AFFINE AND FORMAL ABELIAN GROUP SCHEMES ON 
*p*-POLAR RINGS

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Abstract. We show that the functor of 
*p*-typical co-Witt vectors on commutative algebras over a perfect field 
k of characteristic 
p is defined on, and
in fact only depends on, a weaker structure than that of a 
k-algebra. We call this structure a 
p-polar 
k-algebra. By extension, the functors of points for any
*p*-adic affine commutative group scheme and for any formal group are defined
on, and only depend on, 
p-polar structures. In terms of abelian Hopf algebras,
we show that a cofree cocommutative Hopf algebra can be defined on any 
p-polar 
k-algebra 
P,
and it agrees with the cofree commutative Hopf algebra on a commutative 
k-algebra 
A if 
P is the 
p-polar algebra underlying 
A; a dual result holds for free commutative Hopf algebras on finite 
k-coalgebras.

1. Introduction

Let 
p be a prime. We consider the following generalizations of the notion of a

\textbf{Definition.} Let 
k be a (commutative) ring and 
A a 
k-module. A 
p-polar 
k-algebra structure on 
A is a symmetric 
k-multilinear map \( \mu: A^{\otimes_k p} \to A \) such that

(\text{ASSOC}) \quad \mu(\mu(x_1, \ldots, x_p), x_{p+1}, \ldots, x_{2p-1}) \text{ is } \Sigma_{2p-1}\text{-invariant}

for the permutation action of the symmetric group \( \Sigma_{2p-1} \) on 
x_1, \ldots, x_{2p-1} \in A. We will call a 
p-polar \mathbb{Z}\text{-algebra a } 
p\text{-polar ring.}

A morphism of 
p-polar algebras is the evident structure-preserving map, making


\( p \)-polar algebras into a category \( \text{Pol}_p(k) \). We denote by \( \text{pol}_p(k) \) the full subcategory of 
p-polar algebras which are of finite length as \nk-modules.

Clearly, any \nk-algebra \( R \) gives rise to a \np-polar structure for each \( p \) by restriction, called its \textit{polarization} \( \text{pol}(R) \). If \( R \) is of finite length then so is \( \text{pol}(R) \), so polarization gives functors

\[
\begin{align*}
\text{pol}: \text{Alg}_k &\to \text{Pol}_p(k) \\
\text{pol}: \text{alg}_k &\to \text{pol}_p(k)
\end{align*}
\]

from the categories of 
k-algebras and finite-length \nk-algebras, respectively.

Our main results concern the following categories over a perfect field \( k \) of characteristic \( p \):

\begin{itemize}
  \item the category \( \text{AbSch}_k \) of affine, commutative group schemes ("affine groups"),
  \item anti-equivalent to the category of bicommutative Hopf algebras over \( k \);
\end{itemize}
its full subcategory \( \text{AbSch}^p_k \) of \( p \)-adic groups, i.e. group schemes with values in abelian pro-\( p \)-groups). These correspond to bicommutative Hopf algebras \( H \) that are \( p \)-adic: \( H \cong \text{colim}_n H[p^n] \), where \( H[p^n] \) denotes the kernel of the endomorphism \([p^n]\) of \( H \) representing multiplication by \( p^n \).

- the category \( \text{Fgps}_k \) of affine, commutative, formal group schemes (“formal groups”). These are ind-representable functors from the category \( \text{alg}_k \) of finite-dimensional \( k \)-algebras to abelian groups, as in [Fon77]. The category \( \text{Fgps}_k \) is anti-equivalent to the category of complete Hopf algebras, which is the category of cogroup objects in the category of pro-finite dimensional \( k \)-algebras (with monoidal structure given by the profinitely completed tensor product);

- its full subcategory \( \text{Fgps}^p_k \) of formal \( p \)-group schemes, i.e. formal groups taking values in \( p \)-groups.

The categories \( \text{Fgps}_k \) and \( \text{AbSch}_k \) are anti-equivalent by Cartier duality, represented by taking (continuous) \( k \)-linear duals at the level of (complete) Hopf algebras, and this anti-equivalence restricts to an anti-equivalence between \( \text{Fgps}^p_k \) and \( \text{AbSch}^p_k \).

**Theorem 1.1A.** Let \( k \) be a perfect field of characteristic \( p \) and \( G \in \text{AbSch}^p_k \). Then the functor of points of \( G \) factors through \( \text{pol} \), naturally in \( G \):

\[
\begin{align*}
\text{Alg}_k & \xrightarrow{G} \text{Ab} \\
\text{pol} & \downarrow \quad \quad \downarrow \tilde{G} \\
\text{Pol}_p(k) &
\end{align*}
\]

Note the restriction to \( p \)-adic group schemes. There is also a companion result for formal groups where, interestingly, a similar restriction is not necessary:

**Theorem 1.1F.** Let \( k \) be a perfect field of characteristic \( p \) and \( G \in \text{Fgps}_k \). Then the functor of points of \( G \) factors through \( \text{pol} \), naturally in \( G \):

\[
\begin{align*}
\text{alg}_k & \xrightarrow{G} \text{Ab} \\
\text{pol} & \downarrow \quad \quad \downarrow \tilde{G} \\
\text{pol}_p(k) &
\end{align*}
\]

To prove Theorems 1.1A and 1.1F, we make use of free (formal) group scheme functors:

**Lemma 1.2A.** The forgetful functor \( U : \text{AbSch}^p_k \to \text{Alg}_k^{\text{op}} \) from \( p \)-adic affine groups to \( k \)-algebras has a left adjoint \( \text{Fr} \).

**Lemma 1.2F.** The forgetful functor \( U : \text{Fgps}_k \to (\text{Pro-}\text{alg}_k)^{\text{op}} \) from formal groups to pro-finite dimensional \( k \)-algebras has a left adjoint \( \text{Fr} \).

Using these free functors we show:
**Theorem 1.3A.** Let $k$ be a perfect field of characteristic $p$. Then $Fr$ factors through $pol$:

$$\begin{CD}
\text{Alg}_k^{op} @>{Fr}>> \text{AbSch}_k^p \\
\downarrow_{pol} @. @A{Fr}\�� \\
\text{Pol}_p(k)^{op}
\end{CD}$$

**Theorem 1.3F.** Let $k$ be a perfect field of characteristic $p$. Then $Fr$ factors through $pol$:

$$\begin{CD}
\text{Alg}_k^{op} @>{Fr}>> \text{Fgps}_k \\
\downarrow_{pol} @. @A{Fr}\�� \\
\text{pol}_p(k)^{op}
\end{CD}$$

In the latter theorem, the functor $Fr$ is restricted to the subcategory $\text{alg}_k$ of $\text{Pro-}\text{alg}_k$.

Assuming these theorems, we immediately obtain:

**Proofs of Theorems 1.1A and 1.1F.** Given any $M \in \text{AbSch}_k^p$ and $R \in \text{Alg}_k$ (resp. $M \in \text{Fgps}_k$ and $R \in \text{alg}_k$), we have that

$$M(R) = \text{Hom}(\text{Spec } R, M) = \text{Hom}(Fr(R), M),$$

where the last Hom group is of objects of $\text{AbSch}_k^p$ or $\text{Fgps}_k$, respectively. Since $Fr$ factors through $\text{Pol}_p(k)$ (resp. $\text{pol}_p(k)$), so does $M$. $\square$

In terms of Hopf algebras, this can be reformulated in the following way. Theorem 1.3A says that the cofree cocommutative $p$-adic Hopf algebra functor on $k$-algebras factors through $p$-polar algebras. Using the Cartier equivalence between formal groups and bicommutative Hopf algebras, Theorem 1.3F says that the free commutative Hopf algebra functor on finite-dimensional $k$-coalgebras factors through the opposite category of finite-dimensional $p$-polar $k$-algebras.

Instead of trying to prove Thms. 1.3A and 1.3F directly, we take a detour along Dieudonné functors to the land of Witt vectors. The version of Dieudonné functors we are using (cf. Thm. 5.2) are of the form

$$D: (\text{AbSch}_k)^{op} \rightarrow \text{Dmod}_k^p$$

and

$$D^f: (\text{Fgps}_k)^{op} \rightarrow \text{Dmod}_k^F,$$

and define contravariant equivalences between $\text{AbSch}_k^p$ and $\text{Fgps}_k^p$ and certain categories of $W(k)$-modules with two operations $F$ and $V$, called Frobenius and Verschiebung. Here $W(k)$ denotes the ring of $p$-typical Witt vectors of the field $k$. More generally, let $W_n(R)$ be the group of $p$-typical Witt vectors of length $n$ of a ring $R$.

The Verschiebung $V: W_n(R) \rightarrow W_{n+1}(R)$ gives rise to the group of unipotent co-Witt vectors

$$CW^n(R) = \text{colim}(W_1(R) \xrightarrow{V} W_2(R) \xrightarrow{V} \cdots),$$
an object of $\text{Dmod}_k^p$. It has a completion, $CW(R)$, the group of co-Witt vectors, consisting of possibly infinite negatively graded sequences $(\ldots, a_{-1}, a_0)$ of elements of $R$ almost all of which are nilpotent.

We prove:

**Theorem 1.4.** Let $k$ be a perfect field. Then the functors $W_n$, $CW$, and $CW^u$ from $\text{Alg}_k$ to $\text{Dmod}_k^p$ factor naturally through $\text{Pol}_p(k)$.

The following couple of theorems provide the link between co-Witt vectors and free (formal) groups:

**Theorem 1.5F.** Let $k$ be a perfect field of characteristic $p$ and $R$ a finite-dimensional $k$-algebra. Then there is a natural isomorphism

$$\mathbb{D}^f(\text{Fr}(R)) \cong CW(R).$$

Since the right hand side factors through $\text{pol}_p(k)$ by Thm. 1.4, so does the left hand side. This almost proves Thm. 1.3F – but not quite, since $\mathbb{D}^f$ only provides an equivalence of $\text{Dmod}_k^p$ with the full subcategory $\text{Fgps}_k^p$.

To state the affine version of this theorem, let $\mu_{p^n}(R)$ denote the abelian group of $p$-power torsion elements in $R^n$, for a $k$-algebra $R$.

**Theorem 1.5A.** Let $k$ be a perfect field of characteristic $p$ and $R$ a $k$-algebra. Then there is a natural isomorphism

$$D(\text{Fr}(R)) \cong CW^u(R) \oplus (\mu_{p^n}(R \otimes_k \bar{k}) \otimes W(\bar{k}))^{\text{Gal}(k)}$$

where in the last factor, invariants of the absolute Galois group $\text{Gal}(k)$ acting diagonally on $\bar{k}$ and $W(\bar{k})$ are taken.

Again, Thm. 1.3A follows almost from this theorem; the missing piece in this case is to show that the second summand on the right hand side factors through $p$-polar algebras.

**Overview.** In Section 2, we define $p$-polar $k$-algebras and their properties. Section 3 develops the basic theory of $p$-adic Witt vectors for $p$-polar rings and contains a proof of Thm. 1.4. Section 4 contains a review of the structure of the categories $\text{AbSch}_k$ and $\text{Fgps}_k$ along with the proofs of Lemmas 1.2A and 1.2F. Finally, Section 5 contains the setup of the Dieudonné functors and the proofs of Thms. 1.5F, 1.5A, 1.3A, and 1.3F.

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2. **$p$-POLAR RINGS**

Recall the definition of a $p$-polar algebra $A$ from the introduction. We make the following observations:

- The definition is non-unital in nature. If one were to require the existence of an element $1 \in A$ such that $\mu(1, \ldots, 1, x) = x$, this would make $A$ into a commutative unital ring.
- If $p = 2$ then $A$ is a nonunital algebra.
- The expression in (ASSOC) is $(\Sigma_p \times \Sigma_{p-1})$-equivariant by definition. Given commutativity, the condition is akin to an associative law.
Example 2.1. The multiplication on an ordinary (not necessarily unital) ring $A$ restricts to polar ring structure $\text{pol}(A)$.

Example 2.2. We have that $k[x]_{(j)} = \det k[x^{j(1+(p-1)i)} \mid i \geq 0]$ is a polar subalgebra of $k[x]$ for each $j \geq 0$, and it is not an algebra (unless $j = 0$, in which case it is just $k$, or $j > 0$ and $p = 2$, in which case it is an algebra without unity).

Example 2.3. Let $A = k[x]/(xp)$ and $B = k^{p-1}$ nonunital with trivial multiplication. Then $\mu = 0$, and hence $\text{pol}(A) \cong \text{pol}(B)$ as $p$-polar algebras.

Remark 2.4. For a unital algebra $A$, one can recover $A$ from $\text{pol}(A)$ up to non-canonical isomorphism. Indeed, if $B$ is a $p$-polar algebra of the form $\text{pol}(A)$, there is an element $e$ such that $\mu(e, \ldots, e, x) = x$ for all $x \in B$. For instance, the unity of $A$ is such an element, but $B$ does not preserve that information. One easily checks that an algebra structure on $B$ can be defined by $x \cdot y = \mu(e, \ldots, e, x, y)$. With this algebra structure, multiplication by $e$ gives an algebra isomorphism $A \to B$.

For nonunital algebras, this is not possible, as the previous example illustrates.

Proposition 2.5. Let $A$ be a $p$-polar $k$-algebra and $x_1, \ldots, x_n \in A$. Then there is at most one way to multiply $x_1, \ldots, x_n$ together using $\mu$, and the product exists if and only if $n \equiv 1 \pmod{p-1}$.

Proof. Let us first make the statement more rigorous. Define a multiplication scheme to be a rooted, planar, $p$-ary tree whose leaves are labelled with $x_1, \ldots, x_n$. If a multiplication scheme exists, then it gives a prescription of how to multiply the elements $x_i$ by traversing the tree, applying $\mu$ at every internal vertex. If no multiplication scheme exists, then the elements cannot be multiplied together. Since grafting a basic $p$-ary tree of depth 1 onto an existing tree increases the number of leaves by $p - 1$, a multiplication scheme exists if and only if $n \equiv 1 \pmod{p-1}$.

An equivalence relation on the set of all multiplication schemes is generated by:

1. $M' \sim M$ if $M'$ results from $M$ by permuting the outgoing edges of any internal vertex;
2. $M' \sim M$ if $M'$ results from $M$ by permuting the $2p - 1$ outgoing edges of a subtree of the form, along with their subtrees (illustrated for $p = 3$).

These relations correspond, of course, to the symmetry and axiom (ASSOC). Under this equivalence relation, every multiplication scheme is equivalent to a "left-associative" one, i.e. a scheme where a vertex has a non-leaf subtree only if it is the leftmost among its siblings, such as in the diagram above. In such a left-associative multiplication scheme, all leaves can be permuted without changing the equivalence class. Thus all multiplication schemes are equivalent. 

In light of this proposition, we will unambiguously use monomial notations such as $x^p$ for $\mu(x, \ldots, x)$ or $xy^{p-1}$ for $\mu(x, y, \ldots, y)$ in polar rings.
Example 2.6. Let $S$ be a set. The free $p$-polar ring $P(S)$ on $S$ is given by the sub-$p$-polar algebra of the polynomial ring $\mathbb{Z}[S]$ with generators in $S$ spanned by monomials of length congruent to 1 modulo $p - 1$ (or spanned by all nonconstant monomials if $p = 2$).

Definition 2.7. An ideal in a $p$-polar ring $A$ is a subgroup $I$ such that $\mu(a_1, \ldots, a_{p-1}, i) \in I$ whenever $i \in I$. These are exactly the kernels of homomorphisms of $p$-polar rings.

For a subset $S \subseteq A$, the ideal $(S)$ generated by it is defined to be the smallest ideal containing $S$. If $I$ is an ideal, then $I^p = \langle \mu(I, \ldots, I) \rangle$ is a subideal.

3. Witt vectors of $p$-polar rings

The ($p$-typical) Witt vector functor $\text{Wit}$ and its truncated variants $W_n$ take values in rings and are defined on the category of rings. Since $W_1(A) \cong A$, no information of the input ring is lost. However, if one is only interested in $W(A)$ as an abelian group with Frobenius and Verschiebung operations, one can ask what the minimal required structure on $A$ is. A ring structure is enough; an abelian group structure alone is not. It turns out that the structure is exactly that of a $p$-polar ring. Background on Witt vectors and related constructions can be found in [Hes08, Haz09, Ser68].

Definition. For $0 \leq n \leq \infty$ and a $p$-polar ring $A$, define its set of Witt vectors by

$$W_n(A) = \prod_{i=0}^{n-1} A \quad (\text{and } W(A) = W_\infty(A)).$$

Just as in the classical case, there is a ghost map

$$(3.1) \quad w: W_n(A) \to \prod_{i=0}^{n-1} A \quad (0 \leq n \leq \infty)$$

given by

$$w(a_0, a_1, \ldots) = (a_0, a_0^p + pa_1, a_0^{p^2} + pa_1^p + p^2 a_2, \ldots).$$

We will make use of the following version of Dwork’s lemma for polar rings:

Lemma 3.2 (Dwork). Let $A$ be a $p$-polar ring and $0 \leq n \leq \infty$. Assume that there is a polar ring map $\phi: A \to A$ such that $\phi(a) \equiv a^p \pmod{p}$. Then a sequence $(x_0, x_1, \ldots) \in \prod_{i=0}^{n-1} A$ is in the image of $w$ iff $x_m \equiv \phi(x_{m-1}) \pmod{p^m}$ for all $m \geq 1$.

The classical proof, e.g. as in [Hes08, Lemma 1], works almost without changes. For the reader’s convenience, we include it here.

Proof. As a first step, we show that $a^p \equiv b^p \pmod{p^{m+1}}$ if $a \equiv b \pmod{p^m}$. Indeed, if $b = a + p^m d$ then

$$b^p = (a + p^m d)^p = a^p + \sum_{i=1}^{p} \left( \begin{array}{c} p \\ i \end{array} \right) p^m a^{p-i} d^i$$

by multilinearity and symmetry of $\mu$. Since $\left( \begin{array}{c} p \\ 1 \end{array} \right) = p$, then $p^{m+1} \mid b^p - a^p$. 
In particular, \( \phi(a) p^n \equiv a^{p^n} \pmod{p^{n+1}} \) by induction. Now, since \( \phi \) is a homomorphism of polar rings, we have that

\[
\phi(w_{m-1}(a)) = \sum_{i=0}^{m-1} p^i \phi(a_i)^{p^{m-1-i}} \equiv \sum_{i=0}^{m-1} p^i a_i^{p^{m-1-i}} = w_m(a) \pmod{p^m}.
\]

This shows that if \( (x_i)_{0 \leq i \leq n} \) is in the image of \( w \) then it satisfies the stated congruence. Conversely, suppose the congruence holds. Construct \( a_i \) inductively by first choosing \( a_0 = x_0 \). Having constructed \( a_0, \ldots, a_{m-1} \), observe that

\[
D_m = x_m - \sum_{i=0}^{m-1} p^i a_i^{p^{m-1-i}} \equiv 0 \pmod{p^m}.
\]

Thus choose \( a_m \) such that \( p^m a_m = D_m \). \( \square \)

**Lemma 3.3.** Let \( A \) be a \( p \)-polar ring and \( 1 \leq n \leq \infty \).

1. There is a unique natural \( p \)-polar ring structure on \( W_n(A) \) making the ghost map \( (3.1) \) into a \( p \)-polar ring map.

2. There are unique natural additive maps \( F: W_{n+1}(A) \to W_{n}(A) \) (Frobenius) and \( V: W_{n}(A) \to W_{n+1}(A) \) (Verschiebung) such that the following diagram commutes \((x_{-1} = 0 \text{ by convention})\):

\[
\begin{array}{ccc}
W_n(A) & \xrightarrow{V} & W_{n+1}(A) \\
\downarrow w & & \downarrow w \\
\prod_{i=0}^{n-1} A & \xrightarrow{(x_i)_{i \mapsto (px_i+1)}} & \prod_{i=0}^{n} A
\end{array}
\]

Explicitly, \( V(a_0, \ldots, a_{n-1}) = (0, a_0, \ldots, a_{n-1}) \) and \( F \) is uniquely determined by \( F(a) = a^p \) and \( FV = p \), where \( a = (a_0, \ldots, 0) \) denotes the Teichmüller representative of \( a \in A \) in \( W_n(A) \).

3. If \( A \) is a \( p \)-polar algebra over a perfect field \( k \) of characteristic \( p \) then \( W_n(A) \) is a \( p \)-polar \( W(k) \)-algebra. Denoting the Frobenius \( F \) on \( W(k) \) by \( \text{frob} \) to distinguish it from the Frobenius on \( W_n(A) \), we have that

\[
Fa = \text{frob}(a)F \quad \text{and} \quad Va = \text{frob}^{-1}(a)V \quad \text{for } a \in W(k).
\]

Note that \( \text{frob} \) is bijective since \( k \) is perfect.

**Proof.** For (1), the classical proof (cf. [Hes08, Prop. 2]) works. First consider the free \( p \)-polar ring \( P = P(a_{i,j} \mid 0 \leq i < n, 1 \leq j \leq p) \) of Ex. 2.6 spanned by polynomials of degree congruent to 1 modulo \( p - 1 \) and the \( p \)-polar ring map \( \phi: P \to P \) with \( \phi(a_{i,j}) = a_{i,j}^p \). Write \( a_j \) for the sequence \( (a_{i,j} \mid 0 \leq i < n) \). By Lemma 3.2, \( w(a_1) + w(a_2) - w(a_1) \) and \( w(a_1)w(a_2) \cdots w(a_p) \) are in the image of \( w \). Since \( P \) is torsion free, \( w \) is injective. Thus this defines a unique \( p \)-polar ring structure on \( W_n(P) \). The case for arbitrary \( A \) follows from naturality. Assertions (2) and (3) follow from similar arguments, considering the universal, torsion-free case first and using naturality to deduce the general case. \( \square \)

Now let \( k \) be a perfect field of characteristic \( p \). Denote by \( \mathcal{R} = W(k)(F, V)/(FV - p) \) the noncommutative ring obtained from \( W(k) \) by adjoining two variables \( F, V \) such that \( FV = VF = p \) and such that commutation with scalars is governed by (3.4). Then for a \( p \)-polar \( k \)-algebra \( A \) and \( n \leq \infty \), \( W_n(A) \) becomes an \( \mathcal{R} \)-module by Lemma 3.3.
As in [Fon77, Ch. II] or [BC19, §5.3], we define the group of unipotent co-Witt vectors as the colimit

$$CW^n(A) = \text{colim}(W_0(A) \xrightarrow{V} W_1(A) \xrightarrow{V} \cdots)$$

This works when \(A\) is merely a \(p\)-polar ring by Lemma 3.3.

Define the set of co-Witt vectors as

$$CW(A) = \{(a_i) \in \mathbb{A}^{\leq 0} \mid (a_{-r}, a_{-r-1}, \ldots)_{p^r} = 0 \text{ for some } r, s \geq 0\},$$

where \((a_{-r}, a_{-r-1}, \ldots)\) denotes the ideal (cf. Def. 2.7) in \(A\) generated by the given elements. This functor (as a functor on finite-dimensional \(k\)-algebras) is in fact a formal group [Fon77, §II.4], i.e. ind-representable.

To see that \(CW(A)\) has the structure of an \(\mathcal{R}\)-module even when \(A\) is just a \(p\)-polar ring, containing \(CW^n(A)\) as a submodule, we use and adapt the arguments of [Fon77, §II.1.5].

Define the polynomial \(S_n(x_0, \ldots, x_n, y_0, \ldots, y_n)\) as the \(n\)th component of the Witt vector addition \((x_0, x_1, \ldots, x_n)+(y_0, y_1, \ldots, y_n) \in W(\mathbb{Z}[x_0, \ldots, x_n, y_0, \ldots, y_n]).\)

Note that this polynomial is in fact an element of the free \(p\)-polar ring (cf. Ex. 2.6) \(P(x_1, \ldots, x_n, y_1, \ldots, y_n)\), and hence can be evaluated on elements of \(p\)-polar rings.

**Proposition 3.5.** Let \(A\) be a \(p\)-polar ring and \(a = (a_i), b = (b_i) \in CW(A)\). Then

1. For each \(n \geq 0\), the sequence

   $$S_m(a_{-n-m}, \ldots, a_{-n}, b_{-n-m}, \ldots, b_{-n})$$

   is eventually constant as \(m \to \infty\). Let us call the limit value \(S_{-n}(a, b)\).

2. \(S_{-n}(a, b) \in CW(A)\).

**Proof.** This is proved for commutative rings in [Fon77, Prop. II.1.1].

Since \(a, b \in CW(A)\), there exist natural numbers \(r, s\) such that

$$(a_{-r}, a_{-r-1}, \ldots, b_{-r}, b_{-r-1}, \ldots)_{p^r} = 0.$$

Let

$$R = \mathbb{Z}[x_i, y_i \mid i \leq 0]/(x_r, x_{r-1}, \ldots, y_r, y_{r-1}, \ldots)_{p^r}.$$

Then by [Fon77, Prop. II.1.1], the sequence \(S_m(x_{-n-m}, \ldots, x_{-n}, b_{-n-m}, \ldots, b_{-n})\) is eventually constant as \(m \to \infty\), and its limit lies in \(CW(R)\). In fact, as observed before, it lies in \(CW(P)\), where

$$P = P(x_i, y_i \mid i \leq 0)/(x_r, x_{r-1}, \ldots, y_r, y_{r-1}, \ldots)_{p^r} < R.$$

Since this is the universal case for elements \(a, b\) with the chosen vanishing properties, the claim follows from naturality. \(\square\)

The \(\mathcal{R}\)-module structure on \(CW(A)\) is thus given by

$$a + b = (\ldots, S_{-2}(a, b), S_{-1}(a, b), S_0(a, b)), \quad V a = (\ldots, a_2, a_1), \quad \text{and} \quad F a = (\ldots, a_2^p, a_1^p, a_0^p).$$

**Remark 3.6.** The structure of a \(p\)-polar algebra is the minimal structure needed to define the Witt vector functor; indeed, the \(p\)-polar \(k\)-algebra \(A\) can be reconstructed from the abelian groups \(W_1(A)\) and \(W_2(A)\) together with the Teichmüller map.
the inductively proved identity above sum. The reader may amuse themself by deriving the above equality from Example 3.7 so the prescribed in [\textit{A}]

\[ \sum_{I \subseteq \{1, \ldots, p\}} (-1)^{|I|} t \left( \sum_{i \in I} x_i \right) = (0, x_1 \cdots x_p) \in W_2(A), \]

so the \( p \)-polar structure can be extracted by equating \( V(\tau(\mu(x_1, \ldots, x_p))) \) to the above sum. The reader may amuse themself by deriving the above equality from the inductively proved identity

\[ \sum_{I \subseteq \{1, \ldots, k\}} (-1)^{|I|} t \left( \sum_{i \in I} x_i \right) = (0, \sum_{i_1 + \cdots + i_k = p, i_1, \ldots, i_k \geq 1} \frac{1}{p} p \left( \sum_{i_1, \ldots, i_k} x_1^{i_1} \cdots x_k^{i_k} \right) \]

for \( k \geq 2 \).

**Example 3.7** (Witt vectors of free \( p \)-polar algebras). The ring \( W(F_p[x]) \) is described in [\textit{Bor16}, Exercise 1(10)]. It is a subring of the \( p \)-completed monoid ring \( \Z_p[N[p]^\times] = W(F_p[N[p]^\times]) \) consisting of those series \( \sum a_{\frac{m}{n}}[\frac{m}{n}] \) such that \( p^j \mid a_{\frac{m}{n}} \).

The group \( W(P(x)) \) of the free \( p \)-polar algebra on one generator is the subgroup of power series where \( a_{\frac{m}{n}} = 0 \) unless \( n \equiv 1 \pmod{p-1} \).

**4. Free affine and formal groups**

Continue to let \( k \) be a perfect field of characteristic \( p \). A formal group \( G \) is called connected if its representing pro-\( k \)-algebra \( \O_G \) is a pro-local \( k \)-algebra, and it is called étale if \( \O_G \) is pro-étale as a pro-\( k \)-algebra. An affine group \( G \) is called unipotent if its Cartier dual \( G^* \) is connected, and of multiplicative type if \( G^* \) is étale.

The following splittings are classical:

**Theorem 4.1** ([\textit{Fon77}, §I.7]). The category \( \text{Fgps}_k \) splits naturally as \( \text{Fgps}_k^e \times \text{Fgps}_k^c \), where \( \text{Fgps}_k^e \) and \( \text{Fgps}_k^c \) denote the full subcategories of étale resp. connected formal groups.

Dually, the category \( \text{AbSch}_k \) splits naturally as \( \text{AbSch}_k^m \times \text{AbSch}_k^n \) into the product of its full subcategories of multiplicative-type and of unipotent groups.

Recall that an affine group is \( p \)-adic if it takes values in abelian \( p \)-groups, and that a formal group is a formal \( p \)-group if it takes values in \( p \)-groups.

**Remark 4.2.** If \( G \) is a unipotent affine group then \( G \) is automatically \( p \)-adic; dually, if \( G \) is a connected formal group then \( G \) is automatically a formal \( p \)-group. Since these two statements are Cartier dual to one another, it suffices to show the first. An affine group is unipotent if and only if its representing Hopf algebra \( H \) is conilpotent. In particular, for each \( x \in H \), there exists an \( n \geq 0 \) such that \( V^n(x) = 0 \), where \( V \) denotes the Hopf algebra Verschiebung. Since \([p] = VF = VF \) in any abelian Hopf algebra, with \( F \) the Hopf algebra Frobenius, we have that \( x \in H[p^n] = \ker([p^n]) \). Thus \( H \cong \text{colim}_n H[p^n] \), and \( H \) is \( p \)-adic.

We can now show that the free \( p \)-adic affine group functor \( Fr \) exists:

**Proof of Lemma 1.2A.** The forgetful functor from \( \text{AbSch}_k^p \) to \( \text{Alg}_k^{op} \) factors as

\[ \text{AbSch}_k^p \to \text{AbSch}_k \xrightarrow{U} \text{Alg}_k^{op}, \]
and hence it suffices to construct left adjoints for $J$ and $U$. Since an affine group $G$ is $p$-adic iff $G \cong G_p = \lim_n G/p^n$ as affine group schemes, the functor $J$ has the $p$-completion $G \mapsto \hat{G}_p$ as a left adjoint. On the level of Hopf algebras, the adjoint to the functor $U$ corresponds to the cofree cocommutative Hopf algebra on an algebra $A$, which was constructed in [BC19, Proof of Thm. 1.3].

Let us be more explicit. By Thm. 4.1, $Fr\cong Fr^m \times Fr^n$ splits into a part of multiplicative type and a unipotent part. Since unipotent groups are $p$-adic (Rk. 4.2), $Fr^u$ is represented by the cofree cocommutative conilpotent Hopf algebra on $R$, first constructed by Takeuchi [Tak74]. It is the Hopf algebra of symmetric tensors $\bigoplus_{n \geq 0} (R \otimes k^n) \Sigma_n$.

A group of multiplicative type is $p$-adic if it is isomorphic to $\text{Spec} \overline{k}[M]$ for a $p$-torsion group $M$ after base change to an algebraic closure $\overline{k}$ of $k$. In the case where $k \cong \overline{k}$, the free multiplicative $p$-adic group on $\text{Spec} A$ is represented by $\overline{k}[\mu_p \infty](A)$, and in the general case, it is represented by the Galois invariants (cf. [BC19, Proof of Thm. 1.3])

$$\hat{k}[\mu_p \infty](R \otimes \overline{k})^{\text{Gal}(k)}.$$ 

The proof of Lemma 1.2F follows similar lines.

Proof of Lemma 1.2F. Taking $k$-linear continuous duals of representing objects gives a commutative diagram

$$
\begin{array}{ccc}
Fgps_k & \xrightarrow{U} & (\text{Pro-\ }\text{alg}_k)^{\text{op}} \\
\downarrow G\mapsto \sigma_{G^\ast} & & \downarrow (-)^\ast \\
\text{Hopf}_k & \xrightarrow{U_H} & \text{Coalg}_k,
\end{array}
$$

where $\text{Coalg}_k$ denotes the category of cocommutative coalgebras over $k$ and $U_H$ is the forgetful functor. The left adjoint of $U_H$ is the free commutative Hopf algebra on a cocommutative coalgebra and is constructed in [Tak71].

Again, let us be more explicit. By Thm. 4.1, there is a natural splitting $Fr(R) \cong Fr^e(R) \times Fr^c(R)$ into an étale and a connected part.

Let us describe the $Fr^e(R)$ more concretely, for a finite-dimensional $k$-algebra $R$. The category of étale formal groups is equivalent to the category $\text{Mod}_{\Gamma}$ of abelian groups with a discrete action of the absolute Galois group $\Gamma = \text{Gal}(k)$ (cf. [Fon77, §I.7], [BC19, Thm. 1.6]); this equivalence is simply given by the functor $G \mapsto \text{colim}_{k < k'} G(k')$, where $k'$ runs through the finite extensions of $k$, and the inverse functor is given by $M \mapsto \text{Spf}(\text{map}^\Gamma(M, \overline{k}))$.

Lemma 4.3. Let $R$ be a finite-dimensional $k$-algebra. Then the $\Gamma$-module associated with $Fr^e(R)$ is

$$\mathbb{Z}\langle \text{Hom}_{\text{Alg}_k}(R, \overline{k}) \rangle$$

I thank the anonymous referee for suggesting the following simplified proof:

Proof. Consider the following 2-commutative diagram of functors:

$$
\begin{array}{ccc}
Fgps_k^e & \xrightarrow{U} & (\text{alg}_k^e)^{\text{op}} \\
\downarrow & & \downarrow \\
\text{Mod}_\Gamma & \xrightarrow{U^e} & \text{Set}_\Gamma
\end{array}
$$
where \( \text{alg}^F_k \) denotes the category of finite-dimensional, étale \( k \)-algebras, \( \text{Set}_\Gamma \) the category of discrete \( \Gamma \)-sets, and \( U_{\Gamma} \) the forgetful functor. The vertical arrows are equivalences and given by taking \( \bar{k} \)-valued points. Since the left adjoint of \( U_{\Gamma} \) is the free abelian group functor on a \( \Gamma \)-set \( X \), the result follows. \( \square \)

5. The Dieudonné correspondence and proof of the main theorems

Continue to let \( k \) be a perfect field of characteristic \( p \).

**Definition 5.1.** Denote by \( \text{Dmod}_k \) the category of Dieudonné modules over \( k \). These are \( R \)-modules, i.e. \( W(k) \)-modules with homomorphisms

\[
V: M \to M \quad \text{and} \quad F: M \to M
\]
such that \( FV = VF = p \), \( Fa = \text{frob}(a)F \) and \( aV = V \text{frob}(a) \) for \( a \in W(k) \).

Denote by \( \text{Dmod}^F_k \) the full subcategory of \( p \)-adic Dieudonné modules. These are modules \( M \) such that for all \( x \in M \), the \( W(k) \)-submodule spanned by \( V^i(x) \), for all \( i \), is of finite length.

Dually, let \( \text{Dmod}^F_k \) be the full subcategory of \( F \)-profinite Dieudonné modules, i.e. modules \( M \) that are profinite as \( W(k) \)-modules and have a fundamental system of neighborhoods consisting of \( W(k) \)-modules closed under \( F \).

The following theorem follows from classical Dieudonné theory (cf. [BC19, Section 6], [DG70]):

**Theorem 5.2.** There are equivalences of abelian categories

\[
D: (\text{AbSch}_k^p)\text{op} \to \text{Dmod}^F_k
\]

and

\[
D^f: (\text{Fgps}_k^p)\text{op} \to \text{Dmod}^F_k
\]

These functors are given by

\[
D(G) = \colim_n \text{Hom}_{\text{AbSch}_k}(G, W_n) \oplus \left( \text{Gr}(\mathcal{O}_G \otimes_k \bar{k}) \otimes W(\bar{k}) \right)^\Gamma
\]

and

\[
D^f(G) = \text{Hom}_{\text{Fgps}_k}(G, CW).
\]

Here, \( \text{Gr}(H) \) stands for the group of group-like elements of a Hopf algebra \( H \).

The formal part of this theorem is [BC19, Thm. 6.5], and the affine part is [BC19, Thm. 6.3]. In the latter, if \( G \cong G^u \times G^m \) is the splitting into a unipotent part and a part of multiplicative type (Thm. 4.1), the first summand of \( D(G) \) is \( D(G^u) \) and the second is \( D(G^m) \). The statement in [BC19, Thm. 6.3] actually contains an error for \( D(G^m) \), and we take the opportunity to rectify it here.

**Correction of [BC19, Thm. 6.3].** The proof of \( D(G^u) = \colim_n \text{Hom}_{\text{AbSch}_k}(G, W_n) \) is contained in [Dem86, Chapter III, 6]. Thus let \( G \) be of multiplicative type, and let

\[
D'(G) = I(D^f(G^*)),
\]

where \( G^* \) denotes the formal group Cartier dual to \( G \), and \( I = \text{Hom}_{W(k)}(-, CW(k)) \) denotes Matlis (or Pontryagin) duality between \( W(k) \)-modules and pro-(finite length) \( W(k) \)-modules. Since both dualities are anti-equivalences of categories and \( D^f \) is an equivalence, so is \( D' \). It remains to show that \( D'(G) = D(G) \).

For this, recall (e.g. from [Fon77, §I.7]) that the category of affine groups of multiplicative type over \( k \) is equivalent with the category \( \text{Mod}_{\Gamma} \) of abelian groups
with a discrete action of the absolute Galois group $\Gamma$ via the functor $G \mapsto \text{Gr}(\mathcal{O}_G \otimes_k \bar{k})$.

Write $\overline{CW} = \text{colim}_{k < k'} kCW(k')$ and $\overline{W} = \text{colim}_{k < k'} W(k')$, where $k'$ runs through all finite field extensions of $k$ contained in some algebraic closure $\bar{k}$. Then in terms of these $\Gamma$-modules, the functors $D$ and $D'$ are given, respectively, by

$$D(M) = (M \otimes \overline{W})^\Gamma$$

and

$$D'(M) = I(\text{Hom}_\Gamma(M, \overline{W})) = I(\text{Hom}_\Gamma(M \otimes \overline{W}, \overline{W}))$$

By Galois descent (cf. [Fon77, §III.2]), the category of $\overline{W}$-modules with semilinear $\Gamma$-actions is equivalent to the category of $W(k)$-modules by taking $\Gamma$-fixed points, and hence the last expression equals

$$I(\text{Hom}_{W(k)}((M \otimes \overline{W})^\Gamma, CW(k)) = (M \otimes \overline{W})^\Gamma.$$  \hfill \Box

**Proof of Thm. 1.5F.** For a formal scheme $S$ represented by a profinite algebra $A$ (in particular, for algebras of finite dimension over $k$), we have that

$$D'(\text{Fr}(A)) = \text{Hom}_{F_{\text{gps}}}(\text{Fr}(A), CW) = \text{Hom}_{F_{\text{Sch}}}(\text{Spf}(A), CW) = CW(A).$$

Here $F_{\text{Sch}}$, the category of formal schemes over $k$, is dual to the category of profinite $k$-algebras.

**Proof of Thm. 1.5A.** Let $S$ be an affine scheme $S$ represented by an algebra $A$. Then

$$D(\text{Fr}(A)) = \text{colim} \text{Hom}_{\text{AbSch}}(\text{Fr}(A), W_n) \oplus \left( \mu_{p^n}(A \otimes_k \bar{k}) \otimes W(\bar{k}) \right)^\Gamma$$

$$= \text{colim} W_n(A) \oplus \left( \mu_{p^n}(A \otimes_k \bar{k}) \otimes W(\bar{k}) \right)^\Gamma.$$  \hfill \Box

In order to prove Thm. 1.3A, we need so see that the multiplicative factor in the previous statement is well-defined for $p$-polar algebras. This will follow from the following result about $p$-typical formal group laws.

Recall (e.g. from [Rav86, A2.1.17]) that a 1-dimensional formal group law $F$ over a torsion free $\mathbb{Z}_p$-algebra $R$ is $p$-typical iff its logarithm $\log_F(x) \in (R \otimes \mathbb{Q})[x]$ is of the form $\log_F(x) = \sum_{i \geq 0} l_i x^{p^i}$ (with $l_0 = 1$).

**Lemma 5.3.** Assume that $G \in F_{\text{gps}}$ is the mod-$p$ reduction of a 1-dimensional formal group over a torsion free $\mathbb{Z}_p$-algebra $R$. Then Theorem 1.1F holds for $G$.

**Proof.** Recall (e.g. from [Rav86, A2.1.17]) that a 1-dimensional formal group law $F$ over a torsion free $\mathbb{Z}_p$-algebra $R$ is $p$-typical iff its logarithm $\log_F(x) \in (R \otimes \mathbb{Q})[x]$ is of the form $\log_F(x) = \sum_{i \geq 0} l_i x^{p^i}$ (with $l_0 = 1$). By Cartier’s Theorem [Car67], cf. [Rav86, Thm. A2.1.18], every one-dimensional formal group law is isomorphic to a $p$-typical one, so we may assume that $G$ is the mod-$p$ reduction of a $p$-typical formal group law $F$ over $R$. It is straightforward to see that the (compositionally) inverse power series $\exp_F(x)$ has the form

$$\exp_F(x) = \sum_{i \geq 0} a_i x^{1+i(p-1)}$$

for some $a_i \in R \otimes \mathbb{Q}$, $a_0 = 1$.

Thus both $\log_F$ and $\exp_F$ are in fact elements of the $p$-polar power series algebra

$$\prod_{i \geq 0} (R \otimes \mathbb{Q})(x^{1+i(p-1)}) \subset (R \otimes \mathbb{Q})[x].$$
Hence if $A \in \text{pol}_p(k)$, we can define

$$\tilde{G}(A) = \text{nil}(A)$$

with the group structure

$$x \ast y = \exp_F(\log_F(x) + \log_F(y)),$$

agreeing with $G(R)$ if $A = \text{pol}_p(R)$.

**Corollary 5.4.** The functor $\mu_{p^\infty}: \text{Alg}_k \to \text{Ab}$ factors through $\text{Pol}_p(k)$.

**Proof.** The functor

$$\mu_{p^\infty}(R) = \{x \in R^\times \mid x^{p^n} = 1 \text{ for some } n \geq 0\}$$

is the colimit of the functors $\mu_{p^n}(R)$ represented by $k[y]/(y^{p^n} - 1) \cong k[x]/(x^{p^n})$ where $x = y - 1$. Thus $\mu_{p^n}(R) \cong \text{nil}(R)$ as sets, and the group structure on $\text{nil}(R)$ is the multiplicative one: $x \ast y = x + y + xy$. Thus the restriction of $\mu_{p^\infty}$ to $\text{alg}_k$ is isomorphic to the multiplicative formal group $\hat{G}_m$. Since it is defined over $R = \mathbb{Z}_p$ and one-dimensional, Lemma 5.3 applies and shows that $\mu_{p^\infty}$ factors through finite $p$-polar $k$-algebras.

Now, for an arbitrary $p$-polar $k$-algebra $A$, define

$$\hat{\mu}_{p^\infty} = \text{colim}_{B \leq A} \mu_{p^\infty}(B),$$

where $B$ ranges over the finite-dimensional $p$-polar subalgebras of $A$. Since $\text{nil}(R) = \text{colim}_{B \leq A} \text{nil}(B)$ because finitely generated nilpotent subalgebras of $R$ are finite-dimensional, this is a factorization as required. □

**Proof of Thm. 1.3A.** By Thm. 1.4, $\text{CW}^u: \text{Alg}_k \to \text{Dmod}_k^p$ factors through $\text{Pol}_p(k)$, and by Cor. 5.4, so does $\mu_{p^\infty}: \text{Alg}_k \to \text{Ab}$. It follows that also the functor

$$R \mapsto (\mu_{p^\infty}(R \otimes_k \bar{k}) \otimes W(\bar{k}))^{\text{Gal}(k)}: \text{Alg}_k \to \text{Dmod}_k^p$$

appearing on the right hand side in Thm. 1.5A, factors through $\text{Pol}_p(k)$. Thus, by the said theorem, $R \mapsto D(\text{Fr}(R))$ factors through $\text{Pol}_p(k)$. Since $D$ is an anti-equivalence between $\text{AbSch}_k^p$ and $\text{Dmod}_k^p$, the result follows.

To prove Thm. 1.3F, we need to study the étale part $\text{Fr}^e$ of the free formal group functor more closely.

**Lemma 5.5.** Let $k$ be algebraically closed and let $A$ be a finite, reduced $p$-polar $k$-algebra, i.e. one such that for $x \neq 0$, we have that $x^{p^N} \neq 0$ for all $N \geq 0$. Then $A$ is isomorphic to a finite product of polarizations of $k$.

**Proof.** Suppose $A$ has a unity in the sense of Remark 2.4, i.e. an element $u \in A$ such that $u^{p^{-1}}a = a$ for all $a \in A$. Then, by the quoted remark, $A = \text{pol}(\hat{A})$, where $\hat{A}$ is the algebra structure on $A$ given by $x \cdot y = \mu(u, \ldots, u, x, y)$. Since $\hat{A}$ is reduced and finite, it is a product of finitely many copies of $k$.

For the general case, I claim that $A$ has a nonzero element $e$ such that $e^p = e$.

Choose a nonzero $y \in A$ and $j \geq 1$ such that the powers $y, y^p, \ldots, y^{p^{j-1}}$ are linearly independent and $y^{p^j} = \sum_{i=0}^{j-1} \alpha_i y^{p^i}$ for some $\alpha_i \in k$. Such $y$ and $j$ must
exist by the finiteness of $A$. Since $A$ is reduced, not all $\alpha_i$ are zero. Let $\beta$ be a nonzero root of the polynomial
\[ p(x) = \sum_{i=0}^{j-1} \alpha_i x^{p^j-1} - x \]
and let
\[ e = \sum_{l=0}^{j-1} \left( \sum_{i=0}^{l} (\alpha_i \beta^p)^{p^j-i} \right) y^l. \]
Then it is straightforward to verify that $e^p = e$. Since the $p^j$ are linearly independent, $e$ can only be zero if $\sum_{i=0}^{l} (\alpha_i \beta^p)^{p^j-i} = 0$ for all $l \leq j - 1$. But this can only happen if all $\alpha_i = 0$, contrary to the assumption.

Now consider the map $f : A \to A$ given by $y \mapsto e^{-1}y$. Since $e^p = e$, this map is idempotent and a $p$-polar $k$-algebra endomorphism. Thus
\[ A \cong \ker(f) \times \im(f) \]
as $p$-polar algebras. Note that $\im(f)$ is a nontrivial $p$-polar algebra with unity $e$, so by the previous case, it is a product of copies of $k$. The $p$-polar algebra $\ker(f)$ has smaller dimension over $k$, so we are done by induction.

**Lemma 5.6.** The functor $\text{alg}_k \to \text{Mod}_k$ given by $A \mapsto \mathbb{Z}(\text{Hom}_{\text{alg}_k}(A, \bar{k}))$ factors through $\text{pol}_p(k)$.

**Proof.** Let $\mathcal{E}$ be the full subcategory of $\text{pol}_p(\bar{k})$ of reduced $p$-polar algebras. There is a functor $\text{pol}_p(\bar{k}) \to \mathcal{E}$ given by $A \mapsto A/\text{Nil}(A)$. By Lemma 5.5, all objects of $\mathcal{E}$ are isomorphic to $\bar{k}^n$ for some $n \geq 0$. Consider the set
\[ \text{Hom}_{\text{pol}_p(\bar{k})}(\bar{k}^n, \bar{k}). \]
One easily verifies that a linear map $\bar{k}^n \to \bar{k}$ represented by a row vector $(a_1, \ldots, a_p)$ is a homomorphism of $p$-polar algebras if and only if exactly one $a_i$ is nonzero, and furthermore $a_i \in \mathbf{F}_p$. We now construct a functor to abelian groups,
\[ \Phi : \mathcal{E} \to \text{Ab} \]
by defining $\Phi(\bar{k}^n) = \mathbb{Z}^n$ and for a morphism $\bar{k}^n \to \bar{k}^m$ represented by a matrix $M$, $\Phi(M)$ is the matrix $M$ with every nonzero entry replaced by $1$. Because of the special form of $M$, this is indeed a well-defined functor. Moreover, it carries the $\Gamma$-action on $A \otimes_k \bar{k}$ to an action by permutation matrices on $\Phi(A \otimes_k \bar{k})$. We have that $\Phi(\text{pol}_p(R)) \cong \mathbb{Z}(\text{Hom}_{\text{Alg}_k}(R, k))$ for reduced $R$ over algebraically closed $k$. Now define
\[ \mathcal{F} : \text{pol}_p(k) \to \text{Ab} \]
by $\mathcal{F}(A) = \Phi \left( (A \otimes_k \bar{k})/\text{Nil}(A \otimes_k \bar{k}) \right)$. By the above, this is the desired extension. \qed

**Proof of Thm. 1.3F.** By Theorems 1.4 and 1.5F, the functor $R \mapsto \mathbb{D}^f(\text{Fr}(R))$ factors through $\text{pol}_p(R)$. The Dieudonné functor $\mathbb{D}^f$ is not an equivalence on all formal groups (just on formal $p$-groups), but it is an equivalence between connected formal groups and their image. Thus it remains to show that the étale part $\text{Fr}^e$ of $\text{Fr}$ factors through $\text{pol}_p(k)$. Using the equivalence between étale formal groups and $\Gamma$-modules, this case is covered by Lemma 4.3 together with Lemma 5.6. \qed
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