A NOTE ON BALANCING NON-WIEFERICH PRIMES IN ARITHMETIC PROGRESSIONS

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ABSTRACT. We prove the lower bound for the number of balancing non-Wieferich primes in arithmetic progressions. More precisely, for any given integer \( r \geq 2 \) there are \( \gg \log x \) balancing non-Wieferich primes \( p \leq x \) such that \( p \equiv \pm 1 \pmod{r} \), under the assumption of the abc conjecture for the number field \( \mathbb{Q}(\sqrt{2}) \).

1. INTRODUCTION

Let \( b \geq 2 \) be an integer. An odd prime \( p \) is called a Wieferich prime for base \( b \) if
\[
b^{p-1} \equiv 1 \pmod{p^2}.
\]
Otherwise it is called a non-Wieferich prime for base \( b \). It is well-known that Wieferich prime for base 2 and the first case of Fermat’s last theorem are strongly intertwined [17]. Today, the only known Wieferich primes for base 2 are 1093 and 3511.

A search for the Wieferich prime is one of the long-standing problems in number theory. It is still unknown whether there are finitely or infinitely many Wieferich primes that exist for any base \( b \geq 2 \). However, Silverman [13] established the conditional results on non-Wieferich primes. Assuming the abc conjecture [7], he proved the infinitude of non-Wieferich primes for any base \( b \).

i.e., for any fixed \( b \in \mathbb{Q}^* \), where \( \mathbb{Q}^* = \mathbb{Q} \setminus \{0\} \) and \( b \neq \pm 1 \), if the abc conjecture is true, then
\[
\left| \{ \text{primes } p \leq x : b^{p-1} \equiv 1 \pmod{p^2} \} \right| \gg_x \log x.
\]

In 2013, Graves and Ram Murty [6] improved Silverman’s result to certain arithmetic progressions. They showed that, if \( b \) and \( r \) are positive integers and assume the abc conjecture, then
\[
\left| \{ \text{primes } p \leq x : p \equiv 1 \pmod{r}, b^{p-1} \not\equiv 1 \pmod{p^2} \} \right| \gg \frac{\log x}{\log \log x}.
\]

Then, there has been further enhancement made by Chen and Ding [3].
\[
\left| \{ \text{primes } p \leq x : p \equiv 1 \pmod{r}, b^{p-1} \not\equiv 1 \pmod{p^2} \} \right| \gg \frac{\log x \log \log \log x}{\log \log x}.
\]

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where $M$ is any fixed positive integer. Recently, Ding [4] further strengthened the lower bound as

$$\left| \{ \text{primes } p \leq x : p \equiv 1 \pmod{r}, b^{p-1} \not\equiv 1 \pmod{p^2} \} \right| \gg \log x.$$ 

In this paper, we prove the similar lower bound for non-Wieferich primes in balancing numbers $\{B_n\}$ (defined in Section 2.1), assuming the abc conjecture for the number field $\mathbb{Q}(\sqrt{2})$.

In 1999, Behera and Panda [1] first proposed the concept of balancing numbers and studied their properties. Then, Rout [12] defined the balancing Wieferich primes. i.e., an odd prime $p$ is called a balancing Wieferich prime if it satisfies the congruence

$$B_p \equiv 0 \pmod{p^2},$$

where $\left( \frac{8}{p} \right)$ denotes the Legendre symbol. Otherwise, it is called a balancing non-Wieferich prime. Recently, Wang and Ding [16] proved that there are $\gg \log x$ balancing non-Wieferich primes, assuming the abc conjecture for the number field $\mathbb{Q}(\sqrt{2})$. Earlier, Rout [12] and Dutta et al. [5] proved some lower bounds for the number of balancing non-Wieferich primes $p$ such that $p \equiv 1 \pmod{r}$, where $r \geq 2$ be a fixed integer. However, Wang and Ding [16] remarked that their results had some gaps. To the best of our knowledge, the main theorem in this paper is the first result in this direction which addresses the problem of balancing non-Wieferich primes in arithmetic progressions.

More precisely, we prove the following main theorem:

**Theorem 1.1.** Let $r \geq 2$ be any fixed integer and let $n > 1$ be any integer. Assume that the abc conjecture for the number field $\mathbb{Q}(\sqrt{2})$ is true. Then

$$\left| \{ \text{primes } p \leq x : p \equiv \pm 1 \pmod{r}, B_{p-(\frac{8}{p})} \not\equiv 0 \pmod{p^2} \} \right| \gg_{a,r} \log x.$$ 

2. Preliminaries

2.1. Balancing numbers. The sequence of balancing numbers $\{B_n\}$ is defined by the recurrence relation $B_{n+1} = 6B_n - B_{n-1}$ for $n \geq 1$ with initial conditions $B_0 = 0$ and $B_1 = 1$.

**Definition 2.1.** [1] A positive integer $n$ is called a balancing number if

$$1 + 2 + \ldots + (n - 1) = (n + 1) + (n + 2) + \ldots + (n + k),$$

where $k \in \mathbb{Z}^+$ is called the balancer corresponding to the balancing number $n$.

In other words, $n \in \mathbb{Z}^+$ is a balancing number if and only if $n^2$ is a triangular number. i.e., $8n^2 + 1$ is a perfect square.

The Binet formula for balancing number is

$$B_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$
where \( \alpha = 3 + 2\sqrt{2} \) and \( \beta = 3 - 2\sqrt{2} \).
Throughout this paper, we take \( \alpha = 3 + 2\sqrt{2} \) and \( \beta = 3 - 2\sqrt{2} \). Further we note that, for any prime \( p > 2 \), \( B_p(s) \equiv 0 \pmod{p} \) [10].

2.2. The abc conjecture. The abc conjecture was formulated by Oesterlé [9] and Masser [8]. We state the abc conjecture [7] below:

For any given real number \( \varepsilon > 0 \), there is a constant \( C_\varepsilon \) which depends only on \( \varepsilon \) such that for every triple of positive integers \( a, b, c \) satisfying \( a + b = c \) with \( \gcd(a, b) = 1 \), we have

\[
c < C_\varepsilon \left( \overline{\rad}(abc) \right)^{1+\varepsilon},
\]

where \( \overline{\rad}(abc) = \prod_{p | abc} p \).

We now recall the definition of Vinogradov symbol.

**Definition 2.2.** [15] Let \( f \) and \( g \) are two non-negative functions. If \( f \ll cg \) for some positive constant \( c \), then we write \( f \ll g \) or \( g \gg f \). It is also called **Vinogradov symbol**.

2.2.1. The abc conjecture for number fields [15, 7]. Let \( K \) be an algebraic number field and \( K^* = K \setminus \{0\} \). Let \( V_K \) be the set of primes on \( K \), that is any \( v \in V_K \) is an equivalence class of non-trivial norms on \( K \) (finite or infinite). Let \( \|x\|_v := N_{K/Q}(p)^{-\nu_p(x)} \), if \( v \) (finite) is defined by a prime ideal \( p \) of the ring of integers \( \mathcal{O}_K \) in \( K \) and \( \nu_p \) is the corresponding valuation, where \( N_{K/Q} \) is the absolute value norm. For \( v \) is infinite and let \( \|x\|_v := |\rho(x)|^e \) for all non-conjugate embeddings \( \rho : K \to \mathbb{C} \) with \( e = 1 \) if \( \rho \) is real and \( e = 2 \) if \( \rho \) is complex.

The height of any triple \((a, b, c) \in K^* \) is

\[
H_K(a, b, c) := \prod_{v \in V_K} \max(\|a\|_v, \|b\|_v, \|c\|_v).
\]

The radical of the triple \((a, b, c) \in K^* \) is

\[
\overline{\rad}_K(a, b, c) := \prod_{p \in I_K(a, b, c)} N_{K/Q}(p)^{\nu_p(p)},
\]

where \( p \) is a rational prime with \( p\mathbb{Z} = p \cap \mathbb{Z} \) and \( I_K(a, b, c) \) is the set of all prime ideals \( p \) of \( \mathcal{O}_K \) for which \( \|a\|_v, \|b\|_v, \|c\|_v \) are not equal.

The abc conjecture for algebraic number field \( K \) states that for any \( \varepsilon > 0 \) there exists a positive constant \( C_{K,\varepsilon} \) such that

\[
H_K(a, b, c) \leq C_{K,\varepsilon}(\overline{\rad}_K(a, b, c))^{1+\varepsilon},
\]

for all \( a, b, c \in K^* \) satisfying \( a + b + c = 0 \).
2.3. Cyclotomic polynomial. We now recall the cyclotomic polynomial and some of its properties.

**Definition 2.3.** [11] For any integer \( m \geq 1 \), the \( m^{th} \) cyclotomic polynomial is

\[
\Phi_m(X) = \prod_{\substack{i=1 \\text{gcd}(i,m)=1}}^{m} (X - \zeta_m^i),
\]

where \( \zeta_m \) is the primitive \( m^{th} \) root of unity.

It follows that the recursion formula for cyclotomic polynomial is

\[
X^m - 1 = \prod_{d|m} \Phi_d(X).
\] (2.1)

The following lemma characterizes the prime divisors of \( \Phi_m(\alpha, \beta) \), where

\[
\Phi_m(\alpha, \beta) = \prod_{\substack{i=1 \\text{gcd}(i,m)=1}}^{m} (\alpha - \zeta_m^i \beta).
\]

**Lemma 2.4.** (Stewart [14, Lemma 2]) Let \( (\alpha + \beta)^2 \) and \( \alpha \beta \) be coprime non-zero integers with \( \alpha/\beta \) not a root of unity. If \( m > 4 \) and \( m \neq 6, 12 \) then \( P(m/\gcd(3, m)) \) divides \( \Phi_m(\alpha, \beta) \) to at most the first power. All other prime factors of \( \Phi_m(\alpha, \beta) \) are congruent to \pm 1 (mod \( m \)). Further, if \( m > e^{452.4^{67}} \) then \( \Phi_m(\alpha, \beta) \) has at least one prime factor congruent to \pm 1 (mod \( m \)).

Here, \( P(k) \) denotes the greatest prime factor of \( k \) with the convention that \( P(0) = P(\pm 1) = 1 \). We note that, Yu. Bilu et al. [2] reduced the above lower bound \( e^{452.4^{67}} \) to 30. In the Lemma 2.4 Stewart [14] considered the cyclotomic polynomial

\[
\alpha^m - \beta^m = \prod_{d|m} \Phi_d(\alpha, \beta).
\] (2.2)

Since \( \beta \) is a unit in \( \mathbb{Q}(\sqrt{2}) \), we notice that the prime divisors of \( \Phi_m(\alpha, \beta) \) and the prime divisors of \( \Phi_m(\alpha/\beta) \) are the same. Thus by using above Lemma 2.4 the prime divisors of \( \Phi_m(\alpha/\beta) \) are congruent to \pm 1 (mod \( m \)).

**Lemma 2.5.** (Rout [12, Lemma 2.10]) For any real number \( b \) with \( |b| > 1 \), there exists \( C > 0 \) such that

\[
|\Phi_m(b)| \geq C|b|^\phi(m),
\]

where \( \phi(m) \) is Euler’s totient function.

2.4. Some lemmas. We state some of the important lemmas from [12], [16] and [4].

**Lemma 2.6.** (Rout [12, Lemma 2.12]) Suppose that \( B_n \) factored into \( X_n Y_n \), where \( X_n \) and \( Y_n \) are square-free and powerful part of \( B_n \) respectively. If \( p|X_n \), then

\[
B_{p^k}(z) \not\equiv 0 \pmod{p^2}.
\]
Lemma 2.7. (Rout [12] Lemma 2.9) For any $n \geq 2$, the $n^{th}$ balancing number satisfies the following inequality.

$$\alpha^{n-1} < B_n < \alpha^n.$$  

Lemma 2.8. (Rout [12]) If the abc conjecture for the number field $\mathbb{Q}(\sqrt{2})$ is true, then $Y_{nr} \ll \varepsilon B_{2\varepsilon}^{2\varepsilon}$.  

This result is part of the proof of [12, Theorem 3.1].

Lemma 2.9. (Wang and Ding [16, Lemma 2.4]) If $m < n$, then $\gcd(X'_m, X'_n) = 1$ or a power of $\sqrt{2}$.  

Lemma 2.10. (Ding [4, Lemma 2.5]) For any given positive integers $r$ and $n$, we have

$$\sum_{n \leq x} \frac{\phi(nr)}{nr} = c(r)x + O(\log x),$$

where $c(r) = \prod_p \left(1 - \frac{\gcd(p, r)}{p^2}\right) > 0$ and the implied constant depends on $r$.

3. Main Results

Let $r \geq 2$ be a given fixed integer and let $n > 1$ be any integer.  

Let us take,

$$X'_{nr} = \gcd(X_{nr}, \Phi_{nr}(\alpha/\beta)),$$

$$Y'_{nr} = \gcd(Y_{nr}, \Phi_{nr}(\alpha/\beta)).$$

The following theorem closely follows the proof of [12, Theorem 3.1]. For the purpose of completeness we give the proof.

Theorem 3.1. Assume that the abc conjecture for the number field $\mathbb{Q}(\sqrt{2})$ is true. Then for any $\varepsilon > 0$, $X'_{nr} \gg \varepsilon B_{2(\phi(n) - \varepsilon)}^{2(\phi(n) - \varepsilon)}$.  

Proof. By the recursion formula (2.1) we write,

$$\Phi_{nr}(\alpha/\beta) = \frac{B_{nr}\Phi_1(\alpha/\beta)}{\beta^{nr-1} \prod_{d|nr} \Phi_d(\alpha/\beta)}.$$  

It follows that

$$\Phi_{nr}(\alpha/\beta)|B_{nr}\alpha^{nr-1}.  

That is,

$$\Phi_{nr}(\alpha/\beta)|X_{nr}Y_{nr}\alpha^{nr-1}.  

Since $\alpha$ is a unit in $\mathbb{Q}(\sqrt{2})$, we have $\Phi_{nr}(\alpha/\beta) \not| \alpha$. Thus, $\Phi_{nr}(\alpha/\beta)|X_{nr}Y_{nr}$. As $\gcd(X_{nr}, Y_{nr}) = 1$, we obtain either $\Phi_{nr}(\alpha/\beta)|X_{nr}$ or $\Phi_{nr}(\alpha/\beta)|Y_{nr}$.  


We suppose that $\Phi_{nr}(\alpha/\beta)|X_{nr}$, it follows that $X'_{nr} = \gcd(X_{nr}, \Phi_{nr}(\alpha/\beta)) = \Phi_{nr}(\alpha/\beta)$ and $Y'_{nr} = \gcd(Y_{nr}, \Phi_{nr}(\alpha/\beta)) = 1$. Similar argument for $\Phi_{nr}(\alpha/\beta)|Y_{nr}$ implies that $X'_{nr} = 1$ and $Y'_{nr} = \Phi_{nr}(\alpha/\beta)$. For any of these cases, we finally get

$$X'_{nr}Y'_{nr} = \Phi_{nr}(\alpha/\beta). \tag{3.1}$$

By using Lemma 2.5, we write,

$$|X'_{nr}Y'_{nr}| = |\Phi_{nr}(\alpha/\beta)| \geq C|\alpha/\beta|^{\phi(nr)} \tag{3.2}$$

$$= C|\alpha^{2}\phi(nr)}. \tag{3.3}$$

Since $\{B_{nr}\}$ is a sequence of positive integers, by using Lemma 2.7, we get,

$$X'_{nr}Y'_{nr} \geq C\alpha^{2\phi(nr)} \tag{3.5}$$

$$\geq C(\alpha^{\phi(r)})^{2\phi(n)} \tag{3.6}$$

$$> CB^{2\phi(r)}. \tag{3.7}$$

Now by combining Lemma 2.8 with equation (3.7), we obtain

$$X'_{nr}B_{nr}^{2\epsilon} \gg \epsilon \cdot X'_{nr}Y_{nr} \geq X'_{nr}Y'_{nr} \gg \epsilon \cdot B_{\phi(r)}^{2\phi(n)}$$

$$X'_{nr} \gg \epsilon \cdot B_{\phi(r)}^{2\phi(n) - \epsilon}. \tag{3.10}$$

After simplification, we write,

$$X'_{nr} \gg \epsilon \cdot B_{\phi(r)}^{2\phi(n) - \epsilon}. \tag{3.10}$$

This completes the proof of Theorem 3.1. □

Let us take $T = \{n : X'_{nr} > nr\}$ and $T(x) = |T \cap [1, x]|$. The following lemma is an analogous result of [4, Lemma 2.6] for the balancing sequences.

**Lemma 3.2.** We have $T(x) \gg x$, where the implied constant depends only on $\alpha, r$.

**Proof.** Let $R = \left\{n : \phi(nr) > \frac{2\epsilon(r)}{3}nr\right\}$ and $R(x) = |R \cap [1, x]|$.

By equation (3.5), we have,

$$X'_{nr}Y'_{nr} \gg \alpha^{2\phi(nr)}. \tag{3.8}$$

By using Lemmas 2.7 and 2.8, we obtain,

$$Y'_{nr} \leq Y_{nr} \ll_{\epsilon} B_{nr}^{2\epsilon} < (\alpha^{nr})^{2\epsilon}. \tag{3.9}$$

On substituting equation (3.9) in (3.8), we get,

$$X'_{nr} \gg_{\epsilon} \alpha^{2(\phi(nr) - \epsilon nr)}. \tag{3.10}$$
Let $\varepsilon = \frac{c(r)}{3}$ in (3.10) and we get $X'_{nr} \gg_r \alpha^2 \left( \frac{\phi(nr)}{3} - \frac{c(r)nr}{3} \right)$. For any $n \in R$, we have $\phi(nr) > \frac{2c(r)nr}{3}$. Therefore,

$$X'_{nr} \gg_r \alpha^2 \left( \frac{\phi(nr)}{3} - \frac{c(r)nr}{3} \right) > \alpha \frac{2c(r)nr}{3} > nr.$$

Therefore there exists an integer $n_0$ depending only on $\alpha, r$ such that, if $n \geq n_0$ and $n \in R$, then $X'_{nr} > nr$. Hence we obtain,

$$T(x) = \sum_{n \leq x: X'_{nr} > nr} 1 \geq \sum_{n \leq x: n \geq n_0, n \in R} 1 = \sum_{n \geq n_0: \phi(nr) > 2c(r)nr/3} 1.$$

Since we note that,

$$\sum_{n \leq x: \phi(nr) \leq 2c(r)nr/3} \frac{\phi(nr)}{nr} \leq \sum_{n \leq x: \phi(nr) \leq 2c(r)nr/3} \frac{2c(r)}{3} \leq \frac{2c(r)}{3} x. \quad (3.11)$$

Hence by Lemma 2.10 and equation (3.11) we obtain,

$$T(x) \geq \sum_{n \leq x: n \geq n_0, \phi(nr) > 2c(r)nr/3} 1 \gg \sum_{n \leq x: \phi(nr) > 2c(r)nr/3} \frac{\phi(nr)}{nr} \geq \sum_{n \leq x: \phi(nr) > 2c(r)nr/3} \frac{\phi(nr)}{nr} = \sum_{n \leq x: \phi(nr) \leq 2c(r)nr/3} \frac{\phi(nr)}{nr} - \sum_{n \leq x: \phi(nr) \leq 2c(r)nr/3} \frac{\phi(nr)}{nr} \geq c(r)x + O(\log x) - \frac{2c(r)}{3} x \gg_{\alpha, r} x.$$

This completes the proof of Lemma 3.2. \hfill \Box

3.1. **Proof of Theorem 1.1** The main idea of this theorem is to count number of primes $p$ such that $p$ divides $X'_{nr} \leq x$. For any $n \in T$, it follows that there exists an odd prime $p_n$ such that $p_n | X'_{nr}$ and $p_n \nmid nr$. Since $X'_{nr}, X_{nr}$ and $p_n | X'_{nr}$, by using Lemma 2.6 we obtain

$$B_{p_n, - \left( \frac{a}{p_n} \right)} \neq 0 \pmod{p_n^2}.$$
We note that \( p_n \mid X'_{nr}, X'_{nr} \Phi_{nr}(\alpha/\beta) \) and \( p_n \nmid nr \). Therefore, by using Lemma 2.4 we obtain \( p_n \equiv \pm 1 \pmod{nr} \). Thus for any \( n \in T \), there is a prime \( p_n \) satisfying,
\[
B_{p_n-(\frac{\alpha}{\beta})} \not\equiv 0 \pmod{p_n^2},
\]
\[
p_n \equiv \pm 1 \pmod{nr}.
\]
By Lemma 2.9 we get \( p_n (n \in T) \) are distinct primes. Thus we find that,
\[
|\{ \text{primes } p \leq x : p \equiv \pm 1 \pmod{r}, B_{p-(\frac{\alpha}{\beta})} \not\equiv 0 \pmod{p^2} \}| \geq |\{ n : n \in T, X'_{nr} \leq x \}|.
\]
Since \( X'_{nr} \leq X_{nr} \leq B_{nr} < \alpha^{nr} \), we write
\[
|\{ n : n \in T, X'_{nr} \leq x \}| \geq |\{ n : n \in T, \alpha^{nr} \leq x \}|
\]
\[
= T\left( \frac{\log x}{r \log \alpha} \right).
\]
Hence by Lemma 3.2 we conclude that,
\[
|\{ \text{primes } p \leq x : p \equiv \pm 1 \pmod{r}, B_{p-(\frac{\alpha}{\beta})} \not\equiv 0 \pmod{p^2} \}| \geq T\left( \frac{\log x}{r \log \alpha} \right)
\]
\[
\gg_{\alpha,r} \log x.
\]

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