Polytropes: Implications for Molecular Clouds and Dark Matter

Christopher F. McKee
Departments of Physics and of Astronomy, University of California, Berkeley CA 94720

Abstract. Molecular clouds are supported against their own self-gravity by several different sources of pressure: thermal pressure, mean magnetic pressure, and turbulent pressure. Multi-pressure polytropes, in which each of these pressures is proportional to a power of the density, can account for many of the observed properties of molecular clouds. The agreement with observation can be improved with composite polytropes, in which an isothermal core is embedded in a turbulent envelope. Observed molecular clouds generally have $\gamma_p < 1$, corresponding to a velocity dispersion that increases with scale. For such clouds the ratio of the mean pressure to the surface pressure must be less than 4.

Small, very dense ($n_H \sim 10^{11}$ cm$^{-3}$) molecular clouds have been proposed as models for both dark matter and for extreme scattering events. Insofar as the equation of state in these clouds can be represented by a single polytropic relation, such models conflict with observation. It is possible to contrive composite polytropes that do not conflict with observation, but whether the thermal properties of the clouds are consistent with such structures remains to be determined.

1. Introduction

Understanding star formation requires understanding the interstellar molecular clouds out of which stars form. These molecular clouds are objects of fascinating dynamical complexity, exhibiting highly supersonic motions while at the same time being gravitationally bound (Zuckerman & Evans 1974; Larson 1981). The nonthermal motions help support the clouds against the force of gravity; thermal pressure and magnetic fields also contribute. As is characteristic of turbulent motions, the amplitude of the nonthermal motions increases with scale (Larson 1981). Since the nonthermal motions are largest on the scale of the cloud itself, they lead to substantial changes in the shape of the cloud over time (Ballesteros-Paredes, Vazquez-Semadeni, & Scalo 1999). Attempting to model such a complex system is a daunting task that is only now beginning to be tackled numerically (see Vazquez-Semadeni et al 2000).

In order to treat the structure of molecular clouds analytically, it is necessary to make a number of approximations. First, we assume that the cloud is in a steady state. The steady-state approximation is plausible since the lifetime of molecular clouds is typically about an order of magnitude greater than
their free-fall time (Blitz & Shu 1980). For example, Williams & McKee (1997) have shown that massive stars will destroy a cloud of mass $M \sim 10^6 M_\odot$ by photoionization in $3 \times 10^7$ yr; smaller clouds live longer. By comparison, the typical cloud of that mass has a mean density $\bar{n}_H \simeq 84 M_6^{-1/2}$ cm$^{-3}$ (Solomon et al 1987), corresponding to a free-fall time $t_{ff} = 1.37 \times 10^6 (\bar{n}_H/10^3$ cm$^{-3})^{-1/2}$ yr $= 4.7 \times 10^6 (M/M_6$ $M_\odot)^{1/4}$ yr. Clouds of mass $M \lesssim 10^6 M_\odot$ thus live $\gtrsim 6 t_{ff}$. The steady-state assumption is quite approximate, however, since the process of destroying the cloud by photoionization is very violent.

Since molecular clouds are approximately in a steady state and they are gravitationally bound, it follows that, when averaged over time, they are approximately in hydrostatic equilibrium. For example, the majority of cores in high latitude cirrus clouds observed by Turner (1993) are consistent with being in hydrostatic equilibrium.

Next, we assume that the time-averaged cloud is spherical. Some effects that could lead to non-spherical clouds, including tidal gravitational fields (Scoville & Sanders 1987) and rotation (Goodman et al 1993), are observed to be relatively weak. Some molecular clouds are observed to be highly filamentary (e.g., Alves et al 1998), and it has been suggested that this can be explained by helical magnetic fields (Fiege & Pudritz 2000). Many molecular clouds are not highly filamentary, however; for example, only about 15% of the clouds in the catalog of Solomon et al (1987), which is based on $^{13}$CO observations, have aspect ratios exceeding 2.

### 2. Polytropes

Molecular clouds in the Galaxy are observed to be gravitationally bound, and we are approximating them as being spherical and in hydrostatic equilibrium. Just as in the case of stars, it is convenient to model them as polytropes, which satisfy the equation of hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2}, \quad (1)$$

with

$$P(r) = K_p \rho^{\gamma_p}. \quad (2)$$

The structure of a polytrope is determined by the polytropic index

$$\gamma_p \equiv 1 + \frac{1}{n}, \quad (3)$$

where $n$ is the index used to characterize polytropes in the theory of stellar structure. The stability of a polytrope is determined by $\gamma_p$ and the adiabatic index $\gamma$ that describes how the pressure of a mass element responds to a density perturbation,

$$\delta \ln P \equiv \gamma \delta \ln \rho. \quad (4)$$

Polytropes provided the first quantitative model for stars. In order to have zero pressure at the boundary, which is assumed to be at a finite radius, it is necessary to have $\gamma_p > 6/5$. Stars are stable against gravitational collapse for
\[ \gamma > \frac{4}{3}. \] Since the dynamical timescale for a star is short compared to the heat flow time, the entropy is constant in each mass element during a dynamical perturbation; such perturbations are described as \textit{locally adiabatic} by McKee & Holliman (1999; hereafter MH).

Lynden-Bell & Wood (1968) modeled globular clusters as bounded, isothermal gas spheres (\( \gamma_p = 1 \)). Since there are no internal degrees of freedom, the adiabatic index is \( \gamma = 5/3 \); as a result, the “gas” is non-isentropic. Globular clusters are \textit{globally adiabatic} (MH) since the heat flow time is comparable to the dynamical time of the cluster. As Lynden-Bell & Wood (1968) showed, a pressure-bounded cloud with \( \gamma_p = 1, \gamma = 5/3 \) is subject to core collapse when the density contrast between the center and the surface is too large, \( \frac{\rho_c}{\rho_s} > 389.6 \) (MH).

### 3. Polytropic Models of Molecular Clouds

Molecular clouds are supported by three pressure components, thermal, magnetic, and turbulent. Lizano & Shu (1989) developed the first model that accounted for these three pressure components. They assumed that the gas is isothermal and that the turbulent pressure scales as the logarithm of the density (a “logatrope”); the magnetic field was assumed to be axisymmetric.

MH developed the theory of \textit{multi-pressure polytropes} in which each pressure component has arbitrary values of \( \gamma_p \) and \( \gamma \). Clouds with \( \gamma < \gamma_p \) are convectively unstable, and were not considered. Holliman (1995) showed that the effects of an axisymmetric magnetic field in which the flux is frozen to the gas could be approximated by a gas with \( \gamma = 4/3 \). For a flux-to-mass distribution corresponding to a uniform field in a spherical cloud, \( \gamma_p = 4/3 \); ambipolar diffusion reduces \( \gamma_p \) below 4/3. For the turbulent pressure, MH focused on the case of Alfvénic turbulence, which (at least when it is weak) can be modeled with \( \gamma_p = 1/2, \gamma = 3/2 \) (McKee & Zweibel 1995). Since \( \gamma_p < 1 \) for Alfvén waves, the velocity dispersion \( \sigma \propto \rho^{(\gamma_p-1)/2} \) increases with scale, as observed (Larson 1981). However, since the Alfvén waves are globally adiabatic, they are less effective at providing stability than would be expected for \( \gamma = 3/2 \); in fact, they have the same stability properties as a locally adiabatic component with \( \gamma \leq 1 \). The equation of state for molecular clouds is therefore \textit{soft}: These clouds usually have \( \gamma_p < 6/5 \) unless the magnetic field has \( \gamma_p > 6/5 \) and is sufficiently strong. Furthermore, all the pressure components have \( \gamma \leq 4/3 \) (in the case of Alfvén waves, this is the equivalent locally adiabatic index). As a result, \textit{stable} molecular clouds must be confined by the pressure of the ambient medium, and their properties are determined by conditions at the surface. For example, stable clouds with \( \gamma_p < 4/3 \) satisfy

\[ M \leq 4.555 \frac{\sigma_s^4}{(G^3 P_s)^{1/2}} \]  

(5)

for any value of \( \gamma \), where \( \sigma_s \) is the 1D velocity dispersion at the cloud surface. For \( \gamma_p = 1 \), the coefficient 4.555 is reduced to 1.182, the value for the Bonnor-Ebert sphere. The mean pressure in molecular clouds is limited by the surface pressure: For \( \gamma_p \leq 1 \), it is less than \( 4P_s \). However, the central pressure can
become arbitrarily large compared to the surface pressure if $\gamma$ is sufficiently greater than $\gamma_p$. Holliman (1995) showed that multi-pressure polytropes could successfully account for a number of the observed properties of molecular clouds.

The molecular cloud cores in which low-mass stars form often exhibit central regions that are supported primarily by the pressure of an isothermal gas, with nonthermal motions becoming important only in the envelopes. To model this two component structure it is convenient to introduce composite polytropes (Curry & McKee 2000) of the type used for stars many decades ago (e.g., Milne 1930). Curry & McKee (2000) showed that composite polytropes are very promising as models for low-mass cores: In particular, it is possible to have an isothermal core with a non-isentropic ($\gamma > \gamma_p$) polytropic envelope in which the central temperature and the surface pressure are fixed (as they generally are in practice), but in which the mass is arbitrarily large. Such models are consistent with observations of small NH$_3$ cores in large $^{13}$CO envelopes. Curry (in preparation) has made detailed comparisons of composite polytrope models with the observations.

### 4. Molecular Clouds as Dark Matter

A number of authors have suggested that self-gravitating clouds of cold molecular gas could account for the dark matter in the Galactic halo (Pfenniger, Combes, & Martinet 1994; Pfenniger & Combes 1994; De Paolis et al 1996, 1998; Gerhard & Silk 1996; Combes & Pfenniger 1997). Henriksen & Widrow (1995) and Walker & Wardle (1998; hereafter WW98) went on to suggest that such clouds would have ionized surfaces and could therefore account for the “extreme scattering events” (ESEs) observed by Fiedler et al (1994). Henriksen & Widrow (1995) pointed out that such clouds could cause gravitational microlensing if the clouds had masses of order $0.1 \, M_\odot$. Draine (1998) showed that gas clouds can also act as optical lenses and thereby mimic microlensing.

The clouds considered in the papers by Pfenniger, Combes and Martinet are assumed to be turbulent, with a hierarchical internal structure that terminates on the smallest scale in “clumpuscules” of mass $\sim 10^{-3} \, M_\odot$ and radius $\sim 30$ AU. If the clumpuscules exist near the edge of the cloud, so that they are exposed to typical interstellar conditions, then they are subject to the difficulties described below. In any case, the simulations reviewed in Vazquez-Semadeni et al (2000) show that turbulent, isothermal clouds dissipate their internal kinetic energy in about a dynamical time ($\sim 10^3$ yr), far less than the $\gtrsim 1$ Gyr cloud lifetimes assumed by these authors.

Gerhard & Silk (1996) have shown that clouds embedded in non-baryonic halos can be in stable hydrostatic equilibrium even in the absence of a confining medium. Here we shall argue that it is unlikely that highly pressured, self-gravitating clouds without such halos can exist in the Galactic halo. Valentijn & van der Werf (1999) claim to have detected enough molecular hydrogen in the

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1 The intensity of the interstellar ionizing radiation field is quite weak (Reynolds 1984; Slavin, McKee, & Hollenbach 2000), however, and to date there is no physically self-consistent calculation that demonstrates that the surfaces of these clouds can be sufficiently ionized to account for the ESEs.
nearby spiral NGC 891 to account for the dark matter within the optical disk of that galaxy; the densities they infer ($n_H \lesssim 10^3 \text{ cm}^{-3}$) are orders of magnitude less than those in the models discussed above, however, and do not violate the arguments presented here.

The argument is strongest if the clouds can be represented as single component polytropes. First consider the case in which $\gamma_p \leq 1$. This includes the case of isothermal clouds ($\gamma_p = 1$), such as those considered by DePaolis et al (1998) and by Wardle & Walker (1999).\[^1\] Such clouds must have a mean pressure less than 4 times the ambient pressure (MH). Now, the ambient pressure in the halo is less than that in the disk of the Galaxy, which is $P/k \simeq 2 \times 10^4 \text{ K cm}^{-3}$ (Boulares & Cox 1990; I have omitted the pressure due to cosmic rays, since they penetrate into the cloud). For a given value of the mean cloud density $\bar{n}_H$, a lower limit on the mean pressure of the cloud is given by assuming that the cloud is composed of molecular hydrogen (plus a cosmic abundance of helium) and has a temperature equal to that of the cosmic microwave background, 2.73 K; $\bar{P}/k > 0.6 \times 2.73 \bar{n}_H$. The constraint that this be less than $4 \times 2 \times 10^4 \text{ K cm}^{-3}$ gives an upper limit on the mean cloud density, $\bar{n}_H < 5 \times 10^4 \text{ cm}^{-3}$. (This constraint is comfortably satisfied by galactic molecular clouds—see §1.) If the clouds are polytropes with $\gamma_p < 1$, this constraint rules out the model of WW98, who assumed $\bar{n}_H \sim 10^{12} \text{ cm}^{-3}$. It also rules out much of the parameter space considered by DePaolis et al (1998), $10^4 \text{ cm}^{-3} \lesssim \bar{n}_H \lesssim 10^8 \text{ cm}^{-3}$.

Next, consider polytropes with $1 < \gamma_p < 6/5$. For such clouds, $\bar{n}_H/n_s$ increases toward infinity as $\gamma_p \to 6.5$, but the ratio remains modest unless $\gamma_p$ is very close to 6/5. For example, for $\gamma_p = 1.15$ one finds $\bar{n}_H/n_s < 13.6$. Hence, unless $\gamma_p$ is extremely close to 6/5, such polytropes cannot describe the dense clouds proposed to account for the dark matter.

Finally, consider polytropes with $\gamma_p \geq 6/5$. Such polytropes can have zero pressure at the boundary, like stars do, so the above constraint does not apply. Nonetheless, there are two significant difficulties. First, since $T \propto \rho^{\gamma_p-1}$, the large ratio of the mean density to the surface density in these clouds implies a correspondingly large temperature ratio. The maximum density at the surface is for a molecular gas at 2.73 K, as described above; this gives $n_s \lesssim 1.2 \times 10^4 \text{ cm}^{-3}$. Draine (1998) has calculated polytropic models for clouds of mean density $\bar{n}_H = 6 \times 10^{10} \text{ cm}^{-3}$ with a range of polytropic indexes from $\gamma_p = 11/9 \sim 1.222$ (just above the critical value of 6/5) to $\gamma_p = 5/3$. For $\gamma_p = 11/9$, the cloud is very centrally concentrated: It has a central density $n_c = 6190 \bar{n}_H = 3.7 \times 10^{14} \text{ cm}^{-3}$, corresponding to a central to surface temperature ratio of $T_c/T_s \geq 214$. In the opposite case of $\gamma_p = 5/3$, the cloud is much less centrally concentrated ($n_c/\bar{n}_H = 6.0$), but the temperature ratio is much larger because of the greater value of $\gamma_p$. $T_c/T_s \geq 0.97 \times 10^5$. Setting $T_s = 2.73 \text{ K}$ gives $T_c \geq 580$, $2.6 \times 10^5 \text{ K}$ for $\gamma_p = 11/9$, 6/5, respectively. (These values are larger than Draine’s since he did not impose the requirement that $T_s \geq 2.73K$; to make his models consistent

\[^2\]Wardle (private communication) has pointed out that he and Walker did not assume that the clouds are isothermal, but rather that they could be characterized by a mean temperature. However, since the temperatures they considered are so close to that of the microwave background radiation, which sets a floor on the temperature of each part of the cloud, their models are in fact quite close to being isothermal.
with this constraint, one would either have to drop the assumption that the cloud is a single-component polytrope [see below] or increase $M/R \propto T$ significantly.) Over the entire range of polytropic indexes considered by Draine (1998), it is difficult, if not impossible, to maintain the high central temperatures required.

The dominant heating mechanisms that have been suggested for these clouds are cosmic rays and X-rays (DePaolis et al. 1998 have also suggested heating by embedded binaries, but we shall not consider that mechanism here). In both cases, the heating decreases inward, making it difficult to have a polytropic structure in which the temperature increases inward, as it does for $\gamma_p > 1$. More significantly, the column density to the center of a polytrope with $\gamma_p \geq 6/5$ is so large that the heating rate is substantially reduced there. Cosmic rays are more penetrating that X-rays, so I shall focus on them. For the $\gamma_p = 5/3$ model, I find $\Sigma_c \equiv \int_0^R \rho dr = 59 \text{ g cm}^{-2}$. A lower limit on the interaction cross section of the cosmic rays is the nuclear cross section; for relativistic cosmic rays this is about 0.02 cm$^2$ g$^{-1}$ (Webber, Lee, & Gupta 1992). Thus, in this case (which has already been eliminated by its extremely high central temperature), the cosmic ray intensity is reduced by at least a factor $e$. The case with the lowest central temperature ($\gamma_p = 11/9$) is much more centrally concentrated, and as a result it has such a large central column density ($\Sigma_c = 5000 \text{ g cm}^{-2}$) that it is impossible for cosmic rays to reach the center. We conclude that it is impossible for dense, but non-stellar, gas clouds that obey a polytropic equation of state to exist in the diffuse ISM, much less the Galactic halo.

This conclusion can be circumvented by dropping the assumption that the cloud is a single-component polytrope. A composite polytrope in which the core (with most of the mass) has $\gamma_p > 6/5$ but the envelope (with most of the volume) is isothermal can be developed that is consistent with Draine’s model and has $T_s > 2.73$ K (Draine, private communication). The central column density of these models is similar to that of the single component models, however, so one is restricted to cases with $\gamma_p$ close to $5/3$ if cosmic rays are to be able to heat the central regions of the cloud. Whether such a contrived model is physically realizable remains to be seen.

We see that the condition of hydrostatic equilibrium places severe constraints on the viability of models of dense gas clouds that have been advanced to account for dark matter. Simple polytropic models can be ruled out, but detailed calculations of the structure of such clouds with accurate heating and cooling are necessary to determine if they can exist at the low pressures characteristic of the diffuse ISM.

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