AREA DENSITY AND REGULARITY FOR SOAP FILM-LIKE SURFACES SPANNING GRAPHS

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ABSTRACT. For a boundary configuration $\Gamma$ consisting of arcs and vertices, with two or more arcs meeting at each vertex, we treat the problem of estimating the area density of a soap film-like surface $\Sigma$ spanning $\Gamma$. $\Sigma$ is assumed to be strongly stationary for area with respect to $\Gamma$. We introduce a notion of total curvature $C_{\text{tot}}(\Gamma)$ for such graphs, or nets, $\Gamma$. When the ambient manifold $M^n$ has non-positive sectional curvatures, we show that $2\pi$ times the area density of $\Sigma$ at any point is less than or equal to $C_{\text{tot}}(\Gamma)$. For $n = 3$, these density estimates imply, for example, that if $C_{\text{tot}}(\Gamma) \leq 22.9\pi$, then the only possible singularities of a piecewise smooth $(M, \varepsilon, \delta)$-minimizing set $\Sigma$ are curves, along which three smooth sheets of $\Sigma$ meet with equal angles of $120^\circ$. We also extend these results to allow $M$ to have variable positive curvature.

1. Introduction

The investigation of minimal surfaces has proved extremely fruitful in a wide range of topics in geometry. One of the essential breakthroughs in the subject is the solution of the Plateau problem by Douglas and by Radó, that is, the construction of a disc type minimal surface spanned by a Jordan curve $\Gamma$ in $\mathbb{R}^n$ [D1], [R]. Plateau’s original motivation was, in part, to study the geometry of soap films spanned by variously shaped wires. In particular, it is natural to want to generalize the boundary condition imposed by Douglas and Radó that the wire $\Gamma$ spanning the surface be a Jordan curve, or a union of Jordan curves (cf. [D2], where $\Sigma$ is a branched immersion of higher topological type). In this paper, we will introduce a class of surfaces $\Sigma$ in an ambient manifold $M$, having a piecewise smooth boundary $\Gamma$ which is homeomorphic to a graph, that is, a a finite 1-dimensional polyhedron.
(sometimes called a “net”). Each surface is to satisfy a regularity condition, and is stationary for area under variations induced by one-parameter families of diffeomorphisms of the ambient manifold. This setting allows us to consider surfaces whose induced topology is not locally Euclidean, such as the singular surfaces which may readily be observed in soap film experiments. The main theorems of this paper provide descriptions of the possible singularities of those minimal surfaces in terms of the geometry of the boundary set $\Gamma$.

In a Riemannian manifold $M^n$, we shall consider an embedded graph $\Gamma$ which is a union of arcs $a_k$ meeting at vertices $q_j$, each of which has valence at least two. The valence of a vertex $q$ is the number of times $q$ occurs as an endpoint of the 1-simplices $a_k$. Each 1-simplex $a_k$ is assumed to be $C^2$, and to meet its end points with $C^1$ smoothness; thus there is a well-defined tangent vector $T_k$ to each 1-simplex $a_k$ at a vertex. At a vertex $q_j$ of valence $d$, we consider the contribution to total curvature at $q_j$:

$$
tc(q_j) := \sup_{e \in T_{q_j}M} \left\{ \frac{d}{\ell} \left( \frac{\pi}{2} - \beta^\ell_j(e) \right) \right\}
$$

where $\beta^\ell_j = \beta^\ell_j(e) \in [0, \pi]$ is the angle between the tangent vector $T_\ell$ to $a_\ell$ at $q_j$ and the vector $e$. We define the total curvature of $\Gamma$ as

$$
C_{\text{tot}}(\Gamma) := \int_{\Gamma^{\text{reg}}} |\vec{k}| \, ds + \sum \{tc(q) : q \text{ a vertex of } \Gamma\}.
$$

where $\vec{k}$ is the geodesic curvature vector of $a_i$ as a curve in $M^n$, and $\Gamma^{\text{reg}} = \Gamma \setminus \{\text{vertices}\}$. It should be noted that our definition of total curvature coincides with the standard definition in the case when $\Gamma$ is a piecewise smooth Jordan curve: the integral of the norm of geodesic curvature vector plus the sum of the exterior angles at the vertices. Namely, in that case, every vertex $q$ of the graph $\Gamma$ is of valence two; the supremum in equation (1) is assumed at vectors $e$ lying in the smaller angle between the tangent vectors $T_1$ and $T_2$ to $\Gamma$. Recall that the density of $\Sigma$ at $p$ is

$$
\Theta_{\Sigma}(p) := \lim_{\varepsilon \to 0} \frac{\text{Area}(\Sigma \cap B_\varepsilon(p))}{\pi \varepsilon^2},
$$
provided this limit exists. The type of surface we will consider in this paper is a set $\Sigma$ which is a finite union of $C^2$-smooth open two-dimensional embedded manifolds $\Sigma_i$, $C^1$ up to the boundary $\partial \Sigma_i$, with $\partial \Sigma_i$ piecewise $C^1$. We further impose that the graph $\Gamma$ is a subset of $S := \cup_i \partial \Sigma_i$. The class of such surfaces will be denoted by $S_\Gamma$. Note that given a surface $\Sigma$ in $S_\Gamma$, the density $\Theta_\Sigma(p)$ is a well defined, upper semi-continuous function on $\Sigma$.

Moreover, for $\Sigma$ in the class $S_\Gamma$, we may also write

$$\Theta_\Sigma(p) = \lim_{\varepsilon \to 0} \frac{\text{Length}(\Sigma \cap \partial B_\varepsilon(p))}{2\pi \varepsilon},$$

A surface $\Sigma$ in $S_\Gamma$ is said to be strongly stationary with respect to $\Gamma$ if the first variation of the area of the surface is at most equal to the integral over $\Gamma$ of the length of the component of the variation vector field normal to $\Gamma$ [EWW].

We can now state the main area-density estimate for the case when the ambient space is Euclidean (see Corollary 5 below):

**Area-Density Estimate:** *Let $\Sigma$ in the class $S_\Gamma$ be a strongly stationary surface in $\mathbb{R}^n$ with respect to its boundary set $\Gamma$. Then*

$$2\pi \Theta_\Sigma(p) \leq C_{\text{tot}}(\Gamma).$$

This estimate is a consequence of two inequalities, the first being the comparison of area density of $\Sigma$ and of the cone $C_\Gamma(p)$. Here, and in the remainder of this paper, for a point $p \in \mathbb{R}^n$ and a set $\Gamma_0 \subset \mathbb{R}^n$, we write the cone over $\Gamma_0$ as

$$C_\Gamma(p_0) := \{ p + t(x - p) : x \in \Gamma_0, 0 \leq t \leq 1 \}.$$

In section 6, where $\mathbb{R}^n$ is replaced more generally by a strongly convex Riemannian manifold, $C_\Gamma(p_0)$ will denote the geodesic cone over $\Gamma_0$.

**Theorem 1:** *Given a strongly stationary surface $\Sigma$ in $S_\Gamma$, and a point $p$ in $\Sigma \setminus \Gamma$, let $C_\Gamma(p)$ be the cone spanned by $\Gamma$ with its vertex at $p$. Then we have*

$$\Theta_\Sigma(p) < \Theta_{C_\Gamma(p)}(p)$$
unless \( \Sigma \) is a cone over \( p \) with planar faces.

The second inequality follows from the Gauss-Bonnet formula applied to the double cover of the cone \( C_p(\Gamma) \). (We have not found a useful Gauss-Bonnet formula for general 2-dimensional Riemannian polyhedra in the literature.)

**Theorem 2** (and Corollary 5:)

\[
2\pi \Theta_{C_p(\Gamma)}(p) = -\sum_{k=1}^{n} \int_{a_k} \vec{k} \cdot \nu_C \, ds + \sum_{k=1}^{n} \sum_{j=1,2} \left( \frac{\pi}{2} - \beta_j^k \right) \leq C_{\text{tot}}(\Gamma),
\]

where \( \vec{k} \) is the geodesic curvature vector of \( a_k \) in \( \mathbb{R}^n \), \( \nu_C \) is the outward unit normal vector at \( a_k \subset \partial C_p(\Gamma) \), and \( \beta_j^k \) is the angle between the tangent vector to \( a_k \) at its endpoint \( q_j \) and the line segment from \( q_j \) to \( p \).

The area density estimate \( 2\pi \Theta_{\Sigma}(p) \leq C_{\text{tot}}(\Gamma) \), when \( \Gamma \) is a rectifiable Jordan curve, is a major ingredient of the work by Ekholm, White and Wienholtz [EWW], where it was proven that if \( C_{\text{tot}}(\Gamma) \leq 4\pi \), then every stationary branched minimal surface \( \Sigma \) in \( \mathbb{R}^n \) spanned by \( \Gamma \) is embedded; and that given a compactly supported rectifiable varifold \( \Sigma \) which is strongly stationary with respect to \( \Gamma \) and with area density \( \geq 1 \) on \( \Sigma \setminus \Gamma \), the inequality \( C_{\text{tot}}(\Gamma) < 3\pi \) implies that \( \Sigma \) is smooth in the interior. Therefore one can view the results in this paper as partial extensions of those theorems in [EWW], when the Jordan curve \( \Gamma \) of [EWW] is replaced, more generally, by a graph.

By imposing appropriate upper bounds on the total curvature of the graph \( \Gamma \), we obtain the following statements. We will denote by \( C_Y = 3/2 \) the area density at its vertex of the Y-singularity cone composed of three planes meeting at \( 120^\circ \), and by \( C_T = 6 \cos^{-1}(-1/3) \approx 11.468 \) the area density at its vertex of the T-singularity cone spanned by the one-skeleton of the regular tetrahedron with vertex at its center.
**Theorem 3:** Suppose $\Gamma$ is a graph in $\mathbb{R}^n$ with $C_{\text{tot}}(\Gamma) \leq 2\pi C_Y = 3\pi$, and let $\Sigma$ be a strongly stationary surface relative to $\Gamma$ in the class $S_T$. Then $\Sigma$ is an embedded surface or a subset of the $Y$-singularity cone.

**Theorem 4:** Suppose $\Gamma$ is a graph in $\mathbb{R}^3$ with $C_{\text{tot}}(\Gamma) \leq 2\pi C_T$, and let $\Sigma$ be an $(M, \varepsilon, \delta)$-minimizing set with respect to $\Gamma$ in $S_T$. Then $\Sigma$ is a surface with possibly $Y$-singularities but no other singularities, unless it is a subset of the $T$-stationary cone, with planar faces.

For the definition of $(M, \varepsilon, \delta)$-minimizing sets, see Definition 4 below.

In $\mathbb{R}^3$, there are many known examples of strongly stationary surfaces. In particular, there are exactly ten stationary cones spanned by a graph $\Gamma$ on a unit sphere $[AT]$. Each graph consists of geodesic segments on the sphere meeting in threes at angles of $120^\circ$, including the planar case, where the graph is simply one great circle spanning $\mathbb{R}^2$. By ordering those ten minimal cones with respect to the density at the cone vertex, which is the center of the unit sphere, one has a list of possible tangent cones at the interior points of an $(M, \varepsilon, \delta)$-set $\Sigma$. The first three on the list, that is, the ones with the smallest densities at the vertex, are the plane with its density $1$ where the graph $\Gamma$ is a great circle; the $Y$-singularity cone with its density $C_Y = 3/2$ where $\Gamma$ consists of three semicircles meeting at the north and south poles at angles of $120^\circ$; and the $T$-singularity cone with density $C_T = 6 \cos^{-1}(1/3)$. Recall that those ten cones are stationary, but not minimizing, under interior deformations. Hence given one of those graphs, there may be another surface which is also strongly stationary with respect to the same graph, but has strictly smaller area. Indeed when it comes to soap films, the first three on the list are the only tangent cones experimentally observed in the interior of soap films. This is also true for the mathematical model in terms of 2-rectifiable sets, a result shown by Jean Taylor [T]:

Regularity Theorem for Soap Films: Away from $\Gamma$, an $(M, \varepsilon, \delta)$-minimizing set with respect to $\Gamma$ consists of real analytic surfaces meeting smoothly in threes at 120° angles along smooth curves, with these curves in turn meeting in fours at angles of $\cos^{-1}(-1/3)$.

The singular curves were proved to be $C^{1,\alpha}$ by Taylor [T], and later shown to be real analytic in [KNS]. The class $S_\Gamma$ of surfaces we consider in this paper is chosen so that, given a graph $\Gamma$, we expect to find that every $(M, 0, \delta)$-minimizing set relative to $\Gamma$ is in the regularity class $S_\Gamma$. The ten stationary cones described above are in fact in $S_\Gamma$. However, due to the lack of understanding of boundary regularity of such $(M, 0, \delta)$-minimizing sets, it is not yet known that in general $(M, 0, \delta)$-sets are indeed elements of the class $S_\Gamma$.

In Section 6, we turn our attention to the case where the ambient manifold is of variable curvature. The lack of homogeneity of the ambient space forces us to consider a comparison space of constant sectional curvatures, as was done previously in [CG2]. We consider two classes of Riemannian manifolds $M$ which are strongly convex (not necessarily complete): manifolds with sectional curvature $K_M$ bounded above by $-\kappa^2 \leq 0$, and manifolds with sectional curvature bounded above by $\kappa^2 > 0$. For a Euclidean ambient space, as seen above in Theorem 2, the area density of the surface is bounded above by the total curvature of $\Gamma$. In the variable curvature case, the total curvature of $\Gamma$ is not invariant under diffeomorphisms of $M$ which mimic the homotheties of $\mathbb{R}^n$. Thus, in order to have significance for both large graphs $\Gamma$ and for small ones, $C_{\text{tot}}(\Gamma)$ needs to be replaced in the following manner:

**Area-Density Estimate:** ($K_M \leq -\kappa^2$ case) Let $\Sigma$ be a strongly stationary surface relative to $\Gamma$ in the class $S_\Gamma$ in $M_{K \leq -\kappa^2}$. Then

$$2\pi \Theta_p(\Sigma) \leq C_{\text{tot}}(\Gamma) - \kappa^2 A(\Gamma),$$
where $A(\Gamma)$ is the minimum cone area of all the cones with vertex in the convex hull of the set $\Gamma$.

**Area-Density Estimate:** $(K_M \leq \kappa^2$ case) Let $\Sigma$ be a strongly stationary surface relative to $\Gamma$ in the class $S_\Gamma$ in $M_{K \leq \kappa^2}$. Then

$$2\pi \Theta_p(\Sigma) \leq C_{\text{tot}}(\Gamma) + \kappa^2 \hat{A}(\Gamma),$$

where $\hat{A}(\Gamma)$ is the maximum spherical area of all the cones with vertex in the convex hull of the set $\Gamma$.

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2. **Density and the Regularity of Strongly Stationary Surfaces**

Let $\Gamma \subset \mathbb{R}^n$ be a graph, consisting of immersed arcs $a_i$, which are $C^2$ in the interior and $C^1$ up to their vertices, as in Section 2 we assume that each vertex has valence at least two. Let the class $S_\Gamma$ of singular surfaces be defined as in Section 1 for $\Sigma \in S_\Gamma$, $\Gamma$ is a subset of the one-dimensional part $S \subset \Sigma$. Within the class $S_\Gamma$, we will look at the surfaces satisfying the following property.

**Definition 1.** A rectifiable varifold $\Sigma$ in $\mathbb{R}^n$ is called strongly stationary with respect to $\Gamma$ if for all smooth $\phi : \mathbb{R} \times \mathbb{R}^n$ with $\phi(0, x) \equiv x$, we have

$$\frac{d}{dt} \left( \text{Area}(\phi(t, \Sigma)) + \text{Area}(\phi([0, t] \times \Gamma)) \right) \bigg|_{t=0} \geq 0.$$

The regularity condition on each $\Sigma_i$ guarantees that at almost every point $p$ of $S$, there exists a unit vector $\nu_{\Sigma_i}$, normal to $\Gamma$, tangential to $\Sigma_i$, pointing out of $\Sigma_i$. Hence on each $\Sigma_i$ we have the divergence theorem

$$\int_{\Sigma_i} \text{div}_{\Sigma_i} X T dA = \int_{\partial \Sigma_i} \langle X, \nu_i \rangle ds,$$
where $X^T$ is the tangential (to $\Sigma_i$) component of $X$. Note that the strong stationarity condition implies that $\Sigma \in S$ is stationary, i.e. the first variation of the area vanishes under deformations supported away from $\Gamma$:

$$\frac{d}{dt} \left( \text{Area}(\phi(t, \Sigma)) \right) \bigg|_{t=0} = 0.$$ 

If in addition the vector field $X = \phi'(0, x)$ is supported away from the singular set $S$, then the stationarity condition implies that the interior of each $\Sigma_i$ is minimal, i.e. its mean curvature vector $\vec{H}$ vanishes.

If the vector field $X$ is supported away from $\Gamma$, the stationarity condition implies

$$\int_S \sum_{j \in J \subset I} \langle \nu_{\Sigma_j}(p), X^\perp(p) \rangle ds(p) - \sum_{\Sigma_i} \int_{\Sigma_i} \langle \vec{H}, X^\perp \rangle dA = 0$$

where $J = J(p)$ indexes the collection of surfaces $\Sigma_j$ which meet at a point $p$ in $S \setminus \Gamma$. Note that the second term vanishes since $\vec{H} \equiv 0$. Since the choice of $X$ is arbitrary, it follows that the vector $\times$

$$\nu_{\Sigma}(p) := \sum_{j \in J(p) \subset I} \nu_{\Sigma_j}(p) = 0 \quad \text{almost everywhere on } S \setminus \Gamma,$$

which we call the balancing of $\nu_{\Sigma_i}$ along the singular curves of $\Sigma$, away from $\Gamma$.

The strong stationarity condition of a varifold with respect to $\Gamma$ is equivalent to the existence of an $H^1$-measurable normal (to $\Gamma$) vector field $\nu$ on $\Gamma$ with $\sup |\nu| \leq 1$ such that $\times$

$$\int_{\Sigma} \text{div}_\Sigma X \, dA = \int_{\Gamma} \langle X, \nu \rangle ds$$

for all smooth vector fields $X$ on $\mathbb{R}^n$ (see section 7 of [EWW]). Note that since $X$ is an ambient vector field along $\Sigma$, $\text{div}_\Sigma X$ is the trace on $\Sigma$ of the ambient covarant derivative of $X$.

In our context, that is, when $\Sigma$ is in $S$, the $H^1$-measurable vector field $\nu = \nu_{\Sigma}$ arises as $\times$

$$\nu_{\Sigma}(p) = \sum_{j \in J(p)} \nu_{\Sigma_j}(p)$$
for each $p \in \Gamma$, where $j \in J(p)$ whenever $p \in \overline{\Sigma}_j$.

First we define a surface $\Sigma$ in $S_\Gamma$ to be *locally minimizing relative to* $\Gamma$ at $p$ if for a neighborhood $U$ of $p$, there exists a smaller neighborhood $V$ of $p$ such that for any $\tilde{\Sigma}$, if $\tilde{\Sigma}\setminus V = \Sigma\setminus V$ and $\partial \tilde{\Sigma} = \Gamma$, then $\text{Area}(\tilde{\Sigma}) \geq \text{Area}(\Sigma)$. We are particularly interested in the case $p$ is a point on $\Gamma$.

For intuition, it is useful to understand the relation between strong stationarity and the local minimizing property within the class $S_\Gamma$. First define an $\mathcal{H}^1$ measurable vector field $\nu_\Sigma$ defined on $\Gamma$, as in (5) above. The following proposition may be proved using well-known methods of the calculus of variations.

**Proposition 1.** Suppose that $\Sigma$ is a surface in $S_\Gamma$. Then $\Sigma$ is locally minimizing relative to $\Gamma$ at each point of $\Gamma$ if and only if $|\nu_\Sigma| \leq 1$ $\mathcal{H}^1$-almost everywhere on $\Gamma$, $\nu_\Sigma = 0$ $\mathcal{H}^1$-almost everywhere on a neighborhood of $\Gamma$ in $S\setminus \Gamma$ and the regular parts of $\Sigma$ have vanishing mean curvature vector $\vec{H}$ in some neighborhood of $\Gamma$.

This proposition says that within the class $S_\Gamma$, the local minimizing property relative to $\Gamma$ and stationarity away from $\Gamma$ imply strong stationarity with respect to $\Gamma$. We remark here that strong stationarity is strictly weaker than the locally area minimizing condition. In particular, there are surfaces which are strongly stationary but not locally area minimizing at certain interior points. One such example is the cone $\Sigma \subset \mathbb{R}^3$ spanned by the 1-skeleton $\Gamma$ of a cube, with its vertex at the center of the cube. It is strongly stationary relative to $\Gamma$, but is not locally minimizing at the cone vertex. Namely, there exists a one parameter family of polyhedral surfaces of strictly smaller area, in which a neighborhood of the vertex at the center is replaced by the 2-skeleton of a small cube; the variation is supported in an arbitrarily small neighborhood of $p$ $[\Gamma]$.

Next we introduce the following definition, which, for surfaces in the class $S_\Gamma$, allows us to isolate the two independent parts of the strong stationarity condition. In fact, strong stationarity for surfaces in the class $S_\Gamma$ is equivalent to stationarity in $\mathbb{R}^n\setminus \Gamma$ plus the following boundary condition.
**Definition 2.** \( \Gamma \) is said to be a variational boundary of a surface \( \Sigma \) if there exists an \( \mathcal{H}^1 \) measurable vector field \( \nu_{\Sigma} \) along \( \Gamma \) which is orthogonal to \( \Gamma \), with \( |\nu_{\Sigma}| \leq 1 \) a.e., such that for all smooth vector fields \( X \) defined on \( \mathbb{R}^n \),
\[
\int_{\Sigma} \text{div}_{\Sigma} X^T \, dA = \int_{\Gamma} \langle X, \nu_{\Sigma} \rangle \, ds.
\]

Observe that Definition 1 of strong stationarity refers to ambient derivatives of \( X \), in contrast with Definition 2, which is intrinsic to \( \Sigma \).

Now we are ready to state and prove the main result of this section.

**Theorem 1.** Given a strongly stationary surface \( \Sigma \) in \( S_\Gamma \), and a point \( p \) of \( \Sigma \setminus \Gamma \), let \( C_p(\Gamma) \) be the cone spanned by \( \Gamma \) with its vertex at \( p \). Then we have the following inequality:
\[
\Theta_{\Sigma}(p) < \Theta_{C_p(\Gamma)}(p),
\]
unless \( \Sigma \) is a cone over \( p \) with planar faces, in which case we have equality.

**Proof.** Let \( G(x) \) be the test function \( \log \rho(x) \), where \( \rho(x) = |x - p| \). \( G(x) \) is the Green’s function for the Laplace operator defined on two-dimensional subspaces of \( \mathbb{R}^n \) which contain the point \( p \). On the other hand, on a minimal surface in \( \mathbb{R}^n \), the function \( G(x) \) is subharmonic, as a consequence of the trace formula:
\[
\triangle_{\Sigma} G = \sum_{\alpha=1}^{2} \nabla^2 G(e_\alpha, e_\alpha) + dG(\vec{H}),
\]
where \( \nabla \) is the covariant derivative for the ambient manifold \( \mathbb{R}^n \) (see [CG1]). Thus, we have the following integral estimate:
\[
0 \leq \int_{\Sigma \setminus B_\epsilon(p)} \triangle_{\Sigma_i} G \, dA = \int_{\partial(\Sigma \setminus B_\epsilon(p))} \frac{1}{\rho} \frac{\partial \rho}{\partial n_{\Sigma_i}} \, ds
\]
for each \( i \), where \( \Sigma = \bigcup_{i \in I} \Sigma_i \) is a surface in the class \( S_\Gamma \). The equality is due to the divergence theorem. Note that each boundary \( \partial(\Sigma \setminus B_\epsilon(p)) \) consists of three parts:
\[
\partial(\Sigma \setminus B_\epsilon(p)) = \left( \partial \Sigma_i \cap \Gamma \right) \cup \left( \partial B_\epsilon(p) \cap \Sigma_i \right) \cup \left( \partial \Sigma_i \cap (S \setminus \Gamma) \right),
\]
since $S = \cup \partial \Sigma_i$. Now we sum the inequality above over $i$ and reorganize the boundary terms:

$$0 \leq \int_{\Gamma} \frac{1}{\rho} \frac{\partial \rho}{\partial \nu_{\Sigma}} \, ds + \int_{\partial B_{\varepsilon}(p) \cap \Sigma} \frac{1}{\rho} \frac{\partial \rho}{\partial \nu_{\Sigma}} \, ds + \sum_{i \in I} \int_{\partial \Sigma_i \cap (S \setminus \Gamma)} \frac{1}{\rho} \frac{\partial \rho}{\partial \nu_{\Sigma_i}} \, ds,$$

where $\nu_{\Sigma}$ is as in equation (5). The last term vanishes, since we have the balancing condition among the unit vectors $\nu_{\Sigma_i}$ normal to the edges of $\Sigma_i \cap (S \setminus \Gamma)$, and tangent to $\Sigma_i$, pointing outward of $\Sigma_i$, as a consequence of the (interior) stationarity (3) of $\Sigma$:

$$\sum_{j \in J(p)} \frac{\partial \rho}{\partial \nu_{\Sigma_j}} = \left\langle \nabla \rho, \sum_{j \in J(p)} \nu_{\Sigma_j} \right\rangle = 0,$$

for each $p \in S \setminus \Gamma$, where $J(p)$ is the collection of $j \in I$ with $p \in \Sigma_j$.

As for the second term, note that as $\varepsilon$ goes to zero, $\frac{\partial \rho}{\partial \nu_{\Sigma_i}}$ approaches $-1$ uniformly, and hence

$$\int_{\partial B_{\varepsilon}(p) \cap \Sigma} \frac{1}{\rho} \frac{\partial \rho}{\partial \nu_{\Sigma}} \, ds$$

converges to

$$\lim_{\varepsilon \to 0} \left( -\frac{1}{\varepsilon} \right) \text{Length}(\Sigma \cap \partial B_{\varepsilon}(p)) = -2\pi \Theta_{\Sigma}(p).$$

Therefore we have obtained the following upper bound for the area density of $\Sigma$ at $p$:

$$2\pi \Theta_{\Sigma}(p) \leq \int_{\Gamma} \frac{1}{\rho} \frac{\partial \rho}{\partial \nu_{\Sigma}} \, ds.$$  

(7)

We repeat the argument for the surface Laplacian of $G(x)$, this time replacing $\Sigma$ with the cone $C_p(\Gamma)$ spanned by $\Gamma$ with vertex $p$. Recall that $\Gamma = \cup_j a_j$ where each arc $a_j$ is $C^2$-regular, $C^1$ up to the end points. Denote by $A_j$ the cone $C_p(a_j)$ spanned by $a_j$ with its vertex at $p$. Thus the cone $C_p(\Gamma)$ is the union of all the fans $A_j = A_j \cup \partial A_j$. Observe using (6) that away from the vertex $p$, $G(x)$ is harmonic on $A_j$. Hence we have

$$0 = \int_{A_j \setminus B_{\varepsilon}(p)} \triangle C G(x) \, dA = \int_{\partial(A_j \setminus B_{\varepsilon}(p))} \frac{1}{\rho} \frac{\partial \rho}{\partial \nu_{C}} \, ds.$$
As seen above for $\Sigma$, each boundary $\partial(A_j \setminus B_\varepsilon(p))$ consists of three parts; we sum the equation above over $j$ and reorganize the boundary terms, and find:

$$0 = \int_{\Gamma} \frac{1}{\rho} \frac{\partial \rho}{\partial \nu_C} \, ds + \int_{\partial B_\varepsilon(p) \cap C} \frac{1}{\rho} \frac{\partial \rho}{\partial \nu_C} \, ds + \sum_j \int_{C_p(\partial a_j) \setminus B_\varepsilon(p)} \frac{1}{\rho} \frac{\partial \rho}{\partial \nu_{A_j}} \, ds,$$

where $\nu_C = \nu_{C_p(\Gamma)}$ is defined to be $\sum_j \nu_{A_j}$, with $\nu_{A_j}$ being the unit vector normal to the boundary $\partial A_j$, and tangent to the fan $A_j$, pointing out of $A_j$.

The last term vanishes since the vector $\nu_{A_j}$ and $\nabla \rho$ are perpendicular, which makes $\partial \rho / \partial \nu_{A_j}$ identically zero on $C_p(\partial a_j)$. The second term is equal to $-\text{Length}(C_p(\Gamma) \cap \partial B_\varepsilon(p)) / \varepsilon$ which in turn is equal to $-2\pi \Theta_C(p)$, independent of sufficiently small $\varepsilon > 0$. Therefore we have obtained $\times$

$$2\pi \Theta_C(p) = \int_{\Gamma} \frac{1}{\rho} \frac{\partial \rho}{\partial \nu_C} \, ds.$$  

(8)

Now observe that $\nu_C$ is the unit vector normal to $\Gamma$ most closely aligned with the gradient of $\rho$ along $\Gamma$, while $\nu_{\Sigma}$ is normal to $\Gamma$ with $|\nu_{\Sigma}| \leq 1$, since $\Gamma$ is a variational boundary of $\Sigma$. Hence we have the following inequality: $\times$

$$\frac{\partial \rho}{\partial \nu_C} \geq \frac{\partial \rho}{\partial \nu_{\Sigma}}$$

almost everywhere along $\Gamma$. By integrating, we have

$$\int_{\Gamma} \frac{1}{\rho} \frac{\partial \rho}{\partial \nu_C} \, ds \geq \int_{\Gamma} \frac{1}{\rho} \frac{\partial \rho}{\partial \nu_{\Sigma}} \, ds$$

Combining the inequalities (7), (9) and the equality (8), we finally get $\times$

$$2\pi \Theta_{\Sigma}(p) \leq \int_{\Gamma} \frac{1}{\rho} \frac{\partial \rho}{\partial \nu_{\Sigma}} \, ds \leq \int_{\Gamma} \frac{1}{\rho} \frac{\partial \rho}{\partial \nu_C} \, ds = 2\pi \Theta_C(p).$$

(10)

If equality occurs in (10), then $\Delta_{\Sigma} G \equiv 0$, and the trace formula (6), along with a computation of $\nabla^2 G$, implies that $\nabla \rho$ is tangent to $\Sigma$. Thus each two-dimensional face of $\Sigma$ is both a regular minimal surface and a stationary cone in $\mathbb{R}^n$, and therefore is part of a plane passing through $p$. $\blacksquare$
3. Total Curvatures of Graphs

Let $\Gamma$ be a graph in $\mathbb{R}^n$, consisting of immersed arcs $a_1, a_2, \ldots, a_n$, which are $C^2$ in the interior and $C^1$ up to the vertices $q_1, q_2, \ldots, q_m$. Recall the definition (2) of total curvature $C_{\text{tot}}(\Gamma)$ of a graph $\Gamma$. The definition (1) of $\text{tc}(q)$ for a vertex $q$ of a graph in a manifold is equivalent to the following for a graph in $\mathbb{R}^n$:

**Definition 3.** If $q$ is a vertex of valence $d$ of a graph $\Gamma \subset \mathbb{R}^n$, define the contribution at $q$ to the total curvature of $\Gamma$ as

$$\text{tc}(q) := \sup_{p \in \mathbb{R}^n} \sum_{\ell=1}^{d} \left( \frac{\pi}{2} - \beta_\ell(p) \right)$$

where $\beta_1(p), \ldots, \beta_d(p)$ are the interior angles at $q$ which the $d$ edges of $\Gamma$ make with the line segment from $p$.

The usefulness of these definitions will become clear in section 4 below; see esp. Theorem 2.

It might be noted that even though the geodesic curvature in $\Sigma$ at a smooth point of $\Gamma$ is given by the tangential component of the curvature vector of $\Gamma$, there is no such appropriate vector at a vertex. This is true already at a vertex of degree $d = 2$, that is, for a piecewise smooth Jordan curve.

In this section, we shall collect some observations about $C_{\text{tot}}(\Gamma)$ for specific cases of a graph $\Gamma \subset \mathbb{R}^n$. These will be used for the examples below, but will not be referred to in the proofs of the theorems. As those results are elementary, and some of them previously known, we include brief proofs for the sake of completeness (see [MY] and references therein for more general discussion on minimal network problems.)

Consider a vertex $q$ of $\Gamma$ of valence $d$, and let $T_1, \ldots, T_d$ be the unit tangent vectors to $\Gamma$ at $q$. For a given point $p \in \mathbb{R}^n$, as in Definition 3, we may write $\beta_\ell(p)$ for the angle between $T_\ell$ and the line segment from $q$ to $p$. We shall also (by abuse of notation: compare equation (1)) write this angle as $\beta_\ell(e)$, where $e$ is the unit vector $\frac{p-q}{|p-q|}$. We write $e = e_0 \in S^2$ for a point.
where the sum $\sum_{\ell=1}^{d} \left( \frac{\pi}{2} - \beta_{\ell}(e) \right)$ assumes its maximum value $tc(q)$. Since $e_0$ is also the minimizer of $\sum_{\ell=1}^{d} \beta_{\ell}(e)$, it is the spherical Steiner point of $T_1, \ldots, T_d$. Note that the existence of $e_0$ follows from compactness of $S^2$.

3.1. Valence three.

**Proposition 2.** For all $T_1, T_2$ and $T_3 \in S^2$, there exists $e \in \{T_1, T_2, T_3\}$ so that $\beta_1(e) + \beta_2(e) + \beta_3(e) \leq 4\pi/3$.

**Proof.** $T_1, T_2$ and $T_3$ lie in a small (or great) circle $\gamma$ of $S^2$. Each spherical distance $d(T_i, T_{i+1})$ ($i = 1, 2, 3$ mod 3) is less than (or equal to) the length of the smaller arc of $\gamma$ between $T_i$ and $T_{i+1}$, so their sum is at most the length of $\gamma$, hence $\leq 2\pi$. Renumber $T_1, T_2, T_3$ so that $d(T_2, T_3)$ is the largest of the three distances, and choose $e = T_1$. Then $\beta_1(e) = 0$, while $\beta_2(e), \beta_3(e) \leq \frac{2\pi}{3}$.

**Corollary 1.** For any vertex $q$ of valence $d = 3$, $tc(q) \geq \pi/6$, with equality if and only if the three unit tangent vectors $T_1, T_2$ and $T_3$ at $q$ are balanced: $T_1 + T_2 + T_3 = 0$.

**Proof.** By Proposition 2, $\sup_{e} \sum_{\ell=1}^{3} \left( \frac{\pi}{2} - \beta_{\ell}(e) \right) \geq \frac{3\pi}{2} - \inf_{i} \sum_{\ell=1}^{3} \beta_{\ell}(T_i) \geq \frac{\pi}{6}$.

Now suppose that $tc(q) = \pi/6$. As in the proof of Proposition 2, the unit tangent vectors $T_1, T_2, T_3$ lie on a circle $\gamma \subset S^2$. But $\beta_2(T_1) + \beta_3(T_1) = \sum_{\ell=1}^{3} \beta_{\ell}(T_1) \geq \sum_{\ell=1}^{3} \beta_{\ell}(e_0) = \frac{3\pi}{2} - tc(q) = \frac{4\pi}{3}$, while $d(T_2, T_3) \geq \beta_{\ell}(T_1)$, $\ell = 2, 3$, which implies that $\gamma$ has length $2\pi$. Thus $\gamma$ is a great circle and all of the $d(T_1, T_{i+1}) = \frac{2\pi}{3}$.

In specific situations, it is of interest to compute $tc(q)$ exactly, or even to identify the spherical Steiner point $e_0$. The following lemma is not difficult to prove, using the first variation of the sum of distances on $S^2$.

**Lemma 1.** Suppose a vertex $q$ of $\Gamma$ has valence three, with unit tangent vectors $T_1, T_2, T_3$ to $\Gamma$ at $q$. Let $e_0$ be a Steiner point for $T_1, T_2, T_3$. For $\ell = 1, 2, 3$ choose a minimizing geodesic (great circle) in $S^2$ from $e_0$ to $T_\ell$, and let $\xi_{\ell} \in T_{e_0}S^2$ be the unit tangent vector at $e_0$ to the geodesic. Then
either (1) \( \xi_1 + \xi_2 + \xi_3 = 0 \), that is, the geodesics make equal angles \( 2\pi/3 \) at \( e_0 \); or (2) \( e_0 = T_\ell \) for some \( \ell = 1, 2, 3 \), and the remaining two vectors \( \xi_{\ell+1}, \xi_{\ell+2} \) form an angle \( \geq 2\pi/3 \) (subscripts modulo 3).

For equilateral spherical triangles, one might expect the Steiner point \( e_0 \) of the vertices to be the center of the triangle; however, if the triangle is too large, \( e_0 \) can only be one of the corners of the triangle:

**Corollary 2.** If the vertex \( q \) of \( \Gamma \) has valence 3 and its unit tangent vectors \( T_1, T_2, T_3 \) make equal angles with each other, then

\[
\text{tc}(q) = \begin{cases} 
3 \left( \frac{\pi}{2} - \beta \right) & \text{if } \beta \leq R_0, \\
\frac{3\pi}{2} - 4 \sin^{-1}(\frac{1}{2}\sqrt{3} \sin \beta) & \text{if } \beta \geq R_0;
\end{cases}
\]

where \( 0 \leq \beta \leq \pi/2 \) is the circumradius, the common spherical distance from \( T_\ell \) to the closer center \( N \), of the triangle formed by \( T_1, T_2, T_3 \); and where \( R_0 \approx 1.33458 \) radians is the value of \( \beta \) which makes the two options in formula (11) equal.

**Proof.** It follows from Lemma 1 that a minimizer of \( \sum \beta_\ell \) must be one of the five points \( N, \ -N, T_1, T_2, T_3 \). But \( \sum \beta_\ell(-N) \geq \sum \beta_\ell(N) = 3\beta \), and \( \sum \beta_\ell(T_\ell) = 4s, \ i = 1, 2, 3 \), where \( 2s \) is the side of the equilateral triangle: \( \sin s = \sin \beta \sin(\pi/3) \). But \( 3\beta - 4s \) has the same sign as \( \beta - R_0 \).

### 3.2. Even valence.

**Proposition 3.** If \( T_1, T_2, T_3 \) and \( T_4 \) are points on \( S^2 \), then any of the Steiner points \( e_0 \) must be one of the \( T_\ell \) or one of the six (or more) points of intersection of the two great circles passing through disjoint pairs of the four points \( T_\ell \).

The proof of Proposition 3 will be immediate from the following lemma.

**Lemma 2.** Let \( e_0 \) be a Steiner point for \( T_1, T_2, T_3, T_4 \in S^2 \), and write \( \xi_\ell \in T_{e_0}S^2 \) for the initial unit tangent vector to the minimizing geodesic from \( e_0 \) to \( T_\ell \). If \( e_0 \) is not equal to any of the \( T_\ell \), then after reindexing \( \xi_1, \xi_2, \xi_3, \xi_4 \)
in circular order around the unit circle of $T_{00}S^2$, we have $\xi_1 = -\xi_3$ and $\xi_2 = -\xi_4$.

Proof. We compute the first variation of $\sum_{\ell=1}^{4} \beta_\ell(e)$, and find that $0 = -\sum_{\ell=1}^{4} \langle \xi_\ell, \xi \rangle$ for any $\xi \in T_{00}S^2$. We conclude that the $\xi_\ell$ are balanced: $\star$ (12) $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0$.

Write $\eta_\ell$ for the oriented angle from $\xi_\ell$ to $\xi_{\ell+1}$, $\ell$ modulo 4, with $0 \leq \eta_\ell \leq 2\pi$.

If $\xi_1 = -\xi_3$, then also $\xi_2 = -\xi_4$ according to (12), and we are done. Otherwise, the sum $\xi_1 + \xi_3$ makes the oriented angle $\frac{1}{2}(\eta_1 + \eta_2)$ modulo $\pi$ with $\xi_1$, while the sum $\xi_2 + \xi_4$ makes the angle $\frac{1}{2}(\eta_2 + \eta_3)$ modulo $\pi$ with $\xi_2$. But $\xi_2 + \xi_4 = -(\xi_1 + \xi_3)$, hence $\eta_1 + \frac{1}{2}(\eta_2 + \eta_3) = \frac{1}{2}(\eta_1 + \eta_2) + \pi$ modulo $\pi$, implying that $\eta_1 + \eta_3 = 0$ modulo $2\pi$. But $\eta_1 + \eta_2 + \eta_3 + \eta_4 = 2\pi$ and $\eta_\ell \geq 0$, so this forces either $\eta_1 = \eta_3 = 0$, implying $\xi_1 = \xi_2$ and $\xi_3 = \xi_4$; or $\eta_2 = \eta_4 = 0$, implying $\xi_2 = \xi_3$ and $\xi_4 = \xi_1$. The conclusion now follows from equation (12) in this case as well.

The following lemma has a complex statement but a straightforward demonstration.

Lemma 3. Let $\tilde{\Gamma}$ and $\hat{\Gamma}$ be graphs with a common vertex $\tilde{q} = \hat{q}$. Write $\Gamma$ for the union of $\tilde{\Gamma}$ and $\hat{\Gamma}$, and write $q$ for the common vertex when considered as a vertex of $\Gamma$. Write $\{\tilde{T}_1, \ldots, \tilde{T}_k\}$ for the unit tangent vectors to $\tilde{\Gamma}$ at $\tilde{q}$, and let $\{\hat{T}_1, \ldots, \hat{T}_{d-k}\}$ be the unit tangent vectors to $\hat{\Gamma}$ at $\hat{q}$. Then $tc(q) \leq tc(\tilde{q}) + tc(\hat{q})$. If further $\{\tilde{T}_1, \ldots, \tilde{T}_k\}$ and $\{\hat{T}_1, \ldots, \hat{T}_{d-k}\}$ share the same Steiner point $\tilde{e}_0 = \hat{e}_0$, then the Steiner point $e_0$ of $\{\tilde{T}_1, \ldots, \tilde{T}_k, \hat{T}_1, \ldots, \hat{T}_{d-k}\}$ is equal to both, and $tc(q) = tc(\tilde{q}) + tc(\hat{q})$.

Corollary 3. If a vertex $q$ of $\Gamma$ has an even valence $d$ and the tangent vectors at $q$ occur in antipodal pairs, then $tc(q) = 0$.

Proof. Observe that a vertex $q$ of degree 2 in a straight edge, that is, with $T_2 = -T_1$, has $tc(q) = 0$, with any point of $S^2$ as a Steiner point. The conclusion then follows from Lemma 3 by induction on $d/2$. ■
In contrast with Corollary 2, even valence makes computations easier:

**Corollary 4.** For a regular polygon in $S^2$ with an even number $d$ of sides, the closer center in $S^2$ of the polygon is a Steiner point of the corners $T_1, \ldots, T_d$.

**Proof.** Let $N$ be the closer center (closer than $-N$) of the regular polygon of $d = 2k$ sides, with vertices $T_1, \ldots, T_{2k}$ in order. A Steiner point of two opposite vertices $\{T_i, T_{k+i}\}$ is any point along the minimizing geodesic arc joining them, in particular the midpoint $N$. Now apply Lemma 3 via induction on $k$.

**Proposition 4.** For a vertex $q$ of a graph $\Gamma \subset \mathbb{R}^3$ with unit tangent vectors $T_1, \ldots, T_d$ all lying in a plane through $0$ and making equal angles, an orthogonal unit vector $N$ is a Steiner point if and only if $d$ is even.

**Proof.** If $d$ is even, the conclusion is given by Corollary 4. If $d = 2k + 1$ is odd, then the sum $\sum_{\ell=1}^{d} \beta(\ell(e))$ equals $(2k + 1)\pi/2$ for $e = N$, and equals $(1 + 2 + \cdots + k)4\pi/(2k + 1)$ for $e = T_1$, which is smaller by a difference of $\frac{\pi}{2(2k+1)}$. Thus $N$ cannot be the Steiner point.

### 4. Gauss-Bonnet Formula for Cones

In this section we will prove a Gauss-Bonnet formula for two dimensional cones in $\mathbb{R}^n$. First we quote the following classical result.

**Euler’s Theorem** (see [O]) *For a connected graph $\Gamma'$ with even valence at each vertex, there is a continuous mapping of the circle to $\Gamma'$ which traverses each edge exactly once.*

An immediate consequence of this result is that any connected finite graph $\Gamma$ has a continuous mapping of the circle which traverses each edge exactly twice. Namely, we may apply Euler’s theorem to the graph $\Gamma'$ obtained from $\Gamma$ by doubling each edge and leaving the vertices alone. Note that the new graph $\Gamma'$ has even valence at each vertex.
We shall derive the density formula of Theorem 2 below in three steps, beginning from a well known case.

Suppose first that $\Gamma_0$ is a smooth closed curve in $\mathbb{R}^n$, not necessarily simple, and $p$ a point not on $\Gamma_0$. Without loss of generality (after a suitable scaling centered at $p$), we may assume that $\Gamma_0$ lies outside the unit ball $B_1(p)$ centered at $p$.

Define $\Pi_p$ to be the radial projection to the unit sphere centered at $p$:

$$\Pi_p : \mathbb{R}^n \setminus \{p\} \to \partial B_1(p);$$

$$\Pi_p(x) = p + \frac{x - p}{|x - p|}.$$

Let $A = C_p(\Gamma_0) \setminus B_1(p)$ be the annular region between $\Gamma_0$ and $\Pi_p \Gamma_0$. By the Gauss-Bonnet formula, we have

$$\int_{\partial A} k \cdot \nu_C \, ds + \int_A K \, dA = 2\pi \chi(A) \tag{13}$$

where $k$ is the curvature vector of the graph $\partial A$ in $\mathbb{R}^n$, $\nu_C$ is the outward normal to $\partial A$, $K$ is the Gauss curvature of $A$, and $\chi(A)$ is the Euler characteristic of $A$. For $A$, $K \equiv 0$ and $\chi(A) = 0$. Hence

$$0 = \int_{\partial A} k \cdot \nu_C \, ds$$

$$= \int_{\Pi_p \Gamma_0} \tilde{k} \cdot \nu_C \, ds + \int_{\Gamma_0} \tilde{k} \cdot \nu_C \, ds.$$

For $q \in \Pi_p \Gamma_0$, $\tilde{k}(q)$ is the unit vector from $q$ to $p$, so that the first integral on the last line is equal to the length of $\Pi_p \Gamma_0$, which is also equal to $2\pi \Theta_{C_p(\Gamma_0)}(p)$. Therefore we have for the cone $C_p(\Gamma_0)$ the following equation: $\times$

$$2\pi \Theta_{C_p(\Gamma_0)}(p) = \text{Length}(\Pi_p \Gamma_0) = -\int_{\Gamma_0} \tilde{k} \cdot \nu_C \, ds, \tag{14}$$

where $\nu_C(q)$ is the unit normal vector to $\Gamma_0$ in the plane spanned by the tangent vector at $q$ and the vector $p - q$, and pointing away from the cone vertex $p$. Note that $C_p \setminus \{p\}$ is flat with respect to the induced metric, that is,
locally isometric to $\mathbb{R}^2$. Note further that the integrand $\vec{k} \cdot \nu_C$ is the intrinsic geodesic curvature of $\Gamma_0$ considered as a locally embedded curve in $C_p$.

Next, when $\Gamma'$ is a piecewise smooth immersion of the circle, we generalize the formula above as follows. Let $\Gamma'$ be a union of smooth segments $a_i$, each of which is $C^2$ in the interior and $C^1$ up to the end points $q_{i,0}, q_{i,1}$. We denote $q_{i,j} \sim q_{i',j'}$ if they represent the same point where $a_i$ and $a_{i'}$ meet. Then the cone $C_p(\Gamma')$ can be thought of as a union of fans $A_i(p) = C_p(a_i)$, which is the part of the cone $C_p(\Gamma')$ spanned by $a_i$, with radial edges $pq_{i,0}$ and $pq_{i,1}$. The right hand side of the equation (14) then generalizes as

$$2\pi \Theta_{C_p(\Gamma')}(p) = \text{Length}(\Pi_p \Gamma') = -\sum_i \int_{a_i} \vec{k} \cdot \nu_C \, ds + \sum_{i,j=1,2} \left( \frac{\pi}{2} - \beta_{ij}^k \right)$$

where $\beta_{ij}^k$ is the angle between $a_i$ and $\overrightarrow{pq_{i,j}}$ as they meet at $q_{i,j}$. To see how the last term arises, suppose now that $a_i$ and $a_k$ are the consecutive edges in $\Gamma'$ joined at $q_{i,j} \sim q_{k,j'}$. Then the quantity $(\pi/2 - \beta_{ij}^k) + (\pi/2 - \beta_{j'}^{k'}) = \pi - (\beta_{ij}^k + \beta_{j'}^{k'})$ is the amount the curve $a_i \cup a_k$ turns at $q_{i,j} \sim q_{k,j'}$, when considered as a locally isometrically embedded curve in $\mathbb{R}^2$.

Finally, coming back to the original graph $\Gamma$, Euler’s theorem says that the graph $\Gamma$ with each edge traced twice while its vertices are left intact, which we denoted by $\Gamma'$, can be parameterized by a copy of $S^1$. Write $\Gamma'$ as the union of $a_i'$ where each $a_k$ ($k = 1, \ldots, n$) arises twice as $a_i'$ ($i = 1, \ldots, 2n$), as one goes around $\Gamma'$ once.

Applying the generalized equation (15) when $\Gamma'$ is $\cup_{i=1}^{2n} a_i'$, we obtain the following description of the density of the cone $C_p(\Gamma)$ at $p$.

**Theorem 2.** With the notations as above we have the following.

$$2\pi \Theta_{C_p(\Gamma)}(p) = -\sum_{k=1}^{n} \int_{a_k} \vec{k} \cdot \nu_C \, ds + \sum_{k=1}^{n} \sum_{j=1,2} \left( \frac{\pi}{2} - \beta_{ij}^k \right).$$
Proof. From the preceding discussion, we have

\begin{equation}
2\pi \Theta_{C_p(\Gamma')}(p) = \text{Length}(\Pi_p\Gamma') = -\sum_{i=1}^{2n} \int_{a_i'} \vec{k} \cdot \nu_C \, ds + \sum_{i=1}^{2n} \sum_{j=1,2} \left( \frac{\pi}{2} - \beta_{ij} \right).
\end{equation}

Note that the length of $\Gamma'$ is twice the length of $\Gamma$. Also note that when the edges $a_{i_1}'$ and $a_{i_2}'$ of $\Gamma'$ represent the same edge $a_k$ of $\Gamma$, we have

\begin{equation}
\int_{a_k} \vec{k} \cdot \nu_C \, ds = \int_{a_{i_1}'} \vec{k} \cdot \nu_C \, ds = \int_{a_{i_2}'} \vec{k} \cdot \nu_C \, ds
\end{equation}

independent of the orientations imposed by the Euler circuit. Lastly, over the whole circuit $\Gamma'$, the quantity $\pi/2 - \beta_{ij}$, $(i = 1, \ldots, n; \ j = 1, 2)$ appears twice. The statement of the theorem then follows by dividing both sides of the equation (17) by two.

5. Regularity of Stationary Surfaces

Using the notations from section 2 above, we have the following immediate consequence to (1) the density comparison (Theorem 1) between the area density of a strongly stationary surface $\Sigma$ with respect to $\Gamma$ and that of the cone $C_p(\Gamma)$ over $\Gamma$ with vertex $p$; and (2) the Gauss-Bonnet formula (Theorem 2), which estimates the density of the cone in terms of the total curvature of the graph $\Gamma$:

Corollary 5. The following inequality holds between the area density of a strongly stationary surface $\Sigma$ and the total curvature $C_{\text{tot}}$ of $\Gamma$:

\[ 2\pi \Theta_{\Sigma}(p) \leq C_{\text{tot}}(\Gamma). \]

Proof. We need only observe that in the conclusion of Theorem 2 the right-hand side of equation (16) is bounded above by $C_{\text{tot}}(\Gamma)$.
Proof. At a point $p$ on $\Sigma$, the proof of the above Corollary 5 to the Gauss-Bonnet formula says that

$$\Theta_\Sigma(p) \leq \Theta_{C_\mu(\Gamma)}(p) \leq \frac{1}{2\pi} C_{\text{tot}}(\Gamma) \leq C_Y,$$

where the last inequality is the hypothesis. If $\Theta_\Sigma(p) < C_Y$, we claim that $\Sigma$ is regular at $p$ by the proof of Theorem 7.1 of [EWW]. For the sake of completeness, we reproduce their argument here.

Let $T_p\Sigma$ be the tangent cone at $p$, whose existence and uniqueness is guaranteed by the regularity assumption we impose on the class of surfaces $S_\Gamma$. Then $\Theta_{T_p\Sigma}(x) < \frac{3}{2}$ for all $x$ in the cone since in any minimal cone, the highest density occurs at the vertex. This is because the density function $\Theta_{T_p\Sigma}(x)$ is upper semi-continuous ([Si], §17.8) and constant along radial lines. Now the intersection of $T_p\Sigma$ with the unit sphere is a collection of geodesic arcs, which means that the cone is a polyhedron. At most two faces of the polyhedron $T_p\Sigma$ can meet along a radial edge, since otherwise the density at points along the edge would be $\geq \frac{3}{2}$. This means $T_p\Sigma \cap S^{n-1}$ is a union of complete great circles. Since the density is $< \frac{3}{2}$, there is only one great circle and it has multiplicity 1. By Allard’s regularity theorem ([Al] or [Si]), this means that $\Sigma$ is regular at $p$.

On the other hand, if $\Theta_\Sigma(p) = C_Y$, then equality holds in Theorem 1, implying that $\Sigma$ itself is a cone with vertex $p$ and planar faces. But the $Y$ cone is the unique (up to rotation in $\mathbb{R}^n$) stationary cone having density $3/2$.

As seen above, $3/2$ is the first nontrivial upper bound for the area density above 1, for the class of surfaces we are studying. As for a larger upper bound, we will restrict our attention to the case when the ambient Euclidean space is $\mathbb{R}^3$. There are exactly ten stationary cones in $\mathbb{R}^3$ [AT], where a cone is stationary when its intersection with the unit sphere is a net of geodesics meeting in threes at $120^\circ$. Ordered with respect to the area density $\Theta$ at the vertices of the cones, the first three on the list are the plane with $\Theta = 1$; $Y = \text{three half-planes meeting at } 120^\circ$ with $\Theta = C_Y = 3/2$; and the cone
T spanned by the regular tetrahedron with \( \Theta = C_T = 6 \cos^{-1}(-1/3) \approx 11.4638 \).

In order to state the next result, we need to introduce the following definition \([\text{Alm}]\).

**Definition 4.** Let \( \varepsilon \) be a bound of the form \( \varepsilon(r) = Cr^\alpha \) for some \( \alpha > 0 \), and choose \( \delta > 0 \). We define \( \Sigma \subset R^n \) to be an \((M, \varepsilon, \delta)\)-minimal set with respect to \( \Gamma \subset R^n \) if \( \Sigma \) is 2-rectifiable and if, for every Lipschitz mapping \( \Phi : R^n \to R^n \) with the diameter of the support \( W \) of \( \Phi - \text{id} \) less than \( \delta \),

\[
\mathcal{H}^2(S \cap W) \leq (1 + \varepsilon(r)) \mathcal{H}^2(\Phi(S \cap W)).
\]

We have the following regularity statement in \( R^3 \) for \( \Gamma \) with small total curvature.

**Theorem 4.** Suppose \( \Gamma \) is a graph in \( R^3 \) with \( C_{\text{tot}}(\Gamma) \leq 2\pi C_T \), and let \( \Sigma \) be an \((M, 0, \delta)\)-minimal surface with \( \Gamma \) as its variational boundary in \( S_\Gamma \). Then \( \Sigma \) is a surface with possibly \( Y \) singularities but no other singularities, unless it is a subset of the \( T \) stationary cone, with planar faces.

**Proof.** As in the proof of the previous theorem, for each point \( p \) in \( \Sigma \), we have a series of inequalities

\[
\Theta_\Sigma(p) < \Theta_{C_\mu(\Gamma)}(p) \leq \frac{1}{2\pi} C_{\text{tot}}(\Gamma) \leq C_T,
\]

unless \( \Sigma \) is a cone over \( p \) with planar faces. We now use results in \([\text{T}](\text{II.2 and II.3}), \) which imply that the tangent cone of an \((M, 0, \delta)\)-minimal set \( S \) at \( p \) is area-minimizing with respect to the intersection with the unit sphere centered at \( p \), and that the plane, the \( Y \)-cone and the \( T \)-cone are the only possibilities for the tangent cone. The inequality above implies that the tangent cone \( T_p \Sigma \) can only be the plane or the \( Y \) singularity, since all other stationary singular cones have higher density. If there is a point \( p \) where the tangent cone to \( \Sigma \) is any other cone than the plane or \( Y \), then it can only be the \( T \) stationary cone. But in this case, \( \Theta_\Sigma(p) = C_T \), and \( \Sigma \) itself is a cone over \( p \). It follows that \( \Sigma = T \).\[\square\]
Remark 1. A surface $\Sigma$ in the class $\mathcal{S}_{\Gamma}$ which is $(M,0,\delta)$-minimal with $\Gamma$ as its variational boundary is in particular strongly stationary with respect to $\Gamma$ (See the remark preceding Definition 2.) However note that a cone over the one-skeleton $\Gamma$ of the cube is strongly stationary w.r.t. $\Gamma$, but is not an $(M,0,\delta)$-minimal set.

Remark 2. The previous papers [EWW] and [CG2] had consequences for the knot class of a curve in a 3-dimensional manifold satisfying an inequality on its total curvature. Similar consequences for the isotropy class of a graph would follow from Theorems 3 and 4 if the boundary regularity of an area-minimizing rectifiable set bounded by a graph could be proved.

Example 1. In this example, we shall show that the hypothesis $C_{\text{tot}}(\Gamma) \leq 3\pi$ of Theorem 3 is sharp. Specifically, we shall construct a graph $\Gamma$ in $\mathbb{R}^3$ with $C_{\text{tot}}(\Gamma) = 3\pi$, such that a subset of the minimal cone $Y$, including a nonempty segment of the singular line, is strongly stationary with respect to $\Gamma$.

Recall the description of $Y$ in Section 2 above: $Y$ consists of three half-planes $P_1, P_2, P_3$ meeting along a line $S$, and making equal angles $2\pi/3$ at each point of $S$. Recall also the angle $R_0 = 1.33458$ radians $= 76.466^\circ$ of Corollary 2.

We choose two points $q^\pm$ along $S$, and construct $\Gamma$ as the union of three $C^2$ convex plane arcs $a_\ell$, where $a_\ell$ joins $q^-$ to $q^+$ in the half-plane $P_\ell$, $\ell = 1, 2, 3$, all making an angle $\alpha^\pm$ with $S$ at the endpoint $q^\pm$, where $0 < \alpha^\pm \leq R_0$. Since $a_\ell$ is a convex plane arc, the integral of $|\vec{k}|$ along $a_\ell$ equals $\alpha^+ + \alpha^-$. Using Corollary 2, we may compute that the contribution at $q^\pm$ to the total curvature of $\Gamma$ is $tc(q^\pm) = 3(\pi/2 - \alpha^\pm)$. Thus $C_{\text{tot}}(\Gamma) = 3(\alpha^+ + \alpha^-) + 3(\pi/2 - \alpha^+) + 3(\pi/2 - \alpha^-) = 3\pi$, as claimed.

In Example 1 intuition might lead the reader to expect that every case, with a skinny or fat angle, would give rise to a sharp inequality. In fact, for the case $\alpha^\pm > R_0$, the inequality is not sharp, as follows using Corollary 2.
Example 2. In this example, we shall show that the hypothesis $C_{tot}(\Gamma) \leq 2\pi C_T$ of Theorem 4 is sharp. In fact, we shall show that the cone $\Sigma$ over the one-skeleton $\Gamma$ of the regular tetrahedron itself provides an example.

Let $\alpha_T$ be the angle between an edge $\overline{q_kq_i}$ of $\Gamma$ and $\overline{q_kp}$, $1 \leq k < i \leq 4$, where $p$ is the center of the tetrahedron. Then $\cos(\alpha_T) = \sqrt{2/3}$, so $\alpha_T = 0.61548$ radians, which is less than $R_0 = 1.33458$ radians. This shows, using Corollary 2, that $C_{tot}(\Gamma) = 6\pi - 12\alpha_T$.

On the other hand, we may apply Theorem 2 above to compare the total curvature of $\Gamma$ with the density of $\Sigma$ at the interior singular point $p$. Namely, by Corollary 2, $p$ will be a Steiner point for the unit tangent vectors at each of the four vertices, and the curvature vector $\overrightarrow{k} \equiv 0$ along the regular part of $\Gamma$. In the notation of Theorem 2, all twelve of the interior angles $\beta^j_i$, $1 \leq k \leq 6$, $j = 1, 2$ are equal to $\alpha_T$. Therefore the density $2\pi C_T$ of the cone at $p$ equals

$$\sum_{k=1}^{6} \sum_{j=1,2} \left(\frac{\pi}{2} - \alpha_T\right) = 6\pi - 12\alpha_T = C_{tot}(\Gamma).$$

Example 1 illustrates that the upper bound $3\pi$ for $C_{tot}(\Gamma)$ is achieved for a non-Jordan curve $\Gamma$. The next proposition in turn says that among all the embedded graphs $\Gamma$ which are homeomorphic to the graph of Example 1, $3\pi$ is the sharp lower bound for the total curvature $C_{tot}(\Gamma)$.

Proposition 5. Let $\Gamma$ be an embedding into $\mathbb{R}^3$ of the topological graph with exactly two vertices $q^+$ and three edges $a_1$, $a_2$ and $a_3$, each of which has endpoints $q^+$ and $q^-$. Then $C_{tot}(\Gamma) \geq 3\pi$. Moreover, equality holds if and only if each $a_\ell$ is a convex plane arc with unit tangent vectors $T^\pm_\ell$ at $q^\pm$ satisfying the condition that $\pm e := \pm \frac{q^- - q^+}{|q^- - q^+|}$ is a Steiner point for the three points $T^+_1, T^+_2, T^+_3$ on $S^2$, at both $q^-$ and $q^+$.

Proof. The “if” part of the equality conclusion follows essentially from the discussion of Example 1 above. We have adapted the notation introduced there; further, let $\alpha^\pm_\ell$ be the angle between $T^\pm_\ell$ and the unit tangent vector
\[ \pm \epsilon \text{ at } q^\pm \text{ to the closed line segment } L \text{ joining } q^\pm \text{ to } q^\mp. \] Then \( a_\ell \cup L \) is a closed curve in \( \mathbb{R}^3 \), so by Fenchel’s theorem

\[
2\pi \leq C_{\text{tot}}(a_\ell \cup L) = \int_{a_\ell} |\vec{k}| \, ds + (\pi - \alpha^+_\ell) + (\pi - \alpha^-_\ell).
\]

Thus \( \int_{a_\ell} |\vec{k}| \, ds \geq \alpha^+_\ell + \alpha^-_\ell \), with equality if and only if \( a_\ell \) is a convex planar arc.

Meanwhile, \( \text{tc}(q^\pm) := \sup_p \sum_{\ell=1}^3 \left( \frac{\pi}{2} - \beta^\pm_\ell(p) \right) \geq \sum_{\ell=1}^3 \left( \frac{\pi}{2} - \alpha^\pm_\ell \right) \). Further, equality holds if and only if \( \pm \epsilon \) is a Steiner point on \( S^2 \) for the three points \( T^\pm_1, T^\pm_2, T^\pm_3 \). Therefore,

\[
C_{\text{tot}}(\Gamma) := \sum_{\ell=1}^3 \int_{a_\ell} |\vec{k}| \, ds + \text{tc}(q^+) + \text{tc}(q^-)
\]

\[
\geq \sum_{\ell=1}^3 \left[ (\alpha^+_\ell + \alpha^-_\ell) + (\frac{\pi}{2} - \alpha^+_\ell) + (\frac{\pi}{2} - \alpha^-_\ell) \right] = 3\pi,
\]

with equality if and only if \( a_\ell \) is a convex planar arc and \( \pm \epsilon \) is the Steiner point.

There is a second combinatorial structure for a connected graph \( \Gamma \) with two trivalent vertices and three edges: the “handcuff” consisting of two loops plus an arc joining the vertices of the loops. Similarly to Proposition 5, it may be shown that an embedding of such \( \Gamma \) in \( \mathbb{R}^3 \) must have total curvature at least \( 3\pi \). In fact, it appears likely that the hypothesis of Theorem 3 can hold strictly only for the embedded circle or the two-leafed rose, that is, two circles connected at a point.

The next example will be much more complex than those above.

**Example 3.** In this example, we shall construct a graph \( \Gamma \) with \( C_{\text{tot}}(\Gamma) = 44\pi < 2\pi C_T \), which is sufficiently complicated that the presence of a T-singularity in a strongly stationary surface \( \Sigma \) might appear likely without Theorem 4 above.
Let \( \Gamma \) be the union of eleven congruent (convex) plane ovals. \( \Gamma \) will consist of six horizontal copies in planes \( \{ z = c_k \} \), \( 1 \leq k \leq 6 \), obtained from each other by translation in the \( z \)-direction; and five copies in vertical planes \( \{ y = c_k \} \), \( 7 \leq k \leq 11 \), obtained from each other by translation in the \( y \)-direction. We also assume that each vertical oval meets each horizontal oval twice. For clarity, we assume that each of the eleven ovals includes two unit line segments tangent to the faces \( \{ x = 0 \} \) and \( \{ x = 1 \} \) of the unit cube. In particular, we assume \( 0 < c_1 < c_2 < \cdots < c_6 < 1 \) and \( 0 < c_7 < c_8 < \cdots < c_{11} < 1 \).

Then \( \Gamma \) has 60 vertices \( q_1, \ldots, q_{60} \), each of valence \( d = 4 \), and at each vertex, the unit tangent vectors \( T_1, T_2, T_3, T_4 \) satisfy \( T_3 = -T_1 \) and \( T_4 = -T_2 \). It follows from Corollary 3 that \( tc(q_i) = 0 \), \( 1 \leq i \leq 60 \). Each of the eleven ovals contributes \( 2\pi \) to the total curvature of \( \Gamma_{\text{reg}} \). Therefore \( C_{\text{tot}}(\Gamma) = 44\pi < 2\pi C_T \).

6. Nonzero Ambient Curvature

In this section, we shall indicate the modifications which need to be made to generalize Theorems 1, 2, 3 and 4 above to the case where the ambient space \( \mathbb{R}^n \) is replaced by a manifold \( M^n \) having variable sectional curvatures. In the case of an immersed minimal surface (or a branched immersion) with smooth boundary, the proof was carried out in \([CG2]\); the conclusions in subsection 6.2 however, are more general than those of \([CG2]\), even in the case of a Jordan curve \( \Gamma \), since \([CG2]\) requires constant curvature in the positive case. Many, although not all, of the proofs of \([CG2]\) can be adapted with little change to the present context of singular minimal surfaces which are strongly stationary with respect to a graph \( \Gamma \).

For the rest of this section, let \( M^n \) be a strongly convex Riemannian manifold having sectional curvatures bounded above by either (6.1) a non-positive constant \( -\kappa^2 \); or (6.2) a positive constant \( \kappa^2 \). \( M^n \) is said to be strongly convex if any two points are connected by a unique minimizing geodesic. For example, \( M^n \) might be a complete, simply connected Hadamard-Cartan manifold, or a convex open subset of such a complete manifold, or a...
convex open subset of a ball of radius $\pi/\kappa$ in a complete, simply connected manifold $M^n$ with sectional curvatures $K_M \leq \kappa^2$.

6.1. **Nonpositively Curved Manifold.** Let $M^n$ be a strongly convex Riemannian manifold whose sectional curvatures are bounded above by a non-positive constant $-\kappa^2$. We consider a graph $\Gamma \subset M^n$ and a surface $\Sigma$ in the class $S_\Gamma$ which is strongly stationary with respect to $\Gamma$.

Choose a point $p$ of $\Sigma$. We shall assume that $\Gamma$ is nowhere tangent to the minimizing geodesic from $p$; the general cases of Theorems 5, 6, 7 and 8 below then follow by $C^2$ approximation to $\Gamma$, via the argument on pp. 351–352 of [CG2].

We shall compare $\Sigma$ with the geodesic cone $C = C_p(\Gamma)$, which is formed from the minimizing geodesics joining $p$ to points of $\Gamma$. $C$ may naturally be given the Riemannian metric $ds^2$ induced from $M^n$. However, it should be observed that $C$ with the metric $ds^2$ is not likely to be relevant to the strongly stationary surface $\Sigma$. In fact, $\Sigma$ and the cone $C$ over its boundary inhabit different regions of $M^n$, whose geometries are not related except by an upper bound on curvatures, so that one should not expect any useful comparison between them. For these reasons, we shall endow $C$ with a second metric $\hat{d}s^2$ of constant Gauss curvature $-\kappa^2$, such that the unit-speed geodesics from $p$ to points of $\Gamma$, which generate $C = C_p(\Gamma)$, remain unit-speed geodesics in the metric $\hat{d}s^2$, and so that $\hat{d}s^2$ agrees with $ds^2$ at points of $\Gamma$. For clarity, we shall refer to the cone with this hyperbolic metric as $\hat{C} = \hat{C}_p(\Gamma)$.

More precisely, let $a_j$, $1 \leq j \leq m$, be the smooth arcs of $\Gamma$, and let $A_j = C_p(a_j)$, $1 \leq j \leq m$, be the two-dimensional fans of $C_p(\Gamma)$. On each $A_j$, let $\theta$ be a coordinate which is constant along each of the radial geodesics through $p$, and such that $\rho = \text{dist}(\cdot, p)$ and $\theta$ form a local system of coordinates. We have assumed that $\Gamma$ is nowhere tangent to the radial geodesic, which implies that $\theta$ may be used as a regular parameter along the arc $a_j$. Write $\rho =: r(\theta)$ for the corresponding values of $\rho = \text{dist}_M(p, \cdot)$ along $a_j$, and let $r(\theta)$ be extended to $C_p(\Gamma)$ so that it is constant along each radial geodesic. Then $\rho < r(\theta)$ elsewhere on $A_j$. Note that under our
assumption, there holds $|dr/d\theta| < ds/d\theta$ along $\Gamma$. We may now write the metric $d\tilde{s}^2$ on $A_j$ as

$$d\tilde{s}^2 = d\rho^2 + \left[ \left( \frac{ds}{d\theta} \right)^2 - \left( \frac{dr(\theta)}{d\theta} \right)^2 \right] \frac{\sinh^2 \kappa \rho}{\sinh^2 \kappa \rho} d\theta^2.$$

We may observe that, along any radial geodesic, we have $d\tilde{s}^2 = d\rho^2 = ds^2$. In particular, if arcs $a_j$ and $a_k$ of $\Gamma$ share a common endpoint $q$, then the hyperbolic metrics $d\tilde{s}^2$ defined on the fan $A_j$ and $d\tilde{s}^2$ defined on $A_k$ agree along their common edge, which is the minimizing geodesic from $p$ to $q$. That is, $d\tilde{s}^2$ makes $\tilde{C}$ into a Riemannian polyhedron.

**Theorem 5.** Given a strongly stationary surface $\Sigma$ in $M^n$ of class $S_\Gamma$, and a point $p$ of $\Sigma \setminus \Gamma$, the following inequality holds:

$$\Theta_{\Sigma}(p) \leq \Theta_{\tilde{C}_p(\Gamma)}(p).$$

Moreover, equality implies that $\Sigma$ is a cone with totally geodesic faces of constant Gauss curvature $-\kappa^2$.

**Proof.** The proof is similar to the proof of Theorem 1 with certain modifications. The test function $G(x)$ is taken to be $\log \tanh(\kappa \rho(x)/2)$, rather than $\log \rho(x)$. Since the faces $A_j$ of $\tilde{C}_p(\Gamma)$ are locally isometric to the hyperbolic plane of constant Gauss curvature $-\kappa^2$, with $\rho(x)$ corresponding to the hyperbolic distance from a point, we may readily verify that $G(x)$ is harmonic on the faces of $\tilde{C}_p(\Gamma)$ away from $p$. It follows from the trace formula (6) and the Hessian comparison theorem (p. 4 of [SY]) that $G(x)$ is subharmonic on the faces of $\Sigma$. The factor $1/\rho$ appearing in boundary integrals in the proof of Theorem 1 is replaced by $\frac{\kappa}{\sinh(\kappa \rho)}$, which is the derivative of $G$ with respect to $\rho$. Note that $-\kappa \text{Length}(\tilde{C} \cap \partial B_\varepsilon(p))/\sinh(\kappa \varepsilon)$ is equal to $-2\pi \Theta_{\tilde{C}_p(\Gamma)}(p)$, independent of sufficiently small $\varepsilon > 0$. If $e_n = \nabla \rho$ and $e_1, \ldots, e_{n-1}$ form an orthonormal frame on $M^n \setminus \{p\}$, then by the Hessian comparison theorem $\nabla^2_{e_i e_i} G \geq \frac{\kappa^2 \cosh \kappa \rho}{\sinh^2 \kappa \rho}$ for $i = 1, \ldots, n - 1$, and $\nabla^2_{e_n e_n} G = -\frac{\kappa^2 \cosh \kappa \rho}{\sinh^2 \kappa \rho}$ (See [CG2]). The remainder of the proof is as in the proof of Theorem 1. \qed
**Theorem 6.** Let $\Gamma$ be a graph in $M^n$, and choose $p \in M^n$. Then the cone $\widehat{C} = \widehat{C}_p(\Gamma)$, with the hyperbolic metric $d\widehat{s}^2$, satisfies the density estimate

$$2\pi \Theta \widehat{C}(p) \leq -\sum_{k=1}^{n} \int_{a_k} k \cdot \nu_C \, ds - \kappa^2 \text{Area}(C_p(\Gamma)) + \sum_{k} \sum_{j} \left(\frac{\pi}{2} - \beta^k_j\right),$$

where $\nu_C$ is the outward unit normal vector to $C_p(\Gamma)$; and at a vertex $q_j$ of $\Gamma$, $\beta^k_j$ is the angle between the edge $a_k$ of $\Gamma$ and the minimizing geodesic from $q_j \in \partial a_k$ to $p$.

**Proof.** The proof is similar to the proof of Theorem 4 above. We apply the Gauss-Bonnet formula (13) to the hyperbolic cone $\widehat{C} = \widehat{C}_p(\Gamma)$, and find

$$\int_{\partial B_\varepsilon(p)} K_{\widehat{C}} dA_{\widehat{C}} + \int_{\partial B_\varepsilon(p)} \widehat{k} d\widehat{s} = \int_{\Gamma_{\text{reg}}} \widehat{k} d\widehat{s} + \sum_{k} \sum_{j} \left(\frac{\pi}{2} - \beta^k_j\right) = 0,$$

where $K_{\widehat{C}} \equiv -\kappa^2$ is the Gauss curvature of the faces of $\widehat{C}$; $\widehat{k}$ is the inward geodesic curvature along $\partial \left(\widehat{C} \setminus B_\varepsilon(p)\right)$; and $\beta^k_j$ is the angle formed by the edge $a_k$ of $\Gamma$ and the geodesic edge joining $p$ to $q_j \in \partial a_k$, in the metric $d\widehat{s}^2$.

But along $\partial B_\varepsilon(p) \cap \widehat{C}$, we have $\widehat{k} \equiv -\kappa \coth \kappa \varepsilon$ by a standard computation in the hyperbolic plane. Along $\Gamma$, $d\widehat{s}^2 = ds^2$, so that $\beta^k_j = \beta^k_j$. Further, for each $q \in \Gamma$, there holds $\widehat{k}(q) \leq k(q)$, the geodesic curvature of $\Gamma$ in the cone $C_p(\Gamma)$ with the induced metric $ds^2$ (see Proposition 4 of [CG2]). Thus

$$\kappa \coth \kappa \varepsilon \text{Length}(\partial B_\varepsilon(p) \cap \widehat{C}) \leq -\kappa^2 \text{Area}(\widehat{C} \setminus B_\varepsilon(p)) + \int_{\Gamma_{\text{reg}}} k \, ds + \sum_{k} \sum_{j} \left(\frac{\pi}{2} - \beta^k_j\right).$$

Taking the limit as $\varepsilon \to 0$, we find

$$2\pi \Theta \widehat{C}(p) \leq -\int_{\Gamma_{\text{reg}}} \nu_C \cdot \widehat{k} \, ds + \sum_{k} \sum_{j} \left(\frac{\pi}{2} - \beta^k_j\right) - \kappa^2 \text{Area}(\widehat{C}),$$

since for all $q \in \Gamma$, $k(q) = -\nu_C \cdot \widehat{k}(q)$. Finally, $\text{Area}(\widehat{C}) \geq \text{Area}(C)$, as may be proved by applying Proposition 5 of [CG2] to each face $A_k$ of $C$. □

In order to state the following corollary and the next two theorems, it will be useful to make the following
**Definition 5.** $A(\Gamma)$ is the minimum cone area of $\Gamma$:

$$A(\Gamma) := \min_{p \in H_{cvx}(\Gamma)} \text{Area}(C_p(\Gamma)).$$

Here, the convex hull $H_{cvx}(\Gamma)$ of $\Gamma$ in $M$ is the intersection of closed, locally geodesically convex subsets of $M^n$ which contain $\Gamma$.

**Corollary 6.** For a strongly stationary surface $\Sigma$ in a manifold $M^n$ with sectional curvatures $K_M \leq -\kappa^2$, the area-density estimate holds:

$$2\pi \Theta_\Sigma(p) \leq C_{tot}(\Gamma) - \kappa^2 A(\Gamma).$$

Moreover, equality may only hold when $\Sigma$ is itself a cone over $p$ with totally geodesic faces of constant Gauss curvature $-\kappa^2$.

**Proof.** Recall that Theorem 5 estimates the hyperbolic cone density:

(19) $$2\pi \Theta_{\hat{C}}(p) \leq -\sum_{k=1}^n \int_{\Gamma} \vec{k} \cdot \nu_C ds + \sum_{k} \sum_{j} \left(\frac{\pi}{2} - \beta_k^j\right) - \kappa^2 \text{Area}(C_p(\Gamma)).$$

Since $\Sigma$ must lie in the convex hull $H_{cvx}(\Gamma)$ by the maximum principle, we have $\text{Area}(C_p(\Gamma)) \geq A(\Gamma)$. Also, $|\int_{\Gamma} \vec{k} \cdot \nu_C ds + \sum_{k} \sum_{j} \left(\frac{\pi}{2} - \beta_k^j\right)| \leq C_{tot}(\Gamma)$. Therefore, the right-hand side of inequality (19) is $\leq C_{tot}(\Gamma) - \kappa^2 A(\Gamma)$, while according to Theorem 5 the left-hand side is $\geq 2\pi \Theta_\Sigma(p)$. Moreover, if equality holds, then we must have equality in the conclusion of Theorem 5 implying that $\Sigma$ must be a cone over $p$ with totally geodesic faces of constant Gauss curvature $-\kappa^2$. 

In the following two theorems, the total curvature of $\Gamma$ is “corrected” by subtracting $\kappa^2 A(\Gamma)$. Without this improved hypothesis, Theorems 7 and 8 would have only extremely limited application for $\Gamma$ of large diameter in manifolds $M^n$ of uniformly negative sectional curvature (see Example 2 of [CG2]).

**Theorem 7.** Suppose $\Gamma$ is a graph in $M^n$ with $C_{tot}(\Gamma) - \kappa^2 A(\Gamma) \leq 3\pi$, and let $\Sigma$ be a strongly stationary surface relative to $\Gamma$ in the class $S_\Gamma$. Then $\Sigma$ is either an embedded minimal surface; or, a subset of a singular minimal
cone with an interior edge where three totally geodesic faces, of constant Gauss curvature $-\kappa^2$, meet at equal angles.

**Proof.** Given $p \in \Sigma$, Corollary 6 above implies that

$$2\pi \Theta_{\Sigma}(p) \leq C_{\text{tot}}(\Gamma) - \kappa^2 A(\Gamma).$$

Thus, the present hypothesis implies that $\Theta_{\Sigma}(p) \leq \frac{3}{2}$, and that equality may only hold when $\Sigma$ is a geodesic cone over $p$ and $\Sigma$ has totally geodesic faces of Gaussian curvature $-\kappa^2$ (see Corollary 6). If $\Theta_{\Sigma}(p) < 3/2$, then $\Sigma$ is embedded near $p$. If $\Theta_{\Sigma}(p) = 3/2$, then $\Sigma$ is a geodesic cone, with tangent cone at $p$ congruent to the $Y$ stationary cone, and its faces are totally geodesic with Gauss curvature $\equiv -\kappa^2$. Since $\Sigma$ is a totally geodesic cone of class $S_\Gamma$, it is the exponential image of its tangent cone at $p$. It follows that the exponential map of $M$ at $p$ maps a subset of the $Y$ cone in $T_pM$ onto $\Sigma$.

**Theorem 8.** Suppose $\Gamma$ is a graph in $M^3$ with $C_{\text{tot}}(\Gamma) - \kappa^2 A(\Gamma) \leq 2\pi C_T$, and let $\Sigma$ be an element of the regularity class $S_\Gamma$, which is an $(M, \varepsilon, \delta)$-minimal set with $\Gamma$ as its variational boundary. Then $\Sigma$ is a surface with possibly $Y$ singularities but no other singularities $p$, unless it is a geodesic cone over $p$ with totally geodesic faces of constant Gauss curvature $-\kappa^2$, and having tangent cone at $p$ equal to the $T$ stationary cone.

**Proof.** Choose a point $p \in \Sigma$. Then with respect to a local geodesic coordinate chart centered at $p$, the surface $\Sigma$ is an $(M, \varepsilon, \delta)$-minimal set with $\varepsilon(r) = C r^\alpha$ for some $C > 0$ and $\alpha > 0$. Here we again apply the set of results [11](II.2 and II.3) to conclude that the tangent cone $T_p \Sigma \subset T_p M^3 \cong \mathbb{R}^3$ is area minimizing and that the tangent cone can only be the plane, the $Y$-cone or the $T$-cone.

As in the proof of Theorem 7, we apply Corollary 6 to show that either $\Theta_{\Sigma}(p) < C_T$; or that $\Theta_{\Sigma}(p) = C_T$, and $\Sigma$ is a geodesic cone over $p$ with totally geodesic faces of constant Gauss curvature $-\kappa^2$, which is the image under the exponential map of $M$ at $p$ of the $T$-cone. If $\Theta_{\Sigma}(p) < C_T$, then the tangent cone to $\Sigma$ at $p$ is either a plane or the $Y$ stationary cone. If $T_p \Sigma$ is a
plane, then $\Sigma$ is an embedded surface in a neighborhood of $p$. If $T_p \Sigma$ is the Y stationary cone, then there are Y-type singularities along a curve passing through $p$.

**Remark 3.** In Theorems 7 and 8 the minimum cone area $\mathcal{A}(\Gamma)$ may be replaced by

$$\inf_{\mathcal{H}_{\text{cvx}}(\Gamma)} \text{Area}(\hat{C}_p(\Gamma)),$$

which may be larger (and thus better). See the proof of Theorem 10 below. We have chosen to write Theorems 7 and 8 in terms of the minimum cone area $\mathcal{A}(\Gamma)$, since this quantity seems more closely related to the geometry of $M$. (If $M$ has constant sectional curvature $-\kappa^2$, they are equal.)

### 6.2. Ambient Curvature with Positive Upper Bound.

Throughout this subsection, we shall assume that $M^n$ is a strongly convex Riemannian manifold whose sectional curvatures are bounded above by a positive constant $\kappa^2$. Consider a graph $\Gamma \subset M^n$ and a surface $\Sigma$ of the regularity class $\mathcal{S}_\Gamma$ which is strongly stationary with respect to $\Gamma$.

Choose a point $p$ of $\Sigma$. As in subsection 6.1, we shall assume that $\Gamma$ is nowhere tangent to the minimizing geodesic from $p$. The general cases of the results of this subsection follow by $C^2$ approximation to $\Gamma$.

Since $M^n$ is strongly convex, the unique minimizing geodesic joining $p$ to $q$ varies smoothly as a function of $q$. Therefore, the geodesic cone $C = C_p(\Gamma)$, with the Riemannian metric $ds^2$ induced from $M$, is a Riemannian polyhedron enjoying the same smoothness as $\Gamma$. This cone will be given a second Riemannian metric $d\hat{s}^2$, the spherical metric, so that the faces of the cone have constant Gauss curvature $\kappa^2$, so that the ambient distance $\rho$ to the point $p$ remains equal to the distance in either metric $ds^2$ or $d\hat{s}^2$, and so that at points of $\Gamma$, $d\hat{s}^2 = ds^2$. We may describe the spherical metric at a point $q$ of $C$ as

$$d\hat{s}^2 = d\rho^2 + \frac{\sin^2 \kappa \rho}{\sin^2 \kappa \rho(q)} \left[ \left( \frac{ds}{\rho} \right)_\Gamma^2 - \left( \frac{dr(q)}{\rho} \right)^2 \right].$$

As in subsection 6.1, $r(q)$ denotes $\rho(Q)$, the distance in $M$ from $p$ to the point $Q$ of $\Gamma$ along the radial geodesic from $p$ passing through $q$; also, the
one-form $ds|_{\Gamma}$ has been extended to the cone so that it is invariant under radial deformations. Note that $ds|_{\Gamma} (\partial/\partial \rho) = dr (\partial/\partial \rho) = 0$. We use the notation $\hat{C} = \hat{C}_p(\Gamma)$ for the cone $C$ with this spherical metric $d\hat{s}^2$.

In this section, it will be useful to state theorems in terms of a maximum cone area, rather than the minimum cone area which was of use in subsection 6.1. To account for the positive sectional curvature which may occur in $M$, we will need to add a term $\kappa^2 \hat{A}(\Gamma)$ to the total curvature $C_{\text{tot}}(\Gamma)$. The reader might object that, under certain circumstances, such as when sectional curvatures comparable to $\kappa^2$ appear only in a small part of $M^n$ and large parts of the manifold $M$ actually have nonpositive sectional curvatures, this upper bound may be much larger than the values which need to be considered in Theorems 11 and 12 below. However, when the sectional curvatures of $M$ are nearly equal to the constant $\kappa^2$, the theorems below are nearly sharp.

**Definition 6.** $\hat{A}(\Gamma)$ is the maximum spherical cone area of $\Gamma$:

$$\hat{A}(\Gamma) := \sup_{p \in \mathcal{H}_{\text{cvx}}(\Gamma)} \text{Area}(\hat{C}_p(\Gamma)).$$

**Theorem 9.** Given a strongly stationary surface $\Sigma$ in $M^n$ of class $S_\Gamma$, and a point $p$ of $\Sigma \setminus \Gamma$, the following inequality holds:

$$\Theta_{\Sigma}(p) \leq \Theta_{\hat{C}_p(\Gamma)}(p).$$

Moreover, equality implies that $\Sigma$ is a cone with totally geodesic faces of constant Gauss curvature $\kappa^2$.

**Proof.** Analogous to the proof of Theorem 5 but using $\log \tan(\kappa \rho(x)/2)$ as the test function $G(x)$ in place of $\log \tanh(\kappa \rho(x)/2)$.

**Theorem 10.** Let $\Gamma$ be a graph in $M^n$, and choose $p \in M^n$. Then the cone $\hat{C} = \hat{C}_p(\Gamma)$, with the spherical metric $d\hat{s}^2$, satisfies the density estimate

$$2\pi \Theta_{\hat{C}}(p) \leq -\sum_{k=1}^n \int_{a_k} \vec{k} \cdot \nu_C \, ds + \kappa^2 \text{Area}(\hat{C}_p(\Gamma)) + \sum_k \sum_j \left( \frac{\pi}{2} - \beta_k^j \right),$$
where $\nu_C = \nu_{\hat{C}}$ is the outward unit normal vector to $C_p(\Gamma)$; and $\beta^j_k$ is the angle between the edge $a_k$ of $\Gamma$ and the minimizing geodesic in $M$ from $q_j \in \partial a_k$ to $p$.

**Proof.** The demonstration, which is based on the Gauss-Bonnet formula on $\hat{C}$, is highly analogous to the proof of Theorem 6; the statement has been modified, however, since in the middle term on the right-hand side of equation (18), $\text{Area}(\hat{C})$ was multiplied by the non-positive $-\kappa^2$ and could therefore be replaced in the conclusion of Theorem 6 with the smaller quantity $\text{Area}(C)$. Here, however, the Gauss curvature of $\hat{C}$ is $\kappa^2$, which is positive, so that the spherical area $\text{Area}(\hat{C})$ of the cone must remain on the right-hand side of the inequality.

**Corollary 7.** The area density of a strongly stationary surface $\Sigma$ in a manifold $M^n$ with sectional curvatures $K_M \leq +\kappa^2$ satisfies the inequality:

$$2\pi \Theta_\Sigma(p) \leq C_{\text{tot}}(\Gamma) + \kappa^2 \hat{A}(\Gamma).$$

Moreover, equality may only hold when $\Sigma$ is itself a cone over $p$ with totally geodesic faces of constant Gauss curvature $\kappa^2$.

**Proof.** Theorem 9 estimates the density $\Theta_\Sigma(p) \leq \Theta_{\hat{C}_p(\Gamma)}(p)$. Meanwhile, by Theorem 10,

$$2\pi \Theta_{\hat{C}_p(\Gamma)}(p) \leq \sum_{k=1}^n \int_{a_k} \hat{k} \cdot \nu_C \, ds \tag{20}$$

$$+ \sum_k \sum_j \left( \frac{\pi}{2} - \beta^j_k \right) + \kappa^2 \text{Area} \left( \hat{C}_p(\Gamma) \right). \tag{21}$$

Since $\Sigma$ lies in the convex hull $\mathcal{H}_{\text{cvx}}(\Gamma)$ by the maximum principle, we have $\text{Area} \left( \hat{C}_p(\Gamma) \right) \leq \hat{A}(\Gamma)$. Also, by definition of total curvature, $|\int_{\Gamma_{\text{reg}}} \hat{k} \cdot \nu_C \, ds| + \sum_k \sum_j \left( \frac{\pi}{2} - \beta^j_k \right) \leq C_{\text{tot}}(\Gamma)$. Therefore, $2\pi \Theta_\Sigma(p) \leq C_{\text{tot}}(\Gamma) + \kappa^2 \hat{A}(\Gamma)$. Moreover, if equality holds, then we must have equality in the conclusion of Theorem 9 implying that $\Sigma$ must be a geodesic cone over $p$ with totally geodesic faces of constant Gauss curvature $+\kappa^2$. 

\[ \blacksquare \]
The proofs of our final two theorems are completely analogous to the proofs of Theorems 7 and 8.

**Theorem 11.** Suppose $\Gamma$ is a graph in $M^n$ with $C_{\text{tot}}(\Gamma) + \kappa^2\hat{\mathcal{A}}(\Gamma) \leq 3\pi$, and let $\Sigma$ be a strongly stationary surface relative to $\Gamma$ in the class $S_\Gamma$. Then $\Sigma$ is either an embedded minimal surface or a subset of a singular minimal cone with an interior edge where three totally geodesic faces, of constant Gauss curvature $\kappa^2$, meet at equal angles.

**Theorem 12.** Suppose $\Gamma$ is a graph in $M^3$ with $C_{\text{tot}}(\Gamma) + \kappa^2\hat{\mathcal{A}}(\Gamma) \leq 2\pi C_T$, and let $\Sigma$ be a $(M, 0, \delta)$-minimal set with respect to $\Gamma$ in the regularity class $S_\Gamma$. Then $\Sigma$ is a surface with possibly $Y$ singularities but no other singularities $p$, unless it is a geodesic cone over $p$ with totally geodesic faces of constant Gauss curvature $\kappa^2$, and having tangent cone at $p$ equal to the $T$ stationary cone.

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