GENERALIZED SELECTIVE MODAL ANALYSIS

JULIÁN BARQUÍN *

Abstract. A new approach which generalizes the Selective Modal Analysis (SMA) and algorithms based upon it for solving the generalized eigenvalue problem is described. This approach allows for the systematic consideration of physical properties of the system under study. Two small application cases demonstrate the capabilities of the proposed approach.

Key words. eigenvalues, eigenvectors, eigenspaces, modal analysis

1. Review of SMA. Selective Modal Analysis (SMA) is a physically motivated framework for understanding, simplifying and analyzing complex linear time invariant models of dynamic systems. SMA can focus on selected portions of the structure and behaviour of the system [1, 2].

In many physical systems, it can be readily recognized that some set of modes is associated to a certain set of variables. For instance: in electrical power systems, the electromechanical oscillations are associated to the machines rotors’s angle and speed. Very often, this association is used, explicitly or implicitly, to simplify the mathematical models of the system under study. SMA aims to exploit this relationship in a systematic and rigorous way.

So, let us assume that it is desired to analyze the dynamic system:

\[
\dot{x} = Ax
\]  

The SMA approach is to classify the \( x \) components in relevant (\( r \)) and less-relevant (\( z \)) components. So, possibly after a trivial reordering, it can be written:

\[
x = \begin{bmatrix} r \\ z \end{bmatrix}
\]

Therefore, equation (1.1) can be written as:

\[
\begin{bmatrix} \dot{r} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_{rr} & A_{rz} \\ A_{zr} & A_{zz} \end{bmatrix} \begin{bmatrix} r \\ z \end{bmatrix}
\]  

Let us assume that there is interest in computing and eigenvalue \( \lambda \), and its left and right eigenvectors \( v \) and \( w \).

\[
\lambda v = Av
\]

\[
w^T \lambda = w^T A
\]

The eigenvectors can be partitioned analogously as the states \( x \):

\[
v = \begin{bmatrix} v_r \\ v_z \end{bmatrix} \quad w = \begin{bmatrix} w_r \\ w_z \end{bmatrix}
\]

*Instituto de Investigación Tecnológica, Universidad Pontificia Comillas de Madrid, C/ Alberto Aguilera, 23, E-28015 Madrid, Spain (barquin@iit.upco.es).
It is easy to check that it must be fulfilled that:

\[
\lambda v_r = (A_{rr} + A_{rz}(\lambda - A_{zz})^{-1}A_{sr}) \, v_r
\]  
\[
w_T^{\rho} \lambda = w_T^{\rho} (A_{rr} + A_{rz}(\lambda - A_{zz})^{-1}A_{sr})
\]  

(1.7)

(1.8)

So, the interesting eigenvalue \( \lambda \) is in the spectrum of the matrix \( A_{rr} + A_{rz}(\lambda - A_{zz})^{-1}A_{sr} \). On the other hand, if this mode is strongly correlated with the relevant variables \( r \), it should be expected that the spectrum of \( A_{rr} \) contains an eigenvalue quite similar to \( \lambda \) and, therefore, that the matrix \( A_{rz}(\lambda - A_{zz})^{-1}A_{sr} \) perturbs slightly the desired mode. That suggests the following algorithm:

**Algorithm 1**

**Input:** \( A_{rr}, A_{rz}, A_{sr}, A_{zz} \).

**Output:** \( \lambda, v_r, w_r \).

1. Perform the eigenanalysis of \( A_{rr} \),
2. Select the interesting mode \( 0 \lambda, 0 v_r, 0 w_r \),
3. for \( j = 1, 2, 3, \ldots \) until convergence,
   3.1. Compute \( H^{(j-1)} \lambda = A_{rz}(\lambda - A_{zz})^{-1}A_{sr} \),
   3.2. Perform the eigenanalysis of \( A_{rr} + H^{(j-1)} \lambda \),
   3.3. Select the interesting mode \( j \lambda, j v_r, j w_r \),
4. end

The convergence properties of this algorithm have been studied in [3]. The convergence is controlled by the eigenvectors. Essentially, the algorithm locally converges if and only if it is fulfilled:

\[
| \rho | = \left| \frac{w_T^T v_r}{w_Z^T v_z} \right| > 1
\]  

(1.9)

The number \( \rho \) is called the participation ratio. Notice that, for any eigenvalue, this ratio only depends in the way that the variables have been partitioned in relevant and less relevant.

The former algorithm can be generalized to search for several eigenvalues. The interesting eigenvalues shall be collected in a diagonal matrix \( \Lambda \), and the eigenvectors in matrices \( V \) and \( W \):

\[
\Lambda = \begin{bmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
0 & 0 & \cdots & \cdots \\
0 & 0 & \cdots & \lambda_n \\
\end{bmatrix}
\]

\[
V = [v_1, v_2, \ldots, v_n] \quad W = [w_1, w_2, \ldots, w_n]
\]  

(1.10)

The matrices \( V \) and \( W \) can be also partitioned in relevant and less relevant parts:

\[
V = \begin{bmatrix} V_r \\ V_z \end{bmatrix} \quad W = \begin{bmatrix} W_r \\ W_z \end{bmatrix}
\]  

(1.11)
Algorithm 2

Input: $A_{rr}, A_{rz}, A_{zr}, A_{zz}$.

Output: $\Lambda, V_r, W_r$.

1. Perform the eigenanalysis of $A_{rr}$,
2. Select the interesting modes $0\Lambda, 0V_r, 0W_r$,
3. for $j = 1, 2, 3, \ldots$ until convergence,
   3.1. Compute a matrix $jM$ which fulfills
   
   
   
   
   
   
   where $H(\lambda) = A_{rz}(\lambda - A_{zz})^{-1}A_{zr}$,
3.2. Perform the eigenanalysis of $A_{rr} + jM$,
3.3. Select the interesting modes $j\Lambda, jV_r, jW_r$,
4. end

The convergence conditions of algorithm 2 are also studied in [3], although they are considerably more involved than those of algorithm 1. However, the computational experience shows that the convergence is good if the participation ratios of the interesting modes are high.

From the point of view of the computational effort, the most demanding task of both algorithms is the step 3.1, which requires to solve linear system involving the matrix $\lambda - A_{zz}$. Most SMA applications have been developed for the study of electric power systems [4], where special techniques based upon the peculiar characteristics of these systems have been used to perform efficiently this task.

In addition to algorithms 1 and 2, there are a number of related ones which considerably improve their speed and robustness [5, 6]. However, these algorithms are outside of the scope of this paper.

2. Generalized SMA. Although SMA has been successfully used in applications, it has some drawbacks:

1. Very often, the computation of the desired modes begin with the study of a simplified model, in order to gain an insight on the interesting eigenstructure. Although SMA allows to use subsequently the information on the relevant variables, it does not make use of all the information obtained with the simplified model.

2. There are some problems, specially in continuous media, where it is known the overall shape of the desired modes, but it is impossible to ascribe them to a small number of system variables.

Both kind of problems are treated in this paper examples. The aim of this section is to generalize the classical SMA theory in order to deal with these problems.

So, the problem to solve is the eigenvalue problem:

\begin{align*}
\lambda E v &= A v \\
\omega \dagger E \lambda &= \omega \dagger A
\end{align*}

$A$ and $E$ are $m \times m$ real matrices. The matrix $E$ is a symmetric, possibly singular, projection matrix.
(2.3) \[ E^2 = E = E^T = E^\dagger \]

The superscript \( T \) denotes the transpose and \( \dagger \) the hermitian conjugate. It is assumed that the right eigenvector \( v \) approximately lies in the subspace spanned by \( \{e_1, e_2, \ldots, e_n\} \). Usually, \( n \ll m \). Then, it is defined the matrix

(2.4) \[ \mathcal{E} = [e_1, e_2, \ldots, e_n] \]

In similar way, the left eigenvector \( w \) is assumed to yield, approximately, in the subspace spanned by \( \{f_1, f_2, \ldots, f_n\} \). So, it is defined the matrix

(2.5) \[ \mathcal{F} = [f_1, f_2, \ldots, f_n] \]

Besides, the \( e_i \) and \( f_j \) basis are normalized in order to fulfill the equation:

(2.6) \[ \mathcal{F}^\dagger \mathcal{E} \mathcal{E} = I_n \]

\( I_n \) is the \( n \times n \) identity matrix. This equation can be enforced so long as no vector generated by the basis \( e_i \) or the basis \( f_j \) is included in the kernel of \( E \). This condition shall be assumed in the sequel.

Then, the eigenvectors \( v \) nd \( w \) can be written as:

(2.7) \[ v = \mathcal{E} \alpha + z \]
(2.8) \[ \mathcal{F}^\dagger E z = 0 \]
(2.9) \[ w = \mathcal{F} \beta + y \]
(2.10) \[ \mathcal{E}^\dagger E y = 0 \]

It is easy to show that the above decomposition exists and is unique. Then, after the algebraic manipulations shown in the appendix A, it is found that:

(2.11) \[ \lambda \alpha = A_{rr} \alpha + H(\lambda) \alpha \]

where

(2.12) \[ A_{rr} = \mathcal{F}^\dagger \mathcal{E} \]
(2.13) \[ H(\lambda) = \mathcal{F}^\dagger A \mathcal{P} \left\{ \lambda \mathcal{E} - A + [A, \mathcal{Q}]_+ \right\}^{-1} \mathcal{P} \mathcal{A} \mathcal{E} \]

The matrices \( \mathcal{Q} \) and \( \mathcal{P} \) are idempotent matrices defined by

(2.14) \[ \mathcal{P} = I_m - EE^\dagger E = I_m - \mathcal{Q} \]

and \( [\mathcal{Q}, A]_+ \) is the anti-commutator:
These formulae are the basic ones in selective modal analysis, and can be considered as a generalization of “classical” SMA, as shown in the appendix B. It is also easy to check that $\beta$ is the left eigenvector of $A_{rr} + H(\lambda)$. Notice that the dimension of the matrices $A_{rr}$ and $H(\lambda)$ is $n \ll m$.

It is also noteworthy that

$$z = P \left\{ \lambda E - A + [A, Q]_+ \right\}^{-1} PA^\alpha$$

$$y^\dagger = \beta^\dagger F^\dagger A \left\{ \lambda E - A + [A, Q]_+ \right\}^{-1} P$$

3. Algorithms based on $H(\lambda)$ computation. The aim of this section is to apply the results of the former section in order to obtain workable algorithms, and their convergence conditions. In order to simplify the notation, let us denote by $N(\lambda, Q)$ the matrix

$$N(\lambda, Q) = \{\lambda E - A + [A, Q]_+\}^{-1}$$

so that

$$H(\lambda) = F^\dagger A P \left\{ (j^{-1} \lambda E - A + [A, Q]_+) \right\}^{-1} PA^\epsilon = F^\dagger A P N(\lambda, Q) P A^\epsilon$$

3.1. Linear algorithm. Specifically, let us consider the following generalization of algorithm 1:

| Algorithm 3 |
|--------------|
| **Input:** $E, A, E, F$. |
| **Output:** $\lambda, \alpha, \beta$. |
| 1. Form $A_{rr} = F^\dagger A E$, and perform the eigenanalysis of $A_{rr}$, |
| 2. Select the interesting mode $^0\lambda, ^0\alpha, ^0\beta$, |
| 3. for $j = 1, 2, 3, \ldots$ until convergence, |
| 3.1. Compute $H(j^{-1} \lambda) = F^\dagger A P N(j^{-1} \lambda, Q) P A^\epsilon$. |
| 3.2. Perform the eigenanalysis of $A_{rr} + H(j^{-1} \lambda)$, |
| 3.3. Select the interesting mode $^j\lambda, ^j\alpha, ^j\beta$, |
| 4. end |

The following theorem states the conditions for the local convergence of Algorithm 3:

**Theorem 3.1.** Given an eigenvalue $\lambda$ of the pair $(E, A)$ with associated right and left eigenvectors $v$ and $w$, there is $\delta > 0$ such that if $\|v^\dagger \lambda - \lambda\| < \delta$, algorithm 3 converges to the eigenvalue $\lambda$ whenever

$$|\rho| = \left| \frac{w^\dagger Q v}{w^\dagger (E - Q) v} \right| > 1$$
Furthermore, the error $j\epsilon = j\lambda - \lambda$ fulfills:

\begin{equation}
    j\epsilon = \rho^{-1}j\epsilon + o(j^{-1}\epsilon)
\end{equation}

Proof. The proof is given in the appendix C. □

It is easy to check that $\rho$ just defined is, in the “classical” case, the same $\rho$ defined in equation (1.9).

3.2. Superlinear algorithm. If, in algorithm 3, the right eigenvector $v$ (respectively the left eigenvector $w$) is contained in the span of $\mathcal{E}$ ($\mathcal{F}$), then $z = 0$ ($y = 0$) and $\rho \to \infty$. So, the algorithm could be speeded up if the matrices $\mathcal{E}$ and $\mathcal{F}$ are updated in order that their span contains the last approximation to $v$ and $w$: $j\epsilon v$ and $j\epsilon w$. Therefore, the following algorithm is proposed:

**Algorithm 4**

**Input:** $E, A, 0\mathcal{E}, 0\mathcal{F}$.

**Output:** $\lambda, \alpha, \beta, v, w$.

1. Form $0A_{rr} = 0\mathcal{F}^\dagger A 0\mathcal{E}$, and perform the eigenanalysis of $A_{rr}$,
2. Select the interesting mode $0\lambda, 0\alpha, 0\beta$,
3. for $j = 1, 2, 3, \ldots$ until convergence,
   3.1. Compute $H(j^{-1}\lambda) =$ $j^{-1}\mathcal{F}^\dagger A j^{-1}P N(j^{-1}\lambda, j^{-1}Q) j^{-1}P A j^{-1}\mathcal{E}$,
   3.2. Perform the eigenanalysis of $j^{-1}A_{rr} + H(j^{-1}\lambda)$,
   3.3. Select the interesting mode $j\lambda, j\alpha, j\beta$,
   3.4. Compute $jz = j^{-1}P N(j^{-1}\lambda, j^{-1}Q) j^{-1}P A j^{-1}\mathcal{E} j\alpha$ and $jy\dagger = j\beta j^{-1}\mathcal{F}^\dagger A j^{-1}P N(j^{-1}\lambda, j^{-1}Q) j^{-1}P$,
   3.5. Compute $jv = j^{-1}\mathcal{E} j\alpha + jz$ and $jw\dagger = j\beta j^{-1}\mathcal{F}^\dagger + jy\dagger$,
   3.6. Update $j\mathcal{E}$, $j\mathcal{F}$ in such a way that $j\epsilon v \in \text{span}(j\mathcal{E}), j\epsilon w \in \text{span}(j\mathcal{F})$,
   3.7 Form $jA_{rr} = j\mathcal{F}^\dagger A j\mathcal{E}$,
4. end

A particular case of algorithm 4 is when the matrices $j\mathcal{E}$ and $j\mathcal{F}$ are vectors. Then, these matrices are essentially the estimated eigenvectors. The local convergence properties of the algorithm are, in this case, given by the following theorem:

**Theorem 3.2.** Given an eigenvalue $\lambda$ of the pair $(E, A)$ with associated right and left eigenvectors $v$ and $w$, if $\|N(\lambda, Evw\dagger E)\| < \infty$, there is a neighborhood of $\lambda, v, w$, such that if $0\lambda, 0v, 0w$ belong to it, algorithm 4 converges. Furthermore, it is fulfilled that asymptotically there is a constant $K$ such that

\begin{equation}
    |j\epsilon| \leq K |(j^{-1}\epsilon)|^{1 + \sqrt{2}}
\end{equation}

Proof. The proof is given in the appendix E. □

In the general case, whenever $j\mathcal{E}$ and $j\mathcal{F}$ are not vectors, it is expected that algorithm 4 converges at least so fast. This is because the relevant subspace is bigger, so that the approximation to the eigenstructure can not be worst.
3.3. $E$ and $F$ selection. The computation of $H(\lambda)$ requires to make a selection of the matrices $E$ and $F$. The following theorem can be used for this task:

**Theorem 3.3.**

The matrix $A_{rr} + H(\lambda)$ is invariant under the transformations $E \leftarrow E + (I_m - E)L$, or $F \leftarrow F + (I_m - E)M$, where $L$ or $M$ are arbitrary matrices of the same dimension than $E$ or $F$.

**Proof.**

The proof is provided in appendix D.

In many cases the $E$ matrix can be written as:

$$E = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

$I_r$ is a $r$-dimensional identity matrix. Therefore, the vectors and matrices can be partitioned in dynamic and static parts. Specifically,

$$E = \begin{bmatrix} E_d & E_s \end{bmatrix}$$
$$F = \begin{bmatrix} F_d & F_s \end{bmatrix}$$

The invariance of the matrix $A_{rr} + H(\lambda)$ under the considered transformations means that the value of the static components $E_s$ and $F_s$ is irrelevant in order to compute this matrix.

So, referring to algorithm 4, there are at least two possibilities:

1. To keep the whole eigenvector in step 3.4: update $jE$, $jF$ in such a way that $jv \in \text{span}(jE), jw \in \text{span}(jF)$. Then, as the algorithm converges, the matrix $H(\lambda) \to 0$.

2. To update the matrix as above, but the static components, which are zeroed ($E_s = 0$ and $F_s = 0$). As the matrix converges, the matrix $H(\lambda)$ converges to a non-zero value.

The second possibility can be useful in order to minimize the numerical effort.

4. $H(\lambda)$ computation. From the computational point of view, the most demanding task of the algorithm is the computation of the matrix $H(\lambda)$ or of the matrix $A_{rr} + H(\lambda)$. The purpose of this section is to propose algorithms to deal efficiently with this task.

4.1. $H(\lambda)$ computation using the Shermann-Morrison lemma. $H(\lambda)$ can be written as:

$$H(\lambda) = F^\dagger A P \left\{ \lambda E - A + [A, Q]_+ \right\}^{-1} PA E$$

The basic problem is related to the matrix

$$N(\lambda) = \left\{ \lambda E - A + [A, Q]_+ \right\}^{-1}$$
A problem is that, generally, the matrix $[A, Q]_+\] is not sparse. However, it is possible to obtain an expression of $N(\lambda)$, which allows computations by using only sparse matrices, by means of the Shermann-Morrison lemma. So,

$$N(\lambda) = \left\{ \lambda E - A + [A, Q]_+ \right\}^{-1}$$

$$= \left\{ \lambda E - A + [A, EE^T E]_+ \right\}^{-1}$$

$$= \left\{ \lambda E - A + \left( AE \right) \left( F^T E \right) + \left( EE \right) \left( F^T EA \right) \right\}^{-1}$$

$$= \left\{ \lambda E - A + \eta \phi^\dagger - \eta \phi^\dagger + \left( AE \right) \left( F^T E \right) + \left( EE \right) \left( F^T EA \right) \right\}^{-1}$$

$$= \left\{ \lambda E - A + \eta \phi^\dagger - \left[ \begin{array}{cc} \eta & -AE \end{array} \right] \left[ \begin{array}{c} -EE \\ E \end{array} \right] \right\}^{-1}$$

$$= \left\{ \lambda E - A + \eta \phi^\dagger - \left[ \begin{array}{cc} \eta & -AE \end{array} \right] \right\}^{-1}$$

$$\eta \text{ and } \phi \text{ are two sparse vectors which make sure that the matrix } \lambda E - A + \eta \phi^\dagger \text{ is regular even if } \lambda \text{ is an eigenvalue of the pair } (A, E). \text{ This is going to happen when the SMA algorithm converges. Now, from the Shermann-Morrison lemma:}$$

$$N(\lambda) = \left\{ \lambda E - A + \eta \phi^\dagger \right\}^{-1} + \left\{ \lambda E - A + \eta \phi^\dagger \right\}^{-1} \left[ \begin{array}{cc} \eta & -AE \end{array} \right] \left[ \begin{array}{c} -EE \\ E \end{array} \right] \left( I_{2n+1} - \left[ \begin{array}{c} \phi^\dagger \\ F^T E \\ F^T EA \end{array} \right] \left\{ \lambda E - A + \eta \phi^\dagger \right\}^{-1} \left[ \begin{array}{cc} \eta & -AE \end{array} \right] \right)$$

$$= \left[ \begin{array}{c} \phi^\dagger \\ F^T E \\ F^T EA \end{array} \right] \left\{ \lambda E - A + \eta \phi^\dagger \right\}^{-1}$$

The number $2n + 1$ is usually small. So, it is only required to know the LU factorization of a filled matrix of small $(2n + 1)$ dimension and of the sparse matrix $\lambda E - A + \eta \phi^\dagger$.

**4.2. Composite models.** In power systems analysis, the system to analyze is a set of dynamical subsystems connected through a static relationship. Specifically, there are $l$ subsystems

$$E_k \dot{x}_{MK} = A_k x_{MK} + B_k x_{IK}$$

$$x_{OK} = C_k x_{MK} + D_k x_{IK} \quad k = 1, \ldots, l$$

The variables $x_{MK}$ are the state variables of the $k$-th subsystem, $x_{IK}$ are the input variables and $x_{OK}$ the output variables. It is assumed that the number of output and input variables of each system is equal. It is also assumed that the matrices $E_k$ are symmetric projection real matrices:

$$E_k = E_k^T = E_k^\dagger = E_k^2$$
Let us define the vectors

\[
  x_M = \begin{bmatrix}
    x_{M1} \\
    x_{M2} \\
    \vdots \\
    x_{Ml}
  \end{bmatrix}
  \quad x_I = \begin{bmatrix}
    x_{I1} \\
    x_{I2} \\
    \vdots \\
    x_{II}
  \end{bmatrix}
  \quad x_O = \begin{bmatrix}
    x_{O1} \\
    x_{O2} \\
    \vdots \\
    x_{Ol}
  \end{bmatrix}
\]

In addition to these equations, there is also a static interconnection:

\[
  \begin{bmatrix}
    J_{11} & J_{12} \\
    J_{21} & J_{22}
  \end{bmatrix}
  \begin{bmatrix}
    x_I \\
    x_A
  \end{bmatrix}
  =
  \begin{bmatrix}
    x_O \\
    0
  \end{bmatrix}
\]

\(x_A\) is a set of additional algebraic variables.

For this kind of systems, it is convenient to consider the following \(E\) and \(F\) matrices:

\[
  E = \begin{bmatrix}
    E_M \\
    0 \\
    0
  \end{bmatrix}
  \quad F = \begin{bmatrix}
    F_M \\
    0 \\
    0
  \end{bmatrix}
\]

which mimics the \(x\) structure. Furthermore, \(E_M\) and \(F_M\) are defined on a subsystem basis:

\[
  E_M = \begin{bmatrix}
    E_{M1} & 0 & \ldots & 0 \\
    0 & E_{M2} & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & E_{Ml}
  \end{bmatrix} = \text{diag}(E_{M1}, E_{M2}, \ldots, E_{Ml})
\]

\[
  F_M = \begin{bmatrix}
    F_{M1} & 0 & \ldots & 0 \\
    0 & F_{M2} & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & F_{Ml}
  \end{bmatrix} = \text{diag}(F_{M1}, F_{M2}, \ldots, F_{Ml})
\]

Then, as proved in the appendix, the matrix \(H(\lambda)\) can be computed as

\[
  H(\lambda) = H_A(\lambda) + [(B_r + H_B(\lambda)) \ 0] \begin{bmatrix}
    J_{11} - (D + H_D(\lambda)) & J_{12} \\
    J_{21} & J_{22}
  \end{bmatrix}^{-1} \begin{bmatrix}
    C_r + H_C(\lambda) \\
    0
  \end{bmatrix}
\]

where all the matrices are computed in a subsystem basis:

\[
  A_r = \text{diag}(A_{r1}, \ldots, A_{rl})
\]

\[
  B_r = \text{diag}(B_{r1}, \ldots, B_{rl})
\]

\[
  C_r = \text{diag}(C_{r1}, \ldots, C_{rl})
\]

\[
  D = \text{diag}(D_1, \ldots, D_l)
\]
(4.23) \( H_A = \text{diag}(H_{A1} \ldots H_{A1}) \)
(4.24) \( H_B = \text{diag}(H_{B1} \ldots H_{B1}) \)
(4.25) \( H_C = \text{diag}(H_{C1} \ldots H_{C1}) \)
(4.26) \( H_D = \text{diag}(H_{D1} \ldots H_{D1}) \)

5. **Direct algorithms.** As said above, the most difficult task in order to apply SMA is the computation of the matrix \( H(\lambda) \). However, the proposed algorithms can be formulated without needing to compute this matrix. The purpose of this section is to explain the way of doing it.

5.1. **Single eigenvalue algorithms.** The basic SMA formulae are:

\[
\begin{align*}
\lambda^j & = (A_{rr} + H(j^{-1})^j) \lambda^j \\
\beta^j & = (A_{rr} + H(j^{-1})) \\
\alpha & = \frac{\lambda E - A + [A, Q]}{\lambda E - A + [A, Q]} \\
y & = \beta^j \alpha \\
\end{align*}
\]

as shown in previous sections. Let us also, as above, define the vectors

\[
\begin{align*}
\gamma & = \alpha + \beta \\
\gamma & = \beta \alpha \\
\end{align*}
\]

Then, the following theorem can be stated:

**Theorem 5.1.** Let us assume that equations (5.1-5.4) are fulfilled. Then, the vectors \( \gamma \) and \( \gamma \) fulfill the equation

\[
\begin{align*}
A - j^{-1} \lambda (E - Q) & = \lambda Q \gamma \\
A - j^{-1} \lambda (E - Q) & = \lambda Q \gamma \\
\end{align*}
\]

**Proof.** The proof is provided in Appendix H.

Let us write equation (5.7) in the following equivalent form

\[
\begin{align*}
(A - j^{-1} \lambda E) \gamma & = (\lambda - j^{-1} \lambda) Q \gamma \\
\end{align*}
\]

As \( Q \gamma = 0 \), it is obtained

\[
Q \gamma = E \gamma \alpha
\]

Therefore

\[
(A - j^{-1} \lambda E) \gamma = (\lambda - j^{-1} \lambda) E \gamma \alpha
\]

Let us define
\[ jV = (A - j^{-1} \lambda E)^{-1} EE \]  

So

\[ jv = jVj\alpha (j\lambda - j^{-1}\lambda) \]  

Analogously,

\[ jW^\dagger = F^\dagger E (A - j^{-1} \lambda E)^{-1} \]  
\[ jw^\dagger = (j\lambda - j^{-1}\lambda) j^\beta jW^\dagger \]  

Now, let us consider the matrix \( jW^\dagger (A - j^{-1} \lambda E) jV \).

\[ jW^\dagger (A - j^{-1} \lambda E) jVj\alpha = F^\dagger E jV (j\lambda - j^{-1}\lambda)^{-1} = (j\lambda - j^{-1}\lambda)^{-1} j\alpha \]  

Therefore, \( j\alpha \) is a right eigenvector of this matrix with associated eigenvalue \( (j\lambda - j^{-1}\lambda)^{-1} \). It can be easily shown that \( j\beta \) is the left eigenvector. Note that, because of (5.14), this matrix can be also written as \( F^\dagger E jV \).

Equations (5.12), (5.14) y (5.16) provide an alternative way of performing the SMA iteration. Specifically, algorithm 3 can be also written as:

| Algorithm 5 |
|---|
| **Input:** \( E, A, E, F \). |
| **Output:** \( \lambda, \alpha, \beta \). |
| 1. Form \( A_{rr} = F^\dagger A E \), and perform the eigenanalysis of \( A_{rr} \), |
| 2. Select the interesting mode \( 0\alpha, 0\beta \), |
| 3. for \( j = 1, 2, 3, \ldots \) until convergence, |
| 3.1. Compute \( jV = (A - j^{-1} \lambda E)^{-1} EE \), |
| 3.2. Compute \( jW^\dagger = F^\dagger E (A - j^{-1} \lambda E)^{-1} \), |
| 3.3. Perform the eigenanalysis of \( jM = F^\dagger E jV \), |
| 3.4. Select the interesting mode \( j\alpha, j\beta, (j\lambda - j^{-1}\lambda)^{-1} \), |
| 3.5. Update \( j\lambda \), |
| 4. end |

As this algorithm is essentially the same one that Algorithm 3, its convergence conditions are the same ones. Note also that step 3.2 could be omitted. Analogously, Algorithm 4 can be written as:
Algorithm 6

Input: $E, A, _0^0E, _0^0F$.

Output: $\lambda, \alpha, \beta, v, w$.

1. Form $^0A_{rr} = _0^0F^\dagger A _0^0E$, and perform the eigenanalysis of $^0A_{rr}$.
2. Select the interesting mode $^0\lambda, _0^0\alpha, _0^0\beta$.
3. for $j = 1, 2, 3, \ldots$ until convergence,
   3.1. Compute $\bar{j}V = (A - j^{-1}\lambda E)^{-1}E_{j-1}E$,
   3.2. Compute $\bar{j}W^\dagger = j^{-1}\bar{F}^\dagger E (A - j^{-1}\lambda E)^{-1}$
   3.3. Perform the eigenanalysis of $\bar{j}M = j^{-1}\bar{F}^\dagger E\bar{j}V$,
   3.4. Select the interesting mode $\bar{j}\alpha, \bar{j}\beta, (j\lambda - j^{-1}\lambda)^{-1}$,
   3.5. Update $\bar{j}\lambda$,
   3.6. Update $\bar{j}E, \bar{j}F$ in such a way that $\bar{j}V \in \text{span}(\bar{j}E), \bar{j}W \in \text{span}(\bar{j}F)$,
4. end

Step 3.6 warrants that $\bar{j}v \in \text{span}(\bar{j}E), \bar{j}w \in \text{span}(\bar{j}F)$. The simplest way to achieve that is to set $\bar{j}E = \bar{j}V$ and $\bar{j}F = \bar{j}W$ but, possibly, a normalization constant.

As previously as Algorithm 5, convergence results related to Algorithm 4 can be directly applied to Algorithm 6.

5.2. Several eigenvalues algorithms. In this section algorithm 2 is written in the direct formulation, and a new one which includes relevant subspaces updated is also proposed. However, no convergence results are provided.

Let us firstly define the matrix

$$jV = \left[A - j^{-1}\lambda (E - Q)\right]^{-1}E$$

It is clear, from (5.17) and (5.10):

$$jv = j\lambda\bar{V}j\alpha$$

Analogously, it is possible to write:

$$\bar{W}^\dagger = \bar{F}^\dagger E \left[A - j^{-1}\lambda (E - Q)\right]^{-1}$$

$$j\alpha = j\beta\bar{W}^\dagger j\lambda$$

Let is consider the matrix

$$N^{-1} = \bar{W}^\dagger \left[A - j^{-1}\lambda (E - Q)\right]^{-1}jV = \bar{F}^\dagger E\bar{V} = jW^\dagger E$$

It is fulfilled, because of (5.18):

$$N^{-1}j\alpha = \bar{F}^\dagger E\bar{V}j\alpha = j\lambda^{-1}E_{j}v$$

$$j\lambda^{-1}E_{j}v$$

$$j\lambda^{-1}\alpha$$
Analogously,

\[ j \beta^1 N^{-1} = j \beta^1 \chi^{-1} \]  

(5.25)

Therefore, the matrix \( j N \) contains the sought eigenvalue in its spectrum. On the other hand, it can be shown (see appendix I) that

\[ j N = j M^{-1} + j^{-1} \lambda I_n \]  

(5.26)

where \( I_n \) is the identity matrix. Besides, from (5.1) and (5.2),

\[ H(j^{-1} \lambda) \beta = (j N - A_{rr}) j \alpha \]  

(5.27)

\[ j \beta^1 H(j^{-1} \lambda) = j \beta^1 (j N - A_{rr}) \]  

(5.28)

In fact, a stronger result can be obtained:

**Theorem 5.2.** \( H(\lambda) = N - A_{rr} \)

*Proof.* The proof is provided in the appendix J.

Therefore, it is proposed the following generalization of algorithm 2:

**Algorithm 7**

**Input:** \( E, A, \mathcal{E}, \mathcal{F} \).

**Output:** \( \lambda_k, \alpha_k, \beta_k \).

1. Form \( A_{rr} = \mathcal{F}^\dagger A \mathcal{E} \), and perform the eigenanalysis of \( A_{rr} \),
2. Select the interesting modes \( 0 \lambda_k, 0 \alpha_k, 0 \beta_k \),
3. for \( j = 1, 2, 3, \ldots \) until convergence,
   3.1 For each eigenvalue \( k = 1, \ldots, K \),
      3.1.1. Compute \( j V_k = (A - j^{-1} \lambda_k E)^{-1} EE \),
      3.1.2. Compute \( j M_k = \mathcal{F}^\dagger E j V_k \),
      3.1.3. Compute \( j N_k = j M_k^{-1} + j^{-1} \lambda_k I_n \),
      3.1.4 Compute \( j h_k = (j N_k - A_{rr}) j^{-1} \alpha_k \),
   3.2 Compute a matrix \( j M \) which fullfills \( j M [j^{-1} \alpha_1, \ldots, j^{-1} \alpha_K] = [j h_1, \ldots, j h_K] \)
   3.3 Perform the eigenanalysis of \( A_{rr} + j M \),
   3.4 Select the interesting modes \( j \lambda_k, j \alpha_k, j \beta_k \),
4. end

This algorithm is nothing else that the direct version of algorithm 2, and it reduces to it if the \( \mathcal{E} \) and \( \mathcal{F} \) matrices are chosen as shown in appendix B. A superlinear version of this algorithm, by updating the \( \mathcal{E} \) and \( \mathcal{F} \) matrices, is also proposed:
Algorithm 8

**Input:** $E, A, 0^E, 0^F$.

**Output:** $\lambda_k, \alpha_k, \beta_k$.

1. Form $0^A = 0^F A 0^E$, and perform the eigenanalysis of $0^A$.
2. Select the interesting modes $0^\lambda_k, 0^\alpha_k, 0^\beta_k$.
3. for $j = 1, 2, 3, \ldots$ until convergence,
   3.1 For each eigenvalue $k = 1, \ldots, K$,
      3.1.1. Compute $j^V_k = (A - j^{-1} \lambda_k E)^{-1} E j^{-1} E$,
      3.1.2. Compute $j^W_k = j^{-1} F^\dagger E (A - j^{-1} \lambda_k E)^{-1}$,
      3.1.3. Compute $j^M_k = F^\dagger E j^V_k$,
      3.1.4. Compute $j^N_k = j^M_k + j^{-1} \lambda_k I_n$,
      3.1.5 Compute $j^h_k = (j^N_k - j^{-1} A_{rr}) j^{-1} \alpha_k$,
   3.2 Compute a matrix $j^M$ which fulfills
      $j^M [j^{-1} \alpha_1, \ldots, j^{-1} \alpha_K] = [j^h_1, \ldots, j^h_K]$,
   3.3 Perform the eigenanalysis of $j^{-1} A_{rr} + j^M$,
   3.4 Select the interesting modes $j^\lambda_k, j^\alpha_k, j^\beta_k$,
   3.5 For each eigenvalue $k = 1, \ldots, K$, compute
      $j^v_k = j^V_k j^\alpha_k$, and
      $j^w_k = j^\beta_k j^W_k$,
   3.6 Update $j^E$ and $j^F$ in such a way that
      $j^v_1, \ldots, j^v_K \in \text{span}(j^E)$,
      $j^w_1, \ldots, j^w_K \in \text{span}(j^F)$,
   3.7 Compute $j^A = j^F A j^E$,
   3.8 Compute $j^\alpha_k$ and $j^\beta_k$ by imposing
      $j^E j^\alpha_k = j^v_k$ and $j^F j^\beta_k = j^w_k$,
4. end

Step 3.5 requires to pair each eigenvalue obtained of the eigenanalysis of $A_{rr} + j^M$ with the eigenvalues of the previous iterations. A way to do that is to pair trying to maximize the scalar products $j^\alpha_k j^\alpha_{k_2}$.

6. Numerical tests. The aim of this section is to apply Generalized SMA to two very different problems: the computation of the natural frequencies of a plate and the computation of the electromechanical modes of an electric power system. The code was developed in MATLAB language.

6.1. Natural frequencies of a cross-shaped plate. It is intended to compute a natural frequency in a cross-shaped plate with unequal arms. Mathematically, the problem to solve is

$-\Delta \psi = \omega^2 \psi \quad \psi(x, y) \in \Omega$  \hspace{1cm} (6.1)

where $\Omega$ is the cross-shaped dominion, $\omega$ the natural frequency and $\psi$ the sought mode. In addition, the following boundary conditions must be fulfilled:

$\psi(x, y) = 0 \quad \forall (x, y) \in \partial \Omega$  \hspace{1cm} (6.2)
The previous partial differential equation is approximated by a finite differences scheme. In order to apply algorithms 5, 6, 7 and 8, it is decided that:

1. As it is intended to solve a symmetrical problem, matrices $E$ and $F$ are taken to be real and equal. Furthermore, the hermitian operator $\dagger$ can be substituted by the transpose operator $T$, and the algorithms can be programmed in real, instead of complex, algebra.

2. In algorithms 5 and 7, matrix $E$ is a vector which approximates the sought mode. In algorithms 6 and 8, matrix $E$ is initially that same vector, and it is updated in each iteration to the last mode estimation $\lambda v$.

It was decided to compute the mode corresponding to the upper arm of the cross oscillating against the lower one. The initial mode estimation for algorithms 5 and 6 is shown in figure (6.1), as well as the computed mode. Notice that the algorithms converge to the mode whose shape is closest to the initial assumption, being any initial assumption of the value of the sought eigenvalue largely irrelevant.

The eigenvalue is $\lambda = \omega^2 = 0.1157$. Figure 6.2 shows the evolution of the absolute value of the error $j\lambda - \lambda$ for both algorithms 5 and 6.

To apply algorithms 7 and 8 an additional mode $\omega^2 = 0.1243$, corresponding to the right arm oscillating against the left arm, was computed. Figure 6.3 shows the evolution of the absolute value of the error (algorithm 7 in solid lines and algorithm 8 in dotted lines).

6.2. Electromechanical modes of an electric power system. The electric power system represented in figure 6.4 (a simplified model of the New England electric power system) was analyzed by using SMA. The circles represent electric generators and the lines the electric transmission lines. The electric generators are modelled by 9th to 11th order linear systems, whilst the electric network is modelled as an algebraic constraint. Therefore, the system is a composite system, as explained in the previous section.

The generators input variables $x_{Ik}$ are the axis and quadrature terminal current (two variables per generator), and the output variables $x_{Ok}$ the axis and quadrature...
terminal voltage (two variables per generator). The algebraic variables \( x_A \) are network voltages. The state variables \( x_{Mk} \) are mechanical, electromagnetic and control generator variables.

The following \textit{physical} information is known \textit{a priori}:

1. The most troublesome modes are those called electromechanical modes. These modes are related to the generators angle \( \delta_k \) and speed \( \omega_k \). It is fulfilled that \( \dot{\delta}_k = 120 \pi \omega_k \). The relationship of \( \dot{\omega}_k \) with the rest of the variables is much more complex. However, in a very rough approximation, the subsystem \((\delta_k, \omega_k)\) can be considered a damped pendulum.

2. It is known that the frequencies of the electromechanical eigenvalues are in the order of 1 Hz.

3. The electromechanical modes can be understood as oscillations of one generator or group of generators against other generator or group of generators.

From fact 1, it follows that the \( \delta_k \) and \( \omega_k \) right eigenvector components fulfill:

\[
\lambda v(\delta_k) = 120 \pi v(\omega_k) \tag{6.3}
\]

On the other hand, if the rough simple pendulum model is assumed, it should be fulfilled

\[
\lambda^t w(\omega_k) = 120 \pi w(\delta_k) \tag{6.4}
\]
Besides, it is known, from fact 2, that $\lambda \approx 2\pi i$. So, it is decided that, when applying algorithm 3:

$$E_{Mk} = \begin{bmatrix} 120\pi \\ 2\pi i \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

(6.5)

but a normalization constant. When applying algorithm 4, these ones are the initial values of $E_{Mk}$ and $F_{Mk}$.

It is desired to compute the electromechanical mode corresponding to the generators 1, 2, 3, 8 and 10 oscillating against the 4, 5, 6, 7 and 9 (the East side against the West side). As the reducid matrix $A_r + H(\lambda)$ is a $10 \times 10$ matrix (because each $E_{Mk}$ and $F_{Mk}$ is a vector and there are 10 generators), it is needed to select the relevant eigenvalue and eigenvectors resulting from the factorization of the reduced matrix (see Algorithm 3 and 4). The choosen procedure is as follows:
It is defined an “objective” \( \alpha_o \):

\[
\alpha_o = [1, 1, 1, -1, -1, -1, -1, -1, 1]^T
\]

which represents the generators oscillating as described above. After performing the \( j^{-1}A_r + H(j^{-1}\lambda) \) eigenanalysis it is obtained the matrix spectrum \( j\lambda_k \) and right eigenvectors \( j\alpha_k \), where \( k = 1, \ldots, 10 \) (the generator number). The products

\[
p_k = \alpha_o^\dagger 0E_M^\dagger j^{-1}E_M j\alpha_k
\]

are computed, and it is selected the mode which maximizes \( \|p_k\| \):

\[
k = \arg \max \|p_k\|
\]

\[
\begin{align*}
nj\lambda &= nj\lambda_k \\
nj\alpha &= nj\alpha_k \\
nj\beta &= nj\beta_k
\end{align*}
\]

Note that, in the case of algorithm 3, the matrices \( jE \) are constant and equal to the initial one \( 0E \).

In algorithm 4, the vector \( jv_M \) is computed in each iteration by using (G.14, G.22, G.56, G.57), and also \( jv_M \) by using analogous formulae. The matrices \( E \) and \( F \) are updated according to:

\[
\begin{align*}
jE_M k &= jv_M k \\
jF_M k &= jw_M k
\end{align*}
\]

but, possibly, a normalization constant.

Figure 6.5 shows the modulus and phase of the right and left eigenvectors, and figure 6.6 the modulus and phase of the \( \delta_k \) and \( \omega_k \) components of the eigenvectors. The vertical dotted lines segregates the variables belonging to different generators. Figure 6.7 shows the error evolution. The computed eigenvalue is \(-0.2617 + 6.4017i \).
7. Conclusions. In this article a new approach for solving the generalized eigenvalue problem has been introduced. The introduced algorithms can make efficient use of physical information regarding the shape of the sought eigenvectors.

Appendix A. Proof of the main results.

From (2.1) and (2.7) it is obtained

\[ \lambda E \xi \alpha + \lambda E z = A E \xi \alpha + A z \]

Premultiplying (A.1) by \( F^\dagger \)

\[ \lambda F^\dagger E \xi \alpha + \lambda F^\dagger Ez = F^\dagger A E \xi \alpha + F^\dagger A z \]

But \( F^\dagger Ez = 0 \) and \( F^\dagger E \xi = I_n \). So

\[ \lambda \alpha = F^\dagger A E \xi \alpha + F^\dagger A z \]

Note that, but the last term \( F^\dagger A z \), this equation is an eigensystem of the \( n \times n \) matrix \( F^\dagger A E \). This matrix is usually much smaller than \( A \).

Introducing the matrices defined in (2.14):

\[ \mathcal{P} = I_m - E E^\dagger E = I_m - Q \]

These matrices are idempotent ones (\( \mathcal{P} = \mathcal{P}^2, Q = Q^2 \)). Premultiplying (A.1) by \( \mathcal{P} \), and taking into account that...
Figure 6.6. Right and left eigenvectors. $\delta$ and $\omega$ components.

Figure 6.7. Error evolution.
(A.5) \[ \mathcal{P}E\mathcal{E} = 0 \]
(A.6) \[ \mathcal{P}Ez = Ez \]

it yields

(A.7) \[ \lambda Ez = \mathcal{P}A\mathcal{E}\alpha + \mathcal{P}Az \]

So,

(A.8) \[ (\lambda E - A + QA)z = \mathcal{P}A\mathcal{E}\alpha \]

From this equation, it is possible to solve \( z \) in function of \( \alpha \). In the same way, it is obtained

(A.9) \[ y^\dagger(\lambda E - A + AQ) = \beta^\dagger\mathcal{F}^\dagger\mathcal{A}\mathcal{P} \]

Taking into account that \( Qz = 0 \) and \( y^\dagger Q = 0 \), these formulae can be written in may different ways, The most symmetrical one is:

(A.10) \[ (\lambda E - A + [Q, A]_+)z = \mathcal{P}A\mathcal{E}\alpha \]
(A.11) \[ y^\dagger(\lambda E - A + [Q, A]_+) = \beta^\dagger\mathcal{F}^\dagger\mathcal{A}\mathcal{P} \]

Therefore, from (A.10):

(A.12) \[ z = (\lambda E - A + [A, Q]_+)^{-1}\mathcal{P}A\mathcal{E}\alpha \]
(A.13) \[ y^\dagger = \beta^\dagger\mathcal{F}^\dagger\mathcal{A}\mathcal{P} (\lambda E - A + [A, Q]_+)^{-1} \]

assuming that the inverse matrix exists. Sufficient conditions will be provided in appendix F. On the other hand, it is easy to check that \( \mathcal{P}z = z, y^\dagger\mathcal{P} = y^\dagger \). Therefore

(A.14) \[ z = \mathcal{P} (\lambda E - A + [A, Q]_+)^{-1}\mathcal{P}A\mathcal{E}\alpha \]
(A.15) \[ y^\dagger = \beta^\dagger\mathcal{F}^\dagger\mathcal{A}\mathcal{P} (\lambda E - A + [A, Q]_+)^{-1} \]

and equation (A.3) becomes

(A.16) \[ \lambda\alpha = \mathcal{F}^\dagger A\mathcal{E}\alpha + \mathcal{F}^\dagger\mathcal{A}\mathcal{P} (\lambda E - A + [A, Q]_+)^{-1}\mathcal{P}A\mathcal{E}\alpha \]
(A.17) \[ = A_{rr}\alpha + H(\lambda)\alpha \]

Note also, that by solving \( z \) from (A.8), it is obtained:

(A.18) \[ H(\lambda) = \mathcal{F}^\dagger\mathcal{A}\mathcal{P} (\lambda E - A + QA)^{-1}\mathcal{P}A\mathcal{E} \]
and by solving $y$ from \( A.9 \):

\[
H(\lambda) = \mathcal{F}^\dagger A \mathcal{P} \{ \lambda E - A + AQ \}^{-1} \mathcal{P} AE
\]

These ones are just some few of the many equivalent ways to write $H(\lambda)$. Some of them may be more amenable for computation than \( 2.13 \).

**Appendix B. The relationship with “classical” SMA.**

The formulae \( 2.12, 2.13 \) are just the generalized version of the “classical” SMA (Selective Modal Analysis) formulae, as defined in section 1.

To show the relationship, consider the problem \( 1.1 \). Let also assume that $A, v, w$ are partitioned according \( 1.3, 1.6 \). Therefore, the relevant subspaces in the generalized version are:

\[
E = \begin{bmatrix}
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1 \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0
\end{bmatrix} = \begin{bmatrix} I_n \end{bmatrix} = \mathcal{F}
\]

So

\[
\mathcal{F}^\dagger A \mathcal{E} = \begin{bmatrix} I_n \end{bmatrix} \begin{bmatrix} A_{rr} & A_{rz} \\
A_{zr} & A_{zz} \end{bmatrix} \begin{bmatrix} I_n \\
0 \end{bmatrix} = A_{rr}
\]

as suggested by the notation. On the other hand

\[
\mathcal{E} \mathcal{F}^\dagger = Q = \begin{bmatrix} I_n \\
0 \end{bmatrix} \begin{bmatrix} I_n \\
0 \end{bmatrix} = \begin{bmatrix} I_n & 0 \end{bmatrix}
\]

Therefore

\[
\begin{bmatrix} A, Q \end{bmatrix} = A_{rr} A_{rz} + A_{zr} A_{zz} = 2A_{rr} A_{rz} + A_{zr} A_{zz}
\]

Then,

\[
\lambda - A + [A, Q]_{+} = \begin{bmatrix} \lambda + A_{rr} & 0 \\
0 & \lambda - A_{zz} \end{bmatrix}
\]

\[
\{\lambda - A + [A, Q]_{+}\}^{-1} = \begin{bmatrix} (\lambda + A_{rr})^{-1} & 0 \\
0 & (\lambda - A_{zz})^{-1} \end{bmatrix}
\]
Besides

\[ \mathcal{P} = I_m - Q = \begin{bmatrix} 0 & 0 \\ 0 & I_{m-n} \end{bmatrix} \]

So

\[
\mathcal{P} \mathcal{A} \mathcal{E} = \begin{bmatrix} 0 & 0 \\ 0 & I_{m-n} \end{bmatrix} \begin{bmatrix} A_{rr} & A_{rz} \\ A_{zr} & A_{zz} \end{bmatrix} \begin{bmatrix} I_n \\ 0 \end{bmatrix} \\
= \begin{bmatrix} 0 & 0 \\ 0 & I_{m-n} \end{bmatrix} \begin{bmatrix} A_{rr} \\ A_{zr} \end{bmatrix} = \begin{bmatrix} 0 \\ A_{zr} \end{bmatrix}
\]

(B.7)

\[
\mathcal{F}^\dagger \mathcal{A} \mathcal{P} = \begin{bmatrix} I_n \\ 0 \end{bmatrix} \begin{bmatrix} A_{rr} & A_{rz} \\ A_{zr} & A_{zz} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I_{m-n} \end{bmatrix} \\
= \begin{bmatrix} A_{rr} & A_{rz} \\ 0 & 0 \\ 0 & I_{m-n} \end{bmatrix} \\
= \begin{bmatrix} 0 \\ A_{rz} \end{bmatrix}
\]

(B.8)

And, finally

\[
H(\lambda) = \left( \mathcal{F}^\dagger \mathcal{E} \mathcal{P} \right) \left\{ \lambda - A + [A, Q]_+ \right\}^{-1} \left( \mathcal{P} \mathcal{A} \mathcal{E} \right) \\
= \begin{bmatrix} 0 & A_{rz} \end{bmatrix} \begin{bmatrix} (\lambda + A_{rr})^{-1} & 0 \\ 0 & (\lambda - A_{zz})^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ A_{zr} \end{bmatrix} \\
= A_{rz} \left( \lambda - A_{zz} \right)^{-1} A_{zr}
\]

(B.9)

Appendix C. Proof of theorem 3.1.

Proof. Let us recall \( j \epsilon = j \lambda - \lambda \). \( \lambda \) is in the spectrum of \( A_{rr} + H(\lambda) \) and, because algorithm 3, \( j \lambda \) is in the spectrum of \( A_{rr} + H(j^{-1}\lambda) \). But

\[
A_{rr} + H(j^{-1}\lambda) = A_{rr} + H(\lambda) + \frac{\partial H(\lambda)}{\partial \lambda} j^{-1}\epsilon + o(j^{-1}\epsilon)
\]

By applying a well-known perturbation formula:

\[
j \lambda - \lambda = j \epsilon = \frac{\beta \alpha}{\beta' \alpha'} j^{-1}\epsilon + o(j^{-1}\epsilon)
\]

Let us define

\[
\rho^{-1} = -\frac{\beta \alpha}{\beta' \alpha'}
\]

If \( \lambda \) and \( j^{-1}\lambda \) are close enough, it is possible to neglect the higher order terms \( o(e^{j^{-1}}) \). Then, it is clear that the algorithm converges (\( e^j \to 0 \)) if and only if \( |\rho^{-1}| < 1 \).
On the other hand, from (2.13), (A.14) and (A.15):

\[-\beta^\dagger \frac{\partial H(\lambda)}{\partial \lambda} \alpha = \beta F^\dagger A P \left\{ \lambda E - A + [A, Q]_+ \right\}^{-1} E \left\{ \lambda E - A + [A, Q]_+ \right\}^{-1} \mathcal{F} A E \alpha = y^\dagger E z \]

(C.4)

Therefore

\[\rho = \frac{\beta^\dagger \alpha}{y^\dagger E z} \]

(C.5)

But, premultiplying (2.7) and postmultiplying (2.9) by \( Q \):

\[Qv = QE \alpha + Qz = EE(F^\dagger EE)\alpha + EE(F^\dagger Ez) \]

(C.6)

\[w^\dagger Q = \beta^\dagger F^\dagger + y^\dagger = \beta^\dagger (F^\dagger EE) F^\dagger E + (y^\dagger EE) F^\dagger E \]

(C.7)

So

\[w^\dagger Qv = w^\dagger QQv = \beta^\dagger F^\dagger EE \alpha = \beta^\dagger \alpha \]

(C.8)

On the other hand,

\[w^\dagger Ev = (\beta^\dagger F^\dagger + y^\dagger) E(\epsilon \alpha + z) = \beta^\dagger F^\dagger EE \alpha + \beta^\dagger (F^\dagger Ez) + (y^\dagger EE) \alpha + y^\dagger Ez = \beta^\dagger \alpha + y^\dagger Ez \]

(C.9)

So

\[y^\dagger Ez = w^\dagger (E - Q)v \]

(C.10)

and

\[\rho = \frac{w^\dagger Qv}{w^\dagger (E - Q)v} \]

(C.11)

\[\boxdot\]

Appendix D. Proof of theorem 3.3.
Proof. The invariance with respect to transformations \(\mathcal{E} \leftarrow \mathcal{E} + (I_m - E) \mathcal{L}\) shall be proven, being the other case essentially equal.

Firstly, note that \(\mathcal{Q}\) and \(\mathcal{P}\) are invariant under the transformation

\[
E(E + (I_m - E) \mathcal{L}) \mathcal{F}^\dagger E = E \mathcal{E} \mathcal{F}^\dagger E + E(I_m - E) \mathcal{L} \mathcal{F}^\dagger E = E \mathcal{E} \mathcal{F}^\dagger E
\]

which proves the \(\mathcal{Q}\) invariance. As \(\mathcal{P} = I_m - \mathcal{Q}\), \(\mathcal{P}\) is also invariant. Let us consider now:

\[
\mathcal{P}A(I_m - E) \mathcal{L} = \mathcal{P}A(I_m - E) \mathcal{L} \tag{D.2}
\]

\[
-(I_m - E) \mathcal{L} = \{\lambda E - A + QA\}^{-1} \mathcal{P}A(I_m - E) \mathcal{L} \tag{D.3}
\]

\[
-(I_m - E) \mathcal{L} = \mathcal{P} \{\lambda E - A + QA\}^{-1} \mathcal{P}A(I_m - E) \mathcal{L} \tag{D.4}
\]

\[
-I^\dagger A(I_m - E) \mathcal{L} = \mathcal{F}^\dagger \mathcal{P} \{\lambda E - A + QA\}^{-1} \mathcal{P}A(I_m - E) \mathcal{L} \tag{D.5}
\]

\[
-I^\dagger A(I_m - E) \mathcal{L} = \mathcal{F}^\dagger \mathcal{P} \{\lambda E - A + QA\}^{-1} \mathcal{P}A(I_m - E) \mathcal{L} \tag{D.6}
\]

\[
-I^\dagger A(I_m - E) \mathcal{L} = \mathcal{F}^\dagger \mathcal{P} \{\lambda E - A + QA\}^{-1} \mathcal{P}A(I_m - E) \mathcal{L} \tag{D.7}
\]

It has been used that \(\mathcal{P} = I_m - \mathcal{Q}\) to go from (D.2) to (D.3), that \(E(I_m - E) = 0\) to go from (D.3) to (D.4) and that that \(\mathcal{P}(I_m - E) = (I_m - \mathcal{Q})(I_m - E) = I_m - E\) to go from (D.3) to (D.6).

On the other hand under the considered transformation \(A_{rr} + H(\lambda)\) becomes

\[
\mathcal{F}^\dagger A(E + (I_m - E) \mathcal{L}) + \mathcal{F}^\dagger \mathcal{P} \{\lambda E - A + QA\}^{-1} \mathcal{P}A(E + (I_m - E) \mathcal{L}) = \mathcal{F}^\dagger A \mathcal{E} + \mathcal{F}^\dagger \mathcal{P} \{\lambda E - A + QA\}^{-1} \mathcal{P}A \mathcal{E} + \mathcal{F}^\dagger \mathcal{P} \{\lambda E - A + QA\}^{-1} \mathcal{P}A \mathcal{E} \tag{D.8}
\]

\[
\mathcal{F}^\dagger A(I_m - E) \mathcal{L} + \mathcal{F}^\dagger \mathcal{P} \{\lambda E - A + QA\}^{-1} \mathcal{P}A(I_m - E) \mathcal{L} = \mathcal{F}^\dagger A \mathcal{E} + \mathcal{F}^\dagger \mathcal{P} \{\lambda E - A + QA\}^{-1} \mathcal{P}A \mathcal{E} - \mathcal{F}^\dagger \mathcal{P} \{\lambda E - A + QA\}^{-1} \mathcal{P}A \mathcal{E} \tag{D.9}
\]

It has been used the \(H(\lambda)\) formula \((A.18)\) to write (D.8), and (D.7) to simplify it.

\section*{Appendix E. Proof of theorem \(E.2\)}

The net effect of steps 3.4 and 3.5 is to impose

\[
\mathcal{N}^i v = (I + \mathcal{P}N(\lambda, \mathcal{Q})^i \mathcal{P}A)^i v \tag{E.1}
\]

This is because matrix \(\mathcal{E}\) is proportional to the vector \(\mathcal{E} v\). \(\mathcal{N}^i v\) is a normalization constant. In fact, algorithm 4 is invariant with respect to arbitrary normalizations of vectors \(\mathcal{E} v\) and \(\mathcal{Q} v\) (only the normalized matrices \(\mathcal{E}\), \(\mathcal{F}\), \(\mathcal{Q}\) and \(\mathcal{P}\) are required).

In order to fix the normalization \(\mathcal{N} v\), let us focus in equations \((2.7)\) and \((A.14)\).

\[
v = (I + \mathcal{P}N(\lambda, \mathcal{Q})^i \mathcal{P}A)^i v \tag{E.2}
\]

\(v\) is a right eigenvector. Then, the normalization constant \(\mathcal{N} v\) is implicitly chosen by enacting that this formula is valid for all \(i\), with the same eigenvector (i.e., with
the same phase and absolute value). Of course, there are analogous formulae for the left eigenvectors. Furthermore, it is required that

\[ w^\dagger E v = 1 \]  

(E.3)

Now, from (D.5) it can be deduced that

\[ j^N_v E^{j+1} v = (E + E^j P N(j \lambda, j^Q) E^j P A) E^j v \]  

(E.4)

\[ E v = (E + E^j P N(j \lambda, j^Q) E^j P A) E^j v \]  

(E.5)

\[ E v = E^j v + E^j z \]  

(E.6)

The last line uses the error \( j^z = v - j z \),

\[ j^z = j^P N(j \lambda, j^Q) E^j P A^j v \]  

(E.7)

\[ E = N(j \lambda, j^Q) E^j P A^j v \]  

(E.8)

The last equation follows from (A.12). Let us define the eigenvalue error \( j^\epsilon = \lambda - j \lambda \). It is easy to check that

\[ \frac{\partial N(j \lambda, j^Q)}{\partial \lambda} = -N(j \lambda, j^Q) E N(j \lambda, j^Q) \]  

(E.9)

\[ \frac{\partial^2 N(j \lambda, j^Q)}{\partial \lambda^2} = 2N(j \lambda, j^Q) E N(j \lambda, j^Q) E N(j \lambda, j^Q) \]  

(E.10)

\[ ; \cdots ; \]  

Then substracting (E.5) from (E.4):

\[ j^N_v E^{j+1} v - E v \]  

\[ = E \{ j^P N(j \lambda, j^Q) - j^P N(j \lambda, j^Q) \} E^j P A^j v \]  

\[ = E \{ -j^P N(j \lambda, j^Q) E N(j \lambda, j^Q) j^\epsilon + j^P N(j \lambda, j^Q) E N(j \lambda, j^Q) j^\epsilon^2 + \ldots \} E^j P A^j v \]  

\[ = E \{ -j^P N(j \lambda, j^Q) j^\epsilon + j^P N(j \lambda, j^Q) E N(j \lambda, j^Q) j^\epsilon^2 + \ldots \} E^j P A^j v \]  

\[ = E \{ -j^P N(j \lambda, j^Q) j^\epsilon + j^P N(j \lambda, j^Q) E N(j \lambda, j^Q) j^\epsilon^2 + \ldots \} j^P E^j v \]  

\[ = E \{ -j^P N(j \lambda, j^Q) j^\epsilon + j^P N(j \lambda, j^Q) E N(j \lambda, j^Q) j^\epsilon^2 + \ldots \} j^P E^j z \]  

\[ = E \{ -j^P N(j \lambda, j^Q) j^\epsilon + j^P N(j \lambda, j^Q) E N(j \lambda, j^Q) j^\epsilon^2 + \ldots \} j^P E^j z \]  

\[ = E \{ -j^P N(j \lambda, j^Q) j^\epsilon + j^P N(j \lambda, j^Q) E N(j \lambda, j^Q) j^\epsilon^2 + \ldots \} j^P E^j z \]  

\[ = E \{ -j^P N(j \lambda, j^Q) j^\epsilon + j^P N(j \lambda, j^Q) E N(j \lambda, j^Q) j^\epsilon^2 + \ldots \} j^P E^j z \]  

(E.11)

\[ = E R_z (j^Q, j^\epsilon) E^j z \]

In order to derive the last three equations it has been used that \( j^P j^z = j^z \) and that \( j^P E = (j^P E)^2 \). The last equation is the definition of matrix \( R_z \). This matrix is well defined so long as the matrix \( N(j \lambda, j^Q) \) is bounded and \( j^\epsilon \) is small enough.
Sufficient conditions shall be discussed later on. On the other hand, it is obvious that:

(E.12) \[ R_z(\{ Q, \} ) = P R_z(\{ Q, \} ) = R_z(\{ Q, \} )^j P \]

Let us denote by \( Q \) the matrix formed with the eigenvectors

(E.13) \[ Q = E v w^\dagger E \]

Then

(E.14) \[ E R_z(\{ Q, \} ) Q^j z = E R_z(\{ Q, \} )^j P Q^j z \]
\[ = E R_z(\{ Q, \} )^j P E v w^\dagger E^j z \]
\[ = (w^\dagger E^j z) E R_z(\{ Q, \} )^j P E (v + z) \]
\[ = (w^\dagger E^j z) E R_z(\{ Q, \} ) E^j z \]

So

(E.15) \[ E R_z(\{ Q, \} ) (E - Q)^j z = \left[ 1 - (w^\dagger E^j z) \right] E R_z(\{ Q, \} ) E^j z \]

On the other hand

\[ j N_v E^{j+1} v - E v = j N_v (v - j^{+1} z) - E v \]
\[ = (j N_v - 1) E v - j N_v E^{j+1} z \]
\[ = E R_z(\{ Q, \} ) E^j z \]

(E.16)

So, premultiplying by \( E - Q \), taking into account that \( (E - Q)v = 0 \) and (E.13), it yields

(E.17) \[ (E - Q)^j z = \frac{-(E - Q) R_z(\{ Q, \} ) (E - Q)^j z}{j N_v [1 - (w^\dagger E^j z)]} \]

Premultiplying (E.11)

(E.18) \[ j N_v w^\dagger E^{j+1} v = 1 + w^\dagger E R_z(\{ Q, \} ) E^j z \]
\[ = 1 + \frac{w^\dagger E R_z(\{ Q, \} ) (E - Q)^j z}{[1 - (w^\dagger E^j z)]} \]

But,

(E.19) \[ w^\dagger E^j P = (j w^\dagger + j y^\dagger) E^j P = j y^\dagger E^j P \]
In this formula, it has been used the error $jy = w - jw$. The dual equations of (E.7,E.8) are:

\begin{align*}
\text{(E.20)} & \quad jy^\dagger = jw^\dagger A^\dagger PN(\lambda, jQ)^jP \\
\text{(E.21)} & \quad = jw^\dagger A^\dagger PN(\lambda, jQ)
\end{align*}

Besides

\begin{align*}
\text{(E.22)} & \quad jy^\dagger Q^jP = (jy^\dagger Ev)(w^\dagger E^jP) = (jy^\dagger Ev)(jy^\dagger E^jP) \\
\text{(E.23)} & \quad jy^\dagger (E - Q)^jP = [1 - (jy^\dagger Ev)] jy^\dagger E^jP
\end{align*}

and

\begin{align*}
\text{(E.24)} & \quad w^\dagger E^jP = \frac{jy^\dagger (E - Q)^jP}{[1 - (jy^\dagger Ev)]}
\end{align*}

So, as

\begin{align*}
w^\dagger ER_z (jQ, j\epsilon) & = w^\dagger E^jP R_z (jQ, j\epsilon) \\
& = \frac{jy^\dagger (E - Q)^jP R_z (jQ, j\epsilon)}{[1 - (jy^\dagger Ev)]} \\
& = \frac{jy^\dagger (E - Q) R_z (jQ, j\epsilon)}{[1 - (jy^\dagger Ev)]}
\end{align*}

equation (E.18) yields:

\begin{align*}
\text{(E.26)} & \quad jN_v w^\dagger E^{j+1} v = 1 + \frac{jy^\dagger (E - Q) R_z (jQ, j\epsilon)(E - Q)^jz}{[1 - (jy^\dagger Ev)] [1 - (w^\dagger E^jz)]}
\end{align*}

But

\begin{align*}
\text{(E.27)} & \quad 1 - w^\dagger E^j z = w^\dagger Ev - w^\dagger E^j z = w^\dagger E^j v
\end{align*}

and

\begin{align*}
\text{(E.28)} & \quad 1 - jy^\dagger Ev = w^\dagger Ev - jy^\dagger Ev = jw^\dagger Ev
\end{align*}

On the other hand, from (E.3) and (E.20),

\begin{align*}
\text{(E.29)} & \quad jw^\dagger E^j z = 0 = jy^\dagger E^j v
\end{align*}
So,

\[ j w^\dagger Ev = j w^\dagger E^j v + j z = j w^\dagger E^j v = w^\dagger E^j v \]  

Therefore, from (E.26)

\[ jNc^j w^\dagger E^{j+1}v = 1 + \frac{j y^\dagger (E - Q)R_z(jQ, j\epsilon)(E - Q)^2 z}{(j w^\dagger E^j v)^2} \]  

Substituting in (E.17)

\[ \frac{1}{j+1 w E^j v} (E - Q)^{j+1} z = \frac{-(E - Q)R_z(jQ, j\epsilon)(E - Q)^2 z}{1 + \frac{j y^\dagger (E - Q)R_z(jQ, j\epsilon)(E - Q)^2 z}{(j w^\dagger E^j v)^2}} \]  

Let us define the vectors

\[ j \tilde{z} = \frac{1}{j w E^j v} (E - Q)^j z \]  
\[ j \tilde{y}^\dagger = \frac{1}{j w E^j v} j y^\dagger (E - Q) \]  

Then, equation (E.32) can be written as

\[ j+1 \tilde{z} = \frac{-(E - Q)R_z(jQ, j\epsilon)(E - Q)^2 z}{1 + \frac{j y^\dagger R_z(jQ, j\epsilon)^2 z}{(j w E^j v)^2}} \]  

There is also a dual equation

\[ j+1 \tilde{y}^\dagger = j \tilde{y}^\dagger - \frac{R_z(jQ, j\epsilon)(E - Q)}{1 + \frac{j y^\dagger R_z(jQ, j\epsilon)^2 z}{(j w E^j v)^2}} \]  

Now, let us write the partitipation factor \( j \rho \) in terms of these new error vectors. From (C.11)

\[ j \rho^{-1} = \frac{w^\dagger (E - jQ)v}{w^\dagger jQ v} \]

Taking into account that

\[ jQ = \frac{1}{j w^\dagger E^j v} E^j v^j w^\dagger E \]

it is easy to check that

\[ j \rho^{-1} = \frac{j w^\dagger (E - Q)^j v}{j w^\dagger Q^j v} \]
But

\[ j w^\dagger (E - Q)^j v = (w^\dagger - j y^\dagger) (E - Q) (v - j z) = j y^\dagger (E - Q)^j z \]

because \( w^\dagger (E - Q) = 0 \) and \( (E - Q)v = 0 \). Besides,

\[ j w^\dagger Q^j v = (j w^\dagger E v)(w^\dagger E^j v) = (j w^\dagger E^j v)^2 \]

So

\[ j \rho^{-1} = \frac{j y^\dagger (E - Q)^j z}{(j w^\dagger E^j v)^2} = j \tilde{y}^\dagger j \tilde{z} \]

(Remember that \( (E - Q)^2 = E - Q \)). Convergence results will be proved from formulae \( E.35 \), \( E.36 \) and \( E.42 \).

Some bounds will be derived. The 2-norm, denoted as \( \| \cdot \| \), will be used in the sequel. Firstly, note that

\[ \| E - Q \| \leq 1 \]

In fact, let us consider an arbitrary vector \( \tilde{v} \). As \( E \) is a projection matrix

\[ \| E \tilde{v} \| \leq \| \tilde{v} \| \]

On the other hand, \( \tilde{v} \) can be decomposed in a component lying on the eigenvector \( v \) and a perpendicular component \( v_\perp \):

\[ \tilde{v} = a v + b v_\perp \]

So

\[ \| (E - Q) \tilde{v} \| = \| a (E - Q)v + b (E - Q)v_\perp \| = \| b Ev_\perp \| \leq \| bv_\perp \| \leq \| \tilde{v} \| \]

Let us assume that

\[ \| j \tilde{z} \| \leq \delta_z, \quad \| j \tilde{y} \| \leq \delta_y \]

Then

\[ \| (E - Q)^j z \| \leq \delta_z, \quad \| j y^\dagger (E - Q) \| \leq \delta_y \]

As \( \| E - Q \| \leq 1 \),
(E.49) \[ \| j^2 z \| = \| v - j v \| \leq \delta_z \| j w^\dagger E^j v \| \]

(E.50) \[ \| j^2 y \| = \| w - j w \| \leq \delta_y \| j w^\dagger E^j v \| \]

As, because of (E.29),

(E.51) \[ 1 = w^\dagger E v = (j w^\dagger + j y^\dagger) E (j v + j z) = j w^\dagger E^j v + j y^\dagger E^j z \]

it is fulfilled

(E.52) \[ \| j^2 w^\dagger E^j v \| \leq 1 + \| j^2 y^\dagger E^j z \| \]

Therefore, from (E.49) and (E.50),

(E.53) \[ \| j^2 w^\dagger E^j v \| \leq 1 + \delta_z \delta_y \| j^2 w^\dagger E^j v \|^2 \]

so

(E.54) \[ \| j^2 w^\dagger E^j v \| \leq \frac{1 + \sqrt{1 - 4 \delta_z \delta_y}}{2} \leq 1 \]

And (E.49) and (E.50) yield

(E.55) \[ \| j^2 z \| = \| v - j v \| \leq \delta_z \]

(E.56) \[ \| j^2 y \| = \| w - j w \| \leq \delta_y \]

So

(E.57) \[ \| E^j v \| \leq \| E v \| + \delta_z \]

(E.58) \[ \| j^2 w^\dagger E \| \leq \| w^\dagger E \| + \delta_y \]

Up to now, eigenvectors \( v \) and \( w \) are arbitrary subject to the condition \( w^\dagger E v = 1 \).

In the sequel, they will be chosen such that

(E.59) \[ K = \| E v \| = \| E w^\dagger \| , \ w^\dagger E v = 1 \]

Then,

\[
Q - j Q = E v w^\dagger E - E^j v^\dagger w^\dagger E = E \left( v - j v \right) w^\dagger E + E^j v \left( w^\dagger - j w^\dagger \right) E = E^j z w^\dagger E + E^j v^\dagger y E
\]

(E.60)

So, being \( m \) the dimension of \( Q \),
\[ \| Q - jQ \| = \| E_j z E + E_j v y E \| \]
\[ \leq \| E_j z E \| + \| E_j v y E \| \]
\[ \leq m^2 \left[ \| E_j z \| \| w^1 E \| + \| E_j v \| \| y E \| \right] \]
\[ \leq m^2 \left[ \delta_z K + (K + \delta_z) \delta_y \right] \]
\[ = m^2 \left[ \delta_y + \delta_z \right] K + \delta_z \delta_y ] \]

(E.61)

Let us consider the definition of the matrix \( R_z (jQ, j\epsilon) \). It is clear, because of continuity with respect to \( j\epsilon \), that

(E.62) \[ \exists \delta_{z_1} > 0 \| j\epsilon \| < \delta_{z_1} \Rightarrow \| R_z (jQ, j\epsilon) \| \leq \sqrt{2} \| E_j^j P N (\lambda, jQ) j^j P E \| \| j\epsilon \| \]

But

(E.63) \[ \| E_j^j P N (\lambda, jQ) j^j P E \| \leq \| E_j^j P \| \| N (\lambda, jQ) \| \| j^j P E \| \]
\[ \leq \| E - jQ \| \| N (\lambda, jQ) \| \| E - jQ \| \]
\[ \leq \| N (\lambda, jQ) \| \]

So

(E.64) \[ \exists \delta_{z_1} > 0 \| j\epsilon \| < \delta_{z_1} \Rightarrow \| R_z (jQ, j\epsilon) \| \leq \sqrt{2} \| N (\lambda, jQ) \| \| j\epsilon \| \]

Because of (E.61), \( Q \) and \( jQ \) are close whenever \( j\tilde{z} \) and \( j\tilde{y} \) are small. Therefore,

(E.65) \[ \exists \delta_{z_1} > 0 \| j\tilde{z} \| < \delta_{z_1} \land \| j\tilde{y} \| < \delta_{z_1} \Rightarrow \| N (\lambda, jQ) \| \leq \sqrt{2} \| N (\lambda, Q) \| \]

So,

(E.66) \[ \exists \delta_{z_1} > 0 \land \exists \delta_{z_1} > 0 \| j\epsilon \| < \delta_{z_1} \land \| j\tilde{z} \| < \delta_{z_1} \land \| j\tilde{y} \| < \delta_{z_1} \Rightarrow \]
\[ \| R_z (jQ, j\epsilon) \| \leq 2 \| N (\lambda, Q) \| \| j\epsilon \| \]

Bounds on \( N (\lambda, Q) \) will be provided in the next appendix. Besides, from (E.35) and (E.66)

(E.67) \[ \| j^{+1} \tilde{z} \| \leq \left[ 1 + 2 \| j\tilde{y} \| + R_z (jQ, j\epsilon) j^j z \| \right] \| E - Q \| \| R_z (jQ, j\epsilon) \| \| j\epsilon \| \| j\tilde{z} \| \]
\[ \| j^{+1} \tilde{y} \| \leq \left[ 1 + 2 \| j\tilde{y} \| + R_z (jQ, j\epsilon) j^j z \| \right] \| E - Q \| \| R_z (jQ, j\epsilon) \| \| j\epsilon \| \| j\tilde{y} \| \]

whenever \( \| j\tilde{y} \| + R_z (jQ, j\epsilon) j^j z \| < 1 \). Therefore,

(E.69) \[ \exists \delta_{z_1} > 0 \land \exists \delta_{z_1} > 0 \| j\epsilon \| < \delta_{z_1} \land \| j\tilde{z} \| < \delta_{z_1} \land \| j\tilde{y} \| < \delta_{z_1} \Rightarrow \]
\[ \left\{ \begin{array}{l}
\| j^{+1} \tilde{z} \| \leq 2 \left[ 1 + 4 \| N (\lambda, Q) \| \delta_{z_1} \delta_{\tilde{z}_1} \right] \| N (\lambda, Q) \| \| j\epsilon \| \| j\tilde{z} \|
\| j^{+1} \tilde{y} \| \leq 2 \left[ 1 + 4 \| N (\lambda, Q) \| \delta_{z_1} \delta_{\tilde{z}_1} \right] \| N (\lambda, Q) \| \| j\epsilon \| \| j\tilde{y} \|
\end{array} \right. \]
Let us define

(E.70) \[ \delta_{z_2} = \min\{\delta_{z_1}, \sqrt{\frac{1}{4\|N(\lambda, Q)\|\delta_{e_1}}}, \} > 0 \]

Then,

(E.71) \[ \exists \delta_{e_1} > 0 \land \exists \delta_{z_2} > 0 \backslash \|\epsilon\| < \delta_{e_1} \land \|\hat{z}\| < \delta_{z_2} \land \|\hat{y}\| < \delta_{z_2} \Rightarrow \]
\[ \begin{cases} \|z^{+1}\| \leq 4\|N(\lambda, Q)\|\|\epsilon\|\|\hat{z}\| \\ \|y^{+1}\| \leq 4\|N(\lambda, Q)\|\|\epsilon\|\|\hat{y}\| \\ \|\epsilon^{+1}\| \leq 2\|\hat{z}\|\|\hat{y}\|\|\epsilon\| \end{cases} \]

On the other hand, from (C.2), it is clear that

(E.72) \[ \exists \delta_{e_2} > 0 \land \frac{2}{\|\epsilon\|} < \delta_{e_2} \Rightarrow \frac{1}{4} \leq \|z^{+1}\| \leq 2\|\hat{z}\|\|\hat{y}\|\|\epsilon\| \]

because of (E.42). Let us define

(E.73) \[ \delta_{e_3} = \min\{\delta_{e_1}, \delta_{e_2}\} > 0 \]

Then, from (E.71) and (E.73),

(E.74) \[ \exists \delta_{e_3} > 0 \land \exists \delta_{z_2} > 0 \backslash \|\epsilon\| < \delta_{e_3} \land \|\hat{z}\| < \delta_{e_2} \land \|\hat{y}\| < \delta_{z_2} \Rightarrow \]
\[ \begin{cases} \|z^{+1}\| \leq 4\|N(\lambda, Q)\|\|\epsilon\|\|\hat{z}\| \\ \|y^{+1}\| \leq 4\|N(\lambda, Q)\|\|\epsilon\|\|\hat{y}\| \\ \|\epsilon^{+1}\| \leq 2\|\hat{z}\|\|\hat{y}\|\|\epsilon\| \end{cases} \]

Let us now define

(E.75) \[ \delta_{e_4} = \min\{\delta_{e_3}, \frac{1}{8\|N(\lambda, Q)\|}\} > 0 \land \delta_{z_3} = \min\{\delta_{z_2}, \frac{1}{2}\} \]

Then

(E.76) \[ \exists \delta_{e_4} > 0 \land \exists \delta_{z_3} > 0 \backslash \|\epsilon\| < \delta_{e_4} \land \|\hat{z}\| < \delta_{z_3} \land \|\hat{y}\| < \delta_{z_3} \Rightarrow \]
\[ \begin{cases} \|z^{+1}\| \leq 4\|N(\lambda, Q)\|\|\epsilon\|\|\hat{z}\| \\ \|y^{+1}\| \leq 4\|N(\lambda, Q)\|\|\epsilon\|\|\hat{y}\| \\ \|\epsilon^{+1}\| \leq 2\|\hat{z}\|\|\hat{y}\|\|\epsilon\| \end{cases} \]

which implies that the algorithm converges if initialized in a neighborhood of the solution \((\lambda, v, w)\). In fact, if \(\|\hat{z}\| < \|w^tE^2v\|\delta_{z_3}\|\), then, because of (E.33) \(\|\hat{z}\| < \delta_{z_3}\),

There is an analogous formula for \(\|\hat{y}\|\). In that case, (E.76) applies and the algorithm converges.

In order to assess the quotient convergence factor, let us write (E.74) as
By substituting \( \leq \) by = a majorant succession is obtained. Its asymptotic behaviour is controlled by the dominant eigenvalue of the state matrix, which happens to be \( 1 + \sqrt{2} \). Therefore, it is expected that, asymptotically

\[
\log \| j+1 \tilde{z} \| \leq (1 + \sqrt{2}) \log \| j \tilde{z} \| \tag{E.78}
\]

which completes the proof of the theorem.

**Appendix F. Some bounds on \( N(\lambda, Q) \).**

The purpose of this appendix is to provide some bounds on the matrix \( N(\lambda, Q) \), when \( \lambda \) is an eigenvector of the pair \((E, A)\) and the matrix \( Q \) is formed from its right and left eigenvectors: \( Q = Evw^\dagger E \).

It shall be assumed that the pair \((E, A)\) is solvable [7], that is, the pencil \( \mu E - A \) is regular for all \( \mu \) but a finite number. Of course one of these \( \mu \) is the sought eigenvalue \( \lambda \). Moreover, it shall be assumed that \( \lambda \) is a single eigenvalue.

Let us assume that

\[
N(\lambda, Q)^{-1} = \lambda E - A + AQ \tag{F.1}
\]

is singular. Then, there is a vector \( x \) such that

\[
(\lambda E - A + AQ) x = 0 \tag{F.2}
\]

However, it is possible to write

\[
x = \alpha v + z \, , w^\dagger Ez = 0 \, , w^\dagger Ev = 1 \tag{F.3}
\]

So \( Qz = 0, Qv = Ev \). Then,

\[
\alpha \lambda Ev + (\lambda E - A) z = 0 \tag{F.4}
\]

Premultiplying by the left eigenvector \( w \),

\[
\alpha \lambda = 0 \tag{F.5}
\]

Then, from (F.4), it is obtained that \( (\lambda E - A) z = 0 \). But this is impossible, because it is assumed that \( \lambda \) is a single eigenvalue. Therefore, the matrix \( \lambda E - A + AQ \) is regular and \( N(\lambda, Q) \) is bounded. A similar reasoning can be done if

\[
N(\lambda, Q)^{-1} = \lambda E - A + QA \tag{F.6}
\]
Lastly, let us assume that

\[(F.7) \quad N(\lambda, Q)^{-1} = \lambda E - A + AQ + QA\]

Then, it is obtained that

\[(F.8) \quad \alpha(\lambda E + QA)v + (\lambda E - A + QA)z = 0\]

But \(Av = \lambda Ev\), so

\[(F.9) \quad 2\alpha\lambda Ev + (\lambda E - A + QA)z = 0\]

Premultiplying by \(w^\dagger E\), and taking into account that \(w^\dagger EQ = w^\dagger E\), it is obtained that

\[(F.10) \quad 2\alpha\lambda + \lambda w^\dagger Ez = \alpha\lambda = 0\]

So, from \((F.9)\),

\[(F.11) \quad (\lambda E - A + QA)z = 0\]

which implies that the matrix \(\lambda E - A + QA\) is singular, or that \(z = 0\). As it has been proved that \(\lambda E - A + QA\) is regular, it must be \(z = 0\). Then, the singular vector \(x = v\). From \((F.9)\), it must be \(\lambda Ev = 0\). So:

- If \(\lambda = 0\), then \(N(\lambda, Q)\) is singular, and \(v\) is a singular vector.
- If \(\lambda \neq 0\), then we must have \(Ev = 0\). But \(Ev = \frac{1}{2}Av \neq 0\), because if \(Av = 0\), then \(\lambda = 0\). So, the matrix \(N(\lambda, Q)\) is regular.

**Appendix G. Composite models.**

Let us define:

\[(G.1) \quad x = \begin{bmatrix} x_M \\ x_I \\ x_O \\ x_A \end{bmatrix}\]

The system equations are

\[(G.2) \quad \begin{bmatrix} \text{diag}(E_1, E_2, \ldots, E_l) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \text{diag}(A_1, A_2, \ldots, A_l) & \text{diag}(B_1, B_2, \ldots, B_l) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \dot{x}_M \\ \dot{x}_I \\ \dot{x}_O \\ \dot{x}_A \end{bmatrix}\]
where \( I \) denotes the identity matrix.

This linear dynamic system has eigenvectors which can be partitioned analogously to the variables and equations. Specifically, it is fulfilled:

\[
\lambda E v_M = A v_M + B v_I \quad \text{(G.3)}
\]

\[
0 = C v_M + D v_I - v_O \quad \text{(G.4)}
\]

\[
0 = -J_{11} v_I + v_0 - J_{12} v_A \quad \text{(G.5)}
\]

\[
0 = -J_{21} v_I - J_{22} v_A \quad \text{(G.6)}
\]

\[
\lambda w^\dagger_M E = w^\dagger_M A + w^\dagger_I C \quad \text{(G.7)}
\]

\[
0 = w^\dagger_M B + w^\dagger_I D - \left( w^\dagger_O J_{11} + w^\dagger_A J_{21} \right) \quad \text{(G.8)}
\]

\[
0 = -w^\dagger_I + w^\dagger_O \quad \text{(G.9)}
\]

\[
0 = -w^\dagger_O J_{12} - w^\dagger_A J_{22} \quad \text{(G.10)}
\]

By introducing the vector

\[
\tilde{w}^\dagger_O = w^\dagger_O J_{11} + w^\dagger_A J_{21} \quad \text{(G.11)}
\]

and writing the previous equations in a subsystem basis, it is obtained:

\[
\lambda E v_{Mk} = A_k v_{Mk} + B_k v_{Ik} \quad \text{(G.12)}
\]

\[
v_{Ok} = C_k v_{Mk} + D_k v_{Ik} \quad \text{(G.13)}
\]

\[
\begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} v_I \\ v_A \end{bmatrix} = \begin{bmatrix} v_O \\ 0 \end{bmatrix} \quad \text{(G.14)}
\]

\[
\lambda w^\dagger_{Mk} E_k = w^\dagger_{Mk} A_k + w^\dagger_{Mk} B_k \quad \text{(G.15)}
\]

\[
\tilde{w}^\dagger_{Ok} = w^\dagger_{Mk} C_k + w^\dagger_{Ik} D_k \quad \text{(G.16)}
\]

\[
\begin{bmatrix} w^\dagger_I \\ w^\dagger_A \end{bmatrix} \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} \tilde{w}^\dagger_O \\ 0 \end{bmatrix} \quad \text{(G.17)}
\]

It is easy to check that

\[
\mathcal{F}^\dagger E E = I_m \Rightarrow \mathcal{F}^\dagger_{Mk} E_k E_{Mk} = I_{mk} \quad \forall k \quad \text{(G.18)}
\]

where \( m_k \) is the number of columns of \( E_{Mk} \). Also, the formulae (2.7-2.10) can be written as:

\[
v_{Mk} = E_{Mk} \alpha_{Mk} + z_{Mk} \quad \text{(G.19)}
\]

\[
v_{Ik} = z_{Ik} \quad \text{(G.20)}
\]

\[
v_{Ok} = z_{Ok} \quad \text{(G.21)}
\]

\[
v_A = z_A \quad \text{(G.22)}
\]

\[
\mathcal{F}^\dagger_{Mk} E_k z_{Mk} = 0 \quad \text{(G.23)}
\]

\[
w_{Mk} = \mathcal{F}_{Mk} \beta_{Mk} + y_{Mk} \quad \text{(G.24)}
\]
\[
\begin{align*}
\text{(G.25)} & \quad w_{ik} = y_{ik} \\
\text{(G.26)} & \quad \tilde{w}_{ok} = y_{ok} \\
\text{(G.27)} & \quad w_A = y_A \\
\text{(G.28)} & \quad E_M^\dagger E_k y_M = 0
\end{align*}
\]

and

\[
\alpha = \begin{bmatrix}
\alpha_{M1} \\
\alpha_{M2} \\
\vdots \\
\alpha_{Ml}
\end{bmatrix}, \quad \beta = \begin{bmatrix}
\beta_{M1} \\
\beta_{M2} \\
\vdots \\
\beta_{Ml}
\end{bmatrix}
\]

From \text{(G.12)} and \text{(G.19-G.21)}:

\[
\lambda E_k \varepsilon_{MK} \alpha_{MK} + \lambda E_k z_{MK} = A_k \varepsilon_{MK} \alpha_{MK} + A_k z_{MK} + B_k z_{Ik} \tag{G.30}
\]

Premultiplying by \( F_k^\dagger \), and taking into account \text{(G.23)}:

\[
\lambda \alpha_{MK} = \mathcal{F}^\dagger_{MK} A_k \varepsilon_{MK} \alpha_{MK} + \mathcal{F}^\dagger_{MK} A_k z_{MK} + \mathcal{F}^\dagger_{MK} B_k z_{Ik} \tag{G.31}
\]

Let us define the projections \( \mathcal{P}_k \) and \( \mathcal{Q}_k \) by:

\[
\mathcal{P}_k = I_{mk} - E_k \varepsilon_{MK} F_k^\dagger E_k = I_{mk} - \mathcal{Q}_k
\]

It is easy to check:

\[
\mathcal{P}_k E_k \varepsilon_{MK} = 0 \tag{G.33}
\]

\[
\mathcal{P}_k E_k z_{MK} = E_k z_{MK} \tag{G.34}
\]

Taking the above equations into account, and premultiplying \text{(G.31)} by \( \mathcal{P}_k \):

\[
\lambda E_k z_{MK} = \mathcal{P}_k A_k \varepsilon_{MK} \alpha_{MK} + \mathcal{P}_k A_k z_{MK} + \mathcal{P}_k B_k z_{Ik} \tag{G.35}
\]

So

\[
(\lambda E_k - A_k + \mathcal{Q}_k A_k) z_{MK} = \mathcal{P}_k A_k \varepsilon_{MK} \alpha_{MK} + \mathcal{P}_k B_k z_{Ik} \tag{G.36}
\]

Taking into account that \( \mathcal{P}_k z_{MK} = z_{MK} \) and \( \mathcal{Q}_k z_{MK} = 0 \), the above equation yields:

\[
z_{MK} = \mathcal{P}_k (\lambda E_k - A_k + [\mathcal{Q}_k, A_k]_+)^{-1} \mathcal{P}_k \{ A_k \varepsilon_{MK} \alpha_{MK} + B_k z_{Ik} \} \tag{G.37}
\]

After substitution in equation \text{(G.31)} it yields:
\[ \lambda_M = (A_r + H_A(\lambda)) \alpha_M + (B_r + H_B(\lambda)) z_I \]

where the following matrices have been defined:

\[ A_r = \mathcal{F}_{\lambda}^\dagger A_k \mathcal{E}_{\lambda M} \]
\[ B_r = \mathcal{F}_{\lambda}^\dagger B_k \]
\[ H_A(\lambda) = \mathcal{F}_{\lambda}^\dagger A_k \mathcal{P}_k (\lambda E_k - A_k + [Q_k, A_k]^+)^{-1} \mathcal{P}_k A_k \mathcal{E}_{\lambda M} \]
\[ H_B(\lambda) = \mathcal{F}_{\lambda}^\dagger A_k \mathcal{P}_k (\lambda E_k - A_k + [Q_k, A_k]^+)^{-1} \mathcal{P}_k B_k \]

On the other hand, from equation (4.11), it is obtained:

\[ z_O = C_k \mathcal{E}_{\lambda M} \alpha_M + C_k z_M + D_k z_I \]

After substitution of (G.37) it is obtained:

\[ z_O = (C_r + H_C) \alpha_M + (D_k + H_D) z_I \]

where the following matrices have been defined:

\[ C_r = C_k \mathcal{E}_{\lambda M} \]
\[ H_C(\lambda) = C_k \mathcal{P}_k (\lambda E_k - A_k + [Q_k, A_k]^+)^{-1} \mathcal{P}_k A_k \mathcal{E}_{\lambda M} \]
\[ H_D(\lambda) = C_k \mathcal{P}_k (\lambda E_k - A_k + [Q_k, A_k]^+)^{-1} \mathcal{P}_k B_k \]

Let us define the matrices:

\[ A_r = \text{diag}(A_{r1} \ldots A_{rl}) \]
\[ B_r = \text{diag}(B_{r1} \ldots B_{rl}) \]
\[ C_r = \text{diag}(C_{r1} \ldots C_{rl}) \]
\[ D = \text{diag}(D_1 \ldots D_l) \]
\[ H_A = \text{diag}(H_{A1} \ldots H_{Al}) \]
\[ H_B = \text{diag}(H_{B1} \ldots H_{Bl}) \]
\[ H_C = \text{diag}(H_{C1} \ldots H_{Cl}) \]
\[ H_D = \text{diag}(H_{D1} \ldots H_{Dl}) \]

It is easy to check that \( A_r = \mathcal{F}_{\lambda}^\dagger A_{\mathcal{E}} \), as computed according the general formula. Therefore, the proposed notation is consistent. Note also that all these matrices are diagonal-block matrices, which eases its computation. Then, equations (G.38,G.44) can be written as:

\[ \lambda \alpha = (A_r + H_A(\lambda)) \alpha + (B_r + H_B(\lambda)) z_I \]
\[ z_O = (C_r + H_C(\lambda)) \alpha + (D + H_D(\lambda)) z_I \]
By using equations (G.14) and (G.22), it is obtained:

$$\lambda \alpha = \{ A_r + H_A(\lambda) + \}
\begin{bmatrix}
J_{11} - (D + H_D(\lambda)) & J_{12} \\
J_{21} & J_{22}
\end{bmatrix}^{-1}
\begin{bmatrix}
C_r + H_C(\lambda) \\
0
\end{bmatrix}\}
\alpha
$$

Therefore

$$H(\lambda) = H_A(\lambda) + [(B_r + H_B(\lambda)) 0]
\begin{bmatrix}
J_{11} - (D + H_D(\lambda)) & J_{12} \\
J_{21} & J_{22}
\end{bmatrix}^{-1}
\begin{bmatrix}
C_r + H_C(\lambda) \\
0
\end{bmatrix}
$$

**Appendix H. Proof of theorem 5.1.**

Let us assume that

$$(H.1) \quad [A - j^{-1} \lambda (E - Q)] j v = j \lambda Qj v$$

But

$$(H.2) \quad Qj v = (E \xi F\dagger E)(\xi j + jz) = E \xi j$$

So

$$(H.3) \quad [A - j^{-1} \lambda (E - Q)] (\xi j + jz) = j \lambda E \xi j$$

Premultiplying by $F\dagger$, and taking into account

$$(H.4) \quad F\dagger (E - Q) = F\dagger E - F\dagger EE \xi F\dagger E = 0$$

it yields

$$(H.5) \quad F\dagger A \xi j + F\dagger A' j z = j \lambda j$$

On the other hand, as

$$(H.6) \quad (E - Q) \xi = EE - E \xi F\dagger EE = 0$$

equation (H.3) yields

$$(H.7) \quad AE' j + [A - j^{-1} \lambda (E - Q)] j z = \lambda E \xi j$$

Premultiplying by $P$, and taking into account that $P(E - Q) = E - Q$, that $P E \xi = (E - Q) \xi = 0$, and that $Qj z = 0$, 

This is essentially formula (A.7). So, it is possible to conclude (A.12)

\[ j_\alpha = \mathcal{P} \{ j^{-1} \lambda E - A + [A, Q]_+ \}^{-1} \mathcal{P} A_\alpha \]

and, by substituting in (H.5),

\[ j_\alpha = (A_{rr} + H(j^{-1} \lambda)) j_\alpha \]

Analogous reasoning can be done by using the left eigenvector, which proves the theorem.

**Appendix I. Proof of formula (5.26).**

From the definition (5.17):

\[ J = [A - j^{-1} \lambda (E - EE^\dagger E)]^{-1} EE \]

By using the Sherman-Morrison lemma:

\[
J = \left[ (A - j^{-1} \lambda E) - j^{-1} \lambda EE^\dagger E \right]^{-1} EE \\
= \left\{ (A - j^{-1} \lambda E)^{-1} - j^{-1} \lambda (A - j^{-1} \lambda E)^{-1} \right\} \\
    \left[ I_n + EE^\dagger E^{-1} E^{-1} \right]^{-1} EE \\
= \left\{ Jv - j^{-1} \lambda v \left[ I_n + j^{-1} \lambda E^\dagger V \right]^{-1} E^\dagger V \right\}
\]

Taking into account that \( J^{-1} = F^\dagger E^\dagger J \) and that \( J = F^\dagger E^\dagger V \),

\[
J^{-1} = F^\dagger E^\dagger J \\
= J M - j^{-1} \lambda j M \left[ I_n + j^{-1} \lambda j M \right]^{-1} j M \\
= J M \left[ I_n + j^{-1} \lambda j M \right]^{-1} \left( [I_n + j^{-1} \lambda j M] - j^{-1} \lambda j M \right) \\
= J M \left[ I_n + j^{-1} \lambda j M \right]^{-1} 
\]

So,

\[ J = \left[ I_n + j^{-1} \lambda j M \right] J M^{-1} = J M^{-1} + j^{-1} \lambda I_n \]

**Appendix J. Proof of theorem 5.2.**

Let us consider the equation

\[ [A - \lambda (E - Q)] v = E E \alpha' \]
where $\alpha'$ is an arbitrary vector. Without loss of generality, it can be written:

\[(J.2)\quad v = \mathcal{E}\alpha + z, \mathcal{F}^\dagger E z = 0\]

Premultiplying \[(J.1)\] by $\mathcal{F}^\dagger$,

\[(J.3)\quad \mathcal{F}^\dagger [A - \lambda(E - Q)] v = \alpha'\]

As $\mathcal{F}^\dagger (E - Q) = 0$,

\[(J.4)\quad \mathcal{F}^\dagger A\mathcal{E}\alpha + \mathcal{F}^\dagger A z = \alpha'\]

On the other hand, from \[(J.1)\], as $(E - Q)\mathcal{E} = 0$,

\[(J.5)\quad A v - \lambda (E - Q) z = E\mathcal{E}\alpha'\]

So,

\[(J.6)\quad A\mathcal{E}\alpha + [A - \lambda(E - Q)] z = E\mathcal{E}\alpha'\]

As $\mathcal{P}(E - Q) = E - Q, \mathcal{P}E\mathcal{E} = 0$ and $Q z = 0$, premultiplying by $\mathcal{P}$,

\[(J.7)\quad \mathcal{P} A\mathcal{E}\alpha + \mathcal{P} A z - \lambda E z = 0\]

And, as $Q z = 0$ and $\mathcal{P} z = z$,

\[(J.8)\quad z = \mathcal{P} [\lambda E - A + [A, Q]_+]^{-1} \mathcal{P} A\mathcal{E}\alpha\]

So, from \[(J.5)\],

\[(J.9)\quad \alpha' = \mathcal{F}^\dagger A\mathcal{E}\alpha + \mathcal{F}^\dagger A \mathcal{P} [\lambda E - A + [A, Q]_+]^{-1} \mathcal{P} A\mathcal{E}\alpha\]

\(= (A_{rr} + H(\lambda)) \alpha\)

On the other hand, let us define

\[(J.10)\quad \mathcal{T} = [A - \lambda(E - Q)]^{-1} E\mathcal{E}\]

So,

\[(J.11)\quad v = \mathcal{T} \alpha' = \mathcal{E}\alpha + z\]

Premultiplying by $\mathcal{F}^\dagger E$, and taking into account $\mathcal{F}^\dagger E\mathcal{E} = I_n$,

\[(J.12)\quad \mathcal{F}^\dagger E\mathcal{T} \alpha' = \alpha\]
So

\[(\mathcal{F}^\dagger \mathcal{E} \mathcal{V})^{-1} \alpha = N\alpha = \alpha' \]

Now, \(\alpha'\) is arbitrary. So, as (J.9) and (J.13) hold for any \(\alpha'\), the matrices must be equal:

\[N = A_{rr} + H(\lambda)\]

which proves the claim.

REFERENCES

[1] I. J. Perez-Arriaga, G. Verghese, F. Schweppe, Selective Modal Analysis With Applications to Electric Power Systems, IEEE Trans. on PAS, Vol. PAS-101, No. 9, Sept. 1982.
[2] I. J. Perez-Arriaga, G. C. Verghese, F. L. Pagola, F. C. Schweppe, Developments in Selective Modal Analysis of Small-Signal Stability in Electric Power Systems. Automatica, Vol 26, No 2, pp 215-231, 1990.
[3] I. J. Perez-Arriaga, Selective Modal Analysis With Applications to Electric Power Systems, Ph. D. Thesis, Electrical Engineering, M. I. T., June, 1981.
[4] R. Chiado, J. Soto, J. Corera, L. Rouco, I. J. Perez-Arriaga, SMAS3: A state-of-the-art computer package for analysis of small signal stability in large electric power systems, CIGRE Study Committee 38: Colloquium on Power System Dynamic Performance. Florianopolis (Brasil), 22-23 Septiembre 1993.
[5] J. L. Sancha, I. J. Perez-Arriaga, Selective Modal Analysis of Power System Oscillatory Instability, IEEE Transactions on Power Systems, Vol. PWRS-3, No. 2, May 1988, pp. 429-438.
[6] L. Rouco, I. J. Perez-Arriaga, Multi-area analysis of small signal stability in large electric power systems by SMA, IEEE Transactions on Power Systems, Vol. PWRS-8, No. 3, August 1993, pp. 1257-1265.
[7] E. L. Yip, R. F. Sincovec, Solvability, Controllability and Observability of Continuous Descriptor Systems, IEEE Trans. on Automatic Control, Vol. AC-26, No. 3, June 1981.