POSITIVE SOLUTIONS OF THE DISCRETE ROBIN PROBLEM
WITH $\phi$-LAPLACIAN

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Abstract. In this paper, by using critical point theory, we obtain some sufficient conditions on the existence of infinitely many positive solutions of the discrete Robin problem with $\phi$-Laplacian. We show that, an unbounded sequence of positive solutions and a sequence of positive solutions which converges to zero will emerge from the suitable oscillating behavior of the nonlinear term at infinity and at the zero, respectively. We also give two examples to illustrate our main results.

1. Introduction. Let $Z$ and $\mathbb{R}$ denote the sets of integers and real numbers, respectively. For $a,b \in Z$, define $Z(a) = \{a, a+1, \cdots\}$, and $Z(a,b) = \{a,a+1,\cdots,b\}$ when $a \leq b$.

Consider the following Robin problem of the second order nonlinear difference equation

$$
\begin{cases}
-\Delta (\varphi_p(\Delta u_{k-1})) + q_k \varphi_p(u_k) = \lambda f(k,u_k), & k \in Z(1,T), \\
\Delta u_0 = u_{T+1} = 0,
\end{cases}
$$

(1.1)

where $T$ is a given positive integer, $\Delta$ is the forward difference operator defined by $
\Delta u_k = u_{k+1} - u_k$, $\Delta^2 u_k = \Delta(\Delta u_k)$, $q_k \geq 0$ for all $k \in Z(1,T)$, $\varphi_p$ is a special $\phi$-Laplacian operator [18] given by $\varphi_p(s) = \frac{|s|^{p-2}s}{(1+|s|^p)^{\frac{p-2}{2}}}$ with $p \geq 2$, $\lambda$ is a real positive parameter, and $f(k,\cdot) \in C(\mathbb{R}, \mathbb{R})$ for each $k \in Z(1,T)$.

When $p = 2$, problem (1.1) reduces to the following discrete Robin problem involving the mean curvature operator

$$
\begin{cases}
-\Delta \left( \frac{\Delta u_{k-1}}{\sqrt{1+(\Delta u_{k-1})^2}} \right) + q_k \frac{u_k}{\sqrt{1+u_k^2}} = \lambda f(k,u_k), & k \in Z(1,T), \\
\Delta u_0 = u_{T+1} = 0,
\end{cases}
$$

(1.2)

Difference equations arise in various research fields. Many authors have discussed the existence and multiplicity of solutions for difference problems by using fixed point theory, the method of upper and lower solution techniques, Rabinowitz's
global bifurcation theorem etc., see [1, 4, 12]. In 2003, the variational methods were used to study the existence of periodic solutions and subharmonic solutions of second order difference equations at the first time [11]. Since then, the variational methods have been extensively used to study the existence of periodic solutions and subharmonic solutions [18, 21], homoclinic solutions [13, 14, 22, 24, 25, 26, 27, 28, 29] and boundary value problems [4, 6, 9, 30, 31] of difference equations. We refer to the recent survey articles [3, 10] for the applications of variational methods on difference equations.

In [5], by using critical point theory, the authors consider the existence of infinitely many solutions of the following discrete boundary value problem with p-Laplacian

\[
\begin{aligned}
-\triangle (\phi_p(\triangle u_{k-1})) + q_k \phi_p(u_k) &= \lambda f(k, u_k), \quad k \in Z(1, T), \\
&\quad u_0 = u_{T+1} = 0.
\end{aligned}
\] (1.3)

In [7], the authors considered the special case for \( p = 2 \) of (1.3)

\[
\begin{aligned}
-\triangle^2 u_{k-1} &= \lambda f(k, u_k), \quad k \in Z(1, T), \\
&\quad u_0 = u_{T+1} = 0.
\end{aligned}
\] (1.4)

They obtained the existence of at least two positive solutions for (1.4). In [9], the authors extended the results of [7] to (1.3). Very recently, the authors in [15, 31] considered the existence of positive solutions of the corresponding Dirichlet problem of (1.2) according to the behavior of \( f \) at infinity and at the origin.

Among the boundary value problems of difference equations, there is less work on the Robin problem of the difference equation. In [16], the authors considered the following Robin problem

\[
\begin{aligned}
-\triangle^2 u_{k-1} &= \lambda f(k, u_k), \quad k \in Z(1, T), \\
&\quad u_0 = \triangle u_T = 0.
\end{aligned}
\] (1.5)

By using invariant sets of descending flow and variational methods, the authors established some new sufficient conditions on the existence of sign-changing solutions, positive solutions and negative solutions for the problem (1.5) when the parameter \( \lambda \) belongs to appropriate intervals.

When \( q_k \equiv 0 \), the corresponding Dirichlet problem for (1.2) may be regarded as the discrete analog of the following one-dimensional prescribed curvature problem

\[
\begin{aligned}
-(\varphi_p(u'))' &= \lambda f(t, u), \quad t \in R, \\
u(0) = u(1) &= 0.
\end{aligned}
\] (1.6)

In 2007, Bonheure etc. in [8] discussed the existence and multiplicity of positive solutions of problem (1.6) depending on the behaviour at the origin and at infinity of the potential \( \int_0^u f(t, s)ds \). Their approach is essentially variational and is based on a regularization of the action functional associated with the curvature problem. In [6], Bonanno, Livrea and Mawhin obtained an explicit interval \( \Lambda \) of positive parameters, such that, for every \( \lambda \in \Lambda \), problem (1.6) admits at least one nontrivial nonnegative solution \( u_\lambda \). For the corresponding case of higher dimensions to problem (1.6), we refer to [19].

Compared with differential equations, there is less work on the boundary value problems of difference equations involving the mean curvature operator. Motivated by the recent works [7, 9, 31], in this paper, we will consider the existence of infinitely many positive solutions for problem (1.1) involving \( \phi \)-Laplacian by means of the critical point results in [20]. The main difficult lies in the estimation of the
variational functional involving $\phi$-Laplacian. We find that the suitable oscillating behavior of the nonlinear term $f$ at infinity and at the origin plays an important role. For general background and applications on difference equations, we refer the reader to monographs [2, 17] and the paper [23].

This paper is organized as follows. In section 2, the variational framework associated with problem (1.1) is established, and the abstract critical point theorem is recalled. In section 3, our main results are presented. And we establish a strong maximum principle and obtain the existence of infinitely many positive solutions for problem (1.1) according to the oscillating behavior of $f$ at infinity and at the origin, respectively. In particular, as a special case of problem (1.1) for $p = 2$, we obtain the existence of infinitely many positive solutions of problem (1.2) involving the mean curvature operator for the first time. Finally, in section 4, we present two examples to illustrate our main results.

2. Preliminaries. In this section, we first establish the variational framework associated with problem (1.1). We consider the $T$-dimensional Banach space $S = \{u : Z(0, T + 1) \to R : \triangle u_0 = u_{T+1} = 0\}$ endowed with the norm

$$\|u\|_p := \left(\sum_{k=1}^{T} |\triangle u_k|^p\right)^{\frac{1}{p}}.$$ 

For each $u \in S$, let

$$\begin{align*}
\Phi(u) &= \sum_{k=1}^{T} \left[\sqrt{1 + |\triangle u_k|^p} - 1 + q_k \left(\sqrt{1 + |u_k|^p} - 1\right)\right], \\
\Psi(u) &= \sum_{k=1}^{T} F(k, u_k),
\end{align*}$$

(2.1)

where $F(k, u) = \int_{0}^{u} f(k, \tau) d\tau$ for every $k \in Z(1, T)$. Define

$$I_\lambda(u) = \Phi(u) - \lambda \Psi(u),$$

for $u \in S$. It is easy to show that $\Phi$ and $\Psi$ are two functionals of class $C^1(S, R)$ whose Gâteaux derivatives at the point $u \in S$ are given by

$$\begin{align*}
\Phi'(u)(v) &= \sum_{k=1}^{T} \varphi_p(\triangle u_k) \triangle v_k + \sum_{k=1}^{T} q_k \varphi_p(u_k) v_k, \\
\Psi'(u)(v) &= \sum_{k=1}^{T} f(k, u_k) v_k,
\end{align*}$$

for all $u, v \in S$. Since $\triangle u_0 = u_{T+1} = 0$, we have

$$\begin{align*}
\sum_{k=1}^{T} \varphi_p(\triangle u_k) \triangle v_k &= \sum_{k=1}^{T} \varphi_p(\triangle u_k) v_{k+1} - \sum_{k=1}^{T} \varphi_p(\triangle u_k) v_k \\
&= \sum_{k=1}^{T} \varphi_p(\triangle u_{k-1}) v_k - \sum_{k=1}^{T} \varphi_p(\triangle u_k) v_k \\
&= -\sum_{k=1}^{T} \triangle(\varphi_p(\triangle u_{k-1})) v_k, \\
\end{align*}$$

(2.2)
then,

\[
[\Phi'(u) - \lambda \Psi'(u)](v) = \sum_{k=1}^{T} \left[ -\Delta (\varphi_p(\Delta u_{k-1})) + q_k \varphi_p(u_k) - \lambda f(k, u_k) \right] v_k.
\]

Therefore, the critical points of \( I_{\lambda} \) in \( S \) are exactly the solutions of the problem (1.1).

Let \( X \) be a reflexive real Banach space and let \( I_{\lambda} : X \to \mathbb{R} \) be a function satisfying the following structure hypothesis:

\( (\Lambda) \) \( I_{\lambda}(u) := \Phi(u) - \lambda \Psi(u) \) for all \( u \in X \), where \( \Phi, \Psi : X \to \mathbb{R} \) are two functions of class \( C^1 \) on \( X \) with \( \Phi \) coercive, i.e. \( \lim_{\|u\| \to +\infty} \Phi(u) = +\infty \), and \( \lambda \) is a real positive parameter.

If inf \( X \) \( \Phi < r \), let

\[
\psi(r) := \inf_{u \in \Phi^{-1}(-\infty, r)} \frac{\sup_{v \in \Phi^{-1}(-\infty, r)} \Psi(v) - \Psi(u)}{r - \Phi(u)},
\]

and

\[
\gamma := \liminf_{r \to +\infty} \psi(r), \quad \delta := \liminf_{r \to (\inf_X \Phi)^+} \psi(r).
\]

Clearly, \( \gamma \geq 0 \) and \( \delta \geq 0 \). When \( \gamma = 0 \) or \( \delta = 0 \), in the sequel, we agree to read \( \frac{1}{\gamma} \) or \( \frac{1}{\delta} \) as \( +\infty \).

Now we recall a lemma (Theorem 2.5 of [20]) which will be used to investigate problem (1.1).

**Lemma 2.1.** Assume that the condition \( (\Lambda) \) holds, we have

(a) If \( \gamma < +\infty \), then, for each \( \lambda \in (0, \frac{1}{\gamma}) \), the following alternative holds: either

- \( (a_1) \) \( I_{\lambda} \) possesses a global minimum, or

- \( (a_2) \) there is a sequence \( \{u_n\} \) of critical points (local minima) of \( I_{\lambda} \) such that \( \lim_{n \to +\infty} \Phi(u_n) = +\infty \).

(b) If \( \delta < +\infty \), then, for each \( \lambda \in (0, \frac{1}{\delta}) \), the following alternative holds: either

- \( (b_1) \) there is a global minimum of \( \Phi \) which is a local minimum of \( I_{\lambda} \), or

- \( (b_2) \) there is a sequence \( \{u_n\} \) of pairwise distinct critical points (local minima) of \( I_{\lambda} \), with \( \lim_{n \to +\infty} \Phi(u_n) = \inf_X \Phi \), which weakly converges to a global minimum of \( \Phi \).

3. **Main results.** Let

\[
B := \limsup_{t \to +\infty} t^{-\frac{p}{2}} \sum_{k=1}^{T} F(k, t), \quad D := \limsup_{t \to 0^+} t^{-p} \sum_{k=1}^{T} F(k, t), \quad (3.1)
\]

\[
q_\ast := \min \{q_k : k \in Z(1, T)\}, \quad Q := \sum_{k=1}^{T} q_k.
\]

First, we consider the oscillating behavior of \( f \) at the infinity, we have

**Theorem 3.1.** Assume that there exist two real sequences \( \{a_n\} \) and \( \{b_n\} \), with \( \lim_{n \to +\infty} b_n = +\infty \), such that

\[
(1 + Q) \left( \sqrt{1 + |a_n|^p} - 1 \right) < (1 + q_\ast) \left( \sqrt{1 + T^{1-\frac{p}{2}}b_n^p} - 1 \right), \quad (3.2)
\]

for \( n \in Z(1) \), and

\[
A < \frac{B}{1 + Q}, \quad (3.3)
\]
where
\[ A := \liminf_{n \to \infty} \frac{\sum_{k=1}^{T} \max_{|t| \leq b_n} F(k, t) - \sum_{k=1}^{T} F(k, a_n)}{(1 + q_*) \left( \sqrt{1 + T^{1-p}b_n^p} - 1 \right) - (1 + Q) \left( \sqrt{1 + |a_n|^p} - 1 \right)}. \]

Then, for each \( \lambda \in \left( \frac{1+Q}{B}, \frac{1}{A} \right) \), problem (1.1) admits an unbounded sequence of solutions.

**Proof.** We will use Lemma 2.1 to prove our results. Clearly, (A) is satisfied. Let
\[ r_n = (1 + q_*) \left( \sqrt{1 + T^{1-p}b_n^p} - 1 \right). \]

If \( u \in S \) and \( \Phi(u) = \Phi_1(u) + \Phi_2(u) < r_n \), where
\[ \Phi_1(u) = \sum_{k=1}^{T} \left( \sqrt{1 + |\Delta u_k|^p} - 1 \right), \quad \Phi_2(u) = \sum_{k=1}^{T} q_k \left( \sqrt{1 + |u_k|^p} - 1 \right). \]

Let
\[ v_k = \sqrt{1 + |\Delta u_k|^p} - 1, \]
for \( k \in Z(1,T) \), then \( \sum_{k=1}^{T} v_k = \Phi_1(u) \) and
\[ \sum_{k=1}^{T} |\Delta u_k|^p = \sum_{k=1}^{T} (v_k^2 + 2v_k) \leq \left( \sum_{k=1}^{T} v_k \right)^2 + 2 \sum_{k=0}^{T} v_k = \Phi_1^2(u) + 2\Phi_1(u), \]
which implies that
\[ \Phi_1(u) \geq \sqrt{\|u\|_p^p + 1} - 1. \]

Noticing that, for any \( u \in S \), there exists \( \tau \in Z(1,T) \) such that
\[ \|u\|_\infty := \max \{|u_k| : k \in Z(1,T)\} = |u_\tau| = \sum_{k=\tau}^{T} |\Delta u_k| \leq \sum_{k=1}^{T} |\Delta u_k| \leq T^{1-\frac{p}{2}} \|u\|_p. \]

Thus, we have
\[ \Phi_1(u) \geq \sqrt{T^{1-p}\|u\|_\infty^p + 1} - 1. \]

It is clear that
\[ \Phi_2(u) \geq q_* \left( \sqrt{\|u\|_\infty^p + 1} - 1 \right) \geq q_* \left( \sqrt{T^{1-p}\|u\|_\infty^p + 1} - 1 \right). \]

Therefore
\[ (1 + q_*) \left( \sqrt{1 + T^{1-p}\|u\|_\infty^p} - 1 \right) \leq \Phi_1(u) + \Phi_2(u) < r_n, \]
which implies that
\[ \|u\|_\infty^p < T^{p-1} \left[ \left( \frac{r_n}{1 + q_*} \right)^2 + \frac{2r_n}{1 + q_*} \right] = b_n^p. \]
By the definition of \( \psi \), we have

\[
\psi(r_n) \leq \inf_{u \in \Phi^{-1}(-\infty, r_n)} \frac{\sum_{k=1}^{T} \max_{|t| \leq b_n} F(k, t) - \sum_{k=1}^{T} F(k, u_k)}{(1 + q_n) \left( \sqrt{1 + T^{1-p}b_n} - 1 \right) - \Phi(u)}.
\]

For every \( n \in Z(1) \), let \( w_n \in S \) given by \( (w_n)_k = a_n \) for each \( k \in Z(1, T) \), and \( \triangle(w_n)_0 = (w_n)_{T+1}^0 = 0 \). Then

\[
\Phi(w_n) = (1 + Q) \left( \sqrt{1 + |a_n|^p} - 1 \right) < r_n
\]

by using (3.2). Thus,

\[
\psi(r_n) \leq \frac{\sum_{k=1}^{T} \max_{|t| \leq b_n} F(k, t) - \sum_{k=1}^{T} F(k, (w_n)_k)}{(1 + q_n) \left( \sqrt{1 + T^{1-p}b_n} - 1 \right) - \Phi(w_n)}
\]

(3.4)

\[
= \frac{\sum_{k=1}^{T} \max_{|t| \leq b_n} F(k, t) - \sum_{k=1}^{T} F(k, a_n)}{(1 + q_n) \left( \sqrt{1 + T^{1-p}b_n} - 1 \right) - (1 + Q) \left( \sqrt{1 + |a_n|^p} - 1 \right)}. 
\]

Therefore, by (3.3), we know that \( \gamma \leq \lim inf_{n \to +\infty} \psi(r_n) \leq A < +\infty \).

In order to get the conclusion \((a_2)\) of Lemma 2.1, we need to show \((a_1)\) does not hold. In fact, we will prove that \( I_\lambda \) is unbounded from below by considering two cases: \( B = +\infty \) and \( B < +\infty \). In the case where \( B = +\infty \), let \( \{\alpha_n\} \) be a sequence of positive numbers, with \( \lim_{n \to +\infty} \alpha_n = +\infty \), such that

\[
\sum_{k=1}^{T} F(k, \alpha_n) \geq \frac{2(1 + Q)\alpha_n^2}{\lambda}, \quad \text{for } n \in Z(1).
\]

Defining a sequence \( \{\omega_n\} \) in \( S \) by \( (\omega_n)_k = \alpha_n \) for \( k \in Z(1, T) \) and \( \triangle(\omega_n)_0 = (\omega_n)_{T+1} = 0 \), we have

\[
I_\lambda(\omega_n) = (1 + Q) \left( \sqrt{1 + \alpha_n^p} - 1 \right) - \lambda \sum_{k=1}^{T} F(k, \alpha_n)
\]

\[
\leq (1 + Q)\alpha_n^2 - 2(1 + Q)\alpha_n^2
\]

\[
= -(1 + Q)\alpha_n^2,
\]

this means that \( \lim_{n \to +\infty} I_\lambda(\omega_n) = -\infty \). In the case where \( B < +\infty \), since \( \lambda > \frac{1+Q}{B^2} \), we can take \( \epsilon_0 > 0 \) such that

\[
1 + Q - \lambda(B - \epsilon_0) < 0.
\]

By the definition of \( B \), there exists a sequence of positive numbers \( \{\beta_n\} \) such that \( \lim_{n \to +\infty} \beta_n = +\infty \) and

\[
(B - \epsilon_0)\beta_n^2 \leq \sum_{k=1}^{T} F(k, \beta_n) \leq (B + \epsilon_0)\beta_n^2.
\]
Define a sequence \( \{w_n\} \) in \( S \) by \( (w_n)_k = \beta_n \) for \( k \in Z(1,T) \) and \( \Delta(w_n)_0 = (w_n)_{T+1} = 0 \), we have

\[
I_\lambda(w_n) = (1 + Q) \left( \sqrt{1 + \beta_n^p} - 1 \right) - \lambda \sum_{k=1}^{T} F(k, \beta_n) \leq (1 + Q - \lambda(B - \epsilon_0)) \beta_n^p,
\]

which implies that \( \lim_{n \to +\infty} I_\lambda(w_n) = -\infty \). Combining the above two cases, we see that \( I_\lambda \) is unbounded from below, and by Lemma 2.1, we know the conclusion of Theorem 3.1 holds.

Now, let

\[
A_* := \liminf_{t \to +\infty} \frac{T^{\frac{p+1}{2}} \sum_{k=1}^{T} \max_{|s| \leq t} F(k, s)}{(1 + q_*) t^\frac{p}{2}}.
\]

Then there exists a sequence \( \{b_n\} \) of positive numbers with \( \lim_{n \to +\infty} b_n = +\infty \) such that

\[
\lim_{n \to \infty} \frac{\sum_{k=1}^{T} \max_{|t| \leq b_n} F(k, t)}{(1 + q_*) \left( \sqrt{1 + T^{1-p} d_n^p} - 1 \right)} = A_*.
\]

Taking \( a_n = 0 \) for all \( n \in Z(1) \), by Theorem 3.1, we get the following corollary.

**Corollary 3.1.** If

\[
A_* < \frac{B}{1 + Q}.
\]

Then, for each \( \lambda \in (\frac{1+Q}{B}, \frac{1}{A_*}) \), problem (1.1) admits an unbounded sequence of solutions.

Now, we consider the oscillating behavior of \( f \) at the origin, we have

**Theorem 3.2.** Assume that there exist two real sequences \( \{c_n\} \) and \( \{d_n\} \), with \( d_n > 0 \) and \( \lim_{n \to +\infty} d_n = 0 \), such that

\[
(1 + Q) \left( \sqrt{1 + |c_n|^p} - 1 \right) < (1 + q_*) \left( \sqrt{1 + T^{1-p} d_n^p} - 1 \right),
\]

for \( n \in Z(1) \), and

\[
C < \frac{2D}{1 + Q},
\]

where

\[
C := \liminf_{n \to \infty} \frac{\sum_{k=1}^{T} \max_{|t| \leq d_n} F(k, t) - \sum_{k=1}^{T} F(k, c_n)}{(1 + q_*) \left( \sqrt{1 + T^{1-p} d_n^p} - 1 \right)} - (1 + Q) \left( \sqrt{1 + |c_n|^p} - 1 \right).
\]

Then, for each \( \lambda \in (\frac{1+Q}{2D}, \frac{1}{C}) \), problem (1.1) admits a sequence of nontrival solutions which converges to zero.

**Proof.** Clearly, \( u \equiv 0 \) is a global minimum of \( \Phi \). Let

\[
\nu_n = (1 + q_*) \left( \sqrt{1 + T^{1-p} d_n^p} - 1 \right).
\]

Similar to the proof of the first part of Theorem 3.1, we can show that \( \delta \leq \liminf_{n \to +\infty} \psi(\nu_n) \leq C < +\infty \).
In order to get the conclusion \((b_2)\) of Lemma 2.1, we need to prove that \(u \equiv 0\) is not a local minimum of \(I_\lambda\). To prove this, we consider two cases: \(D = +\infty\) and \(D < +\infty\). In the case where \(D = +\infty\), let \(\{\gamma_n\}\) be a sequence of positive numbers, with \(\lim_{n \to +\infty} \gamma_n = 0\), such that
\[
\sum_{k=1}^{T} F(k, \gamma_n) \geq \frac{(1 + Q) \gamma_n^p}{\lambda}, \quad \text{for } n \in \mathbb{Z}(1).
\]
Defining a sequence \(\{\omega_n\}\) in \(S\) by \((\omega_n)_k = \gamma_n\) for \(k \in \mathbb{Z}(1, T)\) and \((\omega_n)_0 = (\omega_n)_{T+1} = 0\), we have
\[
I_\lambda(\omega_n) = (1 + Q) \left( \sqrt{1 + \gamma_n^2} - 1 \right) - \lambda \sum_{k=1}^{T} F(k, \gamma_n) \leq 1 + Q - \frac{1 + Q}{2} \gamma_n^p < 0.
\]
In the case where \(B < +\infty\), since \(\lambda > \frac{1 + Q}{2D}\), we can take \(\epsilon_0 > 0\) such that
\[
1 + Q - 2\lambda(D - \epsilon_0) < 0.
\]
Then there exists a sequence of positive numbers \(\{\gamma_n\}\) such that \(\lim_{n \to +\infty} \gamma_n = 0\) and
\[
(D - \epsilon_0) \gamma_n^p \leq \sum_{k=1}^{T} F(k, \gamma_n) \leq (D + \epsilon_0) \gamma_n^p.
\]
Define the sequence \(\{\omega_n\}\) in \(S\) as the same with the case where \(D = +\infty\), we have
\[
I_\lambda(\omega_n) = (1 + Q) \left( \sqrt{1 + \gamma_n^2} - 1 \right) - \lambda \sum_{k=1}^{T} F(k, \gamma_n) \leq 1 + Q - \frac{1 + Q}{2} (D - \epsilon_0) \gamma_n^p < 0.
\]
Combining the above two cases, we see that \(u \equiv 0\) is not a local minimum of \(I_\lambda\) and \((b_2)\) of Lemma 2.1 holds. Therefore, problem (1.1) admits a sequence of nontrivial solutions which converges to zero.

Let
\[
C_* := \liminf_{t \to 0^+} \frac{2T^p - 1}{t} \sum_{k=1}^{T} \max_{|s| \leq t} F(k, s).
\]
Then there exists a sequence \(\{d_n\}\) of positive numbers with \(\lim_{n \to +\infty} d_n = 0\) such that
\[
\lim_{n \to +\infty} \frac{\sum_{k=1}^{T} \max_{|t| \leq d_n} F(k, t)}{(1 + q_*) \left( \sqrt{1 + T^{1-q_0} d_n^p - 1} \right)} = C_*.
\]
Taking \(c_n = 0\) for all \(n \in \mathbb{Z}(1)\), by Theorem 3.2, we get the following corollary.

**Corollary 3.2.** If
\[
C_* < \frac{2D}{1 + Q},
\]
then, for each \(\lambda \in \left( \frac{1 + Q}{2D}, \frac{1}{C_*} \right)\), problem (1.1) admits a sequence of nontrivial solutions which converges to zero.
When \( p = 2 \), we can now establish the multiplicity results of solutions for the discrete Robin problem involving the mean curvature operator. By Theorem 3.1 and Theorem 3.2, we have

**Theorem 3.3.** Assume that there exist two real sequences \( \{a_n\} \) and \( \{b_n\} \), with \( \lim_{n \to +\infty} b_n = +\infty \), such that

\[
(1 + Q) \left( \sqrt{1 + a_n^2} - 1 \right) < (1 + q_*) \left( \sqrt{1 + b_n^2} - 1 \right),
\]

for \( n \in \mathbb{Z}(1) \), and

\[
E := \liminf_{n \to \infty} \frac{\sum_{k=1}^{T} \max_{|t| \leq b_n} F(k, t) - \sum_{k=1}^{T} F(k, a_n)}{(1 + q_*) \left( \sqrt{1 + b_n^2} - 1 \right) - (1 + Q) \left( \sqrt{1 + a_n^2} - 1 \right)} < \frac{F}{1 + Q},
\]

where \( F = \limsup_{t \to +\infty} \sum_{k=1}^{T} F(k, t)/t \). Then, for each \( \lambda \in (\frac{1 + Q}{F}, \frac{1}{E}) \), problem (1.2) admits an unbounded sequence of solutions.

**Theorem 3.4.** Assume that there exist two real sequences \( \{c_n\} \) and \( \{d_n\} \), with \( d_n > 0 \) and \( \lim_{n \to +\infty} d_n = 0 \), such that

\[
(1 + Q) \left( \sqrt{1 + c_n^2} - 1 \right) < (1 + q_*) \left( \sqrt{1 + d_n^2} - 1 \right),
\]

for \( n \in \mathbb{Z}(1) \), and

\[
G := \liminf_{n \to \infty} \frac{\sum_{k=1}^{T} \max_{|t| \leq d_n} F(k, t) - \sum_{k=1}^{T} F(k, c_n)}{(1 + q_*) \left( \sqrt{1 + d_n^2} - 1 \right) - (1 + Q) \left( \sqrt{1 + c_n^2} - 1 \right)} < \frac{2H}{1 + Q},
\]

where \( H = \limsup_{t \to 0^+} \sum_{k=1}^{T} F(k, t)/t^2 \). Then, for each \( \lambda \in (\frac{1 + Q}{H}, \frac{1}{G}) \), problem (1.2) admits a sequence of nontrivial solutions which converges to zero.

**Corollary 3.3.** If

\[
\sqrt{T} \sum_{k=1}^{T} \max_{|s| \leq t} F(k, s)
\]

\[
E_* := \liminf_{t \to +\infty} \frac{\max_{|s| \leq t} F(k, s)}{(1 + q_*)t} < \frac{F}{1 + Q}.
\]

Then, for each \( \lambda \in (\frac{1 + Q}{F}, \frac{1}{E_*}) \), problem (1.2) admits an unbounded sequence of solutions.

**Corollary 3.4.** If

\[
\sqrt{T} \sum_{k=1}^{T} \max_{|s| \leq t} F(k, s)
\]

\[
G_* := \liminf_{t \to 0^+} \frac{2T \sum_{k=1}^{T} \max_{|s| \leq t} F(k, s)}{(1 + q_*)t^2} < \frac{2H}{1 + Q}.
\]

Then, for each \( \lambda \in (\frac{1 + Q}{2H}, \frac{1}{G_*}) \), problem (1.2) admits a sequence of nontrivial solutions which converges to zero.
To obtain the positive solutions of problem (1.1), we need the following strong maximum principle.

**Theorem 3.5.** Assume $u \in S$ such that either

$$u_k > 0 \quad \text{or} \quad -\Delta (\varphi_p(\Delta u_{k-1})) + q_k \varphi_p(u_k) \geq 0, \quad (3.15)$$

for all $k \in Z(1, T)$. Then, either $u_k > 0$ for all $k \in Z(1, T)$ or $u \equiv 0$.

**Proof.** Let $j \in Z(1, T)$ and

$$u_j = \min \{ u_k : k \in Z(1, T) \}.$$

If $u_j > 0$, it is easy to see that $u_k > 0$ for all $k \in Z(1, T)$ and the proof is complete.

If $u_j \leq 0$, then $u_j = \min \{ u_k : k \in Z(0, T + 1) \}$. Since $\Delta u_{j-1} = u_j - u_{j-1} \leq 0$ and $\Delta u_j = u_{j+1} - u_j \geq 0$, $\varphi_p(s)$ is increasing in $s$, and $\varphi_p(0) = 0$, we have

$$\varphi_p(\Delta u_j) \geq 0 \geq \varphi_p(\Delta u_{j-1}). \quad (3.16)$$

On the other hand, by (3.15), we see that $-\Delta (\varphi_p(\Delta u_{j-1})) \geq -q_j \varphi_p(u_j) \geq 0$, which implies

$$\varphi_p(\Delta u_j) \leq \varphi_p(\Delta u_{j-1}). \quad (3.17)$$

By combining (3.16) with (3.17), we get $\varphi_p(\Delta u_j) = 0 = \varphi_p(\Delta u_{j-1})$. That is $u_{j+1} = u_{j-1} = u_j$. If $j + 1 = T + 1$, we have $u_j = 0$. Otherwise, $j + 1 \in Z(1, T)$. Replacing $j$ by $j + 1$, we get $u_{j+2} = u_{j+1}$. Continuing this process $T - 1 - j$ times, we have $u_j = u_{j+1} = u_{j+2} = \cdots = u_{T+1} = 0$. Similarly, since we have $u_{j-1} = u_j = 0$, if $j - 1 = 0$ or $j - 1 = 1$, we are done. If $j - 1 \in Z(2, T)$, we replace $j$ by $j - 1$, we get $u_{j-2} = u_{j-1} = 0$. By induction, we see that $u \equiv 0$ and the proof is complete. □

Now, we consider the existence of positive solutions for problem (1.1), we have

**Corollary 3.5.** If $f(k, 0) \geq 0$ for all $k \in Z(1, T)$, and

$$\bar{A} := \liminf_{t \to +\infty} \frac{1}{T} \sum_{k=1}^{T} \max_{0 \leq s \leq t} \int_{0}^{s} f(k, \tau) d\tau < \frac{B}{1 + Q}, \quad (3.18)$$

Then, for each $\lambda \in (\frac{1 + Q}{B}, \frac{1}{A})$, problem (1.1) admits an unbounded sequence of positive solutions.

**Proof.** Let

$$f^*(k, t) = \begin{cases} f(k, t), & \text{if } t > 0, \\ f(k, 0), & \text{if } t \leq 0. \end{cases} \quad (3.19)$$

Noticing that $f(k, 0) \geq 0$, it is easy to see that

$$\max_{0 \leq |s| \leq t} \int_{0}^{s} f^*(k, \tau) d\tau = \max_{0 \leq s \leq t} \int_{0}^{s} f(k, \tau) d\tau,$$

for all $t \geq 0$. By Corollary 3.1, we know that problem (1.1) with $f$ replaced by $f^*$ admits an unbounded sequence of solutions for each $\lambda \in (\frac{1 + Q}{B}, \frac{1}{A})$. And by Theorem 3.5, all these solutions are positive. Therefore, these solutions are solutions of problem (1.1) and the proof of Corollary 3.5 is complete. □

Similar to the proof of Corollary 3.5, we have
Corollary 3.6. If \( f(k,0) \geq 0 \) for all \( k \in Z(1,T) \), and
\[
\bar{C} := \liminf_{t \to 0^+} \frac{2T^{p-1} \sum_{k=1}^{T} \max_{0 \leq s \leq t} \int_{0}^{s} f(k,\tau)d\tau}{(1 + q_k)t^p} < \frac{2D}{1 + Q}.
\] (3.20)
Then, for each \( \lambda \in \left( \frac{1 + Q}{2D}, \frac{1}{\bar{C}} \right) \), problem (1.1) admits a sequence of positive solutions which converges to zero.

For the positive solutions of problem (1.2), we have

Corollary 3.7. If \( f(k,0) \geq 0 \) for all \( k \in Z(1,T) \), and
\[
\bar{E} := \liminf_{t \to +\infty} \frac{\sqrt{T} \sum_{k=1}^{T} \max_{0 \leq s \leq t} \int_{0}^{s} f(k,\tau)d\tau}{(1 + q_k)t} < \frac{F}{1 + Q}.
\] (3.21)
Then, for each \( \lambda \in \left( \frac{1 + Q}{\sqrt{T}}, \frac{1}{\bar{E}} \right) \), problem (1.2) admits an unbounded sequence of positive solutions.

Corollary 3.8. If \( f(k,0) \geq 0 \) for all \( k \in Z(1,T) \), and
\[
\bar{G} := \liminf_{t \to 0^+} \frac{2T \sum_{k=1}^{T} \max_{0 \leq s \leq t} \int_{0}^{s} f(k,\tau)d\tau}{(1 + q_k)t^2} < \frac{2H}{1 + Q}.
\] (3.22)
Then, for each \( \lambda \in \left( \frac{1 + Q}{2H}, \frac{1}{\bar{G}} \right) \), problem (1.2) admits a sequence of positive solutions which converges to zero.

4. Examples. In this section, we give two examples to illustrate our main results.

Example 4.1 Consider the boundary value problem (1.1) with
\[
f(k,u) = \frac{p}{2} |u|^\frac{p-2}{2} \left[ 1 + \epsilon + \sin(\epsilon \ln(|u|^{\frac{p}{2}} + 1)) + \epsilon \cos(\epsilon \ln(|u|^{\frac{p}{2}} + 1)) \right],
\] (4.1)
for \( k \in Z(1,T) \). Then,
\[
F(k,u) = \int_{0}^{u} f(k,\tau)d\tau = (1 + u^{\frac{p}{2}})(1 + \epsilon + \sin(\epsilon \ln(u^{\frac{p}{2}} + 1))) - 1 - \epsilon,
\]
for \( u \geq 0 \). Since \( f(k,u) \geq 0 \) for \( u \geq 0 \), we see that \( F(k,u) \) is increasing in \( u \in [0, +\infty) \). Let
\[
\gamma_n = \left[ \exp \left( \frac{4n + 1}{2\epsilon} \frac{\pi}{2} \right) - 1 \right]^{\frac{2}{p}}, \quad \eta_n = \left[ \exp \left( \frac{4n - 1}{2\epsilon} \frac{\pi}{2} \right) - 1 \right]^{\frac{2}{p}}.
\]
Then \( \lim_{n \to +\infty} \gamma_n = +\infty = \lim_{n \to +\infty} \eta_n \), and
\[
\lim_{n \to +\infty} \frac{F(k,\gamma_n)}{\gamma_n^{\frac{p}{2}}} = 2 + \epsilon, \quad \lim_{n \to +\infty} \frac{\max_{0 \leq s \leq \eta_n} F(k,s)}{\eta_n^{\frac{p}{2}}} = \lim_{n \to +\infty} \frac{F(k,\eta_n)}{\eta_n^{\frac{p}{2}}} = \epsilon,
\]
this implies, by (3.1) and (3.18), that
\[
B \geq (2 + \epsilon)T, \quad \tilde{A} \leq \frac{T^{\frac{p+1}{2}}}{1 + q_* \epsilon}.
\]
Let $0 < \epsilon < 2(1 + q_*)\left((1 + Q)T^{\frac{p-1}{2}} - 1 - q_*\right)^{-1}$, then

$$\frac{T^{\frac{p+1}{2}}}{1 + q_*} \epsilon < \frac{(2 + \epsilon)T}{1 + Q},$$

thus (3.18) holds. By Corollary 3.5, for each $\lambda \in \left(\frac{1 + Q}{(2 + \epsilon)T}, \frac{(1 + q_*)T^{\frac{p-1}{2}}}{\epsilon}\right)$, problem (1.1) admits an unbounded sequence of positive solutions.

**Example 4.2** Consider the boundary value problem (1.1) with

$$f(k, u) = \begin{cases} |u|^{p-2}u(p + p \varepsilon + p \cos(|u|) - \varepsilon \sin(|u|)), & u \neq 0, \\ 0, & u = 0, \end{cases} \quad (4.2)$$

for $k \in \mathbb{Z}(1, T)$, where $\varepsilon$ is a positive number satisfying

$$0 < \varepsilon < \frac{2(1 + q_*)}{(1 + Q)Tp^{-1} - 1 - q_*}. \quad (4.3)$$

Then,

$$F(k, u) = \int_0^u f(k, \tau)d\tau = u^p(1 + \varepsilon + \cos(\varepsilon \ln u)), \quad \text{for } u > 0.$$

Since $f(k, u) \geq 0$ for $u \geq 0$ and $k \in \mathbb{Z}(1, T)$, we see that $F(k, u)$ is increasing in $u \in [0, +\infty)$. Let

$$\xi_n = \exp\left(-\frac{2n\pi}{\varepsilon}\right), \quad \varsigma_n = \exp\left(-\frac{2n\pi + \pi}{\varepsilon}\right).$$

Then $\lim_{n \to +\infty} \xi_n = 0 = \lim_{n \to +\infty} \varsigma_n$, and

$$\frac{F(k, \xi_n)}{\xi_n} = 2 + \varepsilon, \quad \frac{\max_{0 \leq s \leq \varsigma_n} F(k, s)}{\varsigma_n} = \frac{F(k, \varsigma_n)}{\varsigma_n} = \varepsilon,$$

which implies, by (3.1) and (3.20), that

$$D \geq (2 + \varepsilon)T, \quad \bar{C} \leq \frac{2T^p}{1 + q_*}\varepsilon.$$

By (4.3), we see that (3.20) holds. By Corollary 3.6, for each $\lambda \in \left(\frac{1 + Q}{2T(2 + \epsilon)}, \frac{(1 + q_*)T}{2T^p \varepsilon}\right)$, problem (1.1) admits a sequence of positive solutions which converges to zero.

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