Modeling Portfolio Credit Risk Taking into Account the Default Correlations Using a Copula Approach: Implementation to an Italian Loan Portfolio

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Abstract: This work aims to illustrate an advanced quantitative methodology for measuring the credit risk of a loan portfolio allowing for diversification effects. Also, this methodology can allocate the credit capital coherently to each counterparty in the portfolio. The analytical approach used for estimating the portfolio credit risk is a binomial type based on a Monte Carlo Simulation. This method takes into account the default correlations among the credit counterparties in the portfolio by following a copula approach and utilizing the asset return correlations of the obligors, as estimated by rigorous statistical methods. Moreover, this model considers the recovery rates as stochastic and dependent on each other and on the time until defaults. The methodology utilized for coherently allocating credit capital in the portfolio estimates the marginal contributions of each obligor to the overall risk of the loan portfolio in terms of Expected Shortfall (ES), a risk measure more coherent and conservative than the traditional measure of Value-at-Risk (VaR). Finally, this advanced analytical structure is implemented to a hypothetical, but typical, loan portfolio of an Italian commercial bank operating across the overall national country. The national loan portfolio is composed of 17 sub-portfolios, or geographic clusters of credit exposures to 10,500 non-financial firms (or corporates) belonging to each geo-cluster or sub-portfolio. The outcomes, in terms of correlations, portfolio risk measures and capital allocations obtained from this advanced analytical framework, are compared with the results found by implementing the Internal Rating Based (IRB) approach of Basel II and III. Our chief conclusion is that the IRB model is unable to capture the real credit risk of loan portfolios because it does not take into account the actual dependence structure among the default events, and between the recovery rates and the default events. We underline that the adoption of this regulatory model can produce a dangerous underestimation of the portfolio credit risk, especially when the economic uncertainty and the volatility of the financial markets increase.

Keywords: portfolio credit risk; asset correlation; coherent capital allocation; copula function; Monte Carlo simulation; time until default

1. Introduction

Currently, the financial system has become more volatile due to increasing globalization and financial integration. In such a context, financial intermediaries bear a greater amount of risk because of the growing interconnectionedness of financial markets. Various kinds of risks affect the balance-sheets of financial intermediaries, but the most important is credit risk, particularly for commercial banks. For most banks, the loans to the real economy are the largest source of credit risk. For these reasons, banks need to adopt accurate methodologies for consistently assessing the credit risk of their loan portfolio and for allocating economic capital coherently to each counterparty in the
portfolio. Bank capital plays a fundamental role in the safety and soundness of the banking system, since the main role of bank capital is to absorb large unexpected losses.

It is well known that the worth of a portfolio model lies in its ability to take into account the effects of diversification, namely the default correlations among the credit assets in a portfolio (Crouhy et al. 2000, 2014; Gordy 2000).

On the other hand, the regulatory approach based on internal ratings, namely the IRB-model proposed by Basel II and III, founded on the hypothesis of the “portfolio invariant” model (Gordy 2000, 2003), assumes that banking portfolios are perfectly fine grained; namely, that idiosyncratic risks have been diversified away. Consequently, the IRB model calculates the banking capital requirements to cover the unexpected credit losses on banking loans as a function of the characteristics of the borrower and the credit line only, ignoring the empirical portfolio composition and, in particular, the real level of diversification among the credit assets in the portfolio.

More specifically, the two theoretical and restrictive hypotheses underlying Basel’s IRB model are the following:

(i) the infinite granularity of the credit portfolio and, therefore, the asymptotic approximation of the overall portfolio risk to the only non-diversifiable risk. In other words, the loan portfolio is highly diversified.

(ii) the existence of only a single systematic risk factor, and the subsequent quantification of this risk using the correlation between the economic assets of each counterparty in the portfolio and the index of the general economic condition.

The adoption of a portfolio invariant model by Basel II and III offers obvious analytical advantages to regulatory authorities. In particular, this approach permits the calculation of regulatory capital requirements analytically, without considering the real composition and the granularity of the empirical loan portfolios.

On the other hand, real-world portfolios are not perfectly fine grained. The asymptotic assumption might be approximately valid for some large bank portfolios, but could be much less acceptable for portfolios of smaller or more specialized institutions (Gordy and Lütkebohmert 2013).

The IRB model can underestimate the real credit risk amount of undiversified loan portfolios, omitting the contribution of the idiosyncratic risks in the portfolio.

Moreover, Basel sets for the category “corporate exposures” a regulatory value of the correlation between 12% and 24%. The values of the correlation obtained utilizing the regulatory formula, although inclosing an adjustment for small and medium enterprises, appear empirically conservative; that is, too high (de Servigny and Renault 2002; Duellmann and Scheule 2003; Duellmann and Kosiol 2014; Hamerle and Rösch 2006; Sironi and Zazzara 2003; Dietsch and Petey 2004; Lopez 2004; Kitano 2007).

It is well known that asset correlations play a critical role in measuring portfolio credit risk, and in determining both economic and regulatory capital. In a credit portfolio, having many components does not assure good diversification, because the components may be highly correlated to each other, and the default of one may lead to default of the rest of the portfolio. This concept is called concentration risk in credit risk management.

Another reason is the incremental risk. Incremental risk measures the portfolio’s risk sensibility to any changes in the portfolio’s components. Therefore, correlation indicates the movement direction of the portfolio’s assets with each other and with economic events.

An additional purpose for studying the correlations is to achieve a better allocation of assets in the portfolio. Optimal allocation means the minimization of the volatility of the portfolio, which depends on correlations (Mausser and Rosen 2008). Any change in the correlations of the portfolio changes the optimal asset allocation (Mizgier and Pasia 2015). The empirical results (Zhou 2001; de Servigny and Renault 2002) show that the default correlation between the components of the portfolio increases when the market does not perform well, or when there is an event that affects the market adversely. Also, de Servigny and Renault (2002) examined the effect of the time horizon on the default correlation, and showed that the correlation increases with time.
The standard approach for allocating capital in terms of Value-at-Risk (VaR) is founded on the traditional mean-variance approach (Markowitz 1952). However, a lot of recent academic studies (Artzner et al. 1999; Acerbi and Tasche 2002) proved that the mean-variance capital allocation presents many shortcomings. The most important drawback is that the capital amount allocated to the whole portfolio may be greater than the capital amounts allocated to the individual sub-portfolios when the return distributions are not Gaussian (this is the case in credit assets). These problems can be overcome by allocating the capital in terms of Expected Shortfall (ES) as described, for instance, in Kalkbrener et al. (2004). While VaR (or more exactly, the Maximum Loss, ML) can be considered as a quantile of the portfolio loss distribution, the Expected Shortfall is approximately the conditional mean of losses exceeding VaR.

In light of all these considerations, this work aims to illustrate an advanced quantitative methodology capable of measuring the credit risk of the loan portfolio adequately, allowing for diversification effects, and suitable for allocating credit capital coherently (in the sense of Artzner et al. 1999) to each counterparty in the portfolio.

The added value of this research is, therefore, to introduce a new quantitative methodology capable of overcoming the weaknesses of Basel’s IRB model. In order to achieve this goal, the principal tasks of this research are:

(i) Emphasizing that the main disadvantages of the Basel model are its underlying restrictive assumptions.
(ii) Comparing the results in terms of the portfolio’s credit risk measures derived from the new methodology and from the IRB model when we assume both diversified and concentrated loan portfolios.
(iii) Introducing two sound statistical methodologies for estimating the asset return correlations between the obligors in the portfolio.
(iv) Improving the methodological framework of the portfolio credit risk model by assuming a dependence structure between the default events and the recovery rates.
(v) Introducing a coherent methodology for capital allocation that takes into account the non-normality of the portfolio credit loss distribution.

In particular, the quantitative approach used in this paper for estimating the portfolio’s credit risk is of the binomial type, and is based on Monte Carlo Simulation. This methodology takes into account the default correlations among the credit counterparties in the portfolio—following the idea of the copula approach, first developed in Li (2000)—and utilizes the asset return correlations of the obligors, as estimated by means of rigorous statistical methods (Lucas 1995; de Servigny and Renault 2002; Frye 2000; Hamerle and Rösch 2006). Li (2000) first introduced a random variable called time-until-default, which measures the length of time from today until default time, to indicate the survival time of each defaultable obligor. Then, Li (2000) defined the default correlation between two obligors by the correlation between their survival times. Following this original idea, we construct the dependence structure of defaults by means of a one-factor model, generating the scenarios of the times until default’s random vector for the N exposures in the portfolio from the Gaussian copula (Gregory and Laurent 2004). The concept of copula goes back to Sklar (1959). Copula is a function of several variables, and describes, in a powerful way, how joint distribution is linked to its univariate margins. Copula functions are used to combine marginal distributions into a multivariate distribution. They are unique: for any given multivariate distribution (with continuous marginal distributions) there is a unique copula function that represents it. They are also invariant under strictly increasing transformations of the marginal distributions. Moreover, copula functions have long been recognized as a powerful tool for modelling dependence between random variables (Nelsen 1999). The basic idea behind copulas is to separate dependence and the marginal behavior of the univariates. Also, our methodology can consider the recovery rates as stochastic and dependent on each other and on the time until defaults, following the examples of Pykhtin (2003), Tasche (2004), Emmer and Tasche (2004), Gregory and Laurent (2004), and Chabaane et al. (2004).

Moreover, the approach utilized for allocating credit capital coherently (Overbeck 2000; Denault 2001; Kalkbrener et al. 2004; Kalkbrener 2005) estimates the marginal contributions of each
obligor to the overall risk of the loan portfolio in terms of Expected Shortfall (ES), a risk measure that is coherent (in sense of Artzner et al. 1999; Tasche 2002; Acerbi and Tasche 2002) and more conservative than the traditional measure of Value-at-Risk (VaR).

Finally, this advanced analytical structure is implemented to a hypothetical but typical loan portfolio of an Italian commercial bank operating across the country. The national loan portfolio is structured with 17 sub-portfolios or regional clusters of credit exposures to 10,500 non-financial firms (or corporates) belonging to each geo-cluster or sub-portfolio.

The outcomes in term of correlations, portfolio risk measures, and capital allocations obtained from this advanced analytical framework are compared with the results found by implementing the internal rating based approach of Basel II and III.

In particular, the contributions of each geographical sub-portfolio to the credit risk of the overall Italian loan portfolio have been estimated in terms of Value-at-Risk (VaR), Maximum Loss (ML) and Expected Shortfall (ES), calculated for a confidence level of 99.9% over an annual time horizon.

Our chief conclusion is that the IRB model is unable to capture the real credit risk of loan portfolios because it does not take into account the actual dependence structure among the default events, and between the recovery rates and the default events. For this reason, we underline that the adoption of this regulatory model can produce a dangerous underestimation of the portfolio credit risk, especially when the economic uncertainty and volatility of the financial markets increase. The whole paper is structured as follows. Section 2, and Sections 2.1 and 2.2, illustrate the quantitative characteristics of the advanced approach for estimating the credit risk of the loan portfolio consistently. In particular, this binomial (default/non-default) model is based on Monte Carlo Simulation, and takes into account the default correlations among obligors in the portfolio, following the idea of the copula approach first developed by Li (2000). Section 3 describes two sound statistical methods for estimating the asset correlations of obligors in the portfolio consistently. First, we follow Frye’s methodology (2000) for calculating the asset correlation coefficients by factor loadings (namely, the sensitivity coefficients of the obligor asset returns to the changes in the systematic factor), estimated through the maximum likelihood method (MLH). Secondly, we implement an alternative methodology for estimating the correlations, first presented by Lucas (1995) (de Servigny and Renault 2002; Hamerle and Rösch 2006). For comparison purposes, we apply these two robust statistical methodologies to the Italian loan portfolio utilizing the historical time series of default numbers and default rates from 2006 to 2019. These input data can be freely downloaded from the website of the Bank of Italy. Section 4 explains how a dependence structure between recovery rates and default events can be introduced into this credit portfolio model. Section 5 describes a coherent capital allocation technique, emphasizing its peculiarities with reference to the traditional capital allocation scheme founded on Markowitz’s (1952) portfolio theory. In Section 6, we implement the advanced analytical framework to a hypothetical but typical Italian loan portfolio composed of banking credit exposures to 10,500 non-financial Italian firms (or corporates) residing in each of the 17 Italian geographic clusters (regions). Comments and conclusions are reported in Section 7.

2. Credit Portfolio Model and Credit Risk Measures

We assume a loan portfolio with N obligors and a time horizon equal to the longest maturity among the credit assets in the portfolio\(^1\). The random variable (r.v.) \(L\), representing the portfolio loss, is defined following the notation used in Jouanin et al. (2004):

\[
L = \sum_{i=1}^{N} L_i = \sum_{i=1}^{N} E_a D_i \cdot (1 - R_i) \cdot 1\{\tau_i \leq M_i\}
\]  

(1)

Equation (1) denotes a default/non-default model, where \(L_i\) is the r.v. loss for each obligor \(i\), \(E_a D_i\) is the exposure at default of obligor \(i\), \(R_i\) is its recovery rate, \(M_i\) is the maturity of debt of obligor

\(^1\) Suppose we are in time 0, if the longest maturity is \(M_{\text{max}}\), the time horizon is [0,M_{\text{max}}].
The recovery rate $R_i$ may be assumed to be deterministic or stochastic with mean $m_i$ and standard deviation $s_i$, and independent of each other and of their respective times until default $\tau_i$.

The most common assumption about the distributional form of $R_i$ is the Beta $(a, b)$ distribution, with the parameters $a_i$ and $b_i$ estimated by the method of moments, knowing the values of $m_i$ and $s_i$.

Analytically:

\[
a_i = \frac{m_i^2(1 - m_i)}{s_i^2} - m_i, \quad b_i = \frac{m_i^2(1 - m_i)^2}{m_i s_i^2} - (1 - m_i)
\] (2)

The stochastic vector of the times until default $(\tau_1, \ldots, \tau_n)$ has a multivariate cumulative distribution function (c.d.f.), $F$. This may be written by the following copula representation:

\[
F(t_1, \ldots, t_n) = \Pr\{\tau_1 \leq t_1, \ldots, \tau_n \leq t_n\} = C(F_1(t_1), \ldots, F_n(t_n))
\] (3)

In Equation (3), $F_i$ is the marginal c.d.f. of $\tau_i$, and $C$ is the copula function that determines the dependence structure of the multivariate c.d.f. of the times until default vector.

Copula functions are used to combine marginal distributions into a multivariate distribution. They are unique: for any given multivariate distribution (with continuous marginal distributions) there is a unique copula that represents it. They are also invariant under strictly increasing transformations of the marginal distributions. Moreover, copulas have long been recognized as a powerful tool for modelling dependence between random variables (Nelsen 1999). The basic idea behind copulas is to separate the dependence and marginal behavior of the univariates. The most known copulas are the elliptical or standard ones. Important examples in this family of distributions are the Gaussian and Student’s t examples. The unknown c.d.f. $G$ of the r.v. $L$ (portfolio loss) may be estimated by Monte Carlo simulation using the following algorithm:

(1) Generate a determination of $N$ random variables uniformly distributed on $[0,1]$, $(u_1, \ldots, u_N)$ from the copula $C$.

(2) Determine a scenario for the times until default by inverting $(u_1, \ldots, u_N)$ using the margins:

\[
t_i = F_i^{-1}(u_i), i = 1, \ldots, N.
\]

(3) For every obligor $i = 1, \ldots, N$, if $t_i \leq M_i$ we then obtain a loss scenario equal to $EaD_i(1-R_i)$, or equal to 0 otherwise. In the case of stochastic recovery rates, the determination of $R_i$ is generated from a Beta $(a_i, b_i)$ c.d.f.

(4) Add up the losses of the $N$ obligors, obtaining a scenario of the portfolio loss, $L_i$.

(5) Steps from 1 to 4 are repeated a great number of times, $s$.

From the distribution of the portfolio losses obtained by the simulation, we may estimate different risk measures for the loan portfolio, such as the expected loss, $EL$; the maximum loss, $ML$; the Value at Risk, $VaR$; and the Expected Shortfall, $ES$. In particular, portfolio $EL$ is calculated as the mean of the portfolio losses for all $s$ scenarios. Analytically:

\[\text{ES} = \text{mean of portfolio losses for all scenarios.}\]

\[\text{ML} = \text{maximum of portfolio losses for all scenarios.}\]

\[\text{VaR} = \text{quantile of portfolio losses for all scenarios.}\]

\[\text{EL} = \text{mean of portfolio losses for all scenarios.}\]

\[\text{ES} = \text{mean of portfolio losses for all scenarios.}\]

\[\text{ML} = \text{maximum of portfolio losses for all scenarios.}\]

\[\text{VaR} = \text{quantile of portfolio losses for all scenarios.}\]

\[\text{EL} = \text{mean of portfolio losses for all scenarios.}\]
\[
EL = \frac{\sum_{j=1}^{s} L_j}{s}
\]

The portfolio ML at the probability level \( \alpha \), \( ML_{\alpha} \), may be calculated by ordering the \( s \) scenarios of portfolio loss in non-decreasing order and cutting the obtained distribution at the \( \alpha \)-th percentile.

Portfolio Credit VaR, at the probability level \( \alpha \), is calculated as the difference between the ML at the same probability level \( \alpha \) and the EL of the portfolio. Analytically:

\[
\text{VaR}_{\alpha} = ML_{\alpha} - EL
\]

The portfolio ES, at the probability level \( \alpha \), \( ES_{\alpha} \), is calculated as the conditional mean of the portfolio losses exceeding the ML. Analytically:

\[
ES_{\alpha} = ML_{\alpha} + \frac{1}{(1-\alpha) \cdot s} \sum_{j=1}^{s} (L_j - ML_{\alpha})^+
\]

where \((L_j - ML_{\alpha})^+ = L_j - ML_{\alpha} \) if \( L_j - ML_{\alpha} > 0 \); \((L_j - ML_{\alpha})^- = 0 \) if \( L_j - ML_{\alpha} \leq 0 \).

2.1. Determining the Marginal Distributions for the Times until Default

In order to apply the algorithm described in the previous Section 2, it is necessary to give a functional form to the marginal cumulative distribution functions, \( F_i \), for the random variables' times until default, and to estimate their parameters. In order to do this, we have to introduce the hazard rate function, \( h(t) \), defined as follows:

\[
h_i(t) = \lim_{\Delta t \to 0} \frac{\Pr \{ t < \tau_i \leq t + \Delta t | \tau_i > t \}}{\Delta t}
\]

By extending Equation (7), the following is obtained:

\[
h_i(t) = \lim_{\Delta t \to 0} \frac{\Pr \{ t < \tau_i \leq t + \Delta t \}}{\Delta t \Pr \{ \tau_i > t \}} = \frac{\partial}{\partial t} \frac{F_i(t)}{1 - F_i(t)} = -\frac{\partial}{\partial t} \ln(1 - F_i(t))
\]

By solving the differential Equation (8), we obtain

\[
F_i(t) = P_i[\tau_i \leq t] = 1 - \exp \left( -\int_0^t h_i(u) du \right)
\]

If we assume that the time structure of the hazard rate function is flat, that is \( h(t) = h_i \) for each \( t \), we can rewrite Equation (9) as follows:

\[
F_i(t) = P_i[\tau_i \leq t] = 1 - e^{-ht}
\]

The hazard rate function completely characterizes the distribution of the random variable \( \tau_i \). Therefore, the calibration of \( h(t) \) from real data is a core issue. In pricing applications, the hazard rates are usually calibrated using market data such as the quotations of defaultable bonds, asset swap spreads, or Credit Default Swaps. Conversely, for risk management applications, the hazard rates may be calibrated using the probability of default provided by an internal or external credit rating assessment. For instance, if \( q(0, t) \) is the average cumulative default rate over the time horizon \([0, t] \), from Equation (10), we obtain:

5 In this case, a risk neutral measure of \( h \) is obtained.
6 i.e., rating agencies.
7 Usually, since we dispose of the one-year default probabilities, \( t=1 \).
\[ 1 - e^{-h_j(t)} = q_j(0,t) \Rightarrow h_j(t) = -\ln(1-q_j(0,t))/t \] (11)

If we dispose of a term structure of the default probabilities, a piecewise constant functional form for the hazard rates may be assumed. If \( T_1, T_2, \ldots, T_n \) are the nodes of the term structure of the default probabilities (for years), then the hazard rate function may be written in the following way:

\[ h_j(t) = \sum_{i=1}^{m} h_{i,j} 1_{(T_{j-1}, T_j)}(t) \] (12)

In Equation (12), \( h_{i,j} \) are positive constants, \( j = 1, \ldots, m \) and \( 1_{(T_{j-1}, T_j)}(t) = 1 \) if \( t \in (T_{j-1}, T_j] \). This hypothesis implies that the c.d.f. \( F_i(t) \) may be written as follows:

\[ F_i(t) = 1 - \exp\left\{ -\sum_{j=1}^{k} h_{i,j} (T_j - T_{j-1}) \right\} \quad \text{if} \quad k = 1 \text{ if } t \leq T_1 \\
= 2 \text{ if } T_1 < t \leq T_2 \\
= \ldots \\
= m \text{ if } t > T_{m-1} \] (13)

From Equation (13), \( h_{i,1} \) may be estimated using the probability of default over the maturity \( T_1 \); \( h_{i,2} \) may be calibrated using the default probability over the maturity \( T_2 \), known \( h_{i,1} \), and so on. The remaining \( h_{i,j} \) may be calibrated up to time \( T_n \).

2.2. A One-Factor Model for Generating Scenarios from the Gaussian Copula

To apply the algorithm of Section 2, it is necessary to generate scenarios \( (u_1, \ldots, u_N) \) from the generic copula, \( C \). For this purpose, in the academic literature and in practical industry applications, the most utilized copulas are the Gaussian and the Student’s t examples. These copulas are easy to implement and, furthermore, are endowed with a sufficient parameter number for describing the portfolio’s dependence structure effectively. The most important parameter to calibrate is the correlation matrix.

Li (2000) demonstrated that, when we model the c.d.f. in Equation (3) using the Gaussian copula, in the bivariate case, the correlation parameter is equal to the asset correlation between the two counterparties. This result may be extended to the case of the t-copula (see Mashal and Naldi 2002; Meneguzzo and Vecchiato 2004). Therefore, the elements of a correlation matrix with dimensions \( N \times N \) are the asset correlations among the \( N \) obligors in the portfolio.

Nevertheless, the number of obligors for a typical Italian commercial bank is so high as to require high costs in terms of memory space and computational time to implement the Monte Carlo methodology. For this reason, it is convenient to utilize a factorial model with \( J \) clusters in order to simulate scenarios from the Gaussian or the Student’s t copula (Gregory and Laurent 2004). Since \( J \) (the number of clusters) is much lower than \( N \) (the number of obligors), the number of parameters to be estimated and the computational costs will also be much smaller.

We suppose, for instance, to generate scenarios from the Gaussian copula simply by a one-factor model (Merton 1974) which represents the asset return of obligor \( i, Y_i \) for \( i = 1, \ldots, N \). Analytically:

\[ Y_i = b^2_{m(i)} X + \sqrt{1-b^2_{m(i)}} e_i \] (14)

---

8 e.g., we dispose of the probabilities of default over the time horizons 1, 2, 5 and 10 years.
9 We use the term correlation matrix even if it is not completely appropriate in the case of the Student’s t-copula.
10 The clusters may be industrial sectors or geographical areas.
11 The Merton Model is used in Basel’s IRB model.
In Equation (14), \( X \) and \( e \) are independent standard normal random variables\(^{12}\), \( b_{m(i)} \) is the weight of the systematic component \( X \), \( m(i) \) is the relation linking obligor \( i \) to his cluster \( j = m(i), j = 1, \ldots, J \); moreover \( e, \ldots, e \) are independent. In this setting, \( Y_i \) is a standard normal r.v., too.

The weights \( b_{m(i)} \) have been assumed equal for each obligor who belongs to the same cluster. They are calibrated using the asset return correlation intra-cluster. In fact, the asset return correlation between two obligors \( i \) and \( j \) belonging to the same cluster \( k \) is the following:

\[
\rho_k = E[Y_i Y_j] = b_{m(i)} b_{m(j)} = b_k^2
\]  

(15)

In Equation (15) \( b_k = \sqrt{\rho_k} \). Therefore, the asset return correlation between two obligors \( i \) and \( j \) belonging to two different clusters, respectively \( k \) and \( l \), is the following:

\[
\rho_{kl} = E[Y_i Y_j] = b_{m(i)} b_{m(j)} = b_k b_l = \sqrt{\rho_k \rho_l}
\]  

(16)

To generate a scenario from the Gaussian copula, the following algorithm may be applied:

1. Generate \( N + 1 \) independent random variates from the standard normal distribution (they are the determinations of \( X, e, \ldots, e \));
2. Calculate a scenario \( y \) of \( Y_i, i = 1, \ldots, N \);
3. The scenario \( \mu = \Phi(y), i = 1, \ldots, N \), where \( \Phi \) is the standard normal c.d.f., is generated from the Gaussian copula.

To generate a scenario from the Student’s t copula with \( v \) degrees of freedom by a one-factor model (see Frey and McNeil 2003; Wehrspohn 2003), it is sufficient to transform Equation (14) as follows:

\[
Y_i = \sqrt{\frac{v}{W}} \left( b_{m(i)} X + \sqrt{1 - b_{m(i)}^2} e_i \right)
\]  

(17)

In Equation (17) \( X, e, \ldots, e \) are independent standard normal random variables, and \( W \) is a chi-square r.v. with \( v \) degrees of freedom, independent of \( X, e, \ldots, e \). In this case, the algorithm to apply is the following:

1. Generate \( N + 1 \) independent random variates from the standard normal distribution (they are the determinations of \( X, e, \ldots, e \)), and a determination from the chi-square r.v. with \( v \) degrees of freedom, \( W \), independent of \( X, e, \ldots, e \);
2. Calculate a scenario \( y \) of \( Y_i, i = 1, \ldots, N \) using Equation (17);
3. The scenario \( \mu = T_i(y), i = 1, \ldots, N \), where \( T_i \) is the standardized Student’s t c.d.f. with \( v \) degrees of freedom, is generated from the Student’s t-copula with \( v \) degrees of freedom.

The correlation structure implicit in the one-factor model (Equation (14))) is very restrictive. In fact, the correlations of assets belonging to different clusters \( k \) and \( l \) (namely the inter-cluster asset correlations) are implicitly determined by the intra-cluster asset correlations by Equation (16). To get a more complete correlation structure, the following factorial model (Gregory and Laurent 2004) may be used\(^{13}\):

\[
Y_i = b_{m(i)} X_{m(i)} + \sqrt{1 - b_{m(i)}^2} e_i
\]  

(18)

For Equation (18), the same consideration made for Equation (14) holds; moreover, the systematic risk factor, \( X_{m(i)} \), is expressed by a second one-factor model:

\(^{12}\) X may be seen as the return of the macroeconomic factor or the global market index common to the all obligors in the portfolio, representing the systematic factor, \( Y_i \) while \( e \) may be interpreted as the portion of the asset return which is not explained by the systematic factor (that is the specific or idiosyncratic factor).

\(^{13}\) In order to get a model with an even less restricted dependence structure, see Jouanin et al. (2004).
\[ X_j = a_j X + \sqrt{1-a_j^2} e_j, \quad j = 1, \ldots, J \]  

(19)

In Equation (19), \( e_j, Y_j \) are independent standard normal random variables. Therefore, by substituting Equation (19) into Equation (18), we get to:

\[ Y_i = b_{m(i)} a_{m(i)} X + b_{m(i)} \sqrt{1-a_j^2} e_j + \sqrt{1-b_{m(i)}^2} e_i \]  

(20)

Therefore, by the factorial model described in Equation (20), the asset correlation between two obligors belonging to the same cluster \( m(i) = j, j = 1, \ldots, J \) is \( \beta_{m(i)}^2 \). On the contrary, the correlation between two different clusters \( m(i) \) and \( m(j) \), with con \( m(i) \neq m(j) \), is \( \beta_{m(i) m(j)} \). Let us assume, for instance, a correlation structure where the intra-cluster correlations are equal to \( \rho_j, j = 1, \ldots, J \), while all the inter-cluster correlations are equal to \( \rho \). In order to get to this kind of dependence structure, it is sufficient to calibrate the model in Equation (20) in the following way: \( b_j = \sqrt{\rho_j} \) and \( a_j = \sqrt{\rho_j} \), \( j = 1, \ldots, J \).

3. Estimating Asset Correlations

The simplest methodology for estimating the asset return correlation for corporates is to assume them all equal; for instance, to 0.20\(^{14} \). A second methodology, following the last version of Basel’s IRB approach\(^{15} \), determines the asset correlation for each cluster by implementing the regulatory formula in Equation (21):

\[ \rho_i = 0.12 \left( \frac{1 - e^{-50 P(i)}}{1 - e^{-50}} \right) + 0.24 \int \left( \frac{1 - e^{-50 P(i)}}{1 - e^{-50}} \right) \]  

(21)

In Equation (21), \( \rho_i \) is the intra-cluster asset return correlation for cluster \( i, i = 1, \ldots, J \). The asset return correlations between clusters \( i \) and \( j \) are implicitly calculated as \( \rho_{i,j} = \sqrt{\rho_i \rho_j} \).

An alternative methodology for estimating the correlations, adopted first in Lucas (1995), utilizes the historical yearly time series of the default number and of the obligor number for each geo-sectorial cluster\(^{16} \) as input data.

Let \( N(i) \) be the number of obligors at the beginning of year \( j \) in cluster \( i, j = 1, \ldots, n, i = 1, \ldots, J \); let \( S(i) \) be the number of defaults, proceeding from the \( N(i) \) obligors, over year \( j \) in cluster \( i \).

The probability of \( k \) defaults in cluster \( i \) may be assessed as follows:

\[ P_k(i) = \sum_{j=1}^{n} \frac{\binom{S(i)}{k}}{\binom{N(i)}{k}}, \text{ otherwise} \]

\[ P_k(i) = \sum_{j=1}^{n} \frac{\binom{N(i)}{k}}{\binom{S(i)}{k}} \]  

(22)

\(^{14}\) This is the solution adopted in the first version of the IRB model by the Basel Committee.

\(^{15}\) See Basel Committee on Banking Supervision (2003, 2004).

\(^{16}\) These data can be downloaded freely from the web site of Bank of Italy:

www.bancaditalia.it/statistiche/index.html.
The probability of two defaults, the first in cluster \( i \) and the second in cluster \( k \), is the following:

\[
P_2(i,k) = \frac{1}{n} \sum_{j=1}^{n} \frac{S_j(i)S_j(k)}{N_j(i)N_j(k)}
\]  

(23)

The intra-cluster default correlation for cluster \( i \) is:

\[
\rho_D(i) = \frac{P_2(i) - (P_1(i))^2}{P_1(i) - (P_1(i))^2}
\]  

(24)

The default correlation between clusters \( i \) and \( k \) is:

\[
\rho_D(i,k) = \frac{P_2(i,k) - P_1(i)P_1(k)}{\sqrt{[P_1(i) - (P_1(i))^2][P_1(k) - (P_1(k))^2]}}
\]  

(25)

The corresponding asset return correlation may be obtained by solving the following Equation (26)\(^\text{17}\) for \( \rho \):

\[
P_2(i,k) = \int_{-\infty}^{\Phi^{-1}(P_1(i))} \int_{\Phi^{-1}(P_1(i))}^{\infty} \Phi_2(x,y; \rho_{i,k}) dxdy
\]  

(26)

A further methodology (Frey 2000) for estimating the asset return correlation intra-cluster is presented in the following. For application to an Italian loan portfolio, the input data may be represented by the historical time series of the decay or deterioration rates\(^\text{18}\), \( TdD_{i,t} \), downloaded from the website of the Bank of Italy for each year \( t \), \( t = 1, ..., T \), concerning different geo-sectorial clusters \( j \), \( j = 1, ..., J \) and counterparty categories.

The standardized asset return for a generic obligor belonging to cluster \( j \) is assumed to be represented by the one-factor model in Equation (14).

Therefore, estimating the intra-cluster asset return correlation \( \rho_t \) is equivalent to estimating the weight (factor loading) \( b_t \). In fact, it is easy to demonstrate\(^\text{19}\) that \( \rho_t = b_t^2 \). In order to estimate the weights \( b_t \), it is assumed that the number of obligors into each cluster is very high, and that these obligors are homogeneous; that is, all obligors in a cluster get the same downgrading rate and the same factor loading. Therefore, by the Law of Large Numbers (LLN), it is assumed that the downgrading rate observed in year \( t \) is equal to the default probability conditional to the value \( x_t \) of the systematic risk factor \( X \) observed in year \( t \), that is:

\[
TdD_{i,t} = \Pr[X < \Phi^{-1}(TdD_j)|X = x_t] = \Phi\left[\Phi^{-1}(TdD_j) - b_t x_t \sqrt{1 - b_t^2}\right] = g_j(x_t)
\]  

(27)

In Equation (27), \( \Phi \) is the standard normal p.d.f. and \( TdD_j \) is the mean decay rate over the long period in cluster \( j \). Since \( g(x) \) in Equation (27) is a monotonic function of the systematic risk factor \( X \), which is standard normal distributed, the probability density function (p.d.f.) of \( g(X) \) may be written following the Vasicek (2015a, 2015b) formula in this way (Finger 1999, 2001):

\[
f_j(TdD_{i,t}) = \frac{\Phi(\Phi^{-1}(TdD_{i,t}) - \Phi^{-1}(TdD_j))}{\sqrt{1 - b_j^2}} \cdot \frac{\Phi^{-1}(TdD_{i,t}) - \Phi^{-1}(TdD_j)}{b_j \Phi(\Phi^{-1}(TdD_{i,t}))}
\]  

(28)

In Equation (28), \( \Phi \) is the standard normal p.d.f.

To estimate \( b_t \), it is necessary to maximizing the log-likelihood function in Equation (29):
\[
\hat{b}_j = \max_{b_j} \sum_{t=1}^{T} \ln f_j(TdD_{i,j})
\]  
(29)

As a consequence of the one-factor model, it is simple to demonstrate that the asset correlation between two obligors belonging to two different clusters \(i\) and \(j\) is: 
\[
\rho_{i,j} = \rho_i \rho_j.
\]

4. Introducing a Dependence Structure between Recovery Rates and Default Events

So far, we have always supposed the independence among the recovery rates themselves and the times until default. However, in this section, we assume that the recovery rates are correlated to each other and to the times until default by a factorial model. In particular, we follow the approach described in Pykhtin (2003), Tasche (2004), and Gregory and Laurent (2004).

In this context, the portfolio loss, calculated as the sum of the losses of all obligors \(i\) in the portfolio, is driven by two random variables: \(Y_i\), linking up with the times until default, and \(V_i\) driving the recovery rate and hence the loss amount. Both these two random variables may be interpreted, according to the Merton model, as the asset return for obligor \(i\), respectively before, \(Y_i\), and immediately after default, \(V_i\). The economic intuition is that, at the time of default, the recovery rate will be as low as the return on assets is lower immediately after default.

Both the two random variables, \(Y_i\) and \(V_i\), may be expressed through the following factorial model, assuming the correlation among recovery rates and between them and the times until default is constant in each cluster:

\[
\begin{align*}
Y_i &= b_{m(i)} X + \sqrt{1 - b_{m(i)}^2} e_i \\
V_i &= c_{m(i)} X + \sqrt{1 - c_{m(i)}^2} e_i
\end{align*}
\]  
(30)

In the model in Equation (30), \(X, e_i\) and \(e_i, i = 1, \ldots, N\) are independent standard normal random variables; the asset correlations are functions of the factor loadings \(b_{m(i)} > 0\) and \(c_{m(i)} > 0\). Due to calibration problems, it is convenient to assume \(b_{m(i)} = c_{m(i)}\) as in Tasche (2004).

The r.v. \(Y_i\) drives the time until default of each obligor \(i\), as described in the previous sections. On the contrary, the value of the recovery rate \(R_i\), proceeds from the value assumed by \(V_i\). If the obligor \(i\) defaults in scenario \(j\), then a determination of the recovery rate is generated; otherwise, the loss in this scenario is 0. In particular, the obligor \(i\) defaults if \(Y_i < \Phi^{-1}(q_{m(i)}(0,M))\), where \(\Phi\) is the standard normal c.d.f. and \(q_{m(i)}(0,M)\) is the default probability of cluster \(m(i)\) for the maturity \(M\). This kind of event is equivalent to the event \(\tau_i = F_i^{-1}(\Phi(Y_i)) < M_i\), since for Equation (10): \(q_{m(i)}(0,M) = F_i(M_i) = \Pr\{\tau_i \leq M_i\}\). If \(G(x;a,b)\) is the Beta c.d.f., with parameters \(a\) and \(b\) estimated using Equation (2), then the determination of the recovery rate in the default case is the following:

\[
R_i = G^{-1}\left(\Phi(c_{m(i)} X + \sqrt{1 - c_{m(i)}^2} e_i); a, b\right)
\]  
(31)

In Equation (31), the value of the recovery rate is correlated to the default event through the systematic factor \(X\). The factor \(X\) may represent, for instance, a proxy of the general economic condition. In case of a negative economic condition, \(X\) assumes low values; hence, a higher number of defaults (driven by the random variables \(Y_i\)) may happen and the values of the recovery rates (driven by the random variables \(V_i\)) are expected to be lower. The contrary may happen in case of a positive economic condition.

5. Capital Allocation

After calculating the credit portfolio risk measures by the methodology described in the previous sections, it is necessary to allocate the estimated capital among the different obligors or sub-portfolios (clusters).
The typical industry standard solution (Litterman 1996; Overbeck 2000) is to allocate the portfolio VaR among all obligors, or sub-portfolios, proportionally to their covariance: \( \text{Cov}(L_i, L), \ldots, \text{Cov}(L_N, L) \), where \( L_i \) is the r.v. loss of the generic obligor \( i (i = 1, \ldots, N) \) and \( L \) is the r.v. loss of the overall credit portfolio. This capital allocation technique, known as volatility allocation, is the natural choice in the bounds of classical portfolio theory, where risk is measured by the standard deviation. The use of this technique for VaR allocation is correct when all the marginal loss distributions are normal. Unfortunately, this is not the case, mainly for the credit asset portfolios. The capital allocated to a sub-portfolio \( P \) might be greater than the risk capital of \( P \) considered as a stand-alone portfolio (discouraging portfolio diversification). In other words, the capital requirement of a single loan might be greater than its exposure value. On the contrary, a coherent capital allocation scheme has to satisfy the following three properties (Kalkbrener et al. 2004; Kalkbrener 2005):

- The capital allocated to a union of sub-portfolios has to be equal to the sum of the capital amounts allocated to the single sub-portfolios. In particular, the whole portfolio risk capital is the sum of the risk capitals of its sub-portfolios.
- The capital allocated to a sub-portfolio \( X \) belonging to a larger portfolio \( Y \) never has to exceed the risk capital of \( X \) considered as a stand-alone portfolio.
- A small increase of exposition value has to produce a small effect on the risk capital allocated to that exposition.

The capital allocation performed using the Expected Shortfall as a risk measure satisfies the three previous requirements. In other words, the capital allocation by ES is coherent. Analytically, the capital amount allocated to the obligor (or cluster) \( i \) through the ES measure, calculated at the probability level \( \alpha \), is the following:

\[
E \left( L_i \bigg| L > M_{L_{\alpha}}(L) \right) = \frac{1}{1 - \alpha} E \left( L_i \cdot 1_{(L > M_{L_{\alpha}}(L))} \right) \quad (32)
\]

In Equation (32), \( L_i \) is the r.v. loss for obligor \( i \), \( L \) is the r.v. portfolio loss, \( M_{L_{\alpha}}(L) \) is the portfolio Maximum Loss, and \( 1 \) is an r.v., assuming the value 1 if event \( E \) is true and 0 if it is false.

Equation (32) represents the mean contribution of the obligor \( i \) to the portfolio losses exceeding \( M_{L_{\alpha}} \). It may be easily computed by Monte Carlo simulation: first by storing the losses \( L_j \) occurred in the \( j \) scenarios when the portfolio loss \( L_i \) is greater than \( M_{L_{\alpha}} \), and secondly by calculating their conditional mean.

6. Implementation to a Typical Italian Loan Portfolio

In this section, we implement the overall methodology described previously to a typical, but hypothetical, loan portfolio of an Italian commercial bank. The results, in terms of the portfolio credit risk measures and capital allocations, are compared to the ones obtained by the Internal Rating Based (IRB) approach developed by the Basel Committee\(^{20}\). According to Basel’s IRB approach, given a portfolio composed of corporate exposures, the minimum banking capital requirement for a generic obligor \( i \) is the following:

\[
K_i = EaD_i \times LGD_i \times \left( \Phi \left( \frac{1}{\sqrt{1 - \rho_i}} \Phi^{-1}(PD_i) + \frac{\rho_i}{1 - \rho_i} \Phi^{-1}(0.999) \right) \right) \times PD_i \times \frac{1}{(1 - 1.5 \times b(PD_i))} \times (1 + M_i - 2.5) \times b(PD_i)
\]

In the regulatory formula above, \( PD_i \) is the one-year default probability of obligor \( i \), \( EaD_i \) is the banking exposure at default \( i \), \( LGD_i = (1 - R) \) is the loss given default, \( M_i \) is the maturity of the loan given to the generic obligor \( i \) and \( \rho_i \) is the asset return correlation estimated through the regulatory formula (see Equation (21)).

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\(^{20}\) See Basel Committee on Banking Supervision (2003, 2004).
The minimum capital requirement, \( K_i \), is the contribution of the exposure of obligor \( i \) to the credit risk of the loan portfolio; namely, the one-year Credit VaR, adjusted over a time horizon \([0, M]\), at the probability level of 99.9\% for a homogeneous portfolio with infinite granularity\(^2\).

The hypothetical Italian banking portfolio is composed of credit exposures to 10,500 Italian non-financial firms (corporates) residing in each of the 17 different Italian regions. We assume that all obligors belonging to the same Italian region share the same probability of default (that is, the PD of that region or cluster). As a proxy of the long period PD for each cluster, we have utilized the annualized quarterly credit decay rates provided by the Statistic Report of the Bank of Italy, from 06/30/2006 to 09/30/2019, for each Italian regional cluster and the category of non-financial firms\(^2\). The asset return correlations are assumed to be constant for each cluster. We calculate the correlations by both the Basel regulatory formula (Equation (21)) and the two robust statistical methodologies described in Section 3. All the maturities are assumed to be equal to one year. Initially, all the LGDs are assumed to be non-stochastic and constant to 50\%.

In Table 1, we show the main characteristics of the hypothetical Italian loan portfolio. It is composed of 17 clusters (or sub-portfolios) representing the 17 different Italian regions in which the obligors (namely, non-financial firms) reside. In Table 1, we report for each regional cluster \( i \) (\( i = 1, ..., 17 \)), the value of the credit exposure in Euros, \( \text{EaD}_i \); the number of the obligors belonging to each regional cluster, \( N_i \); the probability of default of the cluster, \( PD_i \); and the asset return correlation of each cluster, calculated by both Basel’s formula and by the two selected statistical methods.

**Table 1.** Composition and characteristics of the Italian loan portfolio composed of 17 regional clusters.

| Cluster           | \( \text{EaD}_i \) | \( N_i \) | \( PD_i \) | \( \text{rho (Lucas)} \) | \( \text{rho (MLH)} \) | \( \text{rho (Basel)} \) |
|-------------------|-------------------|----------|---------|------------------------|------------------------|------------------------|
| LIGURIA           | 102,000           | 510      | 3.43\%  | 1.85\%                 | 1.87\%                 | 14.16\%                |
| LOMBARDIA         | 252,000           | 1260     | 3.22\%  | 1.97\%                 | 2.05\%                 | 14.40\%                |
| TRENTINO-ALTO ADIGE | 72,000          | 360      | 2.71\%  | 1.69\%                 | 2.03\%                 | 15.10\%                |
| VENETO            | 142,000           | 710      | 3.09\%  | 1.77\%                 | 1.84\%                 | 14.56\%                |
| FRIULI-VENEZIA GIULIA | 64,000          | 320      | 3.19\%  | 1.58\%                 | 1.62\%                 | 14.43\%                |
| EMILIA-ROMAGNA    | 183,000           | 915      | 3.01\%  | 1.90\%                 | 1.98\%                 | 14.66\%                |
| MARCHE            | 94,000            | 470      | 3.80\%  | 2.68\%                 | 2.56\%                 | 13.79\%                |
| TOSCANA           | 128,000           | 640      | 3.80\%  | 1.92\%                 | 2.00\%                 | 13.80\%                |
| UMBRIA            | 76,000            | 380      | 4.01\%  | 2.45\%                 | 2.45\%                 | 13.62\%                |
| LAZIO             | 231,000           | 1155     | 4.89\%  | 1.68\%                 | 1.68\%                 | 13.04\%                |
| CAMPANIA          | 132,000           | 660      | 5.17\%  | 1.81\%                 | 1.72\%                 | 12.91\%                |
| CALABRIA          | 54,000            | 270      | 5.82\%  | 2.20\%                 | 2.17\%                 | 12.65\%                |
| SICILIA           | 174,000           | 870      | 5.39\%  | 1.91\%                 | 1.77\%                 | 12.81\%                |
| SARDEGNA          | 68,000            | 340      | 4.93\%  | 1.69\%                 | 1.72\%                 | 13.02\%                |
| PIEMONTE E VALLE D’AOSTA | 153,000       | 765      | 3.06\%  | 1.59\%                 | 1.60\%                 | 14.60\%                |
| ABRUZZO E MOLISE  | 84,000            | 420      | 4.83\%  | 2.55\%                 | 2.58\%                 | 13.07\%                |
| PUGLIA E BASILICATA | 91,000          | 455      | 4.35\%  | 2.05\%                 | 1.96\%                 | 13.36\%                |
| **TOTAL**         | **2,100,000**     | **10,500**|          |                        |                        |                        |

Source: our elaboration.

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\(^2\) i.e., all the obligors in the portfolio have the same exposure, the number of obligors is very high, each exposure is very low compared to the total portfolio exposure, \( \text{EaD} \), the VaR is estimated by a one-factor version of the Merton (1974) model.

\(^2\) These data can be freely downloaded by the web site: www.bancaditalia.it.
We may note that the asset return correlations estimated by the maximum likelihood method and by Lucas’s approach are remarkably lower than the ones calculated by Basel’s formula.

We underline that this outcome is coherent with the results of other empirical studies on the Italian, French and German markets (Dietsch and Petey 2004; Duellmann and Scheule 2003; Hamerle and Rösch 2006; Sironi and Zazzara 2003).

The majority of Italian companies are small and medium enterprises (SMEs). It is well known that SMEs are more affected by their specific or idiosyncratic risks than by systematic risks. On the contrary, large firms are much more sensitive to the fluctuations of systematic risk factors such as, for example, the general economic condition.

Moreover, from our empirical results, we do not observe a negative relation between the estimated correlations and the PDs. Conversely, Basel’s correlation formula assumes that the correlation decreases when the PD increases (and vice versa).

In Table 2, we collect the results in terms of capital requirements, K, or portfolio risk contributions for each cluster and for the total portfolio, obtained by implementing Basel’s IRB approach and utilizing both the regulatory correlations and the correlations estimated by maximum likelihood method alternately. We can immediately observe that the values of capital requirements (expressed in both percentages and monetary terms) collapse when we use the estimated correlations (rho MLH) instead of the regulatory ones (rho Basel). Basel’s IRB model is, therefore, extremely sensitive to the value of the asset correlations.

Table 2. Capital requirements, K, for each regional cluster, calculated by the IRB model and by utilizing the regulatory correlations (rho Basel) and the correlations estimated by MLH method (rho MLH) alternatively.

| Cluster             | K% (rho Basel) | K (rho Basel) | K% (rho MLH) | K (rho MLH) |
|---------------------|---------------|--------------|--------------|-------------|
| LIGURIA             | 10.75%        | 10,962       | 3.56%        | 3627        |
| LOMBARDIA           | 10.45%        | 26,340       | 3.49%        | 8806        |
| TRENTINO-ALTO ADIGE | 9.71%         | 6990         | 3.00%        | 2163        |
| VENETO              | 10.27%        | 14,582       | 3.23%        | 4592        |
| FRIULI-VENEZIA GIULIA | 10.41%    | 6661         | 3.17%        | 2030        |
| EMILIA-ROMAGNA      | 10.16%        | 18,590       | 3.26%        | 5959        |
| MARCHE              | 11.25%        | 10,573       | 4.38%        | 4121        |
| TOSCANA             | 11.24%        | 14,388       | 3.97%        | 5086        |
| UMBRIA              | 11.52%        | 8755         | 4.50%        | 3419        |
| LAZIO               | 12.66%        | 29,239       | 4.62%        | 10,678      |
| CAMPANIA            | 13.01%        | 17,175       | 4.88%        | 6444        |
| CALABRIA            | 13.82%        | 7464         | 5.84%        | 3155        |
| SICILIA             | 13.29%        | 23,126       | 5.11%        | 8895        |
| SARDEGNA            | 12.71%        | 8645         | 4.69%        | 3192        |
| PIEMONTE E VALLE D’AOSTA | 10.23%   | 15,646       | 3.05%        | 4666        |
| ABRUZZO E MOLISE     | 12.59%        | 10,577       | 5.35%        | 4498        |
| PUGLIA E BASILICATA | 11.97%        | 10,896       | 4.43%        | 4028        |
| **TOTAL**           | **11.46%**    | **240,610**  | **4.06%**    | **85,359**  |

Source: our elaboration.

Subsequently, we have calculated different portfolio risk measures, namely VaR, ML and ES, for a confidence level of 99.9% for each cluster and the total loan portfolio by implementing the MC simulation model described in Section 2. We have expressed the portfolio losses in monetary terms and in the percentage of the exposure value of the total portfolio and of each cluster. In order to
estimate the portfolio loss distribution, we have generated 100,000 MC scenarios. First, we have supposed that the credit exposure to each obligor is homogenous and equal to 200 euros; in this way, the hypothesis of the infinite granularity of the portfolio is approximated.

The outcomes, in terms of different portfolio risk metrics obtained by utilizing the rho estimated by MLH method, are reported in Table 3.

As expected, the results, in terms of the VaR in Table 3, are not too different from the ones obtained by Basel’s IRB model (see Table 2) and by utilizing the rho estimated by the MLH method.

In fact, from the simulative model we obtain, for example, a value of VaR for the total portfolio equal to 85,098 euros (or 4.05 percent of the total exposure) versus a capital requirement K from the Basel IRB model equal to 85,359 euros (or 4.06 percent of the total exposure).

Also, when we calculate the capital requirements by the two different models, assuming again the granularity of the loan portfolio, but utilizing the regulatory correlations, we find similar outcomes from the two approaches (see Table 4).

Table 3. Portfolio risk measures calculated by the MC simulation model, assuming the granularity of the loan portfolio and correlations estimated by maximum likelihood method.

| Cluster               | VaR 99.9% | %     | ML 99.9% | %     | ES 99.9% | %     |
|-----------------------|-----------|-------|----------|-------|----------|-------|
| LIGURIA               | 3215      | 3.15% | 4791     | 4.70% | 5184     | 5.08% |
| LOMBARDIA             | 8312      | 3.30% | 11,964   | 4.75% | 13,418   | 5.32% |
| TRENTINO-ALTO ADIGE   | 2415      | 3.35% | 3293     | 4.57% | 3365     | 4.67% |
| VENETO                | 4841      | 3.41% | 6816     | 4.80% | 7430     | 5.23% |
| FRIULI-VENEZIA GIULIA | 2365      | 3.70% | 3284     | 5.13% | 3412     | 5.33% |
| EMILIA-ROMAGNA        | 6440      | 3.52% | 8922     | 4.88% | 10,012   | 5.47% |
| MARCHE                | 4444      | 4.73% | 6052     | 6.44% | 6436     | 6.85% |
| TOSCANA               | 5449      | 4.26% | 7637     | 5.97% | 8284     | 6.47% |
| UMBRIA                | 3254      | 4.28% | 4625     | 6.09% | 5513     | 7.25% |
| LAZIO                 | 9371      | 4.06% | 14,450   | 6.26% | 16,431   | 7.11% |
| CAMPANIA              | 6406      | 4.85% | 9475     | 7.18% | 10,206   | 7.73% |
| CALABRIA              | 2765      | 5.12% | 4179     | 7.74% | 4998     | 9.26% |
| SICILIA               | 10,266    | 5.90% | 14,486   | 8.33% | 14,568   | 8.37% |
| SARDEGNA              | 2586      | 3.80% | 4095     | 6.02% | 4684     | 6.89% |
| PIEMONTE E VALLE      |           |       |          |       |          |       |
| D’AOSTA               | 4158      | 2.72% | 6265     | 4.09% | 7482     | 4.89% |
| ABRUZZO E MOLISE      | 4198      | 5.00% | 6026     | 7.17% | 6480     | 7.71% |
| PUGLIA E BASILICATA   | 4613      | 5.07% | 6396     | 7.03% | 6519     | 7.16% |
| TOTAL                 | 85,098    | 4.05% | 122,755  | 5.85% | 134,422  | 6.40% |

Source: our elaboration.

For example, we obtain a total portfolio VaR of 252,211 euros, equal to 12.01 percent of the total exposure from the simulative model (Table 4), versus a capital requirement value K of 240,610 euros for the total portfolio, or 11.46% of the total exposure from the IRB model (Table 2). It is obvious that the capital requirements grow as correlations increase, other conditions being equal.
Table 4. 99.9% VaR, 99.9% ML and 99.9% ES calculated from the simulation model by assuming the granularity of the portfolio and utilizing the regulatory correlations.

| Cluster                  | VaR 99.9% | %    | ML 99.9% | %    | ES 99.9% | %    |
|--------------------------|-----------|------|----------|------|----------|------|
| LIGURIA                  | 11,251    | 11.03% | 12,826   | 12.57% | 28,598   | 28.04% |
| LOMBARDIA                | 25,709    | 10.20% | 29,362   | 11.65% | 83,373   | 33.08% |
| TRENTINO-ALTO ADIGE      | 6185      | 8.59%  | 7062     | 9.81%  | 18,757   | 26.05% |
| VENETO                   | 13,920    | 9.80%  | 15,895   | 11.19% | 41,601   | 29.50% |
| FRIULI-VENEZIA GIULIA    | 6679      | 10.44% | 7598     | 11.87% | 11,975   | 18.71% |
| EMILIA-ROMAGNA           | 17,576    | 9.60%  | 20,058   | 10.96% | 53,219   | 29.08% |
| MARCHE                   | 10,412    | 11.08% | 12,021   | 12.79% | 29,338   | 31.21% |
| TOSCANA                  | 14,484    | 11.32% | 16,671   | 13.02% | 38,106   | 29.77% |
| UMBRIA                   | 8963      | 11.79% | 10,333   | 13.60% | 25,755   | 33.89% |
| LAZIO                    | 33,447    | 14.48% | 38,526   | 16.68% | 43,155   | 18.68% |
| CAMPANIA                 | 19,514    | 14.78% | 22,583   | 17.11% | 36,561   | 27.70% |
| CALABRIA                 | 8788      | 16.27% | 10,202   | 18.89% | 16,884   | 31.27% |
| SICILIA                  | 26,525    | 15.24% | 30,745   | 17.67% | 39,369   | 22.63% |
| SARDEGNA                 | 9604      | 14.12% | 11,112   | 16.34% | 19,470   | 28.63% |
| PIEMONTE E VALLE D’AOSTA | 15,288    | 9.99%  | 17,396   | 11.37% | 43,516   | 28.44% |
| ABRUZZO E MOLISE         | 11,534    | 13.73% | 13,361   | 15.91% | 29,008   | 34.53% |
| PUGLIA E BASILICATA      | 12,334    | 13.55% | 14,117   | 15.51% | 18,726   | 20.58% |
| TOTAL                    | 252,211   | 12.01% | 289,868  | 13.80% | 577,410  | 27.50% |

Source: our elaboration.

Successively, we relax the hypotheses of infinite granularity, or the absence of undiversified idiosyncratic risks in the portfolio, for calculating the capital requirements of a loan portfolio with the same characteristics but with much more concentrated credit exposures. Precisely, we assume that, in each cluster, half of the banking exposure is concentrated towards a single counterparty. For instance, in the case of cluster “Lazio”, 115,500 euros of the total exposure of 231,000 euros are concentrated in a single obligor, while the remaining 115,500 euros are homogeneously distributed among the remaining 1154 obligors in this cluster.

The outcomes, in terms of different portfolio risk measures estimated by the MC simulation model, are reported in Table 5 for the case of asset correlations calculated by the maximum likelihood method, and in Table 6 for the case of asset correlations calculated by Basel’s formula.

Dropping the hypothesis of infinite granularity for the loan portfolio, the results in terms of VaR obtained from the two different models differ strongly, particularly when the correlations are low. In fact, as we have already said, when the asset correlations are close to 1, the portfolio may be thought of as being composed of only one obligor. Therefore, when the correlations are high, a portfolio with high granularity exhibits similar results in terms of VaR to a portfolio with low granularity.

The capital requirements estimated by the simulative model are always greater than those calculated by the IRB model; they are approximately double. In particular, we find a value of VaR equal to 23.69% (see Table 6) for the simulation model versus a value of 11.46% for the Basel model (see Table 2) when we utilize the regulatory correlations, which are typically much higher than the correlations estimated by MLH method.
Table 5. 99.9% VaR, 99.9% ML and 99.9% ES estimated by the MC simulation model, assuming a concentrated portfolio and asset correlations estimated by the maximum likelihood method.

| Cluster                  | VaR 99.9% | %   | ML 99.9% | %   | ES 99.9% | %   |
|--------------------------|-----------|-----|----------|-----|----------|-----|
| LIGURIA                  | 7475      | 7.33 | 9051     | 8.87 | 12,496   | 12.25 |
| LOMBARDIA                | 49,148    | 19.50 | 52,801   | 20.95 | 105,417  | 41.83 |
| TRENTOIN-ALTO ADIGE      | 1830      | 2.54 | 2707     | 3.76 | 730      | 1.01  |
| VENETO                   | 12,998    | 9.15 | 14,973   | 10.54 | 2104     | 1.48  |
| FRIULI-VENEZIA GIULIA    | 1491      | 2.33 | 2410     | 3.77 | 796      | 1.24  |
| EMILIA-ROMAGNA           | 22,352    | 12.21 | 24,835   | 13.57 | 24,690   | 13.49 |
| MARCHE                   | 7402      | 7.87 | 9011     | 9.59 | 7508     | 7.99  |
| TOSCANIA                 | 14,848    | 11.60 | 17,035   | 13.31 | 9980     | 7.80  |
| UMBRIA                   | 4426      | 5.82 | 5796     | 7.63 | 1448     | 1.90  |
| Lazio                    | 50,678    | 21.94 | 55,757   | 24.14 | 70,900   | 30.69 |
| CAMPANIA                 | 18,102    | 13.71 | 21,171   | 16.04 | 25,979   | 19.68 |
| CALABRIA                 | 3300      | 6.11 | 4714     | 8.73 | 3873     | 7.17  |
| SICILIA                  | 27,839    | 16.00 | 32,058   | 18.42 | 34,769   | 19.98 |
| SARDEGNA                 | 4493      | 6.61 | 6002     | 8.83 | 1601     | 2.35  |
| PIEMONTE E VALLE D’AOSTA | 15,680    | 10.25 | 17,788   | 11.63 | 2332     | 1.52  |
| ABRUZZO E MOLISE         | 7765      | 9.24 | 9593     | 11.42 | 7311     | 8.70  |
| PUGLIA E BASILICATA      | 6896      | 7.58 | 8679     | 9.54  | 13,678   | 15.03 |
| TOTAL                    | 256,723   | 12.22 | 294,380  | 14.02 | 325,613  | 15.51 |

Source: our elaboration.

Utilizing low correlations, that is, the correlations obtained by the statistical methods (MLH and Lucas’s model, in our case), the capital requirements from the MC simulative model are always more severe than those attained from the IRB model; they are approximately triple. For example, we find by the simulative model a VaR of 12.22% (see Table 5) versus a capital requirement of 4.06% from the IRB model (see Table 2).

The underestimation of risk and capital is evident when we drop the strong hypothesis of a highly diversified portfolio. For this reason, mostly in the case of undiversified portfolios, coherent capital allocation is the appropriate choice for the purpose of risk management.

Table 6. 99.9% VaR, 99.9% ML and 99.9% ES, estimated by MC simulation model, assuming a concentrated portfolio and asset correlations calculated by regulatory formula.

| Cluster                  | VaR 99.9% | %   | ML 99.9% | %   | ES 99.9% | %   |
|--------------------------|-----------|-----|----------|-----|----------|-----|
| LIGURIA                  | 21,027    | 20.62 | 22,778   | 22.33 | 32,008   | 31.38 |
| LOMBARDIA                | 84,547    | 33.55 | 88,606   | 35.16 | 146,614  | 58.18 |
| TRENTOIN-ALTO ADIGE      | 10,166    | 14.12 | 11,141   | 15.47 | 11,003   | 15.28 |
| VENETO                   | 26,553    | 18.70 | 28,748   | 20.25 | 50,064   | 35.26 |
| FRIULI-VENEZIA GIULIA    | 6915      | 10.81 | 7936     | 12.40 | 6364     | 9.94  |
| EMILIA-ROMAGNA           | 43,261    | 23.64 | 46,019   | 25.15 | 85,417   | 46.68 |
| MARCHE                   | 17,663    | 18.79 | 19,450   | 20.69 | 21,906   | 23.30 |
| TOSCANIA                 | 27,615    | 21.57 | 30,045   | 23.47 | 21,044   | 16.91 |
| UMBRIA                   | 17,732    | 23.33 | 19,255   | 25.34 | 29,415   | 38.70 |
| Lazio                    | 70,481    | 30.51 | 76,124   | 32.95 | 79,155   | 34.27 |
In Table 7, we report the capital allocated to each cluster as a percentage of the total capital, in order to underline the differences between the asset allocations in terms of ML (or, equivalently, VaR) and in terms of ES. In this case, the results in terms of mean-variance (Maximum Loss, ML) and coherent asset allocation (Expected Shortfall, ES) are remarkably different.

Typically for concentrated portfolios, a coherent capital allocation is advisable, given the strong differences deriving from the two different techniques, the traditional (ML) and the coherent (ES). In particular, the capital allocated to the clusters with greater concentration (such as Lombardia, Lazio, and Sicilia) has a greater increase.

Therefore, mostly for concentrated portfolios, the utilization of a coherent capital allocation technique is particularly justified.

Table 7. Results of capital allocation, expressed in terms of ML (traditional allocation) and ES (coherent allocation), assuming concentrated clusters and correlations calculated by both MLH method and Basel’s formula.

| Cluster                  | ML (rho MLH) | ES (rho MLH) | ML (rho Basel) | ES (rho Basel) |
|--------------------------|--------------|--------------|----------------|----------------|
| Liguria                  | 3.07%        | 3.84%        | 4.22%          | 4.85%          |
| Lombardia                | 17.94%       | 32.37%       | 16.43%         | 22.22%         |
| Trentino-Alto Adige      | 0.92%        | 0.22%        | 2.07%          | 1.67%          |
| Veneto                   | 5.09%        | 0.65%        | 5.33%          | 7.59%          |
| Friuli-Venezia Giulia    | 0.82%        | 0.24%        | 1.47%          | 0.96%          |
| Emilia-Romagna           | 8.44%        | 7.58%        | 8.53%          | 12.94%         |
| Marche                   | 3.06%        | 2.31%        | 3.61%          | 3.32%          |
| Toscana                  | 5.79%        | 3.07%        | 5.57%          | 3.28%          |
| Umbria                   | 1.97%        | 0.44%        | 3.57%          | 4.46%          |
| Lazio                    | 18.94%       | 21.77%       | 14.12%         | 11.99%         |
| Campania                 | 7.19%        | 7.98%        | 6.98%          | 4.65%          |
| Calabria                 | 1.60%        | 1.19%        | 2.47%          | 1.33%          |
| Sicilia                  | 10.89%       | 10.68%       | 8.70%          | 7.03%          |
| Sardegna                 | 2.04%        | 0.49%        | 2.99%          | 2.64%          |
| Piemonte e Valle d’Aosta | 6.04%        | 0.72%        | 6.58%          | 3.81%          |
| Abruzzo e Molise         | 3.26%        | 2.25%        | 4.13%          | 3.90%          |
| Puglia e Basilicata      | 2.95%        | 4.20%        | 3.23%          | 3.36%          |

Source: our elaboration.

Conversely, in the case of granular or diversified portfolios, the outcomes in terms of capital allocation derived from the two different techniques, the traditional (ML) and the coherent (ES), seem very similar, especially when the correlations are low (see Table 8). The reason for this is the
adoption of a one-factor model alongside the assumption of high granularity for the loan portfolio. In order to compare the differences between the asset allocation in terms of ML (or, equivalently, VaR), calculated by the mean-variance approach, and in terms of ES, we collect in Table 8 the capital allocated to every cluster in the portfolio as a percentage of the total capital.

From Table 8, we can see that, when the asset correlations are lower, the capital allocation performed by the mean-variance approach (Maximum Loss, ML) is very close to the coherent capital allocation (Expected Shortfall, ES). In fact, if the asset correlation is close to zero, the loss distributions of the single clusters and the whole portfolio distribution converge rapidly towards the Normal distribution. In this case, the mean-variance capital allocation is equivalent to the coherent capital allocation (see, for example, Rockafellar and Uryasev 2000; 2002).

Table 8. Results of capital allocation, expressed in terms of ML (traditional allocation) and ES (coherent allocation), assuming diversified clusters and correlations calculated by both MLH method and Basel’s formula.

| Cluster            | ML (rho ML) | ES (rho ML) | ML (rho Basel) | ES (rho Basel) |
|--------------------|-------------|-------------|----------------|----------------|
| LIGURIA            | 3.90%       | 3.86%       | 4.42%          | 4.95%          |
| LOMBARDIA          | 9.75%       | 9.98%       | 10.13%         | 14.44%         |
| TRENTINO-ALTO ADIGE| 2.68%       | 2.50%       | 2.44%          | 3.25%          |
| VENETO             | 5.55%       | 5.53%       | 5.48%          | 7.20%          |
| FRIULI-VENEZIA GIULIA| 2.68%       | 2.54%       | 2.62%          | 2.07%          |
| EMILIA-ROMAGNA     | 7.27%       | 7.45%       | 6.92%          | 9.22%          |
| MARCHE             | 4.93%       | 4.79%       | 4.15%          | 5.08%          |
| TOSCANA            | 6.22%       | 6.16%       | 5.75%          | 6.60%          |
| UMBRIA             | 3.77%       | 4.10%       | 3.56%          | 4.46%          |
| LAZIO              | 11.77%      | 12.22%      | 13.29%         | 7.47%          |
| CAMPANIA           | 7.72%       | 7.59%       | 7.79%          | 6.33%          |
| CALABRIA           | 3.40%       | 3.72%       | 3.52%          | 2.92%          |
| SICILIA            | 11.80%      | 10.84%      | 10.61%         | 6.82%          |
| SARDEGNA           | 3.34%       | 3.48%       | 3.83%          | 3.37%          |
| PIEMONTE E VALLE D’AOSTA | 5.10%       | 5.57%       | 6.00%          | 7.54%          |
| ABRUZZO E MOLISE   | 4.91%       | 4.82%       | 4.61%          | 5.02%          |
| PUGLIA E BASILICATA| 5.21%       | 4.85%       | 4.87%          | 3.24%          |

Source: our elaboration.

Indeed, we find some differences when the asset correlations are higher. In this case, the loss distributions for each cluster have a slower convergence towards the Normal distribution. It is well known that, when the correlations go to one, all the obligors in the portfolio may be considered as a single obligor. Consequently, when the correlations increase, the hypothesis of infinite granularity tends to fail. In particular, greater differences can be observed in those clusters with higher correlations and with a lower number of obligors.

Now, we relax the hypothesis of deterministic and constant recovery rates. Specifically, we implement the simulative model to the original loan portfolio, but assume the recovery rates to be stochastic and related to each other and with the default event, following the approach of Pykhtin (2003) and Tasche (2004), previously described in Section 4.

In this context, we assume \( b_{i0} = c_{i0} \) with homogenous credit exposures \( i = 1, \ldots, N \), and with all the maturities equal to one year. For each obligor, we adopt a mean recovery rate \( m = 0.5 \) and a standard deviation \( s = 0.2 \). For each cluster in the portfolio, we estimate its own loss distribution, the
respective risk measures, and the capital allocations utilizing both the correlations estimated by the MLH method and the correlations derived from Basel’s formula.

The results are reported in Table 9. Comparing the results in Table 9 with those described previously in Tables 3 and 4, we find that all the credit risk measures are more severe when we drop the hypothesis of deterministic and constant recovery rates to assume stochastic recovery rates related to each other and to the default event. This is particularly true when the correlations are calculated by Basel’s formula. In fact, we have assumed \( b_{(i)} = c_{(i)} \) for \( i = 1, \ldots, N \), and, therefore, the correlations among the recovery rates and between the default events and the recovery rates themselves are higher, as the asset return correlations are also higher. In other words, we obtain greater values of risk measures (ML and ES) when we utilize the correlations calculated by Basel’s formula.
Table 9. 99.9% ML and 99.9% ES, estimated by an MC simulation model, assuming the recovery rates as stochastic and related with the default event and homogeneous portfolios.

| Cluster                  | ML 99.9% (rho MLH) | %  | ES 99.9% (rho MLH) | %  | ML 99.9% (rho Basel) | %  | ES 99.9% (rho Basel) | %  |
|--------------------------|--------------------|----|--------------------|----|--------------------|----|--------------------|----|
| LIGURIA                  | 5375               | 5.27% | 5433               | 5.33% | 29,499            | 28.92% | 37,557           | 36.82% |
| LOMBARDIA                | 13,337             | 5.29% | 14,609             | 5.80% | 78,158            | 31.02% | 108,433          | 43.03% |
| TRENTINO-ALTO ADIGE      | 3679               | 5.11% | 3451               | 4.79% | 16,266            | 22.59% | 24,167           | 33.56% |
| VENETO                   | 7598               | 5.35% | 7946               | 5.60% | 38,587            | 27.17% | 55,300           | 38.94% |
| FRIULI-VENEZIA GIULIA    | 3672               | 5.74% | 4142               | 6.47% | 11,506            | 17.98% | 15,987           | 24.98% |
| EMILIA-ROMAGNA           | 10,139             | 5.54% | 10,888             | 5.95% | 47,863            | 26.15% | 68,440           | 37.40% |
| MARCHE                   | 6931               | 7.37% | 7599               | 8.08% | 28,131            | 29.93% | 39,004           | 41.49% |
| TOSCANA                  | 8584               | 6.71% | 8900               | 6.95% | 39,448            | 30.82% | 53,269           | 41.62% |
| UMBRIA                   | 5202               | 6.84% | 6272               | 8.25% | 26,564            | 34.95% | 34,676           | 45.63% |
| LAZIO                    | 16,500             | 7.14% | 18,319             | 7.93% | 71,067            | 30.76% | 80,739           | 34.95% |
| CAMPANIA                 | 10,972             | 8.31% | 12,151             | 9.21% | 42,368            | 32.10% | 48,838           | 37.00% |
| CALABRIA                 | 4699               | 8.70% | 5582               | 10.34% | 19,865   | 36.79% | 21,415           | 39.66% |
| SICILIA                  | 16,529             | 9.50% | 17,136             | 9.85% | 47,665            | 27.39% | 55,082           | 31.66% |
| SARDEGNA                 | 4692               | 6.90% | 4916               | 7.23% | 23,125            | 34.01% | 27,819           | 40.91% |
| PIEMONTE E VALLE D’AOSTA | 7075               | 4.62% | 7490               | 4.90% | 42,939            | 28.06% | 58,622           | 38.32% |
| ABRUZZO E MOLISE         | 6816               | 8.11% | 7405               | 8.82% | 31,623            | 37.65% | 38,294           | 45.59% |
| PUGLIA E BASILICATA      | 7297               | 8.02% | 7290               | 8.01% | 21,555            | 23.69% | 24,633           | 27.07% |
| TOTAL                    | 139,096             | 6.62% | 149,530             | 7.12% | 616,230          | 29.34% | 792,277          | 37.73% |

Source: our elaboration.
7. Conclusions

From the outcomes of this work, the strong underestimation of portfolio credit risk produced by the IRB model is evident, given its restrictive underlying hypotheses. First, when we drop the assumption of highly diversified portfolios, the estimates of the portfolio risk measures obtained by implementing the advanced quantitative methodology (described in Sections 2 and 3) increase significantly. Also, the results in terms of coherent capital allocation rise considerably. For this reason, mostly in the case of undiversified portfolios, coherent capital allocation is the appropriate choice for efficient credit risk management. Secondly, when we lower the hypothesis of constant and independent recovery rates, we obtain capital requirements more severe than those from the IRB model. Our main conclusion is that the IRB model is unable to capture the real credit risk of a loan portfolio because it does not take into account the real dependence structure among the default events and between the recovery rates and the default events. The adoption of this regulatory model can produce a dangerous underestimation of the portfolio credit risk, particularly when the economic uncertainty and the volatility of the financial markets increase. In summary, the most original findings of this research are the following:

- For Italian SMEs, the asset return correlations estimated by the maximum likelihood method and by Lucas’ approach are remarkably lower than those calculated by Basel’s formula.
- Contrary to the regulatory hypothesis, a negative relation between the estimated correlations and the PDs is not found for Italian SMEs.
- The Basel IRB model, all things being equal, is very and positively influenced by the value of the correlations.
- The credit capital requirements calculated by the IRB model and by the simulative model are quite similar if we maintain the restrictive hypotheses of the regulatory approach.
- After removing the hypothesis of infinite granularity for the loan portfolios, the results in terms of VaR obtained from the two different models differ strongly. The capital requirements estimated by the simulative model are always greater than those calculated by the IRB model, mostly when the correlations are low.
- The underestimation of risk and capital is evident when we drop the strong hypothesis of a highly diversified portfolio. For this reason, mostly in the case of undiversified portfolios, coherent capital allocation is the appropriate choice for purposes of risk management.
- Typically, for concentrated portfolios, coherent capital allocation is advisable, given the strong differences deriving from the two different techniques, the traditional (ML) and the coherent (ES). In particular, the capital allocated to the clusters with greater concentration (such as Lombardia, Lazio, and Sicilia) has a greater increase.
- On the other hand, in the case of granular or diversified portfolios, the outcomes in terms of the capital allocation derived from the two different techniques, ML and ES, seem very similar, especially when the correlations are low. In other words, the mean-variance capital allocation is equivalent to coherent capital allocation.
- The values of the portfolio’s credit risk measures (ML and ES) become more severe when the hypotheses of deterministic and constant recovery rates are dropped. This is particularly true when we utilize the high correlations calculated utilizing Basel’s formula.

One limitation of this advanced methodology is the use of a “pure” or standard Monte Carlo simulation for computing the risk contributions in terms of a coherent risk metric as the Expected Shortfall. These estimates present a very slow convergence towards their true values when traditional Monte Carlo algorithms are used. This problem underlines the necessity of utilizing importance sampling (IS) techniques for estimating the coherent risk contributions efficiently, instead of a simple Monte Carlo simulation model. IS permits us to artificially increase the number of scenarios in the tail of the probability distribution, thus reducing considerably the volatility of the estimate of the tail risk measures. Additional extensions are the possibility of introducing country risk and contagion risk.
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References
Acerbi, Carlo, and Dirk Tasche. 2002. On the coherence of expected shortfall. *Journal of Banking and Finance* 26: 1487–503.

Artzner, Philippe, Freddy Delbaen, Jean-Marc Eber, and David Heath. 1999. Coherent measures of risk. *Mathematical Finance* 9: 203–28.

Basel Committee on Banking Supervision. 2003. *The New Basel Capital Accord*. Consultative Document. Washington, DC: BIS.

Basel Committee on Banking Supervision. 2004. *Modifications to the Capital Treatment for Expected and Unexpected Credit Losses in the New Basel Accord*. Consultative Document. Washington, DC: BIS.

Chabaane, Ali, Jean-Paul Laurent, and Julien Salomon. 2004. Double Impact: Credit Risk Assessment and Collateral Value. *Revue Finance* 25: 157–78.

Crouhy, Michel., Dan Galai, and Robert Mark. 2000. A comparative analysis of current credit risk models. *Journal of Banking and Finance* 24: 59–117.

Crouhy, Michel, Dan Galai, and Robert Mark. 2014. *The Essentials of Risk Management*, 2nd ed. New York: McGraw-Hill.

de Servigny, Arnaud, and Olivier Renault. 2002. *Default Correlation: Empirical Evidence*. New York: Standard and Poor’s Risk Solutions, pp. 1–27.

Denault, Michel 2001. Coherent allocation of risk capital. *Journal of Risk* 4: 1–34.

Dietsch, Michel, and Joël Petey. 2004. Should SME exposures be treated as retail or corporate exposures? A comparative analysis of default probabilities and asset correlations in French and German SMEs. *Journal of Banking and Finance* 28: 773–88.

Duellmann, Klaus, and Philipp Koziol. 2014. Are SME Loans Less Risky than Regulatory Capital Requirements Suggest? *The Journal of Fixed Income* 23: 89–103.

Duellmann, Klaus, and Harald Scheule. 2003. *Determinants of the Asset Correlations of German Corporations and Implications for Regulatory Capital*. Working Paper. Frankfurt: Deutsches Bundesbank.

Emmer, Susanne, and Dirk Tasche. 2004. Calculating Credit Risk Capital Charges with the One-Factor Model. *Journal of Risk* 7: 85–103.

Finger, Christopher C. 1999. Conditional Approaches for CreditMetrics Portfolio Distributions. *CreditMetrics Monitor* 2: 14–33.

Finger, Christopher C. 2001. The One-Factor CreditMetrics Model in the New Basel Capital Accord. *RiskMetrics Journal* 2: 9–18.

Frye, Jon. 2000. Depressing recoveries. *Risk* 13: 106–11.

Frey, Rüdiger, and Alexander McNeil. 2003. Dependent Defaults in Models of Portfolio Credit Risk. *Journal of Risk* 6: 59–92.

Gordy, Michael. 2000. A Comparative Anatomy of Credit Risk Models. *Journal of Banking and Finance* 24: 119–49.

Gordy, Michael. 2003. A Risk-Factor Model Foundation for Ratings-Based bank Capital Rules. *Journal of Financial Intermediations* 12: 199–232.

Gordy, Michael, and Eva Lütkebohmert. 2013. Granularity Adjustment for Regulatory Capital Assessment. *International Journal of Central Banking* 9: 33–71.

Gregory, Jon, and Jean-Paul Laurent. 2004. In the Core of Correlation. *Risk* 87–91.

Hamerle, Alfred, and Daniel Rösch. 2006. Parameterizing Credit Risk Models. *Journal of Credit Risk* 2: 101–22.

Jouanin, Jean-Frédéric, Gaël Riboulet, and Thierry Roncalli. 2004. Financial Applications of Copula Functions. In *Risk Measures for the 21st Century*. Edited by Giorgio P. Szegö. Hoboken: John Wiley & Sons.

Kalkbrener, Michael. 2005. An axiomatic approach to capital allocation. *Mathematical Finance* 15: 425–37.

Kalkbrener, Michael, Hans Lotter, and Ludger Overbeck. 2004. Sensible and Efficient Capital Allocation for Credit Portfolios. *Risk* 19–24.

Kitano, Takashi. 2007. Estimating Default Correlation from Historical Default Data—Maximum Likelihood Estimation of Asset Correlation using Two-Factor Models—. *Transactions of the Operations Research Society of Japan* 50: 42–67.

Li, David X. 2000. On Default Correlation: A Copula Function Approach. *Journal of Fixed Income* 9: 43–54.

Litterman, Robert. 1996. Hot spots [TM] and hedges. *Journal of Portfolio Management* 23: 52–75.
Lopez, Jose A. 2004. The empirical relationship between average asset correlation, firm probability of default and asset size. Journal of Financial Intermediation 13: 265–83.

Lucas, Douglas. 1995. Default correlation and credit analysis. Journal of Fixed Income 4: 76–87.

Markowitz, Harry 1952. Portfolio selection. Journal of Finance 7: 77–91.

Mashal, Roy, Marco Naldi. 2002. Extreme Events and Default Baskets. Risk 119–22.

Mausser, Helmut, and Dan Rosen. 2008. Economic Credit Capital Allocation and Risk Contributions. In Handbooks in Operations Research and Management Science: Financial Engineering. Edited by John Birge and Vadim Linetsky. Amsterdam: North-Holland, vol. 15.

Meneguzzo, Davide, and Walter Vecchiato. 2004. Copula sensitivity in collateralised debt obligations and basket default swaps. In Special Issue: Special Issue on Credit Risk and Credit Derivatives. The Journal of Futures Markets 24: 37–70.

Merton, Robert Cox. 1974. On the pricing of corporate debt: the risk structure of interest rates. Journal of Finance 29: 449–70.

Mizgier, Kamil J., and Joseph M. Pasia. 2015. Multiobjective Optimization of Credit Capital Allocation in Financial Institutions. Central European Journal of Operations Research 24: 801–17.

Nelsen, Roger B. 1999. An Introduction to Copulas. New York: Springer-Verlag.

Overbeck, Ludger 2000. Allocation of economic capital in loan portfolios. In Measuring Risk in Complex Stochastic Systems. volume 147 of Lecture Notes in Statistics. Edited by Jürgen Franke, Wolfgang Härdle and Gerhard Stahl. New York: Springer, pp. 1–17.

Pykhtin, Michael. 2003. Unexpected recovery risk. Risk 16: 74–78.

Rockafellar, Ralph Tyrrell, and Stan Uryasev. 2000. Optimization of Conditional Value-at-Risk. The Journal of Risk 2: 21–41.

Rockafellar, Ralph Tyrrell, and Stan Uryasev. 2002. Conditional Value-at-Risk for general loss distributions. Journal of Banking and Finance 26: 1443–71.

Sironi, Andrea, and Cristiano Zazzara. 2003. The Basel Committee Proposals for a New Capital Accord: Implications for Italian Banks. Review of Financial Economics 12: 99–126.

Sklar, Abe. 1959. Fonctions de Répartition à n Dimensions et Leurs Marges. Paris: Publications de l’Institut de Statistique de l’Université de Paris, pp. 229–31.

Tasche, Dirk. 2002. Expected Shortfall and Beyond. Journal of Banking and Finance 26: 1519–33.

Tasche, Dirk. 2004. The Single Risk Factor Approach to Capital Charges in Case of Correlated Loss Given Default Rates. Discussion paper. Frankfurt: Deutsche Bundesbank.

Vasicek, Oldrich Alfons. 2015a. Probability of loss on loan portfolio. In Handbook Finance, Economics and Mathematics. Chapter 17. Edited by Oldrich Alfons Vasicek. Hoboken: Wiley & Sons.

Vasicek, Oldrich Alfons 2015b. Loan portfolio value. In Handbook Finance, Economics and Mathematics. Chapter 19. Edited by Oldrich Alfons Vasicek. Hoboken: Wiley & Sons.

Wehrspohn, Uwe. 2003. Generalized Asset Value Credit Risk Models and Risk Minimality of the Classical Approach. Available online: http://dx.doi.org/10.2139/ssrn.404920 (accessed on 10 March 2019).

Zhou, Chunsheng. 2001. An Analysis of Default Correlations and Multiple Default. The Review of Financial Studies 14: 555–76.