Approximation systems

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Abstract

We introduce the notion of an approximation system as a generalization of Taylor approximation, and we give some first examples. Next we develop the general theory, including error bounds and a sufficient criterion for convergence. More examples follow. Prerequisites are mostly elementary complex analysis.

1 Introduction

Analytic functions are characterized by the property that they can locally be approximated by a power series, viz. the Taylor series. When we are given an analytic function \( g \) on a domain \( U \), and we denote its derivatives at a point \( x_0 \in U \) by \( a_i := g^{(i)}(x_0) \), then \( g \) is in fact the unique analytic solution for \( g_0 \) in the following infinite system of differential equations:

\[
g_i(x_0) = a_i, \quad g_i'(x) = g_{i+1}(x) \quad (x \in U),
\]

where we assume all \( g_i \) to be analytic on \( U \). We could truncate this infinite set of equations after \( n \) steps, so that we get:

\[
g_i(x_0) = a_i \quad (i \leq n), \quad g_i'(x) = g_{i+1}(x) \quad (x \in U, \ i < n).
\] (1.1)

In terms of integrals this can be restated as:

\[
g_n(x_0) = a_n, \quad g_i(x) = a_i + \int_{x_0}^{x} g_{i+1}(t) \, dt \quad (x \in U, \ i < n).
\]

To solve this system of equations, we can pick any analytic function for \( g_n \) such that \( g_n(x_0) = a_n \), and each choice leads after \( n \) integrations to a solution for \( g_0 \). In particular, when we pick \( g_n \) to be constant \( a_n \), the resulting solution for \( g_0 \) will be the \( n \)-th order Taylor polynomial.

In this paper we will introduce a generalization of the Taylor approximation, based on the following alteration of (1.1):

\[
g_i(x_0) = a_i, \quad g_i'(x) = f_i(g_{i+1}(x), x),
\]

where each \( f_i \) is an analytic function in two variables. To get the generalized equivalent for the \( n \)-th order Taylor approximation, we truncate again these equations after \( n \) steps, so that rewriting them in terms of integrals gives:

\[
g_n(x_0) := a_n, \quad g_i(x) := a_i + \int_{x_0}^{x} f_i(g_{i+1}(t), t) \, dt \quad (x \in U, \ i < n).
\] (1.2)
By picking the constant function \( a_n \) for \( g_n \), we get after \( n \) integrations a solution for \( g_0 \). Like was the case for Taylor approximations, the solution to the truncated sequence of equations (1.2) serves as an approximation for an analytic function \( g \) which satisfies the complete set of differential equations (1.1). As will be shown in Example 2.7, we get the Taylor approximation by simply putting \( f_i(y, x) = y \).

The introduced approximation method is a generalization of Taylor approximation, in the sense that it allows for a possibly non-linear function \( f_i \) at each single integration step. It can also be regarded as a generalization of Picard iteration, in the sense that if we take \( a_i = a \) and \( f_i = f \), i.e., if we take them identical for each \( i \geq 0 \), then (1.1) essentially reduces to a single differential equation. The \( n \)-th order approximation to it then coincides with the \( n \)-th Picard iteration, as will be shown in more detail in Example 2.8.

The first ideas about approximation systems and many of the basic properties and the examples were already formulated by the first author in his BSc Thesis [4]. He introduces a related but different concept called expansion system in his MSc Thesis [5].

The contents of this paper are as follows. In Section 2 we define approximation systems as a framework for approximating analytic functions, and we illustrate their use with some examples. In Section 3 some propositions are proved about the effect of certain coordinate transformations on an approximation system. These will be useful in later sections. Next we prove in Section 4 a general result on the error of the approximations. Section 5 starts with a discussion of the relationship between the coefficients of an approximation system and the derivatives of the approximated function at its basepoint. Next we use this relationship for the development of a criterion that will guarantee the convergence of an approximation system. Section 6 gives some more examples of approximation systems, including estimates of the maximal error. We end the paper with some concluding remarks.

**Notation**

\( B(a, R) \) denotes the open disk with radius \( R \) around \( a \) in \( \mathbb{C} \). By \( \phi^0 \) is meant the \( n \)-fold composition of the function \( \phi \). So \( \phi^0 := \text{Id} \) and \( \phi^{(n+1)} := \phi \circ \phi^n \) for \( n \geq 0 \). We write \( D_1 f \) and \( D_2 f \) for differentiation of a function \( f \) of two real variables with respect to its first and second argument, respectively. We write \( \|f\|_Y \) denotes the sup norm of a function \( f \) on a domain \( Y \).

# 2 Definition and first examples

Let \( U \subset \mathbb{C} \) be a simply connected open set and fix \( x_0 \in U \) (to be called a basepoint of \( U \)). For some given analytic function \( g \) on \( U \) we will define a new way of approximating \( g \) by means of a so-called approximation system. This system will involve a sequence \( \{a_i\}_{i=0}^r \) of complex constants, and a sequence \( \{f_i\}_{i=0}^{r-1} \) of analytic functions in two variables \( f_i : Y_i \to \mathbb{C} \), where \( Y_i = V_i \times U \) for some open set \( V_i \subset \mathbb{C} \) such that \( a_i + 1 \in V_i \) for \( i < r \), and where \( f_i(., x) \) is non-constant for all \( x \in U \). In this setting, \( r \) can be a nonnegative integer, but we will also allow the possibility of \( r \) being infinite, in which case our sequences must be interpreted as \( \{a_i\}_{i=0}^{\infty} \) and \( \{f_i\}_{i=0}^{\infty} \).
Definition 2.1. The pair \( \{a_i\}_{i=0}^r \times \{ f_i \}_{i=0}^{r-1} \) is called an approximation system (shortly AS) of order \( r \) in the basepoint \( x_0 \) for the function \( g \) if there exists a sequence \( \{g_i\}_{i=0}^r \) of analytic functions on \( U \) with \( g_0 := g \) such that \( g_{i+1}(U) \subset V_i \) for all \( i < r \) and the following equations hold:

\[
g_i(x_0) = a_i \quad (i < r + 1), \quad g_i'(x) = f_i(g_{i+1}(x), x) \quad (x \in U, \ i < r).
\]

Then we will also say that the pair \( \{a_i\}_{i=0}^r \times \{ f_i \}_{i=0}^{r-1} \) is an AS for the sequence \( \{g_i\}_{i=0}^r \).

Remark 2.2. Note that according to the definition of an AS, the pair \( \{a_i\}_{i=0}^r \times \{ f_i \}_{i=0}^{r-1} \) describes a set of differential equations which is satisfied by the sequence of functions \( \{g_i\}_{i=0}^r \). Unlike most common systems of differential equations, there aren’t any further relations imposed between functions \( g_i \) and \( g_j \) for different \( i \) and \( j \). The whole set of equations, as described in (2.1), can also be formulated as

\[
g_i(x) = a_i + \int_{x_0}^x f_i(g_{i+1}(t), t) \, dt
\]

for \( x \in U \) and \( i < r \). Now suppose that the pair \( \{a_i\}_{i=0}^r \times \{ f_i \}_{i=0}^{r-1} \) is given as in Definition 2.1 but that functions \( g_i \) and \( g \) are not yet given. Let for some \( n (0 < n < r + 1) \) an analytic function \( g_n : U \to V_{n-1} \) be given with \( g_n(x_0) = a_n \). Then unique analytic functions \( g_i \) \( (i < n) \) exist on a sufficiently small simply connected open neighbourhood \( U_0 \subset U \) of \( x_0 \) such that the pair \( \{a_i\}_{i=0}^n \times \{ f_i \}_{i=0}^{n-1} \) is an AS for \( g := g_0 \) for the sequence \( \{g_i\}_{i=0}^n \). The functions \( g_i \) can recursively be constructed by (2.2) and we put \( g := g_0 \).

Remark 2.3. If in Definition 2.1 only the second part \( \{ f_i \}_{i=0}^{r-1} \) of the approximation system is given, but not the \( a_i \), then we can still have a sequence \( \{g_i\}_{i=0}^r \) satisfying the second equation in (2.1). Next the first equation in (2.1) can be used as a definition for the \( a_i \). With \( \{a_i\}_{i=0}^r \) thus chosen the sequence \( \{g_i\}_{i=0}^r \) will satisfy the full Definition 2.1. For this reason we will often omit explicit values of the \( a_i \), since they will be implied.

Let us now apply the idea of Remark 2.2 by choosing \( g_n \), which we will now call \( g[n] \), identically equal to \( a_n \). So let be given an AS as in Definition 2.1 fix \( n (0 < n < r + 1) \), and recursively define

\[
\begin{align*}
g[n]_i(x) & := a_n, \\
g[n]_i(x) & := a_i + \int_{x_0}^x f_i(g[n]_{i+1}(t), t) \, dt \quad (i < n)
\end{align*}
\]

provided this makes sense because of the inclusions

\[
g[n]_{i+1}(U) \subset V_i \quad (i < n)
\]

being valid. Then put

\[
g[n] := g[n]_0.
\]

We conclude that the pair \( \{a_i\}_{i=0}^n \times \{ f_i \}_{i=0}^{n-1} \) is an approximation system of order \( n \) for \( g[n] \) in the basepoint \( x_0 \). In view of approximation theorems to be given later in this paper, the functions \( g[n] \), as \( n \) gets bigger, often turn out to be good approximations for a function \( g \) as in Definition 2.1. This motivates the following definition.
Definition 2.4. The AS \((\{a_i\}_{i=0}^r, \{f_i\}_{i=0}^{r-1})\) is called a proper approximation system of order \(n\) in the basepoint \(x_0\) if condition (2.4) holds for the functions \(g^{[n]}\) defined by (2.3). If it is a proper approximation system of every order \(n < r + 1\), we call it just a proper approximation system (or shortly PAS).

Because in many cases the basepoint of an AS either can be understood from context or has an abstract value \(x_0\), we will usually not mention it in the following. In case of possible confusion we will write \(g: (U, x_0) \to \mathbb{C}\) in order to emphasize that \(x_0\) is the basepoint.

Remark 2.5. In most cases, the domains can be chosen in such a way that condition (2.4) is satisfied. In general, for given \(n\), we can always shrink \(U\) to a smaller open simply connected neighborhood of \(x_0\) such that (2.4) is satisfied.

Example 2.6. Suppose we have an analytic function \(g: U \to \mathbb{C}\). Then every sequence of invertible analytic functions \(\{f_i\}_{i=0}^{r-1}\) in one variable (i.e. \(f_i(y, x)\) only depends on \(y\)) with sufficiently large range can be used for an approximation system for \(g\). The sequence \(\{g_i\}_{i=0}^r\) is uniquely determined by \(g_0 = g\) and \(g_{i+1} := f_i^{-1} \circ g_i\).

Example 2.7. Taylor approximation
When in Example 2.6 we let \(f_i(y, x) = y\) for all \(i \geq 0\), this gives rise to an AS for the function \(g: U \to \mathbb{C}\), such that \(g_i = g^{(i)}\) (and therefore \(a_i = g^{(i)}(x_0)\)). Because

\[
a_i + \int_{x_0}^x \sum_{k=i+1}^n a_k \frac{t^{k-i-1}}{(k-i-1)!} dt = \sum_{k=i}^n a_k \frac{(x-x_0)^{k-i}}{(k-i)!}
\]

and

\[
g^{[n]}_n(x) = a_n,
\]

we get by induction that

\[
g^{[n]}(x) = g^{[n]}_0(x) = \sum_{k=0}^n a_k \frac{(x-x_0)^k}{k!} = \sum_{k=0}^n g^{(k)}(x_0) \frac{(x-x_0)^k}{k!},
\]

which equals the \(n\)-th order Taylor Approximation. Obviously, the \(f_i\) and \(a_i\) as chosen in this Example give us a PAS of any order in the basepoint \(x_0\).

Example 2.8. Ordinary differential equations
Consider the following ordinary differential equation (ODE) on a domain \(U\):

\[
g'(x) = f(g(x), x), \tag{2.5}
\]

where \(f: V \times U \to \mathbb{C}\) is an analytic function. Suppose we have a solution \(g\) to this differential equation on the domain \(U\) such that \(g(x_0)\) equals a given value \(a \in V\) at a basepoint \(x_0 \in U\) and such that \(g(U) \subset V\). Now with \(f_i := f\) and \(a_i := a\) we obtain an AS for the function \(g\) by just
putting \( g_i := g \) in (2.1). Now, for fixed \( n \) and for \( U \) a sufficiently small open simply connected neighbourhood of \( x_0 \), the Picard iteration scheme (see for instance [1] Ch. 1, §3)

\[
\begin{align*}
g_0^n(x) &:= a, \\
g_i^n(x) &:= a + \int_{x_0}^x f(g_{i+1}^n(t), t) \ dt \quad (i < n) \\
\end{align*}
\]

(2.6)

makes sense with \( g_{i+1}^n(U) \subset V \ (i < n) \). We recognize (2.3) and (2.4) specified for our Example. Thus our choice for the \( f_i \) and \( a_i \) gives us a PAS of any order \( n \) in the basepoint \( x_0 \), but with \( U \) possibly dependent on \( n \). In fact, the general theory of ordinary differential equations tells us that, for \( U \) a sufficiently small neighbourhood of \( x_0 \), we get a PAS for any order and we have uniform convergence of \( g_i^n \) to \( g \) (see also Remark 1.7).

Note that it follows from (2.6) by induction with respect to \( n - i \) that we can say the following about a special case of Definition 2.4

\[
\text{If } f_i = f, a_i = a \text{ for all } i \text{ then } g_i^n = g_i^{n-1} = \ldots = g_0^{n-i} = g_i^{n-i}. \tag{2.7}
\]

**Example 2.9. Functional differential equations**

As a generalization of the ODE (2.5) consider the functional differential equation (FDE)

\[
g'(x) = f(g \circ \phi(x), x), \tag{2.8}
\]

on a domain \( U \), where \( f: V \times U \rightarrow \mathbb{C} \) is an analytic function and \( \phi: U \rightarrow U \) is an analytic endomorphism on \( U \). Assume that \( g \) is a solution of this equation with a given value \( g(x_0) = a \) at a basepoint \( x_0 \in U \), and that \( g(\phi(U)) \subset V \). Also let \( \phi^o \) denote the \( n \)-fold composition of the function \( \phi \). So \( \phi^0 := \text{Id} \) and \( \phi^{o(n+1)} := \phi \circ \phi^o \) for \( n \geq 0 \). Put \( g_i = g \circ \phi^{i} : U \rightarrow V \). Choose \( V_i \subset V_0 := V \) such that \( g(\phi^{o(i+1)}(U)) \subset V_i \). Then we see that the functions

\[
f_i(y, x) := (\phi^i)'(x) f(y, \phi^i(x)) \quad (i \geq 0, \ (y, x) \in V_i \times U)
\]

yield an AS for the functions \( g_0(= g), g_1, g_2, \ldots \). Indeed, (2.1) can be seen to hold as follows. If \( g_i'(x) = f_i(g_{i+1}(x), x) \) for certain \( i \) then

\[
\begin{align*}
g_{i+1}'(x) &= \phi'(x) g_i'(\phi(x)) = \phi'(x) f_i(g_{i+1}(\phi(x)), \phi(x)) \\
&= \phi'(x) (\phi^o)'(\phi(x)) f(g_{i+2}(x), \phi^{o(i+1)}(x)) \\
&= (\phi^{o(i+1)})'(x) f(g_{i+2}(x), \phi^{o(i+1)}(x)) = f_{i+1}(g_{i+2}(x), x).
\end{align*}
\]

Note that \( a_i = g(\phi^o(x_0)) \).

Suppose that moreover \( \phi(x_0) = x_0 \). Then \( a_i = g(x_0) = a \) for all \( i \). Also suppose that the AS obtained above is proper. Then we will show that

\[
g_i^n \circ \phi = g_{i+1}^{n+1}, \quad \text{in particular } \quad g_i^n = g_i^{n-i} \circ \phi^o.
\tag{2.9}
\]

For the proof first observe that the second equation in (2.3) is equivalent to

\[
(g_i^n)'(x) = (\phi^o)'(x) f(g_i^n(x), \phi^o(x)), \quad g_i^n(x_0) = a.
\tag{2.10}
\]
Now assume that $g_j^{[m]} \circ \phi = g_{j+1}^{[m+1]}$ if $m - j < n - i$. (It is certainly true for $m - j = 0$.) Then
\[
(g_j^{[m]} \circ \phi)'(x) = \phi'(x) (g_j^{[m]})'(\phi(x)) = \phi'(x) (\phi \circ i)'(\phi(x)) f(g_i^{[m]}(\phi(x)), \phi \circ (i+1)(x))
\]
\[
= (\phi \circ (i+1))'(x) f(g_i^{[n]}(x), \phi \circ (i+1)(x)) = (g_{i+1}^{[n+1]})(x).
\]
Since also $g_i^{[m]} \circ \phi$ and $g_{i+1}^{[n+1]}$ have the same value at $x_0$, we conclude that (2.9) holds. Furthermore, by (2.10) we see that
\[
(g^{[n]})'(x) = f(g^{[n-1]}(\phi(x)), x).
\]
(2.11)

**Remark 2.10.** Note that (2.11) together with $(g^{[n]})(x_0) = a$ is equivalent with
\[
(g^{[n]})(x) = a + \int_{x_0}^x f(g^{[n-1]}(\phi(t)), t) \, dt.
\]

Define an operator $S$ acting on analytic functions $h$ on $U$ satisfying $h(x_0) = a$ and $h(\phi(U)) \subset V$ as follows:
\[
(Sh)(x) := a + \int_{x_0}^x f(h(\phi(t)), t) \, dt.
\]
Then
\[
g^{[n]} = S(g^{[n-1]}) = S^n(a).
\]
(2.13)

Special cases of proper approximation systems coming from FDE’s with $x_0$ being a fixpoint of $\phi$ are given in Examples 6.1, 6.2, 6.4 and Remarks 6.3, 6.5. There it is possible to generate the approximations $g^{[n]}$ by (2.13) using the operator $S$. Also Example 6.6 is based on a FDE, but as its $\phi$ has no fixpoint, the results there cannot be obtained by iteration of $S$.

It is interesting to compare our operator $S$ given by (2.12) and its iteration with the operator $T$ occurring in Grimm [3, Proof of Theorem 1]. The FDE there is more complicated than (2.8) ($f$ also depending on $g'$ and $\phi$ also depending on $g$), but our FDE can be obtained as a special case. For that case Grimm’s operator $T$ becomes
\[
(Th)(x) := f(a + \int_{x_0}^x h(t) \, dt, x).
\]

Then iterates $T^n(0)$ approximate $g'$ rather than $g$ and, for $h(x_0) = a$, $S$ and $T$ are connected by
\[
(Sh)' = Th', \quad (S^n h)' = T^n h',
\]
or in integral form:
\[
(Sh)(x) = a + \int_{x_0}^x (Th')(t) \, dt, \quad (S^n h)(x) = a + \int_{x_0}^x (T^n h')(t) \, dt.
\]

3 Coordinate transformations

**Proposition 3.1.** Let \( \{a_i\}_{i=0}^{r-1}, \{f_i\}_{i=0}^{r-1} \) be an AS for the sequence of functions \( g_i: (U, x_0) \to \mathbb{C} \) \((i < r + 1)\). Then a biholomorphic coordinate transformation \( \phi: (U, x_0) \to (U, x_0) \) (mapping \( x_0 \) to \( x_0 \)) gives rise to a new AS \( \{a_i\}_{i=0}^{r-1}, \{\overline{f}_i\}_{i=0}^{r-1} \) for the sequence of functions \( \overline{f}_i: (U, x_0) \to \mathbb{C} \) \((i < r + 1)\), where \( \overline{f}_i(y, x) := f_i(y, \phi(x)) \phi'(x) \) and \( \overline{g}_i(x) := g_i \circ \phi(x) \).

*Proof.* Equation \((2.1)\) is satisfied because for \( i < r \) we have
\[
\overline{f}_i(x) = g_i'(\phi(x)) \phi'(x) = f_i(g_{i+1}(\phi(x)), \phi(x)) \phi'(x) = f_i(\overline{g}_{i+1}(x), \phi(x)) \phi'(x) = \overline{f}_i(\overline{g}_{i+1}(x), x).
\]
Moreover, \( \overline{g}_i \) has the same image as \( g_i \), which is contained in \( V_i \), and \( \overline{f}_i \) has domain \( V_i \times \overline{U} \), so that the constructed approximation system indeed satisfies the requirements. \( \square \)

**Remark 3.2.** Let \( \{a_i\}_{i=0}^{r-1}, \{f_i\}_{i=0}^{r-1} \), \( g_i \), \( \phi \) and \( \overline{f}_i \) be as in Proposition 3.1. Assume that \( \{a_i\}_{i=0}^{r-1}, \{\overline{f}_i\}_{i=0}^{r-1} \) is an AS for some sequence \( \overline{g}_i: (\overline{U}, x_0) \to \mathbb{C} \) \((i < r + 1)\), where the \( \overline{g}_i \) are not necessarily as in Proposition 3.1. But if \( g_n = g_{n} \circ \phi \) for some \( n < r + 1 \) then by Remark 2.2 and by the proof of Proposition 3.1 we still have \( g_i \circ \phi = \overline{g}_i \) for all \( i \leq n \).

This Remark leads us immediately to the following Lemma, which will be used in the next section.

**Lemma 3.3.** Let \( \{a_i\}_{i=0}^{r-1}, \{f_i\}_{i=0}^{r-1} \) be a PAS and \( \{a_i\}_{i=0}^{r-1}, \{\overline{f}_i\}_{i=0}^{r-1} \) be an AS for the sequences of functions \( g_i: (U, x_0) \to \mathbb{C} \) and \( \overline{g}_i: (\overline{U}, x_0) \to \mathbb{C} \) \((i < r + 1)\), respectively, such that these are related through a biholomorphic coordinate transformation \( \phi: (\overline{U}, x_0) \to (U, x_0) \) \((i.e., \overline{g}_i(x) = g_i \circ \phi(x) \) and \( \overline{f}_i(y, x) := f_i(y, \phi(x)) \phi'(x) \)). Then \( \overline{g}_i^{[n]} \) is well-defined and equal to \( g_i^{[n]} \circ \phi \) for all \( n < r \) and \( i \leq n \) and the second AS is also proper.

It is also possible to transform the AS by a linear transformation of the first argument of the \( f_i \), as is formulated in the following Proposition. Its proof boils down to just a straightforward calculation like in the proof of Proposition 3.1.

**Proposition 3.4.** Let \( \{a_i\}_{i=0}^{r-1}, \{f_i\}_{i=0}^{r-1} \) be an AS for a sequence of functions \( g_i: (U, x_0) \to \mathbb{C} \) \((i < r + 1)\). Then \( \overline{f}_i(y, x) := f_i(a_i^{-1}(y - b), x) \) \((a \neq 0) \) gives rise to an AS for the sequence of functions \( \overline{g}_i: (U, x_0) \to \mathbb{C} \) \((i < r + 1)\), where \( \overline{g}_i(x) := a g_i(x) + b \). If the former system is proper, then so is the second, and we have: \( \overline{g}_i^{[n]}(x) = a g_i^{[n]}(x) + b \).

It is not possible to formulate a version of Proposition 3.4 involving a non-linear transformation \( \psi_i \) of the first argument of \( f_i \), as is seen from the identity
\[
(\psi_{i-1} \circ g_i)'(x) = \psi_{i-1}'(g_i(x)) f_i(g_{i+1}(x), x) \quad (i > 0).
\]

4 Error estimates of the approximation

In the following propositions, we denote by \( \Gamma(v, w) \) the set of \( C_1 \)-curves in our domain \( U \) connecting \( u \) with \( v \), and we define a metric \( d_U \) on \( U \) by
\[
d_U(v, w) := \inf_{\gamma \in \Gamma(v, w)} \ell(\gamma),
\]
where \( \ell(\gamma) \) is the length of the curve \( \gamma \).
where $\ell(\gamma)$ denotes the length of the path $\gamma$. More generally, let $\ell_i(\gamma)$ denote the length of the initial part of $\gamma \in \Gamma(v, w)$, from the point $v$ to the point $t$. We now prove the following simple lemma, which we will use in Theorem 4.2.

**Lemma 4.1.** Let $\gamma \in \Gamma(v, w)$ be a $C_1$-curve in $U$. Then, for integer $n \geq 0$,

$$\int_{\gamma} d_U(v, t)^n \, dt \leq \frac{\ell(\gamma)^{n+1}}{n+1}.$$  

**Proof.** We have

$$\int_{\gamma} d_U(v, t)^n \, dt \leq \int_{\gamma} \ell_i(\gamma)^n \, dt = \int_{0}^{\ell(\gamma)} s^n \, ds = \frac{\ell(\gamma)^{n+1}}{n+1}. \quad \square$$

**Theorem 4.2.**

A. Let $\{\{a_i\}_{i=0}^{r} \}, \{f_i\}_{i=0}^{r-1}$ be an AS for the sequence $\{g_i\}_{i=0}^{r}$ with $g:=g_0$, where $g_i$ has domain $U$ and $f_i$ has domain $Y_i = V_i \times U$, and where we assume that $V_i$ is convex. Let this AS also be a PAS of order $n < r + 1$. Then

$$|g(x) - g^n(x)| \leq \|D_1 f_0\|_{Y_0} \cdots \|D_1 f_{n-1}\|_{Y_{n-1}} \|g_n - g_n(x_0)\| U \frac{d_U(x_0, x)^n}{n!} \quad (x \in U). \quad (4.1)$$

Here $D_1$ denotes differentiation with respect to the first argument and $\|f\|_{Y}$ denotes the sup norm of a function $f$ on a domain $Y$.

B. Assume that moreover $n < r$. Then

$$|g(x) - g^n(x)| \leq \|D_1 f_0\|_{Y_0} \cdots \|D_1 f_{n-1}\|_{Y_{n-1}} \|f_n\|_{g_{n+1}(U) \times U} \frac{d_U(x_0, x)^{n+1}}{(n+1)!} \quad (x \in U). \quad (4.2)$$

**Proof.**

A. We will show by downward induction with respect to $i$, starting at $i = n$, that for $i \leq n$:

$$|g_i(x) - g^n_i(x)| \leq \|D_1 f_i\|_{Y_i} \cdots \|D_1 f_{n-1}\|_{Y_{n-1}} \|g_n - g_n(x_0)\| U \frac{d_U(x_0, x)^{n-i}}{(n-i)!} \quad (x \in U). \quad (4.3)$$

Then the case $i = 0$ of (4.3) yields (4.1). Clearly, (4.3) holds for $i = n$ because $g^n_i(x) = a_n = g_n(x_0)$. Now suppose that for some $i < n$ (4.3) holds with $i$ replaced by $i+1$. Then for $i$ we have the following string of (in)equalities:

$$|g_i(x) - g^n_i(x)| = \left| \int_{x_0}^{x} (f_i(g_i+1(t), t) - f_i(g^n_i+1(t), t)) \, dt \right|$$

$$\leq \inf_{\gamma \in \Gamma(x_0, x)} \int_{\gamma} |f_i(g_i+1(t), t) - f_i(g^n_i+1(t), t)| \, dt$$

$$\leq \inf_{\gamma \in \Gamma(x_0, x)} \int_{\gamma} \|D_1 f_i\|_{Y_i} |g_{i+1}(t) - g^n_{i+1}(t)| \, dt$$

$$\leq \inf_{\gamma \in \Gamma(x_0, x)} \int_{\gamma} \|D_1 f_i\|_{Y_i} \|D_1 f_{i+1}\|_{Y_{i+1}} \cdots \|D_1 f_{n-1}\|_{Y_{n-1}} \|g_n - g_n(x_0)\| U \frac{d_U(x_0, x)^{n-i-1}}{(n-i-1)!} \, dt.$$
The first equality is by substitution of \((2.2)\) and \((2.3)\). In the second inequality we used the convexity of \(V_i\) and in the third inequality the induction hypothesis. Furthermore, by Lemma 4.1

\[
\inf_{\gamma \in \Gamma(x_0,x)} \int_{\gamma} d\ell(t)^{n-i-1} |dt| \leq \inf_{\gamma \in \Gamma(x_0,x)} \frac{\ell(\gamma)^{n-i}}{(n-i)!} = \frac{dU(x_0,x)^{n-i}}{(n-i)!}.
\]

The last equality follows by continuity and the definition of \(dU\). Combine this result with the earlier inequalities. Then we conclude that \((4.3)\) holds for \(i\) unchanged, and thus the statement follows by induction.

**B.** Now we show by downward induction with respect to \(i\), starting at \(i = n\), that for \(i \leq n\):

\[
|g_i(x) - g_i^n(x)| \leq \|D_1 f_1\|_{n-1} \cdots \|D_1 f_{n-1}\|_{n-1} \|f_n\|_{g_{n+1}(U) \times U} \frac{d(x_0,x)^{n+1-i}}{(n+1-i)!} \quad (x \in U). \tag{4.4}
\]

Then the case \(i = 0\) of \((4.4)\) yields \((4.2)\). Clearly, \((4.4)\) holds for \(i = n\) because

\[
|g_n(x) - g_n^n(x)| = |g_n(x) - g_n(x_0)| = \left| \int_{x_0}^x f_n(g_{n+1}(t),t) \, dt \right| \leq \|f_n\|_{g_{n+1}(U) \times U} d(x_0,x). \tag{4.5}
\]

The proof of the induction step is analogous to what we did in part A. \(\square\)

**Remark 4.3.** Because the sets \(U, V_0, \ldots, V_n\) are open, the sup norms at the right-hand side of \((4.1)\) and \((4.2)\) may possibly be infinite, in which case the theorem becomes a trivial statement. We will often use Theorem 4.2 in cases where the sets \(U, V_0, \ldots, V_n\) are bounded and where the analytic functions under consideration extend to analytic functions on open neighbourhoods of the closures of these sets. Then the sup norms are finite.

**Remark 4.4.** Substitution of the inequality \((4.3)\) for \(|g_n(x) - g_n(x_0)|\) in \((4.1)\) would have given a weaker form of the inequality \((4.2)\), with the denominator \((n+1)!\) replaced by \(n!\).

If the domain \(U\) is starlike with respect to \(x_0\) then \(dU(x_0,x) = |x-x_0|\) for all \(x \in U\). Hence we have the following corollary to Theorem 4.2.

**Corollary 4.5.**

**A.** Assume the same conditions as in Theorem 4.2, and further assume that \(U\) is starlike with respect to the basepoint \(x_0 \in U\). Then

\[
|g(x) - g^n(x)| \leq \|D_1 f_0\|_{n-1} \cdots \|D_1 f_{n-1}\|_{n-1} \|g_n - g_n(x_0)\|_{g_{n+1}(U) \times U} \frac{|x-x_0|^n}{n!} \quad (x \in U). \tag{4.6}
\]

**B.** If moreover \(n < r\) then

\[
|g(x) - g^n(x)| \leq \|D_1 f_0\|_{n-1} \cdots \|D_1 f_{n-1}\|_{n-1} \|f_n\|_{g_{n+1}(U) \times U} \frac{|x-x_0|^{n+1}}{(n+1)!} \quad (x \in U). \tag{4.7}
\]
Example 4.6. Consider the PAS based on the Taylor series as described in Example 2.7. There $f_i = \text{id}$ and $V_i = \mathbb{C}$ for all $i$. Furthermore, $g_n = g^{(n)}$ and $g^{[n]}$ is the $n$-th order Taylor approximation of $g$. Then (4.1) and (4.2) respectively give:

$$|g(x) - g^{[n]}(x)| \leq \|g^{(n)} - g^{(n)}(x_0)\| U \frac{|x - x_0|^n}{n!},$$

$$|g(x) - g^{[n]}(x)| \leq \|g^{(n+1)}\| U \frac{|x - x_0|^{n+1}}{(n+1)!}. $$

These estimates of the remainder term coincide with familiar estimates in Taylor’s Theorem. Thus, it makes good sense to consider this $g^{[n]}$ as an $n$-th order approximation of $g$ which generalizes the $n$-th order Taylor approximation.

Remark 4.7. For a PAS with $r = \infty$ it would be desirable to have that $g^{[n]}(x) \to g(x)$ pointwise or uniform on some neighbourhood of $x_0$ as $n \to \infty$. In the generality of Theorem 4.2 this cannot be concluded from (4.1) or (4.2). However, things improve if we suppose moreover that all functions $f_i$ are the same function $f$ on $Y_i = Y = V \times U$. Then (4.2) yields:

$$|g(x) - g^{[n]}(x)| \leq \|f\| Y \frac{(\|D_1 f\| Y)^n d(x_0, x)^{n+1}}{(n+1)!} \quad (x \in U),$$

(4.8)

which implies uniform convergence on $Y$ if $f$ and $D_1 f$ are bounded on $Y$. Of course, this is still under the assumption that we are dealing with a PAS. In particular, it is required that $g_i^{[n]}(U) \subset V$ for $i < n$. If moreover $a_i = a$ for all $i$ then the requirement simplifies by (2.7) to $g_i^{[m]}(U) \subset V$ for $m < n$. Then, if the AS is such that $g_i = g$ for all $g$ and if we take for $V$ an open disk around $a$ and for $U$ an open disk around $x_0$ such that $f$ and $D_1 f$ are bounded on $V \times U$, then we can shrink $U$ to a sufficiently small open disk around $x_0$ such that (1.8) yields for all $n$ that $g^{[n]}(x) \in V$ for $x \in U$. Of course, this is a classical argument which was used for the convergence proof of Picard iteration, see Example 2.8. The error estimate (4.8) is also classical in the context of Picard iteration, see the last formula in [H Ch. 1, §3].

5 Convergence criterion

Lemma 5.1. Let

$$g_i'(x) = f_i(g_{i+1}(x), x).$$

(5.1)

Then, under assumption of sufficient differentiability of the functions involved, we have for $n \geq 1$ that

$$g_i^{(n+1)}(x) = (D_1 f_i)(g_{i+1}(x), x) g_i^{(n)}(x) + (D_2^n f_i)(g_{i+1}(x), x)$$

$$+ \sum_{m=2}^{n} \sum_{k=1}^{m} (D_1^k D_2^{m-k} f_i)(g_{i+1}(x), x) p_{m,k}(g_{i+1}(x), \ldots, g_i^{(n-m+1)}(x)),$$

(5.2)

where $p_{m,k}$ is a polynomial with nonnegative coefficients.
Proof. This is by induction with respect to $n$, where the case $n = 1$ is:

$$g''_i(x) = (D_1 f_i)(g_{i+1}(x), x) g'_i(x) + (D_2 f_i)(g_{i+1}(x), x).$$

As an immediate corollary we obtain:

**Lemma 5.2.** Assume that for given $f_i$ the derivative $g'_i$ depends on the function $g_{i+1}$ according to (5.1), and assume sufficient differentiability of the functions involved. Then, in a given point $x_0$, the values $g'_i(x_0), g''_i(x_0), \ldots, g^{(n+1)}_i(x_0)$ are completely determined by the values $g_{i+1}(x_0), g'_{i+1}(x_0), \ldots, g^{(n)}_{i+1}(x_0)$.

**Proposition 5.3.** Let $\{a_i\}_{i=0}^r, \{f_i\}_{i=0}^{r-1}$ be an AS. Then:

1. Equations (2.3) completely determine $g^{(j)}(x_0)$ for $i + j < r + 1$ in terms of the AS.

2. For fixed $\{a_i\}_{i=0}^r$ the derivatives $g^{(j)}(x_0) (j = 0, \ldots, r)$ are completely determined by $\{a_i\}_{i=0}^r$.

3. If $r = \infty$ and if the AS is for the sequence $\{g_i\}_{i=0}^\infty$, then the functions $g_i$, and hence $g$, are uniquely determined by the AS.

Proof. We only need to prove part 1. Let $i + j < r + 1$. By Lemma 5.2 we see that $g_i(x_0), g'_i(x_0), \ldots, g^{(j)}(x_0)$ are completely determined by $a_i, g_{i+1}(x_0), g'_{i+1}(x_0), \ldots, g^{(j-1)}_{i+1}(x_0)$. By iteration, $g_i(x_0), g'_i(x_0), \ldots, g^{(j)}(x_0)$ are completely determined by $a_i, a_{i+1}, \ldots, a_{i+j}, g_{i+j}(x_0)$, hence by $a_i, \ldots, a_{i+j}$.

**Corollary 5.4.** Let us have a PAS $\{a_i\}_{i=0}^r, \{f_i\}_{i=0}^{r-1}$ as in Definition 2.4. Then, for $n < r + 1$,

$$g^{[n]}(x_0) = g^{(j)}(x_0) \quad (j \leq n).$$

Also, for fixed $\{f_i\}_{i=0}^{n-1}$, the derivatives $g^{[n]}(x_0) (j = 0, \ldots, n)$ are completely determined by $\{a_i\}_{i=0}^n$.

Proof. Let $n < r + 1$. Then $\{a_i\}_{i=0}^n, \{f_i\}_{i=0}^{n-1}$ is an AS both for $g_i\}_{i=0}^n$ and for $g_i^{[n]}\}_{i=0}^n$. Then the result follows from Proposition 5.3.

**Remark 5.5.** If in the PAS $\{a_i\}_{i=0}^r, \{f_i\}_{i=0}^{r-1}$ for the sequence of functions $g_i$ we have that $x_0 = 0, a_i = 0, f_i(y, x)$ is even in $y$ and in $x$, and $g_i$ is odd, then we can do better and we conclude that for $n < \frac{1}{2}r + 1$

$$g^{[n]}(0) = g^{(j)}(0) \quad (j \leq 2n).$$

Indeed, it then follows by induction from (2.3) that all $g^{[n]}_i$ are odd. Next, by the proof of Proposition 5.3 (where the dependence on the $a_i$ now can be omitted since these are 0) we have for $i + j < r + 1$ and $j$ odd that $g'_0, g^{(3)}_0, \ldots, g^{(j)}_i(0)$ are completely determined by $g^{(3)}_{i+1}(0), g^{(3)}_{i+1}(0), \ldots, g^{(j-2)}_{i+1}(0)$, and hence by iteration by $g^{(3)}_{i+1/2}(j-1)(0)$, and hence completely
determined by the $f_i$. A similar reasoning applies to the $g_i^{[n]}$ for $j \leq 2n - 2i - 1$. Thus (5.4) follows.

Similarly, if in the PAS $\{(a_i)_{i=0}^r, \{f_i\}_{i=0}^{r-1}\}$ for the sequence of functions $g_i$ we have that $x_0 = 0$, $f_i(y, x)$ is odd in $x$, and $g_i$ is even, then we can again conclude that for $n < \frac{1}{2}r + \frac{1}{2}$ (5.4) holds (for $j \leq 2n + 1$). Indeed, it then follows by induction from (2.3) that all $g_i^{[n]}$ are even. Next, by the proof of Proposition 5.3 we have for $i + j < r + 1$ and $j$ even that $g_i(0), g_i^{(2)}(0), \ldots, g_i^{(j)}(0)$ are completely determined by $a_i, g_i(0), g_i^{(2)}(0), \ldots, g_i^{(j-2)}(0)$, and hence by iteration by $a_i, a_i+1, \ldots, a_i+\frac{j}{2}$. A similar reasoning applies to the $g_i^{[n]}$ for $j \leq 2n - 2i$. Thus the claim follows.

We will now generalize to the case of an AS the idea of majorizing a power series in absolute value by a power series with nonnegative coefficients.

**Definition 5.6.** An AS $\{(a_i)_{i=0}^r, \{f_i\}_{i=0}^{r-1}\}$ is called positive if

$$a_i \geq 0 \quad (i < r + 1), \quad (D_0 D_2 f_i)(a_{i+1}, x_0) \geq 0 \quad (i < r, k, l \in \mathbb{Z}_{\geq 0}).$$

A positive AS $\{\tilde{a}_i\}_{i=0}^r, \{\tilde{f}_i\}_{i=0}^{r-1}\}$ is said to dominate an AS $\{(a_i)_{i=0}^r, \{f_i\}_{i=0}^{r-1}\}$ if

$$|a_i| \leq \tilde{a}_i \quad (i < r + 1), \quad |(D_0 D_2 f_i)(a_{i+1}, x_0)| \leq (D_0 D_2 \tilde{f}_i)(\tilde{a}_{i+1}, x_0) \quad (i < r, k, l \in \mathbb{Z}_{\geq 0}).$$

**Proposition 5.7.** If $\{(a_i)_{i=0}^r, \{f_i\}_{i=0}^{r-1}\}$ is a positive AS for $\{g_i\}_{i=0}^r$ then

$$g_i^{(j)}(x_0) \geq 0 \quad (i < r + 1, \ j < r - i + 1). \tag{5.5}$$

If moreover $r = \infty$ then $g_i^{(j)}(x_0) \geq 0$ for all $i$ and $j$.

**Proof.** Let $i < r + 1$ and $j < r - i + 1$. From Lemma [5.1](in particular equation (5.2)) and from the assumptions we see that $g_i^{(j)}(x_0), g_i^{(2)}(x_0), \ldots, g_i^{(j)}(x_0) \geq 0$ if $g_{i+1}^{(j)}(x_0), g_{i+1}^{(2)}(x_0), \ldots, g_{i+1}^{(j-1)}(x_0) \geq 0$. Then (5.5) follows by iteration.

**Proposition 5.8.** Let $\{\tilde{a}_i\}_{i=0}^r, \{\tilde{f}_i\}_{i=0}^{r-1}\}$ be a positive AS for $\{\tilde{g}_i\}_{i=0}^r$ which dominates an AS $\{(a_i)_{i=0}^r, \{f_i\}_{i=0}^{r-1}\}$ for $\{g_i\}_{i=0}^r$.

A. Then

$$|g_i^{(j)}(x_0)| \leq \tilde{g}_i^{(j)}(x_0) \quad (i < r + 1, \ j < r - i + 1). \tag{5.6}$$

B. If moreover, for some $n < r + 1$,

$$|g_i^{(j)}(x_0)| \leq \tilde{g}_i^{(j)}(x_0) \quad (i < n, \ j \in \mathbb{Z}_{\geq 0})$$

then

$$|g_i^{(j)}(x_0)| \leq \tilde{g}_i^{(j)}(x_0) \quad (i \leq n, \ j \in \mathbb{Z}_{\geq 0}).$$

C. Assume moreover that $\{(a_i)_{i=0}^r, \{f_i\}_{i=0}^{r-1}\}$ is a PAS of order $n$ and that $\{(a_i)_{i=0}^\infty, \{f_i\}_{i=0}^{\infty}\}$ extends to a positive AS $\{(\tilde{a}_i)_{i=0}^\infty, \{\tilde{f}_i\}_{i=0}^{\infty}\}$ for $\{\tilde{g}_i\}_{i=0}^\infty$. Then

$$|g_i^{(j)}(x_0)| \leq \tilde{g}_i^{(j)}(x_0) \quad (i \leq n, \ j \in \mathbb{Z}_{\geq 0}). \tag{5.7}$$
Proof.

A. Let $i < r + 1$ and $j < r - i + 1$ and have assumptions as in the statement of A. If $|g_{i+1}^{(k)}(x_0)| \leq \tilde{g}_{i+1}^{(k)}(x_0)$ for $k = 1, \ldots, j - 1$ then it follows by (5.2) that

$$|g_i^{(j)}(x_0)| \leq \sum_{m=1}^{j-1} \sum_{k=0}^{m} \left| \left( D_1^k D_2^{m-k} f_i \right)(a_{i+1}, x_0) \right| p_{m,k} \left( |g_{i+1}^{(j)}(x_0)|, \ldots, |g_{i+1}^{(j-m)}(x_0)| \right)$$

$$\leq \sum_{m=1}^{j-1} \sum_{k=0}^{m} \left( D_1^k D_2^{m-k} \tilde{f}_i \right)(\tilde{a}_{i+1}, x_0) p_{m,k} \left( \tilde{g}_{i+1}^{(j)}(x_0), \ldots, \tilde{g}_{i+1}^{(j-m)}(x_0) \right)$$

$$= \tilde{g}_i^{(j)}(x_0).$$

Then (5.6) follows by iteration as in the proof of Proposition 5.7.

B. The proof is analogous to the proof of A.

C. Let $i \leq n$. Then $|g_i^{[i]}(x_0)| = |a_i| \leq \tilde{a}_i = \tilde{g}_i(x_0)$. Furthermore, for $j > 0$ we have $|g_i^{[j]}(x_0)| = 0 \leq \tilde{g}_i^{(j)}(x_0)$, where the last inequality follows from the last statement of Proposition 5.7. Now (5.7) follows by using B.

Theorem 5.9. Let $\{\tilde{a}_i\}_{i=0}^{\infty}, \{\tilde{f}_i\}_{i=0}^{\infty}$ be a positive AS for $\{\tilde{g}_i\}_{i=0}^{\infty}$ which dominates a PAS $\{a_i\}_{i=0}^{\infty}$, $\{f_i\}_{i=0}^{\infty}$ for $\{g_i\}_{i=0}^{\infty}$. Take $R > 0$ such that $x \in U$ if $|x - x_0| \leq R$. Then $g^{[n]} \rightarrow g$ as $n \rightarrow \infty$, uniformly for $|x - x_0| \leq R$.

Proof. For $|x - x_0| \leq R$ we have

$$|g(x) - g^{[n]}(x)| = \sum_{j=0}^{\infty} \frac{g^{(j)}(x_0) - g^{[n]}^{(j)}(x_0)}{j!} (x - x_0)^j$$

$$= \left| \sum_{j=n+1}^{\infty} \frac{g^{(j)}(x_0) - g^{[n]}^{(j)}(x_0)}{j!} (x - x_0)^j \right| \leq \sum_{j=n+1}^{\infty} \frac{2 \tilde{g}^{(j)}(x_0)}{j!} R^j,$$

where we have used (5.3), (5.6) and (5.7). Now use that the power series of $\tilde{g}(x)$ around $x_0$ has positive coefficients and is convergent for $x = x_0 + R$.

Proposition 5.10. Let $\{a_i\}_{i=0}^{\infty}, \{f_i\}_{i=0}^{\infty}$ be a positive AS for $\{g_i\}_{i=0}^{\infty}$ such that for some $R > 0$ we have $U = B(x_0, R)$ and $B(a_{i+1}, g_{i+1}(x_0 + r)) \subset V_i$ for all $r \in (0, R)$. Then $\{a_i\}_{i=0}^{\infty}, \{f_i\}_{i=0}^{\infty}$ is also a PAS for $\{g_i\}_{i=0}^{\infty}$.

Proof. Fix $n \in \mathbb{Z}_{\geq 0}$ and $r \in (0, R)$. We will show by downward induction with respect to $i$, starting at $i = n$, that $g_i^{[n]}(x) \leq g_i(x)$ if $0 \leq x - x_0 \leq r$ and $|g_i^{[n]}(x) - a_i| \leq g_{i+1}(x_0 + r) - a_{i+1}$ if $|x - x_0| \leq r$. For $i = n$ we have $g_i^{[n]}(x) = a_n$ and the assertions are clear. Suppose the assertions hold for $i$ replaced by $i + 1$. Then

$$g_i^{[n]}(x) = a_i + \int_{x_0}^{x} f_i (g_i^{[n]}(t), t) \, dt.$$
Hence, for \( 0 \leq x - x_0 \leq r \) we have
\[
g_i^{[n]}(x) \leq a_i + \int_{x_0}^{x} f_i(g_{i+1}(t), t) \, dt = g_i(x).
\]
Also, for \( |x| \leq r \) we have
\[
|g_i^{[n]}(x + x_0) - a_i| \leq g_i^{[n]}(x_0 + r) - a_i \leq \int_{x_0}^{x} f_i(g_{i+1}(x_0 + r), t) \, dt = g_i(x_0 + r) - a_i \tag{5.9}
\]

**Remark 5.11.** It is possible to apply Proposition 5.10 to a specific category of FDE’s from Example 2.9. With the notation and conventions from that example, assume that \( a \geq 0 \), \( \phi(x_0) = x_0 \), and that \( \phi \) and \( f \) have nonnegative derivatives (the function \( f \) in both its arguments) in respectively \( x \) and \( (a, x_0) \). Then, it follows by the composition and product formulas for power series, that \( f_i(y, x) = (\phi^i(y))(x) f(y, \phi^i(x)) \) also has nonnegative derivatives in \( (a, x_0) \). Hence the AS associated to our FDE is a positive AS. By Proposition 5.7 it then follows that the solution \( g \) of (2.8) has nonnegative derivatives in \( x \). Now assume that \( U \) equals \( B(x_0, R) \) for certain \( R > 0 \) and that \( g \) and \( \phi \) are analytic on some open set containing the closure of \( B(x_0, R) \). Also assume that for all positive \( r \leq R \) we have
\[
\phi(B(x_0, r)) \subset B(x_0, r).
\]
Letting \( V_i = B(a, g \circ \phi^i(x_0 + R) - a) \), the requirements for Proposition 5.10 are then satisfied, and so the AS associated to our FDE is a PAS of all orders. Therefore, application of Corollary 4.5B yields
\[
|g(x) - g^{[n]}(x)| \leq \|\phi'\|_U \|\phi^{o2}\|_U \cdots \|\phi^{on}\|_U \|D_1 f\|_{Y_0} \cdots \|D_1 f\|_{Y_{n-1}} \|f\|_Y \frac{R^{n+1}}{(n+1)!},
\]
where \( Y_i = B(a, g \circ \phi^i(x_0 + R) - a) \times \phi^i(U) \). Since \( \phi \) is an endomorphism on \( U \) we have
\[
(\phi^{on})^j(x) = \phi'((\phi^{o-1})^j(x)) \phi'((\phi^{o-2})^j(x)) \cdots \phi' \leq \|\phi'\|_U \quad (x \in U).
\]
This implies the more simple estimates
\[
|g(x) - g^{[n]}(x)| \leq \|\phi'\|_U^n \|\phi^{o(n+1)}\|_U \|D_1 f\|_{Y_0} \cdots \|D_1 f\|_{Y_{n-1}} \|f\|_Y \frac{R^{n+1}}{(n+1)!} \tag{5.8}
\]
\[
\leq \|\phi'\|_{B(x_0, R)}^n \|D_1 f\|_{B(a, g(x_0 + R) - a) \times B(x_0, R)} \|f\|_{B(a, g(x_0 + R) - a) \times B(x_0, R)} \frac{R^{n+1}}{(n+1)!}. \tag{5.9}
\]
From (5.9) it is clear that \( g^{[n]}(x) \) converges uniformly to \( g(x) \) on \( B(0, R) \) as \( n \to \infty \).
6 Examples

In some of the examples below we will use the Chebyshev polynomials $T_p$ of the first kind and $U_p$ of the second kind (see \[2\] §10.11, (2), (22), (23)):

$$T_p(\cos \theta) := \cos(p\theta), \quad T_p(x) = \frac{1}{2^p} \sum_{k=0}^{[p/2]} \frac{(-1)^k (p - k - 1)!}{k!(p - 2k)!} (2x)^{p-2k},$$

$$U_p(\cos \theta) := \frac{\sin((p + 1)\theta)}{\sin \theta}, \quad U_p(x) = \sum_{k=0}^{[p/2]} \frac{(-1)^k (p - k)!}{k!(p - 2k)!} (2x)^{p-2k}.$$

We will also use the polynomial $T^+_p$, which is defined as:

$$T^+_p(x) := i^{-p} T_p(ix), \quad T^+_p(x) = \frac{1}{2^p} \sum_{k=0}^{[p/2]} \frac{(p - k - 1)!}{k!(p - 2k)!} (2x)^{p-2k}.$$

We deduce that

$$T^+_p(\sinh x) = T^+_p(i^{-1} \sin(ix)) = i^{-p} T_p(\sin(ix)) = i^{-p} T_p(\cos(\frac{1}{2}p\pi - ix)) = i^{-p} \cos(\frac{1}{2}p\pi - ipx),$$

which equals $\cos(ipx)$ for $p$ even and $i^{-1} \sin(ipx)$ for $p$ odd. Hence

$$T^+_p(\sinh x) = \begin{cases} \cosh(px) & \text{ (p even)}, \\ \sinh(px) & \text{ (p odd)}. \end{cases}$$

We start with some examples which are special cases of an AS determined by a FDE, as has been described in Example 2.9. The underlying FDE’s with their solutions are

$$g'(x) = (g(x/p))^p, \quad g(0) = 1, \quad g(x) = e^x, \quad (6.1)$$

$$g'(x) = T^+_p(g(x/p)), \quad g(0) = 0, \quad g(x) = \sinh x \quad (p \text{ even}), \quad (6.2)$$

$$g'(x) = (-1)^{p/2} T_p(g(x/p)), \quad g(0) = 0, \quad g(x) = \sin x \quad (p \text{ even}),$$

$$g'(x) = \sinh(x/p) U_{p-1}(g(x/p)), \quad g(0) = 1, \quad g(x) = \cosh x,$$

$$g'(x) = - \sin(x/p) U_{p-1}(g(x/p)), \quad g(0) = 1, \quad g(x) = \cos x.$$

Note that in (6.1) and (6.2) with $p$ odd the only analytic solution $g$ of the FDE with $g(0) = 0$ would be the function identically zero.

Example 6.1. Let $p \in \mathbb{Z}_{\geq 1}$, $R > 0$, and consider the AS obtained by the choices $f(y,x) := y^p$, $\phi(x) := x/p$, $x_0 := 0$, $g(x) := e^x$, $U := B(0,R)$, $V_i := B(1,\exp(R/p^{i+1}) - 1) \subset B(0,\exp(R/p^{i+1}))$ in Example 2.9. Note that $\phi$ leaves the basepoint $0$ invariant. Thus $a_i = a = g(0) = 1$ for all $i$. Furthermore,

$$g_i(x) = \exp(x/p^i) \quad (x \in U), \quad f_i(y,x) = p^{-i} y^p \quad ((y,x) \in V_i \times U).$$
By induction we see that \( g_{n-k}^{[n]} \) is a polynomial of degree \( (p^k - 1)/(p - 1) \). Hence \( g^{[n]} \) is a polynomial of degree \( (p^n - 1)/(p - 1) \).

The AS satisfies the conditions of Remark 5.11. Hence, it is proper and by (5.8) we get for \( |x| < R \) the estimate

\[
|g(x) - g^{[n]}(x)| \leq \|\phi''\| \frac{1}{U} \frac{n(n+1)}{R^n} \prod_{i=0}^{n-1} \|D_i f\|_Y \cdot \|f\|_Y \cdot \frac{p^{n+1}}{(n+1)!}
\]

\[
\leq p^{\frac{1}{2}n(n+1)} \prod_{j=0}^{n-1} p \exp((p - 1)R/p^{j+1}) \cdot \exp(R/p^n) \cdot \frac{R^{n+1}}{(n+1)!}
\]

\[
e e^{R} R^{n+1} \frac{p^{n(n-1)} (n+1)!}{n+1},
\]

which converges to 0 as \( n \) goes to infinity. For \( p = 2 \), the first four approximations are given by

\[
g^{[0]}(x) = 1, \quad g^{[1]}(x) = 1 + x, \quad g^{[2]}(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{12},
\]

\[
g^{[3]}(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{7x^4}{192} + \frac{x^5}{192} + \frac{x^6}{2304} + \frac{x^7}{64512}.
\]

Note that formula (5.3) is indeed satisfied: the power series of \( g^{[n]}(x) \) and \( g(x) = e^x \) coincide up to the term with \( x^n \).

**Example 6.2.** Let \( p \in \mathbb{Z}_{\geq 1} \), and consider the AS obtained by the choices \( f(y, x) := T_p^+(y) \), \( \phi(x) := x/p, x_0 := 0, g(x) := \sinh x, U := B(0, R), V_i := B(0, \sinh(R/p^{i+1})) \) in Example 2.9. Note that \( \phi \) leaves the basepoint 0 invariant, so \( a_i = 0 \) for all \( i \). Furthermore,

\[
g_i(x) = \sinh(x/p^i) \quad (x \in U), \quad f_i(y, x) = p^{-i}T_p^+(y) \quad ((y, x) \in V_i \times U).
\]

The AS satisfies the conditions of Remark 5.11. Hence, it is proper and by (5.8) we get for \( |x| < R \) the estimate

\[
|g(x) - g^{[n]}(x)| \leq \|\phi''\| \frac{1}{U} \frac{n(n+1)}{R^n} \prod_{i=0}^{n-1} \|D_i f\|_Y \cdot \|f\|_Y \cdot \frac{p^{n+1}}{(n+1)!}
\]

\[
\leq p^{\frac{1}{2}n(n+1)} \prod_{j=0}^{n-1} p \exp((p - 1)R/p^{j+1}) \cdot \exp(R/p^n) \cdot \frac{p^{n+1}}{(n+1)!}
\]

\[
= \frac{p^{n+1} \sinh R}{p^{2(n-1)} (n+1)!} \cdot \frac{p^{n+1}}{\cosh(R/p^n)} \cdot \prod_{j=1}^{n-1} \tan(R/p^j)
\]

\[
\leq \frac{p^{n+1} e^R}{2 p^{2(n-1)} (n+1)!}.
\]
For $p = 2$, the first four approximations are given by:

$$g^{[0]}(x) = 0, \quad g^{[1]}(x) = x, \quad g^{[2]}(x) = x + \frac{x^3}{6}, \quad g^{[3]}(x) = x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{8064}.$$  

Here, even better than in Example 6.1, $g(x)$ and $g^{[n]}$ (which are both odd functions) have power series which agree up to the (zero) term of degree $2n$. That this holds for all $n \geq 0$ was explained in Remark 5.5.

**Remark 6.3.** By applying a coordinate transformation $\phi: U \rightarrow U$, $\phi(x) = ix$ on the set $U = B(0, R)$ of the previous example, we obtain by Proposition 3.1 and Lemma 3.3 a new PAS with respect to the function sequence $\hat{g}_k(x) = g_k \circ \phi(x) = \sinh(ix/p^k) = i \sin(x/p^k)$, where $\hat{f}_k(y, x) = \phi'(x) f_k(y, \phi(x)) = ip^{-k}T^+_p(y)$. Next by applying Proposition 3.4 with $a = -i$ and $b = 0$, we get a PAS with respect to $\hat{g}_k(x) = \sin(x/p^k)$, where $\hat{f}_k(y, x) = p^{-k} T^+_p(iy) = (-1)^{k/2} p^{-k} T_p(y)$ ($p$ is still assumed to be even). The error estimate of the induced approximations can again be given by Corollary 4.5. It turns out to be equal to the estimate given in Example 6.2.

**Example 6.4.** Consider the AS obtained by the choices $f(y, x) := \sinh(x/p) U_{p-1}(y)$ ($p \geq 2$), $\phi(x) := x/p, x_0 := 0, g(x) := \cosh x, U := B(0, R), V_j := B(1, \cosh(R/p^{j+1}) - 1)$ in Example 6.1. As before, $\phi$ leaves the basepoint 0 invariant, so $a_j = 1$ for all $j$. The AS is a positive system (use 2.10.11(27), 10.9(3))) and it satisfies the conditions of Remark 5.11. Hence by (5.19) we have the estimate

$$|g(x) - g^{[n]}(x)| \leq p^{-\frac{1}{4}n(n+1)} \left( ||D_1 f||_{B(1, \cosh R-1) \times B(0, R)} \right)^n \left( ||f||_{B(1, \cosh R-1) \times B(0, R)} \right) \frac{R^{n+1}}{(n+1)!} \quad (x \in B(0, R)).$$

Again by the positivity of the system we have

$$||f||_{B(1, \cosh R-1) \times B(0, R)} = f(\cosh(R/p), R) = \sinh(R/p) U_{p-1}(\cosh(R/p)) = \sinh R,$$

$$||D_1 f||_{B(1, \cosh R-1) \times B(0, R)} = (D_1 f)(\cosh(R/p), R) = \sinh(R/p) U'_{p-1}(\cosh(R/p)).$$

Also observe that

$$\sinh t U'_{p-1}(\cosh t) = \frac{d}{dt} \left( \frac{\sinh(pt)}{\sinh t} \right) = \frac{\sinh(pt)}{\sinh t} (p \coth(pt) - \coth t) < \frac{\sinh(pt)}{\sinh t} (p - 1) \quad (t > 0),$$

since an elementary analysis yields that $p \coth(pt) - \coth t$ increases from 0 to $p - 1$ as $t$ runs from 0 to $\infty$. Hence

$$\sinh(R/p) U'_{p-1}(\cosh(R/p)) < (p - 1) \frac{\sinh R}{\sinh(R/p)}.$$  

Altogether,

$$|g(x) - g^{[n]}(x)| \leq \frac{R \sinh R}{p^{\frac{1}{2}n(n+1)}(n+1)!} \left( \frac{(p - 1) \sinh R}{\sinh(R/p)} \right)^n \quad (x \in B(0, R)),$$

which converges to 0 as $n$ goes to infinity.
Remark 6.5. Somewhat similarly as in Remark 6.3 we can apply a coordinate transformation \( \phi: x \mapsto ix: U \to U \) to the set \( U = B(0, R) \) of the previous example. Then we obtain by Proposition 5.1 and Lemma 5.3 a new PAS with respect to the function sequence \( \hat{g}_j(x) = g_j \circ \phi(x) = \cos(x/p^j) \), where \( f_j(y, x) = \phi'(x)f_j(y, \phi(x)) = -p^{-j}\sin(x/p)U_{p-1}(y) \). Again, the error estimate of the induced approximations is the same as the estimate given in Example 6.4. For \( p = 2 \) the first four approximations \( g^{[n]}(x) \) in powers of \( \sin(2^{-n-1}x) \) can be computed as follows.

\[
\begin{align*}
g^{[0]}(x) &= 1, \\
g^{[1]}(x) &= 1 - 8 \sin^2 \frac{x}{4}, \\
g^{[2]}(x) &= 1 - 32 \sin^2 \frac{x}{8} + 160 \sin^4 \frac{x}{8} - \frac{512}{3} \sin^6 \frac{x}{8}, \\
g^{[3]}(x) &= 1 - 128 \sin^2 \frac{x}{16} + 2688 \sin^4 \frac{x}{16} - 21504 \sin^6 \frac{x}{16} + \frac{245248}{3} \sin^8 \frac{x}{16} - \frac{2326528}{15} \sin^{10} \frac{x}{16} + \frac{425984}{3} \sin^{12} \frac{x}{16} - \frac{1048576}{21} \sin^{14} \frac{x}{16}.
\end{align*}
\]

Similarly, we can expand \( g(x) = \cos x \) in powers of \( \sin(2^{-n-1}x) \), by using that

\[
\cos(2mt) = (-1)^m \cos(m \pi - 2mt) = (-1)^m T_{2m}(\cos(\frac{1}{2} \pi - t)) = (-1)^m T_{2m}(\sin t),
\]

hence \( \cos x = T_{2n+1}(\sin(2^{-n-1}x)) \) \((n > 0)\).

Both \( g^{[n]} \) and \( g \) are even functions and their expansions in powers of \( \sin(2^{-n-1}x) \) agree up to the (zero) term with power \( 2n + 1 \). This is true for general \( n \) by the second part of Remark 6.5.

There the argument was given for expansions in powers of \( x \). But then the result for expansions in powers of \( \sin(2^{-n-1}x) \) immediately follows.

Example 6.6. Consider the AS obtained by the choices \( U = V = \mathbb{C}, f(y, x) := y, \phi(x) := x + \alpha^{-1}\log \alpha \ (\alpha \in (0, 1)), x_0 := 0, g(x) := e^{\alpha x} \) in Example 5.9. It follows that \( f_i(y, x) = y, g_i(x) = \alpha^i e^{\alpha x}, a_i = \alpha^i \). The AS is proper. One easily verifies that

\[
g^{[n]}_i(x) = \alpha^i \sum_{k=0}^{n-i} \frac{\alpha^k x^k}{k!}, \quad \text{hence} \quad g^{[n]}(x) = \sum_{k=0}^{n} \frac{\alpha^k x^k}{k!}.
\]

Thus the \( n \)-th order approximation coincides with the \( n \)-th order Taylor approximation. Contrary to the previous examples in this section we have here a special case of Example 5.9 where \( \phi \) has no fixpoint and still everything can be worked out explicitly.

Example 6.7. Let \( p \in \mathbb{Z}_{\geq 2} \). Put \( \lambda_{p,0} := 1 \) and \( \lambda_{p,i} := (p^i - 1)/(p^{i+1} - p^i) \) for \( i \geq 1 \). Let \( R \in (0, 1) \). Take

\[
U := D(0, R), \quad x_0 := 0, \quad g_0(x) := \log \left( \frac{1}{1 - x} \right), \quad g_i(x) := \left( \frac{1}{1 - x} \right)^{\lambda_{p,i}} \quad (i \geq 1),
\]

\[
f_i(y, x) := \lambda_{p,i} y^p.
\]

Then we have a positive AS with \( a_i = 1 - \delta_{i,0} \). Choose \( V_i := B(1, (1 - R)^{-\lambda_{p,i}} - 1) \) for the domain of \( f_i \). Then application of Proposition 5.10 yields after some computation that \( \{a_i\}_{i=0}^\infty, \{f_i\}_{i=0}^\infty \)
is a PAS. So by Corollary 4.5B we have for $x \in U$ that

$$|g(x) - g^{[n]}(x)| \leq \|f'_0\|_{V_0} \cdots \|f'_{n-1}\|_{V_{n-1}} \|f_n\|_{V_n} \frac{|x - x_0|^{n+1}}{(n+1)!}$$

$$\leq \prod_{i=0}^{n-1} p \lambda_{p,i} \left((1 - R)^{-\lambda_{p,i}}\right)^{p-1} \lambda_{p,n} \left((1 - R)^{-\lambda_{p,n}}\right)^p \frac{R^{n+1}}{(n+1)!},$$

where we used that $\lambda_{p,i} \leq 1/(p - 1)$ for $i \geq 1$. For $p = 2$ the first four approximations are given by

$$g^{[0]}(x) = 0,\quad g^{[1]}(x) = x,\quad g^{[2]}(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{9x^4}{64} + \frac{3x^5}{64} + \frac{3x^6}{256} + \frac{9x^7}{7168}.$$ 

**Remark 6.8.** In the previous example, clearly $U$ cannot be enlarged so that it contains a disk $D(0, R')$ where $R' > 1$, for the singularity at $x = 1$ cannot lie in the interior of $U$. However, when we allow $U$ to have a non-circular shape, then numerical simulations indicate that the actual domain of convergence can become considerably bigger than the open disk with center 0 and radius 1 (although $x = 1$ still remains a boundary point of course). The precise domain of convergence we have not yet established, but it seems to have a quite irregular shape.

**Remark 6.9.** In all examples in this section, we have been able to calculate the approximations provided by the algorithm explicitly. However, in many cases the approximation $g^{[n]}_i$ is a polynomial whose degree is exponential in the number $n - i$ of steps made. Because the computation time of each new step depends at least linearly on the input size of $g^{[n]}_i$, also the computational effort tends to increase exponentially as the order of the approximation increases. For practical purposes, this can become problematic when $n$ gets large, even when making use of software like Mathematica. It is therefore worth noting that the algorithm suits itself well for numerical implementation. This can be achieved as follows: suppose we want a numerical approximation $\hat{g}^{[n]}$ of $g^{[n]}$ (which in its turn is an approximation of $g$) along a path $\gamma$, going from $x_0$ to $x \in U$. The first step is to partition $\gamma$ into a vector $\bar{\gamma} : \{0, 1, \ldots, N\} \to \text{Im} \gamma$, consisting of only $N + 1$ points from the original path $\gamma$. Let $\Delta_k := \bar{\gamma}(k + 1) - \bar{\gamma}(k)$ for $k = 1, \ldots, N$. Now we construct $\hat{g}^{[n]}_i : \{0, \ldots, N\} \to \mathbb{C}$ as follows:

$$\hat{g}_i^{[n]}(k) = a_n$$

$$\hat{g}_i^{[n]}(0) = a_i$$

$$\hat{g}_i^{[n]}(k) = \hat{g}_i^{[n]}(k - 1) + f_i(\hat{g}_i^{[n]}(k - 1), \bar{\gamma}(k - 1))\Delta_{k-1} \quad (k = 1, \ldots, N, \ i = n - 1, \ldots, 0).$$

The vector $\hat{g}^{[n]} := \hat{g}_0^{[n]}$ then represents a function with value $\hat{g}^{[n]}(k)$ at the point $\bar{\gamma}(k)$. This approximation of $g^{[n]}$ will become better as the partition $\bar{\gamma}$ of $\gamma$ becomes finer. Unlike exact
computation of \( g^n \), computation of \( \bar{g}^n \) carried out up to a certain numerical precision does not get significantly more complicated with each increment in \( n \). Hence, provided that the coefficients \( a_i \) are already given or computed, and that the functions \( f_i \) to be applied do not substantially increase in computational complexity, the computational effort of this numerical implementation tends to be linear in both the depth \( n \) and the partition fineness \( N \), rather than exponential. However, one should also take into account that exact computation gives in principle much more accurate results than the above algorithm. To get an increasingly accurate approximation of a target function \( g \) using the numerical algorithm, it requires besides an increment in \( n \) also a simultaneous increment in \( N \), which makes a comparison between the approaches of course less straightforward.

Another practical advantage of the numerical implementation comes up in cases when \( g^n \) cannot even be expressed in terms of known mathematical functions (cf. [5, Appendix]). Then, numerical calculation is the only option.

**Concluding remarks**

A first remark is about the assumption of analyticity for \( g_i \) and \( f_i \) throughout the paper. This is done for convenience. Many results in this article could just as well be formulated for complex-valued functions with continuous derivatives of sufficiently high order. However, note that the coefficients \( a_i \) only depend on the behavior of \( g \) at \( x_0 \) (just as with the the Taylor approximation). Hence any changes to the function \( g \) outside a neighborhood of \( x_0 \) will not affect the approximations given by the algorithm. This makes the algorithm in particular suitable for analytic functions.

Another assumption which could actually be relaxed, is that the domain \( U \) must be open in \( \mathbb{C} \). This assumption is made primarily for consistency in the conditions under which various theorems are proved. But not surprisingly, the algorithm can just as well be applied, for example, on a closed bounded interval in \( \mathbb{R} \) for a \( C^k \) function \( g \) on the interval with only left respectively right derivatives at the endpoints. If we would have worked with analytic functions on a not necessarily open set \( U \) then this should have been interpreted each time as that the function is analytic on some open neighbourhood of \( U \).

The examples we have discussed in this article mainly focus on approximation systems that arise from a single FDE, within the general framework of Example 2.9. For them we could do explicit symbolic computations and we could prove convergence of the approximations on a certain neighborhood of the basepoint \( x_0 \). The algorithm certainly also works well for examples not coming from a FDE, as we know from numerical simulations. But the main problem there is that it is more difficult to prove specific error bounds, or even to prove convergence at all. In a somewhat different formulation, a couple of such examples are given in the appendix of [5].

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