Abstract

For finite connected graphs $\Gamma$ and $G$, with $\Gamma$ admitting a free involution $\tau$, we characterize the based homotopy classes $\alpha \in [\Gamma,G]$ for which the Borsuk-Ulam property holds in the sense of Gonçalves, Guaschi and Casteluber-Laass, i.e., the homotopy classes $\alpha$ so that each of its representatives $f \in \alpha$ satisfies $f(x) = f(\tau \cdot x)$ for some $x \in \Gamma$. This is attained through a graph-braid-group perspective aided by the use of discrete Morse theory.

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1 Introduction and main result

In its classical formulation, the Borsuk-Ulam Theorem asserts that, for any continuous map

$$f : S^n \to \mathbb{R}^n,$$  

(1)

there is a point $x \in S^n$ so that both $x$ and its antipodal $-x$ have the same image under $f$. Such a phenomenon has been intensively studied in the last 15 years within generalized contexts, namely, for maps $f : M \to N$ between spaces $M$ and $N$, where $M$ admits a free involution. For instance, the case where $M$ ranges over surfaces or suitable families of 3-manifolds is now reasonably well understood [3, 4, 5, 6, 7, 8, 12, 13]. The case where $N$ has non-trivial homotopy information leads to a more refined problem, as the Borsuk-Ulam question can then have different answers for different homotopy classes in $[M,N]$.

Definition 1.1 ([17]). Assume $M$ admits a free involution $\tau$. We say that the Borsuk-Ulam property holds for a homotopy class $\alpha \in [M,N]$ if for every representative $f \in \alpha$ there is a point $x \in M$ such that $f(x) = f(\tau \cdot x)$. If the above condition holds for all homotopy classes in $[M,N]$, we say that the triple $(M,\tau,N)$ satisfies the Borsuk-Ulam property.

We give a complete answer to the Borsuk-Ulam problem in the case where both $M$ and $N$ are 1-dimensional compact connected objects. We will thus focus on maps $f : \Gamma \to G$ between finite connected graphs $\Gamma$ and $G$, addressing the Borsuk-Ulam property with respect to some fixed free involution $\tau$ on $\Gamma$.

Remark 1.2. In the classical situation [11] with $n = 1$, the circle plays no essential role. Indeed, by considering the differences $f(x) - f(\tau \cdot x)$, it can be seen that any map $f : \Gamma \to \mathbb{R}$ satisfies the Borsuk-Ulam property. Such a pleasant situation changes drastically when $\mathbb{R}$ (or an interval, for

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1Unless otherwise noted, spaces are assumed to come equipped with base points which must be preserved by maps between spaces.
that matter) is replaced by a more general graph $G$ which, in what follows, will be assumed not to be homeomorphic to an interval. In particular the configuration spaces $\text{Conf}_2(G)$ and $U\text{Conf}_2(G)$ consisting respectively of pairs $(x_1, x_2)$ and of subsets $\{x_1, x_2\}$ with $x_1 \neq x_2$ are both connected.

The Borsuk-Ulam property for $(\Gamma, \tau, G)$ as above is described next.

**Theorem 1.3.** If $G$ is not homeomorphic to a circle or to an interval, then the Borsuk-Ulam property fails for all homotopy classes in $[\Gamma, G]$, i.e., for every $\alpha \in [\Gamma, G]$ there is a representative $f \in \alpha$ satisfying $f(x) \neq f(\tau \cdot x)$ for all $x \in \Gamma$.

When $G$ is a circle, the behavior of the Borsuk-Ulam property sits in between Remark 1.2 and Theorem 1.3. The explicit answer, given in Theorem 1.4 below, generalizes [17, Proposition 6] and depends on the Euler characteristic $\chi(\Gamma)$. The latter number is even, in view of the free involution $\tau$, and at most 0, since $\Gamma$ is connected. Say $\chi(\Gamma) = -2m$ with $m \geq 0$.

**Theorem 1.4.** If $G$ is homeomorphic to a circle $S^1$, then the Borsuk-Ulam property holds for most of the homotopy classes in $[\Gamma, S^1]$. Explicitly, under a certain identification of $[\Gamma, S^1]$ with $\mathbb{Z}^{2m+1}$, the homotopy classes of maps $\Gamma \to S^1$ for which the Borsuk-Ulam property fails are precisely the $(2m+1)$-tuples $(p, p_1, p_2, p_3, \ldots, p_m, p_m)$ with $p$ odd (and $p_1, \ldots, p_m$ arbitrary).

Observe that Theorems 1.3 and 1.4 can be stated replacing based homotopy classes by free homotopy classes. This is clear in the case of Theorem 1.3, while the case of Theorem 1.4 follows from the fact that $\pi_1(S^1)$ is abelian, so that based homotopy classes and free homotopy classes coincide.

As in [14, 15, 16, 17, 18], we study the Borsuk-Ulam property for graphs through a sharp algebraic model in terms of braid groups. In our case (graphs), the critical information comes from the detailed control of the topological combinatorics associated to graph configuration spaces (both in the ordered and unordered contexts) provided by Farley-Sabalka’s discrete gradient field on Abrams’ homotopy model. All needed details are reviewed in Section 2. Section 3 is devoted to the proof of Theorems 1.3 and 1.4.

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## 2 Graph braid groups via discrete Morse theory

We start by collecting the ingredients we need about Forman’s discrete Morse theory and Farley-Sabalka’s gradient field on Abrams’ discrete model for (ordered and unordered) graph configuration spaces. For details, the reader is referred to [11, 12, 9, 10, 11, 19].

### 2.1 Discrete Morse theory

Let $X$ be a connected finite regular CW complex with cell poset $\mathcal{F}$ partially ordered by inclusion. For a cell $a \in \mathcal{F}$, we use $a^{(p)}$ as a shorthand of $\dim(a) = p$. The Hasse diagram of $\mathcal{F}$, $H_\mathcal{F}$, is thought
of as a directed graph with arrows \( a^{(p+1)} \downarrow b^{(p)} \) oriented from the higher dimensional cell to the lower dimensional cell. Let \( W \) be a partial matching on \( H_F \), i.e., a directed subgraph of \( H_F \) all whose vertices have degree 1. The modified Hasse diagram \( H_F(W) \) is obtained from \( H_F \) by reversing all arrows of \( W \). A reversed edge is denoted as \( b^{(p)} \not\rightarrow a^{(p+1)} \), in which case \( a \) is said to be collapsible and \( b \) is said to be redundant. A path \( \lambda \) in \( H_F(W) \) is a chain of up-going and down-going arrows

\[
a_0 \not\rightarrow b_1 \not\rightarrow a_1 \not\rightarrow \cdots \not\rightarrow b_k \not\rightarrow a_k.
\]

The path \( \lambda \) is said to be a cycle when \( a_0 = a_k \). If there are no cycles, \( W \) is called a gradient field and, in such a case, cells of \( X \) that are neither redundant nor collapsible are said to be critical. In what follows we assume that \( W \) is a gradient field on \( X \).

The subgraph of the one skeleton \( X^{(1)} \) consisting of all vertices and of all collapsible edges forms a maximal forest \( F_X \) with as many components as there are critical 0-cells in \( X \). Add critical edges as needed in order to form a maximal tree \( T_X \). Fix a vertex \( v_0 \in X^{(0)} \) as base point. Collapsing \( T_X \) to \( v_0 \) yields a generating set \( \{ \beta_e \}_e \) for the fundamental group \( \pi_1(X;v_0) \), where \( e \) runs over the set of (arbitrarily oriented) critical 1-cells of \( X \) that are not part of \( T_X \). Explicitly, for each vertex \( u \in X^{(0)} \), let \( \beta_u \) be the unique path in \( T_X \) determined by the ordered sequence of non repeating edges connecting \( v_0 \) to \( u \). Then, for a critical 1-cell \( e \) from \( u_1 \) to \( u_2 \) that is not part of \( T_X \), the loop

\[
\beta_{u_1} * e * \beta_{u_2}
\]

represents the homotopy class \( \beta_e \in \pi_1(X;v_0) \), which will simply be denoted as \( e \in \pi_1(X;v_0) \) — the context clarifies whether we refer to the actual cell or to the corresponding homotopy class. Farley and Sabalka go further describing a set of relations among the homotopy generators that yield a presentation of \( \pi_1(X,v_0) \). The relations depend on the critical 2-cells and the redundant 1-cells. We omit the details as we will not have need to use the relations.

### 2.2 Farley-Sabalka gradient field on Abrams model

Let \( G \) be a finite connected graph. By inserting a few non-essential vertices, we can assume \( G \) is simplicial, i.e., that \( G \) contains no loops nor double edges. Let \( \text{Conf}_2(G) \) denote the ordered configuration space of pairs \( (x,y) \in G^2 \) with \( x \neq y \), and let \( \text{UConf}_2(G) \) denote the orbit space by the involution \( (x,y) \mapsto (y,x) \). Abrams homotopy model \( D_2(G) \) for \( \text{Conf}_2(G) \) is the subcomplex of \( G \times G \) whose cells are the ordered pairs \( c = (c_1, c_2) \) of cells\(^3\) of \( G \) with \( c_1 \cap c_2 = \emptyset \). The orbit complex \( U_2(G) \) resulting from the involution \( (c_1,c_2) \mapsto (c_2,c_1) \) is the corresponding homotopy model for \( \text{UConf}_2(G) \). Thus, cells of \( U_2(G) \) are sets \( c = \{c_1, c_2\} \) of disjoint cells \( c_i \) of \( G \). Both in the ordered and unordered settings:

- the cells \( c_1 \) and \( c_2 \) are called the ingredients of \( c \);
- the dimension of \( c \) is the sum of the dimensions of \( c_1 \) and \( c_2 \);
- the orientation of a 1-dimensional cell with vertex ingredient \( v \) and edge ingredient \( (v_1,v_2) \) (so \( u \neq v_1 < v_2 \neq u \)) will be inherited from that of \( (v_1,v_2) \). For instance, the ordered 1-cell \( ((v_1,v_2),u) \) is oriented from \( (v_1,u) \) to \( (v_2,u) \).

The construction of Farley-Sabalka gradient field on \( D_2(G) \) and of its quotient on \( U_2(G) \) require some preliminary notation. Start by choosing a maximal tree \( T \) of \( G \). Edges of \( G \) outside \( T \) are

\(^3\)A cell of \( G \) is either a vertex or a (closed) edge.
called deleted edges. Fix a planar embedding of $T$ and a root of $T$ (i.e., a vertex of degree 1 in $T$), which is denoted by 0. The rest of the vertices of $G$ are consecutively numbered $1, 2, \ldots$ as we first find them in the walk along $T$ that starts at 0 and that takes the leftmost branch at any given intersection, turning around when a vertex of degree one is reached. An edge $e$ bounded by vertices $u$ and $v$ with $u < v$ is denoted by $e = (u, v)$, and is oriented from $u$ to $v$. Under such conditions we also write $v = \sigma(e)$, the source of $e$, and $u = \tau(e)$, the target of $e$. The source-target notation is compatible with the fact that, by collapsing $T$ down to its root 0, we can think of $\pi_1(G, 0)$ as the free group generated by the deleted edges (each with the vertex-ordering orientation). Note that if $(u, v)$ is non-deleted, $u$ is determined as the vertex adjacent to $v$ in $T$ that is located in the $T$-path leading from $v$ back to 0, so we can safely write $e_v = (u, v)$. In particular, non-deleted edges $e_v$’s will be ordered according to the order of the corresponding $v$’s.

Let $c = (c_1, c_2)$ (or $c = \{c_1, c_2\}$) be a cell of $D_2(G)$ (or of $UD_2(G)$). A vertex ingredient $v = c_i$ of $c$ is said to be critical in $c$ if either $v = 0$ or, else, if replacement of $v$ by $e_v$ in $c$ fails to yield a cell of $D_2(G)$ (or of $UD_2(G)$). Likewise, and edge ingredient $e = c_j$ of $c$ is said to be critical in $c$ if either $e$ is deleted or, else, if $e = e_v$ and there is a vertex ingredient $u$ of $c$ adjacent to $\tau(e)$ with $\tau(e) < u < v$. With such notation, $c$ is critical in Farley-Sabalka gradient field provided $c_1$ and $c_2$ are both critical in $c$. Otherwise, if the smallest of the non-critical ingredients $c_1$ and $c_2$ is:

(i) a vertex $v$, then $c$ is redundant and $c \not\searrow d$, where $d$ is obtained from $c$ by replacing $v$ by $e_v$;

(ii) an edge $e_v$, then $c$ is collapsible and $d \searrow c$, where $d$ is obtained from $c$ by replacing $e_v$ by $v$.

In other words, the only critical 0-cells in $D_2(G)$ are $(0, 1)$ and $(1, 0)$, while $\{0, 1\}$ is the only critical 0-cell in $UD_2(G)$. All other 0-cells are redundant. Likewise, for a 1-cell $c$ with vertex ingredient $u$ and edge ingredient $e$ we have:

(iii) If $e$ is deleted, then $c$ is critical if $u$ is critical in $c$, otherwise $c$ is redundant.

(iv) If $e$ is non-deleted, say $e = e_v$, and

- either $u = 0$ or $v < u$, then $c$ is collapsible;
- $0 < u < v$ with $u$ critical in $c$ (in which case $\tau(e_u) = \tau(e_v)$ is forced), then $c$ is critical;
- $0 < u < v$ with $u$ non-critical in $c$, then $c$ is redundant.

Lastly, a 2-cell $c$ is critical if both of its ingredients are deleted, otherwise $c$ is collapsible.

2.3 Graph braid groups

We now recover the assumption in Remark 1.2 about $G$ not being homeomorphic to an interval, so we can use the facts reviewed above in order to describe generators for

$$P_2(G) = \pi_1(D_2(G)) = \pi_1(Conf_2(G)) \quad \text{and} \quad B_2(G) = \pi_1(UD_2(G)) = \pi_1(UConf_2(G)),$$

the (pure and full, respectively) braid groups of two (ordered and unordered, respectively) non-colliding particles in $G$. In particular, the notation set up in Subsections 2.1 and 2.2 will be in effect throughout the rest of the paper.

The case of $B_2(G)$ is slightly easier as $UD_2(G)$ has a single critical 0-cell, so that the 0-cells and the collapsible 1-cells of $UD_2(G)$ span a maximal tree $UDT$ of the 1-skeleton of $UD_2(G)$. Thus,

\footnotesize

\[\text{[With respect to the vertex-edge ordering discussed in the previous paragraph.}\]
In particular, the collapsible/redundant nature of cells, the effect of the induced monomorphism terms of the generators describe above. Here we only consider the case of $DT$ where $\beta$ in the other five cases.

Assumption in force throughout the rest of the proof, leaving to the reader the fully parallel details after collapsing in order to get a maximal tree $DT$. For our purposes, the required critical 1-cell will have the form $(a, b, c)$ with $b < a < c$. (Note that such an edge goes from $(a, b) \in DT'_d$ to $(a, c) \in DT''_u$.) The explicit values of $a, b, c$ will be spelled out later, depending on the actual graph $G$. All other critical 1-cells $(r, (s, t))$ and $((s, t), r)$ of $D_2(G)$ yield corresponding generators $(r, (s, t)), ((s, t), r) \in P_2(G)$ after collapsing $DT$ to its base point, which is now taken to be $(0, 1)$.

**Proposition 2.1.** The inclusion $\iota: P_2(G) \hookrightarrow B_2(G)$ induced by the 2-fold covering $D_2(G) \twoheadrightarrow UD_2(G)$ is determined on generators by

$$\iota((s, t), r) = \begin{cases} 
{r, (s, t)}, & \text{if } r < s < t; \\
{a, (b, c)}^{-1} \cdot {r, (s, t)}, & \text{if } s < r < t; \\
{a, (b, c)}^{-1} \cdot {r, (s, t)} \cdot {a, (b, c)}, & \text{if } s < t < r; \\
{a, (b, c)}^{-1} \cdot ({r, (s, t)} \cdot {a, (b, c)}), & \text{if } r < s < t; \\
{r, (s, t)} \cdot {a, (b, c)}, & \text{if } s < r < t; \\
{r, (s, t)}, & \text{if } s < t < r.
\end{cases}$$

In particular, $\iota((b, c), a) = {a, (b, c)}^2$.

**Proof.** Since the 2-fold covering projection $D_2(G) \twoheadrightarrow UD_2(G)$ is cellular and preserves the critical/collapsible/redundant nature of cells, the effect of the induced monomorphism $\iota$ is easily readable in terms of the generators describe above. Here we only consider the case of $\iota((s, t), r)$ with $r < s < t$, assumption in force throughout the rest of the proof, leaving to the reader the fully parallel details in the other five cases.

As a path in $D_2(G)$, the edge $((s, t), r)$ goes from $(s, r) \in DT'_d$ to $(t, r) \in DT''_d$. Then the paths $\beta_{(s, r)}$ and $\beta_{(t, r)}$ in [2] are given by

$$\beta_{(s, r)} = \gamma_{(a, c)}^{(0, 1)} \ast (a, (b, c))^{-1} \ast \delta_{(s, r)}^{(a, b)} \quad \text{and} \quad \beta_{(t, r)} = \gamma_{(a, c)}^{(0, 1)} \ast (a, (b, c))^{-1} \ast \delta_{(t, r)}^{(a, b)},$$

where $\gamma$-paths and $\delta$-paths consist of collapsible cells. Explicitly, $\gamma_{(a, c)}^{(0, 1)}$ is the unique simple path in $DT_u$ connecting $D$ to $(a, c)$, while $\delta_{(s, r)}^{(a, b)}$ and $\delta_{(t, r)}^{(a, b)}$ are the unique simple paths in $DT_d$ connecting $(a, b)$ to $(s, r)$ and $(t, r)$, respectively.
The loop \( (2) \) representing \( ((s, t), r) \in P_2(G) \) is then
\[
\gamma_{(a, c)}^{(0, 1)} \ast (a, (b, c))^{-1} \ast \delta_{(s, r)}^{(a, b)} \ast ((s, t), r) \ast \left( \gamma_{(a, c)}^{(0, 1)} \ast (a, (b, c))^{-1} \ast \delta_{(t, r)}^{(a, b)} \right)^{-1},
\]
and the asserted expression for \( \epsilon((s, t), r) \) now follows by noticing that the portions corresponding to \( \gamma \)-paths and \( \delta \)-paths are sent by \( \epsilon \) into \( UDT \) and, so, get squeezed to the base point \( \{0, 1\} \).

From this point on we will think of \( P_2(G) \) as a honest subgroup of \( B_2(G) \), omitting to write the symbol \( \epsilon \) when thinking of an element in \( P_2(G) \) as an element of \( B_2(G) \).

**Corollary 2.2.** For a critical 1-cell \( e \) of \( D_2(G) \) other than \( (a, (b, c)) \), the conjugate
\[
\{a, (b, c)\} \cdot e \cdot \{a, (b, c)\}^{-1} \in P_2(G) \triangleq B_2(G)
\]
is described as follows:

For \( r < s < t \),
\[
\begin{aligned}
\{a, (b, c)\} \cdot (r, (s, t)) \cdot \{a, (b, c)\}^{-1} &= ((b, c), a) \cdot ((s, t), r) \cdot ((b, c), a)^{-1}; \\
\{a, (b, c)\} \cdot ((s, t), r) \cdot \{a, (b, c)\}^{-1} &= (r, (s, t)).
\end{aligned}
\]

For \( s < r < t \),
\[
\begin{aligned}
\{a, (b, c)\} \cdot (r, (s, t)) \cdot \{a, (b, c)\}^{-1} &= ((s, t), r) \cdot ((b, c), a)^{-1}; \\
\{a, (b, c)\} \cdot ((s, t), r) \cdot \{a, (b, c)\}^{-1} &= ((b, c), a), \\
& \quad \text{if } (r, (s, t)) = (a, (b, c)); \\
& = ((b, c), a) \cdot (r, (s, t)), \quad \text{otherwise}.
\end{aligned}
\]

For \( s < t < r \),
\[
\begin{aligned}
\{a, (b, c)\} \cdot (r, (s, t)) \cdot \{a, (b, c)\}^{-1} &= ((s, t), r) ; \\
\{a, (b, c)\} \cdot ((s, t), r) \cdot \{a, (b, c)\}^{-1} &= ((b, c), a) \cdot (s, t) \cdot ((b, c), a)^{-1}.
\end{aligned}
\]

**Proposition 2.3.** If \( G \) is not homeomorphic to an interval, then the morphism \( \theta: B_2(G) \rightarrow \mathbb{Z}_2 \)
induced in fundamental groups by the classifying map of the double covering \( D_2(G) \rightarrow UD_2(G) \) is given on generators by
\[
\theta(\{r, (s, t)\}) = \begin{cases} 1, & \text{if } s < r < t; \\ 0, & \text{otherwise}. \end{cases}
\]

**Proof.** Since \( \theta \) vanishes on \( P_2(G) \), the relations
\[
(r, (s, t)) = \{r, (s, t)\}, \quad \text{for } r < s < t, \\
((s, t), r) = \{r, (s, t)\}, \quad \text{for } s < t < r
\]
force \( \theta(\{r, (s, t)\}) = 0 \) provided \( r < s \) or \( t < r \). On the other hand, for \( s < r < t \), the relation
\[
((s, t), r) = \{r, (s, t)\} \cdot \{a, (b, c)\}
\]
gives \( \theta(\{r, (s, t)\}) = \theta(\{a, (b, c)\}) \), which is forced to be 1, since \( \theta \) is surjective (recall that \( G \) is not an interval, so that \( \text{Conf}_2(G) \) is connected).

**Proposition 2.4.** The morphism \( (p_1)_\#: P_2(G) \rightarrow \pi_1(G) \)
induced in fundamental groups by the projection \( p_1: D_2(G) \rightarrow G \) onto the first coordinate is trivial on generators \( (r, (s, t)) \), while
\[
(p_1)_\#((s, t), r) = \begin{cases} (s, t), & \text{if } (s, t) \text{ is a deleted edge}; \\ 1, & \text{otherwise}. \end{cases}
\]

**Proof.** The conclusion for \( (p_1)_\#((s, t), r) \) when \( r < s < t \) follows from the fact that \( (3) \) is a representing loop for \( ((s, t), r) \in P_2(G) \), and noticing that the \( p_1 \)-image of a collapsible 1-edge lands in the tree \( T \) and, so, it plays no role in \( \pi_1(G) \). The other five cases are treated similarly and are left as an exercise for the reader.
3 Borsuk-Ulam property

Let \((\Gamma, \tau, G)\) be as in Section 1. We assume in effect all hypotheses, ingredients and constructions set up in Subsection 2.3 around \(G\) and its braid groups. Theorem 3.1 below is a “graph” version of [17, Theorem 7], proven with the same argument, using that graph configuration spaces are \(K(\pi, 1)\)’s.

**Theorem 3.1.** The classes \(\alpha \in [\Gamma, G]\) for which the Borsuk-Ulam property fails are precisely those fitting on a commutative diagram of groups

\[
\begin{array}{ccc}
\pi_1(\Gamma) & \xrightarrow{\varphi} & P_2(G) \\
\downarrow{\psi} & & \downarrow{\psi} \\
\pi_1(\Gamma/\tau) & \xrightarrow{\theta_1} & B_2(G) \\
\theta_1 & & \theta_2 \\
\mathbb{Z}_2 & = & \mathbb{Z}_2,
\end{array}
\]

for suitable morphisms \(\varphi\) and \(\psi\). Here the two central group inclusions are induced by the obvious 2-fold covering projections, while morphisms \(\theta_i\) are induced by the corresponding classifying maps. In particular, both downward vertical sequences are short exact.

**Example 3.2.** Assume \(G = T\), a tree (not homeomorphic to an interval). Then \(\theta_2\) is surjective. Since \(\pi_1(\Gamma/\tau)\) is free (see paragraph below), it is possible to choose a lifting \(\psi: \pi_1(\Gamma/\tau) \to B_2(G)\) of \(\theta_1\) along \(\theta_2\). The restricted map \(\varphi: \pi_1(\Gamma) \to P_2(G)\) then completes diagram (4) with \(\alpha_0 \in [\Gamma, G]\) necessarily the unique (trivial) homotopy class. This proves Theorem 1.3 when \(G\) is contractible. Therefore, throughout the rest of the section we assume that \(G \neq T\). In particular \(\pi_1(G) = F(z_1, \ldots, z_k)\), the free group on generators \(z_i = (x_i, y_i)\), where \(\{(x_i, y_i)\}_{i=1, \ldots, k}\) is the set of deleted edges of \(G\) (recall \(x_i < y_i\) for all \(i\), which gives the orientation of the representing loop for \(z_i\) —after collapsing \(T\) to a point). For convenience, we will assume that the deleted edges have been arranged so that \(y_1 < y_2 < \ldots < y_k\).

Ignoring vertices of degree 2, \(\tau\) is forced to act at the level of vertices and edges. Thus \(\Gamma/\tau\) has a natural graph structure. A simple Euler characteristic argument then show that the free groups \(\pi_1(\Gamma)\) and \(\pi_1(\Gamma/\tau)\) have respective ranks \(2m + 1\) and \(m + 1\), where \(m = -\chi(\Gamma)/2 \geq 0\). In particular,

\[
[\Gamma, G] = F(z_1, \ldots, z_k)^{2m+1}.
\]

(5)

More explicitly:

**Lemma 3.3.** It is possible to choose generators \(a, a_1, a_1', a_2, a_2', \ldots, a_m, a_m'\) of \(\pi_1(\Gamma)\) as well as generators \(c, c_1, c_2, \ldots, c_m\) of \(\pi_1(\Gamma/\tau)\) satisfying

\[
a = c^2, \; a_i = c_1, \; a_i' = c c_i^{-1}, \; \theta_1(c) = 1 \in \mathbb{Z}_2, \; \text{and} \; \theta_1(c_i) = 0 \in \mathbb{Z}_2
\]

for \(i = 1, 2, \ldots, m\). In this setting, \(\alpha \in [\Gamma, G]\) is identified under (5) with the tuple

\[
(\alpha_\#(a), \alpha_\#(a_1), \alpha_\#(a_1'), \ldots, \alpha_\#(a_m), \alpha_\#(a_m')).
\]
**Proof.** This lemma is part of the folklore. Since we cannot find an explicit reference, we sketch a proof. Suppose we have an epimorphism \( \theta : \pi_1(\Gamma/\tau) \to \mathbb{Z}_2 \) and let \( \{ e_0, e_1, \ldots, e_m \} \) be an arbitrary base. Assume without loss of generality that \( \theta(e_0) = 1 \in \mathbb{Z}_2 \) and set \( c := e_0 \). If \( \theta(e_i) = 0 \) for \( i > 0 \), then set \( c_i := e_i \). Otherwise, let \( c_i := e_0 \cdot e_i \). Then we have constructed a base \( \{ c, c_1, \ldots, c_m \} \) with the desired \( \theta \)-properties. Now we apply the Reidemeister-Schreier process to find a presentation of \( \ker(\theta) \), using the set of generators \( c, c_1, \ldots, c_m \) and, as Schreier system, \( \{ 1, c \} \). It follows that the kernel has a presentation given by elements as in the statement of the lemma subject to no relation. The result follows. \[ \square \]

**Proof of Theorem 1.4.** It is a standard fact that the right-hand side column in (4) becomes

\[
\pi_1(G) = \mathbb{Z} \xrightarrow{p_1=\text{Id}} \mathbb{Z} \xrightarrow{\theta_1} \mathbb{Z} \xrightarrow{\theta_2=\text{mod-2 proj}} \mathbb{Z}_2.
\]

Consequently, morphisms \( \psi : \pi_1(\Gamma/\tau) \to B_2(G) \) satisfying \( \theta_2 \circ \psi = \theta_1 \) are in one-to-one correspondence with tuples of integer numbers \( (p, q_1, \ldots, q_m) \), with \( p \) odd and each \( q_i \) even, where the correspondence is so that \( \psi(c) = p \) and \( \psi(c_i) = q_i \) for \( 1 \leq i \leq m \). Say \( q_i = 2p_i \). Lemma 3.3 and (6) then imply that the restriction to \( \pi_1(\Gamma) \) of such a \( \psi \) is given by \( \varphi(c) = p \) and \( \varphi(a_i) = p_i = \varphi(a'_i) \) for \( i = 1, 2, \ldots, m \). The result follows. \[ \square \]

The proof of Theorem 1.3 follows the strategy in the previous proof, except that the needed algebraic manipulations are far much subtler, and depend on the results in Subsection 2.3. With this in mind, we assume from this point on that \( G \) is not homeomorphic to a circle (or to a tree, in view of Example 3.2), and pick key elements \( \rho, \lambda_1, \lambda_2, \ldots, \lambda_k \in P_2(G) \) and \( \sigma \in B_2(G) \) as described below. We then set \( \lambda'_i := \sigma \lambda_i \sigma^{-1} \in P_2(G) \).

![Figure 1: Part of the non-linear tree $T$ showing the essential vertex $v$](image1)

When $T$ has an essential vertex $v$. Choose vertices \( v_1, v_2 \) with \( v < v_1 < v_2 \) and \( \tau(e_{v_1}) = v = \tau(e_{v_2}) \). See Figure 2. In this situation we choose the critical 1-cell connecting the trees \( DT_d \) and \( DT_u \) to be \((a, (b, c)) := (v_1, (v_2))\) (see Subsection 2.3), and set

\[
\sigma := \{ (v_1, (v, v_2)) \}, \quad \rho := ((v, v_2), v_1) \quad \text{and} \quad \lambda_i := (x_i + 1, (x_i, y_i)),
\]

for \( 1 \leq i \leq k \) (note that \( x_i + 1 < y_i \) since \( G \) is simplicial). Corollary 2.2 and Propositions 2.1 and 2.4 then give

\[
\rho = \sigma^2, \quad p_1(\rho) = 1, \quad p_1(\lambda_i) = 1 \quad \text{and} \quad p_1(\lambda'_i) = z_i, \quad \text{for} \ i = 1, 2, \ldots, k.
\]

Here and below, we write \( p_1 \) instead of \( (p_1)_\# \). The context clarifies the abuse of notation.
When $T$ is linear. In this situation we choose the critical 1-cell connecting the trees $DT_d$ and $DT_u$ to be $(a, (b, c)) := (x_1 + 1, (x_1, y_1))$. See Figure 2. Recall we have chosen $y_1 < y_i$ for $i > 1$. Furthermore, the condition $x_1 + 1 < y_1$ is forced because $G$ is simplicial. Then set

$$\sigma := \{x_1 + 1, (x_1, y_1)\}, \quad \rho := ((x_1, y_1), x_1 + 1), \quad \lambda_1 := (x_1', (x_1, y_1)) \quad \text{and} \quad \lambda_i := (x_i + 1, (x_i, y_i)),$$

for $2 \leq i \leq k$, where

$$x_1' = \begin{cases} 0, & \text{if } x_1 > 0; \\ y_1 + 1, & \text{if } x_1 = 0 \quad \text{(so } y_1 < n, \text{ even if } k = 1, \text{ since } G \text{ is not homeomorphic to a circle).} \end{cases}$$

Corollary 2.2 and Propositions 2.3 and 2.4 then give

$$\rho = \sigma^2, \quad p_1(\rho) = z_1, \quad p_1(\lambda_i) = z_1 \quad \text{and} \quad p_1(\lambda_i') = z_i z_1^{-1} \quad \text{for } i = 2, 3, \ldots, k. \quad (8)$$

**Proof of Theorem 3.3** Let $\lambda, \lambda'$ and $z$ denote, respectively, the sequences of symbols $(\lambda_1, \ldots, \lambda_k)$, $(\lambda_1', \ldots, \lambda_k')$ and $(z_1, \ldots, z_k)$, and consider an arbitrary sequence of words

$$(w(z), w_1(z), w_1'(z), \ldots, w_m(z), w_m'(z)) \in F(z_1, \ldots, z_k)^{2m+1}. \quad (9)$$

We have to prove that the homotopy class $\alpha \in [\Gamma, G]$ corresponding to (9) under (5) and Lemma 3.3 fits in a commutative diagram (4) for suitable morphisms $\psi$ and $\varphi$.

Assume $T$ has an essential vertex $v$ and consider the setup in (7). The map $\psi: \pi_1(\Gamma/\tau) \to B_2(G)$ defined by

$$\psi(c) = w(\lambda)\sigma \quad \text{and} \quad \psi(c_i) = \sigma w_i(\lambda)\sigma^{-1} w_i'(\lambda), \quad i = 1, \ldots, m,$$

satisfies $\theta_1 = \theta_2 \circ \psi$, in view of Proposition 2.3 and Lemma 3.3. Furthermore, the restricted map $\varphi: \pi_1(\Gamma) \to P_2(G)$ satisfies

$$\varphi(a) = \psi(c)^2 = w(\lambda)\sigma w(\lambda)\sigma = w(\lambda) \cdot \sigma w(\lambda)\sigma^{-1} \cdot \rho,$$

$$\varphi(a_i) = \psi(c_i) = \sigma w_i(\lambda)\sigma^{-1} \cdot w_i'(\lambda),$$

$$\varphi(a_i') = \psi(c_i') = \psi(c_i)^{-1} = w(\lambda)\sigma w_i(\lambda)\sigma^{-1} w_i'(\lambda)\sigma^{-1} w(\lambda)^{-1}$$

$$= w(\lambda)\rho w_i(\lambda)\rho^{-1} \cdot \sigma w_i'(\lambda)\sigma^{-1} \cdot w(\lambda)^{-1}.$$

So

$$p_1(\varphi(a)) = p_1(w(\lambda)) \cdot p_1(\sigma w(\lambda)\sigma^{-1}) \cdot p_1(\rho) = p_1(\sigma w(\lambda)\sigma^{-1}) = p_1(w(\lambda')) = w(z),$$

$$p_1(\varphi(a_i)) = p_1(\sigma w_i(\lambda)\sigma^{-1}) \cdot p_1(\rho) = p_1(\rho) = p_1(\rho) = p_1(\rho) = p_1(\rho) = p_1(\rho) = p_1(\rho) = p_1(\rho) = p_1(\rho) = p_1(\rho) = p_1(\rho) = p_1(w(\lambda')), \quad (9)$$

$$p_1(\varphi(a_i')) = p_1(w(\lambda)\rho w_i(\lambda)\rho^{-1}) \cdot p_1(\sigma w_i'(\lambda)\sigma^{-1}) \cdot p_1(w(\lambda)^{-1}) = p_1(w_i'(\lambda)\sigma^{-1} = p_1(w_i'(\lambda)) = w_i'(z),$$

which yields the result in the case under consideration.

Assume now that $T$ is linear and consider the setup in (5). Write the elements $w(z)z_1^{-1}$, $w_i(z)z_1^{-1}$ and $w_i'(z)z_1^{-1}$ of $F(z_1, z_2, \ldots, z_k)$ as words on the generators $t_1, t_2, \ldots, t_k$ of $F(z_1, z_2, \ldots, z_k)$ given by $t_1 := z_1$ and $t_i := z_i z_1^{-1}$ for $i \geq 2$. Say

$$w(z)z_1^{-1} = \ell(t), \quad w_i(z)z_1^{-1} = \ell_i(t), \quad w_i'(z)z_1^{-1} = \ell_i'(t),$$

where $t$ stands for the tuple $(t_1, t_2, \ldots, t_k)$. The map $\psi: \pi_1(\Gamma/\tau) \to B_2(G)$ defined by

$$\psi(c) = \ell(\lambda)\sigma \quad \text{and} \quad \psi(c_i) = \sigma \ell_i(\lambda)\sigma^{-1} \ell_i'(\lambda)\lambda_1, \quad i = 1, \ldots, m,$$
satisfies $\theta_1 = \theta_2 \circ \psi$, in view of Proposition 2.3 and Lemma 3.3. Furthermore, the restricted map $\varphi: \pi_1(\Gamma) \to P_2(G)$ satisfies

$$\varphi(a) = \psi(c)^2 = \ell(\lambda)\sigma\ell(\lambda)\sigma - \ell(\lambda) \cdot \sigma\ell(\lambda)\sigma^{-1} \cdot \rho,$$

$$\varphi(a_i) = \psi(c_i) = \sigma\ell_i(\lambda)\sigma\lambda_i^{-1}\ell_i(\lambda)\lambda_1 = \sigma\ell_i(\lambda)\sigma^{-1} \cdot \rho\lambda_i^{-1}\ell_i(\lambda)\lambda_1,$$

$$\varphi(a_i') = \psi(c)\psi(c_i)\psi(c)^{-1} = \ell(\lambda)\sigma\cdot \sigma\ell_i(\lambda)\sigma\lambda_i^{-1}\ell_i(\lambda)\lambda_1 \cdot \sigma^{-1}\ell(\lambda)^{-1}$$

$$= \ell(\lambda)\rho\ell_i(\lambda) \cdot \sigma\lambda_i^{-1}\ell_i(\lambda)\lambda_1\sigma^{-1} \cdot \ell(\lambda)^{-1}.$$ 

So

$$p_1(\varphi(a)) = p_1(\ell(\lambda)) \cdot p_1(\sigma\ell(\lambda)\sigma^{-1}) \cdot p_1(\rho) = p_1(\ell(\lambda'))z_1 = \ell(p_1(\lambda'))z_1 = \ell(t)z_1 = w(z),$$

$$p_1(\varphi(a_i)) = p_1(\sigma\ell_i(\lambda)\sigma^{-1}) \cdot p_1(\rho\lambda_i^{-1}\ell_i(\lambda)\lambda_1) = p_1(\sigma\ell_i(\lambda)\sigma^{-1})z_1 = p_1(\ell_i(\lambda'))z_1 = \ell_i(t)z_1 = w_i(z),$$

$$p_1(\varphi(a_i')) = p_1(\ell(\lambda)\rho\ell_i(\lambda)) \cdot p_1(\sigma\lambda_i^{-1}\ell_i(\lambda)\lambda_1\sigma^{-1}) \cdot p_1(\ell(\lambda)^{-1}) = z_1p_1(\sigma\lambda_i^{-1}\ell_i(\lambda)\lambda_1\sigma^{-1})$$

$$= z_1p_1((\lambda_i^{-1}\ell_i(\lambda)\lambda_1^{-1}) = z_1(p_1(\lambda_i^{-1})^{-1} \cdot p_1(\ell_i(\lambda')) \cdot p_1(\lambda_i^{-1}) = z_1z_1^{-1}\ell_i(t)z_1 = w_i(z),$$

which completes the proof.

\[\square\]

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