Perfect quantumness and quantum-noise-free measurements

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To quantify nonclassicality in an operational sense, we fix a positive semidefinite observable describing a given measurement setup and quantify the negativity of its normally ordered version. Perfect quantumness is defined via extension of the result to arbitrary quantum systems, and the route to quantum-noise-free measurements is outlined. As a surprising result, even moderately squeezed states may exhibit perfect quantumness. We propose an implementation of a quantum-noise-free measurement through moderate squeezing.

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The experimental demonstration of fundamental nonclassical effects led to various applications of nonclassical light. As a consequence, the realization of nonclassical states has attracted substantial interest during the last decades. In this context, the quantitative characterization of nonclassical effects is an important problem. Here we seek for the connection of a quantitative characterization of quantum states with potential applications for the suppression of quantum-noise in measurements.

The concept of the distance between two quantum states was introduced by Hillery [1]. He defined the nonclassical distance as a quantitative measure of nonclassicality. Although it is an intuitive approach, in many cases the nonclassical distance is hard to calculate. Another proposal for a measure of nonclassicality was introduced by Lee [2], the so-called nonclassical depth of a quantum state. It is defined by the minimum number of thermal photons admixed to a quantum state, which is needed to destroy its nonclassical effects. From our viewpoint this should be considered as a measure of the robustness or fragility of a nonclassical state, rather than of its quantumness.

More recently, Asbóth et al. proposed to use the amount of entanglement, which can be potentially generated by a nonclassical state, as a measure of nonclassicality [3]. Despite interesting relations between nonclassicality and entanglement [4, 5], a genuine measure of nonclassicality or quantumness should not be defined through a special class of quantum effects. Moreover, an entanglement potential suffers from the difficulty to define a general entanglement measure, cf. e.g. [6, 7].

Qualitatively nonclassicality of a quantum state of a harmonic oscillator is characterized in Quantum Optics by the established notion of nonclassicality for quantum states of harmonic oscillators. These measures are directly based on observable mean values as they are obtained by a chosen experimental setup. This is just what an experimenter needs for a certain experiment. Our approach leads to remarkable perspectives for quantum-noise-free (QNF) measurements based on a manifold of quantum states. As an example we demonstrate that moderately squeezed states can be used to implement QNF measurements.

To introduce an operational measure of nonclassicality, we consider a given experimental setup which is characterized by an arbitrary but fixed operator $\hat{f}$. The resulting quantities $\hat{f}^\dagger \hat{f}$ and $\hat{f}^\dagger \hat{f} :$ are Hermitian operators and hence observables of the chosen setup. Whereas the first observable is positive semidefinite, the second one may attain negative expectation values. We suppose that for a chosen operator $\hat{f}$ the nonclassicality condition $\Pi$ is fulfilled. For example, let us consider a homodyne detection setup, measuring the phase-sensitive quadrature $\hat{x}_\varphi$ of a given radiation mode, cf. e.g. [10]. By choosing $\hat{f} \equiv \Delta \hat{x}_\varphi = \hat{x}_\varphi - \langle \hat{x}_\varphi \rangle$, the condition $\Pi$ represents quadrature squeezing,

$$\langle \Delta \hat{x}_\varphi \rangle < 0 .$$  (3)

More generally, we can choose $\hat{f} \equiv 1 + e^{i(k\hat{x}_\varphi + \phi)}$ for homodyne detection, with an arbitrary but fixed phase $\phi$. Now the quantities $\langle \hat{f}^\dagger \hat{f} \rangle$ and $\langle \hat{f}^\dagger \hat{f} : \rangle$ yield the full information on the characteristic functions of the Wigner and the Glauber-Sudarshan $P$ functions, respectively. Hence we have access to the full information on the quantum state and we may characterize its nonclassicality completely, for details see [11]. Thus, by fixing an observable we do not necessarily restrict the generality.

To quantify the nonclassicality of a given state in a particular experiment, we attempt to properly quantify the negativity that can be attained by the left-hand side (lhs) of the condition $\Pi$. Here and in the following we will assume that the lhs is negative, otherwise there is...
no need for a nonclassicality measure. Let us consider the difference $\Delta$ between the normally ordered and the ordinary expectation values of the chosen observable $\hat{f}^\dagger \hat{f}$,

$$\Delta = \langle \hat{f}^\dagger \hat{f} \rangle - \langle \hat{f}^\dagger \hat{f} \rangle. \quad (4)$$

For a given operator $\hat{f}$ it is straightforward to derive an explicit expression of the quantity $\Delta$ by methods of operator ordering, but this is not needed for the following considerations. By using the fact that $\langle \hat{f}^\dagger \hat{f} \rangle \geq 0$, it is clear that the relation

$$\Delta \leq \langle \hat{f}^\dagger \hat{f} \rangle < 0 \quad (5)$$

holds true. Now we may define the operational relative nonclassicality $R$ of a given quantum state for a chosen measurement scheme as

$$R = \frac{\langle \hat{f}^\dagger \hat{f} \rangle}{\Delta} = \frac{\langle \hat{f}^\dagger \hat{f} \rangle}{\langle \hat{f}^\dagger \hat{f} \rangle - \langle \hat{f}^\dagger \hat{f} \rangle}, \quad (6)$$

which quantifies the negativity of the lhs of the condition (1). Based on this definition, a quantum state exhibits perfect nonclassicality, that is $R = 1$, if the (negative) value of $\langle \hat{f}^\dagger \hat{f} \rangle$ approaches the lower bound according to the relation (3). Hence, the so-defined perfect nonclassicality is attained for

$$\Delta = \langle \hat{f}^\dagger \hat{f} \rangle \iff \langle \hat{f}^\dagger \hat{f} \rangle = 0. \quad (7)$$

In addition, we define that $R \equiv 0$ for $\langle \hat{f}^\dagger \hat{f} \rangle \geq 0$.

Due to the equivalence in Eq. (7), for the operationally defined perfect nonclassicality we no longer need the normal ordering often used in Quantum Optics. By the condition $\langle \hat{f}^\dagger \hat{f} \rangle = 0$ we may define perfect quantumness for an arbitrary quantum system. For a general mixed quantum state, described by the density operator $\hat{\rho} = \sum_\psi p_\psi |\psi\rangle \langle \psi|$, with $p_\psi > 0$ and $\sum_\psi p_\psi = 1$, perfect quantumness requires that

$$\langle \hat{f}^\dagger \hat{f} \rangle = \sum_\psi p_\psi \|\hat{f}|\psi\rangle\|^2 = 0. \quad (8)$$

This condition is fulfilled if and only if

$$\hat{f}|\psi\rangle = 0, \quad (9)$$

for all states $|\psi\rangle$ contained in $\hat{\rho}$. Thus perfect quantumness is attained for any quantum state composed of eigenstates of the operator $\hat{f}$ whose eigenvalues are zero. In such cases the observable $\hat{f}^\dagger \hat{f}$ is totally free of quantum noise. Note that the eigenvalue of zero is not a serious restriction, which becomes clear from the example in Eq. (12). In general one can substitute $\hat{f} \rightarrow \Delta \hat{f} = \hat{f} - \langle \hat{f} \rangle$, together with

$$\Delta \hat{f}|\psi\rangle = 0 \quad (10)$$

replacing Eq. (9).

Let us consider some physical consequences of perfect quantumness. We already renounced our starting assumption to define nonclassicality for a harmonic oscillator by applying normal ordering. Now we may define the appearance of perfect quantumness in a given experimental situation through Eq. (8), and hence by the fact that the accessible observable $\hat{f}^\dagger \hat{f}$ becomes a QNF variable. For this purpose it is sufficient to prepare the system under study in a pure quantum state that fulfills the condition (3). This idea applies to general, i.e. other than harmonic, quantum systems. It opens possibilities to perform high-precision measurements at the ultimate limit of vanishing quantum noise. Given an experimental setup and hence the related operator $\hat{f}$, one may solve Eq. (8) to derive the optimized quantum state for performing QNF measurements.

Now we turn to the prominent example of quadrature squeezing of the harmonic oscillator. Combining Eqs. (4) and (10), the sought perfect quantum state is given by

$$\hat{x}_\varphi|\psi\rangle = x_\varphi|\psi\rangle, \quad (11)$$

which defines the quadrature eigenstate, $|\psi\rangle \equiv |x_\varphi\rangle$, with the eigenvalue being $x_\varphi = \langle \hat{x}_\varphi \rangle$. This reproduces the well-known fact that the quadrature eigenstates are suited for QNF quadrature measurements. The severe difficulty in realizing this situation consists in the fact that these eigenstates represent the limit of infinitely strong squeezing, which would require an infinite amount of energy. Consequently, the perfectly squeezed states are unphysical ones. Nevertheless, experimenters try to generate strongly squeezed states in order to approach this ideal situation. For example, recently a 10 dB reduction of the noise power of radiation has been achieved [12], and even stronger squeezing was realized in the quantized motion of a trapped ion [13]. In this way one can suppress the noise effects in measurements significantly, but one cannot reach the QNF limit.

The question appears whether there is an alternative possibility of using squeezed states for QNF measurements. As we will demonstrate below, the answer to this question is yes! It can be realized by a proper choice of the observable to be measured. Even if the quadrature measurement – in view of the reduction of the quadrature variance of a squeezed state – seems to be the natural choice, we may achieve a better performance and eventually reach the QNF limit as follows. For simplicity, we will deal with a squeezed vacuum state $|0; \nu\rangle$, which obeys the eigenvalue equation

$$(\mu \hat{a}^\dagger + \nu \hat{a})|0; \nu\rangle = 0, \quad \mu^2 - |\nu|^2 = 1. \quad (12)$$

Here, $\nu$ ($\mu$) is a complex (real) parameter which controls the amount of noise reduction of the squeezed vacuum state with respect to the quadrature $\hat{x}_\varphi$ for properly fixed phase $\varphi$. Note that a total suppression of the quadrature noise appears for $|\nu| \rightarrow \infty$. However, as discussed above,
perfect squeezing is not a realistic route towards the realization of perfect quantumness and of QNF measurements. Instead, we may choose the operator \( \hat{f} \) characterizing our measurement device simply as

\[
\hat{f} = \mu \hat{a} + \nu \hat{a}^\dagger.
\]  

(13)

By comparing Eq. (12) with (9), it is obvious that the squeezed vacuum indeed obeys the condition of perfect quantumness for the resulting observable \( ^f \hat{f} \), thus it opens the possibility to implement QNF measurements.

What we still require is an apparatus measuring the observable \( ^f \hat{f} \). It may appear to be counterintuitive that the squeezed vacuum state \( |0; \nu \rangle \) remains perfectly nonclassical even for moderate squeezing, that is for finite \( |\nu| \)-values. To make use of this property, however, one only needs to properly adjust the measurement device to the available squeezed vacuum state.

In the remainder of this contribution we will consider a practical implementation of a QNF measurement by using a squeezed vacuum state with moderate squeezing. For this purpose we will consider the situation for a trapped and laser-driven ion. In this case the vibrational center-of-mass motion of the ion in the trap potential plays the role of the quantized harmonic oscillator. As noted above, the preparation of a motional squeezed state has been realized more than a decade ago [13].

Let us now introduce the required measurement scheme for the observable \( ^f \hat{f} \), with the operator \( \hat{f} \) given by Eq. (13). For this purpose a trapped ion is driven, in the resolved-sideband and the Lamb-Dicke regimes, simultaneously on the first red and blue sidebands, cf. Fig. 1. As indicated in the figure, the couplings on the red and the blue first sidebands are given by the Raby frequencies \( \Omega_r \) and \( \Omega_b \), respectively. We assume that the condition \( |\Omega_r| > |\Omega_b| \) is fulfilled. Note that the unlike driving of the two sidebands is the only needed modification of a known measurement scheme for the determination of the motional quantum state [14], which has already been realized [15]. After a chosen interaction time of the two lasers with the ion, the electronic-state occupation can be tested with a very high quantum efficiency. For this purpose one usually drives a dipole transition between the electronic ground state \( |1\rangle \) and an auxiliary state. The occurrence and the non-occurrence of resonance fluorescence on this transition efficiently detects the system in the state \( |1\rangle \) and \( |2\rangle \), respectively.

The interaction Hamiltonian in the interaction picture is of the form

\[
\hat{H}_{\text{int}} = \frac{\hbar}{2} \hat{A}_{21} e^{i \varphi_r} (|1\rangle \langle \hat{a} | + |\Omega_b| e^{i \Delta \varphi} |\hat{a}^\dagger \rangle) + \text{H.c.},
\]  

(14)

where \( \hat{A}_{ij} = |i\rangle \langle j| \) \((i,j = 1,2)\) is the electronic flip operator, \( \varphi_r \) and \( \Delta \varphi \) are the phase of the red-detuned laser and the phase difference of both lasers, respectively. This

![FIG. 1: (color online). Scheme for a QNF measurement with moderate squeezing. The interaction on the red sideband is stronger than that on the blue one, \( |\Omega_r| > |\Omega_b| \).](image)

Hamiltonian can be rewritten as

\[
\hat{H}_{\text{int}} = \frac{\hbar}{2} \Omega_f \hat{A}_{21} + \text{H.c.},
\]  

(15)

with \( \Omega = e^{i \varphi_r} \sqrt{|\Omega_r|^2 - |\Omega_b|^2} \). The operator \( \hat{f} \) is given by Eq. (13), where

\[
\nu = \frac{|\Omega_b|}{|\Omega|} e^{i \Delta \varphi},
\]  

(16)

together with \( \mu^2 = 1 + |\nu|^2 \) according to Eq. (12). Thus the resulting dynamics is indeed sensitive to the operator \( \hat{f} \) we are interested in.

Now it is straightforward to calculate the time evolution of a trapped ion initially (at time \( t = 0 \)) prepared, for example, in the state \( \hat{\rho}(0) = |2\rangle \langle 2| \otimes \hat{\rho}(0) \), where \( \hat{\rho} \) and \( \hat{\rho} \) denote the vibronic and the vibrational quantum state, respectively. That is, the ion is initially in the upper electronic state and the center-of-mass motion is in an arbitrary mixed quantum state. Let us consider the probability \( p_2(t) = \text{Tr}[|2\rangle \langle 2| \hat{\rho}(t)/2] \), that the ion is in the electronic state \( |2\rangle \) at time \( t \). Note that the trace only refers to the motional degrees of freedom. We need the electronic diagonal element \( \hat{U}_{22} \) of the time evolution operator,

\[
\hat{U}_{22}(t) = \cos \left( \frac{\Omega |t|}{2} \sqrt{|\hat{f}|^2 + 1} \right),
\]  

(17)

where we have used the property \( |\hat{f}|^2 = |\hat{f}^\dagger \hat{f}| + 1 \) of the operator \( \hat{f} \) given in Eq. (13). Note that \( \hat{U}_{22} \) is still an operator in the motional Hilbert space. The time evolution of the occupation probability of the upper electronic state can be easily calculated as

\[
p_2(t) = \frac{1}{2} \left\{ 1 + \text{Tr} \left[ \hat{\rho}(0) \cos \left( \frac{|\Omega| |t| \sqrt{|\hat{f}|^2 + 1}}{2} \right) \right] \right\}.
\]  

(18)
From this result it is obvious that the evolution of the electronic-state occupation sensitively depends on the statistics of the observable \( \hat{f} \hat{f} \) we are interested in.

Let us now consider an ion initially prepared in a motional squeezed vacuum state as given by Eq. (12), \( \hat{\rho}(0) = |0; \nu \rangle \langle 0; \nu | \), together with the choice of \( \hat{f} \) according to Eq. (13). In this case we easily arrive at

\[
p_2(t) = \frac{1}{2} \left[ 1 + \cos \left( |\Omega| t \right) \right].
\]

This represents a completely coherent oscillation, which reflects the QNF property of the observable \( \hat{f} \hat{f} \), which is accessible by our detection scheme. This coherent electronic dynamics clearly displays the striking property of the moderately squeezed states. For any amount of squeezing one may adjust the observable \( \hat{f} \hat{f} \) such that the squeezed state exhibits perfect quantumness. Consequently the implemented detection scheme represents a perfect QNF measurement.

By using a trapped ion, one can realize QNF measurements for a variety of observables. They can be detected by generalizations of the scheme given in Fig. 1. For example, one may drive motional sidebands of higher orders. Under conditions far from the Lamb-Dicke regime, the nonlinear Jaynes-Cummings Hamiltonian applies to each driven sideband [16]. Moreover, the accessible operators can be further generalized by simultaneously driving more than two vibronic transitions and by engineering the vibronic interaction [17].

We also note that the coherent oscillation obtained for a squeezed state is closely related to the coherent dynamics in the standard Jaynes-Cummings interaction [18]. The corresponding interaction Hamiltonian is obtained by setting \( \nu = 0 \) in Eqs. (15) and (13). In this case, a coherent oscillation occurs for the initial preparation of a motional Fock state. This behavior for the Fock state also displays its perfect quantumness, even if for \( \hat{f} = \hat{a} \) the requirement given by the condition (11) cannot be fulfilled. However, by choosing \( \Delta \hat{f} = \hat{a}^\dagger \hat{a} - (\hat{a}^\dagger \hat{a}) \), for \( \hat{f} \rightarrow \Delta \hat{f} \) the condition (11) is fulfilled for all Fock states \(|n \rangle \) with \( n \geq 1 \). For these states the condition (11) for perfect quantumness is clearly fulfilled. Hence, the perfectly coherent Jaynes-Cummings dynamics occurring for all Fock states, for \( n \geq 1 \) represents another special realization of a QNF measurement of the type under study.

The generalization of the method is straightforward. Given any measurement device and the related (positive semidefinite) observable \( \hat{f} \hat{f} \), the optimal quantum state can be calculated as the solution of Eq. (13). Then a possibility of the preparation of such a state must be developed. When this problem can be solved, the QNF measurement can be implemented. Also the extension to the detection of arbitrary Hermitian operators \( \hat{A} \) is easy, the latter are related to the positive semidefinite operators via \( \hat{A} = \hat{f} \hat{f} - \kappa 1, \kappa \in \mathbb{R} \). Such methods may be useful for a manifold of quantum systems, depending on the possibilities to prepare the desired quantum states.

In conclusion we have introduced an operational measure for the nonclassicality or quantumness of a quantum state of the harmonic oscillator. It is based on the negativity of an observable whose classical counterpart is positive semidefinite. The resulting perfect quantumness is related to the feasibility of performing totally quantum-noise-free measurements. As an example, we have demonstrated that a moderately squeezed state of the quantized center-of-mass motion of a trapped ion can display perfect quantumness. An implementation of the corresponding noise-free quantum measurement has been given. We have outlined that the introduced notion of perfect quantumness also applies to other than harmonic quantum systems, and the general strategy of implementing quantum-noise-free measurements has been considered for the case of arbitrary systems.

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