SPECIAL LANGRANGIAN GEOMETRY AND SLIGHTLY DEFORMED ALGEBRAIC GEOMETRY (SPLAG AND SDAG)

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Abstract. The special geometry of calibrated cycles, closely related to mirror symmetry among Calabi–Yau 3-folds, is itself a real form of a new subject, which we call slightly deformed algebraic geometry. On the other hand, both of these geometries are parallel to classical gauge theories and their complexifications. This article explains this parallelism, so that the appearance of invariants of new type in complexified gauge theory (see [D-T] and [T]) can be accompanied by analogous invariants in the theory of special Lagrangian cycles, for which the development is at present much more modest than in gauge theory. We discuss related geometric constructions, arising from mirror symmetry and symplectic geometry.

§1. spLag cycles

We begin by recalling the geometric construction for a pair \( L \subset S \), where \( S \) is a smooth symplectic manifold of dimension \( 2n \) with a given tame almost complex structure \( I \), and \( L \subset S \) a smooth, oriented Lagrangian submanifold (of maximal dimension \( \dim L = n = \frac{1}{2} \dim S \)); this is now quite popular in the set-up of Calabi–Yau threefolds. The structure on \( S \) is an almost Kähler structure, and we say for short that \( S \) is an \( aK \) manifold. Write \( \omega \) for the symplectic form and \( I \) for the almost complex structure on \( S \), giving the Hermitian triple \((\omega, I, g)\).

We now define the Lagrangian Grassmannian \( \Lambda^+(S) = \Lambda^+(TS_p) \) to be the Grassmannian of maximal oriented Lagrangian subspaces in \( TS_p \). Taking this space over every point of \( S \) gives the oriented Lagrangian Grassmannisation of \( TS \)

\[
\pi: \Lambda^+(S) \to S \quad \text{with} \quad \pi^{-1}(p) = \Lambda^+ p.
\]  

(1.1)

Our tame almost complex structure on \( S \) gives each fibre the standard form

\[
\Lambda^+ p = \text{U}(n)/\text{SO}(n)
\]  

(1.2)

This space admits a canonical map

\[
\det: \Lambda^+ p \to \text{U}(1) = S^1_p \quad \text{sending} \ u \in \text{U}(n) \ \text{to} \ \det u \in \text{U}(1) = S^1.
\]  

(1.3)
Recall that the inverse image of the fundamental class of $S^1$ on $\Lambda_\ell p$ is the *universal Maslov class*. Taking this map over every point of $S$ gives the map

$$\det: \Lambda_\ell S \to S^1(L-K), \quad (1.4)$$

where $S^1(L-K)$ is the unit circle bundle of the line bundle $\wedge^n TS = \det TS$, with first Chern class

$$c_1(\det TS) = -K_S, \quad (1.5)$$

where $K_S$ is the canonical class of $S$. Recall that, as a cohomology class, $K_S$ does not depend on the compatible almost complex structure.

Now for every oriented Lagrangian cycle $\mathcal{L} \subset S$, we have the Gauss lift of the embedding $i: \mathcal{L} \to S$ to a section

$$G(i): \mathcal{L} \to \Lambda_\ell S|_{\mathcal{L}} \quad (1.6)$$

sending a point $p \in \mathcal{L}$ to the oriented subspace $T\mathcal{L}_p \subset TS_p$. The composite of this Gauss map with the projection (1.4) gives the map

$$\det \circ G(i): \mathcal{L} \to S^1(L-K)|_{\mathcal{L}} \quad (1.7)$$

Now suppose that the cohomology class of the symplectic form is proportional to the canonical class of $S$, that is,

$$\kappa \cdot [\omega] = K_S \quad \text{for some } \kappa \in \mathbb{Q}; \quad (1.8)$$

then the restriction $\det TS|_{\mathcal{L}}$ is topologically trivial, because the restriction of $[\omega]$ to a Lagrangian $\mathcal{L}$ is zero. Moreover, suppose that $K_S$ has a Hermitian connection $a_K$ with curvature form

$$F_{a_K} = (2\pi i)\omega,$$

where $\omega$ is our symplectic form. Then this connection restricts to a flat connection on $\mathcal{L}$. Let

$$\langle \mathcal{L} \rangle = \{ B \subset S \mid \mathcal{L} \subset B, \pi_1(B) = 1 \text{ and } F_{a}|_{B} = 0 \}$$

be the maximal simply connected submanifold containing $\mathcal{L}$ on which $K_S$ and $a$ restrict to a trivial bundle and a flat connection.

Then on $\langle \mathcal{L} \rangle$ there exists a canonical trivialisation

$$S^1(L-K)|_{\langle \mathcal{L} \rangle} = \langle \mathcal{L} \rangle \times S^1 \quad (1.9)$$

that preserves the Hermitian form and the canonical projection

$$\text{pr}: S^1(L-K)|_{\mathcal{L}} \to S^1. \quad (1.10)$$

Now composing the maps (1.4), (1.6) and (1.10) gives a map

$$m = \text{pr} \circ \det \circ G(i): \mathcal{L} \to S^1 \quad (1.11)$$
Definition 1.1. (1) \( m \) is called the phase map;
(2) \( \mathcal{L} \) is called a special Lagrangian cycle of \( S \) (spLag cycle for short) if \( m(\mathcal{L}) \) is a point or, equivalently, the differential of \( m \) vanishes:

\[
dm = 0. \tag{1.12}
\]

Remark. In this definition, we call \( \mathcal{L} \) a cycle rather than a submanifold, because it may be singular. We really only need the Gauss map (1.6) to be well defined; thus \( \mathcal{L} \) can have nodes, and so on. Thus below we call a cycle any subvariety with regular Gauss map.

Mirror digression. (1.8) holds automatically if \( S \) is a Calabi–Yau manifold, that is, \( K_S = 0 \). In other words, for any Lagrangian submanifold \( \mathcal{L} \)

\[
\langle \mathcal{L} \rangle = S, \tag{1.13}
\]

and we have the map \( m : \mathcal{L} \to S^1 \) (1.11). In this case, the notion of spLag cycle coincides with that in calibrated geometry (see [H-L]).

Recall that a complex orientation of a Calabi–Yau manifold \( X \) is a choice of trivialisation of the canonical line bundle \( L_K \), that is, a holomorphic 3-form \( \theta \). For an oriented Calabi–Yau threefold \( (X, \theta) \), a spLag cycle is a 3-dimensional Lagrangian submanifold \( \mathcal{L} \) such that the restriction

\[
\text{Re} \theta|_{\mathcal{L}} = 0. \tag{1.14}
\]

The local deformation theory of such submanifolds is well understood. The tangent space to the moduli space \( \mathcal{M}_\mathcal{L} \) of such deformations at a submanifold \( \mathcal{L} \) is \( H^1(\mathcal{L}, \mathbb{R}) \), viewed as the space of harmonic 1-forms on \( \mathcal{L} \). This space doesn’t depend on the second quadratic form or others attributes of embeddings. In particular, if \( H^1(\mathcal{L}, \mathbb{R}) = 0 \) then \( \mathcal{L} \) is rigid as a special Lagrangian submanifold. So we can expect that there exists a finite set of such submanifolds \( \{\mathcal{L}_1, \ldots, \mathcal{L}_N\} \) in one cohomology class \([\mathcal{L}_i]\). This subject is quite popular now, and we would like to remark that this construction also works for Fano varieties.

Chern–Simons digression. Let \( \Sigma \) be a compact, smooth, oriented Riemann surface of genus \( g \), and

\[
\pi_1(g) = \langle a_1, \ldots, a_g, b_1, \ldots, b_g \mid \Pi_{i=1}^g [a_i, b_i] = \text{id} \rangle \tag{1.15}
\]

the usual presentation of the fundamental group \( \pi_1(g) \).

The space \( \text{Rep} \pi_1(g) \) is the moduli space of classes of \( \text{SU}(2) \) representations of \( \pi_1(g) \), and is smooth as an orbifold. But, as a manifold, it is singular at the reducible representations

\[
\text{Sing} \text{Rep} \pi_1(g) = \text{Rep} \pi_1(g)^{\text{red}}. \tag{1.16}
\]

The space \( \text{Rep} \pi_1(g) \) contains the special subspace of representations that are trivial on the \( b_i \)

\[
B = \{ \rho \in \text{Rep} \pi_1(g) \mid \rho(b_i) = \text{id} \text{ for } i = 1, \ldots, g \}. \tag{1.17}
\]
As usual,
\[ B^\text{irr} \subset \text{Rep} \pi_1(g)^\text{irr} \]
is the subset of irreducible representations.

Then \( B \) is a Lagrangian suborbifold of \( \text{Rep} \pi_1(g) \) with respect to the canonical symplectic form \( \omega_\Sigma \) (1.26). To apply the phase map to this Lagrangian submanifold \( \mathcal{L} = B \) in \( S = R_g \) we have to remark that
\[ [\omega_\Sigma] = -4 \cdot K_{R_g} \]
(see [N-R], [R]).

Now fixing a complex structure on \( \Sigma \) gives a complex structure on \( R_g \), the Weil–Peterson metric on \( R_g \), which is Kählerian with symplectic form \( \omega_\Sigma \) and a unitary connection on the anticanonical bundle with curvature form \( i \cdot \pi \omega_\Sigma \) and the phase map
\[ m_g: B^\text{irr} \to S^1 = U(1) \]
which can be investigated in the usual way (as in [A-M]).

Bogomolov remarked that there exists a complex structure on \( \Sigma \) for which \( B \) is a spLag cycle. Indeed, consider the complex structure on \( \Sigma \) of genus 2 such that all the cycles \( b_i \) are real. For example if \( \Sigma \) is a double of a pair of pants:

we get a compact complex Riemann surface of genus 2 with a real structure such that cycles \( b_1 \) and \( b_2 \) are real. Then the usual argument implies that the cycle \( B \) must be real. Then one can see that \( B \) is spLag (see 1.14).

It is easy to see that \( B \) is rigid as a spLag cycle. The following question is very interesting:

**Question.** Is there another spLag cycle \( B' \) in the same homology class \( [B] = [B'] \)?

Now there is an absolutely canonical almost Kähler structure on \( R_g \), constructed by Guruprasad and Nilakantan. To describe it, we use the standard presentation (1.15) of \( \pi_1(g) \) and another “dual” presentation given by the following construction (see [G-N]): let
\[ r_i = \Pi_{j=1}^i [a_j, b_j], \quad \text{and set} \quad \alpha_i = r_{i-1} b_i^{-1} r_i^{-1} \quad \text{and} \quad \beta_i = r_i a_i^{-1} r_i^{-1}. \] (1.21)

Then
\[ \pi_1(g) = \langle \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g \mid \Pi_{j=1}^g [a_j, b_j] = 1 \rangle \]
(1.15)'
is another presentation of \( \pi_1(g) \). Sending the generators \( a_i, b_j \) to \( \alpha_i, \beta_j \) gives an involutive automorphism \( W \) of \( \pi_1(g) \), that is, and element \( W \in \text{Mod}_g \) with \( W^2 = \text{id} \).

To describe the tangent space \((TR_g)_\rho\), recall that a SU(2) representation \( \rho \) makes the Lie algebra \( \mathfrak{su}(2) \) into a \( \pi_1(g) \)-module \( \mathfrak{su}(2)_\rho \) by the adjoint action, and, as the tangent space to an orbifold point,

\[
(TR_g)_{[\rho]} = H^1(\pi_1(g), \mathfrak{su}(2)_\rho).
\] (1.22)

The group of cycles of the module \( \mathfrak{su}(2)_\rho \) is the group of skew homomorphisms from \( \pi_1(g) \) to this module, that is

\[
Z^1(\mathfrak{su}(2)_\rho) = \{ u: \pi_1(g) \to \mathfrak{su}(2)_\rho \mid u(g_1 \circ g_2) = u(g_1) + g_1(u(g_2)) \}.
\] (1.23)

Of course, any such \( u \) extends to a \( \mathbb{Z} \)-linear map of the integral group algebra:

\[
u: \mathbb{Z}\pi_1(g) \to \mathfrak{su}(2),
\] (1.24)

and the boundary subspace is

\[
B^1(\mathfrak{su}(2)_\rho) = \{ u: \pi_1(g) \to \mathfrak{su}(2)_\rho \mid u(g) = g(v) - v \text{ for some } v \in \mathfrak{su}(2) \}.
\] (1.25)

Then

\[
(TR_g)_{[\rho]} = Z^1(\mathfrak{su}(2)_\rho)/B^1(\mathfrak{su}(2)_\rho).
\]

The canonical symplectic structure \( \omega_\Sigma \) on \( R_g \) can be defined as a skewsymmetric bilinear form on the tangent space of every class of representations \([\rho]\) by the PU(2) invariant bilinear form on \( Z^1(\mathfrak{su}(2)_\rho) \) given by

\[
\langle u, v \rangle = \sum_{i=1}^g \left[ (u(r_{i-1}^{-1} - b_{i-1}^{-1} \cdot r_i^{-1}), v(a_i)) + (u(a_i^{-1} r_{i-1}^{-1} - r_i^{-1}), v(b_i)) \right]
\] (1.26)

where \(( , )\) is the standard inner product \((m, m) = -\text{tr} m^2\) on \( \mathfrak{su}(2) \).

The inner product on the space of cycles \( Z^1(\mathfrak{su}(2)_\rho) \) (1.23) is given by the formula

\[
G(u, v) = \sum_{i=1}^g \left[ (u(\alpha_i), v(\alpha_i)) + (u(\beta_i), v(\beta_i)) \right]
\] (1.27)

(see [G-N]).

**Proposition 1.1** [G-N]. 1) The inner product (1.27) is positive definite.

2) The restriction of the inner product (1.27) to the orthogonal

\[
B^1(\mathfrak{su}(2)_\rho) = (TR_g)_{[\rho]}^{\perp}
\] (1.28)

defines a special Riemannian metric on \( R^\text{irr}_g \) and on \( R_g \) as an orbifold.
**Definition 1.2.** This metric is called the *Guruprasad metric* (*GN metric* for short).

It can be checked (see [G-N], Proposition 3.1) that the GN metric is compatible with the symplectic form (1.26); this canonical Hermitian structure and the canonical connection on the tangent bundle (the Levi–Civita connection) induce a Hermitian structure and a connection on the determinant line bundle \( L_{-K} \) whose curvature coincides with the symplectic form. Thus the phase map

\[
m_G: B \to S^1
\]

is absolutely canonical.

Now we return to the general phase map.

There is a standard way of describing the differential of the phase map: let

\[
\nabla_{LC}: \Gamma(TS) \to \Gamma(TS \otimes T^*S)
\]

be the Levi–Civita connection, which we restrict to the restriction of the tangent bundle to a Lagrangian cycle \( \mathcal{L} \). Let \( N_{\mathcal{L} \subset S} \) be the normal bundle of \( \mathcal{L} \) in \( S \). Then the Levi–Civita connection (1.30) defines connections

\[
\nabla_{LC}: \Gamma(T\mathcal{L}) \to \Gamma(T\mathcal{L} \otimes T^*\mathcal{L})
\]

\[
\nabla_{LC}: \Gamma(N_{\mathcal{L} \subset S}) \to \Gamma(N_{\mathcal{L} \subset S} \otimes T^*\mathcal{L})
\]

and the tensor

\[
\Pi: T\mathcal{L} \to \text{Hom}(T\mathcal{L}, N_{\mathcal{L} \subset S})
\]

(called the second quadratic form), that is,

\[
\Pi \in \text{End} T\mathcal{L} \otimes N_{\mathcal{L} \subset S}.
\]

The trace component of \( \text{End} T\mathcal{L} = \mathbb{C} \oplus \text{ad} T\mathcal{L} \) gives a section

\[
H \in \Gamma(N_{\mathcal{L} \subset S}),
\]

called the mean curvature section.

Let \( V \) be a vector field on \( \mathcal{L} \). Then pointwise, the value of the differential \( dm_G \) on \( V \) is given by the inner product

\[
dm_G(V_p) = (H_p, I_G(V_p)),
\]

where \( I_G \) is the operator of the Guruprasad almost complex structure.

§2 sdAG cycles (slightly deformed Algebraic Geometric cycles)

The construction of the phase map can be “complexified” to give the complex phase map

\[
m_C: \Sigma \to \mathbb{C}^*
\]

for any middle dimensional cycle \( \Sigma \subset S \), where \( S \) is an almost Kähler manifold.
We illustrate this construction in the case of an aK surface \( S \) (see the beginning of §1). Then over any point \( p \in S \) we have as the fibre of the tangent bundle \( TS_p = \mathbb{C}^2 \) with the standard Hermitian triple \( (\omega_p, I_p, g_p) \), and the oriented Lagrangian Grassmannian is a subspace of the oriented Grassmannian

\[
\Lambda_{\uparrow}p \subset \text{Gr}_\uparrow(2, TS_p). \tag{2.2}
\]

Now we saw that

\[
\Lambda_{\uparrow}p = S^2 \times S^1 \tag{2.3}
\]

(see (1.2)), and

\[
\text{Gr}_\uparrow(2, TS_p) = S^2_- \times S^2_+, \tag{2.4}
\]

so that what we need to see is that the inclusion (2.2) is componentwise, that is,

\[
S^2 = S^2_- \quad \text{and} \quad S^1 \subset S^2_+ \tag{2.5}
\]

and the first identification and the second inclusion carry deep geometric meaning in the theory of Riemannian twistors.

First of all, every oriented plane \( V \subset TS_p \) defines a new complex structure \( I_V \) on \( TS_p \) which is compatible with the Euclidean metric \( g_p \). Thus we have the natural map

\[
I: \text{Gr}_\uparrow(2, TS_p) \to S^2 = \mathbb{P}W^+_p, \tag{2.6}
\]

where \( W^+ = \mathbb{C}^2 \) is the positive spinor space of the Euclidean metric.

The fibre of this map

\[
I^{-1}(I_V) = \mathbb{P}C^2_V = \mathbb{P}W^+_p \tag{2.7}
\]

is the \( I_V \) projective line – that is, complex directions in the complex plane with complex structure \( I_V \). It is easy to see that the decomposition (2.4) is precisely the twistor decomposition

\[
\text{Gr}_\uparrow(2, TS_p) = \mathbb{P}W^-_p \times \mathbb{P}W^+_p. \tag{2.8}
\]

Now for any oriented plane \( V \), consider its \textit{Kähler angle} \( \alpha(V) \), given by the formula

\[
\omega_p|_V = \arccos \alpha \cdot \text{Vol}_{g_p}. \tag{2.9}
\]

The geometric meaning of \( \alpha(V) \) is following: it is the angle between \( V \) and \( I_p(V) \). Sending an oriented plane to the Kähler angle gives a map

\[
\alpha_p: \text{Gr}_\uparrow(2, TS_p) \to [0, \pi] \tag{2.10}
\]

of our Grassmannian to the interval.

The angle \( \alpha(V) \) only depends on the complex structure \( I_V \), so that the map \( \alpha_p \) is the composite

\[
\text{Gr}_\uparrow(2, TS_p) \xrightarrow{pr^+_p} \mathbb{P}W^+_p \xrightarrow{h} [0, \pi] \tag{2.11}
\]

and the last map

\[
h: \mathbb{P}W^+ = S^2 \to [0, \pi] \tag{2.12}
\]
is the standard height function on $S^2$ sending the standard sphere in Euclidean $\mathbb{R}^3$ to the third coordinate:

The fibres of the Kähler angle map $\alpha$ have the following geometric interpretation:

\[ \alpha^{-1}(0) = \mathbb{P}\mathbb{C}^2 = \mathbb{P}W^- \]  \hspace{1cm} (2.13)

is the space of complex tangent directions at a point $p$. In the same vein,

\[ \alpha^{-1}(\pi) = \overline{\mathbb{P}W}^- \]  \hspace{1cm} (2.14)

is the space of complex directions at a point $p$, but with the opposite orientation. Now

\[ \alpha^{-1}(\pi/2) = \Lambda_{\uparrow}p \]  \hspace{1cm} (2.15)

is the Lagrangian Grassmannian (1.1). It is easy to see that the involution changing orientation sends $\alpha$ to $\pi - \alpha$ and preserves the oriented Lagrangian Grassmannian (2.15). So the decomposition of the Lagrangian Grassmannian (2.3) admits the following geometric meaning:

\[ \Lambda_{\uparrow}p = \mathbb{P}W^- \times h^{-1}(\pi/2). \]  \hspace{1cm} (2.16)

So we can see that, in the same way that the fibres of the height map $h$ are the same ($= S^1$) except for two points, the north and south poles, the fibres of the Kähler angle map are the same except for the two spheres

\[ (\mathbb{P}W^- \times 0) \cup (\mathbb{P}W^- \times \pi) \]  \hspace{1cm} (2.17)

Thus, removing these exceptional fibres, we get the map

\[ \text{det}_C: \text{Gr}_{\uparrow}(2, TS_p) \setminus (\alpha^{-1}(0) \cup \alpha^{-1}(\pi)) \xrightarrow{\text{pr}} \mathbb{C}^*, \]  \hspace{1cm} (2.18)

whose restriction to the Lagrangian Grassmannian $\Lambda_{\uparrow}p$ gives the determinant map $\text{det}$ of (1.3):

\[ \text{det}_C |_{\Lambda_{\uparrow}p} = \text{det}. \]  \hspace{1cm} (2.19)
Moreover, the image of the last map is the circle
\[ S^1 = \alpha^{-1}_p(\pi/2) \subset \mathbb{C}^* \subset \mathbb{P}W^+. \] (2.20)

Now we can give another construction for the map \( \det_\mathbb{C} \): we fix any nonzero element \( w \in \Lambda^2 \mathbb{C}^2 \), and call it a complex orientation of \( \mathbb{C}^2 \). Let us fix a standard orthogonal basis \( e_1, e_2, I(e_1), I(e_2) \) in \( \mathbb{C}^2 \) such that
\[ w = (e_1 + I(e_1)) \wedge (e_2 + I(e_2)). \] (2.21)

Now for any oriented plane \( V \in \text{Gr}_\mathbb{C}(2, TS_p) \) consider the oriented orthogonal basis \( v_1, v_2 \) and let \( M_V \) be the linear transformation sending \( e_i \) to \( v_i \) and \( I(e_i) \) to \( I(v_i) \).

It is easy to check the following result: (see for example [H-L])

**Lemma 2.1.**
1) \( M_V \) is \( \mathbb{C} \)-linear transformation \( TS_p \rightarrow TS_p \) that is
\[ M_V \in \text{End}_\mathbb{C}(\mathbb{C}^2) \] (2.22)
is a complex matrix;
2) \( \det_\mathbb{C} = 0 \) iff \( V \) is a complex or an anticomplex direction and
\[ V = \ker M_V; \] (2.23)
3) \( V \) is Lagrangian iff \( M_V \in U(2) \) is unitary matrix.

So sending \( V \) to \( \det M_V \) we get the map
\[ d: \text{Gr}_\mathbb{C}(2, TS_p) \rightarrow \mathbb{C}, \] (2.24)
and it is easy to see the following:

**Lemma 2.2.**
1) \[ |d(V)| \leq 1 \] (2.25)
and
\[ |d(V)| = 1 \iff V \in \Lambda^1_p; \] (2.26)
2) let
\[ V \rightarrow \overline{V} \] (2.27)
be the change of orientation map then
\[ d(V) = -d(\overline{V}). \] (2.28)

Thus the map
\[ d^2: \text{Gr}_\mathbb{C}(2, TS_p) \rightarrow \mathbb{C} \] (2.29)
is the orientation-forgetting map.

Now consider the decomposition
\[ \text{Gr}_\mathbb{C}(2, TS_p) = \text{Gr}_\mathbb{C}(2, TS_p)^+ \cup d^{-1}(S^1) \cup \text{Gr}_\mathbb{C}(2, TS_p)^- \] (2.30)
where
\[ \text{Gr}_\mathbb{C}(2, TS_p)^- = \overline{\text{Gr}_\mathbb{C}(2, TS_p)^+}. \] (2.31)

Then we have two maps
\[ d: \text{Gr}_\mathbb{C}(2, TS_p)^+ \rightarrow \{|z| \leq 1\} \subset \mathbb{C} \] (2.32)
\[-(d)^{-1}: \text{Gr}_\mathbb{C}(2, TS_p)^- \rightarrow \{|z| \geq 1\}\] (2.33)
both of whose restrictions to \( d^{-1}(S^1) \) coincide:
\[ d|_{d^{-1}(S^1)} = (d)^{-1}|_{d^{-1}(S^1)}. \] (2.33)

Thus we get:
Lemma 2.3. 1) These two maps glue to a map

$$\det_C: \text{Gr}_1(2, TS_p) \to \mathbb{P}^1 = S^2. \quad (2.34)$$

2) This map is smooth and equal to the map $pr_+$ of (2.11).

Recall that by the theory of Riemannian twistors, the real structure acts on the projective twistor space $\mathbb{P}W^+$ as the antipodal map $z \to -(z)^{-1}$.

**Globalisation.** Now we have to globalise these constructions, just as we did with the oriented Lagrangian (1.1). First of all, we have the Grassmannisation of the tangent bundle of $S$

$$\text{Gr}_1(2, TS) = \mathbb{P}W^- \times_S \mathbb{P}W^+, \quad (2.35)$$

the bundle of complex quadrics on $S$ which fibrewise is the product of twistor spaces. Of course this bundle contains the Lagrangian Grassmannisation (1.1)

$$\Lambda^\dagger \subset \text{Gr}_1(2, TS), \quad (2.36)$$

and there exists a natural projection

$$\text{Gr}_1(2, TS) \xrightarrow{pr_+} \mathbb{P}W^+. \quad (2.37)$$

The Kähler angle map

$$\alpha: \text{Gr}_1(2, TS) \to [0, \pi] \quad (2.38)$$

(which fibrewise is the map (2.10)) factors as the composite

$$\text{Gr}_1(2, TS) \xrightarrow{pr_+} \mathbb{P}W^+ \xrightarrow{\alpha} [0, \pi] \quad (2.39)$$

and

$$\Lambda^\dagger = \alpha^{-1}(\pi/2). \quad (2.40)$$

Now the spinor bundle $\mathbb{P}W^+$ is the projectivisation

$$\mathbb{P}W^+ = \mathbb{P}(\mathcal{O}_S \oplus \mathcal{O}_S(-K_S)). \quad (2.41)$$

In particular, this projective bundle contains the unitary circle bundle of the anti-canonical line bundle

$$S^1(L_{-K}) \subset \mathbb{P}W^+ \quad (2.42)$$

and the restriction to $\Lambda^\dagger$ of the $pr_+$ map of (2.39) gives the map (1.4).

Now for any oriented 2-dimensional submanifold $\Sigma \subset S$ one can consider the Gauss lift of the embedding $i: \Sigma \to S$ to the section

$$G(i): \Sigma \to \text{Gr}_1(2, TS)|_\Sigma = \mathbb{P}W^- \times_S \mathbb{P}W^+|_\Sigma, \quad (2.43)$$

sending a point $p \in \Sigma$ to the oriented subspace $T\Sigma_p \subset TS_p$. The composite of this Gauss map with the projection $pr_+$ (2.11) defines the map

$$\det_C: \Sigma \to \mathbb{P}W^+|_\Sigma. \quad (2.44)$$

Now let $a_{LC}$ be the Levi–Civita connection on the line bundle $L_{-K}$ and

$$F_{LC} \in \Omega^2(S) \quad (2.45)$$

its curvature, viewed as a 2-form on $S$. 
Definition 2.1. A cycle $\Sigma$ is called \textit{canonically flat} if there exists a simply connected submanifold $B \supset \Sigma$ such that

$$F_{LC}|_B = 0.$$  \hfill (2.46)

If $\Sigma$ is canonically flat, then there exists a canonical trivialisation

$$\mathbb{P}W^+|_{\Sigma} = \Sigma \times S^2$$  \hfill (2.47)

and a canonical projection

$$\text{pr}: \mathbb{P}W^+|_{\Sigma} \rightarrow S^2 = \mathbb{P}^1.$$  \hfill (2.48)

Moreover, the composite of all the maps gives the map

$$m_C = \text{pr} \circ \det_C \circ G(i): \Sigma \rightarrow S^2 = \mathbb{P}^1$$  \hfill (2.49)

\textbf{Definition 2.2.} 1) This map is called the \textit{complex phase map};
2) $\Sigma$ is called a sdAG cycle if $m_C(\Sigma)$ is a point, or equivalently the differential

$$dm_C = 0.$$  \hfill (2.50)

In particular, a spLag cycle is a sdAG cycle. We explain below what “sdAG” stands for.

\textit{Mirror digression.} As before, (2.46) holds automatically if $S$ is a simply connected Calabi–Yau surface, that is, a K3 surface. In this case, again $B = S$. The target space $S^2 = \mathbb{P}^1$ of the complex phase map (2.49) is the space of integrable complex structures compatible with the Calabi–Yau metric $g$ on $S$. Thus any sdAG cycle $\Sigma$ is a complex curve for the complex structure $m_C(\Sigma) \in S^2 = \mathbb{P}^1$ and any spLag cycle is a complex curve for the specific complex structure on $S$. Any such curve must be a fibre of an elliptic pencil or a $-2$-rational curve.

The target space $S^2$ admits the standard complex structure

$$S^2 = \mathbb{C} \cup \infty = D^+ \cup S^1 \cup D^-,$$  \hfill (2.51)

where

$$D^+ = \{ z \mid |z| < 1 \}; \quad D^- = \{ z \mid |z| > 1 \}; \quad S^1 = \{ z \mid |z| = 1 \}.$$  \hfill (2.52)

For any oriented cycle $\Sigma$ we have the following cases:

- $m_C(\Sigma) \subset D^+ \iff \Sigma$ is symplectic;
- $m_C(\Sigma) \subset S^1 \iff \Sigma$ is Lagrangian;
- $m_C(\Sigma) = 0 \iff \Sigma$ is complex or algebraic, AG cycle for short;
- $m_C(\Sigma) = \infty \iff \Sigma$ is anticomplex, antiAG cycle for short;

Thus sdAG is an acronym for \textit{slightly deformed Algebraic Geometric} cycle: every point of the target sphere of the complex phase map can be deformed along its meridian to 0 – the image of the complex phase map of complex cycles.

Suppose $\Sigma$ is a symplectic oriented cycle in $S$. Then the image $m_C(\Sigma) \subset D^+$ is compact, and there is a unique minimal disc containing $m_C(\Sigma)$

$$m_C(\Sigma) \subset D_\Sigma \subset D^+.$$  \hfill (2.53)
Definition 2.3. For a symplectic oriented cycle $\Sigma \subset S$,

1. The centre $c_\Sigma$ of the disc $D_\Sigma$ is called the centre of $\Sigma$;
2. the radius $r_\Sigma$ of the disc $D_\Sigma$ is called the radius of $\Sigma$;
3. $\Sigma$ is $\varepsilon$-AG if $c_\Sigma = 0$ and $r_\Sigma < \varepsilon$. \hfill (2.54)

Donaldson proved the following analogue of Kodaira’s embedding theorem (see [D1]):

Theorem 2.1. Let $L$ be a line bundle on an aK surface $S$ with zero canonical class, and a a Hermitian connection on $L$ whose curvature form is the symplectic form on $S$:

$$ F_a = \frac{i}{2\pi} \omega. \hfill (2.55) $$

Then for any $k \gg 0$ there exists a section $s \in \Gamma(L^k)$ such that

1. the zero set $s^{-1}(0) = \Sigma$ is a smooth oriented symplectic 2-cycle;
2. $c_\Sigma = 0$;
3. $r_\Sigma < 1/\sqrt{k}$.

In particular, there are many smooth symplectic cycles very close to AG cycles.

Now consider an algebraic surface $S$ with canonical class $K_S > 0$ or $K_S < 0$. We would like to deform slightly the algebraic geometry, by considering a distinguished family of submanifolds, namely, the oriented 2-cycles. Moreover these cycles are determined by their first order infinitesimal behavior, that is, the cycles defined by properties of their Gauss lifts (2.29).

Recall that a Spin$^C$ structure is a choice of lift of the projective bundle $\mathbb{P}W^+$ to a vector bundle $W^+$. There are two canonical choices of Spin$^C$ structure:

$$ W^+ = O_S \oplus O_S(\pm K_S), \hfill (2.56) $$

and we have to consider two cases: $K_S > 0$ or $K_S < 0$.

General type ($K_S > 0$). In this case, we consider the lift

$$ W^+(K_S) = O_S \oplus O_S(K_S). $$

Recall that any section $s \in \Gamma(W^+)$ is called a spinor field. In particular, if we consider a nonvanishing section $s$ (or section vanishing along a “divisor”) then we get a section of the projective bundle $\mathbb{P}W^+$, that is, an almost complex structure $I_s$ compatible with our Kähler metric $g$.

In the Kähler (algebraic) case we have a finite dimensional family of holomorphic sections

$$ s \in H^0(O_S \oplus O_S(K_S)) = \mathbb{C} \oplus H^0(O_S(K_S)) \hfill (2.57) $$

We write $p_\theta = \dim H^0(O_S(K_S))$ for the geometric genus of $S$, that is, the complex dimension of $H^{2,0}(S)$. The family of spinor fields (4.23) defines a family of almost complex structures

$$ S^{2p_\theta} = \mathbb{C}^{p_\theta} \cup \{ \infty \setminus \text{point} \}, \hfill (2.58) $$
because the projectivisation
\[ \mathbb{P}(\mathbb{C} \oplus H^0(\mathcal{O}_S(K_S))) = \mathbb{C}^{p_g} \cup \mathbb{P}H^0(\mathcal{O}_S(K_S)) \] (2.59)
where \( \mathbb{C}^{p_g} = \{(1, s)\} \) is the space of nonvanishing sections and \( \mathbb{P}H^0(\mathcal{O}_S(K_S)) = |K_S| \) is the complete canonical linear system, points of which give the same almost complex structure, namely, conjugate to the original complex structure.

Now the Levi–Civita connection gives the Hermitian structure on the canonical bundle \( L_K = \mathcal{O}_S(K_S) \) and, similarly, the Hermitian structure on \( \mathbb{C}^{2p_g} \cup \{\infty \not\text{point}\} \) gives the standard metric on this sphere. Thus we can identify our sphere with the dual sphere
\[ S^{2p_g} = (S^{2p_g})^* \]
So this sphere contains the "equator"
\[ S^{2p_g-1}_c = \{z \mid \|z\| = 1\} \subset \mathbb{C}^{2p_g}. \] (2.60)
Interpreting \( \mathbb{C}^{2p_g} \cup \{\infty \not\text{point}\} \) as the space of sections gives us the embedding
\[ i_{\text{can}}: \mathbb{P}W^+ \to S^{2p_g} \times S \] (2.61)
where, as usual, \( W^+ = W^+(K_S)^* = \mathcal{O}_S \oplus \mathcal{O}_S(-K_S) \). The composite of this embedding and the projection of the trivial bundle \( S^{2p_g} \times S \) to the fibre gives the map
\[ \text{pr}: \mathbb{P}W^+ \to S^{2p_g}. \] (2.62)
Now for any cycle \( \Sigma \subset S \), the composite of the Gauss map (2.43), the projection \( \text{pr}_+ \) (2.44) and the projection (2.48) defines the complex phase map
\[ m_C = \text{pr} \circ \text{pr}_+ \circ G(i): \Sigma \to S^{2p_g}. \] (2.63)
Now in terms of this phase map, one can define the analogues of the special Lagrangian cycles known in the Calabi–Yau case.

**Definition 2.4.** A cycle \( \Sigma \subset S \) is called a sdAG cycle if \( m_C(\Sigma) \) is a point, or equivalently
\[ dm_C = 0. \] (2.64)

The image \( m_C(\Sigma) \in S^{2p_g} \) defines an almost complex structure on \( S \) compatible with the Kähler metric for which \( \Sigma \) is a *pseudoholomorphic curve*. Hence, in particular, if \( \Sigma \) is smooth, its genus is
\[ g(\Sigma) = \frac{1}{2}(\|\Sigma\|^2 + \|\Sigma\| \cdot K_S) + 1. \] (2.65)
Now we can generalise “Lagrangian” properties of cycles:
**Definition 2.5.** 1) A cycle $\Sigma$ is called *weakly Lagrangian* (wLag for short) if

$$m_C(\Sigma) \subset S^{2p_g-1}_e,$$  \hfill (2.66)

where $S^{2p_g-1}_e$ is the equator (2.60);

2) $\Sigma$ is called a spLag cycle, if it is a sdAG cycle and

$$m_C(\Sigma) \in S^{2p_g-1}_e.$$ \hfill (2.67)

The equator divides the target sphere of the complex phase map into upper and lower hemispheres:

$$S^{2p_g} = D^+ \cup S^{2p_g-1}_e \cup D^-,$$ \hfill (2.68)

and the entire catalogue of definitions (such as Definition 2.3), properties and facts (such as Theorem 2.1) can be repeated in this new set-up.

**Remark.** If we start with an aK surface, we should remark that instead of holomorphic spinor fields we should consider Dirac harmonic spinor fields, that is, solutions of the differential equations

$$D_a s = 0,$$

where $D_a$ is the Dirac operator of the Spin$^C$ structure $-K_S$ coupled to the Levi–Civita connection on $W^+$, extended by the connection $a$ on the determinant line bundle $L_{-K}$. More precisely, the solutions of the Seiberg–Witten equations give the target sphere of the complex phase map (2.60), and we can extend all our constructions to aK case.

Finally, it is quite easy to see what to do if $K_S < 0$: we change the sign of the canonical system $K_S \rightarrow -K_S$, getting the sphere

$$S^{2h^0(\mathcal{O}_S(-K_S))}$$ \hfill (2.69)

as the target sphere of the complex phase map. After that, we can repeat all our constructions and definitions. However, recall that

$$h^0(\mathcal{O}_S(K_S)) = p_g = \frac{1}{2}(b_2^+ - 1)$$ \hfill (2.70)

is a purely topological invariant of $S$, but $h^0(\mathcal{O}_S(-K_S))$ isn’t a topological invariant at all. Only for del Pezzo surfaces we can say that

$$h^0(\mathcal{O}_S(-K_S)) \leq 10,$$ \hfill (2.71)

and

$$h^0(\mathcal{O}_S(-K_S)) = 10 - b_2^-.$$ \hfill (2.72)

**Remark.** In the case

$$h^0(\mathcal{O}_S(K_S)) = h^0(\mathcal{O}_S(-K_S)) = 0,$$ \hfill (2.73)

we can’t propose any good way of deforming algebraic geometry. This is the rigid case.
§3. Slightly deformed Algebraic Geometry

Now we want to say why we need to deform “algebraic geometry” in this style. But before starting on such explanations, we have to say what algebraic geometry is, and what kind of questions we would like to discuss in this set-up.

In algebraic geometry, we have a perfect theory of curves, a rather less good theory of surfaces, an even worse theory of threefolds, and so on. But now it is quite reasonable to consider the geometry of pairs \( C \subset S \), where \( C \) is a smooth algebraic curve of genus \( g \) and \( S \) a smooth algebraic regular surface containing \( C \). Every irreducible component \( \mathcal{M} \) of the moduli space of such pairs has a map

\[
f: \mathcal{M} \to M_g \times M_s,
\]

where \( M_g \) is the moduli space of curves of genus \( g \) and \( M_s \) the irreducible component of the moduli space of algebraic surfaces containing \( S \) as a point. Passing over the usual compactification procedure, the fundamental class \([f(\mathcal{M})]\) gives cohomological correspondences

\[
H_*(M_g) \to H^*(M_s) \quad \text{and} \quad H_*(M_s) \to H^*(M_g),
\]

which are the “topological” part of our interest. On the other hand, projections to components of the target space of \( f \) define special algebraic subvarieties of moduli spaces

\[
\text{pr}_g \circ f(\mathcal{M}) \subset M_g \quad \text{and} \quad \text{pr}_s \circ f(\mathcal{M}) \subset M_s,
\]

the first of which is a very interesting subvariety of the moduli space of curves of genus \( g \) and the second usually coincides with the component \( M_s \), and the fibre of this projection

\[
(\text{pr}_s \circ f)^{-1}(S) = |C|
\]

is the complete linear system of curves on \( S \), the natural compactification of the space of all deformations of \( C \) inside \( S \).

**Example 1:** K3 surfaces. Let \( S \) be a K3 surface and suppose that the genus \( g \) of the curve \( C \) is \( \geq 12 \). Then the algebraic cohomology class \([C]\) defines a quasipolarisation of \( S \), and defines a component of moduli spaces \( MK3[C] \) of dimension 19:

\[
\text{pr}_s \circ f(\mathcal{M}) = MK3[C].
\]

On the other hand, the compactified moduli space of deformations of \( C \) inside a fixed K3 surface \( S \) is the projective space

\[
|C| = \mathbb{P}^g.
\]

Thus

\[
\dim_{\mathbb{C}} \mathcal{M} = g + 19.
\]

But, on the other hand, \((\text{pr}_g \circ f)(\mathcal{M})\) is a proper subvariety of the moduli space \( M_g \) of curves of genus \( g \) and

\[
\text{codim}_{\mathbb{C}}((\text{pr}_g \circ f)(\mathcal{M})) = 2g - 22
\]

Thus the algebraic geometric problem is to describe this proper subvariety in the moduli space of curves.

*Amazing observation.* A generic algebraic curve of genus \( \geq 11 \) lying on a K3 surface “remembers” the surface. Recall Mukai’s recipe to reconstruct the K3 surface \( S \) containing a curve \( C \) in terms of the geometry of \( C \) (for odd genus \( g \)).
**Mukai’s recipe.** 1) Consider the moduli space $SU_C(2, K_C)$ of semistable vector bundles on $C$ of rank 2 with $c_1$ the canonical class. Inside this moduli space, consider the Brill–Noether locus

$$SU_C(2, K_C, \frac{1}{2}(g - 1)) = \{ E \in SU_C(2, K_C) \mid h^0(E) \geq \frac{1}{2}(g - 1) \}. \quad (3.9)$$

If $C$ is a general curve in $(\text{pr}_g \circ f)(\mathcal{M})$ then this locus is a K3 surface $S'$.

2) The moduli space

$$M_{S'}(2, \Theta|_{S'}, \frac{1}{2}(g - 5)) = S \quad (3.10)$$

of rank 2 torsion free sheaves with $c_1 = \Theta|_{S'}$ (the restriction of the theta divisor on the moduli space of vector bundles) and $c_2 = \frac{1}{2}(g - 5)$ is a K3 surface, namely the actual one containing $C$.

**Remark.** In this construction, Mukai realises his philosophy: on the moduli space of vector bundles on a curve, the Brill–Noether level plays the role of the second Chern class for vector bundles on a surface.

**Basic classes.** One has the same type of problem when realising special integral 2-dimensional cohomology classes by special geometric submanifolds, for example by (pseudo-)holomorphic curves. The main example is following: the underlying smooth structure of an algebraic surface with $p_g > 0$ determines certain basic 2-dimensional cohomology classes, the Kronheimer–Mrowka and Seiberg–Witten classes:

$$\{ \kappa_i \}_{K-M} = \{ \kappa_i \}_{Z-W} \subset H^2(S, \mathbb{Z}). \quad (3.11)$$

These sets of classes are invariant under the diffeomorphism group and can be realised as algebraic curves in every algebraic structure on $S$. The set of these classes contains the canonical class $K_S$ realised by an effective curve $C$. The genus of this curve is a purely topological invariant of our surface:

$$g(C) = 2\chi + 3\sigma + 1, \quad (3.12)$$

where $\chi$ is the Euler characteristic of $S$ and $\sigma$ its signature. But the moduli of such curves is never generic. Indeed, the normal sheaf of the canonical curve

$$O_C(C) = O_C(\theta); \quad 2\theta = K_C, \quad h^0(O_C(\theta)) = p_g - 1 \quad (3.13)$$

is a theta characteristic on $C$ and if $p_g > 2$, this theta characteristic is special, that is, its theta constant vanishes. In this case one can check that

$$\text{codim}_C(\text{pr}_g \circ f)(\mathcal{M}) = p_g - 2 \quad (3.13')$$

On the other hand, the virtual dimension of the space of solutions of Seiberg–Witten equations for a generic metric equals 0, but for a Kähler metric, its actual geometric dimension equals $2(p_g - 1) = b_2^+ - 2$. Thus Kähler metrics are nongeneric and algebraic geometry isn’t “transversal” for most problems of differential and symplectic geometries, for the theory of quantum multiplications and many others.

To describe the “level of defectiveness” of algebraic geometry, it is enough to remark that the crucial point is the irregularity of the local structure of the theory of deformations of curves inside a fixed algebraic surface:
**Observation 3.1.** On an algebraic surface $S$ with $p_g > 0$, the normal bundle $N_{C \subset S} = \mathcal{O}_C(C)$ of an algebraic curve $C$ is almost always irregular, that is:

$$h^1(N_{C \subset S}) > 0.$$  \hfill (3.14)

More precisely, $h^1(N_{C \subset S}) = 0$ only holds in two cases:

1) $C$ is a fixed component of the complete canonical linear system;
2) $C$ is a multiple fibre of an elliptic pencil.

**Remark.** This property of normal bundles was observed by Castelnuovo and Enriques 100 years ago as “superabundance”: too many curves with respect to points. In modern times it was remarked by Donaldson in his seminal paper [D1].

The geometric meaning of this effect is quite simple: the virtual tangent space to the space $|C|$ of deformations of curve $C$ in $S$ at the point $C$ is

$$T|C|_C = H^0(N_{C \subset S}),$$  \hfill (3.15)

and the space of obstructions is

$$H^1(N_{C \subset S}).$$  \hfill (3.16)

However, there are no genuine obstructions: every infinitesimal deformation extends to a geometric deformation (just like under deformations of Calabi–Yau manifolds): one has the exact sequence

$$H^0(\mathcal{O}_S(C)) \to H^0(N_{C \subset S}) \to H^1(\mathcal{O}_S) = 0.$$  \hfill (3.17)

The final reason for wanting to slightly deform algebraic geometry is purely arithmetic. Every smooth algebraic curve $C$ over $\mathbb{Q}$ is traditionally considered as an algebraic surface fibred over $\text{Spec} \, \mathbb{Z}$, that is, as a pencil of curves. This pattern of thinking predicts to consider rational points of $C$ as sections $s_1, \ldots, s_N$ of this pencil. By the Arakelov theorem squares of these sections are negative. Suppose for a minute that these squares are $-1$. Then one can blow it down to points $p_1, \ldots, p_N$ on the surface $S$ with the curve $C$ (generic fibre) such that the normal bundle

$$N_{C \subset S} = \mathcal{O}_C(p_1 + \cdots + p_N).$$  \hfill (3.18)

If $g(C) > 1$ then $h^1(\mathcal{O}_C(p_1 + \cdots + p_N)) > 0$ and hence

$$N < 2g - 2.$$  \hfill (3.19)

But this estimate is too good to be true. Thus our standard pattern is wrong.

**Remark.** The interesting fact is that this pattern was good enough to prove such “coarse” fact as the Mordell conjecture but using in Miyaoka’s approach to prove Fermat’s last theorem has shown that it was wrong.

What we can do to avoid the speciality of moduli of curves which move in algebraic surfaces, that is, the properness of the image $\text{pr}_g$ projection (3.3)? The strong remedy is to consider all aK surfaces. In this purely symplectic geometry set-up, one has to consider the space of pairs $C \subset S$, where $S$ is any aK surface. Then the flexibility of symplectic geometry gives immediately the result:
Proposition 3.1. For aK surfaces, the image of the projection $p_g$

$$(p_r \circ M) = M_g$$

is the whole moduli space of curves of genus $g$.

Indeed, for any symplectic smooth oriented 2-cycles $\Sigma$ on 4-dimensional symplectic manifold $S$ and any complex structure on $\Sigma$ one can construct a compatible almost complex structure on $S$ such that $\Sigma$ is a pseudoholomorphic curve for this almost complex structure with induced complex structure (see, for example [M. Gromov “Soft and hard symplectic geometry”, Proc. ICM, Berkeley, 1986, AMS, 1987, 81–98]). However we get a problem with the target space $M_s$ of the second projection $p_s$: the “moduli spaces” of aK structures are infinite dimensional. But fortunately, the question of describing fibres of this second projections, that is, the “complete linear systems” of pseudoholomorphic curves on aK surface is correct (because the linearised operator is elliptic).

To be in the finite dimensional set-up, one has to fix some class of finite dimensional subspaces of $M$ of aK structures. And here we want to change tack and to consider the slight deformations of almost complex structures related to Kähler metrics.

We would like to deform algebraic geometry slightly in such a way that the theory of curves will be preserved completely and in a pair $(C \subset S)$ we have to consider, instead of an algebraic curve $C$, a smooth sdAG cycle $\Sigma$ of fixed phase

$$p = m_C(\Sigma) \in S^{2p_s}. \quad (3.20)$$

Every “irreducible” component of the moduli space $M$ of such pairs $(\Sigma \subset S)$ defines a map $f$ (3.1), a cohomological correspondence (3.2) (because every sdAG cycle admits a complex structure) and a subspace (3.3) in the moduli space of curves of genus $g$. In particular one has to describe the global structure of the complete space $|\Sigma|$ of deformations of sdAG cycle $\Sigma$ in $S$. Such type cycles are pseudoholomorphic curves for almost complex structure $S^p$ given by the point $p$ (3.20). Of course, our initial complex structure is $S^0$ and its complex conjugate is $S^\infty$. The local deformation theory of sdAG cycles is quite good, like the theory of local deformations of spLag cycles (see [H-L]).

Now fix an algebraic surface $S$ and consider a smooth sdAG cycle $\Sigma$ of the phase $p$ (3.20). Let

$$\Sigma \in |[\Sigma]|^p \quad (3.21)$$

be the moduli space of deformations of $\Sigma$ as a sdAG cycle. Then there exists a deformation $\delta$-complex such that the tangent space is given by

$$T|\Sigma|^p_\Sigma = H^0_\delta(N_{\Sigma \subset S}) = H^0(\Sigma, N_{\Sigma \subset S}). \quad (3.22)$$

(Recall that $\Sigma$ admits a complex structure, in which the normal bundle $N_{\Sigma \subset S}$ admits a holomorphic structure. The last space is a coherent cohomology group of a holomorphic vector bundle.)
In the same vein, the space of obstructions of infinitesimal deformations of $\Sigma$ as a sdAG cycle is

$$H^1_\delta(N_{\Sigma \subset S}) = H^1(\Sigma, N_{\Sigma \subset S}), \quad (3.23)$$

where the last space is again the space of coherent cohomology of the holomorphic vector bundle $N_{\Sigma \subset S}$ on the complex curve $\Sigma$.

There are two transversality results. The first one was proposed by Donaldson in [D1]:

**Theorem 3.1.** 1) For a generic aK structure, the obstruction space to local deformations of any pseudoholomorphic curve vanishes

$$H^1_\delta(N_{\Sigma \subset S}) = H^1(\Sigma, N_{\Sigma \subset S}) = 0. \quad (3.24)$$

2) The dimension of the local deformations space is given by

$$\dim T[[\Sigma]]^p = H^0_\delta(N_{\Sigma \subset S}) = H^0(\Sigma, N_{\Sigma \subset S}) = 2(\Sigma^2 + 1 - g). \quad (3.25)$$

We can add to this the same type “transversality result”:

**Theorem 3.2.** If $S$ is a surface with $p_g > 0$ then for generic point $p \in S^{2p_g}$ of the target space the complex phase map and any sdAG cycle of the phase $p$ we have equalities (3.24) and (3.25).

The idea of the proof is contained in the same paper of Donaldson [D1]. First of all we would like to consider the whole family

$$\|\Sigma\| = \bigcup_{p \in S^{2p_g}} [[\Sigma]]^p \quad (3.26)$$

of sdAG cycles for all $p \in S^{2p_g}$ and to prove that this family is smooth and of the right dimension (that is, the virtual dimension). After using standard arguments we get the statement of the theorem.

The whole family of sdAG cycles doesn’t fibre over the target of the phase map space because one sdAG cycle can be sdAG in more than one complex structures. But there exists a correspondence, or a space of pairs

$$\mathcal{M} = \{(\Sigma, p) \mid \Sigma \text{ is sdAG with } m_C(\Sigma) = p \} \subset \|\Sigma\| \times S^{2p_g}, \quad (3.27)$$

with two projections

$$\mathcal{M} \xrightarrow{pr_1} S^{2p_g} \text{ and } \mathcal{M} \xrightarrow{pr_e} \|\Sigma\| \quad (3.28)$$

and fibres of the second projection are

$$pr_e^{-1}(p) = [[\Sigma]]^p.$$

**Example.** The canonical system. Consider the canonical class $K_S$ of $S$, and realise it as sdAG cycles for all $p \in S^{2p_g}$. Then one has
Proposition 3.2.

\[ \| K_S \| = |K_S|^0. \]  

(3.27)

In particular, the image of the map \( p_{rg} \circ f(M) \) (3.3) is the same as for canonical curves of the algebraic surface \( S^0 \) (see (3.13)).

To describe the space of pairs (3.27) we must blowup the point \( (1,0) \) in the projective space \( \mathbb{P}(\mathbb{C} \oplus H^0(\mathcal{O}_{S^0}(K_S))) \) (2.43), (2.45):

\[ \sigma: \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(H)) \to \mathbb{P}(\mathbb{C} \oplus H^0(\mathcal{O}_{S^0}(K_S))) \]  

(3.28)

and the second projection of this blown up projective space

\[ \pi: \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-H)) \to |K_S|^0, \]  

(3.29)

where \( \mathcal{O} \) is the structure sheaf of the projective space

\[ \mathbb{P}H^0(\mathcal{O}_{S^0}(K_S)) = |K_S|^0 \]  

(3.30)

and \( \mathcal{O}(-H) \) is the Hopf line bundle on this projective space.

Then it is easy to see that 1) the space of pairs (3.27) is

\[ M = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-H)), \]  

(3.31)

and the map to the target sphere of the complex phase map

\[ \text{pr}_t: M = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(H)) \to S^{2p_g} \]  

(3.32)

is the blowdown of the sections \( \mathbb{P}(\mathcal{O}) \) and \( \mathbb{P}(\mathcal{O}(-H)) \) in the projective bundle (3.31) over the projective space \( |K_S|^0 \):

\[ \text{pr}_t(\mathbb{P}\mathcal{O}) = 0 \in S^{2p_g} \quad \text{and} \quad \text{pr}_t(\mathcal{O}(-H)) = \infty \in S^{2p_g}. \]  

(3.33)

From this picture one get immediately

Proposition 3.3. 1) For any \( p \in S^{2p_g} \setminus (0 \cup \infty) \) on aK surface \( S^p \), the canonical class admits a unique representation as pseudoholomorphic curve

\[ |K_S|^p = \text{a point}; \]  

(3.34)

2) For every curve \( C \in |K_S|^0 \) there exists a 2-sphere

\[ S^2 = \text{pr}_t(\pi^{-1}(C)) \in S^{2p_g} \]  

(3.35)

of almost complex structures in every of which \( C \) is the pseudoholomorphic realisation of the canonical class.
§4. The complex 3-dimensional case

Now we would like to extend the constructions we have described to the case an aK threefold \( X \) given by an Hermitian triple \((\omega, I, g)\). Here we describe simple facts and constructions from linear algebra of such structure over one point. Afterwards, in the next section, we globalise these constructions to the geometry of the tangent bundle and special subspaces of spaces of tangent directions to define new collection of submanifolds and hence, new collections of geometries of complex threefolds.

In the even dimensional case, one has two volume forms as the formula (2.9). In the odd dimensional case, one has to use the special trick to reduce 3 to 2.

Over any point \( p \in X \) we have as the fibre of the tangent bundle \( TX_p = \mathbb{C}^3 \) with the standard Hermitian triple \((\omega_p, I_p, g_p)\) with the constant symplectic form \( \langle \cdot, \cdot \rangle = \omega_p \), the constant Euclidean metric \( g_p \), giving the Hermitian triple \((\omega_p, I_p, g_p)\) and the oriented Lagrangian Grassmannian is a subspace of the oriented Grassmannian

\[
(\Lambda_\ell)_p \subset \text{Gr}_\ell(3, TX_p)
\]

of codimension 3. Moreover, instead of the complex Grassmannian (2.13) one has the subspace

\[
\{ V \in \text{Gr}_\ell(3, TX_p) \mid V \text{ contains a complex direction } z \in \mathbb{P}TX_p \}
\]

of subspace of 3-dimensional planes containing complex directions. Just as in the 2-dimensional case, for every \( V \in \text{Gr}_\ell(3, TX_p) \) there are two possibilities:

\[
\dim \langle V, I(V) \rangle = 6.
\]

This is the general case. In particular every Lagrangian \( V \) has this property. The second case is when

\[
\dim \langle V, I(V) \rangle = 4.
\]

In this case \( V \) contains a complex direction (see (4.2)) and defines the flag

\[
z_V \subset V \subset Z_V
\]

where \( z \) is the unique complex direction in \( V \) and \( Z_V = \langle V, I(V) \rangle \) (4.4).

We can distinguish two components of the set (4.2) by the orientation: the orientation of \( V \) may or may not be compatible with the complex orientation of \( z_V \). Now we can realise these components as level submanifolds of a map of \( \text{Gr}_\ell(3, TX_p) \) to \( S^2 \) (just as in the 2-dimensional case) for complex and anticomplex directions. The following constructions work for any dimensional case. We described its in §2 (see (2.18–2.34)) but for simplicity, we repeat it for the 3-dimensional case again.

**Definition 4.1.** Any nonzero element of \( w \in \bigwedge^3 \mathbb{C}^3 \) is called a complex orientation of \( \mathbb{C}^3 \).

Let \( w \in \bigwedge^3 \mathbb{C}^3 \) be a fixed complex orientation of \( \mathbb{C}^3 = TX_p \). Let us fix some orthogonal basis \( e_1, e_2, e_3, I(e_1), I(e_2), I(e_3) \) of \( TX_p \) such that

\[
w = (e_1 + I(e_1)) \wedge (e_2 + I(e_2)) \wedge (e_3 + I(e_3)).
\]

If we fix the volume form \( w \) such a way that \( |w| = 1 \) then we have to fix a phase \( e^{i\varphi} \) only.

Now in any oriented subspace \( V \in \text{Gr}_\ell(3, TX_p) \), consider an orthogonal basis \( v_1, v_2, v_3 \) and let \( M_V \) be the linear transformation sending \( e_i \) to \( v_i \) and \( I(e_i) \) to \( I(v_i) \). Again it is easy to see the following:
Proposition 4.1. 1) $M_s$ is a $\mathbb{C}$-linear map $TX_p \to TX_p$, that is, a complex matrix in our basis;
2) $\det M_V = 0$ iff $V$ contains a complex direction $z$. Of course $z_V$ is the kernel of $M_V$;
3) $V$ is Lagrangian iff $M_V \in U(3)$.

Again we have the map
$$d: \text{Gr}_\Upsilon(3, TX_p) \to \mathbb{C}$$
(4.7)

sending $V$ to the determinant of $M_V$ with the same inequality (2.25), but the one difference is in (2.26): one has, of course, the following:

Hadamard’s Lemma.

$$V \in (\Lambda_\Upsilon)_p \iff |d(V)| = 1.$$ 
(See for example [H-L], Lemma 1.9.)

Thus the oriented Lagrangian Grassmannian is
$$(\Lambda_\Upsilon)_p = d^{-1}(S^1)$$
(4.8)

However, the dimension of all others fibres is 7, whereas the dimension of $d^{-1}(e^{i\varphi})$ is $5$ – an extremely small number!

Now we can finish our construction. Again we have the decomposition
$$\text{Gr}_\Upsilon(3, TX_p) = \text{Gr}_\Upsilon(3, TX_p)^+ \cup d^{-1}(|d(V)| = 1) \cup \text{Gr}_\Upsilon(3, TX_p)^-$$
(4.9)

with the inversion orientations map (2.31). Recall again that the middle term is $\Lambda_\Upsilon$.

Then the gluing of two maps $d$ on $\text{Gr}_\Upsilon(3, TX_p)^+$ and $-(d)^{-1}$ which are equal on $d^{-1}(|d(V)| = 1)$ gives the map
$$\det_I: \text{Gr}_\Upsilon(3, TX_p) \to \mathbb{P}^1 = S^2_w$$
(4.10)

(just compare with (2.30–2.31)). But the difference between the 2-dimensional and 3-dimensional case is that this map is not smooth along $(\Lambda_\Upsilon)_p$!

To make it smooth we have to “blowup” $(\Lambda_\Upsilon)_p$ inside $\text{Gr}_\Upsilon(3, TX_p)$. We do it after the couple of remarks about complex orientations.

Of course the map $\det_I$ depends on $I$ and on a complex orientation $w$ but the fibres
$$(\det_I)^{-1}(0) \text{ and } (\det_I)^{-1}(\infty)$$
(4.11)
do not depend on a complex orientation. Only the identification of the target space $\mathbb{P}^1 = S^2_w$ depends on $w$.

Indeed the space of complex orientations
$$\{w\} = \mathbb{C}^* \text{ (or } e^{i\varphi} \text{ if } |w| = 1)$$
(4.12)

acts on the complex sphere $S^2 = \mathbb{C} \cup \infty$ by the usual multiplication: for any $\varepsilon \in \mathbb{C}^*$
$$\det_{I, \varepsilon \cdot w} = \varepsilon \cdot \det_{I, w}$$
(4.13)

and one get
**Proposition 4.2.** 1) All fibres of the map $\det_I$ (4.10) over $S^2 \setminus (0 \cup \infty \cup S^1)$ are diffeomorphic:

\[
(\det_I)^{-1}(z) = (\det_I)^{-1}(z') \quad \text{for} \ z, z' \neq 0, \infty, e^{2\pi i \varphi}. \quad (4.14)
\]

Of course one can send the target space $S^2$ of this map to the interval $[0, \pi]$ by the map $h$ (2.12) such a way that one get the “Kähler angle” map:

\[
\alpha_{I,w} = h \cdot \det_{I,w}: \text{Gr}(3, TX_p) \to [0, \pi]. \quad (4.15)
\]

(Just as in the 2-dimensional case).

Moreover, as in the 2-dimensional case,

\[
\dim \alpha_{I}^{-1}(z) = 8, \quad \dim \alpha_{I}^{-1}(e^{i\varphi}) = 5 \quad \text{and} \quad \dim \alpha_{I}^{-1}(0) = 7, \quad (4.16)
\]

and for $\alpha_{I}^{-1}(z)$ (4.14) there exists an extra phase map

\[
\alpha_{I}^{-1}(z) \to S^1, \quad (4.17)
\]

whose restriction to the oriented Lagrangian Grassmannian is equal to the standard phase map

\[
\det: \Lambda^\uparrow(3, TX_p) \to S^1_p \quad (4.18)
\]

(see the beginning of our story in §1).

Every $V \in \text{Gr}(3, TX_p)$ can contain one complex direction only, thus

\[
\text{rank } M_V \geq 2. \quad (4.19)
\]

**Remark.** Hence instead of $M_V$ we can consider $\text{adj} M_V$ and the image

\[
\text{im}(\text{adj} M_V) = z_V; \quad \ker(\text{adj} M_V) = Z_V \quad (4.20)
\]

So if $\det M_V = 0$ then one has the orthogonal decomposition

\[
V = \ker M_V \oplus (\ker M_V)^\perp_V \quad (4.21)
\]

where $\perp_W$ is the orthogonal to subspace of a Euclidean space $W$. Thus $(\ker M_V)^\perp_V$ is an oriented 1-subspace in $V$. But the plane

\[
\langle (\ker M_V)^\perp_V, I((\ker M_V)^\perp_V) \rangle \quad (4.22)
\]

is complex. This is a description of the canonical flag $(z_V \subset V \subset Z_V)$ (4.5).

So one get the map

\[
f: \alpha_I^{-1}(0) \to F^C_{1,2}(\mathbb{P}(TX_p)) = \mathbb{P}T\mathbb{P}(TX_p) \quad (4.23)
\]

to the complex flags of type $(1,2)$ in $TX_p = \mathbb{C}^3$. Obviously a fibre of this map is $S^1$. More precisely, the $\mathbb{P}^1$-bundle which is the target space of the map $f$ (4.23) has the tautological bundle $H$ (the Grothendieck line bundle of a projectivisation) equipped
with the Hermitian metric. The unit circle bundle of this line bundle is the source of (4.23):

$$\alpha_I^{-1}(0) = S^1(H).$$  \hspace{1cm} (4.24)

Recall that this is a description of any fibre of the map $\det_I$ expect for $\{e^{i\varphi}\} = S^1$.

Now if one consider the projection to $\mathbb{P}(TX_p)$ then the fibre of this map over a point $z$ is the space of real rays in the vector space $\mathbb{C}^2 = \mathbb{C}^3/z$. Thus this fibre is the unit 3-sphere

$$\text{pr}_C^{-1}(z) = S^3 \subset \mathbb{C}^3/z.$$ \hspace{1cm} (4.25)

Using the Euler exact sequence it is easy to see that full fibre of $\alpha_I$ over 0 is

$$\alpha_I^{-1}(0) = S^3(T\mathbb{P}(TX_p)(-1)) \hspace{1cm} (4.26)$$

that is, the unit 3-spheres fibration of the twisted tangent bundle $T\mathbb{P}(TX_p)(-1)$ of the complex projectivisation of the tangent space to $X$ at $p \in X$. Of course

$$S^3(T\mathbb{P}(TX_p)(-1)) = S^1(H) \hspace{1cm} (4.27)$$

and fibrewise

$$S^3 \to \mathbb{P}^1 \hspace{1cm} (4.28)$$

is the Hopf fibration.

**Remark.** This is the first difference between the complex dimension 2 case and the 3-dimension case: for 2-dimension case the fibre of $\alpha$ over 0 is just the projectivisation of the tangent space $\mathbb{P}W_-$ at $p$ (see (2.13)).

The second is that the preimage

$$\alpha_{I,w}^{-1}(\pi/2) = (\Lambda^\uparrow)_p \hspace{1cm} (4.29)$$

has extremely small dimensional.

**Definition 4.2.** 1) An oriented 3-dimensional subspace $V \in \text{Gr}_\uparrow(3, TX_p)$ is called 3/2-*pseudoholomorphic* (3/2ps for short) if

$$\alpha_I(V) = 0; \hspace{1cm} (4.30)$$

2) it is called 3/2-*anti-pseudoholomorphic* (3/2aps for short) if

$$\alpha_I(V) = \infty. \hspace{1cm} (4.31)$$

3) a pair $(v \subset V)$ where $v$ is oriented 1-dimensional subspace of $V$ is called *super Lagrangian* (superLag for short) if

$$V \in (\Lambda^\uparrow)_p. \hspace{1cm} (4.32)$$

and super spLag if $V$ is spLag.

**Remark.** 1) We know that the fibres 1) and 2) don’t depend on a complex orientation $w$, but 3) depends on $w$. 
2) For the definition of $\Lambda_\uparrow$ we have to use the form $\omega$ which depends on $I$. So it will be quite correct to write $\Lambda_\uparrow(I)$.

Now we can describe the map $\alpha_I$ (4.7) in other way using the form $\omega$ which is defined by $I$ (and our metric). Let us consider the open set

$$\text{Gr}_\uparrow(3, TX_p)_{\neq 0} = \text{Gr}_\uparrow(3, TX_p) - (\Lambda_\uparrow)_p.$$ (4.33)

defined by

$$V \in \text{Gr}_\uparrow(3, TX_p)_{\neq 0} \iff \omega|_V \neq 0.$$ (4.34)

Then any such $V$ admits the decomposition

$$V = \ker \omega|_V \oplus (\ker \omega|_V)_{\perp V}$$ (4.35)

and we fix the orientation of the plane

$$t_V = (\ker \omega|_V)_{\perp V}$$ (4.36)

such a way that the volume $\omega|_t$ is positive.

Then we get the map

$$\beta_\omega: \text{Gr}_\uparrow(3, TX_p)_{\neq 0} \to \text{Gr}_\uparrow(2, TX_p)$$ (4.37)

sending $V$ to $t_V$ (4.25).

But for $\text{Gr}_\uparrow(2, TX_p)$ we have the classical Kähler angle map

$$\alpha_p: \text{Gr}_\uparrow(2, TX_p) \to [0, \pi]$$ (4.38)

sending a plane $t$ to the same Kähler angle (2.9).

Remark. This construction only depends on the conformal class of $\omega$.

What we have to do now it is just “blowup the oriented Lagrangian Grassmannian $(\Lambda_\uparrow)$ inside the Grassmannian”: consider the space of pairs

$$\text{Gr}_\uparrow(3, TX_p) = \{(v \in V) \mid v \text{ is the kernel of } \omega|_V \}$$ (4.39)

and its projection to $V$:

$$\sigma: \text{Gr}_\uparrow(3, TX_p) \to \text{Gr}_\uparrow(3, TX_p).$$ (4.40)

**Proposition 4.3.**

1. The map $\sigma$ is an isomorphism over $\text{Gr}_\uparrow(3, TX_p)_{\neq 0}$.
2. $$\sigma^{-1}(\Lambda_\uparrow) = S^2(U)$$ (4.41)

is the unit sphere bundle of the universal bundle $U$ over the oriented Lagrangian Grassmannian $\Lambda_\uparrow$. This space is the moduli space of super Lagrangian 3-directions (4.32).
(3) The map (4.36) has a smooth extension to
\[ \beta_\omega : \text{Gr}(3, TX_p) \to \text{Gr}(2, TX_p); \] 
(4.41)
and for any Lagrangian super direction \( v \in V \)
\[ \beta_\omega (v \subset V) = v^\perp, \quad \text{that is}, \quad t_{(v \subset V)} = v^\perp \]
and \( V \) is \( 3/2 - ph \Rightarrow \beta_\omega (V) = z_V. \)

(4)
\[ \beta_\omega \sigma^{-1}(\Lambda_\tau) = \Lambda_\tau(2, TX_p) \subset \text{Gr}(2, TX_p), \]
and any fibre \( \beta_\omega^{-1}(t) \subset S^3(Q_4)_t, \) where \( Q_4 \) is the universal quotient bundle on \( \text{Gr}(2, TX_p) \); let
\[ \mathbb{P}TX_p \subset \text{Gr}(2, TX_p) \]
be the projective plane of complex directions in the tangent space then
\[ \beta_\omega^{-1}(t) = I(t) \cap S^3(Q_4)_t = S^1 \iff t \in \text{Gr}(2, TX_p) - \mathbb{P}TX_p \]
and
\[ \beta_\omega^{-1}(t) = S^3(Q_4)_t \iff t \in \mathbb{P}TX_p. \]

(5) The composite of the maps (4.37) and (4.41) gives a smooth map
\[ \tilde{\alpha}_I : \text{Gr}(3, TX_p) \to [0, \pi]. \] 
(4.42)
Its geometric meaning is quite simple: \( \tilde{\alpha}_I \) sends \( V \) to the Kähler angle of \( t_V \). In particular \( \tilde{\alpha}_I((v \in V)) = \pi/2. \)

Our map depends on the complex structure \( I \). Now, \textit{what happens if we change the initial complex structure slightly?}

In our odd dimensional case, an oriented 3-subspace \( V \in \text{Gr}(3, TX_p) \) doesn’t determine a new complex structure on \( TX_p \) compatible with the metric \( g \). But a flag \( (t \subset V \subset T) \) where \( t \) is an oriented 2-plane in \( V \) and \( T \) is an oriented 4-subspace determines a new complex structure on \( TX_p \) compatible with the metric \( g \). To see this, consider the orthogonal decomposition into 2-dimensional subspaces
\[ TX_p = t \oplus t^\perp \oplus T^\perp \] 
(4.43)
and put the standard complex structure on each 2-subspace of this decomposition. Then one get a new complex structure
\[ I_{(t \subset V \subset W)} \] 
(4.44)
on \( TX_p \) such that \( t \) is a complex direction and \( T \) a complex subspace.
Now the description of the space of all flags \((t \subset V \subset T)\) is:

\[
F_{2,3,4} = S^2(U) \times_{Gr_t(3,TX_p)} S^2(Q_3) \to Gr_t(3,TX_p). \tag{4.45}
\]

where \(U\) is the tautological bundle on the Grassmannian, \(Q\) the universal quotient bundle and \(S^2(*)\) the unit sphere bundle of \(*\).

Every new complex structure (4.44) determines a map

\[
\alpha_{I(t \subset V \subset T)}: Gr_t(3,TX_p) \to [0,\pi] \tag{4.46}
\]

which isn’t smooth along the oriented Lagrangian Grassmannian \(\Lambda_t(I(t \subset V \subset T))\), and a map \(\tilde{\alpha}_{I(t \subset V \subset T)}\) (4.42) which is the regularisation of \(\alpha_I\).

Again one has the subspace of \(3/2\)-pseudoholomorphic oriented 3-subspaces

\[
\alpha_{I(t \subset V \subset W)}^{-1}(0) \subset Gr_t(3,TX_p). \tag{4.47}
\]

and so on.

§5. Complex structures and globalisations

It is now time to describe the space of complex structures on \(TX_p\) compatible with our metric \(g_p\). To do this, consider the complexification of the metric quadric \(g_p^C\) in \(TX_p^C = TX_p \otimes \mathbb{C}\); any compatible complex structure on \(TX_p\) is given by a maximal isotropic subspace \(T^1,0 \subset TX_p^C\) with respect to our quadric \(g_p^C\). For algebraic geometers it is quite convenient to projectivise all geometric objects. So we have

\[
G_p \subset \mathbb{P}^5 = \mathbb{P}TX_p^C \tag{5.1}
\]

where \(G_p\) is the smooth 4-dimensional quadric in \(\mathbb{P}^5 = \mathbb{P}TX_p^C\) of isotropic lines. The projectivisation of \(T^1,0 \subset TX_p^C\) is a projective plane in \(G_p\). There are two systems of planes in any nonsingular quadrics which one can interpret as the Grassmannian of lines in \(\mathbb{P}^3\). Then one system of planes on \(G_p\) is given by points of this \(\mathbb{P}^3\) (as the set of lines through a point) and other one is given by planes (as the set of lines in fixed plane) that is, points of the dual space \((\mathbb{P}^3)^* = \mathbb{P}^3\).

**Proposition 5.1.** These systems of planes on \(G_p\) are distinguished by the orientation of \(X\).

Each of these spaces is the projectivisation of a spinor space at a point \(p \in X\):

\[
\mathbb{P}^3_\pm = \mathbb{P}W^\pm; \quad \mathbb{P}W^+ = (\mathbb{P}W^-). \tag{5.2}
\]

The projective space \(\mathbb{P}TX_p^C\) has a real structure, with respect to which the metric quadric \(G_p\) (4.34) is real:

\[
\theta: G_p \to G_p \tag{5.3}
\]

without fixed points, that is, without real points. Therefore \(\theta\) must send one system of planes on \(G_p\) to the other:

\[
\theta: \mathbb{P}^3_+ \to \mathbb{P}^3_- = (\mathbb{P}^3_+)^*. \tag{5.4}
\]

Thus \(\theta\) determines a projective Hermitian structure on \(\mathbb{P}^3_+\). One gets
Proposition 5.2. The space of complex structures on $TX_p$ compatible with the metric $g_p$ and the orientation of $X$ is $\mathbb{P}^3 = \mathbb{P}W_p^+$. (Just as in the 2-dimensional case).

Now one has the map
\[
c_p: F_{2,3,4} = S^2(U) \times_{Gr(3,TX_p)} S^2(Q) \to \mathbb{P}W_p^+.
\]
(5.5)

sending a flag $(t \subset V \subset W)$ to the complex structure $I_{(t \subset V \subset W)}$ (compare with (2.6)). Here and over there $U$ and $Q$ are the universal subbundle and quotient bundle of the Grassmannian.

Proposition 5.3. The fibre
\[
c_p^{-1}(I_{(t \subset V \subset W)}) = \alpha_{I_{(t \subset V \subset W)}}^{-1}(0).
\]
(5.6)

(See (4.46–4.47)).

Our complex structure $I$ is the point of $\mathbb{P}W_p^+$
\[
I_p \in \mathbb{P}W_p^+.
\]
(5.7)

and the conjugated complex structure
\[
\overline{I} = \theta(I) \in \mathbb{P}W_p^-.
\]
(5.8)

But this point determines the projective plane
\[
(\mathbb{P}^2_I)_p \subset \mathbb{P}W_p^+.
\]
(5.9)

Proposition 5.4. 1) Every pair $(I, I')$ of complex structures on $TX_p$ has unique common complex direction.

2) The plane $\mathbb{P}^2_I$ is canonically isomorphic to the projectivisation $\mathbb{P}TX_p$ of the tangent space (in the complex structure $I$).

3) Every oriented plane $t$ in $TX_p$ defines a line $l_t \in \mathbb{P}W_p^+$, that is, a point of the complex quadric $G_p$ (5.1). This map
\[
s_p: Gr(2,TX_p) \to G_p
\]
(5.10)

is an isomorphism.

Indeed, every pair of points $I, I' \in \mathbb{P}W_p^+$ determine two projective planes
\[
\mathbb{P}T^1_I, \mathbb{P}T^1_{I'} \subset G_p \subset \mathbb{P}TX_p^C
\]
(5.11)

(see (5.1)). These planes in the quadric have one intersection point
\[
\mathbb{P}T^1_I \cap \mathbb{P}T^1_{I'} = z(I, I') \subset \mathbb{P}TX_p^C
\]
(5.12)

which gives the common holomorphic direction. This gives 1).
To get 2), consider the map

\[ z(I, \cdot): \mathbb{P}^2_T \to \mathbb{P}T^{1,0}_I \quad (5.13) \]

sending a plane \( I' \in \mathbb{P}^2_T \) to the intersection point \( z(I, I') \). It is easy to see that this map is an isomorphism.

So one get the following interpretation of the right hand spinor space (5.2)

\[ W^+_p = \mathbb{C} \cdot I \oplus TX_p \quad (5.14) \]

because of the equality \( TX_p = T^{1,0}_I \).

Now for any point \( p \in \mathbb{P}^2_T \) one has the projective line

\[ \langle I, p \rangle \subset \mathbb{P}W^+ \quad (5.15) \]

defined in Proposition 5.5.

**Proposition 5.5.** 1) Every complex structure of the family of complex structures \( \langle I, p \rangle \) has the same complex direction common with \( I \).

2) For every complex direction \( p \) of a complex structure \( I \) the family \( F_I(p) \) of complex structures every of which contains \( p \) as a complex direction is the family (5.15):

\[ F_I(p) = \langle I, p \rangle \subset \mathbb{P}W^+ \quad (5.16) \]

Now we can identify the family of complex structures \( F_I(p) \) (5.16) with the space of complex structures on

\[ \mathbb{C}^2 = \mathbb{R}^4 = T^{1,0}_I / t \quad (5.17) \]

This can be identified with the target space of the map \( \text{pr}_+ \) (2.6) for the complex 2-dimensional case (see §2). That is, we have the identification

\[ \langle I, p \rangle = \mathbb{P}^1_+ \]

for \( TX_p = \mathbb{R}^4 \) and the map

\[ \text{pr}_+: \text{Gr}_t(2, T\mathbb{P}^2_{\theta(I)}) \to \langle I, p \rangle \quad (5.18) \]

Now we have to switch on our form \( \omega \) to get the Kähler angle map (2.11)

\[ \text{Gr}_t(2, T\mathbb{P}^2_{\infty}) \xrightarrow{\text{pr}_+} \langle I, p \rangle \xrightarrow{h} [0, \pi] ; \quad (5.20) \]

see (2.7–2.13).

So under the identification of a point \( p \) of the plane \( \mathbb{P}^2_{\theta(I)} \) to the complex direction \( t \subset T^{1,0}_I \) we get an identification of the family \( F_I(p) \) (5.16) with the family of complex structure \( \langle I, p \rangle = S^2 \) on \( \mathbb{C}^2 = T^{1,0}_I / t \) (5.17).

Thus all the maps (2.12)

\[ h: \langle I, p \rangle \to [0, \pi] \quad (5.21) \]
are restrictions to lines \( \langle I, p \rangle \subset \mathbb{P}^3 \) of the map
\[
\mathbb{P}^3 \xrightarrow{\varpi} S^6_I \xrightarrow{h} [0, \pi],
\]
where \( \varpi \) is the blowdown of the projective plane \( \mathbb{P}^2_\theta \) to the point \( \infty \). Let us denote this composite by
\[
h_I: S^6_I \to [0, \pi]
\]
and call it the height function (for the 3-dimensional case).

This 6-sphere \( S^6 \) is a fibre of the Thom space of the complex vector bundle \( TX^I \) for the initial aK structure on \( X \).

So in the 2-dimensional case we have the map (2.12) \( \mathbb{P}^1_+ = S^2 \xrightarrow{h} [0, \pi] \) and in 3-dimensional case we have the map (5.23) of a fibre of the Thom space
\[
Th(TX^I_p) \xrightarrow{h_I} [0, \pi].
\]

Again we have two points \( I = 0 \) and
\[
\overline{T} = \theta(I) = \infty
\]
such that all its inverse images are points:
\[
h_I^{-1}(0) = I; \quad h_I^{-1}(\infty) = \overline{T}
\]
and all other fibre we can lift by \( \varpi^{-1} \) (5.22) to \( \mathbb{P}^3 \) and project (from \( I \)) to the projective plane \( \mathbb{P}^2_\theta \):
\[
\psi = \varpi^{-1} \cdot \text{pr}_I: h_I^{-1}(\varphi) \to \mathbb{P}^2_\theta.
\]

It is easy to see that this map is surjective and the fibre over an \( I \)-complex direction \( z \in \mathbb{P}TX^I_p \)
\[
\psi^{-1}(z) = h^{-1}(\varphi)
\]
is the actual phase circle of the standard Kähler angle map (2.12) (2-dimensional case)
\[
h: \langle I, z \rangle \to [0, \pi]
\]
sending a complex structure \( I' \in \langle I, z \rangle \) to the complex structure on \( \mathbb{R}^4 = TX/z \) with Kähler angle \( \varphi \).

In particular \( h^{-1}(\pi/2) \) is the circle of complex structure \( \{I'\} \subset \langle I, z \rangle \) such that any \( I' \)-complex direction in \( \mathbb{R}^4 = TX/z \) is Lagrangian (more precisely \( I \)-Lagrangian).

Now in the direct product
\[
\text{Gr}_+(2, TX_p) \times \mathbb{P}^3_+
\]
one has subspace
\[
\bigwedge_{\text{univ}} = \{(t, I) \mid \omega_I|_t = 0\}.
\]
In particular for every \( t \in \text{Gr}_+(2, TX_p) \) we have
\[
\mathcal{I}_t = (t, \mathbb{P}^3_+) \cap \bigwedge_{\text{univ}} \in \mathbb{P}^3_+,
\]
that is, the space of all complex structure for which \( t \) is Lagrangian.

In the same vein, we have subspaces of \( \mathbb{P}^3_+ \):
\[
\mathcal{I}_V = \{I \mid \omega_I|_V = 0\}
\]
and so on. We leave the problem of describing these subspaces as exercises for the reader.
Globalisation. In the 3-dimensional case, our pointwise constructions have “moduli”. Before globalising them, let us bring together all observations coming from our local investigations.

1. First of all, the dependence of the map \( \text{det}_{I,w} \) on \( I \) and on the complex orientation \( w \) (see (4.6)) was described by the formula (4.13).

2. The blowup of the Grassmannian \( \widetilde{\text{Gr}}(3, TX_p) \) (4.39) has the regular map \( \widetilde{\alpha}_I \) (4.42) to the interval \([0, \pi]\) which is submersive over the open interval \((0, \pi)\); that is, all its fibres are diffeomorphic.

3. This map \( \widetilde{\alpha}_I \) (4.42) factors as

\[
\widetilde{\alpha}_I: \widetilde{\text{Gr}}(3, TX_p) \xrightarrow{\text{det}_I} \mathbb{P}^1 \to S^2 \to [0, \pi], \tag{5.33}
\]

where \( \text{det}_I \) is the regularisation of the map (4.10) after the blowup of \( \Lambda^\uparrow \).

The main observation is

**Proposition 5.6.** The map \( \text{det}_I \) (5.33)

\[
\text{Gr}^\uparrow(3, TX_p) \to \mathbb{P}^1 = S^2 \tag{5.34}
\]

is submersive everywhere over the target sphere;

2) All fibres of this map are diffeomorphic and

\[
\text{det}_I^{-1}(z) = \alpha_I^{-1}(0). \tag{5.35}
\]

In particular, as in the 2-dimensional case, the space \( \text{det}_I^{-1}(e^{i\varphi}) \) of super special Lagrangian subspaces is isomorphic to the space of \( 3/2 \)-pseudoholomorphic subspaces:

\[
\text{det}_I^{-1}(e^{i\varphi}) = \alpha_I^{-1}(0). \tag{5.36}
\]

Moreover, as in the 2-dimensional case, for every fibre \( \widetilde{\alpha}_I^{-1}(z), z \neq 0, \infty \) there exists an extra phase map

\[
\alpha_I^{-1}(z) \to S^1, \tag{5.37}
\]

which for the oriented Lagrangian Grassmannian coincides with the standard phase map

\[
\text{det}: \Lambda^\uparrow(3, TX_p) \to S^1_\rho \tag{5.38}
\]
described in §1.

We now globalise these maps. First of all, one has the oriented Grassmannisation of the tangent bundle:

\[
\pi: \text{Gr}^\uparrow(3, TX) \to X \tag{5.39}
\]

which fibrewise is \( \text{Gr}^\uparrow(3, TX_p) \), and the fibration

\[
\tilde{\pi}: \text{Gr}^\uparrow(3, TX) \to X \tag{5.40}
\]
of blown up Grassmannians

$$\sigma: \widetilde{\text{Gr}(3, TX)} \rightarrow \text{Gr}(3, TX)$$

along $\Lambda(3, TX)$.

Secondly, we have to globalise the target 2-spheres. The geometric meaning of formula (4.13) is that the target space of the globalising of the fibre-by-fibre maps $\det_{1, w}$ (4.10) is the projective $\mathbb{P}^1$-bundle

$$\mathbb{P}(\mathcal{O}_X \oplus \mathcal{O}_X(-K_X)).$$

This vector bundle $\mathcal{O}_X \oplus \mathcal{O}_X(-K_X)$ isn’t simple. In particular its automorphism group contains the multiplicative group

$$\mathbb{C}^* \subset \text{Aut}(\mathcal{O}_X \oplus \mathcal{O}_X(-K_X))$$

which preserves the decomposition. Of course we can consider this subgroup as the subgroup of the automorphism group of the line bundle $L_{-K} = \mathcal{O}_X(-K_X)$:

$$\mathbb{C}^* \subset \text{Aut}(L_{-K}).$$

**Definition 5.1.** The choice of an isomorphism

$$\bigwedge^2(\mathcal{O}_X \oplus \mathcal{O}_X(-K_X)) = L_{-K}$$

is called a *global complex orientation* of $X$.

Thus the space of global complex orientations is again $\mathbb{C}^*$.

**Remark.** Actually, we will consider slightly deformed Algebraic Geometry. Thus the initial almost complex structure on $X$ is a holomorphic structure on $X$, and we have the isomorphism

$$\mathbb{C}^* = \text{Aut}(L_{-K})$$

of the group of holomorphic isomorphisms of the holomorphic line bundle $L_{-K}$.

In the general case, this identification is defined up to any gauge transformation of $L_{-K}$.

**Example.** *The CY case.* In this case a global complex orientation is given by a choice of a holomorphic 3-form $\theta$.

We now fix a global complex orientation $w$ of $X$ as in (5.12); then we have the globalisation of the map (4.10)

$$\det_I: \text{Gr}(3, TX) \rightarrow \mathbb{P}(\mathcal{O}_X \oplus \mathcal{O}_X(-K_X))$$

of the fibrations over $X$.

The restriction of this map to the oriented Lagrangian Grassmannisation $\Lambda(3, TX)$ gives the map

$$\det: \Lambda(3, TX) \rightarrow S^1(L_{-K}) \subset \mathbb{P}(\mathcal{O}_X \oplus \mathcal{O}_X(-K_X))$$

(5.47)
This target bundle (a ruled fourfold) has two sections
\[ S_0 = \mathbb{P}(\mathcal{O}_X) \quad \text{and} \quad S_\infty = \mathbb{P}(\mathcal{O}_X(-K_X)), \] (5.48)
and an \( S^1 \)-subbundle
\[ S^1(L_{-K}) \subset \mathbb{P}(\mathcal{O}_X \oplus \mathcal{O}_X(-K_X)). \] (5.49)
and these three subspaces are pairwise disjoint.

The regularisation of the map (5.46) is the map
\[ \tilde{\text{det}}_I: \text{Gr}_\gamma(3, TX) \to \mathbb{P}(\mathcal{O}_X \oplus \mathcal{O}_X(-K_X)), \] (5.50)
which is submersive and smooth.

Finally, over every point \( p \in X \) the space of pairs \((I, w)\) of oriented complex structures is a lifting of the projective space \( \mathbb{P}^3_+ \) to a vector space \( W^+ \):
\[ W^+_p = \{(I, w)\}. \] (5.51)

**Proposition 5.7.** The space of oriented complex structures is the vector bundle
\[ W^+ = \mathcal{O}_X \oplus TX. \] (5.52)

The automorphism group \( \text{Aut} W^+ \) contains \( \mathbb{C}^* \) acting by homotheties on \( TX \). This subgroup can again be identified to \( \mathbb{C}^* \subset \text{Aut} L_{-K} \).

The fibrewise blowdown of the subbundle \( \mathbb{P}TX \) to point gives the Thom bundle
\[ S = TX \cup X_\infty, \] (5.53)
which fibre-by-fibre is a \( S^6 \)-bundle (see (5.22)).

§6. SP\text{LAG} AND SDAG 3-CYCLES

Now for any oriented 3-dimensional submanifold \( Y \subset X \) the embedding \( i: Y \to X \) has a Gauss lifting:
\[ G(i): Y \to \text{Gr}_\gamma(3, TX)|_Y \] (6.1)
sending a point \( y \in Y \) to the oriented subspace \( TY_y \subset TX_y \). The composite of this Gauss map and the map \( \text{det}_I \) (5.46) gives the map
\[ \text{det}_I: Y \to \mathbb{P}(\mathcal{O}_X \oplus \mathcal{O}_X(-K_X))|_Y. \] (6.2)
**Definition 6.1.** 1) An oriented 3-dimensional cycle $Y \subset X$ is 2/3-*pseudoholomorphic* (2/3-ph for short) if
\[
\det_I(Y) \subset S_0
\] (see (5.48));
2) it is called 2/3-*anti-pseudoholomorphic* (2/3-aph for short) if
\[
\det_I(Y) \subset S_\infty.
\]

It is easy to see that $Y$ is Lagrangian iff
\[
\det_I(Y) \subset S^1(L-K).
\]

As before, let $a_{L-C}$ be the Levi–Civita connection on the anticanonical line bundle $L_K$ and
\[
F_{L-C} \in \Omega^2(X)
\] the curvature of this connection as a 2-form on $X$.

**Definition 6.2.** An oriented 3-cycle $Y$ on $X$ is *canonically flat* if there exists a simple connected submanifold $B \supset Y$ such that
\[
F_{L-C}|_Y = 0.
\]

As before, if $Y$ is canonically flat then there exists a canonical trivialisation of the restriction of the target bundle of the map $\det_I$
\[
P(\mathcal{O}_X \oplus \mathcal{O}_X(-K_X))|_Y = Y \times \mathbb{P}^1
\]
and a canonical projection
\[
\text{pr}: P(\mathcal{O}_X \oplus \mathcal{O}_X(-K_X))|_Y \to \mathbb{P}^1 = S^2
\]

The composite of these maps gives the map of a canonically flat cycle $Y$:
\[
m_I = \text{pr} \cdot \det_C \cdot G(i): Y \to S^2.
\]

To use the regularisation of this map, that is, the map (5.50), we have to correct slightly the definition of 3-dimensional cycle: every 3-cycle $Y$ has the (compact) subset
\[
Y_\Lambda = \{ y \in Y \mid TY_y \in \Lambda^3(TX_y) \},
\]
that is, the set of points with Lagrangian tangent space.

Now on $Y \setminus Y_\Lambda$ there is defined the field of directions
\[
0 \to \ker(\omega|_Y) \to TY \xrightarrow{\omega} T^*Y
\]
**Definition 6.3.** A pair \((v,Y)\) where \(v\) is any smooth extension of the field of directions \(\ker \omega|_Y\) from \(Y \setminus Y_\Lambda\) to \(Y\) is called a *supercycle* if this field is parallel with respect to the Levi-Civita connection on \(TY\).

For example, any 3/2-ph cycle can have only one supercycle structure: \(Y_\Lambda = \emptyset\) and \(\omega|_Y\) has to be parallel with respect to \(g|_Y\). We call it s3/2-ph cycle for short and we have to give the

**Warning.** *Not every 3/2-ph cycle is s3/2-ph cycle.*

But for a Lagrangian cycle supercycle structure is given by any character of the fundamental group of \(Y\).

Now for any oriented super submanifold \((v,Y)\) there exists a Gauss lifting of the embedding \(i: Y \to X:\)

\[
G(i): (v,Y) \to \widehat{\text{Gr}}(3, TX)|_Y \tag{6.12}
\]

sending a point \(y \in Y\) to the oriented pair

\[
v_y TY_y \in \widehat{\text{Gr}}(3, TX_y). \tag{6.13}
\]

The composite of this Gauss map and the map \(\det_I\) (5.50) gives the map

\[
\det_C: Y \to \mathbb{P}(\mathcal{O}_X \oplus \mathcal{O}_X(-K_X))|_Y. \tag{6.14}
\]

**Definition 6.5.** 1) An oriented 3-dimensional supercycle \((v,Y)\) is called a *Lagrangian supercycle* if

\[
\det_C(Y) \subset S^1(L-K) \tag{6.15}
\]

(see (5.49));

So for any canonical flat 3-dimensional supercycle \((v,Y)\) the *complex phase map*

\[
m_C = \text{pr} \circ \det_C \circ G(i): Y \to S^2. \tag{6.16}
\]

is well defined as the composite of standard regularised maps.

The target sphere \(S^2\) of this map has two special points

\[
0 = \mathbb{P}^1 \cap S_0 \quad \text{and} \quad \infty = \mathbb{P}^1 \cap S_\infty, \tag{6.17}
\]

and the circle

\[
S^1 = \mathbb{P}^1 \cap S^1(L_K). \tag{6.18}
\]

This sphere admits the standard complex structure, and one has the decomposition (2.37–2.38)

\[
S^2 = \mathbb{C} \cup \infty = D^+ \cup S^1 \cup D^-.
\]

We can extend the list (6.3–6.5) by the following cases:

\[
m_C(Y) \subset D^+ \iff Y \text{ is 2/3-symplectic},
\]

\[
m_C(Y) \subset D^- \iff Y \text{ is 2/3-antisymplectic},
\]

and so on.
Definition 6.4. 1) A canonically flat cycle $Y$ is a \textit{sdAG cycle} if $m_C(Y)$ is a point in the target sphere $S^2$ (6.16), that is, the differential of this map vanishes:

$$dm_C = 0.$$  \hspace{1cm} (6.19)

3) a sdAG cycle $Y$ is a \textit{spLag cycle} if

$$m_C(Y) \in S^1 \subset S^2$$  \hspace{1cm} (6.20)

(see (6.18); it is easy to see that this map forgets super structure);

4) a sdAG cycle $Y$ is called $\alpha$-cycle if

$$\alpha_I(Y) = \alpha \in [0, \pi].$$  \hspace{1cm} (6.21)

It is easy to see that

1) a sdAG cycle $Y$ is 2/3-ph iff

$$m_C(Y) = 0 \in S^2,$$  \hspace{1cm} (6.22)

and is 2/3-aph iff

$$m_C(Y) = \infty$$  \hspace{1cm} (6.23)

(see Definition 5.1).

\textbf{Mirror digression: CY threefolds.} Again, every Lagrangian 3-cycle is canonically flat if $X$ is a simply connected Calabi–Yau threefold. In this case $B = X$ again and \textit{there exists a global complex phase map}

$$m_C: \widehat{Gr}^3(TX) \rightarrow S^2$$  \hspace{1cm} (6.24)

such that for every super 3-cycle $Y$ its complex phase map is the composite of the Gauss map and this universal map.

Moreover every sdAG cycle $Y$ on a CY threefold $X$ defines a complex orientation of $X$, that is, a trivialisation of the canonical line bundle $\wedge^3 TX$. Such a trivialisation is given by a choice of a holomorphic 3-form $\theta$. A pair $(X, \theta)$ is called an \textit{oriented CY threefold}.

A spLag cycle $Y$ on an oriented CY threefold $X$ is Lagrangian with respect to the Kähler form $\omega$ and satisfying the condition

$$\text{Re } \theta|_Y = 0$$  \hspace{1cm} (6.25)

(see [H-L]).

Our aim is to investigate the local deformation theory of super sdAG cycles in complex threefolds just as we did in §3 for the case of aK surfaces (see (3.24–3.28)). Recall that the starting point of this investigation was the theory of complete linear systems of holomorphic curves on algebraic surfaces. The main fact about such deformation theory was Observation 3.1. In 3-dimensional case our experience is the local deformation theory of spLag cycles on CY threefolds.
Such local deformation theory is quite well understood (for a survey, see for example [H]). Let $\mathcal{M}_Y$ be the local deformation space of spLag cycles around a smooth spLag cycle $Y$. Then the tangent space to the moduli space at the point $Y$ is

$$T\mathcal{M}_Y = H^1(Y, \mathbb{R})$$

(6.26)

the space of harmonic 1-forms (in the induced metric) on $Y$.

This space is the space of infinitesimal deformations and the obstructions space for the Kuranishi family of deformations is $H^2(Y, \mathbb{R})$. However, once more, there are no genuine obstructions: every infinitesimal deformation extends to a geometric deformation (just as for holomorphic curves on an algebraic surface $S$ of positive geometric genus, see (3.17)).

On the other hand, if $Y$ is a homology sphere (that is, $H^*(Y) = H^*(S^3)$) then this spLag cycle is rigid.

Now let $\mathcal{SM}_{(v,Y)}$ be the local deformation space of super spLag cycles around a smooth super spLag cycle $(v,Y)$. Then the tangent space to the moduli space at the point $(v,Y)$ is

$$T\mathcal{SM}_{(v,Y)} = H^1(Y, \mathbb{R}) \oplus i \cdot H^1(Y, \mathbb{R}) = H^1(Y, \mathbb{C})$$

(6.27)

as the space of harmonic complex 1-forms (in the induced metric) on $Y$.

This space is the space of infinitesimal deformations and the obstructions space for the Kuranishi family of deformations is $H^2(Y, \mathbb{C})$. However there are no genuine obstructions again: every infinitesimal deformation extends to a geometric deformation.

But now, just as in the 2-dimensional case

the Geometry of special Lagrangian supercycles can be deformed

to the Geometry of sdAG 2/3-pseudoholomorphic supercycles.

Indeed, let $e^{i\varphi}$ be the image of spLag supercycles. Then the space of special super Lagrangian directions is

$$m_{\mathbb{C}}^{-1}(e^{i\varphi}) \subset S^2(U) \subset \widetilde{\Gr}(3, TX),$$

(6.28)

where $m_{\mathbb{C}}$ is the universal complex phase map (2.24), $S^2(U)$ is the unit sphere bundle of the tautological bundle over the Lagrangian Grassmannisation $\Lambda_\tau(3, TX)$ of the tangent bundle to $X$.

Now we can deform this fibre along the meridian to the north pole 0 of the target space of the universal complex phase map (2.24) (see (6.17)). So we get a family of geometries parametrised by the interval $[0, \pi/2]$

$$\mathcal{G} = m_{\mathbb{C}}^{-1}([0, \pi/2])$$

(6.29)

which gives a smooth bordism between the special Lagrangian super Geometry given by the space of super directions (6.28) and the slightly deformed Algebraic Geometry given by the space of 3/2-pseudoholomorphic directions

$$m_{\mathbb{C}}^{-1}(0) \in \Gr(3, TX).$$

(6.30)

This is the explanation why we have the local deformation theory for s3/2-ph cycles as good as the the local deformation theory for spLag supercycles: let $S^3/2\mathcal{M}_Y$ be the local deformation space of s3/2-ph cycles around a smooth s3/2-ph cycle $Y$. Then
Proposition 6.1. 1) The tangent space to the moduli space at the point $Y$ is

$$TS3/2\mathcal{M}_Y = H^1(Y, \mathbb{C});$$  \hspace{1cm} (6.31)

2) this space has a “canonical” complex orientation;
3) there are no genuine obstructions: every infinitesimal deformation extends to a geometric deformation.

For the proof we have to imitate the arguments of Hitchin [H] and McLean in the complex version.

Remark. We should remark that our proposed complex version of the theory of spLag cycles is not contained within symplectic geometry. Indeed, sdAG cycles don’t have to be symplectic in general. We consider the metric as the fundamental object, not the symplectic form $\omega$. Indeed if we are changing the almost complex structure $I_t$ starting with an aK triple $(\omega_0, I_0, g)$ preserving our metric $g$ and such that $\omega_t$ are harmonic, then, for small enough $t$, in the Hermitian triple $(\omega_t, I_t, g)$ the 2-form $\omega_t$ is positive, but after some time it loses this property and we go out of symplectic stuff.

One of the reasons why it is fruitful to go outside symplectic geometry is the main Hitchin construction from [H]. The point is that locally the moduli space $\mathcal{M}_Y$ of deformations of spLag cycles in fixed CY threefold $X$ around $Y$ can be embedded in the space $H^1(Y, \mathbb{R}) \times H^2(Y, \mathbb{R})$ and the image is spLag cycle with respect to the standard symplectic structure (recall that $H^1(Y, \mathbb{R}) = H^2(Y, \mathbb{R})^*$), the standard complex structure which define the metric. This metric isn’t Riemannian but ultrahyperbolic (but the restriction to $Y$ is Riemannian). See [H]. It is easy to see that one can change this triple canonically in such a way that the restriction to $Y$ doesn’t change and the new metric is Riemannian but the image of $\mathcal{M}_Y$ isn’t symplectic, and with respect to this new triple the image of the moduli space is sdAG but not spLag.

So the complex version of the spLag cycles is productive to preserve the standard pattern: “a moduli space of complex submanifolds is a complex manifold” (like complete linear systems on an algebraic surface).

Observation. The moduli space of sdAG supercycles is sdAG.

The last remark in the Calabi–Yau set-up is the following: if you believe in Strominger, Yau and Zaslov’s version of mirror symmetry (see [SYZ]), we can expect the following:

Great Expectations [D]. 1) There exists a natural compactification

$$|Y|$$  \hspace{1cm} (6.32)

of the global moduli space of all s3/2-ph deformations of a smooth s3/2-ph cycle $Y$ in a Calabi–Yau threefold $X$ (we call it a complete linear system of s3/2-ph cycles).

2) The natural complex structure (given by the equality (6.31) is integrable and extends to the complete variety $|Y|$; then

$$Y = T^3 = S^1 \times S^1 \times S^1 \implies |Y| = X' \text{ is a CY threefold.} \hspace{1cm} (6.33)$$
But even outside the Calabi–Yau set-up, these constructions give a beautiful family of Geometries. We conclude our collection of definitions and constructions by extending them to complex threefolds with positive and negative canonical class.

In the set-up of CY threefolds, one can reproduce the following collection of notions: suppose $Y$ is a $2/3$-symplectic oriented cycle in $X$. Then the image $m_C(Y) \subset D_+^+$ is a compact subset and there exists a unique minimal disc containing this image

$$m_C(Y) \subset D_Y \subset D^+$$

(see (2.39)).

Again,

1. the centre $c_Y$ of the disc $D_Y$ is called the centre of $Y$;
2. the radius $r_Y$ of the disc $D_Y$ is called the radius of $Y$;
3. $Y$ is called $\varepsilon$-spLag if

$$c_Y \in S^1 \quad \text{and} \quad r_Y < \varepsilon$$

(6.35)

It would be great to get some analogue of Donaldson’s Theorem 2.1 for $\varepsilon$-spLag cycles.

Now consider a simple connected smooth algebraic threefold $X$ with canonical class $K_X > 0$ (or $K_X < 0$). We would like to deform slightly the $2/3$-ph geometry, by considering a distinguished family of oriented 3-cycles. Again these cycles are determined by their first order infinitesimal behavior, that is, the cycles defined by properties of their Gauss lifts.

We can lift the projective bundle $\mathbb{P}(\mathcal{O}_X \oplus \mathcal{O}_X(-K_X))$ to a vector bundle

$$V_K = \mathcal{O}_X \oplus \mathcal{O}_X(K_X) \quad \text{if } K_X > 0,$$

$$\quad \mathcal{O}_X \oplus \mathcal{O}_X(-K_X) \quad \text{if } K_X < 0.$$  

(6.36)

**General type** ($K_S > 0$). In this case, we consider a nonvanishing section $s$ (or a section vanishing along a “divisor”) to get a section of the projective bundle $\mathbb{P}(\mathcal{O}_X \oplus \mathcal{O}_X(-K_X))$, like $S_0$ and $S_\infty$ (5.48).

In the Kähler (algebraic) case we have a finite dimensional family of holomorphic sections

$$s \in H^0(V_K) = \mathbb{C} \oplus H^0(\mathcal{O}_X(K_X))$$

(6.37)

We write $p_g = \dim H^0(\mathcal{O}_X(K_X))$ for the geometric genus of $X$, that is, the complex dimension of $H^{3,0}(S)$. The family of such sections defines a family of sections of the projectivisation $\mathbb{P}V_K$

$$S^{2p_g} = \mathbb{C}^{p_g} \cup \{\infty \setminus \text{point}\},$$

(6.38)

where $\mathbb{C}^{p_g} = \{(1, s)\}$ is the space of nonvanishing sections and $\mathbb{P}H^0(\mathcal{O}_X(K_X)) = |K_X|$ is the complete canonical linear system, points of which give the same linear subbundle, that is, the same section of the projectivisation. That is, the map

$$\sigma_\infty : \mathbb{P}H^0(\mathcal{O}_X(K_X)) \to S^{2p_g}$$

(6.39)

isn’t holomorphic, and blows the hyperplane $|K_X|$ down to the point $\infty$.  

Now the Levi–Civita connection gives the Hermitian structure on the canonical bundle $L_K = \mathcal{O}_X(K_X)$ and, similarly, the Hermitian structure on $\mathbb{C}^{2p_g} \cup \{\infty \setminus \text{point}\}$ gives the standard metric on this sphere. Thus we can identify our sphere with the dual sphere (see (2.57–2.60)).

So this sphere contains the “equator”

$$S_{e}^{2p_g - 1} = \{z \mid \|z\| = 1\} \subset \mathbb{C}^{2p_g}. \quad (6.40)$$

Again interpretation of $\mathbb{C}^{2p_g} \cup \{\infty \setminus \text{point}\}$ as the space of sections gives us the embedding

$$i_{\text{can}}: \mathbb{P}V_K \rightarrow \mathbb{P}H^0(V_K)^* \times X \quad (6.41)$$

and the projection of the trivial bundle to the fibre gives the map

$$i_{\text{can}}: \mathbb{P}V_K \rightarrow \mathbb{P}H^0(V_K)^* \rightarrow S_{e}^{2p_g}. \quad (6.42)$$

Finally, the composite of this projection and the blowdown (6.39) gives the map

$$\text{pr}: \mathbb{P}V_K \rightarrow S_{e}^{2p_g}. \quad (6.43)$$

Now for any cycle $Y \subset X$, the composite of the Gauss map, the projection $\tilde{\det}_I$ (5.50) and (6.43) defines the complex phase map

$$m_{\mathbb{C}} = \text{pr} \circ \tilde{\det}_I \circ G(i): X \rightarrow S_{e}^{2p_g}. \quad (6.44)$$

Now in terms of this phase map, one can define the analogues of the sdAG and spLag cycles known in the Calabi–Yau case.

**Definition 6.6.** A cycle $Y \subset X$ is called a sdAG cycle if $m_{\mathbb{C}}(Y)$ is a point, or equivalently

$$d m_{\mathbb{C}} = 0. \quad (6.45)$$

**Definition 6.7.** 1) A 3-cycle $Y$ is called weakly Lagrangian (wLag cycle for short) if

$$m_{\mathbb{C}}(\Sigma) \subset S_{e}^{2p_g - 1}, \quad (6.46)$$

where $S_{e}^{2p_g - 1}$ is the equator;

2) $Y$ is called a spLag cycle, if it is a sdAG cycle and

$$m_{\mathbb{C}}(Y) \in S_{e}^{2p_g - 1}. \quad (6.47)$$

The equator divides the target sphere of the complex phase map into upper and lower hemispheres:

$$S_{e}^{2p_g} = D^+ \cup S_{e}^{2p_g - 1} \cup D^-, \quad (6.48)$$

and the entire catalogue of definitions (such as Definition 2.3), properties and facts can be repeated in this new set-up.

**Fano case.** Finally, it is quite easy to see what to do if $K_S < 0$: we change the sign of the canonical system $K_S \rightarrow -K_S$, getting the sphere

$$S_{h^0(\mathcal{O}_S(-K_S))} \quad (6.49)$$

as the target sphere of the complex phase map. After that, we can repeat all our constructions and definitions.
§7. Geometry of 3/2-pseudoholomorphic supercycles

Whereas the system of differential equations for spLag cycles are notoriously complicated (see [H-L]), that for s3/2-ph cycles is much simpler. But instead of describing it, we explain why s3/2 pseudoholomorphic Geometry is a slight deformation of Algebraic Geometry. There is a type of complex oriented threefolds (that is, with trivial canonical class) which are 2-connected, and thus can’t be Kähler. In this case the Geometry of s3/2-ph cycles is the unique tool for investigations. It is proper to show how the s3/2 AG works in this case.

Recall briefly the general properties of such type manifolds. The class of such manifolds was introduced by Miles Reid [R] in the set-up of the investigation of minimal models theory of complex threefolds. Almost at the same time R. Friedman in [F] proposed the basic construction of such manifolds in the set-up of his theory of infinitesimal deformations of singular complex manifolds. So it is resonable to call these complex manifolds Friedman–Reid manifolds (FR threefolds for short) in the same vein as in the Kähler case, we call it Calabi–Yau (CY for short) threefolds.

**Definition 7.1.** A FR threefold is a compact smooth complex threefold $X$ such that

$$H^p(X, \Omega^q) = 0 \quad \text{for } p + q \neq 0, 3, 6,$$

(7.1)

where $\Omega$ is the cotangent bundle of $X$, and $X$ is 2-connected:

$$\pi_n(X) = 0 \quad \text{for } n < 3.$$  

(7.2)

C.T.C. Wall proved that all FR threefolds are diffeomorphic to a connected sum of $g$ copies of $S^3 \times S^3$, where $g$ is any positive integer, which we call the genus of $X$.

The main properties of FR threefolds are just the same as of CY threefolds:

**Main properties** ([C], Proposition 1.3). For a FR threefold $X$,

1. The Hodge spectral sequence

$$E^{p,q}_2 = H^q(X, \Omega^p)$$

(7.3)

degenerates at $E_2$. Thus the cohomology of $X$ has an integral Hodge structure which is pure of weight 3.

2. The local deformation space of $X$ is unobstructed, smooth with the tangent space at $X$ equal to

$$H^1(TX) = H^1(X, \Omega^2) = \mathbb{C}^{g-1}.$$  

(7.4)

3. Let $F^2H^3(X, \mathbb{C})$ be cohomology classes which can be represented by d-closed forms of types (3,0) and (2,1). Then we get the orthogonal decomposition

$$H^3(X, \mathbb{C}) = F^2H^3(X, \mathbb{C}) \oplus \overline{F^2H^3(X, \mathbb{C})}$$

(7.5)

and the Hermitian form

$$(1/2i) \langle \alpha, \overline{\beta} \rangle.$$  

(7.6)

Recall that the signature of this Hermitian form (7.6) is called the signature of $X$.

As usual we fix a complex orientation of $X$ that is a holomorphic 3-form $\theta$ and the pair $(X, \theta)$ is called an oriented FR threefold.
The theory of periods of FR threefolds. In terms of the Hodge structure on \( X \), the existence of an integral 3-cohomology class

\[ [Y] \in H^3(X, \mathbb{Z}) \quad (7.7) \]

which can be realised as a smooth 3/2-pseudoholomorphic cycle \( Y \) gives one equation on the period domain of FR threefolds. Indeed, as for K3 surfaces, in this case

\[ \int_Y \theta = 0. \quad (7.8) \]

**Definition 7.2.** A 3-cohomology class \([Y]\) is called a *polarisation* of \( X \) if there exists a global moduli space \(|Y|\) of all deformations of \( Y \) in \( X \) as s3/2-ph cycles with the (natural) complex structure (we call it a complete linear system) such that

1) the complete linear system \(|Y|\) is an *algebraic variety* and
2) the complex dimension \( \dim |Y| > 2 \) \( (7.9) \)

**Remark.** You can see that the *dimension* of the moduli space works in the same way as *Riemann–Roch formula* plus *adjunction formula* in the case of complex surfaces.

Now we can define the analogue of the Picard lattice for FR threefolds:

**Definition 7.3.** The sublattice

\[ 3/2-\text{Pic}(X) \subset H^3(X, \mathbb{Z}) \quad (7.10) \]

generated by s3/2-ph cycles is called the *3/2-Picard lattice* of \( X \).

In the same vein, using the Hermitian form (7.6), we can define the sublattice of *transcendental cycles*:

\[ \text{Tr}(X) \subset H^3(X, \mathbb{Z}), \quad (7.11) \]

and these constructions are absolutely parallel to the standard constructions for K3 surfaces.

We now return to the local geometry around a smooth s3/2-ph cycle \( Y \).

For any 2/3-ph smooth cycle \( Y \) we have the orthogonal “Hodge decomposition”

\[ TY = \ker(\omega|_Y) \oplus TY_I \quad (7.12) \]

where \( TY_I \) is the complex directions of the tangent bundle of \( Y \).

**Definition 7.4.** A smooth s3/2-ph cycle \( Y \) is called *wrapped* iff there exists a smooth holomorphic curve

\[ C \subset Y \quad (7.13) \]

inside \( Y \).

Now to define a wrapped cycle in terms of the holomorphic curve \( C \), we have to consider the normal bundle

\[ N_{C \subset X} \quad (7.14) \]

of our curve in \( X \). The action of the complex structure operator \( I \) on the subspace \( \ker(\omega|_Y) \) (7.12) defines the subbundle

\[ L_Y = \langle \ker(\omega|_Y), I(\ker(\omega|_Y)) \rangle. \quad (7.15) \]

which is a *complex* subbundle.
Proposition 7.2. Let $Y$ be a holomorphic subbundle of the holomorphic rank 2 bundle $N_{C\subset X}$ (7.14).

Roughly speaking our $3/2$-ph cycle $Y$ wraps the zero section in the line bundle $L_Y$.

Now by the definition

$$3/2\text{-ph cycle } Y \text{ is a supercycle } \implies \deg L_Y = 0 \quad (7.16)$$

and we get

Proposition 7.3. For $3/2$-ph cycle $Y$,

1. the line bundle $L_Y$ admits a flat $U(1)$-connection,
2. that is, there exists the character of the fundamental group of $Y$

$$\rho: \pi(Y) \to U(1) \quad (7.17)$$

which gives our line bundle:

$$L_\rho = L_Y, \quad (7.18)$$

and

3. $Y = S^1(L_\rho) \quad (7.19)$

is the unit circle bundle of this flat $U(1)$-bundle. In particular, topologically

$$Y = C \times S^1. \quad (7.20)$$

Let us return to FR threefolds. Such manifold has no nontrivial line bundles, and has no divisors. But some FR threefolds have an infinity of mutually disjoint rigid elliptic curves. Consider such FR threefold $X$ with such a set $\{C_i\}$ of holomorphic elliptic curves. The normal bundle of any such curve is

$$N_{C\subset X} = \mathcal{O}_C(\xi) \oplus \mathcal{O}_C(-\xi) \quad \text{for } \xi \in \text{Pic}_0(C), \quad (7.21)$$

with

$$h^0(\mathcal{O}_C(\xi)) = h^1(\mathcal{O}_C(\xi)) = h^1(\mathcal{O}_C(-\xi)) = h^0(\mathcal{O}_C(-\xi)) = 0 \quad (7.22)$$

and any line subbundle $L$ of degree 0 is either $\mathcal{O}_C(\xi)$ or $\mathcal{O}_C(-\xi)$. So in this case we have two wrapped $3/2$-ph cycles $Y_+$ and $Y_-$ which topologically are

$$C \times S^1 = T^3 = S^1 \times S^1 \times S^1. \quad (7.23)$$

Then modulo the Great Expectations we have the complex 3-dimensional manifold $|Y_+|$ and the question is:
Typical question. How many other elliptic wrapped $s3/2$-ph cycles are in $|Y_+|$?

Remark. You can see yourself what happens if $N_{C \subset X}$ is a twisted Atiyah bundle (we get one $s3/2$-ph cycle) or the trivial bundle twisted by a point of second order of $Pic_0(C)$ (we get a $S^2$-family of $s3/2$-ph cycles).

We finish with the following construction: $3/2$-ph cycles in $X$ are what we need for the theory of FR structures. Let $X$ be a FR threefold,

$$[Y] \in 3/2\text{-Pic}(X) \quad (7.24)$$

(see (7.10)) and $Y_+$ and $Y_-$ are two $s3/2$ps cycles from this cohomology class.

Definition 7.5. Such two cycles are called K3 cobordant if there exists a smooth 4-submanifold $S$ which is a complex symplectic surface in the sense of §2 of [D2] such that

$$\partial S = Y_+ \cup Y_-;$$

that is, $S$ is a complex surface with a holomorphic symplectic form

$$\theta_2 \in \Omega^{2,0}. \quad (7.25)$$

Of course this construction relates closely to the “complexification” of the diffeomorphism group of $Y_\pm$ and to Nahm’s equations (see [D2]) but on the other hand, this complex symplectic bordism gives a special loop in the intermediate jacobian

$$J^3(X) = F^2 H^3(X, \mathbb{C})/H^3(X, \mathbb{Z}). \quad (7.26)$$

This set-up is closely related to the theory of vector bundles on FR and CY threefolds, and you can consider the paper [T] as the continuation of these considerations.

On the other hand if $Y_\pm$ are wrapped on curves $C_\pm$ the rank 2 vector bundle $E$ on the complex symplectic bordism $S$ such that

$$E|_{C_\pm} = N_{C_\pm \subset X} \quad (7.27)$$

we can consider as the analogue of a rational function between two divisors. What a large new field of Numerical Geometry! So, you can see that this subject is open for investigations and You can move this Geometry Yourself. Good Luck!

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