Geometric momentum for a particle constrained on a curved hypersurface

Q. H. Liu

1School for Theoretical Physics, and Department of Applied Physics, Hunan University, Changsha, 410082, China. Tel/fax 86-731-88820378/86-731-88822332, Email: quanhuiliu@gmail.com

(Dated: May 17, 2013)

A strengthened canonical quantization scheme for the constrained motion on curved surface is proposed with introduction of the second category of fundamental commutation relations between Hamiltonian and positions/momenta, whereas those between positions and moments are categorized into the first. As an \( N-1 \) (\( N \geq 2 \)) dimensional surface is embedded in an \( N \) dimensional Euclidean space, we obtain the geometric momentum \( \mathbf{p} = -i\hbar(\nabla_S + Mn/2) \) where \( \nabla_S \) denotes the gradient operator on the surface and the \( Mn \) is the mean curvature vector. For the surface is the spherical one of radius \( r \), we resolve in a lucid and unambiguous manner a long-standing problem of the geometric potential that proves to be \( V_g = (N - 1)(N - 3)\hbar^2/(8mr^2) \).

PACS numbers: 03.65.-w, Quantum mechanics, 04.62.+v; Quantum fields in curved spacetime, 02.40.-k; Differential geometry.

I. INTRODUCTION

The quantum mechanics for a non-relativistic particle that is constrained to remain on a curved hypersurface attracts much attention [1–14]. As well-known, Dirac’s quantum theory for a constrained motion [15, 16] does not always produce physically significant results and usually exhibits certain difficulties in application [17]. For instance, we do not have a well-defined form for either momentum or Hamiltonian after quantization [4, 5, 12]. In this paper, we propose a strengthened canonical quantization scheme (SCQS) for the constrained motion on the surface. Explicitly, we deal with an \( N-1 \) (\( N \geq 2 \)) dimensional hypersurface \( S^{N-1} \) and its equation is either \( f(x) = 0 \) or parametric form \( x = \{x_i(n)\} \) in the \( N \) dimensional flat Euclidean space \( R^N \), where \( x_i \) (\( i,j,k,\ell = 1,2,3,\ldots,N \)) stand for the Cartesian coordinates and \( u^\mu \) (\( \mu, \nu = 1,2,3,\ldots,N-1 \)) symbolize the local coordinates on the surface. According to Dirac, such an constraint belongs to the second kind [16]. In our approach, we do not quantize the local coordinates \( u^\mu \) and corresponding momenta \( p_\mu \), whereas we treat \( u^\mu \) as parameters and quantize the Cartesian coordinates \( x_i \) and its corresponding momentum \( p_i \).

At first, let us briefly review and comment on the so-called confining potential technique that leads to the well-defined geometric potential [18]. For a particle constrained on the surface, we can establish an effective theory in the following way. First, to formulate the Schrödinger equation in \( R^N \), explicitly in a curved shell of an equal and finite thickness \( \delta \) along normal direction \( n \), and let the intermediate surface of the shell coincide with the prescribed one \( S^{N-1} \). And the particle moves within the range of the same width \( \delta \) due to a confining potential across the surface along the normal direction \( n \), such as one-dimensional parabolic one or simply the square potential well. Second, to take the limit \( \delta \rightarrow 0 \), we have an effective kinetic energy operator different from the well-known one \( -\hbar^2/(2m)\nabla^2_{LB} \) as \([5]\),

\[
-\frac{\hbar^2}{2m} \nabla_{LB}^2 \rightarrow -\frac{\hbar^2}{2m} (\nabla_{LB}^2 + V_g), \quad V_g = \frac{1}{4} (2Tr(k)^2 - (Trk)^2), \tag{1}
\]

where \( V_g \equiv -\hbar^2/(2m)v_g \) is the curvature-induced potential that is usually called as the geometric potential, and \( v_g \) is purely determined by the principal curvatures \( k \) [5]. This approach seems to suffer from a theoretical shortcoming: we do not know why such an establishment of the quantum theory can not be directly on the surface. If so, it predicts a vanishing geometric potential that would contradict the recent experiments [13, 14]. In fact, Dirac’s canonical quantization procedure simply excludes such an attempt, as we see shortly.

Next, let us recall of two differential geometric facts for hypersurfaces: 1. The definition of the mean curvature \( M = Trk = -\partial n_i/\partial x_i \) (rather than a true average \( Trk/(N - 1) \) which is also widely used) [19–22], and the mean curvature vector \( Mn \) which satisfies \( \nabla_S \cdot n = -M \) [19–22], where the surface gradient \( \nabla_S \) is defined by the difference of the usual gradient \( \nabla_N \) in \( R^N \) and its component along the normal direction \( n \partial_n \): \( \nabla_S = (\partial x/\partial u^\mu) \partial^\mu = e_i(\delta_{ij} - n_i n_j)\partial_j \) = \( \nabla_N - n \partial_n \) [21] with \( e_i \) being the unit vector of the ith Cartesian coordinate and \( \delta_{ij} - n_i n_j \) being the orthogonal projection from \( R^N \) to the plane tangential to the \( S^{N-1} \). 2. The Laplace-Beltrami operator \( \nabla_{LB}^2 \) is given by \( \nabla_{LB}^2 = \nabla_S^2 = \partial_i(\delta_{ij} - n_i n_j)\partial_j = \Delta_N + M\partial_n - \partial_n^2 \) with \( \Delta_N \equiv \partial_i\partial_i \) the usual Laplacian operator [19–22], and \( \nabla^2_{LB} x = Mn \) [19–22]. It is evident that \( \nabla_{LB}, \nabla_S, n, \) and \( Mn \) are all geometric invariants. The Einstein summation convention for repeated indices is used throughout the paper.
The organization of the paper is as follows. In section II, the SCQS is proposed, and as a consequence the geometric momentum is given. In section III, by use of the scheme we deal with the geometric potential for a particle on an $N - 1$ dimensional spherical surface and thus resolve a long-standing and highly controversial problem on the form of the geometric potential. Also in section IV, we show that the motion on the surface possesses a dynamical $SO(N, 1)$ group symmetry. In section V, we finally conclude and remark the present approach.

II. SECOND CATEGORY OF FUNDAMENTAL COMMUTATION RELATIONS

In our SCQS, there are two categories of fundamental commutation relations (FCR). The existent one is classified into the first one, which is for positions and momenta ($x_i$, $p_j$): $[x_i, x_j]$, $[x_i, p_j]$, and $[p_i, p_j]$. When constraints are released, these FCR reduce to $[x_i, p_j] = i\hbar\delta_{ij}$ and all other commutators vanishing. This is the fundamental postulate for positions and momenta in quantum mechanics, proposed by Dirac [15, 16], without requiring them to be Cartesian.

It holds true universally if applicable, but is not so practical for, e.g., a system that does have a classical analogue. For sake of the practicality, Dirac immediately developed his FCR with an additional hypothesis that the Hamiltonian is the same function of the canonical coordinates and momenta in the quantum theory as in the classical theory, provided that the Cartesian coordinates must be used [15]. As a consequence, the equation of motion $dO/dt = [O, H]_D$ shows that the motion of a system in quantum mechanics is determined by imposing the algebraic structure between observables $[O_1, O_2]_D$ that preserves in quantum mechanics to the extent possible, i.e., in quantum mechanics $[O_1, O_2] = i\hbar[O_1, O_2]_D$. Fundamentally, we introduce the secondary category of FCR as $[x, H] = i\hbar[x, H]_D$ and $[p, H] = i\hbar[p, H]_D$.

For a system with constraints of the second kind, Dirac’s canonical quantization procedure needs to be further strengthening for sake of the practicality. From this category of the FCR, there are many forms of the momentum $p$ and if substituting these forms of $p$ into Hamiltonian such as $H = p^2/2m + V$, there are various forms of Hamiltonian in quantum mechanics. It is then possibly to hypothesize that the forms of the momentum and the Hamiltonian are strengthened for sake of the practicality. From this category of the FCR, there are many forms of the Hamiltonian such as $H = m/n\delta_{ij}$, $[p, H] = i\hbar[p, H]_D$ would hardly be all satisfied, as illustrated by [12, 27, 28].

As a consequence of this observation, we resort to $R^N$ to deal with $S^{N-1}$ that is embedded in $R^N$ to perform the canonical quantization. For the $N - 1$ dimensional surface, we conveniently choose the equation of surface $f(x) = 0$ such that $|\nabla f(x)| = 1$ so that the normal $n = \nabla f(x)$ at a local point $w^\mu$ and $g_{\mu\nu} \equiv \partial x^\mu/\partial u^\nu \cdot \partial x^\nu/\partial u^\nu = x_{,\mu} \cdot x_{,\nu}$, where $O_{\mu\nu} \equiv \partial O/\partial u^\mu$ and $O^\mu = g^{\mu\nu}O_{,\nu}$ etc. For a particle constrained on the surface, we have a compatible constrained condition [1, 6, 29, 30],

$$n \cdot p = 0. \quad (2)$$

The surface first category of FCR is $[x_i, x_j] = 0$, $[x_i, p_j] = i\hbar\delta_{ij} - n_i n_j$, $[p_i, p_j] = -i\hbar\{n_i n_k, j - n_j n_k, i\}p_k$ Hermitian,

$$[x_i, x_j] = 0, \quad [x_i, p_j] = i\hbar\delta_{ij} - n_i n_j, \quad [p_i, p_j] = -i\hbar\{n_i n_k, j - n_j n_k, i\}p_k \quad \text{Hermitian}, \quad (3)$$

where $O_{\text{Hermitian}}$ stands for a suitable construction of the Hermitian operator of an observable $O$. Because the classical Hamiltonian takes form $H = p^2/2m + V$, we have $[x, H]_D = p/m$, and $[p, H]_D = -m n_{k,j} p_k p_j$. The second category of FCR is then given by,

$$[x, H] = i\hbar[x, H], \quad [H, p] = i\hbar/m (m n_{k,j} p_k p_j) \quad \text{Hermitian}. \quad (4)$$

Once curved surface $f(x) = 0$ becomes flat, both the momentum and the Hamiltonian must assume their usual forms respectively. Thus we ansatz that the quantum mechanics $H$ takes the following form with a potential $V_g$,

$$H = -\hbar^2/(2m)\nabla^2 L_B + V + V_g. \quad (5)$$
III. GEOMETRIC MOMENTUM

First of all, we have the momentum \( \mathbf{p} \) from Eqs. (4) and (5),

\[
p = i \frac{\hbar^2}{2} [\nabla_{LB}^2, \mathbf{x}] = -i \frac{\hbar}{2} (\nabla_{LB}^2 \mathbf{x} + 2\mathbf{x}^\mu \partial_\mu) = -i\hbar (\nabla_S + \frac{M\mathbf{n}}{2}). \tag{6}
\]

We call \( \mathbf{p} \) (6) the geometric momentum for its dependence on the extrinsic curvature \( M \) \[19\text{–}22\].

Second, we demonstrate that the operator version of the constrained condition (2) for the momentum \( \mathbf{p} \),

\[
\mathbf{p} \cdot \mathbf{n} + \mathbf{n} \cdot \mathbf{p} = 0. \tag{7}
\]

It is evident for the action of the vector operator \( \nabla_S \) on the unit normal vector \( \mathbf{n} \) leads to a nonvanishing result as \( \nabla_S \cdot \mathbf{n} = -M \), which exactly cancels \( M \) in \( (\nabla_S + M\mathbf{n}/2) \cdot \mathbf{n} + \mathbf{n} \cdot (\nabla_S + M\mathbf{n}/2) = 0 \) so that we have orthogonal relation (7).

Lastly, we need to show that the first category of the FCR are satisfied with this momentum \( \mathbf{p} \) (6). A verification of the first two FCR in (3) is straightforward. The proof of the last in (3) is also an easy task. The key step is a proper construction of the Hermitian operator of observable \( O \). Only the following naive rule \( (O + O^\dagger)/2 \) \[31\] is used,

\[
\{n_i n_k j p_k\}_{\text{Hermitian}} = \frac{1}{2} (n_i n_k j p_k + p_k n_i n_k j) = n_i n_k j p_k + \frac{1}{2} (-i\hbar) (n_i, k n_j, \epsilon + n_i \partial_j M). \tag{8}
\]

It is applicable in the R.H.S. of the FCR \( \{p_i, p_j\} = -i\hbar \{(n_i n_k j - n_j n_k i)p_k\}_{\text{Hermitian}} \) as,

\[
\{n_i n_k j - n_j n_k i)p_k\}_{\text{Hermitian}} = \{n_i n_k j p_k\}_{\text{Hermitian}} - (i \epsilon \leftrightarrow j). \tag{9}
\]

For a two-dimensional spherical surface, how to measure the geometric momentum (6) is extensively investigated \[32\].

IV. GEOMETRIC POTENTIAL FOR QUANTUM MOTION ON SPHERICAL SURFACE

With the first category of the FCR being used only, we have at least ten choices of \( \alpha(N) \) in \( V_g = \alpha(N)\hbar^2/(2mr^2) \), based upon various understanding of the problem. For instance, on the dependence of \( V_g \) on the dimensions \( N \), we have, 1) \( \alpha(N) = 0 \) \[33, 34\], 2) \( \alpha(N) = (N - 1)/2 \) \[30\], 3) \( \alpha(N) = \left(1 + 4s^2\right)(N - 1)^2/4 \) with \( s \) being a real parameter \[5\], 4) \( \alpha(N) = N^2/4 \) \[35\], 5) \( \alpha(N) = (N - 1)(N + 1)/4 \) \[30\], 6) \( \alpha(N) = (N - 1)/2 \) \[36\], 7) \( \alpha(N) = (N - 3)(N + 1)/4 \) \[37\], 8) \( \alpha(N) = (N - 1)(N - 2)/2 \) \[30\] 9) \( \alpha(N) \) arbitrary \[5\] and 10) \( \alpha(N) = (N - 1)(N - 3)/4 \) \[5\], etc. \[38\] At first sight, these disputant results seem to be rather irrelevant. Common experiments are only capable of detecting energy differences, in which these constants drop out. Cosmology, however, is sensitive to an additive constant \[1, 33\].

Now let us see what \( V_g \) is within our SCQS. The first category of the FCR is \[30, 33\],

\[
[x_i, x_j] = 0, [x_i, p_j] = i\hbar (\delta_{ij} - n_i n_j), [p_i, p_j] = -i\hbar (x_i p_j - x_j p_i)/r^2, \tag{10}
\]

No operator ordering problem occurs in the R.H.S. of \([p_i, p_j]\) because of the Jacobi identity. We see already these relations (10) are automatically satisfied with Cartesian coordinates \( \mathbf{x} \) and geometric momentum \( \mathbf{p} \) (6). Now we examine the remaining FCR in the second category (4) which is given by, with noting relations \( \mathbf{n} = \mathbf{x}/r \) and \( n_{i,j} = (\delta_{ij} - n_i n_j)/r \) so \( n_{i,j} p_k p_j = p^2/r = 2mH/r \),

\[
[H, \mathbf{p}] = i\hbar \frac{\mathbf{x}H + H\mathbf{x}}{r^2}. \tag{11}
\]

On one hand, because the geometric potential \( V_g \) results from the noncommutability of different components of the geometric momentum (6), it depends solely on the geometric invariants as the geometric momentum does. On the other, all principal curvatures for the spherical surface are the same \(-1/r\) and the mean curvature \( M = -(N - 1)/r \). So, the geometric potential \( V_g \) also takes the following form \( \alpha(N)\hbar^2/(2mr^2) \) and the Hamiltonian takes form \( H = -\hbar^2/(2m)\nabla_{LB}^2 + \alpha(N)\hbar^2/(2mr^2) \). We will prove,

\[
V_g = (N - 1)(N - 3) \frac{\hbar^2}{4 2mr^2}, \tag{12}
\]

which is exactly the geometric potential \( V_g \) (1) for the surface under consideration. The proof is as what follows.
We rewrite both $H$ into following form,

$$H = -\frac{\hbar^2}{2m} \nabla^2_{LB} + V_g = \frac{p^2}{2m} - \frac{M^2\hbar^2}{8m} + V_g,$$

(13)

and the quantity $(xH + Hx)$ in the R.H.S. of the FCR (11),

$$xH + Hx = 2xH - \frac{\hbar}{m} p.$$

(14)

The L.H.S. of the (11) $[H, p] = [p^2, p]/(2m)$ is, with repeated use of the first category of FCR (10),

$$\frac{1}{2m} [p^2, p] = \frac{1}{r^2} \left( 2i\hbar x \frac{p^2}{2m} - 2i\hbar x \frac{M^2\hbar^2}{8m} + 2i\hbar x V_g + \frac{\hbar^2}{m} p \right).$$

(15)

Multiplying the results (14) and (15) derived from both sides of (11) by the unit normal vector $\mathbf{n}$ from the left, we obtain (12). Q.E.D.

It is interesting to point out that the geometric momentum $\mathbf{p}$ and the angular momentum $L_{ij} \equiv x_i p_j - x_j p_i$ to form a closed $so(N, 1)$ algebra. Let $P_i \equiv r p_i$, we have from (10),

$$[P_i, P_j] = -i\hbar L_{ij}.$$  

(16)

It is easily to show that the components of the angular momentum satisfies the standard $so(N)$ algebra from its definition of $L_{ij}$ and FCR (10) [1],

$$[L_{ij}, L_{kl}] = -i\hbar (-\delta_{ik} L_{lj} + \delta_{ik} L_{lj} + \delta_{jk} L_{il} - \delta_{jl} L_{ik}).$$

(17)

The commutation relations between $L_{ij}$ and $P_k$ is, with also repeated use of the first category of FCR (10),

$$[L_{ij}, P_k] = i\hbar (\delta_{ik} P_j - \delta_{jk} P_i).$$

(18)

These generators $L_{ij}$ and $P_k$ form a closed $so(N, 1)$ algebra, which reflects a dynamical $SO(N, 1)$ group symmetry beyond its geometrical one $SO(N)$. It implies that there is a dynamical representation that can be used to examine the motion on the spherical surface, as illustrated in [32].

V. CONCLUSIONS AND REMARKS

The usual canonical quantization procedure contains only the first category of the FCR, forming the invariable part of the procedure, therefore universally valid. As widely accepted, this procedure is far from complete, and has free parameters that are sometimes believed to be fixed by the experiments. For a system that has classical analogue whose classical Hamiltonian takes form $H = p^2/2m + V$, the freedom can be fixed by an additional principle that the Cartesian coordinates must be used in passing over to the quantum mechanics such that the second category of the FCR is automatically satisfied.

For a particle constrained to remain on a curved hypersurface, the different components of momentum are not mutually commutable. Thus the procedure of obtaining the quantum Hamiltonian by a simple substitution of an expression of the momentum into the Hamiltonian $H = p^2/2m + V$ has been an issue full of debates, and is therefore questionable. A further strengthening of the quantization procedure is needed, and we propose to use the second category of the FCR to determine the forms of both the quantum momentum and Hamiltonian. The present study shows that there is a universal form of the momentum, the geometric momentum, and there is a concise and lucid way to produce the geometric potential for the spherical surface. Moreover, we demonstrate that there is a dynamical $SO(N, 1)$ group symmetry on the surface beyond the geometrical one $SO(N)$.

There are interesting issues which will be explored in near future: the relation between the geometric momentum and annihilation operators on the $N-1$ dimensional sphere [39], and a possibly universal form of a construction of the operator $(\mathbf{m}k, p_k)_{Hermitean} (4)$ rather than a treatment on the case-by-case basis [27, 28], and the possible influence of the geometric potential on the dark energy as a consequence of embedding our universe in higher dimensional flat space-time, etc.
This work is financially supported by National Natural Science Foundation of China under Grant No. 11175063. The author is grateful to Professor M. Ritoré, Facultad de Ciencias, Universidad de Granada, Spain, for his kind help of the differential geometry of hypersurface.