A counterexample to a nonlinear version of the Krein–Rutman theorem by R. Mahadevan

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Abstract
In this short note we present a simple counterexample to a nonlinear version of the Krein–Rutman theorem reported in [Nonlinear Anal. 11 (2007), 3084–3090]. Correct versions of this theorem, and related results for superadditive maps are also presented.

Keywords: Krein–Rutman theorem, Positive operator, Principal eigenvalue, Convex cone

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1. Introduction
Krein and Rutman in their seminal work [1] have studied linear operators which leave invariant a cone in a Banach space. Recall that an ordered Banach space is a real Banach space $X$ with a cone $K$, a nontrivial closed subset of $X$ satisfying

(a) $tK \subset K$ for all $t \geq 0$, where $tK = \{tx : x \in K\}$;
(b) $K + K \subset K$;
(c) $K \cap (-K) = \{0\}$, where $-K = \{-x : x \in K\}$.

As usual, we write $x \preceq y$ if $y - x \in K$, and $x \prec y$ if $x \preceq y$ and $x \neq y$. When the interior of $K$, denoted as $K^\circ$, is nonempty, we call $X$ a strongly ordered Banach space. We also write $x \prec \prec y$ if $y - x \in K^\circ$.

A continuous map $T : X \to X$ is
1. positive if $T(K) \subset K$;
2. strictly positive if $T(K \setminus \{0\}) \subset K \setminus \{0\}$;
3. strongly positive if $T(K \setminus \{0\}) \subset K^\circ$;
4. order-preserving or increasing if $x \preceq y \implies T(x) \preceq T(y)$;
5. strictly order-preserving if $x \prec y \implies T(x) \prec T(y)$;
6. strongly order-preserving if $x \prec y \implies T(x) \prec \prec T(y)$;
7. homogeneous of degree one, or 1-homogeneous, if $T(tx) = tT(x)$ for all $t \geq 0$.

The following nonlinear extension to the Krein–Rutman theorem was reported in [2]. For a 1-homogeneous map $T : X \to X$ we say that $\lambda \in \mathbb{R}$ is an eigenvalue of $T$ if there exists a nonzero $x \in X$, such that $T(x) = \lambda x$. Recall that a map $T : X \to X$ is called completely continuous if it is continuous and compact.

Theorem 1 ([2, Theorem 2]). Let $T : X \to X$ be an order-preserving, 1-homogeneous, completely continuous map such that for some $u \in K$ and $M > 0$, $MT(u) \supseteq u$. Then there exist $\lambda > 0$ and $\hat{x} \in K$, with $\|\hat{x}\| = 1$, such that $T(\hat{x}) = \lambda \hat{x}$. Moreover, if $K \neq \emptyset$ and $T$ is strongly positive and strictly order-preserving, the following hold.

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(i) \( \hat{x} \) is the unique unit eigenvector in \( K \);
(ii) \( \lambda > |\lambda'| \) for any real eigenvalue \( \lambda' \) of \( T \);
(iii) \( \lambda \) is geometrically simple.

It turns out that the assertions in (i) and (iii) are not true under the assumptions of the theorem. In Section 2 we present a counterexample to the theorem in [2] mentioned above. In Section 3 we review correct versions of this result. With the exception of Section 3.3 concerning superadditive maps, the remaining results in Section 3 are a combination of existing results in the literature, and no originality is claimed.

I also wish to thank the anonymous referee who brought to my attention the work in [3], which employs the notions of semi-strong positivity and semi-strongly increasing maps, and improves upon the results in [4]. It turns out that Theorem 3 in Section 3 is a variation of Theorem 2.3 in [3].

2. A Counterexample

The following is an example of a strongly positive, strictly order-preserving, 1-homogeneous, continuous map \( T \) on \( \mathbb{R}^2 \) that has multiple positive unit eigenvectors.

Example 1. Let \( K = \{ x = (x_1, x_2) \in \mathbb{R}^2 : x_i \geq 0, \ i = 1, 2 \} \). Define
\[
K_1 = \{ x \in K : x_1 > 2x_2 \}, \quad K_3 = \{ x \in K : x_2 > 2x_1 \},
\]
and \( K_2 = K \setminus (K_1 \cup K_3) \). Let
\[
T(x) := \begin{cases} 
\left( \begin{array}{c} 2 \ 2 \\ 1 \ 1 \end{array} \right) x & \text{if } x \in K_1 \\
3x & \text{if } x \in K_2 \\
\left( \begin{array}{c} 1 \ 1 \\ 2 \ 2 \end{array} \right) x & \text{if } x \in K_3.
\end{cases}
\]

It is clear that \( T: K \to K \) is continuous, 1-homogeneous, and strongly positive. Also every element of \( K_2 \) is an eigenvector of \( T \).

It remains to show that \( T \) is strictly order-preserving. We examine all possible cases:

(i) If \( x, y \in K_i, \ i = 1, 2, 3 \), and \( x \prec y \), then it is clear that \( T(x) \prec T(y) \).

(ii) Suppose \( x \in K_1, y \in K_3 \), and \( x \prec y \). Then we must have
\[
2x_2 < x_1 < y_1 < \frac{y_2}{2}.
\]

By (1) we obtain that
\[
x_1 + x_2 < \frac{3x_1}{2} < \frac{3y_1}{2} < \frac{y_1 + y_2}{2}.
\]

Since \( T(x) = \left( \begin{array}{c} 2 \ 2 \\ 1 \ 1 \end{array} \right) (x_1 + x_2) \) and \( T(y) = \left( \begin{array}{c} 1 \ 1 \\ 2 \ 2 \end{array} \right) (y_1 + y_2) \), it follows by (2) that \( T(x) \prec T(y) \). Also, if \( x \succ y \), then \( T(x) \succ T(y) \) by symmetry.

(iii) Suppose \( x \in K_1, y \in K_2 \) and \( x \prec y \). Then we have
\[
2x_2 < x_1 < y_1 \leq 2y_2.
\]

It follows by (3) that \( 2(x_1 + x_2) < 3y_1 \) and \( x_1 + x_2 < 3y_2 \). Therefore \( T(x) \prec T(y) \). On the other hand, if \( x \succ y \), then we have
\[
x_1 > 2x_2 \geq 2y_2 \geq y_1,
\]
and by (4) we obtain \( 2(x_1 + x_2) > 3y_1 \) and \( x_1 + x_2 > 3y_2 \). Therefore \( T(x) \succ T(y) \). Also, by symmetry, if \( x \in K_3 \) and \( y \in K_2 \), then the strictly order-preserving property holds.

It follows by (i)–(iii) that \( T \) is strictly order-preserving.
3. Existence and Uniqueness Results

We denote by $K^*$ the dual cone, i.e., $K^* = \{ x^* \in X^* : \langle x^*, x \rangle \geq 0 \text{ for all } x \in K \}$. The dual cone $K^*$ might not satisfy $K^* \cap (-K^*) = \{0\}$, so is not necessarily a cone. If $X$ is strongly ordered, then $x \in K$ if and only if $(x^*, x) > 0$ for all nontrivial $x^* \in K^*$.

A cone $K \subset X$ is said to be generating if $X = K - K$, and it is said to be total if $X$ equals the closure of $K - K$. A strongly ordered Banach space is always generating. A cone $K \subset X$ is called normal if there exists a positive constant $\gamma$ such that $\|x + y\| \geq \gamma \|x\|$ for all $x, y \in K$.

For a 1-homogeneous, continuous map $T : X \to X$ we define, as in [5, 6],

$$\|T\|_+ := \sup \{ \|Tx\| : x \in K, \|x\| \leq 1 \},$$

$$\tilde{\varrho}_+(T) := \lim_{n \to \infty} \|T^n\|_+^{1/n},$$

$$\mu(x) := \limsup_{n \to \infty} \|T^n(x)\|_+^{1/n},$$

$$\varrho_+(T) := \sup_{x \in K} \mu(x),$$

$$\tilde{r}(T) := \sup \{ \lambda \geq 0 : \exists x \in K \setminus \{0\} \text{ with } T(x) = \lambda x \}. $$

The quantities $\tilde{\varrho}_+(T)$, $\varrho_+(T)$, and $\tilde{r}(T)$, are referred to in [5] as the Bonsall’s cone spectral radius, the cone spectral radius, and the cone eigenvalue spectral radius of $T$, respectively. For a 1-homogeneous, continuous map $T : X \to X$ we always have [5, Proposition 2.1]

$$\tilde{r}(T) \leq \varrho_+(T) \leq \tilde{\varrho}_+(T) < \infty. $$

Also, if $T$ is compact, then $\varrho_+(T) = \tilde{\varrho}_+(T)$ [5, Theorem 2.3]. The equality $\varrho_+(T) = \tilde{\varrho}_+(T)$ also holds in the absence of compactness, provided that $T$ is order preserving and the cone $K$ is normal [5, Theorem 2.2].

We summarize the main hypothesis:

(H1) $T : X \to X$ is an order-preserving, 1-homogeneous, completely continuous map.

3.1. Existence of an eigenvector in $K$ with a positive eigenvalue

Existence of an eigenvector of $T$ in $K$ with a positive eigenvalue, i.e., the existence part of Theorem 1, is asserted in [7, Theorem 3.1] under the following weaker assumption.

(A1) There exist a non-zero $u = v - w$ with $v, w \in K$ and such that $-u \notin K$, a positive constant $M$, and a positive integer $p$ such that $MT^p(u) \geq u$.

On the other hand, is a direct consequence of the more general result in [8, Theorem 2.1] that if $S : X \to X$ satisfies (H1) and

(A2) The orbit $\mathcal{O}(S, x) := \{ S^n(x) : n = 1, 2, \ldots \}$ of some $x \in K$ is unbounded,

then there exist a constant $t_0 \geq 1$ and $x_0 \in K$, with $\|x_0\| = 1$, satisfying $S(x_0) = t_0x_0$. It thus turns out that [7, Theorem 3.1], and hence also the existence part of [2, Theorem 2], are a direct consequence of [8, Theorem 2.1] and the following lemma.

Lemma 2. Suppose $T : X \to X$ satisfies (H1) and (A1). Let $\varepsilon > 0$ be arbitrary, and define $S = (M + \varepsilon)^{1/p}T$. Then $\mathcal{O}(S, v)$ is unbounded.

Proof. We argue by contradiction. If $\mathcal{O}(S, v)$ is bounded, then it is also relatively compact. By the order-preserving property and 1-homogeneity we obtain $S^{kp}(v) \supseteq S^{kp}(u) \supseteq (1 + \varepsilon/M)^k u$. Therefore any limit point $y$ of $\{S^{kp}(v) : k = 1, 2, \ldots \}$ satisfies $y \supseteq (1 + \varepsilon/M)^k u$ for all $k = 1, 2, \ldots$, and since $-u \notin K$ this is not possible. \[\square\]
It then follows by \cite[Theorem 2.1]{8} and Lemma 2 that, under Assumptions (H1) and (A1), there exists \( x_0 \in K \) with \( \|x_0\|_1 = 1 \) and \( \lambda_0 \geq M^{-\nu} \) such that \( T(x_0) = \lambda_0 x_0 \). It is also clear from the above discussion that, under (H1), a necessary and sufficient condition for the existence of a positive eigenvalue with eigenvector in \( K \) is that \( \hat{\varrho}_+(T) > 0 \). We remark here, that the assumption that \( X \) is strongly ordered, \( K \) is normal, and \( \hat{\varrho}_+(T) > 0 \), it is shown in \cite[Proposition 3.1.5]{6} that \( \hat{\varrho}_+(T) = \hat{\varrho}(T) \).

### 3.2. Uniqueness and simplicity of the positive eigenvalue

We define

\[ \sigma_+(T) := \{ \lambda > 0 : T(x) = \lambda x, \ x \in K \setminus \{0\} \} \, . \]

Ogiwara introduced the property of indecomposability \cite[Hypothesis A4]{6} to obtain the following. Suppose that \( X \) is strongly ordered, \( K \) is normal, and \( T \) satisfies (H1) and is indecomposable. Then \( \sigma_+(T) = \{ \lambda_0 \} \), i.e., a singleton, \( \lambda_0 \) is a simple eigenvalue of \( T \), and the corresponding eigenvector lies in \( \hat{K} \) \cite[Theorem 3.1.1, Corollary 3.1.6]{6}.

A significant improvement of the above result can be found in \cite{3}. Chang defines semi-strong positivity of \( T \) in \cite[Definition 4.5]{4} by

\[ \forall x \in \partial K \setminus \{0\}, \exists x^* \in K^* \text{ such that } \langle x^*, T(x) \rangle > 0 = \langle x^*, x \rangle \, . \]

Also \( T \) is called semi-strongly increasing in \cite[Definition 2.1]{3} if

\[ \forall x, y \in X, \text{ with } x - y \in \partial K \setminus \{0\}, \exists x^* \in K^* \text{ such that } \langle x^*, T(x) - T(y) \rangle > 0 = \langle x^*, x - y \rangle \, . \]

Then normality of \( K \) is relaxed to assert the following. If \( X \) is strongly ordered, and \( T \) satisfies (H1) and is semi-strong positive, then there exists a unique positive eigenvalue with eigenvector in \( K \). In addition, if \( T \) is semi-strongly increasing, then the eigenvalue is simple \cite[Theorem 2.3]{3}. It is also shown that the indecomposability hypothesis of Ogiwara is equivalent to the semi-strongly increasing property \cite[Theorem 4.3]{3}.

In the sequel, we only assume that \( X \) is strongly ordered, and that \( T \) is 1-homogeneous and order preserving, and comment on the uniqueness and simplicity of the eigenvalue, provided that \( \sigma_+(T) \neq \emptyset \).

Consider the following hypothesis:

**B1** \( \text{If } x \in \partial K \setminus \{0\}, \text{ then } x - \beta T(x) \notin K \text{ for all } \beta > 0. \)

It is clear that semi-strong positivity implies (B1). On the other hand, it is straightforward to show that if two eigenvectors \( x_0 \) and \( y_0 \) lie in \( K \), then the associated eigenvalues are equal. In turn, it is easy to show that, under (B1), every eigenvector in \( K \) with a positive eigenvalue has to lie in \( K \), and, consequently, that the positive eigenvalue is unique. Also, following for example the proof in \cite[Lemma 3.1.2]{6}, we can show that (B1) implies \( T(\hat{K}) \subset K \).

Next, consider the hypothesis

**B2** \( \text{If } x - y \in \partial K \setminus \{0\}, \text{ then } x - y - \beta (T(x) - T(y)) \notin K \text{ for all } \beta > 0. \)

Clearly, (B2) \( \Rightarrow \) (B1). Also (B2) is weaker than the strong order preserving property. Under (B2), following the argument in the proof of \cite[Theorem 3.1.1]{6}, one readily shows that if there exists a unit eigenvector in \( K \), then it is unique.

We summarize the above assertions in the form of the following theorem.

**Theorem 3.** Let \( X \) be strongly ordered, and \( T : K \to K \) be an order-preserving, 1-homogeneous map with \( \sigma_+(T) \neq \emptyset \).

(i) \( \text{If (B1) holds, then } T(\hat{K}) \subset \hat{K}, \sigma_+(T) \text{ is a singleton, and all eigenvectors lie in } \hat{K}. \)

(ii) \( \text{If (B2) holds, then the unique eigenvalue in } \sigma_+(T) \text{ is simple.} \)
3.3. Superadditive maps

We say that $T : X \to X$ is superadditive (superadditive on $K$) if $T(x + y) \geq T(x) + T(y)$ for all $x, y \in X$ $(x, y \in K)$. It is clear that a (strictly, strongly) positive superadditive map is (strictly, strongly) order-preserving.

**Theorem 4.** Let $T : K \to K$ be a superadditive, 1-homogeneous, completely continuous map such that $\tilde{p}_+(T) > 0$. Then there exists $\lambda_0 > 0$ and $x_0 \in K$ with $\|x_0\| = 1$ such that $T(x_0) = \lambda_0 x_0$. Moreover, if (B1) holds, then $x_0$ is the unique unit eigenvector of $T$ in $K$.

**Proof.** Existence follows from Section 3.1. Suppose $x_0$ and $y_0$ are two distinct unit eigenvectors in $K$. As mentioned in Section 3.2, hypothesis (B1) implies that $x_0, y_0 \in K$, and therefore these eigenvectors have a common eigenvalue $\lambda_0 > 0$. Hence there exists $\alpha > 0$ such that $x_0 - \alpha y_0 \in \partial K \setminus \{0\}$. Since $T$ is superadditive, we obtain

$$T(x_0 - \alpha y_0) \preceq T(x_0) - \alpha T(y_0) = \lambda_0 (x_0 - \alpha y_0).$$

This contradicts (B1) unless $x_0 - \alpha y_0 = 0$. Uniqueness of a unit eigenvector in $K$ follows. □

**Remark 1.** For a superadditive map $T$, we have

$$x - y - \beta T(x - y) \succeq x - y - \beta (T(x) - T(y)).$$

Therefore if $T$ satisfies (B1) it also satisfies (B2).

Let $K_+ := K$, $K_- := -K$, and define

$$\sigma(T) := \{ \lambda \in \mathbb{R} : T(x) = \lambda x, \ x \in X \setminus \{0\} \}.$$

**Corollary 5.** Let $T : X \to X$ be a positive, superadditive, 1-homogeneous, completely continuous map such that $\tilde{p}_+(T) > 0$. Assuming (B1), there exist unique unit eigenvectors $x_+ \in K_+$ and $x_- \in K_-$ with positive eigenvalues $\lambda_+$ and $\lambda_-$, respectively. Moreover, $\lambda_- \geq \lambda_+$. Also, if $\lambda \in \sigma(T)$, then $|\lambda| \leq \lambda_+$.

**Proof.** By Theorem 4, $T$ has a unique eigenvector $x_+ \in K_+$ corresponding to an eigenvalue $\lambda_+ > 0$, and moreover $x_+ \in K_+$. Define $S(x) := -T(-x)$. By superadditivity $-T(-x) \succeq T(x)$, which implies that $S(x_+) \succeq T(x_+) = \lambda_+ x_+$. Therefore (A1) holds for $S$ which implies the existence of a unit eigenvector $x_- \in K_-$ for $S$ with a positive eigenvalue $\lambda_-$. Also property (B1) for $T$ implies that if $x \in \partial K \setminus \{0\}$, then

$$x - \beta S(x) \preceq x - \beta T(x) \notin K,$$

so that property (B1) also holds for $S$. Thus uniqueness of $x_-$ follows by Section 3.2. Let $\alpha > 0$ be such that $x_- + \alpha x_+ \in \partial K_-$. By superadditivity,

$$T(x_- + \alpha x_+) \succeq T(x_-) + \alpha T(x_+) = \lambda_- x_- + \alpha \lambda_+ x_+.$$

By the order-preserving property, we have $\lambda_- x_- + \alpha \lambda_+ x_+ \preceq 0$, which implies that $\lambda_- \geq \lambda_+$.

Suppose $T(x) = \lambda x$ for some $x \in X \setminus (K_+ \cup K_-)$, with $x \neq 0$. Let $\alpha > 0$ be such that $x_+ + \alpha x \in \partial K_+$. Since $T^2$ is order preserving, we have $\lambda_-^2 x_+ + \lambda^2 \alpha x \geq 0$, which is possible only if $\lambda_+ \geq |\lambda|$. □

We would also like to mention here the stability results reported in [9] concerning strongly continuous semigroups of superadditive operators on Banach spaces.

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References

[1] M. G. Krein, M. A. Rutman, Linear operators leaving invariant a cone in a Banach space, Amer. Math. Soc. Translation 1950 (26) (1950) 128.

[2] R. Mahadevan, A note on a non-linear Krein-Rutman theorem, Nonlinear Anal. 67 (11) (2007) 3084–3090. doi:10.1016/j.na.2006.09.062.

[3] K. C. Chang, Nonlinear extensions of the Perron-Frobenius theorem and the Krein-Rutman theorem, J. Fixed Point Theory Appl. 15 (2) (2014) 433–457. doi:10.1007/s11784-014-0191-2.

[4] K. C. Chang, A nonlinear Krein Rutman theorem, J. Syst. Sci. Complex. 22 (4) (2009) 542–554. doi:10.1007/s11424-009-9186-2.

[5] J. Mallet-Paret, R. D. Nussbaum, Eigenvalues for a class of homogeneous cone maps arising from max-plus operators, Discrete Contin. Dyn. Syst. 8 (3) (2002) 519–562. doi:10.3934/dcds.2002.8.519.

[6] T. Ogiwara, Nonlinear Perron-Frobenius problem on an ordered Banach space, Japan. J. Math. (N.S.) 21 (1) (1995) 43–103. doi:10.4099/math1924.21.43.

[7] Y. Cui, J. Sun, A generalization of Mahadevan’s version of the Krein-Rutman theorem and applications to p-Laplacian boundary value problems, Abstr. Appl. Anal. (2012) Art. ID 305279, 14.

[8] R. D. Nussbaum, Eigenvectors of nonlinear positive operators and the linear Krein-Rutman theorem, in: Fixed point theory (Sherbrooke, Que., 1980), Vol. 886 of Lecture Notes in Math., Springer, Berlin, 1981, pp. 309–330.

[9] A. Arapostathis, V. S. Borkar, K. S. Kumar, Risk-sensitive control and an abstract Collatz-Wielandt formula, J. Theoret. Probab. 29 (4) (2016) 1458–1484. doi:10.1007/s10959-015-0616-x.