Direct Flux Gradient Approximation to Close Moment Model for Kinetic Equations

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Abstract

To close the moment model deduced from kinetic equations, the canonical approach is to provide an approximation to the flux function not able to be depicted by the moments in the reduced model. In this paper, we propose a brand new closure approach with remarkable advantages than the canonical approach. Instead of approximating the flux function, the new approach close the moment model by approximating the flux gradient. Precisely, we approximate the space derivative of the distribution function by an ansatz which is a weighted polynomial, and the derivative of the closing flux is computed by taking the moments of the ansatz. Consequently, the method provides us an improved framework to derive globally hyperbolic moment models, which preserve all those conservative variables in the low order moments. It is shown that the linearized system at the weight function, which is often the local equilibrium, of the moment model deduced by our new approach is automatically coincided with the system deduced from the classical perturbation theory, which can not be satisfied by previous hyperbolic regularization framework. Taking the Boltzmann equation as example, the linearization of the moment model gives the correct Navier-Stokes-Fourier law same as that the Chapman-Enskog expansion gives. Most existing globally hyperbolic moment models are re-produced by our new approach, and several new models are proposed based on this framework.

Keywords: Kinetic equation; moment model; global hyperbolicity; conservation law; Maxwellian iteration.

1 Introduction

The kinetic equation is the evolution equation of particle distribution function in phase space, and describes the motion of particles and their interactions with each other or with the background medium. Common examples of the kinetic equation include the Boltzmann equation for rarefied gas, the radiative transfer equation for photon transport, the vlasov equation for plasmas, etc. They have wide applications in fields like rarefied gases, microflow, semi-conductor device simulation, radiation astronomy, optical imaging, and so on. Among the numerous methods developed to solve the kinetic equations [6,4,20,23], moment methods are attractive due to their clear physical interpretation and high efficiency in the transitional regimes [18,31,21,30,9,22,17].

The moment method approximates the original kinetic equation by studying the evolution of a finite number of moments of the distribution function. In gas kinetic theory, it was first proposed in Grad’s seminal paper [18] in 1949, in which the notable Grad’s 13 moment system is also presented. For any moment model, the evolution equations of the lower order moments rely on the higher order moments, hence a moment closure is needed, which is the central problem of the moment method. Different types of moment closures have been developed, resulting in many different kinds of moment models. These models could roughly be divided into two types: the first type are called hyperbolic or first order PDEs, these include Grad’s 13 moment system [18], the maximum entropy moment system [21], approximate maximum entropy moment systems [30], the $P_\gamma$ and $M_N$ models for radiative transfer [35,31], etc. The second kind are parabolic type or second order PDEs, these include regularized moment methods of various kinds [38,36,39] and the diffusion approximation for the radiative transfer equation. This paper

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focus on discussing the first order moment models for their several advantages, as pointed out in [27]. First, as only the first derivatives are computed, an extra order of spatial accuracy can potentially be obtained for a given stencil. Secondly, numerical solution of first order systems are less sensitive to grid irregularities [29], which often occur for practical situations where there are complex geometries, or when adaptive mesh refinement are used. For this first order system of equation, an important property is its hyperbolicity, which is the necessary condition for the local well-posedness of problem with Cauchy data. However, obtaining global hyperbolicity for moment systems is not trivial. In [10] the authors pointed out that Grad’s 13 moment system is not hyperbolic even in any neighbourhood of the Maxwellian. Over the past decade, much effort has been devoted to obtaining globally hyperbolic moment models. Levermore proved in [24] that the maximum entropy moment models are globally hyperbolic. The approximate affordable robust version of the 14 moment system of the hierarchy studied by Levermore, which is proposed by McDonald and Torrilhon in [30], is almost globally hyperbolic. There has also been on-going effort on hyperbolic regularization of Grad-type moment models, such as the hyperbolic regularization proposed by Cai et. al [8, 9], and the quadrature-based regularization method proposed in [22]. Based on understanding of these regularization methods, [11] and [15] proposed a general framework to deduce globally hyperbolic moment systems, where the distribution function is approximated by weighted polynomials. The framework has been applied to various fields to obtain globally hyperbolic moment models in a routine way.

However, for some moment systems, applying the framework proposed in [11, 15] is a naive way. For instance, in [16], a direct application of the framework changes the $M_1$ model, which is already globally hyperbolic and based on which the $MP_N$ model is derived. On the other hand, in [13, 16], it was pointed out that the moment model derived by directly applying the framework in [11, 15] lead to incorrect NSF law and Eddington approximation, when a one-step Maxwellian iteration is applied. There has been some recent progress in this direction. In [16], for the $MP_N$ model of the radiative transfer equation [17], a new hyperbolic regularization was proposed, in which the distribution function and its derivative were approximated in different spaces. The $HMP_N$ model, based on this hyperbolic regularization, no longer had the above disadvantages. This inspires us to propose a new hyperbolic regularization for the moment models of kinetic equation.

In this paper, we propose a new approach to close hyperbolic moment models derived for kinetic equations. We directly approximate the flux gradient to give the moment closure. Precisely, we approximate the derivative of the distribution function by a weighted polynomial. In traditional moment methods, the moment closure is often given by approximating the distribution function, and some of them suffer from lack of hyperbolicity [10, 11, 17]. The new closure approach provide us an improved framework [11, 15] to carry out model reduction for generic kinetic equations. We first prove that using the improved framework, the moment system is globally hyperbolic, as long as the weight function satisfies the basic requirement of positivity. Moreover, for an $N$-th order moment system, moments with orders from 0 to $N - 1$ are conservative, without considering the right hand side. Additionally, the moment model by the improved framework does not change the consequence of Maxwellian iteration, which means the NSF laws are preserved. As mentioned above as comparison, direct application of hyperbolic regularization might not preserve this property [13, 16]. Furthermore, we show that the HME model [8, 9] for the Boltzmann equation, the $P_N$ model, the $M_N$ model, and the $HMP_N$ model for the radiative transfer equation [35, 34, 16, 25] can be regarded as special cases derived by the new approach. Using the new closure approach, we propose new globally hyperbolic moment systems for kinetic equations, such as $HMP_N$ model for monochromatic case and a moment model for the Boltzmann-Peierls equation. The improved framework is extended to 3-D case naturally, where the advantages discussed above are all preserved. Using the improved framework, we derive a new hyperbolic moment model for the Grad’s 13-moment system for quantum gas.

The remaining parts of the paper is organized as follows. In Section 2 we introduce our new closure approach in the 1-D case as the essential ingredients of the improved framework, and some properties are proved. In Section 3 we list some classical moment models and put them into the category of the improved framework, while some new models are proposed. In Section 4 the framework is extended to 3-D case, and the 3-D examples are shown in Section 5. The paper ends with a conclusion in Section 6.
2 Model in one-dimensional case

In this section, we first introduce the moment method for kinetic equation and our new framework in 1-D case, where the kinetic equation is often given by

\[ \frac{\partial f}{\partial t} + v(\xi)\frac{\partial f}{\partial x} = S(f), \quad t \in \mathbb{R}^+, x \in \mathbb{R}, v(\xi) \in \mathbb{R}, \xi \in \mathbb{G} \subset \mathbb{R}. \]  

(2.1)

\( f(t, x, \xi) \) is the distribution function depending on time \( t \), spatial variable \( x \) and velocity-related variable \( \xi \). The \( k \)-th moment is defined as \( E_k(t, x) := \langle \tau(\xi)v^k f \rangle \), where \( \tau(\xi) > 0 \), and

\[ \langle \cdot \rangle := \int_{\mathbb{G}} \cdot d\xi. \]

An \( N \)-th order moment system can be written as

\[ \frac{\partial E_k}{\partial t} + \frac{\partial E_{k+1}}{\partial x} = S_k, \quad 0 \leq k \leq N, \]

(2.2)

where

\[ S_k := \langle \tau(\xi)v^k S \rangle. \]

However, the moment system (2.2) is not closed, since there are \( N + 1 \) equations and \( N + 2 \) variables \( E_0, E_1, \cdots, E_{N+1} \). Thus a moment closure needs to be applied. A typical moment closure is to use a function of \( E_0, E_1, \cdots, E_N \) instead of \( E_{N+1} \) in (2.2), which is given by

\[ E_{N+1} = E_{N+1}(E_0, E_1, \cdots, E_N), \]

(2.3)

and a commonly used method to obtain (2.3) is to give an ansatz \( \hat{f} \), which satisfies

\[ \langle \tau(\xi)v^k \hat{f} \rangle = E_k, \quad 0 \leq k \leq N. \]

(2.4)

Then \( E_{N+1} \) is taken as the \((N + 1)\)-th moment of \( \hat{f} \) to finish the moment closure. Based on this idea, many practical moment models \[18, 24, 41, 1, 17\] for kinetic equations have been proposed.

On the other hand, in order to make (2.2) closed, one only needs to give the moment closure on \( \frac{\partial E_{N+1}}{\partial x} \). This inspires us to take the moment closure as

\[ \frac{\partial E_{N+1}}{\partial x} = \frac{\partial E_{N+1}}{\partial x} \left( E_0, E_1, \cdots, E_N; \frac{\partial E_0}{\partial x}, \frac{\partial E_1}{\partial x}, \cdots, \frac{\partial E_N}{\partial x} \right). \]

(2.5)

It is natural to expect a quasilinear system to be derived, that we require that \( \frac{\partial E_{N+1}}{\partial x} \) relies linearly on \( \frac{\partial E_0}{\partial x}, \cdots, \frac{\partial E_N}{\partial x} \). Since we can use an ansatz \( \hat{f} \) to approximate the distribution function \( f \), we can similarly give an ansatz \( \hat{g} \) on \( \frac{\partial f}{\partial x} \), which satisfies

\[ \langle \tau(\xi)v^k \hat{g} \rangle = \frac{\partial E_k}{\partial x}, \quad 0 \leq k \leq N, \]

(2.6)

and the moment closure (2.5) is then given by

\[ \frac{\partial E_{N+1}}{\partial x} = \langle \tau(\xi)v^{N+1} \hat{g} \rangle. \]

2.1 Model deduction

\[18, 24, 41, 1, 17\] suggest using a weighted polynomial to approximate the distribution function \( f \) to give the moment closure (2.3), which motivates us to use a weighted polynomial to approximate \( \frac{\partial f}{\partial x} \) to give the moment closure (2.5). Precisely, we take a positive weight function \( \omega[\eta] \), where \( \eta \) is a vector composed of some parameters depend on moments \( E_0, E_1, \cdots, E_N \), and the ansatz is given by

\[ \hat{g} = \omega[\eta] \sum_{i=0}^{N} q_i v^i. \]  

(2.7)
Denote $\mathcal{E}_i = \langle \tau(\xi) \omega^{[n]}(\xi) \rangle$ as the $i$-th moment of weight function $\omega^{[n]}$, and matrix $D \in \mathbb{R}^{(N+1) \times (N+1)}$ and $K \in \mathbb{R}^{(N+1) \times (N+1)}$ satisfy $D_{i,j} = \mathcal{E}_{i+j}$, and $K_{i,j} = \mathcal{E}_{i+j+1}$, $0 \leq i, j \leq N$, i.e.

$$D = \begin{bmatrix}
E_0 & E_1 & E_2 & \cdots & E_N \\
E_1 & E_2 & E_3 & \cdots & E_{N+1} \\
E_2 & E_3 & E_4 & \cdots & E_{N+2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
E_N & E_{N+1} & E_{N+2} & \cdots & E_{2N}
\end{bmatrix}, \quad K = \begin{bmatrix}
E_1 & E_2 & E_3 & \cdots & E_{N+1} \\
E_2 & E_3 & E_4 & \cdots & E_{N+2} \\
E_3 & E_4 & E_5 & \cdots & E_{N+3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
E_{N+1} & E_{N+2} & E_{N+3} & \cdots & E_{2N+1}
\end{bmatrix}. \quad (2.8)$$

Since $\tau(\xi) > 0$ and $\omega^{[n]}(\xi) > 0$, it is not difficult to prove that $D$ is symmetric and positive definite, and $K$ is symmetric. Furthermore, denote $g = (g_0, g_1, \cdots, g_N)^T \in \mathbb{R}^{N+1}$, then according to the constraints 2.6, we have

$$Dg = \left( \frac{\partial E_0}{\partial x}, \frac{\partial E_1}{\partial x}, \cdots, \frac{\partial E_N}{\partial x} \right)^T, \quad (2.9)$$

and the moment closure is given by

$$\left( \frac{\partial E_1}{\partial x}, \frac{\partial E_2}{\partial x}, \cdots, \frac{\partial E_{N+1}}{\partial x} \right)^T = Kg = KD^{-1} \left( \frac{\partial E_0}{\partial x}, \frac{\partial E_1}{\partial x}, \cdots, \frac{\partial E_N}{\partial x} \right)^T. \quad (2.10)$$

Notice in (2.10), the gradient of the flux functions rely linearly on $\frac{\partial E_k}{\partial x}$, $k = 0, \cdots, N$, which satisfies our previous requirement. Therefore, the moment system (2.11) can be written as

$$\frac{\partial E}{\partial t} + KD^{-1} \frac{\partial E}{\partial x} = S, \quad (2.11)$$

where $E = (E_0, E_1, \cdots, E_N)^T \in \mathbb{R}^{N+1}$, and $S = (S_0, S_1, \cdots, S_N)^T \in \mathbb{R}^{N+1}$.

### 2.2 Hyperbolicity

The moment system (2.11) is globally hyperbolic, and the eigenvalues can be calculated. We first give this theorem:

**Theorem 1.** Moment system (2.11) is globally hyperbolic.

*Proof. To obtain the hyperbolicity, one only needs to investigate whether $KD^{-1}$ is real diagonalizable. Notice that $D$ is symmetric and positive definite, and $K$ is symmetric, thus $KD^{-1}$ is real diagonalizable. Therefore, moment system (2.11) is globally hyperbolic.*

#### 2.2.1 Orthogonal polynomials

In order to investigate the eigenvalue of (2.11), we first introduce a series of monic orthogonal polynomials $\phi_i^{[n]}(v)$, which satisfy

$$\int_G \tau(\xi) \omega^{[n]}(\xi) \phi_i^{[n]}(v) \phi_j^{[n]}(v) \, d\xi = 0, \quad \text{when} \ i \neq j. \quad (2.12)$$

The Gram-Schmidt procedure to obtain the orthogonal polynomial can be formulated as

$$\phi_0^{[n]} = 1, \quad \phi_i^{[n]} = v^i - \sum_{j=0}^{i-1} \frac{\mathcal{K}_{i,j}}{K_{j,j}} \phi_j^{[n]}, \quad (2.13)$$

where

$$\mathcal{K}_{i,j} := \langle \tau(\xi) \omega^{[n]}(\xi) v^i \phi_j^{[n]} \rangle = \int_G \tau(\xi) \omega^{[n]}(\xi) v^i \phi_j^{[n]} \, d\xi. \quad (2.14)$$

According to the orthogonality (2.12), we have

$$\mathcal{K}_{i,i} = \langle \tau(\xi) \omega^{[n]}(\xi) (\phi_i^{[n]})^2 \rangle > 0, \quad \mathcal{K}_{k,i} = 0, \quad \text{when} \ k < i. \quad (2.15)$$

Furthermore, multiplying $v^k \tau(\xi) \omega^{[n]}(\xi)$ by (2.13), and taking integration with respect to $\xi$ on $G$, $\mathcal{K}_{k,i}$ can be calculated by

$$\mathcal{K}_{k,i} = \mathcal{E}_{k+i} - \sum_{j=0}^{i-1} \frac{\mathcal{K}_{i,j}}{K_{j,j}} \mathcal{K}_{k,j}. \quad (2.16)$$
Therefore, if \( \tau(\xi) > 0 \) and \( \omega^{[n]}(\xi) > 0 \), one can always define the orthogonal polynomials \( \phi_i^{[n]} \) by (2.13). Then, on the eigenvalue of (2.11), we have the following theorem:

**Theorem 2.** The eigenvalues of the moment system (2.11) are the zeros of the \((N+1)\)-th orthogonal polynomial \( \phi_{N+1}^{[n]}(v) \).

**Proof.** Consider the characteristic polynomial of (2.11)

\[
\text{Det}(KD^{-1} - \lambda) = 0,
\]

which can be rewritten as

\[
\text{Det}(K - \lambda D) = 0.
\]

Since \( \lambda \) is an eigenvalue, there exists a vector \( \mathbf{r} = (r_0, r_1, \ldots, r_N)^T \in \mathbb{R}^{N+1} \), satisfying that \( (K - \lambda D)\mathbf{r} = 0 \), i.e.

\[
\sum_{i=0}^{N} (K_{k,i} - \lambda D_{k,i})r_i = 0, \quad 0 \leq k \leq N,
\]

which is

\[
\sum_{i=0}^{N} (E_{k+1}^{i+1} - \lambda E_{k+1}^{i})r_i = 0 = \left( v^k (v - \lambda) \sum_{i=0}^{N} r_i v^i \tau(\xi)\omega^{[n]}(\xi) \right), \quad 0 \leq k \leq N.
\]

Therefore, \( (v - \lambda) \sum_{i=0}^{N} r_i v^i \) is orthogonal to \( v^k \), with respect to \( \tau(\xi)\omega^{[n]}(\xi) \), for any \( 0 \leq k \leq N \). Notice \( (v - \lambda) \sum_{i=0}^{N} r_i v^i \) is a polynomial of \( v \) whose degree is not greater than \( N + 1 \), thus we have \( r_N \neq 0 \), and

\[
(v - \lambda) \sum_{i=0}^{N} r_i v^i = r_N \phi_{N+1}^{[n]}.
\]

Therefore, \( \lambda \) is the root of the orthogonal polynomial \( \phi_{N+1}^{[n]} \). \( \square \)

### 2.3 Comparison with conventional moment system

The conventional moment system, derived by postulating an ansatz for the distribution function to obtain (2.3), can be formulated as a conservation law, without considering the right hand side. Due to the conservation, it is a popular way to deduce a moment system. However, the conventional moment system often suffers from lack of hyperbolicity. As a globally hyperbolic moment system, in this section, we will show some properties of (2.11) in comparison with the conventional moment system.

First, it is direct to show the conservation of the moments with orders from 0 to \( N - 1 \), i.e., we have

**Theorem 3.** In the moment system (2.2), the first \( N \) equations can be written into conservation form.

**Proof.** The first \( N \) equations, corresponding to the moments with orders from 0 to \( N - 1 \), are

\[
\frac{\partial E_k}{\partial t} + \frac{\partial E_{k+1}}{\partial x} = S_k, \quad 0 \leq k \leq N - 1,
\]

therefore, the governing equations of the moments with orders from 0 to \( N - 1 \) can be written into conservation form. \( \square \)

However, compared to the conventional moment system given by (2.3) and (2.4), the framework cannot ensure the conservation property of the entire system. Nevertheless, we have the following theorem:

**Theorem 4.** For the one-dimensional kinetic equation (2.1), if we use the ansatz

\[
\hat{f} = h \left( \sum_{i=0}^{N} \eta_i v^i \right)
\]

(2.15)

to approximate the distribution function \( f \) and derive a moment system \( A_1 \). The resulting system is equivalent to the system (2.11) by taking the weight function

\[
\omega^{[n]} = h' \left( \sum_{i=0}^{N} \eta_i v^i \right),
\]

(2.16)

which is denoted as \( A_2 \). Moreover, the system (2.11) in this situation is in conservation form.
Proof. We write the system derived by taking the ansatz (2.15) into quasilinear form, formulated as
\[
\frac{\partial E}{\partial t} + M \frac{\partial E}{\partial x} = S,
\]  
(2.17)
where the element in the \(i\)-th row and \(j\)-th column, which we hereafter will refer to as the \((i, j)\)-th element of matrix \(M\), is \(M_{i,j} = \frac{\partial E_{i+1}}{\partial E_j}\). Notice the moment closure \(E_{N+1}\) is given by
\[
E_{N+1} = \langle \tau(\xi)v^{N+1} \hat{f} \rangle = \left\langle \tau(\xi)v^{N+1}h\left(\sum_{i=0}^{N} \eta_i v^i\right)\right\rangle,
\]  
(2.18)
and the constraints provided by the first \(N\) moments
\[
E_k = \langle \tau(\xi)v^k \hat{f} \rangle = \left\langle \tau(\xi)v^k h\left(\sum_{i=0}^{N} \eta_i v^i\right)\right\rangle, \quad 0 \leq k \leq N,
\]  
(2.19)
since \(\tau(\xi)\) is independent with \(\eta_i\), thus when \(0 \leq i \leq N, 0 \leq k \leq N + 1,\)
\[
\frac{\partial E_k}{\partial \eta_i} = \frac{\partial}{\partial \eta_i} \left\langle \tau(\xi)v^k h\left(\sum_{i=0}^{N} \eta_i v^i\right)\right\rangle = \left\langle \tau(\xi)v^k h'\left(\sum_{i=0}^{N} \eta_i v^i\right) v^i \right\rangle = \left\langle \tau(\xi)\omega^{[n]} v^{i+k} \right\rangle = E_{k+i}.
\]
Therefore, \(\frac{\partial E_k}{\partial \eta_i}\) is the \((k, i)\)-th element in \(D \ (2.8)\), \(0 \leq i, k \leq N\), and \(\frac{\partial E_k}{\partial E_k}\) is the \((k, i)\)-th element of \(D^{-1}\), since \(D\) is symmetric and positive definite. Furthermore, \(\frac{\partial E_{i+1}}{\partial E_j}\) is \((i, j)\)-th element of \(K\), \(0 \leq i, j \leq N\). Thus,
\[
\frac{\partial E_{i+1}}{\partial E_j} = \sum_{k=0}^{N} \frac{\partial E_{i+1}}{\partial \eta_k} \frac{\partial \eta_k}{\partial E_j} = \sum_{k=0}^{N} K_{i,k} D^{-1}_{k,j} = (KD^{-1})_{i,j}.
\]
Therefore, \(M = KD^{-1}\), and the system (2.17) is equivalent to (2.11).

At last, since (2.17) can be written as
\[
\left\langle \tau(\xi)v^k h \left(\frac{\partial \hat{f}}{\partial t} + v \frac{\partial \hat{f}}{\partial x}\right) \right\rangle = \left\langle (\tau(\xi)v^k S) \right\rangle, \quad 0 \leq k \leq N,
\]  
(2.18)
and (2.11) are in conservation form.

Furthermore, when the number of parameters \(\eta\) is less than \(N\), we have the following theorem

**Theorem 5.** For the one-dimensional kinetic equation (2.1), if we use the weight function
\[
\tilde{\omega}^{[n]} = h \left(\sum_{i=0}^{n} \eta_i v^i\right),
\]  
(2.20)
and make a weighted polynomial \(\hat{f} = \sum_{i=0}^{N} f_i v^i \tilde{\omega}^{[n]}\) to approximate the distribution function \(f\) and derive a moment system \(A_1\), and denote the system (2.11) by taking the weight function
\[
\omega^{[n]} = h' \left(\sum_{i=0}^{n} \eta_i v^i\right),
\]  
(2.21)
as \(A_2\). Then the results of one-step Maxwellian iteration of \(A_1\) and \(A_2\) are the same.

**Proof.** We still use the notation in (2.8), and \(A_2\) can be written as
\[
\frac{\partial E}{\partial t} + KD^{-1} \frac{\partial E}{\partial x} = S.
\]
On the other hand, denote \(\tilde{E}_n = \langle \tau(\xi)v^n \tilde{\omega}^{[n]} \rangle\), and \(\tilde{D}\) as \(\tilde{D}_{i,j} = \tilde{E}_{i+j}\), and \(\tilde{K}\) as \(\tilde{K}_{i,j} = \tilde{E}_{i+j+1}, 0 \leq i, j \leq N\), then \(A_1\) can be written as
\[
\frac{\partial E}{\partial t} + \frac{\partial (KD^{-1} E)}{\partial x} = S.
\]
Furthermore, we can rewrite these two systems by a variable \( w = (\eta, \eta^*) \in \mathbb{R}^{N+1} \), where there is an one-to-one map between \( w \) and \( E \). Additionally, we assume that when \( \tilde{f} \approx \tilde{\omega}^{[n]} \), \( E_k = \tilde{E}_k \), and \( \eta^* = 0 \).

Without loss of generality, we assume that \( A := \frac{\partial E}{\partial w} \) is invertible.

Then \( A_1 \) and \( A_2 \) can be respectively written as

\[
A \frac{\partial w}{\partial t} + \frac{\partial (KD^{-1}E)}{\partial w} \frac{\partial w}{\partial x} = S, \quad A \frac{\partial w}{\partial t} + KD^{-1} A \frac{\partial w}{\partial x} = S,
\]

In order to prove that the one-step Maxwellian iteration of these two systems are equivalent, one only needs to prove that when \( \tilde{f} = \tilde{\omega}^{[n]} \), i.e. \( E_k = \tilde{E}_k \) for \( 0 \leq k \leq N \),

\[
(KD^{-1}A)_{m,l} \big|_{\eta^* = 0} = \frac{\partial (KD^{-1}E)_m}{\partial \eta} \bigg|_{\eta^* = 0}, \quad 0 \leq m \leq N, 0 \leq l \leq n.
\]

The right hand side can be calculated as

\[
\frac{\partial (\sum_{i,j=0}^{N} K_{m,i} D_{i,j}^{-1} E_j)}{\partial \eta} \bigg|_{\eta^* = 0} = \frac{\partial (\sum_{i,j=0}^{N} \tilde{K}_{m,i} \tilde{D}_{i,j}^{-1} E_j)}{\partial \eta} \bigg|_{\eta^* = 0} = \frac{\partial \tilde{K}_{m,0}}{\partial \eta} = K_{m,l}
\]

where when \( \tilde{f} = \tilde{\omega}^{[n]} \), \( E_j = \tilde{E}_j = \tilde{D}_{j,0} \), thus

\[
\left( \sum_{j=0}^{N} D_{i,j}^{-1} E_j \right) \bigg|_{\eta^* = 0} = \begin{cases} 0, & i \neq 0, \\ 1, & i = 0, \end{cases}
\]

and

\[
\frac{\partial \tilde{K}_{i,0}}{\partial \eta} = K_{i,l}, \quad \frac{\partial \tilde{D}_{i,0}}{\partial \eta} = D_{i,l}, \quad 0 \leq i \leq N, 0 \leq l \leq n.
\]

are applied. On the other hand, notice that

\[
A_{i,l} \big|_{\eta^* = 0} = \frac{\partial E_i}{\partial \eta} \bigg|_{\eta^* = 0} = \frac{\partial \tilde{D}_{i,0}}{\partial \eta} \bigg|_{\eta^* = 0} = D_{i,l} \big|_{\eta^* = 0}.
\]

Therefore,

\[
(KD^{-1}A)_{m,l} \big|_{\eta^* = 0} = K_{m,l}, 0 \leq m \leq N, 0 \leq l \leq n,
\]

thus \((2.22)\) holds.

**Remark 1.** The result of Maxwellian iteration for many kinetic equations and their moment models is important. For instance, the one-step Maxwellian iteration of Grad’s 13-moment equation leads to Navier-Stokes-Fourier law \([37, 13]\). However, as pointed out in \([13, 16]\), a direct application of the hyperbolic regularization in \([13, 17]\) may change the result of Maxwellian iteration. According to above theorem, the moment system \((2.11)\) fixes this defect.

### 3 Applications in one-dimensional case

In this section, we first list some existing hyperbolic moment models, some of which can be regarded as special cases of our framework, i.e., they can be written as the form in \((2.11)\) by taking a proper weight function \( \omega^{[n]} \). Then we will also propose some novel hyperbolic moment models based on our new framework.

In order to put moment models into our framework, we need to prove they are equivalent to \((2.11)\). Apart from Theorem 4, in order to judge whether two moment models are the same, we introduce this lemma.

**Lemma 6.** If two one-dimensional \( N \)-th order moment system with \( N + 1 \) equations, denoted as system \( A_1 \) and \( A_2 \), satisfy these two conditions:

1. For each system, the governing equations of the moments with orders from 0 to \( (N - 1) \) can be written in conservation form. In other words, the first \( N \) equations are equivalent to

\[
\frac{\partial E_k}{\partial t} + \frac{\partial E_{k+1}}{\partial x} = S_k, \quad 0 \leq k \leq N - 1.
\]
2. The characteristic polynomial of $A_1$ is the same as the characteristic polynomial of $A_2$.

Then these two moment models are equivalent.

Proof. According to the first condition, the moment system can be written as these quasi-linear forms

$$\frac{\partial E}{\partial t} + A_1 \frac{\partial E}{\partial x} = S \quad \text{and} \quad \frac{\partial E}{\partial t} + A_2 \frac{\partial E}{\partial x} = S,$$

where $E = (E_0, E_1, \cdots, E_N)^T \in \mathbb{R}^{N+1}$, and $S = (S_0, S_1, \cdots, S_N)^T \in \mathbb{R}^{N+1}$. The coefficient matrix $A_1$ and $A_2$ can be written as

$$A_1 = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
c_{1,0} & c_{1,1} & c_{1,2} & c_{1,3} & \cdots & c_{1,N}
\end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
c_{2,0} & c_{2,1} & c_{2,2} & c_{2,3} & \cdots & c_{2,N}
\end{bmatrix}.$$

Therefore, the characteristic polynomials are

$$p_1(\lambda) = \lambda^{N+1} - \sum_{i=0}^{N} c_{1,i} \lambda^i \quad \text{and} \quad p_2(\lambda) = \lambda^{N+1} - \sum_{i=0}^{N} c_{2,i} \lambda^i.$$

Noticing that $p_1(\lambda) = p_2(\lambda)$, we have $c_{1,i} = c_{2,i}, i = 0, 1, \cdots, N$. Thus we have

$$A_1 = A_2,$$

thus $A_1$ and $A_2$ are equivalent.

\[ \square \]

### 3.1 Boltzmann equation

This section considers the one-dimensional Boltzmann equation. The Boltzmann equation depicts the movement of the gas molecules from a statistical point of view \[3, 37\]. It reads

$$\frac{\partial f}{\partial t} + \xi \frac{\partial f}{\partial x} = S(f), \tag{3.1}$$

where $f = f(t, x, \xi)$ is the distribution function, $\xi \in \mathbb{R}$ is the microscopic velocity, and the right-hand side $S$ is the collision term, depicting the interaction between gas particles.

The thermodynamic equilibrium is

$$f_{eq} = \frac{\rho}{\sqrt{2\pi\theta}} \exp\left(\frac{-|\xi - u|^2}{2\theta}\right), \tag{3.2}$$

where $\rho$, $u$ and $\theta$ are macroscopic variables, satisfying that $\rho = \langle f \rangle$, $\rho u = \langle \xi f \rangle$ and $\rho\theta = \langle |\xi - u|^2 f \rangle$.

Furthermore, the commonly used definition of the moments are

$$E_k = \langle \xi^k f \rangle, \quad k \geq 0.$$

Compared with \[2.1\], we have $v(\xi) = \xi$ and $\tau(\xi) = 1$.

#### 3.1.1 One-dimensional HME model

To derive a reduced model from \[3.1\], Grad \[18\] proposed the moment method, in which a weighted polynomial is applied to approximate the distribution function, with the weight function set as $f_{eq}$, i.e.

$$\tilde{\omega}[\eta] = \frac{\rho}{\sqrt{2\pi\theta}} \exp\left(\frac{-1}{2\theta}(\xi - u)^2\right). \tag{3.3}$$

By taking $\eta = \left(\ln\left(\frac{\rho}{\sqrt{2\pi\theta}}, \frac{u^2}{2\theta}, \frac{u}{\theta}, -\frac{1}{2\theta}\right)\right)^T$, The weight function \[3.3\] can be written as the form \[2.20\], with $h(\xi) = \exp(\xi)$.
However, in [32, 10], it was pointed out the Grad’s moment method is not globally hyperbolic, which limits the applications of the Grad’s moment method. Therefore, the hyperbolic moment equation [8, 9] is proposed as the result of a globally hyperbolic regularization of Grad’s moment system.

With respect to the weight function (3.3), the orthogonal polynomial is the generalized Hermite polynomial, formulated as

\[
\text{He}_k[\eta](\xi) = (-1)^k \exp \left( -\frac{(\xi - u)^2}{2\theta} \right) \frac{d^k}{d\xi^k} \exp \left( -\frac{(\xi - u)^2}{2\theta} \right),
\]

(3.4)

it is not difficult to check that

\[
\text{He}_k[\eta](\xi) = \theta^{-k/2} \text{He}_k \left( \frac{\xi - u}{\sqrt{\theta}} \right),
\]

where \( \text{He}_k \) is the \( k \)-th order Hermite polynomial. Therefore, the zeros of \( \text{He}_k[\eta] \) are \( u + ci\sqrt{\theta} \), where \( c_i \) are zeros of Hermite polynomial \( \text{He}_k \), \( 1 \leq i \leq k \). In other words, when we take the weight function as (3.3) in our framework, the eigenvalues of the moment system (2.11) are the zeros of \( \text{He}_N[\eta]+1 \).

On the other hand, according to the discussions in [9], the governing equation of the moments with orders from 0 to \( N - 1 \) (i.e. \( E_k, k \leq N - 1 \)) in HME system can be written in conservation form, and the eigenvalues of the HME system is the zeros of \( \text{He}_N[\eta]+1 \). According to lemma 6, HME system can be put into our framework, by taking the weight function as

\[
\omega[\eta] = h' \left( \sum_{i=0}^N \eta_i \xi^i \right) = \exp \left( \sum_{i=0}^N \eta_i \xi^i \right) = \frac{\rho}{\sqrt{2\pi\theta}} \exp \left( -\frac{1}{2\theta} (\xi - u)^2 \right).
\]

3.1.2 Maximum entropy model

The maximum entropy model [24] makes use of the following ansatz for the distribution function:

\[
\hat{f}(t, x, \xi) = \exp \left( \sum_{i=0}^N \eta_i(t, x) \xi^i \right),
\]

(3.5)

which corresponds to \( h(\xi) = \exp(\xi) \) in Theorem 4. Therefore, according to Theorem 4, take the weight function as

\[
\omega[\eta] = h' \left( \sum_{i=0}^N \eta_i \xi^i \right) = \exp \left( \sum_{i=0}^N \eta_i \xi^i \right),
\]

and we know the two moment models are the same.

3.2 Radiative transfer equation

This section considers the radiative transfer equation (RTE) in slab geometry

\[
\frac{1}{c} \frac{\partial f}{\partial t} + \xi \frac{\partial f}{\partial x} = S(f, T),
\]

(3.6)

where \( c \) is the speed of light, \( \xi \in [-1, 1] \) denotes the cosine of the angle between the photon velocity direction and the \( x \) axis, and the right-hand side \( S \) [7, 25] depicts the interaction between photons and the background medium which has material temperature \( T \).

Furthermore, the commonly used definition of the moments are

\[
E_k = \langle \xi^k f \rangle, \quad k \geq 0.
\]

Compared with (2.1), we have \( v(\xi) = \xi \) and \( \tau(\xi) = 1 \).

3.2.1 \( P_N \) model

The \( P_N \) model [24] suggests to approximate the distribution function by a polynomial, i.e.

\[
\hat{f}(t, x, \xi) = \sum_{i=0}^N \eta_i(t, x) \xi^i.
\]

(3.7)
which corresponds to Theorem 4 with \( h(\zeta) = \zeta \). Therefore, take the weight function as
\[
\omega^{[\eta]} = 1,
\]
then according to Theorem 4 we know that the moment system derived by our framework is the same as the \( P_N \) model.

### 3.2.2 \( M_N \) model

To derive the \( M_N \) model \([31, 24, 14]\) for gray RTE, one uses the following ansatz for the distribution function:
\[
\hat{f}(t, x, \xi) = \left[ \sum_{i=0}^{N} \eta_i(t, x) \xi^i \right]^{-4},
\]
which corresponds to \( h(\zeta) = \zeta^4 \) in Theorem 4.

On the other hand, for the \( M_N \) model for monochromatic radiative transfer, the ansatz to approximate the distribution function is given by
\[
\hat{f}(t, x, \xi) = \left[ \exp \left( \sum_{i=0}^{N} \eta_i(t, x) \xi^i \right) - 1 \right]^{-1},
\]
which corresponds to \( h(\zeta) = \frac{1}{\exp(\zeta) - 1} \) in Theorem 4.

Therefore, according to Theorem 4 we take the weight function for the grey RTE as
\[
\omega^{[\eta]} = \frac{1}{(\sum_{i=0}^{N} \eta_i \xi^i)}^5,
\]
and the weight function for the monochromatic RTE is
\[
\omega^{[\eta]} = \frac{\exp \left( \sum_{i=0}^{N} \eta_i \xi^i \right)}{\left( \exp \left( \sum_{i=0}^{N} \eta_i \xi^i \right) - 1 \right)^2}.
\]

Then the \( M_N \) model for the grey RTE and the monochromatic RTE can also regarded as special cases of our new framework \([2.11]\).

### 3.2.3 \( HMP_N \) model

In \([17]\), the researchers suggest to use a weighted polynomial to approximate the specific intensity \( f \), where the weight function is given by the ansatz of the \( M_1 \) model in grey medium, i.e.
\[
\hat{f} = \frac{1}{(\eta_0 + \eta_1 \xi)} \sum_{i=0}^{N} f_i \xi^i,
\]
where \( \eta_0, \eta_1 \) are determined by the moments \( E_0 \) and \( E_1 \), and \( \frac{\eta_1}{\eta_0} \in (-1, 1) \).

Furthermore, in \([16]\), it was pointed out that the \( MP_N \) model is not globally hyperbolic for \( N \geq 3 \), thus a hyperbolic regularization is needed. Direct application of the framework proposed in \([11, 15]\) results in a globally hyperbolic moment system. However, this system has other physical defects. When \( N = 1 \), this system is different from the \( M_1 \) model. Moreover, this system’s one-step Maxwellian changes the result of the \( MP_N \) model. Therefore, in \([16]\) a new hyperbolic regularization was proposed to fix these defects, and the resulting system is \( HMP_N \) model. Now we claim that the \( HMP_N \) model can also be regarded as an example of our new framework \([2.11]\).

Noticing that in \( MP_N \) model, we use \( \omega^{[\eta]} = 1/(\eta_0 + \eta_1 \xi)^4 \) as the weight function to approximate the specific intensity \( f \). Thus it is natural to use its derivative to approximate \( \frac{\partial f}{\partial x} \), i.e. the weight function is taken as
\[
\omega^{[\eta]} = \frac{1}{(\eta_0 + \eta_1 \xi)^5}.
\]
Actually, the weight function is the same as the weight function we used in the $M_N$ model in Subsection 3.2.2 when $N = 1$.

According to lemma 16, one only need to check

1. According to [16], the governing equations of the moments with orders from 0 to $N - 1$ can be written into conservative form.

2. According to [16], the characteristic speed of the $HMP_N$ model is the zeros of the $(N + 1)$-th orthogonal polynomial with respect to the weight function (3.10).

Therefore, we have the $HMP_N$ model [16] is also a special case of our framework, when the weight function is taken as (3.10). According to Theorem 4, we have that the moment system (2.11) is the same as the $M_1$ model. According to Theorem 5, the Maxwellian iteration of the $HMP_N$ model is the same as the $MP_N$ model.

3.2.4 $HMP_N$ model for monochromatic case

Since the $HMP_N$ model for grey medium leads to a globally hyperbolic models which also preserves nice physical properties [16], it is natural to propose the $HMP_N$ model by taking the weight function as the monochromatic $M_1$ model, i.e.

$$\tilde{\omega}^{[\eta]} = \frac{1}{\exp(\eta_0 + \eta_1 \xi) - 1},$$

(3.11)

which corresponds to (2.21) with $h(\zeta) = \frac{1}{\exp(\zeta) - 1}$.

Taking the weight function as

$$\omega^{[\eta]} = \frac{\exp(\eta_0 + \eta_1 \xi)}{(\exp(\eta_0 + \eta_1 \xi) - 1)^2},$$

(3.12)

one can propose a globally hyperbolic moment system with the first $(N - 1)$-order moments in conservation form. Moreover, according to Theorem 2 we have that the characteristic speeds of the moment system are not greater than the speed of light. Moreover, according to Theorem 4 when $N = 1$, the resulting system is the $M_1$ model. According to Theorem 5 the resulting system preserves the result of the Maxwellian iteration the $MP_N$ model for monochromatic case.

3.3 Boltzmann-Peierls equation

The one-dimensional Boltzmann-Peierls equation [33], which characterizes phonon transport, reads

$$\frac{\partial f}{\partial t} + v(\xi) \frac{\partial f}{\partial x} = S(f),$$

(3.13)

where $v(\xi) = \frac{dw}{d\xi}$, $w(\xi)$ is the dispersion relationship. For instance, if we only consider harmonic interaction between adjacent atoms [19],

$$w(\xi) = 2 \sqrt{\frac{K}{m}} \left| \sin \frac{\xi a}{2} \right|,$$

(3.14)

where $\xi$ is the wave number of lattice vibration, while $K$, $m$ and $a$ are constants. Usually, $\xi \in \mathcal{B} = [-\pi/a, \pi/a]$ is from the first Brillouin zone. $S(f)$ depicts the phonon scattering process.

As a high order extension of the approach in [2], the moments are defined as

$$E_k = \langle w^k f \rangle,$$

i.e. $\tau(\xi) = w(\xi)$ in this case. $E_0$ is proportional to the energy density, and $E_1$ proportional to the heat flux. We consider only the single mode relaxation time approximation [34], then equilibrium distribution function is

$$f_{eq} = \frac{1}{\exp(hw(\xi)/(k_B T)) - 1},$$

(3.15)

where $h$ is the reduced Planck constant, $k_B$ is the Boltzmann constant, and $T$ is temperature defined by energy conservation

$$\left\langle h w \frac{f_{eq} - f}{\tau} \right\rangle = 0,$$

(3.16)
where \( \tilde{\tau} \) is relaxation time.

A natural idea is to use a weighted polynomial to approximate the distribution function \( f \), with the weight function taken as the equilibrium \( f_{eq} \). Some recent works can be found in [2, 3].

Using the conventional moment method, the weight function is taken as

\[
\omega^{[\eta]} = \frac{1}{\exp(\eta_0 w(\xi)) - 1},
\]

and the ansatz is given by

\[
\hat{f} = \tilde{\omega}^{[\eta]} \sum_{i=0}^{\tilde{\eta}} f_i v^i.
\]

The resulting system is then formulated as

\[
\frac{\partial E}{\partial t} + M \frac{\partial E}{\partial x} = S, \tag{3.17}
\]

with \( M_{i,j} = \frac{\partial E_{i+1}}{\partial E_j} \), and \( E_{N+1} \) is given by

\[
E_{N+1} = \langle w(\xi) \hat{f}^{N+1} \rangle.
\]

Under our new framework, the derivative \( \frac{\partial f}{\partial x} \) can be approximated by a weighted polynomial, with

\[
\omega^{[\eta]} = [\exp(\eta_0 w(\xi)) - 1]^{-2} \exp(\eta_0 w(\xi)) w(\xi). \tag{3.18}
\]

Precisely, the ansatz is formulated as

\[
\frac{\partial f}{\partial x} \approx \tilde{g} = \omega^{[\eta]} \sum_{i=0}^{\tilde{\eta}} g_i v^i. \tag{3.19}
\]

According to Theorem 1, the moment system is globally hyperbolic. On the Maxwellian iteration, notice that

\[
\frac{\partial \tilde{\omega}^{[\eta]}}{\partial \eta_0} = \langle w(\xi) \omega^{[\eta]} v^i \rangle = \langle w(\xi) \frac{\partial \omega^{[\eta]}}{\partial \eta_0} v^i \rangle = \langle w(\xi) \omega^{[\eta]} v^i \rangle = \mathcal{E}_i,
\]

and according to proof of Theorem 5, the moment system of the new framework has the same result of Maxwellian iteration with the conventional moment system (3.17).

4 Model in multi-dimensional case

This section considers the D-dimensional kinetic equation, which has the form

\[
\frac{\partial f}{\partial t} + \mathbf{v}(\xi) \cdot \nabla_x f = S(f), \quad t \in \mathbb{R}^+, x \in \mathbb{R}^D, \mathbf{v}(\xi) \in \mathbb{R}^D, \xi \in \mathcal{G} \subset \mathbb{R}^D, \tag{4.1}
\]

where the distribution function \( f = f(t, x, \xi) \), and \( \mathbf{v}(\xi) \in \mathbb{R}^D \) is a function of \( \xi \). The \( k \)-th moment is defined by

\[
E_k(t, x) := \langle \tau(\xi) \psi_k(\mathbf{v}) f \rangle, \quad 0 \leq k \leq M - 1,
\]

where \( \psi_k \) are chosen polynomials of \( \mathbf{v} \), \( M \) is the number of moments considered in the moment system, and \( \tau(\xi) > 0 \).

In different moment models, the basis function \( \psi_k \) could be different. For example, in Grad’s 13-moment model, \( D = 3 \), \( M = 13 \), \( \tau(\xi) = 1 \), \( \mathbf{v}(\xi) = \xi \), and the sequence of \( \psi_k \) is

\[
\psi = (1, \xi_1, \xi_2, \xi_3, \xi_1^2, \xi_1 \xi_2, \xi_1 \xi_3, \xi_2^2, \xi_2 \xi_3, \xi_3^2, ||\xi||^2, ||\xi||^2 \xi_2, ||\xi||^2 \xi_3)^T. \tag{4.3}
\]

In the \( N \)-th order HME model [9], we have \( M = \binom{N+D}{D} \), \( \tau(\xi) = 1 \), \( \mathbf{v}(\xi) = \xi \), and \( \psi_{N(\alpha)} = \xi^\alpha \), where \( \xi^\alpha = \prod_{d=1}^{D} \xi_d^\alpha_d \), and \( \mathcal{N}(\cdot) \) is a map from the multi-index group \( \{ \alpha \in \mathbb{N}^D : |\alpha| \leq N \} \) to the set \( \{0,1,2, \cdots , M-1\} \).

As \( \langle \tau(\xi) f \rangle \) often corresponds to density, hereafter we assume that \( \psi_0 = 1 \).
A moment system can be written as
\[ \frac{\partial E_k}{\partial t} + \sum_{d=1}^{D} \frac{\partial (\tau(\xi) v_d \psi_k)}{\partial x_d} = S_k, \quad 0 \leq k \leq M - 1, \] (4.4)
where
\[ S_k := \langle \tau(\xi) \psi_k S \rangle, \]
It is obvious that the moment system (4.4) is not closed. Thus one has to seek a moment closure. Similar as the 1-D case, we define the moment closure \( \frac{\partial (\tau(\xi) v_d \psi_k)}{\partial x_d} \), \( d = 1, 2, \ldots, D \), by imposing an ansatz on each \( \frac{\partial f}{\partial x_d} \).

### 4.1 Model deduction

Similar to 1-D case, we approximate \( \frac{\partial f}{\partial x_d} \) by a weighted polynomial \( \hat{g}_d \), \( d = 1, 2, \ldots, D \). We take a positive weight function \( \omega^{[n]} \), where \( \eta \) depends on \( E_k \), \( 0 \leq k \leq M - 1 \). Thus for different directions \( x_d \), their weight function is the same. The ansatz is then given by
\[ \hat{g}_d = \omega^{[n]} \sum_{i=0}^{M-1} g_{d,i} \psi_i, \quad 1 \leq d \leq D. \] (4.5)

Denote \( \mathcal{E}_k = \langle \tau(\xi) \psi_{k} \omega^{[n]} \rangle \) is the \( k \)-th moment of weight function \( \omega^{[n]} \). Let matrix \( \mathbf{D} \in \mathbb{R}^{M \times M} \) and \( \mathbf{K}_d \in \mathbb{R}^{M \times M} \), satisfying that \( \mathbf{D}_{i,j} = \langle \tau(\xi) \psi_i \psi_j \omega^{[n]} \rangle \), and \( \mathbf{K}_{d,i,j} = \langle \tau(\xi) v_d \psi_i \psi_j \omega^{[n]} \rangle \), \( 0 \leq i, j \leq M - 1 \).

Similar to 1-D case, since \( \tau(\xi)\omega^{[n]} > 0 \), we have \( \mathbf{D} \) is symmetric and positive definite, and \( \mathbf{K}_d \) is symmetric for all \( d = 1, \ldots, D \). Furthermore, for a given \( 1 \leq d \leq D \), denote \( \mathbf{g}_d \in \mathbb{R}^M \) to satisfy that its \( k \)-th element is \( g_{d,k} \), \( 0 \leq k \leq M - 1 \), then according to the constraints
\[ \langle \tau(\xi) \psi_{k} \hat{g}_d \rangle = \frac{\partial E_k}{\partial x_d}, \quad 0 \leq k \leq M - 1, \] (4.6)
we have
\[ \mathbf{D}\mathbf{g}_d = \left( \frac{\partial E_0}{\partial x_d}, \frac{\partial E_1}{\partial x_d}, \ldots, \frac{\partial E_{M-1}}{\partial x_d} \right)^T, \] (4.7)
and the moment closure is given by
\[ \left( \frac{\partial (\tau(\xi) v_d \psi_0)}{\partial x_d}, \ldots, \frac{\partial (\tau(\xi) v_d \psi_{M-1})}{\partial x_d} \right)^T = \mathbf{K}_d \mathbf{g}_d = \mathbf{K}_d \mathbf{D}^{-1} \left( \frac{\partial E_0}{\partial x_d}, \frac{\partial E_1}{\partial x_d}, \ldots, \frac{\partial E_{M-1}}{\partial x_d} \right)^T. \] (4.8)
Therefore, the multi-dimensional moment system can be written as
\[ \frac{\partial \mathbf{E}}{\partial t} + \sum_{d=1}^{D} \mathbf{K}_d \mathbf{D}^{-1} \frac{\partial \mathbf{E}}{\partial x_d} = \mathbf{S}, \] (4.9)
where \( \mathbf{E} = (E_0, E_1, \ldots, E_{M-1})^T \in \mathbb{R}^M \), and \( \mathbf{S} = (S_0, S_1, \ldots, S_{M-1})^T \in \mathbb{R}^M \).

### 4.2 Hyperbolicity

**Theorem 7.** The moment system (4.9) is globally hyperbolic.

**Proof.** Notice that \( \tau(\xi)\omega^{[n]} > 0 \) implies that \( \mathbf{D} \) is symmetric and positive definite and \( \mathbf{K}_d \) is symmetric for \( d = 1, \ldots, D \), thus any linear combination \( \sum_{d=1}^{D} n_d \mathbf{K}_d \mathbf{D}^{-1} \) is real diagonalizable. Therefore, moment system (4.9) is globally hyperbolic. \( \square \)
4.3 Comparison with the conventional moment system

It is easy to obtain this theorem

**Theorem 8.** In the multi-dimensional moment system, the moment \( \langle \tau(\xi) \psi \rangle \) satisfy an equation in conservation form if \( v_d \phi \) can be linearly expressed by \( \psi, 0 \leq l \leq M - 1, \) for any \( d = 1, 2, \cdots, D. \)

**Remark 2.** When the considered moments are the moments with orders from 0 to \( N, \) i.e. \( \psi_{N(\alpha)} = v^\alpha, \) then the moments with orders from 0 to \( (N - 1), \) \( \langle \tau(\xi) \xi^\alpha f \rangle, |\alpha| \leq N - 1 \) are conservative.

**Remark 3.** In 13-moment system, i.e. \( \psi_k \) is given by (4.3), then

\[
\langle \tau(\xi) f \rangle, \quad \langle \tau(\xi) \xi f \rangle, \quad (\tau(\xi) |\xi|^2 f),
\]

are conservative, which often correspond to density \( \rho, \) momentum \( \rho u, \) and second-order moment \( \rho \theta + |\rho u|^2. \)

However, for the entire system, the conservation form is not always preserved. The following theorem discusses a special case.

**Theorem 9.** For the multi-dimensional kinetic equation (4.11), if we use the ansatz

\[
\dot{f} = \frac{\partial}{\partial t} + \sum_{d=1}^{D} M_d \frac{\partial E}{\partial x_d} = S,
\]

(4.12)

and the constraints provided by the moments satisfying \( \alpha \in I \) are

\[
E_i = \langle \tau(\xi) \psi_i \rangle = \left\langle \tau(\xi) \psi_i h \left( \sum_{k=0}^{M-1} \eta_k \psi_k \right) \right\rangle, \quad 0 \leq i \leq M - 1,
\]

(4.13)

we have that when \( d = 1, 2, \cdots, D \) and \( 0 \leq i, j \leq M - 1, \)

\[
\frac{\partial E_i}{\partial \eta_j} = \frac{\partial \langle \tau(\xi) \psi_i h \left( \sum_{k=0}^{M-1} \eta_k \psi_k \right) \rangle}{\partial \eta_j} = \left\langle \tau(\xi) \psi_i h' \left( \sum_{k=0}^{M-1} \eta_k \psi_k \right) \psi_j \right\rangle = \langle \tau(\xi) \omega^{[\eta]} \psi_i \psi_j \rangle = D_{i,j},
\]

and

\[
\frac{\partial \langle \tau(\xi) v_d \psi_i \rangle}{\partial \eta_j} = \frac{\partial \langle \tau(\xi) v_d \psi_i h \left( \sum_{k=0}^{M-1} \eta_k \psi_k \right) \rangle}{\partial \eta_j} = \left\langle \tau(\xi) v_d \psi_i h' \left( \sum_{k=0}^{M-1} \eta_k \psi_k \right) \psi_j \right\rangle = \langle \tau(\xi) \omega^{[\eta]} v_d \psi_i \psi_j \rangle = K_{d,i,j}.
\]

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Therefore, $\frac{\partial \eta_i}{\partial E_j}$ is the $(i,j)$-th element of $D^{-1}$, since $D$ is symmetric and positive definite. Furthermore, according to

$$\frac{\partial (\tau(\xi)\psi_i \psi_j \tilde{f})}{\partial E_j} = \sum_{k=0}^{M-1} \frac{\partial (\tau(\xi)\psi_i \psi_j \tilde{f})}{\partial \eta_k} \frac{\partial \eta_k}{\partial E_j} = \sum_{k=0}^{M-1} K_{d,i,k} D^{-1} = (K_d D^{-1})_{i,j}.$$  

Therefore, $M_d = K_d D^{-1}$, for any $d = 1, 2, \cdots, D$, and the system (4.12) is equivalent to (4.9).

At last, since (4.12) can be written as

$$(4.12)$$ and (4.9) are in conservation form.

Next, we consider cases where the number of parameters $\eta$ is less than the number of moments $M$.

We calculate the result of one-step Maxwellian iteration for setting $f^{(0)} = \omega^{[n]}$ instead of $f^{(0)} = f_{eq}$. Similar as 1-D case, we have the following theorem.

**Theorem 10.** For the multi-dimensional kinetic equation (4.1), if we use the weight function

$$\omega^{[n]} = h \left( \sum_{i=0}^{n} \eta_i \psi_i \right), \quad (4.15)$$

where $n \leq M - 1$, and construct a weighted polynomial $\tilde{f} = \sum_{i=0}^{n} f_i \psi_i \omega^{[n]}$ to approximate the distribution function $f$ and derive a moment system $A_1$, and denote the system (4.9) by taking the weight function

$$\omega^{[n]} = h' \left( \sum_{i=0}^{n} \eta_i \psi_i \right), \quad (4.16)$$

as $A_2$. Then the results of one-step Maxwellian iteration of $A_1$ and $A_2$ are the same.

**Proof.** Denote $\tilde{D}$ as $\tilde{D}_{i,j} = \langle \tau(\xi)\psi_i \psi_j \omega^{[n]} \rangle$, and $K_d$ as $\tilde{K}_{d,i,j} = \langle \tau(\xi)\psi_i \psi_j \omega^{[n]} \rangle$, $0 \leq i, j \leq M - 1$, then $A_1$ can be written as

$$\frac{\partial E}{\partial t} + \sum_{d=1}^{D} \frac{\partial (K_d D^{-1} E)}{\partial x_d} = S.$$  

Furthermore, we can rewrite these two systems by a variable $w = (\eta, \eta^*) \in \mathbb{R}^M$, where there is an one-to-one map between $w$ and $E$. Additionally, we assume that when $\tilde{f} = \omega^{[n]}$, $\eta^* = 0$. Without loss of generality, we assume that $A := \frac{\partial E}{\partial w}$ is invertible.

Then $A_1$ and $A_2$ can be respectively written as

$$A \frac{\partial w}{\partial t} + \sum_{d=1}^{D} \frac{\partial (K_d D^{-1} E)}{\partial x_d} \frac{\partial w}{\partial x_d} = S, \quad A \frac{\partial w}{\partial t} + \sum_{d=1}^{D} K_d D^{-1} A \frac{\partial w}{\partial x_d} = S.$$  

In order to prove that the one-step Maxwellian iteration of these two systems are equivalent, one only needs to prove that

$$\left. (K_d D^{-1} A)_{m,l} |_{\eta^* = 0} \right|_{\eta^* = 0} = 0, \quad 0 \leq m \leq M - 1, \quad 0 \leq l \leq n, \quad 1 \leq d \leq D. \quad (4.17)$$

The right hand side can be calculated as

$$\left. \frac{\partial \left( \sum_{i,j=0}^{M-1} \tilde{K}_{d,i,j} E_j \right)_{l,m} }{\partial \eta^*} \right|_{\eta^* = 0} = \left. \frac{\partial \left( \sum_{i,j=0}^{M-1} \tilde{K}_{d,i,j} E_j \right) }{\partial \eta^*} \right|_{\eta^* = 0} = \left. \frac{\partial \tilde{K}_{d,m,0} }{\partial \eta^*} \right|_{\eta^* = 0} = K_{d,m,0} |_{\eta^* = 0},$$

where $\psi_0 = 1$, thus

$$\left( \sum_{j=0}^{M-1} \tilde{D}_{i,j} E_j \right)_{\eta^* = 0} = 0, \quad i \neq 0, \quad 1, \quad i = 0.$$  

(4.18)
and
\[ \frac{\partial \tilde{K}_{d,i,0}}{\partial \eta_l} = K_{d,i,l}, \quad 0 \leq i \leq M - 1, 0 \leq l \leq n \]
are applied. On the other hand, notice that
\[ A_{i,l}|_{\eta^* = 0} = \frac{\partial E_{i}}{\partial \eta_l} \bigg|_{\eta^* = 0} = D_{i,l} \bigg|_{\eta^* = 0}, \quad 0 \leq i \leq M - 1, 0 \leq l \leq n. \]
Therefore,
\[ (K_d D^{-1} A)_{m,l}|_{\eta^* = 0} = K_{d,m,l}, \quad 0 \leq m \leq M - 1, 0 \leq l \leq n, 1 \leq d \leq D, \quad (4.19) \]
Thus (4.17) holds.

## 5 Applications in multi-dimensional case

In this section, we will list some existing hyperbolic moment models in multi-dimensional case. They can be regarded as special cases of our new framework. Furthermore, we will also propose some new hyperbolic models.

### 5.1 Boltzmann equation

In Subsection 3.1, we consider Boltzmann equation in 1-D case and verify the equality of our framework and some existing hyperbolic moment models, such as one-dimensional HME and maximum entropy model. In this section, we will prove that in multi-dimensional case, the HME and maximum entropy model can still be included in our new framework (4.9).

First, the multi-dimensional Boltzmann equation reads
\[ \frac{\partial f}{\partial t} + \xi \cdot \nabla_x f = S(f), \quad (5.1) \]
where \( f = f(t, x, \xi) \) is the distribution function, with \( x \in \mathbb{R}^D \), and \( \xi \in \mathbb{R}^D \). Compared with (4.1), we have \( v(\xi) = \xi \).

The collision term \( S(f) \) depicts the interactions between particles, whose form is given by [18, 37]
\[ S(f) = \int_{\mathbb{R}^D} \int_{S^{D-1}} (f'^* f^* - f f^*) B(|\xi - \xi^*|, \sigma) \, dn \, d\xi^*, \]
where \( f, f', f^*, f'^* \) are the shorthand notations for \( f(t, x, \xi), f(t, x, \xi'), f(t, x, \xi^*), \) and \( f(t, x, \xi'^*) \), respectively. \( (\xi, \xi^*) \) and \( (\xi', \xi'^*) \) are the velocities before and after collision, \( n \) represents the collision angle, and \( B(|\xi - \xi^*|, \sigma) \) is the collision kernel.

The local equilibrium is given as Maxwellian [26], formulated as
\[ f_{eq} = \frac{\rho}{\sqrt{2\pi u^2}} \exp \left( -\frac{\|\xi - u\|^2}{2\theta} \right), \]
where
\[ \rho = \langle f \rangle, \quad \rho u = \langle \xi f \rangle, \quad \frac{1}{2} \rho |u|^2 + \frac{D}{2} \rho \theta = \left\langle \frac{\|\xi\|^2}{2} f \right\rangle. \quad (5.2) \]

#### 5.1.1 HME model

The Grad’s moment model [18] suggests to use a weighted polynomial to approximate the distribution function, where the weight function is taken as the local equilibrium,
\[ \tilde{\omega}^{\eta} = \frac{\rho}{\sqrt{2\pi u^2}} \exp \left( -\frac{\|\xi - u\|^2}{2\theta} \right), \]
By taking \( \eta = \left( \ln \left( \frac{\rho}{\sqrt{2\pi u^2}} \right) - \frac{|u|^2}{\theta}, \frac{u}{\theta}, -\frac{1}{2\theta} \right)^T \), one can rewrite the weight function into the form (4.18).
The moments considered in $N$-th order HME model is $E_k$, $0 \leq k \leq M - 1$, where $M = \binom{N + D}{D}$, and

$$E_{N(\alpha)} = (\xi^\alpha f), \quad |\alpha| \leq N.$$  

Compared with (4.2), we have $\tau(\xi) = 1$.

**Remark 4.** In this example, we require that

$$\mathcal{N}(\{ \alpha \in \mathbb{N}^D : |\alpha| \leq N \}) = \{ 0, 1, 2, \cdots, M - 1 \}.$$  

For example, we can let $[9]$,

$$\mathcal{N}(\alpha) = \sum_{i=1}^{D} \left( \sum_{k=D-i+1}^{D} \alpha_k + i - 1 \right).$$

In the following discussion of this paper, we will use $(\cdot)_\alpha$ to represent the $N(\alpha)$ of vector $(\cdot)$, and $(\cdot)_{\alpha,\beta}$ to represent the $(N(\alpha), N(\beta))$-th element of matrix $(\cdot)$.

The ansatz for $f$ is

$$\hat{f} = \sum_{|\alpha| \leq N} f_\alpha \mathcal{H}_\alpha^{[\eta]} = \mathcal{\tilde{w}}^{[\eta]} \sum_{|\alpha| \leq N} f_\alpha \text{He}_\alpha^{[\eta]},$$

where $\mathcal{H}_\alpha^{[\eta]}(\xi) = \text{He}_\alpha^{[\eta]}(\xi) \mathcal{\tilde{w}}^{[\eta]}(\xi)$ is orthogonal basis, and $\text{He}_\alpha^{[\eta]}(\xi)$ is generalized Hermite polynomial, which is orthogonal polynomials with respect to the weight function $\mathcal{\tilde{w}}^{[\eta]}$. Actually, according to [12, 9], we have

$$\text{He}_\alpha^{[\eta]}(\xi) = \prod_{d=1}^{D} \text{He}_\alpha^{[\eta]}(\xi_d) = \prod_{d=1}^{D} \text{He}_{\alpha_d} \left( \frac{\xi_d - u_d}{\sqrt{\theta}} \right),$$

where $\text{He}_\alpha^{[\eta]}$ and $\text{He}_k$ are defined in Subsection 3.3.1.

With a little abuse of the notation, we assume $\text{He}_\alpha^{[\eta]}$ are unit orthogonal polynomials, and $\mathcal{H}_\alpha^{[\eta]}$ is unit orthogonal basis.

**Property 11.** Multi-dimensional HME model is equivalent to the moment model (4.9) by taking the weight function as

$$\omega^{[\eta]} = \frac{\rho}{\sqrt{2\pi \theta}} \exp \left( - \frac{\xi - u^2}{2\theta} \right).$$

**Proof.** According to the statements in [11], multi-dimensional HME system can be written as

$$\mathbf{\tilde{D}} \frac{\partial \mathbf{w}}{\partial t} + \sum_{d=1}^{D} \mathbf{\tilde{M}}_d \mathbf{D}^{-1} \frac{\partial \mathbf{w}}{\partial x_d} = \mathbf{\tilde{S}},$$

(5.3)

where $\mathbf{\tilde{S}} \in \mathbb{R}^M$ is a vector whose $N(\alpha)$-th element is

$$\tilde{S}_\alpha = \langle \text{He}_\alpha^{[\eta]} \mathbf{S} \rangle = \int_{\mathbb{R}^D} \text{He}_\alpha^{[\eta]} \mathbf{S} \, d\xi,$$

Moreover, $\mathbf{w} \in \mathbb{R}^M$ is a vector corresponding to the ansatz of the distribution function $\hat{f}$. Furthermore, we have [11] for $|\alpha|, |\beta| \leq N$ and $d = 1, 2, \cdots, D$,

$$\tilde{D}_{\alpha,\beta} = \int_{\mathbb{R}^D} \text{He}_\alpha^{[\eta]} \frac{\partial \hat{f}}{\partial \omega_\beta} \, d\xi,$$

(5.4)

and

$$\tilde{M}_{d,\alpha,\beta} = \int_{\mathbb{R}^D} \xi_d \text{He}_\alpha^{[\eta]} \text{He}_\beta^{[\eta]} \omega^{[\eta]} \, d\xi.$$  

(5.5)

On the other hand, in our new framework, take the weight function as

$$\omega^{[\eta]} = \frac{\rho}{\sqrt{2\pi \theta}} \exp \left( - \frac{\xi - u^2}{2\theta} \right),$$
and the basis function $\psi_{N(\alpha)} = \xi^\alpha$, $|\alpha| \leq N$, then the resulting system is (4.9). Now we prove that these two systems are the same. First, notice that $H_0^{[n]}$, $|\alpha| \leq N$ and $\psi_{N(\alpha)} = \xi^\alpha$, $|\alpha| \leq N$ are two bases of
\[
\{\xi^\alpha : |\alpha| \leq N\},
\]
there exists a matrix $T \in \mathbb{R}^{M \times M}$, which satisfies
\[
(H_0^{[n]}, H_0^{[n]}, \ldots, H_0^{[n]})^T = T(\xi_0, \xi_1, \ldots, \xi^\alpha)^T,
\]
\[\text{where } \alpha^* \text{ is the last element in } I = \{\alpha \in \mathbb{N}^D : |\alpha| \leq N\}, \text{ i.e. } N(\alpha^*) = M - 1.
\]
According to the definition of $\tilde{S}$ and $S$, we have
\[
\tilde{S} = TS.
\]
Moreover, let matrix $\tilde{D}'$ be defined as
\[
\tilde{D}'_{\alpha, \beta} = \int_{\mathbb{R}^D} \xi^\alpha \frac{\partial \hat{f}}{\partial w_\beta} \, d\xi,
\]
then we have $\tilde{D}' \frac{\partial w}{\partial s} = \frac{\partial E}{\partial s}$, $s = t, x_1, x_2, \ldots, x_D$, and $\tilde{D} = TD'$.
Moreover,
\[
\tilde{M}_d = TK_d T^T,
\]
and
\[
D = T^{-1}T^{-T},
\]
where $D$ and $K_d$ are as defined in (4.9). Therefore, we have
\[
\tilde{M}_d = TK_d^{-1} T^{-1},
\]
and (5.3) can be rewritten as
\[
T \frac{\partial E}{\partial t} + \sum_{d=1}^D TK_d D^{-1} \frac{\partial E}{\partial x_d} = TS,
\]
which is equivalent to (4.9). Therefore, multi-dimensional HME can be included by our framework.

5.1.2 HME model for 13-moment system

Grad’s 13-moment [18] is the most famous model in the Grad-type moment hierarchy, and the HME model for 13-moment system is its hyperbolic regularized version [15]. Different from the Grad’s moment method mentioned in the previous section, the moments considered here do not correspond to $\langle \tau(\xi) \xi^\alpha f \rangle$, with $|\alpha| \leq N$. In this section, we will introduce the HME model for 13-moment system and put it into our new framework (4.9).

In this case, the dimension $D = 3$, and the moments are defined as
\[
E = (\langle f \rangle, \langle \xi_1 f \rangle, \langle \xi_2 f \rangle, \langle \xi_3 f \rangle, \langle \xi_1^2 f \rangle, \langle \xi_1 \xi_2 f \rangle, \langle \xi_1 \xi_3 f \rangle, \langle \xi_2^2 f \rangle, \langle \xi_2 \xi_3 f \rangle, \langle \xi_3^2 f \rangle, \langle ||\xi||^2 \xi_1 f \rangle, \langle ||\xi||^2 \xi_2 f \rangle, \langle ||\xi||^2 \xi_3 f \rangle)^T,
\]
\[\text{which implies that } \tau(\xi) = 1, \text{ and}
\]
\[
\psi = (1, \xi_1, \xi_2, \xi_3, \xi_1^2, \xi_1 \xi_2, \xi_1 \xi_3, \xi_2^2, \xi_2 \xi_3, \xi_3^2, ||\xi||^2 \xi_1, ||\xi||^2 \xi_2, ||\xi||^2 \xi_3)^T.
\]
$\frac{\partial f}{\partial x_d}$ is approximated by the ansatz $\hat{g}_d$, written as
\[
\hat{g}_d = \omega^{[\alpha]} \sum_{i=0}^{M-1} g_i \psi_i^{[\alpha]}.
\]
then the moment system can be written as
\[
\frac{\partial E}{\partial t} + \sum_{d=1}^D K_d D^{-1} \frac{\partial E}{\partial x_d} = S,
\]
(5.8)
where the $(i,j)$-th element of $D$ and $K_d$ are
\[
D_{i,j} = \langle \psi_i^{[\alpha]} \psi_j^{[\alpha]} \rangle \omega^{[\alpha]}, \quad K_{d,i,j} = \langle \xi_d \psi_i^{[\alpha]} \psi_j^{[\alpha]} \rangle \omega^{[\alpha]}.
\]
Using the same technique as in the proof of Property [14] it is not difficult to see that (5.8) is equivalent to the HME13 system proposed in [15].
5.1.3 Maximum entropy model

The moments considered in the full moment maximum entropy model [24] for multi-dimensional case are described by $\psi_{N(\alpha)} = \xi^\alpha$, $|\alpha| \leq N$.

The maximum entropy model uses the following ansatz for the distribution function:

$$\hat{f}(t, x, \xi) = \exp \left( \sum_{k=0}^{M-1} \eta_k(t, x)\psi_k \right),$$  \hspace{1cm} (5.9)

which corresponds to $h(\zeta) = \exp(\zeta)$ in Theorem 9. Therefore, according to Theorem 9, take the weight function as

$$\omega^{[\eta]} = h' \left( \sum_{k=0}^{M-1} \eta_k \psi_k \right) = \exp \left( \sum_{k=0}^{M-1} \eta_k \psi_k \right),$$

and we know the two moment models are the same.

5.2 Three-dimensional radiative transfer equation

Three-dimensional radiative transfer equation is written as

$$\frac{1}{c} \frac{\partial I}{\partial t} + \xi \cdot \nabla_x I = S,$$

where the distribution function (specific intensity) $I = I(t, x, \xi)$, and $t \in \mathbb{R}^+, x \in \mathbb{R}^3$, $\xi \in S^2$. The right hand side $S$ depicts the interactions with other photons and the background medium. In RTE, we have $\tau(\xi) = 1$ and $v(\xi) = \xi$.

In radiative transfer equation, notice that $\|\xi\| = 1$, we have that

$$3 \sum_{d=1}^{3} \langle \xi^\alpha \xi^2_d f \rangle = \langle \xi^\alpha f \rangle, \quad \alpha \in \mathbb{N}^3.$$

Thus the moments $\langle \xi^\alpha f \rangle$, $|\alpha| \leq N$ can be expressed by $\langle \xi^\alpha f \rangle, \alpha \in I$, with

$$I = \{ \alpha \in \mathbb{N}^3 : |\alpha| \leq N, \alpha_3 \leq 1 \},$$

and $M = \#I = (N + 1)^2$. A map from $I$ to $\{0, 1, 2, \cdots, M - 1\}$ can be defined as

$$N(\alpha) = (\alpha_1 + \alpha_2 + \alpha_3)^2 + (\alpha_1 + \alpha_2 + \alpha_3 + 1)\alpha_3 + \alpha_2,$$

and the basis function is $\psi_{N(\alpha)} = \xi^\alpha$, $\alpha \in I$.

5.2.1 $M_N$ model

To derive the $M_N$ model for gray RTE, one uses the following ansatz for the distribution function:

$$\hat{f}(t, x, \xi) = \left( \sum_{k=0}^{M-1} \eta_k(t, x)\psi_k \right)^{-4},$$  \hspace{1cm} (5.11)

which corresponds to $h(\zeta) = \frac{1}{\zeta^2}$ in Theorem 9.

On the other hand, for the $M_N$ model for monochromatic radiative transfer, the ansatz to approximate the distribution function is given by

$$\hat{f}(t, x, \xi) = \left[ \exp \left( \sum_{k=0}^{M-1} \eta_k(t, x)\psi_k \right) - 1 \right]^{-1},$$  \hspace{1cm} (5.12)

which corresponds to $h(\zeta) = \frac{1}{\exp(\zeta) - 1}$ in Theorem 9.
Therefore, according to Theorem 9, we take the weight function for the grey RTE as
\[
\omega[\eta] = \frac{1}{\left(\sum_{k=0}^{M-1} \eta_k \omega_k\right)^5},
\]
and the weight function for the monochromatic RTE is
\[
\omega[\eta] = \frac{\exp\left(\sum_{k=0}^{M-1} \eta_k \psi_k\right)}{\left(\exp\left(\sum_{k=0}^{M-1} \eta_k \psi_k\right) - 1\right)^2}.
\]
Then the \(M_N\) model for the grey RTE and the monochromatic RTE can also regarded as examples of our new framework.

5.2.2 \(HMP_N\) model

In [25], the \(HMP_N\) model was extended to 3D case, and a globally hyperbolic moment model for radiative transfer equation was carried out.

By taking
\[
\omega[\eta] = \frac{1}{(\eta_0 + \eta_1 \xi_1 + \eta_2 \xi_2 + \eta_3 \xi_3)^5},
\]
using the same technique in the proof of Property 11, we know (4.9) is equivalent to 3D \(HMP_N\) model in [25]. Furthermore, according to Theorem 8, the first \((N - 1)\)-order moments can be written as conservation law, and according to Theorem 10, we know the one-step Maxwellian iteration of the 3D \(HMP_N\) model preserves the result of the one-step Maxwellian iteration of the 3D \(MP_N\) model, which will be changed by a direct application of the hyperbolic regularization in [15, 11].

5.3 13-moment model for quantum gas

The quantum Boltzmann equation, as well as the well-known Uehling-Uhlenbeck equation [40], is written as
\[
\frac{\partial f}{\partial t} + \vec{\xi} \cdot \nabla_x f = S(f),
\]
where the collision term is defined as
\[
S(f) = \int_{\mathbb{R}^3} \int_0^{2\pi} \int_0^\pi [(1 - \theta f)(1 - \theta f_*)f f_*' - (1 - \theta f')(1 - \theta f_*)_ff_*'] g \sigma \sin \chi \, d\chi \, d\varepsilon \, d\xi.,
\]
Here \(f, f', f_*, f_*'\) are the shorthand notations for \(f(t, \vec{x}, \vec{\xi}), f(t, \vec{x}, \vec{\xi}'), f(t, \vec{x}, \vec{\xi}_*), f(t, \vec{x}, \vec{\xi}_*)\), respectively. \((\vec{\xi}, \vec{\xi}_*)\) and \((\vec{\xi}', \vec{\xi}'_*)\) are the velocities before and after collision. \(\varepsilon\) is the scattering angle, \(\chi\) is the deflection angle, \(g = |\vec{\xi} - \vec{\xi}_*|\), and \(\sigma\) is the differential cross section. \(\theta = 1, 0, -1\) corresponds to Fermion, classical gas and Boson, respectively. Since classical gas has been studied in the Boltzmann equation in Subsection 5.1 in this example, we focus on Fermion and Boson. Compared with (4.1), we have \(v(\vec{\xi}) = \vec{\xi}\).

The macroscopic variables can be defined as
\[
\rho = \frac{m}{\hbar^3} \int_{\mathbb{R}^3} f \, d\vec{\xi}, \quad \rho \vec{u} = \frac{m}{\hbar^3} \int_{\mathbb{R}^3} \vec{\xi} f \, d\vec{\xi}, \quad p = \frac{m}{3\hbar^3} \int_{\mathbb{R}^3} |\vec{\xi} - \vec{u}|^2 f \, d\vec{\xi},
\]
where \(m\) is the mass of the particle, \(\hbar = h/m\), and \(h\) is Planck’s constant.

The thermodynamic equilibrium is
\[
f_{eq} = \frac{1}{\hat{s}^{-1} \exp\left(\frac{\vec{\xi} - \vec{u}^2}{2RT}\right) + \theta}, \quad (5.13)
\]
where \(\hat{s}\) and \(RT\) is related to \(\rho\) and \(p\) as
\[
\rho = \frac{m}{\hbar^3} \sqrt{2\pi RT^3} \hat{s}^3, \quad p = \frac{m}{\hbar^3} \sqrt{2\pi RT^3} R T \hat{s}^2, \quad (5.14)
\]
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and \( L_i := -\theta L_i ( - \theta ) \) is the polylogarithm. For the special case \( \theta = 0 \), let \( L_i = \delta \).

The considered moments in the quantum Grad’s 13-moment system \( [11] \) are

\[
E = ( \langle f \rangle, \langle x f \rangle, \langle x^2 f \rangle, \langle x f \rangle, \langle f \rangle, \langle x f \rangle, \langle f \rangle, \langle x f \rangle, \langle x f \rangle, \langle f \rangle, \langle x f \rangle, \langle f \rangle )^T,
\]

which implies that \( \tau ( \xi ) = 1 \), and

\[
\psi = ( 1, \xi_1, \xi_2, \xi_3, \xi_1^2, \xi_1, \xi_2, \xi_2^2, \xi_2, \xi_3, \xi_3^2, \parallel \xi \parallel^2 \xi_1, \parallel \xi \parallel^2 \xi_2, \parallel \xi \parallel^2 \xi_3 )^T.
\]

On the other hand, in \( [11] \), the authors proposed an ansatz for the distribution

\[
f_{G13} = f_{eq} \sum_{i=0}^{12} f_i \psi_i,
\]

where \( f_i \) are coefficients. It was pointed out in \( [13] \) that quantum Grad’s 13-moment system is not hyperbolic and a direct application of the classical framework in \( [11] [15] \) will change the NSF law of the quantum Grad’s 13-moment system. Therefore, researchers in \( [13] \) proposed a new regularized 13-moment system, by splitting the expansion into the equilibrium and non-equilibrium part and then applying the framework in \( [11] [13] \). However, this method cannot be extended to higher order moment model, in which case the conservation of most moments cannot be preserved. In this paper, based on the method in Section \( [1] \) we propose a new hyperbolic regularization of the quantum Grad’s 13-moment system.

The weight function is taken as

\[
\omega^{[\eta]} = \frac{\theta^{-1} \exp \left( \frac{\parallel \xi \parallel^2}{2\theta T} \right)}{\left( \theta^{-1} \exp \left( \frac{\parallel \xi \parallel^2}{2\theta T} + \theta \right) \right)^2},
\]

a globally hyperbolic moment system is obtained. According to Theorem \( [8] \) we have the conservation of \( \rho, \rho u \), and \( p \). For higher order moment models, we can get that higher order moments satisfy equations in conservation form, which can not be ensured by the moment system proposed in \( [13] \). On the other hand, according to Theorem \( [10] \) the NSF law of the new moment system is the same as the quantum Grad’s 13-moment system, since \( f_{eq} \) and \( \omega^{[\eta]} \) satisfy the conditions in Theorem \( [10] \).

6 Conclusion

We proposed an improved framework for globally hyperbolic model reduction of kinetic equations by a new closure approach. Different from previous works, most of which derive moment closure by specifying an ansatz for the distribution function, our method directly gives the closing relationship of the flux gradient by approximating the spatial derivative of the distribution function with weighted polynomials. Besides being globally hyperbolic, moment models derived in this way satisfy nice physical properties, such as except for the highest order moments, all the other moments satisfy equations in conservation form. Also, the results of the Maxwellian iteration after the first step remain unchanged, ensuring the models satisfy important properties like the Navier-Stokes-Fourier law. As shown in previous sections, most of the globally hyperbolic moment models developed in the previous literature could be regarded as special cases of this new framework, and we also derived new moment models, such as for the monochromatic radiative transfer equation, phonon Boltzmann equation and quantum gases, which are either the first moment models for such equations with global hyperbolicity, or improvements on previously developed ones. This framework provides us with a new tool to derive in a routine way globally hyperbolic moment models which preserve nice physical properties. Numerical schemes specially designed for this type of moment model will be further investigated.

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References

[1] G. W. Alldredge, R. Li, and W. Li. Approximating the $M_2$ method by the extended quadrature method of moments for radiative transfer in slab geometry. *Kinetic & Related Models*, 9(2), 2016.

[2] Z. Banach and W. Larecki. Nine-moment phonon hydrodynamics based on the modified grad-type approach: formulation. *Journal of Physics A: Mathematical and General*, 37(41):9805, 2004.

[3] Z. Banach and W. Larecki. Nine-moment phonon hydrodynamics based on the modified grad-type approach: hyperbolicity of the one-dimensional flow. *Journal of Physics A: Mathematical and General*, 37(45):11053, 2004.

[4] G. Bird. *Molecular Gas Dynamics and the Direct Simulation of Gas Flows*. Oxford: Clarendon Press, 1994.

[5] L. Boltzmann. Weitere studien über das wärmegleichgewicht unter gas-molekülen. *Wiener Berichte*, 66:275–370, 1872.

[6] J. E. Broadwell. Study of rarefied shear flow by the discrete velocity method. *Journal of Fluid Mechanics*, 19(03):401–414, 1964.

[7] T. A. Brunner. Forms of approximate radiation transport. *Tech. Rep SAND2002-1778*, 2002.

[8] Z. Cai, Y. Fan, and R. Li. Globally hyperbolic regularization of Grad's moment system in one dimensional space. *Comm. Math. Sci.*, 11(2):547–571, 2013.

[9] Z. Cai, Y. Fan, and R. Li. Globally hyperbolic regularization of Grad’s moment system. *Comm. Pure Appl. Math.*, 67(3):464–518, 2014.

[10] Z. Cai, Y. Fan, and R. Li. On hyperbolicity of 13-moment system. *Kinet. Relat. Mod.*, 7(3):415–432, 2014.

[11] Z. Cai, Y. Fan, and R. Li. A framework on moment model reduction for kinetic equation. *SIAM J. Appl. Math.*, 75(5):2001–2023, 2015.

[12] Z. Cai and R. Li. Numerical regularized moment method of arbitrary order for Boltzmann-BGK equation. *SIAM J. Sci. Comput.*, 32(5):2875–2907, 2010.

[13] Y. Di, Y. Fan, and R. Li. 13-moment system with global hyperbolicity for quantum gas. *Journal of Statistical Physics*, 167(5):1280–1302, 2017.

[14] B. Dubroca and J. Feugeas. Theoretical and numerical study on a moment closure hierarchy for the radiative transfer equation. *Comptes Rendus de l’Academie des Sciences Series I Mathematics*, 329(10):915–920, 1999.

[15] Y. Fan, J. Koellermeier, J. Li, R. Li, and M. Torrilhon. Model reduction of kinetic equations by operator projection. *Journal of Statistical Physics*, 162(2):457–486, 2016.

[16] Y. Fan, R. Li, and L. Zheng. A nonlinear hyperbolic model for radiative transfer equation in slab geometry. *SIAM Journal on Applied Mathematics*, 80(6):2388–2419, 2020.

[17] Y. Fan, R. Li, and L. Zheng. A nonlinear moment model for radiative transfer equation in slab geometry. *Journal of Computational Physics*, 404:109128, 2020.

[18] H. Grad. On the kinetic theory of rarefied gases. *Comm. Pure Appl. Math.*, 2(4):331–407, 1949.

[19] Y. Guo and M. Wang. Phonon hydrodynamics and its applications in nanoscale heat transport. *Physics Reports*, 595:1–44, 2015.

[20] C. K. Hayakawa, J. Spanier, and V. Venugopalan. Coupled forward-adjoint Monte Carlo simulations of radiative transport for the study of optical probe design in heterogeneous tissues. *SIAM Journal on Applied Mathematics*, 68(1):253–270, 2007.

[21] J. H. Jeans. Stars, gaseous, radiative transfer of energy. *Monthly Notices of the Royal Astronomical Society*, 78:28–36, 1917.
[22] J. Koellermeier, R. Schaerer, and M. Torrilhon. A framework for hyperbolic approximation of kinetic equations using quadrature-based projection methods. *Kinet. Relat. Mod.*, 7(3):531–549, 2014.

[23] E. W. Larsen and J. E. Morel. Advances in discrete-ordinates methodology. In *Nuclear Computational Science*, pages 1–84. Springer, 2010.

[24] C. D. Levermore. Moment closure hierarchies for kinetic theories. *Journal of Statistical Physics*, 83(5-6):1021–1065, 1996.

[25] R. Li, P. Song, and L. Zheng. A nonlinear moment model for radiative transfer equation. *arXiv preprint arXiv:2005.13142*, 2020.

[26] J. C. Maxwell. On stresses in rarefied gases arising from inequalities of temperature. *Proc. R. Soc. Lond.*, 27(185–189):304–308, 1878.

[27] J. McDonald and M. Torrilhon. Affordable robust moment closures for CFD based on the maximum-entropy hierarchy. *Journal of Computational Physics*, 2013.

[28] R. G. McClarren, T. M. Evans, R. B. Lowrie, and J. D. Densmore. Semi-implicit time integration for PN thermal radiative transfer. *Journal of Computational Physics*, 227(16):7561–7586, 2008.

[29] J. McDonald, J. Sachdev, and C. Groth. Use of the gaussian moment closure for the modelling of continuum and micron-scale flows with moving boundaries. In *Computational Fluid Dynamics 2006*, pages 783–788. Springer, 2009.

[30] J. McDonald and M. Torrilhon. Affordable robust moment closures for CFD based on the maximum-entropy hierarchy. *J. Comput. Phys.*, 251:500–523, 2013.

[31] G. N. Minerbo. Maximum entropy eddington factors. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 20(6):541–545, 1978.

[32] I. Müller and T. Ruggeri. *Rational Extended Thermodynamics, Second Edition*, volume 37 of *Springer tracts in natural philosophy*. Springer-Verlag, New York, 1998.

[33] R. Peierls. On the kinetic theory of thermal conduction in crystals. In *Selected Scientific Papers Of Sir Rudolf Peierls: (With Commentary)*, pages 15–48. World Scientific, 1997.

[34] J.-P. M. Péraud and N. G. Hadjiconstantinou. Efficient simulation of multidimensional phonon transport using energy-based variance-reduced monte carlo formulations. *Physical Review B*, 84(20):205331, 2011.

[35] G. Pomraning. *The equations of radiation hydrodynamics*. Pergamon Press, 1973.

[36] H. Struchtrup. Stable transport equations for rarefied gases at high orders in the Knudsen number. *Phys. Fluids*, 16(11):3921–3934, 2004.

[37] H. Struchtrup. *Macroscopic Transport Equations for Rarefied Gas Flows: Approximation Methods in Kinetic Theory*. Springer, 2005.

[38] H. Struchtrup and M. Torrilhon. Regularization of Grad’s 13 moment equations: Derivation and linear analysis. *Phys. Fluids*, 15(9):2668–2680, 2003.

[39] M. Torrilhon and H. Struchtrup. Regularized 13-moment equations: shock structure calculations and comparison to Burnett models. *J. Fluid Mech.*, 513:171–198, 2004.

[40] E. A. Uehling and G. Uhlenbeck. Transport phenomena in Einstein-Bose and Fermi-Dirac gases. i. *Physical Review*, 43(7):552, 1933.

[41] R. Yano. Semi-classical expansion of distribution function using modified Hermite polynomials for quantum gas. *Physica A: Statistical Mechanics and its Applications*, 416:231–241, 2014.