PPT-inducing, distillation-prohibiting, and entanglement-binding quantum channels

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Entanglement degradation in open quantum systems is reviewed in the Choi-Jamiołkowski representation of linear maps. In addition to physical processes of entanglement dissociation and entanglement annihilation, we consider quantum dynamics transforming arbitrary input states into those that remain positive under partial transpose (PPT-inducing channels). Such evolutions form a convex subset of distillation-prohibiting channels. A relation between the above channels and entanglement-binding ones is clarified. An example of the distillation-prohibiting map $\Phi \otimes \Phi$ is given, where $\Phi$ is not entanglement binding.

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I. INTRODUCTION

The phenomenon of quantum entanglement usually emerges between interacting subsystems in a composite system. The long-distance and long-living forms of such correlations play a vital role in up-to-date quantum technologies [1]. Unavoidable interaction with the environment changes the structure of entanglement [2]. In special cases, the global environment bath can create quantum correlations between the particles of the composite system [3], however, local noises degrade it and impose limitations on achievable entanglement death time in experiments [4, 5]. Bound entangled states are those that are still entangled but cannot be distilled into maximally entangled qubit pairs [6]. Positive partial transpose (PPT) states [7] are known to be undistillable, so it is reasonable to characterize quantum channels resulting in PPT states regardless of the input.

To describe the stages of gradual entanglement degradation, the notions of entanglement annihilation [8] and entanglement dissociation [9] were introduced recently. PPT-inducing and distillation-prohibiting channels can be considered as their generalizations with respect to corresponding entanglement properties (PPT and undistillability). The previously introduced notion of entanglement-binding channel [11] turns out to be a partial case (one-sided realization) of the distillation-prohibiting channel.

The paper is organized as follows.

In Sec. [I] the description of quantum entanglement is briefly reviewed. In Sec. [II] the basic information about quantum channels and their entanglement degradation properties is given. In Sec. [IV] the Choi-Jamiołkowski representation [12] of linear maps and their concatenations is discussed, the structure of Choi matrix is reviewed for entanglement-breaking, entanglement-binding, entanglement-annihilating, and entanglement-dissociating channels. Main results are presented in Sec. [V] where properties of PPT-inducing and distillation-prohibiting channels are studied. In Sec. [VI] brief conclusions are given.

II. QUANTUM ENTANGLEMENT

A quantum state is described by a density operator $\rho$, which is Hermitian, positive semidefinite, and has unit trace. In a composite system, there are several degrees of freedom, say, $A, B, C, \ldots$, and the corresponding Hilbert space has the tensor product structure $\mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^C \otimes \cdots$. Specification of particular degrees of freedom fixes the partitioning structure $A|B|C|\cdots$ and the associated entanglement structure [37]. In experiments and applications, one deals with the accessible degrees of freedom that are naturally related to the used measurement techniques. From now on, these degrees of freedom are supposed to be fixed.

The state $\rho^{ABC\cdots}$ is called fully separable if there exist a probability distribution $\{p_k\}$ and density operators $\rho_k^A, \rho_k^B, \rho_k^C, \ldots$ such that $\rho^{ABC\cdots}$ belongs to the closure of the states $\sum_k p_k \rho_k^A \otimes \rho_k^B \otimes \rho_k^C \otimes \cdots$; otherwise, $\rho^{ABC\cdots}$ is called entangled [10].

An example of a coarse-grained partition is $\mathcal{P} = AB|C|DEF$. The state $\rho^{ABCD\cdots}$ is separable with respect to $\mathcal{P}$ if $\rho^{ABCD\cdots} = \sum_k p_k \rho_k^{AB} \otimes \rho_k^{CD} \otimes \rho_k^{DE\cdots}$. A fully separable state is separable with respect to $AB|C|DEF$ but the converse does not hold in general. It is instructive to remind about the existence of a three-qubit state $\rho^{ABC}$, which is separable with respect to all bipartitions $(A|BC, B|AC, C|AB)$ but is not fully separable, i.e. entangled with respect to tripartition $A|B|C$ [17, 18].

A state, which cannot be written as a convex sum of any separable bipartite states, is usually referred to as genuinely entangled. Under the action of local noises, the genuine entanglement degrades at first to the biseparable form (convex sum of states separable with respect to bipartitions), then to the triseparable form (convex sum of states separable with respect to tripartitions), and so on. Such a process is called entanglement dissociation [10] in analogy to the dissociation of chemical compounds in solvents. The final stage of entanglement evolution is annihilation, when the system state becomes fully separable.

A bipartite state $\rho^{AB}$ is called positive under partial
transpose (PPT) if $\varrho^{AB\top} \equiv I^A \otimes T^B[\varrho^{AB}]$ is positive semidefinite [7]. Here, Id is the identity transformation and $T^B$ is the transposition in some orthonormal basis in $H^B$. Clearly, $\varrho^{AB\top} \geq 0 \iff \varrho^{(A)\top B} \geq 0$ because $\varrho^{(A)\top B}$ is a valid density operator and $T^2 = I^d$. A separable state $\varrho^{AB}$ is necessarily PPT [7], the converse holds if $d^A d^B \leq 6$ [19]. PPT entangled states are known to be undistillable [6, 20], which motivates us to study PPT-inducing quantum evolutions.

III. QUANTUM CHANNELS

Evolution of an open quantum system during the time interval $(0, T)$ can be considered as an input-output relation $\varrho_{t=0}=\Phi[\varrho_{t=0}]$ between the final and initial density matrices. If the system is decoupled from the environment at time $t=0$, then $\Phi$ is a single-valued linear map. Physical reasoning leads to a conclusion that $\Phi$ is a completely positive trace preserving map [21, 22], which we will refer to as quantum channel (see, e.g., [23]).

Suppose a channel $\Phi$ acting on the system $S = ABC\ldots$ and the initial state $\varrho_{\text{in}}$. If the output state $\varrho_{\text{out}}=\Phi[\varrho_{\text{in}}]$ is separable with respect to a fixed partition $P$ for all $\varrho_{\text{in}}$, then $\Phi$ is called entanglement-dissociating with respect to $P$. If $P = A|B|C|\ldots$, i.e. $\varrho_{\text{out}}$ is fully separable for all input states, then $\Phi$ is called entanglement-annihilating [10].

Notions of entanglement dissociation and annihilation do not imply the use of any auxiliary systems. In contrast, the so-called entanglement breaking channels are those for which $\Phi \otimes \text{Id}^{S+\text{anc}}_{\text{in}}$ is separable with respect to partition $S|\text{anc}$ for any initial density operator $\varrho_{\text{in}}^{S+\text{anc}}$, with the dimension of an ancillary Hilbert space being arbitrary [24, 25]. Equivalently, $\Phi^S$ is entanglement-breaking if $\Phi^S \otimes \text{Id}^{\text{anc}}$ is entanglement-dissociating with respect to the bipartition $S|\text{anc}$ for all dimensions $\dim H_{\text{anc}} = 2, 3, \ldots$.

Similarly, if $\Phi \otimes \text{Id}^{S+\text{anc}}_{\text{in}}$ is undistillable (with respect to the system $S$ and the ancillary system) for any initial density operator $\varrho_{\text{in}}^{S+\text{anc}}$, then $\Phi$ is called entanglement binding [11].

IV. CHOJ-JAMIOLKOWSKI REPRESENTATION

In case of finite dimensions, a linear map $\Phi$ acting on a system $S$ can be defined via the so-called Choi-Jamiolkowski isomorphism [12, 13]:

$$\Omega^{SS'}_\Phi = \Phi^S \otimes I^{S'}[|\psi^S_+\rangle\langle\psi^{SS'}_+|], \quad \Phi[X] = d^S \tr_{S'}[\Omega^{SS'}_\Phi(\Omega^S_{\text{out}} \otimes X^T)],$$

where $d^S = \dim H^S = \dim H^{S'}$, $|\psi^S_+\rangle\langle\psi^{SS'}_+|$ is a maximally entangled state shared by system $S$ and its clone $S'$, $\tr_{S'}$ denotes the partial trace over $S'$, $I$ is the identity operator, and $X^T = \sum_{i,j} (j|i)\langle i'|j'|$ is the transposition in some orthonormal basis. The operator $[7]$ is referred to as Choi operator or Choi state (see Fig. 1).

A. Review of properties

The well known results are as follows:

1. $\Phi$ is positive if and only if $\Omega^{SS'}_\Phi$ is block-positive, i.e. $\langle\varphi^S \otimes \chi^{S'}|\Omega^{SS'}_\Phi|\varphi^S \otimes \chi^{S'}\rangle \geq 0$ for all $\varphi, \chi$ [13].

2. $\Phi$ is completely positive (quantum operation) if and only if $\Omega^{SS'}_\Phi \geq 0$ [14].

3. $\Phi$ is entanglement breaking if and only if $\Omega^{SS'}_\Phi \geq 0$ is separable with respect to partition $S|S'$ [23].

4. $\Phi$ is entanglement binding if and only if $\Omega^{SS'}_\Phi \geq 0$ is undistillable (with respect to parties $S$ and $S'$) [11].

A feature of entanglement breaking maps is that their outcome is separable not only for positive inputs (density operators) $\varrho^{S+\text{anc}}$ but also for block-positive inputs $\xi^{S|\text{anc}}$. Since $\tr[\Omega_{\text{pos}}^{SS'}] |\varrho_{\text{ent}}-\text{tr}| \geq 0$, the cone of entanglement breaking maps is dual to the cone of positive maps (see, e.g., [21, 22]).

As far as multipartite composite systems $S = ABC\ldots$ are concerned, the maximally entangled state can be viewed as separable with respect to the partition $AA'|BB'|CC'\ldots$, namely, $|\psi^{SS'}_+\rangle = (d^A d^B d^C \ldots)^{-1/2} \sum_{i=1}^{d^A} \sum_{j=1}^{d^B} \sum_{k=1}^{d^C} \ldots |ij\cdots\rangle \otimes |i'j'\cdots\rangle = |\psi^{AA'}_+\rangle \otimes |\psi^{BB'}_+\rangle \otimes |\psi^{CC'}_+\rangle \otimes \cdots$

If $\Phi$ is a local map, i.e. has the form $\Phi^A \otimes \Phi^B \otimes \Phi^C \otimes \cdots$, then the composite Choi operator reads $\Omega^{SS'}_{\Phi^B,\Phi^C,\ldots} = \Omega^{AA'}_{\Phi^A} \otimes \Omega^{BB'}_{\Phi^B} \otimes \Omega^{CC'}_{\Phi^C} \otimes \cdots$. Positivity of this operator is equivalent to positivity of individual Choi operators. By property 2, $\Phi^A \otimes \Phi^B \otimes \Phi^C \otimes \cdots$ is completely positive if and only if each of the maps $\Phi^A, \Phi^B, \Phi^C, \ldots$ is completely positive. Similarly, by properties 3 and 4, $\Phi^A \otimes \Phi^B \otimes \Phi^C \otimes \cdots$ is entanglement breaking (binding) if and only if each of the maps $\Phi^A, \Phi^B, \Phi^C, \ldots$ is entanglement breaking (binding). Nonetheless, property 1 cannot be extended in analogous way. In fact, the map $\Phi^A \otimes \Phi^B$ is positive if and only if $\Omega^{AA'}_{\Phi^A} \otimes \Omega^{BB'}_{\Phi^B}$, i.e. the composite Choi is block positive with respect to partition $AB|A'B'$. Suppose a positive map $\Psi$ acting on a composite system $ABC\ldots$ $\Psi$ dissociates entanglement with respect to some partition $P(ABC\ldots)$ if and only if

$$\tr\left[\Omega^{AB\ldots A'B'C'}_{\Psi} \left(\sum_{i=0}^{P(ABC\ldots)} \Omega^P_{\Psi} \otimes \varrho^{A'B'C'}_{\Psi}\right)\right] \geq 0 \quad (3)$$

for all block-positive $\Omega^P_{\Psi}$ and density operators $\varrho^{A'B'C'}_{\Psi}$. This result follows immediately from Eq. (2) [10]. However, in order to describe a valid
quantum evolution, \( \Upsilon \) has to be completely positive, i.e. \( \Omega^{ABC...A'B'C'}_{\text{PPT}} \geq 0 \). Choi matrices that satisfy this condition and requirement are precisely entanglement dissociating channels with respect to the partition \( \mathcal{P} \).

The bipartite setting of Eq. \([7]\) reads
\[
\text{tr}[\Omega_{\phi}^{ABA'B'}(\xi_{\text{BP}} \otimes \varrho^{A'B'})] \geq 0 \quad \text{and specifies entanglement annihilating maps } \Phi^{AB}.
\]
The last formula shows that a cone of entanglement annihilating maps is dual to the cone of maps \( \Theta^{AB} \) of the form
\[
\Theta[X] = \sum_{k} \text{tr}[F_{k}X]\xi_{\text{BP}}^{AB} F_{k} \geq 0.
\]

A map \( \Phi^{\dagger} \) is called dual to the map \( \Phi \) if \( \text{tr}[\Phi^{\dagger}[X][Y]] = \text{tr}[X\Phi[Y]] \) for \( X, Y \) from corresponding domains. An observation for the entanglement annihilating map \( \Phi^{AB} \) is that \( \Phi^{\dagger} \) transforms all block-positive operators \( \xi_{\text{BP}}^{AB} \) into positive ones \( \varrho^{AB} \). As a consequence, the concatenation \( \Phi \circ \Phi^{\dagger} \) has to map block-positive operators to separable ones (this is a necessary condition for \( \Phi^{AB} \) to be entanglement annihilating).

Finally, sufficient conditions for entanglement annihilation are as follows \([20]\):

5. If \( \Omega_{\phi}^{ABA'B'} \) can be represented in the form of a convex sum of operators \( \xi_{\text{BP}}^{A'B'} \otimes \varrho^{B} \) and \( \varrho^{A} \otimes \xi_{\text{BP}}^{A'B'} \), then \( \Phi^{AB} \) annihilates entanglement.

6. If \( \Omega_{\phi}^{ABA'B'} \) is a convex sum of separable states of the form \( \varrho^{A'B'} \) and \( \varrho^{B'A'B'} \), then \( \Phi^{AB} \) is entanglement annihilating.

Some key properties of Choi operators for the discussed maps are depicted schematically in Fig. 1.

**B. Concatenation of maps in terms of Choi operators**

A concatenation \( \Phi \circ \Xi \) of two maps \( \Phi : \mathcal{M}_{d} \rightarrow \mathcal{M}_{d} \) and \( \Xi : \mathcal{M}_{d} \mapsto \mathcal{M}_{d} \) is a map such that \( \Phi \circ \Xi[X] \equiv \Phi[\Xi[X]] \).

It is not hard to see that
\[
\Omega_{\phi \circ \Xi} = \sum_{i,j} |m\otimes i\rangle \langle n\otimes j| \Omega_{\phi}[n\otimes j|i\otimes k|\Omega_{\Xi}[j\otimes l](n\otimes l|].
\]

The rule \([4]\) can be treated as a star-product scheme, where the symbols are elements of Choi matrices. The kernel of the star product reads
\[
K(mk; nl; pq, rs; tu, vw) = \delta_{mp}\delta_{qr}\delta_{tv}\delta_{kw}\delta_{wl}.
\]

**V. PPT-INDUCING AND DISTILLATION-PROHIBITING CHANNELS**

A map \( \Phi^{AB} \) transforming operators acting on \( \mathcal{H}^{A} \otimes \mathcal{H}^{B} \) is called PPT-inducing if \( \Phi^{AB}[\varrho^{AB}] \) is positive and PPT with respect to the partition \( A|B \) for all input states \( \varrho^{AB} \).

If in addition \( \Phi^{AB} \) is completely positive and trace preserving, then \( \Phi^{AB} \) is the PPT-inducing quantum channel.

**Proposition 1.** A positive map \( \Phi^{AB} \) is PPT-inducing if and only if \( \Omega^{(AB)}_{\phi} \) is block-positive with respect to the partition \( A|B , A'B' \).

**Proof.** Partial transposition of formula \([2]\) yields
\[
\tilde{\varrho}_{\text{out}}^{(AB)} = \text{tr}_{A'B'}[\Omega^{AB}_{\phi}[A'B'][I_{AB} \otimes \varrho^{(AB)}_{\text{in}}]].
\]

Positivity of \( \tilde{\varrho}_{\text{out}}^{(AB)} \) means \( \text{tr}[\tilde{\varrho}_{\text{out}}^{(AB)}] \geq 0 \) for all density operators \( \tilde{\varrho}^{AB} \), which is equivalent to
Thus, \( \Omega \) is the Schmidt decomposition of an input pure state \( \left| \psi \right\rangle \). The positivity of map \( \Phi \) is essential. Since positive operators are automatically block-positive, we readily obtain the following result.

**Corollary 1.** Suppose a channel \( \Phi^A \otimes \text{Id}^B \) such that \( \Omega^{(A) \otimes B} \geq 0 \), then \( \Phi^A \otimes \text{Id}^B \) is PPT-inducing if and only if \( \Omega^{A' \otimes B'} \) is PPT. Proposition 2. A channel \( \Phi^A \otimes \text{Id}^B \) with \( \dim H^B = \dim H^A \) is PPT-inducing if and only if \( \Omega^{A' \otimes B'} \) is PPT.

**Proof.** Necessity follows from the fact that the map \( \Phi \) should become PPT when acted upon by \( \Phi^A \otimes \text{Id}^B \). Identification of the proper \( d \)-dimensional subspace of \( H^B \) and \( H^A \) implies \( \Omega^{A' \otimes B'} \) is PPT. Sufficiency follows from the tensor product form of the Choi operator, namely, \( \Omega^{A' \otimes B'} = \sum_{\lambda}(\Omega_{\lambda}^{A} \otimes \text{Id}^B) \Lambda_{\lambda}^B \), where the maps \( \Lambda_{\lambda}^B \) are positive and the operations \( \Omega_{\lambda}^{A} \otimes \text{Id}^B \) are PPT-inducing. This fact can be also proven by the rule [3] and Proposition 4.

To some extent, Proposition 2 reproduces the result of Ref. [11] and serves as a sufficient condition for the map \( \Phi \) to be entanglement binding. Examples of PPT but entangled Choi states \( \Omega \) are given in Ref. [11].

A map \( \Phi^A \otimes \text{Id}^B \) that maps density operators to undistillable ones is called distillation-prohibiting. As PPT states cannot be distilled [3], PPT-inducing maps form a convex subset of distillation-prohibiting ones.

The following example illustrates the relation between local entanglement binding channels and distillation-prohibiting ones.

**Example 1.** Consider a local channel \( \Phi_q \otimes \Phi_q \) acting on two qutrits, where \( \Phi_q \) is depolarizing: \( \Phi_q[X] = qX + (1 - q)\text{tr}[X]I_3 \). Suppose \( |\varphi\rangle = \sum_{i=1}^3 \sqrt{\lambda_i} |\varphi_i\rangle |\chi_i\rangle \) is the Schmidt decomposition of an input pure state \( |\varphi\rangle \), i.e., \( \langle \varphi_i | \varphi_j \rangle = \langle \chi_i | \chi_j \rangle = \delta_{ij} \), \( 0 \leq \lambda_i \leq 1 \), and \( \sum_{i=1}^3 \lambda_i = 1 \). Written in the basis \{\( |\varphi_i\rangle |\chi_i\rangle \), \}, the density operator \( \Phi_q \otimes \Phi_q [|\varphi\rangle \langle \varphi |] \) is a sparse matrix, so one can readily find eigenvalues of its partial transpose. For a fixed \( q \), the minimal possible eigenvalue is achieved when \( \lambda_1 = \lambda_2 = \frac{1}{2} \) and \( \lambda_3 = 0 \) (up to the permutation of indexes). Exploring positivity of the minimal eigenvalue, we find that \( \Phi_q \otimes \Phi_q \) is PPT-inducing if and only if \( q \leq \frac{1 + \sqrt{3}}{4 + \sqrt{3}} \approx 0.4766 \). However, \( \Phi_q \) is known to be entanglement binding if and only if \( q \leq \frac{1}{2} \) [32]. A gap between these two values shows that a local channel \( \Phi_1 \otimes \Phi_2 \) can be PPT-inducing (distillation-prohibiting) even if neither of channels \( \Phi_1 \) or \( \Phi_2 \) is entanglement binding.

Assuming that the state \( |\psi\rangle = \frac{1}{\sqrt{3}}(|\varphi\rangle |\chi\rangle + |\varphi\rangle |\chi\rangle + |\varphi\rangle |\chi\rangle) \) of Schmidt rank 2 minimizes the eigenvalues of the partial transpose of \( \Phi_q \otimes \Phi_q [|\psi\rangle \langle \psi |] \) for a general depolarizing map \( \Phi_q : \mathcal{M}_d \mapsto \mathcal{M}_d \). \( \Phi_q[X] = qX + (1 - q)\text{tr}[X]I_3 \), \( q \geq -\frac{1}{\sqrt{d^2 + 1}} \), we readily obtain the following result.

**Conjecture.** \( \Phi_q \otimes \Phi_q \) is PPT-inducing if \( q \leq \frac{1 + \sqrt{3}}{d^2 + 1 + \sqrt{3}} \).

On the other hand, \( \Phi_q \) is entanglement binding if and only if \( q \leq \frac{1}{d + 1} \) [32]. The bound on parameter \( q \) in the above Conjecture was first found in Ref. [29] in connection with the search of robust entangled states.

**VI. CONCLUSIONS**

We have introduced the concepts of PPT-inducing and distillation-prohibiting channels as those that act on a composite system \( AB \) and result in PPT and undistillable output states, respectively. When a one-sided noisy process \( \Phi^A \otimes \text{Id}^B \) is distillation-prohibiting, then our definition naturally leads to the notion of entanglement binding channel \( \Phi^A \). We have characterized PPT-inducing channels and found necessary and sufficient conditions for \( \Phi^A \) to be PPT-inducing in terms of Choi operators; however, distillation-prohibiting channels still need further analysis.

Also, we have demonstrated that \( \Phi_1 \otimes \Phi_2 \) can be PPT-inducing (distillation-prohibiting) even if neither of channels \( \Phi_1 \) or \( \Phi_2 \) is entanglement binding. All these results show that entanglement binding channels are analogues of entanglement breaking ones, whereas distillation-prohibiting channels are analogues of entanglement annihilating ones: in both cases the notion of entanglement is merely replaced by the notion of distillation capability.

Recent progress in the description of the sets of PPT and undistillable states [33, 34] may stimulate a further characterization of PPT-inducing and distillation-prohibiting channels. Future investigation of continuous-variable systems may follow the same line of reasoning as in Refs. [29, 35] because a necessary and sufficient condition for PPT continuous-variable states is known [36].

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[37] For example, a two-body system can be described by either coordinates of individual particles $r_1$ and $r_2$, or the center-of-mass coordinate $R = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}$ and the relative coordinate $r = r_2 - r_1$. The ground state of the hydrogen atom is entangled with respect to the first partition and separable with respect to the second one.

[38] Mind the difference between the notion of dual cones of maps and the notion of dual maps.