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LEARNING LOW-DEGREE FUNCTIONS FROM A LOGARITHMIC NUMBER OF RANDOM QUERIES

ALEXANDROS ESKENAZIS AND PAATA IVANISVILI

Abstract. We prove that every bounded function \( f : [-1, 1]^n \to [-1, 1] \) of degree at most \( d \) can be learned with \( L_2 \)-accuracy \( \epsilon \) and confidence \( 1 - \delta \) from \( \log(\frac{n}{\delta}) e^{-d-1} C d^{3/2} \sqrt{\log d} \) random queries, where \( C > 1 \) is a universal finite constant.

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1. Introduction

Every function \( f : [-1, 1]^n \to \mathbb{R} \) admits a unique Fourier–Walsh expansion of the form

\[
\forall x \in [-1, 1]^n, \quad f(x) = \sum_{S \subseteq \{1, \ldots, n\}} \hat{f}(S) w_S(x),
\]

where \( w_S(x) = \prod_{i \in S} x_i \) and the Fourier coefficients \( \hat{f}(S) \) are given by

\[
\forall S \subseteq \{1, \ldots, n\}, \quad \hat{f}(S) = \frac{1}{2^n} \sum_{y \in \{-1, 1\}^n} f(y) w_S(y).
\]

We say that \( f \) has degree at most \( d \in \{1, \ldots, n\} \) if \( \hat{f}(S) = 0 \) for every subset \( S \) with \( |S| > d \).

1.1. Learning functions on the hypercube. Let \( \mathcal{C} \) be a class of functions \( f : [-1, 1]^n \to \mathbb{R} \) on the \( n \)-dimensional discrete hypercube. The problem of learning the class \( \mathcal{C} \) can be described as follows: given a source of examples \( (x, f(x)) \), where \( x \in [-1, 1]^n \), for an unknown function \( f \in \mathcal{C} \), compute a hypothesis function \( h : [-1, 1]^n \to \mathbb{R} \) which is a good approximation of \( f \) up to a given error in some prescribed metric. In this paper we will be interested in the random query model with \( L_2 \)-error, in which we are given \( N \) independent examples \( (x, f(x)) \), each chosen uniformly at random from the discrete hypercube \( [-1, 1]^n \), and we want to efficiently construct a (random) function \( h : [-1, 1]^n \to \mathbb{R} \) such that \( \|h - f\|_{L_2}^2 < \epsilon \) with probability at least \( 1 - \delta \), where \( \epsilon, \delta \in (0, 1) \) are given accuracy and confidence parameters. The goal is to construct a randomized algorithm which produces the hypothesis function \( h \) from a minimal number \( N \) of examples.

The above very general problem has been studied for decades in computational learning theory and many results are known\(^1\), primarily for various classes \( \mathcal{C} \) of structured Boolean functions \( f : [-1, 1]^n \to [-1, 1] \). Already since the late 1980s, researchers used the Fourier–Walsh expansion (1) to design such learning algorithms (see the survey [14]). Perhaps the most classical of these is the Low-Degree Algorithm of Linial, Mansour and Nisan [12] who showed that for the class \( \mathcal{C}_b^d \) of all bounded functions \( f : [-1, 1]^n \to [-1, 1] \) of degree at most \( d \) there exists an algorithm which produces an \( \epsilon \)-approximation of \( f \) with probability at least \( 1 - \delta \) using \( N = \frac{2n^d}{\epsilon} \log(\frac{2n^d}{\delta}) \) samples. In this generality, the \( O_{\epsilon, \delta, d}(n^d \log n) \) estimate of [12] was the state of the art until the recent work [11] of Iyer, Rao, Reis, Rothvoss and Yehudayoff who employed analytic techniques to derive new bounds on the \( \ell_1 \)-size of the Fourier spectrum of bounded functions.

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\(^1\)We will by no means attempt to survey this (vast) field, so we refer the interested reader to the relevant chapters of O’Donnell’s book [15] and the references therein.
functions (see also Section 3) and used these estimates to show that $N = O_\ell, \delta, d(n^{d-1} \log n)$ examples suffice to learn $c^\ell_b$. The goal of the present paper is to further improve this result and show that in fact $N = O_{\ell, \delta, d}(\log n)$ samples suffice for this purpose.

**Theorem 1.** Fix $\varepsilon, \delta \in (0, 1)$, $n \in \mathbb{N}$, $d \in \{1, \ldots, n\}$ and a bounded function $f : [-1, 1]^n \rightarrow [-1, 1]$ of degree at most $d$. If $N \in \mathbb{N}$ satisfies

$$N \geq \min \left\{ \exp \left( \frac{C d^{3/2} \sqrt{\log d}}{\varepsilon^{d+1}}, 4d n^d \right), \frac{\log \left( \frac{n}{\delta} \right)}{\varepsilon} \right\},$$

where $C \in (0, \infty)$ is a large numerical constant, then $N$ uniformly random independent queries of pairs $(x, f(x))$, where $x \in [-1, 1]^n$, suffice for the construction of a random function $h : [-1, 1]^n \rightarrow \mathbb{R}$ satisfying the condition $\|h - f\|_{L_2} < \varepsilon$ with probability at least $1 - \delta$.

The proof of Theorem 1 relies on some important approximation theoretic estimates going back to the 1930s which we shall now describe (see also [9]). To the best of our knowledge, these tools had not yet been exploited in the computational learning theory literature.

1.2. The Fourier growth of Walsh polynomials in $\ell_2$ or $\ell_1$. Estimates for the growth of coefficients of polynomials as a function of their degree and their maximum on compact sets go back to the early days of approximation theory (see [5]). A seminal result of this nature is Littlewood’s celebrated $\frac{1}{3}$-inequality [13] for bilinear forms which was later generalized by Bohnenblust and Hille [4] for multilinear forms on the torus $\mathbb{T}^n$ or the unit square $[-1, 1]^n$. By means of polarization, one can use this multilinear estimate to derive an inequality for polynomials which reads as follows\(^2\). For every $K \in \{\mathbb{R}, \mathbb{C}\}$ and $d \in \mathbb{N}$, there exists $B_d^K \in (0, \infty)$ such that for every $n \in \mathbb{N}$ and every coefficients $c_n \in K$, where $a \in (\mathbb{N} \cup \{0\})^n$ with $|a| \leq d$, we have

$$\left( \sum_{|a| \leq d} |c_a|^2 \right)^{\frac{1}{2d+1}} \leq B_d^K \max \left\{ \left( \sum_{|a| \leq d} c_a x^a \right) : x \in \mathbb{K}^n \text{ with } \|x\|_{L_\infty(K)} \leq 1 \right\}. \quad (4)$$

Moreover, $\frac{2d}{d+1}$ is the smallest exponent for which the optimal constant in (4) is independent of the number of variables $n$ of the polynomial. The exact asymptotics of the constants $B_d^K$ and $B_d^C$ remain unknown, however it is known that there is a significant gap between $B_d^K$ and $B_d^C$, namely that $\limsup_{d \to \infty} (B_d^K)^{1/d} = 1 + \sqrt{2}$ whereas $B_d^C \leq C^{\sqrt{d} \min d}$ for a finite constant $C > 1$ (see [7, 1, 9, 6, 8] for these and other important advances of the last decade). Restricting inequality (4) to real multilinear polynomials, convexity shows that the maximum on the right-hand side is attained at a point $x \in [-1, 1]^n$, which, in view of (1), makes (4) an estimate for the Fourier–Walsh growth of functions on the discrete hypercube. We shall denote by $B_d^{[1]}$ the corresponding optimal constant (first explicitly investigated by Blei in [3, p. 175]), that is, the least constant such that for every $n \in \mathbb{N}$ and every function $f : [-1, 1]^n \rightarrow \mathbb{R}$ of degree at most $d$,

$$\left( \sum_{S \subseteq [1, \ldots, n]} |f(S)|^{\frac{2d}{d+1}} \right)^{\frac{d+1}{2d}} \leq B_d^{[1]} \|f\|_{L_\infty}^d. \quad (5)$$

The best known quantitative result in this setting is due to Defant, Mastylo and Pérez [8] who showed that $B_d^{[1]} \leq \exp(\kappa \sqrt{d} \log d)$ for a universal constant $\kappa \in (0, \infty)$. The main contribution of this work is the following theorem relating the growth of the constant $B_d^{[1]}$ and learning.

**Theorem 2.** Fix $\varepsilon, \delta \in (0, 1)$, $n \in \mathbb{N}$, $d \in \{1, \ldots, n\}$ and a bounded function $f : [-1, 1]^n \rightarrow [-1, 1]$ of degree at most $d$. If $N \in \mathbb{N}$ satisfies

$$N \geq \frac{e^{8d^2}}{\varepsilon^{d+1}} (B_d^{[1]} + 1)^{2d} \log \left( \frac{n}{\delta} \right), \quad (6)$$

\(^2\)For $a = (a_1, \ldots, a_n) \in (\mathbb{N} \cup \{0\})^n$, we use the standard notations $|a| = a_1 + \cdots + a_n$ and $x^n = x_1^{a_1} \cdots x_n^{a_n}$. 


then given $N$ uniformly random independent queries of pairs $(x, f(x))$, where $x \in \{-1, 1\}^n$, one can construct a random function $h : \{-1, 1\}^n \rightarrow \mathbb{R}$ satisfying $\|h - f\|_{L_2}^2 < \epsilon$ with probability at least $1 - \delta$.

In Section 2 we will prove Theorem 2 and use it to derive Theorem 1. In Section 3 we will present some additional remarks on Boolean analysis and learning, in particular showing that the dependence on $n$ in Theorem 1 is optimal for $\delta = \frac{1}{n}$. Moreover, we shall improve the recent bounds of [11] on the $\ell_1$-Fourier growth of bounded functions of low degree.

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2. Proofs

**Proof of Theorem 2.** Fix a parameter $b \in (0, \infty)$ and denote by

$$N_b \overset{\text{def}}{=} \left\lceil \frac{2}{b^2} \log \left( \frac{2d}{\delta} \binom{n}{k} \right) \right\rceil. \tag{7}$$

Let $X_1, \ldots, X_{N_b}$ be independent random vectors, each uniformly distributed on $\{-1, 1\}^n$. For a subset $S \subseteq \{1, \ldots, n\}$ with $|S| \leq d$ consider the empirical Walsh coefficient of $f$, given by

$$\alpha_S = \frac{1}{N_b} \sum_{j=1}^{N_b} f(X_j) w_S(X_j). \tag{8}$$

As $\alpha_S$ is a sum of bounded i.i.d. random variables and $\mathbb{E}[\alpha_S] = \hat{f}(S)$, the Chernoff bound gives

$$\forall S \subseteq \{1, \ldots, n\}, \quad \mathbb{P}\{|\alpha_S - \hat{f}(S)| > b\} \leq 2 \exp(-N_b b^2/2). \tag{9}$$

Therefore, using the union bound and taking into account that $f$ has degree at most $d$, we get

$$\mathbb{P}\{|\alpha_S - \hat{f}(S)| \leq b, \text{ for every } S \subseteq \{1, \ldots, n\} \text{ with } |S| \leq d\} \geq 1 - 2 \sum_{k=0}^{d} \binom{n}{k} \exp(-N_b b^2/2) \geq 1 - \delta. \tag{10}$$

Fix an additional parameter $a \in (b, \infty)$ and consider the random collection of sets given by

$$S_a \overset{\text{def}}{=} \left\{ S \subseteq \{1, \ldots, n\} : |\alpha_S| \geq a \right\}. \tag{11}$$

Observe that if the event $G_b$ of equation (10) holds, then

$$\forall S \notin S_a, \quad |\hat{f}(S)| \leq |\alpha_S - \hat{f}(S)| + |\alpha_S| \leq a + b \tag{12}$$

and

$$\forall S \in S_a, \quad |\hat{f}(S)| \geq |\alpha_S| - |\alpha_S - \hat{f}(S)| \geq a - b. \tag{13}$$

Finally, consider the random function $h_{a,b} : \{-1, 1\}^n \rightarrow \mathbb{R}$ given by

$$h_{a,b}(x) \overset{\text{def}}{=} \sum_{S \in S_a} \alpha_S w_S(x). \tag{14}$$

Combining (13) with inequality (5), we deduce that

$$|S_a| \overset{(13)}{\leq} (a-b) \frac{2d}{2\pi} \sum_{S \in S_a} |\hat{f}(S)|^{2d/\pi} \leq (a-b) \frac{2d}{2\pi} \sum_{S \subseteq \{1, \ldots, n\}} |\hat{f}(S)|^{2d/\pi} \overset{(5)}{\leq} (a-b) \frac{2d}{2\pi} (B_{d^{\frac{1}{4}}})^{2d/\pi}. \tag{15}$$

Therefore, on the event $G_b$ we have

$$\|h_{a,b} - f\|_{L_2} \overset{(12)}{=} \sum_{S \subseteq \{1, \ldots, n\}} \left| \hat{h}_{a,b}(S) - \hat{f}(S) \right|^2 = \sum_{S \in S_a} |\alpha_S - \hat{f}(S)|^2 + \sum_{S \in S_a} |\hat{f}(S)|^2 \overset{(5) \wedge (15)}{=} (a-b) \frac{2d}{2\pi} (B_{d^{\frac{1}{4}}})^{2d/\pi} \left( (a-b)^{-2d/\pi} b^2 + (a+b)^{-2d/\pi} \right). \tag{16}$$
Choosing $a = b(1 + \sqrt{d + 1})$, we deduce that
\[
\|h_{b(1+\sqrt{d+1}),b} - f\|_{L^2}^2 < (B_d^{[\pm 1]}(d+1)^{-2}\sqrt{d+1} + (d+1)^{2}\sqrt{d+1} + (2 + \sqrt{d+1})\sqrt{d+1}).
\] (17)

Next, we need the technical inequality
\[
(d+1)^{-\frac{2}{\sqrt{d+1}}} + (2 + \sqrt{d+1})\frac{2}{\sqrt{d+1}} \leq (e^d(d+1))^\frac{1}{\sqrt{d+1}} \quad \text{for all } d \geq 1.
\] (18)

Rearranging the terms, it suffices to show that $(2 + \sqrt{d+1})\frac{2}{\sqrt{d+1}} \leq (d+1)^{\frac{1}{\sqrt{d+1}}} (e^d - 1)\sqrt{d+1}$, which is equivalent to \((\frac{2}{\sqrt{d+1}} + 1)^{\frac{2}{\sqrt{d+1}}} \leq (\sqrt{2} + 1)^{\frac{2}{\sqrt{d+1}}} \leq 1 + \frac{3}{d+1} \leq e^{\frac{1}{\sqrt{d+1}} - \frac{1}{d+1}},\) where inequality (*) holds because the left hand side is convex in the variable $\lambda \overset{\text{def}}{=} \frac{2}{\sqrt{d+1}}$ whereas the right hand side is linear and since (*) holds at the endpoints $\lambda = 0, 1$.

Combining (17) and (18) we see that $\|h_{b(1+\sqrt{d+1}),b} - f\|_{L^2}^2 < \epsilon$ holds for $b^2 \leq e^{-5d^{-1/2}} [B_d^{[\pm 1]}]^2 d^d$.

Plugging this choice of $b$ in (7) shows that given $N$ random queries, where
\[
N = \left\lceil \frac{C^d (B_d^{[\pm 1]})^{2d}}{\epsilon^d + 1} \log \left( \frac{2}{\delta} \sum_{k=0}^{d} \binom{n}{k} \right) \right\rceil,
\] (20)

the random function $h_{b(1+\sqrt{d+1}),b}$ satisfies $\|h_{b(1+\sqrt{d+1}),b} - f\|_{L^2}^2 < \epsilon$ with probability at least $1 - \delta$ and the conclusion of the theorem follows from elementary estimates, such as
\[
\sum_{k=0}^{d} \binom{n}{k} \leq \sum_{k=0}^{d} \frac{n^{d}}{k!} = \sum_{k=0}^{d} \frac{n^{d}}{k!} \left( \frac{n}{d} \right)^k \leq \left( \frac{e}{d} \right)^d.
\]

Theorem 1 is a straightforward consequence of Theorem 2.

**Proof of Theorem 1.** Theorem 2 combined with the bound $B_d^{[\pm 1]} \leq \exp(\kappa \sqrt{d \log d})$ of [8] imply the conclusion of Theorem 1 for $\epsilon \geq \frac{\exp(C \sqrt{d \log d})}{n}$, where $C \in (0, \infty)$ is a large universal constant. The case $\epsilon < \frac{\exp(C \sqrt{d \log d})}{n}$ follows from the Low-Degree Algorithm of [12].

3. **Concluding remarks**

We conclude with a few additional remarks on the spectrum of bounded functions defined on the hypercube and corresponding learning algorithms. For a function $f : [-1, 1]^n \to \mathbb{R}$, its Rademacher projection on level $\ell \in \{1, \ldots, n\}$ is defined as
\[
\forall x \in [-1, 1]^n, \quad \text{Rad}_\ell f(x) = \sum_{S \subseteq \{1, \ldots, n\} \mid |S| = \ell} \hat{f}(S)w_S(x).
\] (21)

1. The first main theorem of [11] asserts that if $f : [-1, 1]^n \to \mathbb{R}$ is a function of degree $d$, then
\[
\forall \ell \in \{1, \ldots, d\}, \quad \|\text{Rad}_\ell f\|_{L^2} \leq \begin{cases} \frac{T_d^{[0]}(0)}{\ell} \cdot \|f\|_{L^2}, & \text{if } (d - \ell) \text{ is even} \\ \frac{T_d^{[0]}(0)}{\ell} \cdot \|f\|_{L^2}, & \text{if } (d - \ell) \text{ is odd} \end{cases},
\] (22)

where $T_d(t)$ is the $d$-th Chebyshev polynomial of the first kind, that is, the unique real polynomial of degree $d$ such that $\cos(d\theta) = T_d(\cos \theta)$ for every $\theta \in \mathbb{R}$. Moreover, Iyer, Rao, Reis, Rothvoss and Yehudayoff observed in [11, Proposition 2] that this estimate is asymptotically sharp. We present a simple proof of their inequality (22) (see also [10] for related arguments).
Proof of (22). For any \( f : \{-1,1\}^n \to \mathbb{R} \) consider its harmonic extension on \([-1,1]^n\),
\[
\forall (x_1, \ldots, x_n) \in [-1,1]^n, \quad \hat{f}(x_1, \ldots, x_n) = \sum_{S \subseteq \{1, \ldots, n\}} \hat{f}(S) \prod_{j \in S} x_j.
\tag{23}
\]
By convexity \( ||\hat{f}||_{L^\infty([-1,1]^n)} = ||f||_{L^\infty([-1,1]^n)} \). In particular, the restriction of \( \hat{f} \) on the ray \( t(x_1, \ldots, x_n), \ t \in [-1,1] \), i.e.
\[
\forall t \in \mathbb{R}, \quad h_x(t) \overset{\text{def}}{=} \sum_{S \subseteq \{1, \ldots, n\}} \hat{f}(S)w_S(x)t^{|S|}
\tag{24}
\]
satisfies \( \max_{x \in [-1,1]} |h_x(t)| \leq ||f||_{L^\infty} \) for all \( (x_1, \ldots, x_n) \in \{-1,1\}^n \). Therefore, since \( \deg h_x \leq d \), a classical inequality of Markov (see e.g. [5, p. 248]) gives
\[
|\operatorname{Rad}_\ell f(x)| = \left| \frac{h_x^{(\ell)}(0)}{\ell!} \right| \leq \begin{cases} \frac{|T^{(\ell)}(0)|}{\ell!} \cdot ||f||_{L^\infty} & \text{if } (d - \ell) \text{ is even} \\ \frac{|T^{(\ell)}(0)|}{\ell!} \cdot ||f||_{L^\infty} & \text{if } (d - \ell) \text{ is odd} \end{cases}
\tag{25}
\]
and (22) follows by taking a maximum over all \( x \in \{-1,1\}^n \). \( \square \)

In particular, as observed in [11], inequality (22) implies that if \( f \) has degree at most \( d \) then
\[
\forall \ell \in \{1, \ldots, d\}, \quad \|\operatorname{Rad}_\ell f\|_{L^\infty} \leq \frac{d^{\ell}}{\ell!} \|f\|_{L^\infty}.
\tag{26}
\]

2. The second main theorem of [11] asserts that if \( f : \{-1,1\}^n \to [-1,1] \) is a bounded function of degree at most \( d \), then for every \( \ell \in \{1, \ldots, d\} \) we have
\[
\sum_{S \subseteq \{1, \ldots, n\}} |\operatorname{Rad}_\ell f(S)| = \sum_{S \subseteq \{1, \ldots, n\}, |S| = \ell} |\hat{f}(S)| \leq n^{\frac{\ell+1}{2}} d^{\ell} e^{\frac{\ell}{2}}.
\tag{27}
\]
The Bohnenblust–Hille-type inequality of [8] implies the following improved bound.

**Corollary 3.** Let \( n \in \mathbb{N} \) and \( d \in \{1, \ldots, n\} \). Then, every bounded function \( f : \{-1,1\}^n \to [-1,1] \) of degree at most \( d \) satisfies
\[
\forall \ell \in \{1, \ldots, d\}, \quad \sum_{S \subseteq \{1, \ldots, n\}, |S| = \ell} |\hat{f}(S)| \leq \left( \frac{n}{\ell} \right)^{\frac{\ell+1}{2}} e^{\frac{\ell}{2}} \log \ell \frac{d^{\ell}}{\ell!} \leq n^{\frac{\ell+1}{2}} d^{\ell} e^{\frac{\ell}{2}},
\tag{28}
\]
for some universal constant \( c \in (0,1) \).

**Proof.** Combining Hölder’s inequality with the estimate of [8] and (26) we get
\[
\sum_{S \subseteq \{1, \ldots, n\}, |S| = \ell} |\hat{f}(S)| \leq \left( \frac{n}{\ell} \right)^{\frac{\ell+1}{2}} \left( \sum_{S \subseteq \{1, \ldots, n\}} |\operatorname{Rad}_\ell f(S)| \right)^{\frac{\ell+1}{2}}
\tag{29}
\]
\[
\leq \left( \frac{n}{\ell} \right)^{\frac{\ell+1}{2}} \exp(\kappa \sqrt{\ell} \log \ell) \|\operatorname{Rad}_\ell f\|_{L^\infty}^{(26)} \left( \frac{n}{\ell} \right)^{\frac{\ell+1}{2}} \exp(\kappa \sqrt{\ell} \log \ell) \frac{d^{\ell}}{\ell!}.
\]
The last inequality of (28) follows from (22) and the elementary bound \( \ell^{\frac{\ell}{2}} \leq \left( \frac{\ell}{\ell+1} \right)^{\frac{\ell}{2}} \). \( \square \)

We refer to the recent work [2] for a systematic study of inequalities relating the Fourier growth with various well-studied properties of Boolean functions.

3. It is straightforward to observe (see also [15, Proposition 3.31]) that if \( f : \{-1,1\}^n \to \{-1,1\} \) is a Boolean function and \( h : \{-1,1\}^n \to \mathbb{R} \) is an arbitrary function, then
\[
\|\operatorname{sign}(h) - f\|_{L^2}^2 = 4\mathbb{P}\{|\operatorname{sign}(h) \neq f\} \leq 4\mathbb{P}\{|h - f| \geq 1\} \leq 4\|h - f\|_{L^2}^2
\tag{30}
\]
where we define \( \text{sign}(0) \) as \( \pm 1 \) arbitrarily. Therefore, applying Theorem 1 to a Boolean function, the above algorithm produces a Boolean function \( \hat{h} = \text{sign}(h) \) which is a \( 4\varepsilon \)-approximation of \( f \).

4. In Theorem 1 we showed that bounded functions \( f : \{-1,1\}^n \to [-1,1] \) of degree at most \( d \) can be learned with accuracy at most \( \varepsilon \) and confidence at least \( 1 - \delta \) from \( N = O_{\varepsilon,d}(\log(n/\delta)) \) random queries. We will now show that this estimate is sharp for small enough values of \( \delta \).

**Proposition 4.** Suppose that bounded linear functions \( \ell : \{-1,1\}^n \to [-1,1] \) can be learned with accuracy at most \( 1/n \) and confidence at least \( 1 - \delta \) from \( N \) random queries. Then \( N > \log_2 n \).

**Proof.** By the assumption, for any input \((X_1,y_1), \ldots,(X_N,y_N)\) chosen independently and uniformly from \([-1,1]^n\) there exists a function \( h(X_1,y_1), \ldots,(X_N,y_N) : \{-1,1\}^n \to \mathbb{R} \) such that if \( X_1, \ldots, X_N \) are chosen independently and uniformly from \([-1,1]^n\) and there exists a linear function \( \ell : \{-1,1\}^n \to [-1,1] \) such that \( y_j = \ell(X_j) \) for every \( j \in \{1, \ldots, N\} \), then \( \mathbb{P}(\Omega_\ell) > 1 - \frac{1}{2^N} \), where \( \Omega_\ell \) is the event

\[
\Omega_\ell = \left\{ \mathbb{E}(h(X_1,\ell(X_1)), \ldots, (X_N,\ell(X_N))) - \ell \right\}^2 < \frac{1}{2}. \tag{31}
\]

Let \( X_j = (X_j(1), \ldots, X_j(n)) \) for \( j \in \{1, \ldots, N\} \) and consider the event

\[
\mathcal{W} = \left\{ X_j(1) = X_j(2), \ \forall \ j \in \{1, \ldots, N\} \right\}. \tag{32}
\]

By the independence of the samples, we have \( \mathbb{P}(\mathcal{W}) = \frac{1}{2^n} \). Therefore, if \( N \leq \log_2 n \) and we consider the linear functions \( r_j : \{-1,1\}^n \to [-1,1] \) given by \( r_j(x) = x_i \), then

\[
\mathbb{P}(\Omega_{r_1} \cap \Omega_{r_2} \cap \mathcal{W} \neq \emptyset) > 1 - \frac{1}{n} \geq 1 - \frac{1}{2N} = 1 - \mathbb{P}(\mathcal{W}), \tag{33}
\]

which implies that \( \Omega_{r_1} \cap \Omega_{r_2} \cap \mathcal{W} \neq \emptyset \). Choosing \( X_1, \ldots, X_N \) from this event and denoting by \( h = h(X_1,X_1(1)), \ldots, (X_N,X_N(1)) = h(X_1,X_2), \ldots, (X_N,X_N(2)) \), we deduce from the triangle inequality that

\[
2 = \mathbb{E}(r_1 - r_2)^2 \leq 2\mathbb{E}(h - r_1)^2 + 2\mathbb{E}(h - r_2)^2 < 2 \tag{34}
\]

which is clearly a contradiction. Therefore \( N > \log_2 n \). \( \square \)

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