DEFORMATION OF SEMICIRCULAR AND CIRCULAR LAWS VIA p-ADIC NUMBER FIELDS AND SAMPLING OF PRIMES

Ilwoo Cho and Palle E.T. Jorgensen

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Abstract. In this paper, we study semicircular elements and circular elements in a certain Banach $*$-probability space $(\mathfrak{LS}, \tau^0)$ induced by analysis on the $p$-adic number fields $\mathbb{Q}_p$ over primes $p$. In particular, by truncating the set $\mathcal{P}$ of all primes for given suitable real numbers $t < s$ in $\mathbb{R}$, two different types of truncated linear functionals $\tau_{t_1 < t_2}$, and $\tau^+_{t_1 < t_2}$ are constructed on the Banach $*$-algebra $\mathfrak{LS}$. We show how original free distributional data (with respect to $\tau^0$) are distorted by the truncations on $\mathcal{P}$ (with respect to $\tau_{t < s}$, and $\tau^+_{t < s}$). As application, distorted free distributions of the semicircular law, and those of the circular law are characterized up to truncation.

Keywords: free probability, primes, $p$-adic number fields, Banach $*$-probability spaces, semicircular elements, circular elements, truncated linear functionals.

Mathematics Subject Classification: 11R56, 46L54, 47L30, 47L55.

1. INTRODUCTION

The main purposes of this paper are (i) to construct semicircular elements induced by analysis on the $p$-adic number fields $\mathbb{Q}_p$ over primes $p$, in a certain Banach $*$-probability space $(\mathfrak{LS}, \tau^0)$, (ii) to establish other types of linear functionals $\tau_{t < s}$, and $\tau^+_t < s$ on the Banach $*$-algebra $\mathfrak{LS}$ for suitable real numbers $t < s$ in $\mathbb{R}$, truncating the set $\mathcal{P}$ of all primes, and (iii) to study how our truncations of (ii) affect, or distort the original free-distributional data on $(\mathfrak{LS}, \tau^0)$. To do that, we restrict our interests to the Banach $*$-subalgebra $\mathbb{LS}$ of $\mathfrak{LS}$, generated by the semicircular elements of (i), and the corresponding Banach $*$-probabilistic sub-structure $(\mathbb{LS}, \tau^0)$. Our main results, in particular, characterize how the semicircular law, and the circular law are distorted by our truncations on $\mathcal{P}$. 
In [10] and [6], we constructed and studied *weighted*-semi-circular elements and *semi-circular elements* induced by \( p \)-adic number fields \( \mathbb{Q}_p \), for all \( p \in \mathcal{P} \). We showed there that \( p \)-adic number theory provides *weighted*-semi-circular laws, and the semi-circular law. In this paper, certain “truncated” free-probabilistic information of the free probability of [6] is studied.

1.1. PREVIEW AND MOTIVATION

Relations between *primes* and *operators* have been studied. For instance, we considered in [5] and [4] how primes act on certain *von Neumann algebras* generated by \( p \)-adic and Adelic *measure spaces* as operators. In [3] and [9], primes are regarded as *linear functionals* acting on *arithmetic functions*. Independently, in [8], we studied free-probabilistic structures on *Hecke algebras* \( \mathcal{H}(GL_2(\mathbb{Q}_p)) \), for primes \( p \) (e.g., [2] and [26]). Number-theoretic results motivated such earlier works (see e.g., [11,12], [13–20,23], and [28]).

In [10], the authors constructed \((\text{weighted})\)-semi-circular elements in a certain Banach *-algebra* \( \mathfrak{LS}_p \) induced by the *-algebra* \( \mathcal{M}_p \) of *measurable functions* on a \( p \)-adic number fields \( \mathbb{Q}_p \), for a prime \( p \in \mathcal{P} \). In [6], the first-named author constructed the free product Banach *-probability space* \((\mathfrak{LS}, \tau_0)\) of the Banach *-algebras* \( \{\mathfrak{LS}_p\}_{p \in \mathcal{P}} \) of [10], and studied \((\text{weighted})\)-semi-circular elements of \( \mathfrak{LS} \) as free generators. As application, the asymptotic semi-circular laws “over \( \mathcal{P} \)” are considered in [7].

To make this paper be as self-contained as possible, some main results from [6] will be re-considered below, in short Sections 1 through 7. In this paper, we are interested in the cases where the free product linear functional \( \tau_0 \) on \( \mathfrak{LS} \) of [6] is truncated over \( \mathcal{P} \). How such truncations affect, or distort, the original free-distributional data? Especially, how such truncations distort the semi-circular law on \( \mathfrak{LS} \)? The answers to these questions constitute major parts of our main results. As application, we characterize how our truncations distort the circular law on \( \mathfrak{LS} \).

1.2. OVERVIEW

In Sections 2, we briefly introduce backgrounds of our works. In the short Sections 3 through 7, we construct our Banach *-probability space* \((\mathfrak{LS}, \tau_0)\), and study \((\text{weighted})\)-semi-circular elements induced from \( p \)-adic analysis on \( \mathbb{Q}_p \), for primes \( p \).

In Section 8, we define a free-probabilistic sub-structure \( \mathfrak{LS}_0 = (\mathbb{L} \mathfrak{S}, \tau_0) \) of \((\mathfrak{LS}, \tau_0)\), generated by the free reduced words of \( \mathfrak{LS} \), having “non-zero” free distributions, and study free-probabilistic properties on \( \mathbb{L} \mathfrak{S}_0 \); and then, construct truncated linear functionals of \( \tau_0 \) on \( \mathbb{L} \mathfrak{S} \) to study how free-probabilistic data of such free reduced words are distorted from our truncations on primes, in Section 9.

In Section 10, we provide a different type of truncated linear functionals on \( \mathbb{L} \mathfrak{S} \) over \( \mathcal{P} \) under direct product, and investigate new free-probabilistic structures on \( \mathbb{L} \mathfrak{S} \). Remark that the truncated free probabilistic structures of \( \mathbb{L} \mathfrak{S}_0 \) in Sections 9 and 10 are totally different from each other.

In Section 11, to distinguish-and-emphasize the differences between them, we provide some applications of our main results of Sections 8, 9 and 10; by taking
truncated linear functionals of Sections 9 and 10 on LS. In particular, we show how the circular law is distorted (or affected) by the truncations on $\mathcal{P}$.

Independently, in Section 11, a new type of free random variables is introduced. A free random variable $x$ is said to be followed by the semicircular law in a topological $\ast$-probability space $(A, \psi)$, if (i) $x$ is not self-adjoint, as an operator, and (ii) the free distribution of $x$ is characterized by the joint free moments of $x$ and its adjoint $x^\ast$, satisfying
\[
\psi(x^{r_1}x^{r_2}\ldots x^{r_n}) = \omega_n c_n^2,
\]
for all $(r_1, \ldots, r_n) \in \{1, \ast\}^n$, for all $n \in \mathbb{N}$, where
\[
\omega_k = \begin{cases} 1 & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd,} \end{cases}
\]
and
\[
c_k = \text{the } k\text{-th Catalan number } = \frac{(2k)!}{k!(k+1)!}
\]
for all $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2. PRELIMINARIES
In this section, we offer about background for our work.

2.1. FREE PROBABILITY
For basic free probability, see [27] and [29] (and the cited papers therein). Free probability is the noncommutative operator-algebraic version of classical measure theory (including probability theory) and statistical analysis. As an independent branch of operator algebra theory, it has various applications not only in functional analysis (e.g., [21], [22, 24] and [25]), but also in related fields (e.g., [1] through [10]).

We here use combinatorial free probability of Speicher (e.g., [27]). In the text, without introducing detailed definitions and combinatorial backgrounds, free moments and free cumulants of operators will be computed to verify the free distributions of them. Also, we use free product of $\ast$-probability spaces, without precise introduction.

2.2. ANALYSIS ON $\mathbb{Q}_p$
For more about $p$-adic analysis and Adelic analysis, see e.g., [14, 17, 23, 29] and [28]. In this paper, we use same definitions, and notations of [28]. Let $p \in \mathcal{P}$ be a prime, and let $\mathbb{Q}$ be the set of all rational numbers. Define a non-Archimedean norm $|\cdot|_p$ on $\mathbb{Q}$ by
\[
|x|_p = \left| p^{k/a} \frac{b}{b_p} \right|_p = \frac{1}{p^k},
\]
whenever $x = p^k a/b$, where $k, a \in \mathbb{Z}$, and $b \in \mathbb{Z} \setminus \{0\}$. We call $|\cdot|_p$, the $p$-norm on $\mathbb{Q}$ (as in [28]), for all $p \in \mathcal{P}$. 
The $p$-adic number field $\mathbb{Q}_p$ is the maximal $p$-norm closures in $\mathbb{Q}$, i.e., under norm topology, the set $\mathbb{Q}_p$ forms a Banach space, for $p \in \mathcal{P}$.

All elements $x$ of $\mathbb{Q}_p$ are expressed by

$$x = \sum_{k=-N}^{\infty} x_k p^k, \text{ with } x_k \in \{0, 1, \ldots, p-1\},$$

for $N \in \mathbb{N}$, decomposed by

$$x = -1 \sum_{l=-N}^{-1} x_l p^l + \sum_{k=0}^{\infty} x_k p^k.$$

If $x = \sum_{k=0}^{\infty} x_k p^k$ in $\mathbb{Q}_p$, then we call $x$, a $p$-adic integer. Remark that, $x \in \mathbb{Q}_p$ is a $p$-adic integer, if and only if $|x|_p \leq 1$. So, by collecting all $p$-adic integers in $\mathbb{Q}_p$, one can define the unit disk $\mathbb{Z}_p$ of $\mathbb{Q}_p$.

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}.$$

Under the $p$-adic addition and the $p$-adic multiplication of [28], this Banach space $\mathbb{Q}_p$ forms a field algebraically, i.e., $\mathbb{Q}_p$ is a Banach field.

One can view this Banach field $\mathbb{Q}_p$ as a measure space,

$$\mathbb{Q}_p = (\mathbb{Q}_p, \sigma(\mathbb{Q}_p), \mu_p),$$

where $\sigma(\mathbb{Q}_p)$ is the $\sigma$-algebra of $\mathbb{Q}_p$, consisting of all $\mu_p$-measurable subsets, where $\mu_p$ is a left-and-right additive invariant Haar measure on $\mathbb{Q}_p$, satisfying

$$\mu_p(\mathbb{Z}_p) = 1.$$

If we define

$$U_k = p^k \mathbb{Z}_p = \{p^k x \in \mathbb{Q}_p : x \in \mathbb{Z}_p\},$$

(2.1)

for all $k \in \mathbb{Z}$, then these $\mu_p$-measurable subsets $U_k$’s of (2.1) satisfy

$$\mathbb{Q}_p = \bigcup_{k \in \mathbb{Z}} U_k,$$

and

$$\mu_p(U_k) = \frac{1}{p^k} = \mu_p(x + U_k), \text{ for all } x \in \mathbb{Q}_p,$$

and

$$\ldots \subset U_2 \subset U_1 \subset U_0 = \mathbb{Z}_p \subset U_{-1} \subset U_{-2} \subset \ldots,$$

(2.2)

i.e., the family $\{U_k\}_{k \in \mathbb{Z}}$ of (2.1) forms a basis of the topology for $\mathbb{Q}_p$ (e.g., [28]).

Define now subsets $\partial_k \in \sigma(\mathbb{Q}_p)$ by

$$\partial_k = U_k \setminus U_{k+1}, \text{ for all } k \in \mathbb{Z}.$$  

(2.3)
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We call such \( \mu_p \)-measurable subsets \( \partial_k \) of (2.3), the \( k \)-th boundaries (of \( U_k \)) in \( \mathbb{Q}_p \), for all \( k \in \mathbb{Z} \). By (2.2) and (2.3), one obtains that

\[
\mathbb{Q}_p = \bigsqcup_{k \in \mathbb{Z}} \partial_k,
\]

and

\[
\mu_p(\partial_k) = \mu_p(U_k) - \mu_p(U_{k+1}) = \frac{1}{p^k} - \frac{1}{p^{k+1}},
\]

(2.4)

where \( \sqcup \) means the disjoint union, for all \( k \in \mathbb{Z} \).

Now, let \( \mathcal{M}_p \) be the (pure-algebraic) algebra,

\[
\mathcal{M}_p = \mathbb{C} \{ \chi_S : S \in \sigma(\mathbb{Q}_p) \},
\]

(2.5)

where \( \chi_S \) are the usual characteristic functions of \( \mu_p \)-measurable subsets \( S \) of \( \mathbb{Q}_p \).

So, \( f \in \mathcal{M}_p \) if and only if

\[
f = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S, \quad \text{with } t_S \in \mathbb{C},
\]

(2.6)

where \( \sum \) is the finite sum. Remark that the algebra \( \mathcal{M}_p \) of (2.5) forms a \( \ast \)-algebra over \( \mathbb{C} \), with its well-defined adjoint,

\[
\left( \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S \right)^* \overset{\text{def}}{=} \sum_{S \in \sigma(\mathbb{G}_p)} \bar{t}_S \chi_S,
\]

where \( t_S \in \mathbb{C} \) with their conjugates \( \bar{t}_S \) in \( \mathbb{C} \).

Let \( f \in \mathcal{M}_p \) be in the sense of (2.6) Then one can define the integral of \( f \) by

\[
\int_{\mathbb{Q}_p} f d\mu_p = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \mu_p(S).
\]

(2.7)

Remark that, by (2.5), the integral (2.7) is unbounded on \( \mathcal{M}_p \), i.e.,

\[
\int_{\mathbb{Q}_p} \chi_{\mathbb{Q}_p} d\mu_p = \mu_p(\mathbb{Q}_p) = \infty,
\]

(2.8)

by (2.2).

Note that, by (2.4), if \( S \in \sigma(\mathbb{Q}_p) \), then there exists a unique subset \( \Lambda_S \) of \( \mathbb{Z} \), such that

\[
\Lambda_S = \{ j \in \mathbb{Z} : S \cap \partial_j \neq \emptyset \},
\]

(2.9)

satisfying

\[
\int_{\mathbb{Q}_p} \chi_S d\mu_p = \sum_{j \in \Lambda_S} \int_{\mathbb{Q}_p} \chi_{S \cap \partial_j} d\mu_p = \sum_{j \in \Lambda_S} \mu_p(S \cap \partial_j)
\]
by (2.7)
\[
\leq \sum_{j \in \Lambda_S} \mu_p(\partial_j) = \sum_{j \in \Lambda_S} \left( \frac{1}{p^j} - \frac{1}{p^{j+1}} \right),
\]
(2.10)
by (2.4), for the subset \( \Lambda_S \) of \( \mathbb{Z} \) of (2.9).

Remark again that the right-hand side of (2.10) can be \( \infty \), for instance, \( \Lambda_{Q_p} = \mathbb{Z} \), e.g., see (2.4), (2.7) and (2.8). By (2.10), one obtains the following proposition.

**Proposition 2.1.** Let \( S, \sigma(Q_p) \), and let \( \chi_S \in \mathcal{M}_p \). Then there exist \( r_j \in \mathbb{R} \), such that
\[
0 \leq r_j = \frac{\mu_p(S \cap \partial_j)}{\mu_p(\partial_j)} \leq 1 \text{ in } \mathbb{R}, \text{ for all } j \in \Lambda_S,
\]
and
\[
\int_{Q_p} \chi_S d\mu_p = \sum_{j \in \Lambda_S} r_j \left( \frac{1}{p^j} - \frac{1}{p^{j+1}} \right).
\]
(2.11)

3. FREE-PROBABILISTIC MODELS ON \( \mathcal{M}_p \)

Throughout this section, fix a prime \( p \in \mathcal{P} \), and let \( Q_p \) be the corresponding \( p \)-adic number field, and let \( \mathcal{M}_p \) be the \(*\)-algebra (2.5). In this section, we establish a suitable free-probabilistic model on \( \mathcal{M}_p \). Remark that, since \( \mathcal{M}_p \) is a “commutative” \(*\)-algebra, free probability theory is not needed to be used-or-applied, but, for our purposes, we here construct a free-probability-theoretic model on \( \mathcal{M}_p \) under free-probabilistic language and terminology.

Let \( U_k \) be the basis elements (2.1), and \( \partial_k \), their boundaries (2.3) of \( Q_p \), i.e.,
\[
U_k = p^k \mathbb{Z}_p, \text{ for all } k \in \mathbb{Z},
\]
(3.1)
and
\[
\partial_k = U_k \setminus U_{k+1}, \text{ for all } k \in \mathbb{Z}.
\]

Define a linear functional \( \varphi_p : \mathcal{M}_p \to \mathbb{C} \) by the integration (2.7), i.e.,
\[
\varphi_p(f) = \int_{Q_p} f d\mu_p, \text{ for all } f \in \mathcal{M}_p.
\]
(3.2)
Then, by (2.11), one obtains that
\[
\varphi_p(\chi_{U_j}) = \frac{1}{p^j}, \quad \text{and} \quad \varphi_p(\chi_{\partial_j}) = \frac{1}{p^j} - \frac{1}{p^{j+1}},
\]
since
\[
\Lambda_{U_j} = \{ k \in \mathbb{Z} : k \geq j \}, \quad \text{and} \quad \Lambda_{\partial_j} = \{ j \},
\]
for all \( j \in \mathbb{Z} \), where \( \Lambda_S \) are in the sense of (2.9) for all \( S \in \sigma(Q_p) \). Note that, by (2.8), this linear functional \( \varphi_p \) of (3.2) is unbounded on \( \mathcal{M}_p \).
Definition 3.1. The pair $(M_p, \varphi_p)$ is called the $p$-adic (unbounded-)measure space for \( p \in P \), where $\varphi_p$ is the linear functional (3.2) on $M_p$.

Let $\partial_k$ be the $k$-th boundaries (3.1) of $Q_p$, for all $k \in \mathbb{Z}$. Then, for $k_1, k_2 \in \mathbb{Z}$, one obtains that

$$\chi_{\partial_{k_1}} \chi_{\partial_{k_2}} = \chi_{\partial_{k_1} \cap \partial_{k_2}} = \delta_{k_1, k_2} \chi_{\partial_{k_1}},$$

and hence,

$$\varphi_p (\chi_{\partial_{k_1}} \chi_{\partial_{k_2}}) = \delta_{k_1, k_2} \varphi_p (\chi_{\partial_{k_1}}) = \delta_{k_1, k_2} \left( \frac{1}{p^{k_1}} - \frac{1}{p^{k_1+1}} \right). \quad (3.3)$$

Proposition 3.2. Let $(j_1, \ldots, j_N) \in \mathbb{Z}^N$, for $N \in \mathbb{N}$. Then

$$\prod_{l=1}^N \chi_{\partial_{j_l}} = \delta_{(j_1, \ldots, j_N)} \chi_{\partial_{j_1}} \text{ in } M_p,$$

and hence,

$$\varphi_p \left( \prod_{l=1}^N \chi_{\partial_{j_l}} \right) = \delta_{(j_1, \ldots, j_N)} \left( \frac{1}{p^{j_1}} - \frac{1}{p^{j_1+1}} \right), \quad (3.4)$$

where

$$\delta_{(j_1, \ldots, j_N)} = \left( \prod_{l=1}^{N-1} \delta_{j_l, j_{l+1}} \right) \delta_{j_N, j_1}.$$

Proof. The proof of (3.4) is done by induction on (3.3).

Recall that, for any $S \in \sigma (\mathbb{Q}_p)$,

$$\varphi_p (\chi_S) = \sum_{j \in \Lambda_S} r_j \left( \frac{1}{p^j} - \frac{1}{p^{j+1}} \right), \quad (3.5)$$

for some $0 \leq r_j \leq 1$, for $j \in \Lambda_S$, by (2.11). So, by (3.5), if $S_1, S_2 \in \sigma (\mathbb{Q}_p)$, then

$$\chi_{S_1} \chi_{S_2} = \left( \sum_{k \in \Lambda_{S_1}} \chi_{S_1 \cap \partial_k} \right) \left( \sum_{j \in \Lambda_{S_2}} \chi_{S_2 \cap \partial_j} \right) = \sum_{(k, j) \in \Lambda_{S_1} \times \Lambda_{S_2}} \delta_{k, j} \chi_{S_1 \cap \partial_k} \chi_{S_2 \cap \partial_j} = \sum_{j \in \Lambda_{S_1} \cap \Lambda_{S_2}} \chi_{S_1 \cap \partial_j} \chi_{S_2 \cap \partial_j}, \quad (3.6)$$

where

$$\Lambda_{S_1, S_2} = \Lambda_{S_1} \cap \Lambda_{S_2},$$

by (2.4).
Proposition 3.3. Let $S_l \in \sigma(Q_p)$, and let $\chi_{S_l} \in (\mathcal{M}_p, \varphi_p)$, for $l = 1, \ldots, N$, for $N \in \mathbb{N}$. Let

$$\Lambda_{S_1, \ldots, S_N} = \bigcap_{l=1}^{N} \Lambda_{S_l} \text{ in } \mathbb{Z},$$

where $\Lambda_{S_l}$ are in the sense of (2.9), for $l = 1, \ldots, N$. Then there exist $r_j \in \mathbb{R}$, such that

$$0 \leq r_j \leq 1 \text{ in } \mathbb{R},$$

for all $j \in \Lambda_{S_1, \ldots, S_N}$, and

$$\varphi_p \left( \prod_{l=1}^{N} \chi_{S_l} \right) = \sum_{j \in \Lambda_{S_1, \ldots, S_N}} r_j \left( \frac{1}{p^j} - \frac{1}{p^{j+1}} \right).$$

(3.7)

Proof. The proof of (3.7) is done by the induction on (3.6), and by (3.4). 

4. REPRESENTATIONS OF $(\mathcal{M}_p, \varphi_p)$

Fix a prime $p \in \mathcal{P}$. Let $(\mathcal{M}_p, \varphi_p)$ be the $p$-adic measure space. By understanding $Q_p$ as a measure space, construct the $L^2$-space,

$$H_p \overset{\text{def}}{=} L^2(Q_p, \sigma(Q_p), \mu_p) = L^2(Q_p),$$

(4.1)

over $\mathbb{C}$. Then this $L^2$-space $H_p$ of (4.1) is a well-defined Hilbert space, consisting of all square-integrable elements of $\mathcal{M}_p$, equipped with its inner product $\langle \cdot, \cdot \rangle_2$,

$$\langle f_1, f_2 \rangle_2 \overset{\text{def}}{=} \int_{Q_p} f_1 f_2^* \, d\mu_p,$$

(4.2)

for all $f_1, f_2 \in H_p$, inducing the $L^2$-norm,

$$\|f\|_2 \overset{\text{def}}{=} \sqrt{\langle f, f \rangle_2},$$

for all $f \in H_p$,

where $\langle \cdot, \cdot \rangle_2$ is the inner product (4.2) on $H_p$.

Definition 4.1. We call the Hilbert space $H_p$ of (4.1), the $p$-adic Hilbert space.

By the definition (4.1) of the $p$-adic Hilbert space $H_p$, our $*$-algebra $\mathcal{M}_p$ acts on $H_p$, via an algebra-action $\alpha^p$,

$$\alpha^p(f) (h) = fh,$$

for all $h \in H_p$, for all $f \in \mathcal{M}_p$, i.e., the morphism $\alpha^p$ of (4.3) is a $*$-homomorphism from $\mathcal{M}_p$ to the operator algebra $B(H_p)$ consisting of all bounded linear operators on $H_p$. For instance,

$$\alpha^p(\chi_{Q_p}) \left( \sum_{S \in \sigma(Q_p)} t_S \chi_S \right) = \sum_{S \in \sigma(Q_p)} t_S \chi_{Q_p \cap S} = \sum_{S \in \sigma(Q_p)} t_S \chi_S,$$

(4.4)

for all $h = \sum_{S \in \sigma(Q_p)} t_S \chi_S \in H_p$, with $\|h\|_2 < \infty$, for $\chi_{Q_p} \in \mathcal{M}_p$, even though $\chi_{Q_p} \notin H_p$. 


Indeed, it is not difficult to check that
\[
\alpha^p(f_1f_2) = \alpha^p(f_1)\alpha^p(f_2) \text{ on } H_p, \text{ for all } f_1, f_2 \in \mathcal{M}_p,
\]
\[
(\alpha^p(f))^* = \alpha(f^*) \text{ on } H_p, \text{ for all } f \in \mathcal{M}_p
\]
(4.5)

(e.g., see [6] and [10]).

Denote \( \alpha^p(f) \) by \( \alpha^p_f \), for all \( f \in \mathcal{M}_p \). Also, for convenience, denote \( \alpha^p_S \) simply by \( \alpha^p \), for all \( S \in \sigma(\mathbb{Q}_p) \).

Note that, by (4.4), one has a well-defined operator \( \alpha^p_{\mathbb{Q}_p} = \alpha^p_{\chi_{\mathbb{Q}_p}} \) in \( B(H_p) \), and it satisfies that
\[
\alpha^p_{\mathbb{Q}_p}(h) = h = 1_{H_p}(h), \text{ for all } h \in H_p,
\]
(4.6)

where \( 1_{H_p} \in B(H_p) \) is the identity operator on \( H_p \).

**Proposition 4.2.** The pair \((H_p, \alpha^p)\) is a well-determined Hilbert space representation of \( \mathcal{M}_p \).

**Proof.** It is sufficient to show that \( \alpha^p \) is an algebra-action of \( \mathcal{M}_p \) acting on \( H_p \). But, by (4.5), this linear morphism \( \alpha^p \) of (4.3) is indeed a \( \ast \)-homomorphism from \( \mathcal{M}_p \) into \( B(H_p) \). \( \square \)

For a \( p \)-adic number fields, readers can check other types of representations in e.g., [18] and [20], different from our Hilbert-space representation \((H_p, \alpha^p)\).

**Definition 4.3.** The Hilbert-space representation \((H_p, \alpha^p)\) is said to be the \( p \)-adic (Hilbert-space) representation of \( \mathcal{M}_p \).

Depending on the \( p \)-adic representation \((H_p, \alpha^p)\) of \( \mathcal{M}_p \), one can construct the \( C^* \)-subalgebra \( \mathcal{M}_p \) of \( B(H_p) \) as follows.

**Definition 4.4.** Let \( \mathcal{M}_p \) be the operator-norm closure of \( \mathcal{M}_p \) in the operator algebra \( B(H_p) \), i.e.,
\[
\mathcal{M}_p \overset{df}{=} \overline{\alpha^p(\mathcal{M}_p)} = \overline{\{ \alpha^p_f : f \in \mathcal{M}_p \}}
\]
(4.7)
in \( B(H_p) \), where \( \overline{X} \) mean the operator-norm closures of subsets \( X \) of \( B(H_p) \).

This \( C^* \)-algebra \( \mathcal{M}_p \) of (4.7) is called the \( p \)-adic \( C^* \)-algebra of \((\mathcal{M}_p, \phi_p)\).

By the definition (4.7) of the \( p \)-adic \( C^* \)-algebra \( \mathcal{M}_p \), it is a unital \( C^* \)-algebra, containing its unity (or the unit, or the multiplication-identity) \( 1_{\mathcal{M}_p} = \alpha^p_{\mathbb{Q}_p} \), by (4.6).

5. FREE-PROBABILISTIC MODELS ON \( \mathcal{M}_p \)

Throughout this section, let us fix a prime \( p \in \mathcal{P} \), and let \((\mathcal{M}_p, \phi_p)\) be the corresponding \( p \)-adic measure space, and let \((H_p, \alpha^p)\) be the \( p \)-adic representation of \( \mathcal{M}_p \), inducing the corresponding \( p \)-adic \( C^* \)-algebra \( \mathcal{M}_p \) of (4.7). We here consider suitable (non-traditional) free-probabilistic models on \( \mathcal{M}_p \).
Define a linear functional $\varphi^p_j : M_p \to \mathbb{C}$ by a linear morphism,
\[
\varphi^p_j (a) \overset{\text{def}}{=} \langle a(\chi_{\partial_j}), \chi_{\partial_j} \rangle_2, \quad \text{for all } a \in M_p,
\] (5.1)
for $\chi_{\partial_j} \in H_p$, where $\langle \cdot, \cdot \rangle_2$ is the inner product (4.2) on the $p$-adic Hilbert space $H_p$ of (4.1), and $\partial_j$ are the $j$-th boundaries (3.1) of $\mathbb{Q}_p$, for all $j \in \mathbb{Z}$. It is not hard to check such a linear functional $\varphi^p_j$ on $M_p$ is bounded, since
\[
\varphi^p_j (\alpha_S^p) = \langle \alpha_S^p (\chi_{\partial_j}), \chi_{\partial_j} \rangle_2 = \langle \chi_{S \cap \partial_j}, \chi_{\partial_j} \rangle_2 = \langle \chi_{S \cap \partial_j}, \chi_{\partial_j} \rangle_2
\]
\[
= \int_{\mathbb{Q}_p} \chi_{S \cap \partial_j} d\mu_p = \mu_p (S \cap \partial_j) \leq \mu_p (\partial_j) = \frac{1}{p^j} - \frac{1}{p^{j+1}}
\]
for all $S \in \sigma (\mathbb{Q}_p)$, for any fixed $j \in \mathbb{Z}$.

Remark that, if $a \in M_p$, then
\[
a = \sum_{S \in \sigma (\mathbb{Q}_p)} t_S \alpha_S^p \text{ in } M_p \quad (t_S \in \mathbb{C}),
\]
where $\sum$ is finite or infinite (limit of finite) sum(s) under $C^*$-topology of $M_p$, and hence, the morphisms $\varphi^p_j$ of (5.1) are indeed well-defined bounded linear functionals on $M_p$, for all $j \in \mathbb{Z}$.

**Definition 5.1.** Let $\varphi^p_j$ be bounded linear functionals (5.1) on the $p$-adic $C^*$-algebra $M_p$, for all $j \in \mathbb{Z}$. Then the pairs $(M_p, \varphi^p_j)$ are said to be the $j$-th $p$-adic $C^*$-measure spaces, for all $j \in \mathbb{Z}$.

So, one can get the system
\[
\{(M_p, \varphi^p_j) : j \in \mathbb{Z}\}
\]
of the $j$-th $p$-adic $C^*$-measure spaces $(M_p, \varphi^p_j)$’s.

Note that, for any fixed $j \in \mathbb{Z}$, and $(M_p, \varphi^p_j)$, the unity
\[
1_{M_p} \overset{\text{denote}}{=} 1_{H_p} = \alpha_{\mathbb{Q}_p}^p \text{ of } M_p
\]
satisfies that
\[
\varphi^p_j (1_{M_p}) = \langle \chi_{\partial_j}, \chi_{\partial_j} \rangle_2 = \left\| \chi_{\partial_j} \right\|^2 = \frac{1}{p^j} - \frac{1}{p^{j+1}}.
\]
So, the $j$-th $p$-adic $C^*$-measure space $(M_p, \varphi^p_j)$ is a “bounded” measure space, but not a (classical) probability space, in general.

Now, fix $j \in \mathbb{Z}$, and take the corresponding $j$-th $p$-adic $C^*$-measure space $(M_p, \varphi^p_j)$.

For $S \in \sigma (\mathbb{Q}_p)$, and an element $\alpha_S^p \in M_p$, one has that
\[
\varphi^p_j (\alpha_S^p) = \langle \alpha_S^p (\chi_{\partial_j}), \chi_{\partial_j} \rangle_2 = \langle \chi_{S \cap \partial_j}, \chi_{\partial_j} \rangle_2
\]
\[
= \int_{\mathbb{Q}_p} \chi_{S \cap \partial_j} d\mu_p = \mu_p (S \cap \partial_j) = r_S \left( \frac{1}{p^j} - \frac{1}{p^{j+1}} \right), \quad (5.2)
\]
by (3.7), for some $0 \leq r_S \leq 1$ in $\mathbb{R}$. 


Proposition 5.2. Let \( S \in \sigma(Q_p) \), and \( \alpha^p_S \in (M_p, \varphi^p_S) \), for a fixed \( j \in \mathbb{Z} \). Then there exists \( r_S \in \mathbb{R} \), such that
\[
0 \leq r_S \leq 1 \text{ in } \mathbb{R},
\]
and
\[
\varphi^p_S ((\alpha^p_S)^n) = r_S \left( \frac{1}{p^j} - \frac{1}{p^{j+1}} \right), \text{ for all } n \in \mathbb{N}.
\]  (5.3)

Proof. Remark that the element \( \alpha^p_S \) is a projection in \( M_p \), in the sense that
\[
(\alpha^p_S)^* = \alpha^p_S = (\alpha^p_S)^2, \text{ in } M_p,
\]
and hence,
\[
(\alpha^p_S)^n = \alpha^p_S, \text{ for all } n \in \mathbb{N}.
\]
Thus, we obtain the formula (5.3) by (5.2).

As a corollary of (5.3), we obtain the following results.

Corollary 5.3. Let \( \partial_k \) be the \( k \)-th boundaries (3.1) of \( Q_p \), for all \( k \in \mathbb{Z} \). Then
\[
\varphi^p_j ((\alpha^p_{\partial_k})^n) = \delta_{j,k} \left( \frac{1}{p^j} - \frac{1}{p^{j+1}} \right),
\]  (5.4)
for all \( n \in \mathbb{N} \), for \( k \in \mathbb{Z} \).

6. SEMIGROUP \( \mathcal{C}^* \)-SUBALGEBRAS \( \mathcal{S}_p \) of \( M_p \)

Let \( M_p \) be the \( p \)-adic \( \mathcal{C}^* \)-algebra (4.7) for \( p \in \mathcal{P} \). Take operators
\[
P_{p,j} = \alpha^p_{\partial_j} \in M_p,
\]  (6.1)
for all \( j \in \mathbb{Z} \).

As we have seen in (5.3) and (5.4), these operators \( P_{p,j} \) are projections on the \( p \)-adic Hilbert space \( H_p \) in \( M_p \), for all \( p, j \in \mathbb{Z} \). We now restrict our interests to these projections \( P_{p,j} \) of (6.1).

Definition 6.1. Fix \( p \in \mathcal{P} \). Let \( \mathcal{S}_p \) be the \( \mathcal{C}^* \)-subalgebra
\[
\mathcal{S}_p = \mathcal{C}^* (\{ P_{p,j} \}_{j \in \mathbb{Z}}) = \mathbb{C} \left[ \{ P_{p,j} \}_{j \in \mathbb{Z}} \right] \text{ of } M_p,
\]  (6.2)
where \( P_{p,j} \) are projections (6.1), for all \( j \in \mathbb{Z} \). We call this \( \mathcal{C}^* \)-subalgebra \( \mathcal{S}_p \), the \( p \)-adic boundary \( (\mathcal{C}^*) \)-subalgebra of \( M_p \).

The \( p \)-adic boundary subalgebra \( \mathcal{S}_p \) acts like a diagonal subalgebra of the \( p \)-adic \( \mathcal{C}^* \)-algebra \( M_p \).

Proposition 6.2. Let \( \mathcal{S}_p \) be the \( p \)-adic boundary subalgebra (6.2) of the \( p \)-adic \( \mathcal{C}^* \)-algebra \( M_p \). Then
\[
\mathcal{S}_p \overset{\ast-\text{iso}}{=} \bigoplus_{j \in \mathbb{Z}} (\mathbb{C} : P_{p,j}) \overset{\ast-\text{iso}}{=} \mathbb{C}^{\oplus \mathbb{Z}},
\]  (6.3)
in \( M_p \).
Proof. It suffices to show that the generating projections \( \{P_{p,j}\}_{j \in \mathbb{Z}} \) of the \( p \)-adic boundary subalgebra \( \mathfrak{S}_p \) are mutually orthogonal from each other. But, one can get that
\[
P_{p,j_1} P_{p,j_2} = \alpha_p \left( \chi_{\partial p j_1} \wedge \chi_{\partial p j_2} \right) = \delta_{j_1,j_2} \alpha_p^{\partial_p j_1} = \delta_{j_1,j_2} P_{p,j_1},
\]
in \( \mathfrak{S}_p \), for all \( j_1, j_2 \in \mathbb{Z} \). Therefore, the structure theorem (6.3) holds.

Since the \( p \)-adic boundary subalgebra \( \mathfrak{S}_p \) of (6.2) is a \( C^* \)-subalgebra of \( M_p \), one can naturally obtain the measure spaces,
\[
\mathfrak{S}_{p,j} := (\mathfrak{S}_p, \varphi_p^j), \text{ for all } j \in \mathbb{Z}, \text{ for } p \in \mathcal{P},
\]
where the linear functionals \( \varphi_p^j \) of (6.4) are the restrictions \( \varphi_p^j |_{\mathfrak{S}_p} \) of (5.1), for all \( p \in \mathcal{P}, j \in \mathbb{Z} \).

7. WEIGHTED-SEMICIRCULAR ELEMENTS

Fix \( p \in \mathcal{P} \), and let \( \mathfrak{S}_p \) be the \( p \)-adic boundary subalgebra of the \( p \)-adic \( C^* \)-algebra \( M_p \), satisfying the structure theorem (6.3). Recall that the generating projections \( P_{p,j} \) of \( \mathfrak{S}_p \) satisfy
\[
\varphi_p^j (P_{p,j}) = \frac{1}{p^j} - \frac{1}{p^{j+1}}, \text{ for all } j \in \mathbb{Z},
\]
by (5.3) and (5.4).

Now, let \( \phi \) be the Euler totient function, the arithmetic function,
\[
\phi : \mathbb{N} \to \mathbb{C},
\]
defined by
\[
\phi(n) = |\{k \in \mathbb{N} : k \leq n, \gcd(n,k) = 1\}|,
\]
for all \( n \in \mathbb{N} \), where \( \gcd \) means the greatest common divisor.

By (7.2), one has
\[
\phi(p) = p - 1 = p \left(1 - \frac{1}{p}\right), \text{ for all } p \in \mathcal{P}.
\]
So, we have
\[
\varphi_p^j (P_{p,k}) = \delta_{j,k} \left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right) = \delta_{j,k} \left(\frac{1}{p^j} \left(1 - \frac{1}{p}\right)\right) = \delta_{j,k} \left(\frac{\phi(p)}{p^{j+1}}\right),
\]
by (7.1) and (7.3), for \( P_{p,k} \in \mathfrak{S}_p \), for all \( k \in \mathbb{Z} \).
Now, for a fixed prime $p$, define new linear functionals $\tau^p_j$ on $\mathfrak{S}_p$ by
\[ \tau^p_j = \frac{1}{\varphi(p)} \varphi^p_j, \]
for all $j \in \mathbb{Z}$, where $\varphi^p_j$ are in the sense of (6.4).

Then one obtains new free-probabilistic models of $\mathfrak{S}_p$, \{ \mathfrak{S}_p(j) = \left( \mathfrak{S}_p, \tau^p_j \right) : p \in \mathcal{P}, j \in \mathbb{Z} \}, (7.6)
where $\tau^p_j$ are in the sense of (7.5).

**Proposition 7.1.** Let $\mathfrak{S}_p(j) = \left( \mathfrak{S}_p, \tau^p_j \right)$ be in the sense of (7.6), and let $P_{p,k}$ be generating operators (6.1) of $\mathfrak{S}_p(j)$, for $p \in \mathcal{P}, j \in \mathbb{Z}$. Then
\[ \tau^p_j \left( P_{p,k}^n \right) = \frac{\delta_{j,k}}{p^{n+1}}, \text{ for all } n \in \mathbb{N}, \]
(7.7)

*Proof.* The formula (7.7) is proven by (7.4) and (7.5), since $P_{n} = P_{p,k}$ for all $n \in \mathbb{N}, k \in \mathbb{Z}$. \hfill $\square$

### 7.1. SEMICIRCULAR AND WEIGHTED-SEMICIRCULAR ELEMENTS

Let $(A, \varphi)$ be an arbitrary topological *-probability space ($C^*$-probability space, or $W^*$-probability space, or Banach *-probability space, etc.), equipped with a topological *-algebra $A$ ($C^*$-algebra, resp., $W^*$-algebra, resp., Banach *-algebra, etc.), and a (bounded or unbounded) linear functional $\varphi$ on $A$. If an operator $a \in A$ is regarded as an element of $(A, \varphi)$, we call $a$, a free random variable of $(A, \varphi)$.

**Definition 7.2.** Let $a$ be a self-adjoint free random variable in $(A, \varphi)$. It is said to be semicircular in $(A, \varphi)$, if
\[ \varphi(a^n) = \omega_n c_n, \text{ for all } n \in \mathbb{N}, \]
(7.8)
with
\[ \omega_n = \begin{cases} 1 & \text{if } n \text{ is even}, \\ 0 & \text{if } n \text{ is odd}, \end{cases} \]
for all $n \in \mathbb{N}$, and
\[ c_n = \frac{1}{n+1} \binom{2n}{n} = \frac{1}{n+1} \frac{(2n)!}{(n!)^2} = \frac{(2n)!}{n!(n+1)!} \]
are the $n$-th Catalan numbers, for all $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

It is well-known that, if $k_n(\cdot)$ is the free cumulant on $A$ in terms of a linear functional $\varphi$ (in the sense of [27]), then a self-adjoint free random variable $a$ is semicircular in $(A, \varphi)$, if and only if
\[ k_n(a, a, \ldots, a) = \begin{cases} 1 & \text{if } n = 2, \\ 0 & \text{otherwise}, \end{cases} \]
(7.9)
for all \( n \in \mathbb{N} \). The above equivalent free-distributional data (7.9) of the semicircularity (7.8) is obtained by the Möbius inversion of [27].

Motivated by (7.9), one can define the weighted-emicircularity.

**Definition 7.3.** Let \( a \in (A, \varphi) \) be a self-adjoint free random variable. It is said to be weighted-semicircular in \((A, \varphi)\) with its weight \( t_0 \) (in short, \( t_0 \)-semicircular), if there exists \( t_0 \in \mathbb{C} \times \mathbb{C} \setminus \{0\} \), such that

\[
k_n (a, a, \ldots, a) = \begin{cases} 
  t_0 & \text{if } n = 2, \\
  0 & \text{otherwise,}
\end{cases}
\]

for all \( n \in \mathbb{N} \), where \( k_n (\cdot) \) is the free cumulant on \( A \) in terms of \( \varphi \).

By (7.9) and (7.10), every \( 1 \)-semicircular element is semicircular. By the definition (7.10), and by the Möbius inversion of [27], a self-adjoint free random variable \( a \) is \( t_0 \)-semicircular in \((A, \varphi)\), if and only if there exists \( t_0 \in \mathbb{C} \times \mathbb{C} \setminus \{0\} \), such that

\[
\varphi (a^n) = \omega_n t_0^n c^n_\varphi,
\]

where \( \omega_n \) and \( c^n_\varphi \) are in the sense of (7.8), for all \( n \in \mathbb{N} \).

7.2. TENSOR PRODUCT BANACH ∗-ALGEBRA \( \mathfrak{L} \mathfrak{S}_p \)

Let \( \mathfrak{S}_p (k) = (\mathfrak{S}_p, \tau^p_k) \) be in the sense of (7.6), for \( p \in \mathcal{P}, k \in \mathbb{Z} \). Define now a bounded linear transformations \( c_p \) and \( a_p \) “acting on \( \mathfrak{S}_p \)”, by the linear morphisms satisfying

\[
c_p (P_{p,j}) = P_{p,j+1} \quad \text{and} \quad a_p (P_{p,j}) = P_{p,j-1},
\]

on \( \mathfrak{S}_p \), for all \( j \in \mathbb{Z} \).

By the definition (7.12), these linear transformations \( c_p \) and \( a_p \) are bounded under the operator-norm induced by the \( C^* \)-norm on \( \mathfrak{S}_p \). So, the linear transformations \( c_p \) and \( a_p \) are regarded as Banach-space operators acting “on \( \mathfrak{S}_p \)”, by regarding the \( C^* \)-algebra \( \mathfrak{S}_p \) as a Banach space equipped with its \( C^* \)-norm, i.e., \( c_p \) and \( a_p \) are elements of the operator space \( B (\mathfrak{S}_p) \) consisting of all bounded linear transformations on the Banach space \( \mathfrak{S}_p \).

**Definition 7.4.** The Banach-space operators \( c_p \) and \( a_p \) of (7.12) are called the \( p \)-creation, respectively, the \( p \)-annihilation on \( \mathfrak{S}_p \), for \( p \in \mathcal{P} \). Define a new Banach-space operator \( \mathbf{1}_p \in B (\mathfrak{S}_p) \), by

\[
\mathbf{1}_p = c_p + a_p \text{ on } \mathfrak{S}_p.
\]

We call it the \( p \)-radial operator on \( \mathfrak{S}_p \).

Let \( \mathbf{1}_p \) be the \( p \)-radial operator \( c_p + a_p \) of (7.13) on \( \mathfrak{S}_p \). Construct a closed subspace \( \mathfrak{L}_p \) of \( B (\mathfrak{S}_p) \) by

\[
\mathfrak{L}_p = \overline{C (\mathbf{1}_p)} \text{ in } B (\mathfrak{S}_p),
\]

where \( \overline{\cdot} \) mean the operator-norm-topology closures of all subsets \( \mathcal{Y} \) of \( B (\mathfrak{S}_p) \).
By the definition (7.14), $L_p$ is not only a closed subspace of the topological vector space $B(S_p)$, but also an algebra embedded in $B(S_p)$. On this Banach algebra $L_p$, define the adjoint $*$ by

$$\sum_{k=0}^{\infty} s_k l^k_p \in L_p \mapsto \sum_{k=0}^{\infty} \overline{s_k} l^k_p \in L_p,$$

(7.15)

where $s_k \in \mathbb{C}$ with their conjugates $\overline{s_k} \in \mathbb{C}$ (e.g., [6]).

Then, equipped with the adjoint (7.15), this Banach algebra $L_p$ of (7.14) forms a Banach $*$-algebra.

**Definition 7.5.** Let $L_p$ be a Banach $*$-algebra (7.14) in the operator space $B(S_p)$, for $p \in P$. We call it the $p$-radial (Banach-$*$) algebra on $S_p$.

Let $L_p$ be the $p$-radial algebra (7.14) on $S_p$. Construct now the tensor product Banach $*$-algebra $\mathcal{L}S_p$ by

$$\mathcal{L}S_p = L_p \otimes_{\mathbb{C}} S_p,$$

(7.16)

where $\otimes_{\mathbb{C}}$ means the tensor product of Banach $*$-algebras.

Note that the operators $l^k_p \otimes P_{p,j}$ generate the Banach $*$-algebra $\mathcal{L}S_p$ of (7.16), for all $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $j \in \mathbb{Z}$, where $P_{p,j}$ are the generating projections of (6.1) in $S_p$, with axiomatization:

$l^0_p = 1_{S_p}$, the identity operator on $S_p$,

in $B(S_p)$, satisfying

$$1_{S_p}(T) = T, \text{ for all } T \in S_p,$$

for all $j \in \mathbb{Z}$.

Define now a linear morphism

$$E_p : \mathcal{L}S_p \to S_p$$

by a linear transformation satisfying that

$$E_p \left( l^k_p \otimes P_{p,j} \right) = \left( \frac{p^{j+1}}{\left[ \frac{k}{2} \right] + 1} \right) l^k_p(P_{p,j}),$$

(7.17)

for all $k \in \mathbb{N}_0$, $j \in \mathbb{Z}$, where $\left[ \frac{k}{2} \right]$ is the minimal integer greater than or equal to $\frac{k}{2}$, for all $k \in \mathbb{N}_0$; for example,

$$\left\lfloor \frac{3}{2} \right\rfloor = 2 = \left\lfloor \frac{4}{2} \right\rfloor.$$

By the cyclicity (7.14) of the tensor factor $L_p$ of $\mathcal{L}S_p$, and by the structure theorem (6.3) of the other tensor factor $S_p$ of $\mathcal{L}S_p$, the above morphism $E_p$ of (7.17) is a well-defined bounded surjective linear transformation.

Now, consider how our $p$-radial operator $l_p$ acts on $S_p$. If $c_p$ and $a_p$ are the $p$-creation, respectively, the $p$-annihilation on $S_p$, then

$$c_p a_p(P_{p,j}) = P_{p,j} = a_p c_p(P_{p,j}),$$
Lemma 7.6. Let $c_p, a_p$ be the $p$-creation, respectively, the $p$-annihilation on $\mathfrak{S}_p$. Then
\[c_p^n a_p = (c_p a_p)^n = 1_{\mathfrak{S}_p} = (a_p c_p)^n = a_p^n c_p^n,\]
and
\[c_p^n a_p^{n+1} = a_p^{n+1} c_p^n \text{ on } \mathfrak{S}_p,\]  
(7.19)
for all $n, n_1, n_2 \in \mathbb{N}_0$.

Proof. The formula (7.19) holds by (7.18).

By (7.19), one can get that
\[l_p^n = (c_p + a_p)^n = \sum_{k=0}^{n} \binom{n}{k} c_p^k a_p^{n-k},\]  
(7.20)
with
\[c_p^0 = 1_{\mathfrak{S}_p} = a_p^0,\]
for all $n \in \mathbb{N}$, where
\[\binom{n}{k} = \frac{n!}{k!(n-k)!} \text{ for all } k \leq n \in \mathbb{N}_0.\]

Thus, one obtains the following proposition.

Proposition 7.7. Let $l_p \in \mathfrak{L}_p$ be the $p$-radial operator on $\mathfrak{S}_p$. Then, for all $m \in \mathbb{N},$
(i) $l_p^{2m-1}$ does not contain $1_{\mathfrak{S}_p}$-term,
(ii) $l_p^{2m}$ contains its $1_{\mathfrak{S}_p}$-term, $(2m) \cdot 1_{\mathfrak{S}_p}$.

Proof. The proofs of (i) and (ii) are done by straightforward computations under (7.19) and (7.20). See [6] for more details.

7.3. WEIGHTED-SEMICIRCULAR ELEMENTS $Q_{p,j}$ in $\mathfrak{L}\mathfrak{S}_p$

Fix $p \in \mathcal{P}$, and let $\mathfrak{L}\mathfrak{S}_p = \mathfrak{L}_p \otimes \mathfrak{S}_p$ be the tensor product Banach $*$-algebra (7.16), and let $E_p$ be the linear transformation (7.17) from $\mathfrak{L}\mathfrak{S}_p$ onto $\mathfrak{S}_p$. Throughout this section, fix a generating operator
\[Q_{p,j} = l_p \otimes P_{p,j} \text{ of } \mathfrak{L}\mathfrak{S}_p,\]  
(7.21)
for $j \in \mathbb{Z}$, where $P_{p,j}$ are projections (6.1) generating $\mathfrak{S}_p$.

If $Q_{p,j} \in \mathfrak{L}\mathfrak{S}_p$ is in the sense of (7.21) for $j \in \mathbb{Z}$, then
\[E_p(Q_{p,j}^n) = E_p(l_p^n \otimes P_{p,j}) = \left(\frac{p^{j+1}}{2} \right)^{n+1} \left(\frac{n}{2j} + 1\right) l_p^n (P_{p,j}),\]  
(7.22)
by (7.17), for all $n \in \mathbb{N}$. 

Now, for a fixed $j \in \mathbb{Z}$, define a linear functional $\tau_{p,j}^{0}$ on $\mathfrak{L}\mathfrak{S}_p$ by

$$\tau_{p,j}^{0} = \tau_{p,j} \circ E_p$$

on $\mathfrak{L}\mathfrak{S}_p$,

(7.23)

where $\tau_{p,j}^{0} = \frac{1}{\tau_{p,j}} \tau_{p,j}^{p}$ is in the sense of (7.5).

By the bounded-linearity of both $\tau_{p,j}^{0}$ and $E_p$, the morphism $\tau_{p,j}^{0}$ of (7.23) is a bounded linear functional on $\mathfrak{L}\mathfrak{S}_p$. By (7.22) and (7.23), if $Q_{p,j}$ is in the sense of (7.21), then

$$\tau_{p,j}^{0}(Q_{p,j}^n) = \frac{(p^{j+1})^n+1}{[2]_{+1}^j} \tau_{p,j}^p (l_p^n(P_{p,j})),$$

(7.24)

for all $n \in \mathbb{N}$.

**Theorem 7.8.** Let $Q_{p,j} = l_p \otimes P_{p,j} \in (\mathfrak{L}\mathfrak{S}_p, \tau_{p,j}^{0})$, for a fixed $j \in \mathbb{Z}$. Then

$$\tau_{p,j}^{0}(Q_{p,j}^n) = \omega_n c_{\Xi} \left( p^{2(j+1)} \right)^\frac{n}{2},$$

(7.25)

for all $n \in \mathbb{N}$, where $\omega_n$ are in the sense of (7.11).

**Proof.** The formula (7.25) is obtained by Proposition 7.7 and (7.24). See [10] for details. \qed

8. SEMICIRCULARITY ON $\mathfrak{L}\mathfrak{S}$

For all $p \in \mathcal{P}$, $j \in \mathbb{Z}$, let

$$\mathfrak{L}\mathfrak{S}_p(j) = (\mathfrak{L}\mathfrak{S}_p, \tau_{p,j}^{0})$$

be the measure-theoretic structures of the tensor product Banach $*$-algebra $\mathfrak{L}\mathfrak{S}_p$ of (7.16), and the linear functional $\tau_{p,j}^{0}$ of (7.24).

**Definition 8.1.** We call such pairs $\mathfrak{L}\mathfrak{S}_p(j)$ of (8.1), the $j$-th $p$-adic filter, for all $p \in \mathcal{P}, j \in \mathbb{Z}$.

Let $Q_{p,k} = l_p \otimes P_{p,k}$ be the $k$-th generating elements of the $j$-th $p$-adic filter $\mathfrak{L}\mathfrak{S}_p(j)$ of (8.1), for all $k \in \mathbb{Z}$, for fixed $p \in \mathcal{P}, j \in \mathbb{Z}$. Then they satisfy

$$\tau_{p,j}^{0}(Q_{p,k}^n) = \delta_{j,k} \left( \omega_n \left( p^{2(j+1)} \right)^\frac{n}{2} c_{\Xi} \right),$$

(8.2)

by (7.23) and (7.25), for all $n \in \mathbb{N}$.

For the family

$$\{ \mathfrak{L}\mathfrak{S}_p(j) = (\mathfrak{L}\mathfrak{S}_p, \tau_{p,j}^{0}) : p \in \mathcal{P}, j \in \mathbb{Z} \}$$

of $p$-adic filters of (8.1), define the free product Banach $*$-probability space,

$$\mathfrak{L}\mathfrak{S} \overset{\text{def}}{=} \left( \mathfrak{L}\mathfrak{S}, \tau^{0} \right) \overset{\text{def}}{=} \bigstar_{p \in \mathcal{P}, j \in \mathbb{Z}} \mathfrak{L}\mathfrak{S}_p(j).$$

(8.3)
as in [27] and [29], with

$$\mathcal{LS} = \bigoplus_{p \in \mathcal{P}, j \in \mathbb{Z}} \mathcal{LS}_p,$$

and $$\tau^0 = \bigoplus_{p \in \mathcal{P}, j \in \mathbb{Z}} \tau^0_{p,j}.$$  

Note that the Banach $$\ast$$-probability space $$\mathcal{LS}$$ of (8.3) is a well-defined Banach $$\ast$$-probability space with its free blocks $$\mathcal{LS}_p(j)$$, for all $$p \in \mathcal{P}, j \in \mathbb{Z}$$. For more about free product, see [27] and [29].

**Definition 8.2.** The Banach $$\ast$$-probability space $$\mathcal{LS} = (\mathcal{LS}, \tau^0)$$ of (8.3) is called the free Adelic filterization.

Let $$\mathcal{LS}$$ be the free Adelic filterization (8.3). Then, by (8.2), we obtain a subset

$$\mathcal{Q} = \{Q_{p,j} = l_p \otimes P_{p,j} \in \mathcal{LS}_p(j)\}_{p \in \mathcal{P}, j \in \mathbb{Z}}$$

in $$\mathcal{LS}$$.

Since all entries $$Q_{p,j}$$ of the above family $$\mathcal{Q}$$ are taken from the $$j$$-th $$p$$-adic filters $$\mathcal{LS}_p(j)$$, which are the free blocks of $$\mathcal{LS}$$, they are free from each other in $$\mathcal{LS}$$, for all $$p \in \mathcal{P}, j \in \mathbb{Z}$$. Also, since $$Q^n_{p,j} \in \mathcal{LS}_p(j)$$ in $$\mathcal{LS}$$, for all $$n \in \mathbb{N}$$, they are free reduced words with their lengths-1, and hence,

$$\tau^0 (Q^n_{p,j}) = \tau^0_{p,j} (Q^n_{p,j}) = \omega_n p^{n(j+1)} e^2,$$

by (8.2) and (8.3), for all $$n \in \mathbb{N}$$.

**Lemma 8.3.** Let $$\mathcal{Q}$$ be the subset of the free Adelic filterization $$\mathcal{LS}$$ introduced in the above paragraph. Then all elements $$Q_{p,j} \in \mathcal{Q}$$ are $$p^{2(j+1)}$$-semicircular in $$\mathcal{LS}$$.

**Proof.** As we discussed in the very above paragraphs, it is shown by (7.11), (8.2) and (8.3). □

Recall that a subset $$S$$ of an arbitrary (topological or pure-algebraic) $$\ast$$-probability space $$(A, \varphi)$$ is said to be a free family, if all elements of $$S$$ are mutually free from each other (e.g., [27] and [28]).

**Definition 8.4.** Let $$S$$ be a free family in an arbitrary topological $$\ast$$-probability space $$(A, \varphi)$$. This family $$S$$ is called a free (weighted-)semicircular family, if every element of $$S$$ is (weighted-)semicircular in $$(A, \varphi)$$.

By the above lemma, we obtain the following fact.

**Theorem 8.5.** Let $$\mathcal{LS}$$ be the free Adelic filterization (8.3), and let

$$\mathcal{Q} = \{Q_{p,j} \in \mathcal{LS}_p(j)\}_{p \in \mathcal{P}, j \in \mathbb{Z}} \subset \mathcal{LS},$$

where $$\mathcal{LS}_p(j)$$ are the $$j$$-th $$p$$-adic filters, the free blocks of $$\mathcal{LS}$$. Then this family $$\mathcal{Q}$$ is a free weighted-semicircular family in $$\mathcal{LS}$$.

**Proof.** Let $$\mathcal{Q}$$ be a subset (8.4) of $$\mathcal{LS}$$. Then, by the above lemma, all elements $$Q_{p,j}$$ of $$\mathcal{Q}$$ are $$p^{2(j+1)}$$-semicircular in $$\mathcal{LS}$$, for all $$p \in \mathcal{P}, j \in \mathbb{Z}$$. Also, they are mutually free from each other in $$\mathcal{LS}$$, because all entries $$Q_{p,j}$$ are contained in the mutually distinct free blocks $$\mathcal{LS}_p(j)$$ of $$\mathcal{LS}$$, for all $$p \in \mathcal{P}, j \in \mathbb{Z}$$. Therefore, the family $$\mathcal{Q}$$ forms a free weighted-semicircular family in $$\mathcal{LS}$$. □
Now, take elements
\[ \Theta_{p,j} \overset{\text{def}}{=} \frac{1}{p^{j+1}} Q_{p,j}, \text{ for all } p \in P, j \in \mathbb{Z}, \] (8.5)
in \( \mathfrak{L} \mathfrak{S} \), where \( Q_{p,j} \in \mathbb{Q} \), where \( \mathbb{Q} \) is the free weighted-semicircular family (8.4) in \( \mathfrak{L} \mathfrak{S} \).

Then, by the self-adjointness of \( Q_{p,j} \), these operators \( \Theta_{p,j} \) of (8.5) are self-adjoint in \( \mathfrak{L} \mathfrak{S} \), too, because
\[ p^{j+1} \in \mathbb{R} \text{ in } \mathbb{C}, \]
satisfying \( p^{j+1} = p^{j+1} \), for all \( p \in P, j \in \mathbb{Z} \).

**Theorem 8.6.** Let \( \Theta_{p,j} \in \mathfrak{L} \mathfrak{S}_p(j) \) be free random variables (8.5) of the free Adelic filterization \( \mathfrak{L} \mathfrak{S} \), for all \( p \in P, j \in \mathbb{Z} \). Then the family
\[ \Theta = \{ \Theta_{p,j} \in \mathfrak{L} \mathfrak{S}_p(j) : p \in P, j \in \mathbb{Z} \} \] (8.6)
forms a free semicircular family in \( \mathfrak{L} \mathfrak{S} \).

**Proof.** Let \( \Theta \) be the family (8.6). Then it forms a free family in \( \mathfrak{L} \mathfrak{S} \), because \( \Theta_{p,j} \in \Theta \) are the scalar-product of \( Q_{p,j} \in \mathbb{Q} \), and the family \( \mathbb{Q} \) of (8.4) is a free family in \( \mathfrak{L} \mathfrak{S} \).

Observe now that
\[ \tau^0 (\Theta^n_{p,j}) = \tau^0 \left( \left( \frac{1}{p^{j+1}} \right)^n Q^n_{p,j} \right) \]
\[ = \left( \frac{1}{p^{j+1}} \right)^n \tau^0 (Q^n_{p,j}) = \left( \frac{1}{p^{j+1}} \right)^n \left( \omega_n p^{n(j+1)} c_2 \right) \]
by the \( p^{2(j+1)} \)-semicircularity of \( Q_{p,j} \in \mathbb{Q} \)
\[ = \omega_n c_2, \] (8.7)
for all \( n \in \mathbb{N} \), for all \( p \in P, j \in \mathbb{Z} \).

Thus, all entries \( \Theta_{p,j} \) of the free family \( \Theta \) are semicircular by (7.8) and (8.7). Therefore, this free family \( \Theta \) of (8.6) forms a free semicircular family in \( \mathfrak{L} \mathfrak{S} \).

Define a Banach \( * \)-subalgebra \( \mathfrak{L} \mathfrak{S} \) of \( \mathfrak{L} \mathfrak{S} \) by
\[ \mathfrak{L} \mathfrak{S} \overset{\text{def}}{=} C[Q] \text{ in } \mathfrak{L} \mathfrak{S}, \] (8.8)
where \( Q \) is our free weighted-semicircular family (8.4), and \( Y \) mean the Banach topology closures of subsets \( Y \) of \( \mathfrak{L} \mathfrak{S} \).

Then one can obtain the following structure theorem for the Banach \( * \)-algebra \( \mathfrak{L} \mathfrak{S} \) of (8.8) in \( \mathfrak{L} \mathfrak{S} \).

**Theorem 8.7.** Let \( \mathfrak{L} \mathfrak{S} \) be the Banach \( * \)-subalgebra (8.8) of the free Adelic filterization \( \mathfrak{L} \mathfrak{S} \) generated by the free weighted-semicircular family \( Q \) of (8.4). Then
\[ \mathfrak{L} \mathfrak{S} = \overline{C[Q]} \text{ in } \mathfrak{L} \mathfrak{S}, \] (8.9)
where \( \Theta \) is the free semicircular family (8.6).
Moreover,

$$\mathbb{L} \overset{\ast\text{-iso}}{=} \bigotimes_{\nu \in \Pi, j \in \mathbb{Z}} C[[Q_{\nu,j}]] \overset{\ast\text{-iso}}{=} C \bigotimes_{\nu \in \Pi, j \in \mathbb{Z}} \{Q_{\nu,j}\},$$

(8.10)
in $\mathcal{S}$, where $\ast\text{-iso}$ means “being Banach-$\ast$-isomorphic”, and

$$C[[Q_{\nu,j}]]$$

are Banach $\ast$-subalgebras of $\mathcal{S}_{\nu}(j)$, for all $\nu \in \Pi, j \in \mathbb{Z}$, in $\mathcal{S}$. Here, $\ast$ in the first $\ast$-isomorphic relation of (8.10) is the free-probability-theoretic free product (of [27] and [29]), and $\ast$ in the second $\ast$-isomorphic relation of (8.10) is the pure-algebraic free product (generating noncommutative algebraic free words in $Q$).

**Proof.** Let $\mathbb{L}$ be the Banach $\ast$-subalgebra (8.8) of $\mathcal{S}$. Since the generator set $Q$ of $\mathbb{L}$ is a free family, as an embedded sub-structure of $\mathcal{S}$, we have that

$$\mathbb{L} \overset{\ast\text{-iso}}{=} \bigotimes_{\nu \in \Pi, j \in \mathbb{Z}} C[[Q_{\nu,j}]] \text{ in } \mathcal{S},$$

(8.11)

by (8.3).

Since every free block $C[[Q_{\nu,j}]]$ of (8.11) is generated by a single self-adjoint (weighted-semicircular) element $Q_{\nu,j}$, every operator $T$ of $\mathbb{L}$ is a limit of linear combinations of operator products spanned by the family $Q$ of (8.4), which form noncommutative free reduced words (in the sense of [27] and [29]) in $\mathbb{L}$. Note that every (pure-algebraic) free word in $Q$ has a unique free reduced word in $\mathbb{L}$, as an operator. So, the $\ast$-isomorphic relation (8.11) guarantees that

$$\mathbb{L} \overset{\ast\text{-iso}}{=} C \bigotimes_{\nu \in \Pi, j \in \mathbb{Z}} \{Q_{\nu,j}\},$$

(8.12)

where the free product ($\ast$) in (8.12) is pure-algebraic.

Therefore, by (8.11) and (8.12), the structure theorem (8.10) holds true.

Note now that

$$Q_{\nu,j} = p^{j+1}\Theta_{\nu,j} \in Q, \text{ for all } \nu \in \Pi, j \in \mathbb{Z},$$

by (8.5), where $\Theta_{\nu,j} \in \Theta$ are the semicircular elements of (8.6). So,

$$\mathbb{L} \overset{\text{def}}{=} C[Q] = C[[p^{j+1}\Theta_{\nu,j} : \nu \in \Pi, j \in \mathbb{Z}]] = C[\Theta],$$

(8.13)
in $\mathcal{S}$. Therefore, the equality (8.9) holds by (8.13). □

As a sub-structure of the free Adelic filterization $\mathcal{S}$, one gets the Banach $\ast$-probability space,

$$\left(\mathbb{L}, \tau^0 \overset{\text{def}}{=} \tau^0 |_{\mathcal{L}}\right).$$

(8.14)
**Definition 8.8.** Let $LS$ be the Banach $*$-subalgebra (8.8) of $\mathcal{S}$. Then we call

$$LS_0 = (LS, \tau^0)$$

of (8.14),

the (free) semicircular (Adelic sub-)filterization of the free Adelic filterization $L\mathcal{S}$.

Note that, by (8.3) and (8.10), all elements of the semicircular filterization $LS$ provide possible non-zero free distributions in the free Adelic filterization $L\mathcal{S}$. More precisely, a free reduced word of $L\mathcal{S}$ has its nonzero free distribution, if and only if it is a free reduced words in $\mathbb{Q} \cup \Theta$, if and only if it is contained in $LS_0$. Therefore, we now focus on free probability on the semicircular filterization $LS_0$ of (8.14).

9. TRUNCATED LINEAR FUNCTIONALS $\tau_{t<s}$ ON $LS$

In number theory, one of the most interesting topics is finding the number of primes, or the density of primes, contained in a closed interval $[t_1, t_2]$ of the real numbers $\mathbb{R}$ (e.g., [11–13] and [19]). Motivated by this theory, we consider “suitable” truncated linear functionals on our semicircular filterization $LS_0$ of (8.10).

**9.1. LINEAR FUNCTIONALS $\{\tau_{(t)}\}_{t \in \mathbb{R}}$ on $LS$**

Let $LS_0$ be the semicircular filterization $(LS, \tau^0)$ of the free Adelic filterization $L\mathcal{S}$, where $LS$ is the Banach $*$-subalgebra (8.8) of $L\mathcal{S}$, satisfying (8.10). We now truncate $\tau^0$ on $LS$, in terms of a fixed real number $t \in \mathbb{R}$.

First, recall and remark that

$$\tau^0 = \star \tau^0_{p,j} \text{ on } LS,$$

by (8.3) and (8.14). So, one can sectionize $\tau^0$ in terms of $P$ as follows:

$$\tau^0 = \star \tau^0_p \text{ on } LS,$$

(9.1)

with

$$\tau^0_p = \star \tau^0_{p,j} \text{ on } LS_p, \text{ for } p \in P,$$

where

$$LS_p \overset{def}{=} \star \mathbb{C}[(\Theta_{p,j})] \text{ in } LS \subset L\mathcal{S},$$

(9.2)

for each $p \in P$.

Such a sectionization (9.1) and (9.2) can be done by the structure theorem (8.10) of $LS$ in $L\mathcal{S}$.

By the very constructions (8.14) and (9.2), one can get the following lemma.

**Lemma 9.1.** Let $LS_{p_l}$ be $*$-subalgebras (9.2) of the semicircular filterization $LS_0$, for $l = 1, 2$. Then $LS_{p_1}$ and $LS_{p_2}$ are free in $LS_0$, if and only if $p_1 \neq p_2$ in $P$. 
Proof. The proof of this freeness condition in $\mathbb{L}S_0$ is clear by (8.3), (8.14) and (9.2). Indeed, $p_1 \neq p_2$ in $P$, if and only if all free blocks $\left\{ \mathbb{C}[\{\Theta_{p_1,j}\}] \right\}_{j \in \mathbb{Z}}$ of $\mathbb{L}S_{p_1}$, and those $\left\{ \mathbb{C}[\{\Theta_{p_2,j}\}] \right\}_{j \in \mathbb{Z}}$ of $\mathbb{L}S_{p_2}$ are disjoint from each other in $\mathbb{L}S_0$, if and only if $\mathbb{L}S_{p_1}$ and $\mathbb{L}S_{p_2}$ are free in $\mathbb{L}S_0$ by (8.10).

Fix now $t \in \mathbb{R}$, and define a new linear functional $\tau(t)$ on $\mathbb{L}S$ by

$$\tau(t) \overset{\text{def}}{=} \begin{cases} \star \tau^0_p & \text{on} \quad \mathbb{L}S_p \text{ in } \mathbb{L}S, \\ O & \text{otherwise,} \end{cases}$$

(9.3)

where $\tau^0_p$ are the linear functionals (9.1) on the Banach $*$-subalgebras $\mathbb{L}S_p$ of (9.2) in $\mathbb{L}S$, for all $p \in P$, and $O$ is the zero linear functional, satisfying $O(T) = 0$, for all $T \in \mathbb{L}S$.

By the definition (9.3), one can easily verify that, if $t < 2$ in $\mathbb{R}$, then the corresponding linear functional $\tau(t)$ is defined to the zero linear functional $O$ on $\mathbb{L}S$. From below, if there is no confusion, we simply write the above conditional definition (9.3) by

$$\tau(t) \overset{\text{denote}}{=} \star \tau^0_p \text{ on } \mathbb{L}S,$$

(9.4)

for all $t \in \mathbb{R}$. For example, $$\tau\left(\frac{z}{2}\right) = O, \quad \tau(2.0001) = \tau^0_2, \quad \text{and} \quad \tau(5) = \tau^0_2 \star \tau^0_3 \star \tau^0_5,$$

etc., on $\mathbb{L}S$, in the sense of (9.4) representing (9.3).

**Theorem 9.2.** Let $Q_{p,j} \in \mathcal{Q}$, and $\Theta_{p,j} \in \Theta$ in the semicircular filterization $\mathbb{L}S_0$, for $p \in P$, $j \in \mathbb{Z}$, where $\mathcal{Q}$ is the free weighted-semicircular family (8.4) and $\Theta$ is the semicircular family (8.6), generating $\mathbb{L}S_0$. Let $t \in \mathbb{R}$, and $\tau(t)$, the corresponding linear functional (9.4) on $\mathbb{L}S$. Then

$$\tau(t) (Q^n_{p,j}) = \begin{cases} \omega_n p^{2(j+1)} c^2 & \text{if } t \geq p, \\ 0 & \text{if } t < p, \end{cases} \quad \text{and} \quad \tau(t) (\Theta^n_{p,j}) = \begin{cases} \omega_n c^2 & \text{if } t \geq p, \\ 0 & \text{if } t < p, \end{cases}$$

(9.5)

for all $n \in \mathbb{N}$.

**Proof.** By the $p^{2(j+1)}$-semicircularity of $Q_{p,j} \in \mathcal{Q}$, and the semicircularity of $\Theta_{p,j} \in \Theta$ in $\mathbb{L}S_0$, and by the definition (9.3) or (9.4), one obtains that: if $t \geq p$ in $\mathbb{R}$, then

$$\tau(t) (Q^n_{p,j}) = \tau^0_p (Q^n_{p,j}) = \tau^0_{p,j} (Q^n_{p,j}) = \omega_n p^{2(j+1)} c^2,$$

and

$$\tau(t) (\Theta^n_{p,j}) = \tau^0_p (\Theta^n_{p,j}) = \tau^0_{p,j} (\Theta^n_{p,j}) = \omega_n c^2,$$

by (9.2) and (9.3), for all $n \in \mathbb{N}$. 

If \( t < p \), then

\[
\tau(t) = \bigwedge_{2 \leq q < p} \tau_0^q, \quad \text{or} \quad O, \quad \text{on} \quad \mathbb{L}\mathcal{S}.
\]

So, in such cases,

\[
\tau(t) \left( Q_{p,j}^n \right) = \tau(t) \left( \Theta_{p,j}^n \right) = 0, \quad \text{for all} \quad n \in \mathbb{N},
\]

by (9.3). Therefore, the free-moment data (9.5) for the linear functional \( \tau(t) \) holds.

**Definition 9.3.** Let \( \mathbb{L}\mathcal{S}_0 = (\mathbb{L}\mathcal{S}, \tau^0) \) be the semicircular filterization, and let \( \tau(t) \) be a linear functionals (9.4) on \( \mathbb{L}\mathcal{S} \), for \( t \in \mathbb{R} \). Then the new Banach *-probability spaces,

\[
\mathbb{L}\mathcal{S}_{(t)} \overset{\text{denote}}{=} \left( \mathbb{L}\mathcal{S}, \tau(t) \right), \quad (9.6)
\]

are called the semicircular \( t \)-filterizations of \( \mathbb{L}\mathcal{S}_0 \), for all \( t \in \mathbb{R} \).

Note that if \( t \) is suitable in the sense that \( \tau(t) \neq O \) on \( \mathbb{L}\mathcal{S} \), then the free-probabilistic structure \( \mathbb{L}\mathcal{S}_{(t)} \) of (9.6) is meaningful (or non-trivial).

**Notation and Assumption 9.4.** (in short, NA 9.4, from below) In the following, we will say “\( t \in \mathbb{R} \) is suitable”, if the semicircular \( t \)-filterization “\( \mathbb{L}\mathcal{S}_{(t)} \) of (9.6) is meaningful”, in the sense that \( \tau(t) \neq O \) on \( \mathbb{L}\mathcal{S} \).

Now, let us consider the following concept.

**Definition 9.5.** Let \( (A_k, \varphi_k) \) be Banach *-probability spaces (or \( C^* \)-probability spaces, or \( W^* \)-probability spaces, etc.), for \( k = 1, 2 \). A Banach *-probability space \( (A_1, \varphi_1) \) is said to be free-homomorphic to a Banach *-probability space \( (A_2, \varphi_2) \), if there exists a bounded *-homomorphism

\[
\Phi : A_1 \rightarrow A_2,
\]

such that

\[
\varphi_2 \left( \Phi(a) \right) = \varphi_1 \left( a \right),
\]

for all \( a \in A_1 \). The *-homomorphism \( \Phi \) is called a free-homomorphism.

If \( \Phi \) is a *-isomorphism, then it is called a free-isomorphism; and \( (A_1, \varphi_1) \) and \( (A_2, \varphi_2) \) are said to be free-isomorphic.
By (9.5), we obtain the following free-probabilistic-structural theorem.

**Theorem 9.6.** Let

\[ \mathbb{L}_q = \bigoplus_{j \in \mathbb{Z}} C[\{Q_{q,j}\}] \]

be Banach \(*\)-subalgebras (9.2) of \(\mathbb{L}_s\), for all \(q \in P\), and let \(t \in \mathbb{R}\) be suitable in the sense of NA 9.4. Construct a Banach \(*\)-probability space \(\mathbb{L}^t\) by a Banach \(*\)-probabilistic sub-structure of the semicircular filterization \(\mathbb{L}_s\),

\[ \mathbb{L}^t \overset{\text{def}}{=} \bigoplus_{p \leq t} (\mathbb{L}_p, \tau^0_p) = \left( \bigoplus_{p \leq t} \mathbb{L}_p, \bigoplus_{p \leq t} \tau^0_p \right) \]  

(9.7)

where \(\tau^0_p = \bigoplus_{j \in \mathbb{Z}} \tau^0_p,j\) are in the sense of (9.1), and \(\mathbb{L}_p\) are in the sense of (9.2) in \(\mathbb{L}_s\). Then, for suitable \(t \in \mathbb{R}\)

\[ \mathbb{L}^t \text{ of } (9.7) \text{ is free-homomorphic to } \mathbb{L}_{(t)}. \]  

(9.8)

**Proof.** Let \(\mathbb{L}_{(t)}\) be the semicircular \(t\)-filterization (9.6) of the semicircular filterization \(\mathbb{L}_s\), and let \(\mathbb{L}^t\) be a Banach \(*\)-probability space (9.7), for a fixed suitable \(t \in \mathbb{R}\). Define a bounded linear morphism

\[ \Phi_t : \mathbb{L}^t \rightarrow \mathbb{L}_{(t)}, \]

by the canonical embedding map,

\[ \Phi_t(T) = T \text{ in } \mathbb{L}_{(t)}, \text{ for all } T \in \mathbb{L}^t. \]  

(9.9)

Then it is a well-defined injective bounded \(*\)-homomorphism from \(\mathbb{L}^t\) into \(\mathbb{L}_{(t)}\), by (8.8), (8.11), (9.2) and (9.7).

Therefore, we obtain that

\[ \tau_{(t)}(\Phi(T)) = \tau_{(t)}(T) = \tau^0(T) = \tau^t(T), \]

for all \(T \in \mathbb{L}^t\), where

\[ \tau^t = \bigoplus_{p \leq t} \tau^0_p \text{ on } \mathbb{L}^t, \]

in the sense of (9.7), by (9.5).

It shows that the Banach \(*\)-probability space \(\mathbb{L}^t\) of (9.7) is free-homomorphic to the semicircular \(t\)-filterization \(\mathbb{L}_{(t)}\) of (9.6). Therefore, the statement (9.8) holds by a free-homomorphism \(\Phi_t\) of (9.9).

The above theorem shows that the Banach \(*\)-probability spaces \(\mathbb{L}^t\) of (9.7) are free-homomorphic to the semicircular \(t\)-filterizations \(\mathbb{L}_{(t)}\) of (9.6), for any suitable \(t \in \mathbb{R}\).

**Corollary 9.7.** All free reduced words \(T\) of the semicircular \(t\)-filterization \(\mathbb{L}_{(t)}\), having non-zero free distributions, are contained in the Banach \(*\)-probability space \(\mathbb{L}^t\) of (9.7), whenever \(t\) is suitable. The converse holds true, too.
Deformation of semicircular and circular laws via $p$-adic number fields ...

Proof. The proof of this characterization is done by (9.3), (9.5), (9.7), (9.8), and (9.9).

So, if $T$ are non-zero-free-distribution-having free reduced words of our semicircular $t$-filterization $\mathbb{L}S(t)$, then such operators $T$ are regarded as free random variables of the Banach $*$-probability space $\mathbb{L}S'$ of (9.7).

Remark 9.8. Let $F_n$ be the free groups with $n$-generators, for all $n \in \mathbb{N}_{\infty}^\infty = (\mathbb{N} \setminus \{1\}) \cup \{\infty\}$, and let $L(F_n)$ be the corresponding free group factors (the group von Neumann algebras generated by $F_n$, equipped with their canonical traces), for all $n \in \mathbb{N}_{\infty}^\infty$.

In [25], Radulescu showed that either (9.10) or (9.11) holds, where $L(F_n) \star_{iso} = L(F_\infty)$, for all $n \in \mathbb{N}_{\infty}^\infty$, (9.10)

We do not know which one holds true at this moment.

In our case, we have similar difficulties to check $\mathbb{L}S$ and $\mathbb{L}S'$ are $*$-isomorphic (and hence, free-isomorphic) or not. One thing clear now is that $\mathbb{L}S'$ is free-homomorphic to $\mathbb{L}S(t)$ by (9.8), for any suitable $t \in \mathbb{R}$.

Conjecture 9.9. Let $t \in \mathbb{R}$ be suitable in the sense of NA 9.4, and assume that there are more than one primes less than or equal to $t$. Even though the Banach $*$-algebras $\mathbb{L}S' = \star_{p \leq t} \mathbb{L}S_p$ and $\mathbb{L}S$ are $*$-isomorphic (which we are not sure either), the Banach $*$-probability spaces $\mathbb{L}S'$ and $\mathbb{L}S(t)$ are not free-isomorphic.

9.2. TRUNCATED LINEAR FUNCTIONALS $\tau_{t_1 < t_2}$ ON $\mathbb{L}S$

In this section, we generalize the semicircular $t$-filterizations $\mathbb{L}S(t)$, for suitable $t \in \mathbb{R}$. Throughout this section, let $[t_1, t_2]$ be a closed interval in $\mathbb{R}$, for $t_1 < t_2 \in \mathbb{R}$. For a fixed closed interval $[t_1, t_2]$, define the corresponding linear functional $\tau_{t_1 < t_2}$ on the Banach $*$-algebra $\mathbb{L}S$ by

$$\tau_{t_1 < t_2} = \begin{cases} \star_{t_1 \leq p \leq t_2 \text{ in } P \atop p \leq t} \mathbb{L}S_p \text{ on } \mathbb{L}S, \\ O \text{ otherwise,} \end{cases}$$

(9.12)

where $\tau_p^0$ are the linear functionals (9.1) on the Banach $*$-subalgebras $\mathbb{L}S_p$ of (9.2) in $\mathbb{L}S$, for $p \in P$.

As in Section 9.1, if there is no confusion, we write the conditional definition (9.12) of $\tau_{t_1 < t_2}$ as

$$\tau_{t_1 < t_2} = \star_{t_1 \leq p \leq t_2 \text{ in } P \atop p \leq t} \tau_p^0 \text{ on } \mathbb{L}S.$$

(9.13)
To make a linear functional $\tau_{t_1 < t_2}$ of (9.12) be a non-zero-linear functional on $LS$, the interval $[t_1, t_2]$ must be taken “suitably” in $\mathbb{R}$. For example,

$$\tau_{t_1 < t_2} = O, \text{ whenever } t_2 < 2,$$

and

$$\tau_{8<10} = O, \quad \tau_{14<16} = O, \quad \text{and} \quad \tau_{\frac{7}{2}} = O, \quad \text{etc.,}$$

but

$$\tau_{\frac{2}{3}<8} = \tau_{t_{(8)}} = \tau_{t_2} \ast \tau_{\frac{7}{2}} \ast \tau_{t_5} \ast \tau_{t_7}$$

and

$$\tau_{7<14} = \tau_{t_7} \ast \tau_{t_{11}} \ast \tau_{t_3},$$

on $LS$ in the sense of (9.13), representing (9.12).

It is not difficult to check that the concept of truncated linear functionals $\tau_{t_1 < t_2}$ of (9.12) covers the definition of the linear functionals $\tau_{(t)}$ of (9.3). In particular, if $\tau_{(t)}$ is “suitable” in the sense of NA 9.4, then one may understand

$$\tau_{(t)} = \tau_{s<t}, \text{ for } 2 \geq s < t \in \mathbb{R},$$

with axiomatization:

$$\tau_{p<p} = \tau_{p}^0 \text{ on } LS, \text{ for all } p \in \mathcal{P} \subset \mathbb{R},$$

in the sense of (9.13). Remark that the above axiomatization is only for the case where $p \in \mathcal{P}$.

**Definition 9.10.** Let $[t_1, t_2]$ be a given interval in $\mathbb{R}$, and $\tau_{t_1 < t_2}$, the corresponding linear functional (9.12) on $LS$. Then we call it the $[t_1, t_2]$-(truncated)-linear functional on $LS$. The Banach $*$-probability space

$$\mathbb{L}S_{t_1 < t_2} \text{ denote } (\mathbb{L}S, \tau_{t_1 < t_2}) \quad (9.14)$$

is said to be the semicircular $[t_1, t_2]$-(truncated)-filterization of the semicircular filterization $\mathbb{L}S_0 = (\mathbb{L}S, \tau^0)$.

As we discussed in the above paragraphs, a semicircular $[t_1, t_2]$-filterization $\mathbb{L}S_{t_1 < t_2}$ of (9.14) is “meaningful”, if $t_1 < t_2$ are suitable in $\mathbb{R}$, like in NA 9.4.

**Notation and Assumption 9.11.** (in short, NA 9.11, from below) In the rest of this paper, if we write “$t_1 < t_2$ are suitable in $\mathbb{R}$,” then it means “$\mathbb{L}S_{t_1 < t_2}$ is meaningful”, in the sense that: $\tau_{t_1 < t_2} \neq O$ on $LS$.

**Remark 9.12.** In fact, the study of such “suitability” of $t_1 < t_2$ in $\mathbb{R}$ is to study the density of primes in $[t_1, t_2]$ in number theory. e.g., see [11–13] and [19].

If $t_1 \leq 2$, and if $t_1 < t_2$ is suitable in $\mathbb{R}$, then the semicircular $[t_1, t_2]$-filterization $\mathbb{L}S_{t_1 < t_2}$ of (9.14) is identified with the semicircular $t_2$-filterization $\mathbb{L}S_{(t_2)}$ of (9.6).
Theorem 9.13. Let $t_1 \leq 2$, and $t_2$ is suitable in $\mathbb{R}$ in the sense of NA 9.4.

(i) $\mathbb{L}S_{t_1 < t_2}$ is not only suitable in the sense of NA 9.11, but also it is free-isomorphic to $\mathbb{L}S_{(t_2)}$.

(ii) The Banach $\ast$-probability space $\mathbb{L}S^{t_2}$ of (9.7) is free-homomorphic to $\mathbb{L}S_{t_1 < t_2}$.

Proof. Suppose $t_1 \leq 2$, and $t_2$ is suitable in $\mathbb{R}$ in the sense of NA 9.4. Then $t_1 < t_2$ are suitable in $\mathbb{R}$ in the sense of NA 9.11. Since $t_1$ is assumed to be less than or equal to 2, the linear functional $\tau_{t_1 < t_2} = \tau_{(t_2)}$, by (9.3) and (9.12). So, $\mathbb{L}S_{t_1 < t_2} = (\mathbb{L}S, \tau_{t_1 < t_2}) = (\mathbb{L}S, \tau_{(t_2)}) = \mathbb{L}S_{(t_2)}$.

Therefore, the free-isomorphic relation (i) holds by taking the free-isomorphism as the identity map on $\mathbb{L}S$.

By (9.8), the Banach $\ast$-probability space $\mathbb{L}S^{t_2}$ of (9.7) is free-homomorphic to the semicircular $t_2$-filterization $\mathbb{L}S_{(t_2)}$. Therefore, $\mathbb{L}S^{t_2}$ is free-homomorphic to $\mathbb{L}S_{t_1 < t_2}$, by (i), i.e., the statement (ii) holds. □

The above theorem characterizes the free-probabilistic structures for $\mathbb{L}S_{t_1 < t_2}$, whenever $t_1 \leq 2$, and $t_2$ is suitable, by (i) and (ii). So, we restrict our interests to the cases where $t_1 \geq 2$ in $\mathbb{R}$.

Theorem 9.14. Let $2 \leq t_1 < t_2$ be suitable in $\mathbb{R}$, and let $\mathbb{L}S_{t_1 < t_2}$ be the semicircular $[t_1, t_2]$-filterization of (9.14). Then the Banach $\ast$-probability space

$$\mathbb{L}S^{t_1 < t_2} = \bigoplus_{t_1 \leq p \leq t_2} (\mathbb{L}S_{p}, \tau^0_p) = \bigoplus_{t_1 \leq p \leq t_2} (\mathbb{L}S_{p}, \tau^0_{t_1 < t_2})$$

(9.15)

is free-homomorphic to $\mathbb{L}S_{t_1 < t_2}$ in the semicircular filterization $\mathbb{L}S_0$, i.e.,

$$\mathbb{L}S^{t_1 < t_2}$$

of (9.15) is free-homomorphic to $\mathbb{L}S_{t_1 < t_2}$. (9.16)

Proof. Let $\mathbb{L}S^{t_1 < t_2}$ be in the sense of (9.15) in $\mathbb{L}S_0$, i.e.,

$$\mathbb{L}S^{t_1 < t_2} = \bigoplus_{t_1 \leq p \leq t_2} (\mathbb{L}S_p, \tau^0_{p < t_2})$$

is a free-probabilistic sub-structure of the semicircular filterization $\mathbb{L}S_0$.

By (9.14), one can define the canonical embedding map $\Phi$ from $\mathbb{L}S^{t_1 < t_2}$ into $\mathbb{L}S$, satisfying

$$\Phi(T) = T,$$

for all $T \in \mathbb{L}S^{t_1 < t_2}$.

For any $T \in \mathbb{L}S^{t_1 < t_2}$, one can get that

$$\tau^{t_1 < t_2}(T) = \tau^0(T) = \tau_{t_1 < t_2}(T).$$

Therefore, the Banach $\ast$-probability space $\mathbb{L}S^{t_1 < t_2}$ is free-homomorphic to $\mathbb{L}S_{t_1 < t_2}$ in $\mathbb{L}S$, whenever $2 \leq t_1 < t_2$ are suitable in $\mathbb{R}$. Therefore, the relation (9.16) holds. □

Note again that we are not sure $\mathbb{L}S^{t_1 < t_2}$ and $\mathbb{L}S_{t_1 < t_2}$ are free-isomorphic or not at this moment. But if the conjecture of Section 9.1 is positive, then they may not be free-isomorphic.

Corollary 9.15. Let $T$ be a free reduced word of the semicircular $[t_1, t_2]$-filterization $\mathbb{L}S_{t_1 < t_2}$, and assume that the free distribution of $T$ is not the zero free distribution. Then $T$ is a free random variable of the Banach $\ast$-probability space $\mathbb{L}S^{t_1 < t_2}$ of (9.15).
10. LINEAR FUNCTIONALS $\tau_{t<s}$ on $\mathbb{L}S$ UNDER TRUNCATION ON PRIMES

Throughout this section, let $\mathbb{L}S_0 = (\mathbb{L}S, \tau^0)$ be the semicircular filterization of the free Adelic filterization $\mathbb{L}S$, and assume that $t < s$ be arbitrarily fixed suitable quantities of $\mathbb{R}$ in the sense of NA 9.11. Different from the truncated linear functionals (9.12),

$$\tau_{t<s} = \tau_{t\leq p \leq s}^0$$

on $\mathbb{L}S$, (in the sense of (9.13)), we here introduce and consider a new type of the linear functionals $\tau_{t<s}$ defined by

$$\tau_{t<s} = \sum_{t \leq p \leq s} \tau_p^0$$

on $\mathbb{L}S_p$ in $\mathbb{L}S$,

$$\tau_{t<s} = 0$$

otherwise,

where $\tau_q^0 = \tau_{k \in \mathbb{Z}}^{0, q, k}$ are the linear functionals (9.1) on the Banach $*$-subalgebra $\mathbb{L}S_q$ of (9.2) in $\mathbb{L}S_0$, for all $q \in \mathbb{P}$, where “$\oplus$” is the direct product of Banach $*$-algebras.

If there is no confusion, we write the conditional definition (10.1) simply as

$$\tau_{t<s} = \sum_{t \leq p \leq s} \tau_p^0$$

on $\mathbb{L}S$.

**Definition 10.1.** Let $\tau_{t<s}^+$ be a linear functional (10.1) on $\mathbb{L}S$, for suitable $t < s \in \mathbb{R}$ in the sense of NA 9.11. Then it is called the $[t, s]$-truncated “additive” linear functional on $\mathbb{L}S$. And the corresponding Banach $*$-probability space,

$$\mathbb{L}S_{t<s}^+ = (\mathbb{L}S, \tau_{t<s}^+)$$

is said to be the $[t, s]$-(truncated)-(+)-(semicircular)-filterization of $\mathbb{L}S_0$.

By the definition (10.1), two Banach $*$-probability spaces, the $[t, s]$-filterization $\mathbb{L}S_{t<s}$ of (9.14), and the $[t, s]$-(+)-filterization $\mathbb{L}S_{t<s}^+$ of (10.3) are different free-probabilistic objects in the semicircular filterization $\mathbb{L}S_0$, in general. More precisely, one can get the following result.

**Theorem 10.2.** Let $\mathbb{L}S_{t<s}$ be the $[t, s]$-filterization (9.14), and let $\mathbb{L}S_{t<s}^+$ be the $[t, s]$-(+)-filterization (10.3), for suitable $t < s \in \mathbb{R}$.

(i) If there are multi-primes in $[t, s]$, then $\mathbb{L}S_{t<s}$ and $\mathbb{L}S_{t<s}^+$ are not free-homomorphic.

(ii) If $[t, s]$ contains only one prime $p$, then $\mathbb{L}S_{t<s}$ and $\mathbb{L}S_{t<s}^+$ are free-isomorphic.

**Proof.** First of all, let us prove the statement (ii). Suppose $t < s$ are suitable in $\mathbb{R}$, and assume that $p \in \mathbb{P}$ is the only prime satisfying

$$t \leq p \leq s.$$

Then, by the definitions (9.12) and (10.1), we have

$$\tau_{t<s} = \tau_{t<p} = \tau_p^0 = \tau_{t<s}^+$$

on $\mathbb{L}S$. 
Deformation of semicircular and circular laws via p-adic number fields...

in the sense of (9.13) and (10.2), where \( \tau_{p<s} \) is axiomatized to be \( \tau_p^0 \) on \( \mathbb{LS} \) in the sense of (9.4).

It shows that

\[
\mathbb{LS}_{t<s} = (\mathbb{LS}, \tau_p^0) = \mathbb{LS}_t^+<s.
\]

Therefore, if \( p \) is the only prime in \([t, s]\), then \( \mathbb{LS}_{t<s} \) and \( \mathbb{LS}_t^+<s \) are free-isomorphic in the semicircular filtration \( \mathbb{LS}_0 \), with a free-isomorphism, the identity map on \( \mathbb{LS} \).

Thus, the statement (ii) holds.

Now, assume that there are \( N \)-many primes \( q_1, \ldots, q_N \) are contained in \([t, s]\), where \( N > 1 \) in \( \mathbb{N} \).

Thus,

\[
\tau_{t<s} = \sum_{k=1}^{N} \tau_{q_k}, \quad \text{and} \quad \tau_{t<s}^+ = \sum_{k=1}^{N} \tau_{q_k}^0,
\]

on the Banach \(*\)-algebra \( \mathbb{LS} \) in the sense of (9.13), respectively, (10.2).

Take an arbitrary free reduced word \( T \) with its length-\( n \),

\[
T = Q_{p_1,j_1}^{n_1} Q_{p_2,j_2}^{n_2} \cdots Q_{p_n,j_n}^{n_n},
\]

of \( \mathbb{LS}_0 \) in the free weighted-semicircular family \( \mathbb{Q} \), for \( 1 < n \in \mathbb{N} \), where either

\[
(p_1, \ldots, p_n), \quad \text{or} \quad (j_1, \ldots, j_n)
\]

consists of “mutually distinct” \( p_1, \ldots, p_n \) in \( \mathcal{P} \), respectively, consists of “mutually distinct” \( j_1, \ldots, j_n \) in \( \mathbb{Z} \), for \( n_1, \ldots, n_n \in \mathbb{N} \).

Also, for convenience, assume further that

\[
p_1, \ldots, p_n \in \{q_1, \ldots, q_N\},
\]

and

\[
n_1, \ldots, n_n \in 2\mathbb{N} = \{2n : n \in \mathbb{N}\},
\]

for \( 1 < n \leq N \) in \( \mathbb{N} \).

For any \(*\)-homomorphisms \( \Omega \) from \( \mathbb{LS}_{t<s} \) to \( \mathbb{LS}_t^+ \) (i.e., for any \(*\)-homomorphisms \( \Omega \) on \( \mathbb{LS} \)), the corresponding images \( \Omega(T) \) of the free reduced word \( T \) of (10.4) would be the free reduced word \( T' \) with its length-\( n' \), where

\[
n' \leq n \leq N \in \mathbb{N}.
\]

One may write this image \( T' \) of \( T \) as

\[
T' = Q_{r_1,i_1}^{k_1} Q_{r_2,i_2}^{k_2} \cdots Q_{r_{n'},i_{n'}}^{k_{n'}},
\]

for \( r_1, \ldots, r_n' \in \mathcal{P} \), \( i_1, \ldots, i_{n'} \in \mathbb{Z} \), and \( k_1, \ldots, k_{n'} \in \mathbb{N} \), as a free reduced word of \( \mathbb{LS} \).

Observe now that if \( T \) is in the sense of (10.4), then

\[
\tau_{t<s}(T) = \prod_{k=1}^{n} \left(p_k^{n_k(j_k+1)} c_{n_k}^{1/2}\right) \neq 0.
\]
by (10.5), because all factors of $T$ are mutually free from each other; meanwhile, if $T'$ is in the sense of (10.6), then

$$\tau_{t<s}^+(T') = \begin{cases} 0 & \text{if } n' > 1, \\ \sum_{l=1}^N \delta_{q_l, r_l} \omega_k q_l^{k_l(i_l+1)} c_{l,k} & \text{if } n' = 1, \end{cases}$$

(10.8)

by (10.1).

So, $\mathbb{LS}_{t<s}$ is not free-homomorphic to $\mathbb{LS}_{t<s}^+$ by (10.7) and (10.8).

Similarly, let us take a free reduced word $T$ of (10.4), now in the $[t, s]$-(+)-filterization $\mathbb{LS}_{t<s}^+$, satisfying (10.5). Then, since $N > 1$ in $\mathbb{N}$,

$$\tau_{t<s}^+(T) = 0,$$

more precisely,

$$\tau_{t<s}^+(T^n) = \tau_{t<s}^+((T^*)^n) = \tau_{t<s}^+ (T^{s_1}T^{s_2} \ldots T^{s_n}) = 0,$$

(10.9)

for all $(s_1, \ldots, s_n) \in \{1, \ast\}^n$, for all $n \in \mathbb{N}$. It shows that, as an element of $\mathbb{LS}_{t<s}^+$, the free reduced word $T$, whose length is $N > 1$, follows the zero free distribution. For any $\ast$-homomorphism from $\mathbb{LS}_{t<s}^+$ to $\mathbb{LS}_{t<s}$, the images $T'$ of them (in the sense of (10.6), as elements of $\mathbb{LS}_{t<s}$) satisfy

$$\tau_{t<s}^+ (T) = \delta_{q_1, \ldots, q_N; r_1, \ldots, r_n} \prod_{l=1}^n \left( \omega_k r_l^{k_l(i_l+1)} C_{k,l} \right),$$

(10.10)

by (10.7), where

$$\delta_{q_1, \ldots, q_N; r_1, \ldots, r_n'} = \begin{cases} 1 & \text{if } r_1, \ldots, r_n' \in \{q_1, \ldots, q_N\}, \\ 0 & \text{otherwise}. \end{cases}$$

The formulas (10.9) and (10.10) demonstrate that $\mathbb{LS}_{t<s}^+$ is not free-homomorphic to $\mathbb{LS}_{t<s}$.

Therefore, the $[t, s]$-filterization $\mathbb{LS}_{t<s}$ and the $[t, s]$-(+)-filterization $\mathbb{LS}_{t<s}^+$ are not free-homomorphic from each other, whenever there are multi-primes in $[t, s]$. So, the statement (ii) of this theorem holds true.

By (10.1), we obtain a following free-homomorphic relation.

**Theorem 10.3.** Let $\mathbb{LS}_q$ be in the sense of (9.2) in the semicircular filterization $\mathbb{LS}_0$, for $q \in \mathcal{P}$, and let

$$\mathcal{P}_{[t,s]} = \mathcal{P} \cap [t, s],$$

(10.11)

for suitable $t < s \in \mathbb{R}$. Define a Banach $\ast$-probabilistic sub-structure $\mathbb{LS}_{[t,s]}$ of $\mathbb{LS}_0$ by

$$\mathbb{LS}_{[t,s]} \overset{def}{=} \left( \bigoplus_{p \in \mathcal{P}_{[t,s]}} \mathbb{LS}_p, \tau_{[t,s]} = \sum_{p \in \mathcal{P}_{[t,s]}} \tau_p^0 \right),$$

(10.12)

Then $\mathbb{LS}_{[t,s]}$ of (10.12) is free-homomorphic to the $[t, s]$-(+)-filterization $\mathbb{LS}_{t<s}^+$, in $\mathbb{LS}$. 
Proof. Let $\mathbb{L}_S[t,s]$ be in the sense of (10.12) embedded in the semicircular filterization $\mathbb{L}_S$. Define now a bounded linear transformation

$$\Psi : \mathbb{L}_S[t,s] \to \mathbb{L}_S^+$$

by the canonical embedding map,

$$\Psi(T) = T$$

in $\mathbb{L}_S^+$, for all $T \in \mathbb{L}_S[t,s]$. (10.13)

For any $T \in \mathbb{L}_S[t,s]$, one has that

$$\tau_{t<s}^+ (\Psi(T)) = \tau_{t<s}^+ (T) = \tau_{t<s}^+ \left( \bigoplus_{q \in \mathcal{P}_{[t,s]}} T_q \right)$$

since $T = \Psi(T) \in \mathbb{L}_S[t,s] \subset \mathbb{L}_S^+$, and hence, there exist unique $T_q \in \mathbb{L}_q$, for all $q \in \mathcal{P}_{[t,s]}$, such that $T = \bigoplus_{q \in \mathcal{P}_{[t,s]}} T_q$, and hence, the above formula goes to

$$= \sum_{q \in \mathcal{P}_{[t,s]}} \tau_q^0 (T_q) = \left( \sum_{q \in \mathcal{P}_{[t,w]}} \tau_q^0 \right) \left( \bigoplus_{q \in \mathcal{P}_{[t,s]}} T_q \right)$$

(10.14)

by (10.11) and (10.12). Therefore, the $*$-homomorphism $\Psi$ of (10.13) is free-distribution-preserving by (10.14). Equivalently, it is a free-homomorphism. $\square$

11. APPLICATION: CIRCULARITY ON $\mathbb{L}_S^0$, $\mathbb{L}_S^t<s$, and $\mathbb{L}_S^+ t<s$

Throughout this section, we use same definitions, and notations introduced in previous sections. Let $\mathbb{L}_S^0 = (\mathbb{L}_S, \tau^0)$ be the semicircular filterization in the free Adelic filterization $\mathcal{L}S$, and let $t < s$ be suitable in $\mathbb{R}$ in the sense of NA 9.11, and

$$\mathbb{L}_S[t,s] = (\mathbb{L}_S, \tau_{t<s})$$

are the $[t,s]$-filterization (9.14), respectively, the $[t,s]$-filterization (10.3) of $\mathbb{L}_S^0$.

In this section, we apply our main results of Sections 8, 9 and 10 to the case where we have the operators $X \in \mathbb{L}_S$,

$$X = \frac{1}{\sqrt{2}} \left( \Theta_{p_1,j_1} + i \Theta_{p_2,j_2} \right),$$

(11.1)

where $i = \sqrt{-1}$ in $\mathbb{C}$,

$$\Theta_{p_1,j_1} = \frac{1}{p_l^{j_{l+1}}} Q_{p_1,j_l} \in \Theta, \text{ for all } l = 1, 2,$$

and where either

$$p_1 \neq p_2 \in \mathcal{P}, \text{ or } j_1 \neq j_2 \in \mathbb{Z},$$

(11.2)

where $\Theta$ is the free semicircular family (8.6) generating $\mathbb{L}_S^0$. 

By the condition (11.2), the summands $\Theta_{p_1,j_1}$ and $i\Theta_{p_2,j_2}$ of the operators $X$ of (11.1) are free in the semicircular filtration $\mathbb{L}_0$.

**Definition 11.1.** Let $(A, \psi)$ be an arbitrary topological $*$-probability space, and let $s_1$ and $s_2$ be semicircular elements in $(A, \psi)$. Assume these two semicircular elements $s_1$ and $s_2$ are free in $(A, \psi)$. Then the free random variable

$$x = \frac{1}{\sqrt{2}}(s_1 + is_2) \in (A, \psi),$$

(11.3)

is called the circular element induced by $s_1$ and $s_2$ in $(A, \psi)$ (e.g., [21, 22, 24] and [29]). The free distributions of such circular elements $x$ of (11.3) are called the circular law.

The circular law is characterized by the very semicircularity under free sum (e.g., [21, 22] and [24]). In particular, the circular law is characterized by the joint free-moments of a circular element $x$ of (11.3), and its adjoint $x^*$ under identically-free-distributedness, since $x$ is not self-adjoint in $A$, i.e.,

$$x^* = \frac{1}{\sqrt{2}}(s_1 - is_2) \neq x \in (A, \psi).$$

Recall that two free random variables $a_l$ of topological $*$-probability spaces $(A_l, \psi_l)$, for $l = 1, 2$, are said to be *identically free-distributed*, if

$$\psi_1(a_1^{r_1}a_1^{r_2} \ldots a_1^{r_n}) = \psi_2(a_2^{r_1}a_2^{r_2} \ldots a_2^{r_n}),$$

(11.4)

for all $(r_1, \ldots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$. For instance, if $a_1$ and $a_2$ are self-adjoint in $A_1$, respectively, in $A_2$, then they are identically free-distributed, if and only if

$$\psi_1(a_1^n) = \psi_2(a_2^n),$$

for all $n \in \mathbb{N}$ (e.g., [1] and [29]).

Note that the semicircular law, and the circular law are characterized under identically-free-distributedness universally (different from weighted-semicircular laws). i.e., “all” circular elements (resp., “all” semicircular elements) have the same free distributions, the circular law (resp., the semicircular law).

11.1. CIRCULARITY ON $\mathbb{L}_0$

Let $X$ be an operator (11.1), satisfying the condition (11.2) in the semicircular filterization $\mathbb{L}_0$. Then it is a circular element in $\mathbb{L}_0$ by (11.3).

**Proposition 11.2.** Let $\Theta_{p_1,j_1}, \Theta_{p_2,j_2} \in \Theta$ be semicircular elements of $\mathbb{L}_0$, where either

$$p_1 \neq p_2 \in \mathbb{P}, \text{ or } j_1 \neq j_2 \in \mathbb{Z}.$$

Then the operator $X$,

$$X = \frac{1}{\sqrt{2}}(\Theta_{p_1,j_1} + i\Theta_{p_2,j_2}) \in \mathbb{L}_0$$

(11.5)

is a circular element in $\mathbb{L}_0$. 
Proof. Suppose $\Theta_{p_{1,j_1}}, \Theta_{p_{2,j_2}} \in \Theta$ are semicircular elements of $\mathbb{L}_{S_0}$, and the above condition is satisfied. Then, these two semicircular elements are free in $\mathbb{L}_{S_0}$. So, by the circularity (11.3), the operator $X$ of (11.5) is circular in $\mathbb{L}_{S_0}$. \hfill \Box

11.2. CIRCULARITY ON $\mathbb{L}_{S_0}$ IN $\mathbb{L}_{S_{t<s}}$

Let $t < s$ be suitable real numbers in $\mathbb{R}$ under NA 9.11, and $\mathbb{L}_{S_{t<s}} = (\mathbb{L}_S, \tau_{t<s})$, the corresponding $[t, s]$-filtration of the semicircular filterization $\mathbb{L}_{S_0}$. Let $X$ be an operator (11.1) satisfying the condition (11.2) in the Banach $*$-algebra $\mathbb{L}_S$, and let

$$\mathcal{P}_{[t,s]} = \mathcal{P} \cap [t, s].$$

Before considering the free-distributional data of $X$ in $\mathbb{L}_{S_{t<s}}$, let us introduce the following concept.

Definition 11.3. Let $(A, \psi)$ be an arbitrary topological $*$-probability space, and suppose $x \in (A, \psi)$ is “not” self-adjoint. We will say that the free distribution of $x$ is followed by the semicircular law, if

$$\psi (x^{r_1} x^{r_2} \ldots x^{r_n}) = \omega_n c_n^2,$$

for all $(r_1, \ldots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$.

Suppose a free random variable $x$ is not self-adjoint in a topological $*$-probability space $(A, \psi)$. Then it cannot be a semicircular element by (7.5), (7.8) and (7.9). But, does a free random variable $x$ whose free distribution is followed by the semicircular law in the above sense exist? The following theorem not only characterizes the free distribution of an operator $X$ of (11.1) in the $[t, s]$-filterization $\mathbb{L}_{S_{t<s}}$, but also provides the positive answer of this question.

Theorem 11.4. Let $X$ be a circular element (11.5) in $\mathbb{L}_{S_0}$, and let $\mathcal{P}_{[t,s]} = \mathcal{P} \cap [t, s]$, where $[t, s]$ is a closed interval of $\mathbb{R}$. Then the following assertions hold.

(i) If $p_1, p_2 \in \mathcal{P}_{[t,s]}$, then $X$ is circular in the $[t, s]$-filtration $\mathbb{L}_{S_{t<s}}$.

(ii) If $p_1 \in \mathcal{P}_{[t,s]}$ and $p_2 \notin \mathcal{P}_{[t,s]}$, then the free distribution of $\sqrt{2}X$ is followed by the semicircular law in $\mathbb{L}_{S_{t<s}}$.

(iii) If $p_1 \notin \mathcal{P}_{[t,s]}$ and $p_2 \in \mathcal{P}_{[t,s]}$, then the free distribution of $-i\sqrt{2}X$ is followed by the semicircular law in $\mathbb{L}_{S_{t<s}}$.

(iv) If $p_1 \notin \mathcal{P}_{[t,s]}$ and $p_2 \notin \mathcal{P}_{[t,s]}$, then $X$ has the zero free distribution in $\mathbb{L}_{S_{t<s}}$.

Proof. Suppose first that

$$p_1, p_2 \in \mathcal{P}_{[t,s]}.$$

Then the summands $\Theta_{p_{l,j_l}}$ are free in $\mathbb{L}_{S_{t<s}}$, by Lemma 9.1, for all $l = 1, 2$. So, by (9.16) and (11.5), the operator $X$ is circular in the $[t, s]$-filterization $\mathbb{L}_{S_{t<s}}$, too. So, the statement (i) holds.

Assume that

$$p_1 \in \mathcal{P}_{[t,s]}, \text{ and } p_2 \notin \mathcal{P}_{[t,s]}.$$

and regard $X$ as a free random variable of $\mathbb{L}_{S_{t<s}}$. 

Now, let 
\[ T = \sqrt{2}X = \Theta_{p_1,j_1} + i\Theta_{p_2,j_2} \in \mathbb{LS}_{t<s}. \]

Observe that if there are free reduced words
\[ W_{p_2,j_2} = \Theta_{q_1,j_1} \ldots \Theta_{p_2,j_2} \ldots \Theta_{q_2,j_2} \in \mathbb{LS}_{t<s}, \]
containing at least one free-factor \( \Theta_{q_2,j_2} \) for \( n \in \mathbb{N} \), then
\[ \tau_{t<s}(W_{p_2,j_2}) = 0, \] for all \( N \in \mathbb{N} \),
by (9.12) and (9.13). Therefore, one can get that
\[ \tau_{t<s}(T^n) = \tau_{p_2}^0(\Theta_{p_2,j_2}^n) = \tau_{t<s}((T^*)^n), \]
and
\[ \tau_{t<s}(T^{r_1}T^{r_2} \ldots T^{r_n}) = \tau_{p_1}^0(\Theta_{p_1,j_1}^n), \]
for all mixed \((r_1, \ldots, r_n) \in \{1, *\}^n\), for all \( n \in \mathbb{N} \).

Note that
\[ \Theta_{p_1,j_1} \in \mathbb{LS}_{p_1} \subset \mathbb{LS}_{t<s} \text{ free-homo} \subseteq \mathbb{LS}_0, \]
where “free-homo” means “being free-homomorphic”, and hence, it is semicircular. Therefore, the free distribution of \( T = \sqrt{2}X \) is followed by the semicircular law in \( \mathbb{LS}_{t<s} \), by (11.2). (Remark that this operator \( T \) is not semicircular in \( \mathbb{LS}_{t<s} \), but, the free distribution of \( T \) is followed by the semicircular law.) It shows that the statement (ii) holds.

Let \( p_1 \notin \mathcal{P}_{[t,s]} \) and \( p_2 \notin \mathcal{P}_{[t,s]} \), and let
\[ S = -\sqrt{2}iX = -i\Theta_{p_1,j_1} + \Theta_{p_2,j_2} \in \mathbb{LS}_{t<s}. \]

Then, similar to (11.2), one obtains that
\[ \tau_{t<s}(S^n) = \tau_{p_2}^0(\Theta_{p_2,j_2}^n) = \tau_{t<s}((S^*)^n), \]
and
\[ \tau_{t<s}(S^{r_1}S^{r_2} \ldots S^{r_n}) = \tau_{p_2}^0(\Theta_{p_2,j_2}^n), \]
for all mixed \((r_1, \ldots, r_n) \in \{1, *\}^n\), for all \( n \in \mathbb{N} \). So, like in the proof of (ii), the free distribution of \( S = -\sqrt{2}iX \) is followed by the semicircular law in \( \mathbb{LS}_{t<s} \), by (11.6). Thus, the statement (iii) holds.

Finally, assume that \( p_1 \notin \mathcal{P}_{[t,s]} \), and \( p_2 \notin \mathcal{P}_{[t,s]} \).

Then \( X \notin \mathbb{LS}_{t<s} \), where
\[ \mathbb{LS}_{t<s} = \left( \bigoplus_{q \in \mathcal{P}_{[t,s]}} \mathbb{LS}_q \bigoplus_{q \in \mathcal{P}_{[t,s]}} \tau_q^0 \right) \]
is the Banach \(*\)-probability space (9.15) in \( \mathbb{LS} \). Therefore, by the free-homomorphic relation (9.16), this operator \( X \) has the zero free distribution in the \( [t,s]\)-filterization \( \mathbb{LS}_{t<s} \). Therefore, the statement (iv) holds true. \( \square \)
The above theorem illustrates the difference between original free-distributional data on the semicircular filterization \( \mathbb{L}_0 \), and those on the \([t, s]\)-filterization \( \mathbb{L}_{t<s} \) under suitable truncations for \([t, s]\). In particular, the circularity (11.5) of \( \mathbb{L}_0 \) is affected by the truncations for \([t, s]\) by (i)–(iv).

The following corollary is a direct consequence of the above theorem.

**Corollary 11.5.** Let \( X = \frac{1}{\sqrt{2}} (\Theta_{p_1,j_1} + i\Theta_{p_2,j_2}) \) be a circular element (11.5) of \( \mathbb{L}_0 \). Suppose \( t < s \) are suitable in \( \mathbb{R} \), and assume either

\[
p_1 \notin P_{[t,s]}, \text{ or } p_2 \notin P_{[t,s]} \quad \text{in } P.
\]

Then \( X \) is not circular in \( \mathbb{L}_{t<s} \), i.e., the circular law is distorted by the truncation for \([t, s]\).

**Proof.** Let \( X \in \mathbb{L}_0 \) be a circular element (11.5). Assume that either

\[
p_1 \notin P_{[t,s]}, \text{ or } p_2 \notin P_{[t,s]} \quad \text{in } P.
\]

Then \( X \) is not circular in \( \mathbb{L}_{t<s} \) by (ii)–(iv) of Theorem 11.4.

**11.3. CIRCULARITY OF \( \mathbb{L}_0 \) IN \( \mathbb{L}^+_{t<s} \)**

Let \( \mathbb{L}^+_{t<s} \) be the \([t, s]\)-(+) filtration of the semicircular filterization \( \mathbb{L}_0 \), for suitable \( t < s \) in \( \mathbb{R} \) under NA 9.11, and let \( X \) be a circular element (11.5) of the semicircular filterization \( \mathbb{L}_0 \) under (11.2).

**Lemma 11.6.** Let \( X = \frac{1}{\sqrt{2}} (\Theta_{p_1,j_1} + i\Theta_{p_2,j_2}) \) be a circular element (11.5) in \( \mathbb{L}_0 \). If we regard \( X \) as a free random variable of the \([t, s]\)-(+) filtration \( \mathbb{L}^+_{t<s} \), then one obtains the following free-distributional data.

(i) If \( p_1, p_2 \in P_{[t,s]} = P \cap [t,s] \), then

\[
\tau^+_{t<s}(X^n) = \omega_n \left( \frac{1}{\sqrt{2}} \right)^n (1 + i^n) c_2^n,
\]

and

\[
\tau^+_{t<s}((X^*)^n) = \omega_n \left( \frac{1}{\sqrt{2}} \right)^n (1 + (-i)^n) c_2^n,
\]

for all \( n \in \mathbb{N} \).

(ii) If \( p_1 \in P_{[t,s]} \), and \( p_2 \notin P_{[t,s]} \), then

\[
\tau^+_{t<s}(X^n) = \tau^+_{t<s}((X^*)^n) = \omega_n \left( \frac{1}{\sqrt{2}} \right)^n c_2^n,
\]

for all \( n \in \mathbb{N} \).
(iii) If $p_1 \notin P\{t,s\}$ and $p_2 \in P\{t,s\}$, then
\[
\tau^+_{t<s}(X^n) = \omega_n \left( \frac{i}{\sqrt{2}} \right)^n c_2,
\]
and
\[
\tau^+_{t<s}((X^*)^n) = \omega_n \left( -\frac{i}{\sqrt{2}} \right)^n c_2,
\]
for all $n \in \mathbb{N}$.

(iv) If $p_1 \notin P\{t,s\}$, and $p_2 \notin P\{t,s\}$, then $X$ has the zero free distribution on $\mathbb{L}\mathbb{S}^+_t<s$.

Proof. Suppose $p_1, p_2 \in P\{t,s\}$. Then, by (10.12), (10.13) and (10.14),
\[
\tau^+_{t<s}(X^n) = \tau_{[t,s]} \left( \left( \frac{1}{\sqrt{2}} \Theta_{p_1,j_1} \oplus i\Theta_{p_2,j_2} \right)^n \right)
\]
where $\tau_{[t,s]} = \sum_{q \in P\{t,s\}} \tau_q^0$ is in the sense of (10.12)
\[
= \left( \frac{1}{\sqrt{2}} \right)^n \tau_{[t,s]} \left( \Theta_{p_1,j_1}^n \oplus i^n \Theta_{p_2,j_2}^n \right)
= \left( \frac{1}{\sqrt{2}} \right)^n \left( \tau_{p_1}^0 \left( \Theta_{p_1,j_1}^n \right) + i^n \tau_{p_2}^0 \left( \Theta_{p_2,j_2}^n \right) \right)
= \left( \frac{1}{\sqrt{2}} \right)^n \left( \omega_n c_2 + i^n \omega_n c_2 \right)
\]
by the semicircularity of $\Theta_{p_1,j_1}$ in $\mathbb{L}\mathbb{S}_0$ (and hence, in $\mathbb{L}\mathbb{S}^+_t<s$)
\[
= \omega_n \left( \frac{1}{\sqrt{2}} \right)^n (1 + i^n)c_2,
\]
for all $n \in \mathbb{N}$.

Similarly,
\[
\tau^+_{t<s}((X^*)^n) = \left( \frac{1}{\sqrt{2}} \right)^n \tau_{[t,s]} \left( \left( \Theta_{p_1,j_1} \oplus (-i\Theta_{p_2,j_2}) \right)^n \right)
= \left( \frac{1}{\sqrt{2}} \right)^n \left( \tau_{p_1}^0 \left( \Theta_{p_1,j_1}^n \right) + (-i)^n \tau_{p_2}^0 \left( \Theta_{p_2,j_2}^n \right) \right)
= \omega_n \left( \frac{1}{\sqrt{2}} \right)^n (1 + (-i)^n)c_2,
\]
for all $n \in \mathbb{N}$. Therefore, the statement (i) holds.
Suppose \( p_1 \in \mathcal{P}_{[t,s]} \), and \( p_2 \notin \mathcal{P}_{[t,s]} \). Then

\[
\tau_{t<s}^+(X^n) = \tau_{t,s}
\left( \left( \frac{1}{\sqrt{2}} (\Theta_{p_1,j_1} + i\Theta_{p_2,j_2}) \right)^n \right)
= \left( \frac{1}{\sqrt{2}} \right)^n \tau_{t,s}
\left( \Theta_{p_1,j_1}^n \right)
= \left( \frac{1}{\sqrt{2}} \right)^n \tau_{p_1}^0 \left( \Theta_{p_1,j_2}^n \right)
= \tau_{t<s}^+((X^*)^n),
\]

for all \( n \in \mathbb{N} \). Thus, the statement (ii) holds.

Assume now that \( p_1 \notin \mathcal{P}_{[t,s]} \), and \( p_2 \in \mathcal{P}_{[t,s]} \). Then, similar to the proof of (ii), one can get that

\[
\tau_{t<s}^+(X^n) = \omega_n \left( \frac{i}{\sqrt{2}} \right)^n c_{\frac{s}{2}}^n,
\]

and

\[
\tau_{t<s}^+((X^*)^n) = \omega_n \left( -\frac{i}{\sqrt{2}} \right)^n c_{\frac{s}{2}}^n,
\]

for all \( n \in \mathbb{N} \). It guarantees the statement (iii) holds true.

Finally, assume that \( p_1 \notin \mathcal{P}_{[t,s]} \), and \( p_2 \notin \mathcal{P}_{[t,s]} \). Then, by (10.13) and (10.14), the operator \( X \) has the zero free distribution on \( \mathbb{L}_s^+_{t<s} \). Equivalently, the statement (iv) holds.

By the above lemma, one immediately obtains the following result.

**Theorem 11.7.** Let \( X \) be a circular element (11.5) of the semicircular filterization \( \mathbb{L}_0 \). If \( X \) is regarded as a free random variable of the \([t,s]^-\)-(+) filterization \( \mathbb{L}_s^+_{t<s} \), then \( X \) is not circular in \( \mathbb{L}_s^+_{t<s} \), i.e.,

\[
X \text{ cannot be a circular element in } \mathbb{L}_s^+_{t<s}.
\]

**Proof.** Let \( X \) be given as above in \( \mathbb{L}_s^+_{t<s} \). Then it cannot be circular in \( \mathbb{L}_s^+_{t<s} \), by (i)–(iv) of Lemma 11.6. So, the statement (11.7) is proven.

It shows that a circular element \( X \) of the semicircular filterization \( \mathbb{L}_0 \) cannot be circular in all \([t,s]^-\)-(+) filterizations \( \mathbb{L}_s^+_{t<s} \), whenever \(-\infty < t < s < \infty \) in \( \mathbb{R} \).

### 11.4. DISCUSSION

In Sections 11.1, 11.2 and 11.3, we applied the main results of Sections 8, 9 and 10 to circular elements of the semicircular filterization \( \mathbb{L}_0 \). Especially, the distorted circularity is observed in \( \mathbb{L}_t^+_{t<s} \), and in \( \mathbb{L}_s^+_{t<s} \), where \( t < s \) are suitable in the sense of NA 9.11, i.e., the circularity (11.5) of \( \mathbb{L}_0 \) is affected by our truncations in \( \mathbb{L}_t^+_{t<s} \) by (i)–(iv) of Theorem 11.4, meanwhile, it is distorted by truncations in \( \mathbb{L}_s^+_{t<s} \), by (11.7).

In the middle of studying such distortions, the existence of a certain type of free random variables, whose free distributions are followed by the semicircular law, is shown (in Section 11.2).
Proposition 11.8. There exist a topological $\ast$-probability space $(A, \psi)$, and free random variables $x \in (A, \psi)$, such that:

(i) $x$ is not self-adjoint (and hence, not semicircular),
(ii) the free distribution of $x$ is followed by the semicircular law in the sense that:

$$\psi(x^{r_1} x^{r_2} \ldots x^{r_n}) = \omega_n c_{\frac{n}{2}}$$

for all $(r_1, \ldots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$.

Proof. The proof is done by construction. Let

$$\mathbb{LS}_{t<s} = (\mathbb{LS}, \tau_{t<s})$$

be the $[t,s]$-filterization of the semicircular filterization $\mathbb{LS}_0$, where $t < s$ are suitable in $\mathbb{R}$. Let us take a free random variable

$$T = \Theta_{p_1,j_1} + t\Theta_{p_2,j_2}$$

in $\mathbb{LS}_{t<s}$, for $t \in \mathbb{C}$, where $\Theta_{p,j} \in \Theta$ are two distinct (and hence, free) semicircular elements in $\mathbb{LS}_0$, for $l = 1, 2$, and

$$p_1 \in \mathcal{P}_{[t,s]} = \mathcal{P} \cap [t,s], \text{ and } p_2 \notin \mathcal{P}_{[t,s]}.$$ 

Then, similar to the proofs of (ii) and (iii) of Theorem 11.4, the free distributions of $T$ are characterized by the joint free moments of $\{T, T^*\}$ satisfying

$$\tau_{t<s}(T^{r_1} T^{r_2} \ldots T^{r_n}) = \omega_n c_{\frac{n}{2}},$$

for all $(r_1, \ldots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$.

It guarantees the existence of non-self-adjoint free random variables whose free distributions are followed by the semicircular law.

The above proposition provides an interesting class of free random variables of topological $\ast$-probability spaces. By the Möbius inversion of [27], one can get the following equivalent result of the above proposition.

Corollary 11.9. There exist topological $\ast$-probability spaces $(A, \psi)$, and free random variables $x \in (A, \psi)$, such that

(i) $x$ is not self-adjoint,
(ii) the free distribution of $x$ is followed by the semicircular law in the sense that:

$$k_n^\psi(x^{r_1}, \ldots, x^{r_n}) = \begin{cases} 1 & \text{if } n = 2, \\ 0 & \text{otherwise,} \end{cases}$$

for all $(r_1, \ldots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$, where $k_n^\psi(\cdot)$ is the free cumulant on $A$ in terms of the linear functionals $\psi$.

Proof. The proof of (11.9) is done by (11.8) under the Möbius inversion of [27].
Deformation of semicircular and circular laws via $p$-adic number fields

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Ilwoo Cho
choilwoo@sau.edu

Saint Ambrose University
Department of Mathematics and Statistics
421 Ambrose Hall, 518 W. Locust St.
Davenport, Iowa, 52803, USA
Palle E.T. Jorgensen
palle-jorgensen@uiowa.edu

The University of Iowa
Department of Mathematics
Iowa City, IA 52242-1419, USA

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