Multipole analysis in cosmic topology.

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Abstract. Low multipole amplitudes in the Cosmic Microwave Background CMB radiation can be explained by selection rules from the underlying multiply-connected homotopy. We apply a multipole analysis to the harmonic bases and introduce point symmetry. We give explicit results for two cubic 3-spherical manifolds and lowest polynomial degrees, and derive three new spherical 3-manifolds.

Keywords: Cosmic microwave background, cosmology, harmonic analysis, topology, homotopy, Wigner polynomials, point symmetry

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INTRODUCTION.

Cosmology deals with the large-scale structure of the universe. A global description of this structure employs an averaging and smoothing of all the objects of astrophysics from solar systems to galaxies and clusters of galaxies. The result is a fluid model of the cosmos governed by gravitation, as it was proposed by Einstein [5], see [18] pp. 160-164 and [19] pp. 704-710. This fluid model is governed by Einstein’s differential equations which relate the space-time metric to the energy-momentum tensor. To device such a model one has to make assumptions about the global structure and topology of the underlying space-time manifold. Einstein in his initial analysis of 1917 assumed cosmic 3-space in form of a 3-sphere. His assumption implies an average curvature +1.

The topology of cosmic 3-space has found new attention in relation to the Cosmic Microwave Background (CMB) Radiation, which is supposed to originate from an early stage of the universe. The fluctuations of the incoming CMB radiation are well described by a standard model, except for rather low amplitudes at the lowest multipole orders. This raised the question if multipole selection rules could be due to a particular topology of 3-space.

A familiar paradigm for topology is the Möbius strip. In [16] we explain its topological properties and relate it to the crystallographic space group \textbf{cm}. A cell of this crystal is shown on the left-hand side of Fig. 2. Any spherical topological 3-manifold \( \mathcal{M} \) can be viewed as a prototile on Einstein’s 3-sphere with pairs of faces glued according to the prescriptions of a group of homotopies. Among the spherical 3-manifolds are the Platonic ones. Their homotopies were derived by Everitt [6]. In [15] and papers [12], [13] quoted therein we derived from the homotopy or fundamental group of a Platonic manifold \( \mathcal{M} \) the corresponding groups deck(\( \mathcal{M} \)) of deck operations acting on the 3-sphere. These groups generate by fixpoint-free action the tiling from the prototile. Each group of deck operations for a Platonic 3-manifold \( \mathcal{M} \) is constructed in [15] as a subgroup of...
FIGURE 1. Genesis of the Möbius strip. Left: as rectangular cell of the Möbius crystal \( \text{cm} \) with broken glide reflection line, see Fig. 2, Middle: Twisted by \( \pi \) along the glide line, Right: Bended into a circle and glued by homotopy.

FIGURE 2. Möbius crystal and 8-cell. Left: The red Möbius strip tiles its cover, the plane, into the cells of a Möbius crystal with crystallographic symbol \( \text{cm} \), with a broken glide line and two pointed mirror reflection lines. The vertical edges are glued by homotopy, Fig.1, Right: A spherical red cube tiles its cover, the 3-sphere, into the 8 spherical cubes of the 8-cell, shown in a projection.

a Coxeter group \( \Gamma \). Finally from the unimodular subgroups of the Coxeter groups we construct three new spherical 3-manifolds.

**ACTIONS AND HARMONIC ANALYSIS ON THE 3-SPHERE.**

To study actions on the 3-sphere we pass from the set of four Cartesian coordinates \( x = (x_0, x_1, x_2, x_3) \) in Euclidean 4-space to a matrix description,

\[
\mathbf{u} = \begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{bmatrix}, \quad z_1 = x_0 - ix_3, \quad z_2 = -x_2 - ix_1, \quad z_1\bar{z}_1 + z_2\bar{z}_2 = 1.
\]  

(1)
With the help of the Pauli matrices $\sigma_j$ and $\sigma_0 = e$ and the trace $Tr$ we recover the Cartesian coordinates in the form

$$u = x_0\sigma_0 - i \sum_{j=1}^{3} x_j \sigma_j, \quad x_0 = \frac{1}{2} Tr(u\sigma_0), \quad x_j = \frac{i}{2} Tr(u\sigma_j), \quad j = 1, 2, 3. \quad (2)$$

The Wigner polynomials \[22\], \[4\] are homogeneous polynomials $D_{m_1,m_2}^j(u)$ in the four complex variables $z_1, z_2, \bar{z}_1, \bar{z}_2$ of total degree $2j$, $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, see \[15\] Appendix A. In the familiar Euler half-angles $(\alpha/2, \beta/2, \gamma/2)$, the Wigner polynomials take the form

$$D_{m_1,m_2}^j(\alpha, \beta, \gamma) = \exp(i m_1 \alpha) d_{m_1,m_2}^j(\beta) \exp(i m_2 \gamma), \quad (3)$$

$$-j \leq (m_1, m_2) \leq j.$$

We look at the set of harmonic Wigner polynomials by starting from the integer or half-integer pairs of numbers $(m_1, m_2)$. These pairs of numbers can be viewed as points from two nested lattices on a plane, see Fig.5. For given integer or half-integer values of $(m_1, m_2)$, one finds

$$j = j_0 + \nu, \quad \nu = 0, 1, 2, \ldots, j_0 = \text{Max}(|m_1|, |m_2|). \quad (4)$$

We say that any lattice point $(m_1, m_2)$ in this plane carries a tower, labelled by $\nu$, of Wigner polynomials $D^j$ according to eq. 4. The Wigner polynomials form a complete orthogonal system of polynomial functions on the 3-sphere. Moreover they are harmonic, that is, vanish under the application of the Laplacian acting on functions on Euclidean 4-space, see \[15\]. Therefore they are a basis for harmonic analysis on the 3-sphere.

The isometric rotations of the group $SO(4,R) \sim (SU^l(2,C) \times SU^r(2,C))/K$ can be written in the form $(g_l, g_r)$, $g_l \in SU^l(2,C)$, $g_r \in SU^r(2,C)$ and act on the coordinates $u$ as

$$(g_l, g_r) : u \rightarrow g_l^{-1} u g_r, \quad K = \{(e,e), (-e,-e)\}. \quad (5)$$

The elements of the form $(g, g), g \in SU^C(2,C)$ generate a subgroup $SU^C(2,C)$ acting on $u$ by conjugation. The 3-sphere can be written as the homogeneous space $SO(4,R)/SU^C(2,C)$. We write the action eq. 5 of $SO(4,R)$ on the Wigner polynomials as

$$(T_{(g_l, g_r)} D_{m_1,m_2}^j)(u) = D_{m_1,m_2}^j(g_l^{-1} u g_r). \quad (6)$$

$$= \sum_{m_1', m_2'} D_{m_1',m_2'}^j(u) D_{m_1,m_1'}^j(g_l^{-1}) D_{m_2',m_2}^j(g_r).$$

Here we used the representation property of the Wigner polynomials.

A general pair $(g_l, g_r)$ can always be brought to diagonal form

$$g_l = c_l \delta_l c_l^{-1}, \quad g_r = c_r \delta_r c_r^{-1}. \quad (7)$$

We interprete the transformation

$$u \rightarrow u' = c_l^{-1} u c_r, \quad (8)$$
as a transformation to new coordinates $u'$. In these new coordinates, $(\delta_l, \delta_r)$ are diagonal with diagonal entries $\delta_l : \exp(\pm i\alpha_l/2)$, $\delta_r : \exp(\pm i\alpha_r/2)$, and the action eq. 6 with eq. 7 takes the form

$$D_{m_1, m_2}^j(u) \rightarrow D_{m_1, m_2}^j((\delta_1)^{-1}u\delta_r) = \exp(i(-m_1\alpha_l + m_2\alpha_r))D_{m_1, m_2}^j(u).$$

(9)

Now we can go to a lattice description of the harmonic analysis for the two spherical cubic 3-manifolds: The basis for the harmonic analysis consists of the towers of Wigner $D^j$ polynomials on top of a sublattice in the $(m_1, m_2)$-plane as given in Fig. 5.

THE SPHERICAL CUBIC MANIFOLDS.

As examples we shall choose the two spherical cubic manifolds $N2, N3$. The first homotopy groups of the Platonic spherical polyhedra were given in Everitt [6]. In [15] and work cited therein we construct from Everitt’s results the groups of deck transformations. These act on the 3-sphere and tile it into Platonic polyhedra.

THE DECK GROUPS OF THE CUBIC 3-MANIFOLDS $N2, N3$.

In [6] one finds by an enumeration the homotopic face and edge gluings for this manifold. The tiling of the 3-sphere is the 8-cell shown on the right in Fig 2. The cubic 3-manifold we take as the central spherical cube in the 8-cell. The two different spherical cubic 3-manifolds differ in their face and edge gluing. In Fig. 4 we have marked the faces with numbers 1, 2, 3 by triangles with the colors yellow, blue, and red. The homotopic self-gluing of the initial 3-manifold is converted in the 8-cell tiling into a gluing of neighbour cubes sharing a face. This is illustrated in colors in Fig. 3. In Fig. 4 we use the same color coding to illustrate the two different next neighbour cubes for the two 3-manifolds $N2, N3$.

HARMONIC ANALYSIS ON THE SPHERICAL CUBIC 3-MANIFOLDS $N2, N3$.

Algebraically, the deck operations, being rotations, contain an even number of Weyl reflections and can be written in terms of elements $(g_l, g_r) \in (SU^l(2, \mathbb{C}) \times SU^r(2, \mathbb{C}))$. 
We now construct by projection the linear combinations of Wigner polynomials that span the harmonic analysis on the two cubic 3-manifolds.

For $N_2$, the group $H = \text{deck}(N_2)$ of deck transformations of the 8-cell tiling from [15] is a cyclic group $C_8$. Its generator $g_1 = (g_l, g_r)$ is given in eq. 11. We start from the set of Wigner polynomials and use their representation under $SO(4, R)$, in terms of irreducible representations $D^{j}_j$ of $SU(2, C)$. This allows to apply to them the projection operator to the identity representation of the group $H$,

$$\left( P^h D^j_{m_1 m_2} \right)(u) = \sum_{m'_1 m'_2} D^j_{m'_1 m'_2}(u) \left[ \frac{1}{|H|} \sum_{(g_l, g_r) \in H} D^j_{m_1 m'_1}(g_l^{-1}) D^j_{m'_2 m_2}(g_r) \right].$$  (10)

In general, eq. 10 gives a linear combination of Wigner polynomials. In case of the Platonic manifold $N_2$, $H$ is a cyclic group. By a transformation $u \to u'$ of coordinates we can reduce the action of this cyclic group to diagonal form. We start from the generator $g_1$ of the group deck$(N2) = C_8$,

$$g_1 = (g_l, g_r), g_l = \begin{bmatrix} \bar{a} & 0 \\ 0 & a \end{bmatrix}, g_r = \begin{bmatrix} 0 & \bar{a} \\ -a & 0 \end{bmatrix}, a = \exp(i \frac{\pi}{4}).$$  (11)

We diagonalize $g_r$ as

$$g_r = C \begin{bmatrix} a^2 & 0 \\ 0 & -a^2 \end{bmatrix} C^{-1}, C = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -a \\ a & -1 \end{bmatrix}, C^{-1} = C^t, \det(C) = 1.$$  (12)

Next we define new coordinates $u'$ by

$$u' := uC$$  (13)
TABLE 1. The three generators \( q_i = (g_{li}, g_{ri}) \) of the quaternionic group \( H = \text{deck}(N3) = Q \) as elements of the Coxeter group \( \Gamma \), and the corresponding pairs \( (g_{li}, g_{ri}) \in (SU^l(2,R) \times SU^r(2,R)) \). Products of the matrices \( i, j, k \) follow the standard quaternionic rules.

| \( i \) | \( q_i x \) | \( g_{li} \) | \( g_{ri} \) |
|---|---|---|---|
| 1 | \((x_1, -x_0, x_3, -x_2)\) | \( \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \) | \(-k\) |
| 2 | \((x_2, -x_3, -x_0, x_1)\) | \( \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \) | \(-j\) |
| 3 | \((x_3, x_2, -x_1, -x_0)\) | \( \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \) | \(-i\) |

so that \( g_1 \) acts on \( u' \) by diagonal matrices from left and right as

\[
g_1 : u' \rightarrow \begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix} u' \begin{bmatrix} a^2 & 0 \\ 0 & -a^2 \end{bmatrix} = \delta_l^{-1} u' \delta_r. \tag{14}\]

These relations allow to apply eq. 9. It follows that we can reduce the representation of \( H \) to diagonal form. Invariance under \( H \) now gives a certain linear relation between \( m_1 \) and \( m_2 \). This relation singles out on the full lattice of points \((m_1, m_2)\) all the points of the sublattice with points

\[
m_1 + 2m_2 \equiv 0 \pmod{8}. \tag{15}\]

A basis of this sublattice is \( a_1 = (2, -1), a_2 = (2, 3), \) compare Fig. 5. The harmonic analysis on \( N2 \) is now given by the towers of Wigner polynomials on top of the black sublattice points.

For the cubic 3-manifold \( N3 \) we construct three glue generators \( q_1, q_2, q_3 \) in Table 1 from the homotopy group [15]. The result for three faces is shown on the right of Fig. 4. The deck operations corresponding to the gluings generate the quaternionic group \( H = Q \). [2] p. 134. It acts exclusively by left action on the 3-sphere.

For the harmonic analysis on \( N3 \) we employ the projection eq. 10 for the quaternion group \( Q \),

\[
(P_Q^0 D^j_{m_1,m_2}(u)) = \frac{1}{8} \left[ 1 + (-1)^{2j} \right] \left[ 1 + (-1)^{m_1} \right] D^j_{m_1,m_2}(u) + \delta l^{ij} D^j_{-m_1,m_2}(u). \tag{16}\]

HARMONIC ANALYSIS ON PLATONIC CUBIC 3-MANIFOLDS.

In Fig. 5, we display the sublattices for the two cubic spherical Platonic 3-manifolds. For the manifold \( N3 \) we put the symmetry eq. 16, \((m_1, m_2) \rightarrow (-m_1, m_2)\) as a vertical mirror line. For the harmonic analysis it follows that an orthogonal basis is given by the collection of all towers of Wigner polynomials on top of the sublattice points. Any basis function can be characterized by a sublattice point \((m_1, m_2)\) and by a number \( \nu = 0, 1, 2, \ldots \)
FIGURE 5. Lattice representation of $N_2$ and $N_3$ basis: Any lattice point $(m_1,m_2)$ carries the countable tower $D_{m_1,m_2}(u)$, $j = j_0 + v$, $v = 0, 1, 2, ..., j_0 = \text{Max}(|m_1|,|m_2|)$ of Wigner polynomials. Harmonic analysis on cubic manifolds, left $N_2$, right $N_3$ (with vertical mirror line), selects the towers $D^j(u)$ on sublattices marked by black points with sublattice bases $(a_1,a_2)$. Only these obey the homotopic boundary conditions.

MODELLING INCOMING CMB BY HARMONIC ANALYSIS.

In this section we discuss the algebraic tools for analysing incoming CMB radiation in terms of the harmonic bases for a chosen topology.

Alternative coordinates on $S^3$.

For the harmonic analysis on spherical 3-manifolds we use the spherical harmonics in the form of Wigner polynomials. These polynomials in the coordinates $x$ are often expressed in terms of Euler angle coordinates eq. 3.

An alternative system of polar coordinates is used by Aurich et al. [1]. Here

\[
\begin{align*}
  x_0 &= \cos(\chi), \quad x_1 = \sin(\chi) \sin(\theta) \cos(\phi), \\
  x_2 &= \sin(\chi) \sin(\theta) \sin(\phi), \quad x_3 = \sin(\chi) \cos(\theta), \\
  u &= \begin{bmatrix}
    \cos(\chi) - i \sin(\chi) \cos(\theta), & -i \sin(\chi) \sin(\theta) \exp(-i\phi) \\
    -i \sin(\chi) \sin(\theta) \exp(i\phi), & \cos(\chi) + i \sin(\chi) \cos(\theta)
  \end{bmatrix}
\end{align*}
\]

We shall see in eq. 21 that these polar coordinates are adapted to the analysis of incoming radiation in terms of its direction.
Multipole expansion of spherical harmonics on the 3-sphere.

The CMB radiation as observed is given as a function of polar angles \((\theta, \phi)\) for its direction. To compare with an expansion of the harmonic basis of a given 3-manifold, we must rewrite the Wigner polynomials in terms of polar angles. In terms of representation theory, this can be achieved by reducing the representations of \(SO(4, R)\) into irreducible representations of its subgroup \(SU^C(2, C)\), see eq.19.

We relate our analysis algebraically to this description.

To adapt the Wigner polynomials to a multipole expansion, we transform them for fixed degree \(2j\) by use of Wigner coefficients of \(SU^C(2, C)\), [4] pp. 31-45, into the new harmonic polynomials

\[
\psi_{\beta lm}(u) = \delta_{\beta, 2j+1} \sum_{m_1, m_2} D^j_{m_1, m_2}(u) \langle j - m_1 jm_2 | lm \rangle (-1)^{l-m_1},
\]

\(l = 0, 1, ..., 2j = \beta - 1.\)

Whereas the index \(j\) of the Wigner polynomials can be integer or half-integer, the multipole index \(l\) takes only integer values. For fixed \(l\) we have \(2j \geq l\), and for fixed \(2j\): \(0 \leq l \leq 2j\). Using representation theory of \(SU(2, C)\) it can be shown from eq. 6 that the conjugation action \(u \rightarrow g^{-1} u g\) of the group \(SU^C(2, C)\) acts by a rotation \(R(g)\) only on the coordinate triple \((x_1, x_2, x_3)\), and the new polynomials eq. 18 transform as

\[
(T_{g,g}) \psi_{\beta lm}(u) = \psi_{\beta lm}(g^{-1} u g) = \sum_{m'=-l}^{l} \psi_{\beta lm'}(u) D^l_{m',m}(g),
\]

like the spherical harmonics \(Y^l_m(\theta, \phi)\). We therefore adopt eq. 19 as the action of the usual rotation group for cosmological models covered by the 3-sphere, and eq. 19 qualifies \(l\) as the multipole index of incoming radiation.

The basis transformation eq. 18 is inverted with the help of the orthogonality of the Wigner coefficients [4] to yield

\[
D^j_{m_1, m_2}(u) = \delta_{\beta, 2j+1} \delta_{m, -m_1+m_2} \sum_{l=0}^{2j} \psi_{\beta lm}(u) \langle j - m_1 jm_2 | lm \rangle (-1)^{l-m_1}.
\]

The result eq. 20 can be further elaborated by use of the alternative coordinates \((\chi, \theta, \phi)\) eq. 17. From [1], eqs. 9-17 we find

\[
\psi_{\beta lm}(u) = R_{\beta l}(\chi) Y^l_m(\theta, \phi), \beta = 2j + 1, 2j \geq l,
\]

\[
R_{\beta l}(\chi) = 2^{l+\frac{1}{2}} l! \sqrt{\frac{\beta (\beta - l - 1)!}{\pi (\beta + l)!}} C^{l+1}_{\beta - l - 1}(\cos(\chi)).
\]

where \(C^{l+1}_{\beta - l - 1}\) is a Gegenbauer polynomial. The alternative coordinates admit the separation of the new basis into a part depending on \(\chi\) and a standard spherical harmonic as a function of polar coordinates \((\theta, \phi)\).
TABLE 2. The lowest cubic invariant spherical harmonics $Y^Γ_{l,m}$, expressed by spherical harmonics $Y^l_m$.

| $l$ | $Y^Γ_{l,0}$ |
|-----|-------------|
| 0   | $Y^0_0$    |
| 4   | $\sqrt{\frac{7}{12}} Y^4_0 + \sqrt{\frac{5}{24}} (Y^4_4 + Y^4_{-4})$ |
| 6   | $\sqrt{\frac{1}{12}} Y^6_0 - \sqrt{\frac{7}{12}} (Y^6_2 + Y^6_{-2})$ |
| 8   | $\frac{1}{8\pi} \sqrt{33} Y^8_0 + \frac{1}{3\pi} \sqrt{\frac{21}{2}} (Y^8_4 + Y^8_{-4}) + \frac{1}{2\pi} \sqrt{\frac{195}{2}} (Y^8_8 + Y^8_{-8})$ |

TABLE 3. The lowest $(Q \times s O)$-invariant polynomials $ψ^{0,Γ_1,2j}$ of degree $2j$ on the 3-sphere in the basis, expressed by the cubic invariant spherical harmonics from Table 2. $(Q \times O)$-invariance enforces superpositions of several cubic invariant spherical harmonics.

| $2j$ | $l$ | $ψ^{0,Γ_1,2j} = \sum b_j R_{2j+1}(\chi) Y^Γ_{l,j}(θ, φ)$ |
|------|-----|---------------------------------|
| 0    | 0   | $R_{10} Y^{Γ_{1,0}}$ |
| 4    | 0, 4 | $\sqrt{\frac{3}{2}} R_{50} Y^{Γ_{1,0}} + \sqrt{\frac{5}{2}} R_{54} Y^{Γ_{1,4}}$ |
| 6    | 0, 4, 6 | $\sqrt{2} R_{70} Y^{Γ_{1,0}} - \frac{5}{\pi} R_{74} Y^{Γ_{1,4}} - \sqrt{\frac{21}{2\pi}} R_{76} Y^{Γ_{1,6}}$ |
| 8    | 0, 4, 6, 8 | $\frac{4}{3} \sqrt{\frac{1}{16}} R_{90} Y^{Γ_{1,0}} + \frac{12}{\pi} \sqrt{\frac{3}{2}} R_{92} Y^{Γ_{1,4}} + \frac{8}{3\pi} R_{96} Y^{Γ_{1,6}} + \frac{4}{5} \sqrt{\frac{1}{33\pi}} R_{98} Y^{Γ_{1,8}}$ |

POINT SYMMETRY.

Any Platonic spherical 3-manifold is distinguished by a specific point symmetry group $M$ which stabilizes its center point. There arises the following enigma: The point group stabilizes the center point, but the deck group acts fixpoint-free. So the two groups can never mix. Can they nevertheless be brought together, and what happens to the harmonic analysis?

To examine this question we turn to the cubic spherical manifolds $N2,N3$. Their Coxeter group from [15] has the Coxeter diagram

$$\Gamma = 4 \circ - \circ - \circ - \circ .$$

The point group is the full cubic rotation group $O$ of order $|O| = 24$. The group of deck transformations for $N3$ is $H = \text{deck}(N3) = Q$, the quaternion group with eight elements. The cubic tiling of the 3-sphere is the 8-cell tiling of Einstein’s 3-sphere, see [21] p. 178. The relation between the deck and point groups is addressed in Appendix C of [15]. In [16] one finds selection rules from point symmetry for the multipole orders $6 \geq l \geq 0$.

For the cubic spherical manifold $N3$ we found there:

**Prop 1:** The cubic point group $M = O$ of $N3$ under conjugation leaves invariant the group $H=\text{deck}(N3)=Q$, the quaternion group, and with $Q$ forms a semidirect group $Q \times s O$, which turns out to be $\Sigma\Gamma$, the rotational unimodular subgroup of the Coxeter group, generated by an even number of Weyl reflections.
The relation between point and deck group resembles the case of symmorphic space groups in Euclidean crystallography. There the commutative infinite translation group acts fixpoint-free, and a cubic cell has again the cubic group as its point group. The difference is that, when going from Euclidean 3-space to the 3-sphere, the deck group is finite and no longer commutative.

If we consider first of all only the cubic point group \( O \) as a subgroup of the rotation group \( SO(3, R) \) in Euclidean 3-space, there are well-known results from molecular physics for the multiplicity of its representation in a given representation of \( SO(3, R) \) with angular momentum \( l \), see [17] p. 438. For the identity representation denoted by \( \Gamma_1 \) of \( O \), the lowest non-zero angular momentum is \( l = 4 \). The cubic invariant linear combination of standard spherical harmonics for lowest values of \( l \) are given in Table 2.

Now we wish to include the quaternion group \( H = Q \) of deck transformations. From the semidirect product property it follows that the projectors on the identity representation for \( O \) and \( Q \) commute with one another. This allows for the following procedure: we take a cubic \( O \)-invariant linear combination of spherical harmonics and combine it according to eq. 21 with the lowest possible function of the angle \( \chi \). Then we transform this linear combination back by eq. 20 into Wigner polynomials and apply the projector eq. 16 to \( Q \)-invariant form. Next we transform back with eq. 18 to the basis adapted to the multipole analysis. The resulting linear combination must still be \( O \)-invariant but may contain new \( O \)-invariant linear combinations of spherical harmonics. By use of the cubic invariants from Table 2 we obtain the fully \((Q \times_s O)\)-invariant polynomials of Table 3. The construction requires only the Wigner coefficients of \( SU(2, C) \) and can easily be continued to higher polynomial degree.

For the physics on the cubic spherical manifold with point symmetry, there follows from Table 3 a special and observable property: Different multipole orders of spherical harmonics must be linearly combined to assure the overall invariance.

What happens with the first cubic spherical manifold \( N2 \) under cubic point symmetry? Here we have from [15] the following universality:

**Prop 2:** Any particular homotopy of a regular polyhedron with fixed geometric shape implies a pairwise homotopic boundary condition on its faces. If full rotational symmetry is applied, all the faces and also all their edges are on the same footing. This implies that any particular homotopic boundary condition is automatically fulfilled. For the two cubic spherical manifolds \( N2, N3 \) it follows that the same rules apply to their \( S\Gamma \)-invariant basis whose lowest part we give in Table 3.

The order of the semidirect product group \( Q \times_s O \) is \( |Q \times_s O| = 32 \cdot 8 = 192 \). This is half the order \( |\Gamma| = 384 \) of the Coxeter group. It means that we are projecting to the identity representations of \( S\Gamma \).

### NEW 3-MANIFOLDS FROM UNIMODULAR COXETER GROUPS.

We give a re-interpretation of the results of the previous section. We recall that in [15] we constructed the spherical 3-manifolds from four Coxeter groups generated by Weyl reflections. Table 5 gives the data. In the last section we found that under inclusion of the cubic point group, the group deck(\( N3 \)) extends into the unimodular subgroup \( S\Gamma \) of the cubic Coxeter group \( \Gamma \), generated by an even number of Weyl reflections.
When we introduce in addition to topology the point symmetry of the spherical cube, we can define a fundamental subdomain on the cube under the cubic point group. This subdomain may be taken as the cone, shown in Euclidean form in yellow on Fig. 7. The cone is formed as a double simplex from two Coxeter simplices of the cubic Coxeter group $\Gamma$, with the second simplex the image of the first one under reflection in the Weyl plane perpendicular to Weyl vector $a_1$. In the 8-cell tiling that covers the 3-sphere, the double simplex is a fundamental domain with respect to the unimodular Coxeter group.
FIGURE 8. The new 3-manifold $N10$ as fundamental domain of the unimodular Coxeter group $S\Gamma, \Gamma = \circ - \circ - \circ - \circ$, glued from two Coxeter simplices.

$S\Gamma$, of volume fraction $1/(8 \cdot 24) = 1/192$.

This double simplex on the 3-sphere forms a new topological 3-manifold $N9$ with the group deck($N9$) = $S\Gamma, \Gamma = \circ - \circ - \circ - \circ$. With its small volume fraction it is an attractive candidate for cosmic topology. The first polynomials invariant under deck($N8$) are the entries of Table 2.

Turning to the tetrahedral and octahedral 3-manifolds discussed in [15], their unimodular Coxeter groups admit the analogous construction. The unimodular subgroup for the tetrahedron is the even subgroup $A5 < S5$. Its analysis in [11] takes up work with Marcos Moshinsky [8] on permutational symmetry.

We name the new 3-manifolds $N8, N9$, show their double simplices in Figs. 6, 7, and 8, and give their main data in Table 4, extended from Table 1 in [15]. The double simplices are spherical counterparts to the notion of asymmetric units as used in classical Euclidean crystallography. Since we know the geometry and the deck groups for these new manifolds, it should not be hard to determine the corresponding homotopies. The harmonic analysis for deck($N8$), deck($N9$) still has to be done. The harmonic bases will be invariant in particular under the point group of the tetrahedron and octahedron respectively. The lowest non-zero multipole index is $l = 3$ for the tetrahedron and $l = 4$ for the cubes. If we wish to accommodate lower multipole order we must reduce the point symmetry of the manifold. We conclude:

**Prop 3**: The harmonic analysis on the three new 3-manifolds $N8, N9, N10$ strictly obeys the multipole selection rules given in [15] Table 3 for the tetrahedral and cubic point groups respectively.
TABLE 4. 4 Coxeter groups $\Gamma$, 4 Platonic polyhedra $\mathcal{M}$, 10 groups $H = \text{deck}(\mathcal{M})$ of order $|H|$. $C_n$ denotes a cyclic, $Q$ the quaternion, $\mathcal{T}^*$ the binary tetrahedral, $\mathcal{I}^*$ the binary icosahedral, $\Sigma$ a unimodular Coxeter group. The symbols $Ni$ are generalized from [6].

| Coxeter diagram $\Gamma$ | $|\Gamma|$ | Polyhedron $\mathcal{M}$ | $H = \text{deck}(\mathcal{M})$ | $|H|$ | Reference |
|--------------------------|-----------|------------------------|---------------------------|------|-----------|
| $\circ \circ \circ \circ$ | 120       | tetrahedron $N1$       | $C_5$                     | 5    | [11]      |
| $\circ \circ \circ \circ$ | 384       | cube $N2$              | $C_8$                     | 8    | [12]      |
| $\circ \circ \circ \circ$ | 1152      | octahedron $N4$        | $C_3 \times Q$            | 24   | [13]      |
| $\circ \circ \circ \circ$ | 120 $\cdot$ 120 | dodecahedron $N1'$     | $\mathcal{I}^*$          | 120  | [9], [10] |
| $\circ \circ \circ \circ$ | 384       | double simplex $N8$    | $\Sigma$                  | 60   |           |
| $\circ \circ \circ \circ$ | 1152      | double simplex $N10$   | $\Sigma$                  | 576  |           |

TABLE 5. The Weyl vectors $a_i$ for the four Coxeter groups $\Gamma$ from Table 4 with $\tau := \frac{1 + \sqrt{5}}{2}$.

| $\Gamma$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
|----------|-------|-------|-------|-------|
| $\circ \circ \circ \circ$ | $(0,0,0,1)$ | $(0,0,1,0)$ | $(0,\frac{1}{\sqrt{2}},0,0)$ | $(\frac{1}{\sqrt{2}},0,0,0)$ |
| $\circ \circ \circ \circ$ | $(0,0,1,0)$ | $(0,0,0,0)$ | $(\frac{1}{\sqrt{2}},0,0,0)$ | $(0,0,0,0)$ |
| $\circ \circ \circ \circ$ | $(0,1,0,0)$ | $(0,0,0,1)$ | $(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0)$ | $(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0)$ |
| $\circ \circ \circ \circ$ | $(0,0,1,0)$ | $(0,0,1,0)$ | $(\frac{1}{\sqrt{2}},0,0,0)$ | $(\frac{1}{\sqrt{2}},0,0,0)$ |

CONCLUSION.

On the example of the cubic spherical 3-manifolds, we have explained the construction of the harmonic analysis from topology and its transformation into an expansion for the CMB radiation, ordered by the multipole index $l$. We implemented the additional assumption of point symmetry for spherical manifolds. This assumption yields strong selection rules, including a lowest non-trivial multipole order $l$. Similar rules apply to the other Platonic spherical manifolds analyzed in [15]. These strong selection rules are easier to test from the fluctuation spectrum of the CMB radiation. Moreover we have shown that the inclusion of the unimodular Coxeter groups $\Sigma$ yields 3 new topological 3-manifolds which cover rather small fractions of the 3-sphere.
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