SHORT PROOF OF TWO CASES OF CHVÁTAL’S CONJECTURE

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Abstract. In 1974 Chvátal conjectured that no intersecting family \( \mathcal{F} \) in a downset can be larger than the largest star. In the same year Kleitman and Magnanti proved the conjecture when \( \mathcal{F} \) is contained in the union of two stars, and Sterboul when \( \text{rank}(\mathcal{F}) \leq 3 \). We give short self-contained proofs of these two statements.

1. Introduction

A downset, hereditary set, independence system or (abstract) simplicial complex \( C \) is a family of subsets of some finite ground set closed under taking subsets. Using nomenclature from simplicial complexes we call faces the elements of \( C \) and vertices, edges and triangles, respectively, the faces of sizes 1, 2 and 3. The star of a vertex \( a \), written \( \text{st}_C(a) \), is the family of all faces containing \( a \). It is an example of an intersecting family in \( C \), that is, a set of faces that pairwise intersect.

Chvátal’s 45-year-old conjecture, inspired by the classical result of Erdős, Ko and Rado [3] for the complete, uniform complex \( \binom{[n]}{\leq k} \), states that stars always achieve the maximal cardinality among intersecting families in \( C \):

Conjecture 1.1 (Chvátal [1]). Let \( \mathcal{F} \) be an intersecting family in a simplicial complex \( C \). Then, there exists a vertex \( a \) in \( C \) such that \( |\mathcal{F}| \leq |\text{st}_C(a)| \).

Note that there is no loss of generality in assuming that \( C \) is the smallest downset containing \( \mathcal{F} \) (in other words, \( C \) is generated by \( \mathcal{F} \)). Throughout the article we assume this and give short proofs of the following old and very recent known cases.

Theorem 1.2 (Kleitman and Magnanti [4, Theorem 2]). Let \( \mathcal{F} \) be an intersecting family contained in the union of two stars, \( \mathcal{F} \subset \text{st}(a) \cup \text{st}(b) \). Then \( |\mathcal{F}| \leq \max(|\text{st}(a)|, |\text{st}(b)|) \).

Theorem 1.3 (Sterboul [6, Theorem 2]). Chvátal’s conjecture holds if all elements of \( \mathcal{F} \) have size three or less.

Equivalently, this result settles Chvátal’s conjecture for rank at most three. It was recently reproven by Czabarka, Hurlbert and Kamat [2, Theorem 1.4]. We thank G. Hurlbert for pointing us towards reference [6].

Our proofs are inspired by our recent work with Stump on a related EKR problem [5].
2. Intersecting families contained in two stars

Lemma 2.1. Let $F$ be an intersecting family in $C$. Let $a, b, v$ be three vertices of $C$ and assume that every $B \in F$ with $v \in B$ intersects $\{a, b\}$. Define

$$R_a(v) := \{B \in F : a, v \in B, b \notin B, B \setminus v \notin F\},$$
$$R_b(v) := \{B \in F : b, v \in B, a \notin B, B \setminus v \notin F\}.$$

Then, $F' := F \setminus R_b(v) \cup \{B \setminus v : B \in R_a(v)\}$ is also an intersecting family.

Proof. All sets in $\{B \setminus v : B \in R_a(v)\}$ intersect one another since they all contain $a$. We thus only need to show that every $B_1 \in F \setminus R_b(v)$ intersects every $B_2 \in R_a(v)$ in an element different from $v$. If $v \notin B_1$ this is obvious since $B_1$ and $B_2$ are both in $F$ and thus they meet. If $a \in B_1$ this is obvious too, since then $a \in B_1 \cap B_2$. Hence, assume $B_1$ contains $v$ but not $a$. Our hypotheses imply that $b \in B_1$ and since $B_1 \notin R_b(v)$ we have that $B_1 \setminus v \in F$. Thus, $(B_1 \setminus v) \cap B_2$ indeed meet. \qed

Theorem 1.2 is a direct consequence of the following statement.

Corollary 2.2. Let $F \subseteq C$ be an intersecting family such that $F \subseteq \text{st}(a) \cup \text{st}(b)$ and neither $\text{st}(a)$ nor $\text{st}(b)$ contains $F$. Then there exists an intersecting family $F' \subseteq C$, $F' \subseteq \text{st}(a) \cup \text{st}(b)$, such that either $|F'| > |F|$ or $|F'| = |F|$ but then $F'$ has smaller average size of elements than $F$.

Proof. We first claim that there exists a vertex $v$ in $C$ such that (at least) one of the sets $R_a(v)$ and $R_b(v)$ of the previous lemma is not empty. For this, let $B$ be a minimal face in $F$ containing $a$ but not $b$ (it exists, or else the condition $F \subseteq \text{st}(a) \cup \text{st}(b)$ implies $F \subseteq \text{st}(b)$). If $B = \{a\}$ then $F \subseteq \text{st}(a)$. If $B \neq \{a\}$ then for each $v \in B \setminus a$ we have $R_a(v) \neq \emptyset$.

Assume that either $|R_a(v)| > |R_b(v)|$ or $|R_a(v)| = |R_b(v)|$ and $R_a(v)$ has average size of sets smaller or equal than $|R_b(v)|$. This is no loss of generality since $|R_a(v)| \leq |R_b(v)|$ implies $R_b(v)$ is not empty and we can exchange the roles of $a$ and $b$.

Hence, we have $|F'| = |F| - |R_b(v)| + |R_a(v)| \geq |F|$ with equality only if $|R_a(v)| = |R_b(v)|$. In this case, since $|R_a(v)|$ has average size of sets smaller or equal than $|R_b(v)|$ and we substitute the sets of $R_b(v)$ with sets of size smaller than those of $R_a(v)$, the average size of sets in $F'$ is smaller than in $F$. \qed

3. Intersecting families of rank three

To simplify notation, in what follows we omit braces when referring to a subset of the ground set and write, e. g., $abc$ instead of $\{a, b, c\}$. In part (1) of the following statement, given a triangle $abc \in F$ we say that a second triangle $\tau \in F$ is dangling from $abc$ at one of the vertices $x \in abc$ if $\tau \cap abc = x$.

Lemma 3.1. Let $F$ be an intersecting family consisting only of triangles. If any of the following conditions is satisfied, then there exists an intersecting family of size at least $|F|$ containing an edge or vertex:

1. Some triangle in $F$ has one or no triangles dangling at some vertex;
2. No two triangles in $F$ share an edge;
3. The graph of the complex generated by $F$ is not complete.

Proof. Throughout the proof, let $abc$ be a triangle in $F$.

For part (1), if for some vertex, say $a$, there is only one triangle $\tau \in F$ dangling at $a$, let $F' = F \setminus \{\tau\} \cup \{bc\}$. If there is none, just add $bc$ to $F$. 

For part (2), assume without loss of generality that among the triangles of \( \mathcal{F} \) there are at least as many containing \( a \) than \( b \) or \( c \). Let \( \mathcal{F}' \) consist of the triangle \( abc \) plus all other triangles \( axy \in \mathcal{F} \) together with their edges \( ax \) and \( ay \). Then \( \mathcal{F}' \subseteq \text{st}_c(a) \) and \( |\mathcal{F}'| \geq |\mathcal{F}| \) since all edges \( ax \) and \( ay \) are distinct.

For part (3), let \( c \) and \( v \) be vertices not spanning an edge. Let \( abc \in \mathcal{F} \) be a triangle containing \( c \), and let \( S_v = \{x \in abc : \exists y \text{ with } vxy \in \mathcal{F}\} \). By the hypothesis, \( S_v \subset ab \). The assumption that \( C \) is generated by the faces of \( \mathcal{F} \) implies that \( S_v \neq \emptyset \), so we assume \( a \subset S_v \). If \( S_v = a \) then we add \( ay \) to \( \mathcal{F} \) for each \( axy \in \mathcal{F} \). Hence, assume for the rest that \( S_v = ab \). Note that every element of \( \mathcal{F} \) containing \( v \) must contain either \( a \) or \( b \) since \( \mathcal{F} \) is intersecting. In particular, we can apply Lemma 2.1. If one of \( R_a(v) \) and \( R_b(v) \) is non-empty this yields an intersecting family \( \mathcal{F}' \), \( |\mathcal{F}'| \geq |\mathcal{F}| \), containing edges. If both \( R_a(v) \) and \( R_b(v) \) are empty, the only element of \( \mathcal{F} \) containing \( v \) is \( abc \) and we can add \( ab \) to \( \mathcal{F} \).

**Proof of Theorem 1.3.** Chvátal’s conjecture holds when \( \mathcal{F} \) contains a vertex (trivial) or an edge (Theorem 1.2; note that if \( ab \in \mathcal{F} \), then trivially \( \mathcal{F} \subseteq \text{st}(a) \cup \text{st}(b) \)). Thus, we can assume that \( \mathcal{F} \) consists entirely of triangles and, by Lemma 3.1, that it does not satisfy any of the three conditions listed in that lemma.

In particular, by part (2) of the lemma, \( \mathcal{F} \) contains two triangles \( abc \) and \( abx \) sharing an edge. Observe that all triangles dangling from \( abc \) at \( c \) must contain \( x \) since a triangle dangling at \( c \) and not containing \( x \) does not intersect \( abx \). Moreover:

- **There are exactly two such triangles, say \( cxy \) and \( cxz \).** There are at least two by part (1) of Lemma 3.1. If there is a third triangle \( cxy \) dangling at \( c \), then every triangle in \( \mathcal{F} \) must intersect \( cx \) (otherwise it must contain \( v \), \( y \), and \( z \) and intersect \( abc \), a contradiction). Hence, we can add \( cx \) to \( \mathcal{F} \) and apply Theorem 1.2.

- **The only vertices of \( C \) are \( a, b, c, x, y, z \).** Assume there exists another vertex \( v \in C \). By part (3) of Lemma 3.1 the edge \( cv \) is contained in some triangle \( \tau \in \mathcal{F} \). By the previous item, \( \tau \) is not dangling from \( abc \) at \( c \) so without loss of generality \( \tau = acv \). Now, every triangle \( \sigma \in \mathcal{F} \) dangling at \( b \) must contain both \( v \) and \( x \) or both \( y \) and \( z \). Since the latter is impossible, \( bvx \) is the only possible triangle dangling at \( b \), contradicting part (1) of Lemma 3.1.

Once we know there are exactly six vertices, observe that at most half of the \( \binom{6}{3} = 20 \) triangles on six vertices, one from each complementary pair, can be in \( \mathcal{F} \), so \( |\mathcal{F}| \leq 10 \). But the above implies that \( \text{st}_C(c) \) contains at least the following 10 faces: the three triangles \( abc, cxy, cxz \) plus at least another triangle dangling from \( cxy \) at \( c \), the five edges \( ca, cb, cx, cy, cz \), and \( c \) itself.

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