STRUCTURE OF STATIONARY STRONG IMBALANCED TURBULENCE

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ABSTRACT

In this paper, we systematically study the spectrum and structure of incompressible magnetohydrodynamic turbulence by means of high-resolution direct numerical simulations. We considered both balanced and imbalanced (cross-helical) cases and simulated sub-Alfvénic as well as trans-Alfvénic turbulence. This paper, extends numerics preliminarily reported in Beresnyak & Lazarian. We confirm that driven imbalanced turbulence has a stationary state even for high degrees of imbalance. Our major finding is that the structure of the dominant and subdominant Alfvénic components are notably different. Using the most robust observed quantities, such as the energy ratio, we believe we can reject several existing models of strong imbalanced turbulence.

Key words: ISM: kinematics and dynamics – MHD – turbulence

1. INTRODUCTION

Astrophysical fluids are known to be turbulent from large scales of the order of hundreds of kiloparsecs, as in the case of galaxy clusters (Vogt & Enßlin 2005), to scales as small as hundreds of kilometers. Armstrong et al. (1995) reported a wide range of electron density fluctuations probed by scintillation and scattering techniques. New statistical techniques using Doppler-shifted lines have enabled studies of velocity fluctuations on scales larger than a fraction of a parsec (see Lazarian 2009, and references therein). Recent years have been marked by new understanding of the role that turbulence plays in a number of astrophysical processes (Cho et al. 2003; Elmegreen & Scalo 2004). Most notably, turbulence has drastically changed the paradigms of interstellar medium (ISM) and molecular cloud evolution (Stone et al. 1998; Ostriker et al. 2001; Vázquez-Semadeni et al. 2007; see also review by McKee & Ostriker 2007). Small-scale turbulence has been probed by a variety of approaches such as gyrokinetics, Hall MHD, and electron MHD (Howes et al. 2006; Sckochihiin et al. 2007; Galtier et al. 2003; Cho & Lazarian 2004). This progress calls for better understanding of the fundamentals of turbulence. One reason for doing this is to understand to what extent the turbulence simulated with diffusive computer codes resembles actual turbulence in astrophysical fluids with low diffusivity.

We start with a few general remarks on the relation of fluid dynamics and description of astrophysical fluids. The continuous fluid description has been successful in describing a wide range of physical phenomena. In space, due to the presence of ionizing radiation and cosmic rays (CRs), the medium is almost always ionized to some degree, i.e., is a plasma. Although the dynamics of plasma is complicated on small scales, it can be considered well-conducting continuous fluid on large scales and, therefore, magnetohydrodynamics or MHD is applicable.

One of the most interesting phenomena in fluids is turbulence, a seemingly random stochastic flow which appears spontaneously as long as the microscopic dissipation coefficients such as viscosity or magnetic diffusivity are small (which correspond to large Reynolds or magnetic Reynolds numbers). Turbulence increases dissipation due to so-called turbulent cascade, a nonlinear transfer of energy to smaller scales. As astrophysical fluids are turbulent, this affects dynamics and is manifested in a variety of situations such as reconnection, momentum transfer in accretion disks, etc.

The basic theoretical study of the complicated nonlinear dynamics of turbulence has been concentrated on incompressible turbulence with large Reynolds number (Kolmogorov 1941). While dissipation in plasma might be much more complex than dissipation in molecular gases, the asymptotic large Reynolds number flows the “inertial range” fluctuations do not feel the peculiarities of the dissipation. On the other hand, it is often necessary to account for compressibility of the fluid, as it could be significant. An example of which is the ISM which has sonic Mach numbers in a range of 0.1–10. In this case, the dynamics of Alfvén and slow-mode perturbations on small scales can be considered incompressible (Lithwick & Goldreich 2001), while fast mode has the dynamics of its own (Cho & Lazarian 2002, 2003). On the other hand, even at large scales where compressibility is significant, one can find the unique features of incompressible dynamics (see, e.g., Beresnyak et al. 2005). In other words, the understanding of incompressible MHD turbulence, which is often called Alfvénic turbulence due to the dominant role the Alfvén perturbations play in the cascade, is of utmost importance.

Ideal incompressible MHD equations can be written in the following simple form

\[ \dot{\mathbf{w}}^\pm + \mathbf{S}(\mathbf{w}^+ \cdot \nabla)\mathbf{w}^\pm = 0, \]

where \( \mathbf{S} \) is a solenoidal projection operator and Elsässer variables are defined in terms of velocity \( \mathbf{v} \) and magnetic field in velocity units \( \mathbf{b} = \mathbf{B}/(4\pi \rho)^{1/2} \), as \( \mathbf{w}^+ = \mathbf{v} + \mathbf{b} \) and \( \mathbf{w}^- = \mathbf{v} - \mathbf{b} \). Although fairly idealized, the problem of incompressible MHD turbulence with large Reynolds number has been difficult. First attempts to treat it was an IK model by Iroshnikov (1963) and Kraichnan (1965). They found that the mean magnetic field provided by large scales will, very much unlike hydrodynamics, crucially define the dynamics on small scales. Although this first model was isotropic, it was eventually realized that the dynamics will result in anisotropy (Montgomery & Turner 1981; Shebalin et al. 1983; Higdon 1984). This resulted in Goldreich & Sridhar 1995 (henceforth GS95) model which postulated so-called critical balance, i.e., maximum anisotropy which is allowed under strong interaction. At the same time, the

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concept of the dominant perpendicular cascade, that was used in GS95, has been validated in an analytical perturbative theory of so-called weak Alfvénic turbulence (Galtier et al. 2000, 2002).²

While hydrodynamic turbulence have only one energy cascade, the incompressible MHD turbulence has two, due to the exact conservation of the Elsässer (oppositely going wave packets’) “energies.” This can also be formulated as the conservation of total energy and cross-helicity.³ The situation of zero total cross-helicity has been called “balanced” turbulence, as the amount of oppositely moving wavepackets balance each other, the alternative being “imbalanced” turbulence. Most of the above studies concentrated on the balanced case, and, without exception, the GS95 model, which is the strong cascading model with critical balance, can only be kept self-consistent assuming balanced case. The real life turbulence, however, is almost always imbalanced, such as in situations when one has a strong localized source of perturbations (the Sun for solar wind or central engine for active galactic nucleus (AGN) jets), but also due to inhomogeneity of energy sources for turbulence (supernovas and stellar winds in the ISM) and the tendency of the decaying turbulence to become increasingly more imbalanced with time. Moreover, the purely balanced MHD Alfvénic turbulence cannot be understood as it is, without understanding of the more general imbalanced case. This is due to the fact that turbulence is a stochastic phenomena with all quantities fluctuating, and every piece of turbulence at any given time can have imbalance in it. In this respect, while the mean-field Kolmogorov model can be expanded to include intermittency, the mean field GS95 model cannot.

Imbalanced turbulence or “turbulence with non-zero cross-helicity” has been discussed long ago by a number of authors (Dobrowolny et al. 1980; Matthaeus & Montgomery 1980; Grappin et al. 1983; Pouquet et al. 1988; Biskamp 2003, and references therein). These papers testified that the non-zero cross-helicity modifies the turbulence. Although these studies correctly reproduced separate cascades for energy and cross-helicity, they were based on then popular models of MHD turbulence and later it became evident that these models are problematic. For example, the closure theory of isotropic turbulence (Pouquet et al. 1976) which reproduced IK model can be rejected by both theory and numerics.⁴ Another class of models were based on so-called two-dimensional MHD turbulence that, as we now know, is unable to reproduce important properties of the real three-dimensional turbulence, such as critical balance.

Recently, several models for the strong imbalanced turbulence have been proposed (e.g., Lithwick et al. 2007; Beresnyak & Lazarian 2008; Chandran 2008). As long as the full self-contained analytical model for strong turbulence continues to elude discovery, direct numerical simulations (DNS) will be an inspiration and guidance to theorists. While the Reynolds numbers in those simulations are fairly modest (800–4000 are the best to date), some of the robust quantities can be measured and used as a guidance to reject theories. While in this paper we are fully aware of these limitations, we present the most robust statistical measures, such as total energies, dissipation rates, and second-order structure function (SF; or its equivalent, the spectrum) and the anisotropy derived from it. We leave the study of higher order statistics (and intermittency) to future studies. Our paper is written on the premise that one cannot fully confirm a model using three-dimensional DNS due to their fairly modest resolution, but from these DNS one can collect enough numerical evidence to reject a model.

This paper expands parameter space of the preliminary numerical results reported in Beresnyak & Lazarian (2008). A short introduction to the theories of imbalanced turbulence is given in Section 2. Numerical code, the simulation setup, and the establishment of the stationary state are explained in Section 3. The discussion of the SF calculated parallel to the magnetic field (parallel SF), a quantity which is a key for understanding statistical anisotropy of MHD turbulence is given in Section 4. In this section, we explain various methods to define the local guiding field and the influence to the measurement of the parallel SF. Scale-dependent anisotropies derived from SFs as well as power spectra are described in Section 5. Polarization alignment is briefly discussed in Section 6. Final comparison with models as well as observational data are in Section 7. We summarize our findings in Section 8.

2. THEORETICAL CONSIDERATIONS

The original GS95 model was based on the renormalization rule for Alfvén wave’s frequency called “critical balance.” The necessity of such renormalization can be seen from a rigorous theory of weak Alfvénic turbulence (Galtier et al. 2000, 2002) that predicts so-called perpendicular cascade, i.e., the result that nonlinear interaction of Alfvénic waves conserve these wave’s frequencies and only transverse structure of the wave packet is affected. As perpendicular cascade proceeds to small scales, the applicability of weak interaction breaks down, and Alfvénic turbulence becomes strong. In this situation, GS95 assumed that the frequency of the wavepacket cannot be smaller than the inverse lifetime of the wavepacket, estimated from nonlinear interaction. In their closure model, GS95 have an explicit ad hoc term that allows for the increase of the wave frequency. The scale dependency of this term is based on the assumption of turbulence locality (i.e., there is one characteristic amplitude of perturbation pertaining to each scale). In the imbalanced case, however, we have two such characteristic amplitudes and the choice for frequency renormalization becomes unclear (GS95).⁵ Any theory of strong imbalanced turbulence, which is qualified for serious consideration, must deal with this problem. Let us first demonstrate that a direct generalization of GS95 for imbalanced case does not work, namely, if we assume that the frequency renormalization for one wavepacket is determined by the amplitude of the oppositely moving wavepacket. Indeed, in this situation the wave with small amplitude (say, \( w^- \)) may only weakly perturb large amplitude wave \( w^+ \) and the

² Although a successful analytical theory, the weak Alfvénic turbulence can rarely be applied to real-life MHD turbulence as it would require weak, relatively isotropic fluctuations on the outer scale and a strong mean field. Even when these conditions are satisfied, the strength of the interaction, which can be estimated as \( \frac{\beta B_L}{L} \frac{d_L}{L} \), where \( L / l \) is the anisotropy of the perturbation, will increase down the cascade and break the applicability of the theory. In real life ISM turbulence the perturbations of the magnetic field are of the order of the mean field which makes the turbulence strong from outer scale inwards. For phenomenological treatment of weak Alfvénic turbulence, see also Lazarian & Vishniac (1999).

³ The latter, \( j \times B = \beta d_L \) is a quantity conserved in the absence of dissipation.

⁴ The artificial term for “relaxation of triple correlations,” that was necessary to uphold local isotropy in this model, happen to be larger than real physical nonlinear interaction. Also numerics show that MHD turbulence is locally anisotropic (Cho & Vishniac 2000; Maron & Goldreich 2001; Cho et al. 2002).

⁵ We assume that imbalanced turbulence is “strong” as long as the applicability of weak Alfvénic turbulence breaks down. This requires that at least one component is perturbed strongly. In the imbalanced turbulence, the amplitude of the dominant component is larger, so that in the transition to strong regime the applicability of weak cascading of the subdominant component breaks down first.
frequency of cascaded $w^+$ will conserve. On the other hand, $w^+$ may strongly perturb $w^-$ and $w^-$’s frequency will be determined as $w^- / t^5$. This creates an inconsistency of the local cascade where both wavepackets must have comparable parallel and perpendicular scales. In order to deal with this fundamental inconsistency, a new physical assumption must be adopted. Due to this fact, in the paper, we mostly discuss three models of strong imbalanced turbulence. Lithwick et al. (2007, LGS07), Beresnyak & Lazarian (2008, BL08), Chandran (2008, C08), that clearly state: (1) the problem above, (2) the new physical assumptions being adopted, (3) an internally consistent physical model that follows from these assumptions, and (4) the full predictions of the turbulence spectra and anisotropy. In the discussion section of this paper, we also mention other models that claim to have predictions on the strong imbalanced turbulence. We test these incomplete models by comparison with our numerics whenever possible.

2.1. LGS07 and C08 Models

LGS07 resolves the inconsistency explained above by assuming that the strong wave $w^+$ is also cascaded strongly and its frequency is simply equal to the frequency of the weak wave, i.e., the critical balance for strong wave uses the amplitude of the strong wave itself ($w^+ \Lambda = \nu_4 \lambda_1$). In other words, the anisotropies of the waves are identical. The formulae for energy cascading are strong cascading formulae, i.e.,

$$\epsilon^+ = \frac{(w^+(\lambda))^2 w^+(\lambda)}{\lambda}.$$ 

This leads to the prediction, $w^+/w^- = \epsilon^+ / \epsilon^-$. However, this prediction, together with assumption of the locality of cascading, lead to a contradiction on viscous scale, where the nonlinear cascading rates must smoothly transit into viscous dissipation rates. This requires $w^+ = w^-$ on the dissipation scale, so-called pinning (Lithwick & Goldreich 2003).

C08 is a complicated quantitative theory of advection–diffusion cascading has several rules that determine the diffusion of the waves in the parallel direction, which are analogous to the frequency renormalization of GS95. In effect, in the case of strong turbulence, the C08 rules lead to equal or very close anisotropies for both waves. Unlike LGS07, however, C08 does not have a strong cascading for both waves, but (again, in effect) it has

$$\epsilon^+ = A^\pm (w^-(\lambda))^2 w^+(\lambda),$$

where the coefficients $A^\pm$ depend on the spectral slopes of $w^\pm$. Although the theory of C08 does not explicitly assume local cascading, in effect, it produces such a locality as long as the ratio of dissipation rates (energy fluxes) $\epsilon^+ / \epsilon^-$ is not very large (the critical value is around two), therefore C08 also requires pinning on the viscous scale.

2.2. BL08 Model

BL08 relaxes the assumption of local cascading for the strong component $w^+$, while saying that the $w^-$ is cascaded in a GS95-like way. In BL08 picture, the waves have different anisotropies (see Figure 1) and the $w^+$ wave actually have smaller anisotropy than $w^-$, which is opposite to what a naive application of critical balance would predict. The anisotropies of the waves are determined by

$$w^+(\lambda_1)A^- (\lambda_1) = \nu_4 \lambda_1,$$  

$$w^+(\lambda_2)A^+ (\lambda^*) = \nu_4 \lambda_1,$$  

where $\lambda^* = \sqrt{\lambda_1 \lambda_2}$, and the energy cascading is determined by weak cascading of the dominant wave and strong cascading of the subdominant wave

$$\epsilon^+ = \frac{(w^+(\lambda_2))^2 w^-(\lambda_1)}{\lambda_1} \cdot \frac{w^-(\lambda_1)A^- (\lambda_1)}{\nu_4 \lambda_1} \cdot f(\lambda_1/\lambda_2),$$  

$$\epsilon^- = \frac{(w^-(\lambda_1))^2 w^+(\lambda_1)}{\lambda_1}.$$  

One of the interesting properties of BL08 model is that, unlike LGS07 and C08, it does not produce self-similar (power-law) solutions when turbulence is driven with the same anisotropy for $w^+$ and $w^-$ on the outer scale. BL08, however, claims that, on sufficiently small scales, the initial non-power-law solution will transit into asymptotic power-law solution that has $A^+_0 / A^-_0 = \epsilon^+ / \epsilon^-$ and $\lambda_2/\lambda_1 = (\epsilon^+ / \epsilon^-)^{3/2}$. The range of scales for the transition region was not specified by BL08, but it was assumed that larger imbalance will require larger transition region.

3. NUMERICAL SETUP

Incompressible MHD equations with dissipation and driving are,

$$\partial_t w^\pm + \hat{\nabla}(w^\pm \cdot \nabla)w^\pm = -v_n (-\nabla^2)^n w^\pm + f^\pm,$$

where $n$ is an order of hyperdiffusion, and $f^\pm$ is the driving force, whose root-mean-square (rms) values in arbitrary units are presented in Table 1. These equations were solved by the code

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6 Throughout this paper, we assume that $w^+$ is the larger-amplitude wave. This choice, however, is purely arbitrary and corresponds to the choice of positive versus negative total cross-helicity.

7 LGS07 does not discuss transition to viscous scales.

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Figure 1. Upper: a $w^+$ wavepacket, produced by cascading by $w^-$ wavepacket is aligned with respect to $w^-$ wavepacket, but misaligned with respect to the local mean field on scale $\lambda_1$, by the angle $\dot{\theta}$. Lower: the longitudinal scale $\Lambda$ of the wavepackets, as a function of their transverse scale, $\lambda$: $\Lambda^+$, $\Lambda^-$, $\lambda_1$, $\lambda_2$ are the notations used in this paper. Modified from BL08.
which is similar to one in Cho & Vishniac (2000). The differences include introduction of Elsässer driving, the handling of arbitrary physical sizes of the box, regardless of numerical resolution (i.e., elongated boxes for sub-Alfvénic turbulence) and significant improvements in numerical efficiency. Our code is pseudospectral, i.e., it solves the ordinary differential equation (ODE) in time with finite difference for each spatial Fourier harmonic, the nonlinear term being calculated in real space, while solenoidal projection and dissipation terms were applied in Fourier space. Pseudospectral methods in fluid dynamics has been known since 1980s (Canuto et al., 1988). The strong point of pseudospectral codes is that they allow precise control over dissipation and exact incompressibility. Not only does it relieve worries with respect to grid effects and numerical dissipation, but it also makes possible the use of hyperviscosity and hyperdiffusivity which extends useful inertial interval, although at the expense of increased (compared to physical viscosity) bottleneck effect and a different form of the spectral slopes (for more discussion on spectral slopes, see Berestyak & Lazarian 2009, BL09). The simple version of our pseudospectral code uses periodic boundary conditions. This necessitates a discussion on whether this introduces artificial effects. In the subsequent sections, we show that (1) our numerical boxes have enough parallel size to allow for eddies of the largest parallel size, dictated by dynamics, to exist, (2) at any given time, our box contains large number of independent turbulent realizations (≈40), (3) the dynamical time of the eddy is several times lesser than it takes the eddy to cross the box boundaries.

3.1. Choice of Physical Dimensions, Numerical Resolution, and Driving

The sub-Alfvénic turbulence, where the perturbation strengths $w^\pm$ are smaller than $v_A$ is either weak or strong but anisotropic. The critical anisotropy is determined by the breakdown of the applicability of weak MHD turbulence which happens when $k_0 v_A / k_\parallel \delta v \sim 1$. In most of our simulations, we drive turbulence on outer scale with the same anisotropy for both wave species, so this breakdown is determined by $\delta v$ of the strong wave. We drive turbulence on outer scale in such a manner that the strong interaction establishes on the outer scale. Also, we provided driving for $k = 2...3.5$, which means that the maximum eddy size is several times smaller than the box. This is to ensure that the first turbulent scales $k \approx 4$ have more than enough space in parallel direction, in case that we did not estimate the transition into strong interaction regime correctly and the parallel scale of the cascaded eddies is longer than we expected. The results from Section 5, however, suggest that our choice was correct, with the largest coherent eddy size being around 1/4 of the box size in both parallel and perpendicular directions. We used fully predetermined stochastic driving in both Elsässer variables\(^8\) with a certain amplitude of the force $(f^\pm, \delta v)$, so the energy input was not strictly controlled by the forcing, but rather was calculated during the simulation.\(^9\) In addition, we developed a driving which ensures constant energy input for both components; these tests confirm properties of the imbalanced turbulence that were obtained with fully stochastic driving. We used the latter for most of our simulations.

In studying sub-Alfvénic turbulence, we adopted the approach to increase $v_A$ by increasing $B_0$ and increase the parallel physical size of the box $L$ by the same factor $1/M_A$ without changing the equilibrium value of $\delta v$, so that strong interaction timescale $\lambda/\delta v$ stays constant and similarly the eddy transverse time $\Lambda/\delta v$ also stays constant. Alternatively, one can keep $B_0$ constant, but decrease $\delta v$, but in this case the timescales of sub- and trans-Alfvénic turbulence will be different. Also, it is harder to tune the equilibrium $\delta v$, rather than $B_0$ and $L$.

Note that one can naively assume that due to GS95 anisotropy one needs lower numerical resolution in the parallel direction, approximately by the ratio of the anisotropies on the driving scale and on the dissipation scale, which is $(k_{\perp\text{diss}}/k_{\perp\text{driv}})^{1/3}$ in the GS95 model, and can be a number between 2 and 4 in a high-resolution MHD simulation. For instance, Bigot et al. (2008) used $512 \times 512 \times 64$ numerical resolution. On the second thought, this approach is not evident, since the highest values of $k_0$ in the global reference frame will be determined by field wandering on the outer scale. In other words, the anisotropy in the global frame will be approximately scale independent and the ratio of $k_{\perp\text{diss}}/k_{\perp\text{driv}}$ and $k_{\text{diss}}/k_{\text{driv}}$ will be almost equal, which necessitates the use of NxNxN numerical resolution, i.e., cubes, for both elongated ($M_A < 1$) and cubic ($M_A \approx 1$) physical boxes.

We confirmed this by plotting the parallel and perpendicular spectra in the global frame and saw that the parallel spectrum protrude to almost as far as $k_{\text{max}}/M_A$. Figure 2 shows how energy is distributed on the two-dimensional $k_{\parallel}, k_\perp$ plane (global reference frame). We see that while most of the energy is in GS95 cone, there is also plenty of energy outside of it, especially in the upper right corner which corresponds to maximum space frequencies in both parallel and perpendicular directions. If one decide to significantly cut numerical resolution in parallel direction, he/she would incorrectly describe the dynamics on small scales. In only one of our simulations, A1 (see Table 1), we were able to cut parallel resolution by a moderate factor of 1.5 without sacrificing small parallel scales, due to the relative lack of energy in parallel direction in this particular

\(^8\) Elsässer driving is a preferred way to study the inertial range of sub-Alfvénic turbulence as it simulates the supply of Elsässer energies from larger eddies of the perturbation. It is important to remember that kinetic and magnetic energies are not separately conserved by MHD equations. So when one has a pure velocity driving in a simulation with mean field (as in Cho & Vishniac 2000), he will generate approximately as much magnetic perturbations due to the Alfvén effect, the result being two Alfvén or pseudo-Alfvén waves propagating in opposite directions. These waves, however, would have an artificial correlation (reflected by the fact that at $t = 0 b = 0$). In order to use all degrees of freedom and have better stochasticity one has to drive $w^\pm$ and $w^\parallel$ independently. The mechanisms by which the outer scales of a realistic, say, ISM turbulence are driven are briefly discussed in Sections 7.3 and 7.4.

\(^9\) Some simulations of hydrodynamic turbulence used negative viscosity on large scales to drive turbulence. In MHD, this somewhat unphysical approach does not work because in this case even in the balanced simulations imbalance spontaneously occur and continue to increase without limit.
balanced sub-Alfvénic case. In all other simulations, such a reduction was not possible because most of the k-space was filled with energy. We note that Müller & Grappin (2005), by using 1024 × 1024 × 256 resolution in their balanced sub-Alfvénic simulations have reduced parallel resolution by a factor of 4, which is, most likely, too large.

For all simulations, A1–A8, we used hyperviscosity and hyperdiffusivity of sixth order (k6). This choice was necessitated by the nature of imbalanced turbulence which has shorter inertial range for dominant wave due to fairly large cascading timescale of this wave (see Section 2). With currently available numerical resolutions one cannot see an inertial interval of the strong wave in a simulation with large imbalance and real (k2) diffusivity. Unfortunately, due to the bottleneck effect, hyperdiffusion has affected spectral slopes, although the effect on anisotropy was much less. We refer to BL09 for a comparison of turbulent simulations with normal and hyperviscosity. Due to hyperviscosity, the dissipation scale was fairly small, the dissipation cutoff was around k = 200 (with Nyquist frequency of 384) for balanced simulations and about the same for weak component in imbalanced simulations. The strong component for the most imbalanced simulations, A7 and A8 had a cutoff around k = 100 (Figure 9). Due to hyperviscosity, we cannot uniquely define a Reynolds number of our simulations, however viscous simulations with Re = Re_m ≈ 6000 could provide turbulence inertial ranges that are similar to ours.

Once we have chosen the geometry of our simulation and figured out the extend of the perturbations on the spectral plane, the choice of time step becomes evident. On one hand, for the dissipation term we use integration technique (Cho & Vishniac 2000, see also Maron & Goldreich 2001), and since we do not worry too much about the precision of the dissipation term, it does not limit the time step. On the other hand, the general nonlinear term, containing both B0 and δv can be seen as the sum of linear advection term with velocity v_A and nonlinear advection with δv, δb, etc. In turbulence, that is driven to be strong on the outer scale, these terms will be of the same order if we refer to the outer scale, i.e., the terms will be v_Aδbδk_{L-dr} and δvδvδk_{L-dr}. On the dissipation scale, these terms will be determined by v_Aδvδk_{L-dr} and δvδvδk_{diff}, which are, by the argument in the above paragraph, again on the same order. So we can just use linear advection behavior to estimate the time step. This behavior in k-space is, essentially, a rotation of the phase of the wave, in a manner of exp(ik_3v_A). In order to reproduce this rotation numerically we need k_{jmax}v_Aδt to be smaller than unity, such as around 0.1, so that the code stays stable, since we do not need good precision beyond the dissipation scale where there is no energy.

The average dissipation rates ϵ± reported in Table 1 were calculated using a sum of the work done to the Elsässer fields, i.e., we summed (w± + f±dt) · f±dt at every time step. As our code (its nonlinear part) was energy conserving, we assume that the same amount of energy was, on average, lost to the dissipation term. We also confirmed these values of ϵ± by using third-order Chandrasekhar–Politoano–Pouquet SFs (see, e.g., Biskamp 2003), which quantify nonlinear energy transfer.

3.2. Establishment of the Stationary State

One of the goals of this paper is to demonstrate that a stationary state exists for imbalanced turbulence with rather high degree of imbalance. Note, that the local model of weak Alfvénic turbulence work for imbalances of no more than ϵ+/ϵ− = 2 (Galtier et al. 2000; Lithwick & Goldreich 2003), and the model of strong imbalance turbulence of C08 also requires similar limitation.

The highest imbalance we attempted in our simulation of ϵ+/ϵ− = 16 was essentially limited by the long times of establishment of the stationary state. Note that according to BL08, the dominant wave is cascaded weakly and its cascading times could be very large. Figure 3 shows the total energy evolution for both modes for the ϵ+/ϵ− = 16 case. The full relaxation toward stationary state required around 300 Alfvén times or 50 crossing times.

As high-imbalanced simulations proved to be so computationally expensive, we made a second experiment, which was to take the initial state that was already stationary and to increase the numerical resolution, which allow the spectrum to extend to larger wavenumbers. Note, that our forcing, although stochastically, was predetermined for each particular simulation and did not depend on numerical resolution. Now the question was how fast the spectra will relax to their stationary states. It turned out that the spectrum of the sub-dominant wave relaxed almost instantly, in one dynamical time, which is consistent with BL08, while for the dominant wave the relaxation time was long. Note, the dynamic (kinetic) timescale l/v for this region of k-space for the strong wave was rather small, around 0.3. The relaxation process is shown in Figure 4. It took dt ≈ 60 to get reasonably close to the stationary state. We considered this experiment a success and used the technique of increasing resolution to save computational time.

We also studied longterm evolution of nearly balanced case when we allowed the low-resolution version of A3 to evolve for 500 time units. We did not notice any longterm trends either in three-dimensional spectrum or in any other quantities during this run.

4. PARALLEL STRUCTURE FUNCTION

Unlike perpendicular SF which is largely insensitive to the direction of the local field, the definition of the local field strongly affects the parallel SF, which, in turn, determines the shape of the turbulent eddy. Since the latter is the major object

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10 Time was measured in Alfvénic units, but the size of the box was 2π, thus 2π was the time for an eddy to cross the box. The time for the largest turbulent eddy to cross itself, and also the largest eddy dynamical time (L/ν) was around unity, because the size of the largest eddy was a fraction of around 0.2 or 0.3 of the cube size (see Figures 7 and 8).
of study in this paper we feel that the proper explanation of this point is due.

The simplest way to define parallel SF, $SF_\parallel$, is to take samples along the global mean field. This definition is, however, fairly bad, as it does not take into account field wandering. We expect $SF_\parallel$ defined in such way, along with perpendicular SF to reflect the anisotropy in the global frame, which, by the effects of field wandering, as we argued in Section 2, will be similar to the outer scale anisotropy. Therefore, such definition will effectively erase scale-dependent anisotropy which is the property of GS95-type models.

Another way is to define local magnetic field by averaging over some scale $\lambda$. In this way, the parallel SF becomes a function of two scales, such as

$$SF_\parallel^2(w^\pm, \Lambda, \lambda) = \langle (w^\pm(r - \Lambda b_\lambda/b_\lambda) - w^\pm(r))^2 \rangle_r,$$

where $b_\lambda$ is the magnetic field averaged over scale $\lambda$. We use Gaussian averaging defined as $b_\lambda = 1/\sqrt{2\pi} \int b(r - \mathbf{R}) \exp(-R^2/2\lambda^2) d\mathbf{R}$. In order to reduce such a SF to a function of only $\Lambda$, one can introduce dependency between $\Lambda$ and $\lambda$, and plug in the $\lambda = f(\Lambda)$ in the above equation. This definition of $SF_\parallel$ will be a model-dependent though.

$$SF_\parallel_{\text{model}}^2(w^\pm, \Lambda) = SF_\parallel(w^\pm, \Lambda, f(\Lambda)).$$

For the balanced turbulence, the anisotropy was measured to be close to the one predicted by GS95, i.e., $\Lambda \sim \lambda^{2/3}$. One can, therefore, introduce a reasonable model-dependent $SF_\parallel$ as taking $f(\lambda) = const \cdot \lambda^{3/2}$, where the constant depends on the outer scale of the simulation. As we show below, this definition is almost perfect for balanced turbulence, but the question is whether it does equally well for the imbalanced case.

Let us consider some model-independent ways to determine $SF_\parallel$. Apparently, the first definition using global field is model independent, but fairly bad, it corresponds to taking averaging $\lambda = \infty$. One can also take $\lambda = 0$, i.e., always use local field without any averaging. An interesting model-independent method was used in Maron & Goldreich (2001), where two points were chosen to lie on the same magnetic field line. The distance $\Lambda$ was also calculated along the line.

When we look for anisotropy we normally want to obtain lower values of $SF_\parallel$. According to the eddy ansatz, outlined in Section 2, we receive lower values of $|w^\pm(r - \Lambda n(r)) - w^\pm(r)|$, where $n$ is a unit vector along the eddy. Therefore, the averaging of the field that provides minimum values of $SF_\parallel$ approximates the direction of the eddy alignment better, provided that there is a connection between the field direction and eddy alignment (if there is no such connection, there will be no dependence on the averaging scale $\lambda$).
So, another model independent way to define $SF$ will be

$$\lambda_{\text{max}} = \min \lambda \lambda_{\text{min}}(w^\pm, \Lambda).$$

This definition not only provides us with the value of $\lambda_{\text{max}}$, but, giving $\lambda$ at which minimum is achieved, gives us a hint to how eddies are aligned with respect to the magnetic field.

Figure 5 shows a comparison between different methods to calculate parallel SF. We plotted them relative to $\lambda_{\text{max}}$. In the balanced case, the three methods—“minimal,” “following the field line,” and “model-dependent” work very well, while “global field” method does not work. The latter confirms that turbulent eddies are aligned with respect to local field, not the global field (Cho & Vishniac 2000). In the imbalanced case, the situation is more complicated. For the weak component, all three “good” methods work very well, while for the strong wave there is a systematic error for all methods, except “minimal.” This is due to the fact that in the imbalanced turbulence the strong component ($w^+$) eddies are aligned with respect to much larger scales of the magnetic field (Section 2.2). Since most magnetic field perturbation is provided by the strong wave, it follows that the strong field eddies are aligned with their own field on a larger scale. This directly confirms the prediction of BL08 model. In the bottom panel of Figure 5, the “field line” method gives values that are smaller than “minimal” method. This is due to the fact that in the field line method we measured the distance along magnetic field, and the physical distance was actually shorter; this allowed for smaller than $\lambda_{\text{max}}$ values on the outer scale, where the difference between straight-line distance and along-the-field-line distance is significant.

It turns out (Figure 6) that the averaging scale at which minimum of parallel structure is reached for weak wave, approximately corresponded to its anisotropy, which is consistent with strong cascading hypothesis. But for strong wave, this averaging scale is larger than the perpendicular scale dictated by anisotropy. In other words, the eddies of the strong wave are aligned with respect to the magnetic field which is averaged on larger scale than the eddy’s own perpendicular scale. This is consistent with the BL08 model.

5. SPECTRA AND ANISOTROPIES

We calculated two-dimensional (depending on parallel and perpendicular distances) second-order SFs with respect to the local field using “model-dependent” definition of the local field from previous section. Although this method slightly underestimates anisotropy, according to Figure 5, it works fairly well. The SFs were calculated using all available “stationary state” data cubes, i.e., were averaged over time. The contours of these SFs for balanced simulations A1 and A2 are presented in Figure 7, and for imbalanced simulation A7 in Figure 8. Figure 7 shows SFs for total energy i.e., it is summed over $w^+$ and $w^-$. These figures basically validate our assumptions from Section 3 regarding physical and computational dimensions of the box. We see that, according to expectations, trans-Alfvénic A2 is almost isotropic on outer scale but becomes progressively anisotropic toward small scales, while, as we expected, sub-Alfvénic A1 has approximately 10:1 anisotropy on outer scale, and increases toward small scales. If we decrease anisotropy of A1 by a factor of 10 by rescaling x-axis we almost reproduce A2, the difference is mostly being on the outer scale (this difference is easier to see in Figure 10). Figure 8 shows SFs for two separate components $w^+$ and $w^-$ in the strongly imbalanced case of A7. The anisotropy on outer scale is approximately 10:1 for both components, which validates our choice of computational box. This anisotropy increases toward small scale, but in a different fashion for each component. We see that the anisotropy of strong wave is almost five times smaller on dissipation scales.

Figure 9 shows so-called three-dimensional angle-summed spectra for both components in all simulations. These spectra are obtained by summation of spectra over solid angle for all wavevectors with the same magnitude $k$. It can be related to three-dimensional angle-averaged spectra by dividing by $k^2$. In the sub-Alfvénic cases, A1, A3, A5, and A7, this spectrum is almost identical to the so-called perpendicular spectrum, which takes into account only structures perpendicular to the magnetic field and is the main target of prediction of GS95 model. As they are almost identical we did not have to plot perpendicular spectrum separately. Another definition of spectrum which depends only on the magnitude of the wavevector is the so-called one-dimensional spectrum (see, e.g., Monin & Yaglom 1975). This spectrum is less sensitive to the bottleneck effect. We refer to the paper of Beresnyak & Lazarian (2009, henceforth BL09) for a more thorough comparison between one-dimensional and three-dimensional spectra and discussion on bottleneck effect.

![Figure 6](image-url)  
**Figure 6.** This plot shows values of $\lambda_{\text{avr}}$ at which the minimum of the parallel structure function is reached. Triangles show perpendicular scales $\lambda_{\text{avr}}$ at which the minimum of $SF^2(\pm, \Lambda)$ is reached, while squares show perpendicular averaging scales at which the minimum of $SF^2(\pm, \Lambda)$ is reached. Solid and dotted lines indicate $w^+$ and $w^-$ eddies’ anisotropy which are defined in Section 5 and presented in Figure 11.

![Figure 7](image-url)  
**Figure 7.** Comparison of the SFs from trans-Alfvénic (left) and sub-Alfvénic (right) balanced simulations. Note the difference in x axis between two plots which indicates that A1 is approximately 10 times more anisotropic. Contours indicate SF levels, solid line is a demonstration of GS95 $\Lambda \sim \lambda^{2/3}$ law.
In Figure 9, the two bottom plots have relatively large variation (gray areas) at the end of the $w^+$ spectra. This is due to the fact that in A7 and A8 we barely reached stationary state (for more discussion, see Figure 4 and Section 3.2) in the high-resolution run.

Figure 9 shows spectral slopes between $-1.12$ and $-1.93$ with balanced simulations having slopes as flat as $-1.37$. The GS95 prediction is Kolmogorov’s $-5/3 \approx -1.67$, while Boldyrev (2006) prediction is $-1.5$. The flat slopes observed in real data are most certainly due to rather strong bottleneck effect seen in simulations with hyperviscosity. In the imbalanced case, the predictions are following: LGS07 predicts $-1.67$ slopes for all eight cases; C08 predicts $-1.67$ for balanced cases A1 and A2, A5–A8 are outside of the applicability of his model, A3 and A4 must show very different slopes—approximately $-1$ for weak component and $-3$ for strong component, also C08 predicts pinning on dissipation scales i.e., spectra should converge on dissipation scale. The ratios of the total energies (see Table 1) are predicted as following: C08—A5–A8 are outside of applicability of his model, A3 and A4 should have very large imbalance ($w^+/w^-$) of at least a 1000, while 4–6 is actually observed; LGS07 predicted/observed—A3: 4/5.5, A4: 3/3.9, A5: 55/145, A6: 30/90, A7: 260/1150, A8: 144/1100. We see that in this respect, deviations from LSG07 predictions are small for small imbalances but fairly large for large imbalances. BL08 argues that if one drives turbulence with the same anisotropy on outer scale (as in these simulations) the anisotropies of the components will diverge toward small scales, and this solution will not be self-similar (and not power law). However, BL08 makes predictions regarding local slopes even in this case. This can be seen from Equation (1) which is a classic critical balance between weak wave anisotropy and strong wave amplitude and Equation (4) which is strong cascading of the weak wave. We do not expect relations between slopes based on Equation (4) to hold, because it is strongly influenced by bottleneck effect (BL08 also predicts $-1.67$ slopes for balanced case). However, there is some dependence between energy slope and anisotropy slope, similar to what BL08 predicts. Namely, it follows from Equation (1) that shallower anisotropy slope for $w^-$ means steeper spectral slope for $w^+$ which is observed. Also, from Equation (4), steeper spectral slope for $w^+$ also means shallower spectral slope for $w^-$, which is also observed.

The anisotropy was measured in the following manner. First, the parallel and perpendicular second-order SFs were calculated, then we found equal values of parallel and perpendicular SFs and in this way the mapping or function between independent variables, parallel or perpendicular scales were created. This function is plotted in Figures 10 and 11 with shades of gray indicating rms fluctuations in time. This definition of $\Lambda(\lambda)$
Figure 11. Anisotropies for imbalanced simulations. The mapping of $A(\lambda)$ is explained in Section 5. The difference in anisotropy between $w^+$ and $w^-$ increases with increasing imbalance.

mapping can be understood from two-dimensional plot of SF, e.g., Figure 7, when one follows a contour of SF and finds which parallel scale correspond to particular perpendicular scale. We see that for the imbalanced case, anisotropy curves have different slopes and diverge from outer scale where they are equal (this is dictated by driving) to smaller scales where they are different.

We devoted Section 4 to the discussion of the measurements of the parallel SF which was used in the above definition of the anisotropy curves. Although it might appear that each and every definition produces a different anisotropy curve, the major difference is between global and local definition of the field direction, while all local methods (“field line,” “model-dependent” and “minimal”), also dubbed “good” in Section 4, give very similar results. In fact, these is no perceivable qualitative difference between anisotropy curves obtained by either local methods. This could be explained by Figure 4 middle and bottom panels, where the quantitative differences between methods are small, but on the other hand, the dependence of $SF_\parallel$ on scale is strong ($\sim l_\parallel$). Also, in the middle panel, the difference is mostly by a constant, which will only give a slight shift of the anisotropy plot. All in all, the claim that anisotropy curves will diverge by a factor of 3 to 4 in strongly imbalanced simulations stay true regardless of the “local” method used. We rejected “global field” method, as it does not reveal scale-dependent anisotropy—a ground base of GS95 model. It is worth noting that LGS07, C08, and BL08 use GS95 as a basis and smoothly transit to GS95 in the balanced limit. There is a wealth of theoretical arguments why the SFs have to be measured with respect to the local field (Cho & Vishniac 2000; Maron & Goldreich 2001, etc). We also would like to note that aside from driven simulations described in this paper we also observed a significant difference in $w^+$ and $w^-$ anisotropies in decaying imbalanced simulations.

C08 and LGS07 both predict identical GS95 anisotropy for both modes, which is inconsistent with simulations. BL08 predicts diverging anisotropy, most notably, with stronger wave having smaller anisotropy, which is consistent with simulations. The value of the differences, however, do not reach the asymptotic value of $\epsilon^+ / \epsilon^-$ which may be attributed to the short inertial range.

6. POLARIZATION ALIGNMENT

Aside from energy-type statistics (second-order SFs and spectra), one can measure so-called alignment effects which are, in a sense, deviations from the assumptions of independent randomness of fluctuations included in mean-field models. These were discussed in Boldyrev (2005) and numerically discovered for the first time in Beresnyak & Lazarian (2006). Figure 12 shows two measures of alignment—the angle polarization alignment $AA = \langle |\sin \theta| \rangle$ (dashed line), where $\theta$ is an angle between Elsässer variables perturbations $\delta w^+ = \delta v + \delta b$ and $\delta w^- = \delta v - \delta b$ and “polarization intermittency” $PI = \langle |\delta w^+ \delta w^- \sin \theta| \rangle / (\langle |\delta w^+ \delta w^-| \rangle)$ (solid line).

For the more detailed numerical study of alignment effects we refer to Beresnyak & Lazarian (2006) and BL09. The potential effect of alignment on the energy cascade was discussed in Boldyrev (2005, 2006), although this effect has not yet been convincingly confirmed by numerics (see BL09 for a discussion and references).

7. DISCUSSION

7.1. Comparison with Models

We briefly mentioned the tentative nature of confirming a model in Section 1, we also claimed the ability of numerics to reject some models on the basis of robust quantities. Although a theory can make a wide variety of predictions, only few of those can be effectively attacked by numerics. One of the quantities that is notoriously hard to measure in DNS is the spectral slope of turbulence. A difference between $-3/2$ slope and $-5/3$ can be
masked by a variety of effects such as bottleneck effect, driving, and so on. In contrast, the quantities such as Kolmogorov constant are fairly easy to obtain and quickly converge with increasing resolution. In fact, modest resolutions such as 128^3 give reasonably precise estimates of this constant. This is due to the fact that the total energy and the total dissipation rate are easy to measure, to get a statistical average, and also are free of uncertainties of interpretation. What sort of models can be judged on the basis of these quantities? Such are the models of local cascading where the cascade rate depends only on the characteristic quantities of, say, $w^\pm_l$ on a particular scale $l$ (and, possibly, weakly depend on the spectral slope). In this case, numerics only have to reproduce one or two steps of such cascading to obtain reasonable dissipation rate based on a particular total energy. In this sense, our testing does fairly well, as we mostly consider models of local cascading (LGS07, C08, Perez & Boldyrev 2009; Podesta & Bhattacharjee 2009); at the same time, with 768^3 numerical resolution we reproduce five to six binary steps in $k$-space.

Our numerical data strongly contradict three models of imbalanced turbulence, namely LGS07, C08, and Perez & Boldyrev (2009). In particular, two of these models C08, and Perez & Boldyrev (2009) show gross inconsistencies between observed and predicted energy ratios versus dissipation ratios. Indeed, C08 must have a huge energy ratio (of around 1000) in simulations with $\epsilon^+ / \epsilon^-$ close to two (A3 and A4), while a modest ratio of 4 and 6 is observed. Perez & Boldyrev (2009) does extremely bad in cases with large imbalances. A7 and A8 have an energy ratio of around 1000, while predicted quantities are 16 and 12. LGS07 does a much better job on energy ratios, but still fails the A7 and A8 (large imbalance) tests, see Figure 13.

Furthermore, LGS07 and C08 have predictions regarding eddy anisotropies. Both of these models predict equal anisotropies for $w^+$ and $w^-$ while different anisotropies are observed. Aside from these inconsistencies, we also note that C08 predict pinning at the dissipation scale, which is not observed. C08 also predict that the strong wave has to have steeper spectral slope than the weak wave. This corresponds to numerics qualitatively, but not quantitatively, indeed, according to C08, A3 and A4 must have a huge slope difference of around 2, while the real difference is around 0.12.

Although it is harder to confirm a model rather than to reject a model by direct numerical simulations, we see that there is a qualitative agreement between BL08 and numerics. Most of the features predicted by BL08 are observed in simulations, namely (1) the anisotropies of the waves are different and strong wave anisotropy is smaller, (2) while the weak wave eddies are aligned with respect to the local field on the same scale as the eddy, the strong wave eddies are aligned with respect to a larger-scale field, (Figure 6), (3) the energy imbalance is higher than in the case when both waves are cascaded strongly (Table 1), which suggest that the strong wave is cascaded weakly, (4) the dissipation scales for the weak and the strong waves are different, namely the inertial range for weak wave is longer (Figure 9) which is what predicted by BL08, (5) there is no “pinning” at the dissipation scale which suggests nonlocal cascading.

We note that there is no quantitative agreement between the difference in anisotropies in the “asymptotic power-law solutions” of BL08 and simulations (c.f. Section 2 and Figure 11). This is probably due to the fact that asymptotic power-law solutions have not been established within our inertial range.

Aside from three models with detailed strong imbalanced theories that we described above, LGS07, BL08, and C08, recently two papers also made predictions regarding strong imbalanced turbulence. Perez & Boldyrev (2009) predicted that $(w^+/w^-)^2 = \epsilon^+ / \epsilon^-$ which comes to much stronger contradiction with A7 and A8 than LGS07 and as such should be discarded. We note parenthetically that measurements of the stationary levels of energies for $w^+$ and $w^-$ are the most robust and model independent of all other measurements. Another paper, by Podesta & Bhattacharjee (2009) is a modification of Perez & Boldyrev (2009) which tries to resolve a huge discrepancy of the latter. Unfortunately, their theory has an arbitrary parameter that can be tuned to change the prediction for $(w^+/w^-)^2$ and, therefore, cannot be tested numerically. We also note that two aforementioned papers do not have clearly stated predictions for anisotropy, which also limits our ability to test them.

7.2. Comparison with Earlier Simulations

Maron & Goldreich (2001) and Cho et al. (2002) performed three-dimensional numerical simulations of decaying imbalanced turbulence. In these simulations, the perturbations obtained as a result of balanced driven MHD modeling were separated into oppositely moving flows of Elsässer variables and one of the flows was arbitrary decreased in amplitude. The authors observed the increase of the damping time for the strong component. They also observed the increase of turbulence imbalance as the turbulence was evolving. Naturally, no stationary state was achievable for the imbalanced turbulence induced this way.

7.3. Role of Homogeneous Turbulence Driving

Properties of imbalanced turbulence may depend on how it is driven. All three major models of imbalanced turbulence that we discuss above (LGS07, BL08, and C08) describe imbalanced turbulence driven homogeneously through the volume.

Figure 12. Polarization angle alignment (dashed) and polarization intermittency (see definition in the text). Numbers are power-law slopes determined in the middle of the inertial interval.
The properties of such turbulence may differ if the sources of driving are localized. To illustrate this fact, in Appendix A we discuss a toy model of weak imbalanced turbulence driven inhomogeneously at boundaries. Note that this approach is very different than the DNS approach of the rest of this paper. Being the model of weak turbulence, this toy model describes only a perpendicular cascade, just like its predecessor Lithwick & Goldreich (2003), which considered homogeneous case. However, the stationary states of these inhomogeneous and homogeneous cases substantially differ. We also expect to see substantial differences for the homogeneous and inhomogeneous driving for the strong imbalanced turbulence, but since such a study with the use of the DNS will be complicated and computationally expensive, we defer this discussion to future publications.

7.4. Effects of Compressibility

The simulations presented here are of incompressible MHD turbulence. Not only full compressible MHD equations have more degrees of freedom (incompressible approach exclude fast mode perturbations), but compressibility substantially changes the properties of turbulence when sonic Mach numbers are around unity or higher. However, numerical studies in Cho & Lazarian (2002, 2003) showed that coupling of Alfvénic and magnetosonic waves in strong turbulence is not as strong as was expected, which can be due to the fast non-linear decay of Alfvénic eddies.

In the imbalanced turbulence, the strong wave is long-lived. Therefore, one can expect the imbalanced turbulence to be more affected by coupling of incompressible and compressible motions. Density inhomogeneities present in the compressible fluid can act as mirrors reflecting waves and decreasing the degree of turbulence imbalance. Parametric instabilities (see, e.g., Del Zanna et al. 2001) can develop in the compressible fluid, decreasing the imbalance. Thus, stationary states with high degree of imbalance may not be attainable in compressible fluids. Further research in this direction is necessary.

8. SUMMARY

In the paper above we have performed MHD numerical simulations of the homogeneously driven imbalanced turbulence and have shown that

1. Stationary states exist for rather high degree of imbalance.
2. For large imbalances, the ratio of the amplitudes disagrees with the predictions in LGS07.
3. Rough correspondence of the expectations and measurements are obtained for the model BL08, but more studies and testing is necessary.

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APPENDIX

INHOMOGENEOUS WEAK IMBALANCED TURBULENCE—A TOY PROBLEM.

Although we understand that the models of imbalanced turbulence are in their making and even the case of homogeneous imbalance turbulence is still being investigated, we decided to add this appendix that probes into a different problem, namely, inhomogeneous Alfvénic turbulence.

This is motivated by the fact that imbalanced turbulence often appears in inhomogeneous setting, i.e., when one has a strong localized source of waves, such as the Sun in solar wind turbulence. In this respect, the theory of homogeneous imbalance turbulence should be well applicable to small scales where the timescales are much smaller than outer timescales. However, on larger scales, homogeneity could be broken. In this section, we probe the situation when it is broken. This can be used as a guidance to what extent the observations of the solar wind can be considered as restrictive measurements with respect to the theories of imbalanced turbulence.

Previous studies of weak Alfvénic turbulence (e.g., Galtier et al. 2000; Lithwick & Goldreich 2003) considered spatially homogeneous case, which presume that $w^+$ and $w^-$ are driven homogeneously in space. This assumption is often violated in nature, specifically when we have a strong directional source of perturbations such as the Sun, which drives one component (let us say, $w^+$) and the other component is being generated at a certain distance from the source by means of, e.g., reflection from density inhomogeneities. This problem is closely related to the problem of the imbalance, however, in most of this paper we considered idealized spatially homogeneous case. Dissipation of waves in turbulence (Beresnyak & Lazarian 2008c) is also related to this problem.

We realize, that by relaxing homogeneity we considerably broaden the physical scope of the model, to the point that we might be unable to draw clear conclusions on the nature of inhomogeneous turbulence. In this situation, we preferred to take a first step by considering a toy problem of weak turbulence with two wave sources separated by a certain distance.

One remarkable new property that could arise from such formulation is that the cascading might not reach the dissipation scale. This could happen if, e.g., the sources of the waves are too close to each other. One of the questions that we are asking in this appendix, is whether it is possible that only one wave component, e.g., $w^-$ is dissipated but the other is simply distorted.

We used simplified equations of weak cascading, the diffusive one dimensional $k$-space equations for weak perpendicular
cascade from Lithwick & Goldreich (2003), which were expanded to one spatial dimension by introducing advection in space, making them an advection–diffusion equation with advection happened in real space, while diffusion represented energy cascading in $k$-space. We used open boundaries, so that the non-cascaded waves were allowed to freely escape through them.

By solving those equation numerically, changing the distance between wave sources, we found that the approximate equality of the energies of the waves at the smallest scale which is reached by cascading is rather robust feature, that is produced by cascading itself. Note that Lithwick & Goldreich 2003 assumed that this “pinning” is due to the dissipation term. In our toy model, however, regardless of whether an actual dissipation took place in the system, the spectra were “pinned” on small scales.

Another possibility that was opened by the inhomogeneous formulation was to drive wave sources with arbitrary power and still obtain a stationary state. In the homogeneous formulation the stationary state was not possible if the rate of energy driving $\epsilon^+ / \epsilon^-$ was larger than 2 (Lithwick & Goldreich 2003). Our inhomogeneous toy model can deal with larger imbalances since the waves were allowed to escape through the open boundaries of the box. Figure 14 shows spectrum for both types of waves in such case of strong imbalance, and incomplete cascading mentioned above (energy has not reached dissipation scale). Note, that unlike DNS with periodic boundaries from the main body of this paper, where energy can only be lost through dissipation, in this case energy could be lost due to open boundaries and the stationary state could be achieved without any physical dissipation actually taking place.

Figure 14 demonstrates a feature which was not observed in Lithwick & Goldreich (2003), namely that imbalance reverses on scales an order of magnitude smaller than the driving scale. This unexpected feature is fairly robust in the case of “incomplete” cascading and strong imbalance. This suggests that inhomogeneous imbalanced turbulence could be much more complicated than its homogeneous counterpart and further research is necessary.

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