On Noncommutative Geometric Regularisation

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Abstract

Studies in string theory and in quantum gravity suggest the existence of a finite lower bound to the possible resolution of lengths which, quantum theoretically, takes the form of a minimal uncertainty in positions $\Delta x_0$. A finite minimal uncertainty in momenta $\Delta p_0$ has been motivated from the absence of plane waves on generic curved spaces. Both effects can be described as small noncommutative geometric features of space-time. In a path integral approach to the formulation of field theories on noncommutative geometries, we can now generally prove IR regularisation for the case of noncommutative geometries which imply minimal uncertainties $\Delta p_0$ in momenta.

1. Introduction

As has been known for long, for the resolution of small distances high energetic test particles are needed which through their gravitational effect will disturb eventually significantly the spacetime structure which was tried to be resolved. The problem has been approached from several directions and studies in string theory and quantum gravity suggest that, quantum theoretically, a lower bound to the resolution of distances could take the form of a finite minimal position uncertainty $\Delta x_0$ of the order of the Planck length of $\approx 10^{-35} m$, see e.g. [1]-[4]. On the other hand, on large scales, there is no notion of plane waves or momentum eigenvectors on generic curved spaces. It has therefore been suggested that quantum theoretically there could then exist lower bounds $\Delta p_0$ to the possible determination of momentum [7, 9].

Independent of the suggested mechanisms for the origins of minimal uncertainties (or whether they are intended as a formal regularisation only) both types of effects, i.e. a $\Delta x_0$ or a $\Delta p_0$ can be described as small noncommutative geometric corrections to space-time and/or energy-momentum space [3]-[12].

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Intuitively, the presence of finite minimal uncertainties $\Delta x_0, \Delta p_0$ should have UV and IR regularising effect in field theory. The example of euclidean $\phi^4$-theory on a restricted class of such noncommutative geometries has been studied in detail and both UV and IR regularisation have been shown for this case [7]-[10].

Our aim here is to prove the IR regularity of euclidean propagators $1/(p^2 + m^2c^2)$ for all noncommutative geometries with a minimal uncertainty in momentum $\Delta p_0$, both for $m > 0$ and for $m = 0$.

2. Noncommutative geometries with minimal uncertainties

We consider the possibility of small ‘noncommutative geometric’ corrections to the canonical commutation relations in the associative Heisenberg algebra $\mathcal{A}$ generated by the $x_i, p_j$, see [7]-[10]:

$$[x_i, p_j] = i\hbar(\delta_{ij} + \alpha_{ijkl}x_kx_l + \beta_{ijkl}p_kp_l + ...)$$

and also

$$[x_i, x_j] \neq 0, \quad [p_i, p_j] \neq 0$$

with the involution $x^*_i = x_i, p^*_i = p_i$.

A priori we formulate field theories on generic noncommutative background ‘geometries’ $\mathcal{A}$ which may or may not have certain symmetries, similar to the case of curved background geometries. While nontrivial examples of non Lorentz symmetric non-commutative background geometries have been studied in [7]-[10], Lorentz symmetric examples of suitable noncommutative background geometries were found in [11].

The correction terms necessarily imply new physical features, since unitary transformations are generally commutation relations preserving. Here, for appropriate small $\alpha, \beta$ one obtains ordinary quantum mechanical behaviour at medium scales while the presence of small $\alpha$ and $\beta$ imply modified IR and UV behaviour respectively:

The uncertainty relations, holding in all $\ast$-representations of the commutation relations on some dense domain $D \subset H$ in a Hilbert space $H$, are of the form $\Delta A \Delta B \geq 1/2|\langle[A, B]\rangle|$ so that e.g. $[x_i, x_j] \neq 0$, yields $\Delta x_i \Delta x_j \geq 0$. The noncommutativity implies that the $x_i$ (as well as the $p_i$) can no longer be simultaneously diagonalised. Because of Eqs.1 and the corresponding uncertainty relations there can appear the even more drastic effect that the $x_i$ (as well as the $p_j$) may also not be diagonalisable separately.

Already in one dimension the uncertainty relation (assuming small positive $\alpha, \beta$ with $\alpha\beta < 1/\hbar^2$ and neglecting higher order corrections):

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left(1 + \alpha(\Delta x)^2 + \alpha\langle x \rangle^2 + \beta(\Delta p)^2 + \beta\langle p \rangle^2 \right)$$

implies nonzero minimal uncertainties in $x$- as well as in $p$- measurements: $\Delta x_0 = (1/\beta\hbar^2 - \alpha)^{-1/2}, \quad \Delta p_0 = (1/\alpha\hbar^2 - \beta)^{-1/2}$. For $\alpha = 0$ and a small $\beta$ we cover the
example of an ultraviolet modified uncertainty relation that has been discussed in string theory and quantum gravity, for a review see [4]. For all physical domains $D$, i.e. for all $\ast$-representations of the commutation relations, there are now no physical states in the minimal uncertainty ‘gap’

$$\exists |\psi\rangle \in D : 0 \leq (\Delta x)_{|\psi\rangle} < \Delta x_0$$ (4)

$$\exists |\psi\rangle \in D : 0 \leq (\Delta p)_{|\psi\rangle} < \Delta p_0$$ (5)

where $|\psi\rangle$ generally stands for a normalised element of $D$. Thus, unlike on ordinary geometry, there now do not exist sequences $\{|\psi_n\rangle\}$ of physical states which would approximate point localisations in position or momentum space, i.e. for which the uncertainty would decrease to zero, e.g. $\exists |\psi_n\rangle \in D : \lim_{n \to \infty} (\Delta x)_{|\psi_n\rangle} = 0$.

Technically, the new infrared and ultraviolet behaviour has important consequences for the representation theory, in that e.g. a finite minimal uncertainty $\Delta x_0$ in positions implies that the commutation relations do no longer find a spectral representation of $x$, so that one has to resort to other Hilbert space representations.

The interplay between the functional analysis of the position and the momentum operators has first been studied in [5, 6]. In fact we are giving up (essential) self-adjointness of the $x$ and $p$ operators, to retain only their symmetry. While giving up essential self-adjointness is necessary for the description of the new short distance behaviour, the symmetry is sufficient to guarantee that all physical expectation values are real and also that uncertainties can be calculated applying the usual definition of the standard deviation, e.g. $\Delta x = \langle \psi | (x - \langle \psi | x | \psi \rangle)^2 | \psi \rangle^{1/2}$. Nevertheless, this is a nontrivial step which goes beyond the conventional quantum mechanical treatment, and it also goes beyond Connes’ ‘dictionary’ [14] of how to treat ‘real variables’ on noncommutative geometries.

The key observation is that although self-adjoint extensions and (discrete) diagonalisations of $x$ or $p$ exist in $H$, under the circumstances described, these diagonalisations are not in any common domain, i.e. not in any physical domain, of $x$ and $p$ [5, 6]. Instead there is now the finite uncertainty ‘gap’ separating the physical states from formal $x$ or $p$ eigenstates. For the details and proofs see [5, 6, 10, 11, 12].

Concerning the infrared we remark that due to the correction terms the momenta $p_i$ no longer generate translations on flat space. Under certain conditions, the $p_i$ do generate translations of normal coordinate frames on curved spaces, see [5, 6], in which case the relation between the absence of plane waves (i.e. of $p$-eigenstates) and the presence of a minimal uncertainty in momentum can be explicitly investigated.

The physical states of then maximal localisation have in the meanwhile been extensively studied, first in the special case $\alpha = 0, \beta > 0$, see [11] and recently also in the general (though one-dimensional) case $\alpha, \beta > 0$, see [12]. Explicitly, the physical states $|\phi^{mx}_\xi\rangle$, $|\phi^{mp}_\xi\rangle$ which realise the maximal localisation in positions or momenta obey

$$\Delta x_{|\phi^{mx}_\xi\rangle} = \Delta x_0$$

$$\langle \phi^{mx}_\xi | x | \phi^{mx}_\xi \rangle = \xi$$

$$\langle \phi^{mx}_\xi | p | \phi^{mx}_\xi \rangle = 0$$ (6)
and similarly for $|\phi^{mlp}_\pi\rangle$. E.g. the projection $\langle\phi^{mlp}_\xi|\psi\rangle$ is then the probability amplitude for finding the particle maximally localised around $\xi$. For $\alpha, \beta \to 0$ one recovers the position and the momentum eigenvectors. $n$-dimensional studies are in progress, we here only state one key result, the mutual projection of maximal localisation states:

$$\langle\psi^{mlx}_\xi'|\psi^{mlx}_\xi\rangle = \frac{1}{\pi} \left( \frac{\xi - \xi'}{2\hbar \sqrt{\beta}} - \left( \frac{\xi - \xi'}{2\hbar \sqrt{\beta}} \right)^3 \right)^{-1} \sin \left( \frac{\xi - \xi'}{2\hbar \sqrt{\beta}} \pi \right)$$ (7)

It is the generalisation of the Dirac $\delta$-function which on ordinary geometry would be obtained from projecting maximal localisation states, i.e. then from projecting position eigenstates onto another: $\langle x|x'\rangle = \delta(x - x')$. The nonmultiplicativity of $\delta$-distributions is related to the appearance of ultraviolet divergencies, whereas the behaviour of Eq.7 (note that the singularities of its first factor are cancelled by zeros of the sinus) suggests UV regularity in field theory.

3. Path Integration

The ansatz [6]-[10] for the formulation of field theories on such noncommutative geometries is explained most easily in the simple example of charged euclidean $\phi^4$-theory: The partition function

$$Z[J] := N \int D\phi \ e^{\int d^4x \ \phi^*(\partial_i \partial_i - \mu^2)\phi - \frac{\lambda}{4!} (\phi\phi^*)^2 + \phi J + J^* \phi}$$ (8)

we write in the form

$$Z[J] = N \int D\phi \ e^{-\text{tr} \left( \frac{\partial_i}{\hbar} (p^2 + m^2 c^2) \langle \phi | \phi \rangle - \frac{\lambda}{4!} \langle \phi \phi^* \phi \phi^* \rangle | \phi \rangle + | \langle \phi | J \rangle + | \langle J | \phi \rangle \right)}$$ (9)

where, to make the units transparent, we introduced an arbitrary positive length to render the fields unitless ($l$ could trivially be reabsorbed in the fields).

Eq.8 is recovered from Eq.9 by assuming the ordinary relations $[x_i, p_j] = i\hbar \delta_{ij}$ in $\mathcal{A}$ and by choosing the spectral representation of the $x_i$. We then have as usual $\phi(x) := \langle x | \phi \rangle$ with the scalar product $\langle \phi | \psi \rangle = \int d^4x \phi^*(x) \psi(x)$, i.e. the trace $\text{tr}(q) = \int d^4x \langle x | q | x \rangle$, and the operators acting as $x_i \phi(x) = x_i \phi(x)$, $p_i \phi(x) = -i\hbar \partial_{x_i} \phi(x)$.

The pointwise multiplication

$$\langle \phi_1 \star \phi_2 \rangle(x) = \phi_1(x) \phi_2(x), \quad \text{i.e.} \quad \langle x | \phi_1 \star \phi_2 \rangle = \langle x | \phi_1 \rangle \langle x | \phi_2 \rangle$$ (10)

which expresses point interaction, is (and can also on noncommutative geometries be kept) commutative for bosons. Since fields are in a representation of $\mathcal{A}$, similar to quantum mechanical states, we formally extended Dirac’s bra-ket notation for states to fields. In Eq.8 this yields a convenient notation for the functional analytic structure of the action functional, but of course, the quantum mechanical interpretation does not simply extend, see e.g. [15]. The space $D$ of fields that is formally to be summed
over can be taken to be the dense domain $S_\infty$ in the Hilbert space $H$ of square integrable fields.

Generally, the unitary transformations that map from one Hilbert basis to another have trivial determinant, so that no anomalies are introduced into the field theory and changes of basis can be performed arbitrarily, in the action functional, in the Feynman rules or in the end results of the calculation of $n$-point functions.

Let us now assume that the commutation relations, i.e. $\mathcal{A}$, are represented on a dense domain $D$ spanned by a Hilbert basis of vectors $\{|n\rangle\}$, where $n$ may be discrete, as e.g. in the case of a Bargmann Fock representation or it may be continuous, as in the case e.g. of position or momentum representations, or it may generally be a mixture of both. Keeping this in mind we use the notation for $n$-discrete. The identity operator on $H$ can then be written $1 = \sum_n |n\rangle \langle n|$ and fields and operators are expanded as

$$\phi_n = \langle n|\phi\rangle \quad \text{and e.g.} \quad (p^2 + m^2c^2)_{nn'} = \langle n|p^2 + m^2c^2|n'\rangle$$ (11)

The pointwise multiplication, which we will need for the local interaction then reads

$$* = \sum_{n_i} L_{n_1n_2n_3} |n_1\rangle \otimes |n_2\rangle \otimes |n_3\rangle$$ (12)

In this Hilbert basis the partition function reads, summing over repeated indices:

$$Z[J] = N \int_D D\phi e^{-\frac{l^2}{\hbar^2} \phi_n^* (p^2 + m^2c^2)\phi_n - \frac{\lambda l^4}{4!} L_{n_1n_2n_3}^* L_{n_1n_4n_5} \phi_{n_2}^* \phi_{n_3}^* \phi_{n_4} \phi_{n_5} + J_n^* \phi_n}$$ (13)

Pulling the interaction term in front of the path integral, completing the squares, and carrying out the gaussian integrals yields

$$Z[J] = Ne^{-\frac{\lambda l^4}{4!} L_{n_1n_2n_3}^* L_{n_1n_4n_5} \phi_{n_2}^* \phi_{n_3}^* \phi_{n_4} \phi_{n_5} - \frac{\lambda l^4}{4!} J_n^* (p^2 + m^2c^2)_{nn'} J_{n'}}$$ (14)

The inversion of $(p^2 + m^2c^2)$ is nontrivial and involves a self-adjoint extension in which it can be diagonalised and inverted. This will be investigated below. We obtain the Feynman rules:

$$\Delta_{n_1n_2} = \left(\frac{-\hbar^2}{l^2(p^2 + m^2c^2)}\right)_{n_1n_2} \quad \text{and} \quad \Gamma_{n_1n_2n_3n_4} = -\frac{\lambda l^4}{4!} L_{n_1n_2n_3}^* L_{n_4n_5n_6}$$ (15)

Note that since each vertex attaches to four propagators, $l$ drops out of the Feynman rules, as it should be.

Let us recall that the usual formulation of partition functions, such as e.g. Eq.8, implies that $p^2$ can be represented as the Laplacian on a spectral representation of the $x_i$, so that $p_i$ is represented as $-i\hbar \partial_i$, i.e. it is implied that $[x_i, p_j] = i\hbar \delta_{ij}$. It is crucial that in our formulation of partition functions in abstract form, such as in Eq.9, the commutation relations of the underlying algebra $\mathcal{A}$ are not implicitly fixed and can be generalised, e.g. to the form of Eqs.10,11.
Representing \( A \) on some dense \( D \) in a Hilbert space \( H \) with an e.g. discrete Hilbert basis \( \{ |n\rangle \} \) (the Hilbert space is separable), one straightforwardly obtains the Feynman rules through Eqs. 13-13. Thus, in particular, the formalism allows to explicitly check noncommutative geometries on UV and IR regularisation.

4. IR Regularisation

On ordinary geometry a finite mass term in the propagator \( (p^2 + m^2 c^2)^{-1} \) ensures that, as an operator, it is bounded (we need not specify a Hilbert basis, as we did e.g. in Eq[13]). However, for \( m = 0 \) the operator \( 1/p^2 \) is unbounded, causing infrared divergencies. Indeed, on geometries that imply a minimal uncertainty \( \Delta p_0 \) even the massless propagator \( 1/p^2 \) is as well behaved as if it contained a mass term.

To be precise, we intend to show that for all noncommutative geometric algebras \( A \) of the type of Eqs.18-20 which imply a minimal uncertainty \( \Delta p_0 \) there holds for \( m > 0 \) as well as for \( m = 0 \):

A The operator \( (p^2 + m^2 c^2) := \sum_i p_i p_i + m^2 c^2 \) has exactly one self-adjoint extension \( (p^2 + m^2 c^2)_F \) which is contained in its form domain.

B The operator \( (p^2 + m^2 c^2)_F \) has a unique inverse (the free propagator) which is self-adjoint and defined on the entire Hilbert space \( H \).

C The propagator \( (p^2 + m^2 c^2)^{-1}_F \) is infrared regular, i.e. it is a bounded operator (implying also that its matrix elements are bounded) with bound \( ||(p^2 + m^2 c^2)^{-1}_F|| \leq (n(\Delta p_0)^2 + m^2 c^2)^{-1} \).

D Also propagators that are the inverse to arbitrary other self-adjoint extensions of \( (p^2 + m^2 c^2) \) (for finite deficiency indices) are IR-regular, i.e. they are bounded self-adjoint operators on \( H \).

To see this, let \( A \) be represented on a dense domain \( D \subset H \) in a Hilbert space \( H \). By assumption the momenta \( p_i \) exhibit a minimal uncertainty \( \Delta p_0 > 0 \), i.e. for all states, i.e for all normalised vectors \( |\phi\rangle \in D \) holds \( \Delta p_i|\phi\rangle \geq \Delta p_0 \), so that

\[
\langle \phi|p_i^2|\phi\rangle = \langle \phi|p_i|\phi\rangle^2 + (\Delta p_i|\phi\rangle)^2 \geq (\Delta p_0)^2
\]

and by linearity, for vectors of arbitrary norm:

\[
\langle \phi|p^2|\phi\rangle \geq n||\phi||^2(\Delta p_0)^2
\]

Thus, the operator \( (p^2 + m^2 c^2) \) is a densely defined symmetric positive definite operator (now even for \( m = 0 \)), and therefore has, by a theorem of Friedrich, see e.g. [13-22], a unique self-adjoint extension within its form domain. It has the same lower bound as the original operator. Explicitly, the Friedrich extension \( (p^2 + m^2 c^2)_F \) of \( (p^2 + m^2 c^2) \) has the domain \( D_F = D_{(p^2 + m^2 c^2)^*} \cap H' \), which is the intersection of the domain \( D_{(p^2 + m^2 c^2)^*} \) of the adjoint \( (p^2 + m^2 c^2)^* \) with the Hilbert space \( H' \) obtained by completion of \( D \) with respect to the norm \( ||\phi||' := \langle \phi|p^2 + m^2 c^2|\phi\rangle^{1/2} \) induced by the quadratic form which is defined through the positive definite operator \( (p^2 + m^2 c^2) \).
The range of \((p^2 + m^2c^2)_F\) is \(R((p^2 + m^2c^2)_F) = H\), the inverse \((p^2 + m^2c^2)_F^{-1}\) exists, has the domain \(D_{(p^2+m^2c^2)}^{-1} = \overline{R((p^2 + m^2c^2)_F)} = H\), and is a self-adjoint bounded operator:

\[
||((p^2 + m^2c^2)_F^{-1})|| \leq \frac{1}{n(\Delta p_0)^2 + m^2c^2}\]

(18)

For a constructive proof of the properties of the Friedrich extension see e.g. [2]. To see the invertibility note that, since \((p^2 + m^2c^2)_F\) has the same bound as \((p^2 + m^2c^2)_F\), i.e. 
\[
\forall \phi \in D_F : \langle \phi | (p^2 + m^2c^2)_F | \phi \rangle \geq m^2c^2 + n||\phi||^2(\Delta p_0)^2, \text{ its kernel is empty:} (p^2 + m^2c^2)_F | \phi \rangle = 0 \Rightarrow 0 = \langle \phi | (p^2 + m^2c^2)_F | \phi \rangle \geq m^2c^2 + n||\phi||^2(\Delta p_0)^2.
\]

Due to the Cauchy Schwarz inequality also the matrix elements of \((p^2 + m^2c^2)_F^{-1}\) are bounded:

\[
\forall \phi, |\psi\rangle \in H : \quad ||\langle \phi | (p^2 + m^2c^2)_F^{-1} | \psi \rangle|| \leq \frac{||\phi|| \cdot ||\psi||}{n(\Delta p_0)^2 + m^2c^2} (19)
\]

So far we have shown A-C, i.e. that there exists a canonical inverse \((p^2 + m^2c^2)_F^{-1}\) and that, as a propagator, it does not lead to infrared problems, since it is bounded, also in the case \(m = 0\).

To see D we consider the bi-adjoint \((p^2 + m^2c^2)^{**}\), which is symmetric and closed, as is every bi-adjoint of a densely defined symmetric operator. Due to the existence of one self-adjoint extension, \((p^2 + m^2c^2)_F\), the deficiency indices \((r, r)\) are equal.

We recall that on ordinary geometry the deficiency indices are \((0, 0)\), implying that \((p^2 + m^2c^2)_F\) is the only self-adjoint extension. The deficiency indices can now be nonzero, examples of which are known, see [3] [4] [5] [6]. There then exists a whole family of further self-adjoint extensions \((p^2 + m^2c^2)_f\) (e.g. labeled by \(f\)), and a corresponding family of propagators \((p^2 + m^2c^2)_f^{-1}\) (for invertible \((p^2 + m^2c^2)_f\)) which, in explicit representations, differ by their boundary conditions.

A priori we do not want to exclude these nonstandard propagators (although we exclude as unphysical the case of infinite deficiency indices in which case the propagator would require an infinite set of boundary conditions).

Indeed, also the nonstandard propagators are IR regular. To see this, we note first that also \((p^2 + m^2c^2)^{**}\) is semibound from below by \(n(\Delta p_0)^2 + m^2c^2\) since \((p^2 + m^2c^2)_F\) which is an extension of \((p^2 + m^2c^2)^{**}\) has this property. As seen by the v. Neumann method, the unitary extension of the isometric Cayley transform only involves a finite dimensional mapping of the deficiency spaces and thus all self-adjoint extensions of a closed symmetric operator have the same essential spectrum, see e.g. Thm.8.18 in [18]. Indeed, since \((p^2 + m^2c^2)^{**}\) is closed, symmetric and bounded from below the now interesting part of the spectrum \(\sigma((p^2 + m^2c^2)_f) \cap (-\infty, n(\Delta p_0)^2 + m^2c^2)\) of its self-adjoint extensions consists of isolated eigenvalues only, of total multiplicity \(\leq r\), see e.g. Cor.2 of Thm8.18 in [18]. Thus, for all invertible self-adjoint extensions there exist \(\epsilon > 0\) so that the spectrum is empty in the finite intervall \([-\epsilon, \epsilon]\) i.e. in the neighbourhood of zero. We can therefore conclude the boundedness of the spectra of the inverses to arbitrary invertible self-adjoint extensions of \((p^2 + m^2c^2)\). To be precise,
for all invertible self-adjoint extensions zero is a regular point $0 \in \rho((p^2 + m^2c^2)f)$ since it is not in the spectrum. For self-adjoint operators $A$ there generally holds, see e.g. [18] (Thm5.24), [19] (Thms129.1,2) or [20]-[22]:

$$z \in \rho(A) \iff \exists c > 0, \forall v \in D(A) : \| (z - A)v \| \geq c\|v\| \text{ (and } \Rightarrow \|1/(z - A)\| \leq c^{-1})$$

$$\iff R(z - A) = H$$

(20)

Here, we therefore have:

$$\exists \epsilon > 0, \forall v \in D_f : \| (p^2 + m^2c^2)f.v \| \geq \epsilon\|v\| \text{ and } R((p^2 + m^2c^2)f) = H$$

(21)

Thus, the corresponding propagators are bounded $\| (p^2 + m^2c^2)f^{-1} \| \leq 1/\epsilon$ and are defined on the entire Hilbert space $D_{(p^2 + m^2c^2)}^{-1} = R((p^2 + m^2c^2)f) = H$. Also, the propagators are self-adjoint, as the inverses of self-adjoint operators generally are.

5. Outlook

Concerning the ultraviolet, the same arguments prove of course that e.g. a background Coulomb potential $A_\mu(x) := (q/\sqrt{\sum_i x_i^2})F, 0, 0, 0$ (the square root is well defined since $\sum_i x_i^2$ is positive definite) is bounded in the presence of a minimal uncertainty $\Delta x_0$ in positions. Given a representation of the algebra $A$, the propagator $\Delta = (\sum_i (p_i + eA_i(x))^2 + m^2c^2)_F^{-1}$ can be calculated straightforwardly. Investigations into the ‘local’ gauge principle on geometries with minimal uncertainties should eventually allow to study also dynamical gauge fields and to check for UV regularisation. While in the simpler $\phi^4$-theory detailed studies have been carried out for certain classes of geometries with minimal uncertainties $\Delta x_0$ and $\Delta p_0$, [7]-[10], let us here only remark that for UV-regularisation the structure of the pointwise multiplication $*$ that describes local interaction is crucial. Due to the absence of a position representation, $*$ is nonunique in the case of $\Delta x_0 > 0$. Crucially, an interaction is now observationally local if any formal nonlocality of $*$ is not larger than the scale of the nonlocality $\Delta x_0$ inherent in the underlying space. Thus, intuitively, UV-regularity and strict observational locality become more compatible than on ordinary geometry. There exist ‘quasi-position representations’ [11], built on maximal localisation states, which can be used to establish the locality and causality properties of pointwise multiplications. A detailed study on the special case $\Delta x_0 > 0, \Delta p_0 = 0$ (which allows a convenient momentum space representation) is in preparation [13].

We remark that an alternative approach with a similar motivation, but based on the canonical formulation of field theory is given in [14], see also [17].
References

[1] P. K. Townsend, Phys. Rev. D, Vol. 15, No 10: 2795-2801 (1976)
[2] D. Amati, M. Ciafaloni, G. Veneziano, Phys. Lett. B 216, 41 (1989)
[3] M. Maggiore, Phys. Lett. B 319: 83 (1993)
[4] L. J. Garay, Int. J. Mod. Phys. A10: 145 (1995)
[5] A. Kempf, Proc. XXII DGM Conf. Sept.93 Ixtapa (Mexico), Adv. Appl. Cliff. Alg (Proc. Suppl.) (S1):87 (1994)
[6] A. Kempf, J. Math. Phys. 35 (9): 4483 (1994)
[7] A. Kempf, Preprint DAMTP/94-33, hep-th/9405067
[8] A. Kempf, Czech.J.Phys. (Proc. Suppl.) 44, 11-12: 1041 (1994)
[9] A. Kempf, Preprint DAMTP/96-06, to appear in Proc. Intl. Workshop Lie Theory & Applic. in Physics, Clausthal, Aug 95, World Scientific (1996)
[10] A. Kempf, DAMTP/96-22, hep-th/9602085
[11] A. Kempf, G. Mangano, R.B. Mann, Phys. Rev. D52: 1108-1118 (1995)
[12] A. Kempf, H. Hinrichsen, hep-th/9510144, in press in J. Math. Phys. (1996)
[13] A. Kempf, G. Mangano, in preparation
[14] A. Connes, Noncommutative geometry, AP (1994)
[15] R.P. Feynman, Dirac Memorial Lecture, CUP (1987)
[16] S. Doplicher, K. Fredenhagen, J. E. Roberts, Phys. Lett. B331: 39 (1994)
[17] H. Grosse, C. Klimcik, P. Presnadjer, hep-th/9510177, to appear in Proc. Intl. Workshop on Lie Theory & Applic. in Physics, Clausthal, Aug 95, W.Sc. (1996)
[18] J. Weidmann, Lin. Operatoren in Hilberträumen, (in German), Teubner (1976)
[19] V.I. Smirnov, A Course in Higher Math. V, Pergamon (1964)
[20] N.I. Akhiezer, I.M. Glazman, Theor. Lin. Oper. in Hilbert spaces, Ungar (1963)
[21] K. Maurin, Methods of Hilbert Spaces, Polish Scientific Publishers (1967)
[22] M. Reed, B. Simon, Fourier Analysis, Self-Adjointness, AP (1975)