K3 CARPETS ON MINIMAL RATIONAL SURFACES AND THEIR SMOOTHINGS

PURNAPRAJNA BANGER, JAYAN MUKHERJEE, AND DEBADITYA RAYCHAUDHURY

ABSTRACT. In this article, we study K3 double structures on minimal rational surfaces $Y$. The results show there are infinitely many non-split abstract K3 double structures on $Y = F_e$ parametrized by $\mathbb{P}^1$, countably many of which are projective. For $Y = \mathbb{P}^2$ there exist a unique non-split abstract K3 double structure which is non-projective (see [Dre20]). We show that all projective K3 carpets can be smoothed to a smooth K3 surface. One of the byproducts of the proof shows that unless $Y$ is embedded as a variety of minimal degree, there are infinitely many embedded K3 carpet structures on $Y$. Moreover, we show any embedded projective K3 carpet on $F_e$ with $e < 3$ arises as a flat limit of embeddings degenerating to $2:1$ morphism. The rest do not, but we still prove the smoothing result. We further show that the Hilbert points corresponding to the projective K3 carpets supported on $F_e$, embedded by a complete linear series are smooth points if and only if $0 \leq e \leq 2$. In contrast, Hilbert points corresponding to projective K3 carpets supported on $\mathbb{P}^2$ and embedded by a complete linear series are always smooth. The results in [BGG20] show that there are no higher dimensional analogues of the results in this article.

1. INTRODUCTION

A K3 carpet on a regular surface $Y$ is a locally Cohen-Macaulay double structure on $Y$ with the same invariants as a smooth K3 surface (i.e., regular with trivial canonical sheaf). In this article we study the relationship between deformation theory of double covers and K3 carpets on Hirzebruch surfaces and $\mathbb{P}^2$. A special case of this relationship concerning hyperelliptic K3 surfaces and double structures on rational normal scrolls was studied in [GP97]. Multiple structures arise in a variety of contexts in algebraic geometry. For example, they arise in the study of vector bundles and linear series (see [BM85], [HV85], [M81]), as well as in deformation and moduli problems (see [BGG20], [GGP10], [GGP08a]). One of the interesting features of K3 carpets is that the hyperplane section of K3 carpets supported on scrolls are canonical ribbons. The study of canonical ribbons was first proposed by Bayer and Eisenbud (see [BE95]). It follows from the works of Eisenbud and Green (see [EG95]) that an affirmative answer to Green's conjecture in the case of canonical ribbons will imply Green's conjecture for general curves. The reader is directed to [Deo18], [Dre20], [RS19], and [V05] for some further reading in this direction.

In this article we show that a K3 double structure on a minimal rational surface $Y$ (and $F_1$) is in general not projective, unlike ribbons on curves. The situation in this article resembles more that of K3 double structures arising from unramified K3 double covers of Enriques surfaces than ramified covers of Hirzebruch surfaces embedded as a variety of minimal degree, even though covers that appear in this article are all ramified covers. We show that the non-split K3 carpets on $Y$ are parametrized by the projective line $\mathbb{P}^1$ and the locus that parametrizes the projective K3 carpets is an infinite countable set. This invokes the notion of projective and non-projective K3
surfaces, where the former lie on infinite countably many codimension one families in the moduli space of K3 surfaces. In [GP97] it was shown that there is a unique K3 carpet supported on rational normal scrolls (see [GP97], Proposition 1.7) and that it can be realized as the limit of embedded models of smooth polarized K3 surfaces. The homogeneous ideals of these K3 carpets have been described by Eisenbud and Schreyer in [ES19]. The following result shows the sharp contrast between embedded K3 carpets on minimal and non-minimal degree embeddings of $Y$.

**Theorem 1.1.** (See Theorems 3.1 and 3.3)

1. Suppose $j : S \hookrightarrow \mathbb{P}^n$ be an embedding of a Hirzebruch surface $S$ induced by the complete linear series of $\mathcal{O}_S(1) = \mathcal{O}_S(ac_0 + bf)$ (and hence $b \geq ae + 1$ and $n + 1 = h^0(\mathcal{O}_S(1))$). Let $N + 1 \geq h^0(\mathcal{O}_S(1))$ and let $k : \mathbb{P}^n \hookrightarrow \mathbb{P}^N$ be a linear embedding. Let $i = k \circ j$. Then the K3 carpets on $S$ with an embedding inside $\mathbb{P}^N$ extending the embedding $i$ is parametrized by a non-empty Zariski open subset of the projective space of lines in the vector space $H^0(\mathcal{N}_{S/\mathbb{P}^N} \otimes K_S)$ consisting of nowhere vanishing sections with

$$h^0(\mathcal{N}_{S/\mathbb{P}^N} \otimes K_S) = (N + 1) \frac{(a - 1)}{2}(2b - 2 - ae) + 1.$$ 

2. Suppose that $i : S = \mathbb{P}^2 \hookrightarrow \mathbb{P}^n$ be an embedding induced by the complete linear series $\mathcal{O}_{\mathbb{P}^2}(d)$ (and hence $n + 1 = h^0(\mathcal{O}_{\mathbb{P}^2}(d))$). Let $N + 1 \geq h^0(\mathcal{O}_{\mathbb{P}^2}(d))$ and let $k : \mathbb{P}^n \hookrightarrow \mathbb{P}^N$ be a linear embedding. Let $i = k \circ j$. Then the K3 carpets on $S$ with an embedding inside $\mathbb{P}^N$ that extends the embedding $i$ is parametrized by a non-empty Zariski open subset of the projective space of lines in the vector space $H^0(\mathcal{N}_{S/\mathbb{P}^N} \otimes K_S)$ consisting of nowhere vanishing sections with

$$h^0(\mathcal{N}_{S/\mathbb{P}^N} \otimes K_S) = \frac{1}{2}(N + 1)(d - 1)(d - 2).$$

Note that if in the above theorem we assume that $i : S = \mathcal{F}_e \hookrightarrow \mathbb{P}^N$ is a non-minimal degree embedding induced by the complete linear series of $\mathcal{O}_S(ac_0 + bf)$, we have that the K3 carpets embedded in $\mathbb{P}^N$ supported on $S$ is a non-empty open set inside a projective space of dimension $\frac{1}{4}(a^2 - 1)((2b - ae)^2 - 4) + 1$ which is in contrast to the existence of a unique embedded K3 carpet for the minimal degree embedding as shown in [GP97].

The above theorem brings up new issues unseen for multiple structures on surfaces in general and on minimal degree embeddings of $\mathcal{F}_e$ in particular. First is the existence of abstract K3 double structures on minimal rational surfaces $Y$ (and $\mathcal{F}_1$) and existence of projective K3 double structures among them. The second is the number of embedded K3 double structures on an arbitrary embedding of $Y$ and their smoothing. The existence/non-existence of embedded non-split multiple structures on a given embedded variety and their smoothing is closely related to the deformation theory of finite covers (see [GGP10]). In Section 3, Theorem 4.3 shows that for every abstract projective K3 carpet on $Y = \mathcal{F}_e$ with $0 \leq e \leq 2$, there is an associated double cover that is a K3 surface. We show that this double cover can be deformed to an embedding, thereby smoothing the K3 carpet on $Y$. For $e > 2$, there are no such K3 double covers even though there are K3 double structures on the corresponding Hirzebruch surfaces $\mathcal{F}_e$, but we show these double structures are smoothable, as well.

**Theorem 1.2.** (Theorems 4.3, 4.4, 4.5, 5.1 and 5.2) Let $\mathcal{S}$ be a projective K3 carpet supported on a minimal rational surface $S$. Then $\mathcal{S}$ is smoothable. Moreover, the following results hold.
(a) Let \( \tilde{S} \) be a projective K3 carpet supported on a Hirzebruch surface \( S = \mathbb{F}_e \) and embedded inside \( \mathbb{P}^N \) by the complete linear series of a very ample line bundle. Then \( \tilde{S} \) is a smooth point of its Hilbert scheme if and only if \( 0 \leq e \leq 2 \).

(b) Let \( \tilde{S} \) be a projective K3 carpet supported on \( S = \mathbb{P}^2 \) and embedded inside \( \mathbb{P}^N \) by the complete linear series of a very ample line bundle. Then \( \tilde{S} \) is a smooth point of its Hilbert scheme.

In a recent article [BGG20], it is shown that there are no higher dimensional analogues of these results. In [BGG20] authors introduce the notion of generalized hyperelliptic varieties. It has been shown that the deformations of generalized hyperelliptic polarized Calabi-Yau varieties of dimension \( \geq 3 \) are again generalized hyperelliptic varieties ([BGG20], Theorem 4.5). They also show there are no Calabi-Yau double structures on higher dimensional scrolls, and more generally on higher dimensional projective bundles. However the results in this article show that a general deformation of a generalized hyperelliptic K3 surface is no longer generalized hyperelliptic.

The previous theorem also shows contrast with several earlier results on ribbons and carpets on curves and surfaces respectively: Bânîcă and Manolache, in [BM85], proved that the Hilbert points of ribbons in \( \mathbb{P}^3 \) supported on conics are smooth; Bayer and Eisenbud, in [BE95], proved that the Hilbert points of canonically embedded ribbons on \( \mathbb{P}^1 \) are smooth; and González proved in [G06], the smoothness of the Hilbert point for most ribbons on curves of arbitrary genus. It was also shown in [GGP08b] and [MR20] that K3 carpets on Enriques surfaces and regular K-trivial ribbons on Enriques manifolds represent smooth points of the corresponding Hilbert schemes.

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Convention. We will always work over the complex numbers \( \mathbb{C} \). For a smooth variety \( X \), \( K_X \) denotes the canonical bundle of \( X \). The symbol “\( \sim \)” stands for linear equivalence and the symbol “\( \equiv \)” stands for numerical equivalence of line bundles or divisors.

2. Abstract and embedded K3 carpets

This section is to investigate the existence of of K3 carpets supported on a minimal rational surface \( S \). We will deal with two cases, namely when \( S = \mathbb{F}_e \) and when \( S = \mathbb{P}^2 \). We start with some basic things on ribbons and carpets.

2.1. Ropes, ribbons and K3 carpets. Ropes are multiple structures on a scheme and ribbons are special kinds of ropes. We give the precise definitions below.
Definition 2.1. Let $Y$ be a reduced connected scheme and let $\mathcal{E}$ be a vector bundle of rank $m-1$ on $Y$. A rope of multiplicity $m$ on $Y$ with conormal bundle $\mathcal{E}$ is a scheme $\tilde{Y}$ with $\tilde{Y}_{\text{red}} = Y$ such that $\mathcal{F}_{\tilde{Y}/Y} = \mathcal{E}$ and $\mathcal{F}_{\tilde{Y}/Y} = \mathcal{E}$ as $\mathcal{O}_Y$ modules. If $\mathcal{E}$ is a line bundle then $\tilde{Y}$ is called a ribbon on $Y$.

The following properties of ropes were proven in [BE95] and [G06].

Theorem 2.2. Let $Y$ be a reduced connected scheme and $\mathcal{E}$ be a vector bundle of rank $m-1$ on $Y$.

(1) A rope $\tilde{Y}$ with conormal bundle $\mathcal{E}$ is defined by an element $[e_\mathcal{E}] \in \text{Ext}^1(\Omega_Y, \mathcal{E})$. The rope is split if and only if $[e_\mathcal{E}] = 0$.

Assume further that $Y$ is a smooth variety and $i : Y \hookrightarrow Z$ is a closed immersion into another smooth variety $Z$.

(2) There is an one-to-one correspondence between pairs $(\tilde{Y}, i)$ where $\tilde{Y}$ is a rope with conormal bundle $\mathcal{E}$ and $\tilde{i} : \tilde{Y} \to Z$ is a morphism extending $i : Y \hookrightarrow Z$ and elements $\tau \in \text{Hom}(\mathcal{N}^*_Y, \mathcal{E})$.

(3) If $\tau \in \text{Hom}(\mathcal{N}^*_Y, \mathcal{E})$ corresponds to $(\tilde{Y}, i)$, the $\tilde{i}$ is an embedding if and only if $\tau$ is surjective.

(4) If $\tau \in \text{Hom}(\mathcal{N}^*_Y, \mathcal{E})$ corresponds to $(\tilde{Y}, i)$ then $\tau$ is mapped by the connecting homomorphism onto $[e_\mathcal{E}]$.

K3 carpets were defined as ribbons on surfaces satisfying some additional properties (see [GGP08b], Definition 1.2).

Definition 2.3. A K3 carpet $\tilde{S}$ on a smooth regular surface $S$ is a ribbon on $S$ such that the dualizing sheaf $K_S$ is trivial and $H^1(\mathcal{O}_S) = 0$.

The following characterizing lemma was proven in [GP97], Proposition 1.5.

Lemma 2.4. Let $S$ be a smooth regular surface and let $\tilde{S}$ be a ribbon supported on $S$. Let $\mathcal{L}$ be the dual of the ideal sheaf defining $S$ in $\tilde{S}$. The $\tilde{S}$ is a K3 carpet if and only if $\mathcal{L} \cong K^*_S$.

We refer to [MR20] for a generalization of K3 carpets in higher dimensions.

2.2. K3 carpets on Hirzebruch surfaces. Recall that the Hirzebruch surfaces $S = \mathbb{F}_e$ ($e \geq 0$) are, by definition, the projective bundles $\mathbb{P}(\mathcal{E})$, where $\mathcal{E} = \mathcal{O}_p \oplus \mathcal{O}_p\langle e \rangle$. Let $\pi : S \to \mathbb{P}^1$ be the natural morphism. The line bundles on $S$ are of the form $\mathcal{O}_S(aC_0 + bf)$ where $\mathcal{O}_S(C_0) = \mathcal{O}_{\mathbb{P}^1}(1)$ and $\mathcal{O}_S(f) = \pi^*\mathcal{O}_p\langle 1 \rangle$. The line bundle $\mathcal{O}_S(aC_0 + bf)$ is very ample if and only if $b \geq ae + 1$.

Lemma 2.5. Suppose $j : S \hookrightarrow \mathbb{P}^n$ be an embedding of a Hirzebruch surface $S$ induced by the complete linear series of $\mathcal{O}_S(1) = \mathcal{O}_S(aC_0 + bf)$ (and hence $b \geq ae + 1$ and $n + 1 = h^0(\mathcal{O}_S(1))$). Let $N + 1 \geq h^0(\mathcal{O}_S(1))$ and let $k : \mathbb{P}^n \hookrightarrow \mathbb{P}^N$ be a linear embedding. Let $i = k \circ j$. Let $\pi : \mathbb{F}_e \to \mathbb{P}^1$ be the natural fibration. Then

1. $h^0(\pi^*T_{p1} \otimes K_S) = 0$, $h^1(\pi^*T_{p1} \otimes K_S) = 1$, $h^2(\pi^*T_{p1} \otimes K_S) = 0$.
2. $h^0(T_{S/\mathbb{P}^1} \otimes K_S) = 0$, $h^1(T_{S/\mathbb{P}^1} \otimes K_S) = 1$, $h^2(T_{S/\mathbb{P}^1} \otimes K_S) = 0$.
3. $h^0(T_{p1} \otimes K_S) = (N + 1)h^0((aC_0 + bf) \otimes K_S)$, $h^1(T_{p1} \otimes K_S) = 1$, $h^2(T_{p1} \otimes K_S) = 0$.
4. $h^0(T_S \otimes K_S) = h^1(T_S \otimes K_S) = 0$, $h^2(T_S \otimes K_S) = 2$. 

Proof. Recall that $K_S = \Theta_S(-2C_0 - (e + 2)f)$ is the canonical bundle of $S$.

(1) Note that $\pi^* T_{p^1} = \Theta_S(2f)$, thus $\pi^* T_{p^1} \otimes K_S = \Theta_S(2f) \otimes K_S$. Now, $h^l(\Theta_S(2f) \otimes K_S) = h^{2-l}(\Theta_S(-2f))$ by Serre duality. The assertions follow by using $\pi_*(\Theta_S) = \Theta_{p^1}$ and the projection formula.

(2) We have $T_{S/p^1} = \Theta_S(2C_0 + ef)$. Hence $T_{S/p^1} \otimes K_S = \Theta_S(-2f)$. The assertions again follow by the projection formula and by the fact that $\pi_*(\Theta_S) = \Theta_{p^1}$.

(3) We tensor the restriction of the Euler sequence to $S$ by $K_S$ to get the following

$$0 \to K_S \to \Theta_S(aC_0 + bf)^{\oplus N+1} \otimes K_S \to T_{p^1} \otimes K_S \to 0.$$ 

Since $\Theta_S(aC_0 + bf)$ is very ample, we have that $h^1((aC_0 + bf) \otimes K_S) = h^2((aC_0 + bf) \otimes K_S) = 0$ by Kodaira vanishing. One checks that $h^0(K_S) = 0$, $h^1(K_S) = 0$ and $h^2(K_S) = 1$. The assertions follow by taking the long exact sequence associated to the above short exact sequence.

(4) The assertions follow from the long exact sequence associated to

$$0 \to T_{S/p^1} \otimes K_S \to T_S \otimes K_S \to \pi^*(T_{p^1}) \otimes K_S \to 0.$$

and the previous parts.

We are interested in counting the number of abstract K3 carpets on Hirzebruch surfaces, as well as the number of embedded K3 carpets on the embedded Hirzebruch surfaces. It follows from Theorem 2.2 and Lemma 2.4 that we need the dimension of the cohomology group $H^1(T_S \otimes K_S)$ for the first one, and the dimension of $H^0(N_{S/p^1} \otimes K_S)$ for the second one. To compute their values, we need the following lemma.

Lemma 2.6. In the situation of Lemma 2.5, we have the following

(1) $\pi_*(T_{S/p^1} \otimes K_S) = \Theta_{p^1}(-2)$, $R^1\pi_*(T_{S/p^1} \otimes K_S) = 0$.
(2) $\pi_*(\pi^*(T_{p^1}) \otimes K_S) = 0$, $R^1\pi_*(\pi^*(T_{p^1}) \otimes K_S) = 0$.
(3) $\pi_*(T_S \otimes K_S) = \Theta_{p^1}(-2)$, $R^1\pi_*(T_S \otimes K_S) = \Theta_{p^1}$.
(4) There is an exact sequence

$$0 \to \pi_* \Theta_S((aC_0 + bf) \otimes K_S)^{\oplus N+1}) \to \pi_* (T_{p^1} \otimes K_S) \to R^1\pi_*(K_S) = \Theta_{p^1}(-2) \to 0,$$

and $R^1\pi_*(T_{p^1} \otimes K_S) = 0$.
(5) There is an exact sequence

$$0 \to \Theta_{p^1}(-2) \to \pi_* (T_{p^1} \otimes K_S) \to \pi_* (N_{S/p^1} \otimes K_S) \to \Theta_{p^1} \to 0,$$

and $R^1\pi_*(N_{S/p^1} \otimes K_S) = 0$.

Proof: (1) and (2) are clear since $T_{S/p^1} \otimes K_S = \Theta_S(-2f)$ and $\pi^* T_{p^1} \otimes K_S = \Theta_S(-2C_0 - ef)$. (3) follows using (1) and (2) and applying $\pi_*$ to the exact sequence

$$0 \to \Theta_S(-2f) \to T_S \otimes K_S \to \Theta_S(-2C_0 - ef) \to 0.$$ 

(4) follows by applying $\pi_*$ to the sequence obtained by tensoring the Euler sequence by $K_S$.

$$0 \to K_S \to \Theta_S((aC_0 + bf) \otimes K_S)^{\oplus N+1}) \to T_{p^1} \otimes K_S \to 0$$

(5) follows by applying $\pi_*$ to the following exact sequence

$$0 \to T_S \otimes K_S \to T_{p^1} \otimes K_S \to N_{S/p^1} \otimes K_S \to 0$$

and the previous parts. □
We have seen in the previous lemma that $R^1\pi_*(\mathcal{N}_{\mathbb{P}^N} \otimes K_S) = 0$. Thus, the dimensions of cohomology groups of $\mathcal{N}_{\mathbb{P}^N} \otimes K_S$ can be calculated by pushing it forward to $\mathbb{P}^1$. The following proposition gives an exact sequence where $\pi_*(\mathcal{N}_{\mathbb{P}^N} \otimes K_S)$ fits, and we use that exact sequence to calculate $h^0(\pi_*(\mathcal{N}_{\mathbb{P}^N} \otimes K_S))$.

**Lemma 2.7.** In the situation of Lemma 2.5, we have the following:

1. There is an exact sequence
   
   $$0 \to \pi_*(\mathcal{O}_S((aC_0 + bf) \otimes K_S)^{\oplus N+1}) \to \pi_*(\mathcal{N}_{\mathbb{P}^N} \otimes K_S) \to \mathcal{O}_{\mathbb{P}^1} \to 0.$$  

2. $h^0(\mathcal{N}_{\mathbb{P}^N} \otimes K_S) = (N+1)\frac{(a-1)}{2}(2b-2ae)$. In particular if the embedding $i$ is induced by the complete linear series of $\mathcal{O}_S(aC_0 + bf)$, then $N + 1 = h^0(\mathcal{O}_S(aC_0 + bf))$ and $h^0(\mathcal{N}_{\mathbb{P}^N} \otimes K_S) = \frac{1}{2}(a^2 - 1)(2b - ae)^2 - 4 + 1$. We also have $h^1(\mathcal{N}_{\mathbb{P}^N} \otimes K_S) = h^2(\mathcal{N}_{\mathbb{P}^N} \otimes K_S) = 0$.

**Proof.** (1) Choose an injective morphism $\psi : \mathcal{O}_S(C_0 + bf) \to \mathcal{O}_S(aC_0 + bf)$. Let $h^0(\mathcal{O}_S(C_0 + bf)) = m + 1$. This induces the commutative diagram below.

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & K_S & \longrightarrow & (\mathcal{O}_S(C_0 + bf) \otimes K_S) \otimes H^0(\mathcal{O}_S(C_0 + bf)) & \longrightarrow & T^m_{\mathbb{P}^1} \otimes K_S & \longrightarrow & 0 \\
& & \downarrow \cong & & \downarrow & & \downarrow \alpha & & \\
0 & \longrightarrow & K_S & \longrightarrow & (\mathcal{O}_S(aC_0 + bf) \otimes K_S)^{\oplus N+1} & \longrightarrow & T^m_{\mathbb{P}^1} \otimes K_S & \longrightarrow & 0 \\
\end{array}
\]

Taking $\pi_*$ and noting that $R^1\pi_*(\mathcal{O}_S(C_0 + bf) \otimes K_S) \otimes H^0(\mathcal{O}_S(C_0 + bf)) = 0$ for $i = 1, 2$ and using Lemma 2.6, we have the following commutative diagram.

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \pi_*(T^m_{\mathbb{P}^1} \otimes K_S) & \longrightarrow & R^1\pi_*K_S & \longrightarrow & 0 \\
& & \downarrow \phi = \pi_*\alpha & & \downarrow f & & \downarrow \phi & & \\
0 & \longrightarrow & \pi_*(\mathcal{O}_S(aC_0 + bf) \otimes K_S)^{\oplus N+1} & \longrightarrow & \pi_*(T^m_{\mathbb{P}^1} \otimes K_S) & \longrightarrow & R^1\pi_*K_S & \longrightarrow & 0 \\
\end{array}
\]

Hence we have that $\phi : \pi_*(T^m_{\mathbb{P}^1} \otimes K_S) \to \pi_*(T^m_{\mathbb{P}^1} \otimes K_S) = R^1\pi_*K_S \oplus \pi_*(\mathcal{O}_S(aC_0 + bf) \otimes K_S)^{\oplus N+1}$ and $\phi$ induces an isomorphism between $\pi_*(T^m_{\mathbb{P}^1} \otimes K_S)$ and $R^1\pi_*K_S$.

Now consider the commutative diagram induced by $\phi$.

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & T_S \otimes K_S & \longrightarrow & T^m_{\mathbb{P}^1} \otimes K_S & \longrightarrow & \mathcal{N}_{\mathbb{P}^1} \otimes K_S & \longrightarrow & 0 \\
& & \downarrow \cong & & \downarrow \phi & & \downarrow \phi & & \\
0 & \longrightarrow & T_S \otimes K_S & \longrightarrow & T^m_{\mathbb{P}^1} \otimes K_S & \longrightarrow & \mathcal{N}_{\mathbb{P}^1} \otimes K_S & \longrightarrow & 0 \\
\end{array}
\]

Applying $\pi_*$ and using Lemma 2.6 we get

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_*(T_S \otimes K_S) & \longrightarrow & \pi_*(T^m_{\mathbb{P}^1} \otimes K_S) & \longrightarrow & \pi_*(\mathcal{N}_{\mathbb{P}^1} \otimes K_S) & \longrightarrow & R^1\pi_* (T_S \otimes K_S) & \longrightarrow & 0 \\
& & \downarrow \cong & & \downarrow \phi & & \downarrow \phi & & \\
0 & \longrightarrow & \pi_*(T_S \otimes K_S) & \longrightarrow & \pi_*(T^m_{\mathbb{P}^1} \otimes K_S) & \longrightarrow & \pi_*(\mathcal{N}_{\mathbb{P}^1} \otimes K_S) & \longrightarrow & R^1\pi_* (T_S \otimes K_S) & \longrightarrow & 0 \\
\end{array}
\]

Since the leftmost vertical map is an isomorphism we have that $p$ maps $\pi_*(T_S \otimes K_S)$ isomorphically onto $\phi(\pi_*(T^m_{\mathbb{P}^1} \otimes K_S)) = R^1\pi_*K_S$. Hence we have the exact sequence

\[
0 \to \pi_*(\mathcal{O}_S((aC_0 + bf) \otimes K_S)^{\oplus N+1}) \to \pi_*(\mathcal{N}_{\mathbb{P}^1} \otimes K_S) \to \mathcal{O}_{\mathbb{P}^1} \to 0.
\]
(2) Notice that we have $R^1\pi_*(\mathcal{N}_{S/P}\otimes K_S) = 0$ by Lemma 2.6 and hence

$$h^i(\mathcal{N}_{S/P}\otimes K_S) = h^i(\pi_*(\mathcal{N}_{S/P}\otimes K_S)).$$

Thus, it follows from the exact sequence of (1) that,

$$h^0(\mathcal{N}_{S/P}\otimes K_S) = (N + 1)h^0(\pi_*(\mathcal{O}_S((aC_0 + b f)\otimes K_S))) + 1$$

and $h^1(\mathcal{N}_{S/P}\otimes K_S) = h^2(\mathcal{N}_{S/P}\otimes K_S) = 0$ (to see the vanishings, use $R^1\pi_*(\mathcal{O}_S((aC_0 + b f)\otimes K_S)) = 0$, and Kodaira vanishing).

The assertion follows from the fact that

$$h^0(\mathcal{O}_S((aC_0 + b f)\otimes K_S)) = \frac{(a - 1)}{2}(2b - 2 - ae)$$

and

$$h^0(\mathcal{O}_S(aC_0 + b f)) = \frac{(a + 1)}{2}(2b + 2 - ae).$$

That finishes the proof of the proposition.

\[ \square \]

3. Projective and nonprojective K3 carpets

In contrast to ribbons on curves, not all carpets are projective, even if all of them are proper or even if, as is the case with rational surfaces, they are supported on a projective surface. A natural question to ask about K3 carpets on minimal rational surfaces is whether there exist families of projective K3 carpets. We show this is true in Theorem 3.1 below. Next step is to describe the loci parametrizing K3 carpets on a given minimal rational surface, which we do as well.

As we will see, the situation resembles that of smooth K3 surfaces. We also explore embedded K3 carpets, which naturally is dependent on the embeddings of rational surfaces in projective space. We give precise results on all of the above in the following theorem:

**Theorem 3.1.** In the situation of Lemma 2.5, we have the following:

1. The abstract non-split K3 carpets on $S$ are parametrized by the projective space of lines in the vector space $H^1(T_S\otimes K_S)$ of dimension $h^1(T_S\otimes K_S) = 2$. Hence the non-split abstract K3 carpets on $S$ are parametrized by the projective line $\mathbb{P}^1$.

2. The non-split projective K3 carpets on $S$ are parametrized by a non-empty countable subset of the projective line. They are in $1-1$ correspondence to the classes of primitive ample divisors in $NS(S)$.

3. The K3 carpets on $S$ with an embedding inside $\mathbb{P}^N$ that extends the embedding $i$ is parametrized by a non-empty Zariski open subset of the projective space of lines in the vector space $H^0(\mathcal{N}_{S/\mathbb{P}^N}\otimes K_S)$ consisting of nowhere vanishing sections with

$$h^0(\mathcal{N}_{S/\mathbb{P}^N}\otimes K_S) = (N + 1)\frac{(a - 1)}{2}(2b - 2 - ae) + 1.$$ 

**Proof.** (1) The assertion is clear, $h^1(T_S\otimes K_S)$ has been calculated in Lemma 2.5.

(2) We follow the proof of Theorem 2.5, [GGP08b]. An abstract K3 carpet $\tilde{S}$ on $S$ is given by an element $\tau \in H^{1,1}(S) = H^2(S, \mathbb{C})$. Since the ideal of $S$ in $\tilde{S}$ is of square zero, we have the following exact sequence

$$0 \to K_S \to \mathcal{O}_{\tilde{S}}^* \to \mathcal{O}_{\tilde{S}}^* \to 1.$$
Taking cohomology we get

\[ 0 \to \text{Pic}(\tilde{S}) \xrightarrow{\gamma} \text{Pic}(S) \xrightarrow{\lambda_*} H^2(K_S). \]

Let \( D \) be a divisor in \( \text{Pic}(S) \), then \( \lambda_*(D) = f_D \tau \). The map \( \gamma \) sends a divisor in \( \tilde{S} \) to its restriction to \( S \). Notice that a divisor in \( \tilde{S} \) is ample if and only if its restriction to \( S \) is ample. Hence \( \tilde{S} \) is projective if and only if there exist an ample divisor \( D \in \text{Pic}(S) \) such that \( f_D \tau = 0 \). Now, given a divisor \( D \) in \( S \) we have that the elements \( \tau \in H^{1,1}(S) \) such that \( f_D \tau = 0 \) form a hyperplane \( H_D \) in \( H^{1,1}(S) \). Hence it corresponds to a point with multiplicity one in \( \mathbb{P}(H^{1,1}(S)) = \mathbb{P}^1 \). Hence the projective K3 carpets on \( S \) is a countable set of points in \( \mathbb{P}^1 \) each of which corresponds to a primitive ample divisor in \( NS(S) \). This proves (2).

(3) Using [G06], in order to prove the statement, we need to show that the Zariski open subset of the projective space of lines in the vector space \( H^0(\mathcal{N}_{S/P\mathbb{N}} \otimes K_S) \) consisting of nowhere vanishing sections is non-empty. In other words we want to show that there exist at least one nowhere vanishing section of the vector bundle \( \mathcal{N}_{S/P\mathbb{N}} \otimes K_S \). Consider the exact sequence

\[ 0 \to \mathcal{O}_S \to (\mathcal{O}_S((aC_0 + b f) \otimes K_S))^{\otimes N+1} \to T_{P\mathbb{N} | S} \otimes K_S \to 0. \]

Notice that \( \mathcal{O}_S((aC_0 + b f) \otimes K_S) = \mathcal{O}_S((a-2)C_0 + (b - e - 2) f) \) is globally generated. Indeed, since \( b \geq ae + 1 \), we have

\[ b - e - 2 \geq ae + 1 - e - 2 = ae - e - 1 \geq ae - 2e \]

for \( e \geq 1 \) and hence \( \mathcal{O}_S((aC_0 + b f) \otimes K_S) \) is base point free for \( e \geq 1 \). For \( e = 0 \), \( S = \mathbb{P}^1 \times \mathbb{P}^1 \), hence if \( a \geq 2 \) and \( b \geq 2 \) we still have \( \mathcal{O}_S((aC_0 + b f) \otimes K_S) \) is base point free.

Assume \( a \geq 2 \). From Lemma 2.5, we have that

\[ h^0(T_{P\mathbb{N} | S} \otimes K_S) = (N + 1)h^0(\mathcal{O}_S((aC_0 + b f) \otimes K_S)) = (N + 1) \left( \frac{a - 1}{2} \right) (2b - 2 - ae) \]

which is non-zero since \( a \geq 2 \). Thus, \( T_{P\mathbb{N} | S} \otimes K_S \) is globally generated, since \( \mathcal{O}_S((aC_0 + b f) \otimes K_S)^{\otimes N+1} \) is base point free and it surjects onto \( T_{P\mathbb{N} | S} \otimes K_S \). Consider the sequence

\[ 0 \to T_S \otimes K_S \to T_{P\mathbb{N} | S} \otimes K_S \to \mathcal{N}_{S/P\mathbb{N}} \otimes K_S \to 0. \]

We already have that \( h^0(\mathcal{N}_{S/P\mathbb{N}} \otimes K_S) = (N + 1) \left( \frac{a - 1}{2} \right) (2b - 2 - ae) \) and hence \( h^0(\mathcal{N}_{S/P\mathbb{N}} \otimes K_S) \neq 0 \). Thus, we have that \( \mathcal{N}_{S/P\mathbb{N}} \otimes K_S \) is base point free. Now, \( \text{rank}(\mathcal{N}_{S/P\mathbb{N}} \otimes K_S) = N - 2 \). It is easy to see using \( N + 1 \geq \frac{1}{4}(a + 1)(2b + 2 - ae) \) that, \( N - 2 > 2 \) and hence \( h^0(\mathcal{N}_{S/P\mathbb{N}} \otimes K_S) \) contains a nowhere vanishing section.

It remains to show the existence of a nowhere vanishing section when \( a = 1 \). But this follows from [GP97], Proposition 1.7. That ends the proof. \( \square \)

3.1. K3 carpets on \( \mathbb{P}^2 \). Now we count the number of abstract K3 carpets on \( \mathbb{P}^2 \) and the number of embedded K3 carpets on the \( d \)-uple embeddings of \( \mathbb{P}^2 \).

**Lemma 3.2.** Suppose that \( i : S = \mathbb{P}^2 \hookrightarrow \mathbb{P}^n \) be an embedding \( i \) is induced by the complete linear series \( \mathcal{O}_{\mathbb{P}^2}(d) \) (and hence \( n + 1 = h^0(\mathcal{O}_{\mathbb{P}^2}(d)) \)). Let \( N + 1 \geq h^0(\mathcal{O}_{\mathbb{P}^2}(d)) \) and let \( k : \mathbb{P}^n \hookrightarrow \mathbb{P}^N \) be a linear embedding. Let \( i = k \circ j \).

(1) Then there is an exact sequence,

\[ 0 \to \mathcal{O}_{\mathbb{P}^2}^{\otimes 3}(1) \to \mathcal{O}_{\mathbb{P}^2}^{\otimes N+1}(d) \to \mathcal{N}_{S/P\mathbb{N}} \to 0. \]
(2) \( h^0(\mathcal{N}_{S/\mathbb{P}^N} \otimes K_S) = \frac{1}{2}(N+1)(d-1)(d-2) \). In particular if the embedding \( i \) is induced by the complete linear series of \( \mathcal{O}_{\mathbb{P}^N}(d) \), then \( h^0(\mathcal{N}_{S/\mathbb{P}^N} \otimes K_S) = \frac{1}{2}(d+2)(d+1)(d-1)(d-2) \). We also have \( h^1(\mathcal{N}_{S/\mathbb{P}^N} \otimes K_S) = h^2(\mathcal{N}_{S/\mathbb{P}^N} \otimes K_S) = 0 \).

**Proof.** (1) The following commutative diagram with exact rows and columns shows \( \mathcal{F} \cong \mathcal{N}_{S/\mathbb{P}^N} \).

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \mathcal{O}_{\mathbb{P}^2} & \rightarrow & \mathcal{O}_{\mathbb{P}^2} & \rightarrow & 0 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \mathcal{O}_{\mathbb{P}^2}(1) & \rightarrow & \mathcal{O}_{\mathbb{P}^2}^{\oplus N+1}(d) & \rightarrow & \mathcal{F} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & T_{\mathbb{P}^2} & \rightarrow & T_{\mathbb{P}^N|S} & \rightarrow & \mathcal{N}_{S/\mathbb{P}^N} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}
\]

(2) The assertion follows by tensoring the exact sequence of the previous part with \( K_S \) and using cohomology calculations of the long exact sequence associated to it.

**Theorem 3.3.** In the situation of Lemma 3.2, we have the following.

(1) For \( d \geq 3 \), there exist a unique abstract non-split K3 carpet on \( S \) which is non-projective.

(2) The K3 carpets on \( S \) with an embedding inside \( \mathbb{P}^N \) that extends the embedding \( i \) is parametrized by a non-empty Zariski open subset of the projective space of lines in the vector space \( H^0(\mathcal{N}_{S/\mathbb{P}^N} \otimes K_S) \) consisting of nowhere vanishing sections with

\[
h^0(\mathcal{N}_{S/\mathbb{P}^N} \otimes K_S) = \frac{1}{2}(N+1)(d-1)(d-2).
\]

In particular, there do not exist an embedded K3 carpet for the embedding given by \( d = 2 \), i.e. for the second Veronese embedding.

**Proof.** The proof follows exactly along the same lines of the proof of Theorem 3.1. (1) follows since \( h^1(T_{S} \otimes K_S) = 1 \) and by Poincare duality \( \tilde{f}_D \) is a non-degenerate form on \( H^1(T_{S} \otimes K_S) \) (alternatively, it follows from [Dre20], Theorem 8.0.2). For (2), notice that \( \mathcal{O}_{\mathbb{P}^2}^{\oplus N+1}(d-3) \) is base point free for \( d \geq 3 \) and there is an exact sequence

\[
0 \rightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus 3}(-2) \rightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus N+1}(d-3) \rightarrow \mathcal{N}_{S/\mathbb{P}^N} \otimes K_S \rightarrow 0.
\]

obtained by tensoring the exact sequence of Lemma 3.2, (1) by \( K_S = \mathcal{O}_{\mathbb{P}^2}(-3) \). which implies \( \mathcal{N}_{S/\mathbb{P}^N} \otimes K_S \) is base point free Since \( \text{rank}(\mathcal{N}_{S/\mathbb{P}^N} \otimes K_S) > 2 \), a general section of \( \mathcal{N}_{S/\mathbb{P}^N} \otimes K_S \) is nowhere vanishing.

4. SMOOTHINGS OF ABSTRACT AND EMBEDDED K3 CARPETS

We study the smoothings of the K3 carpets constructed in the previous section. We start with the following technical result (see [G06], Lemma 3.3 and Proposition 3.7).
Lemma 4.1. Let $X$ be a smooth projective variety and $\varphi : X \to \mathbb{P}^N$ be a morphism. Suppose that $\varphi = i \circ \pi$ where $\pi : X \to Y$ is a finite flat morphism with $Y$ smooth and trace zero module $\mathcal{E}$ and $i : Y \hookrightarrow \mathbb{P}^N$ be an embedding. Let $\mathcal{N}_\pi, \mathcal{N}_\varphi, \mathcal{N}_{Y/\mathbb{P}^N}$ be the normal sheaves of the morphisms $\varphi, \pi$, and $i$ respectively. Then there exists an exact sequence

$$0 \to \mathcal{N}_\pi \to \mathcal{N}_\varphi \to \pi^* \mathcal{N}_{Y/\mathbb{P}^N} \to 0.$$  

Taking cohomology and using projection formula we have the following exact sequence

$$0 \to H^0(\mathcal{N}_\pi) \to H^0(\mathcal{N}_\varphi) \xrightarrow{v_1 \oplus v_2} H^0(\mathcal{N}_{Y/\mathbb{P}^N} \oplus \mathcal{E}) \to H^1(\mathcal{N}_\pi)$$

where $v_1 : H^0(\mathcal{N}_\pi) \to H^0(\mathcal{N}_{Y/\mathbb{P}^N})$ and $v_2 : H^0(\mathcal{N}_\varphi) \to H^0(\mathcal{N}_{Y/\mathbb{P}^N} \otimes \mathcal{E})$ are the induced maps.

The following result will be an important technical tool which we will use throughout this section.

Proposition 4.2. In the situation of Lemma 4.1, let $\Phi : \mathcal{X} \to \mathbb{P}^N_T$ be a deformation of $\varphi$ along a smooth pointed affine curve $(T, 0)$. Assume that

(a) $\mathcal{X}$ is irreducible and reduced;

(b) $\mathcal{X}_t$ is smooth, irreducible and projective for all $t \in T$;

(c) $\mathcal{X}_0 = X$ and $\Phi_0 = \varphi$.

Let $\Delta$ be the first infinitesimal neighborhood of $t \in T$. Let $\overline{X} = \mathcal{X}_\Delta$ and $\overline{\varphi} = \Phi_\Delta$. Then $\overline{\varphi}$ naturally defines an element (which we again call $\overline{\varphi}$) in $H^0(\mathcal{N}_\varphi)$. If $v_2(\overline{\varphi}) \in H^0(\mathcal{N}_{Y/\mathbb{P}^N} \otimes \mathcal{E})$ is a nowhere vanishing section of the the vector bundle $\mathcal{N}_{Y/\mathbb{P}^N} \otimes \mathcal{E}$, then the following happens.

(1) After possibly shrinking $T$, we have that for $t \in T, t \neq 0$, the morphism $\Phi_t : \mathcal{X}_t \to \mathbb{P}^N_T$ is an embedding.

(2) The central fibre of the image, $\text{Im}(\Phi)_0$ is a multiple structure on $\text{Im}(\varphi)$ with conormal bundle $\mathcal{E}$. Since $\text{Im}(\Phi)_t \cong \mathcal{X}_t$ and $\mathcal{X}_t$ is smooth, we have that $\text{Im}(\Phi) \to T$ is an embedded smoothing family of $\text{Im}(\Phi)_0$.

Proof. See [GGP13], Proposition 1.4. The last assertion follows from equation 1.9 in the proof of Proposition 1.4, [GGP13].

4.1. Smoothings of K3 carpets on Hirzebruch surfaces. We now prove that projective K3 carpets are smoothable. We first deal with carpets on Hirzebruch surfaces $\mathbb{F}_e$ with $0 \leq e \leq 2$. The reason we treat them first is because these surfaces admit smooth K3 double covers. We will use this result to show that the projective K3 carpets on the remaining Hirzebruch surfaces are smoothable as well, using a degeneration argument. Finally, we show smoothing results on projective K3 carpets on embedded Hirzebruch surfaces given by an arbitrary very ample line bundle.

Theorem 4.3. Let $\tilde{S}$ be a projective K3 carpet supported on $S = \mathbb{F}_e$ with $0 \leq e \leq 2$. Then $\tilde{S}$ is smoothable.
Proof. Let $\bar{J}$ be a very ample line bundle on $\bar{S}$. Consider the embedding $\bar{i}: \bar{S} \hookrightarrow \mathbb{P}^N$ given by the complete linear series of $\bar{J}$. This induces an embedding $i: S \hookrightarrow \mathbb{P}^N$. Let $\bar{J}|_S = J$. Consider the following exact sequence:

$$0 \to K_S \to O_S \to O_S \to 0.$$

Tensoring by $\bar{J}$ we get

$$0 \to K_S \otimes J \to \bar{J} \to J \to 0.$$

Since $H^1(K_S \otimes J) = 0$ we have that after a projection to a linear subspace of codimension $h^0(K_S \otimes J)$, the composition of the embedding followed by the projection is induced by the complete linear series of $J := aC_0 + bf$ and hence satisfies the situation of Lemma 2.5.

Now notice that the complete linear series of $|-2K_S|$ is base point free for $S = F_e$ with $0 \leq e \leq 2$. By Bertini, we can choose a smooth curve section $C \in |-2K_S|$ and consider the double cover $\pi: X \to S$ branched along the smooth curve $C$. It follows that $K_X = O_X$ and $H^1(O_X) = H^1(O_S) \otimes H^1(K_S) = 0$ since the trace zero module of $\pi$ is $K_S$. Thus, $X$ is a $K3$ surface. Now consider the composed morphism $\varphi: X \xrightarrow{\pi} S \to \mathbb{P}^N$. According to Lemma 4.1, we have an exact sequence

$$0 \to H^0(\mathcal{N}_\varphi) \to H^0(\mathcal{N}_\varphi) \to H^0(\mathcal{N}_Y|_{\mathbb{P}^N}) \otimes H^0(\mathcal{N}_Y|_{\mathbb{P}^N} \otimes K_S) \to H^1(\mathcal{N}_\varphi)$$

Observe that $\mathcal{N}_\varphi = -2K_S|$ and it fits into the following exact sequence

$$0 \to -2K_S \to -2K_S \to 0.$$

Note that, since $H^1(-2K_S) = H^2(O_S) = 0$ we have that $H^1(\mathcal{N}_\varphi) = H^1(\mathcal{N}_\varphi) = 0$.

In Theorem 3.1, we showed that nowhere vanishing sections form a non-empty Zariski open subset in the projective space of lines of the vector space $\mathcal{N}_S|_{\mathbb{P}^N} \otimes K_S$ and that $S$ corresponds to such a nowhere vanishing section. This combined with $H^1(\mathcal{N}_\varphi) = 0$ allows us to choose a first order deformation $\tilde{\varphi} \in H^0(\mathcal{N}_\varphi)$ i.e. a deformation $\tilde{\varphi}: \tilde{X} \to \mathbb{P}^N$ over the ring of dual numbers $\Delta$, which maps to $\tilde{S} \in H^0(\mathcal{N}_S|_{\mathbb{P}^N} \otimes K_S)$ under the map $\nu_2$ as defined in Lemma 4.1.

Let $\tilde{L} = \tilde{\varphi}^* (O_{\mathbb{P}^N}(1))$ and $L = \varphi^* (O_{\mathbb{P}^N}(1))$. Then the pair $(\tilde{X}, \tilde{L})$ is a first order deformation of the pair $(X, L)$. Since polarized $K3$ surfaces have a 19 dimensional smooth algebraic formally universal deformation space $D$, one can choose a smooth affine algebraic curve $T$ in $D$ passing through the point $(X, L)$ with tangent vector given by $(\tilde{X}, \tilde{L})$. Pulling back the universal family on $D$ along $T$, we get a pair $(\mathcal{X} \to T, \mathcal{L})$ where $\mathcal{X}$ is smooth, irreducible and projective (since $L$ is ample) after possibly shrinking $T$ and $\mathcal{L}$ is a line bundle on $\mathcal{X}$. Since $T$ is affine, we have that $\mathcal{L}$ induces a morphism $\Psi: \mathcal{X} \to \mathbb{P}^M_T$ by its complete linear series. Let $\psi: X \to \mathbb{P}^M$ and $\bar{\psi}: \bar{X} \to \mathbb{P}^M_{\Delta}$ are the morphisms induced by the complete linear series of $L$ and $\bar{L}$ respectively. Note that $\tilde{\varphi}$ is obtained by composing $\bar{\psi}$ by a projection $\tilde{p}: \mathbb{P}^M_{\Delta} \to \Delta$. Let $p: \mathbb{P}^M \to \mathbb{P}^N$ be the restriction of $\tilde{p}$ to $\mathbb{P}^N$. Note that we have a cartesian diagram.

$$
\begin{array}{ccc}
X & \xrightarrow{\psi} & \mathbb{P}^M \\
\downarrow & & \downarrow \\
\bar{X} & \xrightarrow{\bar{\psi}} & \mathbb{P}^M_{\Delta}
\end{array}
\begin{array}{ccc}
\mathbb{P}^N & \xrightarrow{p} & \mathbb{P}^N \\
\downarrow & & \downarrow \\
\Delta & & \Delta
\end{array}
\Rightarrow
0$$
Since one can lift the projection \( \tilde{p} \) to a projection \( P : \mathbb{P}^M_T \to \mathbb{P}^N_T \) we get the following diagram.

\[
\begin{array}{ccccccccc}
X & \longrightarrow & \tilde{X} & \longrightarrow & \mathcal{X} \\
\downarrow \psi & & \downarrow \tilde{\psi} & & \downarrow \psi \\
\mathbb{P}^M & \longrightarrow & \mathbb{P}^M_{\Delta} & \longrightarrow & \mathbb{P}^M_T \\
\downarrow p & & \downarrow \tilde{p} & & \downarrow p \\
\mathbb{P}^N & \longrightarrow & \mathbb{P}^N_{\Delta} & \longrightarrow & \mathbb{P}^N_T \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Delta & \longrightarrow & T
\end{array}
\]

Let \( \Phi = P \circ \Psi \). Then \( \Phi \) is a deformation of \( \varphi \) which satisfies the conditions of Proposition 4.2. Hence \( \tilde{S} \) has an embedded smoothing by Proposition 4.2.

We are left with the embedded K3 carpets on \( \mathbb{F}_e \), with \( e > 2 \). We treat them below.

**Theorem 4.4.** Let \( \tilde{S} \) be a projective K3 carpet supported on \( S = \mathbb{F}_e \) with \( e > 2 \). Then \( \tilde{S} \) is smoothable.

**Proof.** Let \( \tilde{M} \) be a very ample line bundle on \( \tilde{S} \). Consider the embedding of \( \tilde{i} : \tilde{S} \hookrightarrow \mathbb{P}^N \) given by the complete linear series \( \tilde{M} \). As in Theorem 4.3, the induced embedding \( i : S \hookrightarrow \mathbb{P}^N \) satisfies the situation of Lemma 2.5.

We know that there exist a deformation \( \mathcal{S} \to T \) of \( S = \mathcal{S}_0 = S \) and \( \mathcal{S}_t = S' \cong \mathbb{F}_e \) with \( 0 \leq e \leq 2 \). Given the embedding \( i \), we have the restriction of Euler sequence to \( S \) (here \( \mathcal{O}_S(1) = \mathcal{O}_S(aC_0 + bf) \))

\[
\begin{align*}
0 \to \mathcal{O}_S & \to \mathcal{O}_S(1) \otimes \mathcal{O}_S^{N+1} \to T_{\mathbb{P}^N} |_S \to 0.
\end{align*}
\]

Since \( H^1(\mathcal{O}_S(1)) = H^2(\mathcal{O}_S) = 0 \), we have that \( H^1(T_{\mathbb{P}^N} |_S) = 0 \), consequently the deformation \( \mathcal{S} \to T \) can be realized inside \( \mathbb{P}^N_T \), i.e. we have the following diagram.

\[
\begin{array}{cccccc}
\mathcal{S} & \longrightarrow & \mathbb{P}^N_T \\
\downarrow p & & \downarrow T \\
& &
\end{array}
\]

By lemma 2.7, (2), we have that \( H^1(\mathcal{N}_{\mathcal{S}/\mathbb{P}^N} \otimes K_S) = 0 \). Thus, after possibly shrinking \( T \), \( p_* (\mathcal{N}_{\mathcal{S}/\mathbb{P}^N} \otimes K_{\mathcal{S}/T}) \) is free of rank \( h^0(\mathcal{N}_{\mathcal{S}/\mathbb{P}^N} \otimes K_S) \). Consider the projective bundle

\[
\mathbb{P}(p_* (\mathcal{N}_{\mathcal{S}/\mathbb{P}^N} \otimes K_{\mathcal{S}/T})) \longrightarrow T.
\]

By Theorem 3.1, (3), we see that there exist a open set \( U \subset \mathbb{P}(p_* (\mathcal{N}_{\mathcal{S}/\mathbb{P}^N} \otimes K_{\mathcal{S}/T})) \) which intersects every fibre of \( \pi \) in a nontrivial open set such every \( u \in U \) corresponds to a nowhere vanishing section of \( H^0(\mathcal{N}_{\mathcal{S}/\mathbb{P}^N} \otimes K_{\mathcal{S}}) \). Now \( \tilde{S} \in H^0(\mathcal{N}_{\mathcal{S}/\mathbb{P}^N} \otimes K_S) \) is a nowhere vanishing section and hence is a point in \( U \) in the fibre of \( 0 \in T \). One can then choose a section of \( \pi \) through \( \tilde{S} \) to extend the section \( \tilde{S} \in H^0(\mathcal{N}_{\mathcal{S}/\mathbb{P}^N} \otimes K_S) \) to a section \( \tilde{S}_T \in H^0(\mathcal{N}_{\mathcal{S}/\mathbb{P}^N} \otimes K_{\mathcal{S}/T}) \). Since \( \tilde{S} \in U \), one can assume after possibly shrinking \( T \), that \( \tilde{S}_{T,t} \) is nowhere vanishing for all \( t \in T \). Hence fibrewise each
section $\tilde{S}_T, t \in H^0(\mathcal{N}_T/P_N \otimes K_T)$ gives a $K3$ carpet on $\mathcal{S}_t$ embedded in $\mathbb{P}_T^N$. Since each $\mathcal{S}_t \cong \tilde{S}' \cong \mathbb{F}_e$ for $0 \leq e \leq 2$, we have proved the first part of our claim.

The last statement follows since we just showed that there exist an irreducible component of the Hilbert scheme containing $\tilde{S}$ inside $\mathbb{P}^N$ whose general element is a smooth $K3$ surface. □

4.2. Smoothings of embedded $K3$ carpets on $\mathbb{P}^2$. We treat the embedded $K3$ carpets on the $d$-uple embeddings. The following gives the smoothing result for them.

**Theorem 4.5.** Let $\tilde{S}$ be a projective $K3$ carpet on $S = \mathbb{P}^2$. Then $\tilde{S}$ is smoothable.

**Proof.** The proof is similar to that of Theorem 4.3 using Theorem 3.3. □

The smoothing results and the fact that two $K3$ surfaces are deformation equivalent shows

**Corollary 4.6.** Suppose $\tilde{S}_1$ and $\tilde{S}_2$ be two abstract $K3$ carpets supported on minimal rational surfaces or $\mathbb{F}_1$. Then $\tilde{S}_1$ and $\tilde{S}_2$ are deformation equivalent.

**Remark 4.7.** Note that the abstract split $K3$ carpets on either a Hirzebruch surface $\mathbb{F}_e$ or $\mathbb{P}^2$ are projective by the proof of Theorem 3.1 since the split carpet corresponds to the zero element $\tau \in H^{1,1}(S)$ and $\int_D \tau = 0$ for any ample divisor $D$ over either $\mathbb{F}_e$ or $\mathbb{P}^2$. Hence they are smoothable.

The results in [BGG20] show that there are no higher dimensional analogues of the above results. They introduce the notion of generalized hyperelliptic varieties that unifies various results on deformations of double covers. We now give that definition.

**Definition 4.8.** Let $(X, L)$ a smooth polarized variety with $L$ base-point-free. Let the morphism from $X$ to $\mathbb{P}^N$, induced by $|L|$ be of degree 2 onto its image $i(Y)$ (i is the embedding of $Y$ in $\mathbb{P}^N$). Let the variety $Y$ be smooth and isomorphic to any of these: (1) projective space, (2) a hyperquadric, (3) a projective bundle over $\mathbb{P}^1$. The variety $Y$ is not necessarily embedded by $i$ as a variety of minimal degree in $\mathbb{P}^N$. Then we say that $(X, L)$ is a generalized hyperelliptic polarized variety.

Calabi Yau varieties are higher dimensional analogues of $K3$ surfaces. In [BGG20], they show that there are no Calabi Yau double structures on projective bundles and $\mathbb{P}^N$, in sharp contrast with the results on $K3$ carpets in this article. This together with results on deformations in [BGG20] show that deformations of generalized hyperelliptic Calabi-Yau $n$-fold is again a generalized hyperelliptic $n$-fold.

5. Hilbert Points of $K3$ Carpets

In this section we prove results on the Hilbert point of a projective $K3$ carpet on the Hirzebruch surfaces and $\mathbb{P}^2$. This is in sharp contrast with the results in [GGP08b] on Hilbert points corresponding to $K3$ carpets on an Enriques surfaces, where all the Hilbert points were smooth.

5.1. $K3$ carpets on Hirzebruch surfaces. First we deal with the $K3$ carpets on $\mathbb{F}_e$.

**Theorem 5.1.** Let $\tilde{S}$ be a projective $K3$ carpet supported on $S = \mathbb{F}_e$ embedded in $\mathbb{P}^N$ by the complete linear series of a very ample line bundle. Then $\tilde{S}$ is a smooth point of its Hilbert scheme if and only if $0 \leq e \leq 2$. 
Proof. We know from Theorem 4.3 and Theorem 4.4 that $\tilde{S}$ admits an embedded smoothing. Hence there exists an irreducible component $H$ of the Hilbert scheme containing $\tilde{S}$ such that a general element of $H$ is a smooth $K3$ surface $X$. Let $i : X \to \mathbb{P}^N$ denote the embedding. Let $L = \mathcal{O}_X(1) := i^* \mathcal{O}_{\mathbb{P}^N}(1)$. Then, we have the following exact sequence.

(5.1) 
$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1)^{\oplus N+1} \to T_{\mathbb{P}^N}|_X \to 0.$$ 

Using Kodaira vanishing and $h^2(\mathcal{O}_X) = 1$ we have that $h^1(T_{\mathbb{P}^N}|_X) = 1$. Now consider

(5.2) 
$$0 \to T_X \to T_{\mathbb{P}^N}|_X \to \mathcal{N}_{X/\mathbb{P}^N} \to 0.$$ 

Taking cohomology we get that

(5.3) 
$$h^0(\mathcal{N}_{X/\mathbb{P}^N}) \to H^1(T_X) \to H^1(T_{\mathbb{P}^N}|_X) \to H^1(\mathcal{N}_{X/\mathbb{P}^N}) \to H^2(T_X) = 0 \to 0.$$ 

Note that, since there exist non-projective $K3$ surfaces we have that $H^0(\mathcal{N}_{X/\mathbb{P}^N}) \to H^1(T_X)$ is not surjective. Consequently, $H^1(\mathcal{N}_{X/\mathbb{P}^N}) = 0$. Thus, $X$ is smooth point in its Hilbert scheme. Therefore $\tilde{S}$ is a smooth point of the Hilbert scheme if and only if

$$h^0(\mathcal{N}_{\tilde{S}/\mathbb{P}^N}) = h^0(\mathcal{N}_{X/\mathbb{P}^N}) = (N+1)^2 + 18.$$ 

(Note that $N + 1 = h^0(\mathcal{O}_{\tilde{S}}(1)) = h^0(\mathcal{O}_X(1))$.)

In order to compute $h^0(\mathcal{N}_{\tilde{S}/\mathbb{P}^N})$ we use the following exact sequences;

(5.4) 
$$0 \to \mathcal{N}_{\tilde{S}/\mathbb{P}^N} \otimes K_S \to \mathcal{N}_{\tilde{S}/\mathbb{P}^N} \to \mathcal{N}_{\tilde{S}/\mathbb{P}^N} \otimes \mathcal{O}_S \to 0,$$

(5.5) 
$$0 \to K_S^* \to \mathcal{N}_{\tilde{S}/\mathbb{P}^N} \to \mathcal{H}om(I_{\tilde{S}}/I^2_{\tilde{S}}, \mathcal{O}_S) \to 0,$$

(5.6) 
$$0 \to \mathcal{O}_S \to \mathcal{N}_{\tilde{S}/\mathbb{P}^N} \otimes K_S \to \mathcal{H}om(I_{\tilde{S}}/I^2_{\tilde{S}}, \mathcal{O}_S) \otimes K_S \to 0,$$

(5.7) 
$$0 \to \mathcal{H}om(I_{\tilde{S}}/I^2_{\tilde{S}}, \mathcal{O}_S) \to \mathcal{N}_{\tilde{S}/\mathbb{P}^N} \otimes \mathcal{O}_S \to K_S^{-2} \to 0,$$

(5.8) 
$$0 \to \mathcal{H}om(I_{\tilde{S}}/I^2_{\tilde{S}}, \mathcal{O}_S) \otimes K_S \to \mathcal{N}_{\tilde{S}/\mathbb{P}^N} \otimes K_S \to K_S^* \to 0.$$ 

Using (5.5), we have that $h^1(\mathcal{H}om(I_{\tilde{S}}/I^2_{\tilde{S}}, \mathcal{O}_S)) = h^2(\mathcal{H}om(I_{\tilde{S}}/I^2_{\tilde{S}}, \mathcal{O}_S)) = 0$, and the following;

$$h^0(\mathcal{H}om(I_{\tilde{S}}/I^2_{\tilde{S}}, \mathcal{O}_S)) = h^0(\mathcal{N}_{\tilde{S}/\mathbb{P}^N}) - h^0(K_S^*) + h^1(K_S^*).$$

Using (5.6), we have that $h^1(\mathcal{H}om(I_{\tilde{S}}/I^2_{\tilde{S}}, \mathcal{O}_S) \otimes K_S) = h^2(\mathcal{H}om(I_{\tilde{S}}/I^2_{\tilde{S}}, \mathcal{O}_S) \otimes K_S) = 0$, and

$$h^0(\mathcal{H}om(I_{\tilde{S}}/I^2_{\tilde{S}}, \mathcal{O}_S) \otimes K_S) = h^0(\mathcal{N}_{\tilde{S}/\mathbb{P}^N} \otimes K_S) - 1.$$ 

Using (5.7), we have that $h^1(\mathcal{N}_{\tilde{S}/\mathbb{P}^N} \otimes \mathcal{O}_S) = h^2(\mathcal{N}_{\tilde{S}/\mathbb{P}^N} \otimes \mathcal{O}_S) = 0$, and the following;

$$h^0(\mathcal{N}_{\tilde{S}/\mathbb{P}^N} \otimes \mathcal{O}_S) = h^0(\mathcal{N}_{\tilde{S}/\mathbb{P}^N}) - h^0(K_S^*) + h^1(K_S^*).$$

Using (5.8), we have that $h^1(\mathcal{N}_{\tilde{S}/\mathbb{P}^N} \otimes K_S) = h^1(K_S^*), h^2(\mathcal{N}_{\tilde{S}/\mathbb{P}^N} \otimes K_S) = 0$, and the following;

$$h^0(\mathcal{N}_{\tilde{S}/\mathbb{P}^N} \otimes K_S) = h^0(\mathcal{N}_{\tilde{S}/\mathbb{P}^N} \otimes K_S) - 1 + h^0(K_S^*).$$

Using (5.4), we have that $h^0(\mathcal{N}_{\tilde{S}/\mathbb{P}^N}) - h^1(\mathcal{N}_{\tilde{S}/\mathbb{P}^N}) = h^0(\mathcal{N}_{\tilde{S}/\mathbb{P}^N}) + h^0(\mathcal{N}_{\tilde{S}/\mathbb{P}^N} \otimes K_S) + 24$. It follows from Lemma 2.7 that $h^0(\mathcal{N}_{\tilde{S}/\mathbb{P}^N} \otimes K_S) = (N + 1)h^0(\mathcal{O}_S((aC_0 + b f) \otimes K_S)) + 1$. A straighforward computation gives $h^0(\mathcal{N}_{\tilde{S}/\mathbb{P}^N}) = (N + 1)(h^0(\mathcal{O}_S((aC_0 + b f) \otimes K_S)) - 7$ and hence $h^0(\mathcal{N}_{\tilde{S}/\mathbb{P}^N}) - h^1(\mathcal{N}_{\tilde{S}/\mathbb{P}^N}) = (N + 1)^2 + 18$ (Note that $h^0(\mathcal{O}_S((aC_0 + b f) \otimes K_S))) = N + 1)$. Thus, $\tilde{S}$ is smooth if and only if $h^1(\mathcal{N}_{\tilde{S}/\mathbb{P}^N}) = 0.$
Using (5.4), we get the following exact sequence;

\[ H^1(\mathcal{N}_{S/{\mathbb{P}^N}} \otimes K_S) = H^1(K_S^2) \to H^1(\mathcal{N}_{S/{\mathbb{P}^N}}) \to H^1(\mathcal{N}_{S/{\mathbb{P}^N}} \otimes \mathcal{O}_S) = H^1(K_S^{-2}) \to 0. \]

For \(0 \leq e \leq 2\) both the flanking terms vanish but for \(e > 2\), \(h^1(K_S^{-2}) \neq 0\). Thus, \(\tilde{S}\) is smooth if and only if \(0 \leq e \leq 2\).

5.2. **K3 carpets on \(\mathbb{P}^2\).** We conclude the article by the following result.

**Theorem 5.2.** Let \(\tilde{S}\) be a projective K3 carpet supported on \(S\) and embedded inside \(\mathbb{P}^N\) by the complete linear series of a very ample line bundle. Then \(\tilde{S}\) is a smooth point of its Hilbert scheme.

**Proof.** As in the previous theorem we need to show

\[ h^0(\mathcal{N}_{S/{\mathbb{P}^N}} \otimes K_S) = (N + 1)^2 + 18. \]

Using (5.5), we have that \(h^1(\mathcal{H}om(I\tilde{S}/I^2\tilde{S}, \mathcal{O}_S))) = h^2(\mathcal{H}om(I\tilde{S}/I^2\tilde{S}, \mathcal{O}_S)) = 0\), and the following;

\[ h^0(\mathcal{H}om(I\tilde{S}/I^2\tilde{S}, \mathcal{O}_S)) = (N + 1)h^0(\mathcal{O}_{\mathbb{P}^2}(d)) - 3h^0(\mathcal{O}_{\mathbb{P}^2}(1)) - h^0(\mathcal{O}_{\mathbb{P}^2}(3)). \]

Using (5.6), we have that \(h^1(\mathcal{H}om(I\tilde{S}/I^2\tilde{S}, \mathcal{O}_S) \otimes K_S) = h^2(\mathcal{H}om(I\tilde{S}/I^2\tilde{S}, \mathcal{O}_S) \otimes K_S) = 0\), and

\[ h^0(\mathcal{H}om(I\tilde{S}/I^2\tilde{S}, \mathcal{O}_S) \otimes K_S) = (N + 1)h^0(\mathcal{O}_{\mathbb{P}^2}(d - 3)) - 1. \]

Using (5.7), we have that \(h^1(\mathcal{N}_{S/{\mathbb{P}^N}} \otimes \mathcal{O}_S) = h^2(\mathcal{N}_{S/{\mathbb{P}^N}} \otimes \mathcal{O}_S) = 0\), and the following;

\[ h^0(\mathcal{N}_{S/{\mathbb{P}^N}} \otimes \mathcal{O}_S) = (N + 1)h^0(\mathcal{O}_{\mathbb{P}^2}(d)) - 3h^0(\mathcal{O}_{\mathbb{P}^2}(1)) - h^0(\mathcal{O}_{\mathbb{P}^2}(3)) + (N + 1)h^0(\mathcal{O}_{\mathbb{P}^2}(d)) - 3h^0(\mathcal{O}_{\mathbb{P}^2}(1)) - h^0(\mathcal{O}_{\mathbb{P}^2}(6)). \]

Using (5.8), we have that \(h^1(\mathcal{N}_{S/{\mathbb{P}^N}} \otimes K_S) = h^1(\mathcal{O}_{\mathbb{P}^2}(3))h^2(\mathcal{N}_{S/{\mathbb{P}^N}} \otimes K_S) = h^2(\mathcal{O}_{\mathbb{P}^2}(3))\), and

\[ h^0(\mathcal{N}_{S/{\mathbb{P}^N}} \otimes K_S) = (N + 1)h^0(\mathcal{O}_{\mathbb{P}^2}(d - 3)) - 1 + h^0(\mathcal{O}_{\mathbb{P}^2}(3)). \]

Using (5.4), we have that \(h^0(\mathcal{N}_{S/{\mathbb{P}^N}}) = h^0(\mathcal{N}_{S/{\mathbb{P}^N}} + h^0(\mathcal{N}_{S/{\mathbb{P}^N}} \otimes K_S)\). It follows from Lemma 2.9 that \(h^0(\mathcal{N}_{S/{\mathbb{P}^N}}) = (N + 1)(h^0(\mathcal{O}_{\mathbb{P}^2}(d - 3)) + h^0(\mathcal{O}_{\mathbb{P}^2}(d))) + 18\). The assertion follows since \((N + 1) = h^0(\mathcal{O}_{\mathbb{P}^2}(d - 3)) + h^0(\mathcal{O}_{\mathbb{P}^2}(d))\).

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KANSAS, LAWRENCE, USA

E-mail address: purna@ku.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KANSAS, LAWRENCE, USA

E-mail address: j899m889@ku.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KANSAS, LAWRENCE, USA

E-mail address: debaditya@ku.edu