New time-dependent solutions of viable Horndeski gravity

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Abstract. We generate new spherical and time-dependent solutions of viable Horndeski gravity by disforming a solution of the Einstein equations with scalar field source and positive cosmological constant. They describe dynamical objects embedded in asymptotically FLRW spacetimes and contain apparent horizons and a finite radius singularity that evolve in time in peculiar ways apparently not encountered before in Einstein and “old” scalar-tensor gravity.

Keywords: Horndeski theory, scalar-tensor gravity, exact solutions
1 Introduction

There is currently much research on alternative theories of gravity with motivation ranging from attempts to explain the current acceleration of the universe without an ad hoc dark energy [1, 2] to the realization that, as soon as one tries to quantum-correct general relativity (GR), new fields or higher order terms in the field equations and new degrees of freedom appear that make the resulting theory deviate in essential ways from Einstein gravity.

“First generation” scalar-tensor gravity [3–7] originating with Brans-Dicke theory [3], contains only one extra scalar degree of freedom in addition to the two massless spin two modes of GR and is the prototypical alternative gravity. After the 1998 discovery that the present expansion of the universe is accelerated, an ad hoc and very exotic (i.e., with equation of state parameter \( w \approx -1 \)) dark energy was introduced in GR-based cosmology to explain this phenomenon [8]. This postulate led to the Λ-Cold Dark Matter (ΛCDM) standard model of cosmology. This dark energy introduced overnight to fit observational data is deeply unsatisfactory and has motivated the search for alternative explanations of the cosmic acceleration, reviving the interest in alternative gravity, with the idea that perhaps on large scales gravity is not described by GR but by some alternative theory with extra degrees of freedom [1, 2, 9, 10]. Among several proposals \( f(R) \) gravity, which is a subclass of scalar-tensor gravity, has probably become the most popular for this purpose (see Refs. [11–13] for reviews).

In the last decade, the study of scalar-tensor gravity has gone well beyond first generation, Brans-Dicke-like theories, reviving [14–16] the old Horndeski theory of gravity [17], which was believed to be the most general scalar-tensor theory described by second order field equations [14–16]. This belief was revised when it was discovered that, among higher order scalar-tensor theories beyond Horndeski, imposing a special degeneracy conditions makes the field equations of second order again [18–26]. These theories have come to be known as Degenerate Higher Order Scalar-Tensor (DHOST) theories (see [27, 28] for reviews).

The field equations of Horndeski and DHOST gravity contain many terms and are complicated, hence it is difficult to obtain analytical solutions even in the presence of symmetries. Therefore, the catalogue of exact solutions of the field equations of these theories is rather
slim and probably comprises geometries that are not of high physical relevance but, until our knowledge of analytical solutions expands significantly, one has to live with this shortcoming. The difficulty of solving the field equations directly has led researchers to use the tool known as disformal transformation to generate new solutions from known ones acting as seeds. Probably the majority of the known analytical solutions of Horndeski and DHOST gravity that are not already solutions of the coupled Einstein-Klein-Gordon equations or of “old” scalar-tensor gravity (see [29] for a recent review) have been found using disformal transformations, and efforts have gone into assessing the nature of the solutions generated by a disformal transformation given the properties of the seed solution [30, 31], following similar discussions for conformal transformations [32, 33]. The transformation properties of the various terms composing the DHOST action under disformal transformations, and their inverses, were analyzed in [34], while the Petrov classification of geometries obtained with disformal solutions of DHOST theories, and how these Petrov classes are mapped by disformal transformations, were discussed in [30, 34]. Many of the known solutions are stealth [35, 36, 38–41, 43] and it is only recently that non-stealth solutions have been found [42–46].

Given a solution \((g_{ab}, \phi)\) of GR with scalar field as the matter source, or of scalar-tensor gravity, a disformal transformation maps the metric \(g_{ab}\) into a new one according to [48–51]

\[
g_{ab} \rightarrow \bar{g}_{ab} = \Omega^2(\phi, X)g_{ab} + W(\phi, X) \nabla_a \phi \nabla_b \phi \tag{1.1}
\]

where \(\Omega\) and \(W\) are, in principle, functions of \(\phi\) and of \(X \equiv -\frac{1}{2} \nabla^c \phi \nabla_c \phi\). The conditions

\[
\Omega \neq 0, \quad \Omega^2 - X (\Omega^2)_X - X^2 W_{,X} \neq 0 \tag{1.2}
\]

must hold to ensure invertibility of the map (1.1) [30].

A disformal transformation maps solutions of the coupled Einstein-Klein-Gordon equations into DHOST solutions [30, 34]. Here instead we generate a new family of solutions of viable Horndeski gravity by disforming a solution of the Einstein equations with positive cosmological constant sourced by a minimally coupled scalar field. Apart from cosmology, most of the known geometries describing spherically or axially symmetric objects in DHOST and Horndeski gravity are static or stationary, therefore we attempt to go beyond this restriction to learn more about the nature of gravity at least in the sub-class of Horndeski gravity compatible with a luminal propagation of gravitational waves. Already in GR and in old scalar-tensor theory, moving from stationary to time-dependent objects reveals new and richer phenomenology. For example, black hole event horizons cease being relevant and are replaced by time-dependent apparent horizons which, unfortunately, depend on the foliation [52, 53].

Time-dependent apparent horizons usually appear and annihilate in pairs [55–57, 59] and singularities can be dynamical. Most studies of non-asymptotically flat and dynamical analytical geometries and of their properties seem to have concentrated on spherical objects embedded in Friedmann-Lemaître-Robertson-Walker (FLRW) universes [55, 59, 60].

In this work, we adopt as seed for the disformal transformation a spherically symmetric, time-dependent, and asymptotically FLRW solution of the Einstein equations with positive cosmological constant and a scalar field with exponential potential as the matter source. It is a special case of the Fonarev solution [60] for exponential scalar field potential, with the peculiarity that the geometry is dynamical while the scalar field \(\phi\) is static, which makes the disformal transformation to Horndeski gravity rather manageable (a similar situation

\[\text{The situation is not always so dire, though: for example, in spherical symmetry all spherical foliations give the same apparent horizons [54].}\]
occurs for stealth solutions [29]). Moreover, the geometry is asymptotically FLRW, which distinguishes it from the asymptotically flat solutions populating the (still slim) catalogue of analytical solutions of Horndeski and DHOST gravity [29].

Two solutions are generated in the next section, using two different disformal transformations, and their apparent horizons and singularities are discussed. Section 3 analyses them in more detail; Sec. 3.3 investigates the dynamics of the apparent horizons, while Sec. 4 contains the conclusions.

**Notation:** We follow the conventions of Ref. [47]: the signature of the metric tensor $g_{ab}$ is $(-+++)$, $G$ is Newton’s constant, and units are used in which the speed of light $c$ is unity. Furthermore, again following [47], we employ the “abstract index notation”. Hence, quantities involving Latin letters will represent tensorial objects, whereas the presence of Greek letters will denote the choice of a specific chart.

## 2 Disforming a special case of the Fonarev solution

The Einstein-scalar field equations admit the 3-parameter family of Fonarev solutions given by [60, 61]

$$ds^2_\mathbb{F} = -e^{8\alpha^2at}A^\delta(r)dt^2 + e^{2at}\left[\frac{dr^2}{A^\delta(r)} + A^{1-\delta}(r)r^2d\Omega^2_{(2)}\right],$$

$$\phi(t, r) = \frac{1}{2\sqrt{\pi}}\left[2\alpha at + \frac{1}{2\sqrt{1 + 4\alpha^2}}\ln A(r)\right],$$

where $d\Omega^2_{(2)} \equiv d\vartheta^2 + \sin^2 \vartheta d\varphi^2$ is the line element on the unit 2-sphere,

$$A(r) = 1 - \frac{2m}{r},$$

$$\delta = \frac{2\alpha}{\sqrt{1 + 4\alpha^2}} < 1,$$

$r \geq 2m$, and where $m, \alpha$, and $a$ are constants ($m > 0$ has the dimensions of mass, $a$ those of an inverse length, and $\alpha$ is dimensionless), while the scalar field is subject to the potential [60]

$$V(\phi) = V_0 e^{-8\sqrt{\pi}\alpha\phi}, \quad V_0 = \frac{a^2 (3 - 4\alpha^2)}{8\pi}.$$  

The condition $V \geq 0$ places the restriction $|\alpha| \leq \sqrt{3}/2$ on the dimensionless parameter $\alpha$.

The Fonarev solution is spherically symmetric, time-dependent, and asymptotically FLRW. It describes a time-dependent wormhole or a naked singularity embedded in a FLRW universe (see Refs. [59–61] for discussions). For $\alpha = \pm\sqrt{3}/2$, the potential $V(\phi)$ vanishes and the Fonarev spacetime reduces to the Husain-Martinez-Núñez solution of the Einstein equations with a free scalar field [55]. In the limit $a = 0$, it reduces instead to the Fisher-Janis-Newman-Winicour-Wyman static and spherical scalar field geometry [29, 59, 62–64]. Here we use as a seed solution the special case $\alpha = 0$ of the Fonarev spacetime, which has a time-dependent and asymptotically FLRW geometry but static scalar field:

$$ds^2 = -dt^2 + e^{2at}\left[dr^2 + A(r)r^2d\Omega^2_{(2)}\right],$$

$$\phi(r) = \frac{1}{4\sqrt{\pi}}\ln A(r).$$
In this limit, $\delta = 0$ and the scalar field potential degenerates into a constant. Therefore, Eqs. (2.6) and (2.7) describe a 2-parameter family of solutions of the Einstein equations with positive cosmological constant $\Lambda = 8\pi V_0 = 3a^2$ and sourced by a free scalar field, previously reported in Eqs. (2.27) and (2.28) of Ref. [65]. The areal radius is $R(t, r) = e^{at} r \sqrt{A(r)}$ and $r \geq 2m$ corresponds to real values of the physical radius $\hat{R}$, with $r = 2m$ equivalent to $\hat{R} = 0$. The scalar field diverges as $\hat{R} \to 0^+$ (or $r \to 2m^+$), which is therefore a physical singularity of the theory, except when a wormhole throat is present at a positive radius $\hat{R}$ [60, 61].

Since $\phi$ depends only on the radial coordinate $r$ while $g_{ab}$ depends also on time, this Einstein-scalar field solution is particularly well-suited to generate a new family of solutions of viable Horndeski gravity by means of the disformal transformation

$$g_{ab} \to g_{ab} = \Omega^2 (\phi, \bar{X}) g_{ab} + W (\phi, \bar{X}) \nabla_a \phi \nabla_b \phi$$

(2.8)

with $g_{ab}$ denoting the metric tensor associated with (2.6), $\nabla$ denotes the connection compatible with the metric $g_{ab}$, while

$$\nabla_{\mu} \phi = \frac{m}{2\sqrt{\pi} Ar^2} \delta_{\mu}^r$$

(2.9)

and

$$\bar{X} (t, r) \equiv - \frac{1}{2} g^{ab} \bar{\nabla}_a \phi \bar{\nabla}_b \phi = - \frac{m^2 e^{-2at}}{8\pi r^4 A^2 (r)}.$$  

(2.10)

The new line element obtained from the disformal transformation reads

$$ds^2 = -\Omega^2 dt^2 + \left( e^{2at} \Omega^2 + \frac{Wm^2}{4\pi A^2 r^4} \right) dr^2 + \Omega^2 e^{2at} A^2 r^2 dt (t, r)^2.$$  

(2.11)

If the functions $\Omega$ and $W$ depend only on $\phi$, then $\Omega = \Omega (r)$ and $W = W (r)$. In the following we choose

$$\Omega (\phi) = \frac{1}{\sqrt{A (r)}} = e^{-2\sqrt{\pi} \phi},$$

(2.12)

then the areal radius of the metric $g_{ab}$ associated with the line element (2.11) reads

$$R(t, r) = re^{at}.$$  

(2.13)

Furthermore, set

$$W(r) = \frac{4\pi r^2 A(r)}{m^2} f (r),$$

(2.14)

where the (yet to be specified) function $f (r)$ has the dimensions of a length squared. Inverting the expression of $\phi (r)$ yields

$$r = \frac{2m}{1 - e^{4\sqrt{\pi} \phi}}$$

(2.15)

and

$$W(\phi) = \frac{16 \pi e^{4\sqrt{\pi} \phi}}{(1 - e^{4\sqrt{\pi} \phi})^2} f \left( \frac{2m}{1 - e^{4\sqrt{\pi} \phi}} \right);$$

(2.16)

this choice of $\Omega$ and $W$ automatically satisfies the conditions (1.2).

The disformed line element then reads

$$ds^2 = -\frac{dt^2}{A(r)} + \left[ e^{2at} \frac{f (r)}{r^2} + \frac{dr^2}{A(r)} + R^2 d\Omega^2 (t, r) \right]$$

(2.17)

with $r > 2m$ and $e^{2at} + f (r)/r^2 > 0$. If $f (r)/r^2 \to 0$ as $r \to +\infty$, the geometry is asymptotically de Sitter with comoving time $t$ and Hubble constant $H = a = \sqrt{\Lambda}/3$ for $r \to +\infty$. The scalar field $\phi$, which is left unchanged by the disformal transformation, still diverges as $r \to (2m)^+$ (or $R \to 2m e^{at}$).
2.1 Curvature coordinates

We can rewrite the line element (2.17) in curvature coordinates employing the areal radius \( R(t, r) = r e^{at} \). Substitution of the relation between differentials

\[
dr = e^{-at} (dR - aR dt) \tag{2.18}
\]

into Eq. (2.17) gives

\[
ds^2 = - \left[ 1 - a^2 R^2 \left( 1 + \frac{e^{-2at} f(r)}{r^2} \right) \right] \frac{d t}{A(r)} \left( 1 + \frac{e^{-2at} f(r)}{r^2} \right) dt dR
+ \left( 1 + \frac{e^{-2at} f(r)}{r^2} \right) \frac{dR^2}{A(r)} + R^2 d\Omega^2_2. \tag{2.19}
\]

The cross-term in \( dt dR \) can be eliminated, and the line element diagonalized, by introducing a new time coordinate \( T \) defined by

\[
dT = \frac{1}{F} (dt + \beta dR), \tag{2.20}
\]

where \( \beta(t, R) \) is a function to be determined and \( F(t, R) \) is an integrating factor guaranteeing that \( dT \) is an exact differential and satisfying

\[
\frac{\partial}{\partial R} \left( \frac{1}{F} \right) = \frac{\partial}{\partial t} \left( \frac{\beta}{F} \right). \tag{2.21}
\]

Substituting \( dt = F dT - \beta dR \) in the line element (2.19), one obtains

\[
ds^2 = - \frac{F^2}{A(r)} \left[ 1 - a^2 R^2 \left( 1 + \frac{e^{-2at} f}{r^2} \right) \right] dT^2
+ \frac{2F}{A(r)} \left\{ \beta \left[ 1 - a^2 R^2 \left( 1 + \frac{e^{-2at} f}{r^2} \right) \right] - aR \left( 1 + \frac{e^{-2at} f}{r^2} \right) \right\} dT dR
+ \left\{ -\beta^2 \left[ 1 - a^2 R^2 \left( 1 + \frac{e^{-2at} f}{r^2} \right) \right] + 2aR\beta \left( 1 + \frac{e^{-2at} f}{r^2} \right) + \left( 1 + \frac{e^{-2at} f}{r^2} \right) \right\} \frac{dR^2}{A(r)}
+ R^2 d\Omega^2_2. \tag{2.22}
\]

By choosing

\[
\beta(t, R) = \frac{aR \left( 1 + \frac{e^{-2at} f}{r^2} \right)}{1 - a^2 R^2 \left( 1 + \frac{e^{-2at} f}{r^2} \right)} \tag{2.23}
\]

the line element is diagonalized, becoming

\[
ds^2 = - \left[ 1 - a^2 R^2 \left( 1 + \frac{e^{-2at} f}{r^2} \right) \right] \frac{F^2 dT^2}{A(r)}
+ \frac{\left( 1 + \frac{e^{-2at} f}{r^2} \right)}{A(r) \left[ 1 - a^2 R^2 \left( 1 + \frac{e^{-2at} f}{r^2} \right) \right]} dR^2 + R^2 d\Omega^2_2. \tag{2.24}
\]

Although \( f(r)/r^2 \) and \( A(r) = 1 - 2m/R = 1 - 2m e^{at}/R \) can be expressed in terms of \( R \) using \( r = Re^{-at} \) to eliminate \( r \), the form (2.24) of the line element remains implicit because it contains the old time coordinate \( t \). From the physical point of view, however, it is more interesting to describe the evolution in terms of the comoving observers of the asymptotic FLRW background, who use \( t \) as their time coordinate.
2.2 Apparent horizons

The apparent horizons of a spherically symmetric spacetime are located by the roots of the equation

$$\nabla^c R \nabla_c R = 0,$$

(2.25)

where $R$ is the areal radius (e.g., [58, 59]). Single roots correspond to black hole (or possibly, white hole or cosmological) apparent horizons, while double roots correspond to wormhole throat horizons [58, 59].

For the line element (2.17), this equation becomes

$$\nabla^c R \nabla_c R = g^{tt} \left( \frac{\partial R}{\partial t} \right)^2 + g^{rr} \left( \frac{\partial R}{\partial r} \right)^2 = e^{2at} r^2 A(r) \left[ \frac{1}{r^2 e^{2at} + f(r)} - a^2 \right] = 0. \quad (2.26)$$

The first root, which is a single root and always exists, is given by $r_1 = 2m$ (i.e., $A(r) = 0$) or

$$R_1(t) = 2m e^{at}. \quad (2.27)$$

If $a > 0$, then $R_1(t) \to +\infty$ as $t \to +\infty$. The other roots, if they exist, are the real and positive solutions of

$$r^2 e^{2at} + f(r) - \frac{1}{a^2} = 0 \quad (2.28)$$

and they depend on the choice of the function $f(r)$ in the disformal transformation.
2.3 Singularity

Computing the invariants of the Ricci tensor for the line element (2.17) for a generic function $f(r)$ yields

$$R = \frac{4A(r) \left[r f'(r) - f(r) \left(a^2 f(r) + 2\right)\right] + r A'(r) \left[2f(r) - rf'(r)\right]}{2 \left[r^2 e^{2at} + f(r)\right]^2}$$

$$+ \frac{-2A^2(r) \left[2a^2f(r) + 1\right] + r^2 A(r) A''(r) - r^2 [A'(r)]^2}{A(r) \left[r^2 e^{2at} + f(r)\right]} + 12a^2 A(r)$$

$$+ \frac{2 e^{-2at}}{r^2},$$

(2.29)

$$R_{ab} R^{ab} = \frac{1}{16 \left(r^2 e^{2at} + f\right)^4} \left\{-32a^2 \left[r^2 e^{2at} + f\right] \left[r A'(r) \left[r^2 e^{2at} + f\right] + 2A(r)f\right]^2$$

$$+ \frac{1}{r^4} \left[8 e^{-4at} \left[r^3 e^{2at} A(r) \left(6a^2 r^3 e^{4at} - 2r e^{2at} + f'\right) + 2a^2 e^{2at}\right]$$

$$+ 2r^2 e^{2at} f \left[A(r) \left(5a^2 r^2 e^{2at} - 2\right) + 2\right]$$

$$+ f^2 \left(4a^2 r^2 e^{2at} A(r) + 2\right)\right\}^2$$

$$+ \frac{1}{A^2(r)} \left\{4a^2 A^2(r) \left(2f \left(3r^2 e^{2at} + f\right) + 3r^4 e^{4at}\right) - 2r^2 \left(A'(r)\right)^2 \left[r^2 e^{2at} + f\right]$$

$$+ r A(r) \left(2r A''(r) \left(r^2 e^{2at} + f\right) + A'(r) \left(4r^2 e^{2at} - rf' + 6\right)\right)^2$$

$$+ \left[4A^2(r) \left(f \left(4a^2 r^2 e^{2at} - 2\right) + 3a^2 r^4 e^{4at} + rf'\right) - 2r^2 A'(r)^2 \left(r^2 e^{2at} + f\right)$$

$$+ r A(r) \left(2r A''(r) \left(r^2 e^{2at} + f\right) - A'(r) \left(r \left(4r e^{2at} + f'\right) + 2f\right)\right)\right\}^2\right\}. \quad (2.30)$$

One can immediately conclude that $r_1 = 2m$ (i.e., $A(r) = 0$) is a curvature singularity for (2.17) since both $R$ and $R_{ab} R^{ab}$ diverge there, together with the scalar field (2.2). Note however that as $r \to (2m)^+$ one finds

$$R \simeq -\frac{1}{A(r) \left[r^2 e^{2at} + f(r)\right]},$$

(2.31)

$$R_{ab} R^{ab} \simeq \frac{1}{A^2(r) \left[r^2 e^{2at} + f(r)\right]^2}. \quad (2.32)$$

Unless the denominators in the right-hand sides of Eqs. (2.31) and (2.32) diverge (and they diverge at different rates), these Ricci invariants are certainly divergent. For instance, for $f(r) = -r^2$ the geometry has a curvature singularity at $r_1 = 2m$. Figure 1 illustrates these divergencies for the parameter values $a = m = 1$. 

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Furthermore, other curvature singularity can emerge from the roots of \( e^{2at} + f(r)/r^2 = 0 \), if \( r \neq 2m \).

The singularity surface \( r = 2m \) of the scalar \( \phi \) in (2.7), and consequently of the line element (2.17), is null. In fact, it is defined by the equation \( \psi(r) \equiv r - 2m = 0 \) and the normal vector \( N^\mu \) has components

\[
N_{\mu} \equiv \nabla_{\mu} \psi = \delta_{\mu}^{r},
\]

and norm squared

\[
N_{\mu}N^\mu = \nabla^\mu \psi \nabla_{\mu} \psi = g^{rr} = \frac{A(r)}{e^{2at} + f(r)/r^2},
\]

which vanishes as \( r \to 2m^+ \), therefore \( r = 2m \) is a null surface.

### 3 Two simple choices of \( f(r) \)

Here we consider two concrete choices of the free function \( f(r) \) that produce relatively simple, yet interesting, geometries.

#### 3.1 Choice \( f(r) = 0 \)

If \( f(r) = 0 \), then \( W = 0 \), and the transformation (1.1) reduces to a pure conformal transformation. This case describes a finite radius singularity embedded in a de Sitter universe, which is locally static below its de Sitter horizon [66].

The transformed line element is

\[
ds^2 = -\frac{dt^2}{A(r)} + \frac{e^{2at}}{A(r)} dr^2 + R^2 d\Omega^2_{(2)};
\]
to express it in terms of the areal radius, note that Eq. (2.23) gives

$$\beta(R) = \frac{aR}{1 - a^2R^2}$$

(3.2)

and then Eq. (2.21) for the integrating factor \( F(t, R) \) reduces to \( \partial R F = \beta \partial_t F \), which admits the constant solution \( F = 1 \) and turns the line element into

$$ds^2 = -\frac{(1 - a^2R^2)}{A(r)} dT^2 + \frac{1}{(1 - a^2R^2)A(r)} dR^2 + R^2 d\Omega^2$$

(3.3)

for \( 2m e^{at} < R < 1/a \), where the lower bound on the physical radius is the location of the curvature singularity. This relation implies that it must be \( am < 1/2 \). The transformation between old and new time coordinates can be found explicitly in this case: using \( F = 1 \) and (3.2), Eq. (2.20) is integrated to

$$T(t, R) = t - \frac{1}{2a} \ln (1 - a^2R^2) + \text{const.}$$

(3.4)

The formal (single) root \( R = 1/a \) of the equation \( \nabla^c R \nabla_c R = g_{RR} = 0 \) is reminiscent of the de Sitter horizon of the de Sitter space in which the central object is embedded. (If \( m = 0 \), this line element reduces to the de Sitter one in static coordinates, with Hubble constant \( H = a \).) Accordingly, this apparent horizon is always a null surface. In fact, the normal vector to the apparent horizon defined by \( \psi = R - a^{-1} = 0 \) reads

$$N_\mu \equiv \nabla_\mu \psi = \delta^R_\mu$$

(3.5)

has norm squared

$$N^\alpha N_\alpha = g_{RR} = 0 \quad \text{on} \quad R = a^{-1}. \quad (3.6)$$

Since \( 2m e^{at} < R < 1/a \), the physical spacetime region shrinks as the lower bound \( 2m e^{at} \) to the radius \( R \) grows exponentially while the upper bound \( 1/a \) remains constant.

### 3.2 Choice \( f(r) = -r^2 \)

In the following we assume that \( a > 0 \). If \( f(r) = -r^2 \),

$$W(\phi) = -\frac{64\pi m^2 e^{4\sqrt{\pi} \phi}}{(1 - e^{4\sqrt{\pi} \phi})^4}$$

(3.7)

and the disformed line element

$$ds^2 = -\frac{dt^2}{A(r)} + (e^{2at} - 1) \frac{dr^2}{A(r)} + R^2 d\Omega^2$$

$$= -\frac{1 - a^2R^2}{A(r)} \frac{1 - e^{-2at}}{A(r)} F^2 dT^2 + \frac{(1 - e^{-2at}) dR^2}{A(r) [1 - a^2R^2 (1 - e^{-2at})]} + R^2 d\Omega^2$$

(3.8)

(3.9)

is defined for \( t \geq 0 \) and \( r > 2m \), equivalent to \( R > R_1(t) = 2m e^{at} \). The geometry is not asymptotic to a FLRW universe for large \( r \), however at late times and large \( r \) it approaches a de Sitter space with scale factor \( e^{at} \).
The singularity is, again, described by the equation \( \psi \equiv r - 2m = 0 \). As discussed in Sec. 2.3, this is a null singularity somehow similar to the thunderbolt singularities discussed in the literature \([67–69]\).

Let us describe the dynamics of the singularity and apparent horizon in terms of the time coordinate \( t \), which is the physical time of the comoving observers of the background space (that reduces asymptotically to a de Sitter universe in the limits discussed). Equation (2.26) admits the single root \( R_1(t) = 2m e^{at} \), which describes an expanding singularity, plus the other single root

\[
R_2(t) = \frac{e^{at/2}}{a \sqrt{2 \sinh(at)}} = \frac{e^{a(t)}}{a \sqrt{e^{2at} - 1}}.
\] (3.10)

The physical radius of this apparent horizon is

\[
R_2(t) = \frac{e^{at/2}}{a \sqrt{2 \sinh(at)}} = \frac{e^{at}}{a \sqrt{e^{2at} - 1}}; \tag{3.11}
\]

it converges to the de Sitter horizon \( H^{-1} = 1/a \) of the de Sitter “background” from above as \( t \to +\infty \). As \( t \to 0^+ \), \( R_1 \to 2m \) and \( R_2 \to +\infty \). As time progresses from \( t = 0 \), \( R_1(t) \) increases exponentially while \( R_2(t) \) decreases monotonically from infinity since

\[
\dot{R}_2 = -\frac{e^{-2at}}{(e^{2at} - 1)^{3/2}} < 0 \tag{3.12}
\]

and formally approaches the value \( 1/a \) as \( t \to +\infty \). The expanding singularity and the shrinking apparent horizon meet at the critical time

\[
t_* = \frac{1}{2a} \ln \left( \frac{4a^2 m^2 + 1}{4a^2 m^2} \right) > 0, \tag{3.13}
\]

after which \( R_1 \) becomes larger than \( R_2 \). Since it must be \( R > R_1 \) at all times, this means that the apparent horizon of radius \( R_2(t) \) disappears from spacetime at the time \( t_* \). The physical picture emerging from these considerations is the following (Fig. 2):

- For \( 0 < t < t_* \) there is a finite radius singularity with radius \( R_1(t) < R_2(t) \) which expands exponentially while the larger apparent horizon at \( R_2(t) \) covering it shrinks.

- At \( t = t_* \) the singularity and the apparent horizon meet at the common radius

\[
R_* = \frac{1}{a} \sqrt{4a^2 m^2 + 1} > 2m. \tag{3.14}
\]

- For \( t > t_* \) it is \( R_2(t) < R_1(t) \) and no apparent horizon exists. The singularity is naked.

The behaviour of \( R_1(t) \) and \( R_2(t) \) is illustrated in Fig. 2 for the parameter values \( a = 1/2, 1, \) and \( 3/2 \) (where all parameters are in units of \( m \)). A larger \( a \) leads to smaller \( R_* \) and \( t_* \).

From Eq. (3.10), it is clear that the apparent horizon is the surface of equation

\[
\psi(t, r) \equiv r - \frac{e^{-at/2}}{a \sqrt{2 \sinh(at)}} = 0; \tag{3.15}
\]
the normal vector to this surface is
\[ N_\mu \equiv \nabla_\mu \psi = \frac{e^{at/2}}{2\sqrt{2} \sinh^{3/2}(at)} \delta^t_\mu + \delta^r_\mu \] (3.16)
and has norm squared
\[ N_\mu N_\mu = \frac{(2e^{2at} - 1) \left( 2am\sqrt{e^{2at} - 1} - 1 \right)}{(e^{2at} - 1)^3}. \] (3.17)

For \( t < t_\star \), i.e., when this apparent horizon exists, it is always \( N_c N^c < 0 \) and this apparent horizon is always spacelike.

\[ \text{Figure 2.} \] The behaviour of the physical radii of the singularity (solid curves) and of the apparent horizon (dashed curves) for \( f(r) = -r^2 \) and for three values of the parameter \( a \).

### 3.3 Physical nature of the apparent horizon for \( f(r) = 0 \) and \( f(r) = -r^2 \)

In order to determine the nature of the apparent horizon, we examine the expansions of the congruences of outgoing (+) and ingoing (−) radial null geodesics at the apparent horizon. The line element can be written in the compact form

\[ ds^2 = -\frac{dt^2}{A(r)} + \frac{(e^{2at} - b)}{A(r)} dr^2 + r^2 e^{2at} d\Omega^2, \] (3.1)

where \( b = 0 \) for \( f(r) = 0 \) and \( b = 1 \) for \( f(r) = -r^2 \).

Let \( \lambda \) be an affine parameter along the radial null geodesics and let an overdot denote differentiation with respect to \( \lambda \). The four-tangents \( t^\mu = (\dot{t}, \dot{r}, 0, 0) \) to these radial geodesics
satisfy the normalization $\ell^c \ell_c = 0$, or
\begin{equation}
\ell^2 (e^{2at} - b) - t^2 = 0.
\end{equation}
By introducing the new time coordinate $\bar{t}$ defined by
\begin{equation}
d\bar{t} = \frac{dt}{\sqrt{e^{2at} - b}},
\end{equation}
the line element (2.17) along the radial null geodesics is written in the conformally flat form
\begin{equation}
ds^2 \bigg|_{d\theta = d\varphi = 0} = \frac{(e^{2at} - b)}{A(r)} (-d\bar{t}^2 + dr^2)
\end{equation}
\begin{equation}
= -\frac{(e^{2at} - b)}{A(r)} dudv
\end{equation}
where, in the last line, we have introduced the null coordinates $u \equiv \bar{t} - r$, $v \equiv \bar{t} + r$.

The ingoing and outgoing radial null geodesics, described by $v = \text{const.}$ and $u = \text{const.}$, respectively, have four-tangents
\begin{equation}
\ell_a^{(+)} \equiv -\partial_a u = -\frac{1}{\sqrt{e^{2at} - b}} \delta^0_a + \delta^1_a
\end{equation}
and
\begin{equation}
\ell_a^{(-)} \equiv -\partial_a v = -\frac{1}{\sqrt{e^{2at} - b}} \delta^0_a - \delta^1_a
\end{equation}
in coordinates $(t, r, \theta, \varphi)$, while
\begin{equation}
g^{\alpha\beta} \ell_a^{(+)} \ell_b^{(-)} = -\frac{2A(r)}{e^{2at} - b}.
\end{equation}
In compact form, we have
\begin{equation}
\ell_a^{(\pm)} = -\frac{\delta^0_a \pm \delta^1_a}{\sqrt{e^{2at} - b}}.
\end{equation}
The general expression of the expansion scalars of the congruences of outgoing and ingoing radial null geodesics is
\begin{equation}
\Theta^{(\pm)} = \left[ g^{\alpha\beta} - \ell_a^{(+)} \ell_b^{(-)} + \ell_a^{(-)} \ell_b^{(+)} \right] \nabla_a \ell^{(\pm)}.
\end{equation}
The third term in the right-hand side vanishes if $\ell_a^{(\pm)}$ is globally null; the second term vanishes if the null geodesics are affinely parameterized, which we assume here, leaving
\begin{equation}
\Theta^{(\pm)} = \nabla_c \ell^{c(\pm)}.
\end{equation}
We then have
\begin{equation}
\Theta^{(\pm)} = \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} \ell^{\mu}_{(\pm)} \right)
\end{equation}
\begin{equation}
= \frac{A}{r^2 e^{2at} \sin \vartheta \sqrt{e^{2at} - b}} \partial_\mu \left( r^2 e^{2at} \sin \vartheta \delta^\mu_0 \pm \delta^1_0 r^2 e^{2at} \sin \vartheta \right)
\end{equation}
\begin{equation}
= \frac{2A(r)}{r \sqrt{e^{2at} - b}} \left( ar \pm \frac{1}{\sqrt{e^{2at} - b}} \right).
\end{equation}
Assuming $a > 0$, on the apparent horizon of radius $r_H(t) = \left( |a| \sqrt{e^{2at} - b} \right)^{-1}$ the expansions assume the values

$$
\Theta_{(\pm)}(r_H) = \nabla_c \ell^c_{(\pm)} \bigg|_{r_H} = \frac{2aA(r_H)}{\sqrt{e^{2at} - b}} (1 \pm 1) \quad (3.13)
$$

$$
= \frac{2a}{\sqrt{e^{2at} - b}} \left( 1 - 2am \sqrt{e^{2at} - b} \right) (1 \pm 1) . \quad (3.14)
$$

If $r_H(t) > 2m$ (i.e., $A(r_H(t)) > 0$), in both cases $b = 0$ (or $f(r) = 0$) and $b = 1$ (or $f(r) = -r^2$) it is $\Theta_{(+)}(r_H) > 0$ and $\Theta_{(-)}(r_H) = 0$. Furthermore, the Lie derivative of the expansion of ingoing radial null geodesics with respect to $\ell^c_{(\pm)}$ yields

$$
\mathcal{L}_{\ell_{(\pm)}} \Theta_{(-)} = \ell^0_{(\pm)} \partial_\alpha \Theta_{(-)} = 2a^2 A^2 (r_H) \frac{(2e^{2at} - b)}{(e^{2at} - b)^2} > 0 , \quad (3.15)
$$

which means that the apparent horizon is a cosmological horizon. Therefore, the Fonarev solution (2.1), which describes a wormhole, disforms into a spacetime with a cosmological horizon for $f(r) = 0$ and $f(r) = -r^2$.

4 Conclusions

While most of the solutions of Horndeski gravity are stationary, the ones obtained here by means of disformal transformations are dynamical. The change in the physical nature of spherically symmetric objects (black holes, wormholes, white holes, naked singularities) under static conformal [32, 33] and disformal [30, 31, 34] transformations has been studied in previous literature, but no general theorems are available in the time-dependent case. Here we have considered time-dependent disformal transformations of a specific seed, a special case of the Fonarev scalar field spacetime of GR [60]. The disformation of a white hole or naked singularity (depending on the values of the parameters) does not produce a black hole, which is similar to the results proved for static transformations [30, 31, 34].

The spacetimes thus generated are interesting. For $f(r) = -r^2$ we have a genuine disformal transformation that originates a geometry with a null singularity (reminiscent of thunderbolt singularities [67–69]) which expands and meets a contracting spacelike apparent horizon. The latter then disappears from the spacetime. The expanding singularity is obtained from the disformal transformation of a horizon, and it is well-known that apparent horizons tend to appear and disappear in pairs [55–57, 59], but we are not aware of similar phenomenology in the literature, where a dynamical singularity annihilates an apparent horizon. The lesson is that, in time-dependent dynamical solutions of Horndeski gravity, one should not only expect to find phenomenology of dynamical apparent horizons familiar from GR and “old” scalar-tensor gravity, but also phenomenology not encountered before. We will look further for new behaviours of apparent horizons and dynamical singularities in future works.

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