Errata and Comments for
“Energy of knots and conformal geometry”

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September 6, 2018

Abstract

This article serves as errata of the book “Energy of knots and conformal geometry”, Series on Knots and Everything Vol. 33, World Scientific, Singapore, 304 pages, (2003).

The updated version is available through web, linked from the author’s Home Page: http://www.comp.tmu.ac.jp/knotNRG/indices/indexe.html

The list below would be far from being completed.

Suggestions and comments would be deeply appreciated.

Key words and phrases. Energy, knot, link, Möbius geometry, conformal geometry, cross ratio

1991 Mathematics Subject Classification. Primary 57M25, Secondary 53A30

• Throughout the book, $\binom{n}{p}$ means the number of combination $\binom{n}{p} = \frac{n!}{p!(n-p)!}$. The author is sorry for the Japanese notation.

• Page 3. The second paragraph

“Let $M$ denote $\mathbb{R}^3$ or $S^3$. Two knots $f$ and $f'$ in $M$ are called isotopic if there is an isotopy $h_t : M \rightarrow M$ $(t \in [0,1])$ of the ambient space such that $h_0$ is equal to the identity map and that the map $(x,t) \mapsto (h_t(x),t)$ from $M \times [0,1]$ to itself is a homeomorphism. Then two knots are isotopic if and only if there is an orientation preserving homeomorphism $h$ of $M$ that satisfies $f' = h \circ f$. A knot type $[K]$ a knot $K$ is an isotopy class of $K$."

should be replaced by

“Let $M$ denote $\mathbb{R}^3$ or $S^3$. Two knots $f$ and $f'$ in $M$ are called isotopic if there is an isotopy $h_t : M \rightarrow M$ $(t \in [0,1])$ of the ambient space such that $h_0$ is equal to the identity map, that the map $(x,t) \mapsto (h_t(x),t)$ from $M \times [0,1]$ to itself is a homeomorphism, and that $h_1 \circ f = f'$. Then two knots are isotopic if and only if there is an orientation preserving homeomorphism $h$ of $M$ that satisfies $f' = h \circ f$. In this book, let us call an isotopy class of a knot $K$ a knot type of $[K]$."

• Page 15. The second line from the bottom.

$E_\varepsilon^{(\alpha)}(K) = E_\varepsilon^{(\alpha)}(h) = \int_{S^1} V_\varepsilon^{(\alpha)}(K;x)dx = \int_{S^1} V_\varepsilon^{(\alpha)}(h;s)ds.$

should be replaced by

$E_\varepsilon^{(\alpha)}(K) = E_\varepsilon^{(\alpha)}(h) = \int_K V_\varepsilon^{(\alpha)}(K;x)dx = \int_{S^1} V_\varepsilon^{(\alpha)}(h;s)ds.$

• Page 26. “then $E^{(\alpha)}$” in the first line of Theorem 2.4.1 (2) should be removed.

1 The author can find errors almost every time he has a look at the book.
• Page 26. The 5th line from the bottom, i.e. the second line in the proof of Theorem 2.4.1 (2).
  “Let \( h \) be a knot with \( |h'| = 1 \) and \( b = E^{(a)}(h) \) (\( b > 0 \)).” should be replaced by
  “Let \( h \) be a knot with \( |h'| = 1 \) and \( b = E^{(a)}(h) \) (\( b > 0 \)).”

• Page 34. Definition 3.1. The first line of the formula holds if \( X \neq P \):
  \[
  I_{\Sigma}(X) = \begin{cases} 
  P + \frac{r^2}{|X-P|^2}(X - P) & \text{if } X \in \mathbb{R}^3, \\
  \infty & \text{if } X = P, \\
  P & \text{if } X = \infty.
  \end{cases}
  \]

  should be replaced by
  \[
  I_{\Sigma}(X) = \begin{cases} 
  P + \frac{r^2}{|X-P|^2}(X - P) & \text{if } X \in \mathbb{R}^3\setminus\{P\}, \\
  \infty & \text{if } X = P, \\
  P & \text{if } X = \infty.
  \end{cases}
  \]

• Page 42. Outline of proof of Theorem 3.5.1.
  The convergence means the convergence with respect to the \( C^0 \)-topology.

• Page 43. The first line. “Kusner and J. Sullivan” should be replaced by “Kim and Kusner”.

• The book does not contain the proofs of Theorem 3.6.1 on page 44 and Theorem 3.7.7 on page 53, for which the reader is referred to [FHW] and [He1] respectively.

• Page 45. The right hand side of the 6th line from the bottom.
  \[
  -\varepsilon \left[ (1-\xi) \left\{ |f'|'(s + \varepsilon \xi)| \right\} \right]_0^1 + \varepsilon^2 \int_0^1 (1-\xi) \left\{ |f'|'(s + \varepsilon \xi)| \right\} \, d\xi
  \]

  should be replaced by
  \[
  -\varepsilon \left[ (1-\xi) \left\{ |f'|'(s + \varepsilon \xi)| \right\} \right]_0^1 + \varepsilon^2 \int_0^1 (1-\xi) \left\{ |f'|'(s + \varepsilon \xi)| \right\} \, d\xi
  \]

• Page 54. Section 3.8.
  The main point of [KK] is as follows: As \( r \) approaches 0 or 1, the orbital configurations approach a \( p \)- or \( q \)-fold covered circle, and thus its energy \( E = E^{(2)} \) must go to infinity. Since \( E \) is continuous, it takes on a minimum at some \( r_0 \), and by the symmetric criticality argument, this is actually critical among all variations, not just the orbital ones.

• Page 57. The 4th line from the bottom.
  \[
  \int_{|z|=1} \left( \frac{i(r^2 p^2 + (1-r^2)q^2)}{r^2 z^1 - p(z^p - 1)^2 + (1-r^2)z^1 - q(z^q - 1)^2} - \frac{i}{(z-1)^2} \right) \, ds
  \]

  should be replaced by
  \[
  \int_{|z|=1} \left( \frac{i(r^2 p^2 + (1-r^2)q^2)}{r^2 z^1 - p(z^p - 1)^2 + (1-r^2)z^1 - q(z^q - 1)^2} - \frac{i}{(z-1)^2} \right) \, dz
  \]

• Page 58. The 8th line in subsection 3.9.1.
  \[
  \text{Conf}_{n,m}(K, S^3) = \left\{ (x_1, \cdots, x_{n+m}) \in K^n \times (S^3)^m \mid x_j \neq x_k \ (j \neq k) \right\},
  \]
  \[
  U_D = \left\{ (x_1, \cdots, x_{n+m}) \in \text{Conf}_{n,m}(K, S^3) \mid x_1 < \cdots < x_n \right\}.
  \]

  should be replaced by
  \[
  \text{Conf}_{n,m}(K, \mathbb{R}^3) = \left\{ (x_1, \cdots, x_{n+m}) \in K^n \times (\mathbb{R}^3)^m \mid x_j \neq x_k \ (j \neq k) \right\},
  \]
  \[
  U_D = \left\{ (x_1, \cdots, x_{n+m}) \in \text{Conf}_{n,m}(K, \mathbb{R}^3) \mid x_1 < \cdots < x_n \right\}.
  \]

  i.e. \( S^3 \) should be replaced by \( \mathbb{R}^3 \), as we use Euclidean metric in this subsection.
Page 59, in Definition 3.5.

\[ E_{X, \cos}(K) = \int_{K^4, x_1 - x_2 - x_3 - x_4} \frac{(1 - \cos \theta_{13})(1 - \cos \theta_{24})}{|x_1 - x_3|^2 |x_2 - x_4|^2} \, dx_1 \, dx_2 \, dx_3 \, dx_4, \]

\[ E_{X, \sin}(K) = \int_{K^4, x_1 - x_2 - x_3 - x_4} \frac{\sin \theta_{13} \sin \theta_{24}}{|x_1 - x_3|^2 |x_2 - x_4|^2} \, dx_1 \, dx_2 \, dx_3 \, dx_4, \]

should be replaced by

\[ E_{X, \cos}(K) = \int_{K^4, x_1 - x_2 - x_3 - x_4} \frac{(1 - \cos \theta_{13})(1 - \cos \theta_{24})}{|x_1 - x_3|^2 |x_2 - x_4|^2} \, dx_1 \, dx_2 \, dx_3 \, dx_4, \]

\[ E_{X, \sin}(K) = \int_{K^4, x_1 - x_2 - x_3 - x_4} \frac{\sin \theta_{13} \sin \theta_{24}}{|x_1 - x_3|^2 |x_2 - x_4|^2} \, dx_1 \, dx_2 \, dx_3 \, dx_4, \]

i.e. the order of \(x_2\) and \(x_3\) in “\(x_1 \times x_3 \times x_2 \times x_4\)” should be reversed.

Page 60. Section 3.11. Let me give an idea of the proof of self-repulsiveness of surface energies.

1. Kusner-Sullivan’s \((1 - \cos \theta)^2\) energy \(E_{KS}\)

Suppose a 2-dimensional surface \(M\) in \(\mathbb{R}^3\) (the dimension of the ambient space does not matter) is tangent to itself at a point \(p_0\). Assume that the intersection of \(M\) and a small neighbourhood of \(p_0\) consists of two connected components, say \(S_1\) and \(S_2\), each of which is almost flat. Assume that \(S_1\) and \(S_2\) can be expressed as graphs of functions \(f_1\) and \(f_2\) defined on a subset \(W\) of the common tangent space \(\Pi := T_{p_0}S_1 = T_{p_0}S_2\). We use coordinates of \(\Pi\) so that \(p_0\) is the origin.

Proposition 1

We can find a positive constant \(C\) and a small region \(U\) in \(W\) so that

- Put \(U_i := 2^{-i}U\) (\(i \in \mathbb{N} := \mathbb{N} \cup \{0\}\)), then \(U_i \cap U_j = \emptyset\) if \(i \neq j\).
- Put \(S^1_i := f_1(U_i)\) and \(S^2_i := f_2(U_i)\), then

\[ \int_{S^1_i} \int_{S^2_i} \frac{(1 - \cos \theta)^2}{|x - y|^4} \, dx \, dy \geq C \quad (\forall i \in \mathbb{N}). \]

The claim implies that

\[ E_{KS}(M) \geq \sum_{i=0}^{\infty} \int_{S^1_i} \int_{S^2_i} \frac{(1 - \cos \theta)^2}{|x - y|^4} \, dx \, dy = \infty, \]

which implies the self-repulsiveness of \(E_{KS}\). Here, we used the continuity of \(E_{KS}\) with respect to \(C^2\)-topology, which could be proved anyhow.

The claim would be proved in the following way.

1. The dominant term of the integral can be estimated by up to the quadratic term of \(f_1, f_2\).
2. If \(x_i \in S^1_i\) and \(y_i \in S^2_i\) then

\[ |x_i| \sim |y_i| \sim 2^{-i}, \quad |x_i - y_i| \sim |x_i|^2, \quad |x_i - y_i| \sim |x_i|^2, \quad |\theta(x_i, y_i)| \sim |x_i|, \quad 1 - \cos \theta(x_i, y_i) \sim |x_i|^2, \]

where \(A_i \sim B_i\) means there are positive constants \(C'\) and \(C''\) that are independent of \(i\) such that \(C'A_i \leq B_i \leq C''A_i\) for all \(i\). Since Area\((S^i_1) \sim\) Area\((S^i_2) \sim |x_i|^2\) we have

\[ \int_{S^1_i} \int_{S^2_i} \frac{(1 - \cos \theta)^2}{|x - y|^4} \, dx \, dy \sim 1, \]

which shows the proposition.

2. Auckly-Sadun’s regularized energy \(E_{AS}\)

Let \(S_C\) be a set of closed surfaces \(M\) in \(\mathbb{R}^3\) of class \(C^6\) that are “bounded by a positive constant \(C\) in the sense of \(C^6\)-topology”, to be precise, for any point \(x \in M\), if we express a small neighbourhood of \(x\) of \(M\) as a graph of a function \(f\) on a small neighbourhood \(W_x\) of the origin of the tangent plane \(T_xM\), then \(|\partial^{\alpha} f| \leq C\) on \(W_x\) for any multi-index \(\alpha = (\alpha_1, \alpha_2)\) with \(0 \leq |\alpha| = \alpha_1 + \alpha_2 \leq 5\).

Assume \(M \in S_C\). Let \(k_M\) be the maximum of the principal curvatures of \(M\). Note that \(k_M \leq C\). Put \(R_0 := 1/C\), then \(R_0 \geq 1/k_M\) for any \(M \in S_C\).
For $x \in M$ and $0 < r \leq R_0$, let $U_r(x)$ be the connected component of $M \cap B_r(x)$ that contains $x$, where $B_r(x)$ is the 3-ball with center $x$ and radius $r$.

Recall that

$$E_{AS}(M) := \int_M \left[ \lim_{\varepsilon \to 0} \left( \int_{M,|x-y| \geq \varepsilon} \frac{dy}{|x-y|^4} - \frac{\pi}{\varepsilon^2} + \frac{\pi \Delta(x)}{16} \log (\Delta(x)) + \frac{\pi K(x)}{4} \right) \right] dx,$$

where $\Delta(x) := (\kappa_1 - \kappa_2)^2$ and $K(x) = \kappa_1 \kappa_2$, where $\kappa_1$ and $\kappa_2$ are principal curvatures of $M$ at $x$. Put

$$V(U_r(x); x) := \lim_{\varepsilon \to 0} \left( \int_{U_r(x),|x-y| \geq \varepsilon} \frac{dy}{|x-y|^4} - \frac{\pi}{\varepsilon^2} + \frac{\pi \Delta(x)}{16} \log (\Delta(x)) + \frac{\pi K(x)}{4} \right),$$

then

$$E_{AS}(M) = \int_M V(U_r(x); x) dx + \int_M \left( \int_{M \setminus U_r(x)} \frac{dy}{|x-y|^4} \right) dx.$$

**Lemma 2**

1. There is a constant $C'(r)$ (which might be negative) such that $V(U_r(x); x) \geq C'(r)$ for any $M \in S_C$, for any $x \in M$ and for any $r$ with $0 < r \leq R_0$.

(I think we can take $C'(r)$ to be independent of $r$.)

2. There is a positive constant $A$ such that $\text{Length}(\partial U_r(x)) \geq Ar$ for any $M \in S_C$, for any $x \in M$ and for any $r$ with $0 < r \leq R_0$.

Suppose $M \in S_C$ is close to have a double point. We look for a lower bound of $E_{AS}(M)$ if $M$ satisfies satisfies $M \in S_C$, $\exists p, q \in M$, $0 < \exists r \leq R_0$, $U_r(p) \cap U_r(q) = \emptyset$, $|p - q| = d$, for some positive number $d$, and consider the limit as $d \downarrow 0$.

First note

$$E_{AS}(M) \geq C' \left( \frac{r}{2} \right) \text{Area}(M) + \int_{U_r(p)} \int_{U_r(q)} \frac{dx \, dy}{|x-y|^4}.$$

The second term of the right hand side is bounded below by

$$\int_0^{r/2} \int_0^{r/2} \frac{A s t}{(d + s + t)^4} ds dt,$$

which blows up to $+\infty$ as $d$ goes down to 0, which proves the $C^5$-self-repulsiveness of Auckly-Sadun surface energy.

- Page 67. The 5th line
  If $E^{0,p}(h)$ with $0 < p > 2$ is finite then $h$ cannot have a sharp turn. should be replaced by
  If $E^{0,p}(h)$ with $0 < p > 2$ is bounded then $h$ cannot have a sharp turn.

- Pages 95, Figure 6.7, page 97, Figure 6.11, page 99, Figure 6.15. The caption “Look with the right eye.” should be “Look with the left eye.” (You do not have to close your left eye.)

- There is a misunderstanding about the order of contact. The order of contact in the book should be reduced by 1. The errors can be found on pages 119, 120, 123-125, 184-185.

  - Page 119. Definition 8.2
    “(1) An osculating circle · · · is the circle which is tangent to $K$ at $x$ at least to the third order” should be replaced by
    “(1) An osculating circle · · · is the circle which is tangent to $K$ at $x$ at least to the second order” and
    “(2) An osculating sphere · · · is tangent to $K$ at $x$ at least to the fourth order.” should be replaced by
    “(2) An osculating sphere · · · is tangent to $K$ at $x$ at least to the third order.”
“Proposition 8.3.1(1) An osculating sphere is uniquely determined if the order of tangency of the osculating circle to the knot is just 3.

(2) Suppose the order of tangency of the osculating circle to the knot is just 3. Then \(\cdots\)"

should be replaced by

“Proposition 8.3.1(1) An osculating sphere is uniquely determined if the order of tangency of the osculating circle to the knot is just 2.

(2) Suppose the order of tangency of the osculating circle to the knot is just 2. Then \(\cdots\)"

Page 120. After Proposition 8.3.1.

“When the order of tangency of the osculating circle to the knot is greater than 3, any 2-sphere through the osculating circle is tangent to the knot to the fourth order. But there might be a unique 2-sphere which is tangent to the knot with a higher order of tangency than 4."

should be replaced by

“When the order of tangency of the osculating circle to the knot is greater than 2, any 2-sphere through the osculating circle is tangent to the knot to the third order. But there might be a unique 2-sphere which is tangent to the knot with a higher order of tangency than 3."

Page 120. The 6th line from the bottom (after the formula (8.1)).

“Then \(C\) is tangent to \(K\) at 0 to the fourth order, i.e. \(f^{(3)}(0) = C^{(3)}(0)\), if and only if \(k' = 0\) and \(k\tau = 0\)."

should be replaced by

“Then \(C\) is tangent to \(K\) at 0 to the third order, i.e. \(f^{(3)}(0) = C^{(3)}(0)\), if and only if \(k' = 0\) and \(k\tau = 0\)."

Pages 123-124. Proof of Proposition 8.4.2.

“Case I: \(\cdots\)

If the knot \(K\) is transversal to \(\Sigma\) at \(x\) and if the order of tangency of \(K\) and \(\Sigma\) at \(y\) is 2, then \(K\) must intersect \(\Sigma\) (not necessarily transversally) at a third point \(z\) which is different from both \(x\) and \(y\), and therefore \(\Sigma = \sigma(x, z, y, y)\).

If the order of tangency of \(K\) and \(\Sigma\) at \(y\) is more than or equal to 3, then \(\Sigma = \sigma(x, y, y, y)\).

Case II: Suppose \(x = y\). The order of tangency of \(K\) to \(\Sigma\) at \(x\) is more than or equal to 3.

If it is 3 then \(K\) must intersect \(\Sigma\) (not necessarily transversally) at another point \(z\) (\(z \neq x\)). Then \(\Sigma = \sigma(x, x, x, z)\).

If the order of tangency is more than or equal to 4 then \(\Sigma = \sigma(x, x, x, x)\)."

should be replaced by

“Case I: \(\cdots\)

If the knot \(K\) is transversal to \(\Sigma\) at \(x\) and if the order of contact of \(K\) and \(\Sigma\) at \(y\) is 1, then \(K\) must intersect \(\Sigma\) (not necessarily transversally) at a third point \(z\) which is different from both \(x\) and \(y\), and therefore \(\Sigma = \sigma(x, z, y, y)\).

If the order of contact of \(K\) and \(\Sigma\) at \(y\) is more than or equal to 2, then \(\Sigma = \sigma(x, y, y, y)\).

Case II: Suppose \(x = y\). The order of tangency of \(K\) to \(\Sigma\) at \(x\) is more than or equal to 2.

If it is 3 then \(K\) must intersect \(\Sigma\) (not necessarily transversally) at another point \(z\) (\(z \neq x\)). Then \(\Sigma = \sigma(x, x, x, z)\).

If the order of contact is more than or equal to 3 then \(\Sigma = \sigma(x, x, x, x)\)."

Page 125. The 1st line.

“Since \(\rho(x) > r(\Sigma)\) by the assumption, the knot \(K\) cannot have the tangency of order 3 with \(\Sigma\) at \(x\), and hence \(K\) must lie in \(\cdots\)"

should be replaced by

“Since \(\rho(x) > r(\Sigma)\) by the assumption, the knot \(K\) cannot have the tangency of order 2 with \(\Sigma\) at \(x\), and hence \(K\) must lie in \(\cdots\)"
Page 125. The 5th line.
“then the osculating sphere at \( x \) contains the osculating sphere at \( x \) as the great circle” should be replaced by
“then the osculating sphere at \( x \) contains the osculating circle at \( x \) as the great circle”

Page 141. The 7th and 8th lines from the bottom, i.e. the last two lines of (4) have two errors:
Since \( u_5, v_5 > 1 \) this implies \( u_1 v_1 + \cdots + u_4 v_4 \leq u_5 v_5 - 1 \) which means \( \langle u, v \rangle < -1 \).
should be replaced by
Since \( u_5 v_5 > 1 \) this implies \( u_1 v_1 + \cdots + u_4 v_4 \leq u_5 v_5 - 1 \) which means \( \langle u, v \rangle \leq -1 \).
Furthermore, the proof that \( \langle u, v \rangle \neq -1 \) should be added:
Assume \( \langle u, v \rangle = -1 \). Since \( \langle u, u \rangle = -1 \) we have \( \langle u, u - v \rangle = 0 \). Lemma 9.1.1 (3) implies that \( u - v \) is either space-like or equal to 0. As \( \langle u - v, u - v \rangle = 0 \), \( u - v \) cannot be space-like. Therefore, \( u - v = 0 \), which is a contradiction.

Page 145. The 3rd line. \( S_{nfty}^3 \) should be replaced by \( S_{\infty}^3 \).

Page 145. The bijection in Theorem 9.3.2 can be considered as a modern version of the pentaspherical coordinates in [Dar].

Page 151 last two lines to Page 152 line. (1a). The book misses the description of the case when \( n = 1 \).
When \( n = 1 \) \( P^1 \cap S_1^\infty = \emptyset \) since \( P^1 \) is a time-like line. In this case we may consider the “base sphere” to be \( \emptyset = \partial (P^1 \cap \mathbb{H}^2) \), where \( \partial \mathbb{H}^2 = S_1^\infty \). Let us call \( \mathcal{P} \) or the set of corresponding spheres of dimension 0 a “space-like pencil”.

Page 166. Three lines before Lemma 9.8.2.
\[ \psi : \mathbb{R}_{n+1}^+ \ni (X, r) \mapsto \varphi^{-1} \circ \rho^{-1} (S^n_{\rho^{-1}}(X)) \in A \setminus (\text{Span } \langle Q \rangle)^\perp \]
should be replaced by
\[ \psi : \mathbb{R}_{n+1}^+ \ni (X, r) \mapsto \varphi^{-1} \circ \rho^{-1} (S^n_{\rho^{-1}}(X)) \in A \setminus (\text{Span } \langle Q \rangle)^\perp. \]

Page 167. The 8th line.
\[ \omega_{\mathbb{R}_{n+1}^+} = \frac{1}{r_{n+2}} dx_1 \wedge \cdots \wedge d x_{n+1} \]
should be replaced by
\[ \omega_{\mathbb{R}_{n+1}^+} = -\frac{1}{r_{n+1}} dX_1 \wedge \cdots \wedge dX_n \wedge dr \]

Page 167. The 16th line.
\[ \tilde{\gamma} : \mathbb{R}_{n+1}^+ \ni (X, r) \mapsto \left( \frac{X}{|X|^2 - r^2}, \frac{r}{|X|^2 - r^2} \right) \in \mathbb{R}_{n+1}^+. \]
should be replaced by
\[ \tilde{\gamma} : \mathbb{R}_{n+1}^+ \ni (X, r) \mapsto \left( \frac{X}{|X|^2 - r^2}, \frac{r}{|X|^2 - r^2} \right) \in \mathbb{R}_{n+1}^+. \]

Page 175. Definition 10.1 (2) should be replaced by
“(2) An oriented 2-sphere \( \Sigma \) is called a non-trivial sphere in the strict sense for a knot \( K \) if each connected component of \( \mathbb{R}^3 \setminus \Sigma \) \( (\mathbb{R}^3 \setminus \Sigma) \) contains at least 2 connected components of \( K \setminus (K \cap \Sigma) \).”
• Page 184. Definition 10.6. (1)

\[ C^{(4)}(K) = \left\{ (s, s, s, s) \in \Delta^{(4)} \mid \text{The osculating circle of } K \text{ at } f(s) \text{ is tangent to } K \text{ to the fourth order} \right\} . \]

should be replaced by

\[ C^{(4)}(K) = \left\{ (s, s, s, s) \in \Delta^{(4)} \mid \text{The osculating circle of } K \text{ at } f(s) \text{ is tangent to } K \text{ to the third order} \right\} . \]

• Page 184. Proposition 10.3.3. (1)

"The order of tangency of the osculating circle at \( f(s) \) is exactly 3 if and only if \( f(s) \wedge f'(s) \wedge f''(s) \wedge f'''(s) \neq 0. \"

should be replaced by

"The order of tangency of the osculating circle at \( f(s) \) is exactly 2 if and only if \( f(s) \wedge f'(s) \wedge f''(s) \wedge f'''(s) \neq 0. \"

• Page 185. The 6th line and the 10th line from the bottom (in the Proof of Proposition 10.3.3 (1)).

"(1) Suppose the osculating circle \( C \) is tangent to the knot at \( f(s) \) to the fourth order. \( \cdots \) and hence the osculating circle \( C \) is tangent to the knot at \( f(s) \) to the fourth order. "

should be replaced by

"(1) Suppose the osculating circle \( C \) is tangent to the knot at \( f(s) \) to the third order. \( \cdots \) and hence the osculating circle \( C \) is tangent to the knot at \( f(s) \) to the third order. "

• Page 191. The 7th line.

\[ (I \times I)^* \lambda_R = \lambda_R + \frac{1}{2} d \log (|u|^2) \]

should be replaced by

\[ (I \times I)^* \lambda_R = \lambda_R + \frac{1}{2} d \log (u^2 + v^2) \]

• Page 193. The 4th line in the Proof of Lemma 11.2.2. ‘bas’ should be replaced by ‘basis’.

• Page 193. The 5th line in Lemma 11.2.3. ‘this local coordinate’ should be replaced by ‘these local coordinates’.

• Page 198. Theorem 11.2.7. As a corollary of this theorem, we have

**Corollary** The pull-back \( \omega = \psi^* \omega_0 \) of the canonical symplectic form \( \omega_0 \) of \( T^* S^n \) by \( \psi : S^n \times S^n \setminus \Delta \to T^* S^n \) is invariant under any diagonal action of the Möbius group \( \mathcal{M} \) on \( S^n \times S^n \setminus \Delta \).

• Page 200. The condition (ii) of Theorem 11.2.9 is not necessary.

• Page 203. A remark on Definition 11.4:

The real part of the infinitesimal cross ratio of a knot \( K \) is a smooth 2-form on \( K \times K \setminus \Delta \), but it is not the case with the imaginary part. Since the conformal angle \( \theta_K(x, y) \) is not a smooth function of \( x \) and \( y \) (see Figure 10.1 on page 183), the imaginary part of the infinitesimal cross ratio may have singularity at a pair of points \( (x, y) \in K \times K \setminus \Delta \) where the conformal angle \( \theta_K(x, y) \) vanishes.

• Page 206. The 2nd and 3rd lines

\[ ((T_0 \circ f)^* dx_3)(s_0, t_0) = \left( \frac{d}{ds} (T_0 \circ f)_3 \right)(s_0), \]

\[ ((T_0 \circ f)^* dy_3)(s_0, t_0) = \left( \frac{d}{dt} (T_0 \circ f)_3 \right)(t_0) \]
should be replaced by
\[
(T_0 \circ f)^* dx_3) (s_0, t_0) = \left( \frac{d}{ds} (T_0 \circ f) \right) (s_0) = 0,
\]
\[
(T_0 \circ f)^* dy_3) (s_0, t_0) = \left( \frac{d}{dt} (T_0 \circ f) \right) (t_0) = 0
\]
i.e. = 0 should be added at the end of the both formulae.

- Page 211. The 2nd and 3rd lines
\[
\frac{1}{2} E_0^{(2)} (L_r) = E_0^{(2), \text{mut}} (L_r) = \int_0^{2\pi} \int_0^{2\pi} \frac{dsdt}{2 + 2 \cos (t - s)}
\]
should be replaced by
\[
\frac{1}{2} E_0^{(2)} (L_r) = E_0^{(2), \text{mut}} (L_r) = \int_0^{2\pi} \int_0^{2\pi} \frac{dsdt}{2 - 2 \cos (t - s)}
\]

- Page 218. The 3rd line in the Proof of Lemma 12.3.2
\[
V_{\sin \theta}(K; f(\pm t)), V_{\sin \theta}(K; f(\delta \pm t)) \geq \frac{1}{100} \frac{1}{d + t}
\]
should be replaced by
\[
V_{\sin \theta}(K; f(\pm t)), V_{\sin \theta}(K; f(\delta \pm t)) \leq \frac{1}{100} \frac{1}{d + t}
\]
i.e. the inequality should be reversed.

- Page 218. The 8th line in the Proof of Lemma 12.3.2, i.e. just above Figure 12.2.
\[
\lim_{K \setminus \{x_+ \}} \pi_{x_+} (y) = \hat{x}_+.
\]
should be replaced by
\[
\lim_{K \setminus \{x_+ \}} \pi_{x_+} (I_{x_+} (y)) = \hat{x}_+.
\]

- Page 218. The right picture of Figure 12.2 should be replaced by Figure 1. The radius of the circle, which was \(\frac{1}{400} \frac{1}{d + t}\) in the book, should be replaced by \(\frac{1}{200} \frac{1}{d + t}\). Also, the point \(x_+\) should be closer to the center of the circle, \(\hat{x}_+\), as \(|x_+ - \hat{x}_+| \leq \frac{1}{100} \frac{1}{d + t}\), which is a half of the radius.

- Page 218. The 3rd line from the bottom.
\[
\pi_{x_+} (K_{x_+}) \text{ lies inside the circle on } \Pi_{x_+} \text{ with center } x_+ \text{ and radius } 1/(200(d + t)).
\]
should be replaced by
\[
\pi_{x_+} (K_{x_+}) \text{ lies inside the circle on } \Pi_{x_+} \text{ with center } \hat{x}_+ \text{ and radius } 1/(200(d + t)).
\]

- Page 219. The 8th line (just above Figure 12.3).
\(N_t\) whose meridian disc has radius \((200(d + t))/3\), as illustrated in Figure 12.3.
should be replaced by
\(N_t\) whose meridian disc has radius \((200(d + t))/3\), as illustrated in Figure 12.3.

- Page 219. Figure 12.3 should be replaced by Figure 2. Namely, the radius of the inner circle of the left picture, which was \(\frac{1}{400} \frac{1}{d + t}\) in the book, should be \(\frac{1}{200} \frac{1}{d + t}\), and the radius of the ‘degenerate solid torus’ in the right picture, which was \(\frac{400(d + t)}{3}\) in the book, should be \(\frac{200(d + t)}{3}\).
Figure 1: The right picture of Figure 12.2.

Figure 2: Figure 12.3.
Page 219. The 7th line from the bottom

\[ |f(t) - f(-t)| \leq 2t \ll \frac{400}{3} (d + t) \]

should be replaced by

\[ |f(t) - f(-t)| \leq 2t \ll \frac{200}{3} (d + t) \]

Pae 220. The 2nd line from the bottom

\[ |f(0) - g(t)| \geq \int_0^t (1 - \kappa s) ds = t \left(1 - \frac{\kappa}{2} t\right) > d, \]

should be replaced by

\[ |f(0) - f(t)| \geq \int_0^t (1 - \kappa s) ds = t \left(1 - \frac{\kappa}{2} t\right) > d, \]

Page 226. The last three lines of Remark 13.2.2 should be replaced by

This is because \(1 \leq n C_2 \leq 2n C_4\) when \(n \geq 2\).

Page 229. Line 6 up.

“subarc of \(C_3(r)\) between \(Q_{12}(r)\) and \(P_{23}(r)\)” should be replaced by

“subarc of \(C_3(r)\) between \(Q_{12}(r)\) and \(P_{13}(r)\)”

Page 229. The last line. Page 230. The second line.

The domain of integration “\[ \left[ \frac{1}{2 \sin \theta_K}, \infty \right) \]” should be replaced by “\[ \left[ \frac{1}{2 \sin \theta}, \infty \right) \]”

Page 231. Lines 3-5. “\(\cdots + o(|z|)\)” should be replaced by “\(\cdots + O(|z|)\)”.

Page 236. The 13th line around the middle of the page

\[ \int_{\Sigma_r(X,Y,Z) \cap K_d \neq \emptyset} g(\Sigma_r(X,Y,Z) \cap K_d) dX dY dZ = 2\pi r^2 (K_d), \]

should be replaced by

\[ \int_{\Sigma_r(X,Y,Z) \cap K_d \neq \emptyset} g(\Sigma_r(X,Y,Z) \cap K_d) dX dY dZ = 2\pi r^2 (\tilde{K}_d), \]

Page 236. The 5th line from the bottom

\[ E_{\text{mnts}}(K_d) = \int_{S(K_d)} C_2^2(\Sigma_r(X,Y,Z) \cap K_d) \cdot \frac{1}{r^4} dX dY dZ d\sigma \]

should be replaced by

\[ E_{\text{mnts}}(K_d) = \int_{S(K_d)} C_2^2(\Sigma_r(X,Y,Z) \cap K_d) \cdot \frac{1}{r^4} dX dY dZ d\sigma \]

This implies the 4th line from the bottom because \(m C_2 \geq m - 1\) for \(m \geq 0\).

Page 236. The 8th line (just before Example A.1) should be replaced by

We show that \(I_{tv}\) can detect the unknot.

Namely, it is a conjecture whether \(|csl|\) can detect the unknot or not.
• Page 273.
[CKS2] J. Cantarella, R. B. Kusner, and J. M. Sullivan, *On the minimum ropelength of knots and links*, Preprint.
should be updated to
[CKS2] J. Cantarella, R. B. Kusner, and J. M. Sullivan, *On the minimum ropelength of knots and links*, Invent. Math. 150 (2002), no. 2, 257–286.
[CKS] should be removed as it is same as [CKS1].

• Page 276.
[GLP] M. Gromov, J. Lafontaine, and P. Pnau, *Structures métriques pour les variétés riemanniennes*, Cedic/Fernand Nathan, Paris, 1981.
should be replaced by
[GLP]Gr-La-Pn M. Gromov, J. Lafontaine, and P. Pansu, *Structures métriques pour les variétés riemanniennes*, Cedic/Fernand Nathan, Paris, 1981.

• Page 279.
[Lin] X.-S. Lin, *Knot energies and knot invariants*, J. Differential Geom. 44 (1996), 74–95.
should be replaced by
[Lin] X.-S. Lin, *Knot energies and knot invariants*. *Knot theory and its applications*, Chaos Solitons Fractals 9 (1998) 645–655

• Page 285, Index. The “average crossing number” also appears on page 44.

**Acknowledgement.** The author thanks Rob Kusner for pointing out a couple of mistakes. He also thanks Paul Feehan for helpful comments.