Chen optimal inequalities of CR-warped products of generalized Sasakian space form

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ABSTRACT

Our main objective of this paper is to derive the relationship between the main extrinsic invariant, and the contact CR \(\delta\)-invariant (new intrinsic invariant) on a generic submanifold in trans-Sasakian generalized Sasakian space forms. Further, we find a lower bound of the squared norm of the mean curvature (main extrinsic invariant) in terms of a CR \(\delta\)-intrinsic invariant, and the Laplacian of the warping function for CR-warped products in the same ambient space forms. We also investigate the classifications and triviality of connected, compact CR-warped product manifolds isometrically immersed into the trans-Sasakian generalized Sasakian space forms.

1. Introduction and motivations

It is well known that curvature invariants play the most fundamental role in Riemannian geometry. Curvature invariants provide the intrinsic characteristics of Riemannian manifolds which affect the behaviour in general of the Riemannian manifold. They are the main Riemannian invariants and the most natural ones. They are widely used in the field of differential geometry and in physics also. The innovative work of Kaluza–Klein in general relativity and string theory in particle physics has inspired the mathematicians and physicists to do work on submanifolds of (pseudo-)Riemannian manifolds. Intrinsic and extrinsic invariants are very powerful tools to study submanifolds of Riemannian manifolds.

Theorems which relate intrinsic and extrinsic curvatures invariant always play an important role in mathematical analysis and their applications to physical sciences. The Nash theorem was aimed for in the hope that if Riemannian manifolds could be regarded as Riemannian submanifolds, this would then yield the opportunity to use extrinsic help in the study of Riemannian geometry. There were several reasons why it is so difficult to apply Nash’s theorem. The main reason for this is the lack of control over the extrinsic properties of the submanifolds by the known classical intrinsic invariants. In order to overcome the difficulties, one needs to establish general optimal relationships between the main extrinsic invariants and the new intrinsic invariants for submanifolds. Such invariants and inequalities have many nice applications in several areas in mathematics.

In the early 1990s, B.-Y. Chen came up with new types of Riemannian invariants, called \(\delta\)-invariants (or Chen invariants) on Riemannian manifolds. The \(\delta\)-invariants are not similar to the classical scalar and Ricci curvatures in nature because both of them are the total sum of sectional curvatures on Riemannian manifolds. In contrast, all of the non-trivial \(\delta\)-invariants are derived from the scalar curvature by removing a definite amount of sectional curvatures. He considered the concept of \(\delta\)-invariants in order to find new necessary conditions for the existence of minimal immersions into a Euclidean space of an arbitrary dimension and to obtain applications of the celebrated Nash embedding theorem.

J. A. Schouten, D. V. Dantzig and E. Kähler discovered an important class of the Riemannian manifold which is known as Kähler manifold. A. Kähler manifold is an almost Hermitian manifold \(\mathcal{B}\) with almost complex structure \(J\) if \(J\) is parallel with respect to the Levi–Civita connection \(\nabla\) of \(g\), that is, \(\nabla J = 0\). In [1], Chen defined the CR \(\delta\)-invariant \(\delta(D)\) on a CR-submanifold \(\mathcal{B}\) in a Kähler manifold \(\mathcal{B}\) as follows:

\[
\delta(D)(x) = \tau(x) - \tau(D_x), \quad x \in \mathcal{B},
\]

where \(\tau\) denotes the scalar curvature of \(\mathcal{B}\) and \(\tau(D)\) denotes the scalar curvature of the holomorphic distribution \(D\) of \(\mathcal{B}\). He [1] also established an inequality for anti-holomorphic warped product submanifold \(\mathcal{B} = B^T \times_f B^2\) in a \((r + k)\)-dimensional complex space form \(\mathcal{B}(4\mathcal{C})\) with \(\dim_{\mathcal{C}}(B^T) = r \geq 1\) and \(\dim(B^2) = k \geq 2\).
involving the CR $\delta$-invariant $\delta(D)$
\[
||H||^2 \geq \frac{2(k+2)}{(2r+k)^2(k-1)} \left\{ \delta(D) - \frac{k(\Delta f) + k(k-1)\xi}{f} \right\}.
\]

(2)

He showed that the equality sign of (2) holds at $x \in B$ in if and only if there exists an orthonormal frame $(e_{2r+1}, \ldots, e_n)$ of $\mathcal{D}_x^\perp$ such that the second fundamental form $h$ satisfies

(a) $h_{pp}^q - 3h_{qq}^p = 0$, for $2r+1 \leq p \neq q \leq 2r+k$,
(b) $h_{qt}^q = 0$, for $p, q, t \in \{2r+1, \ldots, 2r+k\}, p \neq q \neq t$.

Faleh et al. [2, 3] proved an optimal inequality for this CR $\delta$-invariant on an holomorphic manifold $B$ in a $(r + k)$-dimensional complex space form $\mathcal{B}(4\zeta)$
\[
\delta(D) \leq \frac{(2r+k)^2}{2} ||H||^2 + \frac{k(4r+k-1)\xi}{2} - \frac{3k^2}{2 k+2} ||H_2||^2,
\]
where $H_2$ is the partial mean curvature vector of $B$, $\text{rank}_c(D) = r$ and $\text{rank}(D^\perp) = k \geq 2$. They also proved that the equality sign holds identically in (3) if and only if the following three conditions are fulfilled:

(a) the partial mean curvature vector $H_1 = 0$,
(b) $h(D, D^\ast) = 0$, and
(c) there exists an orthonormal frame $(e_{2r+1}, \ldots, e_n)$ of $\mathcal{D}_x^\perp$ such that $f$ satisfies

(c.1) $h_{pp}^q - 3h_{qq}^p = 0$, for $2r+1 \leq p \neq q \leq 2r+k$,
(c.2) $h_{qt}^q = 0$, for $p, q, t \in \{2r+1, \ldots, 2r+k\}, p \neq q \neq t$.

The concept of Sasakian manifolds was introduced in the 1960s by S. Sasaki as an odd-dimensional analogue to Kähler manifolds. Recently, Mihai et al. [4] studied Chen’s CR $\delta$-invariant for an odd-dimensional contact CR-submanifold, called contact CR $\delta$-invariant and obtained an optimal estimate for a $(s + 1)$-dimensional generic submanifold in a Sasakian space form $\mathcal{B}(\zeta)$ of constant $\phi$-sectional curvature $\zeta$
\[
\delta(D) \leq k + \frac{(s + 1)^2}{2} ||H||^2 + \frac{k(4r+k-1)(\zeta + 3)}{8} - \frac{3k^2}{2(k+2)} ||H_2||^2,
\]
where $\text{dim}(\mathcal{D}) = 2r + 1$ and $\text{dim}(\mathcal{D}_x^\perp) = k$.

The next result follows the idea of a relationship between the squared norm of mean curvature and the contact CR $\delta$-invariant on CR-warped product submanifolds into the generalized Sasakian space form, that is, included the Laplacian of the warping function. Hence, we prove the our result in following form.

**Theorem 1.1:** Let $B$ be a generic submanifold of a trans-Sasakian generalized Sasakian space form $\mathcal{B}(f_1, f_2, f_3)$. Then
\[
\delta(D) \leq k \left\{ 1 + \frac{(4r+k-1)f_1}{2} \right\} + \frac{(2r+k+1)^2}{2} ||H||^2 - \frac{3k^2}{2(k+2)} ||H_2||^2.
\]
Moreover, the equality case of inequality (5) holds if and only if

(a) $N$ is $D$-minimal,
(b) $N$ is mixed totally geodesic,
(c) there exists an orthonormal basis $(e_{2m_1+1}, \ldots, e_{2m_1+m_2})$ of $\mathcal{D}_x^\perp$ such that

(c.1) $h_{pp}^q - 3h_{qq}^p = 0$, for $2r+1 \leq p \neq q \leq 2r+k$,
(c.2) $h_{qt}^q = 0$, for $p, q, t \in \{2r+1, \ldots, 2r+k\}, p \neq q \neq t$.

The next result follow the idea of a relationship between the squared norm of mean curvature and the contact CR $\delta$-invariant on CR-warped product submanifolds into the generalized Sasakian space form, that is, included the Laplacian of the warping function. Hence, we prove the our result in following form.

**Theorem 1.2:** Let $B = B_1 \times_f B_2$ be a contact CR-warped product submanifold of a trans-Sasakian generalized Sasakian space form $\mathcal{B}(f_1, f_2, f_3)$. Then
\[
||H||^2 \geq \left( \frac{2(k+2)}{(k-1)(2r+k+1)\xi} \delta(D)(x) + k \Delta f \right) - f_1 k(2k+2) \frac{f_1}{(2r+k+1)^2}.
\]

The equality sign in (6) holds at $x \in B$ in if and only if there exists an orthonormal frame $(e_{2r+1}, \ldots, e_n)$ of $\mathcal{D}_x^\perp$ such that the second fundamental form $h$ satisfies
(a) $h_{pp}^q - 3h_{qq}^p = 0$, for $2r + 1 \leq p \neq q \leq 2r +$,
(b) $h_{qt} = 0$, for $p, q, t \in \{2r + 1, \ldots, 2r + k\}, p \neq q \neq t$.

Lastly, we provide some applications of the obtained inequality in the view of Nash's embedding theorem.

The paper as follows: In Section 2, we recall some preliminaries formulas and definitions related to our study. In Section 3, we prove a proposition for generic submanifold into generalized Sasakian space and also given the proof of our first main theorem. In Section 3, we give the definition of warped product manifold and presented the proof of second main theorem. In Section 4, we described some geometric applications of derived results by using compactness and connectedness.

2. Basic formulas and notations

An odd-dimensional smooth manifold $\overline{B}$ has an almost contact metric structure $(\phi, \xi, \eta, g)$ if there exist on $\overline{B}$ a tensor field $\phi$ of type $(1, 1)$, a structure vector field $\xi$, a dual 1-form $\eta$ and a Riemannian metric $g$ such that [6]

$$
\phi^2 = -I + \eta \otimes \xi, \quad \phi \circ \xi = 0,
\eta(\xi) = 1, \quad \eta(\eta) = g(\xi, \xi),
\eta(\phi X, Y) = -g(H, \phi Y),
g(\phi X, \phi Y) = g(X, Y) - \eta(\eta)\eta(Y),
$$

for any $X, Y \in \Gamma(\overline{B})$. The covariant derivative of the tensor field $\phi$ is given by

$$
(\nabla_X \phi) Y = \nabla_X \phi Y - \phi \nabla_X Y,
$$

for any $X, Y \in \Gamma(\overline{B})$. Let $\overline{B}$ be an odd-dimensional contact metric manifold with contact metric structure $(\phi, \xi, \eta, g)$. If the contact metric structure of $\overline{B}$ is normal, then $\overline{B}$ is said to have a Sasakian structure and $\overline{B}$ is known as a Sasakian manifold [6]. It is denoted by $(\overline{B}, \phi, \xi, \eta, g)$. Then for a Sasakian manifold, we have [6]

$$
(\nabla_X \phi) Y = g(\phi X, Y) \xi - \eta(\eta)X,
$$

and

$$
\nabla_X \xi = -\phi X,
$$

for any $X, Y \in \Gamma(\overline{B})$. An almost contact metric structure $(\phi, \xi, \eta, g)$ on $\overline{B}$ is known to be

(a) Kenmotsu structure [5] if

$$
(\nabla_X \phi) Y = g(\phi X, Y) \xi - \eta(\eta)X,
$$

and

$$
\nabla_X \xi = \eta(X)\xi,
$$

for any $X, Y \in \Gamma(\overline{B})$, where $\nabla$ represents the Riemannian connection with respect to the $g$ on $\overline{B}$. Thus, $(\overline{B}, \phi, \xi, \eta, g)$ becomes Kenmotsu manifold.

(b) Cosymplectic structure if

$$
d\phi = 0, \quad d\eta = 0, \quad [\phi, \phi] = 0,
$$

where $d$ is an exterior differential operator. In this case, $(\overline{B}, \phi, \xi, \eta, g)$ is said to be cosymplectic manifold.

It is proved that an almost contact metric structure $(\phi, \xi, \eta, g)$ on $\overline{B}$ is cosymplectic structure if and only if [7]

$$
(\nabla_X \phi) Y = 0,
$$

and

$$
\nabla_X \xi = 0,
$$

for any $X, Y \in \Gamma(\overline{B})$. An almost contact metric structure $(\phi, \xi, \eta, g)$ on $\overline{B}$ is called a trans-Sasakian structure [8] if $(\overline{B} \times \mathbb{R}, J, G)$ belongs to the class $\mathcal{W}_d$ of the Gray–Hervella classification of almost Hermitian manifolds [9], where $J$ is the almost complex structure on $\overline{B} \times \mathbb{R}$ defined by

$$
J(X, t) dt = (\phi X - \hat{\xi}, \eta(X) dt),
$$

for any $X, Y \in \Gamma(\overline{B})$ and smooth functions $\hat{\xi}$ on $\overline{B} \times \mathbb{R}$ and G is the product metric on $\overline{B} \times \mathbb{R}$. This may be expressed by the following condition [8]:

$$
(\nabla_X \phi) Y = \alpha(g(X, Y) \xi - \eta(\eta)X) + \beta(\phi X, Y) \xi - \eta(\eta)\phi X,
$$

for some smooth functions $\alpha$ and $\beta$ on $\overline{B}$ and this trans-Sasakian structure is termed as structure of type $(\alpha, \beta)$. Thus, a trans-Sasakian of type $(0, 0)$ is cosymplectic, of type $(0, \beta)$ is $\beta$-Kenmotsu and of type $(\alpha, 0)$ is $\alpha$-Sasakian.

Remark 2.1: A trans-Sasakian structure of type $(\alpha, \beta)$ is

(a) Sasakian if $\beta = 0, \alpha = 1$;
(b) Kenmotsu if $\alpha = 0, \beta = 1$;
(c) Cosymplectic if $\alpha = \beta = 0$.

In 2004, Alegre et al. [10] introduced and studied the generalized Sasakian space forms. An almost contact metric manifold $(\overline{B}, \phi, \xi, \eta, g)$ is known as a generalized Sasakian space form $\overline{B}(f_1, f_2, f_3)$ if there exist differentiable functions $f_1, f_2, f_3$ such that curvature tensor $\overline{R}$ of $\overline{B}$ is given by [10]

$$
\overline{R}(X, Y, Z, W) = f_1(g(Y, Z)g(X, W) - g(X, Z)g(Y, W)) + f_2(2g(X, \phi Y)g(\phi Z, W) - g(Y, \phi Z)g(\phi X, W)) + f_3(\eta(\eta)g(X, Z)g(Y, W) - g(\eta)g(X, Z)g(Y, W)).
$$
\[ -g(Y, Z)\eta(X)\eta(W), \] (18)

for any \(X, Y, Z, W \in \Gamma(T\bar{B}).\)

**Remark 2.2:** The generalized Sasakian space form extends the concept of Sasakian, Kenmotsu and cosymplectic space forms.

(a) A Sasakian space form is obtained from generalized Sasakian space form with \(f_1 = (c + 3)/4\) and \(f_2 = f_3 = (c - 1)/4\) (see in [4]).

(b) A Kenmotsu space form can be derived from generalized Sasakian space form with \(f_1 = (c - 3)/4\) and \(f_2 = f_3 = (c + 1)/4\) (see in [5]).

(c) A cosymplectic space form is a generalized Sasakian space form with \(f_1 = f_2 = f_3 = c/4\) (see in [7]).

Let us denote the Lie algebra of vector fields in \(B\) is \(\Gamma(T\bar{B}).\) Here \(\nabla\) be the Levi–Civita connection on \(\bar{B}\). As a consequence, the squared norm of warping function \(f\) is defined as we have
\[
||\nabla f||^2 = \sum_{i=1}^{s} (e_i(f))^2,
\]
for a local orthonormal frame \(\{e_1, \ldots, e_s\}\) on \(\bar{B}\). Also, the Laplacian \(\Delta f\) of \(f\) is given by
\[
\Delta f = \sum_{i=1}^{s} \left(\nabla_{e_i} f - e_i(f)\right) = -\sum_{i=1}^{s} g(\nabla_{e_i} grad f, e_i).
\] (20)

Likewise, the Hessian \(\text{Hess}_f\) of \(f\) is defined as
\[
\Delta f = -\sum_{i=1}^{s} \text{Hess}_f(e_i, e_i) = -\text{trace}(\text{Hess}_f).
\] (21)

The theory of submanifolds as follows: Let a Riemannian manifold \(\bar{B}\) and \(B\) be immersed in an odd-dimensional almost contact metric manifold \((\bar{B}, \phi, \xi, \eta, g)\) with induced metric \(g\). The Lie algebra of vector fields in \(B\) and the set of all vector fields normal to \(B\) are, respectively, given by \(\Gamma(T\bar{B})\) and \(\Gamma(T\bar{B})\). Let \(\nabla\) be the induced connection on \(B\). Then the Gauss and Weingarten formulas are, respectively, given below [11]:
\[
\nabla_X Y = \nabla_X Y + h(X, Y), \quad \nabla_X N = -A_N(X) + \nabla_X Y, \quad \text{for any } X, Y \in \Gamma(T\bar{B}) \text{ and } N \in \Gamma(T\bar{B}).
\] (22)

for any \(X, Y \in \Gamma(T\bar{B})\) and \(N \in \Gamma(T\bar{B}).\) Here \(h\) and \(A\) are the bilinear symmetric second fundamental form of \(\bar{B}\) and the shape operator of \(B\), respectively. Both of them are related as [11]
\[
g(h(X, Y), N) = g(A_N(X), Y),
\] (23)

for any \(X, Y \in \Gamma(T\bar{B})\) and \(N \in \Gamma(T\bar{B}).\) We denote the Riemannian curvature tensor fields of \(\bar{B}\) and \(B\) by \(\bar{R}\) and \(R\), respectively. Then the Gauss equation is given by [11]
\[
\bar{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(h(X, Z), h(Y, W)) - g(h(X, W), h(Y, Z)),
\] (24)

for any \(X, Y, Z, W \in \Gamma(T\bar{B}).\) We assume that \(\dim(B) = s\) and \(\dim(\bar{B}) = 2k + 1\). Let \(e_1, \ldots, e_s\) be a local orthonormal frame of \(T_xB\) and \(\{e_{s+1}, \ldots, e_{2k+1}\}\) be a local orthonormal frame of \(T^\perp_xB\); \(x \in B\). Then the mean curvature vector \(H\) of a submanifold \(B\) at \(x\) is defined by
\[
sh = \sum_{a=1}^{s} h(e_a, e_a).
\] (25)

Also, we set \(h^{(l)}_{ab} = g(h(e_a, e_b), e_l), \forall a, b \in \{1, \ldots, s\}, l \in \{s+1, \ldots, 2k+1\}\), and
\[
||h||^2 = \sum_{a, b=1}^{s} g(h(e_a, e_b), h(e_a, e_b)) = \sum_{a, b=1}^{s} ||h(e_a, e_b)||^2.
\] (26)

For any \(X \in \Gamma(T\bar{B})\) and \(N \in \Gamma(T\bar{B})\), respectively, we put [11] \(\phi X = PX + FX\) and \(\phi N = QN + CN\), where \(PX\) and \(FX\) are the tangential and the normal components of \(\phi X\), respectively. Similarly, \(QN\) and \(CN\) are the tangential and the normal components of \(\phi N\), respectively. For their geometric relations, see [11].

A submanifold \(B\) of an almost contact metric manifold \((\bar{B}, \phi, \xi, \eta, g)\) is said to be a contact CR-submanifold [11] of \(\bar{B}\) if there exists a differentiable distribution \(D : x \to D_x \subset T_xB\) such that \(D\) is invariant with respect to \(\phi\) and the orthogonal complementary distribution \(D^\perp\) is anti-invariant with respect to \(\phi\). The tangent bundle \(TB\) has the orthogonal decomposition \(TB = D \oplus D^\perp \oplus \{\xi\}\), where \(\{\xi\}\) is a 1-dimensional distribution which is spanned by \(\xi\). A submanifold \(B\) of \(\bar{B}\) is said to be invariant [11] if \(F \equiv 0\), that is, \(\phi X \in \Gamma(T\bar{B})\), and anti-invariant [11] if \(P \equiv 0\), that is, \(\phi X \in \Gamma(T\bar{B})\), for any \(X \in \Gamma(T\bar{B})\).

A submanifold \(B\) of an almost contact metric manifold \((\bar{B}, \phi, \xi, \eta, g)\) is called a generic submanifold [12] of \(\bar{B}\) if \(\phi T^\perp_xB \subset T_xB\) for all point \(x\) of \(B\) and \(\xi\) is tangent to \(B\). Especially, if \(\phi T^\perp_xB = T_xB \setminus \{\xi\}\), then a generic submanifold \(B\) is an anti-invariant submanifold such that \(2\dim(\bar{B}) - 1 = \dim(\bar{B})\). If \(\dim T^\perp_xB = 1\), that is, if \(B\) is a hypersurface of \(\bar{B}\), then \(B\) is obviously a generic submanifold. A submanifold \(B\) of \(\bar{B}\) has following classification according to [11] as:

(a) \(B\) is a totally umbilical submanifold if its second fundamental form \(h\) satisfies \(h(X, Y) = g(X, Y)H\), for any \(X, Y \in \Gamma(T\bar{B})\), where \(H\) is the mean curvature vector of \(B\) in \(\bar{B}\).

(b) \(B\) is called totally geodesic if \(h(X, Y) = 0\), for any \(X, Y \in \Gamma(T\bar{B})\).

(c) Finally \(B\) is minimal if \(H = 0\).
3. Optimal inequality for generic submanifolds

Let $B$ be a CR-submanifold of a trans-Sasakian manifold $(\overline{B}, \phi, \xi, \eta, g)$ with $(2r + 1)$-dimensional invariant distribution $\mathcal{D}$ and $k$-dimensional anti-invariant distribution $\mathcal{D}^\perp$. Assume that a local orthonormal frame as $\{e_0, e_1, \ldots, e_{2r}, e_{2r+1}, \ldots, e_{2r+k}\}$ on $B$ such that $\{e_0 = \xi, e_1, \ldots, e_{2r}\}$ (or $\{e_0 = \xi, e_1, \ldots, e_{r}, \phi e_1, \ldots, \phi e_r\}$) are in $\mathcal{D}$ and $\{e_{2r+1}, \ldots, e_{2r+k}\}$ are in $\mathcal{D}^\perp$, respectively. Then we put $h^p_{ab} = g(h(e_a, e_b), \phi e_p), \forall a, b \in \{0, \ldots, 2r + k\}$, and $p \in \{2r + 1, \ldots, 2r + k\}$. The two partial mean curvature vectors $h_1$ and $h_2$ of $B$ are, respectively, defined as

$$
(2r + 1)h_1 = \sum_{i=0}^{2r} h(e_i, e_i), \quad (2r + 1)h_2 = \sum_{p=2r+1}^{2r+k} h(e_p, e_p).
$$

In addition, we set $h_1 = h(e_i, e_j), \forall i, j \in \{0, \ldots, 2r\}$, and $h_2 = h(e_p, e_q), \forall p, q \in \{2r + 1, \ldots, 2r + k\}$. Thus,

$$
||h_1||^2 = \sum_{i=0}^{2r} g(h(e_i, e_i), h(e_i, e_i)) = \sum_{i=0}^{2r} ||h(e_i, e_i)||^2,
$$

$$
||h_2||^2 = \sum_{p=2r+1}^{2r+k} g(h(e_p, e_p), h(e_p, e_p)) = \sum_{p=2r+1}^{2r+k} ||h(e_p, e_p)||^2,
$$

where $h_1$ and $h_2$ are the second fundamental form for invariant and anti-invariant submanifold, respectively. By reference [4], a contact CR-submanifold of a trans-Sasakian manifold is said to be $\mathcal{D}$-minimal if $h_1 = 0$, and $\mathcal{D}^\perp$-minimal if $h_2 = 0$.

In order to obtain our main theorem, we need to derive the following result:

**Proposition 3.1:** Let $B$ be a generic submanifold of a trans-Sasakian generalized Sasakian space form $\overline{B}(f_1, f_2, f_3)$. Then

$$
\delta(\mathcal{D})(x) = k \left[ 1 + \frac{(4r + k - 1)f_1}{2} \right] + \frac{(2r + k + 1)}{2} ||H||^2 - \frac{(2r + 1)}{2} ||H_1||^2 - \sum_{i=0}^{2r} \sum_{p=2r+1}^{2r+k} ||h(e_i, e_p)||^2 - \frac{1}{2} ||h_2||^2,
$$

where the notations in the above equation are defined in (25), (27), (29) and (30) as well.

**Proof:** We choose a local orthonormal frame $\{e_0 = \xi, e_1, \ldots, e_{2r}, e_{2r+1}, \ldots, e_{2r+k}\}$ on $B$ as above. Then the scalar curvature $\tau(x)$ of $B$ and $\tau(\mathcal{D}_x)$ of $\mathcal{D}$, $x \in B$ are respectively given by

$$
2\tau(x) = \sum_{0 \leq p \leq 2r} K(e_i, e_j) + 2 \sum_{i=0}^{2r} \sum_{p=2r+1}^{2r+k} K(e_i, e_p) + \sum_{2r+1 \leq p \leq 2r+k} K(e_p, e_q), \quad (32)
$$

On the other hand, we have

$$
2\tau(\mathcal{D}_x) = \sum_{0 \leq p \leq 2r} K(e_i, e_j). \quad (33)
$$

By definition of contact CR $\delta$-invariant and relations (32) and (33), we have

$$
\delta(\mathcal{D})(x) = \sum_{i=0}^{2r} \sum_{p=2r+1}^{2r+k} K(e_i, e_p) + \frac{1}{2} \sum_{2r+1 \leq p \leq 2r+k} K(e_p, e_q) = \sum_{i=0}^{2r} \sum_{p=2r+1}^{2r+k} K(e_i, e_p) + \sum_{i=1}^{2r} \sum_{p=2r+1}^{2r+k} K(e_i, e_p) + \frac{1}{2} \sum_{2r+1 \leq p \leq 2r+k} K(e_p, e_q). \quad (34)
$$

By utilize the Gauss equation (24) and (18), we eliminate all the terms of equation (34) as follows:

$$
\sum_{i=0}^{2r} \sum_{p=2r+1}^{2r+k} K(e_i, e_p) = k, \quad (35)
$$

$$
2r \sum_{i=0}^{2r} \sum_{p=2r+1}^{2r+k} K(e_i, e_p) = 2rkf_1 - \sum_{i=1}^{2r} \sum_{p=2r+1}^{2r+k} ||h(e_i, e_p)||^2 + \sum_{i=1}^{2r} \sum_{p=2r+1}^{2r+k} g(h(e_i, e_i), h(e_p, e_p)), \quad (36)
$$

$$
\frac{1}{2} \sum_{2r+1 \leq p \leq 2r+k} K(e_p, e_q) = \frac{k(k - 1)f_1}{2} - \frac{1}{2} \left\{ \sum_{p,q=2r+1}^{2r+k} ||h(e_q, e_p)||^2 - \sum_{p,q=2r+1}^{2r+k} g(h(e_q, e_q), h(e_p, e_p)) \right\}. \quad (37)
$$

Inserting all equations (35), (36) and (37) into (33), we get

$$
\delta(\mathcal{D})(x) = k \left[ 1 + \frac{(4r + k - 1)f_1}{2} \right] + \sum_{i=1}^{2r} \sum_{p=2r+1}^{2r+k} g(h(e_i, e_i), h(e_p, e_p)) + \frac{2r}{2} \sum_{i=1}^{2r} \sum_{p=2r+1}^{2r+k} g(h(e_i, e_i), h(e_p, e_p))
$$

where the notations in the above equation are defined in (25), (27), (29) and (30) as well.
On simplifying and using the fact that

\[ - \sum_{i=1}^{2r} \sum_{p=2r+1}^{2r+k} \lVert h(e_i, e_p) \rVert^2 + \frac{1}{2} \sum_{p,q=2r+1}^{2r+k} g(h(e_q, e_q), h(e_p, e_p)) = \frac{1}{2} \sum_{p,q=2r+1}^{2r+k} \lVert h(e_q, e_p) \rVert^2. \]  

(38)

On simplifying and using the fact that

\[ h^p_{qt} = h^q_{pt} = h^t_{pq}, \]  

(39)

\[ \forall p, q, t \in \{2r+1, \ldots, 2r+k\}, \]  

the previous equation turns into

\[ \delta(D)(x) = k \left\{ 1 + \frac{(4r + k - 1)f_1}{2} \right\} + \frac{(2r + k + 1)^2}{2} \lVert H \rVert^2 - \frac{(2r + 1)^2}{2} \lVert H_1 \rVert^2 \]  

\[ - \sum_{i=0}^{2r} \sum_{p=2r+1}^{2r+k} \lVert h(e_i, e_p) \rVert^2 - \frac{1}{2} \lVert H_2 \rVert^2. \]  

(40)

This is completes the proposition.

**Proof of Theorem 1.1**

By similar approach as in [4], we derive \((k + 2)\lVert H_2 \rVert^2 - 3k^2\lVert H_2 \rVert^2 \geq 0\). Then we have

\[ \lVert H_2 \rVert^2 \geq \frac{3k^2}{k + 2} \lVert H_2 \rVert^2 \]  

(41)

with the equality case if and only if \( h^p_{pp} - 3h^p_{qq} = 0 \), \( \forall 2r + 1 \leq p \neq q \leq 2r + k \), and \( h^p_{qt} = 0 \), \( \forall p, q, t \in \{2r+1, \ldots, 2r+k\}, p \neq q \neq t \). Taking into account of Proposition 3.1 together with inequality (41), we get our desired result (5). This completes the proof of Theorem.

**3.1. Some consequences**

By using Remark 2.2 (a), we summarize the immediate consequences of Theorem 1.1, given below:

**Corollary 3.1:** Let \( B \) be a generic submanifold of Sasakian space form. Then CR- \( \delta \)-invariant satisfying the inequality

\[ \delta(D) \leq k \left\{ 1 + \frac{(4r + k - 1)(\Xi + 3)}{8} \right\} + \frac{(2r + k + 1)^2}{2} \lVert H \rVert^2 - \frac{3k^2}{2(k + 2)} \lVert H_2 \rVert^2. \]  

(42)

**Remark 3.1:** It should be noted that Corollary 4.1 coincided to Theorem 2.2 in [4]. It means that Theorem 2.2 in [4] is special case of our main Theorem 1.1.

Again Remark 2.2(b) and (5) of Theorem 1.1, we have

**Corollary 3.2:** Let \( B \) be a generic submanifold of Kenmotsu space form. Then we have

\[ \delta(D) \leq k \left\{ 1 + \frac{(4r + k - 1)(\Xi - 3)}{8} \right\} + \frac{(2r + k + 1)^2}{2} \lVert H \rVert^2 - \frac{3k^2}{2(k + 2)} \lVert H_2 \rVert^2. \]  

(43)

Similarly, from (c) of Remark 2.2, we find that

**Corollary 3.3:** Assume that \( B \) is a generic submanifold of cosymplectic space forms. Thus

\[ \delta(D) \leq k \left\{ 1 + \frac{(4r + k - 1)\Xi}{8} \right\} + \frac{(2r + k + 1)^2}{2} \lVert H \rVert^2 - \frac{3k^2}{2(k + 2)} \lVert H_2 \rVert^2. \]  

(44)

Moreover, the equality case of the above inequalities holds if and only if (a), (b) and (c) of Theorem 1.1 are fulfilled.

**4. Warped product manifolds and CR-warped product submanifolds**

The concept of warped product manifolds has many applications in physics. For instance, different models of space-time in general relativity are expressed in terms of warped geometry, and the Einstein field equations and modified field equations have many exact solutions as the warped products. In 1969, the idea of warped product manifolds had been initiated by Bishop and O’Neil [13] with manifolds of negative curvature. These manifolds are the most fruitful and natural generalization of Riemannian product manifolds. Several nice results are available in the literature (see [14] and the references therein).

**Definition 4.1 ([13]):** Let \((B_1, g_1)\) and \((B_2, g_2)\) be two (pseudo)-Riemannian manifolds and \( f > 0 \) be a differentiable function on \( B_1 \). Consider the product \( \pi_1 : B_1 \times B_2 \rightarrow B_1 \) and \( \pi_2 : B_1 \times B_2 \rightarrow B_2 \). Then the warped product \( B = B_1 \times_f B_2 \) is the product manifold \( B_1 \times B_2 \) equipped with the Riemannian structure \( g \) such that

\[ g(X, Y) = g_1(\pi_{1*}X, \pi_{1*}Y) + f^2(\pi_{2*}g_2(\pi_{2*}X, \pi_{2*}Y)), \]  

(45)

for any \( X, Y \in \Gamma(T_{(u,v)}B) \), \( u \in B_1 \) and \( v \in B_2 \), where * is the symbol for the tangent maps. The function \( f \) is called the warping function of the warped product.

A warped product manifold is said to be trivial if its warping function is constant. In this case, the warped product manifold is a Riemannian product manifold. For the trivial warped product manifold \( B = B_1 \times_f B_2 \), submanifolds \( B_1 \) and \( B_2 \) are totally geodesic and totally umbilical of \( B \), respectively. Let \( \dim(B) = s \), \( \dim(B_1) = k \), and \( \dim(B_2) = r \). For unit vectors \( X_1 \) and \( X_2 \) tangent
to $B_1$ and $B_2$, the sectional curvature $K(X_1 \wedge X_2)$ of the plane section spanned by $X_1$ and $X_2$ is

\[
K(X_1 \wedge X_2) = g(\nabla_{X_1} X_2, X_1) - g(\nabla_{X_2} X_1, X_2) = \frac{1}{f} \{ (\nabla_{X_1} X_1) h - X_1^2 h \} = (\nabla_{X_1} X_1) \ln f - X_1^2 \ln f.
\]

If we assume that $\{e_1, \ldots, e_r\}$ on $B$ such that $\{e_1, \ldots, e_r\}$ are in $B_1$ and $\{e_{r+1}, \ldots, e_s\}$ are in $B_2$, then we have

\[
\frac{\Delta f}{f} = \sum_{i=1}^{s} K(e_i \wedge e_j),
\]

for each $j = r + 1, \ldots, s$. From [15], a warped product submanifold $B = B_1 \times_f B_2$ of a Kenmotsu (or cosymplectic) manifold $(\overline{B}, \phi, \xi, \eta, g)$, where $B_1$ is a $(2r+1)$-dimensional invariant submanifold tangent to $\xi$ and $B_2$ is a $k$-dimensional anti-invariant submanifold of $\overline{B}$, is said to be a contact CR-warped product submanifold of $\overline{B}$.

### 4.1. Lower bounds for CR-warped product submanifolds

Chen [16, 17] introduced the notion of a CR-warped product manifold and studied CR-warped products in Kaehler manifold. Later, Hasegawa and Mihai [15] studied contact CR-warped products in Sasakian manifold. We prepare some lemmas for later use.

**Lemma 4.1 ([18]):** Let $B = B_1 \times_f B_2$ be a CR-warped product submanifold of a trans-Sasakian manifold $(\overline{B}, \phi, \xi, \eta, g)$ such that the structure vector field $\xi$ is tangent to $B_1$. Then $g(h(\xi, \xi)) = 0$.

For a CR-warped product $B = B_1 \times_f B_2$ of a trans-Sasakian generalized Sasakian space $(\overline{B}, f_1, f_2, f_3)$, we assume that $\dim(B_1) = 2r + 1$, $\dim(B_2) = k$ and $\dim(B) = s + 1$. Next, we consider an orthonormal frame $\{e_0, e_1, \ldots, e_2r, e_{2r+1}, \ldots, e_{2r+k}\}$ on $B$. Then

\[
\frac{\Delta f}{f} = \sum_{p=0}^{2r} K(e_i \wedge e_p) = \sum_{i=1}^{2r} K(e_i \wedge e_p) + K(\xi \wedge e_p),
\]

for each $p = 2r + 1, \ldots, 2r + k$. Motivated by above result, we give the following proposition which has an important role to prove our main theorem.

**Proposition 4.2:** Let $B = B_1 \times_f B_2$ be a CR-warped product submanifold of a trans-Sasakian generalized Sasakian space form $(\overline{B}, f_1, f_2, f_3)$. Then

\[
\delta(\mathcal{D})(x) = k \left\{ \frac{\Delta f}{f} - 1 + \frac{f_1(k - 1)}{2} \right\} - \frac{1}{2} ||h_2||^2 + \frac{(2r + k + 1)^2}{2} ||H||^2,
\]

where $\Delta f$ is the Laplacian of the warping function $f$.

**Proof:** By similar arguments as in Proposition 3.1, we arrive at

\[
\delta(\mathcal{D})(x) = \sum_{i=0}^{2r} \sum_{p=2r+1}^{2r+k} K(e_i, e_p) + \sum_{2r+1 \leq p < q \leq 2r+k} K(e_p, e_q) = \sum_{i=0}^{2r} \sum_{p=2r+1}^{2r+k} K(e_i, e_p) + \sum_{2r+1 \leq p < q \leq 2r+k} K(e_p, e_q).
\]

Since, $\phi B^\perp = T^\perp B$, then from (24), the last term gives us

\[
\sum_{2r+1 \leq p < q \leq 2r+k} K(e_p, e_q) = \frac{f_1(k - 1)}{2} - \sum_{2r+1 \leq p < q \leq 2r+k} ||h(e_p, e_q)||^2 + \sum_{2r+1 \leq p < q \leq 2r+k} g(h(e_p, e_q), h(e_q, e_q)).
\]

Now, we use Lemma 4.2 together with (51) in (50), we conclude that

\[
\delta(\mathcal{D})(x) = k \left\{ \frac{\Delta f}{f} - 1 + \frac{f_1(k - 1)}{2} \right\} - \frac{1}{2} ||h_2||^2 + \frac{(2r + k + 1)^2}{2} ||H||^2.
\]

Hence, our assertion is derived. The proof of proposition is completed.

**Proof of Theorem 1.2**

If we multiply the relation (49) by a quantity $(k + 2)/(k - 1)$, we deduce that

\[
2 \left(\frac{k + 2}{k - 1} \right) \left\{ k \left\{ \frac{\Delta f}{f} - 1 + \frac{f_1(k - 1)}{2} \right\} - \delta(\mathcal{D})(x) \right\} = \left(\frac{k + 2}{k - 1} \right) ||h_2||^2 - \left(\frac{k + 2}{k - 1} \right) (2r + k + 1)^2 ||H||^2.
\]
Thus, from (54) and (55), we obtain inequality (6) and the CR-warped product submanifold of a Kenmotsu space

\[
\geq (k+2) \left| h_{22} \right|^2 - \frac{(2r+k+1)^2}{k-1} \left| H \right|^2 - (2r+k+1)^2 \left| H \right|^2,
\]

which is equivalent to following:

\[
2 \left[ k+2 \right] \left\{ k \left( \frac{\Delta f}{f} - 1 + \frac{f_1(k-1)}{2} \right) - \delta(D)(x) \right\}
+ (2r+k+1)^2 \left| H \right|^2
= \left( k+2 \right) \left| h_{22} \right|^2 - \frac{(2r+k+1)^2}{k-1} \left| H \right|^2.
\]

By analogy with [1] and the fact that \( A_{\phi X_2} Y_2 = A_{\phi Y_2} X_2 \) for any \( X_2, Y_2 \in \Gamma(D^{\perp}) \), or \( h^p_q = h^p_{q^*} = h^p_{q^*} \) \( \forall p, q, t \in \{2r+1, \ldots, 2r+k\} \), one can derive the following inequality:

\[
\left| \frac{k+2}{k-1} \right| \left| h_{22} \right|^2 - \frac{(2r+k+1)^2}{k-1} \left| H \right|^2 \geq 0.
\]

Thus, from (54) and (55), we obtain inequality (6) and the equality sign holds in (6) if and only if

\[
h^p_{pp} - 3h^p_{qq} = 0, \text{ for } 2r+1 \leq p \neq q \leq 2r+k,
\]

\[
h^p_{qt} = 0, \text{ for } q, t \in \{2r+1, \ldots, 2r+k\}, p \neq q \neq t.
\]

As we wanted to prove. This completes the proof of Theorem.

4.2. Classifications

By using Remark 2.2, we summarize the immediate consequences of Theorem 1.2, given below:

Corollary 4.1: Let \( B = B_1 \times_f B_2 \) be a contact CR-warped product submanifold of a Sasakian space form. Then CR invariant satisfying the following

\[
\left| H \right|^2 \geq \left( \frac{2(k+2)}{(k-1)(2r+k+1)^2} \right) \left\{ \delta(D)(x) + k \frac{\Delta f}{f} \right\}
- \left( \frac{k(k+2)}{4(2r+k+1)^2} \right) \frac{(\zeta+3)(k+2)}{(2r+k+1)^2}.
\]

where \( H \) is mean curvature vector.

Corollary 4.2: Assume that \( B = B_1 \times_f B_2 \) be a contact CR-warped product submanifold of a Kenmotsu space form. Then we have

\[
\left| H \right|^2 \geq \left( \frac{2(k+2)}{(k-1)(2r+k+1)^2} \right) \left\{ \delta(D)(x) + k \frac{\Delta f}{f} \right\}
- \frac{k(k+2)(\zeta+3)(k+2)}{4(2r+k+1)^2}.
\]

Similarly for cosymplectic space form, we find that

Corollary 4.3: Let \( B = B_1 \times_f B_2 \) be a contact CR-warped product submanifold of a cosymplectic space form. Then following inequality holds:

\[
\left| H \right|^2 \geq \left( \frac{2(k+2)}{(k-1)(2r+k+1)^2} \right) \left\{ \delta(D)(x) + k \frac{\Delta f}{f} \right\}
- \frac{\zeta k(k+2)}{4(2r+k+1)^2}.
\]

The equality sign in above inequalities hold at \( x \in B \) if and only if (a) and (b) of Theorem 1.2 are satisfied.

5. Geometric applications

Applications of Theorem 1.2 in view of Nash’s embedding theorem, given below:

Corollary 5.1: Let \( \psi : B_1 \times_f B_2 \rightarrow \overline{B}(f_1, f_2, f_3) \) be an isometric minimal immersion of a CR-warped product submanifold \( B_1 \times_f B_2 \) into a trans-Sasakian generalized Sasakian space form \( \overline{B}(f_1, f_2, f_3) \) whose warping function \( f \) is a harmonic function. Suppose that \( \delta(D) = -1 \), then \( f_1 \geq 0 \).

Corollary 5.2: Let \( \psi : B = B_1 \times_f B_2 \rightarrow \overline{B}(f_1, f_2, f_3) \) be an isometric minimal immersion of a compact and connected CR-warped product submanifold \( B_1 \times_f B_2 \) into a trans-Sasakian generalized Sasakian space form \( \overline{B}(f_1, f_2, f_3) \), then \( B \) is simply a Riemannian product of \((B_1, g_1)\) and \((B_2, f^2 g_2)\) if \( \delta(D) = 2f_1(k-1) - 1 \).

Corollary 5.3: Let \( \psi : B_1 \times_f B_2 \rightarrow \overline{B}(\xi) \) be an isometric minimal immersion of a CR-warped product submanifold \( B_1 \times_f B_2 \) into a Kenmotsu space form \( \overline{B}(\xi) \) whose warping function \( f \) is a harmonic function. Suppose that \( \delta(D) = -1 \), then \( \xi \geq 3 \).

Corollary 5.4: Let \( \psi : B = B_1 \times_f B_2 \rightarrow \overline{B}(\xi) \) be an isometric minimal immersion of a compact and connected CR-warped product submanifold \( B_1 \times_f B_2 \) into a Kenmotsu space form \( \overline{B}(\xi) \), then \( B \) is simply a Riemannian product of \((B_1, g_1)\) and \((B_2, f^2 g_2)\) if \( \delta(D) = k(k-1) (\xi - 3)/2 - 1 \).

Corollary 5.5: Let \( \psi : B_1 \times_f B_2 \rightarrow \overline{B}(\xi) \) be an isometric minimal immersion of a CR-warped product submanifold \( B_1 \times_f B_2 \) into a cosymplectic space form \( \overline{B}(\xi) \) whose warping function \( f \) is a harmonic function. Suppose that \( \delta(D) \geq 0 \), then \( \xi \geq 0 \).

Corollary 5.6: Let \( \psi : B = B_1 \times_f B_2 \rightarrow \overline{B}(\xi) \) be an isometric minimal immersion of a compact and connected CR-warped product submanifold \( B_1 \times_f B_2 \) into a cosymplectic space form \( \overline{B}(\xi) \), then \( B \) is simply a Riemannian product of \((B_1, g_1)\) and \((B_2, f^2 g_2)\) if \( \delta(D) = \xi k(k-1)/2 - 1 \).
6. Conclusion remarks

The $\delta$-invariants are very different in nature from the classical scalar and Ricci curvatures; simply due to the fact that both scalar and Ricci curvatures are total sum of sectional curvatures on a Riemannian manifold. In contrast, almost all of the $\delta$-invariants are obtained from the scalar curvature by throwing away certain amount of sectional curvatures. Curvatures invariants also play key roles in physics. For instance, the magnitude of a force required to move an object at a constant speed, according to Newton’s laws, a constant multiple of the curvature of the trajectory. The motion of a body in a gravitational field is determined, according to Einstein, by the curvatures of space time. Classically, among the Riemannian curvature invariants, people have been studying sectional, scalar and Ricci curvatures in great details (see [19] and references therein). In the present paper, we established a general optimal relationships between extrinsic invariant with the new intrinsic invariants on the submanifolds. Also some of previous results are generalized from this study. As we know generalized Sasakian space form is a class in almost contact metric manifold that generalized three classes, that is, cosymplectic space form, Kenmotsu space form and Sasakian space form see [20–25]. This is the main reason to study CR-warped product submanifold into the generalized Sasakian space form. All the above argument shows that the paper contains interesting applications in mathematical analysis and mathematical physics as well.

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