A wild knot $S^2 \hookrightarrow S^4$ as limit set of a Kleinian Group: Indra’s pearls in four dimensions.

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Abstract
The purpose of this paper is to construct an example of a 2-knot wildly embedded in $S^4$ as the limit set of a Kleinian group. We find that this type of wild 2-knots has very interesting topological properties.

1 Introduction

One geometrical method used to obtain remarkable fractal sets of extreme beauty and complexity having the property of being self-similar (i.e. conformally equal to itself at infinite small scales), is by considering the limit sets of Schottky groups, consisting on finitely generated groups of reflections on codimension one round spheres. As a testimony of such a beauty and complexity, one can consult the wonderful book Indra’s Pearl: The vision of Felix Klein written by D. Mumford, C. Series and D. Wright [18].

The purpose of the present paper is to construct, in the spirit of Indra’s pearls book, an example of a wildly embedded 2-sphere in $S^4$ (i.e. a wild

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2-knot in $\mathbb{S}^4$) obtained as limit set of a Kleinian group.

In section 2, we present the preliminary definitions and results in knot theory and Kleinian groups that we will use in this paper. In section 3, we describe the geometric ideas involved to construct a wild 2-knot, and we give an explicit example of such a group. In section 4, we prove that the limit set obtained in section 3, is a wild 2-knot in $\mathbb{S}^4$. In sections 5, 6 and 7 we give very interesting topological properties in the case where the original arc (see section 2) fibers over the circle. We show that the wild 2-knot also fibers over the circle and we determine its monodromy. In section 8, we lift the action of this Kleinian group to the twistor space of $\mathbb{S}^4$, obtaining a dynamically defined $\mathbb{S}^2 \times \mathbb{S}^2$ wildly embedded in the twistor space $\mathbb{P}_3^3$. I want to thank Prof. Alberto Verjovsky for all the discussions and very important suggestions. I also want to thank Prof. Aubin Arroyo for drawing the beautiful Figures 10 and 11.

### 2 Preliminaries

In 1925 Emil Artin described two methods for constructing knotted spheres of dimension two in $\mathbb{S}^4$ from knots in $\mathbb{S}^3$. The first of them is called suspension. Roughly speaking, this method consists of taking the suspension of $(K, \mathbb{S}^3)$, where $K \subset \mathbb{S}^3$ is a tame knot, to obtain a 2-knot $\Sigma K$ in $\mathbb{S}^4$. By construction, we have that the fundamental group of $\Sigma K$ is easily computable.

The second method is called spinning and uses the rotation process. A way to visualize it is the following. We can consider $\mathbb{S}^2$ as an $\mathbb{S}^1$-family of half-equators (meridians=$\mathbb{D}^1$) such that the respective points of their boundaries are identified to obtain the poles. Then the formula $Spin(\mathbb{D}^1) = \mathbb{S}^2$ means to send homeomorphically the unit interval $\mathbb{D}^1$ to a meridian of $\mathbb{S}^2$ such that $\partial \mathbb{S}^1 = \{0,1\}$ is mapped to the poles and, multiply the interior of $\mathbb{D}^1$ by $\mathbb{S}^1$. In other words, one spins the meridian with respect to the poles to obtain $\mathbb{S}^2$. Similarly, consider $\mathbb{S}^{n+1}$ as an $\mathbb{S}^1$-family of half-equators ($\mathbb{D}^n$) where boundaries are respectivity identified, hence $Spin(\mathbb{D}^n) = \mathbb{S}^{n+1}$ means to send homeomorphically $\mathbb{D}^n$ to a meridian of $\mathbb{S}^{n+1}$ and keeping $\partial \mathbb{D}^n$ fixed, multiply the interior of $\mathbb{D}^n$ by $\mathbb{S}^1$. In particular $Spin(\mathbb{D}^3) = \mathbb{S}^4$. 2
It is this second method that we will use to construct a 2-sphere wildly embedded in $S^4$, so we will give a more detailed description of it.

Consider in $\mathbb{R}^4$ the half-space

$$\mathbb{R}^4_+ = \{(x_1, x_2, x_3, 0) : x_3 \geq 0\}$$

whose boundary is the plane

$$\mathbb{R}^2 = \{(x_1, x_2, 0, 0)\}.$$ 

We can spin each point $x = (x_1, x_2, x_3, 0)$ of $\mathbb{R}^3_+$ with respect to $\mathbb{R}^2$ according to the formula

$$R_\theta(x) = (x_1, x_2, x_3 \cos \theta, x_3 \sin \theta).$$

We define $Spin(X)$ of a set $X \subset \mathbb{R}^3_+$, as

$$Spin(X) = \{R_\theta(x) : x \in X, 0 \leq \theta \leq 2\pi\}.$$

To obtain a knot in $\mathbb{R}^4$, we choose a tame arc $A$ in $\mathbb{R}^3_+$ with its end-points in $\mathbb{R}^2$ and its interior in $\mathbb{R}^3_+ \setminus \mathbb{R}^2$. Then $Spin(A)$ is a 2-sphere in $\mathbb{R}^4$ called a spun knot.

We can think of $A$ as the image of an embedding $A : I \to \mathbb{R}^3_+$ with $A(0) \neq A(1) \in \mathbb{R}^2$, and which will be denoted by the same letter. Then we will say that an arc $A \subset \mathbb{R}^3_+$ is a spinnable arc if it is smooth in every point with contact of infinite order with respect to the normal on its end-points.
It can be proved that the fundamental group of $\text{Spin}(A)$ is isomorphic to $\Pi_1(\mathbb{R}_+^3 - A)$ and by the Seifert-Van Kampen Theorem’s, this is isomorphic to the fundamental group of $A \cup L$ in $\mathbb{R}^3$, where $L \subset \mathbb{R}^3$ is an unknotted segment joining the end-points of $A$ (see [22], [31]).

Our goal is to obtain a wild 2-sphere as the limit set of a conformal Kleinian group. We will give briefly some basic definitions about Kleinian groups.

Let $M\text{ö}b(S^n)$ denote the group of Möbius transformations of the n-sphere $S^n = \mathbb{R}^n \cup \{\infty\}$, i.e. conformal diffeomorphisms of $S^n$ with respect to the standard metric. For a discrete group $G \subset M\text{ö}b(S^n)$ the discontinuity set $\Omega(G)$ is defined as follows

$$\Omega(G) = \{x \in S^n : \text{the point } x \text{ possesses a neighbourhood } U(x) \text{ such that } U(x) \cap g(U(x)) \text{ is empty for all but finite elements } g \in G\}.$$ 

The complement $S^n - \Omega(G) = \Lambda(G)$ is called the limit set (see [9]).

Both $\Omega(G)$ and $\Lambda(G)$ are $G$-invariant, $\Omega(G)$ is open, hence $\Lambda(G)$ is compact.

A subgroup $G \subset M\text{ö}b(S^n)$ is called Kleinian if $\Omega(G)$ is not empty. We will be concerned with very specific Kleinian groups of Schottky type.

We recall that a conformal map $\psi$ on $S^n$ can be extended in a natural way to the hyperbolic space $\mathbb{H}^{n+1}$, such that $\psi|_{\mathbb{H}^{n+1}}$ is an orientation-preserving isometry with respect to the Poincaré metric. Hence we can identify the group $M\text{ö}b(S^n)$ with the group of orientation preserving isometries of hyperbolic $(n + 1)$-space $\mathbb{H}^{n+1}$. This allows us to define the limit set of a Kleinian group through sequences.

A point $x$ is a limit point for the Kleinian group $G$, if there exist a point $z \in S^n$ and a sequence $\{g_m\}$ of distinct elements of $G$, with $g_m(z) \rightarrow x$. The set of limit points is $\Lambda(G)$ (see [14] section II.D).

One way to illustrate the action of a Kleinian group $G$ is to draw a picture of $\Omega(G)/G$. For this purpose a fundamental domain is very helpful. Roughly
speaking, it contains one point from each equivalence class in $\Omega(G)$ (see [10] pages 78-79, [14] pages 29-30).

**Definition 2.1** A fundamental domain $D$ for a Kleinian group $G$ is a codimension-zero piecewise-smooth submanifold (subpolyhedron) of $\Omega(G)$ satisfying the following

1. $\bigcup_{g \in G} g(Cl_{\Omega(G)}D) = \Omega$ ($Cl$ denotes closure).
2. $g(int(D)) \cap int(D) = \emptyset$ for all $g \in G - \{e\}$ ($int$ denotes the interior).
3. The boundary of $D$ in $\Omega(G)$ is a piecewise-smooth (polyhedron) submanifold in $\Omega(G)$, divided into a union of smooth submanifolds (convex polygons) which are called faces. For each face $S$, there is a corresponding face $F$ and an element $g = g_{SF} \in G - \{e\}$ such that $gS = F$ ($g$ is called a face-pairing transformation); $g_{SF} = g_{FS}^{-1}$.
4. Only finitely many translates of $D$ meet any compact subset of $\Omega(G)$.

**THEOREM 2.2** ([10], [14]) Let $D^* = D \cap \Omega/\sim_G$ denote the orbit space with the quotient topology. Then $D^*$ is homeomorphic to $\Omega/G$.

**3 The Construction**

The main idea of this construction is to use the symmetry of the spinning process to find a “packing” (i.e. a cover) of an embedded $S^2$ in $S^4$ consisting of closed round balls of dimension 4, such that the group $\Gamma$ generated by inversions in their boundaries (spheres of dimension 3) is Kleinian, and its limit set is a wild sphere of dimension two.

**Definition 3.1** Let $X \in S^n$. We will say that $E = \bigcup_{i=1}^m B_i$ is a packing for $X$ if this is contained in the interior of $E$, where $B_i$ is a closed ball of dimension $n$ for $i = 1, \ldots, m$.

**Definition 3.2** Let $A$ be a spinnable knotted arc in $\mathbb{R}^3$. A semi-pearl solid necklace subordinate to it, is a collection of consecutive closed round 4-balls $B^1, \ldots, B^n$ such that
1. The end-points of $A$ are the centers of $B^1$ and $B^n$ respectively.

2. The arc $A$ is totally contained in $\bigcup_{i=1}^{n} B^i$.

3. Two consecutive balls are orthogonal; otherwise $B^i \cap B^j = \emptyset$, $j \neq i + 1$.

4. The segment of $A$ lying in the interior of each ball is unknotted.

Each ball $B^i$ is called a solid pearl. Its boundary is a 3-sphere $\Sigma^i$ called a pearl. A semi-pearl necklace subordinate to $A$ is $\bigcup_{i=1}^{n} \Sigma^i$.

Next, we will define a pearl-necklace $\text{Spin}(T)$ subordinate to the 2-knot $\text{Spin}(A)$, with all the requirements needed for the group, generated by reflections on each pearl, to be Kleinian.

**Definition 3.3** Let $A \subset \mathbb{R}^3_+$ be a spinnable knotted arc. A pearl-necklace $\text{Spin}(T)$ subordinate to the tame knot $\text{Spin}(A) \subset S^4$ is constructed in the following way

1. Let $T$ be a semi-pearl necklace subordinate to $A$ consisting of the pearls $\Sigma^0, \Sigma^1, \ldots, \Sigma^{i+1}$. Consider the subset $T = \{\Sigma^1, \ldots, \Sigma^i\}$. Now, in $\text{Spin}(A)$ we will select six isometric copies $R_{2\pi i/6}(A)$ of $A$, called $A_i$, $i = 1, \ldots, 6$, in such a way that each $A_i$ has subordinate a isometric copy $R_{2\pi i/6}(T)$ of $T$, denoted by $T_i$, $i = 1, \ldots, 6$. We will require that $\Sigma^k_i \in T_i$ is orthogonal to the corresponding $\Sigma^k_{i+1} \in T_{i+1}$.

2. At each pole of the knot $\text{Spin}(A)$ we set a pearl $\Sigma_m$, $m = 1, 2$, orthogonal to the pearls of the next and previous levels, $\Sigma^1_i$ and $\Sigma^i_i$, $i = 1, \ldots, 6$, respectively; such that $\Sigma_1 \cap (\bigcap_{s=1}^{i+1} \Sigma^1_s) \neq \emptyset$ and $\Sigma_2 \cap (\bigcap_{s=1}^{i+1} \Sigma^i_s) \neq \emptyset$.

3. At each intersection point (see proposition 3.4) of two consecutive pearls $\Sigma^k_i$, $\Sigma^k_{i+1}$ in $T_i$ and the corresponding $\Sigma^k_{i+1}$, $\Sigma^k_{i+1}$ in $T_{i+1}$, we set a pearl $P^k_i$ which is orthogonal to these four pearls and does not intersect any other, i.e. we require that $P^k_i \cap \Sigma^r_s = \emptyset$ for $r \neq k, k+1$, $s \neq i, i+1$ and $P^k_i \cap \Sigma_m = \emptyset$, for $m = 1, 2$.

4. The intersection $B^k_i \cap \text{Spin}(A)$ is an unknotted disk, where $B^k_i$ is the solid pearl whose boundary is $\Sigma^k_i$. 

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Let \( \text{Spin}(T) = \{\Sigma_1, \ldots, \Sigma_n\} \) be a pearl-necklace. We define the \textit{filling} of \( \text{Spin}(T) \) as \( |\text{Spin}(T)| = \bigcup_{i=1}^{n} B_i \), where \( B_i \) is the round closed 4-ball whose boundary \( \partial B_i \) is the pearl \( \Sigma_i \).

Geometrically, the above definition means that when we rotate \( A \) and \( T \) with respect to \( \mathbb{R}^2 \), we obtain an infinite number of pearls covering \( \text{Spin}(A) \). We shall select a finite number of them keeping a “symmetry”, i.e. we will choose \( T \) such that when we spin a pearl \( \Sigma^k \in T \), we can select six of \( \{R_\theta(\Sigma^k) : 0 \leq \theta \leq 2\pi\} \) in such a way that their centers form a regular hexagon and adjacent pearls are orthogonal (two pearls are orthogonal if the square of the distance between their centers is equal to the sum of the squares of their radii). In other words, we will choose six \( \mathbb{R}^3_+ \) (six pages of the open book decomposition of \( \mathbb{R}^4 \)). As a consequence, in \( \text{Spin}(A) \) we will have six preferential meridians (one for each page). Each meridian \( A_i \) (\( 1 \leq i \leq 6 \)), is a copy of \( A \) and has a semi pearl-necklace \( T_i \) (\( 1 \leq i \leq 6 \)) subordinate to it which is an isometric copy of \( T \). The pearls belong to \( T_i \) will be denoted by \( \Sigma_i^k \), where the superscript \( k = 1, \ldots, l \), indicates its “latitude” and the subscript \( i = 1, \ldots, 6 \), indicates its “meridian” (see Figure 2 and compare [14] page 208).

![Figure 2: Spinning the arc A with six preferential meridians.](image)

At each pole of the knot \( \text{Spin}(A) \) we will set a pearl \( \Sigma_m \) (\( m = 1, 2 \)) orthogonal to the other six of the previous or next level respectively, such that there is no hole among them (see Figure 3), i.e. \( \Sigma_m \cap \Sigma_i^j \cap \Sigma_{i+1}^j \neq \emptyset \) (\( i \in \mathbb{Z}/6\mathbb{Z} \)), for the 3-tuples of indexes (\( m = 1, j = 1, i \)) and (\( m = 2, j = l, i \)).
By standard arguments of Euclidean geometry this sphere always exists.

Figure 3: A pearl set in an end-point of the arc A.

At this point, we have chosen a finite number of pearls. However, we have not proved that Spin(A) is totally covered by them.

**Proposition 3.4** The pearls Σ^k_i, k = 1, ..., l, i = 1, ..., 6 and the two pearls Σ_m, (m = 1, 2) at the poles, totally cover a knot isotopic to Spin(A).

**Proof.**

Firstly, we will verify that the intersection of the pearls Σ^k_i, Σ^{k+1}_i ∈ T_i and the corresponding Σ^{k+1}_{i+1}, Σ^{k+1}_{i+1} ∈ T_{i+1} is not empty.

Let r be the radius of the pearls Σ^k_i and Σ^{k+1}_i with centers c^k_i and c^{k+1}_i respectively. Let R be the radius of the pearls Σ^{k+1}_i and Σ^{k+1}_{i+1} with centers c^{k+1}_i and c^{k+1}_{i+1} respectively (see Figure 4). We know that two pearls are orthogonal if the square of the distance between their centers is equal to the sum of the squares of their radii. Hence

\[d^2(c^k_i, c^{k+1}_i) = 2r^2\]
\[d^2(c^{k+1}_i, c^{k+1}_{i+1}) = 2R^2\]
\[d^2(c^k_i, c^{k+1}_i) = d^2(c^{k+1}_i, c^{k+1}_{i+1}) = r^2 + R^2\]

this implies that

\[d(c^k_i, c^{k+1}_i) = d(c^{k+1}_i, c^{k+1}_{i+1}) = r + R\]
so that the intersection of these four pearls is a point.

Now consider the filling of each $T_i$. We shall prove that there exists a knot isotopic to $\text{Spin}(A)$ which is totally contained in $E := (\cup_{i=1}^{6}|T_i|) \cup_{m=1}^{2} B_m$, where $B_m$ is the closed 4-ball whose boundary is the pearl $\Sigma_m$.

Let $\omega : I \to \mathbb{R}^3_+$ be a parametrization of the arc $A$ by $t \in [0,1]$. Let $\Pi_t \subset \mathbb{R}^4$ be an affine plane parallel to the $zw$-plane. Notice that $\text{Spin}(\omega(t)) \subset \Pi_t$ is a circle for $t \in (0,1)$ and a point for $t = 0, 1$.

Let $t = \epsilon_1 > 0$ be the smallest $t$ for which $\omega(\epsilon_1) \in B_1^1 \cap \Sigma_1$ (remember that $B_1^k$ is the 4-ball such that $\partial B_1^k = \Sigma_1^k$). Let $t = \epsilon_2 > 0$ be the smallest $t$ that satisfies $\omega(\epsilon_2) \in B_1^1 \cap \Sigma_2$. For $t \in [\epsilon_1, \epsilon_2]$, we have that $\omega(t) \in B_1^k$ for some index $k$. Then $S_t := \Pi_t \cap_{i=1}^{6} |\Sigma_i^k|$ is a union of six disks with the property that adjacent disks are either overlapped or tangent. Observe that $\text{Spin}(\omega(t))$ may be not contained in $S_t$ (see Figures 5 and 6).

By an isotopy of $\Pi_t$, we can send $\text{Spin}(\omega(t))$ to a circle $\text{Spin}(\alpha(t))$, which passes through either the middle point of each chord joining the two intersection points of each overlap or the points of tangency of adjacent circles (see Figures 5 and 6). Indeed, this isotopy $\phi_t$ can be constructed radially from a function $\psi_t$ whose graph appears in Figure 7 (both cases). Thus $\phi_t(s,x) = s(\psi_t(x)) + (1-s)x$ is a stable isotopy, i.e. is the identity in the complement of a closed set.
For the pearls at the poles, we have that $\text{Spin}(\omega(t)) \subset B_1$ for $t \in [0, \epsilon_1]$ and $\text{Spin}(\omega(t)) \subset B_2$ for $t \in [\epsilon_2, 1]$. We can transform, by an isotopy, $\omega(t)$ $t \in [0, \epsilon_1] \cup [\epsilon_2, 1]$ in two arcs contained in the interior of the respective balls with the condition that their end-points coincide with $\alpha(\epsilon_1)$, $\omega(0)$ and $\alpha(\epsilon_2)$, $\omega(1)$, respectively.

By the above, we can define a function such that in each level $\omega(t)$ is the previous isotopy. This function depends of the parameter of the isotopy on each level and $t$. Since it is continuous with respect to each variable, it is continuous. Notice that on each level $\omega(t)$, we have that the corresponding isotopy is the identity in the complement of some disk. Hence we can conclude that this function is the identity in the complement of a closed ball.

We can extend this function to an isotopy defined on $S^4$ (see [20]) that sends $\text{Spin}(\omega(t))$ to $\text{Spin}(\alpha(t))$. 
Figure 7: Radial isotopy at level $\omega(t)$.

Therefore, an isotopic knot to $Spin(A)$ is totally covered by the pearls $\Sigma_i^k$, $k = 1, \ldots, l$, $i = 1, \ldots, 6$ and the two pearls at the poles. ■

The intersection of four pearls $\Sigma_i^k$, $\Sigma_i^{k+1} \in T_i$ and $\Sigma_{i+1}^k$, $\Sigma_{i+1}^{k+1} \in T_{i+1}$ is a single point that will be denoted by $p_i^k$ (see Figure 8). We centered at $p_i^k$ a pearl $P_i^k$ orthogonal to these four pearls such that it does not overlap to any other. Notice that this sphere always exists and its construction uses standard Euclidean geometry.

Figure 8: The dotted sphere is the pearl $P_i^k$.

Hence, $Spin(T)$ consists of the pearls $\Sigma_i^k$, $k = 1, \ldots, l$, $i = 1, \ldots, 6$, the two pearls $\Sigma_m$, $m = 1, 2$, at the poles and the pearls $P_i^k$. We will say that $Spin(A)$ is the template of $Spin(T)$.

Consider the group $\Gamma$ generated by reflections through each pearl. To guarantee that the group $\Gamma$ is Kleinian we will use the Poincaré Polyhedron Theorem. This theorem establishes conditions for the group to be discrete.
In practice these conditions are very hard to be verify, but in our case all of them are satisfied automatically from the construction (see [10], [14], [4]).

This theorem also gives us a presentation for the group \( \Gamma \). Suppose that the pearl-necklace \( \text{Spin}(T) \) is formed by the pearls \( \Sigma_j, (j = 1, \ldots, n) \) and we denote by \( I_j \) the reflection with respect to \( \Sigma_j \). Since the dihedral angles between the faces \( F_i, F_j \) are \( \frac{\pi}{n_{ij}} \), where \( n_{ij} \) is either 2 if the faces are adjacent or 0 in other case. Therefore, we have the following presentation of \( \Gamma \)

\[
\Gamma = \langle I_j, j = 1, \ldots, n \mid (I_j)^2 = 1, (I_i I_j)^{n_{ij}} = 1 \rangle
\]

**PROPOSITION 3.5** The group \( \Gamma \) generated by reflections through each pearl, is Kleinian.

*Proof.* By the Poincaré Polyhedron Theorem, we have that \( \Gamma \) is discrete and its fundamental domain is \( S^4 - |\text{Spin}(T)| \). Therefore it is Kleinian. \( \blacksquare \)

The first question to appear is if there exists a pearl-necklace \( \text{Spin}(T) \) for some knot \( \text{Spin}(A) \). In the next theorem we will exhibit a semi-necklace subordinate to an embedded of the trefoil arc \( A \), satisfying all the requirements of the definition 3.3.

**THEOREM 3.6** There exists an embedding of the trefoil arc \( A \) in \( \mathbb{R}^3_+ \) that admits a semi-necklace satisfying all the requirements of the definition 3.3.

*Proof.* By proposition 3.4 it follows that if we have constructed a pearl-necklace \( \text{Spin}(T) \) subordinate to the knot \( \text{Spin}(A) \), it is always possible to find a knot isotopic to \( \text{Spin}(A) \) such that it is totally contained in the interior of \( \text{Spin}(T) \). The group \( \Gamma \) is defined through the pearl-necklace, this means that the pearl-necklace is more fundamental for our purpose than the knot itself. This allows us to consider the trefoil arc \( A \) as a polygonal arc (see Figure 9) obtained joining the centers \( c_k \) of the pearls \( \Sigma_k^i \) whose coordinates appear in the next table.

Observe that the pearl \( \Sigma_i^k \) and the corresponding rotated \( \Sigma_i^{k+1} \) are orthogonal if and only if their radii are equal to the \( z \)-coordinate divided by \( \sqrt{2} \).
Figure 9: A drawing of the trefoil-arc $A$.

| $\Sigma$ | $C_i$ | $R_i$ |
|---|---|---|
| $\Sigma_1$ | $(2426.06421, 2296.89168, .75966995, 0)$ | $.537167778$ |
| $\Sigma_2$ | $(2426.06421, 2296.89168, 2.835126878,0)$ | $2.004737441$ |
| $\Sigma_3$ | $(2426.06421, 2296.89168, 10.58083755,0)$ | $7.481781981$ |
| $\Sigma_4$ | $(2426.06421, 2296.89168, 39.48822332,0)$ | $27.92239049$ |
| $\Sigma_5$ | $(2426.06421, 2296.89168, 147.3720558,0)$ | $104.20778$ |
| $\Sigma_6$ | $(2426.06421, 2296.89168, 550,0)$ | $388.9087297$ |
| $\Sigma_7$ | $(2426.06421, 1746.89168, 550,0)$ | $388.9087297$ |
| $\Sigma_8$ | $(2426.06421, 1196.89168, 550,0)$ | $388.9087297$ |
| $\Sigma_9$ | $(2426.06421, 740, 400,0)$ | $282.8427125$ |
| $\Sigma_{10}$ | $(2226.56071, 597.879, 200,0)$ | $141.4213562$ |
| $\Sigma_{11}$ | $(2126.56071, 397.879, 258.5786444,0)$ | $182.8427129$ |
| $\Sigma_{12}$ | $(2026.56071, 197.879, 200,0)$ | $141.4213562$ |
| $\Sigma_{13}$ | $(1826.56071, 197.879, 200,0)$ | $141.4213562$ |
| $\Sigma_{14}$ | $(1626.56071, 197.879, 200,0)$ | $141.4213562$ |
| $\Sigma_{15}$ | $(1426.56071, 197.879, 200,0)$ | $141.4213562$ |
| $\Sigma_{16}$ | $(1226.56071, 197.879, 200,0)$ | $141.4213562$ |
| $\Sigma_{17}$ | $(1026.56071, 197.879, 200,0)$ | $141.4213562$ |
| $\Sigma_{18}$ | $(826.56071, 197.879, 200,0)$ | $141.4213562$ |
| $\Sigma_{19}$ | $(626.56071, 197.879, 200,0)$ | $141.4213562$ |
| $\Sigma_{20}$ | $(426.56071, 197.879, 200,0)$ | $141.4213562$ |
| $\Sigma_{21}$ | $(426.56071, 390.842826, 186.6225781,0)$ | $131.9620905$ |
| $\Sigma_{22}$ | $(426.56071, 556.1388641, 150,0)$ | $106.0660172$ |
| $\Sigma_{23}$ | $(426.56071, 695.063304, 130,0)$ | $91.92388155$ |
| $\Sigma_{24}$ | $(390, 804.611533, 105,0)$ | $74.24621202$ |
| i   | C_i     | R_i   |
|-----|---------|-------|
| 25  | (420.7470362, 905.0088369, 105, 0) | 74.24621202 |
| 26  | (518.54878, 907.99193, 91.98807, 0) | 65.0453 |
| 27  | (610.5368498, 907.99193, 91.98807, 0) | 63.0453 |
| 28  | (702.5249198, 907.99193, 91.98807, 0) | 65.0453 |
| 29  | (794.5129898, 907.99193, 91.98807, 0) | 65.0453 |
| 30  | (886.5010598, 907.99193, 91.98807, 0) | 65.0453 |
| 31  | (978.4891298, 907.99193, 91.98807, 0) | 65.0453 |
| 32  | (1070.4772, 907.99193, 91.98807, 0) | 65.0453 |
| 33  | (1162.46527, 907.99193, 91.98807, 0) | 65.0453 |
| 34  | (1254.45334, 907.99193, 91.98807, 0) | 65.0453 |
| 35  | (1346.44141, 907.99193, 91.98807, 0) | 65.0453 |
| 36  | (1438.42948, 907.99193, 91.98807, 0) | 65.0453 |
| 37  | (1530.41755, 907.99193, 91.98807, 0) | 65.0453 |
| 38  | (1622.40562, 907.99193, 91.98807, 0) | 65.0453 |
| 39  | (1714.39369, 907.99193, 91.98807, 0) | 65.0453 |
| 40  | (1806.38176, 907.99193, 91.98807, 0) | 65.0453 |
| 41  | (1898.36983, 907.99193, 91.98807, 0) | 65.0453 |
| 42  | (1990.3579, 907.99193, 91.98807, 0) | 65.0453 |
| 43  | (2082.34597, 907.99193, 91.98807, 0) | 65.0453 |
| 44  | (2174.33404, 907.99193, 91.98807, 0) | 65.0453 |
| 45  | (2266.32211, 907.99193, 91.98807, 0) | 65.0453 |
| 46  | (2358.31018, 907.99193, 91.98807, 0) | 65.0453 |
| 47  | (2450.29825, 907.99193, 91.98807, 0) | 65.0453 |
| 48  | (2542.28632, 907.99193, 91.98807, 0) | 65.0453 |
| 49  | (2634.27439, 907.99193, 91.98807, 0) | 65.0453 |
| 50  | (2726.26246, 907.99193, 91.98807, 0) | 65.0453 |
| 51  | (2818.25053, 907.99193, 91.98807, 0) | 65.0453 |
| 52  | (2910.2386, 907.99193, 91.98807, 0) | 65.0453 |
| 53  | (3002.22667, 907.99193, 91.98807, 0) | 65.0453 |
| 54  | (3094.21474, 907.99193, 91.98807, 0) | 65.0453 |
| 55  | (3186.20281, 907.99193, 91.98807, 0) | 65.0453 |
| 56  | (3278.19088, 907.99193, 91.98807, 0) | 65.0453 |
| 57  | (3370.17895, 907.99193, 91.98807, 0) | 65.0453 |
| 58  | (3462.16702, 907.99193, 91.98807, 0) | 65.0453 |
| 59  | (3554.15509, 907.99193, 91.98807, 0) | 65.0453 |
| 60  | (3646.14316, 907.99193, 91.98807, 0) | 65.0453 |
| 61  | (3738.13123, 907.99193, 91.98807, 0) | 65.0453 |
Remark 3.7 The coordinates of the centers of $\Sigma_{76}$, $\Sigma_{77}$ and $\Sigma_{78}$ are rational numbers and their radii are equal to $\frac{1000}{\sqrt{2}}$. We obtained the rest of centers and radii using the equations

$$d^2(C_{k-1}, C_k) = R_{k-1}^2 + R_k^2$$

and

$$d^2(C_{k+1}, C_k) = R_{k+1}^2 + R_k^2$$

Hence, we conclude that all centers and radii of the pearls belong to a finite algebraic extension of the rational numbers.

Let $\Gamma$ be the group generated by reflections $I_j$, through the pearl $\Sigma_j$ ($j = 1, \ldots, n$) of the necklace $Spin(T)$ formed by $n$ pearls. Then $\Gamma$ is a
conformal Kleinian group.

4 Geometric Description of the Limit Set

Let $A$ be a spinnable knotted arc in $S^3$. Consider the 2-knot $\text{Spin}(A) \subset S^4$ and take a pearl-necklace $\text{Spin}(T)$ subordinate to $\text{Spin}(A)$ consisting on $n$ pearls.

Let $\Gamma$ be the group generated by reflections $I_j$ through the pearl $\Sigma_j \in \text{Spin}(T)$. The natural question is: What is its limit set? Recall that to find the limit set of $\Gamma$, we need to find all the accumulation points of orbits. To do that we are going to consider all the possible sequences of elements of $\Gamma$. We will do this in steps:

1. First step: Reflecting with respect to each $\Sigma_j$ ($j = 1, 2, \ldots, n$), a copy of the exterior of $\text{Spin}(T)$ is mapped within it. At the end we obtain a new knot $\text{Spin}(A_1)$, which is in turn isotopic to the connected sum of $n + 1$ copies of $\text{Spin}(A)$ and it is totally covered by $n(n - 2)$ pearls (packing) called $E(T_1)$.

Notice that there exists an isotopy of $S^4$ such that the knot $\text{Spin}(A \# A)$ is sent to $\text{Spin}(A) \# \text{Spin}(A)$. Actually, this remains true for the connected sum of any couple of knotted arcs. Therefore $\text{Spin}(A_1)$ is isotopic to spin of the connected sum of $n + 1$ copies of $A$.

We have that $|\text{Spin}(T)| = |E(T_1)|$. In fact, each pearl of $\text{Spin}(T)$ lies in $E(T_1)$. Hence $|E(T_1)|$ is a closed neighbourhood of $\text{Spin}(A)$. To clarify the above, see Figure 10 for a simpler case, i.e. for an unknotted necklace.

Claim 4.1 $|\text{Spin}(T_1)| = V$ is isotopic to a closed tubular neighbourhood of $\text{Spin}(A)$.

Proof. Let $N$ be a closed tubular neighbourhood of $\text{Spin}(A)$ with the condition that $N \subset \text{Int}(V)$. Since $A$ is a spinnable arc, it follows
that $\text{Spin}(A)$ is smooth. Given $p \in \text{Spin}(A)$, consider the tangent plane $T_p \text{Spin}(A)$ of $\text{Spin}(A)$ at $p$. Let $\Pi^2(p) \subset S^4$ be a 2-sphere totally geodesic with respect to the spherical metric (i.e. radius 1) that passes through $p$ and intersects transversally $T_p \text{Spin}(A)$. Thus $\Pi^2(p)$ cuts each solid pearl $B_i \in |\text{Spin}(T)|$ that contains $p$ in a disk $D_i$. Then $D_p \cup D_i$ is a star-shaped set with respect to $p$. This neighbourhood is contained in a closed disk $B_{R_p}(p)$, where the radius is $R_p = \sup\{d(x,p) : x \in D_p\} + \epsilon$. Notice that $N_p = \Pi^2(p) \cap N \subset D_p$ (see Figure 11).

Hence for each point $p \in \text{Spin}(A)$, we have found a neighbourhood $D_p \subset \Pi^2(p)$ of it which is star-shaped with respect to $p$ and retracts onto $N_p$. This retraction can be constructed in the following way. One draws a ray $r_\theta$ going from $p$ with angle $\theta$. Let $n(\theta)$ be the intersection

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{10}
\caption{An unknotted necklace and the first iteration.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{11}
\caption{A star-shaped neighbourhood of $p$.}
\end{figure}
point of \( r_\theta \) with \( N_p \) and let \( d(\theta) \) be the intersection point of \( r_\theta \) with \( D_p \) (see Figure 13). We can send the segment \( d(\theta) \) to the segment \( n(\theta) \) through the radial isotopy \( \phi_p(t, x) = t\psi_p(x) + (1 - t)x \), where \( \psi_p(s) \) is the unique polygonal function whose graph appears in Figure 12. Observe that this function is the identity beyond a distance \( R_p \) from \( p \).

![Figure 12: The ray \( d(\theta) \) is sent to the ray \( n(\theta) \) by a radial isotopy.](image)

Therefore we have an isotopy defined on \( \Pi^2(p) \) that transforms \( D_p \) to \( N_p \) and is the identity outside of some closed disk \( B_{R_p}(p) \). Since \( \Pi^2(p) \) depends continuously of \( p \), we have an isotopy that sends \( V \) to \( N \). By \([20]\) we can extend this isotopy to \( S^4 \). This isotopy is the identity outside of a closed tubular neighborhood \( Spin(A) \times S^1 \), where the radius of \( S^1 \) is \( \sup\{R_p\} \).

2. Second step: If we consider the action of elements of \( \Gamma \) on \( E(T_1) \), we obtain a new knot \( Spin(A_2) \) totally covered by a packing consisting of \( n(n^2 - 2n + 7) \) pearls, called \( E(T_2) \). The knot \( Spin(A_2) \) is isotopic to the connected sum of \( n^2 + 1 \) copies of \( Spin(A) \). By the above observation, it follows that it is also isotopic to the Spin of the connected sum of \( n^2 + 1 \) copies of \( A \).

Let \( V_1 = E(T_2) - Spin(T) \). Then \( |V_1| \) is connected and is a closed neighbourhood of the 2-knot \( Spin(P_1) \), which is in turn isotopic to the connected sum of \( 2n + 1 \) copies of \( Spin(A) \). By the above claim, \( |V_1| \) is isotopic to a closed tubular neighbourhood of \( Spin(P_1) \). Notice that
$|V_1| \subset |V|$ (see Figure 13).

Figure 13: The dotted pearls form $|V_1|$.

3. $k^{th}$-Step: The action of elements of $\Gamma$ on $E(T_{k-1})$ determines a tame knot $\text{Spin}(A_k)$, which is in turn isotopic to the connected sum of $n^{\frac{(n-1)k-1}{n-2}} + 1$ copies of $\text{Spin}(A)$ and is also isotopic to the Spin of the connected sum of $n^{\frac{(n-1)k-1}{n-2}} + 1$ copies of $A$.

Let $V_{k-1} = E(T_k) - E(T_{k-2})$. Thus $|V_{k-1}|$ is connected and is a closed neighbourhood of the knot $\text{Spin}(P_{k-1})$ which is in turn isotopic to the connected sum of $n^{\frac{(n-1)k-1}{n-2}} + n(n-3)^k + 1$. This neighbourhood consists of $2n(n-3)^k$ pearls and is isotopic to a closed tubular neighbourhood of $\text{Spin}(P_{k-1})$. By construction, $|V_{k-1}| \subset |V_{k-2}|$.

Let $x \in \bigcap_{k=1}^{\infty} |V_k|$. We shall prove that $x$ is a limit point. Indeed, there exists a sequence of closed balls $\{B_m\}$ with $B_m \subset |V_m|$ such that $x \in B_m$ for each $m$. We can find a $z \in S^4 - \text{Spin}(T)$ and a sequence $\{w_m\}$ of distinct elements of $\Gamma$, such that $w_m(z) \in B_m$. Since $\text{diam}(B_m) \to 0$ it follows that $w_m(z)$ converges to $x$. The other inclusion clearly holds. Therefore, the limit set is given by

$$\Lambda(\Gamma, A) = \lim_{k \to \infty} |V_k| = \bigcap_{k=1}^{\infty} |V_k|.$$ 

**Theorem 4.2** The limit set $\Lambda(\Gamma, A)$ is isotopic to $\text{Spin}(\Lambda)$, where $\Lambda$ is a wild arc in the sense of [9], [14], and is contained in each page ($\mathbb{R}^3_+$) of the open book decomposition of $\mathbb{R}^4$. 

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Proof. Let $A$ be a spinnable knotted arc. Construct the 2-knot $Spin(A)$ and take the necklace of $n$-pearls $Spin(T)$ subordinate to $Spin(A)$.

Now consider a semi-pearl necklace $C$ consisting of $n$ consecutive orthogonal round 2-spheres that cover completely to $A$, in which its end-points are the centers of the first pearl, $\Sigma_1$, and the last one, $\Sigma_n$. Construct $Spin(C) = \cup_{0 \leq \theta \leq 2\pi} R_\theta(C)$ (see section 2).

Claim 4.3 $|Spin(T)|$ is isotopic to $|Spin(C)|$.

Indeed, we have already proved that $|Spin(T)|$ is isotopic to a closed tubular neighbourhood of the knot $Spin(A)$. By the same argument, $|Spin(C)|$ is isotopic to a closed tubular neighbourhood of $Spin(A)$. Now two closed tubular neighbourhoods of $Spin(A)$ are isotopic ([8]). This proves the claim.

In the first step of the reflecting process applied to $Spin(T)$, we get a packing $E(T_1)$, of $Spin(A_1)$ formed by pearls. Now, for the case of the semi-necklace $C$, we join the end-points of $A$ by an unknotted curve $L$ obtaining a knot $K$ (see Figure 14).

![Figure 14: The dotted curve L joining the end-points of the arc A.](image)

We complete the semi-pearl necklace $C$ for the knot $K$, with pearls $Z_s$, $s = 1, \ldots, r$, keeping the same conditions on consecutive pearls. This new necklace is called $Z$ (see Figure 15).
Now, we reflect only with respect to each pearl $\Sigma_i$, $i = 1, \ldots, n$, of $C$. Then we obtain a new knot $K^1$ isotopic to the connected sum of $n + 1$ copies of $K$. To return $K^1$ to an arc, we remove the unknotted curve joining the image of the end-points of $A$ under the corresponding reflections. This new arc is called $A^1$ and is totally covered by a set of pearls $C_1$. Observe that $C \subset C_1$ (see Figure 16).

Notice that $\text{Spin}(A^1)$ is isotopic to $\text{Spin}(A_1)$ and $\text{Spin}(C_1)$ is a packing for it. Thus, $\text{Spin}(|C_1|) = \text{Spin}(|C|)$ (where $|C|$ is defined as $|\text{Spin}(T)|$) is a closed neighbourhood of $\text{Spin}(A)$.

In the second step for the necklace $\text{Spin}(T)$, we get a packing $E(T_2)$ of the knot $\text{Spin}(A_2)$. For the semi-necklace $C$, we join again the end-points of
the arc $A^1$ by an unknotted curve, $L_1$, forming again the knot $K^1$ in such a way that when we complete the semi-pearl necklace $C_1$, we add the pearls $Z_s$ $s = 1, \ldots, r$, obtaining the necklace $Z$. We can assume that the end-points of the arc $A^1$ coincide with the centers of the pearls $\Sigma_1$ and $\Sigma_n \in C \subset C_1$, respectively (see Figure 17).

![Figure 17: The pearl-necklace $Z$ subordinate to the knot $K^1$.](image)

Now reflecting only with respect to each pearl of the semi-necklace $C$, i.e. with respect to the pearls $\Sigma_i$ $i = 1, \ldots, n$, we get as in the previous step, the packing $C_2$ of the new arc $A^2$, which is in turn isotopic to the connected sum of $2n + 1$ copies of $K$ minus an unknotted curve $L_2$. When we spin $C_2$ and $A^2$, we obtain the packing $Spin(C_2)$ of $Spin(A^2)$. Define $W_1 = Spin(C_2) - Spin(C) \cup \{ I_j(Z) : j = 1, \ldots, n \}$. Then $|W_1|$ is a closed neighbourhood of a 2-knot $Spin(Q^2)$, which is isotopic to the connected sum of $2n + 1$ copies of $Spin(A)$. So $|V_1|$ and $|W_1|$ are closed neighbourhoods of isotopic 2-knots. Using the same arguments of claim 4.1 and the standard fact that any locally flat embedding of $S^2$ in $S^4$ has trivial normal bundle, it follows that two closed tubular neighbourhoods of isotopic knots are isotopic, hence $|V_1|$ is isotopic to $|W_1|$ and the following diagram commutes

$$
\begin{array}{ccc}
(S^4,|V_1|) & \longrightarrow & (S^4,|V|) \\
\sim & \downarrow & \sim \\
(S^4,|W_1|) & \longrightarrow & (S^4,|W|),
\end{array}
$$

where the row maps are inclusions. Notice that this isotopy is stable, i.e. is the identity on some open in $S^4$, and is orientation-preserving (see [11]).
Inductively, for the $k^{th}$-step we obtain the packings $E(T_k)$ of $\text{Spin}(A_k)$ and $\text{Spin}(C_k)$ of $\text{Spin}(A^k)$. Where $\text{Spin}(A_k)$ is obtained through the reflecting process previously described. The arc $A^k$ is formed applying the reflecting process to $C \subset Z$ subordinate to the knot $K^k$ and removing an unknotted curve $L_k$. Then

$$|V_k| = |E(T_{k+1}) - E(T_{k-1})|$$

and

$$|W_k| = |\text{Spin}(C_{k+1}) - \text{Spin}(C_{k-1}) \cup \{I_{i_1i_2\ldots i_l}(Z) : 1 \leq l \leq k\}|$$

are closed neighborhoods of the knots $\text{Spin}(P_k)$ and $\text{Spin}(Q^k)$ respectively, which are in turn isotopic to the connected sum of $n\left\lceil \frac{(n-1)^{k-1}}{n-2} \right\rceil + n(n-3)^{k-1} + 1$ copies of $\text{Spin}(A)$. Hence, $|V_k|$ is isotopic to $|W_k|$ and the following diagram commutes

$$(\mathbb{S}^4, |V_k|) \longrightarrow (\mathbb{S}^4, |V_{k-1}|)$$

$$\sim \downarrow \sim \downarrow$$

$$(\mathbb{S}^4, |W_k|) \longrightarrow (\mathbb{S}^4, |W_{k-1}|).$$

Observe that this isotopy is stable and orientation-preserving. Summarizing, we have the commutative diagram

$$(\mathbb{S}^4, |V|) \longleftarrow (\mathbb{S}^4, |V_1|) \longleftarrow \cdots \longleftarrow (\mathbb{S}^4, |V_k|) \cdots$$

$$\sim \downarrow \sim \downarrow \sim \downarrow \sim \downarrow$$

$$(\mathbb{S}^4, |W|) \longleftarrow (\mathbb{S}^4, |W_1|) \longleftarrow \cdots \longleftarrow (\mathbb{S}^4, |W_k|) \cdots$$

where the row maps are inclusions and the vertical arrows are orientation-preserving stable isotopies.

The inverse limit in the first row of the above diagram is $(\mathbb{S}^4, \Lambda(\Gamma, A))$ and the inverse limit in the second row is $(\mathbb{S}^4, \text{Spin}(\lim_{\leftarrow k} |W_k|))$. But $\lim_{\leftarrow k} |W_k|$ is a wild arc denoted by $\Lambda(\Gamma)$ (see [14], [10]), i.e. the inverse limit in the second row is $(\mathbb{S}^4, \text{Spin}(\Lambda(\Gamma)))$.

By the universal property of the inverse limit, there exists a homeomorphism of $\mathbb{S}^4$ to $\mathbb{S}^4$ which sends $\Lambda(\Gamma, A)$ to $\text{Spin}(\Lambda(\Gamma))$. This homeomorphism is stable because it coincides with a stable homeomorphism on some open set (see [11]) and is orientation-preserving. This implies that it is isotopic to
Therefore, the knots $\text{Spin}(\Lambda(\Gamma))$ and $\Lambda(\Gamma, A)$ are isotopic. This proves the Theorem. ■

**COROLLARY 4.4** The limit set $\Lambda(\Gamma, A)$ is homeomorphic to $S^2$

*Proof.* By the above theorem, we have that

$\Lambda(\Gamma, A) \cong \text{Spin}(\Lambda(\Gamma)) \cong S^2$.

■

**THEOREM 4.5** Let $\text{Spin}(T)$ be a pearl-necklace subordinate to the non-trivial tame knot $\text{Spin}(A)$. Then $\Lambda(\Gamma, A)$ is wildly embedded in $S^4$.

*Proof.* The fundamental group of $S^4 - \Lambda(\Gamma, A)$ is isomorphic to the fundamental group of the knot obtained joining the end-points of the arc $\Lambda$ by an unknotted curve (see [22]). It is well-known that this fundamental group has no finite representation (see [9], [14]). ■

**Example 4.6** Let $\text{Spin}(T)$ be a pearl-necklace subordinate to $\text{Spin}(A)$ where $A$ is the trefoil arc, $T_{2,3}$. Then

\[ \Pi_1(\text{Spin}(A)) \cong \Pi_1(T_{2,3}) = \{x, y \mid xyx = yxy\} \]

hence

\[ \Pi_1(S^4 - \Lambda(\Gamma, A)) = \{x_1, y_1, \ldots, y_n, \ldots \mid x_1y_1x_1 = y_1x_1y_1, x_1y_2x_1 = y_2x_1y_2, \ldots, x_1y_nx_1 = y_nx_1y_n, \ldots \} \]

\[ \cong (\cdots (\Pi_1(\text{Spin}(A)) *_{x_1} \Pi_1(\text{Spin}(A))) *_{x_1} \cdots *_{x_1} \Pi_1(\text{Spin}(A))) *_{x_1} \cdots \]

is infinitely generated with a infinite number of relations.
5 Hyperbolic Manifolds

The action of $\Gamma$ can be extended to the hyperbolic space $\mathbb{H}^5$ and in this case $\Gamma$ is a subgroup of $\text{Isom}\mathbb{H}^5$, which acts properly and discontinuously on $\mathbb{D}^5 = \mathbb{H}^5 \cup \partial \mathbb{H}^5$. Its fundamental polyhedron is $\mathcal{P} = (\mathbb{H}^5 \cup \partial \mathbb{H}^5) - \widetilde{\text{Spin}}(T)$, where $\widetilde{\text{Spin}}(T)$ is the natural extension of the pearl-necklace to $\mathbb{H}^5$. It is a convex subset and has a finite number of sides, hence $\Gamma$ is geometrically finite (see [2]).

The group $\Gamma$ acts properly and discontinuously on $\mathcal{P}$, then the quotient $\mathcal{M}_{\Gamma} = (\mathbb{D}^5 - \Lambda(\Gamma, A))/\Gamma \cong \mathcal{P}$ (see Theorem 2.2) is a compact orbifold such that its interior is a non-compact hyperbolic manifold of infinite volume and its compactification as a subset of $\mathbb{D}^5$ has boundary which possesses a conformally flat structure given by the action.

For the Kleinian group $\Gamma$ acting on the pearl-necklace $\text{Spin}(T)$, its fundamental domain is $D = S^4 - |\text{Spin}(T)|$. The group $\Gamma$ acts properly and discontinuously on $\mathcal{P}$, hence $\mathcal{P} \cong \Omega(\Gamma)/\Gamma = (S^4 - \Lambda(\Gamma))/\Gamma$ is an orientable, compact, conformally flat 4-orbifold with boundary. Its fundamental group coincides with the fundamental group of the template of $\text{Spin}(T)$.

In the next section, we will describe $(S^4 - \Lambda(\Gamma))/\Gamma$ under the restriction that $\text{Spin}(A)$ is a fibered knot.

Consider now the index-two subgroup $\widetilde{\Gamma} \subset \Gamma$ consisting of even words, i.e. $\widetilde{\Gamma}$ is the orientation preserving index two subgroup of $\Gamma$. Its fundamental polyhedron is $\widetilde{\mathcal{P}} = (\mathbb{H}^5 \cup \partial \mathbb{H}^5 - |\text{Spin}(T)|) \cup (\widetilde{B_j} - I_j(|\text{Spin}(T) - \Sigma_j|))$, where tilde means the natural extensions to the hyperbolic space of both the pearl-necklace and the corresponding reflection map. Since $\widetilde{\mathcal{P}} \subset \mathbb{D}^5$ is a convex subset and has a finite number of sides, it follows that $\widetilde{\Gamma}$ is geometrically finite.

Since $\widetilde{\Gamma}$ acts freely on its domain of discontinuity, then the quotient space $\mathcal{M}_{\widetilde{\Gamma}}^2 = (\mathbb{D}^5 - \Lambda(\widetilde{\Gamma}, A))/\widetilde{\Gamma} \cong \widetilde{\mathcal{P}}/\sim_{\widetilde{\Gamma}}$ is a compact, orientable manifold, such that $\text{Int}(\mathcal{M}_{\widetilde{\Gamma}}^2)$ is a non-compact, orientable hyperbolic manifold of infinite volume. This space as a subset of $(\mathbb{D}^5 - \Lambda(\widetilde{\Gamma}))$, has a boundary which possesses a natural conformally flat structure given by the action.
For the Kleinian group \( \widetilde{\Gamma} \) acting on \( S^4 \), its fundamental domain is \( \tilde{D} = (S^4 - |\text{Spin}(T)|) \cup (B_j - I_j(|\text{Spin}(T) - \Sigma_j|)) \). Since \( \tilde{\Gamma} \) acts freely on \( \Omega(\tilde{\Gamma}) \), we have that \( \Omega/\tilde{\Gamma} \cong \tilde{D} \cap \Omega/\sim_{\tilde{\Gamma}} \) is a compact, orientable, conformally flat 4-manifold with boundary. Its fundamental group is the fundamental group of the knot \( \text{Spin}(A \# A) \).

6 Fibration of \( S^4 - \Lambda(\Gamma) \) over \( S^1 \)

We recall that a mapping \( f : E \to B \) is said to be a locally trivial fibration with fiber \( F \) if each point of \( B \) has a neighbourhood \( U \) and a “trivializing” homeomorphism \( h : f^{-1}(U) \to U \times F \) for which the following diagram commutes

\[
\begin{array}{ccc}
  f^{-1}(U) & \xrightarrow{h} & U \times F \\
  f \downarrow & & \downarrow \text{projection} \\
  U & \xrightarrow{\text{projection}} & U \times F
\end{array}
\]

\( E \) and \( B \) are known as the total and base spaces, respectively. Each set \( f^{-1}(b) \) is called a fiber and is homeomorphic to \( F \). We will be concerned with fibrations with base space \( S^1 \).

**Definition 6.1** A knot or link \( L \) in \( S^3 \) is fibered if there exists a locally trivial fibration \( f : (S^3 - L) \to S^1 \). We require that \( f \) be well-behaved near \( L \). That is, each component \( L_i \) is to have a neighbourhood framed as \( \mathbb{D}^2 \times S^1 \), with \( L_i \cong \{0\} \times S^1 \), in such a way that the restriction of \( f \) to \( (\mathbb{D}^2 - \{0\}) \times S^1 \) is the map into \( S^1 \) given by \( (x, y) \to \frac{y}{|y|} \).

It follows that each \( f^{-1}(x) \cup L, x \in S^1 \), is a 2-manifold with boundary \( L \): in fact a Seifert surface for \( L \) (see [22], page 323).

**Example 6.2** Let \( S^3_\epsilon \subset \mathbb{C}^2 \) be the 3-sphere centered at the origin of radius \( \epsilon \). Let \( V = \{(z_1, z_2) \in \mathbb{C}^2 : z_1^2 + z_2^3 = 0\} \). Then \( S^3_\epsilon \cap V = K \) is the right-handed trefoil knot and the map \( F : S^3_\epsilon - K \to S^1 \) given by \( F(z_1, z_2) = \frac{z_1^2 + z_2^3}{|z_1 + z_2|} \) is a locally trivial fibration with fiber the punctured torus (see [17] section 1, [22] pages 327-333).
LEMMA 6.3 Let $A$ be a spinnable knotted arc. Suppose that the knot $K$, obtained from $A$ joining its end-points by an unknotted curve, fibers over the circle with fiber the surface $S$. Then $\text{Spin}(A)$ fibers over the circle with fiber $S_\theta$, an $\mathbb{S}^1$-family of surfaces $\tilde{S}$ all glued onto a single meridian of $\partial \mathbb{D}^3$ with longitude $\theta$. The interior of $\tilde{S}$ is $S$ and its boundary is a meridian of $\partial \mathbb{D}^3$ (see Figure 18).

Proof. The fibering of the complement of $K$ induces a fibering of $\mathbb{D}^3 - \mathbb{D}^1$ by surfaces $\tilde{S}_\theta$, $\theta \in \mathbb{S}^1$. The interior of $\tilde{S}_\theta$ is $S$ and its boundary is $\partial \tilde{S}_\theta = M^1_\theta$ the meridian of $\partial \mathbb{D}^3$ with longitude $\theta$ (see Figure 18).

Figure 18: Fibering of $\mathbb{D}^3 - \mathbb{D}^1$.

Recall that in the spinning process we multiply the interior of $\mathbb{D}^3$ by $\mathbb{S}^1$ and $\partial \mathbb{D}^3$ stays fixed. Hence, we get a fibering of $\mathbb{S}^4 - \text{Spin}(A)$ by an $\mathbb{S}^1$-family of surfaces $\tilde{S}$ all glued onto the single meridian $M^1_\theta$ of $\partial \mathbb{D}^3$ with longitude $\theta$.

Henceforth, a fibered arc will mean that the knot obtained from it joining its end-points by an unknotted curve, fibers over the circle.

LEMMA 6.4 Let $A$ be a spinnable fibered arc with fiber the surface $S$. Let $\text{Spin}(T)$ be an $n$-pearl necklace subordinate to the tame knot $\text{Spin}(A)$. Let $\Lambda(\Gamma, A)$ be the limit set. Then $\Omega(\Gamma)/\Gamma$ fibers over the circle with fiber $S^{**}$, the closure of the surface $S_\theta$ of the previous lemma.

Proof. Let $\tilde{P} : \mathbb{S}^4 - \text{Spin}(A) \to \mathbb{S}^1$ be the given fibration with fiber the $3$-manifold $S_\theta$. Observe that $\tilde{P} \mid_{\mathbb{S}^4 - \text{Spin}(T)} \equiv P$ is a fibration with fiber $S_\theta$. 

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As we know $\Omega(\Gamma)/\Gamma \cong D^*$. In our case $D^* = S^4 - \text{Spin}(T)$, which fibers over the circle with fiber the closure of $S_\theta$. ■

By the above, to describe $(S^4 - \Lambda(\Gamma, A))/\Gamma$ when the original knot is fibered, we just need to determine its monodromy. It coincides with the knot’s monodromy. Hence we have a complete description of $(S^4 - \Lambda(\Gamma, A))/\Gamma$.

**Lemma 6.5** Let $A$ be a spinnable fibered arc with fiber the surface $S$. Let $\text{Spin}(T)$ be an $n$-pearl necklace subordinate to the tame knot $\text{Spin}(A)$. Let $\tilde{\Gamma}$ be the orientation preserving index two subgroup of $\Gamma$. Let $\Lambda(\tilde{\Gamma}, A)$ be the limit set. Then $\Omega(\tilde{\Gamma})/\tilde{\Gamma}$ fibers over the circle with fiber $S^*$, which is homeomorphic to the connected sum along the boundary of the 3-manifold $S_\theta$ with itself.

**Proof.** We can assume, up to isotopy, that the fiber $S$ cuts each pearl of the semi-necklace corresponding to $A$, in arcs going from one intersection point to another. Hence, we can assume that the fiber $S_\theta$ cuts each pearl $\Sigma_i \in \text{Spin}(T)$ in disks $a_i$, whose boundary is the intersection of $\Sigma_i$ with the adjacent pearls.

When we reflect with respect to $\Sigma_i$ a copy of $S_\theta$, called $S_\theta^i$, is mapped to the interior of $\Sigma_i$ and it is joined to $S_\theta$ along the disk $a_i$. ■

Since $\tilde{\Gamma}$ is a normal subgroup of $\Gamma$, it follows by Lemma 8.1.3 in [27] that $\tilde{\Gamma}$ has the same limit set as $\Gamma$. Therefore $S^4 - \Lambda(\Gamma, A) = S^4 - \Lambda(\tilde{\Gamma}, A)$.

**Theorem 6.6** Let $A$ be a non-trivial spinnable fibered arc. Let $\text{Spin}(T)$ be a pearl-necklace subordinate to the fibered knot $\text{Spin}(A)$. Let $\Gamma$ be the group generated by reflections through the pearls and let $\tilde{\Gamma}$ be the orientation preserving index two subgroup of $\Gamma$. Let $\Lambda(\Gamma, A) = \Lambda(\tilde{\Gamma}, A)$ be the corresponding limit set. Then:

1. There exists a locally trivial fibration $\psi : S^4 - \Lambda(\Gamma, A) \to S^1$, where the fiber $\Sigma_\theta^* = \psi^{-1}(\theta)$ is an $S^1$-family of surfaces $\Sigma$ all glued onto a meridian $\theta$, of $\partial D^3$ (see Lemma 6.3). Where $\Sigma$ is an orientable infinite genus surface with one end.

2. $\Sigma_\theta^* - \Sigma_\theta^* = \Lambda(\Gamma, A)$. 

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Proof. We know that \( \zeta : \Omega(\tilde{\Gamma}) \to \Omega(\tilde{\Gamma})/\tilde{\Gamma} \) is an infinite-fold covering. By the previous lemma, there exists a locally trivial fibration \( \phi : \Omega(\tilde{\Gamma})/\tilde{\Gamma} \to S^1 \) with fiber \( S^* \).

Then \( \psi = \phi \circ \zeta : \Omega(\tilde{\Gamma}) \to S^1 \) is a locally trivial fibration. The fiber is \( \Gamma(S^*) \), i.e. the orbit of the fiber.

We now give another proof. As we know from Theorem 4.1, the knot \( \Lambda(\Gamma, A) \) is isotopic to the knot \( \text{Spin}(\Lambda(\Gamma)) \), where \( \Lambda(\Gamma) \) is a wild arc. Since \( A \) is fibered, so is \( \Lambda(\Gamma) \). In this case the fiber, \( \Sigma \), is an orientable infinite genus surface with one end. Hence \( \text{Spin}(\Lambda(\Gamma)) \) fibers over the circle with fiber \( \Sigma^* \), an \( S^1 \)-family of surfaces \( \Sigma \) all glued onto a meridian \( \theta \), of \( \partial D^3 \) (see Lemma 6.3).

The first part of the theorem has been proved. For the second part, observe that the closure of the fiber is the closure of the \( S^1 \)-family of surfaces \( \Sigma \), i.e. is the closure of an \( S^1 \)-family of ends. As we can see in the Figure 20, each end has as boundary the wild arc \( \Lambda(\Gamma) \). Hence the closure of the fiber is exactly the limit set. Therefore \( \Sigma^* - \Sigma^* = \Lambda(\Gamma, A) \). ■

Remark 6.7
1. This theorem can be generalized to fibered links.

2. This theorem gives an open book decomposition of \( S^4 - \Lambda(\Gamma, A) \), where the “binding” is the wild knot \( \Lambda(\Gamma, A) \), and each “page”, \( \Sigma^* \), is a 3-manifold which fibers over \( S^1 \) and is the \( S^1 \)-family of surfaces \( \Sigma \) all glued onto a meridian \( \theta \), of \( D^3 \) (see Lemma 6.3). Here \( \Sigma \) is an orientable infinite genus surface with one end.

Indeed, this decomposition can be viewed in the following way. For the above theorem, \( S^4 - \Lambda(\Gamma, A) \) is \( \Sigma^* \times [0, 1] \) modulo the identification of the top with the bottom through an identifying homeomorphism. Consider \( \overline{\Sigma^*} \times [0, 1] \) and identify the top with the bottom. This is equivalent to keep \( \partial \overline{\Sigma^*} \) fixes and to spin \( \Sigma^* \times \{0\} \) with respect to \( \partial \overline{\Sigma^*} \) until glue it with \( \Sigma^* \times \{1\} \). Removing \( \partial \overline{\Sigma^*} \) we obtain the open book decomposition.

7 Monodromy
Let \( \text{Spin}(A) \) be a non-trivial fibered tame knot and let \( S \) be the fiber. Since \( S^4 - \text{Spin}(A) \) fibers over the circle, we know that \( S^4 - \text{Spin}(A) \) is a mapping
torus equal to $S \times [0, 1]$ modulo an identifying homeomorphism $\psi : S \to S$ that glues $S \times \{0\}$ to $S \times \{1\}$. This homeomorphism induces a homomorphism

$$\psi_# : \Pi_1(S) \to \Pi_1(S)$$

called the monodromy of the fibration.

Another way to understand the monodromy is through the first return Poincaré map, defined as follows. Let $M$ be connected, compact manifold and let $f_t$ be a flow that possesses a transversal section $\eta$. It follows that if $x \in \eta$ then there exists a continuous function $t(x) > 0$ such that $f_t \in \eta$. We may define the first return Poincaré map $F : \eta \to \eta$ as $F(x) = f_{t(x)}(x)$. This map is a diffeomorphism and induces a homomorphism of $\Pi_1$ called the monodromy (see [29], chapter 5).

For the manifold $S^4 - Spin(A)$, the flow that defines the first return Poincaré map $\Phi$ is the flow that cuts transversally each page of its open book decomposition.

Consider a pearl-necklace $Spin(T)$ subordinate to $Spin(A)$. As we have observed during the reflecting process, $Spin(A)$ and $S$ are copied in each reflection. So the flow $\Phi$ is also copied. Hence, the Poincaré map can be extended in each step, giving us in the end a homeomorphism $\psi : \Sigma_\theta^* \to \Sigma_\theta^*$ that identifies $\Sigma_\theta^* \times \{0\}$ with $\Sigma_\theta^* \times \{1\}$, and induces the monodromy of the wild knot.

From the above, if we know the monodromy of the knot $Spin(A)$ then we know the monodromy of the wild knot $\Lambda(\Gamma, A)$.

By the long exact sequence associated to a fibration, we have

$$0 \to \Pi_1(\Sigma_\theta^*) \to \Pi_1(S^4 - \Lambda(\Gamma, A)) \xrightarrow{\psi} Z \to 0,$$

which has a homomorphism section $\Psi : Z \to (S^4 - \Lambda(\Gamma, A))$. Therefore (1) splits. As a consequence $\Pi_1(S^4 - \Lambda(\Gamma, A))$ is the semi-direct product of $Z$ with $\Pi_1(\Sigma_\theta^*)$. 

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Example 7.1 Let $A$ be the trefoil arc. Consider the knot $\text{Spin}(A)$. Then the fiber $\Sigma^*_\theta$ ($\theta \in S^1$) is an $S^1$-family of punctured torus all glued onto a single meridian (see previous section). The fundamental group of $\Sigma^*_\theta$ is the free group in two generators, $a$ and $b$. Since $\Pi_1(\text{Spin}(A)) \cong \Pi_1(\text{Trefoil knot})$, it follows that the monodromy maps, in both cases, coincide. That is, $\psi_\#$ sends $a \mapsto b^{-1}$ and $b \mapsto ab$. Its order is six up to an outer automorphism (See [22] pages 330-333).

The monodromy in the limit $\psi_\# : \Pi_1(\Sigma^*_\theta) \to \Pi_1(\Sigma^*_\theta)$ is given by $a_i \mapsto b_i^{-1}$ and $b_i \mapsto a_i b_i$, where $\Pi_1(\Sigma^*_\theta) = \{a_i, b_i\}$. So

$$\Pi_1(S^4 - \Lambda(\Gamma, A)) \cong \Pi_1(S^1) \ltimes_{\psi_\#} \Pi_1(\Sigma^*_\theta)$$

$$= \{a_i, b_i, c : a_i * c = b_i^{-1}, b_i * c = a_i b_i\}$$

$$= \{a_i, c : c^{-1}a_i^{-1}c = a_i c^{-1}a_i\}$$

$$= \{a_i, c : c = a_i c a_i c^{-1}a_i^{-1}\}$$

$$= \{a_i, c : c = ca_i c a_i c^{-1}a_i^{-1}\}$$

$$= \{a_i, c : c = ca_i c a_i c^{-1}a_i^{-1}c^{-1}\}.$$

Let $\alpha_i = ca_i c^{-1}$;

$$\{\alpha_i, c : c = \alpha_i c a_i c^{-1}a_i^{-1}\}$$

$$= \{\alpha_i, c : c = \alpha_i c a_i c^{-1}\}$$

This gives another method for computing the fundamental group of a wild 2-knot whose complement fibers over the circle.

COROLLARY 7.2 Let $\text{Spin}(T)$ be a pearl-necklace whose template is a non-trivial tame fibered knot $\text{Spin}(A)$. Then $\Pi_1(\Omega(\Gamma)/\Gamma) \cong \mathbb{Z} \ltimes_{\psi_\#} \Pi_1(\Sigma^*_\theta)$.

8 Kleinian Groups and Twistor Spaces

In this section we will lift the action of the group $\Gamma$ on $S^4$ to its twistorial space, which is complex projective 3-space $P^3_C$. We refer to [24] and [19] for details.

Let us now recall briefly the twistor fibration of $S^4$, also known as the Calabi-Penrose fibration $\pi : P^3_C \to S^4$ (see [19]). There are several equivalent
ways to construct this fibration. A geometric way to describe it is by thinking of $S^4$ as being the quaternionic projective line $P^1_H$, of right quaternionic lines in the quaternionic plane $H^2$ (regarded as a 2-dimensional right $H$-module). That is, for $q := (q_1, q_2) \in H^2$ ($q \neq (0, 0)$) the right quaternionic line passing through $q$ is the linear space $R_q := \langle (q_1 \lambda, q_2 \lambda) \mid \lambda \in H \rangle$. We can identify $H^2$ with $C^4$ via the $R$-linear map given by $(q_1, q_2) \mapsto (z_1, z_2, z_3, z_4)$, where $q_1 = z_1 + z_2j = x_1 + x_2i + x_3j + x_4k$ and $q_2 = z_3 + z_4j = y_1 + y_2i + y_3j + y_4k$. In this notation, $i, j, k$ denote the standard quaternionic units, $z_1 = x_1 + x_2i$, $z_2 = x_3 + x_4i$, $z_3 = y_1 + y_2i$, and $z_4 = y_3 + y_4i$.

Under this identification each right quaternionic line is invariant under right multiplication by $i$. Hence such a line is canonically isomorphic to $C^2$. If we think of $P^3_C$ as being the space of complex lines in $C^4$, then there is an obvious map $\pi : P^3_C \rightarrow S^4$, whose fiber over a point $H \in P^1_H$ is the space of complex lines in the given right quaternionic line $H \cong C^2$; thus the fiber is $P^2_C$.

The group $Conf_+(S^4)$ of orientation preserving conformal automorphisms of $S^4$ is isomorphic to $PSL(2, H)$, the projectivization of the group $2 \times 2$, invertible, quaternionic matrices. This is naturally a subgroup of $PSL(4, C)$, since every quaternion corresponds to a couple of complex numbers. Hence $Conf_+(S^4)$ has a canonical lifting to a group of holomorphic transformations of $P^3_C$, carrying twistor lines into twistor lines.

**Definition 8.1** ([24]) By a twistor Kleinian group we mean a discrete subgroup $G$ of $Aut_{hol}(P^3_C)$ of holomorphic automorhisms, which acts on $P^3_C$ with non-empty region of discontinuity $\Omega(\Gamma)$ and which is a lifting of a conformal Kleinian group acting on $S^4$.

**Remark 8.2** There is no “good” general definition of the discontinuity set $\Omega$ for general groups, hence an appropriate definition must be given in each case (see [12]). We are considering the definition 1.4 of [24], in which $\Omega(G)$ is an open $G$-invariant set and $G$ acts properly and discontinuously on $\Omega(G)$. The space $\Omega(G)/G$ has the quotient topology, and the map $\pi : \Omega \rightarrow \Omega/G$ is continuous and open.

It has been proved in [24] that if $G \subset Conf_+(S^4)$ is a discrete subgroup acting on $S^4$ with limit set $\Lambda$, then its canonical lifting $\tilde{Conf}_+(S^4)$ acts on $P^3_C$ with limit set $\tilde{\Lambda} = \pi^{-1}(\Lambda)$, thus $\tilde{\Lambda}$ is a fibered bundle over $\Lambda$ with fiber.
\[ \mathbb{S}^2. \text{ In } [2], \text{ is also proved that if we restrict the twistor bundle to a proper subset of } \mathbb{S}^4. \]

We consider the Kleinian group \( \Gamma \) such that its limit set is \( \mathbb{S}^2 \) wildly embedded on \( \mathbb{S}^4 \). Then

**THEOREM 8.3** There exists a \( \mathbb{S}^2 \times \mathbb{S}^2 \) wildly embedded in the twistor space \( P_\mathbb{C}^3 \) dynamically defined, i.e. it is the limit set of a complex Kleinian group \( \Gamma \subset Aut_{hol}(P_\mathbb{C}^3) \). ■

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