KILLINGS, DUALITY AND CHARACTERISTIC POLYNOMIALS

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Abstract

In this paper the complete geometrical setting of (lowest order) abelian T-duality is explored with the help of some new geometrical tools (the reduced formalism). In particular, all invariant polynomials (the integrands of the characteristic classes) can be explicitly computed for the dual model in terms of quantities pertaining to the original one and with the help of the canonical connection whose intrinsic characterization is given. Using our formalism the physically, and T-duality invariant, relevant result that top forms are zero when there is an isometry without fixed points is easily proved.
1 Introduction

T-duality (cf.[1][2]) has become a standard tool in the general exploration of the space of vacua in String Theory (apparently best thought of as coming from an 11-dimensional M-theory ), albeit almost always in the somewhat trivial setting of toroidal compactifications.

Although Buscher’s formulas are known not to be exact except in the simplest cases (i.e., generically there are $\alpha'$ corrections to them), it is nevertheless very interesting to get as much information as possible on the “classical” geometry of the dual target space.

In the present paper we generalize (to the case in which there is an n-dimensional abelian group of isometries) and develop further a formalism first discussed by one of us in [3], thus allowing to compute all interesting geometrical quantities of the dual space. Ariadne’s thread will consist in exploiting the residual gauge invariance in adapted coordinates.

As a subproduct all invariant polynomials in the dual space are explicitly determined, yielding some general conclusions on the vanishing of certain topological invariants when they are zero. Some new T-invariants (i.e., scalars under T-duality) also stem from the analysis.

Our results are strongest when all points have trivial isotropy group; that is, when the isometry does not have any fixed points, which in the euclidean signature is the same thing as to say that the Killing vector never has zero modulus.

It is a well known fact than when performing a T-duality transformation, the geometry of the dual space (insofar as this concept makes sense) can be wildly different from the one of the original space. There are two qualifications. First, Buscher’s formulas are expected to receive corrections in all but the simplest cases [5]. Second, the probes to be used to explore the dual geometry are not necessarily the same as the ones to be used in the original one (cf. [6] for some comments on the operator mapping).

It is nevertheless of great interest to determine the “classical” geometry of the dual space in as precise a manner as possible. (cf. [6] for previous attempts in this direction).
2 The Reduced Formalism

In this section we will give a very brief summary about the Reduced Formalism introduced in [3]. We shall assume that the Target Space manifold M is invariant under a n-dimensional abelian group of isometries, and the corresponding Killing vectors will be denoted by \( k_\alpha^\mu \) where \( \mu, \nu = 0, 1, ..., D - 1, a, b = 0, ..., n - 1 \) and D is the dimension of M. We start with two conditions. The first one, we have a set \( \Sigma \) of G-invariant tensors, \( V \), characterized by:

\[
\mathcal{L}_{k_\alpha} V = 0 \tag{1}
\]

which in adapted coordinates \( x_i, x^a \) (\( i = n, ..., D - 1 \)) reduces to \( \partial_a V = 0 \).

The second condition consist on having a connection compatible with the first one, which implies

\[
(\mathcal{L}_{k_\alpha} \nabla)^\sigma_{\mu\nu} = [\mathcal{L}_{k_\alpha}, \nabla\sigma]_{\mu\nu} = K_\alpha^\sigma R^\sigma_{\alpha\mu\nu} + \nabla_\mu K_\nu^\sigma + 2\nabla_\mu (K_\alpha^\alpha T^\sigma_{\alpha\nu}) \equiv 0 \tag{2}
\]

\( R^\sigma_{\alpha\mu\nu} \) is the curvature of the connection \( \Gamma^\Delta_{\lambda\beta} \) and \( T^\sigma_{\mu\nu} \) is the corresponding torsion. (2) is nothing but \( \partial_a \Gamma^\sigma_{\mu\nu} = 0 \) in adapted coordinates.

Given a set of commuting vector fields (in our case, the Killings), \( k_\alpha^\mu \), there always exists a system of coordinates (adapted coordinates) such that \( k_\alpha^\mu = \delta_\alpha^\mu \), i.e., \( k_\alpha \equiv \partial_\alpha \). (cf.[4]). These conditions do not determine completely the system of coordinates; the residual gauge group actually consists in arbitrary compositions of transverse diffeomorphisms \( (x_i' = f_i(x^j)) \) with redefinitions of the ignorable coordinates themselves:

\[
x_i' = x_i \\
x^a' = x^a + \Lambda^a(x^j) \tag{3}
\]

Tensors in \( \Sigma \) transform linearly under (3):

\[
V' = J(\partial\Lambda)V \tag{4}
\]

If we had at our disposal some transverse gauge fields \( A_i^\alpha(x^j) \), i.e., fields transforming under (3) as:
\[ A^\alpha_i(x^j) = A^\alpha_i(x^j) - \partial_i \Lambda^\alpha(x^j) \]  

(5)

Then we could associate the reduced tensor \( v \) to each tensor \( V \) by:

\[ v \equiv J(A)V \]  

(6)

Reduced tensors are invariant under (3), i.e., \( v' = J(A - \partial \Lambda)J(\partial \Lambda)V = v \), because \( J \) provides a representation of the abelian group \( G \) in the space of tensors characterized by \( (\mathcal{L}) \).

It follows simply from (2) y (3) that there exists a reduced covariant derivative

\[ \nabla v \equiv J(A)\nabla V \]  

(7)

corresponding to a reduced connection given by:

\[ \gamma^\rho_{\mu \nu} = J(-A)^\alpha_{\mu}J(-A)^\beta_{\nu}J(A)^\rho_{\delta}(\Gamma^\delta_{\alpha \beta} - \partial_{\alpha}J(A)^\delta_{\beta}) \]  

(8)

With the definitions giving above, (3) and (7), the operation that gives the reduced tensors commutes with the basic operations of the tensor calculus: linear combinations, tensor products, contractions, permutation of indices and covariant derivation. That feature together with its simplicity (see (3)) is the reason to call the whole setting the Reduced Geometry.

In a Riemannian manifold with abelian Killing vectors

\[ G_{\mu \nu} = \begin{pmatrix} G_{ab} & A_{ai} \\ A_{bj} & \hat{\Gamma}_{ij} + A_{ic}A_{jd}G^{cd} \end{pmatrix} \]

with

\[ \mathcal{L}_{k_a}G_{\mu \nu} = 0 \]  

(9)
there is a natural gauge field (5) namely,
\[ A_i^a(x^j) = G^{ab} A_b(x^j) \]  

(10)

where \( G_{ab}G^{bc} = \delta^c_a \).

The reduced Levi-Civita connection, \( \gamma_{\ell-c} \) has a non-zero torsion which is the responsible for the reduced curvature not being simply the curvature of the reduced connection, as can be seen in [3]. In the general case, \( \Gamma = \Gamma_{\ell-c} + H \), the resulting reduced curvature is

\[ \gamma_{\ell-c}^a(x^j) = R(\gamma_{\ell-c} + h)^a_{\lambda\delta\pi} - 2T(\gamma_{\ell-c})^a_{\lambda\delta} (\gamma_{\ell-c} + h)_{a\pi} \]  

(11)

where \( h_{\mu\nu} \) is the reduced tensor corresponding to \( H_{\mu\nu} \) and \( T(\gamma_{\ell-c}) \) is the Levi-Civita reduced torsion.

2.1 Buscher’s formulas for n commuting Killings

The context of most applications of the formalism starts from a two-dimensional non-linear sigma model with target space \( \mathcal{M} \), whose bosonic part is given by:

\[ S = \int (G_{\mu\nu} + B_{\mu\nu}) \partial_+ X^\mu \partial_- X^\nu \]  

(12)

The generalization of Buscher’s transformations to the case where there are \( n \) commuting Killings present (\( L_{k_a} G = 0 \) and \( L_{k_a} B = dW \)) follows easily from the gauging procedure [2]. The resulting dual model has the following backgrounds:

\[ \tilde{Q}^\pm_{ab} = Q^\pm_{ab} \]
\[ \tilde{Q}^\pm_{ai} = \pm Q^\pm_{ab} Q^\pm_{bi} \]
\[ \tilde{Q}_{ij} = Q_{ij} - Q^a_{ai} Q^b_{bj} \]  

(13)

with the conventions \( Q^\pm_{ab} \equiv G_{ab} \pm B_{ab} \), \( Q^\pm_{ai} Q^\pm_{bi} = \delta^a_i \), \( Q^\pm_{ai} \equiv G_{ai} \pm B_{ai} \) and \( Q_{ij} \equiv G_{ij} + B_{ij} \) and \( Q^a_{ai} Q^b_{aj} = \delta_a^b \).

The quotient metric is invariant under T-duality:

\[ \tilde{G}_{ij} = \hat{G}_{ij} \]  

(14)
The three form $H \equiv dB$ is defined in tensorial terms as

$$H_{\mu\nu\rho} \equiv \frac{1}{2} (\nabla_\mu B_{\nu\rho} + \nabla_\nu B_{\rho\mu} + \nabla_\rho B_{\mu\nu})$$ (15)

where the Levi-Civita connection is used to define covariant derivatives. Then, from (6) and (7), the explicit computation yields for its reduced partner

$$h_{abc} = 0; \quad h_{iab} = \frac{1}{2} \partial_i B_{ab};$$

$$h_{aij} = -\frac{1}{2} F^a_{ij} (B) - \frac{A^a_i}{2} \partial_j B_{ab} + \frac{A^a_j}{2} \partial_i B_{ab};$$

$$h_{ijk} = \hat{h}_{ijk}$$ (16)

where $F^a_{ij} (B) \equiv \partial_i B_{aj} - \partial_j B_{ai}$. The reduced three-form $h_{ijk}$ is actually T-invariant.

3 The Canonical Map

The classical string dynamics is governed by the pullback of the generalized connection with torsion $\Gamma^\pm = \Gamma_{lc}^\pm H$, and at one-loop level the beta-functions of the bosonic and $N = 1$ supersymmetric string models are proportional to the Ricci tensor of that generalized connection. Therefore our first interest is to determine the T-dual of the connection $\Gamma^\pm$ (denoted as usual by $\gamma^\pm$ when reduced) in the reduced setting:

$$\Gamma_{\mu\nu}^\pm = \Gamma_{(lc)\mu\nu}^\rho \pm H_{\mu\nu}^\rho$$

$$\gamma_{\mu\nu}^\pm = \gamma_{(lc)\mu\nu}^\rho \pm h_{\mu\nu}^\rho$$ (17) (18)

where the torsion $H_{\mu\nu\sigma} = H_{\mu\nu}^\rho G_\rho^\sigma$ is given by (13). To be specific, the starting point is:

$$\gamma_{ab}^{\pm c} = 0$$

$$\gamma_{ab}^{\pm i} = -\frac{1}{2} \hat{\partial}^{i} Q_{ab}^\pm$$

$$\gamma_{ia}^{\pm b} = \frac{1}{2} G^{bc} \partial_i Q_{ac}^\pm = \gamma_{ai}^{\mp b}$$

$$\gamma_{ai}^{\pm j} = \frac{1}{2} \hat{G}_{jk}^i C_{vka}^\pm = \gamma_{ai}^{\mp j}$$

$$\gamma_{ij}^{\pm a} = -\frac{1}{2} G^{ab} C_{ijb}^\pm$$

$$\gamma_{ij}^{\pm k} = \hat{\gamma}_{ij}^{\pm k} \pm \hat{h}_{ij}^{\pm k} \equiv \hat{\Gamma}_{ij}^k$$ (19)
where $C_{ij}^\pm = F_{ijb}(Q^\pm) + A^d_i \partial_j Q_{bd}^\pm - A^d_j \partial_i Q_{bd}^\pm$.

The T-duals are \[^1\]

$$
\tilde{\gamma}_{\mu\nu}^\pm = t^\mp_{\mu\rho} t^\rho_{\nu\sigma} \gamma^\pm_{\alpha\beta} \tag{20}
$$

for all components except $\tilde{\gamma}_{ia}^\pm = \tilde{\gamma}_{ia}^\pm b$, with

$$
t^\pm_{ib} = \pm Q^ab, \quad t^i_{\pm j} = t^i_{\pm j} = \delta^i_{\pm j}, \quad t^a_{\pm b} = \pm Q^a_{\pm b} \quad \text{otherwise} = 0 \tag{21}
$$

The simplicity of the reduced transformations (20), allow us to built a map between the original ($\Sigma$) and dual ($\tilde{\Sigma}$) geometries, the canonical map, transforming tensors ($V \rightarrow \tilde{V}$) with the property of mapping the corresponding covariant derivatives linearly ($\tilde{\nabla}^\pm \tilde{V} \propto \nabla^\pm V$) :

$$
\tilde{V}^\pm_{\mu_1,...,\mu_t} = (\prod_{r=1}^t T^\mu_{\pm r}^{\nu_r}) (\prod_{s=1}^m T^\pm_{\nu_s}^{\alpha_s}) V^\pm_{\alpha_1,...,\alpha_m} \tag{22}
$$

where the $T^\pm$ and $T^\mp$ can be viewed as a sort of vierbeins relating indices of the initial and dual geometries. The covariant derivatives map linearly but not canonically,

$$
\tilde{\nabla}^\pm \tilde{V}^\pm_{\nu_1,...,\nu_m} = T^\pm_{\rho\lambda} (\prod_{r=1}^t T^\mu_{\pm r}^{\nu_r}) (\prod_{s=1}^m T^\pm_{\nu_s}^{\alpha_s}) \nabla^\pm V^\pm_{\alpha_1,...,\alpha_m} \tag{23}
$$

because the anomaly in the derivative index. The matrices $T^\pm$ and $T^\mp$, first used by Hassan \[^8\] are :

$$
T^\pm_{\mu\nu} = \begin{pmatrix}
\pm Q^a_{\pm b} & 0 \\
-Q^a_{\pm b} & Q^a_{\pm b} \delta^i_j
\end{pmatrix}
$$

\[^1\]It is exceedingly convenient to take advantage of the transformation properties of the combination $s_{ij} = \partial_i Q_{ja}^\pm - (\partial_j Q_{ab}^\pm) A^b_j$ namely, $s_{ij} = \pm \frac{1}{Q^\pm_{ab}} s_{ij} s_{ij}$ with $C_{ij} = s_{ij} a$

6
\[ T_{\pm \nu}^\mu = \begin{pmatrix} \pm Q_{ab}^\pm & \pm Q_{\alpha i}^\pm \\ 0 & \delta^i_j \end{pmatrix} \]

\( \nu \) being the column index and \( \mu \) the row index. The above mentioned anomaly in fact implies that the covariant derivative does not commute with the canonical map, and as a simple corollary the curvatures do not transform simply by trading indices of \( T^\pm \) as we shall see next.

Covariantly constant tensors (with respect to \( \nabla^\pm \)) transform necessarily as (22). This is the case for the metric \( (\nabla^\pm \mu G = \nabla^\pm \bar{G} = 0) \), for the holomorphic complex structures underlying extended supersymmetries \([8]\), p-forms associated to W-algebras \([11]\), and whatever holomorphic covariantly constant tensor we found in our geometry.

There are other tensors with non-canonical transformations, such as the 2-form \( B_{\mu \nu} \) and the 3-form \( H_{\alpha \beta \gamma} \). Torsion is best studied as forming part of the generalized connection. Actually, both equations (22) and (23) together easily yield:

\[ \tilde{\Gamma}_{\mu \nu}^\rho = T_{\mu}^\alpha T_{\nu}^\pm \beta T_{\rho}^\pm \alpha \Gamma_{\pm \alpha}^\pm \beta + (\partial_{\mu} T_{\nu}^\pm \beta) T_{\rho}^\pm \beta \]  

(24)

The target-space connection transforms as a real T-duality connection except for the anomaly in the \( \mu \)-index.
4 Transformation of the Curvature and Canonical Connection

Let us now consider the generalized curvature, $R_{\mu\nu\rho\sigma} = R(\Gamma^{\pm})_{\mu\nu\rho}G_{\lambda\rho}$ which obviously satisfies the symmetry relationships $R_{\mu\nu\rho\sigma} = - R_{\nu\mu\rho\sigma} = - R_{\rho\nu\sigma\mu}$. Working with the transformations of the connection in (20) and using the expression for the reduced curvature (11) we get $2$:

$$\tilde{R}_{\mu\nu\rho\sigma} = \tilde{T}_{\alpha}^{\mu} T_{\beta}^{\nu} T_{\lambda}^{\sigma} T_{\delta}^{\rho} (R^{\pm}_{\alpha\beta\lambda\delta} - 2Q_{ab}^{\alpha\beta} k^{a}_{\lambda} \nabla_{\lambda}^{\mp} k^{b}_{\delta})$$

The fact that there is an inhomogeneous part ($-2Q_{ab}^{\alpha\beta} k^{a}_{\lambda} \nabla_{\lambda}^{\mp} k^{b}_{\delta}$) in the transformation of the curvature turns out to be a useful clue. Actually, when an object does transform inhomogeneously, such as $\tilde{r} = \pm \prod t^{\Delta} (r + \psi)$, the involutive property, $T^{2} = 1$, completely determines the transformation of the inhomogeneous part, $\psi$, namely $\tilde{\psi} = \mp \prod t^{\Delta} \psi$. This fact immediately suggests the definition $w \equiv r + \frac{1}{2} \psi$, which does transform homogeneously.

There is then a quantity, the corrected generalized curvature $W^{\pm}_{\alpha\beta\delta\eta}$ which transforms linearly:

$$W^{\pm}_{\mu\nu\rho\sigma} \equiv R^{\pm}_{\mu\nu\rho\sigma} - Q_{ab}^{\alpha\beta} k^{a}_{\lambda} \nabla_{\lambda}^{\mp} k^{b}_{\delta}$$

$$\tilde{W}^{\pm}_{\mu\nu\rho\sigma} = \tilde{T}_{\alpha}^{\mu} T_{\beta}^{\nu} T_{\lambda}^{\sigma} T_{\delta}^{\rho} W^{\pm}_{\alpha\beta\delta\eta}$$

This correction (26) is minimal because it includes first-derivative terms and therefore it cannot be absorbed in a background’s redefinition.

We have just seen that the transformations of the ordinary connection is not canonical (23). It is possible, however, to define a new connection with canonical transformation properties (24) We shall refer to it as canonical connection:

$$\tilde{\Gamma}^{\pm}_{\mu\nu} = \Gamma^{\pm}_{\mu\nu} - G^{ab} k^{a}_{\lambda} \nabla_{\lambda}^{\mp} k^{b}_{\rho}$$

$$\tilde{\Gamma}^{\pm}_{\mu\nu} = T^{\pm}_{\mu} T^{\pm}_{\nu} T^{\pm}_{\alpha} \Gamma^{\pm}_{\lambda\beta} + (\partial_{\mu} T^{\pm}_{\nu}) T^{\pm}_{\nu}$$

Therefore the canonical covariant derivative, $\nabla_{\mu}^{\pm}$ does now commute with the canonical mapping. This means that a canonical transformation for ten-

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2 $k^{a}_{\mu} \equiv k^{a}_{\nu} G_{\nu\mu}$.

3 We denote as $\prod t^{\Delta}$ the product of matrices $t$ with $\Delta \pm$ or $\mp$ as appropriate.
\[ V^\pm_{\nu_1, \ldots, \nu_l} = (\prod_{A=1}^{m} T^\pm_{\nu A} \beta A) (\prod_{B=1}^{l} T^{\mu B}_{\pm \alpha B}) V^{\alpha_1, \ldots, \alpha_l}_{\beta_1, \ldots, \beta_m} \] (29)

This corresponds with a canonical one for the covariant derivatives:

\[ \nabla^\pm_{\rho} V^\pm_{\nu_1, \ldots, \nu_m} = T^\pm_{\rho \lambda} (\prod_{A=1}^{m} T^\pm_{\nu A} \beta A) (\prod_{B=1}^{l} T^{\mu B}_{\pm \alpha B}) \nabla^\pm_{\lambda} V^{\alpha_1, \ldots, \alpha_l}_{\beta_1, \ldots, \beta_m} \] (30)

The canonical connection is compatible with the metric, \( \nabla^+ \mu G^+_{\nu \rho} = 0 \) provided that the Killing condition \( \mathcal{L}_{k_a} G_{\mu \nu} = 0 \) is satisfied. Moreover, \( \nabla_0 = 0 \) acting on \( \Sigma \), implying \( k^a \tilde{R}^+_{\alpha \sigma \rho} = 0 \). At the end of this section we will give an intrinsic (T-duality independent) characterization of this canonical connection for which the above properties will appear natural.

Also, following simply from the commutativity, we get the canonical transformation of the curvature:

\[ \tilde{R}^\pm_{\rho \sigma \rho} = T^\pm_{\rho} T^\pm_{\nu} T^\pm_{\sigma} T^\pm_{\delta} T^\pm_{\eta} \tilde{R}^\pm_{\alpha \beta \delta \eta} \] (31)

With the canonical connection, the canonical T-duality map commutes with the basic operations of the tensor calculus, i.e., linear combinations, tensor products, permutation and contraction of indices and covariant derivation. In particular it implies that every tensor built from \( \tilde{R}^\pm_{\mu \sigma \rho} \), \( G_{\alpha \beta} \), \( \nabla^\pm_{\eta} \), and any other tensor transforming canonically, transforms canonically. As a corollary, target-space canonical scalars are T-duality scalars (\( \tilde{R}^\pm_{\mu \nu} \), \( \tilde{R}^\pm_{\mu \nu \sigma \rho} \), \( \tilde{R}^\pm_{\mu \nu \sigma \rho} J^{\alpha \rho} \), \( \tilde{R}^\pm_{\mu \nu \sigma \rho} J^{\alpha \rho \mu \sigma} \), \( \tilde{R}^\pm_{\mu \nu \sigma \rho} J^{\alpha \rho \mu \sigma \alpha} \), ...).

In complex manifolds, the presence of additional canonical tensors, i.e., the holomorphic complex structures \( J^+_{\nu} \), allows the construction of another new set of T-duality scalars (\( \tilde{R}^\pm_{\mu \nu \sigma \rho} \), \( \tilde{R}^\pm_{\mu \nu \sigma \rho} J^{\alpha \rho} \), \( \tilde{R}^\pm_{\mu \nu \sigma \rho} J^{\alpha \rho \mu \sigma} \), \( \tilde{R}^\pm_{\mu \nu \sigma \rho} J^{\alpha \rho \mu \sigma \alpha} \), ...).

The definition of canonical connection in [28] was motivated in its very convenient transformation properties under T-duality. Nevertheless, an intrinsic (T-duality independent) characterization can be given for it.

Let us start with our set of abelian (Killing) vectors \( \{ k^\mu_{(a)} \}^4 \) and the vector \( 4k^\mu_{(a)} \equiv k^\mu_{(a)} \).
space K spanned by them. The presence of a metric $G_{\mu\nu}$ on our manifold induces the natural projector on K, $P^\nu_\mu \equiv k_{(a)}^{(a)} k_{(b)}^{(b)} G^{ab}$, with $P^2 = P$ and $P k_{(a)} = k_{(a)}$.

Now, let us take the quotient of the whole space of connections C by the projection $P$, i.e., $\mathcal{C}/P$. Then, two connections $\Gamma^1$ and $\Gamma^2$ belong to the same class on $\mathcal{C}/P$ if $(\Gamma^1 - \Gamma^2)_\mu = P^\nu_\mu L_\nu$ for some matrix valued one-form $L$.

In every class there is an unique covariant derivation $\bar{\nabla}$, with the property

$$\bar{\nabla} k_{(a)} = \mathcal{L}_{k_{(a)}}$$

(32)

Writing $\bar{\nabla} = P^\perp \nabla + P \bar{\nabla}$, the ortogonal component ($P^\perp = 1 - P$) is common to every element on the same class, say $\nabla^\perp$, and the K projection is $(P \bar{\nabla})_\mu = k_{(a)}^{(a)} k_{(b)}^{(b)} G^{ab} \nabla_\nu = k_{(a)}^{(a)} G^{ab} \mathcal{L}_{k_{(b)}}$. Therefore, the barred connection is

$$\nabla_\mu = \nabla_\mu^\perp + k_{(a)}^{(a)} G^{ab} \mathcal{L}_{k_{(b)}}$$

(33)

in every class of $\mathcal{C}/P$.

If we start in a class having a connection compatible with the metric (as it is the case in T-duality (17)), the compatibility of the barred connection trivially implies the Killing condition $\mathcal{L}_{k_{(a)}} G_{\mu\nu} = 0$.

In adapted coordinates, the $\bar{\nabla} k_{(a)} = \mathcal{L}_{k_{(a)}}$ condition means $\bar{\nabla}_a = \partial_a$ and then $\bar{\Gamma}^\rho_{a\nu} = 0$. Taking an arbitrary reference connection in the class, $\bar{\nabla}$ the above conditions imply

$$\bar{\Gamma}^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} - k_{(a)}^{(a)} G^{ab} \nabla^\rho_{\nu} \mathcal{L}_{k_{(b)}}$$

(34)

where the $\mathcal{K}$ operation on connections simply flips the sign of the torsion, $\mathcal{K} : \Gamma^\rho_{(\mu\nu)} \rightarrow \Gamma^\rho_{(\mu\nu)}$ and $\mathcal{K} : \Gamma^\rho_{[\mu\nu]} \rightarrow -\Gamma^\rho_{[\mu\nu]}$. Note that the $\mathcal{K}$ operation transforms our stringy connections $\Gamma^\pm$ one into the other. In that way, the barred connection (33) (34) agree with the T-duality canonical one (28) if we choose the classes which $\Gamma^\pm$ belong to.

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5We remember $G_{ab} = k_{(b)}^{(a)} k_{(a)}^{(b)}$ and $G^{ab} G_{bc} = \delta^a_c$

6$\bar{\nabla} k_{(a)} \equiv k_{(a)}^{(a)} \bar{\nabla}_\mu$
With respect to the reference connection \( \Gamma \), the \textit{barred connection} loses the information about the parallel transport in the Killing’s direction. This \textit{orthogonal projection} is easily seen rewriting (34) in a non-covariant form

\[
\bar{\Gamma}^\rho_{\mu\nu} = P^\perp_{\mu} \Gamma^\rho_{\alpha\nu} - k_{(a)\mu} G^{ab} \partial_b k^\rho_{(b)}
\]  

(35)

As a consequence of the defining properties, \( k_{(a)}^\alpha \bar{\nabla}_\alpha k^\mu_{(b)} = \mathcal{L}_{k_{(a)}} k^\mu_{(b)} = 0 \), implying the \textit{barred geodesics}

\[
\ddot{X}^\mu + \bar{\Gamma}^\mu_{\alpha\beta} \dot{X}^\alpha \dot{X}^\beta = 0
\]

(36)

have the free motion on the Killing’s direction

\[
\dot{X}^\mu = C^a k^\mu_{(a)}(X(\tau)) \quad ; \quad \dot{C}^a = 0
\]

(37)

as a consistent solution. Let us note this motion is allowed for \textit{every} barred connection provided that it projects out the parallel transport in the K direction.

Finally let us remark that \((\bar{\nabla})^k = 0\). Of course, the \((\bar{\nabla})^k\) is in general non-compatible with the metric. As a corollary, we get the useful (see next section) condition

\[
k_{(a)}^\alpha \bar{R}^\sigma_{\alpha\mu\nu} = (\mathcal{L}_{k_{(a)}} \bar{\nabla})^\sigma_{\mu\nu}
\]

(38)

which for the stringy connections means

\[
\bar{R}^\perp_{\alpha\mu\nu} = 0
\]

(39)

5 **Invariant Polynomials**

Invariant polynomials \( P(\Omega) \) are characterized in general (cf [9]) by

\[
P(\Omega) = P(g^{-1}\Omega)g,
\]

(41)

where \( \Omega^\rho_{\alpha} \equiv R(\Gamma)^\rho_{\mu\sigma\nu} dx^\mu \wedge dx^\nu \) is the matrix-valued curvature two-form, which implies that \( dP(\Omega) = 0 \); and moreover, that \( P(\Omega) \) has topologically invariant integrals (on manifolds without boundary). Chern and Simons have proven the specific result that given two different connections, \( \omega \) and \( \omega' \)

\[
P(\Omega') - P(\Omega) = dQ(\omega', \omega)
\]

(40)
where $Q$, the Chern-Simons term, is given by $Q(\omega', \omega) \equiv r \int_0^1 P(\omega' - \omega, \Omega_t, \ldots, \Omega_t)$, $r$ being the degree of the polynomial, and $\Omega_t \equiv d\omega_t + \omega_t \wedge \omega_t$, with $\omega_t \equiv t\omega' + (1 - t)\omega$.

As a consequence, all those polynomials can be determined (up to a total differential, that is in a cohomological sense), using any convenient connection; in our case the canonical connection $\bar{\Gamma}^\pm$ imposes itself naturally, because as we have seen in detail, the corresponding curvature, $\bar{R}_\mu^\pm$ transforms canonically under T-duality.

It is plain that the only non-zero components of any canonical invariant polynomial would be the ones with all indices transverse. This is clear, because $\nabla_a^\pm = 0$ actually implies $\bar{R}_{a\alpha\beta}^\pm = 0$.

In the particular case in which we are considering a Pontryagin characteristic polynomial, appropriate when the curvature lies in the Lie algebra of $O(k)$; $p_j(\Omega) \in \Lambda^{kj}(M)$ we can write $k^a \mu P_{\mu_1 \ldots \mu_{4j-1}}^{(4j)}, \forall a, \nu_1 \nu_{4j-1}$ or, in adapted coordinates,

$$P_{\nu_1 \ldots \nu_{4j-1}}^{(4j)} = 0 \quad \forall a, \nu_1 \nu_{4j-1}$$

Now, $\bar{P}$ transforms canonically:

$$\bar{P}_{\nu_1 \ldots \nu_{4j}}^{(4j)} = \left( \prod_{A=1}^{4j} T_{\mu_A A}^{\pm} \right) \bar{P}_{\nu_1 \ldots \nu_{4j}}^{(4j)}$$

And for the non-vanishing components:

$$\bar{P}_{i_1 \ldots i_{4j}}^{(4j)} = \prod_{A=1}^{4j} \delta_{i_A}^{k_A} \bar{P}_{k_1 \ldots k_{4j}}^{(4j)} = \bar{P}_{i_1 \ldots i_{4j}}^{(4j)}$$

This means that the components of the canonical Pontryagin forms are actually invariant. A glance at (41) implies that Top Forms vanish (because they necessarily include Killing indices, for which $\bar{R}_\mu^\pm = 0$).

This in turn means that if the canonical connection has a global meaning ($\det(G^{ab} \neq 0$) the topological invariants obtained by integrating top forms are necessarily zero, both in the original and in the dual model.

Chern classes are defined for complex manifolds with $\Omega \in gl(k, \mathbb{C})$; $c_j \in \Lambda^{2j}(M)$, but are otherwise similar to the Pontryagin ones from the point of view of the present work.

A well known fact is that for even dimensional manifolds a further $SO(2r)$
invariant polynomial can be defined (the Pfaffian). The corresponding Euler class is essentially the square root of the highest Pontryagin class. The mother of all index theorems is precisely the Gauss-Bonnet theorem, which states that the Euler characteristic \( \chi(M) \) is integral of the Euler class \( e(\tilde{R}^\pm) \). Now we see that the integrand (a top form) is necessarily zero if the group \( G \) acts freely (without fixed points). From (43), this assertion is T-duality invariant.

When there are fixed points, we could define a one parameter family of connections compatible with the metric \( \tilde{g} \) interpolating from \( \tilde{\Gamma}^\pm \) at \( t = 1 \) to \( \Gamma^\pm \) en \( t = 0 \).

\[
\Gamma^\pm_{\mu\nu} (t) = \Gamma^\pm_{\mu\nu} - \frac{tk_\mu}{k^2 + (1-t)^2} \nabla_\nu k^\sigma
\]  

(44)

This is well defined \( \forall t > 0 \) whereas in the limit \( t \to 0 \) the top forms vanish everywhere except perhaps at the fixed points. This means that all the topological information is stored in the fixed points.

In the case at hand this is contained in the well-known theorem by "Poincaré-Hopf" asserting that in a compact manifold \( M \) endowed with a differentiable vector field, \( w \) (Killings in our case) with isolated zeroes, the sum of the corresponding indices \( \iota \) (that is, the Brouwer degree of the mapping \( \hat{k}(x) \equiv \frac{k(x)}{||k(x)||} \)) \( \chi(M) \) is precisely Euler’s characteristic:

\[
\chi(M) = \sum \iota = \sum_{i=0}^{m} (-1)^i \text{rank} H_i(M).
\]  

(45)

(where \( H_i(M) \) stands for the i-th homology group of \( M \).)

As a trivial corollary, only in manifolds with zero Euler Characteristic it is possible to have free Killing actions.

Owing to the fact that we do not have enough control on the topology of the dual manifold, we can not say anything about the topological numbers, which are \textit{integrals} of the invariant polynomials, except, of course, when they vanish. It would also be quite complicated to keep track of all boundary terms in manifolds with non-trivial boundary in order to compute, say, the \( \eta \) invariant.

\( ^7 \)Because of the Killing condition.
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