The mbsts package: Multivariate Bayesian Structural Time Series Models in R

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Abstract The multivariate Bayesian structural time series (MBSTS) model (Qiu et al., 2018) is a general machine learning model that deals with inference and prediction for multiple correlated time series, where one also has the choice of using a different candidate pool of contemporaneous predictors for each target series. The MBSTS model has wide applications and is ideal for feature selection, time series forecasting, nowcasting, inferring causal impact, and others. This paper demonstrates how to use the R package mbsts for MBSTS modeling, establishing a bridge between user-friendly and developer-friendly functions in the package and the corresponding methodology. Object-oriented functions in the package are explained in the way that enables users to flexibly add or deduct some components, as well as to simplify or complicate some settings.

Introduction

Structural time series models are state space models for time series data. They are constructed in terms of components each of which has a direct interpretation. For example, one may consider a decomposition in which a series can be seen as the sum of trend and regression components. The multivariate Bayesian structural time series (MBSTS) model (Qiu et al., 2018) is a generalized version of many structural time series models and is constructed as the sum of a trend component, a seasonal component, a cycle component, a regression component, and an error term, where each component provides an independent and additional effect. Users have flexibility in choosing these components and are free to construct their specific forms, for example adding on a regression component with predictors generated through data mining as that in (Jammalamadaka et al., 2019). The MBSTS model uses the Bayes selection technique via Markov chain Monte Carlo (MCMC) methods to select among a set of contemporary predictors, thus one does not need to commit to a fixed set of predictors. Specifically, the variable selection technique uses a “spike and slab” approach, through which a different set of predictors can be selected in each MCMC iteration. Then important predictors will be selected according to their overall frequency of numbers being selected over the total number of MCMC iterations. The multivariate structure and the Bayesian framework allow the model to take advantage of the association structure among target series.

The MBSTS model and its univariate counterpart, the BSTS model (Scott and Varian, 2014, 2015), have wide applications in causal inference (see, e.g., Brodersen et al. (2015)), health care (see, e.g., Kurz et al. (2019)), spatial analysis (see, e.g., Qiu et al. (2019)), artificial intelligence (see, e.g., Jammalamadaka et al. (2019)), cryptocurrency (see, e.g., Jalan et al. (2019)), medicine (see, e.g., Talaei-Khoei et al. (2019)), airline industry (see, e.g., Talaei-Khoei et al. (2019)), environmental science (see, e.g., Droste et al. (2018)), renewable energy (see, e.g., Jiang et al. (2013)), political analysis (see, e.g., Xu (2017)), social media (see, e.g., Welbers and Opgenhaffen (2018)), and etc. Upon demands from users of the BSTS/MBSTS model in understanding the methodology and establishing a user-friendly interface with sufficient flexibility, the R package mbsts (Qiu and Ning, 2023) is developed, which is available on CRAN at https://cran.r-project.org/web/packages/mbsts/. Our contributions are four-fold:

1. Functional contribution. The mbsts package is the only package that implements the MBSTS model (Qiu et al., 2018). Therefore, it is one that uses Bayesian tools for model fitting, prediction, and feature selection on multivariate correlated time series data, where different contemporaneous predictors could be selected for different target series. The bsts package (Scott, 2021) implements the BSTS model which only works on one-dimensional time series data.

2. User-friendly interface. The mbsts package is developed with the purpose to give users a nice and easy experience. Users of the mbsts package can conduct model training using the mbsts_function function, perform prediction using the mbsts_forecast function, retrieve parameter estimation results using the para_est function, visualize feature selection results using the plot_prob function, diagnose convergence using the plot_cvg function, and visualize time series components training results using the plot_comp function. Different from some machine learning algorithms that have to resort to cloud computing such as Ning and Ionides.
we can rewrite where \( x \) \( \tilde{\text{~}} \) The MBSTS model is constructed as the sum of components and one can flexibly select suitable variables and independent variables which are also called predictors, covariates, or features. The \( \beta \) regression component
\[
R \sim \text{matrix}(c(1.1, 0.7, 0.7, 0.9), \text{nrow}=2, \text{ncol}=2)
\]
\[
R \text{#covariance matrix of target series}
\]
\[
R n \leftarrow 505 \#n: \text{sample size}
\]
\[
R m \leftarrow 2 \#m: \text{dimension of target series}
\]
\[
R \text{cov} \leftarrow \text{matrix}(c(1.1, 0.7, 0.7, 0.9), \text{nrow}=2, \text{ncol}=2)
\]
Regression analysis is a statistical approach for estimating the relationships between dependent variables and independent variables which are also called predictors, covariates, or features. The regression component \( \xi_t = [\xi_t^{(1)}, \ldots, \xi_t^{(m)}]^T \) is written as follows:
\[
\xi_t^{(i)} = \beta_{ti} x_t^{(i)}. \tag{2}
\]
where \( x_t^{(i)} = [x_t^{(i)}, \ldots, x_t^{(i)}]^T \) is the pool of all available predictors at time \( t \) for target series \( y_t^{(i)} \), and \( \beta_t = [\beta_1, \ldots, \beta_{ij}, \ldots, \beta_{ik_t}]^T \) represents corresponding static regression coefficients. Recall that \( \bar{y}_t = [y_t^{(1)}, \ldots, y_t^{(m)}]^T \) and let \( B = [\beta_1, \ldots, \beta_1, \ldots, \beta_m] \) denote a \( k \times m \)-dimensional matrix. Then we can rewrite \( \xi_t \) as
\[
\tilde{\xi}_t = \text{diag}(B^T \tilde{x}_t). \tag{3}
\]
In data, the regression component is generated by setting $B$ and $\tilde{\mathbf{x}}_t$ as follows:

$$B = \begin{bmatrix} 2 & 0 & 2.5 & 0 & 1.5 & -2 & 0 & 3.5 \end{bmatrix}^T,$$

$$\tilde{\mathbf{x}}_t = \begin{bmatrix} x_{t1} & x_{t2} & x_{t3} & x_{t4} & x_{t5} & x_{t6} & x_{t7} & x_{t8} \end{bmatrix}^T,$$

$$x_{t1} \overset{iid}{\sim} \text{N}(5, 5^2), \quad x_{t2} \overset{iid}{\sim} \text{Pois}(10), \quad x_{t3} \overset{iid}{\sim} \text{Pois}(5), \quad x_{t4} \overset{iid}{\sim} \text{N}(-2, 5),$$

$$x_{t5} \overset{iid}{\sim} \text{N}(-5, 5^2), \quad x_{t6} \overset{iid}{\sim} \text{Pois}(15), \quad x_{t7} \overset{iid}{\sim} \text{Pois}(20), \quad x_{t8} \overset{iid}{\sim} \text{N}(0, 10^2).$$

The coding realization for the regression component is given by

R> #Regression component
R> #coefficients for predictors
R> beta<-t(matrix(c(2,-1.5,0,4,2.5,0,0,2.5,1.5,-1,-2,0,0,-3,3.5,0.5),nrow=2,ncol=8))
R> #coefficients for predictors
R> #Regression component

Trend component

A trend is the long-term growth of time series, and it can be further decomposed into two components: level and slope. Level represents the actual mean value of the trend and slope represents the tendency to grow or decline from the trend. The trend component $\tilde{\mu}_t$ is generated by a generalization of the local linear trend model where the slope exhibits stationarity in the form as:

$$\tilde{\mu}_{t+1} = \tilde{\mu}_t + \tilde{\delta}_t + \tilde{u}_t, \quad \tilde{u}_t \overset{iid}{\sim} N_m(0, \Sigma_{\mu}),$$

$$\tilde{\delta}_{t+1} = D + \tilde{\rho}(\tilde{\delta}_t - D) + \tilde{v}_t, \quad \tilde{v}_t \overset{iid}{\sim} N_m(0, \Sigma_{\delta}).$$

Here, $\tilde{\delta}_t$ is a $m$-dimensional vector as short-term slope and $D$ is a $m$-dimensional vector as the long-term slope or in another name as level, which enables the model to incorporate short-term information with long-term information. $\tilde{\rho}$ is a $m \times m$ diagonal matrix with diagonal entries $0 \leq \rho_{ii} \leq 1$ for $i = 1, 2, \ldots, m$, to represent the learning rates at which the local trend is updated for $\{y_t^{(i)}\}_{i=1,2,\ldots,m}$. In the trend component, covariance matrices for error terms $\Sigma_{\mu}$ and $\Sigma_{\delta}$ are assumed to be $m \times m$-dimensional diagonal matrices.

Seasonality is a characteristic of a time series in which the data has regular and predictable changes that recur every period. The seasonal component $\tilde{\tau}_t = [\tau_t^{(1)}, \ldots, \tau_t^{(m)}]^T$ is generated as follows:

$$\tau_t^{(i)} = - \sum_{k=0}^{S_i-2} \tau_{t-k}^{(i)} + w_t^{(i)}, \quad w_t = [w_t^{(1)}, \ldots, w_t^{(m)}]^T \overset{iid}{\sim} N_m(0, \Sigma_{\tau}),$$

where $S_i$ represents the number of seasons for $y_t^{(i)}$ and $\tilde{\tau}_t$ is a $m$-dimensional vector denoting their joint contribution to the observed target time series $y_t$. The model allows for various seasonal components with different periods for each target series $y_t^{(i)}$, such as one can include a seasonal component with $S_i = 7$ to capture the day-of-the-week effect for one target series, and $S_j = 30$ to capture the day-of-the-month effect for another target series. In the seasonal component, the covariance matrix for the error term $\Sigma_{\tau}$ is assumed to be a $m \times m$-dimensional diagonal matrix.

The cyclical effect refers to regular or periodic fluctuations around the trend, revealing a succession of phases of expansion and contraction. In contrast to seasonality that is always of fixed and known periods, a cyclic pattern exists when data exhibits ups and downs that are not of fixed
with an obviously increasing mode. We note that most times series in real life behave more stable
without a cycle component and without a seasonal component. In R
\[
\begin{align*}
  &\texttt{ts.plot(data[,1:2], col = 1:ncol(data[,1:2]))} \\
  &\texttt{#Plot simulated data} \\
  &\texttt{Dtilde=c(-0.1,0.3), Season=c(100,0), vrho=c(0,0.99), lambda=c(0,pi/100))} \\
  &\texttt{data=sim_data(X=X, beta=beta, cov, k=c(8,8), mu=c(1,1), rho=c(0.06,0.08),}
\end{align*}
\]
where
\[
\begin{align*}
  &\texttt{sd_trend=0.5, mean_trend=1, sd_trend=0.5, mean_cycle=20, sd_cycle=0.5.} \\
  &\texttt{R} \\
  &\texttt{#Simulated data} \\
  &\texttt{data=sim_data(X=X, beta=beta, cov, k=c(8,8), mu=c(1,1), rho=c(0.06,0.08),}
\end{align*}
\]
\[
\begin{align*}
  &\texttt{Dtilde=c(-0.1,0.3), Season=c(100,0), vrho=c(0,0.99), lambda=c(0,pi/100))} \\
  &\texttt{data=sim_data(X=X, beta=beta, cov, k=c(8,8), mu=c(1,1), rho=c(0.06,0.08),}
\end{align*}
\]
and
\[
\begin{align*}
  &\texttt{sd_trend=0.5, mean_trend=1, sd_trend=0.5, mean_cycle=20, sd_cycle=0.5.} \\
  &\texttt{R} \\
  &\texttt{#Simulated data} \\
  &\texttt{data=sim_data(X=X, beta=beta, cov, k=c(8,8), mu=c(1,1), rho=c(0.06,0.08),}
\end{align*}
\]
\[
\begin{align*}
  &\texttt{Dtilde=c(-0.1,0.3), Season=c(100,0), vrho=c(0,0.99), lambda=c(0,pi/100))} \\
  &\texttt{data=sim_data(X=X, beta=beta, cov, k=c(8,8), mu=c(1,1), rho=c(0.06,0.08),}
\end{align*}
\]
and
\[
\begin{align*}
  &\texttt{sd_trend=0.5, mean_trend=1, sd_trend=0.5, mean_cycle=20, sd_cycle=0.5.} \\
  &\texttt{R} \\
  &\texttt{#Simulated data} \\
  &\texttt{data=sim_data(X=X, beta=beta, cov, k=c(8,8), mu=c(1,1), rho=c(0.06,0.08),}
\end{align*}
\]
\[
\begin{align*}
  &\texttt{Dtilde=c(-0.1,0.3), Season=c(100,0), vrho=c(0,0.99), lambda=c(0,pi/100))} \\
  &\texttt{data=sim_data(X=X, beta=beta, cov, k=c(8,8), mu=c(1,1), rho=c(0.06,0.08),}
\end{align*}
\]

In data, both $y^{(1)}_t$ and $y^{(2)}_t$ are designed to have a trend component, $\hat{\rho}$ is 0.06 for $y^{(1)}_t$ and 0.08 for $y^{(2)}_t$, and $D$ is $-0.1$ for $y^{(1)}_t$ and 0.3 for $y^{(2)}_t$. In data, we set the mean of each of the diagonal entries equals to 1 and the standard deviation of each of the diagonal entries equals to 0.5. In data, $y^{(1)}_t$ is generated having a seasonal component with seasonality $S_1=100$ and $y^{(2)}_t$ is generated without a seasonal component. In data, we set the mean of each of the diagonal entries equals to 20 and the standard deviation of each of the diagonal entries equals to 0.5. In data, $y^{(1)}_t$ is generated without a cycle component and $y^{(2)}_t$ is generated having a cycle component with the damping factor being 0.99 and the cyclic frequency being $\pi/100$. In data, we set the mean of each of the diagonal entries equals to 0.5 and the standard deviation of each of the diagonal entries equals to 0.5. One can also set the mean and standard deviation of the entries of the covariance matrices of the state components in the sim_data function, whose default is the following: mean_trend=1, sd_trend=0.5, mean_season=20, sd_season=0.5, mean_cycle=20, sd_cycle=0.5.

Now, we plot the simulated data
R> #Plot simulated data
R> ts.plot(data[,1:2],col = 1:ncol(data[,1:2]))
R> legend("topleft", colnames(data[,1:2]),col=1:ncol(data[,1:2]), lty=1, cex=1.1)

From Figure 1, we can see that $y^{(1)}_t$ and $y^{(2)}_t$ are very different, which are generated intentionally to illustrate the strength of the MBSTS model. Clearly, $y^{(1)}_t$ is very hard to predict in that it has frequent and irregular fluctuations, as well as no sign of trend and momentum. $y^{(2)}_t$ is more stable with an obviously increasing mode. We note that most times series in real life behave more stable than $y^{(1)}_t$ and less stable than $y^{(2)}_t$, depending on the time interval and the nature of data. For example, the stock market data is much closer to $y^{(2)}_t$ than $y^{(1)}_t$. We refer to (Qiu et al., 2018) for more numerical and empirical examples.

The two plots in Figure 1 are generated by calling the sim_data function with rho=c(0.06, 0.08) and rho=c(0.66, 0.08), respectively. We can see that both time series have an increasing trend, hence one could set mu=c(1,1) indicating each has a trend component. We can see that with a learning rate of 0.66 the time series increases much faster, giving guidance on tuning the learning rate rho. We can see that the first time series shows a regular periodic mode every 100 time points, which indicates that one could set the seasonality as 100. The second time series does not show any regular mode, hence no seasonality. Instead, it has an irregular fluctuation in the range of 200, hence one could set the cyclic frequency as $\pi/100$. Here, we observe a clear damping effect and hence the damping factor could be set as a number slightly smaller than 1.

## Preparation and the tsc.setting function

In this section, we illustrate the prior distribution setup with the MBSTS model by providing only necessary concepts and formula to understand and use the algorithm. The R script used in the illustration example is covered in the supplement of this paper (illustration.example.R).
Generation of the initial time series components in the \texttt{mbsts} package is through the \texttt{tsc.setting} function. The other input parameters of the \texttt{tsc.setting} function are the following: The trend inclusion parameter $\mu$ and the learning rate parameter $\rho$ for the trend component; The seasonality parameter $S$ for the seasonal component; The damping factor parameter $\nu$ and the frequency parameter $\lambda$ for the cycle component.

\begin{verbatim}
R> # Specify time series components
R> STmodel<-tsc.setting(Ytrain,mu=c(1,1),rho=c(0.06,0.08),S=c(100,0),
+ vrho=c(0,0.99),lambda=c(0,pi/100))
\end{verbatim}

The \texttt{tsc.setting} function is built upon the `SSModel` function in the state-of-the-art package \texttt{KFAS} in (Helske, 2017) and the output of the \texttt{tsc.setting} function is an object of `SSModel` class.

\begin{verbatim}
R> class(STmodel)
[1] "SSModel"
\end{verbatim}

\section*{Model Training}

Model training with the MBSTS model is performed through the \texttt{mbsts} function in the \texttt{mbsts} package. The MBSTS model is the one that successfully used feature selection in multivariate time series analysis. The multivariate structure and the Bayesian framework allow the model to take advantage of the association structure among target series, and enable feature selection and model training be conducted at the same time. Let $Y = [\tilde{y}_1, \ldots, \tilde{y}_n]^T$, and we see that $Y$ is a $n \times m$ matrix. Then set $\tilde{Y} = \text{vec}(Y)$ to transfer $Y$ to be a $nm \times 1$ vector. Similarly, define $M = [\tilde{\mu}_1, \ldots, \tilde{\mu}_n]^T$, $T = [\tilde{\tau}_1, \ldots, \tilde{\tau}_n]^T$, $W = [\tilde{\omega}_1, \ldots, \tilde{\omega}_n]^T$, and $E = [\tilde{\epsilon}_1, \ldots, \tilde{\epsilon}_n]^T$, and set $\tilde{M} = \text{vec}(M)$, $\tilde{T} = \text{vec}(T)$, $\tilde{W} = \text{vec}(W)$, $\tilde{E} = \text{vec}(E)$. Next, we define the regression matrix $X$ and its corresponding coefficient $\beta$

$$
X = \begin{bmatrix}
X_1 & 0 & 0 & \cdots & 0 \\
0 & X_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & X_m 
\end{bmatrix}, \quad \beta = \begin{bmatrix}
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_m
\end{bmatrix}.
$$

\begin{figure}[h]
\centering
\begin{subfigure}{0.45\textwidth}
\includegraphics[width=\textwidth]{figure1a}
\caption{$\rho[1]=0.06$}
\end{subfigure}
\begin{subfigure}{0.45\textwidth}
\includegraphics[width=\textwidth]{figure1b}
\caption{$\rho[2]=0.66$}
\end{subfigure}
\caption{Plot of two target time series.}
\end{figure}
Here, $X$ is of dimension $(nm \times K)$ with $K = \sum_{i=1}^{m} k_i$, and its diagonal block matrix $X_i$ is a $n \times k_i$ matrix, representing all observations of $k_i$ candidate predictors for $y^{(i)}$. Now we are ready to define $\tilde{Y} = Y - \tilde{M} - \tilde{T} - \tilde{W}$, and rewrite the model as:

$$\tilde{Y} = X\beta + \epsilon.$$

Note that, for $i = 1, \cdots, m$, the $k_i$-dimensional vector $\beta_i = [\beta_{i1}, \cdots, \beta_{ij}, \cdots, \beta_{ik_i}]^T$ represents corresponding static regression coefficients for target series $y^{(i)}$, and $\beta$ defined in (9) is a $K$-dimensional vector where $K = \sum_{i=1}^{m} k_i$. Correspondingly, we define $\gamma = [\gamma_1, \cdots, \gamma_m]$, where $\gamma_i = [\gamma_{i1}, \cdots, \gamma_{ik_i}]$ and $\gamma_{ij} = 1_{(\beta_{ij} \neq 0)}$ for $i = 1, \cdots, m$ and $j = 1, \cdots, k_i$. Denote $\beta_\gamma$ as the subset of elements of $\beta$ where $\beta_{ij} \neq 0$, and let $X_\gamma$ be the subset of columns of $X$ where $\gamma_{ij} = 1$.

A common-sense fact, if there are many predictors available then usually only a small portion of those would play a crucial role and the vast majority of regression coefficients would be zero. In the Bayesian paradigm, a natural way to represent this sparsity is through the “spike and slab” technique. The “spike” prior is written as:

$$\gamma \sim \prod_{i=1}^{m} \prod_{j=1}^{k_i} \pi^{\gamma_{ij}}(1-\pi_{ij})^{1-\gamma_{ij}},$$

where $\pi_{ij}$ is the prior inclusion probability of the $j$-th predictor for the $i$-th target time series. One can set $\pi_{ij} = q_i/k_i$ as the expected model size for simplicity, where $q_i$ is the number of expected nonzero predictors for $y^{(i)}$ and $k_i$ is the total number of candidate predictors for $y^{(i)}$; one can also set $\pi_{ij} = 0$ or 1 to ensure certain variables to be excluded or included. In the illustration example, we set $k_i = (8,16)$ as the location index of the last predictor for each target series. We set $\text{pii}$ describe the prior inclusion probabilities $\{\pi_{ij}\}$ of each candidate predictor and initialize $\pi_{ij} = 0.5$ for $i = 1, \cdots, m$ and $j = 1, \cdots, k_i$.

```r
R> # prior parameters setup
R> k<- c(8,dim(Xtrain)[2])
R> pi<- matrix(rep(0.5,dim(Xtrain)[2]),nrow=dim(Xtrain)[2])
```

The coefficient $\beta$ and the covariance matrix $\Sigma_c$ are assumed to be prior independent given $\gamma$,

$$p(\beta, \Sigma_c, \gamma) = p(\beta|\gamma)p(\Sigma_c|\gamma)p(\gamma).$$

The “slab” prior specification is given as below, which is called “slab” because one can choose the prior parameters to make it only very weakly informative (close to flat), conditional on $\gamma$.

$$\beta|\gamma \sim N_K(b_\gamma, (\kappa X_\gamma^T X_\gamma/n)^{-1}), \quad \Sigma_c|\gamma \sim IW(\nu_0, (\nu_0 - m - 1)(1 - R^2)\Sigma_y).$$

(10)

That is, $\beta$ given $\gamma$ is distributed according to the multivariate normal distribution with mean vector $b_\gamma$ and covariance matrix $(\kappa X_\gamma^T X_\gamma/n)^{-1}$, where $b_\gamma$ is the vector of prior means with the same dimension as $\beta_\gamma$ and $\kappa$ is the number of observations worth of weight on the prior mean vector. Here, $\Sigma_c$ given $\gamma$ is distributed according to the inverse Wishart distribution $IW(\nu_0, (\nu_0 - m - 1)(1 - R^2)\Sigma_y)$, where $\nu_0$ is the number of degrees of freedom whose value must be greater than the dimension of $\tilde{y}$ plus one. In the illustration example, since there are 2 target series, we set $\nu_0 = 5$ which is larger than $2 + 1$. We recommend setting $\nu_0$ be the dimension of $\tilde{y}$ plus 2 $\in \{2,3\}$. We use the default that $b_{\gamma} = (0,0)^T$, $\kappa = 0.01$ and $R^2 = 0.8$ ($h=\text{NULL}$, $\kappa = 0.01$ and $R2 = 0.8$ in the `mbest` function).

By the assumption that all components are independent of each other, the prior distributions in multivariate form can be reduced to their univariate counterparts. The prior distributions of the covariance matrices $\Sigma_\mu$ and $\Sigma_\delta$ for the trend component follow the inverse-gamma distribution as

$$\Sigma_\mu \sim IG(w_\mu, W_\mu), \quad \Sigma_\delta \sim IG(w_\delta, W_\delta).$$

The prior distribution of the covariance matrix $\Sigma_\tau$ for the trend component follows the inverse-gamma distribution as

$$\Sigma_\tau \sim IG(w_\tau, W_\tau).$$

The prior distribution of the covariance matrix $\Sigma_\omega$ for the cycle component follows the inverse-gamma distribution as

$$\Sigma_\omega \sim IG(w_\omega, W_\omega).$$

In the illustration example, for simplicity, we use the default value that $w_u = w_a = 0.01$ for $u \in \{\mu, \delta, \tau, \omega\}$ (v = 0.01, $ss = 0.01$ in the `mbest` function).

Model training uses the MCMC approach, which is to sample from a probability distribution
based on constructing a Markov chain that has the desired distribution as its equilibrium distribution. That is, the states of the chain after discarding some steps as “burn-in” data, are used as samples from the desired distribution. Specifically, the MBSTS model uses the Gibbs sampler for feature selection utilizing the classical “spike and slab” prior setup (George and McCulloch, 1997). Gibbs sampler can be seen as a special case of the Metropolis–Hastings algorithm. The point of the Gibbs sampler is that given a multivariate distribution it is simpler to sample from a conditional distribution than to marginalize by integrating over a joint distribution. To implement Gibbs sampler in model training, all necessary conditional probabilities were derived. In the illustration example, we run 400 MCMC iterations while discarding the results of the first 100 iterations.

```r
R> #train a mbsts model
R> mbsts.model<-mbsts_function(Ytrain,Xtrain,STmodel,k1,pii,v0=5,mc=400,burn=100)
```

Each MCMC iteration runs Algorithm 1, where the samples from posterior distributions of the model are generated sequentially.

**Algorithm 1 MBSTS Model Training (Qiu et al., 2018)**

1. Draw the latent state \( \alpha = (\tilde{\beta}, \tilde{\delta}, \tilde{\tau}, \tilde{\omega}) \) from given model parameters and \( \tilde{Y} \), namely \( p(\alpha|\tilde{Y}, \theta, \gamma, \Sigma, \beta) \), using the posterior simulation algorithm from (Durbin and Koopman, 2002).
2. Draw time series state component parameters \( \theta = (\Sigma_u, \Sigma_d, \Sigma_r, \Sigma_o) \) given \( \alpha \), namely simulating \( \theta \sim p(\theta|\tilde{Y}, \alpha) \).
3. Loop over \( i \) in an random order, draw each \( \gamma_i|\gamma_{-i}, \tilde{Y}, \alpha, \Sigma_c \), namely simulating \( \gamma \sim p(\gamma|\tilde{Y}^*, \Sigma_c) \) one by one, using the stochastic search variable selection (SSVS) algorithm from (George and McCulloch, 1997).
4. Draw \( \beta \) given \( \Sigma_c, \gamma, \alpha \) and \( \tilde{Y} \), namely simulating \( \beta \sim p(\beta|\Sigma_c, \gamma, \tilde{Y}^*) \).
5. Draw \( \Sigma_c \) given \( \gamma, \alpha, \beta \) and \( \tilde{Y} \), namely simulating \( \Sigma_c \sim p(\Sigma_c|\gamma, \tilde{Y}^*, \beta) \).

The output of the `mbsts_function` function is an object of the ‘mbsts’ class, which is defined in the `mbsts` package with slots: `Xtrain`, `Ind`, `beta.hat`, `B.hat`, `ob.sig2`, `States`, `st.sig2`, `ki`, `mtrain`, `mtrain`, `Xtrain` contains all candidate predictor series for each target series. `Ind` is a matrix containing MCMC draws of the indicator variable. `beta.hat` is a matrix containing MCMC draws of regression coefficients. `B.hat` is an array generated by combining beta.hat for all target series. `ob.sig2` is an array containing MCMC draws of variance-covariance matrix for residuals. `States` is an array containing MCMC draws of all time series components. `st.sig2` is a matrix containing MCMC draws of variances for time series components. `ki` is a vector of integer values denoting the accumulated number of predictors for the target series. For example, if there are three target series where the first has 8 predictors, the second has 6 predictors, and the third has 10 predictors, then the vector is c(8,14,24). `mtrain` is a numerical value for the number of observations. `mtrain` is a numerical value for the number of response variables.

```r
R> class(mbsts.model)
[1] "mbsts"
attr("package")
[1] "mbsts"
```

### Training results and their associated functions

Recall that, in `data`, the regression component is generated by

\[
\begin{bmatrix}
\beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \beta_7 & \beta_8 \\
2 & 0 & 2.5 & 0 & 1.5 & -2 & 0 & 3.5
\end{bmatrix}
\]

for target time series \( y_t^{(1)} \), and

\[
\begin{bmatrix}
\beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \beta_7 & \beta_8 \\
-1.5 & 4 & 0 & 2.5 & -1 & 0 & -3 & 0.5
\end{bmatrix}
\]

for target time series \( y_t^{(2)} \). Function `plot_prob` in the `mbsts` package, is developed to retrieve information from an object of the ‘mbsts’ class to generate plots for empirical posterior distributions.
of estimated coefficients and indicators, for each target time series. The empirical posterior inclusion probability, as a useful indicator of the importance of one specific predictor, is the proportion of the number of times that the predictor is selected to the total count of MCMC iterations after discarding the “burn-in” iterations. Users can set their desired threshold value of inclusion probability through `prob.threshold`, rename the predictors through `varnames`, and set titles through `title` for each plot. The default threshold value of inclusion probability is 0.8.

R> #title vector for each plot
R> title_new<-c("Inclusion Probabilities for y1",
  + "Inclusion Probabilities for y2")

R> #rename predictors
R> varnames_new<-c("x1", "x2", "x3", "x4", "x5", "x6", "x7", "x8",
  + "x1", "x2", "x3", "x4", "x5", "x6", "x7", "x8")

R> #plot inclusion probability
R> plot_prob(object=mbsts.model,title=title_new,prob.threshold=0.8,
  + varnames=varnames_new)

Figure 2 provides the empirical posterior distribution of estimated indicators and signs of corresponding coefficients. The probability 1 of a specific feature indicates that it was selected in every MCMC iteration after discarding the “burn-in” iterations. We can see that the features with non-zero coefficients all were selected with correct signs indicated.

![Inclusion Probabilities for y1](a) ![Inclusion Probabilities for y2](b)

Figure 2: Empirical posterior distribution of estimated coefficients and indicators. The red color shows positive estimated values of regression coefficients, while blue color displays negative values.

The `para.est` function is developed to retrieve information from an object of the ‘mbsts’ class to provide parameter estimation results for selected predictors. The output is a list: `$index` provides the index of selected predictors according to the threshold value 0.8, `$para.est.mean` provides the estimated parameter values for those selected predictors, `$para.est.sd` provides the standard deviations for parameter estimations.

R> #Generate feature selection and parameter estimation results
R> para.est(object=mbsts.model,prob.threshold=0.8)

$index
[1]  1  3  5  6  8  9 10 12 13 15 16

$para.est.mean
[1]  1.9907  2.4501  1.4931 -1.9473  3.4812 -1.5023  3.9914  2.5143 -1.0007 -2.9951  0.4977

$para.est.sd
[1]  0.0321  0.3245  0.0319  0.1714  0.0072  0.0063  0.0541  0.0300  0.0077  0.0417  0.0018

We conduct convergence diagnosis by calling the `plot_cvg` function, using the index number...
Figure 3: Convergence diagnosis for the first predictor of 400 MCMC iterations after discarding the first 100.

Forecasting

Denote \( \hat{\psi} = (\alpha, \theta, \gamma, \Sigma, \beta) \). Looping through the five steps in Algorithm 1 yields a sequence of draws \( \hat{\psi} \) from a Markov chain with stationary distribution \( p(\hat{\psi}|Y) \) which is the posterior distribution of \( \hat{\psi} \) given \( Y \). After model training, forecasts are based on the posterior predictive distribution. Let \( \hat{Y} \) represent the set of values to be forecast. Samples of \( \hat{Y} \) from \( p(\hat{Y}|\hat{\psi}) \) can be drawn by simply iterating equations (5), (6), (7), (8), and (2) to move forward using initial values of states \( \alpha \) and initial values of parameters \( \theta, \beta \), and \( \Sigma \). For example, for the one-step-ahead forecast, samples were drawn from the multivariate normal distribution with mean equal to \( \hat{\mu} + \hat{\tau} + \hat{\omega} + \hat{\xi} \) and variance equal to \( \Sigma_{\mu} + \Sigma_{\tau} + \Sigma_{\omega} + \Sigma_{\xi} \). The detailed procedure can be seen in Algorithm 2 where the point prediction values could be formed by taking the average of drawn samples at the end.

The `mbsts.forecast` function is developed to retrieve information from an object of the ‘mbsts’ class and an object of the ‘SSModel’ class to forecast. It forecasts multiple steps ahead in the way...
Algorithm 2 Model Forecast (Qiu et al., 2020)

1: Draw the next latent time series states $\alpha_{t+1} = (\tilde{\mu}_{t+1}, \tilde{\delta}_{t+1}, \tilde{\tau}_{t+1}, \tilde{\omega}_{t+1})$ given current latent time series states $\alpha_t = (\tilde{\mu}_t, \tilde{\delta}_t, \tilde{\tau}_t, \tilde{\omega}_t)$ and component parameters $\theta = (\Sigma_{\mu}, \Sigma_{\delta}, \Sigma_{\tau}, \Sigma_{\omega})$, based on equations (5), (6), (7) and (8).

2: Based on indicator variable $\gamma$, compute the regression component given the information about predictors at time $t + 1$ by equation (2).

3: Draw a random error in multivariate normal distribution with variance equal to $\Sigma_{r}$ and sum them up using equation (1).

4: Sum up all the predictions and divide by the total number of MCMC iterations to generate the point prediction.
that once a forecast is produced it is added to the training data as a “fake” data. In the following example, a 5-steps-ahead forecast is generated by setting `steps` in the `mbsts.forecast` function.

```r
R> # make a 5-steps prediction
R> output<-mbsts.forecast(mbsts.model,STmodel,newdata=Xtest,steps=5)
```

The `mbsts.forecast` function has two outputs: `pred.dist` is an array of draws from the posterior distribution for the forecasts, i.e., for each time series it contains a matrix whose row indicates the MCMC iterations after “burn-in” and whose column indicates the steps of forecast; `pred.mean` is a matrix giving the posterior mean of the prediction for each target series.

```r
R> output$pred.mean
          [,1]       [,2]
 [1,] 1316.5353 2019.864
 [2,]  698.9187  1905.985
 [3,]  525.5992  1937.280
 [4,]  987.0031  1904.776
 [5,] 1495.9308  1977.514
```

**Conclusion**

This paper demonstrates how to use the R package `mbsts` for MBSTS modeling. By the multivariate nature of the MBSTS model, the correlations among multiple target series are naturally taken into account, which helps boost the forecasting power. Figure 6 in Qiu et al. (2018) revealed that the higher correlation among multiple target time series, the better performance of the MBSTS model over the univariate BSTS model. Therefore, it is better to model multiple target time series as a whole by MBSTS rather than model them individually by BSTS, especially when strong correlations appear in the multiple target time series. In this paper, we focus on the MBSTS modeling which uses the Bayesian paradigm to reduce dimension, but the frequentist approach could be better under different scenarios (see, e.g., Ning et al. (2021) in Monte Carlo setting).

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