ON THE EVOLUTION OF SCATTERING DATA UNDER PERTURBATIONS OF THE TODA LATTICE

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Abstract. We present the results of an analytical and numerical study of the long-time behavior for certain Fermi-Pasta-Ulam (FPU) lattices viewed as perturbations of the completely integrable Toda lattice. Our main tools are the direct and inverse scattering transforms for doubly-infinite Jacobi matrices, which are well-known to linearize the Toda flow. We focus in particular on the evolution of the associated scattering data under the perturbed vs. the unperturbed equations.

1. Introduction

The purpose of this work is to numerically investigate the long time behavior of solutions to certain perturbations of the completely integrable Toda lattice. In its wider context, this problem lies within a set of fundamental questions concerning the qualitative features of solutions to nonlinear Hamiltonian partial differential or difference equations (PDEs or PΔEs, respectively). It is well known that in these types of equations several different phenomena can appear over long times, among them blow-up, scattering to the free evolution, or the emergence of stable nonlinear structures, such as solitary waves and breather solutions. A detailed description of any such phenomenon often depends on the precise structure of the equation in question. However, there is the common belief, usually referred to as the soliton resolution conjecture, that (in the absence of finite time blow-up) generic solutions can be decomposed at large times into a sum of solitary waves plus a dispersive tail (i.e., radiation).

Loosely speaking, what we will call here solitary waves are special solutions to nonlinear PDEs or PΔEs which travel with constant speed and without changing their profile. Their existence reflects a certain balance between nonlinear and dispersive effects in a given evolution equation. The statement of the soliton resolution conjecture seems to be the natural scenario for the long time behavior of Hamiltonian PDEs or PΔEs, but there are at this time relatively few rigorous results in this direction. Part of the reason for this is that many current mathematical techniques, while extremely powerful, are intrinsically linear. But the very nature of the problem means that we are interested in studying the competition between nonlinearity and dispersion over infinite time intervals.

There is one very well-known example, however, in which the description of the long time asymptotic behavior of solutions is fully known, namely completely integrable equations. These are Hamiltonian equations which have so “many” conserved quantities that they can be “completely” integrated. In the infinite dimensional setting, the definition of complete integrability is not fully set. We will adopt the (highly practical) point of view that a partial differential or difference equation is completely integrable if it can be linearized through some bijective transformation. The transformation of choice is provided by the direct and inverse scattering data associated to the Lax operator of the integrable equation, and it is this scattering transform which can be thought of as a non-linear version of the Fourier transform. One of the advantages of the scattering transform is that it separates, in a very precise, quantitative sense the “parts” of the initial data which lead
to the asymptotic emergence of each soliton, and those that lead to the dispersive tail. This understanding, combined with the nonlinear stationary phase/steepest descent techniques introduced by P. Deift and X. Zhou [6], has yielded a large number of rigorous asymptotic results for various completely integrable PDEs (see, for example, [5] and [7], among many others).

The question we wish to investigate in this paper is whether or not this type of analysis, based on the study of the scattering data, can be extended to apply to certain (non-integrable) perturbations of completely integrable PDEs or PDEs. We are aiming at a more quantitative understanding of the behavior of solutions than is offered by (infinite dimensional) KAM theory. At the same time, we want to work in a situation in which the perturbations are interesting in and of themselves, but which also allow us to minimize the technical challenges (both analytic and numeric) which are not directly related to the question of the evolution of solitary waves under perturbations of the integrable case.

We focus on certain so-called Fermi-Past-Ulam (FPU) lattices, viewed as perturbations of the (completely integrable) Toda lattice. There already exists an extensive body of work, both numerical and analytical, focused on the study of the Toda and FPU lattices, and it is unfortunately impossible to include a comprehensive bibliography here. We wish however to mention the work of G. Friesecke, R. L. Pego and A. D. Wattis (see [14, 15, 16, 17, 18]) as particularly relevant to our study, since they proved the existence of solitary waves for general FPU lattices. Furthermore, we rely heavily on the connection between the Toda lattice and Jacobi matrices, in particular on the scattering theory for Jacobi matrices (see, for example, [30] and [31] and the references therein). Finally, we should mention that the very same perturbed evolution equations we consider (see (2.12) and (2.15)) are also studied in [21]. However, there the authors consider the question of Lieb-Robinson bounds, which provide information on the behavior of solutions for short (vs. arbitrarily long) times.

One can study the behavior of solutions to (not necessarily integrable) PDEs and PDEs in various other asymptotic regimes, (among them, semi-classical and continuum limits), or focus on questions of stability of certain special solutions; these are very well-established and well-studied problems, using a variety of techniques, including some from the completely integrable arsenal. As an example, let us mention the recent and on-going work of B. Dubrovin, T. Claeys, T. Grava, C. Klein and K. McLaughlin, (see, for example, [3, 4, 10]) who, inspired by ideas and conjectures of B. Dubrovin [9], study the behavior of a large class of Hamiltonian perturbations of Burgers’ equation which are required to be integrable up to a certain order in the perturbation parameter.

Our paper is organized in two separate parts. In the first part, comprising Sections 2 and 3, we present the analytical background of our problem, as well as the main results concerning the evolution equations under considerations. These results consist of global-in-time existence and (spatial) decay properties of the solutions on the one hand, as well as set-up and properties of the associated scattering data. In particular, we deduce the evolution equations for the scattering data – which, unsurprisingly, turn out to be nonlinear, nonlocal perturbations of the linear evolution equations that the scattering follow in the integrable, Toda case. The results from Sections 2 and 3 are new, but their proofs are based on fairly standard arguments, adapted to the case at hand. For completeness, and in order to make the paper both self-contained and easier to read, we include all the longer, more technical proofs in the Appendix. The study of the long-time asymptotics of the perturbed equations for the scattering data are known to be highly nontrivial even in the case of dispersive equations (see, for example, the seminal paper [8]). Ultimately, we wish to perform the same type of analysis as in [8] for the equations considered here, but the situation is even more complicated due to the existence, in this case, of solitary, traveling waves.

Thus the numerical part of our project is an essential first step in the study of the long-time asymptotics for solutions of these perturbations of the Toda lattice. In the second, numerical, part of the paper we analyze and report on the long-time behavior of solutions and scattering data for a variety of initial data and of perturbations. Even more results can be found on the project’s
In Section 3 we present our numerical scheme and justify its validity in approximating the actual solutions of the perturbed equations. Section 4 contains our results for initial data given by a Toda soliton, which is then allowed to evolve under the perturbed equation. We study not only the shape of the solution, but also, more importantly, the behavior of the scattering data, with a focus on the eigenvalues. In Section 5 we perform the same analysis for initial data which are without solitons in the Toda lattice – or, equivalently, for initial data whose scattering data do not have any eigenvalues. Finally, Section 6 contains results on so-called clean solitary waves – that is, numerical solutions which correspond to the analytical solitary waves whose existence is known for these perturbed lattices. For these clean solitary waves we compute the scattering data in order to compare them with the long-time asymptotics obtained from non-cleaned initial data, as well as perform numerical experiments to simulate collisions and multi-soliton type solutions.

What we find consistently in our study is that, while the eigenvalues present at time 0 in the scattering data of a solution are, naturally, no longer constant, they do converge very fast to new, slightly perturbed values. However, in addition to this, we also observe the emergence of new eigenvalues from the edges of the absolutely continuous (ac) spectrum. The behavior of these new eigenvalues is more difficult to analyze than that of the persistent, initial eigenvalues, as they do not appear to stabilize at values outside of the ac spectrum. On the physical side, that means that solutions remember, as time goes to infinity, the solitary waves “contained” in their initial data, albeit with slightly modified velocities and amplitudes, but that in addition new, small “bumps” appear instantaneously when the perturbed evolution starts. These new bumps do not appear to approach stationary waves, but rather their speeds and amplitudes seem to decrease with time, which could make them in fact part of the dispersive tails of the solutions. However, our current numerical analysis cannot confidently predict the long time behavior of these new waves (or, equivalently, of their associated eigenvalues), and a different project, based on the numerical study of Riemann-Hilbert problems, is currently investigating this phenomenon.

2. Background and evolution equations

We consider the classical problem of a 1-dimensional chain of particles with nearest neighbor interactions. We will assume throughout that the system is uniform (contains no impurities) and that the mass of each particle is normalized to 1. It is then well-known that the equation that governs the evolution is

\[ \partial_t^2 q_n(t) = V'(q_{n+1}(t) - q_n(t)) - V'(q_n(t) - q_{n-1}(t)), \]

where \( q_n(t) \) denotes the displacement of the \( n \)th particle from its equilibrium position, and \( V \) is the interaction potential between neighboring particles. If \( V'(r) \) is proportional to \( r \), then the interaction is linear, and the solutions are given by linear superpositions of normal modes. In this case there is no transfer of energy between the modes.

The general belief in the early 1950s was that if a nonlinearity is introduced, then energy would flow between the different modes, eventually leading to a stable state of statistical equilibrium. Fermi, Pasta, and Ulam [13] set out to numerically investigate this phenomenon through one of the first-ever computer experiments, which was performed at Los Alamos in 1953. What they found instead was quasiperiodic motion of the system. This phenomenon was later explained through the connection to KdV solitons (see, for example, [22]), and by the discovery in 1972 by M. Toda [33] of what is now called the Toda lattice. It is obtained by setting the potential in (2.1) as:

\[ V_0(r) = e^{-r} + r - 1. \]

We will call \( V_0 \) the Toda potential, and it leads to an explicit form of the evolution equations, namely:

\[ \partial_t^2 q_n(t) = e^{q_n(t) - q_{n+1}(t)} - e^{q_{n-1}(t) - q_n(t)}. \]
One of the remarkable features of this equation, observed by Toda himself, is that it has soliton solutions. For example, the 1-soliton solution for the displacements is given by the 2-parameter family

\[ q_n(t) = \log \left( \frac{1 + \frac{\gamma}{1 - e^{-2k(k-1)-2\kappa \sigma t}}}{1 + \frac{\gamma}{1 - e^{-2k(k-2)\kappa \sigma t}}} \right), \]

where \( \gamma > 0, k > 0 \) is the wave number, \( c = \frac{\sinh k}{k} > 1 \) is the speed of propagation, and \( \sigma = \pm 1 \) is the constant determining the direction of propagation.

But much more is true, as the equation (2.2) turns out to be completely integrable. Indeed, note that the new variables satisfy (2.3)

\[ (p_n)_{n \in \mathbb{Z}} \quad \text{and} \quad (q_n)_{n \in \mathbb{Z}} \quad \text{such that} \]

\[ (p_n)_{n \in \mathbb{Z}} \quad \text{and} \quad (q_n)_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z}). \]

Then the equation (2.2) can be rewritten as a first-order system:

\[
\begin{align*}
\partial_t p_n(t) & = e^{-(q_n(t) - q_{n-1}(t)) - e^{-(q_{n+1}(t) - q_n(t))} \\
\partial_t q_n(t) & = p_n(t),
\end{align*}
\]

for all \( n \in \mathbb{Z} \), and \( t \in \mathbb{R} \). Note first that these can be seen as Hamiltonian equations of motion, with Hamiltonian function:

\[ \mathcal{H}_0(p, q) = \sum_{n \in \mathbb{Z}} \left( \frac{p_n^2}{2} + V_0(q_n) \right). \]

Complete integrability of the Toda lattice was proven by H. Flaschka in 1974 by introducing a change of variables that allowed him to set the system in Lax pair form. Flaschka’s change of variables is given by

\[ a_n = \frac{1}{2} e^{-\frac{q_{n+1}^2 - q_n}{2}} \quad \text{and} \quad b_n = -\frac{1}{2} p_n. \]

Note that the new variables satisfy \( (a_n)_{n \in \mathbb{Z}}, (b_n)_{n \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z}) \), and they obey the evolution equations

\[ \partial_t b_n = -\frac{1}{2} \partial_t p_n = \frac{1}{2} \left[ e^{-(q_n - q_{n-1})} - e^{-(q_{n+1} - q_n)} \right] = 2 \left( a_n - a_{n-1} \right), \]

and

\[ \partial_t a_n = \frac{1}{4} e^{-(q_{n+1} - q_n)^2} \left( -q_{n+1} + q_n \right) = a_n (b_n - b_{n+1}). \]

Introduce the second-order linear difference operators \( L \) and \( P \) defined on \( \ell^2(\mathbb{Z}) \) by

\[ (L f)_n = a_{n-1} f_{n-1} + b_n f_n + a_n f_{n+1} \]

(2.6)

and

\[ (P f)_n = -a_{n-1} f_{n-1} + a_n f_{n+1} \]

(2.7)

and recall that in the standard basis \( L = L((a_n)_{n \in \mathbb{Z}}, (b_n)_{n \in \mathbb{Z}}) \) is a Jacobi matrix (symmetric, tridiagonal with positive off-diagonals) and \( P \) is a bounded skew-adjoint operator, i.e., \( P^* = -P \):

\[ L = \begin{pmatrix} \vdots & b_{n-1} & a_{n-1} & 0 & \vdots \\ a_{n-1} & b_n & a_n & \vdots \\ 0 & a_n & b_{n+1} & \vdots \\ \vdots & \vdots & \vdots & \ddots & \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} \vdots & 0 & a_{n-1} & 0 & \vdots \\ 0 & a_{n-1} & 0 & a_n & \vdots \\ \vdots & \vdots & \vdots & \ddots & \end{pmatrix}. \]

Flaschka’s main observation was that the system of equations (2.6) and (2.7) is equivalent to

\[ \partial_t L = [P, L] = PL - LP; \]

(2.8)
Proposition 2.1. Let $H_\varepsilon$ be the Hamiltonian associated to the perturbed potential $V_\varepsilon$,
\[
H_\varepsilon(p, q) = \sum_{n \in \mathbb{Z}} \frac{p_n^2}{2} + V_\varepsilon(q_{n+1} - q_n).
\]
The equations of motion induced by $H_\varepsilon$ are
\[
\begin{align*}
\partial_t p_n &= e^{-(q_{n+1} + q_{n-1})} - e^{-(q_{n+1} - q_{n})} + \varepsilon \left[ u'(q_{n+1} - q_n) - u'(q_n - q_{n-1}) \right] \\
\partial_t q_n &= p_n.
\end{align*}
\]
In terms of the $a, b$ variables, this translates to
\[
\begin{align*}
\partial_t a_n &= a_n \left[ b_{n+1} - b_n \right] \\
\partial_t b_n &= 2 \left( a_n^2 - a_{n-1}^2 \right) + \frac{\varepsilon}{2} \left( c_{n-1} - c_n \right)
\end{align*}
\]
where
\[
c_n = u' \left( \log \left( \frac{1}{4a_n^2} \right) \right) \quad \text{for all } n \in \mathbb{Z}.
\]
The evolution of the Jacobi matrix $L$ is given by the equation:
\[
\partial_t L = [P, L] + \varepsilon U,
\]
where $U$ is a diagonal matrix with
\[
U_{nn} = \frac{1}{2} \left( c_{n-1} - c_n \right) \quad \text{for all } n \in \mathbb{Z}.
\]
Remark 2.2. We note that one of the conservation laws associated to the Toda lattice still holds true; namely a straightforward calculation shows that $\text{tr}(L - L_0)$ (where $L_0 = L(\{\frac{1}{2}\}, \{0\})$) is the free Jacobi matrix) is conserved in time by the perturbed evolution. It is however no longer true that traces of higher powers of $L - L_0$ are conserved, which is not surprising as we expect the perturbed dynamics to be non-integrable.

We now rewrite (2.12) in terms of the relative displacement coordinates $s_n = q_{n+1} - q_n$:
\[
\begin{align*}
\partial_t p_n &= V'_\varepsilon(s_n) - V'_\varepsilon(s_{n-1}) \\
\partial_t s_n &= p_{n+1} - p_n,
\end{align*}
\]
and present the theorem which states that the system (2.17) is well posed globally in time in 
\( L^2(\mathbb{Z}) \oplus L^2(\mathbb{Z}) \). This result is not new (see, for example, [21]), but we include it for the reader’s convenience. The proof can be found in the Appendix.

**Theorem 1.** Suppose that the perturbed potential \( V_\varepsilon \) is in \( C^2(\mathbb{R}) \), satisfying \( V_\varepsilon(r) \geq 0 \) for all \( r \in \mathbb{R} \), with \( V_\varepsilon(0) = V'_\varepsilon(0) = 0 \), \( V''_\varepsilon(0) > 0 \), and \( V(r) \to +\infty \) as \( |r| \to +\infty \). Then (2.17) is well posed globally in time in \( L^2(\mathbb{Z}) \oplus L^2(\mathbb{Z}) \), and the Hamiltonian

\[
H_\varepsilon(p, s) = \sum_{n \in \mathbb{Z}} \frac{p_n^2}{2} + V_\varepsilon(s_n)
\]

is finite and conserved in time along the solutions \( (p, s) = ((p_n)_{n \in \mathbb{Z}}, (s_n)_{n \in \mathbb{Z}}) \in L^2(\mathbb{Z}) \oplus L^2(\mathbb{Z}) \).

We note that

\[
(p, s) \in L^2(\mathbb{Z}) \oplus L^2(\mathbb{Z}) \quad \text{if and only if} \quad \left( (a_n - \frac{1}{2})_{n \in \mathbb{Z}}, (b_n)_{n \in \mathbb{Z}} \right) \in L^2(\mathbb{Z}) \oplus L^2(\mathbb{Z}),
\]

with \( a_n > 0 \) for all \( n \in \mathbb{Z} \). Moreover, \( p \) and \( s \) are bounded sequences if and only if \( a, b \), and \( \left( \frac{1}{a_n} \right)_{n \in \mathbb{Z}} \) are bounded. In particular, for solutions \( (p(t), s(t)) \) in \( L^2(\mathbb{Z}) \oplus L^2(\mathbb{Z}) \), the bound (A.1) implies that the elements of the sequence \( a(t) \) stay uniformly away from zero, i.e.,

\[
0 < \inf_{t \geq 0} \inf_{n \in \mathbb{Z}} a_n(t).
\]

This remark leads to the following corollary to the Theorem 1.

**Corollary 2.3.** Suppose \( a^0 \) and \( b^0 \) are bounded sequences such that \( a^0 - \frac{1}{2}, b^0 \in L^2(\mathbb{Z}) \) and assume that \( V_\varepsilon \) satisfies the same assumptions as in Theorem 1. Then there exists a unique global solution \( (a(t), b(t)) \) to (2.13), which depends continuously on the initial data \( (a(0), b(0)) = (a^0, b^0) \) and satisfies

\[
(a(t) - \frac{1}{2}, b(t)) \in L^2(\mathbb{Z}) \oplus L^2(\mathbb{Z}) \quad \text{for all} \quad t \geq 0.
\]

Moreover,

\[
\|a(t)\|_{L^\infty} + \|b(t)\|_{L^\infty} \leq C,
\]

for some constant \( C > 0 \) that depends only on the initial data, and

\[
0 < \inf_{t \geq 0} \inf_{n \in \mathbb{Z}} a_n(t).
\]

In fact, more can be shown regarding the solutions \((a, b)\) of the perturbed lattice. We let \( L^1_w(\mathbb{Z}) \) denote the weighted \( L^1 \)-space, with the weight function given by \( n \mapsto 1 + |n| \), and define the Banach space \( X^1_w = L^1_w(\mathbb{Z}) \oplus L^1_w(\mathbb{Z}) \) equipped with the norm

\[
\|(x, y)\|_{w,1} = \sum_{n \in \mathbb{Z}} (1 + |n|) (|x_n| + |y_n|).
\]

Then we have the following result, which establishes further spatial decay for the solutions of the perturbed lattice.

**Theorem 2.** Suppose \( \tilde{a^0} \) and \( b^0 \) are bounded sequences such that

\[
(\tilde{a^0}, b^0) \in X^1_w,
\]

and suppose that \( (a(t), b(t)) \) is the unique global solution of the perturbed lattice (2.13) corresponding to the initial conditions

\[
a(0) = \frac{1}{2} + \tilde{a^0} > 0 \quad \text{and} \quad b(0) = b^0,
\]

under the same assumptions on \( u \) and \( V_\varepsilon \) as in Theorem 1. Then

\[
(a(t) - \frac{1}{2}, b(t)) \in X^1_w.
\]
for all times \( t \geq 0 \).

In the case of the Toda lattice \((\varepsilon = 0)\), this result was proven by G. Teschl [32]. Our proof, which can be found in the Appendix, is a modification of his proof, adapted to deal with the perturbation term in the evolution equation.

Our numerical results, which will be detailed in Sections 4 through 7, rely on the existence of solitary waves in FPU lattices. This was established in the following theorem, due to [18], that states existence of such solutions in FPU lattices for a certain class of potentials.

**Theorem 3** (Theorem 1, [18]). Let \( V \in C^2(\mathbb{R}) \), \( V(r) \geq 0 \) for all \( r \in \mathbb{R} \), \( V(0) = 0 \), and \( V \) be super quadratic on at least one side of the origin, that is to say

\[
\frac{V(r)}{r^2} \text{ increases strictly with } |r| \text{ for all } r \in \Lambda,
\]

where \( \Lambda = (-\infty, 0) \) or \( \Lambda = (0, \infty) \). Then there exists some \( K_0 \geq 0 \) such that for every \( K > K_0 \) the Hamiltonian system \( H(p,q) = \sum_{n \in \mathbb{Z}} \frac{p_n^2}{2} + V(q_{n+1} - q_n) \) possesses a nontrivial traveling wave with finite kinetic energy and with average potential energy \( K \). Moreover, these traveling waves \( q \) have the following properties:

(P1) The are monotone functions, increasing (i.e. expansion waves) if \( \Lambda = \mathbb{R}^+ \) and decreasing (i.e. compression waves) if \( \Lambda = \mathbb{R}^- \).

(P2) They are localized, in the sense that \( q_{n+1} - q_n \to 0 \) as \( n \to \pm \infty \).

(P3) They are supersonic; that is to say their wave speeds \( c \) satisfy \( c^2 > V''(0) \).

In our numerical experiments, we will consider perturbations \( u \), along with the choices of \( \varepsilon > 0 \), so that the interaction potential \( V_\varepsilon \) satisfies both the assumptions of Theorem 1 and the assumptions of Theorem 3. Note that the Toda potential \( V_0 \) already meets these requirements and this fact enables one to consider a variety of the perturbation functions \( u \). For instance, \( u \) is not required to be a non-negative function or to be convex around the origin. In practice, this allows us to conduct numerical experiments with monomials \( u(r) = r^n \), for all \( n \geq 2 \), as well as their linear combinations with positive coefficients, or bounded perturbations such as \( u(r) = 1 - \cos r \). For any potential \( V_\varepsilon \) considered in this study, the super quadratic growth condition is satisfied on \((-\infty, 0)\), and therefore solitary waves obey \( a_n > \frac{1}{2} \) for all \( n \in \mathbb{Z} \).

### 3. Scattering theory for Jacobi matrices

As is well-known, the integrability of the Toda lattice can be exploited via the bijective correspondence between the Lax operator, which in this case is the Jacobi matrix \( L \), and its scattering data. This correspondence goes under the name of direct and inverse scattering theory, and has already been studied in detail. While we do not attempt to give a comprehensive survey of the relevant references, the interested reader may enter the subject, for example, through [30], [31], and the references therein.

Recall that, since \( L \) is a bounded self-adjoint operator, we know that the spectrum \( \sigma(L) \subset \mathbb{R} \) and \( L \) has no residual spectrum. For details, see [25]. Furthermore, if the sequences \((a_n)_{n \in \mathbb{Z}}\) and \((b_n)_{n \in \mathbb{Z}}\) satisfy the hypotheses of Theorem 2 at time \( t=0 \) (and hence at all times \( t \geq 0 \)), then one can further conclude that the spectrum of \( L \) consists of a purely absolutely continuous part,

\[
\sigma_{ac}(L) = [-1, 1]
\]

and a (finite) pure point part

\[
\sigma_{pp}(L) = \{\lambda_j : j = 1, \ldots, N\} \subset (-\infty, -1) \cup (1, \infty).
\]

Furthermore, all the eigenvalues \( \lambda_j \) are simple.
For convenience, we start by mapping the spectral data via the so-called Joukowski transformation:
\[ \lambda = \frac{1}{2} \left( z + \frac{1}{z} \right), \quad z = \lambda - \sqrt{\lambda^2 - 1}, \quad \lambda \in \mathbb{C}, \ |z| \leq 1. \]
Using this parameter \( z \), it is standard to show that for any \( 0 < |z| \leq 1 \) there exist unique Jost solutions \( \varphi_{\pm}(z,n) \), i.e. solutions of
\[ L\varphi_{\pm}(z,\cdot) = \frac{z + z^{-1}}{2} \varphi_{\pm}(z,\cdot), \]
normalized such that
\[ \lim_{n \to \pm \infty} \varphi_{\pm}(z,n) \cdot z^{\mp n} = 1. \]
Moreover, the functions \( z \mapsto z^{\mp n} \varphi_{\pm}(z,n) \) are holomorphic in the domain \( |z| < 1 \) and continuous on \( |z| \leq 1 \). If we focus on the unit circle \( |z| = 1 \) with \( z^2 \neq 1 \), we observe that \( \varphi_{\pm}(z,\cdot), \varphi_{\pm}(z^{-1},\cdot) \) are linearly independent, and hence we can obtain the scattering relations:
\[ T(z)\varphi_{\pm}(z,n) = R_{\mp}(z)\varphi_{\mp}(z,n) + \varphi_{\mp}(z^{-1},n) \]
\[ T(z)\varphi_{\mp}(z,n) = R_{\pm}(z)\varphi_{\pm}(z,n) + \varphi_{\pm}(z^{-1},n) \]
for all \( n \in \mathbb{Z} \), and for \( |z| = 1 \), with \( z^2 \neq 1 \). The transmission coefficient \( T \) and the reflection coefficients \( R_{\pm} \) are related by
\[ |T(z)|^2 + |R_{\mp}(z)|^2 = 1 \quad \text{and} \quad T(z)R_{\pm}(z) + T(z)R_{\mp}(z) = 0. \]
Alternatively, the scattering relations (3.3) and (3.4) can be rewritten as
\[ (\varphi_{\mp}(z,n) \varphi_{\mp}(z^{-1},n)) = (\varphi_{\pm}(z^{-1},n) \varphi_{\pm}(z,n)) \cdot S(z), \]
where \( S(z) \) is the \( 2 \times 2 \) scattering matrix. It can easily be related to the transmission and reflection coefficients:
\[ S(z) = \frac{1}{T(z)} \cdot \begin{pmatrix} 1 & -R_{\pm}(z) \\ R_{\mp}(z) & T(z)^2 - R_{\pm}(z)R_{\mp}(z) \end{pmatrix}. \]

But more is true. Indeed, the transmission coefficient \( T(z) \) has a meromorphic extension inside the entire unit disk \( |z| \leq 1 \), with finitely many simple poles \( \zeta_k \in (-1,0) \cup (0,1), \ k = 1, \ldots, N \). The locations of the poles are related to the eigenvalues of the original Jacobi matrix through the Joukowski relation:
\[ \lambda_k = \frac{\zeta_k + \zeta_k^{-1}}{2} \quad \text{for all} \ k \in \{1, \ldots, N\}. \]
Note that positive \( \zeta_k \)’s correspond to the eigenvalues above 1, while negative \( \zeta_k \)’s correspond to the eigenvalues below -1.

An important description of the locations of the poles of \( T \) is as the points \( z \) inside the unit disk, \( |z| < 1 \), where the Jost solutions \( \varphi_{\mp}(z,\cdot) \) and \( \varphi_{\mp}(z,\cdot) \) are constant multiples of each other – and thus both in \( \ell^2(\mathbb{Z}) \). One can then compute the residue of \( T \) at each simple pole \( \zeta_k \), as follows:
\[ \text{Res}(T; \zeta_k) = -\mu_k \zeta_k \gamma_{k,+} = -\frac{\zeta_k \gamma_{k,-}}{\mu_k}, \]
where
\[ \gamma_{k,\pm} = \frac{1}{\|\varphi_{\pm}(\zeta_k,\cdot)\|_{\ell^2}^2} \]
are the norming constants, and \( \mu_k \) is the associated proportionality constant:
\[ \varphi_{\pm}(\zeta_k,\cdot) = \mu_k \varphi_{\pm}(\zeta_k,\cdot). \]
It is a fundamental fact of scattering theory that Jacobi matrices $L$ whose coefficients decay fast enough (as the hypothesis of Theorem 3) are in bijective correspondence with their scattering data \( \{ R, \zeta_k, \gamma_k \mid 1 \leq k \leq N \} \), where we use the standard convention of setting
\[
R(z) = R_+(z) \quad \text{and} \quad \gamma_k = \gamma_{k,+}.
\]
Note that implicit in this statement is the fact that from $R_+$ and the $\zeta_k$'s and $\gamma_{k,+}$'s one can fully reconstruct $R_-$, $T$ and the $\gamma_{k,-}$'s. From this point onwards we will always use the notation above for $R$ and $\gamma_k$, unless specified otherwise.

Now we take into account the time dependence. The time evolution of the reflection coefficient under the Toda lattice is given by
\[
R(z, t) = R_0(z) e^{t(z-z^{-1})},
\]
where $R_0(z)$ is the reflection coefficient of the initial Jacobi matrix $L(t = 0)$. Under the perturbed lattice evolution, however, additional terms appear. We begin with a technical, but very important, result:

**Theorem 4.** The evolution equation induced by (2.15) on the scattering matrix $S$ defined in (3.6) and (3.7) is given, for all $|z| = 1$, $z \neq \pm 1$, and all $t \in \mathbb{R}$, by
\[
\partial_t S(z, t) + \theta(z) [S(z, t), \sigma_3] = \frac{\varepsilon}{\theta(z)} D(z, t),
\]
where
\[
\theta(z) = \frac{z - z^{-1}}{2},
\]
and
\[
D(z, t) = \sum_{n=-\infty}^{\infty} U_{nn}(t) \begin{pmatrix} \varphi_+(z; n, t) & \varphi_-(z; n, t) \\ -\varphi_+(z; n, t) & -\varphi_-(z; n, t) \end{pmatrix}.
\]

The proof of this result is based on a discrete version of the variation of constants technique, and can once again be found in the Appendix at the end of our paper. While in this case the adaptation of the continuous technique to the discrete setting requires some well-made choices along the way, it is still fairly standard, and similar, for example, to the derivation in [8] of the analogous equation for the reflection coefficient in a perturbed defocusing cubic nonlinear Schrödinger equation.

Rewriting (3.11) as
\[
\partial_t S = -\theta \begin{pmatrix} 0 & -2S_{12} \\ 2S_{21} & 0 \end{pmatrix} + \frac{\varepsilon}{\theta} D,
\]
together with the equation (3.7) for the reflection coefficient in terms of the entries of $S$ leads immediately to the evolution equation for $R$:

**Corollary 3.1.** The evolution equation induced by (2.15) on the (right) reflection coefficient is
\[
\partial_t R(z; t) = 2\theta(z) R(z; t) - \frac{\varepsilon T(z; t)^2}{\theta(z)} \sum_{n=-\infty}^{\infty} U_{nn}(t) \varphi_-(z; n, t)^2,
\]
for all $t \in \mathbb{R}$ and all $|z| = 1$, $z \neq \pm 1$, where $\theta(z)$ was defined in (3.12).

Note that a straightforward integration of (3.14) yields the formula for the reflection coefficient at time $t$:
\[
R(z; t) = e^{(z-z^{-1})t} R(z; 0) - \frac{2\varepsilon}{z - z^{-1}} \int_0^t e^{(z-z^{-1})(t-\tau)} T(z; \tau)^2 \left( \sum_{n=-\infty}^{\infty} U_{nn}(\tau) \varphi_-(z; n, \tau)^2 \right) d\tau.
\]
The evolution equation induced by (2.15) on the left reflection coefficient can also be similarly computed:

\begin{equation}
\partial_t R_-(z; t) = -2\theta(z) R_-(z; t) - \frac{\varepsilon T(z; t)^2}{\theta(z)} \sum_{j=-\infty}^{\infty} (U_{jj}(z; t)\varphi_+(z; j, t))^2.
\end{equation}

The time evolution of the norming constant under the Toda evolution is given by

\begin{equation}
\gamma_k(t) = \gamma_k e^{t(\zeta_k - \zeta_{k-1})},
\end{equation}

where \( \gamma_k = \gamma_k(0) \), while the eigenvalues are constant in time. Under the perturbed lattice, the evolution of the eigenvalues and norming constants is given by:

**Theorem 5.** The evolution equations for the eigenvalues and norming constants under the perturbed lattice are given by:

\begin{equation}
\partial_t \lambda_k(t) = \varepsilon \gamma_k(t) \sum_{n=-\infty}^{\infty} U_{nn}(t)\varphi_+(\zeta_k(t); n, t)^2,
\end{equation}

and

\begin{equation}
\partial_t \gamma_k(t) = 2\theta(\zeta_k(t))\gamma_k(t) - \frac{2\varepsilon \gamma_k^2(t)}{\theta(\zeta_k(t))} \sum_{n=-\infty}^{\infty} \varphi_+(\zeta_k(t); n, t) \sum_{j=n}^{\infty} U_{jj}(t)\varphi_+(\zeta_k(t); j) K(j, n; t),
\end{equation}

where

\begin{equation}
K(j, n; t) = \varphi_+(\zeta_k(t); n, t)\varphi_+(\zeta_k(t)^{-1}; j, t) - \varphi_+(\zeta_k(t)^{-1}; n, t)\varphi_+(\zeta_k(t); j, t).
\end{equation}

Note that we cannot a-priori say that equations (3.17) and (3.18) hold for all time \( t > 0 \) (unlike (3.14)). Rather the equations hold locally in time, assuming that we start at an eigenvalue. Recall, however, that the eigenvalues of our Jacobi matrices are all simple: in particular, this means that eigenvalues cannot cross, but they can stop existing by “entering” the ac spectrum. We do not observe this phenomenon in any of our numerical simulations, but it remains a theoretical possibility.

Before finishing this section, let us note that we do not use the evolution equations of the scattering data for our numerical simulations. Rather, as explained below, we compute the data directly from the (truncation of the) Jacobi matrix \( L(t) \) – this turns out to be easier, numerically, since the equations (3.14) (or, alternatively, (3.1)), (3.17), and (3.18) require full knowledge of not only \( L(t) \), but also the scattering data and the Jost solutions. However, in the simplest case, that of the evolution of the eigenvalues, we numerically compare the results of our direct calculations with the evolution equation (3.17). The results of this comparison can be found on the website of the project.

4. The numerical scheme

Before moving on to presenting the numerical results of our work, we describe the numerical scheme used for solving (2.15) and computing scattering data associated to the (doubly-infinite) Jacobi matrix \( L \).

1http://www.math.uic.edu/~bilman/computing.html
4.1. **Time-stepping for computing** \( L(t) \). We note that \( L, P, \) and \( U \) are discrete operators. Therefore, there is no need for spatial discretization to solve

\[
\partial_t L = [P, L] + \varepsilon U
\]

numerically. To approximate the solutions of (2.15), we truncate the doubly-infinite lattice at particles with indices \( \pm N \in \mathbb{N} \) for some large \( N \), and work with the truncated matrices \( L_N \):

\[
L_N = \begin{pmatrix}
    b_{-N} & a_{-N} & 0 & & & \\
    a_{-N} & b_{-N+1} & a_{-N+1} & & & \\
    & 0 & a_{-N+1} & \ddots & & \\
    & & \ddots & \ddots & \ddots & \\
    & & & \ddots & \ddots & a_{N-2} & 0 \\
    & & & & \ddots & a_{N-2} & b_{N-1} & a_{N-1} \\
    & & & & & 0 & a_{N-1} & b_N
\end{pmatrix},
\]

and accordingly, \( P_N \). Since the solutions \((a, b)\) of (2.13) satisfy

\[
(a(t) - \frac{1}{2}, b(t)) \in \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z}) \quad \text{for all } t \geq 0,
\]

we close the finite system of differential equations for the truncated system by imposing the Dirichlet boundary conditions given by

\[
a_{-(N+1)} - \frac{1}{2} = a_N - \frac{1}{2} = 0 \quad \text{and} \quad b_{\pm(N+1)} = 0,
\]

and consider

\[
\begin{cases}
    \partial_t a_n = a_n(b_{n+1} - b_n) \\
    \partial_t b_n = 2(a_n^2 - a_{n-1}^2) + \varepsilon \left( c_{n-1} - c_n \right),
\end{cases}
\]

for \( n \in \{-N, -N+1, \ldots, N-1, N\} \), subject to the boundary conditions given in (4.2).

To integrate (4.3), we adopt the 4th-order Runge-Kutta time-stepping method. We define the temporal discretization error \( E_{\Delta t} \) at time \( t \) by

\[
E_{\Delta t}(t) = \|L_N(t) - L_N^*(t)\|_{HS},
\]

where \( \Delta t \) is the step size for temporal discretization and \( \| \cdot \|_{HS} \) is the Hilbert-Schmidt norm. When an exact solution is available at hand, \( L_N^* \) stands for the finite truncation of the exact solution to the infinite dimensional problem. Otherwise, \( L_N^* \) is taken to be the solution obtained by choosing \( \Delta t \) extremely small. Table 1 lists the discretization errors and the measured rate of convergence \( \alpha \) of the numerical method

\[
\alpha = \log_2 \left| \frac{E_{2\Delta t}(t)}{E_{\Delta t}(t)} \right|
\]

as the temporal step-size is diminished by half. The underlying experiment is comprised of pure 1-soliton initial data \((k = 0.4)\) which is let to evolve in the Toda lattice \((\varepsilon = 0)\) and a perturbed lattice with with \( u(r) = r^2, \varepsilon = 0.05 \). Figure 1 shows that the growth of the temporal discretization error in time is asymptotically linear.

4.2. **Computing eigenvalues of** \( L(t) \). We approximate the discrete spectrum of \( L(t) \) by computing the eigenvalues of its truncation \( L_N(t) \) using the QR-algorithm at each time-step. Since \( L \) is a discrete operator, no spatial discretization is used to obtain the matrix \( L_N \). To justify accuracy of this method, we begin with some general facts concerning eigenvalues of doubly-infinite (whole-line) Jacobi matrices and eigenvalues of their finite truncations. We omit the proofs of these facts.
\[ \Delta_t E_{\Delta_t}(T), \varepsilon = 0 \quad \alpha \quad \Delta_t E_{\Delta_t}(T), \varepsilon = 0.05 \quad \alpha \]

| \(\Delta t\) | \(E_{\Delta_t}(T), \varepsilon = 0\) | \(\alpha\) | \(\Delta t\) | \(E_{\Delta_t}(T), \varepsilon = 0.05\) | \(\alpha\) |
|---|---|---|---|---|---|
| 4h | 1.3438e-06 | – | 4h | 1.4236e-06 | – |
| 2h | 8.1798e-08 | 4.0381 | 2h | 8.6975e-08 | 4.0328 |
| h | 5.0433e-09 | 4.0196 | h | 5.3727e-09 | 4.0168 |
| \(h/2\) | 3.1303e-10 | 4.0099 | \(h/2\) | 3.3363e-10 | 4.0093 |

**Table 1.** Temporal discretization errors and rate of convergence measured at time \(T = 320\) for one-soliton initial data, \(k = 0.4; h = 0.02, \) and \(N = 2^{13}\)

**Figure 1.** \(E_{\Delta_t}(t)\), with \(\Delta t = 10^{-2}\) and \(N = 2^{14}\)

and refer the reader to the articles they are taken from. As in [19], given any bounded self-adjoint operator \(A\) on \(\ell^2(\mathbb{Z})\), define

\[
\lambda_j^+(A) = \inf_{\phi_1, \ldots, \phi_{j-1}} \sup_{\psi: \psi \perp \phi_j, \psi \in \ell^2(\mathbb{Z}), \|\psi\| = 1} \langle\psi, A\psi\rangle
\]

\[
\lambda_j^-(A) = \sup_{\phi_1, \ldots, \phi_{j}} \inf_{\psi: \psi \perp \phi_j, \psi \in \ell^2(\mathbb{Z}), \|\psi\| = 1} \langle\psi, A\psi\rangle.
\]

(4.4)

From the definitions, we have

\[ \lambda_1^-(A) \leq \lambda_2^-(A) \leq \cdots \leq \lambda_M^-(A) \leq \lambda_1^+(A) . \]

**Proposition 4.1** (p. 111, [19]). The min-max principle [27, Theorem XIII.1] asserts that

1. \( \lim_{n \to +\infty} \lambda_n^+(A) = \sup \sigma_{ess}(A) \) and \( \lim_{j \to +\infty} \lambda_j^-(A) = \inf \sigma_{ess}(A) \).
2. If \( A \) has \( M^+ \) eigenvalues counting multiplicity above the essential spectrum, these eigenvalues are precisely \( \lambda_1^+, \lambda_2^+, \ldots, \lambda_M^+ \), and \( \lambda_j^+ = \sup \sigma_{ess}(A) \) for \( j > M^+ \).
3. If \( A \) has \( M^- \) eigenvalues counting multiplicity below the essential spectrum, these eigenvalues are precisely \( \lambda_1^-, \lambda_2^-, \ldots, \lambda_M^- \), and \( \lambda_j^- = \sup \sigma_{ess}(A) \) for \( j > M^- \).

The following result describes the impact of truncations on eigenvalues of Jacobi matrices.
**Proposition 4.2** (Proposition 2.2, [19]). Let $\Pi$ be an orthogonal projection, and $A$ be a bounded self-adjoint operator on $\ell^2(\mathbb{Z})$. Define $A_\Pi = \Pi A \Pi$, restricted as an operator onto the range of $\Pi$. Then
\[
\lambda_j^+(A_\Pi) \leq \lambda_j^+(A) \quad \text{and} \quad \lambda_j^-(A_\Pi) \geq \lambda_j^-(A),
\]
for $n > 0$.

**Proof.** Changing from $A$ to $A_\Pi$ in (4.4) adds the condition $\psi \in \text{Ran} \, \Pi$. This increases infimums and decreases supremums, hence gives us the desired inequalities. \hfill \Box

**Remark 4.3.** Let $\lambda^-_1 < \lambda^-_2 < \cdots < \lambda^-_{2N+1}$ and $\lambda^+_2 > \cdots > \lambda^+_1$ denote the real simple eigenvalues of $L_N$, labeled in increasing and decreasing order, respectively. Suppose that $L$ has $M^-$ eigenvalues below its ac spectrum and $M^+$ eigenvalues above its ac spectrum. Note that $M^\pm$ are finite. Since $L$ is bounded, the quadratic form of $L_N$ is a restriction of the quadratic form of $L$ to $\mathbb{C}^{2N+1}$. Then by Proposition 4.2 for any $N \in \mathbb{N}^+$, we have
\[
\begin{align*}
\lambda^-_j(L) &\leq \lambda^-_j(L_N) & \text{for } j = 1, 2, \ldots, \min(2N+1, M^-) \\
\lambda^+_j(L) &\leq \lambda^+_j(L_N) & \text{for } j = 1, 2, \ldots, \min(2N+1, M^+),
\end{align*}
\]
as in [27] Theorem XIII.3. Furthermore,
\[
\begin{align*}
-1 &\leq \lambda^-_j(L_N) & \text{for } \min(2N+1, M^-) < j \leq 2N + 1 \\
\lambda^+_j(L_N) &\leq 1 & \text{for } \min(2N+1, M^+) < j \leq 2N + 1.
\end{align*}
\]
This implies that if $L_N$ has $M$ eigenvalues that are strictly less than $\inf \sigma_{ac}(L) = -1$, then $L$ has at least $M$ eigenvalues below its continuous spectrum. Analogously, if $L_N$ has $M$ eigenvalues that are strictly greater than $\sup \sigma_{ac}(L) = 1$, then $L$ has at least $M$ eigenvalues above its continuous spectrum. For a more detailed discussion on this matter, we refer the reader to [27] Theorem XIII.3, the discussion after Proposition 2.4 in [20], or [19].

More is true regarding eigenvalues of finite truncations of $L$. Since eigenvalues of $L_N$ and eigenvalues of its principal submatrix strictly interlace, [29] Proposition 2.1],
\[
\lambda^-_j(L_{N+1}) < \lambda^-_j(L_N) \quad \text{and} \quad \lambda^+_j(L_N) < \lambda^+_j(L_{N+1}) \quad \text{for } j = 1, \ldots, 2N + 1.
\]
(4.7) and (4.7) together imply that eigenvalues of $L_N$ have limit points outside $[-1, 1]$ as $N \to +\infty$. Note that, even for self-adjoint operators, it is not in general true that the set of these limit points is equal to the pure point spectrum of the operator under study. However, in our case, $L$ has additional properties. First, finite truncations of $L$ are also self-adjoint operators. Second, at any time $t \geq 0$, $L(t)$ is a compact perturbation of the discrete free Schrödinger operator $L^0$ that has $a_n \equiv \frac{1}{2}$, $b_n \equiv 0$, and $\sigma(L^0) = \sigma_{ac}(L^0) = [-1, 1]$. As pointed out in [20], these two facts imply:

**Proposition 4.4** (Theorem 2.3, [20]). Let $Z(L)$ denote the set of all limit points of $\bigcup_{N=1}^{\infty} \sigma(L_N)$. Then $Z(L) = \sigma(L)$.

Proposition 4.4 asserts that eigenvalues of $L_N$ that are outside $[-1, 1]$ converge to eigenvalues of $L$ as $N \to +\infty$. Let $E_\lambda(t)$ denote the error in computing an eigenvalue $\lambda$ of $L$, defined by
\[
E_\lambda(t) = |\lambda(L_N(t)) - \lambda^*(t)|,
\]
where $\lambda$ is the computed eigenvalue of a finite truncation $L_N$, and $\lambda^*$ is the eigenvalue $\lambda(L(t))$ of the doubly-infinite operator $L$.

Table [2] displays of $E_\lambda(T)$ for one-soliton initial data, measured at $T = 320$ in the Toda lattice and in the perturbed lattice with $u(r) = r^2$, $\varepsilon = 0.05$. For one-soliton solution of the Toda lattice, the exact eigenvalue $\lambda^*$ is known, $\lambda^*(t) = \pm \cosh(t)$, for $t \geq 0$. In experiments with perturbed lattices, $\lambda^*(t)$ is computed from the reference solution $L^*(t)$. Figure [2] displays $E_\lambda(t)$ over time in the Toda lattice for 1-soliton data with $k = 0.6$, where $\Delta t = 0.01$.  

Table [2] displays...
\[
\begin{array}{|c|c|c|c|c|}
\hline
\Delta t & E_{\lambda}(T), \varepsilon = 0 & E_{\lambda_1}(T), \varepsilon = 0.05 & E_{\lambda_2}(T), \varepsilon = 0.05 & E_{\lambda_3}(T), \varepsilon = 0.05 \\
\hline
4h & 1.262224e-09 & 5.579751e-08 & 1.1505353e-08 & 3.370637e-12 \\
2h & 3.945577e-11 & 2.306966e-09 & 4.037126e-11 & 1.865174e-14 \\
h & 1.227241e-12 & 1.071940e-10 & 4.037126e-11 & 1.865174e-14 \\
\frac{h}{4} & 3.042011e-14 & 5.523581e-12 & 2.511102e-12 & 2.620126e-14 \\
\frac{h}{2} & 1.088019e-14 & 2.873257e-13 & 1.869615e-13 & 2.353672e-14 \\
\end{array}
\]

Table 2. \( h = 0.01, T = 320, \) for 1-soliton initial data with \( k = 0.4, N = 2^{13} \)

4.3. Computing the reflection coefficient \( R(z; t) \). The numerical procedure we adopt for approximating the reflection coefficient in scattering data associated to \( L(t) \) at a time \( t \) is analogous to the one carried out for the Korteweg-deVries equation in [34]. We seek solutions of the form

\[
f(z; n, t) = \varphi_+(z; n, t)z^{-n}, \quad g(z; n, t) = \varphi_-(z; n, t)z^n,
\]

so that we have \( f(z; n, t) \to 1 \) as \( n \to +\infty \) and \( g(z; n, t) \to 1 \) as \( n \to -\infty \). Then \( f \) solves

\[
a_{n-1}z f(z; n - 1) + \left( b_n - \frac{z + z^{-1}}{2} \right) f(z; n) + a_nzf(z; n + 1) = 0,
\]

and \( g \) solves

\[
a_{n-1}z g(z; n - 1) + \left( b_n - \frac{z + z^{-1}}{2} \right) g(z; n) + a_nzg(z; n + 1) = 0.
\]

For any \( z \), \( 4.9 \) can be solved for \( f \) on \( n \geq 0 \) and \( 4.10 \) can be solved for \( g \) on \( n \leq 0 \) by backward substitution method using the appropriate boundary conditions at infinity. Then \( 4.8 \) is inverted to recover \( \varphi_\pm \), and the solutions are matched at \( n = 0 \) to extract the reflection coefficient \( R(z; t) \).

The effect of time-stepping error in \( L \) on this computation is measured in the Toda lattice by

\[
E_{R}(t) = \sqrt{\sum_{z \in \mathbb{T}} |R(z; t) - R(z; 0)e^{(z-z^{-1})t}|^2},
\]

where \( \mathbb{T} \) is a mesh on the unit circle, typically with 1000 mesh points. In Section 7.2 reflection coefficient for clean solitary waves is computed with the choice of \( \Delta t = 0.001 \).
\[ \Delta t \quad E_R(T), \varepsilon = 0, N = 2^{13} \]

| \( \Delta t \) | \( E_R(T) \) |
|----------------|-------------|
| 8h             | 1.610568e-06 |
| 4h             | 1.006489e-07 |
| 2h             | 6.429764e-09 |
| h              | 4.305598e-09 |

Table 3. \( E_R(T) \) at \( T = 1280 \), in the Toda lattice; \( h = 0.001 \)

The underlying experiment for the measurements displayed in Table 3 takes off with the initial data

\[ a_n = \frac{1}{2} + \frac{1}{10} e^{-n^2}, \quad b_n = \frac{1}{10 \cosh(n)}, \]

for which the corresponding reflection coefficient is not identically zero.

5. The numerical results: Soliton initial data

As explained in the discussion following Theorem 3, we consider perturbations \( u \), along with the choices of \( \varepsilon > 0 \), so that the interaction potential \( V_\varepsilon \) satisfies the assumptions of both Theorem 1 (for global well-posedness) and Theorem 3 (existence of solitary waves). Note that the Toda potential \( V_0 \) already meets these requirements and this fact enables one to consider a variety of the perturbation functions \( u \). For instance, \( u \) is not required to be a non-negative function or to be convex around the origin. In practice, this allows us to conduct numerical experiments with monomials \( u(r) = r^n \), for all \( n \geq 2 \), as well as their linear combinations with positive coefficients, or bounded perturbations such as \( u(r) = 1 - \cos r \). For any potential \( V_\varepsilon \) considered in this study, the super quadratic growth condition is satisfied on \(( -\infty, 0)\), and therefore solitary waves obey \( a_n > \frac{1}{2} \) for all \( n \in \mathbb{Z} \).

Results of numerical experiments exhibit no qualitative difference in behavior of the solutions subject to the different choices of perturbations that are mentioned above. In what follows, we only include the results of experiments with \( u(r) = r^2 \) and \( u(r) = r^3 \), as we find those to be more convenient to present here. Results for a variety of perturbation functions are available on the website of this project. Throughout the remaining part of the paper, we refer to \( (a, b) \) as solutions, since the direct and inverse scattering transforms for the Toda lattice are carried out in these coordinates. For 1-soliton data at \( t = 0 \) we use

\[ a_n = \frac{\sqrt{1 + e^{-2k(n-1)}} \sqrt{1 + e^{-2k(n+1)}}}{2 (1 + e^{-2kn})} \quad \text{and} \quad b_n = \frac{e^{-k} - e^{k}}{2} \left( \frac{e^{-2kn}}{1 + e^{-2kn}} - \frac{e^{-2k(n-1)}}{1 + e^{-2k(n-1)}} \right), \]

which are obtained from \( q_n = \log \left( \frac{1 + e^{-2k(n-1)}}{1 + e^{-2k(n-1)}} \right) \) as in (2.3) with \( \gamma = 1 - e^{-2k} \). Behavior of solutions \( a \) and \( b \) under perturbed dynamics are found to be qualitatively identical. Therefore, we present the results only for the sequence \( a \). Results including both of the variables \( a \) and \( b \) are also available on the website of this project.

We proceed with results of numerical experiments, where we commence with initial data that is a pure Toda soliton and let it evolve under the perturbed dynamics.

5.1. Emerging solitary waves. We find that a leading solitary wave emerges from the soliton initial data, followed by a dispersive tail, and that a secondary, counter-propagating wave is generated as soon as \( t \) becomes positive. The leading solitary wave is wider and it has smaller amplitude compared to the soliton initial data. In Figure 3, from \( t = 0 \) to \( t = 25 \), we see a typical occurrence of this phenomenon. The peak of the solution has been truncated to show the details of the dispersion. Figure 3(B) displays the secondary wave propagating towards left, and the bottom portion of the emerging solitary wave propagating towards right, with a dispersive tail that is under

\[ \text{http://www.math.uic.edu/~bilman/computing.html} \]
development. Figure 4 displays the solution in the same experiment at a later time $t = 3000$. As can be seen in Figure 4(A), the leading solitary wave gets separated from the dispersive tail as time elapses. Figure 4(B) shows the counter-propagating wave which spreads and loses amplitude over the course of the entire numerical experiment. We find that this phenomena happens for all other perturbations we consider.

We now turn our attention to the numerical experiments in which we compare solitary waves that emerge from the same initial data under different perturbations. Figure 5 presents the leading solitary waves (in solid-red) plotted against the soliton solution of the Toda lattice (in dashed-blue), at time $t = 3000$. As mentioned earlier, both in Figure 5(A) and (B), amplitude of the leading solitary wave is smaller than that of the Toda soliton it emerges from. On the other hand, the potential with perturbation $u(r) = r^2$ generates a solitary wave that travels faster than the Toda soliton, whereas the potential with $u(r) = r^3$ yields a wave that travels slower than the soliton. This is also not surprising, since a perturbation in the interaction potential introduces a perturbation in the nonlinear dispersion relation (5.3) that governs the propagation speed of these solitary waves, as we shall see further in this section. Note that, for a fixed perturbation, taller solitary waves travel faster than shorter ones do.
In Table 4 we list speed and amplitude pairs that are measured for the leading solitary waves in the same experiments. We measure the speed of propagation by tracking the peak of the solitary wave profile (in the continuous background) over time. To find the peak, we use fourth order polynomial interpolation. Small fluctuations in these computations are due to the fixed spatial grid size of the problem. This method is accurate up to $10^{-4}$ when an error check is performed against the exact height of solitons in the Toda lattice.

| Time  | $u(r) = r^2$ Amplitude | Speed  | $u(r) = r^3$ Amplitude | Speed  | Toda Amplitude | Speed |
|-------|-------------------|-------|-------------------|-------|----------------|-------|
| 5000  | 0.0381            | 1.0737| 0.0365            | 1.0168|                |       |
| 5100  | 0.0381            | 1.0731| 0.0365            | 1.0172|                |       |
| 5200  | 0.0381            | 1.0719| 0.0365            | 1.0177|                |       |
| 5300  | 0.0381            | 1.0735| 0.0365            | 1.0170|                |       |
| 5400  | 0.0382            | 1.0735| 0.0365            | 1.0169| 0.0405         | 1.0268|
| 5500  | 0.0381            | 1.0720| 0.0365            | 1.0174|                |       |
| 5600  | 0.0381            | 1.0730| 0.0365            | 1.0175|                |       |
| 5700  | 0.0381            | 1.0737| 0.0365            | 1.0169|                |       |
| 5800  | 0.0381            | 1.0725| 0.0365            | 1.0170|                |       |
| 5900  | 0.0381            | 1.0723| 0.0365            | 1.0177|                |       |

Table 4. Measured speed and amplitude of the leading solitary wave over long time scales, $\varepsilon = 0.05$.

We now present the results related to time evolution of scattering data associated to the solutions in the numerical experiments discussed above. Given an eigenvalue $\lambda = \frac{1}{2} (\zeta + \zeta^{-1})$ of $L$, and the associated norming constant $\gamma$, the corresponding Toda soliton is given by

$$a_n(t) = \frac{1}{2} \sqrt{1 - \zeta^2 + \gamma(t)\zeta^{2(n-1)}} \sqrt{1 - \zeta^2 + \gamma(t)\zeta^{2(n+1)}}$$

via the inverse scattering transform [24]. The amplitude of this wave is given by

$$\sup_{n \in \mathbb{Z}} \left( a_n(0) - \frac{1}{2} \right) = \frac{|\lambda| - 1}{2} = \frac{(1 - |\zeta|)^2}{4|\zeta|},$$

Figure 5. 1-soliton initial data evolving under perturbed dynamics (solid red) and under the Toda dynamics (dashed blue) for different choices of perturbations, at $t=3000$
as in [21]. Therefore the eigenvalues associated to taller Toda solitons are located farther away from the continuous spectrum. Figure 6 shows that the eigenvalue, which initially corresponds to a Toda soliton, rapidly converges to a new asymptotic constant that is closer to the continuous spectrum. This behavior is consistent with the loss in amplitudes of these waves. Moreover, the influence of the cubic perturbation, which yields the smaller of the emerging solitary waves, accordingly drives the associated eigenvalue closer to the edge of the ac spectrum compared to the case with the quadratic perturbation. Similar behavior is observed for different choices of perturbations or initial data, including one with multiple eigenvalues.

\[ \lambda(t) = x^2 \]

**Figure 6.** Evolution of the eigenvalue associated with 1-soliton initial data for different choices of perturbations, \( \varepsilon = 0.05 \)

Figure 7 displays time evolution of eigenvalues in a numerical experiment where we place two equal sized Toda solitons far away from each other in the spatial domain of integration, and let them evolve towards each other in the perturbed lattice with \( u(r) = r^2 \). As can be seen, both of the eigenvalues (solid-green) converge to new asymptotic values. Figure 7(A) displays the evolution of

\[ \lambda(t) = x^2 \]

**Figure 7.** Eigenvalues for an approximate 2-soliton initial data evolving under the perturbed lattice (solid green) vs. the Toda lattice (dashed blue), \( \varepsilon = 0.05, u(x) = x^2 \)
the eigenvalue for the left-incident wave plotted against the evolution in absence of perturbation (dashed-blue), and Figure 7(B) is the analogous picture for the right-incident wave. The peaks and small oscillations formed over time in the trajectories of these eigenvalues are due to the interactions of the solitary waves with each other, and with the radiation that has been generated. We address the case of interacting solitary waves in greater detail in the next section, but it is worthwhile to note that the eigenvalues revert to their asymptotic values after the interactions.

Next, we study propagation speeds of the emerging solitary waves and their relation to time evolution of the corresponding scattering data. Note that the speed of a Toda soliton in terms of the scattering data is given by

\[ v = -\frac{\theta(\zeta)}{\log(|\zeta|)}, \]

where \( \theta(\zeta) = \frac{1}{2}(\zeta - \zeta^{-1}) \) is defined via \( \gamma(t) = \gamma(0)e^{2\theta(\zeta)t} \), as in (3.16), with \( \gamma(0) = \gamma(0) \). We consider the perturbed lattices with \( u(r) = r^2 \) and \( u(r) = r^3 \), for various values of \( \varepsilon \). Through numerically computing the evolution of the spectral parameter \( \zeta \) and the associated norming constant \( \gamma \), we find that as \( t \to \infty \)

\[ \zeta(t) \sim \zeta_\varepsilon, \]

\[ \log \gamma(t) \sim \log \gamma_\varepsilon + \omega_\varepsilon t, \]

where \( \zeta_\varepsilon, \gamma_\varepsilon, \) and \( \omega_\varepsilon \) are constants that depend on the initial data and the perturbation. In Figure 8 we plot the evolution of the spectral parameter \( \zeta \) from time \( t = 0 \) until \( t = 7000 \), in the same numerical experiments. Clearly, the amount of deviation of \( \zeta(t) \) from the initial value becomes larger as the perturbation size increases.

Figure 9 shows the evolution of \( \log \gamma \), the logarithm of the norming constant, from time \( t = 0 \) until \( t = 7000 \) in the same experiments. In Figure 9(A)-(B) we see that \( \log \gamma(t) \) is asymptotically linear for large values of \( t \), as described in (5.4). Figure 9(C) and (D) provides a closer look at the evolution of \( \log \gamma \) under perturbations \( u(r) = r^2 \) and \( u(r) = r^3 \), with different values of perturbation size \( \varepsilon \). As \( \varepsilon \) increases, \( \log \gamma \) drifts farther away from the trajectory it has in case \( \varepsilon = 0 \). In other words, \( \omega_\varepsilon \) in (5.4) changes monotonically with respect to \( \varepsilon \): increasing for \( u(r) = r^2 \) and decreasing for \( u(r) = r^3 \). Using (5.3) and (5.4), we find that the measured speed of the leading solitary wave in the perturbed lattices is asymptotically given by

\[ v(\omega_\varepsilon, \zeta_\varepsilon) \sim -\frac{\omega_\varepsilon}{2\log(|\zeta_\varepsilon|)}, \]
which is analogous to the expression (5.3). Finally, we note that (5.4) implies

\[ \partial_t \| \varphi_+ (\zeta; t) \|_{L^2}^2 \sim -\omega_\varepsilon \| \varphi_+ (\zeta; t) \|_{L^2}^2, \]

for the Jost solution \( \varphi_+ \), as \( t \to +\infty \).

5.2. Emergence of new eigenvalues. In all of the numerical experiments we consider with 1-soliton initial data, we find that new eigenvalues are pushed out of the continuous spectrum after a small amount of time elapses. By the Remark 4.3, presence of new eigenvalues in the spectrum of the truncated operator implies existence of new eigenvalues in the spectrum of \( L \).

In numerical experiments that are mentioned in this section, the eigenvalue corresponding to the 1-soliton initial data lies in \(( -\infty, -1)\). Figure 10 displays time evolution of new eigenvalues which are being pushed into the opposite side of the continuous spectrum, namely into \(( 1, \infty )\) for \( u(r) = r^2 \) in (A), and \( u(r) = r^3 \) in (B), with \( \varepsilon = 0.05 \). Long-time behavior of these new eigenvalues are in general not clear in the time scales we have been able to conduct our experiments.

Moreover, we also find that there are new eigenvalues being pushed into \( ( -\infty, -1) \), the side of the continuous spectrum that contains the eigenvalue associated to the initial data. Figure 11 displays the evolution of these new eigenvalues coming into the negative side of the ac spectrum in the same numerical experiment, with \( u(r) = r^2 \) in (A), and \( u(r) = r^3 \) in (B).
Figure 10. New eigenvalues emerging from the opposite side of the ac spectrum of \( L, \varepsilon = 0.05 \)

Figure 11. New eigenvalues emerging from the same side of the ac spectrum.

Note that the spatial decay of the solutions \((a, b)\) given in Theorem 2 rules out the possibility of having embedded eigenvalues at the edges \( \lambda = \pm 1 \) of the essential spectrum. Therefore, \( L \) does not have eigenvalues at \( \lambda = \pm 1 \) at \( t = 0 \). What we see here is that the resonances of the Jacobi matrix at \( \lambda = \pm 1 \) are being pulled out by the perturbed dynamics as time evolves, see, for example, [28]. Emergence of new eigenvalues in the spectrum of \( L \) therefore implies that the scattering data associated to \( L \) at a time \( t^* > 0 \) only partially coincide with the scattering data obtained through solving the evolution equations derived in Section 3 from time \( t = 0 \) to \( t = t^* \).

6. Absence of eigenvalues in initial data

In this section we consider the solutions of (2.13) with initial data \((a^0, b^0)\) such that the discrete spectrum of \( L \) is initially empty. In the Toda lattice, such initial data yields a purely dispersive solution \((a, b)\), i.e.

\[
\lim_{t \to \infty} \|a(t) - \frac{1}{2} \|_{L^\infty} + \|b(t)\|_{L^\infty} = 0.
\]

One way to construct such initial data is as follows. Note that if \( 0 < a_n < \frac{1}{2} \) and \( b_n = 0 \) for all \( n \in \mathbb{Z} \), then the quadratic form \( Q_L \) associated to the doubly-infinite Jacobi matrix \( L \) satisfies

\[-1 < Q_L(\phi) < 1, \quad \text{for all } \phi \text{ with } \|\phi\|_{L^2(\mathbb{Z})} = 1.\]
This immediately implies that $L$ has no eigenvalues. In the numerical experiment to be discussed now, we consider a Toda soliton $(\hat{a}, \hat{b})$ with $\|\hat{a} - \frac{1}{2}\|_{L^\infty} < \frac{1}{5}$, and commence with the initial data obtained through setting

$$a_0^n = -(\hat{a}_n - \frac{1}{2}) + \frac{1}{2} = 1 - \hat{a}_n \quad \text{and} \quad b_0^n = 0,$$

which corresponds to reflecting the solitary wave profile $\hat{a}$ vertically and setting the initial velocity of each particle equal to 0. Since $\frac{1}{2} < \hat{a}_n < \frac{3}{5}$ for all $n \in \mathbb{Z}$, we have $0 < a_0^n < \frac{1}{2}$, and hence the Jacobi matrix corresponding to the initial data $(a_0^n, b_0^n)$ has an empty discrete spectrum. Note that this transformation preserves all of the required spatial decay conditions given in Theorem 1 and Theorem 2.

Although we start with an empty discrete spectrum, we find that new eigenvalues emerge from both hand sides of the continuous spectrum under the perturbed dynamics. Figure 12 displays the evolution of two new eigenvalues that emerge from the opposite sides of the ac spectrum. In any case, the new eigenvalues do not seem to converge to asymptotical values that are outside $[-1, 1]$ in the time-scale of the numerical experiments.

7. Clean solitary waves

We now study interactions of solitary waves in the perturbed lattices. To pursue such a study, one needs to have accurate numerical approximations of these solitary wave solutions since an exact formula for such solutions is not available at hand. We are able to generate “clean” solitary waves numerically through the iterative procedure introduced in [2]. We commence with initial data that is a pure 1-soliton solution of the Toda lattice and let it evolve numerically under the perturbed dynamics for a relatively long time. Once a leading solitary wave is separated from the dispersive tail, and from the smaller (hence slower) solitary waves that may emerge, we cut it off by setting the remainder of the solution equal to the free solution $(a \equiv \frac{1}{2}, b \equiv 0)$. Then we place the numerically isolated wave in the middle of the spatial domain of numerical integration and repeat this process. We note that more than one iterations of this procedure were needed in order to obtain an accurate solitary wave.

7.1. Head-on collision of solitary waves. Once we have in hand a good numerical approximation of a solitary wave solution of the perturbed system under study, we set up numerical experiments in which a pair of identical clean solitary waves travel towards each other, and interact in a head-on collision. We study the time evolution of the collision, as well as the time evolution of the eigenvalues (of $L$) associated to these solitary waves in the same time window. As is well

![Figure 12](image-url)
known, Toda solitons exhibit elastic collision, i.e. they retain their shapes and speeds after the collision, and no radiation is produced during their interaction. The situation is slightly different for clean solitary waves in the perturbed lattices, as we shall see in the numerical experiment that is to be discussed now.

Figure 13 displays two equal-sized solitary waves traveling towards each other in the perturbed lattice with the perturbation $u(r) = r^2$ and $\varepsilon = 0.05$, in a time window where the interaction occurs. From Figure 13(A) to (B) the waves propagate towards each other from time $t = 1930$ until $t = 1963$, where they are already in the collision state. At time $t = 1965$, the interacting profile forms a peak as seen in Figure 13(C), and Figure 13(D) displays the wave profiles at $t = 2000$, separating from each other. A closer look at the (spatial) interval between the separating waves in Figure 13(D) reveals that there develops some radiation with very small amplitude. Figure 13(A) displays the solitary waves before the interaction, and Figure 13(B) shows the radiation development between the separating solitary waves. Amplitude of this wave decays with time as is usual for purely dispersive waves.

![Figure 13](image)

**Figure 13.** Interaction of two equal-sized clean solitary waves during head-on collision

Now, we turn our attention to evolution of eigenvalues associated to each clean solitary wave in the numerical experiment described above. As is well known, the entire spectrum of $L$ is conserved under the Toda dynamics. Although this is not the case for the perturbed lattices, once a perturbation is fixed and a clean solitary wave solution of the perturbed system is generated,
the eigenvalue associated to this clean solitary wave is found to remain asymptotically constant in time. However, the situation in case of interacting solitary waves is different. Figure 15(A) and (B) show time evolution of the eigenvalues corresponding to the clean solitary waves that are considered in the preceding paragraph. During the interaction, both eigenvalues deviate from their asymptotic values and move towards the edges of the continuous spectrum $[-1, 1]$. As the collision state comes to an end, eigenvalues converge back to the constant asymptotic values they attain before the collision. In Figure 16, we compare evolution of eigenvalues that are associated to the

solitons in the Toda lattice (dashed-blue), to the leading solitary waves emerging from the soliton initial data (red) in the perturbed lattice, and to the clean solitary waves (green) generated under the same perturbation. Figure 16(A) displays the trajectories of the eigenvalues corresponding to the left incident traveling waves, and (B) displays those corresponding to the right incident waves.

7.2. Reflection coefficient for a clean solitary wave solution. Finally, we numerically study the time evolution of the reflection coefficient in the scattering data that is associated to a single

Figure 14. Head on collision of two clean solitary waves, (A) Pre-collision, (B) Post-collision

Figure 15. Eigenvalues corresponding to two clean solitary waves during their head-on collision; (A) for the left-incident wave, (B) for the right-incident wave

Figure 16. Trajectories of the eigenvalues corresponding to the left incident traveling waves, and (B) displays those corresponding to the right incident waves.
clean solitary wave solution of the perturbed lattice. We find that the reflection coefficient associated to a clean solitary wave solution of the perturbed lattice is nontrivial; and that it exhibits oscillatory behavior, both in time and space, localized near the edges of the ac spectrum, that is, near the points $z = \pm 1$ on the unit circle.

We numerically find that modulus of the reflection coefficient associated to a clean solitary wave solution remains asymptotically constant for all times $t \geq 0$, which is also the case in the pure Toda lattice. Figure 18 displays the absolute value of the reflection coefficient associated to a single clean solitary wave solution of the lattice with $u(r) = r^3$ and $\varepsilon = 0.05$, at time (A) $t = 0$ and (B) $t = 200$. In Figure 17 we plot the same quantities computed for a single clean solitary wave solution of the lattice with quadratic perturbation, that is, the lattice with $u(r) = r^2$ and $\varepsilon = 0.05$.

**Appendix A. Proofs and remarks**

In this appendix, we collect some of the longer proofs of the theorems from the first part of the paper. We include these proofs for completeness, since the methods are in most cases fairly standard, but still needing to be adapted to our case of the perturbation of the Toda lattice.

For convenience, in the following proofs and statements, we will use interchangeably the notations $p = (p_n)_{n \in \mathbb{Z}}$ and $s = (s_n)_{n \in \mathbb{Z}}$ for sequences of momenta and relative displacements, respectively.

**Proof of Theorem 1.** We note that since $V_\varepsilon$ is non-negative and $V_\varepsilon \in C^2(\mathbb{R})$, for any $\delta > 0$, there exists a constant $C_\delta > 0$ such that

$$0 \leq V_\varepsilon(r) \leq C_\delta r^2 \quad \text{and} \quad |V_\varepsilon'(r)| \leq C_\delta |r|$$

for $|r| \leq \delta$. Then for $s \in \ell^2(\mathbb{Z})$ with $\|s\|_{\ell^\infty} \leq \delta$, this implies that

$$\left\| (V_\varepsilon'(s_n))_{n \in \mathbb{Z}} \right\|_{\ell^2} = \sqrt{\sum_{n \in \mathbb{Z}} |V_\varepsilon'(s_n)|^2} \leq C_\delta \|s\|_{\ell^2},$$

and therefore the map

$$s = (s_n)_{n \in \mathbb{Z}} \mapsto (V_\varepsilon'(s_n))_{n \in \mathbb{Z}}$$

is of class $C^1$ from $\ell^2(\mathbb{Z})$ into $\ell^2(\mathbb{Z})$. As the difference operator is also $C^1$ on $\ell^2(\mathbb{Z})$, the right hand side of the system (2.17) is a $C^1$ vector field, and by a standard Picard iteration argument it follows

![Figure 16. Eigenvalues corresponding to solitary waves during their head-on collision: (A) for the left-incident wave, (B) for the right-incident wave](image)
that (2.17) is well posed locally in time in $\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$ (see, for example, [1], Lemma 4.1.6 and 4.1.8). Furthermore, for any $p, s \in \ell^2(\mathbb{Z})$, we have

$$|\mathcal{H}_\varepsilon(p, s)| \leq \frac{1}{2}||p||_{\ell^2}^2 + C_1||s||_{\ell^2}^2 < \infty,$$
for some constant $C_1$ that depends on $\|s\|_{L^\infty}$. Straightforward differentiation therefore shows that $\mathcal{H}_\varepsilon(p(t), s(t))$ is constant in time along the solutions of (2.17) in $\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$.

To prove that (2.17) is globally well-posed in time in $\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$, it is enough to show that the solutions remain bounded in $\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$ over finite time intervals. As $V_\varepsilon(r) \geq 0$ for all $r \in \mathbb{R}$, we have

$$ \|p(t)\|_{\ell^2} \leq 2 \sqrt{\mathcal{H}_\varepsilon(p(t), s(t))} = 2 \sqrt{\mathcal{H}_\varepsilon(p(0), s(0))},$$

and similarly $V_\varepsilon(s_n(t)) \leq \mathcal{H}_\varepsilon(p(0), s(0))$ for all $n \in \mathbb{Z}$, and for all $t$, as long as the solution exists. But this implies that $\|s(t)\|_{L^\infty} \leq M_0$ for some constant $M_0 > 0$, which depends only on the initial data, since $V_\varepsilon(r) \to +\infty$ as $|r| \to +\infty$. Then, by the assumptions on $V_\varepsilon$, there exists another constant $C_{M_0} > 0$ such that $r^2 \leq C_{M_0} V_\varepsilon(r^2)$ for all $|r| \leq M_0$. Therefore

$$ \|s(t)\|_{\ell^2} \leq \sqrt{C_{M_0} \mathcal{H}_\varepsilon(p(s(t)))} \|p(t)\|_{\ell^2} \leq \sqrt{C_{M_0} \mathcal{H}_\varepsilon(p(0), s(0))},$$

and hence the solution $(p(t), s(t))$ remains bounded in $\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$ on any finite time interval. This implies that (2.17) is well-posed globally in time in $\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$. In fact, we have

(A.1) \[ \|s(t)\|_{\ell^2} + \|p(t)\|_{\ell^2} \leq C \] and \[ \|s(t)\|_{L^\infty} + \|p(t)\|_{L^\infty} \leq C \]

for all times $t \geq 0$, where the constant $C > 0$ depends only on the initial data.

**Proof of Theorem 2.** We consider $\tilde{a}(t) = a(t) - \frac{1}{2}$, with $\tilde{a}(0) = \tilde{a}^0$, and study the differential equation which governs the evolution of $(\tilde{a}, b)$:

(A.2) \[ \partial_t \left( \frac{\tilde{a}_n(t)}{b_n(t)} \right) = \frac{a_n(t)(b_{n+1}(t) - b_n(t))}{(2a_n(t) + 1)\tilde{a}_n(t) - (2a_{n-1}(t) + 1)\tilde{a}_{n-1}(t) + \varepsilon \tilde{U}_{n,n}(t)}, \]

subject to initial conditions $(\tilde{a}(0), b(0)) = (\tilde{a}^0, b^0) \in X^1_w$, where

(A.3) \[ \tilde{U}_{n,n}(t) = \frac{1}{2} \left\{ u'( -2 \log (2\tilde{a}_{n-1}(t) + 1) ) - u'( -2 \log (2\tilde{a}_n(t) + 1) ) \right\}, \]

for each $n$ in $\mathbb{Z}$. Let $f(t, (\tilde{a}(t), b(t)))$ denote the right hand side of (A.2). By our assumptions on the initial data, we can choose

$$ \delta = \frac{1}{2} \min_{n \in \mathbb{Z}} \left( \tilde{a}^0_n + \frac{1}{2} \right) $$

so that for any $(x, y)$ in the closed ball $B_\delta = \left\{ (x, y) \in X^1_w : \|(x, y) - (\tilde{a}^0, b^0)\|_{w,1} \leq \delta \right\}$, we have $\min_{n \in \mathbb{Z}} x_n > 0$. Now, since the weight function $n \mapsto 1 + |n|$ satisfies

(A.4) \[ \sup_{n \in \mathbb{Z}} \left\{ \frac{1+|n|}{1+|n+1|}, \frac{1+|n|}{1+|n+1|} \right\} < \infty, \]

the shift operators are bounded with respect to the norm $\|(\cdot)\|_{w,1}$. The multiplication operator with the sequence $a(t)$ is also uniformly bounded from $X^1_w$ into $X^1_w$ as $\|a(t)\|_{L^\infty}$ is bounded uniformly in time. Moreover, since $u \in C^2(\mathbb{R})$ and $r \mapsto \log r$ is Lipschitz continuous on $[\rho, +\infty)$ for any $\rho > 0$, the map

$$ \left( \begin{array}{c} x_n \\ y_n \\ \end{array} \right)_{n \in \mathbb{Z}} \mapsto \frac{1}{2} \left( u' \left( -2 \log (2x_{n-1} + 1) \right) - u' \left( -2 \log (2x_n + 1) \right) \right)_{n \in \mathbb{Z}} $$

is Lipschitz continuous on $B_\delta \subset X^1_w$ by our choice of $\delta$. Therefore, there exists $T > 0$ such that the map

$$ \left( \begin{array}{c} x(t) \\ y(t) \\ \end{array} \right) \mapsto \left( \tilde{a}^0, b^0 \right) + \int_0^t f\left( \tau, (x(\tau), y(\tau)) \right) \, d\tau $$

defined on the Banach space $X_T = [0, T] \times X^1_w$, equipped with the norm

$$ \| \cdot \|_{X_T} = \sup_{t \in [0, T]} \| \cdot \|_{w,1}, $$
is a contraction mapping from the closed ball $B_{δ,T} = [0,T] \times B_{δ} \subset X_T$ into itself, and the Banach Fixed Point Theorem implies that $(\ref{A.5})$ has a unique solution $(\tilde{a}(t), b(t)) \in X_1^2$ for $t \in [0,T]$ with $(\tilde{a}(0), b(0)) = (\bar{a}^0, b^0)$. A straightforward calculation shows that this solution depends continuously on the initial data and $(\bar{a}(t) + \frac{1}{t}, \bar{b}(t))$ coincides with the solution $(a(t), b(t))$ of the perturbed lattice. Now, suppose $(0, t^*)$ is some finite time interval for which the solution to $(\ref{A.2})$ exists. Since the solution satisfies $(\ref{2.18})$ and $(\ref{2.19})$, we have

$$\| (0, \tilde{U}_{mn}(t))_{n \in \mathbb{Z}} \|_{w, 1} \leq \epsilon 6K \| (\tilde{a}(t), b(t)) \|_{w, 1}$$

for any $t \in (0, t^*)$, where $K$ is the product of the Lipschitz constants of $u$ and $r \mapsto \log r$, which depend only on the initial data. Then

$$\| (\tilde{a}(t), b(t)) \|_{w, 1} \leq \| (\bar{a}^0, b^0) \|_{w, 1} + \int_0^t (6(C + \epsilon K) + 3) \| (\tilde{a}(\tau), b(\tau)) \|_{w, 1} d\tau$$

for any time $t \in (0, t^*)$, where $C = \sup_{t \geq 0} \| a(t) \|_{\ell^\infty}$. Using Grönwall’s inequality, we obtain

$$\| (\tilde{a}(t), b(t)) \|_{w, 1} \leq \| (\bar{a}^0, b^0) \|_{w, 1} e^{(6(C + \epsilon K) + 3)t}.$$

Therefore the solutions of $(\ref{A.2})$ remain bounded on finite time intervals and hence they are global in time in $X_1^2$.

In the second part of the appendix we include the proofs of the evolution equations for the scattering data associated to the Jacobi matrices that we are studying.

**Proof of Theorem 4.** We proceed by a discrete version of the variation of constants technique, starting from the usual proof of the time evolution of $R$ in the integrable case. In order to streamline our notation, we consider for each $z$ with $|z| = 1$ two “$\infty \times 2$” matrices:

$$\Psi(z; t) = (\Psi_n(z; t))_{n \in \mathbb{Z}} = (\psi_{n,j}(z; t))_{n \in \mathbb{Z}, j = 1, 2},$$

and

$$\Phi(z; t) = (\Phi_n(z; t))_{n \in \mathbb{Z}} = (\phi_{n,j}(z; t))_{n \in \mathbb{Z}, j = 1, 2},$$

with

$$\psi_{n,1}(z; t) = \varphi_+(z; n, t) \quad \text{and} \quad \psi_{n,2}(z; t) = \varphi_+(z^{-1}; n, t),$$

and

$$\phi_{n,1}(z; t) = \varphi_-(z^{-1}; n, t) \quad \text{and} \quad \phi_{n,2}(z; t) = \varphi_-(z; n, t).$$

Furthermore, in the proofs and some of the statements in this section, we will avail ourselves of the common convention of suppressing the dependence on the $z$ and/or $t$ variables, in order to keep the length of some of the formulas manageable.

We begin by differentiating both sides of the equation

$$(\ref{A.5}) \quad L(t) \Psi(z; t) = \frac{z + z^{-1}}{2} \Psi(z; t)$$

with respect to $t$. Using the evolution equation $(\ref{2.15})$ for $L$ we obtain:

$$(\ref{A.6}) \quad \left( L - \frac{z + z^{-1}}{2} \right) (\partial_t \Psi(z) - P \Psi(z)) = -\epsilon U \Psi(z).$$

For each $n \in \mathbb{Z}$, let $A_n = A_n(z; t)$ be a $2 \times 2$ matrix such that

$$(\ref{A.7}) \quad \partial_t \Psi - P \Psi = \Psi \begin{pmatrix} -\theta & 0 \\ 0 & \theta \end{pmatrix} + (\Psi_n A_n)_{n \in \mathbb{Z}},$$
where \( \theta = \theta(z) \) as in (3.12). We note that the first term on the right hand side is the one obtained in the usual Toda case, while the second term encodes the variation of constants. Plugging (A.7) in (A.6) and simplifying yields

\[
(A.8) \quad \left( L - \frac{z + z^{-1}}{2} \right) \left( \Psi_n A_n \right)_{n \in \mathbb{Z}} = -\varepsilon U \Psi,
\]

Straightforward, if quite lengthy, calculations allow us to simplify the left-hand side, leading to

\[
A.9 \quad a_{n-1} \Psi_{n-1}(A_{n-1} - A_n) - a_n \Psi_{n+1}(A_n - A_{n+1}) = -\varepsilon U_{nn} \Psi_n.
\]

We seek \((A_n)_n\) such that

\[
(A.10) \quad \Psi_n(A_{n+1} - A_n) = 0 \quad \text{for all} \quad n \in \mathbb{Z}.
\]

Then (A.9) becomes

\[
(A.11) \quad -a_n \Psi_{n+1}(A_n - A_{n+1}) = -\varepsilon U_{nn} \Psi_n,
\]

and combining (A.10) and (A.11) into a single matrix equation yields:

\[
(A.12) \quad \begin{pmatrix} -a_n \Psi_{n+1} \\ a_n \Psi_n \end{pmatrix} \begin{pmatrix} A_n - A_{n+1} \end{pmatrix} = \begin{pmatrix} -\varepsilon U_{nn} \Psi_n \\ 0 \end{pmatrix}.
\]

Direct calculations, as well as standard arguments involving the Wronskian of the (linearly independent) Jost solutions \( \varphi_+(z, \cdot) \) and \( \varphi_+(z^{-1}, \cdot) \), show that

\[
(A.13) \quad \det \begin{pmatrix} -a_n \Psi_{n+1} \\ a_n \Psi_n \end{pmatrix} = a_n \frac{z^{-1} - z}{2} = -a_n \theta \neq 0,
\]

and hence we can invert the \(2 \times 2\) matrix \(\begin{pmatrix} -a_n \Psi_{n+1} \\ a_n \Psi_n \end{pmatrix}\) to obtain

\[
(A.14) \quad A_n - A_{n+1} = -\frac{1}{\theta} \begin{pmatrix} \psi_{n,2} & \psi_{n+1,2} \\ -\psi_{n,1} & -\psi_{n+1,1} \end{pmatrix} \times \begin{pmatrix} -\varepsilon U_{nn} \Psi_n \\ 0 \end{pmatrix},
\]

which can be rewritten as

\[
(A.15) \quad A_n - A_{n+1} = \frac{\varepsilon}{\theta} \begin{pmatrix} U_{nn} \psi_{n,1} \psi_{n,2} & U_{nn} \psi_{n,1}^2 \\ -U_{nn} \psi_{n,1}^2 & -U_{nn} \psi_{n,1} \psi_{n,2} \end{pmatrix}.
\]

This leads us to the expression of \(A_n\):

\[
(A.16) \quad A_n = \frac{\varepsilon}{\theta} \sum_{j=n}^{\infty} U_{jj} \begin{pmatrix} \psi_{j,1} \psi_{j,2} & \psi_{j,2}^2 \\ -\psi_{j,1}^2 & -\psi_{j,1} \psi_{j,2} \end{pmatrix}.
\]

We will now obtain the evolution equation for the scattering matrix \( S \). We start with rewriting (A.7) as

\[
(A.17) \quad \partial_t \Psi - P \Psi + \theta \Psi \sigma_3 = (\Psi_n A_n)_{n \in \mathbb{Z}},
\]

and using (3.6) in the form:

\[
(A.18) \quad \Psi_n = \Phi_n S \quad \text{for all} \quad n \in \mathbb{Z}.
\]

For each \( n \in \mathbb{Z} \), this leads to the equation:

\[
(A.19) \quad (\partial_t \Phi_n) S + \Phi_n (\partial_t S) - (P \Phi) _n S + \theta \Phi_n S \sigma_3 = \Phi_n S A_n.
\]

From the expression (A.16) for \( A_n \) and from (A.18) used for all \( j \geq n \), we obtain that

\[
SA_n = \frac{\varepsilon}{\theta} \sum_{j=n}^{\infty} U_{jj} \begin{pmatrix} \psi_{j,1} \phi_{j,2} & \psi_{j,2} \phi_{j,2} \\ -\psi_{j,1} \phi_{j,1} & -\psi_{j,2} \phi_{j,1} \end{pmatrix}.
\]
We now proceed by taking $n \to -\infty$ in (A.19). From the asymptotic properties of the Jost solutions, we see that
\[
\lim_{n \to -\infty} \Phi_n(z) \cdot z^{-n\sigma_3} = (1 \ 1),
\]
where we are using the standard conventions:
\[
\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad z^{\pm n\sigma_3} = \begin{pmatrix} z^{\pm n} & 0 \\ 0 & z^{\pm n} \end{pmatrix}.
\]
Further standard scattering theory arguments show that, as $n \to -\infty$,
\[
\partial_t \Phi_n \to (0 \ 0)
\]
and
\[
(P\Phi_n \cdot z^{-n\sigma_3} \to \theta(z)(1 \ -1).
\]
Using these asymptotics as $n \to -\infty$ in (A.19) yields:
\[
(A.20) \quad \partial_t S - \theta \sigma_3 S + S \theta \sigma_3 = \frac{\varepsilon}{\theta} D,
\]
which can be rewritten as
\[
(A.21) \quad \partial_t S + \theta [S, \sigma_3] = \frac{\varepsilon}{\theta} D.
\]

**Proof of Theorem 5.** The proof of (3.17) is extremely standard. Indeed, let $f_k(t) = \sqrt{\gamma_k(t)} \varphi_+(\zeta_k(t)), t)$ be an $\ell^2$-normalized eigenvector associated to the eigenvalue $\lambda_k(t)$, $\|f_k(t)\|_{\ell^2} = 1$ for all $t \in \mathbb{R}$. Differentiating the equation $Lf_k = \lambda_k f_k$ and then taking the inner product with $f_k$ yields:
\[
\partial_t \lambda_k = \langle f_k, (\partial_t L)f_k \rangle + \langle f_k, (L \partial_t f_k) \rangle - \langle f_k, \lambda_k (\partial_t f_k) \rangle.
\]
The last term on the right-hand side is easily seen to be equal to 0 since the eigenvector $f_k$ is normalized in $\ell^2$. The middle term is also 0, since $L$ is self-adjoint in $\ell^2$, and hence:
\[
\langle f_k, (L \partial_t f_k) \rangle = (L f_k, \partial_t f_k) = \lambda_k (f_k, \partial_t f_k) = 0.
\]
To treat the $(f_k, (\partial_t L)f_k)$ term, we use the evolution equation for $L$, (2.15). From the anti-symmetry of $P$ it follows that $(f_k, Pf_k) = 0$, and hence
\[
(f_k, [P, L]f_k) = 0.
\]
Therefore the only term remaining corresponds to the perturbation, which leads directly to (3.17).

We begin the proof of (3.18) by noting that the variation of constants calculations done in the proof of Theorem 4 did not depend on $|z| = 1$. Further recall the definition of the norming constant as
\[
(A.22) \quad \gamma_k(t) = \frac{1}{\|\varphi_+(\zeta_k(t)); t\|_{\ell^2}}.
\]
Thus we use the first column of (A.17), which translates to:
\[
\partial_t \varphi_+(\zeta_k; n) - P \varphi_+(\zeta_k; n) = -\theta(\zeta_k) \varphi_+(\zeta_k; n) + \frac{\varepsilon}{\theta(\zeta_k)} \times
\]
\[
\sum_{j=n}^{\infty} \left( \varphi_+(\zeta_k; n) U_{jj} \varphi_+(\zeta_k; j) \varphi_+(\zeta_k^{-1}; j) - \varphi_+(\zeta_k^{-1}; n) U_{jj} \varphi_+(\zeta_k; j)^2 \right),
\]

□

Proof of Theorem 5
where the $t$ dependence was suppressed. Now, taking the inner product with $\varphi_+(\zeta_k; \cdot)$ leads to:

$$\partial_t \| \varphi_+(\zeta_k) \|^2_{l^2} = -2\theta(\zeta_k) \| \varphi_+(\zeta_k) \|^2_{l^2} + \frac{2\varepsilon}{\theta(\zeta_k)} \sum_{n \in \mathbb{Z}} \varphi_+(\zeta_k; n) \sum_{j=n}^{\infty} U_{jj} \varphi_+(\zeta_k; j) \times$$

\begin{equation}
(\varphi_+(\zeta_k; n) \varphi_+(\zeta_k^{-1}; j) - \varphi_+(\zeta_k^{-1}; n) \varphi_+(\zeta_k; j)).
\end{equation}

(A.24)

Using (3.19), this becomes

$$\partial_t \| \varphi_+(\zeta_k) \|^2_{l^2} = -2\theta(\zeta_k) \| \varphi_+(\zeta_k) \|^2_{l^2} + \frac{2\varepsilon}{\theta(\zeta_k)} \sum_{n \in \mathbb{Z}} \varphi_+(\zeta_k; n) \sum_{j=n}^{\infty} U_{jj} \varphi_+(\zeta_k; j)K(j, n).$$

The result then follows immediately from (A.22).

□

References

[1] R. Abraham, J. Marsden, T. Ratiu, Manifolds, Tensor Analysis, and Applications, Addison-Wesley Publishing Company, 1993.

[2] J. Bona, M. Chen, A Boussinesq system for two-way propagation of nonlinear dispersive waves, Phys. D 116 (1998), 191–224.

[3] T. Claeyts, T. Grava, Universalities of the break-up profile for the KdV equation in the small dispersion limit using the Riemann-Hilbert approach, Comm. Math. Phys. 286 (2009), no. 3, 979–1009.

[4] T. Claeyts, T. Grava, Painlevé II asymptotics near the leading edge of the oscillatory zone for the Korteweg-de Vries equation in the small-dispersion limit, Comm. Pure Appl. Math. 63 (2010), no. 2, 203–232.

[5] P. Deift, X. Zhou, Perturbation theory for infinite-dimensional integrable systems on the line. A case study, Acta. Math. 188 (2002), 163–262.

[6] P. Deift, X. Zhou, Asymptotics for the Painlevé II equation, Comm. Pure Appl. Math. 48 (1995), no. 3, 277–337.

[7] P. Deift, X. Zhou, Perturbation theory for infinite-dimensional integrable systems on the line. A case study, Acta. Math. 188 (2002), 163–262.

[8] B. Dubrovin, On universality of critical behaviour in Hamiltonian PDEs, (English summary) Geometry, topology, and mathematical physics 59–109, Amer. Math. Soc. Transl. Ser. 2, 224, Amer. Math. Soc., Providence, RI, 2008.

[9] B. Dubrovin, T. Grava, C. Klein, On universality of critical behavior in the focusing nonlinear Schrödinger equation, elliptic umbilic catastrophe and the tritronquée solution to the Painlevé-I equation. (English summary) J. Nonlinear Sci. 19 (2009), no. 1, 57–94.

[10] H. Flaschka, The Toda lattice. I. Existence of integrals, Phys. Rev. B. (3) 9 (1974), 1924–1625.

[11] H. Flaschka, The Toda lattice. II Inverse-scattering solution, Progr. Theoret. Phys. 51 (1974), 703–716.

[12] E. Fermi, J. Pasta, S. Ulam, Studies of nonlinear problems. Collected Works of Enrico Fermi 11, H. Flaschka, The Toda lattice. I Existence of integrals, University of Chicago Press, Chicago, 1965.

[13] G. Friesecke, R. L. Pego, Solitary waves on FPU lattices. I. Qualitative properties, renormalization and continuum limit, Nonlinearity 12 (1999), 1601–1627.

[14] G. Friesecke, R. L. Pego, Solitary waves on FPU lattices. II. Linear implies nonlinear stability, Nonlinearity 15 (2002), 1343–1359.

[15] G. Friesecke, R. L. Pego, Solitary waves on FPU lattices. III. Howland-type Floquet theory, Nonlinearity 17 (2004), 207–227.

[16] G. Friesecke, R. L. Pego, Solitary waves on FPU lattices. IV. Proof of stability at low energy, Nonlinearity 17 (2004), 229–251, 2004.

[17] G. Friesecke, A. D. Wattis, Existence theorem for solitary waves on lattices, Comm. Math. Phys. 161 (1994), 391–418.

[18] D. Hundertmark, B. Simon, Lieb-Thirring inequalities for Jacobi matrices, J. Approx. Theory 118 (2001), 106–130.

[19] E. K. Ifantis, P. N. Panagopoulos, Limit points of eigenvalues of truncated tridiagonal operators, J. Comput. Appl. Math. 133 (2001), 412–422.

[20] U. Islambekov, R. Sims, G. Teschl, Lieb-Robinson Bounds for the Toda Lattice, J. Stat. Phys. 148 (2012), no. 3, 440–479.

[21] M. D. Kruskal, N. J. Zabusky, Interaction of “solitons” in a collisionless plasma and the recurrence of initial states, Phys. Rev. Lett. 15 (1965), 240–243.
[23] H. Krüger, G. Teschl, Long-time asymptotics for the Toda lattice in the soliton region, Math. Z. 262 (2009), 585–602.
[24] H. Krüger, G. Teschl, Long-time asymptotics of the Toda lattice for decaying initial data revisited, Rev. Math. Phys. 21 (2009), 61–109.
[25] S. V. Manakov, Nonlinear Fraunhofer diffraction, Sov. Phys. JETP 38 (1974), 693–696.
[26] M. Reed, B. Simon, Functional Analysis. Vol. 1, Methods of Modern Mathematical Physics, Academic Press, 1972.
[27] M. Reed, B. Simon, Analysis of Operators. Vol. 4, Methods of Modern Mathematical Physics, Academic Press, 1978.
[28] B. Simon, Resonances in one dimension and Fredholm determinants, J. Func. Anal. 178 (2000), 396–420.
[29] B. Simon, Sturm oscillation and comparison theorems, Sturm-Liouville theory, 29–43, Birkhäuser, Basel, 2005.
[30] G. Teschl, Jacobi Operators and Completely Integrable Nonlinear Lattices, Mathematical Surveys and Monographs 72, American Mathematical Society, Rhode Island (1999).
[31] G. Teschl, Almost everything you always wanted to know about the Toda equation, Jahresber. Deutsch. Math.-Verein. 103 (2001), no. 4, 149–162.
[32] G. Teschl, On the spatial asymptotics of solutions of the Toda lattice, Discrete Contin. Dyn. Syst. 27 (2010), no. 3, 1233–1239.
[33] M. Toda, Theory of Nonlinear Lattices, second edition, Springer Series in Solid-State Sciences, 20, Springer-Verlag, Berlin, 1989.
[34] T. Trogdon, S. Olver, B. Deconinck, Numerical inverse scattering for the Korteweg-de Vries and modified Korteweg-de Vries equations, Phys. D 241 (2012), no. 11, 1003–1025.

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