Disk-sphere field duality theorem

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Abstract

This paper presents a new reformulated theorem for fields embedded on a sphere or a disk. We focus in particular on the associated sphere of a disk when closing its only one boundary. We call this the disk-sphere duality theorem for the study of fields topological properties. For that purpose, we use the Poincaré-Hopf theorem and the boundary number theorem to firmly support our developments. In this context, the state of a $n$-symmetry direction field will be analyzed to show that disk-sphere field duality is closely related to the behavior near the disk’s boundary.

Keywords - $N$-symmetry direction field; Topology; Poincaré-Hopf theorem; Boundary number theorem.

1 Introduction

The study of fields on surfaces has a long interest between mechanical, physics and graphics communities. Topological properties of fields on surfaces are directly related to the 2-dimensional manifold’s topology. Applications exists in the graphics field for visualization purposes [1]. For many years, efforts have been spent to find a generalization of the Poincaré-Hopf theorem for higher dimensional manifolds [5, 4].

In this paper, we present topological properties of fields embedded on 2-dimensional manifolds in $\mathbb{R}^3$, especially for a disk along its only one boundary. We focus in particular on the sphere field properties [3]. First, we introduce briefly topological concepts to securely anchor our further developments. Afterwards, we give mathematical field characteristics for a special case in a new reformulated theorem.

2 Topology prerequisites

In this section we provide prerequisites to understand the following developments. A theoretical background in surface topology is needed.

2.1 Poincaré-Hopf theorem And Euler characteristic

The Poincaré-Hopf theorem explains the behavior of fields on compact differentiable manifolds. This theorem is widely used in geometry, physics, economics and other application fields. We give the Poincaré-Hopf theorem for a 2-dimensional manifold $M$:

Definition : Poincaré-Hopf theorem. Let $M$ be a compact differentiable manifold and $\mathbf{d}$ be a 4-symmetry direction field with $n_s$ isolated singularities of indices $I_d$ embedded in vertices $v$. If $M$ has some boundaries, the field must be pointing outward the normal direction along them:
$\sum_{i=1}^{n_s} I_d^i = \chi(M).$  

(1)

Where $\chi(M)$ is the Euler characteristic of the surface $M$. It is a topological invariant, an integer that describes the topological structure relative to the number of boundaries and the genus $g$ of the surface. We are interested only on compact oriented differentiable manifolds possibly with boundaries.

2.2 Field singularities

The singularities of a vector field embedded on a surface are commonly a set of a finite dimension. If we define a vector field such as $d : \mathbb{R}^2 \to \mathbb{R}^2$, the set of zeroes are the singularities, i.e., the set of $d$ that respect : $\{d(x, y) = 0\}$ for each entry. Depending of the symmetry of the field, the singularities can be classified by their index around a neighborhood $\Omega$ of points $P_i$ in place of singularities:

$$I_d(P_i) = \frac{1}{2\pi} \int_{\partial\Omega(P_i)} d\theta.\quad (2)$$

In addition, if we design a cycle $\gamma(s)$ to be equal to the boundary of the neighborhood $\Omega$, we can express the field singularities as:

$$I_d(P_i) = \frac{1}{2\pi} \int_{\gamma(s) = \partial\Omega(P_i)} \kappa_d ds.\quad (3)$$

Where $\kappa_d$ is the field curvature. The following development enable us to define correctly the indices of singularities on the boundaries $\partial M$ of a vector field $d$. Thereafter, the next formulation is given in a generalized form using $n$-dimensional manifolds.

**Definition : Sum of singularity indices.** Let $d$ be a vector field or a $n$-symmetry direction field with isolated zeros on the compact oriented differentiable $n$-dimensional manifold $M$, if $M$ has boundaries $\partial M$, the total index of singularity is defined to be the sum of its indices on the interior and on the boundaries [6]:

$$\sum_{i=1}^{n_s} I_d^i = \sum_{i=1}^{n_i} I_d^i + \sum_{i=1}^{n_b} I_d^i.\quad (4)$$

Where $n_i$ represents the number of singularities embedded on surface whereas $n_b$ represents the number of singularities on boundaries.

2.3 Field turning number

With the previous correct definitions of field singularities, we now describe the number of turns a field $d$ can make along a given cycle $\gamma(s)$. It corresponds to the number of turns the field accomplish in a specific frame [2]. This amount of turns is called the turning number $T_d(\gamma)$ of $d$ along the cycle $\gamma$.

$$T_d(\gamma) = \frac{1}{2\pi} \int_{\gamma} (\kappa_d - \kappa_\gamma) ds.\quad (5)$$

Where $\kappa_\gamma$ is the cycle geodesic curvature. We can reasonably show that the turning number in $\mathbb{R}^2$ can be also expressed with the index of singularity:

$$T_d(\partial\Omega(P_i)) = I_d(P_i) - 1.\quad (6)$$
2.4 Field topological properties

Fields embedded on surfaces can contain relevant invariant information. Turning numbers have fundamental properties which make them useful to compare fields topologies. These information are straightforward to study fields on 2-manifolds. Topology of a field is provided by turning numbers along boundary cycles, homology generators and around singularities [2].

Definition: Field topological equivalence. Two direction fields defined over a surface $M$ are homotopic if and only if they have the same turning numbers along the cycles of their homology generators, boundaries and around singularities, yielding to the following statement:

$$d_1 \equiv_t d_2 \Leftrightarrow \forall \gamma \in H(M) = H_g(M) \cup \partial M, T_{d_1}(\gamma) = T_{d_2}(\gamma).$$  

Where $H_g(M)$ is the set of homology generators of $M$ whereas $\partial M$ is the set of boundary cycles. Singularities are omitted in this formulation. Notice that $\equiv_t$ denotes the topological equivalence.

2.5 Boundary number theorem

Once we have determined turning numbers and topological properties of fields, we can now define the boundary number theorem. This theorem states the behavior of fields near boundaries depending to a topological invariant.

Definition: Boundary number theorem. Let $M$ be a compact differentiable 2-manifold embedded in $\mathbb{R}^3$ with boundaries $\partial M$ and $d$ be a $n$-symmetry direction field, then:

$$T_d(\partial M) = -\chi(M).$$  

Where $\chi(M)$ is the Euler characteristic of the surface $M$ and $\partial M$ is the set of boundaries. We can demonstrate that the boundary turning number theorem is equivalent to the Poincaré-Hopf theorem with a proper definition of the index of singularity [2]. For that purpose, we first generalize the index of singularity for a 2-manifold embedded in $\mathbb{R}^3$ using cycle geodesic curvature $\kappa_\gamma$ and field curvature $\kappa_d$. We then formulate easily (6):

$$I_d(P_i) = T_d(\partial \Omega(P_i)) + 1.$$  

Afterwards, we store all the singularities in the associated closed mesh $M_c$ of $M$ in the new closed boundaries. This lead us to write the following equation involving $b$ boundary components:

$$T_d(\partial M) = T_d(\partial M_c) - \sum_i^b T_d(\partial \Omega(P_i)) = - \sum_i^b (I_d(P_i) - 1) = -\chi(M_c) + b.$$  

$T_d(\partial M_c) = 0$ due to the absence of boundaries.

3 Disk-sphere field duality: opposite turning numbers

This theorem is derived from the boundary turning number theorem and the Poincaré-Hopf theorem. A compact oriented differentiable 2-dimensional manifold $D$ and $S$ are being considered. Let us introduce invariant integers for a topological disk surface $D$:

$$T_d(\partial D) = -\chi(D) = -\chi(D_c) + b = -1.$$  

With $T_d(\partial D)$ the turning number of a $n$-symmetry direction field $d$. $D_c$ is the associated closed mesh of $D$. In case of a topological sphere $S$, this turning number is equal to (consider an empty set $\dim(\partial S) = 0$ of boundary components):

$$T_d(\partial S) = 0 \neq -\chi(S) = -\chi(S_c) + b = -2.$$
Let us introduce all borders in place of singularities. Considering a neighborhood $\Omega$ around a point $P_i \in \partial D$ and also equations (6) and (10), the turning number of the disk boundary can be formulated as:

$$T_d(\partial D) = -I_d(P_i)c + 1 = -1 \Rightarrow I_d(P_i)c = 2.$$ (13)

Where $I_d(P_i)c$ is the index of singularity at $P_i$ on the closed surface $S$ of the associated disk $D$. Previous singularity index $I_d(P_i)c$ can be decomposed into two different parts:

$$I_d(P_i)c = \sum_{j=1}^{2} I_d^j(P_i)c.$$ (14)

One singularity is located on the boundary (i.e., on the associated sphere $S$), the other is embedded in a disk vertex $v$ using sum of singularity indices in equation (4):

$$I_d(P_i)c \in S \& I_d^1(P_i)c = I_d(v) \in D.$$ (15)

Equation (6) is then evaluated for the boundary and also for the disk vertex in order to have the related turning numbers:

$$T_d(\partial \Omega(P_i)) = I_d^1(P_i)c - 1 \& T_d(\partial v) = I_d(v) - 1.$$ (16)

**Definition : Disk-sphere field duality theorem.** For two singularities on a sphere, one embedded in a vertex $v$ and one located at $P_i$, they have opposite turning numbers corresponding to the following duality:

$$T_d(\partial v) = I_d(v) - 1 = -(I_d^1(P_i)c - 1) = -T_d(\partial \Omega(P_i)).$$ (17)

This is due to the definition of the sum of two singularity indices for a sphere $S$ in equation (13). In other words, Poincaré-Hopf theorem violation on a disk is equivalent to define the right index of singularity on the associated sphere. It is possible to define a field without indices of singularity if at least one boundary exists. Therefore violations of Poincaré-Hopf theorem remains possible only if at least one boundary exists. This example states the trade between the boundary number theorem and the Poincaré-Hopf theorem. We use the term "violation" when not taking into account the behavior of the field near boundaries.

4 Conclusion

We have shown that for two singularities on a sphere, they have opposite turning numbers. This is a direct consequence of the Poincaré-Hopf theorem for topological spheres. The presented concepts are a re-formulation of the Poincaré-Hopf index formula, involving fields topological properties such as turning numbers. This was done in a specific case, when converting a disk into its associated topological sphere.

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