A method for putting chiral fermions on the lattice
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We describe a method to put chiral gauge theories on the lattice. Our method makes heavy use of the effective action for chiral fermions in the continuum, which is in general complex. As an example we discuss the chiral Schwinger model.

1. INTRODUCTION

In spite of the well-known importance of chiral gauge theories there is very little nonperturbative information available about such models. Standard nonperturbative techniques seem to fail, in particular the lattice approach faces great difficulties due to the notorious doubling problem. To overcome these difficulties it has been suggested to keep the fermionic degrees of freedom in the continuum and to latlicize only the gauge fields [1]. However, the details of such a procedure have not yet been worked out. In the present note we want to outline a few steps along this line (see also [2]).

2. EFFECTIVE ACTION

First of all we need the effective action for chiral fermions in the background of a continuum gauge field. This has been discussed by several groups of authors [3–5]. We shall mainly follow the approach of Alvarez-Gaumé et al. [3]. Working in euclidean space we choose hermitian $\gamma$-matrices,

$$\gamma_\mu = \gamma_\mu^+,$$

$$\{\gamma_\mu, \gamma_\nu\} = \delta_{\mu,\nu},$$

and define $\gamma_5$ such that

$$\{\gamma_5, \gamma_\mu\} = 0, \gamma_5^2 = 1, \gamma_5^+ = \gamma_5.$$

So the chiral projectors are given by

$$P_\pm = \frac{1}{2} \left( 1 \pm \gamma_5 \right).$$

The gauge fields are taken to be antihermitian:

$$A_\mu^+(x) = -A_\mu(x).$$

We write the chiral Dirac operator as

$$i\hat{D}(A) = i(\bar{\psi}A\psi).$$

Consequently, both chiralities of fermions are present, but the gauge field couples only to one of them. For the vector Dirac operator we use the standard notation

$$i\bar{D}(A) = i(\bar{\psi}A\psi).$$

Formally, i.e. up to regularization, the effective action $W[A]$ can now be expressed in terms of the functional integral

$$e^{-W[A]} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi} \exp \left( -\int d^4x\bar{\psi}(x)i\hat{D}(A)\psi(x) \right) = \det \left( i\hat{D}(A) \right).$$

Since $i\hat{D}(A)$ is not hermitian, $W[A]$ is expected to be complex in general. Up to local counterterms, the real part is given by the effective action for a Dirac fermion interacting with the gauge field $A$,

$$\text{Re}W[A] = -\frac{1}{4} \ln \det(i\bar{D}(A)),$$

and so it is gauge invariant.
For the imaginary part one obtains

\begin{align}
\text{Im}W[A] &= \frac{1}{2i} \int_0^1 d\tau \text{Tr} \gamma_5 \left( \frac{d}{d\tau} i \mathcal{D}(A_\tau) \right) \left( i \mathcal{D}(A_\tau) \right)^{-1} \\
&\quad + \frac{1}{i} \int_0^1 d\tau \text{Tr} \left( \frac{\partial A_\mu}{\partial \tau} J_\mu(A_\tau) \right).
\end{align}

(9)

Here \( \text{Tr} \) denotes the operator trace, \( A_\tau \) interpolates between \( A_0 = 0 \) and \( A_1 = A \) (e.g., \( A_\tau = \tau A \)) and the current \( J_\mu \) accounts for the difference between the covariant and the consistent anomaly. Since we are ultimately only interested in anomaly free models, where contributions from these currents cancel, we shall omit the corresponding terms in the following. (Strictly speaking, the above formula can be derived only for perturbative gauge field configurations, i.e., configurations such that the topological charge vanishes and the Dirac operator has no zero modes. However, one could more generally interpolate between two configurations \( A_0 \) and \( A_1 \) to obtain \( \text{Im}W[A_1] - \text{Im}W[A_0] \).) Note that in eq. (9) only the vector Dirac operator appears.

We can only mention in passing that \( \text{Im}W[A] \) may be expressed in terms of a topological quantity, the spectral asymmetry or \( \eta \)-invariant of a vectorlike Dirac operator in five dimensions (for a review, see \cite{[6]}).

3. METHOD OF SIMULATION

Since the effective action governing the dynamics of the gauge fields has turned out to be complex, one runs into well-known problems when attempting numerical simulations. At least in principle, however, one could incorporate \( \text{Im}W[A] \) in the observable and update the gauge fields with the action

\[ S_G + \text{Re}W = S_G - \frac{1}{2} \ln \text{det}(i \mathcal{D}), \]

(10)

where \( S_G \) denotes the pure gauge field action. Remember that \( \text{Re}W \) is given by a vectorlike fermion action, whose lattice version should be unproblematic.

Irrespective of the details of the simulation method one has to solve the question of how to calculate \( \text{Im}W \) for a given lattice gauge field configuration. A naive procedure would be to replace \( \mathcal{D} \) in eq. (4) by some lattice version of the Dirac operator, e.g. Wilson fermions. However, since \( \text{Im}W \) is related to a topological quantity, the \( \eta \)-invariant, a look at the topological charge might be advisable. In that case such a procedure does not work: An expression for the lattice topological charge constructed out of a few plaquettes only suffers from large perturbative contributions leading to a divergent topological susceptibility in the continuum limit. The geometrical constructions of the lattice topological charge, on the other hand, which avoid this problem, can be viewed as computing the continuum charge from a continuum gauge field derived from the lattice gauge field by a suitable interpolation \cite{[7], [8]}. Therefore we propose to define \( \text{Im}W \) by \( \text{Im}W[A] \) evaluated for this interpolated gauge potential \( A \). A suitable interpolation has been explicitly given in \cite{[8]}, although the construction is not completely trivial as several constraints have to be fulfilled. In particular, gauge covariance has to be guaranteed.

In order to really calculate \( \text{Im}W[A] \) one has to introduce some regularization. From the numerical point of view it seems to be most convenient to use the lattice again, i.e. to compute on finer and finer sublattices of the original lattice anticipating that the results will converge. (The rigorous analytical investigations apply other regularization schemes, e.g. Pauli-Villars.)

So one would proceed as follows. Start from a lattice gauge field \( U(x, \mu) \) on the original lattice, whose spacing is put equal to one. Construct from \( U \) an interpolated continuum gauge field \( A \). This is then used to calculate parallel transporters \( \tilde{U}(x, \mu) \) on a finer lattice of spacing \( \epsilon < 1 \). These link matrices are raised to the power \( \tau \) to arrive at a lattice version of the gauge potential \( A_\tau \). (An alternative consists in computing the parallel transporters on the finer lattice immediately from \( A_\tau \).) As a regularized version of \( \mathcal{D}(A_\tau)^{-1} \) we can then take the propagator of Wilson fermions in the gauge field \( \tilde{U}^\tau \) given by

\[ G(\tilde{U}^\tau|x, y) = \epsilon^{-8} M^{-1}(\tilde{U}^\tau|x, y), \]

(11)

with the fermion matrix

\[ M(U|x, y) \]
\[ = e^{-4 \left\{ \frac{1}{2 \epsilon} \sum_{\mu} (\gamma_{\mu} - r) U(x, \mu) \delta y, x+\hat{\mu} \right.} \]
\[ - \frac{1}{2 \epsilon} \sum_{\mu} (\gamma_{\mu} + r) U^+(y, \mu) \delta y, x-\hat{\mu} \right.} \]
\[ + \frac{4r}{\epsilon} \delta x, y \right\}. \] (12)

Finally one has to calculate
\[ \frac{1}{2i \epsilon} \int_0^1 d\tau e^{\delta} \sum_{x,y} \text{tr} \gamma_5 \left( \frac{d}{d \tau} M(\tilde{U}^\tau | x, y) \right) \times G(\tilde{U}^\tau | x, y) \]
\[ = \frac{1}{2i \epsilon} \int_0^1 d\tau e^{\delta} \sum_{x,\mu} \left\{ \text{tr} \gamma_5 \frac{1}{2 \epsilon} (\gamma_{\mu} - r) \times \ln \tilde{U}(x, \mu) \tilde{U}(x, \mu)^\tau G(\tilde{U}^\tau | x + \hat{\mu}, x) \right. \]
\[ + \text{tr} \gamma_5 \frac{1}{2 \epsilon} (\gamma_{\mu} + r) \ln \tilde{U}(x, \mu) \]
\[ \times \tilde{U}(x, \mu)^{-\tau} G(\tilde{U}^\tau | x, x + \hat{\mu}) \left\} \right. \] (13)
in the limit \( \epsilon \to 0 \), where \( \text{tr} \) denotes the trace over Dirac and internal indices.

**4. THE CHIRAL SCHWINGER MODEL**

As an example consider the chiral Schwinger model, i.e. a chiral U(1) model in two dimensions. On a square of side length \( L \) the fermionic part of the continuum action reads
\[ \int_0^L d^2 x \psi(x) \left( i \slashed{D} - A(x) \right) \psi(x), \] (14)
where now \( \gamma_5 = i \gamma_1 \gamma_2 \) and the potential \( A_\mu(x) \) is real. We assume periodic boundary conditions for the gauge field, whereas the fermion field is taken antiperiodic. Choosing the linear interpolation \( A_\tau = \tau A \) so that
\[ i \slashed{D}(A_\tau) = i \slashed{\partial} - \tau \slashed{A}(x) \] (15)
we get for the relevant contribution to \( \text{Im} W \) (up to regularization)
\[ \frac{1}{2i} \int_0^1 d\tau \text{Tr} \gamma_5 \left( \frac{d}{d \tau} i \slashed{D}(A_\tau) \right) (i \slashed{D}(A_\tau))^{-1} \]
\[ = \frac{i}{2} \int_0^1 d\tau \text{Tr} \gamma_5 \slashed{A} (i \slashed{D}(A_\tau))^{-1} \]
\[ = \frac{1}{2} \int_0^1 d\tau \int_0^L d^2 x \text{tr} \gamma_5 \slashed{A}(x) G(A_\tau | x, x), \] (16)
where the propagator \( G(A | x, y) \) is determined by the equation
\[ \slashed{D}_x(A) G(A | x, y) = \delta(x - y). \] (17)

Since in two dimensions an explicit expression for \( G(A | x, y) \) is available, we can evaluate further. Being only interested in the gauge invariant contributions we fix the Landau gauge,
\[ \partial_\mu A_\mu(x) = 0. \] (18)
The zero momentum mode of the gauge field requires a special treatment, so we define
\[ a_\mu := \frac{1}{L^2} \int_0^L d^2 x A_\mu(x) \] (19)
and assume \(-\pi/L < a_\mu < \pi/L\). Introducing point splitting we find
\[ \int_0^L d^2 x \text{tr} \gamma_5 \slashed{A}(x) G(A_\tau | x, x + \delta) \]
\[ \times \exp \left\{ -i \int_x^{x+\delta} dz A_\mu(z) \right\} \]
\[ = \frac{L}{\pi} \sum_{n \in \mathbb{Z}^2, n \neq 0} (a_2 n_1 - a_1 n_2)/n^2 \]
\[ \times \sin(L(b + \tau a) \cdot n) e^{-n^2 L^2/4} \]
\[ - \frac{4\pi}{L} \sum_{n \in \mathbb{Z}^2} (a_1 n_2 + \frac{i}{2} a_1 - a_2 n_1 - \frac{i}{2} a_2) \]
\[ \times e^{-2\pi n/L + b + \tau a)^2} / (2\pi n/L + b + \tau a)^2 \]
\[ + iL^2 (a_1 \delta_2 - a_2 \delta_1) / (\pi \delta^2) + O(|\delta|), \] (20)
where \( b_1 = b_2 = \pi/L \). Since the divergent term is odd in \( \delta \), symmetric point splitting leads to a finite limit as \( \delta \to 0 \). Only the zero momentum mode of the gauge field contributes in Landau gauge. Note that working in infinite volume one does not have this mode and hence finds \( \text{Im} W = 0 \) in Landau gauge (see, e.g., [4]).

Which results does our lattice recipe produce in this case? Unfortunately, an explicit expression for the lattice fermion propagator in a general background is not available. So we consider the special case of the gauge field configuration
\[ U(x, \mu) = e^{i a_\mu}, \quad -\pi/L < a_\mu < \pi/L, \] (21)
on the original lattice with lattice spacing $1$. It should not come as a surprise that the interpolated continuum gauge field turns out to be $A_\mu(x) = a_\mu$. So we find for the parallel transporters on the sublattice of spacing $\epsilon$

$$\tilde{U}(x, \mu) = e^{i e a_\mu}$$

and consequently

$$\ln \tilde{U}(x, \mu) = ie a_\mu, \quad \tilde{U}(x, \mu)^\tau = e^{i \tau e a_\mu}. \tag{23}$$

The required Wilson fermion propagator can easily be calculated. According to our proposal we then have to compute

$$-ie^2 \sum_{x, \mu} \left\{ \text{tr} \, \gamma_5 \frac{1}{2\epsilon} (\gamma_\mu - r) \ln \tilde{U}(x, \mu) 
\times \tilde{U}(x, \mu)^\tau G(\tilde{U}^\tau | x + \tilde{\mu}, x) 
+ \text{tr} \, \gamma_5 \frac{1}{2\epsilon} (\gamma_\mu + r) \ln \tilde{U}(x, \mu) 
\times \tilde{U}(x, \mu)^{-\tau} G(\tilde{U}^\tau | x + \tilde{\mu}, x) \right\}$$

$$= 2e \sum_k \left[ a_2 \cos(ek_2 + \tau e a_2) \sin(ek_1 + \tau e a_1) 
- a_1 \cos(ek_1 + \tau e a_1) \sin(ek_2 + \tau e a_2) \right] 
\times \left\{ \left[ \sum_{\mu} \left( \cos(ek_\mu + \tau e a_\mu) - 1 \right) \right]^2 
+ \sum_{\mu} \sin^2(ek_\mu + \tau e a_\mu) \right\}^{-1}, \tag{24}$$

where $k$ runs over the momenta appropriate for our finite lattice. In the limit $\epsilon \to 0$ one indeed recovers the expression derived above by the point splitting technique in the continuum. So our recipe for the calculation of $\text{Im}W$ from a lattice gauge field configuration works at least in this (almost trivial) case.

5. OUTLOOK

Obviously, there are many open problems which have to be solved before our proposal can become useful. For example, how can we deal with gauge field configurations leading to zero modes of the Dirac operator $D(A_\tau)$ for some value of $\tau$? According to ref. [3], for every pair of eigenvalues crossing zero, $\text{Im}W$ picks up a contribution $\pm \pi$. This problem has been ignored in our Schwinger model calculation, so there might be additional contributions to $\text{Im}W$ for certain configurations. If such configurations are statistically relevant, $\exp(i \text{Im}W)$ would fluctuate strongly and a Monte Carlo simulation might be hopeless, unless one invents a clever method to update with a complex action. In any case it would be interesting to see how other chiral fermion proposals deal with this problem.

Even the calculation of $\text{Im}W$ for a single configuration poses several numerical challenges. E.g. one has to compute zero-mass fermion propagators in a given background field. Furthermore, an operator trace has to be evaluated, which might be feasible with the help of a stochastic estimator.

Let us close with the remark that the most elegant approach would probably be to find a “geometrical” expression for $\text{Im}W$ based on the relation to the $\eta$-invariant.

REFERENCES

1. L. Alvarez-Gaumé and S. Della Pietra, in Recent developments in quantum field theory, eds. J. Ambjorn, B. J. Durhuus and J. L. Pedersen (North-Holland, Amsterdam, 1985); J. Smit, Nucl. Phys. B (Proc. Suppl.) 4 (1988) 451; A. S. Kronfeld, Nucl. Phys. B (Proc. Suppl.) 4 (1988) 329; V. Vyas, Phys. Lett. B266 (1991) 453; A. I. Karanikas and C. N. Ktorides, MIT-preprint CTP #2010.

2. M. Göckeler and G. Schierholz, talk given at the Workshop on nonperturbative aspects of chiral gauge theories (Rome, March 1992) preprint HLRZ 92-33.

3. L. Alvarez-Gaumé, S. Della Pietra and V. Della Pietra, Phys. Lett. B166 (1986) 177.

4. R. D. Ball and H. Osborn, Phys. Lett. B165 (1985) 410; Nucl. Phys. B263 (1986) 245; R. D. Ball, Phys. Lett. B171 (1986) 435.

5. A. Niemi and G. Semenoff, Phys. Rev. Lett. 55 (1985) 927.

6. R. D. Ball, Phys. Rep. 182 (1989) 1.

7. M. Lüscher, Commun. Math. Phys. 85 (1982) 39.

8. M. Göckeler, A. S. Kronfeld, G. Schier-
holz and U.-J. Wiese, preprint HLRZ 92-34, FERMILAB-PUB-92/194-T, BUTP-92/35.
9. J. Schwinger, Phys. Rev. 128 (1962) 2425.