PANEL COLLAPSE AND ITS APPLICATIONS

MARK F. HAGEN AND NICHOLAS W.M. TOUIKAN

Abstract. We describe a procedure called panel collapse for replacing a CAT(0) cube complex $\Psi$ by a “lower complexity” CAT(0) cube complex $\Psi_*$ whenever $\Psi$ contains a codimension–2 hyperplane that is extremal in one of the codimension–1 hyperplanes containing it. Although $\Psi_*$ is not in general a subcomplex of $\Psi$, it is a subspace consisting of a subcomplex together with some cubes that sit inside $\Psi$ “diagonally”. The hyperplanes of $\Psi_*$ extend to hyperplanes of $\Psi$. Applying this procedure, we prove: if a group $G$ acts cocompactly on a CAT(0) cube complex $\Psi$, then there is a CAT(0) cube complex $\Omega$ so that $G$ acts cocompactly on $\Omega$ and for each hyperplane $H$ of $\Omega$, the stabiliser in $G$ of $H$ acts on $H$ essentially.

Using panel collapse, we obtain a new proof of Stallings’s theorem on groups with more than one end. As another illustrative example, we show that panel collapse applies to the exotic cubulations of free groups constructed by Wise in [44]. Next, we show that the CAT(0) cube complexes constructed by Cashen-Macura in [7] can be collapsed to trees while preserving all of the necessary group actions. (It also illustrates that our result applies to actions of some non-discrete groups.) We also discuss possible applications to quasi-isometric rigidity for certain classes of graphs of free groups with cyclic edge groups. Panel collapse is also used in forthcoming work of the first-named author and Wilton to study fixed-point sets of finite subgroups of $\text{Out}(F_n)$ on the free splitting complex. Finally, we apply panel collapse to a conjecture of Kropholler, obtaining a short proof under a natural extra hypothesis.

Contents

Introduction 1
1. Ingredients 6
2. Panel collapse for single cubes 9
3. Collapsing $\Psi$ along extremal panels 26
4. Applications 29
References 34

Introduction

CAT(0) cube complexes, which generalise simplicial trees in several ways, have wide utility in geometric group theory; making a group act by isometries on a CAT(0) cube complex can reveal considerable information about the structure of the group. The nature of this information depends on where the action lies along a “niceness spectrum”, with (merely) fixed-point freely at one end, and properly and cocompactly at the other.

In this paper, we focus on cocompact (but not necessarily proper) actions on (finite-dimensional but not necessarily locally finite) CAT(0) cube complexes. Examples include

Date: August 23, 2019.
2010 Mathematics Subject Classification. Primary: 20F65; Secondary: 20E08.
Key words and phrases. ends of groups, CAT(0) cube complex, Stallings’s ends theorem, line pattern.

1
actions on Bass-Serre trees associated to finite graphs of groups, but this class also encompasses the large array of groups known to act on higher-dimensional CAT(0) cube complexes satisfying these conditions, e.g. \[29, 8, 42, 32, 18, 20, 25, 27, 41, 4, 34].

This paper highlights a new property of actions on CAT(0) cube complexes: if \( G \) acts on the CAT(0) cube complex \( \Psi \), we say that \( G \) acts **hyperplane-essentially** if for each hyperplane \( H \) of \( \Psi \), the stabiliser \( \text{Stab}_G(H) \) of \( H \) acts essentially on the CAT(0) cube complex \( H \). (Recall that \( G \) acts essentially on \( \Psi \) if, for each halfspace in \( \Psi \), any \( G \)-orbit contains points of that halfspace arbitrarily far from its bounding hyperplane.) Work of Caprace-Sageev shows that, under reasonable conditions on the \( G \)-action, one can always pass to a convex subcomplex of \( \Psi \) on which \( G \) acts essentially, but simple examples (where \( G = \mathbb{Z} \)) show that passing to a convex subcomplex may never yield a hyperplane-essential action.

There are numerous reasons to be interested in hyperplane-essentiality, which is a weak version of “no free faces”. For example, hyperplane-essentiality enables one to apply results guaranteeing that intersections of halfspaces in CAT(0) cube complexes contain hyperplanes (again, under mild conditions on the complex), strengthening the very useful \[6, \text{Lemma } 5.2\]. Access to these lemmas has various useful consequences; for instance, hyperplane-essentiality is used in the forthcoming \[19\] to generalise Guirardel’s core of a pair of splittings of a group \( G \) to a core of multiple cubulations of \( G \).

Other consequences of hyperplane-essentiality arise from one of the typical ways in which it can fail. First, note that if \( \Psi \) is compact and \( G \) acts on \( \Psi \), then the action is essential if and only if \( \Psi \) has no halfspaces, i.e. \( \Psi \) is a single point. Hence, if \( \Psi \) is a CAT(0) cube complex with compact hyperplanes, then a \( G \)-action on \( \Psi \) is hyperplane-essential if and only if \( \Psi \) is a tree.

More generally, suppose that \( G \) acts cocompactly on \( \Psi \), but there is some hyperplane \( H \) so that the action of \( \text{Stab}_G(H) \) on \( H \) is not essential. Then, since \( \text{Stab}_G(H) \) must act cocompactly on \( H \), there exists a hyperplane \( E \) that crosses \( H \) and is extremal in \( H \), in the sense that, for some halfspace \( E^+ \) associated to \( E \), the halfspace \( E^+ \cap H \) of \( H \) lies entirely in the cubical neighbourhood of \( E \cap H \). This leaves the \( G \)-action on \( \Psi \) open to the main technique introduced in this paper, **panel collapse**, which is inspired by an idea in \[38\], in which certain square complexes are equivariantly collapsed to simpler ones. This procedure enables a \( G \)-equivariant deformation retraction from \( \Psi \) to a lower–complexity CAT(0) cube complex \( \Psi_\bullet \). Although \( \Psi_\bullet \) need not be a subcomplex of \( \Psi \), it is a \( G \)-invariant subspace with a natural cubical structure inherited from \( \Psi \), whose hyperplanes extend to those of \( \Psi \). Specifically:

**Theorem A** (Panel collapse, Corollary 3.3). Let \( G \) act cocompactly and without inversions in hyperplanes on the CAT(0) cube complex \( \Psi \). Suppose that for some hyperplane \( H \), the stabiliser of \( H \) fails to act essentially on \( H \) (i.e. \( H \) contains a \( \text{Stab}_G(H) \)-shallow halfspace; equivalently, a shallow halfspace). Then there is a CAT(0) cube complex \( \Psi_\bullet \) such that:

1. \( \Psi_\bullet \subset \Psi \), and each hyperplane of \( \Psi_\bullet \) is a component of a subspace of the form \( K \cap \Psi_\bullet \), where \( K \) is a hyperplane of \( \Psi \).
2. \( \Psi_\bullet \) is \( G \)-invariant and the action of \( G \) on \( \Psi_\bullet \) is cocompact.
3. The action of \( G \) on \( \Psi_\bullet \) is without inversions in hyperplanes.
4. \( \Psi_\bullet \) has strictly lower complexity than \( \Psi \).
5. Each \( g \in G \) is hyperbolic on \( \Psi \) if and only if it is hyperbolic on \( \Psi_\bullet \).
6. If \( \Psi \) is locally finite, then so is \( \Psi_\bullet \).

The complexity of \( \Psi \) is just the number of \( G \)-orbits of cubes of each dimension \( > 1 \), taken in lexicographic order. So, when the complexity vanishes, \( \Psi \) is a tree. A halfspace in a CAT(0) cube complex \( Y \) is **shallow** if it is contained in some finite neighbourhood of its bounding hyperplane, and, given a \( G \)-action on \( Y \), a halfspace is **\( G \)-shallow** if some \( G \)-orbit
intersects the halfspace in a subset contained in a finite neighbourhood of the bounding hyperplane.

From Theorem A, induction on complexity then shows that, if $G$ acts cocompactly on a CAT(0) cube complex $\Psi$, then $G$ acts cocompactly and hyperplane-essentially on some other CAT(0) cube complex $\Omega \subset \Psi$. Moreover, one can then pass to the Caprace-Sageev essential core, as in [6, §3], to obtain a cocompact, essential, hyperplane-essential action of $G$ on a CAT(0) cube complex $\Omega$.

In particular, if the hyperplanes of $\Psi$ were all compact, then the hyperplanes of $\Omega$ are all single points, i.e. $\Omega$ is a tree on which $G$ acts minimally. In other words, we find a nontrivial splitting of $G$ as a finite graph of groups.

The main technical difficulty is that hyperplanes can intersect their $G$-translates (indeed, in our application to Stallings’s theorem, discussed below, this is the whole source of the problem). So, naive approaches involving collapsing free faces cannot work, and this is why our procedure gives a subspace which is not in general a subcomplex.

We now turn to consequences of Theorem A.

**Stallings’s theorem.** Stallings’s 1968 theorem on groups with more than one end is one of the most significant results of geometric group theory:

**Theorem B** ([13] Stallings’s theorem, modern formulation). If $G$ is a finitely generated group, $X$ a Cayley graph corresponding to a finite generating set, and $K$ a compact subgraph such that $X \setminus K$ has at least two distinct unbounded connected components, then $G$ acts nontrivially and with finite edge stabilizers on a tree $T$.

Theorem B is proved as Corollary 4.1 below. The essence of the proof is the fact that if $G$ acts on a cocompactly on a CAT(0) cube complex with compact hyperplanes then, successively applying Theorem A, we can make $G$ act on a tree.

There are numerous proofs of Theorem B. Our proof avoids certain combinatorial arguments (e.g. [11, 16]) and analysis (see [22]). Ours is not the first proof of Stallings’s theorem using CAT(0) cube complexes. In fact, the ideas involved in the original proof anticipate CAT(0) cube complexes to some extent.

Stallings’s original proof (for the finitely presented case) [36, 37] precedes the development of Bass-Serre theory. Instead, Stallings developed so-called bipolar structures. Equipped with Bass-Serre theory, the problem boils down to dealing with a finite separating subset $K$ that intersects some of its $G$-translates.

Dunwoody, in [14], while proving accessibility, gave a beautiful geometric proof of Stallings’s theorem in the almost finitely presented case using the methods of patterns in polygonal complexes. It is relatively easy to turn the separating set $K$ into a track. Using a minimality argument, he shows that it is possible to cut and paste tracks until they become disjoint, preserving finiteness and separation properties.

CAT(0) cube complexes became available via work of Gerasimov and Sageev [33, 17], employing the notion of codimension-1 subgroups. (More generally, one can cubulate a wallspace [31, 9].) Cube complexes are a very natural platform for addressing the intersecting cut-set problem.

Niblo, in [28], gave a proof of Stallings’s theorem using CAT(0) cube complexes, in fact proving something more general about codimension–1 subgroups (see Corollary D.) His method is to cubulate, and then use the cube complex to get a 2-dimensional complex (the 2-skeleton of the cube complex), on which he is then able to use a minimality argument for tracks to get disjoint cut-sets. So, Niblo’s proof uses the CAT(0) cubical action as a way to get an action on a 2–complex with a ready-made system of tracks, namely the traces of the hyperplanes on the 2–skeleton.
Our proof of Theorem B, based on panel collapse, is fundamentally different. Rather than performing surgery on Dunwoody tracks in the 2–skeleton of the cube complex, we collapse the entire complex down to an essential tree with finite edge stabilizers.

Cube complexes associated to line patterns: an example when \( G \) is not finitely generated. Cashen and Macura, in [7], prove a remarkable rigidity theorem which states that to any free group equipped with a rigid line pattern, there is a pattern preserving quasi-isometry to a CAT\((0)\) cube complex, the Cashen-Macura complex, equipped with a line pattern \((X,\mathcal{L})\) so that any line pattern preserving quasi-isometry between two free groups is conjugate to an isometry between the Cashen-Macura complexes.

In their paper the authors ask whether the cut sets of the decomposition used to construct \( X \) can be chosen so that \( X \) is a tree. While not answering this question directly, we show how panel collapse can bring \( X \) to a tree. This gives:

**Theorem C** (Theorem 4.6, c.f. [7, Theorem 5.5]). Let \( F_i, \mathcal{L}_i, i = 0, 1, \) be free groups equipped with a rigid line patterns. Then there are locally finite trees \( T_i \) with line patterns \( \mathcal{L}_i \) and embeddings

\[
F_i \xrightarrow{\iota_i} \text{Isom}(T_i)
\]

inducing cocompact isometric actions \( F_i \circ T_i \), which in turn induce equivariant line pattern preserving quasi-isometries

\[
\phi_i : F_i \rightarrow T_i.
\]

Furthermore for any line pattern preserving quasi-isometry \( q : F_0 \rightarrow F_1 \) there is a line pattern preserving isometry \( \alpha_q \) such that the following diagram of line pattern preserving quasi-isometries commutes up to bounded distance:

\[
\begin{array}{ccc}
(T_0, \mathcal{L}_0) & \xrightarrow{\alpha_q} & (T_1, \mathcal{L}_1) \\
\downarrow \phi_0 & & \downarrow \phi_1 \\
(F_0, \mathcal{L}_0) & \xrightarrow{q} & (F_1, \mathcal{L}_1)
\end{array}
\]

We note that the actions of \( F_i \) on \( T_i \) are free since the quasi-isometries \( \phi_i : F_i \rightarrow T_i \) are equivariant.

This is proved in Section 4.3. By passing to trees, tree lattice methods can be brought to bear and the authors suspect that this result could play an important role in the description of quasi-isometric rigidity in the class of graphs of free groups with cyclic edge groups.

This application illustrates that in Theorem A, we are not requiring \( G \) to be finitely generated. The result holds even when dealing with an uncountable totally disconnected locally compact group of isometries of some cube complex \( X \).

The Kropholler conjecture. Around 1988, Kropholler made the following conjecture: given a finitely generated group \( G \) and a subgroup \( H \leq G \), then the existence of a proper \( H \)–almost invariant subset \( A \) of \( G \) with \( AH = A \) ensures that \( G \) splits nontrivially over a subgroup commensurable with a subgroup of \( H \). This conjecture has been verified under various additional hypotheses [23, 24, 12, 28]; a proof using a different approach to the one taken here appears in [15], although this proof is believed, at the time of writing, to contain a gap\(^1\). Niblo and Sageev have observed that the Kropholler conjecture can be rephrased in terms of actions on CAT\((0)\) cube complexes [30]. In cubical language, the conjecture states that if \( G \) acts essentially on a CAT\((0)\) cube complex with a single \( G \)–orbit of hyperplanes,

\(^1\)Personal communication from Martin Dunwoody and Alex Margolis.
and $H$ is a hyperplane stabilizer acting with a global fixed point on its hyperplane, then $G$ splits over a subgroup commensurable with a subgroup of $H$. In Section 4.4, we prove this under the additional hypothesis that $G$ acts cocompactly on the CAT(0) cube complex in question (making no properness assumptions on either the action or the cube complex). Specifically, we use panel collapse to obtain a very short proof of:

**Corollary D** (Corollary 4.9). Let $G$ be a finitely generated group and $H \leq G$ a finitely generated subgroup with $e(G,H) \geq 2$. Let $\Psi$ be the dual cube complex associated to the pair $(G,H)$, so that $\Psi$ has one $G$–orbit of hyperplanes and each hyperplane stabiliser is a conjugate of $H$. Suppose furthermore that:

- $G$ acts on $\Psi$ cocompactly;
- $H$ acts with a global fixed point on the associated hyperplane.

Then $G$ admits a nontrivial splitting over a subgroup commensurable with a subgroup of $H$.

In the case where $G$ is word-hyperbolic and $H$ a quasiconvex codimension–1 subgroup, this follows from work of Sageev [34], but the above does not rely on hyperbolicity, only cocompactness. A similar statement, [28, Theorem B], was proved byNiblo under the additional hypothesis that hyperplanes are compact. In our case the cocompactness hypothesis and the fact that $H$ acts with a fixed point imply that hyperplanes have bounded diameter but need not be compact. We hope that a future generalisation of panel collapse will enable a CAT(0) cubical proof of the Kropholler conjecture that does not require a cocompact action on the cube complex.

**Other examples and applications.** Theorem A has other applications in group theory. For example, Theorem A is used in a forthcoming paper of Hagen-Wilton to study subsets of outer space and the free splitting complex fixed by finite subgroups of the outer automorphism group of a free group [19]. In that paper, the authors construct, given a group $G$ acting cocompactly, essentially, and hyperplane-essentially on finitely many CAT(0) cube complexes, an analogue of the Guirardel core for those actions. Among other applications, this core can be used to give a new proof of the Nielsen realisation theorem for $\text{Out}(F_n)$, which is done in [19] using Theorem A (without using Stallings’s theorem). Since the original proof relies on Stallings’s theorem [10], it is nice to see that in fact Stallings’s theorem and Nielsen realisation for $\text{Out}(F_n)$ in fact follow independently from the same concrete cubical phenomenon.

Theorem A thus expands the arena in which the techniques from [19] will be applicable, by showing that the hyperplane-essentiality condition can always be arranged to hold. This is of particular interest because of ongoing efforts to define a “space of cubical actions” for a given group. Even for free groups, it’s not completely clear what this space should be, i.e. which cubical actions count as points in this space. However, since known arguments giving e.g. connectedness depend on the aforementioned version of the Guirardel core, one should certainly restrict attention to cubical actions where the hyperplane stabilisers act essentially on hyperplanes. This is what we mean in asserting that Theorem A is aimed at future applications. We investigate this very slightly in the present paper, in Section 4.2, where we show that the exotic cubulations of free groups constructed by Wise in [44] do not have essentially-acting hyperplane stabilisers, and are thus subject to panel collapse.

**Outline of the paper.** Throughout the paper, we assume familiarity with basic concepts from the theory of group actions on CAT(0) cube complexes; see e.g. [35, 43]. In Section 1, we define panels in a CAT(0) cube complex and discuss the important notion of an extremal panel. In Section 2, we describe panel collapse within a single cube, leading to the proof of Theorem A in Section 3. Section 4 contains applications.
Acknowledgements. We thank the organisers of the conference Geometric and Asymptotic Group Theory with Applications 2017, at which we first discussed the ideas in this work. MFH thanks Henry Wilton for discussions which piqued his interest in mutilating CAT(0) cube complexes, and Vincent Guirardel for discussion about possible future generalizations of panel collapse. The authors are also grateful to Mladen Bestvina and Brian Bowditch for bringing up ideas related to the collapse of CAT(0) cube complexes, and to Michah Sageev for encouraging us to apply panel collapse to the Kropholler conjecture. We also thank Alex Margolis for pointing out the reference [28] and the anonymous referee for many helpful comments which improved the exposition and simplified certain proofs. MFH was supported by Henry Wilton’s EPSRC grant EP/L026481/1 and by EPSRC grant EP/R042187/1.

1. Ingredients

1.1. Blocks and panels. Throughout this section, $\Psi$ is an arbitrary CAT(0) cube complex and all hyperplanes, subcomplexes, etc. lie in $\Psi$. Recall that a codimension–$n$ hyperplane in $\Psi$ is the (necessarily nonempty) intersection of $n$ distinct, pairwise intersecting hyperplanes. We will endow $\Psi$ with the usual CAT(0) metric $d_2$ (which we refer to as the $\ell_2$–metric) in which all cubes are Euclidean unit cubes. There is also an $\ell_1$–metric $d_1$, which extends the usual graph-metric on the 1–skeleton. Recall from [21] that a subcomplex is convex with respect to $d_2$ if and only if it is convex with respect to $d_1$ (equivalently, it is equal to the intersection of the combinatorial halfspaces containing it), but we will work with $\ell_2$–convex subspaces that are not subcomplexes.

Recall that convex subcomplexes of CAT(0) cube complexes have the Helly property: if $C_1, \ldots, C_n$ are pairwise-intersecting convex subcomplexes, then $\bigcap_{i=1}^n C_i \neq \emptyset$.

Given $A \subset \Psi$, the convex hull of $A$ is the intersection of all convex subcomplexes containing $A$. In particular, if $A$ is a set of 0–cubes in some cube $c$, then the convex hull of $A$ is just the smallest subcube of $c$ containing all the 0–cubes in $A$.

The word “hyperplane”, unless stated otherwise, means “codimension–1 hyperplane”.

Definition 1.1 (Codimension–$n$ carrier). The carrier $N(H)$ of a codimension $n$ hyperplane $H$ is the union of closed cubes containing $H$. We have, by results in [33],

$$N(H) = H \times \left[ -\frac{1}{2}, \frac{1}{2} \right]^n.$$

If $H$ is a codimension–$n$ hyperplane, then $H = \bigcap_{i=1}^n E_i$, where each $E_i$ is a hyperplane, and $N(H) = \bigcap_{i=1}^n N(E_i)$, where each $N(E_i) \cong E_i \times \left[ -\frac{1}{2}, \frac{1}{2} \right]$.

Definition 1.2 (Blocks, panels, parallel). Let $H$ be a codimension 2 hyperplane. Then the carrier $B(H) = H \times \left[ -\frac{1}{2}, \frac{1}{2} \right]^2$ is called a block. The four closed faces

$$H \times \left[ -\frac{1}{2}, \frac{1}{2} \right] \times \left\{ \pm \frac{1}{2} \right\}, H \times \left\{ \pm \frac{1}{2} \right\} \times \left[ -\frac{1}{2}, \frac{1}{2} \right]$$

are called panels. Panels are parallel if they have empty intersection. Equivalently, panels are parallel if they intersect exactly the same hyperplanes. (More generally, two convex subcomplexes are said to be parallel if they intersect the same hyperplanes; parallel subcomplexes are isomorphic, see [3].) Since carriers of codimension–1 hyperplanes are convex subcomplexes, and each carrier of a codimension–$n$ hyperplane is the intersection of carriers of $n$ mutually intersecting codimension–1 hyperplanes, codimension–$n$ carriers are also convex. From this and the product structure, it follows that panels are convex (justifying the term “parallel”).
If \( P = H \times \left[ -\frac{1}{2}, \frac{1}{2} \right] \times \left\{ \frac{1}{2} \right\} \subset \mathcal{B}(H) \) is a panel, and \( e \) is a 1–cube of \( P \), we say that \( e \) is internal (to \( P \)) if it is not contained in \( H \times \left\{ \pm \frac{1}{2} \right\} \times \left\{ \frac{1}{2} \right\} \). The interior of \( \mathcal{B}(H) \) is

\[
H \times \left( -\frac{1}{2}, \frac{1}{2} \right)^2,
\]

and the inside of \( P = H \times \left[ -\frac{1}{2}, \frac{1}{2} \right] \times \left\{ \frac{1}{2} \right\} \) is \( \text{Int}(P) = H \times \left( -\frac{1}{2}, \frac{1}{2} \right) \times \left\{ \frac{1}{2} \right\} \).

We emphasise that a subcomplex can be a panel in multiple ways, with different inside, according to which hyperplane is playing the role of \( H \). When we talk about a specific panel, it is with a fixed choice of \( H \) in mind. The choice of \( H \) determines the inside of the panel uniquely.

### 1.2. Extremal panels.

**Definition 1.3** (Extremal side, extremal panel). Let \( H, E \subset \Psi \) be distinct intersecting hyperplanes. \( H \cap E \) is extremal in \( H \) if the carrier \( N_H(H \cap E) \) of \( E \cap H \) in \( H \) contains one of the halfspaces \( H \cap \bar{E} \) of \( H \) (here \( \bar{E} \) is one of the components of the complement of \( E \) in \( \Psi \)). The union of cubes of \( H \) contained in \( H \cap \bar{E} \) is called the extremal side \( H^+_{\bar{E}} \) of \( H \) with respect to \( E \). The extremal panel \( P(H^+_{\bar{E}}) \) is the minimal subcomplex of \( \Psi \) containing all cubes intersecting \( H^+_{\bar{E}} \).

If \( H \cap E \) is extremal in \( H \), then \( P(H^+_{\bar{E}}) \) is a panel of \( \mathcal{B}(H \cap E) \) in the sense of Definition 1.2. We say \( H \) abuts \( P(H^+_{\bar{E}}) \) and \( E \) extremalises \( P(H^+_{\bar{E}}) \). Given an extremal panel \( P \) each maximal cube \( c \) of \( P \) is contained in a unique maximal cube of \( \Psi \) and \( c \) has codimension–1 in that cube.

**Lemma 1.4** (Finding extremal hyperplanes). Let \( H \) be a bounded CAT(0) cube complex. Then either \( H \) is a single 0–cube, or \( H \) contains an extremal hyperplane.

**Proof.** Let \( x \in H \) be a 0–cube and let \( K \) be a hyperplane so that \( d = d(x, N(K)) \) is maximal as \( K \) varies over all hyperplanes; such a \( K \) exists because \( d \in \mathbb{N} \) and \( H \) is bounded. By construction, \( K \) cannot separate a hyperplane of \( H \) from \( x \), and thus \( K \) is extremal in \( H \). \( \square \)

Lemma 1.4 immediately yields:

**Corollary 1.5.** Let \( \Psi \) be a CAT(0) cube complex with a compact hyperplane that is not a single point. Then \( \Psi \) contains an extremal panel.

### 1.3. The no facing panels property.

**Definition 1.6** (No facing panels property). We say that \( \Psi \) (respectively, \( \Psi \) and a particular collection \( \mathcal{P} \) of extremal panels) satisfies the no facing panels property if the following holds. Let \( \mathcal{B}(E \cap H) \) and \( \mathcal{B}(E^1 \cap H^1) \) be distinct blocks so that \( P = P(H^\circ_{E\circ}) \) and \( P^1((H^1)^\circ_{E^1}) \) are extremal panels (respectively, in \( \mathcal{P} \)). Suppose that \( P \cap P^1 = \emptyset \). Then \( \mathcal{B}(E \cap H) \) and \( \mathcal{B}(E^1 \cap H^1) \) do not have a common maximal cube.

**Remark 1.7.** The maximal cubes of \( \mathcal{B}(E \cap H) \) have the form \( f \times [-\frac{1}{2}, \frac{1}{2}]^2 \), where \( f \) is a maximal cube of \( E \cap H \). Hence, if \( \mathcal{B}(E \cap H), \mathcal{B}(E^1 \cap H^1) \) have a common maximal cube, it has the form \( f \times [-\frac{1}{2}, \frac{1}{2}]^2 = f^1 \times [-\frac{1}{2}, \frac{1}{2}]^2 \), where \( f \subset E \cap H, f^1 \subset E^1 \cap H^1 \).

Recall that \( G \leq \text{Aut}(\Psi) \) acts without inversions if, for all \( g \in G \) and all hyperplanes \( H \), if \( gH = H \) then \( g \) preserves both of the complementary components of \( H \).

**Lemma 1.8.** Let \( G \leq \text{Aut}(\Psi) \) act without inversions across hyperplanes, let \( P \) be an extremal panel, and let \( \mathcal{P} = G \cdot P \). Then \( \mathcal{P} \) has the no facing panels property.
Proof. Suppose that \( B(E \cap H) \) and \( B(E^1 \cap H^1) \) have a common maximal cube \( c \). Let \( \bar{P}, \bar{P}^1 \) be parallel copies of \( P, P^1 \) in \( B(E \cap H), B(E^1 \cap H^1) \) respectively, so that \( \bar{P}, P \) are parallel and separated by \( E \) (and no other hyperplanes) and \( \bar{P}^1, P^1 \) are parallel and separated only by \( E^1 \).

If \( P \cap P^1 = \emptyset \), then \( P \cap P^1 \) contains a codimension–1 face \( c' \) of \( c \), whose opposite in \( c \), denoted \( c'' \), lies in \( P \). The unique hyperplane separating \( c', c'' \) is \( E \).

By hypothesis, \( P^1 = gP \) for some \( g \in G \), and extremality of \( P^1 \) implies that \( gB(E \cap H) = B(E^1 \cap H^1) \). Hence either \( gE = E^1 \) and \( gH = H^1 \) or \( gE = H^1 \) and \( gH = E^1 \). But \( gH \) intersects \( P^1 \) since \( H \) intersects \( P \), while \( E \) is disjoint from \( P \), so \( gE \) is disjoint from \( P^1 \). Hence \( gE = E^1 \) and \( gH = H^1 \).

Now, since \( B(E^1 \cap H^1) \) contains \( c \) as a maximal cube, \( E \) intersects \( B(E^1 \cap H^1) \). On the other hand, \( c' \) is parallel to a subcomplex of \( E \) (since \( c' \) is a cube of \( \bar{P} \)) while \( c'' \) is also parallel into \( E^1 \) (since \( c'' \) is a cube of \( P^1 \)). But, by the definition of an \( H^1 \)–panel in \( B(E^1 \cap H^1) \), each maximal cube of \( P^1 \) (i.e. codimension–1 face of the maximal cube of the block) is parallel to a subcomplex of a unique hyperplane of \( B(E^1 \cap H^1) \), namely \( E^1 \). Hence \( E = E^1 \), so \( gE = E \). But since \( P \) and \( gP = P^1 \) are separated by \( E \), the element \( g \) acts as an inversion across \( E \), a contradiction. \( \square \)

1.4. Motivating examples. The original motivation for our main construction was the following: knowing that compact CAT(0) cube complexes are collapsible to points, how can one modify an ambient cube complex with compact hyperplanes so as to realize a collapse of the hyperplanes, while keeping the ambient space a CAT(0) cube complex?

First consider a single 3–cube \( c \). If we were to collapse a free face, i.e. to delete an open square in the boundary and the open 3–cube within \( c \), we would have a union of 5 squares giving an “open box”, which is not CAT(0). To preserve CAT(0)ness, we need to “collapse an entire panel”, as follows.

Let \( H, E \) be two distinct hyperplanes in \( c \) and pick a halfspace \( \bar{E} \cap H \). Then the intersection of \( \bar{E} \cap H \) with the 1–skeleton of \( c \) is a pair of 1–cube midpoints. These open 1–cubes will be called internal and the maximal codimension–1 face containing all these is a panel. \( H \) is the abutting hyperplane and \( E \) is the extremalising hyperplane. We obtain a deletion \( D(c) \) by removing all open cubes containing the internal 1–cubes. This amounts to applying Theorem 3.1 once, in the case where \( \Psi \) is the single cube \( c \) and \( P \) consists of a single panel \( P \). Passage from \( c \) to \( D(c) \) can be realised by a strong deformation retraction. See Figure 1.

![Figure 1. Collapsing the panel P of the cube c to obtain D(c). The extremalising hyperplane is E and the abutting hyperplane is H.](image)

The resulting cube complex is again CAT(0). By repeatedly choosing abutting and externalizing hyperplanes, and applying Theorem 3.1 several times in succession, we can collapse the cube down to a tree, as in Figure 2. In this tree, the edge-midpoints arise as intersections of this tree sitting inside \( c \) and the original hyperplanes of \( c \).

Now suppose we wanted to collapse many panels simultaneously (instead of by applying Theorem 3.1 to a single panel, choosing a panel in the resulting complex, and iterating).
Figure 2. Reducing $c$ to a tree via a sequence of panel collapses. In this picture, at each stage a new panel (shaded) is chosen in the current subcomplex of $c$. In all but the first step, the panel in question is a 1–cube. At the final stage, we have a subcomplex of $c$, so the midpoints of edges are contained in hyperplanes of $c$. The bold 1–cubes are dual to the abutting hyperplanes of the panels being collapsed.

The most serious conflict between panels that could arise involves two panels intersecting a common cube in opposite faces, but this is ruled out by the requirement that we draw our panels from a collection with the no facing panels property. The other issue involves two panels that intersect a common cube, which cannot be so easily hypothesised away. This brings us to our second basic example, the situation where $\Phi$ is an infinite CAT(0) square complex with compact hyperplanes that admits a cocompact action by some group $G$.

In this case, hyperplanes are trees and panels are free faces of squares (corresponding to leaves of the hyperplane-trees). Although we will require that the action does not invert hyperplanes (not a real restriction because one can subdivide), it may very well be that some element of $G$ maps a square $c$ to itself, so that $c$ contains two distinct panels $P$ and $gP$. By the no facing panels property, these panels must touch at some corner of $c$. If we wish to $G$-equivariantly modify $\Phi$, then we must collapse both $P$ and $gP$. The resulting $D(c)$ obtained by deleting all the open cubes of $c$ whose interiors lie in the interior of $P$ or $gP$ is disconnected (one of the components is a 0–cube). However, if $h(c)$ is the 0–cube of $c$ in $D(c)$ diagonally opposite the isolated 0–cube, then it can be connected to its opposite $\bar{h}(c) = w$ by a diagonal $S(\bar{h}(c))$. The cubes $h(c), \bar{h}(c)$ are the persistent and salient cubes defined below (in general, they need not be 0–cubes). The resulting subspace $F(c)$ replaces $c$ in the new complex, and we can make these replacements equivariantly. See Figure 3.

The panel collapse construction, described formally in the next section, just generalises these procedures.

2. Panel collapse for single cubes

Let $\Psi$ be a finite-dimensional CAT(0) cube complex and let $P$ be a collection of extremal panels with the no facing panels property. Recall that the inside of $P$ is the union of open cubes of $P$ that intersect the abutting hyperplane. When we refer to the interior of a cube $[-\frac{1}{2}, \frac{1}{2}]^n$, we just mean $(-\frac{1}{2}, \frac{1}{2})^n$. If $c$ is a (closed) cube of $\Psi$ and $P \in P$, then $P \cap c$ is either $\emptyset$ or a sub-cube of $c$, by convexity of $P$. 

The first goal of this section is to define the fundament $F(c)$ of a cube $c$, which is a subspace obtained from $c$ given the intersections $\{c \cap P \mid P \in \mathcal{P}\}$, which are panels of $c$ that must be collapsed. This construction of $F(c)$ must be compatible with the induced collapses and fundaments $F(c')$ of the subcubes $c'$ of $c$. The second goal of this section is to prove the existence of a strong deformation retraction of $c$ onto $F(c)$, compatible with deformation retractions on the codimension–1 faces, and to show that $F(c)$ is again a CAT(0) cube complex.

**Lemma 2.1.** Let $c$ be a cube, let $P \in \mathcal{P}$, and let $e$ be a 1–cube of $c$ that is internal to $P$. Then $c$ has a codimension–1 face $f$ so that every 1–cube of $f$ that is parallel to $e$ is internal to $P$.

In other words, if $c$ is a cube and $P \in \mathcal{P}$, then exactly one of the following holds:

- $c \cap P = \emptyset$;
- the interior of $c$ is contained in the inside of $P$;
- $c \cap P$ is a codimension–1 face $f$ of $c$, and moreover, if $e$ is an edge of $f$ dual to the abutting hyperplane of $P$, then any edge of $f$ parallel to $e$ is in the inside of $P$.

**Proof.** Let $E, H$ be hyperplanes so that $B(P) = B(E \cap H)$, with $H$ dual to the internal 1–cubes of $P$, so $e$ is dual to $H$. By extremality of $P$, we must have that $c \subset B(P)$. If the interior of $c$ is contained in the inside of $P$, we’re done. Otherwise, $E \cap c \neq \emptyset$, and $c$ has two codimension–1 faces, $f, f'$, that are separated by $E$, with $f \subset P$. Thus every 1–cube of $f$ dual to $H$ (i.e. parallel to $e$) is internal to $P$, as required.

The next lemma is immediate from the definition of a panel:

**Lemma 2.2.** Let $c$ be a cube of $\Psi$. If the interior of $c$ lies in the inside of some $P \in \mathcal{P}$, then there is some 1–cube $e$ of $c$ such that $e'$ is internal for $e$ for all 1–cubes $e'$ of $c$ parallel to $e$.

Conversely, if $c$ is a cube, and for some 1–cube $e$ of $c$, every 1–cube parallel to $e$ is internal to $P$, then the interior of $c$ is contained in the inside of $P$.

From Lemma 2.2 and convexity of panels, we obtain:

**Lemma 2.3.** Let $c$ be a cube and let $P \in \mathcal{P}$. Let $e, e'$ be 1–cubes of $c$ that are internal to $P$. Suppose that $e, e'$ do not lie in a common proper sub-cube of $c$. Then the interior of $e$ is contained in the inside of $P$.

Hence, if the interior of $c$ is not contained in the inside of $P$, then all of the 1–cubes of $c$ internal to $P$ are contained in a common codimension–1 face of $c$. 

![Figure 3](image-url)
Definition 2.4 (Internal, external, completely external). A subcube $c$ of $\Psi$ is internal [resp., internal to $P \in \mathcal{P}$] if its interior lies in the inside of some panel in $\mathcal{P}$ [resp., $P$]. Otherwise, $c$ is external. If $c$ contains no 1–cube that is internal, then $c$ is completely external. Completely external cubes are external, but external cubes are not in general completely external. Specifically, $c$ is external but not completely external exactly when $c$ has a codimension–1 face that is internal. (The notions of externality and complete externality coincide for 1–cubes, and every 0–cube is completely external.)

The following two lemmas will be important later in the section, but we state them here since they are just about the definition of an external cube.

Lemma 2.5. Let $f$ be a cube which is the product of external cubes. Then $f$ is external.

Proof. Write $f = f' \times e$, where $f', e$ are external cubes.

If $f$ is not external, then there exists a 1–cube $e'$ such that every 1–cube of $f$ parallel to $e'$ is internal. Since $e$ is external, $e'$ cannot be parallel to $e$. Hence $e'$ is parallel to a 1–cube of $f'$. But since $f'$ is external, every parallelism class of 1–cubes represented in $f'$ has an external representative in $f'$. This proves that $f$ is external.

Lemma 2.6. The intersection $c \cap c'$ of two external cubes $c, c'$ is external.

Proof. If $c' \subset c$, this is obviously true. If $c \cap c'$ is a 0–cube, then this holds since all 0–cubes are external. Suppose now towards a contradiction, that $c \cap c'$ is an internal cube of dimension at least 1. Let $P \in \mathcal{P}$ be a panel containing the interior of $c \cap c'$. Let $H$ be the abutting hyperplane and let $E$ be the extremaliser of $P$. Since $c \cap c'$ is internal to $P$ it cannot contain a 1–cube dual to $E$. The hyperplanes $E$ and $H$ cross $c$, and by extremality $c'$ must lie in the carrier $\mathcal{N}(E)$. Since $c'$ is not contained in $P$, then $c' \cap P$ is a codimension–1 face of $c'$. Hence every 0–cube of $c'$ is contained in a 1–cube dual to $E$. In particular, since $c \cap c'$ contains a 0–cube $v$, the 1–cube $e$ containing $v$ and dual to $E$ lies in $c \cap c'$, contradicting that $c \cap c'$ is internal.

Remark 2.7. In several places, we will use the following consequence of Lemma 2.1. Let $e, e'$ be 1–cubes of a cube $c$ that are parallel (i.e. dual to the same hyperplane $D$). Suppose that every hyperplane of $c$ other than $D$ separates $e, e'$. This is equivalent to the assertion that for each codimension–1 face $c'$ of $c$, at most one of $e, e'$ is contained in $c'$. In this situation, if $e, e'$ are both external, then every 1–cube $e''$ dual to $D$ must also be external. Indeed, suppose that $e''$ is an internal 1–cube dual to $D$. Then there is a panel $P \in \mathcal{P}$ such that $e''$ is internal to $P$ and $e''$ lies in some codimension–1 face $f$ such that every 1–cube of $f$ parallel to $e''$ is internal. Now, either $e$ or $e''$ must lie in $f$, for otherwise both $e, e''$ must be contained in the codimension–1 face parallel to, and disjoint from, $f$. But this is impossible, and hence the externality of $e$ or $e'$ is contradicted by the internality of $e''$.

So far we have not used that $\mathcal{P}$ has the no facing panels property; now it is crucial.

Lemma 2.8. Let $c$ be a cube of $\Psi$. Let $f, f'$ be a pair of parallel codimension–1 faces of $c$ so that there are panels $P, P' \in \mathcal{P}$ with the interiors of $f, f'$ respectively contained in $P, P'$. Then $c$ is internal.

Proof. Let $m$ be a maximal cube containing $c$. Let $H$ be the hyperplane that crosses $c$ (and thus $m$) and separates $f, f'$. Let $n, n'$ be the codimension–1 faces of $m$ that respectively contain $f, f'$ and are separated by $H$.

If $P = P'$, then we are done, by Lemma 2.3. Hence suppose that $P \neq P'$ and $f'$ is not internal to $P$ and $f$ is not internal to $P'$. Then, since $P, P'$ must intersect $m$ in codimension–1 faces, we have that the interior of $n$ lies in the inside of $P$, and the interior of $n'$ lies in the inside of $P'$, and this contradicts the no facing panels property. □
Lemma 2.9 (External cubes have persistent corners). Let $c$ be an external cube of $\Psi$. Then there exists a 0-cube $v$ of $c$, and distinct 1-cubes $e_1, \ldots, e_{\dim c}$, contained in $c$ and incident to $v$, so that each $e_i$ is external.

A 0-cube $v$ as in the lemma is a persistent corner of $c$.

Proof. For each $P \in \mathcal{P}$, Lemma 2.3 implies that $P \cap c$ is either empty or confined to a codimension-1 face $f_P$ of $c$. Identifying $c$ with $f_P \times \left[ -\frac{1}{2}, \frac{1}{2} \right]$ and $f_P$ with $f_P \times \left\{ \frac{1}{2} \right\}$, let $b_P = f_P \times \left\{ -\frac{1}{2} \right\}$, which is a codimension-1 face of $c$. For each $P, P'$ as above, Lemma 2.8 shows that $b_P \cap b_{P'} \neq \emptyset$, so by the Helly property for convex subcomplexes, $\bigcap_P b_P$ is a nonempty sub-cube $c'$ of $c$ (the intersection is taken over all panels in $\mathcal{P}$ contributing an internal 1-cube to $c$).

Let $v$ be a 0-cube of $c'$. By construction, any 1-cube $e$ of $c'$ incident to $v$ is external. Next, suppose that $e$ is a 1-cube of $c$ incident to $v$ that joins $c'$ to a 1-cube not in $c'$. If $e$ is internal to some panel $P$, then $e$ is separated from $b_P$ by a hyperplane of $c$. Since $v \in e$, we have $v \not\in b_P$, a contradiction. Hence $e$ is external. □

Definition 2.10 (Persistent subcube, salient subcube). Let $c$ be an external cube of $\Psi$. Lemma 2.9 implies that $c$ has at least one persistent corner. Let $h(c)$ be the $(\ell_1)$ convex hull of the persistent corners of $c$, i.e. the smallest sub-cube of $c$ containing all of the persistent corners. We call $h(c)$ the persistent subcube of $c$. Let $\tilde{h}(c)$ be the subcube of $c$ with the following properties:

- $\tilde{h}(c)$ is parallel to $h(c)$, i.e. $\tilde{h}(c)$ and $h(c)$ intersect exactly the same hyperplanes;
- the hyperplane $H$ of $c$ separates $h(c)$ from $\tilde{h}(c)$ if and only if $H$ does not cross $h(c)$ and $H$ is dual to some internal 1-cube of $c$.

We call $\tilde{h}(c)$ the salient subcube of $c$. Note that $h(c)$ and $\tilde{h}(c)$ are either equal or disjoint, and have the same dimension. Also, if $c'$ is a sub-cube of $c$, and $h(c) \cap c' \neq \emptyset$, then $c'$ contains a persistent corner of $c$, by minimality of $h(c)$.

Lemma 2.11. Every 0-cube of $h(c)$ is persistent.

Proof. Adopt the notation of Lemma 2.9. So, $c$ has a nonempty subcube $c'$ which is the intersection $\bigcap_{P \in \mathcal{P}} b_P$, where for each panel $P \in \mathcal{P}$, $b_P$ is the codimension-1 face opposite $P \cap c$. The proof of Lemma 2.9 shows that every 0-cube of $c'$ is a persistent corner of $c$.

In particular, $c' \subseteq h(c)$. Now suppose that $v$ is a 0-cube of $c$. Suppose that $v \in P \cap c$ for some $P \in \mathcal{P}$. Then at least one 1-cube of $P \cap c$ incident to $v$ is internal, so $v$ is not a persistent corner of $c$. Hence every persistent corner of $c$ lies in $c'$, so $h(c) \subseteq c'$. Thus $h(c) = c'$. Since every 0-cube of $c'$ is persistent, the lemma follows. □

Remark 2.12 (Equivalent characterisation of persistent subcubes). Given an external cube $c$ of $\Psi$, we can equivalently characterise $h(c)$ and $\tilde{h}(c)$ as follows. As in Lemma 2.9, for each panel $P \in \mathcal{P}$ intersecting $c$, let $f_P$ be the codimension-1 face in which $P$ intersects $c$ and let $b_P$ be the opposite face. Then $h(c) = \bigcap_P b_P$.

2.1. Building the subspace $\mathcal{F}(c)$. Let $c$ be a (closed) cube of $\Psi$. We define a subspace $\mathcal{F}(c)$ of $c$ as follows. First, let $\mathcal{D}(c)$ be the union of all completely external cubes of $c$. In other words, $\mathcal{D}(c)$ is obtained from $c$ by deleting exactly those open sub-cubes whose closures contain internal 1-cubes.

Observe that $\mathcal{D}(c) \cap c' = \mathcal{D}(c')$ for all sub-cubes $c'$ of $c$.

Before defining $\mathcal{F}(c)$, we need the following important fact about $\mathcal{D}(c)$:

Lemma 2.13. If $c$ is an external cube and $c'$ an external sub-cube of $c$, then $\mathcal{D}(c')$ is connected provided $\mathcal{D}(c)$ is connected.
Proof. Let \( d = \dim c \) and \( d' = \dim c' \). If \( d = d' \), then \( c = c' \) and the lemma is immediate. If \( d' = 0 \), then \( \mathcal{D}(c') = c' \), and again the lemma holds. Hence assume that \( 0 < d' < d \).

Let \( \pi : c \to c' \) be the canonical projection, where \( c \) is regarded as \( [-\frac{1}{2}, \frac{1}{2}]^d \) and \( c' \) is regarded as \( [-\frac{1}{2}, \frac{1}{2}]^{d'} \). We first establish some auxiliary claims, the first of which is the special case of the lemma where \( d - d' = 1 \).

**Claim 1.** If \( c \) is external and \( c' \) is a codimension--1 external face, and \( \mathcal{D}(c) \) is connected, then \( \mathcal{D}(c') \) is connected.

**Proof of Claim 1.** Since \( \pi \) restricts to the identity on \( c' \), and \( \mathcal{D}(c') \subseteq \mathcal{D}(c) \), we have \( \mathcal{D}(c') \subseteq \pi(\mathcal{D}(c)) \). We claim that \( \pi(\mathcal{D}(c)) \subseteq \mathcal{D}(c') \). By definition, \( \mathcal{D}(c) \) is the union of the completely external cubes of \( c \), and \( \mathcal{D}(c') \) is the union of the completely external cubes of \( c' \). Hence it suffices to show that \( \pi \) takes each completely external cube \( f \) to a completely external cube.

If \( f \cap c' \neq \emptyset \), then \( f \cap c' \) is a completely external cube of \( c' \). On the other hand, \( \pi(f) = f \cap c' \), so \( \pi(f) \) is completely external.

Otherwise, \( f \) is separated from \( c' \) by some unique hyperplane \( E \), and \( \pi(f) \) is a cube of \( c' \) crossing exactly those hyperplanes that cross \( f \) and \( c' \) and separated from \( f \) by \( E \). If \( \pi(f) \) is not completely external, then there is a 1--cube \( \pi(e) \) of \( \pi(f) \) that is internal. Now, since \( f \) is completely external, every 1--cube \( e \) of \( f \) parallel to \( \pi(e) \) is external.

For each codimension--1 face \( c'' \) containing \( e \), there is no panel \( P \) so that \( P \cap c = c'' \) and the abutting hyperplane for \( P \) is dual to \( E \). By Lemma 2.1.2.3, there is a codimension--1 face \( c''' \) and a panel \( P \) so that \( P \cap c = c''' \) and \( \pi(e) \) is internal to \( P \). Since \( E \) is the only hyperplane separating \( f \) from \( \pi(f) \), the face \( c''' \) cannot cross \( E \), for otherwise it would contain \( e \). Hence \( c''' = c' \). Hence every 1--cube of \( c' \) parallel to \( e \) is internal, so \( c' \) is internal, a contradiction.

We have shown that \( \pi \) restricts on \( \mathcal{D}(c) \) to a surjection \( \mathcal{D}(c) \to \mathcal{D}(c') \). Since \( \mathcal{D}(c) \) is connected and \( \pi \) is continuous, it follows that \( \mathcal{D}(c') \) is connected. \( \square \)

**Conclusion:** Recall that \( c' \) is an external subcube of \( c \). By Claim 1, if \( d - d' = 1 \), then \( \mathcal{D}(c') \) is connected. So, assume that \( d - d' \geq 2 \).

Next, let \( \mathcal{E} \) be the set of 1--cubes \( e \) of \( c \) such that \( c' \cap e \neq \emptyset \), and \( c' \) does not contain \( e \). Note that for each \( e \in \mathcal{E} \), there is a sub-cube \( c' \times e \) of \( c \). Moreover, \( c' \times e \) has codimension at least 1 in \( c \), because \( c' \) has codimension at least 2.

We claim that there exists an external 1--cube \( e \in \mathcal{E} \). There are two cases to consider. First, if \( c' \cap h \neq \emptyset \), then by Lemma 2.11, \( c' \) contains a persistent corner \( v \) of \( c' \). Any 1--cube \( e \) emanating from \( v \) is external, by the definition of a persistent corner, and since such a 1--cube can be chosen so as not to lie in \( c' \), we see that \( \mathcal{E} \) contains an external 1--cube. Second, suppose that \( c' \cap h = \emptyset \) and suppose that every 1--cube in \( \mathcal{E} \) is internal. Let \( P \) be an edge-path in \( c \) joining \( h \) to \( c' \). Then \( P \) must pass through a 1--cube in \( \mathcal{E} \). If every 1--cube in \( \mathcal{E} \) is internal, then \( P \) cannot lie in \( \mathcal{D}(c) \). But since every 0--cube of \( c \) is in \( \mathcal{D}(c) \), and \( \mathcal{D}(c) \) is connected, this gives a contradiction. Thus, in either case, \( \mathcal{E} \) contains an external 1--cube \( e \).

Thus, by Lemma 2.5, \( c' \times e \) is an external sub-cube of \( c \) of codimension \( d - d' - 1 \). By induction on codimension, \( \mathcal{D}(c' \times e) \) is connected. Now, since \( c' \) is a codimension--1 external subcubce of \( c \times e \), Claim 1 implies that \( \mathcal{D}(c') \) is connected, as required.

Now we can define \( \mathcal{F}(c) \) for each cube \( c \) of \( \Psi \). The idea is to construct a subspace of \( c \) that contains all the completely external cubes (in particular, all the 0--cubes) but does not contain the external cubes, and behaves well under intersections of cubes. We will also want \( \mathcal{F}(c) \) to have the structure of a locally CAT(0) cube complex and, in the case where \( c \) is external, \( \mathcal{F}(c) \) will actually be a deformation retract of \( c \).

The case where \( c \) is internal plays only a small role, and is easy to deal with, so we handle it first:
Definition 2.14 ($\mathcal{F}(c)$ for internal $c$). Let $c$ be an internal cube. Then $\mathcal{F}(c) = \mathcal{D}(c)$. (Note that in this case, $\mathcal{D}(c)$ is disconnected by the abutting hyperplane for each panel in $\mathcal{P}$ in which $c$ is internal.)

The definition of $\mathcal{F}(c)$ when $c$ is external is more complicated. The reason is that $\mathcal{D}(c)$ can be disconnected, and in such situations, $\mathcal{F}(c)$ (which must be connected) cannot be taken to be a subcomplex. Accordingly, we divide into cases according to whether $\mathcal{D}(c)$ is connected or not. (Lemma 2.13 above shows that this division into cases is stable under passing to subcubes, which will be important for later arguments by induction on dimension.) The connected case is straightforward:

Definition 2.15 ($\mathcal{F}(c)$ for external $c$, connected case). Let $c$ be an external cube for which $\mathcal{D}(c)$ is connected. Then $\mathcal{F}(c) = \mathcal{D}(c)$. This includes the case where $c$ is completely external, in which case $\mathcal{F}(c) = \mathcal{D}(c) = c$.

Now let $c$ be an external cube that is not completely external.

Lemma 2.9 provides a persistent corner $v \in c$. Let $\bar{h}(c) = h$ be the persistent subcube of $c$. By Lemma 2.11, $h$ is completely external, and is thus $h \subset c$. Let $\bar{h} = \bar{h}(c)$ be the salient subcube of $c$.

If $h$ is a codimension–1 face, and $h = \bar{h}$, then every 1–cube emanating from $\bar{h}$ is external, so every 0–cube is connected to $\bar{h}$ by an external 1–cube, so $\mathcal{D}(c)$ is connected and $\mathcal{F}(c) = \mathcal{D}(c)$ by Definition 2.15.

If $h$ is a codimension–1 face and $h \neq \bar{h}$, then there is a hyperplane $E$ separating $h, \bar{h}$ since $h, \bar{h}$ are parallel and distinct. On the other hand, the definition of $\bar{h}$ implies that some 1–cube dual to $E$ is internal. But since $h$ is codimension–1, Lemma 2.11 implies that every 1–cube dual to $E$ is external, a contradiction.

Hence we can assume that $\dim c - \dim h \geq 2$.

For each codimension–1 face $h'$ of $c$ containing some persistent corner $v_i$ of $c$, the cube $h'$ is spanned by a set of external 1–cubes, so $h'$ is external. By induction on dimension, we have defined $\mathcal{F}(h')$ to be a contractible subspace of $h'$ that contains $\mathcal{D}(h')$ and also contains the convex hull in $h'$ of all persistent corners of $h'$ (in particular, $\mathcal{F}(h')$ contains $h \cap h'$). The base case is where $c$ is a 0–cube, which is necessarily completely external, so $\mathcal{F}(c) = c$.

$\mathcal{F}_0(c)$: **Assembling pieces from codimension–1 faces.** Now let $\text{Faces}'(c)$ be the set of codimension–1 subcubes $c'$ of $c$ containing a persistent corner $v_i$. Let $\mathcal{F}_0(c) = \bigcup_{c' \in \text{Faces}'(c)} \mathcal{F}(c')$. Since $h$ is connected and intersects $\mathcal{F}(c')$ for each $c' \in \text{Faces}'(c)$, the subspace $\mathcal{F}_0(c)$ is connected. (Later, in Lemma 2.19, we will see that $\text{Faces}'(c)$ is precisely the set of external codimension–1 faces of $c$.)

$\mathcal{F}_1(c)$: **Connecting subcubes of $h$ to their opposites in $\bar{h}$.** Recall that $\bar{h}$ is the parallel copy of $h$ in $c$ that is separated from $h$ by exactly those hyperplanes $H$ so that:

- $H$ does not cross $h$;
- at least one 1–cube dual to $H$ is internal.

Let $\kappa \leq d - \dim h$ be the number of hyperplanes separating $h, \bar{h}$. For each completely external cube $w \not\subseteq \bar{h}$, let $S(w)$ be the $\ell_2$–convex hull of $w$ and the cube $\bar{w}$ (a sub-cube of $h$) parallel to $w$ and separated from $w$ by the above hyperplanes.

Equivalently, $c$ has an $\ell_2$–convex subspace $h \times \left[ -\frac{\sqrt{\kappa}}{2}, \frac{\sqrt{\kappa}}{2} \right]$, intersecting exactly those hyperplanes that separate the persistent and salient cubes, and for each completely external subcube $w$ of $\bar{h} = h \times \{ \frac{\sqrt{\kappa}}{2} \}$, let $S(w) = w \times \left[ -\frac{\sqrt{\kappa}}{2}, \frac{\sqrt{\kappa}}{2} \right]$. Also note that $\dim S(w) = \dim w + 1 \leq d - 2$ since $\dim h \leq d - 2$ and $w \not\subseteq h$.

Form $\mathcal{F}_1(c)$ from $\mathcal{F}_0(c)$ by adding $S(w)$ for each completely external cube $w$ of $\bar{h}$.
Remark 2.16. Observe that, if \( D(c) \) is disconnected, then there exists \( w \) as above with \( S(w) \neq \emptyset \). Indeed, the persistent subcube \( h \) is nonempty and \( \bar{h} \) is therefore nonempty, since it is parallel to \( h \). Any 0–cube \( w \) of \( \bar{h} \) is completely external, so by definition \( S(w) \) is a nonempty subspace containing \( w \). Moreover, \( S(w) \) is contained in \( F_1(c) \).

\( F(c) \): Adding the missing cubes from \( D(c) \). To complete the definition of \( F(c) \), we have to ensure that it contains \( D(c) \), in order to support the above inductive part of the definition. In other words, we need to add to \( F_1(c) \) any completely external sub-cube of \( c \) that does not already appear in \( F_1(c) \). Hence let \( F(c) \) be the union of \( F_1(c) \) and any totally external cube of \( c \) not already contained in \( F_1(c) \). Some examples are shown in Figure 4.

**Figure 4.** Three examples of the construction of \( F(c) \) from a set of panels on \( c \), in the case \( \dim c = 3 \). In each case, the faces of \( c \) belonging to panels in \( P \) are shaded, and the internal 1–cubes of \( c \) are bold. The remaining pictures show how \( D(c) \), \( F_0(c) \), \( F_1(c) \), and \( F(c) \) sit inside \( c \). The 1–cubes of \( F(c) \) (which are either 1–cubes of \( c \) or diagonals in \( c \) or its faces) are bold. In the first and last cases, \( h, \bar{h} \) are labelled; in the middle case, \( h \) is the unique 0–cube with 3 incident external 1–cubes, and \( \bar{h} \) is diagonally opposite \( h \) in \( c \).

2.2. Basic properties of the fundament. Before proceeding to a more technical analysis of \( F(c) \), we collect some important properties that capture most of the essence of why \( F(c) \) has been defined as it has.

First, by definition, \( F(c) \) contains every 0–cube of \( c \), since 0–cubes are completely external and \( D(c) \subseteq F(c) \).

Second, \( F(c) \) is “completely external” in the following sense:

**Lemma 2.17.** For each cube \( c \) and each \( P \in P \), we have \( F(c) \cap \text{Int}(P) = \emptyset \).

**Proof.** By definition, \( D(c) \) is disjoint from \( \text{Int}(P) \). On the other hand, any cube of \( \Psi \) containing some \( S(w) \) is external. Thus the lemma follows from the definition of \( F(c) \). \( \square \)

Third, each \( F(c) \) was defined in terms of \( P \) (and the set of 1–cubes internal to panels in \( P \)), from which we immediately get:

**Lemma 2.18.** For each cube \( c \), the subspace \( F(c) \) is uniquely determined by the set of internal 1–cubes of \( c \). In particular, if \( g \in \text{Aut}(\Psi) \) preserves \( P \) (taking insides to insides), then \( F(gc) = gF(c) \) for all cubes \( c \).
Finally, the fundament has been defined so as to be consistent under passing to subcubes:

**Lemma 2.19** (Fundamentals of subcubes). Let $c$ be an external cube. Then both of the following hold:

1. For each external sub-cube $c'$ of $c$,
   \[ F(c) \cap c' = F(c'). \]

2. If $c'$ is a codimension–1 external face of $c$, then $c' \in \text{Faces}'(c)$.

(Recall that $\text{Faces}'(c)$ denotes the set of codimension–1 faces of $c$ that contain persistent corners of $c$.)

Lemma 2.19 has the following important consequence. Let $b, c$ be external cubes. If $b \cap c$ is external, then by the lemma, $F(c) \cap b = F(c) \cap (b \cap c) = F(b \cap c)$, and the same is true with the roles of $b$ and $c$ reversed. Hence $F(c) \cap F(b) = F(b \cap c)$.

**Proof of Lemma 2.19.** We will prove the second assertion first, and use it in the proof of the first assertion.

**Proof of assertion (2):** Suppose that $c' \notin \text{Faces}'(c)$ and $c'$ has codimension 1. Since $\text{Faces}'(c)$ contains a maximal collection of pairwise-intersecting codimension–1 sub-cubes of $c$ (namely, all the faces containing any given persistent corner $v_i$), there exists $c'' \in \text{Faces}'(c)$ so that $c'$ and $c''$ are parallel. Now, if some persistent corner $v_i \in c'$, then $c' \in \text{Faces}'(c)$, a contradiction. So each persistent corner $v_i \in c'$, whence $h = h(c) \subset c'$.

Let $v' \in c'$ be a 0–cube connected by a 1–cube to some $v_i$ (this exists since $c'$ is codimension–1). The 1–cube joining $v_i, v'$ is external, because $v_i$ is a persistent corner. Hence, since $c' \notin \text{Faces}'(c)$, there exists a 1–cube $e$ of $c'$ that is incident to $v'$ and internal to some $P \in \mathcal{P}$. Lemma 2.1 provides a codimension–1 face $c''$ of $c$ so that $e \subset c''$ and every 1–cube of $c''$ parallel to $e$ is internal to $P$. Now, $c''$ cannot contain the 1–cube $\bar{e}$ parallel to $e$ and incident to $v_i$, since $v_i$ is a persistent corner. Hence $c''$ is disjoint from $c'$. (Because any codimension–1 face containing $e$ and intersecting $c'$ must contain $\bar{e}$.) Thus $c'' = c'$, so $c'$ is internal, a contradiction. Thus $c' \in \text{Faces}'(c)$.

**Proof of assertion (1):** We first prove the claim in the case where $c'$ is codimension–1. If $c$ is completely external, then so is $c'$, so $F(c) \cap c' = c \cap c' = c = F(c')$.

Now suppose that $c$ is not completely external. If $D(c)$ is connected, then by definition, $F(c) = D(c)$. Lemma 2.13 implies that $D(c')$ is connected, so $F(c') = D(c')$. It follows immediately from the definition that $D(c') = D(c) \cap c'$, so $F(c) \cap c' = F(c')$.

It remains to consider the case where $D(c)$ is disconnected. By the first part of the lemma, $c' \in \text{Faces}'(c)$. By definition, $F(c) \cap c' \supseteq F(c')$. Lemma 2.20 below implies that for any completely external cube $w$ in $h = h(c)$ so that $S(w)$ intersects $c'$, we have $S(w) \cap c' \subseteq F(c')$. Now, either $F(c) \cap c' = D(c) \cap c' = D(c')$ or $F(c) \cap c' = (D(c) \cap c') \cup (\bigcup_w S(w) \cap c')$, so in either case $F(c) \cap c' \subseteq F(c')$, as required.

Now we complete the proof by arguing by induction on codimension $k = \dim c - \dim c'$. The case $k = 1$ was done above. Let $k \geq 1$. If $c'$ is not contained in some element of $\text{Faces}'(c)$, then $c'$ is a completely external cube that gets added in passing from $F_1(c)$ to $F(c)$.

Indeed, $F(c)$ is the union of $F_0(c)$ together with some completely external cubes and some subspaces of the form $\text{Int}(S(w))$, and the latter are disjoint from $c'$ unless $c'$ is contained in an element of $\text{Faces}'(c)$. So, if $c'$ is external and does not lie in an element of $\text{Faces}'(c)$, then $F(c') \supseteq F(c) \cap c'$. 


Otherwise, \( c' \) is a subcube of some \( l \in \text{Faces}'(c) \). Note that \( \dim l - \dim c' < k \) and \( l \) is external, so by induction on codimension, \( \mathcal{F}(l) \cap c' = \mathcal{F}(c') \). But by the case \( k = 1 \), we have \( \mathcal{F}(c) \cap l = \mathcal{F}(l) \). So \( \mathcal{F}(c) \cap c' = \mathcal{F}(c) \cap (l \cap c') = \mathcal{F}(l) \cap c' = \mathcal{F}(c') \), as required. \( \square \)

The next lemma supported the previous one:

**Lemma 2.20.** Let \( c \) be an external cube and let \( h = h(c), \bar{h} = \bar{h}(c) \). Let \( w \) be a completely external cube of \( h \), and let \( c' \in \text{Faces}'(c) \). Then \( \mathcal{S}(w) \cap c' \subseteq \mathcal{F}(c') \).

**Proof.** Since \( c' \) is external, Lemma 2.9 provides a nonempty persistent subcube \( h' = h(c') \) of \( c' \). As before, let \( \bar{h}' = \bar{h}(c') \) be the salient subcube of \( c' \). (Recall that \( \bar{h}' \) is separated from \( h' \) by those hyperplanes that do not cross \( h' \) and which have at least one dual internal 1-cube in \( c' \).

Note that \( h(c) \cap c' \subseteq h' \) and \( \bar{h}(c) \cap c' \subseteq \bar{h}' \). Note furthermore that \( h \cap c' \) is the projection of \( h \) to \( c' \). Also, each hyperplane \( D' \) in \( c' \) separating \( h, \bar{h}' \) extends to a hyperplane \( D \) of \( c \) separating \( h(c), \bar{h}(c) \). Suppose this was not the case, since \( D' \) is dual to an internal 1-cube, \( D \) must as well. It follows that if \( D \) fails to separate \( h, \bar{h} \) then it must \( D \) must intersect \( h \), but then its projection \( D' \) must intersect the \( h \cap c' \subseteq h' \), which is a contradiction.

Now suppose that \( h(c) \cap c' = \emptyset \). Let \( E \) be a hyperplane separating \( h(c) \) from \( c' \) (so, \( E \) is the unique hyperplane parallel to the codimension–1 face \( c' \) of \( c \)). Since \( E \) does not cross \( h(c) \), and \( h(c) \) is parallel to \( h(c) \), we see that \( E \) also does not cross \( h(c) \). Hence, if \( E \) is dual to an internal 1-cube, then by definition \( E \) separates \( h(c) \) from \( h(c) \), and thus \( h(c) \subseteq c' \). In this case, \( \mathcal{S}(w) \cap c' = \bar{w} \), the completely external subcube of \( h(c) \) diagonally opposite \( w \). On the other hand, \( \bar{w} \subseteq h(c) \cap c' \subseteq h' \subseteq \mathcal{F}(c') \), so the lemma holds in this case.

If \( E \) is not dual to an internal 1–cube, then \( h(c) \) and \( h(c) \) are not separated by \( E \), so \( h(c) \cap c' = \emptyset \). But \( h(c) \) contains all persistent corners of \( c \), so \( c' \) contains no persistent corners of \( c \). Since \( c \in \text{Faces}'(c) \), this is a contradiction.

Otherwise, if \( h(c) \cap c' \neq \emptyset \), then \( h(c) \cap c' \) and \( \bar{h}(c) \cap c' \) are diagonally opposite in \( c' \) in the above sense, and \( \mathcal{S}(w) \cap c' = \mathcal{S}(w \cap c') \subseteq \mathcal{F}(c') \). \( \square \)

### 2.3. Relationship between \( \mathcal{F}_1(c) \) and \( \mathcal{F}(c) \)

Our goal is to create a \( \text{CAT}(0) \) cube complex \( \Psi_* \) from \( \Psi \) by deformation retracting each external cube \( c \) to its fundament, by induction on dimension. In the case where \( \mathcal{D}(c) \) is disconnected, this requires an understanding of the relationship between \( \mathcal{F}(c) \) and \( \mathcal{F}_1(c) \). Lemma 2.21 explains this relationship. In particular, it gives a precise description of how \( \mathcal{F}(c) \) is constructed: it states that all new cubes added to \( \mathcal{F}_1(c) \) are completely external and contain the salient subcube \( \bar{h} \), which must itself be completely external in this case (in general, it can happen that \( \mathcal{F}(c) = \mathcal{F}_1(c) \) and \( \bar{h} \) is not completely external).

The main import of the lemma is the final statement about deformation retractions, which will be used in Lemma 2.23 below (which is where we construct the deformation retraction from \( c \) to \( \mathcal{F}(c) \)).

**Lemma 2.21** (\( \mathcal{F}_1(c) \) versus \( \mathcal{F}(c) \)). Suppose that \( c \) is external. Then \( \mathcal{D}(c) \subseteq \mathcal{F}(c) \). If, in addition, \( \mathcal{D}(c) \neq c \), then all of the following hold, where \( h = h(c) \) and let \( h = \bar{h}(c) \):

1. If \( f \) is a completely external cube of \( \mathcal{F}(c) \) that is not contained in \( \mathcal{F}_0(c) \), then \( f \) has codimension at least 2 in \( c \).
2. If \( f \) is a maximal completely external subcube of \( c \) lying in \( \mathcal{F}(c) \), then either \( f \subseteq \mathcal{F}_1(c) \) or \( f \) contains \( h \). In the latter case, \( h \) must be completely external.
3. If \( \mathcal{D}(c) \) is disconnected, then either \( \mathcal{F}_1(c) = \mathcal{F}(c) \), or \( \bar{h} \) is completely external and \( h \cup \bar{h} \) is contained in a codimension–1 face of \( c \). If \( \mathcal{D}(c) \) is connected, then \( h \cup \bar{h} \) is contained in a codimension–1 face of \( c \).
4. Let \( f_1, \ldots, f_k \) be the maximal completely external cubes of \( c \) that do not lie in \( \mathcal{F}_1(c) \). Then:
• $k \leq 1$.
• $f_1 = h \times s$, where $s$ is a subcube of $c$, none of whose hyperplanes is dual to an internal 1–cube of $c$.
• There exists a cube $h' \subset f_1$, parallel to $\bar{h}$, such that $h'$ is not contained in any $c' \in \text{Faces}'(c)$. Hence $\text{Int}(h')$ is disjoint from $\bigcup_{c' \in \text{Faces}'(c)} c'$. (Here, $\text{Int}(h')$ is understood to denote $h'$ if $\dim h' = 0$ and otherwise denotes the open cube of $c$ whose closure is $h'$.)

In particular, if $\mathcal{D}(c)$ is disconnected (i.e. $\mathcal{F}(c) \neq \mathcal{D}(c)$), then there is a strong deformation retraction from $\left( \bigcup_{c' \in \text{Faces}'(c)} c' \right) \cup S \cup \bigcup_i f_i$ to $\left( \bigcup_{c' \in \text{Faces}'(c)} c' \right) \cup S$, fixing each $c'$, where $S$ is the union of the $S(w)$ over the completely external cubes $w$ of $\bar{h}$.

Proof. By definition, $\mathcal{D}(c)$ is the union of all completely external cubes of $c$. Each such $f \subset c$ either lies in some element of $\text{Faces}'(c)$, and hence in $\mathcal{F}_1(c)$ (by the inductive definition), or is added to $\mathcal{F}_1(c)$ when constructing $\mathcal{F}(c)$. This proves $\mathcal{D}(c) \subseteq \mathcal{F}(c)$ (by construction, the containment is an equality if and only if $\mathcal{D}(c)$ is connected).

Now suppose $\mathcal{D}(c) \neq c$.

Assertion (1): Suppose that the completely external subcube $f$ of $c$ is not contained in any $c' \in \text{Faces}'(c)$. If $f$ is codimension–1, this amounts to saying that $f$ has no persistent corner of $c$. Let $E$ be the hyperplane of $c$ not crossing $f$. Since $f$ is completely external, each of its 0–cubes is a persistent corner of $f$, so each 1–cube dual to $E$ is internal (else $f$ would contain a persistent corner of $c$). Hence $c$ is internal, a contradiction.

Assertion (2): Assume that $f$ is a maximal completely external subcube of $c$ that is not contained in $\mathcal{F}_1(c)$.

We need some preparatory claims:

Claim 2. The cube $h$ is parallel to a sub-cube of $f$, and any hyperplane $H$ with $H \cap f = \emptyset$ must separate $f$ from $h$.

Proof of Claim 2. Let $H$ be a hyperplane crossing $h$, so that by the definition of $h$, there are persistent corners $v, v'$ separated by $H$. Let $k, k'$ be codimension–1 faces of $c$, containing $v, v'$ respectively, with $k, k'$ parallel to $H$. On the one hand $k, k'$ contain persistent corners, and therefore lie in $\text{Faces}'(c)$. On the other hand if $H \cap f = \emptyset$, then $f$ must be a subcube of either $k$ or $k'$ contradicting $f \notin \mathcal{F}_1(c)$. Thus $H \cap f \neq \emptyset$. This shows that, for all hyperplanes $H$, if $H \cap f = \emptyset$, then $H \cap h = \emptyset$. Since every hyperplane crossing $h$ crosses $f$, it follows that $h$ is parallel to a sub-cube $h'$ of $f$.

Moreover, if $H \cap f = \emptyset$, and $h, f$ are not separated by $H$, then $f$ lies in a codimension–1 face of $c$ containing a persistent corner. Hence $f \subset \mathcal{F}_0(c) \subseteq \mathcal{F}_1(c)$, a contradiction. Thus, if $H \cap f = \emptyset$, then $H$ separates $f$ from $h$. ■

Claim 3. Let $H$ be a hyperplane of $c$ such that $H \cap f = \emptyset$ and $H \cap h = \emptyset$. Then $H$ is dual to an internal 1–cube of $c$. Hence $H$ separates $h$ from $\bar{h}$.

Proof of Claim 3. The second conclusion follows from the first, by the definition of $\bar{h}$, so it suffices to prove the first. Suppose that every 1–cube dual to $H$ is external. By Claim 2, $H$ separates $f$ and $h$.

Fix a 1–cube $e$ dual to $H$ and incident to a 0–cube of $f$. Since $e$ is not in $f$, there is a subcube $e \times f$ of $c$, properly containing $f$. (Since $H \cap f = \emptyset$, the dual 1–cube $e$ cannot be a 1–cube of $f$.) Let $k, k'$ be the codimension–1 faces of $c$ parallel to $H$, with $h \subseteq k$ and $f \subseteq k'$. Let $e'$ be a 1–cube of $e \times f$.

If $e' \subset f \times e$ is parallel to $e$, then we can assume that $e'$ is external since it is dual to $H$. Otherwise, if $e' \subset (f \times e) \cap f = f \cap k'$, then $e'$ is external since $f$ is completely external.
The remaining case is where \( e' \subset k \cap (f \times e) \), so \( e' \) is parallel to a 1–cube \( \bar{e}' \) of \( f \). We will choose a specific such \( \bar{e}' \) momentarily.

First, let \( e'' \) be a 1–cube parallel to \( e' \) that contains a 0–cube of \( h \). By Lemma 2.11, \( e'' \) contains a persistent corner and is therefore external.

Now choose \( \bar{e}' \) as above (so, \( e' \) is parallel to \( e' \) and \( e'' \) and lies in \( f \)) such that the distance from \( e'' \) to \( \bar{e}' \) is maximal over all possible such choices.

We saw already that \( e'' \) is external. Since \( f \) is completely external and \( \bar{e}' \) is contained in \( f \), we also have that \( \bar{e}' \) is external.

We claim that \( e'' \) and \( \bar{e}' \) are not contained in a common codimension–1 face of \( c \). Suppose to the contrary that they are. Let \( D \) be the hyperplane parallel to the codimension–1 face containing \( e'', \bar{e}' \). Now, if \( D \cap f = \emptyset \), then \( D \cap h = \emptyset \). Hence, since \( e'' \) contains a 0–cube of \( h \), and \( \bar{e}' \) lies in \( f \), and \( e'' \) and \( \bar{e}' \) lie on the same side of \( D \), we see that \( D \) does not separate \( h \) from \( f \), contradicting Claim 2. Thus \( D \) crosses \( f \). But then \( f \) contains a parallel copy of \( \bar{e}' \) that is separated from \( \bar{e}' \) by \( D \), and no other hyperplane; this parallel copy is thus further from \( e'' \) than \( \bar{e}' \) is, contradicting how \( \bar{e}' \) was chosen. Thus \( e'' \) and \( \bar{e}' \) do not lie in a common codimension–1 face.

Hence, by Lemma 2.1, every 1–cube parallel to \( \bar{e}' \), and in particular \( e' \), is external (see Remark 2.7). Thus \( e \times f \) is a completely external cube, contradicting maximality of \( f \). We conclude that every hyperplane \( H \) separating \( f, h \) is dual to an internal 1–cube.

We can now conclude the proof of assertion (2). Recall that we have assumed that \( h \) is not contained in \( \mathcal{F}_1(c) \). Note that \( \bar{h} \) is contained in \( f \) if and only if, for each hyperplane \( H \), if \( H \) and \( f \) are disjoint, then \( \bar{h} \) and \( f \) are on the same side of \( H \). Indeed, obviously \( \bar{h} \subseteq f \) implies that no hyperplane separates \( \bar{h}, f \). Conversely, suppose that no hyperplane separates \( h \) from \( f \). Then \( \bar{h} \cap f = \emptyset \), so \( \bar{h} \cap f = \bar{h} \) since \( \bar{h} \) is parallel to \( h \) and hence parallel to a subcube of \( f \), by Claim 2.

Now, by Claim 2, if \( H \) is a hyperplane disjoint from \( f \), then \( H \) must separate \( f \) from \( h \). By Claim 3, \( H \) separates \( h \) from \( f \). Hence \( f, h \) are on the same side of \( H \) (namely, the halfspace not containing \( h \)). Thus \( \bar{h} \subseteq f \). Since \( f \) is completely external by hypothesis, the same is true of \( \bar{h} \). This proves assertion (2).

**Assertion (4):** Now let \( f_1, \ldots, f_k \) be the maximal completely external cubes of \( c \) that do not lie in \( \mathcal{F}_1(c) \). If \( k = 0 \), there is nothing to prove, so suppose \( k \geq 1 \). By assertion (2), \( \bar{h} \) is completely external and \( \bar{h} \subseteq f_i \) for all \( i \).

**Claim 4.** If \( \mathcal{D}(c) \) is connected and \( \mathcal{D}(c) \neq c \), then \( c \) has a codimension–1 face \( l \) containing \( h \cup \bar{h} \).

**Proof of Claim 4.** Let \( H \) be a hyperplane. Suppose that \( H \cap h = \emptyset \), so \( H \cap \bar{h} = \emptyset \). If \( H \) is not dual to an internal 1–cube, then \( H \) does not separate \( h \) from \( \bar{h} \), and thus \( h \cup \bar{h} \) lie in a common codimension–1 face. Hence we can assume that for each \( H \), either \( H \) intersects \( h \) (and thus \( \bar{h} \)) or \( H \) is dual to some internal 1–cube.

We can assume there is at least one hyperplane \( H \) disjoint from \( h \) (and thus dual to an internal 1–cube). Otherwise, by the above, every hyperplane intersects \( h \), so \( \mathcal{D}(c) = c \) and \( h = \bar{h} \).

We now show that the above two assumptions imply that \( \mathcal{D}(c) \) is disconnected, yielding a contradiction.

Let \( v \in \bar{h} \) be a 0–cube. Then there is a unique 0–cube \( v' \) such that every hyperplane of \( c \) separates \( v \) from \( v' \). We claim that \( v' \) is a persistent corner.

Let \( w \) be the closest 0–cube (in the graph metric on \( c^{(1)} \)) of \( h \) to \( v \). The 0–cube \( w \) is characterised by the property that a hyperplane \( H \) separates \( v \) from \( w \) if and only if \( H \) separates \( v \) from \( h \). So, the hyperplanes separating \( v \) from \( v' \) fall into two categories: those that separate \( v \) from \( h \), and those that separate \( w \) from \( v' \) but do not separate \( v \) from \( h \).
Since $w \in h$, and each hyperplane separating $w$ from $v'$ crosses $h$, we thus have $v' \in h$. By Lemma 2.11, $v'$ is a persistent corner, as required.

Let $e$ be a 1–cube that contains $v$ and does not lie in $\bar{h}$. Let $e'$ be the parallel 1–cube containing $v'$. Since every hyperplane not dual to $e$ separates $e, e'$, and $e'$ is external (since $v'$ is a persistent corner), if $e$ is external then so is every 1–cube parallel to $e$, by Lemma 2.1 (see Remark 2.7). Hence, since $e$ is dual to a hyperplane separating $h, \bar{h}$, and this hyperplane must be dual to an internal 1–cube, $e$ is internal.

Now, any edge-path $\sigma$ from $h$ to $\bar{h}$ must pass through a 1–cube that contains some $v \in \bar{h}$ and does not lie in $\bar{h}$. We saw above that such a 1–cube must be internal. Hence $\sigma$ is not in $D(c)$. So, since $h, \bar{h} \subset D(c), D(c)$ is disconnected, as required.

In view of the preceding claim, assume that $D(c)$ is disconnected. Then $\mathcal{F}_1(c)$ contains $S(h)$ since $\bar{h}$ is completely external. So, $f_1$ must properly contain $h$, or otherwise we would have $f_1 \subset \mathcal{F}_1(c)$. Hence, since each $f_i$ is a maximal completely external cube, $f_i \not\subset f_1$ and hence $\bar{h} \subset f_1$.

Now, if some hyperplane $E$ satisfies $E \cap f_i = \emptyset$ and does not separate $h, f_i$, then $h, f_i$, and hence $h, \bar{h}$, are on the same side of $E$, and we are done. Hence we can assume that every hyperplane either crosses $f_i$ or separates $h, f_i$.

**Claim 5.** Let $D_i$ be a hyperplane crossing $f_i$ but disjoint from $\bar{h}$. Then every 1–cube dual to $D_i$ is external.

**Proof of Claim 5.** Let $v$ be a persistent corner, and let $v'$ be the 0–cube separated from $v$ by all hyperplanes. Since every hyperplane of $c$ either separates $h, f_i$ or crosses $f_i$, we have $v' \in f_i$. Now, if $e'$ is a 1–cube containing $v'$ and $e$ is parallel to $e'$ and containing $v$, then $e, e'$ are not contained in a common codimension–1 face of $c$. Choose $e, e'$ to be dual to $D_i$. Now, $e$ is external since $v$ is a persistent corner. On the other hand, $e' \subset f_i$, so $e'$ is external. Thus $e, e'$ are external 1–cubes dual to $D_i$ that do not lie in a common codimension–1 face of $c$. Hence, by Lemma 2.1 (via Remark 2.7), every 1–cube dual to $D_i$ is external.

From Claim 5, it follows that $h$ and $\bar{h}$ lie on the same side of $D_i$, whence there is a codimension–1 face containing $h \cup \bar{h}$. Thus, whether or not $D(c)$ is connected, there is a codimension–1 face $l$ containing $h \cup \bar{h}$ provided $k \geq 1$. Let $l'$ be the smallest subcube of $c$ containing $h \cup \bar{h}$, i.e. $l'$ is the intersection of all codimension–1 faces $l$ as above.

Hence $c = l' \times s$, where $s$ is a cube with the property that, for every hyperplane $D$ crossing $s$, all 1–cubes dual to $D$ are external. Indeed, any hyperplane dual to an internal 1–cube either separates $h, \bar{h}$ or crosses $h$, and thus any such hyperplane crosses $l'$.

Fix $i$. Since $f_i \cap l'$ is a subcube of $f_i$, we can write $f_i = (f_i \cap l') \times s'$, where $s'$ is some cube intersecting $f_i \cap l'$ in a single 0–cube. Since the image of $f_i$ under the canonical projection $c \to l'$ is $f_i \cap l'$, $s'$ has trivial projection to $l'$, i.e. $s'$ is a subcube of $s$.

Now, suppose that $s''$ is a proper subcube of $s$, so that $s = s' \times s''$ for some nontrivial cube $s''$. Let $E$ be a hyperplane crossing $s''$. Then $E$ does not separate $h, \bar{h}$, since $E$ does not cross $l'$. Since $E$ does not cross $f_i$, we thus have that $f_i$ lies on the side of $E$ containing $h$, and hence there is a codimension–1 face, parallel to $E$, that contains $f_i$ and $h$. Hence $f_i$ is contained in an element of $\text{Faces}^i(c)$, a contradiction. We conclude that $f_i = (f_i \cap l') \times s$.

For each $i$, we have $f_i \cap l' = h$. Indeed, $h \subset f_i \cap l'$ by our earlier discussion. On the other hand, any hyperplane crossing $l'$ but not $h$ must be dual to an internal 1–cube, while by Claim 5, every hyperplane crossing $f_i$ but not $h$ fails to be dual to an internal 1–cube. Hence every hyperplane crossing $f_i \cap l'$ crosses $h$, so $f_i \cap l' = h$. Thus, for all $i$, $f_i = (f_i \cap l') \times s = h \times s$. So, $k = 1$, and we have a single “extra” completely external cube $f_1$ that is maximal with the property that it is not contained in any element of $\text{Faces}^i(c)$. 
Let $\tilde{h}'$ be the parallel copy of $\tilde{h}$ in $f_1$ that is separated from $\tilde{h}$ by precisely those hyperplanes crossing $s$. Let $c'$ be a codimension–1 face of $c$ containing $\tilde{h}'$. Let $E$ be the hyperplane parallel to $c'$. If $E$ is dual to a 1–cube of $l'$, and $E$ separates $h$ from $\tilde{h}$, then $E$ separates $h$ from $\tilde{h}'$ and, in particular $E$ separates $c'$ from $h$, so $c' \not\in \text{Faces}'(c)$. If $E$ is dual to a 1–cube of $s$, then $E$ cannot separate $h, \tilde{h}$, since all 1–cubes dual to $E$ are external. On the other hand, $E$ separates $\tilde{h}'$ from $h$, and hence from $h$. Thus $E$, as above, separates $h$ from $c'$, so $c'$ is not in $\text{Faces}'(c)$. Otherwise, if $E$ crosses $h$, then $E$ crosses $h$ and hence crosses $h'$. Thus $\tilde{h}' \not\subset c'$.

Thus $\tilde{h}' \subset \bigcap_1 f_i = f_1$ and $\tilde{h}' \not\subset c'$ for any $c' \in \text{Faces}'(c)$. Let $\text{Int}(\tilde{h}')$ be the open cube whose closure is $\tilde{h}'$ (or, if $\tilde{h}'$ is a 0–cube, let $\text{Int}(\tilde{h}') = \tilde{h}'$). If $\text{Int}(\tilde{h}')$ is contained in $\bigcup_{c' \in \text{Faces}'(c)} c'$, then $\text{Int}(\tilde{h}')$, being an open cube or 0–cube, would have to lie in some such $c'$, a contradiction. This completes the proof of assertion (4).

**Assertion (3):** Each part of the assertion was proved above.

The deformation retraction: If $k = 0$, then there is nothing to prove, so suppose that $k \geq 1$, and thus, by the above discussion, $k = 1$. It follows from the above discussion that $\mathcal{S} = \mathcal{S}(h) \subset l'$. We thus have $\mathcal{S} \subset \bigcup_{c' \in \text{Faces}'(c)} c'$. So, we just need to exhibit a deformation retraction of $\left(\bigcup_{c' \in \text{Faces}'(c)} c'\right) \cup f_1$ to $\left(\bigcup_{c' \in \text{Faces}'(c)} c'\right)$.

Let $\bar{h}'$ be the cube from assertion (4). For convenience, let $X = \bigcup_{c' \in \text{Faces}'(c)} c'$ and let $Y = f_1$. So, by assertion (4), $Y = \bar{h}' \times s$.

Clearly $\bar{h}' \times s$ is the unique maximal cube of $Y$ containing $\bar{h}'$. If $\bar{h}'$ lies in some other maximal cube of $X \cup Y$, then $\bar{h}' \subset X$. Since $\text{Int}(\bar{h}') \cap X = \emptyset$, this is impossible. Hence $\bar{h}'$ is contained in a unique maximal cube of $X \cup Y$, i.e. $\bar{h}'$ is a free face of $X \cup Y$. We will perform the standard collapse of this free face to get a deformation retraction onto the space obtained by deleting all open cells whose closures contain the interior of $\bar{h}'$.

Let $y$ be a (closed) cube of $Y$ containing $\bar{h}'$. Then $y$ cannot be contained in a cube of $X$, because such a cube would contain $\text{Int}(\bar{h}') \subset y$. Hence the unique maximal cube of $X \cup Y$ containing $y$ is $\bar{h}' \times s$.

Let $Y_1$ be the subcomplex of $Y$ obtained by removing $\text{Int}(\bar{h}')$ and $\text{Int}(y)$ for every cube $y$ of $Y$ that contains $\bar{h}'$. Passing from $X \cup Y$ to $X \cup Y_1$ amounts to collapsing the free face $\bar{h}'$ of $X \cup Y$, so there is a deformation retraction $X \cup Y \rightarrow X \cup Y$ (fixing $X$) of $X \cup Y$ onto $X \cup Y_1$.

We now collapse $X \cup Y_1$ to $X$ as follows. Let $d$ be a subcube of $\bar{h}'$. Then $\text{Int}(d) \subset X$ if and only if $\text{Int}(d) \times s \subset X$. Indeed, if $\text{Int}(d)$ lies in a codimension–1 face $c'$ containing a persistent corner, then $c'$ is parallel to some hyperplane $E$ crossing $\bar{h}'$ (otherwise $\text{Int}(\bar{h}')$ would lie in $c'$) and hence not crossing $s$. So, $\text{Int}(d) \times s \subset c'$.

Now, for each maximal subcube $d$ of $\bar{h}'$ that lies in $Y_1$ but not in $X$, we have that $d$ is a free face of $Y_1$, contained in the unique maximal cube $d \times s$, and we can now collapse these cubes to get $X \cup Y_2$. Repeating this finitely many times we eventually obtain the subcomplex $X$, as required.

### 2.4. CAT(0)ness and compatible collapse for the $\mathcal{F}(c)$.

We now describe how to deformation retract $c$ to $\mathcal{F}(c)$ compatibly with the corresponding deformation retractions of the other cubes.

**Lemma 2.22.** Let $c$ be an external cube for which $\mathcal{D}(c)$ is connected and let $f$ be a completely external proper subcube of $c$. Then $f$ lies in a codimension–1 external subcube of $c$.

**Proof.** If $c$ is completely external, the claim is obvious, so suppose there is an internal 1–cube. If $f$ is a codimension–1 subcube, then we’re done.

As before, let $h = h(c)$ be persistent subcube of $c$, and let $\bar{h} = \bar{h}(c)$ be salient subcube of $c$. (It is possible that $h = \bar{h}$.)
Suppose that $f$ is a maximal completely external cube and has codimension at least 2, and that $f$ does not lie in any codimension–1 external sub-cube. By Lemma 2.21, we have that $\bar{h} \subset f$ and $h \cup \bar{h} \subset l$ for some external codimension–1 face $l$ containing a persistent corner.

Let $l$ be as above. Since $\bar{h} \subseteq f$ is completely external, $\bar{h} \subset D(l)$. If $f \subseteq l$, we have contradicted that $f$ does not lie in any codimension–1 external face. Hence, since $f \cap l \neq \emptyset$ (it contains $\bar{h}$), we have that $f \cap l$ is a codimension–1 face of $f$.

Now, since $D(c)$ is connected and $l$ is external, Lemma 2.13 implies that $D(l)$ is connected.

Let $D$ be the unique hyperplane of $c$ not crossing $l$. Since $f$ intersects $l$ but does not lie in $l$, we have that $f$ intersects $D$. Now, since $D$ does not separate $h, \bar{h}$, and does not cross $h$ or $\bar{h}$, every 1–cube dual to $D$ is external. By induction on dimension and connectedness of $D(l)$, the completely external cube $f \cap l$ is contained in an external codimension–1 face $t$ of $l$; recall that $t$ contains a persistent corner $v$ of $l$. Let $e$ be the 1–cube of $c$ dual to $D$ and emanating from $v$. Then $e$ is external, so $t \times e$ is a codimension–1 face of $c$ with a persistent corner, $v$, and $f = (f \cap l) \times e$ lies in $t \times e$.

\begin{lemma}
Let $c$ be an external cube. Then there is a strong deformation retraction $\Delta_c : c \times [0,1] \rightarrow c$ so that $\Delta_c(\cdot,0)$ is the identity and $\Delta_c(\cdot,1)$ is a retraction $c \rightarrow F(c)$.

Moreover, if $c'$ is a sub-cube of $c$ which is external and has codimension $\ell$, then the restriction of $\Delta_c$ to $c' \times [0,1]$ coincides with the identity on $c' \times [0,1-\frac{1}{2\ell}]$ and, on $c' \times [1-\frac{1}{2\ell},1]$, restricts to the map given by $(x,t) \mapsto \Delta_{c'}(x,\frac{t}{2\ell}-\frac{1}{2})$.
\end{lemma}

\begin{proof}
Let $\text{Faces}'(c), h = h(c), \tilde{h} = \tilde{h}(c), \{\mathcal{S}(w)\}$ and $\mathcal{F}(c)$ be as in the definition of $\mathcal{F}(c)$.

We will argue by induction on $d = \dim c$. In the base case, $d \leq 1$, then $c$ is external only if $c$ is completely external, in which case $c = \mathcal{F}(c)$ and we are done. More generally, whenever $c$ is completely external, we take $\Delta_c : c \times [0,1] \rightarrow c$ to be projection onto the $c$ factor. Lemma 2.19 implies that $\mathcal{F}(c') = c'$ for each face $c'$ of $c$, whence the “moreover” statement also holds.

Hence suppose that the lemma holds for external cubes of dimension $\leq d - 1$. The inductive step has two parts.

\textbf{The first collapse:} Suppose that $c$ is external but not completely external (since the completely external case was handled above).

Recall that $\text{Faces}'(c)$ denotes the set of external codimension–1 faces $c'$ of $c$ such that $c$ contains a persistent corner. By Lemma 2.19, $\text{Faces}'(c)$ is exactly the set of all codimension–1 external sub-cubes of $c$. Let $\mathcal{G}'(c)$ be the union of the elements of $\text{Faces}'(c)$.

Now, $\text{Faces}'(c)$ is a proper subset of the set of codimension–1 faces of $c$. Otherwise, Lemma 2.1 would imply that $c$ is completely external, a contradiction. On the other hand, $\mathcal{G}'(c)$ is connected, since $\text{Faces}'(c)$ contains a set $c_1', \ldots, c_d'$ of pairwise-intersecting codimension–1 sub-cubes with one in each parallelism class. Note that $\bigcup_{i=1}^{d} c_i'$ is contractible.

For each $i \leq d$, let $c_i'$ be the codimension–1 face parallel to $c_i'$. Let $I$ be the set of $i \leq d$ for which $c_i' \in \text{Faces}'(c)$. Without loss of generality, $I = \{1, \ldots, \ell\}$ for some $\ell < d$. Hence $\mathcal{G}'(c)$ is obtained from $c$ by removing the interior of $c$ together with each open cube that is not contained in $\bigcup_{i=1}^{d} c_i' \cup \bigcup_{i=1}^{\ell} c_i'$. The cubes $\tilde{c}_i, i > \ell$ all contain $\bar{h}$, and $\mathcal{G}'(c)$ is obtained by removing $\text{Int}(c)$ together with a nonempty, connected subspace of the star of $\bar{h}$ which is obtained from a subcomplex of that star by removing its boundary. Hence $\mathcal{G}'(c)$ is contractible.

Let $\mathcal{S}(c)$ be the following subspace of $c$. First, if $D(c)$ is connected, let $\mathcal{S}(c) = \emptyset$. Otherwise, let $h, \bar{h} \subset c$ be the sub-cubes defined above, so that $\mathcal{F}_1(c)$ is obtained by adding $\mathcal{S}(w)$ to $\mathcal{F}_0(c)$ for each completely external sub-cube $w$ of $\bar{h}$. Let $S_1, \ldots, S_t$ be the set of all such $\mathcal{S}(w)$, and let $\mathcal{S}(c) = \bigcup_{i=0}^{t} \mathcal{S}_i$. Let $\mathcal{G}(c) = \mathcal{G}'(c) \cup \mathcal{S}(c)$.
We claim that $G(c)$ is contractible. First, note that $S(c) \cup h$ is contractible and that $h \subset G'(c)$. To show that $G(c) = G'(c) \cup S(c)$ is contractible, it thus suffices to show that $G'(c) \cap (S(c) \cup h)$ is contractible.

For each $d' \in \text{Faces}'(c)$, let $V(d')$ be the (nonempty) set of 0–cubes, each of whose incident 1–cubes in $d'$ is external, and let $h(d')$ be the convex hull of the elements of $V(d')$. In other words, $h(d')$ is the persistent subcube of $d'$ and $V(d')$ is the 0–skeleton of $h(d')$, by Lemma 2.11.

Let $h(d')$ be the cube of $d'$ that is parallel to $h(d')$ and separated from $h(d')$ by exactly those hyperplanes of $d'$ that do not cross $h(d')$ but cross external 1–cubes. Note that $h \cap d' \subseteq h(d')$. Indeed, each 0–cube of $d'$, all of whose incident 1–cubes in $c$ is external, certainly belongs to $V(d')$, and the convex hull of all such 0–cubes is $h \cap d'$.

1. For any codimension–1 face $d'$ of $c$, if $h \subset d'$, then $d'$ is external. Moreover, either $(S(c) \cup h) \cap d' = h$ or $S(c) \cup h \subset d'$. Indeed, if the hyperplane of $c$ not crossing $d'$ does not abut a panel, then the latter holds because $h$ and $h$ are both on the same side of this hyperplane in $d'$; otherwise, the former holds.

2. If $d'$ is a codimension–1 face and $h \cap d' \neq \emptyset$ but $h \not\subset d'$, then again $d'$ is external, and $(S(c) \cup h) \cap d' = S(c) \cap d'$. For each completely external cube $w$ of $h$, we have that $S(w) \cap d'$ is a cube in $S(c)$, where $S(w) \cap d'$ is regarded as a cube (using its product structure).

3. If $h \subseteq d'$, then either $h \subset d'$ or $h \cap d' = \emptyset$, in which case $d'$ is not external. Indeed, if the hyperplane $H$ parallel to $d'$ separates $h$, $h$, then it separates $d'$ from $h$, in which case $d'$ is not external since external faces contain persistent corners.

From the first two statements, any codimension–1 face $d'$ of $c$ intersecting $S(c) \cup h$ is either external or disjoint from $h$. In the latter case, $d'$ must contain $h$, so the third statement implies that $d' \not\subset \text{Faces}'(c)$. It follows that $(S(c) \cup h) \cap G'(c)$ is contractible: just collapse $h$ to a point and observe that $(S(c) \cup h) \cap G'(c)$ becomes the cone on the union of some of the completely external cubes of $h$.

Now, $G(c)$ is a contractible subcomplex of a finite subdivision of $c$, which is a contractible CW complex. Let $G''(c)$ consist of $G(c)$, together with any completely external subcube $F$ of $c$ that is not contained in any element of $\text{Faces}'(c)$. By Lemma 2.21, $G(c)$ is a strong deformation retract of $G''(c)$, so $G''(c)$ is contractible.

Hence $G''(c)$ is a strong deformation retract of $c$, by Whitehead’s theorem [39, 40]. Let $\omega_c : c \times [0,1] \rightarrow c$ be a strong deformation retraction from $c$ to $G''(c)$.

The second collapse: By induction on $d$, for each $d' \in \text{Faces}'(c)$, we have strong deformation retractions $\Delta_{d'} : d' \times [0,1] \rightarrow d'$ with $\Delta_{d'}(-,1)$ a retraction $d' \rightarrow F(d')$.

Let $d', d'' \in \text{Faces}'(c)$. Then $d' \cap d''$ is an external proper sub-cube of $d'$ and $d''$, so by induction on dimension, we have

$$\Delta_{d'}|_{(d' \cap d'') \times \{1\}} = \Delta_{d''}|_{(d' \cap d'') \times \{1\}} = \Delta_{d' \cap d''}(-,1).$$

Pasting thus provides a deformation retraction $\alpha_c : G'(c) \times [0,1] \rightarrow G'(c)$ so that $\alpha_c(-,1)$ is a retraction $G'(c) \rightarrow \bigcup_{d' \in \text{Faces}'(c)} F(d')$. We now show how to extend $\alpha_c$ from a deformation retraction collapsing $G(c)$ to one collapsing $G''(c)$.

If $D(c)$ is connected (so that $F(c) = D(c)$), then by Lemma 2.13, we have $F(c') = D(c') = d' \cap F(c)$ for each $d' \in \text{Faces}'(c)$. We also have $\bigcup_{d' \in \text{Faces}'(c)} F(d') = \bigcup_{d' \in \text{Faces}'(c)} D(c') = D(c) = F(c)$, indeed, if $f$ is a completely external cube of $c$, then $f$ is contained in some codimension–1 external cube, by Lemma 2.22. Hence, in the connected case, $\alpha_c$ is a deformation retraction of $G(c) = G'(c) = G''(c)$ onto $F(c)$.
If \( D(c) \) is disconnected, then \( S(c) \neq \emptyset \). Indeed, the construction of \( F(c) \) ensures that \( F(c) \) contains some \( S(w) \) whenever \( D(c) \) is disconnected (Remark 2.16) and \( S(w) \) is contained in \( S(c) \) by definition.

Now, for each \( S_i \), and each \( c' \in \text{Faces}'(c) \), we have \( S_i \cap c' \subseteq F(c') \), by Lemma 2.20. Hence we can extend the deformation retraction \( \alpha_c \) to a strong deformation retraction \( \alpha_{c'} : [\mathcal{G}'(c) \cup S(c)] \times [0, 1] \rightarrow \mathcal{G}(c) \) with \( \alpha_{c'}(-, 1) \) a retraction \( \mathcal{G}(c) \rightarrow F_1(c) \). Indeed, we just extend each \( \alpha_c(-, t) \) over each \( S_i \cap (c - \mathcal{G}'(c)) \) by declaring it to be the identity. Lemma 2.21 ensures that \( \mathcal{G}''(c) \) differs from \( \mathcal{G}'(c) \) by adding a (possibly empty) set of cubes \( f \) that intersect \( \mathcal{G}'(c) \) in cubes of the various \( F(c'), c' \in \text{Faces}(c) \). Hence we can extend \( \alpha_c \) to a strong deformation retraction \( \alpha_{c'} : \mathcal{G}''(c) \times [0, 1] \rightarrow \mathcal{G}''(c) \) by declaring it to be the identity on each new cube \( f \); this gives the desired deformation retraction from \( \mathcal{G}''(c) \) onto \( F(c) \).

**The composition:** Now let \( \Delta_c : c \times [0, 1] \rightarrow c \) be given by

\[
\Delta_c(x, t) = \begin{cases} 
\omega_c(x, 2t), & t \in [0, \frac{1}{2}] \\
\alpha_c(\omega_c(x, 1), 2(t - \frac{1}{2})), & t \in [\frac{1}{2}, 1].
\end{cases}
\]

This gives the desired strong deformation retraction.

**Compatibility:** Let \( c' \subset c \) be an external sub-cube of \( c \). To prove the “moreover” statement, it suffices to handle the case where \( c' \) is codimension-1; the other cases follow by induction on dimension. We saw above that externality of \( c' \) implies that \( c' \in \text{Faces}'(c) \). Hence, by definition, \( \Delta_c \) restricts on \( c' \times [0, \frac{1}{2}] \) to the identity and on \( c' \times [\frac{1}{2}, 1] \) to the map given by \( (x, t) \mapsto \Delta_c(x, t - \frac{1}{2}) \), as required.

**Corollary 2.24.** For each external cube \( c \), the subspace \( F(c) \) is contractible.

**Proof.** By Lemma 2.23, \( F(c) \) is a deformation retract of \( c \) and is thus contractible.

Our goal is to prove that replacing each external cube \( c \) by \( F(c) \) creates a new CAT(0) cube complex. First, we need:

**Lemma 2.25.** Let \( c \) be an external cube. Then \( F(c) \) is a cube complex whose hyperplanes are components of intersection of \( F(c) \) with hyperplanes of \( c \).

**Proof.** If \( c \) is completely external, then the conclusion is obvious, we therefore assume that \( F(c) \neq c \). The cubes of \( F(c) \) are of two types: sub-cubes of \( c \) belonging to \( F(c) \), or cubes of the form \( S(w) \), where \( w \) is a completely external cube of \( c \) (recall definition of \( F_1(c) \)). To see that this set of cubes makes \( F(c) \) a cube complex, we must check that if \( d, d' \) are cubes of the above two types, then \( d \cap d' \) is a cube of one of the above two types.

For each \( d \) as above, we define an associated sub-cube \( f \) of \( c \) as follows. If \( d \) is a sub-cube of \( c \), then \( d \) is completely external, by Lemma 2.17. In this case, define \( f = d \). Otherwise, if \( d = S(w) \) for some \( w \), let \( f \) be the smallest sub-cube of \( c \) containing the two parallel copies of \( w \) of whose union \( S(w) \) is the \( \ell_2 \) convex hull. Again, \( f \) is external because, by the definition of \( F(c) \), the subspace \( S(w) \) contains a persistent corner of \( c \).

Define \( f' \) analogously for \( d' \). Then \( d \) is a cube of \( F(f) \) and \( d' \) is a cube of \( F(f') \), by Lemma 2.19. First consider the case where \( f = f' \):

**Claim 6.** If \( f = f' \), then \( d = d' \).

**Proof of Claim 6.** If \( d, d' \) are both sub-cubes of \( c \), this is clear. Otherwise, suppose \( d = S(w) \).

This means that, letting \( h, \tilde{h} \subseteq c \) be the persistent, salient subcubes of \( c \) respectively, \( w \) is a completely external cube of \( h, \tilde{w} \subseteq h \) is the parallel copy of \( w \) separated from \( w \) by exactly those abutting hyperplanes of \( c \) not crossing \( h \), and \( S(w) \) is the \( \ell_2 \)-convex hull of \( w \cup \tilde{w} \), and \( f \) is the \( \ell_1 \)-convex hull of \( w \cup \tilde{w} \). Now, \( f \cap h = \tilde{w} \) and \( f \cap \tilde{h} = w \). Indeed, on the one hand, the containment \( \tilde{w} \subseteq f \cap h \) is obvious. On the other hand if \( f \cap h \supseteq \tilde{w} \), then the intersection...
must contain a 1-cube $e$ not in $\bar{w}$ but in $h$. By definition $e$ is dual to a hyperplane not separating $w,\bar{w}$, which means it can’t be on an $\ell_1$ geodesic connecting $w$ and $\bar{w}$.

Thus, any “diagonal” cube $S(w')$ intersecting $f$ has $S(w') \cap f \subseteq S(w)$. So, if $d'=S(w')$ for some $w'$, then $d' \subseteq d$ and, by symmetry, $d \subseteq d'$, i.e. $d = d'$.

The remaining case is where $f = f' = d'$ and $d = S(w)$. By Lemma 2.20 $F(f) = f \cap F(c)$, but since $d'$ is assumed to be completely external $F(f) = f$. We now have that $w \cup \bar{w}$ are not contained in a codimension–1 face of $f$. Since $\bar{w}$ is a subcube of the persistent subcube $h(f)$ and $w$ is a subcube of the salient subcube $h(f)$, it follows by Lemma 2.21, assertion (3), that $D(f)$ is disconnected – contradicting that $f$ is completely external. ■

Next, consider the case where $f \neq f'$.

Claim 7. Let $c$ be an external cube. Let $c''$ be an external sub-cube of $c$. Let $d$ be a cube of $F(c)$. Then $d \cap c''$ is either empty or a cube of $F(c'')$. Hence $d \cap F(c'')$ is either empty or a cube of $F(c'')$.

Proof of Claim 7. Suppose that $d \cap c'' \neq \emptyset$. Then $F(c'') = F(c) \cap c''$ by Lemma 2.19. Hence the cubes of $F(c'')$ are the cubes of $F(c)$ that lie in $c''$. Thus it suffices to check that $d \cap c''$ is a subcube of $d$.

If $d$ is completely external, then $d \cap c''$ is a completely external cube of $c''$, and is thus a cube of $D(c'') \subset F(c'')$, as required.

Next, suppose that $d = S(w)$, where $w$ is some completely external cube of some salient cube $h$. Let $\bar{w}$ be the cube of the corresponding persistent cube $h$ with the property that $S(w)$ is the $\ell_2$–convex hull of $w \cup \bar{w}$. If $w,\bar{w}$ both intersect $c''$, then $S(w \cap c'') = S(w) \cap c'' = d \cap c''$ is a subcube of $d$. Otherwise, $c''$ has nonempty intersection with only one of $w,\bar{w}$. Without loss of generality $c'' \cap w \neq \emptyset$ and $c'' \cap \bar{w} = \emptyset$. Since $c''$ is a subcube of $c$, by definition of $S(w)$, we have $S(w) \cap c'' = w \cap c''$, which is again a subcube of $d$. This proves the claim. ■

Claim 8. Let $c, c'$ be distinct external cubes, neither of which contains the other. Let $d, d'$ be cubes of $F(c), F(c')$ respectively. Then $d \cap d'$ is either empty or is a cube of $F(c)$ and $F(c')$.

Proof of Claim 8. Let $c'' = c \cap c'$, which (if nonempty) is an external sub-cube of $c$ and $c'$ by Lemma 2.6. By Lemma 2.19, $F(c) \cap F(c') = F(c'')$. Hence $d \cap d' = (d \cap F(c'')) \cap (d' \cap F(c''))$.

By Claim 7, $d \cap F(c'')$ and $d' \cap F(c'')$ are subcubes of $F(c'')$. Let $t, t'$ be the subcubes of $c''$ that are the $\ell_1$–convex hulls of $d \cap F(c''), d' \cap F(c'')$ respectively. Note that $t, t'$ are external. So, by Claim 7, $d \cap F(c'')$ and $d' \cap F(c'')$ are cubes of $F(t), F(t')$ respectively. Note that $t, t'$ are proper subcubes of $c, c'$ (and therefore of lower dimension) since they are contained in $c''$, which is a proper subcube of $c$ and $c'$ by hypothesis.

So, if $t \neq t'$, then by induction on dimension, $(d \cap F(c'')) \cap (d' \cap F(c''))$ is a cube of $F(t) \cap F(t')$ (and hence of $F(t) \cap F(t')$). If $t = t'$, then it follows from Claim 6 above that $d \cap F(c'') = d' \cap F(c'')$. ■

Recall that $d, d'$ are cubes of $F(c)$, contained respectively in external cubes $f, f'$ of $c$ defined above. Moreover, we are considering the case where $f \neq f'$. Suppose that neither $f$ nor $f'$ contains the other. Now, $d$ is a cube of $F(f) = F(c) \cap f$ and $d'$ is a cube of $F(f') = F(c) \cap f'$, by Claim 7. Hence $d \cap d'$ is either empty or a cube of $F(f)$, and hence of $F(c)$, by Claim 8.

Finally, consider the case where $f \subseteq f'$. In this case, $d$ is a cube of $F(f)$, by Claim 7. Moreover, $d' \cap f$ is either empty or a cube of $F(f)$, by Claim 7. Since $d \cap d' = d \cap d' \cap f$, either $d \cap d' = \emptyset$, or $d \cap d'$ is the intersection of the cubes $d \cap f$ and $d' \cap f$ of $f$. Since $f$ is a proper face of $c$, induction on dimension yields that $d \cap d'$ is a cube of $F(f)$ and hence of $F(c)$. 


This completes the proof that $\mathcal{F}(c)$ is a cube complex.

**Hyperplanes:** We have shown that $\mathcal{F}(c)$ is a cube complex. The statement about hyperplanes is immediate from the definition of the cubes of $\mathcal{F}(c)$.

**Lemma 2.26.** Let $c$ be an external cube. Then $\mathcal{F}(c)$ is a CAT(0) cube complex.

**Proof.** By Lemma 2.25, $\mathcal{F}(c)$ is a cube complex. By Corollary 2.24, $\mathcal{F}(c)$ is simply connected. So, in view of [5, Theorem 5.20] we just have to check that the given cubical structure on $\mathcal{F}(c)$ has the property that each vertex-link is a flag complex.

Let $v$ be a vertex of $c$. Consider a clique in the link of $v$ in $\mathcal{F}(c)$. Then $v$ is contained in $e_1, \ldots, e_k$, which are (necessarily external) $1$–cubes of $\mathcal{F}(c)$ that are $1$–cubes of $c$, and $v$ is contained in $f_1, \ldots, f_r$, which are $1$–cubes of $\mathcal{F}(c)$ that are “diagonal” (i.e. $1$–cubes of $\mathcal{F}(c)$ but not of $c$). Since we are considering a clique, for all $i, j$, we have that $e_i, e_j$ span a $2$–cube of $\mathcal{F}(c)$, and the same is true for $f_i, f_j$.

Since $c$ is CAT($0$), the $1$–cubes $e_1, \ldots, e_k$ span a $k$–dimensional sub-cube $c'$ of $c$ that contains a collection of $\binom{k}{2}$ intersecting completely external $2$–cubes $s_{ij}$, with $s_{ij}$ spanned by $e_i, e_j$. Then for each $\ell \leq k$, the union of the $s_{ij}$ contains $k$ $1$–cubes parallel to $e_\ell$, all of which are external. Let $f$ be a codimension–$1$ face of $c'$ containing $1$–cubes parallel to $e_\ell$. For each hyperplane $H$ crossing $c'$ which is not dual to $e_\ell$, there is a $1$–cube $e'_\ell(H)$ parallel to $e_\ell$ and separated from $e_\ell$ by $H$. In particular, $f$ contains an external $1$–cube parallel to $e_\ell$. It follows from Lemma 2.1 that $c'$ is completely external, and hence $c' \subset \mathcal{F}(c)$.

Next, note that $\ell \leq 1$. Indeed, by construction, each $2$–cube of $\mathcal{F}(c)$ that is not a sub-cube of $c$ has the form $d \times I$, where $d$ is a $1$–cube of $c$ and $I$ is a diagonal interval. Hence $f_i, f_j$ cannot span a $2$–cube of $\mathcal{F}(c)$, so $\ell \leq 1$.

If $\ell = 0$, the flag condition holds since $c' \subset \mathcal{F}(c)$. Now suppose $\ell = 1$. For each $i$, the $1$–cubes $e_i, f_1$ of $\mathcal{F}(c)$ span a (diagonal) $2$–cube $\sigma_i$ of $\mathcal{F}(c)$. Now, by the definition of $\mathcal{F}(c)$, the cube $\sigma_i$ is the convex hull of $e_i, \bar{e}_i$, where $e_i, \bar{e}_i$ are parallel external $1$–cubes lying in $h(c), \bar{h}(c)$ or vice versa. Indeed, every “diagonal” square of $\mathcal{F}(c)$ is spanned by a “diagonal” $1$–cube and a $1$–cube of $h(c)$. By interchanging the names of $e_i, \bar{e}_i$ if necessary, we can assume that $e_i \subset h(c)$ for all $i$ and $\bar{e}_i \subset \bar{h}(c)$. Hence, by Lemma 2.11, both $0$–cubes of $e_i$ are persistent corners. It follows that $c' \subset h(c)$. Indeed, $c'$ is spanned by $1$–cubes $e_1, \ldots, e_k$, all of whose $0$–cubes are persistent corners, and the convex hull of this set of $0$–cubes is contained in $h(c)$ and, on the other hand, equal to $c'$.

Let $c' = \prod_{i=1}^{k} \bar{e}_i$, so $c' \subset \bar{h}(c)$. Each $0$–cube $v$ of $c'$ is thus a completely external cube of $\bar{h}(c)$, and is thus joined by a diagonal $1$–cube (parallel to $f_1$) to a $0$–cube of $c'$.

Now, if $c'$ is completely external, then by construction, $\mathcal{F}(c)$ contains a cube $c' \times f_1 = \bar{c'} \times f_1$ spanned by $e_1, \ldots, e_k, f_1$, so the link condition is satisfied. So, it suffices to show that $c'$ is completely external.

To that end, it suffices to show that every $1$–cube $\bar{e}$ of $c'$ is external. Let $t$ be the smallest subcube of $c$ containing $c' \cup \bar{e}$. Let $e$ be the $1$–cube of $c'$ that is parallel to $\bar{e}$ and separated from $\bar{e}$ by exactly those hyperplanes separating $c', \bar{e}$.

There exists $i$ such that $e$ is parallel to $e_i$. Let $\bar{e}_i$ be the $1$–cube of $c'$ that is parallel to $e_i$ and separated from $e_i$ by all hyperplanes of $c'$ not crossing $e_i$. Since $c'$ is completely external, the $1$–cube $\bar{e}_i$ is external.

Now, $\bar{e}_i$ is separated from $\bar{e}_i$ by all hyperplanes of $t$ not crossing $\bar{e}_i$. Moreover, since $\bar{e}_i$ lies in $\sigma_i$, it is external. Hence, by Remark 2.7, every $1$–cube of $t$ parallel to $\bar{e}_i$ is external. In particular, $\bar{e}$ is external, as required.

3. **Collapsing $\Psi$ along extremal panels**

We can now prove the main theorem about panel collapse.
Theorem 3.1 (Collapsing extremal panels). Let $Ψ$ be a finite-dimensional CAT(0) cube complex and let $\mathcal{P}$ be a collection of extremal panels with the no facing panels property. Then there is a CAT(0) cube complex $Ψ_0 \subset Ψ$ so that:

- there is a deformation retraction $Ψ \xrightarrow{r} Ψ_0$;
- for each hyperplane $K$ of $Ψ$, the subspace $K \cap Ψ_0$ is the disjoint union of hyperplanes of $Ψ_0$;
- each hyperplane of $Ψ_0$ is a component of $Ψ \cap K$ for some hyperplane $K$ of $Ψ$;
- for each $P \in \mathcal{P}$, the inside of $P$ is disjoint from $Ψ_0$.

Proof. Let $Ψ_0$ be the union over all maximal cubes $c$ of $Ψ$ of the subspaces $F(c)$.

By Lemma 2.17, $Ψ_0$ is disjoint from the inside of each panel in $\mathcal{P}$, i.e. $Ψ_0$ is disjoint from the interior of any cube of $Ψ$ that is internal to a panel in $\mathcal{P}$.

By Corollary 2.24 and Lemma 2.25, each $F(c)$ is a CAT(0) cube complex whose hyperplanes are components of intersection of $F(c)$ with hyperplanes of $Ψ$. The cubes of $F(c)$ are either sub-cubes of $c$ or cubes of the form $w \times \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$ (where $w$ is a sub-cube of $c$), which has dimension at most $d-1$. By Lemma 2.19, if $c, c'$ are intersecting external cubes, then they intersect along a common external face $d''$, and $F(c) \cap F(c') = F(d'')$. (The intersection of distinct external cubes is external since panels in $\mathcal{P}$ are extremal.)

Moreover, if $d, d' \subset F(c), F(c')$ are cubes (including diagonal cubes), the intersection $d \cap d'$ is either empty or a cube of $F(c)$ and $F(c')$, by Lemma 2.26, so $Ψ_0$ is a cube complex. The definition of a hyperplane in a cube complex (a connected union of hyperplanes of cubes that intersects each cube in $\emptyset$ or a hyperplane of that cube), together with the above characterisation of hyperplanes in $F(c)$, establishes the statements about hyperplanes of $Ψ_0$.

To see that $Ψ_0$ is a CAT(0) cube complex, we must first verify that it is simply connected, which we will do by constructing the claimed deformation retraction $Ψ \xrightarrow{r} Ψ_0$. For each maximal cube $c$, any codimension–1 face $c'$ that is not external has the property, by definition, that $\text{Int}(c)$ is contained in the inside of some panel $P \in \mathcal{P}$. Since $P$ is an extremal panel, $c'$ is contained in a unique maximal cube of $Ψ$, namely $c$. Hence the deformation retraction $c \xrightarrow{r} F(c)$ from Lemma 2.23 induces a deformation retraction $Ψ \times [0, 1] \to Ψ$ which is the identity outside of $c$ and agrees on $c$ with $Δ_c$. Lemma 2.23 also implies that these deformation retractions are compatible with intersections of external (and hence maximal) cubes, so the pasting lemma produces the desired deformation retraction $Ψ \xrightarrow{r} Ψ_0$.

It remains to check that $Ψ_0$ is locally CAT(0), i.e. that the link of each 0–cube is a flag complex. Let $v \in Ψ_0$ be a 0–cube, so that $v$ is also a 0–cube of $Ψ$. Let $e_1, \ldots, e_k$ be 1–cubes of $Ψ$ incident to $v$ and lying in $Ψ_0$, and let $f_1, \ldots, f_t$ be “diagonal” 1–cubes of $Ψ$ incident to $v$. Suppose that for all $i, j$, the 1–cubes $e_i, e_j$, and $e_i, f_j$, and $f_i, f_j$ span a 2–cube of $Ψ_0$.

For $i \neq j$, since $f_i, f_j$ span a 2–cube of $Ψ$, they must lie in a common external cube, but this was shown in the proof of Lemma 2.25 to be impossible. Hence $r \leq 1$.

By CAT(0)ness of $Ψ$, the $e_i$ span a $k$–cube $C' \subset Ψ$ which is necessarily external, since it contains a persistent corner. So Lemma 2.25 implies that $C'$ is completely external (and hence in $Ψ_0$). So, if $r = 0$, then we are done.

If $r = 1$, then let $d$ be the $ℓ_1$ convex hull of $f_1$, which is a cube. Then for any $i \leq k$, the fact that $e_i, f_1$ span a 2–cube of $Ψ_0$ implies that $Ψ$ contains the cube $e_i \times d$. Hence there is a cube $c = C' \times d$ that contains $f_1$ and all of the $e_i$. But $Ψ_0 \cap c = F(c)$, so Lemma 2.25 implies that $F(c)$, and hence $Ψ_0$, contains a cube spanned by $f_1, e_1, \ldots, e_k$. This completes the proof that $Ψ_0$ is CAT(0). \(\square\)

3.1. Equivariant panel collapse. Let $Ψ$ be a finite-dimensional CAT(0) cube complex and let $G \leq \text{Aut}(Ψ)$ act cocompactly. (We do not assume that $Ψ$ is locally finite or that the
G–action is proper.) We also require that G acts on Ψ without inversions in hyperplanes, i.e. for each hyperplane H, the stabiliser in G of H stablises both associated halfspaces. This can always be assumed to hold by passing to the first cubical subdivision (see e.g. [21, Lemma 4.2]).

**Definition 3.2** (Complexity). The complexity #(Ψ) of Ψ is the tuple (dim Ψi)i=0i=dimΨ−2, where #i(Ψ) is the number of G–orbits of (dim Ψ − i)–cubes in Ψ. Observe that Ψ is a tree if and only if #(Ψ) = 0. Complexity is ordered lexicographically.

**Corollary 3.3.** Suppose that Ψ contains a hyperplane H and a hyperplane E so that E ∩ H is extremal in H. Then Ψ contains a G–cocompact subspace Ψ∗ such that:

- Ψ∗ is a CAT(0) cube complex;
- For each hyperplane K of Ψ, the subspace K ∩ Ψ∗ is the disjoint union of hyperplanes of Ψ∗;
- Each hyperplane of Ψ∗ is a component of Ψ∗ ∩ K for some hyperplane K of Ψ;
- #(Ψ∗) < #(Ψ) in the lexicographic order on \( \mathbb{N}^{\dim \Psi - 2} \);
- The action of G on Ψ∗ is without inversions;
- Each g ∈ G is a hyperbolic [resp. elliptic] isometry of Ψ∗ if and only if it is a hyperbolic [resp. elliptic] isometry of Ψ.

This holds in particular if G is a group acting cocompactly and without inversions on a CAT(0) cube complex Ψ under either of the following conditions:

- Some hyperplane of Ψ is compact and contains more than one point;
- Ψ contains a hyperplane H so that StabG(H) does not act essentially on H.

Finally if Ψ is locally finite then so is Ψ∗.

**Proof.** The hypotheses provide an extremal panel P; let \( \mathcal{P} = G \cdot P \). Since the action is without inversions, Lemma 1.8 implies that \( \mathcal{P} \) has the no facing panels property. Since \( \mathcal{P} \) is G–invariant, for each cube c and each g ∈ G, Lemma 2.18 implies \( F(gc) = gF(c) \), so the CAT(0) cube complex Ψ∗ from Theorem 3.1 is a G–invariant subspace of Ψ. Theorem 3.1 also says that the hyperplanes of Ψ∗ have the desired form.

Since G acts without inversions on Ψ and hyperplanes of Ψ∗ extend to hyperplanes of Ψ, the action of G on Ψ∗ is without inversions. Indeed, let C be a hyperplane of Ψ∗. Then C is a component of the intersection of Ψ∗ with pairwise-crossing hyperplanes \( H_1, \ldots, H_k \) of Ψ, for some \( k \leq \dim \Psi \). Hence StabG(C) permutes the hyperplanes \( H_k \).

Let \( B = \bigcap_{i=1}^k \mathcal{N}(H_i) \), and observe that StabG(C) acts on B, preserving the collection of extremal panels from \( \mathcal{P} \) in B. The no facing panels property implies that these panels have nonempty intersection, which must be preserved by StabG(C). Hence StabG(C) stabilizes \( \bigcap_i \overline{H_i} \), where \( \overline{H_i} \) is the halfspace of B associated to \( H_i \) that contains a panel in \( \mathcal{P} \) extremalised by \( H_i \). Since \( \bigcap_i \overline{H_i} \) is one of the halfspaces of \( B \cap \Psi^* \) associated to C, we see that no element of StabG(C) acts as an inversion across C.

Since there are finitely many G–orbits of cubes in Ψ, and each contributes a bounded number of cubes to Ψ∗, the action of G on Ψ∗ is cocompact. Finally, let c be a maximal cube of Ψ that intersects the inside of some panel in \( \mathcal{P} \). Then Theorem 3.1 implies that \( F(c) \) has dimension strictly lower than that of c, so \( #(\Psi^*) < #(\Psi) \). Finally, suppose that g ∈ G. By [21, Theorem 1.4], since Ψ is finite-dimensional, either g is hyperbolic (in the \( \ell_1 \) and \( \ell_2 \) metrics) or g fixes a 0–cube (since the action is without inversions). Now, each 0–cube of Ψ lies in Ψ∗, by construction, and the inclusion Ψ∗ → Ψ is G–equivariant, so stabilisers of 0–cubes in Ψ do not change on passing to Ψ∗. Hence g is hyperbolic or elliptic on Ψ∗ according only to whether it was hyperbolic or elliptic on Ψ. This proves the first part of the claim.
Now, let $G$ act cocompactly and without inversions on a CAT(0) cube complex $Ψ$. If $H$ is a hyperplane of $Ψ$ that is compact and contains more than one point, then Corollary 1.5 shows that $Ψ$ has an extremal panel, and we can argue as above.

Similarly, suppose that $Ψ$ contains a hyperplane $H$ so that $\text{Stab}_G(H)$ does not act on $H$ essentially. Then there is a hyperplane $E$ bounding a halfspace $\bar{E} \cap H$ in $H$ so that $\bar{E} \cap H$ and $(Ψ - \bar{E}) \cap H$ are both $\text{Stab}_G(H)$–shallow.

Since $G$ acts cocompactly on $Ψ$, the stabilizer of $H$ acts coboundedly on $H$. To verify this, we will show that there are finitely many $\text{Stab}_G(H)$–orbits of $1$–cubes of $Ψ$ dual to $H$. Indeed, since $G$ acts on $Ψ$ with finitely many orbits of $1$–cubes, there are $1$–cubes $e_1, \ldots, e_k$ dual to $H$ so that every $1$–cube dual to $H$ has the form $ge_i$ for some $i \leq k$ and some $g \in G$. But since $e_i$ is dual to $H$, the $1$–cube $ge_i$ is dual to $gH$. On the other hand, since $ge_i$ is dual to $H$, we have $gH = H$, i.e. $g \in \text{Stab}_G(H)$, as required.

Hence one of $\bar{E} \cap H$ and $(Ψ - \bar{E}) \cap H$ must be contained in a uniform neighbourhood of $E \cap H$. Hence $E$ could be chosen so that $E \cap H$ is extremal in $H$. Thus $Ψ$ has an extremal panel, and we can conclude as above.

The final assertion follows since for each cube $c$, $\mathcal{F}(c)$ has finitely many cubes. \hfill \Box

**Remark 3.4** (Effect of panel collapse on hyperplanes). Let $Ψ$, $G$, and $P$ be as in Corollary 3.3. Let $H$, $E$ be hyperplanes of $Ψ$ so that $E \cap H$ is extremal in $H$. Let $P$ be the extremal panel in $N(E \cap H)$ abutted by $H$ and extremalised by $E$. Then $E \cap Ψ_0$ is contained in the closure of $[E - (E \cap N(E \cap H))] \cup (E \cap H)$ (that is to say, $E$ is replaced by the halfspaces of $E$ induced by $N(E \cap H)$, except that $E \cap H$ might be added back in the event that there were diagonal cubes in $Ψ_0$). This can be seen by inspecting the construction of $Ψ_0$ at the level of individual cubes.

### 4. Applications

#### 4.1. Stallings’s theorem

**Corollary 4.1.** Let $G$ be a finitely generated group with more than one end. Then $G$ splits nontrivially as a finite graph of groups with finite edge groups.

**Proof.** Fix a locally finite Cayley graph $X$ of $G$. Since $G$ has more than one end, there is a ball $K$ so that $X - K$ has at least two components containing points arbitrarily far from $K$. Applying [33], we obtain an action of $G$ on a CAT(0) cube complex $Ψ_0$ whose hyperplanes correspond to $K$ and its translates; moreover, $G$ has no global fixed point in $Ψ_0$. For any $r \geq 0$, there exists $d = d(r)$ so that if $g_1, \ldots, g_k$ are such that $d_X(g_iK, g_jK) \leq r$ for all $i, j$, then there exists $y \in X$ so that each $g_iK$ intersects $B_d(y)$. It follows that $Ψ_0$ has finitely many $G$–orbits of cubes, i.e. $G$ acts cocompactly (and with a single orbit of hyperplanes) on $Ψ_0$. Moreover, the hyperplane-stabilisers are commensurable with the conjugates of the stabiliser of $K$, so they are finite. Hence the hyperplanes of $Ψ_0$ are compact. By passing to the first cubical subdivision, we can assume that $G$ acts without inversions. (Up to here, our argument does not essentially differ from that in [28]; the arguments diverge at the next step.)

Hence, by Corollary 3.3, there is a sequence $\ldots \subseteq Ψ_n \subseteq Ψ_{n-1} \subseteq \ldots \subseteq Ψ_0$ of $G$–CAT(0) cube complexes, so that $\#(Ψ_n) < \#(Ψ_{n-1})$ for all $n \geq 1$ for which $Ψ_n$ is not a tree. Since $\#(Ψ_0)$ is finite, there exists $n$ so that $\#(Ψ_n) = 0$, i.e. $Ψ_n$ is a tree. Moreover, by Corollary 3.3, the stabiliser of a hyperplane in $Ψ_n$ (i.e. an edge stabiliser) is virtually contained in the stabiliser of a hyperplane of $Ψ_0$, and is thus finite. Since $Ψ_n$ is contained in $Ψ_0$, and $G$ did not fix a point in $Ψ_0$, there is no point in $Ψ_n$ fixed by $G$, i.e. the splitting is nontrivial. \hfill \Box
4.2. **Antenna cubulations of free groups.** In [44], Wise proved the following remarkable theorem. Let \( F \) be a finite-rank free group, and let \( H \leq F \) be an arbitrary finitely-generated subgroup of infinite index. Then \( F \) acts freely and cocompactly on a CAT(0) cube complex \( \Psi \), and \( \Psi \) contains a single \( F \)-orbit of hyperplanes, with each hyperplane stabiliser conjugate to \( H \). This gives a profusion of cubulations of \( F \) beyond the obvious 1–dimensional ones. In fact, the hyperplanes of \( \Psi \) are necessarily non-compact when \( H \) is infinite, so \( \Psi \) can be a bit hard to picture: on one hand, it is quasi-isometric to a tree; on the other hand, the separating compact sets one sees cannot be hyperplanes. To illustrate Theorem A, we will explain how to perform panel collapse on these exotic cubulations of \( F \).

We say a CAT(0) cube complex \( \Psi \) with a free, cocompact \( F \)-action is an *antenna cubulation of \( F \)* if it is of the type constructed in [44] (which we discuss in more detail below).

**Proposition 4.2.** Let \( \Psi \) be an antenna cubulation of \( F \). Then \( \Psi \) has a hyperplane \( D \) such that \( \text{Stab}_D(\Psi) \) does not act essentially on \( D \), so \( \Psi \) has an extremal panel \( P \).

From Corollary 3.3, we can then apply panel collapse. The interesting thing about Proposition 4.2 is that it provides an example of panel collapse when the hyperplanes are non-compact but the group is not too complicated.

**Proof of Proposition 4.2.** First we recall the construction of antenna cubulations. Then we apply Theorem A and prove the two statements.

**Antenna cubulations:** Let \( \mathcal{C} \) be a wedge of finitely many oriented circles, labelled by the generators of \( F \), so that \( F \) is identified with \( \pi_1 \mathcal{C} \). Let \( \tilde{\mathcal{C}} \to \mathcal{C} \) be the based cover corresponding to \( H \hookrightarrow F \), and let \( \mathcal{C} \) be the core of \( \tilde{\mathcal{C}} \), which is compact since \( C \) is finitely generated. Since \([F : H] = \infty\), there is a vertex \( v \) of \( \mathcal{C} \) at which the immersion \( \mathcal{C} \to \mathcal{C} \) is not locally surjective.

An *antenna* is a tree \( A = P \cup \bigcup_i T_i \), where \( P \cong [0, n] \) is a tree with \( n \) edges and exactly two leaves, \( T_i \) is a path of length 2 for \( 1 \leq i \leq n \), and \( A \) is formed by identifying the midpoint of each \( T_i \) to the vertex \( i \) of \( P \). We also orient the edges of \( A \) and label them by the generators of \( F \) in such a way as to obtain an immersion \( A \to \mathcal{C} \). We also do this labelling so that:

- the edge \([0, 1]\) of \( P \) is labelled in such a way that attaching \( P \) to \( \tilde{\mathcal{C}} \) by identifying \( v \) with \( 0 \in P \) yields an immersion \( P \cup_{0=v} \tilde{\mathcal{C}} \to \mathcal{C} \) given by the labels and orientations;
- every reduced word of length 2 in the generators of \( F \) appears as the labelling of some \( T_i \) (thus \( n \) is bounded below in terms of the rank of \( F \)). Let \( W = P \cup_{0=v} \mathcal{C} \).

Now add (infinite) trees to \( W \) wherever necessary to obtain an infinite-sheeted cover \( \tilde{\mathcal{C}} \to \mathcal{C} \) homotopy equivalent to \( W \). For each of these finitely many trees, assign it a + or −, and declare \( W \) itself to be −. This induces a partition of the universal cover \( \tilde{\mathcal{C}} \) into two halfspaces (points mapping into + and points mapping into −), and with some care this can be done so that \( H \) is the stabiliser of this wall. At this point, Wise applies Sageev’s construction and, because of the above properties of antennas, is able to show that any axis in \( \tilde{\mathcal{C}} \) for an element of \( F \) is cut by some translate of \( W \). Hence the action on the dual cube complex \( \Psi \) is proper, and it is cocompact for general reasons [34]. It is also not hard to see that the action of \( F \) on \( \Psi \) is essential.

Let \( D \) be a hyperplane of \( \Psi \), which, by construction, has stabiliser \( H \). Since \( H \) stabilised the two halfspaces associated to the wall \( \tilde{W} \), the action of \( F \) on \( \Psi \) is without inversions in hyperplanes (there is no need to subdivide).

**Finding an extremal panel in \( \Psi \):** We can and shall insist that the + and − were assigned so that each tree attached to \( \tilde{\mathcal{C}} \) (as opposed to the antenna), if any, is assigned +.

We will prove that \( H \) does not act essentially on \( D \); the proof of Corollary 3.3 will then show that \( \Psi \) has an extremal panel and panel collapse is in play.
Let $\tilde{W} \subset \tilde{C}$ be a lift of the universal cover of $W$, so that $\tilde{W}$ consists of $\tilde{C}$ together with an antenna $hA$ based at $h\tilde{v}$, for each $h \in H$, where $\tilde{v}$ is a lift of $v$ in $\tilde{C}$ and $A$ is the lift of the antenna at that point.

The construction of $A$ ensures that there exists $g \in F - H$ so that the distinct walls $\tilde{W}, g\tilde{W}$ intersect in the following way:

- $\tilde{W} \cap g\tilde{W} = A \cap gA$;
- $gA \cap A = gT_i$ for some $i$, and $gT_i \subset P$.

Let $\tilde{W}^+, \tilde{W}^-$ denote the subtrees of $\tilde{C}$ projecting to the subgraphs of $\tilde{C}$ labelled $+$ and $-$ respectively. Then (up to relabelling), $\tilde{W}^+ \cap g\tilde{W}^+$ contains $\tilde{C}$ and $g\tilde{C}$, while $\tilde{W}^+ \cap g\tilde{W}^-$, $\tilde{W}^- \cap g\tilde{W}^+$, $\tilde{W}^- \cap g\tilde{W}^+$ are nonempty and bounded.

Hence the walls in $\tilde{C}$ determined by $\tilde{W}, g\tilde{W}$ cross (i.e. all four possible intersections of associated halfspaces are nonempty), so, in the dual cube complex $\Psi$, the hyperplanes $D, gD$ corresponding to these walls also cross. On the other hand, each of $D, gD$ is extremal in the other. Indeed, if $gD \cap D$ were an essential hyperplane of $D$ and vice versa, then all four of the possible intersections of the halfspaces $\tilde{W}^\pm, g\tilde{W}^\pm$ would be unbounded. Hence $H$ does not act essentially on $D$ and, as explained in the proof of Corollary 3.3, $\Psi$ has an extremal panel $P$.

In fact Wise’s antenna construction allows us to perform a finite sequence of panel collapses that reduces the original cubulation to a free action on a tree. In contrast to Proposition 4.2, using multiple hyperplanes, it seems one can construct geometric actions of $F$ on a CAT(0) cube complex $\Psi$ so that $\Psi$ has no compact hyperplanes and every hyperplane stabiliser acts essentially on its hyperplane.

4.3. The Cashen-Macura complexes.

**Definition 4.3.** Let $X$ be a Gromov hyperbolic space. A line pattern $\mathcal{L}$ in $X$ is a collection of quasi-isometry classes of bi-infinite quasi-geodesics in $X$. Given spaces $X_i, \mathcal{L}_i$, equipped with line patterns, for $i = 1, 2$ we say that a quasi-isometry $\phi : X_1 \to X_2$ respects the line patterns if for any representative $l \in \mathcal{L}_1$ the mapping $l \mapsto \phi(l)$ is to a representative of an element of $\mathcal{L}_2$ and induces a bijection $\mathcal{L}_1 \to \mathcal{L}_2$.

Two elements $h, g$ of a hyperbolic group $\Gamma$ are commensurable if there is some $k \in \Gamma$ such that $|(g) : (k^{-1}hhk)| \leq \infty$.

**Definition 4.4.** If $\Gamma$ is a hyperbolic group and $\{g_1, \ldots, g_m\}$ is a tuple of non-commensurable hyperbolic elements, then the line pattern $\mathcal{L}$ generated by $\{g_1, \ldots, g_m\}$ is the collection of quasi-isometry classes of quasi-lines

$$\mathcal{L} = \{ h \cdot \langle g_i \rangle \mid h \in \Gamma, 1 \leq i \leq m \} \text{ (bounded distance)}. $$

A line pattern is called rigid if $\Gamma$ admits no virtually cyclic splittings relative to $\{g_1, \ldots, g_m\}$; equivalently, if for every cut pair $c, c' \subset \partial \Gamma$ there is some $l \in \mathcal{L}$ with endpoints $l^+, l^- \subset \partial \Gamma$ such that $l^+$ is in one connected component of $\partial \Gamma \setminus \{c, c'\}$ and $l^-$ is in a different component, or finally the virtually cyclic JSJ decomposition relative to $\{g_1, \ldots, g_m\}$ is trivial.

In [7] the following remarkable pattern rigidity theorem, which promotes certain quasi isometries to action by isometries on a common space, is proved.

**Theorem 4.5** (Pattern rigidity, see [7, Theorem 5.5]). Let $F_i, \mathcal{L}_i, i = 0, 1$, be free groups equipped with a rigid line patterns. Then there are CAT(0) cube complexes $X_i$ with line patterns $\mathcal{L}_i$ and embeddings

$$F_i \xrightarrow{\iota_i} \text{Isom}(X_i)$$
inducing cocompact isometric actions $F_i \cap X_i$, which in turn induce equivariant line pattern preserving quasi-isometries

$$\phi_i : F_i \to X_i.$$ 

Furthermore for any line pattern preserving quasi-isometry $q : F_0 \to F_1$ there is a line pattern preserving isometry $\alpha_q$ such that the following diagram of line pattern preserving quasi-isometries commutes up to bounded distance:

$$
\begin{array}{ccc}
(X_0, L_0) & \xrightarrow{\alpha_q} & (X_1, L_1) \\
\phi_0 & & \phi_1 \\
(F_0, L_0) & \xrightarrow{q} & (F_1, L_1)
\end{array}
$$

We note that the actions of $F_i$ on $X_i$ are free since the quasi-isometries $\phi_i : F_i \to X_i$ are equivariant.

In [7, §6.4] the authors give an example where the Cashen-Macura complex $X$ given in Theorem 4.5 is not a tree, but observe that it is possible in this example to construct another complex $X'$ satisfying the requirement of the theorem that is a tree. They ask if this can always be done. While not precisely answering their question, which is about a choice of topologically distinguished cut sets, we have the following.

**Theorem 4.6.** Given a free group equipped with a line pattern $F_0, L_0$, there exists a locally finite tree equipped with line pattern $T, L_T$ which satisfies the requirements of the Cashen-Macura complex $X, L$ given in Theorem 4.5. Furthermore $L_T$ can be represented by geodesics.

**Proof.** In [7, §5.2.2] it is shown that in the CAT(0) cube complex $X$ constructed in Theorem 4.5 there is a uniform bound on the number of hyperplanes a given hyperplane $H$ crosses and $X$ is shown to be locally finite. It follows that $X$ has compact hyperplanes. Let $G = \text{Isom}(X)$. By repeatedly applying Corollary 3.3 there a $G$-equivariant deformation retraction from $X$ to a tree $T$. We can take $L_T$ to be the collection of geodesic representatives of the quasi-lines of $L$ in $T$. $(T, L_T)$, by definition, satisfy the necessary requirements. □

Ultimately, given a line pattern preserving quasi-isometry $q : (F_0, L_0) \to (F_1, L_1)$, we would like to construct a virtual isomorphism between $F_0$ and $F_1$ that induces a line pattern preserving quasi-isometry. The appropriate machinery to obtain this result appears to be commensurability of tree lattices: let $T$ be a tree equipped with a line pattern $L$ and denote $G = \text{Isom}(T)$ and $G_L$ the subgroup of isometries of $T$ that preserve $L$. Then the existence of such a virtual isomorphism reduces to asking whether the images of $F_0$ and $F_1$ in $G$ are commensurable within the subgroup $G_L$ of line pattern preserving isometries.

There are examples when $G_L$ is a discrete group (see [7, §6.2]) and therefore finitely generated. In this case the embeddings given in (2) of Theorem 4.5 would give virtual isomorphisms, i.e. isomorphisms between finite index subgroups, between $F_0$ and $F_1$ that preserve line patterns. However in [7, §6.3] an example shows that even if $L$ is a rigid line pattern, $G_L$ may not be a discrete group of automorphisms of $T$. In this case, even if $F_0$ and $F_1$ act cocompactly on $T$, their intersections in $G_L$ could be trivial.

If $H$ is a closed subgroup of $\text{Aut}(T)$, then a discrete subgroup $\Gamma \subset H$ with the same orbits as $H$, i.e. with $H \backslash T = \Gamma \backslash T$ is called a discrete grouping. Discrete groupings are hard to obtain, which is why being able to make Cashen-Macura complexes into trees is advantageous: we are now able to apply the tree lattice techniques in [2]. We finish with an observation that may be useful, in constructing virtual isomorphisms.
Proposition 4.7 \((G_\mathcal{L} is closed and admits a discrete grouping.) Let \(F_0, \mathcal{L}_0\) be a free group equipped with a rigid line pattern. Then the group \(G_\mathcal{L}\) of automorphisms which preserve the line pattern \(\mathcal{L}\) of the Cashen-Macura tree \(T, \mathcal{L}\), given in Proposition 4.6, is a closed subgroup of \(\text{Aut}(T)\) and admits a finitely generated, discrete grouping \(\Phi^0 \leq G_\mathcal{L}\).

Unfortunately we are currently unable to obtain a commensurability result as we do not know if there is a discrepancy between \(G_\mathcal{L}\) and \(G_{G_\mathcal{L}}\) (see [2, Theorem 4.7 (iv)] or [26, Theorem 3].)

Proof. Given \(F_0, \mathcal{L}_0\), let \(T, \mathcal{L}\) be the Cashen-Macura tree constructed in Proposition 4.6. Let \(G = \text{Aut}(T)\) and let \(G_\mathcal{L}\) denote the subgroup that preserves the geodesics in \(\mathcal{L}\).

\(G_\mathcal{L}\) is a closed subgroup of \(G\). Equip \(G\) with the compact open topology. Let \(\gamma \in G \setminus G_\mathcal{L}\). Suppose first that there is some line \(l \in \mathcal{L}\) such that \(\gamma \cdot l \notin \mathcal{L}\). \(\gamma \cdot l\) is still a geodesic in \(T\). Because there is a uniform bound on the intersection of any two distinct lines in \(\mathcal{L}\), it must follow that there is some finite subset \(l_0 \subset l\) such that \(\gamma \cdot l_0\) is not contained within any \(l' \in \mathcal{L}\). Let \(B \subset T\) be a metric ball containing \(l_0\). Then the set \(U\) of isometries that coincide with \(\gamma\) on \(B\) give an open neighbourhood \(\gamma \in U \subset (G \setminus G_\mathcal{L})\) separating \(\gamma\) from \(G_\mathcal{L}\).

Next suppose that for every \(l \in \mathcal{L}\), \(\gamma \cdot l \in \mathcal{L}\) but that \(\gamma \cdot \mathcal{L} \subset \mathcal{L}\). Let \(l \in \mathcal{L} \setminus \gamma \cdot \mathcal{L}\), then \(\gamma^{-1} \cdot l \notin \mathcal{L}\). By the argument of the previous paragraph there is an open set \(\gamma^{-1} \in U \subset (G \setminus G_\mathcal{L})\) separating \(\gamma^{-1}\) from \(G_\mathcal{L}\). Since \(G\) is a topological group, the inversion operation is a homeomorphism \(\gamma^{-1} : G \to G\) that maps \(G_\mathcal{L}\) to itself, since the latter is a subgroup. The image \(U^{-1}\) of \(U\) gives a open neighbourhood \(\gamma \in U^{-1} \subset (G \setminus G_\mathcal{L})\) separating \(\gamma\) from \(G_\mathcal{L}\). It follows that \(G_\mathcal{L}\) is closed in \(G\).

Applying tree lattice techniques. For any subgroup of \(G\) there is a well defined homomorphism to \(\mathbb{Z}_2\) whose kernel does not invert any edges (see [2, §3] or [1, §6.3].) If necessary we therefore pass to index 2 subgroups that do not invert edges of \(T\), but keep our notation. By Theorem 4.5 the group \(F_0\) acts freely on \(T\). By hypothesis the action is also cocompact. \(F_0 \setminus T\) is therefore a finite graph. It follows that \(F_0 \subset G_\mathcal{L}\) is a uniform lattice in the sense of [2, Definition 4.3]. Because \(G_\mathcal{L}\) is closed, [2, Proposition 4.5] then implies that \(G_\mathcal{L}\) is unimodular and [2, Theorem 4.7] implies the existence of a discrete subgroup \(\Phi^0 \subset G_\mathcal{L}\) with \(\Phi^0 \setminus T = G_\mathcal{L} \setminus T\).

Since \(G_\mathcal{L}\) contains \(F_0\), and \(F_0\) acts cocompactly on \(T\), it follows that \(G_\mathcal{L}\), and hence \(\Phi^0\), acts cocompactly on \(T\). Since \(\Phi^0\) is discrete, stabilisers in \(\Phi^0\) of points in the locally finite tree \(T\) are finite, so \(\Phi^0\) acts on \(T\) properly. Hence \(\Phi^0\) is finitely generated, by the Milnor-Švarc lemma. \(\square\)

We end this section with a question that we hope would be of interest to tree lattice experts.

Question 4.8. Let \(\mathcal{L}\) be a rigid line pattern in a tree \(T\) and suppose that \(G_\mathcal{L} \leq \text{Aut}(T)\) is closed and unimodular. Can there be a proper inclusion

\[ G_\mathcal{L} \leq G_{G_\mathcal{L}}, \]

where \(G_{G_\mathcal{L}} \leq \text{Aut}(T)\) is the maximal group with

\[ G_\mathcal{L} \setminus T = G_{G_\mathcal{L}} \setminus T? \]

In [45], Daniel Woodhouse solved the problem that was the original motivation for this question.

4.4. The Kropholler conjecture. We now apply Corollary 3.3 to the following special case of the Kropholler conjecture.
Corollary 4.9 (Kropholler conjecture, cocompact case). Let $G$ be a finitely generated group and $H \leq G$ a subgroup with $e(G, H) \geq 2$. Let $\Psi$ be the dual cube complex associated to the pair $(G, H)$, so that $\Psi$ has one $G$–orbit of hyperplanes and each hyperplane stabiliser is a conjugate of $H$. Suppose that:

- $G$ acts on $\Psi$ cocompactly;
- $H$ acts with a global fixed point on the associated hyperplane.

Then $G$ admits a nontrivial splitting over a subgroup commensurable with a subgroup of $H$.

Proof. Let $D$ be a hyperplane of $\Psi$ with $\text{Stab}_G(D) = H$.

**Bounded hyperplanes:** The fixed-point hypothesis guarantees that the action of $H$ on $D$ is non-essential, so we could apply Corollary 3.3 immediately. In fact, since $H$ acts cocompactly on $D$, and also fixes a point in $D$, we have that $D$ has finite diameter.

Applying Corollary 3.3: Since $D$ is bounded, Corollary 1.5 provides an extremal panel in $\Psi$. At this point, we can subdivide $\Psi$ once if necessary to ensure that the action of $G$ is without inversions. This has the effect of replacing $\text{Stab}_G(D)$ with a subgroup of index at most 2, which will not affect the conclusion. (As usual, we will not subdivide at later stages in the induction, but instead use that the no inversions property of the action persists under panel collapse.)

Corollary 3.3 provides a cocompact, inversion-free $G$–action on a CAT(0) cube complex $\Psi_{\text{•}}$ with $\#(\Psi_{\text{•}}) < \#(\Psi)$, unless $\Psi$ was already a tree. Moreover, the hyperplanes of $\Psi_{\text{•}}$ are components of $\Psi_{\text{•}} \cap gD$ for $g \in G$, and are thus bounded. Furthermore this intersection property implies that, since $G$ acts by permuting the hyperplanes in $\Psi$, a $\Psi_{\text{•}}$–hyperplane stabilizer must lie in the stabilizer of an intersection of $\Psi$–hyperplanes. It follows that each stabiliser of a hyperplane in $\Psi_{\text{•}}$ is virtually contained in a stabiliser of a hyperplane in $\Psi$ and the index is bounded by the dimension of $\Psi$. It thus follows by induction on complexity that $G$ acts on a simplicial tree all of whose edges has stabiliser commensurable with a subgroup of $H$, as required.

□

References

[1] Bass, H. Covering theory for graphs of groups. J. Pure Appl. Algebra 89, 1-2 (1993), 3–47.
[2] Bass, H., and Kulkarni, R. Uniform tree lattices. Journal of the American Mathematical Society 3, 4 (1990), 843–902.
[3] Behrstock, J., Hagen, M., and Sisto, A. Hierarchically hyperbolic spaces, I: Curve complexes for cubical groups. Geometry & Topology 21, 3 (2017), 1731–1804.
[4] Bergeron, N., and Wise, D. T. A boundary criterion for cubulation. American Journal of Mathematics 134, 3 (2012), 843–859.
[5] Bridson, M. R., and Haefliger, A. Metric spaces of non-positive curvature, vol. 319. Springer Science & Business Media, 2011.
[6] Caprace, P.-E., and Sageev, M. Rank rigidity for CAT(0) cube complexes. Geometric and functional analysis 21, 4 (2011), 851–891.
[7] Cashen, C. H., and Macura, N. Line patterns in free groups. Geometry & Topology 15, 3 (2011), 1419–1475.
[8] Chasney, R., and Davis, M. W. Finite $K(\pi, 1)$s for Artin groups. Prospects in Topology (Princeton, NJ, 1994), Princeton University Press, Princeton (1995), 110–124.
[9] Chatterji, I., and Niblo, G. From wall spaces to CAT(0) cube complexes. Internat. J. Algebra Comput. 15, 5-6 (2005), 875–885.
[10] Culler, M. Finite groups of outer automorphisms of a free group. Contributions to group theory 33 (1984), 197–207.
[11] Dunwoody, M., and Krön, B. Vertex cuts. Journal of Graph Theory 80, 2 (2015), 136–171.
[12] Dunwoody, M., and Roller, M. Splitting groups over polycyclic-by-finite subgroups. Bulletin of the London Mathematical Society 25, 1 (1993), 29–36.
[13] Dunwoody, M. J. Cutting up graphs. Combinatorica 2, 1 (1982), 15–23.
[14] Dunwoody, M. J. The accessibility of finitely presented groups. Invent. Math. 81, 3 (1985), 449–457.
[15] Dunwoody, M. J. Structure trees, networks and almost invariant sets. In Groups, graphs and random walks, vol. 436 of London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2017, pp. 137–175.

[16] Evangeliou, A., and Papasoglu, P. A cactus theorem for end cuts. International Journal of Algebra and Computation 24, 01 (2014), 95–112.

[17] Gerasimov, V. N. Semi-splittings of groups and actions on cubings. Algebra, geometry, analysis and mathematical physics (Russian) (Novosibirsk, 1996) (1997), 91–109.

[18] Haglund, F. Isometries of CAT(0) cube complexes are semi-simple. arXiv preprint arXiv:0705.3386 (2007).

[19] Hagen, M. F., and Wilton, H. Guirardel cores for multiple cube complex actions. In preparation (2017).

[20] Hagen, M. F., and Wise, D. T. Cubulating hyperbolic free-by-cyclic groups: the general case. Geometric and Functional Analysis 25, 1 (2015), 134–179.

[21] Kapovich, M. Energy of harmonic functions and Gromov’s proof of Stallings’ theorem. Geometric and Functional Analysis 25, 1 (2015), 134–179.

[22] Niblo, G. A. A geometric proof of Stallings’ theorem on groups with more than one end. Geometriae Dedicata 105, 1 (2004), 61–76.

[23] Nica, B. Cubulating spaces with walls. Algebr. Geom. Topol 4 (2004), 297–309.

[24] Ollivier, Y., and Wise, D. Cubulating random groups at density less than 1/6. Transactions of the American Mathematical Society 363, 9 (2011), 4701–4733.

[25] Sageev, M. Codimension-1 subgroups and splittings of groups. Journal of Algebra 189, 2 (1997), 377–389.

[26] Stallings, J. R. Group theory and three-dimensional manifolds. Yale University Press New Haven, 1971.
[43] Wise, D. T. *From riches to RAAGS: 3-manifolds, right-angled Artin groups, and cubical geometry*, vol. 117. American Mathematical Soc., 2012.

[44] Wise, D. T. Recubulating free groups. *Israel Journal of Mathematics* 191, 1 (2012), 337–345.

[45] Woodhouse, D. J. Revisiting Leighton’s Theorem with the Haar Measure. *arXiv:1806.08196 [math]* (June 2018). arXiv: 1806.08196.

School of Mathematics, University of Bristol, Bristol, UK  
*E-mail address: markfhagen@gmail.com*

Department of Mathematics and Statistics, University of New Brunswick, Fredericton, New Brunswick, Canada  
*E-mail address: ntouikan@unb.ca*