SOME NATURALLY OCCURRING EXAMPLES OF 
\(A_{\infty}\)-BIALGEBRAS

AINHOA BERCIANO\(^1\) AND RONALD UMBLE\(^2\)

Abstract. Let \(p\) be an odd prime. When \(n \geq 3\), we show that each tensor factor \(E \otimes \Gamma\) of \(H^*(\mathbb{Z}, n; \mathbb{Z}_p)\) is an \(A_{\infty}\)-bialgebra with non-trivial structure. We give explicit formulas for the structure maps and the relations among them. Thus \(E \otimes \Gamma\) is a naturally occurring \(A_{\infty}\)-bialgebra.

1. Introduction

Let \(\Omega X\) be a loop space and let \(F\) be a field. In \([12]\), S. Saneblidze and the second author showed that the bialgebra structure on the singular chains \(C^*(\Omega X; F)\) pulls back along a quasi-isomorphism \(f: H = H^*(\Omega X; F) \rightarrow C^*(\Omega X; F)\) to an \(A_{\infty}\)-bialgebra structure on \(H\). However, the global \(A_{\infty}\)-bialgebra structure on \(H\) is quite subtle, and we have yet to completely unravel even the simplest non-trivial examples. This paper takes a first step in this direction.

Let \(p\) be an odd prime and let \(n \geq 3\). Let \(E(v, 2p^i + 1)\) denote the \(\mathbb{Z}_p\)-exterior algebra on a generator \(v\) of dimension \(2p^i + 1\) and let \(\Gamma(w, 2np^i + 2)\) denote the \(\mathbb{Z}_p\)-divided power algebra on a generator \(w\) of dimension \(2np^i + 2\). In \([2]\) and \([3]\), Cartan, Eilenberg and Mac Lane showed that \(H^*(\mathbb{Z}, n; \mathbb{Z}_p)\) factors as an infinite tensor product with infinitely many tensor factors of the form \(E \otimes \Gamma\); for example,

\[H^*(\mathbb{Z}, 3; \mathbb{Z}_p) \approx \bigotimes_{i \geq 0} E(v_i, 2p^i + 1) \otimes \Gamma(w_i, 2p^{i+1} + 2).\]

The main result in this paper is that for each \(i \geq 0\), the factor \(E(v_i, 2m + 1) \otimes \Gamma(w_i, 2mp + 2) \subset H^*(\mathbb{Z}, n; \mathbb{Z}_p)\) (with \(m\) a natural number) is an \(A_{\infty}\)-bialgebra with exactly three non-trivial “structurally compatible” operations, namely, a multiplication \(\mu\), a comultiplication \(\Delta\), and an operation \(\Delta_p: H \rightarrow H \otimes H\) of degree \(p - 2\).

The operations \(\Delta\) and \(\Delta_p\) define the \(A_{\infty}\)-coalgebra structure of \(H\) obtained by the first author using techniques of homological perturbation theory (see \([1]\)). Indeed, this \(A_{\infty}\)-coalgebra structure can be realized as a contraction of the reduced bar construction \(\tilde{B}(\mathbb{Z}[u]/(u^p))\), where \(|u| = 2n\).

Structural compatibility of \(\Delta\) and \(\mu\) is expressed by the fact that \(\Delta\) is an algebra map (\(H\) is a Hopf algebra), and it is natural to ask whether \(\Delta_p\) and \(\mu\) are structurally compatible in some analogous way. Indeed, we prove that \(\Delta_p\) is a “higher derivation” of \(\mu\), and it follows that \(H\) is an \(A_{\infty}\)-bialgebra as defined by S.

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Saneblidze and the second author in [10]. We shall refer to $A_\infty$-bialgebras of the form $(H,\mu,\Delta,\Delta_p)$ as Hopf $A_\infty$-coalgebras, to distinguish them from more general $A_\infty$-bialgebras.

The paper is organized as follows: Section 2 reviews the notion of an $A_\infty$-(co)algebra and the related tilde (co)bar construction. Section 3 reviews the $A_\infty$-coalgebra structure on $E \otimes \Gamma$ mentioned above. Section 4 reviews the construction of the S-U diagonal $\Delta_K$ on the cellular chains of associahedra $C_\ast (K_n)$ given in [9]. In Section 5 we give an exposition of the general notion of a “higher derivation”, i.e., a $\Delta$-derivation with respect to a $\Delta$-compatible family of maps indexed by the faces of a family of polytopes $X=\sqcup_{n\geq 0} X_n$; the ideas in this section are implicit in [10]. We conclude the paper with Section 6, in which we prove Theorem 4: Let $p$ be an odd prime and let $n \geq 1$. Then $E(v_n,2n+1) \otimes \Gamma(w_n,2np+2)$ is a Hopf $A_\infty$-coalgebra over $\mathbb{Z}_p$.

2. $A_\infty$-(CO)ALGEBRAS

We begin with a review of the notion of an $A_\infty$-algebra defined by J. Stasheff in his seminal paper [8], the dual notion of an $A_\infty$-coalgebra, and some related classical constructions. Let $R$ be a commutative ring with identity and let $M$ and $N$ be differential graded $R$-modules (DGMs). If $f : M \to N$ is a map of DGMs, let $f_{i,j} = \hat{1} \otimes f \otimes \hat{1}^{\otimes j} : N^{\otimes i} \otimes M \otimes N^{\otimes j} \to N^{\otimes i+j+1}$. Let $\hat{1}$ and $\hat{1}$ denote the operators that shift homological degree +1 and -1, respectively, and let $\overline{M} = M/M_0$. Let \{(\varphi^k \in Hom^{k-2}(M^{\otimes k}, M))\}_{k \geq 1} be an arbitrary family of operations, and for each $k$ and $n \geq 1$, consider the map of degree -1 given by

$$\sum_{i=0}^{n-k} (\hat{1} \varphi^k \hat{1}^{\otimes k})_{i,n-k-i} : (\hat{1} \overline{M})^{\otimes n} \to (\hat{1} \overline{M})^{\otimes n-k+1}.$$

Define

$$d_{\overline{BM}} = \sum_{1 \leq k \leq n; n \geq k \leq 0 \leq i \leq n-k} (\varphi^k \hat{1}^{\otimes k})_{i,n-k-i} : \overline{B}M \to \overline{B}M,$$

where $\overline{B}M = T^e (\hat{1} M)$ is the tensor coalgebra. Then $(M,n)$ is an $A_\infty$-algebra if $d_{\overline{BM}}^2 = 0$, in which case the operations \{(\varphi^n)\} satisfy the structure relations

$$(-1)^{\ell(i+1)} \varphi^{n-\ell,\ell+1} \varphi\ell,n-\ell-1-i = 0$$

for each $n \geq 1$. The signs in (2.1) were given in [9] and differ from those given by Stasheff in [8]. Either choice of signs induces an oriented combinatorial structure on the associahedra, and these structures are equivalent. The tilde bar construction on $M$ is the DG coalgebra (DGC) $(\overline{B}M, d_{\overline{BM}})$.

Let $K = \sqcup_{n \geq 3} K_n$ denote the disjoint union of associahedra and identify the $A_\infty$-operad with the cellular chains $C_\ast (K)$. Let $e^{n-2}$ denote the top dimensional face of $K_n$ and let $d_{(i,j)} (e^{n-2})$ denote its codimension 1 face corresponding to the parenthesization $x_1 \cdots (x_{i+1} \cdots x_{i+j+1}) \cdots x_n$. If $(M,\varphi^n)_{n \geq 1}$ is an $A_\infty$-algebra, the identifications

$$e^{n-2} \mapsto (-1)^n \varphi^n \text{ and } d_{(i,\ell)} (e^{n-2}) \mapsto \varphi^{n-\ell,\ell+1} \varphi\ell,n-\ell-1-i$$

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induce a chain map
\[ \zeta : \mathcal{A}_\infty \longrightarrow \{ \text{Hom} \left( M^\otimes n, M \right) \}_{n \geq 2} \]
that encodes the given \( \mathcal{A}_\infty \)-algebra structure (see [8], [9] and [10]).

Dually, consider an arbitrary family of operations \( \{ \psi^k \in \text{Hom}^{k-2} (N, N^\otimes k) \} \).
For each \( k \geq 1 \) and \( n \geq 2 \), consider the map of degree \(-1\) given by
\[
\sum_{i=0}^{n-1} \left( \uparrow \otimes \psi^k \uparrow \right)_{i,n-1-i} : (\downarrow N)^\otimes n \rightarrow (\downarrow N)^{\otimes n+k-1}.
\]
Define
\[
d_\tilde{\Omega} = \sum_{1 \leq k \leq n, n \geq 1, 0 \leq i \leq n-1} \left( \uparrow \otimes \psi^k \uparrow \right)_{i,n-1-i} : \tilde{\Omega} N \rightarrow \tilde{\Omega} N,
\]
where \( \tilde{\Omega} C = T^a (\downarrow N) \) is the tensor algebra. Then \( (N, \psi^n)_{n \geq 1} \) is an \( \mathcal{A}_\infty \)-coalgebra if \( d_\tilde{\Omega}^2 = 0 \), in which case the operations \( \psi^k \) satisfy the structure relations
\[
\sum_{0 \leq i \leq n-j-1} (-1)^{j(n+i+1)} \psi_{i,n-j-1-i}^{j+1} \psi_{i,n-j-1-i}^{n-j} = 0
\]
for each \( n \geq 1 \). The tilde cobar construction on \( N \) is the DG algebra (DGA) \( \left( \tilde{\Omega} N, d_\tilde{\Omega} \right) \).

The \( \mathcal{A}_\infty \)-coalgebra structure of a given \( \mathcal{A}_\infty \)-coalgebra \( (N, \psi^n)_{n \geq 1} \) is encoded operadically by the chain map
\[ \xi : \mathcal{A}_\infty \longrightarrow \{ \text{Hom} \left( N, N^\otimes n \right) \}_{n \geq 2} \]
induced by the identifications
\[ \xi : e^{n-2} \mapsto \psi^n \text{ and } \xi : d_{(i,j)} (e^{n-2}) \mapsto \psi_{i,n-j-1-i}^{j+1} \psi_{i,n-j-1-i}^{n-j}. \]

Given a connected DGA \( (A, d, \mu) \), the reduced bar construction of \( A \) is the cofree DGC \( \overline{B}(A) = T^a (\uparrow A) \) with (reduced) coproduct
\[
\Delta_B \left( [a_1] \cdots [a_r] \right) = \sum_{i=1}^{r-1} [a_1] \cdots [a_i] \otimes [a_{i+1}] \cdots [a_r]
\]
and differential \( d_B = d_t + d_a \), where
\[
d_t = \sum_{i=1}^{r} 1^\otimes i-1 \uparrow d \downarrow \otimes 1^r-i \quad \text{and} \quad d_a = \sum_{i=1}^{r-1} 1^\otimes i-1 \uparrow \mu \downarrow \otimes 1^r-i-1.
\]

Dually, given a simply-connected DGC \( (C, d, \Delta) \), The reduced cobar construction of \( C \) is the free DGA \( \overline{\Omega}(C) = T^a (\downarrow C) \) with differential \( d_\tilde{\Omega} = d_t + d_c \), where \( d_t \) is as above and
\[
d_c = \sum_{i=0}^{n-1} 1^\otimes i \uparrow \otimes 1 \downarrow \Delta \downarrow \otimes 1^\otimes n-i-1.
\]
3. The $A_{\infty}$-coalgebra $E \otimes \Gamma$

For $n \in \mathbb{N}$, let $Q_p(u, 2n) = \mathbb{Z}[u] / (u^p)$, where $|u| = 2n$. Recall that the exterior algebra $E(v, 2n - 1)$ on a generator $v$ of degree $2n - 1$ is a Hopf algebra with primitively generated coproduct $\Delta$. As a module, the divided power algebra $\Gamma(w, 2n)$ is generated by $\gamma_i = \gamma_i(w)_{i \geq 1}$, where $\gamma_1(w) = w$, and its algebra structure is defined by

$$\Delta(\gamma_k(u)) = \sum_{i+j=k} \gamma_i(u) \otimes \gamma_j(u).$$

Furthermore, $\Gamma(w, 2n)$ is a Hopf algebra with respect to the coproduct generated by $\Delta(\gamma_k(u)) = \sum_{i+j=k} \gamma_i(u) \otimes \gamma_j(u)$.

In [7], A. Prouté took the first steps toward computing the $A_{\infty}$-coalgebra structure on $H_\ast(K(\pi, n); \mathbb{Z}_p)$. When $p = 2$, he showed that $H$ is a classical coalgebra. But when $p$ is an odd prime, the $A_{\infty}$-coalgebra structure explodes and he only obtained partial results in certain simple cases. Thanks to Eilenberg and Mac Lane [3], there is a contraction (a special type of chain homotopy equivalence)

$$\hat{B}(Q_p(u, 2np)) \to E(v, 2np^i + 1) \otimes \Gamma(w, 2np^{i+1} + 2)$$

by which we obtain the decomposition

$$H_\ast(\mathbb{Z}, 3, \mathbb{Z}_p) \approx \bigotimes_{i \geq 0} E(w_i, 2p^i + 1) \otimes \Gamma(x_i, 2p^{i+1} + 2),$$

(see Theorem 4.5 in [11]). This idea extends inductively to $H_\ast(\mathbb{Z}, n, \mathbb{Z}_p)$ (see Theorem 4.24 of [11]). When $n = 4$, let $n_i = p^{i+1} + 1$; then we have

$$H(\mathbb{Z}, 4; \mathbb{Z}_p) \approx \bigotimes_{i \geq 0} \left[ \Gamma(y_i, 2p^i + 2) \otimes \bigotimes_{j \geq 0} E(a_j, 2n_ip^j + 1) \otimes \Gamma(b_j, 2n_ip^{j+1} + 2) \right];$$

when $n = 5$ we have

$$H(\mathbb{Z}, 5; \mathbb{Z}_p) \approx \bigotimes_{i \geq 0} \left[ \bigotimes_{k \geq 0} E_{i,k} \otimes \Gamma_{i,k} \right] \otimes \bigotimes_{j \geq 0} \left[ \bigotimes_{l \geq 0} E_{i,j,l} \otimes \Gamma_{i,j,l} \right];$$

and so on. In [11], the second author used these decompositions to extend Prouté’s results and obtain

**Theorem 1.** The induced $A_{\infty}$-coalgebra structure map $\Delta_q$ on $H_\ast(K(\mathbb{Z}, n); \mathbb{Z}_p)$ with $n \geq 3$, vanishes whenever $q \neq i(p - 2) + 2$ and $i \geq 0$.

This result follows immediately from

**Theorem 2.** For all $m \in \mathbb{N}$ and every odd prime $p$, the Hopf algebra $H = E(v, 2m + 1) \otimes \Gamma(w, 2np + 2)$ is a non-trivial $A_{\infty}$-coalgebra over $\mathbb{Z}_p$. The induced structure map $\Delta_q : H \to H^{\otimes q}$ is non-trivial if and only if $q = 2, p$. In fact, for $i = 0, 1$ and $\gamma_j = \gamma_j(w)$ we have

$$\Delta_2(v^i \gamma_j) = \sum_{k_0} \sum_{l_0} v^k \gamma_l \otimes v^{i-k} \gamma_{j-l};$$

$$\Delta_p(v^i \gamma_j) = \sum_{k_1 + \cdots + k_p = j - 1} v^{i+1} \gamma_{k_1} \otimes \cdots \otimes v^{i+1} \gamma_{k_p}.$$

The coproduct $\Delta_2$ defined in Formula [3.1] is the induced coproduct on the tensor product of coalgebras and is compatible with the induced multiplication $\mu$ as an algebra map. And as we shall see, $\Delta_p$ is compatible with $\mu$ as a “higher derivation”, which requires a diagonal on cellular chains of associahedra.

4. THE S-U DIAGONAL ON ASSOCIAHEDRA

This section gives a brief review of the S-U diagonals $\Delta_P$ on permutahedra $P = \sqcup_{n \geq 1} P_n$ and $\Delta_K$ on associahedra $K$ (up to sign); for details see [9]. Alternative constructions of $\Delta_K$ were subsequently given by Markl and Shnider [6] and Loday [4].

Let $\underline{n} = \{1, 2, \ldots, n\}, n \geq 1$. A matrix $E$ with entries from $\{0\} \cup \underline{n}$ is a step matrix if:

- Each element of $\underline{n}$ appears as an entry of $E$ exactly once.
- The elements of $\underline{n}$ in each row and column of $E$ form an increasing contiguous block.
- Each diagonal parallel to the main diagonal of $E$ contains exactly one element of $\underline{n}$.

The non-zero entries in a step matrix form a staircase connecting the lower-leftmost and the upper-rightmost entries. Right-shift and down-shift matrix transformations, which include the identity (a trivial shift), act on step matrices and produce derived matrices. Let $a = A_1 | A_2 | \cdots | A_p$ and $b = B_q | B_{q-1} | \cdots | B_1$ be partitions of $\underline{n}$. The pair $a \times b$ is an $(p, q)$-complementary pair (CP) if $B_i$ and $A_j$ are the rows and columns of a $q \times p$ derived matrix. Since faces of $P_n$ are indexed by partitions of $\underline{n}$, and CPs are in one-to-one correspondence with derived matrices, each CP is identified with some product face of $P_n \times P_n$.

**Definition 1.** Define $\Delta_P(e^0) = e^0 \otimes e^0$. Inductively, having defined $\Delta_P$ on $C_*(P_{k+1})$ for all $0 \leq k \leq n - 1$, define $\Delta_P$ on $C_*(P_{n+1})$ by

$$\Delta_P(e^n) = \sum_{(p,q)\text{-CPs } u \times v} \pm u \otimes v,$$

and extend to all of $C_*(P_{n+1})$ multiplicatively with respect to Cartesian product.

Recall that faces of $P_n$ in codimension $k$ are indexed by planar rooted trees with $n + 1$ leaves and $k + 1$ levels (PLTs).

**Example 1.** In terms of PLTs, the diagonal on $C_*(P_3)$ (up to sign) is given by

$$\Delta_P(\Psi) = \Psi \otimes \Psi + \Psi \otimes \Psi \quad + \Psi \otimes \Psi \quad + \Psi \otimes \Psi \quad + \Psi \otimes \Psi \quad + \Psi \otimes \Psi \quad + \Psi \otimes \Psi .$$

The diagonal $\Delta_P$ induces a diagonal $\Delta_K$ on $C_*(K)$. Recall that faces of $K_n$ in codimension $k$ are indexed by planar rooted trees with $n$ leaves and $k + 1$ vertices. Forgetting levels defines the cellular projection $\theta : P_n \rightarrow K_{n+1}$ given by A. Tonks [13]. Thus faces of $P_n$ indexed by PLTs with multiple vertices in some level degenerate under $\theta$, and corresponding generators lie in the kernel of the induced map $\theta : C_*(P_n) \rightarrow C_*(K_{n+1})$. The diagonal $\Delta_K$ is given by $\Delta_K(\theta) = (\theta \otimes \theta)\Delta_P$. 
Example 2. When \( n = 3 \), the components \( 1|23 \otimes 13|2 \) and \( 13|2 \otimes 3|12 \) of \( \Delta P (e^2) \) degenerate under \( \theta \) because the tree corresponding to \( 13|2 \) has two vertices in the top level; equivalently, \( \dim (13|2) = 1 \) whereas \( \dim \theta (13|2) = 0 \). Therefore (up to sign) the diagonal on \( C_* (K_4) \) is given by

\[
\Delta_k (\Psi) = \Psi \otimes \Psi + \Psi \otimes \Psi + \Psi \otimes \Psi + \Psi \otimes \Psi.
\]

5. \( \Delta \)-derivations and \( \Delta_X \)-compatible families

Let \( \{X_n\}_{n \geq 0} \) be a family of polytopes such that \( \dim X_n = n \), let \( X = \sqcup_{n \geq 0} X_n \) and assume that the cellular chains \( C_* (X) \) are equipped with a diagonal approximation \( \Delta_X : C_* (X) \to C_* (X) \otimes C_* (X) \). In this section we introduce the general notion of a \( \Delta \)-derivation homotopy with respect to a \( \Delta_X \)-compatible family of maps. When \( X_n \) is the \( n \)-simplex \( s_n \), our definition agrees with the notion of a high derivation defined by T. Kadeishvili in [4]. When \( X_n \) is the \( n \)-dimensional permutohedron \( P_{n+1} \) or associahedron \( K_{n+2} \), the notion of a \( \Delta \)-derivation with respect to a \( \Delta_X \)-compatible family is encoded in the construction of the biderivative given by S. Saneblidze and the second author (see [10], [11], [12] and [14]).

For each \( n \), let \( n_k \) be the number of \( k \)-faces of \( X_n \) and choose a system of generators \( \{x^k_i\}_{0 \leq k \leq n; 1 \leq i \leq n_k} \) for \( C_* (X_n) \). Let \( X^k_i \) denote the smallest subcomplex of \( X_n \) containing the \( k \)-face associated with \( x^k_i \). Given DGAs \( (A, \mu_A, d_A) \) and \( (B, \mu_B, d_B) \), let \( \Theta : C_* (X_n) \to \text{Hom} (A, B) \) be a map of degree zero and let \( \Theta^k_i = \Theta |_{C_* (X^k_i)} \).

Definition 2. The family of maps

\[ \tilde{\mathfrak{F}}_n = \{ \Theta (x^k_i) \}_{0 \leq k \leq n; 1 \leq i \leq n_k} \]

is \( \Delta_X \)-compatible if each \( \Theta^k_i \) is a chain map commuting the following diagram:

\[
\begin{array}{ccc}
C_k (X^k_i) & \xrightarrow{\Delta_X} & \sum_{p+q=k} C_p (X^k_i) \otimes C_q (X^k_i) \\
\Theta^k_i & \downarrow & \Theta^k_i \otimes \Theta^k_i \\
\text{Hom}^k (A, B) & \xrightarrow{\sum_{p+q=k} \text{Hom}^p (A, B) \otimes \text{Hom}^q (A, B)} & \approx \\
(\mu_A)^* & \downarrow & \approx \\
\text{Hom}^k (A \otimes A, B) & \xrightarrow{(\mu_B)_*} & \text{Hom}^k (A \otimes A, B) \otimes B .
\end{array}
\]

Let \( \tilde{\mathfrak{F}}_n \) be an \( \Delta_X \)-compatible family of maps. The map \( T = \Theta (x^k_i) : A \to B \) associated with the top dimensional cell of \( X \) is a \( \Delta \)-derivation with respect to \( \tilde{\mathfrak{F}}_n \) if the diagram above commutes when \( k = n \). If in addition, \( \Theta \) is a chain map, then \( T \) is a \( \Delta \)-derivation homotopy with respect to \( \tilde{\mathfrak{F}}_n \). There is the dual notion of a \( \Delta \)-coderivation homotopy with respect to a \( \Delta_X \)-compatible family.
Let $\sigma_{n,2} : (H \otimes n)^{\otimes 2} \to (H \otimes 2)^{\otimes n}$ be the canonical permutation of tensor factors. Then for example,

$$\sigma_{3,2}(a_1|a_2|a_3 \otimes b_1|b_2|b_3) = (-1)^{|b_1||a_2|+|b_1||a_3|+|b_2||a_3|} a_1|b_1 \otimes a_2|b_2 \otimes a_3|b_3.$$  

**Example 3.** Set $X_n = K_{n+2}$ and let $\Delta_K$ be the S-U diagonal on associahedra. Given a DGA $(A,d,\mu)$, let $d^{\otimes i}$ denote the free linear extension of $d$ to $A^{\otimes i}$. Choose an arbitrary family of DG module maps $\{\Delta_i \in \text{Hom}^{i-2}(A,A^{\otimes i})\}_{i \geq 2}$. For notational simplicity, identify $\Delta_i$ with the down-rooted $i$-leaf corolla and other down-rooted planar rooted trees with the corresponding compositions in $\text{Hom}(A,A^{\otimes \ast})$.

When $n = 2$, $K_2$ is a point. If $\gamma$ is $\Delta$-derivation homotopy with respect to the empty family $\mathcal{F}_2$, then $\gamma$ is a DGA map, i.e.,

$$d^{\otimes 2}\gamma - \gamma d = 0 \quad \text{and} \quad \mu = \mu \sigma_{2,2}(\gamma \otimes \gamma).$$

When $n = 3$, $K_3$ is an interval. If $\mathcal{F}_3 = \{\gamma, \gamma\}$ is a $\Delta_K$-compatible family of compositions, then $\gamma$ and $\gamma$ are DGA maps. If $\gamma$ is a $\Delta$-derivation homotopy with respect to $\mathcal{F}_3$, then $\gamma$ is a $(\gamma, \gamma)$-derivation homotopy, i.e.,

$$d^{\otimes 3}\gamma + \gamma d = \gamma \gamma \gamma \quad \text{and} \quad \mu = \mu^{\otimes 3}\sigma_{3,2}(\gamma \otimes \gamma + \gamma \otimes \gamma + \gamma \otimes \gamma).$$

In the case of the pentagon $K_4$, assume that $\mathcal{F}_4 = \{\Theta(x^k)\}_{k=0,1}$ is a $\Delta_K$-compatible family. Then

$$\gamma, \gamma, \gamma, \gamma \quad \text{and} \quad \gamma \gamma \gamma \gamma \gamma$$

are DGA maps;

$$\gamma$$ is a $(\gamma, \gamma)$-derivation homotopy;

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If $\gamma$ is a $\Delta$-derivation homotopy with respect to $\mathcal{F}_4$, then

$$d^{\otimes 4}\gamma - \gamma d = \left( \gamma + \gamma + \gamma + \gamma \right) - \left( \gamma + \gamma \right) \quad \text{and} \quad \mu = \mu^{\otimes 4}\sigma_{4,2}\left( \gamma \otimes \gamma + \gamma \otimes \gamma + \gamma \otimes \gamma \right) + \gamma \otimes \gamma - \gamma \otimes \gamma \gamma \gamma \gamma \gamma.$$

### 6. The Hopf $A_\infty$-coalgebra $E \otimes \Gamma$

The restricted class of $A_\infty$-bialgebras we need, called “Hopf $A_\infty$-coalgebras”, are special $A_\infty$-bialgebras defined in terms of the explicit structure relations derived in [14]. For each $q \geq 2$, this amounts to (1) freely extending the structure map $\Delta_q : H \to H^{\otimes q}$ as a $\Delta$-derivation $H^{\otimes 2} \to (H^{\otimes q})^{\otimes 2}$ with respect to the appropriate $\Delta_K$-compatible family of maps, (2) cofreely extending the multiplication $\mu$ as a coalgebra map $(H^{\otimes 2})^{\otimes q} \to H^{\otimes q}$, and (3) checking that these extensions are compatible in the sense that relation (6.1) below holds.

**Definition 3.** A Hopf $A_\infty$-coalgebra is a tuple $(A,d,\mu,\xi)$ with the following properties:

1. The triple $(A,d,\mu)$ is an associative DGA.
Lemma 1. Proof. A standard formula for binomial coefficients gives

\[ \binom{e^{n-2}}{e^{n-2}} = \mu^{\otimes n} \sigma_n \Delta_{e^{n-2}} \Delta_K \left( e^{n-2} \right). \]

Thus \( \Delta_n = \binom{e^{n-2}}{e^{n-2}} \) is a \( \Delta \)-derivation with respect to the standard family of maps indexed by the faces of \( K_n \) in codimension 1. There is the completely dual notion of a Hopf \( A_\infty \)-coalgebra.

Our main result applies following lemma, which follows from Vandermonde’s Identity:

**Theorem 3** (Vandermonde’s Identity). For \( r, s \geq 0 \) and \( 0 \leq k \leq r + s \),

\[ \binom{r + s}{k} = \sum_{i=0}^{k} \binom{r}{i} \binom{s}{k-i}. \]

**Lemma 1.** Let \( R = \mathbb{N} \cup \{0\} \) or \( R = \mathbb{Z}_p \) with \( p \) prime. For all \( i \geq 0 \) and all \( n \)-tuples \((z_1, \ldots, z_n) \in R^n\) we have

\[ \binom{z_1 + \cdots + z_n + 1}{i} = \sum_{s_1 + \cdots + s_n = i-1} \binom{z_1}{s_1} \cdots \binom{z_n}{s_n} + \sum_{t_1 + \cdots + t_n = i} \binom{z_1}{t_1} \cdots \binom{z_n}{t_n}, \]

where we reduce mod \( p \) when \( R = \mathbb{Z}_p \).

**Proof.** A standard formula for binomial coefficients gives

\[ \binom{z_1 + \cdots + z_n + 1}{i} = \binom{z_1 + \cdots + z_n}{i-1} + \binom{z_1 + \cdots + z_n}{i}. \]

Iteratively apply the Vandermonde identity to the right-hand summand and obtain

\[ \binom{z_1 + \cdots + z_n}{i} = \sum_{k_1=0}^{i} \sum_{k_2=0}^{i} \cdots \sum_{k_{n-1}=0}^{i} \binom{z_1}{k_1} \binom{z_2}{k_2-1} \cdots \binom{z_n}{k_n-1}. \]

Note the sum of the lower entries in the \( n \) binomial coefficients of this last expression is \( i \) and set \( t_1 = k_{n-1}, t_2 = k_{n-1} - k_{n-2}, \ldots, t_{n-1} = k_1 - k_2 \) and \( t_n = i - k_1 \). Then expression \( 6.2 \) can be rewritten as

\[ \binom{z_1 + \cdots + z_n}{i} = \sum_{t_1 + \cdots + t_n = i} \binom{z_1}{t_1} \cdots \binom{z_n}{t_n}. \]

**Remark 1.** The formula in Lemma 1 counts the number of ways \( i \) objects can be selected from a collection of \( z_1 + \cdots + z_n + 1 \) objects of \( n+1 \) different colors, one of which is uniquely colored “black” and \( z_i \) of which have the same unique color for each \( i \). The first sum on the right-hand side counts the ways to select \( i \) objects one of which is black; the second sum counts the ways to select \( i \) objects none of which are black.

**Theorem 4.** For each \( i \geq 0 \) and \( n \geq 1 \), let

\[ A_i = E(v_i, 2np^i + 1) \otimes \Gamma(w_i, 2np^{i+1} + 2), \]

let \( \mu = (\mu_E \otimes \mu_T) \sigma_2 \), and for \( j = 2, p \), let \( \Delta_j \) be defined as in \( 6.1 \) and \( 6.2 \). Then \( (A_i, \mu, \Delta_2, \Delta_p) \) is a Hopf \( A_\infty \)-coalgebra over \( \mathbb{Z}_p \).
Proof. For each \( n \geq 1 \) and \( i \geq 0 \), \((A_i, \Delta_2, \mu)\) is a Hopf algebra via \( \mu = (\mu_E \otimes \mu_i)_{i,2,2} \), where 
\[
\mu_E(v_i \otimes v_i) = 0 \quad \text{and} \quad \mu_i(\gamma_i(w_i) \otimes \gamma_j(w_i)) = \binom{i+j}{i} \gamma_{i+j}(w_i),
\]
and \((A_i, \Delta_2, \Delta_p)\) is an \( A_{\infty} \)-coalgebra by Theorem 2 let \( \xi \) be the operadic representation. Since \( A_i \) is coassociative and \( \Delta_j = 0 \) for \( 2 < j < p \), the \( \Delta_K \)-compatible family \( \mathcal{F}_p = \{ 0, f_p = (\Delta_2 \otimes 1^{\otimes p-2}) \cdots (\Delta_2 \otimes 1) \Delta_2 \} \) and the structure relation in (6.1) reduces to 
\[
\Delta_p \mu = \mu^{\otimes p} \sigma_{p,2} (f_p \otimes \Delta_p + \Delta_p \otimes f_p).
\]
Since both sides of relation (6.3) vanish on tensor products involving \( v_i \), it is sufficient to evaluate this relation on \( \gamma_i \otimes \gamma_j = \gamma_i(w_i) \otimes \gamma_j(w_i) \). First, 
\[
\Delta_p \mu (\gamma_i \otimes \gamma_j) = \binom{i+j}{i} \Delta_p (\gamma_{i+j}) = \sum_{z_1 + \cdots + z_p = i+j-1} \binom{i+j}{i} v_i \gamma_{z_1} \otimes \cdots \otimes v_i \gamma_{z_p}
\]
\[
= \sum_{z_1 + \cdots + z_p = i+j-1} \binom{z_1+\cdots+z_p+1}{i} u,
\]
where \( u = v_i \gamma_{z_1} \otimes \cdots \otimes v_i \gamma_{z_p} \), and second, 
\[
\mu^{\otimes p} \sigma_{p,2} (f_p \otimes \Delta_p + \Delta_p \otimes f_p) (\gamma_i \otimes \gamma_j) = \sum_{l_1 + \cdots + l_p = j \atop s_1 + \cdots + s_p = i-1} \binom{l_1+s_1}{s_1} \cdots \binom{l_p+s_p}{s_p} v_i \gamma_{l_1+s_1} \otimes \cdots \otimes v_i \gamma_{l_p+s_p}
\]
\[
+ \sum_{m_1 + \cdots + m_p = j-1 \atop t_1 + \cdots + t_p = i} \binom{m_1+t_1}{t_1} \cdots \binom{m_p+t_p}{t_p} v_i \gamma_{m_1+t_1} \otimes \cdots \otimes v_i \gamma_{m_p+t_p}
\]
\[
= \sum_{z_1 + \cdots + z_p = i+j-1} \left[ \sum_{s_1 + \cdots + s_p = i-1} \binom{z_1}{s_1} \cdots \binom{z_p}{s_p} + \sum_{t_1 + \cdots + t_p = i} \binom{z_1}{t_1} \cdots \binom{z_p}{t_p} \right] u.
\]
\[(6.5)\]
But by Lemma 1 expressions (6.4) and (6.5) are congruent modulo \( p \) and it follows that \( \Delta_p \) is a \( \Delta \)-derivation with respect to \( \mathcal{F}_p \). \( \square \)

Remark 2. For each odd prime \( p \), each \( n \geq 3 \), and each \( i \geq 0 \), the tensor product 
\[
A_i = E(v_i, 2np^l + 1) \otimes \Gamma(w_i, 2np^{l+1} + 2)
\]
with \( \mu \) and \( \Delta_j \) as in (5.3) and (5.4), is a Hopf \( A_{\infty} \)-coalgebra over \( \mathbb{Z}_p \). Therefore, by the Eilenberg-Mac Lane decompositions discussed in Section 3, \( A_i \subset H_* (\mathbb{Z}_n; \mathbb{Z}_p) \) is a naturally occurring example of a Hopf \( A_{\infty} \)-coalgebra for each \( i \geq 0 \).

The Hopf \( A_{\infty} \)-coalgebra structures on the \( A_i \)'s induce a global \( A_{\infty} \)-bialgebra structure on \( H \) as a formal tensor product of \( A_{\infty} \)-bialgebras, but the extent to which this induced formal structure agrees with the geometric global structure mentioned
the introduction is not immediately clear. Nevertheless, such formal structures are relevant. Let $C_n$ denote a finite cyclic group of order $n$. In [15] M. Vejdemo-Johansson and the second author proved that the group cohomology $H^*(C_3; \mathbb{Z}_3)$ is an $A_\infty$-bialgebra dual to $E(v) \otimes \Gamma(w)$ with $|v| = 1$ and $|w| = 2$. And indeed, computing the formal $A_\infty$-bialgebra structure on $H^*(C_3 \times n; \mathbb{Z}_p) = H^*(C_3; \mathbb{Z}_p)^{\otimes n}$ is an interesting problem in pure algebra.

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