Shen’s Processes on Finslerian Connections

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Abstract

In this paper, we discuss the invariant properties of curvatures effected by the Matsumoto’s C or L-process. We find equivalent conditions on curvatures by comparing the difference between the corresponding curvatures of closely related connections. As an application, Matsumoto’s L-process on Randers manifold is studied. Shen connection can not be obtained by using Matsumoto’s processes from other well-known connections. This leads us to two new processes which we call Shen’s C and L-processes. We study the invariant properties of curvatures under the Shen’s processes.

Keywords: Finsler connection, Randers metric, Landsberg metric.

1 Introduction

After Einstein’s formulation of general relativity, Riemannian geometry became fashionable and one of the connections, namely Levi-Civita connection, came to forefront. This connection is both torsion-free and metric-compatible. On the other hand, Finsler geometry is a natural extension of Riemannian geometry. Likewise, connections in Finsler geometry can be prescribed on the pulled-back bundle $\pi^*TM$. Examples of such were proposed by Synge, Taylor, Berwald, Cartan and Chern [1-5],[7],[15]. However, there are four well-known connections in Finsler geometry which may be considered “natural” in some sense: Berwald, Cartan, Hashiguchi and Chern connections. Incidentally in the generic Finslerian setting, it is impossible to have a connection on $\pi^*TM$ which is both torsion-free and compatible with the Riemannian metric induced by Finsler metric.

In [9], Matsumoto introduced a satisfactory and truly aesthetical axiomatic description of Cartan’s connection in the sixties. After the Cartan connection has been constructed, easy processes, baptized by Matsumoto “L-process” and “C-process” yield the Chern, the Hashiguchi and the Berwald connections.

In this paper, we show that the vv-curvature of connections is invariant under the Matsumoto’s L-process. Comparing the corresponding curvatures of related connections obtained by this transformation, we get equivalent conditions on curvatures.

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**Theorem 1.1.** Let \((M, F)\) be a Finsler manifold. Suppose that \(\nabla\) and \(\widetilde{\nabla}\) are two connections on \(M\) and \(\widetilde{\nabla}\) is obtained from \(\nabla\) by Matsumoto’s \(L\)-process. Then we have the following

1. Their \(v\)-\(v\)-curvatures coincide.
2. If their \(h\)-\(h\)-curvatures coincide, then \(F\) is a generalized Landsberg metric. Moreover, if \(M\) is compact, then \(F\) reduces to a Landsberg metric.
3. If their \(h\)-\(h\)-curvatures coincide and \(F\) is of non-zero scalar flag curvature, then \(F\) is a Randers metric.
4. Their \(h\)-\(v\)-curvatures coincide if and only if \(F\) is a Landsberg metric.

It is well known that vanishing \(h\)-\(v\)-curvatures of Cartan and Berwald connections characterize Landsberg metrics and Berwald metrics, respectively. Shen introduces a new connection in Finsler geometry, which vanishing \(h\)-\(v\)-curvature of this connection characterizes Riemannian metrics [13]. On the other hand, the Chern, Berwald, and Hashiguchi connections are obtained from Cartan connection by Matsumoto’s processes, as depicted in the following diagram:

\[
\begin{array}{ccc}
\text{Cartan connection} & \xrightarrow{C\text{-process}} & \text{Chern connection} \\
\downarrow \text{L - process} & & \downarrow \text{L - process} \\
\text{Hashiguchi connection} & \xrightarrow{C\text{-process}} & \text{Berwald connection}
\end{array}
\]

However, Shen connection can not be constructed by Matsumoto’s processes from these well-known connections. Therefore, it is natural to find some kinds of processes on one of these connections, say Chern connection, which yield the Shen connection. Here, we introduce two new processes on connections called Shen’s \(C\) and \(L\)-processes. We show that Shen connection is obtained from Chern connection by Shen’s \(C\)-process. Studying curvature tensors of two connections related by this process leads us to the following theorem.

**Theorem 1.2.** Let \((M, F)\) be a Finsler manifold. Suppose that \(\nabla\) and \(\widetilde{\nabla}\) are two connections on \(M\) and \(\widetilde{\nabla}\) is obtained from \(\nabla\) by Shen’s \(C\)-process. Then we have the following

1. If their \(h\)-\(h\)-curvature coincide, then \(F\) is a Landsberg metric.
2. Their \(h\)-\(v\)-curvature coincide if and only if \(F\) is Riemannian.
3. Their \(v\)-\(v\)-curvature coincide.

Throughout this paper, we set the Cartan connection on Finsler manifolds. The \(h\)- and \(v\)-covariant derivatives are denoted by “;” and “,” respectively. Further, we suppose that the horizontal distribution of connections are the same as Cartan connection’s horizontal distribution.
2 Preliminaries

Let $M$ be an $n$-dimensional $C^\infty$ manifold. Denote by $T_x M$ the tangent space at $x \in M$, and by $TM = \cup_{x \in M} T_x M$ the tangent bundle of $M$.

A Finsler metric on $M$ is a function $F : TM \rightarrow [0, \infty)$ which has the following properties: (i) $F$ is $C^\infty$ on $TM \setminus \{0\}$; (ii) $F$ is positively 1-homogeneous on the fibers of tangent bundle $TM$, and (iii) for each $y \in T_x M$, the following quadratic form $g_y$ on $T_x M$ is positive definite,

$$g_y(u, v) := \frac{1}{2} \left[ F^2(y + su + tv) \right]_{s, t = 0}, \quad u, v \in T_x M.$$ 

Let $x \in M$ and $F_x := F|_{T_x M}$. To measure the non-Euclidean feature of $F_x$, define $C_y : T_x M \times T_x M \times T_x M \rightarrow \mathbb{R}$ by

$$C_y(u, v, w) := \frac{1}{2} \frac{d}{dt} [g_y+tw(u, v)] |_{t=0}, \quad u, v, w \in T_x M.$$ 

The family $C := \{C_y\}_{y \in TM_0}$ is called the Cartan torsion. It is well known that $C=0$ if and only if $F$ is Riemannian. For $y \in T_x M_0$, define mean Cartan torsion $I_y$ by $I_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$ and $u = u^i \frac{\partial}{\partial x^i}|_x$. By Diecke's Theorem, $F$ is Riemannian if and only if $I_y = 0$ [8].

For $y \in T_x M_0$, define the Matsumoto torsion $M_y : T_x M \otimes T_x M \otimes T_x M \rightarrow \mathbb{R}$ by $M_y(u, v, w) := M_{ijk}(y)u^iv^jw^k$ where

$$M_{ijk} := C_{ijk} - \frac{1}{n+1} \left\{ I_i h_{jk} + I_j h_{ik} + I_k h_{ij} \right\},$$

and $h_{ijk} := F F_{i'y'}y' = g_{i'y'} - \frac{1}{n} g_{ij}g^{k}g_{k'q}y^q$. A Finsler metric $F$ is said to be C-reducible if $M_y = 0$. This quantity is introduced by Matsumoto [3]. Matsumoto proves that every Randers metric satisfies that $M_y = 0$. Later on, Matsumoto-Hojo prove that the converse is true too.

**Proposition 2.1.** ([10][11]) A Finsler metric $F$ on a manifold of dimension $n \geq 3$ is a Randers metric if and only if $M_y = 0$, $\forall y \in TM_0$.

The horizontal covariant derivatives of $C$ and $I$ along geodesics give rise to the Landsberg curvature $L_y : T_x M \times T_x M \times T_x M \rightarrow \mathbb{R}$ and mean Landsberg curvature $J_y : T_x M \rightarrow \mathbb{R}$ defined by

$$L_y(u, v, w) := L_{ijk}(y)u^iv^jw^k, \quad \text{and} \quad J_y(u) := J_i(y)u^i,$$

where $L_{ijk} := C_{ijk}s, J_i := I_i(y)s, u = u^i \frac{\partial}{\partial x^i}|_x, v = v^i \frac{\partial}{\partial x^i}|_x$ and $w = w^i \frac{\partial}{\partial x^i}|_x$. The families $L := \{L_y\}_{y \in TM_0}$ and $J := \{J_y\}_{y \in TM_0}$ are called the Landsberg curvature and mean Landsberg curvature. A Finsler metric is called Landsberg metric and weakly Landsberg metric if $L=0$ and $J = 0$, respectively.

The rate of change of $L$ along geodesics is measured by the generalized Landsberg curvature $L_y : T_x M \times T_x M \times T_x M \rightarrow \mathbb{R}$ which is defined by $L_y((u, v, w) := \hat{L}_{ijk}(y)u^iv^jw^k$, where $\hat{L}_{ijk} := L_{ijk}s$. 


The geodesics of Finsler metric $F$ are characterized by the following system of second order ordinary differential equations in local coordinates $\ddot{c} + 2G^i(\dot{c}) = 0$, where $G^i(x, y) := \frac{1}{T^2} [F^2]_{x,y} y^k - [F^2]_{x,y}$. These local functions $G^i$ define a global vector field on $T^0$ as follows

$$ G = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}. $$

For $y \in T_x M_0$, define $B_y : T_x M \otimes T_x M \otimes T_x M \rightarrow T_x M$ by

$$ B_y(u, v, w) := B^{ijkl}(y)u^j v^k w^l \frac{\partial}{\partial x^i}(y), $$

where $B^{ijkl}(y) := \frac{\partial G^i}{\partial y^j |_y} |_x$, $v = v^i \frac{\partial}{\partial x^i}(x)$, and $u = u^i \frac{\partial}{\partial x^i}(x)$. The $B^i_{jkl}(y)$ is called the Berwald curvature. A Finsler metric is called a Berwald metric if $B = 0$ [14]. It is well known that every Berwald metric is a Landsberg metric.

The notion of Riemann curvature is extended to Finsler metrics. For $y \in T_x M_0$, the Riemann curvature $R_y : T_x M \rightarrow T_x M$ is defined by $R_y(u) = R^i_{jk}(y) u^j \frac{\partial}{\partial x^k}$ where

$$ R^i_{jk}(y) := \frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}. $$

Take an arbitrary plane $P \subset T_x M$ (flag) and a non-zero vector $y \in P$ (flag pole), the flag curvature $K(P, y)$ is defined by

$$ K(P, y) := \frac{g_y(R_y(v), v)}{g_y(y, y) g_y(v, v) - g_y(v, y) g_y(v, y)}. $$

We say that a Finsler metric $F$ is of scalar flag curvature if for any $y \in T_x M$, the flag curvature $K = K(x, y)$ is a scalar function on $T^0 M_0$. If $K$ is constant, then $F$ is said to be of constant flag curvature.

Let us consider the pull-back tangent bundle $\pi^*TM$ over $T^0 M_0$ defined by $\pi^*TM = \{(u, v) \in T^0 M_0 \times T^0 M_0 | \pi(u) = \pi(v)\}$. Take a local coordinate system $(x^i)$ in $M$, the local natural frame $(\frac{\partial}{\partial x^i})$ of $T_x M$ determines a local natural frame $\partial_i$ for $\pi^*TM$ the fibers of $\pi^*TM$, where $\partial_i |_v = (v, \frac{\partial}{\partial x^i} | v)$, and $v = y^i \frac{\partial}{\partial x^i} | v \in T^0 M_0$. The fiber $\pi^*TM$ is isomorphic to $T_{\pi(x)} M$ where $\pi(v) = x$. There is a canonical section $\ell$ of $\pi^*TM$ defined by $\ell_v = (v, v)/F(v)$.

Let $TTM$ be the tangent bundle of $TM$ and $\rho$ the canonical linear mapping $\rho : TTM \rightarrow \pi^*TM$ defined by $\rho(\hat{X}) = (z, \pi(\hat{X}))$ where $\hat{X} \in T_z TM_0$ and $z \in T^0 M_0$. The bundle map $\rho$ satisfies $\rho(\frac{\partial}{\partial y^i}) = \partial_i$ and $\rho(\frac{\partial}{\partial x^i}) = 0$. Let $V_z TM$ be the set of vertical vectors at $z$, that is, the set of vectors tangent to the fiber through $z$, or equivalently $V_z TM = ker\rho$, called the vertical space.

Let $\nabla$ be a linear connection on $\pi^*TM$. Consider the linear mapping $\mu_z : T_z TM_0 \rightarrow T_z M_0$ defined by $\mu_z(\hat{X}) = \nabla_{\hat{X}} F\ell$, where $\hat{X} \in T_z TM_0$. The connection $\nabla$ is called a Finsler connection if for every $z \in T^0 M_0$, $\mu_z$ defines an
isomorphism of $V_2 TM_0$ onto $T_{\pi z} M$. Therefore, the tangent space $TTM_0$ in $z$ is decomposed as $T_z TM_0 = H_z TM \oplus V_z TM$, where $H_z TM = \ker \mu_z$ is called the horizontal space defined by $\nabla$. Indeed, any tangent vector $\dot{X} \in T_z TM_0$ in $z$ decomposes to $\dot{X} = H \dot{X} + V \dot{X}$ where $H \dot{X} \in H_z TM$ and $V \dot{X} \in V_z TM$.

The structural equations of the Finsler connection $\nabla$ are

\begin{align}
\mathcal{T}(\dot{X}, \dot{Y}) &= \nabla_{\dot{X}} \dot{Y} - \nabla_{\dot{Y}} \dot{X} - \rho[\dot{X}, \dot{Y}], \\
\Omega(\dot{X}, \dot{Y}) Z &= \nabla_{\dot{X}} \nabla_{\dot{Y}} Z - \nabla_{\nabla_{\dot{X}} \dot{Y}} Z - \nabla_{\nabla_{\dot{Y}} \dot{X}} Z,
\end{align}

where $X = \rho(\dot{X})$, $Y = \rho(\dot{Y})$ and $Z = \rho(\dot{Z})$. The tensors $\mathcal{T}$ and $\Omega$ are called respectively the Torsion and Curvature tensors of $\nabla$. Three curvature tensors are defined by $R(X, Y) := \Omega(H \dot{X}, H \dot{Y})$, $P(\dot{X}, \dot{Y}) := \Omega(H \dot{X}, V \dot{Y})$ and $Q(\dot{X}, \dot{Y}) := \Omega(V \dot{X}, V \dot{Y})$, where $X = \mu(\dot{X})$ and $Y = \mu(\dot{Y})$.

Let $\{e_i\}_{i=1}^n$ be a local orthonormal (with respect to $g$) frame field for the pulled-back bundle $\pi^* TM$ such that $i = \ell$. Let $\{\omega^i\}_{i=1}^n$ be its dual co-frame field. One readily finds that $\omega^i := \frac{\partial}{\partial t^i} dt = \omega$, which is called Hilbert form, and $\omega(\ell) = 1$. Put $\nabla e_i = \omega^j_i \otimes e_j$ and $\Omega e_i = 2 \omega^j_i \otimes e_j$, where $\{\omega^i\}$ and $\{\omega^i_j\}$ are called respectively, the curvature forms and connection forms of $\nabla$ with respect to $\{e_i\}$. By definition $\rho = \omega^i \otimes e_i$ and $\mu := \nabla F \ell = F \omega^{n+i} \otimes e_i$, where $\omega^{n+i} := \omega^i + d (\log F) \delta_i^n$. It is easy to show that $\{\omega^i, \omega^{n+i}\}_{i=1}^n$ is a local basis for $T^*(TM_0)$. In a natural coordinate, we can expand connection forms $\omega^i_j$ as follows

\[\omega^i_j := \Gamma^j_{ik} \, dx^k + F^j_{ik} \, dy^k,\]

where $\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial t^k} = \Gamma^k_{ij} \partial_k$ and $\nabla_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial t^k} = F^k_{ij} \partial_k$. In the rest of the paper, we suppose that all connections satisfy $F^k_{ij} y^i = F^k_{ij} y^j = 0$.

Let $\{e_i, \bar{e}_i\}_{i=1}^n$ be the local basis for $T(TM_0)$, which is dual to $\{\omega^i, \omega^{n+i}\}_{i=1}^n$, i.e., $\bar{e}_i \in HTM, e_i \in VTM$ such that $\rho(\bar{e}_i) = e_i, \mu(\bar{e}_i) = \mu(e_i)$. Then equations (1) and (2) are equivalent to

\begin{align}
d\omega^j - \omega^j \wedge \omega^i &= \frac{1}{2} S^j_{kl} \omega^k \wedge \omega^l + T^j_{kl} \omega^k \wedge \omega^{n+l}, \\
d\omega^i_j - \omega^i_k \wedge \omega^j_k &= \Omega^i_j,
\end{align}

where $T^i_{kl} := \omega^j(T(\bar{e}_i, \bar{e}_k))$ and $S^i_{kl} := \omega^j(T(\bar{e}_k, \bar{e}_l))$. Since the $\Omega^i_j$ are 2-forms on $TM_0$, they can be expanded as

\[\Omega^i_j = \frac{1}{2} R^i_{jkl} \omega^k \wedge \omega^l + P^i_{jkl} \omega^k \wedge \omega^{n+l} + \frac{1}{2} Q^i_{jkl} \omega^k \wedge \omega^{n+l}.
\]

The objects $R$, $P$ and $Q$ are called, respectively, the hh-, hv- and vv-curvature tensors of $\nabla$ with the components $R(\bar{e}_k, \bar{e}_l)e_i = R^j_{kl} e_j$, $P(\bar{e}_k, \bar{e}_l)e_i = P^j_{kl} e_j$ and $Q(\bar{e}_k, \bar{e}_l)e_i = Q^j_{kl} e_j$. By (5), we have $R^j_{kl} = -R^j_{lk}$ and $Q^j_{lk} = -Q^j_{kl}$. 

5
3 Matsumoto’s $C$ and $L$-processes

Matsumoto introduces two processes in connection theory that by them, one can construct the Berwald, Hashiguchi and Chern connections from Cartan connection [9]. The space of all connections makes an affine space modeled on the space of $(1, 2)$-tensors over pulled-back bundle $\pi^*TM$. It means that adding a $(1, 2)$-tensor to a connection makes a new connection. A Finsler metric $F$ gives us two natural $(1, 2)$-tensors with components $C^i_{jk} (=g^{il}C_{ljk})$ and $L^i_{jk} (=g^{il}L_{ljk})$. These two $(1, 2)$-tensors play key role in Matsumoto’s processes, and in what we call Shen’s processes, here. The $C$-processes use Cartan tensor, and the $L$-processes use Landsberg tensor.

Let $(M, F)$ be a Finsler manifold. Suppose that $\nabla$ is a connection with connection forms $\omega^i_{jk}$. We define

$$\tilde{\omega}^i_{jk} := \omega^i_{jk} - C^i_{jk} \omega^n + k.$$  \hspace{1cm} (6)

Then $\tilde{\omega}^i_{jk}$ are connection forms of a connection $\tilde{\nabla}$, that is called the connection obtained from $\nabla$ by Matsumoto’s $C$-process. Similarly, we define

$$\tilde{\omega}^i_{jk} := \omega^i_{jk} + L^i_{jk} \omega^k.$$  \hspace{1cm} (7)

Then $\tilde{\omega}^i_{jk}$ are connection forms of a connection $\tilde{\nabla}$, that is called the connection obtained from $\nabla$ by Matsumoto’s $L$-process. Chern and Hashiguchi connections are obtained from Cartan connection by Matsumoto’s $C$-process, and Matsumoto’s $L$-process, respectively.

3.1 Proof of Theorem 1.1

First, we recall the following well-known result from [8].

Lemma 3.1. Let $(M, F)$ be a Finsler manifold and the Cartan tensor satisfies $C_{ijk} = B_i h_{jk} + B_j h_{ik} + B_k h_{ij}$ such that $y^i B_i = 0$. Then $F$ is a Randers metric.

To prove the Theorem 1.1 we need the following.

Proposition 3.2. Let $(M, F)$ be a generalized Landsberg space. Suppose $c(t)$ is a geodesic. Put $C(t) = C_c(U(t), V(t), W(t))$ where $U(t), V(t)$ and $W(t)$ are the parallel vector fields along $c$. Then following equation holds.

$$C(t) = L(0)t + C(0).$$  \hspace{1cm} (8)

Proof. Let $p$ be an arbitrary point of $M$, $y, u, v, w \in T_p M$ and $c: (-\infty, \infty) \to M$ is the unit speed geodesic passing from $p$ and $\frac{dc}{dt}(0) = y$. If $U(t), V(t)$ and $W(t)$ are the parallel vector fields along $c$ with $U(0) = u, V(0) = v$ and $W(0) = w$, we put

$$L(t) = L_c(U(t), V(t), W(t)).$$
By definition of Landsberg curvature, we have
\[ \mathbf{L}(t) = \mathbf{C}'(t). \] (9)

Let
\[ \bar{\mathbf{L}}(t) = \bar{\mathbf{L}}_e(U(t), V(t), W(t)). \] (10)

From the definition of \( \bar{\mathbf{L}}_e \), we have
\[ \bar{\mathbf{L}}(t) = \mathbf{L}'(t). \] (11)

Since \( F \) is generalized Landsberg metric, then we have
\[ \mathbf{L}'(t) = 0, \] (12)
which implies that \( \mathbf{L}(t) = \mathbf{L}(0) \). By (9), we get the proof. \( \square \)

**Proof of Theorem 1.1** Let \( \bar{\nabla} \) be obtained from \( \nabla \) by Matsumoto’s \( L \)-process
\[ \bar{\omega}_j^i = \omega_j^i + L^i_{jk}\omega^k. \]

Taking an exterior differential of the above relation, yields
\[ d\bar{\omega}_j^i = d\omega_j^i + dL^i_{jk} \wedge \omega_k + L^i_{jk}d\omega^k. \] (13)

On the other hand, we know that
\[ dL^i_{jk} + L^s_{jk}\omega^i_s - L^i_{sk}\omega_j^s - L^i_{js}\omega^s_k = L^i_{jk|s}\omega^s + L^i_{jk,s}\omega^{n+s}, \] (14)
where “\( | \)” and “\( . \)” denote the horizontal and vertical derivative with respect to \( \nabla \). Using (3), (4), (13) and (14), we have
\[ \bar{\Omega}_j^i = \Omega_j^i + \left( L^i_{jk|s}\omega^s + L^i_{jk,s}\omega^{n+s} - L^i_{sk}\omega_j^s + L^i_{js}\omega^s_k + L^i_{js}\omega^k_s \right) \wedge \omega^k \]
\[ - L^i_{jk|u}\left( \frac{1}{2}S^u_{kl}\omega^l + T^u_{kl}(\omega^{n+1}) \wedge \omega^k \right) + L^i_{jk}\omega^s \wedge \omega^k \]
\[ - (\omega_j^k + L^k_{ju}\omega^u) \wedge (\omega^j_i + L^j_{km}\omega^m) + \omega_j^k \wedge \omega^i. \] (15)

Replacing (5) in (15) yields
\[ \bar{R}_j^i_{kl} = R_j^i_{kl} - (L^i_{jk|l} - L^i_{jl|k}) - (L^m_{jk}L^i_{ml} - L^m_{jl}L^i_{mk}) + L^i_{ju}S^u_{kl}, \] (16)
\[ \bar{P}_j^i_{kl} = P_j^i_{kl} - L^i_{jk,l} + L^i_{ju}T^u_{kl}, \] (17)
\[ \bar{Q}_j^i_{kl} = Q_j^i_{kl}. \] (18)

Immediately, we have the proof of part 1. It results that if \( \bar{\nabla} \) is obtained from \( \nabla \) by Matsumoto’s \( L \)-process, then \( \nabla \) is torsion-free if and only if \( \bar{\nabla} \) is torsion-free.

**Proof of part 2.** Let \( \bar{R} = R \). By (16) we have
\[ L^i_{jk|l} = L^i_{jl|k} - L^m_{jk}L^i_{ml} + L^m_{jl}L^i_{mk} + L^i_{ju}S^u_{kl}. \] (19)
Regularity of $\nabla$ results that $y^l |_{y^k} = 0$. Therefore, by contracting with $y^l$, we get $L^i_{jk} |_{y^l} = 0$. By our assumption on connections in this paper, we see that $L^i_{jk} |_{y^l} = L^i_{jk,l} y^l$. Hence $F$ is a generalized Landsberg metric.

Now, suppose that $M$ is a compact manifold. By Proposition 3.2, we have the following

$$C(t) = L(0)t + C(0).$$

Since $M$ is compact then the Cartan tensor is bounded. Using $||C|| < \infty$, and letting $t \to +\infty$ or $t \to -\infty$, we get $L(0) = L(u,v,w) = 0$. It means that $F$ is a Landsberg metric.

**Proof of part 3.** From [14], for Finsler manifolds of scalar flag curvature we have

$$L_{ijk|m} y^m = \frac{-F^2}{3} \{K_i h_{jk} + K_j h_{ik} + K_k h_{ij} + 3KC_{ijk}\}.$$

By part 2, $F$ is a generalized Landsberg metric. Then we get

$$C_{ijk} = \frac{-1}{3R} \{K_i h_{jk} + K_j h_{ik} + K_k h_{ij}\}.$$

By Lemma 3.1, it results that $F$ is a $C$-reducible metric, and by Proposition 2.1 $F$ is a Randers metric.

**Proof of part 4.** Suppose that $\tilde{P} = P$. By (17) we have $L^i_{jk,l} = L^i_{ju} T^u_{kl}$. Contracting with $y^k$ yields $L^i_{jk} = 0$, since Landsberg tensor is positively homogeneous of degree zero and $T^u_{kl} y^k = 0$. This completes the proof.

**Corollary 3.3.** Let $\tilde{\nabla}$ be obtained from $\nabla$ by Matsumoto’s $L$-process. Then the hv-curvature of them under $L$-process is invariant if and only if $\nabla$ coincides with $\tilde{\nabla}$.

It is obvious that any Landsberg metric is generalized Landsberg metric but the converse is still an open problem. Following corollary throws a light into this problem.

**Corollary 3.4.** Let $\tilde{\nabla}$ be obtained from $\nabla$ by Matsumoto’s $L$-process. Suppose that their Riemannian curvature coincide. If their hv-curvature are not equal, then $F$ is a generalized Landsberg metric which is not Landsbergian.

Now, we consider Matsumoto’s $C$-process. By the same argument and technique used in the proof of Theorem 1.1 one can obtain the following theorem.

**Theorem 3.5.** Let $(M,F)$ be a Finsler manifold. Suppose $\nabla$ and $\tilde{\nabla}$ are two connections on $M$. Suppose $\tilde{\nabla}$ is obtained from $\nabla$ by Matsumoto’s $C$-process. Then we have the following

$$\tilde{R}^i_{jk} = R^i_{jk} - C^i_{ju} R^u_{nk},$$

$$\tilde{P}^i_{jk} = P^i_{jk} - C^i_{jul} - C^i_{ju} P^u_{nk},$$

$$\tilde{Q}^i_{jk} = Q^i_{jk} + (C^i_{jul} - C^i_{jul}) + (C^u_{ju} C^i_{uk} - C^i_{juk} C^u_{ul}) - C^u_{ju} Q^u_{nk}.$$
3.2 Matsumoto’s $L$-process on Randers Manifolds

An $(\alpha, \beta)$-metric is a scalar function on $TM$ defined by

$$F := \phi(\frac{\beta}{\alpha})\alpha, \quad s = \frac{\beta}{\alpha},$$

where $\phi = \phi(s)$ is a $C^\infty$ on $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on a manifold $M$. Randers metrics are special $(\alpha, \beta)$-metrics which are closely related to Riemannian metrics defined by $\phi = 1 + s$, i.e., $F = \alpha + \beta$ and have important applications both in mathematics and physics [12].

In this section, we study the Matsumoto’s $L$-process on Randers manifolds equipped with connections whose h-h-torsion vanish. We show that the Riemannian curvature of these connections is invariant under Matsumoto’s $L$-process on a Randers manifold $(M, F)$, if and only if $F$ is a Berwald metric. To prove this result, we need the following.

**Lemma 3.6.** Let $(M, F)$ be a Finsler manifold and $\nabla$ be a connection on $M$ satisfying $g_{ijk} = 0$. Suppose that $\tilde{\nabla}$ is obtained from $\nabla$ by Matsumoto’s $L$-process. Then $R = \tilde{R}$ if and only if the following equations hold

$$L_{iik}L^s_{jl} - L_{isl}L^s_{jk} = 0, \quad (23)$$
$$L_{ijkl} - L_{ijkl} = 0. \quad (24)$$

**Proof.** Fix $k$ and $l$ and put

$$Q_{ij} := L_{ijkl} - L_{ijkl} + L_{isk}L^s_{jl} - L_{isl}L^s_{jk}.$$ 

One can write

$$Q_{ij} := Q^s_{ij} + Q^a_{ij},$$

where

$$Q^s_{ij} := \frac{1}{2}(Q_{ij} + Q_{ji}), \quad \text{and} \quad Q^a_{ij} := \frac{1}{2}(Q_{ij} - Q_{ji}).$$

It is easy to see that $Q_{ij} = 0$ if and only if $Q^s_{ij} = 0$ and $Q^a_{ij} = 0$. On the other hand, we have

$$Q_{ji} = L_{ijll} - L_{ijkl} + L_{jik}L^s_{il} - L_{isl}L^s_{ik}$$
$$= L_{ijll} - L_{ijkl} + L^s_{jk}L_{isel} - L^s_{jl}L_{sik}.$$ 

Hence

$$Q^F_{ij} = L_{ijll} - L_{ijkl},$$

and consequently

$$Q^a_{ij} = L_{isk}L^s_{jl} - L_{isl}L^s_{jk}.$$ 

This proves the Lemma. \qed

Now we are ready to prove the mentioned fact.
Theorem 3.7. Let $F = \alpha + \beta$ be a Randers metric on a manifold $M$ of dimensional $n \geq 3$. Suppose that $\nabla$ has vanishing hh-torsion and $g_{ij}^{\|k} = 0$. Let $\tilde{\nabla}$ be obtained from $\nabla$ by Matsumoto’s $L$-process. Then their hh-curvatures are the same if and only if $F$ is a Berwald metric.

Proof. Using the assumptions $S_{kl}^i = 0$ and $R = \tilde{R}$ in (16) imply that

$$L^i_{jk} - L^i_{jkl} + L^i_{sk}L^s_{jl} - L^i_{sij}L^s_{jk} = 0.$$  \tag{25}

Using the assumption $g_{ij}^{\|k} = 0$, and lowering indices by $g_{ij}$ imply that (25) is equivalent to the following

$$L^i_{jk} - L^i_{jkl} + L^i_{sk}L^s_{jl} - L^i_{sij}L^s_{jk} = 0.$$  \tag{26}

By Lemma 3.6, we have

$$L^i_{sk}L^s_{jl} - L^i_{sij}L^s_{jk} = 0$$  \tag{27}

$$L^i_{ijl} - L^i_{ijkl} = 0.$$  \tag{28}

A direct computation yields

$$h^i_J J_s = J_i,$$  \tag{29}

$$h^i_J h_{js} = h_{ij},$$  \tag{30}

$$g^{ij}h_{ij} = n - 1.$$  \tag{31}

Since $F$ is a Randers metric, then it is C-reducible, i.e.,

$$C_{ijk} = \frac{1}{1 + n} \{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\}.$$  \tag{32}

Taking a horizontal covariant derivative from above relation, we get

$$L^i_{ijk} = \frac{1}{1 + n} \{h_{ij}J_k + h_{jk}J_i + h_{ki}J_j\}.$$  \tag{33}

Substituting (33) into (27), one can obtain

$$\{h_{ij}h_{ki} - h_{jk}h_{li}\}J^s J_s + \{h_{ij}J_k - h_{jk}J_i\}J_i + \{h_{ki}J_l - h_{li}J_k\}J_l = 0.$$  \tag{34}

Contracting (34) with $g^{il}g^{jk}$ and using the relations (29), (30) and (31), we conclude that

$$(n + 1)(n - 2)J_s J_s = 0.$$  \tag{35}

Since $F$ is positive definite and $n > 2$, then we have

$$J_s = 0.$$  \tag{36}

By (33) and (36) we conclude that $F$ is a Landsberg metric. It is proved that $F = \alpha + \beta$ is a Landsberg metric if and only if $F$ is a Berwald metric [8]. This completes the proof.
4 Shen’s $C$ and $L$-processes

Recently, Shen introduced a new torsion-free and almost metric-compatible connection and proved that hv-curvature of his connection vanishes if and only if the Finsler structure is Riemannian [13]. However, the hv-curvature tensor of the Berwald, Cartan, Hashiguchi or the Chern connections does not characterize Riemannian structures. Shen connection can not be constructed by Matsumoto’s processes from Cartan or Chern connection. Therefore it is natural to find a kind of process on Chern connection which yields the Shen connection. This problem leads us to find two new processes which we call them the Shen’s $C$ and $L$-processes.

Let $(M,F)$ be a Finsler manifold. Suppose that $\nabla$ is a connection with connection forms $\omega^{ij}$. We define

$$\tilde{\omega}^{ij} := \omega^{ij} - C^{ij}_{jk}\omega^{k}.$$  \hspace{1cm} (37)

Then $\tilde{\omega}^{ij}$ are connection forms of a connection $\tilde{\nabla}$, that is called the connection obtained from $\nabla$ by Shen’s $C$-process. Similarly, we can define

$$\tilde{\omega}^{ij} := \omega^{ij} - L^{ij}_{jk}\omega^{n+k}.$$ \hspace{1cm} (38)

Then $\tilde{\omega}^{ij}$ are connection forms of a connection $\tilde{\nabla}$, that is called the connection obtained from $\nabla$ by Shen’s $L$-process.

**Theorem 4.1.** Shen connection is obtained from the Chern connection by Shen’s $C$-process.

**4.1 Proof of Theorem 4.1**

Let $\tilde{\nabla}$ be obtained from $\nabla$ by Shen’s $C$-process. Taking exterior differential from (37) yields

$$d\tilde{\omega}^{ij} = d\omega^{ij} - dC^{ij}_{jk} \wedge \omega^{k}. \hspace{1cm} (39)$$

On the other hand we have

$$dC^{ij}_{jk} + C^{s}_{jk}\omega^{i}_{s} - C^{i}_{sk}\omega^{s}_{j} - C^{i}_{js}\omega^{s}_{k} = C^{i}_{jk}|s\omega^{s} + C^{i}_{jk,s}\omega^{n+s}, \hspace{1cm} (40)$$

where “|” and “.” denote the horizontal and vertical derivative with respect to $\nabla$. Substituting (40) into (39), and using (3) and (4) we get

$$\tilde{\Omega}^{i}_{j} = \Omega^{i}_{j} - (C^{i}_{jk}|s\omega^{s} + C^{i}_{jk,s}\omega^{n+s} - C^{s}_{jk}\omega^{i}_{s} + C^{i}_{sk}\omega^{s}_{j} + C^{i}_{js}\omega^{s}_{k}) \wedge \omega^{k}$$

$$- (\omega^{k}_{j} - C^{k}_{jm}\omega^{m}) \wedge (\omega^{i}_{k} - C^{i}_{ki}\omega^{l}) + \omega^{k}_{j} \wedge \omega^{i}_{k} - C^{i}_{jk}\omega^{s} \wedge \omega^{s}_{k}$$

$$+ C^{i}_{jn}(\frac{1}{2}S^{u}_{kl}\omega^{l} + T^{u}_{kl}\omega^{n+l}) \wedge \omega^{k}. \hspace{1cm} (41)$$
Now by decomposing $\tilde{\Omega}^i_j$ and $\Omega^i_j$ as in (5), one can obtain
\begin{equation}
\tilde{R}^i_{j kl} = R^i_{j kl} + (C^m_{jk}C^i_{ml} - C^m_{jl}C^i_{mk}) + (C^i_{jk|l} - C^i_{j||l}k) - C^i_{ju}S^u_{kl}, \tag{42}
\end{equation}
\begin{equation}
P^i_{j kl} = P^i_{j kl} - C^i_{jk,l} - C^i_{jm}T^m_{kl}, \tag{43}
\end{equation}
\begin{equation}
\tilde{Q}^i_{j kl} = Q^i_{j kl}. \tag{44}
\end{equation}

Proof of part 1. Suppose $R = \tilde{R}$. Then from (42) we have
\begin{equation}
C^m_{jk}C^i_{ml} - C^m_{jl}C^i_{mk} = C^i_{jk|l} - C^i_{j||l}k - C^i_{ju}S^u_{kl}. \tag{45}
\end{equation}
Contracting with $y^l$ yields $L^i_{jk} = 0$. It means that $F$ is a Landsberg metric.

Proof of part 2. Suppose that $P = \tilde{P}$. Then from (43) we conclude that
\begin{equation}
C^i_{jk,l} + C^i_{jm}T^m_{kl} = 0. \tag{46}
\end{equation}
Using the positively homogeneities of Cartan tensor, and contracting (46) with $y^l$ yield $C^i_{jk} = 0$. Therefore, by Deicke’s theorem $F$ is Riemannian.

Finally from (44), we see that their vv-curvatures are the same and $\nabla$ is torsion-free if and only if $\tilde{\nabla}$ is torsion-free.

We have some kind of rigidity on Shen’s C-process.

**Corollary 4.2.** Let $\tilde{\nabla}$ be obtained from $\nabla$ by Shen’s C-process. Then the hv-curvature is invariant under Shen’s C-process if and only if $\nabla = \tilde{\nabla}$.

In continue, we study the Shen’s L-process and get the following result.

**Theorem 4.3.** Let $(M, F)$ be a Finsler manifold. Suppose that $\nabla$ and $\tilde{\nabla}$ be two connections on $M$ and $\nabla$ is obtained from $\tilde{\nabla}$ by Shen’s L-process. Then we have the following
\begin{equation}
\tilde{R}^i_{j kl} = R^i_{j kl} - L^i_{ju}R^u_{jk}, \tag{47}
\end{equation}
\begin{equation}
P^i_{j kl} = P^i_{j kl} - L^i_{jl|k} - L^i_{ju}P^u_{jk}, \tag{48}
\end{equation}
\begin{equation}
\tilde{Q}^i_{j kl} = Q^i_{j kl} + (L^i_{jk,l} - L^i_{jl,k}) + (L^u_{jL^i_{uk}} - L^u_{jL^i_{uk}}L^i_{ul}) - L^i_{ju}Q^u_{jk}. \tag{49}
\end{equation}

**Corollary 4.4.** If torsion-free connection $\nabla$ on the Finsler manifold $(M, F)$ remains torsion-free under the Shen’s L-process, then $F$ is a Landsberg metric. Hence, Shen’s L-process acts on the set of all torsion-free connections identically.

**Corollary 4.5.** Let $\nabla$ be obtained from $\tilde{\nabla}$ by Shen’s L-process and $\tilde{\nabla}$ is not torsion-free. If their hv-curvature are equal to zero, then $F$ is a generalized Landsberg metric such that is not Landsbergian.
4.2 Shen’s C-process on Berwald Connection

By Theorem 4.1 applying Shen’s C-process on Chern connection gives Shen connection. It is natural to study effect of Shen’s C-process on the other well-known connections. Here, we study Shen’s C-process on Berwald connection.

**Theorem 4.6.** Let \((M, F)\) be a Finsler manifold. Suppose that \(\nabla\) is the Berwald connection on \(M\) and \(\tilde{\nabla}\) is obtained from \(\nabla\) by Shen’s C-process. Then the hv-curvature of \(\tilde{\nabla}\) vanishes if and only if \(F\) is Riemannian.

**Proof.** The structure equation of \(\tilde{\nabla}\) is given by
\[
d\omega^i = \omega^j \land \omega_j^i, \tag{50}
\]
\[
dg_{ij} = g_{kij}\omega^k_i + g_{ikj}\omega^k_j + 2\{A_{ijk} - L_{ijk}\}\omega^k + 2A_{ijk}\omega^{n+k}, \tag{51}
\]
where \(A_{ijk} = FC_{ijk}\). Differentiating (51) and using (43), (50) and (51) lead to
\[
g_{kij}\Omega^k_i + g_{ikj}\Omega^k_j = -2A_{ijk}\Omega^k_n - 2A_{ijk,l}\omega^k \land \omega^l + 2A_{ijk,l}\omega^{n+k} \land \omega^{n+l} - 2\{A_{ijk,l} - A_{ijk|i}\}\omega^k \land \omega^{n+l} + (L_{ijk|i}\omega^l + L_{ijk,l}\omega^{n+l}) \land \omega^k. \tag{52}
\]
Using (5), yields
\[
R_{ijkl} + R_{jikl} = -2A_{ij}sR_s^{n}klt, \tag{53}
\]
\[
P_{ijkl} + P_{jikl} = -2L_{ijkl} + 2\{A_{ijkl} - A_{ijl|k}\} - 2A_{ij}sP_s^{n}tkl, \tag{54}
\]
\[
A_{ijkl} = A_{ijl,k}. \tag{55}
\]
Permuting \(i, j, k\) in (51) yields
\[
P_{ijkl} = -L_{ijkl} + A_{ijkl} - (A_{ijl|k} + A_{jkl|i} - A_{kil|j}) + A_{kis}P_s^{n}jl - A_{jks}P_s^{n}ik - A_{iks}P_s^{n}kl. \tag{56}
\]
Multiplying (56) with \(y^i\) and using \(P_{njml} = 0\) yield
\[
P_{nijkl} = -A_{ijkl}. \tag{57}
\]
By (57) we get the proof.

4.3 Shen’s C-process on Cartan Connection

Here, we study effect of Shen’s C-process on the Cartan connection.

**Theorem 4.7.** Let \((M, F)\) be a Finsler manifold. Suppose that \(\nabla\) is the Cartan connection on \(M\) and \(\tilde{\nabla}\) is obtained from \(\nabla\) by Shen’s C-process. Then we get

1. If hh-curvature of \(\tilde{\nabla}\) vanishes then \(F\) is a Landsberg metric.

2. The hv-curvature of \(\tilde{\nabla}\) vanishes if and only if \(F\) is Riemannian.
Proof. The structure equation of $\widetilde{\nabla}$ is given by

$$d\omega^i = \omega^j \wedge \omega_j^i - A_{ijkl} \omega^k \wedge \omega^{n+l},$$

(58)

$$dg_{ij} = g_{kj} \omega_i^k + g_{ik} \omega_j^k + 2A_{ijk} \omega^k.$$  

(59)

Differentiating (59) and using (4), (58) and (59) leads to

$$g_{kj} \Omega_i^k + g_{ik} \Omega_j^k = -2(A_{ijk}s \omega^s + 2A_{ijks} \omega^{n+s}) \wedge \omega^k - 2A_{iks} A_{s kl} \omega^k \wedge \omega^{n+l}.$$  

(60)

Using (5), yields

$$R_{ijkl} + R_{jikl} = 2(A_{ijkl} - A_{ijlk}),$$  

(61)

$$P_{ijkl} + P_{jikl} = 2(A_{ijkl} - A_{ijls} A_{s kl}),$$  

(62)

$$Q_{ijkl} + Q_{jikl} = 0.$$  

(63)

If the hh-curvature of $\widetilde{\nabla}$ vanishes, then by (61) we have $A_{ijkl} = A_{ijlk}$ which implies that $F$ is a Landsberg metric.

Now let the hv-curvature of $\widetilde{\nabla}$ vanishes. By (62), we get

$$A_{ijkl} = A_{ijls} A_{s kl}$$  

(64)

Contracting with $y^l$ yields that $F$ is Riemannian. □

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