Deterministic Symmetry Breaking in Ring Networks

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Abstract—We study a distributed coordination mechanism for uniform agents located on a circle. The agents perform their actions in synchronised rounds. At the beginning of each round an agent chooses the direction of its movement from clockwise, anticlockwise, or idle, and moves at unit speed during this round. Agents are not allowed to overpass, i.e., when an agent collides with another it instantly starts moving with the same speed in the opposite direction (without exchanging any information with the other agent). However, at the end of each round each agent has access to limited information regarding its trajectory of movement during this round. We assume that \( n \) mobile agents are initially located on a circle unit circumference at arbitrary but distinct positions unknown to other agents. The agents are equipped with unique identifiers from a fixed range. The location discovery task to be performed by each agent is to determine the initial position of every other agent.

Our main result states that, if the only available information about movement in a round is limited to distance between the initial and the final position, then there is a superlinear lower bound on time needed to solve the location discovery problem. Interestingly, this result corresponds to a combinatorial symmetry breaking problem, which might be of independent interest. If, on the other hand, an agent has access to the distance to its first collision with another agent in a round, we design an asymptotically efficient and close to optimal solution for the location discovery problem.

Index Terms—mobile robots, location discovery, bouncing

I. INTRODUCTION

One of the most studied network topologies in the context of distributed computation, as well as coordination mechanisms for mobile agents, is the ring network [8], [21], [22]. Recently, studies of geometric ring networks were initiated in the context of terrain exploration by agents/robots with limited communication and navigation capabilities [10], [18]. This refers to the concept of swarms, i.e., large groups of limited but cost-effective entities (robots, agents) that can be deployed to perform an exploration in a hard-to-access hostile environment. The usual swarm robot properties include anonymity, negligible dimensions, no explicit communication, and no common coordinate system (cf. [24]). Some of these models assume limited visibility of the surrounding environment and asynchronous operation. In most situations involving such weak robots, the fundamental research question concerns the feasibility of solving a given task (cf. [13], [15]). The cost of the algorithm is usually measured in terms of length of a robot’s walk or the time needed to complete the task. There are several algorithmic solutions providing efficient distributed coordination mechanisms in a variety of models, e.g. [6], [23], [24]. The dynamics of “beads on a ring” and billiard systems is also of independent interest, e.g. [8].

One of the fundamental tasks in ad hoc distributed environments is to determine the actual network topology. This topic was studied in networks modeled as graphs [4], [5], [16], as well as networks deployed in a geometric environment [2], [12], [14], [20]. Most of those solutions work under the assumption that neighbors (in a graph) can exchange messages, or that agents have some visibility allowing them to inspect their nearby neighborhood.

In the case of networks containing swarm robots, communication and visibility capabilities are often severely restricted. Lack of these capabilities in some settings can be overcome by the possibility of agents monitoring their own trajectories, sensing collisions with other agents, or inferring some information from the fact that all agents behave in a fixed regular fashion. Another factor simplifying various tasks might be a restriction on the class of environments or the allowed movement trajectories of agents.

Following [18], [10] we consider a model where the agents operate in synchronised rounds, and they lack direct means of communication. The trajectory of an agent in a given round is represented as a continuous curve that connects the start and the end points of the route adopted by the agent. While moving along their trajectories the agents collide with their immediate neighbours, and information on the exact location of those collisions might be recorded and further processed. When agents are located on a circle, each agent may eventually conclude on the relative location of all agents’ initial positions, even given only limited information about its trajectory, e.g., at specified time intervals. This, in turn, enables other distributed mechanisms based on full synchronisation, e.g. equidistant distribution along the circumference of the circle and an optimal boundary patrolling scheme. Most of the models adopted in the literature on swarms assume that the agents are either almost or entirely oblivious, i.e., throughout the computation process the agents follow a very simple, rarely amendable, routine of actions. Such a scenario is studied in [10], [11], [9], where agents are entirely oblivious but can register all their collisions. (In [11], [9] agents might have different velocities, and in [9] they might have different masses.) In this paper we adopt the model from [18], where even the possibility of an agent tracking its own trajectory is severely limited. (The model we study can also be seen as a variation of that studied in [11].) In order to overcome this weakness, more adaptivity of behavior is allowed. So, the ultimate goal of this line of research is to determine how much information about their trajectories agents need to solve some communication or exploration problems, and how efficiently...
these problems can be solved.

Our focus is on deterministic solutions for these communication and exploration problems for agents having unique IDs, which is necessary for symmetry breaking. However, our results can be applied to randomly chosen IDs from an appropriately chosen range to improve upon the complexity of previous randomized results. Due to space reasons, those adaptations will not be discussed in this paper.

A. Model

A network $A$ is deployed on a circle with circumference one, along which $n$ agents (i.e., the elements of $A$) move and interact in synchronised rounds, where each round lasts one unit of time. The agents do not necessarily share the same sense of direction, i.e., while each agent distinguishes between its own clockwise (C) and anticlockwise (A) directions, agents may not have a coherent view on this. The direction “clockwise” is also called “right”, and we also refer to “antclockwise” as “left”. At the beginning of a round, an agent $a$ assigns one of the values from the set \{idle, right, left\} to its local variable $\text{dir}_a$. When the option “idle” is chosen, the agent starts the round without moving in any direction. In the case that $\text{dir}_a = \text{right}$ or $\text{dir}_a = \text{left}$, the agent starts the round moving at unit speed on the circle in the direction $\text{dir}_a$. We assume that agents are not allowed to overlap each other along the circle. When two agents moving in the opposite directions collide with each other, they instantly start moving with the same speed but in the opposite directions. If an agent $a$ moving in the direction $\text{dir} \in \{\text{right, left}\}$ collides with another agent $a'$ which is currently idle, then $a$ stays idle after the collision and $a'$ immediately starts moving in the direction $\text{dir}$ (i.e., in the same “objective” direction in which $a$ was moving before the collision, irrespective of the fact whether $a$ and $a'$ have consistent senses of direction). The agents cannot leave marks on the ring, they have zero visibility, and they cannot exchange messages. Instead, during each round each agent has access to some (specified) information about its trajectory during this round. This information can be processed or stored for further analysis. Since the agents never overpass, we may assume that the agents are arranged in an implicit (i.e. never disclosed to the agents) periodic order from $a_1$ to $a_n$.

Each agent has access to its relative position at the end of a round; more precisely, it knows the distance $\text{dist}(\cdot)$ to the right (according to its own sense of direction) between its position at the beginning of the round and the position at the end of the round, measured in the agent’s clockwise direction. In other words, there is no “universal” coordinate system on the circle, the distance is measured relative to the starting position of an agent at the start of the round. We distinguish three variants of the model:

- **basic** – an agent is **not** allowed to start a round idle, it has to start moving either in the right or the left direction;
- **lazy** – an agent is allowed to start a round idle, moving right or left;
- **perceptive** (or 1-perceptive) – this is the basic model with the additional feature that an agent gets the value $\text{coll}(\cdot)$ at the end of each round, which is equal to the distance between its position at the beginning of the round and the position of its first collision in that round.

Thus, the basic model is the weakest one. The lazy model extends the basic model by increasing an agent’s freedom in choosing various movement options. The perceptive model, on the other hand, extends the basic model by providing more information about an agent’s own trajectory to itself.

B. Notation and definitions

In this paper we address deterministic algorithms which require (for symmetry breaking) that agents have unique identifiers (IDs). We assume that each ID is a natural number in the set $\{1, \ldots, N\}$ and each agent is aware of the value of $N$. We also consider randomized algorithms, and in this case the agents are uniform and anonymous. That is, they are indistinguishable from other agents; in particular, no IDs are provided in this case.

The actual number of agents is denoted by $n$. In general, we assume that the only information available to agents about $n$ is whether $n$ is odd or even. Additionally, we assume that $N \geq n > 4$.

For an agent $a$, $\text{ID}_a$ denotes the identifier of $a$, and $\text{ID}_a[i]$ denotes the $i$th bit of $\text{ID}_a$. We also assume that at the beginning of each round, each agent $a$ can set a local variable $\text{dir}_a$ with value left, right or idle (only in the lazy model), and the value $\text{dir}_a$ (in general) determines the way in which $a$ starts moving in the next round. For natural numbers $i$ and $j$, let $[i, j] = \{k \in \mathbb{N} | i \leq k \leq j\}$ and let $[i] = [1, i]$.

By *right ring distance* between agents $a$ and $a'$ we mean 1 plus the number of agents on the ring between $a$ and $a'$ going from $a$ to $a'$ in the clockwise direction. The *left ring distance* is defined analogously. If no common sense of direction is established, the right/left distance from the point of view of an agent is measured according to its own sense of direction. Observe that, by the model’s restrictions, the relative order of agents on the ring does not change. Thus, the ring distance between agents does not change during executions of algorithms. For an agent $a$, $N_a(k)$ denotes the set of agents in ring distance at most $k$ from $a$.

Let $S = (S_1, \ldots, S_k)$ be a sequence of subsets of $[N]$. We say that agents execute $S$ in a sequence of $k$ rounds if the agent $a \in [N]$ sets $\text{dir}_a = \text{right}$ in the $i$th round iff $a \in S_i$; otherwise $\text{dir}_a = \text{left}$. Moreover, given a set $A' \subseteq A$ of “marked” agents we say that $S$ is executed on $A'$ if agents from $A'$ set their directions in consecutive rounds according to $S$, while each $a \in A \setminus A'$ sets $\text{dir}_a$ to right in each round.

C. A basic tool

Let an $(n_C, n_A)$-round be any round in which $n_C$ agents start the round clockwise and $n_A$ agents start the round anticlockwise (according to some “objective” sense of direction). A simple but key property of the ring networks was observed in [38].

**Lemma 1.** [38] Assume that the positions of agents $a_1, \ldots, a_n$ at the start of an $(n_C, n_A)$-round are $p_1, \ldots, p_n$. Then, during
By the above lemma, each agent experiences the same shift by \( r \) places in a round. Therefore, we define the rotation index of a round as the number of places by which agents move in that round in the clockwise direction. Thus, the rotation index of an \((n_C - n_A)\)-round is equal to \((n_C - n_A) \mod n\).

In this paper, a single round of computation in which each agent \( a \) starts moving in the direction \( \text{dir}_a \). A reversed single round of computation in which each agent \( a \) starts moving opposite to the direction \( \text{dir}_a \). Note that, after an execution of a single round followed by a reversed single round, each agent \( a \) gets to the position occupied by \( a \) before these two rounds transpired, provided agents do not change their local variables \( \text{dir}_a \) in between the two rounds.

D. Problems considered in the paper and previous results

The main goal of this paper is to evaluate the feasibility and complexity of the location discovery (LD) problem in the models we consider. The location discovery problem is to determine the initial position (i.e., starting position when all agents simultaneously “wake up” to begin the procedure) of every other agent. That is, at the end of an execution of an algorithm, each agent \( a \in A \) should know initial positions of all other agents, with respect to its own initial position.

We consider several problems which turn out to be efficient tools for solving the location discovery problem. Moreover, they are interesting as themselves, since they are useful in designing more complicated communication mechanisms. Below, we define these problems.

Direction agreement. The direction agreement is to agree on which direction is clockwise and which is counterclockwise. That is, at the end of the direction agreement procedure all agents have coherent view on which direction is clockwise, independent of any “objective” sense of direction.

Leader election. The leader election problem is solved when exactly one agent is assigned the status “leader” and all other agents have the status “non-leader”. (Note that we do not require that non-leaders know the ID of the leader or any other information about it.)

Nontrivial move problem. We say that a round is a trivial move if its rotation index belongs to the set \( \{0, n/2\} \) and it is a nontrivial move otherwise. The nontrivial move problem is to assign to each agent \( a \) its direction \( \text{dir}_a \) such that if \( a \) starts a round in the direction \( \text{dir}_a \), then this round is a nontrivial move.

For the direction agreement, leader election, and the nontrivial move problem we use the notion of coordination problems.

As a tool for solutions of other problems, we also consider the emptiness testing problem.

Emptiness testing. Let \( A \subseteq [N] \) denote the set of IDs of agents in the network. Emptiness testing is a protocol which given \( B \subseteq [N] \), determines whether \( B \cap A = \emptyset \). (That is, each agent \( a \in A \) knows \( B \) as an input and it is aware of the fact whether \( A \cap B \neq \emptyset \) at the end of an execution of the protocol.)

The location discovery problem in the basic and perceptive model were studied in [18]. It has been shown that there exists a randomized solution for anonymous networks (i.e., for identical agents without IDs) working in time \( O(n \log^2 n) \) with high probability in the perceivable model. If \( n \) is odd, this solution works also under the assumptions of the basic model. In [10], oblivious algorithms are studied, in which an agent is not allowed to change its direction at the beginning of a round. However, agents have access to positions of all their collisions during a round. It has been shown that, for some initial configurations, the location discovery problem is infeasible in this model. On the other hand, there is a family of initial configurations for which the location discovery can be solved efficiently in (sub)linear time.

E. Our results

In this paper, we examine the complexity of deterministic leader election, nontrivial move, direction agreement, and location discovery problems. We also study the impact on the complexity of these problems of the parity of \( n \), and whether agents initially share the same sense of direction. In all considered settings we obtain results which are optimal or close to optimal (see Tables I and II).

First, we show that the complexity of all coordination problems is asymptotically equal up to an additive \( O(\log N) \) factor. This gives an efficient and simple solution for location discovery when \( n \) is odd (Section III).

The key technical contribution of the paper states that lack of the common sense of direction for even \( n \) substantially changes the complexity of all considered problems, at least in the basic and lazy model. That is, the complexity of all coordination problems and position discovery is superlinear with respect to \( n \) for \( n = O(N^{1-\epsilon}) \) and constant \( \epsilon > 0 \). More precisely, all considered problems require \( \Omega(n \log(N/n)/\log n) \) rounds in this setting (see Table I). The reason for these large lower bounds is that the considered tasks require the solution of a kind of “symmetry-breaking” problem. We define a purely combinatorial notion of a distinguisher (see Section IV) to describe this symmetry-breaking problem which we think might be of independent interest. Using the probabilistic method, we also show that this bound is tight.

For the perceptive model, we provide a construction which solves the nontrivial move problem in \( O(\sqrt{n} \log N) \) rounds, thus the lower bound \( \Omega(n \log(N/n)/\log n) \) does not hold for this case.

We also show that using solutions of the coordination problems considered in the paper, the location discovery problem can be solved in \( n + o(n) \) rounds in the lazy model (or basic model with odd \( n \)) and in \( n/2 + o(n) \) rounds in the perceptive model.
model, provided $\log N = o(\sqrt{n})$ (see the last columns of Tables I and II for details). These results are optimal up to additive $o(n)$ factors (using Lemma 6 described later).

Due to space limitations, proofs are omitted from this conference version. They will be presented in the full version of the paper available on arXiv. The Appendix contains proofs of the results in Section VI to give a flavor of the symmetry-breaking mechanism required for the solution of these coordination problems.

F. Structure of the paper

First, in Section II we provide some basic facts and tools regarding the considered model which will be used throughout the paper. In Section III we establish relationships between asymptotic complexities of coordination problems, summarized in Theorem 7. We also discuss consequences of these reductions when the size $n$ of a network is odd.

In Section IV the complexity of the nontrivial move problem in the basic model is examined. In particular, a superlinear lower bound on the complexity of nontrivial move is shown, and an (almost) matching upper bound is provided. In Section V a construction allowing us to reduce the complexity of location discovery to $n/2 + o(n)$ is described in the perceptive model.

We assume that $n > 4$ in (most of) this paper, and often require that the parity of $n$ is known (e.g., to determine whether location discovery is solvable or not). The problem of determining the parity of $n$ will be discussed in the full version of this paper, as will the case when $n \leq 4$. Our solutions can be applied to build efficient randomized algorithms, but these issues are not discussed in this version.

II. Basic Properties of the Model

In this section we make a few observations regarding features and limitations of the model studied in the paper.

Lemma 2. All agents can determine in $O(1)$ rounds whether a rotation index of a given round is 0, $n/2$, larger than $n/2$ or smaller than $n/2$ (according to their own senses of directions).

For a fixed set of agents $A$, we define the rotation index $RI(B)$ of a set $B$ as the rotation index of a round in which all elements of $B \cap A$ start the round moving right (clockwise) and the remaining agents start the round moving left (anticlockwise). (Note that we assume an objective sense of direction when talking about agents which start a round moving clockwise/anticlockwise.) Thus, $RI(B) = \left( |B| - (n - |B|) \right) \mod n = 2|B| \mod n$. Below, we state some properties which can be proved using similar reasoning to that in the proof of Lemma 2.

Lemma 3. (a) $RI(B) = 0$ if and only if $|B| \in \{0, n/2, n\}$. (b) If $RI(B) \neq 0$, then $0 < |B| < n$. (c) If $RI(B) \neq 0$, and $B = B_1 \cup B_2$ for disjoint $B_1, B_2$, then $RI(B_1) \neq 0$ or $RI(B_2) \neq 0$.

Now, we make an observation regarding information which can be inferred by an agent using the distance between its starting position and the first collision in a round (i.e., $\text{coll}(i)$).

Proposition 4. Assume that an agent $b_0$ starts moving in a round in the direction $\text{dir}_{b_0}$, and let consecutive agents in the direction $\text{dir}_{b_0}$ from $b_0$ be denoted $b_1, \ldots, b_{n-1}$. Moreover, let the geometric distance (on the ring) between $b_{i-1}$ and $b_i$ be $x_{i-1}$. If $b_1, \ldots, b_k$ start the round in the direction $\text{dir}_{b_0}$ for $k < n-1$, and $b_{k+1}$ starts in the opposite direction to $\text{dir}_{b_0}$, then the relative position of the first collision of $b_0$ is equal to $(x_1 + \ldots + x_k)/2$.

A. Lower bounds on the complexity of location discovery

As observed by Friedetzky et al. [18], location discovery cannot be solved in the basic model when $n$ is even.

Lemma 5. [18] It is impossible to solve the location discovery problem in the basic model with even $n$.

The reason of this impossibility result follows from the fact that, when $n$ is even, the rotation index of any round in the basic model is always even. Therefore, an agent can only visit positions of agents having even ring distance from itself.

Below, we state the lower bounds on complexity of the location discovery problem. Intuitively, they follow from the fact that each round gives one linear equation with variables equal to distances between agents in the basic and lazy model, while it provides two linear equations in the perceptive model (as two distances are given to an agent).

Lemma 6. 1) The location discovery problem in the basic and lazy model cannot be solved in less than $n - 1$ rounds in the worst case. 2) The location discovery problem in the perceptive model cannot be solved in less than $n/2$ rounds in the worst case.

III. Reductions Between Considered Problems

In this section we establish reductions between the coordination problems. The results are illustrated in Figures I and II and are summarized in Theorem 7. They work for arbitrary $n$, provided $n > 4$.

![Fig. 1. Complexity of reductions among coordination problems if $n$ is odd or the model is either perceptive or lazy.]

Theorem 7. For each model considered in the paper (basic, lazy, perceptive) the asymptotic complexity of all coordination problems (direction agreement, leader election, nontrivial move) are equal up to an additive term $O(\log N)$. 
A. The setting with the nontrivial move problem solved

In this section, we assume that the nontrivial move problem is solved.

**Lemma 8.** If the nontrivial move problem is solved, the direction agreement problem can be solved in $O(1)$ rounds, also in the case that agents do not have assigned IDs.

The result stated in Lemma 8 is obtained by the direction agreement protocol described in Alg. 1.

**Lemma 9.** Assume that the nontrivial move problem is solved. Then, it is possible to solve the leader election problem in $O(\log N)$ rounds.

The result stated in Lemma 9 is obtained by the leader election protocol described in Alg. 2.

**Algorithm 1** DirAgr($a$)

1. Assign $\text{dir}_a$ as in a nontrivial move
2. **SINGLEROUND**
3. $d_1 \leftarrow \text{dist}()$
4. **SINGLEROUND**
5. $d_2 \leftarrow \text{dist}()$
6. if $d_1 + d_2 > 1$ then
   7. change sense of direction

**Algorithm 2** LeaderWithinMove($a$)

1. Solve the direction agreement problem
2. $X \leftarrow$ all agents starting right in a nontrivial move
3. for $i = 1, \ldots, \log N$ do
   4. $X_0 \leftarrow \{b | b \in X, \text{ID}_b[i] = 0\}$ \quad i.e., set $a \in X_0$ iff $a \in X$ and $\text{ID}_a[i] = 0$
5. if $\text{RI}(X_0) \neq 0$ then
   6. $X \leftarrow X_0$ \quad i.e., set $a \in X$ iff $a \in X_0$
7. else
   8. $X \leftarrow X \setminus X_0$ \quad i.e., set $a \in X$ iff $a \notin X_0$
9. Set the status of $a$ as leader iff $a \in X$.

B. The setting with the chosen leader

In this section, we assume that (exactly) one agent in a network has the status “leader”.

**Lemma 10.** If the leader is chosen, one can solve the nontrivial move problem in $O(1)$ rounds.

**Proof:** Assume that the leader $a$ is chosen. Consider two assignments of directions: (1) $\text{dir}_b = \text{right}$ for each $b \in A$ and (2) $\text{dir}_b = \text{right}$ for each $b \neq a$ and $\text{dir}_a = \text{left}$. The rotation indexes $r_1, r_2$ of such two rounds differ by 2 modulo $n$ (Lemma 11). As $n > 4$, at least one of two numbers which differ by 2 modulo $n$ does not belong to $\{0, n/2\}$. Thus, the nontrivial move problem is solved.

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**Table I** Deterministic solutions in general setting

|                | leader election | nontrivial move | direction agreement | location discovery |
|----------------|-----------------|-----------------|---------------------|--------------------|
| odd $n$        | $O(\log N)$    | $\Theta((N/n)\log N)$ | $O(1)$              | $n + O(\log N)$   |
| basic model, even $n$ | $\Theta(n\log N/n)$ | $\Theta(n\log N/n)$ | $\Theta(n\log N/n)$ | not solvable       |
| lazy model, even $n$ | $\Theta(n\log N/n)$ | $\Theta(n\log N/n)$ | $\Theta(n\log N/n)$ | not solvable       |
| perceptive model, even $n$ | $O(\sqrt{n}\log N)$ | $O(\sqrt{n}\log N)$ | $O(\sqrt{n}\log N)$ | $\frac{n}{2} + O(\sqrt{n}\log^2 N)$ |

**Table II** Deterministic solutions with common sense of direction

|                | leader election | nontrivial move | location discovery |
|----------------|-----------------|-----------------|--------------------|
| odd $n$        | $O(\log N)$    | $\Theta(\log(N/n))$ | $n + O(\log N)$   |
| basic model, even $n$ | $O(\log^2 N)$ | $O(\log^2 N)$ | not solvable       |
| lazy model, even $n$ | $O(\log N)$ | $O(\log N)$ | $n + O(\log N)$ |
| perceptive model, even $n$ | $O(\log N)$ | $O(\log N)$ | $\frac{n}{2} + O(\sqrt{n}\log^2 N)$ |

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**Fig. 2.** Complexity of reductions among coordination problems in the basic model (even $n$).
Corollary 11. If the leader is chosen, one can solve the direction agreement problem in \(O(1)\) rounds.

Proof: Given the leader, we obtain a nontrivial move in \(O(1)\) rounds (Lemma 10). Next, we apply the solution from Lemma 5 to obtain a common sense of direction in \(O(1)\) rounds.

C. The setting with the common sense of direction

In this section we consider the setting that agents have the common sense of direction. We show simple efficient solutions for leader election and nontrivial move in the basic model which rely on the emptiness testing result from the following lemma.

Lemma 12. Assuming all agents share a common sense of direction, the emptiness testing problem can be solved in \(\log N\) rounds in the basic model, and in one round in the lazy and perceptive model. Moreover, if \(n\) is odd, the emptiness testing is solvable in one round in the basic model as well.

With help of the emptiness testing protocol, we devise a solution to the leader election problem. The idea of our solution is based on a binary search approach similar to that from Lemma 9. The main obstacle here is that, without a nontrivial move, the initial set of candidates for the leader is just \(X = A\) and it has size \(n\), thus its rotation index is 0. And, the case that it is split in two subsets \(X_1, X_2\) of size \(n/2\) is indistinguishable from the case that it is split in \(X_1 = X\) and \(X_2 = \emptyset\) (or vice versa), at least on the basis of rotation indexes of appropriate sets. Therefore, we use the more sophisticated emptiness testing from Lemma 12.

Lemma 13. Assuming all agents share common sense of direction, the emptiness testing problem can be solved in \(O(\log^2 N)\) rounds in the basic model (with even \(n\)) and in \(\log N\) rounds in other settings.

An efficient solution for the nontrivial move problem can be easily obtained from Lemma 13 and Lemma 10.

Corollary 14. If all agents have the same sense of direction, the nontrivial move problem can be solved in \(O(\log^2 N)\) rounds in the basic model (with even \(n\)) and in \(\log N\) rounds in other settings.

We note that the nontrivial move problem can also be solved in \(O(\log N)\) rounds in the basic model with even \(n\), thus strengthening the \(O(\log^2 N)\) from Corollary 14 for the basic model, and matching the bound from this corollary for other models. However, the result in the following lemma is weaker, as this is based only on a nonconstructive proof using the probabilistic method (omitted in this conference version).

Lemma 15. If all agents have the same sense of direction, the nontrivial move problem can be solved in \(O(\log N)\) rounds.

D. Application of coordination problems for location discovery

Given the reductions summarized in Figure 1 and Figure 2 (see also Theorem 7), one can simply solve the location discovery problem in the lazy model, irrespective of the parity of \(n\), or in the basic model for odd \(n\). This is the case, since given the common sense of direction and the leader, we can obtain rotation index 1 in the lazy model and 2 in the basic model (all agents but the leader move right at the beginning of a round).

Lemma 16. Assume that (at least) one among the following problems is solved: nontrivial move, leader election, direction agreement. Then, location discovery can be solved in \(n + O(\log N)\) rounds in the lazy model with arbitrary \(n\) and in the basic model with odd \(n\).

Note that the above result for the basic model applies in the stronger perceptive model as well. However, we provide more efficient solutions for this model later.

E. Solutions for the case that \(n\) is odd

The crucial difference between the cases of odd and even \(n\) follows from the following observation: If \(n_C \neq 0\) and \(n_A \neq 0\) in a round then the round is nontrivial in the case of odd \(n\). On the other hand, this is not necessarily the case for even \(n\), as, e.g., \(0 \neq n_C = n_A = n/2\) or \(n_C \in \{\frac{1}{3}n, \frac{4}{7}n\}, n_A = n-n_C\) do not give a nontrivial move.

Proposition 17. The direction agreement problem can be solved in \(O(1)\) time in the basic model, provided \(n\) is odd.

Corollary 18. If the number of agents \(n\) is odd, the leader election problem and the nontrivial move problem can be solved in time \(O(\log N)\). The location discovery problem can be solved in \(n + O(\log N)\) rounds.

There is also a slightly modified variant of a solution for the nontrivial move problem, reducing the complexity from \(O(\log N)\) to \(O(\log(N/n))\).

Proposition 19. The nontrivial move problem can be solved in \(\Theta(\log(N/n))\) time in the basic model with odd \(n\).

IV. BASIC MODEL WITH EVEN \(n\)

It is known (Lemma 5) that the location discovery problem cannot be solved in the basic model when \(n\) is even. However, we can still try to solve other coordination problems. Our results in this section state the their complexity is significantly larger than for the case of odd \(n\). To this aim, we define a related combinatorial problem which we believe can be of independent interest. Proofs are omitted here, but can be found in the Appendix.

First, we define a combinatorial notion of a distinguisher. Then, a relationship between the size of a distinguisher and the complexity of the corresponding nontrivial move problem is established. Finally, tight bounds on the smallest size of distinguishers and the complexity of the nontrivial move problem are showed.

Definition 20. We say that a family \(S = \{S_1, \ldots, S_k\}\) of subsets of \([N]\) is a \((N,n)\)-distinguishers of size \(k\) if for each \(X_1, X_2 \subseteq [N]\) such that \(|X_1| = |X_2| = n\) and \(X_1 \cap X_2 = \emptyset\), there exists \(i \in [k]\) such that \(|S_i \cap X_1| \neq |S_i \cap X_2|\).
**Definition 21.** Let $N \in \mathbb{N}$ and let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a nondecreasing function. A family $\mathcal{S} = S_1, \ldots, S_{f(N,N)}$ of subsets of $[N]$ is a strong $(N,f)$-distinguisher if the prefix $S_1, \ldots, S_{f(N,n)}$ of $\mathcal{S}$ is a $(N,n)$-distinguisher for each $n \leq N$.

The weak nontrivial move problem is to assign to each agent $a$ a direction $\text{dir}_a$ such that if a starts a round in the direction $\text{dir}_a \in \{\text{right}, \text{left}\}$, then the rotation index $r$ in the round is not equal to 0. (A round with the rotation index $n/2$ is treated as a weak nontrivial move, which is not the case in the standard definition of a nontrivial move.)

We first state a reduction between the complexity of the weak nontrivial move problem and the smallest size of a distinguisher.

**Proposition 22.** Let $n > 4$ be an even number and $N \geq n$.

1) Assume that a protocol $A$ solves the weak nontrivial move problem in the basic model in $O(f(N,n))$ rounds when the value of $n$ is known to the agents. Then, there exists a $(N,n/2)$-distinguisher of size $O(f(N,n))$.

2) Assume that a protocol $A$ solves the weak nontrivial move problem in the basic model in $O(f(N,n))$ rounds when the actual value of $n$ is unknown to the agents. Then, there exists a strong $(N,f')$-distinguisher for $f'(N,n/2) = O(f(N,n))$.

We now establish a lower bound on the size of a $(N,n)$-distinguisher in terms of the parameters $N$ and $n$.

**Lemma 23.** If $\mathcal{S}$ is a (standard) $(N,n)$-distinguisher for $N > 2$ and $n \leq N/128$, then the size of $\mathcal{S}$ is $\Omega\left(\frac{n \log(N/n)}{\log n}\right)$.

Our proof uses a notion from [17]:

**Definition 24.** [7], [17] Let $l \leq k \leq n$. A family $\mathcal{F}$ of $k$-subsets (i.e. subsets of size $k$) of $[N]$ is $(N,k,l)$-intersection free if $|F_1 \cap F_2| \neq 1$ for every $F_1, F_2 \in \mathcal{F}$.

**Fact 25.** [7], [17] Let $\mathcal{F}$ be an $(N,k,k/2)$-intersection free family where $k$ is a power of 2 and $2 \leq N/64$. Then, $\log |\mathcal{F}| \leq \frac{11k}{12}\log(N/k)$.

**Corollary 26.** Each algorithm solving the (weak) nontrivial move problem requires $\Omega(n \log(N/n)/\log n)$ rounds in the basic model with known value of $n$.

It can be shown using the probabilistic method that there exists a solution for the nontrivial move problem that nearly matches the lower bound from Corollary 26.

**Theorem 27.** In the basic model, there exist solutions of the nontrivial move problem working in $O(n \log(N/n)/\log n)$ rounds for each $n \in [N]$ and $n > 4$, and also when $n$ is unknown.

**Corollary 28.** The time complexity of the nontrivial move problem, the leader election problem, and the direction agreement problem in the basic model (with even $n$) is $\Theta(n \log(N/n)/\log n)$.

The above result follows from Cor. 28, Th. 27 and Th. 7. Given the relationship between distinguishers and the nontrivial move problem (Prop. 22), the lower bound from Lemma 23 and Cor. 28 we get the following bound.

**Corollary 29.** The size of the smallest $(N,n)$-distinguisher for $N \geq n$ is $\Theta(n \log(N/n)/\log n)$.

For each $N \in \mathbb{N}$, there exists a strong $(N,f)$-distinguisher for some $f(N,n) \in O(n \log(N/n)/\log n)$. Moreover, if $\mathcal{S}$ is a strong $(N,f)$-distinguisher, then $f(N,n) = \Omega(n \log(N/n)/\log n)$.

We note that the bound from Cor. 28 also holds for the lazy model (the proofs are omitted). It follows from the fact that complexities of the weakly non-trivial move in the basic model and the non-trivial in the lazy are asymptotically equal.

**V. PERCEPTIVE MODEL WITHOUT COMMON SENSE OF DIRECTION**

Since the basic model is too weak for the task of position discovery (when $n$ is even), we considered the lazy model. Although one can solve position discovery in this model, the overhead cost for this problem is $\Omega(n \log(N/n)/\log n)$.

In [18], it is shown that position discovery can be solved in the perceptive model (i.e., when the position of the first collision in a round can be detected while each agent has to start the round moving to the right or left). In this section, we inspect efficiency of coordination problems as well as position discovery in this model. First, we show that the perceptive model gives an opportunity to exchange information between neighbors on a ring (Section V-A). Then, we use this feature to build algorithms for the nontrivial move problem which brake the lower bounds working in the basic model and the lazy model (Section V-B). Finally, using these solutions as tools, we provide a solution for the positions discovery problem in time $n/2 + o(n)$ provided $\log N = O(\sqrt{n})$ which is optimal up to the $o(n)$ term (Section V-C).

**A. Communication on a ring**

First, we discuss the following neighbors discovery task in which each agent $a$ should:

- learn (relative) location of its left neighbor Left($a$) and its right neighbor Right($a$);
- determine whether Left($a$) and Right($a$) have the same sense of direction as $a$ has.

Algorithm 3 solves this problem based on the fact that each two IDs differ on at least one bit. (Some calculations performed by agents are not explicitly described in the algorithm, they are discussed later.) In Algorithm 3 each execution of SINGLEROUND is followed by REVERSEDROUND in which each agent starts a round with the direction opposite to its local direction dir. We omit this detail in the pseudocode. However, let us stress here that this gives a guarantee that each agent starts each application of SINGLEROUND at exactly the same position as its position before the execution of the algorithm (so, its distances to neighbours are the same as well).

**Proposition 30.** Algorithm 3 gives solution to neighbors discovery in $O(\log N)$ rounds.
Algorithm 3 NeighborDiscovery(a)
1: $D_{\text{left}} \leftarrow \emptyset$; $D_{\text{right}} \leftarrow \emptyset$ \> distances to collisions
2: for $i = 1, 2, \ldots, \log N$ do
3:  \> for $j \in [0, 1]$ do
4:  \>
5:  \>
6:  \>
7:  \>
8:  \>
9:  \>
10: \>
11: \>
12: \>
13: \>
14: \>

Proposition 31. If each agent knows:
- locations of its neighbors (relative to its initial location);
- AND
- sense of direction of its neighbors (with respect to its own sense of direction);
then each agent can transmit one bit of information to its neighbors in time $O(1)$.

The statement of Prop. 31 can be obtained such that each agent starts round 1 (2, resp.) moving left/right depending on the transferred bit. Then, the distances to the first collision in both rounds give information about the bits of neighbors. Since agents can learn location of their neighbors and their sense(s) of direction in $O(\log N)$ rounds (see Proposition 30), Proposition 31 leads to the following corollary.

Corollary 32. There exists a possibility to exchange one bit of information between each two neighbors in the perceptive model in time $O(1)$, after a $O(\log N)$ preprocessing.

The above corollary gives opportunity to simulate any distributed algorithm on a ring in message passing model (i.e., when each pair of neighbors can exchange a message in one round of computation). However, the time efficiency of such simulations is limited by the fact that only one bit of information is exchanged between neighbors in a round.

Let information dissemination task with parameters $d$ and $p$ be to disseminate a message $m_a$ with $p$ bits by each agent $a$ to all agents in ring distance $\leq d$ from $a$.

Corollary 33. Information dissemination task in which agents are supposed to transmit messages of length $p$ on the ring distance $d$ can be accomplished in time $O(p \cdot d)$.

A solution claimed in the above corollary might we designed such that first all agents transmit own messages, then messages arriving from their left neighbors and finally messages arriving from their right neighbors.

Assume that $A' \subseteq A$ is a set of marked agents such that each agent knows whether it is marked or not and the ring distance between any different $a, a' \in A'$ is at least $d$. Moreover, each $a \in A'$ has a message $M_a$ of size $\leq m$. The sparsed information dissemination task with parameters $A', d$ and $m$ is to deliver the message of each $a \in A'$ to all agents in the ring distance $\leq d$ from $a$. For an agent in $A'$, we denote this task by Diss($M_a, d$). Using the procedure exchanging a bit of information between each pair of neighbors in time $O(1)$, we obtain the following result.

Corollary 34. Sparsed information dissemination task in which agents in distances $\geq d$ are supposed to transmit messages of length $p$ on the ring distance $d$ can be accomplished in time $O(p + d)$.

In a solution to the sparsed information dissemination we have to tackle the fact that an agent has no direct way to convey a message of the type “I have nothing to transmit (yet)”. One can solve this issue by a simple encoding, e.g., 00/11 encodes 0/1, while 01 encodes “no bit to transmit”.

B. Nontrivial Move

As we know, the nontrivial move problem is intuitively to break balance between the number of agents moving clockwise and anticlockwise. In our solution we use $(N, k)$-selective families from [7].

Definition 35. Let $n < N$. A family $F$ of subsets of $[N]$ is $(N, n)$-selective if, for every non empty subset $Z$ of $[N]$ such that $|Z| \leq n$, there is a set $F$ in $F$ such that $|Z \cap F| = 1$.

Clementi et al. [7] showed that for any $N > 2$ and $n \leq N$, there exists an $(N, n)$-selective family of size $O(n \log(N/n))$.

Let a local leader for some fixed number $d$ be an agent $a$ with the largest ID among agents in the ring distance $d$ from $a$. In Algorithm 5 we present a solution to the nontrivial move problem by establishing local leaders for exponentially growing distances $d = 2^k$ and trying to execute $(N, 2^k)$-selective family on those leaders. As the number of local leaders is $\leq n/2^k$, it becomes smaller than $2^k$ for $k > \frac{1}{2} \log n$ and gives a nontrivial move after $O(2^{\frac{1}{2} \log n} \log N) = O(\sqrt{n} \log N)$ rounds.

Algorithm 4 NMoveS(a)
1: $\text{dir}_a \leftarrow \text{right}$; set the status of $a$ as a local leader;
2:  \> SINGLEROUND
3:  \> if the current directions give a nontrivial move: return
4:  \> Establish 1-bit communication \> Cor. 32
5:  \> for $k = 0, 1, 2, \ldots, \log N$ do
6:  \>
7:  \>
8:  \>
9:  \>
10: \>
11: \>
12: \>


Lemma 36. The algorithm NMoveS solves the nontrivial move problem in $O(\sqrt{n} \log N / \log n)$ rounds in the perceptive model.

C. Position Discovery in the perceptive model

In this section we design an efficient solution for the position discovery in the perceptive model. Using results from the previous section and Theorem 7, we can assume that the leader is elected and the common sense of direction is established in $O(\sqrt{n} \log N)$ rounds. Throughout this section, we use a labeling of agents such that $a_1$ is the label of the leader and $a_i$ is the label of the $i$th agent on the ring in the clockwise direction from the leader.

We solve the position discovery problem in two stages. First, each agent determines its right ring distance to the leader (i.e., its label; note that a label denotes the distance to the leader, not ID). In order to achieve this goal in the standard message passing model on a ring, linear time is necessary. In order to perform this task faster, we use arithmetic relationships between distances to collisions ($\text{coll}(i)$) and distances traversed in consecutive rounds ($\text{dist}(i)$). For appropriately designed protocol, an agent in ring distance $\leq d^2$ from the leader will be able to learn its ring distance in $O(d \log N)$ rounds. Then, using the knowledge about ring distances of agents to the leader, the position discovery will be finally solved in the following way. Let $x_1, \ldots, x_n$ be the original distances between agents. Here, we plan movements of agents in such a way that, for each agent and each round, the distance to collision in the round and the distance traveled in the round gives a linear equation over $x_1, \ldots, x_n$ which is linearly independent from equations derived before. In this way each round provides two new equations and $n/2$ rounds are sufficient to determine the actual values of $x_1, \ldots, x_n$, since they give a system of $n$ independent linear equations over $n$ variables.

1) Ring distances: Now, we design the RingDist protocol in which each agent learns its right ring distance to the leader. Throughout this section, ring distance denotes the right ring distance from the leader. We call it a label of an agent and denote the agent in ring distance $i$ by $a_i$.

Let $\text{Shift}(l)$ for $l \in \mathbb{N}$ be a round in which $\text{dir}_{a_i} = \text{right}$ for each $i \in [l]$ and $\text{dir}_{a_i} = \text{left}$ for $i > l$. Moreover, $\text{Shift}(−l)$ is a round with directions of agents opposite to their direction in $\text{Shift}(l)$. Observe that the rotation index of $\text{Shift}(l)$ is equal to

$$(l − (n − l)) \mod n \equiv 2l \mod n.$$ 

RingDist works under assumption that (exactly) one distinguished agent has the status leader (it is denoted $a_1$). Each agent but the leader starts an execution of a protocol with unspecified ring distance. The idea of Algorithm 5 is that the agents gradually learn their ring distances in the following way:

- In the $i$th iteration of the for-loop, the agents $a_k, a_{k+k}, \ldots, a_{k+k^2}$ for $k = 2^i$ learn their ring distances in the following way (see Fig. 3). For each $l > k$, the value of $\text{coll}(i)$ in $\text{Shift}(k/2)$ is equal to $z = (x_t - k + \cdots + x_l - 1)/2$ (see Prop. 4 for $b_0 = a_1$, dir = left and thus $b_l = a_{(l−1) \mod n}$). On the other hand, if one applies $\text{Shift}(−k/2)$ several times, the values of $\text{dist}(i)$ in the $i$th executions of $\text{Shift}(−k/2)$ is equal to $y_j = x_t - jk + \cdots + x_l - (j − 1)k + 1$, since the rotation index of $\text{Shift}(−k/2)$ is equal to $−k$. Using these relationships, we see that there exists $j$ such that $2z = y_1 + \cdots + y_j$ if $l = k + k \cdot j$. This observation is exploited in RingDist in order to determine ring distances of $a_1, \ldots, a_{k+k^2}$ in the $i$th iteration of the main for-loop for $k = 2^i$.

- The remaining agents $a_j$ for $j \leq k + k^2$ learn their distances in the execution of line 8, as each agent knowing its ring distance propagates it in the distance $2k$.

Then, it remains to guarantee that the for-loop is finished when all agents know their ring distances and $2^i = O(\sqrt{n})$. To this aim, we execute CheckCompleteness. Note that the agent $a_n$ knows that it is the last one already at the beginning (without knowing $n$), as it is the left neighbour of the leader. CheckCompleteness is a round in which all agents different from $a_n$ move left, while $a_n$ moves right iff it already knows its own right ring distance (which in turn implies that every other agent knows its ring distance as well). Thus, the rotation index of this round is not zero iff each agent knows its ring distance. In the following, we show more formally that the above described idea works. First, we make an observation following from the definition of Shift (the rotation index of Shift(l) is 2l) and Proposition 4.

Proposition 37. Let $k = 2^i$ for $i \leq \log N$. Assume that agents $a_1, \ldots, a_k$ know their labels before the $i$th iteration of the for-loop (and other agents know that they do not belong to $\{a_1, \ldots, a_k\}$). Then, for $l > k$ the values of $z, y_1, \ldots, y_k$ recorded by the agent $a_i$ satisfy the following conditions in the iteration $i$ of the for-loop:

- $y_j = (x_t - jk + x_t - (j−1)k + 1 \cdots + x_t - (j−k−1)k + 1)/2.$
Lemma 39. Assume that the leader is elected and all agents share common sense of direction. Then, each agent $a$ knows its right distance $k + jk$ from the leader (i.e., $a = a_{k+jk}$).

Lemma 38. The condition $2z = y_1 + \cdots + y_j$ is satisfied for an agent $a \notin \{a_1, \ldots, a_k\}$ if and only if $a$ has the first collision during Pivot($n$); Pivot($n-1$); Pivot($n-2$); $\ldots$; Pivot($n/2$).

Algorithm 5 RingDist($a$)
1: if $a = a_1$: Diss("leader",4) \text{ $\triangleright$ } $a_1$ broadcasts on dist. 4
2: for $i = 1, 2, \ldots, \log N$ do
3:     $k \leftarrow 2^i$
4:     For $j = 1, \ldots, k$: Shift($k/2$); $y_j \leftarrow$ dist()
5:     Repeat $k$ times: Shift($k/2$) \text{ $\triangleright$ } Reverse res. of l. 4
6:     Shift($k$); $z \leftarrow$ coll($)$; Shift($k$)
7:     if $2z = y_1 + \cdots + y_j$ for some $j$ and $a \notin \{a_1, \ldots, a_k\}$:
8:         Set the ring distance of $a$ to $k + jk$; mark $a$
9:         (i.e., $a$ is $a_{k+jk}$ and $a$ is marked)
10: if $a = a_{k+jk}$ for $j \leq k$ and $a$ is marked then
11:     Diss($k + jk$) \text{ (i.e., marked agents}
12:     broadcast their ring dist. on distance $k$)
13: If CheckCompleteness: return

Algorithm 6 Distances($a$)
1: for $i = 1, 2, \ldots, n/2$ do
2:     Convolution($\frac{n}{2} - (2i-1)$)
3:     Pivot($n$); Pivot($n-1$); Pivot($n-2$); $\ldots$; Pivot($n/2$)

Proposition 40. After the for-loop of Algorithm 6, the following conditions hold:

(a) the agent $a_{2i-1}$ can determine the values of $x_1, x_2, \ldots, x_{n-2}$ and $x_{n-1} + x_n$.
(b) the agent $a_{2i}$ can determine the values of $x_1, x_2, \ldots, x_{n-3}, x_n$, and $x_{n-2} + x_{n-1}$ for each $i \in [n/2]$.

The above proposition uses the fact that each execution of Convolution gives information about the sum of two consecutive $x_i$’s (the distance between an agent’s position at the beginning and the end of a round) and about a particular $x_j$ (the distance to the first collision is equal to halve of $x_j$ for some $j$). Now, observe that

- The agent $a_{2i-1}$ has the first collision during Pivot($n$) in distance $x_n/2 + (x_1 + \cdots + x_{2i-2})/2$ for each $i \in [n/4]$. As $a_{2i-1}$ knows $x_1, \ldots, x_{2i-2}$ and $x_{n-1} + x_n$, by Proposition 30(a), it can determine $x_n/2$ from Pivot($n$) and therefore also $x_{n-1}$.
- The agent $a_{2i-1}$ has the first collision during Pivot($n-1$) in distance $x_{n-1}/2 + (x_{n-1-1} + \cdots + x_{n-2})/2$ for each $i \in [n/4 + 1, n/2 - 1]$. As it knows $x_{2i-1}, \ldots, x_{n-2}$ and $x_{n-1} + x_n$, it can determine $x_{n-1}$ from Pivot($n-1$) and then $x_n$ as well.

A similar reasoning works for $a_{n-1}$ as well.

By combining the above with Prop. 40(a), one can conclude that each agent $a_i$ with odd label $i$ knows original positions of all agents. A similar argument applies for even agents and executions of Pivot($n-1$) and Pivot($n-2$), since Prop. 40(b) can be seen as Prop. 40(a) “shifted” by $-1$.

Lemma 41. The protocol Distances (Alg. 6) solves the position discovery problem, provided the leader ($a_1$) is elected, agents share common sense of direction and each agent knows its ring distance.

Theorem 42. The position discovery problem can be solved in the perceptive model in $n/2 + O(\sqrt{n} \log N)$ rounds.

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References

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APPENDIX: PROOFS OMITTED FROM SECTION

Proof of Proposition 22
First, assume that $n$ is known and $\mathcal{A}$ solves the weak nontrivial move problem. Observe that, until the first round of $\mathcal{A}$ with a weak nontrivial move, the only information available to each agent is that its starting position in a round is equal to its position at the end of a round. Thus, its behavior can be defined by a sequence of sets $S_1, S_2, \ldots$, such that the agent $a$ chooses direction right in round $i$ (provided no nontrivial move appeared before) if and only if $a \in S_i$. Let us fix which sense of direction is "correct". Then, consider the situation in which the set of agents $X_1$ with the correct sense of direction and the set of agents $X_2$ with the incorrect sense of direction satisfy $|X_1| = |X_2| = n/2$. Let $m_1 = |X_1 \cap S_i|$, $m_2 = |X_2 \cap S_i|$. Then, the rotation index (mod $n$) in round $i$ is

$$
(2m_1 + 1 - m_2 - n/n = 2m_1 - m_2).$$

And therefore the $i$th round of $\mathcal{A}$ gives a (weak) nontrivial move if and only if $2m_1 + 1 - m_2 \in \{0, n\}$, which implies $m_1 \neq m_2$. On the other hand, $m_1 \neq m_2$ is equivalent to the fact that $S_i$ distinguishes $X_1$ and $X_2$. In conclusion, the sequence $S_1, S_2, \ldots$ defining $\mathcal{A}$ is a $(N, n/2)$-distinguisher.

For unknown $n$, the result follows from the above reasoning and the fact that $\mathcal{A}$ has to tackle arbitrary even $n \leq N$ which reflects the difference between a standard $(N, n)$-distinguisher and its strong counterpart.

Before giving the proof of Lemma 23, we provide a lower bound on the size of a strong $(N, n)$-distinguisher with a simple proof based on a counting argument (a similar bound in another context was given e.g. in [19]). Although this result is subsumed by Lemma 23, we provide it to give some intuition before a more complicated, and less intuitive proof, of Lemma 23.

**Lemma 43.** If $S$ is a strong $(N, n)$-distinguisher for any $N > 4$ and $f : N \times N \to \mathbb{N}$, then $f(N, n) = \Omega \left( \frac{n \log(N/n)}{\log n} \right)$.

**Proof:** First, we show that a strong $(N, n)$-distinguisher $S$ satisfies the property that for each two different sets $X_1, X_2 \subseteq [N]$ such that $|X_1| = |X_2| = n$, there exists $i \leq f(N, n)$ such that $|X_1 \cap S_i| \neq |X_2 \cap S_i|$ (note that $X_1$ and $X_2$ do not have to be disjoint!). Indeed, assume to the contrary that this is not the case for $S$, and thus $|X_1 \cap S_i| = |X_2 \cap S_i|$ for some different sets $X_1, X_2$ of size $n$ and each $i \in [f(N, n)]$. Let $Y_1 = X_1 \setminus X_2$ and $Y_2 = X_2 \setminus X_1$. Then, $Y_1 \cap Y_2 = \emptyset$, $|Y_1| = |Y_2| \leq n$ and $|Y_1 \cap S_i| = |Y_2 \cap S_i|$ for each $i \in [f(N, n)]$. This implies that $S$ is not a strong $(N, f)$-distinguisher, which is a contradiction.

Let $S = (S_1, \ldots, S_k)$ be a strong $(N, f)$-distinguisher. The above observation implies that, for any $X \neq X', X, X' \subseteq [N]$ of size $n$, the sequences $|X \cap S_1|, \ldots, |X \cap S_k|$ and $|X' \cap S_1|, \ldots, |X' \cap S_k|$ are not equal, where $k = f(N, n)$. As each $S_i$ gives at most $n+1$ possible values of $|X \cap S_i|$ for $X \subseteq [N]$ of size $n$, and there are $\binom{n}{k}$ subsets of $[N]$ of size $n$, we obtain
the following bound
\[ k \geq \log_{n+1} \left( \frac{N}{n} \right) = \Omega \left( \frac{\log \left( \frac{N}{n} \right)}{\log(n+1)} \right) = \Omega \left( \frac{n \log(N/n)}{\log n} \right) \]
for \( n > 1 \).

It turns out that the result of Lemma 23 can be strengthened, to give Lemma 24. However, our proof of this fact is much more complicated. It applies techniques from [1], designed for proving lower bounds on size of selective families. We stress here that the lower bound for a strong variant of a distinguisher does not imply an analogous lower bound for a “standard” variant of a distinguisher. As observed in the proof of Lemma 23, the prefix of size \( f(N,n) \) of a stong \((N,f)\)-distinguisher gives an opportunity to “distinguish” each pair of sets of size \( n \). On the other hand, a standard \((N,n)\)-distinguisher is supposed to give a difference only on disjoint sets of size \( n \).

**Proof of Lemma 23**

Let us first stress that the calculations from the previous lemma do not apply here, since a (“standard”) distinguisher does not have to “distinguish” small sets, so it does not have to distinguish non-disjoint sets of size \( n \) either.

Let \( G(V,E) \) be a graph, whose vertices are all \( 2n \)-subsets of \([N]\), where the edges connect vertices corresponding to sets which have exactly \( n \) common elements. That is, \((X_1,X_2) \in E \) for \( X_1,X_2 \in V \) if and only if \(|X_1 \cap X_2| = n\). Let \( \alpha(G) \) and \( \chi(G) \) denote the size of the largest independent set of \( G \) and the chromatic number of \( G \), respectively. We claim that
\[
\log \chi(G) \geq \frac{1}{6} n \log(N/(2n)) \quad \text{and} \quad \log \alpha(G) \leq \frac{22}{12} n \log(N/(2n)).
\]

**Proof of (1):**

We use the fact that \( \chi(G) \geq \frac{|V|}{\alpha(G)} \). Moreover, as each independent set of \( G \) is a \((N,2n, n)\)-intersection free family of sets, Fact 25 implies that
\[
\log \alpha(G) \leq \frac{22}{12} n \log(N/(2n)).
\]

Therefore
\[
\log \chi(G) \geq \log |V| - \log \alpha(G) \\
\geq \log \frac{N}{2n} - \frac{22}{12} n \log(N/(2n)) \quad \text{and} \quad \geq 2n \log(N/(2n)) - \frac{22}{12} n \log(N/(2n)) \\
= \frac{1}{2} n \log(N/(2n)),
\]
which gives (1). In the third inequality, we use the relation \( \binom{n}{2} \geq \left( \frac{n}{2} \right)^b \).

**Proof of (2):**

Let \( S = (S_1, \ldots, S_m) \) be a \((N,n)\)-distinguisher. Observe that for any two sets \( X_1,X_2 \) such that \(|X_1 \cap X_2| = n\) there exists \( S_i \) such that \(|S_i \cap X_1| \neq |S_i \cap X_2|\). In other words for any tuple \((p_1, \ldots, p_m)\), \( p_i \in \{0,2n\} \) the set \( \{ X : \forall i \in [m], |S_i \cap X| = p_i \} \) is independent in \( G \). Therefore, \( \chi(G) \leq (2n+1)^m \). Thus
\[
\log \chi(G) \leq m \log(2n+1),
\]
which proves (2).

Finally, observe that (1) and (2) imply the statement of the lemma.

**Proof of Corollary 26**

The result follows directly from Proposition 22 and Lemma 23.

**Proof of Theorem 27**

Let us choose a sequence \( S \) of sets \( S_1,S_2,\ldots \) probabilistically, such that each \( x \in [N] \) belongs to \( S_i \) with probability \( 1/2 \), where all choices are independent. Then, our algorithm is defined such that, in round \( i \), the agents with IDs in \( S_i \) choose direction right and the other ones choose the direction left. We show that the family \( \mathcal{S} = (S_1, \ldots, S_k) \) chosen in this way gives a protocol solving the nontrivial move problem with positive probability, provided the size \( n \) of the network is smaller than \( N/3 \). That is, the following event holds with positive probability: for each \( X \subseteq [N] \) such that \(|X| < N/3\), the nontrivial move appears during an execution of the prefix of \( S \) of size \( O(n \log(N/n)/\log n) \), where \( n = |X| \). Then we build a sequence \( C \) of size \( O(N/\log N) \) which gives a nontrivial move on each \( X \subseteq [N] \) of size at least \( N/3 \). Thus, by interleaving \( S \) and \( C \), the theorem holds thanks to the probabilistic method.

Let us fix a set of IDs \( A \subseteq [N] \) of size \( n \) and assign sense of directions to them such that \( A = A_c \cup A_l \), where \( A_c \) is the set of agents with correct sense of directions, \(|A_c| = n_c\) and \(|A_l| = n - n_c\). Recall that a round does not give a nontrivial move if and only if it is a \((0,n)\)-round, \((n,0)\)-round, \((n/2,n/2)\)-round, \((3n/4,n/4)\)-round, or a \((n/4,3n/4)\)-round. Then, for a round defined by \( S_1 \) as above, we have:
\[
\begin{align*}
\text{Prob}((n/2, n/2)\text{-round}) &= \frac{1}{2} \sum_{j=0}^{n/2} \binom{n/2}{j} \left( \frac{n-n_c}{n/2-j} \right) \\
\text{Prob}((0, n)\text{-round}) &= \frac{1}{2^{n} (n/2)^n} \sum_{j=0}^{n/2} \binom{n}{j} \left( \frac{n-n_c}{n-j} \right) \\
\text{Prob}((n/2, 0)\text{-round}) &= \frac{1}{2^{n} (n/2)^n} \sum_{j=0}^{n/2} \binom{n}{j} \left( \frac{n-n_c}{n-j} \right) \\
\text{Prob}((n/4, 3n/4)\text{-round}) &= \frac{1}{2} \sum_{j=0}^{n/4} \binom{n/4}{j} \left( \frac{n-n_c}{n/4-j} \right) \\
\text{Prob}((3n/4, n/4)\text{-round}) &= \frac{1}{2^{n} (n/4)^n} \sum_{j=0}^{n/4} \binom{n}{j} \left( \frac{n-n_c}{n-j} \right).
\end{align*}
\]

In the above calculations, we use the relationship that \( \sum_{i=0}^{\min(a,c)} \binom{c}{a} = \binom{c}{a} \) and Stirling’s formula which determines the constant \( c_0 \) in the first row. The above estimations imply that the probability that a round defined by \( S_i \) is a trivial move for \( |A| = n \) is at most \( c_1 / \sqrt{n} \) for some constant \( c_1 \), provided \( n \) is large enough. Let us consider all sets of IDs \( A \) such that \(|A| \in [2^{i-1}, 2^i)\), for \( i \) such that \( 2^i < N/3 \). Let \( k = c_1 \frac{2 \log \binom{n}{2}}{1 - \frac{1}{2^i}} \) for a large enough constant \( c \) whose value will be determined later. By \( E_i \) we denote the event that a sequence of sets \( S_1, \ldots, S_k \) does not give a nontrivial move for all sets \( A \) whose size is in \([2^{i-1}, 2^i)\). Then,
\[
\begin{align*}
\text{Prob}(E_i) &\leq \sum_{d=2^{i-1}}^{2^i} \left( \text{Prob(\text{triv. move on a set of size } d)} \right) \left( \frac{2^i}{d} \right)^2 \\
&\leq \sum_{d=2^{i-1}}^{2^i} \frac{\binom{2^i}{d}^2 c_1}{d^{2i+1/n^2}} \leq c_1 \sum_{d=2^{i-1}}^{2^i} \left( \frac{2^i}{d} \right)^2 \\
&\leq c_1 \sum_{d=2^{i-1}}^{2^i} \left( \frac{1}{d^2} \right) \leq c_1 \sum_{d=2^{i-1}}^{2^i} \frac{1}{d^2} < c_1 \frac{1}{\pi^2}.
\end{align*}
\]
In the above calculations, we use the following facts:

- \( (\frac{N}{d})^{2d} \) is the number of possible choices of sets of size \( d \), and senses of direction of elements of these sets (used in the first inequality);
- \( \text{Prob}(\text{triv. move on a set of size } d) \leq \frac{c_1}{\sqrt{d}} \leq \frac{c_1}{2^{(d-1)/2}} \) (used in the second inequality);
- \( 2^{(i-1)k/2} \geq \frac{(N/2)^c}{(2^i)^3} \) for \( c \geq 3 \) (which follows from the fact that \( k = c\frac{2}{\log(N/2^i)} \); used in the third inequality);
- \( 2^d \leq \left(\frac{N}{d}\right) \) for \( d \leq N/3 \) (used in the third inequality);
- \( \left(\frac{N}{d}\right) \leq \left(\frac{N}{2^i}\right) \) for \( d \leq N/3 \) (used in the fourth inequality);
- \( \left(\frac{N}{2^i}\right) \geq \frac{(N/2^i)^{2^i}}{2^i} \geq 2^i \) if \( 2^i < N/2 \) (used in the fifth inequality).

Let \( i_0 = \lceil \log 4c_1 \rceil + 1 \) and \( i_1 = \lfloor \log(N/3) \rfloor \). The above calculations show that, the union of events \( E_{i_0}, E_{i_0+1}, \ldots, E_{i_1} \) holds with probability \( \sum_{i_0}^{i_1} c_1/2^i < 1/2 \) for \( c > 3 \). Therefore, by the probabilistic method, the sequence \( S \) gives a nontrivial move for each set of IDs of size in \( [2^{i_0}, 2^{i_1}] = [4c_1, N/c] \). It remains to tackle the cases that \( n < 2^{i_0} \) and \( n > 2^{i_1} \).

As for \( n < 2^{i_0} \), note that \( 2^{i_0} \) is a constant independent of \( n \). Thus the number of sets of size \( < 2^{i_0} \) is polynomial wrt \( N \), while the probability that a round gives a nontrivial move for a given set is larger than some positive constant independent of \( N \). Therefore on a sufficiently long prefix of \( S \) of length \( O(\log(N)) = O(n \log(N/n)/\log n) \), the nontrivial move appears with for each set of size \( < 2^{i_0} \) with probability \( 1 - 1/N \).

Now, we consider the case that the size \( n > 2^{i_1} > N/3 \). The number of such sets is upper bounded by \( 2^N \). And, for each such set, each round gives a nontrivial move with probability at least \( c'/\sqrt{N} \) for a constant \( c' \). By a simple calculation, one can show that the nontrivial move appears for each such set on a long enough prefix of \( S \) of size \( O(N/\log N) \) with probability \( 1 - 1/N \). More precisely, on a prefix of \( c'' \log N \), the probability that there is a set without a nontrivial move is smaller than

\[
2^N \left(\frac{c'}{\sqrt{N}}\right)^{c'' \log N} < 1/N
\]

for a large enough constant \( c'' \).