MULTIPOINT SERIES OF GROMOV-WITTEN INVARIANTS OF $\mathbb{CP}^1$

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1. Introduction

The Gromov-Witten theory of $\mathbb{CP}^1$ has an elegant description in terms of an integrable system called the Toda hierarchy (Eguchi and Yang [2], Eguchi, Hori and Xiong [3], Pandharipande [13], Getzler [4] and Okounkov and Pandharipande [10]). In this paper, we derive explicit formulas for the multipoint series of $\mathbb{CP}^1$ in degree 0 from the Toda hierarchy, using the recursions of the Toda hierarchy (Getzler [4]). Since the Toda equation determines the higher degree multipoint series of $\mathbb{CP}^1$ from the degree 0 series, we obtain inductive formulas for the multipoint series in all degrees, along with explicit formulas for the Hodge integrals $\int_{\overline{M}_{g,n,d}} \psi_1^{k_1} \cdots \psi_n^{k_n} \lambda_g$ and $\int_{\overline{M}_{g,n,d}} \psi_1 \cdots \psi_n^{k_n} \lambda_{g-1}$. Let $\overline{M}_{g,n,d} = \overline{M}_{g,n}(\mathbb{CP}^1, d)$ be the moduli space of stable genus $g$, $n$-pointed maps to $\mathbb{CP}^1$ of degree $d$. Let $ev_i : \overline{M}_{g,n,d} \to \mathbb{CP}^1$ denote the morphism

$$ev_i([f : C \to \mathbb{CP}^1, z_1, \ldots, z_n]) = f(z_i)$$

defined by evaluating a stable map $f : C \to \mathbb{CP}^1$ at the $i$th marked point $z_i \in C$. Let $\psi_i \in H^2(\overline{M}_{g,n,d}, \mathbb{Z})$ be the Chern class of the line bundle over $\overline{M}_{g,n,d}$ whose fibre at the point

$$[f : C \to \mathbb{CP}^1, z_1, \ldots, z_n]$$

in the moduli space is the cotangent line $T^*_z C$. Let

$$[\overline{M}_{g,n,d}]^{\text{vir}} \in H_{2(2g-2+2d+n)}(\overline{M}_{g,n,d}, \mathbb{Q})$$

denote the virtual fundamental class.

Let $H \in H^2(\mathbb{CP}^1, \mathbb{Z})$ be the cohomology class Poincaré dual to the homology class of a point. Given $k_i, l_i \in \mathbb{N}$, define

$$\langle \tau_{k_1,Q} \cdots \tau_{k_m,Q} \tau_{l_1,P} \cdots \tau_{l_n,P} \rangle_{g,d} = \int_{[\overline{M}_{g,m+n,d}]^{\text{vir}}} \prod_{i=1}^m \psi_i^{k_i} ev^*_i(H) \cdot \prod_{i=1}^n \psi_i^{l_i} Q$$

The large phase space is the formal affine space with coordinates $\{s_k, t_k\}_{k \geq 0}$. The genus $g$ Gromov-Witten potential $F_g$ of $\mathbb{CP}^1$ is the generating function on the large phase space given by the formula

$$F_g = \sum_{d=0}^{\infty} q^d \sum_{m,n=0}^{\infty} \frac{1}{m! n!} \sum_{k_1, \ldots, k_m, l_1, \ldots, l_n} t_{k_1} \cdots t_{k_m} s_{l_1} \cdots s_{l_n} \langle \tau_{k_1,Q} \cdots \tau_{k_m,Q} \tau_{l_1,P} \cdots \tau_{l_n,P} \rangle_{g,d}.$$ 

The total Gromov-Witten potential is obtained by combining these potentials into a power series

$$F = \sum_{g=0}^{\infty} \varepsilon^{2g} F_g.$$ 

This differs by a factor of $\varepsilon^2$ from the total Gromov-Witten potential of the physics literature.

Denote the constant vector fields $\partial/\partial s_k$ and $\partial/\partial t_k$ on the large phase space by $\partial_{k,P}$ and $\partial_{k,Q}$. We will use the abbreviations $\partial$ and $\partial_Q$ for $\partial_{0,P}$ and $\partial_{0,Q}$. We use the following notation for the partial derivatives of the total potential:

$$\langle \langle \tau_{k_1,Q} \cdots \tau_{k_m,Q} \tau_{l_1,P} \cdots \tau_{l_n,P} \rangle \rangle = \partial_{k_1,Q} \cdots \partial_{k_m,Q} \partial_{l_1,P} \cdots \partial_{l_n,P} F.$$
We now recall the recursions which characterize the Toda hierarchy. Define the functions \( u = \nabla^2 F \) and \( v = \nabla \partial_Q F \) on the large phase space, where
\[
\nabla = \varepsilon^{-1}(e^{e\partial/2} - e^{-e\partial/2}).
\]
The total Gromov-Witten potential \( F \) satisfies the Toda equation (Okounkov and Pandharipande [10])
\[
\partial_Q^2 F = q e^u.
\]
However, this equation yields no information about the degree 0 potential \( \lim_{q \to 0} F \).

The Toda hierarchy is characterized by the following recursions (Getzler [6]): if \( k \) is a positive integer,
\[
D \langle \langle \tau_{k-1}, Q \rangle \rangle = (k + 1) \nabla \langle \langle \tau_k, Q \rangle \rangle, \quad D \langle \langle \tau_{k-1}, P \rangle \rangle = k \nabla \langle \langle \tau_k, P \rangle \rangle + 2 \nabla \langle \langle \tau_{k-1}, Q \rangle \rangle,
\]
where \( D \) denotes the operator: \( D = v \nabla + (e^{e\partial/2} + e^{-e\partial/2}) \partial_Q \). The recursions (1.3) are proved by combining the partial Toda hierarchy on the subspace of the large phase space where \( s_l = 0 \) for \( l \) (Okounkov and Pandharipande [10]) with the Virasoro constraints (Givental [8], Okounkov and Pandharipande [12]).

The multipoint series is closely related to the Gromov-Witten potential \( F \) of \( \mathbb{CP}^1 \). If \( d > 0 \), the degree \( d \) multipoint series \( F^d[y|z] \) is defined by
\[
F^d[y_1, \ldots, y_m|z_1, \ldots, z_n] = \sum_{g=0}^{\infty} \varepsilon^{2g} \sum_{k, l_j} y_1^k \cdots y_m^k z_1^{l_1} \cdots z_n^{l_n} \langle \langle \tau_{k, Q} \cdots \tau_{k, Q, Q} \tau_{1, P} \cdots \tau_{n, P} \rangle \rangle_{g,d}.
\]
If \( m = 0 \), so that there are no \( y \) variables, we write \( F^d[z_1, \ldots, z_n] \) instead of \( F^d[y|z] \).

The multipoint series may also be defined when \( d = 0 \), except that we modify the definition in two exceptional cases, in order to adjust for the fact that the moduli spaces \( \overline{M}_{0,1} \) and \( \overline{M}_{0,2} \) are empty:
\[
F^0[y] = y^{-2} + \sum_{g=1}^{\infty} \varepsilon^{2g} \sum_k y^k \langle \langle \tau_{k, Q} \rangle \rangle_{g,0},
\]
\[
F^0[y|z] = z^{-1} + \sum_{g=1}^{\infty} \varepsilon^{2g} \sum_{k,l} y^k z^l \langle \langle \tau_{k, Q} \tau_{l, P} \rangle \rangle_{g,0}.
\]
When \( d = 0 \), the only nonvanishing multipoint series are \( F^0[y|z_1, \ldots, z_n] \) and \( F^0[z_1, \ldots, z_n] \).

Define auxiliary functions \( L_{a,i}(z) \) for \( a \) and \( i \) in \( \mathbb{N} \), by the formulas
\[
L_{a,0}(z) = \int_0^z y^{a-1} \left( 1 - \frac{\varepsilon y/2}{\tanh(\varepsilon y/2)} \right) dy = -\sum_{k=2}^{\infty} \frac{\varepsilon^k B_k z^{k+a}}{(k+a)k!},
\]
\[
L_{a,i}(z) = i \int_0^z L_{a,i-1}(y) dy = -\sum_{k=2}^{\infty} \frac{il \varepsilon^k B_k z^{k+a+i}}{(k+a) \cdots (k+a+i) k!},
\]
where \( B_k \) is the \( k \)th Bernoulli number.

The following formulas are proved using the recursions (1.3).
Theorem 1.1. Let $Z = z_1 + \cdots + z_n$, and for $I \subset \{1, \ldots, n\}$, let $Z_I = \sum_{i \in I} z_i$. Then

$$F_0[y|z_1, \ldots, z_n] = \left( \frac{\varepsilon(y + Z)/2}{\sinh(\varepsilon(y + Z)/2)} \right) (y + Z)^{n-2},$$

$$F_0[z_1, \ldots, z_n] = -2 \left( \frac{\varepsilon Z/2}{\sinh(\varepsilon Z/2)} \right) \times \times Z^{-1} \left( \sum_{J \subset \{1, \ldots, n\}} \sum_{|J|<n} \left( n - |J| - 1 \right) (Z - Z_J)^{n-|J|-1} L_{|J|-1,i}(Z_J) + L_{n-1,0}(Z) \right).$$

There is an overlap between this paper and [7], where the consequences of the Virasoro constraints for $CP^1$ in degree 0 were analyzed: we showed that the Virasoro constraints determine the two degree 0 multipoint series up to overall constants in each genus. The formula which we prove for $F_0[y|z_1, \ldots, z_n]$ was conjectured in [7] and proved, using localization, in [4], [5]. Our main new result is the explicit identification of the more subtle series $F_0[z_1, \ldots, z_n]$.

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2. Hodge series

Recall the explicit formula for the degree 0 Gromov-Witten invariants of $CP^1$ in terms of Hodge integrals [7]. Let $\overline{M}_{g,n}$ be the moduli space of stable genus $g$, $n$-pointed curves. Let

$$\psi_i = c_1(L_i), \quad \lambda_j = c_j(E),$$

where $L_i$ is the $i$th cotangent line bundle and $E$ is the Hodge bundle on $\overline{M}_{g,n}$. For $h \geq 0$, let

$$F[h] = \sum_{g=h}^{\infty} (-\varepsilon)^g \sum_{n=0}^{1} \frac{1}{n!} \sum_{k_1, \ldots, k_n} s_{k_1} \cdots s_{k_n} \int_{\overline{M}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \lambda_g^{g-h}.$$ (2.1)

Proposition 2.1. The degree 0 Gromov-Witten potential of $CP^1$ is given by the formula:

$$\lim_{g \to 0} F = \sum_{k=0}^{\infty} t_k \frac{\partial F[0]}{\partial s_k} - 2F[1].$$

Let $f$ be a power series in the variables $s_* = \{ s_k \mid k \geq 0 \}$. Denote the vector field $\partial/\partial s_k$ by $\partial_k$, and the vector field $\partial_0$ by $\partial$. Let $\partial(z)$ be the generating function of vector fields

$$\partial(z) = \sum_{k=0}^{\infty} z^k \partial_k.$$

If $n > 0$, let $f[z_1, \ldots, z_n]$ be the generating function

$$f[z_1, \ldots, z_n] = \partial(z_1) \cdots \partial(z_n)f|_{s_* = 0}. $$
Proposition 2.1 yields the following expression for the generating functions of degree 0 Gromov-Witten invariants of $\mathbb{CP}^1$:

\[
\mathcal{F}^0[y_1, z_1, \ldots, z_n] = F^0[y_1, z_1, \ldots, z_n], \quad \mathcal{F}^0[z_1, \ldots, z_n] = -2F^1[z_1, \ldots, z_n].
\]

The string equation in Gromov-Witten theory implies the following differential equation for the Hodge series:

\[
\partial F^h = \delta_{h,0} \left( \frac{\varepsilon^2}{2} - \frac{\varepsilon^2}{24} \right) + \sum_{n=0}^{\infty} s_{n+1} \partial_n F^h.
\]

We obtain the following relations between the series $F^h[z_1, \ldots, z_n]$:

\[
F^h[z_1, \ldots, z_n, 0] = ZF^h[z_1, \ldots, z_n] + \delta_{h,0} \delta_{n,2}.
\]

Let $f^h$ be the generating function $f^h = \nabla F^h$. The string equation (2.3) for $F^h$ implies the string equation for $f^h$:

\[
\partial f^h = \delta_{h,0} s_0 + \sum_{n=0}^{\infty} s_{n+1} \partial_n f^h.
\]

Although we are really more interested in the series $F^h[z_1, \ldots, z_n]$, the recursions (1.3) lead more naturally to formulas for the series $f^h[z_1, \ldots, z_n]$. The string equation gives a formula relating these two sets of series:

**Proposition 2.2.** If $n \geq 3$, then

\[
F^h[z_1, \ldots, z_n] = \left( \frac{\varepsilon Z/2}{\sinh(\varepsilon Z/2)} \right) Z^{-1} f^h[z_1, \ldots, z_n].
\]

**Proof.** We have

\[
f^h[z_1, \ldots, z_n] = \sum_{g=h}^{\infty} \sum_{m=0}^{\infty} \frac{(-\varepsilon)^2 g z^{2m}}{2^{2m}(2m+1)!} \int_{\mathcal{M}_{g,n+2m+1}} \frac{\lambda_{g-h}}{(1-z_1 \psi_1) \ldots (1-z_n \psi_n)}.
\]

By the string equation (2.3),

\[
\int_{\mathcal{M}_{g,n+2m+1}} \frac{\lambda_{g-h}}{(1-z_1 \psi_1) \ldots (1-z_n \psi_n)} = Z^{2m+1} \int_{\mathcal{M}_{g,n}} \frac{\lambda_{g-h}}{(1-z_1 \psi_1) \ldots (1-z_n \psi_n)},
\]

and the result follows on summing over $m$. \qed

### 3. Calculation of $f^0$

We will now derive a formula for the Hodge series $f^0[z_1, \ldots, z_n]$. We actually establish a more general result, for the generating function $e^{x \partial} f^0$, obtained from $f^0$ by translating $s_0$ by $x$.

**Theorem 3.1.** If $n > 1$, $(e^{x \partial} f^0)[z_1, \ldots, z_n] = Z^{n-2} e^{x Z}$.

**Proof.** By Proposition 2.2, we have

\[
\lim_{q \to 0} \nabla \langle \langle \tau_{n,Q} \rangle \rangle = \partial_n f^0 \quad \text{and} \quad \lim_{q \to 0} \partial_Q \langle \langle \tau_{n,Q} \rangle \rangle = 0.
\]

In particular, $\lim_{q \to 0} v = \partial f^0$. Since we work here exclusively in degree 0, we will denote $\partial f^0$ by $v$. The $q \to 0$ limit of the recursion

\[
D \langle \langle \tau_{n-1,Q} \rangle \rangle = (n+1) \nabla \langle \langle \tau_{n,Q} \rangle \rangle
\]
yields the following equation: \( n \partial_{n-1} f[0] = (n+1) \partial_n f[0] \). Thus, we find that

\[
\partial_n f[0] = \frac{v^{n+1}}{(n+1)!}.
\]

Summing over \( n \), we see that

\[
\partial(z) f[0] = z^{-1} (e^{zv} - 1).
\]

We now prove by induction that for \( n > 1 \), \( (n-1)! \partial(\partial(z_2)) \) equals the residue at \( x = 0 \) of

\[
x^{-n} Z^{-1} \exp(Z e^{x \partial} v).
\]

Replacing \( z \) by \( z_1 \) in (3.2) and applying the operator \( \partial(z_2) \) shows that

\[
\partial(z_1) \partial(z_2) f[0] = (\partial(z_2) v) \exp(z_1 v) = \partial v \exp((z_1 + z_2) v),
\]

giving the case \( n = 2 \). The induction step is as follows:

\[
(n-1)! \partial(\partial(z_1) \ldots \partial(z_{n+1})) f[0] = \partial(z_{n+1}) \text{Res}_0(z^{-n} Z^{-1} \exp(Z e^{x \partial} v))
\]

\[
= \text{Res}_0(z^{-n} \partial(\partial(z_{n+1})) e^{x \partial} \exp(Z e^{x \partial} v))
\]

\[
= \text{Res}_0(z^{-n} (e^{x \partial} \partial v) \exp((Z + z_{n+1}) e^{x \partial} v))
\]

\[
= \text{Res}_0(z^{-n} \partial \exp((Z + z_{n+1}) e^{x \partial} v)).
\]

By the string equation (2.3), \( (e^{x \partial} v) |_{s \ast} = 0 \). We then conclude for \( n > 1 \),

\[
(n-1)! (e^{y \partial} f[0])[z_1, \ldots, z_n] = \text{Res}_{x=0}(x^{-n} Z^{-1} \exp((x+y)Z))
\]

\[
= (n-1)! Z^{n-2} e^{yZ}. \quad \Box
\]

The formula for \( F[0][y|z_1, \ldots, z_n] \) in Theorem 1.1 is an immediate consequence of (3.3), (2.2) and (2.4) (which is needed to prove the cases \( n = 0 \) and \( n = 1 \)).

Theorem 3.1 is closely related to results of the paper [5]. By Proposition 2.2, we see that for \( n \geq 3 \),

\[
F[0][z_1, \ldots, z_n] = \left( \frac{\varepsilon Z/2}{\sinh(\varepsilon Z/2)} \right) Z^{n-3}.
\]

Using (2.4), we may also handle the cases with \( n < 3 \). For example, since

\[
F[0][z] = z^{-2} \left( \frac{\varepsilon z/2}{\sinh(\varepsilon z/2)} - 1 \right),
\]

we recover a formula proved in [3]: for \( g > 0 \),

\[
\int_{\mathcal{M}_{0,1}} \psi_1^{2g-2} \lambda_g = (-1)^g (2^{1-2g} - 1) \frac{B_{2g}}{(2g)!}.
\]

We do not know how to derive the above formula directly from the Virasoro constraints, even though in principle, the Virasoro conjecture determines \( F \) from the genus 0 Gromov-Witten potential \( F_0 \) (Dubrovin and Zhang [1]).
4. Calculation of $f^{[1]}$

In this section, we prove the formula of Theorem 1.1 for $\mathcal{F}[z_1, \ldots, z_n]$. We actually work with the series $f^{[1]}[z_1, \ldots, z_n]$, establishing a recursion which determines $f^{[1]}[z_1, \ldots, z_n]$ from $f^{[1]}[z_1, \ldots, z_m]$, $m < n$. The formula for $\mathcal{F}[z_1, \ldots, z_n]$ follows on application of Proposition 2.2 and (2.2).

**Theorem 4.1.**

$$df^{[1]}[z_1, \ldots, z_n] = \sum_{I \subseteq \{1, \ldots, n\}} \frac{\partial}{\partial z} \sum_{n=0}^{\infty} z^n \nabla \langle \langle \tau_{n,I} \rangle \rangle + 2 \sum_{n=0}^{\infty} z^n \nabla \langle \langle \tau_{n,Q} \rangle \rangle.$$

**Proof.** By (1.3), we see

$$\sum_{n=0}^{\infty} z^n \nabla \langle \langle \tau_{n,P} \rangle \rangle = \frac{\partial}{\partial z} \sum_{n=0}^{\infty} z^n \nabla \langle \langle \tau_{n,P} \rangle \rangle + 2 \sum_{n=0}^{\infty} z^n \nabla \langle \langle \tau_{n,Q} \rangle \rangle.$$

From Proposition 2.2, we obtain the formulas:

$$\lim_{q \to 0} \sum_{n=0}^{\infty} z^n \nabla \langle \langle \tau_{n,P} \rangle \rangle = \sum_{k=0}^{\infty} t_k \partial_k \partial(z)f^{[0]} - 2\partial(z)f^{[1]} = e^{zv} \sum_{k=0}^{\infty} t_k \partial_k v - 2\partial(z)f^{[1]},$$

$$\lim_{q \to 0} \sum_{n=0}^{\infty} z^n \nabla \langle \langle \tau_{n,Q} \rangle \rangle = \partial(z)f^{[0]},$$

$$\lim_{q \to 0} \sum_{n=0}^{\infty} z^n \partial_Q\langle \langle \tau_{n,P} \rangle \rangle = \partial(z) \left( \frac{\varepsilon \partial / 2}{\sinh(\varepsilon \partial / 2)} \right) f^{[0]}.$$

We therefore find:

$$\lim_{q \to 0} \sum_{n=0}^{\infty} z^n \nabla \langle \langle \tau_{n,P} \rangle \rangle = \lim_{q \to 0} \sum_{n=0}^{\infty} z^n \left( v \nabla \langle \langle \tau_{n,P} \rangle \rangle + (e^{\varepsilon \partial / 2} + e^{-\varepsilon \partial / 2}) \partial_Q \langle \langle \tau_{n,P} \rangle \rangle \right)$$

$$= ve^{zv} \sum_{k=0}^{\infty} t_k \partial_k v - 2v\partial(z)f^{[1]} + \partial(z) \left( \frac{\varepsilon \partial}{\tanh(\varepsilon \partial / 2)} \right) f^{[0]}.$$

The terms involving the variables $t_k$ cancel in the recursion (4.1), and we are left with the formula:

$$\frac{\partial}{\partial z} (\partial(z)f^{[1]}) = v\partial(z)f^{[1]} + \left( 1 - \frac{\partial / 2}{\tanh(\partial / 2)} \right) \partial(z)f^{[0]}$$

$$= v\partial(z)f^{[1]} + \left( 1 - \frac{e^{\varepsilon \partial / 2}}{\tanh(\varepsilon \partial / 2)} \right) \partial(z) e^{x\partial}f^{[0]}.$$

Substituting $z_i$ for $z$ in this formula and applying the operator $\partial(z_1) \ldots \partial(z_i) \ldots \partial(z_n)$, we obtain:

$$\frac{\partial}{\partial z_i} (\partial(z_1) \ldots \partial(z_i) \ldots \partial(z_n)f^{[1]})$$

$$= \sum_{I \subseteq \{1, \ldots, n\}} \prod_{j \in I} \partial(z_j) v \prod_{j \in I} \partial(z_j)f^{[1]} + \left( 1 - \frac{e^{\varepsilon \partial / 2}}{\tanh(\varepsilon \partial / 2)} \right) \partial(z_1) \ldots \partial(z_n) e^{x\partial}f^{[0]}.$$
Note that for \( n > 0 \), \( v[z_1, \ldots, z_n] = \partial f^{[0]}[z_1, \ldots, z_n] = Z^{n-1} \). Hence, taking \( s_* = 0 \), we see:

\[
\frac{\partial}{\partial z_i} f^{[1]}[z_1, \ldots, z_n] = \sum_{I \subset \{1, \ldots, n\}} (Z - Z_I)^{n-|I|-1} f^{[1]}[z_I] + \left( 1 - \frac{\varepsilon \partial_z/2}{\tanh(\varepsilon \partial_z/2)} \right) x = 0 Z^{n-2} e^{xZ} = \sum_{I \subset \{1, \ldots, n\}} (Z - Z_I)^{n-|I|-1} f^{[1]}[z_I] + Z^{n-2} \left( 1 - \frac{\varepsilon Z/2}{\tanh(\varepsilon Z/2)} \right).
\]

Multiplying by \( dz_i \), summing over \( i \), and using the identity

\[
Z^{n-2} \left( 1 - \frac{\varepsilon Z/2}{\tanh(\varepsilon Z/2)} \right) dZ = Z^{n-1} d\log \left( \frac{\varepsilon Z/2}{\sinh(\varepsilon Z/2)} \right),
\]

the theorem is proved. \( \square \)

The string equation (2.5) implies \( f^{[1]}[0, \ldots, 0] = 0 \). Hence, Theorem 4.1 determines the functions \( f^{[1]}[z_1, \ldots, z_n] \). For example,

\[
f^{[1]}[z] = \log \left( \frac{\varepsilon z/2}{\sinh(\varepsilon z/2)} \right) = -\sum_{k=2}^{\infty} \frac{\varepsilon B_k}{k!} z^k,
\]

which is equivalent to the formula for \( F^{[1]}[z] \) in [4], by Proposition 2.2.

Starting with the above formula for \( f^{[1]}[z] \), Theorem 4.1 yields:

\[
f^{[1]}[z_1, z_2] = -\sum_{k=2}^{\infty} \frac{\varepsilon B_k}{k!} \left( \frac{(z_1 + z_2)^{k+1}}{k+1} + \frac{z_{1}^{k+1} + z_{2}^{k+1}}{k(k+1)} \right).
\]

We now find a general formula for \( f^{[1]}[z_1, \ldots, z_n] \). Let

\[
g[z_1, \ldots, z_n] = \sum_{J \subset \{1, \ldots, n\}} \sum_{|J| < n} \sum_{i=1}^{n-|J|} \left( n - |J| - 1 \right) (Z - Z_J)^{n-|J|-i} L_{|J|-i}(Z_J) + L_{n-1,0}(Z).
\]

**Theorem 4.2.** \( f^{[1]}[z_1, \ldots, z_n] = g[z_1, \ldots, z_n] \).

**Proof.** For \( n = 1 \), both \( f^{[1]}[z_1] \) and \( g[z_1] \) equal \( L_{0,0}(Z) \). We prove the Theorem by showing the functions \( g[z_1, \ldots, z_n] \) satisfy the recursion of Theorem 4.1, which may be restated as:

\[
dg[z_1, \ldots, z_n] - dL_{n-1,0}(Z) = \sum_{I \subset \{1, \ldots, n\}} \sum_{|J| < n} (Z - Z_J)^{n-|J|-1} g[z_I] dZ_I,
\]

where \( g[z_I] = g[z_{i_1}, \ldots, z_{i_{|I|}}] \) for \( I = \{ i_1 < \cdots < i_{|I|} \} \). The left-hand side of (4.2) equals:

\[
\sum_{J \subset \{1, \ldots, n\}} \sum_{|J| < n} \sum_{i=0}^{n-|J|} \left( n - |J| - 1 \right) (Z - Z_J)^{n-|J|-i} L_{|J|-i}(Z_J)((i+1)dz_J + id(Z - Z_J)).
\]

Lemma 4.3 below will be used to handle (4.3); the first of the identities is due to Hurwitz [9], while the second is closely related. (See also Pitman [14].)
Lemma 4.3.

\[(x + y)(x + y + Z)^{n-1} = xy \sum_{I \subset \{1, \ldots, n\}} (x + Z_I)^{|I|-1}(y + Z - Z_I)^{n-|I|-1}\]

\[(x + y + Z)^{n-1} dZ = y \sum_{I \subset \{1, \ldots, n\}} (x + Z_I)^{|I|-1}(y + Z - Z_I)^{n-|I|-1} dZ_I\]

Proof. We require a result from the theory of tree enumeration. A tree \(T\) with vertices \(\{1, \ldots, n\}\) is a subset of the set of pairs \(\{(i, j) \mid 1 \leq i < j \leq n\}\) of cardinality \(n - 1\) such that the associated graph is connected. Let \(d(T, i)\) be the valence of the vertex \(i\) in the tree \(T\). A basic combinatoric identity is:

\[Z^{n-2} = \sum_{\text{trees } T \text{ with vertices } i=1}^{n} \prod_{1 \leq i \leq n} d(T, i)^{-1}.\]

Thus, the summand of the right-hand side of the first identity is obtained by summing over pairs of trees, one of which has vertices \(\{x\} \cup I\), and the other of which has vertices \(\{y\} \cup \{1, \ldots, n\}\setminus I\). Similarly, the coefficient of \(dz_k\) in the summand of the second identity is obtained by summing over pairs of trees, one of which has vertices \(\{x\} \cup \{k\}\), and the other of which has vertices \(\{y\} \cup \{1, \ldots, n\}\setminus I\), subject to the additional condition that \(k \in I\).

Let us prove the first formula. The pairs of trees contributing to the right-hand side may be obtained by taking a tree with vertices \(\{w\} \cup \{1, \ldots, n\}\), and cutting the vertex \(w\) into two vertices \(x\) and \(y\) of valence \(d\) and \(d-e\). If the vertex \(w\) had valence \(d\), and hence weight \(w^{d-1}\), then after being split, the contribution \(x^{e-1}y^{d-e-1}\) is obtained. Summation over \(e\) yields \((xy)^{-1}(x+y)^d\). The left-hand side of the identity is the result of performing the replacement of \(w^{d-1}\) by \((xy)^{-1}(x+y)^d\) on the sum over trees \((w + Z)^{n-1}\).

The proof of the second formula is similar, except that now, we insist that the \(e\) edges emerging from \(x\) include the distinguished edge emerging from \(w\) which lies on the unique path from \(w\) to \(k\). Now, summing over \(e\) yields \(y^{-1}(x+y)^{d-1}\). The left-hand side of the second identity is the result of performing the replacement of \(w^{d-1}\) by \(y^{-1}(x+y)^{d-1}\) on the sum over trees \((w + Z)^{n-1}\). \(\square\)

Taking a derivative of both sides of the first identity of Lemma 4.3 with respect to \(y\) and setting \(y = 0\) yields

\[(x + Z)^{n-1} + (n - 1)x(x + Z)^{n-2} = x \sum_{I \subset \{1, \ldots, n\}} (x + Z_I)^{|I|-1}(Z - Z_I)^{n-|I|-1}\]

\[= x \sum_{I \subset \{1, \ldots, n\}} (x + Z_I)^{|I|-1}(Z - Z_I)^{n-|I|-1} + Z^{n-1}\]

Extracting the coefficient of \(x^i\), \(i > 0\), we see that

\[(4.4) \quad (i + 1) \binom{n - 1}{i} Z^{n-i-1} = \sum_{I \subset \{1, \ldots, n\}}^{0 < |I| < n} \binom{|I| - 1}{i - 1} Z_I^{|I|-i}(Z - Z_I)^{n-|I|-1}.\]

Likewise, taking a derivative of both sides of the second identity with respect to \(y\) and setting \(y = 0\) yields

\[(n - 1)(x + Z)^{n-2} dZ = \sum_{I \subset \{1, \ldots, n\}}^{|I| < n} (x + Z_I)^{|I|-1}(Z - Z_I)^{n-|I|-1} dZ_I.\]
Extracting the coefficient of $x^i$ gives

\[
(4.5) \quad \left(\binom{n-1}{i}\right) Z^{n-i-1} dZ = \sum_{I \in \mathcal{P}} \left(\binom{|I|-1}{i-1}\right) Z^{|I|-i} (Z - Z_I)^{n-|I|-1} dZ_I.
\]

Using formulas (4.4) and (4.5), we easily see that (4.3) equals the right-hand side of (4.2). \qed

5. HIGHER DEGREE MULTIPOLAR SERIES

In this section, we derive an inductive formula from the Toda equation (1.2) for the higher degree multipolar series. In particular, this will show that the multipolar series of the Gromov-Witten theory of $\mathbb{C}P^1$ are integrals of trigonometric functions in all degrees $d$.

From the Toda equation (1.2), we see that if $d > 0$,

\[
\partial^2_{Q} \mathcal{F}^d = \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{d_1 + \ldots + d_k = d-1} \nabla^2 \mathcal{F}^{d_1} \ldots \nabla^2 \mathcal{F}^{d_k}.
\]

Using the divisor equation

\[
\partial_Q \mathcal{F}^d = d \cdot \mathcal{F}^d + \sum_{k=0}^{\infty} s_{k+1} \partial_{k,Q} \mathcal{F}^d,
\]

we see that

\[
\partial^2_{Q} \mathcal{F}^d = d^2 \cdot \mathcal{F}^d + 2d \sum_{k=0}^{\infty} s_{k+1} \partial_{k,Q} \mathcal{F}^d + \sum_{j,k=0}^{\infty} s_{j+1}s_{k+1} \partial_{j,Q} \partial_{k,Q} \mathcal{F}^d.
\]

We conclude that

\[
(5.1) \quad \mathcal{F} = -\frac{2}{d} \sum_{k=0}^{\infty} s_{k+1} \partial_{k,Q} \mathcal{F}^d - \frac{1}{d^2} \sum_{j,k=0}^{\infty} s_{j+1}s_{k+1} \partial_{j,Q} \partial_{k,Q} \mathcal{F}^d + \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \sum_{d_1 + \ldots + d_{\ell} = d-1} \nabla^2 \mathcal{F}^{d_1} \ldots \nabla^2 \mathcal{F}^{d_{\ell}}.
\]

Let $\mathcal{P}_{n,m}$ be the set of partitions with nonempty parts of the set $\{y_1, \ldots, y_m, z_1, \ldots, z_n\}$. For $P \in \mathcal{P}_{n,m}$, let $\{P_1, \ldots, P_{\ell(P)}\}$ denote the parts of $P$. Let $y_{P_i}$ and $z_{P_i}$ denote the $y$ and $z$ variables of the part $P_i$. Let $Y_{P_i}$ and $Z_{P_i}$ denote the sums of the variable sets $y_{P_i}$ and $z_{P_i}$ respectively. We see from (5.1) that for $d > 0$,

\[
(5.2) \quad \mathcal{F}^d[y_1, \ldots, y_m, z_1, \ldots, z_n] =
\]

\[
- \frac{2}{d} \sum_{1 \leq i \leq n} z_i \mathcal{F}^d[y_1, \ldots, y_m, z_i | z_1, \ldots, \hat{z}_i, \ldots, z_n] \\
- \frac{2}{d^2} \sum_{1 \leq i < j \leq n} z_i z_j \mathcal{F}^d[y_1, \ldots, y_m, z_i, z_j | z_1, \ldots, \hat{z}_i, \ldots, \hat{z}_j, \ldots, z_m] \\
+ \frac{1}{d^2} \sum_{P \in \mathcal{P}_{n,m}, d_1 + \ldots + d_{\ell(P)} = d-1} \prod_{i=1}^{\ell(P)} \left( \frac{\sinh(\varepsilon(Y_{P_i} + Z_{P_i})/2)}{\varepsilon(Y_{P_i} + Z_{P_i})/2} \right)^2 \mathcal{F}^d[y_{P_i}|z_{P_i}, 0, 0].
\]

This equation, together with the formulas of Theorem 1.1 for the degree 0 multipolar series, gives explicit formulas for the higher degree multipolar series.
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