An alternative characterisation of universal cells in opetopic $n$-categories

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Abstract

We address the fact that composition in an opetopic weak $n$-category is in general not unique and hence is not a well-defined operation. We define composition with a given $k$-cell in an $n$-category by a span of $(n-k)$-categories. We characterise such a cell as universal if its composition span gives an equivalence of $(n-k)$-categories.

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Introduction

In this paper we give a characterisation of universality in the opetopic theory of $n$-categories.
The opetopic definition of \( n\)-category proceeds in two stages. First, the language for describing \( k\)-cells is constructed. This is the theory of opetopes. Then, a concept of universality is introduced, to deal with composition and coherence. Eventually, we have the following definition:

**Definition 0.1** An opetopic (weak) \( n\)-category is an opetopic set in which

i) Every niche has an \( n\)-universal occupant.

ii) Every composite of \( n\)-universals is \( n\)-universal.

The word ‘composite’ is used in the following sense: given any universal cell, its target cell is said to be a composite of its source cells. Thus in the opetopic theory, composites are not necessarily unique.

In [5] and [4] we examine the various approaches to the theory of opetopes ([1], [6], [7]) and prove that they are equivalent. In the present paper we turn our attention to the second stage of the definition of opetopic \( n\)-category, concerning universality.

There are many ways of characterising universal cells, just as there are many ways of characterising, say, isomorphisms in a category. The original definition given by Baez and Dolan generalises the result ‘\( f \) is an isomorphism if and only if any morphism with the same domain factors through it uniquely’. The characterisation proposed here is motivated by another familiar result in categories, that \( f \) is an isomorphism if and only if composition with \( f \) is an isomorphism.

In a category, “composition with \( f \)” is a function on homsets; however, composition in an opetopic \( n\)-category is not uniquely defined, that is, \( \omega \circ f \) is not a well-defined operation. One way of dealing with this would be to choose composites in order to make \( \omega \circ f \) into a well-defined operation. However, this is not in the spirit of the opetopic definition. To avoid making such choices we instead define “composition with \( f \)” as a span of hom-(\( n-k \))-categories. This “composition span” gives all possible ways of composing with \( f \).

We begin in Section 1 by recalling the definition of universal cells in an opetopic set \( X \), given in [3]. This is our motivation for our new characterisation of universal cells in an \( n\)-category. In Section 2 we show how the composition span is used in this characterisation, and in Section 3 we give the actual construction of the composition span. While we intend that the two notions of universality should coincide when \( X \) is an \( n\)-category, we do not include a proof here as we currently lack an effective method for calculating in arbitrary dimensions. As a gesture towards this result, we include some low-dimensional examples in Section 4 and in Section 5 we prove that the notions do indeed coincide for \( n \leq 2 \). Finally in Section 6 we make some brief concluding remarks.
**Terminology and Notation**

i) Since we are concerned chiefly with weak $n$-categories we continue our previous practice of omitting the word “weak” in general.

ii) In this paper we will avoid any detailed discussion of the language of multicategories and construction of opetopic sets; this has been discussed in detail in our earlier work ([5], [4], [2]). We will adopt the (more practical) method of Hermida, Makkai and Power ([6]), picking one ordering of source elements in order to represent a symmetry class. A $k$-cell has as its source a pasting diagram of $(k - 1)$-cells, and as its target a single $(k - 1)$-cell. For a general $k$-cell we write its source as $a$, say, to indicate a formal composite whose constituent $(k - 1)$-cells may be placed in some order.

iii) Furthermore, we may adopt the following convention for 2-ary cells. A 2-ary $k$-cell $\alpha$ has the form

![Diagram of a 2-ary $k$-cell $\alpha$](image)

where $f$, $g$, and $h$ are $(k - 1)$-cells (and necessarily $k \geq 2$). We write this $k$-cell as

$$\alpha : (f, g) \rightarrow h$$

employing this ordering of the source elements to indicate that $f$ and $g$ are pasted at the target of $g$; we also write $s_1(\alpha) = f$ and $s_2(\alpha) = g$.

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**1 Definition of universality**

We begin by recalling from [3] the definition of a universal cell in an opetopic set.

Let $X$ be an opetopic set.
Definition 1.1 A $k$-cell $\alpha$ is $n$-universal if either $k > n$ and $\alpha$ is unique in its niche, or $k \leq n$ and (1) and (2) below are satisfied:

1. Given any $k$-cell $\gamma$ in the same niche as $\alpha$, there is a factorisation $u : (\beta, \alpha) \to \gamma$

\[
\begin{array}{c}
\ldots \\
\downarrow \\
\vdots \\
\downarrow \\
\alpha \\
\downarrow \\
\beta \\
\end{array}
\xrightarrow{\alpha}
\begin{array}{c}
\ldots \\
\downarrow \\
\vdots \\
\downarrow \\
\gamma \\
\end{array}
\]

2. Any such factorisation is $n$-universal.

Definition 1.2 A factorisation $u : (b, a) \to c$ of $k$-cells is $n$-universal if $k > n$, or $k \leq n$ and (1) and (2) below are satisfied:

1. Given any $k$-cell $b'$ in the same frame as $b$, and any $(k+1)$-cell $v : (b', a) \to c$

\[
\begin{array}{c}
b'
\downarrow \\
y \\
\downarrow \\
\ldots \\
\downarrow \\
b \\
\downarrow \\
u \\
\downarrow \\
c \\
\end{array}
\xrightarrow{\alpha}
\begin{array}{c}
b'
\downarrow \\
a \\
\downarrow \\
\ldots \\
\downarrow \\
v \\
\downarrow \\
c \\
\end{array}
\]

with $b'$ and $a$ pasted in the same configuration as $b$ and $a$ in the source of $u$, there is a factorisation of $(k+1)$-cells $(u, y) \to v$

2. Any such factorisation is itself $n$-universal.
2 An alternative characterisation

We now examine the motivating example in categories. Let \( C \) be a category and \( f : A \to B \) a morphism in \( C \). Then we have a natural transformation

\[
H^f : C(B, -) \to C(A, -)
\]

with components

\[
\circ f : C(B, C) \to C(A, C)
\]

for each \( C \in C \). Then

\[
f \text{ is an isomorphism} \iff H^f \text{ is an isomorphism}
\]

\[
\iff \forall C \in C, \circ f \text{ is an isomorphism}
\]

\[
\iff \text{“composition with } f \text{ is an isomorphism”}
\]

Here “composition with \( f \)” is a function on homsets.

Now let \( X \) be an opetopic \( n \)-category and \( f : a \to b \) a \( k \)-cell in \( X \). Then given any \((k-1)\)-cell \( c \) we have \((n-k)\)-categories \( X(b, c) \) and \( X(a, c) \) whose 0-cells are \( k \)-cells of \( X \) with the appropriate source and target, and whose \( j \)-cells are \((k+j)\)-cells.

Since composition in an opetopic \( n \)-category is not uniquely defined, we cannot expect \( \circ f \) to be a well-defined operation \( X(b, c) \to X(a, c) \). Instead, we will have a span of \((n-k)\)-categories

\[
\begin{array}{ccc}
C_f & \xrightarrow{\sigma_f} & X(b, c) \\
\downarrow^{\tau_f} & & \downarrow \\
X(a, c) & & \\
\end{array}
\]

where \( C_f \) gives all possible ways of composing with \( f \). Here \( \sigma_f \) and \( \tau_f \) are \((n-k)\)-functors i.e. morphisms of the underlying opetopic sets. (\( \sigma \) has more properties that we will not discuss here.)

We then have the following characterisation.

**Definition 2.1** \( A k \)-cell \( f \) is universal iff

i) \( k > n \) and \( f \) is unique in its niche, or

ii) \( k \leq n \) and \( \tau_f \) is an \((n-k)\)-equivalence of \((n-k)\)-categories.

**Definition 2.2** An \( m \)-functor is an \( m \)-equivalence of \( m \)-categories iff

i) it is an \((m-1)\)-equivalence on hom-\((m-1)\)-categories

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ii) it is “essentially surjective on 0-cells” i.e. surjective up to universal 1-cells

We observe (without giving details) that since $\sigma_f$ will be an $(n-k)$-equivalence of $(n-k)$-categories, the above condition for universality will also result in $X(b,c)$ and $X(a,c)$ being $(n-k)$-equivalent.

Furthermore it will follow from the construction of the composition span that in an $n$-category the above definition is equivalent to demanding “on the nose” surjectivity. i.e. $f$ is universal iff $\forall x, y \in C_f$

$$\tau : C_f(x,y) \to X(\tau x, \tau y)$$

is surjective on objects. This is a consequence of the fact that composites of universals are universal in an opetopic $n$-category.

In the next section we construct the composition span itself.

3 Construction of composition span

In this section we give the construction of a composition span; in the next section we give some explicit examples at low dimensions.

Composition of $k$-cells is given by universal $(k+1)$-cells, so in order to construct a composition span for a $k$-cell $f$, we must assume that for all $m > k$ the universal $m$-cells have been defined.

We seek to construct a span of opetopic sets

\[
\begin{array}{ccc}
C_f & \xrightarrow{\sigma_f} & X(b,c) \\
\downarrow & & \downarrow \\
\tau_f & \xrightarrow{\tau_f} & X(a,c)
\end{array}
\]

For convenience we write $C_f = C$, $\sigma_f = \sigma$ and $\tau_f = \tau$. Also put $X(b,c) = X_1$ and $X(a,c) = X_2$. Recall that a morphism $F : A \to B$ of opetopic sets has for each $j \geq 0$ a function

$$F_j : A(j) \to B(j)$$

such that, for each $j \geq 1$ a certain square

\[
\begin{array}{ccc}
A(j) & \xrightarrow{F_j} & B(j) \\
\downarrow & & \downarrow \\
& & 
\end{array}
\]
commutes, ensuring that “underlying shapes are preserved”. So we seek for each $j \geq 0$ functions

\[
\begin{array}{ccc}
C(j) & \xrightarrow{\sigma_j} & X_1(j) \\
\downarrow{\tau_j} & & \downarrow{\tau_j} \\
X_2(j) & & \end{array}
\]

such that for each $j \geq 1$ a certain diagram

\[
\begin{array}{ccc}
C(j) & \xrightarrow{p_j} & X_1(j) \\
\downarrow{p_j} & & \downarrow{\tau_j} \\
X_2(j) & \xrightarrow{\tau_j} & \end{array}
\]

commutes. Then a $j$-cell $\theta \in C(j)$ exhibits $\tau_j(\theta) \in X_2(j)$ as a composite of $f$ with $\sigma_j(\theta) \in X_1(j)$. $p_j$ gives the frame of each $j$-cell in $C$.

- $j = 0$

Put

\[C(0) = \{ u \in \mathcal{U}(k + 1) \mid s_2(u) = f \}\]

where $\mathcal{U}(m)$ is the set of 2-ary universal $m$-cells. Put $\sigma_0 = s_1$ and $\tau_0 = t$.

- $j = 1$

A 1-frame in $C$ has the form $u_1 \rightarrow u_2$. We form the set of occupants of this frame as follows. Write

\[
\begin{align*}
\mathcal{U}_1 &= \{ u \in \mathcal{U}(k + 2) \mid s_1(u) = u_2 \} \\
\mathcal{U}_2 &= \{ u \in \mathcal{U}(k + 2) \mid s_2(u) = u_1 \}
\end{align*}
\]

and form the pullback
• $j > 1$

For higher values of $j$ we construct for each $j$ a pullback over $2^j$ subsets of $U(k + j + 1)$ as follows.

Let $\theta$ be a $(j - 1)$-frame in $C$ with target $\alpha$. $\alpha$ is a $(j - 1)$-cell of $C$ so is a string of $2^{j-1}$ universal $(k + j)$-cells $u_1, \ldots, u_{j-1}$, say. Now write $U = U(k + j + 1)$ and for each $1 \leq i \leq 2^{j-1}$

$$U_i = \{ u \in U(k + j + 1) \mid s_1(u) = u_i \} .$$

For the set of occupants of the frame $\theta$ we form a pullback over $2^j$ sets as follows:

This completes the definition of $C_f$.

4 Some examples at low dimensions

In this section we give some examples of elements of the composition span $C = C_f$ for a 1-cell $f$.

• $j = 0$

$C(0)$ is the set of universal 2-cells of the form
exhibiting $\bar{g}$ as a composite of $f$ and $g$.

- $j = 1$

We form a pullback over

\[
\begin{array}{c}
\begin{array}{c}
U_1 \\
\downarrow s_2 \\
X_1(1)
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
U_2 \\
\downarrow t \\
X_2(1)
\end{array}
\end{array}
\]

So a typical element is of the form $(u_{31}, u_{32})$ with projections as shown below

\[
\begin{array}{c}
\begin{array}{c}
\phi \\
\downarrow s_2 \\
\phi_1
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
\bar{\phi} \\
\downarrow t \\
\phi_1
\end{array}
\end{array}
\]

For example, the following two universal 3-cells exhibit $\bar{\phi}$ as a composite of $f$ with $\phi$; this element of $C(2)$ is in the frame $u_2 \rightarrow u_2'$.

- $j = 2$
We form a pullback over
\[
\begin{array}{cccc}
\mathcal{U}_1 & s_2 & t & s_1 & t & s_2 \\
\mathcal{U} & \mathcal{U} & \mathcal{U} & \mathcal{U} & \mathcal{U} & \mathcal{U} \\
X_1(2) & \phi & \phi_1 & \phi_2 & \phi_3 & \phi \\
\end{array}
\]

A typical element is of the form \((u_{41}, u_{42}, u_{43}, u_{44})\) with projections as shown below

\[
\begin{array}{cccc}
\phi & u_{41} & t & s_2 \\
\phi_1 & u_{42} & t & s_1 \\
\phi_2 & u_{43} & t & s_1 \\
\phi_3 & u_{44} & t & s_2 \\
\overline{\phi} & \phi & \phi_1 & \phi_2 & \phi_3 & \phi \\
\end{array}
\]

exhibiting \(\overline{\phi}\) as a composite of \(f\) with \(\phi\). For example, the following element of \(C(2)\) is in a frame with target \((u_{31}, u_{32})\).
• \( j = 3 \)

Similarly, in \( C(3) \) we have a typical element \((u_{51}, \ldots, u_{58})\):

For example the following element of \( C(3) \) (running over two pages) has target \((u_{41}, u_{42}, u_{43}, u_{44})\):

\[ \phi \]
5 Comparison

We now compare the new characterisation with the original definition for $n \leq 2$, and show that at these low dimensions the notions do indeed coincide. We argue explicitly; at higher dimensions such an approach rapidly becomes unfeasible. An effective method for handling such algebra is urgently needed.

For convenience we refer to the ‘old’ and ‘new’ universal properties as Property 1 (P1) and Property 2 (P2) respectively, and we continue to use all earlier notation.

- $n = 0$

  P1 and P2 clearly coincide for $k$-cells with $k > n$; this deals with all possibilities when $n = 0$.

- $n = 1$

  Let $X$ be an opetopic 1-category. We show that P1 and P2 are equivalent for $k = 1$. Let $f$ be a 1-cell $a \rightarrow b$ in $X$. $f$ has P1 if and only if for all $a \rightarrow c \in X$ there is a unique factorisation

  \[ f \Downarrow u \bar{g} \]

  (see [3]) i.e. $\tau_0$ is an isomorphism $C(0) \rightarrow X_2(0)$, which says precisely that $f$ has P2.

  Note that this argument immediately generalises to all $n = k$.

- $n = 2$

  Let $X$ be an opetopic 2-category. The above arguments deal with $k \geq 2$; we show that the notions coincide when $k = 1$.

  Let $a \rightarrow b$ have P1. We show that

  \[ \tau_f : C_f \rightarrow X(a, c) \]

  is an equivalence of 1-categories.

  i) We show that it is essentially surjective on objects; in fact it is surjective ‘on the nose’. A 0-cell in the codomain is a 1-cell of the form $a \rightarrow c$. Given any such, there certainly is a factorisation

  \[ f \Downarrow u \bar{g} \]

  \[ g \]
so we have $\bar{g}$ such that 

$$\tau_0 : \bar{g} \mapsto g.$$ 

ii) We show that is is ‘locally an isomorphism’. Consider the 1-frame in $C_f$

$$f \Downarrow u_1 \bar{g}_1 \quad \Rightarrow \quad f \Downarrow u_2 \bar{g}_2.$$ 

We show that $\tau$ is an isomorphism on the 0-category (set) induced by this frame. So consider 1-cell in the codomain

$$\xymatrix{ g_1 \ar[d] \ar@/^/[drr]^{\alpha} \ar@/_/[drr]_{\bar{\alpha}} \ar@{.>}[urr] \ar@{.>}[uur] & \quad \Rightarrow \quad & \xymatrix{ g_1 \ar[d] \ar@/^/[drr]^{\alpha} \ar@/_/[drr]_{\bar{\alpha}} \ar@{.>}[urr] \ar@{.>}[uur] & \quad \Rightarrow \quad & \xymatrix{ g_2 \ar[d] \ar@{.>}[urr] \ar@{.>}[uur] & \quad \Rightarrow \quad & \xymatrix{ g_2 \ar[d] \ar@{.>}[urr] \ar@{.>}[uur] & \quad \Rightarrow \quad & \xymatrix{ g_2 \ar[d] \ar@{.>}[urr] \ar@{.>}[uur] & \quad \Rightarrow \quad & \xymatrix{ g_2 \ar[d] \ar@{.>}[urr] \ar@{.>}[uur] }\end{array}}$$

We have

$$\xymatrix{ u_1 \ar@/^/[drr]^{\alpha} \ar@/_/[drr]_{\bar{\alpha}} \ar@{.>}[urr] \ar@{.>}[uur] & \quad \Rightarrow \quad & \xymatrix{ \theta \ar[d] \ar@{.>}[urr] \ar@{.>}[uur] }\end{array}}$$

say, and by P1, $\theta$ induces a unique factorisation

$$\xymatrix{ u_2 \ar@/^/[drr]^{\alpha} \ar@/_/[drr]_{\bar{\alpha}} \ar@{.>}[urr] \ar@{.>}[uur] & \quad \Rightarrow \quad & \xymatrix{ \theta \ar[d] \ar@{.>}[urr] \ar@{.>}[uur] }\end{array}}$$

so we have a unique pre-image of $\alpha$ as required.

The converse follows easily.

6 Conclusions

We conclude that although the outline of the basic syntax of opetopes seems secure, universality is less well understood, and we remain unsure of the ideal form in which it should be defined. The alternative characterisation described in this work seems right in ‘spirit’, but in the end the mathematics that emerges is not as ‘slick’ as might be hoped. It therefore appears that there is much scope for further work in this area.
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