The Probability Distribution of Time to Extinction: A simulation study

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Abstract: The time to extinction (TTE) is a very important topic in some fields of studies; ecology, economics, corporate competition, bacterial sciences and epidemiology. The aim of the present paper is to investigate about the empirical parametric and empirical nonparametric probability distribution of the time to extinction for two related stochastic models; Rosenzweig and Macarthur model and May model. First, we assume the amplitude of $r$ is a random variable with a continuous uniform probability distribution on a closed interval $[r_1, r_2]$. Second, we assume the number of consecutive years during which the amplitude of $r$ remains constant is a random variable with a discrete geometric probability distribution with parameter $p$ which is supported on a positive integer $\mathbb{Z}_+$ and $p \in (0,1)$.

1. Introduction
A species is extinct when the last existing member dies. Extinction therefore becomes a certainty when there are no surviving individuals that can reproduce and create a new generation. A species may become functionally extinct when only a handful of individuals survive, which cannot reproduce due to poor health, age, sparse distribution over a large range, a lack of individuals of both sexes (in sexually reproducing species), or other reasons.

Despite the importance of researching the topic of time to extinction in various topic such as ecology, economics, corporate competition, bacterial sciences and epidemiology, there is little research in this topic specifically.

An approximation is derived by Nasell (1999) [1] for the expected time to extinction in a stochastic model for recurrent epidemics. Saether et al. (2000) [2] discussed the issue of estimation of the time to extinction in an island population of song sparrows. Koeijer et al. (2008) [3] computed the extinction time of a reactivating virus, in particular bovine herpes virus. Kajunguri et al. (2019) [4] Improved estimates for extinction probabilities and times to extinction for populations of tsetse. The issue of Estimating the distribution of time to extinction of infectious diseases in mean-field approaches is considered by Aliee et al. (2020) [5].

Research in the predator-prey models started early in the 19th century. One of the first models to describe the dynamics of the predator-prey interaction was due to (Lotka, 1920) [6]. The model, called the Lotka-Volterra predator-prey model.

In 1963, M. L. Rosenzweig and R. H. Macarthur [7], introduced a model named after them called the Rosenzweig-Macarthur (RM) predator-prey model. Since then several versions of this model was proposed. In this paper we adopt the one proposed in (Alkhayuon et al., 2021) [8], which is given by,

$$\dot{N} = rN \left(1 - \left(\frac{c}{r'}\right)N\right)\left(\frac{N - \mu}{\nu + N}\right) - \frac{aNP}{\beta + N}$$ (1)
\[ \dot{P} = \frac{\chi \alpha P N}{\beta + N} - \delta P \]

In the prey equation, \(-r\mu/v\) is the low-density prey growth rate, \(c\mu/v\) quantities the nonlinear prey birth rate, the term \((N - \mu)/(v + N)\) gives rise to the strong Allee effect that accounts for negative prey growth rate at low prey population density, \(\alpha\) is the saturation predator kill rate, and \(\beta\) is the predator kill half-saturation constant. The ratio \(r/c\) is often referred to as the carrying capacity of the ecosystem. It is the maximum prey population that can be sustained by the environment in the absence of predators [12]. In the predator equation, \(\chi\) represents the prey-to-predator conversion rate, and \(\delta\) is the predator mortality rate.

In 1973, a yet another interesting model due to May [9]. Again we consider the version given by:

\[ \dot{N} = rN \left(1 - \frac{c}{r} N \right) \left(\frac{N - \mu}{v + N} - \frac{\alpha NP}{\beta + N} \right) \]

\[ \dot{P} = sP \left(1 - \frac{qP}{N + \epsilon}\right) \]

The dynamics of the prey is identical to that of the RM model, while the May model adopt a predator dynamics proposed by (Leslie 1948)[10], which considers a logistic growth with carrying capacity \(N + \epsilon\). \(s\) is the low density growth rate of the predator and \(q\) is the minimum prey-to-predator biomass ratio that allows predator population growth and \(\epsilon\) is added to account for the situation of both species extinction[8].

One of the most recent application examples of the May model was to the Canadian lynx and snowshoe hare [11].

### 2. Some Models Analysis

In this paper we are interested in the RM and May models. For the RM model we there are at most four equilibrium point, namely:

\[ e_0 = (0, 0), \quad e_1 = \left(\frac{r}{c}, 0\right), \quad e_2 = (\mu, 0), \quad e_3 = \left(\frac{u - \mu}{v + u}, \frac{\beta + u}{\alpha}\right) \]

where \(u = \frac{2\delta}{\chi \alpha - \delta}\).

To analyse the stability of the equilibria we look at the Jacobean matrix evaluated at the:

![Figure 1](image_url)

**Figure 1:** The behaviour of the RM model for different values of the parameter \(r\), namely 1,5,2,5,3 respectively, and initial value (2,0,1).
\[ J = \begin{bmatrix} \frac{(rN - cN^2)N}{(v + N)^2} + (r - 2cN)\left(\frac{N - \mu}{v + N}\right) - \frac{\alpha P \beta}{(\beta + N)^2} - \frac{\alpha N}{\beta + N} \\ \chi \frac{a \beta P}{(\beta + N)^2} \\ \frac{\chi a N}{\beta + N} - \delta \end{bmatrix} \]

For values of parameters as given in Table (1), and different values of \( r \), Figure 1 shows the behaviour of the RM model.

The May model on the other hand has the following fixed point when \( P = 0 \)
\[ e_0 = (0,0), \quad e_1 = \left(\frac{r}{c}, 0\right), \quad e_2 = (\mu, 0) \]
\( e_3, e_4 \) are obtained by setting \( P = \frac{N + \varepsilon}{q} \) and solving the quadratic equation
\[ N^3 - \left(\frac{r}{c} - \beta + \mu - \frac{\alpha}{qc}\right)N^2 - \left(\frac{r \beta}{c} - \frac{r \mu}{c} + \beta \mu - \frac{\alpha v}{qc} - \frac{\alpha \varepsilon}{qc}\right)N + \frac{r \beta \mu}{c} + \frac{\alpha v \varepsilon}{qc} = 0 \]
For the two positive roots of \( N \). Finally setting \( N = 0 \) in \( P = \frac{N + \varepsilon}{q} \) we obtain \( e_5 = \left(0, \frac{\varepsilon}{q}\right) \).

The Jacobean matrix is given by:
\[ J = \begin{bmatrix} \frac{(rN - cN^2)N}{(v + N)^2} + (r - 2cN)\left(\frac{N - \mu}{v + N}\right) - \frac{\alpha P \beta}{(\beta + N)^2} - \frac{\alpha N}{\beta + N} \\ \frac{sqP^2}{(N + \varepsilon)^2} \\ s - \frac{2sqP}{N + \varepsilon} \end{bmatrix} \]

For values of parameters as given in Table (1), and different values of \( r \), Figure 2 shows the behaviour of the May model.

**3. Time to Extinction**
For both the May and RM models, we are interested in calculating the time to which an extinction occurs for either species. Instead of being a constant, we consider the magnitude of \( r \) to be a uniformly distributed random variable over the interval \([r_1, r_2]\). Furthermore, we consider the number of years at which remains constant to be a geometric random variable with parameter \( p \).

The variability of \( r \) mimics a change in the in the environmental conditions affecting both the predator and the prey. This change can cause the system to tip into extinction for one or both species. Tipping in the system is induced by a bifurcation, rate tipping, noise or phase tipping [8].

![Figure 2](image-url) The behaviour of the RM model for different values of the parameter \( r \), namely 2,2.5,3,3.5 respectively, and initial value (2,0.1).
Table (1): Values of the parameters for the RM and May models

| Parameter | RM   | May   |
|-----------|------|-------|
| $\epsilon$ | 0.19 | 0.229 |
| $\chi$    | 0.004 N.A. |
| $\delta$  | 2.2 N.A. |
| $\beta$   | 1.5 | 0.3 |
| $\alpha$  | 800 | 505 |
| $\mu$     | 0.03 | 0.03 |
| $\nu$     | 0.003 | 0.003 |
| $q$       | N.A. | 205 |
| $s$       | N.A. | 0.85 |
| $\epsilon$ | N.A. | 0.003 |

A bifurcation tipping occurs when the system passes through a bifurcation point as $r$ changes.

Figure 3: The evolution of the May model. In a we show the values taken for $r$, $b$ and $c$ show the time evolution for the predator and the prey respectively. Finally d plots the trajectory of the system. From the figure we see that an extinction occurs for the prey.
Rate tipping occurs when the external forces change too fast, that is when \( r \) changes rapidly. A phase tipping occurs when the system is in the right phases to tip, whereas no tipping occurs from other phases [8].

To calculate the time to extinction, we let the system evolve until it reaches a tipping point. We then record the time it took for the system to reach the tipping point. Error! Reference source not found. and Figure 3 show the time evolution for both the May and RM models respectively.

4. Some statistical tools

In this section we introduce some statistical tools related with our research.

1. Let \( X_1, X_2, \ldots, X_n \) denote a random sample of size \( n \) from a random variable with density \( f(.) \).

The kernel density estimate of \( f \) at the point \( x \) is given by [13],

\[
\hat{f}_h(x) = \frac{1}{nh} \sum_{i=1}^{n} K \left( \frac{x - X_i}{h} \right)
\]

where the smoothing parameter \( h \) is known as the bandwidth and the kernel \( K \) is generally chosen to be a unimodal probability density symmetric about zero. In this case, \( K \) satisfies the following conditions [14,15], \( \int K(y) \, dy = 1 \), \( \int y K(y) \, dy = 0 \) and \( \int y^2 K(y) \, dy > 0 \).

Actually there are a lot of popular choices of kernel function \( K \); Gaussian, Epanechnikov, Rectangular, Triangular, Biweight, Cosine, and Optcosine, and others [13]. Table (a.1) in appendix contains the formulas of considered kernels.

The bandwidth controls the smoothness of the density estimate and highly influence its appearance. Selecting a suitable \( h \) is a pivotal step in estimating \( f(x) \). There has been great advancement in recent years in data-based bandwidth selection for kernel density estimation.

To determine \( h \), we consider the window closing method. This method permits to detect progressively about the shape of the density. The idea beyond the mechanism of “window closing” is to begin with an arbitrary but big \( h \) and compute the resulting density estimate. We then calculate a further series of density estimates by closing the window, i.e. by gradually decreasing \( h \). The incipient estimate based on a large \( h \) will be a very smooth function, but as we decrease \( h \), the estimates will gradually display more detail and become more “eccentric” in form. By examining the manner in which the estimates change as we decrease \( h \), we may be able to detect the point at which the smoothing has been relaxed “too far”, and this will then enable us to select a suitable value for bandwidth. [16]

Figure 4: The evolution of the RM model. In a we show the values taken for \( r, b \) and \( c \) show the time evolution for the predator and the prey respectively. Finally d plots the trajectory of the system. From the figure we see that an extinction occurs in about 44 years.
2. Let \( X_1, X_2, \ldots, X_n \) be independent identically distributed (iid) with common pdf \( f(x; \theta) \), where \( \theta \in \Omega \subseteq \mathbb{R}^p \). The likelihood function and its log are respectively given by \( L(\theta) = \prod_{i=1}^{n} f(x_i; \theta) \) and \( l(\theta) = \log(L(\theta)) = \sum_{i=1}^{n} \ln(f(x_i; \theta)) \). The value that maximizes \( L(\theta) \) or equivalently solves the vector equation \( \left( \frac{\partial}{\partial \theta} \right) l(\theta) = 0 \) is called the maximum likelihood estimator (MLE) [17].

3. The chi-square test is used to test if a sample \( x_1, x_2, \ldots, x_n \) of data came from a population with a specific distribution. The chi-square test is defined for the hypothesis, \( H_0 \): The data follow a specified distribution versus \( H_1 \): The data do not follow a specified distribution. The test statistic is defined as \( \chi^2 = \sum_{i=1}^{k} (O_i - E_i)^2 / E_i \) where the data are divided into \( k \) bins, \( O_i \) is the observed frequency for bin \( i \) and \( E_i \) is the expected frequency for bin \( i \). \( E_i \) is calculated by \( E_i = n(F(Y_u) - F(Y_l)) \), where \( F \) is the cumulative distribution function (cdf) for the distribution being tested, \( Y_u \) and \( Y_l \) are respectively upper and lower limit for class \( i \). The test statistic follows, approximately, a chi-square distribution with \( (k - c) \) degrees of freedom, where \( c \) is the number of estimated parameters for the distribution +1. Therefore \( H_0 \) is rejected if \( \chi^2 > \chi^2_{k-c} \) where \( \chi^2_{k-c} \) is the chi-square critical value with \( (k - c) \) degrees of freedom and significant level \( \alpha \) [18].

4. The Kolmogorov-Smirnov (K-S) test is used to test if a sample \( x_1, x_2, \ldots, x_n \) of data came from a population with a specific distribution. The K-S test is defined for the hypothesis, \( H_0 \): The data follow a specified distribution versus \( H_1 \): The data do not follow a specified distribution. The general steps to run the test are, (a) create an empirical distribution function (EDF) for the sample. (b) specify a parent distribution. (c) calculate the K-S test statistic as, \( D = SUP_{x \in [a,b]} |F_0(x) - F_{data}(x)| \), where \( F_0(x) \) is the cdf of the hypothesized distribution and \( F_{data} \) is the EDF of the observed data. Therefore \( H_0 \) is rejected if \( D \) is greater than the critical value (which are found in the K-S test P-value table) [18].

5. The empirical study
Simulations experiments are conducted to generate Time to Extinction random variable according to the following assumptions,

1. Rosenzweig-Macarthur model and May model are considered.
2. We assume the amplitude of \( r \) is a random variable with a continuous uniform probability distribution on a closed interval \([r_1, r_2]\). For our experiments, we assume that \([r_1, r_2] = [1.63, 3.3] \) and \([r_1, r_2] = [2, 4] \).
3. We assume the number of consecutive years during which the amplitude of \( r \) remains constant is a random variable with a discrete geometric probability distribution with parameter \( p \) which is supported on a positive integer \( \mathbb{Z}_+ \) and \( p \in (0, 1) \). For our experiments, we assume that \( p = 0.2, 0.5, 0.8 \).

So, we obtain 12 sets of data, each set contains 1000 values of EET random variable. The main aim as we stated above is to investigate about the best fitted distribution of EET random variable.

5.1 The parametric inferences of TTE
To investigate about the best fitted parametric distribution of each one of the 12 sets of TTE empirical data which are obtained by simulation experiments,

1. We obtain ML estimations of parameters of 65 distributions based on each set of empirical data, these distributions are Beta, Burr, Burr (4P), Cauchy, Chi-Squared, Chi-Squared (2P), Dagum, Dagum (4P), Erlang, Error, Error Function, Exponential, Exponential (2P), Fatigue Life, Fatigue Life (3P), Frechet, Frechet (3P), Gamma, Gamma (3P), Gen. Extreme Value, Gen. Gamma, Gen. Gamma (4P), Gen. Logistic, Gen. Pareto, Gumbel Max, Gumbel Min, Hypersecant, Inv. Gaussian, Inv. Gaussian (3P), Johnson SB, Kumaraswamy, Laplace, Levy, Levy (2P), Log-Gamma, Log-Logistic, Log-Logistic (3P), Log-Pearson 3, Logistic, Lognormal, Lognormal (3P), Normal, Pareto, Pareto 2, Pearson 5, Pearson 5 (3P), Pearson 6, Pearson 6 (4P), Pert, Phased Bi-Exponential, Phased Bi-Weibull,
Power Function, Rayleigh, Rayleigh (2P), Reciprocal, Rice, Student's t, Triangular, Uniform, Wakeby, Weibull, Weibull (3P), Erlang (3P), Johnson SU and Nakagami.

2. To determine the best distributions that represent the data, we use two test statistics, the first one is chi-squared statistic and the second one is Kolmogorov-Smirnov statistic.

3. Table (2) contains the values of test statistics, best fitted parametric distribution and ML estimations of parameters for each data set according to different default parameters of Geometric and Uniform densities for Rosenzweig-MacArthur model.

4. Table (3) contains the values of test statistics, best fitted parametric distribution and ML estimations of parameters for each data set according to different default parameters of Geometric and Uniform densities for May model.

5. Figures (5-16) represents the pdf of the best fitted parametric distribution of each set of empirical data based on Rosenzweig-MacArthur model.

6. Figures (17-28) represents the pdf of the best fitted parametric distribution of each set of empirical data based on May model.

7. For convenient, the mathematical formulas for best fitted distributions of empirical data are in the appendix.

**Table (2):** Time to Extinction empirical Best fitted parametric distribution based on Rosenzweig and MacArthur model with different

Uniform and Geometric parameters, Kolmogorov Smirnov and Chi-Squared test statistics and ML parameters estimation

| Uniform a and b parameters | Geometric parameter | Test statistic (TS) | TS value | Best fitted distribution | ML estimations of parameters |
|----------------------------|---------------------|--------------------|----------|--------------------------|-----------------------------|
| a = 1, 6, b = 3, 0.2       | Kolmogorov Smirnov  | 0.12141            | Gamma (3P) | \( \hat{\alpha} = 0.52465, \hat{\beta} = 52.299, \hat{\gamma} = 19.94 \) |
|                           | Chi-Squared         | 28.201             | Gamma   | \( \hat{\alpha} = 2.0752, \hat{\beta} = 22.658 \) |
| a = 1, 6, b = 3, 0.5       | Kolmogorov Smirnov  | 0.10433            | Beta   | \( \hat{\alpha}_1 = 0.48091, \hat{\alpha}_2 = 3.608, \hat{\alpha} = 19.916, \hat{\beta} = 265.1 \) |
|                           | Chi-Squared         | 19.5               | Gamma   | \( \hat{\alpha} = 1.8663, \hat{\beta} = 25.765 \) |
| a = 1, 6, b = 3, 0.8       | Kolmogorov Smirnov  | 0.1118             | Gamma (3P) | \( \hat{\alpha} = 0.58648, \hat{\beta} = 52.217, \hat{\gamma} = 19.941 \) |
|                           | Chi-Squared         | 15.063             | Gamma   | \( \hat{\alpha} = 1.9956, \hat{\beta} = 26.683 \) |
| a = 2, 4, b = 4, 0.2       | Kolmogorov Smirnov  | 0.19179            | Log-Logistic (3P) | \( \hat{\alpha} = 1.5089, \hat{\beta} = 1.3885, \hat{\gamma} = 19.52 \) |
|                           | Chi-Squared         | 231.84             | Pareto 2 | \( \hat{\alpha} = 154.73, \hat{\beta} = 3345.7 \) |
| a = 2, 4, b = 4, 0.5       | Kolmogorov Smirnov  | 0.13359            | Burr (4P) | \( \hat{\kappa} = 0.19555, \hat{\alpha} = 5.0003, \hat{\beta} = 0.72435, \hat{\gamma} = 19.397 \) |
|                           | Chi-Squared         | 192.86             | Pareto 2 | \( \hat{\alpha} = 154.73, \hat{\beta} = 3345.7 \) |
| a = 2, 4, b = 4, 0.8       | Kolmogorov Smirnov  | 0.14115            | Burr (4P) | \( \hat{\kappa} = 0.22755, \hat{\alpha} = 4.7917, \hat{\beta} = 0.67633, \hat{\gamma} = 19.445 \) |
|                           | Chi-Squared         | 156.18             | Pareto 2 | \( \hat{\alpha} = 132.07, \hat{\beta} = 2485.3 \) |
Table (3): Time to Extinction empirical Best fitted parametric distribution based on May model with different Uniform and Geometric parameters, Kolmogorov Smirnov and Chi-Squared test statistics and ML parameters estimation

| Geometric parameter | Test statistic (TS) | TS value | Best fitted distribution | ML estimations of parameters |
|---------------------|---------------------|----------|--------------------------|-----------------------------|
|                     |                     |          |                         |                             |
| **Uniform a and b parameters** |                     |          |                          |                             |
|                     | 0.2                 | 0.01298  | Wakeby                   | \( \hat{a} = 128.2, \hat{\beta} = 5.8946, \hat{\gamma} = 66.306, \hat{\delta} = -0.06788, \hat{\xi} = 8.6966 \) |
|                     | Chi-Squared         | 3.5979   | Johnson SB               | \( \hat{\gamma} = 4.5253, \hat{\delta} = 1.7055, \hat{\lambda} = 1464.7, \hat{\xi} = 20.628 \) |
|                     | 0.5                 | 0.02575  | Johnson SB               | \( \hat{\gamma} = 2.1226, \hat{\delta} = 0.91562, \hat{\lambda} = 319.9, \hat{\xi} = 5.1315 \) |
|                     | Chi-Squared         | 14.013   | Burr                     | \( \hat{\gamma} = 1.023, \hat{\delta} = 2.237, \hat{\beta} = 34.745 \) |
|                     | 0.8                 | 0.03818  | Wakeby                   | \( \hat{\delta} = -39.704, \hat{\beta} = 8.5161, \hat{\gamma} = 43.269, \hat{\delta} = -0.06014, \hat{\xi} = 7.5119 \) |
|                     | Chi-Squared         | 40.597   | Gamma                    | \( \hat{\gamma} = 1.3525, \hat{\beta} = 32.646 \) |
|                     | 0.2                 | 0.02108  | Erlang                   | \( \hat{\gamma} = 2, \hat{\beta} = 105.7 \) |
|                     | Chi-Squared         | 12.227   | Johnson SB               | \( \hat{\gamma} = 2.5953, \hat{\delta} = 1.3249, \hat{\lambda} = 1634.4, \hat{\xi} = -22.831 \) |
|                     | 0.5                 | 0.01284  | Johnson SB               | \( \hat{\gamma} = 2.083, \hat{\delta} = 0.96709, \hat{\lambda} = 712.38, \hat{\xi} = 2.0391 \) |
|                     | Chi-Squared         | 8.901    | Gen. Gamma (4P)          | \( \hat{\gamma} = 0.89258, \hat{\delta} = 1.4904, \hat{\beta} = 57.604, \hat{\gamma} = 8.4598 \) |
|                     | 0.8                 | 0.01447  | Gen. Pareto              | \( \hat{\gamma} = 0.03128, \hat{\delta} = 103.2, \hat{\beta} = \hat{\gamma} = 7.5923 \) |
|                     | Chi-Squared         | 4.9785   | Gamma (3P)               | \( \hat{\gamma} = 0.95198, \hat{\beta} = 111.86, \hat{\gamma} = 8.0099 \) |
Figure (5): Gamma (3p) best fitted pdf of time to extinction empirical data based on Rosenzweig-MacArthur model, Uniform (1.6,3) and Geometric (0.2), with K-S statistic value=0.12414 and ML parameters estimation $\delta = 0.52465, \beta = 52.299, \gamma = 19.944$.

Figure (6): Gamma best fitted pdf of time to extinction empirical data based on Rosenzweig-MacArthur model, Uniform (1.6,3) and Geometric (0.2), with chi-squared statistic value=28.204 and ML parameters estimation $\delta = 2.0765, \beta = 73.659$.

Figure (7): Beta best fitted pdf of time to extinction empirical data based on Rosenzweig-MacArthur model, Uniform (1.6,3) and Geometric (0.5), with K-S statistic value=0.10433 and ML parameters estimation $\delta = 5.4891, \beta = 3.698, \gamma = 19.916, \delta = 260.71$.

Figure (8): Gamma best fitted pdf of time to extinction empirical data based on Rosenzweig-MacArthur model, Uniform (1.6,3) and Geometric (0.5), with chi-squared statistic value=19.5 and ML parameters estimation $\delta = 1.8063, \beta = 25.785$.

Figure (9): Gamma (3p) best fitted pdf of time to extinction empirical data based on Rosenzweig-MacArthur model, Uniform (1.6,3) and Geometric (0.8), with K-S statistic value=0.1118 and ML parameters estimation $\delta = 0.98646, \beta = 52.717, \gamma = 19.944$.

Figure (10): Gamma best fitted pdf of time to extinction empirical data based on Rosenzweig-MacArthur model, Uniform (1.6,3) and Geometric (0.8), with chi-squared statistic value=15.063 and ML parameters estimation $\delta = 1.9956, \beta = 26.863$.

Figure (11): Log-Logistic (3p) best fitted pdf of time to extinction empirical data based on Rosenzweig-MacArthur model, Uniform (2.4) and Geometric (0.8), with K-S statistic value=0.19179 and ML parameters estimation $\delta = 1.3897, \beta = 1.3083, \gamma = 19.825$.

Figure (12): Pareto 2 best fitted pdf of time to extinction empirical data based on Rosenzweig-MacArthur model, Uniform (2.4) and Geometric (0.2), with chi-squared statistic value=281.84 and ML parameters estimation $\delta = 154.73, \beta = 33.467$. 
Figure (13): Burr(4P) best pdf of time to extinction empirical data based on Rosensweig-Macarthur model, Uniform (2,4) and Geometric (0.5), with K-S statistic value = 0.13559 and ML parameters estimation \( \alpha = 0.18553, \beta = 0.74433, \gamma = 19.207 \).

Figure (14): Pareto best pdf of time to extinction empirical data based on Rosensweig-Macarthur model, Uniform (2,4) and Geometric (0.5), with chi-squared statistic value = 192.84 and ML parameters estimation \( \alpha = 194.75, \beta = 3845.7 \).

Figure (15): Burr best pdf of time to extinction empirical data based on Rosensweig-Macarthur model, Uniform (2,4) and Geometric (0.5), with K-S statistic value = 0.14115 and ML parameters estimation \( \alpha = 0.22755, \beta = 4.7937, \gamma = 0.00833, \delta = 28.445 \).

Figure (16): Pareto best pdf of time to extinction empirical data based on Rosensweig-Macarthur model, Uniform (2,4) and Geometric (0.5), with ML parameters estimation \( \alpha = 132.07, \beta = 2485.3 \).

Figure (17): Weibull best pdf of time to extinction empirical data based on May model, Uniform (1,6.3) and Geometric (0.2), with K-S statistic value = 0.01298 and ML parameters estimation \( \alpha = 120.3, \beta = 5.8946, \gamma = 66.306, \delta = 0.06788, \zeta = 0.6966 \).

Figure (18): Johnson best pdf of time to extinction empirical data based on May model, Uniform (1,6.3) and Geometric (0.2), with chi-squared statistic value = 3.5979 and ML parameters estimation \( \alpha = 1.8253, \beta = 1.9355, \gamma = 1461.4, \delta = 39.628 \).

Figure (19): Johnson best pdf of time to extinction empirical data based on May model, Uniform (1,6.3) and Geometric (0.5), with K-S statistic value = 0.02575 and ML parameters estimation \( \alpha = 2.3122, \beta = 0.01344, \gamma = 219.9, \delta = 5.3218 \).

Figure (20): Burr best pdf of time to extinction empirical data based on May model, Uniform (1,6.3) and Geometric (0.5), with chi-squared statistic value = 14.013 and ML parameters estimation \( \alpha = 1.033, \beta = 2.397, \gamma = 14.746 \).
5.2 The non-parametric inferences of TTE
To investigate about the best fitted non-parametric distribution of each one of the 12 sets of TTE empirical data which are obtained by simulation experiments,

1. We use 7 kernels Gaussian, Epanechnikov, Rectangular, Triangular, Biweight, Cosine and Optcosine.
2. The optimal Bandwidth is calculated for each of 12 set of data by using the considered method.
3. The maximum of pdf differences between the actual value and estimated value are calculated for each kernel to determine the best kernel density estimation of each one of the 12 sets of TTE empirical data corresponding to the smallest difference.
4. The $x$-values corresponding to maximum of pdf differences are obtained also.
5. Table (4) contains TTE empirical best nonparametric fitted distribution based on Rosenzweig and Macarthur model with different Uniform and Geometric parameters and different kernels.
6. Table (5) contains TTE empirical best nonparametric fitted distribution based on May model with different Uniform and Geometric parameters and different kernels.
7. Figures (29-34) represents the best estimated kernel pdf of each set of empirical data based on Rosenzweig and Macarthur model.
8. Figures (35-40) represents the best estimated kernel pdf of each set of empirical data based on May model.
9. The best kernel to estimate the pdf has been assigned by red line in tables (4) and (5).

6. Summary and conclusion
Prey-predator models are important since they have a lot of applications in different aspects of life. Here Rosenzweig and Macarthur model and May model are considered. We investigate about the empirical parametric and empirical nonparametric probability distribution of the time to extinction by assuming that the amplitude of $r$ is a random variable distributed as continuous uniform on a closed interval $[r_1, r_2]$ and the number of consecutive years’ during which the amplitude of $r$ remains constant is a random variable distributed as discrete geometric with parameter $0 < p < 1$. There are two main conclusions obtained from the empirical study, the first one is that there is no pattern for pdf of TTE random variable according to parametric inferences results. The second main conclusion is that there is no pattern for kernel density estimation of TTE empirical data according to non-parametric inferences results. Therefore, each set of TTE data may assign different behavior from other sets of dat.
Table (4): Time to Extinction empirical best nonparametric fitted distribution based on Rosenzweig and Macarthur model with different Uniform and Geometric parameters and different kernels.

| Geometric parameter | Kernel      | x-value | max. density | Bandwidth |
|---------------------|-------------|---------|--------------|-----------|
| 0.2                 | Gaussian    | 23.33229| 0.02427      |           |
|                     | Epanechnikov| 26.02653| 0.02349      |           |
|                     | Rectangular | 30.87617| 0.02614      |           |
|                     | Triangular  | 21.17690| 0.02379      |           |
|                     | Biweight    | 24.94884| 0.02362      | 6.41319   |
|                     | Cosine      | 24.40999| 0.02367      |           |
|                     | Optcosine   | 25.48769| 0.02348      |           |
| 0.5                 | Gaussian    | 23.16520| 0.02367      |           |
|                     | Epanechnikov| 25.93416| 0.02280      |           |
|                     | Rectangular | 30.36449| 0.02532      | 6.30073   |
|                     | Triangular  | 20.95003| 0.02321      |           |
|                     | Biweight    | 24.82657| 0.02298      |           |
|                     | Cosine      | 24.27278| 0.02303      |           |
|                     | Optcosine   | 25.38036| 0.02281      |           |
| 0.8                 | Gaussian    | 24.77401| 0.01844      | 8.12790   |
|                     | Epanechnikov| 20.44493| 0.95231      |           |
|                     | Rectangular | 28.09410| 0.01794      |           |
|                     | Triangular  | 33.40625| 0.02002      |           |
|                     | Biweight    | 26.76606| 0.01801      |           |
|                     | Cosine      | 26.76606| 0.01805      |           |
|                     | Optcosine   | 28.09410| 0.01793      |           |
| 0.2                 | Gaussian    | 20.44493| 1.01310      | 0.17782   |
|                     | Epanechnikov| 20.44493| 0.95231      |           |
| 0.5                 | Gaussian    | 20.44685| 0.72364      | 0.18344   |
|                     | Epanechnikov| 20.44685| 0.70384      |           |
|                     | Rectangular | 20.44685| 0.64675      |           |
|                     | Triangular  | 20.44685| 0.72386      |           |
|                     | Biweight    | 20.44685| 0.71356      |           |
|                     | Cosine      | 20.44685| 0.71500      |           |
|                     | Optcosine   | 20.44685| 0.70873      |           |
| 0.8                 | Gaussian    | 20.49115| 0.79201      | 0.13032   |
|                     | Epanechnikov| 20.49115| 0.80480      |           |
|                     | Rectangular | 20.49115| 0.73103      |           |
|                     | Triangular  | 20.49115| 0.79912      |           |
|                     | Biweight    | 20.49115| 0.79578      |           |
|                     | Cosine      | 20.49115| 0.79563      |           |
|                     | Optcosine   | 20.49115| 0.80130      |           |

Table (5): Time to Extinction empirical best nonparametric fitted distribution based on May model with different Uniform and Geometric parameters and different kernels.

| Geometric parameter | Kernel      | x-value | max. density | Bandwidth |
|---------------------|-------------|---------|--------------|-----------|
| 0.2                 | Gaussian    | 51.35596| 0.00831      | 12.42979  |
| Function         | Value 1 | Value 2 |
|------------------|---------|---------|
| **a = 2, b = 4** |         |         |
| Gaussian         | 14.94196| 0.01791 |
| Epanechnikov     | 20.40094| 0.01922 |
| Rectangular      | 19.24246| 0.02012 |
| Triangular       | 20.98018| 0.01917 |
| Biweight         | 20.98018| 0.01905 |
| Cosine           | 20.98018| 0.01901 |
| Optcosine        | 20.98018| 0.01917 |
| 0.5              |         |         |
| Gaussian         | 21.55942| 0.01890 |
| Epanechnikov     | 20.40094| 0.01922 |
| Rectangular      | 19.24246| 0.02012 |
| Triangular       | 20.98018| 0.01917 |
| Biweight         | 20.98018| 0.01905 |
| Cosine           | 20.98018| 0.01901 |
| Optcosine        | 20.98018| 0.01917 |
| 0.8              |         |         |
| Gaussian         | 17.94938| 0.01824 |
| Epanechnikov     | 19.75383| 0.02023 |
| Rectangular      | 19.75383| 0.02023 |
| Triangular       | 17.94938| 0.01778 |
| Biweight         | 16.74641| 0.01798 |
| Cosine           | 16.74641| 0.01792 |
| Optcosine        | 17.34790| 0.01812 |
| 0.2              |         |         |
| Gaussian         | 100.90895| 0.00347 |
| Epanechnikov     | 100.90895| 0.00341 |
| Rectangular      | 105.75514| 0.00334 |
| Triangular       | 100.90895| 0.00345 |
| Biweight         | 100.90895| 0.00343 |
| Cosine           | 100.90895| 0.00343 |
| Optcosine        | 100.90895| 0.00342 |
| 0.5              |         |         |
| Gaussian         | 41.50015| 0.00774 |
| Epanechnikov     | 42.79984| 0.00775 |
| Rectangular      | 46.69891| 0.00794 |
| Triangular       | 42.79984| 0.00777 |
| Biweight         | 42.79984| 0.00773 |
| Cosine           | 42.79984| 0.00773 |
| Optcosine        | 42.79984| 0.00774 |
| 0.8              |         |         |
| Gaussian         | 37.59504| 0.00655 |
| Epanechnikov     | 42.05650| 0.00668 |
| Rectangular      | 42.05650| 0.00704 |
| Triangular       | 42.05650| 0.00655 |
| Biweight         | 42.05650| 0.00660 |
| Cosine           | 39.82577| 0.00659 |
| Optcosine        | 42.05650| 0.00665 |
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For convenient, below the mathematical formulas for best fitted distributions of empirical data,

1. It is said that the random variable $X$ have Gamma distribution if it has the pdf,
   \[ f(x) = \frac{x^{\alpha-1}}{\beta^\alpha \Gamma(\alpha)} e^{-x/\beta} \], \quad x > 0 \ , \text{where } \alpha > 0 \text{ is a continuous shape parameter and } \beta > 0 \text{ is a continuous scale parameter.}

2. It is said that the random variable $X$ have Gamma (3P) distribution if it has the pdf,
   \[ f(x) = \frac{(x-r)^{s-1}}{\beta^s \Gamma(s)} e^{-(x-r)/\beta} \], \quad x > r \ , \text{where } \alpha > 0 \text{ is a continuous shape parameter, } \beta > 0 \text{ is a continuous scale parameter and } r \text{ is a continuous location parameter.}

3. It is said that the random variable $X$ have Gen. Gamma (4P) distribution if it has the pdf,
   \[ f(x) = \frac{\kappa(x-r)^{s-1}}{\beta^s \Gamma(s)} e^{-(x-r)/\beta} \], \quad x > r \ , \text{where } \kappa, \alpha > 0 \text{ are continuous shape parameters, } \beta > 0 \text{ is a continuous scale parameter and } r \text{ is a continuous location parameter.}

4. It is said that the random variable $X$ have Erlang distribution if it has the pdf,
   \[ f(x) = \frac{x^{m-1}}{\beta^m \Gamma(m)} e^{-x/\beta} \], \quad x > 0 \ , \text{where } m \text{ is a positive integer shape parameter and } \beta > 0 \text{ is a continuous scale parameter.}

5. It is said that the random variable $X$ have Beta distribution if it has the pdf,
   \[ f(x) = \frac{1}{B(\alpha_1,\alpha_2)} (x-a)^{\alpha_1-1} (b-x)^{\alpha_2-1} \], \quad a \leq x \leq b \ , \text{where } B \text{ is the Beta function, } \alpha_1, \alpha_2 > 0 \text{ are continuous shape parameters and } a, b \text{ are continuous boundary parameters (} a < b \).

6. It is said that the random variable $X$ have Burr distribution if it has the pdf,
   \[ f(x) = \frac{a \kappa(x/\beta)^{s-1}}{\beta (1+(x/\beta)^{a})^{\kappa+1}} \], \quad x > 0 \ , \text{where } \kappa, \alpha > 0 \text{ are continuous shape parameters, } \beta > 0 \text{ is a continuous scale parameter.}

7. It is said that the random variable $X$ have Burr (4P) distribution if it has the pdf,
   \[ f(x) = \frac{a \kappa(x-r/\beta)^{s-1}}{\beta (1+(x-r/\beta)^{a})^{\kappa+1}} \], \quad x > r \ , \text{where } \kappa, \alpha > 0 \text{ are continuous shape parameters, } \beta > 0 \text{ is a continuous scale parameter and } r \text{ is a continuous location parameter.}

8. It is said that the random variable $X$ have Pareto 2 distribution if it has the pdf,
   \[ f(x) = \frac{a \beta^a}{(x+\beta)^{a+1}} \], \quad x > 0 \ , \text{where } \alpha > 0 \text{ is a continuous scale parameter and } \beta > 0 \text{ is a continuous scale parameter.}
9. It is said that the random variable $X$ have Gen. Pareto distribution if it has the pdf,

$$f(x) = \begin{cases} 
\frac{1}{\sigma} \left( 1 + \kappa \frac{(x-\mu)}{\sigma} \right)^{-(1/k+1)}, & \kappa \neq 0 \\
\frac{1}{\sigma} \left( \frac{(x-\mu)}{\sigma} \right)^{-k}, & \kappa = 0 
\end{cases}, \text{where } \kappa \text{ is a continuous shape parameter, } \sigma > 0 \text{ is a continuous scale parameter and } \mu \text{ is a continuous location parameter. The domain of the distribution is, } x \leq \infty \text{ for } k \geq 0 \text{ and } x \leq \mu - \sigma/k \text{ for } k < 0.
$$

10. It is said that the random variable $X$ have Johnson SB distribution if it has the pdf, $f(x) = \frac{\delta}{\lambda \sqrt{2\pi}} z^{-1/2} \left[ 1 - \exp \left( -\frac{1}{2} \left[ y + \delta \ln \left( \frac{z}{1-z} \right) \right] ^2 \right) \right]$, $\xi \leq x \leq \xi + \lambda$, where $z = \frac{x-\xi}{\lambda}$, $\delta > 0$ is a continuous shape parameter, $\lambda > 0$ is a continuous scale parameter, $\xi$ is a continuous location parameter and $y$ is another continuous shape parameter.

11. It is said that the random variable $X$ have Log-Logistic (3P) distribution if it has the pdf, $f(x) = \frac{\alpha}{\beta} \left( \frac{x}{\beta} \right)^{-\alpha} \left( 1 + \left( \frac{x}{\beta} \right)^{-\alpha} \right)^{-2}$, $x \geq y$, where $\alpha > 0$ is a continuous shape parameter, $\beta > 0$ is a continuous scale parameter and $y$ is a continuous location parameter.

12. The Wakeby distribution is defined by the quantile function, $x(F) = \xi + \frac{\alpha}{\beta} \left( 1 - (1 - F)^{\beta/\alpha} \right) - \frac{\delta}{\beta} \left( 1 - (1 - F)^{-\beta/\delta} \right)$, where $\xi \leq x < \infty$ if $\delta \geq 0$ and $y > 0$ and $\xi \leq x < \xi + \alpha/\beta - y/\delta$ if $\delta < 0$ or $y = 0$. The parameters $\xi, \alpha, \beta, y$ and $\delta$ are all continuous and the following conditions are imposed,

$\alpha \neq \beta$ or $y \neq \delta, \beta + \delta > 0$ or $\beta = y = \delta = 0$, if $\alpha = 0$ then $\beta = 0$. if $y = 0$ then $\delta = 0$, $y \geq 0$ and $\alpha + y \geq 0$.

**Appendix (2)**

Table (a.1): Kernels under considerations for the kernel density estimation of $f$ [19]

| Kernel            | Equation                                      | Notes                                                                 |
|-------------------|-----------------------------------------------|----------------------------------------------------------------------|
| Gaussian          | $K(y) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} y^2 \right)$ | $-\infty < y < \infty$                                              |
| Epanechnikov      | $K(y) = \frac{3}{4} (1 - y^2)^{\frac{1}{2}}$ | $\|y\| < 1$ if $\|y\| < 1$ otherwise                                  |
| Rectangular       | $K(y) = 1/2$                                   | $\|y\| \leq 1.$                                                   |
| Triangular        | $K(y) = (1 - |y|)$                             | $|y| \leq 1.$                                                      |
| Biweight(Quartic) | $K(y) = (15/16) \left(1 - y^2\right)^{2}$    | $|y| \leq 1.$                                                     |
| Cosine            | $K(y) = 0.5 (1 + \cos(\pi y))$               | $|y| \leq 1.$                                                    |
| Optcosine         | $K(y) = \frac{\pi}{4} \cos \left( \frac{\pi}{2} y \right)$ | $|y| \leq 1.$                                                 |