The Higgs-boson decay \( H \rightarrow gg \) using the infinite-order scale-setting approach based on the intrinsic conformality

Chu-Tian Gao\(^1\), Xing-Gang Wu\(^1,2\), Xu-Dong Huang\(^1,4\) and Jun Zeng\(^3\)

\(^1\) Department of Physics, Chongqing University, Chongqing 401331, People’s Republic of China
\(^2\) Chongqing Key Laboratory for Strongly Coupled Physics, Chongqing 401331, P.R. China and INPAC, Key Laboratory for Particle Astrophysics and Cosmology (MOE), Shanghai Key Laboratory for Particle Physics and Cosmology, School of Physics and Astronomy, Shanghai Jiao Tong University, Shanghai 200240, People’s Republic of China

(Dated: September 27, 2021)

In this paper, we analyze the total decay width of the Higgs decay channel \( H \rightarrow gg \) up to \( \alpha_s^5 \)-order QCD corrections by using the newly suggested infinite-order scale-setting approach based on intrinsic conformality (PMC\(_\infty\)), which uses the principle of maximum conformality (PMC) as the starting point and ensures the scale invariance at each order. By using the PMC\(_\infty\) approach, the conventional renormalization scale ambiguity in perturbative QCD is eliminated, and the residual scale dependence due to unknown higher-order terms can be highly suppressed. We thus obtain a more accurate pQCD prediction on the decay width, i.e. \( \Gamma(H \rightarrow gg) |_{\text{PMC}\_\infty} = 336.42^{+7.01}_{-4.92} \text{ KeV} \), where the errors are squared averages of all the mentioned error sources.

In quantum chromodynamics (QCD), the Higgs boson plays an important role in precision test of the Standard Model (SM), and it is also helpful for searching the new physics beyond the SM. The Higgs boson decays into two gluons is an important channel for studying the Higgs phenomenology \([1]\). At present, the perturbative QCD (pQCD) corrections of decay width \( \Gamma(H \rightarrow gg) \) have been calculated up to next-to-next-to-next-to-leading order (N\(^4\)LO) \([2–11]\). We are thus facing the opportunity of achieving precise pQCD correction to \( \Gamma(H \rightarrow gg) \).

For the purpose, we need to reduce the pQCD uncertainty as much as possible. Among them, the uncertainty caused by using the guessed renormalization scale with an arbitrary range is usually treated as an important systematic error for theoretical predictions. As required by the renormalization group invariance, a physical observable should be independent to the choice of renormalization scale. In the literature, the principle of maximum conformality (PMC) \([12–15]\) has been suggested to eliminate the such renormalization scale ambiguity. It has been found that the PMC approach works well for dealing with the decay width \( \Gamma(H \rightarrow gg) \) \([16–18]\), e.g. the PMC predictions are scale invariant, scheme independent, and the resultant perturbative series are highly convergent due to the elimination of divergent renormalon terms.

It is noted that the PMC was originally introduced as a multi-scale approach, in which distinct PMC scales at each order are introduced in order to absorb different categories of \( \{ \beta_i \} \)-terms into corresponding \( \alpha_s \). And furthermore, because the same type of \( \{ \beta_i \} \)-terms emerge at different orders, the determined PMC scales are expressed in perturbative form. And then, the precision of the PMC scale for higher-order terms decreases with the increment of perturbative orders since fewer \( \{ \beta_i \} \)-terms are known. Thus the PMC has residual scale dependence due to unknown perturbative terms \([19]\), and if the pQCD convergence of the perturbative series of the PMC scale is weak, such residual scale dependence could become very large \([20]\). This is because the unknown higher-order terms will affect the magnitude of \( \alpha_s \) at each order.

Recently, by further taking the intrinsic conformality (iCF) property into consideration, an infinite-order scale-setting approach called as the PMC\(_\infty\) has been proposed in the literature \([21]\). The PMC\(_\infty\) approach starts from the PMC, the conformal coefficients are the same, but sets the effective scales at each order by requiring all the scale-dependent \( \{ \beta_i \} \)-terms at each order to vanish exactly. Via this way, the newly fixed PMC scales at each orders are no-longer in perturbative form, and the residual scale dependence due to the original perturbative nature of the PMC scales are eliminated. This indicates that the precision of the previous PMC predictions could be further improved. In the present paper, we will reanalyze the decay width \( \Gamma(H \rightarrow gg) \) up to N\(^4\)LO QCD corrections by using the PMC\(_\infty\) approach.

Practically, the decay width of the Higgs decays into two gluons at the \( \alpha_s^5 \)-order level can be expressed as

\[
\Gamma(H \rightarrow gg) = \frac{M_H^3 G_F}{36 \sqrt{2} \pi} \left[ \sum_{k=0}^{4} C_k(\mu_r) a_s^{k+2}(\mu_r) \right],
\]

where Fermi constant \( G_F = 1.16638 \times 10^{-5} \text{ GeV}^{-2} \), \( a_s = \alpha_s/4\pi \) and \( \mu_r \) is the renormalization scale. The perturbative coefficients \( C_k(\mu=0.4(\tilde{M}_H)) \) under the conventional \( \overline{\text{MS}} \)-scheme can be read from Refs.\([2–11]\). As has been argued in Refs.\([16–18]\), it is important to transform them into the ones under the physical momentum space subtraction scheme (mMOM-scheme) \([22–27]\) such that to avoid the ambiguities of fixing the PMC scales with the help of RGE. The mMOM-scheme is gauge dependent \([29]\), and for definiteness, we adopt the Landau
gauge to do our calculation, whose corresponding coefficients $C_k$ can be found in Refs.\cite{17, 18}.

Due to the iCF property, we can divide the $N^4$LO-level total decay width into five conformal subsets, i.e.

$$\Gamma(H \to gg) = \sum_{n=1}^{V} \Gamma_n,$$

which collect together the same type of non-conformal coefficients and ensure the scheme independence of each terms via the commensurate scale relations among different orders \cite{30}. Each conformal subset satisfies the scale invariant condition,

$$\left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s}\right) \Gamma_n = 0.$$ \tag{2}

More explicitly, we have

$$\Gamma_1 = A_{\text{Conf}} \left[ a_2^s(\mu_r) + 2B_{\beta_0}\alpha_0 a_3^s(\mu_r) + \left(3B_{\beta_0}^2 \beta_0^2 + 2B_{\beta_0}\beta_1 a_4^s(\mu_r) + (7B_{\beta_0}^2 - 4B_{\beta_0}^3 + 8B_{\beta_0}\beta_0 a_5^s(\mu_r) + (8B_{\beta_0}^2 - 4B_{\beta_0}\beta_0 a_6^s(\mu_r) + \right) \right], \tag{3}$$

$$\Gamma_1 = B_{\text{Conf}} \left[ a_2^s(\mu_r) + 3C_{\beta_0}\beta_0 a_1^s(\mu_r) + (6C_{\beta_0}^2 \beta_0^2 + 3C_{\beta_0}\beta_1 a_2^s(\mu_r) + (2\beta_0^2 \beta_0 a_3^s(\mu_r) + (12C_{\beta_0}^2 a_4^s(\mu_r) + \right) \right], \tag{4}$$

$$\Gamma_1 = C_{\text{Conf}} \left[ a_2^s(\mu_r) + 4D_{\beta_0}\beta_0 a_1^s(\mu_r) + (10D_{\beta_0}^2 \beta_0^2 + 4D_{\beta_0}\beta_1 a_2^s(\mu_r) + \right) \right], \tag{5}$$

$$\Gamma_1 = D_{\text{Conf}} \left[ a_2^s(\mu_r) + 5E_{\beta_0}\beta_0 a_1^s(\mu_r) \right], \tag{6}$$

$$\Gamma_1 = E_{\text{Conf}} \left[ a_2^s(\mu_r) \right]. \tag{7}$$

Here $A_{\text{Conf}}, B_{\text{Conf}}, C_{\text{Conf}}, D_{\text{Conf}}$ and $E_{\text{Conf}}$ are conformal coefficients, and $B_{\beta_0} = \ln \mu_r^2 / \mu_1^2$, $C_{\beta_0} = \ln \mu_r^2 / \mu_1^2$, $D_{\beta_0} = \ln \mu_r^2 / \mu_2^2$, $E_{\beta_0} = \ln \mu_r^2 / \mu_3^2$. The PMC\textsubscript{$\infty$} scales $\mu_1$, $\mu_2$, $\mu_3$, $\mu_4$, $\mu_5$, which can be fixed by using the scale invariant condition (2). To match the mMOM-scheme perturbative series, the $\beta$-functions under the mMOM-scheme are adopted, whose explicit forms up to five-loop level are available in Ref.\cite{28}. Following the standard PMC\textsubscript{$\infty$} procedures, the conformal coefficients and the PMC\textsubscript{$\infty$} scales can be derived from the known coefficients $C_k$ via a step-by-step manner. For examples, we have $A_{\text{Conf}} = C_0$; The conformal coefficient $B_{\text{Conf}}$ can be determined by setting $n_f = \frac{2}{3}$ to drop off the $\beta_0$ terms in $C_1$, and the PMC\textsubscript{$\infty$} scale $\mu_1$ can be fixed by using the known conformal coefficients $A_{\text{Conf}}, B_{\text{Conf}}$ and the $\{\beta_0\}$-terms of $C_1$; and etc. For convenience, we put all the required conformal coefficients and PMC\textsubscript{$\infty$} scales in the Appendix.

Then, we can transform the original perturbative series (1) into the following conformal series

$$\Gamma(H \to gg) = \frac{M_H^2 G_F}{36\sqrt{2}\pi} \left[ A_{\text{Conf}} a_2^s(\mu_1) + B_{\text{Conf}} a_3^s(\mu_1) + C_{\text{Conf}} a_4^s(\mu_11) + D_{\text{Conf}} a_5^s(\mu_1V) + E_{\text{Conf}} a_6^s(\mu_1V) \right]. \tag{8}$$

The PMC\textsubscript{$\infty$} scales are definite and have no perturbative nature, avoiding the scale ambiguity due to unknown higher-order terms in the perturbative series of the original PMC scales. The determined PMC\textsubscript{$\infty$} scales are

$$\{\mu_1, \mu_11, \mu_111, \mu_1V\} = \{50.1, 46.0, 63.0, 61.3\}(\text{GeV}), \tag{9}$$

which are independent to any choice of renormalization scale and avoid conventional renormalization scale ambiguity. It is interesting that those PMC\textsubscript{$\infty$} scales are around $M_H \exp(-5/6) \sim 54$ GeV, which is suggested by the well-known Gell-Mann Low scheme \cite{31}, in which $\exp(-5/6)$ is a result of the convention that is chosen to define the minimal dimensional regularization scheme. At present, the PMC\textsubscript{$\infty$} scale $\mu_1V$ at the highest order can not be determined, since there is no $\{\beta_1\}$-terms to fix its magnitude. As usual, we adopt $\mu_V = \mu_1V$ \cite{15}, which ensures the scheme independence of the resultant conformal series. Numerically, we have found that due to the the coefficient $E_{\text{Conf}}$ is free of divergent renormalon terms, the magnitude of the final term is negligibly small, and the uncertainty of the total decay width caused by different choice of $\mu_V$ is negligible.

To do the numerical calculation, we take the top-quark pole mass $M_t = 172.5 \pm 0.7$ GeV, and $M_H = 125.25 \pm 0.17$ GeV \cite{32}. The QCD asymptotic scale $\Lambda$ can be determined by using the world average of $\alpha_s$ at the scale $M_Z$, e.g. $\alpha_s(M_Z) = 0.1179 \pm 0.0010$ \cite{32}. As a subtle point, we need to transform the asymptotic scale from the MS-scheme to the mMOM-scheme by using the Celnaster-Gonsalves relation \cite{22-25}.

By setting all input parameters to be their central values, we firstly present the decay width $\Gamma(H \to gg)$ up to different $\alpha_s$-orders under conventional (Conv.) and PMC\textsubscript{$\infty$} scale-setting approaches in Fig. 1. At the $O(\alpha_s^2)$-order level, the perturbative series of $\Gamma(H \to gg)$ does not have $\{\beta_1\}$-terms to fix $\mu_1$, and the PMC\textsubscript{$\infty$} and conventional predictions are the same and both of them are scale dependent. Fig. 1 shows that the decay width $\Gamma(H \to gg)$ under conventional scale-setting approach has a strong dependence on $\mu_r$, which becomes smaller and smaller, when more and more loop terms have been included. Fig. 1 also shows that the decay width $\Gamma(H \to gg)$ at $O(\alpha_s^3)$-order and higher under PMC\textsubscript{$\infty$} scale-setting is independent to any choice of renormalization scale, because of the fact that the scale-dependent nonconformal terms have been eliminated.
shows the relative importance of the \( \mu \) scale-setting approach, the central values are for invariant PMC \( O \) affected by the one-order-higher terms. As for the scale and \( O \) between the two nearby orders becomes smaller when more big dot, the dash-dot line, the dotted line, the dashed line and the solid line are predictions up to \( O(\alpha_s^2) \), \( O(\alpha_s^3) \), \( O(\alpha_s^4) \), and \( O(\alpha_s^6) \), respectively.

FIG. 1. The decay width \( \Gamma(H \rightarrow gg) \) under conventional and PMC scale-setting approaches, respectively. The solid line with big dot, the dash-dot line, the dotted line, the dashed line and the solid line are predictions up to \( O(\alpha_s^2) \), \( O(\alpha_s^3) \), \( O(\alpha_s^4) \), \( O(\alpha_s^6) \), and \( O(\alpha_s^6) \), respectively.

\[
\begin{array}{cccccccccccc}
\hline
n = 2 & n = 3 & n = 4 & n = 5 & n = 6 & \kappa_1 & \kappa_2 & \kappa_3 & \kappa_4 \\
\hline
\Gamma^{O(\alpha_s^2)}_{\text{Conv}} & 219.86^{+34.05}_{-29.50} & 335.46^{+41.34}_{-28.59} & 349.71^{+2.52}_{-1.66} & 340.95^{+1.00}_{-0.67} & 337.45^{+1.94}_{-1.18} & 38\% & 65\% & 0.13\% & 0.13\% & 0.48\% & 0.10\% \\
\Gamma^{O(\alpha_s^2)}_{\text{PMC}} & 219.86 & 389.86 & 342.09 & 334.05 & 336.42 & 77\% & 12\% & 2.4\% & 0.7\% \\
\hline
\end{array}
\]

TABLE I. Results for the decay width \( \Gamma(H \rightarrow gg) \) (in unit: KeV) and \( \kappa_n \) up to different loop corrections under conventional and PMC\(_{\infty}\) scale-setting approaches, respectively. The PMC\(_{\infty}\) predictions are scale independent; While under conventional scale-setting approach, the central values are for \( \mu_r = M_H \), and the errors are for \( \mu_r \in [M_H/2, 2M_H] \).

\[
\begin{array}{cccccccccccc}
\hline
\Gamma^{O(\alpha_s^2)}_{\text{Conv}} & \Gamma^{O(\alpha_s^2)}_{\text{NLO}} & \Gamma^{O(\alpha_s^2)}_{\text{NLO}} & \Gamma^{O(\alpha_s^2)}_{\text{NLO}} & \Gamma^{O(\alpha_s^2)}_{\text{NLO}} & \Gamma^{O(\alpha_s^2)}_{\text{NLO}} & \Gamma^{O(\alpha_s^2)}_{\text{NLO}} \\
\hline
218.66^{+0.75}_{-0.59} & 116.69^{+2.08}_{-1.59} & 14.44^{+2.25}_{-4.16} & -8.70^{+13.84}_{-8.02} & 3.91^{+6.64}_{-3.91} & 337.45^{+1.94}_{-1.18} \\
299.57 & 95.22 & -41.82 & -19.59 & 3.04 & 336.42 \\
\hline
\end{array}
\]

TABLE II. The values (in unit: KeV) of each loop-term (LO, NLO, N\(^2\)LO and N\(^3\)LO) for the four-loop prediction \( \Gamma^{O(\alpha_s^2)} \) under the conventional and PMC\(_{\infty}\) scale-setting approaches, respectively. The PMC\(_{\infty}\) predictions are scale independent; While under conventional scale-setting approach, the central values are for \( \mu_r = M_H \), and the errors are for \( \mu_r \in [M_H/2, 2M_H] \).

Secondly, we present the decay width \( \Gamma(H \rightarrow gg) \) up to different loop QCD corrections under conventional and PMC\(_{\infty}\) scale-setting approaches in Table I. We define a ratio

\[
\kappa_n = \left| \frac{\Gamma^{O(\alpha_s^{n+2})} - \Gamma^{O(\alpha_s^{n+1})}}{\Gamma^{O(\alpha_s^{n+1})}} \right| ,
\]

which indicates how the “known” prediction \( \Gamma^{O(\alpha_s^{n+1})} \) is affected by the one-order-higher terms. As for the scale invariant PMC\(_{\infty}\) series, we have \( \kappa_1 \geq \kappa_2 > \kappa_3 > \kappa_4 \) for any choice of \( \mu_r \), indicating the relative difference between the two nearby orders becomes smaller when more loop terms have been included. This feature is consistent with the perturbative nature of the series and indicates that one can obtain more precise prediction by including more loop terms. As for the scale-dependent conventional series, as shown by Fig. 2, there are crossovers for \( \kappa_{2,3,4} \) within the range of \( \mu_r \in [M_H/2, 2M_H] \), and the ratios vary from 0 to 13\%, 4.8% and 1.0% for \( \kappa_2, \kappa_3 \) and \( \kappa_4 \), respectively. Moreover, to show the perturbative convergence explicitly, we present the magnitudes of each loop terms for the four-loop approximants \( \Gamma^{O(\alpha_s^2)} \) in Table II. Table II shows the relative importance of the (leading-order) LO-terms: (next-to-leading-order) NLO-
terms: \(N^2\text{LO-terms} : N^3\text{LO-terms} : N^4\text{LO-terms for conventional series are}

\[
1 : +53.3^{+13.5\%}_{-16.7\%} : +6.6^{+15.7\%}_{-16.4\%} : -4.0^{+6.9\%}_{-2.1\%} : -1.6^{+0.6\%},
\]

where the central values are for \(\mu_r = M_H\), and the errors are for \(\mu_r \in [\frac{1}{2}M_H, 2M_H]\). The scale dependence for each loop terms are large, but due to the cancellation of scale dependence among different orders, the net scale dependence is small, e.g., \((+0.6\%)\) for \(\mu_r \in [\frac{1}{2}M_H, 2M_H]\), \((+0.6\%)\) for \(\mu_r \in [\frac{1}{4}M_H, 3M_H]\), \((+2.1\%)\) for \(\mu_r \in [\frac{1}{8}M_H, M_H]\) and \((-0.4\%)\) for \(\mu_r \in [\frac{1}{16}M_H, 5M_H]\).

On the other hand, there are no renormalization scale dependence in \(\text{PMC}_\infty\) predictions. For examples, we have also presented the values of each loop-terms (LO, NLO, \(N^2\text{LO}, N^3\text{LO}\) or \(N^4\text{LO}\)) for the four-loop predictions \(\Gamma^{\alpha_s(\mu)}\) under the \(\text{PMC}_\infty\) approach in Table II. At the four-loop level, the \(\text{PMC}_\infty\) series already represents good convergent behavior, and the relative importance of the LO-terms: NLO-terms: \(N^2\text{LO-terms} : N^3\text{LO-terms} : N^4\text{LO-terms becomes}

\[
1 : +31.8\% : -14.0\% : -6.5\% : +1.0\%,
\]

whose magnitudes are scale invariant.

Thirdly, after eliminating the renormalization scale ambiguities, there are still some other error sources for the pQCD prediction of the decay width, such as the \(\alpha_s\) fixed-point error \(\Delta\alpha_s(M_Z)\), Higgs mass uncertainty \(\Delta M_H\), the top-quark pole mass uncertainty \(\Delta M_t\), etc. We observe that

\[
\Gamma|_{\text{Conv.}} = 337.45^{+6.97}_{-6.88} \pm 1.21 + 0.02 \text{ KeV},
\]
\[
\Gamma|_{\text{PMC}_\infty} = 336.49^{+6.90}_{-6.82} + 1.22 + 0.01 \text{ KeV},
\]

where the errors are for \(\Delta\alpha_s(M_Z) = 0.0010\) (which leads to \(\Lambda_{s(\mu=5)}^{\text{MOM}} = 362.0^{+40.7}_{-20.0} \text{ MeV}\), \(\Delta M_H = \pm 0.17 \text{ GeV}\), and \(\Delta M_t = \pm 0.7 \text{ GeV}\), respectively.

As a summary, in the present paper, we have presented a detailed analysis of the Higgs-boson decay \(H \rightarrow gg\) up to \(\alpha_s^4\)-order, and we obtain

\[
\Gamma(H \rightarrow gg)|_{\text{Conv.}} = 337.45^{+7.34}_{-7.08} \text{ KeV},
\]
\[
\Gamma(H \rightarrow gg)|_{\text{PMC}_\infty} = 336.42^{+7.01}_{-6.92} \text{ KeV},
\]

where the errors are squared averages of those from \(\Delta\alpha_s(M_Z)\), \(\Delta M_H\), \(\Delta M_t\) and the uncertainty of the renormalization scale within the region of \([M_H/2, 2M_H]\]. The errors are dominated by \(\Delta\alpha_s(M_Z)\), then followed by the choice of renormalization scale and the accuracy of Higgs mass. If the value of \(\alpha_s(M_Z)\) can be measured accurately to avoid the error from \(\Delta\alpha_s(M_Z)\), we will obtain

\[
\Gamma(H \rightarrow gg)|_{\text{Conv.}} = 337.45^{+2.29}_{-1.69} \text{ KeV},
\]
\[
\Gamma(H \rightarrow gg)|_{\text{PMC}_\infty} = 336.42^{+2.12}_{-1.26} \text{ KeV}.
\]

The Higgs-boson decay \(H \rightarrow gg\) provides another helpful example for the application of \(\text{PMC}_\infty\) scale-setting method to high-energy processes. Up to \(N^4\text{LO} QCD\) corrections, the \(\text{pQCD predictions under the PMC}_\infty\) and conventional scale-setting approaches are consistent with each other. The conventional scale uncertainties are still sizable, which are about \(1\% - 4\%\) (corresponding to \(\Delta\Gamma(H \rightarrow gg) \sim 1 - 4 \text{ KeV}\)) by varying the renormalization scale \(\mu_r\) within the ranges from \([\frac{1}{2}M_H, 2M_H]\) to \([\frac{1}{8}M_H, 5M_H]\). By applying the \(\text{PMC}_\infty\), the \(\alpha_s\) values at lower orders are definitely fixed by the requirement of intrinsic conformality, the conventional renormalization scale ambiguity is eliminated, and the residual scale dependence from the original \(\text{PMC multi-scale-setting approach can also been highly suppressed. Thus a more precise test of the SM can be achieved.}

\textbf{Acknowledgments:} This work was supported in part by the Natural Science Foundation of China under Grant No.11625520 and No.12047564, by the Fundamental Research Funds for the Central Universities under Grant No.2020CQJQY-Z003 and No.2021CDJZYJH-003.

\textbf{APPENDIX: THE CONFORMAL COEFFICIENTS AND PMC}_\infty\text{ SCALES UP TO }\alpha_s^4\text{-ORDER LEVEL}

Applying the \(\text{PMC}_\infty\) scale-setting approach, the perturbative series of the decay width \(\Gamma(H \rightarrow gg)\) under \text{mMOM-scheme is}
\[ \Gamma(H \to gg) = \frac{M_H^2 G_F}{36 \sqrt{2} \pi} \left[ A_{\text{Conf}} a_s^2(\mu_r) + (B_{\text{Conf}} + 2A_{\text{Conf}} B_{\beta_0} \beta_0) a_s^3(\mu_r) + \left( C_{\text{Conf}} + 3B_{\text{Conf}} C_{\beta_0} \beta_0 + 3A_{\text{Conf}} B_{\beta_0}^2 \beta_0^2 \right) + 2A_{\text{Conf}} B_{\beta_0} \beta_1 \right] a_s^4(\mu_r) + \left( D_{\text{Conf}} + 7A_{\text{Conf}} B_{\beta_0}^2 \beta_1 \beta_0 + 4C_{\text{Conf}} D_{\beta_0} \beta_0 + 6B_{\text{Conf}} C_{\beta_0}^2 \beta_0^2 + 4A_{\text{Conf}} B_{\beta_0}^3 \beta_0^3 \right) + 2A_{\text{Conf}} B_{\beta_0} \beta_2 + 3B_{\text{Conf}} C_{\beta_0} \beta_1 \right] a_s^5(\mu_r) + \left( E_{\text{Conf}} + 8A_{\text{Conf}} B_{\beta_0}^2 \beta_2 \beta_0 + \frac{27}{2} B_{\text{Conf}} C_{\beta_0}^2 \beta_1 \beta_0 + 5D_{\text{Conf}} E_{\beta_0} \beta_0 \right) + \frac{47}{3} A_{\text{Conf}} B_{\beta_0}^3 \beta_1 \beta_0 + 10C_{\text{Conf}} D_{\beta_0} \beta_0 + 10B_{\text{Conf}} C_{\beta_0}^3 \beta_0^2 + 5A_{\text{Conf}} B_{\beta_0}^4 \beta_0^2 + 2A_{\text{Conf}} B_{\beta_0} \beta_3 + 3B_{\text{Conf}} C_{\beta_0} \beta_2 + 4A_{\text{Conf}} B_{\beta_0}^2 \beta_1^2 + 4C_{\text{Conf}} D_{\beta_0} \beta_1 \right] a_s^6(\mu_r) \right] + O(a_s^7(\mu_r)), \] (17)

To compare Eq. (1) with Eq. (17), one can determine the conformal coefficients and PMC∞ scales for \( \Gamma(H \to gg) \) up to \( a_s^6 \)-order level via a step-by-step manner, i.e.

\[
A_{\text{Conf}} = C_0, \\
B_{\text{Conf}} = C_1 \left( n_f = \frac{33}{2} \right), \\
C_{\text{Conf}} = C_2 \left( n_f = \frac{33}{2} \right) - 2A_{\text{Conf}} B_{\beta_0} \bar{\beta}_1, \\
D_{\text{Conf}} = C_3 \left( n_f = \frac{33}{2} \right) - 2A_{\text{Conf}} B_{\beta_0} \bar{\beta}_2 - 3B_{\text{Conf}} C_{\beta_0} \bar{\beta}_1, \\
E_{\text{Conf}} = C_4 \left( n_f = \frac{33}{2} \right) - 2A_{\text{Conf}} B_{\beta_0} \bar{\beta}_3 - 3B_{\text{Conf}} C_{\beta_0} \bar{\beta}_2 - 4A_{\text{Conf}} B_{\beta_0}^2 \beta_1^2 - 4C_{\text{Conf}} D_{\beta_0} \bar{\beta}_1
\]

and

\[
\ln \frac{\mu_r^2}{\mu_1^2} = C_1 - B_{\text{Conf}} \beta_1, \\
\ln \frac{\mu_r^2}{\mu_1^2} = \frac{C_2 - C_{\text{Conf}} - 3A_{\text{Conf}} B_{\beta_0}^2 \beta_0^2 - 2A_{\text{Conf}} B_{\beta_0} \beta_1}{3B_{\text{Conf}} \beta_0}, \\
\ln \frac{\mu_r^2}{\mu_1^2} = \frac{C_3 - D_{\text{Conf}} - 7A_{\text{Conf}} B_{\beta_0}^2 \beta_1 \beta_0 - 6B_{\text{Conf}} C_{\beta_0}^2 \beta_0^2 - 4A_{\text{Conf}} B_{\beta_0}^3 \beta_1 \beta_0^2 - 2A_{\text{Conf}} B_{\beta_0} \beta_2 - 3B_{\text{Conf}} C_{\beta_0} \beta_1}{4C_{\text{Conf}} \beta_0}, \\
\ln \frac{\mu_r^2}{\mu_1^2} = \frac{(C_4 - E_{\text{Conf}} - 8A_{\text{Conf}} B_{\beta_0}^2 \beta_2 \beta_0 - \frac{27}{2} B_{\text{Conf}} C_{\beta_0}^2 \beta_1 \beta_0 - \frac{47}{3} A_{\text{Conf}} B_{\beta_0}^3 \beta_1 \beta_0^2 - 10C_{\text{Conf}} D_{\beta_0} \beta_0^2 - 10B_{\text{Conf}} C_{\beta_0}^3 \beta_0^3 - 5A_{\text{Conf}} B_{\beta_0}^4 \beta_0^2 - 2A_{\text{Conf}} B_{\beta_0} \beta_3 - 3B_{\text{Conf}} C_{\beta_0} \beta_2 - 4A_{\text{Conf}} B_{\beta_0}^2 \beta_1^2 - 4C_{\text{Conf}} D_{\beta_0} \beta_1) / 5D_{\text{Conf}} \beta_0}{5D_{\text{Conf}} \beta_0}.
\]

Here \( \bar{\beta}_1 = \beta_1(n_f = \frac{33}{2}) = -107, \bar{\beta}_2 = \beta_2(n_f = \frac{33}{2}) = -2001.29 \) and \( C_k \) are the perturbative coefficients.

[1] A. Djouadi, Phys. Rept. 457, 1 (2008).
[2] T. Inami, T. Kubota, and Y. Okada, Z. Phys. C 18, 69 (1983).
[3] A. Djouadi, M. Spira, and P. M. Zerwas, Phys. Lett. B 264, 440 (1991).
[4] D. Graudenz, M. Spira, and P. M. Zerwas, Phys. Rev. Lett. 70, 1372 (1993).
[5] S. Dawson and R. Kauffman, Phys. Rev. D 49, 2298 (1994).
[6] M. Spira, A. Djouadi, D. Graudenz, and P. M. Zerwas, Nucl. Phys. B 453, 17 (1995).
[7] S. Dawson and R. P. Kauffman, Phys. Rev. Lett. 68, 2273 (1992).
[8] K. G. Chetyrkin, B. A. Kniehl, and M. Steinhauser, Phys. Rev. Lett. 79, 353 (1997).
[9] K. G. Chetyrkin, B. A. Kniehl, and M. Steinhauser, Nucl.
Phys. B 510, 61 (1998).

[10] P. A. Baikov and K. G. Chetyrkin, Phys. Rev. Lett. 97, 061803 (2006).

[11] F. Herzog, B. Ruijl, T. Ueda, J. A. M. Vermaseren, and A. Vogt, JHEP 1708, 113 (2017).

[12] S. J. Brodsky and X. G. Wu, Phys. Rev. D 85, 034038 (2012).

[13] S. J. Brodsky and X. G. Wu, Phys. Rev. Lett. 109, 042002 (2012).

[14] M. Mojaza, S. J. Brodsky, and X. G. Wu, Phys. Rev. Lett. 110, 192001 (2013).

[15] S. J. Brodsky, M. Mojaza, and X. G. Wu, Phys. Rev. D 89, 014027 (2014).

[16] S. Q. Wang, X. G. Wu, X. C. Zheng, J. M. Shen, and Q. L. Zhang, Eur. Phys. J. C 74, 2825 (2014).

[17] D. M. Zeng, S. Q. Wang, X. G. Wu, and J. M. Shen, J. Phys. G 43, 075001 (2016).

[18] J. Zeng, X. G. Wu, S. Bu, J. M. Shen, and S. Q. Wang, J. Phys. G 45, 085004 (2018).

[19] X. C. Zheng, X. G. Wu, S. Q. Wang, J. M. Shen, and Q. L. Zhang, JHEP 10, 117 (2013).

[20] X. G. Wu, J. M. Shen, B. L. Du, X. D. Huang, S. Q. Wang, and S. J. Brodsky, Prog. Part. Nucl. Phys. 108, 103706 (2019).

[21] L. Di Giustino, S. J. Brodsky, S. Q. Wang, and X. G. Wu, Phys. Rev. D 102, 014015 (2020).

[22] W. Celmaster and R. J. Gonsalves, Phys. Rev. D 20, 1420 (1979).

[23] W. Celmaster and R. J. Gonsalves, Phys. Rev. Lett. 42, 1435 (1979).

[24] W. Celmaster and R. J. Gonsalves, Phys. Rev. Lett. 44, 560 (1980).

[25] W. Celmaster and R. J. Gonsalves, Phys. Rev. D 21, 3112 (1980).

[26] J. A. Gracey, J. Phys. A 46, 225403 (2013).

[27] L. von Smekal, K. Maltman, and A. Sternbeck, Phys. Lett. B 681, 336 (2009).

[28] B. Ruijl, T. Ueda, J. A. M. Vermaseren, and A. Vogt, JHEP 1706, 040 (2017).

[29] J. Zeng, X. G. Wu, X. C. Zheng and J. M. Shen, Chin. Phys. C 44, 113102 (2020).

[30] S. J. Brodsky and H. J. Lu, Phys. Rev. D 51, 3652 (1995).

[31] M. Gell-Mann and F. E. Low, Phys. Rev. 95, 1300 (1954).

[32] P.A. Zyla et al. [Particle Data Group], PTEP 2020, 083C01 (2020).