Rotating light, OAM paradox and relativistic complex scalar field

S. C. Tiwari
Institute of Natural Philosophy
1 Kusum Kutir, Mahamanapuri
Varanasi 221005, India

Abstract

Recent studies show that the angular momentum, both spin and orbital, of rotating light beams possesses counter-intuitive characteristics. We present a new approach to the question of orbital angular momentum of light based on the complex massless scalar field representation of light. The covariant equation for the scalar field is treated in rotating system using the general relativistic framework. First we show the equivalence of the U(1) gauge current for the scalar field with the Poynting vector continuity equation for paraxial light, and then apply the formalism to the calculation of the orbital angular momentum of rotating light beams. If the difference between the co-, contra-, and physical quantities is properly accounted for there does not result any paradox in the orbital angular momentum of rotating light. An artificial analogue of the paradoxical situation could be constructed but it is wrong within the present formalism. It is shown that the orbital angular momentum of rotating beam comprising of modes with opposite azimuthal indices corresponds to that of rigid rotation. A short review on the electromagnetism in noninertial systems is presented to motivate a fully covariant Maxwell field approach in rotating system to address the rotating light phenomenon.

PACS numbers:
Key Words: Rotating light, Orbital angular momentum paradox, Gauge invariance, General relativistic approach, Noninertial system.

I. INTRODUCTION

Rotating light beams seem to possess interesting and counter-intuitive properties: frequency shift, angular momentum (AM) opposite to the direction of the beam rotation, and frequency-dependent spin angular momentum (SAM). Recall that the rotation of the fields is inherent in the electromagnetic waves [1], for example, in a circularly polarized plane wave the electric field vector at a fixed point in space rotates at the frequency \( \omega \) of the wave. Rotating light beams, however correspond to the 'forced rotation' [2] employing rotating optical media: rotating half-wave plate for polarization rotation [3] and rotating Dove prism (or a \( \pi \) mode converter) for mode pattern rotation of the paraxial modes of the Laguerre-Gaussian (LG) beams [4, 5]. Until recently the main concern had been with the rotational frequency shifts. Since polarization is associated with SAM, and the LG modes carry orbital angular momentum (OAM) [6] it is logical to investigate the effect of rotation on them. Straightforward generalization of the theory of [6] with the use of time-dependent paraxial wave equation [4] is obtained introducing uniform rotation frequency \( \Omega \) replacing the azimuthal coordinate \( \phi \) to \( \phi - \Omega t \) in the mode function [2]. The vector mode function for the case of the polarization and polychromatic waves are introduced in [7]. The calculated OAM and SAM are found to be oppositely directed to the angular velocity; this is termed 'paradoxical' [2] and counter-intuitive [7]. Alexeyev and Yavorsky [8] also note that the application of Berry’s formula [9] leads to a ‘confusing result’, namely the \( \Omega \)-independent AM. The role of polychromatic nature of light for rotating beams is recognised by all of them, and qualitative arguments based on the notion of photon are presented seeking the resolution of the paradox. A recent work [10] addresses the problem developing quantum optics formalism for rotating light.

Though useful insights have been obtained in the cited literature, there do exist gaps in the understanding of the AM of rotating light. In the present paper we envisage a fresh approach based on the classical relativistic scalar field. It is surprising that apparently unphysical result that SAM depends on frequency is not noticed in the current literature. The fact that spin of photon is independent of frequency, played an important role in the
physical interpretation of the angular Doppler effect \[11\]. Further the Sagnac effect truly belongs to the class of phenomena in which rotation causes frequency shifts. Therefore a brief appraisal of this effect in the context of the controversies associated with rotating light is also presented.

The main contribution of the present work is to develop a relativistic scalar field theory for rotating light. Note that treating a component of the electric field or the vector potential, in some cases, as a complex scalar function is well known \[1\]; typical laser modes of 'doughnut' shape having helical wavefronts (phase singularities) are also describable as complex scalar (electric) field \[12\], and the polarization - a typical vector property of light could be operationally defined based on the intensity (a scalar function) measurements \[13\], see also \[14\]. What differentiates our approach is that a massless complex scalar field \(\Psi\) without recourse to the electromagnetic fields is shown to possess a nice property: exact equivalence of the Noether conserved current corresponding to the gauge symmetry \[15\] with the Poynting vector calculated in the paraxial approximation, Eq.(2.5) of \[16\]. Thus OAM of not only LG modes but that of multipole fields \[1\] follows immediately in this formalism. This allows us to generalize the approach to the covariant form of scalar wave equation in a rotating coordinate system a la general relativity \[17\]. Bichromatic field becomes imperative, and frequency shift follows from the dispersion relation. The expression for the Noether current gives the Poynting vector equivalent without any ambiguity. This allows us to resolve the OAM paradox satisfactorily.

The paper is organized as follows. In the next section key results given in \[16\] are reproduced for a self-contained presentation, and some critical remarks are made concerning AM of light. In Sec.III conserved current associated with the gauge invariance of the complex scalar field is shown to be identical with the Poynting vector for paraxial linearly polarized beam. It is argued that the application of this result to multipole fields indicates its general validity. The covariant form of scalar field equation in a rotating frame is presented in Sec.IV. Assuming cylindrical symmetry and azimuthal dependence of the form \(exp(\imath l \phi)\) dispersion relation is derived. Using the superposition of the scalar fields OAM is calculated from the Noether current. It is remarkable that the OAM of rotating beam could be interpreted in terms of rigid rotation, and there do not arise any counter-intuitive features. In Sec.V the question of OAM paradox is revisited, and resolved. A brief review on the electromagnetic fields in noninertial systems, and its relevance in the present context constitute Sec.VI. The
II. THE PARAXIAL APPROXIMATION AND OPTICAL ANGULAR MOMENTUM

Paraxial rays in optics have been known since long. Paraxial wave equation describes reasonably well the laser beam propagation \[19\]. Assuming monochromatic plane polarized wave propagating along z-axis the electric field can be represented in terms of a scalar function \(u(x, y, z)\) and a phase factor \(e^{ikz-i\omega t}\). In the paraxial approximation the second derivative of \(u\) with respect to \(z\) is neglected compared with \(|k\frac{\partial u}{\partial z}|\) so that the wave equation reduces to

\[
\nabla_t^2 u = -2ik \frac{\partial u}{\partial z}
\]

Here \(\nabla_t^2\) is 2-dimensional Laplacian in the transverse plane and the wave number \(k\) is given by \(\omega = ck\) in free space. Solution of paraxial wave equation, Eq.(1) is inconsistent with the Maxwell equation \[20\]: for an x-polarized wave the electric field must be independent of \(x\), however the lowest mode is Gaussian in \(x\) and \(y\). Lax et al \[20\] find it paradoxical: Experimentally the laser-oscillator modes found in this apparently inconsistent way agree extremely well with those predicted by this theory. A systematic procedure assuming power series expansion of the electromagnetic fields in terms of the ratio of the beam size in transverse plane to the diffraction length in the longitudinal direction is developed by them. Allen et al \[6\] following \[21\] assume the paraxial scalar wave equation for the vector potential instead of the electric field. Let

\[
A = \hat{x}ue^{i(kz-\omega t)}
\]

and the Poynting vector is defined to be

\[
S = \epsilon_0 \mathbf{E} \times \mathbf{B}
\]

The time-averaged Poynting vector for the paraxial light is finally obtained to be

\[
S^{(1)}(1) = \frac{i\omega \epsilon_0}{2} [u \nabla u^* - u^* \nabla u - 2iku^* u \hat{z}]\]

Expression (4) is Eq.(2.5) of \[16\] referred to above in the preceding section and represents the linear momentum density in the beam.
The angular momentum density

\[ j = \epsilon_0 r \times (E \times B) \]  

(5)

can be calculated from Eq.(4). For a cylindrical beam profile having

\[ u(r, \phi, z) = u_0(r, z)e^{il\phi} \]  

(6)

the z-component of the AM density is calculated to be

\[ j_{z}^{(1)} = \epsilon_0 \omega lu^* u \]  

(7)

The energy density in this case is given by the product of the linear momentum density and the velocity of light

\[ w = c \epsilon_0 \omega ku^* u \]  

(8)

Evidently

\[ \frac{j_{z}^{(1)}}{w} = \frac{l}{\omega} \]  

(9)

Since the light beam does not carry SAM expression (7) is interpreted as the orbital angular momentum of light, and putting \( \hbar \) by hand in (9) it is claimed that the light beam carries OAM of \( l\hbar \) per photon.

Generalization to the circularly polarized light is obtained assuming

\[ A = (\alpha \hat{x} + \beta \hat{y})ue^{i(kz-\omega t)} \]  

(10)

and calculating the electric and magnetic field vectors in the paraxial approximation. The Poynting vector is found to be

\[ S = S^{(1)} + S^{(2)} \]  

(11)

\[ S^{(2)} = \frac{i\omega \epsilon_0}{2}(\alpha \beta^* - \beta \alpha^*)\nabla(uu^*) \times \hat{z} \]  

(12)

Expression (12) is interpreted as a spin dependent part of the Poynting vector since the complex quantities \( \alpha, \beta \) determine the polarization of light. Denoting \( i(\alpha \beta^* - \beta \alpha^*) \) by \( \sigma \) that takes values \( \pm 1 \) for left/right circular polarizations the z-component of the angular momentum density is given by the sum of \( j_{z}^{(1)} \) i.e. Eq.(7) and

\[ j_{z}^{(2)} = -\frac{\epsilon_0}{2}\omega r \sigma \frac{\partial|u|^2}{\partial r} \]  

(13)
Integrating angular momentum density and energy density over the transverse plane the ratio of the AM to energy per unit length is obtained to be

\[ \frac{J_z}{W} = \frac{l + \sigma}{\omega} \] (14)

This completes the summary of the main results of [6, 16]. It is pertinent to make certain remarks in the following.

1):- The Hermite-Gaussian (HG) and LG modes in lasers had been extensively studied [12, 19], however the recognition that the LG modes carry OAM in 1992 [6] stimulated enormous activity in this field. Historically Poynting in 1909 [22] using mechanical analogy associated angular momentum transfer to the optical media from the circularly polarized light that was first measured by Beth [23]. In the usual interpretation this experiment is believed to validate the concept of intrinsic spin of photon. Questioning the notion that the spin of electron is some intrinsic quantum property having no classically understandable picture, Ohanian [24] argued that the spin could be related with the circulating energy flow; a nice discussion on the AM of the electromagnetic field is also given by him. Both SAM and OAM are calculated for a quasi-plane wave. An illuminating discussion on the AM of light and the limitations of plane wave approximation can be found in Section 2.7 and Chapter 9 of [14]. Interestingly Problems 6.11 and 6.12 in Jackson’s book [1] capture the essential intricacies of the AM of radiation; note that the expression for the electric field in Problem 6.11 is same as the one obtained using Eq. (10) above in the Allen et al work. The identification of the ratio of the AM to energy in Problem 6.12 is left ambiguous: whether spin or orbital. This ambiguity reappears in Section 16.8 where multipole expansion for a vector plane wave is presented and the value of \( m = \pm 1 \) for a circularly polarized wave is interpreted as \( \pm 1 \) unit of AM per photon along the direction of the wave propagation. Here \( m \) is the index specifying the \( \phi \)-dependence of the spherical harmonics \( Y_{lm}(\theta, \phi) \) of order \( (l, m) \). In the paraxial beam Eq. (12) does indicate circulating energy flow i.e. the presence of \( \nabla |u|^2 \) term in agreement with the suggestion of Ohanian [24]. Though the separation of OAM and SAM is achieved in Eq. (14) a thorough analysis of the role of scalar function in the SAM seems necessary.

2):- Barnett and Allen [25] raise the question whether the AM for the LG modes calculated in [6] is an artefact of the paraxial approximation. Specially the separation of the total AM into the spin and the orbital parts is discussed. In a later paper [26] it is shown that
the AM flux for a light beam could be separated into gauge invariant spin and orbital parts without making paraxial approximation. Let us remember that in the covariant formulation of the electrodynamics [27] conservation laws follow from the invariance of the action functional, and the issue of gauge invariance arises with full complexity. The canonical energy-momentum tensor obtained as a Noether current from the infinitesimal coordinate transformation is not gauge invariant and also not symmetric. Due to the second feature the angular momentum third rank tensor constructed from the canonical energy-momentum tensor is found to be not a conserved quantity. Adding a gauge invariant divergenceless, so called spin energy tensor, to the canonical tensor, the symmetric traceless and gauge invariant energy-momentum tensor is obtained; the corresponding angular momentum is conserved. Therefore, the subtle question [28] is that of the manifest Lorentz covariance and gauge invariance. Fixing a gauge, for example, the radiation gauge in optics breaks the manifest Lorentz covariance. Once we assume the radiation gauge the intricacies of the time-like and longitudinal field excitations (photons) faced in quantum electrodynamics disappear. It is possible to associate physical observables to spin and orbital parts of the AM in such a quantum theory, see e. g. [29].

Regarding the optical angular momentum flux discussed in [26] remarks in [30] set the issue in proper context. The main point is that so long as the surface integral of a divergence term can be made to vanish the angular momentum flux given in [26] could be useful as a physically observable quantity. In some nontrivial cases the surface term may give rise to holonomy akin to Aharonov-Bohm effect; prior to the considerations on the OAM of LG modes [6, 25, 26] angular momentum holonomy, for both OAM and SAM, was postulated to be the physical mechanism for the geometric phases in optics [31]; see Eq. (20) in [31] for the importance of the surface integral term.

Though transfer of both SAM and OAM of the light beams to small particles has been demonstrated experimentally [32] at a single photon level the issue of the separation of spin and orbital parts of the angular momentum is not a settled one since spin seems to possess a metric independent topological attribute.
III. GAUGE INVARIANCE AND ENERGY-MOMENTUM CURRENT OF LIGHT

Pondering over the remarkably nice form (4) of the Poynting vector (inspired by the remarks in [16]) it struck me that $S^{(1)}$ is exactly equivalent to the 3-vector of the conserved current $C^\mu$ associated with the gauge invariance of the complex scalar field [15]. This analogy is discussed in detail as it may have deep significance. First we make few remarks on gauge theories to avoid any confusion. The gauge field theories have electrodynamics as a paradigm. In classical electrodynamics the scalar and vector potentials serve the purpose of mathematical tools as the measurable quantities depend only on the electric and magnetic fields which do not determine the potentials uniquely or alternatively the fields are invariant under the gauge transformation of the potentials, for example, the divergence equation $\nabla \cdot B = 0$ implies that magnetic field derivable from the curl of a vector potential is invariant under the transformation $A \rightarrow A + \nabla \chi$. In quantum theory the wave function is arbitrary up to a phase factor. The requirement of local $U(1)$ (phase) gauge invariance in the Lagrangian formulation leads naturally to the electromagnetic interaction. In the textbooks complex scalar field model is usually discussed to illustrate this. Here we are not considering the gauge theory of electromagnetic interactions. We are interested in the global gauge transformation defined by

$$\Psi \rightarrow e^{-i\alpha} \Psi, \quad \Psi^* \rightarrow e^{i\alpha} \Psi^*$$

for the complex massless scalar field. The action for this field

$$I = \int L d\tau$$

is invariant under the gauge transformation, and the corresponding Noether’s conserved current is given by

$$C^\mu = i(\Psi^* \partial^\mu \Psi - \Psi \partial^\mu \Psi^*)$$

The Lagrangian density for the complex massless scalar field $\Psi$ is given by

$$L = g^{\mu\nu} \partial_\mu \Psi \partial_\nu \Psi^*$$

Here Greek indices run from 0 to 3; $\alpha$ is a real gauge parameter; 4-dimensional volume element $d\tau = \sqrt{-g} dx^4$ where $g$ is the determinant of the metric tensor $g_{\mu\nu}$. In the flat
spacetime $\sqrt{-g} = 1$ and $d\tau = c \, dt \, dV$ where $dV$ is 3-dimensional volume element; $\partial^\mu = (\frac{\partial}{\partial \tau}, - \nabla)$, and $C^\mu = (C^0, \mathbf{C})$.

We make a radical proposition: the scalar field $\Psi$ represents the light beams and the conservation law for $C^\mu$

$$\partial_\mu C^{\mu} = 0 \quad (19)$$

is analogous to the continuity equation satisfied by the Poynting vector

$$\nabla \cdot \mathbf{S} + \frac{\partial w}{\partial t} = 0 \quad (20)$$

Recall that $w$ is the energy density of the electromagnetic field. Thus we make the identification $\mathbf{C} \rightarrow \mathbf{S}$ and $C^0 \rightarrow w/c$.

First we consider its application to the paraxial light beams. Assuming

$$\Psi = u \, e^{i(kz-\omega t)} \quad (21)$$

from Eq.(17) a simple calculation shows that

$$\mathbf{S}^{(1)} = \frac{\omega \epsilon_0}{2} \, \mathbf{C} \quad (22)$$

Evaluating the time-component

$$C^0 = \frac{2\omega}{c} |u|^2 \quad (23)$$

and multiplying it by the factor $\omega \epsilon_0 / 2$ we get the energy density (8). Now the $z$-component of the angular momentum can be easily calculated following [16]. Exact expression for OAM density (7) and the ratio (9) are found. Since the field $\Psi$ is scalar with zero spin the angular momentum is necessarily orbital.

It can be argued that this correspondence is an accidental coincidence for the paraxial beams. To check its validity we apply this formalism to the well known case of multipole radiation [1]. In this case the scalar field satisfies scalar wave equation in spherical coordinates $(r, \theta, \phi)$, for which we assume [1]

$$\Psi = \Psi_0 \, f_l(kr) \, Y_{lm}(\theta, \phi) \, e^{-i\omega t} \quad (24)$$

Here $\Psi_0$ is a constant amplitude factor, $f_l$ is the spherical Hankel function which asymptotically (in the radiation zone) behaves as $|f_l|^2$ tending to $\frac{1}{kr}$. Substituting (24) in Eq. (17) the time component of the gauge current is obtained to be

$$C^0 = \frac{2\omega}{c} |\Psi_0|^2 |f_l|^2 Y^*_{lm} \, Y_{lm} \quad (25)$$
and the $\phi$-component is given by
\[
C_\phi = \frac{2m}{r \sin \theta} |\Psi_0|^2 |f_l|^2 Y^*_m Y_{lm} \tag{26}
\]
The energy of the radiation field in a spherical shell bounded by $r$ and $r + dr$ is calculated from (25)
\[
dw = \frac{2\omega}{c} |\Psi_0|^2 |f_l|^2 r^2 dr \int Y^*_m Y_{lm} \sin \theta d\theta d\phi \tag{27}
\]
The orthonormality of the spherical harmonics give the value of the integral equal to 1. Using the asymptotic value of the Hankel function we get
\[
\frac{dw}{dr} = \frac{2\omega}{k^2} |\Psi_0|^2 \tag{28}
\]
The $z$-component of the angular momentum in the spherical shell can be calculated from $rsin\theta C_\phi$ using the asymptotic value of $|f_l|^2$
\[
\frac{dJ_z}{dr} = \frac{2m}{k^2} |\Psi_0|^2 \tag{29}
\]
It can be checked that the ratio of the AM to energy is $m/\omega$, and coincides exactly with that given by Eq.(16.66) in [1]. Arguments in [1] based on the quantum mechanical angular momentum operator, for example, $z$-component to be $-i\frac{\partial}{\partial \phi}$ indicate the interpretation of the ratio $\frac{m}{\omega} = \frac{m\hbar}{\hbar \omega}$ in terms of $m\hbar$ units of AM per photon of energy $\hbar \omega$ for the multipole radiation of order $(l, m)$. Though such an analogy is quite often made in the literature great caution must be exercised to acribe physical reality to photon or to relate it with quantum theory. An important point overlooked in most discussions is the obscure nature of rotational energy of photon: for arbitrary OAM of $l\hbar$ units and spin $\pm \hbar$ the energy per photon is still $\hbar \omega$.

IV. ROTATING LIGHT

Frequency shifts arising from the cyclic polarization changes due to the rotating waveplates have been related with evolving Pancharatnam phase [33], however a simple energy exchange mechanism between the light and the waveplates is shown to explain the observed frequency shift [34]. The frequency shifts for rotating paraxial beams are also discussed in the literature [16]. The role of angular momentum in geometric phases in the light of recent reports is underlined in [35]. The issue of the angular momentum associated with rotation of
the light has received attention quite recently, and there seem to be controversial theoretical results \[2, 7, 8, 10\].

Interaction of a monochromatic wave with moving or rotating media would in general result changes in its frequency; the problem is that of developing an appropriate theory. In \[4\] two points are made. A time-dependent generalization of the paraxial Eq.(1) is proposed

\[
\nabla_t^2 u = -2ik\left(\frac{\partial}{\partial z} + \frac{\partial}{c\partial t}\right)u
\]

And, for a cylindrical lens rotating at constant frequency \(\Omega\) it is argued that the output beam possesses all frequencies \(\omega + 2n\Omega\), \(n = -\infty \text{ to } \infty\). In \[7\] the transverse electric field is expressed in terms of a vector mode function to incorporate polarization, and the field is assumed to be a superposition of propagating waves with the phases \(\omega_n(t - z/c)\).

For a uniform polarization beam rotating with constant frequency \(\Omega\) polychromatic wave expansion for the scalar field is assumed to be

\[
\sum_l u_l(r, z) e^{il\phi} e^{-i(\omega+\Omega)(t-z/c)}
\]

Specifically for two modes with indices \(\pm l\) and equal amplitudes \(u_l = u_{-l} = u\) the scalar field is

\[
2u(r, z) \cos[l(\phi - \Omega(t - z/c))]e^{-i\omega(t-z/c)}
\]

Bekshaev et al. \[2\] consider rotating beam comprising of LG modes with \(l = \pm 1\), and term it the rotating HG (RHG) beam. If the frequencies are equal such a superposed beam is equivalent to an HG beam with an edge wavefront dislocation.

Now elementary analysis shows that a general solution of the wave equation

\[
\frac{\partial^2 f}{\partial z^2} - \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} = 0
\]

consists of a superposition of the arbitrary functions \(f_1(z - vt)\) and \(f_2(z + vt)\). One can introduce harmonic waves with frequencies \(\omega_n\) and corresponding propagation constants \(k_n\) to expand these functions in terms of polychromatic waves. For an ideal monochromatic wave, by definition, there is only one frequency. In the case of rotating beams, there is such a characteristic frequency i.e. \(\omega\) for the input wave. Since the rotation is affected in the transverse plane, for cylindrical system it would imply the transformation in the azimuthal coordinate \(\phi \rightarrow \phi - \Omega t\); however the propagation along \(z\)-axis is still determined by the characteristic frequency \(k = \omega/c\). Therefore the argument suggesting general polychromatic
expansion for rotating beam seems doubtful. It is also surprising that a vast literature on the electromagnetic fields in the rotating media \[18, 36\] has remained unnoticed in this connection. We present a short review in Sec.VI.

We analyze rotating light using the covariant scalar wave equation

\[
\frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} g^{\mu \nu} \partial_{\nu} \Psi) = 0
\]

Since the rotation frequency is very small it suffices to employ the so called Galilean rotation transformation

\[
r' = r, \ z' = z, \ t' = t, \ \phi' = \phi - \Omega t
\]

Here \((r, \phi, z, t)\) define the inertial system and primed coordinates refer to the uniformly rotating system. Note that (35) breaks the relativistic covariance; there are subtle issues of the laboratory, the corotating, and the instantaneous frames of reference \[17\]. There does exist Trocheris-Takeno transformation \[37\] which is known to be relativistically covariant and has been used to study electromagnetism in rotating media \[38\]. The line element expressed in terms of the unprimed coordinates gives rise to the following nonvanishing metric tensor components

\[
g_{00} = c^2(1 - \beta^2), \ g_{11} = -1, \ g_{22} = -r^2, \ g_{33} = -1, \ g_{02} = g_{20} = \beta rc
\]

where \(\beta = \Omega r/c\). Determinant of the metric is \(\sqrt{-g} = rc\), and the contravariant tensor \(g^{\mu \nu}\) has the components

\[
g^{00} = 1/c^2, \ g^{11} = -1, \ g^{22} = -(1 - \beta^2)/r^2, \ g^{33} = -1, \ g^{02} = g^{20} = \beta/cr
\]

The wave equation (34) assumes the form

\[
\left[ \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) - \frac{(1 - \beta^2)}{r^2} \frac{\partial^2}{\partial \phi^2} - \frac{\partial^2}{\partial z^2} + \frac{2\beta}{cr} \frac{\partial^2}{\partial t \partial \phi} \right] \Psi = 0
\]

Neglecting \(\beta^2\) term it can be seen that this equation is identical with the cylindrical wave equation except the last term. Following the usual prescription let

\[
\Psi = u(r, z)e^{i(-\omega' t + kz + l\phi)}
\]

Assuming that the paraxial equation is satisfied we get the following dispersion relation from (38)

\[
\omega'^2 - 2l\Omega \omega' - k^2 c^2 = 0
\]
There are two possibilities: 1) the propagation constant $k$ along z-axis defines a characteristic frequency of the incoming wave such that $\omega = kc$; the frequency $\omega'$ of the output beam gets shifted, and 2) the propagation constant is shifted by $l\Omega/c$; then the frequency is unaffected by the rotation of the media. Assuming the first possibility we find that unlike the form (30) of ref. 7, also used in [2] the z-dependent phase factor in Eq.(39) is independent of $l\Omega$.

Consider the rotating beam comprising of two modes with indices $\pm l$, and having $u_l = u_{-l} = u$. The scalar field $\Psi$ is a superposition of the two modes

$$\Psi = u[e^{-il\Omega t + il\phi} + e^{il\Omega t - il\phi}] e^{-i\omega t + i k z}$$

Expression (17) for the Noether current gives the energy and linear momentum densities. The metric (36) defining the system implies that the time and azimuthal components of contravariant and covariant current differ. First we calculate the components of (17) for the ordinary derivative operators $\partial_0$, $\partial_\phi$.

$$\bar{C}_0 = 8\omega |u|^2 \cos^2 l(\phi - \Omega t)$$

$$\bar{C}_\phi = 0$$

Compare Eq.(42) with the energy density obtained for similar case in [7]

$$w = 8\varepsilon_0 |u|^2 \cos^2 [l(\phi - \Omega(t - z/c))]$$

Since $k$ does not change with $\Omega$ in our analysis it would appear that $\bar{C}_0$ could be identified with the energy density as proposed in the preceding section. However OAM density calculated in [7] is nonzero while (43) would give its value zero. Let us examine the problem calculating the contravariant current components. Note that for a scalar field, say, $f$ the ordinary derivative $\partial_\mu f$ is a covariant vector, and the contravariant vector is given by $g^{\mu\nu} \partial_\nu f$.

For the metric (37) we get

$$\partial^0 = g^{00} \partial_0 + g^{02} \partial_\phi$$

$$\partial^2 = g^{20} \partial_0 + g^{22} \partial_\phi$$

while r and z components do not change. From (17) using (45) and (46) we finally get

$$C^0_{rot} = \bar{C}_0/c^2$$

$$C^\phi_{rot} = \Omega \bar{C}_0/c^2$$
We have to keep in mind that in tensor analysis we define physical quantities. Here neglecting $\beta^2$ (48) has to be multiplied by $r$ that corresponds to momentum density. The OAM density is obtained to be

$$j_{rot} = r^2 \Omega \bar{C}_0/c^2 \quad (49)$$

Taking $\bar{C}_0$ as energy density, Eq. (49) could be interpreted as AM of a rigid body rotation with angular velocity $\Omega$. However earlier cited works [2, 7, 8] assert that there is no analogy with the rigid body rotation; this contradicts our result. To clarify the issue in the next section we present a detailed discussion.

V. OAM PARADOX

The result obtained in the preceding section is significant: there is no paradoxical feature that was found in the earlier studies [2, 7] on the OAM of rotating light beams; at the same time the circulating energy pattern described by Eq.(42) agrees with the observations as interpreted in [2]. To delineate the physical mechanism responsible for the rotating light we consider a single mode with index $l$ represented by

$$\Psi = u e^{-i\Omega t + il\phi - i\omega t + ikz} \quad (50)$$

Using ordinary derivatives (17) gives

$$C_0^l = 2|u|^2(\omega + l\Omega) \quad (51)$$

$$C_\phi^l = 2l|u|^2 \quad (52)$$

Physical azimuthal component is $C_\phi^l/r$, and the OAM density to energy density ratio is derived to be

$$R = \frac{l}{\omega + l\Omega} \quad (53)$$

Further the contravariant components are evaluated to be

$$C^{0l} = \frac{2|u|^2\omega}{c^2} \quad (54)$$

$$C^{\phi l} = \frac{2l|u|^2}{r^2} + \frac{2\omega|u|^2}{c^2} \quad (55)$$

Physical quantity from (55) is defined multiplying it by $r$, and the OAM density is obtained to be

$$j^{zl} = 2|u|^2[l + \frac{\Omega \omega r^2}{c^2}] \quad (56)$$
Eq. (56) shows that interpreting $C^0_l$ as energy density the OAM density comprises of two parts: the first term on the right side is OAM density corresponding to the nonrotating beam with the azimuthal index $l$, and the second term represents the rigid rotation of the beam with angular velocity $\Omega$.

Doing a naive calculation adding energy density and OAM density for the modes with opposite indices $\pm l$ we get

$$C_0(\text{rot}) = C_0^l + C_0^{-l} = 4|u|^2\omega$$

$$j_z(\text{rot}) = 0$$

Apart from the missing cosine squared term in (57) these are consistent with Eqs. (42) and (43). Suppose we calculate the ratio from (53) for the mode $-l$ and add the two then we find

$$R(\text{rot}) = \frac{l}{\omega + l\Omega} - \frac{l}{\omega - l\Omega} = -\frac{2l^2\Omega}{\omega^2 - l^2\Omega^2}$$

This is an startling result: Eq.(59) bears close resemblance with the main paradoxical feature i. e. Eq.(39) in [7] and Eq.(21) in [2]. In fact, rewriting (59) in the form

$$R(\text{rot}) = \frac{lh}{(\omega + l\Omega)\hbar} - \frac{lh}{(\omega - l\Omega)\hbar}$$

it would seem that following [2, 7] invoking rotating photons with OAM of $\pm lh$ and energy $(\omega \pm l\Omega)\hbar$ a physical interpretation of this paradoxical result could be envisaged. However in our analysis the derivation of (59) is incorrect: it is an artefact of the derivation. Physically meaningful would be the sum of contravariant quantities i. e. using Eq.(56) we find that the OAM arising as a rigid body rotation is consistent with Eq.(49).

The distinction between co-, contra-, and physical vectors and their interrelationship may appear intricate or at times annoying; however this is the limitation imposed by the metric based formalism. One has to be careful about it, for example, in the standard 3 dimensional space in curvilinear coordinate system. Even in flat spacetime geometry treating the time coordinate as $x^0 = t$ or $x^0 = ct$ results into different quantities; in Sec.II it is assumed that $x^0 = ct$, thus $\sqrt{-g} = 1$ whereas in Eq.(35) $x^0 = t$ leading to the appearance of the velocity in odd form subsequently.

To sum up: a new perspective on the OAM of rotating light emerges in the covariant scalar field approach, and though the paradoxical result akin to the one reported in the literature could be obtained it is obviously wrong in the present formalism.
VI. ELECTROMAGNETIC FIELDS IN NONINERTIAL SYSTEMS

The formal structure of the Maxwell equation is relativistically covariant; mathematical aspects are eloquently discussed in Eddington’s book [39]. A consistent application of the covariant formalism to the field equations sheds light on the constitutive relations in the rotating media, and as argued in [36] offers a satisfactory resolution of the Schiff paradox [40]. However there do exist subtle issues [41]. In the limited context of the present paper a short commentary on the radiation in rotating system seems useful. An important example is that of Sagnac effect known since 1913 and reviewed in [18]. To avoid any confusion first we recall that the Sagnac effect occurs in the rotating interferometer experiments. Light emanating from a source is split into two beams which are made to circulate in opposite directions along a closed loop, and recombined to give the interference pattern. If the whole system is set into rotation than the fringe shift relative to the stationary system is observed. The whole system consists of the interferometer, light source and detector, and the medium. Variants of this experiment also show fringe shifts: the medium is stationary while the interferometer rotates, and the medium is rotated keeping the interferometer stationary. Theory of this phenomenon is still controversial [18, 42].

Assuming cylindrical system light propagation is along the circular path in the Sagnac effect in contrast to the rotating beams which travel along the z-axis. However in spite of this difference the consideration of the electromagnetic fields in noninertial frame is crucial and common in both cases. Some of the interesting papers are those of Heer [43], Anderson and Ryon [44], and Post [45]. Heer calculates resonant frequency of a cavity in the rotating system. The constitutive relations used by him are criticized in [44]. A noteworthy result for the cylindrical cavity [43] is the axial modes splitting with frequency

\[ \omega_m = \omega_0^m \pm m\Omega \] (61)

where \( m \) is the azimuthal index (other indices are suppressed) and \( \Omega \) is the rotation frequency along z-axis. Multiplying on both sides by \( \hbar \) and introducing the OAM of photon \( J_z \) this equation is rewritten as

\[ \hbar\omega = \hbar\omega^0 + J_z\Omega \] (62)

Heer calls it Coriolis-Zeeman effect for the photon. Anderson and Ryon develop a generally covariant formalism for an arbitrarily moving medium. In the special case of uniform rotation
the frequency shift found by them does not depend on the relative permeability of the medium contradicting the result of Post \[18\]. In a later article Post \[45\] has analyzed the basic questions related with the problem, and argued that though the physical interpretation of the first order effects in noninertial systems remain unsettled, the experiments on unipolar induction (Kennard-Pegram effect) with different media could throw light on the issue of the constitutive relations. This brief review with a limited and incomplete citations shows that there are open problems in the subject of electromagnetism in rotating media \[18, 38, 45\].

Covariant scalar field approach presented in this article is a step forward in this direction, however for an unambiguous resolution of the problem a consistent covariant formulation of the electromagnetic fields in the rotating media is suggested. Further credence to this suggestion is obtained analyzing the calculation in \[8\]. Authors use quantum mechanical analogy and generalize the formula of \[9\] for quasi-monochromatic light. The main change is contained in the time derivative of a vector reproduced below

\[
\frac{\partial\mathbf{A}}{\partial t} = -i\omega\mathbf{A} + \vec{\Omega} \times \mathbf{A} - \Omega \frac{\partial\mathbf{A}}{\partial \phi} \tag{63}
\]

According to the authors fast time dependence is through the factor $e^{-i\omega t}$, and angular velocity vector is $\vec{\Omega} = (0, 0, \Omega)$. First two terms on the right of (63) are understandable, however the last term is peculiar. To obtain (63) physical arguments are put forward in the paragraph preceding this i. e. Eq.(7) in \[8\]. We examine it in covariant formalism.

The definition of a covariant derivative of a vector \[39\] is

\[
D_\nu A_\mu = \partial_\nu A_\mu - \Gamma^\alpha_{\mu\nu} A_\alpha \tag{64}
\]

Here $\Gamma^\alpha_{\mu\nu}$ is the Christoffel symbol. Setting the indices $\nu = 0; \mu = 1, 2, 3$ in Eq.(64) we get the time component of the covariant derivative for the spatial components of the vector $A_\mu$. In the special case of uniform rotation with the metric (36) Eq.(64) gives expression (63) without the last term. What is the origin of the last term? Besides the covariant derivative of a covariant vector, there are three more quantities \[39\]: covariant derivative of $A^\mu$, and contravariant derivatives of $A_\mu$ and $A^\mu$. Let us calculate the contravariant derivative. We have to use (45) for the operator $D^0$, substitute $g_{02}$ from (37), and evaluate $\Gamma^\alpha_{\mu\nu}$. After a little lengthy calculation we get finally expression (63) if we assume

\[
\frac{\partial\mathbf{A}}{\partial t} = -i\omega\mathbf{A} \tag{65}
\]
It is remarkable that physical intuition led the authors of [8] to the time component of a contravariant derivative. The significance of this identification is two-fold: 1) the approach in [8] is based on quantum mechanical analogy and the generalization of time derivative by them finally results into the conclusion in agreement with [2, 7]. Viewed from the covariant perspective the derivation of electric and magnetic fields using only Eq.(63) for time derivatives is incorrect since these fields are not vectors but the components of a second rank tensor $F_{\mu\nu}$. Thus the final result in [8] is questionable. And, 2) For a conclusive resolution of the issue it becomes imperative that a rigorous generally relativistic covariant treatment of the rotating media be incorporated. Though constitutive relations will be fairly involved and the analysis quite complicated, since fundamental issues are involved such a study will be important.

VII. DISCUSSION AND CONCLUSION

Recent theoretical studies show that rotating light beams possess angular momentum (OAM and SAM) having paradoxical attributes. Attempts have been made to explain such unexpected features, however the physical interpretation still lacks clarity. In this paper the OAM of rotating light is examined afresh. An important ingredient in our approach is the recognition that Poynting vector continuity equation in the case of paraxial beams is equivalent to the Noether gauge current conservation law for a complex scalar field [46]. Postulating scalar field $\Psi$ to represent light without relating it to the electromagnetic fields, a covariant formulation is developed. This allows us to treat uniform rotation adopting the general relativistic framework. The OAM of rotating light is easily calculated using the gauge current $C^\mu$. It is shown that there is no paradox if the distinction between co-, contra-, and physical vectors is taken care of. A significant result is that the OAM of the rotating beam comprising of two modes with opposite azimuthal indices is consistent with the picture of a rigid body rotation. This contradicts the previous literature [2, 7, 8]. The matter is further elucidated in Sec.V where it is shown that one can 'manufacture' a formal paradoxical result, however it is wrong in the present covariant approach.

The interpretation of $C^\mu$ as energy-momentum vector is motivated by the nice form of the Poynting vector (4) for the paraxial light. Though it has been justified showing its applicability to the calculation of the angular momentum of the multipole radiation, there
remain some basic questions to be addressed. In field theory the energy-momentum vector is obtained setting the index $\nu = 0$ in the energy-momentum tensor $T^{\mu\nu}$. On the other hand, the gauge current is associated with the charge of the field such that local gauge invariance naturally leads to the electromagnetic interaction via gauge covariant derivative $\partial^\mu$.

Interestingly for the special case of the complex scalar field we find an important result: the tensor $T^{\mu0}$ for $\Psi$ gives the momentum vector formally equivalent to $C$ assuming $e^{-i\omega t}$ time-dependence for $\Psi$. Note that in flat spacetime $g^{\mu0}$ vanishes for $\mu = 1, 2, 3$ and the expression $\partial^\mu\Psi\partial^\nu\Psi^* + \partial^\nu\Psi\partial^\mu\Psi^*$ in $T^{\mu\nu}$ reduces to the form of $C$. However the time-component $T^{00}$ disagrees with $C^0$. We may think of enlarging the group of symmetries including conformal invariance; one requires energy-momentum tensor to be traceless in that case. One can add a term to $T^{\mu\nu}$ keeping intact the covariant divergence law and derive a new traceless tensor; this can also be done adding a surface term to the Lagrangian (18). The role of gauge and conformal symmetries envisaged here would have wider ramification for the gauge theories and deserves further investigation.

There are two major shortcomings of the present work. First, we have not considered spin angular momentum. In the paraxial approximation assuming Eq. (10) for the vector potential the spin-dependent Poynting vector (12) is derived, see [16]. Nienhuis has generalized it to the rotating polarization beams and considers polychromatic waves. An intriguing discussion in this paper relates with inhomogeneous linear polarization where entangled photons are suggested. In a subsequent work introducing plausible quantized rotating mode operators a quantum theory of rotating light is developed. In Sec.V authors calculate SAM for single photon which is, similar to OAM, found to be counterintuitive. Obviously the scalar field theory is inadequate to address the question of spin of photon, and it is as yet not clear how to generalize our work for this purpose. We may, however point out that spin is intrinsic and possesses the characteristic of metric-independence; the results on single photon found in [10] cannot be accepted without reservation.

Second limitation arises because we do not consider electromagnetic fields: though scalar wave theory does explain a number of properties of light, we cannot be realistic unless electromagnetic fields are considered. To motivate general relativistic treatment of the full Maxwell field theory a short review is presented in the preceding section on the electromagnetism in noninertial systems. Further justification for this kind of approach is sought re-analyzing the theory given in [8]. The main modification in [8] is based on the definition of
the time-derivative of a vector proposed by the authors that is subsequently used in Maxwell equation to calculate electric and magnetic fields. In the present paper we show that the new time-derivative is essentially the time-component of the contra-variant derivative of a vector in the metric space defined by Eq.(36). It is logical to argue that general relativistic formalism is necessary to resolve the controversial issues; not only this perhaps the advances in rotating light experiments may throw light on some outstanding problems in the electromagnetism of rotating media (Sec.VI); an interesting and controversial problem is that of the scattering of radiation from a rotating cylinder discussed by Hillion [38].

In conclusion, equivalence of gauge current for a complex scalar field and energy-momentum vector for paraxial light is shown; a covariant scalar field theory in rotating system is presented to address the issue of the OAM of rotating light, and it is shown that there does not arise any paradox highlighted in the recent literature.

Acknowledgements

I gratefully acknowledge correspondence with Prof. L. Allen, Dr. A. Yu. Bekshaev, and Prof. G. Nienhuis. Library facility of Banaras Hindu University is acknowledged.

[1] J. D. Jackson, Classical Electrodynamics (Wiley, 1962).
[2] A. Yu. Bekshaev, M. S. Soskin, and M. V. Vasnetsov, Opt. Commun. 249, 367 (2005).
[3] B. A. Garetz and S. Arnold, Opt. Commun. 31, 1 (1979).
[4] G. Nienhuis, Opt. Commun. 132, 8 (1996).
[5] J. Courtial, K. Dholakia, D. A. Robertson, L. Allen, and M. J. Padgett, Phys. Rev. Lett. 80, 3217 (1998).
[6] L. Allen, M. W. Beijersbergen, R. J. C. Spreeuw, and J. P. Woerdman, Phys. Rev. A 45, 8185 (1992).
[7] G. Nienhuis, J. Phys. B 39, S529 (2006).
[8] C. N. Alexeyev and M. A. Yavorsky, J. Opt. A Pure and Appl. Opt. 7, 416 (2005).
[9] M. V. Berry, Proc. SPIE 3487, 6 (1998).
[10] S. J. van Enk and G. Nienhuis, Phys. Rev. A 76, 053825 (2007).
[11] B. Garetz, J. Opt. Soc. Am. 71, 609 (1981).
[12] J. H. Vaughan and D. V. Villetts, J. Opt. Soc. Am. 73, 1018 (1983); N. R. Heckenberg, R.
McDuff, C. P. Smith, H. Rubinsztein-Dunlop, and M. J. Wegener, Opt. Quant. Electron. 24, 951 (1992).

[13] G. G. Stokes, Trans. Cambridge Philos. Soc. 9, 399 (1853).

[14] J. W. Simmons and M. J. Guttman, States, Waves and Photons: A Modern Introduction to Light (addison-Wesley, 1970).

[15] I. J. R. Aitchison, An Informal Introduction to Gauge Field Theories (C. U. P. 1982).

[16] L. Allen, M. J. Padgett, and M. Babiker, Prog. Opt. 39, 291 (1999).

[17] C. Moller, The Theory of Relativity (O. U. P., 1972).

[18] E. J. Post, Rev. Mod. Phys. 39, 475 (1967).

[19] A. E. Siegman, Lasers (University Science Books, CA, 1986).

[20] M. Lax, W. H. Louisell, and W. B. McKnight, Phys. Rev. A 11, 1365 (1975).

[21] L. W. Davis, Phys. Rev. A 19, 1177 (1979).

[22] J. H. Poynting, Proc. R. Soc. (London) A 82, 560 (1909).

[23] R. A. Beth, Phys. Rev. 48, 471 (1935).

[24] H. C. Ohanian, Am. J. Phys. 54, 500 (1986).

[25] S. M. Barnett and L. Allen, Opt. Commun. 110, 670 (1994).

[26] S. M. Barnett, J. Opt. B: Quantum and Semiclass Opt. 4, S7 (2002).

[27] E. M. Corson, Introduction to Tensors, Spinors and Relativistic Wave Equations (Blackie and Son, 1953).

[28] S. C. Tiwari, arxiv.org: quant-ph/0609015.

[29] S. J. van Enk and G. Nienhuis, J. Mod. Opt. 41, 963 (1994).

[30] S. C. Tiwari, J. Mod. Opt. 51, 2297 (2004).

[31] S. C. Tiwari, J. Mod. Opt. 39, 1097 (1992).

[32] A. T. O’Neil et al., Phys. Rev. Lett. 88, 053601 (2002); V. Garces-Chavez et al., Phys. Rev. Lett. 91, 093602 (2003).

[33] R. Simon, H. J. Kimble and E. C. G. Sudarshan, Phys. Rev. Lett. 61, 19 (1988).

[34] F. Bretenaker and A. LeFloch, Phys. Rev. Lett. 65, 2316 (1990).

[35] S. C. Tiwari, Optik (in Press).

[36] J. van Bladel, Proc. IEEE 64, 301 (1976).

[37] M. G. Trocheris, Philos. Mag. 40, 1143 (1949); H. Takeno, Prog. Theor. Phys. 7, 367 (1952).

[38] S. Kichenassamy and R. A. Krikorian, J. Math. Phys. 35, 572 (1994); P. Hillion, Phys. Rev.
[39] A. S. Eddington, Mathematical Theory of Relativity (C. U. P. 1924).
[40] L. I. Schiff, Proc. Nat. Acad. Sci. 25, 391 (1939).
[41] E. J. Post, Formal Structure of Electromagnetics (Dover Publications, 1997).
[42] F. Hasselbach and M. Nicklaus, Phys. Rev. A 48, 143 (1993).
[43] C. V. Heer, Phys. Rev. 134, A799 (1964).
[44] J. L. Anderson and J. W. Ryon, Phys. Rev. 181, 1765 (1969).
[45] E. J. Post, Found Phys. 9, 619 (1979).
[46] S. C. Tiwari, J. Mod. Optics (in Press).
[47] P. Ramond, Field Theory: A Modern Primer (Addison-Wesley, IInd Edition, 1990).
[48] R. Jackiw, Phys. Rev. Lett. 41, 1635 (1978).
[49] S. C. Tiwari, J. Math. Phys. 49, 032303 (2008).