Abstract. Let $\Gamma \backslash G/K$ be a compact Hermitian locally symmetric space, where $G$ is simple. We study the components of a de Rham cohomology class of $\Gamma \backslash G/K$, with respect to the Matsushima decomposition, where the class is obtained by taking Poincaré dual of a totally geodesic complex analytic submanifold. Using an extension of the vanishing result of Kobayashi and Oda, we specify the existence of certain components of such cohomology classes when $G = \text{SU}(p, q)$, $5 \leq p \leq q$.

All real Lie algebras and their subspaces will be denoted with a 0 subscript. Dropping of this subscript will denote the complexification.

1. Introduction

Let $G$ be a linear semisimple Lie group with no compact normal subgroups. Let $K$ be a maximal compact subgroup of $G$. Let $\Gamma$ be a torsion-free uniform lattice in $G$. Then $\Gamma \backslash G/K$ is a compact locally symmetric space. Let $\mathfrak{g}_0$ and $\mathfrak{k}_0$ be the Lie algebras of $G$ and $K$ respectively and let $\theta$ be the Cartan involution of $\mathfrak{g}_0$, which fixes $\mathfrak{k}_0$ pointwise. The cohomology of $\Gamma \backslash G/K$ with complex coefficients, can be written as a direct sum of relative Lie algebra cohomologies of $(\mathfrak{g}, K)$ with coefficients in $A_q$, where $q$ are $\theta$-stable parabolic subalgebras of $\mathfrak{g}_0$ and $A_q$ are obtained by cohomological induction on the trivial $q$-module. This is called the Matsushima decomposition. Our aim is to construct concrete cohomology classes of $\Gamma \backslash G/K$ and determine which of their components, with respect to the Matsushima decomposition, are non-zero. A non-zero component implies that the corresponding irreducible unitary representation occur in $L^2(\Gamma \backslash G)$.

The cohomology classes that will be considered are Poincaré duals of totally geodesic submanifolds. We call these geometric classes. In literature these submanifolds are variously called geometric cycles, special cycles or generalized modular symbols. Millson and Raghunathan proved that if a pair of complementary dimensional totally geodesic submanifolds have all the intersection numbers positive, then (going
to a finite cover if necessary) the corresponding geometric classes have non-zero components other than the one corresponding to the trivial representation. Unfortunately, to our knowledge, there is no other result about non-vanishing of components of a geometric class. But since the number of non-zero components can only be finite, we go the roundabout way of ascertaining the representations for which the corresponding components of a geometric class are zero. For this we restrict to the case of compact Hermitian locally symmetric spaces. Here we exploit the additional Hodge bigrading on the cohomology. If the submanifold is complex analytic then the corresponding geometric class is of $(p,p)$-type. On the other hand, if $(p,q)$ denotes the Hodge types of classes in $\mathcal{H}^*(\mathfrak{g}, K; A_q)$, then $q - p$ is constant. Thus a complex analytic geometric class has no $A_q$-component for which the above constant in non-zero. This is the first (and elementary) vanishing result on components of a geometric class that we will use. It was already noted and exploited in [11]. A non-trivial vanishing result was obtained by Kobayashi and Oda in [8]. We generalize this to arrive at the second non-vanishing result (Corollary 4.2 in §4) that we will use:

**Theorem 1.1.** Assume $G$ is simple. Let $G' \subset G$ and $K' := G' \cap K, \Gamma' := G' \cap \Gamma$ be such that $\Gamma' \backslash G' / K'$ is a compact totally geodesic submanifold of $\Gamma \backslash G / K$. Let $\mathfrak{g}'$ be the complexified Lie algebra of $G'$. If $A_q$ is discretely decomposable as a $(\mathfrak{g}', K')$ module, then the Poincaré dual of $\Gamma' \backslash G' / K'$ has no $A_q$-component.

From a computational point of view, this result owes its significance to a simple criterion, given by Kobayashi in [5], [6], for discrete decomposability of $A_q$, when regarded as module over a reductive subalgebra. In fact, using this criterion, Kobayashi and Oshima list in [9] all symmetric pairs $(\mathfrak{g}_0, \mathfrak{g}_0')$ and the modules $A_q$, such that $A_q$ is discretely decomposable as a $(\mathfrak{g}', K')$-module. Our strategy is to list all the cohomologies with $A_q$ coefficients that are non-zero in the dimension of a geometric class and eliminate as many as possible based on the vanishing results. In some cases, this method of elimination leaves out just one $A_q$ coefficient. This allows us to conclude that our geometric class has that particular component. Thus vanishing results allow us to obtain non-vanishing results in certain cases. Following is our main theorem.

**Theorem 1.2.** Let $G = SU(p,q), p \leq q$. Consider the $(\mathfrak{g}, K)$-modules $A_q$, with parameters $\lambda$ and conditions on $p, q$ as given below. (Refer to §5 to see how these parameters are defined.)

1. $\lambda = \epsilon_1 - \epsilon_p$ and $5 \leq p \leq q \leq 2p - 2$. 

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(2) \( \lambda = \epsilon_{p+1} - \epsilon_{p+q} \) and \( 5 \leq p \leq q, p \neq q - 1 \).

For each of these \( A_{q/\lambda} \), there exists a torsion-free uniform lattice \( \Gamma \) in \( G \) and a geometric class in \( H^*(\Gamma \backslash G/K; \mathbb{C}) \) having an \( A_{q/\lambda} \)-component. Hence all these \( A_{q/\lambda} \) occur with non-zero multiplicity in \( L^2(\Gamma \backslash G) \).

Now we give a section wise description. In §2 we give a description of Matsushima’s decomposition, the groups \( H^* (g, K; A_q) \) and the Hodge bigrading on it when \( g_0 \) is associated to a Hermitian symmetric space. In §3 we describe a construction which ensures that given an involution \( \sigma \) of \( G \), there exists a maximal compact subalgebra \( K \) and a uniform torsion-free lattice \( \Gamma \) of \( G \), such that \( \sigma \) induces an involution of \( \Gamma \backslash G/K \) and the Poincaré dual of the fixed point submanifold is not \( G \)-invariant. In §4 we extend Kobayashi and Oda’s vanishing result. In §5 we prove Theorem 1.

2. MATSUSHIMA DECOMPOSITION OF \( H^*(\Gamma \backslash G/K; \mathbb{C}) \)

In this section we wish to describe the de Rham cohomology of \( X_\Gamma := \Gamma \backslash G/K \) in terms of cohomology of \( (g, K) \)-modules. Let us first do this for the universal cover \( X := G/K \). At each point of a manifold, a complex valued differential \( m \) form takes an element of the \( m \)th exterior product of the complexified tangent space and returns a complex number, and it does so in a smooth manner. Since \( X \) is a homogeneous manifold the tangent spaces and their exterior products at each point can be identified with those at \( eK \) via the action of \( G \). Let \( g_0 = k_0 \oplus p_0 \) be the Cartan decomposition with respect to a Cartan involution \( \theta \). The tangent space of \( X \) at \( eK \) can be identified with \( p_0 \). Thus a complex valued differential \( m \) form on \( X \) can be thought of as an element in \( \text{Hom}_K(\wedge^m p, C^\infty(G)) \), where \( C^\infty(G) \) is the space of complex valued smooth functions on \( G \) equipped with the right translation action of \( G \). The \( K \) equivariance takes care of the ambiguity arising from the non-uniqueness of elements of \( G \) that can be chosen for identifying the complexified tangent space at a point with \( p \). Since \( X \) is a cover of \( X_\Gamma \), the differential forms of \( X_\Gamma \) are just \( \Gamma \)-invariant differential forms of \( X \). Thus a complex valued differential \( m \) form of \( X_\Gamma \) can be identified with an element in \( \text{Hom}_K(\wedge^m p, C^\infty(\Gamma \backslash G)) \). Since \( \wedge^m p \) is finite dimensional, its image under a \( K \)-homomorphism must land in the \( K \)-finite part \( C^\infty(\Gamma \backslash G)_K \) of \( C^\infty(\Gamma \backslash G) \). For any \( (g, K) \)-module \( V \), let \( C^*(g, K; V) \) denote the cochain complex of the relative Lie algebra cohomology of \( (g, K) \) with coefficients in \( V \), where the individual cochain groups are defined as \( C^m(g, K; V) = \text{Hom}_K(\wedge^m p, V) \). To summarise the above discussion, we have an isomorphism of chain complexes of
the de Rham complex $\Omega^*(X_\Gamma; \mathbb{C})$ and $C^*(\mathfrak{g},K;C^\infty(\Gamma\backslash G)_K)$. See [2, Chapter VII, Proposition 2.5].

If $M$ is a compact oriented Riemannian manifold then there is an inner product on $\Omega^*(M;\mathbb{C})$ given by $\langle \omega, \eta \rangle = \int_M \omega \wedge *\eta$, where $*$ is the Hodge star operator. On the other hand any relative Lie algebra cohomology with coefficients in a unitary ($\mathfrak{g},K$)-module $V$ has an inner product on it defined as follows. The Killing form on the de Rham complex $\Omega^*(\Gamma\backslash G;\mathbb{C})$ of complex valued harmonic forms. On the other hand any relative Lie algebra cohomology with coefficients in a unitary ($\mathfrak{g},K$)-module $V$ has an inner product on it defined as follows. The Killing form on $\mathfrak{g}_0$ restricts to an inner product on $\mathfrak{p}_0$. The representation on $V$ being unitary, it comes equipped with an inner product. So via the isomorphism $\text{Hom}_K(\wedge^m\mathfrak{p},V) \cong (\wedge^m\mathfrak{p}^* \otimes V)^K$ we get an inner product on $\text{Hom}_K(\wedge^*\mathfrak{p},V)$. In particular if $V = C^\infty(\Gamma\backslash G)_K$, then inner product on $V$ comes from the inclusion $C^\infty(\Gamma\backslash G)_K \subset L^2(\Gamma\backslash G)$. Thus both the vector spaces $\Omega^*(X_\Gamma;\mathbb{C})$ and $C^*(\mathfrak{g},K;C^\infty(\Gamma\backslash G)_K)$ are equipped with inner products. The isomorphism of chain complexes between them is also an isometry.

As in case of de Rham complexes of compact orientable Riemannian manifolds, we may define the Laplace operator $\Delta$ on $C^*(\mathfrak{g},K;V)$, for any ($\mathfrak{g},K$)-module $V$, as $\Delta = d\partial + \partial d$, where $\partial$ is adjoint of $d$ with respect to the inner product. The action of the Laplace operator is equal to that of the Casimir element $c \in U(\mathfrak{g})$. If $V$ is an irreducible ($\mathfrak{g},K$)-module then $H^*(\mathfrak{g},K;V) \neq 0$ only if the action of Casimir element is trivial. In this case $H^*(\mathfrak{g},K;V) = C^*(\mathfrak{g},K;V) = \text{Hom}_K(\wedge^*\mathfrak{p},V)$. See [2, Chapter II, Proposition 3.1].

By Hodge theory $H^*(X_\Gamma,\mathbb{C}) = \mathcal{H}^*(X_\Gamma)$, where $\mathcal{H}^*(X_\Gamma)$ is the space of complex valued harmonic forms. On the other hand $H^*(X_\Gamma,\mathbb{C}) = H^*(\mathfrak{g},K;C^\infty(\Gamma\backslash G)_K)$. The ($\mathfrak{g},K$)-module $C^\infty(\Gamma\backslash G)_K$ is the Harish Chandra module of $L^2(\Gamma\backslash G)$. By [3, Chapter 1, §2], $L^2(\Gamma\backslash G)$ is discretely decomposable as a $G$-module: $L^2(\Gamma\backslash G) \cong \bigoplus_{\pi \in \hat{G}} m(\pi,\Gamma)H_\pi$, where the multiplicities $m(\pi,\Gamma)$ are finite. From all this information one can deduce the Matsushima decomposition:

$$\mathcal{H}^*(X_\Gamma) \cong \bigoplus_{\pi \in \hat{G}_0} m(\pi,\Gamma)\text{Hom}_K(\wedge^*\mathfrak{p},H_{\pi,K}),$$

where $\hat{G}_0 := \{ \pi \in \hat{G} : \pi(c) = 0 \}$, where $c$ is the Casimir element and, by abuse of notation, $\pi$ is also the corresponding action of $U(\mathfrak{g})$ on the Harish-Chandra module $H_{\pi,K}$. See [2, Chapter VII, Corollary 3.3].

**Definition** [8 (2.3.2)] For $\pi \in \hat{G}_0$, we define the $\pi$-component of $\mathcal{H}^*(X_\Gamma) \subset \text{Hom}_K(\wedge^*\mathfrak{p},C^\infty(\Gamma\backslash G)_K)$ to be consisting of those homomorphisms whose images are contained in the $\pi$-isotypical component of $C^\infty(\Gamma\backslash G)_K$. That is, $\omega \in \pi$-component if and only if it is a sum of homomorphisms that factor through $H_{\pi,K}$:
\[ \wedge^q p \xrightarrow{\psi} H_{\pi,K} \xrightarrow{\phi} C^\infty(\Gamma \backslash G)_K, \]

where \( \psi \) is a \( K \)-map and \( \phi \) is a \((g, K)\)-map.

The relative Lie algebra cohomology \( H^*(g, K; H_{\pi,K}) \) is non-zero if and only if \( H_{\pi,K} \) is of the form \( A_q \), where \( q \) is a \( \theta \)-stable parabolic subalgebra of \( g_0 \) and \( A_q \) is the \((g, K)\)-module obtained by cohomological induction on the trivial one dimensional \((q, L \cap K)\)-module. (The subgroup \( L \) is defined below.) See [15, Theorem 4.1]. Let \( q = l + u \), where \( l \) is the Levi part and \( u \) is the nilpotent radical. By definition of \( \theta \)-stable parabolic subalgebra, \( l = q \cap \bar{q} \) and hence \( l_0 := l \cap g_0 \) is a real form of \( l \). Let \( L \) be the connected subgroup of \( G \) with Lie algebra \( l_0 \). By [15, Theorem 3.3], we have

\[ H^k(g, K; A_q) = H^{k-R(q)}(l, L \cap K; \mathbb{C}) = \text{Hom}_{L \cap K}(\wedge^{k-R(q)}(l \cap p), \mathbb{C}), \]

where \( R(q) = \dim(u \cap p) \). The value of the shift \( R(q) \) is 0 if and only if \( q = g \) and in this case \( A_q = \mathbb{C} \), where the action of \((g, K) \) on \( \mathbb{C} \) is trivial. Let us denote this trivial one dimensional representation by \((1, \mathbb{C}) \). Then \( m(1, \Gamma) = 1 \) for all uniform lattice \( \Gamma \). In fact this is the submodule of \( C^\infty(\Gamma \backslash G)_K \) consisting of constant functions. So the corresponding forms are the \( G \)-invariant ones.

**Lemma 2.1.** [8, Lemma 2.4] The only \( \pi \)-component that contributes to the top cohomology group is the 1-component.

Now suppose \( X \) is Hermitian symmetric. Then \( X_\Gamma \) is a Kähler manifold and hence its cohomology has a Hodge bigrading. In this case, relative Lie algebra cohomology of \((g, K) \) with coefficient in any module also has a Hodge bigrading. See [2, Chapter II, Corollary 4.5]. Taking this into account, Matsushima’s decomposition can be written as

\[ H^{p,q}(X_\Gamma) \cong \bigoplus_{\pi \in \mathcal{G}_0} m(\pi, \Gamma)H^{p,q}(g, K; H_{\pi,K}). \]

The complex structure \( J \) on \( p_0 \) induces one on \( l_0 \cap p_0 \). Hence the groups \( H^*(l, L \cap K; \mathbb{C}) \) also have Hodge decomposition. By [15, Proposition 6.19], the Hodge bigradings of \( H^*(g, K; A_q) \) and \( H^*(l, L \cap K; \mathbb{C}) \) are related as

\[ H^{p,q}(g, K; A_q) \cong H^{p-R^+(q),q-R^-(q)}(l, L \cap K; \mathbb{C}), \]

where \( R^+(q) = \dim(u \cap p^+) \) and \( R^-(q) = \dim(u \cap p^-) \). Here \( p^+ \) and \( p^- \) are the \( +i \) and \( -i \) eigenspaces of \( J \) in \( p \). In this case \( L_u/(L_u \cap K) \) is a compact Hermitian symmetric space and hence all its cohomology classes are of Hodge type \((p, p) \). Thus \( H^{p,q}(g, K; A_q) \neq 0 \) if and only if
\[ p - q = R^+(q) - R^-(q). \] We record the particular case of \( p = q \) as a lemma.

**Lemma 2.2.** The vector space \( H^{p,p}(q, K; A_q) \neq 0 \) only if \( A_q \) satisfies \( R^+(q) = R^-(q) \).

### 3. Construction of geometric classes

The construction of non-\( G \)-invariant geometric class in \( H^*(\Gamma \backslash G/K; \mathbb{C}) \), for some torsion-free uniform lattice \( \Gamma \), has reduced to fixing an involution of \( G \), thanks to the results of several people. The first crucial step in this direction was taken by Millson and Raghunathan in [10]. They show that if one can construct pairs of complementary dimensional totally geodesic submanifolds of \( \Gamma \backslash G/K \) such that they intersect in a finite set and the intersection number at each point of this finite set is positive, then going to a finite cover if necessary, the Poincaré duals of these submanifolds are not \( G \)-invariant. See [10, Theorem 2.1]. To construct complementary dimensional submanifolds they start with an involution \( \sigma \) of \( G \) which commutes with a Cartan involution \( \theta \) and assume that there exists a uniform torsion-free arithmetic lattice which is invariant under both \( \sigma \) and \( \theta \). That, given a \( \sigma \), such a Cartan involution and a cocompact lattice will always exist was later proved by Raghunathan in [13] using an earlier construction of Borel [1]. Since \( \sigma \theta = \theta \sigma \), the fixed point subgroup \( K \) of \( \theta \) in \( G \) remains invariant under \( \sigma \). Let \( A(\sigma) \) denote the fixed point subset of a set \( A \) under the action of \( \sigma \). Then \( \Gamma(\sigma) \backslash G(\sigma)/K(\sigma) \) and \( \Gamma(\theta \sigma) \backslash G(\theta \sigma)/K(\theta \sigma) \) are their candidates for submanifolds whose intersection numbers are all positive. Note that \( G(\sigma)/K(\sigma) \) and \( G(\theta \sigma)/K(\theta \sigma) \) are complementary dimensional and intersect at exactly one point. Then a criterion, using the first Galois cohomology of algebraic groups, for \( \Gamma(\sigma) \backslash G(\sigma)/K(\sigma) \) and \( \Gamma(\theta \sigma) \backslash G(\theta \sigma)/K(\theta \sigma) \) to intersect in a finite set with all positive intersection numbers, is given. Using this criterion they obtain non-vanishing results in case of \( G = \text{Sp}(p, q), \text{SU}(p, q) \) and \( \text{SO}(p, q) \). Rohlfs and Schwermer in [14] remove the necessity of this criterion by showing that, under a mild orientability condition, the intersection numbers will always be positive if we go to a finite cover. Their result is proved in a greater generality, but we will not need it here. We summarize this discussion in the form of a theorem.

**Theorem 3.1.**

(1) (Borel [1], Raghunathan [13]) Let \( G \) be a connected linear semisimple Lie group. Let \( \sigma \) be any involution of \( G \). Then there exists a global
Cartan involution $\theta$ of $G$ such that $\sigma \theta = \theta \sigma$ and a cocompact arithmetic lattice $\Lambda$ in $G$ which is invariant under $\sigma$ and $\theta$.

(2) (Rohlfs and Schwermer [14]) With $\sigma, \theta$ as above and $G(\theta) = K$, $\sigma$ induces isometries of $G/K$ and $\Gamma\backslash G/K$ which we again denote by $\sigma$. Under an orientability condition, which is always satisfied if $G/K$ is Hermitian and $\sigma$ is holomorphic, there exists a finite index torsion-free subgroup $\Gamma$ of $\Lambda$, which is again invariant under $\sigma$ and $\theta$, such that $\Gamma(\sigma)\backslash G(\sigma)/K(\sigma)$ and $\Gamma(\theta \sigma)\backslash G(\theta \sigma)/K(\theta \sigma)$ have all their intersection numbers positive.

(3) (Millson and Raghunathan [10]) Replacing $\Gamma$ by a finite index subgroup if necessary, the Poincaré duals of $\Gamma(\sigma)\backslash G(\sigma)/K(\sigma)$ and $\Gamma(\theta \sigma)\backslash G(\theta \sigma)/K(\theta \sigma)$ are not $G$-invariant.

As indicated before we will concentrate on the case where $G/K$ is Hermitian. Assuming $G$ is simple, this happens if and only if $K$ has a non-discrete centre. In fact in this case $Z(K) \cong S^1$ and the complex structure is given by $\text{Ad}(j)|_{p_0}$, where $j \in Z(K)$ such that order of $\text{Ad}(j)$ is 4. See [4, Chapter VII, §6]. Thus an involution $\sigma$ of $G$ is holomorphic if and only if $\sigma$ point-wise fixes $Z(K)$. In this case, both $\Gamma(\sigma)\backslash G(\sigma)/K(\sigma)$ and $\Gamma(\theta \sigma)\backslash G(\theta \sigma)/K(\theta \sigma)$ are complex analytic submanifolds. Hence their Poincaré duals are of $(p,p)$-type. This observation along with Lemma 2.2 yields the following vanishing result.

**Lemma 3.2.** Let $G/K$ be a Hermitian symmetric space and $\sigma$ be an involution of $G$ that keeps $Z(K)$ fixed (pointwise). Let $\omega$ and $\omega'$ be the pair of geometric classes associated to $\sigma$, as explained in Theorem 3.1. If $A_q$ satisfies $R^+(q) \neq R^-(q)$, then $\omega$ and $\omega'$ have no $A_q$-component.

**Remark** Even this simple vanishing result yields non-vanishing ones in quite a few cases. See [11, Theorem 1.1].

4. **A non-trivial vanishing result**

Now we come to the most important vanishing result that we will use. Let $\sigma$ be an involutive automorphism of $G$ which commutes with a Cartan involution $\theta$. Since we will be dealing with only one involution here, we change the notation, for fixed point subset of a set $A$ under $\sigma$, from $A(\sigma)$ to $A'$. Let $\Gamma$ be a $\sigma$-invariant torsion-free uniform lattice in $G$ such that $\Gamma'$ is a lattice in $G'$. (We will always assume that $G'$ is non-compact.) Then $Y_\Gamma := \Gamma'\backslash G'/K'$ is a totally geodesic submanifold of $X_\Gamma := \Gamma\backslash G/K$. Let $i : Y_\Gamma \hookrightarrow X_\Gamma$ be the inclusion map. Let $\mathcal{P}(Y_\Gamma)$ denote the harmonic form that represents the Poincaré dual of $Y_\Gamma$. Given an irreducible unitary representation $(\pi, V_\pi)$ of $(\mathfrak{g}, K)$, on which the Casimir operator acts trivially, Kobayashi and Oda in [8] Theorem
2.8] gave a criterion for $P(Y_Γ)$ to not have a $π$-component. One of the conditions is **discrete decomposability** which we now define.

**Definition** Let $V$ be a unitary $(g, K)$-module. We say that $V$ is **discretely decomposable** if it can be written as a direct sum of irreducible $(g, K)$-modules.

**Remark** In [6, §1], Kobayashi gives a more general definition of discrete decomposability of any $(g, K)$-module and proves that his definition is equivalent to the one above in the unitary case. We work with the above definition since it is more intuitive.

We update Kobayashi and Oda’s result by removing one of the conditions and present it as two separate statements in the form of a Theorem and a Corollary. This is done to bring out the key point of their result. The proof follows their approach.

**Theorem 4.1.** With notations as above, if $\text{Hom}(g', K')(V_π, C) = 0$, where the action of $(g', K')$ on $C$ is trivial, then $P(Y_Γ)$ does not have a $π$-component.

**Corollary 4.2.** Let $G$ be simple. If $1 \neq V_π$ is discretely decomposable as a $(g', K')$-module, then $P(Y_Γ)$ does not have a $π$-component.

**Proof. of Theorem.** Let $\langle \ , \ \rangle$ be the inner product on $H^*(X_Γ)$. Since the components are all mutually orthogonal, it is enough to show that $\langle P(Y_Γ), \omega \rangle = 0$, for all $\omega \in π$-component. By definition of Poincaré dual, $\langle P(Y_Γ), \omega \rangle = \int_{X_Γ} P(Y_Γ) \wedge *\omega = \int_{Y_Γ} \iota^*(*ω)$, where $\iota^*$ is the pull back induced by the inclusion $\iota$. Since the Hodge $*$ operator takes the $π$-component to itself, it is enough to prove that $\int_{Y_Γ} \iota^*(\omega) = 0$ for all $\omega \in π$-component.

Recall from [2] $Ω^*(X_Γ; C) \cong \text{Hom}_K(∧^*p, C^∞(Γ\backslash G)_K)$ and similarly for $Ω^*(Y_Γ; C)$. Let us understand the induced map $\iota^* : \text{Hom}_K(∧^*p, C^∞(Γ\backslash G)_K) \rightarrow \text{Hom}_{K'}(∧^*p', C^∞(Γ\backslash G')_{K'})$. Let $ω : ∧^*p \rightarrow C^∞(Γ\backslash G)_K$ be a $K$-equivariant homomorphism. Then its image under $\iota^*$ is given by the lower horizontal map in the figure below which makes the diagram commutes.

\[
\begin{array}{ccc}
∧^*p & \xrightarrow{ω} & C^∞(Γ\backslash G)_K \\
j & & \iota^* \\
∧^*p' & \xrightarrow{\omega} & C^∞(Γ\backslash G')_{K'}
\end{array}
\]

Here $j : ∧^*p' \rightarrow ∧^*p$ is induced by the natural inclusion $p' \hookrightarrow p$ and $i^* : C^∞(Γ\backslash G)_K \rightarrow C^∞(Γ\backslash G')_{K'}$ is the restriction map. Thus $i^*(ω) = i^* \circ ω \circ j$. Since $ω$ is in the $π$-component, we may assume that it is
of the form \( \phi \circ \psi \), where \( \phi : \wedge^* p \to V_\pi \) is a \( K \)-equivariant map and \( \psi : V_\pi \to C^\infty(\Gamma \setminus G)_K \) is a \((g, K)\)-equivariant map.

\[
\begin{array}{c}
\wedge^* p \\
\downarrow j \\
\wedge^* p' \\
\end{array}
\xrightarrow{\psi} V_\pi
\xrightarrow{\phi} C^\infty(\Gamma \setminus G)_K
\]

If \( i^*(\omega) \) is not a top cohomology class of \( Y \), then \( \int_{Y_\Gamma} i^*(\omega) = 0 \). So let us assume that \( i^*(\omega) \) is a top cohomology class. Now by Lemma 2.1 a top dimensional form is not exact if and only if it has a non-zero \( G' \)-invariant component. Thus \( i^*(\omega) \neq 0 \) means that its image has a component in the trivial one dimensional sub-representation of \( C^\infty(\Gamma' \setminus G')_{K'} \), consisting of constant functions. Let \( \text{pr}_1 : C^\infty(\Gamma' \setminus G')_{K'} \to \mathbb{C} \) be the projection map to the subspace of constant functions. Then the composition \( \text{pr}_1 \circ i^* \circ \phi \circ \psi \circ j \) is a non-trivial homomorphism. In particular \( \text{pr}_1 \circ i^* \circ \phi : V_\pi \to \mathbb{C} \) is a non-trivial \((g', K')\)-homomorphism. But this contradicts our hypothesis. Hence \( \mathcal{P}(Y_\Gamma) \) does not have a \( \pi \)-component.

**Remark** It was brought to our notice that in his 2009 preprint \[7\] Kobayashi had proved that if \( \text{Hom}_{g,K}(V_\pi, C^\infty(G' \setminus G)) = 0 \) then \( \mathcal{P}(Y_\Gamma) \) has no \( \pi \)-component. This is same as Theorem 4.1 since by Frobenius reciprocity \( \text{Hom}_{g,K}(V_\pi, C^\infty(G' \setminus G)) = \text{Hom}_{g',K'}(V_\pi, \mathbb{C}) \). But the proof is different. Kobayashi constructs an intertwining operator \( R : C^\infty(\Gamma \setminus G) \to C^\infty(G' \setminus G) \) by averaging over \( \Gamma' \setminus G' \). Then he shows that \( \int_{Y_\Gamma} i^*(\omega) = \int_{Y_\Gamma} i^*(\phi \circ \psi) = \text{ev}_e \circ R \circ \phi \circ \psi \circ j(\text{vol}) \), where \( \text{ev}_e \) is the evaluation map at \( e \) and \( \text{vol} \) is the oriented norm one element of \( \wedge^{\dim p} p \). Since \( R \circ \phi \in \text{Hom}_{g,K}(V_\pi, C^\infty(G' \setminus G)) = 0 \), we have \( \int_{Y_\Gamma} i^*(\omega) = 0 \).

**Proof. of Corollary.** We will show that if \( V_\pi \) is discretely decomposable as a \((g', K')\)-module, then it satisfies the condition of Theorem 4.1. We have \( V_\pi \cong \bigoplus_{\tau} m_\pi(\tau)V_\tau \), where \( \tau \) runs over irreducible unitary \((g', K')\)-modules and \( m_\pi(\tau) \) are the multiplicities. By [6, Proposition 1.6], \( V_\pi \cong \bigoplus_{\tau} m_\pi(\tau)V_\tau \), where \( V_\tau \) and \( V_\pi \) are the representation spaces of \( G' \) obtained by taking completion of \( V_\tau \) and \( V_\pi \), respectively. Because of discrete decomposability we have \( m_\pi(1) = \dim \text{Hom}_{g',K'}(\mathbb{C}, V_\pi) = \dim \text{Hom}_{g',K'}(V_\pi, \mathbb{C}) \). Let us assume that \( \dim \text{Hom}_{g',K'}(V_\pi, \mathbb{C}) \neq 0 \). Then \((1, \mathbb{C}) = (1, \mathbb{C})\) occurs as a subrepresentation of \((\pi, V_\pi)\). Now
we state a result of Moore [12, Theorem 1]. Suppose that $G'$ is a non-compact subgroup of a simple Lie group $G$. Then $G'$ has the property that if the restriction to $G'$, of any unitary representation of $G$, has a fixed vector then this vector is also fixed under the $G$ action. In our context this means that $\overline{V}_\pi$ has a one dimensional $G$-invariant subspace. But it is given that $V_\pi$, and hence $\overline{V}_\pi$, is irreducible. Thus we arrive at a contradiction. This finishes the proof.

**Remark** For concrete calculations we will only be using the weaker vanishing result of Corollary 4.2. The reason for this is that there is a simple criterion given by Kobayashi to check discrete decomposability, while we are not aware of a simple criterion to check if $\text{Hom}_{G',K'}(V_\pi, \mathbb{C}) = 0$. The only independent application of Theorem 4.1, that we know of, is the result that the non-$G$-invariant part of the Poincaré dual of a to-tally geodesic submanifold of $\Gamma \backslash SO_0(2,p)/SO(2) \times SO(p)$, of the form $\Gamma\backslash SO_0(1,p)/SO(p)$, has Hodge type $(p,0)$ or $(0,p)$. In fact one can specify two $\theta$-stable Borel subalgebras $b_1$ and $b_2$, such that the cohomology class must have a non-zero $A_{b_i}$-component, for at least one $i = 1, 2$. This is shown in [7]. It will be interesting to see more such direct applications of Theorem 4.1.

5. **Proof of Theorem 1.2**

We wish to apply the aforementioned vanishing results to concrete examples of compact Hermitian locally symmetric spaces associated to simple Lie groups. We say $(\mathfrak{g}_0, \mathfrak{g}_0')$ is a *Hermitian symmetric pair* if the centre $Z(\mathfrak{k}_0)$ of $\mathfrak{k}_0$ is non-trivial and $Z(\mathfrak{k}_0) \subset \mathfrak{g}_0'$. For applying Lemma 3.2 we must have that $(\mathfrak{g}_0, \mathfrak{g}_0')$ is a Hermitian symmetric pair. On the other hand for applying Corollary 4.2 we must know for which $\theta$-stable parabolic subalgebras $\mathfrak{q}$, $A_\mathfrak{q}$ is discretely decomposable as a $(\mathfrak{g}_0', K')$-module. This has already been done by Kobayashi and Oshima in [9] based on the criterion given by Kobayashi in [5], [6]. From now on we assume that $(\mathfrak{g}_0, \mathfrak{g}_0')$ is a Hermitian symmetric pair. In this case $\mathfrak{p} = \mathfrak{p}^+ + \mathfrak{p}^-$, where $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are the $+i$ and $-i$ eigenspaces of the complex structure $J$. A $\theta$-stable parabolic subalgebra $\mathfrak{q}$ is called *holomorphic* or *anti-holomorphic* if $\mathfrak{p}^+ \subset \mathfrak{q}$ or $\mathfrak{p}^- \subset \mathfrak{q}$, respectively. By [9, Theorem 4.1 (4)], $A_\mathfrak{q}$ is discretely decomposable as a $(\mathfrak{g}_0', K')$-module if $\mathfrak{q}$ is holomorphic or anti-holomorphic. Note that $\mathfrak{q}$ is holomorphic or anti-holomorphic then the pair $(R^+(\mathfrak{q}), R^-(\mathfrak{q}))$ is of the form $(p,0)$ or $(0,q)$, respectively. Hence both Lemma 3.2 and Corollary 4.2 applies. For the cases where $\mathfrak{q}$ is neither holomorphic nor anti-holomorphic, a list is given in [9, Tables C.3, C.4], of triples $(\mathfrak{g}_0, \mathfrak{g}_0', \mathfrak{q})$ which satisfy that $A_\mathfrak{q}$ is discretely decomposable as a $(\mathfrak{g}_0', K')$-module. (Strictly speaking,
this list includes, rather than consists of, all the cases where \( q \) is neither holomorphic or anti-holomorphic.) Let \((\mathfrak{g}_0, \mathfrak{g}_0')\) be a pair and \( D \) be the family of \( \theta \)-stable parabolic subalgebras \( q \) of \( \mathfrak{g}_0 \), such that \( A_q \) is discretely decomposable as a \((\mathfrak{g}', K')\)-module. Let \( G \) be a linear Lie group with Lie algebra \( \mathfrak{g}_0 \). By Theorem 3.1 any involution \( \sigma \) of \( G \) (and hence of \( \mathfrak{g}_0 \)) induces an involution of \( X_\Gamma = \Gamma \backslash G/K \), for some maximal compact \( K \) and some torsion-free uniform lattice \( \Gamma \) in \( G \), and the Poincaré dual \( \mathcal{P}(Y_\Gamma) \) of the resulting fixed point submanifold \( Y_\Gamma \) is not \( G \)-invariant. Moreover the fixed point submanifold of the involution of \( X_\Gamma \) induced by \( \sigma \theta \) is complementary dimensional to \( Y_\Gamma \) and its Poincaré dual is again \( G \)-invariant. Let \( t \) be the minimum of the complex dimensions of the classes \( \mathcal{P}(Y_\Gamma) \) and its complement. Let \( Q \) be the set of all \( \theta \)-stable parabolic subalgebras \( q \), which satisfy \( R^+(q) = R^-(q) \leq t \). Then by Lemma 3.2 and Corollary 4.2 we can conclude that \( \mathcal{P}(Y_\Gamma) \) has a non-zero \( A_q \)-component for some \( q \in Q \setminus D \). We are only interested in the cases where \( Q \setminus D \) is singleton, so that we get a precise non-vanishing result. This is the idea of proof of Theorem 1.2.

Now let us explain some notations. Let \( \mathfrak{t}_0 \) be a maximal abelian subalgebra of \( \mathfrak{t} \). Fix a subset \( \Phi^+_t \subset \Phi_t \) of positive roots of \((\mathfrak{t}, \mathfrak{t})\). Let \( \Phi_n \) denote the set of weights of the adjoint representation of \( \mathfrak{t} \) on \( \mathfrak{p} \). Let \( \Phi := \Phi_t \cup \Phi_n \). For each \( \alpha \in \Phi \) let \( \mathfrak{g}_\alpha \) denote the corresponding root or weight space. Then any \( \theta \)-stable parabolic subalgebra (up to the equivalence \( q \sim q' \) if \( A_q \cong A_{q'} \)) is of the form \( q_\lambda = \bigoplus_{(\lambda, \alpha) \geq 0} \mathfrak{g}_\alpha \), where \( \lambda \) is in some Euclidean space \( \mathbb{E} \) containing \((it_0)^*\) as a sub-Euclidean space and \( \lambda \) is dominant with respect to \( \Phi_0^+ \). We could have taken \( \mathbb{E} \) to be equal to \((it_0)^*\), but we allow a larger space for ease of calculation. The \( \theta \)-stable parabolic subalgebras will be identified by this parameter \( \lambda \). If we fix a basis \((\epsilon_1, \cdots, \epsilon_m)\) of \( \mathbb{E} \), then \( \lambda \) can be written as \( \sum_{i=1}^m a_i \epsilon_i \). For \( \mathfrak{g}_0 = \mathfrak{su}(p, q) \), we now indicate the Euclidean space \( \mathbb{E} \), a set of positive roots of \((\mathfrak{t}, \mathfrak{t})\) and the set of weights of \( \mathfrak{p} \).

\( \mathfrak{g}_0 = \mathfrak{su}(p, q) \).
\( \mathbb{E} := \mathbb{R}^{p+q} \) with standard orthonormal basis \((\epsilon_1, \cdots, \epsilon_{p+q})\).
\( \Phi_t^+ = \{ \epsilon_i - \epsilon_j : 1 \leq i < j \leq p \text{ and } p + 1 \leq i < j \leq p + q \} \).
\( \Phi_n = \{ \pm (\epsilon_i - \epsilon_j) : 1 \leq i \leq p \text{ and } p + 1 \leq j \leq p + q \} \).
Dominant condition: \( a_1 \geq \cdots \geq a_p \text{ and } a_{p+1} \geq \cdots \geq a_{p+q} \).

In Table 1 we indicate the pairs \((\mathfrak{g}_0, \mathfrak{g}_0')\), where \( \mathfrak{g}_0 = \mathfrak{su}(p, q) \), and the coordinates of \( \lambda \), such that \( A_{\lambda} \) is discretely decomposable as a \((\mathfrak{g}', K')\)-module. This is taken from [9, Table C.3],
Proof. of Theorem 1.2. We retain the notations used before in this section. We consider the pairs in Table 1 with \( k = 1 \). We first deal with the pair in the first row of Table 1 with \( k = 1 \). In this case \( t = q \). Let \( \lambda = \sum_{i=1}^{p+q} a_i \varepsilon_i \) be such that, \( q_\lambda \in Q \setminus D \). Then we must have \( R^+(q_\lambda) = R^-(q_\lambda) \leq q \) and

\[
1 \quad a_1 > a_s > a_p, \text{ for some } p + 1 \leq s \leq p + q.
\]

Fix \( s \). Let \( x, y, z, l, m, n \in \mathbb{N} \) satisfying \( x + y + z = p \) and \( l + m + n = q \), such that, \( a_i > a_s \) for \( 1 \leq i \leq x \) and \( p + 1 \leq i \leq p + l \), \( a_i = a_s \) for \( x + 1 \leq i \leq x + y \) and \( p + l + 1 \leq i \leq p + l + m \), \( a_s > a_i \) for \( x + y + 1 \leq i \leq p \) and \( p + l + m + 1 \leq i \leq p + q \). Note that by (1), \( x, z, m \geq 1 \). Then we have

\[
2 \quad q \geq R^+(q_\lambda) \geq x(m + n) + y = n(x + y) + mx
\]

\[
3 \quad q \geq R^-(q_\lambda) \geq z(l + m) + y = l(y + z) + mz
\]

\[
4 \quad 2q \geq R(q_\lambda) \geq x(m + n) + y(n + l) + z(l + m)
\]

From (4) we get that at least one of \( m + n, n + l \) and \( l + m \) is less than or equal to 2. Let us assume \( l + m \leq 2 \). We will show a contradiction. Since \( n \geq q - 2 \), (2) implies \( R^+(q_\lambda) \geq (q - 2)(x + y) + xm \). If \( x + y \geq 2 \) then \( R^+(q_\lambda) \geq 2(q - 2) + 1 > q \), since \( q \geq 5 \). Thus \( x + y \leq 1 \), which implies \( x = 1 \) and \( y = 0 \). Hence \( z = p - 1 \). From (3) we get \( R^-(q_\lambda) \geq (p - 1)(l + m) + ly \). If \( l + m = 2 \) then \( R^-(q_\lambda) \geq 2(p - 1) > q \), by assumption. Thus \( l + m \leq 1 \) which implies \( l = 0 \) and \( m = 1 \). Hence \( n = q - 1 \). This implies \( R^+(q_\lambda) \geq q \) and \( R^-(q_\lambda) \geq p - 1 \). But then \( R^+(q_\lambda) = q \). This implies \( a_1 > a_{p+1} > a_{p+2} \geq \cdots \geq a_{p+q} \geq a_2 \geq \cdots \geq a_p \). The inequality \( a_{p+q} \geq a_2 \) must be strict since otherwise \( R^-(q) = (p - 1)q > q \). Hence \( R^-(q_\lambda) = p - 1 \). This is a contradiction.

| Hermitian symmetric pairs | coordinates of \( \lambda \) |
|----------------------------|-------------------------------|
| \( \mathfrak{su}(p, q), \mathfrak{su}(k) \oplus \mathfrak{su}(p - k, q) \oplus \mathfrak{u}(1) \), \( q \geq p > k \) | \( a_p \geq a_{p+1} \), \( \exists p + 1 \leq l \leq p + q - 1 \) \( a_l \geq a_1 \) and \( a_p \geq a_{l+1} \), or \( a_{p+q} \geq a_1 \) |
| \( \mathfrak{su}(p, q), \mathfrak{su}(p, q - k) \oplus \mathfrak{su}(k) \oplus \mathfrak{u}(1) \), \( q \geq p, k < q \) | \( a_{p+q} \geq a_1 \), \( \exists 1 \leq l \leq p - 1 \) \( a_l \geq a_{p+1} \) and \( a_{p+q} \geq a_{l+1} \), or \( a_p \geq a_{p+1} \) |

Table 1. Discretely decomposable \( A_{q_\lambda} \) when \( \mathfrak{g}_0 = \mathfrak{su}(p, q) \)
since $R^+(q,\lambda) \neq R^-(q,\lambda)$. Thus $l + m > 2$. Similarly $m + n > 2$. Thus we must have $l + n \leq 2$. So $m \geq q - 2$. Then (2) and (3) implies that $R^+(q,\lambda) \geq (q - 2)x$ and $R^-(q,\lambda) \geq (q - 2)z$. Since $q \geq 5$ we must have $x = 1 = z$. Hence $y = p - 2$. If $l \geq 1$ then (3) implies that $R^-(q,\lambda) \geq p - 1 + q - 2 > q$, which is a contradiction. Hence $l = 0$. Similarly (2) implies $n = 0$. Thus $m = q$ and we have

$$a_1 > a_2 = \cdots = a_{p-1} = a_{p+1} = \cdots = a_{p+q} > a_p.$$ 

(5)

All the parameters $\lambda = \sum_{i=1}^{p+q} a_i$, satisfying (5), give the same $\theta$-stable parabolic subalgebra $q_\lambda$. It satisfies $R^+(q_\lambda) = q = R^-(q_\lambda)$. Thus $Q \setminus D$ is indeed singleton.

For the pair in the second row of Table I with $k = 1$, we repeat the above calculation with $p$ and $q$ interchanged. This time $l + m \leq 2$ or $m + n \leq 2$ does not lead to a contradiction when $q = p + 1$. In that case we have $|Q \setminus D| = 3$. But if we assume $q = p$ or $q > p + 1$, then again we get that $Q \setminus D$ is a singleton. □

**Remark 1.** As a part of [11, Theorem 1.1] it was already proved that, when $G = SU(p,q)$ with $p < q - 1$, there exists a uniform lattice $\Gamma$ and a geometric class in $H^*(\Gamma\backslash SU(p,q)/SU(p) \times SU(q))$ having an $A_q$-component, where $\lambda = \epsilon_{p+1} - \epsilon_{p+q}$.

2. For all Hermitian symmetric pairs $(g_0, g'_0)$ listed in the tables given in [9], except the ones in Table I, either the set $Q \setminus D$ contains more than one elements or the set $Q$ itself is singleton. In the first case we don’t get a precise non-vanishing result while all examples in the second case has already been studied in [11].

3. The statement of Theorem 1.2 also holds when $g_0 = su(2,2)$. It is rather easy to prove using our method.

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**References**

[1] Borel, A. Compact Clifford-Klein forms of symmetric spaces. Topology 2 (1963) 111–122.

[2] Borel, A. and Wallach, N. Continuous cohomology, discrete groups, and representations of reductive groups. Ann. Math. Stud. 94, Princeton Univ. Press, 1980. Second Ed.

[3] Gelfand, I. M. and Pyatetskii-Shapiro, I. I. Theory of representations and theory of automorphic functions. Amer. Math. Soc. Transl. (2) 26 (1963) 173–200.
[4] Helgason, S. *Differential geometry, Lie groups and symmetric spaces*, American Mathematical Society, 2001.

[5] Kobayashi, T. Discrete decomposability of the restriction of $A_q(\lambda)$ with respect to reductive subgroups and its applications, Invent. math. 117 (1994), 181-202.

[6] Kobayashi, T. Discrete decomposability of the restriction of $A_q(\lambda)$ with respect to reductive subgroups III, Invent. math. 131 (1997), 229-256.

[7] Kobayashi, T. A vanishing theorem for modular symbols on locally symmetric spaces II, preprint 2009.

[8] Kobayashi, T. and Oda, T. A vanishing theorem for modular symbols on locally symmetric spaces, Comment. Math. Helv. 73 (1998), 45-70.

[9] Kobayashi, T. and Oshima, Y. Classification of discretely decomposable $A_q(\lambda)$ with respect to reductive symmetric pairs, Adv. Math. 231 (2012) 2013–2047.

[10] Millson, J. J. and Raghunathan, M. S. Geometric construction of cohomology for arithmetic groups. I. Proc. Indian Acad. Sci. (Math. Sci.), 90 (1981) 103–123.

[11] Mondal, A. and Sankaran, P. Geometric cycles in compact locally Hermitian symmetric spaces and automorphic representations. arXiv: 1703.03206.

[12] Moore, C. C. Ergodicity of flows on homogeneous spaces, Amer. J. Math. 88 (1966) 154-178.

[13] Raghunathan, M. S. Arithmetic lattices in semisimple groups. Proc. Indian Acad. Sci. (Math. Sci.), 91 (1982) 133–138.

[14] Rohlfs, J. and Schwermer, J. Intersection numbers of special cycles. J. Amer. Math. Soc. 6 (1993), no. 3, 755–778.

[15] Vogan, D. A. and Zuckerman, G. J. Unitary representations with nonzero cohomology. Compositio Math. 53 (1984), 51–90.

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