Thermal Amplitudes
in DLCQ Superstrings on PP-Waves

Yuji Sugawara
sugawara@hep-th.phys.s.u-tokyo.ac.jp

Department of Physics, Faculty of Science,
University of Tokyo
Hongo 7-3-1, Bunkyo-ku, Tokyo 113-0033, Japan

Abstract

We calculate the thermal partition function of DLCQ superstring on the maximally supersymmetric pp-wave background, which is realized as the Penrose limit of orbifolded $AdS_5 \times S^5$ and known to be dual to the $\mathcal{N} = 2$ “large” quiver gauge theory as shown by S. Mukhi, M. Rangamani and E. Verlinde, hep-th/0204147. Making use of the path-integral technique, we derive the manifestly modular invariant expression and show the equivalence with the free energy of second quantized free superstring on this background. The “virtual strings” wound around the temporal circle play essential roles for realizing the modular invariance and for a simple analysis on the Hagedorn temperature. We also calculate the thermal one-loop amplitudes of open strings under the various backgrounds of the supersymmetric time-like and Euclidean D-branes, and confirm the existence of correct open-closed string duality. Furthermore, we extend these thermodynamical analysis to the 6-dimensional DLCQ pp-waves with general RR and NSNS flux. These superstring vacua are similarly derived from the supersymmetric (half SUSY) and non-supersymmetric orbifolds of $AdS_3 \times S^3 \times M^4$ ($M^4 = T^4$ or $K3$) by the appropriate Penrose limits, giving rise to the SUSY enhancement as in the case of orbifolded $AdS_5 \times S^5$. 
1 Introduction

String theories/M-theory on pp-wave backgrounds [1] have been recently studied with newer motivations. Among other things, it is remarkable that a new superstring vacuum with the maximal SUSY has been discovered and derived from the \( AdS_5 \times S^5 \) background by the Penrose limits [2]. This string vacuum is exactly quantizable in spite of the non-trivial RR-flux by the light-cone GS formalism [3], and provides a powerful tool to investigate the stringy nature of AdS/CFT correspondence beyond the supergravity approximation [4].

From the viewpoints of world-sheet theory, the light-cone string theories on such pp-wave backgrounds with the RR-flux exhibit a peculiar feature, namely, *massive* world-sheets. This apparent lack of conformal symmetry seems to make it quite non-trivial to check whether these string vacua are really consistent, for instance, to check the modular invariance, the open-closed string duality for the cylinder amplitudes (Cardy condition), and so on. In particular, focusing on the problem of open-closed duality, a naive treatment would induce a difficulty, because we cannot take the light-cone gauge \( X^+ \propto \tau \) at the same time for both of the open and closed string channels.

Since the general pp-waves have a translational symmetry along a light-like direction, one can always consider the DLCQ (discrete light-cone quantized) string theories [5] on these backgrounds. In this paper we shall study the DLCQ superstrings on the pp-waves with enhanced SUSY, and analyze the one-loop thermal amplitudes of closed strings and open strings with supersymmetric D-brane backgrounds. Several motivations for this study are in order:

Firstly, as explained in [6] (see also [7]), a nice realization of the DLCQ pp-wave with maximal SUSY is given by considering the Penrose limit of the orbifolded \( AdS_5 \times S^5 \), which is known to be dual to the \( \mathcal{N} = 2 \) quiver gauge theory and has 16 supercharges. The Penrose limit is characterized by picking up a particular configuration of null-geodesic. If we choose it along the fixed point locus, it simply leads to the orbifolded pp-wave that has the same number of supercharges \([8, 9, 10]\)\(^1\). On the other hand, if we place the null-geodesic away from the fixed point locus, we obtain a smooth pp-wave with some compactification along longitudinal directions \([9]\). Moreover, the DLCQ limit is shown to correspond to the “large quiver limit”, which is a certain double scaling limit considering the large \( N \) and the large “size” of quiver diagram (“deconstruction limit” \([12]\)) at the same time \([6, 7]\). In the latter case, which is of primary interest in this paper, the space-time SUSY is maximally enhanced (32 supercharges), because the background has no orbifold singularity and DLCQ does not

\(^1\)The Penrose limits of several orbifolds of \( AdS_5 \times S^5 \) and \( AdS_5 \times T^{1,1} \) that are dual to the \( \mathcal{N} = 1 \) gauge theories have been also studied in \([11]\).
break any supercharges. This is one of the well-known phenomena of SUSY enhancements under the several Penrose limits discussed by many authors (see, for example, [8, 13]).

Secondly, the DLCQ string theory is known to have effectively discretized moduli of worldsheet. Therefore, it seems comparably easy to observe how the modular invariance and the open-closed duality are realized. We will later demonstrate how these consistencies are established in the framework of thermal string theory.

Thirdly, our thermodynamical analysis on DLCQ pp-waves may shed new light on the several attempts for the Matrix string theories [14] describing pp-wave backgrounds [15, 16]. In fact, in the case of flat background, it is known that the free energy of Matrix string theory coincides with that of the second quantized DLCQ superstring [17].

Fourthly, we would like to also mention on the models of 4-dimensional NSNS pp-wave with enhanced SUSY constructed in [18]. These superstring vacua are defined based on the super Nappi-Witten model [19, 20, 21] and arbitrary rational $\mathcal{N} = 2$ SCFT with $c = 9$, being orbifolded by the GSO projection like the Gepner models [22]. We point out that these models have the light-cone momentum discretized by the GSO condition just mentioned, and hence show the feature quite reminiscent of the DLCQ pp-wave.

This paper is organized as follows:

In section 2 we calculate the thermal partition function of IIB superstring on the 10-dimensional DLCQ pp-wave mentioned above. By making use of the path-integral technique we derive the manifestly modular invariant expression, and further confirm that it actually coincides with the free energy of second quantized string theory calculated by the operator formalism defined over the physical Hilbert space. The existence of the “virtual strings” wound around the temporal circle (or, we call it “thermal circle”) is quite important for the modular invariance and a simple analysis on the Hagedorn temperature [23]. In section 3 we analyze the thermal one-loop amplitudes of open strings under the supersymmetric backgrounds of the time-like and Euclidean D-branes. We especially focus on the problem how the open-closed string duality should be understood in the context of thermal string theory on the DLCQ pp-waves. The virtual string sectors again play an essential role. In section 4, we extend our analysis to the cases of the 6-dimensional DLCQ pp-waves with the enhanced SUSY, which are similarly derived from the non-SUSY and half-SUSY orbifolds of $AdS_3 \times S^3$ backgrounds. We give a summary and discussions in section 5.

We should finally comment on some recent works related to this paper. One-loop amplitudes for non-thermal, non-DLCQ string theory on the 10-dimensional pp-wave have been
analyzed for open string in [24], and for closed string in [25]. The thermal partition function for closed string in the non-DLCQ model has been calculated in [26, 27].

2 Thermal Partition Function of DLCQ Superstring on 10-dimensional PP-Wave

2.1 Short Review of the Light-cone GS Superstring on Maximally Supersymmetric PP-Wave

It is familiar that type IIB string on the maximally supersymmetric pp-wave background is canonically quantized in the light-cone GS formalism [3]. We shall start with a brief review of it mainly to prepare the notations.

We introduce the bosonic string coordinates $X^\pm \equiv \frac{1}{\sqrt{2}}(X^9 \pm X^0)$, $X^I$ ($I = 1, \ldots, 8$), and the GS fermions $\theta^A, \bar{\theta}^A$ which are 10-dimensional Majorana-Weyl spinors having the same chirality. The relevant pp-wave geometry is expressed as

$$ds^2 = 2dX^+dX^- - \mu^2(X^I)^2(dX^+)^2 + (dX^I)^2,$$  \hspace{1cm} (2.1)

with the RR 5-form flux

$$F_{+1234} = F_{+5678} \sim \mu.$$  \hspace{1cm} (2.2)

The light-cone gauge is defined by

$$X^+ = \alpha' p^+ \tau, \quad \Gamma^+ \theta = \Gamma^+ \bar{\theta} = 0.$$  \hspace{1cm} (2.3)

We write the remaining 8 component spinors as $S^a, \bar{S}^a$ (with a conventional rescaling) composing the spinor representation $\mathbf{8}_s$ of $SO(8)$ respectively. It is convenient to introduce the chiral representation of $SO(8)$ gamma matrices as

$$\hat{\gamma}^I = \begin{pmatrix} O & \gamma_{ab} \\ \bar{\gamma}_{ab} & O \end{pmatrix}, \quad \{\hat{\gamma}^I, \hat{\gamma}^J\} = 2\delta^{IJ}.$$  \hspace{1cm} (2.4)

The $8 \times 8$ matrices $\gamma^I_{ab}, \bar{\gamma}^I_{ab}$ clearly satisfy

$$\gamma^I_{ab} \bar{\gamma}^J_{bc} + \gamma^I_{ab} \bar{\gamma}^I_{bc} = 2\delta^{IJ}\delta_{ac}, \quad \bar{\gamma}^I_{ab} \gamma^J_{bc} + \bar{\gamma}^I_{ab} \gamma^I_{bc} = 2\delta^{IJ}\delta_{ac}.$$  \hspace{1cm} (2.5)
We can assume all \( \gamma^I \) are real symmetric, resulting that \((\gamma^I)^T = \gamma^I \). The light-cone gauge action is now written as
\[
S = \frac{1}{4\pi\alpha'} \int d^2\sigma \left( \partial_+ X^I \partial_- X^I - m^2 (X^I)^2 \right) + \frac{i}{2\pi} \int d^2\sigma \left( S^a \partial_+ S^a + \tilde{S}^a \partial_- \tilde{S}^a - 2m S^a \Pi_{ab} \tilde{S}^b \right),
\]
where we set \( \Pi = \gamma^1 \bar{\gamma}^2 \gamma^3 \bar{\gamma}^4 \). It is easy to see \( \Pi^T = \Pi, \Pi^2 = 1 \). We denote \( \partial_\pm = \partial_\tau \pm \partial_\sigma \) as usual. The mass parameter \( m \) is defined as \( m = \mu \alpha' p^+ \). The existence of mass terms breaks the conformal symmetry on world-sheet, and moreover the fermionic mass term breaks the global symmetry down to \( SO(4) \times SO(4) \).

The equations of motion are given by
\[
\partial_\pm \partial_- X^I + m^2 X^I = 0,
\]
\[
\partial_+ S^a - m \Pi_{ab} \tilde{S}^b = 0,
\]
\[
\partial_- S^a + m \Pi_{ab} S^b = 0.
\]
(2.7)

The solutions with periodic boundary conditions have the following mode expansions;
\[
X^I(\tau, \sigma) = x_0^I \cos(m \tau) + \frac{\alpha'}{m} p_0^I \sin(m \tau) + \sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{\sqrt{\omega_n}} \left( a_n^I e^{-i(\omega_n \tau - n\sigma)} + a_n^I e^{i(\omega_n \tau - n\sigma)} \right),
\]
(2.8)
\[
P^I(\tau, \sigma) \equiv \frac{1}{2\pi \alpha'} \partial_\tau X^I
\]
\[
= -\frac{m}{2\pi \alpha'} x_0^I \sin(m \tau) + \frac{1}{2\pi} p_0^I \cos(m \tau) - \frac{i}{2\sqrt{2\alpha'} \pi} \sum_{n\neq 0} \sqrt{\omega_n} \left( a_n^I e^{-i(\omega_n \tau - n\sigma)} - a_n^I e^{i(\omega_n \tau - n\sigma)} \right),
\]
(2.9)
\[
S^a(\tau, \sigma) = S_0^a \cos(m \tau) + \Pi_{ab} \tilde{S}^b \sin(m \tau)
+ \sum_{n>0} c_n \left[ S_n^a e^{-i(\omega_n \tau - n\sigma)} + S_n^a e^{i(\omega_n \tau - n\sigma)} + i \frac{\omega_n - n}{m} \Pi_{ab} \left( S_n^b e^{-i(\omega_n \tau + n\sigma)} - S_n^b e^{i(\omega_n \tau + n\sigma)} \right) \right],
\]
(2.10)
\[
\tilde{S}^a(\tau, \sigma) = \tilde{S}_0^a \cos(m \tau) - \Pi_{ab} S^b \sin(m \tau)
+ \sum_{n>0} c_n \left[ S_n^a e^{-i(\omega_n \tau + n\sigma)} + S_n^a e^{i(\omega_n \tau + n\sigma)} - i \frac{\omega_n - n}{m} \Pi_{ab} \left( S_n^b e^{-i(\omega_n \tau - n\sigma)} - S_n^b e^{i(\omega_n \tau - n\sigma)} \right) \right],
\]
(2.11)

where we set
\[
\omega_n \equiv \sqrt{m^2 + n^2}, \quad c_n \equiv \frac{1}{\sqrt{1 + \left( \frac{\omega_n - n}{m} \right)^2}}.
\]
(2.12)
The modifications of mode expansions for more general boundary conditions are quite easy and we do not write them explicitly.

The canonical quantization gives the standard (anti-)commutation relations of harmonic oscillators:

\[ [a_I^m, a_J^n] = \delta_{I,J} \delta_{m,n} , \quad [a_I^m, a_J^n] = [a_I^m, a_J^n] = 0 , \] (2.13)

\[ [x_I^m, p_J^0] = i \delta_{I,J} , \] (2.14)

\[ \{ S_m^a, S_n^{ib} \} = \delta^{a,b} \delta_{m,n} , \quad \{ S_m^a, S_n^{ib} \} = \{ S_m^{ia}, S_n^{ib} \} = 0 , \] (2.15)

\[ \{ S_m^a, S_n^{b} \} = \delta^{a,b} , \quad \{ \tilde{S}_m^a, \tilde{S}_n^b \} = \delta^{a,b} , \quad \{ S_0^a, \tilde{S}_0^b \} = 0 . \] (2.16)

We also introduce the next notations for the zero-mode oscillators in order to diagonalize the zero-mode part of Hamiltonian:

\[ a_0^I \equiv \frac{1}{\sqrt{2m\alpha^\prime}} (mx_0^I + i\alpha^\prime p_0^I) , \quad a_0^{1I} \equiv \frac{1}{\sqrt{2m\alpha^\prime}} (mx_0^I - i\alpha^\prime p_0^I) , \] (2.17)

\[ S_0^a \equiv \frac{1}{2} (1 \pm \Pi)_{ab} \frac{1}{\sqrt{2}} (S_0^b \pm i\tilde{S}_0^b) , \quad S_0^{ia} \equiv \frac{1}{2} (1 \pm \Pi)_{ab} \frac{1}{\sqrt{2}} (S_0^{ib} \mp i\tilde{S}_0^{ib}) , \] (2.18)

which satisfy

\[ [a_0^I, a_0^{1J}] = \delta^{I,J} , \quad [a_0^I, a_0^{1J}] = [a_0^{1I}, a_0^{1J}] = 0 , \] (2.19)

\[ \{ S_0^a, S_0^{b} \} = \delta^{a,b} , \quad \{ S_0^a, S_0^{b} \} = 0 . \] (2.20)

The light-cone Hamiltonian is calculated by the Virasoro constraints:

\[ H_{l.c.} \equiv -p^- = \frac{1}{\alpha^\prime p^+} \sum_{n \in \mathbb{Z}} \omega_n N_n + a(p^+)'(b) + a(p^+)'(f) , \] (2.21)

where \( N_n \) denotes the mode counting operators at the level \( n \):

\[ N_n = a_n^{1I} a_n^I + S_n^{ia} S_n^a , \quad (n \neq 0) \]
\[ N_0 = a_0^{1I} a_0^I + i S_0^{ia} \Pi_{ab} \tilde{S}_0^{ib} + 4 \]
\[ \equiv a_0^{1I} a_0^I + S_0^{ia} S_0^a + S_0^{ia} S_0^a . \] (2.22)

\( a(p^+)'(b), a(p^+)'(f) \) are the normal order constants for bosonic and fermionic sectors respectively which may non-trivially depend on \( p^+ \). In the present set up they should totally cancel because the bosonic and fermionic coordinates satisfy the same boundary condition. We will face more non-trivial situations in which they have different boundary conditions and the cancellation fails. We will separately fix them later in order to calculate the thermal amplitudes.

\[ ^2 \text{We are here taking the convention such that the modes } n > 0 \text{ correspond to the left-mover and } n < 0 \text{ to the right-mover respectively under the conformal limit } m \to 0 \text{ according to [4].} \]
The Fock vacuum \( |0; p^+ \rangle \) is characterized in the standard manner;
\[
a^I_n |0; p^+ \rangle = 0 \quad (\forall I, n), \quad S^a_n |0; p^+ \rangle = 0 \quad (\forall a, \forall n \neq 0), \quad S^a_{\mp} |0; p^+ \rangle = 0 \quad (\forall a).
\]

(2.23)

Since we are now interested in the DLCQ string theory \( [5] \) \( X^- \sim X^- + 2\pi R_- \), the light-cone momentum \( p^+ \) should be quantized as
\[
p^+ = \frac{p}{R_-} \quad (p \in \mathbb{Z}_{>0}).
\]

(2.24)

The Virasoro constraints provide the level matching condition
\[
\sum_{n \in \mathbb{Z}} nN_n = pk \quad (\forall k \in \mathbb{Z}),
\]

(2.25)

for the each winding sector \( \int_0^{2\pi} d\sigma \partial_\sigma X^- = 2\pi k R_- \).

### 2.2 Transverse Partition Functions

Now, we are ready to calculate the toroidal partition function. We first focus on the transverse sector. We so fix the mass parameter \( m \) for the time being. According to the standard treatment, we move to the Euclidean world-sheet by the Wick rotation \( \tau = i\tau_E \), and set
\[
z = i\tau_E - \sigma, \quad \bar{z} = -i\tau_E - \sigma.
\]

(2.26)

resulting the replacement \( \partial_+ \rightarrow -2\partial_\bar{z}, \partial_- \rightarrow 2\partial_z \). We also introduce the next parameterizations \( z = \xi_1 + \tau \xi_2, \bar{z} = \xi_1 + \bar{\tau} \xi_2 \), where \( \tau = \tau_1 + i\tau_2 \) \((\tau_2 > 0)\) denotes the modulus parameter of world-sheet torus. The next formulas are often useful for calculations;
\[
\partial_z = \frac{i}{2\tau_2} (\bar{\tau} \partial_{\xi_1} - \partial_{\xi_2}), \quad \partial_{\bar{z}} = \frac{-i}{2\tau_2} (\tau \partial_{\xi_1} - \partial_{\xi_2}), \quad d^2 z = \tau_2 d\xi_1 d\xi_2.
\]

(2.27)

The transverse partition function is calculated in the way parallel to the standard conformal field theory;
\[
Z^{tr}(\tau, \bar{\tau}; m) = \text{Tr} \left[ (-1)^F e^{-2\pi \tau_2 H + 2\pi i \tau_1 P} \right],
\]

(2.28)

where \( H \equiv \alpha' p^+ H_{1c} \) is the world-sheet Hamiltonian and \( P \equiv \sum_{n \in \mathbb{Z}} nN_n \) is the world-sheet momentum operator. In the conformal limit we of course obtain \( H = L_0 + \bar{L}_0 - 1, \quad P = L_0 - \bar{L}_0 \).

\( F \) denotes the space-time fermion number and the insertion of \( (-1)^F \) is necessary to realize the periodic boundary condition for the GS fermions.
Because our transverse Hilbert space is a free Fock space, the calculation of trace is quite easy except for the evaluation of zero-point energy (or, the normal order constant) with an appropriate regularization. Let us first pick up one complex boson. According to [24, 25, 27], we shall evaluate the regularized zero-point energy as the Casimir energy, which is defined by subtracting the divergent contribution free to the boundary condition. Since the zero-point energy for each of harmonic oscillators $a^I_n, a^{I^+}_n$ is equal to $\omega_n/2$, we can explicitly calculate it as (for the chiral part)

$$\Delta(m) \xlongdef \frac{1}{2} \left( \sum_{n \in \mathbb{Z}} \sqrt{m^2 + n^2} - \int_{-\infty}^{\infty} dk \sqrt{m^2 + k^2} \right)$$

$$= \frac{1}{2} \sum_{n \in \mathbb{Z}} \frac{1}{\Gamma(-1/2)} \int_0^{\infty} dt \frac{t^{-3/2} e^{-t(m^2+n^2)}}{\Gamma(-1/2)} \int_0^{\infty} dt \frac{t^{-3/2} e^{-t(m^2+k^2)}}{\Gamma(-1/2)}$$

$$= -\frac{1}{4} \sum_{n \neq 0} \int_0^{\infty} dt \frac{t^{-2} e^{-tm^2 - \frac{2\pi^2}{9}}}{\Gamma(-1/2)}$$

$$= -\frac{1}{2\pi^2} \sum_{n=1}^{\infty} \int_0^{\infty} ds \frac{e^{-sn^2 - \frac{2\pi^2}{9} n^2}}{\Gamma(-1/2)}$$

where we have used the Poisson resummation formula to derive the third line. This actually converges by the evaluation

$$\sum_{n=1}^{\infty} \left| \int_0^{\infty} ds \ e^{-sn^2 - \frac{2\pi^2}{9} n^2} \right| \leq \sum_{n=1}^{\infty} \left| \int_0^{\infty} ds \ e^{-sn^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6}.$$

The partition function for one complex boson now becomes

$$Z_{\text{boson}}(\tau, \bar{\tau}; m) = \frac{1}{e^{4\pi\tau_2 \Delta(m)} \prod_{n \in \mathbb{Z}} \left(1 - e^{-2\pi\tau_2 \sqrt{m^2 + n^2} + 2\pi i n} \right)^2}.$$  

(2.31)

The regularized zero-point energy (2.29) is justified as follows: Firstly, it has the correct $m \to 0$ limit

$$\lim_{m \to 0} \Delta(m) = -\frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{1}{12} \left( \equiv 2 \times \left(\frac{-1}{24}\right) \right).$$

(2.32)

Moreover, the partition function (2.31) possesses the correct modular properties based on the definition (2.29), as is proved in [24, 25];

$$Z_{\text{boson}}(\tau + 1, \bar{\tau} + 1; m) = Z_{\text{boson}}(\tau, \bar{\tau}; m),$$

$$Z_{\text{boson}}(-1/\tau, -1/\bar{\tau}; m|\tau) = Z_{\text{boson}}(\tau, \bar{\tau}; m).$$

(2.33)

The reason why these modular properties should be correct is most naively explained as follows: The modular transformations preserve the complex structure of torus, but the $S$-transformation $\tau \to -1/\tau$ changes the area as $A \to \frac{1}{|\tau|^2} A$. We now have a unique mass
parameter $m$ that breaks conformal invariance and has the dimension $[\text{length}]^{-1}$. Therefore, the physical contents should not be changed, if the $S$-transformation is compensated by the scale transformation $m \rightarrow |\tau|m$. More rigorous understanding of the modular properties is achieved by the path-integral as discussed in [26]. The Gaussian path-integral is easily evaluated and gives the determinant of the Klein-Gordon operator $-4\partial_z\partial_{\bar{z}} + m^2$. Working with the coordinates $z = \xi_1 + \tau \xi_2$, $\bar{z} = \xi_1 + \bar{\tau}\xi_2$ and making use of (2.27), we can show

$$Z_{\text{boson}}(\tau, \bar{\tau}; m) \sim \prod_{n_1, n_2 \in \mathbb{Z}} \tau_2 \left( \frac{1}{\tau_2} |n_1 - n_2 \tau|^2 + m^2 \right)^{-1},$$

(2.34)

up to a divergent factor that is independent of $\tau, \bar{\tau}, m$ and should be regulated. It manifestly reproduces the modular transformation formulas (2.33). The zero-point energy (2.29) is justified by these reasons.

For later convenience we further introduce the “massive theta function” according to [25];

$$\Theta_{(a,b)}(\tau, \bar{\tau}; m) \overset{\text{def}}{=} e^{4\pi \tau_2 \Delta(m; a)} \prod_{n \in \mathbb{Z}} \left| 1 - e^{-2\pi \tau_2 \sqrt{m^2 + (n+a)^2} + 2\pi i(n+a) + 2\pi ib} \right|^2.$$  

(2.35)

where $a, b$ are arbitrary real parameters. The zero-point energy $\Delta(m; a)$ is similarly defined by

$$\Delta(m; a) \overset{\text{def}}{=} \frac{1}{2} \sum_{n \in \mathbb{Z}} \sqrt{m^2 + (n+a)^2} - \frac{1}{2} \int_{-\infty}^{\infty} dk \sqrt{m^2 + k^2}$$

$$= -\frac{1}{2\pi^2} \sum_{n=-\infty}^{\infty} \int_{0}^{\infty} ds e^{-sn^2 - \frac{2\pi^2}{s^2}} \cos(2\pi na).$$

(2.36)

The function $\Theta_{(a,b)}(\tau, \bar{\tau}; m)$ describes various twisted boundary conditions characterized by $a$ and $b$. Namely, it is easy to see that the partition function for the $d$-components complex massive boson (non-chiral fermion) with the boundary conditions $\phi(z + 2\pi, \bar{z} + 2\pi) = e^{-2\pi i a} \phi(z, \bar{z})$ \footnote{Recall that $z \rightarrow z + 2\pi, \bar{z} \rightarrow \bar{z} + 2\pi$ corresponds to $\sigma \rightarrow \sigma - 2\pi$. The extra minus sign defining this twisted boundary condition is due to this fact.}, $\phi(z + 2\pi \tau, \bar{z} + 2\pi \bar{\tau}) = e^{2\pi ib} \phi(z, \bar{z})$, is calculated as

$$Z(\tau, \bar{\tau}; m) = \text{Tr} \left[ (-1)^d e^{-2\pi \tau_2 H + 2\pi i r_1 \hat{P} + 2\pi i b \hat{h}} \right]$$

$$= \Theta_{(a,b)}(\tau, \bar{\tau}; m)^{-\epsilon d},$$

(2.37)

where $\epsilon = +1$ for the boson and $\epsilon = -1$ for the fermion. In this expression we introduced the momentum operator for the twisted fields

$$\hat{P} = \sum_n \left( (n+a)N_n^{(+)} + (n-a)N_n^{(-)} \right),$$

(2.38)
We also obtain for the twisted boundary conditions for the fermionic coordinates

\[ \hat{h} = \sum_{n} \left( N_n^{(+)} - N_n^{(-)} \right) , \]  

(2.39)

where \( N_n^{(+)} \), \( N_n^{(-)} \) express the mode counting operators associated to the Fourier modes \( e^{\pm i(n+a)\sigma} \), \( e^{\pm i(n-a)\sigma} \) respectively.

\( \Theta_{(a,b)}(\tau, \bar{\tau}; m) \) has the following modular properties

\[ \Theta_{(a,b)}(\tau + 1, \bar{\tau} + 1; m) = \Theta_{(a,b+a)}(\tau, \bar{\tau}; m) , \]
\[ \Theta_{(a,b)}(-1/\tau, -1/\bar{\tau}; m|\tau|) = \Theta_{(b,-a)}(\tau, \bar{\tau}; m) . \]  

(2.40)

We can also show that

\[ \Theta_{(a,b)}(\tau, \bar{\tau}; m) = \Theta_{(-a,-b)}(\tau, \bar{\tau}; m) = \Theta_{(a+r,b+s)}(\tau, \bar{\tau}; m) , \quad (\forall \ r, s \in \mathbb{Z}) , \]
\[ \lim_{m \to 0} \Theta_{(a,b)}(\tau, \bar{\tau}; m) = e^{-\pi a^2} \left( \frac{\theta_1(\tau, a\tau + b)}{\eta(\tau)} \right)^2 . \]  

(2.41)
\[ (2.42) \]

In our present problem the transverse partition function (2.28) is calculated as

\[ Z_{\tau}^{\text{tr}}(\tau, \bar{\tau}; m) = \text{Tr} \left[ (-1)^F e^{-2\pi \tau_2 H + 2\pi i r_1 P} \right] = \frac{\Theta_{(0,0)}(\tau, \bar{\tau}; m)}{\Theta_{(0,0)}(\tau, \bar{\tau}; m)^4} \equiv 1 . \]  

(2.43)

We also obtain for the twisted boundary conditions for the fermionic coordinates

\[ \text{Tr} \left[ e^{-2\pi \tau_2 H + 2\pi i r_1 P} \right] = \frac{\Theta_{(0,1/2)}(\tau, \bar{\tau}; m)^4}{\Theta_{(0,0)}(\tau, \bar{\tau}; m)^4} , \]
\[ \text{Tr}_{H(t)} \left[ (-1)^F e^{-2\pi \tau_2 H + 2\pi i r_1 P} \right] = \frac{\Theta_{(1/2,0)}(\tau, \bar{\tau}; m)^4}{\Theta_{(0,0)}(\tau, \bar{\tau}; m)^4} , \]
\[ \text{Tr}_{H(t)} \left[ e^{-2\pi \tau_2 H + 2\pi i r_1 P} \right] = \frac{\Theta_{(1/2,1)}(\tau, \bar{\tau}; m)^4}{\Theta_{(0,0)}(\tau, \bar{\tau}; m)^4} , \]  

(2.44)

where \( \text{Tr}_{H(t)} \) means the trace over the Hilbert space of the anti-periodic GS fermions \( S^a(z + 2\pi, \bar{z} + 2\pi) = -S^a(z, \bar{z}) \), \( \tilde{S}^a(z + 2\pi, \bar{z} + 2\pi) = -\tilde{S}^a(z, \bar{z}) \).

We finally comment on the result (2.43). In contrast to the flat background the partition function (2.43) does not vanish although we have maximal SUSY. This aspect is understood as follows. We have the 16 kinematical supercharges, which are essentially the zero-modes of GS fermions, and also the 16 dynamical supercharges including the higher level oscillators. The latter commutes with the light-cone Hamiltonian (and hence \( H \), too), but the former does not, contrary to the flat case in which all the supercharges commute with Hamiltonian. In this situation, since the GS partition function (2.28) is a Witten index by definition, we could obtain non-zero contributions from the BPS states, which are annihilated by the dynamical...
supercharges, or equivalently, have the vanishing light-cone energy. In fact, we now have a unique BPS state for each \( p^+ \), namely, the Fock vacuum \(|0; p^+\rangle\) itself, and hence the Witten index should be equal to 1. This property of partition function is already mentioned in [25] and the similar feature in the open string one-loop amplitudes is also pointed out in [24].

One might still ask: Does it mean the non-zero cosmological constant? Cannot we continuously take the flat limit \( \mu \to 0 \) in the level of partition function?

These are really not the cases. Recall that the cosmological constant should be identified with the vacuum energy \( \textit{density} \), and we have a divergent volume since the pp-wave background is non-compact. In the case of flat background the partition function includes the volume factor due to the zero-modes of bosonic coordinates. However, such volume factor does not appear in the present case, since the zero-modes \( a^I_0, a'^I_0, \ldots \) possess non-zero energy due to mass \( m \). In this sense the vacuum energy density surely behaves continuously under the \( \mu \to 0 \) limit (namely, remains zero). In this limit the transverse volume factor \( V_8 \) appears from the bosonic contribution

\[
\lim_{m \to 0} \frac{1}{\Theta_{(0,0)}(\tau, \bar{\tau}; m)^4} \sim V_8 \times \frac{1}{\tau^4} \frac{1}{|\eta(\tau)|^4},
\]

which cancels the denominator to define the vacuum energy density. On the other hand, the fermionic contribution precisely cancels with each other this time, resulting also the vanishing energy density.

### 2.3 Thermal Partition Function

Now, let us proceed to our main subject, the calculation of the thermal partition function for the DLCQ string on pp-wave background. The thermal string theory is defined as the target space with the compactified Euclidean time with the circumference equal to the inverse temperature \( \beta \).

According to [17], we shall employ the path-integral technique for the longitudinal sector. The advantage to do so is that we can most transparently obtain the modular invariant expression of partition function. First of all, we consider the thermal DLCQ string in the flat background as a warm-up. In the Wick rotated space-time \( X^\pm \equiv \frac{1}{\sqrt{2}}(X^0 \pm iX^9) \), the DLCQ string theory \( (X^- \sim X^- + 2\pi R_-) \) is described by the complex identification

\[
X^0_E \sim X^0_E + \sqrt{2\pi R_-} \ , \quad X^9 \sim X^9 + \sqrt{2\pi R_-} ,
\]

and the thermal compactification is defined as

\[
X^0_E \sim X^0_E + \beta ,
\]

\[
2.47
\]
where $\beta$ denotes the inverse temperature. The complex identification (2.46) may sound peculiar, since it makes the world-sheet action complex. Nevertheless, it has been proved in [17] that for the flat background it gives the results equivalent with the operator formalism defined with respect to the original theory of Lorentzian signature, in which only the physical states appear in the calculation. The path-integral approach presented here is justified by this fact.

When calculating the Polyakov path-integral, the contributions from the various topological sectors are most important:

$$X^+(z + 2\pi, \bar{z} + 2\pi) = X^+(z, \bar{z}) + \frac{i\beta}{\sqrt{2}}w,$$

$$X^+(z + 2\pi \tau, \bar{z} + 2\pi \bar{\tau}) = X^+(z, \bar{z}) + \frac{i\beta}{\sqrt{2}}n,$$

$$X^-(z + 2\pi, \bar{z} + 2\pi) = X^-(z, \bar{z}) - \frac{i\beta}{\sqrt{2}}w + 2\pi R_- r,$$

$$X^-(z + 2\pi \tau, \bar{z} + 2\pi \bar{\tau}) = X^-(z, \bar{z}) - \frac{i\beta}{\sqrt{2}}n + 2\pi R_- s,$$

\[(w, n, r, s \in \mathbb{Z}). \quad (2.48)\]

The “instanton” solutions are defined as those which linearly depend on the world-sheet coordinates and are constrained by these boundary conditions. The instanton action is evaluated with the help of (2.27) as

$$S_{\text{inst}}(w, n, r, s) = \frac{\beta^2 |w \tau - n|^2}{4\pi \alpha' \tau_2} + 2\pi \nu \frac{\nu}{\tau_2} \left\{ |\tau|^2 wr - \tau_1 (ws + nr) + ns \right\}, \quad (2.49)$$

where we set $\nu \equiv \frac{\sqrt{2}\beta R_-}{4\pi \alpha'}$. The longitudinal path-integral is evaluated as

$$\frac{V_{\text{l.c.}}}{4\pi^2 \alpha' \tau_2} \times \sum_{w, n, r, s} e^{-S_{\text{inst}}(w, n, r, s)} = \frac{\nu}{\tau_2} \sum_{w, n, r, s} e^{-S_{\text{inst}}(w, n, r, s)}, \quad (2.50)$$

where the $V_{\text{l.c.}} \equiv \sqrt{2}\beta R_-$ is the volume of longitudinal directions. The prefactor $V_{\text{l.c.}}/(4\pi^2 \alpha' \tau_2)$ is given by integrating out the fluctuations around instantons, also taking account of the FP

---

4As well as the Euclidean DLCQ string theory, we obtain the complex world-sheet actions for the Wick rotated theories of general pp-waves (including the DLCQ pp-waves, of course). Such string models seem to be ill-defined as canonically quantized theories based on these complex actions themselves. However, if taking the thermal compactification at the same time, they provide an useful machinery to derive modular invariant amplitudes by the path-integration. Modestly speaking, to adopt the path-integral approach based on these complex world-sheet actions at least has a physical meaning as a conventional method to calculate the free energies of the original pp-wave string theories with Lorentzian signature. We will later confirm the equivalence with the operator formalism, which justifies this approach.
determinant in the standard manner. In other words it corresponds to the factor derived from the Gaussian integral of the zero-mode momenta in the Hamiltonian formalism.

Another non-trivial point is the boundary conditions of fermionic coordinates along the thermal circle \([28]\). The winding numbers \(w, n\) are respectively the spatial and temporal ones. We should hence choose the boundary conditions for GS fermions as

\[
S^a(z + 2\pi \epsilon_1 + 2\pi \epsilon_2 \tau) = (-1)^{\epsilon_1 w + \epsilon_2 n} S^a(z) , \quad \bar{S}^a(\bar{z} + 2\pi \epsilon_1 + 2\pi \epsilon_2 \bar{\tau}) = (-1)^{\epsilon_1 w + \epsilon_2 n} \bar{S}^a(\bar{z}) ,
\]

\((\epsilon_i = 0, 1) . \quad (2.51)\)

This condition is most easily understood by recalling the correct boundary conditions in the thermal field theory of point particles (identified as the \(w = 0\) sector) and further taking account of the consistency with modular invariance.

In this way the desired partition function is calculated in the following form;

\[
Z_{\text{torus}}(\beta) = \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} \nu \sum_{\epsilon_i = 0, 1} \sum_{w \in 2\mathbb{Z} + \epsilon_1} \sum_{n \in 2\mathbb{Z} + \epsilon_2} e^{-S_{\text{inst}}(w, n, r, s)} Z_{\epsilon_1, \epsilon_2}^{tr}(\tau, \bar{\tau}) , \quad (2.52)
\]

\[
Z_{\epsilon_1, \epsilon_2}^{tr}(\tau, \bar{\tau}) = \frac{V_8}{(4\pi^2 \alpha'/\tau_2)^4 |\eta(\tau)|^8} . \quad (2.53)
\]

The subscripts \(\epsilon_1, \epsilon_2\) indicate the twisted boundary conditions of GS fermions \((2.51)\). The calculation of transverse partition function \(Z_{\epsilon_1, \epsilon_2}^{tr}(\tau, \bar{\tau})\) is quite familiar, and we denoted the transverse volume factor as \(V_8\). The integration region \(\mathcal{F}\) is the familiar fundamental domain

\[
\mathcal{F} \overset{\text{def}}{=} \left\{ \tau \in \mathbb{C} ; \tau_2 > 0, |\tau| > 1, |\tau_1| \leq \frac{1}{2} \right\} . \quad (2.54)
\]

In the summation of winding numbers the \(w = n = 0\) sectors lead to a divergent term that should be interpreted as the vacuum energy and does not depend on the parameters \(\beta, R_-\). We shall thus subtract them and assume \((w, n) \neq (0, 0)\) to define the thermal partition function. Substituting the expression \((2.49)\), we readily carry out the summation over \(r, s\), providing a periodic delta function. The result is written as

\[
Z_{\text{torus}}(\beta) = \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} \nu \sum_{\epsilon_i = 0, 1} \sum_{w \in 2\mathbb{Z} + \epsilon_1} \sum_{n \in 2\mathbb{Z} + \epsilon_2} e^{-\frac{\beta^2 |w\nu - n|}{4\pi^2 \alpha'/\tau_2}} \\
\times \tau_2 \delta^{(2)} ((w\nu + iq) \tau - (n\nu + iq)) Z_{\epsilon_1, \epsilon_2}^{tr}(\tau, \bar{\tau}) , \quad (2.55)
\]

which coincides with that given in \([17]\) calculated based on the RNS formalism. The appearance of periodic delta function discretizes the moduli space of torus, which is a characteristic feature of the DLCQ string theory. The partition function \((2.55)\) is manifestly modular invariant, and hence it is a correct form of string amplitude on torus.
Let us turn to the pp-wave case. At first glance we are likely to face an apparent difficulty, since the light-cone gauge $X^+ \propto \tau$ is not compatible with the boundary conditions (2.48) in general. So, one might be afraid that the light-cone gauge quantization would fail in the thermal model. However, we can instead take a natural gauge condition $X^+ = X^+_{w,n,r,s}$, where $X^+_{w,n,r,s}$ denotes the instanton solution for the each topological sector (2.48). We can further consider the rotation of world-sheet coordinates; $z' = e^{i\theta_w,n} z$, $\bar{z}' = e^{-i\theta_w,n} \bar{z}$ with

$$
\cos \theta_w,n = \frac{\tau_1 w - n}{|\tau w - n|}, \quad \sin \theta_w,n = -\frac{\tau_2 w}{|\tau w - n|},
$$

such that

$$
(\partial_{\nu'} + \partial_{\nu''})X^+_{w,n,r,s} \equiv 0, \quad (\partial_{\nu'} - \partial_{\nu''})X^+_{w,n,r,s} \equiv -\frac{\sqrt{2} \beta}{4\pi \tau_2} |\tau w - n|.
$$

Working with the new coordinates $z'$, $\bar{z}'$, we still obtain the quadratic action (2.6) with the mass parameter

$$
m = \frac{\sqrt{2} \mu \beta}{4\pi \tau_2} |\tau w - n| \equiv \hat{\mu} \frac{\nu}{\tau_2} |\tau w - n|,
$$

where we set $\hat{\mu} = \mu \alpha'/R_-$. To be more precise, because the complex structure defined by $z'$, $\bar{z}'$ depends on the choice of $w$, $n$, which may make the quantization problematic, we must go back to the original coordinates $z$, $\bar{z}$ after making the gauge fixing. Fortunately, nothing is changed by this rotation because of the manifest rotational symmetry of the action (2.6). We will later face a more non-trivial situation, in which we need a care for such a rotation of complex coordinates, in the analysis of the 6-dimensional pp-wave.

In this way we have found that the transverse partition function for the each topological sector is evaluated by using the quadratic action (2.6) as in the previous subsection, but with the non-trivial mass parameter $m = \frac{\sqrt{2} \mu \beta}{4\pi \tau_2} |w \tau - n|$ depending on the thermal winding numbers $w$, $n$. The desired partition function is thus calculated as;

$$
Z_{\text{torus}}(\beta) = \int_F \frac{d^2 \tau}{\tau_2^2} \nu \sum_{\epsilon_i = 0,1} \sum_{w, n \in \mathbb{Z}} \sum_{r,s} e^{-S_{\text{inst}}(w,n,r,s)} Z_{\epsilon_1,\epsilon_2}^{\text{tr}} \left( \tau, \bar{\tau}; \frac{\nu}{\tau_2} |w \tau - n| \right). \quad (2.59)
$$

Since the transverse partition functions $Z_{\epsilon_1,\epsilon_2}^{\text{tr}}$ do not depend on the windings $r$, $s$, we can likewise make the summation over them, yielding the same periodic delta function. Recalling the results (2.43), (2.44), we finally obtain

$$
Z_{\text{torus}}(\beta) = \int_F \frac{d^2 \tau}{\tau_2^2} \nu \sum_{\epsilon_i = 0,1} \sum_{w, n \in \mathbb{Z}} \sum_{\nu, q} e^{-\frac{\beta^2 |w \nu - n|^2}{4\pi \alpha' \tau_2^2} - \frac{\partial^2}{4\pi \alpha' \tau_2^2} \delta^{(2)} ((w \nu + i p) \tau - (n \nu + i q))} \times \frac{\Theta_{(\epsilon_1,\epsilon_2)}(\tau, \bar{\tau}; \hat{\mu} |w \nu + i p|)^4}{\Theta_{(0,0)}(\tau, \bar{\tau}; \hat{\mu} |w \nu + i p|)^4}. \quad (2.60)
$$
Here we made use of the replacement of the mass parameter \( \hat{\mu}_\nu \tau^2 | w_\tau - n | \) with the simpler one \( \hat{\mu} | w_\nu + ip | \) due to the constraints

\[
\begin{align*}
\begin{align}
\label{2.61}
\begin{align}
w_\nu \tau_1 - p \tau_2 &= n_\nu, \\
w_\nu \tau_2 + p \tau_1 &= q, 
\end{align}
\end{align}
\end{align}
\]

imposed by the delta function factor. This partition function (2.60) has the manifestly modular invariant form. In fact, we can directly check it by means of the evaluation

\[
\begin{align}
Z_{\epsilon_1, \epsilon_2}^{\varepsilon} (-1/\tau, -1/\bar{\tau}; \hat{\mu} | w_\nu + ip |) \times \frac{\tau_2}{|\tau|^2} \delta^{(2)} ((w_\nu + ip)(-1/\tau) - (n_\nu + iq)) \\
&= Z_{\epsilon_2, \epsilon_1}^{\varepsilon} (\tau, \bar{\tau}; \hat{\mu} | w_\nu + ip |) \times \frac{\tau_2}{|\tau|^2} \delta^{(2)} ((n_\nu + iq) \tau + (w_\nu + ip)) \\
&= Z_{\epsilon_2, \epsilon_1}^{\varepsilon} (\tau, \bar{\tau}; \hat{\mu} | n_\nu + iq |) \times \tau_2 \delta^{(2)} ((n_\nu + iq) \tau + (w_\nu + ip)) \, .
\end{align}
\]

(2.62)

We next present the calculation by operator formalism, which will justify the correctness of our partition function (2.60).

### 2.4 Free Energy of Space-time Theory: Operator Calculation

In general, the free energy (or the grand potential with the vanishing chemical potential) in the thermal ensemble of free string theory is computed as

\[
F = \frac{1}{\beta} \text{Tr} \left[ \left( -1 \right)^F \ln \left( 1 - \left( -1 \right)^F e^{-\beta p^0} \right) \right] \\
\equiv - \sum_{n=1}^{\infty} \frac{1}{\beta n} \text{Tr} \left[ \left( -1 \right)^{(n+1)F} e^{-\beta n p^0} \right] ,
\]

(2.63)

where \( F \) denotes the space-time fermion number (mod 2) and \( p^0 \equiv \frac{1}{\sqrt{2}} (p^+ - p^-) \) is the space-time energy operator. The trace should be taken over the single particle physical Hilbert space on which the on-shell condition and the level matching condition are imposed. The free energy \( F \) should be identified with the one-loop partition function of the first quantized thermal string \( Z_{\text{torus}}(\beta) \) we studied above, by the next simple relation

\[
Z_{\text{torus}} = -\beta F .
\]

(2.64)

The main aim in this subsection is to confirm this relation for the partition function (2.60).

We start with the simple identity derived from the on-shell condition;

\[
p^0 = \frac{1}{\sqrt{2}} (p^+ - p^-) = \frac{1}{\sqrt{2}} \left( \frac{p}{R_-} + \frac{R_-}{\alpha' p} H \right) ,
\]

(2.65)
where \( H \equiv \alpha' p^+ \) denotes the world-sheet Hamiltonian as before. To impose the level matching condition (2.25) \( P(\equiv \sum n N_n) = pk \ (\forall k \in \mathbb{Z}) \), it is convenient to insert the following projection operator into the trace:

\[
\frac{1}{P} \sum_{q \in \mathbb{Z}_p} e^{2\pi i q P}.
\]

We so obtain the following expression from (2.63):

\[
F(\beta) = -\sum_{n,p,q} \frac{1}{\beta np} e^{-\frac{\beta^2 n^2}{4\pi p^2}} \text{Tr} \left[ (-1)^{(n+1)} F e^{-2\pi \tau_2 H + 2\pi i q P} \right].
\]

It is also convenient to introduce the “modulus parameter” \( \tau \equiv \frac{q + i n \nu}{p} \), where we set \( \nu \equiv \frac{\sqrt{2} \beta R}{4\pi \alpha'} \) as before. (2.67) is rewritten as

\[
F(\beta) = -\sum_{n,p,q} \frac{1}{\beta np} e^{-\frac{\beta^2 n^2}{4\pi \alpha' \tau_2^2}} \text{Tr} \left[ (-1)^{(n+1)} F e^{-2\pi \tau_2 H + 2\pi i q P} \right],
\]

where the integers \( n, p(>0), q \) run over the range such that \( \tau \in \mathcal{S} \) with the definition

\[
\mathcal{S} \overset{\text{def}}{=} \left\{ \tau \in \mathbb{C} : \tau_2 > 0, |\tau_1| \leq \frac{1}{2} \right\}.
\]

The trace in this expression (2.68) is already calculated in (2.43), (2.44). We thus finally obtain

\[
F(\beta) = -\sum_{p,q} \left[ \sum_{n:\text{even}} \frac{1}{\beta np} e^{-\frac{\beta^2 n^2}{4\pi \alpha' \tau_2^2}} + \sum_{n:\text{odd}} \frac{1}{\beta np} e^{-\frac{\beta^2 n^2}{4\pi \alpha' \tau_2^2}} \frac{\Theta(0,\frac{1}{2})}{\Theta(0,0)}(\tau, \bar{\tau}; \hat{\mu} p)^4 \right].
\]

Let us now compare this result with the modular invariant partition function (2.60). For this purpose it is easiest to make use of the technique invented in [29]. We first note that (2.60) has the form such as

\[
\mathcal{Z}_{\text{torus}}(\beta) = \sum_{w,n} \int F \frac{d^2 \tau}{\tau_2} f(w,n)(\tau, \bar{\tau}),
\]

where \( w, n \) denote the thermal winding numbers defined in (2.48), and behave as the doublet of \( \text{PSL}(2; \mathbb{Z}) \) under the modular transformations. Moreover, we can always find out a modular transformation setting \( w = 0 \) for arbitrary \( (w, n) \neq (0, 0) \). Therefore, because of the modular invariance, we can simply set \( w = 0 \) in the integrand of (2.60), but must replace the fundamental domain \( \mathcal{F} \) with the larger domain \( \mathcal{S} \) defined above. In summary, the partition
function (2.60) can be rewritten in the following simpler form, although we lose the manifest modular invariance;

\[
Z_{\text{torus}}(\beta) = \int_S \frac{d^2 \tau}{\tau_2^2} \nu \sum_{\epsilon=0,1} \sum_{n \in \mathbb{Z}+\epsilon} \sum_{p,q} e^{-\beta^2 \omega^2 \tau_2} \Theta^{(2)}(i p \tau - (n \nu + i q)) \frac{\Theta(0,\frac{1}{\tau}) (\tau, \bar{\tau}; \mu |p|)^4}{\Theta(0,0)(\tau, \bar{\tau}; \mu |p|)^4}
\]

\[
= \sum_{p,q} \left[ \sum_{n: \text{even}} \frac{1}{np} e^{-\frac{\beta^2 \omega^2}{4 \pi^2 \tau_2}} + \sum_{n: \text{odd}} \frac{1}{np} e^{-\frac{\beta^2 \omega^2}{4 \pi^2 \tau_2}} \frac{\Theta^{(2)}(\tau, \bar{\tau}; \mu p)^4}{\Theta(0,0)(\tau, \bar{\tau}; \mu p)^4} \right].
\]

(2.72)

In the last line we set \( \tau = \frac{q + i n \nu}{p} \), and the summation with respect to \( n, p, q \) should be taken over the range such that \( \tau \in S \). The relation (2.64) is obviously confirmed. Therefore, the validity of the partition function (2.60) has been confirmed.

It is worth pointing out that the first term (the summation over even \( n \)) in (2.72) is "topological" one originating from the Witten index counting the BPS states. This term is absent in the case of flat background. In fact, consider the flat limit \( \mu \to 0 \). There emerges a divergent volume factor \( V_8 \) from the second term and thus the first term becomes negligible.

The partition function per unit volume has the next limit

\[
\lim_{\mu \to 0} \frac{Z_{\text{torus}}(\beta)}{V_8} = \sum_{p,q} \sum_{n: \text{odd}} \frac{1}{np} e^{-\frac{\beta^2 \omega^2}{4 \pi^2 \tau_2}} \frac{1}{(4 \pi^2 \alpha')^4} \frac{1}{|\eta(\tau)|^{16}} \left| \frac{\theta_2(\tau)}{\eta(\tau)} \right|^8.
\]

(2.73)

which is identical to the thermal partition function (per unit volume) in the DLCQ flat background calculated in [17].

We also comment on the decompactification limit \( R_- \to \infty \). To consider it, it is the easiest to start from (2.59). Under this limit the DLCQ windings \( r, s \) decouple, and we obtain

\[
\lim_{R_- \to \infty} \frac{1}{\sqrt{2 \pi R_-}} Z_{\text{torus}}(\beta) = \int_F \frac{d^2 \tau}{\tau_2^2} \frac{\beta}{4 \pi^2 \alpha'} \sum_{\epsilon=0,1} \sum_{\nu \in \mathbb{Z}+\epsilon_1 \atop n \in \mathbb{Z}+\epsilon_2} e^{-\frac{\beta w_{\nu-n}^2}{4 \pi^2 \tau_2}}
\]

\[
\times Z_{\epsilon_1,\epsilon_2}^{\text{tor}} \left( \tau, \bar{\tau}; \frac{\sqrt{2} \mu \beta}{4 \pi \tau_2} |w_{\tau-n}| \right).
\]

(2.74)

Expressing \( Z_{\epsilon_1,\epsilon_2}^{\text{tor}} \) by the appropriate massive theta functions as before, we achieve the modular invariant form of partition function. We can also rewrite it by setting \( w = 0 \) and replacing the integration region \( F \) with \( S \) based on the same argument. Transforming the integration variable as \( p^+ = \frac{\sqrt{2} \beta n}{4 \pi \alpha' \tau_2} \), we obtain

\[
\lim_{R_- \to \infty} \frac{1}{\sqrt{2 \pi R_-}} Z_{\text{torus}}(\beta) = \sum_{\epsilon=0,1} \sum_{n \in \mathbb{Z}+\epsilon, n > 0} \frac{1}{n} \int_0^{\infty} \frac{dp^+}{\sqrt{2 \pi}} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 e^{-\frac{\beta n p^+}{4 \pi \tau_2}} Z_{\epsilon,\epsilon_2}^{\text{tor}} \left( \tau, \bar{\tau}; \mu \alpha' p^+ \right)
\]

(2.75)
where we wrote $\tau = \tau_1 + i\frac{\sqrt{2}\beta n}{4\pi\alpha'p^+}$. The last expression (2.75) is readily compared with the free energy evaluated by the operator formalism and also corresponds to that presented in [26, 27].

2.5 “Virtual String” and Hagedorn Temperature

Let us return to the modular invariant partition function (2.60). As is clear from the above analysis, the sectors of $w = 0$, $p \in \mathbb{Z}_{>0}$ correspond to the states in the physical Hilbert space of the non-thermal string theory (with the Lorentzian signature) and the integer $p$ is precisely identified as the light-cone momentum by the relation $p^+ = p/R_-$. This is a natural correspondence since $p, q$ are actually the “momenta” dual to the DLCQ winding numbers $r, s$. The strings in these physical sectors have the standard mass parameters $m = \alpha'\mu p^+ \equiv \hat{\mu}p$ and satisfy the on-shell condition $-\alpha'p^+p^- = H$, where $H$ is the world-sheet Hamiltonian $H \equiv \sum_{n \in \mathbb{Z}} \omega_n N_n$, as well as the level matching condition (2.25).

However, we still have the sectors with the non-vanishing (spatial) thermal windings $w \neq 0$. They do not correspond to any physical states in the non-thermal theory. In this sense we shall call them the “virtual strings” throughout this paper\(^5\). The world-sheet theories for the virtual strings are likewise described by the world-sheet Hamiltonian $H$, but with the modified mass parameter $m = \hat{\mu}|w\nu + ip|$. The transverse partition function is calculated in the same way and composes the building blocks of the modular invariant amplitude (2.60).

In summary, the manifest modular invariance in (2.60) requires the contributions from the various virtual string sectors $w \neq 0$, while the alternative expression (2.72) only contains the physical string states, although the modular invariance is hidden. This is a general feature of the thermal string theory.

The virtual string states could be tachyonic in spite of the unbroken space-time SUSY, owing to the lack of mass-shell condition in the usual sense. This fact leads to a simple interpretation of Hagedorn temperature [23] as explained in [30, 28]. In fact, we can make use of the analogous argument based on the modular invariance of (2.60) in the previous subsection, but employ the different gauge choice $n = 0$ in (2.60) rather than $w = 0$. In that

\(^5\)The partition function (2.60) including the contributions $w \neq 0$ has been derived by the path-integral calculation based on the complex world-sheet action, as in the flat DLCQ case [17]. So, one should regard them as “virtual” in the doubly meaning. Such virtual winding modes are surely useful to gain the manifest modular invariance and a concise understanding of the thermal instability as discussed here. The physical meaning of the virtual strings resides in this fact.
case the amplitude is UV finite ($\tau_2 \sim 0$), but could have a IR divergence ($\tau_2 \sim +\infty$) due to the tachyonic mode. It is not difficult to see that the leading term is the contribution from the virtual string with $w = 1$, $n = 0$, which has the mass parameter $m = \hat{\mu} \nu \equiv \frac{\sqrt{2} \mu \beta}{4\pi}$. We thus find that

$$Z_{\text{torus}}(\beta) : \text{finite} \iff \frac{\beta^2}{8\pi^2 \alpha'} > 8 \left( \Delta\left(\frac{\sqrt{2} \mu \beta}{4\pi} ; \frac{1}{2} \right) - \Delta\left(\frac{\sqrt{2} \mu \beta}{4\pi} ; 0 \right) \right). \quad (2.76)$$

It is easy to see that the R.H.S of the inequality is always positive and a monotonically decreasing function of $\beta$. Thus we can rewrite it as

$$Z_{\text{torus}}(\beta) : \text{finite} \iff \beta > \beta_H, \quad (2.77)$$

with

$$\frac{\beta_H^2}{8\pi^2 \alpha'} - 8 \left( \Delta\left(\frac{\sqrt{2} \mu \beta_H}{4\pi} ; \frac{1}{2} \right) - \Delta\left(\frac{\sqrt{2} \mu \beta_H}{4\pi} ; 0 \right) \right) = 0. \quad (2.78)$$

the critical temperature $T_H \equiv \beta_H^{-1}$ is no other than the Hagedorn temperature at which the thermal instability occurs. This has the correct limit under $\mu \to 0$

$$\lim_{\mu \to 0} T_H = \frac{1}{\sqrt{8\pi^2 \alpha'}}, \quad (2.79)$$

consistent with the result given in [28].

An alternative interpretation of such thermal instability is presented from the “dual” expression (2.72) that only includes the physical states. This is clearly IR finite, but could be UV divergent due to the rapid growth of massive excitations depending exponentially on the oscillator level.

We further make a few comments.

Firstly, the Hagedorn temperature $T_H$ does not depend on the DLCQ radius $R_\perp$ as in the flat background. In fact, the equation (2.78) is equivalent with those given in the recent papers [26, 27] (with the suitable identification of parameters), in which the analysis for the non-DLCQ thermal model is presented.

Secondly, because the R.H.S of (2.78) is a monotonically decreasing function of $\mu$, it is easy to see that $T_H$ is bounded from below by the value for the flat background ($\mu = 0$);

$$T_H \geq T_{H,\text{flat}} \equiv \frac{1}{\sqrt{8\pi^2 \alpha'}}, \quad (2.80)$$

and diverges under the large $\mu$ limit. This means that the Hagedorn transition does not occur at any finite temperature under this limit. The stringy nature is expected to be lost under the large $\mu$ limit so that the picture of “string bit” becomes a good approximation [31, 15]. Such behavior of Hagedorn temperature is likely to be consistent with this aspect.
3 Thermal Amplitudes of Open Strings in DLCQ PP-Waves

As an extension of our previous analysis, let us consider the thermal ensemble of open strings in the DLCQ pp-waves with supersymmetric D-branes (half BPS D-branes, strictly speaking). We shall only focus on the D-branes in the maximally supersymmetric 10-dimensional pp-wave [32, 33, 24], although the generalization to more general backgrounds (say, the 6-dimensional pp-waves analyzed in the next section) is straightforward (see the papers [34]).

Turning our attention to the open-closed string duality in cylinder amplitudes (or “Cardy condition”), we seem to face a difficulty originating from the light-cone gauge quantization. For example, pick up the time-like D-branes that impose the Neumann boundary condition along the light-cone directions $X^+$, $X^-$. In this case the open string picture is compatible with the light-cone gauge $X^+ = 2\alpha' p^+ \tau$ \(^6\). However, since the boundary conditions for the closed string channel imply $\partial_\tau X^+ = 0$, we cannot choose the usual light-cone gauge. In the case of Euclidean D-brane (or, the D-brane instantons) the situation is opposite. Namely, the closed string description is compatible with the light-cone gauge, while we cannot take it in the open string channel. This naive observation may look puzzling and leads to an apparent discrepancy between the classifications of supersymmetric D-branes given in [33] and [32], where the former is based on the open string picture and the latter is the boundary state approach (closed string picture).

A clear resolution to this puzzle for the thermal model is given by considering the virtual strings we discussed above. Although the virtual strings are not compatible with the light-cone gauge condition, we can consistently define the world-sheet Hamiltonian in the quadratic form by taking the “instanton gauge” as before.

We shall separately discuss the cases of the time-like D-branes and the Euclidean D-branes.

3.1 Thermal Cylinder Amplitude for Time-like D-branes

We first consider the time-like Dp-branes, with which the light-cone coordinates $X^+$, $X^-$

---

\(^6\)It is natural to define the light-cone gauge for open string as $X^+ = 2\alpha' p^+ \tau$ rather than $X^+ = \alpha' p^+ \tau$. The parameter $p^+$ in this expression is really identified as the momentum canonically conjugate to the zero-mode variable $x^-$ in the case of open string, which should be quantized as $p^+ = p/R_-$ ($p \in \mathbb{Z}_{>0}$) in DLCQ. The difference of factor is originating from the simple fact that the spatial direction of open string world-sheet is parametrized as $0 \leq \sigma \leq \pi$, while that of the closed string is done as $0 \leq \sigma < 2\pi$ in our convention.
should satisfy the Neumann boundary conditions. As is shown in [33], the supersymmetry condition (for the half BPS brane) implies that only the cases $p = 3, 5, 7$ are allowed and the branes must be stacked at the origin of transverse plane $X^I = 0$ (if not assuming the extra flux). We only focus on the simplest configurations such that the open strings have the both ends attached to the same D$p$-brane, which are manifestly supersymmetric.

The basic aspects are summarized as follows:

1. **open string channel**

   We have the physical strings compatible with the light-cone gauge condition $X^+ = 2\alpha'p^+\tau \equiv 2\alpha'\frac{p}{R_-}\tau$ and satisfying the mass-shell condition. The world-sheet Hamiltonian $H^{(o)}$ includes the standard mass parameter $m = 2\mu\alpha'p^+ \equiv 2\hat{\mu}p$.

2. **closed string channel**

   The boundary states only contain the virtual string states not compatible with the usual light-cone gauge condition and not satisfying the mass-shell condition. The world-sheet Hamiltonian $H^{(c)}$ includes the mass parameter $m = \hat{\mu}w$ as we will see below.

We begin with the calculation in the open string channel. As in the calculation of closed string partition function, we employ the path-integral approach (especially for the longitudinal sector).

As a preparation we introduce the massive theta functions for the open string amplitudes:

$$\theta_{(a,b)}(t; m) \overset{\text{def}}{=} \sqrt{\Theta_{(a,b)}(it, -it; m)}$$

$$\equiv e^{2\pi t\Delta(m; a)} \prod_{n \in \mathbb{Z}} \left| 1 - e^{-2\pi t\sqrt{m^2 + (a + n)^2 + 2\pi ib}} \right|$$

(3.1)

where $t > 0$ is the open string modulus\(^7\). They have the following modular property

$$\theta_{(a,b)}(1/t; mt) = \theta_{(b,-a)}(t; m) ,$$

(3.2)

and the mass-less limits

$$\lim_{m \to 0} \theta_{(a,b)}(t; m) = e^{-\pi t a^2} \left| \frac{\theta_1(it, iat + b)}{\eta(it)} \right| .$$

(3.3)

We first consider the bosonic amplitudes in the transverse sector. Recall that the NN open string has the zero-modes, while the DD open string does not. The zero-point energy

\(^7\)The “modified f-functions” defined in [24] correspond to $\theta_{(0,0)}(t; m)^{1/2}$, $\theta_{(0,1/2)}(t; m)^{1/2}$, $\theta_{(1/2,1/2)}(t; m)^{1/2}$, and $\theta_{(1/2,0)}(t; m)^{1/2}$. 

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(per one complex boson) is thus evaluated as

\[ m + \sum_{n=1}^{\infty} \sqrt{m^2 + n^2} - \int_{0}^{\infty} dk \sqrt{m^2 + k^2} = \frac{m}{2} + \Delta(m; 0) , \quad (3.4) \]

for the NN strings, and

\[ \sum_{n=1}^{\infty} \sqrt{m^2 + n^2} - \int_{0}^{\infty} dk \sqrt{m^2 + k^2} = -\frac{m}{2} + \Delta(m; 0) , \quad (3.5) \]

for the DD strings. Since we have the \( p-1 \) components of NN strings and \( 9-p \) components of DD strings along the transverse plane, the bosonic amplitude is evaluated as follows [24];

\[ \text{Tr}_{H_b} \left[ e^{-2\pi t H^{(o)}} \right] = \frac{q^m (\frac{p-1}{2} - \frac{9-p}{2})}{(1 - q^m)^p-1} \times \frac{1}{\theta_{(0,0)}(t; m)^4/(1 - q^m)^4} \]

\[ = (2 \sinh(\pi t m))^{5-p} \cdot \frac{1}{\theta_{(0,0)}(t; m)^4} , \quad (3.6) \]

where we wrote \( q \equiv e^{-2\pi t} \).

For the fermionic amplitudes, a careful analysis is needed for the zero-modes. This is presented in [24] and the results are quite simple;

\[ \text{Tr}_{H_f} \left[ (-1)^F e^{-2\pi t H^{(o)}} \right] = \theta_{(0,0)}(t; m)^4 , \]

\[ \text{Tr}_{H_f} \left[ e^{-2\pi t H^{(o)}} \right] = \theta_{(0,1/2)}(t; m)^4 . \quad (3.7) \]

Let us next consider the longitudinal sector. We have various topological sectors characterized by the windings \( n, s \);

\[ X^+(\tau_E + 2\pi t, \sigma) = X^+(\tau_E, \sigma) + i \frac{\beta n}{\sqrt{2}} , \]

\[ X^-(\tau_E + 2\pi t, \sigma) = X^-(\tau_E, \sigma) + i \frac{\beta n}{\sqrt{2}} + 2\pi R_s , \]

\[ \partial_\sigma X^+|_{\sigma=0,\pi} = \partial_\sigma X^-|_{\sigma=0,\pi} = 0 . \quad (3.8) \]

As in the previous analysis, we shall make the “instanton gauge” \( X^+ = X^+_{n,s} \), where the \( X^+_{n,s} \) is the instanton characterized by the above boundary condition (3.8). It makes the world-sheet action quadratic. The mass parameter is evaluated as \( m = \mu \left| \frac{\partial X^+_{n,s}}{\partial \tau_E} \right| \equiv \mu n \nu / t \). The instanton action is also calculated as

\[ S_{\text{inst}}(n, s) = \frac{\beta^2 n^2}{8\pi \alpha'} + 2\pi i \frac{\nu}{2t} n s . \quad (3.9) \]
So, the longitudinal amplitude becomes

$$\frac{V_{l,c}}{8\pi^2\alpha' t} \times \sum_{n,s} e^{-S_{\text{inst}}(n,s)} \equiv \frac{\nu}{\alpha' t} \sum_{n,s} e^{-S_{\text{inst}}(n,s)} . \quad (3.10)$$

where the prefactor is again equal to the Gaussian integral of zero-mode momenta in the Hamiltonian formalism.

Gathering all the things, we achieve the following thermal cylinder amplitude

$$Z^{(o)}_{\text{cylinder}}(\beta; Dp) = \int_0^\infty \frac{dt}{t} \sum_{\epsilon=0,1} \sum_p \sum_{n\in 2Z+\epsilon} e^{-\frac{\beta^2 n^2}{8\pi\alpha' t}} \delta(pt - \nu n/2) \times (2 \sinh(2\pi t\hat{\mu}p))^{5-p} \cdot \frac{\theta_{(0,\hat{\mu})}(t; 2\hat{\mu}p)^4}{\theta_{(0,0)}(t; 2\hat{\mu}p)^4} , \quad (3.11)$$

where $\epsilon$ indicates the thermal boundary condition of GS fermions. Again the DLCQ winding $s$ is dualized into the light-cone momentum $p$, yielding the periodic delta function. Notice that the correct mass parameter $m = 2\mu\alpha'p^+ \equiv 2\hat{\mu}p$ has been successfully reproduced. Performing the modulus integral explicitly, we also obtain

$$Z^{(o)}_{\text{cylinder}}(\beta; Dp) = \sum_{p=1}^\infty \left[ \sum_{n: \text{even}, n > 0} \frac{1}{n} e^{-\frac{\beta^2 n^2}{8\pi\alpha' t}} \cdot (2 \sinh(2\pi t\hat{\mu}p))^{5-p} \right. \left. + \sum_{n: \text{odd}, n > 0} \frac{1}{n} e^{-\frac{\beta^2 n^2}{8\pi\alpha' t}} \cdot (2 \sinh(2\pi t\hat{\mu}p))^{5-p} \cdot \frac{\theta_{(0,\hat{\mu})}(t; 2\hat{\mu}p)^4}{\theta_{(0,0)}(t; 2\hat{\mu}p)^4} \right] , \quad (3.12)$$

where we set $t = \frac{\nu n}{2p}$. The last expression (3.12) should coincide with the open string free energy defined in the same way as (2.63) (up to the factor $-\beta$). In fact, the on-shell condition for the open string is expressed as $H^{(o)} = -2\alpha'p^+ p^-$, and thus we have

$$p^0 = \frac{1}{\sqrt{2 R_-}} p + \frac{R_-}{2\sqrt{2\alpha' p}} H^{(o)} . \quad (3.13)$$

Thanks to this relation, one can immediately confirm their coincidence. In particular (3.12) only includes the contributions from the physical open string states compatible with the light-cone gauge $X^+ = 2\alpha' \frac{p}{R_-}$. Let us next analyze the closed string channel. The wanted boundary state is composed only by the virtual string states that are not compatible with the light-cone gauge and do not
satisfy the mass-shell condition\(^8\). It should have the following structure

\[ |D_p⟩ = \sum_{w,r} N_{w,r} |w, r⟩ \otimes |D_p; \hat{\mu}w\nu⟩^{(tr)}, \tag{3.14} \]

where \( |w, r⟩ \) are the zero-mode eigenstates for the longitudinal directions associated to the topological sectors

\[
\begin{align*}
X^+(\tau_E, \sigma + 2\pi) &= X^+(\tau_E, \sigma) + i\frac{\beta w}{\sqrt{2}}, \\
X^-(\tau_E, \sigma + 2\pi) &= X^-(\tau_E, \sigma) - i\frac{\beta w}{\sqrt{2}} + 2\pi R_r, \\
\partial_{\tau_E} X^+|_{\tau_E=0, \pi T} &= \partial_{\tau_E} X^-|_{\tau_E=0, \pi T} = 0. \tag{3.15}
\end{align*}
\]

\(|D_p; \hat{\mu}w\nu⟩^{(tr)}\) denotes the boundary states describing the transverse part of time-like Dp-brane, which is similarly defined as that for the supersymmetric Euclidean D(p-2)-brane presented in [32], but with the mass parameter \( m = \hat{\mu}w\nu \) instead of \( m = \hat{\mu}p \). The instanton configuration corresponding to (3.15) indeed leads to the mass \( m = \hat{\mu}w\nu \) in the similar manner as before.

Another important modification is the thermal boundary condition of fermionic fields. We must employ the integral modes of fermionic oscillators for the even \( w \), and the half-integral modes for the odd \( w \). \( N_{w,r} \) are the numerical factors which should be determined by the requirements of open-closed duality.

The transverse part yields the standard overlap amplitude;

\[
\langle (D_p; \hat{\mu}w\nu)|(-1)^F e^{-\pi TH^{(c)}}|D_p; \hat{\mu}w\nu⟩^{(tr)} = \begin{cases} 
\frac{\theta_{(0,0)}(T; \hat{\mu}w\nu)^4}{\theta_{(0,0)}(T; \hat{\mu}w\nu)^4} \equiv 1 & (w \in 2\mathbb{Z}), \\
\frac{\theta_{(0,0)}(T; \hat{\mu}w\nu)^4}{\theta_{(1,0)}(T; \hat{\mu}w\nu)^4} & (w \in 2\mathbb{Z} + 1).
\end{cases} \tag{3.16}
\]

The longitudinal part of closed string Hamiltonian \( H^{(c)}_l \) is given as the zero-mode part of standard \( L_0 \) operator, which simply gives

\[
\begin{align*}
\langle w, r|e^{-\pi TH^{(c)}_l}|w', r'⟩ &= \delta_{w,w'}\delta_{r,r'}e^{-S_{\text{inst}}(w,r)}, \\
S_{\text{inst}}(w,r) &= \frac{\beta^2 w^2 T}{8\pi\alpha'} + \pi i T\nu wr. \tag{3.17}
\end{align*}
\]

\(^8\)We now would like to emphasize that this is not a peculiarity of the pp-wave background. In fact, as the simplest example, let us recall the case in which the Neumann boundary conditions are imposed along all the directions on the non-compact flat space-time. In this case, since the boundary state can only include the zero momentum states, the closed string states appearing in it never satisfy the mass-shell condition (except for the massless states with zero momenta). Nevertheless, this boundary state is surely on-shell in the sense that the boundary conformal symmetry is preserved; \( L_n - \hat{L}_{-n} = 0 \). (\(^\prime\)n)
In total, we obtain the cylinder amplitude in the closed string channel

\[
Z_{cylinder}^{(c)}(\beta; Dp) \equiv \int_0^\infty dT \langle Dp \rangle (-1)^F e^{-\pi T \gamma_H^{(c)}} |Dp\rangle \\
= \int_0^\infty dT \sum_{\epsilon=0,1} \sum_{w \in 2\mathbb{Z}+\epsilon} \sum_r \left| \mathcal{N}_{w,r}(T) \right|^2 e^{-\frac{2\pi^2}{\epsilon} \frac{w^2}{8\pi^2} + \pi i T \nu \omega r} \frac{\theta(\frac{\nu}{2})(T; \hat{\mu} w \nu)^4}{\theta(0,0)(T; \hat{\mu} w \nu)^4} .
\] (3.18)

We choose the normalization coefficients \(\mathcal{N}_{w,r}\) as

\[
\mathcal{N}_{w,r} = \sqrt{\nu/2} \cdot (2 \sinh(\pi \hat{\mu} w \nu))^{\frac{5-p}{2}},
\] (3.19)

according to [24]. The prefactor \(\sqrt{\nu/2}\) is the correct one associated to the Neumann boundary conditions along the longitudinal directions. Since \(\mathcal{N}_{w,r}\) does not depend on \(r\), we can explicitly make the summation over \(r\), resulting

\[
Z_{cylinder}^{(c)}(\beta; Dp) = \int_0^\infty dT \sum_{\epsilon=0,1} \sum_{w \in 2\mathbb{Z}+\epsilon} \sum_q e^{-\frac{2\pi^2}{\epsilon} \frac{w^2}{8\pi^2} + \pi i T \nu \omega} \delta \left( \frac{\nu}{2} w T - q \right) \\
\times \nu/2 \cdot (2 \sinh(\pi \hat{\mu} w \nu))^{5-p} \cdot \frac{\theta(\frac{\nu}{2})(T; \hat{\mu} w \nu)^4}{\theta(0,0)(T; \hat{\mu} w \nu)^4} .
\] (3.20)

The closed and open string moduli should be identified by the standard relation \(t = 1/T\). Comparing (3.11) and (3.20), and using the modular property of the massive theta function (3.2), one can find that

\[
Z_{cylinder}^{(o)}(\beta; Dp) = Z_{cylinder}^{(c)}(\beta; Dp) .
\] (3.21)

This equality is no other than the wanted open-closed string duality.

We finally evaluate the Hagedorn temperature based on the cylinder amplitudes. As in the previous analysis of toroidal partition function, it is easiest to evaluate the IR behavior of virtual strings, namely, to study the behavior around \(T \sim \infty\) in \(Z_{cylinder}^{(c)}(\beta; Dp)\) for the present problem. Obviously the dominant term is \(w = 1\), and we find that

\[
Z_{cylinder}^{(c)}(\beta; Dp) : \text{finite} \iff \beta > \beta_H
\] (3.22)

with

\[
\frac{\beta_H^2}{16 \pi^2 \alpha'} - 4 \left( \Delta \left( \frac{\sqrt{2} \mu \beta_H}{4 \pi}; \frac{1}{2} \right) - \Delta \left( \frac{\sqrt{2} \mu \beta_H}{4 \pi}; 0 \right) \right) = 0 .
\] (3.23)

It is equivalent with the equation (2.78) for the closed string sector. Thus the open string sector has the equal Hagedorn temperature as in the case of flat background. Especially, it does not depend on the dimension of brane \(p\).
3.2 Thermal Ensemble of Closed String States Emitted/Absorbed by Euclidean D-branes

We next consider the Euclidean Dp-branes, which impose the Dirichlet condition for $X^+$, $X^-$. We shall express them as D’p to distinguish from the time-like D-branes. The compatibility with supersymmetry implies the allowed values of p are $p = 1, 3, 5$ [32], and again they must be stacked at the origin in the transverse plane.

The roles played by the open and closed string channels are opposite to the previous case:

1. open string channel

We have the virtual strings not compatible with the light-cone gauge. The world-sheet Hamiltonian $H^{(o)}$ includes the mass parameter $m = 2\hat{\mu}w\nu$ as shown below.

2. closed string channel

The boundary states contain the physical string states compatible with the light-cone gauge condition $X^+ = \alpha' p^+ \tau \equiv \alpha' \frac{P}{R_-} \tau$. The world-sheet Hamiltonian $H^{(c)}$ includes the standard mass parameter $m = \mu \alpha' p^+ \equiv \hat{\mu} p$.

We first discuss the closed string channel. Contrary to the previous analysis, it might be ambiguous what type amplitude we should evaluate, because our purpose in this paper is to calculate the thermodynamical trace over the physical states. Generically, the overlap amplitudes of boundary states are not interpreted as a trace over the closed string Hilbert space. Nevertheless, it is quite natural to consider the next “free energy”

$$F(\beta; D'p) \overset{\text{def}}{=} \frac{1}{\beta} \langle D'p|(-1)^F \ln \left(1 - (-1)^F e^{-\beta p^0}\right)|D'p\rangle$$

$$\equiv -\sum_{n=1}^{\infty} \frac{1}{\beta^n} \langle D'p|(-1)^{(n+1)} F e^{-\beta np^0}|D'p\rangle,$$

(3.24)

where $|D'p\rangle$ denotes the boundary state describing the D’p brane localized at $X^+ = X^- = 0$ \(^9\), which should have the structure

$$|D'p\rangle = \sum_{p,h} \mathcal{N}'_p |p^+ = p/R_- \rangle \otimes |p^- (p, h)\rangle \otimes |D'p, h; \hat{\mu}p\rangle^{(tr)},$$

(3.25)

\(^9\)In principle, we can consider the cylinder amplitude in which the ends of open string attached at the D’p branes located at different points along the $X^+$, $X^-$ directions by including the suitable phase factors in $\mathcal{N}'_p$. However, since our purpose is to calculate the trace over the closed string states, we must consider the case of open strings ended at the same brane.
In this expression \(|D'p, h; \hat{\mu}p\rangle^{(tr)}\) is defined by the decomposition of the transverse boundary state of D’p brane \(|D'p; \hat{\mu}p\rangle^{(tr)}\) with respect to the eigenvalue of \(H^{(c)}\):

\[
|D'p; \hat{\mu}p\rangle^{(tr)} = \sum_h |D'p, h; \hat{\mu}p\rangle^{(tr)},
\]

\(H^{(c)}|D'p, h; \hat{\mu}p\rangle^{(tr)} = h|D'p, h; \hat{\mu}p\rangle^{(tr)}.
\]

The longitudinal zero-mode parts \(|p^+ = p/R_−\rangle, |p^-\rangle\) are the eigenstates of light-cone momenta and we assume that the mass-shell condition is satisfied;

\[
\left(\alpha' \frac{p}{R_−} + H^{(c)}\right) |p^+ = p/R_−\rangle \otimes |p^− (p, h)\rangle \otimes |D'p, h; \hat{\mu}p\rangle^{(tr)} = 0,
\]

which uniquely determines \(p^-\) as a function of \(p, h\).

The transverse boundary state \(|D'p; \hat{\mu}p\rangle^{(tr)}\) with the mass parameter \(m = \hat{\mu}p\) is given in [32]. In the present case we have \(p + 1\) NN and \(7 - p\) DD open strings. Therefore, because of the consistency with (3.19), we should choose the normalization coefficients \(N'_p\) as

\[
N'_p = \frac{1}{\sqrt{2\nu}} \cdot (2 \sinh(\pi \hat{\mu}p))^{3-p},
\]

as presented in [24]. The factor \(\frac{1}{\sqrt{2\nu}}\) is the characteristic one for the Dirichlet boundary states which reflects the absence of zero-mode integral in the calculation of open string channel.

The free energy (3.24) can be interpreted as the trace over a subspace of the physical Hilbert space of closed string sector. To be more accurate this subspace is composed of all the single-particle physical states that can be emitted/absorbed by the D’p-brane. We thus regard (3.24) as the free energy for the thermal ensemble of such closed string states.

It is a straightforward calculation to evaluate (3.24) and we obtain

\[
F(\beta; D'p) = -\frac{1}{\beta} \sum_{\beta = 0,1}^\infty \sum_{n:even,n>0} \frac{1}{n} e^{\frac{\beta^2 n^2}{2\pi \alpha' T}} \cdot \frac{1}{2\nu} (2 \sinh(\pi \hat{\mu}p))^{3-p} \\
+ \sum_{n:odd,n>0} \frac{1}{n} e^{\frac{\beta^2 n^2}{2\pi \alpha' T}} \cdot \frac{1}{2\nu} (2 \sinh(\pi \hat{\mu}p))^{3-p} \cdot \frac{\theta_{(0,\frac{1}{2})}(T; \hat{\mu}p)^4}{\theta_{(0,0)}(T; \hat{\mu}p)^4} \right],
\]

where we set \(T = \frac{2n\nu}{p}\). Furthermore, we can rewrite it by a short calculation based on the modular property (3.2) as follows;

\[
F(\beta; D'p) = -\frac{1}{\beta} \sum_{\beta = 0,1}^\infty \sum_{q=1}^\infty \sum_{w \in \mathbb{Z} + e} \int_0^\infty dt e^{-\frac{\beta^2 t^2}{2\nu^2 T}} \delta(2w^2t - q) \\
\times (2 \sinh(2\pi \hat{\mu}w^2t))^{3-p} \cdot \frac{\theta_{(\frac{1}{2},0)}(t; 2\hat{\mu}w^2)^4}{\theta_{(0,0)}(t; 2\hat{\mu}w^2)^4}.
\]

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This expression has a nice interpretation as the virtual open string amplitude possessing the non-vanishing thermal windings. In fact, the longitudinal part is calculated by summing over the instantons associated to the boundary conditions

\[
\begin{align*}
X^+(\tau_E, \sigma = \pi) &= X^+(\tau_E, \sigma = 0) + i \frac{1}{\sqrt{2}} \beta w , \\
X^-(\tau_E, \sigma = \pi) &= X^-(\tau_E, \sigma = 0) - i \frac{1}{\sqrt{2}} \beta w + 2\pi R \cdot r , \\
\partial_{\tau_E} X^+|_{\sigma=0,\pi} &= \partial_{\tau_E} X^-|_{\sigma=0,\pi} = 0 .
\end{align*}
\] (3.31)

The instanton action is equal to

\[
S_{\text{inst}}(w, r) = \frac{\beta^2 w^2}{2 \pi \alpha'} + 4\pi i \nu w r ,
\] (3.32)

leading to the longitudinal amplitude

\[
\sum_{w,r} e^{-S_{\text{inst}}(w,r)} = \sum_{w,q} e^{-\frac{\beta^2 w^2}{2 \pi \alpha'} \delta(2\nu wt - q)} .
\] (3.33)

Notice that we now do not have the zero-mode integral because of the Dirichlet condition.

The mass parameter for the transverse sector is evaluated from this instanton configuration (3.31) as \(m = 2 \hat{\mu} \nu w\). We thus obtain the transverse amplitudes by the calculations similar to (3.6) and (3.7)

\[
\text{Tr}_{Hw} \left[ (-1)^F e^{-2\pi t H(o)} \right] = (2 \sinh(2\pi \hat{\mu} w v t))^{3-p} \frac{\theta_{(0,0)}(t; 2\hat{\mu} w v)^4}{\theta_{(0,0)}(t; 2\hat{\mu} w v)^4} \equiv (2 \sinh(2\pi \hat{\mu} w v t))^{3-p} ,
\]

\[
(w \in 2\mathbb{Z}) ,
\]

\[
\text{Tr}_{Hw} \left[ (-1)^F e^{-2\pi t H(o)} \right] = (2 \sinh(2\pi \hat{\mu} w v t))^{3-p} \frac{\theta(t; 2\hat{\mu} w v)^4}{\theta_{(0,0)}(t; 2\hat{\mu} w v)^4} \quad (w \in 2\mathbb{Z} + 1) ,
\] (3.34)

where \(H_w\) denotes the Hilbert space of virtual open string states with the thermal winding number \(w\).

These results (3.33), (3.34) correctly reproduce the free energy (3.30). Only the non-trivial difference from the standard cylinder amplitude is that the modulus integral is now given as \(\int dt\) instead of \(\int \frac{dt}{t}\). It amounts to that we are now calculating \(\sim \text{Tr} \Delta\) rather than \(\sim \text{Tr} \ln \Delta\), where \(\Delta\) denotes the open string propagator.

The Hagedorn temperature is likewise evaluated by observing the behavior of virtual open string with \(w = 1\);

\[
F(\beta; D'p) : \text{finite} \iff \beta > \beta_H
\] (3.35)
with
\[
\frac{\beta_H^2}{4\pi^2\alpha'} - 4 \left( \Delta \left( \frac{\sqrt{2} \mu \beta_H}{2\pi}, \frac{1}{2} \right) - \Delta \left( \frac{\sqrt{2} \mu \beta_H}{2\pi}, 0 \right) \right) = 0.
\] (3.36)

\(\beta_H\) is again independent of the value \(p\). Moreover, it is easy to see that \(T_H \equiv \beta_H^{-1}\) evaluated by the equation (3.36) is strictly higher than those for (2.78), (3.23). Therefore, we conclude that the existence of Euclidean Dp-branes do not affect the Hagedorn behavior.

4 Thermal Partition Function of DLCQ Superstring on 6-dimensional PP-Wave

4.1 DLCQ PP-Wave as the Penrose limit of Orbifolded \(AdS_3 \times S^3\)

Before analyzing the thermal partition function, let us illustrate how the 6-dimensional DLCQ pp-wave with enhanced SUSY can be derived from the orbifolded \(AdS_3 \times S^3\). This is almost parallel to the 10-dimensional argument [6].

We begin with the familiar system of \(Q_5\) D5 and \(Q_1\) D1, or its transforms by the \(SL(2; \mathbb{Z})\)-duality more generally. The 5-branes are wrapped along the 4-dimensional compact space \(M^4(\equiv T^4\text{ or } K3)\). The near horizon geometry is described by the background \(AdS_3 \times S^3(\times M^4)\) parametrized as

\[
ds^2 = R^2 \left[ -\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, d\phi^2 + d\alpha^2 + \sin^2 \alpha \, d\theta^2 + \cos^2 \alpha \, d\chi^2 \right],
\] (4.1)

with the enhanced SUSY (16 supercharges in the sense of 6-dimension\(^{10}\)).

Let us consider the \(Z_N\)-orbifoldization along the 4-dimensional space transverse to the whole branes, parametrized by complex coordinates \(z_1, z_2\). \(S^3\) in the above near horizon geometry is described as \(|z_1|^2 + |z_2|^2 = R^2\), and hence the relevant geometry is \(AdS_3 \times S^3(\times M^4)\) parametrized as

\[z_1 = R \sin \alpha \, e^{i\theta}, \quad z_2 = R \cos \alpha \, e^{i\chi}.\] (4.2)

It is convenient to recombine \(z_1, z_2\) to a single matrix

\[
Z \equiv \frac{1}{R} \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}.
\] (4.3)

\(^{10}\text{In the sense of 10-dimensional theory we have the 24 supercharges for }M^4 = T^4, \text{ and 16 supercharges for }M^4 = K3.\)
and the isometry of $S^3$ is expressed as $Z \mapsto g_L Z g_R^{-1}$ \((g_L, g_R) \in SU(2)_L \times SU(2)_R\), which corresponds to the R-symmetry group. We have several choices of $Z_N$ action summarized as:

1. The diagonal action $Z_N \subset SU(2)_D \subset SU(2)_L \times SU(2)_R$, namely,

   \[
   Z \mapsto e^{\pi in_3/N} Ze^{-\pi in_3/N},
   \]

   \[
   \iff z_1 \mapsto z_1, \quad z_2 \mapsto e^{2\pi in/N} z_2, \tag{4.4}
   \]

   where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the Pauli matrix. It breaks the full R-symmetry $SU(2)_L \times SU(2)_R$. we hence obtain no SUSY theory.

2. The left action $Z_N \subset SU(2)_L$, namely,

   \[
   Z \mapsto e^{2\pi in_3/N} Z,
   \]

   \[
   \iff z_1 \mapsto e^{2\pi in/N} z_1, \quad z_2 \mapsto e^{2\pi in/N} z_2. \tag{4.5}
   \]

This orbifoldization breaks the $SU(2)_L$ R-symmetry, leaving the half SUSY (8 supercharges as the 6-dimensional theory).

Now, let us consider the following two types of the Penrose limits:

(a) We choose the null-geodesic located along $\alpha = 0$. Namely, we set $r = \rho R, y = \alpha R, x^+ = \frac{1}{2}(t + \chi), x^- = \frac{R^2}{2}(t - \chi)$, and take the large $R$ limit fixing $r,y,x^+,x^-$ to be finite values.

(b) We choose the null-geodesic located along $\alpha = \pi/2$. Namely, we set $r = \rho R, y = (\pi/2 - \alpha)R, x^+ = \frac{1}{2}(t + \theta), x^- = \frac{R^2}{2}(t - \theta)$, and take the large $R$ limit fixing $r,y,x^+,x^-$ to be finite values.

We have several combinations of the choices of $Z_N$-action and the Penrose limits. Their aspects are summarized as follows;

• The combination 1-(a):

The null-geodesic does not lie along the fixed point locus ($\alpha = \pi/2$). We so obtain the smooth 6-dimensional pp-wave:

\[
\text{ds}^2 = -4dx^+dx^- - (r^2 + y^2)(dx^+)^2 + dr^2 + r^2d\phi^2 + dy^2 + y^2d\theta^2, \tag{4.6}
\]

which is compatible with the enhanced SUSY (equal number of supercharges to $AdS_3 \times S^3$). A similar SUSY enhancement in the non-SUSY orbifold is also pointed out in [25].
for the 10-dimensional pp-wave. We can generally turn on the RR-flux and NSNS-flux at the same time,

\[ F_{+12}^{\text{RR}} = F_{+34}^{\text{RR}} \sim \mu , \quad F_{+12}^{\text{NS}} = F_{+34}^{\text{NS}} \sim \gamma , \]

depending on the brane charges we set up.

The orbifoldization (4.4) acts on the new coordinates as

\[ x^+ \mapsto x^+ + \frac{n\pi}{N} , \quad x^- \mapsto x^- + \frac{n\pi R^2}{N} . \]

Therefore, under the “large quiver limit”, which is defined by taking \( R \to \infty \) and \( N \to \infty \) limit at the same time with the ratio \( R^2/N \) fixed to be finite, we obtain the DLCQ pp-wave background;

\[ x^+ : \text{non-compact} , \quad x^- \sim x^- + 2\pi R_- , \]

\[ R_- \stackrel{\text{def}}{=} \frac{R^2}{2N} . \]

- 1-(b):
  The null-geodesic lies along the fixed point locus. We so obtain the non-SUSY pp-wave which has an orbifold singularity in the transverse plane.

- 2-(a) and 2-(b):
  In the case 2-(a), the orbifold action amounts to

\[ x^+ \mapsto x^+ + \frac{n\pi}{N} , \quad x^- \mapsto x^- + \frac{n\pi R^2}{N} , \quad \theta \mapsto \theta + \frac{2n\pi}{N} . \]

Therefore, we again obtain the DLCQ pp-wave with enhanced SUSY (4.9) under the large quiver limit. The case 2-(b) is completely parallel.

We shall concentrate on the DLCQ pp-wave with the enhanced SUSY from now on.

### 4.2 Thermal Partition Function of the 6-dimensional DLCQ pp-wave: Case of \( M^4 = T^4 \)

We first treat the simpler case \( M^4 = T^4 \). The GS action in the light-cone gauge for the background (4.6) with the general flux (4.7) is given as follows ([4], see also [35]):

\[ S = \frac{1}{4\pi\alpha'} \int d^2\sigma \left[ |\partial_\tau Z_i|^2 - |(\partial_\sigma + i\eta)Z_i|^2 - m^2|Z_i|^2 + \partial_+ Y^j \partial_- Y^j \right] \]

\[ - \frac{i}{2\pi} \int d^2\sigma \left[ S(\rho^0 \partial_\tau + \rho^1 \partial_\sigma + \rho^1 \otimes i\eta M - I \otimes mM)S \right] , \]

\[ (4.11) \]
where we set $m = \alpha' \mu p^+, \eta = \alpha' \gamma p^+$. $Z_1 \equiv X^1 + iX^2, Z_2 \equiv X^3 + iX^4$ are the coordinates along the transverse plane in the pp-wave and $Y^j$ are the coordinates of $T^4$. $\rho^0, \rho^1$ denote the 2-dimensional gamma matrices for the world-sheet, explicitly defined by

$$
\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
$$

(4.12)

and the fermionic coordinates are expressed as

$$
S = \begin{pmatrix} S^a \\ \tilde{S}^a \end{pmatrix} \in 8_s \times 8_s, \quad \tilde{S}^a \equiv (\tilde{S}^a, -S^a).
$$

(4.13)

The $8 \times 8$ matrix $M$ is defined in terms of the gamma matrices $\gamma_{ab}^I$, $\tilde{\gamma}_{ab}^I$ introduced before as follows;

$$
M \overset{\text{def}}{=} \frac{i}{2}(\gamma^1 \tilde{\gamma}^2 + \gamma^3 \tilde{\gamma}^4) \equiv i\gamma^1 \tilde{\gamma}^2 \frac{1}{2}(1 - \Pi).
$$

(4.14)

Therefore, we can classify the components of GS fermions as

- $\Pi = +1$: The eigenvalues of $M$ are all zero. We have 8 components of the massless and untwisted GS fermions, which we express as $S^{(0)a_0}, \tilde{S}^{(0)a_0} (a_0 = 1, \ldots, 4)$.

- $\Pi = -1$: The eigenvalues of $M$ are $+1$ and $-1$. They correspond to the 8 components of massive and twisted GS fermions. We denote the 4 components with $M = +1$ ($M = -1$) as $S^{(+a_+}, \tilde{S}^{(+a_+}(a_+ = 1, 2)$ ($S^{(-a_-)}, \tilde{S}^{(-a_-} (a_- = 1, 2)$). The mass and twist parameter are given by $m, \pm \eta$ respectively.

The equation of motion of bosonic coordinates $Z_i$ is given by

$$
\partial^2_\tau Z_i - (\partial_\sigma + i\eta)^2 Z_i + m^2 Z_i = 0.
$$

(4.15)

Making use of the simple redefinition of variable; $\hat{Z}_i \overset{\text{def}}{=} e^{i\eta} Z_i$, this equation reduces to the simpler Klein-Gordon equation

$$
\partial^2_\tau \hat{Z}_i - \partial^2_\sigma \hat{Z}_i + m^2 \hat{Z}_i = 0.
$$

(4.16)

$\hat{Z}_i$ are the free massive complex bosons with the twisted boundary condition

$$
\hat{Z}_i(\tau, \sigma + 2\pi) = e^{2\pi i \eta} \hat{Z}_i(\tau, \sigma),
$$

(4.17)

since the original variable $Z_i$ should be single-valued.
The canonical quantization is most easily defined with respect to the free fields \( \hat{Z}_i \) (and their fermionic counterparts \( \hat{S}_a \), etc.), and the world-sheet Hamiltonian is calculated as

\[
H = \sum_{n \in \mathbb{Z}} \left[ \sqrt{m^2 + (n + \eta)^2 N_n^{(+)}} + \sqrt{m^2 + (n - \eta)^2 N_n^{(-)}} + |n| N_n^{(0)} \right] - a(p^+) , \quad (4.18)
\]

where \( N_n^{(+)}, N_n^{(-)} \) denotes the mode counting operators associated to the Fourier modes \( e^{\pm i(n+\eta)\sigma}, e^{\pm i(n-\eta)\sigma} \) respectively, and \( N_n^{(0)} \) is that for the remaining massless fields \( Y^j, S^{(0)a_0}, \tilde{S}^{(0)a_0} \). For example, the annihilation operators contained in \( \hat{Z}_i \) and the creation operators within \( \hat{Z}_i^\dagger \) are counted by \( N^{(+)n} \). (We again employ the convention such that the positive and negative modes correspond to the left and right-movers respectively, and the zero-modes are suitably defined so as to diagonalize the Hamiltonian.) \( a(p^+) \) again denotes the normal order constant which is evaluated as before.

The DLCQ compactification \( X^- \sim X^- + 2\pi R_- \) leads to the momentum quantization

\[
p^+ = \frac{p}{R_-} , \quad (p \in \mathbb{Z}_{>0}) , \quad (4.19)
\]

and the level matching condition

\[
P \equiv \sum_n n \left( N_n^{(+)}, N_n^{(-)} + N_n^{(0)} \right) = pk , \quad (k \in \mathbb{Z}) , \quad (4.20)
\]

where \( P \) denotes the world-sheet momentum operator associated to the original string coordinates \( Z_i, S^a \), and so on. The momentum associated to the hatted fields is also useful, which is defined as

\[
\hat{P} = \sum_{n \in \mathbb{Z}} (n + \eta) N_n^{(+)}, + \sum_{n \in \mathbb{Z}} (n - \eta) N_n^{(-)} + \sum_{n \in \mathbb{Z}} n N_n^{(0)}
\approx P + \eta \hat{h} , \quad (4.21)
\]

where \( \hat{h} \equiv \sum_{n \in \mathbb{Z}} \left( N_n^{(+)}, - N_n^{(-)} \right) \) is the helicity operator along the transverse plane of pp-wave. It is convenient to further introduce the notations \( \hat{\mu} \equiv \alpha' \mu / R_- \), \( \hat{\gamma} \equiv \alpha' \gamma / R_- \), resulting \( m = \hat{\mu} p, \eta = \hat{\gamma} p \).

According to the similar arguments for the 10-dimensional pp-wave, we can derive the modular invariant expression of thermal partition function by the path-integral calculation. The non-trivial difference is only the existence of twisted boundary conditions for the hatted fields. As before, we take the instanton gauge \( X^+ = X^+_{w,n,r,s} \), with working on the rotated coordinates \( z' = e^{i\theta w.n} z, \bar{z}' = e^{-i\theta w.n} \bar{z} \) defined by (2.56). Thanks to (2.57), the world-sheet action of the transverse coordinates has a quadratic form as (4.11). Especially, the bosonic part is written as (on the Euclidean world-sheet)

\[
S = \frac{1}{8\pi \alpha'} \int d^2 z' \left\{ (2\partial_{z'} - i\eta') Z_i (2\partial_{z'} + i\eta') Z_i + (2\partial_{z'} + i\eta') Z_i^* (2\partial_{z'} - i\eta') Z_i + 2m^2 |Z_i|^2 \right\} . \quad (4.22)
\]
where the mass parameter $m'$ is equal to $\mu \frac{\nu}{\tau_2} |w\tau - n|$ and the twist parameter $\eta'$ is similarly calculated as $\eta' = -\gamma \frac{\nu}{\tau_2} |w\tau - n|$. Introducing the field redefinitions $\hat{Z}_i = Z_i e^{-i\frac{\eta'}{\tau_2} (z' + \bar{z}')}, \hat{Z}_i^* = Z_i^* e^{i\frac{\eta'}{\tau_2} (z' + \bar{z}')}$, the action (4.22) reduces to a simpler form

$$S = \frac{1}{4\pi\alpha'} \int d^2z' \left\{ 2(\partial_{z'} \hat{Z}_i \partial_{\bar{z}'} \hat{Z}_i^* + \partial_{\bar{z}'} \hat{Z}_i \partial_{z'} \hat{Z}_i^*) + m'^2|Z_i|^2 \right\} . \quad (4.23)$$

To perform the quantization we must go back to the original coordinates $z, \bar{z}$. The obtained action also has the same form, since the action (4.23) is rotationally invariant. However, the free fields $\hat{Z}_i, \hat{Z}_i^*$ have non-trivial boundary conditions, because of the relations

$$\hat{Z}_i(z + 2\pi, \bar{z} + 2\pi) = e^{2\pi i\frac{\eta}{\tau_2} (z + \bar{z})} \hat{Z}_i(z, \bar{z}), \quad \hat{Z}_i(z + 2\pi \tau, \bar{z} + 2\pi \bar{\tau}) = e^{2\pi i\frac{\eta}{\tau_2} (z + \bar{z})} \hat{Z}_i(z, \bar{z}). \quad (4.25)$$

In this way we can calculate the desired thermal partition function as the form reminiscent of (2.59);

$$Z_{\text{torus}}(\beta) = \int_\mathcal{F} \frac{d^2\tau}{\tau_2} \nu \sum_{\epsilon_1=0,1} \sum_{w \in \mathbb{Z} + \frac{1}{2}} \sum_{n \in \mathbb{Z} + \frac{1}{2}} e^{-S_{\text{inst}}(w,n,r,s)} Z_{\epsilon_1,\epsilon_2}^{\text{tr}}(\tau, \bar{\tau}; m_{w,n}, \eta^1_{w,n}, \eta^2_{w,n}) \quad (4.26)$$

where the instanton action $S_{\text{inst}}$ is defined in (2.49). The transverse partition function $Z_{\epsilon_1,\epsilon_2}^{\text{tr}}$ depends on the mass parameter

$$m_{w,n} = \mu \frac{\nu}{\tau_2} |w\tau - n| , \quad (4.27)$$

and also the spatial and temporal twist parameters

$$\eta^1_{w,n} = \gamma \frac{\nu(\tau_1 w - n)}{\tau_2} , \quad \eta^2_{w,n} = \gamma \frac{\nu(-|\tau|^2 w + n\tau_1)}{\tau_2} . \quad (4.28)$$

The subscripts $\epsilon_1, \epsilon_2$ again indicate the thermal boundary conditions of the GS fermions. As before, we can readily carry out the summation over $r, s$, since $Z_{\epsilon_1,\epsilon_2}^{\text{tr}}$ only depends on the thermal windings $w, n$. We thus obtain the same periodic delta function

$$\sim \tau_2 \sum_{w,n,p,q} \delta^{(2)}(w\nu + ip)\tau - (n\nu + iq) , \quad (4.29)$$
which imposes the constraints (2.61). Based on this fact we can replace the parameters $m_{w,n}$, $\eta_{w,n}^1$ and $\eta_{w,n}^2$ with the simpler ones:

$$m_{w,n} \to \tilde{\mu}|w\nu + ip|, \quad \eta_{w,n}^1 \to \tilde{\gamma}p, \quad \eta_{w,n}^2 \to -\tilde{\gamma}q.$$  

(4.30)

Notice that the (spatial) twist parameter depends on the light-cone momentum $p^+ = p/R_-$ as $\tilde{\gamma}p$ rather than $\tilde{\gamma}|w\nu + ip|$. This fact is consistent with the existence of spectral flow symmetry in the purely NSNS case $\mu = 0$, as we will later discuss.

$Z_{T^4_{\tilde{\tau}_1, \tilde{\tau}_2}}$ is again expressed by means of the massive theta functions $\Theta_{(a,b)}(\tau, \bar{\tau}; m)$. (Recall (2.37), for example.) We finally obtain

$$Z_{\text{torus}}(\beta) = \int_{\tau} d^2\tau \sum_{\epsilon_i = 0, 1} \sum_{w \in \mathbb{Z} + \epsilon_1} \sum_{n \in \mathbb{Z} + \epsilon_2} e^{-\frac{\beta^2|w\nu + ip|^2}{4\pi^2\tau^2}} \frac{1}{\tau_2} \delta^{(2)}((w\nu + ip)\tau - (n\nu + iq))$$

$$\times Z_T^{(0)}(\tau, \bar{\tau}) \left( \frac{1}{(4\pi^2\alpha'^2 \tau^2)^2} \frac{1}{|\eta(\tau)|^8} \cdot e^{-\pi\tau^2\epsilon_1} \frac{1}{|\eta(\tau)|^8} \left| \frac{\theta_1\left(\tau, \frac{\alpha'}{2}\tau + \frac{\beta}{2}\right)}{\eta(\tau)} \right|^4 \right)$$

(4.31)

where $Z_T^{(0)}$ denotes the (bosonic) zero-mode part of partition function of $T^4$. For example, for the rectangular torus with the radii $R_1, R_2, R_3, R_4$, it is calculated as

$$Z_{T^4}^{(0)}(\tau, \bar{\tau}) = \prod_{i=1}^4 \left[ \frac{2\pi R_i}{\alpha'} \sum_{w_i, n_i} e^{-\frac{\pi R_i^2|w_i\tau - n_i|^2}{\alpha'^2}} \right].$$

(4.32)

As in the previous analysis, we can rewrite (4.31) as a simpler form by setting $w = 0$ based on the modular invariance. A short calculation gives us

$$Z_{\text{torus}}(\beta) = \sum_{p,q,n} \sum_{n \text{ odd}} \frac{1}{np} e^{-\frac{n^2}{4\pi^2\tau^2}} Z_T^{(0)}(\tau, \bar{\tau}) \left( \frac{1}{(4\pi^2\alpha'^2 \tau^2)^2} \frac{1}{|\eta(\tau)|^8} \cdot \frac{1}{|\eta(\tau)|^8} \right) \left| \frac{\theta_2(\tau)}{\eta(\tau)} \right|^4$$

$$\times \frac{\Theta_{(\tilde{\gamma}p, -\tilde{\gamma}q + \frac{1}{2})}(\tau, \bar{\tau}; \tilde{\mu}p)^2}{\Theta_{(\tilde{\gamma}p, -\tilde{\gamma}q)}(\tau, \bar{\tau}; \tilde{\mu}p)^2},$$

(4.33)

where we set $\tau = \frac{q + in\nu}{p}$ and the summation of $p$, $q$ and $n$ run over the range such that $\tau \in \mathcal{S}$. This is again shown to be identical to the free energy of space-time theory (2.63) (up to the factor $-\beta$) calculated by the operator formalism. The check of level matching condition is only the non-trivial task. In fact, in the expression of (4.33) the summation over $q$ acts as the projection operator

$$\frac{1}{p} \sum_q e^{2\pi i q \tilde{\mu} - 2\pi i \tilde{\gamma}q\hat{h}} \equiv \frac{1}{p} \sum_q e^{2\pi i q \hat{P}},$$

(4.34)
which ensures the correct level matching condition (4.20).

Comparing it with (2.72), we notice the absence of the “topological term” including the sum over even \( n \). In fact, the physical spectrum includes the same number of bosonic and fermionic BPS states (see, for example, [36]), resulting the vanishing Witten index. This feature of course reflects the fact that \( T^4 \) has the vanishing Euler number from the view points of dual conformal theory associated to \( Sym^N(T^4) \).

The evaluation of Hagedorn temperature is similarly carried out. We only have to observe the IR behavior of the term with \( w = 1, p = n = 0 \). The result is

\[
Z_{\text{torus}}(\beta) : \text{finite} \iff \beta > \beta_H
\]  

with

\[
\frac{\beta_H^2}{8\pi^2\alpha'} - 4 \left( \Delta \left( \frac{\sqrt{2}\mu\beta_H}{4\pi}; \frac{1}{2} \right) - \Delta \left( \frac{\sqrt{2}\mu\beta_H}{4\pi}; 0 \right) \right) - \frac{1}{2} = 0.
\]  

Notice that the Hagedorn temperature only depends on the RR-flux \( \mu \) and does not on the NSNS one \( \gamma \). Especially, among the \( SL(2; \mathbb{Z}) \)-family of \( (\mu, \gamma) \), the purely NSNS case \( (\mu = 0) \) has the minimal Hagedorn temperature (maximal \( \beta_H \)) which is equal to that for the flat-background.

We finally make a comment on the purely NSNS case. If \( \hat{\gamma} \) is an irrational value, nothing interesting happens. However, in the cases of rational \( \hat{\gamma} \), we gain a periodicity under \( \hat{\gamma}p \to \hat{\gamma}p + r, \hat{\gamma}q \to \hat{\gamma}q + s (r, s \in \mathbb{Z}) \), and also find a “resonance” at the each point of \( \hat{\gamma}p, \hat{\gamma}q \in \mathbb{Z} \). In fact, the twisting disappears at these points. We thus find new zero-modes which arise a divergent volume factor:

\[
\Theta_{(\hat{\gamma}p, -\hat{\gamma}q)}(\tau, \bar{\tau}; 0)^{-2} = \left| \frac{\eta(\tau)}{\theta_1(\tau, \hat{\gamma}p\tau - \hat{\gamma}q)} \right|^4 \sim V_4 \times \frac{1}{|\eta(\tau)|^8}.
\]  

As is already pointed out in [4], these extra zero-modes should correspond to the “long strings” in the original \( AdS_3 \) background [37, 38, 39], and also the periodicity mentioned above reflects the spectral flow symmetry. In the covariant gauge quantization of this background (i.e. the \( H_6 \) super WZW model), the long string modes correspond to the “spectrally flowed type I representations”, which describe the strings freely propagating along the transverse plane, as is discussed in [36] (see also [20, 40]).

### 4.3 Case of \( M^4 = K3 \) : Orbifold Point

We next analyze the more non-trivial case \( M^4 = K3 \). We shall take the simplest orbifold point in the \( K3 \) moduli space, namely, the \( \mathbb{Z}_2 \)-orbifold of \( T^4 \).
The symmetry group acting on the tangent space is
\[ SO(2) \times SO(2) \times SO(4)_{\mathbb{T}^4} \sim U(1) \times U(1) \times \overline{SU}(2)_L \times \overline{SU}(2)_R , \]
and the string coordinates are classified by the representations of this group as follows;
\[ Z_1 \left( Z_1^* \right) : (1, 1)^{+1,0} \left( (1, 1)^{-1,0} \right) , \quad Z_2 \left( Z_2^* \right) : (1, 1)^{0,+1} \left( (1, 1)^{0,-1} \right) , \]
\[ Y^j : (2, 2)^{0,0} , \quad S^{(\pm)} a_\pm, \quad \bar{S}^{(\pm)} a_\pm : (1, 2)^{\pm \frac{1}{2}, \pm \frac{1}{2}} , \quad S^{(0)} a_0, \quad \bar{S}^{(0)} a_0 : (2, 1)^{\pm \frac{1}{2}, \pm \frac{1}{2}} \oplus (2, 1)^{\pm \frac{1}{2}, \pm \frac{1}{2}} , \]
where the superscripts indicate the \( Z_2 \) action. The orbifoldization does not break any SUSY in the sense of 6-dimension,\(^{11}\) leaving the 6-dimensional pp-wave with the enhanced SUSY.

The calculation of partition function is carried out based on the standard orbifold procedure. The result is written as
\[ Z_{\text{torus}} = Z_{\text{torus}}^u + Z_{\text{torus}}^t , \]
where the contribution from the untwisted sector \( Z_{\text{torus}}^u \) is equal to the half of the partition function for the \( T^4 \) case (4.31). The partition function for the twisted sectors is given by
\[ Z_{\text{torus}}^t (\beta) = \frac{16}{2} \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} \nu \sum_{\sigma_{1,2} \neq (0,0)} \sum_{\epsilon \in \{0,1\}} \sum_{w \in \mathbb{Z} + \epsilon_1} \sum_{n \in 2 \mathbb{Z} + \epsilon_2} e^{-\frac{\beta^2 |w - n|^2}{4 \pi^2 \tau_2}} \theta^2 \left( \frac{\pi \tau_2 (\epsilon + \sigma_1)}{2} \right) \left( \frac{\pi \tau_2 (\epsilon + \sigma_2)}{2} \right) \left( \frac{\pi \tau_2 (\epsilon + \sigma_1 + \epsilon_2)}{2} \right) \left( \frac{\pi \tau_2 (\epsilon + \sigma_2 + \epsilon_1 + \epsilon_2)}{2} \right) \theta^2 \left( \frac{\pi \tau_2 (\epsilon + \sigma_1)}{2} \right) \left( \frac{\pi \tau_2 (\epsilon + \sigma_2)}{2} \right) \left( \frac{\pi \tau_2 (\epsilon + \sigma_1 + \epsilon_2)}{2} \right) \left( \frac{\pi \tau_2 (\epsilon + \sigma_2 + \epsilon_1 + \epsilon_2)}{2} \right) \theta (\pi \tau_2 (\epsilon + \sigma_1 + \epsilon_2)) \theta (\pi \tau_2 (\epsilon + \sigma_2 + \epsilon_1 + \epsilon_2)) \theta (\pi \tau_2 (\epsilon + \sigma_1 + \epsilon_2 + \epsilon_1)) \theta (\pi \tau_2 (\epsilon + \sigma_2 + \epsilon_1 + \epsilon_2 + \epsilon_1)), \]
where the numerical factor 16 is due to the number of fixed points.

Let us further rewrite it by setting \( w = 0 \) as before. The twisted sector (4.43) now yields the non-vanishing topological term, which is evaluated as
\[ \frac{16}{2} \sum_{n \text{ : even}} \sum_{p \neq q} \sum_{m \neq n} \frac{1}{p n} e^{-\frac{\beta^2 |p - m|^2}{4 \pi^2 \tau_2}} \sum_{\sigma_{1,2} \neq (0,0)} \sum_{\epsilon \in \{0,1\}} e^{-\frac{\beta^2 |p - m|^2}{4 \pi^2 \tau_2}} . \]
\(^{11}\)In the sense of 10-dimensional theory, it of course breaks the 8 supercharges corresponding to the fermionic zero-modes of \( S^{(0)} a_0, \bar{S}^{(0)} a_0 \), which are the superpartners of \( T^4 \)-coordinates \( Y^j \) with respect to the dynamical supercharges. Hence the number of Killing spinors is reduced from 24 to 16 by the orbifoldization.
The numerical factor 24 is indeed equal to the Witten index and the Euler number of $K^3$ as should be. In fact, we can show that the physical spectrum includes the 24 bosonic BPS states and no fermionic BPS states for each fixed light-cone momentum $p^+ \equiv p/R_-$, as is analyzed in [36] for the purely NSNS case. The desired partition function is finally written as

$$Z_{\text{torus}}(\beta) = 24 \sum_{n: \text{even}} \sum_{p,q} \frac{1}{pn} e^{-\frac{\beta^2 a^2}{4\pi\alpha' \tau^2}}$$

$$+ \frac{1}{2} \sum_{n: \text{odd}} \sum_{p,q} \frac{1}{pn} e^{-\frac{\beta^2 a^2}{4\pi\alpha' \tau^2}} \left[ Z_{T^4}^{(0)}(\tau, \bar{\tau}) \frac{1}{(4\pi^2 \alpha' \tau^2)^2} \frac{1}{\eta(\tau)^8} \left| \frac{\theta_2(\tau)}{\eta(\tau)} \right|^4 \right]$$

$$+ \sum_{(\sigma_1, \sigma_2) \neq (0,0)} \left| \frac{\theta_2 \left( \tau, \frac{a_1}{2} \tau + \frac{a_2}{2} \right)}{\theta_1 \left( \tau, \frac{a_1}{2} \tau + \frac{a_2}{2} \right)} \right|^4 \cdot \frac{\Theta^{(\gamma_p, -\gamma_q + \frac{1}{2})}(\tau, \bar{\tau} ; \hat{\mu} p)^2}{\Theta^{(\gamma_p, -\gamma_q)}(\tau, \bar{\tau} ; \hat{\mu} p)^2}.$$  \hspace{1cm} (4.45)

### 4.4 Case of $M^4 = K^3$ : General Gepner Points

As a consistency check let us consider the general Gepner constructions [22] of $M^4 = K^3$. We focus on the case of purely NSNS flux and employ the RNS formalism, since it is still a difficult problem to work on the models with the general flux in this situation.

In [41] the Gepner models for $K^3$ is studied in detail. The general form of the partition function of $K^3$ non-linear $\sigma$-model is written as

$$Z_{K^3}(\tau, \bar{\tau}) = \frac{1}{2} \sum_{\alpha} \sum_I D_I \left| F_{I}^{(\alpha)}(\tau) \right|^2,$$  \hspace{1cm} (4.46)

which is defined as the diagonal modular invariant with respect to the spin structures $\alpha = \text{NS}, \bar{\text{NS}}, \text{R}, \bar{\text{R}}$. The conformal blocks $F_{I}^{(\alpha)}(\tau)$ are constructed from the characters of $\mathcal{N} = 2$ minimal models, being summed up over the integral spectral flows. The coefficients $D_I$ are the positive integers characterizing the degeneracies of conformal blocks. The overall normalization is determined uniquely so that the “graviton orbit” (the conformal block including the identity representation) has the degeneracy 1.

We also introduce the functions

\begin{align*}
 f^{(\text{NS})}_{(u,v)}(\tau) & \equiv \frac{\theta_3(\tau, u\tau + v)}{\theta_1(\tau, u\tau + v)}, & f^{(\bar{\text{NS}})}_{(u,v)}(\tau) & \equiv \frac{\theta_4(\tau, u\tau + v)}{\theta_1(\tau, u\tau + v)}, \\
 f^{(\text{R})}_{(u,v)}(\tau) & \equiv \frac{\theta_2(\tau, u\tau + v)}{\theta_1(\tau, u\tau + v)}, & f^{(\bar{\text{R}})}_{(u,v)}(\tau) & \equiv \frac{\theta_1(\tau, u\tau + v)}{\theta_1(\tau, u\tau + v)} \equiv 1,
\end{align*}  \hspace{1cm} (4.47)

which are convenient to describe the conformal blocks for the NSNS pp-wave background in the RNS formalism.
The evaluation of thermal partition function is almost parallel. However, we have to make a little modification for the thermal boundary condition of fermionic coordinates, since we are now working with the RNS fermions rather than the GS ones. To this aim it is convenient to introduce the next phase factors depending on the thermal winding numbers $w, n$ and the spin structures:

\[
\kappa(\text{NS}; w, n) \overset{\text{def}}{=} 1 , \quad \kappa(\widetilde{\text{NS}}; w, n) \overset{\text{def}}{=} (-1)^w , \\
\kappa(\text{R}; w, n) \overset{\text{def}}{=} (-1)^n , \quad \kappa(\widetilde{\text{R}}; w, n) \overset{\text{def}}{=} (-1)^{w+n} . \tag{4.48}
\]

When $w = 0$, these phase factors reproduce the correct boundary condition for the fermionic particle theory with finite temperature, and the $w$-dependence is determined by the consistency with the modular invariance [28].

Under these preparations we can write down the modular invariant form of thermal partition function:

\[
Z_{\text{torus}}(\beta) = \frac{1}{4} \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} \nu \sum_{\alpha, \beta} \sum_{I} \sum_{\nu, n, p, q} e^{-\frac{\beta^2 |w\nu-n|^2}{4\pi \alpha'^2} \tau_2 \delta(2)(w\nu + ip)\tau - (n\nu + iq)} \\
\times \kappa(\alpha; w, n)\kappa(\tilde{\alpha}; w, n) \cdot \left( f_{(\gamma p, \gamma q)}^{(\alpha)}(\tau) f_{(\gamma \bar{p}, \gamma \bar{q})}^{(\tilde{\alpha})}(\tau^*) \right)^2 \cdot \epsilon(\alpha)\epsilon(\tilde{\alpha}) D_1 F_I^{(\alpha)}(\tau) F_I^{(\tilde{\alpha})}(\tau), \tag{4.49}
\]

where $\epsilon(\alpha)$ is defined by

\[
\epsilon(\text{NS}) \overset{\text{def}}{=} +1 , \quad \epsilon(\widetilde{\text{NS}}) \overset{\text{def}}{=} -1 , \quad \epsilon(\text{R}) \overset{\text{def}}{=} -1 , \quad \epsilon(\widetilde{\text{R}}) \overset{\text{def}}{=} +1 , \tag{4.50}
\]

which impose the correct GSO projection. We can also rewrite it by setting $w = 0$ as before. For this purpose we first notice the following identity

\[
\sum_{\alpha} \epsilon(\alpha) f_{(u,v)}^{(\alpha)}(\tau) f_{(-u,-v)}^{(\alpha)}(\tau) F_I^{(\alpha)}(\tau) \equiv 0 , \quad (\forall u,v) . \tag{4.51}
\]

This identity generically holds not depending on the detailed structure of Gepner models. It is most easily proved by the general theorem about the character formulas of the “c = 12 extended superconformal algebra” presented in the appendix B of [18] (See also [42, 43, 41]). We also remark the simple relations

\[
f_{(-u,-v)}^{(\alpha)}(\tau) = - f_{(u,v)}^{(\alpha)}(\tau) , \quad (\alpha = \text{NS}, \widetilde{\text{NS}}, \text{R}) , \\
f_{(-u,-v)}^{(\widetilde{\text{R}})}(\tau) = f_{(u,v)}^{(\text{R})}(\tau) (\equiv 1) . \tag{4.52}
\]

We thus obtain

\[
\sum_{\alpha} \epsilon(\alpha) f_{(u,v)}^{(\alpha)}(\tau)^2 F_I^{(\alpha)}(\tau) = 2 f_{(u,v)}^{(\widetilde{\text{R}})}(\tau)^2 F_I^{(\text{R})}(\tau) \equiv 2 F_I^{(\widetilde{\text{R}})}(\tau) , \tag{4.53}
\]

\[
\sum_{\alpha} \epsilon(\alpha) \kappa(\alpha; 0, 2k+1) f_{(u,v)}^{(\alpha)}(\tau)^2 F_I^{(\alpha)}(\tau) = 2 f_{(u,v)}^{(\text{R})}(\tau)^2 F_I^{(\text{R})}(\tau) . \tag{4.54}
\]

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\( F_I^{(R)}(\tau) \) is no other than the Witten index, and it is known [41] that

\[
\sum_I D_I |F_I^{(R)}(\tau)|^2 = \chi(K3) \equiv 24 ,
\]

irrespective of the choice of Gepner models describing \( K3 \).

With the helps of the identities (4.53), (4.54) and (4.55), we finally obtain

\[
Z_{\text{torus}}(\beta) = 24 \sum_{n: \text{even}} \sum_{p,q} \frac{1}{pm} e^{-\frac{\beta^2 n^2}{4\pi\alpha'\tau_2}} + \sum_{n: \text{odd}} \sum_{p,q} \frac{1}{pm} e^{-\frac{\beta^2 n^2}{4\pi\alpha'\tau_2}} \Theta(\hat{\gamma}_p, -\hat{\gamma}_q + \frac{1}{2})(\tau, \bar{\tau}; \hat{\mu}p) \cdot |f^{(R)}_I(\tau)|^4 \cdot \sum_I D_I |F_I^{(R)}(\tau)|^2 ,
\]

where the summation is taken over the range \( \tau \equiv \frac{q + i\nu}{p} \in S \) as before. The topological term is equal to that in our previous result (4.45), which implies the consistency of calculation. The second term is sensitive to the moduli of \( K3 \). Comparing (4.56) with (4.45), we conjecture that the partition function for the general flux \((\mu, \gamma)\) and the general Gepner points of \( K3 \) (4.46) is given by

\[
Z_{\text{torus}}(\beta) = 24 \sum_{n: \text{even}} \sum_{p,q} \frac{1}{pm} e^{-\frac{\beta^2 n^2}{4\pi\alpha'\tau_2}} + \sum_{n: \text{odd}} \sum_{p,q} \frac{1}{pm} e^{-\frac{\beta^2 n^2}{4\pi\alpha'\tau_2}} \cdot \Theta(\hat{\gamma}_p, -\hat{\gamma}_q + \frac{1}{2})(\tau, \bar{\tau}; \hat{\mu}p) \cdot \Theta(\hat{\gamma}_p, -\hat{\gamma}_q)(\tau, \bar{\tau}; \hat{\mu}p) \cdot \sum_I D_I |F_I^{(R)}(\tau)|^2 .
\]

This might be proved by the covariant quantization based on the so-called hybrid formalism developed in [44] (see also [45]), although it is beyond the scope of this paper.

5 Summary and Discussions

In this paper we have calculated the one-loop thermal amplitudes for the closed and open strings on the DLCQ pp-waves with enhanced SUSY. All these amplitudes can be calculated by the operator formalism as the forms that only contain the contributions from the physical states compatible with the standard light-cone gauge. However, the path-integral approach is very useful in order to derive directly the manifestly modular invariant expressions, which include the sectors of “virtual strings” possessing the non-vanishing thermal windings and have the modified mass parameters. The virtual strings yield a simple evaluation of Hagedorn temperature, and further make it possible to achieve the correct open-closed string duality for the cylinder amplitudes.
The existence of Hagedorn behavior is not so surprising and the analysis on it is almost parallel to the flat case, although the equation determining the Hagedorn temperature is affected non-trivially by the mass deformation. However, as a possible direction for future study, it may be interesting to explore the relation to the well-known thermal phase transition between the thermal $AdS$ and the black hole embedded into the $AdS$ space discussed in [46, 47]. Especially, in the case of $AdS_3 \times S^3$ [47], the aspects of thermal phase transition between the thermal $AdS_3$ and the BTZ black hole is finely controlled by the modular transformation on the boundary torus. In this context the inverse temperature $\beta$ should be identified with the modulus $\tau_2$ for the boundary torus, on which the dual $SCFT_2$ is defined. On the other hand, at least for the flat background, the space-time modulus $\beta$ in the thermal DLCQ superstring theory is known to be identified with the world-sheet modulus for the Matrix string theory [17]. Therefore, our thermodynamical analysis on the DLCQ pp-waves would bring a helpful insight in order to understand the aspects of these phase transitions at the stringy level.

The thermodynamical analysis for the dual quiver gauge theory (for the case of 10-dimensional DLCQ pp-wave) given in [6] will be also an important future study. It is interesting to discuss to what extent we can correctly reproduce the Hagedorn temperature based on the perturbative calculation in the large quiver gauge theory. For the 6-dimensional case, the orbifolded $AdS_3 \times S^3$ should be dual to an $\mathcal{N} = (0,4)$ $SCFT_2$. A quiver formulation of the dual $SCFT_2$ based on the symmetric orbifold theory is discussed in [48]. It may be interesting to investigate the large quiver limit of such $SCFT_2$ as the model describing the 6-dimensional DLCQ pp-wave, and perhaps, to discuss the relation with the string bit model [15].

Acknowledgement

I would like to thank Y. Hikida and T. Takayanagi for valuable discussions. This is supported in part by a Grant-in-Aid for the Encouragement of Young Scientists (13740144) from the Japanese Ministry of Education, Culture, Sports, Science and Technology.
Appendix  Some Notations

We here summarize the convention of theta functions. We set $q \equiv e^{2\pi i \tau}$, $y \equiv e^{2\pi iz}$.

\[
\theta_1(\tau, z) = i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n-1/2)^2/2} y^{n-1/2} \equiv 2 \sin(\pi z) q^{1/8} \prod_{m=1}^{\infty} (1 - q^m)(1 - y q^m)(1 - y^{-1} q^m),
\]

\[
\theta_2(\tau, z) = \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2/2} y^{n-1/2} \equiv 2 \cos(\pi z) q^{1/8} \prod_{m=1}^{\infty} (1 - q^m)(1 + y q^m)(1 + y^{-1} q^m),
\]

\[
\theta_3(\tau, z) = \sum_{n=-\infty}^{\infty} q^{n^2/2} y^n \equiv \prod_{m=1}^{\infty} (1 - q^m)(1 + y q^{m-1/2})(1 + y^{-1} q^{m-1/2}),
\]

\[
\theta_4(\tau, z) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2/2} y^n \equiv \prod_{m=1}^{\infty} (1 - q^m)(1 - y q^{m-1/2})(1 - y^{-1} q^{m-1/2}). \quad (A.1)
\]

We also use the standard convention of $\eta$-function;

\[
\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \quad (A.2)
\]
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