ON CERTAIN CUNTZ-PIMSNER ALGEBRAS

ALEX KUMJIAN

Abstract. Let $A$ be a separable unital C*-algebra and let $\pi : A \to L(H)$ be a faithful representation of $A$ on a separable Hilbert space $H$ such that $\pi(A) \cap K(H) = \{0\}$. We show that $O_E$, the Cuntz-Pimsner algebra associated to the Hilbert $A$-bimodule $E = \mathcal{H} \otimes \mathbb{C}A$, is simple and purely infinite. If $A$ is nuclear and belongs to the bootstrap class to which the UCT applies, then the same applies to $O_E$. Hence by the Kirchberg-Phillips Theorem the isomorphism class of $O_E$ only depends on the $K$-theory of $A$ and the class of the unit.

In his seminal paper [Pm], Pimsner constructed a C*-algebra $O_E$ from a Hilbert bimodule over a C*-algebra $A$ as a quotient of a concrete C*-algebra $T_E$, an analogue of the Toeplitz algebra, acting on the Fock space associated to $E$. There has recently been much interest in these Cuntz-Pimsner algebras (or Cuntz-Krieger-Pimsner algebras), which generalize both crossed products by $\mathbb{Z}$ and Cuntz-Krieger algebras, as well as the associated Toeplitz algebras. The structure of these C*-algebras is not yet fully understood, although considerable progress has been made. For example, Pimsner found a six-term exact sequence for the $K$-theory of $O_E$ which generalizes the Pimsner-Voiculescu exact sequence (see [Pm, Theorem 4.8]); conditions for simplicity were found in [Sc2, MS, KPW1, DPW] and for pure infiniteness in [Z].

The purpose of the present note is to analyze the structure of Cuntz-Pimsner algebras associated to a certain class of Hilbert bimodules. Let $A$ be a separable unital C*-algebra and let $\pi : A \to L(H)$ be a faithful representation of $A$ on a separable Hilbert space $H$ such that $\pi(A) \cap K(H) = \{0\}$. Then $E = \mathcal{H} \otimes \mathbb{C}A$ is a Hilbert bimodule over $A$ in a natural way. We show that $O_E$ is separable, simple and purely infinite. If $A$ is nuclear and in the bootstrap class, then the same holds for $O_E$ and thus by the Kirchberg-Phillips theorem the isomorphism class of $O_E$ is completely determined by the $K$-theory of $A$ together with the class of the unit (since $O_E$ is $KK$-equivalent to $A$).

Many examples of Cuntz-Pimsner algebras found in the literature arise from Hilbert bimodules which are finitely generated and projective; in such cases the left action must consist entirely of compact operators. Our examples do not fall in this class; in fact, the left action has trivial intersection with the compacts. And this has some interesting consequences: $O_E \cong T_E$ (see [Pm, Corollary 3.14]) and the natural embedding $A \hookrightarrow O_E$ induces a $KK$-equivalence (see [Pm, Corollary 4.5]).

In §1 we review some basic facts concerning the construction of $T_E$ as operators on the Fock space of $E$ and the gauge action $\lambda : \mathbb{T} \to \text{Aut}(T_E)$. We assume that the left action of $A$ does not meet the compacts $K(E)$ and identify $O_E$ with $T_E$. The fixed point algebra $F_E$, the analogue of the AF-core of a Cuntz-Krieger algebra, contains a canonical descending sequence of essential ideals indexed by $\mathbb{N}$ with trivial intersection. The crossed product $O_E \rtimes \lambda \mathbb{T}$ has a similar collection of essential ideals indexed by $\mathbb{Z}$ on which the dual group of automorphisms acts in a natural way. By Takesaki-Takai duality

$$O_E \otimes K(L^2(\mathbb{T})) \cong (O_E \rtimes \lambda \mathbb{T}) \rtimes \lambda \mathbb{Z};$$
hence, much of the structure of $O_E$ is revealed through an analysis of the double crossed product. In §3 we show that if $E$ is the Hilbert bimodule over $A$ associated to a representation as described above, then for every nonzero positive element $d \in O_E$ there is a $z \in O_E$ so that $z^* dz = 1$; it follows that $O_E$ is simple and purely infinite (see Theorem 2.3). The proof of this proceeds through a sequence of lemmas and is patterned on the proof of [Rø, Theorem 2.1], which is in turn based on a key lemma of Kishimoto (see [Ks, Lemma 3.2]). Our argument uses the version of this lemma found in [OP3, Lemma 7.1] and this requires that we show that the Connes spectrum of the dual action is full (this is also an ingredient in the proof of simplicity found in [DPW]). We invoke a version of a key lemma of Rørdam for crossed products by $\mathbb{Z}$ which arise from automorphisms with full Connes spectrum. The fact that $O_E$ embeds equivariantly into $(O_E \rtimes_A \mathbb{T}) \rtimes_{\chi} \mathbb{Z}$ allows us to apply this lemma to $O_E$. In §4 we use the Kirchberg-Phillips theorem to collect some consequences of this theorem as indicated above and discuss certain connections with reduced (amalgamated) free products.

We fix some notation and terminology. Given a C*-algebra $B$ we let $\hat{B}$ denote its spectrum, that is, the collection of irreducible representations modulo unitary equivalence endowed with the Jacobson topology (see [Pd, §4.1]). If $I$ is an ideal in a C*-algebra $B$, then every irreducible representation of $I$ extends uniquely to an irreducible representation of $B$. This allows one to identify $\hat{I}$ with an open subset of $\hat{B}$, the complement of which consists of the classes of irreducible representations which vanish on $I$. Given a *-automorphism $\beta$ of a C*-algebra $B$, let $\Gamma(\beta)$ denote the Connes spectrum of $\beta$ (see [O, Co] or [Pd, §8.8]): recall that

$$\Gamma(\beta) = \bigcap_H \text{Sp}(\beta|_H)$$

where the intersection is taken over all $\beta$-invariant hereditary subalgebras $H$. A C*-algebra is said to be purely infinite if every hereditary subalgebra contains an infinite projection.

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### 1. Preliminaries

We review some basic facts concerning Cuntz-Pimsner algebras; we shall be mainly interested in those which arise from bimodules for which the left action has trivial intersection with the compacts (see Remark §3). Let $A$ be a C*-algebra.

**Definition 1.1.** (see [Rø, Kk, Fr]) Let $E$ be a right $A$-module. Then $E$ is said to be (a right) pre-Hilbert $A$-module if it is equipped with an $A$-valued inner product $\langle \cdot, \cdot \rangle_A$ which satisfies the following conditions for all $\xi, \eta, \zeta \in E$, $s, t \in \mathbb{C}$, and $a \in A$:

i. $\langle \xi, s\eta + t\zeta \rangle_A = s\langle \xi, \eta \rangle_A + t\langle \xi, \zeta \rangle_A$

ii. $\langle \xi, t\eta \rangle_A = \langle \xi, t \eta \rangle_A a$

iii. $\langle \eta, \xi \rangle_A = \langle \xi, \eta \rangle_A^*$

iv. $\langle \xi, \xi \rangle_A \geq 0$ and $\langle \xi, \xi \rangle_A = 0$ only if $\xi = 0$.

$E$ is said to be a (right) Hilbert $A$-module if it is complete in the norm: $\|\xi\| = \|\langle \xi, \cdot \rangle_A\|^{1/2}$.

Let $E$ be a Hilbert $A$-module. Then $E$ is said to be full if the span of the values of the inner product is dense. The collection of bounded adjointable operators on $E$, $\mathcal{L}(E)$, is a C*-algebra. The closure of the span of operators of the form $\theta_{\xi, \eta}$ for $\xi, \eta \in E$ (where $\theta_{\xi, \eta}(\zeta) = \langle \xi, \zeta \rangle_A$ for $\zeta \in E$) forms an essential ideal in $\mathcal{L}(E)$ which is denoted $\mathcal{K}(E)$. $A$ Hilbert space is a Hilbert module over $\mathbb{C}$.

**Definition 1.2.** Let $E$ be a Hilbert $A$-module and $\varphi : A \to \mathcal{L}(E)$ be an injective *-homomorphism. Then the pair $(E, \varphi)$ is said to be Hilbert bimodule over $A$ (or Hilbert $A$-bimodule).

Pimsner defines the Cuntz-Pimsner algebra $O_E$ as a quotient of the analogue of the Toeplitz algebra, $\mathcal{T}_E$, generated by creation operators on the Fock space of $E$ (see [Pm]). The injectivity
of \( \varphi \) is not really necessary (see \cite[Remark 1.2.1]{Pn}). We will henceforth assume that \( E \) is full (see \cite[Remark 1.2.3]{Pn}).

The Fock space of \( E \) is the Hilbert \( A \)-module
\[
\mathcal{E}_+ = \bigoplus_{n=0}^{\infty} E^\otimes n
\]
where \( E^\otimes 0 = A \) and for \( n > 0 \), \( E^\otimes n \) is the \( n \)-fold tensor product:
\[
E^\otimes n = E \otimes_A \cdots \otimes_A E.
\]
Note that \( \mathcal{E}_+ \) is also a Hilbert \( A \)-bimodule with left action defined by \( \varphi_+(a)b = ab \) for \( a, b \in A = E^\otimes 0 \) and
\[
\varphi_+(a)(\xi_1 \otimes \cdots \otimes \xi_n) = \varphi(a)\xi_1 \otimes \cdots \otimes \xi_n
\]
for \( a \in A \) and \( \xi_1 \otimes \cdots \otimes \xi_n \in E^\otimes n \). Then \( T_E \subset \mathcal{L}(\mathcal{E}_+) \) is the \( C^* \)-algebra generated by the creation operators \( T_\xi \) for \( \xi \in E \) where
\[
T_\xi(a) = \xi a
\]
and
\[
T_\xi(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n.
\]
Observe that \( T_\xi^* T_\eta = \varphi_+(\langle \xi, \eta \rangle A) \) for \( \xi, \eta \in E \). Since \( E \) is full, \( \varphi_+(A) \subset T_E \); let \( \iota : A \hookrightarrow T_E \) denote the embedding. Note that one may define \( T_\xi \) for \( \xi \in E^\otimes n \) in an analogous manner and that we have \( T_\xi^* T_\eta = \iota(\langle \xi, \eta \rangle A) \) for \( \xi, \eta \in E^\otimes n \). There is an embedding \( \iota_n : K(E^\otimes n) \hookrightarrow T_E \) (identify \( K(E^\otimes 0) \) with \( A \)), given for \( n > 0 \) by
\[
\iota_n(\theta_{\xi, \eta}) = T_\xi T_\eta^* \text{ for } \xi, \eta \in E^\otimes n.
\]
Note that such operators preserve the grading of \( \mathcal{E}_+ \) and that there is an embedding \( K(E^\otimes n) \hookrightarrow \mathcal{L}(E^\otimes m) \) for \( m \geq n \). Let \( C_n \) denote the \( C^* \)-subalgebra of \( T_E \) generated by operators of the form \( T_\xi T_\eta^* \) for \( \xi, \eta \in E^\otimes k \) with \( k \leq n \) (by convention \( C_0 = \iota(A) \)). Then the \( C_n \) form an ascending family of \( C^* \)-subalgebras.

**Remark 1.3.** Suppose \( \varphi(A) \cap K(E) = \{0\} \); then the natural map \( C_n \to \mathcal{L}(E^\otimes m) \) is an embedding for \( m \geq n \). By \cite[Corollary 3.14]{Pn} \( T_E \cong \mathcal{O}_E \) and the inclusion \( A \hookrightarrow \mathcal{O}_E \) induces a KK-equivalence (see \cite[Corollary 4.5]{Pn}). Under the isomorphism of \( T_E \) with \( \mathcal{O}_E \), \( \cup_n C_n \) is mapped to \( \mathcal{F}_E \), the analog of the AF core of a Cuntz-Krieger algebra.

For the remainder of this section we shall tacitly assume that \( \varphi(A) \cap K(E) = \{0\} \) and identify \( T_E \) with \( \mathcal{O}_E \).

**Proposition 1.4.** For each \( n \in \mathbb{N} \) the \( C^* \)-subalgebra, \( J_n \), generated by \( \iota_n(K(E^\otimes k)) \) for \( k \geq n \) is an essential ideal in \( \mathcal{F}_E \). We obtain a descending sequence of ideals
\[
J_0 \supset J_1 \supset J_2 \supset \cdots
\]
with \( J_0 = \mathcal{F}_E \) and \( \cap_n J_n = \{0\} \). Furthermore, \( J_n/J_{n+1} \cong K(E^\otimes n) \) (thus \( J_n/J_{n+1} \) is strong Morita equivalent to \( A \)) and the restriction of the quotient map yields an isomorphism \( C_n \cong \mathcal{F}_E/J_{n+1} \).

**Proof.** Given \( n \in \mathbb{N} \) it is clear that \( J_n \) is an ideal (see \cite[Definition 2.1]{Pn}). To see that \( J_n \) is essential it suffices to show that for every \( m \) and nonzero element \( c \in C_m \) there is an element \( d \in K(E^\otimes k) \) for some \( k \geq n \) such that \( c d \neq 0 \). Let \( k \) be an integer with \( k \geq \max(m, n) \); since the map from \( C_m \) to \( \mathcal{L}(E^\otimes k) \) is an embedding for \( k \geq m \), \( c \xi \neq 0 \) for some \( \xi \in E^\otimes k \). Then \( c T_\xi T_\xi^* \neq 0 \) and we take \( d = \theta_{\xi, \xi} \).

The \( J_n \) form a descending sequence of ideals by construction. Since \( \varphi(A) \cap K(E) = \{0\} \), \( C_m \cap J_n = \{0\} \) for \( m < n \). Hence, \( \cap_n J_n = \{0\} \), for \( \mathcal{F}_E \) is the inductive limit of the \( C_m \). Further, for each \( n \) we have
\[
J_n = \iota_n(K(E^\otimes n)) + J_{n+1} \quad \text{and} \quad \iota_n(K(E^\otimes n)) \cap J_{n+1} = \{0\};
\]
it follows that \( J_n/J_{n+1} \cong K(E^\otimes n) \). Finally, since
\[
\mathcal{F}_E = C_n + J_{n+1} \quad \text{and} \quad C_n \cap J_{n+1} = \{0\},
\]
we have \( C_n \cong \mathcal{F}_E/J_{n+1} \). \qed
There is a strongly continuous action
\[ \lambda : \mathbb{T} \to \text{Aut} (\mathcal{O}_E) \]
such that \( \lambda_t(T_\xi) = tT_\xi \). The fixed point algebra under this action is \( \mathcal{F}_E \) and we have a faithful conditional expectation \( P_E : \mathcal{O}_E \to \mathcal{F}_E \) given by
\[ P_E(x) = \int_{\mathbb{T}} \lambda_t(x) dt. \]

Consider the spectral subspaces of \( \mathcal{O}_E \) under this action: for \( n \in \mathbb{Z} \)
\[ (\mathcal{O}_E)_n = \{ x \in \mathcal{O}_E : \lambda_t(x) = t^n x \text{ for all } t \in \mathbb{T} \}. \]

**Remark 1.5.** Note that \( (\mathcal{O}_E)_n \) is the closure of the span of elements of the form \( T_\xi T_\eta^* \) where \( \xi \in E^\otimes k \) and \( \eta \in E^\otimes l \) with \( n = k - l \). For \( n \geq 0 \) and \( x \in (\mathcal{O}_E)_n \) we have \( x^* x \in \mathcal{F}_E \) and \( xx^* \in J_n \).

We may regard \( (\mathcal{O}_E)_n \) as an \( J_n \)-\( \mathcal{F}_E \)-equivalence bimodule (see [Ri]); hence, \( J_n \) is strong Morita equivalent to \( \mathcal{F}_E \) for each \( n \geq 0 \). If we regard \( (\mathcal{O}_E)_1 \) as a Hilbert \( \mathcal{F}_E \)-bimodule we have (cf. [Pm, §2] and [Sc2, §1.4])
\[ (\mathcal{O}_E)_1 \cong E \otimes_A \mathcal{F}_E, \]
where the isomorphism is implemented by the map \( \xi \otimes a \mapsto T_\xi a \). The crossed product \( \mathcal{O}_E \rtimes_{\lambda} \mathbb{T} \) may be identified with the closure of the subalgebra of \( \mathcal{O}_E \otimes \mathcal{K}(\ell^2(\mathbb{Z})) \) consisting of finite sums of the form
\[ \sum x_{ij} \otimes e_{ij} \]
where \( e_{ij} \) are the standard rank one partial isometries in \( \mathcal{K}(\ell^2(\mathbb{Z})) \) and \( x_{ij} \in (\mathcal{O}_E)_{j-i} \).

Let \( \hat{\lambda} : \mathbb{Z} \to \text{Aut} (\mathcal{O}_E \rtimes_{\lambda} \mathbb{T}) \) denote the dual automorphism group.

**Proposition 1.6.** There is an embedding \( \epsilon : \mathcal{F}_E \hookrightarrow \mathcal{O}_E \rtimes_{\lambda} \mathbb{T} \) onto a corner and a collection of essential ideals \( \{I_n\}_{n \in \mathbb{Z}} \) in \( \mathcal{O}_E \rtimes_{\lambda} \mathbb{T} \) satisfying the following conditions:

i. For all \( n \in \mathbb{Z} \), \( \mathcal{F}_E \) is strong Morita equivalent to \( I_n \) and \( A \) is strong Morita equivalent to \( I_n/I_{n+1} \).

ii. For all \( n \geq 0 \), \( \epsilon(J_n) = \epsilon(1) I_n \epsilon(1) \).

iii. \( I_n \subset I_m \) if \( m \leq n \).

iv. \( \cap_n I_n = \{0\} \)

v. \( \cup_n I_n = \mathcal{O}_E \rtimes_{\lambda} \mathbb{T} \)

vi. \( \hat{\lambda}_k(I_n) = I_{n+k} \)

**Proof.** We use the identification of \( \mathcal{O}_E \rtimes_{\lambda} \mathbb{T} \) with a \( C^* \)-subalgebra of \( \mathcal{O}_E \otimes \mathcal{K}(\ell^2(\mathbb{Z})) \) given in Remark 1.3. For each \( n \) let \( I_n \) be the ideal generated by \( p_n = 1 \otimes e_{nn} \). Since \( \mathcal{F}_E = (\mathcal{O}_E)_0 \), it follows that \( \mathcal{F}_E \) is isomorphic to the corner determined by \( p_n \) and thus is strong Morita equivalent to \( I_n \). The desired embedding \( \epsilon : \mathcal{F}_E \hookrightarrow \mathcal{O}_E \rtimes_{\lambda} \mathbb{T} \) is given by \( \epsilon(a) = a \otimes e_{00} \).

Given an element of the form \( a_{mn} = x_{mn} \otimes e_{nn} \) in \( \mathcal{O}_E \rtimes_{\lambda} \mathbb{T} \) with \( m \leq n \), we have
\[ a_{mn}^* a_{mn} = x_{mn}^* x_{mn} \otimes e_{nn} \quad \text{and} \quad a_{mn} a_{mn}^* = x_{mn}^* x_{mn} \otimes e_{mm} \]
with \( x_{mn}^* x_{mn} \in J_{n-m} \); since \( p_n \) may be expressed as a finite sum of elements of the form \( a_{mn}^* a_{mn} \), it follows that \( I_n \subset I_m \) and that \( p_m I_n p_m = J_{n-m} \otimes e_{mm} \). Thus \( \epsilon(J_n) = \epsilon(1) I_n \epsilon(1) \) for all \( n \geq 0 \). Assertion (vi) follows from the fact that \( \hat{\lambda}_k(p_n) = 1 \otimes p_{n+k} \). The remaining assertions follow from Proposition 1.4.\( \square \)
2. $\mathcal{O}_E$ is simple and purely infinite

Let $A$ be a separable unital C*-algebra and let $\pi : A \to \mathcal{L}(\mathcal{H})$ be a faithful representation of $A$ on a separable Hilbert space $\mathcal{H}$.

**Proposition 2.1.** With $A$ and $\pi : A \to \mathcal{L}(\mathcal{H})$ as above,

$$E = \mathcal{H} \otimes \mathcal{C} A$$

is a full Hilbert bimodule over $A$ under the operations

$$\langle \xi \otimes a, \eta \otimes b \rangle_A = \langle \xi, \eta \rangle a^* b, \quad \varphi(a)(\xi \otimes b) = \pi(a)\xi \otimes b$$

for all $\xi, \eta \in \mathcal{H}$ and $a, b \in A$. Moreover, if $\pi(A) \cap \mathcal{K}(\mathcal{H}) = \{0\}$, then $\varphi(A) \cap \mathcal{K}(E) = \{0\}$ and $\mathcal{O}_E \cong \mathcal{T}_E$.

**Proof.** Note that $E = \mathcal{H} \otimes \mathcal{C} A$ is the tensor product of the Hilbert $A$-C-bimodule $\mathcal{H}$ and the Hilbert $\mathcal{C}$-$A$-bimodule $A$ as defined by Rieffel in [R] (see also [L, Ch. 4]). The natural map from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(E) = \mathcal{H} \otimes \mathcal{C} A$ induces an embedding $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H}) \to \mathcal{L}(E)/\mathcal{K}(E)$ (since $\mathcal{K}(\mathcal{H})$ is mapped into $\mathcal{K}(E)$ and the Calkin algebra $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is simple). Hence, if $\pi(A) \cap \mathcal{K}(\mathcal{H}) = \{0\}$, then $\varphi(A) \cap \mathcal{K}(E) = \{0\}$. The last assertion, $\mathcal{O}_E \cong \mathcal{T}_E$, follows by [Pm, Corollary 3.14].

Henceforth, we assume that $\pi(A) \cap \mathcal{K}(\mathcal{H}) = \{0\}$ and identify $\mathcal{O}_E$ with $\mathcal{T}_E$. The aim of this section is to show that $\mathcal{O}_E$ is simple and purely infinite. Simplicity may be proven directly by invoking [Sc2, Theorem 3.9]: if $A$ is unital and $E$ is full, then $\mathcal{O}_E$ is simple if and only if $E$ is minimal and nonperiodic. Lemma 2.3 would then be a consequence of [OP, Theorem 6.5]. We follow a more indirect route patterned on the proof of [Re, Theorem 2.1]: this will also show that $\mathcal{O}_E$ is purely infinite.

**Remark 2.2.** With $E = \mathcal{H} \otimes \mathcal{C} A$ as above, we have $E^\otimes n \cong \mathcal{H}^\otimes n \otimes \mathcal{C} A$ via the map

$$(\xi_1 \otimes a_1) \otimes (\xi_2 \otimes a_2) \otimes \cdots \otimes (\xi_n \otimes a_n) \mapsto (\xi_1 \otimes \pi(a_1)\xi_2 \otimes \cdots \otimes \pi(a_{n-1})\xi_n) \otimes a_n;$$

similarly, if $\sigma : A \to \mathcal{L}(\mathfrak{R})$ is a representation of $A$ on a Hilbert space $\mathfrak{R}$, then

$$E^\otimes n \otimes_A \mathfrak{R} \cong E^{\otimes n-1} \otimes_A \mathcal{H} \otimes \mathfrak{R}.$$ 

Recall that the action of $\mathcal{F}_E$ on Fock space preserves the natural grading. Let $\tilde{\sigma}_n$ denote the representation of $\mathcal{F}_E$ on $E^\otimes n \otimes_A \mathfrak{R}$ given by left action on $E^\otimes n$. Then the restriction of $\tilde{\sigma}_n$ to $C_{n-1}$ is faithful: indeed, this follows from the facts that the natural map

$$\mathcal{L}(E^{\otimes n-1}) \to \mathcal{L}(E^{\otimes n-1} \otimes_A \mathcal{H} \otimes \mathfrak{R}) \cong \mathcal{L}(E^{\otimes n} \otimes_A \mathfrak{R})$$

is an embedding (since $\pi$ is faithful) and that $\tilde{\sigma}_n|_{\mathcal{L}(E^{\otimes n-1})}$ factors through $\mathcal{L}(E^{\otimes n-1})$. Note that $\tilde{\sigma}_n$ is equivalent to the representation of $\mathcal{F}_E$ obtained from $\sigma$ as follows: use the strong Morita equivalence between $A$ and $J_n/J_{n+1}$ to obtain a representation of $J_n/J_{n+1}$ and extend this to a representation of $\mathcal{F}_E$. Since the restriction of $\tilde{\sigma}_n$ to $C_{n-1}$ is faithful, $\ker \tilde{\sigma}_n \subset J_n$ (see Proposition 1.4). It follows that the closure of a point in $\tilde{I}_n - \tilde{I}_{n+1}$ contains the complement of $\tilde{J}_n$. A similar assertion holds for $\mathcal{O}_E \rtimes \mathbb{Z}$: for any $n \in \mathbb{Z}$ the closure of a point in $\tilde{I}_n - \tilde{I}_{n+1}$ contains the complement of $\tilde{I}_n$.

**Lemma 2.3.** With $A$ and $E$ as above, $\Gamma(\hat{\lambda}_1) = \mathbb{T}$ where $\hat{\lambda}$ is the dual action of $\mathbb{Z}$ on $\mathcal{O}_E \rtimes \mathbb{Z}$.

**Proof.** By [OP2, Theorem 4.6] it suffices to find a dense invariant subset of $(\mathcal{O}_E \rtimes \mathbb{Z})^\wedge$ on which $\hat{\lambda}^*_1$ acts freely. That is, we must find an irreducible representation $\sigma$ of $\mathcal{O}_E \rtimes \mathbb{Z}$ such that,

$$\{[\sigma \circ \hat{\lambda}_n] : n \in \mathbb{Z}\},$$

the orbit of the unitary equivalence class of $\sigma$ under $\hat{\lambda}^*$, is dense in $(\mathcal{O}_E \rtimes \mathbb{Z})^\wedge$ and $[\sigma \circ \hat{\lambda}_n] \neq [\sigma \circ \hat{\lambda}_m]$ if $m \neq n$. Let $\sigma_0$ be an irreducible representation of $A$ and use the strong Morita equivalence between $A$ and $I_0/I_1$ to obtain an irreducible representation $\sigma'$ of $I_0/I_1$. Then
σ, the extension of σ to $O_E \rtimes \lambda \mathbb{T}$, is also irreducible. The classes $[\sigma \circ \lambda_n]$ are distinct, for if $m < n$, σ ∘ $\lambda_m$ vanishes on $I_n$. Moreover, for each $n \in \mathbb{Z}$ the closure of $[\sigma \circ \lambda_n]$ in $(O_E \rtimes \lambda \mathbb{T})^\sim$ includes the classes of all irreducible representations which vanish on $I_n$ (since $[\sigma \circ \lambda_n] \in \tilde{I}_n - \tilde{I}_{n+1}$, see Remark 2.2). Hence, $\{[\sigma \circ \lambda_n] : n \in \mathbb{Z}\}$ is dense in $(O_E \rtimes \lambda \mathbb{T})^\sim$.

Using Takesaki-Takai duality we show below that a C*-algebra $D$ equipped with an action $\alpha$ of $\mathbb{T}$ may be embedded equivariantly as a corner in $(D \rtimes \alpha \mathbb{T}) \rtimes \alpha \mathbb{Z}$. This fact is related to Rosenberg’s observation that the fixed point algebra under a compact group action embeds as a corner in the crossed product (see [Rö]).

**Proposition 2.4.** Given a unital C*-algebra $D$ and a strongly continuous action $\alpha : \mathbb{T} \to \text{Aut}(D)$, there is an isomorphism $\psi$ of $D$ onto a full corner of $(D \rtimes \alpha \mathbb{T}) \rtimes \alpha \mathbb{Z}$ which is equivariant in the sense that $\hat{\alpha}_t \circ \psi = \psi \circ \alpha_t$ for all $t \in \mathbb{T}$. Moreover, $\psi(1) \in D \rtimes \alpha \mathbb{T}$.

**Proof.** By Takesaki-Takai duality [Pa, 7.9.3] there is an isomorphism

$$
\gamma : D \otimes \mathcal{K}(L^2(\mathbb{T})) \cong (D \rtimes \alpha \mathbb{T}) \rtimes \alpha \mathbb{Z},
$$

which is equivariant with respect to $\alpha \otimes \text{Ad} \rho$ and $\hat{\alpha}$ (where $\rho$ is the right regular representation of $\mathbb{T}$ on $L^2(\mathbb{T})$). The desired embedding is obtained by finding an Ad $\rho$ invariant minimal projection $p$ in $\mathcal{K}(L^2(\mathbb{T}))$ (cf. [Rö]): set $\psi(d) = \gamma(d \otimes p)$ for $d \in D$. Since $\psi$ is equivariant, $\psi(1)$ is in the fixed point algebra of $\hat{\alpha}$; hence, $\psi(1) \in D \rtimes \alpha \mathbb{T}$.

The following lemma is adapted from [Rö, Lemma 2.4]; the proof is patterned on Rørdam’s but we substitute [OP3, Lemma 7.1] for [Ks, Lemma 3.2].

**Lemma 2.5.** Let $B$ be a C*-algebra and let $\beta$ be an automorphism of $B$ such that $\Gamma(\beta) = \mathbb{T}$ and let $P$ denote the canonical conditional expectation from $B \rtimes \beta \mathbb{Z}$ to $B$. Then for every positive element $y \in B \rtimes \beta \mathbb{Z}$ and $\varepsilon > 0$ there are positive elements $x, b \in B$ such that

$$
\|b\| > \|P(y)\| - \varepsilon, \quad \|x\| \leq 1 \quad \text{and} \quad \|xy - b\| < \varepsilon.
$$

If $y$ is in the corner determined by a projection $p \in B$, then $x, b$ may also be chosen to be in the corner.

**Proof.** As in the proof of [Rö, Lemma 2.4] we may assume (by perturbing $y$ if necessary) that $y$ is of the form

$$
y = y_{-n}u^{-n} + \cdots + y_{-1}u^{-1} + y_0 + y_1u + \cdots + y_nu^n
$$

for some $n$ where $y_j \in B$ and $u$ is the canonical unitary in $B \rtimes \beta \mathbb{Z}$ implementing the automorphism $\beta$; note that $y_0 = P(y)$ is positive. By [OP3, Theorem 10.4] $\beta^k$ is properly outer for all $k \neq 0$. Hence, by [OP3, Lemma 7.1] there is a positive element $x$ with $\|x\| = 1$ such that

$$
\|xy_0x\| > \|y_0\| - \varepsilon, \quad \text{and} \quad \|xy_ku^kx\| = \|xy_k\beta^k(x)\| < \varepsilon/2n
$$

for $0 < |k| \leq n$. Set $b = xy_0x$; then a straightforward calculation yields $\|xy - b\| < \varepsilon$. We now verify the last assertion. Suppose that $y$ is in the corner determined by a projection $p \in B$; we may again assume that $y$ is of the above form. Since $P$ is a conditional expectation onto $B$, $y_0 = P(y)$ is also in the corner determined by $p$. In the proof of [OP3, Lemma 7.1] the positive element $x$ is constructed in the hereditary subalgebra determined by $y_0$; hence we may assume that $x$ and therefore also $b = xy_0x$ lies in the same corner. \(\square\)

Recall that $C_n$ is the C*-subalgebra of $\mathcal{F}_E$ generated by operators of the form $T_\xi T_\eta^*$ for $\xi, \eta \in E^\otimes k$ with $k \leq n$ and that they form an ascending family of C*-subalgebras with dense union. The subspace $E^\otimes n$ is left invariant by $C_n$ and one has an embedding $C_n \hookrightarrow \mathcal{L}(E^\otimes n)$.
Lemma 2.6. Given a positive element \( c \in C_n \) and \( \varepsilon > 0 \), there is \( \xi \in E^{\otimes n} \) with \( \|\xi\| = 1 \) such that \( T_\xi^* c T_\xi \in C_0 \) and \( \|T_\xi^* c T_\xi\| > \|c\| - \varepsilon \).

Proof. The first assertion follows from a straightforward calculation: given \( c \in C_n \) and \( \xi \in E^{\otimes n} \), then \( c\xi \in E^{\otimes n} \) and

\[
T_\xi^* c T_\xi = T_\xi^* T_\xi = \iota(\langle \xi, c\xi \rangle_A) \in C_0.
\]

The second assertion follows from the embedding \( C_n \hookrightarrow \mathcal{L}(E^{\otimes n}) \) and the fact

\[
\|d\| = \sup\{\|\langle \xi, d\xi \rangle_A\| : \xi \in E^{\otimes n}, \|\xi\| = 1\}
\]

for \( d \in \mathcal{L}(E^{\otimes n}) \) positive.

Lemma 2.7. Given a positive element \( a \in A \) and \( \varepsilon > 0 \) with \( \|a\| > \varepsilon \), there is \( \eta \in E \) with \( \|\eta\| \leq (\|a\| - \varepsilon)^{-1/2} \) such that \( T_\eta^* \iota(a) T_\eta = 1 \).

Proof. Let \( f \) be a continuous nonzero real-valued function supported on the interval \([\|a\| - \varepsilon, \|a\|]\) and choose a vector \( \zeta \in \pi(f(a))B \) such that \( \langle \zeta, \pi(a)\zeta \rangle = 1 \); we have

\[
(||a||-\varepsilon)\|\zeta\|^2 \leq ||\langle \zeta, \pi(a)\zeta \rangle|| = 1.
\]

Then \( \eta = \zeta \otimes 1 \in E \) satisfies the desired conditions.

It will now follow that \( O_E \) is simple and purely infinite (cf. proof of [Rø, Theorem 2.1]).

Theorem 2.8. For every nonzero positive element \( d \in O_E \) there is a \( z \in O_E \) so that \( z^*dz = 1 \). Hence, \( O_E \) is simple and purely infinite.

Proof. Let \( d \in O_E \) be a nonzero positive element and choose \( \varepsilon \) so that \( 0 < \varepsilon < \|P(d)\|/4 \). By Proposition 2.4 there is a \( \mathbb{T} \)-equivariant isomorphism \( \psi \) from \( O_E \) onto a corner of \( (O_E \rtimes \Lambda \mathbb{T}) \rtimes \Lambda \mathbb{Z} \) determined by a projection \( p \in O_E \rtimes \Lambda \mathbb{T} \). We now apply Lemma 2.7 to the element \( y = \psi(d) \) and the automorphism \( \beta = \lambda_1 \) (note \( \Gamma(\lambda_1) = \mathbb{T} \) by Lemma 2.3). We identify \( O_E \) with the corner determined by \( p \); note that under this identification \( F_E \) is identified with \( p(O_E \rtimes \Lambda \mathbb{T})p \). There are then positive elements \( x, b \in F_E \) so that

\[
\|b\| > \|P(d)\| - \varepsilon, \quad \|x\| \leq 1 \quad \text{and} \quad \|xdx - b\| < \varepsilon.
\]

Since \( \cup_n C_n \) is dense in \( F_E \) we may assume that \( b \in C_n \) for some \( n \). Hence, by Lemma 2.6 there is \( \xi \in E^{\otimes n} \) with \( \|\xi\| = 1 \) such that

\[
T_\xi^* b T_\xi \in C_0 \quad \text{and} \quad \|T_\xi^* b T_\xi\| > \|b\| - \varepsilon.
\]

Let \( a \) denote the unique element of \( A \) such that \( \iota(a) = T_\xi^* b T_\xi \); then \( \|a\| > \|P(d)\| - 2\varepsilon \) and

\[
\|T_\xi^* xdx T_\xi - \iota(a)\| = \|T_\xi^* (xdx - b) T_\xi\| < \varepsilon.
\]

By Lemma 2.7 there is \( \eta \in E \) such that \( T_\eta^* \iota(a) T_\eta = 1 \) and

\[
\|\eta\| \leq (\|a\| - \varepsilon)^{-1/2} < (\|P(d)\| - 3\varepsilon)^{-1/2} < \varepsilon^{-1/2}.
\]

It follows that

\[
\|T_\eta^* T_\xi^* xdx T_\xi T_\eta - 1\| = \|T_\eta^* (T_\xi^* xdx T_\xi - \iota(a)) T_\eta\| \leq \|T_\xi^* xdx T_\xi - \iota(a)\|(\varepsilon^{-1/2})^2 < 1.
\]

Therefore, \( c = T_\eta^* T_\xi^* xdx T_\xi T_\eta \) is an invertible positive element and we take \( z = xT_\xi T_\eta c^{-1/2} \).
### 3. Applications and concluding remarks

We collect some applications of the above theorem and consider certain connections with the theory of reduced (amalgamated) free product C*-algebras. First we consider criteria under which the Kirchberg-Phillips Theorem applies (see [Ka, Theorem C], [Pi, Corollary 4.2.2]).

**Theorem 3.1.** Let $A$ be a separable nuclear unital C*-algebra which belongs to the bootstrap class to which the uct applies (see [FrS]); let $\pi : A \to \mathcal{L}(\mathcal{H})$ be a faithful representation of $A$ on a separable Hilbert space $\mathcal{H}$ such that $\pi(A) \cap \mathcal{K}(\mathcal{H}) = \{0\}$ and let $E$ denote the Hilbert $A$-bimodule $\mathcal{H} \otimes_{C} A$. Then $O_E$ is a unital Kirchberg algebra (simple, purely infinite, separable and nuclear) which belongs to the bootstrap class. Hence, the Kirchberg-Phillips Theorem applies and the isomorphism class of $O_E$ only depends on $(K_* (A), [1_A])$ and not on the choice of representation $\pi$.

**Proof.** First note that $O_E$ is simple and purely infinite by Theorem 2.8. If $A$ is nuclear, then the argument given in the proof of [DS, Theorem 2.1] shows that $O_E$ must also be nuclear (alternatively, the nuclearity of $O_E$ follows from the structural results discussed in [Pi]). Hence, $O_E$ is a unital Kirchberg algebra. Recall that the inclusion $A \to O_E$ defines a $KK$-equivalence (see [Pi, Corollary 4.5]) which induces a unit-preserving isomorphism $K_* (A) \cong K_* (O_E)$. Hence, if $A$ is in the bootstrap class, then $O_E$ is also. Therefore, the Kirchberg-Phillips Theorem applies and the isomorphism class of $O_E$ only depends on $(K_* (A), [1_A])$. \qed

Let $X$ be a second countable compact space, let $\mu$ be a nonatomic Borel measure with full support and let

$$\pi : C(X) \to \mathcal{L}(L^2(X, \mu))$$

be the representation given by multiplication of functions. Then $\pi$ is faithful and

$$\pi(C(X)) \cap \mathcal{K}(L^2(X, \mu)) = \{0\}.$$ 

Hence, we may apply the above theorem with $A = C(X)$ and $\mathcal{H} = L^2(X, \mu)$.

**Corollary 3.2.** Let $X$ and $\mu$ be as above. Then

$$E = L^2(X, \mu) \otimes_{\mathbb{C}} C(X)$$

is a Hilbert bimodule over $C(X)$ and $O_E$ is a unital Kirchberg algebra. The embedding $C(X) \to O_E$ induces a (unit preserving) $KK$-equivalence. Hence, the isomorphism class of $O_E$ only depends on $(K_* (C(X)), [1_{C(X)}])$ (and not on $\mu$); moreover, if $X$ is contractible, then $O_E \cong O_\infty$.

The following proposition is Theorem 5.6 of [L] (see also [Ka, Theorem 3]); Lance calls this the Kasparov-Stinespring-Gelfand-Naimark-Segal construction.

**Proposition 3.3.** Let $B$ and $C$ be C*-algebras, let $F$ be a Hilbert $C$-module and let $f : B \to \mathcal{L}(F)$ be a completely positive map, then there is a Hilbert $C$-module $E_f$, a *-homomorphism $\varphi_f : B \to \mathcal{L}(E_f)$ and an element $v_f \in \mathcal{L}(F, E_f)$ such that $f(b) = v_f^* \varphi_f(b) v_f$ and $\varphi_f(B)v_f F$ is dense in $E_f$.

I am grateful to D. Shlyakhtenko for the following observation. Let $\mathcal{T}$ denote the “usual” Toeplitz algebra (i.e. $\mathcal{T}_E$ where $E$ is the 1-dimensional Hilbert bimodule over $\mathbb{C}$) and let $g$ denote the vacuum state on $\mathcal{T}$.

**Proposition 3.4.** Let $A$ be a separable unital C*-algebra and let $\pi : A \to \mathcal{L}(\mathcal{H})$ be a faithful representation of $A$ on a separable Hilbert space $\mathcal{H}$ such that $\pi$ has a cyclic vector $\xi \in \mathcal{H}$. Let $f$ denote the vector state $\langle \xi, \cdot \rangle : \mathcal{H}$ and let $\tilde{f}$ denote the corresponding completely positive map from $A$ to $\mathcal{L}(A)$ (given by $\tilde{f}(a) = f(a) 1$). Then $E = E_{\tilde{f}} \cong \mathcal{H} \otimes A$ and $\mathcal{T}_E$ may be realized as a reduced free product (see [A, V]):

$$(\mathcal{T}_E, h) \cong (A, f) * (\mathcal{T}, g)$$
for some state $h$ on $\mathcal{T}_E$.

Proof. This follows from [Sh, Theorem 2.3, Corollary 2.5].

As a result of this observation part (at least) of Corollary 3.2 follows from the existing literature on reduced free products. The simplicity follows from a theorem of Dykema [Dy, Theorem 2]. Criteria for when reduced free products are purely infinite have been found by Choda, Dykema and Rørdam in a series of papers [DR1, DR2, DC]; but none seem to apply generally to the case considered in the corollary.

A theorem of Speicher (see [Sp]) on reduced amalgamated free products (see [V, §5]) and Toeplitz algebras associated to Hilbert bimodules yields a curious stability property of the algebras we have been considering. The following is the version given in [BDS, Theorem 2.4].

**Proposition 3.5.** Suppose that $E_1$ and $E_2$ are full Hilbert bimodules over the C*-algebra $A$. Then

$$\mathcal{T}_{E_1 \oplus E_2} = \mathcal{T}_{E_1} \ast A \mathcal{T}_{E_2}.$$ 

We obtain the following corollary.

**Corollary 3.6.** Let $A$ be a separable nuclear unital C*-algebra which belongs to the bootstrap class to which the UCT applies (see [RS]) and let $\pi : A \rightarrow \mathcal{L}(\mathcal{H})$ be a faithful representation of $A$ on a separable Hilbert space $\mathcal{H}$ such that $\pi(A) \cap \mathcal{K}(\mathcal{H}) = \{0\}$. Let $E$ be the Hilbert bimodule $\mathcal{H} \otimes_C A$. Then

$$\mathcal{O}_E \cong \mathcal{O}_E \ast A \mathcal{O}_E.$$ 

Proof. Observe that $E \oplus E = (\mathcal{H} \oplus \mathcal{H}) \otimes_C A$. Since $\pi \oplus \pi : A \rightarrow \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ is a faithful representation and $(\pi \oplus \pi)(A) \cap \mathcal{K}(\mathcal{H} \oplus \mathcal{H}) = \{0\}$, the result follows follows from Theorem 3.1 and the above proposition.

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Department of Mathematics, University of Nevada, Reno NV 89557, USA

E-mail address: alex@unr.edu