Quantum Entropy Function from $AdS_2/CFT_1$
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Abstract

We review and extend recent attempts to find a precise relation between extremal black hole entropy and degeneracy of microstates using $AdS_2/CFT_1$ correspondence. Our analysis leads to a specific relation between degeneracy of black hole microstates and an appropriately defined partition function of string theory on the near horizon geometry, – named the quantum entropy function. In the classical limit this reduces to the usual relation between statistical entropy and Wald entropy.
1 Introduction and Motivation

One of the successes of string theory has been an explanation of the Bekenstein-Hawking entropy of a class of supersymmetric black holes in terms of microscopic quantum states. In particular Strominger and Vafa [1] computed the Bekenstein-Hawking entropy of a class of extremal supersymmetric black holes in type IIB string theory on $K3 \times S^1$ via the formula

$$S_{BH} = \frac{A}{4G_N}, \quad (1.1)$$

and found agreement with the statistical entropy defined via the relation

$$S_{micro} = \ln d_{micro}. \quad (1.2)$$

In (1.1) $A$ is the area of the event horizon, $G_N$ is the Newton’s constant and we have set $\hbar = c = 1$. In (1.2) $d_{micro}$ is the number of microstates of a D-brane system carrying the same quantum numbers as the black hole. Similar agreement between $S_{BH}$ and $S_{micro}$ has been found in a large class of other extremal black holes. This provides us with a statistical interpretation of the Bekenstein-Hawking entropy.

The initial studies of $S_{BH}$ and $S_{micro}$ was done in the limit of large charges. In this limit the calculation simplifies on both sides. On the black hole side the curvature at the horizon
is small, and we can work with two derivative terms in the full string effective action. On
the microscopic side we can use the asymptotic formula for $d_{micro}$ for large charges instead of
having to compute it exactly. However it is clearly of interest to know if the correspondence
between the black hole entropy and the statistical entropy extends beyond the large charge
limit.

In order to address this problem we need to open two fronts. First of all we need to compute
the degeneracy of states of black holes to greater accuracy so that we can compute corrections
to $S_{micro}$ given in (1.2). Conceptually this is a straightforward problem since $d_{micro}$ is a well
defined number, especially in the case of BPS extremal black holes, since the BPS property gives
a clean separation between the spectrum of BPS and non-BPS states. Technically, counting of
$d_{micro}$ is a challenging problem, although this has now been achieved for a class of black holes
in $\mathcal{N} = 4$ supersymmetric string theories [2,3,4,5]. Significant progress has also been made
for half BPS black holes in a class of $\mathcal{N} = 2$ supersymmetric string theories [6,7,8].

The other front involves understanding how higher derivative corrections / string loop cor-
rections affect the black hole entropy. Wald’s formalism [9,10,11,12] gives a clear prescription
for calculating the effect of tree level higher derivative corrections on the black hole entropy,
and for extremal black holes this leads to the entropy function formalism [13,14,15]. Thus
here there is no conceptual problem, but in order to implement it we need to know the higher
derivative terms in the action.\footnote{Some of these corrections, which can be regarded as the correction to the central charge in the Cardy
formula, have now been computed on both sides and the results match [16,17,18,19,20,21].}

Inclusion of quantum corrections into the computation of black
hole entropy is more challenging both conceptually and tech
ically, and this will be the main
issue we shall try to address.

We restrict our study to extremal black holes with a near horizon geometry of the form
$AdS_2 \times K$, where $K$ is a compact space. We shall argue that the key to defining the quantum
corrected black hole entropy is in the partition function $Z_{AdS_2}$ of string theory on $AdS_2 \times K$
[22,23]. This partition function is divergent due to the infinite volume of $AdS_2$, but we show
in \( \S \) that there is an unambiguous procedure for extracting its finite part. In particular
we argue that the partition function has the form $e^{CL} \times$ a finite part where $C$ is a constant and $L$
is the length of the boundary of regulated $AdS_2$. We define the finite part as the one obtained
by dropping the $e^{CL}$ part and also argue that this is a natural prescription from the point of
view of the dual $CFT_1$.\footnote{An earlier attempt to extract this finite part can be found in [24]. This procedure, which relies heavily on
supersymmetry, is quite different from ours.} We then argue that $AdS_2/CFT_1$ correspondence [25,26,27,28] leads

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\end{itemize}
to the following relation between the $\text{AdS}_2$ partition function and the microscopic degeneracy $d_{\text{micro}}(\vec{q})$ of black holes carrying charge $\vec{q} \equiv \{q_i\}$:

$$d_{\text{micro}}(\vec{q}) = \langle \exp[-i\vec{q} \cdot \oint d\theta A^{(i)}_\theta] \rangle^{\text{finite}}_{\text{AdS}_2}, \quad (1.3)$$

where $\langle \ldots \rangle_{\text{AdS}_2}$ denotes the unnormalized path integral over various fields on euclidean global $\text{AdS}_2$ associated with the attractor geometry for charge $\vec{q}$ and $A^{(i)}_\theta$ denotes the component of the $i$-th gauge field along the boundary of $\text{AdS}_2$. This equation gives a precise relation between the microscopic degeneracy and an appropriate partition function in the near horizon geometry of the black hole. We shall call the right hand side of (1.3) the ‘quantum entropy function’. In the classical limit this reduces to the exponential of the Wald entropy of the black hole. Thus (1.3) is the quantum generalization of the $S_{\text{BH}} = S_{\text{micro}}$ relation in the classical theory. This is the key result of our analysis.

In defining the functional integral over various fields involved in computing the right hand side of (1.3) we need to use appropriate boundary condition on the various fields. For each $U(1)$ gauge field the solution to the classical equations of motion locally has two independent solutions near the boundary; the constant mode and the mode representing the asymptotic value of the electric field (and hence the charge). In $\text{AdS}_2$ the electric field mode is the dominant one near the boundary and hence we hold this fixed as we carry out the functional integral over various fields in $\text{AdS}_2$. In the classical limit the constant mode is determined in terms of the asymptotic electric field by requiring absence of singularity in the interior of $\text{AdS}_2$ whereas in the quantum theory we integrate over this mode. Since the asymptotic electric fields determine the charge carried by the black hole, this boundary condition forces us to work in a fixed charge sector, and as a result the dual $\text{CFT}_1$ also has states of a fixed charge. In special circumstances however it may be possible to define the partition function with fixed values of the constant modes of the gauge field at the boundary. Classical equations of motion together with the requirement of absence of singularities in the interior of $\text{AdS}_2$ sets the asymptotic electric field to be equal to the constant mode; so we shall continue to denote by $e_i$ the boundary values of the constant modes. Quantum mechanically of course we cannot fix both modes; so we have to allow the electric field modes to fluctuate. This leads to a new partition function $Z_{\text{AdS}_2}(\vec{e})$. If we denote by $Z_{\text{AdS}_2}^{\text{finite}}(\vec{e})$ the finite part of this partition function then $\text{AdS}_2/\text{CFT}_1$ correspondence suggests the following relation between
this partition function and the microscopic degeneracies of the boundary theory:

\[ Z_{\text{finite} \text{AdS}^2}(\vec{e}) = \sum_{\vec{q}} d_{\text{micro}}(\vec{q}) e^{-2\pi \vec{e} \cdot \vec{q}}. \] (1.4)

Note that the right hand side now has a sum over different charges since in the definition of the $\text{AdS}^2$ partition function we have allowed the asymptotic electric fields to fluctuate. We show that

1. In the classical limit eq.(1.4) again gives us back the relation equating the classical Wald entropy and statistical entropy.

2. For black holes in type IIA string theory compactified on a Calabi-Yau 3-fold, eq.(1.4) is closely related to the OSV conjecture [29].

However $Z_{\text{finite} \text{AdS}^2}$ appearing in (1.4) may be defined only under special circumstances since here we fix the constant modes of the asymptotic gauge fields and allow the non-normalizable modes corresponding to the asymptotic electric fields to fluctuate. Indeed, in §8 we point out various subtleties with this formula that makes it clear that even when such a partition function can be defined, it probably only makes sense as an asymptotic expansion around the classical limit.

Since the classical counterpart of our results is encoded in the entropy function formalism for computing entropy of extremal black holes [13,14,15], we begin with a lightening review of this formalism in §2. This takes into account the effect of tree level higher derivative corrections on the computation of black hole entropy.

2 The Extremal Limit and the Entropy Function Formalism

Let us consider the metric of a Reissner-Nordstrom black hole in (3+1) dimensions. It is given by

\[ ds^2 = -(1 - a/\rho)(1 - b/\rho)d\tau^2 + \frac{d\rho^2}{(1 - a/\rho)(1 - b/\rho)} + \rho^2 (d\theta^2 + \sin^2 \theta d\phi^2). \] (2.1)

Here $(\tau, \rho, \theta, \phi)$ are the coordinates of space-time and $a$ and $b$ are two parameters labelling the positions of the outer and inner horizon of the black hole respectively ($a > b$). The extremal limit corresponds to $b \to a$. We take this limit keeping the coordinates $\theta, \phi$, and

\[ r \equiv 2 \left( \rho - \frac{a + b}{2} \right) / (a - b), \quad t \equiv (a - b)\tau/2a^2, \] (2.2)
fixed \cite{30,31}. In this limit the metric takes the form:

\[ ds^2 = a^2 \left( -(r^2 - 1)dt^2 + \frac{dr^2}{r^2 - 1} \right) + a^2(d\theta^2 + \sin^2 \theta d\phi^2). \]  

(2.3)

This is the metric of \( AdS_2 \times S^2 \), with \( AdS_2 \) parametrized by \((r, t)\) and \( S^2 \) parametrized by \((\theta, \phi)\). Although in the original coordinate system the horizons coincide in the extremal limit, in the \((r, t)\) coordinate system the two horizons are at \( r = \pm 1 \).

All known extremal black hole solutions have an \( AdS_2 \) factor in their near horizon geometry; furthermore the other near horizon field configurations remain invariant under the \( SO(2,1) \) isometry of \( AdS_2 \). We shall take this to be the definition of an extremal black hole even in theories with higher derivative terms in the action \cite{13,14,32,33}. We can give a uniform treatment of all such extremal black holes by regarding the angular directions as part of compact coordinates. Thus we have an effective two dimensional theory of gravity coupled to (infinite number of) other fields. Among them of particular importance are the \( U(1) \) gauge fields which we shall denote by \( A_{\mu}^{(i)} \). The most general near horizon geometry consistent with the \( SO(2,1) \) isometries of \( AdS_2 \) is given by

\[ ds^2 = v \left( -(r^2 - 1)dt^2 + \frac{dr^2}{r^2 - 1} \right), \quad F_{rt}^{(i)} = e_i, \quad \cdots \]  

(2.4)

where \( F_{\mu\nu}^{(i)} = \partial_\mu A_\nu^{(i)} - \partial_\nu A_\mu^{(i)} \) are the gauge field strengths, \( v \) and \( e_i \) are constants and \( \cdots \) denotes near horizon values of other fields. The important point is that the \( SO(2,1) \) isometry of \( AdS_2 \) allows us to write down the most general near horizon field configuration in terms of some constants \( v, e_i \) etc. Note that we have not explicity displayed the magnetic fields; they represent flux through the sphere labelled by the angular coordinates and are regarded as compactification data in our current viewpoint.

Let us denote by \( \mathcal{L}^{(2)} \) the two dimensional Lagrangian density of this classical string theory including all higher derivative corrections. Since \( \mathcal{L}^{(2)} \) is a scalar, when evaluated on the near horizon geometry \( (2.4) \) it will give a function of the parameters \( v, e_i \) etc. without any dependence on the coordinates \( r \) and \( t \). We now introduce an extra set of variables \( q_i \) in one to one correspondence to the gauge fields and define a function \( \mathcal{E} \) of \( \vec{q} \) and the near horizon parameters as

\[ \mathcal{E}(\vec{q}, v, \vec{e}, \cdots) \equiv 2\pi \left( e_i q_i - v \mathcal{L}^{(2)} \right) \]  

(2.5)

where \( \mathcal{L}^{(2)} \) on the right hand side is evaluated in the near horizon geometry. Using classical equations of motion and Wald’s modified formula for black hole entropy \cite{9,10,11,12} in the presence of higher derivative terms one can show that
1. For a black hole with electric charges $\vec{q}$, all the near horizon parameters are determined by extremizing $E$ with respect to the near horizon parameters:

$$\frac{\partial E}{\partial v} = 0, \quad \frac{\partial E}{\partial e_i} = 0, \quad \cdots \quad (2.6)$$

Since the dependence of $E$ on all the near horizon parameters other than $\vec{e}$ come only through $vL^{(2)}$, we can first extremize $vL^{(2)}$ with respect to these parameters for a fixed $\vec{e}$ and call the result $vL^{(2)}(\vec{e})$. Then we can extremize $E = 2\pi(\vec{e} \cdot \vec{q} - vL^{(2)}(\vec{e}))$ with respect to $e_i$ to find the relation between $\vec{e}$ and $\vec{q}$.

2. The Wald entropy is given by the value of $E$ at this extremum:

$$S_{BH}(\vec{q}) = E \, . \quad (2.7)$$

3. $AdS_2$ Partition Function

A solution to eqs.(2.6) describes an $AdS_2$ background via (2.4). We can analytically continue the solution to euclidean space by defining new coordinates $\theta$ and $\eta$ via

$$t = -i\theta, \quad r = \cosh \eta. \quad (3.1)$$

In these coordinates the solution (2.4) takes the form

$$ds^2 = v \left( d\eta^2 + \sinh^2 \eta \, d\theta^2 \right), \quad F^{(i)}_{\theta \eta} = ie_i \sinh \eta, \quad \cdots \quad (3.2)$$

The metric is non-singular at the point $\eta = 0$ if we choose $\theta$ to have period $2\pi$. Integrating the field strength we can get the form of the gauge field:

$$A^{(i)}_{\mu} dx^\mu = -i \, e_i (\cosh \eta - 1) d\theta = -i \, e_i (r - 1) d\theta. \quad (3.3)$$

Note that the $-1$ factor inside the parenthesis is required to make the gauge fields non-singular at $\eta = 0$.

We can formally define the partition function of string theory in this background as the result of path integral over all the string fields. However due to infinite volume of $AdS_2$ this partition function is $a \text{ priori}$ divergent. We shall now describe an unambiguous procedure for extracting the finite part of this partition function. We begin by describing this procedure in
the classical limit where the path integral over the string fields is saturated by the saddle point corresponding to the classical geometry (3.2). Let \( A \) denote the classical Euclidean action

\[
A = A_{\text{bulk}} + A_{\text{boundary}}
\]

where

\[
A_{\text{bulk}} = - \int dr d\theta \sqrt{\det g} L^{(2)},
\]

evaluated in the \( AdS_2 \) background, and \( A_{\text{boundary}} \) denotes the possible boundary contribution to the action. Then the classical Euclidean partition function is given by

\[
Z_{AdS_2} = e^{-A}.
\]

Since \( AdS_2 \) has infinite volume we need to define (3.5) via suitable regularization. For this we introduce a cut-off at \( \eta = \eta_0 \) or equivalently \( r = r_0 = \cosh \eta_0 \). Then the regularized volume of \( AdS_2 \) is given by

\[
V_{AdS_2} \equiv \int_1^{r_0} dr \int_0^{2\pi} d\theta \sqrt{\det g} = 2\pi v (r_0-1) .
\]

This gives

\[
A_{\text{bulk}} = -(r_0-1) 2\pi v L^{(2)}.
\]

On the other hand \( A_{\text{boundary}} \) can be estimated by making a change of coordinates:

\[
w \equiv r_0 \theta, \quad \xi = \eta_0 - \eta.
\]

\( w \) has period \( 2\pi r_0 \). In this coordinate system the field configuration near the boundary \( r = r_0 \) take the form:

\[
ds^2 = v (d\xi^2 + e^{-2\xi} dw^2) + \mathcal{O}(r_0^{-2}),
\]

\[
A^{(i)}_{\xi w} = -ie_i \left( e^{-\xi} - r_0^{-1} \right) + \mathcal{O}(r_0^{-2}), \quad F^{(i)}_{\xi w} = ie_i e^{-\xi} + \mathcal{O}(r_0^{-2}).
\]

Now \( A_{\text{boundary}} \) can be represented as an integral of a local expression constructed from the metric, the gauge field strength \( F^{(i)}_{\mu \nu} \) and their derivatives, integrated over the boundary coordinate \( w \). Since the field configuration is independent of \( w \), the integration over \( w \) produces an overall multiplicative factor of \( 2\pi r_0 \). On the other hand since the metric and \( F^{(i)}_{\mu \nu} \) (and all other fields) are independent of \( r_0 \) except for corrections of order \( r_0^{-2} \), the integrand appearing in \( A_{\text{boundary}} \) will share the same property. Thus after integration over \( w \) we shall get a term proportional to \( r_0 \) plus corrections of order \( r_0^{-1} \):

\[
A_{\text{boundary}} = -Kr_0 + \mathcal{O}(r_0^{-1}),
\]

8
for some constant $K$. Substituting (3.8) and (3.11) into (3.6) we get

$$Z_{\text{AdS}_2} = e^{r_0 (2\pi v \mathcal{L}^{(2)} + K) - 2\pi v \mathcal{L}^{(2)} + O(r_0^{-1})}.$$  

(3.12)

The term linear in $r_0$ in the exponent is ambiguous since it can be changed by changing the boundary terms. However the finite part in the exponent is independent of the boundary terms and is unambiguous. We shall define

$$Z_{\text{AdS}_2}^{\text{finite}} = e^{-2\pi v \mathcal{L}^{(2)}}.$$  

(3.13)

Our analysis resembles the euclidean action formalism for describing black hole thermodynamics [34, 21, 35]. This formalism was originally developed for non-extremal black holes, but it has been applied to study extremal black holes in [36, 37]. However in these studies one uses the Euclidean action of the full black hole solution and not just its near horizon geometry. Also one cannot directly apply the Euclidean action formalism on an extremal black hole; one needs to begin with the results for a non-extremal black hole and then take the extremal limit at the end. In contrast our analysis has been based purely on the near horizon $\text{AdS}_2$ geometry. As we shall see in §4 this will allow us to interpret $Z_{\text{AdS}_2}$ as the partition function of a dual $\text{CFT}_1$ living on the boundary $r = r_0$.

In quantum theory one defines $Z_{\text{AdS}_2}$ as the path integral over the string fields in $\text{AdS}_2$ weighted by $e^{-A}$ [26, 27]. We can hope to represent the effect of this path integral by a modification of the lagrangian density $\mathcal{L}^{(2)}$ to an appropriate ‘effective Lagrangian density’. In flat space-time the one particle irreducible action is non-local and hence causes an obstruction to expressing the action as an integral over a local Lagrangian density. However the situation in $\text{AdS}_2$ background is better since the curvature of $\text{AdS}_2$ produces a natural infrared cut-off. Thus the quantum $Z_{\text{AdS}_2}$ is expected to be given by an expression similar to (3.12), with $\mathcal{L}^{(2)}$ replaced by $\mathcal{L}^{(2)}_{\text{eff}}$, and $K$ replaced by some other constant $K'$. Thus we have

$$Z_{\text{AdS}_2} = e^{r_0 (2\pi v \mathcal{L}^{(2)}_{\text{eff}} + K') + O(r_0^{-1})} Z_{\text{AdS}_2}^{\text{finite}},$$  

(3.14)

where

$$Z_{\text{AdS}_2}^{\text{finite}} = e^{-2\pi v \mathcal{L}^{(2)}_{\text{eff}}}.$$  

(3.15)

In the above discussion we have glossed over an important issue. In order to properly define the path integral over fields on $\text{AdS}_2$ we need to fix the boundary condition on various fields at $r = r_0$. In $\text{AdS}_{d+1}$ for general $d$ the classical Maxwell equations for a gauge field
near the boundary has two independent solutions. One of these modes represent the constant asymptotic value of the gauge fields, the other one measures the asymptotic electric field or equivalently the charge carried by the solution. Requiring the absence of singularity in the interior of \( AdS_{d+1} \) gives a relation between the two coefficients \[26, 27\]. Thus in defining the path integral over \( AdS_{d+1} \) we fix one of the coefficients and allow the other one to fluctuate. For \( d \geq 3 \) the constant mode of the gauge field is dominant near the boundary; hence it is natural to fix this and allow the mode measuring the charge to be determined dynamically in the classical limit and to fluctuate in the full quantum theory. However for \( d = 1 \) the mode that measures charge is the dominant one near the boundary; thus it is more natural to think of this as a parameter of the boundary CFT and let the constant mode of the gauge field be determined dynamically. This can be seen for example in \((3.3)\) where the term proportional to \( r \) measures the charge and the constant term in the expression for the gauge field is determined in terms of the linear term by requiring the gauge fields to be non-singular at the origin. Thus a more natural definition of the partition function of \( AdS_2 \) will be to fix the coefficient of the linear term in \( r \) and allow the constant term to fluctuate.

We shall reserve the symbol \( Z_{AdS_2} \) for the partition function defined with fixed value of the constant mode of the asymptotic gauge field, and \( \mathcal{L}^{(2)}_{\text{eff}} \) and \( K' \) for the corresponding quantities which appear in the expression for \( Z_{AdS_2} \) via \((3.14)\). Such a partition function may exist only in special circumstances since here we let a non-normalizable mode corresponding to the asymptotic electric field to fluctuate. For the more sensible boundary condition where we fix the asymptotic electric field configuration it will be convenient to introduce the quantity

\[
\hat{Z}_{AdS_2} \equiv \langle \exp[-i q_i \oint d\theta A^{(i)}_{\theta}] \rangle_{AdS_2},
\]

where \( \langle \rangle_{AdS_2} \) denotes the unnormalized path integral over various fields on \( AdS_2 \) with fixed asymptotic values of the electric fields corresponding to the attractor geometry. In the classical theory this is given by multiplying the \( AdS_2 \) partition function \((3.12)\) and \( \exp[-i q_i \oint d\theta A^{(i)}_{\theta}] \) with \( A^{(i)}_{\theta} \) given by its classical value \((3.3)\). This gives

\[
\hat{Z}_{AdS_2} = \exp [r_0(2\pi v \mathcal{L}^{(2)} + K - 2\pi \vec{e} \cdot \vec{q}) + 2\pi(\vec{e} \cdot \vec{q} - v \mathcal{L}^{(2)}) + O(r_0^{-1})].
\]

In the quantum theory arguments similar to the ones given for \( Z_{AdS_2} \) show that \( \hat{Z}_{AdS_2} \) should have the form

\[
\hat{Z}_{AdS_2} = \exp [2\pi C r_0 + O(r_0^{-1})] \langle \exp[-i q_i \oint d\theta A^{(i)}_{\theta}] \rangle_{AdS_2}^{\text{finite}},
\]

where

\[
\langle \text{finite} \rangle_{AdS_2}^{\text{finite}}
\]
where $C$ is a constant and $\left\langle \exp[-iq_i \oint d\theta A^{(i)}] \right\rangle_{AdS_2}^{finite}$ is an $r_0$ independent term. We shall call the latter the ‘quantum entropy function’.

4 AdS$_2$/CFT$_1$ Correspondence: The Macroscopic View

According to [25, 26, 27] string theory on AdS$_{d+1}$ times a compact space is dual to a $d$ dimensional conformal field theory (CFT) living on the boundary of AdS$_{d+1}$. In Euclidean global AdS$_{d+1}$, the boundary S-matrix of string theory can be used to construct the correlation functions of various operators in the $d$-dimensional CFT living on the boundary $S^d$. This provides a constructive definition of the $d$-dimensional CFT. We shall use the same procedure to define a one dimensional CFT living on the boundary of AdS$_2$. In this case the partition function of string theory on AdS$_2$ will be identified as the partition function of the CFT living on the boundary $r = r_0$.

We shall use $w$ defined in (3.9) as the coordinate on the boundary. In this coordinate system the boundary – which is a circle – has period $2\pi r_0$. Thus as $r_0 \to \infty$ the size of the boundary becomes infinite. On the other hand the metric (3.10) near the boundary remains finite in this limit without acquiring an infinite conformal factor; showing that the CFT has a finite ultraviolet cut-off. This is somewhat different from the usual convention where we would use $\theta$ as the coordinate on the boundary so that the boundary will have fixed period $2\pi$, and the $d\theta^2$ term in the metric will have a large conformal factor proportional to $r_0^2$, indicating that $r_0^{-1}$ provides an ultraviolet length cut-off in the CFT. Since the partition function depends on the ratio of the size of the boundary and the ultraviolet length cut-off, it is independent of the convention.

First we consider the case where we fix the asymptotic value of the electric fields $\partial_r A^{(i)}_\theta$ to $-ie_i$ as in (3.3). This fixes the charge of the black hole; thus the dual CFT$_1$ should only contain states carrying a fixed charge $\vec{q}$. AdS$_2$/CFT$_1$ correspondence will relate the partition function on AdS$_2$ to $Tr \exp(-2\pi r_0 H_w)$ in the dual CFT$_1$ where $H_w$ denotes the $w$ translation generator. Since we are interested in taking the $r_0 \to \infty$ limit at the end we need to keep in $H_w$ terms up to order $r_0^{-1}$. Now normally in the presence of a non-zero asymptotic gauge field, $H_w$ will contain a term $iQ_i A^{(i)}_w$ where $Q_i$ is the charge operator conjugate to the gauge field $A^{(i)}_w$. However here the constant mode of the $A^{(i)}_\theta$ – which generates a term proportional to $r_0^{-1}$ in $A^{(i)}_w$, is allowed to fluctuate and does not describe a fixed background. Hence including this term in $H_w$ is not sensible. We can instead consider the modified partition function $\hat{Z}_{AdS_2}$.
introduced in (3.16). Its effect will be to remove the \( iQ_i A^{(i)}_w \) term from the CFT\(_1\) Hamiltonian.\(^3\) We now notice that in the \( w \) coordinate system the field configuration (3.10) near the boundary is independent of \( r_0 \) up to corrections of order \( r_0^{-2} \), except for the \( ie_i/r_0 \) term in \( A^{(i)}_w \). Since the Hamiltonian of CFT\(_1\) is determined by the field configuration on AdS\(_2\) near the boundary and since the Hamiltonian no longer contains any term proportional to the constant mode of the gauge field, it must have the form \( H + \mathcal{O}(r_0^{-2}) \) where \( H \) is \( r_0 \) independent. This allows us to write

\[
\hat{Z}_{\text{AdS}_2} = Tr \left\{ \exp \left\{ -2\pi r_0 \left( H + \mathcal{O}(r_0^{-2}) \right) \right\} \right\}. \quad (4.1)
\]

For large \( r_0 \) the right hand side of (4.1) should reduce to

\[
e^{-2\pi E_0 r_0} d(\vec{q}), \quad (4.2)
\]

where \( E_0 \) is ground state energy and \( d(\vec{q}) \) is the number of ground states. Comparing this with (3.18) we get

\[
E_0 = -C, \quad (4.3)
\]

and

\[
d(\vec{q}) = \left\langle \exp[-iq_i \oint d\theta A^{(i)}_\theta] \right\rangle_{\text{finite AdS}_2}. \quad (4.4)
\]

Note that in writing down (4.2) we have implicitly assumed that \( H \) has a discrete spectrum. We can try to relax this assumption by assuming that \( H \) has a continuous spectrum, and that in the sector with charge \( \vec{q} \) the density of states is given by some function \( f(E, \vec{q}) \). This replaces (4.2) by

\[
\int dE f(E, \vec{q}) e^{-2\pi r_0 E}. \quad (4.5)
\]

Since this must be equal to \( \hat{Z}_{\text{AdS}_2} \) given in (3.18) we can use this to determine the form of \( f(E, \vec{q}) \). Suppose \( E_0 \) is the ground state energy so that \( f(E, \vec{q}) \) vanishes for \( E < E_0 \). This gives

\[
e^{-2\pi r_0 E_0} \int_{E_0}^{\infty} dE f(E, \vec{q}) e^{-2\pi r_0 (E - E_0)} = e^{2\pi r_0 C + \mathcal{O}(r_0^{-1})} \left\langle \exp[-iq_i \oint d\theta A^{(i)}_\theta] \right\rangle_{\text{finite AdS}_2}. \quad (4.6)
\]

\(^3\)The need to consider \( \hat{Z}_{\text{AdS}_2} \) instead of \( Z_{\text{AdS}_2} \) can also be seen directly on the AdS\(_2\) side. In the classical limit the extremization of the action gives the equations of motion if we keep fixed the gauge fields \( A^{(i)}_\theta \) at the boundary. However since here we allow the constant mode of the gauge field to vary, we need to add a boundary term to cancel the term proportional to \( \delta A^{(i)}_\theta \) at the boundary. The \( -iq_i \oint d\theta A^{(i)}_\theta \) term precisely achieves this purpose.
Now \( f(E, \vec{q}) \) depends on \( E \) through the ratio of \( E \) to the ultraviolet length cut-off which has been taken to be of order 1 in our convention. \( f \) cannot depend on \( r_0 \) since by definition \( H \) is \( r_0 \) independent. Comparing the two sides of (4.6) we get

\[
E_0 = -C, \tag{4.7}
\]

and

\[
\int_{E_0}^{\infty} dE \, f(E, \vec{q}) \, e^{-2\pi r_0(E-E_0)} = \left\langle \exp[-i\vec{q}_i \oint d\theta A^{(i)}_{\theta}] \right\rangle_{\text{finite} \, \text{AdS}_2}^{\text{finite}}. \tag{4.8}
\]

Since the large \( r_0 \) behaviour of the integral on the left hand side will be controlled by the behaviour of the integrand near \( E = E_0 \), let us examine the possible behaviour of the integrand near \( E = E_0 \). Since the right hand side of (4.8) is finite, the integral also must be finite and hence \( f(E, \vec{q}) \) cannot grow as \((E - E_0)^{-\alpha} \) with \( \alpha \geq 1 \). On the other hand if it grows as \((E - E_0)^{-\alpha} \) with \( \alpha < 1 \) the result of the integral on the left hand side of (4.8) will behave as \( r_0^{\alpha-1} \) for large \( r_0 \). This vanishes as \( r_0 \to \infty \) in contradiction to what we have on the right hand side of this equation. A simple resolution to this is that \( f(E, \vec{q}) \) has a term

\[
\delta(E - E_0) d(\vec{q}), \quad d(\vec{q}) = \left\langle \exp[-i\vec{q}_i \oint d\theta A^{(i)}_{\theta}] \right\rangle_{\text{finite} \, \text{AdS}_2}, \tag{4.9}
\]

together with possible additive terms of the form \((E - E_0)^{-\alpha} \) with \( \alpha < 1 \) which do not contribute in the large \( r_0 \) limit. (4.9) implies a discrete set of states at \( E = E_0 \) besides a possible continuum. There can also be other terms proportional to \( \delta(E - E_i) \) with \( E_i > E_0 \); the contribution from these terms to (4.8) will be exponentially suppressed. This gives us back (4.4).

It is instructive to see what the above result for the spectrum implies if we change our viewpoint to the more conventional one, and interpret the \( r_0 \to \infty \) limit as taking the ultraviolet length cut-off to zero at fixed size of the boundary. If \( \epsilon \) denotes the energy and \( \tilde{f}(\epsilon, \vec{q}) \) denotes the density of states in this new unit, and if we choose the zero of \( \epsilon \) to be at the ground state, then we have

\[
\epsilon = (E - E_0)r_0, \quad \tilde{f}(\epsilon, \vec{q}) = r_0^{-1}f(E, \vec{q}). \tag{4.10}
\]

This gives

\[
\tilde{f}(\epsilon, \vec{q}) = r_0^{-1}\delta(E - E_0) d(\vec{q}) + r_0^{-1}O((E - E_0)^{-\alpha}) = \delta(\epsilon) d(\vec{q}) + O(r_0^{-1+\alpha} \epsilon^{-\alpha}), \quad \alpha < 1. \tag{4.11}
\]

Thus in the \( r_0 \to \infty \) limit the correction terms vanish, and we conclude that the \( CFT_1 \) living on the boundary contains only zero energy states. \( d(\vec{q}) \) denotes the number of such zero energy
states and is related to the $AdS_2$ partition function via eq.(4.9). Thus the $CFT_1$ relevant
to $AdS_2/CFT_1$ correspondence seems to be quite different from the ones studied in [38, 39].
Examples of quantum systems containing only a finite number of states can be found in [40].

Finally we consider the case where in the computation of the partition function on $AdS_2$
we fix the asymptotic values of the constant mode of the gauge field $A^{(i)}_\theta$ to $i e_i$ (see eq.(3.3))
and allow the electric fields to fluctuate. As already mentioned, such a partition function is
not sensible in a generic situation, but may exist in special cases. In the $w$ coordinate system
the field configuration (3.10) near the boundary is independent of $r_0$ up to corrections of order
$r_0^{-2}$, except for the $i e_i/r_0$ term in $A^{(i)}_w$. Let us denote by $H$ the generator of $w$ translation
in the boundary theory in the $r_0 \to \infty$ limit. Then for finite $r_0$ the generator of $w$ translation
will be given by $H$ plus correction terms suppressed by inverse powers of $r_0$. Of these the $r_0^{-1}$
term will come solely from the $iQ_i A^{(i)}_w$ term that appears in the Hamiltonian. In particular
the $-i e_i/r_0$ term in $A^{(i)}_w$ gives an additive term $e_i Q_i r_0^{-1}$ to the $w$ translation generator where
$Q_i$ is the charge conjugate to $A^{(i)}_\mu$ in the boundary CFT. Thus the partition function of the
boundary CFT is given by

$$Z_{CFT_1} = Tr \left( e^{-2\pi r_0 (H + e_i Q_i r_0^{-1} + O(r_0^{-2}))} \right) = Tr \left( e^{-2\pi r_0 H - 2\pi e_i Q_i + O(r_0^{-1})} \right).$$

(4.12)

In this case the dual $CFT_1$ should in general contain states carrying different charges. For
large $r_0$ the partition function (4.12) for such a $CFT$ should reduce to

$$Z_{CFT_1} = e^{-2\pi E_0 r_0} \sum_{\vec{q}} d(\vec{q}) e^{-2\pi \vec{e} \cdot \vec{q}},$$

(4.13)

where $E_0$ is ground state energy and $d(\vec{q})$ is the number of ground states with $Q_i$ eigenvalue
$q_i$. Equating $Z_{CFT_1}$ given in (4.13) to $Z_{AdS_2}$ given in (3.14) we get

$$E_0 = -\nu \mathcal{L}_{eff}^{(2)} - \frac{K'}{2\pi}, \quad Z_{AdS_2}^{finite} = \sum_{\vec{q}} d(\vec{q}) e^{-2\pi \vec{e} \cdot \vec{q}}.$$  

(4.14)

5 $AdS_2/CFT_1$ Correspondence: The Microscopic View

In the last section we have used the partition function of quantum gravity on $AdS_2$ to define the
partition function of a $CFT_1$ living on the boundary of $AdS_2$. There is however another aspect
of $AdS/CFT$ correspondence that allows us to give an independent description of the boundary
CFT: it is the low energy dynamics of the system of branes which produce the particular $AdS$
space as its near horizon geometry \[25\]. Since the AdS\(_2\) geometry under consideration arises as the near horizon geometry of an extremal black hole, we expect that the corresponding CFT\(_1\) will describe the low energy dynamics of the brane system which describes this black hole. However for all known cases the spectrum of the brane system that produces the black hole has a gap that separates the ground state from the first excited state in any given charge sector. Thus one would expect that in the low energy limit all the excited states will disappear and the CFT\(_1\) will capture information about only the ground states of the brane system. After suitable shift in the energy that sets the ground state energy in each charge sector to zero, we find that CFT\(_1\) contains only zero energy states, in agreement with the results of the last section. Furthermore if \(d_{\text{micro}}(\vec{q})\) denotes the number of such ground states of the brane system in a given charge sector, then identifying CFT\(_1\) with the low energy dynamics of the brane system leads to the identification

\[
d(\vec{q}) = d_{\text{micro}}(\vec{q}). \tag{5.1}
\]

This gives from (4.3)

\[
d_{\text{micro}}(\vec{q}) = \exp[-iq_i \oint d\theta A_{\theta}^{(i)}]_{\text{finite}}. \tag{5.2}
\]

On the other hand if we define the AdS\(_2\) partition function by fixing the constant modes of the gauge fields at the boundary, then we get from (4.14),

\[
Z_{\text{finite}}^{\text{AdS}_2} = \sum_{\vec{q}} d_{\text{micro}}(\vec{q}) e^{-2\pi\vec{e} \cdot \vec{q}}. \tag{5.3}
\]

Eqs. (5.2) and (5.3) reproduce the results (1.3) and (1.4) quoted in the introduction.

6 The Classical Limit

We shall now take the classical limit of (1.3) and show that the result reduces to the statement of equality between Wald entropy and the statistical entropy. For this we note that in the classical limit we can replace the right hand side of (1.3) by the product of the classical AdS\(_2\) partition function (3.12) and \(\exp[-iq_i \oint d\theta A_{\theta}^{(i)}]\) with \(A_{\theta}^{(i)}\) given by its classical value (5.3). After removing terms in the exponential linear in \(r_0\), eq. (1.3) reduces to

\[
d_{\text{micro}}(q) = e^{2\pi q \vec{e} \cdot \vec{e} - 2\pi vL^2(2)}. \tag{6.1}
\]
Taking the logarithm on the two sides we get
\[ \ln d_{\text{micro}}(\vec{q}) = 2\pi (\vec{e} \cdot \vec{q} - v \mathcal{L}^{(2)}) . \] (6.2)
The left hand side gives the statistical entropy whereas the right hand side gives the classical Wald entropy according to (2.5), (2.7).

Next we consider the classical limit of (1.4). First of all in this limit the left hand side is given by (3.13). On the other hand we expect the right hand side to be sharply peaked as a function of \( \vec{q} \) and hence the leading contribution will be given by the value of the summand at its extremum. This leads to the relation
\[ -2\pi v \mathcal{L}^{(2)} = \ln d_{\text{micro}}(\vec{q}) - 2\pi \vec{e} \cdot \vec{q} , \] (6.3)

at
\[ \partial \ln d_{\text{micro}}(\vec{q}) / \partial q_i = 2\pi e_i . \] (6.4)

This is equivalent to
\[ \ln d_{\text{micro}}(\vec{q}) = 2\pi (\vec{e} \cdot \vec{q} - v \mathcal{L}^{(2)}) , \] (6.5)

at
\[ q_i = \frac{\partial}{\partial e_i} (v \mathcal{L}^{(2)}) . \] (6.6)
Comparing this with (2.5)-(2.7) we get
\[ S_{BH} = \ln d_{\text{micro}}(\vec{q}) . \] (6.7)

Before concluding this section we would like to discuss the precise meaning of the classical limit. In any theory this can be done by multiplying \( \mathcal{L}^{(2)} \) by a scale factor \( \lambda \) and then taking \( \lambda \) to be large. In string theory this can be achieved by a redefinition of the dilaton field \( \phi \) and the RR fields. In particular we need to scale \( e^{-2\phi} \) to \( \lambda e^{-2\phi} \), and any RR field \( \psi_{RR} \) to \( \lambda^{1/2} \psi_{RR} \), leaving the NSNS sector fields (other than the dilaton) unchanged. For a black hole in (3+1) dimensions this induces the transformation
\[ p^{NSNS} \rightarrow \lambda p^{NSNS}, \quad e^{NSNS} \rightarrow \lambda e^{NSNS}, \] \[ p^{RR} \rightarrow \lambda^{1/2} p^{RR}, \quad e^{RR} \rightarrow \lambda^{1/2} e^{RR}, \] (6.8)
where \( p \) denotes the magnetic charges (which are hidden arguments of \( Z_{\text{AdS}_2} \)) and \( e \) denotes the electric fields. Under this scaling \( q_i \) computed through (6.6) scales as
\[ q^{NSNS} \rightarrow \lambda q^{NSNS}, \quad q^{RR} \rightarrow \lambda^{1/2} q^{RR} . \] (6.9)
Thus in (1.4) the classical limit will correspond to scaling the magnetic charges and the electric fields in the argument of \( Z_{\text{AdS}_2}^{\text{finite}} \) as in (6.8) and then taking the large \( \lambda \) limit. On the other hand in (1.3) we need to scale \( \bar{p} \) and \( \bar{q} \) in the argument of \( d_{\text{micro}} \) as in (6.8), (6.2).
7 OSV Conjecture and $AdS_2/CFT_1$ Correspondence

$AdS_2/CFT_1$ correspondence suggests a possible route to deriving the OSV conjecture \cite{29} that relates the partition function for BPS states in $\mathcal{N} = 2$ supersymmetric string theories to the topological string partition function. For this we note that the right hand side of (1.4) is the black hole partition function defined in \cite{29}. Thus if we can evaluate $Z_{AdS_2}^{finite}$ we shall compute the black hole partition function. Now the effective action of $\mathcal{N} = 2$ supergravity theories in four dimensions contains a special class of terms, known as F-terms. Since these terms are local and do not require any infrared regulator for their definition, we expect that the structure of these terms should not depend on the background in which they are evaluated, and part of $\mathcal{L}_{eff}^{(2)}$ appearing in (3.15) should be given just by these F-terms evaluated in the attractor geometry. The information about the ‘F-type terms’ can be encoded in a function $F(\{X^I\}, \hat{A})$ – known as the generalized prepotential – of a set of complex variables $X^I$ which are in one to one correspondence with the gauge fields and an auxiliary complex variable $\hat{A}$ related to the square of the graviphoton field strength \cite{44,45}. $F$ is a homogeneous function of degree two in its arguments:

$$F(\{\lambda X^I\}, \lambda^2 \hat{A}) = \lambda^2 F(\{X^I\}, \hat{A}).$$

(7.1)

For a given choice of electric field one finds that the extremum of the effective Lagrangian density computed with the F-term effective action occurs at the attractor point where \cite{46,47,48,16,17,18,19}

$$\hat{A} = -4w^2, \quad 4(\bar{w}^{-1}\bar{X}^I + w^{-1}X^I) = e^I, \quad 4(\bar{w}^{-1}\bar{X}^I - w^{-1}X^I) = -ip^I.$$

(7.2)

Here $w$ is an arbitrary complex parameter and $p^I$ are the magnetic charges carried by the black hole. These magnetic charges have not appeared explicitly in our discussion so far because from the point of view of the near horizon geometry they represent fluxes through compact two cycles and appear as parameters labelling the two (or three) dimensional field theory describing

\footnote{Alternate approaches to analyzing the black hole partition function and deriving the OSV conjecture using AdS/CFT correspondence can be found in \cite{41,42,13,56,37}. The advantage of our approach lies in the fact that since we apply AdS/CFT correspondence on the near horizon geometry with the $AdS_2$ factor, the chemical potentials dual to the charges are directly related to the near horizon electric field, and hence, via the attractor mechanism, to other near horizon field configuration. Furthermore the path integral needs to be performed only over the near horizon geometry where we have enhanced supersymmetry and hence stronger non-renormalization properties. Ref. \cite{24} suggests an approach based purely on the analysis of the partition function in the near horizon geometry; however their approach to dealing with the divergence from infinite volume of $AdS_2$ is quite different from ours.}
the near horizon dynamics. The value of the effective Lagrangian density at the extremum \( (7.2) \) is given by \[49\]

\[ v_{\text{eff}}^{(2)} = 16i \left( w^{-2}F - \bar{w}^{-2}\bar{F} \right). \]  

(7.3)

Note that \( (7.2) \) determines \( X^I \) in terms of the unknown parameter \( w \). However due to the scaling symmetry \( (7.1) \), \( v_{\text{eff}}^{(2)} \) given in \( (7.3) \) is independent of \( w \). Using this scaling symmetry we can choose

\[ w = -8i, \]  

(7.4)

and rewrite \( (7.2) \), \( (7.3) \) as

\[ \hat{A} = 256, \quad X^I = -i(e^I + ip^I), \]  

(7.5)

\[ v_{\text{eff}}^{(2)} = -\frac{i}{4}(F(\{ X^I \}, 256) - \overline{F(\{ X^I \}, 256)}). \]  

(7.6)

Thus the contribution to \( Z_{AdS_2} \) from the F-terms is given by

\[ Z^F_{AdS_2} = e^{-\pi \text{Im} F(\{ p^I - ie^I \}, 256)}. \]  

(7.7)

If we assume that this is the full contribution to \( Z_{AdS_2} \) then eqs.\( (1.4), (7.7) \) leads to the original OSV conjecture \[29\].

It has however been suggested in subsequent papers that agreement with statistical entropy requires modifying this formula by including additional measure factors on the right hand side of \( (7.7) \) \[50, 51, 7, 52\]. A careful analysis of the path integral keeping track of the holomorphic anomaly \[53, 54, 55\] may be able to reproduce these corrections. Some of these corrections are in fact necessary for restoring the duality invariance of the final result for the entropy \[51, 52\].

Note that in trying to prove OSV conjecture from \( AdS_2/CFT_1 \) correspondence we must work with the partition function \( Z_{AdS_2} \). As mentioned earlier this may not always be defined as it requires integrating over fluctuations of the non-normalizable modes representing asymptotic electric fields, but \( \mathcal{N} = 2 \) supersymmetry may help in making \( Z_{AdS_2} \) well defined.

8 Conclusion, Caveats and Open Questions

The concrete achievements of the analysis given above can be summarized as follows.

1. We have been able to give a proper definition of the partition function of a black hole in near horizon attractor geometry of an extremal black hole. Typically such partition func-
tions suffer from infrared divergence, but we have described an unambiguous procedure for extracting its finite part.

2. The usual rules of AdS$_2$/CFT$_1$ correspondence relates the partition function of quantum gravity on AdS$_2$ to the partition function of a CFT$_1$ living on the boundary of AdS$_2$. While we have not given an explicit Lagrangian description of this CFT, the usual rules of AdS/CFT correspondence allows us to identify it as the infrared limit of the quantum mechanics associated with the brane system that describes the black hole. Since in any given charge sector the spectrum has a gap that separates the ground state from the excited states, the Hilbert space of CFT$_1$ consists of only the ground states of this quantum mechanical system.

3. Whether a CFT$_1$ dual to a given attractor geometry contains only the ground states of a given charge associated with the attractor geometry or ground states for all charges depends on whether in the definition of the AdS$_2$ partition function we have fixed the asymptotic values of the electric fields or the asymptotic values of the constant modes of the gauge fields. The former is more natural since electric field mode is the dominant (non-normalizable) mode of the gauge field near the boundary. In this case the dual CFT$_1$ would contain states of a fixed charge, and we are led to (1.3). On the other hand if we fix the asymptotic values of the constant modes of the gauge fields then in the definition of the dual CFT$_1$ we must include sum over states carrying different charges. This leads to (1.4). However since this involves integrating over non-normalizable modes of AdS$_2$, such a partition function may not always be well defined. Nevertheless this partition function is what appears naturally in the OSV conjecture; so in theories with enhanced supersymmetry such partition functions may also be well-defined.

4. In the classical limit both (1.3) and (1.4) reduce to the usual relation between $\ln d_{\text{micro}}(\vec{q})$ and the Wald entropy. Thus based on our analysis one can conclude that the equality of the classical Wald entropy and the statistical entropy of an extremal black hole for large charges is a direct consequence of AdS$_2$/CFT$_1$ correspondence in the classical limit.

Our analysis does not use supersymmetry directly, although supersymmetry is undoubtedly useful in stabilizing the extremal black holes against quantum corrections.

Typically such partition function also suffers from ultraviolet divergence but we expect string theory to cure them. If we have sufficient amount of supersymmetry then these ultraviolet divergences can also cancel due to supersymmetry.
There are however many caveats, many things which require clarification and many open questions. Some of them have already been mentioned earlier, but for the benefit of the reader we shall now summarize these issues.

1. Relation (1.3) requires us to define the partition function on $AdS_2$ for a given set of values of electric charge $\vec{q}$, while relation (1.4) requires us to be able to define it for a given set of values of the constant modes of the asymptotic gauge fields. In the classical limit the latter are equal to the near horizon electric fields $\vec{e}$. Although generically $\vec{e}$ is fixed by $\vec{q}$ and vice versa, in many examples in classical string theory treating $\vec{e}$ as independent variables can be problematic. Consider for example an heterotic string theory compactification. In this case after extremizing the near horizon classical Lagrangian density with respect to all the fields except the dilaton $\phi$, $v\mathcal{L}^{(2)}$ takes the form $e^{-2\phi}f(\vec{e})$ for some function $f(\vec{e})$. Extremization with respect to the dilaton now gives $f(\vec{e}) = 0$, i.e. it gives a constraint among the $e_i$’s instead of fixing the dilaton. This makes defining $Z_{AdS_2}(\vec{e})$ problematic – at least in the classical limit – since for this we need to extremize $v\mathcal{L}^{(2)}$ with respect to all the near horizon parameters other than the electric fields and express the result as a function of $\vec{e}$. If instead we fix $\vec{q}$, then we do not encounter such problems since $\phi$ as well as the $e_i$’s can be computed in terms of $\vec{q}$ via the extremization equation.

Due to this (1.4) seems to be ill defined for such theories, whereas (1.3) does not have such difficulties. This however is not a serious problem as one can use a dual description, exchanging the roles of some of the electric and magnetic charges, where both prescriptions can be made well defined.

2. Eq.(1.4) is close in spirit to the OSV conjecture, but as a result it also shares some of its problems. One of these is that in many supersymmetric string theories the sum over $\vec{q}$ appearing in (1.4) is not convergent since the function $\ln d_{micro}(\vec{q})$, when Taylor expanded around some given $\vec{q}_0$, does not have a negative definite quadratic term. As in the case of OSV proposal, we can define this by formally continuing the sum over $\vec{q}$ to complex values [50]. While this allows us to carry out an asymptotic expansion, it is not clear if this procedure can be made exact. In contrast (1.3), not having any sum over $\vec{q}$, does not suffer from such problems.

3. For an extremal black hole the ground state degeneracy $d_{micro}(\vec{q})$ is not expected to change continuously as we vary the asymptotic moduli. However discrete jumps are
possible and they occur as we cross walls of marginal stability. Thus in order to make (1.4) or (1.3) well-defined we need to decide on a precise prescription for $d_{\text{micro}}(\vec{q})$ that appears in these equations. In (3+1) dimensional flat space-time the natural choice for this seems to be the value $d_{\text{micro}}(\vec{q})$ when the asymptotic moduli are set equal to their attractor values. In this case on the black hole side the contribution to the entropy is known to come solely from single centered black holes with a single $AdS_2$ factor in their near horizon geometry. On the other hand as we cross a wall of marginal stability new multi-centered solutions carrying the same total charge and mass appear, and $d_{\text{micro}}(\vec{q})$ in such a background should count the total degeneracies of single and multi-centered black holes [56, 57, 58, 59, 60, 61, 62, 63, 64]. Since our analysis has been based on the partition function of a single $AdS_2$, it is only natural that $Z_{AdS_2}$ will only contain information about single centered black holes.

4. Although we have followed the spirit of $AdS_{d+1}/CFT_d$ correspondence, the actual implementation of this for $d = 1$ is somewhat different from that for higher $d$. For any $d$ the euclidean version of $AdS/CFT$ correspondence relates the partition function on euclidean global $AdS_{d+1}$ to partition function of $CFT_d$ living on the boundary $S^d$ [26,27]. However only in the special case of $d = 1$ the CFT partition function, being on $S^1$, may be represented as a trace over the Hilbert space of the CFT. This is what we have exploited in our analysis. The time coordinate labelling the coordinate of the boundary circle can be identified (up to a scaling) to the Schwarzschild time of the black hole.

5. For $d \geq 2$ one follows a different route for relating the CFT spectrum to quantum gravity on $AdS_{d+1}$. One uses the exponential map to represent global $AdS_{d+1}$ as $B^d \times \mathbb{R}$ and its boundary as $S^{d-1} \times \mathbb{R}$. The parameter labelling $\mathbb{R}$ is the global $AdS_{d+1}$ time, and this allows us to identify the Hilbert space of states of $CFT_d$ on $S^{d-1} \times \mathbb{R}$ to the Hilbert space of states of quantum gravity on $AdS_{d+1}$, with the Hamiltonian of the $CFT_d$ getting mapped to the generator of global time translation in $AdS_{d+1}$. For $d = 1$ however $S^{d-1}$ corresponds to a pair of disconnected points and hence we have two copies of $CFT_d$, one at each boundary. It has been suggested that the extremal black hole entropy can be related to the entanglement entropy of this pair of $CFT$’s [65]. Is there a relation between this proposal and the one we have presented here? To this end note that according to our proposal in the sector with charge $\vec{q}$ the $CFT_1$ has precisely $d(\vec{q})$ states, all with zero energy. Now if we assume that the ground state of string theory on $AdS_2$, represented
as $B^1 \times \mathbb{R}$, corresponds to a state in the boundary $CFT_1$ where the two copies of the $CFT_1$ living on the two boundaries are maximally entangled, then the entanglement entropy will be given by $\ln d(\vec{q})$, in agreement with the statistical entropy of one of the boundary $CFT_1$’s. Indeed for extremal BTZ black holes precisely such a route to relating the entanglement entropy and statistical entropy was suggested in [65]. It will be interesting to explore this relationship directly in the bulk theory. In the classical limit both proposals reduce to the Wald entropy, but in the quantum theory they may differ.

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