L-SPACE FILLINGS AND GENERALIZED SOLID TORI

THOMAS GILLESPIE

ABSTRACT. Much work has been done recently towards trying to understand the topological significance of being an L-space. Building on work of Rasmussen and Rasmussen, we give a topological characterisation of Floer simple manifolds such that all non-longitudinal fillings are L-spaces. We use this to partially classify L-space twisted torus knots in $S^1 \times S^2$ and resolve a question asked by Rasmussen and Rasmussen.

1. INTRODUCTION

If $M$ is a rational homology $S^3$, we can define the Heegaard Floer homology $\widehat{HF}(M)$, which satisfies $\text{rk}(\widehat{HF}(M)) \geq |H_1(M)|$. It is natural to try and understand the manifolds with the simplest Heegaard Floer homology, those with $\text{rk}(\widehat{HF}(M)) = |H_1(M)|$, which we call L-spaces. A conjecture of Boyer, Gordon and Watson [1] suggests a concise description of such manifolds in terms of $\pi_1(M)$.

Conjecture 1.1 ([1]). An irreducible rational homology 3-sphere is an L-space if and only if its fundamental group is not left-orderable.

A lot of recent work in the subject has gone into studying this conjecture. We follow a path taken by Rasmussen and Rasmussen in [6] who approach the problem from the perspective of Dehn filling. More precisely, if $Y$ is a rational homology $S^1 \times D^2$, we will be interested in classifying the L-space fillings of $Y$, i.e. the set $L(Y) = \{\alpha \in \text{Sl}(Y) \mid Y(\alpha) \text{ is an L-space}\}$ where $\text{Sl}(Y)$ is the set of filling slopes on $\partial Y$. In particular, we will study the case $L(Y) = \text{Sl}(Y) - \{l\}$, where $l$ is the homological longitude.

In [6], Rasmussen and Rasmussen define a subset $D^+_{\tau_0}(Y) \subseteq H_1(Y)$ that (almost) completely determines the set $L(Y)$. We strengthen a result of theirs about the behaviour of manifolds with $L(Y) = \text{Sl}(Y) - \{l\}$.

Theorem 1.2. The following are equivalent

1. $L(Y) = \text{Sl}(Y) - \{l\}$.
2. $Y$ has genus 0 and has an L-space filling.

To prove this theorem, we show that for any $Y$, $D^+_{\tau_0}(Y) = \emptyset$ if and only if $Y$ is a generalized solid torus. This is closely related to the notion of a solid torus-like manifold, introduced by Hanselman and Watson in [4], which is a condition on the bordered Floer homology of the manifold. In particular, all solid torus-like manifolds are generalized solid tori, and if $H_1(Y)$ is torsion-free the notions are equivalent.

This also allows us to simplify a result of Hanselman, Rasmussen, Rasmussen and Watson ([3, Theorem 7]) concerning splicing of manifolds. Given two manifolds with torus boundary, $Y_1$ and $Y_2$, and a map $\varphi : \partial Y_1 \to \partial Y_2$, we can define the spliced manifold $Y_\varphi$ by gluing $Y_1$ and $Y_2$ along their boundaries, by the map $\varphi$.
Corollary 1.3. Suppose \( Y_1 \) and \( Y_2 \) as above are Floer simple and boundary incompressible. Let \( L_i^2 \) be the interior of \( L(Y_i) \subset S(l(Y_i)) \). Then \( Y_\varphi \) is an L-space if and only if \( \varphi_*(L_1^2) \cup L_2^2 = S(l(Y_2)) \).

As another application of this technology, we give a large class of twisted torus knots in \( S^1 \times S^2 \) that have generalised solid torus complements. In [9], Vafaee gives a class of twisted torus knots in \( S^3 \) that are L-space knots, which is extended by Motegi in [5]. We will show how these results can be applied to the study of twisted torus knots in \( S^1 \times S^2 \) with Floer simple complements, and extend the classification.

Theorem 1.4. The twisted torus knot complement \( S^1 \times S^2 \setminus \{ (p,q) \in S^1 \times S^2 \mid (p,q) \text{ is a generalized Floer simple manifold} \} \) is a generalized solid torus if \( (p,q) = 1 \) and \( s \equiv \pm q \ (p) \).

We can use this to produce an set of examples to resolve a question asked in [6].

Corollary 1.5. There exists an infinite set of distinct Floer simple manifolds with the same Turaev torsion.

While studying this problem we also come across an interesting class of two component link complements.

Theorem 1.6. There exists an infinite family of manifolds \( Z_{p,q} \) with \( \partial Z_{p,q} = T^2 \cup T^2 \) such that any filling \( Z_{p,q}(\alpha, \beta) \) with \( b_1(Z_{p,q}(\alpha, \beta)) = 0 \) is an L-space.

Acknowledgements: The author would like to thank his supervisor Jacob Rasmussen for introducing this problem, and for support and helpful conversations throughout.

2. Generalized solid tori

First, we will fix some conventions and notation. Throughout, \( Y \) will be a rational homology \( S^1 \times D^2 \). Let \( \lambda \) be the homological longitude and \( \mu \) a meridian of \( \partial Y \). Write \( H_1(Y) = \mathbb{Z} \oplus T \), \( i : H_1(\partial Y) \rightarrow H_1(Y) \) for the induced inclusion map. Then

\[
\text{Im}(i) = g_Y \mathbb{Z} \oplus \mathbb{Z} / g_Y \mathbb{Z} = g_Y \mathbb{Z} \oplus T' \subset H_1(Y)
\]

where \( i(\mu) \) generates \( g_Y \mathbb{Z} \) and \( \sigma := i(\lambda) \) generates \( T' \).

Fix a projection of the free part \( \phi : H_1(Y) \rightarrow \mathbb{Z} \) and let \( t \) be a generator of \( H_1(Y) / T \). Finally, let \( \iota : H_1(Y) \rightarrow H_1(Y(\lambda)) = \mathbb{Z} \oplus T' \cong \mathbb{Z} \oplus T / T' \) and \( k_Y = |T'| \).

Let \( \tau(Y) \) be the Turaev torsion, which is an element of \( (1-t)^{-1}Z[H_1(Y)] \subset Q(Z[H_1(Y)]) \), where \( t \) generates the free part of \( H_1(Y) \). By writing \( (1-t)^{-1} = \sum_{i=0}^{\infty} t^i \), we can consider \( \tau(Y) \) as an element of the Novikov ring

\[
\Lambda_\varphi[H_1(Y)] = \left\{ \sum_{h \in H_1(Y)} a_h[h] \mid \#\{h \mid a_h \neq 0, \phi(h) < k\} < \infty \text{ for all } k \right\},
\]

i.e. as a formal sum of elements in \( H_1(Y) \). We will normalise \( \tau(Y) \) so that \( a_h = 0 \) for \( \varphi(H) < 0 \) and \( a_0 \neq 0 \). With this representation, we can define \( S[\tau(Y)] = \{ h \in H_1(Y) \mid a_h \neq 0 \} \), the support of \( \tau(Y) \). We say that \( Y \) is Floer simple if \( \mathcal{L}(Y) \) is an interval inside \( S(l(Y)) \).

Finally, we define an analogue of the Alexander polynomial. Let \( \Phi : \Lambda_\varphi[H_1(Y)] \rightarrow \mathbb{Z}[t,t^{-1}] \) be the map induce from \( \varphi \), then set \( \Delta(Y) = (1-t)\Phi(\tau(Y)) \).

Definition 2.1 ([5] Definition 1.5). If \( Y \) is a Floer simple manifold, we define

\[
\mathcal{D}_{\geq 0}(Y) = \{ x - y \mid x \notin S[\tau(Y)], y \in S[\tau(Y)], \phi(x) \geq \phi(y) \} \cap \text{Im} \iota \subset H_1(Y)
\]

and write \( \mathcal{D}_+(Y) \) for the subset of \( \mathcal{D}_{\geq 0}(Y) \) consisting of those elements with \( \phi(h) > 0 \).
A generalized solid torus is a Floer simple manifold $Y$ with $\deg \Delta(Y) < g_Y$. This definition is motivated by the fact that if $Y$ is Floer simple and $\deg \Delta(Y) < g_Y$, then $\mathcal{D}_{\geq 0}(Y) = \emptyset$ (Lemma 2.5), which allows us to more precisely control the behaviour of the polynomials $q_{i,s}$ defined below.

**Theorem 2.3.** Suppose that $Y$ is a Floer simple rational homology $S^1 \times D^2$ and $\mathcal{D}_{>0} = \emptyset$. Then $Y$ is a generalized solid torus.

**Proof.** Let $\Sigma_Y = \sum_{s \in T} s$ and $\Sigma_{Y(\lambda)} = \sum_{s \in T^*} s$. We will need some notation and two lemmas from [8].

**Lemma 2.4.** [8 II.4.5] When $b_1(Y) = 1$ we can define the polynomial part of the torsion

$$[\tau](Y) = \begin{cases} \tau(Y) + (t-1)^{-1}Q_Y(t)\Sigma_Y & \text{if } \partial Y = S^1 \times S^2 \\ \tau(Y) + (t-1)^{-1}(t^{-1} - 1)^{-1}Q_Y(t)\Sigma_Y & \text{if } \partial Y = \emptyset \end{cases}$$

for some polynomial $Q_Y(t)$. Then $[\tau](Y) \in (\frac{1}{2}\mathbb{Z})[H_1(Y)]$, i.e. $[\tau](Y)$ is a polynomial.
Lemma 2.5. [8 VII.1.5] With $Y$ as above,

$$
\iota([\tau(Y)]) = \pm(t^{g_Y} - 1)[\tau](Y(\lambda)) + \Sigma_{Y(\lambda)} Y(t)
$$

for some polynomial $P_Y(t)$.

Write

$$
\tau(Y) = \sum_{s \in T^*} s \sum_{i=0}^{\infty} q_{i,s}(\sigma)t^i
$$

where, since $Y$ is Floer simple, all of the coefficients of $q_{i,s}(\sigma)$ are 1 or 0. Then we can use Turaev’s lemma to show that

$$
\iota([\tau(Y)]) = \sum_{s \in T^*} s \sum_{i=0}^{\infty} q_{i,s}(1)t^i
$$

$$
= \iota([\tau](Y)) + g_Y \Sigma_{Y(\lambda)} \sum_{i=C}^{\infty} t^i
$$

$$
= \pm(t^{g_Y} - 1)[\tau](Y(\lambda)) + \Sigma_{Y(\lambda)} Y(t) + g_Y \sum_{i=C}^{\infty} t^i
$$

So, by taking coefficients of the $s \in T^*$

$$
(t^{g_Y} - 1) | Q_s(t) := \sum_{i=0}^{\infty} q_{i,s}(1)t^i - P_Y(t) - g_Y \sum_{i=C}^{\infty} t^i
$$

Since $D_{>0} = \emptyset$, if the coefficient (in $\tau$) of some $s\sigma^a t^k$ is 1, then the coefficient of $s\sigma' t^k + n g_Y$ must also be 1 for all $\sigma'$ and $k > 0$. So there is at most one value of $n$ such that the polynomial $q_{k+n g_Y,s}(\sigma)$ is not either 0 or $\Sigma_{Y(\lambda)}$. Therefore, the subsequence $q_{k+n g_Y,s}(1)$ must look like $0,0,...,0,q_{k+n g_Y,s}(1),g_Y,g_Y,...$.

We now cut up $Q_s(t)$ into groups of powers of $t$ that correspond to equivalence classes in $\mathbb{Z}/g_Y \mathbb{Z}$, i.e.

$$
Q_{s,k}(t) := \sum_{i=0}^{\infty} q_{k+n g_Y,s}(1)t^{k+n g_Y} - P_{Y,k}(t) - g_Y \sum_{i=C_k}^{\infty} t^{k+n g_Y}
$$

where $C_k = \lfloor (C-k)/g_Y \rfloor$ and $P_{Y,k}(t)$ contains the terms within $P_Y(t)$ of the form $c t^{k+n g_Y}$. Clearly this process preserves divisibility by $t^{g_Y} - 1$, and so we also have divisibility of the following differences (wlog $n_{k,s'} \geq n_{k,s}$)

$$
(t^{g_Y} - 1) | (Q_{s,k}(t) - Q_{s',k}(t))
$$

$$
= q_{k+n g_Y,s}(1)t^{k+n g_Y} + g_Y \sum_{i=n_{k,s}+1}^{n_{k,s'}+1} t^{k+n g_Y} + (g_Y - q_{k+n g_Y,s'}(1)) t^{k+n g_Y}
$$

from which we can conclude that $n_{k,s'} = n_{k,s}$ and $q_{k+n g_Y,s}(1) = q_{k+n g_Y,s'}(1)$ for all $s, s'$.

So if we write $a_i = q_{i,s}(1)$, we get a simple form for the Milnor torsion

$$
\check{\tau}(Y) = pr' \left( \sum_{s \in T^*} s \sum_{i=0}^{\infty} q_{i,s}(\sigma)t^i \right) = k_Y \sum_{i=0}^{\infty} a_i t^i
$$
Lemma 2.6. There is a constant c so that \( \sum_{i=1}^{k} a_{i} \equiv k_{Y}(k + c \cdot g_{Y}) \).

Proof. By [6] Lemma 7.3, \( p(\Delta(t)) = p((t-1)\bar{\tau}(Y)) \sim k_{Y}(1-t^{g_{Y}})/(1-t) \) where \( p : \mathbb{Z}[t] \rightarrow \mathbb{Z}[t]/(t^{g_{Y}} - 1) \), so \( \Delta(t) \) can be obtained from \( k_{Y}(1-t^{g_{Y}}) \) by moves of the form \( f(t) \rightarrow f(t) + t^{i} - t^{g_{Y}-i} \) and \( f(t) \rightarrow t^{i}f(t) \). It is then sufficient to note that if \( f(t)/(1-t) = \sum_{i=0}^{\infty} a_{i}t^{i} \) satisfies the conditions of the lemma, so do \( wf(t) + t^{i} - t^{g_{Y}-i} \) and \( t^{i}f(t) \).

This implies that the subsequence \( (k_{Y}a_{k+n^{g_{Y}}}) \) has the form

\[
0, 0, ..., 0, k_{Y}k, k_{Y}g_{Y}, k_{Y}g_{Y}, ...
\]

so \( \bar{\tau}(Y) \) is obtained from

\[
\bar{\tau}_{0} = k_{Y}(t + 2t^{2} + ... + (g_{Y} - 1)t^{g_{Y}-1} + g_{Y}t^{g_{Y}} + g_{Y}t^{g_{Y}+1} + ...) = k_{Y}t^{g_{Y}-1} \frac{1}{(t-1)^{2}}
\]

by elementary shifts of the form \( k_{Y}(at^{i} + (g_{Y} - a)t^{i+g_{Y}}) \).

Finally, consider the map \( F(Q(t)) = p((1-t)Q(t)) \), \( p : \mathbb{Z}[t] \rightarrow \mathbb{Z}[t]/(t^{g_{Y}} - 1) \). One can check that if \( f(t) - g(t) = k_{Y}(a^{t} + (g_{Y} - a)t^{i+g_{Y}}) \), \( F(f) - F(g) = g_{Y}k_{Y}(t^{i} - t^{i+1}) \). But since \( \bar{\tau}(Y) \) is obtained from \( \bar{\tau}_{0} \) by these shifts, and since \( F(\bar{\tau}_{0}) = k_{Y}\frac{t^{g_{Y}+1}}{(t^{g_{Y}} - 1)} = F(\bar{\tau}(Y)) \), all residue classes must be shifted by the same amount, i.e. \( \bar{\tau}(Y) \sim \bar{\tau}_{0} \). Hence \( Y \) is a generalised solid torus.

We have the following corollary, which follows immediately as an extension of [6] Proposition 1.9 (and which contains Theorem 1.2).

Corollary 2.7. The following are equivalent

1. \( \mathcal{L}(Y) = SI(Y) - \{1\} \).
2. \( Y \) is Floer simple and \( \mathcal{D}_{\geq 0}(Y) = \emptyset \).
3. \( Y \) is Floer simple and has genus 0.
4. \( Y \) has genus 0 and has an L-space filling.

This immediately gives us a result about solid torus-like manifolds, which we define now.

Definition 2.8 ([3] Definition 3.23). A loop-type manifold \( M \) is solid torus like if it is a rational homology solid torus and every loop in the representation of \( CFD(M, \alpha, \beta) \) is solid torus-like. Equivalently, \( \mathcal{D}_{\geq 0}(Y) = \emptyset \).

Corollary 2.9. If \( Y \) is solid torus-like, then it is boundary compressible.

Proof. A solid torus-like manifold has \( \mathcal{D}_{\geq 0}(Y) = \emptyset \), and so \( \mathcal{D}_{\geq 0}(Y) = \emptyset \). Hence there is a generator \( \Sigma \) of \( H_{2}(Y, \partial Y) \) with genus 0.

Also, since \( \mathcal{D}_{\geq 0}(Y) = \emptyset \), we must have that \( q_{i,s}(\sigma) = \sum_{i=0}^{g_{Y}} s^{i} \) for all \( i \) and \( s \), and so \( \Delta(t) = k_{Y}g_{Y}\sum_{i=0}^{\infty} t^{i} \). Therefore \( g_{Y} = 1 \) by [6] Lemma 7.3. So \( \Sigma \) is a compressing disk for \( Y \).

We will now turn our attention to an application to splicing of manifolds. Suppose that \( Y_{1} \) and \( Y_{2} \) are rational homology solid tori, \( \varphi : \partial Y_{1} \rightarrow \partial Y_{2} \) is an orientation reversing diffeomorphism. We say that \( Y_{w} = Y_{1} \cup_{\varphi} Y_{2} \) is obtained by splicing \( Y_{1} \) and \( Y_{2} \) together along \( \varphi \).
**Corollary 2.10.** Suppose $Y_1$ and $Y_2$ as above are Floer simple and boundary incompressible. Let $L_i^\circ$ be the interior of $L(Y_i) \subset SL(Y_i)$. Then $Y_\varphi$ is an L-space if and only if $\varphi_*(L_1^\circ) \cup L_2^\circ = SL(Y_2)$.

*Proof.* By 2.9 the $Y_i$ are not solid torus-like, and we can apply [3, Theorem 7].

3. Twisted Torus Knots

In an effort to find examples of generalised solid tori, we now turn our attention to twisted torus knots. We write $T(p,q;s,r)$ for the knot in figure 2, considered as a knot in $S^1 \times S^2$.

![Figure 2. The twisted torus knot $T(p,q;s,r)$]

In [9], Vafaee studies twisted torus knots $K(p,q;s,r) \subset S^3$, and classifies as subset that have L-space fillings proving

**Theorem 3.1 ([9 Theorem 1]).** For $p \geq 2, k \geq 1, r > 0$ and $0 < s < p$, the twisted torus knot, $K(p,kp \pm 1; s, r)$, is an L-space knot if and only if either $s = p - 1$ or $s \in \{2, p - 2\}$ and $r = 1$.

This theorem can be used to prove the existence class of generalised solid tori as complements of knots in $S^1 \times S^2$.

**Corollary 3.2.** For $p \geq 2, k \geq 1, r > 0$ and $0 < s < p$, the complement of the twisted torus knot, $T(p,kp \pm 1; s, r) \subset S^1 \times S^2$, is a generalised solid torus if either $s \in \{1, p - 1\}$ or $s \in \{2, p - 2\}$ and $r = 1$.

We could similarly extend the results of Motegi to twisted torus knots in $S^1 \times S^2$. Instead, we will give a more direct geometric proof.

**Theorem 3.3.** The twisted torus knot complement $S^1 \times S^2 \setminus T(p,q;s,r)$ is a generalized solid torus if $(p,q) = 1$ and $s \equiv \pm q \ (p)$.

To prove this theorem, we require a lemma concerning the existence of a surface in the knot complement.
Figure 3. The link $L$

Figure 4. A link in $I \times D^2$. Under the $q/p$ gluing, $S_0$ glues to $S_1$
Lemma 3.4. Consider the black tangle \( L \subset I \times D^2 \) in figure 4. There exists an embedded surface \( \Sigma \subset I \times D^2 - \nu(L) \) satisfying the following properties:

1. The boundary of \( \Sigma \) consists of \( q \) parallel copies of \( \{ \frac{1}{2} \} \times \partial D^2 \), \( p \) copies of the longitude of \( L_1 \) and \( B \subset \{0, 1\} \times D^2 \).
2. If we glue \( \{0\} \times D^2 \) to \( \{1\} \times D^2 \) by an automorphism of \( D^2 \) that shifts each strand of \( L \) left \( q \) times (i.e. the gluing that will close \( L_2 \) to form a \( p/q \) torus knot), the boundary components in \( B \) join up to each other to form a surface \( \Sigma' \) with boundary only on \( L_1 \) and \( I \times \partial D^2 \). Moreover, \( \Sigma' \) has genus 0.
3. \( \Sigma \cap S \times [0, 1/2] = \emptyset \)

Proof. We will build \( \Sigma \) inductively. Let \( w = kp + q \), and consider the black tangle \( L \subset I \times D^2 \) in figure 5.

![Figure 5](image-url)

**Figure 5.** A link and an embedded surface in \( I \times D^2 \). The green cylinder contains a reflected copy of figure 4. The pink pipes run through the holes in the blue surface.

Then by induction we can find a surface in a neighbourhood of the first \( p \) strands and \( L_1 \), whose boundary components are \( p \) copies of the longitude of \( L_1 \), \( q \) copies of the longitude of \( L_2 \) and a boundary component \( B \subset \{0, 1\} \times D^2 \). Call this surface \( \Sigma_0 \) and write \( B = B_0 \cup B_1 \) where \( B_i \subset i \times D^2 \). Also note that \( \Sigma_0 \) is ‘surrounded’ by \( L_2 \).

Next, take \( k - 1 \) copies of \( B_1 \times I \) and insert them along the \( k - 1 \) remaining groups of \( p \) strands, i.e. the first one along the strands \( p + 1, \ldots, 2p \), the second along \( 2p + 1, \ldots, 3p \) etc. Along the final \( q \) strands of the braid, we paste a copy of \( B' \times I \), where \( B' \) consists of the components of \( B_0 \) contained inside the circle \( S \). Take the union of all of the pieces that we
have defined so far, and call it \( \Sigma_1 \). Notice that at this stage, the \( p/w \) twisted gluing will correctly match up all boundary components of \( \Sigma_1 \) inside \( \{0,1\} \times D^2 \) (since \( w \equiv q \ (p) \)).

Next we take \( p \) parallel copies of the blue surface in figure [4]. As shown in the picture, each blue surface has one boundary component on \( I \times \partial D^2 \), \( k \) boundary components \( C_i \) containing \( p \) strands, and one boundary component \( E \) containing \( q \) strands.

For \( C_1 \), we run a pipe along the outside of \( \Sigma_i \) to join \( C_1 \) to \( L_2 \), and \( k-1 \) parallel copies of pipes from \( \{1\} \times D^2 \) to \( L_2 \) each running inside the previous pipe, but still outside of \( \Sigma_1 \). An example is shown in figure [3] shown in pink. Similarly for each other \( C_i \), run a pipe from \( C_i \) to \( \{0\} \times D^2 \), and \( k+1-i \) pipes inside that one from \( \{1\} \times D^2 \) to \( \{0\} \times D^2 \), all of them running outside of \( \Sigma_1 \). Repeat this process for each parallel copy of the blue surface. Call this new surface \( \Sigma_2 \), and notice that the boundary components still all match up under a \( p/w \) twisted gluing. Also notice that the boundary of \( \Sigma_2 \) contains \( p \) parallel copies of \( \{1/2\} \times \partial D^2 \) and \( w = q + kp \) copies of the longitude of \( L_2 \).

Finally we run a pipe down from each copy of the final blue boundary component \( E \), and \( p \) parallel pipes joining the red circle \( S \) to \( L_1 \). We know by condition 3 that these pipes don’t intersect \( \Sigma_2 \). These \( p \) boundary components on \( L_1 \) can be paired up with the \( p \) existing boundary components on \( L_1 \), and pushed off. Finally we do an \( \infty \) filling on \( L_1 \) to get our surface \( \Sigma \).

To check the genus, write \( \Sigma'_1 \) for the gluing of \( \Sigma_1 \), and \( \Sigma' \) for the gluing of our new surface \( \Sigma \). Note that \( \Sigma'_1 \) is homeomorphic to the surface that we would get by gluing \( \Sigma_0 \) inside figure [4] and so has genus 0. To obtain \( \Sigma'_2 \), we just added \( p \) copies of the blue surface (and some pipes) all of which are disjoint from \( \Sigma'_1 \), and so \( \Sigma'_2 \equiv \Sigma'_1 \coprod \mathcal{P}D_k^2 \), where \( D_k^2 \) is a \( k \)-punctured disc. So \( \Sigma'_2 \) consists of \( p \) disjoint pieces all of which have genus 0. The final step simply glues one boundary component of each \( D_k^2 \) to a boundary component of \( \Sigma'_1 \), so clearly \( \Sigma' \) has genus 0.

Note finally that condition 3 is clearly satisfied by the construction. \( \square \)

**Lemma 3.5.** Let \( r_i \in \mathbb{Q} \cup \{ \infty \} \), \( r_1 \neq 0 \) and \( r_3 \neq \infty \). Then the filling \( Y_{r_1,0,r_3} \) of the following link \( L \subset S^3 \) (in figure [2]) is an L-space.

**Proof.** Note that \( Y_{-,0,r_3} \) has genus 0, since we can take the surface \( \Sigma \) from lemma 3.4 and cap off the boundary components on \( L_2 \). Also, \( Y_{\infty,0,r_3} \) is an L-space since it is the \( r_3 \) filling of \( T(p,q) \subset S^3 \times S^2 \), which is a generalized solid torus by 3.2. So by theorem 1.2, \( \mathcal{L}(Y_{-,0,r_3}) = \text{SL}(Y_{-,0,r_3}) - \{1\} \), hence \( Y_{r_1,0,r_3} \) is an L-space. \( \square \)

**Proof of theorem 3.3.** \( S^1 \times S^2 \setminus T(p,q;g,r) = S^1 \times S^2 \setminus T(p,q;g',r) = Y_{1/r,0,-s} \), so by the above lemma all of the non-longitudinal fillings are L-spaces. Hence \( S^1 \times S^2 \setminus T(p,q;s,r) \) is a generalized solid torus. \( \square \)

Theorem 1.6 then follows easily by considering the family of manifolds \( Y_{-,0,-} \) above, for all \( p \) and \( q \).

This class of manifolds allows us to resolve a question of Rasmussen and Rasmussen. In [4], Rasmussen and Rasmussen asked if there are infinitely many distinct Floer simple manifolds with the same Turaev torsion. We can build such a family as follows. Consider the manifold \( Y_{-,0,-} \) as above, with \( p = 5 \) and \( q = s = 2 \). Using SnapPy ([2]) we can check that this manifold is hyperbolic, and apply Thurston’s hyperbolic Dehn filling theorem.

**Theorem 3.6** (Thurston’s hyperbolic Dehn filling theorem ([7])). Suppose \( M \) is hyperbolic and \( M(p/q) \) is the filling of one of the boundary components of \( M \). Then

- \( M(p/q) \) is hyperbolic for all but finitely many \( p/q \)
Figure 6. A link in $S^3$ (the top and bottom should be identified)

- $vol(M(p/q)) < vol(M)$ when $M(p/q)$ is hyperbolic
- $vol(M(p/q)) \to vol(M)$ as $p^2 + q^2 \to \infty$

Therefore there is some subsequence $Y_{1/r,0,*}$ that are distinct (since they have different hyperbolic volumes) but have the same $\Delta$. Since, for a Floer simple manifold, there are only finitely many possible torsions corresponding to a choice of $\Delta$, there must therefore be an infinite set of distinct Floer simple manifolds with the same torsion.

References

[1] Steven Boyer, Cameron McA. Gordon, and Liam Watson. On l-spaces and left-orderable fundamental groups. *Mathematische Annalen*, 356(4):1213–1245, 2012.

[2] Marc Culler, Nathan M. Dunfield, and Jeffrey R. Weeks. SnapPy, a computer program for studying the topology of 3-manifolds. Available at http://snappy.computop.org (29/02/2016).

[3] J. Hanselman, J. Rasmussen, S. D. Rasmussen, and L. Watson. Taut foliations on graph manifolds. *ArXiv e-prints*, August 2015.

[4] J. Hanselman and L. Watson. A calculus for bordered Floer homology. *ArXiv e-prints*, August 2015.

[5] K. Motegi. L-space surgery and twisting operation. *ArXiv e-prints*, May 2014.

[6] J. Rasmussen and S. D. Rasmussen. Floer Simple Manifolds and L-Space Intervals. *ArXiv e-prints*, August 2015.

[7] W. P. Thurston. The geometry and topology of 3manifolds. *Lecture notes from Princeton University*, 1978/80.

[8] V. Turaev. *Torsions of 3-dimensional manifolds*, volume 208 of Progress in Mathematics. Birkhauser Verlag, 2002.

[9] F. Vafaee. On the knot Floer homology of twisted torus knots. *ArXiv e-prints*, November 2013.