A straightforward local-search optimization algorithm on the symmetric group

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April 29, 2009

Abstract

Given a real objective function defined over the symmetric group, a direct local-search algorithm is proposed, and its complexity is estimated. In particular for an $n$-dimensional unit vector we are interested in the permutation isometry that acts on this vector by mapping it into a cone of a given angle.

1 Introduction

In several areas of Game Theory and Oil Engineering there appears an optimization problem defined over the group of permutations on an index set. The declination curves [1] on wells, for instance, are important in order to develop exploitation strategies, among them Steam Assisted Gravitational Drainage (SAGD) processes [3]. The well known theories developed by Von Neumann on Game Theory pose also optimization problems whose domain is the symmetric group. A rather complete survey of this kind of optimization procedures in Social Sciences is given in [2].

A set of independent measurements of a certain parameter is realized as a vector in $\mathbb{R}^n$ and when it is normalized (in the statistical sense: it is translated by its average and elongated by the reciprocal of its variance) an unit vector with zero mean and unit variance is obtained. Then the problem consists in finding a permutation of its entries making a given angle with the original normalized vector.

Here we sketch a local-search procedure, with cubic complexity with respect to the dimension of the original vector.

2 Uniform distribution

Let $e_i = (\delta_{ij})_{j=1}^n$ be the $i$-th vector in the canonical basis of $\mathbb{R}^n$, where $\delta_{ij}$ is Kroenecker’s delta. For any non-zero vector $x = (x_1, \ldots, x_n) = \sum_{j=1}^n x_j e_j$, its mean is $E(x) = \frac{1}{n} \sum_{j=1}^n x_j$, its variance is $\text{Var}(x) = \frac{1}{n} \sum_{j=1}^n (x_j - E(x))^2$ and its normalization is

$$z_x = \frac{1}{\sqrt{\text{Var}(x)}}(x - E(x)) 1^n$$

where $1^n$ is the vector in $\mathbb{R}^n$ with constant components 1. Clearly, $E(z_x) = 0$, $\text{Var}(z_x) = 1$ and $z_x$ is an unit vector, with respect to the Euclidean norm. In particular, for the identity vector

$$x = (1, 2, \ldots, n) = \sum_{j=1}^n j e_j.$$  

its mean and variance are

$$E(x) = \frac{1}{n} \sum_{j=1}^n j = \frac{n+1}{2}$$  

$$\text{Var}(x) = \frac{1}{n} \sum_{j=1}^n (j - E(x))^2 = \frac{(n-1)(n+1)}{12}.$$
Let us consider the map identity $R_{\pi}$ for all $\pi$, the normalized vector and its corresponding image by Schwartz's inequality, the inner product between $z = \sum_{j=1}^{n} z_j e_j$, where, for each $j \leq n$:

$$z_j = \frac{1}{\sqrt{n}} \frac{j - E(x)}{\sqrt{Var(x)}} = \frac{3}{(n-1)n(n+1)}(2j-(n+1)).$$

(5)

Let $u = \sqrt{(n-1)n(n+1)} z = \sum_{j=1}^{n} (2j-(n+1)) e_j$.

Then

$$\|u\|_2 = \sqrt{(n-1)n(n+1)}.$$  

(6)

Let us write $\nu_n = \frac{(n-1)n(n+1)}{3}$. From eq. (10), it can be checked that the image of $\Phi$ is included in the discrete set

$$R_n = \left\{ \frac{k}{\nu_n} | k \in \mathbb{Z}, -\nu_n \leq k \leq \nu_n \right\} \subset [-1, 1]$$

(11)

Namely, $\Phi(S_n) \subseteq R_n$, although the inclusion may be proper, for some values of $n$. For instance, by eq. (10), $\forall k \in \mathbb{Z}$,

$$\frac{k}{\nu_n} \in \Phi(S_n) \iff \exists \pi \in S_n : k = \sum_{j=1}^{n} (2j-(n+1))(2\pi(j)-(n+1)).$$

Whenever $n$ is an even integer, each summand $(2j-(n+1))(2\pi(j)-(n+1))$ is odd (as the product of two odd integers), thus their addition will be even. Hence, for $k$ odd, we have $\frac{k}{\nu_n} \notin \Phi(S_n)$.

Besides, we can see that the image $\Phi(S_n)$ is symmetric with respect to $0$, namely, $\forall \pi \in S_n$

$$k = \sum_{j=1}^{n} (2j-(n+1))(2\pi(j)-(n+1)) \implies -k = \sum_{j=1}^{n} (2j-(n+1))(2\pi_{\pi^{-1}}(j)-(n+1)).$$

(12)

Let us consider the following

**Problem 3.1** Given a non-zero $x \in \mathbb{R}^n$ and $r \in [-1, 1]$ find

$$\Pi(x, r) = \text{Arg Min}_{\pi \in S_n} |r - \langle x, \pi(x) \rangle|.$$  

From remark (13), it is enough to assume $r \in [0, 1]$ when $x$ is the identity vector.

Problem 3.1 can be solved by a direct local-search algorithm. Let us consider the elements of $S_n$ as
nodes and let us consider edges of two kinds:

\[(\pi, \rho) \in E_T \iff \exists i, j \in \{1, \ldots, n\} : \rho = (i, j)\pi\]

\[(\pi, \rho) \in E_C \iff \exists i \in \{1, \ldots, n\} : \rho = (i, i')\pi \text{ with } i' = i + 1 \text{ if } i < n \text{ and } n' = 1\]

Let \(S_{nT} = (S_n, E_T)\) and \(S_{nC} = (S_n, E_C)\). The graph \(S_{nT}\) contains as edges all pairs of permutations that differ by an arbitrary transposition, and the graph \(S_{nC}\) contains as edges all pairs of permutations that differ by a transposition of two consecutive indexes. \(S_{nT}\) is a regular graph of \(n!\) nodes of degree \(\binom{n}{2} = \frac{1}{2}n(n-1)\). If we consider any edge as an edge of unit length, then the diameter of \(S_{nT}\) is \(d_{nT} = n - 1 = O(n)\). Instead, \(S_{nC}\) is a regular graph of \(n!\) nodes of degree \(n\) and its diameter is

\[d_{nC} = \frac{1}{4}(n - (n \mod 2))(n + (n \mod 2)) = O(n^2)\]

### 4 Solution procedure

Let \(F : S_n \to \mathbb{R}^+\) be a positive-real valued map defined over the symmetric group. Let \(S_n = (S_n, E)\) be any of \(S_{nT}\) or \(S_{nC}\). A local-search algorithm to minimize \(F\) is sketched at table 1.

In order to solve problem \(\text{Arg Min} \ F\) for any given \(x \in \mathbb{R}^n\), \(r \in [0, 1]\), the objective map \(F_{x,r} : \pi \mapsto |r - \Phi_x(\pi)|\) is considered, where \(\Phi_x\) is given by eq. \(\|x\|\) which in turn involves the normalization \(z_x\) of vector \(x\), given by eq. \(\|x\|\) for the identity vector.

If \(S_n = S_{nT}\), then the graph is regular of degree \(n(n - 1)/2\) and diameter \(n - 1\). Thus step 2.(b) at pseudocode in table 1 entails \(n(n - 1)/2\) evaluations of the objective map, and whole cycle 2, is repeated at most \(n - 1\) times. Thus, in this case the worst case entails \(n(n - 1)^2/2\) evaluations of the objective map.

Instead, if \(S_n = S_{nC}\), then the graph is regular of degree \(n\) and diameter \(d_{nC}\). Thus step 2.(b) at pseudocode in table 1 entails \(n\) evaluations of the objective map, and whole cycle 2, is repeated at most \(d_{nC}\) times. Thus, in this case the worst case entails \(n d_{nC}\) evaluations of the objective map. Since

\[\frac{n(n - 1)^2/2}{n(n - (n \mod 2))(n + (n \mod 2))/4} = 2\left(\frac{\frac{(n - 1)^2}{n(n - (n \mod 2))(n + (n \mod 2))}}{n - \infty}\right)^2\]

### Table 1: Local-search algorithm on the symmetric group.

| Input. | The dimension \(n \in \mathbb{N}\) and the objective map \(F : S_n \to \mathbb{R}^+\). |
|--------|---------------------------------------------------------------------|
| Output. | The permutation \(\pi \in S_n\) such that \(\pi = \text{Arg Min}_{\rho \in S_n} F(\rho)\). |
| 1. | Initialize \(\pi_c := \pi_1\) (the identity permutation); \(v_c := F(\pi_c)\). |
| 2. | repeat |
| (a) | \(\pi_d := \pi_c; v_d := v_c\); |
| (b) | let \(\pi_c\) be the \(E\)-neighbor of \(\pi_d\) minimizing the objective map \(F\) in the \(E\)-neighborhood of \(\pi_d\); |
| (c) | \(v_c := F(\pi_c)\); |
| (d) | if \(v_c \geq v_d\) then \(\pi_c := \pi_d\); |
|     | until \(\pi_c = \pi_d\) |
| 3. | output \(\pi := \pi_c\). |

### References

[1] Kewen Li and Roland N. Horne. A decline curve analysis model based on fluid flow mechanisms. SPE 83470, 2003.

[2] Andrea De Martino and Matteo Marsili. Statistical mechanics of socio-economic systems with heterogeneous agents. Journal of Physics A: Mathematical and General, 39(1–2):R465–R540, 2006.

[3] Nestor V. Queipo, Javier V. Goicochea, and Salvador Pintos. Surrogate modeling-based optimization of SAGD processes. Journal of petroleum science & engineering, 35(1–2):83–93, 2002.