INERTIAL PROXIMAL BLOCK COORDINATE METHOD FOR A
CLASS OF NONSMOOTH SUM-OF-RATIOS OPTIMIZATION
PROBLEMS∗

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Abstract. In this paper, we consider a class of nonsmooth sum-of-ratios fractional optimization
problems with block structure. This model class is ubiquitous and encompasses several important
nonsmooth optimization problems in the literature. We first propose an inertial proximal block
coordinate method for solving this class of problems by exploiting the underlying structure. The
global convergence of our method is guaranteed under the Kurdyka–Lojasiewicz (KL) property and
some mild assumptions. We then identify the explicit exponents of the KL property for three impor-
tant structured fractional optimization problems. In particular, for the sparse generalized eigenvalue
problem with either cardinality regularization or sparsity constraint, we show that the KL exponents
are 1/2, and so, the proposed method exhibits linear convergence rate. Finally, we illustrate our
theoretical results with both analytic and simulated numerical examples.

Key words. fractional program, Kurdyka–Lojasiewicz property, linear convergence, proximal
block coordinate method, sparsity, sum-of-ratios

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1. Introduction. We consider the following nonsmooth and nonconcave frac-
tional maximization problem

(\mathcal{P}) \quad \max_{x=(x_1, \ldots, x_m) \in S := S_1 \times \cdots \times S_m} F(x) := h(x_1, \ldots, x_m) + \sum_{i=1}^{m} \frac{f_i(x_i)}{g_i(x_i)},

where, for each \( i \in \{1, \ldots, m\} \), \( \mathcal{H}_i \) is a finite-dimensional real Hilbert space, \( S_i \) is a
nonempty closed subset of \( \mathcal{H}_i \), \( h: \mathcal{H}_1 \times \cdots \times \mathcal{H}_m \to [-\infty, +\infty] \) is a (possibly) non-
smooth and nonconcave function, and \( f_i, g_i: \mathcal{H}_i \to \mathbb{R} \) are locally Lipschitz functions
such that, for all \( x_i \in S_i \),

(1.1) \quad f_i(x_i) \geq 0 \quad \text{and} \quad g_i(x_i) > 0.

The model problem (\mathcal{P}) covers various important optimization problems arising in
diverse areas, such as the energy efficiency maximization problem and the sparse
generalized eigenvalue problem [30, 32]. On the other hand, it belongs to the class of
so-called sum-of-ratios optimization problems which are known as the most difficult
problems in the fractional programming literature. Obviously, there is an alternative
formulation for (\mathcal{P}) which is obtained by replacing the maximum with minimum.
Although these two formulations are in general independent (due to the nonnegativity
assumption (1.1)), the corresponding algorithmic development can be easily modified

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to suit the other formulation. Therefore, in this paper, we will focus on the maximum formulation. Below, we present a few explicit motivating examples for the model problem \((P)\).

**Example 1.1** (penalization formulation for general sum-of-ratios optimization problem). Consider the classical sum-of-ratios optimization problem

\[
\max_{x \in C} \sum_{i=1}^{m} \frac{f_i(z)}{g_i(z)},
\]

where \(C\) is a bounded polyhedron in \(\mathbb{R}^d\) and, for each \(i \in \{1, \ldots, m\}\), \(f_i\) and \(g_i\) are continuously differentiable functions on \(\mathbb{R}^d\) such that, for all \(z \in C\), \(f_i(z) \geq 0\) and \(g_i(z) > 0\). This, for example, covers the energy efficiency maximization problem discussed in [32], where \(C = \{z \in \mathbb{R}^d_+ : \forall i \in \{1, \ldots, m\}, \ z_i^{\min} \leq z_i \leq z_i^{\max} \text{ and } v^\top z \leq r\} \) with \(0 < z_i^{\min} \leq z_i^{\max} \), \(v \in \mathbb{R}^d_+\), \(r > 0\) and, for each \(i \in \{1, \ldots, m\}\), \(f_i(z) = \log(1 + u_i^\top z + r_i)\) with \(u_i \in \mathbb{R}^d_+ - \{0\}\) and \(r_i \geq 0\), and \(g_i\) is an affine function with positive values on \(C\). Note that (1.2) can be equivalently rewritten as

\[
\max_{x_1, \ldots, x_m \in C} \sum_{i=1}^{m} \frac{f_i(x_i)}{g_i(x_i)} \quad \text{s.t. } x_1 = \cdots = x_m.
\]

Therefore, a plausible alternative optimization formulation for (1.2) becomes

\[
\max_{x_1, \ldots, x_m \in C} -\gamma \sum_{i=2}^{m} \|x_1 - x_i\|^2 + \sum_{i=1}^{m} \frac{f_i(x_i)}{g_i(x_i)},
\]

where \(\gamma > 0\) is a parameter. Direct verification shows that this is a particular case of our model problem \((P)\) with \(h(x_1, \ldots, x_m) = -\gamma \sum_{i=2}^{m} \|x_1 - x_i\|^2\), and \(S = S_1 \times \cdots \times S_m\) with \(S_i = C, i = 1, \ldots, m\).

**Example 1.2** (sparse generalized eigenvalue problem). The generalized eigenvalue problem, which searches for the most dominant eigenvalues (or principal eigenvalues) and corresponding eigenvector, can be written as an optimization problem

\[
\max_{x \in \mathbb{R}^d} \langle \lambda x, \phi(x) \rangle \quad \text{s.t. } \|x\| = 1.
\]

Here, \(A, B\) are symmetric matrices with \(A\) positive semidefinite and \(B\) positive definite, and \(\phi\) is a regularization function which induces sparsity of the solution. Typical choices of \(\phi\) include the \(l_0\) regularization (or cardinality) function given by \(\|x\|_0 = \text{number of } i : x_i \neq 0\), the \(l_1\)-norm given by \(\|x\|_1 = \sum_{i=1}^{d} |x_i|\), and the indicator function of the sparsity set \(C_r = \{x \in \mathbb{R}^d : \|x\|_0 \leq r\}\) with \(r > 0\). For example, in a recent study [30], the authors examined the sparse generalized eigenvalue problem with \(\phi(x) = \delta_{C_r}(x)\), where they proposed a truncated Rayleigh flow method (TRFM) and demonstrated the efficiency of this model problem on classification, correlation analysis and regression. Direct verification shows that the sparse generalized eigenvalue problem is a particular case of our model problem \((P)\) with \(m = 1\), \(h(x) = -\lambda \phi(x)\), \(f_1(x) = x^\top A x\), \(g_1(x) = x^\top B x\), and \(S = \{x \in \mathbb{R}^d : \|x\| = 1\}\).
Example 1.3 (maximizing the sum of a quadratic function and the Rayleigh quotient over the unit sphere). We consider the problem of maximizing the sum of a quadratic function and Rayleigh quotient over the unit sphere

$$\max_{x \in \mathbb{R}^d} x^T W x + \frac{x^T A x}{x^T B x} \quad \text{s.t.} \quad \|x\| = 1,$$

where $A, B$ are positive definite matrices and $W$ is a symmetric matrix. This problem arises in sparse Fisher discriminant analysis, in the context of which it is usually solved iteratively [33]. In particular, $x$ is the desired discriminating vector in cluster analysis, the term $\frac{x^T A x}{x^T B x}$ is known as the Rayleigh quotient (or Fisher information in information science), and the quadratic term $x^T W x$ serves as a local approximation for the sparse penalty term. Direct verification shows that this is a particular case of our model problem ($P$) for $m = 1, h(x) = x^T W x, f_1(x) = x^T A x, g_1(x) = x^T B x$ and $S = \{x \in \mathbb{R}^d : \|x\| = 1\}$.

In the case where $m = 1$ and $h \equiv 0$, problem ($P$) is known as the single ratio fractional programming problem $\max_{x \in S} \frac{f_1(x)}{g_1(x)}$. A classical approach for solving the latter problem is Dinkelbach’s method and its variants (see [10, 12]). In this approach, one typically constructs an iterative scheme which requires finding an optimal solution $x_{n+1}$ of the optimization problem

$$\max_{x \in S} \{f_1(x) - \theta_n g_1(x)\}$$

in each iteration $n$, while $\theta_n$ is updated by $\theta_{n+1} := \frac{f_1(x_{n+1})}{g_1(x_{n+1})}$. For details of this approach, we refer the readers to [10, 12, 14, 27]. However, solving in each iteration an optimization problem of type (1.4) may be as expensive and difficult as solving the original problem in general. Recently, proximal type methods based on Dinkelbach’s approach have been proposed to tackle single ratio fractional programs [8, 9, 21], where each subproblem is much easier to solve and sometimes has closed form solutions.

Unfortunately, in the case of sum-of-ratios fractional programs, that is either $m > 1$ or $h \neq 0$, Dinkelbach’s approach cannot be directly applied anymore. One naive approach is to convert the sum-of-ratios into single ratio’s cases and to apply Dinkelbach’s method. This approach increases the complexity of the function dramatically and leads to numerical methods with poor performance. For example, through this approach, a sum of three linear fractional functions becomes a fractional function whose numerator and denominator are degree 3 nonconvex polynomials, and so, the nice linearity structure is completely lost. Some important steps towards solving sum-of-ratios fractional programs are mainly limited to sum-of-ratios of linear or quadratic fractional programs, and rely on integer programming techniques such as branch and bound and convex relaxation methods, see for example [6, 22, 33]. These approaches, although highly appealing, are much less scalable than the proximal type methods, and cannot directly deal with settings in which nonsmooth functions are involved.

Despite this important progress, it is still no clear whether one can develop proximal methods for solving nonsmooth and nonconcave sum-of-ratios fractional programs ($P$) in the line of [8, 9] for single ratio cases. This forms the basic motivation of our work. Specifically, the contributions of this paper are as follows:

1. In Section 3, we propose an inertial proximal subgradient method for solving the model problem ($P$). We then show that the iterative sequence generated by the proposed method is bounded and any of its limit points is a
stationary point of problem (P) in a suitable sense. This new method can be interpreted as a proximal block coordinate method of Gauss–Seidel type applied to a related non-fractional reformulated problem. We also establish the convergence of the full sequence under the assumption that a suitable merit function satisfies the Kurdyka–Lojasiewicz (KL) property.

(2) In Section 4, we analyze several structured sum-of-ratios fractional programs and obtain the explicit KL exponents of the corresponding desingularization functions in the KL property: sum-of-ratios fractional quadratic programs with spherical constraint, generalized eigenvalue problems with cardinality regularization and generalized eigenvalue problems with sparsity constraints. In particular, we establish that, for the last two classes of fractional programs, the KL exponents are 1/2. As a consequence, we obtain that the proposed numerical method exhibits linear convergence for these two classes of fractional programs.

(3) Finally, we illustrate the proposed method via both analytical and simulated numerical examples in Section 5.

2. Preliminaries. In this section, we recall some basic notations and preliminary results which will be used in this paper. We assume throughout that $\mathcal{H}, \mathcal{H}_1, \ldots, \mathcal{H}_m$ are finite-dimensional real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. The product space $\mathcal{H}_1 \times \cdots \times \mathcal{H}_m$ is also a real Hilbert space endowed with the inner product given by $\langle (x_1, \ldots, x_m), (y_1, \ldots, y_m) \rangle = \sum_{i=1}^{m} \langle x_i, y_i \rangle$. The set of nonnegative integers is denoted by $\mathbb{N}$, the set of real numbers by $\mathbb{R}$, the set of nonnegative real numbers by $\mathbb{R}_+$, and the set of the positive real numbers by $\mathbb{R}_{++}$.

The indicator function of a set $C$ is defined by $\delta_C(x) := 0$ if $x \in C$, and $\delta_C(x) := +\infty$ otherwise. Given an extended-real-valued function $f : \mathcal{H} \to [-\infty, +\infty]$, its domain is defined by $\text{dom} f := \{ x \in \mathcal{H} : f(x) < +\infty \}$. The function $f$ is proper if $\text{dom} f \neq \emptyset$ and it never equals $-\infty$. We say that $f$ is lower semicontinuous if, for all $x \in \mathcal{H}$, $f(x) \leq \liminf_{y \to x} f(y)$, and convex if its epigraph $\{(x, \rho) \in \mathcal{H} \times \mathbb{R} : f(x) \leq \rho \}$ is a convex subset of $\mathcal{H} \times \mathbb{R}$. The function $f$ is said to be weakly convex (on $\mathcal{H}$) if there exists $\alpha \geq 0$ such that $f + \frac{\alpha}{2} \| \cdot \|_2^2$ is a convex function. The smallest constant $\alpha$ such that $f + \frac{\alpha}{2} \| \cdot \|_2^2$ is convex is called the modulus of weak convexity. More generally, $f$ is said to be weakly convex on $S \subseteq \mathcal{H}$ with modulus $\alpha$ if $f + \delta_S$ is weakly convex with modulus $\alpha$. Weakly convex functions form a broad class of functions which covers quadratic functions, convex functions, differentiable functions whose gradient is Lipschitz continuous, and the composition of a convex and Lipschitz continuous function with a $C^1$-smooth mapping whose Jacobian is Lipschitz continuous (see [13, Lemma 4.2]).

Let $f : \mathcal{H} \to [-\infty, +\infty]$ and $x \in \mathcal{H}$ with $|f(x)| < +\infty$. The Fréchet subdifferential of $f$ at $x$ is given by

$$\partial f(x) := \left\{ u \in \mathcal{H} : \liminf_{z \to x} \frac{f(z) - f(x) - \langle u, z - x \rangle}{\| z - x \|} \geq 0 \right\},$$

the limiting subdifferential of $f$ at $x$ is given by

$$\partial_L f(x) := \left\{ u \in \mathcal{H} : \exists x_n \downarrow x, \ u_n \to u \right. \text{ with } u_n \in \partial f(x_n) \right\},$$

and the horizon subdifferential of $f$ at $x$ is given by

$$\partial^\infty f(x) := \left\{ u \in \mathcal{H} : \exists x_n \downarrow x, \ \lambda_n \downarrow 0, \ \lambda_n u_n \to u \right. \text{ with } u_n \in \partial f(x_n) \right\}.$$
Here, the notation $z \xrightarrow{L} x$ means $z \to x$ with $f(z) \to f(x)$. It follows from the above definition that the limiting subdifferential has the following robustness property

$$\partial_L f(x) = \left\{ u \in H : \exists x_n \xrightarrow{L} x, \ u_n \to u \text{ with } u_n \in \partial_L f(x_n) \right\}.$$  

The domain of $\partial_L f$ is $\text{dom} \partial_L f := \{ x \in H : \partial_L f(x) \neq \emptyset \}$. If $f$ is Lipschitz continuous around $x$, then $\partial_L f(x)$ is bounded and $\partial^\infty f(x) = \{ 0 \}$ (see [24, Corollary 1.81]). If $f$ is strictly differentiable at $x$, then $\partial f$ and $\partial_L f$ reduce to the derivative of $f$, denoted by $\nabla f$ (see [24, Corollary 1.82]). If $f$ is convex, then both Fréchet and limiting subdifferentials at $x$ reduce to the classical subdifferential in convex analysis (see [24, Theorem 1.93])

$$\partial f(x) := \{ u \in H : \forall z \in H, \langle u, z - x \rangle \leq f(z) - f(x) \}.$$  

We say that $f$ is regular$^2$ at $x \in H$ if $\hat{\partial} f(x) = \partial f(x)$, and that $f$ is regular on $C \subseteq H$ if it is regular at any $x \in C$. For a proper lower semicontinuous function $f$, it is clear that if $f$ is convex around $x$ or strictly differentiable at $x$, then it is regular at $x$. A nonempty set $S$ in $H$ is regular at $x \in S$ if $\delta_S$ is regular at $x$. We say that $S$ is regular if it is regular at all of its points. From the definition, it can be verified that $C$ is regular if $C$ is a closed and convex set or $C$ is a smooth manifold given by $C = \{ x \in H : g_i(x) = 0, i = 1, \ldots, m \}$, where $g_i$ are smooth functions satisfying the so-called linear independent constraint qualification (that is, $\{ \nabla g_i(x) : i = 1, \ldots, m \}$ are linearly independent for all $x \in C$).

Next, we collect some subdifferential rules and calculations which will be of use in our analysis and whose proofs are given in Appendix A.

**Lemma 2.1 (calculus rules).** Let $f, g : H \to (-\infty, +\infty]$ be proper lower semicontinuous functions and let $x \in \text{dom} f$. Then the following statements hold:

(i) (Separable sum rule) If $f(x) = \sum_{i=1}^m f_i(x_i)$ with $x = (x_1, \ldots, x_m)$, then $\partial_L f(x) = \partial_L f_1(x_1) \times \cdots \times \partial_L f_m(x_m)$ and $f$ is regular at $x$ when each $f_i$ is regular at $x_i$.

(ii) (Sum rule) If $\partial^\infty f(x) \cap (-\partial^\infty g(x)) = \{ 0 \}$, then $\partial_L (f + g)(x) \subseteq \partial_L f(x) + \partial_L g(x)$, where the equality holds when both $f$ and $g$ are regular at $x$, in which case $f + g$ is also regular at $x$. Moreover, if $g$ is strictly differentiable at $x$, then $\partial_L (f + g)(x) = \partial_L f(x) + \nabla g(x)$.

(iii) (Sign rule) If $f$ is Lipschitz continuous around $x$ and $\hat{\partial} f$ is nonempty-valued around $x$, then $\partial_L (-f)(x) \subseteq -\partial_L f(x)$.

(iv) (Quotient rule) Suppose that $f$ and $g$ are Lipschitz continuous around $x$, and $g(x) \neq 0$. If $\hat{\partial} f$ is nonempty-valued around $x$, then

$$\partial_L \left( \frac{-f}{g} \right)(x) \subseteq \frac{-g(x)\partial_L f(x) + \partial_L (f(x)g(x))}{g(x)^2}.$$  

If $f$ is strictly differentiable at $x$, then

$$\partial_L \left( \frac{-f}{g} \right)(x) = \frac{-g(x)\nabla f(x) + \partial_L (f(x)g(x))}{g(x)^2}.$$  

$^1$A function $f$ is strictly differentiable at $x$ if there exists $u \in H$ such that $\lim_{y \to x} \frac{f(y) - f(x) - \langle u, y - x \rangle}{\|y - x\|} = 0$. Clearly, if $f$ is continuously differentiable at $x$, then it is strictly differentiable at $x$.

$^2$This is also referred as lower regular in [24, 25].
and, consequently, \(-f/g\) is regular at \(x\) if and only if \(f(x)g\) is regular at \(x\).

(iii) Chain rule and square root rule) If \(f\) is Lipschitz continuous around \(x\) and \(\theta: \mathbb{R} \to \mathbb{R}\) is continuously differentiable around \(f(x)\), then \(\partial_L(\theta \circ f)(x) = \partial_L(\theta'(f(x))f(x))\). In particular, if \(f\) is Lipschitz continuous around \(x\) and \(f(x) > 0\), then

\[
\partial_L \left( \frac{f}{\sqrt{f}} \right)(x) = \frac{\partial_L f(x)}{2\sqrt{f(x)}} \quad \text{and} \quad \partial_L \left( -\frac{f}{\sqrt{f}} \right)(x) = \frac{\partial_L (-f(x))}{2\sqrt{f(x)}}.
\]

**Lemma 2.2.** Let \(\Lambda := \{x \in \mathbb{R}^d : \|x\| = 1\}\) and \(C_r := \{x \in \mathbb{R}^d : \|x\|_0 \leq r\}\). Given \(x = (x_1, \ldots, x_d) \in \mathbb{R}^d\), set \(\text{supp}(x) := \{j : x_j \neq 0\}\). Then the following statements hold:

(i) \(\forall x \in \mathbb{R}^d, \partial(\|x\|)(x) = \partial_L(\|x\|)(x) = \partial^\infty_L(\|x\|)(x) = \{v : v_j = 0 \text{ if } j \in \text{supp}(x)\}\).

(ii) \(\forall x \in \Lambda, \partial^\infty_L(x) = \partial_L^\infty(x) = \partial^\infty_L(x) = \{tx : t \in \mathbb{R}\}\).

(iii) If \(\|x\|_0 = r\), then \(\partial^\infty_L(x) = \partial_L^\infty(x) = \{v : v_j = 0 \text{ if } j \in \text{supp}(x)\}\) if \(\|x\|_0 < r\), then

\[
\partial^\infty_L(x) = \partial_L^\infty(x) = \{v : v_j = 0 \text{ if } j \in \text{supp}(x) \cup \hat{J}\}\.
\]

(iv) \(\forall x \in \Lambda, \partial(L(\|x\| + \delta_L)) = \partial_L(\|x\|)(x) + \partial_L^\infty_L(x)\).

(v) \(\forall x \in \Lambda \cap C_r, \partial_L(\delta_L + \delta_L) = \partial_L^\infty_L(x) + \partial_L^\infty_L(x)\).

We also need the following two notions of stationary points for problem (\(P\)).

**Definition 2.3.** (Stationary points). We say that \(\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_m) \in S\) is a stationary point for (\(P\)) if \(0 \in \partial^\infty_L(-h + \delta_S)(\tilde{x})\), and a lifted coordinate-wise stationary point for (\(P\)) if, for each \(i \in \{1, \ldots, m\}\),

\[
0 \in \partial^\infty_L(-h + \delta_S)(\tilde{x}_i) + \frac{-g_i(\tilde{x}_i)\partial_L f_i(\tilde{x}_i) + f_i(\tilde{x}_i)\partial_L g_i(\tilde{x}_i)}{g_i(\tilde{x}_i)^2},
\]

where \(\partial^\infty_L\) denotes the subdifferential with respect to the \(x_i\)-variable.

The following lemma whose proof is given in Appendix A provides the relationship between a stationary point and a lifted coordinate-wise stationary point for (\(P\)).

**Lemma 2.4.** (Stationary vs. lifted coordinate-wise stationary points). Let \(\tilde{x} = (\tilde{\pi}_1, \ldots, \tilde{\pi}_m) \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_m\). Suppose that \(-h\) is proper lower semicontinuous, that either \(m = 1\) or \(h\) is strictly differentiable at \(\tilde{x}\), and that, for each \(i \in \{1, \ldots, m\}\), \(f_i\) and \(g_i\) are Lipschitz continuous around \(\tilde{x}_i\), \(f_i(\tilde{x}_i) \geq 0\), and \(g_i(\tilde{x}_i) \neq 0\). Then the following statements hold:

(i) If for each \(i \in \{1, \ldots, m\}\), \(\partial f_i\) is nonempty-valued around \(\tilde{\pi}_i\), then \(\tilde{x}\) is a lifted coordinate-wise stationary point for (\(P\)) whenever \(x\) is a stationary point for (\(P\)).

(ii) If for each \(i \in \{1, \ldots, m\}\), \(f_i\) is strictly differentiable at \(\tilde{\pi}_i\) and either (a) \(g_i\) is strictly differentiable at \(\tilde{x}_i\) or (b) \(g_i\) is regular at \(\tilde{x}_i\) and \(h + \delta_S\) is regular at \(\tilde{x}\), then \(\tilde{x}\) is a lifted coordinate-wise stationary point for (\(P\)) if and only if it is a stationary point for (\(P\)).

### 3. Inertial proximal subgradient method

In this section, we propose an inertial proximal subgradient method for solving the sum-of-ratios optimization prob-
lem (P) and establish the convergence analysis for the proposed method. From now on, we will work under the following assumption.

**Assumption 3.1.** For problem (P), $S$ is a (not necessarily convex) closed set, $-h$ is a proper lower semicontinuous function, and, for each $i \in \{1, \ldots, m\}$, the functions $f_i$ and $g_i$ are locally Lipschitz functions on an open set containing $S_i$. Moreover,

(a) For each $i \in \{1, \ldots, m\}$, $f_i$ is nonnegative on an open set containing $S_i$ and there exists $\alpha_i \geq 0$ such that, for all $x_i, z_i \in S_i$ and all $u \in \partial_L f_i(x_i)$,

$$\left\langle \frac{u}{2 \sqrt{f_i(x_i)}}, z_i - x_i \right\rangle \leq \sqrt{f_i(z_i)} - \sqrt{f_i(x_i)} + \frac{\alpha_i}{2} \|z_i - x_i\|^2,$$

whenever $f_i(x_i) > 0$.

(b) For each $i \in \{1, \ldots, m\}$, $g_i$ is positive on $S_i$ and there exists $\beta_i \geq 0$ such that, for all $x_i, z_i \in S_i$ and all $v \in \partial_L g_i(x_i)$,

$$\langle v, z_i - x_i \rangle \geq g_i(z_i) - g_i(x_i) - \frac{\beta_i}{2} \|z_i - x_i\|^2.$$

**Remark 3.2** (comments for the standing assumption). We note that standing Assumption 3.1 is quite general and, in particular, are satisfied for our motivating examples.

(i) Assumption 3.1(a) is fulfilled if, for each $i \in \{1, \ldots, m\}$, $f_i$ takes nonnegative values on an open set $O_i$ containing $S_i$ and $\sqrt{f_i}$ is weakly convex on $O_i$ with modulus $\alpha_i$. Clearly, this condition is true if $f_i(x_i) = x_i^\top A_i x_i$ for a positive semi-definite matrix $A_i$ (as in the motivating Example 1.3 and Example 1.2) because $\sqrt{f_i}(x_i) = \|A_i^{1/2} x_i\|$ which is convex, where $A_i^{1/2}$ is a symmetric matrix such that $A_i^{1/2} A_i^{1/2} = A_i$.

Assumption 3.1(a) also holds if, for each $i \in \{1, \ldots, m\}$, $S_i$ is compact, $f_i$ takes positive values on an open set $O_i$ containing $S_i$ and $f_i$ is a differentiable function whose gradient is Lipschitz continuous on $O_i$ with modulus $L_i$. Indeed, in this case, letting $r_i := \min_{x_i \in S_i} f_i(x_i) > 0$, a direct verification shows that $\sqrt{f_i}$ is weakly convex with modulus $\alpha_i = \frac{L_i}{2 \sqrt{r_i}} + \frac{1}{4} r_i^{-1} \max_{x_i \in S_i} \|\nabla f_i(x_i)\|^2$. This covers, in particular, the alternative optimization formulation for the energy maximization problem mentioned in the motivating Example 1.1, where $f_i(x_i) = \log(1 + u_i^\top x_i + r_i)$ with $u_i \in \mathbb{R}_+^d \setminus \{0\}$ and $r_i \geq 0$ and, for $i \in \{1, \ldots, m\}$, $S_i = \{x \in \mathbb{R}_+^d : x_i^\min \leq x_i \leq x_i^\max \}$ and $v^\top x \leq r$ with $0 < x_i^\min \leq x_i^\max$, $v \in \mathbb{R}_+^d$ and $r > 0$.

Similarly, Assumption 3.1(b) is satisfied if, for each $i \in \{1, \ldots, m\}$, $g_i$ is positive on $S_i$ and it is a differentiable function whose gradient is Lipschitz continuous with modulus $\beta_i$. Thus, combining these observations, we see Assumption 3.1 are satisfied for the important motivating examples mentioned in the introduction.

(ii) We also notice that the first condition in Assumption 3.1(a) ensures that, if $x_i \in S_i$ and $f_i(x_i) = 0$, then $0 \in \partial_L f_i(x_i)$ for $i \in \{1, \ldots, m\}$.

We now propose our inertial proximal subgradient method for (P). As we will see later on, this method can be seen as a proximal block coordinate method of Gauss–Seidel type applied to an equivalent non-fractional formulation. It is also worthwhile noting that, even when applied to the single-ratio case ($m = 1$ and $h \equiv 0$), our method here is totally different from the proximal type methods in [8, 9] which are based on
Dinkelbach’s approach.

**Algorithm 3.3 (Inertial proximal subgradient method for problem (P)).**

\(\triangleright\) **Step 1.** Choose \(x_{-1} = x_0 = (x_{1,0}, \ldots, x_{m,0}) \in S\) and set \(n = 0\). Let \(\delta \in \mathbb{R}_{++}\) and \(\mathcal{V} \in [0, \delta/2]\).

\(\triangleright\) **Step 2.** Set \(y_n = (y_{1,n}, \ldots, y_{m,n})\) with \(y_{i,n} = \frac{\sqrt{f_i(x_{i,n})}}{y_i(x_{i,n})}\). Choose \(\tau_n \in \mathbb{R}\) such that \(\tau_n \geq \delta + \max_{1 \leq i \leq m} \left\{ \frac{1}{\sqrt{2}} (2y_{i,n} \alpha_i + y_{i,n}^2 \beta_i) \right\}\), where \(\alpha_i\) and \(\beta_i\) are defined in Assumption 3.1. Let \(\nu_n \in [0, \mathcal{V}/\tau_n]\). For each \(i \in \{1, \ldots, m\}\), let \(z_{i,n} = x_{i,n} + \nu_n (x_{i,n} - x_{i,n-1})\), \(u_{i,n} \in \partial f_i(x_{i,n})\), \(v_{i,n} \in \partial g_i(x_{i,n})\), and set

\[
\hat{w}_{i,n} = \begin{cases} 
  y_{i,n} \frac{u_{i,n}}{\sqrt{f_i(x_{i,n})}} - y_{i,n}^2 v_{i,n} & \text{if} \ f_i(x_{i,n}) > 0, \\
  0 & \text{if} \ f_i(x_{i,n}) = 0.
\end{cases}
\]

Denote \(h_{i,n+1}(x_i) := h(x_{i,n+1}, \ldots, x_{i-1,n+1}, x_i, x_{i+1,n}, \ldots, x_{m,n})\) and compute

\[
x_{i,n+1} = \arg\max_{x_i \in S_i} \left( h_{i,n+1}(x_i) - \tau_n \left\| x_i - z_{i,n} - \frac{1}{2\tau_n} {\hat{w}_{i,n}} \right\|^2 \right).
\]

Update \(x_{n+1} = (x_{1,n+1}, \ldots, x_{m,n+1})\).

\(\triangleright\) **Step 3.** If a termination criterion is not met, set \(n = n + 1\) and go to Step 2.

**Remark 3.4 (discussion on the computational costs).** The major computation cost lies in the update of \(x_{n+1}\) in Step 2. The update, for each \(i \in \{1, \ldots, m\}\),

\[
x_{i,n+1} = \arg\max_{x_i \in S_i} \left( h_{i,n+1}(x_i) - \tau_n \left\| x_i - \left( z_{i,n} + \frac{1}{2\tau_n} {\hat{w}_{i,n}} \right) \right\|^2 \right)
\]

is equivalent to computing the proximal operator\(^3\) of \(\frac{1}{2\tau_n} (-h_{i,n+1} + \delta_{S_i})\) at the point \(z_{i,n} + \frac{1}{2\tau_n} {\hat{w}_{i,n}}\). This can be done efficiently in many situations, for example, in the following cases:

(i) if \(h \equiv 0\), then this reduces to the projection onto the set \(S_i\) which, in many cases, has closed forms. This is the case when \(S_i\) is a box, \(S_i\) is a sphere or a ball, \(S_i = \{ x : \|x\|_0 \leq r \}\) for \(r > 0\), \(S_i = \{ x : \|x\| = 1 \}\) and \(\|x\|_0 \leq r \}\) (as in the motivation Example 1.2 of the sparse generalized eigenvalue problem with \(\phi(x)\) being the indicator function of the sparsity set) and \(S_i = \{ X \in \mathbb{R}^{p \times d} : X^\top X = I_d \}\).

(ii) if \(m = 1\), \(h(x) = -\lambda \|x\|_0\) or \(h(x) = -\lambda \|x\|_1 \) with \(\lambda \geq 0\), and \(S = \{ x : \|x\| = 1 \}\) (as in the motivating Example 1.2 of sparse generalized eigenvalue problem with \(\phi(x)\) being the cardinality regularization or \(\ell_1\)-regularization), then the resulting proximal operator can be simplified to \(\arg\min_{x \in \mathbb{R}^d} \{ \|x + a\|^2 - \tau h(x) : \|x\| = 1 \}\) for some \(a \in \mathbb{R}^d\) and \(\tau \geq 0\). This can be further rewritten as \(\arg\min_{x \in \mathbb{R}^d} \{ (2a \cdot x) - \tau h(x) : \|x\| = 1 \}\), which has a closed form solution (see [29, Proposition 6] and [23]).

(iii) if \(h\) is a (possibly) nonconvex quadratic function and \(S_i = \{ x : \|x\| = 1 \}\) (as in the motivating Example 1.3), then the resulting problem is a nonconvex quadratic programming problem with norm constraint which is known as the

---

\(^3\) The **proximal operator** of a function \(\varphi\) at \(x\) is defined by \(\text{Prox}_\varphi(x) = \arg\min_y \{ \varphi(y) + \frac{1}{2\tau} \|y - x\|^2 \}\). 

trust region problem. In this case, this problem can be solved efficiently, for example, by solving a related single generalized eigenvalue problem (see [1]).

(iv) if \( h \) can be expressed as the maximum of finitely many concave quadratic functions, that is, \( h(x) = \max_{1 \leq r \leq p} \{ \frac{1}{2}x^T A_r x + a_r^T x + c_r \} \), where each \( A_r \) is semi-definite (and so, \( h_{i,n+1} \) can also be expressed in this form) and \( S_i \) is a polyhedral set, then this is equivalent to the solving of \( p \) many quadratic programming problems with linear inequality constraints, and so, it can be solved efficiently via quadratic programming solvers. This, in particular, covers the motivating Example 1.1.

Finally, we also remark that Step 2 also requires the availability of a subgradient of \( f_i \) at the current iterate. In general, this requires \( f_i \)'s to have some specific structure. On the other hand, in many important applications, \( f_i \) can be expressed as the maximum/minimum of finitely many continuously differentiable functions, in which case, a subgradient of \( f_i \) is easily obtained.

### 3.1. Interpretation of Algorithm 3.3.

Next, we see that Algorithm 3.3 can be interpreted as a proximal block coordinate method of Gauss-Seidel type applied to the problem

\[
\begin{align*}
(P_1) & \max_{x=(x_1, \ldots, x_m) \in S} h(x) + H(x, y) \quad \text{with } H(x, y) := \sum_{i=1}^m \left[ 2y_i \sqrt{f_i(x_i)} - y_i^2 g_i(x_i) \right].
\end{align*}
\]

We say that \((\overline{x}, \overline{y}) \in S \times \mathbb{R}^m\) is a lifted coordinate-wise stationary point for \((P_1)\) if, for each \( i \in \{1, \ldots, m\} \),

\[
0 \in \partial_L^i (-h + \delta_S)(\overline{x}) + \partial_L^i (-H)(\overline{x}, \overline{y}) \quad \text{and} \quad \overline{y}_i = \sqrt{f_i(\overline{x})}/g_i(\overline{x}),
\]

where the latter is equivalent to \(0 \in \partial_L^i (-H)(\overline{x}, \overline{y})\).

The relationship between lifted coordinate-wise stationary points for \((P)\) and \((P_1)\) is examined in the next lemma with proof in Appendix A. In the case where \( h \equiv 0 \), the following property of global solutions of \((P)\) and \((P_1)\) was mentioned in [6, Theorem 2.2] for problem (1.2) with affine numerators and denominators, and given in [28, Corollary 1] for problem (1.2) with non-affine numerators and denominators. It is worth noting that [28] only provides the non-reformulation for the non-convex problem (1.2) in terms of global solutions, and the numerical algorithms were given only for concave-convex cases (that is, all the numerators are concave and denominators are convex, see [28, Algorithm 1]). Unfortunately, the methods suggested therein are not of the form of splitting algorithms, and there is no convergence guarantee provided for the general setting in this paper covering the motivating examples in the introduction.

#### Lemma 3.5 (fractional vs. non-fractional formulations).

Let \( \overline{x} = (\overline{x}_1, \ldots, \overline{x}_m) \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_m \) and \( \overline{y} = (\overline{y}_1, \ldots, \overline{y}_m) \in \mathbb{R}^m \) with \( \overline{y}_i = \frac{\sqrt{f_i(\overline{x})}}{g_i(\overline{x})} \). Then the following statements hold:

(i) \( \overline{x} \) is a global solution for \((P)\) if and only if \((\overline{x}, \overline{y})\) is a global solution for \((P_1)\), in which case, both problems have the same optimal value.

(ii) Suppose that \(-h\) is proper lower semicontinuous and finite at \( \overline{x} \) and that, for each \( i \in \{1, \ldots, m\} \), \( f_i \) and \( g_i \) are Lipschitz continuous around \( \overline{x}_i \), \( f_i(\overline{x}_i) > 0 \), and \( g_i(\overline{x}_i) > 0 \). Then

(a) If for each \( i \in \{1, \ldots, m\} \), \( \hat{f}_i \) is nonempty-valued around \( \overline{x}_i \), then \( \overline{x} \)
is a lifted coordinate-wise stationary point for \((P)\) whenever \((\mathbf{x}, \mathbf{y})\) is a
lifted coordinate-wise stationary point for \((P_1)\).

(b) If for each \(i \in \{1, \ldots, m\}\), \(f_i\) is strictly differentiable at \(\mathbf{x}\), then \(\mathbf{x}\) is a
lifted coordinate-wise stationary point for \((P)\) if and only if \((\mathbf{x}, \mathbf{y})\) is a
lifted coordinate-wise stationary point for \((P_1)\).

Remark 3.6 (interpretation of Algorithm 3.3 as a block coordinate inertial proximal
algorithm). Suppose that, for each \(i \in \{1, \ldots, m\}\), \(f_i\) is nonnegative and \(g_i\)
is continuously differentiable on an open set containing \(S_i\). We will show that Algorithm
3.3 can be interpreted as a block coordinate inertial proximal subgradient
algorithm. To see this, we recall that, according to Lemma 3.5, problem \((P)\) is
equivalent to \((P_1)\). As, for each \(i \in \{1, \ldots, m\}\), \(y_i \mapsto H_i(x_i, y_i) := 2y_i\sqrt{f_i(x_i)} - y_i^2 g_i(x_i)\) is
a strongly concave one-variable quadratic function which admits a global maximizer
at \(\sqrt{f_i(x_i)} / g_i(x_i)\), one has

\[
y_{n+1} = \arg\max_{\mathbf{y} \in \mathbb{R}^m} \left\{ h(x_{n+1}) + \sum_{i=1}^{m} H_i(x_{i,n+1}, y_i) \right\}
\]

\[
= \arg\max_{\mathbf{y} \in \mathbb{R}^m} \left\{ h(x_{n+1}) + H(x_{n+1}, \mathbf{y}) \right\}.
\]

Let \(i \in \{1, \ldots, m\}\). We see that, if \(f_i(x_{i,n}) > 0\), then

\[
w_{i,n} = y_{i,n} \frac{y_{i,n} - y_{i,n}^2}{\sqrt{f_i(x_{i,n})}} - y_{i,n}^2 v_{i,n} \in \frac{y_{i,n} \partial_L f_i(x_{i,n})}{\sqrt{f_i(x_{i,n})}} - y_{i,n}^2 \nabla g_i(x_{i,n}) = \partial_L^x H_i(x_{i,n}, y_{i,n}).
\]

If \(f_i(x_{i,n}) = 0\), then \(y_{i,n} = 0\), \(w_{i,n} = 0\), and, since \(\sqrt{f_i(x_i)} \geq 0\) on an open set
containing \(S_i\) and \(x_{i,n} \in S_i\), one has \(0 \in \partial_L(\sqrt{f_i})(x_{i,n})\), which implies that

\[
w_{i,n} = 0 \in y_{i,n} \partial_L(\sqrt{f_i})(x_{i,n}) - y_{i,n}^2 \nabla g_i(x_{i,n}) = \partial_L^x H_i(x_{i,n}, y_{i,n}).
\]

So, the update for \(x_{i,n+1} = (x_{1,n+1}, \ldots, x_{m,n+1})\) involves, for \(i \in \{1, \ldots, m\}\),

\[
x_{i,n+1} = \arg\max_{x_i \in S_i} \left\{ h_{i,n+1}(x_i) - \tau_n \left\| x_i - \left( z_{i,n} + \frac{1}{2\tau_n} w_{i,n} \right) \right\|^2 \right\}
\]

\[
\text{with } w_{i,n} \in \partial_L^x H_i(x_{i,n}, y_{i,n}).
\]

Combining the above observations, one sees that Algorithm 3.3 can be regarded as
a block coordinate inertial proximal subgradient algorithm applied to problem \((P_1)\),
where proximal subgradient steps are applied cyclically to the \(\mathbf{x}\)-variable and a direct
maximization step is applied to the \(\mathbf{y}\)-variable.

3.2. Convergence analysis. In this part, we discuss the convergence analysis
for Algorithm 3.3. Let us first start with the subsequential convergence. To do this,
we shall consider the following assumption.

Assumption 3.7. For problem \((P)\), either one of the following holds:
(a) \(m = 1\), \(-h + \delta S\) and \(g_1\) are regular on \(S\), and \(f_1\) is strictly differentiable on
an open set containing \(S\);
(b) \(h\) is strictly differentiable on an open set containing \(S\), \(S\) is regular, and for
each \(i \in \{1, \ldots, m\}\), \(f_i\) is strictly differentiable on an open set containing \(S_i\)
and \(g_i\) is regular on \(S_i\).
(c) $h$ is strictly differentiable on an open set containing $S$ and, for each $i \in \{1, \ldots, m\}$, $f_i$ and $g_i$ are strictly differentiable on an open set containing $S_i$.

It is worth noting that Assumption 3.7 is satisfied with all of our motivation examples. We are now ready to state our first main result as below.

**Theorem 3.8** (subsequential convergence). Let $(x_n)_{n \in \mathbb{N}}$ be the sequence generated by Algorithm 3.3. Suppose that Assumption 3.1 holds, that $h$ is bounded from above on $S$, and that the set $\{x \in S : F(x) \geq F(x_0)\}$ is bounded. Then the following statements hold:

(i) For all $n \in \mathbb{N}$, $F(x_n) - \varpi \|x_n - x_{n-1}\|^2 \leq F(x_{n+1}) - (\delta - \varpi)\|x_{n+1} - x_n\|^2$.

(ii) The sequence $(F(x_n))_{n \in \mathbb{N}}$ is convergent, the sequence $(x_n)_{n \in \mathbb{N}}$ is bounded, and $\sum_{n=0}^{+\infty} \|x_{n+1} - x_n\|^2 < +\infty$.

(iii) Let $\bar{x}$ be a cluster point of $(x_n)_{n \in \mathbb{N}}$ and suppose that $\limsup_{n \to +\infty} \tau_n = \varpi < +\infty$ and that either $m = 1$ or $h$ is continuous on $S \cap \text{dom } h$. Then $\lim_{n \to +\infty} F(x_n) = F(\bar{x})$ and $\bar{x} \in S$ is a lifted coordinate-wise stationary point for $(P)$. If additional Assumption 3.7 holds, then $\bar{x}$ is a stationary point for $(P)$.

**Proof.** (i): Let any $i \in \{1, \ldots, m\}$ and any $n \in \mathbb{N}$. From Step 2 of Algorithm 3.3, we have that $x_{i,n} \in S_i$, $y_{i,n} \geq 0$, and, for all $i \in S_i$,

$$h_{i,n+1}(x_i) - \tau_n \|x_i - z_{i,n} - \frac{1}{2\tau_n} w_{i,n}\|^2 \leq h_{i,n+1}(x_{i,n+1}) - \tau_n \|x_{i,n+1} - z_{i,n} - \frac{1}{2\tau_n} w_{i,n}\|^2,$$

which yields

$$h_{i,n+1}(x_i) - h_{i,n+1}(x_{i,n+1}) \leq -\tau_n \|x_{i,n+1} - x_i\|^2 + \tau_n \|x_i - z_{i,n}\|^2 + \langle w_{i,n}, x_{i,n+1} - x_i \rangle$$

$$= -\tau_n \|x_{i,n+1} - x_{i,n}\|^2 + \tau_n \|x_i - x_{i,n}\|^2 + \langle w_{i,n}, x_{i,n+1} - x_i \rangle$$

$$+ 2\tau_n \nu_n \langle x_{i,n+1} - x_i, x_{i,n} - x_{i,n-1} \rangle,$$

(3.1)

where the last equality follows from the fact that $z_{i,n} = x_{i,n} + \nu_n (x_{i,n} - x_{i,n-1})$. By letting $x_i = x_{i,n}$,

$$h_{i,n+1}(x_i) - h_{i,n+1}(x_{i,n+1}) \leq -\tau_n \|x_{i,n+1} - x_{i,n}\|^2 + \langle w_{i,n}, x_{i,n+1} - x_{i,n} \rangle$$

$$+ 2\tau_n \nu_n \langle x_{i,n+1} - x_{i,n}, x_{i,n} - x_{i,n-1} \rangle.$$

Since $i$ is arbitrary and $2 \langle x_{i,n+1} - x_{i,n}, x_{i,n} - x_{i,n-1} \rangle \leq \|x_{i,n+1} - x_{i,n}\|^2 + \|x_{i,n} - x_{i,n-1}\|^2$, we deduce that

$$h(x_n) - h(x_{n+1}) = \sum_{i=1}^{m} (h_{i,n+1}(x_i) - h_{i,n+1}(x_{i,n+1}))$$

$$\leq -\tau_n \|x_{n+1} - x_n\|^2 + \tau_n \nu_n \|x_{n} - x_{n-1}\|^2$$

$$+ \sum_{i=1}^{m} \langle w_{i,n}, x_{i,n+1} - x_{i,n} \rangle,$$

(3.2)

where the first equality follows from the definition of $h_{i,n+1}$.

Next, we show that

$$\langle w_{i,n}, x_{i,n+1} - x_{i,n} \rangle \leq \frac{f_i(x_{i,n+1})}{g_i(x_{i,n+1})} - \frac{f_i(x_{i,n})}{g_i(x_{i,n})} + (\tau_n - \delta) \|x_{i,n+1} - x_{i,n}\|^2.$$
To see this, let us first consider the case when \( f_i(x_{i,n}) > 0 \). Then \( w_{i,n} = 2y_{i,n} \frac{u_{i,n}}{2\sqrt{f_i(x_{i,n})}} - y_{i,n}v_{i,n} \). Since \( u_{i,n} \in \partial_L f_i(x_{i,n}) \), the assumption on \( \sqrt{f_i} \) gives

\[
(3.3) \quad \left( \frac{u_{i,n}}{2\sqrt{f_i(x_{i,n})}}, x_{i,n+1} - x_{i,n} \right) \leq \sqrt{f_i(x_{i,n+1})} - \sqrt{f_i(x_{i,n})} + \frac{\alpha_i}{2} \| x_{i,n+1} - x_{i,n} \|^2.
\]

Since \( v_{i,n} \in \partial_L g_i(x_{i,n}) \), the assumption on \( g_i \) gives

\[
(3.4) \quad \langle v_{i,n}, x_{i,n+1} - x_{i,n} \rangle \geq g_i(x_{i,n+1}) - g_i(x_{i,n}) - \frac{\beta_i}{2} \| x_{i,n+1} - x_{i,n} \|^2.
\]

Multiplying (3.3) by \( 2y_{i,n} \geq 0 \) and (3.4) by \( -y_{i,n}^2 \leq 0 \) and then adding them we obtain that

\[
(3.5) \quad \langle w_{i,n}, x_{i,n+1} - x_{i,n} \rangle \leq H_i(x_{i,n+1}, y_{i,n}) - H_i(x_{i,n}, y_{i,n}) + \frac{1}{2} (2y_{i,n} \alpha_i + y_{i,n}^2 \beta_i) \| x_{i,n+1} - x_{i,n} \|^2,
\]

where \( H_i(x, y) := 2y_i \sqrt{f_i(x)} - y_i^2 g_i(x) \). On the other hand, if \( f_i(x_{i,n}) = 0 \), then \( y_{i,n} = 0 \) and \( w_{i,n} = 0 \), hence (3.5) still holds. In turn, from (3.5) and the fact that \( y_{i,n+1} \) is the maximizer of \( H_i(x_{i,n+1}, \cdot) \), we derive that

\[
\langle w_{i,n}, x_{i,n+1} - x_{i,n} \rangle \leq H_i(x_{i,n+1}, y_{i,n}) - H_i(x_{i,n}, y_{i,n}) + \frac{1}{2} (2y_{i,n} \alpha_i + y_{i,n}^2 \beta_i) \| x_{i,n+1} - x_{i,n} \|^2
\]

where the last inequality follows by our choice of \( \tau_n \). By combining this with (3.2),

\[
(\delta - \tau_n \nu_n) \| x_{n+1} - x_n \|^2 - \tau_n \nu_n \| x_n - x_{n-1} \|^2
\]

\[
\leq \left[ h(x_{n+1}) + \sum_{i=1}^{m} \frac{f_i(x_{i,n+1})}{g_i(x_{i,n+1})} \right] - \left[ h(x_n) + \sum_{i=1}^{m} \frac{f_i(x_{i,n})}{g_i(x_{i,n})} \right].
\]

Since \( \nu_n \leq \nu/\tau_n \), we get the claimed inequality.

(ii): For all \( n \in \mathbb{N} \), set \( \theta_n := F(x_n) - \nu \| x_n - x_{n-1} \|^2 \). It follows from \( \nu \in [0, \delta/2) \) that \( \delta - 2\nu > 0 \). According to (i), for all \( n \in \mathbb{N} \),

\[
(3.6) \quad \theta_n \leq \theta_{n+1} - (\delta - 2\nu) \| x_{n+1} - x_n \|^2,
\]

and hence \( \{\theta_n\}_{n \in \mathbb{N}} \) is nondecreasing. As \( F \) is bounded from above on \( S \), there exists \( M > 0 \) such that \( \sup_{n \in \mathbb{N}} F(x_n) \leq M \). Then \( \sup_{n \in \mathbb{N}} \theta_n \leq M \), and so \( \theta_n \rightarrow \theta^* \) as \( n \rightarrow +\infty \). Let \( k \in \mathbb{N} \). Summing (3.6) from \( n = 0 \) to \( k \), we have

\[
(\delta - 2\nu) \sum_{n=0}^{k} \| x_{n+1} - x_n \|^2 \leq \theta_{k+1} - \theta_0.
\]

Letting \( k \rightarrow +\infty \), we see that \( \sum_{n=0}^{\infty} \| x_{n+1} - x_n \|^2 < +\infty \). In particular, \( \| x_{n+1} - x_n \| \rightarrow 0 \) as \( n \rightarrow +\infty \). Thus, \( F(x_n) = \theta_n + \nu \| x_n - x_{n-1} \|^2 \rightarrow \theta^* \) as \( n \rightarrow +\infty \).
Next, to see the boundedness of \((x_n)_{n \in \mathbb{N}}\), we observe from the nondecreasing property of \((\theta_n)_{n \in \mathbb{N}}\) that \(F(x_n) \geq \theta_n \geq \theta_0 = F(x_0) - \|x_0 - x_{-1}\|^2 = F(x_0)\), where the last equality follows as \(x_{-1} = x_0\). So, \((x_n)_{n \in \mathbb{N}} \subseteq \{x \in S : F(x) \geq F(x_0)\}\), and hence \((x_n)_{n \in \mathbb{N}}\) is bounded.

(iii): Let \(x = (\bar{x}_1, \ldots, \bar{x}_m)\) be any cluster point of \((x_n)_{n \in \mathbb{N}}\) and let \((x_{k_n})_{n \in \mathbb{N}}\) be a subsequence of \((x_n)_{n \in \mathbb{N}}\) such that \(x_{k_n} \to x\) as \(n \to +\infty\). Then \(x \in S\) and, by the asymptotic regularity, \(x_{k_n} \to x\) as \(n \to +\infty\). Fix any \(i \in \{1, \ldots, m\}\). By the local Lipschitz continuity of \(f_i\) and \(g_i\), we have that \(f_i(x_{i,k_n}) \to f_i(\bar{x}_i), g_i(x_{i,k_n}) \to g_i(\bar{x}_i) > 0\), and, by [24, Corollary 1.81], \((u_{i,k_n})_{n \in \mathbb{N}}\) and \((v_{i,k_n})_{n \in \mathbb{N}}\) are bounded. Noting also that

\[
\begin{align*}
&\quad \quad \quad \quad w_{i,k_n} = \begin{cases} 
\frac{g_i(x_{i,k_n})u_{i,k_n} - f_i(x_{i,k_n})v_{i,k_n}}{g_i(x_{i,k_n})^2} & \text{if } f_i(x_{i,k_n}) > 0, \\
0 & \text{if } f_i(x_{i,k_n}) = 0,
\end{cases} \\
&\quad \quad \quad \quad \in \frac{g_i(x_{i,k_n})\partial_L f_i(x_{i,k_n}) - f_i(x_{i,k_n})\partial_L g_i(x_{i,k_n})}{(g_i(x_{i,k_n}))^2},
\end{align*}
\]

one sees \((w_{i,k_n})_{n \in \mathbb{N}}\) is bounded. Passing to a subsequence if necessary, we can assume that

\[
w_{i,k_n} \to w_i \in \frac{g_i(\bar{x}_i)\partial_L f_i(\bar{x}_i) - f_i(\bar{x}_i)\partial_L g_i(\bar{x}_i)}{g_i(\bar{x}_i)^2} \quad \text{as} \quad n \to +\infty.
\]

Now, replacing \(n\) by \(k_n\) in (3.1), we have, for all \(x_i \in S_i\) and all \(n \in \mathbb{N}\), that

\[
\begin{align*}
&\quad \quad \quad \quad (3.7) \quad h_{i,k_n+1}(x_i) - h_{i,k_n+1}(\bar{x}_i, x_{i,k_n+1}) \\
&\quad \quad \quad \quad \leq -\tau_n \|x_{i,k_n+1} - x_i, x_{i,k_n}\|^2 + \tau_n \|x_i - x_{i,k_n}\|^2 + \langle w_{i,k_n}, x_{i,k_n+1} - x_i \rangle \\
&\quad \quad \quad \quad \quad \quad \quad \quad \quad + 2\tau_n \langle x_{i,k_n+1} - x_i, x_i, x_{i,k_n} - x_{i,k_n-1} \rangle.
\end{align*}
\]

We shall split the proof in two following cases.

Case 1: \(h\) is continuous on \(S \cap \text{dom } h\). Then \(\lim_{n \to +\infty} h(x_{k_n}) = h(x)\), and so \(\lim_{n \to +\infty} F(x_{k_n}) = \lim_{n \to +\infty} F(x_{k_n}) = F(x)\). Letting \(n \to +\infty\) in (3.7), we derive that, for all \(x_i \in S_i\),

\[
\begin{align*}
&\quad \quad \quad \quad h(\bar{x}_1, \ldots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \ldots, \bar{x}_m) - h(x) \leq \|x_i - \bar{x}_i\|^2 + \langle w_i, \bar{x}_i - x_i \rangle,
\end{align*}
\]

which means

\[
\begin{align*}
&\quad \quad \quad \quad \bar{x}_i \in \arg \min_{x_i \in S_i} \{-h(\bar{x}_1, \ldots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \ldots, \bar{x}_m) + \|x_i - \bar{x}_i\|^2 - \langle w_i, \bar{x}_i - x_i \rangle \}
\end{align*}
\]

\[
\begin{align*}
&\quad \quad \quad \quad = \arg \min_{x_i \in S_i} \{-h + \delta_S(\bar{x}_1, \ldots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \ldots, \bar{x}_m) + \|x_i - \bar{x}_i\|^2 - \langle w_i, x_i \rangle \}.
\end{align*}
\]

It follows that

\[
0 \in \partial^*_{L_i}(-h + \delta_S)(\bar{x}) - w_i \subseteq \partial^*_{L_i}(-h + \delta_S)(\bar{x}) + \frac{-g_i(\bar{x}_i)\partial_L f_i(\bar{x}_i) + f_i(\bar{x}_i)\partial_L g_i(\bar{x}_i)}{g_i(\bar{x}_i)^2}.
\]

As this inclusion holds for any \(i \in \{1, \ldots, m\}\), \(x\) is a lifted coordinate-wise stationary point for (P).

Case 2: \(m = 1\). Then (3.7) reduces to, for all \(x \in S\) and all \(n \in \mathbb{N}\),

\[
h(x) - h(x_{k_n+1}) \leq -\tau_n \|x_{k_n+1} - x_{k_n}\|^2 + \tau_n \|x - x_{k_n}\|^2 + \langle w_{k_n}, x_{k_n+1} - x \rangle
\]
\[ + 2\tau_k \nu_k (x_{k_n+1} - x, x_{k_n} - x_{k_n-1}), \]

where \( w_{k_n} \to \bar{w} \in \frac{g_1(x)\partial_2 f_1(x) - f_1(x)\partial_2 g_1(x)}{g_1(x)^2} \) as \( n \to +\infty \). Letting \( x = \bar{x} \) and \( n \to +\infty \), one has \( \liminf_{n \to +\infty} h(x_{k_n+1}) \geq h(\bar{x}) \), which yields \( \lim_{n \to +\infty} h(x_{k_n+1}) = h(\bar{x}) \) due to the lower semicontinuity of \(-h\). By arguing as in Case 1, \( \bar{x} \) is a lifted coordinate-wise stationary point of \((\mathcal{P})\).

Finally, if Assumption 3.7 holds, then Lemma 2.4 implies that \( \bar{x} \) is a stationary point for \((\mathcal{P})\). \( \square \)

We now comment on the assumptions imposed on the previous theorem. In particular, we see that they are quite general, and are all satisfied by our motivating examples.

**Remark 3.9 (comments on the assumptions).** In addition to Assumption 3.1, we also assume in Theorem 3.8 that the objective function \( F \) is bounded from above on the feasible set \( S \) and that \( \{ x \in S : F(x) \geq F(x_0) \} \) is bounded. These assumptions are trivially satisfied in the case when \( S \) is a compact set (as in our three motivating examples). More generally, they are also satisfied in the case when \(-F\) is a coercive function on the set \( S \) (noting that we are considering a maximization formulation), which is a standard assumption in the optimization literature.

Finally, in order to obtain that every cluster point is a stationary point, we also assume that \( \limsup_{n \to +\infty} \tau_n = \tau < +\infty \) and Assumption 3.7 holds. When \( S \) is compact, the first assumption can be easily satisfied with \( \tau_n = \delta + \max_{1 \leq i \leq m} \{ \frac{1}{2} (2y_{i,n} \alpha_i + y^2_{i,n} \beta_i) \} \) and \( \tau = \delta + \max_{1 \leq i \leq m} \{ \frac{2 \alpha_i + M_i}{m \beta_i} \} \), where \( M_i := \max_{x \in S} f_i(x) \) and \( m_i := \min_{x \in S} g_i(x) \). Also, it can be directly verified that Assumption 3.7 is satisfied by our three motivation examples in the introduction.

**Remark 3.10 (convergence to stronger stationary points).** A close inspection of the proof shows that one can obtain a stronger conclusion in Theorem 3.8 for the cluster point \( \bar{x} = (\bar{x}_1, \ldots, \bar{x}_m) \). Indeed, the cluster point \( \bar{x} \) satisfies the following stronger stationarity notion: for each \( i \in \{1, \ldots, m\} \),

\[
(3.8) \quad \bar{x}_i \in \arg\min_{x_i \in S_i} \{ -h(\bar{x}_1, \ldots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \ldots, \bar{x}_m) + \tau \| x_i - \bar{x}_i \|_2^2 - \langle w_i, x_i \rangle \}
\]

for some \( w_i \in \frac{g_i(\bar{x})\partial_2 f_i(\bar{x}) - f_i(\bar{x})\partial_2 g_i(\bar{x})}{g_i(\bar{x})^2} \). This relation implies that \( \bar{x} \) is a lifted coordinate-wise stationary point for \((\mathcal{P})\). Moreover, in the case when \( m = 1 \), and \( f_1, g_1 \) are continuously differentiable, (3.8) reduces to

\[
\bar{x} \in \text{Prox}_{\delta \bar{w} (-h + \delta \bar{w})} \left( \bar{x} + \frac{1}{2\tau} \nabla \left( \frac{f_1}{g_1} \right)(\bar{x}) \right),
\]

which corresponds to the notion of a \( L \)-stationary point with \( L = 2\tau \) [5, Definition 4.8], a stronger notion than the usual one of a stationary point.

We now consider the global convergence of the full sequence generated by Algorithm 3.3. Recall that a proper lower semicontinuous function \( f : \mathcal{H} \to (-\infty, +\infty] \) is said to satisfy the KL property [16, 19] at \( x \in \text{dom} \partial_L f \) if there exist a neighborhood \( U \) of \( x \), \( \eta \in (0, +\infty] \), and a continuous concave function \( \varphi : [0, \eta] \to \mathbb{R}_+ \) such that \( \varphi(0) = 0 \), \( \varphi \) is continuously differentiable with \( \varphi' > 0 \) on \( (0, \eta) \), and, for all \( x \in U \) with \( f(x) < f(x) - f(x) + \eta \),

\[
\varphi'(f(x) - f(x)) \text{ dist}(0, \partial_L f(x)) \geq 1.
\]
If $f$ satisfies the KL property at any $\mathbf{x} \in \text{dom} \partial_{\mathbf{x}} f$, then it is called a KL function. We say that $f$ has the KL property at $\mathbf{x} \in \text{dom} \partial_{\mathbf{x}} f$ with exponent $\alpha$ if it satisfies the KL property at $\mathbf{x} \in \text{dom} \partial_{\mathbf{x}} f$ and the corresponding function $\varphi$ (often referred to as desingularization function) can be chosen as $\varphi(s) = \gamma s^{1-\alpha}$ for some $\gamma \in \mathbb{R}_{++}$ and $\alpha \in [0, 1)$. If $f$ is a KL function and has the same exponent $\alpha$ at any $\mathbf{x} \in \text{dom} \partial_{\mathbf{x}} f$, then it is called a KL function with exponent $\alpha$.

**Theorem 3.11** (global convergence). Let $(\mathbf{x}_n)_{n \in \mathbb{N}}$ be the sequence generated by Algorithm 3.3. Suppose that Assumption 3.1 holds, that $F$ is bounded from above on $S$, that the set $\{ \mathbf{x} \in S : F(\mathbf{x}) \geq F(\mathbf{x}_0) \}$ is bounded, that, for each $i \in \{1, \ldots, m\}$, $f_i$ and $g_i$ are continuously differentiable on an open set containing $S$, and that $G(\mathbf{x}, \mathbf{u}) := -F(\mathbf{x}) + \delta_S(\mathbf{x}) + \gamma \| \mathbf{x} - \mathbf{u} \|^2$ satisfies the KL property at $\mathbf{x}$ for all $\mathbf{x} \in \text{dom} \partial_{\mathbf{x}} (-F + \delta_S)$. Suppose further that $\limsup_{n \to +\infty} \tau_n = \tau < +\infty$, that either $m = 1$ or $h$ is a differentiable function on an open set containing $S$ whose gradient is Lipschitz continuous on $S$, and that there exist $\epsilon, \ell \in \mathbb{R}_{++}$ satisfying

$$\|x_2 - x_1\| \leq \epsilon \implies \left\| \nabla \left( \frac{f_i}{g_i} \right)(x) - \nabla \left( \frac{f_i}{g_i} \right)(x') \right\| \leq \ell \|x_2 - x_1\|.$$  

Then $\sum_{n=0}^{+\infty} \|x_{n+1} - x_n\| < +\infty$ and the sequence $(\mathbf{x}_n)_{n \in \mathbb{N}}$ converges to a stationary point $\mathbf{x}^*$ for (P).

Moreover, if $G$ satisfies the KL property with exponent $\alpha \in [0, 1)$ at $(\mathbf{x}, \mathbf{x})$ for all $\mathbf{x} \in \text{dom} \partial_{\mathbf{x}} (-F + \delta_S)$, then exactly one of the following alternatives holds:

(i) (Finite convergence) $\alpha = 0$ and there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $\mathbf{x}_n = \mathbf{x}^*$.

(ii) (Linear convergence) $\alpha \in (0, \frac{1}{2}]$ and there exist $\gamma \in \mathbb{R}_{++}$ and $\rho \in (0, 1)$ such that, for all $n \in \mathbb{N}$, $\|x_n - x^*\| \leq \gamma \rho^n$ and $|F(x_n) - F(x^*)| \leq \gamma \rho^n$.

(iii) (Sublinear convergence) $\alpha \in (\frac{1}{2}, 1)$ and there exists $\gamma \in \mathbb{R}_{++}$ such that, for all $n \in \mathbb{N}$, $\|x_n - x^*\| \leq \gamma n^{-\frac{1}{2-\alpha}}$ and $|F(x_n) - F(x^*)| \leq \gamma n^{-\frac{2-2\alpha}{2-\alpha}}$.

**Proof.** Let $\mathbf{z}_n := (\mathbf{x}_{n+1}, \mathbf{x}_n)$ for $n \in \mathbb{N}$ and $\Omega$ be the set of cluster points of $(\mathbf{z}_n)_{n \in \mathbb{N}}$. We derive from Theorem 3.8 that the sequence $(\mathbf{z}_n)_{n \in \mathbb{N}}$ in $\mathbf{S} \times \mathbf{S}$ is bounded, that, for all $n \in \mathbb{N}$,

$$G(\mathbf{z}_{n+1}) + \alpha \|\mathbf{x}_{n+2} - \mathbf{x}_{n+1}\|^2 \leq G(\mathbf{z}_n) + \gamma := \delta - 2\gamma > 0,$$

and that, for all $\mathbf{z} \in \Omega$, one has $\mathbf{z} = (\mathbf{x}, \mathbf{x})$ with $\mathbf{x} \in \mathbf{S}$ being a stationary point for (P) and $F(\mathbf{x}_n) \to F(\mathbf{x})$ as $n \to +\infty$. Therefore, $\mathbf{x} \in \text{dom} \partial_{\mathbf{x}} (-F + \delta_S)$ and $G(\mathbf{z}_n) = G(\mathbf{x}_{n+1}, \mathbf{x}_n) = -F(\mathbf{x}_{n+1}) + \gamma \|\mathbf{x}_{n+1} - \mathbf{x}_n\|^2 \to -F(\mathbf{x})$ as $n \to +\infty$.

Now, for all $n \in \mathbb{N}$, since $\partial_L G(\mathbf{z}_n) = (\partial_L (-F + \delta_S)(\mathbf{x}_{n+1}) + 2\gamma(\mathbf{x}_{n+1} - \mathbf{x}_n), 2\gamma(\mathbf{x}_n - \mathbf{x}_{n+1}))^\top$, we have

$$\sqrt{\text{dist}(0, \partial_L G(\mathbf{z}_n))} = \sqrt{\text{dist}(0, \partial_L (-F + \delta_S)(\mathbf{x}_{n+1}) + 2\gamma(\mathbf{x}_{n+1} - \mathbf{x}_n))^2 + (2\gamma)^2\|\mathbf{x}_{n+1} - \mathbf{x}_n\|^2} \leq \text{dist}(0, \partial_L (-F + \delta_S)(\mathbf{x}_{n+1}) + 2\gamma(\mathbf{x}_{n+1} - \mathbf{x}_n)) + 2\gamma\|\mathbf{x}_{n+1} - \mathbf{x}_n\| \leq \text{dist}(0, \partial_L (-F + \delta_S)(\mathbf{x}_{n+1})) + 4\gamma\|\mathbf{x}_{n+1} - \mathbf{x}_n\|. $$
We shall estimate \( \text{dist}(0, \partial_L (-F + \delta_S)(x_{n+1})) \). From Step 2 of Algorithm 3.3 and noting that \( f_i, g_i \) are continuously differentiable on an open set that contains \( S_i \), we have, for all \( i \in \{1, \ldots, m\} \) and all \( n \in \mathbb{N} \), that

\[
0 \in \partial_L \left( -h_{i,n+1} + \delta_{S_i}(x_{i,n+1}) + 2\tau_n (x_{i,n+1} - z_{i,n}) - w_{i,n} \right),
\]

(3.11)

\[
\partial_L \left( -h_{i,n+1} + \delta_{S_i}(x_{i,n+1}) + 2\tau_n (x_{i,n+1} - z_{i,n}) + (w_{i,n+1} - w_{i,n}) \right),
\]

where \( w_{i,n} = g_i(x_{i,n}) - f_i(x_{i,n}) \sum_{i \neq k} g_k(x_{i,n}) - f_i(x_{i,n}) \frac{\delta_{S_i}}{g_i(x_{i,n})} - x_{i,n} \). From Step 2 of Algorithm 3.3 and noting that \( f_i, g_i \) are continuously differentiable on an open set that contains \( S \), with modulus \( L \), we have, for all \( i \in \{1, \ldots, m\} \), \( \lim_{n \to +\infty} \|x_{i,n+1} - x_{i,n}\| = 0 \) (by Theorem 3.8(ii)), there exists \( n_0 \geq 0 \) such that, for all \( n \geq n_0 \),

\[
\tau_n \leq 2\tau \quad \text{and} \quad \|x_{i,n+1} - x_{i,n}\| \leq \varepsilon.
\]

Then, for all \( i \in \{1, \ldots, m\} \) and all \( n \geq n_0 \), we derive from \( \nu_n \leq \bar{\nu}/\tau_n \) that

\[
\|2\tau_n (x_{i,n+1} - z_{i,n})\| = \|2\tau_n (x_{i,n+1} - x_{i,n}) - 2\tau_n \nu_n (x_{i,n} - x_{i,n-1})\|
\]

(3.12)

\[
\leq 4\tau \|x_{i,n+1} - x_{i,n}\| + 2\tau \|x_{i,n} - x_{i,n-1}\|
\]

and from the assumption on \( \nabla(f_i/g_i) \) that

(3.13) \[
\|w_{i,n+1} - w_{i,n}\| = \left\| \nabla \left( \frac{f_i}{g_i} \right) (x_{i,n+1}) - \nabla \left( \frac{f_i}{g_i} \right) (x_{i,n}) \right\| \leq \ell \|x_{i,n+1} - x_{i,n}\|.
\]

We split the discussion into the following cases.

Case 1: \( h \) is a differentiable function on an open set containing \( S \) whose gradient is Lipschitz continuous on \( S \) with modulus \( \ell_h \). Then, it follows from (3.11) that, for all \( i \in \{1, \ldots, m\} \) and all \( n \in \mathbb{N} \),

\[
0 \in -\nabla h_{i,n+1}(x_{i,n+1}) + \partial_L \left( \frac{f_i}{g_i} + \delta_{S_i}(x_{i,n+1}) + 2\tau_n (x_{i,n+1} - z_{i,n}) + (w_{i,n+1} - w_{i,n}) \right),
\]

which yields

\[
- [\nabla_x h(x_{n+1}) - \nabla h_{i,n+1}(x_{i,n+1})] - 2\tau_n (x_{i,n+1} - z_{i,n}) - (w_{i,n+1} - w_{i,n}) \in \partial^*_L (-F + \delta_S)(x_{n+1}).
\]

Combining with (3.12) and (3.13), we deduce that, for all \( n \geq n_0 \),

\[
\text{dist}(0, \partial_L (-F + \delta_S)(x_{n+1}))
\]

\[
\leq \sum_{i=1}^m \text{dist}(0, \partial^*_L (-F + \delta_S)(x_{n+1}))
\]

\[
\leq \sum_{i=1}^m \|\nabla_x h(x_{n+1}) - \nabla h_{i,n+1}(x_{i,n+1})\| + (4\tau + \ell) \sum_{i=1}^m \|x_{i,n+1} - x_{i,n}\|
\]

\[
+ 2\tau \sum_{i=1}^m \|x_{i,n} - x_{i,n-1}\|
\]

\[
\leq m\ell_h \|x_{n+1} - x_n\| + (4\tau + \ell) \sqrt{m} \|x_{n+1} - x_n\| + 2\tau \sqrt{m} \|x_n - x_{n-1}\|,
\]
where the last inequality holds due to the assumption that \( \nabla h \) is Lipschitz continuous with modulus \( \ell_h \) on \( S \). By using (3.10), there exists \( K \in \mathbb{R}_{++} \) such that, for all \( n \geq n_0 \),

\[
\text{dist}(0, \partial_L G(z_n)) \leq K (\|x_{n+1} - x_n\| + \|x_n - x_{n-1}\|).
\]

The first conclusion then follows by applying [9, Theorem 5.1] (with \( I = \{1, 2\} \), \( \lambda_1 = \lambda_2 = 1/2 \), \( \Delta_n = 2K\|x_{n+2} - x_{n+1}\| \), \( \alpha_n = \frac{n}{4\ell^2} > 0 \), \( \beta_n \equiv 1 \), and \( \varepsilon_n \equiv 0 \).

**Case 2:** \( m = 1 \). In this case, we derive from (3.11) that, for all \( n \in \mathbb{N} \),

\[-2\tau_n(x_{n+1} - z_n) - (w_{n+1} - w_n) \in \partial_L (-F + \delta_S)(x_{n+1}).\]

Thus, (3.12) and (3.13) imply that, for all \( n \geq n_0 \),

\[
\text{dist}(0, \partial_L (-F + \delta_S)(x_{n+1})) \leq \|2\tau_n(x_{n+1} - z_n)\| + \|w_{n+1} - w_n\|
\leq (4\tau + \ell)\|x_{n+1} - x_n\| + 2\ell\|x_n - x_{n-1}\|.
\]

Proceeding as in **Case 1**, we also obtain the first conclusion.

As the remaining conclusions are rather standard, we omit the proof here and refer the readers to [2, 9, 17, 20].

As stated in the preceding theorem, the KL exponent of the merit function for the model problem completely determines the convergence rate of the proposed algorithm. On the other hand, finding or estimating the KL exponent of a nonsmooth and nonconvex function is, in general, highly challenging. Some recent progresses in identifying KL exponents for non-fractional problems can be found in [20, 31]. In the next section, we will derive KL exponents of the corresponding merit functions for various classes of structured fractional programming problems.

### 4. KL exponents for structured fractional programs

In this section, we derive the KL exponent of the associated merit functions of three classes of structured fractional programs: sum-of-ratios fractional quadratic programs with spherical constraints, generalized eigenvalue problems with cardinality regularization and generalized eigenvalue problems with sparsity constraints. In particular, we establish that, for the last two classes of fractional programs, the KL exponent is 1/2. As a consequence, the proposed Algorithm 3.3 exhibits linear convergence for these two classes of fractional programs.

We first see that the KL exponent for the merit function associated with (P) can be computed by a merit function associated with the equivalent problem (P1). To do this, we need the following result from [20].

**Lemma 4.1** (cf. [20, Theorem 3.6]). Let \( f \) be a proper lower semicontinuous function. Suppose that \( f \) satisfies the KL property at \( \bar{x} \in \text{dom} \partial_L f \) with exponent \( \alpha \in [\frac{1}{2}, 1) \). Then, for all \( \rho \geq 0 \), \( \tilde{f}(x, u) = f(x) + \rho\|x - u\|^2 \) satisfies the KL property with exponent \( \alpha \) at \( (x, \bar{x}) \).

**Proposition 4.2.** Suppose that Assumption 3.1 holds and that, for each \( i \in \{1, \ldots, m\} \), \( f_i \) and \( g_i \) are continuously differentiable on \( S_i \). Let \( P(x, y) = -h(x) - H(x, y) + \delta_S(x) \), where \( H(x, y) = \sum_{i=1}^m [2y_i \sqrt{f_i(x_i)} - y_i^2 g_i(x_i)] \). Let \( \bar{x} = (\bar{x}_1, \ldots, \bar{x}_m) \in S \) and \( \bar{y} = (\bar{y}_1, \ldots, \bar{y}_m) \in \mathbb{R}^m \) with \( \bar{y}_i = \frac{\sqrt{f_i(\bar{x})}}{g_i(\bar{x})} \). Suppose further that \( h \) is continuous around \( \bar{x} \) and that \( P \) satisfies the KL property with exponent \( \alpha \in [0, 1) \) at
We now consider the following sum-of-ratios fractional quadratic program
\[
\begin{align*}
&\text{max} \quad \sum_{i=1}^{m} \frac{f_i(x_i)}{g_i(x_i)} + \delta_S(x) \\
&\text{s.t.} \quad x_i \geq 0, \quad i = 1, \ldots, m,
\end{align*}
\]
satisfies the KL property with exponent \( \alpha \) at \( \mathbf{x} \). In particular, for all \( \rho \geq 0 \),
\[
G(x, u) := -h(x) - \sum_{i=1}^{m} \frac{f_i(x_i)}{g_i(x_i)} + \rho \|x - u\|^2 + \delta_S(x)
\]
satisfies the KL property with exponent \( \alpha' = \max\{\alpha, \frac{1}{2}\} \) at \( (\mathbf{x}, \mathbf{y}) \).

Proof. As \( P \) satisfies the KL property with exponent \( \alpha \in [0, 1) \) at \( (\mathbf{x}, \mathbf{y}) \in \mathcal{H} \times \mathbb{R}^m \),
there exist \( \delta, \eta, c > 0 \) such that, for all \( (\mathbf{x}, \mathbf{y}) \) with \( \| (\mathbf{x}, \mathbf{y}) - (\mathbf{x}, \mathbf{y}) \| \leq \delta \) and
\( P(\mathbf{x}, \mathbf{y}) < P(\mathbf{x}, \mathbf{y}) + \eta \), one has
\[
\text{dist}(0, \partial L P(\mathbf{x}, \mathbf{y})) \leq \delta \quad \text{and} \quad P(\mathbf{x}, \mathbf{y}) < P(\mathbf{x}, \mathbf{y}) + \eta.
\]
It follows that, for all \( (\mathbf{x}, \mathbf{y}) \) with \( \| (\mathbf{x}, \mathbf{y}) - (\mathbf{x}, \mathbf{y}) \| \leq \delta \) and \( P(\mathbf{x}, \mathbf{y}) < P(\mathbf{x}, \mathbf{y}) + \eta \),
\[
(4.1) \quad \| \text{dist}(0, \partial L P(\mathbf{x}, \mathbf{y})) \|^\frac{1}{2} \geq c^\frac{1}{2} [P(\mathbf{x}, \mathbf{y}) - P(\mathbf{x}, \mathbf{y})].
\]
Here, we drop the condition \( P(\mathbf{x}, \mathbf{y}) < P(\mathbf{x}, \mathbf{y}) \) because (4.1) trivially holds otherwise.

For each \( \mathbf{x} \), let \( y_x = (y_{1, x}, \ldots, y_{m, x}) \) with \( y_{i, x} = \sqrt{\frac{f_i(x_i)}{g_i(x_i)}} \) for \( i = 1, \ldots, m \). Then \( y_x = \mathbf{y} \).
Moreover, by the continuity of \( H \) and \( h \), there exists \( \delta_1 \in (0, \delta) \) such that
for all \( \mathbf{x} \in \mathcal{S} \) with \( \|\mathbf{x} - \mathbf{x}\| \leq \delta_1 \) one has
\( \| (\mathbf{x}, y_x) - (\mathbf{x}, \mathbf{y}) \| \leq \delta \) and \( P(\mathbf{x}, y_x) < P(\mathbf{x}, \mathbf{y}) + \eta \).
Therefore, from (4.1) we derive that, for all \( \mathbf{x} \in \mathcal{S} \) with \( \|\mathbf{x} - \mathbf{x}\| \leq \delta_1 \),
\[
\| \text{dist}(0, \partial L P(\mathbf{x}, y_x)) \|^{\frac{1}{2}} \geq c^\frac{1}{2} [h(x) - H(\mathbf{x}, y_x) + h(\mathbf{x}) + H(\mathbf{x}, \mathbf{y})]
\]
\[
= c^\frac{1}{2} \left[ -h(x) - \sum_{i=1}^{m} \frac{f_i(x_i)}{g_i(x_i)} + h(\mathbf{x}) + \sum_{i=1}^{m} \frac{f_i(x_i)}{g_i(x_i)} \right].
\]
Now, we notice that \( \partial L P(\mathbf{x}, y_x) = (\partial L (-h + \delta_S)(\mathbf{x}) + \partial L S(-H)(\mathbf{x}, y_x),\partial L P(-H)(\mathbf{x}, y_x)) \)
and that, for each \( i \in \{1, \ldots, m\} \),
\[
\partial^L g_i(x_i, y_{i, x}) = \frac{-g_i(x_i) \sum f_i(x_i) + f_i(x_i) \sum g_i(x_i)}{[g_i(x_i)]^2} = \nabla \left( \frac{f_i(x_i)}{g_i(x_i)} \right) (x_i)
\]
and \( \partial^L H_i(x_i, y_{i, x}) = 0 \).
Therefore, \( \partial L P(\mathbf{x}, y_x) = (\partial L P(\mathbf{x}), 0) \), and from here we deduce that, for all \( \mathbf{x} \in \mathcal{S} \)
with \( \|\mathbf{x} - \mathbf{x}\| \leq \delta_1 \),
\[
\| \text{dist}(0, \partial L P(\mathbf{x})) \|^{\frac{1}{2}} \geq c^\frac{1}{2} [\Phi(\mathbf{x}) - \Phi(\mathbf{x})].
\]
So, \( \Phi \) satisfies the KL property with exponent \( \alpha \) at \( \mathbf{x} \), and hence, also with exponent
\( \max\{\alpha, \frac{1}{2}\} \). By using Lemma 4.1, \( G \) satisfies the KL property with exponent
\( \max\{\alpha, \frac{1}{2}\} \) at \( (\mathbf{x}, \mathbf{x}) \). \( \square \)

4.1. Sum-of-ratios fractional quadratic programs with spherical constraint. We now consider the following sum-of-ratios fractional quadratic program
\[
\begin{align*}
\text{(FQP)} \quad \max \quad & \sum_{i=1}^{m} x_i^T A_i x_i + a_i^T x + \sum_{i=1}^{m} x_i^T A_i x_i \\
& \text{s.t.} \quad x_i \geq 0, \quad i = 1, \ldots, m,
\end{align*}
\]
where $A_0$ is a symmetric matrix and, for each $i \in \{1, \ldots, m\}$, $A_i$ and $B_i$ are positive definite matrices. In the special cases of $m = 1$ and $A_0 = 0$, this reduces to the problem of maximizing the sum of a quadratic function and the Rayleigh quotient over the unit sphere (motivating Example 1.3). For this sum-of-ratios fractional quadratic form of maximizing the sum of a quadratic function and the Rayleigh quotient over the unit sphere, we define $\Lambda_i = \{x_i \in \mathbb{R}^d : \|x_i\| = 1\}$, $i \in \{1, \ldots, m\}$, and $\rho \geq 0$. We shall investigate the KL exponent of this merit function. To this end, we use a fundamental result which provides an exponent estimate in the classical Lojasiewicz gradient inequality for polynomials.

**Lemma 4.3** (Lojasiewicz gradient inequality [11, Theorem 4.2]). Let $f$ be a polynomial on $\mathbb{R}^d$ with degree $p \in \mathbb{N}$. Suppose that $f(\mathbf{x}) = 0$. Then there exist constants $\varepsilon, c > 0$ such that, for all $x \in \mathbb{R}^d$ with $\|x - \mathbf{x}\| \leq \varepsilon$, we have

$$\|\nabla f(x)\| \geq c|f(x)|^{1-\tau}, \quad \text{where} \quad \tau = \mathcal{R}(d, p)^{-1} \quad \text{and} \quad \mathcal{R}(d, p) := \begin{cases} 1 & \text{if } p = 1, \\ p(3p - 3)^d & \text{if } p \geq 2. \end{cases}$$

**Theorem 4.4.** Let $\Lambda = \Lambda_1 \times \cdots \times \Lambda_m$, where $\Lambda_i = \{x_i \in \mathbb{R}^d_i : \|x_i\| = 1\}$, $i \in \{1, \ldots, m\}$, with $\sum_{i=1}^m d_i = d$. Consider

$$\Phi(x) = -[x^T A_0 x + a_0^T x] - \sum_{i=1}^m \frac{x_i^T A_i x_i}{x_i^T B_i x_i} + \delta_\Lambda(x),$$

where $A_0$ is a symmetric matrix and, for each $i \in \{1, \ldots, m\}$, $A_i$ and $B_i$ are positive definite matrices. Then $\Phi$ satisfies the KL property with exponent $1 - \tau$, where $\tau = (\mathcal{R}(d + 3m + md, 4))^{-1}$. In particular, for all $\rho \geq 0$,

$$\hat{\Phi}_{FQP}(x, u) = -[x^T A_0 x + a_0^T x] - \sum_{i=1}^m \frac{x_i^T A_i x_i}{x_i^T B_i x_i} + \delta_\Lambda(x) + \rho\|x - u\|^2$$

satisfies the KL property with exponent $1 - \tau$ at $(\mathbf{x}, \mathbf{y})$ for all $\mathbf{x} \in \text{dom} \partial L \Phi$.

**Proof.** From Proposition 4.2 with $S = \Lambda$, $h(x) = x^T A_0 x + a_0^T x$, $f_i(x_i) = x_i^T A_i x_i$, and $g_i(x_i) = x_i^T B_i x_i$, $i \in \{1, \ldots, m\}$, it suffices to show that

$$P(x, y) = -[x^T A_0 x + a_0^T x] - \sum_{i=1}^m \left[ y_i \sqrt{x_i^T A_i x_i} - y_i^2 x_i^T B_i x_i \right] + \delta_\Lambda(x)$$

satisfies the KL property with exponent $1 - \tau$ at $(\mathbf{x}, \mathbf{y}) \in \Lambda \times \mathbb{R}^m$. To do this, let $(\mathbf{x}, \mathbf{y}) \in \Lambda \times \mathbb{R}^m$ and let $\delta, \eta > 0$ be such that, for all $\|(x, y) - (\mathbf{x}, \mathbf{y})\| \leq \delta$, one has $P(\mathbf{x}, \mathbf{y}) < P(x, y) < P(\mathbf{x}, \mathbf{y}) + \eta$. Let $L_i = A_i^{1/2}$ for $i \in \{1, \ldots, m\}$. We can write $P$ as

$$P(x, y) = -[x^T A_0 x + a_0^T x] - \sum_{i=1}^m \left[ y_i \|L_i x_i\| - y_i^2 x_i^T B_i x_i \right] + \sum_{i=1}^m \delta_\Lambda(x_i).$$
As $A_i$ is positive definite, we have $L_i x_i \neq 0$ for all $x_i \in A_i$ and all $i \in \{1, \ldots, m\}$. Let $f_0(x) = x^\top A_0 x + a_0^\top x$. Then, for all $i \in \{1, \ldots, m\}$,

$$\partial_{L_i}^P \mathcal{P}(x, y) = \left\{ -\nabla_x f_0(x) - y_i \frac{A_i x_i}{\sqrt{x_i^\top A_i x_i}} + 2y_i^2 B_i x_i + t_i x_i : t_i \in \mathbb{R} \right\}$$

(Using Lemma 2.2(ii))

and $\partial_{L_i}^P \mathcal{P}(x, y) = -\|L_i x_i\| + 2y_i x_i^\top B_i x_i$,

which imply that

$$\text{dist}(0, \partial_L \mathcal{P}(x, y))^2 = \sum_{i=1}^m \inf_{t_i \in \mathbb{R}} \left\{ \left\| -\nabla_x f_0(x) - y_i \frac{A_i x_i}{\sqrt{x_i^\top A_i x_i}} + 2y_i^2 B_i x_i + t_i x_i \right\|^2 \right\}$$

$$+ \sum_{i=1}^m (-\|L_i x_i\| + 2y_i x_i^\top B_i x_i)^2.$$ 

For all $(x, y) \in \Lambda \times \mathbb{R}^m$ and all $i \in \{1, \ldots, m\}$, one has $\|x_i\| = 1$, and so,

$$\left\| -\nabla_x f_0(x) - y_i \frac{A_i x_i}{\sqrt{x_i^\top A_i x_i}} + 2y_i^2 B_i x_i + t_i x_i \right\|^2$$

$$= \left\| -\nabla_x f_0(x) - y_i \frac{A_i x_i}{\sqrt{x_i^\top A_i x_i}} + 2y_i^2 B_i x_i \right\|^2$$

$$+ 2t_i x_i^\top \left( -\nabla_x f_0(x) - y_i \frac{A_i x_i}{\sqrt{x_i^\top A_i x_i}} + 2y_i^2 B_i x_i \right) + t_i^2,$$

from which we have

$$(4.3) \quad \text{dist}(0, \partial_L \mathcal{P}(x, y))^2 = \sum_{i=1}^m \left\| -\nabla_x f_0(x) - y_i \frac{A_i x_i}{\sqrt{x_i^\top A_i x_i}} + 2y_i^2 B_i x_i + t_{x_i y_i} x_i \right\|^2$$

$$+ \sum_{i=1}^m (-\|L_i x_i\| + 2y_i x_i^\top B_i x_i)^2,$$

where $t_{x_i y_i} := x_i^\top \left( -\nabla_x f_0(x) \right) + y_i x_i^\top A_i x_i + 2y_i^2 x_i^\top B_i x_i$.

Now, let us consider $f : \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{md} \times \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}$ defined by

$$f(x, y, u, \lambda, \mu) = -f_0(x) - \sum_{i=1}^m \left[ y_i (L_i x_i)^\top u_i - y_i^2 x_i^\top B_i x_i \right]$$

$$+ \sum_{i=1}^m \lambda_i(\|u_i\|^2 - 1) + \sum_{i=1}^m \mu_i(\|x_i\|^2 - 1),$$

and let $\hat{f} = f - r$, where $r = f(x, y, u, \lambda, \mu)$ with $u_i = \frac{t_i x_i}{\|L_i x_i\|}$, $\lambda_i = \frac{\|L_i x_i\|}{2}$ and $\mu_i = \frac{t_i x_i}{2}$ for all $i \in \{1, \ldots, m\}$. Clearly, $\hat{f}$ is a polynomial on $\mathbb{R}^{d+3m+md}$ of degree 4. By Lemma 4.3, there exist $\delta_0 > 0$ and $c > 0$ such that, for all $(x, y, u, \lambda, \mu)$ with $\|x, y, u, \lambda, \mu \| < \delta_0$, $\|\nabla f(x, y, u, \lambda, \mu)\| = \|\nabla \hat{f}(x, y, u, \lambda, \mu)\| \geq c|\hat{f}(x, y, u, \lambda, \mu)|^{1 - \tau}$
where \( \tau = \left[ R(d + 3m + md, 4) \right]^{-1} \). Let \( u_x = (u_{1,x}, \ldots, u_{m,x}) \), \( \lambda_{x,y} = (\lambda_{1,x,y}, \ldots, \lambda_{m,x,y}) \), and \( \mu_{x,y} = (\mu_{1,x,y}, \ldots, \mu_{m,x,y}) \) with, for all \( i \in \{1, \ldots, m\} \),

\[
    u_{i,x} = \frac{L_i x_i}{\|L_i x_i\|}, \quad \lambda_{i,x,y} = \frac{y_i \|L_i x_i\|}{2}, \quad \text{and} \quad \mu_{i,x,y} = \frac{t_{x_i,y_i}}{2}.
\]

Shrinking \( \delta > 0 \) if necessary, we can assume that, for all \((x, y) \in \Lambda \times \mathbb{R}^m\) with \( \|(x, y) - (\overline{x}, \overline{y})\| \leq \delta \),

\[
    \|(x, y, u_x, \lambda_{x,y}, \mu_{x,y}) - (\overline{x}, \overline{y}, \mu, \lambda)\| \leq \delta_0,
\]

which implies \( \| \nabla f(x, y, u_x, \lambda_{x,y}, \mu_{x,y}) \|^2 \geq c^2 \| f(x, y, u_x, \lambda_{x,y}, \mu_{x,y}) - f(x, y, \mu, \lambda) \|^{2 (1 - \tau)} \).

Note that, for all \( i \in \{1, \ldots, m\} \),

\[
    \begin{aligned}
    \nabla_x f(x, y, u, \lambda, \mu) &= -\nabla_x f_0(x) - (y_i L_i^\top u_i - 2 y_i^2 B_i x_i) + 2 \mu_i x_i, \\
    \nabla_y f(x, y, u, \lambda, \mu) &= -(L_i x_i)^\top u_i + 2 y_i x_i^\top B_i x_i, \\
    \nabla_u f(x, y, u, \lambda, \mu) &= -y_i L_i x_i + 2 \lambda_i u_i, \\
    \nabla_{\lambda_i} f(x, y, u, \lambda, \mu) &= \|u_i\|^2 - 1, \\
    \nabla_{\mu_i} f(x, y, u, \lambda, \mu) &= \|x_i\|^2 - 1.
    \end{aligned}
\]

Direct verification shows that, for all \((x, y) \in \Lambda \times \mathbb{R}^m\) with \( \|(x, y) - (\overline{x}, \overline{y})\| \leq \delta \) and all \( i \in \{1, \ldots, m\} \), one has

\[
    \begin{aligned}
    \nabla_x f(x, y, u_x, \lambda_{x,y}, \mu_{x,y}) &= -\nabla_x f(x) - \left( y_i \frac{\lambda_{i,x,y}}{\sqrt{A_i x_i}} - 2 y_i^2 B_i x_i \right) + t_{x_i,y_i} x_i, \\
    \nabla_y f(x, y, u_x, \lambda_{x,y}, \mu_{x,y}) &= -\|L_i x_i\|^2 + 2 y_i x_i^\top B_i x_i, \\
    \nabla_u f(x, y, u_x, \lambda_{x,y}, \mu_{x,y}) &= 0, \\
    \nabla_{\lambda_i} f(x, y, u_x, \lambda_{x,y}, \mu_{x,y}) &= 0, \\
    \nabla_{\mu_i} f(x, y, u_x, \lambda_{x,y}, \mu_{x,y}) &= 0
    \end{aligned}
\]

and also

\[
    f(x, y, u_x, \lambda_{x,y}, \mu_{x,y}) = P(x, y) \quad \text{and} \quad f(x, y, \mu, \lambda) = P(x, y) .
\]

These together with (4.3) imply that, for all \((x, y) \in \Lambda \times \mathbb{R}^m\) with \( \|(x, y) - (\overline{x}, \overline{y})\| \leq \delta \) and \( P(x, y) < P(x, y) < P(x, y) + \eta \),

\[
    \text{dist}(0, \partial_L P(x, y)) \geq c |P(x, y) - P(x, y)|^{1 - \tau} = c |P(x, y) - P(x, y)|^{1 - \tau} .
\]

Thus, \( P \) satisfies the KL property with exponent \( 1 - \tau \), and the conclusion follows. \hfill \Box

### 4.2. Generalized eigenvalue problem with cardinality regularization.

Consider the generalized eigenvalue problem with cardinality regularization

\[
    \text{(GEP)} \quad \max_{x \in \mathbb{R}^d} \frac{x^\top A_1 x}{x^\top B_1 x} - \lambda \|x\|_0 \quad \text{s.t.} \quad \|x\| = 1 ,
\]

where \( A_1, B_1 \) are symmetric matrices such that \( A_1 \) is positive semidefinite and \( B_1 \) is positive definite, and \( \lambda > 0 \). For this generalized eigenvalue problem with cardinality
regularization the corresponding merit function for the proposed Algorithm 3.3 takes the form

$$\hat{\Phi}_{GEP}(x,u) = \frac{x^T A x}{x^T B x} + \lambda \|x\|_0 + \delta_\lambda(x) + \rho \|x - u\|^2,$$

with $A = -A_1$ a symmetric matrix, $B = B_1$ is a positive definite matrix, $\Lambda = \{x \in \mathbb{R}^d : \|x\| = 1\}$, and $\rho \geq 0$. Below, we derive the KL exponent of the merit function $\hat{\Phi}_{GEP}$. To this end, we will use the following lemma from [20]. Here we provide an alternative short proof for it.

**Lemma 4.5.** Let $Q$ be a symmetric $d \times d$ matrix. Then there exists $c > 0$ such that, for all $x \in \mathbb{R}^d$, $\|Qx\|^2 \geq c(x^T Q x)$.

**Proof.** Let $Q = U^T \Sigma U$ where $U$ is an orthonormal matrix and $\Sigma = \text{diag}(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix whose diagonal elements are the eigenvalues of $Q$ with $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Let $x \in \mathbb{R}^d$, $y := Ux$, and $I_0 = \{j : \lambda_j \neq 0\}$. Then $x^T Q x = \sum_{j=1}^N \lambda_j y_j^2 = \sum_{j \in I_0} \lambda_j y_j^2$ and

$$\|Qx\|^2 = x^T (Q^T Q)x = (Ux)^T \Sigma^2 (Ux) = \sum_{j=1}^N \lambda_j y_j^2 = \sum_{j \in I_0} \lambda_j y_j^2.$$ Setting $c := \min\{|\lambda_j| : j \in I_0\}$, we see that

$$c(x^T Q x) = \sum_{j \in I_0} c|\lambda_j| y_j^2 \leq \sum_{j \in I_0} c|\lambda_j| y_j^2 \leq \sum_{j \in I_0} |\lambda_j|^2 y_j^2 = \|Qx\|^2,$$

which completes the proof. \hfill \Box

Next we prove that the KL exponent of the merit function $\hat{\Phi}_{GEP}$ is $\frac{1}{2}$. To do this, for an index set $J = \{j_1, \ldots, j_k\} \subseteq \{1, \ldots, d\}$ with $k \leq d$, we denote $x_J := (x_{j_1}, \ldots, x_{j_k})$. Moreover, for two index sets $I, J$, we denote $A_{IJ} = (A_{ij})_{i \in I, j \in J}$.

**Theorem 4.6.** Consider the function $\Phi(x) = \frac{x^T A x}{x^T B x} + \lambda \|x\|_0 + \delta_\lambda(x)$, where $\Lambda = \{x : \|x\| = 1\}$, $A, B$ are symmetric matrices with $B$ positive definite, and $\lambda > 0$. Then $\Phi$ is a KL function with exponent $\frac{1}{2}$. In particular, for all $\rho \geq 0$,

$$\hat{\Phi}_{GEP}(x,u) = \frac{x^T A x}{x^T B x} + \lambda \|x\|_0 + \delta_\lambda(x) + \rho \|x - u\|^2$$

satisfies the KL property with exponent $\frac{1}{2}$ at $(x,x)$ for all $x \in \text{dom} \partial_L \Phi$.

**Proof.** Take any $x \in \text{dom} \partial_L \Phi$. Then $x \in \Lambda$. Let $J = \text{supp}(x)$ and use $|J|$ to denote the cardinality of $J$. Choose $\eta \in (0,1)$ such that, for all $\|x - \bar{x}\| < \eta$,

$$\left| \frac{x^T A x}{x^T B x} - \frac{x^T A \bar{x}}{x^T B \bar{x}} \right| = \frac{\lambda}{4} \text{ and } \eta < \frac{\lambda}{4}.$$ Let $x$ with $\|x - \bar{x}\| < \eta$ and $\Phi(x) < \Phi(\bar{x}) < \Phi(x) + \eta$. We first see that, by shrinking $\eta$ if necessary, one can assume that

$$J = \text{supp}(\bar{x}) = \text{supp}(x).$$ Indeed, by continuity and by shrinking $\eta$ if necessary, one has $\text{supp}(\bar{x}) \subseteq \text{supp}(x)$. Suppose that $\text{supp}(\bar{x}) \subseteq \text{supp}(x)$. Then $\|x\|_0 > \|\bar{x}\|_0$, and so, $\|\bar{x}\|_0 \geq \|\bar{x}\|_0 + 1$. From
our choice of \( x \), one has \( x \in \Lambda \) and
\[
\frac{x^\top A x}{x^\top B x} + \lambda \| x \|_0 < \frac{x^\top A x}{x^\top B x} + \lambda \| x \|_0 < \frac{x^\top A x}{x^\top B x} + \lambda \| x \|_0 + \eta.
\]
This shows that \( \| x \|_0 < \frac{1}{2} + \| x \|_0 \), which is impossible.

Using Lemma 2.2 and noting that \( J = \text{supp}(x) \), we derive that
\[
\partial_{L} \Phi(x) \subseteq \left\{ \frac{2A x (x^\top B x) - 2B x (x^\top A x)}{(x^\top B x)^2} + \lambda v + t x : t \in \mathbb{R},
\begin{align*}
v_j &= 0 \text{ if } j \in J, \text{ and } v_j \in \mathbb{R} \text{ if } j \notin J \right\}
\]
Denoting \([a]_J = (a_j)_{j \in J} \in \mathbb{R}^{|J|} \), this implies that
\[
\text{dist}(0, \partial_{L} \Phi(x)) \geq \inf_{t \in \mathbb{R}} \left\| \frac{2A x (x^\top B x) - 2B x (x^\top A x)}{(x^\top B x)^2} \right\|_J + t x_J.
\]
A direct verification shows that
\[
x^\top \left( \frac{2A x (x^\top B x) - 2B x (x^\top A x)}{(x^\top B x)^2} \right) = 0,
\]
which, together with \( J = \text{supp}(x) \), implies that
\[
(x_J)^\top \left( \frac{2A x (x^\top B x) - 2B x (x^\top A x)}{(x^\top B x)^2} \right)_J = 0.
\]
Therefore,
\[
\text{dist}(0, \partial_{L} \Phi(x)) \geq \left\| \frac{2A x (x^\top B x) - 2B x (x^\top A x)}{(x^\top B x)^2} \right\|_J = \frac{2}{x^\top B x} \| [A x]_J - \frac{x^\top A x}{x^\top B x} [B x]_J \|.
\]
Using \( J = \text{supp}(x) \) again, we have that
\[
[A x]_J = A_{J J} x_J, \ x^\top A x = (x_J)^\top A_{J J} x_J, \ [B x]_J = B_{J J} x_J, \text{ and } x^\top B x = (x_J)^\top B_{J J} x_J,
\]
and hence
\[
\text{dist}(0, \partial_{L} \Phi(x)) \geq \frac{2}{x^\top B x} \left\| A_{J J} x_J - \frac{x^\top A x}{x^\top B x} B_{J J} x_J \right\|.
\]
Now, let \( q(z) = z^\top A_{J J} z - \frac{x^\top A x}{x^\top B x} z^\top B_{J J} z \) for \( z \in \mathbb{R}^{|J|} \). Then
\[
\text{dist}(0, \partial_{L} \Phi(x)) \geq \frac{2}{x^\top B x} \left\| A_{J J} x_J - \frac{x^\top A x}{x^\top B x} B_{J J} x_J \right\|
\geq \frac{2}{x^\top B x} \left( \left\| A_{J J} x_J - \frac{x^\top A x}{x^\top B x} B_{J J} x_J \right\| - \left\| \frac{x^\top A x}{x^\top B x} - \frac{x^\top A x}{x^\top B x} \right\| B_{J J} x_J \right)
= \frac{1}{x^\top B x} \| \nabla q(x_J) \| - \frac{2}{x^\top B x} \| \Phi(x) - \Phi(x_J) \| B_{J J} x_J \|,
\]
where the second inequality follows from the triangle inequality and the last equality holds as \( x, \bar{x} \in \Lambda \) and \( J = \text{supp}(x) = \text{supp}(\bar{x}) \) (and so, \( \| x \|_0 = \| \bar{x} \|_0 \)).
From Lemma 4.5, there exists $c > 0$ such that, for all $\mathbf{z}$, $\|\nabla q(\mathbf{z})\|_2^2 \geq c q(\mathbf{z})$. Indeed, one can set $c := \min_{1 \leq j \leq |J|} \left\{ 4 \lambda_j (A_{JJ} - \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T B \mathbf{x}} B_{JJ}) : \lambda_j (A_{JJ} - \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T B \mathbf{x}} B_{JJ}) \neq 0 \right\}$, where $\lambda_j$ are the eigenvalues of a matrix $Q$. Noting that

$$q(x_j) = \left[ (x_j)^T A_{JJ} x_j - \frac{x_j^T A x_j}{x_j^T B x_j} B_{JJ} (x_j) \right] \frac{1}{x_j^T B x_j}$$

one has

$$\text{dist}(0, \partial L \Phi(x)) \geq \frac{\sqrt{c} q(x_j)^{1/2}}{x_j^T B x_j} - 2 |\Phi(x) - \Phi(\mathbf{z})| \frac{\|B_{JJ} x_j\|}{x_j^T B x_j}$$

$$= [\Phi(x) - \Phi(\mathbf{z})]^{1/2} \left( \frac{\sqrt{c}}{\sqrt{x_j^T B x_j}} - 2 [\Phi(x) - \Phi(\mathbf{z})]^{1/2} \frac{\|B_{JJ} x_j\|}{x_j^T B x_j} \right).$$

Let $c_1 := \min \{ \sqrt{x_j^T B x_j} : x \in \Lambda \}$ and $c_2 := \max \{ \sqrt{x_j^T B x_j} : x \in \Lambda \}$. By shrinking $\eta$ if necessary, we can assume that $\eta \in (0, 1)$ and

$$2 |\Phi(x) - \Phi(\mathbf{z})|^{1/2} \frac{\|B_{JJ} x_j\|}{x_j^T B x_j} \leq 2 \eta^{1/2} \frac{\|B_{JJ} x_j\|}{c_1} \leq \frac{\sqrt{c}}{2 c_2},$$

where the first inequality follows by the fact $\Phi(\mathbf{z}) < \Phi(x) < \Phi(\mathbf{z}) + \eta$. Then, we see that

$$\text{dist}(0, \partial L \Phi(x)) \geq [\Phi(x) - \Phi(\mathbf{z})]^{1/2} \left( \frac{\sqrt{c}}{2 c_2} \right) = \frac{\sqrt{c} [\Phi(x) - \Phi(\mathbf{z})]^{1/2}}{2 c_2}.$$

Thus, $\Phi$ satisfies the KL property with exponent $\frac{1}{2}$. This shows that, according to Lemma 4.1, $\tilde{\Phi}_{GEP}$ satisfies the KL property with exponent $\frac{1}{2}$ at $\mathbf{x}$ for all $\mathbf{x} \in \text{dom} \partial L \Phi$. \hfill \square

### 4.3. Generalized eigenvalue problem with sparsity constraint.

Consider the generalized eigenvalue problem with sparsity constraint

$$(\text{GEPS}) \quad \max_{\mathbf{x} \in \mathbb{R}^d} \frac{\mathbf{x}^T A_1 \mathbf{x}}{\mathbf{x}^T B_1 \mathbf{x}} \quad \text{s.t.} \quad \|\mathbf{x}\| = 1, \|\mathbf{x}\|_0 \leq r,$$

where $A_1, B_1$ are symmetric matrices such that $A_1$ is positive semidefinite and $B$ is positive definite, and $r > 0$. For this generalized eigenvalue problem with sparsity constraint, the corresponding merit function for the proposed Algorithm 3.3 takes the form

$$\tilde{\Phi}_{GEP}(\mathbf{x}, \mathbf{u}) = \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T B \mathbf{x}} + \delta_{\Lambda \cap C_r}(\mathbf{x}) + \rho \|\mathbf{x} - \mathbf{u}\|^2,$$

where $A = -A_1$ is a symmetric matrix, $B = B_1$ is a positive definite matrix, $\Lambda = \{ \mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| = 1 \}$, $C_r = \{ \mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\|_0 \leq r \}$ with $r > 0$, and $\rho \geq 0$. Below, we investigate the KL exponent for this merit function.
Theorem 4.7. Consider the function \( \Phi(x) = \frac{x^TAx}{x^TBx} + \delta_{\Lambda \cap C_r}(x) \), where \( \Lambda = \{ x \in \mathbb{R}^d : \|x\| = 1 \} \), \( C_r = \{ x \in \mathbb{R}^d : \|x\|_0 \leq r \} \) and \( A, B \) are symmetric matrices with \( B \) positive definite. Then \( \Phi \) is a KL function with exponent \( \frac{1}{2} \). In particular, for all \( \rho \geq 0 \),

\[
\hat{\Phi}_{GEP}(x, u) = \frac{x^TAx}{x^TBx} + \delta_{\Lambda \cap C_r}(x) + \rho\|x - u\|^2
\]

satisfies the KL property with exponent \( \frac{1}{2} \) at \((x, x)\) for all \( x \in \text{dom} \partial L \).

Proof. Take any \( \overline{x} \in \Lambda \cap C_r \). We split the proof into two cases: \( \|\overline{x}\|_0 = r \) and \( \|\overline{x}\|_0 < r \).

Case 1: \( \|\overline{x}\|_0 = r \). Let \( \delta > 0 \) and take any \( x \in \Lambda \cap C_r \) with \( \|x - \overline{x}\| \leq \delta \). By shrinking \( \delta \) if necessary, we have \( \text{supp}(x) \subseteq \text{supp}(\overline{x}) \). So, \( \|x\|_0 \geq \|\overline{x}\|_0 = r \). As \( x \in C_r \), we see that \( \|x\| = \|\overline{x}\| = r \) and so, \( \text{supp}(x) = \text{supp}(\overline{x}) \). Then, a similar line of argument as in Theorem 4.6 gives the desired conclusion.

Case 2: \( \|\overline{x}\|_0 < r \). Let \( I = \{ i \in \{1, \ldots, n\} : \text{supp}(x) \subseteq I \} \). Clearly, \( |I| < +\infty \). Let \( \delta > 0 \) and take any \( x \in \Lambda \cap C_r \) with \( \|x - \overline{x}\| \leq \delta \). By shrinking \( \delta \) if necessary, we have \( \text{supp}(x) \subseteq \text{supp}(\overline{x}) \), and so, \( J_x := \text{supp}(x) \in I \). Let \( x \in C_r \) with \( \|x - \overline{x}\| < \eta \) and \( \Phi(x) < \Phi(\overline{x}) + \eta \). From our choice of \( x \), one has \( x \in \Lambda \). Moreover, using Lemma 2.2, a direct computation gives us that

\[
\partial L \Phi(x) \subseteq \left\{ \begin{array}{ll}
2Ax(x^TBx) - 2Bx(x^TAx) & (x^T Bx)^2 \\
\lambda v + tx : t \in \mathbb{R}, \widehat{J} \subseteq \{1, \ldots, n\} \setminus J_x,
\end{array} \right. 
\]

\[
|\widehat{J}| = r - |J_x|, v_i = 0 \text{ if } i \in J_x \cup \widehat{J}, \text{ and } v_i \in \mathbb{R} \text{ if } i \not\in \text{supp}(x) \cup \widehat{J}.
\]

It follows from \( x^T \left( \frac{2Ax(x^TBx) - 2Bx(x^TAx)}{(x^T Bx)^2} \right) = 0 \) that

\[
x_{J_x \cup \widehat{J}} \left( \left[ \begin{array}{c}
\frac{2Ax(x^TBx) - 2Bx(x^TAx)}{(x^T Bx)^2} \\
J_x \cup \widehat{J}
\end{array} \right] \right) = 0.
\]

Thus,

\[
\text{dist}(0, \partial L \Phi(x)) \geq \inf_{t \in \mathbb{R}, J_x \subseteq \{1, \ldots, n\} \setminus J_x, |\widehat{J}| = r - |J_x|} \left\| \left( \begin{array}{c}
2Ax(x^TBx) - 2Bx(x^TAx) \\
(x^T Bx)^2
\end{array} \right)_{J_x \cup \widehat{J}} + tx \right\|_{J_x \cup \widehat{J}}
\]

\[
= \inf_{J_x \subseteq \{1, \ldots, n\} \setminus J_x, |\widehat{J}| = r - |J_x|} \left\| \left[ \begin{array}{c}
2Ax(x^TBx) - 2Bx(x^TAx) \\
(x^T Bx)^2
\end{array} \right]_{J_x \cup \widehat{J}} \right\|.
\]

Using a similar line of argument as in Theorem 4.6, one has

\[
\text{dist}(0, \partial L \Phi(x)) \geq \inf_{J_x \subseteq \{1, \ldots, n\} \setminus J_x, |J| = r} \left\{ \frac{1}{x^T Bx} \|\nabla q_J(x)\| - \frac{2}{x^T Bx} |\Phi(x) - \Phi(\overline{x})| \|B_{J_J} x_{J_J}\| \right\},
\]

where \( q_J(z) = z^T A_J z - \frac{x^T A x}{x^T B x} z^T B_{J_J} z \) for \( z \in \mathbb{R}^{|J|} \). By Lemma 4.5, for each \( J \supseteq J_x \) with \( |J| = r \), there exists \( c_J > 0 \) such that \( \|\nabla q_J(z)\|^2 \geq c_J q_J(z) \). Note that \( \{ J :
\( J \supseteq J_{\mathbf{x}} \) with \(|J| = r \) \( \subseteq \mathcal{I} := \{ J : J \supseteq J_{\mathbf{x}} \text{ with } |J| = r \} \) (as \( \text{supp}(\mathbf{x}) \subseteq \text{supp}(\mathbf{x}) \)) and \(|\mathcal{I}| < +\infty\). So, \( c := \min_{J \in \mathcal{I}} c_J > 0 \). Noting from \( \Phi(\mathbf{x}) - \Phi(\mathbf{x}) > 0 \), for each \( J \supseteq J_{\mathbf{x}} \) with \(|J| = r\), one has

\[
\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}}_{J} = \left[ (\mathbf{x})^\top A_{J,J}(\mathbf{x}) - \frac{\mathbf{x}^\top A_{J,J}(\mathbf{x})}{\mathbf{x}^\top B_{J,J}(\mathbf{x})} \right] \frac{1}{\mathbf{x}^\top B_{J,J}(\mathbf{x})}
\]

Following a similar line of arguments as in Theorem 4.6, we get the desired conclusion.

**Remark 4.8** (linear convergence of Algorithm 3.3). In view of Theorem 3.11, Theorem 4.6, and Theorem 4.7, we see that Algorithm 3.3 exhibits linear convergence when applied to generalized eigenvalue problems with cardinality regularization and generalized eigenvalue problems with sparsity constraints.

5. **Numerical examples.** In the section, we illustrate our proposed method via numerical examples. We first start with an explicit analytic example and use it to demonstrate the behavior of Algorithm 3.3 as well as the effect of the inertial parameters. Then, we examine the performance of the algorithm for the sparse eigenvalue optimization model. All the numerical tests are conducted on a computer with a 2.8 GHz Intel Core i7 and 8 GB RAM, equipped with MATLAB R2015a.

5.1. **Analytic examples.** Consider the problem

\[
(EP) \quad \max_{\mathbf{x} \in \mathbb{R}^m} \left( m + 1 - \sum_{i=1}^{m} x_i \right) \prod_{i=1}^{m} x_i + \gamma \sum_{i=1}^{m} \frac{x_i + 1}{x_i^2 + 2x_i + 5} \quad \text{s.t. } 0 \leq \mathbf{x} \leq 10,
\]

where \( \gamma > 0 \). We first note that, for all \( i \in \{1, \ldots, m\} \), \( x_i^2 + 2x_i + 5 = (x_i + 1)^2 + 2 \geq 4(x_i + 1) \) and that if \( m + 1 - \sum_{i=1}^{m} x_i < 0 \), then \( (m + 1 - \sum_{i=1}^{m} x_i) \prod_{i=1}^{m} x_i \leq 0 \); otherwise, applying the Arithmetic Mean Geometric Mean (AM-GM) inequality to \( (m+1) \) numbers \( (m+1 - \sum_{i=1}^{m} x_i), x_1, \ldots, x_m \) yields \( (m + 1 - \sum_{i=1}^{m} x_i) \prod_{i=1}^{m} x_i \leq 1 \). Direct verification shows that \( \mathbf{x} = (1, \ldots, 1) \) is the global solution of this problem. This example satisfies Assumption 3.1 with \( f_i(x_i) = \gamma(x_i + 1) \), \( g_i(x_i) = x_i^2 + 2x_i + 5 \), \( \alpha_i = \frac{1}{2} \), and \( \beta_i = 2 \) for all \( i \in \{1, \ldots, m\} \). Let \( \gamma = 10 \), \( \mathbf{x}_0 = \mathbf{1}_{-1} \), \( \delta = 1 \), and, for all \( n \in \mathbb{N}, \nu_n = 0, \tau_n = \delta + \max_{1 \leq i \leq m} \left\{ \frac{1}{2} \nu_n \alpha_i + \frac{1}{2} \beta_n \right\} \leq \tau := \delta + \max_{1 \leq i \leq m} \left\{ \frac{1}{2} \nu_n \alpha_i \sqrt{M_i} + \frac{1}{2} \beta_i \frac{M_i}{m_i^2} \right\} \), where \( M_i = \max_{0 \leq x_i \leq 10} f_i(x_i) = 110 \) and \( m_i = \min_{0 \leq x_i \leq 10} g_i(x_i) = 5 \), \( i \in \{1, \ldots, m\} \). Then, for all \( n \in \mathbb{N} \) and all \( i \in \{1, \ldots, m\} \), \( z_i,n = x_i,n + \nu_n(x_i,n - x_i,n-1) = x_i,n \) and \( w_i,n = \frac{\gamma(-x_i,n^2 - 2x_i,n + 3)}{(x_i,n^2 + 2x_i,n + 5)^2} \), and

\[
x_i,n+1 = \argmax_{0 \leq x_i \leq 10} \left\{ x_i (m + 1 - x_i - s_i,n) p_{i,n} - \tau_n \left( x_i - z_i,n - \frac{1}{2\tau_n} w_i,n \right)^2 \right\}
\]
\[ P_{[0,10]} \left( \frac{2\tau_n \left( z_{i,n} + \frac{1}{2\tau_n} w_{i,n} \right)}{2\tau_n + 2p_{i,n}} + (m + 1 - s_{i,n})p_{i,n} \right), \]

where \( s_{i,n} := \sum_{j=1}^{i-1} x_{j,n+1} + \sum_{j=i+1}^{m} x_{j,n} \), \( p_{i,n} := \prod_{j=1}^{i-1} x_{j,n+1} \prod_{j=i+1}^{m} x_{j,n} \), and \( P_C \) denotes the Euclidean projection onto \( C \).

We randomly generate initial points in \([0,10]^m\) and perform Algorithm 3.3. For all the initial points, the algorithm produces a sequence \((x_n)_{n\in\mathbb{N}}\) converging to the global maximizer. Figure 1 depicts the convergence behavior for the case \( m = 2 \) and \( \gamma = 10 \), with initial points \((0,0), (0,1), (1,0), \) and \((10,10)\) by plotting out the Euclidean distance to the solution \((1,1)\) per iteration.

**Effect of the inertial parameters.** We now illustrate the behavior of Algorithm 3.3 by varying the inertial parameters. To do this, we fix \( m = 2 \) and \( \gamma = 10 \) and an \( \alpha \in (0,1) \). We set \( \nu_n = \alpha \frac{\delta}{2\tau_n} < \frac{\delta}{2\tau_n} \). Starting with the initialization \( x_0 = (10,10) \), we then run Algorithm 3.3 with different values for \( \alpha \in [0,1) \). Figure 2 depicts the distance, in the log scale, between the sequence of iterates \((x_n)_{n\in\mathbb{N}}\) and the solution \( x = (1,1) \), for \( \alpha \in \{0,0.3,0.6,0.9\} \). As one can see from the figure, as \( \alpha \) increases and approaches 1, the algorithm tends to converge faster.

**5.2. Sparse generalized eigenvalue problems.** As another illustration of our algorithm, following [30], we consider a sparse generalized eigenvalue problem that arises from binary classification using sparse Fisher discriminant analysis. Consider \( p \) observations \( z_1, \ldots, z_p \) with \( z_i \in \mathbb{R}^d \), \( i \in \{1, \ldots, p\} \), each of which belongs to one of two distinct classes. Let \( I_k \subseteq \{1, \ldots, p\} \) contain the indices of the observations in
class $k$, with $p_k = |I_k|$, $k = 1, 2$, and $p_1 + p_2 = p$. Let $\tilde{\mu}_k = \frac{1}{p_k} \sum_{i \in I_k} z_i$, for $k = 1, 2$. The so-called within-class and between-class covariance matrices are given by

$$
V_w = \frac{1}{p} \sum_{k=1}^{2} \sum_{i \in I_k} (z_i - \tilde{\mu}_k)(z_i - \tilde{\mu}_k)^\top \text{ and } V_b = \frac{1}{p} \sum_{k=1}^{2} p_k \tilde{\mu}_k \tilde{\mu}_k^\top.
$$

The classification problem using sparse Fisher discriminant analysis (SFDA) then seeks a low dimensional projection of the observations such that the between-class variance is large relative to the within-class variance. Mathematically, it solves

$$
\text{max } \frac{x^\top V_b x}{x^\top V_w x} - \lambda \phi(x) \quad \text{s.t. } ||x|| = 1,
$$

where $\phi$ is a regularization function inducing sparsity, and $\lambda > 0$. This is a sparse generalized eigenvalue problem with $A = V_b$ and $B = V_w$. Here, we consider two specific sparse regularization functions: $\phi(x) = ||x||_0$, and $\phi(x) = \delta_{C_r}(x)$ with $C_r = \{x \in \mathbb{R}^d : ||x||_0 \leq r\}$ and $r > 0$.

In the case where $\phi(x) = \delta_{C_r}(x)$, [30] proposed a truncated Rayleigh flow method (TRFM) for solving the above sparse generalized eigenvalue problem and showed the linear convergence of this method when the initial point $x_0$ is close enough to a global solution. We note that, in general, it is hard to theoretically guarantee whether an initial point $x_0$ is chosen to be close enough to a global solution, in order to ensure the convergence of the algorithm. On the other hand, Algorithm 3.3 can be applied to (SFDA) with both $\phi(x) = ||x||_0$ and $\phi(x) = \delta_{C_r}(x)$, and Remark 4.8 shows that Algorithm 3.3 converges linearly regardless of the choice of the initial points.

### 5.2.1. Sparsity constrained case

In this subsection, we consider the generalized eigenvalue problem with sparsity constraints, that is, (SFDA) with $\phi(x) = \delta_{C_r}(x)$. In this setting, Algorithm 3.3 reads as

$$
x_{n+1} \in P_{\Lambda \cap C_r}\left( z_n + \frac{1}{2\tau_n} \frac{x_n^\top V_b x_n}{(x_n^\top V_w x_n)^2} \begin{bmatrix} x_n^\top V_w x_n & V_b x_n - V_w x_n \\ x_n^\top V_b x_n & x_n^\top V_b x_n - V_w x_n \end{bmatrix} \right) \\
\text{with } z_n = x_n + \nu_n(x_n - x_{n-1}).
$$

It is known that, for all $a = (a_1, \ldots, a_d) \in \mathbb{R}^d$, $(P_{C_r}(a))_i = a_i$ for the $r$ largest components in absolute value of $a$, and $(P_{C_r}(a))_i = 0$ otherwise. Then

$$
P_{\Lambda \cap C_r}(a) = \begin{cases} \{ \frac{\mathbf{v}}{||\mathbf{v}||} : \mathbf{v} \in P_{C_r}(a) \} & \text{if } a \neq 0, \\ \Lambda \cap C_r & \text{if } a = 0. \end{cases}
$$

This can be seen, for example, by noting that $P_{\Lambda \cap C_r}(a) = \arg\min\{ \frac{1}{2}||x - a||^2 : x \in \Lambda \cap C_r \} = \arg\min\{ (a, x) : x \in \Lambda \cap C_r \}$, and applying [23, Proposition 13].

In our simulation, we adopt the same setting as in [30]: we set $\mu_1 = 0, \mu_2 = (\mu_{2,1}, \ldots, \mu_{2,4})^\top$ with $\mu_{2,j} = 0.5$ for $j \in \{2, 4, \ldots, 40\}$ and $\mu_{2,j} = 0$ otherwise. Let $\Sigma$ be a block diagonal covariance matrix with five blocks, each of dimension $(d/5 \times d/5)$. The $(j, j')$-th element of each block takes the value $0.8|j - j'|$. As explained in [30], this covariance structure is intended to mimic the covariance structure of gene expression data. The observation data are simulated as $z_i \sim \mathcal{N}(\mu_k, \Sigma)$ for $i \in I_k$, $k = 1, 2$.

We use our proposed inertial proximal subgradient method (Algorithm 3.3) and the truncated Rayleigh flow method (TRFM) for solving (SFDA) with $\phi(x) = \delta_{C_r}(x)$, where we set $r = 50$, $p_1 = p_2 = 500$, $p = p_1 + p_2 = 1000$, and $d = 2000$. 


• For Algorithm 3.3, we use the initial point \( x_0 = (1/\sqrt{r}, \ldots, 1/\sqrt{r}, 0, \ldots, 0) \in \mathbb{R}^d \). Direct verification shows that Assumption 3.1 is satisfied with \( \alpha_1 = 0 \) and \( \beta_1 = 2\lambda_{\max}(V_w) \). So, by Remark 3.9, we can set \( \delta = 1 \), \( \tau_n = 1 + \frac{x_n^\top V_w x_n}{\lambda_{\max}(V_w)} \), \( \bar{\nu} = 0.4999 < \frac{\delta}{2} \) and \( \nu_0 = \frac{\bar{\nu}}{\tau_0} \). We stop the algorithm when either the iterations reach the maximum iteration number 6000 or the quantity \( \|x_{n+1} - x_n\| \) is less than \( 10^{-6} \).

• For (TRFM), we use the same initial point \( x_0 \) as in Algorithm 3.3. We also use the same termination criteria as in Algorithm 3.3.

We run TRFM and Algorithm 3.3 for 50 trials. Table 1 summarizes the output of the two methods by listing the average value for

(i) the objective value of the computed solution;

(ii) the CPU time measured in seconds;

(iii) the number of iterations used (round to the nearest integer).

Table 1

|                   | Objective value of computed solution | CPU time   | Number of iterations |
|-------------------|-------------------------------------|------------|----------------------|
| TRFM              | 12.2932                             | 6.9976     | 1083                 |
| Algorithm 3.3     | 12.5461                             | 4.8148     | 555                  |

From Table 1, one can see that Algorithm 3.3 is competitive with the TRFM method and produces a solution with better quality in terms of the final objective value (note that (SFDA) is a maximization problem). Moreover, Algorithm 3.3 also uses less CPU time and number of iterations. As an illustration, we also plot \( \|x_n - x^*\| \) against the number of iterations \( n \), in logarithmic scale, where \( x^* \) is the approximated solution produced by the corresponding algorithm. Figure 3 supports the theoretical finding that Algorithm 3.3 exhibits linear convergence in this case.

![Fig. 3. Euclidean distance between \( x_n \) and \( x^* \) in every iteration](image)

5.2.2. Sparse generalized eigenvalue problem with cardinality regularization. In this subsection, we consider the generalized eigenvalue problem with cardinality regularization, that is, (SFDA) with \( \phi(x) = \|x\|_0 \). In this setting, Al-
Algorithm 3.3 reads as
\[
x_{n+1} = \arg\max_{\|x\|=1} \left\{ -\lambda \|x\|_0 - \tau_n \left\| x - z_n - \frac{1}{2\tau_n} w_n \right\|^2 \right\}
\]
\[
= \arg\max_{\|x\|=1} \left\{ -\lambda \|x\|_0 + \langle 2\tau_n z_n + w_n, x \rangle \right\}
\]
with \( \lambda > 0, z_n = x_n + \nu_n (x_n - x_{n-1}) \), and
\[
w_n = \frac{x_n^\top V_b x_n}{(x_n^\top V_w x_n)^2} \left[ \frac{x_n^\top V_w x_n}{x_n^\top V_b x_n} V_b V_n - V_b x_n \right].
\]
We note that, for each \( a \in \mathbb{R}^d \), the optimization problem \( \arg\max_{\|x\|=1} \left\{ -\lambda \|x\|_0 + \langle a, x \rangle \right\} \) has a closed form solution [29, Proposition 6]. In our numerical experiment, we set \( \lambda = 0.035 \). We also generate the data as in the previous subsection, using the same initial point, parameters \( \tau_n, \nu_n \) and \( \delta \), and employing the same termination criteria.

We run Algorithm 3.3 for 50 trials. Table 2 summarizes the output of the method where the meanings of the items are the same as in the previous subsection.

| Objective value of computed solution | Number of iterations |
|--------------------------------------|----------------------|
| 13.7196 | 3.1013 | 1074 |

We also plot out Euclidean distance between \( x_n \) and \( x^* \) per iteration in log scale (Figure 4), which supports the theoretical finding that Algorithm 3.3 exhibits linear convergence for this problem.

Appendix A. Proof of Lemmas 2.1, 2.2, 2.4, and 3.5.

Proof of Lemma 2.1. (i): This is given in [26, Proposition 10.5].
(ii): This follows from [26, Corollary 10.9].
(iii): This is an application of [25, Corollary 3.4] with \( \varphi_1 \equiv 0 \) and \( \varphi_2 = f \).
In view of (A.2) and (A.3), if
\[
\partial_L \left( \frac{-f}{g} \right)(\mathbf{x}) = \partial_L \left( -g(\mathbf{x})f + f(\mathbf{x})g(\mathbf{x}) \right) \subseteq \partial_L \left( -g(\mathbf{x})f(\mathbf{x}) \right) + \partial_L \left( f(\mathbf{x})g(\mathbf{x}) \right).
\]
Assume that \(\hat{\partial}f\) is nonempty-valued around \(\mathbf{x}\). Then, if \(g(\mathbf{x}) > 0\), \(\partial_L \left( -g(\mathbf{x})f(\mathbf{x}) \right) = g(\mathbf{x})\partial_L \left( -f(\mathbf{x}) \right) \subseteq -g(\mathbf{x})\partial_L \left( f(\mathbf{x}) \right)\) due to (iii). If \(g(\mathbf{x}) < 0\), then \(-g(\mathbf{x}) > 0\) and \(\partial_L \left( -g(\mathbf{x})f(\mathbf{x}) \right) = -g(\mathbf{x})\partial_L \left( f(\mathbf{x}) \right)\). Thus, we obtain the desired inclusion.

Now, assume that \(f\) is strictly differentiable at \(\mathbf{x}\). Then, by combining (A.1) with
the last assertion in (ii), \(\partial_L \left( \frac{-f}{g} \right)(\mathbf{x}) = -g(\mathbf{x})\hat{\partial}f(\mathbf{x}) + \partial_L \left( f(\mathbf{x})g(\mathbf{x}) \right)\). On the other hand, we have from [15, Corollaries 1.12.2 and 1.14.2] that \(\partial_L \left( \frac{-f}{g} \right)(\mathbf{x}) = -g(\mathbf{x})\hat{\partial}f(\mathbf{x}) + \partial_L \left( f(\mathbf{x})g(\mathbf{x}) \right)\).
The remaining conclusion follows from these two equalities.

(v): The chain rule is given in [24, Theorem 1.110(ii)]. The two square root rules follow by letting \(\theta(t) = \sqrt{t}\) and \(\theta(t) = -\sqrt{t}\), respectively.

Proof of Lemma 2.2. (i): The formula for Fréchet and limiting subdifferentials of \(\| \cdot \|_0\) can be found in [18, Section 3]. The formula for the horizon subdifferential can be verified directly.

(ii): This follows by a direct verification.

(iii): The limiting subdifferential formula for \(\delta_C\) can be found in [4, Theorem 3.9]. The formula for the horizon subdifferential can be verified directly.

(iv)\&(v): We deduce from (i), (ii), and (iii) that, for all \(\mathbf{x} \in \mathbb{A}\), \(\| \cdot \|_0\) and \(\delta_\Lambda\) are regular at \(\mathbf{x}\) and \((-\partial^\infty_L(\| \cdot \|_0)(\mathbf{x})) \cap \partial^\infty_L \delta_\Lambda(\mathbf{x}) = \{0\}\), and that, for all \(\mathbf{x} \in \mathbb{A} \cap C_r\), \((-\partial^\infty_L \delta_C(\mathbf{x}))(\mathbf{x})) \cap \partial^\infty_L \delta_\Lambda(\mathbf{x}) = \{0\}. The conclusions then follow from Lemma 2.1(ii).

Proof of Lemma 2.4. Let us first consider the case when \(h\) is strictly differentiable at \(\mathbf{x}\). By Lemma 2.1(ii), \(\partial_L \left( -h + \delta_S \right)(\mathbf{y}) = -\nabla h(\mathbf{x}) + \partial_L \delta_S(\mathbf{x})\). Since \(\delta_S(\mathbf{x}) = \delta_{S_1}(x_1) + \cdots + \delta_{S_m}(x_m)\), we learn from Lemma 2.1(i) that \(\partial_L \delta_S(\mathbf{x}) = \partial^{\delta_{S_1}}_L \delta_S(\mathbf{x}) \times \cdots \times \partial^{\delta_{S_m}}_L \delta_S(\mathbf{x})\), and so
\[
\partial_L \left( -h + \delta_S \right)(\mathbf{y}) = \partial^{\delta_{S_1}}_L \left( -h + \delta_S \right)(\mathbf{y}) \times \cdots \times \partial^{\delta_{S_m}}_L \left( -h + \delta_S \right)(\mathbf{y}).
\]
This equality is obvious in the case when \(m = 1\). Next, since \(F(\mathbf{x}) = h(\mathbf{x}) + \sum^{m}_{i=1} f_i(x_i)\) with each \(\frac{f_i}{g_i}\) Lipschitz continuous around \(\mathbf{y}_i\), again using Lemma 2.1(i)\&(ii), we have that
\[
\partial_L \left( -F + \delta_S \right)(\mathbf{x}) \subseteq \partial_L \left( -h + \delta_S \right)(\mathbf{x}) + \partial_L \left( -\sum^m_{i=1} \frac{f_i}{g_i} \right)(\mathbf{x})
\]
\[
= \partial^{\delta_{S_1}}_L \left( -h + \delta_S \right)(\mathbf{x}) \times \cdots \times \partial^{\delta_{S_m}}_L \left( -h + \delta_S \right)(\mathbf{x}) + \partial_L \left( -\sum^m_{i=1} \frac{f_i}{g_i} \right)(\mathbf{y}_1) \times \cdots \times \partial_L \left( -\frac{f_m}{g_m} \right)(\mathbf{y}_1).
\]
(i): Assume that, for each \(i \in \{1, \ldots, m\}\), \(\hat{\partial}f_i\) is nonempty-valued around \(\mathbf{y}_i\). Then, by Lemma 2.1(iv), for each \(i \in \{1, \ldots, m\}\),
\[
\partial_L \left( -\frac{f_i}{g_i} \right)(\mathbf{y}_i) \subseteq -\frac{g_i(\mathbf{y}_i)\partial_L f_i(\mathbf{y}_i) + f_i(\mathbf{y}_i)\partial_L g_i(\mathbf{y}_i)}{g_i(\mathbf{y}_i)^2}.
\]
In view of (A.2) and (A.3), if \(\mathbf{x}\) is a stationary point for \((\mathcal{P})\), then it is a lifted coordinate-wise stationary point for \((\mathcal{P})\).

(ii): By Lemma 2.1(i),(ii)\&(iv), the inclusions in (A.2) and (A.3) can be replaced by equalities. The conclusion then follows.
Proof of Lemma 3.5. For each $i \in \{1, \ldots, m\}$, set $H_i(x_i, y_i) := 2y_i\sqrt{f_i(x_i)} - y_i^2g_i(x_i)$.

(i): This follows from the observation that

$$\max_{y \in \mathbb{R}^m} H(x, y) = \sum_{i=1}^m \max_{y_i \in \mathbb{R}} H_i(x_i, y_i) = \sum_{i=1}^m H_i \left( x_i, \frac{\sqrt{f_i(x_i)}}{g_i(x_i)} \right) = \sum_{i=1}^m f_i(x_i) / g_i(x_i).$$

(ii): Assume that, for each $i \in \{1, \ldots, m\}$, $\partial f_i$ is nonempty-valued around $\mathfrak{y}_i$. Then, since $f_i(\mathfrak{y}_i) > 0$ and $\mathfrak{y}_i \geq 0$, we have from Lemma 2.1(ii), Lemma 2.1(v), and then Lemma 2.1(iii) that

$$\partial^x \mathfrak{L}(-H)(x, y) = \partial^x \mathfrak{L}(-H_i)(\mathfrak{y}_i, \mathfrak{y}_i) \subseteq \mathfrak{y}_i \frac{\partial L(-f_i(\mathfrak{y}_i))}{\sqrt{f_i(\mathfrak{y}_i)}} + \mathfrak{y}_i^2 \frac{\partial L g_i(\mathfrak{y}_i)}{g_i(\mathfrak{y}_i)} \mathfrak{y}_i \mathfrak{y}_i$$

$$\subseteq -\mathfrak{y}_i \frac{\partial L f_i(\mathfrak{y}_i)}{\sqrt{f_i(\mathfrak{y}_i)}} + \mathfrak{y}_i^2 \frac{\partial L g_i(\mathfrak{y}_i)}{g_i(\mathfrak{y}_i)} \mathfrak{y}_i = -g_i(\mathfrak{y}) \frac{\partial L f_i(\mathfrak{y}_i)}{g_i(\mathfrak{y}_i)} f_i(\mathfrak{y}_i) \frac{g_i(\mathfrak{y}_i)}{g_i(\mathfrak{y}_i)}.$$

As a result, if $(x, y)$ is a lifted coordinate-wise stationary point for $(\mathcal{P}_1)$, then $x$ is a lifted coordinate-wise stationary point for $(\mathcal{P})$.

Now, assume that, for each $i \in \{1, \ldots, m\}$, $f_i$ is strictly differentiable at $\mathfrak{y}_i$. Then the inclusions in (A.4) become equalities, and the conclusion follows.

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