Some new bounds for the Hadamard product and the Fan product of matrices

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Abstract

If $A$ and $B$ are nonnegative matrices, a sharp upper bound on the spectral radius $\rho(A \circ B)$ for the Hadamard product of two nonnegative matrices is given, and the minimum eigenvalue $\tau(A \ast B)$ of the Fan product of two $M$-matrices $A$ and $B$ is discussed. In addition, we also give a sharp lower bound on $\tau(A \circ B^{-1})$ for the Hadamard product of $A$ and $B^{-1}$. Several examples, illustrating that the given bound is stronger than the existing bounds, are also given.

Key words: Hadamard product; Nonnegative matrices; Spectral radius; Fan product; M-matrix; Inverse $M$-matrix; Minimum eigenvalue

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1 Introduction

In this paper, for a positive integer $n$, $N$ denotes the set $\{1, 2, \ldots, n\}$. $\mathbb{R}^{n \times n}$ denotes the set of all $n \times n$ real matrices and the set of all $n \times n$ complex matrices is denoted by $\mathbb{C}^{n \times n}$. Let $A = (a_{ij})$ and $B = (b_{ij})$ be two real $n \times n$ matrices. We write $A \geq B (> B)$ if $a_{ij} \geq b_{ij} (> b_{ij})$ for all $i, j \in N$. If $A \geq 0 (> 0)$, we say that $A$ is a nonnegative (positive) matrix. The spectral radius of $A$ is denoted by $\rho(A)$. If $A$ is a nonnegative matrix, the Perron-Frobenius theorem guarantees that $\rho(A) \in \sigma(A)$, where $\sigma(A)$ is the set of all eigenvalues of $A$. In addition, define $\tau(A) \triangleq \min\{\lambda | \lambda \in \sigma(A)\}$, and denote by $\mathcal{M}_n$ the set of nonsingular $M$-matrices (see [1]).

For $n \geq 2$, an $n \times n$ matrix $A$ is said to be reducible if there exists a permutation matrix $P$ such that

$$P^TAP = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}.$$
where $B$ and $D$ are square matrices of order at least one. If no such permutation matrix exists, then $A$ is called irreducible. If $A$ is a $1 \times 1$ complex matrix, then $A$ is irreducible if and only if its single entry is nonzero (see [2]).

According to Ref. [2], a matrix $A$ is called an $M$-matrix, if there exists an $n \times n$ nonnegative real matrix $P$ and a nonnegative real number $\alpha$ such that $A = \alpha I - P$, and $\alpha \geq \rho(P)$, where $\rho(P)$ denotes the spectral radius of $P$ and $I$ is the identity matrix. Moreover, if $\alpha > \rho(P)$, $A$ is called a nonsingular $M$-matrix; if $\alpha = \rho(P)$, we call $A$ a singular $M$-matrix.

In addition, a matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is called $Z$-matrix if all of whose off-diagonal entries are negative, and denoted by $A \in \mathbb{Z}_n$. For convenience, the following simple facts are needed (see Problems 16, 19 and 28 in Section 2.5 of [3]):

(1) $\tau(A) \in \sigma(A)$;
(2) If $A, B \in \mathcal{M}_n$, and $A \geq B$, then $\tau(A) \geq \tau(B)$;
(3) If $A \in \mathcal{M}_n$, then $\rho(A^{-1})$ is the Perron eigenvalue of the nonnegative matrix $A^{-1}$, and $\tau(A) = \frac{1}{\rho(A^{-1})}$ is a positive real eigenvalue of $A$.

Let $A$ be an irreducible nonsingular $M$-matrix. It is well known that there exist positive vectors $u$ and $v$ such that $Au = \tau(A)u$ and $v^T A = \tau(A)v^T$, where $u$ and $v$ are right and left Perron eigenvectors of $A$, respectively.

The Hadamard product of $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ and $B = (b_{ij}) \in \mathbb{C}^{n \times n}$ is defined by $A \circ B = (a_{ij}b_{ij}) \in \mathbb{C}^{n \times n}$.

For two real matrices $A, B \in \mathcal{M}_n$, the Fan product of $A$ and $B$ is denoted by $A \star B = C = [c_{ij}] \in \mathcal{M}_n$ and is defined by

$$c_{ij} = \begin{cases} -a_{ij}b_{ij}, & \text{if } i \neq j, \\ a_{ii}b_{ii}, & \text{if } i = j. \end{cases}$$

We define: for any $i, j, l \in N$,

$$r_i = \frac{|a_{ii}|}{|a_{il}| - \sum_{k \neq l, i} |a_{ik}|}, \quad l \neq i; \quad r_i = \max_{i \neq i} \{r_{ii}\}, \quad i \in N;$$

$$s_{ji} = \frac{|a_{ji}| + \sum_{k \neq j, i} |a_{jk}|r_k}{|a_{jj}|}, \quad j \neq i; \quad s_i = \max_{j \neq i} \{s_{ji}\}, \quad i \in N;$$

throughout the paper.

For two nonnegative matrices $A, B$, we will exhibit a new upper bound for $\rho(A \circ B)$, a new lower bound on the eigenvalue $\tau(A \star B)$ for the Fan product and a new lower bound on the eigenvalue $\tau(A \circ B^{-1})$ for the hadamard product in this paper.
2 An upper bound for the spectral radius of the Hadamard product of two nonnegative matrices

In ([3], p. 358), there is a simple estimate for $\rho(A \circ B)$: if $A, B \in \mathbb{R}^{n \times n}$, $A \geq 0$, and $B \geq 0$, then

$$\rho(A \circ B) \leq \rho(A)\rho(B). \quad (2.1)$$

Fang [9] gave an upper bound for $\rho(A \circ B)$, that is,

$$\rho(A \circ B) \leq \max_{1 \leq i \leq n} \left\{ 2a_{ii}b_{ii} + \rho(A)\rho(B) - b_{ii}\rho(A) - a_{ii}\rho(B) \right\}, \quad (2.2)$$

which is sharper than the bound $\rho(A)\rho(B)$ in ([3], p. 358).

Recently, Liu [1] improved the above results, have

$$\rho(A \circ B) \leq \max_{i \neq j} \frac{1}{2} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} + [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 \right. \right.$$

$$\left. + 4(\rho(A) - a_{ii})(\rho(B) - b_{ii})(\rho(A) - a_{jj})(\rho(B) - b_{jj})]^{\frac{1}{2}} \right\}. \quad (2.3)$$

Firstly, we give some lemmas in this section.

**Lemma 2.1 (Perron-Frobenius theorem)([3]).** If $A$ is an irreducible nonnegative matrix, there exist positive vectors $u$, such that $Au = \rho(A)u$.

**Lemma 2.2 ([3]).** If $A, B \in \mathbb{C}^{n \times n}$, $D$ and $E$ are positive diagonal matrices, then

$$D(A \circ B)E = (DAE) \circ B = (DA) \circ (BE) = (AE) \circ (DB) = A \circ (DBE).$$

**Lemma 2.3 (Brauer’s theorem).** Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ ($n \geq 2$), then all the eigenvalues of $A$ lie inside the union of $\frac{n(n-1)}{2}$ ovals of Cassini, i.e.,

$$B(A) = \bigcup_{i,j=1; i \neq j}^n \left\{ z \in \mathbb{C} : |z - a_{ii}||z - a_{jj}| \leq \left( \sum_{k \neq i} |a_{ki}| \right) \left( \sum_{k \neq j} |a_{kj}| \right) \right\}. \quad (2.4)$$

Obviously, if we denote $C = D^{-1}AD$, $D = diag(d_1, d_2, \cdots, d_n)$, $d_i > 0$, then $C$ and $A$ have the same eigenvalues, we obtain that all the eigenvalues of $A$ lie in the region:

$$\bigcup_{i,j=1; i \neq j}^n \left\{ z \in \mathbb{C} : |z - a_{ii}||z - a_{jj}| \leq \left( \sum_{k \neq i} \frac{d_k}{d_i} |a_{ik}| \right) \left( \sum_{k \neq j} \frac{d_i}{d_j} |a_{ji}| \right) \right\}. \quad (2.5)$$

Next, we present a new estimating formula on the upper bound of $\rho(A \circ B)$.

**Theorem 2.1** If $A = (a_{ij})$ and $B = (b_{ij})$ are nonnegative matrices, $s_i = \max_{j \neq i} \{ a_{ij} \}$,
\[ t_i = \max_{j \neq i} \{b_{ij}\}, \text{ then} \]

\[ \rho(A \circ B) \leq \max_{i \neq j} \frac{1}{2} \left\{ a_{ii} b_{ii} + a_{jj} b_{jj} + [(a_{ii} b_{ii} - a_{jj} b_{jj})^2 + 4t_i s_j (\rho(A) - a_{ii})(\rho(B) - b_{jj})]^{1/2} \right\}. \]  \hfill (2.6)

**Proof.** It is evident that the inequality (2.6) holds with the equality for \( n = 1 \). Therefore, we assume that \( n \geq 2 \) and divide two cases to prove this problem.

**Case 1.** Suppose that \( A \circ B \) is irreducible. Obviously \( A \) and \( B \) are also irreducible. By Lemma 2.1, there exists positive vectors \( u = (u_1, u_2, \cdots, u_n) \) and have

\[ (D^{-1}AD)u = \rho(D^{-1}AD)u = \rho(A)u, \]

where \( D = \text{diag}(d_1, d_2, \cdots, d_n), d_i > 0, \) then

\[ \sum_{j \neq i} a_{ij} d_j u_j d_i u_i = \rho(A) - a_{ii}. \]

Define \( U = \text{diag}(u_1, u_2, \cdots, u_n), C = (DU)^{-1}A(DU), \) then we have that

\[ C = \begin{pmatrix} a_{11} & d_{12} u_1 a_{12} & \cdots & d_{1n} u_1 a_{1n} \\ \frac{d_{12} u_1}{d_1 u_2} a_{12} & a_{22} & \cdots & d_{1n} u_2 a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d_{1n} u_1}{d_n u_2} a_{1n} & d_{2n} u_2 a_{2n} & \cdots & a_{nn} \end{pmatrix} \]

is an irreducible nonnegative matrix and

\[ C \circ B = (m_{ij}) = \begin{pmatrix} a_{11} b_{11} & d_{12} u_1 a_{12} b_{12} & \cdots & d_{1n} u_1 a_{1n} b_{1n} \\ \frac{d_{12} u_1}{d_1 u_2} a_{21} b_{21} & a_{22} b_{22} & \cdots & d_{1n} u_2 a_{2n} b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d_{1n} u_1}{d_n u_2} a_{1n} b_{1n} & d_{2n} u_2 a_{2n} b_{2n} & \cdots & a_{nn} b_{nn} \end{pmatrix}. \]

By Lemma 2.2,

\[ (DU)^{-1}(A \circ B)(DU) = (DU)^{-1}A(DU) \circ B = C \circ B, \]

i.e., \( \rho(A \circ B) = \rho(C \circ B). \)

By the inequality (2.4) and \( \rho(A \circ B) \geq a_{ii} b_{ii} \) (see [5]), for any \( j \neq i \in N, \)
we have
\[
(\rho(A \circ B) - a_{ii}b_{ii})(\rho(A \circ B)) - a_{jj}b_{jj}) \leq \sum_{k \neq i} |m_{ik}| \sum_{l \neq j} |m_{jl}|
\]
\[
= \sum_{k \neq i} \frac{d_{ik}a_{kk}b_{kk}}{d_{ii}} \sum_{l \neq j} \frac{d_{lj}a_{jj}b_{jj}}{d_{jj}}
\]
\[
\leq \left( \max_{k \neq i} \{b_{ik}\} \sum_{k \neq i} \frac{d_{ik}a_{kk}}{d_{ii}} \right) \left( \max_{l \neq j} \{a_{jl}\} \sum_{l \neq j} \frac{d_{lj}b_{jj}}{d_{jj}} \right)
\]
\[
\leq \max_{k \neq i} \{b_{ik}\}(\rho(A) - a_{ii}) \max_{l \neq j} \{a_{jl}\}(\rho(B) - b_{jj})
\]
\[
= t_i s_j (\rho(A) - a_{ii}) (\rho(B) - b_{jj})
\]
Thus, by solving the quadratic inequality (2.7), we have that
\[
\rho(A \circ B) \leq \frac{1}{2} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} + [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4t_i s_j (\rho(A) - a_{ii})(\rho(B) - b_{jj})]^{\frac{3}{2}} \right\}
\]
\[
\leq \max_{i \neq j} \frac{1}{2} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} + [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4t_i s_j (\rho(A) - a_{ii})(\rho(B) - b_{jj})]^{\frac{3}{2}} \right\}
\]
i.e., the conclusion (2.6) holds.

**Case 2.** If \( A \circ B \) is reducible. We may denote by \( P = (p_{ij}) \) the \( n \times n \) permutation matrix \( (p_{ij}) \) with
\[
p_{12} = p_{23} = \cdots = p_{n-1,n} = p_{n,1} = 1,
\]
the remaining \( p_{ij} \) zero, then both \( A + \varepsilon P \) and \( B + \varepsilon P \) are nonnegative irreducible matrices for any sufficiently small positive real number \( \varepsilon \). Now we substitute \( A + \varepsilon P \) and \( B + \varepsilon P \) for \( A \) and \( B \), respectively in the previous Case 1, and then letting \( \varepsilon \to 0 \), the result (2.6) follows by continuity. \( \square \)

**Remark 2.1.** Next, we give a comparison between the upper bound in the inequality (2.3) and the upper bound in the inequality (2.6). Without loss of generality, if \( t_i + b_{ii} \geq \rho(B) \), \( s_j + a_{jj} \geq \rho(A) \), \( i, j = 1, \cdots, n \), then we have \( t_is_j \geq (\rho(B) - b_{ii})(\rho(A) - a_{jj}) \).

Thus, the upper bound in the inequality (2.6) is better than the upper bound in the inequality (2.3).

**Example 2.1.** Let \( A \) and \( B \) be the same as in Example 1 from [1]:
\[
A = (a_{ij}) = \begin{pmatrix}
4 & 1 & 0 & 2 \\
1 & 0.05 & 1 & 1 \\
0 & 1 & 4 & 0.5 \\
1 & 0.5 & 0 & 4
\end{pmatrix}, \quad B = (b_{ij}) = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{pmatrix}.
\]

By direct calculation, \( \rho(A \circ B) = 5.7339 \).

According to (2.1), we have
\[
\rho(A \circ B) \leq \rho(A)\rho(B) = 22.9336.
\]

If we apply (2.2) and (2.3), we get
\[
\rho(A \circ B) \leq \max_{1 \leq i \leq 4} \left\{ 2a_{ii}b_{ii} + \rho(A)\rho(B) - a_{ii}\rho(B) - b_{ii}\rho(A) \right\} = 17.1017,
\]
and

\[
\rho(A \circ B) \leq \max_{i \neq j} \frac{1}{2} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} + [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 \\
+ 4(\rho(A) - a_{ii})(\rho(B) - b_{jj})(\rho(A) - a_{jj})(\rho(B) - b_{jj})]^{\frac{1}{2}} \right\} = 11.6478.
\]

If we apply Theorem 2.1, we obtain that

\[
\rho(A \circ B) \leq \max_{i \neq j} \frac{1}{2} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} + [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 \\
+ 4t_i s_j (\rho(A) - a_{ii})(\rho(B) - b_{jj})]^{\frac{1}{2}} \right\} = 8.1897.
\]

The example shows that the bound in Theorem 2.1 is better than the existing bounds. In addition, by the Theorem 2.1 and [1], we also have the following corollary:

**Corollary 2.1** Let \( A \) and \( B \) be nonnegative matrices, then

\[
|\det(A \circ B)| \leq \left( \rho(A \circ B) \right)^n \leq \max_{i \neq j} \frac{1}{2^n} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} + [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 \\
+ 4t_i s_j (\rho(A) - a_{ii})(\rho(B) - b_{jj})]^{\frac{1}{2}} \right\}^n \leq \max_{i \neq j} \frac{1}{2^n} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} + [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 \\
+ 4(\rho(A) - a_{ii})(\rho(B) - b_{ii})(\rho(A) - a_{jj})(\rho(B) - b_{jj})]^{\frac{1}{2}} \right\}^n.
\]

3 **Inequalities for the Fan product of two \( M \)-matrices**

It is known (p.359, [3]) that the following classical result is given: if \( A, B \in \mathbb{R}^{n \times n} \) are \( M \)-matrices, then

\[
\tau(A \star B) \geq \tau(A)\tau(B). \tag{3.1}
\]

In 2007, Fang improved (3.1) in the Remark 3 of Ref. [9] and gave a new lower bound for \( \tau(A \star B) \), that is

\[
\tau(A \star B) \geq \min_{1 \leq i \leq n} \left\{ b_{ii}\tau(A) + a_{ii}\tau(B) - \tau(A)\tau(B) \right\}. \tag{3.2}
\]

Subsequently, Liu et al.[1] gave a sharper bound than (3.2), i.e.,

\[
\tau(A \star B) \geq \frac{1}{2} \min_{i \neq j} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} - [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 \\
+ 4(b_{ii} - \tau(B))(a_{ii} - \tau(A))(b_{jj} - \tau(B))(a_{jj} - \tau(A))]^{\frac{1}{2}} \right\}. \tag{3.3}
\]

In addition, by the definition of Fan product, the following lemma holds:
Lemma 3.1 ([1]). If \( A, B \in \mathbb{C}^{n \times n} \) be nonsingular M-matrices, \( D \) and \( E \) are positive diagonal matrices, then
\[
D(A \ast B)E = (DAE) \ast B = (DA) \ast (BE) = (AE) \ast (DB) = A \ast (DBE).
\]

Next, we give a new lower bound on the minimum eigenvalue \( \tau(A \ast B) \) of the Fan product of nonsingular M-matrices.

**Theorem 3.1** If \( A = (a_{ij}) \) and \( B = (b_{ij}) \) are nonsingular M-matrices, \( s_i = \max_{j \neq i} |a_{ij}| \), \( t_i = \max_{j \neq i} |b_{ij}| \), then
\[
\tau(A \ast B) \geq \min_{i \neq j} \frac{1}{2} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} - [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4t_i s_j (a_{ii} - \tau(A))(b_{jj} - \tau(B))]^{\frac{1}{2}} \right\}. \tag{3.4}
\]

**Proof.** It is clear that the (3.4) holds with the equality for \( n = 1 \).

We next assume \( n \geq 2 \) and divide two cases to prove this problem.

**Case 1.** Suppose that \( A \ast B \) is irreducible. Obviously \( A \) and \( B \) are also irreducible. By [5], there exists positive vectors \( u = (u_1, u_2, \cdots, u_n) \) such that
\[
(D^{-1}AD)u = \tau(D^{-1}AD)u = \tau(A)u,
\]
where \( D = \text{diag}(d_1, d_2, \cdots, d_n), d_i > 0 \), and then
\[
a_{ii} - \sum_{j \neq i} \frac{|a_{ij}|d_j u_j}{d_i u_i} = \tau(A).
\]

Define \( U = \text{diag}(u_1, u_2, \cdots, u_n), C = (DU)^{-1}A(DU), \) we have that
\[
C = \begin{pmatrix}
    a_{11} & \frac{d_2 u_2}{d_1 u_1} a_{12} & \cdots & \frac{d_n u_n}{d_1 u_1} a_{1n} \\
    \frac{d_1 u_1}{d_2 u_2} a_{21} & a_{22} & \cdots & \frac{d_n u_n}{d_2 u_2} a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    \frac{d_1 u_1}{d_n u_n} a_{n1} & \frac{d_2 u_2}{d_n u_n} a_{n2} & \cdots & a_{nn}
\end{pmatrix}
\]
is an irreducible nonsingular \( M \)-matrix, then
\[
C \ast B = (m_{ij}) = \begin{pmatrix}
    a_{11}b_{11} & \frac{d_2 u_2}{d_1 u_1} a_{12} b_{12} & \cdots & \frac{d_n u_n}{d_1 u_1} a_{1n} b_{1n} \\
    \frac{d_1 u_1}{d_2 u_2} a_{21} b_{21} & a_{22} b_{22} & \cdots & \frac{d_n u_n}{d_2 u_2} a_{2n} b_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    \frac{d_1 u_1}{d_n u_n} a_{n1} b_{n1} & \frac{d_2 u_2}{d_n u_n} a_{n2} b_{n2} & \cdots & a_{nn} b_{nn}
\end{pmatrix}.
\]

By the Lemma 3.1,
\[
(DU)^{-1}(A \ast B)(DU) = (DU)^{-1}A(DU) \ast B = C \ast B,
\]
i.e., $\tau(A \star B) = \tau(C \star B)$.

In addition, by the inequality (2.4) and $0 \leq \tau(A \star B) \leq a_{ii}b_{ii}$ (see [5]), for any $j \neq i \in N$, we have

$$|\tau(A \star B) - a_{ii}b_{ii}||\tau(A \star B) - a_{jj}b_{jj}| \leq \sum_{k \neq i} |m_{ik}| \sum_{l \neq j} |m_{jl}|$$

$$= \sum_{k \neq i} \left| \frac{d_{ik}u_k}{d_{ii}} \right| \sum_{l \neq j} \left| \frac{d_{lj}u_l}{d_{jj}} \right|$$

$$\leq \left( \max_{k \neq i} \left| b_{ik} \right| \sum_{k \neq i} \left| \frac{d_{ik}u_k}{d_{ii}} \right| \right) \left( \max_{l \neq j} \left| a_{jl} \right| \sum_{l \neq j} \left| \frac{d_{lj}u_l}{d_{jj}} \right| \right)$$

$$\leq \max_{k \neq i} \left| b_{ik} \right| (a_{ii} - \tau(A)) \max_{l \neq j} \left| a_{jl} \right| (b_{jj} - \tau(B))$$

Thus, by solving the quadratic inequality (3.5), we have that

$$\tau(A \star B) \geq \frac{1}{2} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} - [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4t_i s_j (a_{ii} - \tau(A))(b_{jj} - \tau(B))]^{\frac{1}{2}} \right\}$$

$$\geq \min_{i \neq j} \frac{1}{2} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} - [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4t_i s_j (a_{ii} - \tau(A))(b_{jj} - \tau(B))]^{\frac{1}{2}} \right\}.$$

i.e., the conclusion (3.4) holds.

**Case 2.** If $A \star B$ is reducible. It is well known that a matrix in $Z_n$ is a nonsingular $M$-matrix if and only if all its leading principal minors are positive (see condition (E17) of Theorem 6.2.3 of [5]). We denote by $P = (p_{ij})$ the $n \times n$ permutation matrix with $p_{12} = p_{23} = \cdots = p_{n-1,n} = p_{n,1} = 1$, the remaining $p_{ij}$ zero, then both $A - \varepsilon P$ and $B - \varepsilon P$ are irreducible nonsingular $M$-matrices for any sufficiently small positive real number $\varepsilon$. Now we substitute $A - \varepsilon P$ and $B - \varepsilon P$ for $A$ and $B$, respectively in the previous Case 1, and then letting $\varepsilon \to 0$, the result (3.4) follows by continuity. $\square$

**Remark 3.1.** Similarly, we give a comparison between the lower bound in the inequality (3.3) and the lower bound in the inequality (3.4). If $a_{jj} \geq \tau(A) + s_j$, $b_{ii} \geq \tau(B) + t_i$, $i, j = 1, \cdots, n$, then $(a_{jj} - \tau(A))(b_{ii} - \tau(B)) \geq s_j t_i$ for all $i \neq j$. Thus, the lower bound in the inequality (3.4) is better than the lower bound in the inequality (3.3).

In addition, from Theorem 3.1 and [5], we may get the following corollary.

**Corollary 3.1.** If $A$, $B$ are nonsingular $M$-matrices, then

$$|\det(A \star B)| \geq \left( \tau(A \star B) \right)^n$$

$$\geq \min_{i \neq j} \frac{1}{2^n} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} - [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4t_i s_j (a_{ii} - \tau(A))(b_{jj} - \tau(B))]^{\frac{1}{2}} \right\}^n$$

$$\geq \min_{i \neq j} \frac{1}{2^n} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} - [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4(a_{ii} - \tau(A))(b_{jj} - \tau(B))(a_{jj} - \tau(A))(b_{jj} - \tau(B))]^{\frac{1}{2}} \right\}^n.$$

**Example 3.1 ([1]).** Let $A$ and $B$ be the nonsingular $M$-matrices:
\[ A = (a_{ij}) = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & -0.5 \\ -0.5 & -1 & 2 \end{pmatrix}, \quad B = (b_{ij}) = \begin{pmatrix} 1 & -0.25 & -0.25 \\ -0.5 & 1 & -0.25 \\ -0.25 & -0.5 & 1 \end{pmatrix}. \]

By (3.1), we have
\[ \tau(A \ast B) \geq \tau(A)\tau(B) = 0.1854. \]

If we use the inequalities (3.2) and (3.3), then we get
\[ \tau(A \ast B) \geq \min_{1 \leq i \leq 3} \left\{ a_{ii} \tau(B) + b_{ii} \tau(A) - \tau(A)\tau(B) \right\} = 0.6980, \]
and
\[ \tau(A \ast B) \geq \min_{i \neq j} \left\{ \frac{1}{2} \left( a_{ii}b_{ii} + a_{jj}b_{jj} - [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4(a_{ii} - \tau(A))(b_{ii} - \tau(B))(a_{jj} - \tau(A))(b_{jj} - \tau(B)) \right)^{\frac{1}{2}} \right\} = 0.7655. \]

If we apply Theorem 3.1, we obtain that
\[ \tau(A \ast B) \geq \min_{i \neq j} \left\{ \frac{1}{2} \left( a_{ii}b_{ii} + a_{jj}b_{jj} - [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4t_is_j(a_{ii} - \tau(A))(b_{jj} - \tau(B)) \right)^{\frac{1}{2}} \right\} = 0.8002. \]

In fact, \( \tau(A \ast B) = 0.8819. \) The example shows that the bound in Theorem 3.1 is better than the existing bounds.

### 4 A bound for the Hadamard product of M-matrix and an inverse M-matrix

Now, we consider the lower bound of \( \tau(A \circ B^{-1}) \), for \( A = (a_{ij}), B = (b_{ij}) \in \mathcal{M}_n \) and \( B^{-1} = (\beta_{ij}) \).

Firstly, in [3], Horn and Johnson gave the classical results
\[ \tau(A \circ B^{-1}) \geq \tau(A) \min_{1 \leq i \leq n} \beta_{ii}. \quad (4.1) \]

Subsequently, Huang [8] gave new bound for \( \tau(A \circ B^{-1}) \), that is,
\[ \tau(A \circ B^{-1}) \geq \frac{1 - \rho(J_A)\rho(J_B)}{1 + \rho^2(J_B)} \min_{1 \leq i \leq n} \frac{a_{ii}}{b_{ii}}, \quad (4.2) \]

where \( \rho(J_A) \) and \( \rho(J_B) \) are the spectral radius of the Jacobi iterative matrices \( J_A \) and \( J_B \), respectively.
In 2008, Li [10] improved the above results as follows.

\[ \tau(A \circ B^{-1}) \geq \min_i \frac{b_{ii} - s_i \sum_{j \neq i} |b_{ji}|}{a_{ii}}. \]  

(4.3)

Recently, Chen [11] improved the result and gave a new lower bound for \( \tau(A \circ B^{-1}) \):

\[ \tau(A \circ B^{-1}) \geq \min_{i \neq j} \frac{1}{2} \left\{ a_{ii} \beta_{ii} + a_{jj} \beta_{jj} - \left[ (a_{ii} \beta_{ii} - a_{jj} \beta_{jj})^2 + 4a_{ii} a_{jj} \beta_{ii} \beta_{jj} \rho^2 (J_A \rho^2 (J_B))^\frac{1}{2} \right] \right\}. \]  

(4.4)

In this section, we give a lower bound of \( \tau(A \circ B^{-1}) \) for \( M \)-matrix and inverse \( M \)-matrix, which improves the above bounds.

**Lemma 4.1** ([12]). If \( A = (a_{ij}) \in M_n \), there exists a positive diagonal matrix \( D \) such that \( D^{-1}AD \) is a strictly row diagonally dominant \( M \)-matrix.

**Lemma 4.2** ([12]). If \( A = (a_{ij}) \in M_n \), and \( D = \text{diag}(d_1, d_2, \ldots, d_n) \), \( d_i > 0 \ (i \in N) \), then \( D^{-1}AD \) is also an \( M \)-matrix.

**Lemma 4.3** ([12]). If \( A, B \in M_n \), then \( B \circ A^{-1} \) is also an \( M \)-matrix.

**Lemma 4.4** ([10]). If \( A = (a_{ij}) \) be a strictly diagonally dominant \( M \)-matrix by rows, then for \( A^{-1} = (\alpha_{ij}) \), we have

\[ \alpha_{ji} \leq \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{kj}| r_k}{a_{jj}} \alpha_{ii}, \quad \text{for all } j \neq i. \]

**Theorem 4.1** If \( A = (a_{ij}) \) and \( B = (b_{ij}) \) are two nonsingular \( M \)-matrices and \( B^{-1} = (\beta_{ij}) \), \( s_i = \max_{j \neq i} |a_{ij}| \), then

\[ \tau(A \circ B^{-1}) \geq \min_{i \neq j} \frac{1}{2} \left\{ a_{ii} \beta_{ii} + a_{jj} \beta_{jj} - \left[ (a_{ii} \beta_{ii} - a_{jj} \beta_{jj})^2 + 4s_i s_j \beta_{ii} \beta_{jj} (a_{ii} - \tau(A)) (b_{jj} - \tau(B)) \right] \right\}. \]  

(4.5)

**Proof.** If \( A \) is an \( M \)-matrix, by Lemmas (4.1-4.2), there exists a positive diagonal matrix \( D \) such that \( D^{-1}AD \) is a strictly diagonally dominant \( M \)-matrix by rows.

**Case 1.** Suppose that \( A \circ B^{-1} \) is irreducible. Obviously \( A \) and \( B \) are also irreducible. Since \( A - \tau(A)I \) is an irreducible nonsingular \( M \)-matrix, then \( a_{ii} - \tau(A) > 0, \forall i \in N \), and there exists a positive vector \( u = (u_1, u_2, \ldots, u_n) \) such that

\[ Au = \tau(A)u, \]

where \( u = \text{diag}(u_1, u_2, \ldots, u_n) \), \( u_i > 0 \), and then

\[ a_{ii} + \sum_{j \neq i} \frac{a_{ji} u_j}{u_i} = \tau(A). \]
Define $U = \text{diag}(u_1, u_2, \cdots, u_n)$, $C = U^{-1}AU$, then we have that

$$
C = (\bar{a}_{ij}) = U^{-1}AU = 
\begin{pmatrix}
\frac{a_{11}}{u_1} & \frac{a_{12}u_2}{u_1} & \cdots & \frac{a_{1n}u_n}{u_1} \\
\frac{a_{21}u_2}{u_1} & \frac{a_{22}}{u_1} & \cdots & \frac{a_{2n}u_n}{u_1} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{a_{n1}u_n}{u_1} & \frac{a_{n2}u_2}{u_1} & \cdots & \frac{a_{nn}}{u_1}
\end{pmatrix}
$$

is an irreducible nonsingular $M$-matrix.

By Lemma 2.2,

$$
U^{-1}(A \circ B^{-1})U = (U^{-1}AU) \circ B^{-1} = C \circ B^{-1},
$$
i.e., $\tau(A \circ B^{-1}) = \tau(C \circ B^{-1})$.

By the inequality (2.4) and $0 \leq \tau(A \ast B) \leq a_{ii}b_{ii}$ (see [5]), for any $j \neq i \in N$, we have

$$
|\tau(A \circ B^{-1}) - a_{ii}b_{ii}| = \sum_{k \neq i} |\bar{a}_{ki}| \sum_{l \neq j} |\bar{a}_{lj}| |\beta_{kj} - \tau(A)|
\leq \sum_{k \neq i} |\bar{a}_{ki}| |\beta_{ki}| \sum_{l \neq j} |\bar{a}_{lj}| |\beta_{lj}|
\leq \sum_{k \neq i} |\bar{a}_{ki}||s_{ki}| |\beta_{jj}| \sum_{l \neq j} |\bar{a}_{lj}|s_{lj}|\beta_{jj}|
\leq \sum_{k \neq i} |\bar{a}_{ki}||s_{ki}| |\beta_{jj}| \sum_{l \neq j} |\bar{a}_{lj}|s_{lj}|\beta_{jj}|
= s_is_j\beta_{ii}\beta_{jj}(a_{ii} - \tau(A))(a_{jj} - \tau(A)).
$$

Thus, by solving the quadratic inequality (4.6), we obtain that

$$
\tau(A \circ B^{-1}) \geq \min_{i \neq j} \left\{ \frac{1}{2} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} - [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4s_is_j\beta_{ii}\beta_{jj}(a_{ii} - \tau(A))(a_{jj} - \tau(A))]^{\frac{1}{2}} \right\} \right\}
\geq \min_{i \neq j} \left\{ \frac{1}{2} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} - [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4s_is_j\beta_{ii}\beta_{jj}(a_{ii} - \tau(A))(a_{jj} - \tau(A))]^{\frac{1}{2}} \right\} \right\}.
$$
i.e., the conclusion (4.5) holds.

**Case 2.** If $A \circ B^{-1}$ is reducible, then one denotes by $P = (p_{ij})$ the $n \times n$ permutation matrix with

$$
p_{12} = p_{23} = \cdots = p_{n-1,n} = p_{n,1} = 1,
$$
the remaining $p_{ij}$ zero, then both $A - \varepsilon P$ and $B - \varepsilon P$ are irreducible nonsingular $M$-matrices for any sufficiently small positive real number $\varepsilon$. Now we substitute $A - \varepsilon P$ and $B - \varepsilon P$ for $A$ and $B$, respectively from the previous Case, and then letting $\varepsilon \to 0$, the result (2.6) follows by continuity. □

**Example 4.1** ([11]). Let $A$ and $B$ be nonsingular $M$-matrices:
\[ A = (a_{ij}) = \begin{pmatrix} 1 & -0.5 & 0 & 0 \\ -0.5 & 1 & -0.5 & 0 \\ 0 & -0.5 & 1 & -0.5 \\ 0 & 0 & -0.5 & 1 \end{pmatrix}, \quad B = (b_{ij}) = \begin{pmatrix} 4 & -1 & -1 & -1 \\ -2 & 5 & -1 & -1 \\ 0 & -2 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{pmatrix}. \]

By direct calculation, \( \tau(A \circ B^{-1}) = 0.2148. \)

According to (4.1), we have

\[ \tau(A \circ B^{-1}) \geq \tau(A) \min_{1 \leq i \leq n} \beta_{ii} = 0.07. \]

If we apply (4.2) and (4.3), we get

\[ \tau(A \circ B^{-1}) \geq \frac{1 - \rho(J_A \rho(J_B))}{1 + \rho^2(J_B)} \min_i \frac{b_{ii}}{a_{ii}} = 0.0707, \]

and

\[ \tau(A \circ B^{-1}) \geq \min_i \frac{b_{ii} - s_i \sum_{j \not= i} |b_{ji}|}{a_{ii}} = 0.08. \]

According to (4.4)

\[ \tau(A \circ B^{-1}) \geq \min_{i \not= j} \frac{1}{2} \left\{ a_{ii} \beta_{ii} + a_{jj} \beta_{jj} - [(a_{ii} \beta_{ii} - a_{jj} \beta_{jj})^2 + 4a_{ii}a_{jj} \beta_{ii} \beta_{jj} \rho^2(J_A)\rho^2(J_B)]^{1/2} \right\} = 0.1524. \]

If we apply Theorem 4.1, we obtain that

\[ \tau(A \circ B^{-1}) \geq \min_{i \not= j} \frac{1}{2} \left\{ a_{ii} \beta_{ii} + a_{jj} \beta_{jj} + [(a_{ii} \beta_{ii} - a_{jj} \beta_{jj})^2 + 4s_i s_j \beta_{ii} \beta_{jj} (a_{ii} - \tau(A))(a_{jj} - \tau(A))]^{1/2} \right\} = 0.1929. \]

The example shows that the bound in Theorem 4.1 is better than the existing bounds.

5 **Inequalities for the Fan product of several \( M \)-matrices**

Firstly, let us recall the following lemmas.

**Lemma 5.1 ([7]).** Let \( A \) be an irreducible nonsingular \( M \)-matrix, if \( AZ \geq kZ \) for a nonegative nonzero vector \( Z \), then \( k \leq \tau(A) \).
Lemma 5.2 ([6]). Let $x_j = (x_j(1), \ldots, x_j(n))^T \geq 0$, $j \in \{1, 2, \ldots, m\}$, if $P_j > 0$ and $
abla \frac{1}{P_k} \geq 1$, then we have

$$\sum_{i=1}^{n} \prod_{j=1}^{m} x_j(i) \leq \prod_{i=1}^{n} \left\{ \sum_{j=1}^{m} [x_j(i)]^{P_j} \right\}^{\frac{1}{P_j}}.$$  \hspace{1cm} (5.1)

Next, according to these results, we expand the inequality (3.2) of the Fan product of two matrices to the Fan product of several matrices. One can obtain the following result:

Theorem 5.1 For any matrices $A_k \in M_n$, and positive integers $P_k$ with $\sum_{k=1}^{m} \frac{1}{P_k} \geq 1$, $k \in \{1, 2, \ldots, m\}$, we have that

$$\tau(A_1 \ast A_2 \cdots \ast A_m) \geq \min_{1 \leq i \leq n} \left\{ \prod_{k=1}^{m} A_k(i, i) - \prod_{k=1}^{m} [A_k(i, i)^{P_k} - \tau(A_k^{(P_k)})]^{\frac{1}{P_k}} \right\}. \hspace{1cm} (5.2)$$

Proof. It is quite evident that the (5.2) holds with the equality for $n = 1$. Below we assume that $n \geq 2$.

Case 1. Let $A_1 \ast A_2 \cdots \ast A_m$ be an irreducible nonsingular $M$-matrix, thus $A_k$ is irreducible, $k \in \{1, 2, \ldots, m\}$, we can obtain that $A_k^{(P_k)}$ is also irreducible. Let $u_k^{(P_k)} = (u_k(1)^{P_k}, \ldots, u_k(n)^{P_k})^T > 0$ be a right Perron eigenvector of $A_k^{(P_k)}$, and $u_k = (u_k(1), \ldots, u_k(n))^T > 0$, thus for any $i \in N$, we have that

$$A_k^{(P_k)} u_k^{(P_k)} = \tau(A_k^{(P_k)}) u_k^{(P_k)},$$

$$A_k(i, i)^{P_k} u_k^{(P_k)} = \sum_{j \neq i} [A_k(i, j)^{P_k} u_k^{(P_k)}] = \tau(A_k^{(P_k)}) u_k^{(P_k)},$$

and

$$\sum_{j \neq i} [A_k(i, j)^{P_k} u_k^{(P_k)}] = (A_k(i, i)^{P_k} - \tau(A_k^{(P_k)})) u_k^{(P_k)}. \hspace{1cm} (5.3)$$

Denote $C = A_1 \ast A_2 \cdots \ast A_m$, $Z = u_1^T u_2 \cdots u_m = (Z(1), \ldots, Z(n))^T > 0$, thus $Z(i) = \prod_{k=1}^{m} u_k(i)$. By the Lemma 5.2 and (5.3), we get that

$$(CZ)_i = \left( \prod_{k=1}^{m} A_k(i, i) \right) Z(i) - \left( \sum_{j \neq i} \prod_{k=1}^{m} [A_k(i, j)] \right) Z(j)$$

$$= \left( \prod_{k=1}^{m} A_k(i, i) \right) Z(i) - \sum_{j \neq i} \prod_{k=1}^{m} [A_k(i, j)] Z(j)$$

$$\geq \left( \prod_{k=1}^{m} A_k(i, i) \right) Z(i) - \prod_{k=1}^{m} \left\{ \sum_{j \neq i} [A_k(i, j)] Z(j) \right\}^{\frac{1}{P_k}} \quad \text{(by the equality (5.3))}$$

$$= \left( \prod_{k=1}^{m} A_k(i, i) \right) Z(i) - \prod_{k=1}^{m} \left\{ [A_k(i, i)^{P_k} - \tau(A_k^{(P_k)})] u_k^{(P_k)} \right\}^{\frac{1}{P_k}}$$

$$= \left( \prod_{k=1}^{m} A_k(i, i) - \prod_{k=1}^{m} [A_k(i, i)^{P_k} - \tau(A_k^{(P_k)})] \right)^{\frac{1}{P_k}} Z(i).$$

According to the Lemma 5.1, we obtain that

$$\tau(A_1 \ast A_2 \cdots \ast A_m) \geq \min_{1 \leq i \leq n} \left\{ \prod_{k=1}^{m} A_k(i, i) - \prod_{k=1}^{m} [A_k(i, i)^{P_k} - \tau(A_k^{(P_k)})]^{\frac{1}{P_k}} \right\}. \hspace{1cm} \Box$$
Case 2. If $A_1 \star A_2 \cdots \star A_m$ is reducible, where $A_i$ ($i = 1, 2, \cdots, m$) are nonsingular $M$-matrices. Similarly, let $P = (p_{ij})$ be the $n \times n$ permutation matrix with $p_{12} = p_{21} = \cdots = p_{n-1,n} = p_{n,1} = 1$, the remaining $p_{ij}$ zero, then $A_k - \varepsilon P$ is an irreducible nonsingular $M$-matrix for any chosen positive real number $\varepsilon$. Note that $A_k - \varepsilon P$ is a continuous function on $\varepsilon$. Now we substitute $A_k - \varepsilon P$ for $A_k$, in the previous Case 1, and then letting $\varepsilon \to 0$, the result (5.2) follows by continuity. □

Remark 4.1. If we take $m = 2$ in Theorem 4.1, one can obtain the following results:

- If $p_1 = p_2 = 1$, $A_1 = A = (a_{ij})$, $A_2 = B = (b_{ij})$, we have that
  \[ \tau(A \star B) \geq \min_{1 \leq i \leq n} \left\{ a_{ii}b_{ii} - (a_{ii} - \tau(A))(b_{ii} - \tau(B)) \right\}, \]
  which is just the inequality (3.2).

- If $p_1 = p_2 = 2$, $A_1 = A = (a_{ij})$, $A_2 = B = (b_{ij})$, then
  \[ \tau(A \star B) \geq \min_{1 \leq i \leq n} \left\{ a_{ii}b_{ii} - [a_{ii}^2 - \tau(A \star A)]\frac{1}{2}[b_{ii}^2 - \tau(B \star B)]\frac{3}{2} \right\}. \tag{5.4} \]

In addition, by using the inequalities of arithmetic and geometric means, we may obtain that
\[ a_{ii}^2\tau(B \star B) + b_{ii}^2\tau(A \star A) \geq 2a_{ii}b_{ii}[\tau(A \star A)\tau(B \star B)]\frac{1}{2}, \]
so
\[ (a_{ii}^2 - \tau(A \star A))(b_{ii}^2 - \tau(B \star B)) \leq \left\{ a_{ii}b_{ii} - [\tau(A \star A)\tau(B \star B)]\frac{1}{2} \right\}^2. \tag{5.5} \]

Since for any $A, B \in M_n$, $\tau(A \star B) \geq \tau(A)\tau(B)$ (see [1] or (3.1)), then, by (5.5), we have that
\[ a_{ii}b_{ii} - [(a_{ii}^2 - \tau(A \star A))(b_{ii}^2 - \tau(B \star B))]^{\frac{3}{2}} \geq [\tau(A \star A)\tau(B \star B)]^{\frac{3}{2}} \geq \tau(A)\tau(B). \]

That is, the bound in (5.2) is better than the bound in (3.1).

- If $p_1 = 1, p_2 = 2$, $A_1 = A = (a_{ij})$, $A_2 = B = (b_{ij})$, then we get
  \[ \tau(A \star B) \geq \min_{1 \leq i \leq n} \left\{ a_{ii}b_{ii} - [a_{ii} - \tau(A)][b_{ii}^2 - \tau(B \star B)]\frac{1}{2} \right\}. \]

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