MINIMAL VOLUME $k$-POINT LATTICE $d$-SIMPLICIES

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Abstract. We show via triangulations that for $d \geq 3$ there is exactly one class (under unimodular equivalence) of nondegenerate lattice simplices in $\mathbb{R}^d$ with minimal volume and $k$ interior lattice points.

1. Introduction

A $d$-polytope $P$ is a polytope of dimension $d$. If its vertex set $\mathcal{V}(P)$ is a subset of $\mathbb{Z}^d$, then $P$ is a lattice $d$-polytope. If in addition $|\mathcal{V}(P)| = d + 1$, then $P$ is a lattice $d$-simplex. The convex hull of $\mathcal{P} = \{v_1, \ldots, v_n\} \subset \mathbb{Z}^d$, denoted by $\text{conv}(\mathcal{P})$, is a lattice polytope with at most $n$ vertices and dimension at most $d$. This notation will be used loosely; for convenience, we use $\text{conv}(P, v)$ to mean $\text{conv}(\mathcal{V}(P) \cup \{v\})$ when it is clear $P$ is a polytope and $v$ is a point. As used in [22], we say that $P$ is clean if $\partial P \cap \mathbb{Z}^d = \mathcal{V}(P)$, where $\partial P$ is the boundary of $P$. If in addition $\text{int}(P)$, the interior of $P$, contains $k$ lattice points, then $P$ is a clean $k$-point lattice polytope. If $k = 0$, then the polytope is empty. We use $\mathcal{S}_d^k$ to denote the collection of clean $k$-point lattice $d$-simplices. Unless otherwise stated, all polytopes are taken to be convex $d$-polytopes.

Reznick proved in [22] and [23] that any lattice tetrahedron with at least one clean face is unimodularly equivalent to some $T_{a,b,n}$, the lattice tetrahedron with vertex set

$$\{ (0, 0, 0), (1, 0, 0), (0, 1, 0), (a, b, n) \}$$

where $(a, b, n) \in \mathbb{Z}^3$ and $0 < a, b < n$. Reznick also classified the set of clean 1-point tetrahedra, up to equivalence under unimodular transformations, using barycentric coordinates. Very recently, Bey, Henk, and Wills proved in [2] that if $P$ is a lattice $d$-polytope, not necessarily clean, and $P$ has $k$ interior lattice points, then for $d \geq 1$, the volume of $P$ satisfies

$$\text{Vol}(P) \geq \frac{1}{d!} (dk + 1). \tag{1}$$

Moreover, they showed that for $k = 1$, equality holds if and only if $P$ is unimodularly equivalent to the simplex $S_d(1)$, where

$$S_d(k) = \text{conv} \left( e_1, \ldots, e_d, -k \sum_{i=1}^d e_i \right)$$

and $e_i$ denotes the $i$-th unit point. This is not true for $d = 2$ and $k > 2$. We will show that equality holds in (1) for all $k > 0$ if and only if $d \geq 3$ and $P$ is unimodularly equivalent to $S_d(k)$. We first prove that if $T \in \mathcal{S}_d^k$ and $\text{Vol}(T) = \frac{1}{d!} (dk + 1)$, then the interior points lie on a line passing through some vertex of $T$. We then show that such simplices are unimodularly equivalent to $T_{a_1, \ldots, a_d}$, the $d$-simplex.

Date: April 21, 2008.
whose vertex set consists of the origin, the points $e_i$ ($1 \leq i \leq d - 1$), and the point $(a_1, \ldots, a_d)$, where $a_j = dk$ for $1 \leq j < d$ and $a_d = dk + 1$. Finally, we will show that $S_d(k) \in \mathcal{J}_k^d$ and $\text{Vol}(S_d(k)) = \frac{1}{d!}(dk + 1)$.

2. Preliminaries

The following definitions are taken from [1], [7], [17], and [24]. A $j$-flat is a $j$-dimensional affine subspace of $\mathbb{R}^d$. Points, lines, and planes are 0-flats, 1-flats, and 2-flats, respectively. The affine hull of a set $\mathcal{P} \subset \mathbb{R}^d$, denoted by $\text{aff}(\mathcal{P})$, is the intersection of all flats containing $\mathcal{P}$. Equivalently, $\text{aff}(\mathcal{P})$ is the smallest flat containing $\mathcal{P}$. We say $\mathcal{P}$ is in general position if no $j + 2$ points of $\mathcal{P}$ lie in a $j$-flat, where $j < d$. A hyperplane

$$H = \{x \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} = b\}$$

is a $(d - 1)$-flat. If $P$ is a $d$-polytope, then $H$ is a supporting hyperplane of $P$ if $P$ lies entirely on one side of $H$. A face of $P$ is an intersection $P \cap H$, where $H$ is a supporting hyperplane. If we allow degenerate hyperplanes, then $P$ is a face of $P$ corresponding to $H = \mathbb{R}^d$; $\emptyset$ is also a face of $P$ corresponding to a hyperplane that does not meet $P$. A $j$-face of $P$ is a $j$-dimensional face of $P$. A $(d - 1)$-face is a facet, a 1-face is an edge, and a 0-face is a vertex. One property of $d$-polytopes is that any $j$-face of $P$ is contained in at least $d - j$ facets of $P$.

**Lemma 1.** [7] Section 3.1] If $0 \leq i < j \leq d - 1$ and if $P$ is a $d$-polytope, each $i$-face of $P$ is the intersection of the family of $j$-faces of $P$ containing it. There are at least $j + 1 - i$ such faces.

We generally use capital letters to denote $d$-polytopes. In particular $P$ and $Q$ are $d$-polytopes, and $S$ and $T$ are $d$-simplices. Capital script letters will generally denote sets of $d$-simplices. In particular, $\mathcal{P}$ is a set of points (0-simplices).

3. Triangulations and Refinements

Borrowing from [15], we let $\mathcal{P}$ denote a set of $n$ distinct points in $\mathbb{R}^d$, where $n \geq d + 1$ and $d \geq 2$. Assume $\mathcal{P}$ does not lie entirely in a hyperplane. Let $P = \text{conv}(\mathcal{P})$. A triangulation, $\mathcal{T}$, of $\mathcal{P}$ (or of $P$ with the dependence on $\mathcal{P}$ understood) is a set of nondegenerate $d$-simplices $\{T_i\}$ with the following properties.

(a) All vertices of each simplex are members of $\mathcal{P}$.
(b) The interiors of the simplices are pairwise disjoint.
(c) Each facet of a simplex is either on the boundary of $P$, or else is a common facet of exactly two simplices.
(d) Each simplex contains no points of $\mathcal{P}$ other than its vertices.
(e) The union of $\{T_i\}$ is $\mathcal{P}$ and the union of $T_i$ is $P$.

Since each $d$-simplex has volume at least $\frac{1}{d!}$, one immediate consequence is

$$\text{Vol}(P) \geq \frac{1}{d!} |\mathcal{T}|. \tag{2}$$

To prove [1], Bey, Henk, and Wills showed $P$ with $k$ interior lattice points can be decomposed into at least $dk + 1$ nondegenerate $d$-subpolytopes. Any $d$-polytope must contain a $d$-simplex as a subpolytope, so (2) still holds if $\mathcal{T}$ is replaced by this decomposition. We will present a slight variation of their theorem and its proof by using triangulations.
**Definition.** Let \( P \) be a lattice \( d \)-polytope. A *lattice triangulation* \( \mathcal{T} \) of \( P \) is a triangulation of some set \( \mathcal{P} \subset \mathbb{Z}^d \) such that \( \mathcal{V}(\mathcal{P}) \subseteq \mathcal{P} \subseteq \mathcal{P} \cap \mathbb{Z}^d \). Note that if \( \mathcal{V}(\mathcal{P}) \subseteq \mathcal{P}, \mathcal{P}' \subseteq \mathcal{P} \cap \mathbb{Z}^d \) and \( \mathcal{P} \neq \mathcal{P}' \), we still have \( \text{conv}(\mathcal{P}) = P = \text{conv}(\mathcal{P}') \).

On the other hand, the triangulation \( \mathcal{T} \) of \( P \) and the triangulation \( \mathcal{T}' \) of \( P' \) are necessarily different. Fortunately, we can reconstruct the vertex set of \( \mathcal{T} \), denoted by \( \mathcal{V}(\mathcal{T}) \), by taking the union of all vertices of all \( \mathcal{T} \in \mathcal{T} \). Thus \( \mathcal{V}(\mathcal{T}) = \mathcal{P} \) and \( \mathcal{V}(\mathcal{T}') = \mathcal{P}' \).

**Definition.** Let \( P \) be a lattice polytope and \( \mathcal{T} \) be a lattice triangulation \( P \). We say \( \mathcal{T}' \) is a *refinement* of \( \mathcal{T} \) (and write \( \mathcal{T} \prec \mathcal{T}' \)) provided \( \mathcal{T}' \) is a lattice triangulation of \( P \), \( \mathcal{V}(\mathcal{T}) \subset \mathcal{V}(\mathcal{T}') \), and for all \( \mathcal{T}' \in \mathcal{T}' \) there exists \( \mathcal{T} \in \mathcal{T} \) such that \( \mathcal{T}' \subseteq \mathcal{T} \).

We say that \( \mathcal{T} \) is a *full lattice triangulation* if \( \mathcal{V}(\mathcal{T}) = P \cap \mathbb{Z}^d \). Otherwise we say \( \mathcal{T} \) is a *partial lattice triangulation*.

Naturally, triangulations of \( P \subset \mathbb{R}^d \) partition \( \text{conv}(\mathcal{P}) \) into simplices. On the other hand, there exist partitions of \( \text{conv}(\mathcal{P}) \) that satisfy all but condition (d) in the definition of a triangulation.

![Figure 1. Partitions of P into triangles](image)

**Example 1.** Let \( \mathcal{P} = \{e_1, e_2, -e_1, -e_2\} \subset \mathbb{R}^2 \) and let \( P = \text{conv}(\mathcal{P}) \). Note that \( P \) is a 1-point lattice polygon, and its interior lattice point is the origin. Three possible partitions \( \mathcal{D}_1, \mathcal{D}_2, \) and \( \mathcal{D}_3 \) of \( P \) into triangles are shown in Figure 1. Note that \( \mathcal{D}_1 \) is a triangulation of \( \mathcal{P} \cup \{(0,0)\} \) and a full lattice triangulation of \( P \). However, \( \mathcal{D}_1 \) is not a triangulation of \( \mathcal{P} \). The middle partition \( \mathcal{D}_2 \) is both a triangulation of \( \mathcal{P} \) and a partial lattice triangulation of \( P \) (viewed as a triangulation of \( \mathcal{V}(P) = \mathcal{P} \)). The partition \( \mathcal{D}_3 \) on the right is not a triangulation since it fails condition (d).

The following theorem, the proof of which can be found in the appendix of [1], guarantees the existence of a triangulation of the vertex set of a polytope.

**Theorem 2.** [1, Theorem 3.1] Every convex polytope \( P \) can be triangulated using no new vertices. That is, there exists a triangulation of \( \mathcal{V}(P) \).

Let \( P \) be a clean, non-empty lattice \( d \)-polytope, and suppose \( w \in \text{int}(P) \cap \mathbb{Z}^d \). We construct a basic lattice triangulation \( \mathcal{T}_w \) of \( P \) in the following manner. Theorem 2 guarantees that each facet \( F \) of \( P \), as a \((d-1)\)-polytope, has a (lattice) triangulation \( \mathcal{T}_F \) of \( \mathcal{V}(F) \). Let \( \mathcal{F} \) be the set of facets of \( P \) and let \( \mathcal{B} \) be the set

\[
\mathcal{B} = \bigcup_{F \in \mathcal{F}} \mathcal{T}_F
\]
of \((d-1)\)-simplices. Finally, let \(\mathcal{T}_w = \{ \text{conv}(S, w) : S \in \mathcal{B} \} \). It is easy to check that \(\mathcal{T}_w\) is a lattice triangulation of \(P\). Note that

\[
|\mathcal{T}_w| = |\mathcal{B}| \geq |\mathcal{T}| \geq d + 1.
\]

In particular, if \(T \in \mathcal{I}_w^d\), where \(k \geq 1\), and \(F_1, \ldots, F_{d+1}\) are the facets of \(T\), then a basic lattice triangulation \(\mathcal{T}_w\) is simply the convex hull of the facets of \(T\) with an interior lattice point \(w\) of \(T\). Moreover, \(\mathcal{T}_w\) is a refinement of the trivial triangulation \(\mathcal{T}_0 = \{ T \}\). The main idea in proving the collinearity property of \(\int(T) \cap \mathbb{Z}^d\) is to start with the trivial triangulation \(\mathcal{F}_0 = \{ T \}\) and obtain a sequence \(\mathcal{F}_1 \prec \cdots \prec \mathcal{F}_k\) of refinements such that \(\mathcal{F}_k\) is a full triangulation of \(T\) and \(|\mathcal{F}_k| \geq d k + 1\), and then show that noncollinearity forces \(|\mathcal{F}_k| > d k + 1\). The following lemma appears as an assertion in the proof of \([1]\) in \([2]\). We prove it here since it is crucial in computing \(|\mathcal{F}_i| - |\mathcal{F}_{i-1}|\).

**Lemma 3.** Suppose \(P\) is a clean, non-empty, lattice \(d\)-polytope. Let \(\mathcal{T}\) be a partial triangulation of \(P\). If \(S\) is a \(j\)-face of some \(T \in \mathcal{T}\), and \(\mathcal{V}(S) \cap \int(P) \cap \mathbb{Z}^d \neq \emptyset\), then \(S\) is contained in at least \(d + 1 - j\) simplices in \(\mathcal{T}\).

**Proof.** Let \(w \in \mathcal{V}(S) \cap \int(P) \cap \mathbb{Z}^d \). By Lemma \([1]\) \(S\) is contained in at least \(d - j\) facets of \(T\). Moreover \(w\) is a vertex of each facet of \(T\) containing \(S\). It follows that these facets are not contained in facets of \(P\) and are therefore shared by exactly two simplices in \(\mathcal{T}\). On the other hand any two simplices in \(\mathcal{T}\) intersect in at most one common facet. Thus \(S\) is contained in at least \(d - j\) other simplices in \(\mathcal{T}\). ■

**Theorem 4.** Let \(P\) be a clean, non-empty, lattice \(d\)-polytope. Let \(\mathcal{T}\) be a partial triangulation of \(P\). For any \(w \in \mathcal{V}(S) \cap \int(P) \cap \mathbb{Z}^d\) there exists a refinement \(\mathcal{T}'\) of \(\mathcal{T}\) such that \(\mathcal{V}(\mathcal{T}') = \mathcal{V}(\mathcal{T}) \cup \{ w \}\) and \(|\mathcal{T}'| \geq |\mathcal{T}| + d\).

**Proof.** Since \(\mathcal{T}\) is a partial triangulation, there exists an interior lattice point \(w\) of \(P\) such that \(w \not\in \mathcal{V}(\mathcal{T})\). Moreover, \(w\) must lie in the relative interior of some \(j\)-face \((1 \leq j \leq d)\), say \(S\), of some simplex in \(\mathcal{T}\). Note that \(S\) is in fact a \(j\)-simplex. Let \(\mathcal{V}(S) = \{ v_1, \ldots, v_{j+1} \}\) and consider the basic triangulation \(\mathcal{T}_w\) of \(S\) into \(j\)-simplices, where

\[
\mathcal{T}_w = \{ \text{conv}(\mathcal{V}(S) \cup \{ w \}\setminus\{v_i\}) : 1 \leq i \leq j + 1 \}.
\]

Clearly \(\mathcal{T}_w\) is a refinement of \(S\) into \(j\)-simplices. This refinement of \(S\) induces a refinement of any \(d\)-simplex containing \(S\). More precisely, if \(T \in \mathcal{T}\) contains \(S\), then the set

\[
\{ \text{conv}(\mathcal{V}(\mathcal{T}) \cup \{ w \}\setminus\{v_i\}) \}
\]

is a lattice triangulation of \(T\). Since \(w \in \int(S)\) and \(P\) is clean, \(S \not\subset \partial P\). Thus \(\mathcal{V}(S) \cap \int(P) \cap \mathbb{Z}^d \neq \emptyset\). By Lemma \([4]\) there are at least \(d + 1 - j\) simplices in \(\mathcal{T}\) containing \(S\) as a \(j\)-face. Now consider \(\mathcal{T}\) with all such \(d\)-simplices in \(\mathcal{T}\) replaced with their respective induced triangulations and take this to be \(\mathcal{T}'\). By construction, \(w\) is contained in a simplex \(T' \in \mathcal{T}'\) if and only if \(w \in \mathcal{V}(T')\). Hence \(\mathcal{T}'\) is a refinement of \(\mathcal{T}\) such that \(\mathcal{V}(\mathcal{T}') = \mathcal{V}(\mathcal{T}) \cup \{ w \}\) and

\[
|\mathcal{T}'| \geq |\mathcal{T}| + (d + 1 - j)(j + 1) - (d + 1 - j) = |\mathcal{T}| + (d + 1 - j)j.
\]

Finally,

\[
(d + 1 - j)j - d = (d - j)(j - 1) \geq 0 \quad \text{for} \quad 1 \leq j \leq d,
\]

which implies \((d + 1 - j)j \geq d\). ■
Corollary 5. If \( T \in \mathcal{K}_k^d \) and \( \mathcal{F}_0 = \{ T \} \), then there exists a sequence
\[
\mathcal{F}_0 < \cdots < \mathcal{F}_k
\]
of refinements of \( \mathcal{F}_0 \) such that \( \mathcal{F}_k \) is a full triangulation of \( T \) and \( |\mathcal{F}_k| \geq dk + 1 \). Moreover, \( \text{Vol}(T) \geq \frac{1}{d!}(dk + 1) \).

**Proof.** For \( k = 0 \), then the sequence consists only of \( \mathcal{F}_0 \). If \( k > 0 \) then \( \mathcal{F}_0 = \{ T \} \) is a partial triangulation of \( T \). Let \( w_1, \cdots, w_k \) be an arbitrary enumeration of the interior lattice points of \( T \). For \( 1 \leq i \leq k \), we refine \( \mathcal{F}_{i-1} \) into \( \mathcal{F}_i \), using Theorem 4 with \( w = w_i \). After \( k \) refinements, \( \mathcal{F}_k \) is a full triangulation of \( T \), and \( |\mathcal{F}_k| \geq |\mathcal{F}_0| + dk = dk + 1 \). Lastly, (2) implies \( \text{Vol}(T) \geq \frac{1}{d!}(dk + 1) \).

Corollary 6. Suppose \( T \in \mathcal{K}_k^d \) where \( k \geq 2 \). Let \( w_1, \cdots, w_k \) be an arbitrary enumeration of \( \text{int}(T) \cap \mathbb{Z}^d \), and let \( \mathcal{T} \) be the corresponding refinement sequence guaranteed by Corollary 4. If \( w_{i+1} \) lies in a \( j \)-face of a simplex in \( \mathcal{T}_i \), where \( i > 0 \) and \( 1 < j < d \), then \( |\mathcal{T}_{i+1}| - |\mathcal{T}_i| > d \) and \( \text{Vol}(T) > \frac{1}{d!}(dk + 1) \).

**Proof.** This follows immediately from Theorem 4 and (3) and (4).

In the context of Corollary 5 (4) implies that for each \( \mathcal{T}_i \) in (4), \( |\mathcal{T}_i| \geq di + 1 \). Since equality in (4) holds if and only if \( j = 1 \) or \( j = d \), \( |\mathcal{T}_i| = di + 1 \) if and only if for each \( i \), \( w_{i+1} \) lies in the relative interior of an edge in \( \mathcal{T}_i \), or in the relative interior of a simplex in \( \mathcal{T}_i \). Note that this must be true for any ordering of the \( w_i \)'s.

Finally, to prove (1) for a general non-empty lattice \( d \)-polytope \( P \), we start with a basic triangulation of \( P \) and obtain a refinement sequence similar to that of Corollary 6.

Corollary 7. [2, Theorem 1.2] If \( P \) is a lattice \( d \)-polytope with \( k \geq 0 \) interior lattice points, then there exists a sequence
\[
\mathcal{F}_1 < \cdots < \mathcal{F}_k
\]
of lattice triangulations of \( P \) such that \( \mathcal{F}_1 \) is a basic triangulation of \( P \), \( \mathcal{F}_k \) is a full triangulation of \( P \) and \( |\mathcal{F}_k| \geq dk + 1 \). Moreover, \( \text{Vol}(P) \geq \frac{1}{d!}(dk + 1) \).

**Proof.** If \( k = 0 \), take \( \mathcal{F}_1 \) to be the triangulation guaranteed by Theorem 2. If \( k \geq 1 \), then consider an arbitrary enumeration \( w_1, \cdots, w_k \) of the the interior points of \( P \). Let \( \mathcal{F}_1 \) be the basic triangulation \( \mathcal{F}_{w_1} \) of \( T \). Either \( k = 1 \) and \( \mathcal{F}_1 \) is a full triangulation, or we can apply Theorem 4 as in the proof of Corollary 5 to obtain a refinement sequence
\[
\mathcal{F}_1 < \cdots < \mathcal{F}_k
\]
such that \( |\mathcal{F}_k| \geq dk + 1 \). Finally, (2) implies \( \text{Vol}(P) \geq \frac{1}{d!}(dk + 1) \).

In general, we need not start with a basic triangulation of \( P \). It is easy to check that so long as \( \mathcal{F} \) is a partial triangulation such that
\[
|P \cap \mathbb{Z}^d| - |\mathcal{F}(\mathcal{F})| = k - j \quad \text{and} \quad |\mathcal{F}| \geq dj + 1,
\]
then we can still refine \( \mathcal{F} \) into a full triangulation with at least \( dk + 1 \) simplices by applying Theorem 4. This is in fact equivalent to the inductive step in [2], with triangulations replaced by decompositions into \( d \)-subpolytopes. The proof of Corollary 7 is otherwise essentially the same as that in [2].
4. Properties of $d + 2$ Points in $\mathbb{R}^d$

In the context of Corollary 13 if we can show $j = d$ implies $\text{Vol}(T) > \frac{1}{d}(dk + 1)$ as well, then any two interior lattice points of $T$ must be collinear with some vertex of $T$. As a base case, we first consider the possible configurations of any $T \in \mathcal{P}_d^2$, where $d \geq 3$. Suppose $T$ has interior lattice points $w_1$ and $w_2$, and let $\mathcal{V}(T) = \{v_1, \cdots, v_d+1\}$. The sets

$$\mathcal{P}_i = \mathcal{V}(T) \setminus \{v_i\} \cup \{w_1, w_2\}$$

are all sets of $d + 2$ points not contained in a hyperplane. Many properties of such sets are discussed in [3, 8, 13, 15, 17, 18, and 19]. Two relevant and well known results in the theory of convex bodies are Carathéodory’s theorem [4] and Radon’s theorem [19].

Carathéodory’s Theorem. If $\mathcal{P} = \{v_1, \cdots, v_{d+1}\} \subset \mathbb{R}^d$ is not contained in a hyperplane, then every $x \in \mathbb{R}^d$ can be expressed as

$$x = \sum_{i=1}^{d+1} \alpha_i v_i, \quad \text{where} \quad \alpha_i \in \mathbb{R}, \quad v_i \in \mathcal{P}, \quad \text{and} \quad \sum_{i=1}^{d+1} \alpha_i = 1.$$

The coefficients $\alpha_i$ in Carathéodory’s theorem are the barycentric coordinates of $x$ relative to $\text{conv}(\mathcal{P})$. If $P = \text{conv}(\mathcal{P})$ is a $d$-simplex, then the barycentric coordinates of $x$ relative to the simplex $P$ are the numbers $\alpha_1, \cdots, \alpha_{d+1}$ satisfying

$$\begin{pmatrix}
\alpha_1 & \alpha_2 & \cdots & \alpha_{d+1} \\
1 & 1 & \cdots & 1
\end{pmatrix} \begin{pmatrix}
v_1 & 1 \\
v_2 & 1 \\
\vdots & \vdots \\
v_{d+1} & 1
\end{pmatrix} = \begin{pmatrix}
x & 1 \\
1 \times (d+1)
\end{pmatrix}.$$

For each $i$, the sign of $\alpha_i$ indicates the position of $x$ relative to the hyperplane $H_i$ containing the facet of $P$ opposite vertex $v_i$. That is, $\alpha_i > 0$ when $v_i$ and $x$ are on the same side of $H_i$, $\alpha_i < 0$ if $v_i$ and $x$ are on opposite sides of $H_i$, and $\alpha_i = 0$ if $x$ lies in $H_i$. The barycentric coordinates of $x$ relative to $T$ are all positive if and only if $x \in \text{int}(P)$. Thus any point $x \in \mathcal{P}_i$ (cf. (6)), can be described in terms of its barycentric coordinates relative to the simplex $\text{conv}(\mathcal{P}_i)$.

Radon’s Theorem. If $\mathcal{A}$ is a set of $k \geq d + 2$ points in $\mathbb{R}^d$, then there exist a partition $\{\mathcal{A}_1, \mathcal{A}_2\}$ of $\mathcal{A}$ such that $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ and $\text{conv}(\mathcal{A}_1) \cap \text{conv}(\mathcal{A}_2) \neq \emptyset$.

The partition in Radon’s theorem is called a Radon partition of $\mathcal{A}$. A Radon partition in $\mathcal{A}$ is a Radon partition of a subset of $\mathcal{A}$. Let $\{\mathcal{A}_1, \mathcal{A}_2\}$ and $\{\mathcal{A}_1', \mathcal{A}_2'\}$ be Radon partitions of $\mathcal{A}$. Then $\{\mathcal{A}_1, \mathcal{A}_2\} \text{ extends } \{\mathcal{A}_1', \mathcal{A}_2'\}$ provided $\mathcal{A}_1' \subseteq \mathcal{A}_1$. In [8], Hare and Kennely introduced the notion of a primitive Radon partition, a Radon partition that is minimal with respect to extension. An immediate consequence is that if $\{\mathcal{A}_1, \mathcal{A}_2\}$ is a Radon partition of $\mathcal{A}$, then there exists a primitive Radon partition in $\mathcal{A}$ such that $\{\mathcal{A}_1, \mathcal{A}_2\}$ extends it. Breen proved in [3] that $\{\mathcal{A}_1, \mathcal{A}_2\}$ is a primitive Radon partition in $\mathcal{A}$ if and only if $\mathcal{A}_1 \cup \mathcal{A}_2$ is in general position in $\mathbb{R}^{|\mathcal{A}_1| + |\mathcal{A}_2|}$. Recall that $\mathcal{A}_1 \cup \mathcal{A}_2$ is in general position if no $j + 2$ points in this union lie in a $j$-flat for all $j < |\mathcal{A}_1| + |\mathcal{A}_2|$. Peterson proved in [17] that the Radon partition of $d + 2$ points in general position in $\mathbb{R}^d$ is unique, and both Breen and Peterson showed that if $\mathcal{A}_1 \cup \mathcal{A}_2$ is in general position in $\mathbb{R}^{|\mathcal{A}_1| + |\mathcal{A}_2|}$, then $\text{conv}(\mathcal{A}_1) \cap \text{conv}(\mathcal{A}_2)$
is a single point. Lastly, Proskuryakov proved in [18] that if \( \mathcal{P} \subset \mathbb{R}^d \) is a set of \( d + 2 \) points in general position, then two points will lie in the same component of the (unique) Radon partition of \( P \) if and only if they are separated by the hyperplane through the remaining \( d \) points. Kosmak also proved this result in [13] using affine varieties. These properties of Radon partitions are equivalent to Lawson’s First and Second Theorems from [15].

**Lawson’s First Theorem.** [15] Theorem 1] Let \( \mathcal{P} = \{v_1, \ldots, v_{d+2}\} \subset \mathbb{R}^d \) and suppose \( \mathcal{P} \) does not lie entirely in any hyperplane. There is a partition of \( \mathcal{P} \) into three sets \( \mathcal{A}_0, \mathcal{A}_1, \) and \( \mathcal{A}_2 \), and \( \alpha_i \in \mathbb{R} \), satisfying

\[
\sum_{v_i \in \mathcal{A}_1} \alpha_i v_i = \sum_{v_i \in \mathcal{A}_2} \alpha_i v_i,
\]

\[
\sum_{v_i \in \mathcal{A}_1} \alpha_i = \sum_{v_i \in \mathcal{A}_2} \alpha_i = 1,
\]

\[
\alpha_i > 0 \quad \text{if} \quad v_i \in \mathcal{A}_1 \cup \mathcal{A}_2.
\]

The numbers \( \alpha_i \) are uniquely determined by the set \( \mathcal{P} \). We set \( \alpha_i = 0 \) if \( v_i \in \mathcal{A}_0 \). The sets \( \mathcal{A}_0 \) and \( \{\mathcal{A}_1, \mathcal{A}_2\} \) are also unique.

Since the sets \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) in Lawson’s First Theorem form the unique primitive Radon partition in \( \mathcal{P} \), the point

\[
\text{conv}(\mathcal{A}_1) \cap \text{conv}(\mathcal{A}_2) = \sum_{v_i \in \mathcal{A}_1} \alpha_i v_i = \sum_{v_i \in \mathcal{A}_2} \alpha_i v_i,
\]

is in the relative interior of \( \text{conv}(\mathcal{A}_1) \) and \( \text{conv}(\mathcal{A}_2) \) by (7), (8), and (9). These sets also determine the possible triangulations of \( \text{conv}(\mathcal{P}) \).

**Lawson’s Second Theorem.** [15] Theorem 2] Let \( \mathcal{P} = \{v_1, \ldots, v_{d+2}\} \subset \mathbb{R}^d \), and let \( T_i \) be the simplex with vertex set \( \mathcal{V}(T_i) = \mathcal{P}\setminus\{v_i\} \). There are at most two distinct triangulations of \( P = \text{conv}(\mathcal{P}) \), namely

\[
\mathcal{T}_1 = \{T_i : v_i \in \mathcal{A}_1\} \quad \text{and} \quad \mathcal{T}_2 = \{T_i : v_i \in \mathcal{A}_2\},
\]

where the sets \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are as defined in Lawson’s First Theorem. The set \( \mathcal{T}_j \) is a valid triangulation if and only if \( |\mathcal{A}_j| > 1 \), where \( j \in \{1, 2\} \).

**Corollary 8.** [15] Corollary 1] In the context of building triangulations, an enumeration of all possible configurations of \( d + 2 \) points in \( \mathbb{R}^d \), not lying in any hyperplane, is given by all of the possible ways of assigning values to \( |\mathcal{A}_0|, |\mathcal{A}_1|, \) and \( |\mathcal{A}_2| \) satisfying

\[
|\mathcal{A}_2| \geq |\mathcal{A}_1| \geq 1,
\]

\[
|\mathcal{A}_2| \geq 2,
\]

\[
|\mathcal{A}_0| \geq 0,
\]

\[
|\mathcal{A}_0| + |\mathcal{A}_1| + |\mathcal{A}_2| = d + 1.
\]

Consider the possible sets of five points in \( \mathbb{R}^3 \) such that the convex hull is a non-degenerate polyhedron with two different triangulations. Lawson’s Second Theorem implies \( |\mathcal{A}_1| \geq 2 \). Inequalities (10), (11), (12), and (13) imply either

\( (|\mathcal{A}_0|, |\mathcal{A}_1|, |\mathcal{A}_2|) = (0, 2, 3) \) or \( (|\mathcal{A}_0|, |\mathcal{A}_1|, |\mathcal{A}_2|) = (1, 2, 2) \).
Example 2. Suppose $\mathcal{P} = \{v_1, \ldots, v_5\} \subset \mathbb{R}^3$ is in general position. As shown on the left in Figures 2 and 3 let $P = \text{conv}(\mathcal{P})$. In the context of Lawson’s First Theorem, $\mathcal{A}_0 = \emptyset$, $\mathcal{A}_1 = \{v_1, v_5\}$, and $\mathcal{A}_2 = \{v_2, v_3, v_4\}$. The polyhedron $P$ is a "bipyramid" with triangulations

$$\mathcal{T}_1 = \{ \text{conv}(v_1, v_2, v_3, v_4), \text{conv}(v_2, v_3, v_4, v_5) \}$$

and

$$\mathcal{T}_2 = \{ \text{conv}(v_1, v_2, v_3, v_5), \text{conv}(v_1, v_2, v_4, v_5), \text{conv}(v_1, v_3, v_4, v_5) \}$$

by Lawson’s Second Theorem.

![Figure 2. P (left) and the two simplices of $\mathcal{T}_1$ (right)](image)

![Figure 3. P (left) and the three simplices of $\mathcal{T}_2$ (right)](image)
Example 3. Consider $\mathcal{P} = \{v_1, \ldots, v_5\} \subset \mathbb{R}^3$ and $P = \text{conv}(\mathcal{P})$ shown on the left in Figure 4. The polyhedron $P$ is a nondegenerate pyramid whose base is a planar quadrilateral with vertex set $\{v_2, v_3, v_4, v_5\}$. In the context of Lawson’s First Theorem, $\mathcal{A}_0 = \{v_1\}$, $\mathcal{A}_1 = \{v_2, v_4\}$ and $\mathcal{A}_2 = \{v_3, v_5\}$. Lawson’s Second Theorem implies $P$ has two triangulations

$$\mathcal{T}_1 = \{ \text{conv}(v_1, v_3, v_4, v_5), \text{conv}(v_1, v_2, v_3, v_5) \}$$

and

$$\mathcal{T}_2 = \{ \text{conv}(v_1, v_2, v_4, v_5), \text{conv}(v_1, v_2, v_3, v_4) \}.$$ 

Each triangulation is induced by the triangulation (in dimension 2) of the base.

![Figure 4. $P$ (left) and both of its triangulations (right)](image-url)

In both Examples 2 and 3, $|\mathcal{A}_1| = 2$, and $\text{conv}(\mathcal{A}_1 \cup \mathcal{A}_2)$ can be viewed as two simplices joined at a common facet. Moreover, the line segment formed by the vertices not in the common facet intersects that facet at a point. Such a configuration is called a bipyramid. In general, a $d$-polytope $P$ is a $d$-bipyramid if $P$ is the convex hull of a line segment $L$ and a $(d-1)$-simplex $S$ such that the intersection $L \cap S$ is a single point contained in $\text{int}(L) \cap \text{int}(S)$. A 1-bipyramid is simply a line segment with a point in its interior (or the convex hull of three collinear points). In Example 2, $P$ itself is a 3-bipyramid, whereas in Example 3, $P$ contains a 2-bipyramid (its base). The following result is a direct consequence of Lawson’s theorems.

**Corollary 9.** Let $P$ be a $d$-polytope with $d + 2$ vertices. If there exists a subset $\mathcal{P} \subseteq \mathcal{V}(P)$ such that $\text{conv}(\mathcal{P})$ is a $j$-bipyramid, where $2 \leq j \leq d$, then $\mathcal{V}(P)$ has exactly triangulations, one with cardinality 2 and another with cardinality $j$. 
5. Existence of Bipyramids in $T \in \mathcal{K}_k$

For $k \geq 2$, under what conditions will $T \in \mathcal{K}_k$ contain a bipyramid? Consider the following construction of possible bipyramids from a $d$-simplex. Let $S$ be a $d$-simplex, and let $\mathcal{V}(S) = \{v_1, \ldots, v_{d+1}\}$. For any point $x$ exterior to $S$, let $(\alpha_i)$ be the barycentric coordinates of $x$ relative to $S$. By the definition of a bipyramid, $P = \text{conv}(S, x)$ is a bipyramid if and only if exactly one $\alpha_i$ is negative, and the remainder are positive. If we allow for, say, $j$ of the $\alpha_i$ to be zero, then $P$ will contain a $(d-j)$-bipyramid.

The geometric interpretation is as follows. Let $H_i$ be the hyperplane containing the facet of $S$ opposite $v_i$. Then $P$ is a $d$-bipyramid if $x$ and $S$ are on opposite sides of exactly one $H_i$, and $x$ is on the same side of the remaining hyperplanes as $S$. If $x$ is contained in $j$ of these hyperplanes, then $P$ contains a $(d-j)$-bipyramid.

**Theorem 10.** Suppose $T \in \mathcal{K}_k$, where $k \geq 2$. Let $w_1$ and $w_2$ be any two interior lattice points of $T$, and let $\mathcal{T}_{w_1} = \{T_i\}$ be the basic triangulation of $T$ with respect to $w_1$. If $w_2$ lies in the relative interior of a simplex in $\mathcal{T}_{w_1}$, say $T_n$, then there exist numbers $\alpha_i$ such that

$$w_2 = \alpha_1 w_1 + \alpha_n v_n + \sum_{i=2}^{d+1} \alpha_i v_i,$$

$$\alpha_1 > 0, \alpha_n < 0, \text{ and } \alpha_i \geq 0 \text{ otherwise}.$$

**Proof.** Let $\mathcal{V}(T) = \{v_1, \ldots, v_{d+1}\}$ and $F_i$ be the facet of $T$ opposite the vertex $v_i$. We may assume without loss of generality that $T_i = \text{conv}(F_i, w_1)$ and $n = d + 1$ (i.e. $w_2 \in \text{int}(T_{d+1})$). Let $L$ be the line through $v_{d+1}$ and $w_1$.

**Case 1:** If $w_2$ lies on $L$, then $w_1$ is necessarily between $v_{d+1}$ and $w_2$. There exists $\alpha > 1$ such that

$$w_2 = \alpha_1 w_1 + \alpha_n v_n + \alpha w_1 + (1 - \alpha)v_{d+1}.$$  

Choose $\alpha_1 = \alpha$, $\alpha_{d+1} = 1 - \alpha$, and $\alpha_i = 0$ for $2 \leq i \leq d + 1$.

**Case 2:** Suppose $w_2$ does not lie on $L$. Let $x = L \cap F_{d+1}$. If $x \not\in \text{int}(F_{d+1})$, then the line segment $v_{d+1}x$, which contains $w_1$, would be contained in a facet of $T$ and contradict the cleanliness of $T$. Thus $x \in \text{int}(F_{d+1})$ and consequently $x$ cannot be a lattice point. Since $x \in \text{int}(F_{d+1})$ if and only if there exists $\beta_i$ such that

$$x = \sum_{i=1}^d \beta_i v_i, \quad \beta_i > 0, \text{ and } \sum_{i=1}^d \beta_i = 1.$$  

the set $\mathcal{D} = \mathcal{V}(T_{d+1}) \cup \{x\} \not\subset \mathbb{Z}^d$ can be partitioned into

$$\mathcal{A}_0 = \{w_1\}, \quad \mathcal{A}_1 = \{x\}, \quad \text{and} \quad \mathcal{A}_2 = \mathcal{V}(F_{d+1})$$  

according to Lawson’s First Theorem. By Lawson’s Second Theorem, $\mathcal{D}$ has exactly one (non-lattice) triangulation

$$\mathcal{T} = \{ \text{conv}(\mathcal{V}(T_{d+1}) \setminus \{v_i\} \cup \{x\}) : 1 \leq i \leq d \}. $$  

Since $T_{d+1} = \text{conv}(\mathcal{D})$ and $w_2 \in \text{int}(T_{d+1})$, $w_2$ must be in some (non-lattice) simplex in $\mathcal{T}$. Without loss of generality, suppose

$$w_2 \in \text{conv}(v_2, \ldots, v_d, w_1, x).$$
Then there exist $\gamma_i$ such that $\gamma_i \geq 0$ for $1 \leq i \leq d + 1$,

$$
\sum_{i=1}^{d+1} \gamma_i = 1, \quad \text{and} \quad w_2 = \gamma_1 w_1 + \sum_{i=2}^{d} \gamma_i v_i + \gamma_{d+1} x.
$$

Since $w_2$ cannot lie in any face of $T$, $\gamma_1 > 0$. Similarly, $\gamma_{d+1} > 0$ since $w_2$ is not in any face of $T_{d+1}$. Furthermore, the assumption that $w_2$ does not lie on $L$ implies one of the remaining $\gamma_i$ ($2 \leq i \leq d$) must also be positive. Since $w_1$ lies between $v_{d+1}$ and $x$ on $L$, there exists $\mu > 1$ such that

$$
x = v_{d+1} + \mu(w_1 - v_{d+1}) = \mu w_1 + (1 - \mu)v_{d+1}.
$$

It follows that

$$
w_2 = (\gamma_1 + \mu \gamma_{d+1}) w_1 + \sum_{i=2}^{d} \gamma_i v_i + (1 - \mu)\gamma_{d+1} v_{d+1}.
$$

Let

$$
\alpha_1 = \gamma_1 + \mu \gamma_{d+1}, \quad \alpha_{d+1} = (1 - \mu)\gamma_{d+1}, \quad \text{and} \quad \alpha_i = \gamma_i \quad \text{for} \quad 2 \leq i \leq d.
$$

Note that

$$
\sum_{i=1}^{d+1} \alpha_i = [\mu + (1 - \mu)] \gamma_{d+1} + \sum_{i=1}^{d} \gamma_i = 1,
$$

$\alpha_1 > 0$, and $\alpha_{d+1} < 0$. The remaining $\alpha_i$ are nonnegative, and at least one is positive since at least one of the $\gamma_i$ is positive for $2 \leq i \leq d$.

The geometric interpretation of Theorem 10 is that for some simplex $T_m \neq T_n$ in $\mathcal{T}_w$, $\conv(T_m, w_2)$ contains a $j$-bipyramid $P$, and $\mathcal{V}(P)$ includes $w_1$ and $w_2$. In terms of triangulations, $\mathcal{V}(T) \cup \{w_1, w_2\}$ has two triangulations, provided $j > 1$.

**Corollary 11.** Suppose $T \in \mathcal{F}_k^d$, where $k \geq 2$. If $w_2$ lies in the relative interior of a simplex in the basic triangulation $\mathcal{F}_w$ of $T$, then the set $\mathcal{V}(T) \cup \{w_1, w_2\}$ has two triangulations provided $w_1$ and $w_2$ are not collinear with any $v \in \mathcal{V}(T)$.

**Proof.** Let $\mathcal{F}_1 = \mathcal{F}_w = \{T_i\}$. Without loss of generality, suppose $w_2 \in \text{int}(T_{d+1})$. Let $\mathcal{F}_{w_2}$ be the basic triangulation of $T_{d+1}$ with respect to $w_2$. One triangulation of $\mathcal{V}(T) \cup \{w_1, w_2\}$ is the refinement $\mathcal{F}_2$ of $\mathcal{F}_1$ guaranteed by Theorem 4 (with $w = w_2$), where

$$
\mathcal{F}_2 = \mathcal{F}_w \setminus \{T_{d+1}\} \cup \mathcal{F}_{w_2}.
$$

Let $S_i$ be the facet of $T_{d+1}$ opposite $v_i$ for $1 \leq i \leq d$, and let $S_{d+1}$ be the facet of $T_{d+1}$ opposite $w_1$. Theorem 4 guarantees there exists $1 \leq m \leq d$ such that

$$
\conv(T_m, w_2) = \conv \left( T_m \cup \conv(S_m, w_2) \right).
$$

contains a $j$-bipyramid. Note that both $T_m$ and $\conv(S_m, w_2)$ are simplices in $\mathcal{F}_2$. Since $w_1$ and $w_2$ are not collinear with any $v \in \mathcal{V}(T)$, $T$ is clean, and no three vertices of $T$ are collinear, it follows that $j > 1$. Corollary 3 implies $\mathcal{V}(T_m) \cup \{w_2\}$ has two triangulations, one of which is contained in $\mathcal{F}_2$. That is,

$$
\conv(T_m, w_2) = T_m \cup \conv(S_m, w_2).
$$

Thus $\mathcal{V}(T) \cup \{w_1, w_2\}$ has two triangulations.  


Among all possible $d$-bipyramids that have two triangulations, 2-bipyramids are unique in that their triangulations have the same cardinality. The existence of lattice 2-bipyramids within lattice $d$-simplices has further implications. Note that 2-bipyramids are simply convex planar quadrilaterals.

**Lemma 12.** Suppose $T \in S_k^d$ where $k \geq 2$. Let $w_1$ and $w_2$ be any two of the interior points of $T$. Let $Q = \text{conv}(w_1, w_2, v, v')$ where $v, v' \in V(T)$. If $Q$ is a planar quadrilateral, then the opposing edges of $Q$ cannot be parallel.

**Proof.** Without loss of generality, suppose $w_1v$ and $w_2v'$ are edges of $Q$. Let $H_1$ be the hyperplane containing the facet of $T$ opposite $v'$, and let $H_2$ be the hyperplane parallel to $H_1$ and containing $v'$. Since $w_1$ is an interior point of $T$, it must lie between $H_1$ and $H_2$. If $w_1v \parallel w_2v'$, then $w_2$ must be opposite of $w_1$ relative to both $H_1$ and $H_2$. This is impossible since $w_2$ would lie outside of $T$ as shown in the left figure above. If $w_1w_2 \parallel vv'$, edge $w_1w_2$ of $Q$ must be shorter than edge $vv'$ since $w_1w_2$ is in the interior of $T$. Then $w = v' - (w_2 - w_1)$, or $w = v + (w_2 - w_1)$, is a lattice point on the edge $vv'$, as shown above on the right, which is also impossible as $T$ is clean. 

**Lemma 13.** Suppose $Q = \text{conv}(v_1, v_2, v_3, v_4)$ is a planar lattice quadrilateral. If the opposing edges of $Q$ are not parallel, then interior of $Q$ contains a lattice point $w \neq v_i$ for $1 \leq i \leq 4$. Moreover, $w$ lies in the interior of a triangle whose edge set is a subset of the edges and diagonals of $Q$.

**Proof.** Consider the pairs of adjacent edges of $Q$. Without loss of generality, suppose $v_1v_2$ and $v_2v_3$ are edges of $Q$ such that the triangle $\Delta = \text{conv}(v_1, v_2, v_3)$ has minimal area. The lines containing each edge of $\Delta$ partition the plane into 7 different regions as shown below (left). Since $Q$ is a convex quadrilateral, $v_4$ can only lie in the three unshaded regions. Without loss of generality, suppose $v_4$ lies in region $R$ as shown above. Let $H_1$ be the line through $v_1v_2$ and $H_2$ be the line through $v_3$ such that $H_2 \parallel H_1$. Similarly, let $L_1$ be the line through $v_2v_3$ and $L_2$
be the line through $v_1$ such that $L_2 \parallel L_1$. (See right-side figure above.) These four lines divide $R$ into four regions. Since $Q$ has no parallel edges, $v_4$ cannot lie on any one of these four lines. If $v_4$ lies in region I, then $\text{conv}(v_2, v_3, v_4)$ is a triangle of smaller area than $\Delta$ (they both share the same leg $v_2v_3$, whereas the distance between $v_4$ and $L_1$ is shorter than the distance between $v_1$ and $L_1$). Similarly, $v_4$ cannot lie in regions III or IV. Thus $v_4$ must lie in region II. Then $w = v_1 + (v_3 - v_2)$ is a lattice point contained in interior of $\text{conv}(v_1, v_3, v_4)$. 

6. Collinearity Property of Minimal Volume $T \in \mathcal{I}_k^d$

We can now prove the collinearity property of $\text{int}(T) \cap \mathbb{Z}^d$, where $T \in \mathcal{I}_k^d$, $d \geq 3$, and $\text{Vol}(T) = \frac{1}{d!}(dk + 1)$.

**Theorem 14.** Suppose $T \in \mathcal{I}_k^d$, where $d \geq 3$ and $\text{Vol}(T) = \frac{1}{d!}(dk + 1)$. Any two interior lattice points of $T$ are collinear with a vertex of $T$.

**Proof.** The claim is vacuously true for $k \leq 1$, so suppose $k \geq 2$. If $k = 2$ then let $w_1$ and $w_2$ be the two interior lattice points of $T$. Otherwise let $w_1$ and $w_2$ be any two interior lattice points of $T$. Suppose $w_1$ and $w_2$ are not collinear with any $v \in \mathcal{V}(T)$. That is, suppose there does not exist a lattice 1-bipyramid consisting of $w_1$, $w_2$, and any $v \in \mathcal{V}(T)$. Let $\mathcal{F}_1 = \mathcal{F}_{w_1} = \{T_1\}$, where $\mathcal{F}_{w_1}$ is the basic triangulation of $T$ with respect to $w$. By assumption, $w_2$ must lie in the relative interior of either a simplex in $\mathcal{F}_1$, or a $j$-face of a simplex in $\mathcal{F}_1$, where $1 < j < d$.

**Case 1:** If $w_2$ lies in the relative interior of a $j$-face of a simplex in $\mathcal{F}_1$, where $1 < j < d$, then Corollary 6 implies $\text{Vol}(T) > \frac{1}{d!}(dk + 1)$, a contradiction.

**Case 2:** If $w_2$ lies in the relative interior of a simplex in $\mathcal{F}_1$, then we may assume without loss of generality that $w \in \text{int}(T_{d+1})$. Let $\mathcal{F}_{w_2}$ be the basic triangulation of $T_{d+1}$ with respect to $w_2$. The refinement $\mathcal{F}_2$ of $\mathcal{F}_1$ obtained by applying Theorem 4 with $w = w_2$ is $\mathcal{F}_2 = \mathcal{F}_{w_1} \setminus \{T_{d+1}\} \cup \mathcal{F}_{w_2}$.

Theorem 10 implies $T$ contains a lattice $j$-bipyramid $P$, where $2 \leq j \leq d$ and $\{w_1, w_2\} \subset \mathcal{V}(P)$. If $j = 2$, then $P = \text{conv}(v, v', w_1, w_2)$ is convex planar lattice quadrilateral, where $\{v, v'\} \subset \mathcal{V}(T)$. Lemmas 12 and 13 imply $P$ contains a lattice point $w_3 \notin \{w_1, w_2\}$. This is a contradiction if $k = 2$. If $k > 2$, then the second part of Lemma 13 implies $w_3$ lies in the relative interior of some triangle $\Delta_i$, where

$$\Delta_1 = \text{conv}(w_1, w_2, v), \quad \Delta_2 = \text{conv}(w_1, w_2, v'), \quad \Delta_3 = \text{conv}(w_1, v, v'), \quad \Delta_4 = \text{conv}(w_2, v, v').$$

Since $d \geq 3$, and each $\Delta_i$ is a 2-face of simplices in $\mathcal{F}_2$, $\text{Vol}(T) > \frac{1}{d!}(dk + 1)$ by Corollary 6. This contradicts the mimal volume property of $T$. On the other hand, if $j > 2$, then $\mathcal{V}(T) \cup \{w_1, w_2\}$ has two possible triangulations $\mathcal{F}_2$ and $\mathcal{F}_2'$, where

$$|\mathcal{F}_2'| = |\mathcal{F}_1| + d - 2 + j > 2d + 1.$$ 

Subsequent applications of Theorem 4 starting with $\mathcal{F}_2'$, and 2 imply

$$\text{Vol}(T) > \frac{1}{d!}(dk + 1).$$

This is again a contradiction. Thus any two points in $\text{int}(T) \cap \mathbb{Z}^d$ must be collinear with some $v \in \mathcal{V}(T)$. 

$\blacksquare$
Corollary 15. Suppose $T \in \mathcal{S}_k^d$, where $d \geq 3$ and $\text{Vol}(T) = \frac{1}{dk}(dk+1)$. The points $\text{int}(T) \cap \mathbb{Z}^d$ are collinear with some $v \in \mathcal{V}(T)$.

Proof. The claim is vacuously true for $k \leq 1$. Theorem 14 implies the claim also holds for $k = 2$. Suppose $k \geq 3$. It suffices to show the claim holds for any three points in $\text{int}(T) \cap \mathbb{Z}^d$. Let $w_1, w_2$, and $w_3$ be any three points in $\text{int}(T) \cap \mathbb{Z}^d$. Theorem 14 implies there exists $v \in \mathcal{V}(T)$ such that $v_1, w_2$, and $v$ are collinear. Similarly, there exists $v' \in \mathcal{V}(T)$ such that $w_1, v_3$, and $v'$ are collinear. We need to show $v = v'$. Suppose $v \neq v'$. Let $L$ be the line through $v_1, w_2$, and $v$, and let $L'$ be the line through $w_1, v_3$, and $v'$. Without loss of generality, suppose $w_2$ is between $w_1$ and $v$ on $L$.

Case 1: If $w_3$ lies between $w_1$ and $v'$ on $L'$, then $P = \text{conv}(v, v', w_2, w_3)$ is a convex planar quadrilateral contained within $\text{conv}(w_1, v, v')$. Lemmas 12 and 13 imply $\text{int}(P)$ contains a lattice point $w_4 \notin \{w_1, w_2, w_3\}$. Since $P \subset \text{conv}(w_1, v, v')$ and $d \geq 3$, Corollary 6 implies $\text{Vol}(T) > \frac{1}{dk}(dk+1)$.

Case 2: If $w_1$ lies between $w_3$ and $v'$ on $L'$, then $w_2 \in \text{int}(\text{conv}(w_3, v, v'))$. Again, since $d \geq 3$, Corollary 6 implies $\text{Vol}(T) > \frac{1}{dk}(dk+1)$. Thus $v = v'$, and all points in $\text{int}(T)$ and $v$ are collinear.

Corollary 16. Suppose $T \in \mathcal{S}_k^d$ and $d \geq 3$. Let $L$ be the line through $\text{int}(T) \cap \mathbb{Z}^d$. The consecutive points on $L$ are evenly spaced.

Proof. Let $v = L \cap \mathcal{V}(T)$ and let $w \in \text{int}(T)$ be the lattice point closest to $v$. Since any line can be described as a linear combination of two points, any $w' \in L$ can be expressed as

$$w' = v + \alpha(w - v),$$

where $\alpha \in \mathbb{R}$. It suffices to show that for any $w' \in L \cap T \cap \mathbb{Z}^d$, the $\alpha$ in (13) is an integer and $0 \leq \alpha \leq k$. If $\alpha < 0$, then $v \in \text{int}(\text{conv}(w, w')) \subset \text{int}(T)$, which is impossible. If $\alpha = 0$ then $w' = v$. Let $[\alpha]$ and $\{\alpha\}$ denote the integer and fractional parts of $\alpha$, respectively. If $\{\alpha\} \neq 0$, then

$$x = w' - [\alpha](w - v) = v + \{\alpha\}(w - v) \in \mathbb{Z}^d$$

lies between $v$ and $w$ on $L$, which contradicts our assumption that $w$ is closest to $v$ on $L$. Hence $\alpha \in \mathbb{Z}$. Finally, if $\alpha > k$, then $T$ has more than $k$ interior points, which is also impossible.

Corollary 17. If $T \in \mathcal{S}_k^d$, $\text{Vol}(T) = \frac{1}{dk}(dk+1)$, and $d \geq 3$, then $T \cap \mathbb{Z}^d$ has a unique triangulation.

Proof. Let $\mathcal{V}(T) = \{v_1, \ldots, v_{d+1}\}$. Without loss of generality, suppose $v_{d+1}$ is collinear with $\text{int}(T) \cap \mathbb{Z}^d$. Let $w_1$ be the point in $\text{int}(T) \cap \mathbb{Z}^d$ closest to $v_{d+1}$ and

$$\mathcal{W} = \{w_i : w_i = v_{d+1} + i(v_1 - v_{d+1}), \ 0 \leq i \leq k \}. $$

Let $S_{k+1, d} = \text{conv}(v_1, \ldots, v_d, w_k)$. For $1 \leq i \leq k$ and $1 \leq j \leq d$, let

$$S_{i,j} = \text{conv}(\mathcal{V}(T) \backslash \{v_j, v_{d+1}\} \cup \{w_i, w_{i-1}\}). $$

It is easy to check that $\mathcal{T} = \{S_{i,j}\}$ is a triangulation of $T \cap \mathbb{Z}^d$. In fact, this triangulation corresponds to the refinement sequence in Corollary 5.

Let $\mathcal{T}'$ be another full triangulation of $T$. Consider any $S \in \mathcal{T}'$. Since $\mathcal{T}'$ is a full triangulation, $S$ is necessarily empty and clean. These two conditions force

$$1 \leq |\mathcal{V}(S) \cap \mathcal{W}| \leq 2.$$
Case 1: If $|\mathcal{V}(S) \cap \mathcal{W}| = 1$, then there exist $v \in \mathcal{V}(T)$ and $w \in \mathcal{W}$ such that $S = \text{conv}(\mathcal{V}(T) \setminus \{v\} \cup \{w\})$.

Since $S$ is empty, it follows that $w = w_k$, $v = v_{d+1}$, and $S = S_{k+1,d}$.

Case 2: If $|\mathcal{V}(S) \cap \mathcal{W}| = 2$, then there exist $\{v, v'\} \subset \mathcal{V}(T)$ and $\{w, w'\} \subset \mathcal{W}$ such that $S = \text{conv}(\mathcal{V}(T) \setminus \{v, v'\} \cup \{w, w'\})$.

Without loss of generality, suppose $w$ is closer to $w_0$ than $w'$ is to $w_0$. Since $S$ is clean,

\begin{equation}
(w, w') = (w_{i-1}, w_i)
\end{equation}

for some $1 \leq i \leq k$. Moreover, $w_0 = v_{d+1}$ is a vertex of $S$ if and only if $i = 1$ in (15) since $S$ must also be empty. Thus $S = S_{i,j}$ for some appropriate $j$, and $\mathcal{T}' \subset \mathcal{T}$. Consequently, $\mathcal{T}' = \mathcal{T}$.

\section{Unimodular Transformations}

A unimodular transformation $f : \mathbb{Z}^d \to \mathbb{Z}^d$ is an affine map of the form

\[ f(v) = v \cdot M + u, \]

where $u, v \in \mathbb{Z}^d$, $M$ is a $d \times d$ matrix with entries in $\mathbb{Z}$, and $\det(M) = \pm 1$. A translation is a unimodular transformation in which $M = I_d$, where $I_d$ is the $d \times d$ identity matrix. If $\det(M) = \pm 1$, $M$ is invertible, then $M^{-1}$ is unimodular, and $f$ is one-to-one. Thus $P$ is a lattice $d$-polytope if and only if $f(P)$ is a lattice $d$-polytope with the same number of vertices. Using barycentric coordinates, it is easy to check that $f(w)$ is an interior point of $f(P)$ if and only if $w$ is an interior point of $P$. Similarly, $f(w)$ is a boundary point of $f(P)$ if and only if $w$ is a boundary point of $P$ (cf. [23]). We say $P_1$ and $P_2$ are equivalent lattice $d$-polytopes and write $P_1 \simeq P_2$ if there exists a unimodular transformation $f$ such that $P_1 = f(P_2)$.

\textbf{Lemma 18.} Let $T$ be a $d$-simplex with $\text{Vol}(T) = \frac{1}{d!}$. Then $T$ is equivalent to $T_{1,\ldots,1}$, the $d$-simplex whose vertex set consists of the origin, $e_i$ for $1 \leq i \leq d-1$, and the point $(1, \ldots, 1)$.

\textbf{Proof.} Let $v_1, \ldots, v_{d+1}$ be an enumeration of the vertices of $T$. If necessary, translate $T$ so that one of its vertices is the origin. Without loss of generality, suppose $v_{d+1} = 0$. Let

\[ P = \text{conv}(v_1, \ldots, v_{d+1}, \sum_{i=1}^{d} v_i). \]

Note that $P$ is a parallelepiped containing $T$, and $\text{Vol}(P) = \pm \det(A)$, where $A$ is a $d \times d$ matrix whose entry $a_{i,j}$ is the $j$-th coordinate of $v_i$. Since

\[ \frac{1}{d!} = \text{Vol}(T) = \frac{1}{d!} \text{Vol}(P) = \frac{1}{d!} \cdot |\det(A)|, \]

$\det(A) = \pm 1$ and $A$ is invertible. Thus, there exists a unique, unimodular, $d \times d$ matrix $M$ such that $AM = I_d$. Thus $f : \mathbb{Z}^d \to \mathbb{Z}^d$ defined by $f(v) = v \cdot M$ is a unimodular transformation such that and $f(v_i) = e_i$ for $1 \leq i \leq d$ and $f(v_{d+1}) = 0$. Let $g : \mathbb{Z}^d \to \mathbb{Z}^d$ be the unimodular transformation given by $g(v) = v \cdot M'$ where

\[ M' = \begin{bmatrix} I_{d-1} & 0 \\ 1 & 1 \end{bmatrix}. \]
It is easy to check that $g(e_i) = e_i$ for $1 \leq i \leq d - 1$, $g(e_d) = (1, \ldots , 1)$, and $g(0) = 0$. Then the composition $g \circ f$, where $(g \circ f)(v) = v \cdot \mathbf{M} \cdot \mathbf{M}'$, is also a unimodular transformation, and $(g \circ f)(T) = T_{1, \ldots , 1}$.

Let $T_{a_1, \ldots , a_d}$ denote the $d$-simplex whose vertex set consists of the origin, the unit points $e_1, \ldots , e_{d-1}$, and the point $(a_1, \ldots , a_d) \in \mathbb{Z}^d$. What can we say about $T_{a_1, \ldots , a_d}$? If $a_d < 0$, then we can apply $f : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$, where

$$f(v) = v \cdot \begin{bmatrix} \mathbf{I}_{d-1} & 0 \\ 0 & -1 \end{bmatrix}.$$  

Note that $f$ is simply a reflection in the $d$-th coordinate. By reflection, if necessary, we can take $a_d$ to be nonnegative. Moreover, $a_d$ is determined by the volume of $T_{a_1, \ldots , a_d}$ since

$$0 < d! \cdot \text{Vol}(T_{a_1, \ldots , a_d}) = \pm \det \begin{bmatrix} a_1 & \cdots & a_d \\ \mathbf{I}_{d-1} & \cdots & 0 \end{bmatrix} = |a_d|.$$  

For the remaining $a_i$, consider the integers $b_i$ such that such that

$$0 \leq a_i + b_i a_d < a_d,$$

and the transformation $g : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$, where

$$g(v) = v \cdot \begin{bmatrix} \mathbf{I}_{d-1} & 0 \\ b_1 & \cdots & b_{d-1} \\ 0 & \cdots & 1 \end{bmatrix}.$$  

All the vertices of $T_{a_1, \ldots , a_d}$ excluding $(a_1, \ldots , a_d)$ are fixed under $g$. On the other hand, $g$ sends $(a_1, \ldots , a_d)$ to $(a_1', \ldots , a_d')$ where $a_d' = a_d$ and $a_i' = a_i + b_i a_d$ for $1 \leq i \leq d - 1$. Since $T_{a_1, \ldots , a_d}$ is not contained in any hyperplane, $a_i > 0$ and $a_i' > 0$. Thus the class of all $T_{a_1, \ldots , a_d} \in \mathcal{S}_k^d$ is represented by $T_{a_1', \ldots , a_d'}$ where

$$0 < a_i' < a_d' = a_d.$$  

Lastly, let $w \in \text{int}(T_{a_1, \ldots , a_d})$ and let $(\lambda_i)$ be the barycentric coordinates of $w$ relative to $T_{a_1, \ldots , a_d}$, where

$$w = \lambda_{d+1} \cdot 0 + \lambda_d \cdot (a_1, \ldots , a_d) + \sum_{i=1}^{d-1} \lambda_i e_i.$$  

The $d$-th coordinate of $w$ is $\lambda_d \cdot a_d$, and for $1 \leq i \leq d - 1$, the $i$-th coordinate is

$$\lambda_i + \lambda_d \cdot a_i = [\lambda_d \cdot a_i].$$  

We can conclude that every point in $\text{int}(T_{a_1, \ldots , a_d}) \cap \mathbb{Z}^d$ is completely determined by its $d$-th coordinate. Equivalently, no two interior lattice points can have the same $d$-th coordinate.

**Theorem 19.** Let $d \geq 3$ and $k \geq 1$. If $T \in \mathcal{S}_k^d$, and $\text{Vol}(T) = \frac{1}{d!} (dk + 1)$, then $T \simeq T_{a_1, \ldots , a_d}$, where $(a_1, \ldots , a_{d-1}, a_d) = (dk, \ldots , dk, dk + 1)$.

**Proof.** Let $\mathcal{Y}(T)$, $v_{d+1}$, $\mathcal{W}$, $S_{i,j}$, and $\mathcal{S}$ be as in the proof of Corollary [17]. For $S_{1,d} \in \mathcal{S}$, Lemma [18] implies there exists a unimodular transformation $f$ such that $f(v_{d+1}) = 0$, $f(v_1) = (1, \ldots , 1)$, and $f(v_j) = e_j$ for $1 \leq j \leq d - 1$. Let $f(v_d) = (a_1, \ldots , a_d)$. 


We may assume $0 < a_j \leq a_d$ by (19). Since $\text{Vol}(T) = \frac{1}{d}(dk + 1)$ and unimodular transformations preserve volume, (17) implies $a_d = dk + 1$. Unimodular transformations also preserve interior lattice points, so $f(w_1) \in \text{int}(f(T)) \cap \mathbb{Z}^d$, and $f(w_1)$ is collinear with $0$. Corollary (10) implies $f(w_i) = (i, \cdots, i)$. If $(\lambda_{i,j})$ are the barycentric coordinates of $f(w_i)$, where

$$f(w_i) = \lambda_{i,d+1} \cdot 0 + \lambda_{i,d} \cdot (a_1, \cdots, a_d) + \sum_{j=1}^{d-1} \lambda_{i,j} \cdot e_j,$$

then (20) implies

$$\lambda_{i,j} + \lambda_{i,d} \cdot a_j = i.$$

Note that $0 < \lambda_{i,j} < 1$, and

$$f(w_i) \in \text{int}(f(T)) \cap \mathbb{Z}^d \iff w_i \in \text{int}(T) \cap \mathbb{Z}^d \iff 1 \leq i \leq k.$$

Since

$$\lambda_{i,d} \cdot a_j = (i - 1) + 1 - \lambda_{i,j} > (i - 1)\lambda_{i,d} + 1 - \sum_{j=1}^{d+1} \lambda_{i,j} > i \cdot \lambda_{i,d},$$

$a_j > i$ for all $i$, which implies $a_j > k \geq 1$. The final step is to show that $a_j = dk$

Consider the interior point $f(w_k) = (k, k, \cdots, k)$ and let $(\lambda_j)$ be the barycentric coordinates of $f(w_k)$. Clearly $\lambda_d = \frac{k}{dk+1}$. By (20), we have

$$\lambda_j = k - a_j \cdot \lambda_d = k - \frac{a_j \cdot k}{dk+1}.$$

Since $\sum_{j=1}^{d+1} \lambda_j = 1$, it follows that

$$0 < \sum_{j=1}^{d} \lambda_j = (d-1)k - \frac{k}{dk+1} (a_1 + \cdots + a_d-1) - 1 < 1.$$

Using elementary algebraic manipulation, we obtain

$$1 \leq (kd+1)(d-1) - (a_1 + \cdots + a_d-1 - 1) \leq d,$$

which implies

(21) \hspace{1cm} a_1 + \cdots + a_d-1 \geq kd(d-1).

Finally, $a_j < a_d = dk + 1$ and (21) imply $a_j = dk$ for $1 \leq j \leq d$. 

**Corollary 20.** If $T \in \mathcal{S}_k^d$, then $\text{Vol}(T) = \frac{1}{d}(dk + 1)$ if and only if $T \simeq S_d(k)$.

**Proof.** We first show that $\text{Vol}(S_d(k)) = \frac{1}{d!} \cdot |\text{det}(A)|$, 

\begin{align*}
\text{Vol}(S_d(k)) &= \frac{1}{d!} \cdot |\text{det}(A)|. 
\end{align*}
where
\[
A = \begin{bmatrix}
e_2 - e_1 \\
e_3 - e_1 \\
\vdots \\
e_d - e_1 \\
(-k, -k, \ldots, -k) - e_1
\end{bmatrix} = \begin{bmatrix}
-1 \\
-1 \\
\vdots \\
-k \\
-k \cdots -k
\end{bmatrix} \ 	ext{and} \ \mathbf{I}_{d-1}.
\]

Let \( A' \) be the matrix obtained by adding \( k \) times the sum of the first \( d - 1 \) rows of \( A \) to row \( d \) of \( A \). That is,
\[
A' = \begin{bmatrix}
-1 & \mathbf{1}_{d-1} \\
-1 & \mathbf{1}_{d-1} \\
\vdots & \vdots \\
-k & \mathbf{1}_{d-1}
\end{bmatrix}.
\]

It is easy to check that \( |\det(A)| = |\det(A')| = dk + 1 \). If \( T \simeq S_d(k) \), then \( \text{Vol}(T) = \text{Vol}(S_d(k)) = \frac{1}{d!}(dk + 1) \). On the other hand, if \( \text{Vol}(T) = \frac{1}{d!}(dk + 1) \), then \( T \simeq S_d(k) \) by Theorem 19 provided \( S_d(k) \) is clean. Since \( \text{Vol}(S_d(k)) = \frac{1}{d!}(dk + 1) \) and any lattice triangulation of \( \mathcal{V}(S_d(k)) \cup \mathcal{I}(S_d(k)) \cap \mathbb{Z}^d \) contains at least \( dk + 1 \) simplices, \( \mathcal{S} \) and Corollary \( \mathcal{S} \) imply \( S_d(k) \) cannot have any lattice points on its boundary except for its vertices.

Let \( f : \mathbb{Z}^d \to \mathbb{Z}^d \) be the translation
\[
f(v) = v \cdot \mathbf{1}_d + k \cdot \sum_{i=1}^{d} e_i = v + (k, k, \ldots, k),
\]
and let \( g : \mathbb{Z}^d \to \mathbb{Z}^d \) be the transformation \( g(v) = v \cdot \mathbf{M} \) where
\[
\mathbf{M} = \begin{bmatrix}
1 - k & -k & -k & \cdots & -k & -k \\
-k & 1 - k & -k & \cdots & -k & -k \\
-k & -k & 1 - k & \cdots & -k & -k \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-k & -k & -k & \cdots & 1 - k & -k \\
(d - 1)k & (d - 1)k & (d - 1)k & \cdots & (d - 1)k & (d - 1)k + 1
\end{bmatrix}.
\]

That is, if \( m_{i,j} \) is the entry in the row \( i \) and column \( j \) of \( \mathbf{M} \), then
\[
m_{i,j} = \begin{cases}
1 - k, & i = j \neq d, \\
(d - 1)k, & j = d, i \neq d \\
(d - 1)k + 1, & i = j = d \\
-k, & \text{otherwise}.
\end{cases}
\]

Consider the image of \( \mathcal{V}(S_d(k)) \) under the composition \( g \circ f \). For \( v \in \mathcal{V}(S_d(k)) \), the \( j \)-th coordinate of \( v \) under \( g \circ f \) is simply the dot product of \( v + (k, \ldots, k) \) with the \( j \)-th column of \( \mathbf{M} \). It is easy to check that
\[
(g \circ f)(v) = \begin{cases}
0, & v = -k \sum_{i=1}^{d} e_i, \\
e_i, & v = e_i, 1 \leq i \leq d - 1, \\
(a_1, \ldots, a_d), & v = e_d.
\end{cases}
\]
The composition \( g \circ f \) is in fact a unimodular transformation which maps \( S_d(k) \) to \( T_{a_1, \ldots, a_d} \), where the \( a_i \) are as in Theorem \( 19 \).

8. **Counterexamples in \( \mathbb{R}^2 \)**

The collinearity property of the interior lattice points does not hold for all clean triangles in \( \mathbb{R}^2 \). Pick’s theorem states that if \( P \) is a lattice polygon, then

\[
\text{Vol}(P) = k + \frac{b}{2} - 1
\]

where \( k = |\text{int}(P) \cap \mathbb{Z}^2| \) and \( b = |\partial P \cap \mathbb{Z}^2| \). For clean triangles, (22) reduces to

\[
\text{Vol}(P) = k + \frac{1}{2}.
\]

Thus all clean lattice triangles satisfy the minimal volume condition. In [22], Reznick proved that any \( T \in S^2_k \) is equivalent to some \( T_{a, 2k+1} \), where \( 0 < a < 2k+1 \). Using this representation for clean lattice triangles, Reznick then showed that the number of equivalence classes of clean \( k \)-point lattice triangles increases with \( k \). Thus there exist clean lattice triangles whose interior points are not collinear.

Consider the triangle \( \Delta_{p,q} \) in \( \mathbb{R}^2 \) with vertex set \( \{ (-1, 0), (0, q), (p, -1) \} \), where \( 1 \leq p \leq q \) and \( \gcd(p, q + 1) = 1 \). It is easy to check that \( \Delta_{p,q} \) is clean. We first compute the number of interior points of \( \Delta_{p,q} \). Using determinants,

\[
\text{Vol}(\Delta_{p,q}) = \frac{q^2(p + 1)}{2}.
\]

Pick’s theorem implies

\[
|\text{int}(\Delta_{p,q}) \cap \mathbb{Z}^2| = \frac{q^2(p + 1)}{2}.
\]

For \( p = 1 \), the interior points of \( \Delta_{p,q} \) are all on the line \( x = 0 \). However, for \( p > 1 \), this is not the case as they are covered by the lines \( x = i \) \( (0 \leq i \leq p - 1) \). Since these lines are parallel, they cannot be contained within any one line. Incidentally, this collection of counterexamples is related to the following summation identity.

**Proposition 21.** If \( (p, q) \in \mathbb{Z}_+^2 \) and \( \gcd(q + 1, p) = 1 \), then

\[
\sum_{i=0}^{p-1} \left\lfloor q - \frac{i(q + 1)}{p} \right\rfloor = \frac{q^2(p + 1)}{2}.
\]

**Proof.** This is a simple counting argument using (23), the fact that the lines \( x = i \) \( (0 \leq i \leq p - 1) \) cover \( \text{int}(\Delta_{p,q}) \cap \mathbb{Z}^2 \), and the fact that there are

\[
\left\lfloor q - \frac{i(q + 1)}{p} \right\rfloor
\]

lattice points on the line \( x = i \) and \( \text{int}(\Delta_{p,q}) \). \[\square\]

The counterexample above can be generalized to a triangle with vertex set

\[
\{ (-r, 0), (0, q), (p, -1) \},
\]

where \( p, q, \) and \( r \) are positive integers and \( \gcd(r, q) = \gcd(p, q + 1) = 1 \). The corresponding identity below is similar to (24).
Proposition 22. If \((p, q, r) \in \mathbb{Z}_+^3\) and \(\gcd(q, r) = \gcd(q + 1, p) = 1\), then

\[
\sum_{i=1}^{p-1} \left\lfloor \frac{i q}{r} \right\rfloor + \sum_{i=0}^{p-1} \left( q - \frac{i(q+1)}{p} \right) = \frac{q(p+r)+(r-1)}{2}.
\]

Pick’s theorem itself is a counterexample since only in \(\mathbb{R}^2\) is the volume of a lattice polytope \(P\) an equality in terms of the number of lattice points on \(\partial P\) and in \(\text{int}(P)\). Reeve’s tetrahedra \([20]\) have vertices

\[
\{(0,0,0), (1,0,0), (0,1,0), (1,1,n)\},
\]

where \(n \in \mathbb{Z}_+\). These tetrahedra are clearly clean and empty, yet their volume increases with \(n\). Reeve concluded that the volume of lattice of a polyhedron \(P\) cannot be expressed solely in terms of \(b = |\partial P \cap \mathbb{Z}^3|\) and \(k = |\text{int}(P) \cap \mathbb{Z}^3|\). However, Reeve was able to find and analogue to Pick’s theorem by considering sublattices.

Let \(L_n^d\) denote the lattice consisting of all points in \(\mathbb{R}^d\) whose coordinates are multiples of \(\frac{1}{n}\). Note that \(L_n^d = \mathbb{Z}^d\). For a lattice \(d\)-polytope \(P\), let

\[
b_n = b_n(P) = |\partial P \cap L_n^d| \quad \text{and} \quad k_n = k_n(P) = |\text{int}(P) \cap L_n^d|.
\]

Reeve showed in \([21]\) that

\[
2n(n^2-1) \cdot \text{Vol}(P) = b_n - nb_1 + 2(k_n - nk_1) + (n-1)[2\chi(P) - \chi(\partial P)],
\]

where \(n \geq 2\) and \(\chi\) is the Euler characteristic. Reeve also conjectured a similar formula for \(d = 4\). Soon after, Macdonald \([16]\) not only confirmed Reeve’s conjecture, but generalized Reeve’s formulas to

\[
(d-1)d! \cdot \text{Vol}(P) = \sum_{i=1}^{d-1} (-1)^{i-1} \binom{d-1}{i-1} (b_{d-i} + 2k_{d-i})
\]

\[
+ (-1)^{d-1}[2\chi(P) - \chi(\partial P)].
\]

In 1996, Kolodzieczyk \([11]\) showed that \((26)\) has the following alternate form in which only the numbers \(k_n\) of interior lattice points in the sublattice \(L_n\) are required.

\[
d! \cdot \text{Vol}(P) = \sum_{i=0}^{d-1} (-1)^i \binom{d}{i} k_{d-i} + (d)\chi(P) - \chi(\partial P).
\]

For a \(d\)-polytope \(P\) all of whose vertices join \(d\) edges, \(\chi(P) - \chi(\partial P) = (-1)^d\) and \([21]\) assumes the form

\[
d! \cdot \text{Vol}(P) = \sum_{i=0}^{d-1} (-1)^i \binom{d}{i} k_{d-i} + 1.
\]

Recently, Kolodzieczyk showed in \([12]\) that Pick-type formulas exist even when considering only the points in \(L_n \cap P\) whose coordinates are odd multiples of \(\frac{1}{n}\).

Though no formula for the volume of a lattice polytope in terms of the number of its boundary and interior lattice points exist, there does exist a Pick-type inequality with sharp bounds on the volume of a polyhedron. Any polyhedron in \(\mathbb{R}^3\) that does not intersect itself has an associated planar graph. We can use elementary graph theory and Euler’s formula to obtain a lower bound on the volume of lattice polyhedra.
Proposition 23. Let \( P \subset \mathbb{R}^3 \) be a convex lattice polyhedron with \( b \) lattice points on the boundary and \( k \geq 1 \) interior lattice points. Then

\[
\text{Vol}(P) \geq \frac{2b + 3k - 7}{6}.
\]

Proof. By (2), it suffices to show \( P \) has a lattice triangulation \( \mathcal{T} \) satisfying

\[
|\mathcal{T}| \geq 2b + 3k - 7.
\]

Any refinement sequence starting with any basic triangulation of \( P \) will suffice. Consider the graph \( G \) whose vertex set \( V(G) = \partial P \cap \mathbb{Z}^3 \) and whose edge set consists of the edges of \( \partial P \). Since \( P \) is convex, \( G \) is planar. (Just embed \( G \) into \( S^2 \) and then map onto \( \mathbb{R}^2 \).) Thus every triangulation of \( G \) has \( 2 \cdot |\mathcal{V}(G)| - 4 \) faces, implying \( \partial P \) can be triangulated into \( 2b - 4 \) triangles. Since \( k \geq 1 \), \( P \) can be triangulated into \( 2b - 4 \) subtetrahedra using any interior lattice point. The remaining interior points refine this triangulation, via Theorem [31] into a full triangulation having at least \( 2b - 4 + 3(k - 1) \) subtetrahedra. \( \blacksquare \)

This bound is sharp (consider any clean, non-empty lattice tetrahedron). A similar result is proved (independently) in [5, Lemma 3.5.5] by De Loera, Rambau, and Santos Leal. While their proof does not assume that \( P \) is nonempty, it does assume that \( P \cap \mathbb{Z}^3 \) is in general position. They also showed that any polyhedron with \( n \) vertices has a triangulation consisting of at most \( 2n - 7 \) tetrahedra.

9. Future Research

What can we say about an upper bound for the volume of polyhedra? Hensley proved in [9] that the volume of any lattice \( d \)-polytope is bounded above by a function in terms of \( d \) and the number of interior lattice points. Ziegler improved Hensley’s results and showed in [14] that the volume of a \( k \)-point lattice \( d \)-polytope \( P \) satisfies

\[
\text{Vol}(P) \leq k \cdot [7(k + 1)]^d 2^{d+1}.
\]

In \( \mathbb{R}^3 \), a few simple computations seem to indicate that the upper bound on the volume of clean tetrahedra is linear in \( k \). Moreover, maximal-volume tetrahedra also appear to have collinear interior lattice points. Unlike the minimal-volume tetrahedra, however, the line containing the interior points does not pass through any vertex. Based on these computations, we make the following conjectures.

Conjecture 1. If \( T \in \mathcal{S}^3_k \), then \( \text{Vol}(T) \leq \frac{1}{4}(12k + 8) \).

Conjecture 2. If \( T \in \mathcal{S}^3_k \) and \( \text{Vol}(T) = \frac{1}{4}(12k + 8) \), then \( T \simeq T_{2k+1,4k+3,12k+8} \). If \( k > 1 \), then \( \text{int}(T) \cap \mathbb{Z}^3 \) is a set of collinear points that does not pass through any vertex of \( T \).

These conjectures are true for \( k = 1 \), as proved independently by Kasprzyk [10] and Reznick [22]. It may be possible to gain some new insight on the relationship between lattice points of a lattice \( d \)-polytope and its volume using Ehrhart theory. For example, it is well known that every convex lattice \( d \)-polytope is associated with an Ehrhart polynomial of degree \( d \). Moreover, the leading coefficient of the associated polynomial is precisely the volume of the \( d \)-polytope (cf. [1], [6]).
10. Acknowledgements

I would like to thank my thesis advisor Bruce Reznick for having introduced me to lattice polytopes. I am also grateful to Phil Griffith, former director of graduate studies of the math department at the University of Illinois in Champaign-Urbana, who was directly responsible for my opportunity to work with Bruce Reznick. I also would like to thank Matthias Beck and Sinai Robins, the authors of [1], for their enlightening course on lattice point enumeration in Banff, Canada.

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