INTEGRAL TRANSFORMS CONNECTING THE HARDY SPACE WITH BARUT-GIRARDELLO SPACES

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ABSTRACT. We construct a one parameter family of integral transforms connecting the classical Hardy space with a class of weighted Bergman spaces called Barut-Girardello spaces.

1 INTRODUCTION

The paper deal with we the construction of a one-parameter family of integral transforms that connect the classical Hardy space $H^2_+$ ($\mathbb{R}$) of complex-valued square integrable functions $f(x)$ on the real line, whose Fourier transform are supported by the positive real semi-axis with Barut-Girardello spaces ([1, p.51]) which are weighted Bergman space, denoted $\mathfrak{F}_{\sigma}(\mathbb{C})$, consisting of analytic functions $\varphi(z)$ on the complex plane $\mathbb{C}$, that are square integrable with respect to the measure $|z|^{2\nu-1}K_{\frac{1}{2}-\sigma}(2|z|)d\lambda(z)$, $K_{\nu}(.)$ is the MacDonald function and $d\lambda$ being the planar Lebesgue measure and $\sigma = \frac{1}{2}, 1, \frac{3}{2}, 2, \cdots$ is a parameter.

The essence of our method consists on a coherent states analysis. Precisely, we will exploit some known results in [2, pp.59-62] to construct a class of coherent states of Barut-Girardello type which belong to the Hardy space and solve its identity. Therefore, the associated coherent states transform turns out to be the integral transform we are concerned with.

In the next section we recall briefly a well known formalism of coherent states with their corresponding coherent state transforms. In section 3, some basic facts on Hardy spaces are reviewed. Section 4 is devoted to the definition of the Barut-Girardello spaces. In Section 5, we establish an integral transform linking the Hardy space with Barut-Girardello spaces.

2 COHERENT STATES

Let us recall a well known general formalism ([3, pp.72-76]). Let $(X, \mu)$ be a measure space and let $A_2(x) \subset L^2(X, \mu)$ be a closed subspace of infinite dimension. Let $\{\Phi_n\}_{n=0}^{\infty}$ be an orthogonal basis of $A_2(x)$ satisfying, for arbitrary $\xi \in X$,
\begin{align}
\omega(\xi) := \sum_{n=0}^{\infty} \frac{\Phi_n(\xi)^2}{\rho_n} < +\infty, \tag{2.1}\end{align}

where \( \rho_n := \| \Phi_n \|_{L^2(X)}^2 \). Define

\begin{align}
K(\xi, \zeta) := \sum_{n=0}^{\infty} \frac{\Phi_n(\xi) \overline{\Phi_n(\zeta)}}{\rho_n}, \quad \xi, \zeta \in X. \tag{2.2}\end{align}

Then, \( K(\xi, \zeta) \) is a reproducing kernel, \( A_2(x) \) is the corresponding reproducing kernel Hilbert space and \( \omega(\xi) = K(\xi, \xi), \xi \in X \).

Let \( \mathcal{H} \) be another Hilbert space with \( \dim \mathcal{H} = \infty \) and \( \{ \phi_n \}_{n=0}^{\infty} \) be an orthonormal basis of \( \mathcal{H} \). Therefore, define a coherent state as a ket vector \( | \xi > \in \mathcal{H} \) labeled by a point \( \xi \in X \) as

\begin{align}
| \xi > := (\omega(\xi))^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\Phi_n(\xi)}{\sqrt{\rho_n}} \phi_n, \tag{2.3}\end{align}

We rewrite (2.3) using Dirac’s bra-ket notation as

\begin{align}
< q | \xi > = (\omega(\xi))^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\Phi_n(\xi)}{\sqrt{\rho_n}} \phi_n(q). \tag{2.4}\end{align}

By definition, it is straightforward to show that \( < \xi | \xi > = 1 \) and the coherent state transform \( T : \mathcal{H} \rightarrow A_2(x) \subset L^2(X, \mu) \) defined by

\begin{align}
T[\phi](\xi) := (\omega(\xi))^{\frac{1}{2}} < \xi | \phi > \tag{2.5}\end{align}

is an isometry. Thus, for \( \phi, \psi \in \mathcal{H} \), we have

\begin{align}
< \phi | \psi >_{\mathcal{H}} = < T[\phi] | T[\psi] >_{L^2(X)} = \int_X d\mu(\xi) \omega(\xi) < \phi | \xi > < \xi | \psi > \tag{2.6}\end{align}

and thereby we have a resolution of the identity

\begin{align}
1_{\mathcal{H}} = \int_X d\mu(\xi) \omega(\xi) | \xi > < \xi |, \tag{2.7}\end{align}

where \( \omega(\xi) \) appears as a weight function.

**Remark 2.1.** The formula (2.3) can be considered as a generalization of the canonical coherent states:

\begin{align}
| z > := e^{-\frac{1}{2}|z|^2} \sum_{k=0}^{\infty} \frac{z^k}{\sqrt{k!}} \phi_k, z \in \mathbb{C}, \tag{2.8}\end{align}

with \( \{ \phi_k \}_{k=0}^{\infty} \) being an orthonormal basis consisting of eigenstates of the harmonic oscillator. Here, the space \( A_2 \) is the Bargmann space of holomorphic functions on \( \mathbb{C} \) which are square integrable with respect to the Gaussian measure \( e^{-|z|^2} d\lambda(z) \) and \( \omega(z) \propto e^{\frac{1}{2}|z|^2}, z \in \mathbb{C} \).
3 The Barut-Girardello Space

In [1, p.51], Barut and Girardello have considered a countable set of Hilbert spaces $F_{\sigma}(C), \sigma > 0$ with $2\sigma = 1, 2, 3, \ldots$, whose elements are analytic functions $\varphi$ on $C$. For each fixed $\sigma$, the inner product is defined by

$$\langle \varphi, \psi \rangle_{\sigma} := \int_{C} \varphi(z) \overline{\psi(z)} \, d\mu_{\sigma}(z), \quad (3.1)$$

where

$$d\mu_{\sigma}(z) := \frac{2}{\pi \Gamma(2\sigma)} r^{2\sigma - 1} K_{\frac{1}{2} - \sigma} (2r) \, rd\theta, \quad z = re^{i\theta} \in C, \quad (3.2)$$

with the MacDonald function $K_{\nu}(.)$ defined by [4, p.78]:

$$K_{\nu}(\xi) = \frac{1}{2\pi} \left[ I_{-\nu}(\xi) - I_{\nu}(\xi) \right], \quad (3.3)$$

$I_{\nu}(.)$ denotes the modified Bessel function given by the series

$$I_{\nu}(\xi) = \sum_{m=0}^{+\infty} \frac{\left( \frac{\xi}{2} \right)^{\nu+2m}}{m! \Gamma(\nu+m+1)}. \quad (3.4)$$

Precisely, $F_{\sigma}(C)$ consists of entire functions $\varphi$ with finite norm $\|\varphi\|_{\sigma} = \sqrt{\langle \varphi, \varphi \rangle_{\sigma}} < +\infty$. Note also that if $\varphi(z)$ is an entire function with power series $\sum_n c_n z^n$, then the norm in terms of the expansion coefficients is given by

$$\|\varphi\|_{\sigma} = \left( (\Gamma(2\sigma))^{-1} \sum_{n=0}^{+\infty} |c_n|^2 n! \Gamma(2\sigma+n) \right)^{\frac{1}{2}}. \quad (3.5)$$

Every set of coefficients $(c_n)$ for which the sum (3.5) converges defines an entire function $\varphi \in F_{\sigma}(C)$. An orthonormal set of vectors in $F_{\sigma}(C)$ is given by:

$$\Phi_{n,\sigma}(z) := (\Gamma(2\sigma))^{\frac{1}{2}} \frac{z^n}{\sqrt{n! \Gamma(2\sigma+n)}}, \quad n = 0, 1, 2, \ldots, z \in C. \quad (3.6)$$

The reproducing kernel of the Hilbert space $F_{\sigma}(C)$ can be obtained as the confluent hypergeometric limit function $0F_1$ as

$$K_{\sigma}(z, w) = \sum_{n=0}^{+\infty} \frac{1}{(2\sigma)_n} \frac{(zw)^n}{n!} = 0F_1(2\sigma; zw). \quad (3.7)$$

Recall that ([4, p.100]):

$$0F_1(\eta; u) = \sum_{n=0}^{+\infty} \frac{1}{(\eta)_n} \frac{u^n}{n!}. \quad (3.8)$$
in which \((a)_n\) denotes the Pochhammer’s symbol defined by \((a)_0 := 1\) and
\[
(a)_n := \prod_{j=1}^{n} (a + j - 1) = a (a + 1) \cdots (a + j - 1) = \frac{\Gamma (a + n)}{\Gamma (a)} \tag{3.9}
\]
Making use of the relation ([4], p.100):
\[
_{0}F_{1} \left( \nu + 1; -\frac{1}{4} \xi^2 \right) = \Gamma (\nu + 1) \left( \frac{1}{2} \xi \right)^{-\nu} J_{\nu} (\xi), \tag{3.10}
\]
\(J_{\nu} (\cdot), \nu \in \mathbb{R}\) being the Bessel function given by
\[
J_{\nu} (\xi) = \sum_{m=0}^{+\infty} \frac{(-1)^m (\frac{1}{2} \xi)^{2m+\nu}}{m! \Gamma (m+\nu+1)}. \tag{3.11}
\]
as well as the relation:
\[
I_{\nu} (u) = \exp \left( -\frac{1}{2} u \pi i \right) J_{\nu} \left( e^{\frac{1}{2} \pi i} u \right), \tag{3.12}
\]
for \(\nu = 2\sigma - 1\) and \(\xi = 2 |z|\), we can write the diagonal function \(\omega_{\sigma} (z) := \mathcal{R}_{\sigma} (z, z)\) of the reproducing kernel of \(\mathcal{H}_{\sigma} (\mathbb{C})\) as
\[
\omega_{\sigma} (z) = \Gamma (2\sigma) |z|^{1-2\sigma} I_{2\sigma-1} (2 |z|), z \in \mathbb{C}. \tag{3.13}
\]

4 \hspace{1em} The Hardy space

The Hardy space \(\mathcal{H} (\mathbb{H}^{+})\) on the upper half of the complex plane \(\mathbb{H}^{+} := \{ z = x + iy, x \in \mathbb{R}, y > 0 \}\) consists of all functions \(F(z)\) analytic on \(\mathbb{H}^{+}\) such that
\[
\sup_{y > 0} \int_{\mathbb{R}} |F (x + iy)|^2 dx < +\infty. \tag{4.1}
\]
Any function \(F (x + iy)\) has a unique boundary value \(f(x)\) on the real line \(\mathbb{R}\), i.e.,
\[
\lim_{y \to 0} F (x + iy) = f(x) \tag{4.2}
\]
which is square integrable on \(\mathbb{R}\). Thus, a function \(F \in \mathcal{H} (\mathbb{H}^{+})\) uniquely determines a function \(f \in L^2 (\mathbb{R})\). Conversely, any function \(F\) can be recovered from its boundary values on the real line by mean of the Cauchy integral as follows
\[
F (z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(x)}{x - z} dx, \tag{4.3}
\]
f(\(x\)) being the function representing the boundary values of \(F(z)\). The linear space of all functions \(f(x)\) is denoted by \(\mathcal{H}_{\sigma}^2 (\mathbb{R})\). Since there is one to one correspondence between functions in \(\mathcal{H}_{\sigma}^2 (\mathbb{C})\) and their boundary values in \(\mathcal{H}_{\sigma}^2 (\mathbb{R})\), we identify these two spaces.
Moreover, using a Paley-Wiener theorem ([6, p.175]) one can characterize Hardy functions \( f \in H^2_+ (\mathbb{R}) \) by the fact that their Fourier transforms

\[
\mathcal{F} [f] (t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-itx} f(x) dx
\]  

(4.4)

are supported in \( \mathbb{R}^+ = [0, +\infty) \). That is,

\[
H^2_+ (\mathbb{R}) = \{ f \in L^2 (\mathbb{R}), \mathcal{F} [f] (t) = 0, \forall t < 0 \}.
\]  

(4.5)

This last definition appear in the context of the wavelets analysis [7].

Now, since \( \mathcal{F} \) is a linear isometry from \( L^2 (\mathbb{R}) \) onto \( L^2 (\mathbb{R}^+) \) under which the Hardy space \( H^2_+ (\mathbb{R}) \) is mapped onto the space \( L^2 (\mathbb{R}^+) \) which admits the complete orthonormal system of functions given in terms Laguerre polynomial \( L_n^{(\alpha)} (t) \) as

\[
L_n^{(\alpha)} (t) := \left( \frac{n!}{\Gamma (n + \alpha + 1)} \right)^{\frac{1}{2}} t^{n} e^{-\frac{1}{2} t} I_n^{(\alpha)} (t), \quad \alpha > -1,
\]  

(4.6)

the application of the inverse Fourier transform to the Laguerre functions in (4.6), \( \hat{I}_n^{(\alpha)} (x) := \mathcal{F}^{-1} \{ t \mapsto I_n^{(\alpha)} (t) \} (x) \), generates a class of orthonormal rational functions which are complete in \( H^2_+ (\mathbb{R}) \). The obtained functions can be found in the book of J.R. Higgins [2, p.62] and are of the form:

\[
\left[ \frac{a^{1+\alpha} \Gamma (1 + n + \alpha)}{n! 2\pi} \right] \frac{1}{2} \Gamma (1 + \alpha) \left( \frac{a}{2} \right)^{-(1+\alpha)} 2F_1 \left( -n, \frac{\alpha}{2} + 1, \alpha + 1; \frac{2a}{2ix + a} \right)
\]  

\times (ix + \frac{a}{2})^{-(1+\alpha)} 2F_1 \left( -n, \frac{\alpha}{2} + 1, \alpha + 1; \frac{2a}{2ix + a} \right)
\]  

where \( 2F_1 \) is the Gauss hypergeometric function defined by ([8, p.64]):

\[
2F_1 (a, b, c; \xi) = \sum_{n=0}^{+\infty} \frac{(a)_n (b)_n \xi^n}{(c)_n n!}.
\]  

(4.8)

For our purpose, we take \( a = -1 \) and we set \( \alpha + 1 = 2\sigma \) and we will be dealing with

\[
\phi_n^\sigma (x) := \left( \frac{\Gamma (\sigma + \frac{1}{2}) \Gamma (2\sigma + n)}{2^{2\sigma} \sqrt{\pi} \Gamma (2\sigma) \Gamma (\sigma) n!} \right)^{\frac{1}{2}} \left( \frac{1}{2} - ix \right)^{-(\sigma + \frac{1}{2})} 2F_1 \left( -n, \sigma + \frac{1}{2}, \sigma; \frac{1}{2} - ix \right)
\]  

\times (\frac{1}{2} - ix)^{-(\sigma + \frac{1}{2})} 2F_1 \left( -n, \sigma + \frac{1}{2}, \sigma; \frac{1}{2} - ix \right)
\]  

(4.9)

as a complete orthonormal system of rational functions in the Hardy space \( H^2_+ (\mathbb{R}) \).

**Remark 4.1.** In the particular case \( \sigma = \frac{1}{2} \), the orthonormal basis \( \phi_n^{\frac{1}{2}} (x) \) have been discussed in [9] in connection with the Hardy filter.
5 Coherent states in the Hardy space

Now, we combine the two basis \( \left( \phi_n^\sigma (x) \right)_n \) in (4.9) and \( \left( \Phi_n, \sigma (z) \right)_n \) in (3.6) according to definition (2.3) to construct for every fixed parameter \( \sigma > 0 \) with \( 2\sigma = 1, 2, 3, \cdots \), a set of coherent states \( | z, \sigma > \in \mathbb{C} \) labeled by points \( z \) of the complex plane \( \mathbb{C} \) and belonging to the Hardy space \( \mathcal{H}_{+}^{2} (\mathbb{R}) \) as

\[
| z, \sigma > := (\omega_\sigma (z))^{-\frac{1}{2}} \sum_{n=0}^{\infty} \Phi_{\sigma,n} (z) \phi_n^\sigma .
\]

(5.1)

with the following precisions:

- \( (X, \sigma) = (\mathbb{C}, |z|^{2\sigma-1}K_{2\sigma-2} (2|z|) d\lambda (z)), d\lambda (z) \) being the Lebesgue measure on \( \mathbb{C} \)
- \( A_2 := \mathfrak{F}_\sigma (\mathbb{C}), \sigma > 0 \) with \( 2\sigma = 1, 2, 3, \cdots \) denotes the Barut-Girardello space
- \( \omega_\sigma (z) = \Gamma (2\sigma)|z|^{1-2\sigma}I_{2\sigma-2} (2|z|), z \in \mathbb{C} \) as in (3.13).
- \( \Phi_{\sigma,n} (z), n = 0, 1, 2, \cdots \) are the basis elements given by (3.6).
- \( \mathcal{H} := \mathcal{H}_{+}^{2} (\mathbb{R}) \) is the Hilbert space carrying the coherent states
- \( \phi_n^\sigma (x), n = 0, 1, 2, \cdots \) is the orthonormal basis in (4.9).

From (5.1), the coherent states \( | z, \sigma > \) are defined by their wave functions through the series expansion

\[
\langle x | \sigma, z > = (\omega_\sigma (z))^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{(2\sigma)_n \Gamma (n+1)}} \phi_n^\sigma (x)
\]

(5.2)

Explicitly, we have that

\[
\langle x | \sigma, z > = \left( \Gamma (2\sigma)|z|^{1-2\sigma}I_{2\sigma-2} (2|z|) \right)^{-\frac{1}{2}}
\]

\[
\times \left( \frac{1}{2} - ix \right)^{-(\sigma + \frac{1}{2})} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{(2\sigma)_n \Gamma (n+1)}}
\]

\[
\times \left( \frac{\Gamma (\sigma + \frac{1}{2}) \Gamma (2\sigma + n)}{2^{2\sigma} \sqrt{\pi} \Gamma (2\sigma) \Gamma (\sigma) n!} \right)^{\frac{1}{2}} 2F_1 \left( -n, \sigma + \frac{1}{2}, 2\sigma; \frac{1}{2} - ix \right)
\]

(5.3)

We write the Gauss hypergeometric function \( 2F_1 \) in terms of the Meixner polynomial \( [5] \ p.346) as

\[
2F_1 \left( -n, -u, b; 1 - \frac{1}{c} \right) = M_n \left( u, b; c \right)
\]

(5.4)

Next we make use of the relation \( [5] \ p.349)\)

\[
\sum_{n=0}^{\infty} M_n \left( u, b; c \right) \frac{\zeta^n}{n!} = \mathcal{E} \left( -u, b, \left( \frac{1 - c}{c} \right) \zeta \right)
\]

(5.5)
where for $1F_1(\cdot)$ is the Kummer’s function ([10, p.262])

$$1F_1(a, \beta; z) := \frac{\Gamma(\beta)}{\Gamma(a)} \sum_{j=0}^{\infty} \frac{\Gamma(\beta + j) z^j}{\Gamma(a + j) j!}$$

and

$$\zeta = z, u = -\left(\sigma + \frac{1}{2}\right), b = 2\sigma, c = -\frac{1}{2} - ix \quad (5.6)$$

Therefore, we get that

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} 2F_1\left(\frac{-n, \sigma + \frac{1}{2}, 2\sigma}{\frac{1}{2} - ix}\right) = e^\zeta 1F_1\left(\sigma + \frac{1}{2}, 2\sigma, \frac{-z}{\frac{1}{2} - ix}\right) \quad (5.7)$$

and the wave functions of these coherent states

$$<x|z, \sigma> = \left(\frac{\Gamma(\sigma + \frac{1}{2})}{2^\sigma \sqrt{\pi} \Gamma(\sigma)}\right)^{\frac{1}{2}} \left(\frac{\Gamma(2\sigma)|z|^{1-2\sigma} I_{2\sigma-1}(2|z|)}{2^\sigma \Gamma(\sigma)}\right)^{-\frac{1}{2}} e^{\zeta} \quad (5.8)$$

$$\times \left(\frac{1}{2} - ix\right)^{-\left(\sigma + \frac{1}{2}\right)} 1F_1\left(\sigma + \frac{1}{2}, 2\sigma, \frac{-z}{\frac{1}{2} - ix}\right)$$

Now, according to (2.5) the coherent state transform corresponding to these coherent states is the isometry mapping the Hilbert space $H^2_\sigma(\mathbb{R})$ into the weighted Bergman space $\mathfrak{H}_\sigma(\mathbb{C})$

$$T_\sigma : H^2_\sigma(\mathbb{R}) \to \mathfrak{H}_\sigma(\mathbb{C}) \quad (5.9)$$

defined, according to (4.1) by

$$T_\sigma[f](z) := (\omega_\sigma(z))^\frac{1}{2} < z, \sigma | f > \quad (5.10)$$

Explicitly,

$$T_\sigma[f](z) = \int_{\mathbb{R}} \mathcal{K}_\sigma(z, x) \overline{f(x)} dx, f \in H^2_\sigma(\mathbb{R}), z \in \mathbb{C} \quad (5.11)$$

with the kernel function

$$\mathcal{K}_\sigma(z, x) := \frac{1}{2^\sigma \pi^\frac{3}{2}} \left(\frac{\Gamma(\sigma + \frac{1}{2})}{\Gamma(\sigma)}\right)^\frac{1}{2} \left(\frac{1}{2} - ix\right)^{-\left(\sigma + \frac{1}{2}\right)} e^\zeta 1F_1\left(\sigma + \frac{1}{2}, 2\sigma, \frac{-z}{\frac{1}{2} - ix}\right) \quad (5.12)$$

Finally, recalling (2.7) one can write that the coherent states $|z, \sigma>$ labeled by points $z \in \mathbb{C}$ solve the identity of the Hardy space $H^2_\sigma(\mathbb{R})$ as

$$1_{H^2_\sigma(\mathbb{R})} = \int_{\mathbb{C}} d\lambda(z) \omega_\sigma(z) |z, \sigma> <z, \sigma|.$$  (5.13)
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