DIOPHANTINE APPROXIMATION OF POLYNOMIALS OVER $\mathbb{F}_q[t]$ SATISFYING A DIVISIBILITY CONDITION

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Abstract. Let $\mathbb{F}_q[t]$ denote the ring of polynomials over $\mathbb{F}_q$, the finite field of $q$ elements. We prove an estimate for fractional parts of polynomials over $\mathbb{F}_q[t]$ satisfying a certain divisibility condition analogous to that of intersective polynomials in the case of integers. We then extend our result to consider linear combinations of such polynomials as well.

1. Introduction

In 1927, Vinogradov [12] proved the following result, confirming a conjecture of Hardy and Littlewood [3]. Let $\| \cdot \|$ denote the distance to the nearest integer.

Theorem 1.1. For every positive integer $k$, there exists an exponent $\theta_k > 0$ such that

$$\min_{1 \leq n \leq N} \| \alpha n^k \| \ll n^{-\theta_k}$$

for any positive integer $N$ and real number $\alpha$.

A brief history and introduction to the topic is given in [7, Section 1], which we paraphrase here. Vinogradov showed that one could take $\theta_k = \frac{1}{k^2} - \frac{1}{k+1} - \epsilon$ for any $\epsilon > 0$. In particular, one can take $\theta_2 = 2/5 - \epsilon$. Heilbronn [4] improved this to $\theta_2 = 1/2 - \epsilon$. The best result to date is due to Zaharescu [14], who showed we can take $\theta_2 = 4/7 - \epsilon$, though his method is not applicable to higher powers. It is an open conjecture that we can choose $\theta_2$ (and more generally $\theta_k$) to be $1 - \epsilon$.

Natural generalizations of Vinogradov’s result have been made. Davenport [2] obtained an analogue of Theorem 1.1 when $n^k$ is replaced by a polynomial $f(n)$ of degree $k$ without a constant term (the corresponding bound being uniform in the coefficients of $f$ and depending only on $k$). Notably, the best bound is due to Wooley, who showed that we can choose $\theta_k = \frac{1}{4k(k-2)} - \epsilon$ for $k \geq 4$, as a consequence of his recent breakthrough [13] on Vinogradov’s mean value theorem. We note that Vinogradov’s result has also been generalized to simultaneous approximation, where we consider multiple polynomials at once. However, we focus on the single polynomial case in this paper and we refer the reader to [7] for more information on simultaneous approximation.

In contrast, Lê and Spencer put more emphasis on the qualitative side of these problems in [7]. They were interested in generalizing Theorem 1.1 in the following manner. For instance, is it possible to replace $n^k$ in Theorem 1.1 with a polynomial $h \in \mathbb{Z}[x]$? That is, for which polynomials $h \in \mathbb{Z}[x]$ do we have

$$\min_{1 \leq n \leq N} \| \alpha h(n) \| \ll_n N^{-\theta}$$

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for some $\theta = \theta(h)$, uniformly in $\alpha$ and $N$? By the result of Davenport [2] mentioned in
the previous paragraph, this is the case if $h$ is without a constant term, but apparently these
are not all the polynomials satisfying this property. By considering $\alpha = 1/q$, we see that in
order for such a bound to exist, $h$ must have a root modulo $q$ for every $q \in \mathbb{Z}^+$. Clearly, this
condition is satisfied by polynomials without constant terms. Lê and Spencer proved that
this condition is also sufficient.

**Theorem 1.2.** [7 Theorem 3] Let $h$ be a polynomial in $\mathbb{Z}[x]$ with the property that for every
$q \neq 0$, there exists $n_q \in \mathbb{Z}$, $0 \leq n_q < q$, such that $h(n_q) \equiv 0 \pmod{q}$. Then there is an
exponent $\theta > 0$ depending only on the degree of $h$ such that

$$
\min_{1 \leq n \leq N} \| \alpha h(n) \| \ll_h N^{-\theta}
$$

for any positive integer $N$ and real number $\alpha$.

Our goal in this paper is to consider analogous problems of qualitative nature over $\mathbb{F}_q[t]$, where $\mathbb{F}_q$ is a finite field of $q$ elements, taking the approach of Lê and Spencer in [7]. However,
before we can state our results we need to introduce notation, some of which we take from
the material in [3 Section 1]. We denote the characteristic of $\mathbb{F}_q$, a positive prime number,
by $\text{ch}(\mathbb{F}_q) = p$. Let $K = \mathbb{F}_q(t)$ be the field of fractions of the polynomial ring $\mathbb{F}_q[t]$. For
$f/g \in K$, we define the norm $|f/g| = q^{\deg f - \deg g}$ (with the convention that $\deg 0 = -\infty$).
The completion of $K$ with respect to this norm is $K_\infty = \mathbb{F}_q((1/t))$, the field of formal Laurent series in $1/t$. In other words, every element $\alpha \in K_\infty$ can be written as $\alpha = \sum_{i=-\infty}^n a_i t^i$ for
some $n \in \mathbb{Z}$ and $a_i \in \mathbb{F}_q (i \leq n)$. Therefore, $\mathbb{F}_q[t], K,$ and $K_\infty$ play the roles of $\mathbb{Z}, \mathbb{Q},$ and $\mathbb{R},$
respectively. Let

$$
T = \left\{ \sum_{i=-\infty}^{-1} a_i t^i : a_i \in \mathbb{F}_q \ (i \leq -1) \right\},
$$

which is the analogue of the unit interval $[0, 1]$.

For $\alpha = \sum_{i=-\infty}^n a_i t^i \in K_\infty$, if $a_n \neq 0$, we define $\text{ord} \alpha = n$. We say $\alpha$ is rational if $\alpha \in K$ and
irrational if $\alpha \not\in K$. We define $\{ \alpha \} = \sum_{i=-\infty}^{-1} a_i t^i \in T$ to be the fractional part of $\alpha$. We refer
to $a_{-1}$ as the residue of $\alpha$, denoted by $\text{res} \alpha$. We now define the exponential function
on $K_\infty$. Let $\text{tr} : F_q \rightarrow F_p$ denote the familiar trace map. There is a non-trivial additive character $e_q : F_q \rightarrow \mathbb{C}^\times$ defined for each $a \in F_q$ by taking $e_q(a) = e^{2\pi i (\text{tr}(a)/p)}$. This character
induces a map $e : K_\infty \rightarrow \mathbb{C}^\times$ by defining, for each element $\alpha \in K_\infty$, the value of $e(\alpha)$ to be
$e_q(\text{res} \alpha)$. For $N \in \mathbb{Z}^+$, we write $G_N$ for the set of all polynomials in $\mathbb{F}_q[t]$ whose degree are
less than $N$.

Given $j, r \in \mathbb{Z}^+$, we write $j \leq_p r$ if $p \nmid \binom{r}{j}$. By Lucas’ Theorem, this happens precisely
when all the digits of $j$ in base $p$ are less than or equal to the corresponding digits of $r$. From
this characterization, it is easy to see that the relation $\leq_p$ defines a partial order on $\mathbb{Z}^+$. If
$j \leq_p r$, then we necessarily have $j \leq r$. Let $K \subseteq \mathbb{Z}^+$. We say an element $k \in K$ is maximal
if it is maximal with respect to $\leq_p$, that is, for any $r \in K$, either $r \leq_p k$ or $r$ and $k$ are not
comparable. Following the notation of [6], we define the shadow of $K$, $S(K)$, to be

$$
S(K) = \left\{ j \in \mathbb{Z}^+ : j \leq_p r \text{ for some } r \in K \right\}.
$$

We also define

$$
K^* = \left\{ k \in K : p \nmid k \text{ and } p^v k \notin S(K) \text{ for any } v \in \mathbb{Z}^+ \right\}.
$$
Given \( f(u) \in \mathbb{K}_{\infty}[u] \), we mean by \( f(u) \) is supported on a set \( \mathcal{K} \subseteq \mathbb{Z}^+ \) that \( f(u) = \sum_{r \in \mathcal{K} \cup \{0\}} \alpha_r u^r \), where \( 0 \neq \alpha_r \in \mathbb{K}_{\infty} \). As explained in the remark of [6] Theorem 12, the non-zero coefficient \( \alpha_k \) for \( k \in \mathcal{K}^* \) which is maximal in \( \mathcal{K} \), plays the role of the leading coefficient of the polynomial. This is, in a sense, the “true” \( \mathbb{F}_q[t] \) analogue of the leading coefficient.

**Example 1.3.** Let \( p > 3 \). The polynomial \( f_1(u) = c_{2p^2+p}u^{2p^2+p} + c_{3p+1}u^{3p+1} + c_p u^p + c_2 u^2 + c_1 u^1 + c_0 \), where each \( c_j \neq 0 \), is supported on \( \mathcal{K}_1 = \{2p^2+p, 3p+1, p, 2, 1\} \). We can verify that

\[
S(\mathcal{K}_1) = \{2p^2 + p, 2p^2, p^2 + p, p^2, p, 3p + 1, 2p + 1, p + 1, 3p, 2p, 2, 1\}
\]

and

\[
\mathcal{K}_1^* = \{3p + 1\}.
\]

We are now in position to state one of our main results. The following theorem is an analogue of Theorem 1.2

**Theorem 1.4.** Let \( h(u) = \sum_{r \in \mathcal{K} \cup \{0\}} c_r u^r \) be a polynomial supported on a set \( \mathcal{K} \subseteq \mathbb{Z}^+ \) with coefficients in \( \mathbb{F}_q[t] \). Suppose \( c_k \neq 0 \) for some \( k \in \mathcal{K}^* \). Suppose further that for every \( g \) in \( \mathbb{F}_q[t] \\setminus \{0\} \), there exists an \( m_g \in \mathbb{G}_{\deg \mathcal{K}} \) such that \( h(m_g) \equiv 0 \pmod{g} \). Then there exist \( \theta = \theta(\mathcal{K}, q, \deg h) > 0 \) and \( N_0 = N_0(\mathcal{K}, q, h, \theta) \in \mathbb{Z}^+ \) such that for any \( N > N_0 \), we have

\[
\min_{x \in \mathbb{G}_N} \text{ord} \{\beta h(x)\} \leq -\theta N
\]

uniformly in \( \beta \in \mathbb{K}_{\infty} \).

A set \( \mathcal{H} \subseteq \mathbb{F}_q[t] \\setminus \{0\} \) is said to be *van der Corput* if the sequence \((a_x)_{x \in \mathbb{F}_q[t]} \subseteq \mathbb{K}_{\infty}\) is equidistributed in \( \mathbb{T} \) (defined analogously as in the case of \( \mathbb{R} \)), whenever the sequence \((a_{x+h} - a_x)_{x \in \mathbb{F}_q[t]} \) is equidistributed in \( \mathbb{T} \) for each \( h \in \mathcal{H} \). We remark that given a polynomial \( h(u) \) that satisfies the hypothesis of Theorem 1.4, the set \( \{h(x) : x \in \mathbb{F}_q[t]\} \\setminus \{0\} \) is van der Corput [6] Theorem 23, a topic which we do not get into in our current chapter. We instead refer the reader to [5] and [6] for more information on this topic.

It is clear that any polynomial \( h(u) \in \mathbb{F}_q[t][u] \) such that \((u - a)|h(u)\) for some \( a \in \mathbb{F}_q[t] \) satisfies the hypothesis of Theorem 1.4. However, polynomials in \( \mathbb{F}_q[t][u] \) with roots in \( \mathbb{F}_q[t] \) are not the only elements satisfying this condition.

**Example 1.5.** Let \( p = 5 \) and consider \( h(u) = (u^2 - t)(u^2 - (t + 1))(u^2 - (t^2 + t)) \in \mathbb{F}_5[t][u] \). Then \( h(u) \) does not have a root in \( \mathbb{F}_5[t] \), but it has a root modulo \( g \) for every \( g \) in \( \mathbb{F}_5[t] \\setminus \{0\} \).

We postpone the proof of this statement to [6]. Lé and Spencer also proved the following theorem in [7].

**Theorem 1.6.** [7] Theorem 6] Suppose the polynomials \( h_1, ..., h_L \) of distinct degrees are such that any linear combination of them with integer coefficients has a root modulo \( q \) for any \( q \in \mathbb{N} \). Let \( \alpha_1, ..., \alpha_L \in \mathbb{R} \). Then there is an exponent \( \theta > 0 \) (depending at most on \( h_1, ..., h_L \)) such that

\[
\min_{1 \leq n \leq N} \|\alpha_1 h_1(n) + ... + \alpha_L h_L(n)\| \ll N^{-\theta}
\]

uniformly in \( \alpha_1, ..., \alpha_L, N \).
Suppose we have polynomials \( h_1, \ldots, h_L \in \mathbb{F}_q[t][u] \), where \( h_j(u) = \sum_{r \in \mathcal{K}_j \cup \{0\}} c_{j,r} u^r \), and \( \mathcal{K}_j \subseteq \mathbb{Z}^+ \) (1 ≤ j ≤ L). Let \( \mathcal{K} = \mathcal{K}_1 \cup \ldots \cup \mathcal{K}_L \). We define the \( \mathcal{K}^* \)-portion of \( h_j \) as

\[
h^*_j(u) := \sum_{r \in \mathcal{K}_j \cap \mathcal{K}^*} c_{j,r} u^r.
\]

We say the \( \mathcal{K}^* \)-portion of \( (h_j)_{j=1}^L \) is linearly independent if \( h^*_1, \ldots, h^*_L \) are linearly independent over \( \mathbb{K} \). We also define a slightly stronger notion, the maximal \( \mathcal{K}^* \)-portion of \( h_j \) as

\[
h^\text{max}_j(u) := \sum_{r \in \mathcal{K}_j \cap \mathcal{K}^*} c_{j,r} u^r.
\]

We say the maximal \( \mathcal{K}^* \)-portion of \( (h_j)_{j=1}^L \) is linearly independent if \( h^\text{max}_1, \ldots, h^\text{max}_L \) are linearly independent over \( \mathbb{K} \). Clearly, if the maximal \( \mathcal{K}^* \)-portion of \( (h_j)_{j=1}^L \) is linearly independent, then the \( \mathcal{K}^* \)-portion of \( (h_j)_{j=1}^L \) is linearly independent. We give an example of these notions below.

**Example 1.7.** Let \( p > 3 \). Consider polynomials \( f_1(u) = c_{2p^2+p} u^{2p^2+p} + c_{3p+1} u^{3p+1} + c_p u^p + c_2 u^2 + c_1 u + c_0 \) and \( f_2(u) = c'_{p^3+3p+1} u^{p^3+3p+1} + c'_{p+1} u^{p+1} + c'_{2p+1} u^{2p+1} \) in \( \mathbb{F}_q[t][u] \), where each \( c_j \) and \( c'_j \) are non-zero elements of \( \mathbb{F}_q[t] \). In other words, \( f_1(u) \) and \( f_2(u) \) are supported on \( \mathcal{K}_1 = \{2p^2 + p, 3p + 1, p, 2, 1\} \) and \( \mathcal{K}_2 = \{p^3 + 3p + 1, p^2 + 1, 2p + 1\} \), respectively. Thus we let

\[
\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2 = \{p^3 + 3p + 1, 2p^2 + p, p^2 + 1, 3p + 1, 2p + 1, p, 2, 1\},
\]

and we can verify that \( \mathcal{K}^* = \{p^3 + 3p + 1, 3p + 1\} \).

It follows that

\[
f^*_1(u) = c_{3p+1} u^{3p+1} \quad \text{and} \quad f^*_2(u) = c'_{p^3+3p+1} u^{p^3+3p+1},
\]

which are clearly linearly independent over \( \mathbb{K} \). Therefore, \( \mathcal{K}^* \)-portion of \( (f_1, f_2) \) is linearly independent. However, note \( p^3 + 3p + 1 \) is maximal in \( \mathcal{K} \), but not \( 3p + 1 \). Thus we have

\[
f^\text{max}_1(u) = 0 \quad \text{and} \quad f^\text{max}_2(u) = c'_{p^3+3p+1} u^{p^3+3p+1},
\]

and consequently, the maximal \( \mathcal{K}^* \)-portion of \( (f_1, f_2) \) is not linearly independent.

The following theorem is an analogue of Theorem 1.6.

**Theorem 1.8.** Let \( h_j \in \mathbb{F}_q[t][u] \) be supported on a set \( \mathcal{K}_j \subseteq \mathbb{Z}^+ \) (1 ≤ j ≤ L), and let \( \mathcal{K} = \mathcal{K}_1 \cup \ldots \cup \mathcal{K}_L \). Suppose any linear combination of them with \( \mathbb{F}_q[t] \) coefficients has a root modulo \( g \) for any \( g \in \mathbb{F}_q[t] \setminus \{0\} \). Suppose further that the \( \mathcal{K}^* \)-portion of \( (h_j)_{j=1}^L \) is linearly independent. Then there exist \( \theta = (\mathcal{K}, q, \max_{1 \leq j \leq L} \deg h_j) > 0 \) and \( N_0 = N_0(\mathcal{K}, q, \theta, h_1, \ldots, h_L) \in \mathbb{Z}^+ \) such that for any \( N > N_0 \), we have

\[
\min_{x \in \mathbb{G}_N} \text{ord} \{ \beta_1 h_1(x) + \ldots + \beta_L h_L(x) \} \leq -\theta N
\]

uniformly in \( \beta_1, \ldots, \beta_L \in \mathbb{K}_\infty \).

We give an example of a system of polynomials \( (h_1, h_2) \subseteq \mathbb{F}_q[t][u] \) that satisfies the hypothesis of Theorem 1.8 in Example 1.2. We note that these polynomials \( h_1(u) \) and \( h_2(u) \) do not have a common root in \( \mathbb{F}_5[t] \), but they do have \( h(u) \) from Example 1.5 as a common factor. There may well be examples of systems \( (h_1, \ldots, h_L) \) without a common factor such that any linear combination of them with \( \mathbb{F}_q[t] \) coefficients has a root modulo \( g \) for any \( g \in \mathbb{F}_q[t] \setminus \{0\} \), but we do not have such an example in hand at this time.
We also prove an analogue of [7, Theorem 7] in Theorem 4.1 which is a (partial) generalization of Theorem 1.8. However, we defer stating the result to Section 4 in order to avoid introducing further notation here.

The organization of the rest of the paper is as follows. In Section 2, we introduce some notation and notions required to carry out our discussions in the setting over \( \mathbb{F}_q[t] \). In Section 3 we prove lemmas involving basic linear algebra utilized in the proof of our main results given in Section 4. We provide the proof of the statement in Example 1.5 in A. We note that Lé and Spencer generalized [7, Theorem 7], which Theorem 4.1 is an analogue of, and obtained results on simultaneous approximation [7, Theorems 4 and 8]. However, due to complications that arose during our attempt from certain arguments in linear algebra and geometry of numbers in the setting over \( \mathbb{F}_q[t] \), at present time we decided to leave generalizing Theorem 4.1 in a similar manner as a possible future work. Finally, in the case when the polynomials in question do not have constant terms, a more general result is available due to Spencer and Wooley [1]. We note that their result does not require the extra hypothesis on the coefficients as in this paper.

2. Preliminaries

Suppose a system of polynomials \((h_1, \ldots, h_L)\) satisfies the following,

Condition \((\star)\): For every \( g \in \mathbb{F}_q[t] \setminus \{0\} \), there exists \( m_g \in \mathbb{F}_q[t] \) such that \( h_i(m_g) \equiv 0 \pmod{g} \) for \( i = 1, \ldots, L \).

In the case of \( \mathbb{Z} \) (in place of \( \mathbb{F}_q[t] \)), such a system of polynomials satisfying the analogous condition is called jointly interesective polynomials.

We have the following analogue of [1] Proposition 6.1, which we omit the proof of.

**Lemma 2.1.** A system of polynomials \((h_1, \ldots, h_L)\) in \( \mathbb{F}_q[t][u] \) satisfies Condition \((\star)\) if and only if there exists a polynomial \( d \in \mathbb{F}_q[t][u] \), which has a root modulo \( g \) for every \( g \in \mathbb{F}_q[t] \setminus \{0\} \), and \( d|h_i \) (\( 1 \leq i \leq L \)) over \( \mathbb{F}_q[t] \).

Let \( w \) be a monic irreducible polynomial in \( \mathbb{F}_q[t] \). Let \( \lambda_N \) be the canonical projection from \( \mathbb{F}_q[t]/w^{N+1}\mathbb{F}_q[t] \) to \( \mathbb{F}_q[t]/w^N\mathbb{F}_q[t] \). For each \( w \), we define the projective limit

\[
\lim_N \mathbb{F}_q[t]/w^N\mathbb{F}_q[t] = \left\{ (x_i)_{i \in \mathbb{N}} \in \prod_{i=1}^{\infty} \mathbb{F}_q[t]/w^i\mathbb{F}_q[t] : \lambda_i(x_{i+1}) = x_i, i = 1, 2, \ldots \right\}.
\]

Take \( \bar{x} = (x_i)_{i \in \mathbb{N}} \in \lim_N \mathbb{F}_q[t]/w^N\mathbb{F}_q[t] \). We say that \( \bar{x} \) is a solution to the equation \( f(u) = 0 \), if \( \bar{x} \) satisfies

\[
f(x_i) \equiv 0 \pmod{w^i}
\]

for all \( i \in \mathbb{N} \).

We have the following lemma, which its proof follows closely that of the \( p \)-adic integers, for example see [9, Chapter II, Proposition 1.4].

**Lemma 2.2.** Let \( f \) be a polynomial in \( \mathbb{F}_q[t][u] \) and \( w \) a monic irreducible in \( \mathbb{F}_q[t] \). Then \( f \) has a root modulo \( w^N \) for every \( N \in \mathbb{N} \) if and only if the equation \( f(u) = 0 \) has a solution in \( \lim_N \mathbb{F}_q[t]/w^N\mathbb{F}_q[t] \).
We leave the following lemma as an exercise for the reader.

**Lemma 2.3.** Let \( f \) be a polynomial in \( \mathbb{F}_q[t][u] \). Then \( f \) has a root modulo \( g \) for every \( g \in \mathbb{F}_q[t]\backslash\{0\} \) if and only if for every monic irreducible \( w \), the equation \( f(u) = 0 \) has a solution in \( \lim_{N \to \infty} \mathbb{F}_q[t]/w^N\mathbb{F}_q[t] \).

Corresponding to any system of polynomials \( (h_1, ..., h_L) \) satisfying Condition (*), there exists \( d \in \mathbb{F}_q[t][u] \) satisfying the conditions of Lemma 2.1. Given a monic irreducible \( w \), by Lemma 2.3 we know there exists \( (r_w) \in \lim_{N \to \infty} \mathbb{F}_q[t]/w^N\mathbb{F}_q[t] \) which is a solution to \( d(u) = 0 \), in other words \( d(r_w) \equiv 0 \pmod{w^j} \) and \( r_{w_j} \equiv r_{w_{j+1}} \pmod{w^j} \) for all \( j \in \mathbb{N} \). We fix such a solution for each \( w \). Suppose we are given \( g = a \prod_{i=1}^{T} w_i^{S_i} = ag_1 \), where the \( w_i \)'s are distinct monic irreducibles in \( \mathbb{F}_q[t] \) and \( a \in \mathbb{F}_q \). By the Chinese Remainder Theorem, we define \( r_g \) to be the unique element in \( \mathbb{F}_q[t]/(g_1) \) such that \( r_g \equiv r_{w_i}^{S_i} \pmod{w_i^{S_i}} \) for \( 1 \leq i \leq T \). Since \( d(r_g) \equiv 0 \pmod{w_i^{S_i}} \) for \( 1 \leq i \leq T \), it follows that \( d(r_g) = 0 \pmod{g} \). Suppose we have \( y = b \prod_{i=1}^{T} w_i^{S_i} \), where \( S_i \leq S_1 \) and \( b \in \mathbb{F}_q \), so that \( y \equiv g \). Then since \( r_g \equiv r_{w_i}^{S_i} \equiv r_{w_i}^{S_i} \pmod{w_i^{S_i}} \) for \( 1 \leq i \leq T \), we obtain \( r_g \equiv r_y (\text{mod } y) \). Finally, for \( a \in \mathbb{F}_q \) we let \( r_a = 0 \).

Therefore, corresponding to any system of polynomials \( (h_1, ..., h_L) \) satisfying Condition (*), we can associate a sequence \( (r_x)_{x \in \mathbb{F}_q[t]\backslash\{0\}} \subseteq \mathbb{F}_q[t] \) such that for any \( m, y \in \mathbb{F}_q[t]\backslash\{0\} \), \( r_y \in \mathcal{G}_{\text{ord } y}, r_{my} \equiv r_y (\text{mod } y) \), and

\[
(2.1) \quad h_j(r_y) \equiv 0 (\text{mod } y) \quad (1 \leq j \leq L).
\]

We note that the approach to define the sequence \( (r_x)_{x \in \mathbb{F}_q[t]\backslash\{0\}} \) here was taken from [8], which deals with the case of \( \mathbb{Z} \).

For any element \( \alpha \in \mathbb{K}_\infty \), it is easy to see that

\[
\text{ord } \{\alpha\} = \min_{z \in \mathbb{F}_q[t]} \text{ord } (\alpha - z),
\]

where the minimum is achieved when \( z = \alpha - \{\alpha\} \), the integral part of \( \alpha \). Also for \( \alpha_1, ..., \alpha_L \in \mathbb{K}_\infty \), we have

\[
(2.2) \quad \text{ord } \left\{ \sum_{j=1}^{L} \alpha_j \right\} \leq \text{ord } \left( \sum_{j=1}^{L} \alpha_j - \sum_{j=1}^{L} (\alpha_j - \{\alpha_j\}) \right) = \text{ord } \left( \sum_{j=1}^{L} \{\alpha_j\} \right) \leq \max_{1 \leq j \leq L} \text{ord } \{\alpha_j\}.
\]

**Lemma 2.4.** Let \( \beta_1, \beta_2, ..., \beta_R \in \mathbb{K}_\infty \) and suppose \( \text{ord } \{\beta_j\} \geq -M \) \((1 \leq j \leq R)\). Then there exists \( x \in \mathcal{G}_M \backslash \{0\} \) such that

\[
\left| \sum_{j=1}^{R} e(x; \beta_j) \right| \geq \frac{R}{q^M - 1}.
\]

**Proof.** For \( \alpha \in \mathbb{K}_\infty \), we have by [10, Lemma 7]

\[
(2.3) \quad \sum_{x \in \mathcal{G}_M} e(x\alpha) = \begin{cases} \qquad q^M, & \text{if } \text{ord } \{\alpha\} < -M, \\ \quad 0, & \text{if } \text{ord } \{\alpha\} \geq -M. \end{cases}
\]
Since \( \text{ord} \{ \beta_j \} \geq -M \) (1 \( \leq j \leq R \)), we have
\[
\sum_{j=1}^{R} \sum_{x \in G_M} e(x\beta_j) = 0.
\]
Therefore, it follows that
\[
\sum_{x \in G_M \setminus \{0\}} \left| \sum_{j=1}^{R} e(x\beta_j) \right| \geq R,
\]
from which we obtain our result. \( \square \)

We invoke the following result from [6]. The theorem allows us to estimate certain coefficients of a polynomial \( f(u) \) by an element in \( \mathbb{K} \) when the exponential sum of \( f(u) \) is sufficiently large.

**Theorem 2.5.** [6, Theorem 15] Let \( f(u) = \sum_{r \in \mathbb{K} \cup \{0\}} \alpha_r u^r \) be a polynomial supported on a set \( \mathcal{K} \subseteq \mathbb{Z}^+ \) with coefficients in \( \mathbb{K}_\infty \). Then for any \( k \in \mathcal{K}^* \), there exist constants \( c_k, C_k > 0 \), depending only on \( \mathcal{K} \) and \( q \), such that the following holds: suppose that for some \( 0 < \eta \leq c_k N \), we have
\[
\left| \sum_{x \in \mathbb{G}_N} e(f(x)) \right| \geq q^{N-\eta}.
\]
Then for any \( \epsilon > 0 \) and \( N \) sufficiently large in terms of \( \mathcal{K} \), \( \epsilon \) and \( q \), there exist \( a_k, g_k \in \mathbb{F}_q[t] \) such that
\[
\text{ord} (g_k\alpha_k - a_k) < -kN + \epsilon N + C_k \eta \quad \text{and} \quad \text{ord} g_k \leq \epsilon N + C_k \eta.
\]

We have the following corollary where we replace the polynomial \( g_k \in \mathbb{F}_q[t] \) and constants \( c_k, C_k > 0 \) in the statement of Theorem 2.5 with \( g \in \mathbb{F}_q[t] \) and \( c, C > 0 \), which are independent of the choice of \( k \in \mathcal{K}^* \), respectively.

**Corollary 2.6.** Let \( f(u) = \sum_{r \in \mathbb{K} \cup \{0\}} \alpha_r u^r \) be a polynomial supported on a set \( \mathcal{K} \subseteq \mathbb{Z}^+ \) with coefficients in \( \mathbb{K}_\infty \). There exist constants \( c, C > 0 \), depending only on \( \mathcal{K} \) and \( q \), such that the following holds: suppose that for some \( 0 < \eta \leq cN \), we have
\[
\left| \sum_{x \in \mathbb{G}_N} e(f(x)) \right| \geq q^{N-\eta}.
\]
Then for any \( \epsilon > 0 \) and \( N \) sufficiently large in terms of \( \mathcal{K} \), \( \epsilon \) and \( q \), there exists \( g \in \mathbb{F}_q[t] \) such that
\[
\text{ord} \{ g\alpha_k \} < -kN + \epsilon N + C \eta \quad (k \in \mathcal{K}^*) \quad \text{and} \quad \text{ord} g \leq \epsilon N + C \eta.
\]
**Proof.** For each \( k \in \mathcal{K}^* \), let \( c_k, C_k \) be the constants, depending only on \( \mathcal{K} \) and \( q \), and \( a_k, g_k \) be the polynomials from the statement of Theorem 2.5. Let \( c = \min_{k \in \mathcal{K}^*} c_k \) and \( C = \max_{k \in \mathcal{K}^*} C_k \). We let \( g = \prod_{k \in \mathcal{K}^*} g_k \) and \( C' = |\mathcal{K}^*|C \). Since \( \text{ord} g_k \leq \epsilon N + C_k \eta \) \((k \in \mathcal{K}^*)\), it follows that
\[
\text{ord} g \leq |\mathcal{K}^*| \epsilon N + C' \eta.
\]
We also obtain
\[
\text{ord} \{ g\alpha_k \} \leq \text{ord} \left( g\alpha_k - a_k \prod_{j \in \mathcal{K}^* \setminus \{k\}} g_j \right) \leq -kN + |\mathcal{K}^*| \epsilon N + C' \eta.
\]
\( \square \)
We note that all of our main results, Theorems 1.4, 1.8 and 4.1, rely on Corollary 2.6, which explains the reason for our assumptions on the coefficients of the polynomials in these theorems.

3. Basic Linear Algebra

In this section, we prove lemmas involving basic linear algebra which are utilized in the proofs of our main results. Given a polynomial \( f(u) \in \mathbb{K}_\infty[u] \), we use the notation \([f]_i\) to mean the \(u^i\) coefficient of \(f\). We have the following lemma, which is an analogue of [7, Lemma 1].

**Lemma 3.1.** Suppose \( d, s \in \mathbb{F}_q[t] \), \( d \neq 0 \), and \( f_1, \ldots, f_L \in \mathbb{F}_q[t][u] \) with \( \deg f_1 < \ldots < \deg f_L \). There exist polynomials \( g_1, \ldots, g_L \in \mathbb{F}_q[t][u] \), depending on \( d \) and \( s \), and an \( L \times L \) matrix \( A \) with entries in \( \mathbb{F}_q[t] \) satisfying the following properties:

1. \( A \left( \begin{array}{c} f_1(du + s) \\ \vdots \\ f_L(du + s) \end{array} \right) = \left( \begin{array}{c} g_1(u) \\ \vdots \\ g_L(u) \end{array} \right) \)

2. \( A \) is lower triangular with entries in \( \mathbb{F}_q[t] \). All its diagonal entries are equal to a constant \( c \in \mathbb{F}_q[t] \) depending only on \( f_1, \ldots, f_L \). In fact, every entry of \( A \) is dependent at most on \( s \) and \( f_1, \ldots, f_L \).

3. We have \([g_j]_{\deg g_j} = 0\) if \( i \neq j \). Also, \( \deg g_j = \deg f_j \) and \([g_j]_{\deg g_j} = cd^{\deg f_j}[f_j]_{\deg f_j}\) for all \( 1 \leq j \leq L \).

**Proof.** Let \( A' = (a_{i,j}) \) be a lower triangular matrix with all entries on the main diagonal equal to 1. For each \( 1 \leq i \leq L \), one can successively select elements in \( \mathbb{K} \), \( a_{i,i-1}, \ldots, a_{i,1} \) so that in the polynomial

\[
h_i(u) = a_{i,1}f_1(du + s) + a_{i,2}f_2(du + s) + \ldots + a_{i,i-1}f_{i-1}(du + s) + f_i(du + s),
\]

the coefficient of \( u^{\deg f_j} \) is 0 for every \( j < i \). We prove by induction that \( a_{i,j} \) \( (j < i) \) depend only on \( s \) and \( f_1, \ldots, f_L \), and that their denominators depend only on \( f_1, \ldots, f_L \). Fix \( 1 \leq i \leq L \). For the base case \( j = i - 1 \), we have

\[
0 = [h_i]_{\deg f_{i-1}} = [a_{i,i-1}f_{i-1}(du + s) + f_i(du + s)]_{\deg f_{i-1}}
\]

\[
= a_{i,i-1}[f_{i-1}]_{\deg f_{i-1}}d^{\deg f_{i-1}} + \sum_{l=\deg f_{i-1}}^{\deg f_i} [f_i]_l \binom{l}{\deg f_{i-1}} d^{\deg f_{i-1}} s^{l-\deg f_{i-1}}.
\]

By rearranging the last equality above, we obtain the following equation

\[
a_{i,i-1} = \frac{-1}{[f_{i-1}]_{\deg f_{i-1}}} \sum_{l=\deg f_{i-1}}^{\deg f_i} [f_i]_l \binom{l}{\deg f_{i-1}} s^{l-\deg f_{i-1}},
\]
which we deduce our base case from. Suppose the statement holds for \( j_0 < j < i \). Then we have by similar calculations as above and the induction hypothesis that

\[
0 = [h_i]_{\deg f_{j_0}} \\
= [a_{i,j_0} f_{j_0}(du + s) + ... + a_{i,i-1} f_{i-1}(du + s) + f_i(du + s)]_{\deg f_{j_0}} \\
= a_{i,j_0} d^{\deg f_{j_0}} [f_{j_0}]_{\deg f_{j_0}} + d^{\deg f_{j_0}} \bar{a},
\]

where \( \bar{a} \in \mathbb{K} \) depends only on \( s \) and \( f_1, ..., f_L \), and its denominator depends only on \( f_1, ..., f_L \).

We then obtain our claim for \( \bar{a} \) by rearranging the last equation displayed above. Let \( c \in \mathbb{F}_q[t] \) be the common denominator of the non-zero entries in \( \mathcal{A} \); the matrix \( \mathcal{A} = c \mathcal{A}' \) and the polynomials \( g_j(u) = ch_j(u) \) \((1 \leq j \leq L)\) satisfy the desired properties.

By Lemma 3.1 we obtain Lemmas 3.2 and 3.3 which involve polynomials with \( \mathcal{K}^* \)-portion and maximal \( \mathcal{K}^* \)-portion, respectively, that are linearly independent.

**Lemma 3.2.** Let \( h_j \in \mathbb{F}_q[t][u] \) be supported on a set \( \mathcal{K}_j \subseteq \mathbb{Z}^+ \(1 \leq j \leq L)\), and let \( \mathcal{K} = \mathcal{K}_1 \cup ... \cup \mathcal{K}_L \). Suppose the \( \mathcal{K}^* \)-portion of \( (h_j)_{j=1}^L \) is linearly independent. Let \( \beta_1, ..., \beta_L \in \mathbb{K}_\infty \). Then we can find an \( L \times L \) matrix \( \mathcal{T} \) with entries in \( \mathbb{F}_q[t] \) and \( g_j \in \mathbb{F}_q[t][u] \(1 \leq j \leq L)\) with the following properties:

1. \( g_j \) is a polynomial supported on a subset of \( \mathcal{K} \).

2. \( \mathcal{T} \left( \begin{array}{c} h_1(u) \\ \vdots \\ h_L(u) \end{array} \right) = \left( \begin{array}{c} g_1(u) \\ \vdots \\ g_L(u) \end{array} \right) \).

3. There exist \( T_j \in \mathcal{K}^* \(1 \leq j \leq L) \) such that \([g_i]_{T_j} = 0 \) if \( i \neq j \).

4. There exist \( \gamma_j \in \mathbb{K}_\infty \(1 \leq j \leq L) \) such that

\[
\beta_1 h_1(u) + ... + \beta_L h_L(u) = \gamma_1 g_1(u) + ... + \gamma_L g_L(u).
\]

**Proof.** By the hypothesis, the polynomials \( \{h_j^*\}_{j=1}^L \) are linearly independent over \( \mathbb{K} \). Therefore, we can find an \( L \times L \) invertible matrix \( \mathcal{B} \) with entries in \( \mathbb{K} \) such that

\[
\mathcal{B} (h_1^*, ..., h_L^*)^T = (b_1, ..., b_L)^T,
\]

where \( b_j \in \mathbb{F}_q[t][u] \) with coefficients supported on a subset of \( \mathcal{K}^* \) and \( \deg b_1 < ... < \deg b_L \). Let \( \mathcal{A} \) and \( g_1', ..., g_L' \) be the matrix and polynomials, respectively, obtained by applying Lemma 3.1 to the polynomials \( b_1, ..., b_L \) with \( d = 1 \) and \( s = 0 \). It follows that the polynomials \( g_j' \) have coefficients supported on a subset of \( \mathcal{K}^* \). Let \( T_j = \deg g_j' = \deg b_j \in \mathcal{K}^*(1 \leq j \leq L) \). Also let

\[
(g_1'', ..., g_L'')^T = \mathcal{A} \mathcal{B} (h_1 - h_1^*, ..., h_L - h_L^*)^T
\]

and

\[
g_j := g_j' + g_j'' \(1 \leq j \leq L).\]

Let \( c_j \) be the common denominator of the coefficients of \( g_j \in \mathbb{K}[u] \(1 \leq j \leq L) \), \( c' \) be the common denominator of the matrix \( \mathcal{A} \mathcal{B} \), and \( c = c' \prod_{j=1}^L c_j \). By construction, we see that \( c g_j \) is a polynomial in \( \mathbb{F}_q[t][u] \) with coefficients supported on a subset of \( \mathcal{K} \),

\[
(c g_j)^* = c(g_j') = c g_j'.
\]
and

\[(cg_1(u), ..., cg_L(u))^T = cA B (h_1(u), ..., h_L(u))^T.\]

Since \([g_i']_{T_j} = 0\) if \(i \neq j\), it follows that \([cg_i]_{T_j} = 0\) if \(i \neq j\). Let

\[(\gamma_1, ..., \gamma_L) = (\beta_1, ..., \beta_L) (cA B)^{-1}.\]

Then we have

\[\gamma_1cg_1(u) + ... + \gamma_Lcg_L(u) = \beta_1h_1(u) + ... + \beta_Lh_L(u).\]

Therefore, we see that the matrix \(cAB\) and the polynomials \(cg_j\) \((1 \leq j \leq L)\) satisfy the desired properties.

Let \(f(u) = \sum_{r \in \mathcal{K} \cup \{0\}} \alpha_r u^r\) be a polynomial supported on a set \(\mathcal{K} \subseteq \mathbb{Z}^+\) with coefficients in \(\mathbb{K}_\infty\). For any \(r \in \mathcal{K}\) and \(y, s \in \mathbb{F}_q[t]\), we have

\[(y + s)^r = \sum_{j \leq r} \binom{r}{j} y^j s^{r-j} + s^r.\]

Therefore, for a fixed \(s\), if \(k\) is maximal in \(\mathcal{K}\), then there exist \(\alpha'_j = \alpha'_j(\{\alpha_r\}_{r \in \mathcal{K}}; s) \in \mathbb{K}_\infty\) \((j \in S(\mathcal{K}) \setminus \{k\})\) and \(\alpha'_0 = \alpha'_0(\{\alpha_r\}_{r \in \mathcal{K} \cup \{0\}}; s) \in \mathbb{K}_\infty\) such that

\[f(y + s) = \alpha_k(y + s)^k + \sum_{r \in \mathcal{K} \setminus \{k\}} \alpha_r(y + s)^r + \alpha_0 = \alpha_k y^k + \sum_{j \in S(\mathcal{K}) \setminus \{k\}} \alpha'_j y^j + \alpha'_0.\]

In other words, the \(y^k\) coefficient of \(f(y)\) and \(f(y + s)\) are the same. Therefore, it follows that if \(k_1, ..., k_M\) are maximal in \(\mathcal{K}\), then

\[(3.1) \quad f(y + s) = \sum_{i=1}^{M} \alpha_{k_i} y^{k_i} + \sum_{j \in S(\mathcal{K}) \setminus \{k_1, ..., k_M\}} \alpha'_j y^j + \alpha'_0.\]

**Lemma 3.3.** Let \(h_j \in \mathbb{F}_q[t][u]\) be supported on a set \(\mathcal{K}_j \subseteq \mathbb{Z}^+\) \((1 \leq j \leq L)\), and let \(\mathcal{K} = \mathcal{K}_1 \cup ... \cup \mathcal{K}_L\). Suppose the maximal \(\mathcal{K}^*\)-portion of \((h_j)_{j=1}^L\) is linearly independent. Let \(\beta_1, ..., \beta_L \in \mathbb{K}_\infty\) and \(s, d \in \mathbb{F}_q[t]\) with \(d \neq 0\). Then we can find \(g_j \in \mathbb{F}_q[t][u] (1 \leq j \leq L)\), depending on \(d\) and \(s\), and an \(L \times L\) matrix \(T\) with entries in \(\mathbb{F}_q[t]\) with the following properties:

1. \(g_j\) is a polynomial supported on a subset of \(S(\mathcal{K})\) and every entry of \(T\) depends only on \(h_1, ..., h_L\).

2. \(T \begin{pmatrix} h_1(du + s) \\ \vdots \\ h_L(du + s) \end{pmatrix} = \begin{pmatrix} g_1(u) \\ \vdots \\ g_L(u) \end{pmatrix} \).

3. For \(x \in \mathbb{F}_q[t]\), we have

\[\text{ord } g_j(x) \leq \left(\max_{1 \leq j \leq L} \text{deg } h_j\right) \text{ord } (dx + s) + D,\]

where \(D\) is some constant dependent only on \(h_1, ..., h_L\).

4. There exist \(T_j \in \mathcal{K}^* \ (1 \leq j \leq L)\) such that \(T_j\) is maximal in \(\mathcal{K}\) and \([g_i]_{T_j} = 0\) if \(i \neq j\). Moreover, we have \([g_j]_{T_j} = \bar{c}_j d^{T_j}\) for some \(\bar{c}_j \in \mathbb{F}_q[t]\) dependent only on \(h_1, ..., h_L\).
(5) There exist $\gamma_j \in \mathbb{K}_\infty$ (1 ≤ $j$ ≤ $L$) such that
\[ \beta_1 h_1(du + s) + \ldots + \beta_L h_L(du + s) = \gamma_1 g_1(u) + \ldots + \gamma_L g_L(u). \]

Proof. Let $h_j(u) = \sum_{r \in \mathcal{K}_j \cup \{0\}} c_{j,r} u^r$ for 1 ≤ $j$ ≤ $L$. We also let $\mathcal{H}_j = \{ r \in \mathcal{K}_j \cap \mathcal{K}^* : r \text{ is maximal in } \mathcal{K} \}$ so that $h_j^{\max}(u) = \sum_{r \in \mathcal{H}_j} c_{j,r} u^r$. Let $\mathcal{H} = \mathcal{H}_1 \cup \ldots \cup \mathcal{H}_L$. We have by (3.1), the maximality condition of $r \in \mathcal{H}_j$, that
\[ h_j(du + \bar{s}) = \sum_{r \in \mathcal{H}_j} c_{j,r} u^r + \sum_{v \in (\mathcal{S}(\mathcal{K}_j) \setminus \mathcal{H}_j) \cap \mathcal{H}_j} c'_{j,v} u^v + c'_{j,0} \]
for some $c'_{j,0}, c'_{j,v} \in \mathbb{F}_q[t]$ (1 ≤ $j$ ≤ $L, v \in \mathcal{S}(\mathcal{K}_j) \setminus \mathcal{H}_j$). For any $l \neq j$, we have $(\mathcal{S}(\mathcal{K}_j) \setminus \mathcal{H}_j) \cap \mathcal{H}_l = \emptyset$. Suppose $v \in (\mathcal{S}(\mathcal{K}_j) \setminus \mathcal{H}_j) \cap \mathcal{H}_l$. Since $v$ is maximal in $\mathcal{K}$, the only way $v$ can be an element of $\mathcal{S}(\mathcal{K}_j)$ is if $v \in \mathcal{K}_j$. However, this forces $v \in \mathcal{H}_j$ which is a contradiction. Therefore, we can in fact write $h_j(du + \bar{s})$ as
\[ h_j(du + \bar{s}) = h_j^{\max}(du) + \sum_{v \in (\mathcal{S}(\mathcal{K}_j) \setminus \mathcal{H}_j) \cap \mathcal{H}_j} c'_{j,v} u^v + c'_{j,0}. \]

By the hypothesis, the polynomials $\{h_j^{\max}\}_{j=1}^L$ are linearly independent over $\mathbb{K}$. Therefore, we can find an $L \times L$ invertible matrix $B$ with entries in $\mathbb{F}_q[t]$ such that
\[ B \begin{pmatrix} h_1^{\max}(du) \\ \vdots \\ h_L^{\max}(du) \end{pmatrix} = (b_1, \ldots, b_L)^T, \]
where $b_j \in \mathbb{F}_q[t][u]$ (1 ≤ $j$ ≤ $L$) with coefficients supported on a subset of $\mathcal{H}$ and $\deg b_1 < \ldots < \deg b_L$. The entries of the matrix $B$ and the polynomials $b_1, \ldots, b_L$ are dependent only on $h_1^{\max}, \ldots, h_L^{\max}$. Clearly we have
\[ B \begin{pmatrix} h_1^{\max}(du) \\ \vdots \\ h_L^{\max}(du) \end{pmatrix} = (b_1(du), \ldots, b_L(du))^T. \]
Let $A$ and $g_1', \ldots, g_L'$ be the matrix and polynomials, respectively, obtained by applying Lemma 3.1 to the polynomials $b_1, \ldots, b_L$ with $s = 0$ and $d$. It follows that the coefficients of $g_j'$ (1 ≤ $j$ ≤ $L$) are supported on a subset of $\mathcal{H}$. Note by (2) of Lemma 3.1 the entries of $A$ depend only on $h_1, \ldots, h_L$. Let $T_j = \deg g_j' = \deg b_j \in \mathcal{H}$ (1 ≤ $j$ ≤ $L$). We have by (3) of Lemma 3.1 that $[g_j'][T_j] = cdT_j[b_j][T_j]$ for some $c \in \mathbb{F}_q[t]$ dependent only on $b_1, \ldots, b_L$, and $[g_j'][T_j] = 0$ if $i \neq j$. Let
\[ (g_1''(du), \ldots, g_L''(du))^T = AB \begin{pmatrix} h_1(du + \bar{s}) - h_1^{\max}(du) \\ \vdots \\ h_L(du + \bar{s}) - h_L^{\max}(du) \end{pmatrix}. \]
We define polynomials $g_j$ by
\[ g_j := g_j' + g_j'' \quad (1 \leq j \leq L), \]
then we have
\[ (g_1(u), \ldots, g_L(u))^T = AB \begin{pmatrix} h_1(du + \bar{s}) \\ \vdots \\ h_L(du + \bar{s}) \end{pmatrix}. \]
By construction, we see that $g_j$ and $g_j''$ are polynomials in $\mathbb{F}_q[t][u]$ with coefficients supported on a subset of $\mathcal{S}(\mathcal{K})$ and a subset of $\mathcal{S}(\mathcal{K}) \setminus \mathcal{H}$, respectively. Then (4) of this lemma follows by the fact that $[g_i'][T_j] = [g_j'][T_j]$ (1 ≤ $i, j$ ≤ $L$).

Let
\[ (\gamma_1, \ldots, \gamma_L) = (\beta_1, \ldots, \beta_L) \quad (AB)^{-1}. \]
Then we have
\[ \gamma_1 g_1(u) + \ldots + \gamma_L g_L(u) = \beta_1 h_1(du + \bar{s}) + \ldots + \beta_L h_L(du + \bar{s}). \]
Finally, recall from above that the entries of matrices $A$ and $B$ are dependent only on $h_1, ..., h_L$. Then (3) of this lemma follows easily from (3.3).

4. Proof of the Main Results

We have collected enough material in the previous sections to prove our main results of the paper. We begin this section by proving Theorems 1.4 and 1.8.

**Proof of Theorem 1.4.** Let $\beta$ be an arbitrary element in $K_\infty$. Let $M = \lceil \theta N \rceil + 1$, where $\theta$ is a sufficiently small positive number to be chosen later. We prove by contradiction that for any $N$ sufficiently large,

$$\min_{x \in G_N} \text{ord} \{ \beta h(x) \} \leq -M \leq -\theta N.$$ 

Suppose for some $N$ sufficiently large in terms of $K$, $q$, $\theta$, and $\text{ord} c_k$, we have $\text{ord} \{ \beta h(x) \} > -M$ for all $x \in G_N$. Then by Lemma 2.4, there exists $y \in G_M \backslash \{0\}$ such that

$$\left| \sum_{x \in G_N} e^y (\beta h(x)) \right| \geq \frac{q^N}{q^M - 1} > q^{N-M}.$$ 

It follows by Corollary 2.6 that for $\theta < c$ there exists $g \in \mathbb{F}_q[t]$ such that $\text{ord} g < CM$ and $\text{ord} \{ gy\beta z \} \leq -kN + CM$ for some constants $c, C > 0$, depending only on $K$ and $q$. By the hypothesis, we know there exists $x \in G_{\text{ord}(gy\beta z)}$ such that $h(x) \equiv 0 \pmod{gy\beta z}$. Since $N$ is sufficiently large, by taking $\theta < 1/(C + 1)$ we have

$$\text{ord} x < \text{ord} (gy\beta z) < CM + M + \text{ord} c_k \leq N.$$ 

We denote by $D$ some constant dependent only on $h$. We have

$$\text{ord} \{ \beta h(x) \} \leq \min_{z \in \mathbb{F}_q[t]} \text{ord} \left( \frac{\beta h(x)}{gy\beta z} \right)$$

$$= \text{ord} \left( \frac{h(x)}{gy\beta z} \right) + \text{ord} \{ gy\beta z \}$$

$$\leq D + (\text{ord} g + \text{ord} y)(\deg h - 1) + \text{ord} \{ gy\beta z \}$$

$$\leq D + (CM + M)(\deg h - 1) + CM - kN$$

$$= D + (C + 1)(\deg h - 1) + C)M - kN.$$ 

Suppose

$$\theta < \min \left\{ \frac{1}{(C + 1)(\deg h - 1) + C + 1}, \frac{1}{C + 1} \right\}.$$ 

Then for $N$ sufficiently large in terms of $D$, we obtain from above that $\text{ord} \{ \beta h(x) \} \leq -M$, which is a contradiction.

**Proof of Theorem 1.8.** Let $\beta_1, ..., \beta_L$ be arbitrary elements in $K_\infty$. Let $M = \lceil \theta N \rceil + 1$ and $\theta$ be a sufficiently small positive number to be chosen later. To obtain contradiction, suppose for some $N$ sufficiently large in terms of $K$, $q$ and $\theta$, we have

$$\text{ord} \{ \beta_1 h_1(x) + ... + \beta_L h_L(x) \} > -M$$

for all $x \in G_N$.

Let $T$ and $g_1, ..., g_L$ be the matrix and polynomials, respectively, obtained by applying Lemma 3.2 to the polynomials $h_1, ..., h_L$. We also have by (3) and (4) of Lemma 3.2 that
there exist $T_j \in \mathcal{K}^*$ ($1 \leq j \leq L$) such that $[g_j]_{T_j} = 0$ if $i \neq j$, and $\gamma_j \in \mathbb{K}_\infty$ ($1 \leq j \leq L$) such that
\[
\beta_1h_1(u) + \ldots + \beta_Lh_L(u) = \gamma_1g_1(u) + \ldots + \gamma_Lg_L(u).
\]
Hence for all $x \in \mathbb{G}_N$, we have
\[
\text{(4.1)} \quad \text{ord } \{\gamma_1g_1(x) + \ldots + \gamma_Lg_L(x)\} > -M.
\]
Then, by Lemma 2.3, there exists $y \in \mathbb{G}_M \setminus \{0\}$ with
\[
\left| \sum_{x \in \mathbb{G}_N} e(y\gamma_1g_1(x) + \ldots + y\gamma_Lg_L(x)) \right| \geq \frac{q^N}{q^M - 1} > q^{N-M}.
\]
Let $f(u) = y\gamma_1g_1(u) + \ldots + y\gamma_Lg_L(u)$, and suppose it is supported on $\hat{\mathcal{K}} \subseteq \mathbb{Z}^+$. We can verify that each $T_j \in (\hat{\mathcal{K}})^*$. Applying Corollary 2.4 with $f(u)$, we obtain that for $\theta < c$ there exists $g \in \mathbb{F}_q[t]$ such that $\text{ord } g < CM$ and
\[
\text{(4.2)} \quad \text{ord } \{g[f]_{T_j}\} = \text{ord } \{gy\gamma_j[g_j]_{T_j}\} \leq CM - T_jN \quad (1 \leq j \leq L),
\]
for some constants $c, C > 0$ depending only on $\mathcal{K}$ and $q$.

Let $v = gy \prod_{j=1}^L [g_j]_{T_j}$ and let $D$ be some constant dependent only on $g_1, \ldots, g_L$. Consequently, $D$ is dependent only on $h_1, \ldots, h_L$. Note the actual value of $D$ may vary from line to line during calculations. Then
\[
\text{ord } v \leq D + \text{ord } gy \leq D + CM + M,
\]
and we make sure $\text{ord } v < N$ by taking $N$ sufficiently large with respect to $D$. Thus for all $1 \leq j \leq L$, we have
\[
\text{ord } \{v\gamma_j\} = \min_{z \in \mathbb{F}_q[t]} \text{ord } (v\gamma_j - z) \leq \min_{z \in \mathbb{F}_q[t]} \left( v\gamma_j - \prod_{i \neq j} [g_i]_{T_i} \right) = \min_{z \in \mathbb{F}_q[t]} \left( \prod_{i \neq j} [g_i]_{T_i} \right) + \text{ord } \{gy\gamma_j[g_j]_{T_j} - z\} = \sum_{i \neq j} \text{ord } ([g_i]_{T_i}) + \text{ord } \{gy\gamma_j[g_j]_{T_j}\} \leq D + CM - T_jN,
\]
where we used (4.2) to obtain the last inequality. It follows that if we let $a_j = (v\gamma_j - \{v\gamma_j\}) \in \mathbb{F}_q[t]$, then we have
\[
\text{ord } \left( \gamma_j - \frac{a_j}{v} \right) \leq D + CM - T_jN - \text{ord } v \quad (1 \leq j \leq L).
\]
Recall each $g_j$ is a linear combination over $\mathbb{F}_q[t]$ of $h_1, \ldots, h_L$. Thus by the hypothesis, we know there exists $n \in \mathbb{G}_{\text{ord } v}$ such that
\[
a_1g_1(n) + \ldots + a_Lg_L(n) \equiv 0 \pmod{v}.
\]
Clearly, we have $\max_{1 \leq j \leq L} \deg g_j \leq \max_{1 \leq j \leq L} \deg h_j$. Thus we obtain
\[
\begin{align*}
\ord \left\{ \sum_{j=1}^{L} \gamma_j g_j(n) \right\} &\leq \ord \left( \sum_{j=1}^{L} \gamma_j g_j(n) - \frac{1}{v} \sum_{j=1}^{L} a_j g_j(n) \right) \\
&= \ord \left( \sum_{j=1}^{L} \left( \gamma_j - \frac{a_j}{v} \right) g_j(n) \right) \\
&\leq \max_{1 \leq j \leq L} \ord \left( \left( \gamma_j - \frac{a_j}{v} \right) g_j(n) \right) \\
&\leq \max_{1 \leq j \leq L} CM - T_j N + (\ord v)(\deg g_j - 1) + D \\
&\leq \max_{1 \leq j \leq L} CM - T_j N + (CM + M)(\deg g_j - 1) + D \\
&\leq -N + CM + (C + 1) \left( \max_{1 \leq j \leq L} \deg h_j - 1 \right) M + D.
\end{align*}
\]

Suppose $\theta$ is sufficiently small in terms of $C$ and $\max_{1 \leq j \leq L} \deg h_j$. Then it is not too difficult to see that the final quantity obtained above is less than or equal to $-M$ for $N$ sufficiently large, which contradicts (4.1.1). Therefore, there exists some $m \in \mathbb{G}_N$ such that
\[
\ord \left\{ \beta_1 h_1(m) + \ldots + \beta_L h_L(m) \right\} \leq -M \leq -\theta N.
\]

Recall from Section 2 that corresponding to any system of polynomials $(h_1, \ldots, h_L)$ satisfying Condition (\textast), we can associate a sequence $(r_x)_{x \in \mathbb{F}_q[t] \setminus \{0\}}$ such that (2.1) is satisfied. We prove the following theorem.

**Theorem 4.1.** Let $h_j \in \mathbb{F}_q[t][u]$ be supported on a set $\mathcal{K}_j \subseteq \mathbb{Z}^+$ $(1 \leq j \leq L)$, and let $\mathcal{K} = \mathcal{K}_1 \cup \ldots \cup \mathcal{K}_L$. Suppose the system $(h_j)_{j=1}^L$ satisfies Condition (\textast) and that the maximal $\mathcal{K}^*$-portion of $(h_j)_{j=1}^L$ is linearly independent. Then there exist $\theta = \theta(\mathcal{K}, q, \max_{1 \leq j \leq L} \deg h_j), \sigma = \sigma(h_1, \ldots, h_L) > 0$ and $N_0 = N_0(\mathcal{K}, q, \theta, \sigma, h_1, \ldots, h_L) \in \mathbb{Z}^+$ such that the following holds when $N > N_0$. Given any $d \in \mathbb{F}_q[t]$ with $ord d < [\sigma N]$, and $\beta_1, \ldots, \beta_L$ in $\mathbb{K}_\infty$, there exists $n \in \mathbb{G}_N$ such that $n \equiv r_d \pmod{d}$ and
\[
\ord \left\{ \beta_1 h_1(n) + \ldots + \beta_L h_L(n) \right\} \leq -\theta N.
\]

Before we prove the theorem, we give an example of a system $(h_1, h_2) \subseteq \mathbb{F}_5[t][u]$ that satisfies the hypothesis of Theorem 4.1 where $h_1(u)$ and $h_2(u)$ do not share a common root in $\mathbb{F}_5[t]$.

**Example 4.2.** Let $p = 5$, and consider $h_1(u) = (u + 1)(u^2 - t)(u^2 - (t + 1))(u^2 - (t + 1))$ and $h_2(u) = u^{20}(u^2 - t)(u^2 - (t + 1))(u^2 - (t + 1))$ in $\mathbb{F}_5[t][u]$. It is clear that $h_1(u)$ and $h_2(u)$ do not share a common root in $\mathbb{F}_5[t]$. It follows from Example 1.3 and Lemma 2.1 that the system $(h_1, h_2)$ satisfies Condition (\textast). The polynomials $h_1(u)$ and $h_2(u)$ are supported on $\mathcal{K}_1 = \{7, 6, 5, 4, 3, 2, 1\}$ and $\mathcal{K}_2 = \{26, 24, 22, 20\}$, respectively. We can verify that $h_1^{max}(u) = u^7$ and $h_2^{max}(u) = u^{26} - (t^2 + 3t + 1)u^{24}$, which are clearly linearly independent over $\mathbb{K}$. Therefore, the maximal $\mathcal{K}^*$-portion of $(h_1, h_2)$ is linearly independent, and it follows that $(h_1, h_2)$ satisfies the hypothesis of Theorem 4.1.

It is also easy to see that $(h_1, h_2)$ satisfies the hypothesis of Theorem 4.1.
Proof. Let $\beta_1, \ldots, \beta_L$ be arbitrary elements in $\mathbb{K}_\infty$ and $d$ be an arbitrary element in $\mathbb{G}_{[\sigma N]}$. Let $\theta$ and $\sigma$ be positive real numbers sufficiently small to be chosen later, and let $M = [(\theta N)] + 1$. Suppose for some $N$ sufficiently large in terms of $\mathcal{K}$, $q$, $\sigma$, and $\theta$, we have

$$\text{ord}\left\{\beta_1 h_1(dx + r_d) + \ldots + \beta_L h_L(dx + r_d)\right\} > -M$$

for all $x \in \mathbb{G}_{[1-(\sigma)N]}$. Let $T$ and $g_1, \ldots, g_L$ be the matrix and polynomials, respectively, obtained by applying Lemma 3.3 to the polynomials $h_1, \ldots, h_L$ with $s = r_d$ and $d$. By (4) of Lemma 3.3, we know there exist $T_j \in \mathcal{K}^\ast$ ($1 \leq j \leq L$) such that $T_j$ is maximal in $\mathcal{K}$, $|g_1|_{T_j} = 0$ if $i \neq j$, and $|g_j|_{T_j} = \tilde{c}_j d^{T_j}$ for some $\tilde{c}_j \in \mathbb{F}_q[t]$ dependent only on $h_1, \ldots, h_L$. We also know there exist $\gamma_j \in \mathbb{K}_\infty$ ($1 \leq j \leq L$) such that

$$\beta_1 h_1(du + r_d) + \ldots + \beta_L h_L(du + r_d) = \gamma_1 g_1(u) + \ldots + \gamma_L g_L(u).$$

Thus we have

$$\text{ord}\{\gamma_1 g_1(x) + \ldots + \gamma_L g_L(x)\} > -M$$

for all $x \in \mathbb{G}_{[1-(\sigma)N]}$. By Lemma 2.4, there exists $y \in \mathbb{G}_M \setminus \{0\}$ such that

$$\left|\sum_{x \in \mathbb{G}_{[1-(\sigma)N]}} e(y \gamma_1 g_1(x) + \ldots + y \gamma_L g_L(x))\right| \geq \frac{q^{[(1-(\sigma)N)]}}{q^M - 1} > q^{N-(\sigma+\theta)N}.$$  

Let $f(u) = y \gamma_1 g_1(u) + \ldots + y \gamma_L g_L(u)$, and suppose it is supported on $\mathcal{K} \subseteq \mathbb{Z}^\ast$. We can verify that each $T_j \in (\mathcal{K})^\ast$. Applying Corollary 2.6 with $f(u)$, we obtain that for $\sigma + \theta < c$ there exists $g \in \mathbb{F}_q[t]$ such that $\text{ord} g < C(\sigma + \theta)N$ and

$$\text{ord}\{g[f]_{T_j}\} = \text{ord}\{gy\gamma_j[g_j]_{T_j}\} \leq \sigma(\sigma + \theta)N - T_j[1-(\sigma)N] \quad (1 \leq j \leq L),$$

for some constants $c, C > 0$ depending only on $\mathcal{K}$ and $q$. Let $v = gy \prod_{j=1}^L [g_j]_{T_j}$ and let $D$ be some constant dependent only on $h_1, \ldots, h_L$ (note the actual value of $D$ may vary from line to line during calculations). We define $T' = \sum_{1 \leq j \leq L} T_j$. Then

$$\text{ord} v \leq \text{ord} gy + T'\text{ord} d + D \leq C(\sigma + \theta)N + M + T'[\sigma N] + D.$$ 

In particular, we have $\text{ord} v < [(1-(\sigma)N)]$ for $N$ sufficiently large with respect to $D$ and $\theta, \sigma$ sufficiently small.

For simplicity denote $n = r_v \in \mathbb{G}_{\text{ord} v}$, then $h_j(n)$ is divisible by $v$ for any $1 \leq j \leq L$. We also have $n \equiv r_d \pmod{d}$, because $d|v$. Each $g_j(u)$ can be written as an $\mathbb{F}_q[t]$-linear combination of the polynomials $h_1(du + r_d), \ldots, h_L(du + r_d)$. Thus if we write $n = dw + r_d$ for some $w \in \mathbb{G}_{\text{ord} v}$, then $g_j(w)$ is divisible by $v$ for any $1 \leq j \leq L$. Let $H = \max_{1 \leq j \leq L} \text{deg} h_j$. Then it follows that

$$\text{ord}\{\gamma_j g_j(w)\} \leq \min_{z \in \mathbb{F}_q[t]} \text{ord}\left(\gamma_j g_j(w) - \frac{z g_j(w)}{v}\right)$$

$$= \text{ord}\left(\frac{g_j(w)}{v}\right) + \min_{z \in \mathbb{F}_q[t]} \text{ord}\left(v\gamma_j - z\right)$$

$$= \text{ord}\left(\frac{g_j(w)}{v}\right) + \text{ord}\{v\gamma_j\}$$

$$\leq D + H(\text{ord} d + \text{ord} w) - \text{ord} v + \text{ord}\{v\gamma_j\},$$

where $D$ is a constant depending only on $q$ and $n$. Therefore

$$\text{ord}\{\gamma_j g_j(w)\} \leq D + H(\text{ord} d + \text{ord} w) - \text{ord} v + \text{ord}\{v\gamma_j\}.$$
where the last inequality is obtained via (3) of Lemma 3.3. We also have by similar calculations as in (4.3) that for \( 1 \leq j \leq L \),

\[
\text{ord} \{ v_j \} = \min_{z \in \mathbb{F}_q[t]} \text{ord} \left( v_j - z \right)
\]

\[
\leq \min_{z \in \mathbb{F}_q[t]} \text{ord} \left( v_j - z \prod_{i \neq j} [g_i]_{T_i} \right)
\]

\[
= \text{ord} \left( \prod_{i \neq j} [g_i]_{T_i} \right) + \min_{z \in \mathbb{F}_q[t]} \text{ord} \left( g_j g_j [g_j]_{T_j} - z \right)
\]

\[
= \sum_{i \neq j} \text{ord} \left( [g_i]_{T_i} \right) + \text{ord} \left\{ g_j g_j [g_j]_{T_j} \right\}
\]

\[
= \left( \sum_{i \neq j} \text{ord} \bar{c}_i + T_i \text{ord} d \right) + \text{ord} \left\{ g_j g_j [g_j]_{T_j} \right\}
\]

(4.7)

\[
\leq T' \text{ord} d + D + C(\sigma + \theta)N - T_j [(1 - \sigma)N],
\]

where we used (4.5) to obtain the last inequality. Therefore, we have by (2.2), (4.6), and (4.7) that

\[
\text{ord} \left\{ \sum_{j=1}^{L} \gamma_j g_j (w) \right\}
\]

\[
\leq \max_{1 \leq j \leq L} \text{ord} \left\{ \gamma_j g_j (w) \right\}
\]

\[
\leq T' \text{ord} d + D + C(\sigma + \theta)N - [(1 - \sigma)N] \min_{1 \leq j \leq L} T_j + H(\text{ord} d + \text{ord} w) - \text{ord} v
\]

\[
\leq \sigma T' N + D + C(\sigma + \theta)N - [(1 - \sigma)N] \min_{1 \leq j \leq L} T_j + H(\sigma N + \text{ord} v) - \text{ord} v
\]

\[
\leq \sigma T' N + D + C(\sigma + \theta)N - [(1 - \sigma)N] + \sigma H N + H(C(\sigma + \theta)N + M + \sigma T' N).
\]

Suppose \( \theta \) is sufficiently small in terms of \( C \) and \( H \), and also that \( \sigma \) is sufficiently small in terms of \( C, T' \) and \( H \). Then for \( N \) sufficiently large, the final quantity obtained above is less than or equal to \(-M\), which contradicts (4.4). Therefore, there exists \( x \in \mathbb{G}_{[(1 - \sigma)N]} \) such that \( m = dx + r_d \in \mathbb{G}_N \) and

\[
\text{ord} \left\{ \beta_1 h_1 (m) + \ldots + \beta_L h_L (m) \right\} \leq -M \leq -\theta N.
\]

\[\square\]

**Appendix A.**

In this section, we prove the statement presented in Example 1.5. We let \( p = 5 \) and let \( h(u) = (u^2 - t)(u^2 - (t+1))(u^2 - (t^2 + t)) \in \mathbb{F}_5[t][u] \). First, since \( \mathbb{F}_5[t] \) is a unique factorization domain, it is clear that \( h(u) \) has a root in \( \mathbb{F}_5[t] \) if and only if at least one of \( u^2 - t, u^2 - (t+1), \) and \( u^2 - (t^2 + t) \) has a root in \( \mathbb{F}_5[t] \). It can be verified easily that none of the three polynomials have a root in \( \mathbb{F}_5[t] \). Therefore, \( h(u) \) does not have a root in \( \mathbb{F}_5[t] \).

In order to prove that \( h(u) \) has a root modulo \( g \) for every \( g \) in \( \mathbb{F}_5[t]\backslash\{0\} \), we use the following version of the Hensel’s lemma.
Lemma A.1. [10, Lemma 25] Suppose char \((\mathbb{F}_q) = p \nmid K\), and we have \(w, z_0, z \in \mathbb{F}_q[t]\), where \(w\) is irreducible, \(w \nmid z_0\), and
\[z^K \equiv z_0 \pmod{w^A}.
\]
Then, for \(B > A\), there exists \(z' \in \mathbb{F}_q[t]\) such that
\[(z')^K \equiv z_0 \pmod{w^B} \quad \text{and} \quad z' \equiv z \pmod{w^A}.
\]

By the quadratic reciprocity law in \(\mathbb{F}_5[t]\), given any irreducible \(\pi \in \mathbb{F}_5[t]\) that is not \(t\) or \(t+1\), we know that either \(t^2 + t\) is a quadratic residue modulo \(\pi\), or one of \(t\) and \(t+1\) is a quadratic residue modulo \(\pi\). Suppose \(\pi = t+1\), then we have
\[2^2 \equiv -1 \equiv t \pmod{\pi}.
\]
On the other hand, if \(\pi = t\), then we have
\[1^2 \equiv 1 \equiv t + 1 \pmod{\pi}.
\]
By the Hensel’s lemma above, it follows that given any \(L \geq 1\), one of \(u^2 - t\), \(u^2 - (t+1)\), and \(u^2 - (t^2 + t)\) has a root modulo \(\pi^L\). In other words, \(h(u)\) has a root modulo \(\pi^L\). It then follows from the Chinese Remainder Theorem that \(h(u)\) has a root modulo \(g\) for every \(g\) in \(\mathbb{F}_5[t] \setminus \{0\}\).

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References

[1] V. Bergelson, A. Leibman, and E. Lesigne, Intersective polynomials and the polynomial Szemerédi theorem, Adv. Math. 219 (2008), no. 1, 369-388.
[2] H. Davenport, On a theorem of Heilbronn, Quart. J. Math. Oxford Ser. (2) 18 (1967), 399 - 344.
[3] G. H. Hardy and J. E. Littlewood, Some problems of diophantine approximation Part I. The fractional part of \(n^k\theta\), Acta Math. 37 (1914), no. 1, 155-191.
[4] H. Heilbronn, On the distribution of the sequence \(n^2\theta \pmod{1}\), Quart. J. Math. Oxford Ser. 19 (1948), 249 - 256.
[5] T.H. Lê, Problems and results on intersective sets, to appear in Proceedings of Combinatorial and Additive Number Theory 2011.
[6] T.H. Lê and Y.-R. Liu, Equidistribution of polynomial sequences in function fields, with applications, arXiv:1311.0892.
[7] T.H. Lê and C. V. Spencer, Intersective polynomials and diophantine approximation, Internat. Math. Res. Notices (2014), no.5, 1153-1173.
[8] J. Lucier, Intersective sets given by a polynomial, Acta Arith. 123 (2006), 57-95.
[9] J. Neukirch, Algebraic number theory. Translated from the 1992 German original and with a note by Norbert Schappacher. With a foreword by G. Harder, Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), 322. Springer-Verlag, Berlin, 1999. xviii+571 pp.
[10] R. M. Kubota, Waring’s problem for \(\mathbb{F}_q[x]\), Dissert. Math. (Rozprawy Mat.) 117 (1974), 60pp.
[11] C. V. Spencer and T. D. Wooley, Diophantine inequalities and quasi-algebraically closed fields, Israel J. Math. 191 (2012), 721-738.
[12] I. M. Vinogradov, Analytischer Beweis des Satzes über die Verteilung der Bruchteile eines ganzen Polynoms, Bull. Acad. Sci. USSR (6) 21 (1927), 567-578.
[13] T. D. Wooley, Vinogradov’s mean value theorem via efficient congruencing, Ann. of Math. 175 (2012), no. 3, 1575 - 1627.
[14] A. Zaharescu, *Small values of $n^2 \alpha \pmod{1}$*, Invent. Math. **121** (1995), no. 2, 379 - 388.

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