ON THE DIVISION PROBLEM FOR THE WAVE MAPS EQUATION

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Abstract. We consider Wave Maps into the sphere and give a new proof of small data global well-posedness and scattering in the critical Besov space, in any space dimension $n \geq 2$. We use an adapted version of the atomic space $U^2$ as the single building block for the iteration space. Our approach to the so-called division problem is modular as it systematically uses two ingredients: atomic bilinear (adjoint) Fourier restriction estimates and an algebra property of the iteration space, both of which can be adapted to other phase functions.

1. Introduction

Let $(\mathbb{R}^{1+n}, \eta)$ be the Minkowski space-time with metric $(\eta_{\alpha\beta}) = \text{diag}(-1, 1, \ldots, 1)$ and $M$ be a smooth manifold with Riemannian metric $g$. Formally, a wave map is a map $\phi : \mathbb{R}^{1+n} \to M$ which is a critical point of the Lagrangian

$$\mathcal{L}(\phi) = \frac{1}{2} \int_{\mathbb{R}^{1+n}} \langle \partial^\alpha \phi, \partial_\alpha \phi \rangle_g dt dx.$$

Space-time coordinates are denoted by $(t, x)$, we use the standard summation convention, raise indices according to $\partial^\alpha = \eta^{\alpha\beta} \partial_\beta$, and write $\Box = -\partial^\alpha \partial_\alpha = \partial^2_t - \Delta$ for the d’Alembertian. In the extrinsic formulation, assuming that $M$ is a submanifold of some Euclidean space $\mathbb{R}^m$, a wave map is a solution $\phi : \mathbb{R}^{1+n} \to M \subset \mathbb{R}^m$ to

$$\Box \phi = -S(\phi)(\partial^\alpha \phi, \partial_\alpha \phi), \quad (1.1)$$

where $S(p) : T_p M \times T_p M \to (T_p M)^\perp$ is the second fundamental form at $p \in M$. For the purposes of this paper, the important point is that the Wave Maps equation (1.1) takes the form of a nonlinear wave equation with null structure, more specifically

$$\Box \phi = \phi(|\nabla \phi|^2 - |\partial_t \phi|^2) \quad (1.2)$$

in the case of the target manifold $M = S^2 \subset \mathbb{R}^3$. We remark that for classical (smooth) solutions to the equation (1.2) one can drop the target constraint because if the initial data $(\phi(0), \partial_t \phi(0)) : \mathbb{R}^n \to \mathbb{R}^3$ satisfy $|\phi(0)| = 1$ and $\partial_t \phi(0) \cdot \phi(0) = 1$, one can prove $|\phi(t)| = 1$ for all $t$.

Solutions can be rescaled according to $\phi(t, x) \to \phi(\lambda t, \lambda x)$. Therefore $\dot{H}^{\frac{n}{2}}(\mathbb{R}^n)$ is the critical Sobolev regularity for global well-posedness, which barely fails to control the $L^\infty$-norm. Wave Maps conserve the energy

$$E(\phi) = \frac{1}{2} \int_{\mathbb{R}^n} |\partial_t \phi|^2 + |\nabla \phi|^2 dx,$$
therefore the space dimension \( n = 2 \) is the energy-critical dimension. It turned out that, even in the case of small initial data, the Cauchy problem is challenging to solve in the critical Sobolev space, in particular in low space dimensions \( n = 2, 3 \).

For instance, the problem cannot be solved iteratively in Fourier restriction norms only \[13\]. In the (smaller) critical Besov space \( \dot{B}^{\frac{1}{2}}_{2,1}(\mathbb{R}^n) \) the breakthrough global well-posedness and scattering result in dimension \( n = 2, 3 \) was obtained by Tataru \[31\].

The example of Nirenberg \[11, \text{p. } 45\] shows that the scalar model problem \( \Box u = \partial^\alpha u \partial_\alpha u \) is ill-posed in \( H^{\frac{1}{2}}(\mathbb{R}^n) \). In the critical Sobolev space the Wave Maps problem exhibits a quasilinear behaviour and a renormalization is necessary. In the case \( M = S^2 \) this was solved by Tao \[29\], later by Krieger \[18\] for the hyperbolic plane target, and for more general targets by Klainerman-Rodnianski \[12\] and Tataru \[33\].

Building on the small data results, Sterbenz-Tataru \[27, 26\] and Krieger-Schlag \[19\] solved the global regularity problem in the energy-critical dimension \( n = 2 \) for initial data below the threshold given by nontrivial harmonic maps of lowest energy (if any). We refer the reader to \[25, 30, 32, 16, 9\] for more comprehensive introductions to various aspects of the theory of Wave Maps and further references.

In this paper, we revisit the problem of iteratively solving the Cauchy problem associated with (1.2) in the critical Besov space for small data, which was solved first in \[31\]. This problem is also known as the \textit{division problem}. Citing \[32, \text{p. } 195\] (see also \[17\]), the name stems “from the fact that in Fourier space the parametrix \( \Box^{-1} \) for the wave equation is essentially the division by” the symbol \(|\xi|^2 - \tau^2\) of the d’Alembertian, which fails to be locally integrable. In essence, it consists in constructing a function space \( S^{\frac{n}{2}} \) with the properties

\[
S^{\frac{n}{2}} \cdot S^{\frac{n}{2}} \rightarrow S^{\frac{n}{2}} \text{ and } \Box^{-1}(S^{\frac{n}{2}} \cdot \Box S^{\frac{n}{2}}) \rightarrow S^{\frac{n}{2}},
\]

see Theorem 2.2 below for a precise statement. The division problem arises on the level of the Littlewood-Paley pieces and we do not address the \textit{summation problem}, which is the second ingredient for a proof of global well-posedness in the critical Sobolev space and was solved first in \[29\]. We emphasize that a solution of the division problem is crucial for all later developments on Wave Maps mentioned above and the original construction \[31\] has been successfully used, adapted and refined in related problems, such as \[6, 3, 4, 20, 22\], among others. Further, the division problem is universal in the sense that it crucially arises in many other nonlinear dispersive evolution equations at the critical regularity or for global-in-time problems, such as for Schrödinger maps \[5\].

One of the key difficulties in the solution of the division problem originates in the fact that, even for solutions \( u \) on the unit frequency scale, it is impossible to obtain global-in-time control \( \Box u \) in \( L^1_t L^2_x \). Instead, in \[31\] Tataru introduces characteristic (or null) coordinate frames \( (t_\Theta, x_\Theta) \), for unit vectors \( \Theta \) on the cone, and \( t_\Theta = \Theta \cdot (t, x) \). If \( u \) is Fourier localized in a transversal direction, it is possible to control \( \Box u \) in \( L^1_{t_\Theta} L^{\infty}_{x_\Theta} \). Then, by an involved construction of an atomic function space in addition to the standard Fourier restriction space he succeeds in proving the requires estimates alluded to above, which rest on certain bilinear estimates in \( L^2_{t,x} \).

Due to its complexity we do not describe the solution to the division problem of \[31\] in more detail here but refer to \[31\] instead. In a first version of \[31\] Tataru took a different route, based on the space of functions of bounded \( p \)-variation \( V^p \).
and its predual $U^q$ [24]. In this construction, the space for solutions is an atomic space, where the atoms are normalized step-functions and each step solves the homogeneous wave equation. However, as pointed out by Nakanishi, there was a serious problem in the proof of the crucial bilinear $L^2_{t,x}$ estimates. Instead, Tataru abandoned the approach via $U^p$ and $V^p$ and developed the null frame spaces instead. The null frame space construction is custom-made for the application to the Wave Maps problem, for which it proved very successful, but the functional analysis is delicate and adaptations to closely related problems require new ideas [5, 4].

Around the same time as [31] there have been significant advances on the Fourier restriction problem for the cone. The key fact is that by passing to the bilinear setting it is possible to use both curvature and transversality properties of the cone. Indeed, in dimension $n \geq 2$, Wolff [35] proved that for every $p > p_n := \frac{n+3}{n+1}$, the bilinear (adjoint) Fourier restriction estimate

$$\|e^{-i|\nabla|}\langle f e^{-i|\nabla|} g \rangle\|_{L^p_t L^p_x (\mathbb{R}^{1+n})} \lesssim \|f\|_{L^2_t L^2_x} \|g\|_{L^2_t L^2_x},$$

holds true, provided that the Fourier-supports of $f$ and $g$ are angularly separated and contained in the unit annulus. Shortly after, Tao [28] proved this estimate in the endpoint case $p = p_n$.

In the present paper, we prove Tataru’s original conjecture to be true: it is possible to use $U^2$ as the only building block in the solution of division problem by using recent advances on the bilinear (adjoint) Fourier restriction estimates as the key new ingredient. Specifically, for $u : \mathbb{R}^{1+n} \to \mathbb{R}^3$, we consider the model problem

$$\Box u = u(|\nabla u|^2 - |\partial_t u|^2)$$

$$(u, \partial_t u)(0) = (f, g)$$

for small initial data in $(f, g) \in \dot{B}^{\frac{3}{2}}_{2,1}(\mathbb{R}^n) \times \dot{B}^{\frac{3}{2}}_{2,1}(\mathbb{R}^n)$, for $n \geq 2$, and provide a new proof of global well-posedness and scattering.

While we present our approach with a focus on this specific problem, we emphasize that it is modular. It is straight-forward to adapt the function space $U^2$ to any linear propagator. By the standard resonance analysis of (1.3), the mapping properties (1.4) can be reduced to two independent building blocks, both of which are new. Firstly, if one of the factors has Fourier support far from the characteristic set, then the required estimates are consequences of bilinear Fourier multiplier estimates of the following toy estimates:

$$\|fg\|_{V^2} \lesssim \|f\|_{L^\infty_t L^2_x} \|g\|_{V^2}, \quad \|fg\|_{U^2} \lesssim \|f\|_{L^\infty_t L^2_x} \|g\|_{U^2}$$

for $g$ with high temporal frequency, see Subsection 6.2. Secondly, if all factors have Fourier support close to the characteristic set, we exploit atomic versions of bilinear (adjoint) Fourier restriction estimates, which are available for general phases under transversality and curvature conditions, see Section 7 and [7]. To put things into perspective, let us mention that since the spaces $U^p$ and $V^p$ have been introduced in the PDE context in [14], the theory of these spaces has been developed in [10, 16, 15], among others. In parallel, since the seminal works of Wolff [35] and Tao [28], there have been advances in the theory of bilinear Fourier restriction estimates by Bejenaru [2], Lee-Vargas [21], the first named author [7], among others. Here, we are able to connect these two lines of research and provide a systematic and modular solution of the division problem.
The paper is organized as follows. After introducing some notation, we describe the solution to the division problem in Section 2, see Theorem 2.2. Also, we introduce the function spaces and provide a proof of small data global well-posedness and scattering for the Wave Maps equation. In Section 3 we establish properties of the iteration space, prove product estimates in far cone regions and the bilinear $L^2$ estimates, and give a proof of Theorem 2.2. Section 4 is devoted to the basic properties of the critical function spaces $U^p$ and $V^p$, such as embedding properties, almost orthogonality and duality statements. In Section 5 we provide characterizations of $U^p$ which are crucial for applications to PDE. In Section 6 we establish results concerning convolution and multiplication in $U^p$ and $V^p$ and introduce the adapted function spaces. Finally, in Section 7 we prove the bilinear restriction estimate in adapted function spaces which is used in Section 3.

1.1. Notation. Let $S(\mathbb{R}^n)$ the Schwartz space and $S'(\mathbb{R}^n)$ the space of tempered distributions. Given a function $f \in L^2(\mathbb{R}^n)$, we let $\hat{f}(\xi)$ denote the Fourier transform, and if $u \in L^2_{t,x}(\mathbb{R}^{1+n})$, we let $\hat{u}(\tau, \xi)$ denote the space-time Fourier transform.

Let $P_d$ denote a smooth (spatial) cutoff to the Fourier region $|\xi| \approx \lambda$. Similarly, we take $P_d^{(t)}$ denote the (temporal) Fourier projection to the set $|\tau| \approx d$. The Fourier multipliers $P_{\leq \lambda}$ and $P_{\leq d}^{(t)}$ are defined similarly to restrict to spatial frequencies $|\xi| \lesssim \lambda$, and temporal frequencies $|\tau| \lesssim d$ respectively. We often use the shorthand $P_{\lambda} u = u_{\lambda}$. Let $C_d$ and $C_d^{(t)}$ restrict to the space-time Fourier regions $||\tau| - |\xi|| \approx d$ and $|\tau \pm |\xi|| \approx d$ respectively. Note that we may write $C_d^{(t)} = e^{\pm \tau} e^{\pm \xi}$.

Define $C_\alpha$ to be a finitely overlapping collection of caps $\kappa \subset S^{-1}_{n-1}$ of radius $\alpha$ which cover the sphere, and take $\angle(\xi, \eta)$ denote the angle between $\xi, \eta \in \mathbb{R}^n \setminus \{0\}$. Let $Q_\mu$ be a collection of finitely overlapping cubes of diameter $\mu$ which form a cover of $\mathbb{R}^n$. We denote the corresponding Fourier cutoffs to caps $\kappa \in C_\alpha$ and cubes $q \in Q_\mu$ as $R_\kappa$ and $P_q$ respectively.

For $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$ and $f \in S'(\mathbb{R}^n)$, let

$$\|f\|_{\dot{B}_{p,q}^s} = \left( \sum_{\lambda \in 2^N} \lambda^{sq} \|f_\lambda\|_{L^q(\mathbb{R}^n)}^q \right)^{\frac{1}{q}},$$

with the obvious modification for $q = \infty$. Further, let $\dot{B}_{p,q}^s$ denote the space of all $f \in S'(\mathbb{R}^n)$ satisfying

$$f = \sum_{\lambda \in 2^N} f_\lambda \text{ in } S'(\mathbb{R}^n) \text{ and } \|f\|_{\dot{B}_{p,q}^s} < +\infty.$$  

It is well-known (see [1]) that, if either $s < n/p$ or $s = n/p$ and $q = 1$, then $\dot{B}_{p,q}^s$ is a Banach space. We have the continuous embedding $\dot{B}_{2,1}^s \subset C_0(\mathbb{R}^n)$.

2. A SOLUTION TO THE DIVISION PROBLEM

2.1. Definition of the spaces $U^p$ and $V^p$. Let

$$P = \{\tau = (t_j)_{j=1}^N \mid N \in \mathbb{N}, \ t_j \in \mathbb{R}, \ t_j < t_{j+1}\}$$

be the set of partitions, i.e. finite increasing sequences. For a partition $\tau = (t_j)_{j=1,...,N}$, let

$$I_\tau = \{[t_1, t_2), \ldots, [t_{N-1}, t_N), [t_N, \infty)\},$$

be the set of intervals.
i.e. left-closed disjoint intervals associated with $\tau$. Let $1 \leq p < \infty$. We say that $u$ is a $U^p$-atom if there exists $\tau \in \mathcal{P}$ such that $u(t) = \sum_{I \in \mathcal{I}_x} 1_I(t) f_I$ is a step function satisfying

$$\left( \sum_{I \in \mathcal{I}_x} ||f_I||_p^p \right)^{\frac{1}{p}} = 1.$$ 

The atomic space $U^p$ is then defined to be

$$U^p = \left\{ \sum_{j \in \mathbb{N}} c_j u_j \big| (c_j) \in \ell^1, \ u_j \text{ is a } U^p \text{ atom } \right\},$$

with the induced norm

$$\|u\|_{U^p} = \inf_{u = \sum_{j \in \mathbb{N}} c_j u_j} \sum_{j \in \mathbb{N}} |c_j|.$$ 

Functions in $u \in U^p$ are bounded, have one-sided limits everywhere and are right-continuous with $\lim_{t \to -\infty} u(t) = 0$ in $L^2(\mathbb{R}^n)$.

Closely related to the atomic spaces $U^p$, are the $V^p$ spaces of finite $p$-variation. Given a function $v : \mathbb{R} \to L^2(\mathbb{R}^n)$, we define

$$|v|_{V^p} = \sup_{(t_j)_{j=1}^N} \left( \sum_{j=1}^{N-1} \|v(t_{j+1}) - v(t_j)\|_{L^2}^p \right)^{\frac{1}{p}}.$$ 

and $\|v\|_{V^p} = \|v\|_{L^p L^2} + |v|_{V^p}$. If $|v|_{V^p} < \infty$, then $v$ has one-sided limits everywhere including $\pm \infty$. Let $\hat{V}^p$ be the space of all right-continuous functions $v$ such that $|v|_{V^p} < \infty$ and $\lim_{t \to -\infty} v(t) = 0$ in $L^2(\mathbb{R}^n)$. Then, $\|v\|_{V^p} \leq 2|v|_{V^p}$ for all $v \in \hat{V}^p$.

The spaces $V^p$ were introduced by Wiener [34] while a discrete version of the atomic $U^p$ spaces appeared in work of Pisier-Xu [24]. In the context of PDE, the spaces $U^p$ and $V^p$ were used in unpublished work of Tataru as a replacement for the endpoint $X^{\frac{a}{d}, \frac{a}{d}}$ type spaces, and developed in more detail by Koch-Tataru [14]. We leave a more complete discussion of the spaces $U^p$ and $V^p$ till Section 4 below, further properties can also be found in [10] [14] [15].

2.2. The solution space. The solution space $S^{\frac{a}{d}}$ for the Wave Maps equation is based on an adapted version of the atomic space $U^2$. More precisely, we define $U^2_{\pm} = e^{\pm i|\nabla|} U^2$ with the obvious norm

$$\|u\|_{U^2_{\pm}} = ||e^{\pm i|\nabla|} u||_{U^2}.$$ 

Elements of $U^2_{\pm}$ should be thought of as being close to solutions to the linear half-wave equation. In fact, atoms in $U^2_{\pm}$ are piecewise solutions to $(-i\partial_t \pm |\nabla|) u = 0$. Let $S$ be the collection of all $u \in C_b(\mathbb{R}; L^2_x)$ with $|\nabla|^{-1} \partial_t u \in C_b(\mathbb{R}; L^2_x)$ such that $u \pm i|\nabla|^{-1} \partial_t u \in U^2_{\pm}$ with the norm

$$\|u\|_S = \|u + i|\nabla|^{-1} \partial_t u\|_{U^2_{\pm}} + \|u - i|\nabla|^{-1} \partial_t u\|_{U^2_{\pm}}.$$ 

The space $S$ will contain the frequency localised pieces of the wave map $u$. To define the full solution space $S^{\frac{a}{d}}$ we take

$$S^{\frac{a}{d}} = \{ u \in C_b(\mathbb{R}, \hat{B}^{\frac{a}{d}}_{2,1}) \mid P\lambda u \in S \}$$

with the norm

$$\|u\|_{S^{\frac{a}{d}}} = \sum_{\lambda \in 2^\mathbb{Z}} \lambda^{\frac{s}{d}} \|P\lambda u\|_S.$$
The space $S^\pm$ is a Banach space, this is an immediate consequence of the fact that the subspace of continuous functions in $U^2_{\pm}$ is closed. Since we may write

$$u = \frac{1}{2}(u + i|\nabla|^{-1}\partial_t u) + \frac{1}{2}(u - i|\nabla|^{-1}\partial_t u),$$
$$\partial_t u = \frac{1}{2i}|\nabla|(u + i|\nabla|^{-1}\partial_t u) - \frac{1}{2i}|\nabla|(u - i|\nabla|^{-1}\partial_t u)$$

(2.1)

it is clear that we have bounds

$$\|u_\lambda\|_{L^\infty_t L^2_x} + \|\partial_t u_\lambda\|_{L^\infty_t H^{-1}} \lesssim \|u_\lambda\|_S$$

and

$$\|u\|_{L^\infty_t B^\pm_{2,1}} + \|\partial_t u\|_{L^\infty_t B^\pm_{2,1}} \lesssim \|u\|_{S^\pm}.$$

Let

$$V(t)(f, g) = \cos(t|\nabla|)f + \frac{\sin(t|\nabla|)}{|\nabla|}g$$

be the propagator for the homogeneous wave equation. Further, let $\chi(t) \in C^\infty$ with $\chi(t) = 1$ for $t \geq 0$, and $\chi(t) = 0$ for $t < -1$.

**Lemma 2.1.** For all $f_\lambda, g_\lambda \in L^2$, we have $\chi(t|\nabla|)V(t)(f_\lambda, g_\lambda) \in S$ and

$$\|\chi(t|\nabla|)V(t)(f_\lambda, g_\lambda)\|_S \lesssim \|f_\lambda\|_{L^2} + \lambda^{-1}\|g_\lambda\|_{L^2}.$$  

(2.2)

**Proof.** Let $v = V(t)(f_\lambda, g_\lambda)$ and $w = \chi(t|\nabla|)V(t)(f_\lambda, g_\lambda)$. Then,

$$v \pm i|\nabla|^{-1}\partial_t v = e^{\mp t|\nabla|}(f_\lambda \pm i|\nabla|^{-1}g_\lambda),$$

and therefore

$$w \pm i|\nabla|^{-1}\partial_t w = \chi(t|\nabla|)e^{\mp t|\nabla|}(f_\lambda \pm i|\nabla|^{-1}g_\lambda) \pm i\chi'(t|\nabla|)v.$$  

Due to (4.1) we have

$$\|\chi(t|\nabla|)e^{\mp t|\nabla|}(f_\lambda \pm i|\nabla|^{-1}g_\lambda)\|_{U^2_\pm} = \|\chi(t|\nabla|)(f_\lambda \pm i|\nabla|^{-1}g_\lambda)\|_{L^2} \lesssim \|\chi'(t|\xi|)|\xi|(\hat{f}_\lambda \pm i|\xi|^{-1}\hat{g}_\lambda)\|_{L^1_t L^2_\xi} \lesssim \|f_\lambda\|_{L^2} + \lambda^{-1}\|g_\lambda\|_{L^2},$$

similarly,

$$\|\chi'(t|\nabla|)v_\lambda\|_{U^2_\pm} = \|\chi'(t|\nabla|)e^{\pm t|\nabla|}v_\lambda\|_{L^2} \lesssim \|\partial_t (\chi'(t|\xi|)e^{\pm t|\xi|}\hat{\chi}_\xi)\|_{L^1_t L^2_\xi} \lesssim \|\partial_t (\chi'(t|\xi|)e^{\pm t|\xi|}|\xi|\hat{\chi}_\xi) + \frac{\sin(t|\xi|)}{|\xi|}\hat{\chi}_\xi|\xi|\|_{L^1_t L^2_\xi} \lesssim \|f_\lambda\|_{L^2} + \lambda^{-1}\|g_\lambda\|_{L^2},$$

and therefore $w \in S$ with the required bound. \hfill \square

We prove that the space $S^\pm$ is a solution to the division problem. More precisely, if we define

$$\square^{-1} F(t) = 1_{[0, \infty)}(t) \int_0^t \frac{\sin((t-s)|\nabla|)}{|\nabla|} F(s) ds$$

to be the solution to the wave equation $\square u = F$ on $[0, \infty) \times \mathbb{R}^n$ with vanishing data at $t = 0$, then we have the following.
Theorem 2.2. Let $\lambda_0, \lambda_1, \lambda_2 \in 2^\mathbb{Z}$. If $u_{\lambda_1}, v_{\lambda_2} \in S$, then $P_{\lambda_0}(u_{\lambda_1}, v_{\lambda_2}) \in S$ and
\[
\lambda_0^2 \| P_{\lambda_0}(u_{\lambda_1}, v_{\lambda_2}) \|_S \lesssim (\lambda_1 \lambda_2) \| u \|_S \| v \|_S. \tag{2.3}
\]
Moreover, if in addition we have $\Box v_{\lambda_2} \in L_{t,loc}^1 L_x^2$, then $\Box^{-1} P_{\lambda_0}(u_{\lambda_1}, v_{\lambda_2}) \in S$ and
\[
\lambda_0^2 \| \Box^{-1} P_{\lambda_0}(u_{\lambda_1}, v_{\lambda_2}) \|_S \lesssim (\lambda_1 \lambda_2)^2 \| u \|_S \| v \|_S. \tag{2.4}
\]

Remark 2.3. The estimates can be easily summed up. However, a summation problem arises when one aims at solving the Wave Maps equation in the critical Sobolev space $\dot{H}^{\frac{n}{2}} \times \dot{H}^{\frac{n}{2} - 1}$. The $\ell^1$ summation over frequencies has to be replaced with an $\ell^2$ sum, which would require obtaining the stronger bounds and a renormalization. Such estimates are known in the null frame based solution space, see [29, equations (21), (27)] and [30, equation (4.3)] (the proof can be found directly following (129) in [29]). We do not pursue this issue here.

2.3. Small data GWP for the Wave Maps equation. Let $Q_0(u, v) = -\partial^\alpha u \cdot \partial^\alpha v = \partial_h u \cdot \partial_v v - \partial_{\bar{z}} u \cdot \partial_{\bar{v}} v$.

It is well-known that this is a null form, i.e. it satisfies the identity
\[
2Q_0(u, v) = \Box (uv) - \Box uv - u \Box v. \tag{2.5}
\]

Then, (1.2) can be written as
\[
\Box u = u Q_0(u, u).
\]

Theorem 2.2 together with (2.5) and a standard fixed point argument can be used to construct a solution to the Wave Maps equation (1.2) in the space $S^{\frac{n}{2}}$ provided that the initial data $(f, g)$ are sufficiently small in $\dot{B}^{\frac{n}{2}}_{2,1}(\mathbb{R}^n) \times \dot{B}^{\frac{n}{2} - 1}_{2,1}(\mathbb{R}^n)$. In more detail, set
\[
S^{\frac{n}{2}}_0 = \{ u \in S^{\frac{n}{2}} : \forall \lambda \in 2^\mathbb{Z} : \Box u_{\lambda} \in L_{t,loc}^1 L_x^2(\mathbb{R}^{1+n}) \}.
\]

The purpose of this subset of $S^{\frac{n}{2}}$ is to ensure that all nonlinear expressions are a-priori well-defined. Define the map $\mathcal{T} : S^{\frac{n}{2}}_0 \to S^{\frac{n}{2}}_0$ by
\[
\mathcal{T}[u](t) = \chi(t|\nabla|) V(t)(f, g) + \Box^{-1}(u Q_0(u, u)).
\]

Theorem 2.2 and summation implies that
\[
\| \Box^{-1}(u Q_0(v, w)) \|_{S^{\frac{n}{2}}} \lesssim \| \Box^{-1}(uvw) \|_{S^{\frac{n}{2}}} + \| \Box^{-1}(uvw) \|_{S^{\frac{n}{2}}} + \| u Q_0(u, w) \|_{S^{\frac{n}{2}}} + \| \Box^{-1}(uwv) \|_{S^{\frac{n}{2}}} \lesssim \| u \|_{S^{\frac{n}{2}}} \| v \|_{S^{\frac{n}{2}}} \| w \|_{S^{\frac{n}{2}}},
\]

where we have used (2.5) in the first, (2.4) in the second and (2.3) in the third inequality. By Lemma 2.1 and summation we obtain
\[
\| \mathcal{T}[u] \|_{S^{\frac{n}{2}}} \lesssim \| (f, g) \|_{\dot{B}^{\frac{n}{2}}_{2,1} \times \dot{B}^{\frac{n}{2} - 1}_{2,1}} + \| u \|_{S^{\frac{n}{2}}}^3,
\]
\[
\| \mathcal{T}[u] - \mathcal{T}[v] \|_{S^{\frac{n}{2}}} \lesssim \left( \| u \|_{S^{\frac{n}{2}}}^2 + \| v \|_{S^{\frac{n}{2}}}^2 \right) \| u - v \|_{S^{\frac{n}{2}}}.
\]

Therefore, $\mathcal{T}$ is a contraction in a small ball in $S^{\frac{n}{2}}_0$. This implies the existence of a fixed point in $S^{\frac{n}{2}}$ as the latter space is complete. Then, the restriction of the fixed point to the interval $[0, \infty)$ is a generalized solution of (1.2). Clearly, if in addition the initial data are $C^\infty$, we obtain a classical solution to (1.4). Scattering
is an immediate consequence of (2.1) and the existence of one-sided limits in $U^2$. Indeed, it implies the existence of

$$\lim_{t \to \infty} e^{\pm it|\nabla|} (u(t) \pm i|\nabla|^{-1} \partial_t u(t)) =: f_\pm \in \dot{B}_{2,1}^\beta(R^n).$$

Now,

$$f_\infty := \frac{1}{2}(f_+ + f_-) \in \dot{B}_{2,1}^\beta(R^n)$$

and

$$g_\infty := \frac{1}{2}|\nabla|(f_- - f_+)^{-} \in \dot{B}_{2,1}^{-\beta}(R^n)$$

satisfy

$$\lim_{t \to \infty} \|u(t) - V(t)(f_\infty, g_\infty)\|_{L_t^\infty \dot{B}_{2,1}^\beta} + \lim_{t \to \infty} \|\partial_t u(t) - \partial_t V(t)(f_\infty, g_\infty)\|_{L_t^\infty \dot{B}_{2,1}^{-\beta}} = 0.$$

In fact, the result is slightly stronger. For instance, the embedding $U^2 \subset V^2$ implies that the quadratic variation is finite. The analogous results on $(-\infty, 0]$ follow from time reversibility.

3. Multilinear estimates

3.1. Properties of the iteration space $S$. In this subsection we describe the main properties of the space $S$ which are needed for the solution of the division problem. The results collected here are for the most part consequences of properties of the $U^p$ and $V^p$ spaces. To aid the reader, we state (and for the most part give a proof of) these properties in Sections [1] [5] and [5].

We start by defining a weak version of the space $S$, which is based on $V^2$ rather than $U^2$. Let $S_w$ denote the collection of all right continuous functions $v$ such that

$$\|v\|_{S_w} = \|v\|_{V^2 + V^2} = \inf_{v = v^+ + v^-} \left( \|v^+\|_{V^2} + \|v^-\|_{V^2} \right) < \infty$$

where we define $\|\phi\|_{V^2} = \|e^{\pm it|\nabla|}\phi\|_{V^2}$. It is clear that

$$\|u\|_{S_w} + \|\nabla|^{-1} \partial_t u\|_{S_w} \leq \|u + i|\nabla|^{-1} \partial_t u\|_{V^2} + \|u - i|\nabla|^{-1} \partial_t u\|_{V^2} \lesssim \|u\|_S,$$  (3.1)

and

$$\|u\|_{U^2 + U^2} + \|\nabla|^{-1} \partial_t u\|_{U^2 + U^2} \lesssim \|u\|_S.$$  (3.2)

Thus the norm on $S_w$ is weaker than that on $S$ and $S$ is slightly stronger than just have taking $u, |\nabla|^{-1} \partial_t u \in U^2 + U^2$. However, for space-time frequencies localised to a fixed dyadic distance from the cone, these spaces are all closely related, see Part (ii) of Lemma 3.3 below. In particular, in some sense the differences in these spaces only appear when considering frequency regions of the form $\{||\tau| - |\xi|| \lesssim \mu\}$.

The space $S$ satisfies the following properties.

Lemma 3.1.

(i) (Dual pairing) Let $\psi \in S$ and $\phi \in S_w$ with $|\nabla|^{-1} \Delta \phi, \partial_t \phi, |\nabla| \phi \in L^1_tL^2_x$.

Then

$$\left| \int_R \langle |\nabla|^{-1} \Delta \phi, \psi \rangle_{L^2} dt \right| \lesssim \|\phi\|_{S_w} \|\psi\|_S.$$

(ii) ($X^{0,\frac{1}{2}\infty}$ control) Let $d, \lambda \in 2\mathbb{Z}$. Then

$$\|C_d u \lambda\|_{L^2_t L^\infty_x} \approx d^{-\frac{1}{2}} \frac{\lambda}{d + \lambda} \|C_d u \lambda\|_S \approx d^{-\frac{1}{2}} \|C_d u \lambda\|_{S_w}.$$
(iii) (Square sum bounds) Let $\epsilon > 0$. For any $0 < \alpha \leq 1$ and $\lambda \geq \mu > 0$ we have
\[
\left( \sum_{q \in \mathcal{Q}_\mu} \| P_q u_\lambda \|_S^2 \right)^{1/2} \lesssim \| u_\lambda \|_S, \quad \left( \sum_{\kappa \in \mathcal{C}_\mu} \| R_\kappa u_\lambda \|_S^2 \right)^{1/2} \lesssim \| u_\lambda \|_S.
\]

(iv) (Uniform disposability) For any $d \in 2^\mathbb{Z}$ we have
\[
\| Ca v \|_S + \| C_{\leq d} v \|_S \lesssim \| u \|_S, \quad \| C_d v \|_{S_w} + \| C_{\leq d} v \|_{S_w} \lesssim \| v \|_{S_w}.
\]

Proof. We start with the proof of (i), which is a consequence of an approximation argument, together with the $U^2$ and $V^2$ version
\[
\left| \int_R \langle \partial_t v, u \rangle_{L^2} dt \right| \lesssim \| v \|_{V^2} \| u \|_{U^2} \tag{3.3}
\]
which holds provided that $\partial_t v \in L^1_t L^2_x$, see Theorem 4.4 below. The definition of the space $S_w$, implies that we can write $\phi = \phi_+ + \phi_-$ with $\phi_\pm \in V^2_x$. However we have a slight difficulty as the functions $\phi_\pm$ do not necessarily inherit the smoothness or integrability properties of $\phi$. To address this problem, we define the phase space localisation operator $P_{\leq d} \phi = P_{\leq d}^t P_{\leq d} (\rho_d \phi)$ where $\rho \in C_0^\infty$ with $\rho(t) = 1$ on $|t| \leq 1$, and we take $\rho_d(t) = \rho(t/2^d)$. The assumptions on $\phi$ imply that $\| (|\nabla|^{-1} \langle 1 - P_{\leq d} \phi \rangle \|_{L^1_t L^2_x} \to 0$ as $d \to \infty$. Thus it suffices to bound the dual pairing with $\phi$ replaced with $P_{\leq d} \phi$. To this end, as $P_{\leq d} \phi_\pm$ is now smooth and integrable, we have
\[
\left| \int_R \langle |\nabla|^{-1} \langle 1 - P_{\leq d} \phi_\pm \rangle (\psi) \rangle_{L^2} dt \right| = \left| \int_R \langle (-i\partial_t \pm |\nabla|) P_{\leq d} \phi_\pm, \psi \pm i|\nabla|^{-1} \partial_t \psi \rangle_{L^2} dt \right|
\]
\[
= \left| \int_R \langle \partial_t (e^{\pm it|\nabla|} P_{\leq d} \phi_\pm), e^{\pm i|\nabla|} (\psi \pm i|\nabla|^{-1} \partial_t \psi) \rangle_{L^2} dt \right|
\]
\[
\leq \| P_{\leq d} \phi_\pm \|_{V^2_x} \| \psi \pm i|\nabla|^{-1} \partial_t \psi \|_{U^2_x}
\]
where we applied (3.3). If we now observe that
\[
e^{\pm it|\nabla|} P_{\leq d}^t (\phi(t)) = \int_R d^{-1} \chi(\frac{s}{d}) e^{\pm is|\nabla|} e^{\pm i(t-s)|\nabla|} \phi(t-s) ds
\]
for some $\chi \in L^1$ with $\| \chi \|_{L^1(\mathbb{R})} \leq 1$, a short computation gives the bound
\[
\| P_{\leq d} \phi \|_{V^2_x} \lesssim \| \phi \|_{V^2_x}.
\]
Consequently (ii) follows.

To prove (iii), we first observe that $\| |\tau| - |\xi| \| \lesssim |\tau \pm |\xi| \|$. Now, if $C_d v_\lambda = v_+ + v_-$, from Theorem 6.2 and (6.6)
\[
\| C_d v_\lambda \|_{L^2_t x} \lesssim \| C^+_{\leq d} v_+ \|_{L^2_t x} + \| C^-_{\leq d} v_- \|_{L^2_t x} \lesssim d^{-\frac{1}{2}} (\| v_+ \|_{V^2_x} + \| v_- \|_{V^2_x}),
\]
proving $\| C_d u_\lambda \|_{L^2_t x} \lesssim d^{-\frac{1}{2}} \| C_d u_\lambda \|_{S_w}$. If $d \lesssim \lambda$, this also implies
\[
\| C_d u_\lambda \|_{L^2_t x} \lesssim d^{-\frac{1}{2} - \frac{\lambda}{d + \lambda}} \| C_d u_\lambda \|_S.
\]
On the other hand, to obtain the high modulation gain in the region $d \gg \lambda$, we use the above together with the fact that the $S$ norm controls the time derivative, to see that
\[ \| C_d u_\lambda \|_{L_{t,x}^2} \approx \frac{\lambda}{d} \| C_d \partial_t |\nabla|^{-1} u_\lambda \|_{L_{t,x}^2} \lesssim d^{-\frac{1}{2}} \frac{\lambda}{d + \lambda} \| \partial_t |\nabla|^{-1} C_d u_\lambda \|_{S_w} \]
\[ \lesssim d^{-\frac{1}{2}} \frac{\lambda}{d + \lambda} \| C_d u_\lambda \|_{S} \]

where we used (3.1). This completes the proof of all inequalities in (ii). For the converse inequalities, let \( P_{\pm}^{(t)} \) denote the temporal Fourier multiplier with symbol \( \mathbb{1}_{(\pm \tau \geq 0)}(\tau) \). Then the identity \( C_d u_\lambda = C_{\geq d}^+ P_{d}^{(t)} C_d u_\lambda + C_{\geq d}^+ P_d^{(t)} C_d u_\lambda \) implies that

\[ \| C_d (u_\lambda + i |\nabla|^{-1} \partial_t u_\lambda) \|_{L_{t,x}^2} \]
\[ \lesssim \| C_d^+ P_+^{(t)} C_d (u_\lambda + i |\nabla|^{-1} \partial_t u_\lambda) \|_{L_{t,x}^2} + \| C_{d+\lambda}^+ P_+^{(t)} C_d (u_\lambda + i |\nabla|^{-1} \partial_t u_\lambda) \|_{L_{t,x}^2} \]
\[ \lesssim d^{\frac{1}{2}} \| C_d^+ P_+^{(t)} C_d (u_\lambda + i |\nabla|^{-1} \partial_t u_\lambda) \|_{L_{t,x}^2} \]
\[ + (d + \lambda)^{\frac{1}{2}} \| C_{d+\lambda}^+ P_+^{(t)} C_d (u_\lambda + i |\nabla|^{-1} \partial_t u_\lambda) \|_{L_{t,x}^2} \]
\[ \lesssim d^{\frac{1}{2}} \frac{d + \lambda}{\lambda} \| C_d u_\lambda \|_{L_{t,x}^2} + (d + \lambda)^{\frac{1}{2}} \frac{d}{\lambda} \| C_d u_\lambda \|_{L_{t,x}^2} \approx d^{\frac{1}{2}} \frac{d + \lambda}{\lambda} \| C_d u_\lambda \|_{L_{t,x}^2}. \]

Since an identical argument gives the \( U_{d}^2 \) version, we conclude that

\[ \| C_d u_\lambda \|_{L_{t,x}^2} \gtrsim d^{-\frac{1}{2}} \frac{\lambda}{d + \lambda} \| C_d u_\lambda \|_{S}. \]

For the \( S_w \) version, we observe that by definition

\[ \| C_d u_\lambda \|_{S_w} \lesssim \| C_d P_{d}^{(t)} v_\lambda \|_{V_{d}^2} + \| C_d^+ P_d^{(t)} v_\lambda \|_{V_{d}^2} \]
\[ \lesssim \| C_{\geq d}^+ C_d P_{d}^{(t)} v_\lambda \|_{V_{d}^2} + \| C_{\geq d}^+ C_d P_d^{(t)} v_\lambda \|_{V_{d}^2} \lesssim d^{\frac{1}{2}} \| C_d v_\lambda \|_{L_{t,x}^2} \]

as required.

To prove (iii), the square sum bounds, we observe that the \( S \) case follows immediately from the square sum control in \( U^2 \), namely Proposition 4.3 below.

For \( \lambda \), it suffices to show that

\[ \| C_d \phi_\lambda \|_{U_{d}^2} + \| C_{\geq d} \phi_\lambda \|_{U_{d}^2} \lesssim \| \phi_\lambda \|_{U_{d}^2}, \quad \| C_d \phi_\lambda \|_{V_{d}^2} + \| C_{\geq d} \phi_\lambda \|_{V_{d}^2} \lesssim \| \phi_\lambda \|_{V_{d}^2}. \]

After writing \( C_{\geq d}^{\pm} = e^{-i|\nabla|} P_{\geq d}^{\pm} e^{i|\nabla|} \) and \( C_d^{\pm} = e^{-i|\nabla|} P_{d}^{(t)} e^{i|\nabla|} \) where \( P_{\geq d}^{(t)} \) and \( P_d^{(t)} \) are smooth temporal cutoffs to \( |\tau| \leq d \) and \( |\tau| \approx d \), respectively, the fact that convolution with \( L_{t,x}^1 \) kernels is bounded in \( U^2 \) and \( V^2 \) (see Lemma 6.1 below), implies that

\[ \| C_d \phi_\lambda \|_{U_{d}^2} + \| C_{\geq d} \phi_\lambda \|_{U_{d}^2} \lesssim \| \phi_\lambda \|_{U_{d}^2}, \quad \| C_d \phi_\lambda \|_{V_{d}^2} + \| C_{\geq d} \phi_\lambda \|_{V_{d}^2} \lesssim \| \phi_\lambda \|_{V_{d}^2}. \] (3.4)

Thus our goal is to replace \( C_d^{\pm} \) in (3.4) with \( C_d \). We only prove the \( U_{d}^2 \) case, as the \( V_{d}^2 \) case is similar. For the \( C_d \) multipliers, we observe that after decomposing

\[ C_d \phi_\lambda = C_{\geq d} C_d \phi_\lambda + (1 - C_{\geq d}) C_d \phi_\lambda = C_{\geq d} C_d \phi_\lambda + C_{\geq \lambda} (1 - C_{\geq d}) C_d \phi_\lambda \]

the standard Besov embedding in Theorem 6.2 gives

\[ \| C_d \phi_\lambda \|_{U_{d}^2} \lesssim d^{\frac{1}{2}} \| C_{\geq d} C_d \phi_\lambda \|_{L_{t,x}^2} + \lambda^{\frac{1}{2}} \| C_{\geq \lambda} (1 - C_{\geq d}) C_d \phi_\lambda \|_{L_{t,x}^2} \]
\[ \lesssim \sup_{d'} (d')^{\frac{1}{2}} \| C_{d'} \phi_\lambda \|_{L_{t,x}^2} \lesssim \| \phi_\lambda \|_{U_{d}^2}. \]
On the other hand, for the $C_{<d}$ multipliers, we first write $C_{<d} = C^+_{<d} + C^+_{\geq d}$. The first term is immediate by (3.4). For the second, we note that

$$C^+_{\geq d} C_{<d} \phi_\lambda = C^+_{\geq d} C_{<d} \phi_\lambda + C^+_{= d} C_{<d} \phi_\lambda = C^+_{= d} C_{<d} \phi_\lambda + C^+_{\geq d} C^+_{= d} C_{<d} \phi_\lambda$$

and apply the reasoning used in the $C_d$ case. □

For later use, we note that (ii) in Lemma 3.1 implies that for any $d \lesssim \lambda$ we have the bounds

$$\|\Box C_d \psi_\lambda \|_{L^2_{t,x}} + \|\Box C_{d} \phi_\lambda \|_{L^2_{t,x}} \lesssim d^2 \lambda \|\phi_\lambda\|_S.$$  (3.5)

The norm $\|\cdot\|_S$ also controls the Strichartz type spaces $L^p_t L^q_x$. In fact, as the $S$ norm is based on $U^2$, essentially any estimate for the free wave equation which involves $L^p_t$ with $p \geq 2$ implies a corresponding bound for functions in $S$. However, somewhat surprisingly, we make no use of Strichartz estimates in the proof of Theorem 2.2, and instead rely on bilinear $L^2_t L^4_x$ estimates together with the $X^{s,b}$ type bound (3.5).

The space $S$ is constructed using the atomic space $U^2$. Although this definition is convenient for proving properties of functions in $S$, it is a challenging problem to determine precisely when a general function $u \in C_0(R, L^2_x)$ belongs to $S$. This problem is closely related to the difficult question of characterising elements of $U^p$. We also give a slightly different statement of the $U^p$ characterisation result in Theorem 5.1 as well as a self-contained proof closely following that given by Koch-Tataru [13]. Restated in terms of the solution space $S$, the conclusion is the following.

**Theorem 3.2 (Characterisation of $S$).** Let $(\psi, \partial_t \psi) \in C_0(R; L^2_x \times \dot{H}^{-1}_x)$ with $\|\psi(t), \partial_t \psi(t)\|_{L^2_t \times \dot{H}^{-1}_x} \to 0$ as $t \to -\infty$. If

$$\sup_{\phi \in C_0^\infty} \left| \int_R \langle \Box \phi, |\nabla|^{-1} \psi \rangle_{L^2_x} dt \right| < \infty$$

then $\psi \in S$ and

$$\|\psi\|_S \lesssim \sup_{\phi \in C_0^\infty} \left| \int_R \langle \Box \phi, |\nabla|^{-1} \psi \rangle_{L^2_x} dt \right|.$$ 

**Proof.** Theorem 5.1 implies that if $u \in L^\infty_t L^2_x$ with $\|u(t)\|_{L^2_x} \to 0$ as $t \to -\infty$, and

$$\sup_{v \in C_0^\infty(R; L^2_x)} \left| \int_R \langle \partial_t v, \psi \rangle_{L^2_x} dt \right| < \infty,$$

then $u \in U^2$. To translate this statement to $S$, we first observe that as in the proof of (3) in Lemma 5.1, we have

$$\sup_{v \in C_0^\infty} \left| \int_R \langle \partial_t v, e^{\mp it|\nabla|} (\psi \pm i |\nabla| v) \rangle_{L^2_x} dt \right| = \sup_{v \in C_0^\infty} \left| \int_R \langle e^{\mp it|\nabla|} v, |\nabla|^{-1} \psi \rangle_{L^2_x} dt \right|$$

$$\leq \sup_{\phi \in C_0^\infty} \left| \int_R \langle \Box \phi, |\nabla|^{-1} \psi \rangle_{L^2_x} dt \right|.$$ 

Consequently, since $\|\psi \pm i |\nabla|^{-1} \partial_t \psi\|_{L^2} \to 0$ as $t \to -\infty$, we conclude from the characterisation of $U^2$ that $\psi \pm i |\nabla|^{-1} \partial_t \psi \in U^2$ and moreover that the claimed bound holds. □
Recall that we have defined the inhomogeneous solution operator $\Box^{-1}F$ as

$$\Box^{-1}F = 1_{[0,\infty)}(t) \int_0^t \left| \nabla \right|^{-1} \sin \left( (t-s) \left| \nabla \right| \right) F(s) \, ds.$$ 

Applying Theorem 3.2 to the special case $\psi = \Box^{-1}F$ gives the following.

**Corollary 3.3 (Energy Inequality).** Let $F \in L^1_{t,\text{loc}} \dot{H}^{-1}$ with

$$\sup_{\phi \in C_0^\infty} \left| \int_0^\infty \langle \phi, |\nabla|^{-1}F \rangle_{L^2_t} \, dt \right| < \infty.$$ 

Then $\Box^{-1}F \in S$ and

$$\|\Box^{-1}F\|_S \lesssim \sup_{\phi \in C_0^\infty} \left| \int_0^\infty \langle \phi, |\nabla|^{-1}F \rangle_{L^2_t} \, dt \right|.$$ 

**Proof.** We would like to apply Theorem 3.2 to $\Box^{-1}F$. We start by observing that the definition of $\Box^{-1}F$ and fact that $F \in L^1_{t,\text{loc}} \dot{H}^{-1}$ implies that for any $\phi \in C_0^\infty$ we have

$$\left| \int_\mathbb{R} \langle \phi, |\nabla|^{-1} \Box^{-1}F \rangle_{L^2_t} \, dt \right| = \left| \int_0^\infty \langle \phi, |\nabla|^{-1}F \rangle_{L^2_t} \, dt \right|$$

as well as $\Box^{-1}F, |\nabla|^{-1} \partial_t (\Box^{-1}F) \in C(\mathbb{R}, L^2)$ and $\Box^{-1}F(t) = |\nabla|^{-1} \partial_t (\Box^{-1}F)(t) = 0$ for $t < 0$. In view of Theorem 3.2 the required conclusion would follow provided that $\|\Box^{-1}F \pm i |\nabla|^{-1} \partial_t (\Box^{-1}F)\|_{L_t^r L^s_x} < \infty$. To this end, we observe that

$$\|\Box^{-1}F \pm i |\nabla|^{-1} \partial_t \Box^{-1}F\|_{L_t^r L^s_x}$$

$$= \left\| 1_{[0,\infty)}(t) \int_0^t e^{i(t-s)\left| \nabla \right|} |\nabla|^{-1}F(s) \, ds \right\|_{L_t^r L^s_x}$$

$$= \sup_{\phi \in C_0^\infty} \left| \int_0^\infty \left( \int_0^t e^{i(t-s)\left| \nabla \right|} |\nabla|^{-1}F(s) \, ds \right) dt \right|$$

$$= \sup_{\phi \in C_0^\infty, \|\phi\|_{L_t^r L^s_x} \leq 1} \left| \int_0^\infty \left( \int_s^\infty e^{i(s-t)\left| \nabla \right|} \phi(t) \, dt \right) \left| \nabla \right|^{-1}F(s) \, ds \right|.$$ 

If we let $\phi_\pm(s) = e^{\pm i|\nabla|} \int_s^\infty e^{i(t-s)|\nabla|} \phi(t) \, dt$, then the definition of the $V^2_\pm$ norm gives

$$\|\phi_\pm\|_{V^2_\pm} \lesssim \|\phi\|_{L_t^1 L^2_x} \leq 1$$

and hence we conclude that

$$\|\Box^{-1}F \pm i |\nabla|^{-1} \partial_t \Box^{-1}F\|_{L_t^r L^s_x} \lesssim \sup_{\phi \in C_0^\infty, \|\phi\|_{V^2_\pm} \leq 1} \left| \int_0^\infty \langle \phi, |\nabla|^{-1}F \rangle_{L^2_t} \, dt \right|$$

$$\lesssim \sup_{\phi \in C_0^\infty, \|\phi\|_{S_u} \leq 1} \left| \int_0^\infty \langle \phi, |\nabla|^{-1}F \rangle_{L^2_t} \, dt \right|,$$

where the last line follows from the definition of $\| \cdot \|_{S_u}$. Therefore,

$$\|\Box^{-1}F \pm i |\nabla|^{-1} \partial_t (\Box^{-1}F)\|_{L_t^r L^s_x} < \infty,$$

as required. \qed
Remark 3.4. It is possible to remove the time cutoff \( \chi_{[0, \infty)}(t) \) in Corollary 3.3 by using an approximation argument. More precisely, provided that \( F \in L^1_{t, \text{loc}} H^{-1} \), we have

\[
\sup_{\|\phi\|_{S_w} \leq 1} \left| \int_0^\infty \langle \phi, |\nabla|^{-1} F \rangle_{L^2_x} dt \right| \leq \sup_{\|\phi\|_{S_w} \leq 1} \left| \int_\mathbb{R} \langle \phi, |\nabla|^{-1} F \rangle_{L^2_x} dt \right|.
\]

This follows by writing \( \phi(t) = \chi(\epsilon^{-1}t)\phi(t) + (1 - \chi(\epsilon^{-1}t))\phi(t) \) with \( \chi(t) \in C^\infty \), \( \chi(t) = 1 \) for \( t \geq 1 \), and \( \chi(t) = 0 \) for \( t < \frac{1}{\epsilon} \), and noting that \( \chi(\epsilon^{-1}t)\phi(t) \in C^\infty_0(0, \infty) \) and

\[
\left| \int_0^\infty \langle (1 - \chi(\epsilon^{-1}t))\phi(t), |\nabla|^{-1} F \rangle_{L^2_x} dt \right| \to 0
\]
as \( \epsilon \to 0 \) since \( F \in L^1_{t, \text{loc}} H^{-1} \).

3.2. Product estimates in far cone regions. In this section we give the key estimates to control the product of two functions in the spaces \( S \) and \( S_w \) in the special case where one of the functions has high modulation, or in other words, is far from the cone. This estimate is a consequence of a general product high-low type product estimate in adapted versions \( U^p \) and \( V^p \). To state the product estimate we require, we start by defining the (spatial) bilinear Fourier multiplier

\[
\mathcal{M}[f, g](x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi - \eta) \hat{g}(\eta) m(\xi - \eta, \eta) d\eta e^{ix \cdot \xi} d\xi \tag{3.6}
\]

where \( m : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C} \).

**Theorem 3.5** (High-low product estimate: wave case). Let \( d \in 2\mathbb{Z}^2 \). Suppose that \( m : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C} \) with

\[
|\pm_0|\xi + \eta| - \pm_1|\xi| - \pm_2|\eta|| \leq d
\]

for \((\xi, \eta) \in \text{supp } m\). If \( u \in V^2_{\pm_1} \) with \( \text{supp } \tilde{u} \subset \{|\tau \pm_1|\xi|| \geq 4d\} \) and \( v \in V^2_{\pm_2} \) then \( \mathcal{M}[u, v] \in V^2_{\pm_1} \) with

\[
\|\mathcal{M}[u, v]\|_{V^2_{\pm_1}} \lesssim \|m(\xi, \eta)\|_{L^\infty_{\xi} L^1_{\eta}} \|u\|_{V^2_{\pm_1}} \|v\|_{V^2_{\pm_2}}.
\]

If in addition we have \( u \in U^2_{\pm_1} \) and \( v \in U^2_{\pm_2} \), then \( \mathcal{M}[u, v] \in U^2_{\pm_1} \) and

\[
\|\mathcal{M}[u, v]\|_{U^2_{\pm_1}} \lesssim \|m(\xi, \eta)\|_{L^\infty_{\xi} L^1_{\eta}} \|u\|_{U^2_{\pm_1}} \|v\|_{U^2_{\pm_2}}.
\]

**Proof.** This is a direct consequence of Theorem 6.8 and a rescaling argument. Roughly the point is that, after applying Plancherel, for the \( V^2 \) estimate, by definition, we need to prove that

\[
\left\| \int_{\mathbb{R}^n} e^{it(\pm_0|\xi| - \pm_1|\xi - \eta| - \pm_2|\eta|)} m(\xi - \eta, \eta) \phi(t, \xi - \eta) \psi(t, \eta) d\eta \right\|_{V^2} \lesssim \|m(\xi, \eta)\|_{L^\infty_{\xi} L^1_{\eta}} \|\phi\|_{V^2} \|\psi\|_{V^2}.
\]

However, the Fourier support assumptions imply that \( \mathcal{F}_t(\phi) \subset \{|\tau| \geq 4d\} \) (and thus \( \phi \) has high temporal frequency), and \( e^{it(\pm_0|\xi| - \pm_1|\xi - \eta| - \pm_2|\eta|)} \) has low temporal frequency. Hence, using the standard heuristic that the derivative only falls on the high frequency term, we see that the \( V^2 \) norm only hits the product \( \phi(t) \psi(t) \), and we can simply place the exponential factor in \( L^\infty_t \). This argument is made precise in Subsection 6.2. \( \Box \)
The high-low product estimate in Theorem 3.5 is also true for general phases, see Theorem 6.6 and Theorem 6.8 below. In particular, Theorem 3.5 is a consequence of a general property of $U^p$ and $V^p$ spaces, and the precise nature of the solution operator $e^{|\xi|t|\nabla|}$ plays no role.

Adapting Theorem 3.5 to the solution spaces $S$ and $S_w$ gives the following.

**Theorem 3.6.** For all $\lambda_0, \lambda_1, \lambda_2 \in 2\mathbb{Z}$ we have

\[
\|P_{\lambda_0}(C_{\gg \mu}u_{\lambda_1}v_{\lambda_2})\|_{S_w} \lesssim \mu^{\frac{1}{2}} \left( \frac{\min\{\lambda_1, \lambda_2\}}{\mu} \right)^{\frac{1}{2}} \|u_{\lambda_1}\| \|v_{\lambda_2}\|_{S_w},
\]

(3.7)

\[
\|P_{\lambda_0}(C_{\gg \mu}u_{\lambda_1}v_{\lambda_2})\|_{S} \lesssim \mu^{\frac{1}{2}} \left( \frac{\min\{\lambda_1, \lambda_2\}}{\mu} \right)^{\frac{1}{2}} \|u_{\lambda_1}\| \|v_{\lambda_2}\|_{S},
\]

(3.8)

where $\mu = \min\{\lambda_0, \lambda_1, \lambda_2\}$.

**Proof.** Let $\lambda_{\text{max}} = \max\{\lambda_0, \lambda_1, \lambda_2\}$. By dropping $C_{\gg \mu}$ and supposing throughout that the Fourier support of $u_{\lambda_1}$ or $v_{\lambda_2}$ is contained in $\{|\tau| - |\xi| \gg \mu\}$, we may assume that $\lambda_1 \geq \lambda_2$. We claim

\[
\|P_{\lambda_0}(u_{\lambda_1}v_{\lambda_2})\|_{V_{\pm}^2} \lesssim \mu^{\frac{1}{2}} \left( \frac{\lambda_{\text{max}}}{\mu} \right)^{\frac{1}{2}} \|u_{\lambda_1}\|_{V_{\pm}^2} v_{\lambda_2} \|v_{\lambda_2}\|_{V_{\pm}^2} + v_{\lambda_2} \|v_{\lambda_2}\|_{V_{\pm}^2},
\]

(3.9)

\[
\|P_{\lambda_0}(u_{\lambda_1}v_{\lambda_2})\|_{U_{\pm}^2} \lesssim \mu^{\frac{1}{2}} \left( \frac{\lambda_{\text{max}}}{\mu} \right)^{\frac{1}{2}} \|u_{\lambda_1}\|_{U_{\pm}^2} v_{\lambda_2} \|v_{\lambda_2}\|_{U_{\pm}^2} + U_{\lambda_2} \|v_{\lambda_2}\|_{U_{\pm}^2}.
\]

(3.10)

In addition, if $M(f,g)(x)$ is defined as in (3.6) with

\[
m(\xi, \eta) = (|\xi + \eta| - |\xi| - |\eta|) 1_{\{\xi = \pm \lambda_1, \eta = \pm \lambda_2, |\xi + \eta| \approx \lambda_0\}}(\xi, \eta)
\]

we claim

\[
\|M(u_{\lambda_1}, v_{\lambda_2})\|_{L_{\pm}^2} \lesssim \mu^{\frac{1}{2}} \left( \frac{\lambda_{\text{max}}}{\mu} \right)^{\frac{1}{2}} \min\{\lambda_1, \lambda_2\} \|u_{\lambda_1}\|_{L_{\pm}^2} + L_{\lambda_2} \|v_{\lambda_2}\|_{L_{\pm}^2} + L_{\lambda_2} \|v_{\lambda_2}\|_{L_{\pm}^2}.
\]

(3.11)

Assuming these claims for the moment, we now give the proof of the bounds (3.7) and (3.8). Concerning (3.7), we observe that in the case $\lambda_1 \approx \lambda_2$, (3.7) boils down to (3.9). On the other hand, if $\lambda_1 \gg \lambda_2$, we can directly apply Theorem 3.5 with symbol $m(\xi, \eta) = 1_{\{|\xi + \eta| \approx \lambda_0, |\xi = \pm \lambda_1, |\eta = \pm \lambda_2\}}$ and obtain

\[
\|P_{\lambda_0}(u_{\lambda_1}v_{\lambda_2})\|_{S_w} \lesssim \|P_{\lambda_0}(u_{\lambda_1}v_{\lambda_2})\|_{V_{\pm}^2} \lesssim \mu^{\frac{1}{2}} \|u_{\lambda_1}\|_{V_{\pm}^2} v_{\lambda_2} \|v_{\lambda_2}\|_{V_{\pm}^2}.
\]

since $\pm \lambda_1 \approx |\xi| \approx \pm \lambda_2 \approx |\eta|$ for $(\xi, \eta) \in \text{supp } m$ and $\|m\|_{L_{\xi}^{\infty}L_{\nabla}^{\infty}L_{\xi}^{\infty}} \lesssim \mu^{\frac{1}{2}}$.

For the proof of (3.8), it is enough to show

\[
\| (1 + i|\nabla|^{-1}\partial_{t_1})P_{\lambda_0}(u_{\lambda_1}v_{\lambda_2})\|_{U_{\pm}^2} \lesssim \mu^{\frac{1}{2}} \left( \frac{\min\{\lambda_1, \lambda_2\}}{\mu} \right)^{\frac{1}{2}} \|u_{\lambda_1}\|_{S} \|v_{\lambda_2}\|_{S}.
\]

We decompose the left hand side into

\[
(1 + i|\nabla|^{-1}\partial_{t_1})(u_{\lambda_1}v_{\lambda_2}) = (1 + i|\nabla|^{-1}\partial_{t_1})u_{\lambda_1}v_{\lambda_2} - i|\nabla|^{-1}M(|\nabla|^{-1}\partial_{t_1}u_{\lambda_1}, v_{\lambda_2})
\]

\[+ i|\nabla|^{-1}(u_{\lambda_1}\partial_{t_1}v_{\lambda_2}) - i|\nabla|^{-1}(|\nabla|^{-1}\partial_{t_1}u_{\lambda_1}|\nabla|v_{\lambda_2}).
\]

For the first term we directly apply Theorem 3.5 as above and, since $\lambda_1 \geq \lambda_2$, obtain

\[
\| (1 + i|\nabla|^{-1}\partial_{t_1})u_{\lambda_1}v_{\lambda_2}\|_{U_{\pm}^2} \lesssim \mu^{\frac{1}{2}} \|u_{\lambda_1}\|_{S} \|v_{\lambda_2}\|_{S}.
\]
For the second term we apply (3.11) and obtain
\[ \left\| \nabla^{-1} \mathcal{M}(\nabla^{-1} \partial_t u_{\lambda_1}, v_{\lambda_2}) \right\|_{U_2^\mu} \lesssim \mu \left( \frac{\lambda_{\max}}{\mu} \right)^{\frac{2}{n}} \lambda_{\max} \left\| \nabla^{-1} \partial_t u_{\lambda_1} \right\|_{U_2^\mu + U_2^\mu} \left\| v_{\lambda_2} \right\|_{U_2^\mu + U_2^\mu} \lesssim \mu \left( \frac{\lambda_{\max}}{\mu} \right)^{\frac{2}{n}} \| u_{\lambda_1} \|_S \| v_{\lambda_2} \|_S. \]

For the terms in the second line, we apply (3.10) and obtain
\[ \left\| \nabla^{-1} (u_{\lambda_1} \partial_t v_{\lambda_2}) \right\|_{U_2^\mu} \lesssim \frac{\lambda_{\max}}{\lambda_0} \mu \left( \frac{\lambda_{\max}}{\mu} \right)^{\frac{2}{n}} \| u_{\lambda_1} \|_{U_2^\mu + U_2^\mu} \left\| \nabla^{-1} \partial_t v_{\lambda_2} \right\|_{U_2^\mu + U_2^\mu} \]

and similarly
\[ \left\| \nabla^{-1} (|\nabla|^{-1} \partial_t u_{\lambda_1} \nabla v_{\lambda_2}) \right\|_{U_2^\mu} \lesssim \frac{\lambda_{\max}}{\lambda_0} \mu \left( \frac{\lambda_{\max}}{\mu} \right)^{\frac{2}{n}} \left\| \nabla^{-1} \partial_t u_{\lambda_1} \right\|_{U_2^\mu + U_2^\mu} \left\| v_{\lambda_2} \right\|_{U_2^\mu + U_2^\mu}, \]

so that
\[ \left\| \nabla^{-1} (u_{\lambda_1} \partial_t v_{\lambda_2}) \right\|_{U_2^\mu} + \left\| \nabla^{-1} (|\nabla|^{-1} \partial_t u_{\lambda_1} \nabla v_{\lambda_2}) \right\|_{U_2^\mu} \lesssim \mu \left( \frac{\lambda_{\max}}{\mu} \right)^{\frac{2}{n}} \| u_{\lambda_1} \|_S \| v_{\lambda_2} \|_S, \]

and the proof of (3.8) is complete.

It remains to prove (3.9), (3.10) and (3.11). We start by observing that the standard \( U^2 \) Besov embedding (see Theorem 6.2 below) gives
\[ \left\| C_{\lesssim \lambda_{\max}} P_{\lambda_0} (u_{\lambda_1} v_{\lambda_2}) \right\|_{U_2^\mu} \lesssim \frac{\lambda_0}{\lambda_{\max}} \left\| P_{\lambda_0} (u_{\lambda_1} v_{\lambda_2}) \right\|_{L^2_{x, t}}. \]

Now, if \( u_{\lambda_1} \) is away from the cone, we have by (11) in Lemma 3.1
\[ \left\| P_{\lambda_0} (u_{\lambda_1} v_{\lambda_2}) \right\|_{L^2_{x, t}} \lesssim \mu \left( \frac{\lambda_{\max}}{\mu} \right)^{\frac{2}{n}} \| u_{\lambda_1} \|_{L^2_{x, t}} \| v_{\lambda_2} \|_{L^2_{x, t}} \lesssim \mu \left( \frac{\lambda_{\max}}{\mu} \right)^{\frac{2}{n}} \| u_{\lambda_1} \|_{S_w} \| v_{\lambda_2} \|_{S_w}, \]

and similarly, if \( v_{\lambda_2} \) is away from the cone, we have
\[ \left\| P_{\lambda_0} (u_{\lambda_1} v_{\lambda_2}) \right\|_{L^2_{x, t}} \lesssim \mu \left( \frac{\lambda_{\max}}{\mu} \right)^{\frac{2}{n}} \| u_{\lambda_1} \|_{L^\infty L^2_{x, t}} \| v_{\lambda_2} \|_{L^2_{x, t}} \lesssim \mu \left( \frac{\lambda_{\max}}{\mu} \right)^{\frac{2}{n}} \| u_{\lambda_1} \|_{S_w} \| v_{\lambda_2} \|_{S_w}. \]

In summary, we have
\[ \left\| C_{\lesssim \lambda_{\max}} P_{\lambda_0} (u_{\lambda_1} v_{\lambda_2}) \right\|_{U_2^\mu} \lesssim \lambda_{\max} \left( \frac{\lambda_{\max}}{\mu} \right)^{\frac{2}{n}} \| u_{\lambda_1} \|_{S_w} \| v_{\lambda_2} \|_{S_w}. \quad (3.12) \]

On the other hand, we note that by Theorem 3.5 and the uniform disposability from (11) in Lemma 3.1
\[ \left\| C_{\gg \lambda_{\max}} P_{\lambda_0} (u_{\lambda_1} v_{\lambda_2}) \right\|_{V_{x, t}^2} \lesssim \| P_{\lambda_0} (C_{\gg \lambda_{\max}} u_{\lambda_1} v_{\lambda_2}) \|_{V_{x, t}^2} + \| P_{\lambda_0} (C_{\lesssim \lambda_{\max}} u_{\lambda_1} C_{\gg \lambda_{\max}} v_{\lambda_2}) \|_{V_{x, t}^2} \lesssim \mu \left( \frac{\lambda_{\max}}{\mu} \right)^{\frac{2}{n}} \| u_{\lambda_1} \|_{S_w} \| v_{\lambda_2} \|_{S_w}. \]

This, together with the estimate (3.12) and the embedding \( U_{x, t}^2 \subset V_{x, t}^2 \), finishes the proof of (3.9). The proof of (3.10) follows from the same argument, by using the \( U^2 \)-estimate in Theorem 3.5 and the embedding \( U_{x, t}^2 \subset S_w \) instead. This argument also proves (3.11), because Theorem 3.5 allows for multipliers, and due to the obvious fixed-time multiplier bound in \( L^2_x \) in (3.12). \( \Box \)
3.3. Bilinear $L^2_{t,x}$ Estimates. In this section we give the second key bilinear bound that is required for the proof of Theorem 2.2. Similar to the previous section, the key bilinear input is a special case of a general bilinear restriction estimate in $L^2_{t,x}$, which holds not just for the wave equation, but also the case of general phases. On the other hand, in contrast to the previous section, the bilinear estimate we prove here is much more delicate, as it handles the case where both functions, as well as their product, is close to the light cone.

We start with some motivation. Let $\lambda_1 \geq 1$ and define

$$\Lambda_1 = \{ |\xi - e_1| < \frac{1}{100}\}, \quad \Lambda_2 = \{ |\xi + \lambda e_2| < \frac{1}{100}\lambda \}$$

with $e_1 = (1, 0, \ldots, 0)$ and $e_2 = (0, 1, 0, \ldots, 0)$. For free solutions, an application of Plancheral followed by Hölder implies that if $\operatorname{supp} \hat{f} \subset \Lambda_1$ and $\operatorname{supp} \hat{g} \subset \Lambda_2$, then we have the bilinear estimate

$$\|e^{-it|\nabla|}f e^{-it|\nabla|}g\|_{L^2_{t,x}} \lesssim \|f\|_{L^2_t} \|g\|_{L^2_t}.$$  \hspace{1cm} (3.13)

The atomic definition of $U^2$ then easily implies that

$$\|uv\|_{L^2_{t,x}} \lesssim \|u\|_{L^2_t} \|v\|_{L^2_t}$$

for $\operatorname{supp} \hat{u} \subset \Lambda_1$ and $\operatorname{supp} \hat{v} \subset \Lambda_2$. Although this estimate is potentially useful, the fact that both functions must be placed into $U^2$, means that it is far to weak to be able to deduce the estimates required to solve the division problem in Theorem 2.2. In fact this lack of a good $V^2$ replacement for (3.13), was a key motivation in developing the null frame spaces of Tataru, which were constructed to solve this issue (among others).

Recently however, in work of the first author [7], it was shown that it is possible to deduce a $U^2 \times V^2$ version of (3.13) provided that the low frequency term is placed in $U^2$. It turns out that the bilinear estimate for free solutions given in (3.13) is insufficient for this purpose, essentially since it does not exploit any dispersive properties of free waves. Instead, the argument given in [7], shows that the $U^2 \times V^2$ estimate can be reduced to a richer property of free waves, namely the bilinear estimates satisfied by wave tables. The wave table construction efficiently exploits both transversality and curvature, and was introduced by Tao [28] in the proof of the endpoint bilinear restriction estimate for the cone. In the case of the wave equation, the conclusion is the following.

**Theorem 3.7 ([7] Theorem 1.7 and Theorem 1.10]).** Let $2 \leq a \leq b < n + 1$. Let $0 < \lambda_1 \leq \lambda_2$, $0 < \alpha \leq 1$, and $\kappa, \kappa' \in C_\alpha$ with $\angle(\pm \kappa, \kappa') \approx \alpha$. If $u \in U^a_+$ and $v \in U^b_+$ then

$$\|R_\alpha u_{\lambda_1} R_{\kappa'} v_{\lambda_2}\|_{L^2_{t,x}} \lesssim \alpha^{\frac{n-1}{2}} \lambda_1^{\frac{n-1}{2}} \left(\frac{\lambda_2}{\lambda_1}\right)^{(n+1)(\frac{4}{b} - \frac{1}{2})} \|R_\alpha u_{\lambda_1}\|_{L^2_t} \|R_{\kappa'} v_{\lambda_2}\|_{L^2_t}.$$  

If $\alpha \approx 1$ and $\lambda_1 \approx \lambda_2$, for cubes $q, q' \in Q_\mu$, with $0 < \mu \lesssim \lambda_1$ we have stronger bound

$$\|R_\alpha P_q u_{\lambda_1} P_{q'} R_{\kappa'} v_{\lambda_2}\|_{L^2_{t,x}} \lesssim \mu^{\frac{n-1}{2}} \|R_\alpha P_q u_{\lambda_1}\|_{L^2_t} \|R_{\kappa'} P_{q'} v_{\lambda_2}\|_{L^2_t}.$$  

**Proof.** In the special case $\alpha \approx 1$, the bounds are a consequence of the wave table construction introduced by Tao in [28], together with an induction on scales argument. In Section 7 we give the details of this argument by following [7] in the case of the cone. In the case $0 < \alpha \lesssim 1$, the proof requires a slightly more general wave table decomposition, see the proof of [7] Theorem 1.7 and Theorem 1.10] and Remark 7.2. \hfill $\square$
Remark 3.8. The argument developed in [7] in fact shows that the bound in Theorem 3.7 can be generalised to the full bilinear range, and moreover holds for general phases under suitable curvature and transversality assumptions. See Section 7 for further discussion.

After using the standard embedding $V^2 \subset U^b$, we see that
\[
\| R_{\omega u_{\mu}} R_{\varepsilon' v_{\lambda}} \|_{L^2_{t,x}} \lesssim \alpha^{\frac{\alpha - 3}{2}} \mu^\frac{\alpha - 4}{2} \| R_{\omega u_{\mu}} \|_{U^2_{t,x}} \| R_{\varepsilon' v_{\lambda}} \|_{V^2_{t,x}}.
\]
In particular, we have a bilinear $L^2_{t,x}$ estimate with the high frequency term in $V^2$, without any high frequency loss.

If we combine Theorem 3.7 with an analysis of the resonant set, we obtain the following key bilinear estimate.

**Theorem 3.9** (Main Bilinear $L^2_{t,x}$ bound). Let $d, \lambda_0, \lambda_1, \lambda_2 \in 2^\mathbb{Z}$ and $\epsilon > 0$. If $\lambda_1 \leq \lambda_2$, $\mu = \min\{\lambda_0, \lambda_1\}$, and $d \lesssim \mu$ we have the bilinear estimates
\[
\| C_d \rho_0 (C_{\leq d} u_{\lambda_1} C_{\leq d} v_{\lambda_2}) \|_{L^2_{t,x}} \lesssim d^{\frac{\alpha - 3}{2}} \mu^\frac{\alpha - 4}{2} \lambda_1^\frac{1}{2} (\frac{\lambda_2}{d \mu})^\epsilon \| u_{\lambda_1} \|_{S} \| v_{\lambda_2} \|_{S}.
\]
(3.14)

\[
\| C_d \rho_0 (C_{\leq d} u_{\lambda_1} C_{\leq d} v_{\lambda_2}) \|_{L^2_{t,x}} \lesssim d^{\frac{\alpha - 3}{2}} \mu^\frac{\alpha - 4}{2} \lambda_1^\frac{1}{2} (\frac{\lambda_2}{d \mu})^\epsilon \| u_{\lambda_1} \|_{S} \| v_{\lambda_2} \|_{S}.
\]
(3.15)

\[
\| C_d \rho_0 (C_{\leq d} u_{\lambda_1} C_{\leq d} v_{\lambda_2}) \|_{L^2_{t,x}} \lesssim d^{\frac{\alpha - 3}{2}} \mu^\frac{\alpha - 4}{2} \lambda_1^\frac{1}{2} (\frac{\lambda_2}{d \mu})^\epsilon \| u_{\lambda_1} \|_{S} \| v_{\lambda_2} \|_{S}.
\]
(3.16)

On the other hand, if $d \gg \mu$, we have
\[
\| C_d \rho_0 (C_{\leq d} u_{\lambda_1} C_{\leq d} v_{\lambda_2}) \|_{L^2_{t,x}} \lesssim \mu^{\frac{\alpha - 3}{2}} (\frac{\lambda_1}{\mu})^\epsilon \| u_{\lambda_1} \|_{S} \| v_{\lambda_2} \|_{S}.
\]
(3.17)

The first bound in the previous theorem, (3.14), is a direct consequence of the corresponding estimate for free solutions. In particular, it is sharp. On the other hand, due to the fact that we now only have $V^2_2$ control over $v_{\lambda_2}$, the bounds (3.15), (3.16) and (3.17) require the bilinear restriction estimates contained in Theorem 3.7. The key point is that we have no high-frequency $\lambda_2$ loss, provided that we place the low frequency term $u_{\lambda_1}$ into $U^2_2$ (i.e. the $S$ norm). The only loss (3.15) appears when $\lambda_1 \approx \lambda_2 \approx \lambda_0$, which is in general an easier case to deal with. On the other hand, placing both $u_{\lambda_1}$ and $v_{\lambda_2}$ into $V^2$ causes an $\epsilon$ loss in the high frequency $\lambda_2$, and thus is only useful in certain special cases.

To reduce Theorem 3.7 to the bilinear restriction estimates in Theorem 3.9, we need to show that the waves $u$ and $v$ are transverse. This is a consequence of the following.

**Lemma 3.10** (Resonance bound for full cone). Let $d, \lambda_0, \lambda_1, \lambda_2 \in 2^\mathbb{Z}$. Assume that $(\tau, \xi), (\tau', \eta) \in \mathbb{R}^{1+n}$ satisfy $|\xi| \approx \lambda_1$, $|\eta| \approx \lambda_2$, $|\xi + \eta| \approx \lambda_0$ and
\[
|\tau| - |\xi| \ll d, \quad |\tau'| - |\eta| \ll d, \quad |\tau + \tau'| - |\xi + \eta| \ll d.
\]
If $d \lesssim \min\{\lambda_0, \lambda_1, \lambda_2\}$, then
\[
\angle(\text{sgn}(\tau)\xi, \text{sgn}(\tau')\eta) \approx \left(\frac{d \lambda_0}{\lambda_1 \lambda_2}\right)^\frac{1}{2},
\]
\[
\angle(\text{sgn}(\tau + \tau') (\xi + \eta), \text{sgn}(\tau)\xi) \lesssim \left(\frac{d \lambda_2}{\lambda_0 \lambda_1}\right)^\frac{1}{2},
\]
\[
\angle(\text{sgn}(\tau + \tau') (\xi + \eta), \text{sgn}(\tau')\eta) \lesssim \left(\frac{d \lambda_2}{\lambda_0 \lambda_2}\right)^\frac{1}{2}.
\]
On the other hand, if $d \gg \min\{\lambda_0, \lambda_1, \lambda_2\}$ then in fact $\text{sgn}(\tau) = \text{sgn}(\tau')$ and

$$\angle(\xi, \eta) \approx 1, \quad d \approx \max\{\lambda_0, \lambda_1, \lambda_2\}, \quad \lambda_0 \ll \lambda_1 \approx \lambda_2.$$  

Proof. We first observe that since

$$||\tau + \tau'| - |\text{sgn}(\tau)||\xi| + \text{sgn}(\tau')||\eta||| \leq ||\tau| - |\xi|| + ||\tau' - |\eta||| \ll d$$

and

$$||(\text{sgn}(\tau)||\xi| + \text{sgn}(\tau')||\eta||)^2 - |\xi + \eta|^2| \approx \lambda_1 \lambda_2 \angle(\text{sgn}(\tau)||\xi|, \text{sgn}(\tau')||\xi||')$$

we have

$$d \approx \||\text{sgn}(\tau)||\xi| + \text{sgn}(\tau')||\eta||| - |\xi + \eta|| \approx \frac{\lambda_1 \lambda_2 \angle(\text{sgn}(\tau)||\xi|, \text{sgn}(\tau')||\eta||)}{||\text{sgn}(\tau)||\xi| + \text{sgn}(\tau')||\eta||| + |\xi + \eta||}. \quad (3.18)$$

If $\lambda_0 \approx \max\{\lambda_1, \lambda_2\}$, then (3.18) already gives $d \lesssim \min\{\lambda_0, \lambda_1, \lambda_2\}$ and the claimed orthogonality bound, so it remains to consider the case $\lambda_0 \ll \lambda_1 \approx \lambda_2$. We first consider the interactions where $\text{sgn}(\tau) = -\text{sgn}(\tau')$ which implies that

$$||\text{sgn}(\tau)||\xi| + \text{sgn}(\tau')||\eta||| + |\xi + \eta|| \approx \lambda_0, \quad \lambda_1 \lambda_2 \angle(\xi, -\eta) \approx |\xi + \eta|^2 - ||\xi|| - |\eta||^2 \lesssim \lambda_0^2.$$

Consequently the claimed bounds again follow immediately from (3.18). On the other hand, if $\text{sgn}(\tau) = \text{sgn}(\tau')$ then as $\angle(\xi, -\eta) \lesssim \frac{d}{\lambda_1} \ll 1$ we must have $\angle(\xi, \eta) \approx 1$ and hence (3.18) implies that

$$d \approx \frac{\lambda_1 \lambda_2 \angle(\xi, \eta)}{\lambda_1} \approx \lambda_1$$

as required. It only remains to control the angle between $\xi + \xi'$ and $\xi$. To this end, an analogous computation to that used to deduce (3.18) gives

$$\lambda_0 \angle(\text{sgn}(\tau + \tau')(\xi + \eta), \xi) \lesssim d(||\text{sgn}(\tau + \tau')||\xi + \eta| - \text{sgn}(\tau)||\xi|| + |\eta||)$$

which suffices unless we have $\lambda_0 \approx \lambda_1 \gg \lambda_2$ and $\text{sgn}(\tau + \tau') = -\text{sgn}(\tau)$. But this implies that

$$d \approx ||\tau + \tau'| - |\xi + \eta|| \approx ||\text{sgn}(\tau)||\xi| + \text{sgn}(\tau')||\eta||| - \text{sgn}(\tau + \tau')||\xi + \eta||| \approx \lambda_0$$

which contradicts the previous computation which showed that we must have $d \lesssim \lambda_2$. Hence the case $\lambda_0 \approx \lambda_1 \gg \lambda_2$ and $\text{sgn}(\tau + \tau') = -\text{sgn}(\tau)$ cannot occur, and consequently we deduce the correct angle bound between $\xi, \eta$ and $\eta$. \hfill \Box

Lemma 3.11 (Lower bound on resonance). Let $d, \lambda_0, \lambda_1, \lambda_2 \in 2\mathbb{Z}$. Assume that $$(\tau, \xi), (\tau', \eta) \in \mathbb{R}^{1+n}$$ satisfy

$$||\xi|| \approx \lambda_1, \quad ||\eta|| \approx \lambda_2, \quad ||\xi + \eta|| \approx \lambda_0$$

and

$$||\tau| - |\xi|| \lesssim d, \quad ||\tau' - |\eta|| \lesssim d, \quad ||\tau + \tau'| - |\xi + \eta|| \lesssim d.$$

Then,

$$\angle(\text{sgn}(\tau)||\xi|, \text{sgn}(\tau')||\eta||) + \angle(\text{sgn}(\tau)||\xi|, \text{sgn}(\tau + \tau')(\xi + \eta))$$

$$+ \angle(\text{sgn}(\tau')||\eta||, \text{sgn}(\tau + \tau')(\xi + \eta)) \lesssim \left(\frac{d}{\min\{\lambda_0, \lambda_1, \lambda_2\}}\right)^{\frac{1}{2}}$$

and

$$\angle(\text{sgn}(\tau)||\xi|, \text{sgn}(\tau')||\eta||) \lesssim \left(\frac{d}{\min\{\lambda_0, \lambda_1, \lambda_2\}}\right)^{\frac{1}{2}}.$$
Proof. We start with the observation
\[ |\text{sgn}(\tau)|\xi| + \text{sgn}(\tau')|\eta| - \text{sgn}(\tau + \tau')|\xi + \eta| \lesssim d. \]
The left hand side is bounded below by
\[ \left| |\text{sgn}(\tau)|\xi| + \text{sgn}(\tau')|\eta| - |\xi + \eta| \right| \approx \frac{|\xi||\eta|L^2(\text{sgn}(\tau)|\xi|, \text{sgn}(\tau')|\eta|)}{|\text{sgn}(\tau)|\xi| + \text{sgn}(\tau')|\eta| + |\xi + \eta|}, \]
which implies the bound on the first summand. The bounds on the other two summands follow similarly. \(\square\)

We now give the proof of Theorem 3.9.

Proof of Theorem 3.9. Let \(\lambda_0, \lambda_1, \lambda_2 \in 2^\mathbb{Z}\) with \(\lambda_1 \leq \lambda_2\), and take \(\mu = \min\{\lambda_0, \lambda_1, \lambda_2\}\). It is enough to consider the case \(u_{\lambda_1} \in U_{d}^+\) (or \(V_{d}^+\)), and \(v_{\lambda_2} \in U_{d}^+\) (or \(V_{d}^+\)). Suppose that \(d \lesssim \mu\) and note that we can write
\[ C_{\leq d}u_{\lambda_1} = C_{\leq d}^+u_{\lambda_1} + C_{\leq d}^-u_{\lambda_1} = C_{\leq d}^+u_{\lambda_1} + C_{\leq d}^-C_{\lambda_1}u_{\lambda_1}. \]
Applying Hölder’s inequality we deduce that
\[ \|C_dP_{\lambda_0}(C_{\leq d}^-u_{\lambda_1}C_{\leq d}v_{\lambda_2})\|_{L^2_{t,x}} \lesssim \mu^{\frac{1}{2}}\|C_{\lambda_1}u_{\lambda_1}\|_{L^2_{t,x}}\|v_{\lambda_2}\|_{L^\infty_{t,x}L^2_{x}} \lesssim \mu^{\frac{1}{2}}\lambda_1^{-\frac{1}{2}}\|u_{\lambda_1}\|_{U_2^+}\|v_{\lambda_2}\|_{V_2^+} \quad (3.19) \]
which suffices if \(n = 2, 3\) (clearly we can choose an exponent larger than 4 if necessary). To obtain a slightly sharper bound, we can decompose into caps/cubes before applying Hölder, namely, letting \(\alpha = (\frac{d\mu}{\lambda_1\lambda_2})^\frac{1}{2}\) and applying Lemma 3.10 we have
\[ \|C_dP_{\lambda_0}(C_{\leq d}^-u_{\lambda_1}C_{\leq d}v_{\lambda_2})\|_{L^2_{t,x}} \lesssim \left( \sum_{\kappa,\kappa' \in \mathcal{C}_\alpha, q,q' \in \mathcal{Q}_\mu} \|C_dP_{\lambda_0}(R_{\kappa}P_{q}C_{\leq d}^+u_{\lambda_1}R_{\kappa'}P_{q'}C_{\leq d}v_{\lambda_2})\|_{L^2_{t,x}}^2 \right)^{\frac{1}{2}} \lesssim \mu^{\frac{1}{2}}(\mu d)^{\frac{1}{4}}\sum_{\kappa,\kappa' \in \mathcal{C}_\alpha, q,q' \in \mathcal{Q}_\mu} \|R_{\kappa}P_{q}C_{\lambda_1}u_{\lambda_1}\|_{L^2_{t,x}}\|R_{\kappa'}P_{q'}v_{\lambda_2}\|_{L^\infty_{t,x}L^2_{x}} \lesssim \mu^{\frac{1}{2}}(\mu d)^{\frac{1}{4}}\lambda_1^{-\frac{1}{2}}\|u_{\lambda_1}\|_{U_2^+}\|v_{\lambda_2}\|_{V_2^+} \]
where we used the fact that since \(\lambda_1 \leq \lambda_2\) we have \(\lambda_1 \alpha = (d\mu)^{\frac{1}{2}}\).
Interpolating with (3.19), we obtain an estimate which clearly suffices in higher dimensions as well. After noting the identity
\[ C_{\leq d}v_{\lambda_2} = C_{\leq d}^+v_{\lambda_2} + C_{\leq d}^-C_{\lambda_2}v_{\lambda_2} \]
a similar argument to the above reduces the problem to proving the bounds
\[ \|C_dP_{\lambda_0}(C_{\leq d}^+u_{\lambda_1}C_{\leq d}^+v_{\lambda_2})\|_{L^2_{t,x}} \lesssim d^{-\frac{3}{2}}\mu^{-\frac{1}{2}}\lambda_1^{\frac{1}{2}}\|u_{\lambda_1}\|_{U_2^+}\|v_{\lambda_2}\|_{V_2^+}, \]
\[ \|C_dP_{\lambda_0}(C_{\leq d}^+u_{\lambda_1}C_{\leq d}v_{\lambda_2})\|_{L^2_{t,x}} \lesssim d^{-\frac{3}{2}}\mu^{-\frac{1}{2}}\lambda_1^{\frac{1}{2}}\left(\frac{\lambda_1^2}{\mu d}\right)^{\frac{1}{2}}\|u_{\lambda_1}\|_{U_2^+}\|v_{\lambda_2}\|_{V_2^+}, \]
\[ \|C_dP_{\lambda_0}(C_{\leq d}^-u_{\lambda_1}C_{\leq d}v_{\lambda_2})\|_{L^2_{t,x}} \lesssim d^{-\frac{3}{2}}\mu^{-\frac{1}{2}}\lambda_1^{\frac{1}{2}}\left(\frac{\lambda_1\lambda_2}{\mu d}\right)^{\frac{1}{2}}\|u_{\lambda_1}\|_{V_2^+}\|v_{\lambda_2}\|_{V_2^+} \quad (3.20) \]
We now exploit Lemma 3.10 and orthogonality. Let \(\alpha = (\frac{d\mu}{\lambda_1\lambda_2})^\frac{1}{2}\), and \(\beta = (\frac{\lambda_1\lambda_2}{d\mu})^\frac{1}{2}\).
Note that since we assume \(\lambda_1 \leq \lambda_2\), we have \(\lambda_1 \alpha = (d\mu)^{\frac{1}{2}}\) and \(\lambda_0 \beta = (d\mu)^{\frac{1}{2}}\). An
application of Lemma 3.10 and orthogonality implies that after decomposing into caps, we have

\[
\|C_dP_{\lambda_0}(C_{\ll d}^{\pm}u_{\lambda_1}C_{\ll d}^{\pm}v_{\lambda_2})\|_{L^2_{t,x}}^2 \\ \lesssim \sum_{\kappa \in C_{\alpha}} \sum_{q,q' \in \mathbb{Q}_\mu} \left( \sum_{\kappa' \in C_{\alpha}} \|C_dP_{\lambda_0}(C_{\ll d}^{\pm}R_{\kappa}P_qu_{\lambda_1}C_{\ll d}^{\pm}R_{\kappa'}P_{q'}v_{\lambda_2})\|_{L^2_{t,x}} \right)^2.
\]

(3.21)

The standard bilinear \(L^2_{t,x}\) bound for free solutions, together with the disposability of the \(C_{\ll d}\) multipliers, gives for any cubes \(q,q' \in \mathbb{Q}_\mu\) and caps \(\kappa,\kappa' \in C_{\alpha}\) with \(|\kappa \mp \kappa'| \approx \alpha\) the estimate

\[
\|C_dP_{\lambda_0}(C_{\ll d}^{\pm}R_{\kappa}P_qu_{\lambda_1}C_{\ll d}^{\pm}R_{\kappa'}P_{q'}v_{\lambda_2})\|_{L^2_{t,x}} \lesssim d^{\frac{a-c}{3} \mu} \lambda_1^{\frac{1}{3}} \|R_{\kappa}P_qu_{\lambda_1}\|_{U^2_{1,t,x}} \|R_{\kappa'}P_{q'}v_{\lambda_2}\|_{U^2_{1,t,x}}.
\]

Therefore, from (3.21) and the \(U^2\) square sum bound, we deduce that

\[
\|C_dP_{\lambda_0}(C_{\ll d}^{\pm}u_{\lambda_1}C_{\ll d}^{\pm}v_{\lambda_2})\|_{L^2_{t,x}} \lesssim d^{\frac{a-b}{3} \mu} \lambda_1^{\frac{1}{3}} \|u_{\lambda_1}\|_{U^2_{1,t,x}} \|v_{\lambda_2}\|_{U^2_{1,t,x}}.
\]

(3.22)

To replace \(U^2\) with \(V^2\), we apply Theorem 3.7. We first suppose that \(\lambda_1 \approx \lambda_2\). After applying Lemma 3.10 and decomposing into caps of size \(\alpha\), Theorem 3.7 implies that for any \(2 \leq a \leq b < n + 1\)

\[
\|C_dP_{\lambda_0}(C_{\ll d}^{\pm}u_{\lambda_1}C_{\ll d}^{\pm}v_{\lambda_2})\|_{L^2_{t,x}} \lesssim \sum_{\kappa \in C_{\alpha}} \|C_{\ll d}^{\pm}R_{\kappa}u_{\lambda_1}C_{\ll d}^{\pm}R_{\kappa}v_{\lambda_2}\|_{L^2_{t,x}} \\ \lesssim \alpha^{\frac{a-b}{3} \lambda_1^{\frac{1}{3}}} \sum_{\kappa \in C_{\alpha}} \|R_{\kappa}u_{\lambda_1}\|_{U^b_{1,t,x}} \|R_{\kappa}v_{\lambda_2}\|_{U^b_{1,t,x}} \\ \lesssim \alpha^{\frac{a-b}{3} - (n-1)(1-\frac{1}{d^\mu} - \frac{1}{d\mu}) \lambda_1^{\frac{1}{3}}} \left( \sum_{\kappa \in C_{\alpha}} \|R_{\kappa}u_{\lambda_1}\|_{U^b_{1,t,x}}^\frac{1}{b} \right)^{\frac{b}{b}} \left( \sum_{\kappa \in C_{\alpha}} \|R_{\kappa}v_{\lambda_2}\|_{U^b_{1,t,x}}^\frac{1}{b} \right)^{\frac{b}{b}} \\ \lesssim d^{\frac{a-b}{3} \mu} \lambda_1^{\frac{1}{3}} \left( \frac{\lambda_1}{\mu} \right)^{\frac{1}{2}} \left( \frac{\lambda_1^2}{d\mu} \right)^{(n-1)(1-\frac{1}{d^\mu} - \frac{1}{d\mu})} \|u_{\lambda_1}\|_{U^b_{1,t,x}} \|v_{\lambda_2}\|_{U^b_{1,t,x}}.
\]

(3.23)

Together (3.22) and (3.23) together with the standard \(V^2\) interpolation argument give (3.18) in the case \(\lambda_1 \approx \lambda_2\). On the other hand, if \(\lambda_1 \ll \lambda_2\), we decompose into caps of size \(\beta = (\frac{d}{\lambda_1})^{\frac{1}{2}}\) and again apply Lemma 3.10 and Theorem 3.7 to deduce
that for any $2 \leq a < b < n + 1$
\[
\| C_d P_{\lambda_0} (C^+_{\leq d} u_{\lambda_1} C^+_{\leq d} v_{\lambda_2}) \|_{L^2_{t,x}} \leq \left( \sum_{\kappa \in C_\beta} \sup_{\kappa' \in C_\beta} \| C^+_{\leq d} R_{\kappa} u_{\lambda_1} C^+_{\leq d} R_{\kappa'} v_{\lambda_2} \|_{L^2_{t,x}}^2 \right)^{\frac{1}{2}},
\]
\[
\leq \beta^{\frac{n-1}{2}} \frac{\lambda_2}{\lambda_1} (n+1) \left( \frac{1}{\lambda_1} \right)^{(n+1)} \left( \sum_{\kappa \in C_\beta} \| R_{\kappa} u_{\lambda_1} \|_{L^+_{t,x}} \right)^{\frac{1}{2}} \| u_{\lambda_2} \|_{V^2_x}.
\]
Choosing $a = 2$, we get the $U^2 \times V^2$ estimate. Taking $a$ sufficiently close to 2 gives the $V^2 \times V^2$ estimate.

It remains to consider the case $d \gg \mu$. In light of Lemma 3.10, the left hand side is only nonzero if $\lambda_0 \ll \lambda_1 \approx \lambda_2$ and $d \approx \lambda_1$, and moreover, we have the identity
\[
C_d P_{\lambda_0} (C^+_{\leq d} u_{\lambda_1} C^+_{\leq d} v_{\lambda_2}) = C_d P_{\lambda_0} (C^+_{\leq d} u_{\lambda_1} C^+_{\leq d} v_{\lambda_2}) + C_d P_{\lambda_0} (C^-_{\leq d} u_{\lambda_1} C^-_{\leq d} v_{\lambda_2}).
\]
To estimate the $L^2_{t,x}$ norm of the first term in (3.24), if $\pm = +$, we decompose into caps and apply Theorem 3.7 which gives
\[
\| C_d P_{\lambda_0} (C^+_{\leq d} u_{\lambda_1} C^+_{\leq d} v_{\lambda_2}) \|_{L^2_{t,x}} \leq \sum_{\kappa, \kappa' \in C_\beta} \sum_{|q-q'| \approx \mu} \| R_{\kappa} P_q u_{\lambda_1} R_{\kappa'} P_q' v_{\lambda_2} \|_{L^2_{t,x}} \leq \mu^{\frac{n-1}{2}} \sum_{q, q' \in \mathbb{Q}_\mu} \| P_q u_{\lambda_1} \|_{L^+_{t,x}} \| P_{q'} v_{\lambda_2} \|_{L^+_{t,x}} \leq \mu^{\frac{n-1}{2}} \left( \frac{\lambda_1}{\mu} \right)^{(n-1)} \| u_{\lambda_1} \|_{L^+_{t,x}} \| v_{\lambda_2} \|_{L^+_{t,x}}.
\]
If $\pm = -$, we have $C^+_{\leq d} v_{\lambda_2} = C^-_{\leq \lambda_2} C^+_{\leq d} v_{\lambda_2}$ and
\[
\| C_d P_{\lambda_0} (C^-_{\leq d} u_{\lambda_1} C^+_{\leq d} v_{\lambda_2}) \|_{L^2_{t,x}} \leq \mu^{\frac{n-1}{2}} \| u_{\lambda_1} \|_{L^+_{t,x}} \| C^-_{\leq \lambda_2} v_{\lambda_2} \|_{L^2_{t,x}} \leq \mu^{\frac{n-1}{2}} \left( \frac{\mu}{\lambda_2} \right)^{\frac{1}{2}} \| u_{\lambda_1} \|_{L^+_{t,x}} \| v_{\lambda_2} \|_{V^2_x}.
\]
To estimate the $L^2_{t,x}$ norm of the second term in (3.24), we use $C^-_{\leq d} u_{\lambda_1} = C^-_{\leq \lambda_1} C^-_{\leq d} u_{\lambda_1}$ and obtain
\[
\| C_d P_{\lambda_0} (C^-_{\leq d} u_{\lambda_1} C^-_{\leq d} v_{\lambda_2}) \|_{L^2_{t,x}} \leq \mu^{\frac{n-1}{2}} \| u_{\lambda_1} \|_{L^+_{t,x}} \| C^-_{\leq \lambda_1} v_{\lambda_2} \|_{L^2_{t,x}} \leq \mu^{\frac{n-1}{2}} \left( \frac{\mu}{\lambda_1} \right)^{\frac{1}{2}} \| u_{\lambda_1} \|_{L^+_{t,x}} \| v_{\lambda_2} \|_{V^2_x},
\]
which implies the claimed estimate as $\lambda_1 \approx \lambda_2$ in this case. 

3.4. **Proof of Theorem 2.2** We now combine the high-low product estimates in Theorem 3.6 together with the bilinear $L^2_{t,x}$ estimate in Theorem 3.9 and show that the space $S$ solves the division problem. To simplify the proof, we start by giving the following consequence of the bilinear $L^2_{t,x}$ bound, which is used to control the close cone interactions in Theorem 2.2.
Lemma 3.12. Let $\epsilon > 0$ and $\lambda_0, \lambda_1, \lambda_2 \in 2\mathbb{Z}$ with $\mu = \min\{\lambda_0, \lambda_1, \lambda_2\}$. If $v_{\lambda_0}, u_{\lambda_1} \in S$ and $w_{\lambda_2} \in S_w$, then

$$\left| \int_{\mathbb{R}_{1+n}^+} C_{\leq \mu} v_{\lambda_0} C_{\leq \mu} u_{\lambda_1} \square C_{\leq \mu} w_{\lambda_2} \, dx \, dt \right| \lesssim \mu^{\frac{n+1}{2}} \lambda_2 (\min\{\lambda_0, \lambda_1\})^{\frac{4}{3}} \|v_{\lambda_0}\|_S \|u_{\lambda_1}\|_S \|w_{\lambda_2}\|_{S_w}. \tag{3.25}$$

Similarly, if $u_{\lambda_1} \in S$ and $v_{\lambda_0}, w_{\lambda_2} \in S_w$, then

$$\left| \int_{\mathbb{R}_{1+n}^+} C_{\leq \mu} v_{\lambda_0} C_{\leq \mu} u_{\lambda_1} \square C_{\leq \mu} w_{\lambda_2} \, dx \, dt \right| \lesssim \mu^{\frac{n+1}{2}} \lambda_2 (\min\{\lambda_0, \lambda_1\})^{\frac{4}{3}} \|v_{\lambda_0}\|_S \|u_{\lambda_1}\|_S \|w_{\lambda_2}\|_{S_w}. \tag{3.26}$$

Proof. We start by proving the bounds

$$\sum_{\mu \leq \lambda} \left| \int_{\mathbb{R}_{1+n}^+} C_{\leq \mu} v_{\lambda_0} C_{\leq \mu} u_{\lambda_1} \square C_{\leq \mu} w_{\lambda_2} \, dx \, dt \right| \lesssim \mu^{\frac{n+1}{2}} \lambda_2 (\min\{\lambda_0, \lambda_1\})^{\frac{4}{3}} \|v_{\lambda_0}\|_S \|u_{\lambda_1}\|_S \|w_{\lambda_2}\|_{S_w} \tag{3.27}$$

and, for every $\epsilon > 0$,

$$\sum_{\mu \leq \lambda} \left| \int_{\mathbb{R}_{1+n}^+} C_{\leq \mu} v_{\lambda_0} C_{\leq \mu} u_{\lambda_1} \square C_{\leq \mu} w_{\lambda_2} \, dx \, dt \right| \lesssim \mu^{\frac{n+1}{2}} \lambda_2 \left(\frac{\min\{\lambda_0, \lambda_1\}}{\mu}\right)^{\frac{4}{3}} \|v_{\lambda_0}\|_S \|u_{\lambda_1}\|_S \|w_{\lambda_2}\|_{S_w}. \tag{3.29}$$

Let $\beta = (\frac{d}{\mu})^{\frac{1}{2}}$. The bound (3.27) follows by decomposing into caps of radius $\beta$, applying Lemma 3.11 together with Lemma 3.1 and observing that

$$\int_{\mathbb{R}_{1+n}^+} C_{\mu} v_{\lambda_0} C_{\mu} u_{\lambda_1} \square C_{\mu} w_{\lambda_2} \, dx \, dt \lesssim \beta^{\frac{n+1}{2}} \mu^{\frac{n+1}{2}} \sum_{\kappa, \kappa' \in C_{\beta}} \sup_{\kappa'' \in \mathcal{C}_{\beta}} \left\| R_{\kappa} v_{\lambda_0} \right\|_{L^2_t L^\infty_x} \left\| R_{\kappa'} C_{\mu} u_{\lambda_1} \right\|_{L^2_t L^2_x} \left\| C_{\mu} w_{\lambda_2} \right\|_{L^2_t L^2_x}$$

$$\lesssim \beta^{\frac{n+1}{2}} \mu^{\frac{n+1}{2}} \lambda_2 \left(\|v_{\lambda_0}\|_S \|u_{\lambda_1}\|_S \|w_{\lambda_2}\|_{S_w}. \right.$$
Consequently summing up over modulation $d \lesssim \mu$, this gives (3.27). The proof of (3.28) is similar. More precisely, again decomposing into caps of radius $\beta$ gives

$$
\int_{\mathbb{R}^{1+n}} C_{\leq d}v_{\lambda_0} C_{\leq d}u_{\lambda_1} \Box C_{\leq d}w_{\lambda_2} \, dx \, dt
\lesssim \beta^{\frac{n+1}{2}} \mu^{\frac{n}{2}} \sum_{\kappa, \kappa', \kappa'' \in C_{\beta}} \left\| R_{\kappa} v_{\lambda_0} \right\|_{L^\infty_t L^2_x} \left\| R_{\kappa'} C_{\leq d}u_{\lambda_1} \right\|_{L^2_t L^2_x} \left\| C_{\leq d} R_{\kappa''} w_{\lambda_2} \right\|_{L^2_t x}
\lesssim \beta^{\frac{n+1}{2}} \mu^{\frac{n}{2}} \sup_{\kappa \in C_{\beta}} \left\| R_{\kappa} v_{\lambda_0} \right\|_{L^\infty_t L^2_x} \left\| C_{\leq d}u_{\lambda_1} \right\|_{L^2_t L^2_x} \left\| C_{\leq d}w_{\lambda_2} \right\|_{L^2_t x}
\lesssim d^{\frac{n+1}{2}} \mu^{\frac{n+1}{2}} \lambda_2 \left\| v_{\lambda_0} \right\|_{S_w} \left\| u_{\lambda_1} \right\|_S \left\| w_{\lambda_2} \right\|_{S_w}.
$$

Consequently summing up over modulation $d \lesssim \mu$, we have (3.28). To prove (3.29), we first observe that Theorem 3.9 implies that for every $\epsilon > 0$

$$
\left\| P_{\lambda_1} C_{\leq d}v_{\lambda_0} C_{\leq d}u_{\lambda_1} \right\|_{L^2_t x}
\lesssim d^{\frac{n+3}{2}} \mu^{\frac{n+1}{2}} \left( \min\{\lambda_0, \lambda_1\} \right)^{\frac{1}{2}} \left( \frac{\lambda_1 \min\{\lambda_0, \lambda_1\}}{d\mu} \right)^{\epsilon} \left\| v_{\lambda_0} \right\|_{S_w} \left\| u_{\lambda_1} \right\|_S \left\| w_{\lambda_2} \right\|_{S_w},
$$

where we can put $\epsilon = 0$ if $\left\| v_{\lambda_0} \right\|_{S_w}$ is replaced by $\left\| v_{\lambda_0} \right\|_S$. Hence an application of Hölder’s inequality gives

$$
\int_{\mathbb{R}^{1+n}} C_{\leq d}v_{\lambda_0} C_{\leq d}u_{\lambda_1} \Box C_{\leq d}w_{\lambda_2} \, dx \, dt
\lesssim \left\| C_{\leq d} P_{\lambda_2} \left( C_{\leq d}v_{\lambda_0} C_{\leq d}u_{\lambda_1} \right) \right\|_{L^\infty_t L^2_x} \left\| C_{\leq d}w_{\lambda_2} \right\|_{L^2_t x}
\lesssim d^{\frac{n+1}{2}} \mu^{\frac{n+1}{2}} \lambda_2 \left( \min\{\lambda_0, \lambda_1\} \right)^{\frac{1}{2}} \left( \frac{\lambda_1 \min\{\lambda_0, \lambda_1\}}{d\mu} \right)^{\epsilon} \left\| v_{\lambda_0} \right\|_{S_w} \left\| u_{\lambda_1} \right\|_S \left\| w_{\lambda_2} \right\|_{S_w},
$$

where, again, we can put $\epsilon = 0$ if $\left\| v_{\lambda_0} \right\|_{S_w}$ is replaced by $\left\| v_{\lambda_0} \right\|_S$. Summing up over modulation $d \lesssim \mu$, this gives (3.29).

In order to finally prove (3.25) and (3.26), we decompose the product into

$$
C_{\leq \mu} v_{\lambda_0} C_{\leq \mu} u_{\lambda_1} \Box C_{\leq \mu} w_{\lambda_2} = \sum_{d \leq \mu} C_{d}v_{\lambda_0} C_{d}u_{\lambda_1} \Box C_{d}w_{\lambda_2} + \sum_{d \leq \mu} C_{<d}v_{\lambda_0} C_{d}u_{\lambda_1} \Box C_{<d}w_{\lambda_2}.
$$

Estimate (3.27) takes care of the first term and estimate (3.28) yields the required bound for the second term. Further, we write

$$
\sum_{d \leq \mu} C_{d}v_{\lambda_0} C_{d}u_{\lambda_1} \Box C_{d}w_{\lambda_2} = \sum_{d \leq \mu} C_{d}v_{\lambda_0} C_{d}u_{\lambda_1} \Box C_{d}w_{\lambda_2} + \sum_{d \leq \mu} C_{<d}v_{\lambda_0} C_{d}u_{\lambda_1} \Box C_{d}w_{\lambda_2} + \sum_{d \leq \mu} C_{d}v_{\lambda_0} C_{<d}u_{\lambda_1} \Box C_{d}w_{\lambda_2},
$$

and now estimate (3.29) gives an acceptable bound for the first term. For the second and the third term we again use estimate (3.27) and (3.28), in addition to the uniform disposability of the modulation projections. \(\square\)

We now come to the proof of Theorem 2.2.
Proof of Theorem 2.2. We start with the proof of the algebra property, (2.3). Let \( \mu = \min\{\lambda_0, \lambda_1, \lambda_2\} \) and decompose the product into

\[
 u_{\lambda_1} v_{\lambda_2} = (u_{\lambda_1}, v_{\lambda_2} - C_{\leq \mu} u_{\lambda_1} C_{\leq \mu} v_{\lambda_2}) + C_{\gg \mu} (C_{\leq \mu} u_{\lambda_1} C_{\leq \mu} v_{\lambda_2}) + C_{\leq \mu} (C_{\leq \mu} u_{\lambda_1} C_{\leq \mu} v_{\lambda_2}).
\]

For the first term, at least one of \( u_{\lambda_1} \) or \( v_{\lambda_2} \) must have modulation at distance \( \gg \mu \) from the cone, hence an application of Theorem 3.6 gives

\[
 \|P_0(u_{\lambda_1}, v_{\lambda_2} - C_{\leq \mu} u_{\lambda_1} C_{\leq \mu} v_{\lambda_2})\|_S \lesssim \mu^\frac{\mu}{2} \left( \frac{\min\{\lambda_1, \lambda_2\}}{\mu} \right)^{\frac{\mu}{2}} \|u_{\lambda_1}\|_S \|v_{\lambda_2}\|_S.
\]

For the second term in (3.30), Lemma 3.10 implies that we have the identity

\[
 C_{\gg \mu} P_0 (C_{\leq \mu} u_{\lambda_1} C_{\leq \mu} v_{\lambda_2}) = C_{\gg \lambda_{\max}} P_0 (C_{\leq \mu} u_{\lambda_1} C_{\leq \mu} v_{\lambda_2})
\]

where \( \lambda_{\max} = \max\{\lambda_0, \lambda_1, \lambda_2\} \). Hence Lemma 3.1 and Theorem 3.9 give

\[
 \|C_{\gg \mu} P_0 (C_{\leq \mu} u_{\lambda_1} C_{\leq \mu} v_{\lambda_2})\|_S \lesssim \lambda_{\max} \frac{\lambda_{\max}}{\lambda_0} \|C_{\gg \mu} P_0 (C_{\leq \mu} u_{\lambda_1} C_{\leq \mu} v_{\lambda_2})\|_{L^2_t L^\infty_x}
\]

\[
 \lesssim \lambda_{\max} \frac{\lambda_{\max}}{\lambda_0} \mu^{\frac{\mu}{2}} \left( \frac{\min\{\lambda_1, \lambda_2\}}{\mu} \right)^{\frac{\mu}{2}} \|u_{\lambda_1}\|_S \|v_{\lambda_2}\|_S
\]

provided we take \( \epsilon > 0 \) sufficiently small, and \( n \geq 2 \). For the last term in (3.30), we apply (3.25) in Lemma 5.12 together with the characterisation of \( S \) in Lemma 3.1 to deduce that

\[
 \|C_{\leq \mu} (C_{\leq \mu} u_{\lambda_1} C_{\leq \mu} v_{\lambda_2})\|_S \lesssim \sup_{\phi \in C_0^\infty} \left| \int_R \langle \nabla \Delta C_{\leq \mu} \phi_{\lambda_0}, C_{\leq \mu} u_{\lambda_1} C_{\leq \mu} v_{\lambda_2} \rangle \right| dt
\]

\[
 \lesssim \mu^{\frac{\mu}{2}} (\min\{\lambda_1, \lambda_2\})^{\frac{\mu}{2}} \|u_{\lambda_1}\|_S \|v_{\lambda_2}\|_S.
\]

Therefore (2.3) follows.

We now turn to the proof of (2.4). The argument is in some sense a dual version of the argument used to prove the algebra property (2.3). An application of the characterisation of \( S \) in (3.1), together with the invariance under complex conjugation of \( S \) and \( S_w \), reduces the problem to proving that

\[
 \left| \int_R \langle \phi_{\lambda_0} u_{\lambda_1}, \nabla v_{\lambda_2} \rangle \right| dt \lesssim \left( \frac{\lambda_1 \lambda_2}{\lambda_0} \right)^{\frac{\mu}{2}} \lambda_0 \|\phi_{\lambda_0}\|_{S_w} \|u_{\lambda_1}\|_S \|v_{\lambda_2}\|_S
\]

for all \( \phi \in C_0^\infty \). The first step in the proof of (3.31) is to decompose into the far cone and close cone regions

\[
 \phi_{\lambda_0} u_{\lambda_1} = (\phi_{\lambda_0} u_{\lambda_1} - C_{\leq \mu} \phi_{\lambda_0} C_{\leq \mu} u_{\lambda_1}) + C_{\gg \mu} (C_{\leq \mu} \phi_{\lambda_0} C_{\leq \mu} v_{\lambda_2}) + C_{\leq \mu} (C_{\leq \mu} \phi_{\lambda_0} C_{\leq \mu} u_{\lambda_1}).
\]

(3.32)
For the first term in (3.32), at least one of \( \phi_{\lambda_0} \) or \( u_{\lambda_1} \) must have modulation \( \gg \mu \). Hence Theorem 3.6 and Lemma 3.1 gives

\[
\left| \int_{\mathbb{R}} \langle (\phi_{\lambda_0} u_{\lambda_1} - C_{\leq \mu} \phi_{\lambda_0} C_{\leq \mu} u_{\lambda_1}), \Box u_{\lambda_2} \rangle_{L_2^2} dt \right| \lesssim \lambda_2 \| P_{\lambda_1} (\phi_{\lambda_0} u_{\lambda_1} - C_{\leq \mu} \phi_{\lambda_0} C_{\leq \mu} u_{\lambda_1}) \|_{S_u} \| u_{\lambda_2} \|_S \\
\lesssim \mu \frac{2}{\min\{\lambda_0, \lambda_1\}} \lambda_2 \| \phi_{\lambda_0} \|_{S_u} \| u_{\lambda_1} \|_S \| u_{\lambda_2} \|_S \\
\lesssim \left( \frac{\lambda_1 \lambda_2}{\lambda_0} \right)^{\frac{2}{\mu}} \| \phi_{\lambda_0} \|_{S_u} \| u_{\lambda_1} \|_S \| u_{\lambda_2} \|_S.
\]

On the other hand, to bound the second term in (3.32), we note that Lemma 3.10 gives the identity

\[
C_{\geq \mu} P_{\lambda_2} (C_{\leq \mu} \phi_{\lambda_0} C_{\leq \mu} u_{\lambda_1}) = C_{\geq \lambda_{\max}} P_{\lambda_2} (C_{\leq \mu} \phi_{\lambda_0} C_{\leq \mu} u_{\lambda_1})
\]

and hence Theorem 3.9 together with Lemma 3.1 gives

\[
\left| \int_{\mathbb{R}} \langle C_{\geq \mu} P_{\lambda_0} (C_{\leq \mu} \phi_{\lambda_0} C_{\leq \mu} u_{\lambda_1}), \Box u_{\lambda_2} \rangle_{L_2^2} dt \right| \lesssim \lambda_{\max} \lambda_2 \left| \int_{\mathbb{R}} \langle C_{\geq \mu} P_{\lambda_0} (C_{\leq \mu} \phi_{\lambda_0} C_{\leq \mu} u_{\lambda_1}), u_{\lambda_2} \rangle_{L_2^2} dt \right| \\
\lesssim \mu \frac{2}{\min\{\lambda_0, \lambda_1\}}^{\frac{1}{\mu}} \lambda_2 \lambda_{\max} \| \phi_{\lambda_0} \|_{S_u} \| u_{\lambda_1} \|_S \| u_{\lambda_2} \|_S \\
\lesssim \left( \frac{\lambda_1 \lambda_2}{\lambda_0} \right)^{\frac{2}{\mu}} \| \phi_{\lambda_0} \|_{S_u} \| u_{\lambda_1} \|_S \| u_{\lambda_2} \|_S.
\]

Finally, to bound the last term in (3.32), we apply the close cone bound in Lemma 3.12 and conclude that

\[
\left| \int_{\mathbb{R}} \langle C_{\geq \mu} P_{\lambda_0} (C_{\leq \mu} \phi_{\lambda_0} C_{\leq \mu} u_{\lambda_1}), \Box u_{\lambda_2} \rangle_{L_2^2} dt \right| \\
\lesssim \mu \frac{2}{\min\{\lambda_0, \lambda_1\}}^{\frac{1}{\mu}} \lambda_2 \left( \frac{\min\{\lambda_0, \lambda_1\}}{\mu} \right)^{\epsilon} \left( \frac{\lambda_1}{\mu} \right)^{\epsilon} \| \phi_{\lambda_0} \|_{S_u} \| u_{\lambda_1} \|_S \| u_{\lambda_2} \|_S \\
\lesssim \left( \frac{\lambda_1 \lambda_2}{\lambda_0} \right)^{\frac{2}{\mu}} \| \phi_{\lambda_0} \|_{S_u} \| u_{\lambda_1} \|_S \| u_{\lambda_2} \|_S.
\]

Therefore we obtain (3.31). \( \square \)

4. The spaces \( U^p \) and \( V^p \)

In this section we briefly review some of the fundamental properties of the spaces \( U^p \) and \( V^p \), in particular we discuss embedding properties, almost orthogonality principles, and the dual pairing. The material we present in this section can essentially be found in \([10, 14, 15]\) and thus we shall be somewhat brief but self-contained (up to the proof of Theorem 4.1). Using the notation introduced in Subsection 2.1 a function \( u : \mathbb{R} \rightarrow L^2(\mathbb{R}^n) \) is a step function with partition \( \tau \in \mathcal{P} \) if for each \( I \in \mathcal{I}_\tau \) there exists \( f_I \in L^2(\mathbb{R}^n) \) such that

\[
u(t) = \sum_{I \in \mathcal{I}_\tau} 1_I(t) f_I.
\]

By definition, all step functions vanish for sufficiently negative \( t \). Define \( \mathfrak{S} \) to be the collection of all step functions with partitions \( \tau \in \mathcal{P} \), and let \( \mathfrak{S}_0 \) denote those
elements of \( \mathcal{S} \) with compact support. In other words, step functions in \( \mathcal{S}_0 \) vanish on the final interval \([t_N, \infty)\).

The normalisation condition at \( t = -\infty \) implies that \( \mathcal{S} \subset V^p \). However step functions are not dense in \( V^p \); this follows by considering an example of Young \[36\] that shows that \( V^p \not\subset U^p \[15\] Theorem B.22]. On the other hand the set of step functions, \( \mathcal{S} \), is dense in \( U^p \), this follows by noting that \( U^p \) atoms are step functions, and if \( u = \sum c_j u_j \) is an atomic decomposition of \( u \in U^p \), then letting \( \phi_N = \sum_{j \leq N} c_j u_j \) we get a sequence of step functions such that

\[
\|u - \phi_N\|_{U^p} \leq \sum_{j \geq N} |c_j| \to 0
\]
as \( N \to \infty \).

Recall that

\[
|v|_{V^p} = \sup_{\{t_j\}_{j=1}^{N-1} \in \mathcal{P}} \left( \sum_{j=1}^{N-1} \| v(t_{j+1}) - v(t_j) \|_{L^2}^p \right)^{1/p}.
\]

A computation shows that for any \( v : \mathbb{R} \to L^2 \) and \( t_0 \in \mathbb{R} \) we have

\[
2^{-1}(\|v(t_0)\|_{L^2}^p + |v|_{V^p}^p) \leq \|v\|_{V^p} \leq 2(\|v(t_0)\|_{L^2}^p + |v|_{V^p}^p).\]

In particular, if \( v \in V^p \), then letting \( t_0 \to -\infty \), we see that \( |\cdot|_{V^p} \) and \( \|\cdot\|_{V^p} \) are equivalent norms on \( V^p \). The definition of the spaces \( U^p \) and \( V^p \) implies that they are both Banach spaces and convergence with respect to \( \|\cdot\|_{U^p} \) or \( \|\cdot\|_{V^p} \) implies uniform convergence (i.e. in \( \|\cdot\|_{L^\infty L^2} \)).

### 4.1. Embedding properties

For \( 1 \leq q < r < \infty \) we have the continuous embeddings \( U^q \subset V^q \subset V^p \). An example due to Young \[36\] (see \[16\] Lemma 4.15]) shows that \( V^q \not\subset U^q \). On the other hand, for \( p < q \) we do have \( V^p \subset U^q \).

**Theorem 4.1.** Let \( 1 \leq p < q < \infty \) and \( v \in V^p \). There exists a decomposition 

\[
v = \sum_{j=1}^{\infty} v_j \quad \text{such that} \quad \|v_j\|_{U^q} \leq 2^{j(\frac{q}{p}-1)}|v|_{V^p}.
\]

In particular, the embedding \( V^p \subset U^q \) holds.

**Proof.** The proof can be found in \[14\] pp. 255–256], \[10\] p. 923 or \[15\]. Proof of Theorem B.18].

**Remark 4.2.** To gain some intuition into the \( U^p \) and \( V^p \) spaces, we note the following properties:

(i) (\( C^\infty_0 \subset U^p, V^p \)) Clearly we have \( C^\infty_0 \subset V^1 \). Consequentially an application of Theorem 4.1 implies that for \( p > 1 \) we have \( C^\infty_0 \subset U^p \). More generally, if \( \partial_t u \in L^1 L^2 \) and \( \lim_{t \to -\infty} u(t) = 0 \), then \( u \in V^1 \subset U^p \) and

\[
\|u\|_{U^p} \lesssim \|u\|_{V^1} \lesssim \|\partial_t u\|_{L^1 L^2} \quad (4.1)
\]

by Theorem 4.1

(ii) (Approximation of \( C^\infty_0 \) functions in \( V^p \), \( p > 1 \)) The example of Young shows that \( \mathcal{S} \) is not dense in \( V^p \). On the other hand, \( \mathcal{S} \) is dense in the closure of \( V^p \cap C^\infty_0 \) for \( p > 1 \). This follows by observing that given \( \tau \in \mathcal{P} \) and defining

\[
u_\tau = \sum_{j=1}^N \chi_{[t_j, t_{j+1})} u(t_j) \quad \text{(with} \ t_{N+1} = \infty) \text{we have}
\]

\[
|u - u_\tau|_{V^p} \lesssim \left( \sup_j \int_{t_j}^{t_{j+1}} \|\partial_t u\|_{L^2} \right)^{p-1} \int_{\mathbb{R}} \|\partial_t u\|_{L^2} \, dt.
\]
(iii) (Counterexample for $p = 1$) If we suppose that $u(t) = \rho(t)f$ with $f \in L^2$ and $\rho$ monotone on an interval $(a, b)$, then $u$ cannot be approximated by step functions using the $V^1$ norm. As a consequence $u \notin U^1$. In particular, if

$$
\rho(t) = \begin{cases} 
  t & t \in [0, 1) \\
  0 & t \notin [0, 1)
\end{cases}
$$

then $u \in U^p$ but not in $U^1$. In fact, a similar example shows that $C_0^\infty \not\subset U^1$.

4.2. **Almost orthogonality.** The definition of the spaces $U^p$ and $V^p$ implies that they satisfy a one sided version of the standard almost orthogonality property.

**Proposition 4.3** (Almost orthogonality in $U^p$ and $V^p$). Let $M_1, M_2 \in [0, \infty]$. For $k \in \mathbb{N}$, let $T_k : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be a linear and bounded operator (acting spatially) such that

$$
M_1 \|f\|_{L^2} \leq \left( \sum_{k \in \mathbb{N}} \|T_k f\|_{L^2}^2 \right)^{\frac{1}{2}} \leq M_2 \|f\|_{L^2} \quad \text{for all } f \in L^2.
$$

If $1 \leq p \leq 2$, then for all $u \in U^p$ we have the bound

$$
\left( \sum_{k \in \mathbb{N}} \|T_k u\|_{U^p}^2 \right)^{\frac{1}{2}} \leq M_2 \|u\|_{U^p}.
$$

On the other hand, if $p \geq 2$, then for all $v \in V^p$ we have the bound

$$
\left( \sum_{k \in \mathbb{N}} \|T_k v\|_{V^p}^2 \right)^{\frac{1}{2}} \leq M_1 \left( \sum_{k \in \mathbb{N}} \|T_k v\|_{V^p}^2 \right)^{\frac{1}{2}}.
$$

**Proof.** We start with the $U^p$ bound. It is enough to consider the case where $u = \sum_{1 \leq m \leq N} \mathbb{1}_{[t_m, t_{m+1}]}(t)f_k$ is a $U^p$-atom. Then by definition of the $U^p$ norm, together with the assumption $1 \leq p \leq 2$, we have

$$
\left( \sum_{k \in \mathbb{N}} \|T_k u\|_{U^p}^2 \right)^{\frac{1}{2}} \leq \left( \sum_{k \in \mathbb{N}} \left( \sum_{1 \leq m \leq N} \|T_k f_m\|_{L^2}^p \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}
$$

$$
\leq \left( \sum_{1 \leq m \leq N} \left( \sum_{k \in \mathbb{N}} \|T_k f_m\|_{L^2}^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \leq M_2,
$$

as required.

Concerning the $V^p$ bound, we let $\tau = (t_j)_{j=1}^N \in \mathcal{P}$ be any partition. We compute

$$
\left( \sum_{j=1}^{N-1} \left\| \sum_{k \in \mathbb{N}} T_k v(t_{j+1}) - \sum_{k \in \mathbb{N}} T_k v(t_j) \right\|_{L^2}^p \right)^{\frac{1}{2}}
$$

$$
\leq \left( \sum_{j=1}^{N-1} \left\| \sum_{k \in \mathbb{N}} (v(t_{j+1}) - v(t_j)) \right\|_{L^2}^p \right)^{\frac{1}{2}}
$$

$$
\leq M_1 \left( \sum_{j=1}^{N-1} \left( \sum_{k \in \mathbb{N}} \|T_k(v(t_{j+1}) - v(t_j))\|_{L^2}^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}
$$

$$
\leq M_1 \left( \sum_{k \in \mathbb{N}} \|T_k v\|_{V^p}^2 \right)^{\frac{1}{2}},
$$

as required. \qed
4.3. The dual pairing. Due to the atomic definition of $U^p$, it can be very difficult to estimate $\|u\|_{U^p}$ for $u \in U^p$. For instance, even the question of estimating the $U^p$ norm of a step function is a nontrivial problem. However, there is a duality argument that can reduce this problem to estimating a certain bilinear pairing between elements of $U^p$ and $V^p$. In the following we give two possible versions of this pairing, a discrete version and a continuous version. More precisely, given a step function $w \in S$ with partition $(t_j)_{j=1}^N \in P$, and a function $u : \mathbb{R} \to L^2$, we define the dual pairing

$$B(w, u) = \langle w(t_1), u(t_1) \rangle_{L^2} + \sum_{j=2}^N \langle w(t_j) - w(t_{j-1}), u(t_j) \rangle_{L^2}.$$  

The bilinear pairing $B$ is well defined, and sesquilinear in $w \in S$ and $u$. The linearity in $u$ is immediate, while the remaining properties follow by observing that if $w \in S$ is a step function with partition $\tau = (t_j) \in P$, that is also a step function with respect to another partition $\tau' = (t'_j) \in P$, then a computation using the fact that $w(t) = 0$ for $t < \max\{t_1, t'_1\}$ gives

$$\langle w(t_1), u(t_1) \rangle_{L^2} + \sum_{j=2}^N \langle w(t_j) - w(t_{j-1}), u(t_j) \rangle_{L^2} = \langle w(t'_1), u(t'_1) \rangle_{L^2} + \sum_{j=2}^{N'} \langle w(t'_j) - w(t'_{j-1}), u(t'_j) \rangle_{L^2}.$$  

It is not so difficult to show that if $w \in S$ and $u \in U^p$ then

$$|B(w, u)| \leq \|v\|_{V^q} \|u\|_{U^p},$$

we give the details of this computation in Theorem 4.4 below. Thus $B(w, u)$ gives a way to pair (step) functions in $V^q$ with $U^p$ functions. Alternative pairings are also possible, for instance we can also use the continuous pairing

$$\int_{\mathbb{R}} \langle \partial_t \phi, u \rangle_{L^2} dt$$

for $\partial_t \phi \in L^1_t L^2_x$ and $u \in U^p$. Clearly (4.2) and $B$ are closely related, since if $w \in S$ and $u \in C^1$ we have

$$B(w, u) = \int_{\mathbb{R}} \langle v, \partial_t u \rangle dt$$

where the step function

$$v(t) = 1_{(-\infty, t_1)}(t)w(t_1) - \sum_{j=1}^{N-1} 1_{[t_j, t_{j+1})}(t)[w(t_N) - w(t_{j-1})].$$

If we had $v \in C^1$, then integrating by parts would give (4.2). More general pairings are also possible, this relates to the more general Stieltjes integral, and using a limiting argument, the definition of $B(w, u)$ can be extended from $w \in S$ to elements $w \in V^p$, see [10]. However for our purposes, it suffices to work with the pairings $B$ and (4.2) as defined above. The pairings are useful due to the following.

**Theorem 4.4 (Dual pairing for $U^p$).** Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. If $u \in U^p$ and $\partial_t \phi \in L^1_t L^2_x$ we have

$$\left| \int_{\mathbb{R}} \langle \partial_t \phi, u \rangle_{L^2} dt \right| \leq \|\phi\|_{V^q} \|u\|_{U^p}.$$
and
\[
\|u\|_{U^p} = \sup_{w \in \mathcal{S}, |w|_{V^q} \leq 1} |B(w, u)| = \sup_{\partial_t \phi \in C^0_0} \left| \int_{\mathbb{R}} \langle \partial_t \phi, u \rangle_{L^2} dt \right|.
\]

**Proof.** We start by proving that for every \( u \in U^p \) we have
\[
\sup_{\partial_t \phi \in L^1_t L^2_x} \left| \int_{\mathbb{R}} \langle \partial_t \phi, u \rangle_{L^2} dt \right| \leq \sup_{w \in \mathcal{S}, |w|_{V^q} \leq 1} |B(w, u)| \leq \|u\|_{U^p}. \quad (4.3)
\]

By definition of the \( U^p \) norm, it is enough to consider the case where \( u \in \mathcal{S} \) is a \( U^p \)-atom. The second inequality in (4.3) follows by observing that if \( w \in \mathcal{S} \) with partition \( \tau = (t_j)_{j=1}^N \in \mathcal{P} \), and we let \( E = \{ t_j \in \tau \mid u(t_j) = u(t_{j+1}) \} \) and \( \tau^* = (t^*_k)_{k=1}^{N^*} = \tau \setminus E \) (i.e. we remove points from \( \tau \) where the step function \( u \) is constant) then by definition of the bilinear pairing we have
\[
|B(w, u)| = \left| \langle w(t_1), u(t_1) \rangle_{L^2} + \sum_{j=2}^N \langle w(t_j) - w(t_{j-1}), u(t_j) \rangle_{L^2} \right|
\]
\[
= \left| \langle w(t^*_1), u(t^*_1) \rangle_{L^2} + \sum_{j=2}^{N^*} \langle w(t^*_j) - w(t^*_{j-1}), u(t^*_j) \rangle_{L^2} \right|
\]
\[
\leq \left( |w(t^*_1)|^2_{L^2} + \sum_{j=2}^{N^*} \|w(t^*_j) - w(t^*_{j-1})\|_{L^2}^2 \right)^{1/2} \left( \sum_{j=1}^{N^*} \|u(t^*_j)\|_{L^p}^p \right)^{1/2} \leq |w|_{V^q}.
\]

where the last line follows by noting that \( w(t) = 0 \) for \( t < t^*_1 \) and \( u \) is a \( U^p \) atom together with the definition of the partition \( \tau^* \). On the other hand, to prove the first inequality in (4.3), we observe that if \( u = \mathbb{1}_{[s_{N'}, \infty)}(t)u(s_{N'}) + \sum_{k=1}^{N'-1} \mathbb{1}_{[s_k, s_{k+1})}(s)u(s_k) \) and \( T > s_{N'} \), we have
\[
\int_{-\infty}^T \langle \partial_t \phi, u \rangle_{L^2} dt = \langle \phi(T), 0, u(s_{N'}) \rangle_{L^2} + \sum_{j=1}^{N'} \langle \phi(s_{j+1} - \phi(s_j), u(s_j))_{L^2}
\]
\[
= \langle w(s_1), u(t_1) \rangle_{L^2} + \sum_{j=2}^{N'} \langle w(s_j) - w(s_{j-1}), u(t_j) \rangle_{L^2}
\]
where we define \( w \in \mathcal{S} \) as
\[
w(t) = \sum_{j=1}^{N'-1} \mathbb{1}_{[s_j, s_{j+1})}(t)\left( \phi(s_{j+1}) - \phi(s_j) \right) + \mathbb{1}_{[s_{N'}, \infty)}(t)\left( \phi(T) - \phi(s_1) \right).
\]

Since \( |w|_{V^q} \leq |\phi|_{V^q} \) we conclude that
\[
\left| \int_{\mathbb{R}} \langle \partial_t \phi, u \rangle_{L^2} dt \right| \leq \int_{-\infty}^\infty \|\partial_t \phi(t)\|_{L^2} \|u\|_{L^\infty_x L^2_t} + |\phi|_{V^q} \sup_{w \in \mathcal{S}, |w|_{V^q} \leq 1} |B(w, u)|.
\]

Hence letting \( T \to \infty \) and using the assumption \( \partial_t \phi \in L^1_t L^2_x \) the bound (4.3) follows.

It remains to show that for \( u \in U^p \) we have the bound
\[
\|u\|_{U^p} \leq \sup_{\partial_t \phi \in L^1_t L^2_x} \left| \int_{\mathbb{R}} \langle \partial_t \phi, u \rangle_{L^2} dt \right|.
\]
Since the set of step functions $\mathcal{S} \subset U^p$ is dense, it suffices to consider the case $u = \mathbb{1}_{[t,\infty)}(t)f_N + \sum_{j=1}^{N-1} \mathbb{1}_{[t_j,t_{j+1})}(t)f_j \in \mathcal{S}$. An application of the Hahn-Banach Theorem implies that there exists $L \in (U^p)^*$ such that

$$\|u\|_{U^p} = T(u), \quad \sup_{\|h\|_{U^p} \leq 1} |L(h)| = 1. \quad (4.5)$$

Note that given $f \in L^2$ and fixed $t \in \mathbb{R}$, we have $\mathbb{1}_{[t,\infty)}f \in U^p$ and $\|\mathbb{1}_{[t,\infty)}f\|_{U^p} \leq \|f\|_{L^2}$. In particular, the map $f \mapsto L(\mathbb{1}_{[t,\infty)}f)$ is a linear functional on $L^2$. Consequently, by the Riesz Representation Theorem, there exists a function $\psi : \mathbb{R} \to L^2$ such that for every $t \in \mathbb{R}$ and $f \in L^2$ we have

$$L(\mathbb{1}_{[t,\infty)}f) = \langle \psi(t), f \rangle_{L^2}.$$

By construction, we see that

$$\|u\|_{U^p} = T(u) = \sum_{j=1}^{N-1} L(\mathbb{1}_{[t_j,t_{j+1})}f_j) + L(\mathbb{1}_{[t,N,\infty)}f_N) = \sum_{j=1}^{N-1} \langle \psi(t_j) - \psi(t_{j+1}), f_j \rangle_{L^2} + \langle \psi(t_N), f_N \rangle.$$

Let $\rho \in C_0^\infty(-1,1)$ with $\int_{\mathbb{R}} \rho = 1$, and define

$$\phi_\epsilon(t) = -\int_{\mathbb{R}} \frac{1}{\epsilon} \rho\left(\frac{t-s}{\epsilon} + 1\right)w(t)dt$$

with

$$w(t) = \mathbb{1}_{(-\infty,t_1)}(t)\psi(t_1) + \sum_{j=1}^{N-1} \mathbb{1}_{[t_j,t_{j+1})}(t)\psi(t_j).$$

Then provided we choose $\epsilon > 0$ sufficiently small (depending on $(t_j) \in \mathcal{P}$), we have $\phi_\epsilon(t_j) = \psi(t_j)$ for $j = 1, \ldots, N$, and $\phi_\epsilon(t) = 0$ for $t > t_N + 2\epsilon$ and consequently

$$\|u\|_{U^p} = \sum_{j=1}^{N-1} \langle \psi(t_j) - \psi(t_{j+1}), f_j \rangle_{L^2} + \langle \psi(t_N), f_N \rangle = \int_{\mathbb{R}} \langle \partial_\epsilon \phi_\epsilon, u \rangle_{L^2} dt.$$

Therefore, since $\partial_\epsilon \phi_\epsilon \in C_0^\infty$ and $|\phi_\epsilon|_{V^q} \leq |w|_{V^q}$, it only remains to show that $|w|_{V^q} \leq 1$. To this end, we start by observing that

$$|w|_{V^q} \leq \sup_{(s_j) \in \mathcal{P}} \left( \sum_{j=1}^{N'-1} \|\psi(s_{j+1}) - \psi(s_j)\|_{L^2}^q + \|\psi(s_{N'})\|_{L^2}^q \right)^{\frac{1}{q}}. \quad (4.6)$$

Fix $(s_j)_{j=1}^{N'} \in \mathcal{P}$ and define $v \in \mathcal{S}$ as

$$v(t) = \mathbb{1}_{[s_{N'},\infty)}(t) \frac{\alpha \psi(s_{N'})}{\|\psi(s_{N'})\|_{L^2}^q} + \sum_{j=1}^{N'-1} \mathbb{1}_{[s_j,s_{j+1})}(t) \frac{\alpha [\psi(s_j) - \psi(s_{j+1})]}{\|\psi(s_{j+1}) - \psi(s_j)\|_{L^2}^q}$$

with

$$\alpha = \left( \|\psi(s_{N'})\|_{L^2}^q + \sum_{1 \leq j \leq N' - 1} \|\psi(s_{j+1}) - \psi(s_j)\|_{L^2}^q \right)^{\frac{1}{q} - 1}. $$
Then \( v \) is a \( U^p \) atom, and by construction, we have
\[
L(v) = \langle \psi(sN'), v(sN') \rangle_{L^2} + \sum_{j=1}^{N'-1} \langle \psi(s_j) - \psi(s_{j+1}), v(s_j) \rangle_{L^2}
\]
\[
= \left( \|\psi(s_{N'})\|_{L^2}^2 + \sum_{j=1}^{N'-1} \|\psi(s_{j+1}) - \psi(s_j)\|_{L^2}^2 \right)^{\frac{1}{2}}.
\]
Therefore, from (4.5) and (4.6) we conclude that
\[
|w|_{V^q} \leq \sup_{v \in \mathcal{G}} \|L(v)\|_{L^1} \leq 1
\]
as required.

Remark 4.5. It is possible to replace the conditions \( w \in \mathcal{G} \) and \( \partial_s \phi \in C_0^\infty \) in Theorem 4.4 with the compactly supported functions \( w \in \mathcal{G}_0 \) and \( \phi \in C_0^\infty \). More precisely, provided that \( u(t) \to 0 \) as \( t \to -\infty \), we have
\[
\sup_{\|w\|_{V^q} \leq 1} |B(w, u)| \leq 2 \sup_{\|w\|_{V^q} \leq 1} |B(w, u)| \quad (4.7)
\]
and
\[
\sup_{\partial_s \phi \in C_0^\infty |\phi|_{V^q} \leq 1} \left| \int_{\mathbb{R}} \langle \partial_T \phi, u \rangle_{L^2} dt \right| \leq 2 \sup_{\phi \in C_0^\infty |\phi|_{V^q} \leq 1} \left| \int_{\mathbb{R}} \langle \partial_T \phi, u \rangle_{L^2} dt \right|. \quad (4.8)
\]
The factor of 2 arises as we potentially add another jump in the step function \( w \) at \( t = +\infty \) by imposing the compact support condition. To prove the discrete bound (4.7), we note that if \( w = \sum_{j=0}^{N-1} \mathbb{1}_{(t_j, t_{j+1})} f_j + \mathbb{1}_{(t_{N-1}, t_N)} f_N \) and we take \( w_T = \mathbb{1}_{(t_{N-1}, t_N)} f_N + \mathbb{1}_{(t_{N-1}, t_N)} (f_{N-1} - f_N) \) then for any \( u : \mathbb{R} \to L^2 \) we have
\[
B(w, u) = \langle f_N, u(T) \rangle_{L^2} + B(w_T, u).
\]
Since \( w_T \in \mathcal{G}_0 \), and \( |w_T|_{V^q} \leq 2|w|_{V^q} \), we conclude that if \( u(t) \to 0 \) as \( t \to -\infty \), then
\[
\sup_{\|w\|_{V^q} \leq 1} |B(w, u)| \leq 2 \sup_{\|w\|_{V^q} \leq 1} |B(w, u)|
\]
which implies (4.7). To prove (4.8), we first take a map \( \rho : \mathbb{R} \to \mathbb{R} \) such that
\[
\int_{\mathbb{R}} |\partial_s \rho| dt \leq 1, \quad \rho(t) = 1 \text{ for } t > 1, \quad \text{and } \rho(t) = 0 \text{ for } t < -1, \quad \text{and let } \rho_T(t) = \rho(t - T).
\]
Given \( \partial_s \phi \in C_0^\infty \), we take
\[
\phi_T(t) = \rho_T(t)(\phi(t) - \phi_\infty)
\]
where we take \( \phi_\pm = \lim_{t \to \pm \infty} \phi(t) \), note that \( \phi(\pm t) = \phi_\pm \) for \( t > 0 \) sufficiently large, since \( \partial_s \phi \in C_0^\infty \). Then \( \phi_T \in C_0^\infty \), \( \partial_s \phi_T = \partial_s \phi \) for \( t > T + 1 \), and for all \( T < 0 \) sufficiently negative we have
\[
|\phi_T|_{V^q} \leq \| (1 - \rho_T)(\phi - \phi_\infty) \|_{V^q} + |\phi - \phi_\infty|_{V^q}
\]
\[
\leq \| (1 - \rho_T)(\phi_{-\infty} - \phi_\infty) \|_{V^q} + |\phi|_{V^q} \leq \| \phi_{-\infty} - \phi_\infty \|_{L^2} + |\phi|_{V^q} \leq 2|\phi|_{V^q}
\]
where we used the bound
\[
\left( \sum_j |\rho_T(t_{j+1}) - \rho_T(t_j)|^q \right)^{\frac{1}{q}} \leq \int_{\mathbb{R}} |\partial_s \rho| dt = 1.
\]
Therefore, provided that $\partial_t \phi \in C_0^\infty$ and $|\phi|_{V^q} \leq 1$, we have
\[
\left| \int_\mathbb{R} (\partial_t \phi, u) dt \right| \leq \int_{-\infty}^{T+1} \|\partial_t \phi - \partial_t \phi_T\|_{L^2} \|u(t)\|_{L^2} dt + 2 \sup_{\psi \in C_0^\infty, |\psi|_{V^q} \leq 1} \left| \int_\mathbb{R} (\partial_t \psi, u)_{L^2} dt \right|.
\]
Consequently, since $u(t) \to 0$ as $t \to -\infty$, (4.8) follows by letting $T \to -\infty$.

5. Two Characterisations of $U^p$

In this section we consider the problem of determining if a general function $u : \mathbb{R} \to L^2$ belongs to $U^p$. If we apply the definition of $U^p$, this requires finding an atomic decomposition of $u$, which in general a highly non-trivial problem. An alternative approach is suggested by Theorem 4.4. More precisely, since the two norms defined by the dual pairings in Theorem 4.4 are well-defined for general functions $u : \mathbb{R} \to L^2$, we can try to use the finiteness of these quantities to characterise $U^p$. Recent work of Koch-Tataru [15] shows that this is possible by using the discrete pairing $B(w, u)$. In this section we adapt the argument used in [15], and show that it is also possible to characterise $U^p$ using the continuous pairing $\int_\mathbb{R} (\partial_t \phi, u)_{L^2} dt$.

Following [15], let $u : \mathbb{R} \to L^2$ and define the semi-norms
\[
\|u\|_{\tilde{U}^p_{dis}} = \sup_{v \in \mathcal{S}, |v|_{V^p} \leq 1} |B(v, u)| \in [0, \infty]
\]
and its continuous counterpart
\[
\|u\|_{\tilde{U}^p_{cts}} = \sup_{\phi \in C_0^\infty, |\phi|_{V^q} \leq 1} \left| \int_\mathbb{R} (\partial_t \phi, u)_{L^2} dt \right| \in [0, \infty].
\]
Note that both $\|\cdot\|_{\tilde{U}^p_{dis}}$ and $\|\cdot\|_{\tilde{U}^p_{cts}}$ are only norms after imposing the normalisation $u(t) \to 0$ as $t \to -\infty$. Our goal is to show that both $\|\cdot\|_{\tilde{U}^p_{dis}}$ and $\|\cdot\|_{\tilde{U}^p_{cts}}$ can be used to characterise $U^p$. To make this claim more precise, let $\tilde{U}^p_{dis}$ be the collection of all right continuous functions $u : \mathbb{R} \to L^2$ satisfying the normalising condition $u(t) \to 0$ (in $L^2$) as $t \to -\infty$, and the bound $\|u\|_{\tilde{U}^p_{dis}} < \infty$. Similarly, we take $\tilde{U}^p_{cts}$ to be the collection of all right continuous functions such that $u(t) \to 0$ as $t \to -\infty$ and $\|u\|_{\tilde{U}^p_{cts}} < \infty$.

**Theorem 5.1** (Characterisation of $U^p$). Let $1 < p < \infty$. Then $U^p = \tilde{U}^p_{dis} = \tilde{U}^p_{cts}$.

We give the proof of Theorem 5.1 in Subsection 5.2 below. Roughly, since Theorem 4.4 already shows that the norms are equivalent for step functions, and $\mathcal{S}$ is dense in $U^p$, it is enough to show that $\mathcal{S}$ is also dense in the spaces $\tilde{U}^p_{dis}$ and $\tilde{U}^p_{cts}$. The key step is a density argument which we give for a general bilinear pairing satisfying certain assumptions, this argument closely follows that given in [15] Appendix B for the special case of the discrete pairing.

5.1. A general density result. Let $X \subset V^q$ be a subspace and $B_{rc} \subset L^p_{\text{cts}} L^2_x$ denote the set of bounded right-continuous ($L^2_x$ valued) functions $u : \mathbb{R} \to L^2$. Let $\mathfrak{B}(v, u) : X \times B_{rc} \to \mathbb{C}$ be a sesquilinear form. For $u \in B_{rc}$, define
\[
\|u\|_{\tilde{U}^p_{\mathfrak{B}}} = \sup_{v \in X, |v|_{V^p} \leq 1} |\mathfrak{B}(v, u)| \in [0, \infty]
\]
implies that there exists \( w \) such that

\[ \text{supp } w \subset (a,b) \]

and bounded intervals implies that is enough to prove that for every \( u \)

\[ \text{supp } u \subset (a,b) \]

 fails, then there exists \( \tau \) such that

\[ \| u - u_{\tau} \| \leq C \left( \sum_{j=0}^{N} \| u \|_{\tilde{U}_{B}^{p}(t_{j},t_{j+1})}^{N} \right)^{\frac{1}{p}}, \]  

(5.1)

where we take \( t_{0} = -\infty \), \( t_{N+1} = \infty \), and define the step function

\[ u_{\tau} = \sum_{j=1}^{N} \mathbb{1}_{[t_{j},t_{j+1})}(t)u(t_{j}) \in \mathcal{S}. \]

(A2) If \( v \in X \) and \( \epsilon > 0 \), there exists a step function \( w \in \mathcal{S} \) such that

\[ |v - w|_{\mathcal{V}} < \epsilon. \]

Under the above assumptions, the set of step functions \( \mathcal{S} \) is dense in \( \tilde{U}_{B}^{p} \).

**Theorem 5.2.** Let \( 1 < p < \infty \) and assume that (A1) and (A2) hold. Then for every \( u \in \tilde{U}_{B}^{p} \) and \( \epsilon > 0 \), there exists \( \tau \in \mathbb{P} \) such that

\[ \| u - u_{\tau} \| \leq \epsilon. \]

We start by proving two preliminary results.

**Lemma 5.3.** Let \( 1 < p < \infty \) and \( \| u \|_{\tilde{U}_{B}^{p}} < \infty \). For every \( \epsilon > 0 \) there exists a partition \( (t_{k})_{k=1}^{N} \in \mathbb{P} \) such that

\[ \sup_{0 \leq k \leq N} \| u \|_{\tilde{U}_{B}^{p}(t_{k},t_{k+1})} < \epsilon \]

where \( t_{0} = -\infty \) and \( t_{N+1} = \infty \).

**Proof.** Let \( \epsilon > 0 \). Since \( \| \cdot \|_{\tilde{U}_{B}^{p}(a,b)} \) decreases as \( a \nearrow b \), the compactness of closed and bounded intervals implies that is enough to prove that for every \( -\infty < t^{*} \leq \infty \) and \( -\infty \leq t_{*} < \infty \) we can find \( t_{1} < t^{*} \) and \( t_{2} > t_{*} \) such that

\[ |u|_{\tilde{U}_{B}^{p}(t_{1},t^{*})} + |u|_{\tilde{U}_{B}^{p}(t_{*},t_{2})} < \epsilon. \]  

(5.2)

We only prove the first bound, as the second one is similar. Suppose that (5.2) fails, then there exists \( -\infty < t^{*} \leq \infty \) and \( \epsilon > 0 \) such that for every \( t_{1} < t^{*} \) we have

\[ |u|_{\tilde{U}_{B}^{p}(t_{1},t^{*})} > \epsilon. \]  

(5.3)

Let \( T_{1} < t^{*} \). By definition, (5.3) together with the fact that \( X \) is a subspace of \( V^{q} \), implies that there exists \( w_{1} \in X \) such that \( \text{supp } w_{1} \subset (T_{1},t^{*}) \), \( |w_{1}|_{V^{q}} \leq 1 \) (with \( \frac{1}{p} + \frac{1}{q} = 1 \)) and

\[ \mathcal{B}(w_{1},u) \geq \frac{\epsilon}{2}. \]
Define functions \( \mathcal{B}(w, u) \geq \varepsilon \).

In particular, at least one of the terms \( |w_j|_{V^\alpha} \leq 1 \), \( \supp w_j \subset (T_j, T_{j+1}) \), \( \mathcal{B}(w_j, u) \geq \varepsilon \).

Continuing in this manner, for every \( N \in \mathbb{N} \) we obtain a sequence \( T_1 < T_2 < \cdots < T_{N+1} < t^* \) and functions \( w_j \in X \) such that

\[
|w_j|_{V^\alpha} \leq 1, \quad \supp w_j \subset (T_j, T_{j+1}), \quad \mathcal{B}(w_j, u) \geq \varepsilon.
\]

If we now let \( w = \sum_{j=1}^{N} w_j \), then using the disjointness of the supports of the \( w_j \) we have \( |w|_{V^\alpha} \lesssim (\sum_{1 \leq j \leq N} |w_j|_{V^\alpha})^\frac{1}{2} \lesssim N^{\frac{1}{2}} \) and hence

\[
\frac{\varepsilon}{2} N \leq \sum_{j=1}^{N} \mathcal{B}(w_j, u) = \mathcal{B}(w, u) \leq |w|_{V^\alpha} \|u\|_{\mathcal{B}_u} \lesssim N^{\frac{1}{2}} \|u\|_{\mathcal{B}_u}.
\]

Letting \( N \to \infty \) we obtain a contradiction. Therefore (5.2) follows.

The second result gives the existence of functions \( w \in X \) satisfying a number of crucial properties.

**Lemma 5.4.** Let \( \frac{1}{p} + \frac{1}{q} = 1 \) with \( 1 < p < \infty \) and assume that (A1) holds. Let \( u \in \mathcal{U}^p_u \) and \( \tau = (s_k)_{k=1}^N \in \mathbb{P} \) with \( \|u - u_\tau\|_{\mathcal{B}_u} > 0 \). There exists \( w \in X \) such that \( |w|_{V^\alpha} \leq 1 \), we have the \( L^\infty \) bound

\[
\|w\|_{L^\infty L^2} \lesssim \|u - u_\tau\|_{\mathcal{B}_u} \sup_{0 \leq k \leq N} \|u\|_{\mathcal{B}_u(s_k, s_{k+1})}^{1-p} \|u\|_{\mathcal{B}_u(s_k, s_{k+1})}^{p-1},
\]

and the lower bound

\[
\mathcal{B}(w, u) \gtrsim \|u - u_\tau\|_{\mathcal{B}_u}.
\]

**Proof.** We begin by observing that by (A1) and the assumption \( \|u - u_\tau\|_{\mathcal{B}_u} > 0 \), we have

\[
\sum_{k=0}^{N} \|u\|_{\mathcal{B}_u(s_k, s_{k+1})}^{p} > 0. \tag{5.4}
\]

In particular, at least one of the terms \( \|u\|_{\mathcal{B}_u(s_k, s_{k+1})} \neq 0 \). For \( k = 0, \ldots, N \), we define functions \( w_k \in X \) as follows. If \( \|u\|_{\mathcal{B}_u(s_k, s_{k+1})} = 0 \), we take \( w_k = 0 \). On the other hand, if \( \|u\|_{\mathcal{B}_u(s_k, s_{k+1})} > 0 \), we take \( w_k \in X \) such that \( \supp w_k \subset (s_k, s_{k+1}) \), \( |w_k|_{V^\alpha} \leq 1 \), and

\[
\mathcal{B}(w_k, u) \geq \frac{1}{2} \|u\|_{\mathcal{B}_u(s_k, s_{k+1})}.
\]

Such a function \( w_k \in X \) exists by definition of \( \| \cdot \|_{\mathcal{B}_u(s_k, s_{k+1})} \). We now let \( w = \sum_{k=0}^{N} \alpha \|u\|_{\mathcal{B}_u(s_k, s_{k+1})}^{p-1} w_k \) with

\[
\alpha = 2^{-\frac{1}{p}} \left( \sum_{k=0}^{N} \|u\|_{\mathcal{B}_u(s_k, s_{k+1})}^{p} \right)^{-\frac{1}{p}}.
\]

Note that (5.3) implies that \( \alpha < \infty \), and at least one of the functions \( w_k \) is non-zero, thus \( w \in X \setminus \{0\} \). By construction, the step functions \( w_k \) have separated supports,
and hence we have the $V^q$ bound
\[
|w|_{V^q}^q \lesssim 2^{q-1} \alpha^q \sum_{k=0}^N \left\| u \right\|_{U_{s_k}^{s_k+1}(s_k)}^{(p-1)} |w_k|_{V^q}^q \lesssim 2^{q-1} \alpha^q \sum_{k=0}^N \left\| u \right\|_{U_{s_k}^{s_k+1}(s_k)}^p = 1.
\]

Moreover, via (A1), we have the $L^\infty_t$ bound
\[
\|w\|_{L^\infty_tL^2} \lesssim \alpha \sup_{0 \leq k \leq N} \left\| u \right\|_{U_{s_k}^{s_k+1}(s_k)}^{p-1} |w_k|_{V^q} \lesssim \|u - u_\tau\|_{\tilde{U}^p_{s_k} U^p_{s_k+1}}^{1-p} \sup_{0 \leq k \leq N} \left\| u \right\|_{U_{s_k}^{s_k+1}(s_k)}^{p-1}
\]
and the lower bound
\[
\mathcal{B}(w, u) = \sum_{k=0}^N \alpha \left\| u \right\|_{U_{s_k}^{s_k+1}(s_k)}^{p-1} \mathcal{B}(w_k, u) \\
\geq \frac{\alpha}{2} \sum_{k=0}^N \left\| u \right\|_{U_{s_k}^{s_k+1}(s_k)}^{p} = 2^{-\frac{1}{p}} \left( \sum_{k=0}^N \left\| u \right\|_{U_{s_k}^{s_k+1}(s_k)}^{p} \right)^{\frac{1}{p}} \gtrsim \|u - u_\tau\|_{\tilde{U}^p_{s_k}}.
\]

We now come to the proof of Theorem 5.2.

**Proof of Theorem 5.2.** We argue by contradiction. Let $u \in \tilde{U}^p_{s_k}$ and suppose there exists $\epsilon_0 > 0$ such that for every $\tau \in \mathcal{P}$ we have
\[
\|u - u_\tau\|_{\tilde{U}^p_{s_k}} \geq \epsilon_0. \tag{5.5}
\]
We claim that (5.5) implies that for each $N \in \mathbb{N}$ there exists $w_N \in X$ such that
\[
|w_N|_{V^q} \lesssim (2N)^{\frac{q}{2}}, \quad \mathcal{B}(w_N, u) \gtrsim \frac{1}{4} N \epsilon_0. \tag{5.6}
\]
But, similar to Lemma 5.3, this gives a contradiction as $N \to \infty$ since (5.6) implies that
\[
\frac{1}{4} N \epsilon_0 \lesssim \mathcal{B}(w_N, u) \lesssim \|w_N|_{V^q} \|u\|_{\tilde{U}^p_{s_k}} \lesssim (2N)^{\frac{q}{2}} \|u\|_{\tilde{U}^p_{s_k}}.
\]
Thus it only remains to show that (5.5) implies (5.6). To this end, note that (5.5) together with (A1), Lemma 5.3 and Lemma 5.4 implies that for every $\delta > 0$ there exists $w' \in X$ such that
\[
|w'|_{V^q} \lesssim 1, \quad \|w'\|_{L^\infty_tL^2} \lesssim \delta, \quad B(w', u) \gtrsim \epsilon_0. \tag{5.7}
\]
Taking $\delta = 1$ gives $w_1$ satisfying (5.6). Suppose we have $w_N$ satisfying (5.6). Let $\epsilon^* > 0$. By (A2), we have a step function $w'_N \in \mathcal{S}$ such that $|w_N - w'_N|_{V^q} < \epsilon^*$. Choose $w' \in X$ such that (5.7) holds with
\[
\delta < \epsilon^* \min_{t_1 < t_2} \frac{\|w'_N(t_1) - w'_N(t_2)\|_{L^2}}{w'_N(t_1) \neq w'_N(t_2)}
\]
(this quantity is non-zero since $w'_N$ is a step function) and define $w_{N+1} = w_N + w'$ in $X$. To check the required $V^q$ bound, suppose that $\tau = (t_j) \in \mathcal{P}$ and take
\[
\tau' = \{t_j \in \tau \mid w'_N(t_j) = w'(t_j+1)\}, \quad \tau'' = \{t_j \in \tau \mid w'_N(t_j) \neq w'_N(t_j+1)\}.
\]
Then
\[
\sum_{j=1}^{N-1} \|(w_N' + w')(t_j + 1) - (w_N' + w')(t_j)\|_{L^2}^q
\leq \sum_{t_j \in \tau'} \|w'(t_{j+1}) - w'(t_j)\|_{L^2}^q + (1 + 2\epsilon^*)^q \sum_{t_j \in \tau''} \|w_N'(t_{j+1}) - w_N'(t_j)\|_{L^2}^q
\leq 1 + (1 + 2\epsilon^*)^q 2N
\]
and consequently, by choosing \(\epsilon^* > 0\) sufficiently small we have
\[
|w_{N+1}|_{\mathcal{V}_t} \leq |w_N - w_N'|_{\mathcal{V}_t} + |w_N' + w'|_{\mathcal{V}_t} \leq \epsilon^* + (1 + 2N(1 + 2\epsilon^*))^{\frac{1}{q}} \leq (2(N + 1))^{\frac{1}{q}}.
\]
On the other hand, we have
\[
\mathcal{B}(w_{N+1}, u) = \mathcal{B}(w_N, u) + \mathcal{B}(w', u) \gtrsim N\epsilon_0 + \frac{1}{4}\epsilon_0.
\]
Consequently \(w_{N+1}\) satisfies (5.6) as required. \(\square\)

5.2. Proof of Theorem 5.1

Let \(1 < p < \infty\). An application of Theorem 5.1 implies that for every \(u \in \mathcal{S} \subset U^p\) we have
\[
\|u\|_{U^p} = \|u\|_{U^p_{cts}} = \|u\|_{U^p_{cts}}^{\text{cts}}.
\]
Since \(\mathcal{S}\) is a dense subset of \(U^p\), to prove Theorem 5.1 it suffices to show that the set of step functions \(\mathcal{S}\) is also dense in \(U^p_{cts}\) and \(U^p_{cts}\). In view of Theorem 5.2, the density of \(\mathcal{S}\) in \(U^p_{cts}\) and \(U^p_{cts}\) would follow provided that the conditions (A1) and (A2) hold true. It is clear that (A2) holds in the discrete case by definition, in the continuous case we use Remark 4.5. On the other hand, the proof of the localisation condition (A1) is more involved. Let \(-\infty \leq a < b \leq \infty\) and define local versions of the norms \(\|\cdot\|_{U^p_{cts}}^{\text{cts}}\) and \(\|\cdot\|_{U^p_{cts}}\) by taking
\[
\|u\|_{U^p_{cts}(a,b)} = \sup_{v \in \mathcal{C}(a,b)} |B(v, u)|
\]
and
\[
\|u\|_{U^p_{cts}(a,b)} = \sup_{v \in \mathcal{C}(a,b)} \left| \int_{\mathbb{R}} \langle \partial_t v, u \rangle_{L^2} dt \right|.
\]
We have reduced the proof of Theorem 5.1 to the following.

Lemma 5.5. Let \(1 < p < \infty\) and \(\tau \in \mathcal{P}\). If \(u \in L^\infty_t L^2_x\) is right continuous with \(u(t) \to 0\) as \(t \to -\infty\), and we define \(u_\tau = \sum_{j=1}^N 1_{[t_j, t_{j+1})} u(t_j)\) and \(t_0 = -\infty\), \(t_{N+1} = \infty\), then
\[
\|u - u_\tau\|_{U^p_{cts}} \leq 4 \left( \sum_{j=0}^{N} \|u\|_{U^p_{cts}(t_j, t_{j+1})}^p \right)^{\frac{1}{p}}, \quad (5.8)
\]
and
\[
\|u - u_\tau\|_{U^p_{cts}} \leq 4 \left( \sum_{j=0}^{N} \|u\|_{U^p_{cts}(t_j, t_{j+1})}^p \right)^{\frac{1}{p}}. \quad (5.9)
\]
Proof. We start by proving something similar in the $V^q$ case. Suppose that $w \in V^q$ and $\tau = (t_j)_{j=1}^N \in \mathbf{P}$. Let $t_0 = -\infty$ and $t_{N+1} = \infty$. For $j = 0, \ldots, N$ let $I_j \subset (t_j, t_{j+1})$ be a left closed and right open interval, and define $w_j = 1_{I_j}(w - w(a_j))$ where $a_j \in (t_j, t_{j+1}]$. We claim that

$$
\sum_{j=0}^N |w_j|_{V^q}^q \leq 2^{q+1} |w|_{V^q}.
$$

To prove the claim, we observe that for every $\epsilon > 0$ there exists $(s^{(j)}_k)_{k=1}^{N_j} \in \mathbf{P}$ such that $|w_j|_{V^q} \leq \epsilon + \sum_{k=1}^{N_j} \|w_j(s^{(j)}_k) - w_j(s^{(j)}_{k-1})\|_{L^2}^q$. Without loss of generality, we may assume that $s^{(j)}_1 = t_j, s^{(j)}_N = t_{j+1}$ for $1 \leq j \leq N - 1$, and $s^{(0)}_{N_0} = t_1, s^{(N)}_1 = t_N$. The definition of $w_j$ then gives

$$
\sum_{k=1}^{N_j-1} \|w_j(s^{(j)}_k) - w_j(s^{(j)}_{k+1})\|_{L^2}^q \\
\leq 2^{q-1} \sum_{k=1}^{N_j-1} \|\left(1_{I_j}(s^{(j)}_k) - 1_{I_j}(s^{(j)}_k)\right)(w(s^{(j)}_k) - w(a_j))\|_{L^2}^q \\
+ 2^{q-1} \sum_{k=1}^{N_j-1} \|w(s^{(j)}_k) - w(s^{(j)}_{k+1})\|_{L^2}^q \\
\leq 2^{q-1} \left( \|w(s^{(j)}_{k_{min}}) - w(a_j)\|_{L^2}^q + \|w(s^{(j)}_{k_{max}}) - w(a_j)\|_{L^2}^q \right) \\
+ 2^{q-1} \sum_{k=1}^{N_j-1} \|w(s^{(j)}_k) - w(s^{(j)}_{k+1})\|_{L^2}^q
$$

for some $t_j \leq s^{(j)}_{k_{min}} < s^{(j)}_{k_{max}} \leq t_{j+1}$. Summing up over $j$, and choosing $\epsilon > 0$ sufficiently small, we then obtain (5.10).

We now turn the proof of (5.8). Let $\tau = (t_j)_{j=1}^N \in \mathbf{P}$ and $v \in \mathcal{S}, \epsilon > 0$, and for $j = 1, \ldots, N - 1$ define

$$
v_j(t) = 1_{[t_j + \epsilon, t_{j+1} - \epsilon]}(t)(v(t) - v(t_{j+1} - \epsilon)),
$$

and

$$
v_0(t) = 1_{[-\epsilon, t_1]}(t)(v(t) - v(t_1 - \epsilon)), \quad v_N(t) = 1_{[t_N + \epsilon, \infty)}(t)(v(t) - v(\infty))
$$

where we let $v(\infty) = \lim_{t \rightarrow \infty} v(t)$, note that $v(t) = v(\infty)$ for all sufficiently large $t$ since $v \in \mathcal{S}$. By construction and the bound (5.10), we have $v_j \in \mathcal{S}, \text{supp} v_j \subset (t_j, t_{j+1})$, and $\sum_{j=0}^N |v_j|_{V^q}^q \leq 2^{q+1} |v|_{V^q}^q$. Suppose that we can show that for all $\delta > 0$, by choosing $\epsilon = \epsilon(u, v, \delta, \tau) > 0$ sufficiently small we have for all $j = 0, \ldots, N$

$$
|B(1_{[t_j, t_{j+1}]}v - v_j, u - u_\tau)| \leq \delta.
$$

(5.11)
Then, by linearity and definition of the bilinear pairing $B$, we have $B(v, u - u_\tau) = 0$ and hence

$$|B(v, u - u_\tau)| \leq (N + 1) \max_j |B(\mathbb{1}_{[t_j, t_{j+1}]} v - v_j, u - u_\tau)| + \sum_{j=0}^N |B(v_j, u)|$$

$$\leq (N + 1)\delta + \sum_{j=0}^N |v_j| |\nu| |\nu| U_{t_j,t_{j+1}}^{p/2}$$

$$\leq (N + 1)\delta + 4|v| |\nu| \left(\sum_{j=0}^N |u| U_{t_j,t_{j+1}}^{p/2}\right)^{1/2}.$$  

Since this holds for every $\delta > 0$, (5.8) follows. Thus it remains to prove (5.11). Since $v \in \mathcal{S}$ is a step function with $v(-\infty) = 0$, provided that we choose $\epsilon > 0$ sufficiently small, a computation gives for $j = 1, \ldots, N$ the identities

$$\mathbb{1}_{[t_j, t_{j+1}]}(t)v(t) - v_j(t) = \mathbb{1}_{[t_j, t_j+\epsilon]}(t)v(t_j) + \mathbb{1}_{[t_j+\epsilon, t_{j+1}]}(t)v(t_{j+1} - \epsilon),$$

and

$$\mathbb{1}_{(-\infty,t_1]}(t)v(t) - v_0(t) = \mathbb{1}_{[-\epsilon,t_1]}(t)v(t_1 - \epsilon).$$

Hence by definition we have for $j = 1, \ldots, N$

$$B(\mathbb{1}_{[t_j, t_{j+1}]}v - v_j, u - u_\tau) = \langle v(t_{j+1} - \epsilon) - v(t_j), u(t_j + \epsilon) - u(t_j)\rangle,$$

and

$$B(\mathbb{1}_{(-\infty,t_1]}v - v_0, u - u_\tau) = \langle v(t_1 - \epsilon), u(-\epsilon^{-1})\rangle.$$  

Therefore (5.11) follows from the right continuity of $u$ and the normalisation condition on $u$ at $-\infty$. This completes the proof of (5.8).

We now turn to the proof of (5.9). Let $\partial_t \phi \in C_0^\infty$, $\epsilon > 0$, and define

$$\phi_j(t) = \int_{\mathbb{R}} \frac{1}{\epsilon} \rho\left(\frac{s}{\epsilon}\right) \mathbb{1}_{I_j}(t - s) ds (\phi(t) - \phi(t_{j+1}))$$

with $I_j = [t_j + \epsilon, t_{j+1} - \epsilon)$ for $j = 1, \ldots, N - 1$, $I_0 = [-\epsilon^{-1}, t_1 - \epsilon)$, $I_N = [t_N + \epsilon, \epsilon^{-1}]$, and we take $\rho \in C_0^\infty(-1, 1)$ with $\int_{\mathbb{R}} \rho = 1$. Then clearly $\phi_j \in C_0^\infty$, supp $\phi_j \subset (t_j, t_{j+1})$, and a short computation using the bound (5.10) gives

$$\sum_j \|\phi_j\|_{V_q} \leq 2q+1 |\phi|_{V_q}. $$  

Consequently, similar to the discrete case above, it is enough to show that for every $\delta > 0$ we can find an $\epsilon = \epsilon(\phi, u, \tau, \delta) > 0$ such that

$$\left|\int_{t_j}^{t_{j+1}} (\partial_t \phi - \partial_t \phi_j, u - u_\tau) dt\right| \leq \delta. $$  

(5.12)

This is a consequence of the right continuity of $u$. More precisely, if $j = 1, \ldots, N - 1$, after writing

$$\partial_t \phi(t) - \partial_t \phi_j(t) = \left(1 - \int_{\mathbb{R}} \frac{1}{\epsilon} \rho\left(\frac{s}{\epsilon}\right) \mathbb{1}_{I_j}(t - s) ds\right) \partial_t \phi(t)$$

$$- \frac{\phi(t) - \phi(t_{j+1})}{\epsilon} \int_{\mathbb{R}} \frac{1}{\epsilon} \partial_s \rho\left(\frac{s}{\epsilon}\right) \mathbb{1}_{I_j}(t - s) ds$$
we see that
\[
\left| \int_{t_j}^{t_{j+1}} (\partial_t \phi - \partial_t \phi_j, u - u_t) dt \right| \\
\lesssim \epsilon \| \partial_t \phi \|_{L^p_t L^2_x} \| u \|_{L^p_t L^2_x} + \epsilon^{-1} \| \phi \|_{L^p_t L^2_x} \int_{t_j}^{t_{j+3\epsilon}} \| u(t) - u(t_j) \|_{L^2} dt \\
+ \epsilon^{-1} \| u \|_{L^p_t L^2_x} \int_{t_{j+1} - 3\epsilon}^{t_{j+1}} \| \phi(t) - \phi(t_j) \|_{L^2} dt 
\]
and hence (5.12) follows by the right continuity of \( u \) provided we choose \( \epsilon \) sufficiently small. A similar argument proves the cases \( j = 0 \) and \( j = N \). \( \square \)

6. Convolution, multiplication, and the adapted function spaces

In this section we record a number of key properties of the \( U^p \) and \( V^p \) spaces that have been used frequently throughout this article. Namely, in Subsection 6.1 we prove a basic convolution estimate together with the standard Besov embedding
\[
\dot{B}^\frac{1}{p,1}_p \subset V^p \subset U^p \subset \dot{B}^\frac{1}{p,\infty}_p,
\]
up to normalisation at \( t = -\infty \). This is well-known, see [23, 14, 16]. In Subsection 6.2 we prove an important high-low product type estimate in \( U^p \) and \( V^p \), which gives as a special case the crucial product estimates used in Section 6. Finally, in Subsection 6.3 we consider the adapted functions spaces \( U^p_0 \) and \( V^p_0 \).

6.1. Convolution and the Besov Embedding. We use the notation
\[
f *_R g(t) = \int_R f(s) g(t-s) ds
\]
to signify the convolution in the \( t \) variable.

**Lemma 6.1.** Let \( 1 < p < \infty \) and \( \phi(t) \in L^1_t(\mathbb{R}) \). For all \( u \in U^p \) and \( v \in V^p \) we have \( \phi *_R u \in U^p \), \( \phi *_R v \in V^p \), and the bounds
\[
\left\| \phi *_R u \right\|_{U^p} \leq 2 \| \phi \|_{L^1_t} \| u \|_{U^p}, \quad \left\| \phi *_R v \right\|_{V^p} \leq \| \phi \|_{L^1_t} \| v \|_{V^p}.
\]

**Proof.** We first observe that since \( u \) and \( v \) are right continuous and decay to zero as \( t \to -\infty \), the convolutions \( \phi *_R u \) and \( \phi *_R v \) also satisfy these conditions. The proof of the \( V^p \) bound is immediate, and hence \( \phi * v \in V^p \). To show that \( \phi *_R u \in U^p \), we apply Theorem 5.1 and observe that for any \( \psi \in C^\infty_0 \), we have
\[
\left| \int_{\mathbb{R}} (\partial_t \psi(t), \phi *_R u(t))_{L^2_x} dt \right| = \left| \int_{\mathbb{R}} \left\langle \int_{\mathbb{R}} (\partial_t \psi(t+s) \phi(t) dt, u(s) \right\rangle_{L^2_x} ds \right| \\
\leq \left\| \int_{\mathbb{R}} \psi(t+s) \phi(t) dt \right\|_{V^p} \| u \|_{U^p} \\
\leq \| \psi \|_{V^p} \| \phi \|_{L^1_t} \| u \|_{U^p}.
\]
\( \square \)

A similar argument shows that the space-time convolution with an \( L^1_{t,x} 1^{1+n} \) kernel is also bounded on \( U^p \) and \( V^p \).

The spaces \( V^p \) and \( U^p \) are closely related. In fact, for functions which have temporal Fourier support in an annulus centered at the origin, the \( U^p \) and \( V^p \) norms are equivalent.
Theorem 6.2. Let $1 < p < \infty$ and $d > 0$. If $u \in L^p_xL^2_t$ with $\text{supp } \mathcal{F}_t u \subset \{ \frac{d}{100} \leq |\tau| \leq 100d \}$ then $u \in U^p$ and

$$
\|u\|_{U^p} \approx \|u\|_{V^p} \approx \|u\|_{L^p_xL^2_t}.
$$

Conversely, if $u \in U^p$ and $\text{supp } \mathcal{F}_t u \subset \{ \frac{d}{100} \leq |\tau| \leq 100d \}$, then $u \in L^p_xL^2_t$.

Remark 6.3. Theorem 6.2 can also be stated in terms of the Besov spaces $\dot{B}^{\frac{p}{p}}_{p,\infty}$ and $\dot{B}^{\frac{p}{p}}_{p,1}$. More precisely, let $\rho \in C^\infty_0(\{2^{-1} < \tau < 2\})$ such that $\sum_{d \in \mathbb{Z}^d} \rho(\frac{\tau}{d}) = 1$ for $\tau > 0$ and define $P^{(t)}_d = \rho \left( \frac{\cdot - \langle t \rangle}{d} \right)$. If $1 < p < \infty$ and $v \in V^p$ then

$$
\sup_{d \in \mathbb{Z}^d} d^\frac{p}{p} \|P^{(t)}_d v\|_{L^p_xL^2_t} \lesssim |v|_{V^p}.
$$

On the other hand, if $u(t) \to 0$ in $L^2_t$ as $t \to -\infty$ and

$$
\sum_{d \in \mathbb{Z}^d} d^\frac{p}{p} \|P^{(t)}_d u\|_{L^p_xL^2_t} < \infty,
$$

then $u \in U^p$ and

$$
\|u\|_{U^p} \lesssim \sum_{d \in \mathbb{Z}^d} d^\frac{p}{p} \|P^{(t)}_d u\|_{L^p_xL^2_t}.
$$

The first claim follows directly from Theorem 6.2 and the disposability of the multipliers $P^{(t)}_d$ which is a consequence of Lemma 6.1. To prove the second claim, in view of Theorem 6.2, the sum converges in the Banach space $U^p$, and so we have $\sum_d P^{(t)}_d u \in U^p$. Since $u(t)$ and $\sum_d P^{(t)}_d u(t)$ can only differ by a polynomial, and both vanish at $-\infty$, we conclude that $u(t) = \sum_d P^{(t)}_d u(t) \in U^p$ and the claimed bound.

The proof of Theorem 6.2 as well as the proof of the product estimates contained in the next section, requires the following observation.

Lemma 6.4. For any $s \in \mathbb{R}$, $(t_j)_{j=1}^N \in \mathbb{P}$, and $g \in V^p$ we have

$$
\sum_{j=1}^N m^p_j \|g(t_j) - g(t_j - s)\|_{L^2_x} \leq 2(1 + |s|) |g|_{V^p}
$$

where $m_j = \min\{t_{j+1} - t_j, 1\}$.

Proof. It is enough to consider $s > 0$. Let

$$
J_k = \{ j \in \{1, \ldots, N\} : t_j \in [sk, (k+1)s) \}.
$$

Then,

$$
\sum_{j=1}^N m^p_j \|g(t_j) - g(t_j - s)\|_{L^2_x} = \sum_{k \in \mathbb{Z}} \sum_{j \in J_k} m^p_j \|g(t_j) - g(t_j - s)\|_{L^2_x}
$$

and, as $m_j \leq 1$,

$$
\sum_{j \in J_k} m^p_j \|g(t_j) - g(t_j - s)\|_{L^2_x} \leq \left( \sum_{j \in J_k} m_j \right) \max_{j \in J_k} \|g(t_j) - g(t_j - s)\|_{L^2_x}
$$

$$
\leq (1 + s) \|g(t_{j_k}) - g(t_{j_k} - s)\|_{L^2_x}
$$
where \( t_{j_k} \in J_k \) is chosen such that
\[
\|g(t_{j_k}) - g(t_{j_k} - s)\|_{L^2_x} = \max_{j \in J_k} \|g(t_j) - g(t_j - s)\|_{L^2_x}.
\]
Now,
\[
\sum_{k \in \mathbb{Z}, J_k \neq \emptyset} \|g(t_{j_k}) - g(t_{j_k} - s)\|_{L^2_x}^p \lesssim \sum_{k \text{ even}} \|g(t_{j_k}) - g(t_{j_k} - s)\|_{L^2_x}^p + \sum_{k \text{ odd}} \|g(t_{j_k}) - g(t_{j_k} - s)\|_{L^2_x}^p \leq 2|v|_{V^p},
\]
because for \( k = 2m \) even we have
\[
t_{j_{2m}} < t_{j_{2(m+1)}} - s < t_{j_{2(m+1)}} \quad ^1
\]
hence the above points form a partition, and a similar argument applies to \( k = 2m + 1 \) odd. In summary, we have
\[
\sum_{j=1}^{N-1} m_j^p \|g(t_j) - g(t_j - s)\|_{L^2_x}^p \leq 2(1 + s)|g|_{V^p}.
\]
\[\square\]

We now come to the proof of Theorem 6.2.

**Proof of Theorem 6.2.** After rescaling, it is enough to consider the case \( d = 1 \). Let \( \rho \in C^\infty(\mathbb{R}) \) with \( \operatorname{supp} \hat{\rho} \subset \{|\tau| \leq \frac{1}{100}\}, \|(1 + |s|)\rho(s)\|_{L^1} \lesssim 1 \) and \( \mathcal{F}_1(\rho)(0) = 1 \). The temporal Fourier support assumption implies that
\[
u(t) = \int_{\mathbb{R}} \rho(s)[u(t) - u(t-s)]ds
\]
and hence an application of Lemma 6.4 gives
\[
\|u\|_{L^p_t L^2_x} \lesssim \int_{\mathbb{R}} |\rho(s)||u(t-s) - u(t)|\|_{L^p_t L^2_x} ds \lesssim 2 \int_{\mathbb{R}} |\rho(s)| \left( \sum_{j \in \mathbb{N}} \|u(t_j - s) - u(t_j)\|_{L^2_x}^p \right)^{\frac{1}{p}} ds \lesssim 4 \|(1 + |s|)\rho(s)\|_{L^1} \|u\|_{V^p} \lesssim \|u\|_{V^p}
\]
where we choose \( t_j = t_j(s) \in [j, j+1) \) such that
\[
\sup_{t \in [j, j+1)} \|u(t-s) - u(t)\|_{L^2_x} \leq 2\|u(t_j - s) - u(t_j)\|_{L^2_x}.
\]

It remains to show that \( u \in U^p \) and the bound \( \|u\|_{U^p} \lesssim \|u\|_{L^p_t L^2_x} \). The assumptions on \( u \) imply that \( u \) is right continuous and \( \|u(t)\|_{L^2} \to 0 \) as \( t \to -\infty \). Thus we apply Theorem 5.1 and observe that, using the \( V^p \) case proved above, we have for \( \phi \in C^\infty_0 \)
\[
\left| \int_{\mathbb{R}} (\partial_t \phi, u)_{L^2_x} dt \right| \lesssim \|1_{1 \leq \tau \leq 100}(-i\partial_t)\|_{L^2_x} \|u\|_{L^p_t L^2_x} \lesssim \|1_{1 \leq \tau \leq 100}(-i\partial_t)\|_{L^2_x} \|u\|_{L^p_t L^2_x} \lesssim |\phi|_{V^q} \|u\|_{L^p_t L^2_x}
\]
where \( \frac{1}{p} + \frac{1}{q} = 1 \). Hence \( u \in U^p \) and the claimed bounds follow. \[\square\]
6.2. Stability under multiplication. In the following, given \( u, v : \mathbb{R} \rightarrow L^2_x \), we let \( \hat{u}(t) \) denote the Fourier transform in \( x \) (i.e. with respect to the \( L^2 \) variable), and \( \mathcal{F}_t u(\tau) \) be the Fourier transform in the \( t \) variable. Our goal is to find conditions under which the product \( uv \) belongs to either \( U^p \) or \( V^p \). One possibility is the following.

**Lemma 6.5.** Let \( 1 \leq p < \infty \). Let \( f, g : \mathbb{R} \rightarrow L^2_x \) be bounded and satisfy \( \text{supp}(\mathcal{F}_t f) \subset (-1,1) \) and \( \text{supp}(\mathcal{F}_t g) \subset \mathbb{R} \setminus (-4,4) \). If \( g \in V^p \), then \( fg \in V^p \) and

\[
\|fg\|_{V^p} \lesssim \|f\|_{L^\infty_{t,x}} \|g\|_{V^p}.
\]

On the other hand, if \( g \in U^p \), then \( fg \in U^p \) and

\[
\|fg\|_{U^p} \lesssim \|f\|_{L^\infty_{t,x}} \|g\|_{U^p}.
\]

Under the support assumptions of the previous lemma, it is clear that for \( s > 0 \) we have the Sobolev product inequality \( \|fg\|_{H^s} \lesssim \|f\|_{L^\infty} \|g\|_{H^s} \). In particular, the previous lemma should be thought of as the \( U^p \) and \( V^p \) version of the standard heuristic that derivatives can essentially always be taken to fall on the high frequency term.

We now turn to the proof of Lemma 6.5.

**Proof of Lemma 6.5.** We start with the \( V^p \) bound under the weaker assumption that we only have \( \text{supp} \mathcal{F}_t g \subset \mathbb{R} \setminus (-3,3) \). Clearly, it is enough to prove the bound for \( |fg|_{V^p} \). Let \( \tau = (t_j)_{j=1}^N \in \mathcal{P} \) be any partition. Then

\[
\left( \sum_{j=1}^{N-1} \|fg(t_{j+1}) - fg(t_j)\|_{L^2_x}^p \right)^{\frac{1}{p}} \leq \left( \sum_{j=1}^{N-1} \|f(t_{j+1})(g(t_{j+1}) - g(t_j))\|_{L^2_x}^p \right)^{\frac{1}{p}}
\]

\[+ \left( \sum_{j=1}^{N-1} \|f(t_{j+1}) - f(t_j)\|_{L^2_x}^p \right)^{\frac{1}{p}}\),

and due to

\[
\left( \sum_{j=1}^{N-1} \|f(t_{j+1})(g(t_{j+1}) - g(t_j))\|_{L^2_x}^p \right)^{\frac{1}{p}} \leq \|f\|_{L^\infty_{t,x}} \|g\|_{V^p}
\]

it is enough to prove

\[
\left( \sum_{j=1}^{N-1} \|f(t_{j+1}) - f(t_j)\|_{L^2_x}^p \right)^{\frac{1}{p}} \lesssim \|f\|_{L^\infty_{t,x}} \|g\|_{V^p}.
\]

The hypothesis on the Fourier-supports of \( f \) and \( g \) implies that there exists \( \rho \in \mathcal{S}(\mathbb{R}) \) such that

\[
f = \rho * f \quad \text{and} \quad \rho * (f(\cdot - b)g) = 0 \quad \text{for all} \quad b \in \mathbb{R}.
\]

Consequently, we have the identity

\[
(f(t_{j+1}) - f(t_j))g(t_j) = \int_{\mathbb{R}} \rho(s)(f(t_{j+1} - s) - f(t_j - s))(g(t_j) - g(t_j - s))ds.
\]
Now, since
\[
\|f(a) - f(b)\|_{L^\infty_x} = \left\| \int_a^b (\rho(a - s) - \rho(b - s)) f(s) ds \right\|_{L^\infty_x}
\]
\[
= \left\| \int_a^b \rho'(t - s) f(s) ds dt \right\|_{L^\infty_x} \leq |a - b| \|\rho'\| \|f\|_{L^\infty_x},
\]
if we let \(m_j = \min\{t_{j+1} - t_j, 1\}\), an application of Lemma 6.4 gives
\[
\left( \sum_{j=1}^{N-1} \|(f(t_{j+1}) - f(t_j))g(t_j)\|_{L^p_{x,t}} \right)^{\frac{1}{p}} \]
\[
\lesssim \|f\|_{L^\infty_x} \int R |\rho(s)| \left( \sum_{j=1}^{N-1} m_j \|g(t_j) - g(t_j - s)\|_{L^p_x} \right)^{\frac{1}{p}} ds
\]
\[
\lesssim \|f\|_{L^\infty_x} \|g\|_{V^p} \int R |\rho(s)|(1 + |s|)^{\frac{1}{2}} ds
\]
\[
\lesssim \|f\|_{L^\infty_x} \|g\|_{V^p}
\]
and hence (6.2) follows.

To prove the \(U^p\) version, we observe that since \(g \in U^p\) and \(f\) is smooth and bounded, the product \(fg\) is right continuous and satisfies the normalising condition \((fg)(t) \to 0\) in \(L^2_x\) as \(t \to -\infty\). Consequently, by Theorem 5.1, it suffices to show that
\[
\int_R \langle \partial_t \phi, f g \rangle_{L^2_x} dt \lesssim \|\phi\|_{V^q} \|f\|_{L^\infty} \|g\|_{U^p}
\]
(6.2)
where \(\partial_t \phi \in C^{1}_{0}\) and \(\frac{1}{q} + \frac{1}{p} = 1\). Applying the Fourier support assumption, together with the \(V^p\) case of the product estimate proved above, we see that after writing \(\partial_t \phi f = \partial_t (\phi f) - \partial_t^{-1} \partial_t (\phi \partial_t f),\)
\[
\left| \int_R \langle \partial_t \phi, f g \rangle dt \right| = \left| \int_R \langle \partial_t \phi^{(t)}_{\geq 3} \phi, f g \rangle dt \right|
\]
\[
\lesssim \|\phi^{(t)}_{\geq 3} \|_{V^q} \|g\|_{U^p} + \|\phi^{(t)}_{\geq 3} \|_{V^q} \|\partial_t^{-1} g\|_{U^p}
\]
\[
\lesssim \|\phi\|_{V^q} \left( \|f\|_{L^\infty} \|g\|_{U^p} + \|\partial_t f\|_{L^\infty} \|\partial_t^{-1} g\|_{U^p} \right)
\]
where \(\phi^{(t)}_{\geq 3}\) is a temporal Fourier projection to the set \(\{ |\tau| \geq 3\}\), and we used Theorem 4.1. The required bound (6.2) then follows by the boundedness of convolution operators on \(U^p\) and \(V^p\) together with the Fourier support assumptions on \(f\) and \(g\).

In applications to PDE, in particular to the wave maps equation, we require a more general version of the previous lemma which includes a spatial multiplier. To this end, for \(\phi : R \to L^\infty_x L^2_y + L^\infty_y L^2_x\) and \(u, v \in L^\infty_t L^2_x\), we define
\[
\mathcal{T}_\phi[u, v](t, x) = \int_R \phi(t, x - y, y) u(t, x - y) v(t, y) dy.
\]
An application of Fubini and Hölder shows that \(\mathcal{T}_\phi(t, x, y) : L^2 \times L^2 \to L^2\) with the fixed time bound
\[
\|\mathcal{T}_\phi[u, v]\|_{L^\infty_t L^2_x} \leq \left( \sup_{t \in R} \|\phi(t, x, y)\|_{L^\infty_x L^2_y + L^\infty_y L^2_x} \right) \|u\|_{L^\infty_t L^2_x} \|v\|_{L^\infty_t L^2_x}.
\]
Adapting the proof of Lemma 6.5 under certain temporal Fourier support conditions on $\phi$ and functions $u, v \in V^p$ (or $U^p$), we can show that $T_\phi[u, v] \in V^p$ (or $U^p$).

**Theorem 6.6.** Let $1 < p < \infty$. Let $\phi(t, x, y) : \mathbb{R} \to L^\infty \subset L^2_y + L^\infty_y L^2_z$ continuous and bounded with $\text{supp } F_t \phi \subset (-1, 1)$. If $u, v \in V^p$ and $\text{supp } F_t u \subset \mathbb{R} \setminus (-4, 4)$, then we have

$$\|T_\phi[u, v]\|_{V^p} \lesssim \left( \sup_{t \in \mathbb{R}} \|\phi(t, x, y)\|_{L^\infty_x L^2_y + L^\infty_y L^2_z} \right) \|u\|_{V^p} \|v\|_{V^p}.$$  

Moreover, if in addition we have $u, v \in U^p$, then $T_\phi[u, v] \in U^p$ and

$$\|T_\phi[u, v]\|_{U^p} \lesssim \left( \sup_{t \in \mathbb{R}} \|\phi(t, x, y)\|_{L^\infty_x L^2_y + L^\infty_y L^2_z} \right) \|u\|_{U^p} \|v\|_{U^p}.$$  

**Proof.** Let $S := \sup_{t \in \mathbb{R}} \|\phi(t, x, y)\|_{L^\infty_x L^2_y + L^\infty_y L^2_z}$. We start with the $V^p$ case. As in the proof of Lemma 6.5, we can reduce to considering the case when the difference falls on $\phi$, thus our goal is to show that for $(t_j)^N \in \mathbb{P}$, we have

$$\sum_{j=1}^{N-1} \left\| \int_{\mathbb{R}^n} (\phi(t_{j+1}, x - y, y) - \phi(t_j, x - y, y)) u(t_j, x - y) v(t_j, y) dy \right\|_{L^2_x(\mathbb{R}^n)}^p \lesssim S^p \|u\|_{V^p}^p \|v\|_{V^p}^p.$$  

Following the argument used in Lemma 6.5 we see that the temporal Fourier support assumption implies the identity

$$\left( \phi(t_{j+1}, z, y) - \phi(t_j, z, y) \right) u(t_j, z)$$

$$= \int_{\mathbb{R}} \rho(s) \left( \phi(t_{j+1}, s - z, y) - \phi(t_j, s - z, y) \right) \left( u(t_j, z) - u(t_j - s, z) \right) ds$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \rho(s) \left( \phi(t_{j+1}, s - s') - \phi(t_j, s - s') \right) \phi(s', z, y) \left( u(t_j, z) - u(t_j - s, z) \right) ds\, ds'$$

for some $\rho \in \mathcal{S}(\mathbb{R})$ with $F_t \rho = 1$ on $\text{supp } F_t \phi$. Consequently we see that

$$\left\| \int_{\mathbb{R}^n} (\phi(t_{j+1}, x - y, y) - \phi(t_j, x - y, y)) u(t_j, x - y) v(t_j, y) dy \right\|_{L^2_x} \lesssim S \int_{\mathbb{R}} \rho(s) \|\phi(t_{j+1} - s') - \phi(t_j - s')\|_{L^2_x} \|u(t_j) - u(t_j - s)\|_{L^2_x} \|v(t_j)\|_{L^2_x} ds\, ds'$$

$$\lesssim S \int_{\mathbb{R}} |\rho(s)| m_j \|u(t_j) - u(t_j - s)\|_{L^2_x} ds \|v\|_{L^\infty_x L^2_z} \|u\|_{U^p} \|v\|_{U^p}.$$  

with $m_j = \min\{t_{j+1} - t_j, 1\}$. Hence (6.3) follows from Minkowski’s inequality and Lemma 6.4.

We now turn to the proof of the $U^p$ case. Since $u, v \in U^p$ and $\phi$ is continuous, $T_\phi(u, v)$ is right continuous and converges to zero as $t \to -\infty$. Consequently, applying the Besov embedding in Remark 6.3 we have

$$\|P_d^{(t)} T_\phi(u, v)\|_{U^p} \lesssim \sum_{d \leq 4} d^p \|P_d^{(t)} T_\phi(u, v)\|_{U^p} \lesssim \|T_\phi(u, v)\|_{L^p_x L^2_z} \lesssim S \|u\|_{L^p_x L^2_z} \|v\|_{L^\infty_x L^2_z} \lesssim S \|u\|_{U^p} \|v\|_{U^p}.$$  

For the remaining term, we note that an application of Theorem 5.1 reduces the problem to proving the bound

$$\left| \int_{\mathbb{R}} \langle \partial \psi, T_\phi[u, v] \rangle_{L^2} dt \right| \lesssim S \|\psi\|_{V^p} \|u\|_{U^p} \|v\|_{U^p}.$$
where \( \psi, \mathcal{F}_t \psi \in C^\infty \) and \( \text{supp} \mathcal{F}_t \psi \subset \mathbb{R} \setminus (-4, 4) \). We write the left hand side as

\[
\left| \int_\mathbb{R} \langle \partial_t \psi, T_\varphi[u, v] \rangle_{L^2_y} dt \right|
= \left| \int_\mathbb{R} \langle \partial_t \psi(t, x), \phi(t, x - y, y) \rangle \langle u(t, x - y) v(t, y) \rangle_{L^2_y} dt \right|
\leq \left| \int_\mathbb{R} \langle \psi(t, x), \partial_t \phi(t, x - y, y) \rangle \langle u(t, x - y) v(t, y) \rangle_{L^2_y} dt \right|
+ \left| \int_\mathbb{R} \langle \partial_t \psi(t, x) \phi(t, x - y, y), u(t, x - y) v(t, y) \rangle_{L^2_y} dt \right|
\tag{6.4}
\]

To bound the first term in (6.4), we use the temporal support assumption on \( \phi \) to write \( \partial_t \phi(t) = \int_\mathbb{R} \partial_t \rho(s) \phi(t - s) ds \) with \( \|\partial_t \rho\|_{L^1_y} \lesssim 1 \), hence again applying the Besov embedding in Remark 6.3, we obtain

\[
\left| \int_\mathbb{R} \langle \psi(t, x), \partial_t \phi(t, x - y, y) \rangle \langle u(t, x - y) v(t, y) \rangle_{L^2_y} dt \right|
\leq \|\psi\|_{L^p_t L^2_y} \int_\mathbb{R} \|\partial_t \rho(s)\| \int_\mathbb{R} \|\phi(t - s, x - y, y) u(t, x - y) v(t, y)\|_{L^2_y} ds
\lesssim S\|\psi\|_{V'\psi'} \|u\|_{L^p_t L^2_y} \|v\|_{L^p_t L^2_y} \lesssim S\|\psi\|_{V'\psi'} \|u\|_{U^p} \|v\|_{U^p}
\]

where we used the temporal Fourier support assumptions on \( \psi \) and \( u \).

On the other hand, to bound the second term in (6.4), we first observe that it is enough to consider the case where \( u \) and \( v \) are \( U^p \) atoms with partitions \( \tau \) and \( \tau' \). Let \((t_j)_{j=1}^N = \tau \cup \tau'\). Computing the integral in time, we deduce that

\[
\int_\mathbb{R} \langle \partial_t (\psi(t, x) \phi(t, x - y, y)), u(t, x - y) v(t, y) \rangle_{L^2_y} dt
= \sum_{j=1}^{N-1} \langle \psi(t_{j+1}, x) \phi(t_{j+1}, x - y, y) - \psi(t_j, x) \phi(t_j, x - y, y), u(t_j, x - y) v(t_j, y) \rangle_{L^2_y}
- \langle \psi(t_N), T_\varphi[u, v](t_N) \rangle_{L^2_y}
= \sum_{j=1}^{N-1} \langle \psi(t_{j+1}, x) \phi(t_{j+1}, x - y, y) - \psi(t_{j+1}, x - y, y), u(t_j, x - y) v(t_j, y) \rangle_{L^2_y}
+ \sum_{j=1}^{N-1} \langle \psi(t_{j+1}) - \psi(t_j), T_\varphi[u, v](t_j) \rangle_{L^2_y} - \langle \psi(t_N), T_\varphi[u, v](t_N) \rangle_{L^2_y}.
\]

Applying Hölder’s inequality and the fixed time convolution bound, we have

\[
\sum_{j=1}^{N-1} \|\psi(t_{j+1}) - \psi(t_j)\|_{L^2_y} \|u(t_j)\|_{L^p_x} \|v(t_j)\|_{L^p_y} \lesssim S \left( \sum_{j=1}^{N-1} \|\psi(t_{j+1}) - \psi(t_j)\|_{L^2_y} \right)^{p'/2} \left( \sum_{j=1}^{N-1} \|u(t_j)\|_{L^p_x} \|v(t_j)\|_{L^p_y} \right)^{1/p}
+ S\|\psi\|_{L_t^p L_y^2} \|u\|_{L_t^p L_y^2} \|v\|_{L_t^p L_y^2}
\lesssim S\|\psi\|_{V'\psi'}
\]

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where we used the fact that \( u \) and \( v \) are \( U^p \) atoms and \( (t_j)_{j=1}^N = \tau \cup \tau' \). Consequently, it only remains to prove that

\[
\sum_{j=1}^{N-1} \left| \left\langle \psi(t_{j+1}, x) \left( \overline{\phi}(t_{j+1}, x - y, y) - \phi(t_j, x - y, y) \right), u(t_j, x - y) v(t_j, y) \right\rangle_{L^2_y} \right| \lesssim S\| \psi \|_{V^{p'}}.
\]

(6.5)

This follows by adapting the proof of the \( V^p \) case above, namely, we first use the temporal Fourier support assumption to obtain the identity

\[
\psi(t_{j+1}, x) \left( \overline{\phi}(t_{j+1}, x - y, y) - \phi(t_j, x - y, y) \right) = \int_{\mathbb{R}} \rho(s) \left( \overline{\phi}(t_{j+1} - s, x - s, y) - \phi(t_j - s, x - s, y) \right) \psi(t_{j+1} - s, x) ds
\]

and hence

\[
\left| \left\langle \psi(t_{j+1}, x) \left( \overline{\phi}(t_{j+1}, x - y, y) - \phi(t_j, x - y, y) \right), u(t_j, x - y) v(t_j, y) \right\rangle_{L^2_y} \right| \lesssim S\| u(t_j) \|_{L^2} \| v(t_j) \|_{L^2} \int_{\mathbb{R}} |\rho(s)| m_j \| \psi(t_j) - \psi(t_j - s) \|_{L^2} ds.
\]

Summing up, applying Hölder’s inequality together with Lemma \ref{lem:holder_embedding} we then deduce (6.5).

\[ \square \]

6.3. Adapte\textsuperscript{d} \( U^p \) and \( V^p \). Given a phase \( \Phi : \mathbb{R}^n \to \mathbb{R} \) (measurable and of moderate growth), we define the adapted function spaces \( U^p_\Phi \) and \( V^p_\Phi \) as

\[
U^p_\Phi = \left\{ u \mid e^{it\Phi(-i\nabla)} u(t) \in U^p \right\}, \quad V^p_\Phi = \left\{ v \mid e^{it\Phi(-i\nabla)} v(t) \in V^p \right\},
\]

as in \cite{14, 10}. As in the case of \( U^p \) and \( V^p \), elements of \( U^p_\Phi \) and \( V^p_\Phi \) are right continuous, and approach zero as \( t \to -\infty \). With the norms

\[
\| u \|_{U^p_\Phi} = \| t \mapsto e^{it\Phi(-i\nabla)} u(t) \|_{U^p}, \quad \text{resp. } \| v \|_{V^p_\Phi} = \| t \mapsto e^{it\Phi(-i\nabla)} v(t) \|_{V^p},
\]

these spaces \( U^p_\Phi \) and \( V^p_\Phi \) are Banach spaces. They are constructed to contain perturbations of solutions to the linear PDE

\[-i\partial_t u + \Phi(-i\nabla) u = 0.\]

In particular, for any \( T \in \mathbb{R} \) and \( f \in L^2 \), we have \( 1_{[T, \infty)}(t) e^{-it\Phi(-i\nabla)} f \in U^p_\Phi \) (and \( V^p_\Phi \)). Note that the cutoff \( 1_{[T, \infty)} \) is essential here due to the normalisation at \( t = -\infty \) (of course one could also choose a smooth cutoff \( \rho(t) \in C^\infty \) instead).

The results obtained above for the \( U^p \) and \( V^p \) spaces can all be translated to the setting of the adapted function spaces \( U^p_\Phi \) and \( V^p_\Phi \). For instance, left and right limits always exist in \( U^p \) and \( V^p \), an application of Theorem \ref{thm:embedding} implies that for \( p < q \) we have the embedding \( V^p_\Phi \subset U^p_\Phi \), and Theorem \ref{thm:regularity} gives the embedding

\[
\| u \|_{V^p_\Phi} \approx \| u \|_{U^p_\Phi} \approx d^n \| u \|_{\mathcal{L}^p_t L^q_x}
\]

for all \( u \in L_t^p L_x^q \) with \( \text{supp} \, \tilde{u} \subset \{ (\tau, \xi) \in \mathbb{R}^{1+n} \mid |\tau + \Phi(\xi)| \approx d \} \). In particular, we have \( u \in U^p_\Phi \). Similarly, Theorem \ref{thm:stability} and Theorem \ref{thm:semigroup} imply that

\[
\| u \|_{U^p_\Phi} = \sup_{\| v \|_{V^p_\Phi} \leq 1} \left| \int_{\mathbb{R}} \left( -i\partial_t v + \Phi(-i\nabla) v, u \right)_{L^2} dt \right|
\]
and if \(-i\partial_t \psi + \Phi(-i \nabla) \psi = F\), then

\[
\|u(t)\|_{L^p_x} \approx \|\psi(0)\|_{L^2_x} + \sup_{\|\psi\|_{L^2_x} \leq 1} \int_0^\infty \langle v, F \rangle_{L^2_x} dt
\]

in the sense that if the right-hand side is finite, and \(u\) is right continuous with \(u(t) \to 0\) as \(t \to -\infty\), then \(u, \psi \in U^p_{\Phi}\). Clearly, in view of Remark 6.5 for reasonable phases \(\Phi\) we can take the sup over \(v \in C_0^\infty\) with \(|v|_{L^2_x} \leq 1\) instead. These properties show why the adapted function spaces are well adapted to studying the PDE \(-i\partial_t \psi + \Phi(-i \nabla) \psi = F\). Further properties of the adapted function spaces are also known, see for instance the interpolation estimates in 10-14-15, and the vector valued transference type arguments in 78.

We now give two applications of the product estimates in Subsection 6.3. The first is an application of Lemma 6.5 to remove the solution operators \(e^{-it\Phi(-i \nabla)}\) in certain high modulation regimes.

**Proposition 6.7.** Let \(1 < p < \infty, \Omega \subset \mathbb{R}^n\) measurable and \(\Phi : \Omega \to \mathbb{R}\) such that \(|\Phi(\xi)| \leq 1\) for \(\xi \in \Omega\) and assume that \(\text{supp} \hat{u} \subset \{ |\tau| \geq 5 \} \times \Omega\). If \(u \in V^p\) then

\[
\|u\|_{V^p} \approx \|u\|_{V^p_x}.
\]

Similarly, if \(u \in U^p\) we have

\[
\|u\|_{U^p} \approx \|u\|_{U^p_x}.
\]

**Proof.** In the \(V^p\) case, an application of Plancherel shows that it suffices to prove the inequality \(\|e^{it\Phi(\xi)} \hat{u}\|_{V^p} \lesssim \|\hat{u}\|_{V^p}\). The support assumption on \(\hat{u}\) implies that we may write \(e^{it\Phi(\xi)} \hat{u}(t, \xi) = e^{it\Phi(\xi)} 1_{\Omega}(\xi) \hat{u}(t, \xi)\). Since \(|\Phi(\xi)| \leq 1\) for \(\xi \in \Omega\), we conclude that \(\text{supp} \mathcal{F}[e^{it\Phi(\xi)} 1_{\Omega}(\xi)] \subset \{ |\tau| \leq 1 \}\). Thus an application of Lemma 6.5 gives

\[
\|e^{it\Phi(\xi)} \hat{u}\|_{V^p} = \|e^{it\Phi(\xi)} 1_{\Omega} \hat{u}\|_{V^p} \lesssim \|e^{it\Phi(\xi)} 1_{\Omega} \|_{L^\infty_t} \|\hat{u}\|_{V^p} \lesssim \|\hat{u}\|_{V^p}
\]

as required. An identical argument gives the \(U^p\) case.

The second is a reformulation of Theorem 6.6 Define the spatial bilinear Fourier multiplier

\[
\mathcal{M}[u, v](x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} m(\xi - \eta, \eta) \hat{u}(\xi - \eta) \hat{v}(\eta) e^{ix \cdot \xi} d\eta d\xi.
\]

**Theorem 6.8.** Let \(1 < p < \infty\). Let \(m : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}\) and \(\Phi_j, j = 0, 1, 2\) be real-valued phases such that

\[
|\Phi_0(\xi + \eta) - \Phi_1(\xi) - \Phi_2(\eta)| \leq 1
\]

for all \((\xi, \eta) \in \text{supp} m\). If \(u \in V^p_{\Phi_1}\) with \(\text{supp} \hat{u} \subset \{ |\tau + \Phi_1(\xi)| \geq 4 \}\) and \(v \in V^p_{\Phi_2}\), then \(\mathcal{M}[u, v] \in V^p_{\Phi_0}\) and

\[
\|\mathcal{M}[u, v]\|_{V^p_{\Phi_0}} \lesssim \|m(\xi, \eta)\|_{L^\infty_t L^2_q + L^\infty_t L^2_q} \|u\|_{V^p_{\Phi_1}} \|v\|_{V^p_{\Phi_2}}.
\]

If in addition \(u \in U^p_{\Phi_1}\) and \(v \in U^p_{\Phi_2}\), then \(\mathcal{M}[u, v] \in U^p_{\Phi_0}\) and

\[
\|\mathcal{M}[u, v]\|_{U^p_{\Phi_0}} \lesssim \|m(\xi, \eta)\|_{L^\infty_t L^2_q + L^\infty_t L^2_q} \|u\|_{U^p_{\Phi_1}} \|v\|_{U^p_{\Phi_2}}.
\]
Proof: We begin by observing that \( \|\psi\|_{V^p} = \|\hat{\psi}\|_{V^p} \) and, via Theorem 5.1, \( \|\psi\|_{U^p} \approx \|\hat{\psi}\|_{U^p} \).

Let \( u_1(t, \xi, \eta) = e^{-i \Phi_1(\xi-\eta)} \hat{u}(t, \xi) \) and \( v_2(t, \xi, \eta) = e^{-i \Phi_2(\xi-\eta)} \hat{v}(t, \xi) \), then supp \( \mathcal{F}_t u_1 \subset \{ |\tau| \geq 4 \} \), and it suffices to prove that
\[
\left\| \int_{\mathbb{R}^n} e^{i \Phi_0(t, \xi - \eta, \eta)} m(\xi - \eta, \eta) u_1(\xi - \eta) v_2(\eta) d\eta \right\|_{V^p} \lesssim \|m\|_{L^{\infty}_t L^2_x + L^\infty_t L^2_x} \|u_1\|_{V^p} \|v_2\|_{V^p}.
\]

But this is a consequence of Theorem 6.6 after noting that
\[
\int_{\mathbb{R}^n} e^{i \Phi_0(t, \xi - \eta, \eta)} m(\xi - \eta, \eta) u_1(\xi - \eta) v_2(\eta) d\eta = \mathcal{T}_\phi[u_1, v_2]
\]
with \( \phi(t, \xi, \eta) = e^{i \Phi_0(t, \xi + \eta) - \Phi_1(\xi - \eta)} m(\xi, \eta) \), and supp \( \mathcal{F}_t \phi \subset \{ |\tau| \leq 1 \} \). The proof of the \( U^p \) bound follows from an identical application of Theorem 6.6 \( \square \)

Typically the multiplier \( m(\xi, \eta) = \mathbb{1}_{\Omega_0}(\xi + \eta) \mathbb{1}_{\Omega_1}(\xi) \mathbb{1}_{\Omega_2}(\eta) \) is a cutoff to some frequency region, in which case we have
\[
\|m\|_{L^{\infty}_t L^2_x + L^\infty_t L^2_x} \lesssim (\min\{ |\Omega_0|, |\Omega_1|, |\Omega_2| \})^{\frac{1}{2}}.
\]
Clearly more involved examples are also possible.

7. The Bilinear Restriction Estimate in Adapted Function Spaces

Let \( \lambda \geq 1 \) and define
\[
\Lambda_1 = \{ |\xi - e_1| < \frac{1}{100} \}, \quad \Lambda_2 = \{ |\xi + e_2| < \frac{1}{100} \lambda \}
\]
with \( e_1 = (1, 0, \ldots, 0) \) and \( e_2 = (0, 1, 0, \ldots, 0) \). In this section, we give the proof of the following theorem.

Theorem 7.1. Let \( \frac{1}{n+1} < \frac{1}{a} \leq \frac{1}{b} \leq \frac{1}{a} + \frac{1}{\lambda} \geq \frac{1}{b} \) and \( \lambda \geq 1 \). Assume that \( u \in U_+^a \) and \( v \in U_-^b \) with supp \( \hat{u} \subset \Lambda_1 \) and supp \( \hat{v} \subset \Lambda_2 \). Then
\[
\|uv\|_{L_{t,x}^{\infty}} \lesssim \lambda^{(n+1)(\frac{1}{a} - \frac{1}{b})} \|u\|_{U_{t,x}^a} \|v\|_{U_{t,x}^b}.
\]

This is the case \( \alpha \approx 1 \) of Theorem 3.7. In fact, the general case can essentially be reduced to this case.

Remark 7.2 (Proof of the general case of Theorem 3.7). To obtain the \( L_{t,x}^{2,\infty} \) estimate stated in Theorem 3.7, we need a small angle version of Theorem 7.1. More precisely, we need to show that if \( \kappa, \kappa' \in \mathbb{C}_a \) with \( \angle(\kappa, \pm \kappa') \approx \alpha \), then provided that supp \( \hat{u} \subset \{ |\xi| \approx 1, \frac{1}{|\xi|} \in \kappa \} \) and supp \( \hat{v} \subset \{ |\xi| \approx \lambda, \frac{1}{|\xi|} \in \kappa' \} \) we have
\[
\|uv\|_{L_{t,x}^{2,\infty}} \lesssim \alpha^{\frac{n-2}{2}} \lambda^{(n-1)(\frac{1}{a} - \frac{1}{b})} \|u\|_{U_{t,x}^{a/2}} \|v\|_{U_{t,x}^{b/2}}.
\]

As in \( \square \) Section 2.3, after rotating and rescaling, this bound is equivalent to showing that
\[
\|uv\|_{L_{t,x}^{2,\infty}} \lesssim \lambda^{(n-1)(\frac{1}{a} - \frac{1}{b})} \|u\|_{L_t^{2,\infty} L_x^{a/2}} \|v\|_{L_t^{2,\infty} L_x^{b/2}}
\]
where the phase becomes \( \Phi(\xi) = \alpha^{-2}(\xi_1^2 + \alpha^2|\xi'|)^{\frac{1}{2}} \), and the functions \( u \) and \( v \) now have Fourier support in the rectangles \( \{ \xi_1 \approx 1, \xi_2 \approx 1, |\xi''| \ll 1 \} \) and \( \{ \pm \xi_1 \approx \lambda, |\xi'| \ll \lambda \} \). It is easy to check that the phase \( \Phi \) behaves essentially the same as \( |\xi| \). In particular, an analogue of the wave table construction Tao, Theorem 4.3, holds with \( |\nabla| \) replaced with \( \Phi \), which together with the argument given below, gives the small angle case of the bilinear \( L_{t,x}^{2,\infty} \) estimate. See \( \square \) for the details.
Remark 7.3 (The general bilinear restriction estimate). Theorem 7.1 is a special case of a bilinear restriction type estimate for general phases. More precisely, suppose that $1 < q, r \leq 2$ and $a, b \geq 2$ with

$$\frac{1}{q} + \frac{n + 1}{2r} < \frac{n + 1}{2}, \quad \frac{1}{(n+1)q} < \frac{1}{b} \leq \frac{1}{a} < \frac{1}{2}, \quad \frac{1}{\min\{q, r\}} \leq \frac{1}{a + b}.\$$

Then provided that the phases $\Phi_j$ satisfy suitable curvature and transversality properties, we have

$$\|uv\|_{L^q_t L^r_x} \leq C\|u\|_{U^a_q} \|v\|_{U^b_r}$$

(7.1)

where the constant is given by

$$C \approx \mu^{n+1-\frac{n+1}{q}} \frac{2}{(n+1)q} \frac{1}{\min\{q, r\}} \left(\frac{\mathcal{H}_1}{\mathcal{H}_2}\right)^{\frac{1}{2q} + (n+1)(\frac{1}{b} - \frac{1}{a})} \left(\frac{\mathcal{V}_{\max}}{\mu \mathcal{H}_1}\right)^{(1 - \frac{1}{b} - \frac{1}{a})},$$

with $\mu = \min\{\text{diam}(\tilde{v}), \text{diam}(\tilde{v})\}$, $\mathcal{H}_j = \|\nabla^2 \Phi_j\|_{L^\infty}$, $\mathcal{V}_{\max} = \sup \|\Phi_1(\xi) - \Phi_2(\eta)\|$, and $\mathcal{H}_2 \leq \mathcal{H}_1$. This estimate is essentially sharp, see [7, Theorem 1.7] for a more precise statement. Note that Theorem 3.7 is a special case of the bilinear restriction estimate (7.1), together with a rescaling argument, see the proof of Theorem 1.10 in [7].

In the remainder of this section we give the proof Theorem 7.1 assuming only Tao’s wave table construction for free waves [28, Proposition 15.1]. More precisely, in Subsection 7.1, we introduce some additional notation, and state the wave table construction of Tao [28], as well as an averaging of cubes lemma. In Subsection 7.2, following closely the arguments in [7], we show how the wave table construction for free waves implies a key localised bilinear estimate for atoms. Finally in Subsection 7.3, we run the induction on scales argument, and complete the proof of Theorem 7.1.

7.1. Notation and the wave table construction. We use same notation as in [7]. Given a set $\Omega \subset \mathbb{R}^n$, a vector $\mathbf{h} \in \mathbb{R}^n$, and a scalar $c > 0$, we let $\Omega + \mathbf{h} = \{x + \mathbf{h} \mid x \in \Omega\}$ denote the translation of $\Omega$ by $\mathbf{h}$, and define $\text{diam}(\Omega) = \sup_{x, y \in \Omega} |x - y|$ and $\Omega + c = \{x + y \mid x \in \Omega, |y| < c\}$ to be the Minkowski sum of $\Omega$ and the ball $\{|x| < c\}$.

Let $R \geq 1$ and $0 < \epsilon \leq 1$. The constant $R$ denotes the large space-time scale and $0 < \epsilon \leq 1$ is a small fixed parameter used to control the various error terms that arise. All cubes in this article are oriented parallel to the coordinate axis. Let $Q$ be a cube side length $R$, and take a subscale $0 < r \leq R$. We define $Q_r(Q)$ to be a collection of disjoint subcubes of width $2^{-j_0}R$ which form a cover of $Q$, where $j_0$ is the unique integer such that $2^{-1-j_0}R < r \leq 2^{-j_0}R$ (in other words we divide $Q$ up into smaller cubes of equal diameter $2^{-j_0}R$). Thus all cubes in $Q_r(Q)$ have side lengths $\approx r$, and moreover, if $r \leq r' \leq R$ and $q \in Q_r(Q)$, $q' \in Q_{r'}(Q)$ with $q \cap q' \neq \emptyset$, then $q \in Q_{r'}(q')$. To estimate various error terms which arise, we need to create some separation between cubes. To this end, following Tao, we introduce the following construction. Given $0 < \epsilon \ll 1$ and a subscale $0 < r \leq R$, we let

$$I^{r, \epsilon}(Q) = \bigcup_{q \in Q_{r}(Q)} (1 - \epsilon)q,$$

Note that we have the crucial property, that if $q \in Q_r(Q)$, and $(t, x) \notin I^{r, \epsilon}(Q) \cap q$, then $\text{dist}((t, x), q) \geq \epsilon r$. Given sequences $(\epsilon_m)$ and $(r_m)$ with $\epsilon_m > 0$ and $0 <
functions \( u_C \) for each \( C \geq r_m \leq R \), we define

\[
X[Q] = \bigcap_{m=1}^M I^{m-r_m}.
\]

Thus cubes inside \( X[Q] \) are separated at multiple scales. An averaging argument allows us to move from a cube \( Q_R \), to the set \( X[Q] \) but with a larger cube \( Q \).

**Lemma 7.4** ([28] Lemma 6.1). Let \( R > 0, \epsilon = \sum_{j=1}^M \epsilon_j \leq 2^{-(n+2)} \), and \( r_m \leq R \). For every cube \( Q_R \) of diameter \( R \), there exists for each \( F \in L^2_{t,x}(Q_R) \) we have

\[
\|F\|_{L^2_{t,x}(Q_R)} \leq (1 + 2^{n+2}\epsilon)\|F\|_{L^2_{t,x}(X[Q])}.
\]

All functions in the following are vector valued, thus maps \( u : \mathbb{R}^{1+n} \to l^2_v(\mathbb{Z}) \) where \( l^2_v(\mathbb{Z}) \) is the set of complex valued sequences with finitely non-zero components. The vector valued nature of the waves plays a key role in the induction on scales argument. A function \( u \in L^\infty_t L^2_x \) is a \( \pm \)-wave, if \( u = e^{\mp i|\nabla|t}f \) for some (vector valued) \( f \in L^2_x \). An atomic \( \pm \)-wave is a function \( v \in L^\infty_t L^2_x \) such that \( v = \sum_I \frac{1}{1} \cdot v_I \) with the intervals \( I \) forming a partition of \( \mathbb{R} \), and each \( v_I \) is a \( \pm \)-wave (i.e. we take \( e^{\pm i|\nabla|t}v \in \mathfrak{S} \)). Given an atomic \( \pm \)-wave \( v = \sum_I \frac{1}{1} \cdot v_I \), we let

\[
||v||_{L^2_v} = \left( \sum_I ||v_I||_{L^2_v}^2 \right)^{\frac{1}{2}}.
\]

For a subset \( \Omega \subset \mathbb{R}^{1+n} \) we let \( \mathfrak{1}_\Omega \) denote the indicator function of \( \Omega \). Let \( E \) be a finite collection of subsets of \( \mathbb{R}^{1+n} \), and suppose we have a collection of \( (l^2_v\text{-valued}) \) functions \( (u^{(E)})_{E \in \mathcal{E}} \). We then define the associated **quilt**

\[
[u^{(\mathcal{E})}](t,x) = \sum_{E \in \mathcal{E}} \mathfrak{1}_E(t,x)|u^{(E)}(t,x)|.
\]

This notation was introduced by Tao [28], and plays a key technical role in localising the product \( uv \) into smaller scales.

Following the argument in [7], which adapts the proof of Tao to the setting of atomic waves, our goal is to show that Theorem 7.1 is a consequence of the following wave table construction of Tao.

**Theorem 7.5** ([Wave Tables [28] Proposition 15.1]). Fix \( n \geq 2 \). There exists a constant \( C > 0 \), such that, for all \( 0 < \epsilon \ll 1, \lambda \geq 1, R \geq 100\lambda \), all cubes \( Q \) of diameter \( R \), and \( + \)-waves \( F \), and \( \pm \)-waves \( G \) with

\[
\supp \hat{F} \subset \Lambda_1 + \frac{1}{100}, \quad \supp \hat{G} \subset \Lambda_2 + \frac{1}{100},
\]

there exists for each \( B \in \mathcal{Q}_i(\mathfrak{Q}(Q)) \), a \( + \)-wave \( \mathcal{W}_{1,\epsilon}^{(B)} = \mathcal{W}_{1,\epsilon}^{(B)}[F; G, Q] \), and a \( \pm \)-wave \( \mathcal{W}_{2,\epsilon}^{(B)} = \mathcal{W}_{2,\epsilon}^{(B)}[G; F, Q] \) satisfying

\[
F = \sum_{B \in \mathcal{Q}_i(\mathfrak{Q}(Q))} \mathcal{W}_{1,\epsilon}^{(B)}, \quad G = \sum_{B \in \mathcal{Q}_i(\mathfrak{Q}(Q))} \mathcal{W}_{2,\epsilon}^{(B)}.
\]

the Fourier support condition

\[
\supp \hat{\mathcal{W}}_{1,\epsilon}^{(B)} \subset \supp \hat{F} + R^{-\frac{1}{2}}, \quad \supp \hat{\mathcal{W}}_{2,\epsilon}^{(B)} \subset \supp \hat{G} + (\lambda R)^{-\frac{1}{2}},
\]
the energy estimates

\[
\left( \sum_{B \in \mathcal{Q}_B(Q)} \| \mathcal{W}_{1,\varepsilon}^{(B)} \|^2_{L_t^\infty L_x^2} \right)^{1/2} \lesssim (1 + C \varepsilon) \| F \|_{L_t^\infty L_x^2},
\]

and the bilinear estimates

\[
\| (F, G) \|_{L_t^\infty L_x^2(\mathcal{Q})} \lesssim C \varepsilon \frac{R}{\lambda} \| F \|_{L_t^\infty L_x^2} \| G \|_{L_t^\infty L_x^2}.
\]

Remark 7.6. The notation used in Theorem 7.5 differs somewhat from that used in [28]. For a proof in the general phase case, see [7, Theorem 9.3]. It is important to note that Theorem 7.5 is purely a statement about free waves, and does not involve any atomic structure.

Remark 7.7. The construction of the wave table \( \mathcal{W}_{1,\varepsilon} \) relies on a wave packet decomposition of \( u \). Roughly speaking, we take

\[
\mathcal{W}_{1,\varepsilon}^{(B)} = \sum_T c_T F_T
\]

where \( F_T \) is a wave packet concentrated in the tube \( T \), and the (real-valued) coefficients \( c_T = c_T(G, B) \) are given by

\[
c_T \approx \left( \frac{\| G \|^2_{L_t^\infty L_x^2(B \cap T)} \| G \|^2_{L_t^\infty L_x^2(Q \cap T)}}{\| G \|^2_{L_t^\infty L_x^2(Q \cap T)}} \right)^{1/2}.
\]

Thus \( \mathcal{W}_{1,\varepsilon}^{(B)} \) contains the wave packets \( F_T \) of \( F \), such that \( G \mid_T \) is concentrated on the smaller cube \( B \). Since \( \mathcal{W}_{1,\varepsilon}^{(B)} \) is a sum of wave packets of \( F \), the support conclusion and fact that it is a \(+\)-wave essentially follow directly. Similarly, ignoring the constant and using the (almost) orthogonality of the wave packet decomposition, we have

\[
\sum_B \left\| \mathcal{W}_{1,\varepsilon}^{(B)} \right\|^2_{L_t^\infty L_x^2} \lesssim \sum_T \sum_B \| F_T \|^2_{L_t^\infty L_x^2} \| G \|^2_{L_t^\infty L_x^2(B \cap T)} \| G \|^2_{L_t^\infty L_x^2(Q \cap T)} \lesssim \sum_T \| F_T \|^2_{L_t^\infty L_x^2} \| G \|^2_{L_t^\infty L_x^2(Q \cap T)} \lesssim \| F \|^2_{L_t^\infty L_x^2}.
\]

Improving the constant here to \( 1 + C \varepsilon \) requires an improved wave packet decomposition introduced by Tao. Finally, the bilinear estimate exploits the fact that \( |u| - |\mathcal{W}_{1,\varepsilon}^{(B)}| \) on the cube \( B \), only contains wave packets such that \( G \mid_T \) is not concentrated on \( B \). This is essentially a non-pigeon holed version of the argument of Wolff [35].

7.2. From wave tables to a bilinear estimate for atoms. In this section, we give the proof of the following consequence of Theorem 7.5.
Theorem 7.8. Let $\frac{1}{n+1} < \frac{1}{a} \leq \frac{1}{2}$, $0 < \epsilon \ll 1$, and $Q_R$ be a cube of diameter $R \geq 100\lambda$. Then for any atomic $\pm$-wave $u = \sum_{I \in \mathcal{I}} \mathbb{1}_I u_I$, and any atomic $\pm$-wave $v = \sum_{J \in \mathcal{J}} \mathbb{1}_J v_J$ with

$$\text{supp } \hat{u} \subset \Lambda_1 + \frac{1}{100}, \quad \text{supp } \hat{v} \subset \Lambda_2 + \frac{1}{100},$$

there exist a cube $Q$ of diameter $2R$ such that for each $I \in \mathcal{I}$ and $J \in \mathcal{J}$ we have a decomposition

$$u_I = \sum_{B \in \mathcal{Q}_{2R}^+(Q)} u_I^{(B)}, \quad v_J = \sum_{B' \in \mathcal{Q}_{2R}^+(Q)} v_J^{(B')}$$

where $M \in \mathbb{N}$ with $4^{M-1} \leq \lambda < 4^M$, and $u_I^{(B)} = \sum_{I \in \mathcal{I}} 1_I u_I^{(B)}$ is an atomic $\pm$-wave, $v_J^{(B')} = \sum_{J \in \mathcal{J}} 1_J v_J^{(B')}$ is an atomic $\pm$-wave, with the support properties

$$\text{supp } \hat{u}_I^{(B)} \subset \text{supp } \hat{u} + 2\left(\frac{2R}{\lambda}\right)^{-\frac{1}{2}}, \quad \text{supp } \hat{v}_J^{(B')} \subset \text{supp } \hat{v} + 2\left(\frac{2R}{\lambda}\right)^{-\frac{1}{2}}.$$ Moreover, for any $a_0, b_0 \geq 2$ we have the energy bounds

$$\left( \sum_{B \in \mathcal{Q}_{2R}^+(Q)} \| u_I^{(B)} \|_{\ell_0^2 \ell_2^2}^{a_0} \right)^{\frac{1}{a_0}} \leq (1 + C\epsilon) \| u \|_{\ell_0^2 \ell_2^2}$$

$$\left( \sum_{B' \in \mathcal{Q}_{2R}^+(Q)} \| v_J^{(B')} \|_{\ell_0^2 \ell_2^2}^{b_0} \right)^{\frac{1}{b_0}} \leq (1 + C\epsilon) \| v \|_{\ell_0^2 \ell_2^2}$$

and the bilinear estimate

$$\| uv \|_{L_t^1 L_x^\infty(Q_R)} \leq (1 + C\epsilon) \left\| \left[ u^{(1)} \right] \left[ v^{(1)} \right] \right\|_{L_t^1 L_x^\infty(Q)} + C\epsilon^{-C \lambda(n+1)(\frac{1}{2} - \frac{1}{b})} \left( \frac{R}{\lambda} \right)^{\frac{n+1}{2} + 1 (\frac{1}{2} - \frac{1}{b}) - \frac{n+1}{2}} \| u \|_{\ell_0^2 \ell_2^2} \| v \|_{\ell_0^2 \ell_2^2}$$

where the constant $C$ depends only on the dimension $n$, and the exponents $a, b$.

Proof. Let $C_0$ denote the constant appearing in Theorem 7.5. An application of Lemma 7.4 implies that there exists a cube $Q$ of radius $2R$ such that

$$\| uv \|_{L_t^1 L_x^\infty(Q_R)} \leq (1 + C\epsilon) \| uv \|_{L_t^1 L_x^\infty(X[Q])}$$

where we take

$$X[Q] = \bigcap_{m=1, \ldots, M} I_{\epsilon=4^{-m}2R}(Q), \quad \epsilon_m = 4^{\delta(m-M)}\epsilon$$

and $\delta > 0$ is some fixed constant to be chosen later (which will depend only on the dimension $n$, the exponent $b$, and the constant $C_0$). Let $V = (v_J)_{J \in \mathcal{J}}$, perhaps after relabeling, we have $V : \mathbb{R}^{1+n} \to \ell_2^2(\mathbb{Z})$. In particular, $V$ is a $\pm$-wave such that $|v| \leq |V|$ and $\| V \|_{L_t^1 L_x^\infty} = \| v \|_{\ell_0^2 \ell_2^2}$. We now repeatedly apply the wave table construction in Theorem 7.3 with $G = V$. More precisely, given $B_1 \in \mathcal{Q}_{2R}^+(Q)$ we let

$$u_{B_1}^{(B_1)} = W_{1,\epsilon_1}(u_I; V, Q)$$

and assuming we have constructed $u_{B_m}^{(B_m)}$ with $B_m \in \mathcal{Q}_{2R}^+(Q)$, we define for $B_{m+1} \in \mathcal{Q}_{2R}^+(B_m)$

$$u_{B_{m+1}}^{(B_{m+1})} = W_{1,\epsilon_{m+1}}(u_{B_m}; V, B_m)$$
Let \( \mathcal{U} \) to \( B \) from an application of Theorem 7.5 the Fourier supports satisfy \( C \) (thus we apply Theorem 7.5 with \( F \parallel \text{energy bound} \ v \) wave and that \( B \) for \( u \) by a similar argument to the \( v \) extend this to the atomic waves, we simply take \( \lambda \) as a result of Theorem 7.5 the Fourier supports satisfy

\[
\supp u^{(B)} \subset \supp u^{(M-1)} + \left( \frac{2R}{4^{M-1}} \right)^{-
\frac{1}{2}} \supset \supp \hat{u} + \sum_{m=1}^{M-1} \left( \frac{2R}{4^{m-1}} \right)^{-
\frac{1}{2}} \subset \supp \hat{u} + 2 \left( \frac{2R}{\lambda} \right)^{-
\frac{1}{2}}.
\]

On the other hand, the energy inequality follows by exchanging the order of summation, using the fact that \( a \geq 2 \), and repeatedly applying the energy estimate in Theorem 7.5

\[
\left( \sum_{B \in \mathcal{Q}} \| u^{(B)} \|_{L^2_x L^2_t} \right)^{\frac{1}{2}} \leq \left( \sum_{I \in \mathcal{I}} \left( \sum_{B \in \mathcal{Q}} \| u^{(B)} \|_{L^\infty_t L^2_x} \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \leq (1 + C_0 \epsilon_M) \left( \sum_{I \in \mathcal{I}} \left( \sum_{B \in \mathcal{Q}} \| u^{(B)} \|_{L^\infty_t L^2_x} \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \leq \Pi_{m=1}^M (1 + C_0 \epsilon_m) \left( \sum_{I \in \mathcal{I}} \| u_I \|_{L^\infty_t L^2_x} \right)^{\frac{1}{2}} \leq (1 + C\epsilon) \| u \|_{L^2_t L^2_x}
\]

where \( C \) depends only on \( \delta \) and \( C_0 \). The next step is to decompose \( v = \sum_{I \in \mathcal{I}} \mathbb{1}_I(t) v_I \). Let \( U = (u^{(B)}_{I})_{I \in \mathcal{I}, B \in \mathcal{Q} \mathcal{M}} \). Then again, perhaps after relabeling, \( U : \mathbb{R}^{1+n} \rightarrow \ell^2(\mathbb{Z}) \), and hence \( U \) is a \pm-wave with the pointwise bound \( |u^{(B)}| \leq |U| \) and the energy bound \( \| U \|_{L^\infty_t L^2_x} \leq (1 + C\epsilon) \| u \|_{L^2_t L^2_x} \). We now decompose each \( v_I \) relative to \( U \) and the cube \( Q \), in other words we apply Theorem 7.5 and take for every \( B' \in \mathcal{Q} \mathcal{M} \)

\[
v^{(B')}_{I} = \mathcal{W}_{2,\epsilon}^{(B')} (v_I; U, Q)
\]

and finally define \( v^{(B')} = \sum_{I \in \mathcal{I}} \mathbb{1}_I(t) v^{(B')}_{I} \). It is clear that \( v^{(B')} \) is an atomic \pm-wave and that \( v^{(B')} \) satisfies the correct Fourier support conditions. Furthermore, by a similar argument to the \( u^{(B)} \) case, the required energy inequality also holds.

We now turn to the proof of the bilinear estimate. After observing that

\[
\| uv \|_{L^2_{x,t} (X | Q)} \leq \left( \| u^{(1)} \|_{L^2_{x,t} (X | Q)} \| v^{(1)} \|_{L^2_{x,t} (X | Q)} \right)
+ \left( \| (|u| - |u^{(1)}|) v \|_{L^2_{x,t} (X | Q)} + \| (|v| - |v^{(1)}|) u \|_{L^2_{x,t} (X | Q)} \right)
\]

we can apply Theorem 7.5 with \( F = u^{(B)}_{I,m} \, G = V, \, \epsilon = \epsilon_m, \) and \( Q = B_m \). To extend this to the atomic waves, we simply take \( u^{(B)}_m = \sum_{I \in \mathcal{I}} \mathbb{1}_I(t) u^{(B)}_{I,m} \). Finally, for \( B \in \mathcal{Q} \mathcal{M} \), we let \( u^{(B)} = u^{(B)}_M \). Clearly \( u^{(B)} \) is again an atomic \pm-wave, and from an application of Theorem 7.5 the Fourier supports satisfy

\[
\supp u^{(B)} \subset \supp u^{(M-1)} + \left( \frac{2R}{4^{M-1}} \right)^{-
\frac{1}{2}} \supset \supp \hat{u} + \sum_{m=1}^{M-1} \left( \frac{2R}{4^{m-1}} \right)^{-
\frac{1}{2}} \subset \supp \hat{u} + 2 \left( \frac{2R}{\lambda} \right)^{-
\frac{1}{2}}.
\]
an application of Hölder’s inequality implies that it is enough to show that for \( \frac{1}{n+1} < \frac{1}{\theta} \leq \frac{1}{2} \), and \( \frac{1}{2} < \frac{1}{\theta} \leq \frac{1}{2} \) we have

\[
\| (|u| - \lfloor u \rfloor) v \|_{L^p_t L^q_x(X(Q))} + \| |u| - \lfloor u \rfloor \|_{L^p_t L^q_x(X(Q))} \lesssim \epsilon^{-C} \left( \frac{R}{\lambda} \right)^{\frac{1}{2} - \frac{n+1}{2 \theta}} \lambda^{(n+1)(\frac{1}{\theta} - \frac{1}{2})} \| v \|_{\ell^p L^q} \| v \|_{\ell^p L^q}.
\]

We start by estimating the first term. The point is to interpolate between the “bilinear” \( L^2_t \) estimate given in Theorem 7.5 which decays in \( R \), and a “linear” \( L^2_t \) estimate which can lose powers of \( R \), but gains in the summability of the intervals \( I \) and \( J \). We first observe that by construction, Theorem 7.5 implies that

\[
\| (\lfloor u \rfloor_{m-1} - \lfloor u \rfloor_{m}) v \|_{L^p_t L^q_x(X(Q))} \leq \sum_{I \in I} \sum_{b \in B_{m-1} \in Q} \sup_{\tilde{a} \in I,m \in Q} \left( \| (|u|_{m-1} - \lfloor u \rfloor_{m-1}) v \|_{L^p_x L^q_t} \right)^2 \lesssim C_0 \epsilon^{-2} \left( \frac{R}{\lambda} \right)^{\frac{1}{2} - \frac{n+1}{2 \theta}} \\sup_{\tilde{a} \in I,m \in Q} \left( \| (|u|_{m-1} - \lfloor u \rfloor_{m-1}) v \|_{L^p_x L^q_t} \right)^2 \lesssim \epsilon^{-2} \left( \frac{R}{\lambda} \right)^{\frac{1}{2} - \frac{n+1}{2 \theta}} \| u \|_{\ell^p L^q_x} \| u \|_{\ell^p L^q_x}.
\]

where we used the definition of \( \epsilon_m \) and \( M \). On the other hand, to obtain the linear \( L^2_t \) bound, we start by noting that for any \( 1 \leq m \leq M \), and \( a_0 \geq 2 \) we have

\[
\left( \sum_{B \in Q_{m,a_0}^{(B)}} \| u(B) \|_{L^p_t L^q_x}^2 \right)^{\frac{1}{2}} \lesssim \left( \sum_{B \in Q_{m,a_0}^{(B)}} \| u_m(B) \|_{\ell^p L^q_x}^2 \right)^{\frac{1}{2}} \lesssim \lambda^{(n+1)(\frac{1}{2} - \frac{n+1}{2 \theta})} \| u \|_{\ell^p L^q_x}.
\]

Therefore, an application of Hölder’s inequality gives for any \( a_0 \geq 2 \)

\[
\| (\lfloor u \rfloor_{m-1} - \lfloor u \rfloor_{m}) v \|_{L^p_t L^q_x(X(Q))} \lesssim \left( \frac{R}{4m} \right)^{\frac{1}{2}} \left( \sum_{B \in B_{m-1}} \| u(B_{m-1}) v \|_{L^p_t L^q_x(B_{m-1})}^2 + \sum_{B \in B_{m}} \| u(B_m) v \|_{L^p_t L^q_x(B_m)}^2 \right)^{\frac{1}{2}} \lesssim \left( \sum_{B \in B_{m-1}} \| u(B_{m-1}) \|_{L^p_t L^q_x}^2 \right)^{\frac{1}{2}} \| v \|_{L^p_t L^q_x} \lesssim \left( \frac{R}{\lambda} \right)^{\frac{1}{2} - \frac{1}{\theta}} \lambda^{(n+1)(\frac{1}{2} - \frac{n+1}{2 \theta})} \| u \|_{\ell^p L^q_x} \| v \|_{\ell^p L^q_x}.
\]
where we used the fact that $v^{(B')}$ has Fourier support contained in a set of diameter $1$. Interpolating between (7.3) and (7.5) then gives for any $\frac{1}{3} < \frac{1}{m} \leq \frac{1}{2}$,

$$\left\| \left[ \left( u_{m-1}^{(c)} \right) - \left( u_{m}^{(c)} \right) \right] v \right\|_{L^2_t L^m_x(X[Q])} \leq \epsilon^{-C} \frac{R}{\lambda} \left[ \frac{\alpha}{2} + \frac{\alpha + 1}{2} \lambda^{(n+1)\left(\frac{1}{2} - \frac{1}{m}\right)} \right] \|v\|_{L^m_t L^2_x} \|v\|_{L^2_t L^2_x}$$

where $\delta^* = \frac{\alpha + 1}{2} - \frac{1}{2} - 2\delta C_0 \frac{1}{4}$. Consequently, provided that $\frac{1}{m+1} < \frac{1}{5} \leq 1 - \frac{1}{5}$, and we choose $\delta$ sufficiently small depending only on $C_0$, $b$, and $n$, we have $\delta^* > 0$. Thus by telescoping the sum over $m$ and letting $u_0^{(Q)} = u$, we deduce that

$$\| (u - [u^{(c)}]) v \|_{L^2_t L^m_x(X[Q])} \leq \sum_{m=1}^M \| (u_{m-1}^{(c)} - u_m^{(c)}) v \|_{L^2_t L^m_x(X[Q])} \leq \epsilon^{-C} \frac{R}{\lambda} \left[ \frac{\alpha}{2} + \frac{\alpha + 1}{2} \lambda^{(n+1)\left(\frac{1}{2} - \frac{1}{m}\right)} \right] \|u\|_{L^m_t L^2_x} \|v\|_{L^2_t L^2_x}.$$

It only remains to estimate the second term on the left hand side of (7.2). To this end, applying the definition of $v^{(B')}$ together with Theorem 7.5, we have

$$\| (u^{(c)}) (v - [v^{(c)}]) \|_{L^2_t L^m_x(X[Q])} \leq \sum_{J \in \mathcal{J}} \| U (|v| - |v^{(c)}|) \|_{L^2_t L^m_x(I^J, \frac{\epsilon}{2}(Q))} \leq \epsilon^{-2C} \left( \frac{R}{\lambda} \right)^{-\frac{\alpha}{2}} \|u\|_{L^m_t L^2_x} \|v\|_{L^2_t L^2_x}. \quad (7.6)$$

On the other hand an application of Hölder’s inequality together with the energy estimates and (7.4) gives for any $a_0 \geq 2$

$$\| (u^{(c)}) (|v| - |v^{(c)}|) \|_{L^2_t L^m_x(X[Q])} \leq \| (u^{(c)}) \|_{L^2_t L^m_x(X[Q])} \leq \left( \sum_{B \in Q} \| u^{(B)} \|_{L^2_t L^m_x(B)} \right)^{\frac{1}{2}} \sup_{J \in \mathcal{J}} \left( \| v_J \|_{L^m_t L^2_x} + \| v^{(c)}_J \|_{L^m_t L^2_x} \right) \leq \left( \frac{R}{\lambda} \right)^{\frac{1}{2}} \lambda^{(n+1)(\frac{1}{2} - \frac{1}{m})} \|u\|_{L^m_t L^2_x} \|v\|_{L^m_t L^2_x}. \quad (7.7)$$

Therefore, interpolating between (7.0) and (7.7), we deduce that for $\frac{1}{5} < \frac{1}{m} \leq \frac{1}{2}$, we have

$$\| (u^{(c)}) (|v| - |v^{(c)}|) \|_{L^2_t L^m_x(X[Q])} \leq \epsilon^{-C} \left( \frac{R}{\lambda} \right)^{\frac{1}{2}} \lambda^{(n+1)(\frac{1}{2} - \frac{1}{m})} \|u\|_{L^m_t L^2_x} \|v\|_{L^m_t L^2_x}$$

and consequently (7.2) follows.

**7.3. The Induction on Scales Argument.** Here we apply Theorem 7.8 and give the proof of Theorem 7.1. We start with the following definition.

**Definition 7.9.** Given $R > 0$, we let $A(R) > 0$ denote the best constant such that for all cubes $Q$ of diameter $R \geq 100\lambda$, and all atomic $\pm$-waves $u$, and atomic $\pm$-waves $v$ such that

$$\text{supp } u \subset \Lambda_1 + 4 \left( \frac{R}{\lambda} \right)^{-\frac{1}{2}}, \quad \text{supp } v \subset \Lambda_2 + 4 \left( \frac{R}{\lambda} \right)^{-\frac{1}{2}}$$

we have

$$\|uv\|_{L^2_t L^m_x(X[Q])} \leq A(R) \|u\|_{L^m_t L^2_x} \|v\|_{L^2_t L^2_x}. \quad (7.8)$$
It is clear that \( A(R) \lesssim R^{\frac{1}{2}} \). Our goal is to show that in fact we have \( A(R) \lesssim \lambda^{(n+1)(\frac{1}{2} - \frac{1}{b})} \) for all \( R \geq 100 \lambda \). This is a consequence of an induction on scales argument, using the following bounds.

**Proposition 7.10** (Induction Bounds). There exists \( C > 0 \) such that for all \( R \geq 100 \lambda \) and \( 0 < \epsilon < \frac{1}{100} \) we have

\[
A(R) \lesssim C \lambda^{(n+1)(\frac{1}{2} - \frac{1}{b})} \left( \frac{R}{\lambda} \right)^{10} \tag{7.8}
\]

and

\[
A(2R) \lesssim (1 + C \epsilon) A(R) + C \epsilon C \lambda^{(n+1)(\frac{1}{2} - \frac{1}{b})} \left( \frac{R}{\lambda} \right) \left( \frac{2R}{\lambda} \right)^{\frac{1}{2} - \frac{1}{b}} \tag{7.9}
\]

**Proof.** Let \( Q \) be a cube of diameter \( 2R \), and let \( u = \sum_{I \in \mathcal{I}} 1_I(t) u_I \) be an atomic +wave, and \( v = \sum_{J \in \mathcal{J}} 1_J(t) v_J \) be an atomic ±-wave satisfying the support conditions

\[
\text{supp } \hat{u} \subset A_1 + 4 \left( \frac{2R}{\lambda} \right)^{-\frac{1}{2}}, \quad \text{supp } \hat{v} \subset A_2 + 4 \left( \frac{2R}{\lambda} \right)^{-\frac{1}{2}}.
\]

An application of Theorem \[\text{(7.8)}\] gives a cube \( Q' \) of diameter \( 4R \), and atomic waves \((u^{(B)})_{B \in Q} \subset \mathcal{Q}(Q), (v^{(B')}))_{B' \in Q}(Q)\) such that

\[
\|uv\|_{L^2_{t,x}(Q')} \lesssim (1 + C \epsilon) \left[ \|u^{(j)}\|_{L^2_{t,x}(Q')} \right] + C \epsilon C \lambda^{(n+1)(\frac{1}{2} - \frac{1}{b})} \left( \frac{R}{\lambda} \right) \left( \frac{2R}{\lambda} \right)^{\frac{1}{2} - \frac{1}{b}} \tag{7.10}
\]

and the support properties

\[
\text{supp } \hat{u}^{(B)} \subset \text{supp } \hat{u} + 2 \left( \frac{4R}{\lambda} \right)^{-\frac{1}{2}} \subset A_1 + 4 \left( \frac{R}{\lambda} \right)^{-\frac{1}{2}}
\]

and similarly \( \text{supp } \hat{v} \subset A_2 + 4 \left( \frac{R}{\lambda} \right)^{-\frac{1}{2}} \).

To prove \( \|uv\| \), we let \( B' \in Q_R(Q) \) and define the atomic +wave \( U^{(B')} = \sum_{I \in \mathcal{I}} 1_I(t) U^{(B')}_{I} \) with \( U^{(B')}_{I} = (u^{(B')})_{B \in Q} \subset \mathcal{Q}(Q) \). Then for every \( B' \in Q_R(Q) \) we have an atomic +wave \( U^{(B')} \) and an atomic ±-wave \( v^{(B')} \) satisfying the correct support assumptions to apply the definition of \( A(R) \). Thus

\[
\|u^{(j)}\|^2_{L^2_{t,x}(Q')} \lesssim \left( \sum_{B' \in Q_R(Q')} \|U^{(B')} v^{(B')}\|^2_{L^2_{t,x}} \right)^{\frac{1}{2}}
\]

\[
\leq A(R) \left( \sum_{B' \in Q_R(Q')} \|U^{(B')} v^{(B')}\|^2_{L^2_{t,x}} \right)^{\frac{1}{2}} \leq A(R) \left( \sum_{B' \in Q_R(Q')} \|v^{(B')}\|^2_{L^2_{t,x}} \right)^{\frac{1}{2}} \leq (1 + C \epsilon) A(R)
\]

where the second line used the assumption \( \frac{1}{a} + \frac{1}{b} \geq \frac{1}{2} \) and the last applied the energy inequalities in Theorem \[\text{(7.8)}\]. Therefore the induction bound \( \text{(7.9)} \) follows from an application of \( \text{(7.10)} \).

We now turn to the proof of \( \text{(7.8)} \). We begin by observing that again using the bound \( \text{(7.10)} \) with \( \epsilon \approx 1 \) it is enough to prove that for every \( \frac{1}{b} \leq \frac{1}{a} \leq \frac{1}{2} \) we have the quilt bound

\[
\|u^{(j)}\|^2_{L^2_{t,x}(Q')} \lesssim \lambda^{(n+1)(\frac{1}{2} - \frac{1}{b})} \left( \frac{R}{\lambda} \right) \left( \frac{2R}{\lambda} \right)^{\frac{1}{2} - \frac{1}{b}} \|u\|_{L^2_{t,x}} \|v\|_{L^2_{t,x}} \tag{7.11}
\]
But this follows by observing that since \( u(\cdot) = \sum_B 1_B |u(B)| \) is localised to cubes of diameter \( \frac{1}{\lambda} \), an application of Hölder’s inequality together with the energy estimates implies that

\[
\| [u(\cdot)] [v(\cdot)] \|_{L^2_{t,x}(Q')} \lesssim \lambda^{(n+1)(\frac{1}{2} - \frac{1}{a})} (R \lambda^2) \left( \sum_B \| u(B) \|_{L^p L^2} \right) \left( \sup_{B'} \| v(B') \|_{L^p L^2} \right) \lesssim \lambda^{(n+1)(\frac{1}{2} - \frac{1}{a})} \| u \|_{L^a L^2} \| v \|_{L^\infty L^2}.
\]

\[\square\]

We now come to the proof of Theorem \ref{thm:7.1}.

**Proof of Theorem \ref{thm:7.1}** Let \( C \) denote the constant in Proposition \ref{prop:7.10}. Let \( R = 2^k 100 \lambda \) and \( \epsilon_k = 2^{-\delta k} \) with \( 0 < \delta < \frac{1}{C} \left( \frac{n-1}{2} - \left( \frac{1}{2} - \frac{1}{a} \right) \frac{n+1}{2} \right) \). Then an application of \( \ref{eq:7.9} \) gives

\[
A(2^{k+1} 100 \lambda) \lesssim (1 + C 2^{-k\delta}) A(2^k 100 \lambda) + C \lambda^{(n+1)(\frac{1}{2} - \frac{1}{a})} \| u \|_{L^a L^2} \| v \|_{L^\infty L^2}.
\]

Since both exponents decay in \( k \), after \( k \) applications, we deduce that

\[
A(2^{k+1} 100 \lambda) \lesssim A(100 \lambda) + \lambda^{(n+1)(\frac{1}{2} - \frac{1}{a})} \lesssim \lambda^{(n+1)(\frac{1}{2} - \frac{1}{a})}
\]

where we used the initial induction bound \( \ref{eq:7.8} \). Hence Theorem \ref{thm:7.1} follows. \[\square\]

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