Quantum magnetism of bosons with synthetic gauge fields in one-dimensional optical lattices: a Density Matrix Renormalization Group study

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In this paper, we provide a comprehensive study of the quantum magnetism in the Mott insulating phases of the 1D Bose-Hubbard model with abelian or non-abelian synthetic gauge fields, using the Density Matrix Renormalization Group (DMRG) method. We focus on the interplay between the synthetic gauge field and the asymmetry of the interactions, which give rise to a very general effective magnetic model: a XYZ model with various Dzyaloshinskii-Moriya (DM) interactions. The properties of the different quantum magnetic phases and phases transitions of this model are investigated.

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I. INTRODUCTION

Recently, significant effort has been devoted to the realization of synthetic gauge fields for electrically neutral atoms [1–3]. By suitably coupling the atoms to laser fields, experimentalists have successfully created both abelian (effective magnetic fields [4, 5]) and non-abelian gauge potentials (effective spin-orbit coupling [6]) in ultracold atomic systems, where the neutral atoms subjected to synthetic gauge fields exhibit a variety of interesting phenomena, including the Hofstadter fractal spectrum [7–9], spin-orbit coupled Bose-Einstein condensates [6, 10–17] (in which the spin degrees of freedom are given by different internal states, that we will call ‘species’ in the following), as well as spin-orbit coupled degenerate fermi gases [18–20]. While most of these studies focus on the weakly interacting regime, taking advantage of the high-tunability of the parameters of the Hamiltonian, the addition of an optical lattice enables us to investigate the strongly correlated Mott insulating phases in the presence of gauge fields, where the interplay between the strong interactions and synthetic gauge fields can give rise to exotic quantum magnetism that is difficult to access in solid state physics [21–30].

When the optical lattice is deep enough and the system is driven deep into the Mott insulating phase, the charge fluctuations are suppressed and the physics can be captured by an effective magnetic superexchange model. In the absence of synthetic gauge field, it is well known that the effective Hamiltonian is described by an anisotropic Heisenberg model (XXZ model) [31, 32], where the anisotropy is determined by the asymmetry of the interactions in spin or quasi-spin space (the ratio between the inter-species and intra-species interaction strength). Introducing synthetic gauge fields into the Mott insulating phases, as we will show below, will give rise to a Dzyaloshinskii-Moriya (DM) interaction [33, 34], which is strongly reminiscent of its counterpart in strongly correlated electronic materials e.g. in the cuprate superconductor YBa$_2$Cu$_3$O$_6$ [35, 36] or in low-dimensional magnetic materials [37, 38]. In electronic materials, the spin-independent interaction (Coulomb interaction) has the consequence that the leading magnetic superexchange model is an isotropic Heisenberg model. It is known that for the 1D isotropic Heisenberg model, the additional DM interaction can be gauged away by performing a spin rotation [39]. However, for ultracold bosons with spin degrees of freedom, the situation is different: the interspecies and intra-species scattering lengths can be tuned within a broad range using Feshbach resonances [40, 41] and tuning the lattice depth. It leads to an asymmetry of the interactions as well as an anisotropy in the Heisenberg model, where the DM interaction can no longer be gauged away.

In this paper, we provide a comprehensive understanding of the quantum magnetism of the Mott insulating phases of the 1D Bose-Hubbard model with both abelian and non-abelian synthetic gauge fields using the Density Matrix Renormalization Group (DMRG) method [42, 43]. We will show that the interplay between the synthetic gauge field and the asymmetry of the interactions can give rise to a XYZ model with different DM interactions, which is one of the most general forms for a 1D spin-1/2 quantum magnetic model with two-site nearest neighbor interactions. We explore the phase diagram of this model, and analyze different quantum phases and phase transitions in this model.

II. MODEL AND HAMILTONIAN

We consider here an interacting two-component gas of bosons in a one-dimensional lattice, subject to a spin-dependent artificial magnetic field and a synthetic spin-orbit coupling of Rashba type. It is described by the Hamiltonian

\[ \mathcal{H} = \mathcal{H}_{\text{FLUX}} + \mathcal{H}_{\text{SOC}} + \mathcal{H}_U. \]
Figure 1: Correlation functions in the ground state of Hamiltonian (5). Shown are \( \langle S^x_i S^x_j \rangle \) (orange plus signs), \( \langle S^y_i S^y_j \rangle \) (gray crosses) and \( \langle S^z_i S^z_j \rangle \) (green dots) for \( \beta = \pi/8 \) and (a) \( U'/U = 1.5 \) and (b) \( U'/U = 0.75 \). The inset of (b) shows \( \langle S^z_i S^z_j \rangle \) in log/log scale.

The kinetic part \( \mathcal{H}_{\text{FLUX}} + \mathcal{H}_{\text{SOC}} \) reads

\[
\mathcal{H}_{\text{FLUX}} = -t \cos \alpha \sum_j \left[ b_{j+1,\uparrow}^\dagger b_{j,\uparrow} e^{i\beta} + b_{j+1,\downarrow}^\dagger b_{j,\downarrow} e^{-i\beta} \right] + \text{h.c.,}
\]

\[
\mathcal{H}_{\text{SOC}} = -t \sin \alpha \sum_j \left[ b_{j+1,\uparrow}^\dagger b_{j,\downarrow} - b_{j+1,\downarrow}^\dagger b_{j,\uparrow} \right] + \text{h.c.,}
\]

(2)

where \( t \) is the hopping amplitude, \( b_{j,\uparrow,\downarrow} \) is the bosonic annihilation operator, \( j \) is the site index, and \( \uparrow, \downarrow \) denotes the two bosonic species (‘spin’ degree of freedom), \( \beta \) represents the strength of a spin-dependent magnetic field, where the different bosonic species feel opposite magnetic fields, and \( \alpha \) denotes the strength of the 1D Rashba spin-orbit coupling, which allows spin-flipping tunneling. The interaction part of the Hamiltonian reads

\[
\mathcal{H}_I = \frac{U}{2} \sum_j \left[ n_{j,\uparrow}(n_{j,\uparrow} - 1) + n_{j,\downarrow}(n_{j,\downarrow} - 1) \right]
+ U' \sum_j n_{j,\uparrow} n_{j,\downarrow},
\]

(3)

where \( U \) (respectively \( U' \)) represents the strength of the intra- (resp. inter-) species interaction.

When the interactions are strong enough to drive the system at unit filling into a Mott insulating phase, the charge fluctuations are suppressed, and the physics is captured by an effective magnetic model. Using the following spin-1/2 representation [31] \( S^z_j = n_{j,\uparrow} - n_{j,\downarrow}, \quad S^x_j = b_{j,\uparrow}^\dagger b_{j,\downarrow} + b_{j,\downarrow}^\dagger b_{j,\uparrow}, \quad S^y_j = -i(b_{j,\uparrow}^\dagger b_{j,\downarrow} - b_{j,\downarrow}^\dagger b_{j,\uparrow}), \) the leading terms of the effective super-exchange Hamiltonian can be derived as:

\[
\mathcal{H}_S = \sum_j \left[ \sum_{a=x,y,z} J_a S^a_j S^a_{j+1} + D \cdot (S_j \times S_{j+1}) \right].
\]

(4)

The Heisenberg terms are anisotropic in the three directions (XYZ model):

\[
J_x = J_0 [\sin^2 \alpha - \cos^2 \alpha \cos(2\beta)]
\]

\[
J_y = -J_0 [\sin^2 \alpha + \cos^2 \alpha \cos(2\beta)]
\]

\[
J_z = J_0 (-2U'/U + 1) [\cos^2 \alpha - \sin^2 \alpha]
\]

with \( J_0 = 4t^2/U' \). The parameter \( U'/U \) characterizes the asymmetry of the interactions, and \( U'/U = 1 \) represents an SU(2) symmetric interactions in spin space. The Dzyaloshinskii-Moriya (DM) interaction \([33, 34]\) is characterized by a three-dimensional vector \( \mathbf{D} \) with:

\[
D_x = J_0 \frac{U'}{U} \sin(2\alpha) \sin \beta, \quad D_y = J_0 \frac{U'}{U} \sin(2\alpha) \cos \beta
\]

and

\[
D_z = J_0 \cos^2 \alpha \sin(2\beta).
\]

The effective magnetic model given in Eq. (4) is one of the most general forms for a 1D spin-1/2 quantum magnetic model with two-site nearest neighbor interactions (even though there are only three independent parameters \( \alpha \), \( \beta \) and \( U'/U \)).

III. BOSE-HUBBARD HAMILTONIAN WITH SPIN-DEPENDENT MAGNETIC FIELD

Let us first focus on a relatively simple case in which only an abelian synthetic gauge field is present, \( \alpha = 0 \), which is directly relevant to current experiments with ultra-cold atoms \([8, 9, 44]\). If the different (spin) species felt the same magnetic field, the magnetic field would have no effect on the superexchange magnetic Hamiltonian; we thus focus on the case in which spin-\( \uparrow \) and \( \downarrow \) bosons feel an opposite magnetic field. For \( \alpha = 0 \), Eq. (4) reduces to an anisotropic Heisenberg model (XXZ) with...
a DM interaction along the $z$-direction:

$$\mathcal{H}_a = J_0 \sum_j \left[ -\cos(2\beta) \left( S_j^x S_{j+1}^x + S_j^y S_{j+1}^y \right) \right. \tag{5}$$

$$+ \left. \left(-2 \frac{U}{U'} + 1\right) S_j^z S_{j+1}^z + \sin(2\beta)(S_j^x S_{j+1}^y - S_j^y S_{j+1}^x) \right].$$

Since the anisotropy of the Heisenberg model and the DM vector are in the same direction, the DM interaction can be gauged away by performing a rotation of the local spin basis for $S_j$ around the $z$-axis by an angle $2\beta \in \{x, y, z\}$ in the ground state of Eq. (5) for $\beta = \pi/8$ and $U'/U = 1.5$. For $0 < U'/U < 1$ the ground state of Hamiltonian (5) is a gapless phase that ensues from the XY phase of the XXZ model, with algebraically decaying correlations. However, due to the rotation from the mapping, the correlations in the $x$ and $y$ directions will exhibit spiral order, with a period of $\pi/\beta$ sites. This is shown in Fig. 1(b) where the spin-spin correlations are plotted for $\beta = \pi/8$ and $U'/U = 0.75$: $\langle S_j^x S_{j+1}^x \rangle$ and $\langle S_j^y S_{j+1}^y \rangle$ oscillate with a period of 8 sites.

IV. BOSE-HUBBARD HAMILTONIAN WITH RASHBA SOC

We now study the case of a non-abelian synthetic gauge field. We consider the Rashba type of spin-orbit coupling in 1D, which has been implemented experimentally for atoms in the continuum [6, 11], and there are proposals for schemes to implement it on lattices [2, 48]. In the case $\beta = 0$, equation (4) simplifies to:

$$\mathcal{H}_a = J_0 \sum_j \left[ -\cos(2\alpha) S_j^x S_{j+1}^x - S_j^y S_{j+1}^y \right. \tag{6}$$

$$+ \left. \left(-2 \frac{U'}{U} + 1\right) \cos(2\alpha) S_j^z S_{j+1}^z \right. \tag{6}$$

$$+ \left. \frac{U'}{U} \sin(2\alpha)(S_j^x S_{j+1}^y - S_j^y S_{j+1}^x) \right]. \tag{7}$$

The phase diagram of this model – that we will explain in detail in the rest of the paper – is presented in Fig. 2.

We only consider the region $0 < \alpha < \pi/4$, since the rest can be deduced by simple transformations. Indeed Eq. (7) is $\pi$-periodic in $\alpha$, and if $\pi/4 < \alpha < \pi/2$, by setting $\alpha' = \pi/2 - \alpha$ and $S_j^x = (-1)^j S_j^y$, $S_j^y = S_j^y$, and $S_j^z = (-1)^j S_j^z$, we recover Hamiltonian (7) with $0 < \alpha < \pi/4$. In some particular cases (signaled by plain lines in Fig. 2), Eq. (7) can be mapped onto a XXZ model, in which case the phases are known ab-initio, as in Sec. III. However, for general values of $U'/U$ and $\alpha$, the DM term cannot be gauged away, and this model can hardly be handled analytically. Here, we therefore explore the phase diagram numerically, by computing the ground state of Hamiltonian (7) by the Density-Matrix Renormalization Group (DMRG) method [42, 43].

A. $U' > U$ – Ising to spiral phase transition

Let us first analyze the region $U' > U$ of the phase diagram, where we have identified two different phases. The spin-spin correlation functions of these two phases $\langle S_a^\alpha S_b^\alpha \rangle$ in the ground state (with $\alpha \in \{x, y, z\}$) are presented in figure 3. To get some insight, we first focus on several special points (lines) in the phase diagram.

Firstly, along the $\alpha = 0$ axis, Eq. (7) becomes a XXZ
model:
\[ \mathcal{H}_{a}^{\alpha=0} = J_{0} \sum_{\langle i,j \rangle} \left[ -(S_{j}^{x}S_{j+1}^{x} + S_{j}^{y}S_{j+1}^{y}) + \left( -\frac{U'}{U} + 1 \right) S_{j}^{z}S_{j+1}^{z} \right]. \]  
(8)

For \( U'/U > 1 \) the ground state is therefore an Ising state with perfect ferromagnetic ordering along the \( z \) direction [46]. For small values of \( \alpha \), the term \( -S_{j}^{z}S_{j+1}^{z} \) still dominates in \( \mathcal{H}_{a} \) and the ground state is still a perfect ferromagnetic phase (a gapped Ising-type phase) as shown in the dark blue region in phase diagram of Fig. 2. This can be numerically verified by the spin-spin correlation functions as shown in Fig. 3(a), where we can observe that in this regime the ground state exhibits perfect ferromagnetic order in the \( z \) direction with \( \langle S_{i}^{z}S_{j}^{z} \rangle = 1/4 \) for any \( \langle i - j \rangle \).

Secondly, if we focus on the line \( \alpha = \pi/4 \), the Hamiltonian of Eq. (7) is given by
\[ \mathcal{H}_{a}^{\alpha=\pi/4} = J_{0} \sum_{\langle i,j \rangle} \left[ -S_{j}^{y}S_{j+1}^{y} + \frac{U'}{U}(S_{j}^{x}S_{j+1}^{x} - S_{j}^{z}S_{j+1}^{z}) \right]. \]  
(9)

It maps onto \( \mathcal{H}_{a}^{\alpha=\pi/4} = J_{0} \sum_{\langle i,j \rangle} -S_{j}^{y}S_{j+1}^{y} + \frac{U'}{U}(S_{j}^{x}S_{j+1}^{x} + S_{j}^{z}S_{j+1}^{z}) \) after a rotation of the local basis of each spins by \( j \pi/2 \) around axis \( y \). Along that axis, the system is therefore in a gapless ’XY phase’ [46] [the symmetry plane being \((x,z)\) here]. It features algebraic decay of the correlations, and a spiral order with a 4-site period along \( y \), due to the rotation of the mapping. This picture does not qualitatively change when \( \alpha \) is close to \( \pi/4 \), in which case the DM term dominates in Eq. (7), and we find a spiral phase around the \( y \) direction in the region of

Fig. 2 shaded in pink. The correlations also decay algebraically [see Fig. 3(b)], signaling a gapless Luttinger-Liquid phase [49]. Moreover, we find that the correlations \( \langle S_{i}^{y}S_{j}^{y} \rangle \) in the \( x \) and \( z \) directions oscillate with the same period, which shows spiral order. Note that the spiral does not have the same amplitude in the \( x \) and \( z \) directions, due to the anisotropy in the exchange term in Eq. (7).

In the limit \( U'/U \to \infty \), the Hamiltonian in Eq. (7) reduces to
\[ \mathcal{H}_{a}^{U'/U \to \infty} = J_{0} \sum_{\langle i,j \rangle} \left[ -2 \cos(2\alpha) S_{j}^{x}S_{j+1}^{x} + \sin(2\alpha)(S_{j}^{y}S_{j+1}^{y} - S_{j}^{z}S_{j+1}^{z}) \right], \]  
(10)

which can be solved exactly by performing a rotation in spin space and using Jordan-Wigner transformation [49, 50]. Eq. (10) can be mapped to a (quasi-) non-interacting spinless fermion Hamiltonian, with the dispersion relation:
\[ \epsilon_{k}^{\pm} = J_{0} \frac{U'}{U} \left[ \frac{\tan(2\alpha)}{2} \sin(k) \pm \frac{1}{2} \right], \]  
(11)

where \( k \) is the wavevector in units of reciprocal lattice spacing. One immediately finds a phase transition between a gapped phase at small values of \( \alpha \) and a gapless phase for higher values, with the phase transition point \( \alpha_{c} = \pi/8 \). This value is reported as a dotted black line in Fig. 2.

For finite values of \( U'/U \), the problem can no longer been solved analytically, we have thus computed the group state properties with DRMG. The magnetization \( m^{z} = \lim_{|i-j|\to\infty} \langle S_{i}^{z}S_{j}^{z} \rangle \) is plotted in Fig. 4(a) as a function of \( \alpha \) for fixed \( U'/U = 1.5 \). It shows that for

![Figure 3: Correlation functions in the ground state of Hamiltonian (7) obtained by DMRG calculations. Shown are \( \langle S_{i}^{x}S_{j}^{x} \rangle \) (orange plus signs), \( \langle S_{i}^{y}S_{j}^{y} \rangle \) (gray crosses) and \( \langle S_{i}^{z}S_{j}^{z} \rangle \) (green dots) for (a) \( \alpha = \pi/20 \) and \( U'/U = 1.5 \) and (b) \( \alpha = \pi/10 \) and \( U'/U = 1.5 \). The inset of (b) show \( \langle S_{i}^{y}S_{j}^{y} \rangle \) in log/log scale. For the sake of clarity those points are identified in Fig. 2 by crosses.](image-url)
critical phase (see below); hence the topology of the right part of the phase diagram in Fig. 2.

**B. \( U' \simeq U \) – Incomplete ferromagnet to spiral phase transition**

In the case \( U' = U \), Hamiltonian (7) reads

\[
\hat{\mathcal{H}}_g^{U'=U} = J_0 \sum_j \left[ -\cos(2\alpha) \left( S_j^x S_{j+1}^x + S_j^y S_{j+1}^y \right) - S_j^z S_{j+1}^z \right.
\]

\[
\left. + \sin(2\alpha) \left( S_j^+ S_{j+1}^y - S_j^y S_{j+1}^+ \right) \right],
\]

and a rotation of the local basis of each spin by \( 2j\alpha \) around axis \( y \) permits to gauge out the \( \mathcal{D} \) term [24], and the model reduces to an isotropic ferromagnetic Heisenberg model \( \hat{\mathcal{H}}_g^{U'=U} = -S_j^+ S_{j+1}^- \). In that case, the ground state is ferromagnetic, but, following Goldstone’s theorem, it is gapless, as it breaks a continuous symmetry of the XXZ model [46].

Figure 5 displays \( m^\parallel = \sqrt{\lim_{|i-j| \to \infty} \langle S_i^x S_j^x \rangle} \), \( m^\perp \) and the ordering wavevector \( Q \) in the spiral phase as a function of \( U'/U \) with fixed \( \alpha = \pi/5 \) [Fig. 5 (a)] and \( \pi/10 \) [Fig. 5 (b)] respectively. In both cases we find that \( U'/U = 1 \) is indeed the critical point of a first-order transition between the spiral phase and a ferromagnetically ordered phase with \( m^\parallel \neq 0 \). This type of phase, that we call incomplete ferromagnet (iFM) in the following since \( m^\parallel < 1/4 \), is found in the top/ left corner of the phase diagram (see light blue-shaded region in Fig. 2), a region in which the Hamiltonian is dominated by \( -S_j^y S_{j+1}^y \). Figure 6(a) shows the spin-spin correlation functions at \( U'/U = 0.2 \) and \( \alpha = \pi/10 \). We indeed find a reduced magnetization in the \( y \)-direction: \( \lim_{|i-j| \to \infty} \langle S_i^y S_j^y \rangle \simeq 0.062 < 1/4 \). We also observe that the correlations along the \( x \) and \( z \) directions decay exponentially, indicating a gapped phase. The nature of the iFM state can be understood in the limit \( U'/U = 0 \), where Eq. (7) becomes

\[
\hat{\mathcal{H}}_g^{U'=0} = J_0 \sum_j \left[ \cos(2\alpha) \left( -S_j^x S_{j+1}^x + S_j^z S_{j+1}^z \right) - S_j^y S_{j+1}^y \right].
\]

(13)

After the rotation of each spins \( S_j \) by \( j\pi \) around \( y \)-axis: \( S_j^x = (-1)^j S_{j+1}^x \), \( S_j^y = (-1)^j S_{j+1}^y \) and \( S_j^z = S_{j+1}^z \), the model maps to an antiferromagnetic (AF) XXZ model in the easy-axis phase (with preferred axis \( y \)). Unlike for the ferromagnetic XXZ model, it is known that the perfect Néel phase along \( y \)-axis is not the ground state of the AF-XXZ model in the easy-axis phase. Therefore, after the above spin rotation, the ground state of Eq. (13) exhibit ferromagnetic order with reduced magnetization along the \( y \)-axis.
α = π/5 and (b) constant α = π/10. Those cuts are highlighted in Fig. 2 by dashed gray lines. The magnetization is extracted from correlation functions in the ground state with $m^{y,z} = \sqrt{\lim_{|i-j| \to \infty} \langle S_i^y S_j^z \rangle}$ and $Q$ from large-distance fits of $\langle S_i^x S_j^x \rangle$ and $\langle S_i^z S_j^z \rangle$ by $\cos(Q|i-j|)/|i-j|^\gamma$.

Figure 6: Correlation functions in the ground state of Hamiltonian (7). Shown are $\langle S_i^x S_j^x \rangle$ (orange plus signs), $\langle S_i^y S_j^y \rangle$ (gray crosses) and $\langle S_i^z S_j^z \rangle$ (green dots) for (a) $\alpha = \pi/10$ and $U'/U = 0.2$ and (b) $\alpha = 0$ and $U'/U = 0.2$. The insets show the same data in log/linear scale for (a), and in log/log scale for (b). For the sake of clarity those points are identified in Fig. 2 by crosses.

We therefore find that the transition from the ifFM phase to the incommensurate spiral phase is a first-order transition, as previously found for the Ising to spiral phase transition (see Sec. IV A). As can be seen in Figs. 5(a2) and (b2), the onset of incommensurability also coincides with the transition point. However, in that case, we find that the ordering wavevector discontinuously jumps from 0 (in the commensurate ifFM phase) to a finite value $Q = 2\alpha$. This value stems from the mapping introduced below Eq. (12). In Fig. 5(a2) we observe that the value of $Q$ is rather constant in the spiral phase along a cut at $\alpha = \pi/5$, however the other fit parameter $\gamma$ depends on $U'/U$ (not shown). In the cut at $\alpha = \pi/10$ [Fig. 5(b2)] however, $Q$ decreases and reaches 0 at $U'/U \simeq 1.75 \pm 0.01$, at the point where the magnetization $m^z$ jumps to 0.5, indicating that the transition to the Ising phase described above takes place.

C. $U' < U$ – Incomplete ferromagnet to XY phase transition

Eventually, in the limit $\alpha = 0$ [see Eq. (8)], the ground state for $0 < U'/U < 1$ is a XY phase. In this phase, all correlation functions decay algebraically, as shown in Fig. 6(b) for $U'/U = 0.2$ and $\alpha = 0$. It indeed has a
gapless excitation spectrum. We expect that a small \( \alpha \) will not change the nature of this gapless phase. Therefore, there should be a phase transition to the gapped iFM phase when increasing \( \alpha \). Such a gapless phase with respect to spin excitations is also predicted in Ref. [30]

VI. ACKNOWLEDGMENTS

In this paper, we have studied the quantum magnetism of the Mott insulating phases found in the strongly interacting limit of the 1D Bose-Hubbard model with both abelian and non-abelian synthetic gauge fields. In the abelian case (spin-dependent magnetic field) which is relevant to current experiments with cold atoms, we found that the ground state exhibits a spiral quasi long-range order in the regime \( U'/U < 1 \). In the non-abelian case (Rashba spin-orbit coupling), we have studied the phase diagram of the effective Hamiltonian, where we identified four phases with different magnetic textures: two gapped phases with complete (Ising) and incomplete (iFM) ferromagnetic order and two gapless phases with and without spiral quasi long-range order. We have found that the transitions between the ferromagnetic phases and the spiral phase are both first order, and the emergence of the incommensurability in the spiral phase coincides with the phase transition. The ordering wavevector is continuous at the Ising to spiral phase transition, whereas it is discontinuous at the iFM to spiral one. Finally, in the regime \( U'/U < 1 \), there is a continuous phase transition from an XY phase to a gapped iFM phase, for small \( \alpha \).

Here, we have focused on situations in which only one component of the DM interaction is non-zero. However, the general model given in Eq. (4) allows more than one component for the DM-vector \( \mathbf{D} \), in the case in which both the abelian and the non-abelian synthetic gauge fields are present at the same time. This may give rise to richer quantum magnetic phases and deserves to be explored in the future. We have focused on the 1D case. However, DM interaction are also very interesting in 2D, as they can give rise to exotic topological magnetic textures such as vortex and skyrmion crystals, that could also be addressed with ultracold atoms [24–26].

Note – During the completion of this manuscript, we became aware of the work reported in Ref. [51], which addresses a similar topic.
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