A NOTE ON THE FOURTH MOMENT OF DIRICHLET L-FUNCTIONS

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Abstract. We prove an asymptotic formula for the fourth power mean of Dirichlet L-functions averaged over primitive characters to modulus \( q \) and over \( t \in [0, T] \) which is particularly effective when \( q \geq T \). In this range the correct order of magnitude was not previously known.

1. Introduction

For \( \chi \) a Dirichlet character \((\text{mod } q)\), the moments of \( L(s, \chi) \) have many applications, for example to the distribution of primes in the arithmetic progressions to modulus \( q \). The asymptotic formula of the fourth power moment in the \( q \)-aspect has been obtained by Heath-Brown \([1]\), for \( q \) prime, and more recently by Soundararajan \([5]\) for general \( q \). Following Soundararajan’s work, Young \([7]\) pushed the result much further by computing the fourth moment for prime moduli \( q \) with a power saving in the error term. The problem essentially reduces to the analysis of a particular divisor sum. To this end, Young used various techniques to estimate the off-diagonal terms.

In the case that the \( t \)-aspect is also included, a result of Montgomery \([2]\) states that

\[
\sum_{\chi \, (\text{mod } q)}^* \int_0^T |L(\frac{1}{2} + it, \chi)|^4 dt \ll \varphi(q)T(\log qT)^4
\]

for \( q, T \geq 2 \), where \( \sum_{\chi \, (\text{mod } q)}^* \) indicates that the sum is restricted to the primitive characters modulo \( q \). As we shall see, the upper bound is too large by a factor \( (q/\varphi(q))^5 \).

A second result of relevance is due to Rane \([4]\). After correcting a misprint it states that

\[
\sum_{\chi \, (\text{mod } q)}^* \int_T^{2T} |L(\frac{1}{2} + it, \chi)|^4 dt = \frac{\varphi^*(q)T}{2\pi^2} \prod_{p|q} \frac{(1 - p^{-1})^3}{1 + p^{-1}} (\log qT)^4 + O(2^{\omega(q)} \varphi^*(q)T(\log qT)^3(\log \log 3q)^5),
\]

where \( \varphi^*(q) \) is the number of primitive characters modulo \( q \) and \( \omega(q) \) is the number of distinct prime factors of \( q \). This can only give an asymptotic relation when \( 2^{\omega(q)} \leq \log q \), which holds for some values of \( q \), but not others. Finally we mention the work of Wang \([6]\), where an asymptotic formula is proved for \( q \leq T^{1-\delta} \), for any fixed \( \delta > 0 \).

The goal of the present note is to establish an asymptotic formula, valid for all \( q, T \geq 2 \), as soon as \( q \to \infty \).
**Theorem 1.** For \( q, T \geq 2 \) we have, in the notation above,

\[
\sum_{\chi (\bmod q)}^\ast \int_0^T |L(\frac{1}{2} + it, \chi)|^4 dt
\]

\[
= \left( 1 + O \left( \frac{\omega(q)}{\log q} \sqrt{\varphi(q)} \right) \right) \varphi^*(q) T \prod_{p \nmid q} \frac{(1 - p^{-1})^3}{(1 + p^{-1})} (\log q T)^4 + O(q T (\log q T)^{\frac{7}{2}}).
\]

Our proof uses ideas from the works of Heath-Brown \([1]\) and Soundararajan \([5]\), but there is extra work to do to handle the integration over \( t \).

**Remark 1.** It is possible, with only a little more effort, to extend the range to cover all \( T > 0 \). In this case the term \( \varphi^*(q) T \) in the main term remains the same, as does the factor \( q T \) in the error term, but one must replace \( \log q T \) by \( \log q (T + 2) \) both in the main term and in the error term.

**Remark 2.** One may readily verify that our result provides an asymptotic formula, as soon as \( q \to \infty \), with an error term which saves at least a factor \( O((\log \log q)^{-1/2}) \).

**Remark 3.** The literature appears not to contain a precise analogue of this for the second moment. However Motohashi \([3]\) has considered a uniform mean value in \( t \)-aspect. He proved that if \( \chi \) is a primitive character modulo a prime \( q \), then

\[
\int_0^T |L(\frac{1}{2} + it, \chi)|^2 dt = \frac{\varphi(q) T}{q} \left( \log \frac{q T}{2 \pi} + 2 \gamma + 2 \sum_{p \nmid q} \frac{\log p}{p - 1} \right) + O((q^{1/3} T^{1/3} + q^{1/2}) (\log q T)^4),
\]

for \( T \geq 2 \). This provides an asymptotic formula when \( q \leq T^{2-\delta} \), for any fixed \( \delta > 0 \). Our theorem does not give a power saving in the error term, but it yields an asymptotic formula without any restrictions on \( q \) and \( T \).

2. Auxiliary Lemmas

**Lemma 1.** Let \( \chi \) be a primitive character \( (\bmod q) \) such that \( \chi(-1) = (-1)^a \) with \( a = 0 \) or \( 1 \). Then we have

\[
|L(\frac{1}{2} + it, \chi)|^2 = 2 \sum_{a, b \geq 1} \frac{\chi(a) \overline{\chi(b)}}{\sqrt{ab}} \left( \frac{a}{b} \right)^{-it} W_a \left( \frac{\pi ab}{q}; t \right),
\]

where

\[
W_a(x; t) = \frac{1}{2\pi i} \int_{(2)} \frac{\Gamma(\frac{1}{4} + \frac{it}{2} + \frac{s}{2} + \frac{a}{2}) \Gamma(\frac{1}{4} - \frac{it}{2} + \frac{s}{2} + \frac{a}{2})}{\Gamma(\frac{1}{4} + \frac{it}{2} + \frac{s}{2})^2} e^{\pi x z - \frac{z^2}{4}} dz.
\]

**Proof.** Let

\[
I := \frac{1}{2\pi i} \int_{(2)} \frac{\Lambda(\frac{1}{2} + it + z, \chi) \overline{\Lambda(\frac{1}{2} - it + z, \chi)}}{\Gamma(\frac{1}{4} + \frac{it}{2} + \frac{s}{2})^2} e^{\pi z^2/4} dz,
\]

where

\[
\Lambda(\frac{1}{2} + s, \chi) = \left( \frac{q}{\pi} \right)^{s/2} \Gamma \left( \frac{1}{4} + \frac{s}{2} + \frac{a}{2} \right) L(\frac{1}{2} + s, \chi).
\]

We recall the functional equation

\[
\Lambda(\frac{1}{2} + s, \chi) = \frac{\tau(\chi)}{i^a \sqrt{q}} \Lambda(\frac{1}{2} - s, \overline{\chi}).
\]
Hence, moving the line of integration to \( \Re z = -2 \) and applying Cauchy’s Theorem, we obtain \( |L(\frac{1}{2} + it, \chi)|^2 = 2I \). Finally, expanding \( L(\frac{1}{2} + it + z, \chi)L(\frac{1}{2} - it + z, \chi) \) in a Dirichlet series and integrating termwise we obtain the lemma.

We decompose \( |L(\frac{1}{2} + it, \chi)|^2 \) as \( 2(A(t, \chi) + B(t, \chi)) \), where

\[
A(t, \chi) = \sum_{ab \leq Z} \frac{\chi(a)\overline{\chi(b)}}{\sqrt{ab}} \left( \frac{a}{b} \right)^{-it} W_a\left( \frac{\pi ab}{q}; t \right),
\]

and

\[
B(t, \chi) = \sum_{ab > Z} \frac{\chi(a)\overline{\chi(b)}}{\sqrt{ab}} \left( \frac{a}{b} \right)^{-it} W_a\left( \frac{\pi ab}{q}; t \right),
\]

with \( Z = qT/2^{\omega(q)} \). In the next two sections, we evaluate the second moments of \( A(t, \chi) \) and \( B(t, \chi) \) after which our theorem will be an easy consequence.

The function \( W_a(x; t) \) approximates the characteristic function of the interval \( [0, |t|] \).

Indeed, we have the following.

**Lemma 2.** The function \( W_a(x; t) \) satisfies

\[
W_a(x; t) = \begin{cases} 
O((\tau/x)^2) & \text{for } x \geq \tau, \\
1 + O((x/\tau)^{1/4}) & \text{for } 0 < x < \tau,
\end{cases}
\]

and

\[
\frac{\partial}{\partial t} W_a(x; t) \ll \begin{cases} 
\tau^{-1}(\tau/x)^2 & \text{for } x \geq \tau, \\
\tau^{-1}(x/\tau)^{1/4} & \text{for } 0 < x < \tau,
\end{cases}
\]

where \( \tau = |t| + 2 \).

**Proof.** The first estimate is a direct consequence of Stirling’s formula, while for the second one merely shifts the line of integration to \( \Re z = -1/4 \) before employing Stirling’s formula. To handle the derivative one proceeds as before, differentiates under the integral sign and uses the estimate

\[
\frac{\Gamma'(w)}{\Gamma(w)} = \log w + O(|w|^{-1}),
\]

which holds for \( 1/8 \leq \Re w \leq 2 \). \( \square \)

The next lemma concerns the orthogonality of primitive Dirichlet characters.

**Lemma 3.** For \((mn, q) = 1\), we have

\[
\sum_{\chi \mod q}^* \chi(m)\overline{\chi(n)} = \sum_{k|(q,m-n)} \phi(k)\mu(q/k).
\]

Moreover

\[
\sum_{\chi \mod q}^* \chi(m)\overline{\chi(n)} = \frac{1}{2} \sum_{k|(q,m-n)} \phi(k)\mu(q/k) + \frac{(-1)^a}{2} \sum_{k|(q,m+n)} \phi(k)\mu(q/k).
\]

**Proof.** This follows from [3, page 27]. \( \square \)

To handle the off-diagonal term we shall use the following bounds.
Lemma 4. Let $k$ be a positive integer and $Z_1, Z_2 \geq 2$. If $Z_1 Z_2 \leq k^{10}$ then
\[ E := \sum_{Z_1 \leq a < 2Z_1} \sum_{Z_2 \leq c < 2Z_2} \sum_{\begin{array}{c} ac \equiv bd \pmod{k} \\ ac \neq bd \\ (abcd,k) = 1 \end{array}} \frac{1}{|\log \frac{ac}{bd}|} \ll \left( \frac{Z_1 Z_2}{k} \right)^{1+\varepsilon} \]
for any fixed $\varepsilon > 0$, while if $Z_1 Z_2 > k^{10}$ then
\[ E \ll \frac{Z_1 Z_2}{k} (\log Z_1 Z_2)^3. \tag{1} \]

Proof. We note that in each case the contribution of the terms with $|\log \frac{ac}{bd}| > \log 2$ is satisfactory, by the corresponding lemma of Soundararajan [5; Lemma 3]. Thus, by symmetry, it is enough to consider the terms with $bd < ac \leq 2bd$. We shall show how to handle the terms in which $ac \equiv bd \pmod{k}$, the alternative case being dealt with similarly. We write $n = bd$ and $ac = kl + bd$ and observe that $kl \leq bd$. We deduce that $n \leq 2\sqrt{Z_1 Z_2}$ and $1 \leq l \leq 2\sqrt{Z_1 Z_2}/k$. Since $\log \frac{ac}{bd} \gg k/n$ the contribution of these terms to $E$ is
\[ \ll \frac{1}{k} \sum_{l \leq 2\sqrt{Z_1 Z_2}/k} \frac{1}{l} \sum_{n \leq 2\sqrt{Z_1 Z_2}/k} nd(n)d(kl+n). \]

We estimate the sum over $n$ using a bound from Heath-Brown’s paper [11; (17)]. This shows that the above expression is
\[ \ll \frac{Z_1 Z_2 (\log Z_1 Z_2)^2}{k} \sum_{l \leq 2\sqrt{Z_1 Z_2}/k} \frac{1}{l} \sum_{d|l} d^{-1} \ll \frac{Z_1 Z_2}{k} (\log Z_1 Z_2)^3. \]

This suffices to complete the proof. The reader will observe that when $Z_1 Z_2 \leq k^{10}$ it is only the terms with $|\log \frac{ac}{bd}| > \log 2$ which prevent us from achieving the bound $E$.

Finally we shall require the following two lemmas [5; Lemmas 4 and 5].

Lemma 5. For $q \geq 2$ we have
\[ \sum_{\substack{n \leq x \\ (n,q) = 1}} \frac{1}{n} = \frac{\varphi(q)}{q} (\log x + O(1 + \log \omega(q))) + O\left( \frac{2^{\omega(q)} \log x}{x} \right). \]

Lemma 6. For $x \geq \sqrt{q}$ we have
\[ \sum_{\substack{n \leq x \\ (n,q) = 1}} \frac{2^{\omega(n)}}{n} \ll \left( \frac{\varphi(q)}{q} \right)^2 (\log x)^2, \]
and
\[ \sum_{\substack{n \leq x \\ (n,q) = 1}} \frac{2^{\omega(n)} \left( \log \frac{x}{n} \right)^2}{n} = \left( 1 + O\left( \frac{1 + \log \omega(q)}{\log q} \right) \right) \frac{(\log x)^4}{12 \zeta(2)} \prod_{p|q} \frac{1 - 1/p}{1 + 1/p}. \]
3. The main term

Applying Lemma 3 we have

$$\sum_{\chi(\text{mod } q)} \star \int_0^T A(t, \chi)^2 dt = M + E,$$

where

$$M = \frac{\varphi^* (q)}{2} \sum_{a=0,1} \sum_{\substack{ab, cd \leq Z \\ ac=bd}} \frac{1}{\sqrt{abcd}} \int_0^T W_a \left( \frac{\pi ab}{q}; t \right) W_a \left( \frac{\pi cd}{q}; t \right) dt,$$

and

$$E = \sum_{k|q} \varphi(k) \mu(q/k) E(k),$$

with

$$E(k) = \sum_{a=0,1} \sum_{\substack{ab, cd \leq Z \\ ac=\pm bd \pmod{k} \\ ac \neq bd}} \frac{1}{\sqrt{abcd}} \int_0^T \left( \frac{ac}{bd} \right)^{-it} W_a \left( \frac{\pi ab}{q}; t \right) W_a \left( \frac{\pi cd}{q}; t \right) dt.$$
From Lemma 2 we have \( W_a(\pi g^2 n/q; t) = 1 + O(g^{1/2}(n/qt)^{1/2}) \), whence

\[
M = \varphi^*(q) T \sum_{n \leq Z} \frac{2^{\omega(n)}}{n} \left( \sum_{g \leq \sqrt{Z/n}} \frac{1}{g} + O(1) \right)^2.
\]

We split the terms \( n \leq Z \) into the cases \( n \leq Z_0 \) and \( Z_0 < n \leq Z \), where \( Z_0 = Z/9^{\omega(q)} \). In the first case, from Lemma 5 the sum over \( g \) is

\[
= \frac{\varphi(q)}{2q} \log \frac{Z_0}{n} + O(1 + \log \omega(q)),
\]

since the first error term in Lemma 5 dominates the second. Hence the contribution of such values of \( n \) to \( M \) is

\[
\varphi^*(q) T \left( \frac{\varphi(q)}{2q} \right)^2 \sum_{n \leq Z_0} \frac{2^{\omega(n)}}{n} \left( \log \frac{Z_0}{n} \right)^2 + O(\omega(q) \log Z).
\]

Here we use the fact that \( q/\varphi(q) \ll 1 + \log \omega(q) \). This estimate will be employed a number of times in what follows, without further comment. In view of Lemma 6 the contribution from terms with \( n \leq Z_0 \) is now seen to be

\[
\frac{\varphi^*(q) T}{8\pi^2} \prod_{p|q} \left( \frac{1 - 1/p}{1 + 1/p} \right)^3 (\log Z_0)^4 \left( 1 + O\left( \frac{\omega(q)}{\log q} \right) \right).
\]

For \( Z_0 \leq n \leq Z \), we extend the sum over \( g \) to all \( g \leq 3^{\omega(q)} \) that are coprime to \( q \). By Lemma 5, this sum is \( \ll \omega(q) \varphi(q)/q \). Hence the contribution of these terms to \( M \) is

\[
\ll \varphi^*(q) T \left( \omega(q) \frac{\varphi(q)}{q} \right)^2 \sum_{Z_0 \leq n \leq Z} \frac{2^{\omega(n)}}{n} \ll \varphi^*(q) T \left( \frac{\varphi(q)}{q} \right)^4 \omega(q)^2 (\log Z)^2.
\]

Combining this with (2) and (3) we obtain

\[
\sum_{\chi(\mod q)}^* \int_0^T A(t, \chi)^2 dt = \left( 1 + O\left( \frac{\omega(q)}{\log q} \right) \right) \frac{\varphi^*(q) T}{8\pi^2} \prod_{p|q} \left( \frac{1 - 1/p}{1 + 1/p} \right)^3 (\log q T)^4.
\]

4. The error term

We have

\[
\sum_{\chi(\mod q)}^* \int_0^T B(t, \chi)^2 dt \leq \sum_{\chi(\mod q)} \int_0^T B(t, \chi)^2 dt
\]

\[
= \frac{\varphi(q)}{2} \sum_{a=0}^1 \sum_{ab,\text{coprime}>Z} \frac{1}{\sqrt{abcd}} \int_0^T \left( \frac{ac}{bd} \right)^{-it} W_a \left( \frac{\pi ab}{q}; t \right) W_a \left( \frac{\pi cd}{q}; t \right) dt.
\]

Using Lemma 2 and integration by parts, the integral over \( t \) is

\[
\ll \frac{1}{\log \frac{ac}{bd}} \left( 1 + \frac{ab}{qT} \right)^{-2} \left( 1 + \frac{cd}{qT} \right)^{-2}.
\]
if \(ac \neq bd\), and is
\[
\ll T \left(1 + \frac{ab}{qT}\right)^{-2} \left(1 + \frac{cd}{qT}\right)^{-2}
\]
if \(ac = bd\). Hence the right hand side of (5) is \(O(R_1 + R_2)\), where
\[
R_1 = \varphi(q)T \sum_{ab,cd > Z} \frac{1}{abcd} \left(1 + \frac{ab}{qT}\right)^{-2} \left(1 + \frac{cd}{qT}\right)^{-2},
\]
and
\[
R_2 = \varphi(q) \sum_{ab,cd > Z} \frac{1}{abcd |\log ac|} \left(1 + \frac{ab}{qT}\right)^{-2} \left(1 + \frac{cd}{qT}\right)^{-2}.
\]

To estimate \(R_2\), we again break the terms into dyadic ranges \(Z_1 \leq ab < 2Z_1\) and \(Z_2 \leq cd < 2Z_2\), where \(Z_1, Z_2 > Z\). By Lemma 4, the contribution of each such block is
\[
\ll \frac{\varphi(q)}{\sqrt{Z_1 Z_2}} \left(1 + \frac{Z_1}{qT}\right)^{-2} \left(1 + \frac{Z_2}{qT}\right)^{-2} \frac{Z_1 Z_2}{q} (\log Z_1 Z_2)^3.
\]

Summing over all the dyadic ranges we obtain
\[
R_2 \ll \varphi(q)T (\log qT)^3. \tag{6}
\]

To handle \(R_1\) we argue as in the previous section. We write \(a = gr\), \(b = gs\), \(c = hs\) and \(d = hr\), where \((r, s) = 1\), and we put \(n = rs\). Then
\[
R_1 \ll \varphi(q)T \sum_{(n,q)=1} \frac{2^\omega(n)}{n} \left(\sum_{g > \sqrt{Z/n}} \frac{1}{g} \left(1 + \frac{g^2 n}{qT}\right)^{-2}\right)^2. \tag{7}
\]

We split the sum over \(n\) into the ranges \(n \leq qT\) and \(n > qT\). In the first case, the sum over \(g\) is
\[
\ll 1 + \sum_{\sqrt{Z/n} \leq g \leq \sqrt{qT/n}} \frac{1}{g}.
\]
When \(n \leq Z_0\) this is
\[
\ll \frac{\varphi(q)}{q} \omega(q).
\]
by Lemma 5. In the alternative case \(n > Z_0\) we extend the sum over \(g\) to include all \(g \leq 3^\omega(q)\) that are coprime to \(q\). Lemma 5 then gives the same bound as before. Thus the contribution of the terms \(n \leq qT\) to (7), using Lemma 6, is
\[
\ll \varphi(q)T \left(\frac{\varphi(q)}{q} \omega(q)\right)^2 \sum_{n \leq qT} \frac{2^\omega(n)}{n} \ll qT \left(\frac{\varphi(q)}{q}\right)^5 \omega(q)^2 (\log qT)^2. \tag{8}
\]
In the remaining case \( n > qT \), the sum over \( g \) in (7) is \( O(q^2T^2/n^2) \). Hence the contribution of such terms is 
\[
\ll \varphi(q)T \sum_{n > qT} \frac{2^{\omega(n)}}{n} \frac{q^4T^4}{n^4} \ll \varphi(q)T \log qT.
\]

In view of (6) and (8) we now have 
\[
\sum_{\chi \pmod q}^* \int_0^T B(t, \chi)^2 \, dt \ll qT \left( \frac{\varphi(q)}{q} \right)^5 \omega(q)^2(\log qT)^2 + \varphi(q)T(\log qT)^3. \tag{9}
\]

5. Deduction of Theorem 1

From Lemma 1 we have 
\[
\sum_{\chi \pmod q}^* \int_0^T |L(\frac{1}{2} + it, \chi)|^4 \, dt = 4 \sum_{\chi \pmod q}^* \int_0^T (A(t, \chi)^2 + 2A(t, \chi)B(t, \chi) + B(t, \chi)^2) \, dt.
\]
The first and third terms on the right hand side are handled by (4) and (9). Also, by Cauchy’s inequality we have
\[
\sum_{\chi \pmod q}^* \int_0^T A(t, \chi)B(t, \chi) \, dt \leq \left( \sum_{\chi \pmod q}^* \int_0^T A(t, \chi)^2 \, dt \right)^{\frac{1}{2}} \left( \sum_{\chi \pmod q}^* \int_0^T B(t, \chi)^2 \, dt \right)^{\frac{1}{2}}.
\]
Hence (4) and (9) also yield an estimate for the cross term. Combining these results leads to the theorem.

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