Star-shaped acceptability indexes

Marcelo Brutti Righi∗
Federal University of Rio Grande do Sul
marcelo.righi@ufrgs.br

Abstract

We propose the star-shaped acceptability indexes as generalizations of both the approaches of Cherny and Madan (2009) and Rosazza Gianin and Sgarra (2013) in the same vein as star-shaped risk measures generalize both the classes of coherent and convex risk measures in Castagnoli et al. (2022). We characterize acceptability indexes through star-shaped risk measures and star-shaped acceptance sets as the minimum of a family of quasi-concave acceptability indexes. Further, we introduce concrete examples under our approach linked to Value at Risk, risk-adjusted reward on capital, reward-based gain-loss ratio, and monotone reward-deviation ratio.

Keywords: Star-shaped acceptability indexes, star-shaped risk measures, performance measures, risk adjusted return on capital, gain-loss ratio.

1 Introduction

The task of assessing the performance of financial investments is central. Since the beginning of modern financial theory, indexes such as the Sharpe ratio have been considered to assess the trade-off between risk and return. In the last decade, performance has been analyzed through acceptability indexes since the seminal paper of Cherny and Madan (2009). In that paper, the authors present an axiomatic theory with properties that a performance measure must fulfill. That is, better results are desired, diversification is beneficial, and the scale does not affect the decision. Further, they explore the connection to coherent risk measures and their acceptance sets in the sense of Artzner et al. (1999) and Delbaen (2002). The main feature is that a performance measure possesses the desired properties if and only if it can be represented by a family of acceptability indexes linked to coherent risk measures, representing some level of acceptability.

This axiomatic framework contemplates a wide number of specific examples that are used both in literature and industry. Many authors study specific measures. See Chen et al. (2011), Schuhmacher and Breuer (2014) and Zhitlukhin (2019) for the relation with Sharpe ratio, Biagini and Pinar (2013), Zakamouline (2014) and Voelzke (2015) regarding Gain-loss ratio (GLR), Zakamouline (2010) for the risk-adjusted return on capital (RAROC), and Mondal

∗We are grateful for the financial support of CNPq (Brazilian Research Council) projects number 302369/2018-0 and 407556/2018-4.
(2020) for the so-called upside beta. Furthermore, extensions of the initial approach in Cherny and Madan (2009) are conducted: Bielecki et al. (2013), Bielecki et al. (2014) and Bielecki et al. (2016) to a conditional dynamic framework; Kountzakis and Rossello (2020) for stochastic processes, and Zeng et al. (2019) for a multivariate context. Moreover, other advances are discussed, such as qualitative robustness in Rossello (2015) and optimization in Kováčová et al. (2022).

One of the properties demanded in these papers is scale invariance, which assures that acceptability does not change with position size. However, the size of a financial position can affect its performance. This topic is relevant in financial market descriptions and their intrinsic dynamics. For risk measures theory, positive homogeneity plays this role. Föllmer and Schied (2002), and Frittelli and Rosazza Gianin (2002) argue against this property and the consequent sub-additivity assumptions adopted in the framework of coherent risk measures by introducing the class of convex risk measures. Further discussion on the relationship between risk measures and liquidity risk is done in Acerbi and Scandolo (2008), and Lacker (2018).

In order to incorporate this feature in the theory of performance measures and acceptability indexes, Rosazza Gianin and Sgarra (2013) relax the scale invariance property. Furthermore, the authors represent their framework’s acceptability indexes under convex risk measures. This kind of convex acceptability family is also considered in Drapeau and Kupper (2013) and Frittelli et al. (2014), where the key property for the functional is quasi-concavity. Such framework is extended to the conditional dynamic context in Biagini and Bion-Nadal (2014).

Despite the role of liquidity, there is also debate on the benefits of diversification. More specifically, convexity, which is equivalent to sub-additivity under positive homogeneity, of risk measures can be misleading. See Dhaene et al. (2008) for instance. This issue could be present when risk is allocated competitively, when a non-concave utility function is involved in risk assessment, or when capital requirements are aggregated in robust ways that do not preserve convexity. In this sense, Castagnoli et al. (2022) propose the class of star-shaped risk measures, which can be understood as a generalization for positive homogeneity and convexity. The nomenclature comes from the star-shaped property of the generated acceptance set. This class allows for the most used non-convex risk measure, the Value at Risk (VaR), to be in the same class as the convex risk measures. Liebrich (2021) explores allocations of star-shaped risk measures. In contrast, Moresco and Righi (2022) relate them to the broader class of monetary risk measures, Herdegen and Khan (2024) applies it to sensitivity to large losses and regulatory arbitrage, Laeven et al. (2023) relates to law invariance, Tian and Wang (2023) explores a dynamic framework, while Nie et al. (2024) considers a set-valued setup.

Based on this discussion, there would be a gain in generality by considering indexes that are not restricted to the coherent or convex cases in order to be able to avoid both scale invariance, which is linked to liquidity issues, and quasi-concavity, which is connected to the role for diversification. The reasoning for star-shapedness as a sensible property is that any scaled reduction is possible if a position is acceptable at some level. More specifically, the meaning of this property is evident: reducing the exposure to an acceptable position at some level cannot make it unacceptable at this same level. For instance, the change of quasi-concavity by star-shapedness is motivated by the lack of empirical support for the general feasibility of mergers. Thus, star-shapedness penalizes the concentration of risk and the ensuing liquidity
problems. However, unlike quasi-concavity, star-shapedness does not take a position on the effects of merge-and-downsize strategies.

Under this context, our first objective and contribution is to extend the main representation results of acceptability indexes for this framework of star-shapedness, which is based on the maximum of a family of quasi-concave acceptability indexes and, equivalently, by an increasing family of star-shaped risk measures. Our second contribution is to propose concrete non-trivial classes of star-shaped acceptability indexes, studying their representation in the light of our main results. Such representations are crucial for practical implementation and are non-trivial from a technical standpoint. In practical matters, an agent’s acceptability is based on some family of utilities or risk measures, and a regulator or decision maker considers some family of individual acceptability indexes. Both situations are covered in our paper. There is no paper dealing with this setup.

Section 2 exposes notations, definitions, and background material. In Section 3, we propose the star-shaped acceptability indexes as generalizations of both the approaches of Cherny and Madan (2009) and Rosazza Gianin and Sgarra (2013) in the same vein as star-shaped risk measures generalize both the classes of coherent and convex risk measures in Castagnoli et al. (2022). We expose the main theoretical results, with an emphasis on Theorem 3.3, which characterize acceptability indexes through star-shaped risk measures, star-shaped acceptance sets, and as the minimum of some family of quasi-concave acceptability indexes. These results are the building block for the subsequent sections and examples in the paper.

In Section 4 we use the fact that a direct way to produce acceptability indexes is to consider an increasing family of star-shaped risk measures/star-shaped acceptance sets to study the most used risk measure in literature, the VaR, which is not contemplated in the coherent and convex cases. From VaR, we also consider distortions in the Choquet integral format. Expected Shortfall (ES) is a prominent example, but there are also ones with non-concave distortions. This kind of distortion-based acceptability indexes are mainly used for pricing and hedging purposes, especially in the literature on conic finance. See the book of Madan and Schoutens (2016) for a review. However, they are less used as tools for performance, mostly due to their lack of interpretation. Nonetheless, in our context, the intuition is for a risk-averse family expectation, which are risk measures that give more weight to losses and increase risk aversion. Furthermore, we use this setup to characterize law invariant or second-order stochastic dominance, preserving star-shaped acceptability indexes. This issue is connected to consistent risk measures as proposed by Mao and Wang (2020).

In Section 5, we explore acceptability indexes based on ratios of reward and risk measures. In subsection 5.1, we focus on the widespread performance measure risk-adjusted return in capital (RAROC), which is a ratio between the return, measured as expectation, and a risk measure. We make two extensions/generalizations for this function. First, we allow the risk measure in the denominator to be star-shaped. Second, we extend the numerator from expectation to some reward measure, defined as the negative of some risk measure. In subsection 5.2 we explore another reward/risk criterion: a ratio acting as a performance measure. The gain-loss ratio (GLR), a ratio between the expectation of both positive and negative parts, that Bernardo and Ledoit (2000) propose as an alternative to the Sharpe ratio, which can
lead to good deals. We consider a related acceptability formulation for the GLR that is not quasi-concave but a star-shaped acceptability index. In subsection 5.3, we consider another possibility for a ratio that is the one between reward and deviation measures, in the sense of Rockafellar et al. (2006), Rockafellar and Uryasev (2013) and Righi (2019). Such a structure lacks monotonicity. Zhitlukhin (2019) considers a monotone version of the Sharpe ratio, which we then extend this approach into our framework of star-shaped acceptability indexes by allowing a star-shaped reward in the numerator and a deviation in the denominator.

In section 6, we evaluate the acceptability indexes described above for a simple position in a spot market portfolio. The sample considered is composed of stocks from the S&P100 index. Such illustration allows us to explore the numerical magnitudes and the types of values one may expect in both absolute and relative matters. This illustration is also useful for empirical problems such as pricing under the physical measure since if an agent has access to historical data, he/she may compute the price that attains a particular level of acceptability. We consider examples of star-shaped acceptability indexes mainly based on quantiles and their most usual quasi-concave or coherent counterparts. Results make it possible to verify that they present similar average values, indicating that they can be used for performance evaluation, producing the same evaluations but with more theoretical generality. Nonetheless, regarding the changes in distinct sub-samples, there are larger discrepancies for the quasi-concave/coherent performance measures concerning the star-shaped ones, reflecting some robustness for the latter class.

2 Background material

We consider a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). All equalities and inequalities are in the \(\mathbb{P}\)-a.s. sense. Let \(L^0 = L^0(\Omega, \mathcal{F}, \mathbb{P})\) and \(L^\infty = L^\infty(\Omega, \mathcal{F}, \mathbb{P})\) be the spaces of (equivalence classes under \(\mathbb{P}\)-a.s. equality of) finite and essentially bounded random variables, respectively. When not explicit, we consider in \(L^\infty\) its norm topology. We define \(1_A\) as the indicator function for an event \(A \in \mathcal{F}\). We identify constant random variables with real numbers. We denote by \(X_n \to X\) convergence in the \(L^\infty\) essential supremum norm \(\|\cdot\|_\infty\), whereas \(\lim_{n \to \infty} X_n = X\) indicates \(\mathbb{P}\)-a.s. convergence.

The notation \(X \succeq Y\), for \(X, Y \in L^\infty\), indicates second-order stochastic dominance, that is, \(E[f(X)] \leq E[f(Y)]\) for any increasing convex function \(f : \mathbb{R} \to \mathbb{R}\).

Let \(\mathcal{P}\) be the set of all probability measures on \((\Omega, \mathcal{F})\). We denote by \(E_Q[X] = \int_{\Omega} XdQ\), \(F_{X, Q}(x) = Q(X \leq x)\), and \(F_{X, Q}^{-1}(\alpha) = \inf\{x : F_{X, Q}(x) \geq \alpha\}\), the expected value, the (increasing and right-continuous) probability function, and its left quantile for \(X \in L^\infty\) concerning \(Q \in \mathcal{P}\). We write \(X \overset{Q}{\succeq} Y\) when \(F_{X, Q} = F_{Y, Q}\). We drop subscripts indicating probability measures when \(Q = \mathbb{P}\). Furthermore, let \(\mathcal{Q} \subset \mathcal{P}\) be the set of probability measures \(Q\) that are absolutely continuous with respect to \(\mathbb{P}\), with Radon–Nikodym derivative \(\frac{dQ}{d\mathbb{P}}\).

We begin by defining the main object of this paper, the acceptability indexes and their properties. These properties are detailed in Cherny and Madan (2009), and Rosazza Gianin and Sgarra (2013).

**Definition 2.1.** A functional \(\alpha : L^\infty \to [0, \infty]\) is an acceptability index. It may have the following properties:

(i) Monotonicity: if \(X \succeq Y\), then \(\alpha(X) \geq \alpha(Y)\), \(\forall X, Y \in L^\infty\).
(ii) Quasi-concavity: \( \alpha(\lambda X + (1 - \lambda)Y) \geq \min\{\alpha(X), \alpha(Y)\}, \forall X, Y \in L^\infty, \forall \lambda \in [0, 1]. \)

(iii) Scale invariance: \( \alpha(\lambda X) = \alpha(X), \forall X \in L^\infty, \forall \lambda > 0. \)

(iv) Star-shapedness: \( \alpha(0) = \infty \) and \( \alpha(\lambda X) \leq \alpha(X), \forall X \in L^\infty, \forall \lambda \geq 1. \)

(v) Fatou continuity: if \( \lim_{n \to \infty} X_n = X \) and \( \alpha(X_n) \geq x \), then \( \alpha(X) \geq x \) for any \( x \geq 0, \forall \{X_n\}_{n=1}^\infty \) bounded in \( L^\infty \) norm and for any \( X \in L^\infty \).

(vi) Weak expectation consistency: \( \alpha(C) = 0, \forall C \in \mathbb{R} \) with \( C < 0. \)

(vii) Expectation consistency: if \( E[X] < 0 \), then \( \alpha(X) = 0 \) and if \( E[X] > 0 \), then \( \alpha(X) > 0, \forall X \in L^\infty. \)

(viii) Arbitrage consistency: if \( X \geq 0 \) and \( \mathbb{P}(X > 0) > 0 \), then \( \alpha(X) = \infty, \forall X \in L^\infty. \)

(ix) SSD consistency: if \( X \succeq Y, \) then \( \alpha(X) \geq \alpha(Y), \forall X, Y \in L^\infty. \)

(x) Law invariance: if \( F_X = F_Y, \) then \( \alpha(X) = \alpha(Y), \forall X, Y \in L^\infty. \)

An acceptability index is called coherent if it satisfies (i), (ii), (iii), (v), and (vi); quasi-concave if it satisfies (i), (ii), (iv), (v) and (vi); star-shaped if it fulfills (i), (iv), (v) and (vi); weak expectation, expectation, arbitrage, or SSD consistent if it satisfies, respectively (vi), (vii), (viii) and (ix); and law invariant if it satisfies (x).

Remark 2.2. In our star-shaped approach, and also on that of Rosazza Gianin and Sgarra (2013), \( 0 \) is acceptable at any level, i.e., \( \alpha(0) = \infty. \) Intuitively, this means that a null payoff, which is morally the same as doing nothing, is always acceptable. This acceptability is natural in the framework of monetary risk measures since, in that case, under normalization \( (\rho(0) = 0), \) the null position is always acceptable. This property is important since it implies, under Monotonicity, that if \( X \geq 0, \) then \( \alpha(X) \geq \alpha(0) = \infty. \) In particular, Arbitrage consistency is satisfied. This definition is distinct from the approach in Cherny and Madan (2009), where most examples fulfill \( \alpha(0) = 0, \) which under Monotonicity is stronger than Weak expectation consistency. In order to unify both approaches, it is necessary to make mild adjustments, for instance, to assume Weak expectation consistency, as is the case for our acceptance indexes, which causes no harm. Furthermore, Proposition 3.2 shows that under both quasi-concavity or scale invariance, presents in the approach of Cherny and Madan (2009), we have star shapedness, i.e., in such contexts, it is sufficient to have \( \alpha(0) = \infty. \)

A key aspect of acceptability indexes is their representation under risk measures. In this sense, we now expose the definition of this concept. We refer to Delbaen (2020) and Föllmer and Schied (2016) for more details regarding these properties.

Definition 2.3. A risk measure is a functional \( \rho: L^\infty \to \mathbb{R}. \) It may have the following properties:

(i) (Anti-)Monotonicity: If \( X \leq Y, \) then \( \rho(X) \geq \rho(Y), \forall X, Y \in L^\infty. \)

(ii) Translation invariance: \( \rho(X + C) = \rho(X) - C, \forall X \in L^\infty, \forall C \in \mathbb{R}. \)
(iii) Convexity: \( \rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda)\rho(Y), \forall X, Y \in L^\infty, \forall \lambda \in [0, 1]. \)

(iv) Positive homogeneity: \( \rho(\lambda X) = \lambda \rho(X), \forall X \in L^\infty, \forall \lambda \geq 0. \)

(v) Star-shapedness: \( \rho(\lambda X) \geq \lambda \rho(X), \forall X \in L^\infty, \forall \lambda \geq 1. \)

(vi) Law invariance: If \( F_X = F_Y, \) then \( \rho(X) = \rho(Y), \forall X, Y \in L^\infty. \)

(vii) Fatou continuity: If \( \lim_{n \to \infty} X_n = X, \) then \( \rho(X) \leq \liminf_{n \to \infty} \rho(X_n), \forall X \in L^\infty. \)

A risk measure is called monetary if it satisfies (i) and (ii); convex if it is monetary and satisfies (iii); coherent if it is convex and satisfies (iv); law invariant if it satisfies (vi), and Fatou continuous if it attends (vi). Unless otherwise stated, we assume that risk measures are normalized in the sense that \( \rho(0) = 0. \) Its acceptance set is defined as \( \mathcal{A}_\rho = \{ X \in L^\infty : \rho(X) \leq 0 \}. \)

A fundamental concept for both risk measures and acceptability indexes is the acceptance set. We now define this concept and expose some properties. Propositions 2.1 and 2.2 of Artzner et al. (1999), Propositions 4.6 and 4.7 of Föllmer and Schied (2016), and Proposition 2 of Castagnoli et al. (2022) expose a direct interplay between the properties of risk measures and acceptance sets.

**Definition 2.4.** An acceptance set is a subset \( \mathcal{A} \subseteq L^\infty. \) It may fulfill the following properties:

(i) Monotonicity: if \( X \in \mathcal{A} \) and \( X \leq Y, \) then \( Y \in \mathcal{A}, \forall X, Y \in L^\infty. \)

(ii) Monetarity: \( \inf \{ m \in \mathbb{R} : m \in \mathcal{A} \} = 0. \)

(iii) Convexity: if \( X \) and \( Y \in \mathcal{A}, \) then \( \lambda X + (1 - \lambda)Y \in \mathcal{A}, \forall X, Y \in L^\infty, \forall \lambda \in [0, 1]. \)

(iv) Conicity: if \( X \in \mathcal{A}, \) then implies \( \lambda X \in \mathcal{A}, \forall X \in L^\infty, \forall \lambda \geq 0. \)

(v) Star-shapedness: if \( X \in \mathcal{A}, \) then \( \lambda X \in \mathcal{A}, \forall X \in L^\infty, \forall \lambda \in [0, 1]. \)

An acceptance set is called monetary if it satisfies (i) and (ii); convex if it is monetary and satisfies (iii); coherent if it is convex and satisfies (iv); star-shaped if it is monetary and satisfies (v). Its induced risk measure is given by \( \rho_\mathcal{A}(X) = \inf \{ m \in \mathbb{R} : X + m \in \mathcal{A} \}, \forall X \in L^\infty. \)

We have the following main representations in the literature for coherent and quasi-concave acceptability indexes.

**Theorem 2.5** (Theorem 1 in Cherny and Madan (2009), Proposition 3 in Rosazza Gianin and Sgarra (2013)). A functional \( \alpha : L^\infty \to [0, \infty] \) is a quasi-concave (resp. coherent) acceptability index if and only if there is an increasing family \( \{ \rho_x \}_{x > 0} \) of Fatou continuous convex (resp. coherent) risk measures such that the representation holds

\[
\alpha(X) = \sup \{ x > 0 : \rho_x(X) \leq 0 \} = \sup \{ x > 0 : X \in \mathcal{A}_{\rho_x} \}, \forall X \in L^\infty, \tag{2.1}
\]

with the convention \( \sup \emptyset = 0. \)
3 Main Theory

We begin by exposing preliminary results regarding basic properties fulfilled by star-shaped acceptability indexes. We first show alternative equivalent formulations for star-shapedness.

**Proposition 3.1.** Let \( \alpha : L^\infty \to [0, \infty) \) with \( \alpha(0) = \infty \). The following are equivalent:

(i) \( \alpha(\lambda X) \leq \alpha(X), \ \forall \ X \in L^\infty, \ \forall \ \lambda \geq 1. \)

(ii) \( \alpha(\lambda X) \geq \alpha(X), \ \forall \ X \in L^\infty, \ \forall \ \lambda \in [0, 1]. \)

(iii) \( \lambda \to \alpha(\lambda X) \) is decreasing \( \forall \ X \in L^\infty. \)

**Proof.** We have for any \( X \in L^\infty \) the following chain of implications:

(i) \( \implies \) (ii). Let \( \lambda \in [0, 1]. \) If \( \lambda = 0 \) the result is trivial. If \( \lambda > 0, \) then \( \frac{1}{\lambda} \geq 1. \) Hence, we have \( \alpha(X) = \alpha(\frac{1}{\lambda} \lambda X) \leq \alpha(\lambda X). \)

(ii) \( \implies \) (iii). Let \( \lambda_1 \geq \lambda_2 \geq 0. \) If \( \lambda_1 = 0 \) the result is trivial. If \( \lambda_1 > 0, \) then \( 0 \leq \frac{\lambda_2}{\lambda_1} \leq 1. \)

We then have that \( \alpha(\lambda_2 X) = \alpha\left(\frac{\lambda_2}{\lambda_1} \lambda_1 X\right) \geq \alpha(\lambda_1 X). \)

(iii) \( \implies \) (i). Let \( \lambda \geq 1. \) Then, we directly have that \( \alpha(X) = \alpha(1X) \geq \alpha(\lambda X). \)

We now prove an interplay of Star-shapedness to Scale invariance and Quasi-concavity. In particular, coherence implies quasi-concavity, which in turn implies star-shapedness. Further, under quasi-supperaditivity, i.e. \( \alpha(X + Y) \geq \alpha(X) \wedge \alpha(Y) \) for any \( X, Y \in L^\infty, \) the three classes coincide.

**Proposition 3.2.** Let \( \alpha : L^\infty \to [0, \infty] \) with \( \alpha(0) = \infty. \) Then, both Quasi-concavity and Scale invariance implies Star-shapedness. The converse is true if \( \alpha \) is quasi-superadditive.

**Proof.** Let \( \lambda \in [0, 1] \) and \( X, Y \in L^\infty. \) If \( \alpha \) is quasi-concave, then

\[ \alpha(\lambda X) = \alpha(\lambda X + (1 - \lambda)0) \geq \alpha(X) \wedge \alpha(0) = \alpha(X), \]

which, according to Proposition 3.1, is star-shapedness. We have \( \alpha(\lambda X) = \alpha(X) \) for Scale invariance. For the converse, we have

\[ \alpha(\lambda X + (1 - \lambda)Y) \geq \alpha(\lambda X) \wedge \alpha((1 - \lambda)Y) \geq \alpha(X) \wedge \alpha(Y), \]

which is Quasi-concavity. For Scale invariance, let \( k \in \mathbb{N}. \) Then, \( \alpha(kX) \geq \alpha(X) \geq \alpha(kX). \)

Thus, \( \lambda \to \alpha(\lambda X) \) is constant and equal to \( \alpha(X) \) in \( \mathbb{N}, \) which jointly to Proposition 3.1 implies it is constant in \( \mathbb{R}_+. \)

In the context of star-shaped acceptability indexes, we have the following Theorem. It generalizes the result in Theorem 2.5 in a sense; it provides an interplay through representation among star-shaped acceptability indexes, star-shaped risk measures, star-shaped acceptance sets, and quasi-concave acceptability indexes.

**Theorem 3.3.** Let \( \alpha : L^\infty \to [0, \infty] \) and assume the convention \( \sup \emptyset = 0. \) Then the following claims are equivalent:
(i) \( \alpha \) is a star-shaped acceptability index.

(ii) there is a decreasing family of star-shaped acceptance sets \( \{A_x\}_{x>0} \) closed in the a.s. convergence of bounded sequences such that the representation holds

\[
\alpha(X) = \sup \{x > 0 : X \in A_x\}, \ \forall \ X \in L^\infty.
\]  

(iii) there is an increasing family \( \{\rho_x\} \) of Fatou continuous star-shaped risk measures such that the representation holds

\[
\alpha(X) = \sup \{x > 0 : \rho_x(X) \leq 0\}, \ \forall \ X \in L^\infty.
\]  

Moreover, each of these equivalent conditions implies the following

(iv) there is a family \( \{\alpha_i\}_{i \in I} \) of quasi-concave acceptability indexes such that the representation holds

\[
\alpha(X) = \max_{i \in I} \alpha_i(X), \ \forall \ X \in L^\infty.
\]  

Such family can be chosen as the quasi-concave acceptability indexes dominated by \( \alpha \), i.e.,

\[
I = \{\beta : L^\infty \to [0, \infty] : \beta \text{ is quasi-concave acceptability index and } \beta \leq \alpha\}.
\]

Proof. (i) \( \implies \) (ii). Define for any \( x > 0 \) the set

\[
A_x = \{X \in L^\infty : \alpha(X) \geq x\}.
\]

It is clear that \( A_x \subseteq A_y \) for any \( x > y \). Moreover, from the Monotonicity of \( \alpha \), we have that each \( A_x \) is monotone. Furthermore, due to Star-shapedness of \( \alpha \), \( A_x \) is star-shaped. Finally, from Weak expetation consistency, we get that \( \inf_{n \to \infty} x_n = X \). Hence, \( \alpha(X_n) \geq x \). From Fatou continuity of \( \alpha \), we obtain that \( \alpha(X) \geq x \), which implies \( X \in A_x \).

(ii) \( \implies \) (iii). Define for any \( x > 0 \) the map

\[
\rho_x(X) = \rho_{A_x}(X) = \inf \{m \in \mathbb{R} : X + m \in A_x\}, \ \forall \ X \in L^\infty.
\]

We now show that \( \{\rho_x\}_{x>0} \) is an increasing family of Fatou continuous star-shaped risk measures. Since \( \{A_x\}_{x>0} \) is decreasing, we have that \( \{\rho_x\}_{x>0} \) is increasing. Proposition 2 in Castagnoli et al. (2022) implies that each \( \rho_x \) is a star-shaped risk measure and \( A_x = A_{\rho_x} = \{X \in L^\infty : \rho_x(X) \leq 0\} \). Hence, \( \alpha(X) \geq x \) if and only if \( \rho_x(X) \leq 0 \). For Fatou continuity of \( \rho_x \), let \( \{X_n\} \) be bounded such that \( \lim_{n \to \infty} X_n = X \) and \( \rho_x(X_n) \leq m \) for any \( n \in \mathbb{N} \). By Translation invariance of \( \rho_x \), we get \( \rho_x(X_n + m) \leq 0 \). Thus \( \{X_n + m\} \subseteq A_x \). By closedness in the a.s. convergence of bounded sequences, we obtain \( X + m \in A_x \). This is equivalent to \( \rho_x(X) \leq m \), which implies \( \rho(X) \leq \liminf_{n \to \infty} \rho_x(X_n) \).

(iii) \( \implies \) (i). Let \( \alpha \) be given by (3.2). Monotonicity follows since for any \( X \geq Y \), we have \( \rho_x(X) \leq \rho_x(Y) \) for any \( x > 0 \). For Star-shapedness, let \( \lambda > 1 \). Then \( \rho_x(\lambda X) \geq \lambda \rho_x(X) \) for any...
\[ \rho. \] This fact implies

\[ \alpha(\lambda X) = \sup \{ x > 0 : \rho_x(\lambda X) \leq 0 \} \leq \sup \{ x > 0 : \rho_x(X) \leq 0 \} = \alpha(X). \]

Further, since \( \rho_x(0) = 0 \) for any \( x \) we directly have

\[ \alpha(0) = \sup \{ x > 0 : \rho_x(0) \leq 0 \} = \sup \{ x > 0 \} = \infty. \]

For Weak expectation consistency, let \( C \in \mathbb{R} \) such that \( C < 0 \). By Monotonicity, we get

\[ 0 \leq \alpha(C) = \sup \{ x > 0 : C \geq 0 \} = 0. \]

Regarding to Fatou continuity, let \( \{X_n\} \) be bounded such that \( \lim_{n \to \infty} X_n = X \) and \( \alpha(X_n) \geq x \). This is equivalent to \( \rho_x(X_n) \leq 0 \) for any \( n \in \mathbb{N} \). By Fatou continuity of the family \( \{\rho_x\} \) we get \( \rho_x(X) \leq \liminf_{n \to \infty} \rho_x(X_n) \leq 0 \). Then, we obtain \( \alpha(X) \geq x \).

This is equivalent to \( \alpha(X) \geq \limsup \alpha(X_n) \).

(iii) \( \implies \) (iv). Let \( \alpha : L^\infty \to [0, \infty] \) be represented under the monotone star-shaped families \( \{\rho_x\}_{x > 0} \). For any \( x > 0 \) and any \( Y \in L^\infty \), we define

\[ A_{Y,x} = cl^*(\text{conv} \{Y + \rho_x(Y) \} \cup \{0\}) + L^\infty_+ \]

where \( cl^* \) means the closure regarding weak* topology. Furthermore, note that if \( X \in A_{Y,x} \), then \( X = \lim_{n \to \infty} \lambda_n(Y + \rho_x(Y) + Z_n) \) for some \( \{Z_n\} \subseteq L^\infty_+ \) bounded and \( \{\lambda_n\} \subseteq [0,1] \). This fact implies that

\[ \rho_x(X) \leq \liminf_{n \to \infty} \lambda_n(Y + \rho_x(Y) + Z_n) \leq \liminf_{n \to \infty} \lambda_n(\rho_x(Y) - \rho_x(Y)) = 0. \]

Thus, \( A_{Y,x} \subseteq A_{\rho_x} \). We then have that \( A_{Y,x} \) is convex and weak* closed since \( L^\infty_+ \) is a convex cone, monotone since \( A_{Y,x} + L^\infty = A_{Y,x} \), and fulfills the condition \( \inf \{m \in \mathbb{R} : m \in A_{Y,x} \} = 0 \). Then, \( \rho_{A_{Y,x}} \) is a Fatou continuous convex risk measure, see Theorem 4.33 in Föllmer and Schied (2016), such that \( \rho_{A_{Y,x}} \geq \rho_x \) for any \( Y \in L^\infty \) and any \( x > 0 \). Nonetheless, since \( X + \rho_x(X) \in A_{X,x} \) we get

\[ \rho_{A_{X,x}}(X) = \inf \{m \in \mathbb{R} : X + m \in A_{X,x} \} \leq \rho_x(X). \]

Hence, we obtain that \( \rho_{A_{X,x}}(X) = \rho_x(X) \), which immediately implies

\[ \rho_x(X) = \min_{Y \in L^\infty} \rho_{A_{Y,x}}(X), \forall X \in L^\infty, \forall x > 0. \]

Now, for any \( Y \in L^\infty \) we define

\[ \alpha_Y(X) = \sup \{ x > 0 : \rho_{A_{Y,x}}(X) \leq 0 \}. \]

By Theorem 2.5, we have that each \( \alpha_Y \) defines a quasi-concave acceptability index. Furthermore, for any \( X \in L^\infty \) we have that

\[ \alpha_Y(X) \leq \sup \{ x > 0 : \rho_x(X) \leq 0 \} = \alpha(X) = \sup \{ x > 0 : \rho_{A_{X,x}}(X) \leq 0 \} = \alpha_X(X). \]
Hence, we have that 
\[ \alpha(X) = \max_{Y \in L^\infty} \alpha_Y(X), \ \forall \ X \in L^\infty. \]
Moreover, let \( I = \{ \beta: L^\infty \to [0, \infty]: \beta \) is quasi-concave acceptability index and \( \beta \leq \alpha \} \). We have that \( \alpha(X) \geq \sup_I \beta(X) \). Since \( \alpha_X \in I \), we have that 
\[ \alpha(X) = \max_{I} \beta(X), \ \forall \ X \in L^\infty. \]

**Remark 3.4.** The set \( I \) in the representation (3.3) is not unique. Nonetheless, under relaxation, we have some unique results. For any set \( I \) of quasi-concave acceptability indexes, define its relaxation as 
\[ I^* = \left\{ \beta: L^\infty \to [0, \infty]: \beta \text{ is quasi-concave acceptability index and } \beta \leq \max_{I} \beta_i \right\}. \]
Thus, if \( \alpha = \max_{I_1} \beta_i = \max_{I_2} \beta_i \), then we directly have that 
\[ I_1^* = I_2^* = \left\{ \beta: L^\infty \to [0, \infty]: \beta \text{ is quasi-concave acceptability index and } \beta \leq \alpha \right\}. \]
In particular, these sets are dependent on \( \alpha \) and not on a specific representation. Moreover, we can recover both \( A_x \) and \( \rho_x \) from such representation since we have that 
\[ A_x = \{ X \in L^\infty: \exists i \in I \text{ s.t. } \alpha_i(X) \geq x \} = \bigcup_{I} \{ X \in L^\infty: \alpha_i(X) \geq x \} = \bigcup_{I} A_x^i. \]
\[ \rho_x(X) = \inf \left\{ m \in \mathbb{R}: \exists i \in I \text{ s.t. } \rho_x^i(X) \leq m \right\} = \inf \left( \inf_{I} \rho_x^i(X), \infty \right) = \inf_{I} \rho_x^i(X). \]

The converse implication of (iv) \( \Rightarrow \) (iii) in Theorem 3.3 is not assured as Fatou continuity (in fact, any upper semicontinuity) of a family of maps is not in general preserved by the supremum operation. Nonetheless, by dropping Fatou continuity, we have an interesting characterization result of a monotone, star-shaped, and weak expectation consistent functional.

**Proposition 3.5.** Let \( \alpha: L^\infty \to [0, \infty] \) and assume the convention \( \sup \emptyset = 0 \). Then, \( \alpha \) is monotone, star-shaped, and weak expectation consistent if and only if there is a family \( \{ \alpha_i \}_{i \in I} \) of monotone, quasi-concave and weak expectation consistent acceptability indexes such that the representation holds 
\[ \alpha(X) = \max_{i \in I} \alpha_i(X), \ \forall \ X \in L^\infty. \]
Such family can be chosen as the quasi-concave acceptability indexes dominated by \( \alpha \), i.e. \( I = \{ \beta: L^\infty \to [0, \infty]: \beta \) is quasi-concave and monotone acceptability index and \( \beta \leq \alpha \} \).

**Proof.** The claim follows as (iii) \( \Rightarrow \) (iv) in Theorem 3.3 by considering the acceptance sets \( A_{Y,x} = \text{conv} \{ \{Y + \rho_x(Y)\} \cup \{0\} \} + L^\infty_+. \) The converse implication is straightforward. \( \square \)
Remark 3.6. In the previous proposition, weak expectation consistency is important to achieve the result. This claim can be directly compared with Theorem 5 in Castagnoli et al. (2022) where translation invariance is assumed, and Theorem 5.1 of Han et al. (2022) where one should rely on quasistar-shapedness (see Remark 4.7) instead of star-shapedness when weak expectation consistency is replaced by normalization.

In the conditions of Theorem 3.3, we can recover the original results of Theorem 1 in Cherny and Madan (2009) and Proposition 3 in Rosazza Gianin and Sgarra (2013). We have a Corollary regarding this topic, which has been stated without proof.

Corollary 3.7. Let \( \alpha : L^\infty \rightarrow [0, \infty] \) and assume the convention \( \sup \emptyset = 0 \). If the equivalent properties in Theorem 3.3 hold, then we have the following:

(i) \( \alpha \) has Scale invariance if and only if \( \{ A_x \}_{x>0} \) is composed by cones if and only if each member in \( \{ \rho_x \}_{x>0} \) fulfills Positive Homogeneity.

(ii) \( \alpha \) has Quasi-concavity if and only if \( \{ A_x \}_{x>0} \) is composed by convex sets if and only if each member in \( \{ \rho_x \}_{x>0} \) fulfills Convexity.

We end this section by exposing some practical examples where our main results are directly applicable. The main focus is to exemplify where both the maximum acceptability indexes and the representation over star-shaped risk measures/utilities are useful.

Example 3.8. Consider a financial institution with \( K \in \mathbb{N} \) business lines (or even a regulation system with \( K \) institutions), and the manager of each business line adopts quasi-concave acceptability indexes \( \alpha_i(X) \), \( i = 1, \ldots, K \) to assess the performance from a given global position \( X \). It is interesting to resume this \( K \)-dimensional information into one number to determine whether \( X \) is deemed acceptable for such an institution. The most conservative choice can be obtained by \( \alpha_{\min}(X) = \min_{i \in \{1, \ldots, K\}} \alpha_i(X) \), in a way that the most pessimistic acceptability is taken in order to satisfy all business lines. It is easy to verify that such a function is also quasi-concave. Despite this solution being cautious, it can lead to a pathological situation where more rigorous business lines may be concerned that this approach ignores what the less cautious ones classify as acceptability. It would generate a moral hazard problem by incentivizing less rigorous business lines to support risky asset purchases with high acceptability. The funds cover the possible losses on the more cautious business lines. In order to circumvent this situation, the institution could decide to delegate the decision to the business line with the larger acceptability index while assuring only limited coverage for losses. Thus, the corresponding acceptability index would be \( \alpha_{\max}(X) = \max_{i \in \{1, \ldots, K\}} \alpha_i(X) \), which is, by Theorem 3.3, star-shaped but not quasi-concave.

Example 3.9. A context where star-shaped acceptability indexes are necessary is non-concave utilities. Let \( \{ u_x \}_{x>0} \) represent concave utility functions with aversion parameter \( x \). The interpretation is that \( u_x \) represents more risk-averse attitudes/preferences as \( x \) increases. In this sense, one can build for some position \( X \) an acceptability index based on expected utility criteria over some threshold as \( \alpha(X) = \sup \{ x > 0 : E[u_x(X)] \geq K \} \). This map is quasi-concave. Concavity of \( u_x \) corresponds to a strong form of risk aversion, while more flexible utility functions with local convexities have been well documented, see Landsberger and Meilijson.
(1990) for instance. In this case, we consider the broader class of utility functions such that \( \lambda \to \frac{u_\lambda(y)}{y} \) decreasing on \((0, \infty)\) for any \( y \in \mathbb{R} \), which is equivalent to star-shapedness of \( u_x \). This condition, which is weaker than concavity, implies that the acceptability index defined as \( \alpha(X) = \sup \{ x > 0 : E[u_x(X)] \geq K \} \) is, by Theorem 3.3, star-shaped but not quasi-concave.

**Example 3.10.** This example follows similar reasoning as that exposed in Castagnoli et al. (2022). Assume that a supervising agency consists of a board of experts \( i \in \mathcal{I} \), which proposes a quasi-concave acceptability index \( \alpha_i \). Suppose the agency has to aggregate these opinions on some global portfolio \( X \) for an institution. In that case, it will have \( \alpha_f(X) = f(\alpha_{\mathcal{I}}(X)) \), where \( \alpha_{\mathcal{I}}(X) = \{ \alpha_i(X), i \in \mathcal{I} \} \) and \( f \) is some aggregation map, see Righi (2023) for details on combination of risk measures. When \( f \) is any weighting average under some weight scheme \( \mu \), one gets, under the necessary measurability properties, \( \alpha_f(X) = \int_\mathcal{I} \alpha_i(X) d\mu \). This map is quasi-concave, as is the \( \alpha_{\min} \) from previous paragraphs. However, many other choices for \( f \), such as median, general order statistics, and supremum, are typically not quasi-concave but lead to star-shaped acceptability indexes. In the following section 4, we expose and study some concrete cases of acceptability indexes based on quantiles.

### 4 Value at Risk based acceptability indexes

Let VaR be defined as \( \text{VaR}^p(X) = -F_X^{-1}(p), p \in [0, 1] \), with \( \mathcal{A}_\text{VaR}^p = \{ X \in L^\infty : \mathbb{P}(X < 0) \leq p \} \). We then consider \( \rho_x = \text{VaR}^{1-x} \), which is increasing in \( x \), as a building block. It is also possible to consider more general formulations such as \( \rho_x = \text{VaR}^u(x) \) for a decreasing function \( u : \mathbb{R}_+ \to [0, 1] \). Nonetheless, this paper focuses on this specific choice for the ratio since it is the simpler decreasing transformation \( u \) from non-negative numbers to the unit interval. Moreover, it is easily handled, and such a formulation naturally appears in other contexts.

**Definition 4.1.** The VaR based acceptability index is a functional \( \alpha_{\text{VaR}} : L^\infty \to [0, \infty] \) defined as

\[
\alpha_{\text{VaR}}(X) = \sup \left\{ x > 0 : \text{VaR}^{1-x}(X) \leq 0 \right\}.
\]

Since VaR is a Fatou continuous monetary risk measure that fulfills positive homogeneity, it is star-shaped and, by Theorem 3.3, \( \alpha_{\text{VaR}} \) is a star-shaped acceptability index. By Remark 3.7 and Proposition 4.5, we also have that this index is Scale and Law Invariant. Furthermore, we get that the acceptance sets in the representation (3.2) are as

\[
\mathcal{A}_x = \mathcal{A}_{\text{VaR}^{1-x}} = \left\{ X \in L^\infty : \text{VaR}^{1-x}(X) \leq 0 \right\} = \left\{ X \in L^\infty : \mathbb{P}(X < 0) \leq \frac{1}{1+x} \right\}.
\]

Combinations of VaR at distinct levels once such combination function fulfills some properties; see Righi (2023) for details. A typical formulation is the one for distortion risk measures, which are defined through Choquet integrals as

\[
\rho_g(X) = \int_{-\infty}^0 (g(\mathbb{P}(-X \geq x) - 1)dx + \int_0^\infty g(\mathbb{P}(-X \geq x))dx,
\]

where \( g : [0, 1] \to [0, 1] \) is a distortion, which is normalized and increasing. Such measures are law-invariant, monetary, and positively homogeneous. In atomless probability spaces, the
convexity of $\rho$ is equivalent to $g$ being concave. Furthermore, if $g_1 \geq g_2$, then $\rho_{g_1} \geq \rho_{g_2}$. Thus, we have a star-shaped acceptability index by taking an increasing family of distortions, not necessarily concave.

**Definition 4.2.** The distortion based acceptability index is a functional $\alpha_G: L^\infty \rightarrow [0, \infty]$ defined as

$$\alpha_G(X) = \sup \{x > 0: \rho_{g_x}(X) \leq 0\}, \quad (4.2)$$

where $G = \{g_x\}_{x>0}$ is an increasing family of distortion functions.

Possible choices for $G$ are $g_x(y) = 1_{\{y \geq \frac{1}{1+x}\}}$, in which case we recover $\rho_x = \text{VaR}^{\frac{1}{1+x}}$. A concave choice is regarding the well-known Expected Shortfall (ES), which is a Fatou continuous law-invariant coherent risk measure defined as $\text{ES}^p(X) = \frac{1}{p} \int_0^p \text{VaR}^s(X) ds$, $p \in (0, 1]$ and $\text{ES}^0(X) = \text{VaR}^0(X) = -\text{ess inf} X$. We have $\mathcal{A}_{\text{ES}^p} = \{X \in L^\infty : \int_0^p \text{VaR}^s(X) ds \leq 0\}$ and $\mathcal{Q}_{\text{ES}^p} = \{Q \in \mathcal{Q} : \frac{dQ}{dx} \leq \frac{1}{p}\}$. By taking the distortion $g_x(y) = y(1+x) \land 1$ we obtain $\rho_x = \text{ES}^{\frac{1}{1+x}}$.

In this situation we have

$$A_x = \mathcal{A}_{\text{ES}^{\frac{1}{1+x}}} = \left\{X \in L^\infty: \text{ES}^{\frac{1}{1+x}}(X) \leq 0\right\} = \left\{X \in L^\infty: \int_0^{1+x} \text{VaR}^s(X) ds \leq 0\right\}.$$

Other properties of Definition 2.1 are also preserved in the chain of implications in Theorem 3.3. We now focus on Law Invariance and consistency (Monotonicity) concerning SSD. If the space is atomless, Law invariance is inherited by Theorem 2.1 in Mao and Wang (2020).

**Proposition 4.3.** In the conditions of Theorem 3.3 we have:

(i) $\alpha$ is law invariant if and only if $\{A_x\}_{x>0}$ is law invariant ($X \in A_x$ and $Y \sim X$ implies $Y \in A_x$) if and only if $\{\rho_x\}_{x>0}$ is law invariant if $\{\alpha_i\}_{i \in \mathcal{I}}$ are law invariant.

(ii) $\alpha$ is consistent to SSD if and only if $\{A_x\}_{x>0}$ is consistent to SSD ($X \in A_x$ and $Y \succeq X$ implies $Y \in A_x$) if and only if $\{\rho_x\}_{x>0}$ is consistent to SSD if and only if $\{\alpha_i\}_{i \in \mathcal{I}}$ are consistent to SSD.

**Proof.** For (i), the equivalences arise by following similar steps as those in the proof of (iii) $\implies$ (i) of Theorem 3.3 by choosing $\mathcal{A}'_{Y,x} = \{X \in L^\infty: X \sim Y, Y \in A_Y\}$.

Regarding (ii), the steps are similar to those regarding usual Monotonicity in a.s. partial order in Theorem 3.3 by taking $\mathcal{A}'_{Y,x} = \{X \in L^\infty: X \succeq Y, Y \in A_Y\}. \qed$

**Remark 4.4.** It is possible that $\alpha$ be law invariant but $\{\alpha_i\}_{i \in \mathcal{I}}$ are not all law invariant. The intuitive reason is that law-invariant quasi-concave acceptability indexes respect SSD order, so by taking a maximum, one arrives at an acceptability that is consistent with SSD order. Not all law-invariant star-shaped acceptability indexes respect SSD order, as for the VaR-based one.

Under this discussion, we can have the following result for law invariant acceptability indexes in atomless probability spaces directly influenced by distortion-based acceptability indexes, particularly VaR-based ones.

**Proposition 4.5.** Under the equivalent conditions in Theorem 3.3 on an atomless probability space, we have that:
\[ \alpha(\theta) = \sup \{ x : \inf_{g \in G_x} \sup_{y > 0} \{ \text{VaR}^{\theta(x)}(X) - g\left(\frac{1}{1+y}\right) \} \leq 0 \}, \forall X \in L^\infty. \]  

(ii) \( \alpha \) is SSD consistent if and only if there is an increasing family of star-shaped sets \( \{G_x\}_{x>0} \) of decreasing functions \( g : (0, 1) \to \mathbb{R} \) with \( g(1^-) \geq 0 \) such that

\[ \alpha(X) = \sup \{ x > 0 : \inf_{g \in G_x} \sup_{y > 0} \{ \text{ES}^{\theta(x)}(X) - g\left(\frac{1}{1+y}\right) \} \leq 0 \}, \forall X \in L^\infty. \]

Proof. For (i), by Proposition 4.3, we have that \( \alpha \) is law invariant if and only if the family \( \{\rho_x\}_{x>0} \) is composed of law invariant star-shaped risk measures. Theorem 12 in Castagnoli et al. (2022) states that \( \rho : L^\infty \to \mathbb{R} \) is a law invariant star-shaped risk measure if and only if there is a star-shaped set \( G \) of decreasing functions \( g : (0, 1) \to \mathbb{R} \) with \( g(1^-) \geq 0 \) such that

\[ \rho(X) = \inf_{g \in G} \sup_{\alpha \in (0,1)} \{ \text{VaR}^{\alpha}(X) - g(\alpha) \}, \forall X \in L^\infty. \]

Thus, we take \( G_x \) as the set of functions \( g \) that represents the star-shaped risk measure \( \rho_x \). Note that since \( \{\rho_x\}_{x>0} \) is increasing, the family \( \{G_x\}_{x>0} \) must be decreasing.

Regarding (ii), by Proposition 4.3 we have that \( \alpha \) is SSD consistent if and only if the family \( \{\rho_x\}_{x>0} \) is composed by star-shaped SSD consistent risk measures. Moreover, Theorem 11 in Castagnoli et al. (2022) states that \( \rho \) is SSD consistent star-shaped risk measure if and only if there is a star-shaped set \( G \) of decreasing functions \( g : (0, 1) \to \mathbb{R} \) with \( g(1^-) \geq 0 \) such that

\[ \rho(X) = \inf_{g \in G} \sup_{\alpha \in (0,1)} \{ \text{ES}^{\alpha}(X) - g(\alpha) \}, \forall X \in L^\infty. \]

Again, we take \( G_x \) as the set of maps that represents the star-shaped risk measure \( \rho_x \).

We end this section with an example of a star-shaped risk measure that is neither positively homogeneous nor convex, the Benchmark loss VaR (LVaR) introduced in Bignozzi et al. (2020). It is defined as \( \text{LVaR}_\theta(X) = \sup_{t \in \mathbb{R}_+} \{ \text{VaR}^{\theta(t)}(X) - t \} \), where \( \theta : \mathbb{R}_+ \to [0, 1] \) is increasing and right-continuous. In this case we have \( \mathcal{A}_{\text{LVaR}} = \{ X \in L^\infty : \mathbb{P}(X < -t) \leq \theta(t) \text{ } \forall t \in \mathbb{R}_+ \} \). By making \( \theta = \alpha \) one recovers the \( \text{VaR}^{\alpha} \). By considering an increasing family of maps \( \{\theta_x : \mathbb{R}_+ \to [0,1]\}_{x>0} \) that are increasing and right-continuous we are able to define \( \rho_x = \text{LVaR}_{\theta_x} \) and generate a star-shaped acceptability index.

**Definition 4.6.** The LVaR based acceptability index is a functional \( \alpha_{\text{LVaR}} : L^\infty \to [0, \infty] \) defined as

\[ \alpha_{\text{LVaR}}(X) = \sup \{ x > 0 : \text{LVaR}^{\theta_x}(X) \leq 0 \}. \]
We get that the acceptance sets in the representation (3.2) are as
\[
\mathcal{A}_x = \mathcal{A}_{LVaR_x} = \{X \in L^\infty : P(X < -t) \leq \theta_x(t) \forall t \in \mathbb{R}_+\}.
\]

Remark 4.7. From the idea of LVaR, it might be interesting to discuss another similar form of variation of VaR. One of those forms is the \(\Lambda VaR\), introduced in Frittelli et al. (2014). This map is defined for any \(X\) as \(\Lambda VaR(X) = -\inf\{x \in \mathbb{R} : P(X \leq x) \geq \Lambda(x)\}\), for some function \(\Lambda : \mathbb{R} \to [0, 1]\) that is not constantly 0. By making \(\Lambda = \alpha\), we recover the traditional VaR.

Despite being very interesting, \(\Lambda VaR\) is not star-shaped. Nonetheless, it is quasi-star-shaped in the sense that \(\Lambda VaR(\lambda X + (1 - \lambda)t) \leq \max\{\Lambda VaR(X), \Lambda VaR(t)\}, \forall X \in L^\infty, \forall t \in \mathbb{R}, \forall \lambda \in [0, 1]\). It will be an interesting extension to study the acceptability indexes based on \(\Lambda VaR\), such as \(X \mapsto \alpha_{\Lambda VaR}(X) = \sup\{x > 0 : \Lambda x VaR(X) \leq 0\}\), where \(\{\Lambda^x\}\) is an increasing family of maps. One can even think of other more general classes of Quasi-star-shaped risk measures. Due to parsimony, we left this pursuit for future study.

5 Ratio based acceptability indexes

A very relevant kind of acceptability index is based on performance measures, typically ratios between some gain or return and a risk measure. In this section, we expose and study the representations of some concrete cases of acceptability indexes based on ratios.

5.1 Risk adjusted reward on capital

A widespread performance measure is RAROC, which is a ratio between the return, measured as expectation, and a risk measure as
\[
RAROC(X) = \begin{cases} 
\frac{E[X]}{\rho(X)} & \text{if } E[X] > 0 \text{ and } \rho(X) > 0, \\
0 & \text{if } E[X] \leq 0 \text{ and } \rho(X) > 0, \\
\infty & \text{if } \rho(X) \leq 0.
\end{cases}
\]

RAROC is a coherent or quasi-concave acceptability index if \(\rho\) is coherent or convex. This quantity is very useful for performance and even regulation. We now generalize it to our framework by considering a reward measure \(\mu\), defined as some risk measure’s negative. We keep the nomenclature of risk measures for reward measures \(\mu\). The expectation \(E[X]\) is a reward measure. The reasoning for \(\mu\) is the possibility to consider a more conservative gain in the numerator, with \(\mu(X) \leq E[X]\) for any \(X \in L^\infty\), for instance. In this case, we have \(\alpha_{\mu, \rho} \leq RAROC\). Thus, a possible choice is a quantile to-quantile ratio with \(\mu(X) = -VaR^{p_1}(X)\) and \(\rho(X) = VaR^{p_2}(X)\) with \(0 \leq p_2 \leq p_1 \leq 1\).

Definition 5.1. Let \(\mu, \rho : L^\infty \to \mathbb{R}\) be reward and risk measures, respectively. Then the acceptability index they generate, called risk-adjusted reward on capital, is a functional \(\alpha_{\mu, \rho} : L^\infty \to \mathbb{R}\).
result is trivially obtained if \( \rho = 0 \).

Proof. Let \( \rho \) be a Fatou continuous star-shaped reward and risk measures. Then \( \alpha_{\mu, \rho} \) is a star-shaped acceptability index. Moreover, \( \rho \geq -\mu \) is necessary and sufficient for it can be represented under (3.1) and (3.2) by

\[
\alpha_{\mu, \rho}(X) = \begin{cases} 
\frac{\mu(X)}{\rho(X)} & \text{if } \mu(X) > 0 \text{ and } \rho(X) > 0, \\
0 & \text{if } \mu(X) \leq 0 \text{ and } \rho(X) > 0, \\
\infty & \text{if } \rho(X) \leq 0.
\end{cases} \tag{5.1}
\]

Proposition 5.2. Let \( \mu, \rho : L^\infty \to \mathbb{R} \) be, respectively, Fatou continuous star-shaped reward and risk measures. Then \( \alpha_{\mu, \rho} \) is a star-shaped acceptability index. Moreover, \( \rho \geq -\mu \) is necessary and sufficient for it can be represented under (3.1) and (3.2) by

\[
\rho_x(X) = -\frac{1}{1 + x} \mu(X) + \frac{x}{1 + x} \rho(X), \quad \forall X \in L^\infty, \tag{5.2}
\]

\[
A_x = \left\{ X \in L^\infty : \frac{\mu(X)}{\rho(X)} \geq x \right\}. \tag{5.3}
\]

Proof. Regarding Monotonicity, for any \( X \geq Y \) we have \( \mu(X) \geq \mu(Y) \) and \( \rho(X) \leq \rho(Y) \). The result is trivially obtained if \( \rho(X) \leq 0 \). For the case \( \rho(X) > 0 \), if \( \mu(X) \leq 0 \), the result is also trivial. Otherwise, we thus get

\[
\alpha_{\mu, \rho}(X) = \frac{\mu(X)}{\rho(X)} \geq \frac{\mu(Y)}{\rho(Y)} = \alpha_{\mu, \rho}(Y).
\]

Regarding to Star-shapedness, for any \( \lambda \geq 1 \) and any \( X \in L^\infty \) we have that \( \mu(\lambda X) \leq \lambda \mu(X) \) and \( \rho(\lambda X) \geq \lambda \rho(X) \). The result is trivial if \( \rho(X) \leq 0 \), which is the case for \( X = 0 \). If \( \rho(X) > 0 \) we then get

\[
\alpha_{\mu, \rho}(\lambda X) = \frac{\mu(\lambda X)}{\rho(\lambda X)} \leq \frac{\lambda \mu(X)}{\lambda \rho(X)} = \alpha_{\mu, \rho}(X).
\]

Weak expectation consistency is obtained directly from the definition. For Fatou continuity, let \( \{ X_n \} \subseteq L^\infty \) be bounded such that \( \lim_{n \to \infty} X_n = X \). Then, both \( \mu(X) \geq \limsup_{n \to \infty} \mu(X_n) \) and \( \rho(X) \leq \liminf_{n \to \infty} \rho(X_n) \). We then obtain

\[
\alpha_{\mu, \rho}(X) = \frac{\mu(X)}{\rho(X)} \geq \frac{\limsup_{n \to \infty} \mu(X_n)}{\liminf_{n \to \infty} \rho(X_n)} = \lim_{n \to \infty} \left( \sup_{k \geq n} \frac{\mu(X_k)}{\inf_{k \geq n} \rho(X_k)} \right) \geq \limsup_{n \to \infty} \frac{\mu(X_n)}{\rho(X_n)} = \limsup_{n \to \infty} \alpha_{\mu, \rho}(X_n).
\]

Let \( \{ \rho_x \}_{x > 0} \) be a family defined as

\[
\rho_x(X) = -\frac{1}{1 + x} \mu(X) + \frac{x}{1 + x} \rho(X), \quad \forall X \in L^\infty.
\]

It is easy to verify that it is a star-shaped risk measure. Further, it increasing in \( x \) since we
have the following for any $X \in L^\infty$:

\[
\frac{\partial \rho_x(X)}{\partial x} = (1 + x)^{-2} (\mu(X) - x \rho(X)) + (1 + x)^{-1} \rho(X) \\
\geq (1 + x)^{-2} (-\rho(X) - x \rho(X)) + (1 + x)^{-1} \rho(X) \\
= (1 + x)^{-1} \rho(X) - (1 + x)^{-1} \rho(X) = 0.
\]

If $\rho(X) > 0$, then for any $x > 0$ we have that

\[
\alpha_{\mu, \rho}(X) \geq x \iff \mu(X) \geq x \rho(X) \iff \frac{1}{1 + x} \mu(X) + \frac{x}{1 + x} \rho(X) \leq 0 \\
\iff \rho_x(X) \leq 0.
\]

If $\rho(X) \leq 0$, then $\alpha_{\mu, \rho}(X) = \infty$, which assures $\alpha_{\mu, \rho}(X) \geq x$ for any $x > 0$. If $\rho(X) > 0$, then $\mu(X) \geq -\rho(X) \geq 0 \geq x \rho(X)$. In this case we obtain that

\[
\rho_x(X) = \frac{1}{1 + x} \mu(X) + \frac{x}{1 + x} \rho(X) \leq 0, \forall x > 0.
\]

Thus, $\alpha_{\mu, \rho}(X) = \infty = \sup \{x > 0 : \rho_x(X) \leq 0\}$. Finally, we get that

\[
A_x = A_{\rho_x} = \{X \in L^\infty : -\mu(X) + x \rho(X) \leq 0\} = \left\{ X \in L^\infty : \frac{\mu(X)}{\rho(X)} \geq x \right\}.
\]

\[\square\]

### 5.2 Gain-Loss ratio

Bernardo and Ledoit (2000) propose the gain-loss ratio (GLR), which in its usual coherent acceptability form is defined as

\[
GLR(X) = \begin{cases} 
\frac{E[X]}{E[X^-]} & \text{if } E[X] > 0 \text{ and } E[X^-] > 0, \\
0 & \text{if } E[X] \leq 0 \text{ and } E[X^-] > 0, \\
\infty & \text{if } E[X^-] = 0.
\end{cases}
\]

This map is a coherent acceptability index, fulfilling consistencies regarding Expectation and SSD. It can be represented under expectiles. A $p$-expectile of $X$ is the unique solution $y$ of $pE[(X - y)^+] = (1 - p)E[(X - y)^-]$ for $p \in [0, 1]$. The risk measure expectile Value at risk ($EVaR^p$), which is the negative of $p$-expectile, is Fatou continuous and coherent for $p \leq \frac{1}{2}$. More specifically, we can take $\rho_x = EVaR^{\frac{1}{2p}}$ since

\[
A_{EVaR^p} = \left\{ X \in L^\infty : \frac{E[X^+]}{E[X^-]} \geq \frac{1 - p}{p} \right\}.
\]
A related acceptability formulation for the GLR is

\[ GLR(X) = \begin{cases} 
E[X^+] & \text{if } E[X^-] > 0, \\
\frac{E[X^+]}{E[X^-]} & \text{if } E[X^-] = 0.
\end{cases} \]

This formulation is not quasi-concave due to convexity of the numerator. However, it is a star-shaped acceptability index. Hence, it can be represented under Theorem 3.3 by \( \rho_x = EVaR^{\mu,\rho} \), which is not convex for \( x < 1 \), but it is star-shaped for any \( x \).

We propose generalizing this version of GLR. The reasoning for \( \mu \) and \( \rho \) is similar to RAROC since it brings the possibility of considering a more conservative estimate. For instance, we could consider with \( \mu(X) \leq E[X] \) and \( \rho(X) \geq E[-X] \) for any \( X \in L^\infty \) and, thus, we would have \( \alpha_{\mu,\rho} \leq GLR \).

**Definition 5.3.** Let \( \mu, \rho : L^\infty \to \mathbb{R} \) be reward and risk measures, respectively. Then the acceptability index they generate, called reward-based gain-loss ratio, is a functional \( \alpha_{GLR,\mu,\rho} : L^\infty \to [0, \infty] \) defined as

\[ \alpha_{GLR,\mu,\rho}(X) = \begin{cases} 
\mu(X^+) & \text{if } \rho(-X^-) > 0, \\
\frac{\mu(X^+)}{\rho(-X^-)} & \text{if } \rho(-X^-) = 0.
\end{cases} \] (5.4)

We have to adjust the sign for the loss in \( -X^- \) to obtain a positive value due to our pattern for Monotonicity. The quantile-to-quantile ratio is also a possibility here. We now prove properties for this acceptability index.

**Proposition 5.4.** Let \( \mu, \rho : L^\infty \to \mathbb{R} \) be, respectively, Fatou continuous star-shaped reward and risk measures. Then \( \alpha_{GLR,\mu,\rho} \) is a star-shaped acceptability index. Moreover, it can be represented under (3.2) by

\[ \rho_x(X) = -\sup \{ y \in \mathbb{R} : \mu((X - y)^+) = x \rho((-X - y)^-) \}, \forall X \in L^\infty. \] (5.5)

**Proof.** Monotonicity follows since for any \( X \geq Y \), we have both \( X^+ \geq Y^+ \) and \( X^- \leq Y^- \). Then \( \mu(X^+) \geq \mu(Y^+) \) and \( \rho(-X^-) \leq \rho(-Y^-) \). The result is trivially obtained if \( \rho(-X^-) = 0 \).

For the case \( \rho(-X^-) > 0 \), we thus get

\[ \alpha_{GLR,\mu,\rho}(X) = \frac{\mu(X^+)}{\rho(-X^-)} \geq \frac{\mu(Y^+)}{\rho(-Y^-)} = \alpha_{GLR,\mu,\rho}(Y). \]

Regarding to Star-shapedness, from definition \( \alpha_{GLR,\mu,\rho}(0) = \infty \). Further, for any \( \lambda \geq 1 \) and any \( X \in L^\infty \) we have that \( \mu(\lambda X^+) \leq \lambda \mu(X^+) \) and \( \rho(-\lambda X^-) \geq \lambda \rho(-X^-) \). If \( \rho(-X^-) = 0 \), the result is trivial. If \( \rho(-X^-) > 0 \) we then get

\[ \alpha_{GLR,\mu,\rho}(\lambda X) = \frac{\mu(\lambda X^+)}{\rho(-\lambda X^-)} \leq \frac{\lambda \mu(X^+)}{\lambda \rho(-X^-)} = \alpha_{GLR,\mu,\rho}(X). \]

Weak expectation consistency is obtained directly from the definition. For Fatou continuity, let \( \{X_n\} \subseteq L^\infty \) be bounded such that \( \lim_{n \to \infty} X_n = X \). Then \( \lim_{n \to \infty} |X_n| = |X| \). Since \( 2X^+ = |X| + X^+ \) and \( 2X^- = |X| + X^- \), we have

\[ |\alpha_{GLR,\mu,\rho}(X)| = \rho_x(X) = \sup \{ y \in \mathbb{R} : \mu((X - y)^+) = x \rho((-X - y)^-) \}, \forall X \in L^\infty. \]
and \(2X^{-} = |X| - X\) we have that both \(\lim_{n \to \infty} X_{n}^{+} = X^{+}\) and \(\lim_{n \to \infty} X_{n}^{-} = X^{-}\). Thus, \(\mu(X^{+}) \geq \limsup_{n \to \infty} \mu(X_{n}^{+})\) and \(\rho(-X^{-}) \leq \liminf_{n \to \infty} \rho(-X_{n}^{-})\). We then obtain

\[
\alpha_{GLR,\mu,\rho}(X) \geq \lim_{n \to \infty} \left( \frac{\sup_{k \geq n} \mu(X_{k}^{+})}{\inf_{k \geq n} \rho(-X_{k}^{-})} \right) \geq \limsup_{n \to \infty} \frac{\mu(X_{n}^{+})}{\rho(-X_{n}^{-})} = \limsup_{n \to \infty} \alpha_{GLR,\mu,\rho}(X_{n}).
\]

Furthermore, since \(A_{x} = \{X \in L^{\infty} : \alpha_{GLR,\mu,\rho}(X) \geq x\}\) is star-shaped acceptance set closed in weak* topology we have that \(\rho_{A_{x}}\) is a Fatou continuous star-shaped risk measure. By Monotonicity and Translation invariance we have that \(g_{X} : \mathbb{R} \to \mathbb{R}\) defined as

\[
g_{X}(y) = \mu((X - y)^{+}) - x\rho(-(X - y)^{-})
\]

is non-increasing, surjective and continuous. Then, \(\{y \in \mathbb{R} : g_{X}(y) = 0\} \neq \emptyset\) and

\[
\inf\{y \in \mathbb{R} : g_{X}(y) \geq 0\} = \inf\{y \in \mathbb{R} : g_{X}(y) = 0\}.
\]

We get for any \(X \in L^{\infty}\) that

\[
\rho_{A_{x}}(X) = \inf\{m \in \mathbb{R} : \mu((X + m)^{+}) \geq x\rho(-(X + m)^{-})\}
\]

\[
= \inf\{-y \in \mathbb{R} : g_{X}(y) \geq 0\}
\]

\[
= \inf\{-y \in \mathbb{R} : g_{X}(y) = 0\}
\]

\[
= -\sup\{y \in \mathbb{R} : g_{X}(y) = 0\} = \rho_{x}(X).
\]

Since \(A_{\rho_{x}} = A_{\rho_{A_{x}}} = A_{x}\), by Theorem 3.3 we get the claim. \(\square\)

**Remark 5.5.** We have that \(\alpha_{GLR,\mu,\rho}\) inherits Scale and Law invariance and SSD consistency from \(\mu\) and \(\rho\). When \(\rho(X) = -\mu(X) = -E[X]\) the set \(\{y \in \mathbb{R} : g_{X}(y) = 0\}\) is a singleton and we recover \(\rho_{x} = EV aR^{1+\varepsilon}\). For the general case, there is no guarantee that \(\{y \in \mathbb{R} : g_{X}(y) = 0\}\) is a singleton. A possible situation is when \(\mu\) and \(\rho\) are strictly monotone.

### 5.3 Reward to deviation ratio

Another possibility for a ratio is the one between reward and deviation measures, which we now define.

**Definition 5.6.** A functional \(D : L^{\infty} \to \mathbb{R}_{+}\) is a deviation measure. It may fulfill the following properties:

(i) **Non-negativity:** For all \(X \in L^{\infty}\), \(D(X) = 0\) for constant \(X\) and \(D(X) > 0\) for non-constant \(X\);

(ii) **Translation insensitivity:** \(D(X + C) = D(X), \forall X \in L^{\infty}, \forall C \in \mathbb{R}\);

(iii) **Convexity:** \(D(\lambda X + (1 - \lambda)Y) \leq \lambda D(X) + (1 - \lambda)D(Y), \forall X, Y \in L^{\infty}, \forall \lambda \in [0, 1]\);

(iv) **Positive homogeneity:** \(D(\lambda X) = \lambda D(X), \forall X \in L^{\infty}, \forall \lambda \geq 0;\)
(v) Star-shapedness: \( D(\lambda X) \geq \lambda D(X) \), \( \forall X \in L^\infty \), \( \forall \lambda \geq 1 \).

(vi) Fatou continuity: If \( \lim_{n \to \infty} X_n = X \) implies that \( D(X) \leq \liminf_{n \to \infty} D(X_n) \), \( \forall \{X_n\}_{n=1}^\infty \) bounded in \( L^\infty \) norm and for any \( X \in L^\infty \).

A deviation measure \( D \) is called proper if it fulfills (i) and (ii); convex if it is proper and respects (iii); generalized (also called coherent) if it is convex and fulfills (iv); star-shaped if it is proper and fulfills (v); Fatou continuous if it respects (vi).

However, a ratio between reward and deviation measures lacks Monotonicity. To see this, let \( \mu, D : L^\infty \to \mathbb{R} \) be star-shaped reward and deviation measures, respectively. Let \( X > 0 \) bounded away from zero, \( m > 0 \) and \( Y = \lambda X - m \), where \( \lambda \geq \frac{\|X+m\|_\infty}{\|X\|_\infty} \geq 1 \). Note that \( \frac{X+m}{X} \in L^\infty \) since

\[
\left\| \frac{X+m}{X} \right\|_\infty \leq 1 + m \left\| \frac{1}{X} \right\|_\infty \leq 1 + m \frac{1}{\|X\|_\infty} < \infty.
\]

Thus, \( Y \geq X \) while the ratio possesses the converse order as

\[
\frac{\mu(Y)}{D(Y)} = \frac{\mu(\lambda X - m)}{D(\lambda X - m)} \leq \frac{\mu(\lambda X)}{D(\lambda X)} \leq \frac{\mu(X)}{D(X)}.
\]

Hence, it does not fit in the framework of acceptability indexes. We then study a monotone version.

**Definition 5.7.** Let \( \mu, D : L^\infty \to \mathbb{R} \) be reward and deviation measures, respectively. Then the acceptability index they generate, called monotone reward-deviation ratio, is a functional \( \alpha_{\mu, D} : L^\infty \to [0, \infty] \) defined as

\[
\alpha_{\mu, D}(X) = \begin{cases} 
\sup_{Y : X, D(Y) > 0} \frac{\mu(Y)}{D(Y)} & \text{if } \mu(X) > 0 \text{ and } D(X) > 0, \\
0 & \text{if } (\mu(X) \leq 0 \text{ and } D(X) > 0) \text{ or } (\mu(X) < 0 \text{ and } D(X) = 0), \\
\infty & \text{if } \mu(X) \geq 0 \text{ and } D(X) = 0.
\end{cases}
\]

(5.6)

Here, we consider three cases to refine financial reasoning since deviations do not take negative values. We now explore properties and representations for this ratio.

**Proposition 5.8.** If \( \mu, D : L^\infty \to \mathbb{R} \) are, respectively, Fatou continuous star-shaped reward and deviation measures, then \( \alpha_{\mu, D} \) is a star-shaped acceptability index. Moreover, it can be represented under (3.1) and (3.2) by

\[
\rho_x(X) = \inf_{Y : X, D(Y) > 0} \{-\mu(Y) + xD(Y)\},
\]

(5.7)

\[
\mathcal{A}_x = \left\{ X \in L^\infty : \frac{\mu(X)}{D(X)} \geq x \right\} + L^\infty.
\]

(5.8)

**Proof.** Monotonicity follows since for any \( X \geq Z \), the cases \( \mu(Z) \leq \mu(X) \leq 0 \) and \( \mu(X) \geq 0 \geq \mu(Z) \) are trivial. Therefore, let \( \mu(X) \geq \mu(Z) > 0 \). Then, we have that \( \{Y \in L^\infty : Y \leq Z\} \subseteq \)
For normalization, we have
\[ \rho \leq Y \]
Star-shapedness follows since for any \( \lambda \geq 1 \) and any \( X \in L^\infty \) we have it trivially for the case \( \mu(\lambda X) \leq \mu(X) \leq 0 \). Otherwise, we get that
\[ \alpha_{\mu, D}(\lambda X) = \sup_{Y \leq \lambda X, D(Y) > 0} \frac{\mu(Y)}{D(Y)} = \sup_{Z \leq \lambda X, D(Z) > 0} \frac{\mu(Z)}{D(Z)} = \alpha_{\mu, D}(X). \]

Regarding to Star-shapedness, by definition \( \alpha_{\mu, D}(0) = \infty \). Further, for any \( \lambda \geq 1 \) and any \( X \in L^\infty \) we have it trivially for the case \( \mu(\lambda X) \leq \mu(X) \leq 0 \). Otherwise, we get that
\[ \alpha_{\mu, D}(\lambda X) = \sup_{Y \leq \lambda X, D(Y) > 0} \frac{\mu(Y)}{D(Y)} = \sup_{Z \leq \lambda X, D(Z) > 0} \frac{\mu(Z)}{D(Z)} = \alpha_{\mu, D}(X). \]

Weak expectation consistency is obtained directly from the definition. For Fatou continuity, let \( \{X_n\} \subseteq L^\infty \) be bounded such that \( \lim_{n \to \infty} X_n = X \) and \( \alpha_{\mu, D}(X_n) \geq x \) for any \( n \in \mathbb{N} \). Then, we have that for any \( \epsilon > 0 \) there is \( \{Y_n^\epsilon\} \subseteq L^\infty \) such that, for any \( n \in \mathbb{N} \), \( Y_n^\epsilon \leq X_n \), \( D(Y_n^\epsilon) > 0 \) and
\[ \frac{\mu(Y_n^\epsilon)}{D(Y_n^\epsilon)} > x - \epsilon. \]

Note that we can take \( \{Y_n^\epsilon\} \) bounded below since \( \mu \) is monetary. In fact, take \( Y_n^\epsilon \geq K > (x - \epsilon)D(Y_n^\epsilon) \). This implies, jointly to boundedness of \( \{X_n\} \), that \( \{Y_n^\epsilon\} \) is bounded. Moreover, \( Y_n^\epsilon = \lim_{n \to \infty} Y_n^\epsilon \leq \lim_{n \to \infty} X_n = X \). From Fatou continuity of both \( \mu \) and \( D \) we get that
\[ \alpha_{\mu, D}(X) \geq \lim_{\epsilon \downarrow 0} \frac{\mu(Y_n^\epsilon)}{D(Y_n^\epsilon)} \geq \lim_{\epsilon \downarrow 0} \sup_{n \to \infty} \frac{\mu(Y_n^\epsilon)}{D(Y_n^\epsilon)} \geq \lim_{\epsilon \downarrow 0} (x - \epsilon) = x. \]

For the representation under star shaped risk measures, we start showing that the family \( \{\rho_x\}_{x > 0} \), defined as
\[ \rho_x(X) = \inf_{Y \leq X, D(Y) > 0} \{-\mu(Y) + xD(Y)\} \]
is composed by star-shaped risk measures and is increasing in \( x \). Since the deviations do not take negative values, \( x \to \rho_x \) is increasing. For Monotonicity we have that if \( X \geq Z \) we have that \( \{Y \in L^\infty : Y \leq Z\} \subseteq \{Y \in L^\infty : Y \leq X\} \). Then \( \rho_x(X) \leq \rho_x(Z) \). Translation invariance follows since for any \( m \in \mathbb{R} \) we have
\[ \rho(X + m) = \inf_{Y - m \leq X, D(Y) > 0} \{-\mu(Y) + xD(Y)\} \]
\[ = \inf_{Z \leq X, D(Z) > 0} \{-\mu(Z + m) + xD(Z + m)\} \]
\[ = \inf_{Z \leq X, D(Z) > 0} \{-\mu(Z) + xD(Z)\} + m = \rho_x(X) + m. \]

Star-shapedness follows since for any \( \lambda \geq 1 \) and any \( X \in L^\infty \) we have
\[ \rho_x(\lambda X) = \inf_{Y \leq X, D(Y) > 0} \{-\mu(\lambda Y) + xD(\lambda Y)\} \geq \inf_{Y \leq X, D(Y) > 0} \{\lambda(-\mu(Y) + xD(Y))\} = \lambda \rho_x(X). \]

For normalization, we have \( \rho_x(0) = \rho_x(0^2) \leq 0 \rho_x(0) = 0 \). On the other hand, \( \mu(Y) \leq 0 \) for any \( Y \leq 0 \). Thus, \( -\mu(Y) + xD(Y) \geq -\mu(Y) \geq 0 \) for any \( Y \leq 0 \). By taking the infimum over those \( Y \leq 0 \) such that \( D(Y) > 0 \), we conclude that \( \rho_x(0) \geq 0 \). Hence \( \rho_x(0) = 0 \). For Fatou continuity,
let \( \{X_n\} \subseteq L^\infty \) be bounded such that \( \lim_{n \to \infty} X_n = X \) and \( \rho_x(X_n) \leq y \) for any \( n \in \mathbb{N} \). Then, we have that for any \( \epsilon > 0 \) there is \( \{Y'_n\} \subseteq L^\infty \) such that, for any \( n \in \mathbb{N} \), \( Y'_n \leq X_n \), \( D(Y'_n) > 0 \) and 

\[ -\mu(Y'_n) + xD(Y'_n) < y + \epsilon. \]

Note that we can take \( \{Y'_n\} \) bounded below, which implies, jointly to boundedness of \( \{X_n\} \), that \( \{Y_n\} \) is bounded. Moreover, \( Y^\epsilon = \lim_{n \to \infty} Y'_n \leq \lim_{n \to \infty} X_n = X \). From Fatou continuity of both \( \mu \) and \( D \) we get that

\[ \rho_x(X) \leq \lim_{\epsilon \downarrow 0} \left\{ \liminf_{n \to \infty} (\mu(Y'_n) + xD(Y'_n)) \right\} \leq \lim_{\epsilon \downarrow 0} (y + \epsilon) = y. \]

We now show that \( \{\rho_x\}_{x > 0} \) represents \( \alpha_{\mu,D} \). Since \( \mu \) is monotone, the supremum in the definition of \( \alpha_{\mu,D} \) and the infimum in \( \rho_x \) can be considered on those \( Y \) such that \( D(Y) \leq D(X) \) without harm. For any \( X \in L^\infty \) such that one of the conditions \( D(X) > 0 \), or \( \mu(X) < 0 \) and \( D(X) = 0 \) is satisfied, we have that \( \alpha_{\mu,D}(X) \geq x \) if and only if for any \( \epsilon > 0 \) there is \( Y_\epsilon \leq X \) with \( D(Y_\epsilon) \in (0, D(X)] \) such that \( \mu(Y_\epsilon) > D(Y_\epsilon)(x - \epsilon) \). Thus, for any \( \epsilon > 0 \) we get that

\[ \rho_x(X) \leq -\mu(Y_\epsilon) + xD(Y_\epsilon) < \epsilon D(Y_\epsilon) \leq \epsilon D(X). \]

By taking the limits for \( \epsilon \downarrow 0 \) we get \( \rho_x(X) \leq 0 \). If \( \mu(X) \geq 0 \) and \( D(X) = 0 \), then \( X \in \mathbb{R}_+ \) and \( \alpha_{\mu,D}(X) = \infty \), which assures \( \alpha_{\mu,D}(X) \geq x \) for any \( x > 0 \). On the other hand

\[ \rho_x(X) \leq -\mu(X) + xD(X) \leq 0, \quad \forall x > 0. \]

Thus, \( \alpha_{\mu,D}(X) = \infty = \sup\{x > 0 : \rho_x(X) \leq 0\} \).

Finally, for the family of acceptance sets we have that if \( X \in \left\{ X \in L^\infty : \frac{\mu(X)}{D(X)} \geq x \right\} + L^\infty_+ \), then \( X = Z + K \), where \( \frac{\mu(Z)}{D(Z)} \geq x \) with and \( K \in L^\infty_+ \). Thus, \( X \geq Z \), which implies

\[ \sup_{Y \leq X, D(Y) > 0} \frac{\mu(Y)}{D(Y)} \geq \sup_{Y \leq Z, D(Y) > 0} \frac{\mu(Y)}{D(Y)} \geq \frac{\mu(Z)}{D(Z)} \geq x. \]

Thus, \( X \in \mathcal{A}_x \). For the converse inclusion, let \( X \in \mathcal{A}_x \). Then for any \( n \in \mathbb{N} \) there is \( Y_n \leq X \) with \( D(Y_n) > 0 \) such that

\[ \frac{\mu(Y_n)}{D(Y_n)} > x - \frac{1}{n}. \]

Note that we can take \( \{Y_n\} \) bounded below, which implies that this sequence is bounded. Thus, \( Y = \lim_{n \to \infty} Y_n \leq X \). Then we get

\[ \frac{\mu(Y)}{D(Y)} \geq \limsup_{n \to \infty} \frac{\mu(Y_n)}{D(Y_n)} \geq x. \]

Hence, \( X \in \left\{ X \in L^\infty : \frac{\mu(X)}{D(X)} \geq x \right\} + L^\infty_+ \). \( \square \)

### 6 Illustration

We now focus on evaluating the performance measures described above for a simple position in a spot market portfolio. Following Cherny and Madan (2009), this is useful because we get an idea about the numerical magnitudes of measures and the types of values one may expect to
see for them. In this same vein, the illustration allows an understanding of the relative values of the various measures as they are all computed for the same data series. Moreover, due to this illustration, there is a direct application for pricing under the physical measure because the acceptability indexes we consider here represent star-shaped levels of law invariant acceptance sets. More specifically, if an agent can access only historical data from net asset values (returns), he/she may compute the price that attains a particular level of acceptability. From that, it is possible to price a position to attain a level of acceptability comparable to that observed in the historical data.

We consider examples of star-shaped acceptability measures as proposed in the previous sections and their most usual quasi-concave or coherent counterparts. More specifically, we consider VaR and ES based acceptability indexes; RAROC as $E[X]/ES^{0.05}(X)$ as well as the star-shaped reward version (RAROC SS) with a quantile to quantile ratio $-VaR^{0.50}(X)/VaR^{0.05}(X)$. The choice for $VaR^{0.50}$ is to reflect the median, which is star-shaped but not convex; GLR in the usual version $E[X^+]/E[-X^-]$ and the star-shaped reward version (GLR SS) computed as $-VaR^{0.50}(X^+)/VaR^{0.05}(-X^-)$; reward to deviation ratio (RDR), where we consider both a coherent $E[X]/ES^{0.05}(X-E[X])$ as star-shaped (RDR SS) quantile $-VaR^{0.50}(X)/(VaR^{0.05}(X)-VaR^{0.50}(X))$. The denominators here are an ES deviation and an inter-quartile range. In addition, we consider combinations: the minimum one, linked to the robust acceptability index that preserves quasi-concavity, and the median and maximum ones, which only preserve star-shapedness.

The sample comprises 93 stocks of the S&P 100 from January 5, 2010, to October 22, 2021, totaling 2973 observations. Due to the lack of data availability for the entire period, some stocks were excluded from the sample. We adopted the S&P 100 composition in October 2021. We use log returns from daily closing prices (adjusted to splits and dividends). The analysis was conducted during the full sample period, but we also split the portfolio evaluation into sub-periods to consider different market momentum and conditions. We divided the out-of-sample period into four sub-samples (number of observations in parenthesis): 2010 to 2013 (1005), 2014 to 2016 (756), 2017 to 2019 (754), and 2020 onward (457). The period between 2010 to late 2021 allows us to analyze different conditions and market events, including the Greek default from 2010 to 2011, oil price swings in 2014, Brexit voting in 2016, the US-China trade war that started in 2018 and the beginning of the Covid-19 pandemic in 2020.

Each of the following Tables 1 to 5 refers to some of the studied windows and brings descriptive statistics from the considered acceptability index computed through the 93 assets. Furthermore, we also compute the same descriptive statistics for the mean returns of these same assets. Results indicate some clear patterns that we mention in the following.

For magnitude, both $\alpha VaR$ and GLR present the larger values, with a mean lightly above the unity. This result can be explained in the case of VaR because daily log returns have central tendency measures close to zero. Thus, since the median occurs with significance level $p = 0.5$, linked to acceptability level $x = 1$, one expects this pattern for VaR-based acceptability indexes. Similar reasoning explains the values for GLR since it represents a ratio of positive and negative parts expectations. The deviation from unity follows the typically negative skewness of daily financial returns.
Table 1: Results for the period 1 - 2010 to 2013

|      | Mean | Stdev | Skewness | Kurtosis | Minimum | Maximum |
|------|------|-------|----------|----------|---------|---------|
| Returns | 0.00 | 0.02  | -0.00    | 4.92     | -0.09   | 0.09    |
| VaR   | 1.11 | 0.07  | 0.52     | 0.80     | 0.95    | 1.35    |
| ES    | 0.02 | 0.01  | 0.40     | 0.58     | 0.00    | 0.05    |
| RAROC | 0.02 | 0.01  | 0.07     | 0.13     | 0.00    | 0.05    |
| RAROC_SS | 0.03 | 0.02  | 0.39     | -0.34    | 0.00    | 0.09    |
| GLR   | 1.16 | 0.07  | -0.22    | 0.84     | 0.93    | 1.34    |
| GLR_SS| 0.03 | 0.02  | 0.47     | -0.15    | 0.00    | 0.08    |
| RDR   | 0.02 | 0.01  | 0.01     | 0.11     | 0.00    | 0.05    |
| RDR_SS| 0.03 | 0.02  | 0.31     | -0.48    | 0.00    | 0.08    |
| Min   | 0.02 | 0.01  | 0.22     | -0.20    | 0.00    | 0.04    |
| Median| 0.03 | 0.02  | 0.57     | 0.25     | 0.00    | 0.08    |
| Max   | 1.17 | 0.07  | -0.04    | 0.64     | 0.95    | 1.35    |

Table 2: Results for the period 2 - 2014 to 2016

|      | Mean | Stdev | Skewness | Kurtosis | Minimum | Maximum |
|------|------|-------|----------|----------|---------|---------|
| Returns | 0.00 | 0.01  | -0.03    | 5.80     | -0.08   | 0.08    |
| VaR   | 1.10 | 0.08  | 0.39     | -0.23    | 0.96    | 1.30    |
| ES    | 0.01 | 0.01  | 1.35     | 2.72     | 0.00    | 0.04    |
| RAROC | 0.02 | 0.01  | 1.40     | 4.66     | 0.00    | 0.07    |
| RAROC_SS | 0.03 | 0.02  | 0.51     | -0.28    | 0.00    | 0.09    |
| GLR   | 1.11 | 0.08  | 1.52     | 5.83     | 0.97    | 1.49    |
| GLR_SS| 0.03 | 0.02  | 0.40     | -0.73    | 0.00    | 0.07    |
| RDR   | 0.02 | 0.01  | 1.26     | 3.98     | 0.00    | 0.06    |
| RDR_SS| 0.03 | 0.02  | 0.42     | -0.46    | 0.00    | 0.08    |
| Min   | 0.01 | 0.01  | 1.32     | 2.65     | 0.00    | 0.04    |
| Median| 0.03 | 0.02  | 0.51     | -0.25    | 0.00    | 0.08    |
| Max   | 1.14 | 0.08  | 1.26     | 3.65     | 0.99    | 1.49    |

Table 3: Results for the period 3 - 2017 to 2019

|      | Mean | Stdev | Skewness | Kurtosis | Minimum | Maximum |
|------|------|-------|----------|----------|---------|---------|
| Returns | 0.00 | 0.01  | -0.23    | 6.74     | -0.08   | 0.08    |
| VaR   | 1.18 | 0.10  | 0.11     | 0.35     | 0.86    | 1.46    |
| ES    | 0.02 | 0.01  | 0.62     | -0.05    | 0.00    | 0.06    |
| RAROC | 0.02 | 0.01  | 0.06     | -0.72    | 0.00    | 0.05    |
| RAROC_SS | 0.05 | 0.02  | 0.25     | -0.30    | 0.00    | 0.11    |
| GLR   | 1.15 | 0.10  | -0.07    | -0.44    | 0.87    | 1.37    |
| GLR_SS| 0.05 | 0.03  | 0.21     | -0.40    | 0.00    | 0.11    |
| RDR   | 0.02 | 0.01  | 0.02     | -0.77    | 0.00    | 0.05    |
| RDR_SS| 0.04 | 0.02  | 0.14     | -0.38    | 0.00    | 0.10    |
| Min   | 0.02 | 0.01  | 0.51     | -0.31    | 0.00    | 0.05    |
| Median| 0.04 | 0.02  | 0.21     | -0.35    | 0.00    | 0.10    |
| Max   | 1.20 | 0.10  | -0.05    | 0.63     | 0.87    | 1.46    |

The remaining performance measures have a value close to zero because the numerators in the ratios that define them are based either on mean or median, which is very close to zero. At the same time, the denominator assumes values that are much larger and linked to the tails,
Table 4: Results for the period 4 - 2020 to late 2021

|        | Mean | Sdev | Skewness | Kurtosis | Minimum | Maximum |
|--------|------|------|----------|----------|---------|---------|
| Returns | 0.00 | 0.02 | 0.18     | 9.61     | -0.13   | 0.15    |
| VaR    | 1.11 | 0.12 | 0.20     | -0.02    | 0.86    | 1.44    |
| ES     | 0.01 | 0.01 | 1.35     | 2.03     | 0.00    | 0.05    |
| RAROC  | 0.02 | 0.01 | 0.47     | -0.50    | 0.00    | 0.05    |
| RAROC_SS| 0.03 | 0.02 | 0.58     | -0.41    | 0.00    | 0.10    |
| GLR    | 1.14 | 0.09 | 0.32     | -0.48    | 0.92    | 1.36    |
| GLR_SS | 0.03 | 0.02 | 0.64     | -0.20    | 0.00    | 0.10    |
| RDR    | 0.02 | 0.01 | 0.42     | -0.55    | 0.00    | 0.05    |
| RDR_SS | 0.03 | 0.02 | 0.49     | -0.56    | 0.00    | 0.09    |
| Min    | 0.01 | 0.01 | 1.23     | 1.43     | 0.00    | 0.05    |
| Median | 0.03 | 0.02 | 0.70     | -0.21    | 0.00    | 0.09    |
| Max    | 1.16 | 0.10 | 0.62     | -0.04    | 0.99    | 1.44    |

Table 5: Results for the whole period - 2010 to late 2021

|        | Mean | Sdev | Skewness | Kurtosis | Minimum | Maximum |
|--------|------|------|----------|----------|---------|---------|
| Returns | 0.00 | 0.02 | 0.07     | 13.21    | -0.15   | 0.16    |
| VaR    | 1.12 | 0.05 | 0.13     | -0.15    | 1.00    | 1.25    |
| ES     | 0.01 | 0.01 | 0.49     | -0.65    | 0.00    | 0.03    |
| RAROC  | 0.02 | 0.01 | 0.16     | -0.92    | 0.01    | 0.03    |
| RAROC_SS| 0.03 | 0.01 | 0.03     | -0.18    | 0.00    | 0.07    |
| GLR    | 1.14 | 0.05 | 0.15     | -0.94    | 1.04    | 1.24    |
| GLR_SS | 0.03 | 0.01 | 0.08     | -0.09    | 0.00    | 0.06    |
| RDR    | 0.02 | 0.01 | 0.14     | -0.92    | 0.01    | 0.03    |
| RDR_SS | 0.03 | 0.01 | -0.04    | -0.19    | 0.00    | 0.06    |
| Min    | 0.01 | 0.01 | 0.40     | -0.48    | 0.00    | 0.03    |
| Median | 0.03 | 0.01 | 0.21     | -0.18    | 0.00    | 0.06    |
| Max    | 1.15 | 0.05 | 0.04     | -0.79    | 1.04    | 1.25    |

such as extreme quantiles or ES. \( \alpha_{ES} \) also exhibits this magnitude despite not being a ratio because the larger value \( ES^p(X) \) may assume \(-E[X]\), which is already very close to zero. In fact, except for \( \alpha_{VaR} \) and GLR, the acceptability indexes have assumed a zero value for at least some stocks in the sample, as indicated by the minimum statistic.

The standard deviations follow the same pattern, with larger values for \( \alpha_{VaR} \) and GLR, naturally explained by their also larger magnitudes. Moreover, in a general way, we report a predominance of positive values for skewness and negative values for kurtosis of the acceptability indexes. This pattern indicates a concentration of values around the global mean with more probability of occurrence above the mean.

Regarding the sub-samples, crisis periods, such as those in Tables 1 and 4, expose a tendency to reduce the mean value and skewness of acceptability. This pattern partially reflects the higher volatility and the more frequent occurrence of losses in this period. In sub-samples related to steady periods, as in Tables 2 and 3, we realize the presence of more acceptability indexes with a positive kurtosis, indicating the larger concentration of values far from the mean. The frequent occurrence of stocks can partially explain this result and the good performance during this period.
Comparing the star-shaped acceptability indexes with their quasi-concave/coherent counterparts makes it possible to verify that they present similar average values. This pattern indicates that star-shaped acceptability indexes can be used for performance evaluation, producing the same evaluations as the quasi-concave/coherent ones but with more theoretical generality.

Nonetheless, the two groups present discrepancies regarding the changes in distinct sub-samples observed in the previous paragraphs. More specifically, such changes are more prominent and have larger discrepancies for the quasi-concave/coherent performance measures concerning the star-shaped ones, reflecting some robustness from the latter type. This result is related to the fact that the functionals used to define the quasi-concave/coherent ones, expectation and ES, are more sensitive to changes in data than quantiles, which we have considered the star-shaped ones.

References

Acerbi, C., Scandolo, G., 2008. Liquidity risk theory and coherent measures of risk. Quantitative Finance 8, 681–692.

Artzner, P., Delbaen, F., Eber, J., Heath, D., 1999. Coherent measures of risk. Mathematical Finance 9, 203–228.

Bernardo, A., Ledoit, O., 2000. Gain, loss, and asset pricing. Journal of Political Economy 108, 144–172.

Biagini, S., Bion-Nadal, J., 2014. Dynamic quasi-concave performance measures. Journal of Mathematical Economics 55, 143–153.

Biagini, S., Pinar, M.C., 2013. The best gain-loss ratio is a poor performance measure. SIAM Journal on Financial Mathematics 4, 228–242.

Bielecki, T., Cialenco, I., Drapeau, S., Karliczek, M., 2016. Dynamic assessment indices. Stochastics An International Journal of Probability and Stochastic Processes 88, 1–44.

Bielecki, T., Cialenco, I., Iyigunler, I., Rodriguez, R., 2013. Dynamic conic finance: Pricing and hedging in market models with transaction costs via dynamic coherent acceptability indices. International Journal of Theoretical and Applied Finance 16, 1350002.

Bielecki, T., Cialenco, I., Zhang, Z., 2014. Dynamic coherent acceptability indices and their applications to finance. Mathematical Finance 24, 411–441.

Bignozzi, V., Burzoni, M., Munari, C., 2020. Risk measures based on benchmark loss distributions. Journal of Risk and Insurance 87, 437–475.

Castagnoli, E., Cattelan, G., Maccheroni, F., Tebaldi, C., Wang, R., 2022. Star-shaped risk measures. Operations Research.

Chen, L., He, S., Zhang, S., 2011. When all risk-adjusted performance measures are the same: in praise of the sharpe ratio. Quantitative Finance 11, 1439–1447.
Cherny, A., Madan, D., 2009. New measures for performance evaluation. Review of Financial Studies 22, 2371–2406.

Delbaen, F., 2002. Coherent risk measures on general probability spaces, in: Sandmann, K., Schönbucher, P.J. (Eds.), Advances in Finance and Stochastics: Essays in Honour of Dieter Sondermann. Springer Berlin Heidelberg, pp. 1–37.

Delbaen, F., 2020. Monetary Utility Functions. Lecture Notes: University of Osaka.

Dhaene, J., Laeven, R.J.A., Vanduffel, S., Darkiewicz, G., Goovaerts, M.J., 2008. Can a coherent risk measure be too subadditive? The Journal of Risk and Insurance 75, 365–386.

Drapeau, S., Kupper, M., 2013. Risk preferences and their robust representation. Mathematics of Operations Research 38, 28–62.

Föllmer, H., Schied, A., 2002. Convex measures of risk and trading constraints. Finance and stochastics 6, 429–447.

Föllmer, H., Schied, A., 2016. Stochastic Finance: An Introduction in Discrete Time. 4 ed., de Gruyter.

Frittelli, M., Maggis, M., Peri, I., 2014. Risk measures on p(r) and value at risk with probability/loss function. Mathematical Finance 24, 442–463.

Frittelli, M., Rosazza Gianin, E., 2002. Putting order in risk measures. Journal of Banking & Finance 26, 1473–1486.

Han, X., Wang, Q., Wang, R., Xia, J., 2022. Cash-subadditive risk measures without quasi-convexity. arXiv:2110.12198.

Herdegen, M., Khan, N., 2024. ρ -arbitrage and ρ-consistent pricing for star-shaped risk measures.

Kountzakis, C.E., Rossello, D., 2020. Acceptability indices of performance for bounded càdlàg processes. Stochastics 92, 1043–1063.

Kováčová, G., Rudloff, B., Cialenco, I., 2022. Acceptability maximization. Frontiers of Mathematical Finance 1, 219–248.

Lacker, D., 2018. Liquidity, risk measures, and concentration of measure. Mathematics of Operations Research 43, 813–837.

Laeven, R.J.A., Gianin, E.R., Zullino, M., 2023. Law-invariant return and star-shaped risk measures. arXiv:2310.19552.

Landsberger, M., Meilijson, I., 1990. Lotteries, insurance, and star-shaped utility functions. Journal of Economic Theory 52, 1–17.

Liebrich, F.B., 2021. Risk sharing under heterogeneous beliefs without convexity .

Madan, D., Schoutens, W., 2016. Applied Conic Finance. Cambridge University Press.
Mao, T., Wang, R., 2020. Risk aversion in regulatory capital principles. SIAM Journal on Financial Mathematics 11, 169–200.

Mondal, D., 2020. Upside beta ratio: A performance measure for potential-seeking investors. International Journal of Theoretical and Applied Finance 23, 2050014.

MoreSCO, M.R., Righi, M.B., 2022. On the link between monetary and star-shaped risk measures. Statistics & Probability Letters 184, 109345.

Nie, B., Tian, D., Jiang, L., 2024. Set-valued star-shaped risk measures. arXiv:2402.18014.

Righi, M., 2019. A composition between risk and deviation measures. Annals of Operations Research 282, 299–313.

Righi, M., 2023. A theory for combinations of risk measures. Journal of Risk 25, 25–60.

Rockafellar, R., Uryasev, S., 2013. The fundamental risk quadrangle in risk management, optimization and statistical estimation. Surveys in Operations Research and Management Science 18, 33–53.

Rockafellar, R., Uryasev, S., Zabarankin, M., 2006. Generalized deviations in risk analysis. Finance and Stochastics 10, 51–74.

Rosazza Gianin, E., Sgarra, C., 2013. Acceptability indexes via 'g-expectations': An application to liquidity risk. Mathematics and Financial Economics 7, 457–475.

Rossello, D., 2015. Ranking of investment funds: Acceptability vs robustness. European Journal of Operational Research 245, 828–836.

Schuhmacher, F., Breuer, W., 2014. When all risk-adjusted performance measures are the same: in praise of the sharpe ratio - a comment. Quantitative Finance 14, 775–776.

Tian, D., Wang, X., 2023. Dynamic star-shaped risk measures and g-expectations. arXiv:2305.02481.

Voelzke, J., 2015. Weakening the gain-loss-ratio measure to make it stronger. Finance Research Letters 12, 58–66.

Zakamouline, V., 2010. On the consistent use of var in portfolio performance evaluation: A cautionary note. The Journal of Portfolio Management 37, 92–104.

Zakamouline, V., 2014. Portfolio performance evaluation with loss aversion. Quantitative Finance 14, 699–710.

Zeng, X., Chen, Y., Hu, Y., 2019. Acceptability indexes for portfolio vectors. Mathematical Problems in Engineering 2019, 2196563.

Zhitlukhin, M., 2019. Monotone Sharpe Ratios and Related Measures of Investment Performance. Springer International Publishing, Cham. pp. 637–665.