Properties of fractional integral operators involving the three-parameters Mittag-Leffler function in the kernels with respect to another function

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Abstract: This paper aims to investigate properties associated with fractional integral operators involving the three-parameters Mittag-Leffler function in the kernels with respect to another function. We prove that the Cauchy problem and the Volterra integral equation are equivalent. We find a closed-form to the solution of the Cauchy problem using successive approximations method and ψ-Caputo fractional derivative.

Keywords: ψ-Riemann-Liouville fractional integral, ψ-Caputo fractional derivative, three-parameters Mittag-Leffler function, general fractional integral operators

1 Introduction

In the last years, the number of integral and differential operators has increased a lot [1, 15, 17, 21, 24]. Some of these operators contain in its kernels the so-called special functions, for example: hypergeometric function [10], Meijer G-function, Fox H-function [20] and three-parameters Mittag-Leffler function. In 1971, Prabhakar introduced in the kernel of the Riemann-Liouville fractional integral the three-parameters Mittag-Leffler function, [16]. In 2002, Kilbas et al. investigated an integrodifferential equation involving the Riemann-Liouville fractional derivative and the fractional integral operator developed by Prabhakar, [11]. In 2004, the same authors, proved some properties associated with the generalized operator defined by Prabhakar, [12]. Srivastava and Tomovski, in 2009, proposed the fractional integral operator, which contains in its kernel the generalized Mittag-Leffler function, [23]. Garra et al., in 2014, defined the Hilfer-Prabhakar fractional derivative, this fractional derivative is a generalization of Hilfer derivative in which Riemann-Liouville fractional integral is replaced by Prabhakar fractional integral, [7]. In 2016, Dorrego introduced the k-Mittag-Leffler function in the kernel of k-Riemann-Liouville fractional integral, [5]. Recently, in 2018, Sousa and Capelas de Oliveira defined fractional integral operators containing in their kernel the Mittag-Leffler function [22] and, Baleanu and Fernandez introduced a fractional derivative involving this same function in its kernel [4]. Atangana and Baleanu proposed the so-called AB fractional derivative operators which contain in the kernel the one-parameter Mittag-Leffler function, [3]. Based in these fractional operators it was proposed fractional integral operators which contain in its kernel the Mittag-Leffler function with respect to another function, [24].

Numerous applications have emerged from these operators, among which we can mention: Zhao and Sun applied the Caputo type fractional derivative with a Prabhakar-like kernel to discuss the anomalous relaxation model and its solution, [27] and Göriska et al. published a note about the work of Zhao and Sun, [9]; Garra and Garrappa used fractional operators containing in its kernels the Prabhakar function to present their applications in dielectric models of Havriliak-Negami type, [6]; Giusti discussed some generalities of relaxation processes involving Prabhakar derivatives, [8]; Sandev presented analytical results related to the generalized Langevin equation.
with regularized Prabhakar derivative operator, [19]; Yavuz et al. used AB fractional derivative operators to solved time-fractional partial differential equations, [26]. Xiao et al. discussed in their book applications of fractional derivatives with nonsingular kernels in viscoelasticity, [25].

The main objective of this work is to prove some properties associated with the fractional operator proposed by Yang and use it in a fractional differential equation. The structure of the paper is as follows: In Section 2 we present the space of functions used throughout the text and some basic definitions and properties associated with \( \psi \)-Riemann-Liouville fractional integral and \( \psi \)-Caputo fractional derivative. Section 3 is devoted to the study of some theorems and lemmas related to fractional integral operators involving the three-parameters Mittag-Leffler function in the kernels with respect to another function. In Section 4 we discuss the equivalence between the Cauchy problem and the Volterra integral equation. We find a closed-form to the solution of the Cauchy problem using successive approximations method and \( \psi \)-Caputo fractional derivative of order \( \beta \), where \( n - 1 < \beta < n \), \( n \in \mathbb{N} \), subject to the initial conditions. Finally, in Section 5 we define, inverse operators of fractional integral operators involving the three-parameters Mittag-Leffler function in the kernels with respect to another function. Concluding remarks close the paper.

2 Preliminaries

In this section, we present definitions of weighted and continuous spaces of functions [13]. This section contains, also, definitions and properties associated with the \( \psi \)-Riemann-Liouville fractional integral and the \( \psi \)-Caputo fractional derivative, [1].

Let \( \Omega = [a, b] \) \((0 < a < b < \infty)\) be a finite interval of the real axis \( \mathbb{R} \) and \( n \in \mathbb{N}_0 = \{0, 1, \ldots\} \). We denote by \( C^n(\Omega) \) a space of functions which are \( n \) times continuously differentiable on \( \Omega \) with the norm
\[
\|f\|_{C^n(\Omega)} = \sum_{k=0}^{n} \|f^{(k)}\|_{C(\Omega)} = \sum_{k=0}^{n} \max_{x \in \Omega} |f^{(k)}(x)|, \quad n \in \mathbb{N}_0.
\]
In particular, for \( n = 0 \), \( C^0(\Omega) = C(\Omega) \) is the space of the continuous function \( f \) on \( \Omega \) with the norm defined by
\[
\|f\|_{C(\Omega)} = \max_{x \in \Omega} |f(x)|.
\]
The weighted space \( C_{\nu,\psi}[a, b] \) of functions \( f \) given on \((a, b)\) with \( \nu \in \mathbb{R} \) \((0 \leq \nu < 1)\) is
\[
C_{\nu,\psi}(\Omega) = \{ f : (a, b) \to \mathbb{R} ; (\psi(x) - \psi(a))^\nu f(x) \in C(\Omega) \},
\]
with the norm
\[
\|f\|_{C_{\nu,\psi}(\Omega)} = \|(\psi(x) - \psi(a))^\nu f(x)\|_{C(\Omega)} = \max_{x \in \Omega} |(\psi(x) - \psi(a))^\nu f(x)|. \quad (1)
\]
If \( \nu = 0 \), we have \( C_{0,\psi}(\Omega) = C(\Omega) \).

**Definition 1.** [16] Let \( \alpha, \gamma, \rho, z \in \mathbb{R} \) with \( \rho > 0 \). The three-parameters Mittag-Leffler function is given by
\[
E_{\rho,\alpha}^\gamma(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\rho k + \alpha)} \frac{z^k}{k!}, \quad (2)
\]
where \( (\gamma)_k \) is the Pochhammer symbol defined as follow
\[
(\gamma)_k = \begin{cases} 1, & \text{for } k = 0 \\ \gamma(\gamma + 1) \cdots (\gamma + k - 1), & \text{for } k = 1, 2, \ldots, \end{cases} \quad (3)
\]
or, in terms of a quotient of gamma functions,
\[(\gamma)_k = \frac{\Gamma(\gamma + k)}{\Gamma(\gamma)}. \quad (4)\]

**Definition 2.** Let \(x, y \in \mathbb{R}\) with \(x > 0\) and \(y > 0\). The beta function, \(B(x, y)\), is defined by the Euler integral of the first kind
\[
B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt.
\]
or, in terms of gamma function
\[
B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.\quad (5)
\]

**Definition 3.** Let \(0 < \alpha \in \mathbb{R}\), \(\Omega = [a, b]\) be a finite or infinite interval, \(f\) an integrable function defined on \(\Omega\) and \(\psi \in C(\Omega)\) an increasing function such that \(\psi'(x) \neq 0\), for all \(x \in \Omega\). The left- and right-sided \(\psi\)-Riemann-Liouville fractional integrals of order \(\alpha\) of \(f\) on \(\Omega\) are defined by
\[
\Gamma^{\alpha;\psi}_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} f(t) dt \quad (6)
\]
and
\[
\Gamma^{\alpha;\psi}_{b-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \psi'(t) (\psi(t) - \psi(x))^{\alpha-1} f(t) dt, \quad (7)
\]
respectively. For \(\alpha \to 0\), we have
\[
\Gamma^{0;\psi}_{a+} f(x) = \Gamma^{0;\psi}_{b-} f(x) = f(x).
\]

**Definition 4.** Let \(0 < \alpha \in \mathbb{R}\), \(n \in \mathbb{N}\), \(I\) is the interval \(-\infty \leq a < b \leq \infty\), \(f, \psi \in C^n(I)\) two functions such that \(\psi\) is increasing and \(\psi'(x) \neq 0\), for all \(x \in I\). The left- and right-sided \(\psi\)-Caputo fractional derivatives of \(f\) of order \(\alpha\) are given by
\[
\mathcal{C}D^{\alpha;\psi}_{a+} f(x) = \Gamma^{n-\alpha;\psi}_{a+} \left(\frac{1}{\psi'(x)} \frac{d^n}{dx^n}\right) f(x)
\]
and
\[
\mathcal{C}D^{\alpha;\psi}_{b-} f(x) = \Gamma^{n-\alpha;\psi}_{b-} \left(-\frac{1}{\psi'(x)} \frac{d^n}{dx^n}\right) f(x),
\]
respectively, where
\[
n = [\alpha] + 1 \quad \text{for} \quad \alpha \not\in \mathbb{N}, \quad n = \alpha \quad \text{for} \quad \alpha \in \mathbb{N}.
\]
To simplify notation, we will use the abbreviated notation
\[
f^{[\alpha]}(x) = \left(\frac{1}{\psi'(x)} \frac{d^n}{dx^n}\right) f(x).
\]

**Property 1.** Let \(f \in C^n[a, b]\), \(0 < \alpha \in \mathbb{R}\) and \(\delta > 0\),
1. \(f(x) = (\psi(x) - \psi(a))^{\delta-1}\), then
\[
\Gamma^{\alpha;\psi}_{a+} f(x) = \frac{\Gamma(\delta)}{\Gamma(\alpha + \delta)} (\psi(x) - \psi(a))^{\alpha + \delta - 1}.
\]
2. \(\Gamma^{\alpha;\psi}_{a+} \mathcal{C}D^{\alpha;\psi}_{a+} f(x) = f(x) - \sum_{k=0}^{n-1} f^{[k]}(a) \frac{\psi(x) - \psi(a)^{k}}{k!}, \) where \(n - 1 < \alpha < n\) with \(n \in \mathbb{N}\).
3 Main results

In this section, we present properties associated with fractional integral operators involving the three-parameters Mittag-Leffler function in the kernels with respect to another function [24]. These operators were motivated by \( \psi \)-Riemann-Liouville fractional integrals containing in its kernels the three-parameters Mittag-Leffler function.

**Definition 5.** [24] Let \( \alpha, \gamma, \rho, \omega \in \mathbb{R} \) with \( \alpha > 0 \) and \( \rho > 0 \) and let \( \Omega = [a, b] \) be a finite or infinite interval of the real axis \( \mathbb{R} \), \( f \) an integrable function defined on \( \Omega \) and \( \psi \in C(\Omega) \) an increasing function such that \( \psi'(x) \neq 0 \), for all \( x \in \Omega \). The left- and right-sided fractional integral operators involving the three-parameters Mittag-Leffler function in the kernels with respect to another function are defined by

\[
E_{\rho, \alpha, \omega}^{\gamma, \psi} f(x) = \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma, \psi} [\omega(\psi(x) - \psi(t))]^\rho f(t) \, dt
\]

and

\[
E_{\rho, \alpha, \omega}^{-\gamma, \psi} f(x) = \int_x^b \psi'(t) (\psi(t) - \psi(x))^{\alpha-1} E_{\rho, \alpha}^{-\gamma, \psi} [\omega(\psi(t) - \psi(x))]^\rho f(t) \, dt,
\]

respectively. In particular, if \( \gamma = 0 \) we have the fractional integrals given by Eq. (6) and Eq. (7), this is,

\[
E_{\rho, 0, \omega}^{0, \psi} f(x) = E_{\alpha+}^{0, \psi} f(x) \quad \text{and} \quad E_{\rho, 0, \omega}^{-0, \psi} f(x) = E_{\beta-}^{0, \psi} f(x).
\]

If \( \alpha \to 0 \) and \( \gamma = 0 \), we have

\[
E_{\rho, 0, \omega}^{0, \psi} f(x) = f(x) \quad \text{and} \quad E_{\rho, 0, \omega}^{-0, \psi} f(x) = f(x).
\]

We prove some properties of the left-sided fractional operator \( E_{\rho, \alpha, \omega}^{\gamma, \psi} \). The corresponding results for \( E_{\rho, \alpha, \omega}^{-\gamma, \psi} \) can be derived analogously. The first result yields the linearity of the integral operators \( E_{\rho, \alpha, \omega}^{\gamma, \psi} \) and \( E_{\rho, \alpha, \omega}^{-\gamma, \psi} \).

**Theorem 1.** Let \( \alpha, \gamma, \rho, \omega \in \mathbb{R} \), with \( \alpha > 0 \) and \( \rho > 0 \). Also let \( f \) and \( g \) two functions and \( \lambda, \mu \) are arbitrary real constants, then

\[
E_{\rho, \alpha, \omega}^{\gamma, \psi} [\lambda f(x) \pm \mu g(x)] = \lambda E_{\rho, \alpha, \omega}^{\gamma, \psi} f(x) \pm \mu E_{\rho, \alpha, \omega}^{\gamma, \psi} g(x)
\]

and

\[
E_{\rho, \alpha, \omega}^{-\gamma, \psi} [\lambda f(x) \pm \mu g(x)] = \lambda E_{\rho, \alpha, \omega}^{-\gamma, \psi} f(x) \pm \mu E_{\rho, \alpha, \omega}^{-\gamma, \psi} g(x).
\]

Proof. The result follows from the fact these integral operators are linear. \( \square \)

The second result consists in calculating the fractional integrals \( E_{\rho, \alpha, \omega}^{\gamma, \psi} \) and \( E_{\rho, \alpha, \omega}^{-\gamma, \psi} \) of a power function.

**Lemma 1.** Let \( \alpha, \beta, \gamma, \rho, \omega \in \mathbb{R} \), with \( \alpha > 0 \), \( \beta > 0 \) and \( \rho > 0 \). Then,

\[
E_{\rho, \alpha, \omega}^{\gamma, \psi} [(\psi(x) - \psi(a))^{\beta-1}] = \Gamma(\beta)(\psi(x) - \psi(a))^{\alpha+\beta-1} E_{\rho, \alpha}^{\gamma, \psi} [\omega(\psi(x) - \psi(a))]^\rho
\]

and

\[
E_{\rho, \alpha, \omega}^{-\gamma, \psi} [(\psi(b) - \psi(x))^{\beta-1}] = \Gamma(\beta)(\psi(b) - \psi(x))^{\alpha+\beta-1} E_{\rho, \alpha}^{-\gamma, \psi} [\omega(\psi(b) - \psi(x))]^\rho.
\]
Proof. From Definition 5 we can write

\[ E_{p, \alpha, \omega}^{\gamma \psi}[\int_{x}^{a} ((\psi(x) - \psi(a))^{\beta-1} = \int_{a}^{x} \psi'(t) \sum_{k=0}^{\infty} \frac{\omega^k (\gamma)_{k} - \psi(t) - \psi(t))^{\rho k + \alpha - 1}}{k!} (\psi(t) - \psi(a))^{\beta-1} dt. \]

Making the change of variable \( \tau = \frac{\psi(t) - \psi(a)}{\psi(x) - \psi(a)} \) and since, the entire Mittag-Leffler function is uniformly convergent, we have interchange the order of integration and summation, to get

\[ E_{p, \alpha, \omega}^{\gamma \psi}[\int_{x}^{a} ((\psi(x) - \psi(a))^{\beta-1} = \sum_{k=0}^{\infty} \frac{\omega^k (\gamma)_{k}}{\Gamma(\rho k + \alpha k)} (\psi(x) - \psi(a))^{\rho k + \alpha + \beta - 1} \int_{0}^{1} (1 - \tau)^{\rho k + \alpha - 1} \tau^{\beta - 1} d\tau, \]

which, in accordance with Eq. (5), yields Eq. (9). \( \square \)

The following assertion, which yields the boundedness of the fractional integration operator \( E_{p, \alpha, \omega}^{\gamma \psi} \), holds on \( C_{\nu, \psi} \).

**Theorem 2.** Let \( \alpha, \gamma, \rho, \omega \in \mathbb{R} \) with \( \alpha > 0 \) and \( b > a \). The operators \( E_{p, \alpha, \omega}^{\gamma \psi} \) and \( E_{p, \alpha, \omega}^{\gamma \psi} \) are bounded on \( C_{\nu, \psi} \)

\[ \|E_{p, \alpha, \omega}^{\gamma \psi} f\|_{C_{\nu, \psi}[a,b]} \leq M \|f\|_{C_{\nu, \psi}[a,b]} \quad \text{and} \quad \|E_{p, \alpha, \omega}^{\gamma \psi} f\|_{C_{\nu, \psi}[a,b]} \leq M \|f\|_{C_{\nu, \psi}[a,b]}, \]

where

\[ M = \|((\psi(b) - \psi(a))^\alpha E_{p, \alpha, \omega}^{\gamma \psi} \omega (\psi(b) - \psi(a))^{\rho})\| \] (10)

**Proof.** According to Eq. (11), we have

\[ \|E_{p, \alpha, \omega}^{\gamma \psi} f\|_{C_{\nu, \psi}[a,b]} = \|((\psi(x) - \psi(a))^{\nu} E_{p, \alpha, \omega}^{\gamma \psi} f\|_{C_{\nu, \psi}[a,b]} = \max_{x \in [a,b]} |((\psi(x) - \psi(a))^{\nu} E_{p, \alpha, \omega}^{\gamma \psi} f| \leq \|f\|_{C_{\nu, \psi}[a,b]} \max_{x \in [a,b]} \int_{a}^{x} |\psi'(t) (\psi(x) - \psi(t))^{\alpha - 1} E_{p, \alpha, \omega}^{\gamma \psi} \omega (\psi(x) - \psi(t))^{\rho}| dt = \|f\|_{C_{\nu, \psi}[a,b]} \max_{x \in [a,b]} |((\psi(x) - \psi(a))^{\alpha} E_{p, \alpha, \omega}^{\gamma \psi} \omega (\psi(x) - \psi(a))^{\rho})| \leq \|f\|_{C_{\nu, \psi}[a,b]} M \|\nu\|_{C_{\nu, \psi}[a,b]}, \]

where \( M \) is given by Eq. (10). Consider the following relation with \( x \in \mathbb{R} \), \( x > 0 \) and \( (x + b) \in \mathbb{R} \setminus \{0, -1, -2, \ldots\} \).\( \square \)

\[ \lim_{x \to \infty} \frac{\Gamma(x + a)}{\Gamma(x + b)} x^{b-a} = 1. \]

We denote by \( c_k \) the \( k \)th term of the series in Eq. (10), then by ratio test the series converges,

\[ \lim_{k \to \infty} \frac{c_{k+1}}{c_k} = \lim_{k \to \infty} \frac{\omega(\gamma + k) \Gamma(p k + \alpha + 1)}{(k + 1) \Gamma(p k + \rho + \alpha + 1)} |(\psi(b) - \psi(a))^\rho | = \lim_{k \to \infty} \left[ \frac{(\gamma + k)}{(k + 1)(p k)^\rho} \right] |\omega| |(\psi(b) - \psi(a))^\rho | \to 0. \]

\( \square \)

The next result shows the composition of the fractional integral of a function with respect to another function \( E_{p_a}^{\gamma \psi} \) and fractional integral operator involving the three-parameters Mittag-Leffler function in the kernel with respect to another function \( E_{p, \alpha, \omega}^{\gamma \psi} \)
Theorem 3. Let $\alpha, \beta, \gamma, \rho, \omega \in \mathbb{R}$ with $\alpha > 0$, $\beta > 0$ and $\rho > 0$. Then,
\begin{align}
E_{\rho, \alpha, \omega; a}^{\gamma; \psi} I_{a}^{\beta; \psi} f(x) &= E_{\rho, \alpha, \omega; a}^{\gamma; \psi} f(x) = \frac{1}{\Gamma(\beta)} \int_{a}^{x} \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma; \psi} [\omega(\psi(x) - \psi(t))^\rho] \times \psi(t) (\psi(x) - \psi(t))^{\beta-1} f(u) du \, dt.
\end{align}

Applying the Dirichlet formula to interchange the order of integration, we obtain
\begin{align}
E_{\rho, \alpha, \omega; a}^{\gamma; \psi} I_{a}^{\beta; \psi} f(x) &= \frac{1}{\Gamma(\beta)} \int_{a}^{x} \psi'(u) f(u) du \int_{u}^{x} \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} \times E_{\rho, \alpha}^{\gamma; \psi} [\omega(\psi(x) - \psi(t))^\rho] (\psi(t) - \psi(u))^{\beta-1} dt,
\end{align}
and by changing the variable $\tau = \frac{\psi(t) - \psi(u)}{\psi(x) - \psi(u)}$, in the above second integral and rearranging, we find that
\begin{align}
E_{\rho, \alpha, \omega; a}^{\gamma; \psi} I_{a}^{\beta; \psi} f(x) &= \frac{1}{\Gamma(\beta)} \int_{a}^{x} \psi'(u) f(u) du \sum_{k=0}^{\infty} (\psi(x) - \psi(u))^{\rho k + \alpha + \beta - 1} \frac{\omega^k(\gamma)_k}{\Gamma(\rho k + \alpha + \beta)} \frac{1}{(1 - \tau)^{\rho k + \alpha + \beta - 1}} d\tau.
\end{align}

By Eq. (5), we can write
\begin{align}
E_{\rho, \alpha, \omega; a}^{\gamma; \psi} I_{a}^{\beta; \psi} f(x) &= \int_{a}^{x} \psi'(u) (\psi(x) - \psi(u))^{\alpha + \beta - 1} E_{\rho, \alpha + \beta}^{\gamma; \psi} [\omega(\psi(x) - \psi(u))^\rho] f(u) du

&= E_{\rho, \alpha + \beta, \omega; a}^{\gamma; \psi} f(x).
\end{align}

The proof of $E_{\rho, \alpha, \omega; a}^{\beta; \psi} E_{\rho, \alpha, \omega; a}^{\gamma; \psi} f(x) = E_{\rho, \alpha + \beta, \omega; a}^{\gamma; \psi} f(x)$ is similar. \qed

The following assertion for fractional integral operator involving the three-parameters Mittag-Leffler function in the kernel with respect to another function is the validity of the semigroup property.

Theorem 4. Let $\alpha, \beta, \gamma, \nu, \rho, \sigma, \omega \in \mathbb{R}$ with $\alpha > 0$, $\nu > 0$ and $\rho > 0$. Then,
\begin{align}
E_{\rho, \alpha, \omega; a}^{\gamma; \psi} I_{a}^{\sigma; \psi} f(x) &= E_{\rho, \alpha, \nu, \omega; a}^{\gamma + \sigma; \psi} f(x) = E_{\rho, \nu, \omega; a}^{\gamma; \psi} E_{\rho, \alpha, \omega; a}^{\sigma; \psi} f(x) \tag{12}
\end{align}
and
\begin{align}
E_{\rho, \alpha, \omega; b}^{\gamma; \psi} I_{b}^{\sigma; \psi} f(x) &= E_{\rho, \alpha + \nu, \omega; b}^{\gamma + \sigma; \psi} f(x) = E_{\rho, \nu, \omega; b}^{\sigma; \psi} E_{\rho, \alpha, \omega; b}^{\gamma; \psi} f(x).
\end{align}

Proof. Considering the Definition [5] we have
\begin{align}
E_{\rho, \alpha, \omega; a}^{\gamma; \psi} I_{a}^{\sigma; \psi} f(x) &= \int_{a}^{x} \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} E_{\rho, \alpha}^{\gamma; \psi} [\omega(\psi(x) - \psi(t))^\rho] \times \int_{a}^{t} \psi'(u) (\psi(t) - \psi(u))^{\nu-1} E_{\rho, \nu}^{\sigma; \psi} [\omega(\psi(t) - \psi(u))^\rho] f(u) du \, dt.
\end{align}
Interchanging the order of integration, we can write
\[
E_{\rho,\alpha,\omega; a}^{\gamma, \psi} + E_{\rho,\nu,\omega; a}^{\sigma, \psi} f(x) = \int_a^x \psi'(u) f(u) du \int_a^x \psi(t)(\psi(x) - \psi(t))^{\alpha - 1} E_{\rho,\alpha}^{\gamma}[\omega(\psi(x) - \psi(t))] \times (\psi(t) - \psi(u))^{\nu - 1} E_{\rho,\nu}^{\sigma}[\omega(\psi(t) - \psi(u))] dt.
\]

Taking the same variable change as Theorem 3 we obtain
\[
E_{\rho,\alpha,\omega; a}^{\gamma, \psi} + E_{\rho,\nu,\omega; a}^{\sigma, \psi} f(x) = \int_a^x \psi'(u) f(u)(\psi(x) - \psi(u))^{\alpha + \nu - 1} E_{\rho,\alpha}^{\gamma}[\omega(\psi(x) - \psi(u))] E_{\rho,\nu}^{\sigma}[\omega(\psi(x) - \psi(u))] \times \left[ \int_0^1 (1 - \tau)^{\rho k + \alpha - 1 - \tau^{m + \nu - 1}} d\tau \right] du.
\]

From Definition 2 we find
\[
E_{\rho,\alpha,\omega; a}^{\gamma, \psi} + E_{\rho,\nu,\omega; a}^{\sigma, \psi} f(x) = \int_a^x \psi'(u) f(u) (\psi(x) - \psi(u))^{\alpha + \nu - 1} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{\omega^{k+m}(\gamma)(\rho \kappa)(\omega(\psi(x) - \psi(u)))^{\rho(k+m)}}{k! m! \Gamma(\rho(k + m) + \alpha + \nu)} du.
\]

Let \( k \rightarrow k - m \), then
\[
E_{\rho,\alpha,\omega; a}^{\gamma, \psi} + E_{\rho,\nu,\omega; a}^{\sigma, \psi} f(x) = \int_a^x \psi'(u) f(u) (\psi(x) - \psi(u))^{\alpha + \nu - 1} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{\omega^{k}(\gamma)(\rho \kappa)(\omega(\psi(x) - \psi(u)))^{\rho k}}{(k - m)! m! \Gamma(\rho k + \alpha + \nu)} du
\]
\[
= \int_a^x \psi'(u) f(u) (\psi(x) - \psi(u))^{\alpha + \nu - 1} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{k \omega^{k}(\gamma)(\rho \kappa)(\omega(\psi(x) - \psi(u)))^{\rho k}}{\Gamma(\rho k + \alpha + \nu) k!} du.
\]

Using the relation
\[
(\gamma + \sigma)k = \sum_{m=0}^{k} \binom{k}{m} (\gamma)_{k-m}(\sigma)^m,
\]
we have
\[
E_{\rho,\alpha,\omega; a}^{\gamma, \psi} + E_{\rho,\nu,\omega; a}^{\sigma, \psi} f(x) = \int_a^x \psi'(u) f(u) (\psi(x) - \psi(u))^{\alpha + \nu - 1} \sum_{k=0}^{\infty} \frac{(\gamma + \sigma)k}{\Gamma(\rho k + \alpha + \nu)} [\omega(\psi(x) - \psi(u))]^{\rho k} du
\]
\[
= \int_a^x \psi'(u) f(u) (\psi(x) - \psi(u))^{\alpha + \nu - 1} E_{\rho,\alpha + \nu}^{\gamma+\sigma}[\omega(\psi(x) - \psi(u))] du
\]
\[
= E_{\rho,\alpha + \nu,\omega; a}^{\gamma+\sigma, \psi} f(x).
\]

The proof of \( E_{\rho,\nu,\omega; a}^{\sigma, \psi} + E_{\rho,\alpha,\omega; a}^{\gamma, \psi} f(x) = E_{\rho,\alpha + \nu,\omega; a}^{\gamma+\sigma, \psi} f(x) \) goes along similar lines. \( \square \)

### 4 Cauchy problem

In this section, we convert the initial value problem for the differential equation (Cauchy problem) into an equivalent Volterra integral equation. We obtain the solution of the Cauchy problem using the successive approximations.

**Theorem 5.** Consider the initial value problem:

\[
\begin{align*}
\mathcal{D}_a^\frac{\gamma}{\nu} u(x) &= f(x, u(x)), \quad x \in [a, b], \\
\left( \frac{1}{\psi'(x)} \frac{d}{dx} \right)^i u(x) \bigg|_{x=a} &= b_i, \quad i = 0, 1, \ldots, n - 1, 
\end{align*}
\]

where
1. \( 0 < \beta \notin \mathbb{N} \) and \( n = [\beta] + 1 \),
2. \( b_i, i = 0, 1, \ldots, n - 1 \), are fixed reals,
3. \( u \in C^{n-1}[a, b] \) such that \( C \mathcal{D}^{\beta, \psi}_{a+} u \) exists and is continuous in \([a, b]\),
4. \( f : [a, b] \times \mathbb{R} \to \mathbb{R} \) is continuous.

The Cauchy problem (13) is equivalent to the following Volterra integral equation

\[
\begin{equation} \label{eq:14}
    u(x) = \sum_{i=0}^{n-1} b_i \frac{(\psi(x) - \psi(a))^i}{i!} + \mathcal{I}^{\beta, \psi}_{a+} f(x, u(x)).
\end{equation}
\]

If \( f \) is Lipschitz continuous with respect to the second variable, then exists a unique solution to problem (13) on interval \([a, a + h] \subseteq [a, b]\).

From Theorem 5 we have the following lemma and theorem as particular cases.

**Lemma 2.** Let \( \alpha, \beta, \gamma, \rho, \lambda, \omega \in \mathbb{R} \) with \( \alpha > 0 \), \( \beta > 0 \) and \( \rho > 0 \). We consider the Cauchy problem with initial conditions:

\[
\begin{cases}
    C \mathcal{D}^{\beta, \psi}_{a+} u(x) = \lambda \mathcal{E}^{\gamma, \psi}_{\rho, \alpha, \omega, a+} u(x) + f(x), \\
    \left. \left( \frac{1}{\psi'(x)} \frac{d}{dx} \right)^i u(x) \right|_{x=a} = b_i, \quad (b_i \in \mathbb{R}; i = 0, 1, \ldots, n - 1).
\end{cases}
\]  \( \tag{15} \)

We suppose that \( f \in C([a, b], a \leq x \leq b) \), then by Theorem 5, the Cauchy problem (15) is equivalent in the space \( C^{n-1}[a, b] \) to the Volterra integral equation of the second kind

\[
\begin{equation} \label{eq:16}
    u(x) = \sum_{i=0}^{\infty} b_i \frac{(\psi(x) - \psi(a))^i}{i!} + \lambda \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha + \beta - 1} \mathcal{E}^{\gamma, \psi}_{\rho, \alpha, \omega, a+} \big[ \omega(\psi(x) - \psi(t)) \big] u(t) dt \\
    + \frac{1}{\Gamma(\beta)} \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\beta - 1} f(t) dt.
\end{equation}
\]

**Proof.** From Eq.(14) with \( f(x, u(x)) = \lambda \mathcal{E}^{\gamma, \psi}_{\rho, \alpha, \omega, a+} u(x) + f(x) \), we have

\[
    u(x) = \sum_{i=0}^{n-1} b_i \frac{(\psi(x) - \psi(a))^i}{i!} + \lambda \mathcal{I}^{\beta, \psi}_{a+} \mathcal{E}^{\gamma, \psi}_{\rho, \alpha, \omega, a+} u(x) + \mathcal{I}^{\beta, \psi}_{a+} f(x).
\]

According to Definition 3 and Theorem 3 we obtain Eq.(16). \( \square \)

**Theorem 6.** Let \( \alpha, \beta, \gamma, \rho, \lambda, \omega \in \mathbb{R} \) with \( \alpha > 0 \), \( \beta > 0 \) and \( \rho > 0 \). The solution of Eq.(16) is given by

\[
\begin{equation} \label{eq:17}
    u(x) = \sum_{i=0}^{n-1} b_i \frac{(\psi(x) - \psi(a))^i}{i!} + \lambda \sum_{j=0}^{\infty} \lambda^j \mathcal{E}^{\gamma, \psi}_{\rho, j(\alpha + \beta) + \beta, \omega, a+} [\omega(\psi(x) - \psi(a))]^j \\
    + \sum_{j=0}^{\infty} \lambda^j \mathcal{E}^{\gamma, \psi}_{\rho, j(\alpha + \beta) + \beta, \omega, a+} f(x).
\end{equation}
\]

**Proof.** According to successive approximations method, we set

\[
    u_0(x) = \sum_{i=0}^{\infty} b_i \frac{(\psi(x) - \psi(a))^i}{i!} \quad (18)
\]
and
\[ u_m(x) = u_0(x) + \lambda E^{\beta;\psi}_{\rho,\alpha,\omega,a} u_{m-1}(x) + I_{\alpha+}^\beta f(x), \quad m \in \mathbb{N}. \] (19)

Using Eq. (19) with \( m = 1, 2, \ldots \), we find
\[
\begin{align*}
 u_1(x) & = u_0(x) + \lambda E^{\beta;\psi}_{\rho,\alpha,\omega,a} u_0(x) + I_{\alpha+}^\beta f(x) \\
 u_2(x) & = u_0(x) + \lambda E^{\beta;\psi}_{\rho,\alpha,\omega,a} u_0(x) + \lambda E^{\beta;\psi}_{\rho,\alpha,\omega,a} u_0(x) + I_{\alpha+}^\beta f(x) \\
 & = u_0(x) + \lambda E^{\beta;\psi}_{\rho,\alpha,\omega,a} u_0(x) + \lambda E^{\beta;\psi}_{\rho,\alpha,\omega,a} u_0(x) + \lambda^2 E^{2\beta;\psi}_{\rho,2(\alpha+),\omega,a} u_0(x) + \lambda E^{\beta;\psi}_{\rho,\alpha,\omega,a} u_0(x) \\
 & = u_0(x) + \lambda E^{\beta;\psi}_{\rho,\alpha,\omega,a} u_0(x) + \lambda^2 E^{2\beta;\psi}_{\rho,2(\alpha+),\omega,a} u_0(x) + \lambda^3 E^{3\beta;\psi}_{\rho,3(\alpha+),\omega,a} u_0(x) + \lambda E^{\beta;\psi}_{\rho,\alpha,\omega,a} u_0(x) \\
 & \vdots \\
 u_m(x) & = u_0(x) + \sum_{j=1}^{m-1} \lambda^j E^{j;\psi}_{\rho,j(\alpha+),\omega,a} u_0(x) + \sum_{j=1}^{m-1} \lambda^j E^{j;\psi}_{\rho,j(\alpha+),\omega,a} f(x) + I_{\alpha+}^\beta f(x).
\end{align*}
\]

From Eq. (18), Lemma 1 and the particular case of Definition 5 we have
\[
 u_m(x) = \sum_{i=0}^{n-1} b_i(\psi(x) - \psi(a))^i \sum_{j=0}^{m-1} \lambda^j E^{j;\psi}_{\rho,j(\alpha+),\omega,a} + \sum_{j=0}^{m-1} \lambda^j E^{j;\psi}_{\rho,j(\alpha+),\omega,a} f(x).
\]

Taking \( m \to \infty \) follows Eq. (17).

### 4.1 Particular case

To conclude this section, we consider the following particular case of the Cauchy problem, Eq. (15), taking \( f(x) = \xi(\psi(x) - \psi(a))^\mu-1 E_{\rho,\mu}^\sigma [\omega(\psi(x) - \psi(a))^\rho] \).

**Theorem 7.** Let \( \alpha, \beta, \gamma, \mu, \rho, \sigma, \xi, \lambda, \omega \in \mathbb{R} \) with \( \alpha > 0, \beta > 0, \rho > 0 \) and \( \mu > 0 \). Then, the Cauchy problem
\[
\begin{cases}
 C^{\beta;\psi}_{\alpha+} u(x) = \lambda E^{\gamma;\psi}_{\rho,\alpha,\omega,a} u(x) + \xi(\psi(x) - \psi(a))^\mu-1 E_{\rho,\mu}^\sigma [\omega(\psi(x) - \psi(a))^\rho], \\
 \left( \frac{1}{\psi'(x)} \frac{d}{dx} \right)^i u(x) \bigg|_{x=a} = b_i, \quad (b_i \in \mathbb{R}; i = 0, 1, \ldots, n-1)
\end{cases}
\] (20)

admits a unique solution \( u(x) \in C^{n-1}[a, b] \), given by
\[
 u(x) = \sum_{i=0}^{n-1} b_i(\psi(x) - \psi(a))^i \sum_{j=0}^{\infty} \lambda^j (\psi(x) - \psi(a))^j(\alpha+)^{\beta+i} E^{j;\psi}_{\rho,j(\alpha+),\omega,a} + \xi(\psi(x) - \psi(a))^\beta+\mu-1 \sum_{j=0}^{\infty} \lambda^j (\psi(x) - \psi(a))^j(\alpha+)^{\beta+i} E^{j;\psi}_{\rho,j(\alpha+),\omega,a} + \mu \omega(\psi(x) - \psi(a))^\rho].
\] (21)
Proof. Using Lemma \[2\] with \(f(x) = \xi(x - x(a))^{\mu-1}E_{\rho,\mu}^\sigma[\omega(x - x(a))\rho]\) and the linearity property, Theorem \[1\] we have

\[
\begin{align*}
    u(x) &= \sum_{i=0}^{n-1} b_i (x - x(a))^i \sum_{j=0}^\infty \lambda^j (x - x(a))^{j(\alpha+\beta)} E_{\rho,j+\beta,\sigma}^{\gamma} \omega(x - x(a))^{\rho} \\
    &+ \xi \sum_{j=0}^\infty \sum_{i=0}^{n-1} E_{\rho,j+\beta,\sigma}^{\gamma} \omega(x - x(a))^{\rho} \{ (x - x(a))^{j(\alpha+\beta)} E_{\rho,\mu}^\sigma \omega(x - x(a))^{\rho} \}.
\end{align*}
\]

Eq. (\*) can be proved directly by using the proof of Theorem \[3\] which yields Eq. (21). If \(\psi(x) = x\), we recover the result presented in \[11\]. \(\square\)

5 The inverse operator

In this section, we construct the left inverse operator \(D_{a+}^{\gamma,\psi}\) of the operator \(E_{\rho,\mu}^\sigma\).

Definition 6. Let \(\alpha, \beta, \gamma, \mu, \rho, \omega \in \mathbb{R}\) with \(\alpha > 0\) and \(\rho > 0\). We define the left inverse operators \(D_{a+}^{\gamma,\psi}\) and \(D_{b-}^{\gamma,\psi}\) of the operators \(E_{\rho,\mu}^{\psi,\omega,a+}\) and \(E_{\rho,\mu}^{\psi,\omega,b-}\), respectively, as follows:

\[
D_{a+}^{\gamma,\psi} f(x) = C_{\alpha}^{\gamma,\psi} f_{\rho,\alpha-a,\omega,\sigma} + f(x)
\]

and

\[
D_{b-}^{\gamma,\psi} f(x) = C_{\alpha}^{\gamma,\psi} f_{\rho,\alpha-b,\omega,\sigma} - f(x).
\]

In fact, we have

\[
D_{a+}^{\gamma,\psi} f(x) = C_{\alpha}^{\gamma,\psi} f_{\rho,\alpha,a,\omega,\sigma} - f(x) = C_{\alpha}^{\gamma,\psi} f_{\rho,\alpha-a,\omega,\sigma} + f(x) = C_{\alpha}^{\gamma,\psi} f_{\rho,\alpha,\omega,\sigma} + f(x).
\]

6 Concluding remarks

In this work, we proved some properties associated with a fractional integral operator involving the three-parameters Mittag-Leffler function in the kernel with respect to another function. We proved that a Cauchy problem is equivalent to the Volterra integral equation of the second kind, established the solution in its closed-form for that, we used the method of successive approximations. Finally, the inverse operators of the \(E_{\rho,\mu}^{\psi,\omega,a+}\) and \(E_{\rho,\mu}^{\psi,\omega,b-}\) were defined.

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