A Fully Polynomial Parameterized Algorithm for Counting the Number of Reachable Vertices in a Digraph

Naoto Ohsaka*
NEC Corporation

Abstract
We consider the problem of counting the number of vertices reachable from each vertex in a digraph $G$, which is equal to computing all the out-degrees of the transitive closure of $G$. The current (theoretically) fastest algorithms run in quadratic time; however, Borassi has shown that this problem is not solvable in truly subquadratic time unless the Strong Exponential Time Hypothesis fails [Inf. Process. Lett., 116(10):628–630, 2016]. In this paper, we present an $O(f^3n)$-time exact algorithm, where $n$ is the number of vertices in $G$ and $f$ is the feedback edge number of $G$. Our algorithm thus runs in truly subquadratic time for digraphs of $f = O(n^{1-\epsilon})$ for any $\epsilon > 0$, i.e., the number of edges is $n$ plus $O(n^{1-\epsilon})$, and is fully polynomial fixed parameter tractable, the notion of which was first introduced by Fomin, Lokshtanov, Pilipczuk, Saurabh, and Wrochna [ACM Trans. Algorithms, 14(3):34:1–34:45, 2018]. We also show that the same result holds for vertex-weighted digraphs, where the task is to compute the total weights of vertices reachable from each vertex.

1 Introduction
Consider the following problem concerning reachability on graphs. Given a digraph $G$, count the number of vertices reachable from each vertex. This problem is known by the name of DESCENDANT COUNTING [Coh97], and it coincides with computing all the out-degrees of the transitive closure of $G$. Computation of the size of the transitive closure has several applications, including query optimization [LN95, LNS90], sparse matrix multiplication [Coh98], and social network analysis, wherein it is used as a subroutine in identifying the most influential set of individuals in a social network [CWY09, KSNM10, OAYK14].

The current (theoretically) fastest algorithms for DESCENDANT COUNTING have (at least) quadratic time complexity: they explicitly construct the transitive closure of $G$ in $O(nm)$ time [Pur70, Ebe81] by running a breadth-first search or in $O\left(nn^{\log_2 m/n} + n^2\right)$ time [BVW08] by using a sophisticated data structure for sparse graphs; for dense graphs, this can be done in $\tilde{O}(n^{\omega})$ time$^1$ [ABPR78] through fast matrix multiplication. Here, $n$ is the number of vertices in $G$, $m$ is the number of edges in the $G$, and $\omega < 2.3728639$ [LG14] is the exponent of matrix multiplication. Note that Cohen’s celebrated approximation algorithm [Coh97] estimates the number

\*naoto.ohsaka@gmail.com

$^1\tilde{O}(g)$ denotes $O(g \log^c g)$ for some positive integer $c$. 
of reachable vertices within a factor of \((1 \pm \epsilon)\) with high probability and runs in \(O(\epsilon^{-2}n \log n)\) time, which is almost linear. Unfortunately, it has been proven by Borassi [Bor16] that any exact algorithm that runs in truly subquadratic time, i.e., in \(O(n^{2-\epsilon})\) time for any \(\epsilon > 0\), refutes the Strong Exponential Time Hypothesis (SETH) [IP01]. The SETH states that for any \(\epsilon > 0\) there exists some integer \(k \geq 3\) such that \(k\)-SATISFIABILITY on \(n\) variables cannot be solved in \(O(2^{(1-\epsilon)n})\) time. In particular, the same result is still true for sparse acyclic digraphs (\(m = O(n)\)).

Nevertheless, in this study, we quest for truly subquadratic time as well as exact algorithms for Descendant Counting. To circumvent the quadratic time barrier, we follow the framework of parameterized algorithms. Given a parameter \(k\) in addition to the input size, a problem is referred to as fixed parameter tractable (FPT) if it is solvable in \(g(k) \cdot |I|^{O(1)}\) time, where \(g\) is some computable function depending only on parameter \(k\) and \(|I|\) is the input size, e.g., \(|I| = n + m\) in our case. While FPT algorithms have been actively studied for NP-hard problems (see, e.g., [CFK+15]), the concept of “FPT inside P” has opened up a new exciting line of research [GMN17].

Our contribution is that we present an exact parameterized algorithm for Descendant Counting that has running time \(O(f^3n)\), where \(n\) is the number of vertices in \(G\) and \(f\) is the feedback edge number of \(G\). The feedback edge number is the minimum number of edges, the removal of which renders the underlying undirected graph acyclic; this parameter has been used to develop parameterized algorithms for graph problems in \(P\), e.g., Maximum Matching [MNN20], Betweenness Centrality [BDK+18], Hyperbolicity [FKM+19], Triangle Listing [BFNN19], and Diameter [BN19]. Hence, for “very tree-like” digraphs having \(f = O(n^{3-\epsilon})\) for any \(\epsilon > 0\), i.e., the number of edges is bounded by \(m = n + O(n^{3-\epsilon})\), our algorithm runs in \(O(n^{2-\epsilon})\)—truly subquadratic—time and thus outperforms the current fastest algorithms described above.\(^2\) On the other hand, if it holds that \(f = \Omega(n^{3+\epsilon})\), which would be the case for real-world networks, the proposed algorithm requires more than \(O(nm)\) time. Furthermore, the dependence of the time complexity on parameter \(f\) is polynomial; such an algorithm, introduced by Fomin, Lokshtanov, Pilipczuk, Saurabh, and Wrochna [FLS+18], is called fully polynomial FPT. We also show that the same result holds for vertex-weighted digraphs, where the task is to compute the total weights of vertices reachable from each vertex.

We here stress that some graph parameters do not admit truly subquadratic time, fully polynomial FPT algorithms for Descendant Counting: Ogasawara [Oga18] showed that under the SETH, a \(k^{O(1)}n^{2-\epsilon}\)-time algorithm does not exist for any \(\epsilon > 0\), where \(k\) denotes the treewidth of \(G\); the same hardness applies to the case where \(k\) is the feedback vertex number of \(G\), which is the minimum number of vertices the removal of which renders the underlying undirected graph acyclic, because Ogasawara used Borassi [Bor16]’s reduction which constructs a digraph whose feedback vertex number is \(O(\log n)\).

2 Preliminaries

Notations and Definitions. For a digraph \(G = (V, E)\), let \(V(G)\) and \(E(G)\) denote the vertex set \(V\) and the edge set \(E\) of \(G\), respectively. Throughout this paper, all the digraphs are simple; i.e.,

\(^2\)Even when \(G\) is a polytree, i.e., \(m = n - 1\), the naive algorithms show quadratic time complexity, because the transitive closure of \(G\) can be of size \(O(n^2)\). In contrast, our algorithm no longer constructs the transitive closure.
they have no self-loops and no multi-edges. For a subset of vertices \( S \subseteq V(G) \), the subgraph induced by \( S \) is denoted by \( G[S] \). A digraph is said to be acyclic if it contains no directed cycles, to be weakly connected if the underlying undirected graph is connected, and to be strongly connected if every vertex can reach every other vertex. A polyforest is a digraph, the underlying undirected graph of which is a forest. A path \( P := (v_0, v_1, \ldots, v_\ell) \) is a digraph with vertex set \( V(P) := \{v_0, v_1, \ldots, v_\ell\} \) and edge set \( E(P) := \{(v_0, v_1), (v_1, v_2), \ldots, (v_\ell-1, v_\ell)\} \), where \( v_0, v_1, \ldots, v_\ell \) are distinct. For a digraph \( G \) and for an edge \((u, v)\) and a path \( P \), we denote \( G + (u, v) := (V(G), E(G) \cup \{(u, v)\}) \) and \( G + P := (V(G) \cup V(P), E(G) \cup E(P)) \). Given a vertex weighting \( a : V(G) \to \mathbb{R} \), we denote by \( a_v \) the weight for vertex \( v \) and abuse notation by writing \( a(S) = \sum_{v \in S} a_v \) for vertex set \( S \subseteq V(G) \).

For a digraph \( G \), the reachability set of vertex \( v \), denoted \( R_G(v) \), is defined as the set of vertices reachable from \( v \) on \( G \) (including \( v \) itself), and the reachability number of vertex \( v \) is defined as \( r_G(v) = |R_G(v)| \). For a vertex weighting \( a \), the weighted reachability number of vertex \( v \) is defined as \( r_{G,a}(v) = a(R_G(v)) \). We formally define the descendant counting problem and its vertex-weighted version below.

**Problem 2.1 (Descendant Counting).** Given a digraph \( G \), the task is to compute the reachability number \( r_G(v) \) for each vertex \( v \) in \( G \).

**Problem 2.2 (Weighted Descendant Counting).** Given a digraph \( G \) and a vertex weighting \( a : V(G) \to \mathbb{R} \), the task is to compute the weighted reachability number \( r_{G,a}(v) \) for each vertex \( v \) in \( G \). In particular, the case where \( a_v = 1 \) for all \( v \in V(G) \) corresponds to **Descendant Counting**.

A feedback edge set in a digraph is a set of edges, the deletion of which renders the digraph a polyforest. The feedback edge number is defined as the minimum size of any feedback edge set. That is, the feedback edge number of \( G \) is equal to \( |E(G)| - |V(G)| + c \), where \( c \) is the number of weakly connected components in \( G \), which takes 1 if \( G \) is entirely weakly connected.

**Remark 2.3.** Our definition of a feedback edge set coincides with that for an undirected graph. For a digraph, the feedback arc set, the deletion of which renders the digraph acyclic, is usually adopted. We adopt the present definition, because solving **Descendant Counting** in truly subquadratic time for acyclic digraphs still falsifies the SETH.

The condensation \( G^{\text{SCC}} \) of a digraph \( G \) is defined as the digraph obtained from \( G \) by contracting each strongly connected component. Formally, the vertices in \( G^{\text{SCC}} \) are the strongly connected components in \( G \), and there exists an edge from a vertex \( C \) to another vertex \( C' \) in \( G^{\text{SCC}} \) if and only if there exists an edge \((u, v) \in E(G) \) such that \( u \in C, v \in C' \). Note that the condensation is acyclic. Let \( \pi : V(G) \to V(G^{\text{SCC}}) \) denote a mapping from vertices in \( G \) to vertices in \( G^{\text{SCC}} \); i.e., \( \pi(v) = C \) whenever \( v \in C \in V(G^{\text{SCC}}) \). We abuse notation by writing \( \pi((u, v)) = (\pi(u), \pi(v)) \) for edge \((u, v) \) and \( \pi(S) = \{\pi(e) \mid e \in S\} \) for set \( S \).

**Warm-up: Linear-time Algorithm for a Polyforest.** Let us first consider the case where the input graph is a polyforest \( T \). Noting that the weighted reachability number \( r_{T,a}(v) \) of vertex \( v \) is the sum of the weighted reachability numbers over its out-neighbors plus \( a_v \), i.e.,

\[
r_{T,a}(v) = a_v + \sum_{w : (v, w) \in E(T)} r_{T,a}(w),
\]

(1)
we are able to determine $r_{T,a}(v)$ in a bottom-up fashion. Algorithm 2.1 shows the precise pseudo-code. Because the topological ordering can be found in linear time [Tar76], Algorithm 2.1 runs in $O(n)$ time. This linear-time algorithm is used as a subroutine, as described in the following section.

3 Fully Polynomial FPT Algorithm for Bounded Feedback Edge Number

We now consider a general digraph of feedback edge number $f$. Unlike in the case of a polyforest, reachability numbers cannot be written as the sum of reachability numbers, because out-neighbors’ reachability sets can overlap each other. Our strategy is based on incremental update; i.e., (1) we first delete $f$ edges and solve (Weighted) Descendant Counting on the resulting polyforest, and (2) we then revert the $f$ edges serially and update the reachability number.

3.1 Efficient Incremental Update on Acyclic Digraphs

Let $G$ be an acyclic digraph of feedback edge number $f$ and $(s,t) \in V(G) \times V(G)$ be an edge not in $G$. Assume that inserting $(s,t)$ does not render $G$ cyclic. Given the reachability number $r_G$ for $G$, we obtain the reachability number $r_{G+(s,t)}$ for $G+(s,t)$ as follows. Let $R^\triangledown$ denote the set of vertices that can reach $s$ on $G$ and $R_\triangle$ denote the set of vertices reachable from $t$ on $G$. Here, $R^\triangledown$ and $R_\triangle$ are disjoint as $G+(s,t)$ is acyclic. We also remark that only vertices in $R^\triangledown$ can newly reach some vertices in $R_\triangle$. Obviously, we can update reachability numbers for such vertices by merely running a breadth-first search, which, however, consumes quadratic time in the worst case.

To bypass this inefficiency, further definitions are required. We say that a vertex $b \in R^\triangledown$ is a boundary if there exists a path from $b$ to some vertex of $R_\triangle$ that does not pass through $(s,t)$ and the internal vertices of which do not touch $R^\triangledown$. We define the boundary set as the set of all boundaries and denote it by $B \subseteq R^\triangledown$. See Figure 1. For a vertex $v \in R^\triangledown$, we call $R_G(v) \cap B$ the restricted boundary set for $v$. Observe that the set difference of $R_{G+(s,t)}(v)$ and $R_G(v)$ for $v \in V(G)$ can be expressed by using $R_\triangle$ and $R_G(b)$’s for $b \in B$:

$$R_{G+(s,t)}(v) \setminus R_G(v) = \begin{cases} R_\triangle \setminus \bigcup_{b \in R_G(v) \cap B} R_G(b) & \text{if } v \in R^\triangledown, \\ \emptyset & \text{otherwise.} \end{cases} \quad (2)$$

It is determined that $r_{G+(s,t)}(v) = r_G(v) + |R_\triangle \setminus \bigcup_{b \in R_G(v) \cap B} R_G(b)|$ for vertex $v \in R^\triangledown$. The boundary set has the following convenient properties to compute Eq. (2) efficiently, whose proofs are

---

**Algorithm 2.1** $O(n)$-time algorithm for a polyforest.

**Input:** polyforest $T$ and vertex weighting $a$.

1: find a topological ordering of $V(T)$.
2: for all $v \in V(T)$ in reverse topological order do
3: \hspace{1em} $r(v) \leftarrow a_v + \sum_{w: (v,w) \in E(T)} r(w)$.
4: \hspace{1em} return $r$. 

---
Figure 1: Illustration of $R^\triangledown$, $R_\triangle$, and $B$. Paths from $b_1, b_2, b_3, b_4$ to $s$ and paths from $t$ to $c, c', c''$ are omitted for simplicity. Each vertex in $B$ can reach $R_\triangle$ without passing through $(s, t)$. Because vertex $v$ can reach $b_1$ and $b_2$ and their reachable sets, it can reach vertices of $R_\triangle \setminus (R_G(b_1) \cup R_G(b_2))$ for the first time on $G + (s, t)$.

defered to Section 3.3.

**Lemma 3.1.** Let $G$ be an acyclic digraph of feedback edge number $f$ and $(s, t)$ be an edge not in $G$, and assume $G + (s, t)$ acyclic. Then, the boundary set $B$ has at most $f$ vertices; i.e., $|B| \leq f$.

**Lemma 3.2.** Let $G$ be an acyclic digraph of feedback edge number $f$ and $(s, t)$ be an edge not in $G$, and assume $G + (s, t)$ acyclic. Then, the collection of the restricted boundary set for $v \in R^\triangledown$ has at most $2f$ distinct sets; i.e.,

$$|\{R_G(v) \cap B \mid v \in R^\triangledown\}| \leq 2f. \tag{3}$$

Lemma 3.1 tells us that we can explicitly compute the reachability set for all boundaries in $O(f(n + m))$ time. By Lemma 3.2, we have that the space to store $\bigcup_{b \in R_G(v) \cap B} R_G(b)$ in Eq. (2) for all $v \in R^\triangledown$ is bounded by $2f \cdot |R_G(\cdot)| = O(fn)$. As a by-product, we can say that the collection of the set differences between $R_{G+(s,t)}(v)$ and $R_G(v)$ for all $v \in R^\triangledown$ are of cardinality at most $2f$; i.e., $|\{R_{G+(s,t)}(v) \setminus R_G(v) \mid v \in R^\triangledown\}| \leq 2f$.

### 3.2 Algorithm Description for Acyclic Digraphs

Algorithm 3.1 shows the precise pseudocode of our algorithm that, given an acyclic digraph $G$ of feedback edge number $f$ and a vertex weighting $a : V(G) \to \mathbb{R}$, computes the reachability number $r_{G,a}$ for all vertices in $G$. We first construct a polyforest $T$ on $V(G)$ by deleting $f$ edges from $G$ and invoke Algorithm 2.1 on $T$ with $a$ to obtain its weighted reachability number $r_{T,a}$. Let $E(G) \setminus E(T) = \{(s_1, t_1), \ldots, (s_f, t_f)\}$ be the $f$ edges to be reverted to $T$ in an arbitrary order. Let $G^{(0)} = T$ and define $G^{(i)} = G^{(i-1)} + (s_i, t_i)$ for each $i \in \{1, \ldots, f\}$. For each $i \in \{0, 1, \ldots, f\}$, let $r^{(i)}$ be the weighted reachability number for $G^{(i)}$; in particular, we have that $r^{(0)} = r_{T,a}$. 

5
Algorithm 3.1 $O(f^3 n + f^2 m)$-time algorithm for an acyclic digraph of feedback edge number $f$.

**Input:** acyclic digraph $G$ of feedback edge number $f$ and vertex weighting $a$.

1. compute a polyforest $T$ on $V(G)$, the edge set of which is obtained by removing $f$ edges from $G$.
2. invoke Algorithm 2.1 on $T$ with $a$ to obtain $r_{T,a}$.
3. let $G^{(0)} \leftarrow T$ and $r^{(0)} \leftarrow r_{T,a}$.
4. let $E(G) \setminus E(T) = \{(s_1, t_1), \ldots, (s_f, t_f)\}$ in any order.
5. for $i = 1$ to $f$
6.   compute set $R^\triangledown$ of vertices that can reach $s_i$ on $G^{(i-1)}$.
7.   compute set $R_\bigtriangleup$ of vertices reachable from $t_i$ on $G^{(i-1)}$.
8.   compute boundary set $B \subseteq R^\triangledown$.
9.   declare empty set $\text{MARK}[v]$ for all $v \in R^\triangledown$.
10. for all $b \in B$
11.   for all $v \in R^\triangledown$ that can reach $b$ on $G^{(i-1)}$
12.     $\text{MARK}[v] \leftarrow \text{MARK}[v] + b$.
13. $R(b) \leftarrow$ reachability set of $b$ w.r.t. $G^{(i-1)}$.
14. declare empty trie $\text{GAIN}$.
15. for all $v \in R^\triangledown$
16.   if $\text{MARK}[v]$ is not found in $\text{GAIN}$ then
17.     $\text{GAIN}[\text{MARK}[v]] \leftarrow a(R_\bigtriangleup \setminus \bigcup_{b \in \text{MARK}[v]} R(b))$.
18. $r^{(i)}(v) \leftarrow r^{(i-1)}(v) + \text{GAIN}[\text{MARK}[v]]$ for all $v \in R^\triangledown$.
19. $r^{(i)}(v) \leftarrow r^{(i-1)}(v)$ for all $v \in V(G) \setminus R^\triangledown$.
20. $G^{(i)} \leftarrow G^{(i-1)} + (s_i, t_i)$.
21. return $r^{(f)}$.

The remaining part of the algorithm consists of $f$ rounds, which reverts each of the $f$ edges serially. Given the current $r^{(i-1)}$ at the beginning of the $i$-th round with $i \in \{1, \ldots, f\}$, we calculate $r^{(i)}$ as follows. As in the previous section, let $R^\triangledown$ be the set of vertices that can reach $s_i$ on $G^{(i-1)}$ and $R_\bigtriangleup$, the set of vertices reachable from $t_i$ on $G^{(i-1)}$. We compute the boundary set $B$ with regard to $G^{(i-1)}$ and $(s_i, t_i)$. This computation can be done by running a breadth-first search starting from $R_\bigtriangleup$ that does not go forward whenever it touches $R^\triangledown$ on the transposed counterpart of $G^{(i-1)}$. Then, for each boundary $b \in B$, we mark vertices in $R^\triangledown$ that can reach $b$ and compute the reachability set of $b$ in $G^{(i-1)}$, denoted by $R(b)$. Let $\text{MARK}[v]$ be the set of boundaries marked for $v$; i.e., $\text{MARK}[v] = R_{G^{(i-1)}}(v) \cap B$. We know that for every $v \in R^\triangledown$, $R_{G^{(i-1)}}(v) \setminus R_{G^{(i-1-1)}}(v)$ is equal to $R_\bigtriangleup \setminus \bigcup_{b \in \text{MARK}[v]} R(b)$ because of Eq. (2). Hence, we declare an empty trie $\text{GAIN}$ for storing $a(R_\bigtriangleup \setminus \bigcup_{b \in \text{MARK}[v]} R(b))$ with key $\text{MARK}[v]$ for every $v \in R^\triangledown$. It should be noted that we can omit the reachability set computation for some vertices, because $\text{MARK}[u] = \text{MARK}[v]$ may hold for $u \neq v$. Finally, we compute $r^{(i)}(v)$ as $r^{(i-1)}(v) + \text{GAIN}[\text{MARK}[v]]$ for $v \in R^\triangledown$ and $r^{(i-1)}(v)$ for $v \in V(G) \setminus R^\triangledown$ and insert $(s, t)$ into $G^{(i-1)}$ to obtain $G^{(i)}$. Having completed the $f$ rounds, we return $r^{(f)}$. 
3.3 Correctness and Time Complexity

We now verify the correctness and the time complexity of Algorithm 3.1. To this end, we first validate Eq. (2).

Observation 3.3. Let \( G \) be an acyclic digraph and \((s,t)\) be an edge not in \( G \), and assume \( G + (s,t) \) acyclic. Then, for any vertex \( v \), Eq. (2) is correct.

Proof. The case where \( v \not\in R^\triangledown \) is obvious; we prove for \( v \in R^\triangledown \). We show that a vertex \( x \in V(G) \) is in the set on the left hand side in Eq. (2) if and only if \( x \) is in the set on the right hand side in Eq. (2). First, assume that \( x \in R_{G+(s,t)}(v) \setminus R_G(v) \). We have that any path from \( v \) to \( x \) on \( G + (s,t) \) must pass through \((s,t)\). Obviously, \( x \in R_{\Delta} \). We also have that \( x \not\in R_G(b) \) for any \( b \in R_G(v) \cap B \), because otherwise \( x \) can reach \( v \) via such \( b \) without passing through \((s,t)\), which results in a contradiction. Consequently, \( x \in R_{\Delta} \setminus \bigcup_{b \in R_G(v) \cap B} R_G(b) \). Now, assume that \( x \in R_{\Delta} \setminus \bigcup_{b \in R_G(v) \cap B} R_G(b) \). We have that \( x \) is reachable from \( t \) (on \( G \)). Thus, \( v \) can reach \( x \) on \( G + (s,t) \) by passing through \((s,t)\), i.e., \( x \in R_{G+(s,t)}(v) \). On the other hand, \( v \) cannot reach \( x \) on \( G \), because to exit \( R^\triangledown \) without passing through \((s,t)\), a path starting from \( v \) must touch some \( b \in R_G(v) \cap B \); i.e., \( x \not\in R_G(v) \). Consequently, \( x \in R_{G+(s,t)}(v) \setminus R_G(v) \), which completes the proof.

We now prove the two lemmas on the boundary set.

Proof of Lemma 3.1. We prove the statement by a contradiction. Suppose that the boundary set \( B \) contains at least \( f + 1 \) vertices, say, \( b_1, b_2, \ldots, b_{f+1} \). Without loss of generality, we can assume that these vertices are sorted in reverse topological order, so that \( b_i \) cannot reach \( b_j \) whenever \( i < j \).

We construct a sequence of \( f + 2 \) subgraphs of \( G \), denoted by \( G_0, G_1, \ldots, G_{f+1} \), as follows. The initial graph \( G_0 \) consists of a single edge \((s,t)\); i.e., \( G_0 = \{(s,t)\} \). For each \( i \in \{1, \ldots, f + 1\} \), \( G_i \) is obtained from \( G_{i-1} \) by adding the following three paths: (1) a path \( P_{b_i}^s \) on \( G[R^\triangledown] \) from \( b_i \) to \( s \), (2) a path \( P_{b_i}c_i \) from \( b_i \) to some vertex \( c_i \in R_{\Delta} \) the internal vertices of which belong to neither \( R^\triangledown \) nor \( R_{\Delta} \), and (3) a path \( P_{c_i} \) on \( G[R_{\Delta}] \) from \( t \) to \( c_i \). See Figure 2. Note that the three paths are edge-disjoint, and \( f(G_i) = |E(G_i)| - |V(G_i)| + 1 \) for each \( G_i \), where \( f(\cdot) \) denotes a feedback edge number, as it is weakly connected.

We now show that the feedback edge number increases by at least one from \( G_{i-1} \) to \( G_i \). Let \( x \) and \( y \) be the vertex next to \( b_i \) with regard to the two paths \( P_{b_i}^s \) and \( P_{b_i}c_i \), respectively. Then, we divide \( P_{b_i}^s \) into an edge \((b_i, x)\) and a subpath \( P_{xs} \) from \( x \) to \( s \) and divide \( P_{b_i}c_i \) into an edge \((b_i, y)\) and a subpath \( P_{yc_i} \) from \( y \) to \( c_i \). See Figure 2. It is easy to see that \( G_{i-1} + P_{xs} + P_{c_i} + P_{yc_i} \) is weakly connected. Then, consider the addition of \((b_i, x)\) and \((b_i, y)\) to \( G_{i-1} + P_{xs} + P_{yc_i} \), which yields \( G_i \). Observing that \( x \) and \( y \) have already been added and \( x \neq y \) (because \( P_{b_i}^s \) and \( P_{b_i}c_i \) are edge-disjoint), we can ensure that this addition preserves the weak connectivity and increases the number of vertices by one for \( b_i \) and the number of edges by two for \((b_i, x)\) and \((b_i, y)\); i.e., \( f(G_i) \geq f(G_{i-1}) + 1 \). Hence, \( f(G_{f+1}) \geq f + 1 + f(G_0) = f + 1 \), from which it follows that \( f(G) \geq f(G_{f+1}) \geq f + 1 \), a contradiction.

Proof of Lemma 3.2. The proof is by contradiction. Suppose that at least \( 2f + 1 \) vertices in \( R^\triangledown \) have distinct restricted boundary sets; i.e., \(|\{R_G(v) \cap B \mid v \in R^\triangledown\}| \geq 2f + 1 \). We bound from below
for every $v$ in linear time, because spanning tree computation and Algorithm 2.1 are completed in linear time. Proof.

To prove correctness of Algorithm 3.1. To this end, we show that it holds that $r^{(i)} = r^{(i-1)}$ for all $i \in \{0, 1, \ldots, f\}$ by induction on $i$. The base case $i = 0$ is clear. Now, assume that $r^{(i-1)} = r^{(i-1)}$, where $i \in \{1, \ldots, f\}$. In the $i$-th iteration, $\text{MARK}[v]$ is equal to $R_G(v) \cap B$ for every $v \in R^\triangledown$. Thus, by construction of $\text{GAIN}$, it holds that $r^{(i)}(v) = r^{(i-1)}(v) + a(R^\triangle \cup \bigcup_{b \in R_G(v) \cap B} R_G(b))$ for every $v \in R^\triangledown$, which is equal to $r^{(i-1)}(v) + a(R_G(v) \setminus R_G(v)) = r^{(i-1)}(v)$ by the assumption and Observation 3.3, which completes the inductive step.

We now bound the running time of Algorithm 3.1. $T$ and $r_{T,a}$ (steps 1–2) can be constructed in linear time, because spanning tree computation and Algorithm 2.1 are completed in linear time, because spanning tree computation and Algorithm 2.1 are completed in linear time. Proof.

To prove correctness of Algorithm 3.1. To this end, we show that it holds that $r^{(i)} = r^{(i-1)}$ for all $i \in \{0, 1, \ldots, f\}$ by induction on $i$. The base case $i = 0$ is clear. Now, assume that $r^{(i-1)} = r^{(i-1)}$, where $i \in \{1, \ldots, f\}$. In the $i$-th iteration, $\text{MARK}[v]$ is equal to $R_G(v) \cap B$ for every $v \in R^\triangledown$. Thus, by construction of $\text{GAIN}$, it holds that $r^{(i)}(v) = r^{(i-1)}(v) + a(R^\triangle \cup \bigcup_{b \in R_G(v) \cap B} R_G(b))$ for every $v \in R^\triangledown$, which is equal to $r^{(i-1)}(v) + a(R_G(v) \setminus R_G(v)) = r^{(i-1)}(v)$ by the assumption and Observation 3.3, which completes the inductive step.

We now bound the running time of Algorithm 3.1. $T$ and $r_{T,a}$ (steps 1–2) can be constructed in linear time, because spanning tree computation and Algorithm 2.1 are completed in linear time, because spanning tree computation and Algorithm 2.1 are completed in linear time.
time. We now show that each of the $f$ rounds (steps 6–20) consumes $O(f^2 n + fm)$ time. First, we can compute $R^\triangledown, R_\triangle$, and $B$ (steps 6–8) in $O(n + m)$ time by running three breadth-first searches starting from $s_t, t_t$, and $R_\triangle$ on $G^{(i-1)}$, respectively. By Lemma 3.1, $B$ contains at most $f$ vertices. Thus, $\text{MARK}$ can be constructed (steps 10–13) in $O(|B| \cdot (n + m)) = O(f(n + m))$ time. Note that each $\text{MARK}[v]$ contains at most $f$ vertices of $B$. Then, observing that there are at most $2f$ distinct sets in $\text{MARK}$ by Lemma 3.2, we can ensure that the algorithm reaches step 17 at most $2f$ times and compute the right hand side of step 17 in $O(fn)$ time by taking a union over (at most) $f$ sets. Each search for $\text{GAIN}$ (step 16) and each update of $\text{GAIN}$ (step 17) can be done in $O(f)$ time because the length of a binary representation of a key (i.e., any subset of $B$) is $O(f)$. It is thus determined that the construction of $\text{GAIN}$ (steps 15–17) consumes $O(fn)$ time. We can obviously update the reachability number and the digraph (steps 18–20) in time $O(fn)$. Accordingly, the entire time complexity is bounded from above by $O(f^3 n + f^2 m)$. The space complexity is obvious. 

\section{Solution for General Digraphs}

We finally use Algorithm 3.1 to solve \textsc{Weighted Descendant Counting} for arbitrary digraphs of bounded feedback edge number. For the sake of completeness, we prove the following observation.

\begin{observation}
If a digraph $G$ has a feedback edge number of at most $f$, then so does the condensation $G^{\text{SCC}}$.
\end{observation}

\begin{proof}
Let $F$ be a feedback edge set of size (at most) $f$ of $G$. Then, the edge set $F' := \pi(F)$, which is of size at most $f$, is a feedback edge set of $G^{\text{SCC}}$. This is because $G^{\text{SCC}} - F'$ is equal to the graph $(\pi(V(G)), \pi(E(G) - F))$, which is a polyforest by definition of $F$.
\end{proof}

\begin{corollary}
For a digraph $G$ of feedback edge number $f$ and a vertex weighting $a : V(G) \rightarrow \mathbb{R}$, \textsc{Weighted Descendant Counting} can be solved in $O(f^3 n)$ time and $O(fn)$ space, where $n = |V(G)|$.
\end{corollary}

\begin{proof}
Given $G$ and $a$, we first construct its condensation $G^{\text{SCC}}$ in linear time [Tar72] and a vertex weighting $b : V(G^{\text{SCC}}) \rightarrow \mathbb{R}$ such that $b_C = a(C)$ for each strongly connected component $C$ of $G$. We then invoke Algorithm 3.1 on $G^{\text{SCC}}$ with $b$, which consumes at most $O(f^3 |V(G^{\text{SCC}})| + f^2 |E(G^{\text{SCC}})|)$ time because of Theorem 3.4 and Observation 3.5. We finally compute the weighted reachability number for $G$ by $r_{G,a}(v) = r_{G^{\text{SCC}},b}(\pi(v))$ for each $v \in V(G)$, where $\pi : V(G) \rightarrow V(G^{\text{SCC}})$ is a mapping from vertices in $G$ to vertices in $G^{\text{SCC}}$. The whole time complexity is thus bounded by $O(f^3 |V(G)|)$ since $|E(G)| \leq |V(G)| + f$. The space complexity is obvious.
\end{proof}

\section*{Acknowledgments}

The author would like to thank Tomoaki Ogasawara for providing their master’s thesis [Oga18] and the anonymous reviewers for their valuable comments and suggestions.
References

[ABPR78] Leonard M. Adleman, Kellogg S. Booth, Franco P. Preparata, and Walter L. Ruzzo. Improved time and space bounds for boolean matrix multiplication. *Acta Inform.*, 11(1):61–70, 1978.

[BDK+18] Matthias Bentert, Alexander Dittmann, Leon Kellerhals, André Nichterlein, and Rolf Niedermeier. An adaptive version of Brandes’ algorithm for betweenness centrality. In *ISAAC*, pages 36:1–36:13, 2018.

[BFNN19] Matthias Bentert, Till Fluschnik, André Nichterlein, and Rolf Niedermeier. Parameterized aspects of triangle enumeration. *J. Comput. Syst. Sci.*, 103:61–77, 2019.

[BN19] Matthias Bentert and André Nichterlein. Parameterized complexity of diameter. In *CIAC*, pages 50–61, 2019.

[Bor16] Michele Borassi. A note on the complexity of computing the number of reachable vertices in a digraph. *Inf. Process. Lett.*, 116(10):628–630, 2016.

[BVW08] Guy E. Blelloch, Virginia Vassilevska, and Ryan Williams. A new combinatorial approach for sparse graph problems. In *ICALP*, pages 108–120, 2008.

[CFK+15] Marek Cygan, Fedor V. Fomin, Łukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh. *Parameterized algorithms*. Springer, 2015.

[Coh97] Edith Cohen. Size-estimation framework with applications to transitive closure and reachability. *J. Comput. Syst. Sci.*, 55(3):441–453, 1997.

[Coh98] Edith Cohen. Structure prediction and computation of sparse matrix products. *J. Comb. Optim.*, 2(4):307–332, 1998.

[CWY09] Wei Chen, Yajun Wang, and Siyu Yang. Efficient influence maximization in social networks. In *KDD*, pages 199–208, 2009.

[Ebe81] Jürgen Ebert. A sensitive transitive closure algorithm. *Inf. Process. Lett.*, 12(5):255–258, 1981.

[FKM+19] Till Fluschnik, Christian Komusiewicz, George B. Mertzios, André Nichterlein, Rolf Niedermeier, and Nimrod Talmon. When can graph hyperbolicity be computed in linear time? *Algorithmica*, 81(5):2016–2045, 2019.

[FLS+18] Fedor V. Fomin, Daniel Lokshtanov, Saket Saurabh, Michal Pilipczuk, and Marcin Wrochna. Fully polynomial-time parameterized computations for graphs and matrices of low treewidth. *ACM Trans. Algorithms*, 14(3):34:1–34:45, 2018.

[GMN17] Archontia C. Giannopoulou, George B. Mertzios, and Rolf Niedermeier. Polynomial fixed-parameter algorithms: A case study for longest path on interval graphs. *Theor. Comput. Sci.*, 689:67–95, 2017.
[IP01] Russell Impagliazzo and Ramamohan Paturi. On the complexity of k-SAT. *J. Comput. Syst. Sci.*, 62(2):367–375, 2001.

[KSNM10] Masahiro Kimura, Kazumi Saito, Ryohei Nakano, and Hiroshi Motoda. Extracting influential nodes on a social network for information diffusion. *Data Min. Knowl. Discov.*, 20(1):70–97, 2010.

[LG14] François Le Gall. Powers of tensors and fast matrix multiplication. In *ISSAC*, pages 296–303, 2014.

[LN95] Richard J. Lipton and Jeffrey F. Naughton. Query size estimation by adaptive sampling. *J. Comput. Syst. Sci.*, 51(1):18–25, 1995.

[LNS90] Richard J. Lipton, Jeffrey F. Naughton, and Donovan A. Schneider. Practical selectivity estimation through adaptive sampling. In *SIGMOD*, pages 1–11, 1990.

[MNN20] George B. Mertzios, André Nichterlein, and Rolf Niedermeier. The power of linear-time data reduction for maximum matching. *Algorithmica*, 82(12):3521–3565, 2020.

[OAYK14] Naoto Ohsaka, Takuya Akiba, Yuichi Yoshida, and Ken-ichi Kawarabayashi. Fast and accurate influence maximization on large networks with pruned monte-carlo simulations. In *AAAI*, pages 138–144, 2014.

[Oga18] Tomoaki Ogasawara. Fully polynomial FPT algorithms for polynomial-time solvable graph problems. Master’s thesis, The University of Tokyo, Jan. 2018.

[Pur70] Paul Purdom. A transitive closure algorithm. *BIT Numerical Mathematics*, 10(1):76–94, 1970.

[Tar72] Robert Endre Tarjan. Depth-first search and linear graph algorithms. *SIAM J. Comput.*, 1(2):146–160, 1972.

[Tar76] Robert Endre Tarjan. Edge-disjoint spanning trees and depth-first search. *Acta Inform.*, 6(2):171–185, 1976.