RAMIFIED CLASS FIELD THEORY AND DUALITY OVER FINITE FIELDS

RAHUL GUPTA, AMALENDU KRISHNA

Abstract. We prove a duality theorem for the $p$-adic étale motivic cohomology of a variety $U$ which is the complement of a divisor on a smooth projective variety over $\mathbb{F}_p$. This extends the duality theorems of Milne and Jannsen-Saito-Zhao. The duality introduces a filtration on $H^1_{\text{ét}}(U, \mathbb{Q}/\mathbb{Z})$. We identify this filtration to the classically known Matsuda filtration when the reduced part of the divisor is smooth. We prove a reciprocity theorem for the idele class groups with modulus introduced by Kerz-Zhao and Rülling-Saito. As an application, we derive the failure of Nisnevich descent for Chow groups with modulus.

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1. Introduction

The objective of this paper is to study the duality and reciprocity theorems for non-complete smooth varieties over finite fields and draw consequences. Below we describe the contexts and statements of our main results.

1.1. The duality theorem. Let $k$ be a finite field of characteristic $p$ and $X$ a smooth projective variety of dimension $d$ over $k$. Let $W^r_m\Omega^r_{X,\log}$ be the logarithmic Hodge-Witt sheaf on $X$, defined as the image of the dlog map from the Milnor $K$-theory sheaf $K^M_{r,X}$ to the $p$-typical de Rham-Witt sheaf $W^r_m\Omega^r_X$ in the étale topology. Milne \cite{Milne} proved that there is a perfect pairing of cohomology groups

$$H^i_{\text{ét}}(X, W^r_m\Omega^r_{X,\log}) \times H^{d+1-i}_{\text{ét}}(X, W^d_m\Omega^d_X, \mathbb{Z}/p^m) \rightarrow \mathbb{Z}/p^m. \tag{1.1}$$

By \cite{Kerry} Theorem 8.4, there is an isomorphism $H^i_{\text{ét}}(X, W^r_m\Omega^r_{X,\log}) \cong H^{i+r}(X, \mathbb{Z}/p^m(r))$, where the latter is the $p$-adic étale motivic cohomology due to Suslin-Voevodsky. Milne's

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etale cohomology is the classical Poincaré duality for etale cohomology for a prime \( \ell \) where \( H^p \) support such that a paper) of such a duality theorem is to the study of the mod-
 
yet an open problem. Recall that one of the applications (which is the interest of this k

difinitions (see \([13]\)) for the \( \ell \)-adic etale cohomology in the form

\[
H^i_\text{et}(U, \mathbb{Z}/\ell^n(r)) \times H^{2d+1-i}_c(U, \mathbb{Z}/\ell^n(d-r)) \to \mathbb{Z}/\ell^n,
\]

where \( H^i_\text{et}(U, \mathbb{Z}/\ell^n(j)) \) is the \( \ell \)-adic etale cohomology of \( U \) with compact support.

However, there is no known \( p \)-adic analogue of the etale cohomology of \( U \) with compact support such that a \( p \)-adic analogue of \((1.2)\) could hold. Construction of this duality is yet an open problem. Recall that one of the applications (which is the interest of this paper) of such a duality theorem is to the study of the mod-

dual fundamental group of \( U \), which is in general a very complicated object. A duality theorem such as above would allow one to study this group in terms of a more tractable etale cohomology of U with compact support.

In \([22]\), Jannsen-Saito-Zhao proposed an approach in a special case when \( U \) is the complement of a divisor on a smooth projective variety \( X \) over \( k \) such that \( D_{\text{red}} \) is a simple normal crossing divisor. They constructed a relative version of the logarithmic Hodge-Witt sheaves on \( X \), denoted by \( W_m \Omega^r_{X,D,\log} \). Using these sheaves, they showed that there is a semi-perfect pairing (see Definition \(8.1)\)

\[
H^i_{\text{ét}}(U, W_m \Omega^r_{U,\log}) \times \lim_{\to n} H^{d+1-i}_{\text{ét}}(X, W_m \Omega^d_{X[D,\log]}) \to \mathbb{Z}/p^m
\]

of topological abelian groups, where the first group has discrete topology and the second has profinite topology. This pairing is perfect when \( i = 1 \) and \( r = 0 \).

In this paper, we propose a different approach to the \( p \)-adic duality for \( U \). This new approach has the advantage that it allows \( D \) to be an arbitrary divisor on \( X \). This is possible by virtue of the choice of the relative version of the logarithmic Hodge-Witt sheaves. Instead of using the Hyodo-Kato de Rham-Witt sheaves with respect to a suitable log structure considered in \([22]\), we use a more ingenious version of the relative logarithmic Hodge-Witt sheaf, which we denote by \( W_m \Omega^r_{X,D,\log} \). The latter is defined to be the kernel of the canonical surjection \( W_m \Omega^r_{X,\log} \to W_m \Omega^r_{D,\log} \). Our main result on the \( p \)-adic duality for \( U \) is roughly the following. We refer to Theorem \(8.6)\) for the precise statement.

**Theorem 1.1.** Let \( X \) be a smooth projective variety of dimension \( d \) over a finite field \( k \) of characteristic \( p \). Let \( D \subset X \) be an effective Cartier divisor with complement \( U \). Let \( r, i \geq 0 \) and \( m \geq 1 \) be integers. Then there is a semi-perfect pairing of topological abelian groups

\[
H^i_\text{et}(U, W_m \Omega^r_{U,\log}) \times \lim_{\to n} H^{d+1-i}_{\text{et}}(X, W_m \Omega^d_{(X,D),\log}) \to \mathbb{Z}/p^m.
\]

This pairing is perfect if \( D_{\text{red}} \) is a simple normal crossing divisor, \( i = 1, r = 0 \) and one of the conditions \( \{ d \neq 2, k \neq \mathbb{F}_2 \} \) holds.

We show that Theorem \(1.1)\) recovers the duality theorem of Jannsen-Saito-Zhao if \( D_{\text{red}} \) is a simple normal crossing divisor.

As a consequence of Theorem \(1.1)\) we obtain a filtration \( \{ \text{fil}_{nD} H^1(K) \}_n \), where \( K \) is the function field of \( X \) and \( H^1(K) \) is a shorthand for \( H^1_\text{et}(K, \mathbb{Q}/\mathbb{Z}) \) (see \(8.2)\). We let \( \text{fil}_D H^1(K) \) be the subgroup of \( H^1(K) \) introduced in \([13]\) Definition 7.12. This coincides with the filtration defined in \([29]\) Definition 2.9 by \([16]\) Theorem 1.2. \( \text{fil}_D H^1(K) \) can be described as the subgroup of continuous characters of the absolute Galois group of \( K \) whose Artin conductors (see \([37]\) Definition 3.2.5) at the generic points of \( D \) are bounded by the multiplicities of \( D \) in \( D_{\text{red}} \). This is a more intricate subgroup of \( H^1(K) \)
than \( \text{fil}_D^{\text{et}} H^1(K) \) because the latter can be described in terms of the simpler objects such as the cohomology of the relative logarithmic Hodge-Witt sheaves. On the other hand, \( \text{fil}_D H^1(K) \) determines the ramification theory of finite étale coverings of \( U \). It is therefore desirable to know if and when these two filtrations agree. Our next result is the following.

**Theorem 1.2.** Let \( X \) be a smooth projective variety of dimension \( d \) over a finite field \( k \) of characteristic \( p \). Let \( D \subset X \) be an effective Cartier divisor such that \( D_{\text{red}} \) is regular. Assume that either \( d \neq 2 \) or \( k \neq \mathbb{F}_2 \). Then

\[
\text{fil}_D H^1(K) = \text{fil}_D^{\text{et}} H^1(K).
\]

### 1.2. The reciprocity theorem

The purpose of reciprocity theorems in class field theory over a perfect field is to be able to represent the abelianized étale fundamental groups of varieties over the field in terms of idele class groups which are often described in terms of explicit sets of generators and relations. Let us assume that the base field \( k \) is finite of characteristic \( p \). In this case, such a reciprocity theorem for smooth projective varieties over \( k \) is due to Kato-Saito [25] which describes the abelianized étale fundamental group in terms of the Chow group of 0-cycles.

For the more intricate case of a smooth quasi-projective variety \( U \) which is not complete, an approach was introduced by Kato-Saito [26] whose underlying idea is to study the so called ‘étale fundamental group of \( X \) with modulus \( D \)’, where \( D \subset X \) is a fixed divisor which is supported on the complement of \( U \) in a normal compactification \( U \subset X \). This group characterizes the finite étale coverings of \( U \) whose ramification along \( X \setminus U \) is bounded by \( D \) in a certain sense. There are various ways to make sense of this bound on the ramification, and they give rise to several definitions of the étale fundamental group with modulus. It turns out that depending on what one wants to do, each of these has certain advantage over the others.

Kato and Saito were able to describe \( \pi_1^{\text{ab}}(U) \) in terms of the limit (over \( D \)) of the idele class groups with modulus \( H^d_{\text{nis}}(X, K^M_{d,(X,D)}) \), where \( d = \dim(X) \). In [16, Theorems 1.1], it was shown that \( H^d_{\text{nis}}(X, K^M_{d,(X,D)}) \) describes \( \pi_1^{\text{div}}(X, D) \) for every \( D \) if we let \( \pi_1^{\text{div}}(X, D) \) be the Pontryagin dual to the Matsuda filtration \( \text{fil}_D H^1(K) \). It was also shown in loc. cit. that \( \pi_1^{\text{div}}(X, D) \) coincides with the fundamental group with modulus \( \pi_1^{\text{ab}}(X, D) \), introduced earlier by Deligne and Laumon [34] if \( X \) is smooth. The latter was shown to coincide (along the degree zero parts) with the Chow group of 0-cycles with modulus \( CH_0(X|D) \) by Kerz-Saito [29] (when \( p \neq 2 \)) and Binda-Krishna-Saito [3].

If we use Kato’s Swan conductor instead of Matsuda’s Artin conductor to bound the ramification in terms of a divisor supported away from \( U \), we are led to a different notion of the abelianized étale fundamental group with modulus which we denote by \( \pi_1^{\text{abk}}(X, D) \). This is defined as the Pontryagin dual to the subgroup \( \text{fil}_D^{\text{et}} H^1(K) \). The latter is the subgroup of continuous characters of the absolute Galois group of \( K \) whose Swan conductors (defined by Kato [24]) at the generic points of \( D \) are bounded by the multiplicities of \( D \) in \( D_{\text{red}} \).

One can now ask if \( \pi_1^{\text{abk}}(X, D) \) could be described by an idele class group with modulus, similar to the \( K \)-theoretic idele class group of Kato-Saito and the cycle-theoretic idele class group of Kerz-Saito. Our next result solves this problem.

Let \( R^M_{r,X} \) be the improved Milnor \( K \)-theory sheaf of Gabber and Kerz [27]. Let \( R^M_{r,X|D} \) be the relative Milnor \( K \)-theory sheaf, defined locally as the image of the map \( K^M_{1,(X,D)} \hat{\otimes}_{\mathbb{Z}} j_* \mathcal{O}_U^* \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} j_* \mathcal{O}_U^* \rightarrow R^M_{r,X} \). We refer to Lemma 3.1 for the proof that this map is defined. This sheaf was considered by Rülling-Saito [12] when \( X \) is smooth and \( D_{\text{red}} \) is a simple
normal crossing divisor. There are degree maps \( \deg: H^d_{\text{nis}}(X, \mathcal{R}_{d,X}^M) \to \mathbb{Z} \) (see §3.1) and 
\( \deg': \pi_1^{\text{ab}}(X, D) \to \mathbb{Z} \). We let \( H^d_{\text{nis}}(X, \mathcal{R}_{d,X}^M) = \ker(\deg) \) and \( \pi_1^{\text{ab}}(X, D) = \ker(\deg') \).

**Theorem 1.3.** Let \( X \) be a normal projective variety of dimension \( d \) over a finite field. Let \( D \subset X \) be an effective Cartier divisor whose complement is regular. Then there is a continuous reciprocity homomorphism

\[
\rho'_{X|D}: H^d_{\text{nis}}(X, \mathcal{R}_{d,X}^M) \to \pi_1^{\text{ab}}(X, D)
\]

with dense image such that the induced map

\[
\rho'_{X|D}: H^d_{\text{nis}}(X, \mathcal{R}_{d,X}^M)_0 \to \pi_1^{\text{ab}}(X, D)_0
\]

is an isomorphism of finite groups.

Let \( H^i_M(X|D, \mathbb{Z}(j)) \) be the motivic cohomology with modulus. This is defined as the Nisnevich hypercohomology of the sheafified cycle complex with modulus \( j(X|D, 2j - \bullet) \), introduced in [4]. Using Theorem 1.3 and [42, Theorem 1], we obtain the following.

**Corollary 1.4.** Assume in Theorem 1.3 that \( X \) is regular and \( D_{\text{red}} \) is a simple normal crossing divisor. Then there is an isomorphism of finite groups

\[
cyc'_{X|D}: H^{2d}_M(X|D, \mathbb{Z}(d))_0 \cong \pi_1^{\text{ab}}(X, D)_0.
\]

This result can be viewed as the cycle-theoretic description of the \( \pi_1^{\text{ab}}(X, D) \), analogous a similar result for \( \pi_1^{\text{div}}(X, D) \), proven in [29], [3] and [16].

### 1.3. Failure of Nisnevich descent for Chow group with modulus

Recall that the classical higher Chow groups of smooth varieties satisfy the Nisnevich descent in the sense that the canonical map \( CH^i(X, j) \to H^{2i-j}_M(X, \mathbb{Z}(i)) \) is an isomorphism for \( i, j \geq 0 \), where the latter is the Nisnevich hypercohomology of the sheafified (in Nisnevich topology) Bloch’s cycle complex. However, this question for the higher Chow groups with modulus is yet an open problem. Some cases of this were verified by Rülling-Saito [42, Theorem 3]. As an application of Corollary 1.3 we prove the following result which provides a counterexample to the Nisnevich descent for the Chow groups with modulus. This was one of our motivations for studying the reciprocity for \( \pi_1^{\text{ab}}(X, D) \).

**Theorem 1.5.** Let \( X \) be a smooth projective surface over a finite field. Let \( D \subset X \) be an effective Cartier divisor such that \( D_{\text{red}} \) is a simple normal crossing divisor. Then the canonical map

\[
CH_0(X|D) \to H^4_M(X, \mathbb{Z}(2))
\]

is not always an isomorphism.

Using Theorem 1.5 one can show that the Nisnevich descent for the Chow groups with modulus fails over infinite fields too.

Recall that the map in Theorem 1.5 is known to be an isomorphism if \( X \) is a curve. This is related to the fact that for a Henselian discrete valuation field \( K \) with perfect residue field, the Swan conductor and the Artin conductor agree for the characters of the absolute Galois group of \( K \). In this context, we also remark that one could attempt to define an analogue of the Deligne-Laumon fundamental group with modulus \( \pi_1^{\text{ab}}(X, D) \) by using the Brylinski-Kato filtration along all integral curves in \( X \) not contained in \( D \). However, the resulting fundamental group will coincide with \( \pi_1^{\text{div}}(X, D) \) and not yield anything new for the same reason as above.
1.4. Overview of proofs. We give a brief overview of our proofs. The main idea behind the proof of our duality theorem is the observation that the naive relative logarithmic Hodge-Witt sheaves are isomorphic to the naive relative mod-$p$ Milnor $K$-theory sheaves (in the sense of Kato-Saito) in the pro-setting. The hope that the étale cohomology of relative mod-$p$ Milnor $K$-theory sheaves are the correct objects to use for duality originated from our results in previous papers which showed that the Nisnevich cohomology of the Kato-Saito relative Milnor $K$-theory yields a reciprocity isomorphism without any condition on the divisor.

To implement the above ideas, we need to prove many results about the relative logarithmic Hodge-Witt sheaves and their relation with various versions of relative Milnor $K$-theory. One of these is a pro-isomorphism between the relative logarithmic Hodge-Witt sheaves and the twisted de Rham-Witt sheaves. The latter are easier objects to work with because duality for them follows from the Grothendieck-Serre coherent duality. The next step is to construct higher Cartier operators on the twisted de Rham-Witt sheaves. This allows us to construct the relative version of the two-term complexes considered by Milne [39]. The proof of the duality theorem is then reduced to coherent duality using an induction procedure.

To prove Theorem 1.2, we use our duality theorem and the reciprocity theorem of [16] to reduce it to proving an independent statement that the top Nisnevich cohomology of the relative Milnor $K$-theory sheaf coincides with the corresponding étale cohomology if the reduced divisor is regular (see Theorem 1.9).

To prove Theorem 1.3, we follow the strategy we used in the proof of a similar result for $\pi_1^\text{div}(X, D)$ in [16]. But there are some new ingredients to be used. The first key ingredient is a result of Kerz-Zhao [30] which gives an idelic presentation of $H^d_{\text{nis}}(X, \mathcal{F}_{d,X|D})$. The second key result is a theorem of Kato [24] which gives a criterion for the characters of the absolute Galois group of a Henselian discrete valuation field to annihilate various subgroups of the Milnor $K$-theory of the field under a suitable pairing (see Theorem 9.1). What remains after this is to prove Proposition 9.3.

We prove various results about relative Milnor $K$-theory in sections 2, 3 and 4. We study the relative logarithmic Hodge-Witt sheaves and prove some results about their cohomology in § 5. We introduce the Cartier operators on the twisted de Rham-Witt sheaves in § 6. We construct the pairing for the duality theorem and prove its perfectness in sections 7 and 8. The reciprocity theorem is proven in § 9 and the failure of Nisnevich descent for the Chow groups with modulus is shown in § 10.

1.5. Notation. We shall work over a field $k$ of characteristic $p > 0$ throughout this paper. We let $\text{Sch}_k$ denote the category of separated and essentially of finite type schemes over $k$. The product $X \times_{\text{Spec}(k)} Y$ in $\text{Sch}_k$ will be written as $X \times Y$. We let $X^{(q)}$ (resp. $X_{(q)}$) denote the set of points on $X$ having codimension (resp. dimension) $q$. We let $\text{Sch}_k^{\text{zar}}$ (resp. $\text{Sch}_k^{\text{nis}}$, resp. $\text{Sch}_k^{\text{ét}}$) denote the Zariski (resp. Nisnevich, resp. étale) site of $\text{Sch}_k$. We let $\epsilon: \text{Sch}_k^{\text{ét}} \to \text{Sch}_k^{\text{nis}}$ denote the canonical morphism of sites. If $\mathcal{F}$ is a sheaf on $\text{Sch}_k^{\text{nis}}$, we shall denote $\epsilon^* \mathcal{F}$ also by $\mathcal{F}$ as long as the usage of the étale topology is clear in a context. For $X \in \text{Sch}_k$, we shall let $\psi: \mathcal{X} \to X$ denote the absolute Frobenius morphism.

For an abelian group $A$, we shall write $\text{Tor}_1^Z(A, \mathbb{Z}/n)$ as $\tilde{n} A$ and $A/n A$ as $A/n$. The tensor product $A \otimes \mathbb{Z} B$ will be written as $A \otimes B$. We shall let $A(p')$ denote the subgroup of elements of $A$ which are torsion of order prime to $p$. We let $A(p)$ denote the subgroup of elements of $A$ which are torsion of order power of $p$. 
2. Relative Milnor $K$-theory

In this section, we recall several versions of relative Milnor $K$-theory sheaves and establish some relations among their cohomology.

For a commutative ring $A$, the Milnor $K$-group $K_r^M(A)$ (as defined by Kato [23]) is the $r$-th graded piece of the graded ring $K^M_r(A)$. The latter is the quotient of the tensor algebra $T_r(A^x)$ by the two-sided graded ideal generated by the homogeneous elements $a_1 \otimes \cdots \otimes a_r$ such that $r \geq 2$, and $a_i + a_j = 1$ for some $1 \leq i \neq j \leq r$. The residue class of $a_1 \otimes \cdots \otimes a_r \in T_r(A^x)$ in $K^M_r(A)$ is denoted by the Milnor symbol $\underline{a} = \{a_1, \ldots, a_r\}$. Given an ideal $I \subset A$, the relative Milnor $K$-theory $K^M_r(A, I)$ is defined as the kernel of the restriction map $K^M_r(A) \to K^M_r(A/I)$.

Let $\widetilde{K}^M_r(A)$ denote the $r$-th graded piece of the graded ring $\widetilde{K}^M_r(A)$, where the latter is the quotient of the tensor algebra $T_r(A^x)$ by the two-sided graded ideal generated by the homogeneous elements $a_1 \otimes a_2$ such that $a_1 + a_2 = 1$. We let $\widetilde{K}^M_r(A, I)$ be the kernel of the restriction map $\widetilde{K}^M_r(A) \to \widetilde{K}^M_r(A/I)$. It is clear that there is a natural surjection $\widetilde{K}^M_r(A) \twoheadrightarrow K^M_r(A)$. This is an isomorphism if $A$ is a local ring with infinite residue field (see [27] Proposition 2). This says that a similar thing holds also for the relative $K$-theory.

**Lemma 2.1.** Let $A$ be a local ring and let $I \subset A$ be an ideal. Then the following hold.

1. $K^M_r(A, I)$ and $\widetilde{K}^M_r(A, I)$ are generated by the Milnor symbols $\{a_1, \ldots, a_r\}$ such that $a_i \in K^M_1(A, I)$ for some $1 \leq i \leq r$.

2. The canonical map $\widetilde{K}^M_r(A, I) \to K^M_r(A, I)$ is surjective. This is an isomorphism if $A$ has infinite residue field.

**Proof.** The assertion (1) for $K^M_r(A, I)$ is [26] Lemma 1.3.1] and the proof of (1) for $\widetilde{K}^M_r(A, I)$ is completely identical to that of $K^M_r(A, I)$. The second part of (2) follows from the corresponding result in the non-relative case mentioned above. To prove the first part of (2), we fix an integer $r \geq 1$. Let $\underline{a} = \{a_1, \ldots, a_r\} \in \widetilde{K}^M_r(A)$ be such that $a_i \in \widetilde{K}^M_1(A, I) = K^M_1(A, I)$ for some $1 \leq i \leq r$. It is then clear that $\underline{a} \in \widetilde{K}^M_r(A, I)$. Using this observation, our assertion follows directly from item (1) for $K^M_r(A, I)$.

For a ring $A$ as above, we let $\widetilde{R}^M_*(A)$ denote the improved Milnor $K$-theory defined independently by Gabber and Kerz [27]. For an ideal $I \subset A$, we define $\widetilde{R}^M_*(A, I)$ to be the kernel of the map $\widetilde{R}^M_*(A) \to \widetilde{R}^M_*(A/I)$. Let $K_*(A)$ denote the Quillen $K$-theory of $A$. We state the following facts as a lemma and refer to [27] for their source.

**Lemma 2.2.** There are natural maps

$$K^M_*(A) \xleftarrow{\alpha_A} \widetilde{R}^M_*(A) \xrightarrow{\beta_A} \widetilde{R}^M_*(A) \xrightarrow{\gamma_A} K_*(A),$$

where $\alpha_A$ is always surjective and $\beta_A$ is surjective if $A$ is local. These two maps are isomorphisms if $A$ is a field or a local ring with infinite residue field.

Let $X$ be a Noetherian scheme and $\nu: D \to X$ a closed immersion. We shall say that $(X, D)$ is a modulus pair if $D$ is an effective Cartier divisor on $X$. This Cartier divisor may be empty. If $\mathcal{P}$ is a property of schemes, we shall say that $(X, D)$ satisfies $\mathcal{P}$ if $X$ does so. We shall say that $(X, D)$ has dimension $d$ if $X$ has Krull dimension $d$.

Given a Noetherian scheme $X$ and an integer $r \geq 1$, let $\mathcal{K}^M_r, X$ denote the sheaf on $X_{\text{nis}}$ whose stalk at a point $x \in X$ is $K^M_r(O^h_{X, x})$. We let $\mathcal{K}^M_r, (X, D)$ be the kernel of the restriction map $\mathcal{K}^M_r, X \to \mathcal{K}^M_r, D := \iota_* \mathcal{K}^M_r, D$. We shall usually refer to $\mathcal{K}^M_r, (X, D)$ as the Kato-Saito relative Milnor $K$-sheaves. We define the sheaves $\mathcal{K}^M_r, (X, D)$ and $\widetilde{\mathcal{K}}^M_r, (X, D)$ in an analogous way. We let $\mathcal{K}_{r, X}$ be the Quillen $K$-theory sheaf on $X_{\text{nis}}$. 

Lemma 2.3. Let $X$ be a reduced Noetherian scheme of Krull dimension $d$ and let $D \subset X$ be a nowhere dense closed subscheme. Then the canonical map

$$H^d_{\text{nis}}(X, \overline{\mathcal{K}}^r_{r,(X,D)}/n) \to H^d_{\text{nis}}(X, \mathcal{K}^r_{r,(X,D)}/n)$$

is an isomorphism for every integer $n \geq 0$.

Proof. We look at the commutative diagram

$$(2.2) \quad \overline{\mathcal{K}}^M_{r,(X,D)} \hookrightarrow \overline{\mathcal{K}}^M_{r,X}$$

$$\quad \downarrow \quad \quad \downarrow$$

$$\quad \mathcal{K}^M_{r,(X,D)} \hookrightarrow \mathcal{K}^M_{r,X}.$$  

Lemma 2.1 says that the left vertical arrow is surjective. We have seen above that the kernel of the right vertical arrow is supported on a nowhere dense closed subscheme of $X$ (this uses reducedness of $X$). Hence, the same holds for the left vertical arrow. But this easily implies the lemma when $n = 0$.

For $n \geq 1$, we use the commutative diagram

$$H^d_{\text{nis}}(X, \overline{\mathcal{K}}^M_{r,(X,D)}) \xrightarrow{\cong} H^d_{\text{nis}}(X, \overline{\mathcal{K}}^M_{r,(X,D)}) \xrightarrow{\cong} H^d_{\text{nis}}(X, \overline{\mathcal{K}}^M_{r,(X,D)}/n) \to 0$$

$$\quad \downarrow \quad \quad \downarrow$$

$$H^d_{\text{nis}}(X, \mathcal{K}^M_{r,(X,D)}) \xrightarrow{\cong} H^d_{\text{nis}}(X, \mathcal{K}^M_{r,(X,D)}) \xrightarrow{\cong} H^d_{\text{nis}}(X, \mathcal{K}^M_{r,(X,D)}/n) \to 0.$$ 

It is easily seen that the two rows are exact. The $n = 0$ case shows that the left and the middle vertical arrows are isomorphisms. It follows that the right vertical arrow is also an isomorphism. \qed

For $D \subset X$ as above and $n \geq 1$, let

$$(2.3) \quad \overline{\mathcal{K}}^M_{r,(X,D)}/n := \overline{\mathcal{K}}^M_{r,(X,D)}/(\overline{\mathcal{K}}^M_{r,(X,D)} \cap n\overline{\mathcal{K}}^M_{r,X}) = \text{Image}(\overline{\mathcal{K}}^M_{r,X,D}) \to \overline{\mathcal{K}}^M_{r,X}/n).$$

Since the map $n\overline{\mathcal{K}}^M_{r,X} \to n\overline{\mathcal{K}}^M_{r,D}$ is surjective, it easily follows that there are exact sequences

$$(2.4) \quad 0 \to \overline{\mathcal{K}}^M_{r,(X,D)}/n \to \overline{\mathcal{K}}^M_{r,X}/n \to \overline{\mathcal{K}}^M_{r,D}/n \to 0;$$

$$n\overline{\mathcal{K}}^M_{r,D} \to \overline{\mathcal{K}}^M_{r,(X,D)}/n \to \overline{\mathcal{K}}^M_{r,(X,D)}/n \to 0.$$ 

We let $\overline{\mathcal{K}}^M_{r,(X,D)}/n := \overline{\mathcal{K}}^M_{r,(X,D)}/(\overline{\mathcal{K}}^M_{r,(X,D)} \cap n\overline{\mathcal{K}}^M_{r,X})$. It is then clear that the two exact sequences of (2.3) hold for the improved Milnor $K$-sheaves as well.

The following is immediate from (2.4).

Lemma 2.4. Let $X$ be a Noetherian scheme of Krull dimension $d$ and let $D \subset X$ be a nowhere dense closed subscheme. Then the canonical map

$$H^d_{\text{nis}}(X, \overline{\mathcal{K}}^M_{r,(X,D)}/n) \to H^d_{\text{nis}}(X, \overline{\mathcal{K}}^M_{r,(X,D)}/n)$$

is an isomorphism for all $n \geq 1$.

To proceed further, we need the following result about the sheaf cohomology. Given a field $k$ and a topology $\tau$ on Spec$(k)$, let $cd_\tau(k)$ denote the cohomological dimension of $k$ for torsion $\tau$-sheaves.
Lemma 2.5. Let $k$ be a field and $X \in \text{Sch}_k$. Let $\mathcal{F}$ be a torsion sheaf on $X_{\text{nis}}$ such that $\mathcal{F}_x = 0$ for every $x \in X(q)$ with $q > 0$. Then $H^2_\tau(X, \mathcal{F}) = 0$ for $q > \text{cd}_\tau(k)$ and $\tau \in \{\text{nis, ét}\}$. In particular, given an exact sequence of Nisnevich sheaves

$$0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0,$$

the induced map $H^2_\tau(X, \mathcal{F}') \to H^2_\tau(X, \mathcal{F}'')$ is an isomorphism for $q > \text{cd}_\tau(k)$ and $\tau \in \{\text{nis, ét}\}$.

Proof. We fix $\tau \in \{\text{nis, ét}\}$. Let $J$ be the set of finite subsets of $X(0)$. Given any $\alpha \in J$, let $S_\alpha \subset X$ be the 0-dimensional reduced closed subscheme defined by $\alpha$ and let $U_\alpha = X \setminus S_\alpha$. Let $\iota_\alpha: S_\alpha \to X$ and $j_\alpha: U_\alpha \to X$ be the inclusions. As $J$ is cofiltered with respect to inclusion, there is a cofiltered system of short exact sequences of $\tau$-sheaves

$$(2.5) \quad 0 \to (j_\alpha)_!(\mathcal{F}|_{U_\alpha}) \to \mathcal{F} \to (\iota_\alpha)_*(\mathcal{F}|_{S_\alpha}) \to 0.$$ 

It is easily seen from our assumption that $\lim_{\alpha \in J} (j_\alpha)_!(\mathcal{F}|_{U_\alpha}) = 0$ so that the map $\mathcal{F} \to \lim_{\alpha \in J} (\iota_\alpha)_*(\mathcal{F}|_{S_\alpha})$ is an isomorphism. Since $H^*_\tau(S_\alpha, \mathcal{F}|_{S_\alpha}) \cong H^*_\tau(X, (\iota_\alpha)_*(\mathcal{F}|_{S_\alpha}))$, we are done by [8, Proposition 58.89.6].

Lemma 2.6. Let $k$ be a field and $(X, D)$ a reduced modulus pair in $\text{Sch}_k$. Assume that $X$ is of pure dimension $d \geq 1$ and $\tau \in \{\text{nis, ét}\}$. Let $n \geq 1$ be an integer and

$$(2.6) \quad H^d_\tau(X, \overline{\mathcal{K}}^M_{r,(X,D)}) \to H^d_\tau(X, \overline{\mathcal{K}}^M_{r,(X,D)});$$

$$(2.7) \quad H^d_\tau(X, \overline{\mathcal{K}}^M_{r,(X,D)}/n) \to H^d_\tau(X, \overline{\mathcal{K}}^M_{r,(X,D)}/n)$$

the canonical maps. Then the following hold.

1. If $k$ is infinite, then both maps are isomorphisms.
2. If $d = 1$ and $\tau \leq 1$, then both maps are isomorphisms.
3. If $d \geq 2$ and $\tau = \text{nis}$, then both maps are isomorphisms.
4. If $k$ is finite, $d \geq 3$ and $\tau = \text{ét}$, then (2.7) is an isomorphism.
5. If $k \neq \mathbb{F}_2$ is finite, $d = 2$, $\tau \leq 2$ and $\tau = \text{ét}$, then (2.7) is an isomorphism.

Proof. The items (1) and (2) are clear from Lemma 2.2. To prove (3) and (4), we let $n \geq 1$ and consider the commutative diagram

$$
\begin{array}{ccccccc}
H^{d-1}_\tau(X, \overline{\mathcal{K}}^M_{r,X}/n) & \to & H^{d-1}_\tau(D, \overline{\mathcal{K}}^M_{r,D}/n) & \to & H^d_\tau(X, \overline{\mathcal{K}}^M_{r,(X,D)}/n) & \to & H^d_\tau(D, \overline{\mathcal{K}}^M_{r,D}/n) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{d-1}_\tau(X, \overline{\mathcal{K}}^M_{r,X}/n) & \to & H^{d-1}_\tau(D, \overline{\mathcal{K}}^M_{r,D}/n) & \to & H^d_\tau(X, \overline{\mathcal{K}}^M_{r,(X,D)}/n) & \to & H^d_\tau(D, \overline{\mathcal{K}}^M_{r,D}/n) \\
\end{array}
$$

with $d \geq 2$. The two rows are exact by (2.4). It suffices to show that all vertical arrows except possibly the middle one are isomorphisms. But this follows easily from Lemmas 2.2 and 2.5 if we observe that $\text{cd}_{\text{nis}}(k) = 0$ for $k$ arbitrary and $\text{cd}_{\text{ét}}(k) = 1$ for $k$ finite (see [2, Exposé X]). The proof of (2.6) for $\tau = \text{nis}$ is identical if we observe that Lemma 2.5 holds even if the sheaf $\mathcal{F}$ is not torsion when $\tau = \text{nis}$.
We now prove (5). We only consider the case \( r = 2 \) as the other cases are trivial. We look at the commutative diagram of Nisnevich (or étale) sheaves

\[
0 \rightarrow \overline{K}_r^{M, (X, D)}/n \rightarrow \overline{K}_r^{M, X}/n \rightarrow \overline{K}_r^{M, D}/n \rightarrow 0
\]

(2.8)

We now recall from [27, Proposition 10(6)] that for a local ring \( A \) containing \( k \), the map \( \overline{K}_2^M(A) \rightarrow K_2(A) \) is an isomorphism. Using this, it follows from [31, Corollary 3.3] that \( \overline{K}_2^M(A) \) is generated by the Milnor symbols \( \{a, b\} \) subject to bilinearity, Steinberg relation, and the relation \( \{a, b\} = \{-b, a\} \). Since \( \text{Ker}(\overline{K}_2^M(A) \rightarrow \overline{K}_2^M(A)) \) surjects onto \( \text{Ker}(\overline{K}_2^M(A)/n \rightarrow \overline{K}_2^M(A)/n) \), it follows that the latter is generated by the images of the symbols of the type \( \{a, b\} + \{-b, a\} \). Since \( A^* \rightarrow (A/I)^* \) is surjective for any ideal \( I \subset A \), we conclude that the map

\[
\text{Ker}(\overline{K}_2^M(A)/n \rightarrow \overline{K}_2^M(A)/n) \rightarrow \text{Ker}(\overline{K}_2^M(A/I)/n \rightarrow \overline{K}_2^M(A/I)/n)
\]

is surjective. We have therefore shown that \( \text{Ker}(\alpha_X) \rightarrow \text{Ker}(\alpha_D) \) in (2.8). Subsequently, a diagram chase shows that

\[
\overline{K}_r^{M, (X, D)}/n \rightarrow \overline{K}_r^{M, (X, D)}/n
\]

(2.10)
is a surjective map of Nisnevich (or étale) sheaves. We let \( \mathcal{F} \) denote the kernel of this map. A diagram similar to (2.2) (with \( K_r^{M, X} \) replaced by \( \overline{K}_r^{M, X} \)) and Lemma 2.2 together show that \( \mathcal{F}_x = 0 \) for every \( x \in X(q) \) with \( q > 0 \). We now apply Lemma 2.5 to finish the proof. \( \square \)

Combining Lemmas 2.3, 2.4 and 2.6 we get the following key result that we shall use.

**Proposition 2.7.** Let \( k \) be a field and \((X, D)\) a reduced modulus pair in \( \text{Sch}_k \). Assume that \( X \) is of pure dimension \( d \geq 2 \). Then there are natural isomorphisms

\[
H^d_{\text{nis}}(X, K_r^{M, (X, D)}) \cong H^d_{\text{nis}}(X, \overline{K}_r^{M, (X, D)});
\]

\[
H^d_{\text{nis}}(X, K_r^{M, (X, D)}/n) \cong H^d_{\text{nis}}(X, \overline{K}_r^{M, (X, D)}/n)
\]

for every integer \( n \geq 1 \). If \( d = 1 \), these isomorphisms hold if either \( k \) is infinite or \( r \leq 1 \).

3. The idele class group of Kerz-Zhao

Let \( A \) be a local domain with fraction field \( F \) and let \( I = (f) \) be a principal ideal, where \( f \in A \) is a non-zero element. Let \( A_f \) denote the localization \( A[f^{-1}] \) obtained by inverting the powers of \( f \) so that there are inclusions of rings \( A \hookrightarrow A_f \hookrightarrow F \). We let \( \overline{K}_1^M(A/I) = K_1^M(A, I) \) and for \( r \geq 2 \), we let \( \overline{K}_r^M(A/I) \) denote the image of the canonical map of abelian groups

\[
K_1^M(A, I) \otimes (A_f)^\times \otimes \cdots \otimes (A_f)^\times \rightarrow K_r^M(F),
\]

(3.1)

induced by the product in Milnor \( K \)-theory, where the tensor product is taken \( r \) times. These groups coincide with the relative Milnor \( K \)-groups of Rülling-Saito (see [42, Definition 2.4 and Lemma 2.1]) when \( A \) is regular and \( \text{Spec}(A/f)_{\text{red}} \) is a normal crossing divisor on \( \text{Spec}(A) \). We shall denote the associated sheaf on an integral scheme \( X \) with an effective Cartier divisor \( D \) by \( \overline{K}_r^{M, X|D} \).
We let \( E_\tau(A, I) \) be the image of the canonical map \( K_1^M(A, I) \to K_1^M(F) \). This is the same as the image of the map \( \tilde{K}_r^M(A, I) \to K_r^M(F) \) by Lemma 2.1. The following result will be important for us.

**Lemma 3.1.** We have the following.

1. The composite map
   \[
   \tilde{K}_r^M(A|I) \to K_r^M(F) \xrightarrow{\partial} \bigoplus_{ht(p)=1} K_{r-1}^M(k(p))
   \]
   is zero. In particular, \( \tilde{K}_r^M(A|I) \subset \tilde{K}_r^M(A) \) if \( A \) is regular and equicharacteristic or a Henselian discrete valuation ring.

2. There is an inclusion \( E_\tau(A, I) \subset \tilde{K}_r^M(A|I) \) which is an equality if \( I \) is a prime ideal.

**Proof.** We let \( p \subset A \) be a prime ideal of height one such that \( f \not\in p \). To prove (1), it suffices to show that the composite \( \tilde{K}_r^M(A_0|I_p) \to K_r^M(F) \xrightarrow{\partial} K_{r-1}^M(k(p)) \) is zero, where \( \partial \) is the boundary map defined in [23, § 1]. We can therefore assume that \( \dim(A) = 1 \) and \( f \not\in A' \). By the definition of \( \partial \), we can assume that \( A \) is normal. But one easily checks in this case that \( \tilde{K}_r^M(A|I) \subset \tilde{K}_r^M(A) \) (see [22, Proposition 2.8]) and \( \partial(\tilde{K}_r^M(A)) = 0 \) (see [7, Proposition 2.7]). The second part of (1) follows from [27, Proposition 10] and [35, Theorem 5.1].

We now prove (2). One checks using Lemma 2.1 that \( E_\tau(A, I) \subset \tilde{K}_r^M(A|I) \). Suppose now that \( I \) is prime and consider an element \( \alpha = \{1 + af, u_2, \ldots, u_r\} \), where \( a \in A \) and \( u_i \in (Af)^X \) for \( i \geq 2 \). Since \( I \) is prime, it is easy to check that \( u_i = a_i f^{n_i} \) for \( a_i \in A^X \) and \( n_i \geq 0 \). Using the bilinearity and the Steinberg relations, it easily follows from [23, Lemma 1] that \( \alpha \in E_\tau(A, I) \). \( \square \)

Suppose that \( k \) is any field and \( X \in \text{Sch}_k \) is regular. Let \( D \subset X \) be an effective Cartier divisor. It follows from Lemma 3.1 that there are inclusions \( \tilde{K}_r^M(X|D) \subset \tilde{K}_r^M(X|X) \supset \tilde{K}_r^M(X|X) \) for \( n \geq 1 \). We let \( \tilde{K}_r^M(X|D)/m = \tilde{K}_r^M(X|D)/(\tilde{K}_r^M(X|X) \cap m\tilde{K}_r^M(X|X)) \).

**Proposition 3.2.** Let \( k \) be a field and \( X \in \text{Sch}_k \) a regular scheme of dimension \( d \geq 1 \). Let \( D \subset X \) be a simple normal crossing divisor. Let \( \tau \in \{\text{nis, \text{ét}}\} \) and \( m \geq 1 \). Then the conclusions (1) \( \sim \) (5) of Lemma 2.6 hold for the maps

\[
\{H^d_\tau(X, \tilde{K}_r^M(X|X))\}_n \to \{H^d_\tau(X, \tilde{K}_r^M(X|X))\}_n;
\]

\[
\{H^d_\tau(X, \tilde{K}_r^M(X|D)/m)\}_n \to \{H^d_\tau(X, \tilde{K}_r^M(X|D)/m)\}_n
\]

of pro-abelian groups.

**Proof.** We let \( \mathcal{E}_r(X,nD) = \text{Image}(\tilde{K}_r^M(X,nD)) \to \tilde{K}_r^M(X|X) \) and \( \mathcal{E}_r(X,nD)/m = \text{Image}(\tilde{K}_r^M(X,nD)) \to \tilde{K}_r^M(X|X)/m \). It is easy to check using [42, Proposition 2.8] that the inclusions \( \{\mathcal{E}_r(X,nD)\}_n \to \{\tilde{K}_r^M(X|D)/m\}_n \) and \( \{\mathcal{E}_r(X,nD)/m\}_n \to \{\tilde{K}_r^M(X|D)/m\}_n \) are pro-isomorphisms. The rest of the proof is now completely identical to that of Lemma 2.6. \( \square \)

Let us now assume that \( k \) is any field and \( X \in \text{Sch}_k \) is an integral scheme of dimension \( d \geq 1 \). We shall endow \( X \) with its canonical dimension function \( d_X \) (see [28]). Assume that \( D \subset X \) is an effective Cartier divisor. Let \( F \) denote the function field of \( X \) and \( U = X \setminus D \). We refer the reader to [26, § 1] or [15, § 2] for the definition and all properties of Parshin chains and their Milnor \( K \)-groups that we shall need.

We let \( \mathcal{P}_{U/X} \) denote the set of Parshin chains on the pair \((U \subset X)\) and let \( \mathcal{P}^\max_{U/X} \) be the subset of \( \mathcal{P}_{U/X} \) consisting of maximal Parshin chains. If \( P \) is a Parshin chain on
Lemma 2.5 therefore implies the desired result. □

X of dimension \( d_X(P) \), then we shall consider \( K^M_{d_X}(P)(k(P)) \) as a topological abelian group with its canonical topology if \( P \) is maximal (see [15, § 2.5]). Otherwise, we shall consider \( K^M_{d_X}(P)(k(P)) \) as a topological abelian group with its discrete topology. If \( P = (p_0, \ldots, p_d) \) is a maximal Parshin chain on \( X \), we shall let \( P' = (p_0, \ldots, p_{d-1}) \).

Recall from [15, Definition 3.1] that the idele group of \( (U \subset X) \) is defined as

\[
I_{U/X} = \bigoplus_{P \in P_{\text{max}}^{U/X}} K^M_{d_X}(P)(k(P)).
\]

We consider \( I_{U/X} \) as a topological group with its direct sum topology. We let

\[
I(X|D) = \text{Coker}
\left( \bigoplus_{P \in P_{\text{max}}^{U/X}} K^M_{d_X}(P)(\mathcal{O}_{X,P'}^h|I_D) \to I_{U/X} \right),
\]

where \( I_D \) is the extension of the ideal sheaf \( \mathcal{I}_D \subset \mathcal{O}_X \) to \( \mathcal{O}_{X,P'}^h \) and the map on the right is induced by the composition \( \bar{\mathcal{R}}^M_{d_X}(P)(\mathcal{O}_{X,P'}^h|I_D) \to K^M_{d_X}(P)(k(P)) \to I_{U/X} \) for \( P \in P_{\text{max}}^{U/X} \).

We consider \( I(X,D) \) a topological group with its quotient topology. Recall from [15, § 3.1] that \( I(X,D) \) is defined analogous to \( I(X|D) \), where we replace \( \bar{\mathcal{R}}^M_{d_X}(P)(\mathcal{O}_{X,P'}^h|I_D) \) by \( K^M_{d_X}(P)(\mathcal{O}_{X,P'}^h|I_D) \).

We let \( Q_{U/X} \) denote the set of all \( Q \)-chains on \( (U \subset X) \). Recall that the idele class group of the pair \( (U \subset X) \) is the topological abelian group

\[
C_{U/X} = \text{Coker}
\left( \bigoplus_{Q \in Q_{U/X}} K^M_{d_X}(Q)(k(Q)) \to I_{U/X} \right)
\]

with its quotient topology. We let

\[
C(X|D) = \text{Coker}
\left( \bigoplus_{Q \in Q_{U/X}} K^M_{d_X}(Q)(k(Q)) \to I(X|D) \right).
\]

This is a topological group with its quotient topology. This idele class group was introduced by Kerz-Zhao in [30, Definition 2.1.4]. We let \( C(X,D) \) be the cokernel of the map \( \bigoplus_{Q \in Q_{U/X}} K^M_{d_X}(Q)(k(Q)) \to I(X,D) \). One checks using part (2) of Lemma 3.1 that there is a commutative diagram

\[
\begin{array}{ccc}
I_{U/X} & \longrightarrow & I(X,D) \\
\downarrow & & \downarrow \\
C_{U/X} & \longrightarrow & C(X,D)
\end{array}
\]

If \( \dim(X) = d \), we let \( C_{KS}(X|D) = H^d_{\text{nis}}(X, \mathcal{R}^M_{d_X}(X|D)) \) and \( C^{\text{et}}_{KS}(X|D) = H^d_{\text{et}}(X, \bar{\mathcal{R}}^M_{d_X}(X|D)) \).

We let \( C_{KS}(X,D) = H^d_{\text{nis}}(X, K^M_{d_X}(X,D)) \) and \( C^{\text{et}}_{KS}(X,D) = H^d_{\text{et}}(X, K^M_{d_X}(X,D)) \).

**Lemma 3.3.** The canonical map \( C_{KS}(X,D) \to C_{KS}(X|D) \) is surjective. It is an isomorphism if \( D \subset X \) is a reduced Cartier divisor.

**Proof.** It is clear that the stalks of the kernel and cokernel of the map \( K^M_{r_X}(X,D) \to \bar{\mathcal{R}}^M_{r_X}(X|D) \) vanish at the generic point of \( X \). It follows moreover from Lemma 3.1 that these stalks vanish at all points of \( X \) with dimension \( d-1 \) or more if \( D \) is reduced. A variant of Lemma 2.3 therefore implies the desired result. □
Let $Z_0(U)$ be the free abelian group on $U_{(0)}$. It is clear from the definitions of $C_{KS}(X|D)$ and $C(X|D)$ that there is a diagram

$$
\begin{array}{ccc}
Z_0(U) & \rightarrow & C(X|D) \\
\downarrow & & \downarrow \nabla \\
& C_{KS}(X|D),
\end{array}
$$

where the left vertical arrow is induced by the inclusion of the Milnor $K$-groups of the length zero Parshin chains on $(U \subset X)$ into $I_{U/X}$, and the diagonal arrow is induced by the coniveau spectral sequence for $C_{KS}(X|D)$. The following is [30, Theorem 3.1.1] whose proof is obtained by simply repeating the proof of [28, Theorem 8.2] mutatis mutandis (see [15, Theorem 3.8]).

**Theorem 3.4.** Assume that $U$ is regular. Then one has an isomorphism

$$
\phi_{X|D}: C(X|D) \xrightarrow{\cong} C_{KS}(X|D)
$$

such that (3.7) is commutative.

3.1. **Degree map for $C_{KS}(X|D)$**. Let $k$ be any field and $X \in \text{Sch}_k$ an integral scheme of dimension $d \geq 1$. Assume that $D \subset X$ is an effective Cartier divisor. Let $\text{CH}^F(X)$ denote the Chow group of $r$-dimensional cycles on $X$ as in [9, Chapter 1]. It follows from Lemma 3.1 and [23, Proposition 1] that

$$
\begin{array}{ccc}
\overline{K}^M_{r,X|D} & \rightarrow & \bigoplus_{x \in X^{(0)}} (t_x)_*(K^M_r(k(x))) \\
\downarrow & & \downarrow \\
\cdots & \rightarrow & \bigoplus_{x \in X^{(d-1)}} (t_x)_*(K^M_{r-d+1}(k(x))) \xrightarrow{\partial} \bigoplus_{x \in X^{(d)}} (t_x)_*(K^M_{r-d}(k(x))) \rightarrow 0
\end{array}
$$

is a complex of Nisnevich sheaves on $X$. An elementary cohomological argument (see [16, § 2.1]) shows that by taking the cohomology of the sheaves in this complex, one gets a canonical homomorphism

$$
\nu_{X|D}: C_{KS}(X|D) \rightarrow \text{CH}^F_0(X).
$$

If $X$ is projective over $k$, there is a degree map $\deg: \text{CH}^F_0(X) \rightarrow \mathbb{Z}$. We let $\deg: C_{KS}(X|D) \rightarrow \mathbb{Z}$ be the composition $C_{KS}(X|D) \xrightarrow{\nu_{X|D}} \text{CH}^F_0(X) \xrightarrow{\deg} \mathbb{Z}$. We let $C_{KS}(X|D)_0$ be the kernel of the composite map $C_{KS}(X,D) \xrightarrow{\nu_{X|D}} \text{CH}^F(X) \xrightarrow{\deg} \mathbb{Z}$.

4. **Comparison theorem for cohomology of $K$-sheaves**

Let $k$ be a perfect field of characteristic $p > 0$ and $X \in \text{Sch}_k$ an integral regular scheme of dimension $d$. Let $D \subset X$ be an effective Cartier divisor. Let $U = X \setminus D$ and $F = k(X)$. Let $m, r \geq 1$ be integers. A combination of [22, Theorem 1.1.5] and [30] Theorems 2.2.2, 3.3.1 implies the following.

**Theorem 4.1.** Assume that $k$ is finite and $D_{\text{red}}$ is a simple normal crossing divisor. Then the canonical map

$$
H^d_{\text{nis}}(X, \overline{K}^M_{d,X|D}/p^m) \rightarrow H^d_{\text{et}}(X, \overline{K}^M_{d,X|D}/p^m)
$$

is an isomorphism.

**Corollary 4.2.** Under the assumptions of Theorem 4.1, the canonical map

$$
\{H^d_{\text{nis}}(X, \overline{K}^M_{d,(X,nD)}/p^m)\}_n \rightarrow \{H^d_{\text{et}}(X, \overline{K}^M_{d,(X,nD)}/p^m)\}_n
$$

is an isomorphism if either $k \neq \mathbb{F}_2$ or $d \neq 2$. 
Proof. Combine Lemma 2.6, Proposition 3.2 and Theorem 4.1.

Our goal in this section is to prove an analogue of Theorem 4.1 for the cohomology of $K_d^M(X,D)$ when $D_{red}$ is regular. This will be used in the proof of Theorem 1.2.

4.1. Filtrations of Milnor $K$-theory. Let $A$ be a regular local ring essentially of finite type over $k$ and $I = (f)$ a principal ideal, where $f \in A$ is a non-zero element such that $R = A/I$ is regular. Let $A_f$ denote the localization $A[f^{-1}]$ obtained by inverting the powers of $f$. Let $F$ be the quotient field of $A$ so that there are inclusions of rings $A \hookrightarrow A_f \hookrightarrow F$. For an integer $n \geq 1$, let $I^n = (f^n)$.

We let $E_r(A, I^n)$ be the image of the canonical map $\hat{K}_r^M(A, I^n) \to \hat{K}_r^M(A) \in K_r^M(F)$. One easily checks that there is a commutative diagram (see (2.3))

\[
\begin{array}{ccc}
\hat{K}_r^M(A, I^n) / p^n & \xrightarrow{\cong} & \hat{K}_r^M(A) / p^n K_r^M(A) \\
\hat{K}_r^M(A) & \xrightarrow{\cong} & \hat{K}_r^M(A) / p^n K_r^M(A) \\
\hat{K}_r^M(A) / p^n & \xrightarrow{\cong} & \hat{K}_r^M(A) / p^n K_r^M(A),
\end{array}
\]

where the existence of the left-most bottom vertical arrow and the right-most bottom horizontal inclusion follows from [42, Proposition 2.8] (see also [43, Lemma 3.8]).

We consider the filtrations $F_{m,r}^n(A)$ and $G_{m,r}^n(A)$ of $\hat{K}_r^M(A) / p^n$ by setting $F_{m,r}^n(A) = E_r(A, I^n) + p^n \hat{K}_r^M(A)$ and $G_{m,r}^n(A) = \hat{K}_r^M(A) / p^n K_r^M(A)$. We then have a commutative diagram of filtrations

\[
\begin{array}{ccc}
E_r(A, I^n) & \xrightarrow{\cong} & \hat{K}_r^M(A) / p^n \\
F_{m,r}^n(A) & \xrightarrow{\cong} & G_{m,r}^n(A) & \xrightarrow{\cong} & \hat{K}_r^M(A) / p^n.
\end{array}
\]

Lemma 4.3. One has $\hat{K}_r^M(A) / p^{n+1} \subseteq E_r(A, I^n)$ and $G_{m,r}^n(A) \subseteq F_{m,r}^n(A)$. Furthermore, $E_r(A, I) = \hat{K}_r^M(A)$ and $F_{m,r}^1(A) = G_{m,r}^1(A)$.

Proof. This is immediate from [42, Proposition 2.8].

We let $H_{m,r}^n(A) = \hat{K}_r^M(A) / p^n K_r^M(A)$. Note that the canonical map $\hat{K}_r^M(A) \to \hat{K}_r^M(A)$ is surjective. We therefore have a surjective map $G_{m,r}^n(A) \to H_{m,r}^n(A)$.

Lemma 4.4. The canonical map

\[
\frac{G_{m,r}^n(A)}{G_{m,r}^{n+1}(A)} \to \frac{H_{m,r}^n(A)}{H_{m,r}^{n+1}(A)}
\]

is an isomorphism.

Proof. The map in question is clearly surjective since its both sides are quotients of $\hat{K}_r^M(A) / p^n$. So we need to prove only the injectivity. Using Lemma 4.3 and an easy
reduction, one can see that it suffices to show that the map
\[
\overline{K}^M_f(A_f) \rightarrow \overline{K}^M(A)
\]
is injective.

Since \(\overline{K}^M_f(A_f)\) has no \(p\)-torsion, a snake lemma argument shows that the previous injectivity is equivalent to the injectivity of the map \(\overline{K}^M(A)/p^m \rightarrow \overline{K}^M_f(A_f)/p^m\). We now look at the composition \(\overline{K}^M(A)/p^m \rightarrow \overline{K}^M_f(A_f)/p^m \rightarrow K^M_f(F)/p^m\). It suffices to show that this composition is injective. But this is an easy consequence of [27, Proposition 10(8)] and [11, Theorem 8.1].

We now consider the map
\[
(4.3) \quad \rho : \Omega_{R}^{-1} \oplus \Omega_{R}^{-2} \rightarrow \overline{R}^M_f(A|I^n)/\overline{K}^M_f(A|I^{n+1})
\]
having the property that
\[
\rho(ad\log b_1 \wedge \cdots \wedge d\log b_{r-1}, 0) = \{1 + \overline{a}f^n, \overline{b}_1, \ldots, \overline{b}_{r-1}\};
\]
\[
\rho(0, ad\log b_1 \wedge \cdots \wedge d\log b_{r-2}) = \{1 + \overline{a}f^n, \overline{b}_1, \ldots, \overline{b}_{r-2}, f\},
\]
where \(\overline{a}\) and \(\overline{b}_i\)’s are arbitrary lifts of \(a\) and \(b_i\)’s, respectively, under the surjection \(A \rightarrow R\).

It follows from [5, §4, p. 122] that this map is well defined. It is clear from (4.2) and Lemma 4.3 that \(\rho\) restricts to a map
\[
(4.4) \quad \rho : \Omega_{R}^{-1} \rightarrow E_f(A, I^n)/\overline{K}^M_f(A|I^{n+1}).
\]

This map is surjective by Lemma 2.1.

Let \(\psi : R \rightarrow R\) be the absolute Frobenius morphism. Let \(Z_i\Omega_{R}^{-1}\) be the unique \(R\)-submodule of \(\psi^i \Omega_{R}^{-1}\) such that the inverse Cartier operator (see [20, Chap. 0, § 2]) induces an \(R\)-linear isomorphism \(C^{-1} : Z_i\Omega_{R}^{-1} \cong Z_i\Omega_{R}^{-1}/d\Omega_{R}^{-2}\), where we let \(Z_1\Omega_{R}^{-1} = \text{Ker}(\Omega_{R}^{-1} \rightarrow \Omega_{R}^0)\). We let \(B_i\Omega_{R}^{-1} = d\Omega_{R}^{-2}\) and let \(B_i\Omega_{R}^{-1} (i \geq 2)\) be the unique \(R\)-submodule of \(\psi^i \Omega_{R}^{-1}\) such that \(C^{-1} : B_{i-1}\Omega_{R}^{-1} \rightarrow B_i\Omega_{R}^{-1}/d\Omega_{R}^{-2}\) is an isomorphism of \(R\)-modules.

**Lemma 4.5.** Let \(n = n_1 p^s\), where \(s \geq 0\) and \(p \nmid n_1\). Then \(\rho\) induces isomorphisms
\[
\rho_{m,r}^n : \frac{\psi^m \Omega_{R}^{-1}}{Z_m \Omega_{R}^{-1}} \cong \frac{F_{m,r}^n(A)}{G_{m,r}^{n+1}(A)} \quad \text{if } m \leq s;
\]
\[
\rho_{m,r}^n : \frac{\psi^m \Omega_{R}^{-1}}{B_s \Omega_{R}^{-1}} \cong \frac{F_{m,r}^n(A)}{G_{m,r}^{n+1}(A)} \quad \text{if } m > s.
\]

**Proof.** Once it exists, \(\rho_{m,r}^n\) is clearly surjective in both cases. We therefore need to prove its existence and injectivity. These are easy consequences of various lemmas in [5, §4] (see also [22, Proposition 1.1.9]) once we have Lemma 4.3. The proof goes as follows.

When \(m \leq s\), we look at the diagram
\[
\begin{array}{ccc}
\psi^m \Omega_{R}^{-1} \oplus \psi^m \Omega_{R}^{-2} & \xrightarrow{\rho_{m,r}^n} & \frac{F_{m,r}^n(A)}{G_{m,r}^{n+1}(A)} \\
\downarrow & & \downarrow \\
\frac{\psi^m \Omega_{R}^{-1}}{Z_m \Omega_{R}^{-1}} & \xrightarrow{\rho_{m,r}^n} & \frac{F_{m,r}^n(A)}{G_{m,r}^{n+1}(A)} \\
\end{array}
\]

We now consider the map
\[
(4.3) \quad \rho : \Omega_{R}^{-1} \oplus \Omega_{R}^{-2} \rightarrow \overline{R}^M_f(A|I^n)/\overline{K}^M_f(A|I^{n+1})
\]

where the bottom right isomorphism is by Lemma 4.4. The right vertical arrow is clearly injective. The composite arrow on the bottom is an isomorphism by [5, (4.7), Remark 4.8]. It follows that the top horizontal arrow exists and it is injective.

When \( m > s \), we look at the commutative diagram

\[
\begin{array}{ccc}
\psi^1 \Omega_r^{-1} & \rightarrow & F_{m,r}(A) \\
B \subset H_R & \oplus & B \subset H_R \\
\psi^1 \Omega_r^{-2} & \rightarrow & \mathrm{Coker}(\theta) \\
& \rightarrow & \frac{G^n_{m,r}(A)}{G^{n+1}_{m,r}(A)} \rightarrow \frac{H^n_{m,r}(A)}{H^{n+1}_{m,r}(A)},
\end{array}
\]

where \( \theta \) is defined in [5, Lemma 4.5]. The map \( \rho \) on the bottom is again an isomorphism. Hence, it suffices to show that the left vertical arrow is injective. But this is immediate from the definition of \( \theta \) (see loc. cit.) since \( C^{-s} \) is an isomorphism.

\[ \square \]

### 4.2. The comparison theorem

Let \( k \) be a perfect field of characteristic \( p > 0 \). Let \( X \in \mathbf{Sch}_k \) be a regular scheme of pure dimension \( d \geq 1 \) and let \( D \subset X \) be a regular closed subscheme of codimension one. Let \( \mathcal{F}^n_{m,r,X} \) be the sheaf on \( X_{\text{nis}} \) such that for any point \( x \in X \), the stalk \( (\mathcal{F}^n_{m,r,X})_x \) coincides with \( F^n_{m,r}(\mathcal{O}^h_{X,x}) \), where \( I = I_D \mathcal{O}^h_{X,x} \). The corresponding étale sheaf is defined similarly by replacing the Henselization \( \mathcal{O}^h_{X,x} \) by the strict Henselization \( \mathcal{O}^h_{X,x} \). We define the Nisnevich and étale sheaves \( \mathcal{G}^n_{m,r,X} \) in an identical way, using the groups \( G^n_{m,r}(A) \) for stalks.

**Lemma 4.6.** The change of topology map

\[
H^i_{\text{nis}}(X, \mathcal{F}^1_{m,r,X}/\mathcal{F}^n_{m,r,X}) \to H^i_{\text{ét}}(X, \mathcal{F}^1_{m,r,X}/\mathcal{F}^n_{m,r,X})
\]

is an isomorphism for every \( i \geq 0 \).

**Proof.** We consider the exact sequence of Nisnevich (or étale) sheaves

\[
0 \to \mathcal{F}^n_{m,r,X} \mathcal{G}^1_{m,r,X} \mathcal{G}^{n+1}_{m,r,X} \mathcal{F}^1_{m,r,X} \mathcal{F}^n_{m,r,X} \to 0,
\]

where we have replaced \( \mathcal{G}^1_{m,r,X} \) by \( \mathcal{F}^1_{m,r,X} \) in the quotient on the right using Lemma 4.3. Working inductively on \( n \), it follows from [22, Proposition 1.1.9] that the middle term in \( \mathcal{F} \) has identical Nisnevich and étale cohomology. It suffices therefore to show that the same holds for the left term in \( \mathcal{F} \) too.

Since this isomorphism is obvious for \( i = 0 \), it suffices to show using the Leray spectral sequence that \( R^i \epsilon_* (\mathcal{F}^n_{m,r,X}/\mathcal{G}^{n+1}_{m,r,X}) = 0 \) for \( i > 0 \). Equivalently, we need to show that if \( A = \mathcal{O}^h_{X,x} \) for some \( x \in X \) and \( f \in A \) defines \( D \) locally at \( x \), then \( H^i_{\text{ét}}(A, F^n_{m,r}(A)/G^{n+1}_{m,r}(A)) = 0 \) for \( i > 0 \). But this is immediate from Lemma 4.5. \( \square \)

**Lemma 4.7.** Assume that \( k \) is finite. Then the change of topology map

\[
H^d_{\text{nis}}(X, \mathcal{F}^n_{m,d,X}) \to H^d_{\text{ét}}(X, \mathcal{F}^n_{m,d,X})
\]

is an isomorphism.

**Proof.** We first assume \( n = 1 \). Using Lemma 4.3, we can replace \( \mathcal{F}^1_{m,d,X} \) by \( \mathcal{G}^1_{m,d,X} \). By [22, Theorem 1.1.5] and [30, Theorems 2.2.2, Proposition 2.2.5], the dlog map \( \mathcal{G}^1_{m,d,X} \to W_m \mathcal{O}^d_{X,\log} \) is an isomorphism in Nisnevich and étale topologies. We conclude by [30, Theorem 3.3.1].
We now assume $n \geq 2$ and look at the exact sequence

$$0 \to F_{m,d,X}^n \to F_{m,d,X}^1 \to F_{m,d,X}^1 \to 0.$$ 

We saw above that the middle term can be replaced by $W_m^\Omega_{X,\log}^d X$. We can now conclude the proof by Lemmas 4.6 and the independent statement Lemma 5.6 via the 5-lemma. □

**Lemma 4.8.** Assume that $k$ is finite. Then the change of topology map

$$H^d_{\text{nis}}(X, \overline{K}_d^{M,(X,nD)/p^m}) \to H^d_{\text{ét}}(X, \overline{K}_d^{M,(X,nD)/p^m})$$

is an isomorphism if either $d \neq 2$ or $k \neq \mathbb{F}_2$.

**Proof.** In view of Lemma 4.7, it suffices to show that the canonical map (see (4.1))

$$H^d_{\tau}(X, \overline{K}_d^{M,(X,nD)/p^m}) \to H^d_{\text{ét}}(X, F_{m,d,X})$$

is an isomorphism for $\tau \in \{\text{nis, ét}\}$. Using Lemma 2.5 and (4.1), the proof of this is completely identical to that of Lemma 2.6. □

The following is the main result we were after in this section. We restate all assumptions for convenience.

**Theorem 4.9.** Let $k$ be a finite field of characteristic $p$. Let $X \in \text{Sch}_k$ be a regular scheme of pure dimension $d \geq 1$ and let $D \subset X$ be a regular closed subscheme of codimension one. Assume that either $d \neq 2$ or $k \neq \mathbb{F}_2$. Then there is a natural isomorphism

$$H^d_{\text{nis}}(X, \overline{K}_d^{M,(X,nD)/p^m}) \cong H^d_{\text{ét}}(X, \overline{K}_d^{M,(X,nD)/p^m})$$

for every $m,n \geq 1$.

**Proof.** Combine Lemmas 2.3, 2.4, 2.6 and 4.8. □

**Corollary 4.10.** Under the assumptions of Theorem 4.9, there is a commutative diagram

$$H^d_{\text{nis}}(X, \overline{K}_d^{M,(X,nD)/p^m}) \to H^d_{\text{ét}}(X, \overline{K}_d^{M,(X,nD)/p^m})$$

for $m,n \geq 1$ in which the horizontal arrows are isomorphisms.

**Proof.** In view of Theorems 4.1 and 4.9, we only have to explain the vertical arrows. But their existence follows from Lemmas 2.3, 2.4, 2.6 and the diagram (4.1). □

**Remark 4.11.** The reader may note that Theorem 4.1 holds whenever $D_{\text{red}}$ is a simple normal crossing divisor, but this is not the case with Theorem 4.9. The main reason for this is that while $\overline{K}_d^{M,(X,nD)}$ satisfies cdh-descent for closed covers [22], the same is not true for $\overline{K}_d^{M,(X,D)}$. This prevents us from using an induction argument on the number of irreducible components of $D$.

5. **Relative logarithmic Hodge-Witt sheaves**

In this section, we shall recall the relative logarithmic Hodge-Witt sheaves and show that they coincide with the Kato-Saito relative Milnor $K$-theory with $\mathbb{Z}/p^m$ coefficients in a pro-setting. We fix a field $k$ of characteristic $p > 0$ and let $X \in \text{Sch}_k$. 
5.1. The relative Hodge-Witt sheaves. Let \( \{ W_m\Omega_X^i \}_{m \geq 1} \) denote the pro-complex of de Rham-Witt (Nisnevich) sheaves on \( X \). This is a pro-complex of differential graded algebras with the structure map \( R \) and the differential \( d \). Let \( [-]: \mathcal{O}_X \to W_m\mathcal{O}_X \) be the multiplicative Teichmüller homomorphism. Recall that the pro-complex \( \{ W_m\Omega_X^i \}_{m \geq 1} \) is equipped with the Frobenius homomorphism of graded algebras \( F: W_m\Omega_X^r \to W_{m-1}\Omega_X^r \) and the additive Verschiebung homomorphism \( V: W_m\Omega_X^r \to W_{m+1}\Omega_X^r \). These homomorphisms satisfy the following properties which we shall use frequently.

1. \( FV = p = VF \).
2. \( FdV = d, dF = pFd \) and \( pdV = Vd \).
3. \( F(d[a]) = [a^{p-1}]d[a] \) for all \( a \in \mathcal{O}_X \).
4. \( V(F(x)y) = xV(y) \) and \( V(xy) = VxVy \) for all \( x \in W_m\mathcal{O}_X^i, y \in W_m\mathcal{O}_X^j \).

Let \( \text{fil}^i W_m\Omega_X^i = V^i W_m\Omega_X^i + d V^{i-1} W_m\Omega_X^{i-1} \), \( \text{fil}^i W_m\Omega_X^i = \text{Ker}(R^{i-1}) \) and \( \text{fil}^0 W_m\Omega_X^i = \text{Ker}(R^i) \) for \( 0 \leq i \leq r \). By [19, Lemma 3.2.4] and [20, Proposition I.3.4], we know that \( \text{fil}^i W_m\Omega_X^i = \text{fil}^i W_m\mathcal{O}_X^i \), \( \text{fil}^{i+1} W_m\mathcal{O}_X^i \subseteq \text{fil}^i W_m\mathcal{O}_X^i \) and the last inclusion is an equality if \( X \) is regular.

Let \( ZW_m\Omega_X^i = \text{Ker}(d: W_m\Omega_X^i \to W_m\Omega_X^{i+1}) \) and \( BW_m\Omega_X^i = \text{Image}(d: W_m\Omega_X^{i-1} \to W_m\Omega_X^i) \). We set \( H^n(W_m\Omega_X^i) = ZW_m\Omega_X^i/BW_m\Omega_X^i \).

Let \( \psi: X \to X \) denote the absolute Frobenius morphism. One then knows that \( d: \psi_* W_1\Omega_X^i \to \psi_* W_1\Omega_X^{i+1} \) is \( \mathcal{O}_X \)-linear. In particular, the sheaves \( \psi_* ZW_1\Omega_X^i, \psi_* BW_1\Omega_X^i \) and \( \psi_* H^n(W_1\Omega_X^i) \) are quasi-coherent on \( X_{\text{nis}} \). If \( X \) is \( F \)-finite, then \( \psi \) is a finite morphism, and therefore, \( \psi_* ZW_1\Omega_X^i, \psi_* BW_1\Omega_X^i \) and \( \psi_* H^n(W_1\Omega_X^i) \) are coherent sheaves on \( X_{\text{nis}} \).

Let \( D \subset X \) be a closed subscheme defined by a sheaf of ideals \( I_D \) and let \( nD \subset X \) be the closed subscheme defined by \( I_D^n \) for \( n \geq 1 \). We let \( W_m\Omega_X^i(X, D) = \text{Ker}(W_m\Omega_X^i \to \psi_* W_m\Omega_X^i(D)) \), where \( \psi: X \to X \) is the inclusion. We shall often use the notation \( W_m\Omega_X^i(A, I) \) if \( X = \text{Spec}(A) \) and \( D = \text{Spec}(A/I) \). If \( D \) is an effective Cartier divisor on \( X \) and \( n \in \mathbb{Z} \), we let \( W_m\mathcal{O}_X(nD) = \{ f^{-n} \cdot W_m\mathcal{O}_X \subset j_* \mathcal{O}_U, \text{where } f \in \mathcal{O}_X \text{ is a local equation of } D \text{ and } j: U \to X \} \) is the inclusion of the complement of \( D \) in \( X \). One knows that \( W_m\mathcal{O}_X(nD) \) is a sheaf of invertible \( W_m\mathcal{O}_X \)-modules. We let \( W_m\Omega_X^i(nD) = W_m\Omega_X^i \otimes_{W_m\mathcal{O}_X} W_m\mathcal{O}_X(nD) \). It is clear that there is a canonical \( W_m\mathcal{O}_X \)-linear map \( W_m\Omega_X^i(-D) \to W_m\Omega_X^i(X, D) \). The reader is invited to compare the following with a weaker statement [10, Proposition 2.5].

**Lemma 5.1.** If \( (X, D) \) is a modulus pair and \( m \geq 1 \) an integer, then the canonical map of pro-sheaves

\[
\{ W_m\Omega_X^i(-nD) \}_n \to \{ W_m\Omega_X^i(X, nD) \}_n
\]

on \( X_{\text{nis}} \) is surjective. If \( X \) is furthermore regular, then the map \( W_m\Omega_X^i(-D) \to W_m\Omega_X^i(X, D) \) is injective.

**Proof.** The first part follows directly from Lemma 5.2 below. We prove the second part. Let \( U = X \setminus D \) and \( j: U \to X \) the inclusion. Using the Néron-Popescu approximation and the Gersten resolution of \( W_m\Omega_X^i \) (when \( X \) is smooth), proven in [12, Proposition II.5.1.2], it follows that the canonical map \( W_m\Omega_X^i \to j_* W_m\mathcal{O}_U^i \) is injective (see [33, Proposition 2.8]). Since \( W_m\mathcal{O}_X(-D) \) is invertible as a \( W_m\mathcal{O}_X \)-module, there are natural isomorphisms \( j_* W_m\Omega_U^i \otimes_{W_m\mathcal{O}_X} W_m\mathcal{O}_X(-D) \to j_* (W_m\Omega_U^i \otimes_{W_m\mathcal{O}_X} j^* W_m\mathcal{O}_X(-D)) \cong j_* W_m\Omega_U^i \) (see [17, Exercise II.5.1.4]). Using the invertibility of \( W_m\mathcal{O}_X(-D) \) again, we conclude that the canonical map \( W_m\Omega_X^i(-D) \to j_* W_m\Omega_U^i \) is injective. Since this map factors through \( W_m\Omega_X^i(-D) \to W_m\Omega_X^i(X, D) \to W_m\Omega_X^i(X) \), the lemma follows.

**Lemma 5.2.** Let \( A \) be a commutative \( \mathbb{F}_p \)-algebra and \( I = (f) \subseteq A \) a principal ideal. Let \( m \geq 1 \) be any integer and \( n = p^m \). Then the inclusion \( W_m\Omega_X^i(A, I^n) \subseteq W_m\Omega_X^i(A) \) of \( W_m(A) \)-modules factors through \( W_m\Omega_X^i(A, I^n) \subseteq (f) \cdot W_m\Omega_X^i(A) \) for every \( r \geq 0 \).
Corollary 5.3. If \( \omega = a_0 \otimes \cdots \otimes a_r \in W_m \Omega^n_X(A, I^n) \) is the \( W_m \Omega^n_X(A, I^n) \)-submodule of \( W_m \Omega^n_X(A, I^n) \) generated by the de Rham-Witt forms of the type \( a_0 da_1 \wedge \cdots \wedge da_r \), where \( a_i \in W_m \Omega^n_X(A, I^n) \) for all \( i \) and \( a_i \in W_m \Omega^n_X(A, I^n) \) for some \( i \). We let \( \omega_i = a_0 da_1 \wedge \cdots \wedge da_r \in W_m \Omega^n_X(A, I^n) \) such that \( a_i \in W_m \Omega^n_X(A, I^n) \) for some \( i \neq 0 \). We can assume after a permutation that \( i = 1 \). We let \( \omega' = da_2 \wedge \cdots \wedge da_r \). We can write \( \omega_i = \sum_{i=0}^{m-1} V^i([f^n]_{m-i}[a_i']_{m-i}) \) for some \( a_i' \in \Lambda \). It follows that \( \omega = \sum_{i=0}^{m-1} a_i dV^i([f^n]_{m-i}[a_i']_{m-i}) \wedge \omega' \).

We fix an integer \( 0 \leq i \leq m-1 \) and let \( \omega_i = a_0 dV^i([f^n]_{m-i}[a_i']_{m-i}) \wedge \omega' \). It suffices to show that each \( \omega_i \in W_m \Omega^n_X(\Lambda) \), where \( X = \text{Spec} (A) \) and \( \Lambda = \text{Spec} (A/I) \). However, we have

\[
\omega_i = a_0 dV^i([f^n]_{m-i}[a_i']_{m-i}) \wedge \omega' = a_0 dV^i([f^n]_{m-i}[a_i']_{m-i}) \wedge \omega' = [f^n]_{m-i} a_0 dV^i(a_i') - p^{m-i}[f^n]_{m-i} a_0 dV^i(a_i') \wedge \omega'.
\]

It is clear that the last term lies in \( W_m \Omega^n_X(\Lambda) \). This finishes the proof. \( \square \)

The following result will play a key role in our exposition.

**Corollary 5.3.** If \( (X, D) \) is a regular modulus pair, then the canonical map of pro-sheaves

\[
\{ W_m \Omega^n_X(-n\Lambda) \}_n \to \{ W_m \Omega^n_{(X, n\Lambda)} \}_n
\]
on \( \Lambda \) is an isomorphism for every \( m \geq 1 \).

### 5.2. The relative logarithmic sheaves

In the remaining part of §5 our default topology will be the étale topology. All other topologies will be mentioned specifically. Let \( k \) be a field of characteristic \( p > 0 \) and \( \Lambda \subseteq X \) a closed immersion in \( \text{Sch}_k \). Recall from \cite{20} that \( W_m \Omega^n_{X, \log} \) is the étale subsheaf of \( W_m \Omega^n_X \) which is the image of the map \( d\log: \overline{F}_{r, X}^M/p^m \to W_m \Omega^n_X \), given by \( d\log([a_1, \ldots, a_r]) = d\log([a_1]_m) \wedge \cdots \wedge d\log([a_r]_m) \). It is easily seen that this map is always surjective. One knows (e.g., see \cite{36} Remark 1.6) that this map in fact factors through \( d\log: \overline{F}_{r, X}^M/p^m \to W_m \Omega^n_{X, \log} \). Moreover, this map is multiplicative. The naturality of the \( d\log \) map gives rise to its relative version

\[
d\log: \overline{F}_{r, (X, D)}^M/p^m \to W_m \Omega^n_{(X, D), \log} := \text{Ker}(W_m \Omega^n_{X, \log} \to W_m \Omega^n_{D, \log})
\]

which is a morphism of étale sheaves of \( \overline{F}_{r, X}^M/p^m \)-modules as \( r \geq 0 \) varies.

Recall that the Frobenius map \( F: W_m \Omega^n_X \to W_m \Omega^n_X \) sends \( \text{Ker} (R) \) into the subsheaf \( dV^{m-1} \Omega^{r-1}_X \) so that there is an induced map \( F: W_m \Omega^n_X \to W_m \Omega^n_X/dV^{m-1} \Omega^{r-1}_X \). We let \( \pi: W_m \Omega^n_X \to W_m \Omega^n_X / dV^{m-1} \Omega^{r-1}_X \) denote the projection map. Since \( R - F: W_m \Omega^n_X \to W_m \Omega^n_X \) is surjective by \cite{20} Proposition I.3.26, it follows that \( \pi - F \) is also surjective. Indeed, this surjectivity is proven for smooth schemes over \( k \) in loc. cit., but then the claim follows because \( X \) can be seen locally as a closed subscheme of a regular scheme and a regular scheme is a filtered inductive limit of smooth schemes over \( k \), by Néron–Popescu desingularisation. We thus get a commutative diagram

\[
0 \to W_m \Omega^n_{X, \log} \to W_m \Omega^n_X \xrightarrow{\pi - F} W_m \Omega^n_X / dV^{m-1} \Omega^{r-1}_X \to 0
\]

\[
0 \to W_m \Omega^n_{D, \log} \to W_m \Omega^n_D \xrightarrow{\pi - F} W_m \Omega^n_D / dV^{m-1} \Omega^{r-1}_D \to 0.
\]

The middle and the right vertical arrows are surjective by definition and the two rows are exact by loc. cit. and \cite{40} Corollary 4.2. The surjectivity of the left vertical arrow
follows by applying the dlog map to the surjection $\widehat{K}^M_{r,X} \to \widehat{K}^M_{r,D}$. Taking the kernels of the vertical arrows, we get a short exact sequence

\begin{equation}
0 \to W_m\Omega^r_{(X,D),\log} \to W_m\Omega^r_{(X,D)} \to \frac{W_m\Omega^r_{(X,D)}}{W_m\Omega^r_{(X,D)} \cap dV^m\Omega^{-1}_X} \to 0.
\end{equation}

The following property of the relative dlog map will be important to us.

**Lemma 5.4.** Assume that $k$ is perfect and $X$ is regular. Then the map of étale pro-sheaves

\[ d\log: \{\widehat{K}^M_{r,(X,nD)}/p^m\}_n \to \{W_m\Omega^r_{(X,nD),\log}\}_n \]

is an isomorphism for every $m, r \geq 1$.

**Proof.** We consider the commutative diagram of pro-sheaves:

\begin{equation}
0 \to \{\widehat{K}^M_{r,(X,nD)}/p^m\}_n \to \widehat{K}^M_{r,X}/p^m \to \{\widehat{K}^M_{r,nD}/p^m\}_n \to 0
\end{equation}

The middle vertical arrow is an isomorphism by the Bloch-Gabber-Kato theorem for fields [5] Corollary 2.8 and the proof of the Gersten conjecture for Milnor $K$-theory (27 Proposition 10) and logarithmic Hodge-Witt sheaves (13 Théorème 1.4). The right vertical arrow is an isomorphism by [36] Theorem 0.3. Since the rows are exact, it follows that the left vertical arrow is also an isomorphism. This finishes the proof. \qed

We now prove a Nisnevich version of Lemma 5.4. For any $k$-scheme $X$, we let $W_m\Omega^r_{X,\log, nis}$ be the image of the map of Nisnevich sheaves $d\log: \widehat{K}^M_{r,X}/p^m \to W_m\Omega^r_{X}$, given by $d\log([a_1, \ldots, a_r]) = d\log[a_1]_m \wedge \ldots \wedge d\log[a_r]_m$ (see [36] Remark 1.6) for the existence of this map. The naturality of this map gives rise to its relative version

\begin{equation}
\text{dlog: } \{\widehat{K}^M_{r,(X,nD)}/p^m\}_n \to W_m\Omega^r_{(X,nD),\log, nis} := \text{Ker}(W_m\Omega^r_{X,\log, nis} \to W_m\Omega^r_{D,\log, nis})
\end{equation}

if $D \subseteq X$ is a closed immersion.

**Lemma 5.5.** Assume that $k$ is perfect and $X$ is regular. Then the map of Nisnevich pro-sheaves

\[ d\log: \{\widehat{K}^M_{r,(X,nD)}/p^m\}_n \to \{W_m\Omega^r_{(X,nD),\log, nis}\}_n \]

is an isomorphism for every $m, r \geq 1$.

**Proof.** The proof is completely identical to that of Lemma 5.4, where we only have to observe that the middle and the right vertical arrows are isomorphisms even in the Nisnevich topology, by the same references that we used for the étale case. \qed

**Lemma 5.6.** Assume that $k$ is a finite field and $X$ is regular of pure dimension $d \geq 1$. Then the canonical map

\[ H^i_{\text{nis}}(X, \widehat{K}^M_{d,X}/p^m) \to H^i_{\text{ét}}(X, \widehat{K}^M_{d,X}/p^m) \]

is an isomorphism for $i = d$ and surjective for $i = d - 1$.

**Proof.** We have seen in the proofs of Lemmas 5.4 and 5.5 that for every $r \geq 0$, there are natural isomorphisms

\begin{equation}
(\widehat{K}^M_{r,X}/p^m)_{\text{nis}} \cong W_m\Omega^r_{X,\log, nis} \quad \text{and} \quad (\widehat{K}^M_{r,X}/p^m)_{\text{ét}} \cong W_m\Omega^r_{X,\log}.
\end{equation}

Using these isomorphisms, the lemma follows from [30] Propositions 3.3.2, 3.3.3. The only additional input one has to use is that $H^0_{\text{ét}}(X, W_m\Omega^d_{X,\log}) = 0$ if $x \in X^{(a)}$ and $b < a$. 

Lemma 5.7. There is a short exact sequence

\[ 0 \to W_{m-1}^{r}\Omega^r_{X,\log} \xrightarrow{p^i} W_m^{r}\Omega^r_{X,\log} \xrightarrow{R^{m-i}} W_i^{r}\Omega^r_{X,\log} \to 0 \]

of sheaves on \( X \) in Nisnevich and étale topologies.

Proof. The étale version of the lemma is already known by [6, Lemma 3]. To prove its Nisnevich version, we can use the first isomorphism of (5.6). This reduces the proof to showing that tensoring the exact sequence

\[ 0 \to \mathbb{Z}/p^{m-i} \xrightarrow{p^i} \mathbb{Z}/p^m \xrightarrow{\pi_{m-i}} \mathbb{Z}/p^i \to 0 \]

with \( \mathcal{R}^M_{r,X} \) yields an exact sequence

\[ (5.7) \quad 0 \to \mathcal{R}^M_{r,X}/p^{m-i} \xrightarrow{p^i} \mathcal{R}^M_{r,X}/p^m \xrightarrow{\pi_{m-i}} \mathcal{R}^M_{r,X}/p^i \to 0 \]

of Nisnevich sheaves on \( X \). But this follows directly from the fact that \( \mathcal{R}^M_{r,X} \) has no \( p^i \)-torsion (see [11, Theorem 8.1]).

Lemma 5.8. There is a short exact sequence

\[ 0 \to \{ W_{m-i}^{r}\Omega^r_{nD,\log} \}_n \xrightarrow{p^i} \{ W_m^{r}\Omega^r_{nD,\log} \}_n \xrightarrow{R^{m-i}} \{ W_i^{r}\Omega^r_{nD,\log} \}_n \to 0 \]

of pro-sheaves on \( D \) in Nisnevich and étale topologies.

Proof. The argument below works for either of Nisnevich and étale topologies. We shall prove the lemma by modifying the proof of [10, Theorem 4.6]. The latter result says that there is an exact sequence

\[ (5.8) \quad \{ W_m^{r}\Omega^r_{nD,\log} \}_n \xrightarrow{p^i} \{ W_m^{r}\Omega^r_{nD,\log} \}_n \xrightarrow{R^{m-i}} \{ W_i^{r}\Omega^r_{nD,\log} \}_n \to 0. \]

It suffices to show that the first arrow in this sequence has a factorization

\[ (5.9) \quad \{ W_m^{r}\Omega^r_{nD,\log} \}_n \xrightarrow{R^i} \{ W_{m-i}^{r}\Omega^r_{nD,\log} \}_n \xrightarrow{p^i} \{ W_m^{r}\Omega^r_{nD,\log} \}_n. \]

To prove this factorization, we look at the commutative diagram

\[ (5.10) \quad \begin{array}{ccc}
\{ W_m^{r}\Omega^r_{nD,\log} \}_n & \xrightarrow{p^i} & \{ W_m^{r}\Omega^r_{nD,\log} \}_n \\
\{ W_m^{r}\Omega^r_{nD} \}_n & \xrightarrow{R^i} & \{ W_{m-i}^{r}\Omega^r_{nD,\log} \}_n \\
\{ W_m^{r}\Omega^r_{nD} \}_n & \xrightarrow{p^i} & \{ W_m^{r}\Omega^r_{nD} \}_n.
\end{array} \]

It follows from [10, Proposition 2.14] that the map \( p^i \) at the bottom has a factorization (see § 5.1 for the definitions of various filtrations of \( W_m^{r}\Omega^r_{nD} \))

\[ (5.11) \quad \{ W_m^{r}\Omega^r_{nD} \}_n \xrightarrow{R^i} \{ W_{m-i}^{r}\Omega^r_{nD,\log} \}_n \xrightarrow{p^i} \{ W_m^{r}\Omega^r_{nD} \}_n. \]

Since the image of \( \{ W_m^{r}\Omega^r_{nD,\log} \}_n \) under \( R^i \) is \( \{ W_{m-i}^{r}\Omega^r_{nD,\log} \}_n \) (note the surjectivity of the second arrow in (5.8)), it follows at once from (5.11) that the map \( p^i \) on the top in (5.10) has a factorization as desired in (5.9). This concludes the proof. \( \square \)
Proposition 5.9. There is a short exact sequence
\[ 0 \to \{W_{m-i}\Omega^r_{(X,nD),\log}\}_n \xrightarrow{p^i} \{W_m\Omega^r_{(X,nD),\log}\}_n \xrightarrow{R^{m-i}} \{W_i\Omega^r_{(X,nD),\log}\}_n \to 0 \]
of pro-sheaves on \(X\) in Nisnevich and étale topologies.

Proof. Combine the previous two lemmas and use that the maps \(W_m\Omega^r_{X,\log} \to W_m\Omega^r_{nD,\log}\) and \(W_m\Omega^r_{X,nis} \to W_m\Omega^r_{nD,nis}\) are surjective.

Applying the cohomology functor, we get

Corollary 5.10. There is a long exact sequence of pro-abelian groups
\[ \cdots \to H^j_{et}(X,W_{m-i}\Omega^r_{(X,nD),\log}) \xrightarrow{p^i} H^j_{et}(X,W_m\Omega^r_{(X,nD),\log}) \xrightarrow{R^{m-i}} H^j_{et}(X,W_i\Omega^r_{(X,nD),\log}) \to \cdots \]
The same also holds in the Nisnevich topology.

5.3. Some cohomology exact sequences. Let us now assume that \(k\) is a finite field and \(X \in \textbf{Sch}_k\) is regular of pure dimension \(d \geq 1\). For any \(p^m\)-torsion abelian group \(V\), we let \(V^* = \text{Hom}_{Z/p^m}(V, Z/p^m)\). Let \(D \subset X\) be an effective Cartier divisor with complement \(U\). We let \(F^j_{m,r}(n) = H^j_{et}(X,W_m\Omega^r_{(X,nD),\log})\) and \(F^j_{m,r}(U) = \lim_n F^j_{m,r}(n)\). Each group \((F^j_{m,r}(n))^*\) is a profinite abelian group (see [41] Theorem 2.9.6).

Lemma 5.11. The sequence
\[ \cdots \to F^j_{m-1,r}(U) \to F^j_{m,r}(U) \to F^j_{1,r}(U) \to F^j_{m-1,r}(U) \to \cdots \]
is exact.

Proof. We first prove by induction on \(m\) that \(\lim_n F^j_{m,r}(n) = 0\). The \(m = 1\) case follows from Lemma 8.3. In general, we break the long exact sequence of Corollary 5.10 into short exact sequences. This yields exact sequences of pro-abelian groups
\[ (5.12) \quad 0 \to \text{Image}(\partial^1) \to \{F^j_{m-1,r}(n)\}_n \to \text{Ker}(R^{m-1}) \to 0; \]
\[ (5.13) \quad 0 \to \text{Ker}(R^{m-1}) \to \{F^j_{m,r}(n)\}_n \to \text{Image}(R^{m-1}) \to 0; \]
\[ (5.14) \quad 0 \to \text{Image}(R^{m-1}) \to \{F^j_{1,r}(n)\}_n \to \text{Image}(\partial^1) \to 0. \]

Lemma 8.4 says that \(\{F^j_{1,r}(n)\}_n\) is a pro-system of finite abelian groups. This implies by (5.14) that each of \(\text{Image}(R^{m-1})\) and \(\text{Image}(\partial^1)\) is isomorphic to a pro-system of finite abelian groups. In particular, the pro-systems of (5.14) do not admit higher derived lim functors. On the other hand, \(\lim_n F^j_{m-1,r}(n) = 0\) by induction. It follows from (5.12) that \(\lim_n \text{Ker}(R^{m-1}) = 0\). Using (5.13), we get \(\lim_n F^j_{m,r}(n) = 0\).

It follows from what we have shown is that the above three short exact sequences of pro-abelian groups remain exact after we apply the inverse limit functor. But this implies that the long exact sequence of Corollary 5.10 also remains exact after applying the inverse limit functor. This proves the lemma.

Lemma 5.12. There is a long exact sequence
\[ \cdots \to \lim_n (F^j_{i,r}(n))^* \xrightarrow{(R^{m-i})^*} \lim_n (F^j_{m,r}(n))^* \xrightarrow{(p^i)^*} \lim_n (F^j_{m-i,r}(n))^* \to \cdots \]
of abelian groups for every \(i \geq 0\).
Proof. This is an easy consequence of Corollary 5.10 using the fact that the Pontryagin dual functor (recalled in §7.4) is exact on the category of discrete torsion abelian groups (see [11, Theorem 2.9.6]) and the direct limit functor is exact on the category of ind-abelian groups. We also have to note that the Pontryagin dual of a discrete $p^n$-torsion abelian group $V$ coincides with $V^*$. □

Lemma 5.13. Assume further that $D \subset X$ is a simple normal crossing divisor. Then the group $\lim_{n} F^d_{m,a(n)}$ is profinite and the canonical map

$$\lim_{n} (F^d_{m,a(n)})^* \rightarrow (\lim_{n} F^d_{m,a(n)})^*$$

is an isomorphism of topological abelian groups if either $k \neq \mathbb{F}_2$ or $d \neq 2$.

Proof. By [28, Theorem 9.1], Proposition 2.7, Corollary 4.2, and Lemma 5.4, $\{F^d_{m,a(n)}\}_n$ is isomorphic to a pro-abelian group $\{E_n\}_n$ such that each $E_n$ is finite and the map $E_{n+1} \rightarrow E_n$ is surjective. We can therefore apply Lemma 7.3. □

6. Cartier map for twisted de Rham-Witt complex

In this section, we shall prove the existence and some properties of the Cartier homomorphism for the twisted de Rham-Witt complex. We fix a perfect field $k$ of characteristic $p > 0$ and a modulus pair $(X, D)$ in $\text{Sch}_k$ such that $X$ is connected and regular of dimension $d \geq 1$. Let $c : D \rightarrow X$ and $j : U \rightarrow X$ be the inclusions, where $U = X \setminus D$. Let $F$ denote the function field of $X$. We also fix integers $m \geq 1$, $n \in \mathbb{Z}$ and $r \geq 0$. Let $W_m \mathcal{O}_X(nD)$ be the Nisnevich sheaf on $X$ defined in §5.1. We begin with following result describing the behavior of the Frobenius and Verschiebung operators on $W_n \mathcal{O}_X(nD)$. We shall then use this result to describe these operators on the full twisted de Rham-Witt complex. Until we talk about the topology of $X$ again in this section, we shall assume it to be the Nisnevich topology.

Lemma 6.1. We have

1. $F(W_{m+1} \mathcal{O}_X(nD)) \subseteq W_m \mathcal{O}_X(pmD)$.
2. $V(W_m \mathcal{O}_X(pmD)) \subseteq W_{m+1} \mathcal{O}_X(nD)$.
3. $R(W_{m+1} \mathcal{O}_X(nD)) \subseteq W_m \mathcal{O}_X(nD)$.

Proof. We can check the lemma locally at a point $x \in X$. We let $A = \mathcal{O}_{X,x}$ and let $f \in A$ define $D$ at $x$. The lemma is easy to check when $n \leq 0$. We therefore assume $n \geq 1$. For $w \in W_{m+1}(A)$, we have $F(w[f^{-n}]) = F(w) \cdot F([f^{-n}]) = F(w) \cdot [f^{-pm}] \in W_m \mathcal{O}_{X,x}(pmD)$. This proves (1). For (2), we use the projection formula to get $V(w'[f^{-pn}]) = V(w') \cdot F([f^{-an}]) = V(w') \cdot [f^{-n}] \in W_{m+1} \mathcal{O}_{X,x}(nD)$ for $w' \in W_m(A)$. The part (3) is obvious. □

Lemma 6.2. For $i \geq 1$, there is a short exact sequence

$$0 \rightarrow W_{m-i} \mathcal{O}_X (p^{i}nD) \xrightarrow{V^i} W_m \mathcal{O}_X(nD) \xrightarrow{R^{m-i}} W_i \mathcal{O}_X(nD) \rightarrow 0.$$
We shall now extend Lemma 6.1 to the de Rham-Witt forms of higher degrees. One knows using the Gersten conjecture for the de Rham-Witt sheaves due to Gros [12] that the map $W_m\Omega_X^r \to W_m\Omega_{K}^r$ is injective. It follows that the canonical map $W_m\Omega_X^r \to j_*W_m\Omega_U^r$ is injective too. Using the invertibility of the $W_m\mathcal{O}_X$-module $W_m\mathcal{O}_X(nD)$, we get an injection

\[ W_m\Omega_X^r(nD) \xrightarrow{(6.2)} W_m\mathcal{O}_X(nD) \otimes_{W_m\mathcal{O}_X} W_m\Omega_X^r \xrightarrow{\varphi} W_m\mathcal{O}_X(nD) \otimes_{W_m\mathcal{O}_X} j_*W_m\Omega_U^r. \]

**Lemma 6.3.** Let $q \geq 1$ be a positive integer. Then the inclusions $W_m\mathcal{O}_X(qnD) \hookrightarrow W_m\mathcal{O}_X(q(n+1)D)$ induce an isomorphism

\[ \lim_{n \geq 1} W_m\mathcal{O}_X(qnD) \xrightarrow{\approx} j_*W_m\mathcal{O}_U. \]

**Proof.** We shall prove the lemma by induction on $m$. For $m = 1$, the statement is clear. We now assume $m \geq 2$ and use the commutative diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{\sim} & \lim_{n \geq 1} W_{m-1}\mathcal{O}_X(qnD) \xrightarrow{V} \lim_{n \geq 1} W_m\mathcal{O}_X(qnD) \xrightarrow{\varphi} \lim_{n \geq 1} W_1\mathcal{O}_X(qnD) \xrightarrow{R^{m-1}} 0 \\
0 & \xrightarrow{j_*W_{m-1}\mathcal{O}_U} & j_*W_m\mathcal{O}_U \xrightarrow{V} j_*W_m\mathcal{O}_U \xrightarrow{R^{m-1}} j_*W_1\mathcal{O}_U \xrightarrow{0}.
\end{array}
\]

It is easy to check that the bottom row is exact. The top row is exact by Lemma 6.2. The right vertical arrow is an isomorphism by $m = 1$ case and the left vertical arrow is an isomorphism by induction on $m$ (with $q$ replaced by $qp$). The lemma follows. \(\square\)

**Corollary 6.4.** For $q \geq 1$, the canonical map $W_m\Omega_X^r(qnD) \to W_m\Omega_X^r(q(n+1)D)$ is injective and the map $\lim_{n \geq 1} W_m\Omega_X^r(nD) \xrightarrow{j_*} j_*W_m\Omega_U^r$ is an isomorphism of $W_m\mathcal{O}_X$-modules.

**Proof.** The injectivity claim follows from (6.2). To prove the isomorphism, we take the tensor product with the $W_m\mathcal{O}_X$-module $W_m\Omega_X^r$ on the two sides of the isomorphism in Lemma 6.3. This yields an isomorphism $\lim_{n \geq 1} W_m\Omega_X^r(qnD) \cong (j_*W_m\mathcal{O}_U) \otimes_{W_m\mathcal{O}_X} W_m\Omega_X^r$.

On the other hand, the étale descent property (see [20, Proposition I.1.14]) of the de Rham-Witt sheaves says that the canonical map $(j_*W_m\mathcal{O}_U) \otimes_{W_m\mathcal{O}_X} W_m\Omega_X^r \to j_*W_m\Omega_U^r$ is an isomorphism. The corollary follows. \(\square\)

In view of Corollary 6.4, we shall hereafter consider the sheaves $W_m\Omega_X^r(nD)$ as $W_m\mathcal{O}_X$-submodules of $j_*W_m\Omega_U^r$.

**Lemma 6.5.** The $V$, $F$, and $R$ operators of $j_*W_m\Omega_U^r$ satisfy the following.

1. $F(W_m\Omega_X^r(nD)) \subseteq W_{m-1}\Omega_X^r(pmD)$.
2. $V(W_m\Omega_X^r(pmD)) \subseteq W_{m+1}\Omega_X^r(nD)$.
3. $R(W_m\Omega_X^r(nD)) \subseteq W_m\Omega_X^r(nD)$.

**Proof.** We have $F([f^{-n}]\omega) = F(\omega)F([f^{-n}]) \in W_{m-1}\Omega_X^r \cdot W_{m-1}\mathcal{O}_X(pmD) = W_{m-1}\Omega_X^r(pmD)$, where the first inclusion holds by Lemma 6.1. We have $V([f^{-n}]\omega) = V(F([f^{-n}])\omega) = [f^{-n}]V(\omega) \in W_{m+1}\mathcal{O}_X(nD) \cdot W_{m+1}\Omega_X^r = W_{m+1}\Omega_X^r(nD)$, where the second inclusion holds again by Lemma 6.1. The last assertion is clear. \(\square\)

**Lemma 6.6.** The multiplication by $p$ map $p: W_{m+1}\Omega_X^r(nD) \to W_{m+1}\Omega_X^r(nD)$ has a factorization

\[ W_{m+1}\Omega_X^r(nD) \xrightarrow{R} W_m\Omega_X^r(nD) \xrightarrow{p} W_{m+1}\Omega_X^r(nD). \]
in the category of sheaves of $W_{m+1}\mathcal{O}_X$-modules. This factorization is natural in $X$.

**Proof.** Since $R$ is $W_{m+1}\mathcal{O}_X$-linear, we get an exact sequence (see § 5.1)

$$0 \rightarrow (V^mW_1\Omega^r_X + dV^mW_1\Omega^r_{X-1}) \cdot W_{m+1}\mathcal{O}_X(nD) \rightarrow W_{m+1}\Omega^r_X(nD) \xrightarrow{R} W_m\Omega^r_X(nD) \rightarrow 0.$$  

On the other hand, we note that $p: W_{m+1}\Omega^r_X \rightarrow W_{m+1}\Omega^r_{X'}$ is also a $W_{m+1}\mathcal{O}_X$-linear homomorphism. Since $W_{m+1}\mathcal{O}_X(nD)$ is an invertible $W_{m+1}\mathcal{O}_X$-module, we get an exact sequence

$$0 \rightarrow \text{Ker}(p) \cdot W_{m+1}\mathcal{O}_X(nD) \rightarrow W_{m+1}\Omega^r_X(nD) \xrightarrow{p} W_{m+1}\Omega^r_X(nD).$$

Since $\text{Ker}(p) = \text{Ker}(R)$ as $X$ is regular (see § 5.1), we get $\text{Ker}(p) \cdot W_{m+1}\mathcal{O}_X(nD) = (V^mW_1\Omega^r_X + dV^mW_1\Omega^r_{X-1}) \cdot W_{m+1}\mathcal{O}_X(nD)$. The first part of the lemma now follows. The naturality is clear from the proof.

**Lemma 6.7.** The square

$$\begin{array}{ccc}
W_m\Omega^r_X & \xrightarrow{p} & W_{m+1}\Omega^r_X \\
\downarrow j^* & & \downarrow j^*
\end{array}
\quad j_*W_m\Omega^r_U \xrightarrow{\tilde{p}} j_*W_{m+1}\Omega^r_U
$$

is Cartesian.

**Proof.** We can check this locally. So let $x \in j_*W_m\Omega^r_U$ be such that $p(x) \in W_{m+1}\Omega^r_X$. Since $j$ is affine and $W_m\Omega^r_X$ is an $W_m\mathcal{O}_X$-module, it follows that $j_*W_m\Omega^r_U \xrightarrow{j_*R} j_*W_m\Omega^r_U$ is surjective. We can therefore find $\tilde{x} \in j_*W_m\Omega^r_U$ such that $R(\tilde{x}) = x$. In particular, $p\tilde{x} = p(x) \in W_{m+1}\Omega^r_X$. We thus get $VF(\tilde{x}) = p\tilde{x} \in W_{m+1}\Omega^r_X$. Since

$$(j_*VF_m\Omega^r_U) \cap W_{m+1}\Omega^r_X = \text{Ker}(F^m: W_{m+1}\Omega^r_X \rightarrow j_*\Omega^r_U) = \text{Ker}(F^m: W_{m+1}\Omega^r_X \rightarrow \Omega^r_X)$$

$$= VW_m\Omega^r_X,$$

it follows that there exists $y' \in VW_m\Omega^r_X$ such that $VF(\tilde{x}) = V y'$.

Since $\text{Ker}(V: j_*W_m\Omega^r_U \rightarrow j_*W_{m+1}\Omega^r_U) = j_*FdV^m\Omega^r_U$ (see [20 I.3.21.1.4]), it follows that there exists $z' \in j_*\Omega^r_{U-1}$ such that $F dV^m(z') = F(\tilde{x}) - y'$. Equivalently, $F(\tilde{x} - dV^m(z')) = y' \in W_m\Omega^r_X$. Since

$$j_*FW_{m+1}\Omega^r_U \cap W_m\Omega^r_X = \text{Ker}(F^{m-1}d: W_m\Omega^r_X \rightarrow j_*\Omega^r_U) = \text{Ker}(F^{m-1}d: W_m\Omega^r_X \rightarrow \Omega^r_X+1)$$

$$= FW_{m+1}\Omega^r_X,$$

we can find $y'' \in W_{m+1}\Omega^r_X$ such that $F(\tilde{x} - dV^m(z')) = F(y'')$.

Since $\text{Ker}(F: j_*W_{m+1}\Omega^r_U \rightarrow j_*W_{m+1}\Omega^r_U) = j_*V\Omega^r_U$ (see [20 I.3.21.1.2]), we can find $z'' \in j_*\Omega^r_{U'}$ such that $V_m(z'') = \tilde{x} - dV^m(z') - y''$. Equivalently, $\tilde{x} - y'' = V^m(z'') + dV^m(z')$. On the other hand, we have

$$V^m j_*\Omega^r_U + dV^m j_*\Omega^r_{U-1} = \text{Ker}(R: j_*W_m\Omega^r_U \rightarrow j_*W_m\Omega^r_{U'}).$$

We thus get $x = R(\tilde{x}) = R(y'')$. Since $y'' \in W_{m+1}\Omega^r_X$, we get $x \in W_m\Omega^r_X$. This proves the lemma.

It is easy to see that for $n \in \mathbb{Z}$, the differential $d: j_*W_m\Omega^r_X \rightarrow j_*W_{m+1}\Omega^r_{X+1}$ restricts to a homomorphism $d: W_m\Omega^r_X(nD) \rightarrow W_{m+1}\Omega^r_{X+1}(n+1)D$ such that the composite map $d^2: W_m\Omega^r_X(nD) \rightarrow W_m\Omega^r_{X+2}((n+2)D)$ is zero by Corollary 6.4. The map $d: W_m\Omega^r_X(p^m nD) \rightarrow W_m\Omega^r_{X+1}(p^m n+1)D$ actually factors through $d: W_m\Omega^r_X(p^m nD) \rightarrow W_m\Omega^r_{X+1}(p^m nD)$ as one easily checks. In particular, $W_m\Omega^r_X(p^m nD)$ is a complex for every $m \geq 1$ and $n \in \mathbb{Z}$.
Let
\[ Z_1 W_m \Omega_X^r(nD) = (j_s Z_1 W_m \Omega_U^r) \cap W_m \Omega_X^r(nD) = j_s \text{Ker}(F^{m-1}d) \cap W_m \Omega_X^r(nD) \]
\[ = \text{Ker}(F^{m-1}d: W_m \Omega_X^r(nD) \to j_s \Omega_U) = \text{Ker}(F^{m-1}d: W_m \Omega_X^r(nD) \to \Omega_X^{r+1}(p^{m-1}(n + 1)D)) \]
where the third equality is by the left exactness of \( j_s \) and the last equality by Lemma 6.3.
If \( m = 1 \), we get
\[ (6.7) \quad Z_1 W_1 \Omega_X^r(nD) = Z \Omega_X^r(nD) = \text{Ker}(d: \Omega_X^r(nD) \to \Omega_X^{r+1}((n + 1)D)). \]
We let \( B \Omega_X^r(nD) = \text{Image}(d: \Omega_X^r((n - 1)D) \to \Omega_X^r(nD)). \)

**Proposition 6.8.** There exists a homomorphism \( C: Z_1 W_m \Omega_X^r(pmD) \to W_m \Omega_X^r(nD) \) such that the diagram
\[ (6.8) \quad \begin{array}{ccc}
Z_1 W_m \Omega_X^r(pmD) & \xrightarrow{V} & W_{m+1} \Omega_X^r(nD) \\
& & \\
\xrightarrow{C} & & \xrightarrow{p} \\
& \xrightarrow{\varphi} & \xrightarrow{\varphi}
\end{array} \]
is commutative. The map \( C \) induces an isomorphism of \( \mathcal{O}_X \)-modules
\[ C: \mathcal{H}^r(\psi_\ast W_1 \Omega_X^r(pmD)) \xrightarrow{\sim} W_1 \Omega_X^r(nD). \]

**Proof.** We consider the diagram
\[ (6.9) \quad \begin{array}{ccc}
Z_1 W_m \Omega_X^r(pmD) & \xrightarrow{V} & W_{m+1} \Omega_X^r(nD) \\
& & \\
\xrightarrow{\varphi} & & \xrightarrow{\varphi}
\end{array} \]
where the vertical arrows are inclusions from Corollary 6.4. The right square exists and
commutes by Lemma 6.3. The big outer square clearly commutes. It suffices therefore
to show that the right square is Cartesian.

We have a commutative diagram of \( W_{m+1} \mathcal{O}_X \)-modules
\[ (6.10) \quad \begin{array}{ccc}
W_{m+1} \Omega_X^r(nD) & \xrightarrow{\varphi} & W_{m+1} \Omega_X^r(nD) \\
& & \\
\xrightarrow{j_s} & & \xrightarrow{j_s}
\end{array} \]
\[ j_s \xrightarrow{\varphi} j_s \xrightarrow{\varphi} j_s \xrightarrow{\varphi} 0. \]
The top sequence is clearly exact and the bottom sequence is exact by the Serre vanishing
because all sheaves are \( W_{m+1} \mathcal{O}_X \)-modules and \( j \) is an affine morphism. By Lemma 6.3
it suffices to show that the right vertical arrow is injective.

To prove this injectivity, we note that the top row of (6.10) is same as the sequence
\[ W_{m+1} \Omega_X^r \otimes \mathcal{O}(nD) \xrightarrow{\varphi \otimes 1} W_{m+1} \Omega_X^r \otimes \mathcal{O}(nD) \to W_{m+1} \Omega_X^r / p \otimes \mathcal{O}(nD) \to 0, \]
where \( \mathcal{O} = W_{m+1} \mathcal{O}_X \). Similarly, we have
\[ j_s W_{m+1} \Omega_U^r / p \otimes \mathcal{O}(nD) \cong j_s (W_{m+1} \Omega_U^r / p \otimes W_{m+1} \mathcal{O}_U) \]
\[ j_s W_{m+1} \Omega_U^r / p \otimes W_{m+1} \mathcal{O}_U \cong j_s W_{m+1} \Omega_U^r / p, \]
where the first isomorphism follows from the projection formula for \( \mathcal{O} \)-modules using the
fact that \( \mathcal{O}(nD) \) is an invertible \( \mathcal{O} \)-module (see [17, Exercise II.5.1]).
Moreover, it is clear that the right vertical arrow in (6.10) is the map

\[ W_{m+1}\Omega^n_X/p \otimes O(nD) \xrightarrow{j^* \otimes 1} (j_* W_{m+1}\Omega^n_U/p) \otimes O(nD). \]

Since \( O(nD) \) is an invertible \( O \)-module, it suffices to show that the map \( j^*:W_{m+1}\Omega^n_X/p \to j_* W_{m+1}\Omega^n_U/p \) is injective. But this follows from Lemma 6.7 since \( j^* \) in (6.6) is injective. This proves the first part of the lemma.

We now prove the second part for which we can assume \( m = 1 \). We know classically that the map \( C \) on \( \Omega^n_X \) induces an \( O_X \)-linear isomorphism \( C: \mathcal{H}^r(\psi_*\Omega^n_X) \cong \Omega^n_X \). Taking its inverse, we get an \( O_X \)-linear isomorphism

\[ C^{-1}: \Omega^n_X \to \mathcal{H}^r(\psi_*\Omega^n_X). \]

Since \( \psi_*\Omega^n_X \in D^+(\mathrm{Coh}_X) \), we see that \( \mathcal{H}^r(\psi_*\Omega^n_X) \) is a coherent \( O_X \)-module. In particular, we get an isomorphism

\[ C^{-1}: \Omega^n_X(nD) \cong \mathcal{H}^r(\psi_*\Omega^n_X)(nD). \]

On the other hand, we have

\[ \mathcal{H}^r(\psi_*\Omega^n_X)(nD) \cong \mathcal{H}^r((\psi_*\Omega^n_X)(nD)) \cong \mathcal{H}^r(\psi_*(\Omega^n_X \otimes O_X \psi^*(O(nD)))) \cong \mathcal{H}^r(\psi_*(\Omega^n_X(\pm nD))). \]

This proves the second part. \( \square \)

Lemma 6.9. We have the following.

1. \( \ker(F^{m-1}:W_m\Omega^n_X(nD) \to \Omega^n_X(p^{m-1}nD)) = VW_{m+1}\Omega^n_X(\pm nD). \)
2. \( Z_1W_m\Omega'^n_X(p\pm nD) = FW_{m+1}\Omega'^n_X(nD). \)

Proof. We first prove (1). It is clear that the right hand side is contained in the left hand side. We prove the other inclusion. It suffices to show that

\[ (j_* VW_{m-1}\Omega^n_U) \cap W_m\Omega^n_X(nD) \subset VW_{m+1}\Omega^n_X(\pm nD). \]

We can check this locally. So let \( D \) be defined by \( f \in O_X \) and let \( y = Vx = [f^n]x \), where \( x \in j_*W_m\Omega^n_U \) and \( \omega \in W_m\Omega^n_X \). This yields \( \omega = [f^n]Vx = V(F([f^n])x) = V([f^n]x) \). This implies that \( \omega \in W_m\Omega^n_X \cap j_* VW_{m-1}\Omega^n_U \). On the other hand, we have

\[ W_m\Omega^n_X \cap j_* VW_{m-1}\Omega^n_U = \ker(F^{m-1}:W_m\Omega^n_X \to j_*\Omega^n_U) = \ker(F^{m-1}:W_m\Omega^n_X \to \Omega^n_X) \]

\[ = VW_{m-1}\Omega^n_X. \]

We thus get \( \omega \in VW_{m-1}\Omega^n_X \). Let \( y' \in W_m\Omega^n_X \) be such that \( \omega \inVy' \). This yields \( \omega = [f^{-n}]\omega = [f^{-n}]Vy' = V(F([f^{-n}]y')) = V([f^{-n}]y') \in VW_{m-1}\Omega^n_X(\pm nD) \). This proves (1).

We now prove (2). Since \( FW_{m+1}\Omega'^n_X(nD) \subset j_* FW_{m+1}\Omega'^n_U \cap W_m\Omega'^n_X(p\pm nD) \) by Lemma 6.3 it follows that \( FW_{m+1}\Omega'^n_X(nD) \subset Z_1W_m\Omega'^n_X(\pm nD) \). We show the other inclusion.

We let \( z \in Z_1W_m\Omega'^n_X(\pm nD) \) so that \( z = Fx = [f^{-p\pm}]\omega \) for some \( x \in j_* W_{m+1}\Omega^n_U \) and \( \omega \in W_m\Omega'^n_X \). We can then write \( \omega = [f^{p\pm}]Fx = F([f^n]x) \). This implies that \( \omega \in W_m\Omega'^n_X \cap j_*FW_{m+1}\Omega^n_U \). On the other hand, we have

\[ W_m\Omega'^n_X \cap j_* FW_{m+1}\Omega^n_U = \ker(F^{m-1}d:W_m\Omega'^n_X \to j_*\Omega'^n_U) = \ker(F^{m-1}d:W_m\Omega^n_X \to \Omega'^n_X) \]

\[ = FW_{m+1}\Omega^n_X. \]

We can thus write \( \omega = Fx' \) for some \( x' \in W_{m+1}\Omega^n_X \). This gives \( z = [f^{-p\pm}]\omega = [f^{-p\pm}]Fx' = F([f^{-n}]x') \in FW_{m+1}\Omega^n_X(nD) \). This proves (2). \( \square \)
7. The pairing of cohomology groups

We fix a finite field $k$ of characteristic $p$ and an integral smooth projective scheme $X$ of dimension $d \geq 1$ over $k$. Let $D \subset X$ be an effective Cartier divisor with complement $U$. Let $\nu: D \hookrightarrow X$ and $j: U \hookrightarrow X$ be the inclusions. In this section, we shall establish the pairing for our duality theorem for the $p$-adic étale cohomology of $U$. We fix integers $m, n \geq 1$ and $r \geq 0$. We shall consider the étale topology throughout our discussion of duality.

7.1. The complexes. We consider the complex of étale sheaves

$$\mathcal{F}_{n}^{\bullet} = [\{Z_{1}W_{m}\Omega_{X}^{r}(nD) \xrightarrow{1-C} W_{m}\Omega_{X}^{r}(nD)\}].$$

The differential of this complex is induced by the composition

$$Z_{1}W_{m}\Omega_{X}^{r}(nD) \hookrightarrow Z_{1}W_{m}\Omega_{X}^{r}(pnD) \xrightarrow{C} W_{m}\Omega_{X}^{r}(nD),$$

where the last map is defined by virtue of Proposition 6.8.

We now consider the map $F: W_{m+1}\Omega_{X}^{r}(nD) \rightarrow W_{m}\Omega_{X}^{r}(-pnD)$ whose existence is shown in Lemma 6.3. We have shown in (7.2) that $F(Ker(R)) = F((V^{m}W_{1}\Omega_{X}^{r} + dV^{m}W_{1}\Omega_{X}^{r-1}) \cap W_{m+1}\Omega_{X}^{r}(nD)) \subset dV^{m-1}\Omega_{X}^{r-1} \cap W_{m}\Omega_{X}^{r}(-pnD)$. It follows that $F$ induces a map $\mathcal{F}: W_{m}\Omega_{X}^{r}(nD) \rightarrow W_{m}\Omega_{X}^{r}(-pnD)/(dV^{m-1}\Omega_{X}^{r-1} \cap W_{m}\Omega_{X}^{r}(-pnD))$. We denote the composition

$$W_{m}\Omega_{X}^{r}(nD) \rightarrow \frac{W_{m}\Omega_{X}^{r}(-pnD)}{dV^{m-1}\Omega_{X}^{r-1} \cap W_{m}\Omega_{X}^{r}(-pnD)} \rightarrow \frac{W_{m}\Omega_{X}^{r}(nD)}{dV^{m-1}\Omega_{X}^{r-1} \cap W_{m}\Omega_{X}^{r}(nD)}$$

also by $\mathcal{F}$ and consider the complex of étale sheaves

$$W_{m}\mathcal{H}_{n}^{\bullet} = [\{W_{m}\Omega_{X}^{d}(nD) \xrightarrow{1-C} W_{m}\Omega_{X}^{d}\}],$$

where the map $C$ (see Proposition 6.8 for $n = 0$) is defined because $Z_{1}W_{m}\Omega_{X}^{d} = W_{m}\Omega_{X}^{d}$.

7.2. The pairing of complexes. We consider the pairing

$$\langle \cdot, \cdot \rangle: Z_{1}W_{m}\Omega_{X}^{r}(nD) \times W_{m}\Omega_{X}^{d-r}(-nD) \rightarrow W_{m}\Omega_{X}^{d}$$

by letting $(w_{1}, w_{2})_{1} = w_{1} \wedge w_{2}$. This is defined by the definition of the twisted de Rham-Witt sheaves.

We define a pairing $(\langle \cdot, \cdot \rangle)_{2}: Z_{1}W_{m}\Omega_{X}^{r}(nD) \times W_{m}\Omega_{X}^{d-r}(-nD) \rightarrow W_{m}\Omega_{X}^{d}$ by $(w_{1}, w_{2})_{2} = -C(w_{1} \wedge w_{2})$. We claim that $C(w_{1} \wedge w_{2}) = 0$ if $w_{2} \in dV^{m-1}\Omega_{X}^{d-r-1} \cap W_{m}\Omega_{X}^{d-r}(-nD)$. To prove it, we write $w_{1} = F(w_{1}')$ for some $w_{1}' \in j_{*}W_{m+1}\Omega_{U}^{d}$. This gives $V(w_{1} \wedge j^{*}w_{2}) = V(F(w_{1}') \wedge j^{*}w_{2}) = w_{1}' \wedge j^{*}Vw_{2} = 0$, where the last equality holds because $VdV^{m-1}\Omega_{X}^{d-r-1} \subset pdV^{m-1}\Omega_{X}^{d-r-1}$ and $p\Omega_{X}^{d-r-1} = 0$. In particular, $j^{*}(w_{1} \wedge w_{2}) \in Ker(V: W_{m}\Omega_{U}^{d} \rightarrow W_{m+1}\Omega_{U}^{d}) = FdV^{m-1}\Omega_{U}^{d-1} = dV^{m-1}\Omega_{U}^{d-1} = Ker(C_{U})$, where $C_{U}$ is the Cartier map for $U$ (see [21] Chapitre III for the last equality). It follows that $j^{*}C(w_{1} \wedge w_{2}) = C_{U} \circ j^{*}(w_{1} \wedge w_{2}) = 0$. Since $W_{m}\Omega_{X}^{d} \rightarrow j_{*}W_{m+1}\Omega_{U}^{d}$, the claim follows. Using the claim, we get a pairing

$$\langle \cdot, \cdot \rangle: Z_{1}W_{m}\Omega_{X}^{r}(nD) \times \frac{W_{m}\Omega_{X}^{d-r}(-nD)}{dV^{m-1}\Omega_{X}^{d-r-1} \cap W_{m}\Omega_{X}^{d-r}(-nD)} \rightarrow W_{m}\Omega_{X}^{d}.$$

We define our third pairing of étale sheaves

$$\langle \cdot, \cdot \rangle: W_{m}\Omega_{X}^{r}(nD) \times W_{m}\Omega_{X}^{d-r}(-nD) \rightarrow W_{m}\Omega_{X}^{d}$$
by \(\langle w_1, w_2 \rangle_3 = w_1 \wedge w_2\).

**Lemma 7.1.** The above pairings of étale sheaves give rise to a pairing of two-term complexes of sheaves

\[
\langle \cdot, \cdot \rangle: W_m \mathcal{F}_n^{r, \bullet} \times W_m \mathcal{G}_n^{d-r, \bullet} \rightarrow W_m \mathcal{H}^{d, \bullet}.
\]

**Proof.** By [38 § 1, p.175], we only have to show that

\[
(1 - C)(w_1 \wedge w_2) = (1 - C)(w_1) \wedge w_2 - C(w_1) \wedge (\pi - F)w_2
\]

for all \(w_1 \in Z_1W_mO_X^n(nD)\) and \(w_2 \in W_mO_X^n(-nD)\). But this follows from the equalities

\[
\begin{align*}
(1 - C)(w_1 \wedge w_2) &= w_1 \wedge w_2 - C(w_1) \wedge w_2 \\
&= w_1 \wedge w_2 - C(w_1) \wedge C(w_2) \\
&= (1 - C)(w_1) \wedge w_2 - C(w_1) \wedge (C - 1)(w_2) \\
&= (1 - C)(w_1) \wedge w_2 - C(w_1) \wedge (\pi - F)(w_2),
\end{align*}
\]

where \(w = 1\) holds because \(C \circ F = \text{id}\) (see [44 Proposition 1.1.4]). \(\square\)

### 7.3. The pairing of étale cohomology.

Using Lemma 7.1, we get a pairing of hypercohomology groups

\[
\mathbb{H}^i_{\text{ét}}(X, W_m \mathcal{F}_n^{r, \bullet}) \times \mathbb{H}^{d+1-i}_{\text{ét}}(X, W_m \mathcal{G}_n^{d-r, \bullet}) \rightarrow \mathbb{H}^{d+1}_{\text{ét}}(X, W_m \mathcal{H}^{d, \bullet}).
\]

By [44 Proposition 1.1.7] and [39 Corollary 1.12], there is a quasi-isomorphism \(W_mO_X^{\log, \emptyset} \rightarrow W_m \mathcal{H}^{d, \bullet}\), and a bijective trace homomorphism \(\text{Tr}: H^{d+1}_{\text{ét}}(X, W_m \Omega^d_X) \rightarrow \mathbb{Z}/p^m\). We thus get a pairing of \(\mathbb{Z}/p^m\)-modules

\[
\mathbb{H}^i_{\text{ét}}(X, W_m \mathcal{F}_n^{r, \bullet}) \times \mathbb{H}^{d+1-i}_{\text{ét}}(X, W_m \mathcal{G}_n^{d-r, \bullet}) \rightarrow \mathbb{Z}/p^m.
\]

Since this pairing is compatible with the change in values of \(m\) and \(n\), we get a pairing of ind-abelian (in first coordinate) and pro-abelian (in second coordinate) groups

\[
\{\mathbb{H}^i_{\text{ét}}(X, W_m \mathcal{F}_n^{r, \bullet})\}_n \times \{\mathbb{H}^{d+1-i}_{\text{ét}}(X, W_m \mathcal{G}_n^{d-r, \bullet})\}_n \rightarrow \mathbb{Z}/p^m.
\]

It follows from Corollary 6.4 that \(\lim_{\rightarrow n} W_m \mathcal{F}_n^{r, \bullet} \rightarrow j_*([Z_1W_mO_{U}^{1-C} \rightarrow W_mO_{U}^{r}])\). Since \(j\) is affine, we get

\[
\lim_{\rightarrow n} W_m \mathcal{F}_n^{r, \bullet} \rightarrow Rj_*( [Z_1W_mO_{U}^{1-C} \rightarrow W_mO_{U}^{r}] ) \cong Rj_*( W_mO_{U}^{r} ).
\]

In order to understand the pro-complex \(\{W_m \mathcal{G}_n^{r, \bullet}\}_n\), we let

\[
W_m \mathcal{G}_n^{r, \bullet} = \left( W_mO_{(X,nD)}^{r} \xrightarrow{\pi - F} \frac{W_mO_{(X,nD)}^{r}}{dV^{m-1}\Omega^{r-1}_X \cap W_mO_{(X,nD)}^{r}} \right).
\]

This complex is defined by the exact sequence (5.3). It follows then from Corollary 5.3 that the canonical inclusion \(\{W_m \mathcal{G}_n^{r, \bullet}\}_n \rightarrow \{W_m \mathcal{G}_n^{r, \bullet}\}_n\) is an isomorphism of pro-complexes of sheaves. Using the exact sequence (5.3) and Lemma 5.4, we get that there is a canonical isomorphism of pro-complexes

\[
dlog: \{K_{r, (X,nD)}/p^m\}_n \xrightarrow{\cong} \{W_mO_{(X,nD), \log}^{r} \rightarrow \{W_m \mathcal{G}_n^{r, \bullet}\}_n\}.
\]

We conclude that (7.11) is equivalent to the pairing of ind-abelian (in first coordinate) and pro-abelian (in second coordinate) groups

\[
\{\mathbb{H}^i_{\text{ét}}(X, W_m \mathcal{F}_n^{r, \bullet})\}_n \times \{H^{d+1-i}_{\text{ét}}(X, W_m \Omega_{(X,nD), \log}^{d-r})\}_n \rightarrow \mathbb{Z}/p^m.
\]
7.4. Continuity of the pairing. Recall from §5.3 that for a $\mathbb{Z}/p^n$-module $V$, one has $V^* = \text{Hom}_{\mathbb{Z}/p^n}(V, \mathbb{Z}/p^n)$. For any profinite or discrete torsion abelian group $A$, let $A^\vee = \text{Hom}_{\text{cont}}(A, \mathbb{Q}/\mathbb{Z})$ denote the Pontryagin dual of $A$ with the compact-open topology (see [15, § 7.4]). If $A$ is discrete and $p^n$-torsion, then we have $A^\vee = A^*$. We shall use the following topological results.

**Lemma 7.2.** Let $\{A_n\}_n$ be a direct system of discrete torsion topological abelian groups whose limit is $A$ with the direct limit topology. Then the canonical map $\lambda: A^\vee \to \varprojlim_n A_n^\vee$ is an isomorphism of profinite groups.

**Proof.** The fact that $\lambda$ is an isomorphism of abelian groups is well known and elementary. We show that $\lambda$ is a homeomorphism. Since its source and targets are compact Hausdorff, it suffices to show that it is continuous. We let $B_n = \text{Image}(A_n \to A)$. Then, we have a surjection of finite ind-groups $\{A_n\}_n \to \{B_n\}_n$ whose limits coincides with $A$. We thus get maps $A^\vee \xrightarrow{\lambda'} \varprojlim_n B_n^\vee \xrightarrow{\lambda''} \varprojlim_n A_n^\vee$ such that $\lambda = \lambda'' \circ \lambda'$. Since $\lambda''$ is clearly continuous, we have to only show that $\lambda'$ is continuous. We can therefore assume that each $A_n$ is a subgroup of $A$. Lemma now follows from [15] Lemma 2.9.3 and Theorem 2.9.6. □

**Lemma 7.3.** Let $\{A_n\}_n$ be an inverse system of discrete torsion topological abelian groups. Let $A$ be the limit of $\{A_n\}_n$ with the inverse limit topology. Then any $f \in A^\vee$ factors through the projection $\lambda_n: A \to A_n$ for some $n \geq 1$. In particular, the canonical map $\lim_n A_n^\vee \xrightarrow{\gamma} A^\vee$ is continuous and surjective. This map is an isomorphism if the transition maps of $\{A_n\}_n$ are surjective.

**Proof.** Since $f: A \to \mathbb{Q}/\mathbb{Z}$ is continuous and its target is discrete, it follows that $\text{Ker}(f)$ is open. By the definition of the inverse limit topology, the latter contains $\text{Ker}(\lambda_n)$ for some $n \geq 1$. Letting $A'_n = A/\text{Ker}(\lambda_n)$, this implies that $f$ factors through $f'_n: A'_n \to \mathbb{Q}/\mathbb{Z}$. Since $A_n$ is discrete, the map $(A_n)^\vee \to (A'_n)^\vee$ is surjective by [15] Lemma 7.10. Choosing a lift $f_n$ of $f'_n$ under this surjection, we see that $f$ factors through $A \xrightarrow{\lambda_n} A_n \xrightarrow{f_n} \mathbb{Q}/\mathbb{Z}$. The continuity of $\eta$ is equivalent to the assertion that the map $A_n^\vee \to A^\vee$ is continuous for each $n$. But this is well known since $\lambda_n$ is continuous. The remaining parts of the lemma are now obvious. □

**Remark 7.4.** The reader should note that the map $\lim_n A_n^\vee \to A^\vee$ may not in general be injective even if each $A_n$ is finite.

For $n \geq 1$, we endow each of $H_{\varprojlim}^i(X, W_m \mathcal{F}_n^{\cdot \vee})$ and $H_{\varprojlim}^i(X, W_m \Omega_{(X,nD),\log}^{d-r})$ with the discrete topology. We endow $\varprojlim_n H_{\varprojlim}^i(X, W_m \Omega_{(X,nD),\log}^{d-r})$ with the inverse limit topology and $\varinjlim_n H_{\varprojlim}^i(X, W_m \mathcal{F}_n^{\cdot \vee}) \cong H_{\varprojlim}^i(U, W_m \Omega_{U,\log}^{d-r})$ with the direct limit topology. Note that the latter topology is discrete.

If we let $x \in \varprojlim_n H_{\varprojlim}^i(X, W_m \mathcal{F}_n^{\cdot \vee})$, then we can find some $n \gg 0$ such that $x = f_n(x')$ for some $x' \in H_{\varprojlim}^i(X, W_m \mathcal{F}_n^{\cdot \vee})$, where $f_n: H_{\varprojlim}^i(X, W_m \mathcal{F}_n^{\cdot \vee}) \to H_{\varprojlim}^i(X, W_m \mathcal{F}_n^{\cdot \vee})$ is the canonical map to the limit. This gives a map

$$\langle x, \cdot \rangle: \varprojlim_n H_{\varprojlim}^{d+1-i}(X, W_m \Omega_{(X,nD),\log}^{d-r}) \to \mathbb{Z}/p^m$$

which sends $y$ to $\langle x', \pi(y) \rangle$ under the pairing (7.10), where $\pi$ is the composite map

$$\lim_n H_{\varprojlim}^{d+1-i}(X, W_m \Omega_{(X,nD),\log}^{d-r}) \cong \varinjlim_n H_{\varprojlim}^{d+1-i}(X, W_m \mathcal{G}_n^{d-r} \cdot \vee) \xrightarrow{\varprojlim_n g_n} H_{\varprojlim}^{d+1-i}(X, W_m \mathcal{G}_n^{d-r} \cdot \vee).$$

One checks that (7.15) is defined. Since $H_{\varprojlim}^i(U, W_m \Omega_{U,\log}^{\cdot \vee})$ is profinite, this shows that the map (see Lemma 7.2)

$$\theta_m: \lim_n H_{\varprojlim}^{d+1-i}(X, W_m \Omega_{(X,nD),\log}^{d-r}) \to H_{\varprojlim}^i(U, W_m \Omega_{U,\log}^{\cdot \vee})$$

is a isomorphism of profinite groups. □
is continuous. Since \( H^i_{ct}(U,W_m \Omega^r_{U,\log}) \) is discrete, the map
\[
\vartheta_m: H^i_{ct}(U,W_m \Omega^r_{U,\log}) \to (\lim_{\longleftarrow n} H^{d+1-i}_{ct}(X,W_m \Omega^{d-r}_{(X,nD),\log}))^\vee
\]
is clearly continuous. We have thus shown that after taking the limits, \( (7.14) \) gives rise to a continuous pairing of topological abelian groups
\[
H^i_{ct}(U,W_m \Omega^r_{U,\log}) \times \lim_{\longleftarrow n} H^{d+1-i}_{ct}(X,W_m \Omega^{d-r}_{(X,nD),\log}) \to \mathbb{Q}/\mathbb{Z}.
\]
Since each \( H^{d+1-i}_{ct}(X,W_m \Omega^{d-r}_{(X,nD),\log}) \) is \( p^m \)-torsion and discrete, we get the following.

**Proposition 7.5.** There is a continuous pairing of topological abelian groups
\[
H^i_{ct}(U,W_m \Omega^r_{U,\log}) \times \lim_{\longleftarrow n} H^{d+1-i}_{ct}(X,W_m \Omega^{d-r}_{(X,nD),\log}) \to \mathbb{Z}/p^m.
\]
Equivalently, there is a continuous pairing of topological abelian groups
\[
H^i_{ct}(U,W_m \Omega^r_{U,\log}) \times \lim_{\longleftarrow n} H^{d+1-i}_{ct}(X,W_m \Omega^{d-r}_{(X,nD),\log}) \to \mathbb{Z}/p^m.
\]
These pairings are compatible with respect to the canonical surjection \( \mathbb{Z}/p^m \to \mathbb{Z}/p^{m-1} \) and the inclusion \( \mathbb{Z}/p^{m-1} \to \mathbb{Z}/p^m \).

It follows from \( (7.14) \) that the map \( \vartheta_m \) in \( (7.17) \) has a factorization
\[
H^i_{ct}(U,W_m \Omega^r_{U,\log}) \xrightarrow{\theta'_m} \lim_{\longleftarrow n} H^{d+1-i}_{ct}(X,W_m \Omega^{d-r}_{(X,nD),\log})^\vee \xrightarrow{\vartheta'_m} \lim_{\longleftarrow n} H^{d+1-i}_{ct}(X,W_m \Omega^{d-r}_{(X,nD),\log})^\vee,
\]
where \( \vartheta'_m \) is the canonical map. This map is continuous and surjective by Lemma \( 7.3 \). The map \( \theta'_m \) is continuous since its source is discrete. Our goal in the next section will be to show that the maps \( \theta_m \) and \( \vartheta'_m \) are isomorphisms.

### 8. Perfectness of the Pairing

We continue with the setting of § 7. Our goal in this section is to prove a perfectness statement for the pairing \( (7.14) \). To make our statement precise, it is convenient to have the following definition.

**Definition 8.1.** Let \( \{A_n\}_n \) and \( \{B_n\}_n \) be ind-system and pro-system of \( p^m \)-torsion discrete abelian groups, respectively. Suppose that there is a pairing
\[
\{A_n\}_n \times \{B_n\}_n \to \mathbb{Z}/p^m.
\]
We shall say that this pairing is continuous and semi-perfect if the induced maps
\[
\theta: \lim_{\longleftarrow n} B_n \to (\lim_{\longleftarrow n} A_n)^\vee \quad \text{and} \quad \theta': \lim_{\longleftarrow n} A_n \to \lim_{\longleftarrow n} B_n^\vee
\]
are continuous and bijective homomorphisms of topological abelian groups. Recall here that \( (\lim_{\longleftarrow n} A_n)^\vee \cong \lim_{\longleftarrow n} A_n^\vee \) by Lemma \( 7.2 \). We shall say that \( (\ast) \) is perfect if it is semi-perfect, the surjective map (see Lemma \( 7.3 \)) \( \lim_{\longleftarrow n} B_n^\vee \to (\lim_{\longleftarrow n} B_n)^\vee \) is bijective and \( \theta \) is a homeomorphism. Note that perfectness implies that \( \theta' \) is also a homeomorphism.

The following is easy to check.

**Lemma 8.2.** The pairing \( (\ast) \) is perfect if it is semi-perfect and \( \{B_n\}_n \) is isomorphic to a surjective inverse system of compact groups.

Our goal is to show that \( (7.14) \) is semi-perfect by induction on \( m \geq 1 \). We first consider the case \( m = 1 \). We shall prove this case using our earlier observation that \( W_m \Omega^r_X(p^m n D) \) is a complex for every \( m \geq 1 \) and \( n \in \mathbb{Z} \).

In this case, we have \( Z_1 \Omega^r_X(p n D) = \text{Ker}(d: \Omega^{r+1}_X(p n D) \to \Omega^{r+1}_X(p D)) \) by \( (6.7) \). We claim that the inclusion \( d \Omega^{r+1}_X(-p n D) \to \Omega^{r+1}_X(-p D) \cap d \Omega^{r+1}_X \) is actually a bijection when
Equivalently, the map $\Omega^r_X(-pd)/d\Omega^r_{X}(-pd) \to \Omega^r_X/d\Omega^r_{X}$ is injective. Since $\psi$ is identity on the topological space $X$, the latter injectivity is equivalent to showing that the map

$$\frac{(\psi_*\Omega^r_X)(-nD)}{d(\psi_*\Omega^r_X)(-nD)} \cong \frac{\psi_*\Omega^r_X(-nD)}{d\psi_*\Omega^r_X(-nD)} \to \psi_*\Omega^r_X/d\psi_*\Omega^r_X$$

is injective. Using a snake lemma argument, it suffices to show that the map $\psi_*Z_1\Omega^r_X \to \psi_*Z_1\Omega^r_X \otimes_{O_X} O_{nD}$ is surjective. But this is obvious.

**Lemma 8.3.** For $n \in \mathbb{Z}$, the sheaves $\psi_*Z_1\Omega^r_X(pmD)$ and $\psi_*\Omega^r_X(pmD)/d\Omega^r_X(pmD)$ are locally free $O_X$-modules. For $n \geq 1$, the map

$$\psi_*Z_1\Omega^r_X(pmD) \to \text{Hom}_{O_X}(\psi_*\Omega^d_{X}(-pd)/d\Omega^d_X(pmD), \Omega^d_X),$$

induced by (7.5), is an isomorphism.

**Proof.** We first note that

$$\psi_*Z_1\Omega^r_X(pmD) \cong \psi_*(\text{Ker}(\Omega^r_X(pmD) \to \Omega^{r+1}_X(pmD)))$$

$$\cong \text{Ker}(\psi_*\Omega^r_X(pmD) \to \psi_*\Omega^{r+1}_X(pmD))$$

$$\cong \text{Ker}(\psi_*\Omega^r_X(nD) \to (\psi_*\Omega^{r+1}_X)(nD))$$

$$\cong (\text{Ker}(\psi_*\Omega^r_X \to \psi_*\Omega^{r+1}_X))(nD)$$

$$\cong (\psi_*Z_1\Omega^r_X)(nD).$$

Since $\psi_*$ also commutes with $\text{Coker}(d)$, we similarly get

$$\psi_*(\Omega^r_X(pmD)/d\Omega^r_X(pmD)) \cong (\psi_*\Omega^r_X/d\psi_*\Omega^r_X)(nD).$$

In the exact sequence

$$0 \to \psi_*Z_1\Omega^r_X \to \psi_*\Omega^r_X \to \psi_*\Omega^{r+1}_X \to \psi_*\Omega^{r+1}_X/d\psi_*\Omega^r_X \to 0$$

of $O_X$-linear maps between coherent $O_X$-modules, all terms are locally free by [38, Lemma 1.7]. It therefore remains an exact sequence of locally free $O_X$-modules after tensoring with $O_X(nD)$ for any $n \in \mathbb{Z}$. The first part of the lemma now follows by using (8.2) and (8.3) (for different values of $r$).

To prove the second part, we can again use (8.2) and (8.3). Since $O_X(nD)$ is invertible, it suffices to show that $(\psi_*Z_1\Omega^r_X(nD) \to \text{Hom}_{O_X}(\psi_*\Omega^d_{X}(-pd)/d\psi_*\Omega^d_X(pmD), \Omega^d_X(nD)))$ is an isomorphism. But the term on the right side of this map is isomorphic to the sheaf $\text{Hom}_{O_X}(\psi_*\Omega^d_{X}/d\psi_*\Omega^d_X, \Omega^d_X(nD))$ via the canonical isomorphism

$$\text{Hom}_{O_X}(A, B) \otimes_{O_X} \mathcal{E} \cong \text{Hom}_{O_X}(A, B \otimes_{O_X} \mathcal{E}),$$

which locally sends $(f \otimes b) \to (a \mapsto f(a) \otimes b)$ if $\mathcal{E}$ is locally free. We therefore have to only show that the map $\psi_*Z_1\Omega^r_X \to \text{Hom}_{O_X}(\psi_*\Omega^d_{X}/d\psi_*\Omega^d_X, \Omega^d_X(nD))$ is an isomorphism. But this follows from [38, Lemma 1.7].

**Lemma 8.4.** The groups $H^i_{\ell}(X, W_1^r \mathcal{F}^*_{n})$ and $H^i_{\ell}(X, \Omega^r_{(X,nD),log})$ are finite for $i, r \geq 0$.

**Proof.** Using (5.3), the finiteness of $H^i_{\ell}(X, \Omega^r_{(X,nD),log})$ is reduced to showing that the group $H^i_{\ell}(X, \frac{\Omega^r_{(X,nD)}}{\Omega_{(X,nD)}/d\Omega_X})$ is finite. By (5.2), it suffices to show that $H^i_{\ell}(Z, \frac{\Omega^r_{Z}}{\Omega_{Z}/d\Omega_X})$ is finite if $Z \in \{X, nD\}$. Since $k$ is perfect, the absolute Frobenius $\psi$ is a finite morphism. In particular, $\psi_*$ is exact. It suffices therefore to show that $H^i_{\ell}(Z, \frac{\psi_*\Omega^r_{Z}}{\psi_*\Omega^r_{Z}/d\Omega_X})$ is finite. But this is clear because $k$ is finite, $Z$ is projective over $k$ and $\psi_*\Omega^r_{Z}/\psi_*\Omega^r_{Z}/d\Omega_X$ is coherent. The finiteness of $H^i_{\ell}(X, W_1^r \mathcal{F}^*_{n})$ follows by a similar argument because $\psi_*Z_1\Omega^r_X(nD)$ is coherent.
Lemma 8.5. The pairing
\[ \{\mathbb{H}^i_{\text{et}}(X, W_1 \mathcal{F}_{n}^\bullet)\}_n \times \{\mathbb{H}^{d+1-i}_{\text{et}}(X, W_1 \mathcal{G}_{n}^{d-r, \bullet})\}_n \to \mathbb{Z}/p \]
is a perfect pairing of the ind-abelian and pro-abelian finite groups.

Proof. The finiteness follows from Lemma 8.4. For perfectness, it suffices to prove that without using the pro-systems, the pairing in question is a perfect pairing of finite abelian groups if we replace \( n \) by \( pn \). We shall show the latter. For an \( \mathbb{F}_p \)-vector space \( V \), we let \( V^* = \text{Hom}_{\mathbb{F}_p}(V, \mathbb{F}_p) \).

Using the definitions of various pairings of sheaves and complexes of sheaves, we have a commutative diagram of long exact sequences
\[ \cdots \to H^i_{\text{et}}(X, Z_i \Omega^r_X(pnD)) \to H^i_{\text{et}}(X, \Omega^r_X(pnD)) \to \mathbb{H}^i_{\text{et}}(X, W_1 \mathcal{F}_{pn}^\bullet) \to \cdots \]
\[ \cdots \to H^{d+1-i}_{\text{et}}(X, \Omega^{d-r}_X(\cdot(-pnD)))^* \to H^{d+1-i}_{\text{et}}(X, \Omega^{d-r}_X(-pnD))^* \to \mathbb{H}^{d+1-i}_{\text{et}}(X, W_1 \mathcal{G}_{pn}^{d-r, \bullet})^* \to \cdots \]

The exactness of the bottom row follows from Lemma 8.4 and [15, Lemma 7.10]. Using Lemma 8.3 and Grothendieck duality for the structure map \( X \to \text{Spec} \mathbb{F}_p \), the map
\[ H^i_{\text{et}}(X, \psi_*(Z_i \Omega^r_X(pnD))) \to H^d_{\text{et}}(X, \psi_*(-\Omega^{d-r}_X(pnD))^*) \]
is an isomorphism. Since \( \psi_* \) is exact, it follows that the map
\[ H^i_{\text{et}}(X, Z_i \Omega^r_X(pnD)) \to H^d_{\text{et}}(X, -\Omega^{d-r}_X(\cdot(-pnD))^*) \]
is an isomorphism. By the same reason, the map
\[ H^i_{\text{et}}(X, \Omega^r_X(pnD)) \to H^d_{\text{et}}(X, \Omega^{d-r}_X(-pnD))^* \]
is an isomorphism. Using (8.4), (8.5) and (8.6), we conclude that the right vertical arrow in (8.1) is an isomorphism. An identical argument shows that the map \( \mathbb{H}^{d+1-i}_{\text{et}}(X, W_1 \mathcal{G}_{pn}^{d-r, \bullet})^* \to \mathbb{H}^i_{\text{et}}(X, W_1 \mathcal{F}_{pn}^\bullet)^* \) is an isomorphism. This finishes the proof of the perfectness of the pairing of the hypercohomology groups.

We can now prove the main duality theorem of this paper.

Theorem 8.6. Let \( k \) be a finite field and \( X \) a smooth and projective scheme of pure dimension \( d \geq 1 \) over \( k \). Let \( D \subset X \) be an effective Cartier divisor with complement \( U \). Let \( m \geq 1 \) and \( i, r \geq 0 \) be integers. Then
\[ \{\mathbb{H}^i_{\text{et}}(X, W_m \mathcal{F}_{n}^\bullet)\}_n \times \{\mathbb{H}^{d+1-i}_{\text{et}}(X, W_m \mathcal{G}_{n}^{d-r, \bullet})\}_n \to \mathbb{Z}/p^m \]
is a semi-perfect pairing of ind-abelian and pro-abelian groups.

Proof. We have shown already (see (7.18)) that the pairing is continuous after taking limits. We need to show that the maps \( \theta_m \) (see (7.16)) and \( \theta'_m \) (see (7.20)) are isomorphisms of abelian groups. We shall prove this by induction on \( m \geq 1 \).

We first assume \( m = 1 \). Then Lemma 8.5 implies that the map
\[ \theta_1: \{\mathbb{H}^{d+1-i}_{\text{et}}(X, W_1 \mathcal{G}_{n}^{d-r, \bullet})\}_n \to \{\mathbb{H}^i_{\text{et}}(X, W_1 \mathcal{F}_{n}^\bullet)^\vee\}_n \]
is an isomorphism of pro-abelian groups. Taking the limit and using (7.13), we get an isomorphism
\[ \theta_1: F^{d+1-i}_{1,r}(U) \cong \lim_{n \to \infty} \mathbb{H}^i_{\text{et}}(X, W_1 \mathcal{F}_{n}^\bullet)^\vee. \]
But the term on the right is same as $H^d_\text{et}(U, W_r \Omega^r_{U, \log})$ by (7.12) and Lemma 7.2. Lemma 8.8 also implies that $\theta'_1$ is an isomorphism. This proves $m = 1$ case of the theorem.

We now assume $m \geq 2$ and recall the definitions of $F^i_{m,r}(n)$ and $F^i_{m,r}(U)$ from §5.3.

We consider the commutative diagram

\[
\begin{array}{ccccccccc}
\cdots & \longrightarrow & F^d_{m-1,i}(U) & \longrightarrow & F^d_{m,i}(U) & \longrightarrow & F^d_{1,i}(U) & \longrightarrow & \cdots \\
& & \downarrow{\theta_{m-1}} & & \downarrow{\theta_m} & & \downarrow{\theta_1} & & \\
\cdots & \longrightarrow & H^i_\text{et}(U, W_{m-1}\Omega^r_{U, \log}) & \longrightarrow & H^i_\text{et}(U, W_m\Omega^r_{U, \log}) & \longrightarrow & H^1_\text{et}(U, W_1\Omega^r_{U, \log}) & \longrightarrow & \cdots
\end{array}
\]

The top row is exact by Lemma 5.11. The bottom row is exact by Lemma 5.7 and [15, Theorem 9.1] and the equivalence of (7.18) and (7.19). □

**Remark 8.7.** In [22, Theorem 4.1.4] and [44, Theorem 3.4.2], a pairing using a logarithmic version of the pro-system \( \{H^i_{\text{et}}(X, W_m^{d-r}(X_n D), \log)\}_n \) is given under the assumption that $D_{\text{red}}$ is a simple normal crossing divisor. The authors in these papers say that their pairing is perfect. Although they do not explain their interpretations of perfectness, what they actually prove is the semi-perfectness in the sense of Definition 8.1 according to our understanding.

**Corollary 8.8.** The pairing of Theorem 8.6 is perfect if $D_{\text{red}}$ is a simple normal crossing divisor, $i = 1$, $r = 0$ and one of the conditions \( d \neq 2, k \neq \mathbb{F}_2 \) holds.

**Proof.** Combine Theorem 8.6, Lemma 8.2, Corollary 4.2, Proposition 2.7, [26, Theorem 9.1] and the equivalence of (7.18) and (7.19). □

We let $C^d_{K_S}(X, D; m) = H^d_{\text{et}}(X, \overline{K}^M_{d,(X,D)}/p^m)$. We shall study the following special case in the next section.

**Corollary 8.9.** Under the assumptions of Theorem 8.6, the map

\[ \theta_m: \lim_n C^d_{K_S}(X, nD; m) \rightarrow \pi^\text{ab}_1(U)/p^m \]

is a bijective and continuous homomorphism between topological abelian groups. This is an isomorphism of topological groups under the assumptions of Corollary 8.8.

### 8.1. The comparison theorem

We shall continue with the assumptions of Theorem 8.6. We fix an integer $m \geq 1$. Let $\pi^\text{ab}_1(U)$ be the abelianized étale fundamental group of $U$ and $\pi^\text{adiv}_1(X, D)$ the co-1-skeleton étale fundamental group of $X$ with modulus $D$, introduced in [15, Definition 7.5]. The latter characterizes finite abelian covers of $U$ whose ramifications are bounded by $D$ at each of its generic point, where the bound is given by means of Matsuda’s Artin conductor. There is a natural surjection $\pi^\text{ab}_1(U) \twoheadrightarrow \pi^\text{adiv}_1(X, D)$.

Let $C_{K_S}(X, D) = H^d_{\text{nis}}(X, \overline{K}^M_{d,(X,D)})$ and $C_{K_S}(X, D; m) = H^d_{\text{nis}}(X, \overline{K}^M_{d,(X,D)}/p^m) \cong C_{K_S}(X, D)/p^m$. By Proposition 2.7, we have

\[ C_{K_S}(X, D; m) \xrightarrow{\cong} H^d_{\text{nis}}(X, \overline{K}^M_{d,(X,D)}/p^m). \]

We let $\tilde{C}_{U/X} = \lim_n C_{K_S}(X, D)$. The canonical map $\tilde{C}_{U/X}/p^m \rightarrow \lim_n C_{K_S}(X, D; m)$ is an isomorphism by [18, Lemma 5.9]. The groups $C_{K_S}(X, D)$ and $\tilde{C}_{K_S}(X, D; m)$ have the discrete topology while $\tilde{C}_{U/X}$ and $\tilde{C}_{U/X}/p^m$ have the inverse limit topology. The
groups $\pi_1^{ab}(U)$ and $\pi_1^{\text{div}}(X, D)$ have the profinite topology. By [15, Theorem 1.2], there is a commutative diagram of continuous homomorphisms of topological abelian groups

\[
\begin{array}{ccc}
\tilde{C}_{U/X} & \xrightarrow{\rho_{U/X}} & \pi_1^{ab}(U) \\
\downarrow & & \downarrow \\
C(X, D) & \xrightarrow{\rho_{X/D}} & \pi_1^{\text{div}}(X, D).
\end{array}
\]

The horizontal arrows are injective with dense images. They become isomorphisms after tensoring with $\mathbb{Z}/p^m$ by [16, Corollary 5.15].

We let $\tilde{C}_{U/X}^{\text{ét}}(m) = \varprojlim_n C_{K_S}^{\text{ét}}(X, nD; m)$. By Corollary 8.9 we have a bijective and continuous homomorphism between topological abelian groups $\rho_{U/X}^{\text{ét}} : \tilde{C}_{U/X}^{\text{ét}}(m) \to \pi_1^{ab}(U)/p^m \cong H_1^{\text{êl}}(U, \mathbb{Z}/p^m)^\ast$. We therefore have a diagram

\[
\begin{array}{ccc}
\tilde{C}_{U/X}^{\text{ét}}(m) & \xrightarrow{\rho_{U/X}^{\text{ét}}} & \pi_1^{ab}(U)/p^m \\
\downarrow & & \downarrow \\
\tilde{C}_{U/X}^{\text{ét}}(m) & \xrightarrow{\rho_{U/X}^{\text{ét}}} & \pi_1^{ab}(U)/p^m
\end{array}
\]

of continuous homomorphisms, where $\eta$ is the change of topology homomorphism. We wish to prove the following.

**Proposition 8.10.** The diagram (8.10) is commutative.

**Proof.** It follows from [16, Theorem 1.2] that $\rho_{U/X}$ is an isomorphism of profinite groups. Since the image of the composite map $\mathbb{Z}_0(U) \xrightarrow{\text{cyc}_U/X} \tilde{C}_{U/X}/p^m \xrightarrow{\rho_{U/X}} \pi_1^{ab}(U)/p^m$ is dense by the generalized Chebotarev density theorem, it follows that the image of cyc$_U/X$ is also dense in $\tilde{C}_{U/X}/p^m$. Since $\pi_1^{ab}(U)/p^m$ is Hausdorff, it suffices to show that $\rho_{U/X} \circ \text{cyc}_U/X = \rho_{U/X}^{\text{ét}} \circ \eta \circ \text{cyc}_U/X$. Equivalently, we have to show that for every closed point $x \in U$, one has that $(\rho_{U/X}^{\text{ét}} \circ \eta \circ \text{cyc}_U/X)([x])$ is the image of the Frobenius element under the map $\text{Gal}(\overline{k}/k(x)) \to \pi_1^{ab}(U)/p^m$. But this is well known (e.g., see [30, Theorem 3.4.1]).

8.2. A new filtration of $H_1^{\text{êl}}(U, \mathbb{Q}_p/\mathbb{Z}_p)$. We keep the assumptions of Theorem 8.6. By (7.13) and Theorem 8.8, we have the isomorphism of abelian groups

\[
\theta_m^{\text{êl}} : H_1(U, \mathbb{Z}/p^m) \xrightarrow{\cong} \varprojlim_n C_{K_S}^{\text{ét}}(X, nD; m)^\vee,
\]

where $H_1(U, \mathbb{Z}/p^m)$ denote the étale cohomology $H_1^{\text{êl}}(U, \mathbb{Z}/p^m)$. We let

\[
\text{fil}_D^{\text{êl}} H_1(U, \mathbb{Z}/p^m) = (\theta_m^{\text{êl}})^{-1}(\text{Image}(C_{K_S}^{\text{ét}}(X, nD; m)^\vee \to \varprojlim_n C_{K_S}^{\text{ét}}(X, nD; m)^\vee)).
\]

We set

\[
\text{fil}_D^{\text{êl}} H_1(U, \mathbb{Q}_p/\mathbb{Z}_p) = \lim_{\to m} \text{fil}_D^{\text{êl}} H_1(U, \mathbb{Z}/p^m).
\]

It follows that $\{\text{fil}_D^{\text{êl}} H_1(U, \mathbb{Q}_p/\mathbb{Z}_p)\}_n$ defines an increasing filtration of $H_1(U, \mathbb{Q}_p/\mathbb{Z}_p)$. This is an étale version of the filtration $\text{fil}_D H_1(U, \mathbb{Q}_p/\mathbb{Z}_p)$ defined in [15, definition 7.12]. This new filtration is clearly exhaustive. We do not know if $\text{fil}_D H_1(U, \mathbb{Q}_p/\mathbb{Z}_p)$ and $\text{fil}_D^{\text{êl}} H_1(U, \mathbb{Q}_p/\mathbb{Z}_p)$ are comparable in general. We can however prove the following.

**Theorem 8.11.** Let $k$ be a finite field and $X$ a smooth and projective scheme of pure dimension $d \geq 1$ over $k$. Let $D \subset X$ be an effective Cartier divisor with complement $U$.
such that $D_{\text{red}}$ is a simple normal crossing divisor. Assume that either $d \neq 2$ or $k \neq \mathbb{F}_2$. Then one has

$$\text{fil}_D^\ell H^1(U, \mathbb{Z}/p^m) \subseteq \text{fil}_D H^1(U, \mathbb{Z}/p^m) \quad \text{and} \quad \text{fil}_D^\ell H^1(U, \mathbb{Q}_p/\mathbb{Z}_p) \subseteq \text{fil}_D H^1(U, \mathbb{Q}_p/\mathbb{Z}_p)$$

as subgroups of $H^1(U, \mathbb{Z}/p^m)$ and $H^1(U, \mathbb{Q}_p/\mathbb{Z}_p)$, respectively. These inclusions are equalities if $D_{\text{red}}$ is regular.

**Proof.** The claim about the two inclusions follows directly from the definitions of the filtrations in view of Proposition 8.10 and Corollary 8.8.

Moreover, Corollary 8.9 yields that $\rho_{U/X}^\ell$ is an isomorphism of profinite topological groups. Hence, there exists a unique quotient $\pi_1^\ell(X, nD; m)$ of $\pi_1^\text{ab}(U)/p^m$ such that the diagram

\[
\begin{array}{ccc}
\tilde{\mathcal{C}}_{U/X}^\ell(m) & \xrightarrow{\rho_{U/X}^\ell} & \pi_1^\text{ab}(U)/p^m \\
\downarrow & & \downarrow \\
\pi_1^\ell(X, nD; m) & \xrightarrow{\rho_{nD}^\ell} & \pi_1^\text{div}(X, nD)/p^m \\
\end{array}
\]

commutes and the horizontal arrows are isomorphisms of topological groups. One knows that $\text{fil}_{nD}^\ell H^1(U, \mathbb{Z}/p^m) = (\pi_1^\text{div}(X, nD)/p^m)^\vee$. By Theorem 4.9 it is easy to see that $\text{fil}_{nD}^\ell H^1(U, \mathbb{Z}/p^m) \cong \pi_1^\ell(X, nD; m)^\vee$ when $D_{\text{red}}$ is regular.

More precisely, if $D_{\text{red}}$ is a simple normal crossing divisor and either $d \neq 2$ or $k \neq \mathbb{F}_2$, we have the following diagram.

\[
\begin{array}{ccc}
H^1(U, \mathbb{Z}/p^m) & \xrightarrow{\rho^\ell} & \lim_n \mathcal{C}_{K_S}^\ell(X, nD; m)^\vee \\
\downarrow & & \downarrow \\
\text{fil}_D^\ell H^1(U, \mathbb{Z}/p^m) & \xrightarrow{\cong} & \mathcal{C}_{K_S}^\ell(X, nD; m)^\vee \\
\downarrow & & \downarrow \\
\text{fil}_D H^1(U, \mathbb{Z}/p^m) & \xrightarrow{\cong} & C_{K_S}(X, nD; m)^\vee.
\end{array}
\]

Assume now that $D_{\text{red}}$ is regular. By Theorem 4.9 it then follows that the transition maps in the pro-system $\{\mathcal{C}_{K_S}^\ell(X, nD; m)\}_n$ are surjective. In particular, the right top vertical arrow is injective and hence the middle horizontal dotted arrow is a honest arrow such that the top square commutes. It follows from Proposition 8.10 that the bottom square commutes as well. By the definition of $\text{fil}_D^\ell H^1(U, \mathbb{Z}/p^m)$, it is clear that the middle horizontal arrow is now surjective. To prove its injectivity, it suffices to show that the right bottom vertical arrow is injective. But this follows from Theorem 4.9 $\square$

9. **Reciprocity theorem for $C_{K_S}(X|D)$**

In this section, we shall prove the reciprocity theorem for the idele class group $C_{K_S}(X|D)$. Before going into this, we recall the definition of some filtrations of $H^1_r(K, \mathbb{Q}/\mathbb{Z})$ for a Henselian discrete valuation field $K$.

9.1. **The Brylinski-Kato and Matsuda filtrations.** Let $K$ be a Henselian discrete valuation field of characteristic $p > 0$ with ring of integers $\mathcal{O}_K$, maximal ideal $m_K \neq 0$ and residue field $\mathfrak{f}$. We let $H^0(A) = H^3_r(A, \mathbb{Q}/\mathbb{Z}(q-1))$ for any commutative ring $A$. The generalized Artin-Schreier sequence gives rise to an exact sequence

\[
0 \to \mathbb{Z}/p^r \to W_r(K) \xrightarrow{1-F} W_r(K) \xrightarrow{\vartheta} H^1_r(K, \mathbb{Z}/p^r) \to 0,
\]
where \( F((a_{r-1}, \ldots, a_0)) = (a^p_{r-1}, \ldots, a^p_0) \). We write the Witt vector \((a_{r-1}, \ldots, a_0)\) as \( \mathbf{a} \) in short. Let \( v_K: K^* \to \mathbb{Z} \) be the normalized valuation. Let \( \delta_r \) denote the composite map \( W_r(K) \xrightarrow{\partial} H^1_{fil}(K, \mathbb{Z}/p^r) \to H^1(K) \). Then one knows that \( \delta_r = \delta_{r+1} \circ V \). For an integer \( m \geq 1 \), let \( \text{ord}_p(m) \) denote the \( p \)-adic order of \( m \) and let \( r' = \min(r, \text{ord}_p(m)) \). We let \( \text{ord}_p(0) = -\infty \).

For \( m \geq 0 \), we let
\[
(9.2) \quad \fil_m^b W_r(K) = \{ ap^i v_K(a_i) \geq -m \}; \quad \text{and}
(9.3) \quad \fil_m^s W_r(K) = \fil_m^b W_r(K) + V^{r-r'}(\fil_m^b W_r(K)).
\]

We have \( \fil^b W_r(K) = \fil^s W_r(K) = W_r(\mathcal{O}_K) \). We let \( \fil^b W_r(K) = \fil^s W_r(K) = 0 \).

For \( m \geq 1 \), we let
\[
(9.4) \quad \fil_{m-1}^b H^1(K) = H^1(K)\{p'\} \bigoplus \bigcup_{r \geq 1} \delta_r(\fil_{m-1}^b W_r(K)) \quad \text{and}
\]
\[
\fil_m^s H^1(K) = H^1(K)\{p'\} \bigoplus \bigcup_{r \geq 1} \delta_r(\fil_m^s W_r(K)).
\]

Moreover, we let \( \fil_0^s H^1(K) = \fil_1^b H^1(K) = H^1(\mathcal{O}_K) \), the subgroup of unramified characters. The filtrations \( \fil^b H^1(K) \) and \( \fil^s H^1(K) \) are due to Brylinski-Kato [24] and Matsuda [37], respectively. We refer to [15] Theorem 6.1 for the following.

**Theorem 9.1.** The two filtrations defined above satisfy the following relations.

1. \( H^1(K) = \bigcup_{m \geq 0} \fil_m^b H^1(K) = \bigcup_{m \geq 0} \fil_m^s H^1(K) \).
2. \( \fil_m^s H^1(K) \subset \fil_m^b H^1(K) \subset \fil_{m+1}^s H^1(K) \) for all \( m \geq 1 \).
3. If \( m \geq 1 \) such that \( \text{ord}_p(m) = 0 \), then \( \fil_{m-1}^b H^1(K) = \fil_m^s H^1(K) \). In particular, \( \fil_0^b H^1(K) = \fil_1^s H^1(K) \), which is the subgroup of tamely ramified characters.

For integers \( m \geq 0 \) and \( r \geq 1 \), let \( U_m K_r^M(K) \) be the subgroup \( \{1 + \mathfrak{m}_K^m, K^* \} \) of \( K_r^M(K) \). We let \( U'_m K_r^M(K) \) be the subgroup \( \{1 + \mathfrak{m}_K^m, \mathcal{O}_K^*, \ldots, \mathcal{O}_K^* \} \) of \( K_r^M(K) \). It follows from [15] Lemma 6.2 that \( U_{m+1} K_r^M(K) \subset U'_m K_r^M(K) \subset U_m K_r^M(K) \) for every integer \( m \geq 0 \). If \( K \) is a \( d \)-dimensional Henselian local field (see [13] § 5.1), there is a pairing
\[
(9.5) \quad \{,\}: K_d^M(K) \times H^1(K) \to H^{d+1}(K) \cong \mathbb{Q}/\mathbb{Z}.
\]

The following result is due to Kato and Matsuda (when \( p \neq 2 \)). We refer to [13] Theorem 6.3 for a proof.

**Theorem 9.2.** Let \( \chi \in H^1(K) \) be a character. Then the following hold.

1. For every integer \( m \geq 0 \) we have that \( \chi \in \fil_m^b H^1(K) \) if and only if \( \{\alpha, \chi\} = 0 \) for all \( \alpha \in U_{m+1} K_d^M(K) \).
2. For every integer \( m \geq 1 \) we have that \( \chi \in \fil_m^s H^1(K) \) if and only if \( \{\alpha, \chi\} = 0 \) for all \( \alpha \in U'_m K_d^M(K) \).

We have the inclusions \( K \hookrightarrow K^{sh} \to \overline{K} \), where \( \overline{K} \) is a fixed separable closure of \( K \) and \( K^{sh} \) is the strict Henselization of \( K \). We shall use the following key result in the proof of our reciprocity theorem.

**Proposition 9.3.** For \( m \geq 0 \), the canonical square
\[
\begin{array}{ccc}
\fil_m^b H^1(K) & \to & H^1(K) \\
\downarrow & & \downarrow \\
\fil_m^b H^1(K^{sh}) & \to & H^1(K^{sh})
\end{array}
\]
is Cartesian.

**Proof.** Since $\text{fil}^{b_{k}}H^{1}(K)$ is an exhaustive filtration of $H^{1}(K)$ by Theorem 9.11, it suffices to show that for every $m \geq 1$, the square

$$
\begin{array}{ccc}
\text{fil}^{b_{k}}H^{1}(K) & \longrightarrow & \text{fil}^{b_{k}}H^{1}(K) \\
\downarrow & & \downarrow \\
\text{fil}^{b_{k}}H^{1}(K^{sh}) & \longrightarrow & \text{fil}^{b_{k}}H^{1}(K^{sh})
\end{array}
$$

is Cartesian. Equivalently, it suffices to show that for every $m \geq 1$, the map

$$
\phi_{m}^{*}: \text{gr}^{b_{k}}H^{1}(K) \to \text{gr}^{b_{k}}H^{1}(K^{sh}),
$$

induced by the inclusion $\phi: K \hookrightarrow K^{sh}$, is injective.

We fix $m \geq 1$. By [21] Corollary 5.2, there exists an injective (non-canonical) homomorphism $\text{rsw}_{\pi_{K}}: \text{gr}^{b_{k}}H^{1}(K) \to \Omega_{1}^d \otimes \bar{\mathfrak{f}}$. By Theorem 5.1 of loc. cit., this refined swan conductor $\text{rsw}_{\pi_{K}}$ depends only on the choice of a uniformizer $\pi_{K}$ of $K$. Since $\mathcal{O}^{sh}_{K}$ is unramified over $\mathcal{O}_{K}$, we can choose $\pi_{K}$ to be a uniformizer of $K^{sh}$ as well. It therefore follows that for all $m \geq 1$, the diagram

$$
\begin{array}{ccc}
\text{gr}^{b_{k}}H^{1}(K) & \overset{\text{rsw}_{\pi_{K}}}{\longrightarrow} & \Omega_{1}^d \otimes \bar{\mathfrak{f}} \\
\downarrow & & \downarrow \\
\text{gr}^{b_{k}}H^{1}(K^{sh}) & \overset{\text{rsw}_{\pi_{K}}}{\longrightarrow} & \Omega_{1}^d \otimes \bar{\mathfrak{f}}
\end{array}
$$

is commutative, where $\bar{\mathfrak{f}}$ is a separable closure of $\mathfrak{f}$. Since the horizontal arrows in the above diagram are injective, it suffices to show that the natural map $\Omega_{1}^d \to \Omega_{1}^d \otimes \bar{\mathfrak{f}}$ is injective. But this is clear. \qed

### 9.2. Logarithmic fundamental group with modulus

Let $k$ be a finite field of characteristic $p$ and $X$ an integral projective scheme over $k$ of dimension $d \geq 1$. Let $D \subset X$ be an effective Cartier divisor with complement $U$. Let $K = k(\eta)$ denote the function field of $X$. We let $C = D_{\text{red}}$. We fix a separable closure $\overline{K}$ of $K$ and let $G_{K}$ denote the absolute Galois group of $K$. Recall the following notations from [26, § 3.3] or [15, § 2.3].

Assume that $X$ is normal. Let $\lambda$ be a generic point of $D$. Let $K_{\lambda}$ denote the Henselization of $K$ at $\lambda$. Let $P = (p_{0}, \ldots, p_{d-2}, \lambda, \eta)$ be a Parshin chain on $(U \subset X)$. Let $V \subset K$ be a $d$-DV which dominates $P$. Let $V = V_{0} \subset \cdots \subset V_{d-2} \subset V_{d-1} \subset V_{d} = K$ be the chain of valuation rings in $K$ induced by $V$. Since $X$ is normal, it is easy to check that for any such chain, one must have $V_{d-1} = \mathcal{O}_{X, \lambda}$. Let $V'$ be the image of $V$ in $k(\lambda)$. Let $\overline{V}_{d-1}$ be the unique Henselian discrete valuation ring having an ind-étale local homomorphism $V_{d-1} \to \overline{V}_{d-1}$ such that its residue field $E_{d-1}$ is the quotient field of $(V')^{h}$. Then $V^{h}$ is the inverse image of $(V')^{h}$ under the quotient map $\overline{V}_{d-1} \to E_{d-1}$. It follows that its function field $Q(V^{h})$ is a $d$-dimensional Henselian discrete valuation field whose ring of integers is $\overline{V}_{d-1}$ (see [26, § 3.7.2]). It then follows that there are canonical inclusions of discrete valuation rings

$$
\mathcal{O}_{X, \lambda} \hookrightarrow \overline{V}_{d-1} \hookrightarrow \mathcal{O}^{sh}_{X, \lambda}.
$$

Moreover, we have (see the proof of [26, Proposition 3.3])

$$
\mathcal{O}^{h}_{X, P'} \cong \prod_{V \in \mathcal{V}(P)} \overline{V}_{d-1},
$$

(9.6)
where $\mathcal{V}(P)$ is the set of $d$-DV’s in $K$ which dominate $P$. As an immediate consequence of Proposition 9.3, we therefore get the following.

**Corollary 9.4.** For every $m \geq 0$, the square

$$
\begin{array}{ccc}
\fil_{m}^{bk} H^1(K_\lambda) & \rightarrow & H^1(K_\lambda) \\
\downarrow & & \downarrow \\
\fil_{m}^{bk} H^1(Q(V^h)) & \rightarrow & H^1(Q(V^h))
\end{array}
$$

is Cartesian.

Let $\text{Irr}_C$ denote the set of all generic points of $C$ and let $C_\lambda$ denote the closure of an element $\lambda \in \text{Irr}_C$. We write $D = \sum_{\lambda \in \text{Irr}_C} n_\lambda C_\lambda$. We can also write $D = \sum x \overline{\{x\}}$, where $n_x = 0$ for all $x \in U$. We allow $D$ to be empty in which case we write $D = 0$. For any $x \in X^{(1)} \cap C$, we let $\overline{K}_x$ denote the quotient field of the $\mathfrak{m}_x$-adic completion $\overline{\mathcal{O}}_{X,x}$ of $\mathcal{O}_{X,x}$. Let $\mathcal{O}^{sh}_{X,x}$ denote the strict Henselization of $\mathcal{O}_{X,x}$ and let $K^{sh}_x$ denote its quotient field. Then it is clear from the definitions that there are inclusions

$$
(9.8) \quad K \hookrightarrow K_x \hookrightarrow K^{sh}_x \rightarrow \overline{K} \text{ and } K \rightarrow K_x \rightarrow \overline{K}_x.
$$

**Definition 9.5.** Let $\fil_{D}^{bk} H^1(K)$ denote the subgroup of characters $\chi \in H^1(K)$ such that for every $x \in X^{(1)}$, the image $\chi_x$ of $\chi$ under the canonical surjection $H^1(K) \twoheadrightarrow H^1(K_\lambda)$ lies in $\fil_{n_\lambda-1}^{bk} H^1(K_\lambda)$. It is easy to check that $\fil_{D}^{bk} H^1(K) \subset H^1(U)$. We let $\fil_{D}^{bk} H^1(U, \mathbb{Z}/m) = H^1_{et}(U, \mathbb{Z}/m) \cap \fil_{D}^{bk} H^1(K)$.

Recall that $H^1(K)$ is a torsion abelian group. We consider it a topological abelian group with discrete topology. In particular, all the subgroups $H^1(U)$ (e.g., $\fil_{D}^{bk} H^1(K)$) are also considered as discrete topological abelian groups. Recall from [15, Definition 7.12] that $\fil_{D} H^1(K)$ is a subgroup of $H^1(U)$ which is defined similar to $\fil_{D}^{bk} H^1(K)$, where we only replace $\fil_{n_\lambda-1}^{bk} H^1(K_\lambda)$ by $\fil_{n_\lambda}^{ms} H^1(K_\lambda)$.

**Definition 9.6.** We define the quotient $\pi_{1}^{abk}(X, D)$ of $\pi_{1}^{ab}(U)$ to be the Pontryagin dual of $\fil_{D}^{bk} H^1(K) \subset H^1(U)$, i.e.,

$$
\pi_{1}^{abk}(X, D) := \text{Hom}_{cont}(\fil_{D}^{bk} H^1(K), \mathbb{Q}/\mathbb{Z}) = \text{Hom}(\fil_{D}^{bk} H^1(K), \mathbb{Q}/\mathbb{Z}).
$$

Since $H^1_{et}(U, \mathbb{Z}/p^m) = \rho^m H^1(U, \mathbb{Q}/\mathbb{Z})$, it follows that

$$
\pi_{1}^{abk}(X, D)/p^m \cong \text{Hom}_{cont}(\fil_{D}^{bk} H^1_{et}(U, \mathbb{Z}/p^m), \mathbb{Q}/\mathbb{Z}).
$$

Since $\fil_{D}^{bk} H^1(K)$ is a discrete topological group, it follows that $\pi_{1}^{abk}(X, D)$ is a profinite group. Moreover, [41, Theorem 2.9.6] implies that

$$
(9.9) \quad \pi_{1}^{abk}(X, D)^{\vee} \cong \fil_{D}^{bk} H^1(K) \text{ and } (\pi_{1}^{abk}(X, D)/p^m)^{\vee} \cong \fil_{D}^{bk} H^1_{et}(U, \mathbb{Z}/p^m).
$$

**Lemma 9.7.** The quotient map $\pi_{1}^{ab}(U) \rightarrow \pi_{1}^{abk}(X, D)$ factors through $\pi_{1}^{adiv}(X, D)$.

**Proof.** This is a straightforward consequence of Theorem [9.1] and [15, Theorem 7.16] once we recall that

$$
(9.10) \quad \pi_{1}^{adiv}(X, D) = \text{Hom}(\fil_{D} H^1(K), \mathbb{Q}/\mathbb{Z}).
$$

\[ \square \]

**Remark 9.8.** Using [11 § 3], one can mimic the construction of [15 § 7] to show that $\pi_{1}^{abk}(X, D)$ is isomorphic to the abelianization of the automorphism group of the fiber functor of a certain Galois subcategory of the category of finite étale covers of $U$. Under this Tannakian interpretation, $\pi_{1}^{abk}(X, D)$ characterizes the finite étale covers of $U$ whose ramifications are bounded at each generic point of $D$ by means of Kato’s Swan conductor.
9.3. Reciprocity for \( C_{KS}(X|D) \). We shall continue with the setting of §9.2. Before we prove the reciprocity theorem for \( C_{KS}(X|D) \), we recall the construction of the reciprocity map for \( C_{IJ} \) from [15, §5.4]. We let \( P = (p_0, \ldots, p_s) \) be any Parshin chain on \( X \) with the condition that \( p_s \in U \). Let \( X(P) = \overline{\{p_s\}} \) be the integral closed subscheme of \( X \) and let \( U(P) = U \cap X(P) \). We let \( V \subset k(p_s) \) be an \( s \)-DV dominating \( P \). Then \( Q(V^h) \) is an \( s \)-dimensional Henselian local field. By [15, Proposition 5.1], we therefore have the reciprocity map

\[
\rho_{Q(V^h)}: K^M_s(Q(V^h)) \rightarrow \text{Gal}(Q(V^h)/Q(V^h)) \cong \pi_1^{ab}(\text{Spec}(Q(V^h))).
\]

Taking the sum of these maps over \( \mathcal{V}(P) \) and using [15, Lemma 5.10], we get a reciprocity map

\[
\rho_P: K^M_s(k(P)) \rightarrow \pi_1^{ab}(\text{Spec}(k(P))) \rightarrow \pi_1^{ab}(U),
\]

where last map exists because \( p_s \in U \). Taking sum over all Parshin chains on the pair \( (U \subset X) \), we get a reciprocity map \( \rho_{U|X}: I_{U|X} \rightarrow \pi_1^{ab}(U) \). By [13, Theorem 5.13, Proposition 5.15], this descends to a continuous homomorphism of topological groups

\[
\rho_{U|X}: C_{U|X} \rightarrow \pi_1^{ab}(U).
\]

Recall that \( \pi_1^{ab}(X,D)_0 \) is the kernel of the composite map \( \pi_1^{ab}(X,D) \rightarrow \pi_1^{ab}(X) \rightarrow \hat{\mathbb{Z}} \). One defines \( \pi_1^{abk}(X,D)_0 \) similarly. One of the main results of this paper is the following.

**Theorem 9.9.** There is a continuous homomorphism

\[
\rho'_{X|D}: C(X|D) \rightarrow \pi_1^{abk}(X,D)
\]

with dense image such that the diagram

\[
\begin{array}{ccc}
C_{U|X} & \xrightarrow{\rho_{U|X}} & \pi_1^{ab}(U) \\
\downarrow \rho'_{U|X} & & \downarrow \rho'_{U|X} \\
C(X|D) & \xrightarrow{\rho'_{X|D}} & \pi_1^{abk}(X,D)
\end{array}
\]

is commutative. If \( X \) is normal and \( U \) is regular, then \( \rho'_{X|D} \) induces an isomorphism of finite groups

\[
\rho'_{X|D}: C(X|D)_0 \cong \pi_1^{abk}(X,D)_0.
\]

**Proof.** We first show the existence of \( \rho'_{X|D} \). Its continuity and density of its image will then follow by the corresponding assertions for \( \rho_{U|X} \), shown in [15]. In view of (9.13), we only have to show that if \( \chi \) is a character of \( \pi_1^{abk}(X,D) \), then the composite \( \chi \circ \rho'_{X|D} \circ \rho_{U|X} \) annihilates \( \text{Ker}(\rho'_{X|D}) \). By (3.3), we only need to show that \( \chi \circ \rho'_{X|D} \circ \rho_{U|X} \) annihilates the image of \( \hat{K}^1_{1/3}(\mathcal{O}^k_{X,P}|I_D) \rightarrow C_{U|X} \), where \( P \) is any maximal Parshin chain on \( (U \subset X) \). But the proof of this is completely identical to that of [15, Theorem 8.1], only difference being that we have to use part (1) of Theorem 9.2 instead of part (2).

We now assume that \( X \) is normal and \( U \) is regular. In this case, we can replace \( C(X|D) \) by \( C_{KS}(X|D) \) by Theorem 3.3. We show that \( \rho'_{X|D} \) is injective on all of \( C_{KS}(X|D) \). By Lemmas 3.3 and 9.7, there is a diagram

\[
\begin{array}{ccc}
\text{fil}^{bk} D H^1(K) & \xrightarrow{\rho'^{\gamma}_{X|D}} & C_{KS}(X|D)^\gamma \\
\downarrow \alpha' & & \downarrow \beta' \\
\text{fil} D H^1(K) & \xrightarrow{\rho^{\gamma}_{X|D}} & C_{KS}(X,D)^\gamma,
\end{array}
\]

...
whose vertical arrows are injective. It is clear from the construction of the reciprocity maps that this diagram is commutative.

To show that \( \rho'_{X|D} \) is injective, it suffices to show that \( \rho'_{X|D} \) is surjective (see [15 Lemma 7.10]). We fix a character \( \chi \in C_{KS}(X | D)_{\nu} \) and let \( \overline{\chi} = \beta'(\chi) \). Since \( \rho'_{X|D} \) is surjective by [16 Theorem 1.1], we can find a character \( \chi' \in \text{fil}_D H^1(K) \) such that \( \overline{\chi} = \rho'_{X|D}(\chi') \). We need to show that \( \chi' \in \text{fil}^{abk}_D H^1(K) \).

We fix a point \( x \in \text{Irr}_C \) and let \( \chi'_x \) be the image of \( \chi' \) in \( H^1(K_x) \). We need to show that \( \chi'_x \in \text{fil}^{abk}_{n_x} H^1(K_x) \), where \( n_x \) is the multiplicity of \( D \) at \( x \). By Corollary 9.11 it suffices to show that for some maximal Parshin chain \( P = (p_0, \ldots, p_{d-2}, x, \eta) \) on \( (U \subset X) \) and \( dDV \) \( V \subset K \) dominating \( P \), the image of \( \chi'_x \) in \( H^1(Q(V^h)) \) lies in the subgroup \( \text{fil}^{abk}_{n_x} H^1(Q(V^h)) \). But this is proven by repeating the proof of [16 Theorem 1.1] mutatis mutandis by using only one modification. Namely, we need to use part (1) of Theorem 9.2 instead of part (2).

The surjectivity of \( \rho'_{X|D} \) on the degree zero part follows because the top horizontal arrow in the commutative diagram

\[
C_{KS}(X, D)_0 \xrightarrow[\rho'_{X|D}]{\rho} \pi_{1, \text{div}}^{\text{abk}}(X, D)_0
\]

is surjective by [16 Theorem 1.1]. The finiteness of \( C_{KS}(X | D)_0 \) follows because \( \beta \) is surjective by Lemma 3.3 and \( C_{KS}(X, D)_0 \) is finite by [16 Theorem 4.8]. This concludes the proof.

Using Theorem 9.9 and [15 Lemma 8.4], we get the following.

**Corollary 9.10.** Under the additional assumptions of Theorem 9.9, the map

\[
\rho'_{X|D}: C_{KS}(X | D)/m \to \pi_{1, \text{abk}}^{\text{abk}}(X, D)/m
\]

is an isomorphism of finite groups for every integer \( m \geq 1 \).

### 9.4. Filtration of Kerz-Zhao

Let \( k \) be a finite field of characteristic \( p \) and \( X \) an integral regular projective scheme over \( k \) of dimension \( d \geq 1 \). Let \( D \subset X \) be an effective Cartier divisor with complement \( U \) such that \( D_{\text{red}} \) is a simple normal crossing divisor. Let \( K \) denote the function field of \( X \). By [22 Theorems 1.1.5, 4.1.4], there is an isomorphism

\[
(9.17) \quad \lambda_m: H^1_{\text{et}}(U, \mathbb{Z}/p^m) \cong \varprojlim_n H^1_{\text{et}}(X, \mathbb{K}_{d, X|D}^{\text{M}}/p^n)^{\nu}.
\]

Each group \( H^1_{\text{et}}(X, \mathbb{K}_{d, X|D}^{\text{M}}/p^n)^{\nu} \) is finite by [30 Theorem 3.3.1] and Corollary 9.10. It follows that (9.17) is an isomorphism of discrete torsion topological abelian groups.

Since \( H^1_{\text{et}}(X, \mathbb{K}_{d, X|D}^{\text{M}}/p^n)^{\nu} \to \lim_{\longrightarrow} H^1_{\text{et}}(X, \mathbb{K}_{d, X|D}^{\text{M}}/p^n)^{\nu} \), we can define

\[
\text{fil}^{abk}_D H^1_{\text{et}}(U, \mathbb{Z}/p^m) := (\lambda^{-1}_m^{-1})(H^1_{\text{et}}(X, \mathbb{K}_{d, X|D}^{\text{M}}/p^n)^{\nu}).
\]

This filtration was defined by Kerz-Zhao [30 Definition 3.3.7]. We let \( \text{fil}^{abk}_{D, \text{et}} H^1_{\text{et}}(U, \mathbb{Q}_p / \mathbb{Z}_p) = \lim_{\longrightarrow} \text{fil}^{abk}_{D, \text{et}} H^1_{\text{et}}(U, \mathbb{Z}/p^m) \) and \( \text{fil}^{abk}_{D, \text{et}} H^1(K) = H^1(K) \{ p \} \oplus \text{fil}^{abk}_{D, \text{et}} H^1(U, \mathbb{Q}_p / \mathbb{Z}_p) \).

Using [30 Theorem 3.3.1] and Corollary 9.10 we get the following logarithmic version of Theorem 8.11 (without assuming that \( D_{\text{red}} \) is regular). This identifies the filtration due to Kerz-Zhao to the one induced by the Brylinski-Kato filtration.

**Corollary 9.11.** As subgroups of \( H^1(K) \), one has

\[
\text{fil}^{abk}_{D, \text{et}} H^1(K) = \text{fil}^{abk}_D H^1(K).
\]
10. Counterexamples to Nisnevich descent

We shall now prove Theorem 1.5. Let \( k \) be a finite field of characteristic \( p \) and \( X \) an integral regular projective scheme of dimension two over \( k \). Let \( C \subset X \) be an integral regular curve with complement \( U \). Let \( K = k(X) \) and \( \mathfrak{f} = k(\lambda) \), where \( \lambda \) is the generic point of \( C \). Let \( K_\lambda \) be the Henselian discrete valuation field with ring of integers \( \mathcal{O}_{X, \lambda}^h \) and residue field \( \mathfrak{f} \). Fix a positive integer \( m_0 = p^r m' \), where \( r \geq 1 \) and \( p \nmid m' \). Assume that \( p \neq 2 \). Then Matsuda has shown (see the proof of [37, Proposition 3.2.7]) that there is an isomorphism

\[
\eta^\prime: \frac{\text{fil}^{\text{ms}}_{m_0} H^1(K_\lambda)}{\text{fil}^{\text{bk}}_{m_0-1} H^1(K_\lambda)} \cong B_r \Omega^1_\mathfrak{f},
\]

where \( B_r \Omega^1_\mathfrak{f} \) is an increasing filtration of \( \Omega^1_\mathfrak{f} \), recalled in the proof of Lemma 4.4. Suppose that \( B_1 \Omega^1_\mathfrak{f} = d(\mathfrak{f}) \subset \Omega^1_\mathfrak{f} \) is not zero. Then \( B_r \Omega^1_\mathfrak{f} \neq 0 \) for every \( r \geq 1 \). Hence, the left hand side of (10.1) is not zero. Since the restriction map \( \delta_\lambda: H^1(K) \rightarrow H^1(K_\lambda) \) is surjective, it follows that we can find a continuous character \( \chi \in H^1(K) \) such that \( \delta_\lambda(\chi) \in \text{fil}^{\text{ms}}_{m_0} H^1(K_\lambda) \setminus \text{fil}^{\text{bk}}_{m_0-1} H^1(K_\lambda) \).

We let \( U' \subset U \) be the largest open subscheme where \( \chi \) is unramified and let \( C' = X \setminus U' \) with the reduced closed subscheme structure. Since \( X \) is a surface, we can find a morphism \( f: X' \rightarrow X \) which is a composition of a sequence monoidal transformations such that the reduced closed subscheme \( f^{-1}(C') \) is a simple normal crossing divisor. In particular, \( E_0 \rightarrow C \) is an isomorphism if we let \( E_0 \) be the strict transform of \( C \). We let \( E = f^{-1}(C') \) with reduced structure. Note that there is a finite closed subset \( T \subset C' \) such that \( f^{-1}(X \setminus T) \rightarrow X \setminus T \) is an isomorphism.

We let \( E = E_0 + E_1 + \cdots + E_s \), where each \( E_i \) is integral. We let \( \lambda_i \) be the generic point of \( E_i \) so that \( \lambda_0 = \lambda \). Let \( \lambda_0 : H^1(K) \rightarrow \mathbb{Z} \) be the Artin conductor (see [37, Definition 3.2.5]). Theorem 9.1 implies that \( \lambda_0(\chi) = m_0 \). We let \( m_i = \lambda_0(\chi) \) for \( i \geq 1 \) and define \( D' = \sum_{i=0}^{s} m_i E_i \). It is then clear that \( \chi \in \text{fil}_D H^1(K) \setminus \text{fil}^{\text{bk}} H^1(K) \). We have thus found a smooth projective integral surface \( X' \) and an effective Cartier divisor \( D' \subset X' \) with the property that \( D'_\text{red} \) is a simple normal crossing divisor and \( \text{fil}_D H^1(K) \setminus \text{fil}^{\text{bk}} H^1(K) \neq 0 \) if \( K \) is the function field of \( X' \). It follows that \( \text{Ker}(\pi_1^{\text{div}}(X', D') \rightarrow \pi_1^{\text{ab}}(X', D')) \neq 0 \).

Equivalently, \( \text{Ker}(\pi_1^{\text{div}}(X', D')_0 \rightarrow \pi_1^{\text{ab}}(X', D')_0) \neq 0 \).

We now look at the the commutative diagram

\[
\begin{array}{ccc}
\text{CH}_0(X'|D')_0 & \xrightarrow{cyc_{X,D'}} \ & \pi_1^{\text{div}}(X', D')_0 \\
\downarrow & & \downarrow \\
H^1_M(X'|D', \mathbb{Z}(2))_0 & \xrightarrow{cyc_{X,D'}} & \pi_1^{\text{ab}}(X', D')_0.
\end{array}
\]

The top horizontal arrow is an isomorphism by [16, Theorems 1.2, 1.3] and the bottom horizontal arrow is an isomorphism by Corollary 1.4. We conclude that the left vertical arrow is surjective but not injective.

To complete the construction of a counterexample, what remains is to find a pair \((X, C)\) such that \( d(\mathfrak{f}) \subset \Omega^1_\mathfrak{f} \) is not zero. But this is an easy exercise. For instance, take \( X = \mathbb{P}^2_k \) and \( C \subset \mathbb{P}^2_k \) a coordinate hyperplane. Then \( \mathfrak{f} = k(t) \) is a purely transcendental extension of degree one and \( d(t) \) is a free generator of \( \Omega^1_\mathfrak{f} \). We remark that we had assumed above that \( p \neq 2 \), but this condition can be removed using the proof of [15, Theorem 6.3].
Let \((X', D')\) be as above and let \(F = k(t)\) be a purely transcendental extension of degree one. We let \(\overline{X} = X'_F\) and \(\overline{D} = D'_F\). Then we have a commutative diagram

\[
\begin{array}{ccc}
\text{CH}_0(X'|D') & \longrightarrow & \text{CH}_0(\overline{X}|\overline{D}) \\
\downarrow & & \downarrow \\
H^4_M(X'|D', \mathbb{Z}(2)) & \rightarrow & H^4_M(\overline{X}|\overline{D}, \mathbb{Z}(2)),
\end{array}
\]

where the horizontal arrows are the flat pull-back maps. It easily follows from the proof of [32, Proposition 4.3] that the top horizontal arrow is injective. It follows that the right vertical arrow is not injective. This shows the failure of Nisnevich descent for the Chow groups with modulus over infinite fields too.

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