Regularity for minimizers of non-autonomous non-quadratic functionals in the case $1 < p < 2$: an a priori estimate

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September 27, 2018

Abstract

We establish an a priori estimate for the second derivatives of local minimizers of integral functionals of the form

$$F(v, \Omega) = \int_{\Omega} f(x, Dv(x)) \, dx,$$

with convex integrand with respect to the gradient variable, assuming that the function that measures the oscillation of the integrand with respect to the $x$ variable belongs to a suitable Sobolev space. The novelty here is that we deal with integrands satisfying subquadratic growth conditions with respect to gradient variable.

AMS Classifications. 49N60; 35J60; 49N99.

Key words and phrases. Local minimizers; A priori estimate; Sobolev coefficients.

1 Introduction

In this paper we consider integral functionals of the form

$$F(v, \Omega) = \int_{\Omega} f(x, Dv(x)) \, dx,$$  \hspace{1cm} (1.1)

where $\Omega \subset \mathbb{R}^n$ is a bounded open set, $f : \Omega \times \mathbb{R}^{N \times n} \to \mathbb{R}$ is a Carathéodory map, such that $\xi \mapsto f(x, \xi)$ is of class $C^2(\mathbb{R}^{N \times n})$, and for an exponent $p \in (1, 2)$ and some constants $L, \alpha, \beta > 0$ the following conditions are satisfied:
\[
\frac{1}{L} |\xi|^p \leq f(x, \xi) \leq L (1 + |\xi|^p), \quad (1.2)
\]
\[
\langle D_\xi f(x, \xi) - D_\xi f(x, \eta), \xi - \eta \rangle \geq \alpha \left(1 + |\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}} |\xi - \eta|^2, \quad (1.3)
\]
\[
|D_\xi f(x, \xi) - D_\xi f(x, \eta)| \leq \beta \left(1 + |\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}} |\xi - \eta|. \quad (1.4)
\]

For what concerns the dependence of the energy density on the \(x\)-variable, we shall assume that the function \(D_\xi f(x, \xi)\) is weakly differentiable with respect to \(x\) and that \(D_x(D_\xi f) \in L^q(\Omega \times \mathbb{R}^{N \times n})\), for some \(q > n\).

By the point-wise characterization of the Sobolev functions due to Hajlasz ([18]) this is equivalent to assume that there exists a nonnegative function \(g \in L^q_{\text{loc}}(\Omega)\) such that
\[
|D_\xi f(x, \xi) - D_\xi f(y, \xi)| \leq (g(x) + g(y)) |x - y| \left(1 + |\xi|^2\right)^{\frac{p-2}{2}} \quad (1.5)
\]
for all \(\xi \in \mathbb{R}^{N \times n}\) and for almost every \(x, y \in \Omega\).

The regularity properties of minimizers of such integral functionals have been widely investigated in case the energy density \(f(x, \xi)\) depends on the \(x\)-variable through a continuous function both in the superquadratic and in the subquadratic growth case. In fact, it is well known that the partial continuity of the vectorial minimizers can be obtained with a quantitative modulus of continuity that depends on the modulus of continuity of the coefficients (see for example [1, 12, 14] and the monographs [13, 17] for a more exhaustive treatment). For regularity results under general growth conditions, that of course include the superquadratic and the subquadratic ones, we refer to [8, 9].

Recently, there has been an increasing interest in the study of the regularity under weaker assumptions on the function that measures the oscillation of the integrand \(f(x, \xi)\) with respect to the \(x\)-variable.

This study has been successfully carried out when the oscillation of \(f(x, \xi)\) with respect to the \(x\)-variable is controlled through a coefficient that belongs to a suitable Sobolev class of integer or fractional order and the assumptions (1.2)–(1.5) are satisfied with an exponent \(p \geq 2\).

Actually, it has been shown that the weak differentiability of the partial map \(x \mapsto f(x, \xi)\) transfers to the gradient of the minimizers of the functional (1.1) (see [4, 10, 11, 15, 19]) as well as to the gradient of the solutions of non linear elliptic systems (see [5, 6, 7, 20]) and of non linear systems with degenerate ellipticity (see [16]).

As far as we know, no higher differentiability results are available for vectorial minimizers under the so-called subquadratic growth conditions, i.e. when the assumptions (1.2)–(1.5) hold true for an exponent \(1 < p \leq 2\) in case of Sobolev coefficients. The aim of this paper is to start the study of the higher differentiability properties of local minimizers of integral functional (1.1) under subquadratic growth condition. More precisely, we...
shall establish the following a priori estimate for the second derivatives of the local minimizers.

**Theorem 1.1.** Let \( u \in W^{2,p}_{\text{loc}}(\Omega; \mathbb{R}^N) \) be a local minimizer of the functional \( F(v, \Omega) \) under the assumptions (1.2)–(1.5). If \( q \geq 2n/p \), than the following estimate

\[
\|D^2u\|_{L^p(B_r)} \leq C(\alpha, \beta, p, n) \left( \|Du\|_{L^p(B_R)} + \|g\|_{L^q(B_R)} \right)
\]

(1.6) holds true for every \( 0 < r < R \) such that \( B_R \subset \Omega \) with \( C = C(\alpha, \beta, p, n) \).

The main tool to establish previous result is the use of the so called difference quotient method and a double iteration to reabsorb terms with critical summability. Respect to previous papers on this subject, new technical difficulties arise since we are dealing with the subquadratic growth case.

## 2 Preliminary results

In this section we shall collect some results that will be useful to achieve our main result. In this section we recall some standard definitions and collect several lemmas that we shall need to establish our results. We shall follow the usual convention and denote by \( C \) or \( c \) a general constant that may vary on different occasions, even within the same line of estimates. Relevant dependencies on parameters and special constants will be suitably emphasized using parentheses or subscripts. All the norms we use on \( \mathbb{R}^n, \mathbb{R}^N \) and \( \mathbb{R}^{n \times N} \) will be the standard Euclidean ones and denoted by \( |\cdot| \) in all cases. In particular, for matrices \( \xi, \eta \in \mathbb{R}^{n \times N} \) we write \( \langle \xi, \eta \rangle := \text{trace}(\xi^T \eta) \) for the usual inner product of \( \xi \) and \( \eta \), and \( |\xi| := (\langle \xi, \xi \rangle)^{1/2} \) for the corresponding Euclidean norm. When \( a \in \mathbb{R}^N \) and \( b \in \mathbb{R}^n \) we write \( a \otimes b \in \mathbb{R}^{n \times N} \) for the tensor product defined as the matrix that has the element \( a_r b_s \) in its \( r \)-th row and \( s \)-th column.

For a \( C^2 \) function \( f : \Omega \times \mathbb{R}^{n \times N} \to \mathbb{R} \), we write

\[
D\xi f(x, \xi)[\eta] := \frac{d}{dt}\bigg|_{t=0} f(x, \xi + t\eta) \quad \text{and} \quad D\xi\xi f(x, \xi)[\eta, \eta] := \frac{d^2}{dt^2}\bigg|_{t=0} f(x, \xi + t\eta)
\]

for \( \xi, \eta \in \mathbb{R}^{n \times N} \) and for almost every \( x \in \Omega \).

With the symbol \( B(x, r) = B_r(x) = \{ y \in \mathbb{R}^n : |y - x| < r \} \), we will denote the ball centered at \( x \) of radius \( r \) and

\[
(u)_{x_0,r} = \int_{B_r(x_0)} u(x) \, dx,
\]

stands for the integral mean of \( u \) over the ball \( B_r(x_0) \). We shall omit the dependence on the center when it is clear from the context.
2.1 An auxiliary function

As usual, we shall use the following auxiliary function

\[ V_p(\xi) := \left(1 + |\xi|^2\right)^{\frac{p-2}{4}} \xi, \quad \text{for all } \xi \in \mathbb{R}^{N \times n}. \]  

(2.1)

2.2 Some useful lemmas

The following result is proved in [1], and will be useful to estimate the \(L^p\) norm of \(D^2 u\), using the \(L^2\) norm of the gradient of \(V_p(Du)\).

**Lemma 2.1.** For every \( \gamma \in \left(\frac{1}{2}, 0\right) \) and \( \mu \geq 0 \) we have

\[
(2\gamma + 1)|\xi - \eta| \leq \frac{|(\mu^2 + |\xi|^2)\gamma \xi - (\mu^2 + |\eta|^2)\gamma \eta|}{(\mu^2 + |\xi|^2 + |\eta|^2)\gamma} \leq \frac{c(k)}{2\gamma + 1}|\xi - \eta|, 
\]

(2.2)

for every \( \xi, \eta \in \mathbb{R}^k \).

**Lemma 2.2.** For every \( \gamma \in \left(\frac{1}{2}, 0\right) \) we have

\[
c_0(\gamma)(1 + |\xi|^2 + |\eta|^2)\gamma \leq \int_0^1 (1 + |t\xi + (1 - t)\eta|^2)|^\gamma dt \leq c_1(\gamma)(1 + |\xi|^2 + |\eta|^2)\gamma, 
\]

(2.3)

for every \( \xi, \eta \in \mathbb{R}^k \).

The next lemma can be proved using an iteration technique, and will be very useful in the following, where we will refer to this as Iteration Lemma.

**Lemma 2.3** (Iteration Lemma). Let \( h : [\rho, R] \to \mathbb{R} \) be a nonnegative bounded function, \( 0 < \theta < 1 \), \( A, B \geq 0 \) and \( \gamma > 0 \). Assume that

\[
h(r) \leq \theta h(d) + \frac{A}{(d - r)^\gamma} + B
\]

for all \( \rho \leq r < d \leq R_0 \). Then

\[
h(\rho) \leq \frac{c(A)}{(R_0 - \rho)^\gamma} + cB,
\]

where \( c = c(\theta, \gamma) > 0 \).

For the proof we refer to [17, Lemma 6.1].
2.3 Finite difference and difference quotient

In what follows, we denote, for every function $f$, $h \in \mathbb{R}$, and being $e_s$ the unit vector in the $x_s$ direction,

$$\tau_{s,h}f(x) := f(x + he_s) - f(x)$$

defines the finite difference operator.

Here we recall some properties of the finite difference, that will be useful in the following.

**Proposition 2.4.** Let $f$ and $g$ be two functions such that $f, g \in W^{1,p}(\Omega, \mathbb{R}^n)$ with $p \geq 1$, and let us consider the set

$$\Omega_{|h|} := \{x \in \Omega : \text{dist}(x, \partial\Omega) > |h|\}.$$

Then the following properties hold:

1. $\tau_{s,h}f \in W^{1,p}(\Omega_{|h|})$ and $D_i(\tau_{s,h}f) = \tau_{s,h}(D_i f)$;

2. if at least one of the functions $f$ or $g$ has support contained in $\Omega_{|h|}$, then

$$\int_{\Omega} f \tau_{s,h}g \, dx = \int_{\Omega} g \tau_{s,-h}f \, dx;$$

3. we have

$$\tau_{s,h}(fg)(x) = f(x + he_s)\tau_{s,h}g(x) + g(x)\tau_{s,h}f(x).$$

The following lemmas describe fundamental properties of finite differences and difference quotients of Sobolev functions.

**Lemma 2.5.** If $0 < \rho < R$, $|h| < \frac{R-\rho}{2}$, $1 < p < +\infty$, $s \in \{1, \ldots, n\}$ and $f, D_s f \in L^p(B_R)$, then

$$\int_{B_{\rho}} |\tau_{s,h}f(x)|^p \, dx \leq |h|^p \int_{B_R} |D_s f(x)|^p \, dx.$$ 

Moreover, for $\rho < R$, $|h| < \frac{R-\rho}{2}$,

$$\int_{B_{\rho}} |f(x + he_s)|^p \, dx \leq c(n, p) \int_{B_R} |f(x)|^p \, dx.$$ 

**Lemma 2.6.** Let $f : \mathbb{R}^n \to \mathbb{R}^N$, $f \in L^p(B_R)$ with $1 < p < +\infty$. Suppose that there exist $\rho \in (0, R)$ and $M > 0$ such that
\[ \sum_{s=1}^{n} \int_{B_{\rho}} |\tau_{s,h} f(x)|^p dx \leq M |h|^p \]

for every \( h < \frac{R - \rho}{s} \). Then \( f \in W^{1,p}(B_R, \mathbb{R}^N) \). Moreover
\[
|Df|_{L^p(B_{\rho})} \leq M.
\]

## 3 Proof of Theorem 1.1

It is well known that every local minimizer of the functional (1.1) is a weak solution \( u \in W^{1,p}(\Omega, \mathbb{R}^N) \) of the corresponding Euler-Lagrange system, i.e.
\[
\text{div} A(x, Du(x)) = 0,
\]
where we set
\[
A^\alpha_i(x, \xi) := D\xi^\alpha f(x, \xi), \quad \text{for all } \alpha = 1, \ldots, N \text{ and } i = 1, \ldots, n.
\]

Assumptions (1.2), (1.3), (1.4), can be written as
\[
\langle A(x, \xi) - A(x, \eta), \xi - \eta \rangle \geq \alpha |\xi - \eta|^2 (1 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}},
\]
\[
|A(x, \xi) - A(x, \eta)| \leq \beta |\xi - \eta| (1 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}}
\]
for every \( \xi, \eta \in \mathbb{R}^{n \times N} \) and for almost every \( x \in \Omega \).

Conceiving the dependence on the \( x \)-variable, assumption (1.5) translates into the following
\[
|A(x, \xi) - A(y, \xi)| \leq (g(x) + g(y)) |x - y| (1 + |\xi|^2)^{\frac{p-1}{2}},
\]
for every \( \xi, \eta \in \mathbb{R}^{N \times n} \) and for almost every \( x, y \in \Omega \).

**Proof of Theorem 1.1.** Let us fix a ball \( B_R(x_0) = B_R \) of radius \( R \in (0, \text{dist}(x_0, \partial \Omega)) \), and consider \( \frac{R}{2} < r < \tilde{s} < s < \lambda r < R < 1 \), with \( 1 < \lambda < 2 \). Let’s test the equation (3.1) with the function \( \varphi = \tau_{s,-h}(\eta^2 \tau_{s,h} u) \), where \( \eta \in C_0^{\infty}(B_1) \) is a cut off function such that \( \eta = 1 \) on \( B_{\tilde{s}} \), \( |D\eta| \leq \frac{\tilde{s}}{2} |z|^{-1} \).

With this choice of \( \varphi \), and by (2.2) of Proposition 2.4, we get
\[
\int_{B_R} \langle \tau_{s,h} A(x, Du(x)), D(\eta^2)(\tau_{s,h} u(x)) \rangle dx = 0.
\]

After some manipulations, and dropping the vector \( e_s \) to simplify the notations, we can write the last equivalence as follows
\[
I_0 := \int_{B_R} \langle A(x + h, Du(x + h)) - A(x + h, Du(x)), \eta^2(x)D(\tau_{s,h}u(x)) \rangle \, dx \\
= - \int_{B_R} \langle A(x + h, Du(x)) - A(x, Du(x)), \eta^2(x)D(\tau_{s,h}u(x)) \rangle \, dx \\
- \int_{B_R} \langle \tau_{s,h}A(x, Du(x)), 2\eta(x)D\eta(x) \otimes \tau_{s,h}u(x) \rangle \\
= - \int_{B_R} \langle A(x + h, Du(x)) - A(x, Du(x)), \eta^2(x)D(\tau_{s,h}u(x)) \rangle \, dx \\
- \int_{B_R} \langle A(x, Du(x)), \tau_{s,-h} \left(2\eta(x)D\eta(x) \otimes \tau_{s,h}u(x) \right) \rangle \\
= - \int_{B_R} \langle A(x + h, Du(x)) - A(x, Du(x)), \eta^2(x)D(\tau_{s,h}u(x)) \rangle \, dx \\
- \int_{B_R} \langle A(x, Du(x)), \tau_{s,-h} \left(2\eta(x)D\eta(x) \otimes \tau_{s,h}u(x) \right) \rangle \, dx \\
- \int_{B_R} \langle A(x, Du(x)), 2\eta(x)D\eta(x) \otimes \tau_{s,-h} \left(\tau_{s,h}u(x) \right) \rangle \, dx : I + II + III.
\]

Previous equality implies that
\[
I_0 \leq |I| + |II| + |III|.
\] (3.6)

In order to estimate the integral \(|I|\), we use the hypothesis (3.5) and Young’s inequality, as follows
\[
|I| \leq c|h| \int_{B_R} \eta^2(x) \left(g(x) + g(x + h)\right) \left(1 + |Du(x)|^2\right)^{\frac{n+1}{2}} |D\tau_{s,h}u(x)| \, dx \\
\leq c|h| \int_{B_R} \eta^2(x) \left(g(x) + g(x + h)\right) \left(1 + |Du(x)|^2 + |Du(x + h)|^2\right)^{\frac{n+1}{2}} |D(\tau_{s,h}u(x))| \, dx \\
\leq \varepsilon \int_{B_R} \eta^2(x) |D(\tau_{s,h}u(x))|^2 \left(1 + |Du(x)|^2 + |Du(x + h)|^2\right)^{\frac{n+1}{2}} \, dx \\
+ c_\varepsilon |h|^2 \int_{B_R} \eta^2(x) \left(g^2(x) + g^2(x + h)\right) \left(1 + |Du(x)|^2 + |Du(x + h)|^2\right)^{\frac{n+2}{2}} \, dx.
\] (3.7)

Now, we estimate \(|II|\) by (3.4) and the properties of \(\eta\) thus obtaining
\[
|II| \leq \frac{c|h|}{(t - s)^2} \int_{B_t} \left(1 + |Du(x)|^2\right)^{\frac{n+1}{2}} |\tau_{s,h}u(x)| \, dx \\
\leq \frac{c|h|}{(t - s)^2} \left(\int_{B_t} \left(1 + |Du(x)|^2\right)^{\frac{p}{2}} \right)^{\frac{n+1}{p}} \left(\int_{B_t} |\tau_{s,h}u(x)|^p \, dx \right)^{\frac{1}{p}},
\]
where, in the last inequality, we used Hölder’s inequality. By virtue of Lemma 2.5, we obtain
\[ |II| \leq \frac{c|h|^2}{(t-s)^2} \int_{B_{t,s}} (1 + |Du(x)|^2) \frac{p-1}{p} |\tau_{s,h}Du(x)| \, dx. \] (3.8)

The term $|III|$ is estimated using the hypothesis (3.4), the properties of $\eta$, Hölder’s inequality and Lemma 2.5, as follows
\[ |III| \leq \frac{c}{t-s} \left( \int_{B_t} (1 + |Du(x)|^2)^\frac{p-1}{p} \left( \int_{B_t} |\tau_{s,h}Du(x)|^p \, dx \right)^{\frac{1}{p}} \right), \] (3.9)

where in the last inequality we used Lemma 2.5 and (2.1) of Proposition 2.4. By the assumption (3.3), we get
\[ |I_0| \geq c(p, \alpha) \int_{B_R} \eta^2(x) (1 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{p-2}{2}} |\tau_{s,h}Du(x)|^2 \, dx. \] (3.10)

Inserting estimates (3.7), (3.8), (3.9) and (3.10) in (3.6), we obtain
\[ c(p, \alpha) \int_{B_R} \eta^2(x) (1 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{p-2}{2}} |\tau_{s,h}Du(x)|^2 \, dx \]
\[ \leq \varepsilon \int_{B_R} \eta^2(x) |D(\tau_{s,h}u(x))|^2 (1 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{p-2}{2}} \, dx \]
\[ + c_e|h|^2 \int_{B_R} \eta^2(x) (g^2(x) + g^2(x+h)) (1 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{p-2}{2}} \, dx \]
\[ + \frac{c|h|^2}{(t-s)^2} \int_{B_{t,s}} (1 + |Du(x)|^2)^\frac{p-1}{p} \left( \int_{B_{t,s}} |\tau_{s,h}Du(x)|^p \, dx \right)^{\frac{1}{p}} \] (3.11)

Choosing $\varepsilon = \frac{c(p, \alpha)}{2}$ in previous estimate, we can reabsorb the first integral in the right hand side by the left hand side thus getting
\[
\int_{B_R} \eta^2(x) \left(1 + |Du(x)|^2 + |Du(x + h)|^2\right)^{p-2} |\tau_{s,h}Du(x)|^2 \, dx \\
\leq c |h|^2 \int_{B_R} \eta^2(x) \left(g^2(x) + g^2(x + h)\right) \left(1 + |Du(x)|^2 + |Du(x + h)|^2\right)^{\frac{p}{2}} \, dx \\
+ \frac{c|h|^2}{(t - s)^2} \int_{B_{3r}} \left(1 + |Du(x)|^2\right)^{\frac{p}{2}} \\
+ \frac{c}{t - s} \left(\int_{B_t} \left(1 + |Du(x)|^2\right)^{\frac{p}{2}}\right)^{\frac{p-1}{p}} \left(\int_{B_{3r}} \left|\tau_{s,h}Du(x)\right|^p \, dx\right)^{\frac{1}{p}},
\]

with \( c = c(\alpha, \beta, p, n) \). Dividing previous estimate by \(|h|^2\) and using Lemma 2.1, we have

\[
\int_{B_R} \eta^2(x) |\tau_{s,h}(V_p(Du))|^2 \frac{\left|\tau_{s,h}(V_p(Du))\right|^2}{|h|^2} \, dx \\
\leq c \int_{B_R} \eta^2(x) \left(1 + |Du(x)|^2 + |Du(x + h)|^2\right)^{p-2} \frac{|\tau_{s,h}Du(x)|^2}{|h|^2} \, dx \\
\leq c \int_{B_R} \eta^2(x) \left(g^2(x) + g^2(x + h)\right) \left(1 + |Du(x)|^2 + |Du(x + h)|^2\right)^{\frac{p}{2}} \, dx \\
+ \frac{c}{(t - s)^2} \int_{B_{3r}} \left(1 + |Du(x)|^2\right)^{\frac{p}{2}} \\
+ \frac{c}{t - s} \left(\int_{B_t} \left(1 + |Du(x)|^2\right)^{\frac{p}{2}}\right)^{\frac{p-1}{p}} \left(\int_{B_{3r}} \left|\tau_{s,h}Du(x)\right|^p \, dx\right)^{\frac{1}{p}},
\]

Now, by Hölder’s inequality and Lemma 2.1, we get

\[
\int_{B_R} \eta^2(x) |\tau_{s,h}(V_p(Du))|^p \frac{\left|\tau_{s,h}(V_p(Du))\right|^p}{|h|^p} \, dx \\
\leq \int_{B_R} \eta^2(x) \left|\tau_{s,h}(V_p(Du))\right|^p \left(1 + |Du(x)|^2 + |Du(x + h)|^2\right)^{\frac{p(2 - p)}{2}} \\
\leq \left(\int_{B_R} \eta^2(x) \left|\tau_{s,h}(V_p(Du))\right|^2 \frac{\left|\tau_{s,h}(V_p(Du))\right|^2}{|h|^2} \right)^{\frac{p}{2}} \left(\int_{B_R} \eta^2(x) \left(1 + |Du(x)|^2 + |Du(x + h)|^2\right)^{\frac{p}{2}} \, dx\right)^{\frac{2 - p}{p}},
\]

and therefore, combining (3.13) and (3.14), we have
\[
\begin{align*}
\int_{B_R} \eta^2(x) \frac{|\tau_{s,h}Du(x)|^p}{|h|^p} dx & \\
\leq & c \left\{ \int_{B_R} \eta^2(x) \left( g^2(x) + g^2(x + h) \right) \left( 1 + |Du(x)|^2 + |Du(x + h)|^2 \right)^{\frac{p}{2}} dx \\
+ & \frac{c}{(t - \bar{s})^2} \int_{B_{\lambda r}} (1 + |Du(x)|^2)^{\frac{p}{2}} \\
+ & \frac{c}{t - \bar{s}} \left( \int_{B_t} (1 + |Du(x)|^2)^{\frac{p-1}{p}} \left( \int_{B_{\lambda r}} \frac{|\tau_{s,h}Du(x)|^p}{|h|^p} dx \right) \right)^{\frac{1}{p}} \\
\cdot \left\{ \int_{B_R} \eta^2(x) \left( 1 + |Du(x)|^2 + |Du(x + h)|^2 \right)^{\frac{2-p}{2}} dx \right\}^{\frac{2-p}{2-p}} \\
+ & c \left( 1 + 1 \left( \frac{1}{t - \bar{s}} \right)^2 + 1 \left( \frac{1}{t - \bar{s}} \right)^{p-1} \right) \int_{B_{\lambda r}} \left( 1 + |Du(x)|^2 \right)^{\frac{p}{2}} dx \\
+ & \frac{1}{2} \int_{B_{\lambda r}} \frac{|\tau_{s,h}Du(x)|^p}{|h|^p} dx.
\end{align*}
\]

Using Young’s inequality with exponents \( \frac{2}{p} \) and \( \frac{2}{2-p} \) and the properties of \( \eta \), we have

\[
\begin{align*}
\int_{B_R} \eta^2(x) \frac{|\tau_{s,h}Du(x)|^p}{|h|^p} dx & \\
\leq & c \int_{B_R} \eta^2(x) \left( g^2(x) + g^2(x + h) \right) \left( 1 + |Du(x)|^2 + |Du(x + h)|^2 \right)^{\frac{p}{2}} dx \\
+ & \left( 1 + \frac{c}{(t - \bar{s})^2} \right) \int_{B_{\lambda r}} (1 + |Du(x)|^2)^{\frac{p}{2}} \\
+ & \left( \int_{B_t} (1 + |Du(x)|^2)^{\frac{p-1}{p}} \left( \int_{B_{\lambda r}} \frac{|\tau_{s,h}Du(x)|^p}{|h|^p} dx \right) \right)^{\frac{1}{p}} \\
\cdot \left\{ \int_{B_R} \eta^2(x) \left( 1 + |Du(x)|^2 + |Du(x + h)|^2 \right)^{\frac{2-p}{2}} dx \right\}^{\frac{2-p}{2-p}} \\
+ & c \left( 1 + 1 \left( \frac{1}{t - \bar{s}} \right)^2 + 1 \left( \frac{1}{t - \bar{s}} \right)^{p-1} \right) \int_{B_{\lambda r}} \left( 1 + |Du(x)|^2 \right)^{\frac{p}{2}} dx \\
+ & \frac{1}{2} \int_{B_{\lambda r}} \frac{|\tau_{s,h}Du(x)|^p}{|h|^p} dx.
\end{align*}
\]
\[
\int_{B_{\lambda r}} \frac{|\tau_{s,h} Du(x)|^p}{|h|^p} \, dx \leq \frac{1}{2} \int_{B_{\lambda r}} \frac{|\tau_{s,h} Du(x)|^p}{|h|^p} \, dx \\
+ c \int_{B_{\lambda r}} g^2(x) \, dx + c \int_{B_{\lambda r}} g^2(x) |Du(x)|^p \, dx \\
+ c \left( 1 + \frac{1}{(t-\tilde{s})^2} + \frac{1}{(t-\tilde{s})^{p-1}} \right) \int_{B_{\lambda r}} \left( 1 + |Du(x)|^2 \right)^{\frac{p}{2}} \, dx.
\] (3.18)

Since the previous estimate holds for every \( r < \tilde{s} < t < \lambda r \), the Lemma 2.3 implies

\[
\int_{B_r} \frac{|\tau_{s,h} Du(x)|^p}{|h|^p} \, dx \leq c \int_{B_{\lambda r}} g^2(x) \, dx + c \int_{B_{\lambda r}} g^2(x) |Du(x)|^p \, dx \\
+ c \left( 1 + \frac{1}{r^2(\lambda-1)^2} + \frac{1}{r^{p-1}(\lambda-1)^{p-1}} \right) \int_{B_{\lambda r}} \left( 1 + |Du(x)|^2 \right)^{\frac{p}{2}} \, dx.
\] (3.19)

and so, by Lemma 2.5

\[
\int_{B_r} |D^2 u|^p \, dx \leq c \int_{B_{\lambda r}} g^2(x) \, dx + c \int_{B_{\lambda r}} g^2(x) |Du(x)|^p \, dx \\
+ c \left( 1 + \frac{1}{r^2(\lambda-1)^2} + \frac{1}{r^{p-1}(\lambda-1)^{p-1}} \right) \int_{B_{\lambda r}} \left( 1 + |Du(x)|^2 \right)^{\frac{p}{2}} \, dx.
\] (3.20)

To go further in the estimate, we have to study the term

\[
\int_{B_{\lambda r}} g^2(x) |Du(x)|^p \, dx,
\] (3.21)

and to do this, our first step is to apply Hölder’s inequality with exponents \( \frac{q}{2} \) and \( \frac{q}{q-2} \), thus obtaining

\[
\int_{B_{\lambda r}} g^2(x) |Du(x)|^p \, dx \leq \left( \int_{B_{\lambda r}} g^q(x) \, dx \right)^{\frac{2}{q}} \left( \int_{B_{\lambda r}} |Du(x)|^{pq} \, dx \right)^{\frac{2}{pq}}. \] (3.22)

Now we observe that, by Sobolev’s embedding Theorem, if \( u \in W^{2,p}_{\text{loc}}(\Omega) \), then \( Du \in L^{q'}_{\text{loc}}(\Omega) \) for all \( q' \in [p,p^*) \), where \( p^* = \frac{np}{n-p} \). So, the second integral in the right hand side of (3.22), converges for \( \frac{pq}{q-2} \leq \frac{np}{n-p} \), that is \( q \geq \frac{2n}{p} \).

We have to distinguish between two cases.
Regularity for Minimizers

Case I. \( \frac{pq}{q-2} = \frac{np}{n-p} \).

In this case we have \( q = \frac{2n}{p} \), then, by Sobolev’s inequality,

\[
\left( \int_{B_{\lambda r}} g^q(x) dx \right)^{\frac{2}{q}} \left( \int_{B_{\lambda r}} |Du(x)|^{\frac{pq}{q-2}} dx \right)^{\frac{q-2}{q}} = \left( \int_{B_{\lambda r}} g^q(x) dx \right)^{\frac{2}{q}} \left( \int_{B_{\lambda r}} |Du(x)|^{\frac{np}{n-p}} dx \right)^{\frac{n-p}{np}}
\leq c \left( \int_{B_{\lambda r}} g^q(x) dx \right)^{\frac{2}{q}} \int_{B_{\lambda r}} (|D^2u|^p + |Du|^p) dx.
\]

(3.23)

By the absolute continuity if the integral, there exists \( R_0 > 0 \) such that, for every \( R < R_0 \), we have

\[
c \left( \int_{B_R} g^q(x) dx \right)^{\frac{2}{q}} < \frac{1}{2}.
\]

(3.24)

For this choice of \( R \), joining (3.20), (3.22), (3.23), (3.24), we get:

\[
\int_{B_r} |D^2u(x)|^p dx \leq c \int_{B_{\lambda r}} g^2(x) dx + \frac{1}{2} \int_{B_{\lambda r}} |D^2u(x)|^p dx + \left( c + \frac{c}{r^2(\lambda-1)^2} + \frac{c}{re^{p-1}(\lambda-1)^{p-1}} \right) \int_{B_{\lambda r}} \left( 1 + |Du(x)|^2 \right)^{\frac{p}{2}} dx.
\]

(3.25)

Case II. \( \frac{pq}{q-2} < \frac{np}{n-p} \).

In this case we have \( q > \frac{2n}{p} \).

Since \( u \in W^{2,p}_{\text{loc}} \), then \( Du \in W^{1,p}_{\text{loc}}(\Omega) \) and \( D^2u \in L^p_{\text{loc}}(\Omega) \). Recalling that, by Sobolev’s embedding Theorem, we have \( W^{1,p}_{\text{loc}}(\Omega) \hookrightarrow L^{p'}_{\text{loc}}(\Omega) \) for all \( p' \in [p, p^*] \), where \( p^* = \frac{np}{n-p} \), we have, for a constant \( c = c(n,p) \),

\[
\|Du\|_{W^{1,p}_{\text{loc}}(\Omega)} \leq c \|Du\|_{L^{p'}(\Omega)} \leq c \left( \|Du\|_{L^p(B_{\lambda r})} + \|D^2u\|_{L^p(B_{\lambda r})} \right).
\]

(3.26)

Now since, for \( q' = \frac{pq}{q-2} \) we have \( p < q' < p^* \), then \( L^{q'}_{\text{loc}}(\Omega) \hookrightarrow L^{p'}_{\text{loc}}(\Omega) \), then

\[
\|Du\|_{L^{p'}(\Omega)} \leq c \|Du\|_{L^{q'}(\Omega)}.
\]

(3.27)

Joining (3.26) and (3.27), we get

\[
\left( \int_{B_{\lambda r}} |Du(x)|^{\frac{pq}{q-2}} dx \right)^{\frac{q-2}{pq}} \leq c \left( \int_{B_{\lambda r}} (|D^2u|^p + |Du|^p) dx \right)^{\frac{1}{p}}.
\]

(3.28)
that is
\[
\left( \int_{B_{\lambda r}} |Du(x)|^{\frac{pq}{q-2}} dx \right)^{\frac{q-2}{q}} \leq c \left( \int_{B_{\lambda r}} (|D^2 u|^p + |D^u|^p) dx \right).
\]
So we obtain
\[
\left( \int_{B_{\lambda r}} g^q(x)dx \right)^{\frac{2}{q}} \left( \int_{B_{\lambda r}} |Du(x)|^{\frac{pq}{q-2}} dx \right)^{\frac{q-2}{q}} \leq c \left( \int_{B_{\lambda r}} g^q(x)dx \right)^{\frac{2}{q}} \left( \int_{B_{\lambda r}} (|D^2 u|^p + |D^u|^p) dx \right)
\]
and by the absolute continuity of the integral, as in the previous case, choosing the value of $r$ opportune, we get an estimate like (3.25) in this case too.

Since (3.25) holds for all $r$ and for all $\lambda \in (1,2)$, setting $\rho = r$, $R_0 = \lambda r$, $\gamma = \frac{p}{p-1}$ and
\[
h(\rho) = \int_{B_{\rho}} |D^2 u(x)| dx,
\]
by Lemma 2.3 we have
\[
\|D^2 u\|_{L^p(B_\rho)} \leq c(\alpha, \beta, p, n) \left( \|Du\|_{L^p(B_{\lambda r})} + \|g\|_{L^q(B_{\lambda r})} \right).
\]
Since $q \geq \frac{2n}{p} > 2$, we have $L^q_{loc}(\Omega) \hookrightarrow L^2_{loc}(\Omega)$, and by (3.29) we get
\[
\|D^2 u\|_{L^p(B_\rho)} \leq C(\alpha, \beta, p, n) \left( \|Du\|_{L^p(B_{\lambda r})} + \|g\|_{L^q(B_{\lambda r})} \right),
\]
that is (1.6).

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