Boundary S-matrix for the Tricritical Ising Model

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Abstract

The Tricritical Ising model perturbed by the subleading energy operator $\Phi_{(\frac{3}{4})}$ was known to be an Integrable Scattering Theory of massive kinks [14] and in fact preserves supersymmetry. We consider here the model defined on the half-plane with a boundary and computed the associated factorizable boundary S-matrix. The conformal boundary conditions of this model were identified and the corresponding S-matrices were found. We also show how some of these S-matrices can be perturbed and generate “flows” between different boundary conditions.

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1. Introduction

Quantum field theory of systems with boundaries have received much attention in recent years due to substantial advances in several aspects of this subject. One successful area of research is in the study of s-wave scattering of electrons from impurities [1] employing the technology of boundary conformal field theory [2]. Another fruitful avenue of work is the calculation of the boundary S-matrix for the elastic reflection of particles off the boundary [3]. In the context of quantum spin chains, this matrix is known as the $K$ matrix [4, 5]. Such $K$ matrix facilitates the generalization of the quantum inverse scattering method to the case of an open spin chain [6]-[9]. While for the case of integrable perturbation of conformal field theory, the boundary S-matrix is factorizable if integrable boundary conditions are chosen [10]. By applying the technique of thermodynamic Bethe ansatz with the boundary S-matrix, many physical results relating to boundary phenomena can be obtained [11]-[13].

The current work concerns the boundary S-matrix for the tricritical Ising model. As is known, perturbing this model in the bulk by the sub-leading energy operator $\nu'$ generates a flow from the $c = \frac{7}{10}$ conformal field theory down to a field theory of massive kinks [14]. \(^1\) In fact the resultant theory is supersymmetric and the supercharges can be explicitly constructed as integrals of motion [14]. In the present work, we consider this model defined on the half-plane with a boundary. The general S-matrix describing the factorizable scattering of kinks from the wall is derived by solving the corresponding

\(^1\) The same perturbation with an opposite sign in the coupling constant generates a massless flow down to the Ising fixed point [15, 16]
“boundary Yang-Baxter equation”. The conformal boundary conditions of the model can be classified following the proposal of [2]. This is done by using the RSOS picture of the Tricritical Ising model. The S-matrix associated with the conformal boundary conditions were found and we also studied the perturbation of these boundary conditions by a relevant boundary field.

2. Backround on the Tricritical Ising Model

The 2-Dimensional Tricritical Ising Model is of interest to physicists because it describes tricritical phenomena in a variety of microscopic models [17]. In particular, it is the simplest known statistical model to exhibit Supersymmetry [18], and the fact that it can be realized experimentally [19] makes it an important model to study.

Several lattice realization of this tricritical behavior is possible, among them the Blume-Capel Quantum Spin Chain [20] and the $A_4$ Restricted Solid-on-Solid (RSOS) model [21]. The microscopic model that we will consider here is the Ising model with annealed vacancies [22]. The classical Hamiltonian is

$$H = -\beta \sum_{<i,j>} \sigma_i \sigma_j - \mu \sum_i (\sigma_i)^2,$$

where $i,j$ label the lattice sites, each of the variables $\sigma_i$ takes three values $-1,0,1$ (0 represents the vacant site) and the first sum in (1) is made over nearest neighbor pairs. Concentration of the vacancies is controlled by the chemical potential $\mu$. The phase di-
agram of this system in coordinates of temperature $T$ and $e^{-\mu}$ is shown (schematically) in Fig.1. There are two phases separated by a line AB of phase transition. For low vacancies and low temperature, we have an ordered phase I with spontaneously broken $Z_2$ symmetry. On the other side of the transition line we have a disordered phase II with unbroken symmetry. The point B is the usual critical point of the Ising model, and the solid segment BT corresponds to the second order transition belonging to the Ising model universality class. The dotted segment AT is a line of first order phase transition. The point T where the two critical segments joined smoothly is the tricritical point of the system. This model belongs to the universality class of the Landau-Ginzburg $\varphi^6$-theory at this tricritical point [23], so the phase diagram could also be understood in terms of the Landau-Ginzburg effective potential of this system [14]. This effective potential (specific free energy) is plotted as the function of the order parameter ($< \sigma >$) of this system in Fig.2 for various regions of the phase diagram. The ground state is degenerate in phase I, while in phase II it is non-degenerate. There is only one vacuum on the line BT, typical of a second order transition. Note that along the first order transition line AT, the vacuum is in fact three-fold degenerate.

At the tricritical point T, the system is described by the conformal field theory (CFT) with central charge $c = \frac{7}{10}$ [24]. There are six irreducible representations of the Virasoro algebra, and the corresponding conformal dimensions $\Delta_{(r,s)}$ of the primary field $\phi_{(r,s)}$ are organized into a Kac table in Fig.3.
The fields in this theory can be classified according to their properties under the $Z_2$ spin-reversal transformation $\sigma \rightarrow -\sigma$ (this is a symmetry of the microscopic model (1)). In particular we have four even fields: the identity $I \equiv \Phi_{(0)}$, the leading energy density $\epsilon \equiv \Phi_{(\frac{1}{10})}$, the subleading energy (vacancy) density $\epsilon' \equiv \Phi_{(\frac{3}{5})}$ and the irrelevant field $\epsilon'' \equiv \Phi_{(\frac{7}{16})}$. Here we use the short-hand notation $\Phi_{(\Delta)}$ for the scalar field $\Phi_{(\Delta, \bar{\Delta})}$ with $\Delta = \bar{\Delta}$. There are two odd fields under spin-reversal: the leading spin field $\sigma \equiv \Phi_{(\frac{3}{2})}$, and the subleading spin field $\sigma' \equiv \Phi_{(\frac{3}{2})}$.

In addition to the scalar fields, this conformal field theory also possesses fermion fields $G(z) \equiv \Phi_{(\frac{1}{4}, 0)}$ and $\bar{G}(\bar{z}) \equiv \Phi_{(0, \frac{3}{4})}$. The currents $G(z)$ ($\bar{G}(\bar{z})$) together with the stress tensor $T(z)$ ($\bar{T}(\bar{z})$) generate the Neveu-Schwartz-Ramond algebra, hence this model exhibits superconformal symmetry [18]. We can alternatively classify the fields in this CFT according to this extended symmetry. For example, in the Neveu-Schwartz sector of this CFT we have the superfield

$$\Phi_{(\frac{1}{10})}(z, \theta, \bar{z}, \bar{\theta}) = \epsilon + \theta \bar{\psi} + \bar{\theta} \psi + \theta \bar{\theta} \epsilon',$$

where $\psi \equiv \Phi_{(\frac{1}{4}, \frac{1}{10})}$, and $\bar{\psi} \equiv \Phi_{(\frac{1}{10}, \frac{1}{4})}$ are fermion and antifermion fields respectively. All the component fields of $\Phi_{(\frac{1}{10})}$ are mutually local and, together with the identity operator $I$ and its descendants, they constitute the Neveu-Schwartz sector of this CFT. In this sector, Kramers-Wannier duality transformation acts as a second $Z_2$ symmetry under
which $\Phi_{(\frac{1}{10})} \rightarrow -\Phi_{(\frac{1}{10})}$; $\theta \rightarrow -\theta$. More specifically, under duality transformation, the signs of $\epsilon, \Psi$, and $\bar{G}$ are changed, while $I, \epsilon', \bar{\Psi}$, and $G$ are unaltered.

The spin fields $\sigma$ and $\sigma'$ provide the Ramond representation of the superconformal symmetry [18]. The duality transformation converts them into the disorder fields $\mu$ and $\mu'$ respectively [25], where $\mu = G_{0}\sigma, \mu' = G_{0}\sigma'$. Here $G_{0}$ is the zero-mode of $G$ in the Ramond sector. Hence the Operator Product Algebras generated by $(I, \sigma, \sigma', \epsilon, \epsilon')$ and $(I, \mu, \mu', -\epsilon, \epsilon')$ are isomorphic under duality. In other words, the Tricritical Ising Model is a self-dual system.

3. Supersymmetric perturbation of the Tricritical Ising Model

In the $c = \frac{7}{10}$ conformal field theory, the energy densities and spin fields are relevant operators. Perturbation of the bulk theory by these operators were studied numerically in [26]. This work confirms the conjecture that perturbations by the operators $\epsilon, \epsilon'$ and $\sigma'$ are individually integrable [27]. In particular, the transition line AB in Fig.1 can be generated by perturbing the tricritical fixed point Hamiltonian by the vacancy density $\epsilon'$ [15]. Thus the scaling region around the point T can be described by the field theory with the “action"

$$H_{\lambda} = H_{(\frac{7}{10})} + \lambda \int \Phi_{(\frac{4}{5})} d^2 x,$$  \hspace{1cm} (2)

where $H_{(\frac{7}{10})}$ is the “action” of the bulk CFT. The renormalization group (RG) flow from T down to the Ising fixed point B is given by (2) with $\lambda > 0$ and can be described by a massless scattering field theory [16]. For $\lambda < 0$, (2) is an integrable massive field theory.
and is associated with the first order transition line AT. According to [27], this model possesses an infinite number of bosonic Integrals of Motion, and we shall concentrate on the case with \( \lambda < 0 \). Furthermore, because \( \Phi(3) \) is the upper component of the superfield \( \Phi(1/8) \), so the perturbation (5.2) does not destroy the global supersymmetry of \( H(7/10) \). In fact the supercharges

\[
Q = \int [G dz + 4\lambda \bar{\psi} d\bar{z}]; \quad \bar{Q} = \int [\bar{G} d\bar{z} + 4\lambda \psi dz]
\]

are fractional spin integrals of motion in the perturbed theory [14]. They were used in [14] to construct the factorizable scattering S-matrix given below.

Along the line AT, the vacuum is three-fold degenerate, thus the particle spectrum of (2) with \( \lambda < 0 \) must contain “kinks” separating the domains of different vacua. As shown in Fig.4a, if we label the vacua as \(-1, 0, 1\), then there are four sorts of kinks of mass \( m \): \( K_{0,+1}, K_{0,-1}, K_{1,0}, K_{-1,0} \). An asymptotic N-kink scattering state, say an in-state

\[
|K_{\sigma_0,\sigma_1}(\theta_1)K_{\sigma_1,\sigma_2}(\theta_2)K_{\sigma_2,\sigma_3}(\theta_3)\ldots K_{\sigma_{N-1},\sigma_N}(\theta_N)>_{in}
\]

(3)

where \( \theta_i \) is the rapidity of the \( i \)-th kink, can be thought of as the sequence of \( N+1 \) domains of vacua \( \sigma_0, \sigma_1, \ldots, \sigma_N \) placed along the x-axis, with the kink \( K_{\sigma_i,\sigma_{i+1}} \) separating two neighboring domains \( \sigma_i \) and \( \sigma_{i+1} \). The neighboring vacua must satisfy the “admissibility condition” \( |\sigma_i - \sigma_{i+1}| = 1 \). Thus in (3) the vacua +1 and −1 must be separated by the vacuum 0. The out-state of N-kink scattering is related to the in-state (3) by the factorizable S-matrix built up from the two kink S-matrix. This S-matrix consists of four different amplitudes (shown in Fig.4b) \( A_0(\theta), A_1(\theta), B_0(\theta), B_1(\theta) \), defined as

\[
|K_{0,s}(\theta_1)K_{s,0}(\theta_2)>_{in} = A_0(\theta_{12})|K_{0,s}(\theta_2)K_{s,0}(\theta_1)>_{out} + A_1(\theta_{12})|K_{0,-s}(\theta_2)K_{-s,0}(\theta_1)>_{out};
\]

(4a)
\[ |K_{s,0}(\theta_1)K_{0,s}(\theta_2) >_{in} = B_0(\theta_{12})|K_{s,0}(\theta_2)K_{0,s}(\theta_1) >_{out}; \]  
\[ (4b) \]
\[ |K_{s,0}(\theta_1)K_{0,-s}(\theta_2) >_{in} = B_1(\theta_{12})|K_{s,0}(\theta_2)K_{0,-s}(\theta_1) >_{out}; \]  
\[ (4c) \]

where \( s = +1 \) or \(-1\), and \( \theta_{12} = \theta_1 - \theta_2 \). These amplitudes are given in [14] as

\[ A_0 = \cosh(\frac{\theta}{4})A(\theta); \quad A_1 = -i\sinh(\frac{\theta}{4})A(\theta); \]
\[ B_0 = \cosh(\frac{\theta}{4} - \frac{i\pi}{4})B(\theta); \quad B_1 = \cosh(\frac{\theta}{4} + \frac{i\pi}{4})B(\theta). \]  
\[ (5) \]

The “minimal” expressions for \( A(\theta) \) and \( B(\theta) \) are

\[ A(\theta) = e^{\exp(\gamma\theta)}S(\theta); \quad B(\theta) = \sqrt{2}\exp(-\gamma\theta)S(\theta), \]  
\[ (6) \]

where \( \exp(2\pi i\gamma) = 2 \) and

\[ S(\theta) = \prod_{k=1}^{\infty} \frac{\Gamma(k - \frac{\theta}{2\pi i})\Gamma(k - \frac{1}{2} + \frac{\theta}{2\pi i})}{\Gamma(k + \frac{\theta}{2\pi i})\Gamma(k + \frac{1}{2} - \frac{\theta}{2\pi i})}. \]  
\[ (7) \]

Note that these amplitudes do not exhibit any poles in the “physical strip”, hence there are no bounded state in the bulk scattering theory.

4. The General Boundary S-Matrix

In this section we consider the perturbed Tricritical Ising model in the semi-infinite plane \( \{(x, y) : x \leq 0; -\infty < y < \infty\} \). The general symbolic action with general boundary condition has the form

\[ H_{\lambda, \Phi_B} = H_{\frac{\lambda}{10} + CBC} + \lambda \int_{-\infty}^{\infty} dy \int_{-\infty}^{0} \Phi_{\frac{\lambda}{2}}(x, y)dx + \int_{-\infty}^{\infty} \Phi_B(y)dy, \]  
\[ (8) \]
where $H_{1/4}^{+} + C^\text{BC}$ is the CFT action on the half-space with Conformal Boundary Conditions (CBC) on the boundary $x = 0$. Here we assume a choice for the boundary operator $\Phi_B$ can be made such that the integrability of the bulk theory is preserved. Therefore the field theory (8) can be described by a factorizable scattering theory of kinks with the boundary. Since the bulk theory contains three degenerate vacua, we can expect the boundary to exist in one or more of these vacuum states. In the general scenario, the boundary states need not be degenerate, and some of these states can appear as boundary bound states in the scattering process [10]. If some of the boundary vacua are degenerate, then the scattering of kinks with the boundary can change its state.

Let us denote the “boundary creating operator” by $B_a$ where $a \in \{-1, 0, 1\}$ labels the boundary vacuum state. The Fock space of the boundary scattering theory consists of the scattering in-states

$$|K_{\sigma_1,\sigma_2}(\theta_1)K_{\sigma_2,\sigma_3}(\theta_2)...K_{\sigma_N,a}B_a >_{in}$$

(9)

with $\theta_1 > \theta_2 > \ldots > \theta_N > 0$. The out-state of the scattering is again related to the in-state (9) by a S-matrix built up from the two-kink S-matrix (4) and the one-kink boundary scattering S-matrix. The boundary S-matrix comprised of six elementary amplitudes defined as

$$|K_{+1,0}(\theta)B_0 >_{in} = R_+(\theta)|K_{+1,0}(-\theta)B_0 >_{out}; \quad (10a)$$

$$|K_{-1,0}(\theta)B_0 >_{in} = R_- (\theta)|K_{-1,0}(-\theta)B_0 >_{out}; \quad (10b)$$

$$|K_{0,+1}(\theta)B_{+1} >_{in} = P_+(\theta)|K_{0,+1}(-\theta)B_{+1} >_{out} + V_+(\theta)|K_{0,-1}(-\theta)B_{-1} >_{out}; \quad (10c)$$

$$|K_{0,-1}(\theta)B_{-1} >_{in} = P_- (\theta)|K_{0,-1}(-\theta)B_{-1} >_{out} + V_-(\theta)|K_{0,+1}(-\theta)B_{+1} >_{out}, \quad (10d)$$
and the associated space-time diagrams are illustrated in Fig.5. Note that in the general case we do not assume a priori that the boundary will respect the spin-reversal symmetry ($\sigma \rightarrow -\sigma$) of the bulk theory.

Factorizability of the scattering theory (8) means that the boundary amplitudes (10) have to satisfy the Boundary Yang-Baxter equation [3, 10] (shown in Fig.6) and results in the following set of functional equations:

\begin{align*}
R_+ A_0 R_+ A_1 + R_+ A_1 R_- A_0 &= R_- A_1 R_+ A_0 + R_- A_0 R_- A_1; \\
P_+ B_0 V_+ B_1 + V_+ B_1 P_- B_0 &= V_+ B_0 P_+ B_0 + P_- B_1 V_+ B_0; \\
P_+ B_1 V_+ B_0 + V_+ B_0 P_- B_0 &= V_+ B_1 P_+ B_1 + P_- B_0 V_+ B_1; \\
P_+ B_1 P_+ B_1 + V_+ B_0 V_- B_1 &= P_+ B_1 P_+ B_1 + V_- B_0 V_+ B_1; \\
P_+ B_0 P_+ B_0 + V_+ B_1 V_- B_0 &= P_+ B_0 P_+ B_0 + V_- B_1 V_+ B_0,
\end{align*}

(11a, 11b, 11c, 11d, 11e)

(the other equations are obtained by interchanging $+ \leftrightarrow -$ in (11)) where each term in the equations has the arguments $R_i(\theta_1)S_j(\theta_2 + \theta_1)R_k(\theta_2)S_l(\theta_2 - \theta_1)$. Notice that the equation for $R_+$ and $R_-$ decouples from the equations for the other amplitudes. This is to be expected since the scattering associated with $R_+$ and $R_-$ does not involve changes in the boundary state. However if the amplitudes (11) exhibit “boundary bound states”, then the “boundary bound-state bootstrap equation” will mix the various amplitudes. This will be discussed in Section 6. Solution to these equations can be obtained by elementary methods and has the form

\begin{align*}
R_+(\theta) &= (1 + Ash^{\theta/2})M(\theta); \\
R_-(\theta) &= (1 - Ash^{\theta/2})M(\theta);
\end{align*}

(12a)
\begin{align}
  P_+(\theta) &= (X + Y \sin \theta) N(\theta); \quad P_-(\theta) = (X - Y \sin \theta) N(\theta); \\
  V_+(\theta) &= k_+ \sinh \frac{\theta}{2} N(\theta); \quad V_-(\theta) = k_- \sinh \frac{\theta}{2} N(\theta),
\end{align}

where \( A, X, Y, k_+ \) and \( k_- \) are constants depending on the boundary condition. It is worth noting that \( R_+ \) and \( R_- \) contains one free parameter \( A \), while the other amplitudes involve three independent parameters (we can always absorb a constant into \( N(\theta) \) when needed).

The normalization functions \( N(\theta) \) and \( M(\theta) \) can be determined by the boundary unitarity and cross-unitarity constraint for the boundary S-matrix. For this model, the unitarity conditions assume the form

\begin{align}
  R_+(\theta)R_+(-\theta) &= 1; \\
  P_+(\theta)V_+(-\theta) + V_+(\theta)P_-(\theta) &= 0; \\
  P_+(\theta)P_+(-\theta) + V_+(\theta)V_-(\theta) &= 1;
\end{align}

and other equations with the substitution \(+ \leftrightarrow -\) in (13).

To obtain the cross-unitarity condition, we have to analytically continue the boundary S-matrix (10) into the cross channel [10], resulting in the following equations

\begin{align}
  R_+(\frac{\pi i}{2} - \theta) &= A_0(2\theta)R_+(\frac{\pi i}{2} + \theta) + A_1(2\theta)R_-(\frac{\pi i}{2} + \theta); \\
  P_+(\frac{\pi i}{2} - \theta) &= B_0(2\theta)P_+(\frac{\pi i}{2} + \theta); \\
  V_+(\frac{\pi i}{2} - \theta) &= B_1(2\theta)V_+(\frac{\pi i}{2} + \theta),
\end{align}

and similar equations under the interchange \(+ \leftrightarrow -\) in (14). The unitarity conditions (13) and (14) reduce to

\begin{align}
  M(\theta)M(-\theta) &= \frac{1}{1 - A^2 \sinh^2 \frac{\theta}{2}}; \\
\end{align}
\[
M\left(\frac{\pi i}{2} - \theta\right) = e^{4\gamma \theta} B_0(2\theta) M\left(\frac{\pi i}{2} + \theta\right); \quad (15b)
\]
\[
N(\theta)N(-\theta) = \frac{1}{X^2 - Y^2 \sinh^2 \theta - k_+ k_- \sinh^2 \theta}; \quad (15c)
\]
\[
N\left(\frac{\pi i}{2} - \theta\right) = B_0(2\theta) N\left(\frac{\pi i}{2} + \theta\right), \quad (15d)
\]
for the normalization functions \(N(\theta)\) and \(M(\theta)\). The solution to these equations requires analytic information about the scattering amplitudes (12) which are governed by the boundary condition. More specifically we need to determine the existence, if any, of “physical poles” (ie poles in the physical strip) in these amplitudes. To this end we will describe in the next sections the various conformal boundary conditions in this model and their associated scattering S-matrices.

5. Conformal Boundary Conditions of \(c = \frac{7}{10}\) CFT

A variety of boundary conditions are possible for a given bulk theory, and they can be categorized into universality classes much like the case of the bulk theory. Each universality class of boundary conditions are characterized by a boundary fixed point [28] which is invariant under conformal transformations. Various conformal fixed points can be connected by renormalization group trajectories. The conformal boundary conditions for \(c = \frac{7}{10}\) may be classified following the work in [2]. Conformal invariance of the boundary is equivalent to the requirement that the two components \(T\) and \(\bar{T}\) of the stress tensor be equal on the boundary. If we denote a conformal boundary state by \(|a\rangle\), then it must be annihilated by \((L_n - \bar{L}_n)\) for each \(n\). Solutions to this constraint were given in [29], and
they are linear combinations of the states

$$|j> = \sum_N |j, N > \otimes |\tilde{N} >,$$

(16)

where $j$ labels a highest weight representation of the algebra of the $L_n$, and $|j, N >$ is an orthonormal basis in this representation space. Similarly, $|j, \tilde{N} >$ forms an orthonormal basis in the $j$ representation of the algebra of $\tilde{L}_n$. 2 The boundary states corresponding to “physical” boundary conditions can be constructed following the procedure in [2] and have the form

$$|\tilde{0} > = C(|0 > + \eta |\frac{1}{16} > + \eta |\frac{3}{5} > + \sqrt{2}|\frac{7}{16} > + \sqrt{2}|\frac{3}{80} >);$$

$$|\frac{1}{16} > = C[\eta^2|0 > - \eta^{-1}|\frac{1}{10} > - \eta^{-1}|\frac{3}{5} > + \eta^2|\frac{3}{2} > - \sqrt{2}\eta^2|\frac{7}{16} > + \sqrt{2}\eta^{-1}|\frac{3}{80} >];$$

$$|\frac{3}{5} > = C[\eta^2|0 > - \eta^{-1}|\frac{1}{10} > - \eta^{-1}|\frac{3}{5} > + \eta^2|\frac{3}{2} > + \sqrt{2}\eta^2|\frac{7}{16} > - \sqrt{2}\eta^{-1}|\frac{3}{80} >];$$

$$|\frac{3}{2} > = C(|0 > + \eta |\frac{1}{16} > + \eta |\frac{3}{5} > + |\frac{3}{2} > - \sqrt{2}|\frac{7}{16} > - \sqrt{2}|\frac{3}{80} >);$$

$$|\frac{3}{80} > = \sqrt{2}C[\eta^2|0 > - \eta |\frac{1}{10} > + \eta |\frac{3}{5} > - |\frac{3}{2} >] = \frac{1}{\sqrt{2}}[D(|\tilde{0} > + |\frac{3}{2} >)];$$

$$|\frac{3}{5} > = \sqrt{2}C[\eta^2|0 > + \eta^{-1}|\frac{1}{10} > - \eta^{-1}|\frac{3}{5} > - \eta^2|\frac{3}{2} >] = \frac{1}{\sqrt{2}}[D(|\frac{1}{10} > + |\frac{3}{80} >)],$$

(17)

where $C = \sqrt{\frac{\sin \frac{\pi}{5}}{\sqrt{\sqrt{5}}}}$ and $\eta = \sqrt{\frac{\sin \frac{\pi}{5}}{\sin \frac{2\pi}{5}}}$. Here $D$ denotes the Kramers-Wannier duality transform. Thus we can regard the state $|\frac{3}{10} >$ as the dual boundary condition of $|\tilde{0} >$ and $|\frac{3}{2} >$. Similarly $|\frac{3}{80} >$ is dual to $|\frac{3}{5} >$ and $|\frac{1}{10} >$.

To interpret the boundary conditions associated with the states in (17), it will be illustrative to refer to the $A_4$ RSOS lattice realization of this model [21]. Consider a diagonal square lattice where the degree of freedom at each site can take on four values
$l_i = 1, 2, 3$ and 4. The value of $l_i$ is further constrained by the requirement $|l_i - l_j| = 1$ for nearest neighbor sites $i$ and $j$. In addition there is a ferromagnetic next-nearest-neighbor interaction and a single-site interaction. Thus the lattice divides into two sublattices I and II, where the values of $l_i$ are odd in one sublattice, and even in the other sublattice. It is clear that there are three degenerate ground states as shown in Fig.9. To make contact with the order parameter $<\sigma>$ of the Ising model with vacancies (1), we shall label the ground states in Fig.9 by $-1, 0$ and $+1$. The Hamiltonian of the RSOS model is symmetric under the global $Z_2$ transform $l_i \rightarrow 5 - l_i$, which corresponds to the spin-reversal symmetry of the model (1). The three ground states will become identical in the bulk at the tricritical point. However the nature of the order parameter at the boundary will depend on the specific boundary condition. As shown in [2], the state $|\tilde{\Delta}(1, s)\rangle$ corresponds to the boundary condition where all boundary degrees of freedom are fixed to the value $s$, this is illustrated in Fig.10. For the state $|\tilde{0}\rangle$, the boundary degrees of freedom are fixed to 1, and hence the neighboring state must be in the state 2. Thus in the continuum limit, the order parameter at the boundary will be in the $-1$ vacuum, and we shall denote this boundary condition by $(-)$. The boundary state $|\tilde{1}\rangle$ corresponds to fixing the the boundary degree of freedom to 2, and the neighboring sites can be in either states 1 or 3. In this case the order parameter at the boundary may be in the $-1$ or 0 vacuum. This boundary condition where the $-1$ and 0 vacua are degenerate will be labeled $(-0)$. We can similarly associated $|\tilde{2}\rangle$ with the boundary condition $(0+)$ where the 0 and $+1$ vacua are degenerate at the boundary. In the same way, the boundary condition $(+)$ where the order parameter is fixed to the $+1$ vacuum will correspond to the boundary state $|\tilde{3}\rangle$. One
can regard (−) and (+) as “fixed” boundary conditions for the Ising model with vacancies (1).

In the work of [30], they identified the boundary states |Δ_{(r,1)}⟩ as the boundary condition where the boundary degrees of freedom are fixed to the value  \( r \), while the neighboring sites must be in the state  \( r + 1 \) (for  \( 1 \leq r \leq 3 \)). For the boundary state |\tilde{\Delta}_{16}⟩, this means that the boundary will be in state 2 and the neighboring sites will be in state 3. This of course fixes the order parameter to be in the vacuum 0 in the continuum limit, and we denote this the (0) boundary condition. A few comments about this boundary condition is in order. Using the states in (17), the partition function of the model on the cylinder with boundaries at both ends can be computed. Let us define the modular parameter as 
\[ q \equiv e^{2\pi i \tau} \]
where \( \tau = \frac{iR}{2L} \) on a cylinder of length \( L \) and circumference \( R \). The partition functions involving the boundary condition (0) include

\[ Z_{(0)(0)}(q) = \chi_0(q) + \chi_{\frac{3}{2}}(q), \quad (18a) \]

and

\[ Z_{(-)(0)}(q) = \chi_{\frac{7}{16}}(q). \quad (18b) \]

The result (18a) was conjectured earlier in [31] for the Tricritical Ising partition function with microscopic “free” boundary spins, and later confirmed by direct calculation [30] and numerically in [32, 33]. Furthermore (18b) was also obtained in [33] by considering the partition function with “mixed” boundary condition, ie microscopic “fixed” and “free” boundary conditions. Thus we concluded that the boundary condition (0) is associated with the lattice Tricritical Ising model where the microscopic boundary spins are not
constrained. Such lattice boundary conditions are often referred to in the literature as the “free” boundary condition, even in the conformal field theory.

Lastly we consider the conformal boundary state $| \tilde{3}_{80} \rangle$ which has not previously been identified. Note that it is dual to the boundary conditions $(-0)$ and $(0+)$, and we conjecture that it corresponds to the case where the boundary can exist in all three vacua. In other words, like in the bulk theory, the three vacua are also degenerate on the boundary. We shall call such boundary condition “degenerate” and label it as $(d)$. Since a great deal of “fine-tuning” of the boundary parameters are required to achieve three-fold degeneracy of the vacua, we expect the boundary condition $(d)$ to be unstable under perturbation by relevant boundary operators. This boundary condition carries the $\tilde{3}_{80}$ representation of the Virasoro algebra, so the boundary operators that can appear along such a boundary is given by the Operator Product Expansion of the primary field $\Phi_{(\tilde{3}_{80})}$ with itself [18]. From the operator product

$$\Phi_{(\tilde{3}_{80})}\Phi_{(\tilde{3}_{80})} = [\Phi_{(0)}] + [\Phi_{(\frac{1}{10})}] + [\Phi_{(\frac{2}{5})}] + [\Phi_{(\frac{3}{2})}],$$

we see that two relevant boundary fields $\psi_{(\frac{1}{10})}$ and $\psi_{(\frac{3}{2})}$ can appear for this boundary condition. Perturbing by these relevant operators could generate renormalization group flow from the boundary condition $(d)$ down to a more “stable” boundary fixed point. To lend support for this speculation, we computed the boundary ground state degeneracy (the so-called “g-factor” [34]) from the boundary states (17) and they are in order of decreasing magnitude

$$g(d) = \sqrt{2}\eta^2 C; \quad g(-0) = g(0+) = \eta^2 C,$$

$$g(0) = \sqrt{2}C; \quad \text{and} \quad g(-) = g(+) = C.$$
The value of $g$ should decrease along renormalization group trajectories, and in this respect, $(d)$ would be the most "unstable" conformal boundary condition. Similar comments can be made about the two-fold degenerate boundary conditions $(-0)$ and $(0+)$ with the next-largest value of $g$. One can repeat the analysis for the associated boundary operator for these boundary conditions, and we found that they both admit the relevant boundary operator $\psi_{(\frac{\pi}{4})}$. Thus we expect that the perturbation of $(-0)$ and $(0+)$ by this operator would generate RG flow down to other conformal boundary conditions. This is studied in the next section.

6. Boundary S-matrix for the Conformal Boundary Conditions

Boundary conditions can in general be changed by altering the coupling constants of model on the boundary. A simple example would be the introduction of a "magnetic field" which favors a particular direction of lattice spins. In the continuum limit, this magnetic field would shift the specific free energy of one of the vacua and destroy the ground state degeneracy. Let us consider the massive Tricritical Ising model with the conformal boundary condition $(-0)$ and perturbed by the relevant boundary operator $\psi_{(\frac{\pi}{4})}$. The symbolic action for the corresponding field theory is (8) with

$$H_{\frac{\pi}{4}+C BC} = H_{\frac{\pi}{4}+(-0); \Phi_B = h\psi_{(\frac{\pi}{4})}},$$ (19)

and $h$ is some boundary coupling constant with dimension $[length]^{-\frac{1}{2}}$. Since the boundary operator has the same dimension as the bulk perturbing operator $\Phi_{(\frac{\pi}{4})}$, this boundary perturbation is integrable [10]. The boundary field $\psi_{(\frac{\pi}{4})}$ couples to the boundary spins for
the boundary condition \((-0)\) and would alter the specific free energy of the \(-1\) vacuum. Thus for \(h \neq 0\), the ground state will no longer be degenerate.

To be more specific, let us treat the case of a RG flow from \((-0)\) down to the “fixed” boundary condition \((-)\). For \(h > 0\) we expect the vacuum \(-1\) to be the ground state of the boundary, while the vacuum \(0\) is an excited state. The associated boundary scattering S-matrix will involve the amplitude \(P_-(\theta)\), and since the boundary cannot exist in the vacuum \(+1\), we have \(k_- = k_+ = 0\) in this case. The Yang-Baxter equation (11) becomes an identity for this boundary condition, and the amplitude \(P_-(\theta)\) will be determined by the unitarity conditions (13) and (14). The solution can be written in the form

\[
P_-(\theta) = P_{\xi CDD}^C(\theta)P_{\min}(\theta),
\]

where \(P_{\min}(\theta)\) is the minimal solution of the equations

\[
P_{\min}(\theta)P_{\min}(-\theta) = 1; \quad (21a)
\]

\[
P_{\min}(\frac{i\pi}{2} - \theta) = B_0(2\theta)P_{\min}(\frac{i\pi}{2} + \theta), \quad (21b)
\]

with no poles in the physical strip, and has the expression

\[
P_{\min}(\theta) = \prod_{k=1}^{\infty} \frac{\Gamma(k - \frac{\theta}{2\pi i})^2 \Gamma(k - \frac{1}{4} + \frac{\theta}{2\pi i}) \Gamma(k + \frac{1}{4} + \frac{\theta}{2\pi i}) \Gamma(k + \frac{1}{4} - \frac{\theta}{2\pi i}) \Gamma(k - \frac{1}{4} - \frac{\theta}{2\pi i})}{\Gamma(k + \frac{\theta}{2\pi i})^2 \Gamma(k + \frac{1}{4} - \frac{\theta}{2\pi i}) \Gamma(k - \frac{1}{4} + \frac{\theta}{2\pi i})}.
\]

The other factor

\[
P_{\xi CDD}^C(\theta) = \frac{\sin\xi - i\sinh\theta}{\sin\xi + i\sinh\theta}, \quad (23)
\]

is a CDD factor [35] which exhibits a pole at \(\theta = i\xi\). When this pole lies inside the “physical strip”, \(0 \leq \xi \leq \frac{\pi}{2}\), it could be interpreted as a “boundary bound state” as shown
in Fig.11. If we denote $e_a$ to be the specific energy (per unit boundary length) of the boundary state $|B_a>$, then

$$e_0 - e_{-1} = m \cos \xi,$$

(24)
defines the binding energy of the bound state. Clearly the parameter $\xi$ is related to the boundary coupling constant $h$, and we can analyzes the effect of the RG flow by varying the value of $\xi$. Furthermore we can consider scattering of this bound state with other kinks by the “boundary bound state bootstrap equation” [10] ³

$$R_+(\theta) = B_1(\theta - i \xi)P_-(\theta)B_1(\theta + i \xi);$$  

(25a)

and

$$R_-(\theta) = B_0(\theta - i \xi)P_-(\theta)B_0(\theta + i \xi),$$  

(25b)
as illustrated in Fig.12. This gives two more scattering amplitudes which can occur when the boundary is in the vacuum state 0. Equation (25) can be simplified to a form resembling the general solution (12):

$$R_+(\theta) = \frac{1}{2}(\cos\frac{\xi}{2} + ish\frac{\theta}{2})B(\theta - i \xi)B(\theta + i \xi)P_{CDD}^\xi(\theta)P_{\min}(\theta);$$  

(26a)

and

$$R_-(\theta) = \frac{1}{2}(\cos\frac{\xi}{2} - ish\frac{\theta}{2})B(\theta - i \xi)B(\theta + i \xi)P_{CDD}^\xi(\theta)P_{\min}(\theta).$$  

(26b)

For the boundary condition $(0)$ the vacua $-1$ and 0 are degenerate on the boundary and we expect the boundary to admit a bound state at $\theta = \frac{i \pi}{2}$. Thus the amplitudes (20) and

³ Similar “fusion” procedure was used in [36] to derive the boundary S-matrix for the Sine-Gordon model at the supersymmetric points.
with \( \xi = \frac{\pi}{2} \) constitute the boundary S-matrix for this unperturbed boundary condition \((-0)\). When we “turn on” the boundary perturbation \((h > 0)\), \((e_0 - e_{-1}) > 0\) and the amplitude \(P_-(\theta)\) will exhibit a pole in the physical strip at \(\theta = i\xi\) for \(0 < \xi < \frac{\pi}{2}\). One can think of the bound state as formally “gluing” a kink \(K_{0,-1}\) of rapidity \(i\xi\) (interpreted as a fluctuating domain wall) to the boundary \(|B_{-1} >\). As we increase the perturbation, \(\xi \to 0\), the bound state becomes weakly bound and the domain wall develops large fluctuations which propagate well into the bulk. At \(\xi = 0\), the domain wall has zero rapidity and is no longer bounded to the boundary.

For \(-\frac{\pi}{2} < \xi \leq 0\), the pole leaves the physical strip. In this regime \((e_0 - e_{-1}) > m\), so there is no “boundary bound state” and \(-1\) remains as the only stable ground state of the boundary. At \(\xi = -\frac{\pi}{2}\), the poles and zeros of \(P_{\xi}^{CDD}(\theta)\) become two-fold degenerate. As we increase \(h\) further, \(\xi\) develops an imaginary component: \(\xi = -\frac{\pi}{2} + i\vartheta\) and the amplitude \(P_-(\theta)\) exhibits two poles at \(\theta = -\frac{i\pi}{2} \pm \vartheta\). These poles depart to infinity when \(h\) gets increasingly large. In this limit, the energy difference between the two vacua \((e_0 - e_{-1})\) becomes infinitely large and we expect the resultant boundary condition to be the “fixed” case \((-\)\). Thus the amplitude \(P_-\) will become the boundary S-matrix for the \((-\)) boundary condition in this limit, and we found

\[
R_{(-)}(\theta) = P_{min}(\theta). \tag{27}
\]

It is not surprising that this “flow” from a two-fold degenerate boundary condition down to a unique ground state resembles closely the “flow” from free to fixed boundary condition in the Ising model [10].

It is interesting to consider the perturbed action \((8)\) for \(h < 0\). In this case we expect
the perturbation to “raise” the specific boundary energy of the $-1$ vacuum, and the vacuum 0 would now become the unique ground state. This situation corresponds to the analytic continuation of $\xi$ above the degenerate point $\frac{\pi}{2}$. In the regime $\frac{\pi}{2} < \xi < \pi$, the difference in boundary energies become

$$ (e_{-1} - e_{0}) = m\cos\zeta > 0, \quad (28) $$

where $\zeta = \pi - \xi$. Hence we can regard the $-1$ vacuum as a boundary excited state and it could appear as a boundary bound state in the scattering S-matrix. Indeed one can check that the amplitude $R_-(\theta)$ now possesses a bound state pole in the physical strip at $\theta = i\zeta$. The other amplitude $R_+(\theta)$ does not have this pole, consistent with the fact that the boundary cannot be in the vacuum state $+1$. The analysis in this regime mirrors that of the domain $0 \leq \xi \leq \frac{\pi}{2}$, except that the role of $-1$ and 0 are interchanged. At $\xi = \pi$, the bound state becomes unbounded and 0 emerges as the only stable boundary ground state. As we decrease the value of $h$ further such that $\xi > \pi$, $R_+(\theta)$ and $R_-(\theta)$ are the only physical boundary scattering amplitudes and there is no more bound state pole. When the RG flow reaches the point for which $\xi = 2\pi$, the poles and zeros in the factor $(\cos\frac{\xi}{2} \pm ish\frac{\theta}{2})B(\theta + \xi)B(\theta - \xi)P_{\xi}^{CD}(\theta)$ (common to both $R_+$ and $R_-$) become two-fold degenerate. Further perturbation would introduce an imaginary component to $\xi$: $\xi = 2\pi + i\varphi$. The amplitudes $R_+(\theta)$ and $R_-(\theta)$ would correspondingly exhibit two poles at $\theta = 2\pi i \pm \varphi$. These poles will separate and depart to infinity when $h$ approach negative infinity. Under this RG flow, we expect the boundary specific energy $e_{-1}$ to become increasingly large, and thus the boundary will be “fixed” to the stable ground state 0. This is the $(0)$ boundary condition and in this limit $R_+$ and $R_-$ will both be
reduced to the boundary S-matrix for (0):

\[ R_{(0)}(\theta) = e^{-2\gamma \theta} P_{\text{min}}(\theta). \]  

The RG flows from (−0) boundary condition down to (−) and (0) are shown schematically in Fig.13. Of course we could also consider the perturbation of the (0+) boundary condition by \( \psi_{\frac{3}{5}} \) and we expect it to generate the same flow pattern and conformal boundary S-matrices. It is not easy to relate the perturbation coupling \( h \) to the S-matrix parameter \( \xi \). However, based upon the pattern of flow of the scattering amplitudes, we conjecture the relation in this case to be

\[ h \sim m^2 \left[ \cos \left( \frac{2\xi}{5} + \frac{\pi}{5} \right) - \cos \frac{2\pi}{5} \right]. \]  

As shown in Fig.13, by applying the duality transformation to the RG flows from (−0), we obtained the RG flows from the (d) boundary condition down to either (0) or a superposition of (−) and (+). Note that the field \( \psi_{\frac{3}{5}} \) does not change sign under duality. The symbolic action of this dual theory is (8), where \( H_{\frac{3}{5}+C BC} = H_{\frac{3}{5}+(d)} \) is the CFT action with conformal boundary condition (d). For this new RG flow, the point \( h = 0 \) corresponds to the boundary condition (d) where all three vacua are degenerate. This boundary condition respects the full symmetry of the bulk theory, so the associated boundary S-matrix should be invariant under spin-reversal. Thus we have \( A = Y = 0 \) and \( k_+ = k_- \) in the general solution (12), and denotes

\[ R_+ = R_- = R; \quad P_+ = P_- = P \quad \text{and} \quad V_+ = V_- = V. \]

When this conformal boundary condition is perturbed with \( h > 0 \), the boundary energies \( e_+ \) and \( e_- \) are both higher than \( e_0 \), and hence the vacua −1 and +1 are the excited states
of the boundary. Here we shall assume the perturbation does not destroy the $Z_2$ spin-reversal symmetry of $(d)$. The amplitude $R(\theta)$ can be obtained by solving the unitarity constraints (13) and (14). It must exhibit a bound state pole in the physical strip, say at $i\nu$, corresponding to these excited boundary states, and has the form

$$R(\theta) = e^{-2\gamma \theta} P_v^{CDD}(\theta) P_{\text{min}}(\theta);$$

(31)

where $P_v^{CDD}(\theta)$ is the CDD factor (23). The other amplitudes are again related to $R(\theta)$ by the “boundary bound state bootstrap equation”

$$P(\theta) = [A_0(\theta - \nu) A_0(\theta + \nu) + A_1(\theta - \nu) A_1(\theta + \nu)] R(\theta),$$

(32a)

and

$$V(\theta) = [A_0(\theta - \nu) A_1(\theta + \nu) + A_1(\theta - \nu) A_0(\theta + \nu)] R(\theta).$$

(32b)

The expressions for $P$ and $V$ can be simplified to

$$P(\theta) = c_h \frac{\nu}{2} e^{-2\gamma \theta} A(\theta - \nu) A(\theta + \nu) P_v^{CDD}(\theta) P_{\text{min}}(\theta);$$

(33a)

$$V(\theta) = -i s_h \frac{\theta}{2} e^{-2\gamma \theta} A(\theta - \nu) A(\theta + \nu) P_v^{CDD}(\theta) P_{\text{min}}(\theta),$$

(33b)

which are the general solution (12) with $X = c_h \frac{\nu}{2}$ and $k_+ = k_- = -i$.

For the boundary condition $(d)$, the three vacua are degenerate and all scattering amplitudes should possess a pole at $\theta = \frac{i\pi}{2}$. The associated boundary S-matrix is given by (31) and (33) with $\nu = \frac{\pi}{2}$. Perturbing this boundary condition in (8) with $h > 0$ would lower $e_0$, while the other vacua $-1$ and $+1$ will appear as bound states in the amplitude $R$ at $\theta = i\nu$. In the domain $-\frac{\pi}{2} < \nu < 0$, there is no bound state, and $R$ remains as the only physical amplitude. Under further RG flow, $\nu$ develops an imaginary component
\( v = -\frac{\pi}{2} + iv_0 \) and travels to infinity. One can check that in this limit, \( R \) reduces to the amplitude \( R_{(0)} \) for the boundary condition (0).

Perturbing \( (d) \) with \( h < 0 \) raises the energy \( e_0 \) above \( e_+ \) and \( e_- \). For \( \frac{\pi}{2} < v < \pi \) the vacuum 0 enters as the bound state in \( P \) and \( V \). In the regime \( \pi < v < 2\pi \), the bound state leaves the physical strip, with +1 and −1 remaining as the only stable boundary ground states. At \( v = 2\pi \), the poles and zeros of the \( v \)-dependent factors of both \( P \) and \( V \) become two-fold degenerate. Further RG flow would introduce an ever increasing imaginary part to \( v \), i.e. \( v = 2\pi + iv' \), and drives it to infinity. In this limit, the amplitude \( V \) approaches zero, while \( P \) flows to the “fixed” boundary condition amplitude (27). This means that the scattering theory decouples into two equivalent sectors, and the boundary becomes a superposition of the pure boundary states \( |B_{+1} \rangle \) and \( |B_{-1} \rangle \). Since this RG flow pattern is the same as the flow from the \((-0)\) boundary condition, we expect the parameter \( v \) to be related to the coupling \( h \) in the same manner as (30).

7. Discussion

In this work we computed the boundary S-matrix associated with the conformal boundary conditions of the Tricritical Ising model. The interpolating S-matrix which describe the renormalization group flow between various boundary conditions was also found. However we have not yet found an integrable boundary perturbation which give rise to the general S-matrix (12). It would be interesting to determine this more general perturbation. Since the general solution (12) contains more than one boundary param-
eter, it is conceivable that the associated perturbation involves more than one boundary operator, and some of them would break the spin-reversal symmetry. Another interesting follow-up to this work would be to consider the massless limits of the S-matrix. Since the interpolating S-matrix contains a boundary parameter, its massless limit might be non-trivial. This information would be relevant to the interpretation of the $c = \frac{7}{10}$ conformal field theory as a massless scattering theory [16, 37].

Finally, we have not discussed in this work the possibility of a supersymmetric boundary condition [36]. Since the degenerate boundary condition (d) respects the full symmetry of the bulk theory, it is plausible that it might also preserve supersymmetry. One would like to check if any other conformal boundary states (or a linear combination of them) could also be supersymmetric. Furthermore, if some of the conformal boundary states do possess supersymmetry, it would be interesting to see if it could survive under perturbation, for example, by the boundary field $\psi(\frac{3}{5})$. At the level of the boundary scattering theory, one could check supersymmetry by determining the appropriate linear combination of supercharges $Q$ and $\bar{Q}$ which survive as an integral of motion. This combination should commute with the supersymmetric boundary S-matrix, and could possibly involve the topological charge of the massive theory [36].

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Figure Captions

Fig.5.1. Phase diagram of the Tricritical Ising model.

Fig.5.2. Landau-Ginzburg effective potential for the various regions of the phase diagram.

Fig.5.3. Kac table of conformal dimensions for $c = \frac{7}{10}$.

Fig.5.4. Space-time diagrams of the four variety of the kinks (a) and their bulk scattering amplitudes (b).

Fig.5.5. Space-time diagrams for the General Boundary Scattering Amplitudes.

Fig.5.6. Figure of the “Boundary Yang-Baxter” equation satisfied by the boundary S-matrix. Here $a, b, c \in \{+1, -1\}$.

Fig.5.7. Space-time diagram for the Boundary Unitarity equation with $a, d \in \{+1, -1\}$.

Fig.5.8. Figures of the Cross-Unitarity condition with $a, b \in \{+1, -1\}$.

Fig.5.9. The three ground states of the RSOS model and the associated vacuum states in the continuum limit.

Fig.5.10. Various conformal conditions of the Tricritical Ising model in terms of the RSOS picture.

Fig.5.11. Bound state pole exhibit by the amplitude $P_-$.

Fig.5.12. Figure of the “Boundary Bound State Bootstrap” equation with $a \in \{+1, -1\}$.

Fig.5.13. Schematic diagram for the renormalization group flow from the $(−0)$ and $(d)$ boundary conditions. Here $D$ denotes duality transformation.

Fig.5.14. Figure of the “Boundary Bound State Bootstrap” equation for the pertur-
bation of the degenerate boundary condition. Here $a, b \in \{+1, -1\}$. 
Fig. 1
Fig. 2
Fig. 5
\[ \sum_s \theta_1 \theta_2 - \theta_2 - \theta_1 = \sum_s \theta_1 \theta_2 \]

Fig. 6
\[ \sum_{s} \delta^{d}_{a} = 1 \]

**Fig. 7**

\[ \sum_{s} \delta^{d}_{a} = 1 \]

**Fig. 8**
