NUMERICAL RANK OF SINGULAR KERNEL FUNCTIONS

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Abstract. We study the rank of sub-matrices arising out of kernel functions, \( F(x, y) : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R} \), where \( x, y \in \mathbb{R}^d \), that have a singularity along \( x = y \). Such kernel functions are frequently encountered in a wide range of applications such as \( N \) body problems, Green’s functions, integral equations, geostatistics, kriging, Gaussian processes, etc. One of the challenges in dealing with these kernel functions is that the corresponding matrix associated with these kernels is large and dense and thereby, the computational cost of matrix operations is high. In this article, we prove new theorems bounding the numerical rank of sub-matrices arising out of these kernel functions. Under reasonably mild assumptions, we prove that the rank of certain sub-matrices is rank-deficient in finite precision. This rank depends on the dimension of the ambient space and also on the type of interaction between the hyper-cubes containing the corresponding set of particles. This rank structure can be leveraged to reduce the computational cost of certain matrix operations such as matrix-vector products, solving linear systems, etc. We also present numerical results on the growth of rank of certain sub-matrices in 1D, 2D, 3D and 4D, which, not surprisingly, agrees with the theoretical results.

Key words. Numerical rank, Singular kernel, Chebyshev interpolation

AMS subject classifications. 65F55, 65D12, 65R20, 65D05, 65R10

1. Introduction. Matrices arising out of kernel functions are encountered frequently in many applications such as integral equations [12, 26], electromagnetic scattering [32], Gaussian process regression [31], machine learning [20], radial basis function interpolation [23], kernel density estimation [35], etc. These matrices are typically large and dense, since the kernel functions are not compactly supported. Due to this, storing these matrices and performing matrix operations such as matrix-vector products, solving linear systems, matrix factorizations, etc. are challenging. However, such matrices, especially ones arising out of an underlying application, possess some structure, which is leveraged to store and perform matrix operations. One such structure, which is frequently exploited, in the context of matrices arising out of kernel functions is its rank-structuredness. We do not seek to review the entire literature on rank-structuredness of matrices arising out of kernel functions. We direct the interested readers to selected important developments on this front [37, 9, 21, 8, 15, 4, 16, 2, 3, 5, 28, 6, 27, 1, 30, 13, 11, 29].

One of the most frequently encountered rank-structuredness for matrices arising out of kernel functions is the hierarchical low-rank structure. The first works along these lines were the Barnes-Hut algorithm [9] and the Fast Multipole Method [21] (from now on abbreviated as FMM), which reduced the computational complexity of performing \( N \) body simulations from \( O(N^2) \) to \( O(N \log N) \) and \( O(N) \) respectively, for a given accuracy. One of the main highlights of the Fast Multipole Method is that the far-field interactions can be efficiently approximated by a degenerate kernel. The above two algorithms can be easily interpreted as matrix-vector products and the fact that the far-field interactions can be approximated by degenerate kernel approximation is equivalent to stating that sub-matrices corresponding to the far-field interactions can be well-represented using low-rank matrices. This interpretation has led to a new class of fast matrix-vector product algorithms leveraging the low-rank structure of certain sub-matrices. Matrices that possess such a hierarchical low-rank structure are termed as hierarchical matrices and various hierarchical low-rank algorithms have been developed for accelerating matrix-vector products [21, 15,
These hierarchical structures have also been leveraged to construct fast direct solvers [8, 13, 3, 25].

However, all the works on hierarchical low-rank matrices so far have only studied the rank of sub-matrices corresponding to the far-field interactions [10, 11, 21, 15, 36, 22, 14]. In this article, we study the rank of nearby interactions in \( d \) dimensions, where \( d \in \mathbb{Z}^+ \). More precisely, we prove that these matrices are rank deficient in finite precision and we obtain the growth in rank between any two hyper-cubes in \( d \)-dimension, containing particles in its interior, that share a hyper-surface of at-most \( d' \)-dimension, where \( d' \in \{0, 1, 2, \ldots, d - 1\} \).

To the best of our knowledge, we believe that this is the first work to formally study the rank of all nearby interactions in any dimension for generic kernel functions having singularity along the diagonal. We would like to point out that this work was inspired by our earlier work, HODLR2D [27], where we obtained rank bounds for nearby interactions in two dimensions for the kernel function, \( \log \left( \| x - y \| \right) \). It is also worth noting that some of the hierarchical matrices such as HODLR [7], HSS [34, 13], HBS [18], \( \mathcal{H} \)-matrices [24] with weak admissibility rely on representing certain or all nearby interactions as low-rank matrices. We hope our work will be of relevance in these cases and will also lead to newer faster algorithms.

2. Preliminaries. In this section, we state the notations, definitions and lemmas, we will be using in this article. We also state the main theorem (Theorem 2.1) that we will be proving in this article.

- Let \( Y \subset \mathbb{R}^d \) be a compact hyper-cube in \( d \)-dimensions with \( N = n^d \) charges (or sources) uniformly distributed in its interior, where \( n \in \mathbb{N} \).
- Let \( R_Y \) be the set of locations of these sources, i.e., \( R_Y = \{ y_1, y_2, \ldots, y_N \} \) and \( R_Y \subset \text{interior}(Y) \).
- Let \( Q_Y \) be the set of charges, i.e., \( Q_Y = \{ q_1, q_2, \ldots, q_N \} \), where \( q_j \) is located at \( y_j \) and \( Q = \sum_{j=1}^{N} |q_j| \).
- \( \text{diam}(Y) = \sup \{ \| \hat{u} - \hat{v} \|_2 : \hat{u}, \hat{v} \in Y \} \)
- We will be sub-dividing the hyper-cube \( Y \) hierarchically (using an appropriate adaptive tree) into smaller hyper-cubes to prove the theorems. We define \( Y_k \) to be the subdivision of domain \( Y \) at level \( k \) in the tree and \( Y_{k,j} \) be the \( j^{th} \) subdivision of \( Y_k \). Note that we have \( Y = \bigcup_k Y_k = \bigcup_{k,j} Y_{k,j} \). Further, we will enforce \( Y_{a,r} \cap Y_{b,s} \neq \emptyset \) iff \( a = b \) & \( r = s \). Equivalently, \( Y_{k,j} \)’s form a partition of the hyper-cube \( Y \). Let \( Q_k \) and \( Q_{k,j} \) be the sum of absolute value of the charges in \( Y_k \) and \( Y_{k,j} \) respectively. We define \( Y_{k+1} = Y \setminus \bigcup_{k=1}^{\kappa} Y_k \), \( \kappa \sim \log_{2^d}(N) \).
- Let \( X \subset \mathbb{R}^d \) be another compact hyper-cube in \( d \)-dimension containing \( T \) distinct target points in its interior. Further, we will only consider cases where \( \text{interior}(X) \cap \text{interior}(Y) = \emptyset \).
- Let \( R_X \) be the set of locations of these targets, i.e., \( R_X = \{ x_1, x_2, \ldots, x_T \} \) and \( R_X \subset \text{interior}(X) \).
- For any two hyper-cubes \( U \) and \( V \) in \( \mathbb{R}^d \), we define \( \text{dist}(U, V) = \inf \{ \| \hat{u} - \hat{v} \|_2 : \hat{u} \in U \text{ and } \hat{v} \in V \} \).
• Let \( \tau_{[y, \bar{y}]} : \mathbb{C} \to \mathbb{C} \) such that \( \tau_{[y, \bar{y}]}(z) = \left( \frac{y + \bar{y}}{2} \right) + \left( \frac{y - \bar{y}}{2} \right) z \), where \( y, \bar{y} \in \mathbb{C} \).

Essentially this map scales and shifts the interval \([-1, 1]\) to the interval \([y, \bar{y}]\).

• **Bernstein ellipse:** The standard Bernstein ellipse is an ellipse whose foci are at \((-1, 0)\), \((1, 0)\) and its parametric equation is given by

\[
B_\rho = \text{interior} \left( \left\{ \frac{\rho e^{it} + \rho^{-1} e^{-it}}{2} \in \mathbb{C} : t \in [0, 2\pi) \right\} \right)
\]

for a fixed \( \rho > 1 \).

• **Generalized Bernstein ellipse:** Let

\[
V = [y_1, \bar{y}_1] \times [y_2, \bar{y}_2] \times \cdots \times [y_d, \bar{y}_d]
\]

be a hyper-cube with side length \( r \), i.e., \(|\bar{y}_k - y_k| = r \) for all \( k \in \{1, 2, \ldots, d\} \).

The generalized Bernstein ellipse on a hyper-cube \( V \) is denoted as \( B(V, \rho') \) with \( \rho' \in (1, \infty)^d \) and is given by

\[
B(V, \rho') = \tau_{[y_1, \bar{y}_1]}(B_{\rho_1}) \times \tau_{[y_2, \bar{y}_2]}(B_{\rho_2}) \times \cdots \times \tau_{[y_d, \bar{y}_d]}(B_{\rho_d})
\]

\( \tau_{[y, \bar{y}]}(B_{\rho_k}) \) scales the standard Bernstein ellipse whose foci are at \(-1\) and 1 to an ellipse whose foci are \( y_k \) and \( \bar{y}_k \), where \( \rho_k > 1 \) for \( k \in \{1, 2, \ldots, d\} \).

• \( T_k(x) \) is Chebyshev polynomial of the first kind, i.e., \( T_k(x) = \cos \left( k \cos^{-1}(x) \right) \) for \( k \geq 0 \).

• Chebyshev nodes of order \( p \) are the points \( y^k = \cos \left( \frac{\pi k}{p} \right), k \in \{0, 1, 2, \ldots, p\} \)

• \( \sum^\prime \) indicates that the first and last summand are halved.

• The potential at the target \( x_i \in R_X \subseteq X \) due to the sources in \( Y \) is given by

\[
\phi_i = \sum_{j=1}^{N} F(x_i, y_j) q_j
\]

where \( F(x, y) : \text{interior}(X) \times \text{interior}(Y) \to \mathbb{R} \) is the kernel function.

• The above can be written in matrix-vector form as \( \phi = Kq \) where \( \phi \in \mathbb{R}^{T \times 1}, q \in \mathbb{R}^{N \times 1} \) and \( K \in \mathbb{R}^{T \times N} \). The matrix \( K \) will be called as the kernel matrix throughout this article. The \((i, j)^{th}\) entry of \( K \) is given by \( K(i, j) = F(x_i, y_j) \).

• **Analytic continuation assumption:** Let \( U, V \) be compact hyper-cubes in \( \mathbb{R}^d \) such that \( \text{dist}(U, V) > 0 \). We say that \( F : U \times V \to \mathbb{R}^d \) is analytically continuable on \( V \) if there exists a generalized Bernstein ellipse \( B(V, \rho') \) and an analytic function \( F_a : U \times B(V, \rho') \to \mathbb{C}^d \) such that the restriction of \( F_a \) onto \( U \times V \) is same as \( F \).

• Let \( K_k \in \mathbb{R}^{T \times N} \) be defined by: \( K_k(u, v) = \begin{cases} K(u, v) & \text{if } y_v \in Y_k \\ 0 & \text{otherwise} \end{cases} \)

where \( k \in \{1, 2, \ldots, \kappa, \kappa + 1\} \)

• Let \( K_{k,j} \in \mathbb{R}^{T \times N} \) be defined by: \( K_{k,j}(u, v) = \begin{cases} K(u, v) & \text{if } y_v \in Y_{k,j} \\ 0 & \text{otherwise} \end{cases} \)

where \( k \in \{1, 2, \ldots, \kappa\} \) is the level and \( j \) is indicative of the \( j^{th} \) box at level \( k \).

• \( M = \sup_{x \in X, y \in B(Y, \rho')} |F_a(x, y)|, \rho' \in (1, \alpha)^d \) for some \( \alpha > 1 \).
The main theorem that we prove in this article is given below as Theorem 2.1.

**Theorem 2.1.** For a given \( \delta > 0 \), there exists a matrix \( \tilde{K} \) with rank \( p_\delta \) such that \( \| K q - \tilde{K} q \|_\infty < \delta \), where \( p_\delta \in O \left( \mathcal{R}(N) \left( \log \left( \frac{\mathcal{R}(N) A}{\delta} \right) \right)^d \right) \)

1. If \( X \) and \( Y \) are far away (far-field \( d \)-hyper-cubes (domains)), i.e., \( \text{dist}(X,Y) \geq \text{diam}(Y) \) then \( \mathcal{R}(N) = 1 \) and \( A = MQ \).

   For example, if \( Y := [0,r]^d \) and \( X := [-2r,-r] \times [0,r]^{d-1} \). then the interaction between \( X \) and \( Y \) is a far-field interaction.

2. If \( X \) and \( Y \) share a vertex (vertex sharing \( d \)-hyper-cubes (domains)) then \( \mathcal{R}(N) = \log_{2e}(N) \) and \( A = \max_{k,j} M_{k,j} Q_{k,j} \).

   For example, if \( Y := [0,r]^d \) and \( X := [-r,0]^d \) then the interaction between \( X \) and \( Y \) is vertex sharing interaction.

3. If \( X \) and \( Y \) share a hyper-surface of dim \( d' \in \{1,2,\ldots,d-1\} \) \((d')-hyper-surface sharing \( d \)-hyper-cubes (domains)), then \( \mathcal{R}(N) = N^{d'/d} \) and \( A = \max_{k,j} M_{k,j} Q_{k,j} \).

   For example, if \( Y := [0,r]^d \) and \( X := [-r,0]^{d-d'} \times [0,r]^d \) then the interaction between \( X \) and \( Y \) is a \( d' \)-hyper-surface sharing interaction \( d' \in \{1,2,\ldots,d-1\} \). For instance, if \( d = 3 \) and \( d' = 2 \) then it will be the face sharing case in 3D.

Before we prove the theorem in \( d \)-dimension (see Section 6), for pedagogical reasons, we first prove it in one dimension (Section 3), two dimension (Section 4) and three dimension (Section 5). We also plot the ranks of the different interactions in different dimensions for the following four kernel functions.

1. \( F_1(x,y) = \frac{1}{r} \)
2. \( F_2(x,y) = \log(r) \)
3. \( F_3(x,y) = \frac{r}{r} \)
4. \( F_4(x,y) = H^{(1)}_2(r) \), i.e., the Hankel function. (Note that this kernel is complex valued with its real part being a smooth function and imaginary part having a singularity along the diagonal \( y = x \). Even though the theorem we have proved is only for real valued kernel functions, it can be easily adapted to Hankel function as well.)

where \( r = \| x - y \|_2 \), \( x \in \text{interior}(X) \) and \( y \in \text{interior}(Y) \).

- **Numerical Rank of a matrix.** For a given \( \epsilon > 0 \), the \( \epsilon \)-rank of \( K \in \mathbb{R}^{T \times N} \), denoted by \( p_\epsilon \), is defined as follows.

\[
(2.5) \quad p_\epsilon = \max\{k \in \{1,2,\ldots,\min\{T,N\} : \sigma_k > \epsilon \sigma_1\}
\]

where \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\min\{T,N\}} \geq 0 \) are the singular values of \( K \). In our plots, we have set \( \epsilon = 10^{-12} \).

- **Polynomial interpolation error in different dimension.** We will be
using interpolation to prove the main theorem (Theorem 2.1). The main tool which we will be relying on is the following results Lemma 2.2 and Lemma 2.3.

**Lemma 2.2.** Let a function $f$ analytic in $[-1, 1]$ be analytically continuable to the open Bernstein ellipse $B_\rho$ with $\rho > 1$, where it satisfies $|f(y)| \leq M$ for some $M > 0$, then for each $n \geq 0$ its Chebyshev interpolants $P_n(f)(y) = \sum_{j=0}^p c_j T_j(y)$ satisfy

\[(2.6) \quad \|f - P_n\|_\infty \leq \frac{4M\rho^{-p}}{\rho - 1}\]

where $c_j = \frac{2}{p} \sum_{k=0}^p n f(y^k) \cos \left(\frac{j\pi k}{p}\right)$

The proof of this lemma is given in [33]. Note that $P_p(f)(y)$ can be written as

$$P_p(f)(y) = \sum_{j=0}^p f(y^j) L_j(y)$$

where $L_j(y)$ is the Lagrange polynomial given by $L_j(y) = \prod_{k=0, k \neq j}^p \left(\frac{y - y^k}{y^j - y^k}\right)$

Now we discuss a multi-dimension extension of the above lemma given by [17, 19]. Let $f : V \mapsto \mathbb{R}$ with $V = [-1, 1]^d \subset \mathbb{R}^d$. Let $\tilde{n} = (p_1, p_2, \ldots, p_d)$ with $p_i \in \mathbb{N}_0$ for $i = 1, 2, \ldots, d$. The interpolation with $\prod_{i=1}^d (p_i + 1)$ summands is given by

$$P_{\tilde{n}}(f)(y) := \sum_{j \in J} c_j T_j(y)$$

where $J = \{(j_1, j_2, \ldots, j_d) \in \mathbb{N}_0^d : j_i \leq p_i\}$, $T_j(y_1, y_2, \ldots, y_d) = \prod_{i=1}^d T_{j_i}(y_i)$ and

\[(2.7) \quad c_j = \left(\prod_{i=1}^d \frac{2\delta_{0<j_i<p_i}}{p_i}\right) \sum_{k_1=0}^{p_1} \ldots \sum_{k_d=0}^{p_d} n f(y^k) \prod_{i=1}^d \cos \left(\frac{j_i\pi k_i}{p_i}\right)\]

$k = (k_1, k_2, \ldots, k_d) \in J$ and $y^k = (y^{k_1}, y^{k_2}, \ldots, y^{k_d})$ with $y^{k_i} = \cos \left(\frac{\pi k_i}{p_i}\right)$, $k_i = 0, 1, \ldots, p_i$, $i = 1, 2, \ldots, d$ and $j \in J$

**Lemma 2.3.** Let $f : V \mapsto \mathbb{R}$ has an analytic extension to some generalized Bernstein ellipse $B(V, \rho')$, for some parameter vector $\rho' \in (1, \infty)^d$ and $\|f\|_{\infty} = \max_{y \in B(V, \rho')} |f(y)| \leq M$. Then

\[(2.8) \quad \|f - P_{\tilde{n}}\|_{\infty} \leq \sum_{m=1}^d 4M \left(\frac{\rho_m^{p_m}}{p_m - 1}\right) + \sum_{i=2}^d 4M \frac{\rho_i^{p_i}}{\rho_i - 1} \left(2^{l-1}(l-1) + 2^{l-1} - 1\right) \frac{\prod_{j=1}^{l-1} \left(1 - \frac{1}{\rho_j}\right)}{\prod_{j=1}^{l-1} \left(1 - \frac{1}{\rho_j}\right)}\]
The proof is given by Glau et al. [19] using induction. In our case, we will be working with \( p_1 = p_2 = \cdots = p_d = p \). Further, for the sake of simplicity, we set \( \rho = \min\{\rho_i : i = 1, 2, \ldots, d\} \).

\[
\| f - P_n \|_\infty \leq 4M \frac{\rho^{-p}}{\rho - 1} \left( \sum_{m=1}^{d} 1 + \sum_{l=2}^{d} \frac{2^{l-1} (l - 1) + 2^{l-1} - 1}{l - 1} \right)
\]

We have \( V_d = \left( d + \sum_{l=2}^{d} 2^{l-1} (l - 1) + 2^{l-1} - 1 \right) \left( 1 - \frac{1}{\rho} \right) \), it does not depend upon \( p \).

Therefore from (2.9)

\[
\| f - P_n \|_\infty \leq 4M V_d \frac{\rho^{-p}}{\rho - 1}
\]

We use (2.10) to prove our higher dimensional \((d > 1)\) results. Note that

\[
P_n(f)(y) = \sum_{j \in J} f(y^j) R_j(y)
\]

where \( R_j(y) \) is given by \( R_j(y) = \prod_{i=1}^{d} L_{j_i}(y_i) \).

- **Approximation of a kernel matrix via Chebyshev interpolation:**
  The function \( F(x,y) \) is interpolated using Chebyshev nodes along \( y \). The interpolant is given by \( P_n(F)(x,y) = \sum_{j \in J} c_j(x) T_j(y) \). More precisely

\[
\tilde{F}(x,y) \approx \sum_{k \in J} F(x,y^k) R_k(y)
\]

where \( R_k \) is Lagrange basis. Let \( \tilde{K} \) be the matrix corresponding to the function \( \tilde{F} \), i.e., \( \tilde{K}(i,j) = \sum_{k \in J} F(x_i,y^k) R_k(y_j) = \tilde{F}(x_i,y_j) \). The matrix \( \tilde{K} \) will be the approximation to the kernel matrix \( K \). Similarly, \( \tilde{K}_k \) and \( \tilde{K}_{k,j} \) be the approximation of the submatrices \( K_k \) and \( K_{k,j} \) respectively.

**3. Rank growth of different interactions in 1D.** Using the Lemma 2.2, we discuss the rank of far-field and vertex sharing interactions as shown in Figure 1a. \( I \) and \( V \) are the far-field and vertex sharing domains of \( Y \) respectively.

![Fig. 1: Different interactions in 1D](image-url)
3.1. Rank growth of far-field domains. Here we will assume far-field interaction means the interaction between two domains which are one line-segment away. In Figure 1b the lines \( Y = [0, r] \) and \( X = [-2r, -r] \) are one line-segment away. We choose \((p + 1)\) Chebyshev nodes in the domain \( Y \) to obtain the polynomial interpolation of \( F(x, y) \) along \( y \). Let \( \tilde{K} \) be the approximation of the matrix \( K \). Then by Lemma 2.2 the error in matrix-vector product at \( i^{th} \) component

\[
\left| \left( Kq - \tilde{K}q \right)_i \right| \leq \frac{4MQ\rho^{-p}}{\rho - 1}
\]

Hence, setting the error to be less than \( \delta \) (for some \( \delta > 0 \)), we have

\[
p = \left\lceil \log \left( \frac{4MQ\delta}{\log(\rho)} \right) \right\rceil
\]

Hence, the rank of \( \tilde{K} \) scales \( O \left( \log \left( \frac{MQ}{\delta} \right) \right) \) with \( \left\| Kq - \tilde{K}q \right\|_\infty < \delta \). The numerical rank plots of the far-field interaction of the four functions as described in Section 2 are shown in Figure 2 and tabulated in Table 1.

![N vs Numerical rank plot of far-field interaction in 1D](image)

Table 1: Numerical rank \((p_{\epsilon})\) with \((\epsilon = 10^{-12})\) of different kernel functions of far-field interaction in 1D

| \( N \) | \( F_1(x, y) \) | \( F_2(x, y) \) | \( F_3(x, y) \) | \( F_4(x, y) \) |
|-------|---------------|---------------|---------------|---------------|
| 1000  | 7             | 7             | 8             | 8             |
| 5000  | 7             | 7             | 8             | 8             |
| 10000 | 7             | 7             | 8             | 8             |
| 15000 | 7             | 7             | 8             | 8             |
| 20000 | 7             | 7             | 8             | 8             |
| 25000 | 7             | 7             | 8             | 8             |
| 30000 | 7             | 7             | 8             | 8             |
| 40000 | 7             | 7             | 8             | 8             |

3.2. Rank growth of vertex sharing domains. Consider a pair of vertex sharing domains \( Y = [0, r] \) and \( X = [-r, 0] \) of length \( r \) as shown in Figure 3a. The domain \( Y \) is hierarchically sub-divided using an adaptive binary tree as shown in Figure 3b. Then we have,

\[
Y = \bigcup_{k=1}^{\infty} Y_k = \bigcup_{k=1}^{\infty} Y_k = \left[ 0, \frac{r}{2^k} \right] \cup \bigcup_{k=1}^{\infty} \left[ \frac{r}{2^k}, \frac{r}{2^{k-1}} \right]
\]

![Vertex sharing domain and subdivision in 1D](image)
where \( \kappa \sim \log_2(N) \) and \( Y_{\kappa+1} = \left[ \begin{array}{c} 0 \\frac{\rho}{2^\kappa} \end{array} \right] \) having one particle. Let \( \tilde{K} \) be the approximation of the matrix \( K \) and \( \tilde{K}_k \) be the approximation of \( K_k \), then \( \tilde{K} = \sum_{k=1}^{\kappa} \tilde{K}_k + K_{\kappa+1} \).

The rank of \( K_{\kappa+1} \) is one. Then by Lemma 2.2 the \( i^{th} \) component error in matrix-vector product at level \( k \) is

\[
\left| (K_iq - \tilde{K}_iq) \right| \leq \frac{4M_kQ_k\rho^{-p_k}}{\rho - 1}
\]

(3.2)

Now choosing \( p_k \) such that the above error is less than \( \delta_1 \) (for some \( \delta_1 > 0 \)), we obtain

\[
p_k = \left( \left\lceil \log \left( \frac{4M_kQ_k}{\rho - 1} \right) \right\rceil \right) \Rightarrow \left| (K_iq - \tilde{K}_iq) \right| < \delta_1.
\]

Let \( p_l = \max\{p_k : k = 1, 2, \ldots, \kappa \} \) Hence, we get

\[
\left| (K_iq - \tilde{K}_iq) \right| \leq \sum_{i=1}^{\kappa} \left| (K_iq - \tilde{K}_iq) \right| < \kappa \delta_1
\]

(3.3)

and rank of \( \tilde{K} \) is bounded above by \( (1 + \kappa p_l) = \left( 1 + \kappa \left( \left\lceil \log \left( \frac{4M_kQ_k}{\rho - 1} \right) \right\rceil \right) \). If we choose \( \delta_1 = \frac{\delta}{1 + \kappa} \), then \( \left| (Kq - \tilde{K}q) \right| < \delta \) with rank of \( \tilde{K} \) bounded above by

\[
1 + \kappa \left( \left\lceil \log \left( \frac{4M_kQ_k}{\rho - 1} \right) \right\rceil \right) = 1 + \log_2(N) \left( \left\lceil \log \left( \frac{4M_kQ_k}{\rho - 1} \right) \right\rceil \right).
\]

So the rank of \( \tilde{K} \) scales \( \mathcal{O} \left( \log_2(N) \log \left( \frac{M_kQ_k \log_2(N)}{\delta} \right) \right) \) with

\[
\left\| Kq - \tilde{K}q \right\|_\infty < \delta.
\]

The numerical rank plots of the vertex sharing interaction of the four functions as described in Section 2 are shown in Figure 4 and tabulated in Table 2.

| \( N \) | \( F_1(x,y)/F_2(x,y)/F_3(x,y)/F_4(x,y) \) |
|---|---|
| 1000 | 22 | 20 | 22 | 22 |
| 5000 | 27 | 23 | 27 | 26 |
| 10000 | 29 | 25 | 29 | 27 |
| 15000 | 30 | 26 | 30 | 28 |
| 20000 | 31 | 26 | 31 | 29 |
| 25000 | 31 | 27 | 31 | 30 |
| 30000 | 32 | 27 | 32 | 30 |
| 40000 | 33 | 27 | 33 | 31 |

Table 2: Numerical rank \( (p_k) \) with \( (\epsilon = 10^{-12}) \) of different kernel functions of vertex sharing interaction in 1D

![Fig. 4: N vs Numerical rank plot of vertex sharing interaction in 1D](image)

4. Rank growth of different interactions in 2D. We discuss the rank of far-field, vertex sharing and edge sharing interactions as shown in Figure 5a. \( I, V \) and \( E \) are far-field, vertex sharing and edge sharing domains of \( Y \) respectively.
4.1. **Rank growth of far-field domains.** Let $X$ and $Y$ be square boxes of length $r$ and the distance between them is also $r$ as shown in Figure 5b. We choose a $(p+1) \times (p+1)$ tensor product grid on Chebyshev nodes in the box $Y$ to obtain the polynomial interpolation of $F(x,y)$ along $y$. Let $\tilde{K}$ be the approximation of the matrix $K$. Then by (2.10) the $i^{th}$ component error in matrix-vector product is

\[
\left| \left( Kq - \tilde{K}q \right) \right|_i \leq 4MQV_{\omega} \frac{\rho^{-p}}{\rho - 1}
\]

Hence, setting the error to be less than $\delta$ (for a given $\delta > 0$), we have $p = \left\lceil \frac{\log(4MQV_{\omega})}{\log(\rho)} \right\rceil$

It gives $(p+1)^2 = \left( 1 + \left\lceil \frac{\log(4MQV_{\omega})}{\log(\rho)} \right\rceil \right)^2$

So, the rank of $\tilde{K}$ scales $O\left( \left( \log \left( \frac{MQ}{\delta} \right) \right)^2 \right)$ with $\left\| Kq - \tilde{K}q \right\|_\infty < \delta$. The numerical rank plots of the far-field interaction of the four functions as described in Section 2 are shown in Figure 6 and tabulated in Table 3.

| $N$ | $F_1(x,y)$ | $F_2(x,y)$ | $F_3(x,y)$ | $F_4(x,y)$ |
|-----|------------|------------|------------|------------|
| 1600 | 42         | 21         | 48         | 49         |
| 2500 | 42         | 21         | 48         | 49         |
| 5625 | 42         | 21         | 47         | 48         |
| 10000| 42         | 21         | 47         | 48         |
| 22500| 42         | 19         | 47         | 47         |
| 40000| 42         | 19         | 46         | 47         |

Table 3: Numerical rank ($p$) with $(\epsilon = 10^{-12})$ of different kernel functions of far-field interaction in 2D.
4.2. Rank growth of vertex sharing domains. Consider two vertex sharing square domains of length $r$, $X$ and $Y$ as shown in Figure 7a. The box $Y$ is hierarchically sub-divided using an adaptive quad tree as shown in Figure 7b.

\[
Y = \bigcup_{k=1}^{\infty} Y_k = \bigcup_{k=1}^{2^{2^k}-1} \bigcup_{j=1}^{\kappa} Y_{k,j} = \bigcup_{k=1}^{\kappa} \bigcup_{j=1}^{3} Y_{k,j}
\]

where $\kappa \sim \log_4(N)$ and $Y_{k+1}$ having one particle. Let $\tilde{K}$ be the approximation of the kernel matrix $K$. We choose a $(p + 1) \times (p + 1)$ tensor product grid on Chebyshev nodes in the box $Y$ to obtain the polynomial interpolation of $F(x,y)$ along $y$.

Let $\tilde{K}_{k,j}$ be approximation of the matrix $K_{k,j}$, then approximation of matrix $K$

\[
\tilde{K} = \sum_{k=1}^{\kappa} \sum_{j=1}^{3} \tilde{K}_{k,j} + K_{k+1}
\]

Then by (2.10) the $i^{th}$ component error in matrix-vector product at $j^{th}$ subdivision of the level $k$ is given by (pick $p_{k,j}$ such that the error is less than $\delta_1$)

\[
\left| \left( K_{k,j}q - \tilde{K}_{k,j}q \right)_i \right| \leq 4M_{k,j}Q_{k,j}V_2^2 \rho^{-p_{k,j}}
\]

Now choosing $p_{k,j}$ such that the above error is less than $\delta_1$ (for some $\delta_1 > 0$), we obtain

\[
p_{k,j} = \left\lfloor \log \frac{4M_{k,j}Q_{k,j}V_2^2}{\rho^{-p_{k,j}}} \right\rfloor \Rightarrow \left| \left( K_{k,j}q - \tilde{K}_{k,j}q \right)_i \right| < \delta_1.
\]

Let $p_{l,m} = \max\{p_{k,j} : k = 1, 2, \ldots, \kappa \text{ and } j = 1, 2, 3\}$. So, at level $k$ the error in matrix-vector product at $i^{th}$ component is

\[
\left| \left( K_{k}q - \tilde{K}_{k}q \right)_i \right| \leq \sum_{j=1}^{3} \left| \left( K_{k,j}q - \tilde{K}_{k,j}q \right)_i \right| < 3\delta_1
\]

with rank of $\tilde{K}_k$ is bounded above by $3(1 + p_{l,m})^2$. Hence, we get

\[
\left| \left( Kq - \tilde{K}q \right)_i \right| = \left| \left( (K_1 + K_2 + \cdots + K_{\kappa})q - (\tilde{K}_1 + \tilde{K}_2 + \cdots + \tilde{K}_{\kappa})q \right)_i \right| \leq \sum_{k=1}^{\kappa} \left| \left( K_{k}q - \tilde{K}_{k}q \right)_i \right| < 3\kappa \delta_1
\]
and rank of $\tilde{K}$ bounded by $1 + 3\kappa(1 + p_{l,m})^2 = 1 + 3\kappa \left(1 + \left[\log\left(\frac{4M_{l,m}Q_{l,m}V^2}{\delta_{l,p(m-1)}}\right)\right]^2\right)$.

If we choose $\delta_1 = \frac{\delta}{3\kappa}$, then $\left|\left(Kq - \tilde{K}q\right)_i\right| < \delta$ with rank of $\tilde{K}$ bounded above by

$$1 + 3\kappa \left(\left[\log\left(\frac{12\log_4(N)M_{l,m}Q_{l,m}V^2}{\delta}\right)\right]^2\right)^2 = 1 + 3\kappa \left(\log(N)\left(\log\left(\frac{M_{l,m}Q_{l,m}V^2}{\delta}\right)\right)^2\right)^2$$

So the rank of $\tilde{K}$ scales $O\left(\log_4(N)\left(\log\left(\frac{M_{l,m}Q_{l,m}V^2}{\delta}\right)\right)^2\right)$ with

$$\left\|Kq - \tilde{K}q\right\|_{\infty} < \delta.$$ The numerical rank plots of the vertex sharing interaction of the four functions as described in Section 2 are shown in Figure 8 and tabulated in Table 4.

Table 4: Numerical rank ($p_{\epsilon}$) with ($\epsilon = 10^{-12}$) of different kernel functions of vertex sharing interaction in 2D

| $N$  | $F_1(x,y)$ | $F_2(x,y)$ | $F_3(x,y)$ | $F_4(x,y)$ |
|------|------------|------------|------------|------------|
| 1600 | 81         | 34         | 81         | 94         |
| 2500 | 87         | 36         | 91         | 102        |
| 5625 | 96         | 39         | 101        | 114        |
| 10000| 104        | 41         | 108        | 122        |
| 22500| 112        | 44         | 118        | 135        |
| 40000| 119        | 45         | 124        | 143        |

Fig. 8: $N$ vs Numerical rank plot of vertex sharing interaction in 2D

(a) Edge sharing square boxes

(b) Hierarchical subdivision

Fig. 9: Rank of edge sharing boxes in 2D

### 4.3. Rank growth of edge sharing domains.

Consider two edge sharing boxes $X$ and $Y$ as shown in Figure 9a. Further, subdivide the left side box of domain $Y$ hierarchically as shown in Figure 9b.

(4.6) \[
Y = \bigcup_{k=1}^{\infty} Y_k = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{2^k} Y_{k,j} = Y_{\kappa+1} + \bigcup_{k=1}^{\infty} Y_{k,j}
\]

where $\kappa \sim \log_4(N)$ and $Y_{\kappa+1}$ having $2^\kappa = \sqrt{N}$ particles. Let $\tilde{K}_{k,j}$ be approximation of $K_{k,j}$, then approximation of matrix $K$ is

(4.7) \[
\tilde{K} = \sum_{k=1}^{\kappa} \sum_{j=1}^{2^k} \tilde{K}_{k,j} + K_{\kappa+1}
\]
Then by (2.10) the \(i^{th}\) component error in matrix-vector product at \(j^{th}\) subdivision of the level \(k\) is given by (pick \(p_{k,j}\) such that the error is less than \(\delta_1\))

\[
(A.8) \quad \left| \left(K_{k,j}\mathbf{q} - \tilde{K}_{k,j}\mathbf{q}\right)_i \right| \leq 4M_{k,j}Q_{k,j}V_2 \frac{\rho^{-p_{k,j}}}{\rho - 1}
\]

Now choosing \(p_{k,j}\) such that the above error is less than \(\delta_1\) (for some \(\delta_1 > 0\)), we obtain

\[
p_{k,j} = \left\lceil \log \left( \frac{4M_{k,j}Q_{k,j}V_2}{\delta_1(p-1)} \right) \log(p) \right\rceil.
\]

Let \(p_{l,m} = \max\{p_{k,j} : k = 1, 2, \ldots, \kappa \text{ and } j = 1, 2, \ldots, 2^k\}\)

The rank of \(\tilde{K}\) is bounded above by

\[
\sqrt{N} + \sum_{k=1}^{\kappa} 2^k \left( 1 + \left\lceil \log \left( \frac{4M_{l,m}Q_{l,m}V_2}{\delta_1(p-1)} \right) \log(p) \right\rceil \right)^2
\]

\[
= \sqrt{N} + \left( 2^{\kappa+1} - 2 \right) \left( 1 + \left\lceil \log \left( \frac{4M_{l,m}Q_{l,m}V_2}{\delta_1(p-1)} \right) \log(p) \right\rceil \right)^2
\]

\[
= \sqrt{N} + \left( 2\sqrt{N} - 2 \right) \left( 1 + \left\lceil \log \left( \frac{4M_{l,m}Q_{l,m}V_2}{\delta_1(p-1)} \right) \log(p) \right\rceil \right)^2
\]

Hence the rank of \(\tilde{K}\) is \(O\left( \sqrt{N} \left( \log \left( \frac{M_{l,m}Q_{l,m}}{\delta_1(p-1)} \right) \right)^2 \right)\) and the error in matrix-vector product

\[
(A.10) \quad \left| \left(K\mathbf{q} - \tilde{K}\mathbf{q}\right)_i \right| < \sum_{k=1}^{\kappa} 2^k \delta_1 = \left( 2^{\kappa+1} - 2 \right) \delta_1 < 2\sqrt{N}\delta_1, \text{ as } \kappa \sim \log_4(N)
\]

If we choose \(\delta_1 = \frac{\varepsilon}{2\sqrt{N}}\), then \(\left| \left(K\mathbf{q} - \tilde{K}\mathbf{q}\right)_i \right| < \delta\). Hence rank of \(\tilde{K}\) scales

\[
O\left( \sqrt{N} \left( \log \left( \frac{\sqrt{N}M_{l,m}Q_{l,m}}{\delta} \right) \right)^2 \right)\text{ with } \left\| K\mathbf{q} - \tilde{K}\mathbf{q} \right\|_\infty < \delta. \text{ The numerical rank plots of the edge sharing interaction of the four functions as described in Section 2 are shown in Figure 10 and tabulated in Table 5.}
\]

| \(N\) | \(p_1\) | \(p_2\) | \(p_3\) | \(p_4\) |
|---|---|---|---|---|
| 1000 | 116 | 99 | 220 | 241 |
| 2500 | 266 | 120 | 269 | 296 |
| 5000 | 582 | 172 | 388 | 434 |
| 10000 | 945 | 223 | 502 | 570 |
| 22500 | 717 | 323 | 727 | 842 |
| 40000 | 936 | 423 | 949 | 1112 |

**Table 5:** Numerical rank \((p_\varepsilon)\) with \(\varepsilon = 10^{-12}\) of different kernel functions of edge sharing interaction in 2D

**Fig. 10:** \(N\) vs Numerical rank plot of edge sharing interaction in 2D
5. Rank growth of different of interactions in 3D. We discuss the rank of far-field, vertex sharing, edge sharing and face sharing interactions as shown in Figure 11a. \( I, V, E \) and \( F \) be the far-field, vertex sharing, edge sharing and face sharing domains of the cube \( Y \) respectively.

5.1. Rank growth of far-field domains. Let \( X \) and \( Y \) be cubes of size \( r \) separated by a distance \( r \) as shown in Figure 11b. We choose a \((p + 1) \times (p + 1) \times (p + 1)\) tensor product grid of Chebyshev nodes in the box \( Y \) to obtain the polynomial interpolation of \( F(x, y) \) along \( y \). Let \( \tilde{K} \) be the approximation of the matrix \( K \). Then by (2.10) the \( i^{th} \) component error in matrix-vector product is given by

\[
\left| (Kq - \tilde{K}q) \right| i \leq 4MQV_3 \frac{\rho^{-p}}{\rho - 1}
\]

Hence, setting the above error to be less than \( \delta > 0 \), we have \( p = \left\lceil \frac{\log(4MQV_3)}{\log(\rho)} \right\rceil \). It gives \((p + 1)^3 = \left( 1 + \left\lceil \frac{\log(4MQV_3)}{\log(\rho)} \right\rceil \right)^3 \).

Hence, the rank of \( \tilde{K} \) scales \( \mathcal{O} \left( \left( \frac{M}{\delta} \right)^3 \right) \) and \( \| Kq - \tilde{K}q \|_\infty < \delta \). The numerical rank plots of the far-field interaction of the four functions as described in Section 2 are shown in Figure 12 and tabulated in Table 6.

| \( N \) | \( p_1(x, y) \) | \( p_2(x, y) \) | \( p_3(x, y) \) | \( p_4(x, y) \) |
|-------|----------------|----------------|----------------|----------------|
| 1000  | 149            | 188            | 163            | 238            |
| 3375  | 148            | 191            | 160            | 249            |
| 8000  | 147            | 190            | 158            | 246            |
| 15625 | 143            | 188            | 156            | 243            |
| 27000 | 143            | 186            | 156            | 241            |
| 42675 | 141            | 186            | 154            | 241            |
| 64000 | 140            | 185            | 154            | 241            |

Table 6: Numerical rank \( (p_\epsilon) \) with \( \epsilon = 10^{-12} \) of different kernel functions of far-field interaction in 3D

5.2. Rank growth of vertex sharing domains. Consider two vertex sharing cubes \( X \) and \( Y \) as shown in Figure 13. The cube \( Y \) is hierarchically sub-divided using

![Fig. 11: Different interactions in 3D](image)

![Fig. 12: \( N \) vs Numerical rank plot of far-field interaction in 3D](image)
an adaptive oct tree as shown in Figure 13. For a better view click here.

\[ Y = \bigcup_{k=1}^{\infty} Y_k = \bigcup_{k=1}^{2^3-1} \bigcup_{j=1}^{2} Y_{k,j} = \bigcup_{k=1}^{\kappa} \bigcup_{j=1}^{7} Y_{k,j} \]

where \( \kappa \sim \log_8(N) \) and assume one particle is in \( Y_{\kappa+1} \). Let \( \tilde{K} \) be the approximation of the kernel matrix \( K \). We choose a \( (p+1) \times (p+1) \times (p+1) \) tensor product grid on Chebyshev nodes in the box \( Y \) to obtain the polynomial interpolation of \( F(x,y) \) along \( y \).

Let \( \tilde{K}_{k,j} \) be approximation of \( K_{k,j} \), then approximation of the kernel matrix \( K \) is

\[ \tilde{K} = \sum_{k=1}^{\kappa} \sum_{j=1}^{7} \tilde{K}_{k,j} + K_{\kappa+1} \]

Then by (2.10) the \( i \)th component error in matrix-vector product at \( j \)th subdivision of the level \( k \) is given by

\[ \left| \left( K_{k,j}q - \tilde{K}_{k,j}q \right)_i \right| \leq 4M_{k,j}Q_{k,j}V_3 \frac{\rho^{p_{k,j}}}{\rho - 1} \]

Now choosing \( p_{k,j} \) such that the above error is less than \( \delta_1 \) (for some \( \delta_1 > 0 \), we obtain

\[ p_{k,j} = \left[ \log \left( \frac{4M_{k,j}Q_{k,j}V_3}{\delta_1} \right) \right] \log(\rho) \implies \left| \left( K_{k,j}q - \tilde{K}_{k,j}q \right)_i \right| < \delta_1 \text{ with rank of } \tilde{K}_{k,j} \text{ bounded above by } (1 + p_{k,j})^3. \]

Let \( p_{l,m} = \max \{ p_{k,j} : k = 1, 2, \ldots, \kappa \text{ and } j = 1, 2, \ldots, 7 \} \).

\[ \left| \left( K_{k}q - \tilde{K}_{k}q \right)_i \right| = \left| \sum_{j=1}^{7} \left( K_{k,j}q - \tilde{K}_{k,j}q \right)_i \right| \leq \sum_{j=1}^{7} \left| \left( K_{k,j}q - \tilde{K}_{k,j}q \right)_i \right| < 7\delta_1 \]

with rank of \( \tilde{K}_{k} \) bounded above by \( 7(1 + p_{l,m})^3 \). Hence, we get

\[ \left| \left( K_{k}q - \tilde{K}_{k}q \right)_i \right| \leq \sum_{k=1}^{\kappa} \left| \left( K_{k}q - \tilde{K}_{k}q \right)_i \right| < 7\kappa\delta_1 \]
and rank of $\tilde{K}$ bounded by $1 + 7\kappa(1 + p_{l,m})^3 = 1 + 7\kappa \left(1 + \left[\frac{\log(4M_{l,m}Q_{l,m} V_3)}{\log(p)}\right]\right)^3$

If we choose $\delta_1 = \frac{\delta}{10\kappa}$, then $\left\|Kq - \tilde{K}q\right\|_\infty < \delta$ with rank of $\tilde{K}$ bounded above by

$1 + 7\kappa \left(\left[\frac{28\log(N)M_{l,m}Q_{l,m} V_3}{\log(p)}\right]\right)^3$

So the rank of $\tilde{K}$ scales $O(\log(N))$ with $\left\|Kq - \tilde{K}q\right\|_\infty < \delta$. The numerical rank plots of the vertex sharing interaction of the four functions as described in Section 2 are shown in Figure 14 and tabulated in Table 7.

![Fig. 14: $N$ vs Numerical rank plot of vertex sharing interaction in 3D](image1)

5.3. Rank growth of edge sharing domains. Consider edge sharing cubes $X$ and $Y$ as shown in Figure 15. The cube $Y$ is hierarchically sub-divided as shown in Figure 15. For a better view click here. $Y = \bigcup_{k=1}^{\infty} Y_k = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{2^k} Y_{k,j} = Y_{\kappa+1} \bigcup_{k=1}^{\kappa} \bigcup_{j=1}^{2^k} Y_{k,j}$,

where $\kappa \sim \log(N)$ and $Y_{\kappa+1}$ having $2^\kappa = N^{1/3}$ particles. Let $\tilde{K}_{k,j}$ be approximation of $K_{k,j}$, then approximation of matrix $K$ is $\tilde{K} = \sum_{k=1}^{\kappa} \sum_{j=1}^{3\cdot 2^k} \tilde{K}_{k,j} + K_{\kappa+1}$. Then by (2.10)
the $i^{th}$ component error in matrix-vector product is

$$\left| \left( K_{k,j} \mathbf{q} - \tilde{K}_{k,j} \mathbf{q} \right) \right|_i \leq 4 M_{k,j} Q_{k,j} V^3 \frac{\rho^{-p_{k,j}}}{\rho - 1}$$

Now choosing $p_{k,j}$ such that the above error is less than $\delta_1$ (for some $\delta_1 > 0$), we obtain

$$p_{k,j} = \left\lceil \log \left( \frac{4 M_{k,j} Q_{k,j} V^3}{\delta_1 (\rho - 1)} \right) \log(\rho) \right\rceil$$

$$\implies \left| \left( K_{k,j} \mathbf{q} - \tilde{K}_{k,j} \mathbf{q} \right) \right|_i < \delta_1.$$ 

Let $p_{l,m} = \max \{ p_{k,j} : k = 1, 2, \ldots, \kappa \text{ and } j = 1, 2, \ldots, 3 \cdot 2^k \}$. The rank of $\tilde{K}$ is bounded above by

$$\begin{align*}
1 + \sum_{k=1}^{\kappa} 3 \cdot 2^k \left( 1 + \left\lceil \log \left( \frac{4 M_{l,m} Q_{l,m} V^3}{\delta_1 (\rho - 1)} \right) \log(\rho) \right\rceil \right)^3 \\
= 1 + 3 \cdot 2^{\kappa+1} - 2 \left( 1 + \left\lceil \log \left( \frac{4 M_{l,m} Q_{l,m} V^3}{\delta_1 (\rho - 1)} \right) \log(\rho) \right\rceil \right)^3 \\
= 1 + 3 \left( 2N^{\frac{1}{2}} - 2 \right) \left( 1 + \left\lceil \log \left( \frac{4 M_{l,m} Q_{l,m} V^3}{\delta_1 (\rho - 1)} \right) \log(\rho) \right\rceil \right)^3
\end{align*}$$

Hence the rank $\tilde{K} \in \mathcal{O} \left( N^{\frac{1}{2}} \left( \log \left( \frac{M_{l,m} Q_{l,m}}{\delta_1} \right) \right)^3 \right)$ of and the error in matrix-vector product

$$\left| \left( K \mathbf{q} - \tilde{K} \mathbf{q} \right) \right|_i < \sum_{k=1}^{\kappa} 3 \cdot 2^k \delta_1 = 3 \left( 2^{\kappa+1} - 2 \right) \delta_1 < 6 N^{\frac{1}{2}} \delta_1,$$ as $\kappa \sim \log_2(N)$

If we choose $\delta_1 = \frac{\delta}{6N^{\frac{1}{2}}}$ then $\left| \left( K \mathbf{q} - \tilde{K} \mathbf{q} \right) \right|_i < \delta$. Hence the rank of $\tilde{K}$ scales as

$$\mathcal{O} \left( N^{1/3} \left( \log \left( \frac{N^{1/3} M_{l,m} Q_{l,m}}{\delta} \right) \right)^2 \right)$$

with $\| K \mathbf{q} - \tilde{K} \mathbf{q} \|_\infty < \delta$. The numerical rank plots of the edge sharing interaction of the four functions as described in Section 2 are shown in Figure 16 and tabulated in Table 8.

| $N$  | $p_{1(x,y)} p_{2(x,y)} p_{3(x,y)} p_{4(x,y)}$ |
|------|-----------------------------------------------|
| 1000 | 189, 200, 262, 302                             |
| 3375 | 271, 371, 249, 458                             |
| 8000 | 345, 466, 366, 600                             |
| 15625| 416, 552, 442, 741                             |
| 27000| 483, 637, 512, 877                             |
| 42875| 546, 711, 580, 1006                            |
| 64000| 602, 783, 649, 1136                            |

Table 8: Numerical rank $(p_{\epsilon})$ with $\epsilon = 10^{-12}$ of different kernel functions of edge sharing interaction in 3D

**Fig. 16:** $N$ vs Numerical rank plot of edge sharing interaction in 3D
5.4. Rank growth of face sharing domains. Consider two face sharing cubes $X$ and $Y$ as shown in Figure 17. The cube $Y$ is hierarchically sub-divided as shown Figure 17. For a better view click here.

\begin{equation}
Y = \bigcup_{k=1}^{\infty} Y_k = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{(2^k)^k} Y_{k,j} = Y_{k+1} \bigcup_{k=1}^{\kappa} 4^k \bigcup_{j=1}^{\kappa} Y_{k,j}
\end{equation}

where $\kappa \sim \log_8(N)$ and $Y_{k+1}$ having $4^\kappa = N^{2/3}$ particles. Let $\tilde{K}_{k,j}$ be approximation of $K_{k,j}$, then approximation of matrix $K$ is $\tilde{K} = \sum_{k=1}^{\kappa} \sum_{j=1}^{4^k} \tilde{K}_{k,j} + K_{k+1}$. Then by (2.10) the $i^{th}$ component error in matrix-vector product at $j^{th}$ subdivision of the level $k$ is given by (pick $p_{k,j}$ such that the error is less than $\delta_1$)

\begin{equation}
\left| \left( K_{k,j}q - \tilde{K}_{k,j}q \right)_i \right| \leq 4M_{k,j}Q_{k,j}V_3 \frac{p_{k,j}}{\rho - 1}
\end{equation}

Now choosing $p_{k,j}$ such that the above error is less than $\delta_1$ (for some $\delta_1 > 0$), we obtain

$p_{k,j} = \left[ \log \left( \frac{4M_{k,j}Q_{k,j}V_3}{\delta_1 (\rho - 1)} \right) \right] \Rightarrow \left| \left( K_{k,j}q - \tilde{K}_{k,j}q \right)_i \right| < \delta_1$. Let $p_{l,m} = \max \{ p_{k,j} : k = 1, 2, \ldots, \kappa \text{ and } j = 1, 2, \ldots, 4^k \}$. The rank of $\tilde{K}$ is bounded above by

\begin{equation}
1 + \sum_{k=1}^{\kappa} 4^k \left( 1 + \left[ \log \left( \frac{4M_{l,m}Q_{l,m}V_3}{\delta_1 (\rho - 1)} \right) \right] \right)^3
\end{equation}

\begin{equation}
= 1 + \frac{4}{3} \left( 4^\kappa - 1 \right) \left( 1 + \left[ \log \left( \frac{4M_{l,m}Q_{l,m}V_3}{\delta_1 (\rho - 1)} \right) \right] \right)^3
\end{equation}

\begin{equation}
= 1 + \frac{4}{3} \left( N^{\frac{2}{3}} - 1 \right) \left( 1 + \left[ \log \left( \frac{4M_{l,m}Q_{l,m}V_3}{\delta_1 (\rho - 1)} \right) \right] \right)^3
\end{equation}
Hence, the rank of $\tilde{K} \in \mathcal{O} \left( N^{\frac{2}{3}} \left( \log \left( \frac{M_{l,m}Q_{l,m}}{\delta} \right) \right)^{\frac{3}{2}} \right)$ and the error in matrix-vector product

\begin{equation}
(5.13) \quad \left| \left( Kq - \tilde{K}q \right)_i \right| < \sum_{k=1}^{\kappa} 4^k \delta_1 = \frac{4}{3} \left( N^{\frac{2}{3}} - 1 \right) \delta_1 < \frac{4}{3} N^{\frac{2}{3}} \delta_1, \text{ as } \kappa \sim \log_8(N)
\end{equation}

If we choose $\delta_1 = \frac{3\delta}{4N^{\frac{2}{3}}}$ then $\left| \left( Kq - \tilde{K}q \right)_i \right| < \delta$. Hence, from (5.12) rank of $\tilde{K}$ scales $\mathcal{O} \left( N^{2/3} \left( \log \left( \frac{N^{2/3}M_{l,m}Q_{l,m}}{\delta} \right) \right)^{3/2} \right)$ with $\| Kq - \tilde{K}q \|_\infty < \delta$. The numerical rank plots of the face sharing interaction of the four functions as described in Section 2 are shown in Figure 18 and Table 9.

| $N$ | $F(x,y)F(x,y)F(x,y)F(x,y)$ |
|-----|---------------------------|
| 1000 | 312 427 315 494 |
| 3375 | 610 815 614 964 |
| 8000 | 1003 1314 1012 1623 |
| 15625 | 1491 1901 1563 2443 |
| 27000 | 2077 2641 2092 3419 |
| 42675 | 2764 3465 2781 4556 |
| 64000 | 3547 4373 3564 5751 |

Table 9: Numerical rank ($p_\epsilon$) with $\epsilon = 10^{-12}$ of different kernel functions of face sharing interaction in 3D

Fig. 18: $N$ vs Numerical rank plot of face sharing interaction in 3D

6. Rank growth of different interaction in $d$ dimensions. In this section, we discuss and prove the main theorem, Theorem 2.1. We also plot the numerical ranks of different interactions in four dimensions ($d = 4$), i.e., far-field, vertex ($d' = 0$) sharing interaction, one dimensional hyper-surface ($d' = 1$ or the "edge") sharing interaction, two dimensional hyper-surface ($d' = 2$ or the "face") sharing interaction and three dimensional hyper-surface ($d' = 3$ or the "cube") sharing interactions.

6.1. Rank growth of far-field domains. Let $Y = [0, r]^d$ and $X = [-2r, -r] \times [0, r]^{d-1}$ be two hyper-cubes, which are one hyper-cube away. We choose a $(p + 1)^d$ tensor grid of Chebyshev nodes inside the hyper-cube $Y$. We will interpolate $F(x,y)$ on the Chebshev grid along $y$. Let $\tilde{K}$ be the approximation of the matrix $K$. Then by (2.10) the $i^{th}$ component error in matrix-vector product is given by

\begin{equation}
(6.1) \quad \left| \left( Kq - \tilde{K}q \right)_i \right| \leq 4MQV_d \frac{p^{-p}}{\rho - 1}
\end{equation}

Hence, setting the error to be less than $\delta$ (for some $\delta > 0$), we have $p = \left[ \log \left( \frac{4MQV_d}{\delta \rho} \right) \right]$. Since the rank of the matrix $\tilde{K}$ is $(p + 1)^d$, we have that the rank of $\tilde{K}$ to be bounded by $\left( 1 + \left[ \log \left( \frac{4MQV_d}{\delta \rho} \right) \right] \right)^d$. Hence, the rank of $\tilde{K}$ scales $\mathcal{O} \left( \left( \log \left( \frac{MQ}{\delta} \right) \right)^d \right)$ with $\| Kq - \tilde{K}q \|_\infty < \delta$. The numerical rank of the far-field interaction of the four functions (as described in Section 2) are plotted in Figure 19 and tabulated in Table 10.
Now choosing (6.4)

\[
\text{of level } \kappa \text{ where } i \text{ then approximation of matrix } K \text{ is given by}
\]

(6.3) \[
\tilde{K} = \sum_{k=1}^{n} \sum_{j=1}^{2d-1} \tilde{K}_{k,j} + K_{k+1}
\]

Then by (2.10) the \(i^{th}\) component error in matrix-vector product at the \(j^{th}\) subdivision of level \(k\) is given by

(6.4) \[
\left| \left( K_{k,j}q - \tilde{K}_{k,j}q \right)_i \right| \leq 4M_{k,j}Q_{k,j}V_d \rho^{-p_{k,j}} \rho - 1
\]

Now choosing \(p_{k,j}\) such that the above error is less than \(\delta_1\) (for some \(\delta_1 > 0\)), we obtain

\[
p_{k,j} = \left[ \frac{\log(4M_{k,j}Q_{k,j}V_d)}{\log(\rho)} \right] \implies \left( K_{k,j}q - \tilde{K}_{k,j}q \right)_i < \delta_1 \text{ with rank of } \tilde{K}_{k,j} \text{ bounded above by } (1 + p_{k,j})^d.
\]

Let \(p_{l,m} = \max\{p_{k,j} : k = 1, 2, \ldots, \kappa \text{ and } j = 1, 2, \ldots, 2d - 1\} \).

(6.5) \[
\left| \left( Kq - \tilde{K}q \right)_i \right| = \sum_{j=1}^{2d-1} \left| \left( K_{k,j}q - \tilde{K}_{k,j}q \right)_i \right| < \sum_{j=1}^{2d-1} \left( K_{k,j}q - \tilde{K}_{k,j}q \right)_i < (2^d - 1) \delta_1
\]

with rank of \(\tilde{K}\) bounded above by \((2^d - 1)(1 + p_{l,m})^d\). Hence, we get

(6.6) \[
\left| \left( Kq - \tilde{K}q \right)_i \right| = \left| \left( (K_1 + K_2 + \cdots + K_\kappa)q - (\tilde{K}_1 + \tilde{K}_2 + \cdots + \tilde{K}_\kappa)q \right)_i \right| \\
\leq \sum_{k=1}^{\kappa} \left| \left( K_kq - \tilde{K}_kq \right)_i \right| < (2^d - 1) \kappa \delta_1
\]
and rank of $\tilde{K}$ is bounded above by $1 + (2^d - 1) \kappa (1 + p_{l,m})^d$

$= 1 + (2^d - 1) \kappa \left(1 + \left[ \frac{\log \left(\frac{\log(M_{l,m}Q_{l,m}V_d)}{\log(\rho)}\right)}{(2^d-1)^d} \right] \right)$. If we choose $\delta_1 = \frac{\delta}{(2^d - 1) \kappa}$, then

$\left| \left(Kq - \tilde{K}q \right) \right|_i < \delta$ with rank of $\tilde{K}$ bounded above by

$1 + (2^d - 1) \log_{2^d}(N) \left(\frac{\log \left(\frac{\log(M_{l,m}Q_{l,m}V_d)}{\log(\rho)}\right)}{(2^d-1)^d} \right)$. Hence the rank of $\tilde{K}$

scales $O \left( \log_{2^d}(N) \left(\log \left(\frac{A \log_{2^d}(N)}{\delta} \right) \right)^d \right)$ with $\left\| Kq - \tilde{K}q \right\|_\infty < \delta$. The numerical rank plots of the vertex sharing interaction of the four functions as described in Section 2 are shown in Figure 20 and tabulated in Table 11.

| No. of particles $(N)$ | Numerical rank $(p_\epsilon)$ |
|------------------------|-------------------------------|
| 1296                   | 369                           |
| 2400                   | 446                           |
| 4096                   | 506                           |
| 10000                  | 582                           |
| 20736                  | 643                           |
| 38416                  | 690                           |
| 59025                  | 714                           |

Table 11: Numerical rank $(p_\epsilon)$ with $\epsilon = 10^{-12}$ of different kernel functions of vertex sharing interaction in 4D

6.3. **Rank growth of different hyper-surface interaction.** Consider two identical hyper-cubes $Y = [0, r]^d$ and $X = [-r, 0]^{d-d'} \times [0, r]^{d'}$ of dim $d$ which share a hyper-surface of dim $d'$. In contrast, if $d = 3$ then $d' = 1$ implies edge-sharing blocks (edge of a cube is a line segment which has dim 1) and $d' = 2$ implies face-sharing blocks (face of a cube is a square which has dim 2). In the previous sub-section we have discussed the case $d' = 0$, i.e. vertex sharing. Now here we discuss the cases $1 \leq d' \leq d - 1$. Further, subdivide the hyper-cube $Y$ hierarchically using an adaptive $2^d$ tree. At level 1, we have $(2^d - 2^{d'})$ finer hyper-cubes that do not share any hyper-surface with $X$. In general, at level $k$, we have $(2^d - 2^{d'}) (2^{d'})^{k-1}$ finer hyper-cubes that do not share any hyper-surface with $X$.

$$Y = \bigcup_{k=1}^{\infty} Y_k = Y_{k+1} \bigcup_{k=1}^{\kappa} \bigcup_{j=1}^{(2^d-2^{d'})} Y_{k,j} = Y_{k+1} \bigcup_{k=1}^{\kappa} \bigcup_{j=1}^{(2^d-d')} Y_{k,j}$$

and $\kappa \sim \log_{2^d}(N)$ with $Y_{k+1}$ having $(2^{d'})^\kappa = N^{d'/d}$ particles. Let $\tilde{K}_{k,j}$ be approximation of $K_{k,j}$, then approximation of matrix $K$ is given by $\tilde{K} = \sum_{k=1}^{\kappa} \sum_{j=1}^{(2^d-d'-1)} \beta (2^{d'})^k \tilde{K}_{k,j} + K_{k+1}$, where $\beta = (2^d-d'-1)$. Then by (2.10) the $i^{th}$ component error in matrix-vector
product at \( j^{th} \) subdivision of the level \( k \) is given by (pick \( p_{k,j} \) such that the error is less than \( \delta_1 \))

\[
(6.8) \quad \left| \left( K_{k,j}q - \tilde{K}_{k,j}q \right)_i \right| \leq 4M_{k,j}Q_{k,j}V_d \frac{\rho^{-p_{k,j}}}{\rho - 1}
\]

Now choosing \( p_{k,j} \) such that the above error is less than \( \delta_1 \) (for some \( \delta_1 > 0 \)), we obtain \( p_{k,j} = \left( 1 + \frac{\log \left( \frac{4M_{l,m}Q_{l,m}V_d}{\delta_1(\rho - 1)} \right)}{\log(\rho)} \right) \) \( \Rightarrow \left| \left( K_{k,j}q - \tilde{K}_{k,j}q \right)_i \right| < \delta_1 \) with rank of \( \tilde{K}_{k,j} \) is bounded above by \( (1 + p_{k,j})^d \). Let \( p_{l,m} = \max \{ p_{k,j} : k = 1, 2, \ldots, \kappa \) and \( j = 1, 2, \ldots, \beta \left( 2^d \right)^\kappa \}. \) The rank of \( \tilde{K} \) is bounded above by

\[
N^{d'/d} + \sum_{k=1}^\kappa \beta \left( 2^d \right)^k \left( 1 + \frac{\log \left( \frac{4M_{l,m}Q_{l,m}V_d}{\delta_1(\rho - 1)} \right)}{\log(\rho)} \right)^d
\]

\[
= N^{d'/d} + \beta \left( 2^d \left( \frac{2^d}{2^d - 1} \right) \right)^\kappa \left( 1 + \frac{\log \left( \frac{4M_{l,m}Q_{l,m}V_d}{\delta_1(\rho - 1)} \right)}{\log(\rho)} \right)^d
\]

\[
\leq N^{d'/d} + \beta \left( 2^d \left( \frac{2^d}{2^d - 1} \right) \right)^\kappa \left( 1 + \frac{\log \left( \frac{4M_{l,m}Q_{l,m}V_d}{\delta_1(\rho - 1)} \right)}{\log(\rho)} \right)^d
\]

\[
= N^{d'/d} + \beta \frac{2^d N^{d'/d}}{(2^d - 1)} \left( 1 + \frac{\log \left( \frac{4M_{l,m}Q_{l,m}V_d}{\delta_1(\rho - 1)} \right)}{\log(\rho)} \right)^d
\]

Hence the rank of \( \tilde{K} \) scales as \( \mathcal{O} \left( N^{d'/d} \left( \log \left( \frac{M_{l,m}Q_{l,m}}{\delta_1} \right) \right)^d \right) \) and the error in matrix-vector product is bounded above by

\[
(6.10) \quad \left| \left( Kq - \tilde{K}q \right)_i \right| < \sum_{k=1}^\kappa \beta \left( 2^d \right)^k \delta_1 = \beta \left( 2^d \left( \frac{2^d}{2^d - 1} \right) \right)^\kappa \delta_1 < \beta 2^d N^{d'/d} \frac{\delta}{(2^d - 1) \delta_1}
\]

If we choose \( \delta_1 = \frac{\delta (2^d - 1)}{\beta 2^d N^{d'/d}} \) then \( \left| \left( Kq - \tilde{K}q \right)_i \right| < \delta \). Therefore the rank of \( \tilde{K} \) scales

\[
\mathcal{O} \left( N^{d'/d} \left( \log \left( \frac{AN^{d'/d}}{\delta} \right) \right)^d \right) \text{ with } \left\| Kq - \tilde{K}q \right\|_\infty < \delta.
\]

The numerical rank plots of the different hyper-surface sharing interaction of the four functions as described in Section 2 are shown in Figures 21 to 23 and tabulated in Tables 12 to 14.
Remark 1. In the proofs, we need a hierarchy of hyper-cubes so that at each level, the interaction is at least one hyper-cube away. This is done to ensure that $\rho$ is constant at all levels.

Remark 2. One can find tighter upper bound for the error (therefore, tighter upper bound for the rank), if we have a multipole/eigen function expansion as in [10]. Our theorem is applicable for any kernel function under reasonably mild assumptions.

7. Conclusion. We have proved a theorem (Theorem 2.1) regarding the rank growth for different interaction matrices in any dimension. As a consequence of the theorem, we see that the rank of not just the far-field but also the vertex sharing interaction matrices do not scale with any power of $N$. This explains why HODLR, HSS,
and $\mathcal{H}$ matrices with weak admissibility work exceedingly well for one-dimensional problems. Our recent work [27] relies on this fact to construct fast matrix-vector product for matrices arising out of kernel functions in two dimensions. This work could be leveraged to construct newer faster algorithms. A suitable hierarchical representation leveraging the low-rank of nearby interactions could be used to construct direct solvers [8] as well as iterative solvers (to accelerate the matrix-vector product).

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