W-GRAPH IDEALS

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Abstract. We introduce a concept of a W-graph ideal in a Coxeter group. The main goal of this paper is to describe how to construct a W-graph from a given W-graph ideal. The principal application of this idea is in type A, where it provides an algorithm for the construction of W-graphs for Specht modules.

1. Introduction

Let \((W, S)\) be a Coxeter system and \(\mathcal{H}(W)\) its Hecke algebra over \(\mathbb{Z}[q, q^{-1}]\), the ring of Laurent polynomials in the indeterminate \(q\). There are certain representations of \(\mathcal{H}(W)\) whose structure can be encoded by combinatorial objects called W-graphs, introduced by Kazhdan and Lusztig in [10]. A W-graph provides a compact way of providing all the information needed to construct the representation. Moreover, from the work of Gyoja, [6], it is known that if \(W\) is a finite Weyl group then all irreducible \(\mathcal{H}(W)\)-modules can be realized as modules carried by W-graphs. However, the problem of explicitly describing these W-graphs is not completely solved.

In [10] Kazhdan and Lusztig constructed a special basis for \(\mathcal{H}(W)\), using a family of polynomials in \(q\) with integer coefficients. These polynomials, now known as the Kazhdan-Lusztig polynomials, are parametrized by pairs of elements of \(W\), and are defined by a recursive procedure. The Kazhdan-Lusztig basis gives the regular representation of \(\mathcal{H}(W)\) a W-graph structure. Moreover, Kazhdan and Lusztig showed that W-graphs may be split into cells, which are themselves W-graphs, thus potentially providing a means of decomposing the regular representation. In type A the cells in the regular W-graph yield irreducible representations; however, constructing W-graphs for the irreducible representations has to date been computationally challenging because of the large number of Kazhdan-Lusztig polynomials that must be calculated.

In [3] Deodhar gave a generalization of the Kazhdan-Lusztig construction, using parabolic Kazhdan-Lusztig polynomials relative to a standard parabolic subgroup \(W'\) to give W-graph structures to \(\mathcal{H}(W')\)-modules induced from certain one-dimensional \(\mathcal{H}(W')\)-modules. This raises the question whether W-graphs for other classes of representations may be constructed similarly, and to do so is one of the main objectives of our project. We introduce the concept of a W-graph ideal in \((W; \leq_L)\) (where \(\leq_L\) is the the partial order such that \(u \leq_L v\) if and only if \(l(vu^{-1}) = l(v) - l(u)\)) and give a Kazhdan-Lusztig like algorithm to produce, for any such ideal \(\mathcal{I}\), a W-graph with vertices indexed by the elements of \(\mathcal{I}\).

Our main focus is on \(\mathcal{H}(W_n)\), the Hecke algebra of type \(A_{n-1}\). Of course in this case the Weyl group, \(W_n\), is isomorphic to the symmetric group of degree \(n\), and its representation theory (and that of \(\mathcal{H}(W_n)\)) is deeply connected with the combinatorics of tableaux. The irreducibles are parametrized by partitions of \(n\), and for each
Let \((W, S)\) be a Coxeter system. Thus \(W\) is a group generated by a set \(S\) subject to defining relations of the form

\[(ss')^m(s, s') = 1 \quad \text{for all } s, s' \in S\]

where \(m(s, s') = m(s', s)\) is a positive integer or \(\infty\) and \(m(s, s') = 1\) if and only if \(s = s'\). (A relation \((ss')^\infty = 1\) is regarded as vacuously true.) Elements of \(S\) are called simple reflections, and the cardinality of \(S\) is called the rank of the system. It turns out that in all cases that \(m(s, s')\) equals the order of \(ss'\) in \(W\).

Let \(l\) be the length function defined on \(W\); that is, if \(w \in W\) then \(l(w)\) is the minimal \(k\) such that \(w = s_1s_2 \cdots s_k\) for some elements \(s_1, s_2, \ldots, s_k \in S\). If \(w = s_1s_2 \cdots s_k\) and \(l(w) = k\), then \(s_1s_2 \cdots s_k\) is said to be a reduced expression for \(w\). If \(W\) is finite then there is a unique longest element in \(W\); we shall denote it by \(w_S\).

Define \(T = \{ w^{-1}sw \mid s \in S, w \in W \}\) (the set of reflections in \(W\)). The following partial orders are defined on \(W\).

**Definition 2.1** (Bruhat order). The Bruhat order \(\leq\) is the transitive closure of the relation \(\rightarrow\) given by \(u \rightarrow w\) if \(l(u) \leq l(w)\) and \(w = tu\) for some \(t \in T \cup \{1\}\).

**Definition 2.2** (Weak order). The left weak order \(\leq_L\) is the transitive closure of the relation \(\rightarrow\) given by \(u \rightarrow w\) if \(l(u) \leq l(w)\) and \(w = su\) for some \(s \in S \cup \{1\}\). If \(u \leq_L w\), we say that \(u\) is a suffix of \(w\).
Observe that \( u \leq_L w \) implies \( u \leq w \). It is well known that if \( W \) is finite then \( u \leq_L w_S \) for all \( u \in W \), where \( w_S \) is the maximal length element of \( W \).

We shall employ the customary conventions that \( w \geq u \) means the same thing as \( u \leq w \) and that \( u < w \) means \( u \leq w \) and \( u \neq w \), and so forth.

The following property of the Bruhat order is standard (see [9, Section 7.4]).

**Lemma 2.3.** Let \( s \in S \) and \( u, w \in W \) satisfy \( u < sw \) and \( w < sw \). Then \( u \leq w \) if and only if \( u \leq sw \), and \( u \leq w \) if and only if \( su \leq sw \).

Let \( J \) be an arbitrary subset of \( S \) and \( W_J \) the subgroup of \( W \) generated by \( J \); such subgroups are called standard parabolic subgroups of \( W \). It can be shown that \((W_J, J)\) is a Coxeter system. The length function on \( W_J \) relative to the generating set \( J \) coincides with the restriction of the length function on \( W \) (see [9, Section 5.5]), and the Bruhat and weak orders on \( W_J \) are the restrictions of the corresponding orders on \( W \) (see [9, Section 5.10]). Each left coset of \( W_J \) in \( W \) contains a unique element of \( D_J = \{ w \in W \mid l(ws) > l(w) \text{ for all } s \in J \} \), and \( l(du) = l(d) + l(u) \) for all \( u \in W_J \) and \( d \in D_J \). The set \( D_J \) is called the set of distinguished (or minimal) left coset representatives in \( W \) for the subgroup \( W_J \) (see [9, Section 1.10]). If \( W_J \) is finite then we denote the longest element of \( W_J \) by \( w_J \). If \( W \) is finite then we let \( d_J \) be the unique element in \( D_J \cap w_S W_J \); then \( D_J = \{ w \in W \mid w \leq_L d_J \} \) (see [3, Lemma 2.2.1]).

**Lemma 2.4.** [3, Lemma 2.1 (iii)] Let \( J \subseteq S \). For each \( s \in S \) and each \( w \in D_J \), exactly one of the following occurs:

(i) \( l(sw) < l(w) \) and \( sw \in D_J \);
(ii) \( l(sw) > l(w) \) and \( sw \notin D_J \);
(iii) \( l(sw) > l(w) \) and \( sw \notin D_J \), and \( w^{-1}sw \in J \).

We shall find it convenient to make use of the following definition.

**Definition 2.5.** If \( X \subseteq W \), let \( \text{Pos}(X) = \{ s \in S \mid l(xs) > l(x) \text{ for all } x \in X \} \).

Thus \( \text{Pos}(X) \) is the largest subset \( J \) of \( S \) such that \( X \subseteq D_J \).

3. HECKE ALGEBRAS

Let \( A = \mathbb{Z}[q, q^{-1}] \), the ring of Laurent polynomials with integer coefficients in the indeterminate \( q \), and let \( A^+ = \mathbb{Z}[q] \). Let \((W, S)\) be a Coxeter system. Then the corresponding Hecke algebra, denoted \( \mathcal{H}(W) \), is the associative algebra over \( A \) generated by the elements \( \{ T_s \mid s \in S \} \) subject to the following defining relations:

\[
T_s^2 = 1 + (q - q^{-1})T_s \quad \text{for all } s \in S,
\]

\[
T_sT_{s'} = T_{s'}T_s \quad \text{for all } s, s' \in S,
\]

where in the second of these there are \( m(s, s') \) factors on each side, \( m(s, s') \) being the order of \( ss' \) in \( W \). We remark that the traditional definition has \( T_s^2 = q + (q - 1)T_s \) in place of the first relation above; our version is obtained by replacing \( q \) by \( q^2 \) and dividing the generators by \( q \).

It is well known that \( \mathcal{H}(W) \) is \( A \)-free with an \( A \)-basis \( \{ T_w \mid w \in W \} \) and multiplication satisfying

\[
T_sT_w = \begin{cases} 
T_{sw} & \text{if } l(sw) > l(w), \\
T_{sw} + (q - q^{-1})T_w & \text{if } l(sw) < l(w),
\end{cases}
\]

for all \( s \in S \) and \( w \in W \).
If $J \subseteq S$ then $\mathcal{H}(W_J)$, the Hecke algebra associated with the Coxeter system $(W_J, J)$, is isomorphic to the subalgebra of $\mathcal{H}(W)$ generated by $\{ T_s \mid s \in J \}$. We shall identify $\mathcal{H}(W_J)$ with this subalgebra.

4. $W$-GRAPHS

Let $\mathcal{H} = \mathcal{H}(W)$ be the Hecke algebra associated with the Coxeter system $(W, S)$. Let $a \mapsto \overline{a}$ be the involutory automorphism of $A = \mathbb{Z}[q, q^{-1}]$ defined by $q \mapsto q^{-1}$. This extends to an involution on $\mathcal{H}$ satisfying

$$T_s = T_s^{-1} = T_s - (q - q^{-1}) \quad \text{for all } s \in S.$$

A $W$-graph is a triple $(V, \mu, \tau)$ consisting of a set $V$, a function $\mu: V \times V \to \mathbb{Z}$ and a function $\tau$ from $V$ to the power set of $S$, subject to the requirement that the free $A$-module with basis $V$ admits an $\mathcal{H}$-module structure satisfying

$$T_v = \begin{cases} -q^{-1}v & \text{if } s \in \tau(v) \\ qv + \sum_{u \in V \mid s \in \tau(u)} \mu(u, v)u & \text{if } s \notin \tau(v), \end{cases}$$

for all $s \in S$ and $v \in V$.

The set $V$ is called the vertex set of the $W$-graph, and there is a directed edge from a vertex $v$ to $u$ if and only if $\mu(u, v) \neq 0$. We may regard the integer $\mu(u, v)$ as the weight of the edge from $v$ to $u$, and the set $\tau(v)$ as the colour of the vertex $v$.

Since the $\mathcal{H}$-module $AV$ is $A$-free it admits a unique $A$-semilinear involution $\alpha \mapsto \overline{\alpha}$ such that $\overline{\alpha} = v$ for all elements $v$ of the basis $V$. It follows from (4.1) that for all $s \in S$ and $v \in V$,

$$T_{\overline{s}}v = \begin{cases} -qv & \text{if } s \in \tau(v) \\ q^{-1}v + \sum_{u \in V \mid s \in \tau(u)} \mu(u, v)u & \text{if } s \notin \tau(v), \end{cases}$$

and hence $T_{\overline{h}}\alpha = \overline{T_{\alpha}}$ for all $h \in \mathcal{H}$ and $\alpha \in AV$.

5. $W$-GRAPH IDEALS

Let $(W, S)$ be a Coxeter system and $\mathcal{H}$ the associated Hecke algebra. Let $\mathcal{I}$ be an ideal in the poset $(W, \leq_L)$; that is, $\mathcal{I}$ is a subset of $W$ such that every $u \in W$ that is a suffix of an element of $\mathcal{I}$ is itself in $\mathcal{I}$. This condition implies that $\text{Pos}(\mathcal{I}) = S \setminus \mathcal{I} = \{ s \in S \mid s \notin \mathcal{I} \}$ (see Definition 2.3). Let $J$ be a subset of $\text{Pos}(\mathcal{I})$, so that $\mathcal{I} \subseteq D_J$. For each $w \in \mathcal{I}$ we define the following subsets of $S$:

$$SA(w) = \{ s \in S \mid sw > w \text{ and } sw \in \mathcal{I} \},$$
$$SD(w) = \{ s \in S \mid sw < w \},$$
$$WA_J(w) = \{ s \in S \mid sw > w \text{ and } sw \in D_J \setminus \mathcal{I} \},$$
$$WD_J(w) = \{ s \in S \mid sw > w \text{ and } sw \notin D_J \}.$$
and \( WD(w) \) rather than \( WA_J(w) \) and \( WD_J(w) \). We also define the sets of descents and ascents of \( w \) by \( D_J(w) = SD(w) \cup WD_J(w) \) and \( A_J(w) = SA(w) \cup WA_J(w) \).

Remark. It follows from Lemma 2.4 that
\[
WA_J(w) = \{ s \in S \mid sw \notin \mathcal{I} \text{ and } w^{-1}sw \notin J \}
\]
and
\[
WD_J(w) = \{ s \in S \mid sw \notin \mathcal{I} \text{ and } w^{-1}sw \in J \},
\]

since \( sw \notin \mathcal{I} \) implies that \( sw > w \) (given that \( \mathcal{I} \) is an ideal in \( (W, \leq_L) \)). Note also that \( J = WD_J(1) \).

Definition 5.1. With the above notation, the set \( \mathcal{I} \) is said to be a \( W \)-graph ideal with respect to \( J \) if the following hypotheses are satisfied.

(i) There exists an \( \mathcal{A} \)-free \( \mathcal{H} \)-module \( \mathcal{I} = \mathcal{I}(\mathcal{I}, J) \) possessing an \( \mathcal{A} \)-basis \( B = \{ b_w \mid w \in \mathcal{I} \} \) on which the generators \( T_s \) act by
\[
T_s b_w = \begin{cases} 
  b_{sw} & \text{if } s \in SA(w), \\
  b_{sw} + (q - q^{-1})b_w & \text{if } s \in SD(w), \\
  -q^{-1}b_w & \text{if } s \in WD_J(w), \\
  q b_w - \sum_{y \in \mathcal{I}, y < w} r^{*}_{y,w} b_y & \text{if } s \in WA_J(w),
\end{cases}
\]
for some polynomials \( r^{*}_{y,w} \in q \mathcal{A}^+ \).

(ii) The module \( \mathcal{I} \) admits an \( \mathcal{A} \)-semilinear involution \( \alpha \mapsto \overline{\alpha} \) satisfying \( \overline{b_1} = b_1 \) and \( \overline{h} = \overline{h} \) for all \( h \in \mathcal{H} \) and \( \alpha \in \mathcal{I} \).

We shall show in Section 7 below that if \( \mathcal{I} \) is a \( W \)-graph ideal with respect to \( J \) then the associated module \( \mathcal{I}(\mathcal{I}, J) \) is isomorphic to a \( W \)-graph module. Moreover, the \( W \)-graph can be constructed by an algorithm that depends only on \( \mathcal{I} \) and \( J \). Hence \( \mathcal{I}(\mathcal{I}, J) \) is determined up to isomorphism by \( \mathcal{I} \) and \( J \).

Remark. As we shall see in Section 8 below, it is quite possible for an ideal \( \mathcal{I} \) to be a \( W \)-graph ideal with respect to two different subsets \( J \) of \( \text{Pos}(\mathcal{I}) \), corresponding to two \( W \)-graph modules that are not isomorphic. So the set \( J \) is an important part of the definition of a \( W \)-graph ideal.

Definition 5.2. If \( \Lambda \subseteq W \) and \( \mathcal{I} = \{ u \in W \mid u \leq_L w \text{ for some } w \in \Lambda \} \) is a \( W \)-graph ideal then we call \( \Lambda \) a \( W \)-graph determining set, and we call \( w \in W \) a \( W \)-graph determining element if \( \{ w \} \) is a \( W \)-graph determining set.

The simplest example of a \( W \)-graph determining element is \( w_S \), the maximal length element of a finite Coxeter group \( W \), with \( J \) the empty subset of \( S \). The \( W \)-graph we obtain is the Kazhdan-Lusztig \( W \)-graph corresponding to the regular representation of \( W \). More generally, if \( J \) is an arbitrary subset of \( S \) then \( d_J \), the minimal length element of the left coset \( w_S W_J \), is a \( W \)-graph determining element with respect to \( J \) and also with respect to \( \emptyset \). In both cases \( \mathcal{I} = D_J \), and we recover Deodhar’s parabolic analogues of the Kazhdan-Lusztig construction. See Section 8 below for the details.

6. An example

The general algorithm for constructing \( W \)-graphs from \( W \)-graph ideals is deferred to the next section. In the current section we present a motivational example.

Let $W_n$ be the Coxeter group of type $A_{n-1}$, which we identify with the the symmetric group on $[1, n]$, the set of integers from 1 to $n$, by identifying the simple reflections $s_1, s_2, \ldots, s_{n-1}$ with the transpositions $(1, 2), (3, 4), \ldots, (n-1, n)$ (respectively). We use a left-operator convention for permutations, writing $wi$ for the action of $w \in W_n$ on $i \in [1, n]$. It is well known that if $t = (i, j) \in W_n$ is an arbitrary transposition, with $i < j$, and $w \in W_n$ is an arbitrary permutation, then $wt < w$ if and only if $wi > wj$ and $tw < w$ if and only if $w^{-1}i > w^{-1}j$; moreover, $l(w)$ is the number of pairs $(i, j) \in [1, n] \times [1, n]$ such that $i < j$ and $wi > wj$.

Since our example will involve Young diagrams and tableaux, we need to start by recalling some basic definitions and establishing our notation.

A sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is called a partition of $n$ if $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$ and $\lambda_1 \geq \cdots \geq \lambda_k$. The $\lambda_i$ are called the parts of $\lambda$. We define $P(n)$ to be the set of all partitions of $n$. For each $\lambda = (\lambda_1, \ldots, \lambda_k) \in P(n)$ we define

$$[\lambda] = \{ (i, j) \mid 1 \leq j \leq \lambda_i \text{ and } 1 \leq i \leq k \},$$

and refer to this as the Young diagram of $\lambda$. Pictorially $[\lambda]$ is represented by a left-justified array of boxes with $\lambda_i$ boxes in the $i$-th row; the pair $(i, j) \in [\lambda]$ corresponds to the $j$-th box in the $i$-th row. Thus the Young diagram of $\lambda = (4, 2, 2)$ looks like this:

```
+---+---+
|   |   |
+---+---+---+
|   |   |
```

If $\lambda$ is a partition of $n$ then a $\lambda$-tableau is a bijection $t: [\lambda] \to [1, n]$. In other words, $t$ is a one to one correspondence between the boxes of the Young diagram $[\lambda]$ and the integers from 1 to $n$. Of course $t$ can be conveniently described by writing the number $t(i, j)$ in the box $(i, j)$, for all $(i, j) \in [\lambda]$. For each $i \in [1, n]$ we define row$_i(t)$ and col$_i(t)$ to be the row index and column index of $i$ in $t$ (so that $t^{-1}(i) = (\text{row}_i(t), \text{col}_i(t))$). We define Tab($\lambda$) to be the set of all $\lambda$-tableaux, and we let $t^\lambda$ be the specific $\lambda$-tableau given by

$$t_\lambda(i, j) = j + \sum_{h=1}^{i-1} \lambda_h$$

for all $(i, j) \in [\lambda]$. That is, the numbers 1, 2, ..., $\lambda_1$ fill the first row of $[\lambda]$ in order from left to right, then the numbers $\lambda_1 + 1, \lambda_1 + 2, \ldots, \lambda_1 + \lambda_2$ similarly fill the second row, and so on.

We define $t_\lambda$ to be the $\lambda$-tableau that is the transpose of the $\lambda'$-tableau $t^{\lambda'}$, where $\lambda'$ is the partition dual to $\lambda$. Thus $t_\lambda$ is the unique standard $\lambda$-tableau whose columns consist of sequences of consecutive numbers, while $t^\lambda$ is the unique standard $\lambda$-tableau whose rows consist of sequences of consecutive numbers. We shall find it convenient to define box$_\lambda(i) = t_{\lambda'}^{-1}(i)$; thus box$_\lambda(i)$ is the box of $[\lambda]$ such that $i$ is in box$_\lambda(i)$ in $t_\lambda$. We say that box$_\lambda(i)$ is “earlier” than box$_\lambda(j)$ if $i < j$.

It is clear that for any fixed $\lambda \in P(n)$ the group $W_n$ acts on the set of all $\lambda$-tableaux, via $(wt)(i, j) = w(t(i, j))$ for all $(i, j) \in [\lambda]$, for all $\lambda$-tableaux $t$ and all $w \in W_n$. Moreover, the map from $W_n$ to Tab($\lambda$) defined by $w \mapsto wt_\lambda$ for all $w \in W_n$ is bijective. We use this bijection to transfer the partial orders defined in Definitions 2.1 and 2.2 from $W_n$ to Tab($\lambda$). Thus if $t_1, t_2$ are arbitrary $\lambda$-tableaux and we write $t_1 = w_1t_\lambda$ and $t_2 = w_2t_\lambda$ with $w_1, w_2 \in W_n$, then by definition $t_1 \leq t_2$
Proof. Observe that $i < w$ if and only if $w_1 \leq w_2$, and $t_1 \leq t_2$ if and only if $w_1 \leq w_2$. Similarly, if $t = wt_\lambda$ is an arbitrary $\lambda$-tableau, where $w \in W_n$, then we define $l(t) = l(w)$.

For later reference, we note the following trivial result.

Lemma 6.1. Let $w \in W_n$ and let $t = wt_\lambda$ be the corresponding $\lambda$-tableau. If $i \in [1, n-1]$ then $l(s_i t) > l(t)$ if and only if either $\text{col}_t(i) < \text{col}_t(i+1)$ or $\text{row}_t(i) = \text{col}_t(i+1)$ and $\text{row}_t(i) < \text{row}_t(i+1)$.

Proof. Observe that $w^{-1}i = w^{-1}(t(\text{row}_t(i), \text{col}_t(i))) = t_\lambda(\text{row}_t(i), \text{col}_t(i))$ and similarly $w^{-1}(i+1) = t_\lambda(\text{row}_t(i+1), \text{col}_t(i+1))$. Since $t_\lambda(j, k) < t_\lambda(j', k')$ if and only if either $k < k'$ or $k = k'$ and $j < j'$, the condition that $\text{col}_t(i) < \text{col}_t(i+1)$ or $\text{col}_t(i) = \text{col}_t(i+1)$ and $\text{row}_t(i) < \text{row}_t(i+1)$ is equivalent to $w^{-1}i < w^{-1}(i+1)$. Since this in turn is equivalent to $l(s_i w) > l(w)$, the result follows. □

A $\lambda$-tableau $t$, where $\lambda \in P(n)$, is said to be column standard if its entries increase down the columns, that is, if $t(i, j) < t(i+1, j)$ whenever $(i, j) \in [\lambda]$ and $(i+1, j) \in [\lambda]$. Similarly, $t$ is said to be row standard if its entries increase along the rows, that is, if $t(i, j) < t(i, j+1)$ whenever $(i, j) \in [\lambda]$ and $(i, j+1) \in [\lambda]$. A standard tableau is a tableau that is both column standard and row standard. We write $\text{CSTD}(\lambda)$, $\text{RSTD}(\lambda)$ and $\text{STD}(\lambda)$ for the sets of all column standard tableaux, row standard tableaux and standard tableaux for $\lambda$.

Given $\lambda \in P(n)$ we define $J_\lambda$ to be the subset of $S$ consisting of those simple reflections $s_i = (i, i+1)$ such that $i$ and $i+1$ lie in the same column of $t_\lambda$, and we define $W_\lambda$ to be the standard parabolic subgroup of $W_n$ generated by $J_\lambda$. Thus, by our convention, $W_\lambda$ is the column stabilizer of $t_\lambda$ rather than the row stabilizer of $t^\lambda$. Moreover, the set of minimal left coset representatives for $W_\lambda$ in $W_n$ is the set

$$D_\lambda = \{ d \in W_n \mid d_i < d(i+1) \text{ whenever } s_i \in J_\lambda \}$$

since the condition $d_i < d(i+1)$ is equivalent to $l(ds_i) > l(d)$. It follows that $\{ dt_\lambda \mid d \in D_\lambda \}$ is precisely the set of column standard $\lambda$-tableaux.

Now suppose that $t \in \text{STD}(\lambda)$ and $t \neq t^\lambda$. Choose $i$ to be the least integer whose position in $t$ is not the same as its position in $t^\lambda$, and let $j = t(\text{row}_t(i), \text{col}_t(i))$, the number whose position in $t$ is the position of $i$ in $t^\lambda$. If $h = \text{row}_t(j)$ then the number $j - 1$ cannot appear to the left of $j$ in the $h$-th row of $t$, or in any earlier row, since these positions are occupied by the numbers from 1 to $i-1$. Hence, since $t$ is standard, it follows that $\text{row}_t(j+1) > \text{row}_t(j)$ and $\text{col}_t(j+1) < \text{col}_t(j)$. In particular, since $j-1$ and $j$ are not in the same row of $t$ or the same column of $t$, the tableau obtained from $t$ by swapping the positions of $j - 1$ and $j$ is still standard. That is, $s_{j-1}t \in \text{STD}(\lambda)$. But by Lemma 6.1 above we see that $l(s_{j-1}t) > l(t)$, and therefore $t < s_{j-1}t$. So $t$ is not maximal in the ordering $<_L$, and it follows that $t^\lambda$ is the unique maximal standard $\lambda$-tableau relative to $<_L$.

Similarly, if $t \in \text{STD}(\lambda)$ and $s_jt < t$ for some $j \in [1, n-1]$, then $t$ has $j + 1$ in an earlier box than $j$, and since $t$ is standard we see that $\text{row}_t(j+1) > \text{row}_t(j)$ and $\text{col}_t(j+1) < \text{col}_t(j)$. Thus $s_jt \in \text{STD}(\lambda)$. So if $t'$ is any $\lambda$-tableau such that $t' <_L t$ then $t'$ is standard. Hence we obtain the following result (see [4, Lemma 1.5]).

Lemma 6.2. Let $\lambda \in P(n)$ and define $v_\lambda \in W_n$ by the requirement that $t^\lambda = v_\lambda t_\lambda$. Then $\text{STD}(\lambda) = \{ wt_\lambda \mid w \leq_L v_\lambda \} = \{ t \in \text{Tab}(\lambda) \mid t \leq_L t^\lambda \}$. 
For \( t \in \text{STD}(\lambda) \), define
\[
\begin{align*}
\text{SA}(t) &= \{ i \in [1, n-1] \mid \text{col}_t(i) < \text{col}_t(i+1) \text{ and } \text{row}_t(i) \neq \text{row}_t(i+1) \}, \\
\text{SD}(t) &= \{ i \in [1, n-1] \mid \text{col}_t(i) > \text{col}_t(i+1) \}, \\
\text{WA}(t) &= \{ i \in [1, n-1] \mid \text{row}_t(i) = \text{row}_t(i+1) \}, \\
\text{WD}(t) &= \{ i \in [1, n-1] \mid \text{col}_t(i) = \text{col}_t(i+1) \}.
\end{align*}
\]

Observe that if \( \mathcal{I} \) is the left ideal of \((W_n, \leq_L)\) generated by \( v_\lambda \) and if \( J = J_\lambda \), then for each \( w \in \mathcal{I} \) the sets \( \text{SA}(w), \text{SD}(w), \text{WA}(w) \) and \( \text{WD}(w) \) as defined in Section 3 coincide with the sets \( \text{SA}(wt_\lambda), \text{SD}(wt_\lambda), \text{WA}(wt_\lambda) \) and \( \text{WD}(wt_\lambda) \) as defined above.

Let \( \mathcal{H}_n = \mathcal{H}(W_n) \) be the Hecke algebra of \( W_n \). Thus \( \mathcal{H}_n \) is generated by elements \( T_1, T_2, \ldots, T_{n-1} \) satisfying \( T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \) for all \( i \in [1, n-2] \) and \( T_i T_j = T_j T_i \) for all \( i, j \in [1, n-1] \) with \( |i - j| > 1 \), as well as \( T_i^2 = 1 + (q - q^{-1})T_i \) for all \( i \in [1, n-1] \). Let \( \lambda \in P(n) \) and let \( S^\lambda \) be the Specht module for \( \mathcal{H}_n \) corresponding to \( \lambda \). It follows from results proved in [12] Chapter 3 that \( S^\lambda \) has an \( \mathcal{A} \)-basis \( (b_t \mid t \in \text{STD}(\lambda)) \) such that for all \( i \in [1, n-1] \) and \( t \in \text{STD}(\lambda) \),
\[
T_ib_t = \begin{cases} 
    b_{s,t} & \text{if } i \in \text{SA}(t), \\
    b_{s,t} + (q - q^{-1})b_t & \text{if } i \in \text{SD}(t), \\
    -q^{-1}b_t & \text{if } i \in \text{WD}(t), \\
    qb_t - \sum_{s < t} r_s^{(i)}b_s & \text{if } i \in \text{WA}(t),
\end{cases}
\]

where the \( r_s^{(i)} \) in the last equation are in \( \mathcal{A} \), but are not easy to describe explicitly.

The basis \( (b_t \mid t \in \text{STD}(\lambda)) \) is known as the standard basis of \( S^\lambda \). Note that our hypotheses and conventions are slightly different from those used in [12], and hence our formulas above are also slightly different from those in [12]. More explanation of (6.1) will be given below.

Let \( \mathcal{F} \) be the field of fractions of \( \mathcal{A} \), and write \( \mathcal{F}S^\lambda \) for the \( \mathcal{F} \)-module obtained from \( S^\lambda \) by extension of scalars. In this context we can obtain the simpler semi-normal form of the representation: \( \mathcal{F}S^\lambda \) has an \( \mathcal{F} \)-basis \( (b'_t \mid t \in \text{STD}(\lambda)) \) such that for all \( i \in [1, n-1] \) and \( t \in \text{STD}(\lambda) \),
\[
T_ib'_t = \begin{cases} 
    -q^{-1}b'_t & \text{if } i \in \text{WD}(t), \\
    qb'_t & \text{if } i \in \text{WA}(t), \\
    p_1(d; q)b'_t + p_2(d; q)b'_{s,t} & \text{otherwise},
\end{cases}
\]

where \( d = (x_1 - y_1) - (x_2 - y_2) \) if the row and column indices of \( i \) and \( i + 1 \) in \( t \) are, respectively, \( x_1 \) and \( y_1 \) and \( x_2 \) and \( y_2 \), and
\[
p_1(d; q) = (q^2 - 1)/(q - q^{2d+1}), \quad p_2(d; q) = (1 - q^{2d+2})/(q - q^{2d+1}).
\]

A proof of the validity of these formulas can be found in the paper of Ariki and Koike, [11] Theorem 3.7. Note that we are using a variant of \( \mathcal{H}_n \) in which the eigenvalues of the generators \( T_i \) are \( q \) and \( q^{-1} \), whereas Ariki and Koike use the traditional \( q \) and \( -1 \); hence to convert the formulas Ariki and Koike give to the ones that are appropriate for our context it was necessary to replace \( q \) by \( q^2 \) and \( T_i \) by \( qT_i \).
The seminormal form suffers the drawback that it gives matrix coefficients that are not integral. The standard basis and the W-graph basis \( (c_t | t \in \text{STD}(\lambda)) \) give integral representations but no (currently known) simple formulae for all the matrix coefficients. All three bases are related by triangular basis changes, with \( c_{t\lambda} = b_{t\lambda} = b'_{t\lambda} \). (This vector spans the 1-dimensional subspace of \( S^\lambda \) consisting of those \( v \) such that \( T_w v = (-q)^{-\ell(w)}v \) for all \( w \in W_\lambda \).

Using the seminormal form of the representation it can easily be shown that \( FS^\lambda \) admits a semilinear involution \( v \mapsto \overline{v} \) satisfying \( \overline{hv} = h\overline{v} \) for all \( h \in H_n \) and all \( v \in S^\lambda \). Indeed, if \( v \in S^\lambda \) then \( v = \sum_{t \in \text{STD}(\lambda)} a_t b'_t \) for some coefficients \( a_t \in F \), and we define \( \overline{v} = \sum_{t \in \text{STD}(\lambda)} \overline{a_t} b'_t \). Then for all \( i \in [1, n-1] \) and \( t \in \text{STD}(\lambda) \),

\[
\overline{T_i b'_t} = T_i b'_t + (q^{-1} - q)b'_t = T_i b'_t
\]

since \( p_1(d; q) + (q^{-1} - q) = p_1(d; q^{-1}) \) and \( p_2(d; q) = p_2(d; q^{-1}) \). It follows by linearity that \( T_i \overline{v} = \overline{T_i v} \) for all \( i \in [1, n-1] \) and all \( v \in FS^\lambda \), and this gives the desired result since the \( T_i \) generate \( H_n \).

Now for our example. We take \( n = 7 \) and let \( \lambda = (3,3,1) \), a partition of 7 giving a Specht module of dimension 21. The 21 standard tableaux \( t_1, t_2, \ldots, t_{21} \) are listed in order below.

\[
\begin{array}{cccccccc}
1 & 4 & 6 & 1 & 3 & 6 & 1 & 2 \\
2 & 5 & 7 & 2 & 5 & 7 & 3 & 5 \\
3 & 4 & 1 & 2 & 4 & 5 & 5 & 1 \\
1 & 2 & 4 & 3 & 5 & 7 & 2 & 4 \\
3 & 5 & 7 & 4 & 5 & 7 & 2 & 5 \\
4 & 7 & 6 & 5 & 7 & 2 & 6 & 7 \\
5 & 6 & 4 & 7 & 6 & 5 & 3 & 4 \\
6 & 7 & 5 & 6 & 4 & 7 & 3 & 5 \\
7 & 6 & 7 & 5 & 6 & 4 & 7 & 3 \\
\end{array}
\]

Note that we have chosen a total ordering of \( \text{STD}(\lambda) \) that is consistent with the partial ordering \( \leq \), in the sense that if \( i \leq j \) then \( t_i \leq t_j \). Let \( b_1, b_2, \ldots, b_{21} \) be the standard basis elements corresponding (respectively) to \( t_1, t_2, \ldots, t_{21} \). We shall construct a new basis \( c_1, c_2, \ldots, c_{21} \) such that for all \( j \),

\[
c_j = b_j - q \sum_{i \prec j} f_{i,j} c_i
\]

for certain \( f_{i,j} \in \mathbb{Z}[q] \), to be defined recursively. In terms of this new basis the action of the algebra will be as follows:

\[
T_k c_j = \begin{cases} 
-q^{-1} c_j & \text{if } k \in D(t_j), \\
q c_j + \sum_{i \in R(k,j)} \mu_{i,j} c_i & \text{if } k \in WA(t_j), \\
q c_j + c_h + \sum_{i \in R(k,j)} \mu_{i,j} c_i & \text{if } k \in SA(t_j), 
\end{cases}
\]

where \( h \) is defined by \( s_h t_j = t_h \), the set \( R(k,j) \) consists of all \( i < j \) such that \( k \) is a descent of \( t_i \), and \( \mu_{i,j} \) is the constant term of \( f_{i,j} \).

These conditions easily yield formulae for the \( c_j \), as listed below. To start the process, \( c_1 = b_1 \) is given. Now to find \( c_2 \), we first find a strong descent \( r \) of \( t_2 \); in this case, the only choice is \( r = 3 \). By the third formula above we must have
\( T_3c_1 = qc_1 + c_2 \), and thus \( c_2 = T_3b_1 - qc_1 = b_2 - qc_1 \). In general, to find \( c_h \) given that the earlier \( c_j \)'s have already been found, first find \( k \in SD(t_h) \), and let \( t_j = s_k t_h \). Then

\[
\begin{align*}
    c_h &= T_k c_j - qc_j - \sum_{i \in R(k,j)} \mu_{i,j} c_i \\
    &= T_k \left( b_j - q \sum_{i < j} f_{i,j} c_i \right) - qc_j - \sum_{i \in R(k,j)} \mu_{i,j} c_i \\
    &= b_h - qc_j - q \sum_{i < j} f_{i,j} T_k c_i - \sum_{i \in R(k,j)} \mu_{i,j} c_i,
\end{align*}
\]

which can be expressed in the form \( b_h - q \sum_{i < j} f_{i,h} c_i \) by using the formulas for evaluating \( T_k c_i \). The crucial point is that the coefficient of each \( c_j \) in \(-qf_{i,j} T_k c_i\) will be a polynomial divisible by \( q \) unless \( i \in R(k,j) \), in which case \(-qf_{i,j} T_k c_i = f_{i,j} c_i\), and the constant term \( \mu_{i,j} c_i \) is cancelled by one of the terms in the second sum. In this way all the terms in the second sum also disappear.

For example, having found \( c_2 \), to find \( c_3 \) we first observe that \( 2 \) is a descent of \( t_3 \) and \( s_2 t_3 = t_2 \), giving

\[
c_3 = b_3 - qe_2 - q \sum_{i < 2} f_{i,2} T_2 c_i - \sum_{i \in R(2,2)} \mu_{i,2} c_i.
\]

Since \( f_{1,2} = \mu_{1,2} = 1 \) and \( 2 \in D(t_1) \) we find that \(-qf_{1,2} T_2 c_1 = c_1 = \mu_{1,2} c_1\), leaving \( c_3 = b_3 - qe_2 \). After similarly calculating that \( c_4 = b_4 - qe_2 \), the calculation for \( c_5 \) proceeds as follows. Since \( r = 2 \) is a descent of \( t_5 \) with \( s_2 t_5 = t_2 \),

\[
c_5 = b_5 - qe_4 - q T_2 c_2 - \sum_{i \in R(2,4)} \mu_{i,4} c_i,
\]

\[
eq b_5 - qe_4 - q(qe_2 + c_3 + c_1) - 0,
\]

since the fact that \( 2 \notin D(t_2) \) means that \( R(2,4) \) is empty, and \( T_2 c_2 = qe_2 + c_3 + c_1 \) (since \( 2 \in D(t_1) \) and \( \mu_{1,2} = 1 \), and \( s_2 t_2 = t_3 \)). As a further example, the calculations involved in deriving the formula for \( c_{21} \) are given below.

\[
\begin{align*}
    c_1 &= b_1, \\
    c_2 &= b_2 - q c_1, \\
    c_3 &= b_3 - q c_2, \\
    c_4 &= b_4 - q c_2, \\
    c_5 &= b_5 - q c_4 - q c_3 - q^2 c_2 - q c_1, \\
    c_6 &= b_6 - q c_1, \\
    c_7 &= b_7 - q c_6 - q c_2 - q^2 c_1, \\
    c_8 &= b_8 - q c_7 - q c_3 - q^2 c_2, \\
    c_9 &= b_9 - q c_7 - q^2 c_6 - q c_4 - q^2 c_2 - q c_1, \\
    c_{10} &= b_{10} - q c_9 - q c_8 - q^2 c_7 - q c_5 - q^2 c_4 - q^2 c_2 - q^3 c_1 - q^2 c_2, \\
    c_{11} &= b_{11} - q c_{10} - q^2 c_9 - q^2 c_5 - q c_4 - q^3 c_1, \\
    c_{12} &= b_{12} - q c_7 - q c_4 - q^3 c_2, \\
    c_{13} &= b_{13} - q c_{12} - q c_8 - q^2 c_7 - q c_6 - q c_5 - q^2 c_4 - q^2 c_3 - q^3 c_2 - q^2 c_1.
\end{align*}
\]
\[ c_{14} = b_{14} - q c_{12} - q c_9 - q^2 c_7 - q^2 c_4 - q^3 c_2 - q^2 c_1, \]
\[ c_{15} = b_{15} - q c_{14} - q c_{13} - q^2 c_{12} - q c_{10} - q^2 c_9 - q^3 c_7 - q^2 c_6 \]
\[ - q^2 c_5 - q^3 c_4 - q^3 c_3 - q^4 c_2 - q^3 c_1, \]
\[ c_{16} = b_{16} - q c_{15} - q c_{14} - q^2 c_{13} - q c_{12} - q c_{11} - q^2 c_{10} - q^3 c_9 - q c_8 \]
\[ - q^2 c_7 - q^3 c_6 - q^3 c_5 - q^4 c_4 - q^3 c_1, \]
\[ c_{17} = b_{17} - q c_{12} - q^2 c_7, \]
\[ c_{18} = b_{18} - q c_{17} - q c_{13} - q^2 c_{12} - q^2 c_8 - q^3 c_7 - q^2 c_6, \]
\[ c_{19} = b_{19} - q c_{17} - q c_{14} - q^2 c_{12} - q^2 c_9 - q^3 c_7 - q c_4, \]
\[ c_{20} = b_{20} - q c_{19} - q c_{18} - q^2 c_{17} - q c_{15} - q^2 c_{14} - q^3 c_{13} - q^4 c_2 - q^2 c_10 \]
\[ - q^3 c_9 - q^3 c_8 - q^4 c_7 - q^3 c_6 - q c_5 - q^2 c_4 - q^2 c_1, \]
\[ c_{21} = b_{21} - q c_{20} - q^2 c_{19} - q^2 c_{18} - q c_{17} - q c_{16} - q^2 c_{15} - q^3 c_{14} - q^3 c_{13} \]
\[ - q^2 c_{12} - q^2 c_{11} - q^3 c_{10} - q^4 c_9 - q^2 c_8 - q^3 c_7 - q^4 c_6 - q^2 c_5 \]
\[ - (q^3 + q) c_4 - q c_3 - q^2 c_2 - q^3 c_1. \]

Here are the calculations for \( c_{21} \). We have \( s_3 t_{21} = t_{20} \); so
\[
c_{21} = b_{21} - q c_{20} - \sum_{i < 20} f_i T_3 c_i - \sum_{i \in R(3,20)} \mu_{i,20} c_i
\]
\[ = b_{21} - q c_{20} - q T_3 c_{19} - q T_3 c_{18} - q^2 T_3 c_{17} - q T_3 c_{15} - q^2 T_3 c_{14} \]
\[ - q^2 T_3 c_{13} - q^3 T_3 c_{12} - q^2 T_3 c_{10} - q^3 T_3 c_{9} - q^3 T_3 c_{8} - q^4 T_3 c_{7} \]
\[ - q^3 T_3 c_{6} - q T_3 c_{5} - q^2 T_3 c_{4} - q^2 T_3 c_{1} - \sum_{i \in R(3,20)} \mu_{i,20} c_i. \]

Now 3 is a descent of \( t_{17}, t_{12}, t_8, t_7 \) and \( t_4 \); so
\[
q^2 T_3 c_{17} - q^3 T_3 c_{12} - q^3 T_3 c_{8} - q^4 T_3 c_{7} - q^2 T_3 c_{4}
\]
\[ = q c_{17} + q^2 c_{12} + q^2 c_8 + q^3 c_7 + q c_4\]
and we see also that the sum \( \sum_{i \in R(3,20)} \mu_{i,20} c_i \) has no nonzero terms. Turning to
the other terms in the expression for \( c_{21} \), the coefficient of \( q \) in the formula for \( c_{19} \)
tells us that \( \mu_{17,19} = \mu_{14,19} = \mu_{4,19} = 1 \), and thus
\[
- q T_3 c_{19} = - q(q c_{19} + c_{17} + c_4)
\]
since \( s_3 t_{19} \) does not exist, and 3 is in \( D(t_{17}) \) and \( D(t_4) \) but not \( D(t_{14}) \). Similarly
\[
- q T_3 c_{18} = - q(q c_{18} + c_{17})
\]
\[
- q T_3 c_{15} = - q(q c_{15} + c_{16})
\]
\[
- q^2 T_3 c_{14} = - q^2(q c_{14} + c_{12})
\]
\[
- q^2 T_3 c_{13} = - q^2(q c_{13} + c_{12} + c_8)
\]
\[
- q^2 T_3 c_{10} = - q^2(q c_{10} + c_{11} + c_8)
\]
\[
- q^3 T_3 c_{9} = - q^3(q c_9 + c_7 + c_4)
\]
\[
- q^3 T_3 c_{6} = - q^3(q c_6 + c_7)
\]
\[
- q T_3 c_5 = - q(q c_5 + c_4 + c_3)
\]
\[
- q^2 T_3 c_1 = - q^2(q c_1 + c_2).
and we leave it to the reader to check that when these formulas together with (5.3) and (6.4) above are substituted into our expression for $c_{21}$ the answer is as given previously.

The above example is meant to illustrate a procedure that will work for all Specht modules. Although it is clear enough that the procedure will produce a basis $(c_t \mid t \in \STD(\lambda))$ such that the formulas in Equation (6.2) hold, it is not clear that the these formulas define a representation of $H$. The proof that they do relies on Proposition 6.3 below, which is proved in [15]. The algorithm has been implemented using the computational algebra system MAGMA [2], and in particular has been used in the case $\lambda = (5, 5, 3, 3)$ to confirm the result of McLaran and Warrington [13] that in this case 5 occurs as an edge-weight in the $W$-graph. \footnote{The magma files used can be obtained from \url{http://www.maths.usyd.edu.au/u/bobh/magma/}, or from \url{http://magma.maths.usyd.edu.au/magma/extra/}. It is planned to include them in the next release of MAGMA.}

We now briefly indicate how to adapt the discussion of the standard basis of $S^\lambda$ given in [12] to yield the formulas in (6.1) above. It follows from Corollary 3.4, Corollary 3.21 and Proposition 3.22 of [12] that the Specht module $S^\lambda$ (defined immediately after Corollary 3.21) has a basis $(m_t \mid t \in \STD(\lambda))$ such that

$$T_i m_t = \begin{cases} m_{s,t} & \text{if } i \in SD(t), \\ qm_{s,t} + (q-1)m_t & \text{if } i \in SA(t), \\ qm_t & \text{if } i \in WA(t), \\ -m_t + \sum_{s < t} a_{s,t}^{(i)} m_s & \text{if } i \in WD(t), \end{cases}$$

where the elements $a_{s,t}^{(i)}$ are polynomials in $q$. Note that [12] employs the traditional definition of $H_n$, so that to make the above formulas compatible with our definitions we should replace $q$ by $q^2$ and $T_i$ by $qT_i$. After this we use the automorphism of $H_n$ given by $T_i \mapsto -T_i^{-1} = -T_i$ to define a new action, obtaining a module that we call the dual Specht module. This gives

$$-T_i m_t = \begin{cases} q^{-1}m_{s,t} & \text{if } i \in SD(t), \\ qm_{s,t} + (q-q^{-1})m_t & \text{if } i \in SA(t), \\ qm_t & \text{if } i \in WA(t), \\ -q^{-1}m_t + q^{-1} \sum_{s < t} a_{s,t}^{(i)} m_s & \text{if } i \in WD(t), \end{cases}$$

where now the $a_{s,t}^{(i)}$ are polynomials in $q^2$. We apply these formulas for the module corresponding to the partition $\lambda'$ dual to $\lambda$. This dualises again, swapping ascents and descents, and giving a module that has a basis $(m_t \mid t \in \STD(\lambda))$ (not the same as the $m_t$'s we started with) satisfying

$$T_i m_t = \begin{cases} -q^{-1}m_{s,t} & \text{if } i \in SA(t), \\ -qm_{s,t} - (q-q^{-1})m_t & \text{if } i \in SD(t), \\ -qm_t & \text{if } i \in WD(t), \\ q^{-1}m_t - q^{-1} \sum_{s < t} a_{s,t}^{(i)} m_s & \text{if } i \in WA(t). \end{cases}$$

We now define $b_t = (-q)^{(i)}m_t$. Applying the involution $v \mapsto \overline{v}$ to both sides of the above formulas and multiplying through by $(-q)^{(i)}$ yields (6.1) above.
The following proposition was proved in the second author’s PhD thesis [14] Theorem 6.3.4. A shorter proof is presented in [15], a recently submitted followup to the present paper.

**Proposition 6.3.** The elements \( r_{st}^{(i)} \) appearing in (6.1) are polynomials in \( q \) with zero constant term.

In fact Theorem 6.3.4 of [14] was stronger, saying that the \( r_{st}^{(i)} \) are divisible by \( q^2 \). But the weaker version is sufficient for our present needs.

Given that Proposition 6.3 is true, it follows that \( \nu_\lambda \) satisfies all the hypotheses in Definition 6.1 and is a \( W \)-graph determining element relative to \( J_\lambda \). According to the theory presented in the next section, it follows that \( S^\lambda \) has a \( W \)-graph basis \( \{ c_t \mid t \in \text{STD}(\lambda) \} \) which can be computed by means of the algorithm illustrated above.

7. Constructing the \( W \)-Graph from a \( W \)-Graph Ideal

We return now to the situation described in Section 5 above, and let \( \mathcal{I} \) be a \( W \)-graph ideal with respect to \( J \subseteq S \). By Definition 6.1 there is an \( H \)-module \( \mathcal{I} \) possessing an \( A \)-basis \( B = \{ b_w \mid w \in \mathcal{I} \} \) on which the generators of \( H \) act via the formulas in Definition 5.1. Moreover, there is an \( A \)-semilinear involution \( v \mapsto \overline{v} \) on \( \mathcal{I} \) satisfying \( \overline{b_1} = b_1 \) and \( \overline{hv} = \overline{h} \overline{v} \) for all \( h \in H \) and \( v \in \mathcal{I} \).

**Lemma 7.1.** For each \( w \in \mathcal{I} \) there exist coefficients \( r_{y,w} \in A \), defined for \( y \in \mathcal{I} \) and \( y < w \), such that \( \overline{b_w} - b_w = \sum r_{y,w} b_y \) (summation over \( \{ y \in \mathcal{I} \mid y < w \} \)).

**Proof.** This is obvious when \( w = 1 \) since \( \overline{b_1} - b_1 = 0 \). Proceeding inductively, suppose that \( w \in \mathcal{I} \) and \( w \neq 1 \), and choose \( s \in S \) such that \( w = su \) for some \( u < w \). Then \( u \in \mathcal{I} \), and by the inductive hypothesis there exist \( r_{z,u} \in A \) with \( \overline{b_u} - b_u = \sum_{z < u} r_{z,u} b_z \). Moreover, \( s \in \text{SA}(u) \), and so \( T_s b_u = b_{su} = b_w \). Thus

\[
\overline{b_w} - b_w = \overline{T_s b_u} - T_s b_u = (\overline{T_s} - T_s)b_u + \overline{T_s}(\overline{b_u} - b_u) = (q^{-1} - q)b_u + \sum_{z < u} r_{z,u}(T_s - (q - q^{-1}))b_z.
\]

Clearly \( (q^{-1} - q)b_z \) is in the \( \mathcal{I} \)-module spanned by \( \{ y \in \mathcal{I} \mid y < w \} \) whenever \( z \leq u \), and so it will suffice to show that \( T_s b_z \) is in this module whenever \( z \in \mathcal{I} \) and \( z < u \). The formulas in Definition 5.1 describe how to express \( T_s b_z \) as an \( A \)-linear combination of elements \( b_x \) for \( x \in \mathcal{I} \), and our task is simply to check that every \( x \) that occurs satisfies \( x < w \).

The result is immediate if \( s \) is a weak descent of \( z \), since in this case the only \( x \) that occurs is \( x = z \), and \( z < u < w \). If \( s \) is a strong descent of \( z \) then \( x = z \) or \( x = sz \), and in this case \( sz < z \). So again \( x \leq z < u < w \), as required.

If \( s \) is a strong ascent of \( z \) then the only \( x \) that occurs is \( x = sz \). Since \( z < u \) it follows from Lemma 2.3 that \( sz < su = w \), giving the required result.

Finally, if \( s \) is a weak ascent of \( z \) then \( T_s b_z \) is a linear combination of \( b_x \) and \( \{ b_x \mid x \in \mathcal{I} \) and \( x < sz \} \). So either \( x = z < w \) or else \( x < sz < su = w \) by Lemma 2.3.

Our aim is to construct a \( W \)-graph basis for \( \mathcal{I} \). To do this we mimic the proof of Proposition 2 in Lusztig [11].
Lemma 7.2. The module $\mathcal{S}$ has a unique $\mathcal{A}$-basis $C = (c_w \mid w \in \mathcal{S})$ such that for all $w \in \mathcal{S}$ we have $\overline{c_w} = c_w$ and

$$b_w = c_w + q \sum_{y < w} q_y w c_y$$

for certain polynomials $q_y w \in \mathcal{A}^+$. 

Proof. Clearly (7.1) holds for $w = 1$ if and only if $c_1 = b_1$, and defining $c_1 = b_1$ also ensures that $\overline{c_1} = c_1$, since $\overline{b_1} = b_1$ is given.

Now suppose that $w \neq 1$, and assume, inductively, that for all $y < w$ there exists a unique element $c_y \in \mathcal{S}$ such that (7.1) holds and $\overline{c_y} = c_y$. Then Lemma 7.1 gives

$$b_w - \overline{b_w} = \sum_{y < w} r_{y, w} c_y$$

for some coefficients $r_{y, w} \in \mathcal{A}$, and applying the involution $v \mapsto \overline{v}$ we see that $\overline{r_{y, w}} = -r_{y, w}$ for all $y < w$, since (7.1) and linear independence of the elements $b_y$ ensure linear independence of the elements $\overline{c_y}$. So the coefficient of $q^n$ in $r_{y, w}$ must be zero, and for $n > 0$ the coefficient of $q^{-n}$ must be the negative of the coefficient of $q^n$. Hence $r_{y, w} = q s_{y, w} - \overline{q s_{y, w}}$ for a uniquely determined $s_{y, w} \in \mathcal{A}^+$. Moreover, $q_{y, w} = s_{y, w}$, gives the unique solution to $b_w = c_w + q \sum_{y < w} q_{y, w} c_y$ with $q_{y, w} \in \mathcal{A}^+$ and $\overline{c_y} = c_y$. So there is a unique element $c_y$ satisfying our requirements, and the induction is complete. $\square$

Throughout the remainder of this section we let the elements $c_w$ and the polynomials $q_{y, w}$ be defined so that the conditions of Lemma 7.2 are satisfied. We also define $\mu_{y, w}$ to be the constant term of $q_{y, w}$.

Theorem 7.3. Let $s \in S$ and $w \in \mathcal{S}$. Then

$$T_s c_w = \begin{cases} -q^{-1} c_w & \text{if } s \in D(w), \\ q c_w + \sum_{y \in R(s, w)} \mu_{y, w} c_y & \text{if } s \in WA(w), \\ q c_w + c_s w + \sum_{y \in R(s, w)} \mu_{y, w} c_y & \text{if } s \in SA(w), \end{cases}$$

where the set $R(s, w)$ consists of all $y \in \mathcal{S}$ such that $y < w$ and $s \in D(y)$.

Proof. Suppose first that $w = 1$. If $s \notin \mathcal{I}$ then either $s \in WD(1)$ (if $s \in J$) or $s \in WA(1)$ (if $s \notin J$), and since $c_1 = b_1$ it follows from the formulas in Definition 5.1 that

$$T_s c_1 = \begin{cases} -q^{-1} c_1 & \text{if } s \in WD(1), \\ q c_1 & \text{if } s \in WA(1). \end{cases}$$

Since the set $R(s, 1)$ is obviously empty, the formulas in the statement of the theorem hold in these two cases. If $s \in \mathcal{I}$ then clearly $s \in SA(1)$ since $s 1 < 1$ is impossible, and in this case Definition 5.1 gives $T_s b_1 = b_s$. So

$$b_s - \overline{b_s} = T_s c_1 - \overline{T_s c_1} = (T_s - \overline{T_s}) c_1 = (q - q^{-1}) c_1.$$

Thus $q_1 = 1$ and (7.1) becomes $b_s = c_s + q c_1$, giving

$$T_s c_1 = b_s = q c_1 + c_s,$$

as required.
Proceeding by induction, suppose now that $w > 1$, and consider first the case that $s \in \text{SD}(w)$. Then $y = sw < w$, and $s \in \text{SA}(y)$; so the inductive hypothesis gives
\[ T_sc_y = qc_y + c_w + \sum_{x \in R(s,y)} \mu_{x,y}c_x, \]
which can be rewritten as
\[ c_w = (T_s - q)c_y - \sum_{x \in R(s,y)} \mu_{x,y}c_x. \]
Since $T_s(T_s - q) = -q^{-1}(T_s - q)$ and $T_sc_x = -q^{-1}c_x$ for all $x \in R(s,y)$ (by the inductive hypothesis), it follows that
\[ T_sc_w = -q^{-1}c_w, \]
as required.

Now consider the case that $s \in \text{WD}(w)$. Definition 5.1 gives
\[ (T_s + q^{-1})b_w = 0, \]
and so by (7.1),
\[ (T_s + q^{-1})c_w = -q \sum_{y < w} q_{y,w}(T_s + q^{-1})c_y. \]
If $y < w$ then $(T_s + q^{-1})c_y = 0$ if $s \in D(y)$, while if $s \notin D(y)$ then
\[ (T_s + q^{-1})c_y = (q + q^{-1})c_y + v_y \]
for some $v_y$ in $\mathcal{S}_s^-$, the subspace of $\mathcal{S}$ spanned by $\{c_x \mid s \in D(x)\}$. Hence
\[ (T_s + q^{-1})c_w = \left(-q \sum_{y \in Y} q_{y,w}(q^2 + 1)c_y\right) + v \]
for some $v \in \mathcal{S}_s^-$, where $Y = \{y \mid y < w \text{ and } s \notin D(y)\}$. Since the map $v \mapsto \nabla$ fixes $(T_s + q^{-1})c_w$ it follows that when $(T_s + q^{-1})c_w$ is expressed as a linear combination of the the basis elements $c_y$, all the coefficients are fixed. Hence
\[ (q^2 + 1)q_{y,w} = (q^2 + 1)q_{y,w} \]
for all $y < w$ such that $s \notin D(y)$. But since $(q^2 + 1)q_{y,w}$ is a polynomial in $q$ this forces it to be a constant, and hence forces $q_{y,w} = 0$. So all the terms on the right-hand side of (7.2) disappear, and
\[ T_sc_w = -q^{-1}c_w \]
as required. Note that this argument has shown that for all $s$ such that $s \in \text{WD}(w)$, the right-hand side of (7.1) involves only elements $c_y$ such that $s \in D(y)$.

Suppose next that $s \in \text{SA}(w)$, so that $w < sw \in \mathcal{S}$. By Lemma 7.2 and Definition 5.1 we have
\[ T_sc_w = T_sb_w - q \sum_{y < w} q_{y,w}T_sc_y \]
\[ = b_{sw} - q \sum_{y < w} q_{y,w}T_sc_y \]
\[ = c_{sw} + q \sum_{y < sw} q_{y,sw}c_y - q \sum_{y < w} q_{y,w}T_sc_y. \]
Applying the inductive hypothesis to evaluate the \( T_s c_y \) in the second sum gives

\[
(T_s - q)c_w = c_{sw} - qc_w + q \sum_{y<s,w} q_{y,s,w}c_y + \sum_{y \in R(s,w)} q_{y,w}c_y
\]

\[
- q \sum_{y \in R'(s,w)} q_{y,w} \left( qc_y + c_{sy} + \sum_{x \in R(s,y)} \mu_{x,y}c_x \right)
\]

where \( c_{sy} \) is to be interpreted as zero if \( s \in WA(y) \), and we have written \( R'(s,w) \) for the set of all \( y < w \) such that \( s \notin D(y) \). Now since there are no negative powers of \( q \) appearing in any of the coefficients on the right hand side, but \( (T_s - q)c_w = (T_s - q)c_w \), we deduce that all the coefficients must simply be integers, and the positive powers of \( q \) must cancel out. So

\[
(T_s - q)c_w = c_{sw} + \sum_{y \in R(s,w)} \mu_{y,w}c_y,
\]

where \( \mu_{y,w} \) is the constant term of \( q_{y,w} \), as required.

As a by-product of the above calculations we have shown that

\[
- qc_w + q \sum_{y<s,w} q_{y,s,w}c_y + \sum_{y \in R(s,w)} (q_{y,w} - \mu_{y,w})c_y
\]

\[
= q \sum_{y \in R(s,w)} q_{y,w} \left( qc_y + c_{sy} + \sum_{x \in R(s,y)} \mu_{x,y}c_x \right),
\]

whenever \( w < sw \in \mathcal{I} \). We shall return to this below, and use it to obtain a recursive formula for the polynomials \( q_{y,w} \).

Finally, suppose that \( s \in WA(w) \). By (i) of Definition 5.1 this gives

\[
(T_s - q)b_w = - \sum_{y<w} r_{y,w}^s b_y,
\]

for some \( r_{y,w}^s \in qA^+ \), so that by Lemma 7.2

\[
(T_s - q)c_w + q \sum_{y<w} q_{y,w}(T_s - q)c_y = - \sum_{y<w} r_{y,w}^s \left( c_y + q \sum_{x<y} q_{x,y}c_x \right).
\]

Hence \( (T_s - q)c_w \) is equal to

\[
\sum_{y \in R(s,w)} q_{y,w}(q^2 + 1)c_y - \sum_{y \in R'(s,w)} qq_{y,w} \left( c_{sy} + \sum_{x \in R(s,y)} \mu_{x,y}c_x \right)
\]

\[
- \sum_{y<s,w} r_{y,w}^s \left( c_y + q \sum_{x<y} q_{x,y}c_x \right),
\]

where again \( c_{sy} \) is interpreted as 0 if \( s \in WA(y) \). Since \( (T_s - q)c_w = (T_s - q)c_w \) it follows again that all terms involving positive powers of \( q \) must cancel out; this includes all of \( \sum_{y<s,w} r_{y,w}^s(c_y + q \sum_{x<y} q_{x,y}c_x) \) since \( r_{y,w}^s \in qA^+ \). Hence

\[
(T_s - q)c_w = \sum_{y \in R(s,w)} \mu_{y,w}c_y,
\]

where \( \mu_{y,w} \) is the constant term of \( q_{y,w} \), as required. \( \square \)

Returning now to (7.3), which holds whenever \( w < sw \in \mathcal{I} \), we proceed to derive the promised recursive formula for the polynomials \( q_{y,w} \).
Observe first that \( c_w \) does not occur on the right hand side of (7.3) or in the last sum on the left hand side; hence it follows that \( q_{w,sw} = 1 \). Next, examining the coefficients of \( c_z \) when \( z \in \mathcal{R}'(s,w) = \{ z < w \mid s \notin D(z) \} \) gives \( q_{z,sw} = q_{z,w} \) in this case. (Note that when \( z \notin w \) and \( s \notin D(z) \) the conditions \( z < w \) and \( z < sw \) are equivalent, by Lemma 2.3.) Finally, suppose that \( z < sw \) and \( s \in D(z) \). If \( z \notin w \) then \( z = sy \) for some \( y \in \mathcal{R}'(s,w) \), and the coefficient of \( c_z \) on the right hand side of (7.3) is \( q_{qs,z,w} \), while on the left hand side it is \( q_{z,sw} \). Thus \( q_{z,sw} = q_{z,w} \) in this case. If \( z < w \) and \( s \in SD(z) \) then \( sz \in \mathcal{R}'(s,w) \), and we see that \( c_z \) occurs on the right hand side of (7.3) as \( csy \) when \( y = sz \), and also occurs in the sums \( \sum_{x \in \mathcal{R}(s,y)} \mu_x y c_x \) for those \( y \in \mathcal{R}'(s,w) \) such that \( z < y \). Thus the coefficient of \( c_z \) on the right hand side of (7.3) is \( q_{qs,z,w} + q \sum_{y} \mu_y q_{y,w} \), where the sum is over all \( y \in \mathcal{I} \) such that \( z < y < w \) and \( s \notin D(y) \). On the left hand side of (7.3) the coefficient of \( c_z \) is \( q_{z,sw} + (q_{z,w} - \mu_z,w) \). Hence

\[
q_{z,sw} = -q^{-1}(q_{z,w} - \mu_z,w) + q_{sz,w} + \sum_y \mu_y q_{y,w}
\]

where the sum is over all \( y \in \mathcal{I} \) such that \( z < y < w \) and \( s \notin D(y) \). If \( z < w \) and \( s \in WD(z) \) then we obtain the same formula without the \( q_{sz,w} \) term.

We have proved the following result.

**Corollary 7.4.** Suppose that \( w < sw \in \mathcal{I} \) and \( y < sw \). If \( y = w \) then \( q_{y,sw} = 1 \), and if \( y \neq w \) we have the following formulas:

(i) \( q_{y,sw} = q_{y,w} \) if \( s \in A(y) \),

(ii) \( q_{y,sw} = -q^{-1}(q_{y,w} - \mu_y,w) + q_{sy,w} + \sum_x \mu_y x q_{x,w} \) if \( s \in SD(y) \),

(iii) \( q_{y,sw} = -q^{-1}(q_{y,w} - \mu_y,w) + \sum_x \mu_y x q_{x,w} \) if \( s \in WD(y) \),

where \( q_{y,w} \) and \( \mu_y,w \) are regarded as 0 if \( y \notin w \), and in (ii) and (iii) the sums extend over all \( x \in \mathcal{I} \) such that \( y < x < w \) and \( s \notin D(x) \).

The following result follows easily from Corollary 7.4 by induction on \( l(w) - l(y) \).

**Proposition 7.5.** Let \( y < w \in \mathcal{I} \). Then the degree of \( q_{y,sw} \) is at most \( l(w) - l(y) - 1 \).

Now let \( \mu: C \times C \to \mathbb{Z} \) be given by

\[
\mu(c_y, c_w) = \begin{cases} 
\mu_y,w & \text{if } y < w \\
\mu_w,y & \text{if } w < y \\
0 & \text{otherwise,}
\end{cases}
\]

and let \( \tau \) from \( C \) to the power set of \( S \) be given by \( \tau(c_w) = D(w) \) for all \( y \in \mathcal{I} \).

**Theorem 7.6.** The triple \((C, \mu, \tau)\) is a W-graph.

**Proof.** In view of Theorem 7.3 it suffices to show that for all \( w \in \mathcal{I} \) and \( s \in S \), if \( s \in WA(w) \) then the set \( \{ y \in \mathcal{I} \mid s \in D(y) \text{ and } \mu(c_y, c_w) \neq 0 \} \) contains no elements \( y > w \), while if \( s \in SA(w) \) then the only such element is \( sw \), and \( \mu(c_{sw}, c_w) = 1 \).

Accordingly, suppose that \( w < y \in \mathcal{I} \) with \( \mu_{w,y} \neq 0 \), and suppose that \( s \in D(y) \cap A(w) \). As noted in the proof of Theorem 7.3 if \( s \in WD(y) \) then \( q_{z,y} = 0 \) for all \( z < y \) with \( s \in A(y) \); in particular, \( q_{w,y} = 0 \), contradicting \( \mu_{w,y} \neq 0 \). Hence \( s \in SD(y) \). Now define \( x = sy \), so that \( x < sz \in \mathcal{I} \), and observe by Corollary 7.4 (i) that \( q_{w,xx} = q_{w,x} \) if \( w \neq x \). Since this contradicts \( \mu_{w,y} \neq 0 \) we conclude that \( w = x \), and \( q_{w,xx} = 1 \), by Corollary 7.4. So \( y = sw \) and \( \mu_{w,sw} = 1 \), as required. □
Proposition 7.7. Let the bases $B = (b_w \mid w \in \mathcal{J})$ and $C = (c_w \mid w \in \mathcal{J})$ be as in Lemma 7.2 above. Then there exist polynomials $p_{y,w} \in \mathcal{A}^+$ such that $c_w = b_w - q \sum_{y < w} p_{y,w}b_y$ for all $w \in \mathcal{J}$, and the constant term of $p_{y,w}$ is $\mu_{y,w}$.

Proof. It follows readily from Equation (7.1) that the required polynomials $p_{y,w}$ are given recursively by

$$
(7.7) \quad p_{y,w} = q_{y,w} - \sum_{y < x < w} q_{y,x}q_{x,w} \quad \text{if } y < w,
$$

whence the constant term of $p_{y,w}$ equals that of $q_{y,w}$. ∎

For our final theoretical result of this section, we show that if $\mathcal{J}$ is a $W$-graph ideal that is generated by a single ($W$-graph determining) element, then in Part (i) of Definition 5.1 in the case $s \in WA_f(w)$, the sum $\sum_{y \in \mathcal{J}, y < s} r_{y,w}^sb_y$ can be replaced by the simpler $\sum_{y \in \mathcal{J}, y < s} r_{y,w}b_y$.

Lemma 7.8. Suppose that $x, y, v, w \in W$ and $s \in S$ satisfy

1. $xy = vw$ and $l(xy) = l(x) + l(y) = l(v) + l(w)$,
2. $sw > w$ and $vs > v$,
3. $y \leq sw$.

Then $y \leq w$.

Proof. Assume that $x, y, v, w$ and $s$ satisfy the stated hypotheses. If $sy > y$ then the desired conclusion follows immediately from the hypotheses $y \leq sw$ and $sw > w$, by Lemma 2.3. So we may assume that $sy < y$. With this extra hypothesis, we use induction on $l(w)$ to prove the result.

If $l(w) = 0$ then the hypothesis (3) becomes $y \leq s$, and since $sy < y$ it follows that $y = s$. So $l(x) + l(y) = l(v) + l(w)$ becomes $l(x) = l(v) - 1$, and $xy = vw$ becomes $xs = v$, which together contradict the hypothesis $vs > v$. So the result is vacuously true in this case.

Now suppose that $l(w) > 0$ and that the result holds in all cases corresponding to shorter $w$. Choose $r \in S$ such that $w' = wr < w$. Note that since

$$
l(w') + 1 = l(w) < l(sw) = l(sw') \leq l(sw') + 1
$$

and also

$$
l(sw') \leq l(w') + 1 = l(w) < l(sw) = l(sw')\n$$

it follows that $w' < sw'$ and $sw' < sw'$.

Suppose first that $yr > y$. By hypothesis (1),

$$
l(xyr) = l(vwr) = l(vw') \leq l(v) + l(w') = l(v) + l(w) - 1 = l(xy) - 1,
$$

and so $xyr = x'y$ for some $x'$ with $l(x') = l(x) - 1$, by the Exchange Condition. Moreover, since $sw' < sw'$ and $x' = x'$, it follows from Lemma 2.3 and the hypothesis $y < sw'$ that $y \leq sw'$. So now we have

(1') $x'y = vw'$ and $l(x'y) = l(x') + l(y) = l(v) + l(w')$,
(2') $sw' > w'$ and $vs > v$,
(3') $y \leq sw'$,

and since $l(w') < l(w)$ the inductive hypothesis gives $y \leq w'$. But $w' < w$; so $y \leq w$ in this case.

It remains to consider the case $yr < y$. Put $y' = yr$, and observe that

$$
l(x'y) \leq l(x) + l(y') = l(x) + l(y) - 1 = l(xy) - 1 \leq l(xyr) = l(xy'),
$$

and since $l(xy') < l(x)$ and $l(x) \leq l(xy)$ it follows from Lemma 2.3 that $y \leq sw'$.

(1') $x'y = vw'$ and $l(x'y) = l(x') + l(y) = l(v) + l(w')$,
(2') $sw' > w'$ and $vs > v,$
(3') $y \leq sw'$,

and since $l(w') < l(w)$ the inductive hypothesis gives $y \leq w'$. But $w' < w$; so $y \leq w$ in this case.
so that \( l(xy') = l(x) + l(y') \). The same argument gives \( l(vw') = l(v) + l(w') \). And since \( sw' < sw'w = sw \) and \( y' < y'w = y \), it follows from the hypothesis \( y < sw \) and Lemma 2.8 that \( y' < sw' \). So now we have

\[
\begin{align*}
(1') \quad & x'y = vw' \text{ and } l(xy') = l(x) + l(y') = l(v) + l(w'), \\
(2') \quad & sw' > w' \text{ and } vs > v, \\
(3') \quad & y' \leq sw',
\end{align*}
\]

and since \( l(w') < l(w) \) the inductive hypothesis gives \( y' \leq w' \). Since \( y' < y'w = y \) and \( w' < w'r = w \), this yields \( y < w \), by Lemma 2.8.

**Proposition 7.9.** Suppose that \( u \in W \) and \( J = \{ w \in W \mid w \leq_L u \} \) is a W-graph ideal with respect to \( J \). With all the notation as in Definition 5.1, if \( w \in J \) and \( s \in WAJ(w) \), then every \( y \in J \) with \( y < sw \) satisfies \( y \leq w \).

**Proof.** Suppose that \( w \in J \) and \( s \in WAJ(w) \), and that \( y \in J \) with \( y < sw \). Since \( w \) and \( y \) are in \( J \) they are both suffixes of \( u \), and so there exist \( x, v \in W \) with \( u = xy = vw \) and \( l(u) = l(x) + l(y) = l(v) + l(w) \). If \( v' = vs < v \) then \( u = (v's)w = v'(sw) \), showing that \( sw \leq_L u \) since

\[
l(u) = l(v') + l(w) = l(v'w) + 1 + l(sw).
\]

Since this contradicts the assumption that \( s \in WAJ(w) \), it follows that \( vs > v \), and hence all the hypotheses of Lemma 7.8 are satisfied. So \( y \leq w \), as required. □

The recursive nature of Corollary 7.4 makes it relatively straightforward to implement calculation of the polynomials \( q_{y,w} \) (and hence the W-graph edge-weights \( \mu_{y,w} \)) using a computational algebra program. We outline one possible way to do this.

Assume that the elements of \( J \) are listed as \( w_1, w_2, \ldots, w_d \), where \( i \leq j \) implies that \( l(w_i) \leq l(w_j) \), and let \( S = \{ s_1, s_2, \ldots, s_n \} \). The input to the process is an array \( \text{tab} \) such that \( \text{tab}[i,j] = j \) if \( s_i \) is a weak ascent of \( w_j \) and \( \text{tab}[i,j] = -j \) if \( s_i \) is a weak descent of \( w_j \), while \( \text{tab}[i,j] = k \) if \( s_i \) is a strong ascent or strong descent of \( w_j \) and \( s_i w_j = w_k \). It is convenient to precompute another array \( \text{descents} \) such that

\[
\text{descents}[j] = \{ i \mid \text{tab}[i,j] < j \}.
\]

We can now define a function \( Q \) such that \( Q(j,k) \) returns the polynomial \( q_{y,z} \) if \( y = w_j < z = w_k \), and returns 0 otherwise.

If \( j \geq k \) then \( Q(j,k) \) immediately returns 0. Otherwise the set \( \text{descents}[k] \) is searched for an \( s \) with \( \text{tab}[s,k] = m > 0 \); note that since \( m < k \) the value of \( Q(j,m) \) can be used in the calculation of \( Q(j,k) \). By (i) of Corollary 7.4, \( Q(j,k) \) can be set equal to \( q * Q(j,m) \) if \( s \) is not in \( \text{descents}[j] \). If \( s \) is in \( \text{descents}[j] \) then (ii) of Corollary 7.4 is applicable if \( \text{tab}[s,j] > 0 \), while (iii) is applicable if \( \text{tab}[s,j] < 0 \). Interpreting \( Q(\text{tab}[s,j],m) \) as zero in this latter case, the formula for \( Q(j,k) \) becomes

\[
Q(j,k) = ((\mu(j,m) - Q(j,m))/q) + Q(\text{tab}[s,j],m) + \text{Sum}
\]

where \( \mu(j,m) \) is the constant term of \( Q(j,m) \), and \( \text{Sum} \) denotes the sum of the values \( \mu(j,i) * Q(i,m) \) for \( i \) in the range \( j < i < m \).

8. The Kazhdan-Lusztig and Deodhar Constructions

Since every \( u \in W \) occurs as a suffix of the longest element \( w_S \), the ideal of \( (W, \leq_L) \) generated by \( w_S \) is the whole of \( W \). We seek to show that \( w_S \) is a W-graph determining element, or, equivalently, that \( W \) is a W-graph ideal. We are forced
to let $J = \emptyset$ so that the requirement $W \subseteq D_J$ is satisfied, and this means that the sets $WA_J(w)$ and $WD_J(w)$ are empty for all $w \in W$. Hence to show that $w_S$ is a $W$-graph determining element we need to produce an $H$-module with an $\mathcal{A}$-basis $(b_w \mid w \in W)$ such that for all $s \in S$ and $w \in W$,

$$T_s b_w = \begin{cases} b_{sw} & \text{if } sw > w \\ b_{sw} + (q - q^{-1})b_w & \text{if } sw < w. \end{cases}$$

The module must also admit an $\mathcal{A}$-semilinear involution such that $T_w b_{l_1} = T_w b_{l_1}$ for all $w \in W$. Since these conditions are obviously satisfied if we put $b_w = T_w$ for all $w \in W$, the required module is the left regular module $H$. Thus our construction in Section 7 will produce a $W$-graph basis of $H$, and combining Propositions 7.5 and 7.7 yields the following result.

**Proposition 8.1.** The Hecke algebra $H$ has a $W$-graph basis $(c_w \mid w \in W)$ such that $c_w = c_w$ and $c_w = T_w - \sum_{y < w} q_p y, w T_y$ for all $w \in W$, where $p_{y, w}$ is a polynomial of degree at most $l(w) - l(y) - 1$ and the $W$-graph edge-weight $\mu_{y, w}$ is the constant term of $p_{y, w}$.

Converting the traditional version of $H$ as used in [10] to our version requires replacing $q$ by $q^2$, after which the $T_w$ of [10] becomes $q^{l(w)} T_w$ in our context. So the formula in [10] Theorem 1.1], when converted to our context, becomes

$$C_w = \sum_{y \leq w} (-1)^{l(w) - l(y) - 1} q^{l(w) - 2l(y)} P_{y, w}^* (q^{(y)/y})$$

where $P_{y, w}^*$ is obtained from the Kazhdan-Lusztig polynomial $P_{y, w}$ by replacing $y$ by $q^{-2}$. Since $P_{y, w}$ is 1 the coefficient of $T_w$ on the right hand side of this expression is 1, and since $P_{y, w}$ is a polynomial of degree at most $\frac{1}{2}(l(w) - l(y) - 1)$ when $y < w$ we see that the coefficient of $T_y$, namely $(-q)^{l(w) - l(y)} P_{y, w}^*$, is a polynomial in $q$ with zero constant term. Since also $C_w = c_w$, from which we can deduce a simple relationship between our polynomials $p_{y, w}$ and the Kazhdan-Lusztig polynomials.

**Proposition 8.2.** The polynomials $p_{y, w}$ appearing in Proposition 8.1 are related to the Kazhdan-Lusztig polynomials $P_{y, w}$ via

$$p_{y, w} = (-q)^{l(w) - l(y) - 1} P_{y, w}^* .$$

where $P_{y, w}^*$ is obtained from $P_{y, w}$ by replacing $y$ by $q^{-2}$. In particular, the coefficient of $q^{\frac{1}{2}(l(w) - l(y) - 1)}$ in $P_{y, w}$ is $(-1)^{l(w) - l(y) - 1} \mu_{y, w}$.

Note that Kazhdan and Lusztig show that $\mu_{y, w} \neq 0$ only if $l(w) - l(y) - 1$ is even.

Turning now to Deodhar’s construction, let $J$ be an arbitrary subset of $S$ and let $d_J$ be the longest element of $D_J$ (which is the shortest element of $w_S W_J$). An element $u \in W$ is a suffix of $d_J$ if and only if $u \in D_J$, and so the ideal $\mathcal{I}$ of $(W, \leq_L)$ generated by $d_J$ coincides with $D_J$. Clearly $\text{Pos}(\mathcal{I}) = J$. We shall show that $\mathcal{I} = D_J$ is a $W$-graph ideal with respect to $J$, and also that it is a $W$-graph ideal with respect to $\emptyset$. We consider the latter case first.

Since $D_\emptyset = W$, it follows from the definitions in Section 5 that if $w \in \mathcal{I}$ then $SA(w) = \{ s \in S \mid sw > w \text{ and } sw \in D_J \}$ and $SD(w) = \{ s \in S \mid sw < w \}$, while $WD_\emptyset(w) = \{ s \in S \mid sw \notin D_\emptyset \} = \emptyset$ and

$$WA_\emptyset(w) = \{ s \in S \mid sw \in D_\emptyset \setminus D_J \} = \{ s \in S \mid sw = wt \text{ for some } t \in J \}$$

where $d_J$ the longest element of $D_J$ (which is the shortest element of $w_S W_J$). An element $u \in W$ is a suffix of $d_J$ if and only if $u \in D_J$, and so the ideal $\mathcal{I}$ of $(W, \leq_L)$ generated by $d_J$ coincides with $D_J$. Clearly $\text{Pos}(\mathcal{I}) = J$. We shall show that $\mathcal{I} = D_J$ is a $W$-graph ideal with respect to $J$, and also that it is a $W$-graph ideal with respect to $\emptyset$. We consider the latter case first.
by Lemma 2.3. We proceed to construct an $\mathcal{H}$-module $\mathcal{J}$ satisfying the requirements of Definition 5.1 (Our module $\mathcal{J}$ is essentially the module $M^J$ in [3], in the case $u = q$, the only differences being due to our non-traditional definition of $\mathcal{H}$.) Let $\mathcal{H}_J$ be the Hecke algebra associated with the Coxeter system $(W_J, J)$, and recall that $\mathcal{H}_J$ can be identified with the subalgebra of $\mathcal{H}$ spanned by $\{T_u \mid u \in W_J\}$. There is an $\mathcal{A}$-algebra homomorphism $\psi: \mathcal{H}_J \rightarrow \mathcal{A}$ such that $\psi(T_u) = q^{l(u)}$ for all $u \in W_J$, and this can be used to give $\mathcal{A}$ the structure of an $\mathcal{H}_J$-module, which we denote by $\mathcal{A}_\psi$. Since $\mathcal{H}$ is obviously an $(\mathcal{H}, \mathcal{H}_J)$-bimodule, the tensor product $\mathcal{J}_\psi = \mathcal{H} \otimes_{\mathcal{H}_J} \mathcal{A}_\psi$ is a (left) $\mathcal{H}$-module, and it is straightforward to show that it is $\mathcal{A}$-free with basis $B = \{b_w \mid w \in D_J\}$ defined by $b_w = T_w \otimes 1$ for all $w \in D_J$.

Let $w \in D_J$ and $s \in S$. If $s \in \text{SA}(w)$ then $l(sw) > l(w)$, and so

$$T_s b_w = T_s(T_w \otimes 1) = (T_sT_w) \otimes 1 = T_{sw} \otimes 1 = b_{sw}$$

since $sw \in D_J$. If $s \in \text{SD}(w)$ then $l(sw) < l(w)$, and so

$$T_s b_w = (T_sT_w) \otimes 1 = (T_{sw} + (q - q^{-1})T_w) \otimes 1 = b_{sw} + (q - q^{-1})b_v$$

since again $sw \in D_J$. There are no weak descents, and if $s \in \text{WA}(w)$ then there is a $t \in J$ with $sw = wt$, and we find that

$$T_s b_w = (T_wT_t) \otimes 1 = (T_wT_t) \otimes 1 = T_w \otimes \psi(T_t) = q b_w.$$ 

So the action of the generators $\{T_s \mid s \in S\}$ on the basis $B$ is in accordance with the requirements of Definition 5.1 (ii) (with all the polynomials $r_{\alpha,w}$ being zero), and it only remains to check that $\mathcal{J}_\psi$ admits an $\mathcal{A}$-semilinear involution satisfying the requirements of Definition 5.1 (ii). We include a proof here for the sake of completeness, although the result is proved in [3].

We show that the unique $\mathcal{A}$-semilinear map $\mathcal{J}_\psi \rightarrow \mathcal{J}_\psi$ satisfying $\bar{T}_w = T_w \otimes 1$ for all $w \in D_J$ has the required properties. Note first that $\psi(T_u) = \psi(T_u) = \psi(T_w)$ for all $u \in W_J$. Now if $x \in W$ is arbitrary then we may write $x = wu$ for some $w \in D_J$ and some $u \in W_J$, and we find that

$$\bar{T}_x = \bar{T}_u \bar{T}_w \otimes 1 = \bar{T}_w \otimes \psi(T_u) = \psi(T_u)(T_w \otimes 1) = \psi(T_u)\bar{T}_w \otimes 1 = (\bar{T}_wT_u) \otimes 1.$$ 

Hence $\bar{k} \otimes 1 = \bar{k} \otimes 1$ for all $k \in \mathcal{H}$, and so

$$\bar{h}(k \otimes 1) = (\bar{h}k) \otimes 1 = \bar{h}k \otimes 1 = (\bar{h} \bar{k}) \otimes 1 = \bar{h}(k \otimes 1)$$

for all $h, k \in \mathcal{H}$. So $\bar{h}\alpha = \bar{h}\alpha$ for all $h \in \mathcal{H}$ and $\alpha \in \mathcal{J}_\psi$, as required.

Since the requirements of Definition 5.1 have all been met, the construction in Section 7 above produces a $W$-graph basis in the module $\mathcal{J}_\psi$. This basis corresponds to the basis of $M^J$ in Proposition 3.2 (iii) of [3] (in the case $u = q$). Deodhar’s polynomials $P^J_{y,w}$ and our polynomials are related by the obvious modification of (8.1) above.

**Proposition 8.3.** The $\mathcal{H}$-module $\mathcal{J}_\psi$ has a $W$-graph basis $(c_w \mid w \in D_J)$ such that $c_w = c_w$ and $c_w = b_w - \sum_{y < w} q p^J_{y,w} b_y$ for all $w \in W$, where $p^J_{y,w}$ is a polynomial of degree at most $l(w) - l(y) - 1$ and the $W$-graph edge-weight $\mu_{y,w}$ is the constant term of $p^J_{y,w}$. The polynomials $p^J_{y,w}$ are related to Deodhar’s polynomials $P^J_{y,w}$ via

$$p^J_{y,w} = (-q)^{l(w) - l(y) - 1} P^*_y,$$

where $P^*_y$ is obtained from $P^J_{y,w}$ by replacing $q$ by $q^{-2}$. 


The proof that $\mathcal{I} = D_J$ is a $W$-graph ideal with respect to $J$ is very similar to the proof just given. We find that

\[
\begin{align*}
SA(w) &= \{ s \in S \mid sw > w \text{ and } sw \in D_J \}, \\
SD(w) &= \{ s \in S \mid sw < w \}, \\
WA_J(w) &= \{ s \in S \mid sw \in D_J \setminus D_J \} = \emptyset,
\end{align*}
\]

and

\[
WD_J(w) = \{ s \in S \mid sw \notin D_J \} = \{ s \in S \mid sw = wt \text{ for some } t \in J \}.
\]

Thus the weak ascents of the previous case are now weak descents, and vice versa. The corresponding $H$-module is $\mathcal{I}_0 = H \otimes H_J A_\phi$, where $A_\phi$ is $A$ made into an $H_J$-module via the homomorphism $\phi: H_J \rightarrow A$ that satisfies $\phi(T_u) = (-q)^{l(u)}$ for all $u \in W_J$. This corresponds to $M^J$ in (3) in the case $u = -1$. We again define $b_w = T_w \otimes 1$ for all $w \in D_J$, and this time we find that

\[
T_s b_w = \begin{cases} 
  b_{sw} & \text{if } w \in SA(w), \\
  b_{sw} + (q - q^{-1})b_w & \text{if } w \in SD(w), \\
  -q^{-1}b_w & \text{if } w \in WD_J(w)
\end{cases}
\]

in accordance with the requirements of Definition 5.1. The proof that $\mathcal{I}_0$ admits an $A$-semilinear involution with the required properties is exactly as in the previous case. Again the $W$-graph basis given by our construction is essentially the same as the basis of $M^J$ in Proposition 3.2 (iii) of [3] (now in the case $u = -1$).

**Proposition 8.4.** The $H$-module $\mathcal{I}_0$ has a $W$-graph basis $(c_w \mid w \in D_J)$ such that $c_w = c_w$ and $c_w = b_w - \sum_{y < w} q p_{y,w} b_y$ for all $w \in W$, where $p_{y,w}$ is a polynomial of degree at most $l(w) - l(y) - 1$ and the $W$-graph edge-weight $\mu_{y,w}$ is the constant term of $p_{y,w}$. The polynomials $p_{y,w}^J$ are related to Deodhar’s polynomials $p_{y,w}$ via

\[
p_{y,w}^J = (-q)^{l(w)-l(y)-1} p_{y,w}^*.
\]

where $p_{y,w}^*$ is obtained from $p_{y,w}$ by replacing $q$ by $q^{-2}$.

We remark that the above constructions are special cases of a more general construction to be described in the next section. If $J \subseteq S$ and $J = J_1 \cup J_2$, where no element of $J_1$ is conjugate in $W_J$ to any element of $J_2$, then $H_J$ has a one-dimensional module on whose basis element $b_1$ the generators $T_s$ of $H_J$ act as follows:

\[
T_s b_1 = \begin{cases} 
  -q^{-1}b_1 & \text{if } s \in J_1, \\
  qb_1 & \text{if } s \in J_2.
\end{cases}
\]

Thus the subset of $W_J$ consisting of the identity element alone is a $W_J$-graph ideal with respect to $J_1$, with $D(1) = WD(1) = J_1$ and $A(1) = WA(1) = J_2$. By Theorem 4.2 below it follows that $D_J$ is a $W$-graph ideal with respect to $J_1$. Deodhar’s two constructions correspond to the cases $J_1 = \emptyset$ and $J_1 = J$.

9. **Induced $W$-graph ideals**

Let $K \subseteq S$, and let $H_K$ be the Hecke algebra associated with the Coxeter system $(W_K, K)$, identified with a subalgebra of $H$ as in Section 8 above. Suppose that $\mathcal{I}_0 \subseteq W_K$ is a $W_K$-graph ideal with respect to $J \subseteq K$, and let $\mathcal{I}_0 = \mathcal{I}(\mathcal{I}_0, J)$ be
the corresponding $\mathcal{H}_K$-module. Thus $\mathcal{J}_0$ has an $\mathcal{A}$-basis \((b^0_t \mid z \in \mathcal{J}_0)\) such that for all \(t \in K\) and \(z \in \mathcal{J}_0\),

\[
T_t b^0_z = \begin{cases} 
  b^0_{tz} & \text{if } t \in \text{SA}(K, z), \\
  b^0_{tz} + (q - q^{-1})b^0_z & \text{if } t \in \text{SD}(K, z), \\
  -q^{-1}b^0_z & \text{if } t \in \text{WD}_J(K, z), \\
  qb^0_z - \sum_{y \in \mathcal{J}_0 \cap z<tz} r^t_{y,z} b^0_y & \text{if } t \in \text{WA}_J(K, z),
\end{cases}
\]

(9.1) for some \(r^t_{y,z} \in q\mathcal{A}^+\), where the descent and ascent sets are given by

- \(\text{SA}(K, z) = \{ t \in K \mid tz > z \text{ and } tz \in \mathcal{J}_0 \}\),
- \(\text{SD}(K, z) = \{ t \in K \mid tz < z \}\),
- \(\text{WA}_J(K, z) = \{ t \in K \mid tz \notin \mathcal{J}_0 \text{ and } z^{-1}tz \notin J \}\),
- \(\text{WD}_J(K, z) = \{ t \in K \mid tz \notin \mathcal{J}_0 \text{ and } z^{-1}tz \notin J \}\).

Furthermore, \(\mathcal{J}_0\) admits an \(\mathcal{A}\)-semilinear involution \(\alpha \mapsto \overline{\alpha}\) satisfying \(\overline{\alpha}^2 = b^0_t = b^0_t\) and \(\overline{h} = \overline{h}^\alpha\) for all \(h \in \mathcal{H}_K\) and \(\alpha \in \mathcal{J}_0\).

We shall show that \(\mathcal{J} = D_K\mathcal{J}_0 = \{ dz \mid d \in D_K \text{ and } z \in \mathcal{J}_0 \}\) is a W-graph ideal with respect to \(J\). The corresponding \(\mathcal{H}\)-module \(\mathcal{I}(\mathcal{J}, J)\) is \(\mathcal{I} = \mathcal{H} \otimes_{\mathcal{H}_K} \mathcal{J}_0\).

**Lemma 9.1.** The set \(\mathcal{J}\) defined above is an ideal of \((W, \leq_L)\).

**Proof.** In view of Definition 2.2 it suffices to show that \(sw \in D_K\mathcal{J}_0\) whenever \(s \in S\) and \(w \in D_K\mathcal{J}_0\) satisfy \(l(sw) < l(w)\).

Let \(w = dz\), where \(d \in D_K\) and \(z \in \mathcal{J}_0\). Let \(s \in S\), and suppose that \(l(sw) < l(w)\). If \(sd \in D_K\) then trivially \(sw = (sd)z \in D_K\mathcal{J}_0\). Now suppose that \(sd \notin D_K\). By Lemma 2.4 this gives \(sd = dt\) for some \(t \in K\), and since \(z \in \mathcal{J}_0 \subseteq W_K\) we see that \(tz \in W_K\). Hence, since \(d \in D_K\),

\[
l(tz) = l(dtz) - l(d) = l(sdz) - l(d) = l(sw) - l(d) \leq l(w) - l(d) = l(dz) - l(d) = l(z),
\]

Since \(t \in K\) and \(z \in \mathcal{J}_0\), and \(\mathcal{J}_0\) is an ideal of \((W_K, \leq_L)\), it follows that \(tz \in \mathcal{J}_0\). Hence \(sw = d(tz) \in D_K\mathcal{J}_0\) in this case also, as required. \(\square\)

For each \(w \in \mathcal{J}\) the sets of strong ascents, strong descents, weak ascents and weak descents of \(w\) relative to \(\mathcal{J}\) and \(J\) are defined as in Section 3 above. Note that each \(w \in W\) is uniquely expressible as \(dz\) with \(d \in D_K\) and \(z \in W_K\), and \(w \in \mathcal{J}\) if and only if \(z \in \mathcal{J}_0\). Moreover,

\[
\mathcal{J} = \bigoplus_{d \in D_K} T_d \mathcal{H}_K \otimes_{\mathcal{H}_K} \mathcal{J}_0 = \bigoplus_{d \in D_K} T_d \otimes \mathcal{J}_0
\]

and it follows that \(\mathcal{J}\) is \(\mathcal{A}\)-free with \(\mathcal{A}\)-basis \((T_d \otimes b^0_z \mid d \in D_K \text{ and } z \in \mathcal{J}_0)\). We define \(b_w = T_d \otimes b^0_z\) whenever \(w = dz\) as above, and proceed to show that for each \(s \in S\) and \(w \in \mathcal{J}\) the generator \(T_s\) of \(\mathcal{H}\) acts on the basis element \(b_w\) in accordance with Definition 3.1.

Let \(w = dz\), where \(d \in D_K\) and \(z \in \mathcal{J}_0\), and let \(s \in \text{SA}(w)\), so that \(w < sw \in \mathcal{J}\). Suppose first that \(sd \notin D_K\), so that \(d < sd = dt\) for some \(t \in K\), by Lemma 2.4. Then \(tz \in W_K\), and since \(d(tz) = sw \in D_K\mathcal{J}_0\), it follows that \(tz\) must be in \(\mathcal{J}_0\). Moreover, since \(l(w) < l(sw)\),

\[
l(tz) = l(d(tz)) - l(d) = l(sw) - l(d) > l(w) - l(d) = l(dz) - l(d) = l(z),
\]
and therefore $t \in \text{SA}(K, z)$. By (9.1) above it follows that
\[ T_s b_w = T_s T_d \otimes b_0^1 = T_d T_t \otimes b_0^1 = T_d \otimes T_t b_0^1 = T_d \otimes b_0^1 = b_{dt} = b_{sw} \]
in accordance with Definition 5.1. It remains to show that this same equation holds if $sd \in D_K$, and in this case we find that
\[ b_{sw} = b_{(sd)z} = T_{sd} \otimes b_0^1 = T_s T_d \otimes b_0^1 = T_s b_w, \]
as required.

Suppose now that $s \in \text{SD}(w)$, where $w = dz$ as above, so that $sw < w$. Suppose first that $sd \notin D_K$, so that $d < sd = dt$ for some $t \in K$, by Lemma 2.3. Then $tz \in W_K$, and since $l(w) < l(sw)$ it follows that
\[ l(tz) = l(d(tz)) - l(d) = l(sw) - l(d) < l(w) - l(d) = l(dz) - l(d) = l(z), \]
whence $t \in \text{SD}(K, z)$. By (9.1),
\[ T_s b_w = T_s T_d \otimes b_0^1 = T_d T_t \otimes b_0^1 = T_d \otimes T_t b_0^1 = T_d \otimes b_0^1 = (q - q^{-1})b_0^1 + (q - q^{-1})b_{dt} = b_{dtz} = b_{sw} + (q - q^{-1})b_{dw} \]
in accordance with Definition 5.1. It remains to show that this same equation holds if $sd \in D_K$. In this case $b_{sw} = b_{(sd)z} = T_{sd} \otimes b_0^1$, and we also find that $l(sd) = l((sd)z) - l(z) = l(sw) - l(dz) = l(z) - l(d) = l(d)$. So
\[ T_s b_w = T_s T_d \otimes b_0^1 = T_d \otimes (q - q^{-1})T_d \otimes b_0^1 = (T_{sd} \otimes b_0^1) + (q - q^{-1})(T_d \otimes b_0^1) = b_{sw} + (q - q^{-1})b_{dw} \]
as required.

Next, suppose that $s \in WD_J(w)$, where $w = dz$ as above, so that $sw \notin J$ and $w^{-1}sw \in J$. Since $sw = (sd)z$ and $z \in J_0$, the fact that $sw \notin J = D_KJ_0$ means that $sd \notin D_K$, and so $sd = dt$ for some $t \in K$, by Lemma 2.3. Moreover, $z^{-1}tz = z^{-1}d^{-1}sdz = w^{-1}sw \in J$, so that $t \in WD_J(K, z)$. By (9.1),
\[ T_s b_w = T_s T_d \otimes b_0^1 = T_d T_t \otimes b_0^1 = T_d \otimes T_t b_0^1 = T_d \otimes b_0^1 = (q - q^{-1})b_0^1 = T_d \otimes (-q^{-1})b_0^1 = -q^{-1}b_{dw} \]
in accordance with Definition 5.1.

Finally, suppose that $s \in WA_J(w)$, where $w = dz$ as above, so that $sw \notin J$ and $w^{-1}sw \notin J$. As in the preceding case it follows that $sd \notin D_K$, and $sd = dt$ for some $t \in K$, but now $z^{-1}tz = w^{-1}sw \notin J$. So $t \in WA_J(K, z)$, and by (9.1) it follows that
\[ T_s b_w = q b_0^1 - \sum_y r_{y,z} b_0^1 \text{ for some polynomials } r_{y,z} \in qA^+ \]  
(defined whenever $y < tz$ and $y \in J_0$). Hence
\[ T_s b_w = T_s T_d \otimes b_0^1 = T_d T_t \otimes b_0^1 = T_d \otimes T_t b_0^1 = T_d \otimes (q b_0^1 - \sum_y r_{y,z} b_0^1) \]
\[ = q(T_d \otimes b_0^1) - \sum_y r_{y,z}(T_d \otimes b_0^1) = qb_w - \sum_y r_{y,z} b_{dy} \]
where the sums range over $y \in J_0$ such that $y < tz$. Since $y \in J_0$ and $y < tz$ imply that $dy \in D_KJ_0 = J$ and $dy < dtz = sw$ (by Lemma 2.3 and an induction on $l(d)$), we conclude that in this case also the requirements of Definition 5.1 are satisfied.

To complete the proof that $J$ is a $W$-graph ideal with respect to $J$ it remains only to show that $J$ admits a semilinear involution $\alpha \mapsto \overline{\alpha}$ such that $\overline{h \alpha} = \overline{h} \overline{\alpha}$ for all $h \in H$ and $\alpha \in J$. The proof is very similar to the corresponding proofs
in Section 8 above: we set $T_d \otimes b_0^z = T_d \otimes b_0^z$ for all $d \in D_K$ and $z \in \mathcal{I}_0$, using semilinearity to extend the definition to the whole of $\mathcal{I}$. We omit further details.

The discussion above enables us to state the following theorem.

**Theorem 9.2.** Let $K \subseteq S$ and suppose that $\mathcal{I}_0 \subseteq W_K$ is a $W_K$-graph ideal with respect to $J \subseteq K$, and let $\mathcal{I} = D_K \mathcal{I}_0$ be the corresponding $H_K$-module. Then $\mathcal{I} = D_K \mathcal{I}_0$ is a $W_K$-graph ideal with respect to $J$, the corresponding $H$-module $\mathcal{I}(J, K)$ being isomorphic to $H \otimes_{H_K} \mathcal{I}_0$.

**Remark.** In the situation of Theorem 9.2 the assumption that $\mathcal{I}_0$ is a $W_K$-graph ideal in $(W_K, \leq_L)$ implies, by the construction in Section 7 that $\mathcal{I}_0$ is isomorphic to an $H_K$-module arising from a $W_K$-graph. By [8, Theorem 5.1] it follows that the induced module $\mathcal{I}$ is isomorphic to a $W$-graph module. Theorem 9.2 yields an alternative construction of the induced $W$-graph in this special case that the $W_K$-graph in question comes from a $W_K$-graph ideal in $(W_K, \leq_L)$.

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