A TWO-WAREHOUSE PROBABILISTIC MODEL WITH PRICE DISCOUNT ON BACKORDERS UNDER TWO LEVELS OF TRADE-CREDIT POLICY

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ABSTRACT. It is impossible in this competitive era to assess the demand for items in advance. So, it is essential to refer to a stochastic demand function. In this paper, a probabilistic inventory model for deteriorating items is unfolded. Here, the supplier as well as the retailer adopt the trade-credit policy for their customers with the aim of promoting the market competition. Shortages are included into the model, and when stock on hand is zero, the retailer offers a price discount to those customers who are willing to back-order their demands. We consider two different warehouses in which the first one is an Own Warehouse (OW) where the deterioration is constant over time and the other is a Rented Warehouse (RW), and where the deterioration rate follows a Weibull distribution. An algorithm is provided for finding the solutions of the formulated model. Global convexity of the cost function is established which shows that our model has a unique solution.

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The proposed model is very helpful for any supplier or retailer to finalize the optimal ordering policy. Beside of this, we target to increase the total profit for retailer by reducing the corresponding total inventory cost. The theoretical concept is justified with the help of some numerical examples. A sensitivity analysis of the optimal solution with respect to the major parameters is also provided in order to stabilize our model. We finalize the paper through a conclusion and a preview onto possible future studies.

1. **Introduction.** In classical inventory models, we usually assume that the warehouse, namely, the Own Warehouse (OW), has an unlimited capacity to store items. However, busy markets like super market, municipality market, multiplex, etc., face a limited storage capacity. Therefore, when an attractive price discount is accessible, or the demand of any items is very high, or procuring cost for any goods is higher than the other cost related to inventory, or there are some problems in frequent procurement, at any such a critical moment, managers decide to hire an additional warehouse, namely, the Rented Warehouse (RW), on a rental basis for storing a large amount of items at a time. The RW may be located a bit away from the OW, but it has more facilities than the OW with unlimited capacities. We assume that the holding cost in the RW is greater than OW and, therefore, the purchased items are stored firstly in the OW and only the excess amount of items are stored in the RW. The actual service to the customer is done at the OW only, and then immediately the items are transferred from the RW to the OW. Therefore, it is necessary and important also to investigate the influence of the warehouse capacity in various inventory-policy problems.

In the traditional inventory model, one assumes that the demand function is known and pre-determined. But in a present-day market scenario, it is quite difficult to assess the demand function exactly. When a company is introducing a product, it is rather natural that sometimes the demand is very high, and sometimes it be quite low. In this situation, we can add a positive impulse into the inventory system by considering a probabilistic demand function.

Trade credit is an agreement between the supplier and the retailer where the supplier offers a certain fixed period within which the retailer has to pay the supplier instead of paying instantly after buying. The retailer can earn interest and accumulate revenue within that period, but if the retailer fails to pay in the course of the offered period, the supplier charges an extra money. The retailer can offer the same policy to the customer, obviously his offered period is less than that the offered period of the supplier. This policy is known as two-level of trade-credit policy. Hence, paying later for any items reduces the stock holding cost and also encourages both the retailer and customer to buy more. Thereafter, trade credit takes a significant and growing part in the managerial life of both supplier and retailer.

Deterioration also means a loss of utility or loss of marginal value of goods, which results in a decreased usefulness of the original items. Deterioration is normally caused by decay, obsolescence, vaporization, poor preservation technology, damage, spoilage, dryness, etc. The rate of deterioration is less for some products such as hardware, glass ware, metal items, toys, etc., but it is very high for volatile liquids, fruits, fish, vegetables, medicine, blood commodities, etc. Therefore, deterioration of items plays a crucial role in the determination of inventory model, and it needs to be examined carefully.

Shortage is a very common feature, especially, when the demand is of probabilistic type. When shortage occurs, some customers may prefer to accept backorders if the reputation of the company is very high or the customers are very much in favor
of a particular product, such as fashionable goods as shoes, cosmetics and clothes. However, in general, many customers prefer to leave for another company. In that situation, if an attractive price discount on backorders can be offered, a customer will stay in the system for more advantage. Hence, price discount on backorders can guarantee a profit in stock-out period and have a significant value in determining an inventory model.

Herewith, each term described above plays an important role in the formulation of this paper.

The main motivations of our work are as follows:

- We consider a stochastic inventory model with probabilistic demand function, as it is impossible for a businessman to determine the demand before starting the business.
- A two-warehouse inventory model is included here, where in an OW, a constant deterioration is considered, while in an RW a Weibull-distributed deterioration is taken into account. The model is designed based on real-life situations, because a single-storage warehouse with limited capacity is inadequate for storing all the products and, as a result an RW is needed.
- The supplier and the retailer both offer a fixed-credit period which will encourage the supplier’s selling, and the retailer can take an advantage to reduce the cost and to increase the profit.
- Shortages are taken into account to design the model in more realistic meanings.
- Price discount on backorders is offered, since it is a novel strategy to attract more customers.

The rest of the paper is organized as follows: In Section 2, a literature review on research about inventory model is presented. The problem definition is described in Section 3 which consists of two subsections. In the first part, i.e., in Subsection 3.1, notations which are used throughout our paper are presented, and in the second part, i.e., in Subsection 3.2, assumptions which are highly necessary for our model are given. Section 4 discusses the mathematical model of the proposed inventory problem, and a corresponding solution procedure is also derived in this section including an algorithm. Section 5 contains three numerical examples to illustrate the proposed model. Section 6 gives a sensitivity analysis with illustrated figures of our proposed methodology under parametric variation. Section 7 presents concluding remarks, and it proposes new pathways of future investigation.

2. Review on research. In classical inventory models, we mainly assume warehouse problems based on a single-storage facility. But, in the field of inventory management, when a large amount of units of items is purchased, it cannot be stored completely in the existing warehouse situated in the market due to its limited capacity. Then, excess units of stock are kept in an RW located at some little distance from an OW. This situation arises, e.g., when the management gets an attractive price discount for purchasing grain, or when the acquisition costs are higher than using an RW. This criterion leads two-warehouse inventory models to receive a great deal of attention, both in research and in practice by various authors in recent years. This type of model was firstly discussed by Hartely [17]. Then, Sarma [33] developed a model with infinite production rate, but without shortages. Further, Goswami and Chaudhuri [15], reflected on those models with or without shortages, taking into account time-dependent demand, which is a linearly
increasing factor with respect to time. Correcting and modifying the assumptions of Goswami and Chaudhuri [15], Bhunia and Maiti [5] analyzed the same inventory model, presented a sensitivity analysis graphically on the optimal average cost and also considered the cycle length for the variations of the demand parameter. Yang [40] described a two-warehouse model for an item with constant demand and shortages under inflation. Chung and Huang [7] worked on a two-warehouse inventory model for deteriorating items under trade credit and the assumption that the deterioration rates in the two warehouses are the same. Bhunia et al. [4] elaborated on a two-warehouse inventory model with permissible delay in payment and partial backlogging. Palanivel et al. [24] formulated a two-warehouse model with stock dependent demand with the effect of inflation. A two-warehouse model with fuzzy demand and fuzzy deterioration was executed by Shabani et al. [34]. Yang and Chang [41] established a two-warehouse model with partial backlogging under the effect of inflation. Recently, Jaggi et al. [18] developed a two-warehouse model with permissible delay in payment, where they allowed imperfection. Kaliraman et al. [20] derived a two-warehouse model with exponential demand.

There exist many probabilistic inventory models, published by different authors, who proposed various demand patterns. A few of those which are rather effective and relevant for our investigation are mentioned here: Datta and Pal [11] considered both deterministic and probabilistic versions of power demand patterns with a variable rate of deterioration. De and Goswami [12] discussed a probabilistic model for deteriorating items. Shah [35] addressed a probabilistic order-level system with delay in payments and lead time. Singh et al. [37] formulated a probabilistic inventory model with permissible delay in payments. Shah and Shah [36] developed a probabilistic inventory model with discrete-time factor for deteriorating items, and they also offered permissible delay in payments.

In a traditional inventory model, it is usually presumed that the retailer has to pay for the items to the supplier as soon as the items are received. In practice, however, the supplier offers the retailer a certain fixed-credit period without interest to actively encourage the market competition. The retailer can sell the goods, accumulate revenue and earn interest at the end of the prescribed trade-credit period. In the recent two decades, the effect of permissible delay in payments on various inventory systems has attracted more attention by numerous researchers. Goyal [16] first explored a single-item Economic Order Quantity (EOQ) model under permissible delay in payments. Aggarwal and Jaggi [1] extended Goyal’s model [16] to the case with deteriorating items. All those previously published models discussed a trade credit and assumed that the supplier would offer to the retailer a trade credit, but the retailer will not offer exactly the same trade credit to his or her customers. Pervin et al. [25] presented and solved an EOQ model with trade-credit policy. Many researchers such as Benkherouf [3], Chung and Liao [8], who published methods and results in this area which extended a new horizon by implementing their research.

Deterioration is a very common phrase in the context of inventory. Therefore, scientists are always engaged to modify their model for controlling deterioration. Ghare and Schrader [13] were the first to investigate a deteriorating inventory model without shortage. Then, Covert and Philip [9] extended Ghare-Schrader [13] model, whereby they assumed a two-parametric Weibull distribution and addressed a variable rate of deterioration. Pervin et al. [26] analyzed a deteriorating inventory model, and they also modified their model by taking into account declining demand.
under trade-credit policy. Recently, Pervin et al. [27] derived an integrated model with variable holding cost under trade-credit policy. A multi-item deteriorating inventory model with trade-credit policy was elaborated by Pervin et al. [28].

The occurrence of shortages in an inventory system is a natural phenomenon. We can easily notice that many products of famous brands or fashionable goods, such as certain branded gum shoes, clothes, hi-fi equipment and jewelry, may create a situation in which customers think that it will be better to wait for backorders at the time period in which shortages occur, because maybe the other company will not provide the same quality of items. There is another cause to wait for backorder, namely, a price discount from the retailer to the customers; many authors like Ouyang et al. [23] formulated their models based on this fact. Ghoreishi et al. [14] derived their work on delay in payments, inflation and selling price-dependent demand. Lashgari et al. [22] assumed partial up-stream and down-stream payment in a three-level supply chain. Pervin et al. [29] formulated and solved a deteriorating inventory model with stock-dependent demand and variable holding cost. Here, most of the research works implied shortages. Shortages can also guarantee a profit if delay in payment is offered; this is reflected in the papers considered next. Sarkar et al. [31] analyzed a deteriorating model of seasonal products with stock-dependent consumption rate and allowed shortages. An EOQ model with linear trend in demand is analyzed by Chakrabarty et al. [6]. Das et al. [10] prepared a two-warehouse production model for deteriorating inventory items with stock-dependent demand under inflation. Ray and Chaudhuri [30] designed an EOQ model with stock-dependent demand and shortages. The price discount on backorders during stock-out period can make customers become more interested to wait for their desired items. This thought was reflected by various authors, among them some were Sarkar et al. [32] and Kurdhi et al. [21]. Taleizadeh and Noori-daryan [38] built up a model on pricing and production policies with rework process. A comparative study between our contribution with respect to other publications is given in Table 1. For a short survey we present the contribution of the existing researches related with our study in the compact form of the following list:

1976: Hartley [17] first introduced a two-warehouse inventory model.
1983: Sarma [33] developed a two-warehouse model with infinite production rate, but without shortages.
1992: Goswami and Chaudhuri [15] reflected on those models, taking into account time-dependent demand, which is a linearly increasing factor with respect to time.
1994: Bhunia and Maiti [5] analyzed the previous inventory model, presented a sensitivity analysis graphically on the optimal average cost and also considered the cycle length for the variations of the demand parameter.
2006: Yang [40] described a two-warehouse model for an item with constant demand and shortages under inflation.
2007: Chung and Huang [7] worked on a two-warehouse inventory model for deteriorating items under trade credit and the assumption that the deterioration rates in the two warehouses are the same.
2013: Yang and Chang [41] established a two-warehouse model with partial backlogging under the effect of inflation.
2014: Bhunia et al. [4] elaborated on a two-warehouse inventory model with permissible delay in payment and partial backlogging.
Table 1: Previous works of different authors in this field including our work.

| Author(s)                  | Two warehouse | Probabilistic demand | Trade credit | Deteriorations | Shortage | Price discount |
|----------------------------|---------------|----------------------|--------------|----------------|----------|---------------|
| Datta and Pal (1988)       | ✓             | ✓                    | ✓            |                |          |               |
| Bhunia and Maiti (1994)    | ✓             | ✓                    | ✓            |                |          |               |
| Shah and Shah (1998)       | ✓             | ✓                    | ✓            |                |          |               |
| Shah (1997)                | ✓             |                      | ✓            |                |          |               |
| Palanivel et al. (2016)    | ✓             |                      | ✓            |                |          |               |
| Jaggi et al. (2017)        | ✓             | ✓                    | ✓            |                |          |               |
| Benkherouf (1997)          | ✓             | ✓                    | ✓            | ✓              |          |               |
| Singh et al. (2010)        | ✓             | ✓                    | ✓            | ✓              |          |               |
| Kaliraman et al. (2017)    | ✓             | ✓                    | ✓            |                |          |               |
| Bhunia et al. (2014)       | ✓             | ✓                    | ✓            | ✓              |          |               |
| Chung and Liao (2004)      | ✓             | ✓                    | ✓            |                | ✓        |               |
| Yang (2006)                | ✓             | ✓                    | ✓            |                | ✓        |               |
| De and Goswami (2009)      | ✓             | ✓                    | ✓            |                | ✓        |               |
| Chung and Huang (2007)     | ✓             | ✓                    | ✓            |                |          |               |
| Kurdhi et al. (2015)       | ✓             |                      | ✓            |                |          |               |
| Pervin et al. (2018)       | ✓             | ✓                    | ✓            |                |          |               |
| Pervin et al. (2017)       | ✓             | ✓                    | ✓            | ✓              |          |               |
| Goyal (1985)               | ✓             | ✓                    | ✓            |                |          |               |
| Yang and Chang (2013)      | ✓             | ✓                    | ✓            |                |          |               |
| Sarkar et al. (2015)       | ✓             | ✓                    | ✓            |                |          |               |
| Ray and Chaudhuri (1997)   | ✓             | ✓                    | ✓            |                |          |               |
| Hariga (1995)              | ✓             | ✓                    | ✓            |                | ✓        |               |
| Our paper                  | ✓             | ✓                    | ✓            | ✓              | ✓        |               |

2016: Palanivel et al. [24] formulated a two-warehouse model with stock-dependent demand with the effect of inflation.
2016: Shabani et al. [34] executed a two-warehouse model with fuzzy demand and fuzzy deterioriation.
2017: Jaggi et al. [18] developed a two-warehouse model with permissible delay in payment and imperfection.
2017: Kaliraman et al. [20] derived a two-warehouse model with exponential demand.

From the above literature review, from the schematic view of Table 1 and a quick review, one can observe that no researchers until now considered a two-warehouse inventory model with probabilistic demand and price discount on backorders under two levels of trade-credit policy. This interesting fact attracts us to consider that type of model, and we also derive some remarkable results which will make our model turn out to be an appealing one.

3. Problem description. Consider a two-warehouse inventory model with probabilistic demand and price discount on backorders under two levels of trade-credit policy, as shown in Figure 1 and Figure 2. The retailer purchases a certain amount of products from the supplier and keeps it stored at an OW. Since the demand for the products follows a probabilistic distribution, it may increase at any time. To meet the demand of the customers, the retailer purchases extra amounts and also rents a warehouse to store the excess amount of an OW. Suppose a product is available in the market at a lower price, and the retailer can store the product for the future with a higher price for earning more. (This is known as stock market business.) But, at the time of supplying the demand to the customer, the items are delivered from RW first, then from OW, because the holding cost in RW is higher than that of OW. Here, the supplier offers the retailer a certain credit period to pay
the amount and, within that credit period, no extra charge is paid by the retailer. However, after that certain period, if the retailer is unable to pay, then an extra charged is demanded by the supplier to the retailer. The retailer also applies the same policy to the customer. If the demand for any items increases suddenly, then shortages can happen in an OW; to maintain a smooth relationship between the retailer and the customer, the retailer offers a price discount to those customers who are willing to backorder their demand. In order to design the model and solve the formulated problem, we use a classical optimality condition and then describe an easy algorithm to clarify our model. The convex nature of the cost function and the optimality condition simply show the global optimality of our derived solution.

The main aim of this paper is to increase the total gain of the system by reducing the total cost, while taking into account the important aspects of probabilistic demand. The following notations and assumptions are used throughout the paper to develop and solve our mathematical model.

3.1. Notations.

- $A$: Ordering cost per order;
- $c$: Unit purchasing cost per item;
- $T_c$: Unit transportation cost per item;
- $T_k$: Time-interval end points ($k=0,1,2,\ldots,n-1$) with $T_{k-1} < T_k$, where $T_0 = 0$;
- $T_n$: Time at which the inventory level reaches to $W$, where $W$ is fixed capacity of OW;
- $T$: Length of cycle time (a decision variable);
- $T'$: Stochastic end time of the inventory cycle, where $T' \geq T$;
- $s$: Unit selling price per item;
- $D(t)$: Demand rate at any time $t$;
- $I_1(t)$: Inventory level at any time $t$ in the OW when shortages do not occur;
- $I_2(t)$: Inventory level at any time $t$ in the OW when shortages occur;
- $I_3(t)$: Inventory level at any time $t$ in the RW;
- $\delta$: Fraction of the demand during the stock-out period that will be back ordered, where $0 \leq \delta < 1$;
- $h_o$: Holding cost per unit time in the OW;
- $h_r$: Holding cost per unit time in the RW, where $h_r > h_o$;
- $I_c$: Interest, which can be earned per unit of time (i.e., per $ per year) by the retailer;
- $I_c$: Interest charges in stocks per unit of time (i.e., per $ in stocks per year) by the supplier;
- $M$: Credit period in years offered by the supplier;
- $N$: Trade-credit period in years offered by the retailer;
- $TC$: Total relevant cost function per unit of time (a decision variable);
- $TC_p$: Marginal profit per unit;
- $TC_1$: Price discount on unit backordered offered;
- $b_0$: Upper bound on backorder ratio, where $0 \leq b_0 \leq 1$ for consistency.

3.2. Assumptions.
3.2.1: The demand rate $x$, during any scheduling period, follows a probability density function (p.d.f.) $f(x|T)$, $a(T) \leq x \leq b(T)$ ($b(T)$ is a decision variable) with $\mu(T) = E(x|T) = \int_{a(T)}^{b(T)} x f(x|T) = RT$. Here, $\mu(T)$ is the mean demand during the time interval $[0, T]$ and $R = \mu(T)/T$ denotes the average expected demand per unit time during a cycle. We also assume that the p.d.f. $f(x|T)$ of demand $x$ during the time interval $[0, T]$ is sufficiently well-behaved so that all the expected costs corresponding to our model exist. Furthermore, we suppose that the distribution function of demand is stationary over the planning horizon.

3.2.2: Lead time is 0 and replenishment rate is finite.

3.2.3: Deterioration rate $P(t)$ in the OW is constant, i.e., $P(t) = \theta$, where $\theta$ is a constant such that $0 < \theta < 1$.

3.2.4: Deterioration rate $G(t)$ in the RW follows a two-parameter Weibull distribution, i.e., $G(t) = \alpha \beta t^{\beta-1}$, where $0 < \alpha \leq 1$, $\beta \geq 1$ and $\alpha$ is the shape parameter and $\beta$ is the scale parameter.

3.2.5: Shortages are allowed and demand is partially backlogged during the stock-out period.

3.2.6: The OW has a fixed capacity of $W$ units, and the RW has an unlimited capacity.

3.2.7: The holding costs in the RW are higher than those in the OW.

3.2.8: Less deteriorating items are to be preserved in OW and more deteriorating items are to be stored in RW, because the manager decides to deliver items from RW first, and then from OW.

3.2.9: The fixed-credit period $M$ offered by the supplier to the retailer is always greater than the fixed-credit period $N$ offered by the retailer to the customer.

3.2.10: If $T \geq M$, then the retailer settles the account at time $M$ and as the offered period $M$ is less than the credit period $T$, the retailer pays for the interest charges on items in stock with rate $I_c$ over the interval $[M, T]$. If $T < M$, then the retailer settles the account at time $M$, and as the offered period $M$ is greater than the credit-period length $T$, there is no interest charge for the retailer in stock during the whole cycle. On the other hand, if $M > N$, the retailer can accumulate revenue and earn interest during the period from $N$ to $M$ with rate $I_e$ under trade-credit policy.

3.2.11: There is no reparation or replacement of deteriorated units during the planning horizon. The item will be withdrawn from the stock immediately after it has become deteriorated.

3.2.12: During the stock-out period, the fraction of the demand $\delta$ is directly proportional to the price discount $TC_1$, offered by the retailer. Thus, $\delta = \frac{b_0}{TC_p}TC_1$, where $0 \leq TC_1 \leq TC_p$.

4. Mathematical model and solution procedure. Following the assumptions, described in Section 3, we develop a mathematical model.

A company purchases $S$ units of items, out of which $W$ ($W < S$) units are kept in the OW and $(S - W) = Q$ units are stored in the RW. Initially, the stocks of the RW are not used for meeting the demand until the stock level drops to $W - K$ units at the end of $T_1$. At this stage, $K$ ($K \leq W$) units are transported from the RW to the OW. As a result, the stock level of the OW again becomes $W$ and the stocks of the OW are used to meet further demands.
This process continues until the stock level in the RW is completely exhausted. After the last shipment, only \( W \) units are taken to satisfy the demand during the interval \([T_{n-1}, T_n]\), and then the shortages occur and are completely backlogged during the interval \([T_n, T']\), where \( T' \) is the end point of the inventory cycle. The graphical representation of our described model is shown in Figure 1 and Figure 2. The mathematical formulation of the above process is given by

\[
d'I_1(t) + P(t)I_1(t) = -D(t) \quad (T_{k-1} \leq t \leq T_k), \quad I_1(T_{k-1}) = W. \tag{1}
\]

By using the values of \( P(t) \) from Assumption 3.2.3 and \( D(t) \) from Assumption 3.2.1, equation 1 becomes:

\[
d'I_1(t) + \theta I_1(t) = -\frac{x}{T}. \tag{2}
\]

Now, using boundary condition, i.e., \( I_1(T_{k-1}) = W \), the solution of equation 2 turns out to be

\[
I_1(t) = We^{-\theta t} - \frac{x}{\theta T} \left( e^{\theta t} - 1 \right). \tag{3}
\]

Since shortages are allowed in OW, we use the terminal condition \( I_1(T) = 0 \) when \( x = b(T) \). That is, as shortages are allowed in our model, so, when the demand will be highest, then we can only address an inventory whose minimum value is 0, not lower than that. Herewith, we get:

\[
W = \frac{b(T)}{\theta T} (e^{\theta T} - 1). \tag{4}
\]

Using equation 4, equation 3 takes the form

\[
I_1(t) = \frac{e^{-\theta t}}{\theta T} \left[ b(T)(e^{\theta T} - 1) - x(e^{\theta t} - 1) \right]. \tag{5}
\]

When shortages occur within some time interval \([T_n, T']\), then the mathematical model is given as follows:

\[
d'I_2(t) = -\frac{D(T)}{1 + \delta(T' - t)} \quad (T_n = T \leq t \leq T'). \tag{6}
\]

By using the value of \( D(T) \), equation 6 becomes

\[
d'I_2(t) = -\frac{x}{T + \delta T(T' - t)} \quad (T \leq t \leq T'). \tag{7}
\]

Now, the solution of equation 7 in the interval \([T, T']\) looks so:

\[
I_2(t) = \frac{x}{\delta T} \log(T + \delta T(T' - t)). \tag{8}
\]

The RW system can be represented subsequently:

\[
d'I_3(t) = -G(t)I_3(t) \quad (T_{k+1} \leq t \leq T_n), \quad I_3(T_n) = W. \tag{9}
\]

Using the value of \( G(t) \) from Assumption 3.2.4, the above equation attains this form:

\[
d'I_3(t) = -\alpha \beta t^{(\beta - 1)} I_3(t), \quad I_3(T_n) = W. \tag{9}
\]
Now, employing the terminal condition, the solution of equation 9 becomes:

$$I_3(t) = W e^{-\alpha t^3}.$$ 

The average expected inventory in the system per unit time is:

$$\frac{1}{T} \int_0^T E(I_1(t)) dt = \frac{1}{\theta^2 T^2} [b(T)(e^{\theta T} + e^{-\theta T} - 2) + \mu(T)(1 - \theta T - e^{-\theta T})].$$

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**Figure 1.** Characteristic path of the OW.

The elements comprising the retailer’s profit function per cycle are listed below:

4.1: Annual ordering cost (OC) = A.

4.2: For the OW, inventory is available in the system during $T_k \leq t \leq T_{k+1}$ ($k = 0, 1, 2, \ldots, n$). Hence, holding cost (HC) during the period is:

$$\sum_{k=0}^{n-1} \frac{h_o}{T} \int_{T_k}^{T_{k+1}} I_1(t) dt$$

$$= \sum_{k=0}^{n-1} \frac{h_o}{\theta^2 T^2} \left[ b(T)(e^{\theta T} + e^{-\theta T} - 2) + \mu(T)(1 - \theta T - e^{-\theta T}) \right],$$

where $T_m = \Delta T := (T_{k+1} - T_k)$ ($k = 0, 1, 2, \ldots, n$).
4.3: For the RW, inventory is available in the system during

\[ T_k \leq t \leq T_{k+1} \quad (k = 0, 1, 2, \ldots, n-1) \].

Hence, holding cost (HC) during the period is given by:

\[
\sum_{k=0}^{n-2} \frac{h_r}{T} \int_{T_k}^{T_{k+1}} I_3(t)dt = \sum_{k=0}^{n-2} \frac{h_r}{\alpha \beta T T_m} \left[ e^{-\alpha T_m^\beta} - e^{-\alpha \beta T_m^\beta} \right],
\]

with \( T_m = \Delta T := (T_{k+1} - T_k) \quad (k = 0, 1, 2, \ldots, n-1) \).

4.4: Number of deteriorated units (DC) during a cycle is equal to

\[
W - \mu(T) - \int_0^T E(I_2(t))dt = \frac{1}{\theta^2 - T^2} \left[ b(T)(e^{\theta T} + e^{-\theta T} - 2) + \mu(T)(1 - \theta T - e^{-\theta T}) \right].
\]

4.5: Shortage occurs during the period of \( T_n \leq t \leq T' \). Hence, the shortage cost (SC) during the period is equal to

\[
\frac{s}{T} \int_{T_n}^{T'} (-I_2(t))dt = \frac{s}{T} \cdot x \cdot (T' - T_n) \cdot \left[ \log T - \log(T + \delta T(T' - T_n)) \right].
\]

4.6: Due to shortage during time interval \([T_n, T']\), not all of the customers are interested to wait for the coming lot size, which may cause a loss in profit.
Hence, *lost-sale cost (LSC)* is introduced by

\[
\frac{s}{T} \int_{T_n}^{T'} (1 - \delta)D(t)dt = \frac{s \cdot (1 - \delta) \cdot x \cdot (T' - T_n)}{T^2}.
\]  

(14)

4.7: *Backorder cost (BC)* is defined as

\[
\frac{\delta}{T} \int_{T_n}^{T'} I_2(t)dt = \frac{\delta \cdot x \cdot (T' - T_n) \cdot \left[ \log T - \log(T + \delta(T' - T_n)) \right]}{T^2}.
\]

(15)

4.8: The cost incurred during inventory transferred from the RW to the OW at times \(T_k\) \((k = 0, 1, 2, ..., n - 1)\) is known as *transportation cost (TC)* and is defined by

\[
\sum_{k=0}^{n-1} \frac{T_c}{T} \int_{T_k}^{T_{k+1}} I_1(t)dt = \frac{T_c}{T} \sum_{k=0}^{n-1} \left[ 4(T_{k+1} - T_k)(e^{\theta(T_{k+1} - T_k)} - e^{-\theta(T_{k+1} - T_k)}) - x(T_{k+1} - T_k) \right.
\]

\[
- 4(T_{k+1} - T_k)e^{-\theta(T_{k+1} - T_k)}.\]

(16)

4.9: From our considered assumptions and based on the values of \(N\) and \(M\), there are three possible alternative cases, which are as follows:

**Case 4.9.1.** \(M \leq T\). The retailer will earn an interest (*interest earned IE*) per unit time:

\[
\frac{sI_c}{T} \left[ \frac{\mu(T)}{T} \frac{N(M - N)}{N} + \int_0^{M-N} \frac{\mu(T)}{T} dt \right] = \frac{sI_c}{T} R \frac{(M^2 - N^2)}{2T},
\]

(17)

and he or she will pay the following interest (*interest charged IC*) per unit time:

\[
\frac{cI_c}{T} \int_M^T E(I_1(t))dt = \frac{cI_c}{T} \int_M^T \frac{1}{\theta^2T^2} \left[ b(T)(e^{\theta T} + e^{-\theta T} - 2) + \mu(T)(1 - \theta T - e^{-\theta T}) \right] dt
\]

\[
= \frac{2cI_c}{\theta^2T^2} \left[ b(T)(e^{\theta(T-M)} - 1) - (b(T) - \mu(T))(e^{-\theta M} - e^{-\theta T}) - \theta \mu(T)(T - M) \right].
\]

(18)

**Case 4.9.2.** \(N \leq T < M\).
Retailer’s credit interest (interest earned IE) per unit time during the interval is given as follows:

\[
\frac{sI_e}{T} \left[ \frac{\mu(T)}{T} N(T - N) + \int_0^{T-N} \frac{\mu(T)}{T} t \, dt + \mu(T)(M - T) \right] = \frac{sI_e R}{2T} \left( 2MT - T^2 - N^2 \right).
\]

The retailer will pay an interest (interest charged IC) per unit time during this period with value 0.

**Case 4.9.3.** \( 0 < T < N \).

Retailer receives an interest (interest earned IE) per unit time during the interval, given as follows:

\[
\frac{sI_e}{T} \left[ \frac{\mu(T)}{T} T(M - N) \right] = sI_e R(M - N),
\]

and he or she will pay an interest (interest charged IC) per unit time during this period of value 0.

Then, the total relevant cost per unit of time for the retailer can be expressed as

\[
TC = OC + HC + PC + DC + SC + BC + IC - IE - LSC.
\]

Therefore, inserting the values of the terms according to equations 10-20, the following case-wise results are achieved:

\[
TC_1(T) = \frac{A}{T} + \frac{b_o}{\theta^2 T^2} \left[ b(T)(e^{\theta T} + e^{-\theta T} - 2) + \mu(T)(1 - \theta T - e^{-\theta T}) \right] + \frac{b_r}{T \alpha \beta T_k^2 + 1} \left[ e^{-\alpha T_k^\beta} - e^{-\alpha T_k^\beta - 1} \right] + \frac{2}{\theta^2 T^2} \left[ b(T)(e^{\theta T} + e^{-\theta T} - 2) \right] + \mu(T)(1 - \theta T - e^{-\theta T}) - \frac{s(1 - \delta) x(T' - T_n)}{T^2} - \frac{s}{T} \cdot x \cdot (T' - T_n).
\]

\[
\left[ \log T - \log(T + \delta T(T' - T_n)) \right] + \frac{\delta}{T} \cdot x \cdot (T' - T_n).
\]

\[
\left[ \log T - \log(T + \delta T(T' - T_n)) \right] + \frac{T_c}{T} \sum_{k=0}^{n-1} [4(T_{k+1} + T_k) + (e^{\theta(T_{k+1} - T_k)} - e^{-\theta(T_{k+1} - T_k)}) \cdot x(T_{k+1} - T_k) - 4(T_{k+1} - T_k)e^{-\theta(T_{k+1} - T_k)} + \frac{2cI_e}{\delta^2 T^3} \left[ b(T)(e^{\theta(T - M)} - 1) - (b(T) - \mu(T)) \right] + (e^{-\theta M} - e^{-\theta T}) - \theta \mu(T)(T - M)] - \frac{sI_e R}{2T}(M^2 - N^2).
\]

\[(22)\]

If we denote

\[
\Delta TC_1(T) := TC_1(T + 1) - TC_1(T),
\]

then an (approximate) necessary optimality condition for \( TC_1(T) \) to be minimal at \( T = T_1 \) is

\[
\Delta TC_1(T - 1) \leq 0 \leq \Delta TC_1(T),
\]

\[(24)\]
where $T$ and $T_1$ must be non-negative integers. When $T = T_1$, the account is to be settled exactly at the time of placing the next order. For the given value of $T$, the total cost of the system can be obtained from equation 22.

Furthermore, we obtain:

$$TC_2(T) = \frac{A}{T} + \frac{h^r}{\theta^2T^2} \left[ b(T)(e^{\theta T_m} + e^{-\theta T_m} - 2) + \mu(T)(1 - \theta T_m - e^{-\theta T_m}) \right]$$

$$+ \frac{h^r}{T \alpha \beta T_k^3} \left[ e^{-\alpha T_k^3} - e^{-\alpha T_{k+1}} \right] + \frac{2}{\theta^2T^2} \left[ b(T)(e^{\theta T} + e^{-\theta T} - 2) \right]$$

$$+ \mu(T)(1 - \theta T - e^{-\theta T}) - \frac{s(1 - \delta)x(T' - T_n)}{T^2} - \frac{s}{T} \cdot x \cdot (T' - T_n);$$

$$[\log T - \log(T + \delta T(T' - T_n))] + \frac{s}{T} \cdot x \cdot (T' - T_n).$$

(25)

Finally, we get:

$$TC_3(T) = \frac{A}{T} + \frac{h^r}{\theta^2T^2} \left[ b(T)(e^{\theta T_m} + e^{-\theta T_m} - 2) + \mu(T)(1 - \theta T_m - e^{-\theta T_m}) \right]$$

$$+ \frac{h^r}{T \alpha \beta T_k^3} \left[ e^{-\alpha T_k^3} - e^{-\alpha T_{k+1}} \right] + \frac{2}{\theta^2T^2} \left[ b(T)(e^{\theta T} + e^{-\theta T} - 2) \right]$$

$$+ \mu(T)(1 - \theta T - e^{-\theta T}) - \frac{s(1 - \delta)x(T' - T_n)}{T^2} - \frac{s}{T} \cdot x \cdot (T' - T_n);$$

$$[\log T - \log(T + \delta T(T' - T_n))] + \frac{s}{T} \cdot x \cdot (T' - T_n).$$

(28)

If we set

$$\Delta TC_2(T) := TC_2(T + 1) - TC_2(T),$$

then the optimality condition at $T = T_2$ is given by

$$\Delta TC_2(T - 1) \leq 0 \leq \Delta TC_2(T),$$

(27)

where $T$ and $T_2$ must be non-negative integers. When $T = T_2$, the account is to be settled exactly at the time of placing the next order. For the given value of $T$, the total cost of the system can be obtained from equation 25.

Finally, we get:

$$TC_3(T) = \frac{A}{T} + \frac{h^r}{\theta^2T^2} \left[ b(T)(e^{\theta T_m} + e^{-\theta T_m} - 2) + \mu(T)(1 - \theta T_m - e^{-\theta T_m}) \right]$$

$$+ \frac{h^r}{T \alpha \beta T_k^3} \left[ e^{-\alpha T_k^3} - e^{-\alpha T_{k+1}} \right] + \frac{2}{\theta^2T^2} \left[ b(T)(e^{\theta T} + e^{-\theta T} - 2) \right]$$

$$+ \mu(T)(1 - \theta T - e^{-\theta T}) - \frac{s(1 - \delta)x(T' - T_n)}{T^2} - \frac{s}{T} \cdot x \cdot (T' - T_n);$$

$$[\log T - \log(T + \delta T(T' - T_n))] + \frac{s}{T} \cdot x \cdot (T' - T_n).$$

(29)

If we define

$$\Delta TC_3(T) := TC_3(T + 1) - TC_3(T),$$

then the optimality condition at $T = T_3$ is given by

$$\Delta TC_3(T - 1) \leq 0 \leq \Delta TC_3(T),$$

(30)
where \( T \) and \( T_3 \) must be non-negative integers. When \( T = T_3 \), the account is to be settled exactly at the time of placing the next order. For the given value of \( T \), the total cost of the system can be obtained from equation 28.

To determine the optimal solution, we follow the subsequent procedure:

**Algorithm:**

**Step 1:** Find \( T^*_1 \) which satisfies equation 24. If \( M \leq T^*_1 \), then find \( TC_1(T^*_1) \) from equation 22.

**Step 2:** Find \( T^*_2 \) which satisfies equation 27. If \( N \leq T^*_2 < M \), then calculate \( TC_2(T^*_2) \) from equation 25.

**Step 3:** Find \( T^*_3 \) which satisfies equation 30. If \( 0 < T^*_3 < N \), then derive \( TC_3(T^*_3) \) from equation 28.

**Step 4:** The optimal solution is \((T^*_0, W_0, TC_0)\), where \( T^*_0 \) is the optimal value of \( T \) in association with \( T^*_1, T^*_2 \) and \( T^*_3 \) for the objective function \( TC_0 \), where \( TC_0 = \min\{TC_1(T^*_1), TC_2(T^*_2), TC_3(T^*_3)\} \), and \( W_0 = \frac{b(T^*_0)}{\theta T^*_0} (e^{\theta T^*_0} - 1) \).

5. **Numerical examples.** The following numerical examples are given to illustrate our proposed model and method. The parametric values taken in Example 1, Example 2 and Example 3 are related with the cost function \( TC_1, TC_2 \) and \( TC_3 \), respectively.

**Example 1.** Let us assume \( A = 400$/order, \( h_o = 60$/unit, \( h_r = 70$/unit, \( \theta = 0.8, \alpha = 0.05, \beta = 3.00, M = 0.5 \) months, \( N = 0.4 \) months, \( R = 10 \) units/weeks, \( x = 50 \) units/days, \( s = 10$/order, \( \delta = 0.08, T_e = 80$/order, \( c = 50$/unit, \( I_c = 0.15$/year, I_o = 0.14$/year, W = \) capacity of OW = 800 units. We consider time values for different numbers of cycles, i.e., \( T_1, T_2, T_3, T_4, T_5 \), respectively, as 2.0 days, 2.5 days, 3.0 days, 3.5 days, 4.0 days, etc. Then, fulfilling equation 24, we get the value as \( T^* = 0.0880 \) years, \( b(T^*) = 521 \) units/day and \( T^*_n = 0.330 \) years, and by applying the solution procedure of Section 5, we find the value of \( TC_1(T^*) \) from equation 22 as \( TC_1(T^*) = 1923.41\$. In other words, the total cycle time will be 0.0880 years if the maximum demand in a day will be 521 units, then the total cost of the system will be 1923.41\$. We have used software Mathematica for deriving the results. Now, we conclude that both warehouses will be empty after 0.570 years as RW vanishes at 0.330 years.

**Example 2.** Consider that \( A = 400$/order, \( h_o = 60$/unit, \( h_r = 70$/unit, \( \theta = 0.8, \alpha = 0.05, \beta = 3.00, M = 0.9 \) years, \( N = 0.8 \) years, \( R = 10 \) units/weeks, \( x = 50 \) units/days, \( s = 10$/order, \( \delta = 0.09, T_e = 80$/order, \( c = 50$/unit, \( I_c = 0.15$/year, I_o = 0.14$/year, W = \) capacity of OW = 1000 units. We take time values for different numbers of cycles, i.e., \( T_1, T_2, T_3, T_4, T_5 \), respectively, as 2.5 days, 3.0 days, 3.5 days, 4.0 days, 4.5 days, and so on. Then, by solving equation 27, we obtain the value of as \( T^* = 0.0927 \) years and \( b(T^*) = 570 \) units/days and \( T^*_n = 0.412 \) years. By utilizing the solution procedure from Section 5, we derive the value of \( TC_2(T^*) \) from equation 25 as \( TC_2(T^*) = 2152.00\$. In other words, the total cycle time will be 0.627 years if the maximum demand in a day will 570 units, then the total cost of the system will be 2152.00\$. We have employed software Mathematica for deriving the results and we conclude that both the two warehouses will be empty after 0.452 years as RW vanishes at 0.412 years.

**Example 3.** Let us choose \( A = 400$/order, \( h_o = 60$/unit, \( h_r = 70$/unit, \( \theta = 0.8, \alpha = 0.05, \beta = 3.00, M = 0.9 \) years, \( N = 0.6 \) years, \( R = 10 \) units/weeks,
\(x = 50\) units/days, \(s = 10\$/order, \(\delta = 0.06, T_c = 80\$/order, c = 50\$/unit, I_e = 0.15\$/year, I = 0.14\$/year, W = \text{capacity of OW} = 1500\) units. We further presume time values for different numbers of cycles, i.e., \(T_1, T_2, T_3, T_4, T_5\), respectively, as 3.0 days, 3.5 days, 4.0 days, 4.5 days, 5.0 days, etc. Now, by evaluating equation 30, we find the value as \(T^* = 0.0875\) years, \(b(T^*) = 685\) units/days and \(T_n^* = 0.425\) years. Employing the aforementioned solution procedure of Section 5, we calculate the value of \(TC_3(T^*)\) from equation 28 by \(TC_3(T^*) = 2886.00\$. In other words, the total cycle time will be 0.0875 years if the maximum demand in a day will be 685 units, then the total cost of the system will be 2886.00$. We have used software Mathematica for deriving the results and we conclude that the warehouses will be void after 0.481 years as RW becomes 0 at 0.425 years.

**Remark:** Examples 1, 2 and 3 consist of the same parametric values, except for the values of OW capacity and trade-credit period. Based on the three examples, Example 1 shows the minimum total cost under a trade-credit condition. The values of parametric changes are perfectly shown in Table 2. From Table 2, it is observed that if the trade-credit period increases, the cycle length and the total cost decrease. This implies that the retailer should procure more quantity to avoid higher interest charge, which eventually results in higher profit. It is also noticeable that if the OW capacity increases, the total cost decreases. This also turns beneficial for the retailer under trade-credit condition.

**Table 2: Effect of change in capacity of OW and trade-credit period.**

| \(W\) | \(M\) | \(N\) | \(T^*\) | \(b(T^*)\) | \(TC\) | Case |
|---|---|---|---|---|---|---|
| 0.5 | 0.4 | 0.088 | 521 | 1923.41 | \(M \leq T\) | 
| 800 | 0.9 | 0.8 | 0.097 | 560 | 1979.05 | \(N \leq T < M\) |
| 0.6 | 0.9 | 0.109 | 582 | 1992.16 | 0 < \(T < N\) |
| 1000 | 0.9 | 0.8 | 0.0927 | 570 | 2152.00 | \(N \leq T < M\) |
| 0.6 | 0.9 | 0.0989 | 590 | 2203.12 | 0 < \(T < N\) |
| 1500 | 0.9 | 0.8 | 0.0857 | 647 | 2539.11 | \(N \leq T < M\) |
| 0.6 | 0.9 | 0.0875 | 685 | 2886.00 | 0 < \(T < N\) |

**6. Sensitivity analysis.** We now analyze the effects of changes in the system parameters \(A, s, c, M, N, I_e, R, h_o, h_r, \alpha, \beta, \delta, \theta\), depending on the optimal values of \(T^*, b(T^*)\) and the optimal cost \(TC\). The sensitivity analysis is performed by changing each of the parameter by +50\%, +20\%, -20\% and -50\%, addressing one parameter varied at a time and keeping the remaining parameters fixed at that moment. The computational results prepared for Example 1 are shown in Table 3.

On the basis of these results, the observations derived are taken into account to show the validity of our model. If we consider Examples 2 and 3, then, by applying the same procedure, we can conduct our sensitivity analysis with respect to the parameters, and we may represent it by tables. However, for not exceeding the length of the paper, we only state Example 1 here.

The following observations are made by us based on Table 3:

(i) As ordering cost \(A\) increases, the replenishment cycle time, \(T^*\), increases and the total optimal cost, \(TC(T^*)\), increases.

(ii) \(TC, T^*\) are more sensitive with regard to any change of the values of \(\alpha\) and \(\beta\).
Table 3: Sensitivity analysis for different parameters involved in Example 1.

| Parameter | % change | value | $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T^*$ | $h(T^*)$ | TC | % change of TC |
|-----------|----------|-------|-------|-------|-------|-------|-------|-----------|----|----------------|
| $A$       | +50      | 600   | 1.5   | ...   | ...   | 0.274 | 594   | 2284.24   |    | +36.64        |
|           | -20      | 320   | 1.5   | 2     | ...   | 0.240 | 573   | 2170.33   |    | +28.41        |
|           | -50      | 200   | 1.5   | 2.3   | 3     | 0.173 | 529   | 1981.07   |    | -0.85         |
| $h_a$     | +50      | 90    | 1.5   | ...   | ...   | 0.198 | 468   | 2478.21   |    | +27.53        |
|           | -20      | 72    | 1.5   | 2     | ...   | 0.183 | 450   | 2356.34   |    | +21.23        |
|           | -50      | 30    | 1.5   | 2.5   | 3     | 0.166 | 426   | 2087.60   |    | +9.39         |
| $h_r$     | +50      | 105   | 1.5   | ...   | ...   | 0.098 | 537   | 3798.26   |    | +47.07        |
|           | -20      | 84    | 1.5   | 2     | ...   | 0.082 | 522   | 3523.65   |    | +31.67        |
|           | -50      | 56    | 1.5   | 2.5   | ...   | 0.076 | 517   | 3247.43   |    | +23.81        |
| $\theta$  | +50      | 90    | 1.5   | ...   | ...   | 0.076 | 510   | 3068.11   |    | +12.37        |
|           | -20      | 72    | 1.5   | 2.5   | ...   | 0.076 | 502   | 2984.00   |    | +10.22        |
|           | -50      | 56    | 1.5   | 2.5   | 3     | 0.076 | 502   | 2984.00   |    | +10.22        |
| $\alpha$  | +50      | 150   | 1.5   | ...   | ...   | 0.098 | 537   | 3798.26   |    | +47.07        |
|           | -20      | 120   | 1.5   | 2     | ...   | 0.082 | 522   | 3523.65   |    | +31.67        |
|           | -50      | 80    | 1.5   | 2.5   | ...   | 0.076 | 517   | 3247.43   |    | +23.81        |
| $c$       | +50      | 75    | 1.5   | ...   | ...   | 0.076 | 537   | 3798.26   |    | +47.07        |
|           | -20      | 60    | 1.5   | 2     | ...   | 0.082 | 522   | 3523.65   |    | +31.67        |
|           | -50      | 45    | 1.5   | 2.5   | ...   | 0.076 | 517   | 3247.43   |    | +23.81        |
| $s$       | +50      | 150   | 1.5   | ...   | ...   | 0.098 | 537   | 3798.26   |    | +47.07        |
|           | -20      | 120   | 1.5   | 2     | ...   | 0.082 | 522   | 3523.65   |    | +31.67        |
|           | -50      | 80    | 1.5   | 2.5   | ...   | 0.076 | 517   | 3247.43   |    | +23.81        |
| $W$       | +50      | 1200  | 1.5   | ...   | ...   | 0.240 | 573   | 2170.33   |    | +28.41        |
|           | -20      | 960   | 1.5   | 2     | ...   | 0.240 | 573   | 2170.33   |    | +28.41        |
|           | -50      | 640   | 1.5   | 2.5   | ...   | 0.240 | 573   | 2170.33   |    | +28.41        |
| $x$       | +50      | 75    | 1.5   | ...   | ...   | 0.082 | 522   | 3523.65   |    | +31.67        |
|           | -20      | 60    | 1.5   | 2     | ...   | 0.082 | 522   | 3523.65   |    | +31.67        |
|           | -50      | 45    | 1.5   | 2.5   | ...   | 0.076 | 517   | 3247.43   |    | +23.81        |
(iii) For a larger value of unit selling price, $s$, and unit cost price, $c$, we get a smaller value of the optimal cycle time, $T^*$, and of the optimal annual total cost, $TC(T^*)$.

(iv) We can also see that under a higher value of the rate of interest earned, $I_e$, the annual total relevant cost, $TC(T^*)$, will be at a minimum.

(v) The holding costs in OW and RW, $h_o$ and $h_r$, increase, respectively, the cycle time, $T^*$, decreases, whereas the optimal annual total cost, $TC(T^*)$, increases. When $h_r$ increases, the total cost of the system rises more rapidly by an increase of $h_o$.

(vi) The values of $T^*$ and $b(T^*)$ increase when raising the value of the backlogging rate, $\delta$, and the value of total cost, $TC(T^*)$, decreases with increasing this value.

(vii) $TC$, $T$ and $t_1$ are less sensitive when changing the value of $s$.

(viii) The number of cycles increases, the net profit decreases, but the net profit per unit time is found to be maximal for $n = 1$. Consecutively, the net profit per unit time diminishes by increasing the number of cycles.

(ix) When the retailer’s warehouse capacity $W$ is increasing, then the optimal replenishment cycle time $T^*$ will decrease, and the relevant total costs, $TC(T^*)$, will diminish.

(x) When the probabilistic demand $x$ increases, the optimal replenishment cycle time $T^*$ and the relevant total costs, $TC(T^*)$, decrease together.

Based on Table 3, the following managerial insights are settled:

(a) When the trade-credit periods $M$ and $N$ increase, cycle lengths and optimal ordering period decrease, along with the total cost which decreases, too. This indicates that trade credit is beneficial for an economic ordering policy. Therefore, the retailer should procure less quantity to enjoy the benefit of trade credit. It is also remindful that the retailer should order less but more frequently if the interest payable rate is very high.

(b) As the values of $\alpha$ and $\beta$ are very sensitive with respect to total cost $TC$, therefore, it is advisable that the retailer should place small orders but more frequently under deteriorating conditions to avoid loss due to deterioration though the ordering cost can be high.

(c) When the capacity of OW rises, then the total cycle length increases as well as the total cost of the system decreases. This important fact is compatible with our assumption that the holding cost of RW is greater than the holding cost of OW.

(d) When ordering cost increases, the total cost of the system also increases. So, the manager should order a greater quantity at a time to avoid a higher charge against ordering cost, if the deterioration cost will be less than the ordering cost.

(e) It is also found that for low backorder cost, it will be beneficial for the inventory manager to offer the customers a high discount on backorders.

(f) It is really a noteworthy observation that when the rate of selling price is changing, this does not change the total cost effectively, but when the cost price rises, to minimize the total cost, the retailer is forced to order less.

(g) If the probabilistic demand rate increases, then the total cost decreases. So, to maintain the profit of the system, it will be an important managerial task for the retailer to increase the demand at any cost.
Table 3 shows the slight change of total cost with respect to respective parameters. But, in Table 4, the combined effect on total cost with respect to major (important) parameters are displayed. The following managerial insights are stated based on Table 4:

(h) It is worth mentioning that the total cost will be lowest if one considers the joint effect of trade-credit policy, probabilistic demand and partial backordering in compare to the case when this effect is assumed separately. It is really attractive to researchers and practitioners that the cycle length also diminishes when the joint effect is observed.

Table 4: Sensitivity analysis for the combined effect of total cost involved in Example 1.

| % change | x | M | N | δ  | \( T^* \) | \( b(T^*) \) | TC | % change of TC |
|----------|---|---|---|----|--------|-----------|----|----------------|
| +50      | 75| 0.725| 0.6 | 0.12 | 0.165  | 610       | 2685.00 | +25.16         |
| +40      | 70| 0.70 | 0.56 | 0.112| 0.137  | 589       | 2576.27 | +32.00         |
| +30      | 65| 0.65 | 0.52 | 0.104| 0.114  | 576       | 2450.08 | +23.76         |
| +20      | 60| 0.6  | 0.48 | 0.096| 0.105  | 558       | 2329.18 | +21.34         |
| +10      | 55| 0.55 | 0.44 | 0.088| 0.098  | 543       | 2249.71 | +19.47         |
| 0        | 50| 0.5  | 0.4  | 0.08 | 0.089  | 522       | 1923.46 | ...            |
| -10      | 45| 0.45 | 0.36 | 0.072| 0.068  | 516       | 1879.34 | +13.32         |
| -20      | 40| 0.4  | 0.32 | 0.064| 0.062  | 511       | 1794.11 | -11.06         |
| -30      | 35| 0.35 | 0.28 | 0.056| 0.058  | 504       | 1720.57 | +5.48          |
| -40      | 30| 0.30 | 0.24 | 0.048| 0.051  | 497       | 1685.00 | -2.21          |
| -50      | 25| 0.25 | 0.2  | 0.04 | 0.048  | 483       | 1649.27 | -0.23          |

Figure 3 shows the required inventory model with respect to replenishment cycle length \( T \), maximum demand \( b(T) \) and total cost \( TC \), based on Example 1. The function displayed in Figure 3 is a strictly convex function; this is compatible with our assumptions. Furthermore, Figure 3 expresses that the cost function is a strictly convex function with respect to replenishment cycle length \( T \) and maximum demand \( b(T) \). The value of the cost function displayed in Figure 3 is the minimum value, and it is also satisfied by the value of Example 1. Further indication of Figure 3 are validity and stability of our proposed model. Figure 4 and Figure 5 exhibits the convexity nature of total cost when the joint effect of partial backorder, probabilistic demand and trade credit is regarded. Here, the changes of the total cost with respect to the corresponding parameters are shown by the following figures (cf. Figure 6- Figure 9). We would like to point out that furthermore a careful analytical Sensitivity Analysis can be conducted in the ways presented, e.g., in the works such as Bank et al. [2], who presented a variety of results and methods on nonlinear parametric optimization. Jongen and Weber [19] treated a general model class from parametric nonlinear programming. Weber [39] defined an optimization model on a goal in integral form and investigated it by the topology of parametric optimal control.

7. Concluding remarks and future research directions. In our paper, we have considered a two-warehouse inventory model with probabilistic demand and price discount on backorders under two levels of trade-credit policy. Here, we have included two kinds of deteriorations which are different in an OW, compared to an RW, due to their different preservation environments: a constant deterioration in an OW and a Weibull-distribution deterioration in an RW. Since the demand function of a major part of items is not fixed, we have interpreted the demand function in
Figure 3. Convexity of the function of total cost.

Figure 4. Convexity nature of total cost in case of joint effect.
Figure 5. Convexity nature of total cost in case of joint effect.

Figure 6. Variation of total cost $TC$ with respect to demand factor $x$. 
a probabilistic sense. Actually, not only the retailer but also the supplier offers a trade-credit policy to stimulate the sales and earning of revenue. Moreover, shortages are allowed and during that period, price discount on backorders is permitted to attract customers for more backorders and more profit.

In addition, the holding cost in an RW is greater than in an OW for a better preserving facility for deteriorating items. As a result, the rate of deterioration in an RW is less than that of an OW. In order to reduce the total inventory cost, the stocks of an RW are cleared first; then, the stocks from an OW are emptied. Due to the preserving condition of warehouses, items which are stored in a warehouse gradually lose their utility. Thus, deterioration can take place, and this realistic phenomenon is incorporated within our model. As it is impossible for a businessman to determine the demand for any item, a stochastic demand function has been included here. Stochastic demand function leads to a variability of demand, which implies a high potential of possible demand and also achieves a higher cost reduction. That is why the stochastic demand function makes our model a valuable one.

From our sensitivity analysis, we observed that as the number of cycles increased, and the total relevant cost corresponding to the cycles increased, too, whereas the net profit of the system decreased. From our analysis, we also concluded that the net profit per unit time is at a maximum when \( n = 1 \); so we stated that the net profit of the system is diminished when enhancing the number of cycles. Furthermore, we observed that for a higher rate of selling price, if the retailer wants a benefit from the trade-credit policy, he or she has to order less. The retailer should get a higher

\[ \text{Figure 7. Variation of total cost } TC \text{ with respect to shape parameter } \alpha. \]
Figure 8. Variation of total cost $TC$ with respect to ordering cost $A$.

Figure 9. Variation of total cost $TC$ with respect to OW capacity $W$. 
benefit from the permissible delay if he or she earned a large rate of interest from the trade-credit policy. We have seen that when the holding cost of the total system is increasing; to maintain a profit of the system, the retailer shortens the cycle time and reduces the order quantity at a time. It will be very profitable for the retailer to have an own warehouse with unlimited capacity. When shortages occur, a company may lose profit from some disappointed customers; however, for the well-behavior and reputation of the company, some customers prefer to wait until a backorder can be made. We also found out that at low backorder cost, it will be beneficial for the inventory manager to offer the customers a high discount on backorders. Moreover, it is disclosed by our analysis that backorder discount on price may create a good effect on customers during a stock-out period. Thus, the company can earn some profit. We have seen that for a higher value of the retailer’s warehouse capacity, he or she gets a higher value of the total cost. This implies that if a retailer orders a quantity less, but more frequently, this reduces the costs when the retailer has an own warehouse.

Consequently, our analysis of the numerical examples demonstrates that ordering cost and holding cost are the core factors for increasing the total cost of an inventory system.

To the very best of our knowledge, this is the first work on two-warehouse policy with probabilistic demand and price discount on backorders under the effect of trade-credit policy. So, our investigation may have a crucial impact on the selling and pricing model in this modern age. Our model is very much applicable for food grains like rice, paddy, wheat, etc., as the demand for these considered food grains increases with respect to time for a fixed-time planning horizon. It is applicable for other items where the demand is totally and linearly dependent with time. This model also helps the inventory management to deal with items having a probabilistic demand. Our contribution can be applied for seasonable and fashionable goods which are marketed within a fixed-time period.

As future studies, we can construct an extended model by introducing stochastic inflation, deteriorating cost, stock-dependent demand and permissible delay in payments. Adding some multi-raw materials, multi-product or defective production rate can transfer our model to an integrated model will be a possible and interesting extension. In addition to the above advances, our model can provide an interesting research area with an impact on selling defective items at a lower price on demand. Further, we can advance the work by collecting some real data from a particular industry, and applying our new technique on them. This can also allow for some more realistic managerial insights, based on involving actual data. Eventually, we would like to analytically evaluate the optimization problem of this paper in terms of sensitivity or stability of the optimal solutions and of any additional Kuhn-Tucker points, both in the parametric (univariate) sense and in the multivariate sense. We welcome the interested scientific community to this and related research and application.

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A TWO-WAREHOUSE PROBABILISTIC MODEL

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