Measuring distance between quantum states on a quantum computer

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Abstract
We propose protocols for determining the distances in Hilbert space between pure and mixed quantum states prepared on a quantum computer. In the case of pure quantum states, the protocol is based on measuring the square of modulus of scalar product between certain states. Determination of the distance between mixed quantum states is reduced to measuring the squares of modules of scalar products between all pure states included in the mixed states. In addition, we develop a protocol that allows one to determine the speed of evolution of the spin system simulated by a quantum computer. We apply these protocols to measure distances and speeds of evolution of different quantum systems implemented on the ibmq-santiago quantum computer.

Keywords Distance between quantum states · pure state · Mixed state · Speed of evolution · Quantum computer

1 Introduction
The concept of a distance between quantum states in Hilbert space [1–3] has found its application in different fields of quantum mechanics related to the evolution of quantum systems [4–19], quantum entanglement [8,20–30], quantum computations [31–35], etc. It was shown that the distance, which the quantum system passes during the evolution in the Hilbert space, is related to the integral of the uncertainty of energy that in turn defines the speed of evolution [4]. This distance is defined by the expression

$$s = \int_0^\tau \sqrt{g_{tt}} \, dt,$$

(1)
where
\[ g_{tt} = \gamma^2 \langle \psi(t) | (\Delta H)^2 | \psi(t) \rangle, \] (2)
and \( \tau \) is a period of time. Here, \( \Delta H = H - \langle \psi(t) | H | \psi(t) \rangle \) is the energy uncertainty and \( | \psi(t) \rangle = \exp(-iHt) | \psi_I \rangle \) is the state which the system described by the Hamiltonian \( H \) achieves during the time \( t \) having started from the initial state \( | \psi_I \rangle \), and \( \gamma \) is a scale parameter. We put \( \hbar = 1 \), which means that the energy is measured in the frequency units. From equation (1), it follows that the speed of quantum evolution has the form
\[ v = \sqrt{g_{tt}}. \] (3)

This expression is called the Anandan–Aharonov relation [4]. The distance defined by expression (1) is obtained from the Fubini–Study metric [7,36–39] for two neighboring pure quantum states separated by an infinitesimal period of time. Indeed, the Fubini–Study distance [1–3,38,40] between two pure states \( | \psi_1 \rangle \) and \( | \psi_2 \rangle \) is defined by the expression
\[ d_{FS} (| \psi_1 \rangle, | \psi_2 \rangle) = \gamma \sqrt{1 - | \langle \psi_1 | \psi_2 \rangle |^2}. \] (4)

Then, for two neighboring pure states \( | \psi(t) \rangle \) and \( | \psi(t + dt) \rangle \) separated by the period of time \( dt \) the square of the Fubini–Study distance up to the second order in \( dt \) takes form [36]
\[ ds^2 = g_{tt} dt^2. \] (5)

From this equation, it is easy to obtain an expression that allows one to define the distance (1) that the system passes during the time \( \tau \). Note that in some way equation (5) can be derived from the Wootters distance [41]
\[ d^W (| \psi_1 \rangle, | \psi_2 \rangle) = \gamma \arccos | \langle \psi_1 | \psi_2 \rangle |. \] (6)

minimal distance [42]
\[ d^{min} (| \psi_1 \rangle, | \psi_2 \rangle) = \gamma \sqrt{2(1 - | \langle \psi_1 | \psi_2 \rangle |)}, \] (7)
or the definition of another distance between pure states (see, for instance, [1,43]). In general, these distances are different; however, for neighboring pure quantum states they coincide.

There are many definitions of the distance between mixed states in the physical literature: the Jauch–Misra–Gibson distance [44,45], the trace distance proposed by Hillery [46,47], the Bures–Uhlmann distance [48,49], the Hilbert–Schmidt distance [1–3,5,45,50,51]. The most convenient for calculations is the Hilbert–Schmidt distance. It is based on the Hilbert–Schmidt norm \( \| A \|_2 \equiv \sqrt{\text{Tr}(A^+A)} \). This distance
between two mixed states $\rho_1$ and $\rho_2$ is defined as follows

$$d_{HS}(\rho_1, \rho_2) = \gamma' \|\rho_1 - \rho_2\|_2 = \gamma' \sqrt{\text{Tr} (\rho_1 - \rho_2)^2} = \gamma' \sqrt{\text{Tr} \rho_1^2 + \text{Tr} \rho_2^2 - 2\text{Tr} \rho_1 \rho_2},$$

(8)

where $\gamma'$ is a scale parameter. In the case of pure states, the Hilbert–Schmidt distance turn into the Fubini–Study distance (4) with $\gamma' = \gamma/\sqrt{2}$. It is important to note that Hilbert–Schmidt distance is often used in quantum optics [1,52,53].

In this paper, we propose protocols that allow one to determine the distance in Hilbert space between pure quantum states and define the speed of evolution of the quantum system prepared on a quantum computer (Sec. 2). Using this protocol, in Sec. 3 we obtain results for different quantum states and systems prepared on the ibmq-santiago quantum computer. Namely, we measure the speed of evolution and distances between different states of spin-$1/2$ in the magnetic field (Subsec. 3.1), the distance between the Schrödinger cat and factorized states (Subsec. 3.2), and speed of evolution and distances between states achieved during the evolution of a spin-$1/2$ chain described by the Ising model (Sect. 3.3). In addition, we develop and test a protocol which allows measuring the distance between mixed quantum state prepared on a quantum computer (Sec. 4). Conclusions are presented in Sec. 5.

2 Protocol for determining the distance between pure quantum states

The Fubini–Study (4), Wootters (6) and minimal (7) distances contain the modulus of the scalar product of states $|\psi_1\rangle$ and $|\psi_2\rangle$. The problem is to find a method that allows us to measure this modulus on a quantum computer. Let us represent these states as a transformation of the initial state $|0\rangle = |00\ldots0\rangle$ under the action of the unitary operators $U_i$ as follows $|\psi_i\rangle = U_i |0\rangle$, where $|0\rangle$ is the projection of the qubit on the positive direction of the $z$-axis. We use such a representation because basically the initial state of quantum computers has the form $|0\rangle$. We obtain

$$|\langle \psi_1 | \psi_2 \rangle|^2 = |\langle 0 | U_1^+ U_2 | 0 \rangle|^2 = |\langle 0 | \psi \rangle|^2,$$

(9)

where $U_i^+$ is the conjugate transpose of $U_i$ and $|\psi\rangle = U_1^+ U_2 |0\rangle$. The problem of determination of the distance between pure quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$ is reduced to measuring the probability corresponding to the reduction of $|\psi\rangle$ state into the $|0\rangle$ state. The protocol for measuring this probability is shown in Fig. 1. Firstly, we are preparing the state $|\psi\rangle$ by applying the unitary operators $U_2$ and $U_1^+$ to the initial state $|0\rangle$, and then we make measurements of each qubit on $z$-axis.

We can also measure the speed of quantum evolution (3) on a quantum computer, which in turn allows us to determine the path that the system takes in Hilbert space during the period of time $\tau$ (1). The speed of evolution is defined by the component of metric tensor (2). Using the fact that the operator of evolution $\exp(-i H t)$ and
Fig. 1 Quantum circuit for measuring the square of the modulus of scalar product between state $|\psi_1\rangle$ and $|\psi_2\rangle$ (9), which are performed by the $U_1$ and $U_2$ operators, respectively.

Hamiltonian $H$ mutually commute, the $g_{tt}$ component can be rewritten in the following form:

$$g_{tt} = \gamma^2 \left( \langle \psi_I | H^2 | \psi_I \rangle - \langle \psi_I | H | \psi_I \rangle^2 \right),$$  \hfill (10)

where $|\psi_I\rangle$ is the initial state prepared on a quantum computer by applying the certain unitary operator $U_I$ to the state $|0\rangle$ as follows: $|\psi_I\rangle = U_I|0\rangle$. As we can see, the mean values of $H$ and $H^2$ in the state $|\psi_I\rangle$ should be measured. For this purpose, we represent the Hamiltonian in the form $H = \sum_{\alpha} h_{\alpha} U_{h_{\alpha}}$, where $h_{\alpha}$ are real parameters which determine the Hamiltonian and $U_{h_{\alpha}}$ are the Hermitian operators which satisfy the condition $U_{h_{\alpha}}^2 = 1$. It should be noted that operators $U_{h_{\alpha}}$ determine the interactions in the system and they are represented by the basis gates of a quantum computer. Then, the mean values in equation (10) take the form

$$\langle \psi_I | H^2 | \psi_I \rangle = \sum_{\alpha, \beta} h_{\alpha} h_{\beta} \langle \psi_I | U_{h_{\alpha}} U_{h_{\beta}} | \psi_I \rangle,$$

$$\langle \psi_I | H | \psi_I \rangle = \sum_{\alpha} h_{\alpha} \langle \psi_I | U_{h_{\alpha}} | \psi_I \rangle,$$  \hfill (11)

The quantum computer provides the measurements of each qubit on the basis $|0\rangle$, $|1\rangle$ which consists of the eigenstates of $\sigma^z$ operators. This means that the $U_{h_{\alpha}}$ operators should be expressed by the $\sigma^z$ operators. For this purpose, each of the qubits of the system should be rotated as follows: if certain qubit $i$ in the term of Hamiltonian is defined by $\sigma^x_i$, $\sigma^y_i$ Pauli operator it should be rotated as follows:

$$\sigma^x = e^{-i \frac{\pi}{4} \sigma^y} \sigma^z e^{i \frac{\pi}{4} \sigma^y}, \quad \sigma^y = e^{i \frac{\pi}{4} \sigma^z} \sigma^z e^{-i \frac{\pi}{4} \sigma^z}.$$

(12)

As a result, mean values in equation (11) take the form

$$\langle \psi_I | U_{h_{\alpha}} U_{h_{\beta}} | \psi_I \rangle = \langle \tilde{\psi}_R^{\alpha \beta} | \sum_{\alpha} \sigma^z_{\alpha} \tilde{\psi}_R^{\alpha \beta} \rangle,$$

$$\langle \psi_I | U_{h_{\alpha}} | \psi_I \rangle = \langle \tilde{\psi}_R^\alpha | \sum_{\alpha} \sigma^z_{\alpha} \tilde{\psi}_R^\alpha \rangle,$$  \hfill (13)
Fig. 2 Quantum circuit for measuring the mean values (13). Operator $U_I$ provides preparation of the initial state $|\psi_I\rangle$. To provide the measurements of each qubit on the basis $|0\rangle, |1\rangle$ of a quantum computer, the $U_\alpha$ operators should be expressed by the $\sigma^z$ operators. For this purpose, operator $R$ provides rotations of each qubit (12) so that the operator $U_\alpha$ transforms into the operator $\Sigma^z_\alpha$. Finally, the quantum computer provides measurements of each qubit on the $z$-axis and the result is written into the classical register $c$.

where $|\tilde{\psi}_I^{R_{\alpha\beta}}\rangle, |\tilde{\psi}_I^{R_{\alpha}}\rangle$ are the states reached from the state $|\psi_I\rangle$ by the rotation of certain qubits that the operators $U_{h_{\alpha}}$ transform into the operators $\Sigma^z_\alpha$ which consists of the compositions of the $\sigma^z_i$ Pauli operators. The method for the determination of mean values of the $\Sigma^z_\alpha$ operator is described in detail in papers [54–56]. This means the value in the state $|\psi\rangle$ is defined as follows:

$$\langle \psi | \Sigma^z_\alpha | \psi \rangle = p_+ - p_-,$$

(14)

where $p_\pm$ are the probabilities that correspond to the mean values $\pm 1$. The protocol, which allows measuring the mean value, is shown in Fig. 2.

Let us apply these protocols to determine the distance between certain quantum states and values of the speed of evolution of some quantum systems.

### 3 Determining the distance between pure quantum states on a quantum computer

In this section, we test our protocols on the ibmq-santiago quantum computer. We determine the distance between different pure quantum states prepared on this device. In addition, we simulate the time-evolution and measure its speed in the case of a single spin in the magnetic field and spin system described by the Ising model. The ibmq-santiago is a 5-qubit quantum device designed by IBM (Fig. 3). It can be used freely through the cloud service called the IBM Q Experience [57]. To perform the quantum circuits, this computer uses a controlled-NOT gate, the identity, $R_z(\phi)$, $\sigma_x$ and $\sqrt{\sigma_x}$ single-qubit gates [58]. The $R_z(\phi)$ gate corresponds to rotating the qubit state around the $z$ axis by the angle $\phi$. The ibmq-santiago allows us to prepare the $\sqrt{\sigma_x}$ quantum gate with fidelity $> 99.6\%$, the controlled-NOT gate with fidelity $> 93\%$. Almost all qubits are read with fidelity $> 98\%$. 
Fig. 3 The ibmq-santiago quantum device consists of five superconducting qubits and has a linear structure. Each pair of qubits connected by the bidirectional arrow can be driven by the controlled-NOT operator in a way that each of the qubits can be both a control and a target.

3.1 Spin-1/2 in the magnetic field

The spin-1/2 in the magnetic field is defined by the Hamiltonian

\[ H = \frac{\omega}{2} \sigma \cdot n, \]  

(15)

where \( \omega \) define the value of interaction between spin and magnetic field, \( \sigma = \sigma^x \mathbf{i} + \sigma^y \mathbf{j} + \sigma^z \mathbf{k} \) is the spin-1/2 operator, \( n = n_x \mathbf{i} + n_y \mathbf{j} + n_z \mathbf{k} \) is the unit vector which defines the direction of the magnetic field. Having started from the initial state \( |\psi_I\rangle \), the evolution of such a system can be expressed by the state vector \[ |\psi\rangle = e^{-i \frac{\omega t}{2} \sigma \cdot n} |\psi_I\rangle = \cos \left( \frac{\omega t}{2} \right) I - i \sin \left( \frac{\omega t}{2} \right) \sigma \cdot n |\psi_I\rangle, \]  

(16)

where \( I \) is an identity operator. Depending on the direction of the magnetic field and time of evolution, the system can reach an arbitrary one-qubit state

\[ |\psi(\theta, \phi)\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i \phi} \sin \frac{\theta}{2} |1\rangle, \]  

(17)

where \( \theta \in [0, \pi] \), \( \phi \in [0, 2\pi] \) are some real parameters which determine the state. This state can be achieved from the initial state \( |0\rangle \) during the time \( t \) in the case of the perpendicular orientation of the magnetic field with respect to these states. Then, the parameters of state take the values \( \theta = \omega t \) and \( \phi = \arctan \left( \frac{n_y}{n_x} \right) - \frac{\pi}{2} \). On the ibmq-santiago quantum computer, state (17) can be achieved by applying the \( U(\theta, \phi, \lambda) \) gate to the state \( |0\rangle \), where \( \lambda \) is a real parameter which can take the values from the range \( \lambda \in [0, 2\pi] \). This gate is represented by the basis gates of the ibmq-santiago quantum computer as follows

\[ U(\theta, \phi, \lambda) = R_z(\phi + \pi) \sqrt{\sigma^x} R_z(\theta - \pi) \sqrt{\sigma^x} R_z(\lambda). \]  

(18)

In the basis \( |0\rangle, |1\rangle \), this gate reads

\[ U(\theta, \phi, \lambda) = \begin{pmatrix} \cos \frac{\theta}{2} & -e^{i \lambda} \sin \frac{\theta}{2} \\ e^{i \phi} \sin \frac{\theta}{2} & e^{i (\lambda + \phi)} \cos \frac{\theta}{2} \end{pmatrix}. \]  

(19)
Let us study the distance between two arbitrary quantum states of spin-1/2. Due to the symmetry properties of the state-space of spin-1/2, we can calculate the distance between \( |0\rangle \) and (17) states. Recall those different definitions of distances (4), (6), (7) contain the modulus of scalar product between certain states. In our case of spin-1/2, the square of this value has the following form:

\[
|\langle \psi_1 | \psi_2 \rangle|^2 = |\langle 0 | \psi(\theta, \phi) \rangle|^2 = \cos^2 \frac{\theta}{2}.
\]

(20)

This is the probability to measure state (17) on state \( |0\rangle \). Using the fact that \( |\psi(\theta, \phi)\rangle = U(\theta, \phi, \lambda)|0\rangle \), the following value \( |\langle 0 | U(\theta, \phi, \lambda)|0\rangle|^2 \) should be measured. Because probability (20) does not depend on parameters \( \phi \) and \( \lambda \), we set them to zero. Thus, the protocol for determining probability (20) is shown in Fig. 1 with \( U_1 = I \) and \( U_2 = U(\theta, 0, 0) \). On the ibmq-santiago quantum computer, we measure this probability for different angles \( \theta \), which changes in the range from 0 to \( 2\pi \) with the step \( \pi/20 \). Here and further in the paper to obtain enough statistics, for each value we provide 1024 measurements on the quantum computer. We substitute the results in expressions for distances (4), (6), (7). The dependencies of distances on parameter \( \theta \) are shown in Fig. 4. Since we measure the distances in the case of one qubit, the results obtained on the quantum computer are in good agreement with the theoretical prediction.

Now let us study the speed of evolution of spin-1/2 in the magnetic field (15). In this case \( h_\alpha = \omega/2, U_\alpha = \sigma \cdot n \) and we obtain

\[
\langle \psi_1 | H^2 | \psi_1 \rangle = \frac{\omega^2}{4},
\]
Fig. 5 Dependence of speed of evolution of spin-1/2 in the magnetic field on the angle between directions of magnetic field and vector of the initial state. The solid line shows the theoretical prediction, and the dots correspond to the value of speed obtained on the ibmq-santiago quantum computer

\[
\langle \psi_I | H | \psi_I \rangle = \frac{\omega}{2} \langle \psi_I | \sigma \cdot n | \psi_I \rangle, \tag{21}
\]

As we can see, to determine the speed of evolution of spin-1/2 in the magnetic field only the mean value of the \( \sigma \cdot n \) operator should be measured. For this purpose, we rotate the qubit to direct the magnetic field along the \( z \)-axis which corresponds to the transformation of \( \sigma \cdot n \) into \( \sigma^z \). Then, we can use expressions (13), (14) with \( \Sigma^z_\alpha = \sigma^z \) for determination of the mean value, where state \( | \tilde{\psi}_I^{R_\alpha} \rangle \) has the form (17). Thus, this mean value takes the form

\[
\langle \psi(\theta, \phi) | \sigma^z | \psi(\theta, \phi) \rangle = |\langle \psi(\theta, \phi) | 0 \rangle|^2 - |\langle \psi(\theta, \phi) | 1 \rangle|^2. \tag{22}
\]

Here, we use the fact that \( \sigma^z = |0\rangle \langle 0 | - |1\rangle \langle 1 | \). The problem reduces to the measurement of this mean value which should be substituted into (22) and then into (10) and (3). Making theoretical calculations, we obtain \( \langle \psi(\theta, \phi) | \sigma^z | \psi(\theta, \phi) \rangle = \cos \theta \) and \( g_{\text{tt}} = \gamma^2 \omega^2 \sin^2 \theta / 4, v = \gamma \omega | \sin \theta | / 2 \). On the ibmq-santiago we measure mean value (22) for different \( \theta \). In Fig. 5, we show the results for the speed of evolution and compare them with theoretical predictions.
3.2 Distance between Schrödinger cat and factorized states

In this subsection, we study the distance between 5-qubit states, namely the Schrödinger cat state

$$|\psi_{\text{cat}}\rangle = \frac{1}{\sqrt{2}} (|00000\rangle + |11111\rangle)$$  \hspace{1cm} (23)$$

and factorized state

$$|\psi_{\text{fact}}\rangle = \prod_{i=1}^{5} |\psi(\theta, \phi)\rangle_i,$$  \hspace{1cm} (24)$$

where $|\psi(\theta, \phi)\rangle_i$ is a single-qubit state defined by the expression (17). Since the definition of distances between pure states (4), (6), (7) contains the same modulus of the scalar products between certain states measured by a quantum computer, farther in the paper we study only the Fubini–Study distance (4). This definition contains the square of the modulus of scalar product between pure states which for the $|\psi_{\text{cat}}\rangle$ and $|\psi_{\text{fact}}\rangle$ takes the form

$$|\langle \psi_{\text{cat}} | \psi_{\text{fact}} \rangle|^2 = \frac{1}{2} \left( \cos^{10} \frac{\theta}{2} + \sin^{10} \frac{\theta}{2} + 2 \cos^{5} \frac{\theta}{2} \sin^{5} \frac{\theta}{2} \cos 5\phi \right).$$ \hspace{1cm} (25)$$

In Fig. 6, we show the protocol for measuring this value on a quantum computer.

We take measurements for different value of $\theta$ in the case of $\phi = 0$ (Fig. 7a) and for different values of $\phi$ in the case of $\theta = \pi/2, 3\pi/8, \pi/4$ and $\pi/8$ (Fig. 7b). We compare these dependencies with theoretical ones. As we can see, the closer the angle $\theta$ is to the value $\pi/2$ and $3\pi/2$, the more accurate the quantum computer performs the measurements. This fact follows the form definition of the $U(\theta, \phi, \lambda)$ gate (18). In the general case, this gate is performed by five basis operators. However, in the case of $\theta =
\[ U(\pm \pi/2, \phi, \lambda) = R_z(\phi \pm \pi/2)\sqrt{\sigma^x} R_z(\lambda \mp \pi/2), \]
which in turn reduced the error of this gate. Thus, the preparation of the five-qubit factorized state \(|\psi_{fact}\rangle\) with \(\theta = \pi/2, 3\pi/2\) requires ten basis operators less than in the general case.

### 3.3 Ising model

Based on the structure of the ibmq-santiago quantum device (see Fig. 3), the Ising model with the nearest neighbor interaction between spins can be simulated. We examine the distance which separates the initial state and any other state achieved during the evolution of such a system. Hamiltonian of the Ising model with the nearest neighbor interaction has the form

\[ H = \frac{J}{4} \sum_{i=1}^{4} \sigma_i^z \sigma_{i+1}^z, \]

where \(J\) defines the value of interaction between spins. Due to the fact that all terms in the Hamiltonian mutually commute, the evolution of this system can be expressed as follows:

\[
|\psi(\chi, \theta, \phi)\rangle = e^{-iHt} |\psi_{fact}\rangle = \prod_{i=1}^{4} e^{-i \frac{J}{8} \sigma_i^z \sigma_{i+1}^z} |\psi_{fact}\rangle, \]

where \(\chi = Jt\) is a parameter that depends on time and has a period of \(4\pi\). \(|\psi_{fact}\rangle\) is defined by expression (24). Each of the terms in the evolution operator can be performed on a quantum computer using two controlled-NOT operators and one \(R_z(-\chi/2)\) operator. Based on expression (9) in Fig. 8, we represent a quantum circuit that allows measuring the square of the modulus of the scalar product between initial state (24) and state which is achieved during the evolution (27). Because this value does not depend on \(\phi\)

\[
|\langle \psi_{fact} | \psi(\chi, \theta, \phi) \rangle|^2 = \cos^8 \frac{\chi}{4} + \sin^8 \frac{\chi}{4} \cos^4 \theta
+ \cos^4 \frac{\chi}{4} \sin^4 \frac{\chi}{4} \left( 9 \cos^8 \theta + 2 \cos^6 \theta - 7 \cos^4 \theta + 2 \cos^2 \theta \right)
+ \cos^6 \frac{\chi}{4} \sin^2 \frac{\chi}{4} \left( 10 \cos^4 \theta - 6 \cos^2 \theta \right)
+ \cos^2 \frac{\chi}{4} \sin^6 \frac{\chi}{4} \left( 4 \cos^8 \theta + 2 \cos^6 \theta - 2 \cos^4 \theta \right)
\]

(28)

we put \(\phi = 0\). On the ibmq-santiago quantum computer for different initial states, we measure the dependence of this value on parameter \(\chi\). Using equation (4), we calculate the Fubini–Study distance between initial state (24) and state which is achieved during
Fig. 7 Dependence of the Fubini–Study distance between Schrödinger cat (23) and factorized (24) states on state parameters in the case of five qubits. In figure (a), the dependence of the distance on angle $\theta$ in the case of $\phi = 0$ is shown. In turn, in figure (b) the dependencies of the distances on angle $\phi$ for different values of $\theta$ are represented. The solid lines show the theoretical prediction, and the dots correspond to the results obtained on the ibmq-santiago quantum device.

the evolution at the moment of time $t$ (27). In Fig. 9, we demonstrate these results. As in the previous case, we obtain the best coincidence of the results with the theoretical prediction for $\theta = \pi/2$. 
The quantum circuit allows one to measure the square of the modulus of the scalar product (9) between initial state (24) and state which is achieved during the evolution (27). To calculate this modulus, we conjure the state which is achieved during the evolution (27). A set of operators \( U(\theta, 0, 0) \) determine the initial state with \( \phi = \lambda = 0 \), the operator \( U(\theta, \pi, \pi) \) is the conjugate transpose to the operator \( U(\theta, 0, 0) \), and the unit consisting of two controlled-NOT operators and \( R_z(-\chi/2) \) generates the Ising interaction between certain spins. The quantum computer provides measurements of each qubit on the basis \(|0\rangle, |1\rangle\), and the result is written into the classical register \( c \).

Finally, substituting Hamiltonian (26) with initial state (24) into expression (10) we obtain

\[
g_{t t} = \frac{\gamma^2 J^2}{16} \left( 4 + \sum_{i \neq j=1}^{4} \langle \psi_{fact} | \sigma_i^z \sigma_{i+1}^z \sigma_j^z \sigma_{j+1}^z | \psi_{fact} \rangle ight)^2 \right) \\
= \frac{\gamma^2 J^2}{16} \left( 4 + 6 \cos^2 \theta - 10 \cos^4 \theta \right). \tag{29}
\]
As we can see, to determine the speed of evolution the two- and four-spin correlation functions should be measured. Due to the structure of Hamiltonian (26), the operators substituted into equations (11) already have the form $U_{h} = \sigma_i^z \sigma_{i+1}^z$. To measure these correlations on a quantum computer, we use expressions (13) and (14) with $|\tilde{\psi}^R_{\alpha} \rangle = |\psi_{\text{fact}} \rangle$. Since the speed of evolution does not depend on parameter $\phi$, on the ibmq-santiago device we obtain the dependence of the speed on parameter $\theta$ (Fig. 10).

4 Protocol for measuring the distance between mixed quantum states

In this section, we describe the protocol for determining the Hilbert–Schmidt distance between mixed quantum states (8) prepared on a quantum computer. An arbitrary mixed quantum state $\rho_i$ which consists of a set of pure quantum states $|\psi_{\alpha}^{(i)} \rangle$ with appropriate weights $\omega_{\alpha}^{(i)}$ can be expressed as follows:

$$\rho_i = \sum_{\alpha} \omega_{\alpha}^{(i)} |\psi_{\alpha}^{(i)} \rangle \langle \psi_{\alpha}^{(i)}|,$$

where $\sum_{\alpha} \omega_{\alpha}^{(i)} = 1$. The traces that included equation (8) can be represented as follows:
where $|k\rangle$ is a set of the basis vectors which defines the Hilbert space of the states $\rho_i$ and $\sum_k |k\rangle\langle k|$ is an identity operator defined in this space. As we can see from (31), the problem of determination of the distance between mixed quantum states is reduced to the problem of the determination of the squares of modules of scalar products between all pure quantum states included by the mixed states. For this purpose, the protocol described in section 2 is used. The results with appropriate products of weight factors are substituted into expressions (31) and then into equation (8).

As an example, let us measure on the ibmq-santiago quantum computer the distance between the following quantum states:

$$\rho_1 = \left( \cos\frac{\theta}{2}|00000\rangle + \sin\frac{\theta}{2}|11111\rangle \right) \left( \cos\frac{\theta}{2}\langle 00000| + \sin\frac{\theta}{2}\langle 11111| \right)$$

$$\rho_2 = \frac{1}{4}|00000\rangle\langle 00000| + \frac{3}{4}|11111\rangle\langle 11111|.$$  

Thus, we want to define the distance between the pure state $\rho_1$ and mixed state $\rho_2$ consisting of $|\psi_1^{(2)}\rangle = |00000\rangle$, $|\psi_2^{(2)}\rangle = |11111\rangle$ pure states with weight factors $\omega_1^{(2)} = 1/4$, $\omega_2^{(2)} = 3/4$, respectively. Here, the problem is reduced to the determination of all squares of modules between pure states $\cos\frac{\theta}{2}|00000\rangle + \sin\frac{\theta}{2}|11111\rangle$, $|00000\rangle$, $|11111\rangle$. Using equation (8) for states (32), we obtain

$$d^{HS}(\rho_1, \rho_2) = \gamma' \sqrt{\frac{5}{8} + \cos \frac{\theta}{2}}.$$  

In Fig. 11, we compare the results obtained on the ibmq-santiago quantum computer with theoretical ones. Since in the case of mixed quantum states we measure the squares of modules of scalar products between all pure states included in these states, the errors accumulate from all measurements. In turn, this leads to the worse coincidence of the measurement results with the theory than in the case of pure states.
Fig. 11  The dependence of the Hilbert–Schmidt distance between $\rho_1$ and $\rho_2$ states (32) on state parameter $\theta$. The solid line shows the theoretical prediction, and the dots correspond to the results obtained on the ibmq-santiago quantum device.

5 Conclusions

We have proposed the protocol that allows one to define the distance between pure states prepared on a quantum computer. To determine the distance between certain states, the measurement results on the initial state $|0\rangle$ of the quantum computer are enough to take. This fact makes our protocol easy to use and practical for calculations. In addition, we have proposed the method for determining the speed of evolution of a quantum system simulated on a quantum computer. This method is based on measurement energy uncertainty which is included in the well-known Anandan–Aharonov relation (3). The problem is reduced to the measurement of the mean values of spins and correlation functions of spins (13). We have applied our methods to different pure quantum states and systems prepared on the ibmq-santiago quantum computer. As an example, we have determined the distances between pure states of spin-$1/2$ in the magnetic field. Depending on the direction of the magnetic field to the initial state, we have measured the speed of evolution of such a system. We have also applied our protocol to determine the distance between five-qubit pure states. Namely, we determine the distance between the Schrödinger cat state (23) and factorized state (24). We have also simulated the evolution of the system defined by the Ising Hamiltonian (26). The distances between the initial state and states achieved during the evolution have been measured. In addition, depending on the initial state the speed of evolution of such system has been obtained.

Finally, we have developed the protocol to measure the distance between mixed quantum states prepared on a quantum computer. This protocol is based on the determination of the Hilbert–Schmidt norm (8). We have shown that the distance between
two mixed states is represented by the squares of the modules of scalar products between the pure quantum states included by the mixed states (32). We have applied this protocol for the determination of the distance between two states defined by density matrices (32) prepared on the ibmq-santiago quantum computer. Despite the fact that the measurements are performed for all possible scalar products between pure states, the experimental results are in good agreement with theoretical predictions.

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Declarations

Conflict of interest The author declares that he has no conflict of interest.

References

1. Dodonov, V.V., Man’ko, O.V., Man’ko, V.I., Wünsche, A.: Energy-sensitive and “classical-like” distances between quantum states. Phys. Scr. 59, 81 (1999)
2. Bengtsson, I., Życzkowski, K.: Geometry of Quantum States. Cambridge University Press, Cambridge (2006)
3. Tkachuk, V.M.: Fundamental Problems of Quantum Mechanic. Ivan Franko National University of Lviv, Lviv (2011). [in Ukrainian]
4. Anandan, J., Aharonov, Y.: Geometry of quantum evolution. Phys. Rev. Lett. 65, 1697 (1990)
5. Anandan, J.: A geometric approach to quantum mechanics. Found. Phys. 21, 1265 (1991)
6. Abe, S.: Quantized geometry associated with uncertainty and correlation. Phys. Rev. A 48, 4102 (1993)
7. Kolodrubetz, M., Sels, D., Mehta, P., Polkovnikov, A.: Geometry and non-adiabatic response in quantum and classical systems. Phys. Rep. 697, 1 (2017)
8. Brody, D.C., Hughston, L.P.: Geometric quantum mechanics. J. Geom. Phys. 38, 19 (2001)
9. Kuzmak, A.R., Tkachuk, V.M.: The quantum brachistochrone problem for two spins-1/2 with anisotropic Heisenberg interaction. J. Phys. A 46, 155305 (2013)
10. Kuzmak, A.R., Tkachuk, V.M.: The quantum brachistochrone problem for an arbitrary spin in a magnetic field. Phys. Lett. A 379, 1233 (2015)
11. Carlini, A., Hosoya, A., Koike, T., Okudaira, Y.: Time-optimal quantum evolution. Phys. Rev. Lett. 96, 060503 (2006)
12. Brody, D.C., Hook, D.W.: On optimum Hamiltonians for state transformations. J. Phys. A 39, L167 (2006)
13. Frydryszak, A.M., Tkachuk, V.M.: Quantum brachistochrone problem for a spin-1 system in a magnetic field. Phys. Rev. A 77, 014103 (2008)
14. Russell, B., Stepney, S.: Zermelo navigation and a speed limit to quantum information processing. Phys. Rev. A 90, 012303 (2014)
15. Chenu, A., Beau, M., Cao, J., del Campo, A.: Quantum simulation of generic many-body open system dynamics using classical noise. Phys. Rev. Lett. 118, 140403 (2017)
16. Deffner, S., Campbell, S.: Quantum speed limits: from Heisenberg’s uncertainty principle to optimal quantum control. J. Phys. A 50, 453001 (2017)
17. Krynytskyi, Y.S., Kuzmak, A.R.: Geometry and speed of evolution for a spin-s system with long-range zz-type Ising interaction. Ann. Phys. 405, 38 (2019)
18. Frydryszak, A.M., Gieysztor, M., Kuzmak, A.R.: Probing the geometry of two-qubit state space by evolution. Quantum Inf. Process. 18, 84 (2019)
19. Laba, H.P., Tkachuk, V.M.: Geometric characteristics of quantum evolution: curvature and torsion. Cond. Matt. Phys. 20, 13003 (2017)
20. Shimony, A.: Degree of entanglement, Ann. N.Y. Acad. Sci. 755, 675 (1995)
21. Wei, T.C., Goldbart, P.M.: Geometric measure of entanglement and applications to bipartite and multipartite quantum states. Phys. Rev. A 68, 042307 (2003)
22. Chen, L., Aulbach, M., Hajdusek, M.: Comparison of different definitions of the geometric measure of entanglement. Phys. Rev. A 89, 042305 (2014)
23. Frydryszak, A.M., Samar, M.I., Tkachuk, V.M.: Quantifying geometric measure of entanglement by mean value of spin and spin correlations with application to physical systems. Eur. Phys. J. D 71, 233 (2017)
24. Kus, M., Życzkowski, K.: Geometry of entangled states. Phys. Rev. A 63, 042305 (2001)
25. Duan, L.M., Cirac, J.I., Zoller, P.: Geometric manipulation of trapped ions for quantum computation science. 292, 1695 (2001)
26. Zu, C., et al.: Experimental realization of universal geometric quantum gates with solid-state spins. Nature 514, 72 (2014)
27. Avron, J.E., Kenneth, O.: Entanglement and the geometry of two qubits. Ann. Phys. 324, 470 (2009)
28. Kuzmak, A.R., Tkachuk, V.M.: Geometry of a two-spin quantum state in evolution. J. Phys. A 49, 045301 (2016)
29. Kuzmak, A.R.: Quantum state geometry and entanglement of two spins with anisotropic interaction in evolution. J. Geom. Phys. 116, 81 (2017)
30. Kuzmak, A.R.: Entanglement and quantum state geometry of a spin system with all-range Ising-type interaction. J. Phys. A. 51, 175305 (2018)
31. Nielsen, M.A., Dowling, M.R., Gu, M., Doherty, A.C.: Optimal control, geometry, and quantum computing. Phys. Rev. Phys. Rev. A 73, 062323 (2006)
32. Nielsen, M.A.: A geometric approach to quantum circuit lower bounds. Quant. Inf. Comput. 6, 213 (2006)
33. Nielsen, M.A., Dowling, M.R., Gu, M., Doherty, A.C.: Quantum computation as geometry. Science 311, 1133 (2006)
34. Khaneja, N., Heitmann, B., Spörl, A., Yuan, H., Schulte-Herbrüggen, T., Glaser, S.J.: Quantum gate design metric. arXiv:quant-ph/0605071, (2006)
35. Li, B., Yu, Z.-H., Fei, S.-M.: Geometry of quantum computation with qutrits. Sci. Rep. 3, 2594 (2013)
36. Abe, S.: Quantum-state space metric and correlations. Phys. Rev. A 46, 1667 (1992)
37. Page, D.N.: Geometrical description of Berry’s phase. Phys. Rev. A 36, 3479(R) (1987)
38. Kobayashi, S., Nomizu, K.: Fundations of Differential Geometry, vol. 2. Wiley, New York (1969)
39. Ozawa, T., Goldman, N.: Extracting the quantum metric tensor through periodic driving. Phys. Rev. B 97, 201117(R) (2018)
40. Bargmann, V.: On unitary ray representations of continuous groups. Ann. Math. 59, 1 (1954)
41. Wootters, W.K.: Statistical distance and Hilbert space. Phys. Rev. D 23, 357 (1981)
42. Pati, A.K.: Relation between “phases” and “distance” in quantum evolution. Phys. Lett. A 159, 105 (1991)
43. Ravicule, M., Casas, M., Plastino, A.: Information and metrics in Hilbert space. Phys. Rev. A 55, 1695 (1997)
44. Jauch, J.M., Misra, B., Gibson, A.G.: On the asymptotic condition of scattering theory. Helv. Phys. Acta 41, 513 (1968)
45. Dieks, D., Veltkamp, P.: Distance between quantum states, statistical inference and the projection postulate. Phys. Lett. A 97, 24 (1983)
46. Hillery, M.: Nonclassical distance in quantum optics. Phys. Rev. A 35, 725 (1987)
47. Hillery, M.: Total noise and nonclassical states. Phys. Rev. A 39, 2994 (1989)
48. Bures, D.: An extension of Kakutani’s theorem on infinite product measures to the tensor product of semifinite W*-algebras. Trans. Am. Math. Soc. 135, 199 (1969)
49. Uhlmann, A.: The “transition probability” in the state space of a *-algebra. Rep. Math. Phys. 9, 273 (1976)
50. von Baltz, R.: Distance between quantum states and the motion of wave packets. Europ. J. Phys. 11, 215 (1990)
51. Życzkowski, K., Slomczynski, W.: The Monge metric on the sphere and geometry of quantum states. J. Phys. A 34, 6689 (2001)
52. Knöll, L., Orlowski, A.: Distance between density operators: applications to the Jaynes-Cummings model. Phys. Rev. A 51, 1622 (1995)
53. Dodonov, V.V., Reno, M.B.: Classicality and anticlassicality measures of pure and mixed quantum states. Phys. Lett. A 308, 249 (2003)
54. Kuzmak, A.R., Tkachuk, V.M.: Detecting entanglement by the mean value of spin on a quantum computer. Phys. Lett. A 384, 126579 (2020)
55. Gnatenko, Kh.P., Tkachuk, V.M.: Entanglement of graph states of spin system with Ising interaction and its quantifying on IBM’s quantum computer. Phys. Lett. A 396, 127248 (2021)
56. Kuzmak, A.R., Tkachuk, V.M.: Measuring entanglement of a rank-2 mixed state prepared on a quantum computer. Eur. Phys. J. Plus 136, 564 (2021)
57. IBM Q Experience. https://quantum-computing.ibm.com
58. Cross, A.W., Bishop, L.S., Smolin, J.A., Gambetta, J.M.: Open quantum assembly language. arXiv:1707.03429 (2017)
59. Boscain, U., Mason, P.: Time minimal trajectories for a spin 1/2 particle in a magnetic field. J. Math. Phys. 47, 062101 (2006)
60. Boozer, A.D.: Time-optimal synthesis of SU(2) transformations for a spin-1/2 system. Phys. Rev. A 85, 012317 (2012)

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