Reflected backward SDEs with two barriers under monotonicity and general increasing conditions

Mingyu Xu *
Departement des Mathématiques, Université du Maine, Le Mans France;
Department of Financial Mathematics and Control science, School of Mathematical Science
Fudan University, Shanghai, 200433, China.

Abstract In this paper, we prove the existence and uniqueness result of the reflected BSDE with two continuous barriers under monotonicity and general increasing condition on $y$, with Lipschitz condition on $z$.

Keywords: Reflected backward stochastic differential equation, monotonicity condition, comparison theorem, Dynkin game.

1 Introduction
Nonlinear backward stochastic differential equations (BSDEs in short) were firstly introduced by Pardoux and Peng (1990), who proved the existence and uniqueness of adapted solutions, when the coefficient $f$ is Lipschitz in $(y,z)$ uniformly in $(t,\omega)$, with square-integrability assumptions on the coefficient $f(t,\omega,y,z)$ and terminal condition $\xi$. Later, Pardoux (1999) and Briand, Delyon, Hu, Pardoux and Stoica (2003) studied the solution of a BSDE with a coefficient $f(t,\omega,y,z)$ that satisfies only monotonicity, continuity and general increasing growth conditions with respect to $y$, and Lipschitz on $z$. That is, for some real number $\mu \in \mathbb{R}$, $k \geq 0$ and some continuous increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$:

\begin{align}
|f(t,y,z)| &\leq |f(t,0,z)| + \varphi(|y|), \\
(y - y')(f(t,y,z) - f(t,y',z)) &\leq \mu |y - y'|^2, \\
|f(t,y,z) - f(t,y',z')| &\leq k |z - z'|.
\end{align}

Reflected backward stochastic differential equations (RBSEs in short) with one lower barrier were studied by El Karoui, Kapoudjian, Pardoux, Peng and Quenez (1997), in one dimension. The solution is constrained to remain above a continuous lower-boundary process with the help of a continuous increasing process. Later, Cvitanic and Karatzas (1996) studied the backward stochastic differential equation with two barriers. A solution to such equation associated to a terminal condition $\xi$, a coefficient $f(t,\omega,y,z)$ and two barriers $L$ and $U$, is a triple $(Y,Z,K)$ of adapted processes, valued in $\mathbb{R}^{1+d+1}$, which satisfies

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds + K^+_T - K^-_T - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T \text{ a.s.}$$

$L_t \leq Y_t \leq U_t$, $0 \leq t \leq T$ and $\int_0^T (Y_s - L_s)dK^+_s = \int_0^T (Y_s - U_s)dK^-_s = 0$, a.s. In this case, a solution $Y$ has to remain between the lower boundary $L$ and upper boundary $U$, almost surely. This is

*Email: xvmingyu@gmail.com
achieved by the cumulative action of two continuous, increasing reflecting processes $K^\pm$, which act in a minimal way when $Y$ attempts to cross barriers. And the authors proved the existence and uniqueness of the solution when $f(t, \omega, y, z)$ is Lipschitz on $(y, z)$ uniformly in $(t, \omega)$ and when $L < U$ on $[0, T]$ and there exists a different supermartingale between $L$ and $U$ (Mokobodski’s assumption in Dynkin game). Furthermore they established the connection between solution $Y$ and the value of Dynkin games (certain stochastic games of stopping). Then in [8], the existence of a solution was proved when $f$ is only continuous with linear growth in $(y, z)$, but in the case when one obstacle is smooth. Later, Lepeltier and San Martin used the penalization method to prove the existence of a solution to such equation, with same assumption on one obstacle is smooth. Later, Lepeltier and San Martin used the penalization method to prove the existence of a solution to such equation, with same assumption on $f$ as in [8], without extra smoothness of the barriers, i.e. when $L$ and $U$ are continuous, $L < U$ on $[0, T]$, and Mokobodski’s assumption.

More recently, Lepeltier, Matoussi and Xu proved the existence and uniqueness of the solution to the reflected BSDE with one lower continuous barrier under the assumption (1), and give the uniqueness and existence of the solution, which is obtained by approximation. In this paper, we consider the reflected BSDE with two continuous barrier under the assumption (1), and give the uniqueness and existence of the solution, which is obtained by approximation.

The paper is organized as following: In subsection 2.1, we present notations and assumptions; then we prove the main results of this paper, the existence and uniqueness of the solution in subsection 2.2; in subsection 2.3 we prove an important theorem for the existence in five steps. Finally, in section 3, we prove several comparison theorems with respect to RBSDE with one or two barriers, which are used in the proof of existence.

2 RBSDE’s with two continuous barriers

2.1 Assumptions and notations

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, and $B = (B_1, B_2, \cdots, B_d)'$ be a $d$-dimensional Brownian motion defined on a finite interval $[0, T]$, $0 < T < +\infty$. Denote by $\{\mathcal{F}_t; 0 \leq t \leq T\}$ the natural filtration generated by the Brownian motion $B$:

$$\mathcal{F}_t = \sigma\{B_s; 0 \leq s \leq t\},$$

where $\mathcal{F}_0$ contains all $P$–null sets of $\mathcal{F}$.

We denote the following notations. For any given $n \in \mathbb{N}$, let us introduce the following spaces:

$$\mathbf{L}^2_n(\mathcal{F}_t) = \{\xi: n\text{-dimensional } \mathcal{F}_t\text{-measurable random variable, s.t. } E(|\xi|^2) < +\infty\},$$

$$\mathbf{H}^2_n(0, T) = \{\psi: n\text{-dimensional } \mathcal{F}_t\text{-predictable process on the interval } [0, T],$$

$$\text{s.t. } E \int^T_0 \|\psi(t)\|^2 dt < +\infty\},$$

$$\mathbf{S}^2_n(0, T) = \{\psi: n\text{-dimensional } \mathcal{F}_t\text{-progressively measurable continuous process on the interval } [0, T],$$

$$\text{s.t. } E(\sup_{0 \leq t \leq T} \|\psi(t)\|^2) < +\infty\},$$

$$\mathbf{A}^2(0, T) = \{K: \text{real valued } \mathcal{F}_t\text{-adapted increasing continuous process, s.t. } K(0) = 0,$$

$$E(K(T)^2) < +\infty\},$$

$$\mathbf{VF}^2(0, T) = \{V: \text{real valued } \mathcal{F}_t\text{-adapted continuous process with finite variation, s.t. } V = K^+ - K^-, \text{ with } K^\pm \in \mathbf{A}^2(0, T)\}.$$

Finally, we shall denote by $\mathcal{P}$ the $\sigma$-algebra of predictable sets on $[0, T] \times \Omega$. In the real–valued case, i.e., $n = 1$, these spaces will be simply denoted by $\mathbf{L}^2(\mathcal{F}_t)$, $\mathbf{H}^2(0, T)$ and $\mathbf{S}^2(0, T)$, respectively.

Let us consider the reflected backward stochastic differential equation with monotonic condition in $y$ on a fixed time interval; we need the following assumptions:
**Assumption 2.1.** A final condition $\xi \in L^2(\mathcal{F}_T)$.

**Assumption 2.2.** A coefficient $f: \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, satisfying for some continuous increasing function $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$, real numbers $\mu \in \mathbb{R}$ and $k > 0$:

(i) $f(\cdot, y, z)$ is progressively measurable, $\forall (y, z) \in \mathbb{R} \times \mathbb{R}^d$;

(ii) $|f(t, y, z)| \leq |f(t, 0, z)| + \varphi(|y|)$, $\forall (t, y, z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$, a.s.;

(iii) $E\left[\int_0^T |f(t, 0, 0)|^2 dt\right] < \infty$;

(iv) $|f(t, y, z) - f(t, y', z)| \leq k|z - z'|$, $\forall (t, y) \in [0, T] \times \mathbb{R}$, $z, z' \in \mathbb{R}^d$, a.s.

(v) $(y - y')(f(t, y, z) - f(t, y', z)) \leq \mu |y - y'|^2$, $\forall (t, z) \in [0, T] \times \mathbb{R}^d$, $y, y' \in \mathbb{R}$, a.s.

(vi) $y \to f(t, y, z)$ is continuous, $\forall (t, z) \in [0, T] \times \mathbb{R}^d$, a.s.

**Assumption 2.3.** Two barriers $L_t, U_t$, which are $\mathcal{F}_t$-progressively measurable continuous processes, defined on the interval $[0, T]$, satisfying

(i) $$E[\varphi^2(\sup_{0 \leq t \leq T} (e^{\mu t}(L_t)^+))] < \infty, \quad E[\varphi^2(\sup_{0 \leq t \leq T} (e^{\mu t}(U_t)^-))] < \infty,$$

$(L)^+, (U)^- \in S^2(0, T)$, and $L_T \leq \xi \leq U_T$, a.s., where $(L)^+$ (resp. $(U)^-$) is the positive part (resp. negative) part of $L$ (resp. $U$).

(ii) there exists a process $J_t = J_0 + \int_0^t \phi \sigma dB_s - V_t^+ + V_t^-$, $J_T = \xi$ with $\phi \in H^2_d(0, T)$, $V^+, V^- \in A^2(0, T)$, s.t.

$$L_t \leq J_t \leq U_t, \quad 0 \leq t \leq T.$$ (iii) $L_t < U_t$, a.s., for $0 \leq t < T$.

Now we introduce the definition of the solution of RBSDE with two barriers $L$ and $U$.

**Definition 2.1** We say that $(Y_t, Z_t, K_t)_{0 \leq t \leq T}$ is a solution of the backward stochastic differential equation with two continuous reflecting barriers $L(\cdot)$ and $U(\cdot)$, terminal condition $\xi$ and coefficient $f$, which is denoted as RBSDE$(\xi, f, L, U)$, if the following hold:

1. $Y \in S^2(0, T)$, $Z \in V^2(0, T)$, and $K \in VF^2(0, T)$, $K = K^+ - K^-$, where $K^\pm \in A^2(0, T)$.

2. $Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + K_t^+ - K_t^-$, $0 \leq t \leq T$ a.s.

3. $L_t \leq Y_t \leq U_t$, $0 \leq t \leq T$, a.s.

4. $\int_0^T (Y_s - L_s) dK_s^+ = \int_0^T (Y_s - U_s) dK_s^- = 0$, a.s.

Actually, a general solution of our RBSDE$(\xi, f, L, U)$ would satisfy the assumptions (1) to (4). The state-process $Y(\cdot)$ is forced to stay between the barrier $L(\cdot)$ and $U(\cdot)$, thanks to the cumulation action of the reflection processes $K^+ (\cdot)$ and $K^- (\cdot)$ respectively, which act only when necessary to prevent $Y(\cdot)$ from crossing the respective barrier, and in this sense, its action can be considered minimal, i.e. the integrability assumption (4). From the fact that $K^\pm \in A^2(0, T)$ is continuous and (2), it follows that $Y$ is continuous.

**Remark 2.1** We have an analogue result of Proposition 4.1 in [4]. Precisely, the square-integrable solution $Y$ of the RBSDE$(\xi, f, L, U)$ is the value of the Dynkin game problem, whose payoff is

$$R_t(\sigma, \tau) = \int_0^{\sigma \wedge \tau} f(s, Y_s, Z_s) ds + \xi 1_{\{\sigma \wedge \tau = \tau\}} + L_\tau 1_{\{\tau < T, \tau \leq \sigma\}} + U_\sigma 1_{\{\sigma < \tau\}},$$

and a saddle-point $(\tilde{\sigma}_t, \tilde{\tau}_t) \in T_t \times T_t$ is given by

$$\tilde{\sigma}_t = \inf\{s \in [t, T); Y_s = U_s\} \wedge T,$$

$$\tilde{\tau}_t = \inf\{s \in [t, T); Y_s = L_s\} \wedge T.$$
2.2 Main results

Our main results in this paper is following:

**Theorem 2.1** Under the assumptions 2.1, 2.2 and 2.3, the RBSDE(ξ, f, L, U) has the unique solution (Y, Z, K), which satisfies definition 2.1 (1)-(4).

**Proof.** Uniqueness. Suppose that the triples (Y, Z, K) and (Y', Z', K') are two solutions of the RBSDE(ξ, f, L), i.e. satisfy (1)-(4) of definition 2.1. Set ΔY = Y - Y', ΔZ = Z - Z', ΔK = ΔK - ΔK', with ΔK⁺ = K⁺ - K'⁺, ΔK⁻ = K⁻ - K'⁻. Applying Itô’s formula to ΔY² on the interval [t, T], and taking expectation on both sides, it follows

\[ E|ΔY_t|^2 + E \int_t^T |ΔZ_s|^2 ds ≤ 2(k^2 + μ)E \int_t^T ΔY_s^2 ds + \frac{1}{2}E \int_t^T |ΔZ_s|^2 ds, \]

in view of monotonic assumption on y, Lipschitz assumption on z, and \( \int_t^T ΔY_s dΔK_s ≤ 0 \). We get

\[ E|ΔY_t|^2 ≤ 2(k^2 + μ)E \int_t^T ΔY_s^2 ds. \]

From the Gronwall’s inequality, it follows \( E|ΔY_t|^2 = E|Y_t - Y'_t|^2 = 0, 0 ≤ t ≤ T \), i.e. \( Y_t = Y'_t \) a.s.; then we have also \( E \int_0^T |ΔZ_s|^2 ds = E \int_0^T |Z_s - Z'_s|^2 ds = 0 \), from which follows \( K_t = K'_t \).

Existence. We firstly present the following existence theorem when \( f \) does not depend on \( z \), which will be proved a little later.

**Theorem 2.2** Suppose that \( ξ, f \) and \( L, U \) satisfy assumption 2.1, 2.2 and 2.3, then for any process \( Q ∈ H^2_d(0, T) \), there exists a unique triple of progressively measurable processes \( \{(Y_t, Z_t, K_t)_{0 ≤ t ≤ T}\} ∈ S^2(0, T) × H^2_d(0, T) × VF^2(0, T) \), with \( K = K^+ - K^- \), \( (K_t^+)_{0 ≤ t ≤ T} ∈ A^2(0, T) \), which satisfies 2.1 (1), (3), (4) and

\[ Y_t = ξ + \int_t^T f(s, Y_s, Q_s)ds + K_T^- - K_T^+ - (K_T^+ - K_T^-) - \int_t^T Z_s dB_s, 0 ≤ t ≤ T. \]

Thanks to Theorem 2.2 we can construct a mapping \( Φ \) from \( S \) into itself, where \( S \) is defined as the space of the progressively measurable processes \( \{(Y_t, Z_t); 0 ≤ t ≤ T\} \), valued in \( \mathbb{R} × \mathbb{R}^d \) which satisfy (1) as follows.

Given \( (P, Q) ∈ S \), \( (Y, Z) = Φ(P, Q) \) is the unique solution of following RBSDE

\[ Y_t = ξ + \int_t^T f(s, Y_s, Q_s)ds + K_T - K_t - \int_t^T Z_s dB_s, \]

i.e., if we define the process

\[ K_t = Y_t - Y_0 - \int_0^t f(s, Y_s, Q_s)ds + \int_0^t Z_s dB_s, 0 ≤ t ≤ T, \]

then the triple \( (Y, Z, K) \) satisfies definition 2.1 (1)-(4), with \( f(s, y, z) = f(s, y, Q_s) \).

Consider another element of \( S \), and define \( (Y', Z') = Φ(P', Q') \); set

\[ ΔP = P - P', ΔQ = Q - Q', ΔY = Y - Y', ΔZ = Z - Z', ΔK = K^+ - K^-, ΔK^+ = K^+ - K'^+, ΔK^- = K^- - K'^-. \]
We apply the Itô’s formula to $e^{\gamma t} |\Delta Y_t|^2$ on the interval $[t, T]$, for $\gamma > 0$,
\[ e^{\gamma t} E |\Delta Y_t|^2 + E \int_t^T e^{\gamma s} (\gamma |\Delta Y_s|^2 + |\Delta Z_s|^2) ds \]
\[ \leq 2(k^2 + \mu) E \int_t^T e^{\gamma s} |\Delta Y_s|^2 ds + \frac{1}{2} E \int_t^T e^{\gamma s} |\Delta Q_s|^2 ds, \]

since $\int_t^T e^{\gamma s} \Delta Y_s d\Delta K_s = \int_t^T e^{\gamma s} \Delta Y_s d\Delta K^+_s - \int_t^T e^{\gamma s} \Delta Y_s d\Delta K^-_s \leq 0$. Hence, if we choose $\gamma = 1 + 2(k^2 + \mu)$, it follows
\[ E \int_t^T e^{\gamma s} (|\Delta Y_s|^2 + |\Delta Z_s|^2) ds \leq \frac{1}{2} E \int_t^T e^{\gamma s} |\Delta Q_s|^2 ds \]
\[ \leq \frac{1}{2} E \int_t^T e^{\gamma s} (|\Delta P_s|^2 + |\Delta Q_s|^2) ds. \]

Consequently, $\Phi$ is a strict contraction on $\mathcal{S}$ equipped with the norm
\[ ||(Y, Z)||_\gamma = \left( E \int_0^T e^{\gamma s} (|Y_s|^2 + |Z_s|^2) ds \right)^{\frac{1}{2}}, \]
and has a fixed point, which is the unique solution of the RBSDE($\xi, f, L, U$). □

### 2.3 Proof of theorem 2.2

Now we prove the theorem 2.2 in several steps for the existence of solution. We write $f(s, y)$ for $f(s, y, Q_s)$. First we note that the triple $(Y, Z, K)$ solves the RBSDE($\xi, f, L, U$), $K = K^+ - K^-$, if and only if
\[ (Y_t, Z_t, K^+_t, K^-_t) := (e^{\lambda t} Y_t, e^{\lambda t} Z_t, \int_0^t e^{\lambda s} dK^+_s, \int_0^t e^{\lambda s} dK^-_s) \]

solves the RBSDE($\bar{\xi}, \bar{f}, \bar{L}, \bar{U}$), where
\[ (\bar{\xi}, \bar{f}(t, y), \bar{L}, \bar{U}) = (\xi e^{\lambda T}, e^{\lambda t} f(t, e^{-\lambda t} y) - \lambda y, e^{\lambda T} L_t, e^{\lambda T} U_t). \]

If we choose $\lambda = \mu$, then the coefficient $\bar{f}$ satisfies the same assumptions in assumption 2.2 as $f$, but with assumption 2.2-(v) replaced by
\[ (v') \quad (y - y')(f(t, y, z) - f(t, y', z)) \leq 0. \]

Since we are in 1-dimensional case, $(v')$ means that $f$ is decreasing on $y$. From another part the barriers $\bar{L}, \bar{U}$ satisfies:

(i'):
\[ E[\sup_{0 \leq t \leq T} (\bar{L}_t)^+] < \infty, E[\varphi^2(\sup_{0 \leq t \leq T} (\bar{L}_t)^+)] = E[\varphi^2(\sup_{0 \leq t \leq T} (e^{\mu t} (L_t)^+))] < \infty, \]
\[ E[\sup_{0 \leq t \leq T} (\bar{U}_t)^-] < \infty, E[\varphi^2(\sup_{0 \leq t \leq T} (\bar{U}_t)^-)] = E[\varphi^2(\sup_{0 \leq t \leq T} (e^{\mu t} (U_t)^-))] < \infty. \]

In the following, we shall work with assumption 2.2’ which is assumption 2.2 with $(v)$ replaced by $(v')$ and assumption 2.3’ which is assumption 2.2 with (i') instead of (i).

**Proof of Theorem 2.2.** First, let us recall the assumptions on the coefficient $f$:
Assumption 2.4. For \( y \in \mathbb{R}, s \in [0,T], \)

(i) \(|f(s,y)| \leq |f(s,0,0)| + k|Q_s| + \varphi(|y|);  
(ii) \( E \int_0^T |f(t,0)|^2 dt < \infty;  
(iii) \( (y-y')(f(s,y) - f(s,y')) \leq 0;  
(iv) \( y \rightarrow f(s,y) \) is continuous, a.s.

We point out that we always denote by \( c > 0 \) a constant whose value can be changed line by line. The proof will be done by five steps as following.

- Using a penalization method we prove the existence under the assumption

\[
|\xi| + \sup_{0 \leq t \leq T} |f(t,0)| + \sup_{0 \leq t \leq T} L^+_t + \sup_{0 \leq t \leq T} U^-_t \leq c. \tag{3}
\]

- Approximating the lower barrier \( L \), we prove the existence under the assumption that \( L \) satisfies assumption 2.3’-(i) and the bounded assumption on \( \xi, f(t,0) \) and \( \sup_{0 \leq t \leq T} U^-_t \).

- Like above step, we approximate the upper barrier \( U \) to prove the existence under assumption 2.3’ and \( \xi \) and \( f(t,0) \) satisfy

\[
|\xi|^2 + \sup_{0 \leq t \leq T} |f(t,0)|^2 \leq c. \tag{4}
\]

- By approximation, we prove the existence of the solution under the assumption \( \xi \geq c, \inf_{0 \leq t \leq T} f(t,0) \geq c. \)

- Finally, we prove the existence of the solution under the assumption \( \xi \in L^2(\mathcal{F}_T), f(t,0) \in H^2(0,T) \), by approximation.

In each step, we use monotonic property of approximation solutions to get the convergence.

**Step 1.** Consider the penalization equations with respect to the two barriers \( L, U \), for \( m, n \in \mathbb{N}, \)

\[
Y^{m,n}_t = \xi + \int_t^T f(s, Y^{m,n}_s)ds + m \int_t^T (Y^{m,n}_s - L)^- ds - n \int_t^T (U_s - Y^{m,n}_s)^- ds - \int_t^T Z^{m,n}_s dB_s. \tag{5}
\]

Set \( f_{m,n}(s,y) = f(s,y) + m(y-L)^- - n(U_s-y)^- \), obviously, \( f_{m,n} \) satisfies the condition of Proposition 2.4 in [13]. So by the Proposition 2.4 in [13], there exists \((Y^{m,n}_t, Z^{m,n}_t)_{0 \leq t \leq T}\), which is the solution of [5]. Denote \( K^{m,n,} = m \int_0^T (Y^{m,n}_s - L)^- ds, K^{m,n,-} = n \int_0^T (U_s - Y^{m,n}_s)^- ds. \)

Now let us do the uniformly a priori estimation of \((Y^{m,n}, Z^{m,n}, K^{m,n,}, K^{m,n,-})\).

**Lemma 2.1** There exists a constant \( C_0 \) independent of \( n \), such that

\[
E[ \sup_{0 \leq t \leq T} |Y^{m,n}_t|^2 + \int_0^T |Z^{m,n}_s|^2 ds + (K^{m,n,}_T)^2 + (K^{m,n,-}_T)^2 ] \leq C_0.
\]

**Proof.** Consider the RBSDE\((\xi, f, L)\) with one lower barrier \( L \); due to theorem 2.3 in [11], it admits a unique solution \((\overline{Y}_t, \overline{Z}_t, \overline{K}_t)_{0 \leq t \leq T} \in S^2(0,T) \times H^2_0(0,T) \times A^2(0,T)\), which satisfies

\[
\overline{Y}_t = \xi + \int_t^T f(s, \overline{Y}_s)ds + \overline{K}_T - \overline{K}_t - \int_t^T \overline{Z}_s dB_s, \tag{6}
\]
\( \mathbf{Y}_t \geq L_t, 0 \leq t \leq T, \int_0^T (\mathbf{Y}_s - L_s) d\mathbf{K}_s = 0. \) In order to compare (6) and (3), we consider the penalization equation associated with the RBSDE (6), for \( m \in \mathbb{N}, \)

\[
\mathbf{Y}_t^m = \xi + \int_t^T f(s, \mathbf{Y}_s^m) ds + m \int_t^T (L_s - \mathbf{Y}_s^m)^+ ds - \int_t^T Z_s^m dB_s. \tag{7}
\]

Comparing (5) and (7), we get \( Y_t^{m,n} \leq \mathbf{Y}_t^m, \forall t \in [0, T], n \in \mathbb{N}. \) Thank to the convergence result of step 1 and step 2 in the proof of theorem 2.3 in [11], i.e. \( \mathbf{Y}_t^m \rightarrow \mathbf{Y} \) in \( S^2(0, T). \) So we get for any \( m, n \in \mathbb{N}, t \in [0, T], Y_t^{m,n} \leq \mathbf{Y}_t. \)

Similarly, we consider the RBSDE(\( \xi, f, U \)) with one upper barrier \( U \). There exists \( (Y_t, Z_t, K_t)_{0 \leq t \leq T} \in S^2(0, T) \times H^2_0(0, T) \times A^2(0, T), \) which satisfies

\[
Y_t = \xi + \int_t^T f(s, Y_s) ds - (K_T - K_t) - \int_t^T Z_s dB_s, \tag{8}
\]

\( Y_t \leq U_t, 0 \leq t \leq T, \int_0^T (\mathbf{Y}_s - U_s) d\mathbf{K}_s = 0. \) By the penalization equation associated with (3) and the comparison theorem, we deduce that \( Y_t^{m,n} \geq \mathbf{Y}_t, \) for any \( m, n \in \mathbb{N}, t \in [0, T]. \) Then we get, with the results of the step 1 in the proof of theorem 2.3 [11],

\[
\sup_{0 \leq t \leq T} |Y_t^{m,n}| \leq \max\{ \sup_{0 \leq t \leq T} |Y_t|, \sup_{0 \leq t \leq T} |Y_s| \} \leq C. \tag{9}
\]

In the following, notice that assumption 2.4-(iii) implies that \( f \) is decreasing on \( y, \) for \( s \in [0, T], \) so \( f(s, Y_s) \geq f(s, Y_s^{m,n}) \geq f(s, \mathbf{Y}_s), \) with the square-integrable results of (6) and (8), it follows

\[
|f(s, Y_s^{m,n})| \leq \max\{ |f(s, Y_s)|, |f(s, \mathbf{Y}_s)| \} \leq C. \tag{10}
\]

To get the estimation of \( (K^{m,n,\prime}, K^{m,n,\prime}, Z^{m,n}) \), we apply Itô’s formula to \( (Y^{m,n})^2 \), then

\[
E(Y_t^{m,n})^2 + E \int_t^T |Z_s^{m,n}|^2 ds \\
\leq E[\mathbb{E}] + E \int_t^T |Y_s^{m,n}|^2 ds + E \int_t^T |f(s, 0)|^2 ds + \frac{1}{\alpha} E[\sup_{0 \leq t \leq T} (L^+_s)^2] + \frac{1}{\alpha} E[\sup_{0 \leq t \leq T} (U^-_s)^2] \\
+ \alpha E[m \int_t^T (L_s - Y_s^{m,n})^+ ds] + \alpha E[n \int_t^T (U_s - Y_s^{m,n})^- ds],
\]

for some \( \alpha > 0 \), in view of

\[
\int_t^T Y_s^{m,n}(L_s - Y_s^{m,n})^+ ds = \int_t^T L_s(L_s - Y_s^{m,n})^+ ds - \int_t^T ((L_s - Y_s^{m,n})^+)^2 ds \leq \int_t^T L_s(L_s - Y_s^{m,n})^+ ds, \text{ and}
\]

\[
\int_t^T Y_s^{m,n}(U_s - Y_s^{m,n})^- ds \leq \int_t^T U_s(U_s - Y_s^{m,n})^- ds. \text{ So}
\]

\[
E \int_t^T |Z_s^{m,n}|^2 ds \leq C + \alpha E[m \int_t^T (L_s - Y_s^{m,n})^+ ds] + E[n \int_t^T (U_s - Y_s^{m,n})^- ds] \tag{11}.
\]

We need to prove that there exists a constant \( C \) independent of \( m, n \) such that for any \( 0 \leq t \leq T \)

\[
E[m \int_t^T (L_s - Y_s^{m,n})^+ ds] + E[n \int_t^T (U_s - Y_s^{m,n})^- ds] \leq C + 8E \int_t^T |Z_s^{m,n}|^2 ds.
\]
In fact, let us consider the stopping time

\[ \tau_1 = \inf\{r \geq t | Y_{r}^{m,n} \geq U_r\} \land T, \quad \sigma_1 = \inf\{r \geq \tau_1 | Y_{r}^{m,n} = L_r\} \land T, \]

\[ \tau_2 = \inf\{r \geq \sigma_1 | Y_{r}^{m,n} = U_r\} \land T, \]

and so on. Since \( L < U \) on \([0, T]\), and \( L \) and \( U \) are continuous, then when \( k \to \infty \), we have \( \tau_k \not
\to T \), \( \sigma_k \not
\to T \). Obviously \( Y_{m,n} \geq L \) on the interval \([\tau_k, \sigma_k]\), so we get

\[ Y_{\tau_k}^{m,n} = Y_{\sigma_k}^{m,n} + \int_{\tau_k}^{\sigma_k} f(s, Y_{s}^{m,n})ds - n \int_{\tau_k}^{\sigma_k} (Y_{s}^{m,n} - U_s)^+ds - \int_{\tau_k}^{\sigma_k} Z_s^{m,n}dB_s. \]

On the other hand

\[ Y_{\tau_k}^{m,n} \geq J_{\tau_k}, \text{ on } \{\tau_k < T\}, Y_{\tau_k}^{m,n} = J_{\tau_k} = \xi, \text{ on } \{\tau_k = T\}, \]

\[ Y_{\sigma_k}^{m,n} \leq J_{\sigma_k}, \text{ on } \{\sigma_k < T\}, Y_{\sigma_k}^{m,n} = J_{\sigma_k} = \xi, \text{ on } \{\sigma_k = T\}, \]

and these inequalities imply that for all \( k \), the following holds

\[ n \int_{\tau_k}^{\sigma_k} (Y_{s}^{m,n} - U_s)^+ds \leq J_{\sigma_k} - J_{\tau_k} + \int_{\tau_k}^{\sigma_k} f(s, Y_{s}^{m,n})ds - \int_{\tau_k}^{\sigma_k} Z_s^{m,n}dB_s \]

\[ \leq \int_{\tau_k}^{\sigma_k} (\phi_s - Z_s^{m,n})dB_s + V_{\sigma_k}^- - V_{\tau_k}^- + \int_{\tau_k}^{\sigma_k} |f(s, Y_{s}^{m,n})|ds. \]

Notice that on the interval \([\sigma_k, \tau_{k+1}],[Y_{s}^{m,n} \leq U_s\); we obtain by summing in \( k \)

\[ n \int_{t}^{T} (Y_{s}^{m,n} - U_s)^+ds \leq \int_{t}^{T} ((\phi_s - Z_s^{m,n})(\sum_{k} 1_{[\tau_k, \sigma_k]}(s))dB_s + V_T^- + \int_{t}^{T} |f(s, Y_{s}^{m,n})|ds. \]

By squaring and taking the expectation, with (10), we get

\[ E[n \int_{t}^{T} (Y_{s}^{m,n} - U_s)^+ds]^2 \]

\[ \leq 4E \int_{t}^{T} |\phi_s|^2ds + 4E \int_{t}^{T} |Z_s^{m,n}|^2ds + 2E[(V_T^-)^2] + 2E(\int_{t}^{T} |f(s, Y_{s}^{m,n})|ds)^2 \]

\[ \leq C + 4E \int_{t}^{T} |Z_s^{m,n}|^2ds, \]

in the same way, we obtain

\[ E[m \int_{t}^{T} (L_s - Y_{s}^{m,n})^+ds]^2 \leq C + 4E \int_{t}^{T} |Z_s^{m,n}|^2ds. \] (13)

By (12) and (13), and (11), with \( \alpha = \frac{1}{16} \), it follows

\[ E \int_{t}^{T} |Z_s^{m,n}|^2ds \leq C, \]

(14)

then

\[ E[(K_T^{m,n,+})^2 + (K_T^{m,n,-})^2] \leq C. \] (15)
Let \( m \rightarrow \infty \), due to the convergence results in step 1 of the proof in [11], \( Y^{m,n} \rightarrow Y^n \) in \( S^2(0,T), K^{m,n,+} \rightarrow K^{n,+} \) in \( A^2(0,T) \), and \( Z^{m,n} \rightarrow Z^n \) in \( H^2_d(0,T) \), where \( (Y^n, Z^n, K^{n,+}) \) is the solution of the one lower barrier RBSDE \((\xi, f_n, L)\), with \( f_n(s, y) = f(s, y) - n(y - U_s)^+ \). So

\[
Y^n_t = \xi + \int_t^T f(s, Y^n_s)ds + K^{n,+}_t - K^{n,+}_t - n \int_t^T (Y^n_s - U_s)^+ds - \int_t^T Z^n_sdB_s,
\]

\( Y^n_t \geq L_t, 0 \leq t \leq T, \int_0^T (Y^n_s - L_s)dK^n_s = 0 \). Thank to the uniform estimations, which we got as above, we know that there exists a constant \( C \) independent of \( n \) and \( t \), s.t.

\[
\sup_{0 \leq t \leq T} (Y^n_t)^2 + f(t, Y^n_t) \leq C, \tag{16}
\]

and

\[
E \int_0^T |Z^n_s|^2 ds + E[(K^{n,+}_T)^2] + E[(K^{n,-}_T)^2] \leq C \tag{17}
\]

where \( K^{n,+}_t = n \int_0^t (Y^n_s - U_s)^+ds \). Then by the comparison theorem 4.3 in [11], we deduce that \( Y^n_t \searrow Y_t \), for \( t \in [0, T] \), as \( n \rightarrow \infty \), and by the dominated convergence theorem

\[
E \int_0^T (Y^n_s - Y_s)^2ds \rightarrow 0, \text{ as } n \rightarrow \infty. \tag{18}
\]

Then we want to prove the convergence of \((Z^n)\) in \( H^2_d(0,T) \). For this, we need the following lemma, which is analogue as Lemma 4 in [12]. With (10), (9) and (3), we can easily get it, so we omit the proof.

**Lemma 2.2**

\[
\lim_{n \rightarrow \infty} E(\sup_{0 \leq t \leq T} ((Y^n_t - U_t)^+)^2) = 0. \tag{19}
\]

For \( n, p \in \mathbb{N} \), applying Itô’s formula to \(|Y^n - Y^p|^2\), and taking the expectation, then

\[
E(Y^n_t - Y^p_t)^2 + E \int_t^T |Z^n_s - Z^p_s|^2 ds
\]

\[
\leq 2E \int_t^T (Y^n_s - U_s)^+dK^{p,-}_s + 2E \int_t^T (Y^p_s - U_s)^+dK^{n,-}_s
\]

\[
\leq 2(E[\sup_{0 \leq t \leq T} (Y^n_s - U_s)^+]^2)^{\frac{1}{2}}(E(K^{p,-}_T)^2)^{\frac{1}{2}} + 2(E[\sup_{0 \leq t \leq T} (Y^p_s - U_s)^+]^2)^{\frac{1}{2}}(E(K^{n,-}_T)^2)^{\frac{1}{2}},
\]

since \( \int_t^T (Y^n_s - Y^p_s)d(K^{n,+} - K^{p,+}) \leq 0 \). So by (19) and (17), as \( n, p \rightarrow \infty \), \( E \int_t^T |Z^n_s - Z^p_s|^2 ds \rightarrow 0 \), which implies \( \{Z^n\} \) is a Cauchy sequence in \( H^2_d(0,T) \). So there exists a process \( Z \in H^2_d(0,T) \), s.t., as \( n \rightarrow \infty \),

\[
E \int_t^T |Z^n_s - Z_s|^2 ds \rightarrow 0.
\]

Moreover by Itô’s formula, we have

\[
E[\sup_{0 \leq t \leq T} |Y^n_t - Y^p_t|^2] \leq 2E \int_t^T (Y^n_s - U_s)^+dK^{p,-}_s + 2E \int_t^T (Y^p_s - U_s)^+dK^{n,-}_s
\]

\[
+ 2E[\sup_{0 \leq t \leq T} \int_t^T |Z^n_s - Z^p_s||Y^n_s - Y^p_s|d B_s].
\]
By Burkholder-Davis-Gundy inequality and \([19]\), we get, as \(n, p \to \infty\)
\[
E[\sup_{0 \leq t \leq T} |Y^n_t - Y^p_t|^2] \to 0,
\]
i.e. \(Y^n \to Y\), in \(S^2(0, T)\).

By the convergence of \(Y^n_t\), i.e. \(Y^n_t \to Y_t\), \(0 \leq t \leq T\), and the fact that \(f(s, y)\) is continuous and decreasing in \(y\), we get \(f(s, Y^n_s) \to f(s, Y_s),\ 0 \leq s \leq T\). Moreover \(|f(s, Y^n_s)| \leq C\). Using the monotonic convergence theorem, we deduce that
\[
E \int_0^T [f(t, Y^n_t) - f(t, Y_t)]^2 dt \to 0,
\]
i.e. the sequence \(\{f(\cdot, Y^n)\}\) is also a Cauchy sequence in \(H^2(0, T)\).

Now we consider the convergence of the increasing processes \((K^{n,+}\) and \((K^{n,-}\). By the comparison theorem 4.3 in \([11]\), we get \(K_t^{n,+} \geq K_t^{p,+},\ K_t^{n,+} - K_t^{n,-} \geq K_t^{p,+} - K_t^{p,-}\), for \(0 \leq s \leq t \leq T\). So for \(0 \leq t \leq T\), \(K_t^{n,+} \to K_t^+\), with \(E[(K^{n,+}_t)^2] \leq C\), we get that \(E[(K_t^+)^2] \leq C\). Furthermore, \(K_T^{n,+} - K_T^{p,+} \geq K_T^+ - K_T^{p,+}\), which follows
\[
E[\sup_{0 \leq t \leq T} (K_t^{n,+} - K_t^{p,+})^2] \leq E[(K_T^+ - K_T^{p,+})^2] \to 0,
\]
so \(K^{n,+} \to K^+\) in \(A^2(0, T)\). On the other hand, since \((Y^n, Z^n, K^{n,+}, K^{n,-}\) satisfies
\[
Y^n_t = \xi + \int_t^T f(s, Y^n_s) ds + K_T^{n,+} - K_T^{n,-} - (K_T^+ - K_T^-) - \int_t^T Z^n_s dB_s,
\]
and we can rewrite it in the following form
\[
K_t^{n,-} = Y^n_t - Y^n_0 + \int_0^t f(s, Y^n_s) ds + K_t^{n,+} - \int_0^t Z^n_s dB_s.
\]
Without losing the generality, for \(p < n\), with BDG inequality, we get
\[
E[\sup_{0 \leq t \leq T} (K_t^{n,-} - K_t^{p,-})^2]
\leq 5E[\sup_{0 \leq t \leq T} (Y^n_t - Y^p_t)^2] + 5(Y^n_0 - Y^p_0)^2 + 5TE(\int_0^T (f(s, Y^n_s) - f(s, Y^p_s))^2 ds)
\leq 5E[(K_T^{n,+} - K_T^{p,+})^2] + CE \int_0^T (Z^n_s - Z^p_s)^2 ds
\to 0,
\]
i.e. there exists a process \(K^- \in A^2(0, T)\), s.t. \(K^{n,-} \to K^-\) in \(A^2(0, T)\), and the limit \((Y, Z, K^+, K^-)\) satisfies
\[
Y_t = \xi + \int_t^T f(s, Y_s) ds + K_T^+ - K_t^+ - (K_T^- - K_t^-) - \int_t^T Z_s dB_s.
\]

Since for \(n \in \mathbb{N}, \ Y^n_t \geq L_t,\ 0 \leq t \leq T,\) so \(Y_t \geq L_t\). The last is to check (4) of definition \([21]\). Since \((Y^n, K^{n,+}, K^{n,-})\) tends to \((Y, K^+, K^-)\) uniformly in \(t\) in probability, then the measure \(dK^{n,+}\) converges to \(dK^+\) weakly in probability, so
\[
\int_0^T (Y^n_t - L_t) dK_t^{n,+} \to \int_0^T (Y_t - L_t) dK_t^+,
\]
in probability as \( n \to \infty \). Obviously \( \int_0^T (Y_t - L_t) dK_t^+ \geq 0 \), On the other hand, for each \( n \in \mathbb{N} \), \( \int_0^T (Y^n_t - L_t) dK^{n,+}_t = 0 \). Hence
\[
\int_0^T (Y_t - L_t) dK_t^+ = 0, \text{ a.s.}
\]
Similarly, we have \( \int_0^T (Y_t - U_t) dK_t^- = 0 \). Consequently the triple \( (Y, Z, K^+, K^-) \) is solution of the RBSDE\((\xi, f, L, U)\), under the assumptions (3). \( \square \)

**Step 2.** In this step, we consider the case of a barrier \( L \) which satisfies the assumption 2.3'-(i):
\[
E[\varphi^2(\sup_{0 \leq t \leq T} (L_t^+))] < \infty,
\]
and \( L^+ \in S^2(0, T) \), but we still assume that for some \( C > 0 \),
\[
|\xi| + \sup_{0 \leq t \leq T} |f(t, 0)| + \sup_{0 \leq t \leq T} (U_t)^- \leq C. \tag{21}
\]
For \( n \in \mathbb{N} \), set \( L^n = L \land n \), then \( \sup_{0 \leq t \leq T}(L^n_t) \leq n \) and \( L^n_t \leq L_t \); so assumption 2.3'-
(ii), (iii) are satisfied and by the step 1, we know that there exists a triple \( (Y^n, Z^n, K^n) \), with \( K^n = K^{n,+} - K^{n,-} \), which satisfies
\[
Y^n_t = \xi + \int_t^T f(s, Y^n_s) ds + K^{n,+}_T - K^{n,+}_t - (K^{n,-}_T - K^{n,-}_t) - \int_t^T Z^n_s dB_s, \tag{22}
\]
\( L^n_t \leq Y^n_t \leq U_t, 0 \leq t \leq T \), and \( \int_0^T (Y^n_t - L^n_t) dK^{n,+}_t = \int_0^T (Y^n_t - U_t) dK^{n,-}_t = 0 \).
Consider the solution \((\overline{Y}, \overline{Z}, \overline{K})\) of one lower barrier RBSDE\((\xi, f, L)\) and the solution \((\mathcal{Y}, \mathcal{Z}, \mathcal{K})\) of the super barrier RBSDE\((\xi, f, U)\), in fact these two equations can be considered as the following two barriers RBSDE\((\xi, f, \overline{L})\) and RBSDE\((\xi, f, \overline{U})\), where \( \overline{L} = -\infty \), \( \overline{U} = +\infty \). By the comparison theorem \( \mathcal{3.3} \) it follows that \( \overline{Y}_t \leq Y^n_t \leq \overline{Y}_t, 0 \leq t \leq T \). So
\[
E[\sup_{0 \leq t \leq T} |Y^n_t|^2] \leq \max\{E[\sup_{0 \leq t \leq T} |\overline{Y}_t|^2], E[\sup_{0 \leq t \leq T} |\mathcal{Y}_t|^2] \} \leq C. \tag{23}
\]
Since \( L^n_t \leq L^{n+1}_t, 0 \leq t \leq T \), thanks to the comparison theorem \( \mathcal{3.3} \) \( Y^n_t \not\geq Y_t, 0 \leq t \leq T \). From the above estimate and Fatou’s lemma, we get
\[
E[\sup_{0 \leq t \leq T} (Y_t)^2] \leq C. \tag{24}
\]
And
\[
E\int_0^T |Y^n_t - Y_t|^2 dt \to 0, \text{ as } n \to \infty, \tag{25}
\]
follows from the dominated convergence theorem.
Notice that \( f \) is decreasing on \( y \), then \( f(t, \overline{Y}_t) \leq f(t, Y^n_t) \leq f(t, \mathcal{Y}_t), 0 \leq t \leq T \), and with the integral property of \( \overline{Y} \) and \( \mathcal{Y} \), we have
\[
E[\int_0^t f(s, Y^n_s) ds]^2] \leq \max\{E[(\int_0^t f(s, \overline{Y}_s) ds)^2], E[(\int_0^t f(s, \mathcal{Y}_s) ds)^2] \} \leq C. \tag{26}
\]
In order to prove the convergence of \((Z^n, K^n)\), we first need a-priori estimations. We apply the Itô formula to \(|Y^n_t|^2\) on the interval \([t, T]\),

\[
E |Y^n_t|^2 + E \int_t^T |Z^n_s|^2 \, ds
\leq E |\xi|^2 + E \int_t^T |Y^n_s|^2 \, ds + E \int_t^T |f(s,0)|^2 \, ds + (\alpha + \beta)E \sup_{0 \leq t \leq T} |Y^n_t|^2
\]

\[
+ \frac{1}{\alpha} E[(K^{n,+}_T - K^{n,+}_t)^2] + \frac{1}{\beta} E[(K^{n,-}_T - K^{n,-}_t)^2],
\]

where \(K^n = K^{n,+} - K^{n,-}\). We first use the comparison theorem to estimate \(K^{n,-}\). Consider the linear RBSDE\((\xi, f(s, L_s^-), L, U)\), by existence results of \([1]\), we know there exists \((\tilde{Y}, \tilde{Z}, \tilde{K}^+, \tilde{K}^-) \in S^2(0, T) \times H^2_0(0, T) \times A^2(0, T) \times A^2(0, T)\) satisfying

\[
\tilde{Y}_t = \xi + \int_t^T f(s, (L_s^-)) \, ds + \tilde{K}^+_T - \tilde{K}^+_t - (\tilde{K}^-_T - \tilde{K}^-_t) - \int_t^T \tilde{Z}_s dB_s,
\]

\[
L_t \leq \tilde{Y}_t \leq U_t, \quad \int_0^T (\tilde{Y}_t - L_t) \, d\tilde{K}^+_t = \int_0^T (\tilde{Y}_t - U_t) \, d\tilde{K}^-_t = 0.
\]

Then we have the following lemma, which will be proved in Appendix.

**Lemma 2.3** For \(0 \leq s \leq t \leq T\), \(K^{n,-}_t - K^{n,-}_s \leq \tilde{K}^-_t - \tilde{K}^-_s\), and \(K^{n,-}_T \leq \tilde{K}^-_T\).

Now we have

\[
E[(K^{n,-}_T)^2] \leq E[(\tilde{K}^-_T)^2] \leq C.
\]

We rewrite the RBSDE\((\xi, f, L^n, U)\) \([22]\),

\[
K^{n,+}_T - K^{n,+}_t = Y^n_t - \xi - \int_t^T f(s, Y^n_s) \, ds + (K^{n,-}_T - K^{n,-}_t) + \int_t^T Z^n_s dB_s,
\]

hence

\[
E(K^{n,+}_T - K^{n,+}_t)^2 \leq 5E|Y^n_t|^2 + 5E|\xi|^2 + 5E \left( \int_t^T f(s, Y^n_s) \, ds \right)^2
\]

\[
+ 5E[(K^{n,-}_T - K^{n,-}_t)^2] + 5E \int_t^T |Z^n_s|^2 \, ds
\]

\[
\leq C + 5E \int_t^T |Z^n_s|^2 \, ds.
\]

Then we substitute \((28)\) into \((27)\), set \(\alpha = 10\), \(\beta = 1\), and with \((21)\) and \((23)\), it follows

\[
E(K^{n,+}_T)^2 + E \int_t^T |Z^n_t|^2 \, ds \leq C.
\]

Now for \(n, p \in N, n \geq p\), then \(L^n_t \geq L^p_t\), \(0 \leq t \leq T\). We apply the Itô’s formula to \(|Y^n_t - Y^p_t|^2\) on the interval \([t, T]\), and take expectation

\[
E(|Y^n_t - Y^p_t|^2) + E \int_t^T |Z^n_s - Z^p_s|^2 \, ds \leq 2E \int_t^T (L^n_s - L^p_s) dK^{n,+}_s - 2E \int_t^T (L^n_s - L^p_s) dK^{p,+}_s
\]

\[
\leq 2E \int_t^T (L^n_s - L^p_s) dK^{n,+}_s,
\]

12
in view of $\int_0^T (Y^n_t - Y^p_t) dB_t \geq 0$. Since $L_t - L^p_t \downarrow 0$, for each $t \in [0,T]$, and $L_t - L^p_t$ is continuous, by the Dini’s theorem, the convergence holds uniformly on the interval $[0,T]$, i.e.

$$E[\sup_{0 \leq t \leq T} (L_t - L^p_t)^2] \to 0, \text{ as } n \to \infty.$$  \hspace{1cm} (30)

Then with (28),

$$E \int_0^T |Z^n_s - Z^p_s|^2 ds \leq 2(E(\sup_{0 \leq t \leq T} (L^n_t - L^p_t)^2) \frac{1}{2} (E((K^n_s + K^p_s))^2)^{\frac{1}{2}} \leq C(E(\sup_{0 \leq t \leq T} (L^n_t - L^p_t)^2))^{\frac{1}{2}} \to 0,$$

as $n,p \to \infty$, i.e. $\{Z^n\}$ is a Cauchy sequence in the space $H^2_d(0,T)$, and there exists a process $Z \in H^2_d(0,T)$, s.t. as $n \to \infty$,

$$E \int_0^T |Z^n_s - Z_s|^2 ds \to 0. \hspace{1cm} (31)$$

Furthermore from Itô’s formula, we have

$$\sup_{0 \leq t \leq T} |Y^n_t - Y^p_t|^2 \leq 2 \sup_{0 \leq t \leq T} \int_t^T (L^n_s - L^p_s) d(K^n_s + K^p_s) + 2 \sup_{0 \leq t \leq T} \left| \int_t^T (Y^n_s - Y^p_s)(Z^n_s - Z^p_s) dB_s \right|.$$ 

Taking the expectation on the both sides, by BDG inequality and (29), we get

$$E \sup_{0 \leq t \leq T} |Y^n_t - Y^p_t|^2 \leq 2E\left[ \int_0^T (L^n_s - L^p_s) dK^n_s \right] + CE \int_0^T (Y^n_s - Y^p_s)^2 (Z^n_s - Z^p_s)^2 ds \leq C(2E(\sup_{0 \leq t \leq T} (L^n_t - L^p_t)^2))^\frac{1}{2} \leq 2 \sup_{0 \leq t \leq T} |Y^n_t - Y^p_t|^2 + CE \int_0^T |Z^n_s - Z^p_s|^2 ds.$$ 

Hence, by (31) and (30), as $n,p \to \infty$,

$$E \sup_{0 \leq t \leq T} |Y^n_t - Y^p_t|^2 \to 0, \hspace{1cm} (32)$$

i.e. $\{Y^n\}$ is a Cauchy sequence in the space $S^2(0,T)$, which implies that there exists a process $\nu \in S^2(0,T)$, s.t. as $n \to \infty$,

$$E \sup_{0 \leq t \leq T} |Y^n_t - \nu_t|^2 \to 0. \hspace{1cm} (33)$$

Moreover, since $f$ is continuous and decreasing on $y$, with $Y^n_t \nearrow Y_t$,

$$f(t,Y^n_t) - f(t,Y_t) \searrow 0, \hspace{1cm} 0 \leq t \leq T. \hspace{1cm}$$

By the monotonic limit theorem, we get $\int_0^T [f(t,Y^n_t) - f(t,Y_t)] dt \searrow 0$, and with (26), it follows

$$E[(\int_0^T f(t,Y_t) dt)^2] \leq C,$$

then

$$E[(\int_0^T (f_n(t,Y^n_t) - f(t,Y_t)) dt)^2] \to 0, \hspace{1cm} (34)$$
as \( n \to \infty \).

From corollary [3.1] we know that for \( \forall t \in [0,T] \), \( K_{t}^{n,-} \) is increasing with respect to \( n \), and with \( E[(K_{t}^{n,-})^{2}] \leq C \), there exists \( K_{t}^{n,-} \) such that \( K_{t}^{n,-} \not\to K_{t}^{-} \) in \( L^{2}(\mathcal{F}_{t}) \). Since for each \( t \in [0,T] \), 
\[ E[(K_{t}^{n,+})^{2}] \leq C, \]
the sequence \( (K_{t}^{n,+}) \) has weak limit \( K_{t}^{+} \) in \( L^{2}(\mathcal{F}_{t}) \), with \( E[(K_{t}^{+})^{2}] \leq C \). Then for \( 0 \leq t \leq T \), \( (Y,Z,K_{t},K_{t}^{+},K_{t}^{-}) \) satisfies

\[
Y_{t} = \xi + \int_{t}^{T} f(s,Y_{s})ds + K_{t}^{+} - K_{t}^{-} - \int_{t}^{T} Z_{s}dB_{s}.
\]

(35)

We will then prove that the convergence of \( \{K_{t}^{n,+}\} \) and \( \{K_{t}^{n,-}\} \) also holds in strong sense. First, we consider \( \{K_{t}^{n,-}\} \), for \( n, p \in \mathbb{N} \), with \( n \geq p, \) since \( L_{t}^{n} \geq L_{t}^{p} \), by corollary [3.1], we have for \( 0 \leq s \leq t \leq T \), \( K_{t}^{n,-} - K_{s}^{n,-} \geq K_{t}^{p,-} - K_{s}^{p,-} \). So \( 0 \leq K_{t}^{n,-} - K_{t}^{p,-} \leq K_{t}^{-} - K_{T}^{-} \), and it follows immediately by letting \( n \to \infty \)

\[
0 \leq K_{t}^{-} - K_{t}^{p,-} \leq K_{T}^{-} - K_{T}^{p,-}.
\]

This inequality yields as \( p \to \infty \),

\[
E \sup_{0 \leq t \leq T} \left| K_{t}^{-} - K_{t}^{p,-} \right|^{2} \leq E \left| K_{T}^{-} - K_{T}^{p,-} \right|^{2} \to 0.
\]

(36)

Then we consider the term \( \{K_{t}^{n,+}\} \). For this we rewrite (22) and (35) in the forward form:

\[
K_{t}^{n,+} = Y_{0}^{n} - Y_{t} - \int_{0}^{t} f(s,Y_{s})ds + K_{t}^{n,-} + \int_{0}^{t} Z_{s}^{n}dB_{s},
\]

\[
K_{t}^{+} = Y_{0} - Y_{t} - \int_{0}^{t} f(s,Y_{s})ds + K_{t}^{-} + \int_{0}^{t} Z_{s}dB_{s},
\]

so consider the difference and take expectation on the both sides, by the BDG inequality, and \( f(s,Y_{s}) \geq f(s,Y_{s}) \), it follows

\[
E[ \sup_{0 \leq t \leq T} \left| K_{t}^{n,+} - K_{t}^{+} \right|^{2} ] \leq 5 |Y_{0}^{n} - Y_{0}|^{2} + 5E[ \sup_{0 \leq t \leq T} |Y_{t}^{n} - Y_{t}|^{2} ] + 5E(\int_{0}^{T} |f(s,Y_{s}) - f(s,Y_{s})|ds)^{2}
\]

\[
+ 5E \sup_{0 \leq t \leq T} \left| K_{t}^{n,-} - K_{t}^{-} \right|^{2} + CE \int_{0}^{T} |Z_{s}^{n} - Z_{s}|^{2} ds.
\]

Then by (33), (34), (36) and (31), we deduce that

\[
E[ \sup_{0 \leq t \leq T} \left| K_{t}^{n,+} - K_{t}^{+} \right|^{2} ] \to 0.
\]

The last thing to check is that (3) and (4) are also satisfied. Since for each \( n \in \mathbb{N} \), \( L_{t}^{n} \leq Y_{t}^{n} \leq U_{t}, \) \( 0 \leq t \leq T \), with \( Y_{t}^{n} \not\to Y_{t} \) and \( L_{t}^{n} \not\to L_{t} \), then \( L_{t} \leq Y_{t} \leq U_{t} \). From another part, the processes \( K_{t}^{n,+} \) and \( K_{t}^{n,-} \) are increasing, so the limit \( K_{t}^{+} \) and \( K_{t}^{-} \) are also increasing. Notice that \( (Y^{n}, K_{t}^{n,+}, K_{t}^{n,-}) \) tends to \( (Y, K^{+}, K^{-}) \) uniformly in \( t \) in probability, so the measure \( dK_{t}^{n,+} \) (resp. \( dK_{t}^{n,-} \)) converges to \( dK^{+} \) (resp. \( dK^{-} \)) weakly in probability. So

\[
\int_{0}^{T} (Y_{t} - L_{t})dK_{t}^{n,+} \to \int_{0}^{T} (Y_{t} - L_{t})dK_{t}^{+} , \int_{0}^{T} (Y_{t}^{n} - U_{t})dK_{t}^{n,-} \to \int_{0}^{T} (Y_{t} - U_{t})dK_{t}^{-}.
\]
in probability as \( n \to \infty \). Obviously \( \int_0^T (Y_t - U_t) dK_t^- \leq 0 \). On the other hand, for each \( n \in \mathbb{N} \),
\[
\int_0^T (Y_t^n - U_t) dK_t^{n,-} = 0.
\]
Hence
\[
\int_0^T (Y_t - U_t) dK_t^- = 0, \text{ a.s.}
\]
For the lower barrier, since \( L^n \) converges to \( L \) in \( S^2 \), as \( n \to \infty \), we have
\[
E \int_0^T (Y_t^n - L^n_t) dK_t^{n,+} + E \int_0^T (Y_t - L_t) dK_t^+
\]
\[
= E \int_0^T (Y_t^n - Y_t) dK_t^{n,+} + E \int_0^T (Y_t - L_t) d(K_t^{n,+} - K_t^+) + E \int_0^T (L_t - L^n_t) dK_t^{n,+}
\]
\[
\leq C(E[ \sup_{0 \leq t \leq T} (Y_t^n - Y_t)^2]) \frac{1}{n} + E \int_0^T (Y_t - L_t) d(K_t^{n,+} - K_t^+) + C(E[ \sup_{0 \leq t \leq T} (L_t - L^n_t)^2]) \frac{1}{n} \to 0.
\]
Since \( Y_t \geq L_t \), then \( \int_0^T (Y_t - L_t) dK_t^+ \geq 0 \), while \( E \int_0^T (Y_t^n - L^n_t) dK_t^{n,+} = 0 \), so \( E \int_0^T (Y_t - L_t) dK_t^+ = 0 \), then \( \int_0^T (Y_t - L_t) dK_t^+ = 0 \). □

**Step 3.** In this step, we study the general case for \( L \) and \( U \), when assumption 2.3’ is satisfied:
\[
E[\varphi^2(\sup_{0 \leq t \leq T} (L_t)^+)] + E[\varphi^2(\sup_{0 \leq t \leq T} (U_t)^-)] < \infty,
\]
\( L^+, U^- \in S^2(0,T) \). But we still assume that for some \( C > 0 \),
\[
|\xi| + \sup_{0 \leq t \leq T} |f(t,0)| \leq C.
\]

For \( n \in \mathbb{N} \), set \( U^n = U \vee (-n) \); then \( \sup_{0 \leq t \leq T} (U^n)^- \leq n \) and \( U^n \geq U \), so assumption 2.3’-(ii), (iii) are satisfied, and by the step 2, we know that there exists a triple \( (Y^n, Z^n, K^n) \), with \( K^n = K^{n,+} - K^{n,-} \), which satisfies
\[
Y_t^n = \xi + \int_t^T f(s, Y^n_s) ds + K^{n,+}_T - K^{n,-}_T - (K^{n,-}_t - K^{n,+}_t) - \int_t^T Z^n_s dB_s, \tag{38}
\]
\( L_t \leq Y_t^n \leq U^n_t, 0 \leq t \leq T \), and \( \int_0^T (Y_t^n - L_t) dK_t^{n,+} = \int_0^T (Y_t^n - U^n_t) dK_t^{n,-} = 0 \).

Like in step 2, we consider the solution \((\bar{Y}, \bar{Z}, \bar{K})\) of the one lower barrier RBSDE(\( \xi, f, L \)) and the solution \((\bar{Y}, \bar{Z}, \bar{K})\) of the one super barrier RBSDE(\( \xi, f, U \)). Then by the comparison theorem \[33\] it follows that \( \sum_{t \leq T} \leq Y_t^n \leq \bar{Y}_t, 0 \leq t \leq T \). So
\[
E[ \sup_{0 \leq t \leq T} |Y_t^n|^2 ] \leq \max\{ E[ \sup_{0 \leq t \leq T} |\bar{Y}_t|^2 ], E[ \sup_{0 \leq t \leq T} |\bar{Y}_t|^2 ] \} \leq C. \tag{39}
\]
Since \( U_t^{n+1} \leq U_t^n, 0 \leq t \leq T \), thanks to the comparison theorem \[33\] \( Y^n_t \searrow Y_t, 0 \leq t \leq T \). From \[39\] and Fatou’s lemma, we get
\[
E[ \sup_{0 \leq t \leq T} (Y_t)^2 ] \leq C, \tag{40}
\]
and
\[
E \int_0^T |Y_t^n - Y_t|^2 dt \to 0, \text{ as } n \to \infty, \tag{41}
\]
which follows from the dominated convergence theorem.
Notice that \( f \) is decreasing on \( y \), then \( f(t, \underline{Y}_t) \leq f(t, \underline{Y}_t^n) \leq f(t, \overline{Y}_t), \ 0 \leq t \leq T, \) and with the integral property of \( \underline{Y} \) and \( \overline{Y} \), we have
\[
E[\int_0^T f(s, Y_s^n) ds]^2 \leq \max \{E[\int_0^T f(s, \underline{Y}_s) ds]^2], E[\int_0^T f(s, \overline{Y}_s) ds]^2] \leq C. \tag{42}
\]

Then we use again the comparison theorem for the estimation of \( K_t^{n,+} \). Consider the linear RBSDE\((\xi, f(s, U_s^n), L, U)\), by results of [1], we know that there exists \( (\tilde{Y}, \tilde{Z}, \tilde{K}^+, \tilde{K}^-) \in S^2(0, T) \times H_d^2(0, T) \times \mathbb{A}^2(0, T) \times \mathbb{A}^2(0, T) \) satisfying the following:
\[
\tilde{Y}_t = \xi + \int_t^T f(s, U_s^n) ds + \tilde{K}^+_T - \tilde{K}^-_t - (\tilde{K}^-_T - \tilde{K}^-_t) - \int_t^T \tilde{Z}_s dB_s,
\]
\[
L_t \leq \tilde{Y}_t \leq U_t, \int_0^T (\tilde{Y}_t - L_t) d\tilde{K}^+_t = \int_0^T (\tilde{Y}_t - U_t) d\tilde{K}^-_t = 0.
\]

We admits for a instant the following lemma, which will be proved later.

**Lemma 2.4** For \( 0 \leq s \leq t \leq T, \ K_t^{n,+} - K_s^{n,+} \leq \tilde{K}_t^+ - \tilde{K}_s^+, \) and \( K_t^{n,+} \leq \tilde{K}_T^+ \).

Now we have
\[
E[(K_t^{n,+})^2] \leq E[(\tilde{K}_t^+)^2] \leq C,
\]
Then apply the Itô’s formula to \( |Y_t^n|^2 \) on the interval \([t, T]\), by the same method as in step 2, we have the following estimates
\[
E[(K_t^{n,-})^2] + E \int_0^T |Z_t^n|^2 ds \leq C. \tag{43}
\]

Since \( U_t^n - U_t \downarrow 0 \), for each \( t \in [0, T] \), and \( U_t^n - U_t \) is continuous, by the Dini’s theorem again, the convergence holds uniformly on the interval \([0, T]\), i.e.
\[
E[\sup_{0 \leq t \leq T} (U_t^n - U_t)^2] \to 0, \text{ as } n \to \infty. \tag{44}
\]

Now we are in the same situation as step 2. With the same arguments, we deduce that there exists processes \( Y \in S^2(0, T), Z \in H_d^2(0, T), K^+ \in A^2(0, T), K^- \in A^2(0, T), \) s.t. as \( n \to \infty, \)
\[
E[\sup_{0 \leq t \leq T} |Y_t^n - Y_t| + \int_0^T |Z_t^n - Z_s|^2 ds + \sup_{0 \leq t \leq T} |K_t^{n,+} - K_t^+|^2 + \sup_{0 \leq t \leq T} |K_t^{n,-} - K_t^-|^2 ] \to 0,
\]
which satisfies
\[
Y_t = \xi + \int_t^T f(s, Y_s) ds + K^+_T - K^-_t - (K^-_T - K^-_t) - \int_t^T Z_s dB_s. \tag{45}
\]

The last thing to check is that (3) and (4) of definition [2,1] are satisfied. Since for each \( n \in \mathbb{N}, \)
\( L_t \leq Y_t^n \leq U_t^n, \ 0 \leq t \leq T, \) with \( Y_t^n \downarrow Y_t \) and \( U_t^n \downarrow U_t, \) then \( L_t \leq Y_t \leq U_t. \) On the other hand, the processes \( K^{n,+} \) and \( K^{n,-} \) are increasing, so the limit \( K^+ \) and \( K^- \) are also increasing. Notice that \( (Y^n, K^{n,+}, K^{n,-}) \) tends to \( (Y, K^+, K^-) \) uniformly in \( t \) in probability, and \( U^n \) converges to \( U \) in \( S^2, \) as \( n \to \infty, \) similarly as step 2, we get
\[
\int_0^T (Y_t - K_t)dK_t^+ = \int_0^T (Y_t - U_t)dK_t^- = 0, \hspace{1em} \text{a.s.}
\]
Step 4. In this step, we will partly relax the bounded assumption for \( \xi \) and \( f(t,0) \). We only suppose that for a constant \( c \),
\[
\xi \geq c \quad \text{and} \quad \inf_{0 \leq t \leq T} f(t,0) \geq c. \tag{46}
\]
We approximate \( \xi \) and \( f(t,0) \) by a sequence whose elements satisfy the bounded assumption in step 3, as following: for \( n \in \mathbb{N} \), set
\[
\xi_n = \xi \wedge n, \quad f_n(t,y) = f(t,y) - f(t,0) + f(t,0) \wedge n.
\]
Obviously, \((\xi_n, f_n)\) satisfies the assumptions of the step 3, and since \( \xi \in L^2(\mathcal{F}_T) \), \( f(t,0) \in H^2(0,T) \), then
\[
E[|\xi^n - \xi|^2] \to 0, \quad E \int_0^T |f(t,0) - f_n(t,0)|^2 \to 0, \tag{47}
\]
as \( n \to \infty \).

From the results in step 3, for each \( n \in \mathbb{N} \), there exists \((Y^n_t, Z^n_t, K^n_t)_{0 \leq t \leq T} \in \mathbb{S}^2(0,T) \times H^2_d(0,T)\times \mathbf{VF}^2(0,T) \), with \( K^n = K^{n,+} - K^{n,-} \), which is the unique solution of the RBSDE \((\xi^n, f_n, L, U)\), i.e.
\[
Y^n_t = \xi^n + \int_t^T f_n(s, Y^n_s) ds + K^{n,+}_t - K^{n,-}_t - \int_t^T Z^n_s dB_s, \tag{48}
\]
\[
L_t \leq Y^n_t \leq U_t, \quad \int_0^T (Y^n_t - L_t) dK^{n,+}_t = \int_0^T (Y^n_t - U_t) dK^{n,-}_t = 0.
\]
Like in step 3, we consider the solution \((\overline{Y}, \overline{Z}, \overline{K})\) of one lower barrier RBSDE \((\xi, f, L)\) and the solution \((\underline{Y}, \underline{Z}, \underline{K})\) of one super barrier RBSDE \((\underline{\xi}, f, U)\), where \( \underline{\xi} \) is the negative part of \( \xi \), \( f(t,y) = f(t,y) - f(t,0) + (f(t,0))^− \). Then we can take the RBSDE \((\xi, f, L)\) (resp. RBSDE \((\underline{\xi}, f, U)\)) as a RBSDE with two barriers associated to the parameters \((\xi, f, L, \overline{U})\) (resp. \((\underline{\xi}, f, \underline{L}, U)\)) where \( \overline{U} = \infty \) and \( \underline{L} = -\infty \). By the comparison theorem 3.3, since
\[
\xi \geq \xi^n \geq \underline{\xi}, \quad f(t,y) \geq f_n(t,y) \geq \underline{f}(t,y),
\]
it follows that
\[
\underline{Y}_t \leq Y^n_t \leq \overline{Y}_t, \quad 0 \leq t \leq T.
\]
So
\[
E[\sup_{0 \leq t \leq T} |Y^n_t|^2] \leq \max\{E[\sup_{0 \leq t \leq T} |\overline{Y}_t|^2], E[\sup_{0 \leq t \leq T} |\underline{Y}_t|^2]\} \leq C.
\]
Then by the comparison theorem 3.6 since for all \((s,y) \in [0,T] \times \mathbb{R}, n \in \mathbb{N}, \xi_1 \leq \xi_n, f_1(s,y) \leq f_n(s,y)\), we have \( K^{1,+}_t \geq K^{n,+}_t \geq 0 \) for \( 0 \leq t \leq T \), so \( E[(K^{1,+}_t)^2] \leq E[(K^{n,+}_t)^2] \leq C \). Following the same steps, we deduce that
\[
E[\int_0^T f(s,Y^n_s) ds] + E \int_0^T |Z^n_s|^2 ds + E[(K^{1,-}_t)^2] + E[(K^{n,-}_t)^2] \leq C. \tag{49}
\]
Due to the comparison theorem 3.3 since for all \((s,y) \in [0,T] \times \mathbb{R}, n \in \mathbb{N}, \xi_n \leq \xi_{n+1}, f_n(s,y) \leq f_{n+1}(s,y)\), we have \( Y^n_t \leq Y^{n+1}_t, \ 0 \leq t \leq T, \ a.s. \) Hence
\[
Y^n_t \uparrow Y_t, \ 0 \leq t \leq T, \ a.s. \tag{50}
\]
Applying Itô formula to $|Y_t^n - Y_t^p|^2$, for $n, p \in \mathbb{N}$, $n \geq p$, on $[t, T]$, we get
\[
E|Y_t^n - Y_t^p|^2 + E\int_t^T |Z_s^n - Z_s^p|^2 \, ds \\
\leq E|\xi^n - \xi|^2 + E\int_t^T |Y_s^n - Y_s^p|^2 \, ds + E\int_t^T |f_n(s, 0) - f_p(s, 0)|^2 \, ds,
\]
since $\int_t^T (Y_s^n - Y_s^p)d(K_s^{n+} - K_s^{p+}) - \int_t^T (Y_s^n - Y_s^p)d(K_s^{n-} - K_s^{p-}) \leq 0$. Hence from Gronwall's inequality and (47), we deduce
\[
\sup_{0 \leq t \leq T} E|Y_t^n - Y_t^p|^2 \to 0, \quad E\int_0^T |Z_s^n - Z_s^p|^2 \, ds \to 0. \tag{51}
\]
Consequently there exists $(Z_t)_{0 \leq t \leq T} \in \mathcal{H}_d^2(0, T)$, s.t.
\[
E\int_0^T |Z_s^n - Z_s|^2 \, ds \to 0. \tag{52}
\]

Using again Itô formula, taking sup and the expectation, in view of the BDG inequality, $Y_t^n \geq Y_t^p$, assumption 2.4-(iii) and $f_n(t, 0) \geq f_p(t, 0)$, we get
\[
E\left[ \sup_{0 \leq t \leq T} |Y_t^n - Y_t^p|^2 \right] \leq E|\xi^n - \xi|^2 + 4TE\int_0^T |f_n(s, 0) - f_p(s, 0)|^2 \, ds + \frac{1}{4} E\sup_{0 \leq t \leq T} |Y_s^n - Y_s^p|^2 \\
+ \frac{1}{4} E\left[ \sup_{0 \leq t \leq T} |Y_t^n - Y_t^p|^2 \right] + cE\int_0^T |Z_s^n - Z_s^p|^2 \, ds.
\]

From (47) and (51), it follows $E[\sup_{0 \leq t \leq T} |Y_t^n - Y_t^p|^2] \to 0$, as $n, p \to \infty$, i.e. the sequence $\{Y^n\}$ is a Cauchy sequence in the space $\mathcal{S}^2(0, T)$. Consequently, with (50), we have $Y \in \mathcal{S}^2(0, T)$ and
\[
E\left[ \sup_{0 \leq t \leq T} |Y_t^n - Y_t|^2 \right] \to 0. \tag{53}
\]

By the comparison theorem 3.6 since for all $(s, y) \in [0, T] \times \mathbb{R}$, $n \in \mathbb{N}$, $\xi_n \leq \xi_{n+1}$, $f_n(s, y) \leq f_{n+1}(s, y)$, we have $K_t^{n+} \geq K_t^{n+1+} \geq 0$, and $0 \leq K_t^{n-} \leq K_t^{n+} - K_t^{-}$ for $0 \leq t \leq T$, so
\[
K_t^{n+} \searrow K_t^+, \quad K_t^{-} \nearrow K_t^-,
\]
with (49), by the monotonic limit theorem, it follows that $K_t^{n+} \to K_t^+$, $K_t^{n-} \to K_t^-$ in $\mathcal{L}^2(\mathcal{F}_t)$, and $E[(K_t^{n+})^2 + (K_t^{-})^2] < \infty$, moreover, $(K_t^+)_{0 \leq t \leq T}$ and $(K_t^-)_{0 \leq t \leq T}$ are increasing.

Notice that since $f(t, y)$ is decreasing and continuous in $y$, and $Y_t^n \nearrow Y_t$, we have $f(t, Y_t^n) \searrow f(t, Y_t)$. Then by the monotonic limit theorem, $\int_0^t f(s, Y_s^n) \, ds \searrow \int_0^t f(s, Y_s) \, ds$. With (49), it follows that $\int_0^t f(s, Y_s^n) \, ds \to \int_0^t f(s, Y_s) \, ds$ in $\mathcal{L}^2(\mathcal{F}_t)$, as $n \to \infty$.

Now we need to prove that the convergence of $\{K_t^{n+}\}$ and $\{K_t^{-}\}$ holds in a stronger sense. Using again the comparison theorem 3.6 since for all $(s, y) \in [0, T] \times \mathbb{R}$, $n, p \in \mathbb{N}$, with $n \geq p$, $\xi_p \leq \xi_n$, $f_p(s, y) \leq f_n(s, y)$, we have for $0 \leq s \leq t \leq T$,
\[
K_t^{p+} - K_t^{-} \geq K_t^{n+} - K_t^{-} \geq 0,
\]
Then let $n \to \infty$, for $t \in [0, T]$, $K_t^{p+} - K_t^+ \geq K_t^{n+} - K_t^+ \geq 0$. So as $n \to 0$,
\[
E\sup_{0 \leq t \leq T} |K_t^{p+} - K_t^+|^2 \leq E|K_t^{p+} - K_t^+|^2 \to 0.
\]
Similarly, we have \( E \sup_{0 \leq t \leq T} \left| K^-_t - K^-_p \right|^2 \leq E \left| K^-_T - K^-_p \right|^2 \to 0. \)

It remains to check if \((Y_t, Z_t, K_t)_{0 \leq t \leq T}\) satisfies (3) and (4) of the definition \([2.1]\). Since \(L_t \leq Y_t \leq U_t, 0 \leq t \leq T\), then letting \(n \to \infty\), \(L_t \leq Y_t \leq U_t, 0 \leq t \leq T\), a.s.. Furthermore \((Y^n, K^{n,+})\) tends to \((Y, K^+)\) uniformly in \(t\) in probability, as \(n \to \infty\), then the measure \(dK^{n,+} \to dK^+\) weakly in probability, as \(n \to \infty\), i.e. \( \int_0^T (Y^n_t - L_t) dK^{n,+}_t \to \int_0^T (Y_t - L_t) dK^+_t \), in probability. While \(L_t \leq Y_t \leq U_t, 0 \leq t \leq T\), so \( \int_0^T (Y_t - L_t) dK^+_t \geq 0 \). On the other hand \( \int_0^T (Y^n_t - L_t) dK^{n,+}_t = 0 \), so \( \int_0^T (Y_t - L_t) dK^+_t = 0 \). Similarly, \( \int_0^T (Y_t - U_t) dK^-_t = 0 \), i.e. the triple \((Y_t, Z_t, K_t)_{0 \leq t \leq T}\) is the solution of \(\text{RBSDE}(\xi, f, L)\), under the assumption \([16]\). \(\square\)

**Step 5.** Now we consider a terminal condition \(\xi \in L^2(\mathcal{F}_T)\) and a coefficient \(f\) which satisfies assumption \(2.4\). For \(n \in \mathbb{N}\), set

\[
\xi_n = \xi \vee (-n), f_n(t, y) = f(t, y) - f(t, 0) + f(t, 0) \vee (-n).
\]

Obviously, \((\xi^n, f^n)\) satisfies the assumptions of the step 4, and since \(\xi \in L^2(\mathcal{F}_T), f(t, 0) \in H^2(0, T)\), then

\[
E[|\xi^n - \xi|^2] \to 0, E \int_0^T |f(t, 0) - f_n(t, 0)|^2 \to 0,
\]

as \(n \to \infty\).

From the results in step 3, for each \(n \in \mathbb{N}\), there exists \((Y^n_t, Z^n_t, K^n_t)_{0 \leq t \leq T} \in S^2(0, T) \times H^2(0, T) \times V^2(0, T)\), with \(K^n = K^{n,+} - K^{n,-}\), which is the unique solution of the RBSDE\((\xi^n, f_n, L, U)\). Like in step 4, we consider the solution \((Y, Z, K)\) of the one lower barrier RBSDE\((\xi^+, f, L, U)\), where \(\xi^+\) is the positive part of \(\xi\), \(\xi^+(t, y) = f(t, y) - f(t, 0) + (f(t, 0))^+\), and the solution \((Y, Z, K)\) of the one super barrier RBSDE\((\xi, f, U)\). Then we can take the RBSDE\((\xi^+, f, L)\) (resp. RBSDE\((\xi, f, U)\)) as a RBSDE with two barriers associated to the parameters \((\xi^+, f, L, U)\) (resp. \((\xi, f, L, U)\)), where \(\bar{U} = \infty\) and \(L = -\infty\). Thanks to the comparison theorem \([3.3]\) we have that

\[
E[\sup_{0 \leq t \leq T} |Y^n_t|^2] \leq \max\{E[\sup_{0 \leq t \leq T} |Y_t|^2], E[\sup_{0 \leq t \leq T} |Y^n_t|^2]\} \leq C.
\]

For \(n, p \in \mathbb{N}\), with \(n \geq p\), we have \(\xi^n \leq \xi^p\) and \(f_n(t, y) \leq f_p(t, y), \forall (t, y) \in [0, T] \times \mathbb{R}\). From approximations for \(\xi^n, \xi^p, f_n(t, y)\) and \(f_p(t, y)\) as following:

\[
\xi^{n,m} := \xi^n \wedge m, \xi^{p,m} := \xi^p \wedge m
\]

\[
f_{n,m}(t, y) = f_n(t, y) - f_n(t, 0) + f_n(t, 0) \wedge m = f(t, y) - f(t, 0) + (f(t, 0) \vee (-n)) \wedge m,
\]

\[
f_{p,m}(t, y) = f_p(t, y) - f_p(t, 0) + f_p(t, 0) \wedge m = f(t, y) - f(t, 0) + (f(t, 0) \vee (-p)) \wedge m,
\]

then the parameters satisfy the assumptions in theorem \([3.6]\) and

\[
\xi^{n,m} \leq \xi^{p,m}, f_{n,m}(t, y) \leq f_{p,m}(t, y).
\]

Consider the solution \((Y^{n,m}, Z^{n,m}, K^{n,m})\) (resp. \((Y^{p,m}, Z^{p,m}, K^{p,m})\)) of the RBSDE\((\xi^{n,m}, f_{n,m}, L, U)\) (resp. \((\xi^{p,m}, f_{p,m}, L, U)\)); by the comparison theorem \([3.6]\) for \(0 \leq s \leq t \leq T\), we have \(K^{n,m,-}_t - K^{n,m,-}_s \leq K^{p,m,-}_t - K^{p,m,-}_s\). Then by the convergence results in step 4, let \(m \to \infty\), we get

\[
K^{n,-}_t - K^{n,-}_s \leq K^{p,-}_t - K^{p,-}_s, \text{ for } n \geq p.
\]

So we have \(0 \leq K^{n,-}_t \leq K^{p,-}_t\), then \(E[(K^{n,-}_t)^2] \leq E[(K^{p,-}_t)^2] \leq C\). By the same method as previous step, we deduce that

\[
E[\int_0^T f(s, Y^n_s) ds]^2 + E[(K^{n,-}_T)^2] + E \int_0^T |Z^n_s|^2 ds \leq C.
\]
Now we are in the same situation as in step 4, and following the same method, we get that the sequence \((Y_t^n, Z_t^n, K_t^{n,+}, K_t^{n,-})\) converge to \((Y_t, Z_t, K_t^+, K_t^-)\) as \(n \to \infty\), in \(S^2(0,T) \times H^2_d(0,T) \times A^2(0,T) \times A^2(0,T)\), and \((Y_t, Z_t, K_t^+, K_t^-)\) is the solution to the RBSDE\((\xi, f, L, U)\). □

3 Appendix

3.1 Proofs of Lemmas

In this subsection, we prove lemma 2.3 and lemma 2.4, which play important roles in previous section.

**Proof of Lemma 2.3.** Obviously \(f(s, (L_s)^-) \in H^2(0,T)\), in view of assumption 2.3’. Consider for \(m, n \in \mathbb{N}\), the following RBSDEs with one lower barrier,

\[
\tilde{Y}^m_t = \xi + \int_t^T f(s, (L_s)^-)ds - m \int_t^T (\tilde{Y}^m_s - U_s)^+ds + \tilde{K}^{m,+}_t - \tilde{K}^{m,-}_t - \int_t^T \tilde{Z}^m_sdB_s,
\]

and

\[
\tilde{Y}^m_t \geq L_t, \int_0^T (\tilde{Y}^m_t - L_t)d\tilde{K}^{m,+}_t = 0,
\]

Since \(Y_t^{m,n} \geq L_t^n \geq (L_t)^-\), we get \(f(t, Y_t^{m,n}) \leq f(t, L_t^n) \leq f(t, (L_t)^-).\) Then for \(m, n \in \mathbb{N}\), \(\forall t \in [0,T]\)

\[
f(t, Y_t^{m,n}) - m(Y_t^{m,n} - U_t)^+ \leq f(t, (L_t)^-) - m(Y_t^{m,n} - U_t)^+, \quad L_t^n \leq L_t.
\]

By the general comparison theorem for RBSDE with one barrier theorem \[3.1\] it follows \(Y_t^{m,n} \leq \tilde{Y}^m_t, \forall t \in [0,T].\) Denote \(\tilde{K}^{m,-}_t = m \int_0^t (\tilde{Y}^m_s - U_s)^+ds\) and \(K^{m,-}_t = m \int_0^t (Y_s^{m,n} - U_s)^+ds\), then we get for \(0 \leq s \leq t \leq T\)

\[
K^{m,-}_t - K^{m,n,-}_s \leq \tilde{K}^{m,-}_t - \tilde{K}^{m,-}_s.
\] \hspace{1cm} (55)

Thanks to the convergence result in step 1 and in \[12\], notice that \((L^n)^+\) is bounded, we know that as \(m \to \infty, \tilde{K}^{m,-}_t \to K^-_t, K^{m,n,-}_t \to K^{n,-}_t\), in \(L^2(\mathcal{F}_t)\). Here \(K^-_t\) and \(K^{n,-}_t\) are increasing processes with respect to the upper barrier \(U\) of the solution of the RBSDE\((\xi, f, (L_t^-), U)\) and RBSDE\((\xi, f, L^n, U)\), respectively. Then from (55), we deduce that for \(0 \leq s \leq t \leq T\)

\[
K^{n,-}_t - K^{m,n,-}_s \leq K^-_t - K^-_s.
\]

It follows immediately that \(K^{n,-}_T \leq K^-_T\). □

**Proof of Lemma 2.4.** Obviously \(f(s, U_s^+) \in H^2(0,T)\), in view of assumption 2.3’-(i). Consider the following RBSDEs with one barrier, for \(n, m, p \in \mathbb{N}\), with \(U^n = U \cup (-n),\)

\[
\tilde{Y}^{n,m,p}_t = \xi + \int_t^T f(s, (U_s)^+)ds + m \int_t^T (\tilde{Y}^{n,m,p}_s - L^p_s)^-ds - (\tilde{K}^{n,m,p,-}_t - \tilde{K}^{n,m,p,-}_t) - \int_t^T \tilde{Z}^{n,m,p}_sdB_s,
\]

and

\[
\tilde{Y}^{n,m,p}_t \leq U^n_t, \int_0^T (\tilde{Y}^{n,m,p}_t - U^n_t)d\tilde{K}^{n,m,p,-}_t = 0,
\]

\[
\tilde{Y}^{n,m,p}_t = \xi + \int_t^T f(s, (U_s)^+)ds + m \int_t^T (\tilde{Y}^{n,m,p}_s - L^p_s)^-ds - (\tilde{K}^{n,m,p,-}_t - \tilde{K}^{n,m,p,-}_t) - \int_t^T \tilde{Z}^{n,m,p}_sdB_s,
\]

and

\[
\tilde{Y}^{n,m,p}_t \geq U^n_t, \int_0^T (\tilde{Y}^{n,m,p}_t - U^n_t)d\tilde{K}^{n,m,p,-}_t = 0,
\]

\[
\tilde{Y}^{n,m,p}_t = \xi + \int_t^T f(s, (U_s)^+)ds + m \int_t^T (\tilde{Y}^{n,m,p}_s - L^p_s)^-ds - (\tilde{K}^{n,m,p,-}_t - \tilde{K}^{n,m,p,-}_t) - \int_t^T \tilde{Z}^{n,m,p}_sdB_s,
\]

and

\[
\tilde{Y}^{n,m,p}_t \geq U^n_t, \int_0^T (\tilde{Y}^{n,m,p}_t - U^n_t)d\tilde{K}^{n,m,p,-}_t = 0.
\]
Notice that from general comparison theorem for RBSDE with one barrier \(3.1\), we have
\[
Y^{n,m,p}_t = \xi + \int_t^T f(s, Y^{n,m,p}_s) ds + m \int_t^T (Y^{n,m,p}_s - L^p_s)^- ds - (K^{n,m,p}_T - K^{n,m,p}_t) - \int_t^T Z^{n,m,p}_s dB_s,
\]
\[
Y^{n,m,p}_t \leq U^n_t, \int_0^T (Y^{n,m,p}_t - U^n_t) dK^{n,m,p}_t = 0.
\]
Since \(Y^{n,m,p}_t \leq U^n_t \leq (U_t)^+\), by monotonic property of \(f\), we get \(f(t, Y^{n,m,p}_t) \geq f(t, (U_t)^+)\). So
\[
f(t, (U_t)^+) + m(Y^{n,m,p}_t - L^p_t)^- \leq f(t, Y^{n,m,p}_t) + m(Y^{n,m,p}_t - L^p_t)^-, U_t \leq U^n_t,
\]
from general comparison theorem for RBSDE with one barrier \(3.1\) we have \(Y^{n,m,p}_t \geq \tilde{Y}^{n,m,p}_t\). Set
\[
K^{n,m,p,+}_t = m \int_0^t (Y^{n,m,p}_s - L^p_s)^- ds, \quad \tilde{K}^{n,m,p,+}_t = m \int_0^t (Y^{n,m,p}_s - L^p_s)^- ds,
\]
for \(0 \leq s \leq t \leq T\)
\[
K^{n,m,p,+}_t - K^{n,m,p}_t \leq \tilde{K}^{n,m,p,+}_t - \tilde{K}^{n,m,p}_t.
\]
Notice that \((L^p)^+\) and \((U^n)^-\) are bounded, by convergence results in step 1, and the convergence result in \(12\), as \(m \to \infty\), for \(0 \leq s \leq t \leq T\), we have
\[
K^{n,p,+}_t - K^{n,p}_t \leq \tilde{K}^{n,p,+}_t - \tilde{K}^{n,p}_t,
\]
where \(K^{n,p,+}_t\) and \(\tilde{K}^{n,p,+}_t\) are the increasing processes corresponding to lower barrier \(L^p\) for RBSDE\((\xi, f, L^p, U^n)\) and RBSDE\((\xi, f(t,(U_t)^+), L^p, U^n)\), respectively. Then thanks to the convergence result in step 2 for the approximation of lower barrier \(L\), we have that as \(p \to \infty\),
\[
K^{n,p,+}_t \to K^{n,+}_t\text{ and } \tilde{K}^{n,p,+}_t \to \tilde{K}^{n,+}_t \text{ in } L^2(\mathcal{F}_t).
\]
Here \(K^{n,+}_t\) (resp. \(\tilde{K}^{n,+}_t\)) is the increasing process corresponding to lower barrier \(L\) for RBSDE\((\xi, f, L, U^n)\) (resp. RBSDE\((\xi, f(t,(U_t)^+), L, U^n)\)). It follows for \(0 \leq s \leq t \leq T\)
\[
K^{n,+}_t - K^{n,+}_s \leq \tilde{K}^{n,+}_t - \tilde{K}^{n,+}_s.
\]
Finally by comparison theorem \(3.2\) since \(U_t \leq U^n_t, \forall t \in [0, T]\), we get
\[
\tilde{K}^{+}_t - \tilde{K}^{+}_s \geq \tilde{K}^{n,+}_t - \tilde{K}^{n,+}_s.
\]
where \(\tilde{K}^{+}_t\) is the increasing process corresponding to lower barrier \(L\) for RBSDE\((\xi, f(t,(U_t)^+), L, U^n)\). So for \(0 \leq s \leq t \leq T\)
\[
K^{n,+}_t - K^{n,+}_s \leq \tilde{K}^{+}_t - \tilde{K}^{+}_s.
\]
Specially, \(K^{n,+}_T \leq \tilde{K}^{+}_T\).

### 3.2 Comparison theorems

First we need a general comparison theorem for the RBSDE with one lower barrier.

**Theorem 3.1 (General case for RBSDE’s)** Suppose that the parameters \((\xi^1, f^1, L^1)\) and \((\xi^2, f^2, L^2)\) satisfy assumption 2.1-2.3-(i). Let the triples \((Y^1, Z^1, K^1)\), \((Y^2, Z^2, K^2)\) be respectively the solutions of the RBSDE\((\xi^1, f^1, L^1)\) and RBSDE\((\xi^2, f^2, L^2)\), i.e.
\[
Y^i_t = \xi^i + \int_t^T f^i(s, Y^i_s, Z^i_s) ds + K^i_T - K^i_t - \int_t^T Z^i_s dB_s,
\]
\[
Y^i_t \geq L^i_t, \quad 0 \leq t \leq T, \quad \int_0^T (Y^i_t - L^i_t) dK^i_t = 0, \quad i = 1, 2.
\]
Assume in addition the following: \(\forall t \in [0, T]\),
\[
\xi^1 \leq \xi^2, f^1(t, Y^1_t, Z^1_t) \leq f^2(t, Y^1_t, Z^1_t), L^1_t \leq L^2_t
\]
then \(Y^1_t \leq Y^2_t, \text{ for } t \in [0, T]\).
Theorem 3.2 Suppose that the parameters \((\xi^1, f^1, L^1, U^1)\) and \((\xi^2, f^2, L^2, U^2)\) satisfy the following conditions: for \(i = 1, 2,\)

(i) \(\xi^i \in L^2(\mathcal{F}_T);\)

(ii) \(f^i\) satisfy assumption 2.2-(i), (iii), (vi) and a Lipschitz condition in \((y, z)\) uniformly in \((t, \omega), i.e. there exists a constant \(k\) such that, for \(y, y' \in \mathbb{R}, z, z' \in \mathbb{R}^d,\)
\[
|f^i(t, y, z) - f^i(t, y', z')| \leq k(|y - y'| + |z - z'|);
\]

(iii) \(L^i\) and \(U^i\) are real-valued, \(\mathcal{F}_t\)-adapted, continuous with \((L^i)^+, (U^i)^- \in S^2(0, T).\)

Let \((Y^i, Z^i, K^{i, +}, K^{i, -})\) be the solution of the RBSDE\((\xi^i, f^i, L^i, U^i), i.e.
\[
Y^i_t = \xi^i + \int_t^T f^i(s, Y^i_s, Z^i_s)ds + K^{i, +}_T - K^{i, -}_t - \int_t^T Z^i_s dB_s,
\]

\(Y^i_t \geq L^i_t, 0 \leq t \leq T,\) and \(\int_0^T (Y^i_s - L^i_s)dK^{i, +}_s = \int_0^T (Y^i_s - U^i_s)dK^{i, -}_s = 0,\)

Moreover, we assume \(\forall (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d,\)
\[
\xi^1 \leq \xi^2, f^1(t, y, z) \leq f^2(t, y, z),
\]

(57)

Then we have: for \(0 \leq t \leq T,\)

(i) If \(L^1 = L^2, U^1 = U^2,\) then \(Y^1_t \leq Y^2_t, K^{1, +}_t \geq K^{2, +}_t, K^{1, -}_t \leq K^{2, -}_t,\) and \(dK^{1, +} \geq dK^{2, +}, dK^{1, -} \leq dK^{2, -};\)

(ii) If \(L^1 \leq L^2, U^1 = U^2,\) then \(Y^1_t \leq Y^2_t, K^{1, -}_t \leq K^{2, -}_t,\) and \(dK^{1, -} \leq dK^{2, -};\)

(iii) If \(L^1 = L^2, U^1 \leq U^2,\) then \(Y^1_t \leq Y^2_t, K^{1, +}_t \geq K^{2, +}_t,\) and \(dK^{1, +} \geq dK^{2, +}.\)

Proof. (i) Set \(L := L^1 = L^2, U := U^1 = U^2,\) and consider the penalization equations for \(m, n \in \mathbb{N}, i = 1, 2,\)
\[
Y^{m,n,i}_t = \xi^i + \int_t^T f^i(s, Y^{m,n,i}_s, Z^{m,n,i}_s)ds + m \int_t^T (Y^{m,n,i}_s - L_s)^-ds - n \int_t^T (Y^{m,n,i}_s - U_s)^+ds - \int_t^T Z^{m,n,i}_s dB_s.
\]
By comparison theorem for BSDEs, since $\xi^1 \leq \xi^2$ and
\[
f^1(t, y, z) + m(y - L_t)^- - n(y - U_t)^+ \leq f^2(t, y, z) + m(y - L_t)^- - n(y - U_t)^+,
\]
we have $Y^{m,n,1}_t \leq Y^{m,n,2}_t$, $\forall t \in [0, T]$. Denote $K^{m,n,+,i}_t = m \int_0^t (Y^{m,n,i}_s - L_s)^- ds$, then for $0 \leq s \leq t \leq T$,
\[
K^{m,n,1,+}_t - K^{m,n,1,+}_s \geq K^{m,n,2,+}_t - K^{m,n,2,+}_s.
\]
From the convergence results in [12], which also holds for Lipschitz function, we have for $0 \leq t \leq T$,
\[
\lim_{m \to \infty} \lim_{n \to \infty} Y^{m,n,i}_t = Y^i_t, \quad \lim_{m \to \infty} \lim_{n \to \infty} K^{m,n,+,i}_t = K^{i,+}_t,
\]
in $L^2(\mathcal{F}_t)$, where $Y^i, K^{i,+}, K^{i,-}$ are elements of the solution of RBSDE$(\xi^i, f^i, L, U)$. Consequently,
\[
Y^1_t \leq Y^2_t, K^{1,+}_t - K^{1,+}_s \geq K^{2,+}_t - K^{2,+}_s;
\]
if we especially set $s = 0$, we get $K^{1,+}_t \geq K^{2,+}_t$. Similarly $K^{1,-}_t \leq K^{2,-}_t$.

(ii) Set $U := U^1 = U^2$, we consider the penalized reflected BSDE’s, for $n \in \mathbb{N}$, $i = 1, 2$,
\[
Y^{n,i}_t = \xi^i + \int_0^T f^i(s, Y^{n,i}_s, Z^{n,i}_s) ds + K^{n,+}_t - K^{n,+}_s - n \int_t^T (Y^{n,i}_s - U_s)^+ ds - \int_t^T Z^{n,i}_s dB_s,
\]
\[
Y^{n,i}_t \geq L^i_t, \int_0^T (Y^{n,i}_t - L^i_t) dK^{n,+}_t = 0.
\]
Since $\forall t \in [0, T]$,
\[
\xi^1 \leq \xi^2, \quad f^1(t, y, z) - n(y - U_t)^+ \leq f^2(t, y, z) - n(y - U_t)^+, \quad L^1_t \leq L^2_t
\]
by the comparison theorem for RBSDE with one barrier, we have $Y^{n,1}_t \leq Y^{n,2}_t$. Let $K^{n,-}_t = n \int_0^t (Y^{n,i}_s - U_s)^+ ds$, then for $0 \leq s \leq t \leq T$,
\[
K^{n,1,-}_t - K^{n,1,-}_s \leq K^{n,2,-}_t - K^{n,2,-}_s.
\]
Thanks to the convergence result in [12], which still works for Lipschitz functions, we have for $0 \leq s \leq t \leq T$,
\[
Y^1_t \leq Y^2_t, K^{1,-}_t - K^{1,-}_s \leq K^{2,-}_t - K^{2,-}_s;
\]
if we especially set $s = 0$, we get $K^{1,+}_t \geq K^{2,+}_t$, $K^{1,-}_t \leq K^{2,-}_t$.

(iii) The proof is in the same as (ii), so we omit it. □

We next prove a comparison theorem for RBSDE with two barriers in a general case.

**Theorem 3.3 (General case for RBSDE’s)** Suppose that the parameters $(\xi^1, f^1, L^1, U^1)$ and $(\xi^2, f^2, L^2, U^2)$ satisfy assumption 2.1, 2.2 and 2.3. Let $(Y^1, Z^1, K^{1,+}, K^{1,-}), (Y^2, Z^2, K^{2,+}, K^{2,-})$ be respectively the solutions of the RBSDE$(\xi^1, f^1, L^1, U^1)$ and RBSDE$(\xi^2, f^2, L^2, U^2)$ as definition 2.7. Assume in addition the following: $\forall t \in [0, T]$
\[
\xi^1 \leq \xi^2, \quad f^1(t, Y^1_t, Z^1_t) \leq f^2(t, Y^1_t, Z^1_t), \quad L^1_t \leq L^2_t, \quad U^1_t \leq U^2_t,
\]
then $Y^1_t \leq Y^2_t$, for $t \in [0, T]$.
Proof. Applying Ito’s formula to \( [(Y^1 - Y^2)^+]^2 \) on the interval \([t, T]\), and taking expectation on the both sides, we get immediately

\[
E[(Y_t^1 - Y_t^2)^+]^2 + E \int_t^T 1_{\{Y_s^1 > Y_s^2\}} |Z_s^1 - Z_s^2|^2 ds \\
\leq 2E \int_t^T 1_{\{Y_s^1 > Y_s^2\}} (Y_s^1 - Y_s^2)(f^2(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2)) ds \\
\leq \frac{1}{2}E \int_t^T 1_{\{Y_s^1 > Y_s^2\}} |Z_s^1 - Z_s^2|^2 ds + (2\mu + 4k^2)E \int_t^T [(Y_s^1 - Y_s^2)^+]^2 ds,
\]

in view of (58) and the Lipschitz condition and monotonic condition on \( f^2 \), and the fact that

\[
\int_t^T (Y_s^1 - Y_s^2)^+ d(K_s^{1+} - K_s^{2+}) \leq 0, \int_t^T (Y_s^1 - Y_s^2)^+ d(K_s^{1-} - K_s^{2-}) \geq 0,
\]

which is similar to reflected BSDE with one barrier. Hence

\[
E[(Y_t^1 - Y_t^2)^+]^2 \leq (2\mu + 4k^2)E \int_t^T [(Y_s^1 - Y_s^2)^+]^2 ds,
\]

and from Gronwall’s inequality, we deduce \((Y_t^1 - Y_t^2)^+ = 0, 0 \leq t \leq T. \square\)

From the convergence of penalization equations, we get the following comparison theorem.

**Theorem 3.4 (Special case)** Suppose that \( f^1(s, y), f^2(s, y) \) satisfy assumption 2.4, and \( \xi^i, f^i(\cdot, 0), L, U, i = 1, 2 \) satisfies the bounded assumption [3]. The two triples \((Y^1, Z^1, K^1), (Y^2, Z^2, K^2)\) are respectively the solutions of the RBSDE\((\xi^1, f^1, L, U)\) and RBSDE\((\xi^2, f^2, L, U)\) as definition 2.1. If we have

\[
\xi^1 \leq \xi^2, \text{ and } f^1(t, y) \leq f^2(t, y), \forall (t, y) \in [0, T] \times \mathbb{R},
\]

then \( Y_t^1 \leq Y_t^2, K_t^{1+} \geq K_t^{2+}, K_t^{1-} \leq K_t^{2-}, \) for \( t \in [0, T], \) and \( dK_t^{1+} \geq dK_t^{2+}, dK_t^{1-} \leq dK_t^{2-}. \)

Proof. We consider the penalized equations relative to the RBSDE\((\xi^i, f^i, L, U)\), for \( i = 1, 2, n \in \mathbb{N}, \)

\[
Y_t^{m,n,i} = \xi^i + \int_t^T f^i(s, Y_s^{m,n,i}) ds + n \int_t^T (Y_s^{m,n,i} - L_s)^- ds - m \int_t^T (Y_s^{m,n,i} - U_s)^+ ds - \int_t^T Z_s^{m,n,i} dB_s.
\]

For each \( m, n \in \mathbb{N}, \)

\[
f^{m,n,1}(s, y) = f^1(s, y) + n(y - L_s)^- - m(y - U_s)^+, \quad f^{m,n,2}(s, y) = f^2(s, y) + n(y - L_s)^- - m(y - U_s)^+,
\]

and \( \xi^1 \leq \xi^2. \) So by the comparison theorem in [13], we get

\[
Y_t^{m,n,1} \leq Y_t^{m,n,2}, 0 \leq t \leq T.
\]

Since \( K_t^{m,n,i,+} = n \int_0^t (Y_s^{m,n,i} - L_s)^- ds, \) then we deduce, for \( 0 \leq s \leq t \leq T, \)

\[
K_t^{m,n,1,+} - K_t^{m,n,2,+} \geq K_t^{m,n,2,+} - K_t^{m,n,2,+},
\]

By the convergence results of the step1, we know that the inequalities hold for \( 0 \leq s \leq t \leq T: \)

\[
Y_t^1 \leq Y_t^2, K_t^{1+} - K_t^{2+} \geq K_t^{2+} - K_t^{2+},
\]

Particularly, set \( s = 0, \) we get \( K_t^{1+} \geq K_t^{2+}. \) Symmetrically, \( K_t^{1-} \leq K_t^{2-}. \square\
Corollary 3.1 Suppose that \( f^1(s, y), f^2(s, y) \) satisfy assumption 2.4, and \( \xi^i, f^i(\cdot, 0), L^i, U^i, i = 1, 2 \) satisfies (3). The two triples \((Y^1, Z^1, K^1), (Y^2, Z^2, K^2)\) are respectively the solutions of the RBSDE\((\xi^1, f^1, L^1, U^1)\) and RBSDE\((\xi^2, f^2, L^2, U^2)\), with \( K^i = K^i_+ - K^i_-, i = 1, 2 \). In addition, we assume
\[
\xi^1 \leq \xi^2, \text{ and } f^1(t, y) \leq f^2(t, y), \forall (t, y) \in [0, T] \times \mathbb{R},
\]
then for \( 0 \leq t \leq T \),
(i) If \( L_t^1 \leq L_t^2, U_t^1 = U_t^2 \), then \( Y_t^1 \leq Y_t^2, K_t^1 \leq K_t^2 \), and \( dK_t^1 \leq dK_t^2 \);
(ii) If \( L_t^1 = L_t^2, U_t^1 \leq U_t^2 \), then \( Y_t^1 \leq Y_t^2, K_t^1 \geq K_t^2 \), and \( dK_t^1 \geq dK_t^2 \).

Proof. (i) To simplify symbols, we denote \( U = U^1 = U^2 \). For \( n \in \mathbb{N} \), we consider the following RBSDE with one barrier \( L^i, i = 1, 2 \).
\[
Y_t^{n,i} = \xi^i + \int_t^T f^i(s, Y_s^{n,i})ds + K_T^{n,i} - K_t^{n,i} - n \int_t^T (Y_s^{n,i} - U_s)ds - \int_t^T Z_s^{n,i}dB_s,
\]
\[
Y_t^{n,i} \geq L_t^{i, n}, \int_0^T (Y_t^{n,i} - L_t^{i, n})dt = 0.
\]
Since \( \xi^1 \leq \xi^2, f^1(t, y) \leq f^2(t, y), L_t^1 \leq L_t^2 \), by general comparison theorem of RBSDE with one barrier, we know \( Y_t^{n,1} \leq Y_t^{n,2} \). Denote \( K_t^{n,1} = \int_0^T (Y_s^{n,1} - U_s)ds, K_t^{n,2} = \int_0^T (Y_s^{n,2} - U_s)ds \), then for \( 0 \leq s \leq t \leq T \),
\[
K_t^{n,1} - K_t^{n,2} \leq K_t^{n,2} - K_t^{n,2}.
\]
Thanks to the convergence result of step 1 of theorem 2.2, it follows immediately that for \( 0 \leq s \leq t \leq T \),
\[
Y_t^1 \leq Y_t^2 \text{ and } K_t^1 \leq K_t^2.
\]
Especially with \( s = 0 \), we get \( K_t^1 \leq K_t^2 \).
(ii) follows similarly as (i), so we omit it. □

Theorem 3.5 Suppose that \( f^1(s, y), f^2(s, y) \) satisfy assumption 2.4, \( \xi^i, f^i(\cdot, 0), U^i, i = 1, 2 \) satisfies (2.1), and \( L^i \) satisfies assumption 2.3'. The two groups \((Y^1, Z^1, K^1), (Y^2, Z^2, K^2)\) are respectively the solutions of the RBSDE\((\xi^1, f^1, L^1, U^1)\) and RBSDE\((\xi^2, f^2, L^2, U^2)\), as definition 2.7. Moreover, assume
\[
\xi^1 \leq \xi^2, f^1(t, y) \leq f^2(t, y), \forall (t, y) \in [0, T] \times \mathbb{R},
\]
then for \( 0 \leq t \leq T \),
(i) If \( L_t^1 = L_t^2 \) and \( U_t^1 = U_t^2 \), then \( Y_t^1 \leq Y_t^2, K_t^1 \geq K_t^2, K_t^1 \leq K_t^2, \) and \( dK_t^1 \geq dK_t^2 \);
(ii) If \( L_t^1 \leq L_t^2, U_t^1 = U_t^2 \), then \( Y_t^1 \leq Y_t^2, K_t^1 \leq K_t^2, \) and \( dK_t^1 \leq dK_t^2 \);
(iii) If \( L_t^1 = L_t^2, U_t^1 \leq U_t^2 \), then \( Y_t^1 \leq Y_t^2, K_t^1 \geq K_t^2, \) and \( dK_t^1 \geq dK_t^2 \).

Proof. Like in the step 2 of the proof of theorem 2.2, we approximate the barrier \( L^i \) by super bounded barrier \( L^{n,i} \), where \( L^{n,i} = L^i \land n \).
(i) Set \( L := L^1 = L^2, U := U^1 = U^2, \) and \( L^n = L \land n \). Then consider the RBSDE\((\xi, f^i, L^n, U)\), for \( i = 1, 2 \),
\[
Y_t^{n,i} = \xi^i + \int_t^T f^i(s, Y_s^{n,i})ds + K_T^{n,i} - K_t^{n,i} - (K_T^{n,i} - K_t^{n,i}) - \int_t^T Z_s^{n,i}dB_s,
\]
\[
L_t^n \leq Y_t^{n,i} \leq U_t, \int_0^T (Y_s^{n,i} - L_s)dB_s = \int_0^T (Y_s^{n,i} - U_s)dB_t = 0.
\]
Since 
\[ \xi^1 \leq \xi^2, f^1(t,y) \leq f^2(t,y), \]
from comparison theorem 3.4 we have for \( 0 \leq s \leq t \leq T \),
\[ Y_n^{1} \leq Y_n^{2}, K_n^{1,1} - K_n^{1,1} \geq K_n^{2,1} - K_n^{2,1}. \]

Thanks to the convergence results in step 2 of the proof for theorem 2.2 we get that
\[ Y_t^{1} \leq Y_t^{2}, K_t^{1,1} - K_t^{1,1} \geq K_t^{2,1} - K_t^{2,1}, \quad \text{for } 0 \leq s \leq t \leq T. \]

Especially, with \( s = 0 \), we get \( K_t^{1,1} \geq K_t^{2,1} \). Similarly \( K_t^{1,-} \leq K_t^{2,-} \), for \( t \in [0,T] \).

(ii) Set \( U := U^1 = U^2 \), and \( L_{n,i} = L_i \land n \). Then we consider the solutions \((Y_{n,i}, Z_{n,i}, K_{n,i})\) of the RBSDEs \((\xi^i, f^i, L_{n,i}, U)\), for \( i = 1, 2 \).

Since 
\[ \xi^1 \leq \xi^2, f^1(t,y) \leq f^2(t,y), L_{n,1}^1 \leq L_{n,2}^1, \]
from corollary 3.1 we have for \( 0 \leq s \leq t \leq T \), \( Y_n^{1} \leq Y_n^{2} \), and \( K_n^{1,1} - K_n^{1,1} \leq K_n^{2,1} - K_n^{2,1} \).

Then by the convergence results in step 2 of the proof for theorem 2.2 it follows that
\[ Y_t^{1} \leq Y_t^{2}, K_t^{1,1} - K_t^{1,1} \leq K_t^{2,1} - K_t^{2,1}, \quad \text{for } 0 \leq s \leq t \leq T. \]

Especially, with \( s = 0 \), we get \( K_t^{1,1} \leq K_t^{2,1} \), for \( t \in [0,T] \).

(iii) The proof is similar to (ii), which follows from corollary 3.1 and by the convergence results in step 2 of the proof for theorem 2.2 so we omit it. \( \square \)

**Theorem 3.6** Suppose that \( f^1(s,y), f^2(s,y) \) satisfy assumption 2.4, \( \xi^1, f^1(-,0) \), for \( i = 1, 2 \) satisfies the bounded assumption \((\mathcal{H})\), and \( L^1 \) and \( U^i \) satisfy assumption 2.3’. The two groups \((Y^1, Z^1, K^1), (Y^2, Z^2, K^2)\) are respectively the solutions of the RBSDE(\(\xi^1, f^1, L^1, U^1\)) and RBSDE(\(\xi^2, f^2, L^2, U^2\)). Moreover, assume
\[ \xi^1 \leq \xi^2, f^1(t,y) \leq f^2(t,y), \forall (t,y) \in [0,T] \times \mathbb{R}, \]
then for \( t \in [0,T] \),
(i) If \( L_1 = L_2 \) and \( U_1 = U_2 \), then \( Y_t^1 \leq Y_t^2, K_t^{1,1} \geq K_t^{2,1}, K_t^{1,-} \leq K_t^{2,-}, \) and \( dK_t^{1,1} \geq dK_t^{2,1}, dK_t^{1,-} \leq dK_t^{2,-} \);
(ii) If \( L_1 \leq L_2 \) and \( U_1 = U_2 \), then \( Y_t^1 \leq Y_t^2, K_t^{1,1} \leq K_t^{2,1}, \) and \( dK_t^{1,1} \leq dK_t^{2,1} \);
(iii) If \( L_1 = L_2 \) and \( U_1 \leq U_2 \), then \( Y_t^1 \leq Y_t^2, K_t^{1,-} \geq K_t^{2,-}, \) and \( dK_t^{1,-} \geq dK_t^{2,-} \).

**Proof.** Like in theorem 3.5 we approximate the barrier \( U \) by lower bounded barrier \( U^n \), where \( U^n = U \lor (-n) \), then the results follow from the comparison theorem 3.5 and the convergence results of step 3 in the proof of theorem 2.2 so we omit it. \( \square \)

**Theorem 3.7** Suppose that for \( i = 1, 2 \), \( \xi^i \) satisfies assumption 2.1, \( f^i \) does not depend on \( z \) and satisfies assumption 2.2, \( L^1 \) and \( U^i \) satisfy assumption 2.3. The two triples \((Y^1, Z^1, K^1), (Y^2, Z^2, K^2)\) are the solutions of the RBSDE(\(\xi^1, f^1, L^1, U^1\)) and RBSDE(\(\xi^2, f^2, L^2, U^2\)), respectively. Moreover, assume for \( (t,y) \in [0,T] \times \mathbb{R}, \)
\[ \xi^1 \leq \xi^2, f^1(t,y) \leq f^2(t,y), \quad \text{and} \quad f^1(t,0) = f^2(t,0), \]
then for \( t \in [0,T] \),
(i) If \( L_1 = L_2 \) and \( U_1 = U_2 \), then \( Y_t^1 \leq Y_t^2, K_t^{1,1} \geq K_t^{2,1}, K_t^{1,-} \leq K_t^{2,-}, \) and \( dK_t^{1,1} \geq dK_t^{2,1}, dK_t^{1,-} \leq dK_t^{2,-} \);
(ii) If \( L_1 \leq L_2 \) and \( U_1 = U_2 \), then \( Y_t^1 \leq Y_t^2, K_t^{1,-} \leq K_t^{2,-}, \) and for \( dK_t^{1,-} \leq dK_t^{2,-} \);
(iii) If \( L_1 = L_2 \) and \( U_1 \leq U_2 \), then \( Y_t^1 \leq Y_t^2, K_t^{1,1} \geq K_t^{2,1}, \) and for \( dK_t^{1,1} \geq dK_t^{2,1} \).
Proof. (i) Set $L := L^1 = L^2$, $U := U^1 = U^2$. Like in the proof of the theorem 2.2 for $i = 1, 2$, set

$$
(Y^i_t, Z^i_t, K^i_t, K^i_s) := (e^{\lambda t} Y^i_t, e^{\lambda t} Z^i_t, \int_0^t e^{\lambda s} dK^i_s, \int_0^t e^{\lambda s} dK^i_s).
$$

Then it’s easy to check that for $i = 1, 2$, $(Y^i_t, Z^i_t, K^i_t, K^i_s)_{0 \leq t \leq T}$ is the solution of the RBSDE $(\xi^i, f^i, \mathcal{L}, \mathcal{U})$, where

$$
(\xi^i, f^i(t, y), \mathcal{L}, \mathcal{U}) = (e^{\lambda T} \xi^i, e^{\lambda T} f^i(t, e^{-\lambda y} - \lambda y, e^{\lambda T} L_t, e^{\lambda T} U_t)).
$$

If we assume $\lambda = \mu$, then $(\xi^i, f^i, \mathcal{L}, \mathcal{U})$ satisfies assumption 2.1, 2.4 and 2.3'. Since the transform keeps the monotonicity, the results are equivalent to

$$
Y^i_t \leq Y^2_t, K^i_{t-} \geq K^2_{t-}, K^i_t - K^2_t \geq 0,
$$

for $0 \leq s \leq t \leq T$. Then we make the approximations

$$
\xi^m, n, i := \xi^i \wedge m := (\xi^i \vee (-n)) \wedge m,
$$

$$
\tilde{T}_{m, n}(t, y) := T_{m, n}(t, y) - T_{m, n}(t, 0) \wedge m,
$$

and

$$
\xi^m, n, 1 \leq \xi^m, n, 2, \text{ and } \tilde{T}_{m, n}(t, y) \leq \tilde{T}_{m, n}(t, y), \text{ for } (t, y) \in [0, T] \times \mathbb{R},
$$

in view of $\tilde{T}(t, 0) = f^1(t, 0) = f^2(t, 0) = \tilde{T}(t, 0)$. Using the comparison theorem 3.3(i), we have for $0 \leq s \leq t \leq T$

$$
\tilde{T}_t \leq \tilde{T}_t, K^m, n, 1, - K^m, n, 1, - K^m, n, 2, + - K^m, n, 2, +.
$$

By the convergence results in the step 4 and step 5 of the proof of theorem 2.2, let $m \to \infty$, then $n \to \infty$, we get for $0 \leq s \leq t \leq T$

$$
Y^i_t \leq Y^2_t, K^i_{t-} \geq K^2_{t-}, K^i_t - K^2_t \geq 0.
$$

Especially, with $s = 0$, it follows $K^i_{t-} \geq K^2_{t-}$. Similarly $K^i_{t-} \leq K^2_{t-}$.

(ii) and (iii) are from comparison theorem 3.3(ii) and (iii), with approximation as in (i), so we omit it. □

Acknowledgement Author thanks professor Jean-Pierre Lepeltier for his helps when author worked on this paper.

References

[1] Alario Nazaret M., Lepeltier, J.P. and Marchal, B., 1982. Dynkin games. Lecture Notes in Control and Inform. Sci. 43. (Springer, Berlin), 23-42.

[2] Bismut, J.M., 1977. Sur un problème de Dynkin. Z.Wahrsch. Verw. Gebiete 39 31-53.
[3] Briand, Ph., Delyon, B., Hu, Y., Pardoux, E. and Stoica, L. (2003) $L_p$ solutions of BSDEs, *Stochastic Process. Appl.* 108, 109-129.

[4] Cvitanic, J. and Karatzas, I., 1996. Backward Stochastic Differential Equations with Reflection and Dykin Games, Ann. Probab. 24, no 4, 2024–2056.

[5] Elliott, R., 1976. The existence of value in stochastic differential games. SIAM JCO 14(1), 85-94.

[6] El Karoui, N., Kapoudjian, C., Pardoux, E., Peng, S. and Quenez, M.C., 1997. Reflected Solutions of Backward SDE and Related Obstacle Problems for PDEs, Ann. Probab. 25, no 2, 702–737.

[7] El Karoui, N., Peng, S. and Quenez, M.C., 1997. Backward stochastic differential equations in Finance. Math. Finance, 7, 1-71.

[8] Hamadène, S., Lepeltier, J.P. and Matoussi, A. 1997. Double barrier backward SDEs with continuous coefficient. In: El Karoui, N. and Mazliak, L., (Eds.), Backward Stochastic differential Equations. Pitman Research Notes in Mathematics Series 364, pp. 161-177.

[9] Hamadène, S., Lepeltier, J.P. and Peng, S., 1997. BSDE with continuous coefficients and stochastic differential games. In: El Karoui, N. and Mazliak, L., (Eds.), Backward Stochastic differential Equations. Pitman Research Notes in Mathematics Series 364, pp. 115-128.

[10] Lepeltier, J.P. and Maingueneau, M.A., 1984. Le jeu de Dynkin en théorie générale sans l’hypothèse de Mokoboski. Stochstics. 13, 25-44.

[11] Lepeltier, J.P., Matoussi, A. and Xu, M. (2005) Reflected Backward stochastic differential equations under monotonicity and general increasing growth conditions. Advanced Applied Probability, 37, 1-26.

[12] Lepeltier, J.P. and San Martín, J., 2004. Backward SDE’s with two barriers and continuous coefficient. An existence result. Journal of Applied Probability, vol. 41, 162-175.

[13] Pardoux, E., 1999. BSDE’s, weak convergence and homogenization of semilinear PDE’s in Nonlinear analysis, Differential Equations and Control, F. H. Clarke & R. J. Stern Eds , pp. 503-549, Kluwer Acad. Pub.

[14] Pardoux, E. and Peng, S., 1990. Adapted solutions of Backward Stochastic Differential Equations. Systems Control Lett. 14, 51-61.

[15] Revuz, D. and Yor, M., 1991. Continuous martingales and Brownian motion. Springer, New York.