A FINITE ANALOGUE OF THE RING OF ALGEBRAIC NUMBERS

JULIAN ROSEN

ABSTRACT. We construct an analogue of the ring of algebraic numbers, living in a quotient of the product of all finite fields of prime order. We use this ring to deduce some results about linear recurrent sequences.

1. INTRODUCTION

A period is a complex number given as the integral of an algebraic function over a region defined by algebraic inequalities. The set of all periods is a countable subring of $\mathbb{C}$ containing $\mathbb{Q}$ (see [8] for an overview of periods). Several recent works (e.g. [3, 4, 5, 10, 11, 13]) consider “finite” analogues of certain periods (finite multiple zeta values, finite multiple polylogarithms, etc.) living in the ring

$$A := \prod_p \mathbb{Z}/p\mathbb{Z} \bigoplus_p \mathbb{Z}/p\mathbb{Z},$$

which was introduced by Kontsevich ([7], §2.2). An element of $A$ is a prime-indexed sequence $(a_p)_p$, with $a_p \in \mathbb{Z}/p\mathbb{Z}$, and two sequences are equal if they agree for all sufficiently large $p$. Every non-zero integer is invertible modulo $p$ for all sufficiently large $p$, so there is a diagonal embedding $\mathbb{Q} \hookrightarrow A$.

1.1. Results. The purpose of this paper is to define a countable $\mathbb{Q}$-subalgebra $P^0_A \subset A$ that is a finite analogue of $\mathbb{Q} \subset \mathbb{C}$. This algebra is not the integral closure of $\mathbb{Q}$ inside $A$, which has continuum cardinality.

Our first main result is three equivalent characterizations of $P^0_A$.

Theorem 1.1. The following subsets of $A$ are equal.

1. The set of elements $(a_p \mod p)_p$, where $a_0, a_1, a_2, \ldots \in \mathbb{Q}$ is a recurrent sequence (that is, a sequence satisfying a linear recurrence relation with constant coefficients).

2. The set of elements $(g(\phi_p) \mod p)_p$, where $L/\mathbb{Q}$ is a finite Galois extension, $g : \text{Gal}(L/\mathbb{Q}) \to L$ satisfies $g(\sigma \tau \sigma^{-1}) = \sigma(g(\tau))$, and $\phi_p$ is the Frobenius at $p$.

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This is independent of the representative of the Frobenius conjugacy class (see §2).
(3) The set of $\mathbb{Q}$-linear combinations of matrix coefficients for the $A$-valued Frobenius automorphism

$$F_A : L \otimes A \to L \otimes A,$$

defined by Definition 4.1, as $L$ ranges over all number fields.

The equivalence of (1) and (2) is Theorem 2.2, and the equivalence of (2) and (3) is Theorem 4.2.

**Definition 1.2.** We define $P_0^A \subset A$ to be the set given by Theorem 1.1.

The Skolem-Mahler-Lech theorem says that if $(a_n)$ is a recurrent sequence, the set \( \{ n : a_n = 0 \} \) is has finite symmetric difference with a finite union of arithmetic progressions. As a consequence of Theorem 1.1, we obtain an analogue of Skolem-Mahler-Lech for the set of primes \( \{ p : a_p \equiv 0 \mod p \} \). A set $P$ of primes is called Frobenian (cf. [12], §3.3) if there is a finite Galois extension $L/\mathbb{Q}$ and a union of conjugacy classes $C \subset \text{Gal}(L/\mathbb{Q})$ such that $P$ has finite symmetric difference with the set of rational primes whose Frobenius conjugacy class is in $C$. The Chebotarev density theorem implies that the natural density of a Frobenian set exists and is a rational number.

**Corollary 1.3.** A set $P$ of primes is Frobenian if and only if there exists a recurrent sequence $(a_n)$ such that

$$P = \{ p : a_p \equiv 0 \mod p \}.$$ 

Unlike the Skolem-Mahler-Lech Theorem, Corollary 1.3 is effective: given the recurrence relation satisfied by $(a_n)$ and a list of initial values, there is a finite algorithm to determine the number field $L$, union of conjugacy classes $C \subset \text{Gal}(L/\mathbb{Q})$, and the finite exceptional set.

We also prove some results about polynomial equations satisfied by elements of $P_0^A$. The first of these results implies that $P_0^A$ is an integral extension of $\mathbb{Q}$.

**Theorem 1.4.** Suppose $\alpha \in P_0^A$. Then there exists a non-zero polynomial $f(x) \in \mathbb{Q}[x]$ such that $f(\alpha) = 0$, and every such $f(x)$ has a rational root.

**Remark 1.5.** The Fibonacci sequence $F_n$ is known to satisfy the congruence $F_p \equiv \left( \frac{p}{5} \right) \mod p$ for every prime $p$, where $\left( \frac{p}{5} \right)$ is a Legendre symbol. Thus $f(F_p) \equiv 0 \mod p$ for $p \geq 7$, where $f(x) = x^2 - 1 \in \mathbb{Q}[x]$. Theorem 1.4 implies that every recurrent sequence satisfies an analogous identity for some $f$, which necessarily has a rational root.

We also prove a result about the density of the set of primes $p$ for which $f(a_p) \equiv 0 \mod p$, when $(a_p) \in P_0^A$ and $f(x) \in \mathbb{Q}[x]$.

**Theorem 1.6.** For $f(x) \in \mathbb{Q}[x]$, we have

$$\sup_{(a_p) \in P_0^A} \delta \left( \{ p : f(a_p) \equiv 0 \mod p \} \right) = \delta \left( \{ p : f \text{ has a root mod } p \} \right),$$
where $\delta$ denotes natural density. Moreover if $f(x)$ has no rational roots, then there is no element of $P_0^A$ realizing the supremum.

In §4 we explain the analogy between $P_0^A \subset A$ and $\mathbb{Q} \subset \mathbb{C}$, and the relationship with periods.

2. Functions on a Galois group

Let $L/\mathbb{Q}$ be a finite Galois extension, with ring of integers $O_L$ and Galois group $\Gamma := \text{Gal}(L/\mathbb{Q})$.

Definition 2.1 ([9], §2). We define $A(L)$ to be the set of functions $g : \Gamma \rightarrow L$ satisfying

$$g(\sigma \tau^{-1}) = \sigma(g(\tau))$$

for all $\sigma, \tau \in \Gamma$, which is a commutative $\mathbb{Q}$-algebra under pointwise addition and multiplication.

For $g \in A(L)$, let $p$ be a rational prime unramified in $L$ that is coprime to the denominators of all values of $g$. Let $\mathfrak{P}$ be a prime of $L$ over $p$, with Frobenius element $\phi_\mathfrak{P} \in \Gamma$. It follows from (2.1) that the residue class

$$g(\phi_\mathfrak{P}) \mod \mathfrak{P}$$

is fixed by $\phi_\mathfrak{P}$, so (2.2) is an element of $\mathbb{Z}/p\mathbb{Z} \subset O_L/\mathfrak{P}$. It can be checked that the value of $g(\phi_\mathfrak{P}) \mod \mathfrak{P}$ is independent of the choice of $\mathfrak{P}|p$ (see [9], §4), and we write $g(\phi_\mathfrak{P}) \mod p$ for this residue class in $\mathbb{Z}/p\mathbb{Z}$. We leave $g(\phi_\mathfrak{P}) \mod p$ undefined for the finitely many primes that are either ramified in $L$ or are not coprime to the denominators of $g$.

The following result gives equivalence of conditions (1) and (2) in the statement of Theorem 1.1.

Theorem 2.2. An element of $A$ has the form $(a_p \mod p)_p$ for some recurrent sequence $(a_n)$ if and only if that element of $A$ can be written $(g(\phi_\mathfrak{P}) \mod p)$ for some finite Galois extension $L/\mathbb{Q}$ and some $g \in A(L)$.

Proof. $(\Rightarrow)$ Let $(a_n)$ be a recurrent sequence. Then there exist column vectors $u, v$ and an invertible matrix $M$, with entries in $\mathbb{Q}$, such that

$$a_n = u^T M^n v$$

for all $n \in \mathbb{Z}$. There is a Jordan-Chevalley decomposition

$$M = M_{ss}M_u,$$

where $M_{ss}$ is semi-simple, $M_u$ is unipotent, and $M_{ss}$ commutes with $M_u$. For every prime $p$ larger than the size of $M_u$ that is coprime to all denominators appearing in $M_u$, the $p$-th power $M_u^p$ is congruent to the identity matrix modulo $p$, and if in addition $p$ is coprime to denominators appearing in $u$ and $v$, then

$$a_p \equiv u^T M_{ss}^p v \mod p.$$
Let $L$ be a finite Galois extension of $\mathbb{Q}$ over which $M_{\text{ss}}$ diagonalizes, let $\lambda_1, \ldots, \lambda_k \in L$ be the eigenvalues of $M_{\text{ss}}$, and write $\Gamma = \text{Gal}(L/\mathbb{Q})$. Using the Jordan normal form of $M_{\text{ss}}$, it follows from (2.3) that there are elements $b_1, \ldots, b_k \in L$ such that
\[
a_p \equiv \sum_i b_i \lambda_i^p \mod p,
\]
and $\Gamma$ permutes the pairs $b_i, \lambda_i$, i.e. the element
\[
\alpha := \sum_i b_i \otimes \lambda_i \in L \otimes_{\mathbb{Q}} L
\]
is invariant under the diagonal action of $\Gamma$. There is a canonical isomorphism
\[
\varphi : L \otimes_{\mathbb{Q}} L \to \text{Hom}(\Gamma, L),
\]
\[
x \otimes y \mapsto (\sigma \mapsto x\sigma(y)),
\]
taking the $\Gamma$-invariant elements of $L \otimes L$ to $A(L)$, and we let $g = \varphi(\alpha) \in A(L)$. If $p$ is a rational prime unramified in $L$ coprime to every denominator of the values of $g$, then for every prime $P$ of $L$ over $p$,
\[
g(\phi_P) = \sum_i b_i \phi_P(\lambda_i)
\]
\[= \sum_i b_i \lambda_i^p \mod \mathfrak{P}
\]
\[\equiv a_p \mod \mathfrak{P}.
\]
Thus we have $(a_p \mod p) = (g(\phi_p) \mod p)$.

$(\iff)$ Suppose $g \in A(L)$ is given, and let
\[
\varphi^{-1}(g) = \sum_i b_i \otimes \lambda_i \in (L \otimes L)^\Gamma,
\]
where we may choose $b_i, \lambda_i$ such that the pairs $(b_i, \lambda_i)$ are permuted by $\Gamma$. Then the sequence
\[
a_n := \sum_i b_i \lambda_i^n
\]
is recurrent, and takes values in $\mathbb{Q}$ because the pairs $(b_i, \lambda_i)$ are permuted by $\Gamma$. By the computation above, we see that
\[
a_p \equiv g(\phi_p) \mod p
\]
for all sufficiently large $p$. This completes the proof. \hfill \square

Remark 2.3. Let $L/\mathbb{Q}$ be a finite Galois extension. In the language of motives, the ring $A(L)$ defined in §2 is the ring of de Rham motivic periods of $\text{Spec} L$ (see [2], §1.2, and [9], §5). There is a ring homomorphism
\[
\text{per}_A : A(L) \to A,
\]
\[g \mapsto (g(\phi_p) \mod p)_p,
\]
which is an example of an $A$-valued period map (see [10], §5).
3. Proofs of the theorems

In this section we prove Corollary 1.3, and Theorems 1.4 and 1.6.

Proof of Corollary 1.3. Suppose \((a_n)\) is a recurrent sequence. Let \(L/\mathbb{Q}\) and \(g \in A(L)\) be as in the statement of Theorem 2.2. Then for all primes \(p\) unramified in \(L\) coprime to the numerators and denominators of all non-zero values of \(L\) and all \(\mathfrak{p}|p\), we have

\[ a_p \equiv 0 \mod p \iff g(\phi_p) = 0. \]

So we may take \(C = \{\sigma \in \text{Gal}(L/\mathbb{Q}) : g(\sigma) = 0\}\), which is a union of conjugacy classes by (2.1).

Conversely, suppose \(L/\mathbb{Q}\) and \(C \subset \text{Gal}(L/\mathbb{Q})\) are given. Let \(g \in A(L)\) be the characteristic function of \(C\), and let \(a_n\) be a recurrent sequence such that

\[ a_p \equiv g(\phi_p) \mod p \]

for all but finitely many \(p\) (which exists by Theorem 2.2). Then \(\{p : a_p \equiv 0 \mod p\}\) coincides with \(\{p : \phi_p \subset C\}\) up to a finite set. We can scale the sequence \((a_n)\) through by a constant rational number to modify \(\{p : a_p \equiv 0 \mod p\}\) by any finite set. This completes the proof. \(\square\)

Proof of Theorem 1.4. Suppose \((a_p)_p \in \mathcal{P}_A^0\) is given. By Theorem 2.2, we can find \(L/\mathbb{Q}\) and \(g \in A(L)\) such that \(a_p \equiv g(\phi_p) \mod p\) for all sufficiently large \(p\). Since \(A(L)\) is a finite-dimensional \(\mathbb{Q}\)-algebra, there is a non-zero \(f(x) \in \mathbb{Q}[x]\) such that \(f(g) = 0\), which implies

\[ f(a_p) \equiv f(g(\phi_p)) \equiv 0 \mod p \]

for all sufficiently large \(p\). We can scale \(f(x)\) by a rational constant to make (3.1) hold for all \(p\).

Now suppose we are given \(f(x) \in \mathbb{Q}[x]\) with \(f(a_p) \equiv 0 \mod p\) for all \(p\). There are infinitely many primes \(p\) that split completely in \(L\), and for all but finitely many of these \(p\), we have

\[ f(a_p) \equiv f(g(1)) \equiv 0 \mod p, \]

where 1 is the identity element. Since (3.2) holds for arbitrarily large \(p\), it follows that \(f(g(1)) = 0\). Finally, (2.1) implies that \(g(1) \in \mathbb{Q}\), so we conclude \(f(x)\) has a rational root. \(\square\)

Before proving Theorem 1.6, we need some preliminary results. Suppose \(f(x) \in \mathbb{Q}[x]\) is monic, let \(L/\mathbb{Q}\) be a finite Galois extension over which \(f(x)\) splits into linear factors, and define \(\Gamma := \text{Gal}(L/\mathbb{Q})\). Let \(\alpha_1, \ldots, \alpha_n\) be the roots of \(f\) in \(L\), and for \(1 \leq i \leq n\) set \(\Gamma_i = \text{Gal}(L/\mathbb{Q}(\alpha_i)) \subset \Gamma\).

Lemma 3.1. Let \(p\) be a rational prime unramified in \(L\) that is coprime to the denominators of coefficients of \(f\). Then \(f(x)\) has a root modulo \(p\) if and
only if the Frobenius conjugacy class \( \phi_p \subset \Gamma \) is contained in
\[
S_1 := \bigcup_i \Gamma_i.
\]

**Proof.** There is a root of \( f(x) \) in \( \mathbb{Z}/p\mathbb{Z} \) if and only if for some (equivalently, every) prime \( \mathfrak{p} \) of \( L \) over \( p \), there is some \( i \) for which \( \alpha_i \mod \mathfrak{p} \) is in \( \mathbb{Z}/p\mathbb{Z} \subset \mathcal{O}_L/\mathfrak{p} \). Now, \( \alpha_i \mod \mathfrak{p} \) is in \( \mathbb{Z}/p\mathbb{Z} \) if and only if \( \phi_{\mathfrak{p}}(\alpha_i) = \alpha_i \), which happens if and only if \( \phi_{\mathfrak{p}} \in \Gamma_i \). So \( f \) has a root in \( \mathbb{Z}/p\mathbb{Z} \) if and only if there exists \( \mathfrak{p} | p \) with \( \phi_{\mathfrak{p}} \in \bigcup \Gamma_i \). Since \( \bigcup \Gamma_i \) is closed under conjugation, this is equivalent to the condition that \( \phi_p \subset \bigcup \Gamma_i \). \( \square \)

We also need the following fact.

**Lemma 3.2.** Define a set
\[
S_2 := \bigcup_i \{ \sigma \in \Gamma : C_\Gamma(\sigma) \subset \Gamma_i \},
\]
where \( C_\Gamma(\sigma) \) is the centralizer of \( \sigma \) inside \( \Gamma \). The for every \( g \in A(L) \), we have
\[
(3.3) \quad \{ \sigma \in \Gamma : f(g(\sigma)) = 0 \} \subseteq S_2,
\]
and there exists \( g \in A(L) \) for which (3.3) is an equality of sets.

**Proof.** If \( f(g(\sigma)) = 0 \), then \( g(\sigma) = \alpha_i \) for some \( i \). By (2.1), \( g(\sigma) \) is fixed by \( C_\Gamma(\sigma) \), so we must have \( C_\Gamma(\sigma) \subset \text{Gal}(L/\mathbb{Q}(\alpha_i)) = H_i \). This proves the containment (3.3). To show that we can choose \( g \in A(L) \) for which (3.3) is equality, let \( \sigma_1, \ldots, \sigma_k \in \Gamma \) be a system of conjugacy class representatives. For \( 1 \leq j \leq k \), if there does not exist \( i \) for which \( C_\Gamma(\sigma_j) \subset H_i \), then define \( g \) to be 0 on the conjugacy class of \( \sigma_j \). If there does exist \( i \), the define \( g \) on the conjugacy class of \( \sigma_j \) by
\[
g(\tau \sigma_j \tau^{-1}) = \tau(\alpha_i).
\]
We have \( g \in A(L) \) and \( \{ \sigma : f(g(\sigma)) = 0 \} = S_2 \). \( \square \)

We also need a group-theoretic fact about wreath products.

**Lemma 3.3.** Let \( \Gamma \) and \( A \) be finite groups, with \( A \) abelian, and consider the wreath product
\[
\Gamma' := A^\Gamma \rtimes \Gamma.
\]
Let \( \pi : \Gamma' \to \Gamma \) be the projection. Then at least
\[
\left( 1 - \frac{|\Gamma|^2}{|A|} \right) |\Gamma'|
\]
elements \( \xi \in \Gamma' \) satisfy
\[
(3.4) \quad \pi(C_{\Gamma'}(\xi)) \subset \langle \pi(\xi) \rangle.
\]
Proof. We identify elements of $\Gamma'$ with pairs $(\varphi, \sigma)$, where $\varphi : \Gamma \to A$ and $\sigma \in \Gamma$. Under this identification, multiplication in $\Gamma'$ is given by

$$(\varphi, \sigma) \circ (\psi, \tau) = (\varphi + \psi \circ R_\sigma, \sigma \tau)$$

(3.5)

(here $R_\sigma : \Gamma \to \Gamma$ is right multiplication by $\sigma$). A direct computation shows that $(\varphi, \sigma), (\psi, \tau) \in \Gamma'$ commute if and only if $\sigma$ and $\tau$ commute and

$$\varphi - \varphi \circ R_\tau = \psi - \psi \circ R_\sigma.$$ 

For $\eta : \Gamma \to A$, there exists $\psi : \Gamma \to A$ with $\eta = \psi - \psi \circ R_\sigma$ if and only if

$$\text{ord}(\sigma) - 1 \sum_{n=0}^{\text{ord}(\sigma)-1} \eta \circ R_{\sigma^n} = 0.$$ 

(3.6)

Combining (3.5) and (3.6), we see that, if $\varphi, \sigma,$ and $\tau$ are fixed, then there exists $\psi$ such that $(\varphi, g)$ and $(\psi, h)$ commute if and only if

$$\text{ord}(\sigma) - 1 \sum_{n=0}^{\text{ord}(\sigma)-1} (\varphi \circ R_{\sigma^n} - \varphi \circ R_{\sigma^n}) = 0.$$ 

(3.7)

For each $\sigma, \tau \in \Gamma$, define a group homomorphism

$$\chi_{\sigma, \tau} : A^\Gamma \to A,$$

$$\varphi \mapsto \text{ord}(\sigma) - 1 \sum_{n=0}^{\text{ord}(\sigma)-1} (\varphi(\sigma^n \tau) - \varphi(\sigma^n)).$$

If $\tau \notin \langle \sigma \rangle$, then the elements $\sigma^n$ and $\sigma^n \tau$ are all distinct. In this case $\chi_{\sigma, \tau}$ is seen to be surjective, and the kernel of $\chi_{\sigma, \tau}$ has index $|A|$ in $A^\Gamma$. It follows from (3.7) that, for fixed $\tau, \sigma$ with $\tau \notin \langle \sigma \rangle$, there are at most $|A|^{|\Gamma|-1}$ functions $\varphi : \Gamma \to A$ for which $\tau \in \pi(C_{\Gamma^2}(\langle \sigma, \varphi \rangle))$. Taking the union over all $\sigma, \tau \in \Gamma$ with $\tau \notin \langle \sigma \rangle$, we find that the number of elements $\xi \in \Gamma'$ for which (3.4) does not hold is at most

$$|\Gamma^2| A^{|\Gamma|-1}.$$ 

This completes the proof. \qed

We are now ready to prove Theorem 1.6.

Proof of Theorem 1.6. Suppose $f(x) \in \mathbb{Q}[x]$. It is obvious that

$$\delta \left( \left\{ p : f(a_p) \equiv 0 \mod p \right\} \right) \leq \delta \left( \left\{ p : f \text{ has a root mod } p \right\} \right)$$ 

(3.8)

for all $(a_p)_p \in \mathbb{P}_A^0$. We need to show that the inequality (3.8) is strict if $f(x)$ has no rational roots, and that we can choose $(a_p)_p \in \mathbb{P}_A^0$ to make (3.8) arbitrarily close to an equality.
Suppose $f(x)$ has no rational roots. For $(a_p)_p \in \mathcal{P}_A^0$, let $L/\mathbb{Q}$ and $g \in A(L)$ be such that $a_p \equiv g(\phi_p) \mod p$, and let $\Gamma \supset S_1 \supset S_2$ be as in the statements of Lemmas 3.1 and 3.2. By the Chebotarev density theorem,

$$\delta \left( \{ p : f \text{ has a root modulo } p \} \right) = \frac{\#S_1}{\#\Gamma},$$

$$\max_{g \in A(L)} \delta \left( \{ p : f(g(\phi_p)) \equiv 0 \mod p \} \right) = \frac{\#S_2}{\#\Gamma}.$$ 

We get strictness of (3.8) because the identity element of $\Gamma$ is in $S_1$ but not in $S_2$.

To show that (3.8) is sharp, we pass from $L$ to an extension $L'/L$ with the property that in $\text{Gal}(L'/\mathbb{Q})$, most elements have small centralizers (in a sense to be made precise). For $L'/\mathbb{Q}$ a finite Galois extension containing $L$, write $\Gamma' = \text{Gal}(L'/\mathbb{Q})$ and $\pi : \Gamma' \to \Gamma$ for the restriction map. Let $\Gamma'_i = \pi^{-1}(\Gamma_i) = \text{Gal}(L'/\mathbb{Q}(\alpha_i)) \subset \Gamma'$, for $i = 1, \ldots, n$. If an element $\sigma \in \Gamma'_i$ satisfies

$$\pi(C_{\Gamma'}(\sigma)) \subset \langle \pi(\sigma) \rangle,$$

then $C_{\Gamma'}(\sigma) \subset \Gamma'_i$. We will show that for every $\epsilon > 0$, we can choose the $L'/L$ so that (3.9) holds for at least $(1 - \epsilon)|\Gamma'_i|$ elements $\sigma$ of $\Gamma'$. This will prove the theorem.

Let $\epsilon > 0$ be given, and choose a positive integer $r$ such that

$$\frac{|\Gamma|^2}{2^r} < \epsilon.$$ 

Let $p_1, \ldots, p_r$ be distinct rational primes that split completely in the Hilbert class field of $L$. For each $i$, let $\beta_i \in \mathcal{O}_L$ be a generator for a (necessarily degree 1 and principal) prime of $L$ over $p_i$. Let $L'$ be the extension of $L$ obtained by adjoining a square root of $\sigma(\beta_i)$ for all $\sigma \in \Gamma$ and $1 \leq i \leq r$. Then $\Gamma'$ is isomorphic to a wreath product

$$\Gamma' \cong A^\Gamma \rtimes \Gamma,$$

with $A = (\mathbb{Z}/2)^r$. The result now follows from Lemma 3.3.
Definition 4.1. The \( A \)-valued Frobenius automorphism is the \( A \)-algebra automorphism \( F_{A,L} \) of \( L \otimes Q A \) induced by \( F_{p,L} \) in the \( p \)-th factor.

If we choose a basis for \( L \) as a \( Q \)-vector space, we can represent \( F_{A,L} \) by a square matrix with entries in \( A \), and the \( Q \)-span of the matrix entries does not depend on the choice of basis.

Theorem 4.2. For each finite Galois extension \( L/Q \), the \( Q \)-span of the matrix entries for \( F_{A,L} \) is equal to the set of elements \( (g(\phi_p) \mod p)_p \in A \) for \( g \in A(L) \).

Proof. The \( Q \)-span of matrix coefficients for \( F_{A,L} \) is the image of the map

\[
L^\vee \otimes_Q L \to A, \quad \varphi \otimes y \mapsto (\varphi(y^p) \mod p)_p.
\]

Here \( L^\vee \) is the \( Q \)-linear dual of \( L \). The trace form induces an isomorphism of \( L \) with \( L^\vee \), so the image of (4.1) is equal to the image of

\[
L \otimes L \to A, \quad x \otimes y \mapsto \left( \sum_{\sigma \in \Gamma} \sigma(xy^p) \right) \mod p,
\]

where \( \Gamma = \text{Gal}(L/Q) \).

It follows from the proof of Theorem 2.2 that \( \{ (g(\phi_p) \mod p)_p \} \) is equal to the image of the map

\[
(L \otimes L)^\Gamma \to A, \quad \sum_i x_i \otimes y_i \mapsto \left( \sum_i x_i y_i^p \right) \mod \mathfrak{P}_p,
\]

where for each \( p \) we have chosen a prime \( \mathfrak{P} \) of \( L \) over \( p \). The result now follows from the fact that

\[
L \otimes L \to (L \otimes L)^\Gamma, \quad x \otimes y \mapsto \sum_{\sigma} \sigma(x) \otimes \sigma(y)
\]

is surjective. \( \square \)

The algebraic de Rham cohomology of Spec(\( L \)) (which we view as a 0-dimensional algebraic variety over \( Q \)) is identified with \( L \). Thus \( P_p^0 \) is the \( Q \)-span of the matrix coefficients for the isomorphism

\[
H^0_{dR}(\text{Spec}(L)) \otimes A \sim H^0_{dR}(\text{Spec}(L)) \otimes A,
\]

for \( L \) ranging over the finite Galois extensions of \( Q \). If instead we look at de Rham-Betti comparison isomorphism

\[
H^0_{dR}(\text{Spec}(L)) \otimes C \sim H^0_B(\text{Spec}(L)) \otimes C
\]
for varying $L$, the $\mathbb{Q}$-span of the matrix coefficients is $\overline{\mathbb{Q}}$. For this reason $P^0_A \subset A$ is analogous to $\overline{\mathbb{Q}} \subset \mathbb{C}$. By contrast, the integral closure of $\mathbb{Q}$ inside $A$ is uncountable.

4.2. Positive dimension. The characterization of $P^0_A$ as matrix coefficients of the $A$-valued Frobenius can be generalized to produce elements of $A$ from varieties of positive dimension. If $X$ is a variety defined over $\mathbb{Q}$ and $i \geq 0$ is an integer, the algebraic de Rham cohomology $H^i_{dR}(X)$ is finite-dimensional vector space over $\mathbb{Q}$, and for all sufficiently large $p$ there is a distinguished automorphism $F_{p,X}$:

$$F_{p,X} : H^i_{dR}(X) \otimes \mathbb{Q}_p \sim \rightarrow H^i_{dR}(X) \otimes \mathbb{Q}_p$$

coming from crystalline cohomology (see [6]). Matrix coefficients for $F_{p,X}$ with respect to a $\mathbb{Q}$-basis are (one type of) $p$-adic periods of $X$. Each matrix coefficient for $F_{p,X}$ is $p$-integral for all sufficiently large $p$, so reduction modulo $p$ (for all large $p$ at once) gives an element of $A$. These elements are called $A$-valued periods in [10]. It is convenient to assemble the maps $F_{p,X}$ to form an $A$-valued Frobenius map

$$F_{A,X} : H^i_{dR}(X) \otimes A \rightarrow H^i_{dR}(X) \otimes A,$$

whose matrix coefficients are $A$-valued periods (the map $F_{A,X}$ is no longer an isomorphism). Details can be found in [10], §6.

If we instead use the de Rham-Betti comparison isomorphism

$$comp_X : H^i_{dR}(X) \otimes \mathbb{C} \sim \rightarrow H^i_B(X) \otimes \mathbb{C},$$

matrix coefficients are the ordinary (complex) periods of $X$. So in this analogy $A$ corresponds to $\mathbb{C}$, and $F_{A,X}$ corresponds to $comp_X$. Define $P_A \subset A$ (resp. $P_C \subset \mathbb{C}$) to be the $\mathbb{Q}$-span of the matrix coefficients for $F_{A,X}$ (resp. $comp_X$), as $X$ ranges through all varieties over $\mathbb{Q}$. By taking $X$ to have dimension 0 we see that $P_A^0 \subset P_A$ and $\overline{\mathbb{Q}} \subset P_C$.

The period conjecture of Grothendieck (see [1], §7.5) would imply that there is a $\mathbb{Q}$-algebra homomorphism

$$\Delta : P_C \rightarrow P_C \otimes_{\mathbb{Q}} P_A.$$

Concretely, fix a variety $X$ and bases for $H^i_{dR}(X)$ and $H^i_{dR}(X)$, say of length $n$. Write $F_{A,X}$ and $comp_X$ as matrices $(\alpha_{i,j}) \in M_n(A)$ and $(\beta_{i,j}) \in M_n(\mathbb{C})$, respectively. The map $\Delta$ is then given by

$$\Delta(\beta_{i,j}) = \sum_{k=1}^n \beta_{i,k} \otimes \alpha_{k,j} \in P_C \otimes_{\mathbb{Q}} P_A.$$

A priori the right hand side of (4.2) might depend on $X$, $i$, and $j$, but the the period conjecture implies that in fact the right hand side depends only on the value $\beta_{i,j} \in P_C$.

Every algebraic number occurs as a matrix coefficient for $comp_X$ for some 0-dimensional $X$. Since the $A$-valued periods of this $X$ are in $P^0_A$, this implies
\( \Delta \) takes \( \mathbb{T} \subset P_C \) into \( P_\mathcal{A}^0 \otimes_\mathbb{Q} P_C \). So the truth of the period conjecture would imply that if we see an algebraic number as a complex period of an arbitrary variety, we will also see elements of \( P_\mathcal{A}^0 \) in the \( \mathcal{A} \)-valued periods of that variety.

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E-mail address: julianrosen@gmail.com