Notes on the spheroidal harmonic multipole moments of gravitational radiation

L. London

MIT-Kavli Institute for Astrophysics and Space Research and LIGO Laboratory, 77 Massachusetts Avenue, 37-664H, Cambridge, MA 02139, USA

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The estimation of gravitational radiation’s multipole moments is a central problem in gravitational wave theory, with essential applications in gravitational wave signal modeling and data analysis. This problem is complicated by most astrophysically relevant systems’ not having angular modes that are analytically understood. A ubiquitous workaround is to use spin weighted spherical harmonics to estimate multipole moments; however, these are only related to the natural modes of non-spinning spacetimes, thus obscuring the behavior of radiative modes when the source has angular momentum. In such cases, radiative modes are spheroidal in nature. Here, common approaches to the estimation of spheroidal harmonic multipole moments are unified under a simple framework. This framework leads to a new class of spin weighted spheroidal harmonic functions. Adjoint-spheroidal harmonics are introduced and used to motivate the general estimation of spheroidal harmonic multipole moments via bi-orthogonal decomposition.

I. INTRODUCTION

Central to gravitational wave detection and the inference of source parameters is the representation of gravitational radiation in terms of multipole moments [1, 2]. By construction, these functions of time or frequency allow the radiation’s angular dependence to be given by spin weighted harmonic functions. This leaves the radiation itself to be represented as a sum over harmonic functions, whereby each term is weighted by a different multipole moment. The choice of representation, namely the choice of which harmonic functions to use, is not unique. Only the radiation’s spin weight must be respected [3]. And while there are multiple appropriate spin weighted functions, only one set of harmonic functions correspond to the system’s natural modes.

Spin-weighted spherical harmonics are perhaps the most commonly used functions for describing the angular behavior of gravitational radiation [4, 5]. They are the simplest known functions appropriate for representing gravitational radiation, and their completeness and orthonormality on the sphere make them straightforward to use. However, their application in gravitational wave theory lacks a single physical origin [4, 6]. They are the natural scalar harmonics associated with the symmetric-trace-free formulation of gravitational waves [4]. They are also the eigenfunctions of Einstein’s equations linearized around the Schwarzschild metric [6]. The two examples are linked by the requirement that the harmonics are consistent with the spin weight ($s = -2$) of gravitational radiation with minimal additional assumptions [5].

However, the latter example supports the fact that spin weighted spherical harmonics only correspond to the natural modes of spherically symmetric spacetimes [5, 6]. When applied to non-spherically symmetric systems such as a spinning black hole (BH), or a binary black hole (BBH) system, spherical harmonic multipole moments are not directly related to the system’s natural modes. While this poses no impediment to representing gravitational waves, it is known to complicate the morphology of gravitational wave signal models, and obscure the underlying physics of BBH merger and ringdown [7–10]. Fig. (1) provides examples of how features from the dominant spherical harmonic moment, with $(\ell, m) = (2, 2)$, may spuriously present in others due to basis choice rather than underlying physics. Examples such as these drive ongoing interest in representing gravitational waves, particularly those from BBH merger and ringdown, using harmonics that are, as closely as possible, related to the system’s natural modes [7, 8, 11, 12].

The simplest additional physical effect to include beyond spherical symmetry is angular momentum. A general theory of perturbed spinning spacetimes may be ripe for cultivation [13], but for now it is clear from the study of single perturbed spinning BHs that the system’s natural modes correspond not to a spherical harmonic representation, but a spheroidal harmonic one [6, 7, 11, 12, 14, 15]. To date, spheroidal harmonics have often not been used for representing gravitational radiation, in part for technical reasons. They are generally the non-orthogonal eigenfunctions of a non-hermitian operator. The spectral expansion possible with spherical harmonics and used ubiquitously in gravitational wave theory cannot be done with the spheroidals in the same way. The matter is further complicated by the potential importance of gravitational wave overtone modes, which are the gravitational equivalent of (nearly) degenerate quantum states.

Here, we will see how these complications can be overcome. Common methods for the ad-hoc estimation of spheroidal harmonic multipole moments are shown to be not necessarily equivalent interpretations of a single linear representation. The relative benefits and implications of each method are discussed. This discussion is followed by the presentation of a general method to compute spheroidal harmonic multipole moments via the introduction of adjoint-spheroidal harmonics and their application in bi-orthogonal decomposition.

A. Overview

We begin in Sec. (II) with a review of spherical and spheroidal harmonic representations of gravitational radiation. This section lays the groundwork for this work’s key results by collecting common linear fitting methods for estimating spheroidal harmonic multipole moments in a unified framework. In Sec. (II A) we are introduced to the estimation of spheroidal multipoles via least-squares fitting. In Section III we are introduced to the adjoint-spheroidal harmonics for Kerr, and in Sections (III A 1-III C 2) we en-
counter a general method for spheroidal harmonic decomposition. In Sec. (III B) we are introduced to a matrix representation for which the adjoint-harmonics are eigenvectors. In Sec. (III C) we address issue of completeness, and motivate spheroidal harmonic decomposition with overtone-subspaces. In Sec. (III D) we see example adjoint-spheroidal harmonics for Kerr. Lastly, in Sec. (IV) we summarize this work’s limitations, open problems, and potential applications.

B. Resources for this work

The quantitative results of this work may be reproduced using routines from the openly available Python package, positive [16]. Of principle use are the Kerr Quasi-Normal Mode (QNM) frequencies and the spheroidal harmonics. Both of which may be determined using, for example, Leaver’s analytic representation [6]. In positive, the QNM frequencies may be accessed via positive.leaver. Similarly, positive contains multiple inter-consistent routines for calculating the central objects of current interest, the spheroidal harmonic functions. These may be accessed via positive.slm, which uses Leaver’s representation, and positive.slmcg, which uses a spherical harmonic representation. This work’s central result, namely the adjoint-spheroidal harmonics, may be accessed via positive.aslmcg.

II. PRELIMINARIES

The most documented example of spacetime angular momentum’s effect on gravitational radiation’s multipole moments is linear “mode-mixing” during BH ringdown, where the natural time domain modes damp away with one of a discrete set of QNM frequencies [8, 12, 17–21]. The mixing in question is between the canonical spherical harmonic multipole moments, and the system’s natural spheroidal modes.

Fig. (1) shows two examples of mode-mixing for non-precession BBH cases in which the dominant quadrupole, having spherical harmonic indices \((\ell, m) = (2, 2)\), mixes with other multipole moments which have the same azimuthal index, \(m\). Low frequencies correspond to late inspiral where each multipole amplitude is well approximated by a power-law. Intermediate and high frequencies, where the displayed amplitudes transition from one power-law to another steeper one, correspond to merger and ringdown. In the cases presented, we see in the \((\ell, m) = (3, 2)\) multipole moment prominent high-frequency features that are due to mixing from its \((2, 2)\) counterpart, while the \((2, 2)\) multipole experiences mostly minute mixing not visible on the scales presented. These mixing features are most prominent during merging-ringdown.

The 3:2 mass ratio case shows a \((3, 2)\) multipole moment with a significant but localized jump around the \((2, 2)\) mode’s natural frequency. The 8:1 mass-ratio case illustrates that mixing can take the form of a non-localized leaking of power between multipoles throughout the binary’s coalescence. This case’s \((3, 2)\) multipole moment shows approximately power-law decay before a sudden drop in power at the the \((2, 2)\) mode’s natural frequency. Unlike the previous case, here we see no appreciable rise in multipole power shortly before the \((2, 2)\) mode’s natural frequency. However, we do see a feature at the expected \((3, 2)\) mode’s natural frequency that is nearly an order of magnitude lower than its \((2, 2)\) counterpart. Thus, rather than a localized feature, this case sees all of its visible inspiral and merger dominated by mixing.

In this section we will specify exactly what’s meant by mode-mixing and review known linear methods for \(wn\)-mixing multipole content. We begin by addressing how spherical and spheroidal bases present different pictures of gravitational wave multipole moments.

A. Spherical and Spheroidal Pictures

Gravitational wave observatories detect a linear combination of gravitational wave strain’s polarizations \(h_+\) and \(h_\times\). In both spherical and spheroidal pictures, a useful shorthand for the gravitational wave strain takes the form

\[
h = h_+ - i h_\times
\]

where \(h_+\) and \(h_\times\) are the observable gravitational wave polarizations. From this starting point gravitational wave theory poses two starting points for representing the gravitational wave strain in terms of a sum over multipole moments. The spherical harmonic expansion,

\[
h = \frac{1}{r} \sum_{\ell m} h_{\ell m} Y_{\ell m}(\theta, \phi),
\]

Figure 1. Numerical relativity examples of spherical-spheroidal mixing in frequency domain amplitudes of radiative spherical harmonic multipole moments. Moments for spin weight 2 spherical harmonics \((\ell, m) = (2, 2)\) (solid grey) and \((\ell, m) = (3, 2)\) (dashed black) are shown. Text boxes mark the horizontal location of select quasi-normal mode frequencies for each remnant black hole. The \((2, 1)\) label denotes apparent 2nd order modes at twice the frequency of the \((\ell, m) = (2, 1, 0)\) 1st order modes [7]. (Top Panel) Mass ratio 3:2 binary black hole coalescence with initially non-spinning components [22]. (Bottom Panel) Mass ratio 8:1 binary black hole coalescence with initial dimensionless component spins of 0.85 aligned with the orbital angular momentum [23, 24].
and its potential spheroidal harmonic counterpart,
\[ h = \frac{1}{r} \sum_{\ell m} h_{\ell m} Y_{\ell m}(\theta, \varphi; \gamma_{\ell m}) \, . \]  

In Eqs. (2-3), \( r \) is the physical source’s luminosity distance, \( \theta \) is the spherical polar angle defined in an flat source centered frame, and \( \varphi \) is the usual spherical polar azimuthal angle. Azimuthal and polar indices follow the usual relationships: \(|s| \leq \ell, |s| \leq \ell, |m| \leq \ell \) and \(|m| \leq \ell \). The overtone index \( n \) spans all non-negative integers. In Eq. (2), \(-2Y_{\ell m}(\theta, \varphi)\) is the spin weighted spherical harmonic,
\[ -2Y_{\ell m}(\theta, \varphi) = -2Y_{\ell m}(\theta) e^{im\phi} , \]
and \( h_{\ell m} \) is its time or frequency domain multipole moment [3, 25, 26]. In Eq. (3), \(-2S_{\ell m}(\theta, \varphi; \gamma_{\ell m})\) is the spheroidal harmonic,
\[ -2S_{\ell m}(\theta, \varphi; \gamma_{\ell m}) = -2S_{\ell m}(\theta; \gamma_{\ell m}) e^{im\phi} , \]
and \( S_{\ell m} \) is its multipole moment [6, 18, 27]. Each spheroidal harmonic depends on a spin-frequency parameter, \( \gamma_{\ell m} \). In the case of a perturbed spinning BH, this dimensionless parameter is the BH spin, \( \alpha \), times one of the BH’s complex valued QNM frequencies, \( \tilde{\omega}_{\ell m} \).

We will at times adopt a slightly different notation for convenience and brevity. We will drop the spin weight labels from the harmonics; while we will only consider outgoing gravitational radiation corresponding to spin weight -2, many aspects of our discussion apply to all spin weights. We will be centrally concerned with the \( \theta \) dependence of each harmonic; thus, \( Y_{\ell m} \) and \( S_{\ell m} \) will refer to \( Y_{\ell m}(\theta) \) and \( S_{\ell m}(\theta; \gamma_{\ell m}) \). As done above, we will denote spheroidal harmonic indices with an overbar, but we will at times use a compound index such as \( \bar{\ell}m \) to serialize the relevant values of \( s, \ell \) and \( m \). Most often this serialization will correspond to situations where only \( \bar{\ell} \) is variable, and in some cases \( \bar{\ell} \) simply will be used. Similarly we will use a compound index such as \( k \) to serialize spherical indices \( s, \ell \) and \( m \). At present, indices \( (s, \ell, m) \) will be used. In later sections, bracket notation, \( (\ldots) \), will be adopted to simplify various expressions.

We are now positioned to consider how information from one multipolar picture with that of the other. Noting that Eqs. (4-5)’s complex exponentials \( e^{im\phi} \) are orthogonal in \( m \), it is wise to consider sets of like \( m \) or \( \bar{m} \),
\[ h = \frac{1}{r} \sum_{\bar{m}} h_{\bar{m}} e^{im\phi} \]  
where upon considering the spherical and spheroidal representations together, we have that
\[ h_{\bar{m}} = \sum_{\ell} h_{\ell \bar{m}} Y_{\ell \bar{m}}(\theta) \]  
\[ h_{\bar{m}} = \sum_{\ell m} h_{\ell m} S_{\ell m}(\theta; \gamma_{\ell m}) . \]

Plainly, \( h_{\bar{m}} = h_{\bar{m}} \) if \( m = \bar{m} \).

In Eq. (6) we have distilled the multipolar structure of \( h \) into moments that depend only on the azimuthal moments, or \( m \)-poles, \( h_{m} \). And in Eqs. (7-8) it is the \( m \)-poles that set the stage for representing gravitational radiation in spherical or spheroidal harmonics.

Eq. (7) follows directly from the fact that spherical harmonics are complete and orthonormal in the standard way
\[ \int_{0}^{\pi} Y_{\ell m}^{*}(\theta) Y_{\ell^\prime m}(\theta) \sin(\theta) d\theta = \delta_{\ell \ell^\prime} , \]
meaning that spherical harmonic multipole moments can be computed by projection
\[ h_{\ell m}(t) = \int_{0}^{\pi} h_{mn}(t, \theta) Y_{\ell m}^{*}(\theta) \sin(\theta) d\theta . \]

Eq. (10) encapsulates the spherical harmonic’s core use. Despite their not generally being the natural physical harmonics for gravitationally radiating systems, they enable the simple calculation of multipole moments.

Spherical harmonics can lack this trait,
\[ \int_{0}^{\pi} S_{\ell m}^{*}(\theta; \gamma_{\ell m}) S_{\ell^\prime m}(\theta; \gamma_{\ell m}) \sin(\theta) d\theta \neq \delta_{\ell \ell^\prime} , \]
meaning that their multipole moments may not be computed in the same way. This is the case when \( \gamma_{\ell m} \) is complex valued, as happens during the non-stationary inspiral-merger of compact objects, or during the acquiescence of perturbed BHs into their stationary state.

Equations (7-10) allow us to express spherical harmonic multipole moments in terms of spheroidal ones. This follows from inputting Eq. (8)’s right-hand-side into Eq. (10),
\[ h_{\ell m} = \sum_{\ell m} \sigma_{\ell mn} h_{mn} , \]

where \( \sigma_{\ell mn} \) are the spherical-spheroidal mixing coefficients studied in Refs. [8, 18, 28–30]
\[ \sigma_{\ell mn} = \int_{0}^{\pi} Y_{\ell m}^{*}(\theta) S_{mn}(\theta; \gamma_{\ell m}) \sin(\theta) d\theta . \]

Eq. (12) has played a central role in the estimation of spheroidal harmonic multipole moments, given a set of spherical ones. It says that spherical harmonic multipole moments are linearly mixed with spheroidal ones in a way that’s weighted by the spherical-spheroidal mixing coefficients. From a modeling perspective, Eq. (12) provides a simple linear model with an infinite number of terms, and thus infinite order.

However, there is good reason to consider a reduced number of terms in Eq. (12). Just as the removal of BH spin reduces a Kerr BH to a Schwarzschild one, the spheroidal harmonics reduce to the spherical ones. This requires that \( \sigma_{\ell mn} \) be proportional to \( \gamma_{\ell m} \) when \( \bar{\ell} \neq \ell [29] \). Generally, it is well known that
\[ \sigma_{\ell mn} \propto \gamma_{\ell m}^{\bar{\ell} - \ell} \]
as can be shown by standard perturbation theory arguments [18, 29, 31]. So while the spheroidals are not generally orthogonal, they are approximately orthogonal for small values of \( \gamma_{\ell m} \). This reasoning underpins linear modeling approaches for un-mixing the spheroidal multipoles from spherical ones [7, 8, 12, 32, 33].
B. Linear regression of ringdown’s spheroidal multipole moments

Long before the first Numerical Relativity (NR) simulations of coalescing BHs ([34, 35]), it was appreciated that the ringdown of NR’s spherical multipoles would be well approximated by a sum of spheromonic QNMs,

$$h_{t nth}(t) \approx \sum_{l mn} e^{i\omega_{l mn} t} B_{l mn} \sigma_{l nth/mn} ,$$

and that a greater understanding of QNM excitation could assist tests of General Relativity (GR) [36–40]. The complex valued QNM frequencies are composed of a real valued central frequency \(\omega_{l mn}\) and positive damping time \(\tau_{l mn}\),

$$\omega_{l mn} = \omega_{l mn} + i/\tau_{l mn} .$$

The QNM amplitudes \(B_{l mn}\) are determined by the binary’s component masses and spins. Many early numerical studies used nonlinear fitting to model spheroidal QNMs within spherical multipoles (e.g. [17, 40–44]); however, these methods often disregarded mode-mixing, meaning that the effect was either not modeled, or modeled poorly. In cases where mode-mixing was broached, it was at times not clear which terms in Eq. (12) were relevant. Reference [7] was perhaps the first to apply iterative-regression and linear-least-squares fitting in the basis of QNMs to the problem, thereby making use of each QNM contribution is maximal [7, 12]. This approach may be advantageous if NR data contains non-stationary noise that is localized in frequency away from QNM values [7].

Time and frequency domain regression are sensitive to systematic deviations from the QNM ansatz. Deviations may take the form of noise that impacts QNM frequencies or, more likely, lingering effects from merger that are nonlinear, or perhaps due to linear but non-stationary dynamics [48]. In this there is a significant risk that estimates of \(B_k\) may differ between different choices for the start and end of ringdown when it should not [7, 33, 49, 50]. Further, TD and FD regression use basis functions that are over-complete, meaning that if \(K\) basis functions are assumed, there likely exists a different set of \(K\) basis functions that produces a fit of similar quality [7, 51].

While the situation is helped by the discrete nature of the QNM frequencies, consistency checks must be used to verify that estimates of \(B_k\) are consistent with the predictions of linear perturbation theory [7, 17]. This is typically performed by making use of each \(B_k\) appearing in different spherical moments. To probe this point it is useful to acknowledge that \(B_k\) from different \(h_{f nth}\) may not be identical. We do so by relabeling \(B_k\) as \(B_k^{(i)}\). Using all indices for clarity, we wish to consider two different ringdowns

$$h_{22}(t) \approx B_{220}^{(2)} \sigma_{22220} \tau_{22020} \tau_{22020} e^{i\omega_{22020} t} + B_{220}^{(32)} \sigma_{32220} e^{i\omega_{32220} t} + ...$$

and

$$h_{32}(t) \approx B_{320}^{(32)} \sigma_{32220} \tau_{32020} \tau_{32020} e^{i\omega_{32020} t} + B_{320}^{(32)} \sigma_{32220} e^{i\omega_{32220} t} + ... .$$

Table I. Linear regression methods for estimating spheroidal multipole content from numeric spherical harmonic multipole moments. Methods shown only apply to ringdown. Nonlinear approaches not shown. \(h_j(t_k)\) is the discrete Fourier transform of \(h_j(t)\).

| Method | \(y_{kn}\) | \(Q_{nk}\) | \(a_k\) | References |
|---|---|---|---|---|
| TD Regression | \(h_j(t_k)\) | \(\exp(i\omega_{jk} t_k)\) | \(B_k \sigma_{jk}\) | [20, 32, 33] |
| FD Regression | \(\tilde{h}_j(\omega_k)\) | \(i/(\omega_k - \omega_j)\) | \(B_k \sigma_{jk}\) | [7, 12] |
| Change of Basis | \(\tilde{h}_j(\omega_k)\) | \(\sigma_{nk} iB_k/(\omega_k - \omega_n)\) | | [8, 10] |

Table I. Linear regression methods for estimating spheroidal multipole content from numeric spherical harmonic multipole moments. Methods shown only apply to ringdown. Nonlinear approaches not shown. \(h_j(t_k)\) is the discrete Fourier transform of \(h_j(t)\).

The angular domain approach amounts to a change of basis and will be referred to thsly. The structure of each approach is summarized in Table (I).

FD regression uses the damped sinusoidal behavior predicted by perturbation theory as a set of basis functions. The functions correspond to damped sinuosids with QNM frequencies labeled with the same \(m\) but different \(\ell\) and potentially different \(n\) [32, 33]. This method benefits from its conceptual simplicity, but it is perhaps the most susceptible to numerical noise that can be present throughout ringdown before becoming dominant as ringdown’s amplitude diverges towards a simulation’s noise floor.

FD regression takes a similar approach, but may be designed to evade the effects of numerical noise by only focusing on the central frequencies \(\omega_k\) predicted by perturbation theory. In this frame, FD regression’s \(Q_{nk}\) is restricted to \(\omega_n\) that are members of the set populated by \(\omega_k\), meaning that the method only uses frequency domain values for which each QNM contribution is maximal [7, 12]. This approach may be advantageous if NR data contains non-stationary noise that is localized in frequency away from QNM values [7].

Linear multipoles often disregarded mode-mixing, meaning that the effect was either not modeled, or modeled poorly. In cases where mode-mixing was broached, it was at times not clear which terms in Eq. (12) were relevant. Reference [7] was perhaps the first to apply iterative-regression and linear-least-squares fitting in the basis of QNMs to the problem, thereby addressing mode-mixing and which QNM terms are relevant. Since, other studies have used similar linear modeling techniques [8, 12, 32, 33]. Nonlinear approaches have found widespread use in gravitational wave signal modeling (e.g. [45–47]), but here it is useful to review what linear approaches can teach us about generic spheroidal harmonic decomposition.

We start with a small shift in perspective. The five spherical and spheroidal indices present in Eq. (15) encode information about the problem’s spatial information, but in essence they communicate that spherical harmonic moments are a one-dimensional sum over \(K\) spheroidal ones

$$h_j(t) \approx \sum_{k} \sigma_{jk} B_k e^{i\omega_{jk} t} .$$

In Eq. (17), \(\sigma\) denotes the discrete sampling of NR data. The starting point of linear methods for estimating \(B_k\) is to recognize that Eq. (17) may be framed as a linear matrix equation: a vector \(y\) of spherical harmonic multipole information being equal to a matrix \(\hat{Q}\) acting on a vector \(\bar{a}\) of spheroidal information

$$\bar{y} = \hat{Q} \bar{a} .$$

This implies that the unknown vector of spheroidal harmonic information \(\bar{a}\) may be determined if the pseudo-inverse of \(\hat{Q}\) exists,

$$\bar{a} = (\hat{Q}^T \hat{Q})^{-1} \hat{Q}^T \bar{y} .$$

In Eq. (19), \(\hat{Q}^T\) is the conjugate-transpose of \(\hat{Q}\).

Different linear methods for estimating spherical multi-pole content differ by their definition of \(\hat{Q}\) and \(\bar{a}\). The differences are motivated by whether the method seeks to unmix spherical moments from time, frequency, or angular domain data. We will refer to the time and frequency domain approaches as TD and FD regression. The angular domain approach amounts to a change of basis and will be referred to thsly. The structure of each approach is summarized in Table (I).
By construction, the Change of Basis approach passes the above inner-product ratios check. This approach was first applied in Ref. [10] to model the ringdowns of initially non-spinning BBH remnants. While Table (I) associates the vector of spheroidal information \( \tilde{\alpha} \) with the frequency domain form of QNM terms, this method requires no such association. As a result, physically meaningful interpretations of Change of Basis results hinge on the appropriate application of \( \gamma_k = \tilde{a} \tilde{\omega}_k \) which parameterized the spheroidal harmonics, and ultimately informs each \( \sigma_m^k \).

In the case of ringdown, where \( a \) and \( \tilde{\omega}_k \) are well defined, the accuracy of Change of Basis results is limited by the available number of NR spherical harmonic multipole moments. This number is typically small due to limited numerical resolution, causing this approach to be applied to the \((2,2)\) and \((3,2)\) multipoles with \( \tilde{Q} \) being a \( 2 \times 2 \) matrix [8, 10]. It is known that inner-product ratios can be non-negligible for approximately \( |\ell - \ell'| \leq 2 \) ([19, 28, 29]), suggesting that estimation of a general spheroidal moment may require five spherical harmonic moments for robust accuracy. However this criteria is necessarily relaxed for cases where adjacent harmonics cannot exist as demanded by \( \ell \geq |s| \) and \( \ell \geq |s| \) [3, 14, 25, 27].

In making no assumption about the time or frequency domain behavior of the spheroidal moment, the Change of Basis approach implicitly assumes that there exists an underlying spheroidal harmonic representation that is spectrally complete. While unproven in Refs. [8, 10], this assumption has been supported by standing results from TD and FD regression [7, 8, 12, 33, 50]. Despite this numerically empirical support, spectral completeness and the closely related concept of spectral decomposition are not guaranteed.

### III. SPHEROIDAL HARMONIC DECOMPOSITION

Since the first applications of spin weighted spherical and spheroidal harmonics to gravitational wave physics, mathematical developments in quantum mechanics have precipitated new and potentially relevant concepts [52–57]. Of principle relevance here are dual or what we will refer to as adjoint functions, and their role in bi-orthogonal decomposition [52, 53, 56].

In this section we apply these concepts to the spheroidal harmonics, with particular emphasis on the spheroidal harmonics of Kerr BHs. We discuss how orthogonality and bi-orthogonality result from the properties of these operators’ adjoints [52, 56]. We detail a special case in which the spin weighted spheroidal harmonics with complex \( \gamma_k \) display an elementary kind of bi-orthogonality. And we generalize this special case to physical scenarios in which the spheroidal harmonic spin-frequency parameters vary with \( \ell \). In this we develop the adjoint-spheroidal harmonics as a generalization of the regular spheroids’ complex conjugates.

Lastly, we arrive at an algorithm for the practical spheroidal harmonic decomposition of gravitational radiation using adjoint-spheroidal harmonics derived from an overtone-subspace.

#### A. Orthogonality & Bi-Orthogonality

The properties of spin weighted harmonics are closely related to the properties of the differential operator for which they are eigen-functions. For spinning BHs, this differential operator \( \mathcal{L}_k \) is the polar part of Einstein’s equations linearized about the Kerr metric [6, 15, 58]. This operator’s eigen-relationhiop is satisfied by the spheroidal harmonics

\[
\mathcal{L}_k S_k = -A_k S_k,
\]

with

\[
\mathcal{L}_k = \left( s(1-s) + (u\gamma_k - s^2) \right) + \frac{(m + su)^2}{1 - u^2} \partial_u (1 - u^2) \partial_u .
\]

In Eq. (22), \( A_k \) is the spheroidal harmonic eigenvalue, often referred to a separation constant [6, 18]. In Eq. (23), \( u = \cos(\theta) \), and for a Kerr BH of mass \( M = 1 \), dimensionless angular momentum \( a = S/M^2 \) and modal frequency \( \tilde{\omega}_k \), we have that \( \gamma_k = a \tilde{\omega}_k \). Whether the harmonics possess any kind of orthogonality depends centrally on the properties of \( \mathcal{L}_k \) or, equivalently, its matrix representation. For that contemplation it is useful to recall that, given a linear differential operator, say \( \mathcal{L}_k \), its adjoint operator, \( \mathcal{L}_k^\dagger \), is defined by the requirement that

\[
\langle p | \mathcal{L}_k q \rangle = \langle \mathcal{L}_k^\dagger p | q \rangle ,
\]

where the bra-ket \( \langle \cdot | \cdot \rangle \) is an infinite dimensional inner-product (an integral). The concept of adjoint operators will play a central role in this section as we briefly review the orthogonality properties of spherical and spheroidal harmonics.

#### 1. Orthogonality of the spin weighted spherical harmonics

The spin weighted spherical harmonics emerge from Eqs. (22-23) when \( \gamma_k = 0 \)

\[
\mathcal{L}_a = \left( s + \frac{(m + su)^2}{1 - u^2} \right) + \partial_u (1 - u^2) \partial_u .
\]

It is useful to represent \( \mathcal{L}_a \)’s matrix elements using bra and ket notation. In this perspective, we will use the spherical harmonics as basis vectors, and equate the spherical ket, \( |Y_k \rangle \), with the spherical harmonic function \( Y_k(u) \). Similarly, we will equate the spherical harmonic bra, \( \langle Y_k | \), with the complex conjugate \( Y_k^\ast \).

Since all terms in Eq. (24) are real, so are \( Y_k \), therefore conjugation in \( \langle Y_k | \) is a superficial but standard notation. As is also standard, we will denote inner-product of two functions, \( p(u) \) and \( q(u) \), on \( u \in [-1,1] \) using the bra-ket,

\[
\langle p | q \rangle = \int_{-1}^{1} p(u)^* q(u) du .
\]
With Eqs. (24-25) we have all we need to write the matrix elements of \( \mathcal{L}_o \) in the basis of its eigenfunctions, \( \langle Y_j \mid \mathcal{L}_o \mid Y_k \rangle \). And with the linear differential form of \( \mathcal{L}_o \), known, we are able to use definition of the operator’s adjoint, to arrive at multiple representations of \( \mathcal{L}_o \)’s matrix elements

\[
\langle Y_j \mid \mathcal{L}_o \mid Y_k \rangle = -A_{j}^{(o)} \langle Y_j \mid Y_k \rangle = -\langle \mathcal{L}_o Y_j \mid Y_k \rangle = -A_{j}^{(0)} \langle Y_j \mid Y_k \rangle .
\]

In the first line of Eq. (26) we have applied the eigenvalue relationship given by Eq. (22), and we have used the fact that all quantities involved are real valued. Here we denote the eigenvalue as \( A_{j}^{(o)} \) to distinguish it from the spheroidal eigenvalue \( A_k \). In Eq. (26)’s second line, we have used the definition of the adjoint operator, \( \mathcal{L}_o^\dagger \), and applied the fact that \( \mathcal{L}_o = \mathcal{L}_o^\dagger \), as can be shown by imposing \( \langle Y_j \mid \mathcal{L}_o \mathcal{L}_o^\dagger Y_k \rangle = \langle \mathcal{L}_o^\dagger Y_j \mid Y_k \rangle \) along with integration by parts. Equating the first and last lines of Eq. (26) yields

\[
(A_{k}^{(o)} - A_{j}^{(0)}) \langle Y_j \mid Y_k \rangle = 0 ,
\]

or rather, if \( j \neq k \), then \( \langle Y_j \mid Y_k \rangle = 0 \). Thus, \( \langle Y_j \mid Y_k \rangle \propto \delta_{jk} \), and normalization of the harmonics means that

\[
\langle Y_j \mid Y_k \rangle = \delta_{jk} .
\]

In Eqs. (24-28) we see that the hermiticity of \( \mathcal{L}_o \) requires the orthogonality of its eigenfunctions, and thus diagonality of its matrix representation (in the appropriate basis).

In turn, the orthogonality of the spin weighted spherical harmonics enable us to resolve the identity via

\[
\hat{a} = \sum_{jk} |Y_j \rangle (Y_j \mid Y_k) Y_k = \sum_{l} |Y_l \rangle (Y_l \mid h_m) \quad \text{(29)}
\]

such that, for example, the gravitational wave strain’s \( \bar{m} \)-poles, \( h_m = |h_m \rangle \) (Eq. 6), may be represented as

\[
|h_m \rangle = \hat{a} |h_m \rangle = \sum_{l} |Y_l \rangle (Y_l \mid h_m) .
\]

Upon comparing Eq. (30) and Eq. (7) it is clear that \( h_{\bar{m}} = \langle Y_l \mid h_m \rangle \), just as expected from Eq. (10).

The convergence of Eq. (30) is assured by the completeness of the spin weighted spherical harmonics. And together, Eqs. (24-30) illustrate standard pedagogical arguments for how properties of the differential operator, \( \mathcal{L}_o \), are reflected in its eigenfunctions.

It is well known that the spheroidal harmonics do exhibit orthogonality, but only when \( \gamma_k \) is real valued, a scenario applicable to the perturbative inspiral of binary systems [14, 31]. However, during non-perturbative inspiral and perturbative ringdown, when \( \gamma_k \) is complex, the spheroidals do not exhibit orthogonality. In such cases a slightly different perspective is useful.

2. Bi-orthogonality of the spheroidal harmonics:

A special case

The spheroidal spin-frequency parameter \( \gamma_k \) plays the role of a dial, tuning solutions of Eq. (23) between zero and extreme sphericidity. However it is more appropriate to think...
of this parameter as not one but two dials, one controlling the real part of \( \gamma_k \) and another its imaginary part. This imaginary part is set by the dissipative nature of gravitational radiation \([58, 59]\). And it is this imaginary part that makes \( L_k \) non-hermitian. In the same way that \( L_k \)'s hermiticity can be demonstrated using the definition of the adjoint along with integration by parts, it may also be demonstrated that if \( \gamma_k \) is complex, then

\[
L_k^\dagger = L_k^\ast .
\]  

(31)

The spheroidal harmonics of Kerr, and likely more general spacetimes, are interesting not in that \( \gamma_k \) is complex, but rather in that they are coupled to an external, radial, equation \([6, 18]\). It is this coupling to another spatial dimension that gives additional structure to the space of spheroidal harmonics by way of the QNM frequencies.

For nonspinning BHs, these \( \ell \) and \( m \) dependent frequencies are determined by the differential system’s radial equation and the boundary conditions imposed on its solutions. For spinning systems, the radial and angular equations are related by the appearance of \( \dot{\omega}_k \) in Eq. (23)’s potential term, meaning that \( \dot{\omega}_k \) may be thought of as external inputs for which \( A_k \) and \( S_k \) may be determined. This has the effect of skewing the angular equation’s dependence on the polar index, \( \ell \), and thereby the compound label \( k = (\ell, m, n) \).

Rather than a single differential equation with eigenfunctions labeled in \( \ell \) and \( m \), we are now faced with a different differential equation for each \( \dot{\omega}_k = \dot{\omega}_k \exp \), each with a distinct solution space. While these equations only differ by their different QNM frequencies, this difference plays a central role in the full physical problem’s structure.

Before addressing the full physical problem of spheroidal harmonics, let us first probe the structure of each \( L_k \)’s solution space by considering the case of a spheroidal harmonic equation that has a complex spin-frequency parameter that is constant with \( k \),

\[
\gamma_k = \gamma .
\]

For this special case, the spheroidal operator is

\[
L = \left( s(1 - s) + (u \gamma - s)^2 \right) + \partial_u (1 - u^2) \partial_u .
\]

(32)

As in Eq. (22), the eigenfunctions of \( L \) are the spheroidal harmonics, but they are explicitly those for which harmonics of different \( k \) are parameterized by the same \( \gamma \).

Importantly, as \( \gamma \) no longer plays an active role, different eigenfunctions of \( L \) need only be labeled by spherical harmonic indices

\[
\bar{k} = (\bar{\ell}, \bar{m}) .
\]

We will use these indices to distinguish the spheroidal harmonics of this special case, \( S_{\bar{k}} = S_{\bar{k}}(\theta) \), from the physical harmonics discussed elsewhere in this work.

We may now follow the template established for the spherical harmonics by considering the matrix elements of \( L \). However, unlike with the spherical harmonics, we must take care to use the eigenfunctions of \( L \) as well as those of \( L^\dagger \),

\[
L' \tilde{S}_{\bar{j}}(u) = -\bar{A}_\bar{k} \tilde{S}_{\bar{j}}(u) .
\]

In Eq. (33), \( \bar{A}_\bar{k} \) is the adjoint-eigenvalue and \( \tilde{S}_{\bar{j}} \) is the adjoint-eigenfunction.

With this tool in hand, we may consider the appropriate matrix representation of \( L \) in the heterogeneous basis of adjoint and non-adjoint eigenfunctions,

\[
\langle \tilde{S}_{\bar{j}} \mid L \mid S_{\bar{k}} \rangle = -A_k \langle \tilde{S}_{\bar{j}} \mid S_{\bar{k}} \rangle = -A_k \langle S_{\bar{j}}^\dagger \mid S_{\bar{k}} \rangle
\]

= \langle L' \tilde{S}_{\bar{j}} \mid S_{\bar{k}} \rangle = \langle L' S_{\bar{j}}^\dagger \mid S_{\bar{k}} \rangle
\]

= -A_k \langle S_{\bar{j}}^\dagger \mid S_{\bar{k}} \rangle .
\]

(34)

In Eq. (34) we have used the fact that \( L^\dagger S_{\bar{j}}^\dagger = -A_k S_{\bar{j}}^\dagger \), meaning that for our special case \( \bar{S}_{\bar{k}} = S_{\bar{k}}^\dagger \). The subtractions of Eq. (34)’s first line from its last yields an analog of the spherical harmonic orthogonality statement (Eq. 27),

\[
(A_k - A_j) \langle S_{\bar{j}}^\dagger \mid S_{\bar{k}} \rangle = 0 .
\]

(35)

As with Eq. (28), we conclude that when \( j \neq \bar{k} \)

\[
\langle S_{\bar{j}}^\dagger \mid S_{\bar{k}} \rangle = \int_{-1}^{1} S_{\bar{j}}(u; \gamma) S_{\bar{k}}(u; \gamma) du = \delta_{\bar{j} \bar{k}} .
\]

(36)

Thus, under the standard inner-product Eq. (25), our special case’s spheroidal harmonics are not orthogonal with themselves via \( \langle S_{\bar{j}} \mid S_{\bar{k}} \rangle \), but instead they are bi-orthogonal with their complex conjugates via \( \langle S_{\bar{j}}^\dagger \mid S_{\bar{k}} \rangle \).

In Eqs. (31-36), like its spherical harmonic counterpart (Eqs. 24-28), we see that our special case leads to spectral decomposition through projection

\[
\tilde{\mu} = \sum_{\bar{k}} |S_{\bar{j}}^\dagger \rangle \langle S_{\bar{j}}^\dagger | S_{\bar{k}} \rangle .
\]

(37)

The last equality of Eq. (37) follows from the invariance of \( \tilde{\mu} \) under complex conjugation, and confers that we may either use the spheroidal harmonics or the adjoint spheroidals as basis vectors.

Equations (32-37) may inform what we might expect from the full physical problem. From the perspective of Eq. (32), we might expect the full problem’s adjoint functions to be a generalization of the conjugate spheroidals. We have noted that the full problem is informed by many operators, \( L_k \). Equation (34) implies that the matrix representation for this list of operators has rows that are informed by different spheroidal harmonics at their respective \( \gamma_k \), and columns informed by new adjoint-spherical harmonics that would simply be the conjugates of the regular spheroidal harmonics, if not for the dependence of the spin-frequency parameter on \( k \). However, we might also foresee a lingering complication of size. The spheroidal overtone index implies that no one-to-one association can be made between spherical harmonics and their physical spheroidal counterparts.

3. Bi-orthogonality of physical spheroidal harmonics

In the last section we saw that, for each spin-frequency parameter (or in the case of Kerr, each QNM frequency), there exists a single set of spheroidal functions corresponding to
In Eqs. (40-41) we have chosen to associate vector and matrix elements \( u \) represent any function on spheroidal harmonics to be determined. Given Eq. (38), may be the adjoint-spheroidal harmonics, and learn about the matrix elements of \( \hat{X} \) that adjoint-spheroidals exist, and proceed to construct them using information about their non-adjoint counterparts. Along the way, we will determine an algorithm for calculating the adjoint-spheroidal harmonics, and learn about the matrix representations of their effective operator.

For the fiducial case of Kerr spheroidals, we assume that adjoint functions exist, and allow resolution of the identity via

\[
\hat{I} = \sum_k |S_k\rangle\langle S_k| .
\]  

In Eq. (38), \( S_k \) are the spheroidal harmonics corresponding to different spin-frequencies \( \gamma_k = a \tilde{\omega}_k \), and \( \tilde{S}_k \) are the adjoint-spheroidal harmonics to be determined. Given Eq. (38), may represent any function on \( u \in [-1, 1] \) in the basis of adjoint spheroidals. It is convenient to represent the spherical harmonics in this way

\[
|Y_j\rangle = \sum_k |\tilde{S}_k\rangle\langle S_k| Y_j\rangle .
\]  

In Eq. (39) we may once again notice a linear matrix equation,

\[
\vec{g} = \hat{X}^* \vec{b} ,
\]  

with vector and matrix elements

\[
y_j = |Y_j\rangle , \quad X_{kj} = \langle Y_j|S_k\rangle , \quad \text{and} \quad b_k = |\tilde{S}_k\rangle .
\]  

In Eqs. (40-41) we have chosen to associate \( \langle S_k| Y_j\rangle \) with elements of \( \hat{X}^* \) to be consistent with the definition of spheroidal-spheroidal inner-products given in Eq. (13).

If the inverse of \( \hat{X} \) exists, then the vector of adjoint-spheroidals is

\[
\vec{b} = \hat{X}^* \vec{g}
\]  

with

\[
\hat{Z} = \hat{X}^{-1} .
\]

That is, we should expect that the adjoint-spheroidal harmonics should have a spherical harmonic representation with mixing coefficients, \( \langle Y_j|\tilde{S}_k\rangle \), given by the inverse matrix’s elements \( Z_{kj} \)

\[
|\tilde{S}_k\rangle = \sum_j Z_{kj} |Y_j\rangle .
\]

Equation (44) prescribes how to calculate the adjoint-spheroidals: if the matrix of spherical-spheroidal inner-products (Sec. II A) is non-singular, then the adjoint-spheroidal to spherical inner-products may be calculated via matrix inverse. Equivalently, if Eq. (44)’s sum is over \( J \) spherical harmonics, then Eq. (42) is a way of solving \( J \times J \) linear equations for \( Z_{kj} \). That is an exercise to the effect of imposing bi-orthogonality harmonic by harmonic.

B. Matrix representation of physical adjoint-spheroidal operator

Although Eq. (44) implies that evaluation of the adjoint-spheroidals requires the calculation of regular spheroidals for use in \( \langle S_k| Y_j\rangle \), it is informative to review why this is not the case. To that end let us return to our contemplation of the spherical harmonic equation where \( \gamma_k \) is decoupled from the polar index. Unlike in the previous section (Sec. III A 2), we wish to continue writing \( \gamma_k \) rather than \( \gamma \). To do so self-consistently, we will denote each associated spherical harmonic as \( |\mathcal{S}_{\ell' \gamma}| \), where our original notation is recovered when \( (\ell, m) = (\ell, m) \) in \( |S_{\ell m}, \gamma_{\ell m}| \).

\[
|\tilde{S}_k\rangle = |S_{\ell m}, \gamma_{\ell m}| .
\]
Thus we may write each harmonic’s eigen-relationship as
\[ \mathcal{L}_k \mathcal{S}_j; \gamma_k = -A_j(\gamma_k) \mathcal{S}_j; \gamma_k. \]

In a spherical harmonic basis, this becomes
\[ \sum_\beta \langle Y_\beta | \mathcal{L}_k | Y_\beta \rangle \langle Y_\beta | \mathcal{S}_j; \gamma_k = -A_j(\gamma_k) \langle Y_\beta | \mathcal{S}_j; \gamma_k. \] \tag{45} \]

Equation (45) communicates a host of useful information. When considered for many values of \( \ell \) it says that the inner-products, \( \langle Y_\beta | \mathcal{S}_j; \gamma_k \rangle \), compose \( \mathcal{L}_k \)’s eigenvectors (in the spherical harmonic basis). From that perspective, Eq. (45) may be compactly re-written as
\[ \hat{L}_k \hat{X}_k = -\hat{X}_k \hat{A}_k. \] \tag{46} \]

where the related matrix elements are (as read from Eq. 45)
\[ L_\beta \mu k = \langle Y_\beta | \mathcal{L}_k | Y_\mu \rangle, \]
\[ X_\beta \mu k = \langle Y_\beta | \mathcal{S}_j; \gamma_k \rangle, \]
\[ A_\beta \mu k = A_\beta (\gamma_k) \delta_{\beta \mu}. \] \tag{47, 48, 49} \]

Equations (46-49) are useful, in part, because they provide a manner of calculating spherical-spheroidal inner-products directly from the spheroidal operator \( \mathcal{L}_k \): Write the spheroidal operator in the spherical harmonic basis, and then find its eigenvectors and eigenvalues, and then use the eigenvectors to evaluate the spheroidal harmonics as a sum over spherical ones \([30, 31]\). Equations (46-49) are also useful because they allow \( \hat{L}_k \) to be expressed in terms of its eigenvectors and eigenvalues,
\[ \hat{L}_k = -\hat{X}_k \hat{A}_k \hat{X}_k^{-1}. \] \tag{50} \]

Equation (50) provides a blueprint for using information about eigenvalues and eigenvectors to determine their underlying operator.

Returning now to the full problem, where we have in Eqs. (41-44) matrices of inner-products akin to \( \hat{X}_k \) and \( \hat{X}_k^{-1} \). In the same way that we have arrived at Eq. (50), we might use \( \hat{X} \) (Eq. 41) and \( \hat{Z} \) (Eq. 42), along with spheroidal harmonic eigenvalues to determine the matrix whose eigenvectors span the columns of \( \hat{Z} \). If we refer to the matrix of interest as \( \hat{L}^z \), then we expect that
\[ \hat{L}^z \hat{Z}^* = -\hat{Z}^* \hat{A}^z. \] \tag{51} \]

In Eq. (51), \( \hat{L}^z \) is the matrix representation of a kind of adjoint-spheroidal operator. This should not be confused with the matrix representation of the standard adjoint operator given in Eqs. (31-32). The physical adjoint-spheroidals are underpinned by multiple differential operators, thus it may be appropriate to consider them to be eigenfunctions of what we will call a heterogeneous adjoint, \( \mathcal{L}_k^z \), where
\[ \mathcal{L}_k^z |S_\ell \rangle = -A_k^z |S_\ell \rangle. \]

In this framing, \( \hat{L}^z \) is the matrix representation of a differential operator \( \mathcal{L}_k^z \) in the spin weighted spherical harmonic basis. In Eq. (51), \( \hat{A} \) is a diagonal matrix composed of spheroidal harmonic eigenvalues, \( A_k \). Equation (51) allows the expression of \( \hat{L}^z \) via
\[ \hat{L}^z = -\left( \hat{Z} \hat{A} \hat{Z}^{-1} \right)^*. \] \tag{52} \]

Equations (44-52) begin to shed light on the adjoint-spheroidal harmonics of physical systems for which angular modes are coupled to radial ones. Equation (44) provides a method to calculate the adjoint-spheroidal harmonics, and Eq. (52) informs the differential operator for which the adjoint-spheroidals are eigenfunctions. However these equations also stress a lingering complication. The adjoint-spheroidal harmonics can only be defined if the matrix of spherical-spheroidal inner-product, \( \hat{X} \), is non-singular. In other words, physical adjoint-spheroidals and their underlying operator are only well defined if the spheroidal harmonics are linearly independent.

C. Practical spheroidal harmonic decomposition with overtone subspaces

A revealing exercise is to consider an infinitesimally small region around zero spin-frequency. For Kerr BHs, this is in effect a region around the Schwarzschild limit. We previously encountered this limit in Eqs. (23-24), which illustrate that at
zero spin-frequency, the spheroidal harmonics reduce to the spherical ones. We now use this limit to delve further into how non-zero spin activates the spheroidal harmonic’s spin-frequency dependence. Even an infinitesimally small spin-frequency brings a significant structural change in the space of harmonics associated with QNMs. For Kerr, the space of spherical harmonics in \( \ell \) and \( m \) increases in size due to QNM overtones labeled in \( n \); hence, the cardinality of the Kerr spheroidal harmonics is larger than that of the sphericals. Even for infinitesimally small values of BH spin, we cannot choose a single spherical harmonic and associate it with only one spheroidal harmonic.

Put oppositely, in the non-spinning limit, sets of spheroidal harmonics with varying \( n \) but fixed \( \ell \) and \( m \) become populated with an infinite number of effectively identical spherical harmonics. Clearly, in this limit, the spheroidal harmonics are not linearly-independent, and the inner-product matrices discussed in last section are not invertible. In the exactly zero spin case, it is well known that Schwarzschild overtones with labels \((\ell, \bar{m}, n)\) are naturally related to a single spherical harmonic with labels \((\ell, \bar{m})[6]\).

Beyond the zero spin-frequency limit, a naive handling of overtones might spell trouble for the existence of the last section’s adjoint-spheroidals. For Kerr, when attempting to write the spherical harmonics as a sum over adjoint-spheroidal ones \((\text{Eq. 39})\), it is well known that the inner-products between spherical harmonics and the spheroidals of common \((\ell, m)\) but different \( n \) are near unity for all physical BH spins \([12, 29]\). In the non-spinning limit, they must be exactly unity. This causes the right-hand-side of \text{Eq. (39)} to diverge in general.

Similarly, if one were to manually attempt a term-by-term approach to solve for \( \langle S_{\ell m} | Y \rangle \) \text{(see below Eq. 44)}, one would quickly conclude that the presence of overtones makes them over-determined. This means that adjoint-spheroidal harmonics may exist only if a one-to-one correspondence can be made between spherical harmonics and a linearly independent subspace of spheroidal ones.

In short, something must be done about the overtones.

1. Projection onto Overtone Subspaces

One ostensibly natural choice is to consider a fixed \( n \) \text{(i.e. fixed overtone) subspace}. A slight change of notation is useful to facilitate this change in perspective. We will use \( n \) to denote the single overtone label chosen for subspace decomposition. This \( n \) will be shared by all spheroidal harmonics in the subspace. And we will use \( n’ \) to denote a general overtone index; that is, \( n \) may only take on one value while \( n’ \) may be any non-negative integer. In addition, \( S_{\ell mn} \) will denote a member of the spheroidal subspace spanned in \( \ell \) and \( m \), and \( S_{\ell mn}^\prime \) will refer to an adjoint-harmonic derived from the linear independence of this space \text{(Eq. 44)}. Lastly, effective spheroidal harmonic multipole moments of the overtone subspace will be denoted \( h_{\ell mn}^\prime \), and the intrinsic spheroidal harmonic multipole moments \text{(i.e. those corresponding directly to Eq. 8)} will be denoted \( h_{\ell mn}^\prime \). These choices facilitate the rewriting of the \( m \)-poles \text{(Eq. 6)} as

\[
h_{\ell mn} = \sum_{\ell’ n} h_{\ell mn}^\prime |S_{\ell’ mn}^\prime\rangle = \sum_{\ell} h_{\ell mn} |S_{\ell’ mn}\rangle
\]

with the effective spheroidal multipole moment, \( h_{\ell mn}^\prime \), being

\[
h_{\ell mn}^\prime = \langle S_{\ell mn} | h_{\ell mn} \rangle = \sum_{\ell’} h_{\ell mn}^\prime \langle S_{\ell mn}^\prime | S_{\ell’ mn} \rangle
\]

In \text{Eq. (54)}, spheroidal harmonic decomposition with an overtone-subspace amounts to projecting out collections of overtones with like \( m \).

In \text{Eq. (53)}’s second line, we see the application of \text{Eq. (38)}’s conjugate form

\[
\hat{a} = \sum_{k} |S_{k}\rangle\langle S_{k}^\dagger|.
\]

Thus the effective spheroidal harmonic multipole moment, \( h_{\ell mn}^\prime \), is simply the inner-product between an adjoint-spheroidal and an \( m \)-pole.

2. Intrinsic & effective spheroidal multipole moments

To extract more from \text{Eq. (54)} it is useful to separate its last line into three parts.

\[
h_{\ell mn}^\prime = h_{\ell mn}^\prime + \sum_{n’ n_a} h_{\ell mn}^\prime \langle S_{\ell mn}^\prime | S_{\ell mn}^\prime \rangle + \sum_{\ell’ n’ n_a} h_{\ell’ mn}^\prime \langle S_{\ell’ mn}^\prime | S_{\ell mn}^\prime \rangle .
\]

The first part is \text{Eq. (55)}’s first term. It is simply the term for which \( \ell’ = \ell \) and \( n’ = n \). This is the term for which \( \langle S_{\ell mn}^\prime | S_{\ell mn}^\prime \rangle = 1 \), making it likely to dominate over others. We might next consider the remaining terms for which \( n’ = n \); however, the construction of the overtone-subspace’s adjoint-harmonics requires these terms to be zero. The second part collects terms for which \( \ell’ = \ell \), but \( n’ \neq n \). The similarity of the spheroidal harmonics for different overtone index suggests that this will be the next dominant part. Convergence of this subseries requires that the amplitude of successive overtone contributions must, after some value of \( n’ \), generally decrease faster than \( 1/n’ \). Lastly, we are left with terms for which neither \( \ell’ = \ell \) nor \( n’ = n \). Following the same reasoning applied to previous cases, these terms are likely to contribute the least to the effective multipole moment.

It can now be illustrated that in the zero spin-frequency \text{(e.g Schwarzschild) limit, Eq. (55) along with the confluence of spherical and spheroidals yield that}

\[
h_{\ell mn} = \lim_{a \to 0} h_{\ell mn}^\prime = h_{\ell mn}^\prime + \sum_{n’ n_a} h_{\ell mn}^\prime .
\]

Consequently, the use of overtone-subspaces is naturally consistent with the zero-spin limit where spherical harmonic decomposition is most appropriate and naturally insensitive to overtone number.

D. Example: Kerr adjoint-spheroids and their operator

When applied to the Kerr spheroidals, the content and results of previous sections communicate the following. For a BH of mass \( M \), dimensionless spin \( a = S/M^2 \), and QNM frequencies \( \omega_{\ell mn} \), its modes have angular functions given by the
spheroidal harmonics, $S_{\ell m n}$. The space of these harmonics is related to the radial structure of the spacetime in such a way that each $S_{\ell m n}$ corresponds to a different spheroidal harmonic operator. Each of these operators is parameterized by a complex quantity $Y_{\ell m n} = \tilde{\omega} S_{\ell m n}$. The complex nature of each operator’s potential means that the operators themselves are not hermitian. And the potential relevance of overtone modes, labeled in $n$, means that for every spherical harmonic, there are potentially an infinite number of spheroidal ones with the same $\ell$ and $m$. A linear algebraic analysis of this harmonic structure enables the calculation of adjoint-spheroidal harmonics for Kerr, if a particular overtone subspace is applied. When used in conjunction with the regular spheroidal harmonics, adjoint-spheroidals enable the calculation of effective spheroidal multipole moments via using bi-orthogonal decomposition. And the algebraic structure of the adjoint-harmonics is closely related to the matrix representation of the operator for which they are eigenfunctions. In this section, we present results for the Kerr adjoint-spheroidals and their matrix operator when only fundamental ($n = 0$) QNMs are considered.

Figure (2) compares the spheroidal harmonics with their adjoint counterparts for BH spin of $a = 0.7$ and QNM indices $(\ell, m, n) = (2, 2, 0)$ and $(3, 2, 0)$. This figure’s spheroidal harmonics were calculated by numerically solving Eq. (46)’s eigenvalue problem. A byproduct of this method is the matrix of spheric-al-spheroidal inner-products (Eq. 41). This matrix was inverted for the calculation of Fig. (2)’s adjoint-spheroidal harmonics (Eq. 44). In Fig. (2)’s left panels we see that the spheroidals and their adjoint functions differ non-trivially in amplitude. In the right panels we see that the phases of the spheroidals and their adjoint functions differ approximately, by a minus sign and a constant offset.

Figure (3) visualizes the non-orthogonality and bi-orthogonality of the adjoint and non-adjoint spheroidals for an azimuthal index $m = 2$ and BH spin $a = 0.7$. Inner-products were computed according to Eq. (25). The diagonal structure of Fig. (3) left and central panels is indicative of how spheroidal-spheroidal inner-products scale with $\gamma_k$ (e.g. Eq. 14). In Fig. (3)’s right panel, bi-orthogonality is signaled by the purely diagonal nature of $\langle S_{\ell m n} | S_{\ell m n} \rangle$.

Figure (4) visualizes the matrix representations for heterogeneous adjoint operators in the spin weighted $s = -2$ spherical harmonic basis. Related matrix representations were calculated according to Eq. (52). Figure (4)’s left panel shows this matrix operator’s elements for a BH spin of $a = 0.01$, a value for which the operator should have approximately the same form as its spherical harmonic counterpart: approximately tridiagonal, and generally pentadiagonal [18, 19, 29, 31]. In Fig. (4)’s central panel, we see $L^2$ for a BH spin of $a = 0.1$. For this and similar spins, the operator’s structure still mirrors its spheroidal counterpart. In Fig. (4)’s right-most panel, we see that for moderate and high spins the operators structure appears to depart from its spheroidal counterpart by being defined by more than five diagonal bands.

Together, Figures (2-4) provide examples of this work’s central results.

IV. DISCUSSION & CONCLUSIONS

When seeking to represent gravitational radiation in terms of its multipole moments, there has been a tension. While it has been most practical to represent gravitational radiation in terms of spin weighted spherical harmonics, it is simultaneously understood that a system’s intrinsic radiative modes are those most closely related to the system’s physical dynamics. The modes of gravitationally radiating systems can be difficult to define, and when they can be defined, mathematical complications have perhaps limited their use. The prototypical example is that of Kerr QNMs’ being mixed in the spherical harmonic multipole moments of NR. This case presents complications that are likely common to the radiative modes of many gravitationally radiating systems with angular momentum: The differential equation defining each mode’s angular behavior is non-hermitian, and parametrically coupled to the mode’s radial behavior. This causes the modes’ angular harmonics to be non-orthogonal, and defined by not one but an infinite number of differential operators. Further, the potential presence of overtone modes is incompatible with spectral decomposition.

The work presented here address these complications. We have shown that spheroidal harmonic differential equations with complex potentials display a basic kind of bi-orthogonality. They are orthogonal, not with themselves, but instead with their complex conjugates (Sec. III A 2). We have shown that bi-orthogonality in the full physical problem motivates the definition of adjoint-spheroidal harmonics (Sec. III A 3). We have seen that the adjoint-spheroidal harmonics are the eigenfunctions of a heterogeneous adjoint operator (Sec. III B). And we have discussed the required use of an overtone-subspace if spectral decomposition is to be practical (Sec. III C 2). We have seen example adjoint-spheroidals for Kerr (Fig. 2), and we have demonstrated their bi-orthogonality with the regular spheroidals (Sec. III D & Fig. 3). We have demonstrated that the related heterogeneous adjoint operator’s matrix representation (Fig. 4) can be computed directly from the analysis of the usual spherical harmonic differential equations, and that the structure of this operator is manifestly consistent with the small spin limit (Sec. III D). In these points, we have presented formal arguments and practical tools towards the general spherical harmonic representation of gravitational radiation. But more remains to be shown, and further questions are spurred.

Regarding the potential importance of overtones, this work may be used to support the following conclusions. For the Kerr remnants of BBH mergers, overtone modes cannot be computed directly via decomposition, despite their coupling to spheroidal harmonics (Sec. III C 1). They can only be investigated via time or frequency domain fitting, which poses a host of challenges at the intersection of modeling and physics: the space of damped sinusoids is overcomplete (Sec. II B), and the proximity of overtones to merger increases the chances of their being conflated with non-stationary effects [48]. Thus the potential importance of overtones must be subjected to consistency tests akin to those discussed in Sec. (II B). It remains to be shown whether overtones from numerical BBH remnants can pass this manner of test [7, 20, 33, 60–62]. The potential instability of all overtone solutions would seem to make the passing of such a test vital [63].
Many aspects of the presented work may be refined and expanded upon. For example, the presented analysis relies heavily on an equivalence between linear differential operators and their matrix representations. This approach results in infinite dimensional matrices, such as $L^1$ (Eq. 52), that must truncated for practical computations. Consequently, presented algorithms for adjoint-spheroidal are non-perturbative but limited by the largest spherical harmonic index considered. For the harmonics shown in Fig. (2) we have enforced that $|\ell - \ell'| \leq 8$. When compared to the spherical harmonics computed from Leaver's analytic representation [6], this choice yields a typical residual error less than 0.01%. Rather than working with numerical matrices, one could work with an analytic approximant to the spherical harmonics [18, 30]. Similarly, one could work with the analytic form of the spherical harmonic operator’s matrix form, and use approximate schemes for its eigenvectors [18, 29]. This has not been done here in favor of presenting high accuracy tools of potential use for gravitational wave signal modeling, including the decomposition of NR data. Future investigations may expand on the analytic properties of the adjoint-spheroidal harmonics and their operators.

Similarly, we have only briefly discussed the spectral decomposition of gravitational radiation into effective spheroidal moments using an overtone subspace. A multi-faceted investigation into potential applications is needed, but beyond the current scope. This too may be expanded upon in future work.

Each of these potential investigations carries new and potentially useful questions. Does the analytic structure of adjoint-spheroidal harmonics inform the broader non-hermitian nature of Einstein’s equations? How should the spin-frequency parameter be defined in systems where mass and spin are radiated non-adiabatically? And can the answer to these questions inform yet unprobed aspects of BBH merger for which the adjoint-spheroidal likely apply?

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