‘Gauging’ time reversal symmetry in tensor network states

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It is well known that unitary symmetries can be ‘gauged’, i.e. defined to act in a local way, which leads to a corresponding gauge field. Gauging, for example, the charge conservation symmetry leads to electromagnetic gauge fields. It is an open question whether an analogous process is possible for time reversal which is an anti-unitary symmetry. Here we discuss a route to gauging time reversal symmetry that applies to gapped quantum ground states. We show how time reversal can be applied locally and also describe time reversal symmetry twists which act as gauge fluxes through nontrivial loops in the system. The procedure is based on the tensor network representation of quantum states which provides a notion of locality for the wave function coefficient. As with unitary symmetries, gauging time reversal provides useful access to the physical properties of the system. We show how topological invariants of certain symmetry protected topological phases in $D = 1, 2$ are readily extracted using these ideas and also discuss how they help capture a subtle distinction between time reversal symmetric $Z_2$ gauge theories.

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\section{I. Introduction}

For condensed matter systems with global symmetry, coupling to the corresponding gauge field provides a useful access to the physical properties of the system. For example, in systems with charge conservation ($U(1)$) symmetry, coupling to the electromagnetic field\cite{Chen:2010} and measuring the induced charge or current is a direct probe of the low energy excitations of the system. In gapped systems without low energy excitations, coupling to gauge field and introducing gauge fluxes to the system creates finite energy excitations which can reveal important information about the topological order of the system. For example, in fractional quantum Hall systems with $U(1)$ symmetry, inserting a magnetic flux $\Phi$ results in the accumulation of charge $\sigma_{xy}$ around the flux and hence is a direct measure of the quantized Hall conductance. When the continuous $U(1)$ symmetry is broken down to a discrete symmetry, discrete fluxes can be introduced. For example, in superconductors where the $U(1)$ symmetry breaks down to $Z_2$, fluxes in multiples of $hc/2e$ can penetrate the system. One of the most important properties of the chiral $p + ip$ superconductor is that each $hc/2e$ flux contains a Majorana zero mode.\cite{Fu:2008} Similarly, systems with nonabelian symmetries can be coupled to nonabelian gauge fields.\cite{Chen:2010} More generally, coupling symmetry protected topological (SPT) phases or symmetry enriched topological (SET) phases results in nontrivial responses (like nontrivial statistics, symmetry fractionalization, etc.) and provides an important tool in distinguishing these phases.\cite{Chen:2010, Xie:2012}

Can time reversal symmetry be similarly gauged? The anti-unitary nature of time reversal symmetry has made such a notion hard to define and we can see how a straight-forward generalization from unitary symmetries fails. In coupling systems with global unitary symmetries to gauge fields, we first find the action of the symmetry on the local Hilbert spaces in the system. For $U(1)$ symmetry it would be phase factor on local charges and for $SU(2)$ symmetry it would be local rotation of spins. Then extra degrees of freedom – the gauge field – are introduced into the system which transform under the local symmetry and couple to the original degrees of freedom in such a way that the total system is now invariant under arbitrary local actions of the symmetry. In trying to implement the same procedure for time reversal symmetry, we fail at the first step – we do not know how to define a local action of time reversal! Global time reversal symmetry involves not only unitary transformations on local Hilbert spaces (like inverting spins) but also a complex conjugation operation on the coefficient of each basis state.

\begin{align}
T \sum_{i_1,...,i_N} C_{i_1,...,i_N} |t_1,...,i_N\rangle \\
= \sum_{i_1,...,i_N} C_{i_1,...,i_N}^* U_1 \otimes ... \otimes U_N |t_1,...,i_N\rangle
\end{align}

Therefore, to couple a system to time reversal gauge field, first we need to define how complex conjugation acts on $C_{i_1,...,i_N}$ locally. (The local action of the unitary part is straightforwardly defined.) Should we consider $C_{i_1,...,i_N}$ to be localized on a particular lattice site or distributed over the entire system? Different ways to divide the coefficient would result in different local actions of time reversal. If we designate $C_{i_1,...,i_N}$ to be localized on site 1, then local action of time reversal on site 1 takes complex conjugation of $C_{i_1,...,i_N}$ while local actions on other sites do not. If we think of $C_{i_1,...,i_N}$ as composed of various parts on different lattice sites, then time reversal on a subsys-
tem takes complex conjugation of the corresponding part and not the others.

For a product state such as

\[ |\psi\rangle = \prod_k (a_k^0|0\rangle + a_k^1|1\rangle) = \sum_{i_1,\ldots,i_N} a_{i_1}^{i_N}|i_1,\ldots,i_N\rangle \]

it is natural to divide \( C_{i_1,\ldots,i_N} \) into \( N \) parts \( a^{i_k}, k = 1,\ldots,N \) and associate them with each site \( k \). Acting time reversal locally on site \( k \) then involves taking complex conjugation on \( a^{i_k} \). With this definition of local time reversal symmetry action, we can see that if the total state \( |\psi\rangle \) is invariant under global time reversal action, then it is also invariant under local time reversal action (up to a phase factor). This is similar to the action of unitary symmetries on a product symmetric state.

How to divide the wave function coefficient \( C_{i_1,\ldots,i_N} \) in a many-body entangled state? The tensor network representation of many-body entangled states provides a very natural way to do so. The tensor network representation describes a many-body wave function in terms of a set of local tensors \( T^{i_k} \).

\[ |\psi\rangle = \sum_{i_1,\ldots,i_N} tTr \left( T^{i_1}\cdots T^{i_N} \right) |i_1,\ldots,i_N\rangle \]

where \( tTr \) denotes tensor contraction. We can then think of the tensors \( T^{i_k} \) as local pieces of the total coefficient \( C_{i_1,\ldots,i_N} \) and define local action of complex conjugation on site \( k \) as taking complex conjugation of \( T^{i_k} \). Combined with the local action of the unitary part of time reversal, we obtain a local way of implementing time reversal symmetry. In this paper, we focus on gapped short range correlated quantum states which can be well described using the tensor network formalism. The local action of time reversal symmetry action on anyons has been discussed in Ref.[12].

Is this a valid and useful definition? First we notice that, such a local action of time reversal leads to similar changes on a gapped symmetric quantum state as that induced by the local action of unitary symmetries. Imagine applying a unitary symmetry to a subregion in a gapped symmetric quantum state. Both deep inside and outside the region, the state should remain invariant. The only change in the state happens at the border of the region of symmetry action, as shown in Fig.1. To see how this can be true in our definition of local time reversal symmetry action on tensor network states, we note that global time reversal symmetry involves complex conjugation on all the tensors \( T^{i_k} \) and unitary operations \( \prod_k U_k \) on all the physical degrees of freedom. The tensors at site \( k \) (\( T^{i_k} \)) may change into \( T^{i_k} \) under complex conjugation and \( U_k \). However, if the state is invariant under global time reversal symmetry action, then the changes in \( T^{i_k} \) should cancel with that coming from neighboring sites. Because of this, if we apply time reversal symmetry locally (complex conjugation and \( U_k \) on sites in a subregion), then tensors both inside and outside the subregion remain effectively invariant while tensors along the border can change. Therefore, intuitively, this definition of local time reversal symmetry action changes the quantum state in a way we would expect.

More concretely, we are going to show in the following sections that using such a definition of local time reversal symmetry action, we can extract topological invariants from the gapped symmetric quantum states and hence identify the symmetry related topological order.

In section II we discuss the 1D case in terms of matrix product states and present a way to insert time reversal fluxes through a 1D ring. In 1D there are two different time reversal symmetry protected topological (SPT) phases[13–15]. We demonstrate how these two phases can be distinguished from each other using the projective composition rule of time reversal twists induced by the time reversal fluxes. That is, two time reversal twists compose into identity up to a universal phase factor characterizing the underlying SPT order of the state. In 1D, the distinction between different time reversal SPT phases has been well understood in the matrix product formalism[13–15]. Our discussion is just a reinterpretation of that procedure in terms of local time reversal symmetry action and time reversal twists.

In section III we move onto the 2D case, where we define local action of time reversal in tensor product states and discuss how to insert time reversal fluxes through nontrivial loops in the systems. To facilitate discussion, we first review the procedure for the unitary \( Z_2 \) symmetry and demonstrate how topological invariants of the 2D \( Z_2 \) SPT phases can be extracted from the projective composition rules of \( Z_2 \) symmetry twist lines. Then we study a 2D state with trivial time reversal SPT order, a 2D SPT phase with \( Z_2 \times Z_2^T \) symmetry and a 2D \( Z_2 \) gauge theory with time reversal symmetry and see how the topological invariants of these phases can be extracted similarly. In particular, we find that for the trivial time reversal SPT state the projective composition rule for the time reversal fluxes are all trivial, as expected. For the \( Z_2 \times Z_2^T \) SPT state and the \( Z_2 \) gauge theory with time reversal symmetry, a nontrivial \((-1)\) phase factor can appear which is related to the \( T^2 = -1 \) transformation law of the \( Z_2 \) fluxes in the bulk.

In section IV we summarize what we have learned and discuss open problems in gauging time reversal symmetry.

II. IN 1D MATRIX PRODUCT STATES

The matrix product state (MPS) representation of 1D gapped quantum states provides a natural way to divide the wave function coefficient into local pieces. The matrix prod-
uct state representation reads
\[ |\psi\rangle = \sum_{i_1,i_2,...,i_N} Tr \left( A^{i_1} A^{i_2} ... A^{i_N} \right) |i_1,i_2,...,i_N\rangle \]  
\[ (4) \]
with \( D \times D \) matrices \( A^i \). We call the \( i \)'s the physical indices and the left and right indices of \( A \)'s the inner indices. Similar terminology is used for tensor product states discussed later.

The local action of time reversal symmetry on matrix product states has been discussed extensively in the study of 1D symmetry protected topological phases. Here we review the procedure and discuss the notion of time reversal flux and time reversal twists based on such a definition.

Suppose that the global time reversal symmetry action is \( U \otimes U K \), where \( K \) denotes complex conjugation in the \( |i\rangle \) basis. The global action of time reversal reads
\[ T|\psi\rangle = \sum_{i_1,i_2,...,i_N} Tr \left( A^{i_1} A^{i_2} ... A^{i_N} \right)^* U \otimes U ... U |i_1,i_2,...,i_N\rangle \]
\[ (5) \]
Then acting time reversal locally on a single site in the matrix product state changes the matrices to
\[ \tilde{A}^i = \sum_j U_{ij}^T (A^j)^* \]
\[ (6) \]
Note that if this is applied to all sites, then it is equivalent to applying time reversal globally. If the state \( |\psi\rangle \) is short range correlated and time reversal symmetric, then the MPS representation satisfies
\[ \tilde{A}^i = \sum_j U_{ij}^T (A^j)^* = M A^i M^{-1} \]
\[ (7) \]
with an invertible matrix \( M \).

If we apply time reversal to a segment of site \( m \) to \( m + n \), then the matrices on all sites remain invariant except for those on site \( m \) and \( m + n \). On site \( m \), the matrices change to
\[ \tilde{A}^i = M A^i \]
\[ (8) \]
On site \( m + n \), the matrices change into
\[ \tilde{A}^i = A^i M^{-1} \]
\[ (9) \]

For example, consider the dimer state with two spin 1/2's per site and the spin 1/2's on neighboring sites pair into singlets \( |01\rangle - |10\rangle \). The MPS representation of the state contains matrices
\[ A^{00} = -|1\rangle \langle 0|, A^{01} = -|1\rangle \langle 1|, A^{10} = |0\rangle \langle 0|, A^{11} = |0\rangle \langle 1| \]
\[ (10) \]
If time reversal symmetry acts as
\[ T = i \sigma_y \otimes ... \otimes i \sigma_y K \]
\[ (11) \]
Then acting time reversal on each site changes the matrices as
\[ \tilde{A}^i = (i Y) A^i (-i Y) \]
\[ (12) \]
Here we use \( X, Y \) and \( Z \) to denote Pauli matrices on the inner indices.

It is known that the time reversal SPT order can be extracted from \( M \) by
\[ M^* M = \beta = \pm 1 \]
\[ (13) \]
where \( \beta = 1 \) in the trivial SPT phase and \( \beta = -1 \) in the nontrivial one (including the dimer state). Now we are going to reinterpret this in terms of time reversal fluxes and time reversal twists.

With the definition of local time reversal symmetry action, we can discuss how to insert time reversal flux through a 1D ring, in analogy to unitary symmetries. Let's first recall how the procedure works for unitary symmetries.

Consider, for example, a system with \( U(1) \) symmetry \( e^{i \theta_1} \otimes e^{i \theta_2} ... \otimes e^{i \theta_N} \). WLOG, consider a Hamiltonian with two-body interactions \( H = \sum_k h_{k,k+1} \). Inserting a \( \phi \) flux through the one dimensional ring corresponds to changing the Hamiltonian term on the boundary as
\[ h_{N,1} \rightarrow e^{i \phi} h_{N,1} e^{-i \phi} \]
\[ (14) \]
For all the SPT and SET phases we are considering in this paper, the ground states satisfy that all the local reduced density matrices are determined by local Hamiltonian terms and not affected by terms far away (this is the TQO-2 condition used in the definition of topological order in Ref \[ 13 \]. Therefore the two ground states \( |\psi\rangle \) and \( |\psi\rangle_\phi \) without and with flux should have the following relation:

1. Away from the boundary, \( |\psi\rangle \) and \( |\psi\rangle_\phi \) should look the same.

2. Near the boundary, \( |\psi\rangle_\phi \) should look the same as \( e^{i \phi} \otimes ... \otimes e^{i \phi} |\psi\rangle \) or \( e^{-i \phi} \otimes ... \otimes e^{-i \phi} |\psi\rangle \) with \( 1 << m << N \).

Here by ‘look the same’ we mean that the two states have the same reduced density matrix locally.

For time reversal symmetry, we do not know how to couple the system to fluxes on the Hamiltonian level. However, we can couple the symmetric gapped ground state to time reversal fluxes in a way similar to unitary symmetries. Denote the state not coupled / coupled to a time reversal flux as \( |\psi\rangle \) and \( |\psi\rangle_{\tau} \). We expect that

1. Away from the boundary, \( |\psi\rangle \) and \( |\psi\rangle_{\tau} \) should look the same.

2. Near the boundary, \( |\psi\rangle_{\tau} \) should look the same as acting time reversal locally from site \( m \) to \( N \) or from site 1 to \( m \) on \( |\psi\rangle \), with \( 1 << m << N \).

As we discussed above, in short range correlated matrix product states, local time reversal symmetry action changes the representing matrices only near the two ends of the local region. Therefore, if we change the matrices only at one point, for example at the boundary of the 1D ring, this would correspond to inserting a time reversal flux through the 1D ring.

In particular, to insert a time reversal flux, we can change the matrices at site \( N \) to
\[ A^i \rightarrow A^i M^{-1} \]
\[ (15) \]
Or we can change the matrices at site 1 to

\[ A^i \rightarrow MA^i \quad (16) \]

The resulting state indeed has the property discussed above when compared to the original state without flux and contains a time reversal symmetry twist on the boundary.

The usefulness of this definition becomes evident when we compose two time reversal symmetry twists and extract universal properties of the SPT order from the procedure. Suppose that we insert two time reversal fluxes by changing the matrices on site 1 twice. This should be equivalent to a state without time reversal flux. However, as we will see, two time reversal twists may differ from zero twist by an important phase factor. On inserting the first flux, the matrices on site 1 are changed to \( A^i \rightarrow MA^i \). On inserting the second flux, the \( A_i \) part undergoes the change \( A^i \rightarrow MA^i \) again. Moreover, because we are considering time reversal fluxes, we need to take complex conjugation of the first \( M \). Therefore, the total change to \( A^i \) on site 1 is

\[ A^i \rightarrow M^*MA^i = \beta A^i \quad (17) \]

with \( \beta = M^*M = \pm1 \). Therefore, the composition of two time reversal twists is equivalent to zero twist up to a phase factor of \( \beta \). From this projective composition rule of time reversal twists, we recover the topological invariant \( \beta \) characterizing the SPT order of the state. The result remains the same if we insert flux by changing the matrices on site \( N \) as \( A^i \rightarrow A^iM^{-1} \).

This corresponds exactly to the procedure of extracting SPT order from the MPS representation of a gapped symmetric state. Here we are merely reinterpreting the procedure as finding the projective composition rule of time reversal twists induced by time reversal fluxes. It follows from previous discussions that the topological invariant extracted (\( \beta \)) is independent of the gauge choice we make for the matrix product representation. In particular, we can change the gauge choice of the MPS representation by an invertible matrix \( N \). Then inserting a single flux corresponds to changing the matrices on site 1 by \( N^*MN^{-1} \) and

\[ (N^*MN^{-1})^* (N^*MN^{-1}) = M^*M = \beta \quad (18) \]

We want to note that this procedure applies to any matrix product state, not only the fixed point ones with zero correlation length as shown in the dimer state example above.

III. IN 2D TENSOR PRODUCT STATES

Now, we are ready to generalize the procedure to 2D. First we will review in section III.A how everything works for unitary symmetries. In particular, we are going to review how the \( Z_2 \) symmetry acts locally on the tensor product representation of a gapped symmetric state and how topological invariants of the SPT order can be extracted from the projective composition rules of the \( Z_2 \) symmetry twist lines. After this preparation, we will move on to define how time reversal symmetry acts locally on tensor product states and discuss how to insert time reversal fluxes through nontrivial loops in the system in section III.B. In particular, we are going to discuss in detail three examples in sections III.B, III.C, III.D and demonstrate how topological invariants of the phases can be extracted from the projective composition rules of time reversal twist lines.

A. Review: local action and twists of unitary symmetry in 2D

Consider the 2D states with trivial and nontrivial \( Z_2 \) symmetry protected topological order.\(^{14,15,16,17}\) We use the form of wave function similar to that in Ref.\(^4\) The system lives on a honeycomb lattice, where each lattice site contains three spin 1/2’s (in state \( |0 \rangle \) or \( |1 \rangle \), as shown in Fig.\(^2\). In the ground state, the six spin 1/2’s around a plaquette are either all in the \( |0 \rangle \) state or all in the \( |1 \rangle \) state, forming \( Z_2 \) domains. A state with trivial \( Z_2 \) SPT order can be obtained as an equal weight superposition of all \( Z_2 \) domain configurations

\[ |\psi_0\rangle = \sum_C |C\rangle \quad (19) \]

where \( C \) denotes \( Z_2 \) domain configurations. A state with nontrivial \( Z_2 \) SPT order takes the form

\[ |\psi_1\rangle = \sum_C (-)^{NC} |C\rangle \quad (20) \]

where \( N_C \) counts the number of domain walls in the configuration. Obviously, these two wave functions are symmetric under the \( Z_2 \) symmetry action of flipping \( Z_2 \) domains \( |0 \rangle \leftrightarrow |1 \rangle \). Moreover, it has been shown that the \( Z_2 \) fluxes, once gauged, have bosonic statistics in the trivial phase and semion statistics in the nontrivial phase.\(^8\) We will see how this is reflected in the projective composition rule of the \( Z_2 \) symmetry twist lines as discussed below.

![FIG. 2. Wave function of 2D states on the honeycomb lattice with \( Z_2 \) SPT order: each lattice site contains three spin 1/2’s; the six spin 1/2’s around each plaquette are either all in the \( |0 \rangle \) state or all in the \( |1 \rangle \) state. The \( Z_2 \) symmetry flips between the \( |0 \rangle \) and \( |1 \rangle \) state.](image)
network representation for $|\psi_0\rangle$ can be obtained from that in Fig.3 simply by removing the phase factors $i$ and $-i$ (all terms have coefficient 1.)

> It is interesting to see how the $Z_2$ symmetry acts locally on the tensors. Obviously the tensors for $|\psi_0\rangle$ and $|\psi_1\rangle$ are not invariant under local action of $Z_2$ symmetry $|0\rangle \leftrightarrow |1\rangle$, but the transformed tensors differ from the original ones by unitary transformations on the inner indices. This relation is shown in Fig.4, where for $|\psi_0\rangle$, $\alpha = \bar{\alpha} = 1$ and for $|\psi_1\rangle$

$$\alpha = |00\rangle\langle 00| + i|01\rangle\langle 01| + i|10\rangle\langle 10| + |11\rangle\langle 11|$$

$$\bar{\alpha} = |00\rangle\langle 00| - i|01\rangle\langle 01| - i|10\rangle\langle 10| + |11\rangle\langle 11|$$

(21)

$\sigma_x$ denotes the spin flip of physical degrees of freedom while $X$ denotes the same operator but on the inner indices. In the following discussion, we call $X \otimes X\alpha$ and $X \otimes X\bar{\alpha}$ the ‘inner symmetry operators’. Using the relation given in Fig.4 it is straightforward to see that the state is invariant under global $Z_2$ symmetry action and changes only along the border if $Z_2$ symmetry is applied to a subregion. This is because changes to the inner indices cancel if the index lies within the subregion.

> Now we can couple the state to $Z_2$ symmetry fluxes. Consider a $Z_2$ SPT state on a torus. Inserting a flux through a nontrivial loop of the torus results in a symmetry twist line on the torus along the other nontrivial loop. Assume WLOG that the Hamiltonian of the system contains only two-body interactions. Creating symmetry twist lines corresponds to taking all terms in the Hamiltonian $h_{mn}$ that are divided by the twist line and changing them to

$$h_{mn} \rightarrow \sigma^m_x h_{mn} \sigma^m_x$$

(22)

Denote the ground state without / with the twist line as $|\psi\rangle$ and $|\psi\rangle_{Z_2}$. Because for the systems under consideration here, local reduced density matrices of the ground states are all determined by local Hamiltonian terms, we expect that

1. Away from the twist line, $|\psi\rangle$ and $|\psi\rangle_{Z_2}$ should look the same.

2. Near the twist line, $|\psi\rangle_{Z_2}$ should look like $\prod_{r \in R} \sigma^r_x |\psi\rangle$ where $R$ is a large region with the twist line as part of the border.

In this way, we can discuss symmetry fluxes in terms of the ground state, instead of the Hamiltonian.

The tensor product representation of the ground state provides a particularly simple way to find $|\psi\rangle_{Z_2}$. The tensor product representation of $|\psi\rangle_{Z_2}$ can be obtained from that of $|\psi\rangle$ by inserting the inner symmetry operators along the twist line. For simplicity of discussion, we combine every two sites on the A, B sub-lattices and map the system to square lattice structure. Each tensor now has four inner indices. The inner symmetry operators are $X \otimes X\alpha$ for the up and left indices (inherited from sub-lattice A) and $X \otimes X\bar{\alpha}$ for the down and right indices (inherited from sub-lattice B).

In a square lattice with periodic boundary condition as shown in Fig.5 threading a $Z_2$ flux in the $x$ direction (through the nontrivial loop in the $y$ direction) corresponds to inserting inner symmetry operators, $X \otimes X\alpha$ or $X \otimes X\bar{\alpha}$, along the nontrivial loop in the $x$ direction. Threading a $Z_2$ fluxes in the $x$ direction (through the nontrivial loop in the $y$ direction) corresponds to inserting inner symmetry operators, $X \otimes X\alpha$ or $X \otimes X\bar{\alpha}$, along the twist line of nontrivial loop in the $x$ direction. Composing two twist lines in the same direction should be equivalent to having no flux in this direction. However, as we will see, this equivalence is true only up to a phase factor, which is a topological invariant characterizing the underlying SPT order. A similar procedure of inserting symmetry / gauge twist lines and applying modular transformations to extract topological invariants from the states is discussed in Refs.23 and 24.

Consider first a state with $Z_2$ twist lines only in the $x$ direction. Composing two twist lines, we find that the inner symmetry operator on each inner index compose into

$$\left( X \otimes X\alpha \right) \left( X \otimes X\alpha \right) = I \quad \text{for trivial } \alpha$$

$$= Z \otimes Z \quad \text{for nontrivial } \alpha$$

(23)

With trivial $\alpha$, it is obvious that two $Z_2$ twist lines compose into zero. With nontrivial $\alpha$, which is illustrated in Fig.5, the inner symmetry operators on each inner index do not compose into $I$. However, acting $Z \otimes Z$ on all inner indices along a loop does not change the state at all. This is a special inner symmetry of the tensor product representation given in Fig.4. Therefore, for both the trivial and nontrivial $Z_2$ SPT states, we see that composing two twist lines in the $x$ direction is equivalent to having no twist line, with no extra phase factor.

Nontrivial phase factors can arise when we compose two twist lines in the $x$ direction in the presence of a twist line in the $y$ direction. For the trivial SPT state, the phase factor is...
of two \(Z\) is, the projective phase factor coming from the composition from the twist line composition process discussed above. That still 1 and does not change due to the \(y\) direction twist line. However, for the nontrivial SPT state, we can see from Fig.6 that a \(-1\) phase factor arises. The inner symmetry operators along the \(x\) direction compose into \(Z \otimes Z\) on each inner index. While the \(Z \otimes Z\) operators in the middle of the loop keeps the tensor product state invariant, the one on the boundary results in a \(-1\) phase factor due to the presence of the \(X \otimes X\) operator in the \(y\) direction. Therefore, for the nontrivial SPT state, the composition of twist lines in the \(x\) direction is projective (with a \(-1\) phase factor) in the presence of a twist line in the \(y\) direction.

![Figure 5](image)  
**FIG. 5.** In the nontrivial \(Z_2\) SPT state, composing two \(Z_2\) twist lines in the \(x\) direction is equivalent to having no twist lines in the state. The tensor product representation of the state follows from that given in Fig.3.

We can interpret this \(\pm 1\) phase factor as related to the \(Z_2^2\) symmetry transformation on each \(Z_2\) flux locally. For the \(Z_2\) SPT state, we know that if we create \(Z_2\) fluxes in the bulk of the system, they have bosonic or fermionic exchange statistics in the trivial and nontrivial SPT state. Equivalently, we can say that each \(Z_2\) flux carries 0 or 1/2 \(Z_2\) charge in these two phases. That is, each \(Z_2\) flux gets a 1 or \(i\) phase factor under the local action of \(Z_2\) symmetry. If we apply the \(Z_2\) symmetry locally twice on a \(Z_2\) flux, we expect to get a phase factor of 1 and \(-1\) respectively, which is exactly what we obtained from the twist line composition process discussed above. That is, the projective phase factor coming from the composition of two \(Z_2\) twist lines in the \(x\) direction in the presence of a \(Z_2\) twist line in the \(y\) direction can be interpreted as the \(Z_2^2\) transformation on each \(Z_2\) flux locally.

To understand this connection, we can imagine cutting the system open along the nontrivial loop in the \(x\) direction and turn the torus into a cylinder. Due to the existence of the \(Z_2\) twist line in the \(y\) direction, \(Z_2\) fluxes are present at the top and bottom end of the cylinder. Now inserting two \(Z_2\) twist lines in the \(x\) direction at the bottom end of the cylinder as show in Fig.6 is effectively measuring the \(Z_2^2\) quantum number of one \(Z_2\) flux. Note that if we actually apply \(Z_2\) symmetry \(\prod \sigma_x\) twice in a region near the bottom end, we will not be able to see the nontrivial phase factor because \((\prod \sigma_x)^2\) is always equal to 1. Similar situations happen with time reversal symmetry in the examples discussed in later sections.

![Figure 6](image)  
**FIG. 6.** In the nontrivial \(Z_2\) SPT state, composing two \(Z_2\) twist lines in the \(x\) direction in the presence of a \(Z_2\) twist line in the \(y\) direction is equivalent to a state with only a \(y\) direction twist line up to a \(-1\) phase factor. The tensor product representation of the state follows from that given in Fig.3.

We can interpret this \(\pm 1\) phase factor as related to the \(Z_2^2\) symmetry transformation on each \(Z_2\) flux locally. For the \(Z_2\) SPT state, we know that if we create \(Z_2\) fluxes in the bulk of the system, they have bosonic or fermionic exchange statistics in the trivial and nontrivial SPT state. Equivalently, we can say that each \(Z_2\) flux carries 0 or 1/2 \(Z_2\) charge in these two phases. That is, each \(Z_2\) flux gets a 1 or \(i\) phase factor under the local action of \(Z_2\) symmetry. If we apply the \(Z_2\) symmetry locally twice on a \(Z_2\) flux, we expect to get a phase factor of 1 and \(-1\) respectively, which is exactly what we obtained from the twist line composition process discussed above. That is, the projective phase factor coming from the composition of two \(Z_2\) twist lines in the \(x\) direction in the presence of a \(Z_2\) twist line in the \(y\) direction can be interpreted as the \(Z_2^2\) transformation on each \(Z_2\) flux locally.

To understand this connection, we can imagine cutting the system open along the nontrivial loop in the \(x\) direction and turn the torus into a cylinder. Due to the existence of the \(Z_2\) twist line in the \(y\) direction, \(Z_2\) fluxes are present at the top and bottom end of the cylinder. Now inserting two \(Z_2\) twist lines in the \(x\) direction at the bottom end of the cylinder as show in Fig.6 is effectively measuring the \(Z_2^2\) quantum number of one \(Z_2\) flux. Note that if we actually apply \(Z_2\) symmetry \(\prod \sigma_x\) twice in a region near the bottom end, we will not be able to see the nontrivial phase factor because \((\prod \sigma_x)^2\) is always equal to 1. Similar situations happen with time reversal symmetry in the examples discussed in later sections.

Similar to the 1D case, this result applies not only to the fixed point tensors shown in this section, but also to those away from the fixed point. The topological properties of the inner symmetry operators remain the same when the tensors are perturbed away from the fixed point form.

Finally, we want to comment that there are two differences between the 1D time reversal SPT example and the 2D \(Z_2\) SPT example: 1. in composing time reversal twists we need to take complex conjugation on one of the twist while composing unitary \(Z_2\) twists does not involve such a step; 2. to see the nontrivial phase factor, for the 1D SPT state we are simply composing two twists while for the 2D case we are composing twists in the presence of another twist in the orthogonal direction and thus investigating the relation between them. In our following study of 2D phases with time reversal symmetry, these two features need to be combined.

### B. Local action and twists of time reversal in 2D

Now we are ready to generalize this procedure to time reversal symmetry in 2D. In 2D, there is only a trivial time reversal symmetric SPT phase and we use it as an example to illustrate the basic ideas. In section III C and III D, we discuss the more interesting cases of nontrivial SPT phases with \(Z_2 \times Z_2^T\) symmetry and \(Z_2\) gauge theories with time reversal symmetry.

Consider a 2D square lattice with four spin 1/2’s per site and the four spin 1/2’s at the corners of each plaquette form an entangled state \(|0000⟩ + |1111⟩\). The total wave function is hence

\[|ψ⟩ = \prod_\square (|0000⟩ + |1111⟩)\]  

where the product is over all plaquettes \(\square\). Time reversal symmetry acts by first taking complex conjugation, and then applying \(σ_x\) to each spin. The state is obviously short range entangled and time reversal invariant. We can think of the four spins around a plaquette as a time reversal domain. The wave function is hence an equal weight superposition of all domain configurations. Of course, there are simpler states with the same SPT order, but here we use this form of short range entangled wave function because this is the standard form of SPT wave function and all SPT states can be written in this way. (The \(Z_2\) SPT state in Eq.19 and 20 can also be put into this form with a local basis transformation.)

To define local time reversal symmetry action, we need the tensor product representation of the state, which gives us a way to divide the coefficient of the wave function into local
The tensors at each site can be chosen as shown in Fig. 7.

\[
\sum_{i,j,k,m=0,1} i j k m
\]

FIG. 7. Tensor representing 2D trivial SPT state with time reversal symmetry. The labels in circles are physical indices and the labels at the end of the links are inner indices.

With the tensor product representation, we can now define local action of time reversal symmetry as: 1. taking complex conjugation of the tensors. 2. acting \( \sigma_x K \) on the physical basis. The induced ‘inner symmetry operators’ on the tensors are \( X \otimes X \) on each inner index, as shown in Fig. 8. From this relation, we can see that the state is invariant under global time reversal symmetry action, because the inner symmetry operators cancel with each other if time reversal is applied globally. Moreover, acting time reversal locally in a subregion changes only the tensors along the border.

FIG. 8. Acting time reversal on one site induces ‘inner symmetry operators’ \( X \otimes X \) on the inner indices.

Knowing how time reversal acts locally on the state, we can insert twist lines and couple the state to time reversal fluxes through the nontrivial loops in the system. Similar to the \( Z_2 \) case discussed previously, we expect that

1. Away from the twist line, the states with and without time reversal flux should look the same.

2. Near the twist line, the state with time reversal flux should look like the state without flux but with time reversal symmetry acting locally in region \( R \), where \( R \) is a large region with the twist line as part of the border.

For the time reversal symmetric state discussed above, this can be realized by inserting inner symmetry operators \( X \otimes X \) into the inner indices along a nontrivial loop (in the \( x \) or \( y \) direction). We expect the projective composition rule of time reversal twist lines to reflect the universal topological properties of the state. Note that because we are considering time reversal twists here, when composing two inner symmetry operators, we should take complex conjugation of the first one. It is easy to see that two copies of the inner symmetry operators \( X \otimes X \) naturally compose into identity, therefore time reversal twist lines in this state do not have nontrivial projective composition, as we would expect for a state with trivial SPT order.

We want to make a comment about how the gauge choice for the tensor product representation affects the result. For unitary symmetries discussed in the previous section, changing the gauge of the tensors does not affect the projective composition rule at all. Suppose that we change the gauge of the tensors by an invertible matrix \( N \). The inner symmetry operators all change by conjugation with \( N \) and \( N^{-1} \) and their composition and commutation relations remain the same.

For the time reversal symmetry discussed in this section, changing the gauge of the tensor leaves the result almost invariant, except for one subtlety: there are certain gauge choices of the tensors such that local time reversal symmetry action results in a null state. Suppose that we change the gauge of the upper index of the tensor in Fig. 7 by \( e^{iX} \otimes I \). Then applying time reversal symmetry locally on one site leads to inner symmetry operators \( iI \otimes X \) on the upper index and \( X \otimes X \) on the other indices. When we try to connect the tensors on different sites and find the total wave function after local time reversal symmetry action, we find that the resulting state has zero amplitude. Therefore, we need to exclude these possibilities and require that local time reversal symmetry action always results in a nonzero state.

As long as this nonzero condition is satisfied, the gauge change of the tensors does not affect the composition of time reversal twist lines. For the time reversal symmetric trivial SPT state discussed above, changing the gauge by an invertible matrix \( N \) changes the inner symmetry operators to \( N^*(X \otimes X)N^{-1} \). Two such operators composed together is still the identity

\[
(N^*(X \otimes X)N^{-1})^*N^*(X \otimes X)N^{-1} = I
\]

Therefore, our discussion is independent of the gauge choice of the tensor product representation (as along as local time reversal symmetry action results in a nonzero state).

C. Example: 2D SPT with \( Z_2 \times Z_2^2 \) symmetry

Now let’s study a more interesting example: a 2D SPT state with \( Z_2 \times Z_2^2 \) (time reversal) symmetry. The 2D SPT phases with \( Z_2 \times Z_2^2 \) symmetry have a \( Z_2 \times Z_2 \) classification.\(^{(12)}\)\(^{(13)}\)\(^{(14)}\)\(^{(15)}\)\(^{(16)}\)\(^{(17)}\)\(^{(18)}\)\(^{(19)}\)\(^{(20)}\)\(^{(21)}\)

The trivial and nontrivial SPT order with pure \( Z_2 \) symmetry accounts for the first \( Z_2 \) in the classification. The second \( Z_2 \) in the classification corresponds to the \( Z_2 \) symmetry fluxes in the bulk of the state transforming as \( T^2 = \pm 1 \) under time reversal.\(^{(22)}\)\(^{(23)}\)\(^{(24)}\)\(^{(25)}\) (This kind of ‘local Kramer degeneracy’ has also been studied in Refs.\(^{(12)}\)\(^{(13)}\)\(^{(14)}\)\(^{(15)}\)\(^{(16)}\)\(^{(17)}\)\(^{(18)}\)\(^{(19)}\)\(^{(20)}\)\(^{(21)}\)) To distinguish states with trivial and nontrivial SPT order under the \( Z_2 \) part of the symmetry, we can insert \( Z_2 \) twist lines along the nontrivial loops and study their projective composition rules as discussed in section \(^{(22)}\)\(^{(23)}\)\(^{(24)}\)\(^{(25)}\). In this section, we are going to show how to determine whether \( T^2 = 1 \) or \(-1 \) on each \( Z_2 \) symmetry flux from the projective composition rule of the time reversal twist lines in the presence of a unitary \( Z_2 \) twist line.
Consider a $Z_2 \times Z_2^2$ SPT state on the square lattice with two sets of spin 1/2 degrees of freedom $\sigma$ (solid circles in Fig. 9) which we call the $Z_2$ spins and $\tau$ (dashed circles in Fig. 9) which we call the time reversal spins. The wave function is a product of two parts: the $Z_2$ part $|\psi_{Z_2}\rangle$ and the time reversal part $|\psi_T\rangle$

$$|\psi\rangle = |\psi_{Z_2}\rangle \otimes |\psi_T\rangle \quad (26)$$

The $Z_2$ part formed by the $\sigma$ spins takes the same form as discussed in section III A which can have either trivial or non-trivial $Z_2$ SPT order. $Z_2$ symmetry acts as $\sigma_x$ on each $Z_2$ spin. The local $Z_2$ symmetry action, the $Z_2$ symmetry twist line and their projective composition rules follow directly from section III A for both the trivial and nontrivial $Z_2$ SPT order. The time reversal part $|\psi_T\rangle$ of the wave function is formed by the $\tau$ spins and is an equal weight superposition of time reversal domain configurations, as discussed in section III B. The tensor product representation of the time reversal part is hence the same as that given in Fig. 7. The tensor product representation of the total wave function is a product of the $Z_2$ part and the time reversal part.

![FIG. 9. Local symmetry action on the tensor representing the SPT state with $Z_2 \times Z_2^2$ symmetry. The solid ($Z_2$) part of the tensor follows from that given in section III A and the dashed (time reversal) part of the tensor is the same as that in Fig. 7. Local $Z_2$ and time reversal symmetry induce changes to the tensors as shown on the left and right hand side of this figure.](image)

To understand how the nontrivial $\eta$ is related to the $\mathcal{T}^2 = -1$ transformation of the $Z_2$ fluxes, we can think of the time reversal $\tau$ spin involved in each triangle as living between the $Z_2$ domains formed by the two $\sigma$ spins as shown in Fig. 9. From the definition of $\eta$ we see that, if $\tau$ is on a $Z_2$ domain wall, then time reversal acts on it as $i\tau_y K$ which squares to $-1$. If $\tau$ is not on a $Z_2$ domain wall, then time reversal acts on it as $\tau_x K$ which squares to 1. Along the domain wall, the $\mathcal{T}^2 = -1 \tau$ spins form time reversal singlets and the total wave function is a superposition of all $Z_2$ configurations with domain walls decorated by the time reversal singlets. When $Z_2$ fluxes are inserted in the bulk of the system, the $Z_2$ domain wall ends, leaving un-paired $\mathcal{T}^2 = -1 \tau$ spins at each of the flux points. Therefore, with nontrivial $\eta$, each $Z_2$ flux in the bulk transforms as $\mathcal{T}^2 = -1$ under time reversal symmetry.

This feature can be identified from the projective composition rule of the time reversal twist lines in the state. First, let us find the local time reversal symmetry action on the representing tensors of the state. The local time reversal symmetry action on the tensors is given by taking complex conjugation on the tensor and applying $\tau_x \eta$ and complex conjugation on the physical basis states. Fig. 8 illustrates the inner symmetry operators induced by the local time reversal symmetry action on the tensor, which reads

$$\left( X^\tau \otimes X^\tau \right) \cdot \eta^\tau \sigma \cdot \alpha^\sigma \sigma (\text{or} \bar{\alpha}^\sigma \sigma) \quad (28)$$

The $\alpha^\sigma \sigma (\text{or} \bar{\alpha}^\sigma \sigma)$ part comes from taking complex conjugation of the $Z_2$ part of the tensor. On the left hand side of the figure, we also illustrate the inner symmetry operator induced by local $Z_2$ symmetry action, which is

$$\left( X^\sigma \otimes X^\sigma \right) \cdot \alpha^\sigma \sigma \quad (29)$$

as discussed in section III A.

Suppose that we compose two time reversal twist lines along the $x$ direction. The composition of two inner symmetry operators gives

$$[\left( X^\tau \otimes X^\tau \right) \cdot \eta^\tau \sigma \sigma \cdot \alpha^\sigma \sigma] \left( X^\tau \otimes X^\tau \right) \cdot \eta^\tau \sigma \sigma \cdot \alpha^\sigma \sigma]^* = I \quad \text{for trivial} \ \eta$$

$$= Z^\sigma \otimes Z^\sigma \quad \text{for nontrivial} \ \eta$$

(30)

The result is similar to the composition of two $Z_2$ twist lines discussed in Eq. 23. Therefore, if the system has only two time reversal twist lines in the $x$ direction, then their composition is equivalent to having no twist line. However, if we are composing two time reversal twist lines in the $x$ direction in the presence of a $Z_2$ symmetry twist line in the $y$ direction, a $-1$ phase factor arises for nontrivial $\eta$. In section III A, we interpreted the projective phase factor in the composition of two $Z_2$ twist lines in the presence of another $Z_2$ twist line as the $Z_2^2$ value of a $Z_2$ flux. Similarly, here we interpret the projective phase factor in the composition of two time reversal twist lines in the presence of another $Z_2$ twist line as the $\mathcal{T}^2$ value of a $Z_2$ flux.

Note that the projective phase factor in the composition of two time reversal twist lines depends only on $\eta$ and not on $\alpha$. $\alpha$ is responsible for the projective phase factor in the composition of two $Z_2$ twist lines which is related to the first $Z_2$ classification while $\eta$ is related to the second.
Finally we want to comment that this result is independent of the gauge choice of the representing tensors, as long as local time reversal action leads to a nonzero state. This is similar to the cases discussed in the previous section.

D. Example: 2D $Z_2$ gauge theory with time reversal symmetry

In the previous sections, we studied symmetry protected topological phases with short range entanglement. In this section, we study an example of long range entangled state and demonstrate that the notions of local time reversal action and time reversal twist lines still work. In particular, we are going to gauge the $Z_2$ symmetry in the examples discussed in the last section and study $Z_2$ gauge theories with time reversal symmetry. These are examples of the so-called 'symmetry enriched topological' (SET) orders.

Above we have seen that there are four $(Z_2 \times Z_2)$ SPT phases in 2D with $Z_2 \times Z_2^T$ symmetry. Note, even with just $Z_2$ symmetry, there are two phases. In that case, gauging the $Z_2$ symmetry leads to two distinct topological orders, the topological order of the usual $Z_2$ gauge theory ($Z_2$ topological order like in the toric code state) and the twisted $Z_2$ gauge theory (double semion topological order). Now, in the presence of time reversal symmetry, the SPT classification implies that there is an additional two fold distinction. For the usual $Z_2$ topological order this is readily interpreted as arising when each bosonic $Z_2$ flux transforms projectively (as a Kramers doublet with $T^2 = -1$) under time reversal symmetry. A natural identification of the fourth phase is that the semions in the doubled semion model transform as Kramers pairs under time reversal symmetry. However, time reversal action on semions turns them into anti-semions, so it is unclear how to discuss the $T^2$ value on the semions and interpret the defining feature of this phase. We will see below how this distinction can be made precise by studying the time reversal symmetry twist lines in the ground states. At the end of this section, we will discuss what kind of physical measurements can be used to identify the fourth phase.

A duality transformation from domain degrees of freedom to domain wall degrees of freedom maps a $Z_2$ symmetric state to a $Z_2$ gauge theory. The tensor product representation of the double semion $Z_2$ gauge theory obtained with this mapping from the nontrivial $Z_2$ SPT state is given in Fig. 10. The $Z_2$ gauge theory obtained from the trivial $Z_2$ SPT state is the 'toric code' state and the tensor product representation is similar to the cases discussed in the previous section.

While for the $Z_2$ SPT states, the representing tensors change by the inner symmetry operators $(X \otimes X)\alpha$ and $(X \otimes X)\bar{\alpha}$ under local $Z_2$ action, the tensors for the $Z_2$ gauge theories are invariant under the inner symmetry operators, as shown in Fig. 11.

To write down a $Z_2$ gauge theory with extra time reversal structure, we introduce time reversal spins $\tau$ into the state such that the tensors representing the state is composed of the $Z_2$ part given above and the time reversal part given in Fig. 7. Global time reversal symmetry action contains three parts: 1. taking complex conjugation in the $|0\rangle$, $|1\rangle$ basis of the $Z_2$ and time reversal spins 2. acting $\tau_x$ on all the time reversal spins 3. applying phase factors $\eta'$ in the $|0\rangle$, $|1\rangle$ basis to each pair of $Z_2$ and time reversal spin connected by red arrows in Fig. 12.

$\eta'$ involves one $Z_2$ spin and one time reversal spin and has two possibilities:

$$\eta' = 1 \text{ for all states or, } \eta' = -1 \text{ on } |11\rangle, \eta' = 1 \text{ otherwise} \quad (31)$$

Note that $\eta'$ can be obtained from $\eta$ in Eq. 27 by changing the $Z_2$ domain degrees of freedom to the $Z_2$ domain wall degrees of freedom.

Local time reversal symmetry action then takes complex conjugation of the tensors and applies $\tau_x$, $\eta'$ and complex conjugation to the physical degrees of freedom. The transformation of the tensor under local time reversal action is shown in the right part of Fig. 12. The induced inner symmetry operator is the same as that in the $Z_2 \times Z_2^T$ SPT state. The left part of Fig. 12 shows an inner invariance of the tensor which is related to the $Z_2$ gauge theory.

Now we can create time reversal twist lines by inserting $(X^\tau \otimes X^\tau^*) \cdot \eta'^x \cdot \alpha^\sigma$ or $(X^\tau \otimes X^\tau^*) \cdot \eta'^x \cdot \bar{\alpha}^\sigma$ into the inner indices along a nontrivial loop. Moreover, we can create $Z_2$ gauge twist lines by inserting $(X^\sigma \otimes X^\sigma^*) \cdot \alpha^\sigma$ or $(X^\sigma \otimes X^\sigma^*) \cdot \bar{\alpha}^\sigma$ into the inner indices along a nontrivial loop. Note that this way of creating $Z_2$ gauge twist lines is equivalent to using the string operators defined in Ref. 28. Following similar calculations as in the last section, we can find the projection composition rules for the time reversal twist lines.
In particular, a nontrivial phase factor of $-1$ arises when we compose two time reversal twist lines in the $x$ direction in the presence of a $Z_2$ gauge twist line in the $y$ direction when $\eta(\gamma)$ is nontrivial.

This result looks totally similar to what we obtained for the $Z_2 \times Z_2^T$ SPT states. However, we want to discuss two differences between the SPT and the gauged version.

First, in the gauged version, because the tensor remains invariant under the action of $(X^\sigma \otimes X^\sigma) \cdot \alpha^{\sigma\sigma}$ and $(X^\sigma \otimes X^\sigma) \cdot \bar{\alpha}^{\sigma\sigma}$ on the inner indices, the inner symmetry operator induced by local time reversal symmetry action is not unique. In particular, we can redefine the inner symmetry operator for time reversal as the combination of the $(X^\sigma \otimes X^\sigma) \cdot \eta^{\sigma\sigma} \cdot \alpha^{\sigma\sigma}$ (or $(X^\sigma \otimes X^\sigma) \cdot \eta^{\sigma\sigma} \cdot \bar{\alpha}^{\sigma\sigma}$) part and the $(X^\sigma \otimes X^\sigma) \cdot \bar{\alpha}^{\sigma\sigma}$ (or $(X^\sigma \otimes X^\sigma) \cdot \alpha^{\sigma\sigma}$) part. However, direct calculation shows that, the time reversal twist line given by the modified inner symmetry operator satisfies the same projective composition rule.

Secondly, we want to discuss how to distinguish the different phases through measurable effects in the SPT case and the gauge theory case respectively. Our procedure of inserting twist lines is based on the tensor network representation of the state and it is a nature question to ask whether the topological invariants extracted from this process reflect real distinctions between phases. Indeed for the four SPT phases discussed in the last section and the four gauge theories discussed in this section, we can always find physical ways to distinguish them.

The most straight-forward way to distinguish SPT phases is through the gapless edge states between them. As was shown in Ref. [19] SPT phases associated with inequivalent group cocycles always have protected gapless degenerate edge excitations between them. In this way, we can be sure that the four $Z_2 \times Z_2^T$ SPT phases distinguished through twist line compositions are indeed different phases.

For the gauge theories, we can no longer use gapless edge states to distinguish them. Because in all the four phases we consider there is a trivial bosonic $Z_2$ charge, the edge of the phases can always be gapped by condensing this $Z_2$ charge on the boundary. Instead, we can use the statistics and the time reversal transformation rule of the $Z_2$ fluxes to distinguish the phases. To distinguish the toric code type and the double semion type of $Z_2$ gauge theory, we can measure the statistics of the $Z_2 \times Z_2$ gauge flux. To identify the $T^2$ transformation rule of the $Z_2$ fluxes, we can put the system on a cylinder, create a $Z_2$ flux at each end and measure the degeneracy. One might worry whether this procedure could work for the double semion $Z_2$ gauge theory, because the $Z_2$ flux in this phase is a semion (or an anti-semion) and breaks time reversal symmetry automatically. Therefore, it seems not well defined to talk about the $T^2$ value for each $Z_2$ flux. However, notice that a semion is mapped into an anti-semion (and vice versa) under time reversal which differs by a bosonic $Z_2$ charge. If the bosonic $Z_2$ charge is condensed on each end of the cylinder, then time reversal symmetry is still preserved when we create $Z_2$ fluxes at each end. This can be explicitly seen using the string operators given in Ref. [28]. The time reversal partner of the string operator for the semion ($W_s$) is that for the anti-semion ($W_s$) which differs by the string operator of the $Z_2$ charge ($W_c$).

\[ W_s \xrightarrow{T} W_s = W_s W_c \]  

Therefore, if the state on the cylinder is a $+1$ eigenstate of the string operator $W_c$, creating $Z_2$ charges at each end, i.e. the $Z_2$ charge is condensed at each end, then the state remains time reversal invariant after the application of $W_s$. Then we can ask about the $T^2$ transformation at each end of the cylinder and measure it through Kramer’s degeneracy. Combining this with the statistics allows us to distinguish all four gauge theories.

IV. CONCLUSION AND OPEN QUESTIONS

In summary, we have proposed a way to ‘gauge’ time reversal symmetry in the tensor network representation of many-body entangled quantum states. First we define a local action of time reversal symmetry in the tensor network states. The tensor network representation provides a natural way to divide the global wave function coefficient into local pieces. Based on this division, we can define the local action of complex conjugation, which is the key in defining local actions of anti-unitary symmetries. Then we can discuss how to introduce time reversal twists induced by time reversal fluxes through the nontrivial loops in the system. Moreover, by composing the time reversal twists, we can extract topological invariants of the phase from the projective composition rules. In particular, we demonstrate how this works for 1D time reversal SPT phases, which is a re-interpretation of the procedure used to determine the SPT order from the matrix product state representation. Moreover, we studied a 2D time reversal symmetric state with trivial SPT order, 2D SPT states with $Z_2 \times Z_2^T$ SPT order and 2D $Z_2$ gauge theories with time reversal symmetry. For the 2D time reversal symmetric state with trivial SPT order, we find that all the projective composition rules are trivial, as it should be for a trivial SPT state. For the 2D SPT states with $Z_2 \times Z_2^T$ SPT order, the projective composition rules of the $Z_2$ and time reversal symmetry twist lines allow us to distinguish all four phases in the $Z_2 \times Z_2$ classification. For the
$Z_2$ gauge theories with time reversal symmetry, we also find four different types of projective composition rules for the $Z_2$ gauge twist lines and the time reversal symmetry twist lines. This allows us to distinguish the four phases with $Z_2$ fluxes being bosonic or semionic and transforming as $T^2 = \pm 1$ respectively.

This work just represents our first attempt at gauging time reversal symmetry and many questions remain. First of all, our discussion is totally based on quantum states. Is there a way to gauge time reversal symmetry on the Hamiltonian of the system? For unitary symmetries, we know how to do this. Of course, for time reversal symmetry, we can start from the gauged ground state and construct the corresponding gauged Hamiltonian. However, if we do not know the ground state, do we have a generic way to gauge the Hamiltonian? If we know how to do this, we can gauge time reversal symmetry not only in gapped systems, but in gapless systems as well.

Secondly, we ask if there is a dynamical time reversal gauge theory. In this paper, we are only discussing non-dynamical configurations of time reversal twists. Can we promote it to a dynamical gauge theory by defining a time reversal gauge field and introducing quantum dynamics to it, as can be done for unitary symmetries like $Z_2$?

Moreover, we want to know what is the general procedure for determining time reversal protected and time reversal enriched topological orders using time reversal twists. In particular, we want to know how to identify time reversal related topological orders in 3D. In 3D, time reversal invariant SPT phases have a $Z_2 \times Z_2$ classification. The group cohomology classification gives one $Z_2^{[1]}$ and the other $Z_2$ is beyond the cohomology classification. One might expect, based our experience in 1D and 2D, that local time reversal operations in these 3D states leads to time reversal twist membranes, from which topological invariants of the SPT orders can be extracted. However, as we argue below, a time reversal twist membrane does not exist for the nontrivial phase in the beyond cohomology $Z_2$.

The nontrivial phase in the beyond cohomology $Z_2$ is special in how time reversal symmetry acts on the surface of the system. The system has half quantized surface thermal Hall effect. That is, we can break time reversal symmetry in opposite ways on the left and right hand side of the surface and find a gapless edge state with chiral central charge $c_- = 8$ between the gapped left and right half. Because time reversal symmetry on the surface maps the state on the left hand side to that on the right hand side, it maps between 2D states with different chirality. This can only be accomplished by a large quantum circuit with circuit depth scaling linearly with the size of the 2D surface. As we can see from all the other examples discussed in this paper, the inner symmetry operators induced by local symmetry actions on the tensors takes very similar form to the symmetry transformations on the edge / surface of the state. For all the other examples, the symmetry action on the edge / surface, hence the inner symmetry operators, are composed of a few layers of unitary operators. However, for the beyond cohomology SPT state, the surface symmetry transformation cannot take this form. Therefore, we expect that the transformation induced by local symmetry action on the tensors cannot be described by simple inner symmetry operators on each inner index, hence complicating the discussion of time reversal twist membranes in the state.

For the within cohomology $Z_2$, a time reversal twist membrane can indeed be found. We leave it to future work to study how to extract topological invariants from it.

Finally, one may ask the question – can gapped quantum ground states be generally represented as tensor network states? This is rigorously known to be true only in 1D. The question of representing chiral 2D states with tensor networks was recently discussed, although the tensor networks have power law decaying correlation. Here we are interested in time reversal invariant, and hence non-chiral, phases. For example, can topological insulators and superconductors with time reversal symmetry be represented as short range correlated tensor network states? If so, will the gauged time reversal analysis discussed here give us a deeper understanding of these states? We leave these interesting questions for future work.

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