THE SCHWINGER MODEL AND
PERTURBATIVE CHARGE SCREENING

PAUL HOYER

Department of Physics
University of Helsinki, Helsinki, Finland

ABSTRACT

The well-known exact solution of the massless Schwinger model can be simply obtained by perturbing around a vacuum without the Dirac sea of filled negative energy states. The unusual vacuum structure changes the sign of the $i\epsilon$ prescription at the negative energy pole of the free fermion propagator. Consequently, all convergent fermion loop integrals vanish. The logarithmically divergent two-point loop gives a mass $e/\sqrt{\pi}$ to the photon, which turns into the free pointlike boson of the Schwinger model. We discuss the possiblity of using corresponding expansions to describe charge screening effects in the massive Schwinger model and the confinement phenomenon of QCD.
1. Introduction

In this paper we show how the well-known exact solution of massless QED$_2$ – the Schwinger model$^1$ – can be simply obtained in perturbation theory. We consider the perturbative expansion around a vacuum without a Dirac sea, i.e., with the negative energy states left unfilled. All fermion loop diagrams vanish, except the one with only two current insertions. Due to its logarithmic singularity the two-point loop gives a mass $M = e/\sqrt{\pi}$ to the photon. The screening of electric charge is complete and local, resulting in the dynamics of a free, pointlike massive boson.

The interpretation of this result is quite simple. Since the empty vacuum has no filled (negative energy) states, there is no real or virtual pair production. This suppresses the fermion degrees of freedom that are present in expansions around the Dirac vacuum. The non-vanishing, pointlike contribution of the two-point loop depends on the regularization of its logarithmic singularity. We regularize by defining the empty vacuum as the limit of one where the fermion states are filled only for energies $E \leq -\Lambda$, with $\Lambda \to \infty$. The pointlike contribution of the loop is independent of $\Lambda$, and can be obtained from the standard, gauge-invariant regularization of the two-point function for $\Lambda = 0$.

The quantitative success of this approach to the Schwinger model suggests that charge screening effects can perhaps be analogously described using perturbation theory also when the screening distance is finite, as in the massive Schwinger model and in QCD$_4$. For a screening distance of order $1/\Lambda$, all components of the vacuum state with $|\vec{p}| > \Lambda$ should then be standard, i.e., the positive (negative) energy states with large momentum should be empty (filled). To preserve charge conjugation symmetry, the $|\vec{p}| \leq \Lambda$ part of the vacuum wave function can be taken
as a superposition of two orthogonal states, one with all (positive and negative energy) states empty and another one with all these states filled.

In the Feynman rules, a change in the occupation number of a vacuum state implies reversing the $i\epsilon$ prescription at the corresponding (positive or negative energy) pole of the free fermion propagator. Changing the $i\epsilon$ prescription only for $|\vec{p}| \leq \Lambda$ leaves the short distance behavior, and hence the renormalization properties, of the theory intact. A similar modification can be applied also to the gluon propagator, even though we do not explicitly know the gluonic vacuum wave functional that would correspond to this prescription. We shall comment on some aspects of the perturbative expansion that results when such changes in the $i\epsilon$ prescription of the propagators are made.

The connection between the filling of the negative energy states in the perturbative vacuum and the form of the free propagator can be seen in the Schrödinger picture of the Grassmann path integral\(^2,3\). The standard Dirac vacuum, which has all negative energy states filled, has the wave function

$$V_F(t) = \exp \left[ -\frac{1}{2} \int \frac{d^3\vec{p}}{(2\pi)^3} \bar{\psi}(t,-\vec{p}) \vec{\gamma} \cdot \vec{p} + m \frac{E_p}{E_p - i\epsilon} \psi(t,\vec{p}) \right]$$

(1.1)

Here $\bar{\psi}$ and $\psi$ are the standard Grassmann variables describing the fermion fields and $E_p = \sqrt{\vec{p}^2 + m^2}$. The Dirac vacuum implies the standard Feynman propagator,

$$S_F(p) = i \frac{\not{p} + m}{(p^0 - E_p + i\epsilon)(p^0 + E_p - i\epsilon)}$$

(1.2)

which describes the propagation of free physical fermions. Particles of positive energy propagate forward in time, while those of negative energy propagate backwards, and are interpreted as forward moving anti-particles of positive energy.
In a regime like that in the Schwinger model, or in the confinement region of QCD, the electrons (or quarks) are not, however, simply related to the physically relevant degrees of freedom. We may then consider also other choices of vacua, and propagators, than those given by (1.1) and (1.2). One can in fact see that the Dirac vacuum functional (1.1) is not suitable for the physics of the Schwinger model. The filling of all states with \( E < 0 \) leads to a non-local wave function in \( x \)-space, given by the Fourier transform of \( (\vec{p} \cdot \vec{\gamma} + m)/E \) in (1.1). Since in the exact solution of the Schwinger model charge is screened locally, the long-range correlations of the Dirac vacuum suggest that this is a poor starting point for a perturbative expansion. Nevertheless, even though the usual perturbation theory requires summing many diagrams, it has in fact also been used to derive some of the Schwinger model results \(^4\).

When none of the negative energy states are filled the vacuum functional is

\[
V_E(t) = \exp \left[ -\frac{1}{2} \int \frac{d^3\vec{p}}{(2\pi)^3} \bar{\psi}(t, -\vec{p}) \gamma^0 \psi(t, \vec{p}) \right] \tag{1.3}
\]

which is local also in coordinate space. The corresponding free fermion propagator differs from the Feynman propagator (1.2) only in the \( i\epsilon \) prescription at the negative energy pole,

\[
S_E(p) = i \frac{\gamma^0 + m}{(p^0 - E_p + i\epsilon)(p^0 + E_p + i\epsilon)} \tag{1.4}
\]

Hence both positive and negative energy fermions propagate only forward in time,

\[
S_E(t, \vec{p}) = \theta(t) \frac{1}{2E_p} \left[ (\gamma^0 + m) \exp(-iE_pt) + (\gamma^0 - m) \exp(iE_pt) \right] \tag{1.5}
\]

where \( p^0 = E_p \). This means that all (non-local) fermion loops vanish, as the fermion must propagate backwards in time in some part of the loop. In particular,
there are no free fermion pair production thresholds. The empty vacuum (1.3) may thus be a good starting point for a perturbative expansion in situations where asymptotic free fermion states do not exist, as in the Schwinger model.

In a free fermion vacuum we can choose to fill the negative energy states, or to leave them empty, separately for each momentum $\vec{p}$. In theories such as QCD, or in QED$_2$ with a finite coupling constant to mass ratio $e/m$ (the massive Schwinger model), we expect the charge screening radius to be finite, say $\mathcal{O}(1/\Lambda)$, where $\Lambda$ is some momentum scale ($\Lambda \simeq \Lambda_{QCD} \simeq 200$ MeV for QCD). Then it is natural to use the mixed propagator

$$S_\Lambda(p) = S_E(p)\theta(|\vec{p}| \leq \Lambda) + S_F(p)\theta(|\vec{p}| > \Lambda) \quad (1.6)$$

The ultraviolet behavior of the theory is thus unaffected by the long-distance charge screening effects, as is physically reasonable.

In the next section we solve the Schwinger model starting from the mixed propagator (1.6), then taking $\Lambda \to \infty$. In section 3 we discuss some properties of the perturbative expansion for massive QED$_2$ and QCD$_4$ using a mixed propagator like (1.6) with finite $\Lambda$. Our conclusions are summarized in section 4.

2. The Massless Schwinger Model

The QED$_2$ partition function is

$$Z = \int \mathcal{D}(A) \exp \left\{ i \int d^2x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - e \frac{\delta}{\delta \vec{\zeta}^\dagger} \frac{\delta}{\delta \zeta} \right] \right\} Z_f[\zeta, \bar{\zeta}] \bigg|_{\zeta = \bar{\zeta} = 0} \quad (2.1)$$

where $Z_f$ is the free fermion generating functional, expressed in terms of the sources
ζ, ¯ζ of the fermion fields ¯ψ, ψ:

$$Z_f[ζ, ¯ζ] = \exp \left[ \frac{d^2 p}{(2\pi)^2} ¯ζ(-p)S(p)ζ(p) \right]$$  \hspace{1cm} (2.2)

The $iε$ prescription of the free propagator $S(p)$ depends on the wave function of the perturbative vacuum used as a boundary condition at $t = ±∞$. Here we wish to show that the physics of the massless Schwinger model is obtained using the unconventional propagator (1.4).

Since both poles of the propagator (1.4) are in the lower half $p^0$ plane, the loop momentum integral of any convergent fermion loop vanishes (the $p^0$ contour can be closed in the upper half plane). Hence the only potentially non-vanishing fermion contribution to (2.1) is from the loop with two photon insertions, which is logarithmically divergent in 1+1 dimensions. Summing these contributions gives

$$Z = \int D(A) \exp \left\{ i \int d^2 x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{i}{2} \int d^2 x d^2 y A_\mu(x) L_E^{\mu\nu}(x - y) A_\nu(y) \right] \right\}$$  \hspace{1cm} (2.3)

where $L_E^{\mu\nu}$ is the two-point fermion loop

$$L_E^{\mu\nu}(x - y) = -(-ie)^2 Tr[γ^\mu S_E(x - y)γ^\nu S_E(y - x)]$$  \hspace{1cm} (2.4)

This loop can give a finite contribution only owing to its logarithmic divergence in momentum space, and we may thus expect that it is proportional to $δ^2(x - y)$. This can be seen more explicitly from the expression of the massless propagator (1.5) in coordinate space:

$$S_E(x^0, x^1; m = 0) = θ(x^0)\left[ \frac{1}{2}(γ^0 + γ^1)δ(x^0 + x^1) + \frac{1}{2}(γ^0 - γ^1)δ(x^0 - x^1) \right]$$  \hspace{1cm} (2.5)

In the loop (2.4), $θ(x^0 - y^0)θ(y^0 - x^0)$ forces $x^0 = y^0$, while the $δ$-functions in the propagator (2.5) then ensure $x^1 = y^1$. Hence the Schwinger model boson, viewed
as an $f \bar{f}$ composite, has a pointlike wave function. This feature is known from studies of the QED$_2$ bound state wave functions in the $m \to 0$ limit.

Since the full contribution to the loop (2.4) comes from the local point $x = y$, its value (and therefore also the mass of the free boson) is dependent on the regularization procedure. We shall use the standard regularization which preserves gauge invariance. We calculate the loop using the mixed propagator (1.6), and take $\Lambda \to \infty$ at the end. Hence the ultraviolet contribution is always evaluated with the standard Feynman $i \epsilon$ prescription. This is equivalent to keeping a finite mass in evaluating the loop. The screening radius $1/\Lambda$ for massive QED$_2$ can be estimated as the distance at which the linear potential energy $V(1/\Lambda) = \frac{1}{2} e^2/\Lambda$ equals the mass $2m$ of a produced fermion pair,

$$\Lambda \simeq e^2/4m$$  \hfill (2.6)

For our present purposes the precise value of $\Lambda$ is not important, only the property $\Lambda \to \infty$ as $m/e \to 0$.

For a finite screening radius $1/\Lambda$ we use the propagator (1.6), and thus consider

$$L_{\Lambda}^{\mu\nu}(x - y) = -(-ie)^2 Tr[\gamma^\mu S_{\Lambda}(x - y)\gamma^\nu S_{\Lambda}(y - x)]$$  \hfill (2.7)

We first calculate the standard, gauge-invariant expression for the two-point function $L_{\Lambda}^{\mu\nu}$ when $\Lambda = 0$, \textit{i.e.}, using Feynman propagators. This takes care of the ultraviolet regularization. The expression for $L_{\Lambda}^{\mu\nu}$ at any finite $\Lambda$ is then unambiguous, since the loop momentum integral is modified only over a finite range $|p^1| < \Lambda$. 

7
The momentum space expression for $L_{\mu\nu}^0$ can be written as

$$L_{\mu\nu}^0(q) = e^2 \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} \frac{dk}{2\pi} Tr \left[ \gamma^\mu S_F(t, k) \gamma^\nu S_F(-t, k) \right] \exp(iq^0t) + C^{\mu\nu} \quad (2.8)$$

where $k_\pm \equiv k \pm q^1/2$. From (1.2) we have

$$S_F(t, p^1) = \frac{1}{2E} \left[ \theta(t)(p^0 + m)e^{-iEt} - \theta(-t)(p^0 - m)e^{iEt} \right] \quad (2.9)$$

where $p^0 = E = \sqrt{(p^1)^2 + m^2}$. It is straightforward to check that the integral in (2.8) actually is convergent. Due to the logarithmic divergence of other integral representations of $L_{\mu\nu}^0$, the integral in (2.8) needs to give the usual gauge-invariant definition of $L_{\mu\nu}^0$ only up to a $q$-independent constant $C^{\mu\nu}$, as indicated in (2.8).

Some details of the evaluation of the integral in (2.8) are given in the Appendix. The result is

$$L_{\mu\nu}^0(q) = \frac{ie^2}{\pi} q^2 \left( -g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) \int_0^1 \frac{dx}{m^2 - q^2x(1-x)} - i\epsilon + \frac{ie^2}{\pi} g^{\mu1} g^{\nu1} + C^{\mu\nu} \quad (2.10)$$

Hence gauge invariance demands

$$C^{\mu\nu} = -i \frac{e^2}{\pi} g^{\mu1} g^{\nu1} \quad (2.11)$$

The expression for $L_{\mu\nu}^0$ given by (2.8) is then the usual one which is directly obtained in a gauge-invariant procedure such as dimensional regularization.

The screened loop $L_{\mu\nu}^\Lambda(q)$ is given by (2.8) with the Feynman propagators $S_F$ replaced with the screened propagators $S_\Lambda$ of (1.6). Since $S_E(t, p) \propto \theta(t)$ according to (1.5), the $k$-integral of $L_{\mu\nu}^\Lambda$ is restricted to the domain where at least one of the
propagators is a Feynman propagator. When both propagators are of the Feynman type, \( |k_+| > \Lambda \) and \( |k_-| > \Lambda \), the integral is convergent as before, and the condition on the integration range forces this part of the integral to vanish as \( \Lambda \to \infty \). The finite integration range where \( |k_+| > \Lambda \) but \( |k_-| < \Lambda \), and vice versa, gives a non-vanishing contribution in the \( \Lambda \to \infty \) limit. As shown in the Appendix, the result is

\[
\lim_{\Lambda \to \infty} L_{\Lambda}^{\mu\nu}(q) = -i \frac{e^2}{\pi} (-g^{\mu\nu} + \frac{q^{\mu} q^{\nu}}{q^2})
\]  

(2.12)

Substituting this into the partition function (2.3) we obtain

\[
Z = \int \mathcal{D}(A) \exp \left\{ i \int d^2x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{e^2}{2\pi} A_\mu(x) \left( -g^{\mu\nu} + \frac{\partial^{\mu} \partial^{\nu}}{\partial^2} \right) A_\nu(x) \right] \right\}
\]  

(2.13)

This agrees with the well-known exact solution of the massless Schwinger model\(^1\).

3. Finite Screening Lengths in Perturbation Theory

The solution of the massless Schwinger model presented above has some interesting aspects:

(i) Somewhat paradoxically, it shows that perturbation theory can give simple and correct results even in a strong coupling \((e/m \to \infty)\) regime, provided the expansion is made around the proper vacuum state.

(ii) The derivation suggests an immediate generalization to higher dimensions.

Consider, then, a system with finite screening length, such as the massive Schwinger model or QCD in \(3 + 1\) dimensions. The straightforward generalization of the method employed in Section 2 is to use perturbation theory with screened propagators of the form (1.6), with a momentum scale \( \Lambda \) of order (2.6) in QED\(_2\) and
$\Lambda \simeq \Lambda_{QCD}$ in QCD$_4$. Here we wish to make some preliminary remarks concerning the properties of such an expansion. More careful studies will show whether the effects of confinement can indeed be described so simply, using perturbation theory. Related ideas, concerning changes in the occupation of low momentum positive and negative energy vacuum states, have been put forward in Ref. 6.

3.1. Quark and Gluon Vacuum Functionals

The free fermion vacuum functional which leads to the screened fermion propagator (1.6) is according to (1.1) and (1.3) given by $V_{\Lambda}^{\pm}(t)$, where

$$V_{\Lambda}^{\pm}(t) = \exp \left\{ -\frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \bar{\psi}(t, -\p) \left[ \theta(|\p| > \Lambda) \frac{\p \cdot \hat{\gamma} + m}{E_p} \right. \right.$$

$$\left. \left. \pm \theta(|\p| \leq \Lambda) \gamma^0 \right] \psi(t, \p) \right\}$$

(3.1)

The vacua $|\Omega_{\Lambda}^{\pm}\rangle$ described by the wave functionals (3.1) have their positive and negative energy states for $|\p| \leq \Lambda$ all empty ($|\Omega_{\Lambda}^{+}\rangle$) or filled ($|\Omega_{\Lambda}^{-}\rangle$). Charge symmetry considerations suggest that we choose the linear superposition

$$|\Omega_{\Lambda}\rangle = \frac{1}{\sqrt{2}} \left( |\Omega_{\Lambda}^{+}\rangle + |\Omega_{\Lambda}^{-}\rangle \right)$$

(3.2)

as the ground state for a perturbative expansion when the physical screening length is $1/\Lambda$.

The states $|\Omega_{\Lambda}^{+}\rangle$ and $|\Omega_{\Lambda}^{-}\rangle$ have different occupation numbers over a finite range of $\p$, and are therefore orthogonal$^2$. This orthogonality ensures that there are only diagonal contributions to Green functions at any finite order of perturbation theory,
\[ \langle \Omega_\Lambda | \mathcal{O} | \Omega_\Lambda \rangle = \frac{1}{2} \left[ \langle \Omega_\Lambda^+ | \mathcal{O} | \Omega_\Lambda^+ \rangle + \langle \Omega_\Lambda^- | \mathcal{O} | \Omega_\Lambda^- \rangle \right] \tag{3.3} \]

for all operators \( \mathcal{O} \). Thus all the \(|\vec{p}| \leq \Lambda\) propagators in a given Feynman diagram have the same \(+i\epsilon\) or \(−i\epsilon\) prescription at both \((p^0 = \pm E_p)\) poles. The full result is then obtained by averaging the \(+i\epsilon\) prescription at both poles of each propagator (as in (1.4)) with the \(−i\epsilon\) prescription, Feynman diagram by Feynman diagram.

It is natural to postulate the same \(i\epsilon\) prescription for describing the screening of the color charge of gluons. Thus, for example, the \(\langle \Omega_\Lambda^+ | \mathcal{O} | \Omega_\Lambda^- \rangle\) contribution to (3.3) is obtained with a screened gluon propagator analogous to (1.6),

\[ D_\Lambda(p) = D_E(p)\theta(|\vec{p}| \leq \Lambda) + D_F(p)\theta(|\vec{p}| > \Lambda) \tag{3.4} \]

where in Feynman gauge

\[ iD_E^{\mu\nu}(p) = \frac{-ig^{\mu\nu}}{(p^0 - E_p + i\epsilon)(p^0 + E_p + i\epsilon)} \tag{3.5} \]

and \(iD_F(p)\) is the usual Feynman propagator. We do not explicitly know what gluon vacuum functional would give the propagator (3.4). A free gaussian functional leads uniquely to a gluon propagator of the Feynman form\(^3\). However, the consequences of using screened gluon propagators (3.4) can be studied even without knowing the corresponding structure of the gluon vacuum.

The fact that the propagators (1.6) and (3.4) differ from the standard Feynman ones only at low momenta \(|\vec{p}| \leq \Lambda\) suggests that the short distance structure and renormalizability of the screened theory remain conventional. This is obviously important for both theoretical and phenomenological reasons. On the other hand, low energy production thresholds of free quarks and gluons are eliminated, as we shall see below.
3.2. GLUON AND QUARK MASSES

Consider again the two-point fermion loop contribution (2.7), now for QCD. The standard result, using Feynman propagators (Λ = 0) for SU(N) is

\[ L_{0}^{\mu\nu}(q) = iN\delta^{ab} \left( -g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) q^2 \omega_{0}(q^2) \]  \hspace{1cm} (3.6)

\[ \omega_{0}(q^2) = -\frac{g^2}{2\pi^2} \int_{0}^{1} dx (1-x)\log[1-x(1-x)q^2/m^2] \]  \hspace{1cm} (3.7)

where \( a, b \) are the color indices of the external currents. In the current rest frame (\( q = 0 \)), the expression for the loop with screened propagators (1.6) is

\[ L_{\Lambda}^{\mu\nu}(q) = L_{0}^{\mu\nu} + iN\delta^{ab} g^2 \int \frac{d^4k}{(2\pi)^3} \theta(\Lambda^2 - k^2) Tr[\gamma^\mu (\not{k} + m)\gamma^\nu (\not{k} + m)] \]
\[ \times \left[ \frac{\delta(k^0_+ + E_k)}{k^0_+ + E_k} + \frac{\delta(k^0_- + E_k)}{k^0_- + E_k} \right] \]  \hspace{1cm} (3.8)

where \( k_\pm = k \pm \frac{1}{2}g \). The difference between \( L_{\Lambda} \) and \( L_{0} \) in (3.8) is due to the \( i\epsilon \) prescription of the quark propagator poles at \( k^0_\pm = -E_k = -\sqrt{k^2 + m^2} \) for \( |\vec{k}| \leq \Lambda \), as given by the integral. Defining \( \omega_{\Lambda} \) as in (3.6) we get (for \( q = 0 \))

\[ \omega_{\Lambda} = \omega_{0} + \frac{4g^2}{3\pi^2 q^2} \int_{0}^{\Lambda} \frac{dk}{E_k} \frac{k^2(2k^2 + 3m^2)}{(q^2 - 4E_k^2 + i\epsilon)} \]  \hspace{1cm} (3.9)

It is straightforward to deduce from (3.7) and (3.9) that

\[ \text{Im} \omega_{\Lambda} = \begin{cases} 
0 & \text{for } q^2 < 4(\Lambda^2 + m^2) \\
\text{Im} \omega_{0} & \text{for } q^2 > 4(\Lambda^2 + m^2) 
\end{cases} \]  \hspace{1cm} (3.10)

This result is a direct consequence of the screened quark propagators (1.6). For \( |\vec{k}| \leq \Lambda \) the poles in the loop energy \( k^0 \) all lie in the lower half plane, preventing
a pinch of the integration contour and hence also an imaginary part. From a
phenomenological point of view, the absence of an imaginary part in the low $q^2$
region reflects the absence of physical quark degrees of freedom. For large $q^2$, $\text{Im} \omega_\Lambda$
is as expected given by standard perturbation theory.

Just as in the Schwinger model, the screened propagators give rise to a finite
mass for the gauge boson. After an iteration of the quark loop contribution, the
denominator of the gluon propagator is $q^2(1 + \omega_\Lambda)$. According to (3.9), $\omega_\Lambda$ has a
pole at $q^2 = 0$, shifting the $q^2 = 0$ pole of the free propagator to $q^2 = m_g^2$, where
$\omega_\Lambda(q^2 = m_g^2) = -1$. For $\Lambda >> m$ the second term in (3.9) dominates, and we
obtain approximately, for small $g^2$,

$$m_g^2 \simeq \frac{g^2}{3\pi^2} \Lambda^2$$

(3.11)

A non-vanishing gluon mass implies a finite range for the color interaction, as is
appropriate for a screened charge. In a quantitative calculation we should of course
include the contribution of the gluon loop. Using a screened gluon propagator (3.4),
the analytic properties of the gluon loop would be similar to those of the quark
loop. In particular, its imaginary part would again vanish for $q^2 < 4\Lambda^2$.

In an analogous way, the radiative corrections to the quark propagator give
rise to a finite quark mass of $\mathcal{O}(g^2\Lambda)$, even if the bare quark mass vanishes. Hence
chiral symmetry is broken, and one may consider the applicability of the Nambu-
Jona-Lasinio method$^7$ in the present framework.
3.3. Boost Non-Invariance

The $\theta$-functions in the screened quark and gluon propagators (1.6), (3.4) depend on the absolute value of the 3-momentum and hence on the reference frame. It is thus clear that the Green functions will not be boost invariant order by order in perturbation theory.

The non-Lorentz covariance of perturbation theory derives from our choice of vacuum functional. For $|\vec{p}| > \Lambda$ the vacuum (3.2) includes the usual Dirac sea of negative energy fermion pairs, while for $|\vec{p}| \leq \Lambda$ both the positive and negative energy states are either empty ($|\Omega^+\Lambda\rangle$) or filled ($|\Omega^-\Lambda\rangle$). Insofar as the full perturbative expansion at least formally represents the complete, Lorentz-invariant theory, one may hope that boost invariance is restored at higher orders.

A well-known example of how Lorentz invariance requires summing perturbative diagrams to arbitrary order is provided by the Bethe-Salpeter equation for a light fermion bound to a particle with large mass $M^8$. The bound state equation has non-invariant projection operators on the positive energy states. The relativistically invariant Dirac equation is obtained in the $M \to \infty$ limit only after the inclusion of kernels of arbitrarily high order, thus allowing any number of fermion pairs in the Fock states. It is also interesting to note that the Dirac equation is, on the other hand, obtained directly from the lowest order, single photon exchange kernel when one expands around the empty vacuum (1.3) (which from the standard point of view already contains an infinite number of pairs)$^3$.

The dynamics of bound states is very dependent on the choice of frame. The description of hard scattering processes in terms of perturbative QCD is generally formulated in a frame where the hadrons have large momenta. It is only in this
frame that their constituents can be treated as quasifree partons carrying a measurable fraction of the total momentum. In the formulation of perturbation theory discussed above, high momentum ($|\vec{p}| > \Lambda$) quarks and gluons are treated as physical particles, obeying the rules of ordinary perturbation theory. Slow partons – the constituents of hadrons at rest or wee partons of fast hadrons – are “screened”, in analogy to the fermions of the Schwinger model. The consequent loss of explicit boost invariance is, at least superficially, in accord with conventional wisdom.

3.4. A PSEUDO THRESHOLD SINGULARITY

The fermion loop $L^{\mu\nu}_\Lambda(q)$ in (2.7) has both a threshold singularity in the region $q^2 > 0$, and a pseudothreshold singularity when $q^2 < 0$. The latter exists in a frame with center-of-mass motion ($\vec{q} \neq 0$) when one of the propagators in the loop has momentum above $\Lambda$, and the other one momentum below $\Lambda$. There can then be a pinch between the negative energy pole of the Feynman propagator and the negative energy pole of the $|\vec{p}| \leq \Lambda$ propagator (1.4), since they have opposite $i\epsilon$ prescriptions. This corresponds to the excitation of a fermion in the filled Dirac sea to one of the empty negative energy states. The pseudothreshold singularity is relevant, e.g., in deep inelastic scattering (since $q^2 < 0$), and describes a final state consisting of a “physical” antifermion with $|\vec{p}| > \Lambda$ and a wee, negative energy fermion with $|\vec{p}| \leq \Lambda$. The charge symmetric process with a fast fermion is obtained from the $|\Omega^+\Lambda\rangle$ vacuum component in (3.2). Some momentum transfer from a target particle is of course required to materialize the jet (which starts off with $q^2 < 0$).
4. Summary

In this paper we have solved the Schwinger model in the strong coupling limit \((e/m \rightarrow \infty)\), based on perturbation theory around a vacuum without the Dirac sea. In such a vacuum fermion pair production is suppressed, and the sole non-vanishing contribution is given by the two-point fermion loop, due to its logarithmic singularity. This expansion is thus much more useful for understanding the physics of the Schwinger model than the equivalent standard perturbation theory\(^4\).

The different vacuum structure manifests itself only in the \(i\epsilon\) prescription at the negative energy pole of the free fermion propagator. Hence the method can be generalized in a straightforward way both to higher dimensions and to boson propagators. We briefly discussed some aspects of such a method for treating the physics of systems with a finite charge screening length, such as the massive Schwinger model and QCD. Several interesting features emerge, including finite gluon and quark masses, which signal the breakdown of gauge and chiral symmetry. More work will be required to establish whether such an approach can lead to a self-consistent and useful description of charge screening by perturbative methods.

Acknowledgement: I am grateful to S. J. Brodsky, H. Hansson, C. S. Lam and H. B. Nielsen for helpful discussions. I particularly thank J. Grundberg for pointing out an error in the manuscript.
APPENDIX

Here we would like to give the (straightforward) derivation of Eqs. (2.10) and (2.12). Substituting the Feynman propagator (2.9) into (2.8) and doing the $t$-integral gives

$$L_{0}^{\mu\nu}(q) = -\frac{ie^{2}}{8\pi} \int_{-\infty}^{\infty} \frac{dk}{E_{+}E_{-}} \left\{ \frac{Tr[\gamma^{\mu}(\not{k}_{+} + m)\gamma^{\nu}(\not{k}_{1} - m)]}{q^{0} - E_{+} - E_{-} + i\epsilon} - \frac{Tr[\gamma^{\mu}(\not{k}^{1}_{+} - m)\gamma^{\nu}(\not{k}_{-} + m)]}{q^{0} + E_{+} + E_{-} - i\epsilon} \right\} + C^{\mu\nu} \quad (A.1)$$

where $k^{0}_{\pm} = E_{\pm} = \sqrt{k_{\pm}^{2} + m^{2}}$, with $k_{\pm} = k \pm q^{1}/2$. The traces are $O(1)$ for $k \to \pm \infty$, thus ensuring the convergence of the $k$-integral.

We express the energies and momenta appearing in (A.1) in terms of a “center-of-mass” momentum $p$ as follows:

$$E_{\pm} = E_{p} \cosh \zeta \pm p \sinh \zeta$$

$$\pm k_{\pm} = E_{p} \sinh \zeta \pm p \cosh \zeta \quad (A.2)$$

where $E_{p} = \sqrt{p^{2} + m^{2}}$ and $\sinh \zeta = q^{1}/2E_{p}$. Denoting the traces in (A.1) by $Tr_{1}^{\mu\nu}$ and $Tr_{2}^{\mu\nu}$, respectively, we find

$$Tr_{1}^{00} = Tr_{2}^{00} = 4m^{2} \sinh^{2} \zeta$$

$$Tr_{1}^{11} = Tr_{2}^{11} = 4m^{2} \cosh^{2} \zeta \quad (A.3)$$

$$Tr_{1}^{01} = -Tr_{2}^{01} = 4m^{2} \sinh \zeta \cosh \zeta$$

Finally, we choose as the new integration variable $x = (E_{p} + p)/2E_{p}$. Inserting (A.2), (A.3) and the jacobian

$$\frac{1}{E_{+}E_{-}} \frac{dk}{dx} = \frac{2E_{p}}{m^{2} \cosh \zeta} \quad (A.4)$$

into (A.1) we get (2.10).
To get (2.12), we start from the expression for the screened loop $L^{\mu\nu}_\Lambda$ of (2.7) given by (2.8), with the Feynman propagators $S_F$ replaced with the mixed propagators $S_\Lambda$ of (1.6). The integration range can be divided into four domains, according to $|k|\pm\Lambda$ being $>\Lambda$ or $<\Lambda$. When $|k_+|<\Lambda$ and $|k_-|<\Lambda$ both propagators are of the $S_E$ type (1.5), and the integral vanishes due to $\theta(t)\theta(-t) = 0$. For $|k_+|>\Lambda$ and $|k_-|>\Lambda$ both propagators are $S_F$, and the integrand is as in (A.1). Since the traces given by (A.3) do not grow with $|k|$, the integral is convergent and vanishes in the limit $\Lambda \to \infty$. Hence

$$L^{\mu\nu}_\Lambda(q) = e^2 \int dt \int \frac{dk}{2\pi} \exp(iq^0 t) \left\{ \theta(|k_+|>\Lambda)\theta(|k_-|<\Lambda)Tr \left[ \gamma^\mu S_F(t,k_+)\gamma^\nu S_E(-t,k_-) \right] ight. $$

$$+ \theta(|k_+|<\Lambda)\theta(|k_-|>\Lambda)Tr \left[ \gamma^\mu S_E(t,k_+)\gamma^\nu S_F(-t,k_-) \right] $$

$$+ C^{\mu\nu} + O(1/\Lambda^2) \tag{A.5}$$

where $C^{\mu\nu}$ is given by (2.11). Taking $q^1>0$ and substituting (1.5) and (2.9) into (A.5), the first term on the r.h.s. becomes,

$$L^{\mu\nu}_{\Lambda a}(q) = i e^2 \int_{-\infty}^{\infty} dt \int \frac{dk}{2\pi} \exp(iq^0 t) \left\{ \frac{Tr \left[ \gamma^\mu (\not{k}_+ - m)\gamma^\nu (\not{k}_- + m) \right]}{q^0 + E_+ + E_- - i\epsilon} \right.$$

$$+ \frac{Tr \left[ \gamma^\mu (\not{k}_+ - m)\gamma^\nu (\not{k}_- - m) \right]}{q^0 + E_+ - E_- - i\epsilon} \right\} \tag{A.6}$$

Since the integral is over a finite range in $k$, it can be non-vanishing in the limit $\Lambda \to \infty$ only if the integrand is finite. The first term in the integrand of (A.6) behaves like $1/k^3$ for large $k$, and hence contributes $O(1/\Lambda^2)$ to the integral. Since

$$E_+ - E_- = \sqrt{(k + \frac{1}{2}q^1)^2 + m^2} - \sqrt{(k - \frac{1}{2}q^1)^2 + m^2} = q^1 + O(1/k^2) \tag{A.7}$$

18
we get

\[ L_{\Lambda a}^{\mu
u}(q) = ie^2 \frac{\Lambda^+ q^1}{8\pi} \int d^4 k \frac{Tr \left[ \gamma^\mu (\gamma^0 + \gamma^1) \gamma^\nu (\gamma^0 + \gamma^1) \right]}{q^0 + q^1 - i\epsilon} + O(1/\Lambda^2) \]

\[ = ie^2 \frac{q^1}{2\pi q^0 + q^1 - i\epsilon} D^{\mu\nu} + O(1/\Lambda^2) \] (A.8)

where \( D^{00} = D^{11} = -D^{10} = -D^{01} = 1 \). The second term in (A.5) similarly gives (for all \( \mu, \nu \))

\[ L_{\Lambda b}^{\mu
u}(q) = -ie^2 \frac{q^1}{2\pi q^0 + q^1 + i\epsilon} + O(1/\Lambda^2) \] (A.9)

Summing all contributions in (A.5) gives the Lorentz and gauge invariant result

\[ L_{\Lambda}^{\mu
u}(q) = L_{\Lambda a}^{\mu
u}(q) + L_{\Lambda b}^{\mu
u}(q) + C^{\mu\nu} + O(1/\Lambda^2) \]

\[ = -ie^2 \frac{q^1}{2\pi} \left( -g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) + O(1/\Lambda^2) \] (A.10)

which is (2.12).

REFERENCES

1. J. Schwinger, Phys. Rev. 128 (1962) 2425; J. H. Lowenstein and J. A. Swieca, Ann. Phys. (N.Y.) 68 (1971) 172; A. Casher, J. Kogut and L. Susskind, Phys. Rev. D10 (1974) 732; S. Coleman, R. Jackiw and L. Susskind, Ann. Phys. (N.Y.) 93 (1975) 267; S. Coleman, Ann. Phys. (N.Y.) 101 (1976) 239; R. Jackiw, “Topological Investigations of Quantized Gauge Theories”, in S. Treiman, et. al., “Current Algebras and Anomalies”, World Scientific, Singapore 1985.

2. R. Floreanini and R. Jackiw, Phys. Rev. D37 (1988) 2206.
3. P. Hoyer, Int. J. Mod. Phys. **A4** (1989) 963 and **A4** (1989) 4535.

4. G. McCartor, Z. Phys. **C36** (1987) 329.

5. T. Eller, H.-C. Pauli and S. J. Brodsky, Phys. Rev. **D35** (1987) 1493, and references therein.

6. V. N. Gribov, University of Lund preprint LU TP 91-7 (March 1991).

7. Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122** (1961) 345.

8. S. J. Brodsky, in Brandeis Lectures 1969, Vol. 1, p. 95 (edited by M. Chrétien and E. Lipworth, Gordon and Breach, N.Y. (1971)); W. Dittrich, Phys. Rev. **D1** (1970) 3345; A. R. Neghabian and W. Glöckle, Can. J. Phys. **61** (1983) 85.