathbb{Z}_2$ (fermion) characters; the implications of this fact for the relationship between ordinary fermions and $\mathbb{Z}_{16}$ parafermions are yet to be investigated. Similar relationships exist as well for other values of $K$.

Therefore, we have organized this paper in such a way that it can be read without a detailed understanding of fractional superstrings per se. In Sect. 2.1 we provide a brief introduction to the $\mathbb{Z}_K$ parafermion theory and review the definitions and properties of the $\mathbb{Z}_K$ parafermion characters (or level-$K$ string functions). In Sect. 2.2 we then discuss the role of the $\mathbb{Z}_K$ parafermion theory in the fractional superstring, along the way introducing some of our new identities and discussing the fractional-superstring contexts in which they arise. Sect. 3 is simply a list of all of these new identities, and in Sect. 4 we prove these identities using some powerful results from the theory of modular functions. The proofs of the $K > 2$ identities exactly mirror the proofs of the well-known $K = 2$ special cases, and accordingly we have kept the discussion in Sect. 4 sufficiently general so that it is clear how additional $[K_1, K_2]$ identities may be obtained and proven. A reader unconcerned with fractional superstrings can skip Sect. 2.2, but it seems that the fractional superstring framework provides interesting physical interpretations for many of these new identities.
2. \( Z_K \) PARAFERMIONS AND FRACTIONAL SUPERSTRINGS

In this section we first review the definitions and properties of the \( Z_K \) parafermion characters (or level-\( K \) string functions) which appear in our new identities. We then provide a brief but self-contained introduction to the fractional superstring idea, presenting many of these new identities in the physical contexts in which they arise and discussing some of their implications.

2.1. THE \( Z_K \) PARAFERMION CHARACTERS

The \( Z_K \) parafermion theory\(^{[1,4]}\) is closely related to, and in fact can be derived from, the \( SU(2)_K \) Wess-Zumino-Witten (WZW) theory\(^{[5]}\). As is well-known, the \( SU(2)_K \) WZW theory can be viewed as a tensor product of two independent theories: the first is that of a free \( U(1) \) boson compactified on a circle of radius \( 2\sqrt{K} \), and the remaining \( SU(2)_K/U(1) \) coset theory is the \( Z_K \) parafermion theory. It therefore follows that the characters of these remaining \( Z_K \) parafermion fields can be obtained from the full \( SU(2)_K \) characters by appropriately factoring out the \( U(1) \) boson characters. We now review precisely how this is done.

We begin by considering the chiral \( SU(2)_K \) WZW theory\(^{[5]}\). This theory consists of holomorphic primary fields \( \Phi^j_m(z) \) which can be organized into \( SU(2) \) representations labelled by \( j \in Z/2 \), where \( 0 \leq j \leq K/2 \) and \( |m| \leq j \) with \( j - m \in Z \). Since \( SU(2)_K \) always has a \( U(1) \) subgroup which can be bosonized as a free boson \( \varphi(z) \) compactified on a circle of radius \( 2\sqrt{K} \), we can correspondingly factor the primary
fields as
\[
\Phi^j_m(z) = \phi^j_m(z) \exp \left\{ \frac{m}{\sqrt{K}} i \varphi(z) \right\}.
\] (2.1)

The \(\phi^j_m(z)\) are therefore primary fields of the coset \(SU(2)_K/U(1)\) theory, i.e., of the \(\mathbb{Z}_K\) parafermion theory. The parafermion fields \(\phi^j_m\) have conformal dimensions \(h^{2j}_{2m}\) where
\[
h_n^j = \frac{\ell(\ell + 2)}{4(K + 2)} - \frac{n^2}{4K} \quad \text{for } |n| \leq \ell,
\] (2.2)

and it is convenient to extend the definition of the parafermion fields outside the range \(|m| \leq j\) via the identifications
\[
\phi^j_m = \phi^j_{m+K} = \phi^{K/2-j}_{-(K/2-m)}.\] (2.3)

The fusion rules of these parafermion fields \(\phi^j_m\) follow from those of the \(SU(2)_K\) theory:
\[
[\phi^j_{m_1}] \times [\phi^j_{m_2}] = \sum_{j=|j_1-j_2|}^{r} [\phi^j_{m_1+m_2}]
\] (2.4)

where \(r = \min(j_1 + j_2, K - j_1 - j_2)\) and where the sectors \([\phi^j_m]\) include the primary fields \(\phi^j_m\) and their parafermion descendants. Note that a given field \(\phi^j_m\) may appear multiple times in the theory.
The $SU(2)_K$ characters $\chi_\ell(\tau, z)$ for spin $j = \ell/2$ are given in Ref. [3]:

$$\chi_\ell(\tau, z) = \frac{\Theta_{\ell+1, K+2}(\tau, z) - \Theta_{-\ell-1, K+2}(\tau, z)}{\Theta_{1,2}(\tau, z) - \Theta_{-1,2}(\tau, z)} \quad (2.5)$$

where the classical (Jacobi-Riemann) $\Theta$-functions are defined by

$$\Theta_{n,L}(\tau, z) \equiv \sum_{s \in \mathbb{Z} + n/2L} \exp\left\{2\pi i L(s^2 \tau - sz)\right\} \quad \text{for } n \in \mathbb{Z} \pmod{2L}. \quad (2.6)$$

These $\Theta$-functions have relatively simple properties under modular transformations

$$\tau \rightarrow \gamma \tau \equiv \frac{a \tau + b}{c \tau + d}, \quad (2.7)$$

where $\gamma \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an element of the homogeneous modular group $\tilde{\Gamma} \equiv SL(2, \mathbb{Z})$ if $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. The modular group $\Gamma \equiv \tilde{\Gamma}/\{\pm 1\}$ is generated by $T: \tau \rightarrow \tau + 1$ and $S: \tau \rightarrow -1/\tau$, and under the action of these two generators we have

$$\Theta_{n,L}(\tau + 1, z) = \exp\left\{\frac{2\pi i n^2}{4L}\right\} \Theta_{n,L}(\tau, z), \quad (2.8)$$

and

$$\Theta_{n,L}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \sqrt{-\frac{i \pi}{2n}} \exp\left\{\frac{\pi i L z^2}{\tau}\right\} \sum_{n' = 0}^{2L-1} \exp\left\{-\frac{\pi i nn'}{L}\right\} \Theta_{n',L}(\tau, z). \quad (2.9)$$

Here the square root indicates the branch with non-negative real part. From these results the modular transformation properties of the full $SU(2)_K$ characters $\chi_\ell(\tau, z)$ can be obtained.
We are primarily interested in obtaining the characters $Z_{2m}^{2j}$ of the parafermion sectors $[\phi_m^j]$, for the string functions $c_n^\ell$ are simply related to these characters via

$$Z_n^\ell(\tau) = \eta(\tau) c_n^\ell(\tau)$$  \hspace{1cm} (2.10)$$

where $\ell \equiv 2j$, $n \equiv 2m$, and where $\eta$ is the Dedekind $\eta$-function:

$$\eta(\tau) \equiv q^{1/24} \prod_{n=1}^\infty (1 - q^n) = \sum_{n=0}^\infty (-1)^n q^{3(n-1/6)^2/2}$$  \hspace{1cm} (2.11)$$

with $q \equiv \exp\{2\pi i \tau\}$. However, since the $Z_K$ parafermion theory is equivalent to the $SU(2)_K/U(1)$ coset theory, the parafermion characters $Z_n^\ell$ can be obtained by expanding the full $SU(2)_K$ characters $\chi(\tau, z)$ in a basis of $U(1)$ characters:

$$\chi(\tau, z) = \sum_{n=-\ell}^{2K-\ell-1} Z_n^\ell(\tau) \frac{\Theta_n, K(\tau, z)}{\eta(\tau)} = \sum_{n=-\ell}^{2K-\ell-1} c_n^\ell(\tau) \Theta_n, K(\tau, z).$$  \hspace{1cm} (2.12)$$

Here $\Theta_n, K(\tau, z)/\eta(\tau)$ are the characters of a $U(1)$ boson compactified at radius $2\sqrt{K}$. Eq. (2.12), then, can be taken as a definition of the string functions $c_n^\ell$.

Explicit expressions for the $c_n^\ell$ can be extracted from (2.5), (2.6), and (2.12). The original formula obtained by Kač and Peterson\cite{6} is

$$c_n^\ell(\tau) = [\eta(\tau)]^{-3} \sum_{x,y} \text{sign}(x) q^{x^2(K+2) - \ell^2 K}$$  \hspace{1cm} (2.13)$$

where the prime on the summation indicates that $x$ and $y$ must be chosen so that three conditions are satisfied: (1) $-|x| < y \leq |x|$; (2) either $x$ or $(1/2 - x)$ must equal
For many calculational purposes, however, a useful alternative expression is\(^7\)

\[
\ell \in \mathbb{Z} \quad \text{and} \quad (\ell + 1)/[2(K + 2)] \mod 1; \quad \text{and} \quad (3) \text{ either } y \text{ or } (\frac{1}{2} + y) \text{ must equal } n/(2K) \mod 1.
\]

The string functions exhibit the symmetries

\[
c_n^{\ell} = c_{-n}^{\ell} = c_{K-n}^{\ell} = c_{n+2K}^{\ell},
\]

as a consequence of which for any \(K\) we are free to choose a “basis” of string functions \(c_n^{\ell}\) where \(0 \leq \ell \leq K\) and \(0 \leq n \leq n_{\text{max}}\), where \(n_{\text{max}}\) equals \(\ell\) if \(\ell \leq K/2\), and \(\ell - 2\) otherwise. Note also that the \(K\)-dependence of the string functions is suppressed in this notation. From (2.14), then, we see that the string functions \(c_n^{\ell}\) all take the general form \(q^{H_n^{\ell}}(1 + ...)\) where within the parentheses all powers of \(q\) are non-negative integers, and where

\[
H_n^{\ell} \equiv h_n^{\ell} + \frac{1}{4(K + 2)} - \frac{1}{8} = h_n^{\ell} - \frac{K}{8(K + 2)}.
\]
under $T$ they transform as eigenfunctions:

$$c_n^\ell(\tau + 1) = \exp\left\{2\pi i H_n^\ell\right\} c_n^\ell(\tau), \quad (2.17)$$

and under $S$ they mix among themselves:

$$c_n^\ell(-1/\tau) = \frac{1}{\sqrt{-i\tau}} \frac{1}{\sqrt{K(K+2)}} \sum_{\ell' = 0}^{K} \sum_{n' = -K+1 \atop \ell' - n' \in 2\mathbb{Z}} b(\ell, n, \ell', n') c_{n'}^{\ell'}(\tau) \quad (2.18)$$

where the mixing coefficients $b(\ell, n, \ell', n')$ are

$$b(\ell, n, \ell', n') = \exp\left\{\frac{i\pi nn'}{K}\right\} \sin\left\{\frac{\pi(\ell + 1)(\ell' + 1)}{K + 2}\right\}. \quad (2.19)$$

The first square root in (2.18) once again indicates the branch with non-negative real part. It is often convenient to define the linear combinations $d_n^{\ell\pm} \equiv c_n^\ell \pm c_n^{K-\ell}$ when $\ell$ and $n$ are even: if $K \in 4\mathbb{Z}$, then these $d^{\pm}$-functions are also eigenfunctions of $T$, and (2.18) implies that that the $d^+$-functions close exclusively amongst themselves under $S$. The $d^-$-functions, on the other hand, close exclusively with themselves and with the odd $(\ell, n)$ string functions under $S$. We will see that for $K > 2$ all of our new identities involve only the $d_n^\ell \equiv d_n^{\ell+}$ functions.

As expected, the string functions $c_n^\ell$ reduce to the better-known Dedekind $\eta$-function (2.11) and the Jacobi $\vartheta$-functions for the $K = 1$ and $K = 2$ special cases respectively. In particular, for $K = 1$ there is only one independent string
function $c_0^0$, and since the “$Z_1$ parafermion” theory $SU(2)_1/U(1)$ consists of only the identity field $\phi_0^0 = 1$, we immediately find

$$K = 1 : \quad Z_0^0 = \eta c_0^0 = 1 \quad \Rightarrow \quad c_0^0 = \eta^{-1}.$$ (2.20)

Similarly, for $K = 2$, there are precisely three independent string functions $c_0^0$, $c_1^1$, and $c_2^2$, and these can be expressed in terms of the three non-vanishing Jacobi $\vartheta$-functions

$$\vartheta_2(\tau) \equiv 2 q^{1/8} \prod_{n=1}^{\infty} (1 + q^n)^2 (1 - q^n)$$
$$\vartheta_3(\tau) \equiv \prod_{n=1}^{\infty} (1 + q^{n-1/2})^2 (1 - q^n)$$
$$\vartheta_4(\tau) \equiv \prod_{n=1}^{\infty} (1 - q^{n-1/2})^2 (1 - q^n)$$ (2.21)

via the relations

$$K = 2 : \quad \begin{cases} 
2 (c_1^1)^2 &= \vartheta_2/\eta^3 \\
(c_0^0 + c_2^2)^2 &= \vartheta_3/\eta^3 \\
(c_0^0 - c_2^2)^2 &= \vartheta_4/\eta^3.
\end{cases}$$ (2.22)

We thus see that for $K = 1$ and $K = 2$, the string functions reproduce the boson and fermion characters respectively; indeed, (2.22) reflects the fact that the $Z_2$ parafermion theory is equivalent to that of a free fermion in two dimensions. For $K > 2$, on the other hand, the string functions are the characters of parafermions, and as such they are not a priori related in these simple ways to the boson and fermion characters. However, we will see that our $[K,1]$ and $[K,2]$ series of identities nevertheless provide such unexpected relations. These new identities arise naturally in the partition functions of fractional superstrings, and it is in such theories that they find possible physical interpretations.
2.2. Fractional Superstrings

We now provide a brief introduction to the theory of fractional superstrings.[2] Since our purpose here is to motivate many of our new string-function identities and discuss some of the physical contexts in which they arise, our treatment will focus primarily on fractional-superstring partition functions. A fuller treatment of these and other aspects of the fractional superstring can be found in Refs. [2] and [8].

As indicated in the Introduction, the basic idea behind the construction of the fractional superstring is simple.[2] Let us first recall the special case of the superstring. The worldsheet structure of the superstring theory is closely related to the $SU(2)_2$ WZW theory: the worldsheet superpartner of the coordinate boson $X^\mu$ is a Majorana fermion $\psi^\mu$ which can be described by the $SU(2)_2/U(1)$ coset theory, and the spacetime coordinate $X^\mu$ can be interpreted as the remaining $U(1)$ boson but with its radius of compactification relaxed to infinity. (The spacetime index $\mu$ runs from 0 to $D_c - 1$.) This boson-decompactification procedure destroys the $SU(2)_2$ symmetry of the original WZW model, but its superconformal symmetry survives and exists on the worldsheet.

The fractional superstring theory is related in the same way to the $SU(2)_K$ WZW theory for $K \geq 2$. The coset theory $SU(2)_K/U(1)$ is the $\mathbb{Z}_K$ parafermion theory,[1,4] and once again we obtain the spacetime coordinate field $X^\mu$ by completely decompactifying the remaining WZW $U(1)$ boson. Replacing the supercurrent for $K > 2$ is a new chiral current[5,9] whose conformal dimension is $(K+4)/(K+2)$; these new
currents have fractional spin, and transform the bosonic $X^\mu$ fields to fractional-spin fields on the worldsheet. It is natural to refer to this remaining worldsheet symmetry as a fractional superconformal symmetry,$[^{10}]$ and to the strings based on these worldsheet fractional supersymmetries as fractional superstrings.

We are interested in constructing the one-loop partition functions $Z$ that such fractional superstring theories must have; in particular, we focus here on the “Type II” fractional superstring in which both the left-moving and right-moving worldsheet theories exhibit a level-$K$ fractional supersymmetry. We can therefore consider, for simplicity, only the holomorphic components of $Z$; these are the terms arising from the left-moving worldsheet degrees of freedom. As discussed in Sect. 2.1, for each $K$ such terms will be products of the characters of free worldsheet bosons $X^\mu$ and worldsheet $Z_K$ parafermions $(\phi^j_m)^\mu$: each coordinate boson $X^\mu$ contributes to the partition function a factor of $1/\eta$ (the character of an infinite-radius boson), while its fractional superpartner (the corresponding worldsheet parafermion) contributes a factor of $\eta c_n^\ell$. Thus, the net holomorphic contribution to the fractional-superstring partition function from each spacetime dimension is a factor of one string function $c_n^\ell$. Even though the fractional superstring theory is formulated in $D_c$ spacetime dimensions (i.e., even though the full worldsheet structure of the fractional superstring is a tensor product of $D_c$ copies of the individual boson/parafermion theories), the large gauge symmetry of critical string theory is expected to remove all time-like and longitudinal components, thereby leaving a spectrum of physical states arising from the excitations of fields corresponding to only the $D_c - 2$ transverse dimensions.$[^{2,8}]$ This is completely analogous to the case of ordinary $K = 2$ superstring theory, in which there are only $D_c - 2 = 8$ “effective”
dimensions giving rise to propagating fields (as is evident, for example, in light-cone gauge). The holomorphic components of $K$-fractional superstring partition functions therefore consist of $(D_c - 2)$ factors of level-$K$ string functions, and take the general form $c^{D_c-2}$.

Let us now consider the contribution to the partition function from some of the low-lying states of the fractional superstring; this will, as a byproduct, yield one method of determining the critical dimension $D_c$. Recall that in general an expansion $\sum_n a_n q^n$ of the holomorphic factors within the partition function $Z$ indicates the net number $a_n$ of states with (left-moving) mass $M^2 = n$ contributing to that term. As explained in Refs. [2] and [8], the left-moving bosonic vacuum state of the $D_c$-dimensional fractional superstring corresponds to $(c_0^0)^{D_c-2}$. Since (2.16) indicates that this term takes the form

$$ (c_0^0)^{D_c-2} \sim q^{H_{\text{tach}}} (1 + \ldots) \quad \text{where} \quad H_{\text{tach}} = \frac{(D_c - 2)K}{8(K + 2)}, \quad (2.23) $$

we see that for $D_c > 2$ the bosonic ground state is tachyonic. This is analogous to the case of ordinary Type II superstrings, in which the bosonic ground state is tachyonic with $M^2 = H_{\text{tach}} = -\frac{1}{2}$. Similarly, the left-moving vector state in the fractional superstring theory contributes to the first term in the expansion of

$$ (c_0^0)^{D_c-3}(c_0^2) \sim q^{H_{\text{grav}}} (1 + \ldots) \quad \text{where} \quad H_{\text{grav}} = \frac{2}{K+2} - \frac{(D_c - 2)K}{8(K + 2)} \quad (2.24) $$

Since this vector state is the left-moving component of the graviton, the fractional superstring theory will therefore be a theory of gravity (i.e., contain a massless
graviton) only if $H_{\text{grav}} = 0$, or

$$D_c - 2 = \frac{16}{K}.$$  \hspace{1cm} (2.25)

Thus, for $K = 2, 4, 8,$ and 16 we have the integer critical spacetime dimensions $D_c = 10, 6, 4,$ and 3 respectively.

Physically, we are interested only in fractional superstring theories which are tachyon-free. This requires that all tachyonic states be projected out of the physical spectrum, as occurs in physically sensible superstring and heterotic string theories. Hence, when constructing partition functions $Z_K$ for the closed “Type II” $K$-fractional superstrings, we seek modular-invariant combinations of terms of the form $e^{D_c - 2} = e^{16/K}$ in which the massless sector (2.24) is present but the tachyonic sector (2.23) is absent.

For the $K = 2$ (superstring) theory, of course, there exists such a unique tachyon-free modular-invariant solution:

$$Z_2(\tau) = \tau_2^{-4} |A_2|^2.$$  \hspace{1cm} (2.26)

Here $\tau_2$ is the imaginary part of $\tau$, and $A_2$ is the modular-invariant combination

$$A_2 = 8(c_0^0)^7(c_0^3) + 56(c_0^0)^5(c_0^2)^3 + 56(c_0^0)^3(c_0^2)^5 + 8(c_0^0)(c_0^2)^7 - 8(c_1^1)^8.$$  \hspace{1cm} (2.27)

This can be translated into a more familiar form by recalling the equivalences (2.22) between the $K = 2$ string functions and the Jacobi $\vartheta$-functions; using these results, we find that $A_2$ can be rewritten as

$$A_2 = \frac{1}{2} \eta^{-12} \left( \vartheta_3^4 - \vartheta_4^4 - \vartheta_2^4 \right) = \frac{1}{2} \Delta^{-1/2} J$$  \hspace{1cm} (2.28)

where $\Delta \equiv \eta^{24}$ and $J$ is the vanishing Jacobi factor (1.2).
We can construct solutions satisfying our requirements for the $K > 2$ cases as well.

For $K = 4$, we find one tachyon-free modular-invariant partition function $^2$

$$Z_4(\tau) = \tau_2^{-2} \left( |A_4|^2 + 12 |B_4|^2 \right)$$  \hspace{1cm} (2.29)

where $A_4$ and $B_4$ are the combinations

$$A_4 = 4(c_0^4 + c_0^8)(c_0^8 - 4(c_0^4)^4 - 4(c_0^4)^4 + 32(c_0^2)(c_0^4)^3),$$

$$B_4 = 8(c_0^4 + c_0^8)(c_0^8)^2 + 4(c_0^4 + c_0^8)(c_0^4)(c_0^4) - 4(c_0^2)^2(c_0^2)^2. \hspace{1cm} (2.30)$$

Unlike $A_2$, which was by itself modular-invariant, $A_4$ and $B_4$ mix amongst themselves under modular transformations; under $T$ we find that $A_4$ and $B_4$ transform as eigenfunctions with eigenvalues $+1$ and $-1$ respectively, whereas under $S$ we find

$$S : \begin{pmatrix} A_4 \\ B_4 \end{pmatrix} \rightarrow -\frac{1}{\tau^2} \begin{pmatrix} 1/2 & 3 \\ 1/4 & -1/2 \end{pmatrix} \begin{pmatrix} A_4 \\ B_4 \end{pmatrix}. \hspace{1cm} (2.31)$$

Similarly, for the $K = 8$ closed “Type II” fractional superstring, we find the unique tachyon-free modular-invariant partition function $^2$

$$Z_8(\tau) = \tau_2^{-1} \left( |A_8|^2 + |B_8|^2 + 2 |C_8|^2 \right)$$  \hspace{1cm} (2.32)

where we now have the three combinations

$$A_8 = 2(c_0^4 + c_0^8)(c_0^4 + c_0^8) - 2(c_0^4)^2 - 2(c_0^4)^2 + 8(c_0^0c_4^8),$$

$$B_8 = 4(c_0^4 + c_0^8)(c_0^4) + 4(c_0^4 + c_0^8)(c_0^8) - 4(c_0^2c_4^4), \hspace{1cm} (2.33)$$

$$C_8 = 4(c_2^4 + c_2^8)(c_2^8 + c_6^8) - 4(c_2^4)^2.$$

These combinations are eigenfunctions of $T$ with eigenvalues $+1$, $-1$, and $-i$ re-
spectively, and under $S$ they mix as follows:

$$S : \begin{pmatrix} A_8 \\ B_8 \\ C_8 \end{pmatrix} \rightarrow \frac{i}{\tau} \begin{pmatrix} 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & -1 \\ 1/2 & -1/2 & 0 \end{pmatrix} \begin{pmatrix} A_8 \\ B_8 \\ C_8 \end{pmatrix}. \quad (2.34)$$

For the $K = 16$ fractional superstring there also exists a solution:

$$Z_{16} = \tau_2^{-1/2} \left( |A_{16}|^2 + 4 |C_{16}|^2 \right), \quad (2.35)$$

where our combinations are

$$A_{16} = c_0^2 + c_0^{14} - c_8^8 - 2c_8^{14},$$

$$C_{16} = c_4^2 + c_4^{14} - c_4^8. \quad (2.36)$$

These combinations have eigenvalues $+1$ and $-i$ under $T$, and under $S$ they mix as follows:

$$S : \begin{pmatrix} A_{16} \\ C_{16} \end{pmatrix} \rightarrow \frac{1}{\sqrt{-i\tau}} \frac{1}{2\sqrt{2}} \begin{pmatrix} 2 & 4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} A_{16} \\ C_{16} \end{pmatrix}. \quad (2.37)$$

It turns out that each of the partition functions $Z_K$ in this series has a remarkable property: viewed as a function of $q \equiv e^{2\pi i\tau}$, each vanishes identically. We shall prove this assertion in Sect. 4. In the $K = 2$ (superstring) case, we see from (2.26) and (2.28) that this vanishing is equivalent to the Jacobi identity

$$J \equiv \vartheta_3^4 - \vartheta_2^4 - \vartheta_4^4 = 0, \quad (2.38)$$

and indeed it is well-known that the particle spectrum of this Type II superstring exhibits a spacetime supersymmetry. Thus, the famous Jacobi identity (2.38) is
the reflection (at the partition-function level) of this underlying spacetime supersymmetry. In analogous fashion, we interpret the vanishing of each $Z_K$ for $K > 2$ as a sign of spacetime supersymmetry in the fractional-superstring spectrum of states: the contribution to $Z_K$ from each bosonic state at every mass level in the theory is cancelled by the (equal but opposite) contribution from a corresponding fermionic state. Since these $K > 2$ partition functions $Z_K$ have been written as the sums of squares, it follows that the separate string-function combinations $A_K$, $B_K$, and $C_K$ must each independently also vanish as functions of $q$:

$$A_4 = B_4 = A_8 = B_8 = C_8 = A_{16} = C_{16} = 0 . \quad (2.39)$$

These string-function identities (2.39) are therefore the $K > 2$ analogues of the ($K = 2$) Jacobi identity, and we shall prove this series of new Jacobi-like identities in Sect. 4.

We emphasize that in spite of (2.39), one-loop modular invariance continues to require that the individual terms $|A_K|^2$, $|B_K|^2$, and $|C_K|^2$ appear together in our partition functions $Z_K$ in the combinations given in (2.29), (2.32), and (2.35). As in the superstring, one-loop modular invariance not only guarantees multi-loop modular invariance, but is also required for the internal consistency of the theory. In fact, since (2.39) is the reflection of spacetime supersymmetry in the fractional-superstring spectrum, such cancellations must be distinguished from those due to “internal” (GSO-like) projections which act between states with the same statistics and which thereby actually remove physical states from the theory.
Given that these partition functions $Z_K$ vanish, we now turn to an examination of the individual terms within each $Z_K$. In particular, since this vanishing is a reflection of spacetime supersymmetry, we focus on determining the spacetime statistics of the particles contributing to each of the terms in $Z_K$.

Let us first recall the well-understood $K = 2$ case. In the $K = 2$ superstring partition function (where we are restricting ourselves to the left-moving holomorphic sector only), we have seen that the expression $A_2$ is proportional to $\Delta^{-1/2} J$, and it is well-known that within this factor $\Delta^{-1/2} J$ the terms $\Delta^{-1/2}(\vartheta_3^4 - \vartheta_4^4)$ represent the contributions from spacetime bosonic states (i.e., from the worldsheet Neveu-Schwarz sector). Likewise, the remaining term $\Delta^{-1/2} \vartheta_2^4$ within $\Delta^{-1/2} J$ represents the contributions from spacetime fermionic states (i.e., from the worldsheet Ramond sector). Using the relations (2.22) between the Jacobi $\vartheta$-functions and the string functions, we see that the fermionic Ramond sector contributes only to the term $(e^1_1)^8$ within $A_2$, while the remaining terms within $A_2$ receive contributions from only the bosonic Neveu-Schwarz sector. Having thus distinguished these contributions in terms of the $K = 2$ string functions, we can now easily discern which of the worldsheet $Z_2$ “parafermion” primary fields $\phi^j_m$ are responsible for bosonic or fermionic statistics: since the character of each field $\phi^j_m$ is $\eta c_{2m}$, we see that the parafermion fields giving rise to spacetime fermionic states are $\phi^{1/2}_m$, while those giving rise to spacetime bosonic states are $\phi^j_m$ with quantum number $m = 0$. Thus, in light-cone gauge, spacetime bosons will have vertex operators proportional to

$$B \sim \prod_{i=1}^{8} (\phi^j_{0i})$$

(2.40)

where $j_i = 0$ or 1, and spacetime fermions will have vertex operators proportional
to

\[ F \sim \prod_{i=1}^{8} \left( \phi_{m_i}^{1/2} \right) \]  

(2.41)

where \( m_i = \pm 1/2 \). Note that we are suppressing the contributions to the vertex operators from the worldsheet bosons \( X^\mu \). Since the worldsheet bosons give rise only to states with spacetime bosonic statistics, this suppression does not affect the identification of the statistics of vertex operators. Similarly, any of the above parafermion primary fields \( \phi^i_m \) may be replaced by one of its descendant fields without altering the statistics.

It is straightforward to demonstrate that the above vertex operator assignments satisfy a number of self-consistency checks. First, we can recall the correspondence between the \( \mathbb{Z}_2 \) parafermion fields and the free fermion fields of the Ising model

\[
\begin{align*}
\phi_0^0 &= 1 , \\
\phi_0^1 &= \psi , \\
\phi_{1/2}^1 &= \sigma , \\
\phi_{-1/2}^{1/2} &= \sigma^{\dagger} ;
\end{align*}
\]

(2.42)

here \( \psi \) is the Majorana fermion field and \( \sigma \) and \( \sigma^{\dagger} \) are the spin fields. These worldsheet spin fields and their descendants create the (Ramond) spacetime fermionic states from the vacuum, and the Majorana and identity fields similarly create the (Neveu-Schwarz) spacetime bosonic states. This is therefore consistent with the above vertex-operator assignments. More compellingly, we can check that the \( \mathbb{Z}_2 \) parafermion algebra itself reproduces the correct spin-statistics selection rules. Recall that according to the fusion rules (2.4), the \( m \)-quantum number of the parafermion field is additive. From this and the field identifications (2.3), it follows that
our vertex operators (2.40) and (2.41) satisfy the following selection rules under fusion:

\[ F \times F \to B \]
\[ F \times B \to F \]
\[ B \times B \to B . \quad (2.43) \]

This is, of course, in accordance with the required spin-statistics connection.

These considerations can easily be generalized to the \( K > 2 \) cases: here the analogues of (2.40) and (2.41) are

\[ B \sim \prod_{i=1}^{D_c-2} (\phi^0_j) \quad \text{and} \quad F \sim \prod_{i=1}^{D_c-2} (\phi^j_{\pm K/4}) , \quad (2.44) \]

and from the equivalences (2.3) and the fusion rules (2.4) we see that the selection rules (2.43) are again satisfied.

This identification of the spacetime spin and statistics of fractional-superstring states is compatible with the \( A_K \) parts of the partition functions \( Z_K \) given above, for each term within each \( A_K \) can readily be identified as the contribution from either a spacetime bosonic state or a spacetime fermionic state. It is therefore straightforward to decompose each \( A_K \) into two pieces (just as was done for the \( K = 2 \) Jacobi factor \( J \) ), writing

\[ A_K = A^b_K - A^f_K \quad (2.45) \]

where \( A^b_K \) (respectively \( A^f_K \)) contains the terms in which all string functions are of
the form $c_0^\ell$ (respectively $c_{K/2}^\ell$):

\begin{align*}
K = 2: \quad A_2^b &\equiv 8(c_0^0)^7(c_0^2) + 56(c_0^0)^5(c_0^2)^3 + 56(c_0^0)^3(c_0^2)^5 + 8(c_0^0)(c_0^2)^7 \\
A_2^f &\equiv 8(c_1^1)^8 \\
K = 4: \quad A_4^b &\equiv 4(c_0^0 + c_4^4)^3(c_0^2) - 4(c_0^2)^4 \\
A_4^f &\equiv 4(c_2^2)^4 - 32(c_2^4)^3(c_2^2) \\
K = 8: \quad A_8^b &\equiv 2(c_0^0 + c_8^8)(c_0^2 + c_0^0) - 2(c_0^4)^2 \\
A_8^f &\equiv 2(c_4^4)^2 - 8(c_4^8)(c_4^8) \\
K = 16: \quad A_{16}^b &\equiv c_0^2 + c_0^{14} - c_0^8 \\
A_{16}^f &\equiv c_8^8 - 2c_8^{14} .
\end{align*}

This separation of terms is also consistent with our previous observations. First, the term we identified as containing a massless vector particle in (2.24) is of the form $\prod_i (c_0^{\ell_i})$, in agreement with the form of the boson vertex operators given in (2.44); note that it appears with a positive sign in $A_K^b$. Next, we observe that according to the above identification, massless fermions must correspond to the first term in the $q$-expansion of

$$
\left( c_{K/2}^{K/2} \right)^{16/K} \sim q^0 (1 + ...) ; \quad (2.47)
$$

fortunately this term is present within each of our expressions $A_K^f$, and indeed it can be shown\cite{2} that the physical-state conditions on the massless state in (2.47) yield the massless Dirac equation. The Fermi statistics carried by these spacetime spinorial states is reflected in the negative sign with which this term appears in each $A_K$ (or the positive sign within each $A_K^f$), and in fact the coefficient for
this term in each case gives precisely the counting of physical states necessary for
a Majorana and/or Weyl fermion in the critical spacetime dimension $D_c$ of the
fractional superstring. Furthermore, note that this left-moving spin-1/2 massless
fermionic state in (2.47) can be tensored with the right-moving spin-1 massless
bosonic state in (2.24) (and vice versa) to form a massless spin-3/2 gravitino.
Consistency would then require that the closed fractional-superstring spectrum
exhibit an $N = 2$ spacetime supersymmetry. The vanishing of each $A_K$, $B_K$, and $C_K$ (or the equality of their respective bosonic and fermionic parts) is indeed
consistent with this conclusion.

One new feature for $K > 2$ is the presence of additional terms within $A_K^b$ and
$A_K^f$ which appear with the “wrong” signs (i.e., negative signs within the individual
pieces $A_K^b$ and $A_K^f$). Although these flipped signs might initially seem to contradict
our identification of the statistics of states, it turns out that the expressions $A_K^f$
as defined in (2.46) satisfy the following identities:

$$
\begin{align}
K = 2 : & \quad A_2^f = 8 \left( \prod_{n=1}^{\infty} \frac{1 + q^n}{1 - q^n} \right)^8 \\
K = 4 : & \quad A_4^f = 4 \left( \prod_{n=1}^{\infty} \frac{1 + q^n}{1 - q^n} \right)^4 \\
K = 8 : & \quad A_8^f = 2 \left( \prod_{n=1}^{\infty} \frac{1 + q^n}{1 - q^n} \right)^2 \\
K = 16 : & \quad A_{16}^f = \left( \prod_{n=1}^{\infty} \frac{1 + q^n}{1 - q^n} \right).
\end{align}
$$

(2.48)

These new identities will be proven in Sect. 4. Note that since $A_K = A_K^b - A_K^f = 0$,
the bosonic terms $A_K^b$ also have the above $q$-expansions. Since (2.48) implies that
the coefficients in the $q$-expansions of each $A^K_b$ and $A^K_f$ are all positive, we are led to view the minus signs in $A^K_b$ and $A^K_f$ as “internal” projections (or cancellations) of degrees of freedom in the fractional superstrings. Furthermore, the pattern inherent in this series of identities (matching smoothly as it does onto the well-understood $K = 2$ case) also suggests that our statistics identification is indeed correct.

From the definitions of $\eta$ and $\vartheta_2$, Eqs. (2.11) and (2.21) respectively, it follows that

$$\left( \prod_{n=1}^{\infty} \frac{1 + q^n}{1 - q^n} \right)^2 = \frac{\vartheta_2}{2 \eta^3} = (c_1^1)^2$$

(2.49)

where $c_1^1$ is a level $K = 2$ string function. Thus the identities (2.48) can be viewed as relating the $K > 2$ parafermion characters (string functions) to $K = 2$ fermion characters (Jacobi $\vartheta$-functions), and indeed these identities are just some of the identities in our $[K, 2]$ series. The full set is collected in Sect. 3.

One immediate consequence of (2.48) is that the level-by-level counting of physical degrees of freedom in each of the fermionic $A^K_f$ sectors in the fractional superstring is identical to that of the Ramond sector of the usual superstring, except with spacetime dimension $D_c = 2 + 16/K$ instead of 10. At first glance, this observation might seem to imply that the $A^K$ spectrum of the fractional superstring can itself be equivalently described by the worldsheet Majorana-fermion and free-boson theories of the ordinary ($K = 2$) superstring, without any need for worldsheet parafermions. Indeed, the spectrum of the spacetime fermionic sectors $A^K_f$ of the fractional superstring can be generated by $D_c - 2 = 16/K$ pairs of free worldsheet bosons $X^\mu$ and worldsheet Ramond fermions $\psi^\mu$, as (2.48) and (2.49) jointly
suggest, and this is of course simply the operator content of the Ramond sector in ordinary superstring theory. However, consider the counting of the spacetime \textit{bosonic} states (Neveu-Schwarz sector) for the superstring

\[ K = 2 : \quad A_2^b \equiv \frac{1}{2} \left[ (c_0^0 + c_0^2)^8 - (c_0^0 - c_0^2)^8 \right] = 8 \left( \frac{\vartheta_3^4 - \vartheta_4^4}{16 \eta^{12}} \right). \tag{2.50} \]

Here \( \sqrt{\vartheta_3/\eta} \) and \( \sqrt{\vartheta_4/\eta} \) are the characters of worldsheet Majorana fermions obeying Neveu-Schwarz boundary conditions, and it is for this reason that we have written \( \vartheta_3^4 - \vartheta_4^4 \) rather than \( \vartheta_2^4 \) in (2.50). From (2.48), then, we can write the \( K > 2 \) generalizations of (2.50) as

\[
K = 4 : \quad A_4^b = 4 \left( \frac{\vartheta_3^4 - \vartheta_4^4}{16 \eta^{12}} \right)^{1/2}
\]

\[
K = 8 : \quad A_8^b = 2 \left( \frac{\vartheta_3^4 - \vartheta_4^4}{16 \eta^{12}} \right)^{1/4}
\]

\[
K = 16 : \quad A_{16}^b = \left( \frac{\vartheta_3^4 - \vartheta_4^4}{16 \eta^{12}} \right)^{1/8}.
\] \tag{2.51}

The right sides of (2.51) cannot be interpreted as the characters of any tensor products of bosons and Majorana fermions because of the presence of the fractional exponents for \( K > 2 \). Therefore, an attempted description of the \( A_K \) part of the fractional-superstring spectrum in terms of worldsheet bosons and fermions fails, and indeed it is only the introduction of \textit{parafermions} on the worldsheet which allows us to generate the bosonic-sector partition functions given in (2.51).

Thus far we have focused exclusively on the \( A_K \) sectors of the fractional superstrings, and we have seen that obtaining the desired fractional exponents in (2.51) \( (i.e., \) reducing the critical dimension from \( D_c = 10 \) to \( D_c = 2 + 16/K \)) is achieved
through the introduction of parafermions on the string worldsheet. There is, however, a price that must be paid. Let us now rewrite the entire expression $A_K = A^K_\beta - A^K_\delta$ in terms of Jacobi $\vartheta$-functions: using (2.51) for $A^K_\beta$, and using (2.48) and (2.49) jointly for $A^K_\delta$, we find

$$A_K \propto \Delta^{-1/K} \left\{ (\vartheta_3^4 - \vartheta_4^4)^{2/K} - \vartheta_2^{8/K} \right\}$$

(2.52)

where $\Delta \equiv \eta^{24}$. Thus, we see that only for $K = 2$ is $A_K$ by itself modular-invariant; for $K > 2$ we find that $A_K$ does not close into itself under $S$, but rather requires the introduction of additional sectors (such as those giving rise to $B_K$ and $C_K$) in order to achieve modular-invariant partition functions. This is the underlying reason why these additional expressions appeared naturally in our partition functions $Z_K$. Since these new sectors contain only massive states (i.e., states with masses at the Planck scale), we consider their introduction a small price to pay for the ability to decrease the critical spacetime dimension in string theory. Furthermore, we have seen that these additional expressions $B_K$ and $C_K$ also vanish as functions of $q$, and therefore the introduction of these additional sectors preserves the spacetime supersymmetry at all mass levels of the theory.

These additional sectors, however, seem to contain much of the spacetime physics which is intrinsically new to fractional superstrings. To see this, let us consider the spacetime statistics of the states appearing in these sectors. The fields corresponding to terms in the $B_K$ sector are themselves products of parafermion fields, half of which have quantum number $m = 0$ and half of which have quantum number
\( m = K/4 \). Therefore, according to our previous discussion, their vertex operators will have the following form in light-cone gauge:

\[
Q_B \sim \prod_{i=1}^{D_c/2 - 1} \left( \phi_0^i \phi_{\pm K/4}^i \right).
\]

(2.53)

According to (2.44), these states are therefore neither fermions nor bosons in \( D_c \) spacetime dimensions. A similar situation exists for the \( C_K \) sectors. These sectors consist of terms of the form \( \prod_i (c_i^{j_i}) \) [where we recall the identities (2.15)], and in light-cone gauge these naturally correspond to vertex operators of the form

\[
Q_C \sim \prod_{i=1}^{D_c - 2} \left( \phi_{\pm K/8}^i \right).
\]

(2.54)

Once again, such states cannot be interpreted as bosons or fermions in \( D_c \) spacetime dimensions.

In order to gain some insight into the properties of these states, we can calculate the fusion rules that these additional vertex operators \( Q_B \) and \( Q_C \) satisfy. It turns out that these rules depend on the level \( K \) of the fractional superstring, since the number of parafermion fields within each vertex operator depends on the critical dimension \( D_c \) (and hence on \( K \)). For the \( K = 4 \) superstring, we have only the \( A_4 \) and \( B_4 \) sectors: the \( A_4 \) sector can be decomposed into bosonic and fermionic pieces satisfying the algebra (2.43), as we have seen, and the \( B_4 \) sector introduces
the additional vertex-operator fusion rules:

\[ Q_B \times B \rightarrow Q_B \]
\[ Q_B \times F \rightarrow Q_B \]
\[ Q_B \times Q_B \rightarrow B \text{ or } F \text{ or } Q_B . \]

(2.55)

These selection rules are therefore suggestive of “spin-quarter” statistics for \(B_4\)-sector particles, since the fusing of two identical \(B_4\)-sector particles can result in an \(A_4\)-sector fermion. Similarly, for the \(K = 8\) superstring, we have three sectors: \(A_8, B_8,\) and \(C_8\). While the \(A_8\) sector again leads to the fusion rules (2.43), the \(B_8\) sector now introduces the (slightly modified) fusion rules:

\[ Q_B \times B \rightarrow Q_B \]
\[ Q_B \times F \rightarrow Q_B \]
\[ Q_B \times Q_B \rightarrow B \text{ or } F \]

(2.56)

and the \(C_8\) sector introduces the additional rules:

\[ Q_C \times B \rightarrow Q_C \]
\[ Q_C \times F \rightarrow Q_C \]
\[ Q_C \times Q_B \rightarrow Q_C \]
\[ Q_C \times Q_C \rightarrow B \text{ or } F \text{ or } Q_B . \]

(2.57)

Once again, therefore, the \(B_8\)-sector fusion rules suggest “spin-quarter” statistics, and the \(C_8\)-sector rules seem to indicate “spin-eighth” statistics. For the \(K = 16\) string, on the other hand, there are only an \(A_{16}\) and a \(C_{16}\) sector. In this case the
vertex operators contain only one parafermion field, and while the $A_{16}$ sector again leads to (2.43), the $C_{16}$ sector now yields the fusion rules:

$$Q_C \times B \rightarrow Q_C$$
$$Q_C \times F \rightarrow Q_C$$
$$Q_C \times Q_C \rightarrow B \text{ or } F.$$  \hspace{1cm} (2.58)

Thus in this instance it is the $C_{16}$ sector which seems to suggest “spin-quarter” statistics.

Even though this discussion has focused on only the left-moving sectors of the fractional superstrings, these considerations indicate that it is likely that the $B_K$ and $C_K$ sectors break the usual spin-statistics connection in the critical dimension. (This is not necessarily true of the $K = 16$ string, since fractional statistics are allowed in three spacetime dimensions.) This means that, for the fractional superstrings to be consistent, either Lorentz invariance, quantum mechanics, or locality must be broken in some way in the critical dimension.

An important point following from all of these fusion rules is that the tree-level scattering of particles in the $A_K$ sector can involve only other fields in the $A_K$ sector as intermediate states. Since only the $A_K$ sectors contain the massless states, we see that the $B_K$ and $C_K$ sectors make no contribution to the semi-classical low-energy physics of the fractional superstrings. Therefore, since Lorentz invariance and the spin-statistics connection appear to hold in the $A_K$ sector, the semi-classical low-energy physics predicted by the fractional superstrings will indeed be the familiar Yang-Mills and gravity theories (plus corrections proportional to powers of the string tension, as in the usual superstring). At the string loop level, however,
fields in the $B_K$ or $C_K$ sectors can contribute to the scattering amplitudes of the massless particles in the $A_K$ sector. Thus, it is only the quantum effects of the fractional superstring which render the massless (i.e., potentially observable) physics inconsistent with the traditional spin-statistics connection. Since the lowest-lying states in the $B_K$ and $C_K$ sectors have masses at the Planck scale, we expect the quantum corrections to the massless-sector gravity and Yang-Mills theories to be suppressed by factors of the Planck mass relative to the low-energy scale (at least for sufficiently weak string coupling).

These violations of the spin-statistics connection, though potentially weak, may nevertheless be important qualitative signals of stringy behavior. Indeed, there are many possible mechanisms through which these $B_K$ and $C_K$ sectors might lead to violations of the usual spin-statistics relation. First, as mentioned above, the spin-statistics connection can be invalidated by sacrificing locality. Since strings are extended objects, it is possible that the massive states in the $B_K$ and $C_K$ sectors correspond to extended states that cannot be interpreted as elementary particles. Indeed, the possibility of exotic statistics due to the extended nature of strings has been discussed in Ref. [12]. This proposal relies on the non-trivial nature of the motion group in string-configuration space in four spacetime dimensions. Another possibility is that the states in the $B_K$ and $C_K$ sectors are soliton-like objects, extended objects formed from the $A_K$-sector fields. The slow fall-off of massless scalar or vector fields could allow the violation of the usual statistics selection rules, similar to what occurs in monopole-fermion systems.$^{[13]}$

On the other hand, the spin-statistics theorem can be avoided at the expense of Lorentz invariance. Lorentz invariance is a symmetry known to hold only at
distances much larger than the Planck scale. As we have seen, these fractional superstring theories lead to a breaking of Lorentz invariance only at the Planck scale and only at the quantum loop level. Such a breaking may actually exist, and simply not be observable experimentally at the present time.

Alternatively, even if the massive-sector breaking of Lorentz invariance in the critical dimension $D_c$ is strong even at low energies, a lower-dimensional Lorentz invariance might still survive. In this scenario, the spacetime symmetry group of the $B_K$ and $C_K$ sectors would have a $D$-dimensional Lorentz subgroup with $D < D_c$, providing a mechanism for spontaneously compactifying from the critical dimension $D_c$ down to $D$ dimensions. Let us see explicitly how this might occur.$^8$ Recall the form of the $B_K$-sector vertex operators given in (2.53). While we have seen that these states cannot be interpreted as either bosons or fermions in $D_c$ spacetime dimensions, they may well have bosonic or fermionic interpretations in spacetime dimensions $D < D_c$. For example, a bosonic interpretation for $Q_B$ is possible if the parafermion fields with quantum number $m = 0$ within (2.53) are viewed as the (fractional) superpartners of the $D < D_c$ worldsheet coordinate bosons, with the remaining $m \neq 0$ parafermion fields viewed as part of an “internal” worldsheet theory resulting from spacetime compactification. In fact, it is shown in Ref. [8] that a theory fully consistent with Lorentz invariance and the spin-statistics connection is possible for the $K = 4$ and $K = 8$ cases if the spacetime dimensions are compactified respectively to $D = 4$ and $D = 3$. Furthermore, such a compactification scheme in the $K = 4$ case may also permit the construction of four-dimensional fractional-superstring models containing chiral spacetime fermions.$^8$
It is therefore evident that there is much potentially new physics to be discovered within fractional superstring theory, whether a possible Planck-scale breakdown of Lorentz invariance, the appearance of exotic statistics due to the extended (non-local) nature of strings, or a “self-compactification” required for internal consistency. While the interpretations of many of these effects have yet to be resolved, the important point is that fractional superstrings offer a unique and concrete framework in which to explore these issues.
3. LIST OF NEW STRING-FUNCTION IDENTITIES

In this section we gather together all of the new string-function identities to be proven in Sect. 4. As discussed in the Introduction, we refer to an identity as a \([K_1, K_2]\) identity if it relates string functions \(c^\ell_m\) of level \(K_1\) to string functions of level \(K_2\). Recall that the level \(K = 1\) string function is equivalent to the Dedekind \(\eta\)-function, and that the level \(K = 2\) string functions are equivalent to the Jacobi \(\vartheta_1\)-functions [these relations are given in (2.20) and (2.22)]. If we define the level \(K = 0\) function \(c^0_0 \equiv 0\), then our new identities come in three distinct series: these are the \([K,0]\), \([K,1]\), and \([K,2]\) series, for \(K = 2, 4, 8, \) and \(16\). These identities are listed below.

3.1. FIRST SERIES: THE \([K,0]\) IDENTITIES

This series of identities generalizes the famous Jacobi “supersymmetry” identity on fermion characters:

\[
J \equiv \vartheta_3^4 - \vartheta_2^4 - \vartheta_1^4 = 0 .
\]  

(3.1)

In terms of \(K = 2\) string functions, (3.1) is equivalent to

\[
A_2 \equiv 8 \left( c^0_0 \right)^7 (c^2_0)^2 + 56 \left( c^0_0 \right)^5 (c^2_0)^3 + 56 \left( c^0_0 \right)^3 (c^2_0)^5 + 8 \left( c^0_0 \right) (c^2_0)^7 - 8 \left( c^1_1 \right)^8
\]

\[
= \frac{1}{2} \Delta^{-1/2} J = 0
\]

(3.2)

where \(\Delta \equiv \eta^{24}\). The Jacobi identity (3.1) can therefore be regarded as the \([2,0]\) special case, and the analogous \([K,0]\) Jacobi-like “supersymmetry” identities for
the parafermion characters at higher levels $K$ are as follows. For $K = 4$, we define

$$A_4 \equiv 4 (c_0^4 + c_0^8)(c_0^2) - 4 (c_0^2)^4 - 4 (c_2^2)^4 + 32 (c_2^2)(c_2^4)^3,
$$

$$B_4 \equiv 8 (c_0^4 + c_0^8)(c_0^2)(c_2^2)^2 + 4 (c_0^2 + c_0^8)^2 (c_2^2)(c_2^4) - 4 (c_0^2)^2 (c_2^2)^2,$$

and for $K = 8$, we define

$$A_8 \equiv 2 (c_0^8 + c_0^8)(c_0^4) - 2 (c_0^4)^2 - 2 (c_4^4)^2 + 8 (c_4^4c_4^8),
$$

$$B_8 \equiv 4 (c_0^4 + c_0^8)(c_4^4) + 4 (c_0^2 + c_0^8)(c_4^8) - 4 (c_0^4c_4^8),
$$

$$C_8 \equiv 4 (c_2^8 + c_2^8)(c_2^4 + c_2^8) - 4 (c_4^4)^2.$$ (3.3)

Similarly, for $K = 16$, we define

$$A_{16} \equiv c_0^2 + c_0^{14} - c_0^8 - c_8^8 + 2 c_8^{14},
$$

$$C_{16} \equiv c_4^2 + c_4^{14} - c_4^8.$$ (3.5)

Then each of these expressions also vanishes as a function of $q$:

$$A_2 = A_4 = B_4 = A_8 = B_8 = C_8 = A_{16} = C_{16} = 0.$$ (3.6)

These new “Jacobi-like” identities therefore form the $[K, 0]$ series. Note that while the Jacobi identity is the unique such identity for $K = 2$, for each level $K > 2$ there are in fact several independent Jacobi-like $[K, 0]$ identities.

### 3.2. Second Series: The $[K, 2]$ Identities

This series of identities relates the $Z_K$ parafermion characters to the ordinary fermion characters, and are the higher-$K$ analogues of the $K = 2$ “identities” $\Delta^{-1/2} \vartheta_i^4 = \Delta^{-1/2} \vartheta_i^4$ for $i = 2, 3, 4$. 

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For $K = 4$, we define the following quantities:

\[
A^b_4 \equiv 4 (d_0^6)^2 c_0^2 - 4 (c_0^2)^4 \\
A^f_4 \equiv 4 (c_2^4)^4 - 32 (c_2^4)^3 c_2^2 \\
C_4 \equiv 4 c_0^2 (c_2^4)^3 - 12 d_0^6 (c_2^4)^2 c_2^2 - 8 c_0^2 (c_2^4)^3 = 4 q^{1/4} (1 + ...) \\
D_4 \equiv (d_0^6)^3 c_2^2 + 6 (d_0^6)^2 c_0^2 c_2^4 - 4 (c_0^2)^3 c_2^2 = q^{-1/4} (1 + ...) ,
\]

where $d^\ell_n \equiv c_n^\ell + c_n^{K-\ell}$. Note that $A^b_4 - A^f_4 = A_4$. Then our $[4, 2]$ identities, which contain the $[4, 0]$ Jacobi-like identities $A_4 = B_4 = 0$ as a subset, are as follows:

\[
\begin{align*}
A^b_4 &= \vartheta_2^2 / \eta^6 \\
A^f_4 &= \vartheta_2^2 / \eta^6 \\
B_4 &= 0 \\
C_4 &= \frac{1}{2} (\vartheta_3^2 - \vartheta_4^2) / \eta^6 \\
D_4 &= \frac{1}{2} (\vartheta_3^2 + \vartheta_4^2) / \eta^6 .
\end{align*}
\]

As a consequence of (3.8), we have

\[
(A^b_4 + A^f_4)^2 - 16 C_4 D_4 = 4 \Delta^{-1/2} J = 0 .
\]

Similarly, for $K = 8$, we define

\[
\begin{align*}
A^b_8 &\equiv 2 d_0^6 d_0^2 - \frac{1}{2} (d_0^4)^2 \\
A^f_8 &\equiv \frac{1}{2} (d_4^8)^2 - 2 d_4^6 d_4^8 \\
E_8 &\equiv \frac{1}{2} d_2^5 d_4^4 - d_2^7 d_4^4 - d_4^4 d_2^8 = 2 q^{3/8} (1 + ...) \\
F_8 &\equiv \frac{1}{2} d_0^6 d_2^2 + \frac{1}{2} d_0^2 d_2^8 - d_0^4 d_2^4 = q^{-1/8} (1 + ...) .
\end{align*}
\]

Once again $A^b_8 - A^f_8 = A_8$. Then our $[8, 2]$ identities, which contain the $[8, 0]$
Jacobi-like identities $A_8 = B_8 = C_8 = 0$ as a subset, are as follows:

$$
A^b_8 = \vartheta_2/\eta^3 \\
A^f_8 = \vartheta_2/\eta^3 \\
B_8 = C_8 = 0 \\
E_8 = \frac{1}{2} (\vartheta_3 - \vartheta_4)/\eta^3 \\
F_8 = \frac{1}{2} (\vartheta_3 + \vartheta_4)/\eta^3 .
$$

(3.11)

As a consequence of (3.11), we have

$$(E_8 + F_8)^4 - (E_8 - F_8)^4 - 16 (A^b_8 + A^f_8)^4 = \Delta^{-1/2} J = 0 .$$

(3.12)

Similarly, for $K = 16$, we define the five quantities:

$$
A^b_{16} \equiv d^2_0 - \frac{1}{2} d^8_0 \\
A^f_{16} \equiv \frac{1}{2} d^8_0 - d^{14}_8 \\
C_{16} \equiv d^{14}_4 - \frac{1}{2} d^8_4 \\
E_{16} \equiv \frac{1}{2} d^8_6 - d^{14}_6 = q^{7/16}(1 + ...) \\
F_{16} \equiv d^2_2 - \frac{1}{2} d^8_2 = q^{-1/16}(1 + ...) ;
$$

(3.13)

here too $A^b_{16} - A^f_{16} = A_{16}$. Then our $[16, 2]$ identities, which likewise contain the $[16, 0]$ Jacobi-like identities $A_{16} = C_{16} = 0$ as a subset, are as follows:

$$
A^b_{16} = \sqrt{\vartheta_2/(2\eta^3)} = c^1_1 \\
A^f_{16} = \sqrt{\vartheta_2/(2\eta^3)} = c^1_1 \\
C_{16} = 0 \\
E_{16} = \frac{1}{2} \left( \sqrt{\vartheta_3/\eta^3} - \sqrt{\vartheta_4/\eta^3} \right) = c^2_0 \\
F_{16} = \frac{1}{2} \left( \sqrt{\vartheta_3/\eta^3} + \sqrt{\vartheta_4/\eta^3} \right) = c^0_0
$$

(3.14)

where on the right sides of these equations the string functions are at level $K = 2$. 

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As a consequence of (3.14), we have

\[(E_{16} + F_{16})^8 - (E_{16} - F_{16})^8 - \frac{1}{16}(A_{16}^b + A_{16}^f)^8 = \Delta^{-1/2} J = 0. \] (3.15)

The identities (3.14) are in fact quite remarkable, for they are linear relations indicating that the differences of certain \(K = 16\) string functions are nothing but the \(K = 2\) string functions. The full consequences of these relations between the \(\mathbb{Z}_2\) fermionic characters and the \(\mathbb{Z}_{16}\) parafermionic characters are yet to be explored.

### 3.3. Third Series: The \([K, 1]\) Identities

This new series of identities generalizes the \(\vartheta\)-function identity

\[\vartheta_2 \vartheta_3 \vartheta_4 = 2 \eta^3\] (3.16)

to levels \(K > 2\), thereby relating the \textit{parafermion} characters \(c^\ell_n\) to the boson character \(\eta\).

This series of identities actually starts at \(K = 1\), where we have the relation (2.20):

\[\eta c^0_0 = 1.\] (3.17)

For \(K = 2\), as mentioned, we have the identity

\[\vartheta_2 \vartheta_3 \vartheta_4 = 2 \eta^3,\] (3.18)

which can be written in terms of the \(K = 2\) string functions as

\[\eta^3 Q_2 = 1\] (3.19)
where

\[ Q_2 \equiv c_1 \left[ (c_0^0)^2 - (c_0^2)^2 \right] = \eta^{-3} \sqrt{\frac{\varphi_2 \varphi_3 \varphi_4}{2 \eta^3}}. \tag{3.20} \]

[Thus (3.19) is in fact equivalent to the square root of (3.18), and thereby contains the extra information about the sign of the square root.] Eqs. (3.17) and (3.19) are the first two identities in our \([K, 1]\) series, each of the form \(\eta^p \sum (c)^p = 1\) for some power \(p\). The corresponding \([K, 1]\) identities for \(K > 2\) are as follows. For \(K = 4\), we have

\[ \eta^2 Q_4 = 1 \tag{3.21} \]

where

\[ Q_4 \equiv (c_0^0 + c_0^4) c_2^2 - 2 c_0^2 c_4^4, \tag{3.22} \]

and for \(K = 8\), we have

\[ \eta^3 Q_8 = 1 \tag{3.23} \]

where

\[ Q_8 \equiv c_4^4 (c_0^2 + c_0^6) \left[ (c_0^0 + c_0^8) - (c_2^0 + c_2^8) \right] \\
+ 2 c_2^4 \left[ c_4^8 (c_0^2 + c_0^6) - c_4^6 (c_0^0 + c_0^8) \right] \\
- 2 c_0^4 \left[ c_4^8 (c_2^2 + c_2^6) - c_4^6 (c_2^0 + c_2^8) \right]. \tag{3.24} \]

Note that this expression for \(Q_8\) can be rewritten in a variety of forms due to the \([8, 0]\) Jacobi-like identities \(A_8 = B_8 = C_8 = 0\). Similarly, for \(K = 16\), we define
the string-function combinations $d_\ell^n \equiv c_\ell^n + c_{n-\ell}^K$. Then our corresponding [16,1]
identity is

$$\eta^3 Q_{16} = 1 ,$$

(3.25)

where $2Q_{16}$ is the following quantity:

$$- 2d_0^4 d_2^4 d_2^{10} - 3d_0^6 d_6^6 d_6^6 + 3d_0^6 d_6^6 d_6^6 + 2d_0^6 d_6^{12} d_6^{16} - d_0^2 d_2^2 d_2^2$$

$$- 4d_0^6 d_2^6 d_2^6 - d_0^8 d_6^8 d_6^8 + d_0^8 d_6^8 d_6^8 + 4d_0^8 d_6^{14} d_6^{14} + d_0^2 d_6^{14} d_6^{14}$$

$$+ 2d_0^4 d_2^4 d_2^4 - 3d_0^{14} d_6^{14} d_2^2 - 2d_0^4 d_6^6 d_6^{12} + 3d_0^4 d_6^{16} d_6^{16} + 3d_0^4 d_2^4 d_2^4$$

$$- 2d_0^6 d_6^6 d_2^{10} + 2d_0^6 d_6^6 d_6^6 - 3d_0^6 d_6^{12} d_6^{12} - d_0^8 d_6^2 d_2^2 + d_0^8 d_6^{14} d_6^{14} .$$

(3.26)

This expression for $Q_{16}$ can also be rewritten in many different forms by using the
[16,0] Jacobi-like identities $A_{16} = C_{16} = 0$. Thus, defining $Q_1 \equiv c_0^0 = \eta^{-1}$ for
$K = 1$, we have

$$Q_1^3 = Q_2 = Q_1 Q_4 = Q_8 = Q_{16} .$$

(3.27)

3.4. Comments On Other Identities

As indicated in the Introduction, these $[K,0]$, $[K,1]$, and $[K,2]$ identities are un-
doubtedly only some of the general $[K_1,K_2]$ identities which exist. Of course,
implicit in the above identities are $[K_1,K_2]$ relations which do not involve the
$K = 1$ or $K = 2$ string functions; for example, we have the identities (3.9), (3.12),
(3.15), and (3.27), as well as additional identities such as

$$\pm(C_4 \pm D_4) = (E_8 \pm F_8)^2 = (E_{16} \pm F_{16})^4 ,$$

$$\frac{1}{4} (A_4^4 + A_4^4)^2 = (C_8 + D_8)^4 + (E_{16} - F_{16})^8 ,$$

(3.28)

$$Q_{16}^6 = \frac{1}{2} (A_8^4 + A_8^4) (F_8^2 - E_8^2) (Q_4)^6 .$$

However, these identities are not independent of those in the three preceding series.
We also point out that there can exist several distinct series of a given \([K_1, K_2]\) type. For example, although the above three series are all of the form

\[
\sum (c_{K_1})^p = \sum (c_{K_2})^p
\]

(3.29)

where \(p\) is an arbitrary power, there also exist more general “mixed” identities of the form

\[
\sum \left\{ (c_{K_1})^p (c_{K_2})^q \right\} = \sum \left\{ (c_{K_1})^p (c_{K_2})^q \right\}
\]

(3.30)

which cannot be rewritten in the form (3.29) and which are indeed independent of all such identities. For example, consider the following three mixed identities involving the \(K = 2\) and \(K = 4\) string functions (where we have distinguished the two sets of string functions by writing the \(K = 2\) string functions in terms of the Jacobi \(\vartheta\)-functions):

\[
A_4^1 \left( \vartheta_3^2 + \vartheta_4^2 \right) = 8 D_4 \vartheta_2^2 ,
\]

\[
\left[ (d_2^2)^4 + 4 d_2^2 (d_1^4)^3 \right] \left( \vartheta_3^2 + \vartheta_4^2 \right) = \left[ (d_0^4)^3 d_2^2 + 4 (d_0^4)^3 d_2^2 \right] \vartheta_2^2 ,
\]

\[
\left[ 4 (d_0^4)^2 d_2^4 - 2 (d_0^4)^2 (d_2^4)^2 \right] \left( \vartheta_3^2 - \vartheta_4^2 \right) = \left[ -4 (d_0^4)^2 d_0^4 d_2^4 - (d_0^4)^3 d_2^2 \right] \vartheta_2^2 .
\]

(3.31)

While the first of these identities follows directly from (3.8), the remaining two are in fact new identities independent of any presented thus far.

It thus appears that there are many unexpected identities involving string functions at different levels \(K\), implying a rich set of relations between the characters of different \(\mathbb{Z}_K\) parafermions (and suggesting various as-yet-undiscovered relationships between the different \(\mathbb{Z}_K\) parafermion theories themselves). In Sect. 4 we
will prove the identities in our three series, and discuss how others, such as those in (3.31), might be obtained.
4. PROOFS OF THE IDENTITIES

In this section we prove the series of identities listed in Sect. 3. We will find that the proofs of the new $K > 2$ cases exactly mirror the traditional proofs for the known $K = 2$ special cases, thus demonstrating that each series of identities shares the same underlying mathematical basis. Our proofs make use of some fundamental and powerful results from the theory of modular functions, suitably generalized so as to be appropriate for the $K > 2$ cases. In the first part of this section, therefore, we provide a review of these results from modular function theory, ultimately quoting a theorem upon which our proofs rest. The second part of this section then contains the proofs of our identities. We have kept our discussion sufficiently general throughout in the hope that methods of obtaining and proving additional $[K_1, K_2]$ identities will become self-evident.

4.1. RESULTS FROM MODULAR FUNCTION THEORY

We first provide a review of those aspects of modular function theory which will be relevant for the proofs of our identities. For more details, we refer the reader to any of the standard modular function theory references; in particular, our approach is based upon those of Refs. [14] and [15].

The homogeneous modular group is $\tilde{\Gamma} \equiv SL(2, \mathbb{Z})$, the group formed by the set of $2 \times 2$ matrices with integer entries and unit determinant under matrix multiplication. The inhomogeneous modular group $\Gamma$ (the so-called “modular group”) is the quotient group $\Gamma \equiv PSL(2, \mathbb{Z}) \equiv \tilde{\Gamma}/\{\pm 1\}$, the subgroup of $\tilde{\Gamma}$ in which every
matrix $A \in \Gamma$ is identified with the matrix $-A$. We shall need to consider various subgroups of $\Gamma$. Three series of subgroups of $\Gamma$ may be defined as follows:

\[
\Gamma(N) : \quad \left\{ A \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid a \equiv d \equiv 1 \pmod{N}, \ b \equiv c \equiv 0 \right\},
\]

\[
\Gamma_1(N) : \quad \left\{ A \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid a \equiv d \equiv 1 \pmod{N}, \ c \equiv 0 \right\},
\]

\[
\Gamma_0(N) : \quad \left\{ A \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid c \equiv 0 \right\},
\]

where $\equiv \pmod{N}$ signifies equality modulo $N$. We thus see that the elements of $\Gamma(N)$, $\Gamma_1(N)$, and $\Gamma_0(N)$ are respectively of the forms $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ modulo $N$ (where the asterisk indicates the absence of any defining relation), and therefore $\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset \Gamma$ for $N > 1$. For $N = 1$ these groups each equal $\Gamma$. Such groups are called **congruence subgroups** of $\Gamma$, with $\Gamma(N)$ called the **principal congruence subgroups**; in general a congruence subgroup $\Gamma' \subset \Gamma$ is said to be of **level** $N$ if $\Gamma(N) \subset \Gamma' \subset \Gamma$. (This level $N$ bears no relation to the Kač-Moody level $K$.) Only the principal congruence subgroups $\Gamma(N)$ are normal subgroups of $\Gamma$, and in fact this series of subgroups will concern us the most. Note that $\Gamma(N)$ is isomorphic to $\Gamma/SL(2, \mathbb{Z}_N)$, where $\mathbb{Z}_N$ is the set of integers modulo $N$. In particular, $SL(2, \mathbb{Z}_1) \approx \{1\}$ since all elements of $SL(2, \mathbb{Z})$ are, modulo 1, isomorphic to the identity.

Each of the congruence subgroups $\Gamma'$ in (4.1) has finite index in $\Gamma$:

\[
[\Gamma : \Gamma'] \equiv \dim \Gamma/\Gamma' < \infty,
\]
and straightforward number-theoretic arguments\textsuperscript{[14,15]} yield the values:

\[
\begin{align*}
[\Gamma : \Gamma(N)] &= \epsilon_N N^3 \prod_{p|N} (1 - p^{-2}) \\
[\Gamma : \Gamma_1(N)] &= \epsilon_N N^2 \prod_{p|N} (1 - p^{-2}) \quad (4.3) \\
[\Gamma : \Gamma_0(N)] &= N \prod_{p|N} (1 + p^{-1})
\end{align*}
\]

where the products are taken over all primes \(p > 1\) dividing \(N\) and where

\[
\epsilon_N \equiv \begin{cases} 
1 & \text{for } N = 1, 2 \\
1/2 & \text{for } N > 2.
\end{cases} \quad (4.4)
\]

These factors of \(\epsilon_N\) in (4.3) reflect the fact that \(1^{\frac{N}{N}} = 1\) for \(N = 1, 2\), but not for \(N > 2\). From (4.3), therefore, we find

\[
[\Gamma : \Gamma(N)] = \begin{cases} 
1 & \text{for } N = 1 \\
6 & \text{for } N = 2 \\
24 & \text{for } N = 4.
\end{cases} \quad (4.5)
\]

For any congruence subgroups \(\Gamma'\) with finite index in \(\Gamma\), we can identify two generators \(\gamma_i \in \Gamma' \ (i = 1, 2)\) and a set of coset representatives \(\alpha_j \in \Gamma \ (j = 1, ..., [\Gamma : \Gamma'])\). The \(\gamma_i\) are generators in the sense that every element of \(\Gamma'\) can be written as a “word” in \(\gamma_1\) and \(\gamma_2\). The set of coset representatives (also called a transversal) contains one element from each coset of \(\Gamma / \Gamma'\); therefore only one of these coset representatives is in \(\Gamma'\) itself, and we are free to choose this representative to be
\( \alpha_1 = 1 \). At level \( N = 1 \), we have only the full modular group \( \Gamma \): its two generators are

\[ S \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (4.6) \]

and its sole “coset” representative is of course \( \alpha_1 = 1 \). At higher levels there are more possibilities. At levels \( N = 2 \) and \( N = 4 \), for example, we find the following generators and transversals for the principal congruence subgroups:

\[ \Gamma(2) : \quad \text{generators : } T^2, ST^{-2}S \]

\[ \text{transversal : } \{1, S, T, ST, TS, T^{-1}ST\} \]

\[ \Gamma(4) : \quad \text{generators : } T^4, ST^4S \]

\[ \text{transversal : } \{X, T^2X, ST^2SX, T^2ST^2SX\} \]

where \( X \equiv \{1, S, T, ST, TS, TST\} \), whereas for the \( \Gamma_0 \) subgroups we find:

\[ \Gamma_0(2) : \quad \text{generators : } T, ST^2S \]

\[ \text{transversal : } \{1, S, ST\} \] \quad (4.8)

\[ \Gamma_0(4) : \quad \text{generators : } T, ST^{-4}S \]

\[ \text{transversal : } \{1, S, ST, ST^2, ST^3, ST^2S\} . \]

Note that \( \Gamma_0(N) \) and \( \Gamma_1(N) \) contain \( T \) for all \( N \); therefore one of the generators of these groups is always \( T \). This is not the case for the principal subgroups \( \Gamma(N) \), which contain \( T \) only for \( N = 1 \). As we shall see, we will be primarily interested in those subgroups for which \( T \) is not a generator.

* These are actually right transversals (representatives of right cosets). For the \( \Gamma(N) \) subgroups the right and left transversals coincide because \( \Gamma(N) \) is a normal subgroup of \( \Gamma \) for every \( N \).
The modular group \( \Gamma \) is isomorphic to the set of linear-fractional transformations
\[
\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{Z},
\] (4.9)
of \( \tau \in H \) (where \( H \) is the complex upper half-plane); indeed, we can identify the transformation (4.9) with the group element \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \). A fundamental domain \( \mathcal{F} \equiv \mathcal{F}[\Gamma] \) for \( \Gamma \), therefore, is a set of points \( \tau \in H \) such that no two are related by a transformation of the form (4.9). It is conventional to choose this domain to be contiguous and symmetric about the \( \tau_2 \) axis (where \( \tau_1 \) and \( \tau_2 \) are respectively the real and imaginary parts of \( \tau \)):
\[
\mathcal{F} \equiv \mathcal{F}[\Gamma] \equiv \{ \tau \in \mathbb{C} \mid \tau_2 > 0, \ |\tau_1| \leq \frac{1}{2}, \ |\tau| \geq 1 \}.
\] (4.10)

A fundamental domain \( \mathcal{F}[\Gamma'] \) corresponding to any subgroup \( \Gamma' \subset \Gamma \) must therefore be larger than \( \mathcal{F}[\Gamma] \), and is in fact the totality of points obtained by acting upon each point in \( \mathcal{F}[\Gamma] \) with each of the coset representatives (including 1) of \( \Gamma' \):
\[
\mathcal{F}[\Gamma'] \equiv \bigcup_{j=1}^{[\Gamma:\Gamma']} \alpha_j \mathcal{F}[\Gamma].
\] (4.11)

One typically chooses the transversal \( \{\alpha_j\} \) in such a way that with the choice (4.10), the domain \( \mathcal{F}[\Gamma'] \) in (4.11) is contiguous.

There are certain points in the complex upper half-plane \( H \) which are called cusp points: these are the point \( \tau_\infty \equiv (0, \infty) \equiv i\infty \), along with the set \( Q \) (i.e., the points with \( \tau_2 = 0 \) and rational values of \( \tau_1 \)). We shall need to distinguish those cusp points which are \( \Gamma' \)-inequivalent for a given congruence subgroup \( \Gamma' \subset \Gamma \) (i.e.,
those cusp points not related by a transformation in \( \Gamma' \). At level \( N = 1 \) the only congruence subgroup is \( \Gamma \) itself, and indeed it is a simple matter to see that all of the cusp points are in the same \( \Gamma \)-equivalence class. The set of \( \Gamma \)-inequivalent cusp points therefore contains only one element, and the typical choice \( \tau_\infty \) is consistent with the choice of fundamental domain (4.10). Given this choice, it follows that at any higher level \( N > 1 \), all \( \Gamma' \)-inequivalent cusps for \( \Gamma' \subset \Gamma \) must be in the set \( \{ \alpha_i \tau_\infty \} \) where \( \{ \alpha_i, i = 1, \ldots, [\Gamma : \Gamma'] \} \) is the transversal of \( \Gamma' \) in \( \Gamma \). Such points \( \alpha_i \tau_\infty \) for \( i > 1 \) are all \( \in \mathbb{Q} \). However, even the points in this restricted set are not necessarily \( \Gamma' \)-inequivalent: in general two such points \( \alpha_i \tau_\infty \) and \( \alpha_j \tau_\infty \) are \( \Gamma' \)-inequivalent if and only if there does not exist an integer \( n \) such that \( \alpha_j T^n \alpha_i^{-1} \in \Gamma' \).

It turns out, for example, that for \( \Gamma' = \Gamma_0(p) \) with \( p \) a prime number there are only two \( \Gamma' \)-inequivalent cusp points (0 and \( \tau_\infty \)), while for \( \Gamma' = \Gamma_0(p^2) \) there are \( p + 1 \) such cusps points: 0, \( \tau_\infty \), and \(-1/(rp)\) for \( r = 1, \ldots, p - 1 \). A listing of the \( \Gamma' \)-inequivalent cusp points for the congruence subgroups at levels \( N = 2 \) and \( N = 4 \) which will be relevant to our later discussion is as follows:

\[
\begin{align*}
\Gamma_0(2) : & \quad \text{cusps} : 0, \tau_\infty \\
\Gamma_0(4) : & \quad \text{cusps} : 0, \tau_\infty, -1/2 \\
\Gamma(2) : & \quad \text{cusps} : 0, \tau_\infty, -1 \\
\Gamma(4) : & \quad \text{cusps} : 0, \tau_\infty, -1, -1/2, 1, 3/2, 2.
\end{align*}
\]

(4.12)

Such points are called cusp points of \( \Gamma' \) because in each case fundamental domains \( \mathcal{F}[\Gamma'] \) can be chosen whose shapes are cusp-like at each of these points (and “cusp”-like at \( \tau_\infty \)).
We can now rigorously define modular functions, modular forms, and cusp forms with respect to these general groups $\Gamma' \subseteq \Gamma$. First we consider the full modular group $\Gamma$ (i.e., level $N = 1$). A modular function of weight $k \in 2\mathbb{Z}$ with respect to $\Gamma$ is defined to be a function $f(\tau)$ satisfying two conditions. First, it must have an expansion in powers of $q \equiv \exp\{2\pi i \tau\}$ of the form

$$f(\tau) = \sum_{n \in \mathbb{Z}} a_n q^n$$ \hspace{1cm} (4.13)

where there exists an $m \in \mathbb{Z}$ such that $a_n = 0$ for all $n < m$ (i.e., there can be at most finitely many non-zero values of $a_n$ with $n < 0$). This condition therefore ensures that $f(\tau)$ is meromorphic at $\tau = \tau_\infty$ (i.e., $q = 0$). Second, $f(\tau)$ must satisfy

$$f(\gamma \tau) = (c\tau + d)^k f(\tau)$$ \hspace{1cm} (4.14)

for all $\gamma \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $\tau \in \mathbb{H}$. It is convenient to define the stroke operator $[\alpha]$ for any $\alpha \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$: this operator $[\alpha]$ transforms a modular function $f$ of weight $k$ to $f[\alpha]$, where

$$(f[\alpha])(\tau) \equiv (c\tau + d)^{-k} f(\alpha \tau) .$$ \hspace{1cm} (4.15)

With this notation, then, (4.14) becomes the requirement that

$$f[\gamma] = f \quad \text{for all } \gamma \in \Gamma ,$$ \hspace{1cm} (4.16)

and it is clear that any function $f$ invariant under the two $\Gamma$-generators $[S]$ and $[T]$ therefore satisfies (4.16). Note that the set of modular functions of a given weight $k$ forms a complex vector space, and that the product of two modular functions of weights $k_1$ and $k_2$ respectively is a modular function of weight $k_1 + k_2$. 

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Any such modular function $f$ which additionally satisfies $a_n = 0$ for all $n < 0$ in (4.13) is said to be a modular form with respect to $\Gamma$. Modular forms are therefore holomorphic (rather than merely meromorphic) at $\tau = \tau_\infty$, remaining finite at $q = 0$. If in fact $a_0 = 0$ as well, so that the modular form $f$ actually vanishes at $q = 0$, then $f$ is called a cusp form. The complex vector spaces of modular forms and cusp forms of weight $k$ with respect to $\Gamma$ are respectively denoted $M_k[\Gamma]$ and $S_k[\Gamma]$.

It is straightforward to generalize these definitions to congruence subgroups at higher levels $N > 1$. For any such subgroup $\Gamma'$, a modular function $f$ of weight $k \in 2\mathbb{Z}$ with respect to $\Gamma'$ must again satisfy two conditions. The natural generalization of (4.16) is the requirement that

$$f[\gamma] = f$$

for all $\gamma \in \Gamma'$,  

(4.17)

and once again it is sufficient to demonstrate that $f[\gamma_i] = f$ for the two generators $\gamma_i$ of $\Gamma'$ in order to demonstrate (4.17). The generalization of (4.13), on the other hand, is a bit more subtle. Eq. (4.13) was the requirement that $f$ be meromorphic at $q = 0$ (i.e., at $\tau = \tau_\infty$, the cusp point of $\Gamma$). For subgroups $\Gamma'$ at higher levels $N > 1$, we therefore analogously require that $f$ be meromorphic at each of the $\Gamma'$-inequivalent cusp points of $\Gamma'$. This meromorphicity is not determined by evaluating $f(\tau)$ as $\tau$ approaches each of these cusp points, however; in fact, since each of these additional cusp points is $\in \mathbb{Q}$, we have $|q| = 1$ at these points and therefore a straightforward $q$-expansion of $f$ does not converge. Instead, meromorphicity at the cusps is defined as follows. We have seen that this set of $\Gamma'$-inequivalent cusp
points can be written as \( \{ \alpha_i \tau_\infty \} \) where these \( \alpha_i \) are among (but not necessarily all of) the coset representatives of \( \Gamma' \) in \( \Gamma \). Let \( s \) denote the set of these \( \alpha_i \) (hence \( s \) is a subset of the transversal). The analogue of (4.13) is therefore the requirement that for each \( \alpha_i \in s \), we can perform a \( q \)-expansion

\[
f[\alpha_i] = \sum_{n \in \mathbb{Z}/N} a_n q^n \tag{4.18}
\]

where, as before, there exists an \( m \in \mathbb{Z}/N \) such that \( a_n = 0 \) for all \( n < m \). Note that \( n \) and \( m \) can now take values in the larger set \( \mathbb{Z}/N \) (rather than \( \mathbb{Z} \) itself). Also note that for \( i > 1 \) we have \( \alpha_i \notin \Gamma' \), so (4.18) is in general quite stringent.

It is clear that at level \( N = 1 \), (4.18) reduces to (4.13), for in this case \( s = \{ \alpha_1 \} = \{ 1 \} \). For \( N > 1 \), the \( i = 1 \) case of (4.18) implies meromorphicity at \( \tau_\infty \), and the \( i > 1 \) cases imply meromorphicity at each of the remaining \( \Gamma' \)-inequivalent cusps \( \in \mathbb{Q} \). Note that (4.18) is in fact sufficient to imply that \( f \) is actually meromorphic at all of the cusp points of \( \Gamma' \).

Once again, if indeed \( a_n = 0 \) for all \( n < 0 \) and each \( \alpha_i \in s \) (so that \( f \) is holomorphic at each \( \Gamma' \)-cusp, remaining finite), then \( f \) is deemed a modular form of weight \( k \) with respect to \( \Gamma' \). If additionally \( a_0 = 0 \) for each \( \alpha_i \in s \) (so that \( f \) vanishes at each cusp), then \( f \) is a cusp form with respect to \( \Gamma' \). The complex vector spaces of such weight-\( k \) modular forms and cusp forms are denoted \( M_k[\Gamma'] \) and \( S_k[\Gamma'] \) respectively; note that \( M_k[\Gamma] \subseteq M_k[\Gamma'] \) for all \( \Gamma' \subseteq \Gamma \), as well as the property \( M_{k_1}[\Gamma']M_{k_2}[\Gamma'] \subseteq M_{k_1+k_2}[\Gamma'] \).

We are now in a position to state the fundamental theorem\(^{[15]} \) upon which the proofs of our identities rest.
**Theorem:** Let \( \Gamma' \subseteq \Gamma \) be any level-\( N \geq 1 \) congruence subgroup of the modular group \( \Gamma \), and let \( M_k[\Gamma'] \) denote the space of modular forms of weight \( k \in 2\mathbb{Z} \) with respect to \( \Gamma' \). Then the sizes of such spaces depend on \( k \) as follows:

- \( \dim M_k[\Gamma'] = 0 \) for all \( k < 0 \).
- \( \dim M_k[\Gamma'] = 1 \) for \( k = 0 \).
- For \( k > 0 \) a general formula exists as well. Although we will not require these \( k > 0 \) results for the proofs of our specific identities, we include the following two special cases which are likely to be useful in proving additional \([K_1, K_2]\) identities:

  - For level \( N = 1 \) (i.e., \( \Gamma' = \Gamma \)), we have
    \[
    \dim M_k[\Gamma] = \begin{cases} 
    [k/12] & \text{for } k \equiv 2 \\
    [k/12] + 1 & \text{otherwise}
    \end{cases}
    \] (4.19)
    where \([x]\) is the greatest integer \( \leq x \), and \( \equiv \) signifies equality modulo 12.
  - For the principal congruence subgroups \( \Gamma(N) \) at levels \( N > 1 \), we have instead:
    \[
    \dim M_k[\Gamma(N)] = \frac{(k - 1)N + 6}{12N} [\Gamma : \Gamma(N)]
    \] (4.20)
    where the index \([\Gamma : \Gamma(N)]\) is given in (4.3).
Thus, for example, in our cases of interest this theorem tells us that for $\Gamma' = \Gamma$ we have
\[
\dim M_k[\Gamma'] = \begin{cases} 
0 & \text{for } k < 0 \text{ or } k = 2 \\
1 & \text{for } k = 0, 14 \text{ and } 4 \leq k \leq 10, 
\end{cases} \quad (4.21)
\]
whereas for $\Gamma' = \Gamma(N)$ we have
\[
\dim M_k[\Gamma(N)] = \begin{cases} 
0 & \text{for all } N > 1, k < 0 \\
1 & \text{for all } N > 1, k = 0 \\
\frac{1}{2}k + 1 & \text{for } N = 2, k > 0 \\
2k + 1 & \text{for } N = 4, k > 0.
\end{cases} \quad (4.22)
\]
These dimensions are important, for they tell us the number of “basis” modular forms in terms of which any modular form of weight $k$ can be expressed as a polynomial. For example, since $\dim M_0[\Gamma'] = 1$ for all congruence subgroups $\Gamma' \subseteq \Gamma$, and since $f = 1$ is a valid $\Gamma'$-modular form of weight $k = 0$, all $\Gamma'$-modular forms of weight $k = 0$ must be constants:
\[
M_0[\Gamma'] = \mathbb{C} \quad (4.23)
\]
where $\mathbb{C}$ is the space of complex numbers. Similarly, since $\dim M_k[\Gamma'] = 0$ for all $k < 0$, all $\Gamma'$-modular forms of negative weight must vanish identically:
\[
f \in M_k[\Gamma'] \implies f = 0 \quad \text{for all } k < 0. \quad (4.24)
\]
Likewise, for $k = 4, 6, 8, 10, \text{ or } 14$, we see that all $f \in M_k[\Gamma]$ must be multiples of a single function $E_k$; these “basis functions” $E_k(\tau)$ form the so-called Eisenstein series.$^{[14-16]}$

Finally, we remark that equally powerful results can be obtained for the spaces of cusp forms $S_k[\Gamma']$, as well as for the cases when $k$ is odd (and in fact half-integral). However, the above results will be sufficient for the proofs of our series of identities.
4.2. Proofs of the Identities

Given the theorem presented in the last subsection, it is relatively straightforward to prove each of the series of identities listed in Sect. 3. The basic idea of each proof is the same: we demonstrate that our string-function expressions are modular forms of a given weight with respect to an appropriate congruence subgroup $\Gamma' \subseteq \Gamma$, whereupon the theorem allows us to conclude the claimed identity. The primary subtleties involve properly formulating the identities and identifying the relevant congruence subgroups.

Since all of our identities involve combinations of the Dedekind $\eta$-function, the Jacobi $\vartheta_i$-functions, and the string functions $c^\ell_n$, let us first recall how these functions transform under the modular group. Under $T$, their respective transformation rules take the forms:

$$\eta(\tau + 1) = \alpha \eta(\tau)$$
$$\vartheta_i(\tau + 1) = \sum_j \alpha_{ij} \vartheta_j(\tau)$$
$$c^\ell_n(\tau + 1) = \alpha'^\ell c^\ell_n(\tau)$$

(4.25)

(Where $\alpha$, $\alpha'$, and $\alpha_{ij}$ indicate various phases and mixing matrices), while under $S$ we have

$$\eta(-1/\tau) = \sqrt{\tau} \beta \eta(\tau)$$
$$\vartheta_i(-1/\tau) = \sqrt{\tau} \sum_j \beta_{ij} \vartheta_j(\tau)$$
$$c^\ell_n(-1/\tau) = (\sqrt{\tau})^{-1} \sum_{\ell',n'} \beta_{\ell n'}^{\ell'} c^\ell_{n'}(\tau)$$

(4.26)

(Where again the $\beta$’s represent various phases and mixing matrices). Thus, we see that even though these functions themselves are not invariant under $S$ and $T$, they
each transform \textit{covariantly} under the modular group, filling out (in the case of \( \vartheta_i \) and \( c_n^\ell \)) representations of the modular group with dimensions greater than 1. Furthermore, we see from (4.25) and (4.26) that the \( \eta \)-function and \( \vartheta_i \)-functions transform with positive modular weight \( k = 1/2 \), while the string functions \( c_n^\ell \) transform with negative modular weight \( k = -1/2 \).

Given these observations, it is straightforward to determine the modular weights of each of our identities listed in Sect. 3. For the first series of identities, we see that each expression \( A_K, B_K, \) or \( C_K \) contains \( 16/K \) string-function factors; indeed, each of these identities takes the general form

\[
\sum (c)^{16/K} = 0 .
\]  

Thus, for any Kač-Moody level \( K \), the level-\( K \) identity in the first series has modular weight \( k = -8/K \). Similarly, the second series of identities involves combinations of all three of our functions (\( \eta, \vartheta, \) and \( c \)); however, for each level \( K \), the level-\( K \) identity always takes the general form

\[
\sum (c)^{16/K} + \sum \left( \frac{\vartheta}{\eta^3} \right)^{8/K} = 0 ,
\]

from which it follows once again that the level-\( K \) identity in this series has modular weight \( k = -8/K \). A similar situation exists for the third series as well: each of these identities takes the general form

\[
\eta^p \sum (c)^p = 1
\]

for some integer power \( p \), and therefore for any \( K \) these identities have modular weight \( k = 0 \).
Hence it is clear that all of these identities can be made to follow from the theorem given in the last subsection, provided each can be rewritten in such a manner that their left sides are modular forms of appropriate weight \( k \) with respect to a congruence subgroup \( \Gamma' \subseteq \Gamma \).

**First series – the \([K, 0]\) identities:**

We begin by proving the \( K = 2 \) identity \( A_2 = 0 \), as given in Sect. 3. This is of course the famous Jacobi \( \vartheta \)-function identity, and our subsequent proofs will be generalizations of this proof. Recall that \( A_2 \) can be rewritten in terms of \( \vartheta \) and \( \eta \) functions as \( \frac{1}{2} \Delta^{-1/2} J \) where \( \Delta = \eta^{24} \) and \( J \equiv \vartheta_3^4 - \vartheta_2^4 - \vartheta_4^4 \). Thus it is clear that \( A_2 \) has modular weight \( k = -8/K = -4 \), and since \( A_2[S] = A_2[T] = A_2 \), we see that \( A_2 \) is a modular function with respect to the full modular group \( \Gamma \). It is also straightforward to check that \( A_2 \) is tachyon-free: by this we mean that in a \( q \)-expansion

\[
A_2 = \sum_n a_n q^n \tag{4.30}
\]

we find \( a_n = 0 \) for all \( n < 0 \). It then follows that \( A_2 \in M_{-4}[\Gamma] \), whereupon the theorem in the last subsection gives the result \( A_2 = 0 \). Note that since \( A_2 = \frac{1}{2} \Delta^{-1/2} J \), and since it is easy to check that \( \Delta^{-1/2} \neq 0 \), this result implies the Jacobi identity \( J = 0 \). Note that it would have been more difficult to prove this latter identity directly, for even though the dimension of \( M_2[\Gamma] \) is zero, \( J \) itself is not invariant under \( S \) and \( T \) (indeed, one finds \( J[S] = J[T] = -J \)). The extra factor of \( \Delta^{-1/2} \), which appears naturally in the definition of \( A_2 \), absorbs this unwanted minus sign and leads to the simple proof \( A_2 = 0 \).
Let us now proceed to the $K = 4$ case: we wish to prove $A_4 = B_4 = 0$. Unlike the $K = 2$ case, $A_4$ and $B_4$ are not each invariant under the stroke operators $[S]$ and $[T]$; rather, they together fill out a two-dimensional representation of $\Gamma$:

$$
\begin{pmatrix}
A_4 \\
B_4
\end{pmatrix}
[S] = e^{i\pi}
\begin{pmatrix}
1/2 & 3 \\
1/4 & -1/2
\end{pmatrix}
\begin{pmatrix}
A_4 \\
B_4
\end{pmatrix}
$$

$$
\begin{pmatrix}
A_4 \\
B_4
\end{pmatrix}
[T] = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
A_4 \\
B_4
\end{pmatrix},
\tag{4.31}
$$

and we wish to prove that $A_4$ and $B_4$ individually vanish. This is the reason it is necessary to consider the congruence subgroups $\Gamma' \subset \Gamma$.

Let us first prove $A_4 = 0$. It is clear that $A_4$ is not modular-invariant under $[S]$, so the full modular group $\Gamma$ cannot be the relevant group in this case. Instead, let us consider $\Gamma_0(2)$. From the mixing matrices (4.31) we have $A_4[T] = A_4$, as well as

$$
A_4[ST^2S] = e^{i\pi} \left( \frac{1}{2} A_4 + 3 B_4 \right) [T^2S]
$$

$$
= e^{i\pi} \left( \frac{1}{2} A_4 + 3 B_4 \right) [S]
$$

$$
= (e^{i\pi})^2 A_4 = A_4.
\tag{4.32}
$$

The second equality follows from the fact that $A_4$ and $B_4$ are both invariant under $T^2$; note also that in general $f[\alpha\beta] = (f[\alpha])[\beta]$. Thus, the first condition (4.17) for $A_4$ to be a $\Gamma_0(2)$ modular form is satisfied: $A_4$ is invariant under all $\gamma \in \Gamma_0(2)$. We now must show that the second condition (4.18) is satisfied as well. We see from (4.12) that for $\Gamma' = \Gamma_0(2)$ there are only two $\Gamma'$-independent cusp points, $\tau_\infty$ and $0$, and therefore the transversal subset $s$ is only $\{1, S\}$. It is clear that $A_4[1] = A_4$ has a $q$-expansion of the proper form (4.18) (with $m = 0$), and similarly we see that $A_4[S] = e^{i\pi}(\frac{1}{2} A_4 + 3B_4)$ also has a $q$-expansion of the proper form (with $m = 0$, 57
since both $A_4$ and $B_4$ are tachyon-free). It therefore follows that $A_4 \in M_{-2}[\Gamma_0(2)]$, whereupon we obtain the identity $A_4 = 0$.

Note that there are indeed other ways we might have obtained this result. For instance, let us instead consider the congruence subgroup $\Gamma' = \Gamma_0(4)$. It is straightforward to check that $A_4$ is invariant under both generators $[T^4]$ and $[ST^{-4}S]$, and for this congruence subgroup the cusp points are $\tau_\infty$, 0, and $-1/2$. We have already shown that $A_4$ has the proper behavior at the first two of these cusp points; let us therefore focus on the third. Note that $-1/2 = \alpha \tau_\infty$ where $\alpha \equiv \left( \begin{smallmatrix} -1 & 1 \\ 2 & -3 \end{smallmatrix} \right) \in \Gamma$. We therefore must examine $A_4[\alpha]$. Since it turns out that $\alpha = ST^2ST^{-1}$, we have

$$
A_4[ST^2ST^{-1}] = e^{i\pi} \left( \frac{1}{2} A_4 + 3 B_4 \right) [T^2ST^{-1}]
= e^{i\pi} \left( \frac{1}{2} A_4 + 3 B_4 \right) [ST^{-1}]
= (e^{i\pi})^2 A_4 [T^{-1}] = A_4.
$$

Therefore $A_4$ has the same behavior at the cusp point $-1/2$ as it does at $\tau_\infty$, and since $A_4$ is tachyon-free (i.e., since $A_4$ remains finite at $\tau_\infty$), we find $A_4 \in M_{-2}[\Gamma_0(4)]$. This again implies the conclusion $A_4 = 0$. In fact, one can similarly demonstrate that $A_4 \in M_{-2}[\Gamma(2)]$ and $A_4 \in M_{-2}[\Gamma(4)]$, each of which leads as well to this result.

Having proven $A_4 = 0$, we find that there are two ways to prove $B_4 = 0$. The first is indirect but simpler: since

$$
A_4[S] = e^{i\pi} \left( \frac{1}{2} A_4 + 3 B_4 \right) = 0,
$$

we must have $B_4 = 0$. A more direct method (not relying on the identity $A_4 = 0$) is to construct an independent proof along the above lines. Note that since $B_4$
is not invariant under \([T]\), we cannot consider any subgroups for which \(T\) is a generator; we are therefore restricted to consideration of the principal congruence subgroups \(\Gamma(N)\). It is straightforward to demonstrate that \(B_4 \in M_{-2}[\Gamma(N)]\) where \(N\) is either 2 or 4, and therefore \(B_4 = 0\). In fact, \(B_4\) is a cusp form with respect to these groups \(\Gamma(N)\), since \(B_4\) a priori has a \(q\)-expansion of the form \(q^h(1 + ...)\) where \(h > 0\).

Let us now collect together the essential ingredients in these proofs, in order to frame a general argument. First, the set of string-function expressions \(\{A_K, B_K, \ldots\}\) must be closed under \([S]\) and \([T]\), forming a multi-dimensional representation \(R_K\) of the modular group \(\Gamma\); furthermore, a congruence subgroup \(\Gamma' \subseteq \Gamma\) must be identified such that each member of the representation \(R_K\) is itself invariant under the generators of \(\Gamma'\) (i.e., each member must separately comprise a one-dimensional representation of \(\Gamma'\)). Second, this entire representation \(R_K\) must transform under the modular group with negative even modular weight \(k\). Third, all members of \(R_K\) must have \(q\)-expansions with finitely many non-zero coefficients \(a_n\) with \(n < 0\). This third condition is needed in order to insure that each element \(\in R_K\) is meromorphic at all of the cusp points of \(\Gamma'\), for any one member of \(R_K\) will be meromorphic at all the cusp points of \(\Gamma'\) if and only if all of the members of \(R_K\) are meromorphic at the one cusp point \(\tau_\infty\) (because the set \(s\) always contains at least 1 and \(S\)). These three conditions then guarantee that each string-function expression in the representation \(R_K\) is itself a modular function with respect to \(\Gamma'\). If each member of the representation is also tachyon-free, then each is a modular form with respect to \(\Gamma'\) and hence must vanish identically.
Let us now see how to apply this general argument to the $K = 8$ case as we attempt to prove $A_8 = B_8 = C_8 = 0$. The mixing matrices of these expressions under the stroke operators $[S]$ and $[T]$ are as follows:

\[
\begin{pmatrix}
A_8 \\
B_8 \\
C_8
\end{pmatrix}
\begin{bmatrix}
1/2 & 1/2 & 1 \\
1/2 & 1/2 & -1 \\
1/2 & -1/2 & 0
\end{bmatrix}
\begin{pmatrix}
A_8 \\
B_8 \\
C_8
\end{pmatrix} = e^{i\pi/2} 
\begin{pmatrix}
A_8 \\
B_8 \\
C_8
\end{pmatrix} \tag{4.35}
\]

\[
\begin{pmatrix}
A_8 \\
B_8 \\
C_8
\end{pmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -i
\end{bmatrix}
\begin{pmatrix}
A_8 \\
B_8 \\
C_8
\end{pmatrix}. 
\]

There are immediately two problems. First, we see that this $K = 8$ representation has an odd modular weight $k = -8/K = -1$; our theorem applies only to the cases $k \in 2\mathbb{Z}$. Second, no member of this representation is invariant under any of the congruence subgroups: for example, under $\Gamma_0(4)$ we find $A_8[T] = A_8$ but

\[
A_8[ST^{-4}S] = e^{i\pi/2} \left( \frac{1}{2} A_8 + \frac{1}{2} B_8 + C_8 \right) [T^{-4}S]
= e^{i\pi/2} \left( \frac{1}{2} A_8 + \frac{1}{2} B_8 + C_8 \right) [S] 
= (e^{i\pi/2})^2 A_8 = -A_8. \tag{4.36}
\]

There are two ways to solve these difficulties. One possibility is to extend the theorem presented in Sect. 4.1 to apply to odd $k$ and modular functions with so-called multiplier systems (i.e., phases such as the unwanted sign appearing above). Such extensions can indeed be made; in this relatively simple case, for example, we can instead choose to prove the modified identities $A_8/\eta^6 = B_8/\eta^6 = C_8/\eta^6 = 0$. These modified identities would then have modular weight $k = -2$, and the extra $\eta$-functions absorb the unwanted sign. A simpler approach, however, (and one which generalizes more easily to other situations) is to prove instead the identities $(A_8)^2 =
$$(B_8)^2 = (C_8)^2 = 0,$$ for such identities also have an even modular weight $k = -2$ and simultaneously avoid such unwanted signs. (We are essentially enlarging our $K = 8$ representation: $R_8 \rightarrow R'_8 \equiv R_8 \otimes R_8$.) If these quadratic identities can be proven for all $\tau$, then of course the linear results $A_8 = B_8 = C_8 = 0$ immediately follow. To prove these quadratic identities, we follow the procedure outlined above: either choice $\Gamma' = \Gamma_0(4)$ or $\Gamma(4)$ suffices for proving $(A_8)^2 = 0$, and $\Gamma(4)$ suffices for independently proving $(B_8)^2 = 0$ and $(C_8)^2 = 0$. It is of course possible to deduce $B_8 = C_8 = 0$ from the result $A_8 = 0$ as we did for the $K = 4$ case: in the present case the analogue of (4.34) is

$$A_8[S] = e^{i\pi/2} \left( \frac{1}{2} A_8 + \frac{1}{2} B_8 + C_8 \right) = 0 ,$$

(4.37)

and this implies the weaker result $B_8 + 2C_8 = 0$. However, $B_8 \sim q^{1/2}(1 + ...)$ and $C_8 \sim q^{3/4}(1 + ...)$, where inside the parentheses all $q$-exponents are integral. Therefore, $B_8$ and $C_8$ must each vanish separately. Note that in this $K = 8$ case we are compelled to consider congruence subgroups of levels $N \in 4\mathbb{Z}$ only. This occurs because the $K = 8$ representation includes a sector $C_8$ with quarter-integer powers of $q$. Since in general the generators of $\Gamma(N)$ are of the form $T^N$ and $ST^\pm NS$, the choice $N = 4$ is the smallest level $N$ for which each element in the $K = 8$ representation $R_8$ is $\Gamma(N)$-invariant.

The same procedure applies for the $K = 16$ case as well; here we wish to prove
\( A_{16} = C_{16} = 0 \), and the relevant mixing matrices are as follows:

\[
\begin{pmatrix}
A_{16} \\
C_{16}
\end{pmatrix}
[S] = e^{i\pi/4} \frac{1}{2\sqrt{2}} \begin{pmatrix}
2 & 4 \\
1 & -2
\end{pmatrix}
\begin{pmatrix}
A_{16} \\
C_{16}
\end{pmatrix}
\]

\[
\begin{pmatrix}
A_{16} \\
C_{16}
\end{pmatrix}
[T] = \begin{pmatrix}
1 & 0 \\
0 & -i
\end{pmatrix}
\begin{pmatrix}
A_{16} \\
C_{16}
\end{pmatrix}
\]

\[(4.38)\]

Once again we find that we must enlarge our representation

\[ R_{16} \to R'_{16} \equiv R_{16} \otimes R_{16} \otimes R_{16} \otimes R_{16} \]

\[(4.39)\]

and prove instead the identities \((A_{16})^4 = (C_{16})^4 = 0\); similarly, we must choose choose the level \(N = 4\) due to the presence of the \(C_{16}\) sector in \(R_{16}\). As usual, the subgroup \(\Gamma(4)\) suffices in general for proving that each member of \(R'_{16}\) vanishes, and we can instead make the choice \(\Gamma_0(4)\) in the case of the \(A_{16}\) sector (which is invariant under \(T\)). Once again the proof that any one member of \(R_{16}\) vanishes is sufficient to prove that all vanish, provided each has a different eigenvalue under \(T\).

Note that instead of enlarging the representation as in \((4.39)\), it would also have been possible in this case to divide our original representation by \(\eta^2\); however, this would have necessitated constructing a proof using congruence subgroups of level \(N = 8\).

**Second series – the \([K,2]\) identities:**

The second series of identities is closely related to the first and in fact contains the first series as a subset. Recall that for each Kac-Moody level \(K \in \{2, 4, 8, 16\}\), there exists a string-function expression \(A_K \sim q^0(1 + \ldots)\) which, according to the first series of identities, vanishes identically. In this second series of identities we
show that the separate bosonic and fermionic pieces of $A_K$ (denoted $A^K_b$ and $A^K_f$, respectively) can each be written in terms of Jacobi $\vartheta$-functions; these $\vartheta$-function expressions for $A^K_b$ and $A^K_f$ are of course equal, since $A_K = A^K_b - A^K_f = 0$. As a by-product, we also obtain additional string-function expressions \{\(C_K, D_K, \ldots\}\} which can be easily expressed in terms of Jacobi $\vartheta$-functions as well.

Let us first consider the $K = 4$ identities in this series, as given in (3.8): here we have separated $A_4$ into its separate bosonic and fermionic contributions $A^b_4$ and $A^f_4$ as in (3.7). While we know that $A_4$ and $B_4$ together fill out a two-dimensional representation of the modular group with weight $k = -2$, we find that the individual pieces $A^b_4$ and $A^f_4$ do not close separately into only themselves and $B_4$. Rather, in order to construct a representation of $\Gamma$ containing $A^b_4$ and $A^f_4$ as separate members, we must introduce the two additional string-function expressions $C_4$ and $D_4$ defined in (3.7). Together, the set \{\(A^b_4, A^f_4, B_4, C_4, D_4\}\} indeed fills out a complete representation $R_c$ with weight $k = -2$; each member of $R_c$ transforms as an eigenfunction under $[T]$:

\[
\begin{pmatrix} A^b_4 \\ A^f_4 \\ B_4 \\ C_4 \\ D_4 \end{pmatrix} [T] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & -i \end{pmatrix} \begin{pmatrix} A^b_4 \\ A^f_4 \\ B_4 \\ C_4 \\ D_4 \end{pmatrix}, \tag{4.40}
\]

and under $[S]$ they close into each other:

\[
\begin{pmatrix} A^b_4 \\ A^f_4 \\ B_4 \\ C_4 \\ D_4 \end{pmatrix} [S] = \frac{e^{i\pi}}{4} \begin{pmatrix} 1 & -1 & 6 & -4 & 4 \\ -1 & 1 & -6 & -4 & 4 \\ 1 & -1 & -2 & 0 & 0 \\ -1 & -1 & 0 & 2 & 2 \\ 1 & 1 & 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} A^b_4 \\ A^f_4 \\ B_4 \\ C_4 \\ D_4 \end{pmatrix}. \tag{4.41}
\]
As a check, note that these matrices indeed contain the \((A_4, B_4)\) mixing matrices (4.31) as the appropriate submatrices.

Unlike the previous representations we have considered, not all members of this representation \(R_c\) individually vanish, for while all are modular \textit{functions} with respect to an appropriate \(\Gamma' \subset \Gamma\), not all are modular \textit{forms}. Since the quantity \(D_4\) is not tachyon-free, it clearly has the wrong behavior at the cusp point \(\tau = \tau_\infty\); furthermore, since the \([S]-\)transforms of \(A_4^b, A_4^f, C_4, \) and \(D_4\) each separately involve \(D_4\), each of these quantities has incorrect (\textit{i.e.}, tachyonic) behavior at the \(\Gamma'\) cusp point \(\tau = 0\). Indeed, only \(B_4\) and the difference \(A_4 = A_4^b - A_4^f\) are free of this tachyonic behavior at both cusp points \(\tau = \tau_\infty\) and \(\tau = 0\), so of the five quantities in the above representation \(R_c\) only \(B_4\) is itself a proper \(\Gamma'\)-modular form. Therefore, in order to construct identities for these five individual quantities, it is necessary to build a \textit{new} representation involving them in such a manner that \textit{each} member is a tachyon-free modular form.

It turns out that this is not hard to do. As we have seen in (4.25) and (4.26), the \(\vartheta\)- and \(\eta\)-functions fill out valid representations of the modular group, and indeed the three quantities \(\{\eta^{-6}\vartheta_2^2, \eta^{-6}\vartheta_3^2, \eta^{-6}\vartheta_4^2\}\) fill out such a representation with modular weight \(k = -2\). Let us take various linear combinations of these quantities, promoting them to the “five”-dimensional representation

\[
R_\vartheta \equiv \left\{ \frac{\vartheta_2^2}{\eta^6}, \frac{\vartheta_2^2}{\eta^6}, 0, \frac{1}{2} \left( \frac{\vartheta_3^2 - \vartheta_4^2}{\eta^6} \right), \frac{1}{2} \left( \frac{\vartheta_3^2 + \vartheta_4^2}{\eta^6} \right) \right\}. \tag{4.42}
\]

Written this way, this five-dimensional representation \(R_\vartheta\) with weight \(k = -2\) has two very important properties. First, its mixing matrices under \([S]\) and \([T]\) are (or...
can be chosen to be) the same as those in (4.40) and (4.41) for $R_c$; indeed, in this respect the two representations $R_c$ and $R_\vartheta$ transform identically. More importantly, however, it is easy to verify that the tachyonic terms within the fifth member of the $\vartheta$-function representation $R_\vartheta$ are the same as those of the fifth member $D_4$ of the string-function representation $R_c$; additionally, all of the other members of $R_\vartheta$ are themselves tachyon-free. Therefore, subtracting the two representations, $i.e.,$

\[
R \equiv R_c - R_\vartheta = \begin{pmatrix}
A_4^b - \vartheta_2^2 / \eta^6 \\
A_4^f - \vartheta_2^2 / \eta^6 \\
B_4 \\
C_4 - \frac{1}{2} (\vartheta_3^2 - \vartheta_4^2) / \eta^6 \\
D_4 - \frac{1}{2} (\vartheta_3^2 + \vartheta_4^2) / \eta^6
\end{pmatrix}, \quad (4.43)
\]

yields a five-dimensional representation $R$ in which each member is tachyon-free.

It is then a straightforward matter to demonstrate that each member of this new representation $R$ is indeed a modular form with respect to a congruence subgroup $\Gamma' \subset \Gamma$ of level $N \in 4\mathbb{Z}$ (e.g., each member is $\in M_{-2}[\Gamma(4)]$), whereupon it follows that each vanishes identically. This, then, establishes the identities (3.8).

Note that the existence of such identities relies on the existence of a $\vartheta$-function representation $R_\vartheta$ with the desired modular weight $k$, the desired transformation matrices $[S]$ and $[T]$, and the required tachyonic behaviors of its members. Such a representation does not always exist. It is indeed fortuitous, however, that such representations do exist for each value of $K$, yielding all of the identities in this second series.

The derivations of the other identities in this series proceed in analogous fashion. For the $K = 8$ case, we find that $A_8^b$ and $A_8^f$ are members of the six-dimensional
representation \( R_c \equiv \{A^b_8, A^f_8, B_8, C_8, E_8, F_8\} \) [where these six quantities are defined in and (3.4) and (3.10)]; under \([S]\) and \([T]\) quantities mix as follows:

\[
\begin{pmatrix}
A^b_8 & A^f_8 & B_8 & C_8 & E_8 & F_8
\end{pmatrix}
= \frac{e^{i\pi/2}}{4}
\begin{pmatrix}
1 & -1 & 1 & 2 & -4 & 4 \\
-1 & 1 & -1 & -2 & -4 & 4 \\
2 & -2 & 2 & -4 & 0 & 0 \\
2 & -2 & -2 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 2 & 2 \\
1 & 1 & 0 & 0 & 2 & 2
\end{pmatrix}
\begin{pmatrix}
A^b_8 & A^f_8 & B_8 & C_8 & E_8 & F_8
\end{pmatrix}
\]  

(4.44)

and

\[
\begin{pmatrix}
A^b_8 & A^f_8 & B_8 & C_8 & E_8 & F_8
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -i & 0 & 0 \\
0 & 0 & 0 & 0 & e^{3\pi i/4} & 0 \\
0 & 0 & 0 & 0 & 0 & e^{-\pi i/4}
\end{pmatrix}
\begin{pmatrix}
A^b_8 & A^f_8 & B_8 & C_8 & E_8 & F_8
\end{pmatrix}
\]  

(4.45)

These matrices of course contain the \((A_8, B_8, C_8)\) mixing matrices (4.35) as the appropriate submatrices, and these six quantities are again modular functions rather than modular forms due to the presence of the tachyonic sixth quantity \(F_8\). To compensate for this, we introduce the corresponding six-dimensional \(R_\phi\) representation with weight \(k = -1\)

\[
R_\phi \equiv \left\{ \frac{\vartheta_2}{\eta^3}, \frac{\vartheta_2}{\eta^3}, 0, 0, \frac{1}{2} \left( \frac{\vartheta_3 - \vartheta_4}{\eta^3} \right), \frac{1}{2} \left( \frac{\vartheta_3 + \vartheta_4}{\eta^3} \right) \right\}
\]  

(4.46)

and construct the tachyon-free representation \(R' \equiv (R_c - R_\phi) \otimes (R_c - R_\phi)\). It can be proven that each member of \(R'\) is a modular form of even weight \(k = -2\) with respect to a congruence subgroups at level \(N \in 8\mathbb{Z}\), whereupon the identities (3.11) immediately follow.
The $K = 16$ case is similar. Here the expressions $A_{b}^{16}$ and $A_{f}^{16}$ are members of the five-dimensional representation $R_{c} \equiv \{ A_{b}^{16}, A_{f}^{16}, C_{16}, E_{16}, F_{16} \}$ [where these five quantities are defined in (3.13)]; under $[S]$ and $[T]$ these quantities mix as follows:

$$
\begin{pmatrix}
    A_{b}^{16} \\
    A_{f}^{16} \\
    C_{16} \\
    E_{16} \\
    F_{16}
\end{pmatrix}
 =
\frac{e^{i\pi/4}}{2\sqrt{2}}
\begin{pmatrix}
    1 & -1 & 2 & -2 & 2 \\
    -1 & 1 & -2 & -2 & 2 \\
    1 & -1 & -2 & 0 & 0 \\
    -1 & -1 & 0 & \sqrt{2} & \sqrt{2} \\
    1 & 1 & 0 & \sqrt{2} & \sqrt{2}
\end{pmatrix}
\begin{pmatrix}
    A_{b}^{16} \\
    A_{f}^{16} \\
    C_{16} \\
    E_{16} \\
    F_{16}
\end{pmatrix} \quad (4.47)
$$

and

$$
\begin{pmatrix}
    A_{b}^{16} \\
    A_{f}^{16} \\
    C_{16} \\
    E_{16} \\
    F_{16}
\end{pmatrix}
 =
\begin{pmatrix}
    1 & 0 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 & 0 \\
    0 & 0 & -i & 0 & 0 \\
    0 & 0 & 0 & e^{7\pi i/8} & 0 \\
    0 & 0 & 0 & 0 & e^{-\pi i/8}
\end{pmatrix}
\begin{pmatrix}
    A_{b}^{16} \\
    A_{f}^{16} \\
    C_{16} \\
    E_{16} \\
    F_{16}
\end{pmatrix} \quad (4.48)
$$

These matrices of course contain the $(A_{16}, C_{16})$ mixing matrices (4.38) as the appropriate submatrices. Once again we find that only the fifth quantity $F_{16}$ is tachyonic, and again there exists an appropriate compensating five-dimensional $R_{\vartheta}$ representation:

$$
R_{\vartheta} \equiv \left\{ \sqrt{\frac{\vartheta_2}{2\eta^3}}, \sqrt{\frac{\vartheta_2}{2\eta^3}}, 0, \frac{1}{2} \left( \sqrt{\frac{\vartheta_3}{\eta^3}} - \sqrt{\frac{\vartheta_4}{\eta^3}} \right), \frac{1}{2} \left( \sqrt{\frac{\vartheta_3}{\eta^3}} + \sqrt{\frac{\vartheta_4}{\eta^3}} \right) \right\}
$$

$$
= \left\{ c_{1}^{0}, c_{1}^{1}, 0, c_{0}^{2}, c_{0}^{0} \right\} \quad (4.49)
$$

where in the second line the string functions are at level $K = 2$. This representation $R_{\vartheta}$ transforms under $[S]$ and $[T]$ with the same mixing matrices as $R_{c}$; in particular
under $[S]$ the $K = 2$ string functions satisfy

$$
\begin{pmatrix}
  c_0^0 \\
  c_0^2 \\
  c_1^1
\end{pmatrix}
= [S] = \frac{e^{i\pi/4}}{2}
\begin{pmatrix}
  1 & 1 & \sqrt{2} \\
  1 & 1 & -\sqrt{2} \\
  \sqrt{2} & -\sqrt{2} & 0
\end{pmatrix}
\begin{pmatrix}
  c_0^0 \\
  c_0^2 \\
  c_1^1
\end{pmatrix}.
\quad (4.50)
$$

Since only $c_0^0$ is tachyonic (and in fact has the same tachyonic terms as $F_{16}$), the entire representation $R \equiv R_c - R_\theta$ is tachyon-free. The identities (3.14) then follow by building the tensor-product representation $R' \equiv R \otimes R \otimes R \otimes R$ and considering congruence subgroups with levels $N \in 16\mathbb{Z}$.

Finally, we note that this second series of identities can also be proven in a slightly different manner. Once we have constructed the appropriate representations $R_c$ and $R_\theta$ for each $K$, rather than prove $R_c - R_\theta = 0$ it is possible to instead demonstrate $R_c/R_\theta = 1$ (where $R_c/R_\theta$ denotes the representation formed by dividing each member of $R_c$ by the corresponding member of $R_\theta$). It is clear that $R_c/R_\theta$ is itself a valid representation of the modular group with modular weight $k = 0$, and by the usual arguments it is straightforward to demonstrate that each member of this “quotient” representation is, for example, $\in M_0[\Gamma']$ for an appropriately chosen $\Gamma' \subset \Gamma$. From the theorem presented in the previous subsection it then follows that each member of this quotient representation is a constant, and one can verify from the $q^0$ term in a $q$-expansion of each that this constant is 1 (because the relative normalization of $R_c$ and $R_\theta$ is fixed by the requirement that they have the same tachyonic structure).
**Third series – the $|K, 1|$ identities:**

This series of identities is actually the simplest to prove. Recall that the Dedekind $\eta$-function satisfies

$$\eta[S] = e^{-i\pi/4} \eta, \quad \eta[T] = e^{i\pi/12} \eta,$$

and that each identity in this series is of the form

$$\eta^p Q_K = 1$$

where $p$ is a given power and $Q_K$ is a sum of terms each containing $p$ factors of level-$K$ string functions. Since in each case $Q_K$ satisfies

$$Q_K[S] = e^{i\pi/4} Q_K, \quad Q_K[T] = e^{-i\pi/12} Q_K,$$

and since in each case the product $\eta^p Q_K$ is tachyon-free, it follows that

$$\eta^p Q_K \in M_0[\Gamma] = \mathbb{C}.$$ 

Overall normalizations have been chosen in each case such that this constant is always 1. Note that this proof yields the familiar $\vartheta$-function identity $\vartheta_2 \vartheta_3 \vartheta_4 = 2\eta^3$ in the $K = 2$ special case, and just as easily yields its higher $K > 2$ string-function generalizations. Thus, we see once again how all of our series of identities provide the natural generalizations of their known $K = 2$ special cases.
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