Spin interactions in mesons in strong magnetic field

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Abstract
Spin interactions in relativistic quark-antiquark system in magnetic field is considered in the framework of the relativistic Hamiltonian, derived from the QCD path integral. The formalism allows to separate spin-dependent terms from the basic spin-independent interaction, contained in the Wilson loop, and producing confining and gluon exchange interaction. As a result one obtains relativistic spin-spin interaction $V_{ss}$, generalizing its nonrelativistic analog. It is shown, that in large magnetic field $eB$, $V_{ss}$ modifies and produces hyperfine shifts which grow linearly with $eB$ and preclude the use of perturbation theory. We also show, that tensor forces for $eB \neq 0$ are active in all meson states, but do not grow with $eB$.

1 Introduction
The spin-dependent terms in the interaction of two fermions in QED have a long history [1, 2, 3, 4], where spin terms were deduced based on Pauli or Dirac form of Hamiltonians. In particular the hyperfine (hf) interaction term was introduced by Fermi in [3] in the nonrelativistic form. Later on the QED perturbation theory was used to consider hf interaction in higher orders [5] see [7] for a review, and also in presence of magnetic field (m.f.) [8].

An important tool for the higher order relativistic treatment of bound states in QED was Bethe-Salpeter equation (BSE) [9], or its instantaneous Salpeter form [10].
It is the purpose of the present paper to consider spin-dependent effects in the dynamics of the quark-antiquark ($q\bar{q}$) mesons in strong m.f. The physical interest of such systems is due to high magnetic fields, which are expected in neutron stars, early universe and heavy-ion collisions, see [11] for discussion and references.

In such QCD systems, as mesons or baryons, the main part of dynamics is of nonperturbative (np) origin, e.g. the confinement interaction $V_{\text{conf}}$, which cannot be described by perturbative propagator particle exchanges. In addition the relative time problem existing in BSE is more fundamental in QCD, where the vacuum field correlations with period $\lambda$, yielding confinement (and all strong interactions, including spin-dependent ones) are shorter in time, than physical particle exchanges, so that each quark in meson propagates in the already time-averaged vacuum [12]. Therefore to establish the fine and hf structure of strong interactions, one must use instead of BSE another relativistic formalism, namely the relativistic path integrals method, based on the so-called Fock-Feynman-Schwinger Representation [13, 14]. Recently a new form of this method was derived in [15], where a new integral representation of the $q\bar{q}$ Green’s function was given.

Basically, the main part of the strong dynamics (non spin-dependent) is given by the Wilson loop average and the methods, based on FFSR, give all meson, baryon, glueball and hybrid properties in good agreement with experimental data, see [16], [17] for reviews.

Of special importance for us here are spin-dependent forces in QCD (which we shall test applying high m.f.). The latter have been derived for large quark masses in [18], and in the framework of our method for any masses in [12, 19, 20, 21, 22, 23].

The most important development in the study of the spin-dependent (SD) interactions consists of three advancements:

1. Definition of perturbative and np, SD contributions and expressions for them in terms of the standard field correlators. This was done for large masses in [12, 19].

2. Derivation of SD terms in the relativistic systems for low and zero mass quarks, done in [16, 20] and [21].

3. Definition of SD terms for nonzero temperature and in the deconfining QCD phase, which was done in [22].
4. Reexamining of the SD forces and suppression of spin-orbit forces in hadrons by the string motion, [23].

In the present paper we perform calculation of SD forces in the relativistic $q\bar{q}$ system in the presence of m.f. $eB$, which strongly modifies these forces, and in particular yields a linear growth in $eB$ for the hf term. We also show, that m.f. creates deformation of the meson shape and thus stipulates tensor forces in the originally spherical meson.

A possible contradiction of these effects with the positivity property of the Green’s function is formulated and possible outcomes are discussed.

In the same formalism we calculate the selfenergy correction to the meson mass in strong m.f.

The paper is organized as follows: in section 2 we define spectral properties of Green’s function in m.f., in section 3 we derive the $q\bar{q}$ Green’s function and relativistic Hamiltonians in m.f. from FFSR, in section 4 we concentrate on SD forces in m.f. and write down explicit expressions for them. The section 5 is devoted to tensor forces in m.f. The last section contains discussion of the results and an outlook. One appendix contains details of derivations.

2 Absence of pair creation in color Euclidean and magnetic fields

Consider quark-antiquark system in the external magnetic field, interacting with color vacuum Euclidean and perturbative fields, so that the covariant derivative is

$$D_\mu \equiv D_\mu(A^{(e)}_\mu, A^{(o)}_\mu t^a) = \partial_\mu - ieA^{(e)}_\mu(x) - igA^{(o)}_\mu t^a,$$

where $A^{(e)}_\mu$ can be decomposed into the magnetic (“Euclidean”) part $A^{(B)}_\mu$ and the electric part, e.g.

$$A^{(e)}_\mu = A^{(B)}_\mu + A^{(e)}_\mu^{\text{pert}},$$

$$\mathbf{A}^{(B)} = \frac{1}{2}(\mathbf{x} \times \mathbf{B}), \quad A^{(B)}_4 = 0$$

for the constant magnetic field $\mathbf{B}$ along $z$ axis. Then the partition function averaged over nonperturbative vacuum can be written as

$$\langle Z \rangle_A = \langle \int D\bar{\psi}D\psi \exp(-i\bar{\psi}(m + \hat{D})\psi) \rangle_A,$$
where the averaging over gluonic fields is implied,
\[
\langle K \rangle_A = \int K \exp \left( -\frac{1}{4} \int (F^a_{\mu\nu} F^a_{\mu\nu}) d^4 x \right) DA,
\] (4)
and we disregard for simplicity gauge fixing and ghost terms.

Integration over \( D\bar{\psi}D\psi \) in (3) yields the standard answer (where the proper renormalization is implied),
\[
\langle Z \rangle_A = \left\langle \exp \left( \frac{1}{2} \text{tr} \ln (m^2 - \hat{D}^2) \right) \right\rangle_A = \left\langle \exp \left( \frac{1}{2} \text{tr} \int_0^{\infty} \frac{ds}{s} e^{-s(m^2 - \hat{D}^2)} \right) \right\rangle_A. \] (5)

Now the question of stability of the vacuum in the given external fields can be associated with the nonnegativity of the operator \((m^2 - \hat{D}^2)\), since otherwise negative eigenvalues of this operator would provide imaginary part in the exponent of (5), which implies finite probability of pair creation, as it is clearly seen in the Schwinger expression for the pair creation in the constant electric (nonEuclidean) field [24].

In what follows we show that in purely Euclidean (colorelectric or colormagnetic) fields and in the magnetic field (external and perturbative) the operator \((m^2 - \hat{D}^2)\) is nonnegative and hence pair creation is absent and vacuum stability is ensured.

To this end consider the Euclidean operator
\[
iD_\mu \gamma_\mu = \gamma_\mu (i\partial_\mu + eA_\mu^{(e)} (x) + gA_\mu^a t^a), \] (6)
with Euclidean and hermitian \( \gamma \) matrices, \( \gamma_4 \equiv \beta; \gamma_i = -i\beta \alpha_i; \gamma_i^+ = \gamma_i \), and with hermitian \( A_i^{(e)} \) and \( A_4 \equiv A_4^{a} t^a, A_4^+ = A_4 \), while \( A_4^{(e)} \equiv 0 \), so that
\[
(iD_\mu \gamma_\mu)^+ = iD_\mu \gamma_\mu, \] (7)
and for the eigenfunctions \( u_n \) and eigenvalues \( \lambda_n \) we have
\[
iD_\mu \gamma_\mu u_n = \lambda_n u_n, \quad \lambda_n \text{ real}. \] (8)

Hence
\[
-\hat{D}^2 u_n = (iD_\mu \gamma_\mu)^2 u_n = \lambda_n^2 u_n, \quad \lambda_n^2 \geq 0, \] (9)
and \( \| m^2 - \hat{D}^2 \| \geq m^2 \), which implies vacuum stability and no pair creation for any \( m \geq 0 \).
In terms of the quark propagator $S = \frac{i}{D+m}$, one has representations

$$iS = \sum_n \frac{u_n(x)u_n^+(y)}{\lambda_n - im}, \quad \left( \frac{1}{m^2 - \hat{D}^2} \right)_{xy} = \sum_n \frac{u_n(x)u_n^+(y)}{m^2 + \lambda_n^2}. \quad (10)$$

Writing $u_n = \begin{pmatrix} \varphi_n \\ \chi_n \end{pmatrix}$, one obtains the following equations

$$-\hat{D}^2 + \sigma(gE - gH - eB)(\varphi_n - \chi_n) = \lambda_n^2(\varphi_n - \chi_n),$$

$$-\hat{D}^2 - \sigma(gE + gH + eB)(\varphi_n + \chi_n) = \lambda_n^2(\varphi_n + \chi_n), \quad (11)$$

where $\hat{D}^2 = \left( \partial_\mu - ieA^{(e)}_\mu \right)^2 - gA_\mu^2 = (A_\mu^\alpha t_\alpha)^2$, and $A_\mu, E, H$ correspond to the color fields e.g. $A_\mu \equiv A_\mu^a t_a$, while $A^{(e)}_\mu$ is the electromagnetic field, and $B$ is the external magnetic field. From the system (11) one can see, that for hermitian Euclidean fields $A^{(e)}_\mu, A_\mu, E, H, B$ the eigenvalues $\lambda_n^2$ are real and the Green’s function $\frac{1}{m^2 - \hat{D}^2}$ has only positive set of eigenvalues.

Note, that in real electric field $A^{(e)}_0, A^{(e)}_4 = iA^{(e)}_0$ and the property of nonnegativity of $(-\hat{D}^2)$ is violated, implying possible quark pair creation, while in absence of $A^{(e)}_0$, but with real vacuum color field $A_4^+ = A_4$ and hence real colorelectric vacuum field $E^{\text{vac}}_4 \sim \partial_i A_4$, vacuum is stable with known vacuum condensate ($(E^{\text{vac}}_i)^2 + (H^{\text{vac}}_i)^2$). However, the quasizero modes with small $\lambda_n$ can accumulate, implying Chiral Symmetry Breaking (CSB), signalled by the Banks-Casher formula [25], see [26] for review and discussion of CSB from this point of view. As we shall see, perturbative colormagnetic interactions in the lowest order, in the external magnetic field can violate positivity condition (9), signalling the divergence of the perturbative series.

3 Relativistic $q\bar{q}$ Green’s function in magnetic field in the path-integral form

The quark Green’s function in magnetic field can be written as

$$S_q(x,y) = (m + \hat{D})^{-1}_{xy} = (m - \hat{D})_{x}(m^2 - \hat{D}^2)^{-1}_{xy}, \quad (12)$$

where

$$m^2 - \hat{D}^2 = m^2 - D^2 - g\sigma_{\mu\nu}F_{\mu\nu} - e\sigma_{\mu\nu}F^{(e)}_{\mu\nu} \quad (13).$$
and $D_\mu$ is given in (10), while

$$\sigma_{\mu\nu} F_{\mu\nu} = \begin{pmatrix} \sigma H & \sigma E \\ \sigma E & \sigma H \end{pmatrix}, \quad \sigma_{\mu\nu} F_{\mu\nu}^{(e)} = \begin{pmatrix} \sigma B & 0 \\ 0 & \sigma B \end{pmatrix}$$

(14)

One can use for $S_q$ the path-integral form [13, 15]

$$S_q(x, y) = (m - \hat{D})_x \int_0^\infty ds D^4 z \Phi^{(F)}_z(x, y) e^{-K}$$

(15)

where

$$\Phi^{(F)}_z(x, y) = P A \exp \left( i e \int_y^x A^{(e)}_i dz_i + ig \int_y^x A_\mu dz_\mu + e \sigma_{\mu\nu} \int_0^s F^{(e)}_{\mu\nu} d\tau + g \sigma_{\mu\nu} \int_0^s F_{\mu\nu} d\tau \right),$$

(16)

$$K = \int_0^s \left[ m^2 + \frac{1}{4} \left( \frac{dz_\mu}{d\tau} \right)^2 \right] d\tau.$$

(17)

Then the $q\bar{q}$ Green's function can be written as

$$G_{q_1\bar{q}_2}(x, y) = \int_0^\infty ds_1 \int_0^\infty ds_2 (D^4 z^{(1)})_{xy} (D^4 z^{(2)})_{xy} \langle \hat{T} W_\sigma(A) \rangle A W_\sigma(A^{(e)}) \rangle,$$

(18)

where

$$W_\sigma(A) = P \exp (ig \int_C A_\mu(z) dz_\mu + g \int_0^{\tau_1} \sigma_{\mu\nu} F_{\mu\nu}(\tau_1) d\tau_1 - g \int_0^{\tau_2} \sigma_{\mu\nu} F_{\mu\nu}(\tau_2) d\tau_2),$$

(19)

$$W_\sigma(A^{(e)}) = \exp (ie_1 \int_y^x A^{(e)}_\mu dz^{(1)}_\mu - ie_2 \int_y^x A^{(e)}_\mu dz^{(2)}_\mu + e_1 \int_0^{\tau_1} d\tau_1 (\sigma_{\mu\nu} F_{\mu\nu}^{(e)}) - e_2 \int_0^{\tau_2} d\tau_2 (\sigma_{\mu\nu} F_{\mu\nu}^{(e)})),$$

(20)

$$\hat{T} = \frac{1}{4} tr(\Gamma_1(m_1 - \hat{D}_1)\Gamma_2(m_2 - \hat{D}_2)) \exp(-K_1 - K_2),$$

(21)

and $\Gamma_1 = \gamma_\mu$, $\Gamma_2 = \gamma_\nu$ for vector currents, while $\Gamma_1 = \Gamma_2 = \gamma_5$ for pseudoscalars, and the symbol $tr$ implies summation over color and Dirac indices and refers to all terms.

At this point we introduce new variables $\omega_i$ in the path integral [15], defined via the connection between the proper time $\tau_i$ and the real Euclidean
time $t_i^E = z_4(\tau_i)$ (see details in Appendix 1 of [15]), and integrating over time fluctuations using

$$s_i = \frac{T}{2\omega_i}, \quad ds_i = -\frac{Td\omega_i}{2\omega_i^2}, \quad d\tau_i = \frac{dt_i^E}{2\omega_i},$$

$$\int_0^\infty ds_i (D^4 z^{(i)}) \Phi(x, y) e^{-K} = T \int_0^\infty \frac{d\omega_i}{2\omega_i^2} (D^3 z^{(i)}) x y e^{-K(\omega)} \langle \Phi(x, y) \rangle_{\Delta z_4}. \tag{22}$$

where $\langle \rangle_{\Delta z_4}$ means the averaging over time fluctuations, which can be written in terms of the averaged Wilson line [15]

$$\langle \Phi(x, y) \rangle_{\Delta z_4} = \sqrt{\frac{\omega_i}{2\pi T}} \Phi(x, y), \tag{23}$$

and $K(\omega)$ is

$$K(\omega) = \int_0^T dt_E \left( \frac{\omega}{2} + \frac{m^2}{2\omega} + \frac{\omega}{2} \left( \frac{dz}{dt_E} \right)^2 \right) \tag{24}$$

$$T = |x_4 - y_4|.$$
\begin{equation}
\exp \sum_{n=1}^{\infty} \frac{(ig)^n}{n!} \int d\pi(1) \ldots \int d\pi(n) \langle F(1) \ldots F(n) \rangle,
\end{equation}

where \( d\pi_{\mu\nu} \equiv ds_{\mu\nu} + \sigma^{(1)}_{\mu\nu} d\tau_1 - \sigma^{(2)}_{\mu\nu} d\tau_2 \), and \( ds_{\mu\nu} \) is an area element of the minimal surface, which can be constructed using straight lines, connecting the points \( z^{(1)}_\mu(t) \) and \( z^{(2)}_\nu(t) \) on the paths of \( q_1 \) and \( \bar{q}_2 \) at the same time \( t \) [12, 28]. Note, that \( z^{(1)}_4(t) = z^{(2)}_4(t) = t \). Then the spin-independent part of the exponent reduces to the confinement term \( V_{\text{conf}}(r) \) plus color Coulomb potential \( V_{\text{Coul}} \), while spin-dependent part \( V_{\text{SD}} \) depends also on proper time variables \( \tau_1, \tau_2 \), (see [12, 19] for derivation and discussion). For the case of zero quark orbital momenta with the minimal surface, discussed above, one obtains a simple answer for \( \langle W_\sigma(A) \rangle_A \), which we shall derive below.

The average \( \langle \ldots \rangle \) stands for connected correlators, for example, for the bilocal correlator, \( \langle F(1)F(2) \rangle = \langle F(1) \rangle \langle F(2) \rangle - \langle F(1)F(2) \rangle \), and \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \) is the vacuum field strength. Obviously, due to the \( O(4) \) rotational invariance and colour neutrality of the vacuum, \( \langle \langle F \rangle \rangle = \langle F \rangle = 0 \).

In the Gaussian approximation for the vacuum, when only the lowest, bilocal correlator is retained, one has, with the accuracy of a few per cent (see Ref. [21, 23] for the discussion):

\begin{equation}
\langle TrW(C) \rangle \propto \exp \left[ -\frac{1}{2} \int S d\pi_{\mu\nu}(x) d\pi_{\lambda\rho}(x') D_{\mu\nu,\lambda\rho}(x-x') \right],
\end{equation}

where

\begin{equation}
D_{\mu\nu,\lambda\rho}(x-x') \equiv \frac{g^2}{N_c} \langle \langle TrF_{\mu\nu}(x)\Phi(x, x')F_{\lambda\rho}(x')\Phi(x', x) \rangle \rangle.
\end{equation}

This bilocal correlator of gluonic fields can be expressed through only two gauge-invariant scalar functions \( D(u) \) and \( D_1(u) \) as [30]

\begin{equation}
D_{\mu\nu,\lambda\rho}(u) = (\delta_{\mu\lambda}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\lambda})D(u) + \frac{1}{2} \left[ \frac{\partial}{\partial u_\mu}(u_\lambda\delta_{\nu\rho} - u_\rho\delta_{\nu\lambda}) + \left( \mu \leftrightarrow \nu \right) \left( \lambda \leftrightarrow \rho \right) \right] D_1(u).
\end{equation}

The correlator \( D(u) = D(u_0, [u]) \) contains a nonperturbative part and it is responsible for confinement: the QCD string formation at large interquark separations. Indeed \( \int ds_{u_4}(u) \int ds_{v_4}(v) D(u-v) = S \int d^2(u-v) D(u-v) \), where \( S \) is the total area between the averaged quark trajectories. The fundamental
string tension can be calculated from the area law asymptotics of\(^{(28)}\) for large area loop and is expressed as a double integral:

\[
\sigma = \int d^2(u - v) D(u - v) = 2 \int_0^\infty d\nu \int_0^\infty d\lambda D(\nu, \lambda). \tag{31}
\]

Using\(^{(19)}\) with \(\sigma_{\mu \nu} F_{\mu \nu} \equiv 0\), one obtains spin-independent terms in the \(q \bar{q}\) interaction\(^{(30)}\):

\[
V_0(r) = V_{\text{conf}}(r) + V_{\text{OGE}}(r), \tag{32}
\]

\[
V_{\text{conf}}(r) = 2r \int_0^r d\lambda \int_0^\infty d\nu D(\lambda, \nu) \rightarrow \sigma r, \tag{33}
\]

\[
V_{\text{OGE}} = \int_0^r d\lambda \int_0^\infty d\nu D_{1\text{pert}}(\lambda, \nu) = -\frac{4\alpha_s}{3r}, \tag{34}
\]

where we keep only perturbative part of \(D_{1}\) and to the lowest order \(D_{1\text{pert}}(\lambda, \nu) = \frac{16\alpha_s}{3\pi(\lambda^2 + \nu^2)^2}\). We now turn to the spin-dependent terms of the \(q \bar{q}\) interaction, \(V_{SD}\), and we shall be interested only in the zero orbital moment states for simplicity, hence only spin-spin interaction term \(V_{ss}\) will be treated below.

The spin-dependent terms in the interquark interaction are generated by the combination \(\sigma_{\mu \nu} F_{\mu \nu}\) present in Eq.\(^{(14)}\) and therefore one needs correlators of the colour-electric and colour-magnetic fields, as well as mixed terms, separately. They immediately follow from the general expression\(^{(30)}\) and read\(^{(31)}\):

\[
\frac{g^2}{N_c} \langle \langle \text{Tr} E_i(x) \Phi E_j(y) \Phi^\dagger \rangle \rangle = \delta_{ij} \left( D^E(u) + D_1^E(u) + u^2 \frac{\partial D^E}{\partial u^2} \right) + u_i u_j \frac{\partial D^E}{\partial u^2},
\]

\[
\frac{g^2}{N_c} \langle \langle \text{Tr} H_i(x) \Phi H_j(y) \Phi^\dagger \rangle \rangle = \delta_{ij} \left( D^H(u) + D_1^H(u) + u^2 \frac{\partial D^H}{\partial u^2} \right) - u_i u_j \frac{\partial D^H}{\partial u^2},
\]

\[
\frac{g^2}{N_c} \langle \langle \text{Tr} H_i(x) \Phi E_j(y) \Phi^\dagger \rangle \rangle = \varepsilon_{ijk} u_k \frac{\partial D^{EH}}{\partial u^2}, \tag{35}
\]

where \(u_\mu = x_\mu - y_\mu\), \(u^2 = u_\mu u_\mu\). We keep here the superscripts \(E\) and \(H\) in the correlators \(D\) and \(D_1\) in order to distinguish in principle the electric and magnetic parts of the correlators and thus to be able to consider a nonzero temperature \(T\) and to distinguish Euclidean and Minkowskian contributions. Indeed, while \(D^{E} = D^{H}\) and \(D_1^{E} = D_1^{H}\) at \(T = 0\), at higher temperatures they behave differently. In particular, above the deconfinement temperature, \(T > T_c\), the electric correlator \(D^E\) disappears, whereas \(D_1^E\) and the magnetic correlators survive.
It is clear from (14), that in the norelativistic limit, when in (17) in
\[
\int \sigma_{ \mu \nu} F_{\mu \nu} d\tau_i = \frac{d\tau_i}{2m_i}
\]
onlyxspace
only the upper left corner of (14), i.e. \((\sigma H)\), will contribute to \(V_{ss}\). The corresponding derivation was done in [12, 19] and gives
\[
V_{ss}(r) = \frac{\sigma_1 \sigma_2}{12m_1 m_2} V_4(r) + \frac{3(\sigma_1 r)(\sigma_2 r) - \sigma_1 \sigma_2 r^2}{12m_1 m_2 r^2} V_3(r) \quad (36)
\]
where
\[
V_3(r) = -\int_{-\infty}^{\infty} d\nu r^2 \frac{\partial D_{pert}^1(r, \nu)}{\partial r^2} = \frac{4\alpha_s}{r^3}, \quad (37)
\]
\[
V_4(r) = \int_{-\infty}^{\infty} d\nu \left( 3 D_{pert}^1(r, \nu) + 2 r^2 \frac{\partial D_{pert}^1(r, \nu)}{\partial r^2} \right) = \frac{32\pi\alpha_s}{3}\delta^{(3)}(r), \quad (38)
\]

It is our purpose below to calculate \(V_{ss}\) in the relativistic \(q\bar{q}\) system and in arbitrarily large magnetic field \(B\), and to this end we shall use below first the path integral formalism [14, 15], deriving the general structure of the \(q\bar{q}\) Green’s function, and in the next section we shall determine how relativistic \(V_{ss}\) expressions depend on \(B\).

First we need to find the the Euclidean action \(S_{E q_1 q_2}\) in terms of \(\omega_1, \omega_2\) and common time \(t^E\) of the \(q\bar{q}\) system at \(t^E_1 = t^E_2 = t^E\). To this end we define the Euclidean Lagrangian \(L_{E q_1 q_2}\). We write\nx
\[
d \tau_i = 2 \omega_i \ddot{z}_i, \quad k = 1, 2, 3.
\]
Then all terms in the exponents in (19), (20) and (21) can be represented as \(\exp(- \int dt^E L_{E q_1 q_2})\) and thus we arrive at the following action
\[
S_{E q_1 q_2} = \int_0^{T_E} dt^E \left[ \frac{\omega_1 + \omega_2}{2} + \sum_i \left( \frac{\omega_i}{2} (\dot{z}_i)^2 \right) - ie_k A_{k}^{(e)} \dot{z}_i^{(i)} + \frac{m_1^2}{2\omega_1} + \frac{m_2^2}{2\omega_2} + e_1 \frac{\sigma_1 B}{2\omega_1} + e_2 \frac{\sigma_2 B}{2\omega_2} + \sigma |z^{(1)} - z^{(2)}| - \frac{4}{3} \frac{\alpha_s}{|z^{(1)} - z^{(2)}|} \right] + S_F, \quad (39)
\]
where \(S_F\) contains \((\sigma F)\) terms. Here \(A_{k}^{(e)}\) is the \(k\)-th component of the QED vector potential, \(\sigma\) is the QCD string tension and the contribution of terms \((\sigma_1 F), (\sigma_2 F)\) is separated in \(S_F\). The next step is the transition to the Minkowski metric and the construction of the Hamiltonian. This is easy, since confinement is already expressed in terms of the string tension. We
have \( \exp(- \int L^{E} dt^{E}) \rightarrow \exp(i \int L^{M} dt^{M}) \), \( t^{E} \rightarrow it^{M} \), and

\[
H_{q_{1}q_{2}} = \sum_{i} \dot{z}_{k}^{(i)} p_{k}^{(i)} - L_{M}, \quad p_{k}^{(i)} = \frac{\partial L^{M}}{\partial \dot{z}_{k}^{(i)}} = \omega_{i} \dot{z}_{k}^{(i)} + e_{i} A_{k}^{(e)}. \tag{40}
\]

As a result one obtains (back in the Euclidean time \( T = |x_{4} - y_{4}| \)),

\[
G(x, y) = \frac{T}{2\pi} \int_{0}^{\infty} \frac{d\omega_{1}}{\omega^{3/2}} \int_{0}^{\infty} \frac{d\omega_{2}}{\omega^{3/2}} \sum_{n=0}^{\infty} 4trY(x) e^{-H_{q_{1}q_{2}}T} |y\rangle, \tag{41}
\]

\[
H_{q_{1}q_{2}} = H_{0} + H_{\sigma} + W, \tag{42}
\]

\[
W = V_{\text{conf}} + V_{\text{OGE}} + \Delta M_{SE} + \Delta M_{ss}, \tag{43}
\]

where

\[
H_{0} = \sum_{i=1}^{2} \left( p^{(i)} - \frac{1}{2} (B \times z^{(i)}) \right)^{2} + m_{i}^{2} + \omega_{i}^{2}, \tag{44}
\]

\[
H_{\sigma} = - \frac{e_{1}\sigma_{1}B}{2\omega_{1}} - \frac{e_{2}\sigma_{2}B}{2\omega_{2}}. \tag{45}
\]

Here the terms \( \Delta M_{SE} \) and \( \Delta M_{ss} \) are produced by \( S_{F}^{E} \), and we shall find them as a first order correction. But before that we must treat the \( \omega_{i} \) dependence either in the path integral (41), or the Hamiltonian (42). In the path integral \( \omega_{i} \) play the role of quark energy parameters, and one can use the spectral decomposition in (41) to rewrite it as

\[
G(x, y) = \frac{T}{2\pi} \int_{0}^{\infty} \frac{d\omega_{1}}{\omega^{3/2}} \int_{0}^{\infty} \frac{d\omega_{2}}{\omega^{3/2}} \sum_{n=0}^{\infty} 4trY(x) e^{-H_{n}T} |n\rangle |y\rangle, \tag{46}
\]

At large \( T \) one can use the stationary point method, and one defines \( \omega_{i}^{(0)} \) from the extremum values of \( M_{n}^{(0)}(\omega_{1}, \omega_{2}) \), namely for the Hamiltonian \( \bar{H} \),

\[
H_{0} + H_{\sigma} + V_{\text{conf}} + V_{\text{OGE}} = \bar{H}; \quad \bar{H}\Psi = M_{n}^{(0)}\Psi, \tag{47}
\]

and \( \omega_{i}^{(0)} \) is defined from the condition

\[
\frac{\partial M_{n}^{(0)}(\omega_{1}, \omega_{2})}{\partial \omega_{i}}|_{\omega_{i}=\omega_{i}^{(0)}} = 0, \quad i = 1, 2. \tag{48}
\]
To have an idea of the possible meson masses and the values of $\omega_i^{(0)}$, which we shall use below, it is instructive to consider as in [12] the main part of the Hamiltonian, i.e.

$$\tilde{H} = H_0 + H_\sigma + V_{\text{conf}}, \quad \tilde{H}\tilde{\psi} = \tilde{M}\tilde{\psi}$$  \hspace{1cm} (49)

and replace $V_{\text{conf}}$ by the quadratic term,

$$V_{\text{conf}}(r) = \sigma r \rightarrow \tilde{V}_{\text{conf}}(r) = \sigma \left( \frac{r^2}{\gamma} + \gamma \right).$$  \hspace{1cm} (50)

where $\gamma$ is the variational parameter, yielding some 5% accuracy in the replacement (50). Then the resulting mass $\tilde{M}$ can be found explicitly as

$$\tilde{M}(\omega_1, \omega_2, \gamma) = \varepsilon_{n_{\perp}, n_z} + \frac{m_1^2 + \omega_1^2 - e_1 B \sigma_z}{2\omega_1} + \frac{m_2^2 + \omega_2^2 - e_2 B \sigma_z}{2\omega_2},$$  \hspace{1cm} (51)

where

$$\varepsilon_{n_{\perp}, n_z} = \frac{1}{2\tilde{\omega}} \left[ \sqrt{e^2 B^2 + \frac{4\sigma \omega}{\gamma} (2n_{\perp} + 1)} + \sqrt{\frac{4\sigma \omega}{\gamma} (n_z + \frac{1}{2})} \right] + \frac{\gamma \sigma}{2},$$  \hspace{1cm} (52)

As a result one can estimate the masses $\tilde{M}$ and $\omega_i^{(0)}$ at large m.f., since the basic pattern is defined by relative signs of $eB$ terms in $\varepsilon_{n_{\perp}, n_z}$ and $H_\sigma$.

Indeed, for $eB \gg \sigma$ one can write $\tilde{M} \approx \sum_{i=1,2} \frac{m_i^2 + \omega_i^2 + |e_i B| - e_i \sigma_i B}{2\omega_i}$ and

$$\omega_i^{(0)} \approx \Omega_i \equiv \sqrt{m_i^2 + |e_i B| - e_i \sigma_i B} + O(\sqrt{\sigma}), \quad \tilde{M} \approx \Omega_1 + \Omega_2.$$  \hspace{1cm} (53)

Thus, for the neutral meson with $e_2 = -e_1$, and $\sigma_{1z}, \sigma_{2z} = (++)$, $\omega_{++}$ is growing as $\sqrt{|e_1 B|}$, while for the $(+-)$ state $\omega_{+-}$ is tending to a constant.

One can also find the wave function

$$\tilde{\psi}(\eta) = \frac{1}{\sqrt{\pi^3/2 r_\perp^2 r_0^2}} \exp \left( -\frac{\eta_{\perp}^2}{2r_\perp^2} - \frac{\eta_z^2}{2r_0^2} \right), \quad \eta = z^{(1)} - z^{(2)},$$  \hspace{1cm} (54)

and

$$r_\perp = \frac{2}{eB} \left( 1 + \frac{4\sigma \tilde{\omega}}{\gamma_0 e^2 B^2} \right)^{-1/2}, \quad r_0 = \left( \frac{\gamma}{\sigma \tilde{\omega}} \right)^{1/4}.$$  \hspace{1cm} (55)

At large $eB \gg \sigma$, one has $r_\perp^2 \approx \frac{2}{eB}, r_0 \approx \text{const} \approx \frac{1}{\sqrt{\sigma}},$ and hence

$$|\tilde{\psi}(0)|^2 \approx \frac{eB \cdot \sqrt{\sigma}}{2\pi^{3/2}}, \quad (eB \gg \sigma).$$  \hspace{1cm} (56)
This is the focussing effect of m.f., which is most important in SD forces, as well as in other processes \[34\].

It is important to stress, that we have kept in $\bar{H}$ only those terms, which are the main part of interaction, and therefore in $M_n^{(0)}$ and $\omega^{(0)}$ they are treated to all orders, i.e. exactly. However, the terms $V_{ss}$ and $\Delta M_{SE}$ are considered only as a perturbation, and therefore one should substitute there the values $\omega^{(0)}_i$ obtained from (47), (48), where $V_{ss}$, $\Delta M_{SE}$ do not enter. The Hamiltonians $\bar{H}$ are considered in [15], and below we shall derive both $V_{ss}$ and $\Delta M_{SE}$ in the relativistic $\bar{q}q$ system.

4 The quark-antiquark spin-dependent interaction in strong magnetic field

The advantage of representation (18), (25) lies in the fact, that the only place, where the Dirac $\gamma$ matrices enter, is the local term $(m - \bar{D})$, and it can be assembled in the factor $Y$ with due care, while all the rest nontrivial spin dependence is contained in the $(\sigma F)$ and $(\sigma B)$ factors (14). We shall demonstrate below in this section, that the correlators $(\sigma^{(i)} F)(\sigma^{(k)} F)$ with $i = k$ define $\Delta M_{SE}$, while those with $i \neq k$, define $V_{ss}$.

Consider the Taylor expansion in powers of the color spin interaction $g\sigma_{\mu\nu}F_{\mu\nu} \equiv g(\sigma F), \quad m^2 - \bar{D}^2 = m^2 - D^2 - g(\sigma F), \quad \bar{D}_\mu = D_\mu - e(\sigma F^{(e)}),$

$$\frac{1}{m^2 - \bar{D}^2} = \frac{1}{m^2 - D^2 - g(\sigma F)} = \frac{1}{m^2 - D^2} + \frac{1}{m^2 - D^2}g(\sigma F)\frac{1}{m^2 - D^2} + \quad (57)$$

One can define in (57) the selfenergy correction to the mass,$$
\Delta m^2(x, y) = -g(\sigma F)_x \left( \frac{1}{m^2 - D^2} \right)_{xy} g(\sigma F)_y. \quad (58)$$

$$\bar{\Delta} m^2 = \int d^4(x - y)\Delta m^2(x, y); \quad \Delta m^2(x, y) = -g^2\sigma_\cdot \sigma_k(\langle H_i H_k \rangle + \langle E_i E_k \rangle)_{xy} G_0(x, y) \quad (59)$$

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where
\[ G_0(x, y) = \left( \frac{1}{m^2 - D^2} \right)_{xy}, \]  
while
\[ \frac{g^2}{N_c} (H_i(x)H_k(y) + E_i(x)E_k(y)) = 2D(x - y)\delta_{ik}. \]  

It was shown in [32] that in absence of magnetic field and in the limit of small \( m \) and small vacuum correlation length \( \lambda \), one can replace \( \left( \frac{1}{m^2 - D^2} \right)_{xy} \) by the free propagator \( \frac{1}{(4\pi)^2(x-y)^2} \), yielding
\[ \bar{\Delta}m^2 = -\frac{3\sigma}{\pi}, \]  
then the correction (58) yields for the total mass \( M_n^{(0)} \), \( \Delta M_n = \sum_i \bar{\Delta}m^2_i \), for zero mass \( q \) and \( \bar{q} \)
\[ M_n^{(0)}(\omega_0) \rightarrow M_n^{(0)}(\omega_0) - \frac{3\sigma}{\pi\omega_0}. \]  
Note, that \( \omega \) plays the role of the integration variables in (25), (41) and is defined from the stationary point condition (48), where \( M^{(0)}_n \) does not include \( \Delta m^2 \). However for large \( m \) and small \( |x - y| \lesssim \lambda \) one should multiply (62) with the coefficient \( \eta(m\lambda) < 1 \) calculated in [32].

Consider now the case of constant magnetic field \( B \) along \( z \) axis. One can calculate the effect of magnetic field on the selfenergy correction, to this end expand
\[ (\sigma F) \frac{1}{m^2 - D^2 - e\sigma_3 B} (\sigma F) = (\sigma F) \frac{m^2 - D^2 + e\sigma_3 B}{(m^2 - D^2)^2 - (eB)^2} (\sigma F) \rightarrow \frac{\sigma F(G_+ + G_-)\sigma F}{2}. \]  
where \( G_{+,/-} = (m^2 - D^2 \pm eB)_{xy}^{-1} \), and \( m^2 - D^2 \approx m_i^2 + 2|e_iB| \) in one of the Green’s functions \( G_+, G_- \) and the corresponding Green’s function will contribute \( \frac{3\sigma}{2\pi\omega_0} \eta(eB) \), where \( \eta(eB) \equiv \eta(\sqrt{2|e_iB| + m_i^2\lambda}) \) is the coefficient, introduced in [32], e.g. \( \eta(0) = 1, \eta(5 \text{ GeV})^2 = 0.03 \), see appendix of [32] for explicit expression. The final expression for \( \Delta M_n \) can therefore be written as
\[ \Delta M_n = -\frac{3\sigma}{2\pi\omega_0}(1 + \eta(eB)), \]  
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where $\omega_0$ corresponds to the Green’s function $G$ of a given quark, when spin terms ($\sigma F$) are absent. One can see, that at large $eB$ the selfenergy correction numerator decreases approximately twice, as compared to $eB = 0$, $m = 0$, since $\eta(eB \gg \sigma) \rightarrow 0$.

We now turn to the $(q\bar{q})$ Green’s function and write it in the form

$$\int G_{q\bar{q}}(x,y)d^3(x-y) = tr\langle [\Gamma_1(m_1 - i\hat{p}_1)\Gamma_2(m_2 - i\hat{p}_2)]\rangle,$$  \hspace{1cm} (66)

where we have taken into account, that $(\hat{\partial} - ig\hat{A})$ acting in (66) on the Wilson loop, can be replaced by the momentum operator \[33\].

For the case, when $H$ does not contain $\gamma_\mu$ matrices, noncommuting with $(m_i - i\hat{p}_i)$, one can rewrite (66) as in \[33\], but now taking into account spin and isospin nonconservation in m.f., one must keep the possible eigenvalue dependence on the spin projections,

$$\int G_{q\bar{q}}(x,y)d^3(x-y) = \sum_{n,\nu}(\varepsilon_r \otimes \varepsilon_r)\nu \frac{(M_n^{(\nu)}f_n^{(\nu)})^2 e^{-M_n^{(\nu)}T}}{2M_n},$$  \hspace{1cm} (67)

with $\varepsilon_\gamma = \varepsilon_1 = 1$, $\varepsilon_V \equiv \varepsilon^{(k)}_\mu$, and, and the index $\nu$ denotes a specific polarization and charge component of quark and antiquark, e.g. $\nu = 1$ for $(\bar{u}u), < - + |$ component. As a result the quark decay constant of the $\gamma_\nu$ state is

$$(f_{\Gamma\nu}^n)^2 = \frac{N_c\langle Y_{\Gamma\nu}\rangle|\psi_n^{(\nu)}(0)|^2}{\bar{\omega}_1\bar{\omega}_2M_n^{(\nu)}\xi_\nu},$$  \hspace{1cm} (68)

where $\xi_\nu$ occurs due to $\omega_i$ integrations in (44), see appendix of \[15\], and

$$\langle Y_{\Gamma\nu}\rangle = \frac{1}{4}tr(\Gamma^{\nu}(m_1 - i\hat{p}_1)\Gamma^{\nu}(m_2 - i\hat{p}_2)).$$  \hspace{1cm} (69)

In the case, when $H$ contain spin-dependent terms, and in addition depends on magnetic field $B$, one should be more careful with the ordering of operators ($\sigma F$), $\sigma B$ in $H$ and projectors $(m_1 - i\hat{p}_1), (m_2 - i\hat{p}_2)$.

Correspondingly, $(m - \hat{D})$ can be rewritten as

$$(m_q - \hat{D})_x \rightarrow (m_1 - i\hat{p}_1), \hspace{0.5cm} (m_{\bar{q}} - \hat{D})_x = m_2 - i\hat{p}_2,$$  \hspace{1cm} (70)

where $p_1 = (i\omega_1, p)$, $p_2 = -(i\omega_2, -p)$ and we take into account, that $D_\mu(x)$ acting on $\Phi_z(x,y)$ yields $\partial_\mu \rightarrow i\sigma_\mu$.

$$m_1 - i\hat{p}_1 = m_1 + \omega_1\gamma_4 - i\sigma p\gamma = \begin{pmatrix} m_1 + \omega_1 & -i\sigma p \\ i\sigma p & m_1 - \omega_1 \end{pmatrix}$$  \hspace{1cm} (71)
At this point we have at least two possibilities for the relative ordering of factors \((m_q^2 - \hat{D}^2)\) and \((m_q - \hat{D})\) in \(G_{qq}\). We shall define this ordering as
\[
G_{qq}(x, y) = \langle \text{tr}[\Gamma_1 S_q(x, y) \Gamma_2 S_q(y, x)] \rangle_A =
\]
\[
= \langle \text{tr}[\Gamma_1(m_q - \hat{D}) x (m_q^2 - \hat{D}^2)_{xy} \Gamma_2(m_q - \hat{D}) y (m_q^2 - \hat{D}^2)_{yx}] \rangle_A \quad (72)
\]
We shall show below, that this ordering yields correct results for spinless and spin-dependent parts, which can be checked in the nonrelativistic limit, whereas other orderings lead to wrong answers. Now
\[
m_2 - i\hat{p}_2 = m_2 - \omega_2 g - i\gamma = \begin{pmatrix} m_2 - \omega_2 & -i\sigma p \\ i\sigma p & m_2 + \omega_2 \end{pmatrix}. \quad (73)
\]
If is clear, that in the nonrelativistic situation, \(p \ll m, \omega = \sqrt{p^2 + m^2} \rightarrow m\), the product \((\Gamma(m_1 - i\hat{p}_1) \Gamma(m_2 - i\hat{p}_2))\) tends to the nonrelativistic projector
\[
\begin{pmatrix} 4m_1 m_2 & 0 \\ 0 & 0 \end{pmatrix}. \quad (74)
\]
We now turn to the spin-dependent terms and make expansion of \(m_1^2 - \hat{D}^2 = m_1^2 - D_\mu^2 - e\sigma B - g(\sigma F)\) in powers of \(g(\sigma F)\):
\[
(m_1^2 - \hat{D}_1) \equiv \Delta_B + \Delta_B g(\sigma F) \Delta_B, \quad \Delta_B \equiv (m_1^2 - D_\mu^2 - e_1 \sigma B)^{-1} \quad (75)
\]
\[
(m_2^2 - \hat{D}^2) \equiv \Delta_B - \Delta_B g(\sigma F) \Delta_B, \quad \Delta_B \equiv (m_2^2 - D_\mu^2 - e_2 \sigma B)^{-1} \quad (76)
\]
and \((72)\) can be rewritten keeping only one spin-spin term as
\[
G_{qq}(x, y) = G_{qq}^{(0)}(x, y) - \langle \text{tr}[\Gamma(m_1 - i\hat{p}_1) \Lambda_B \bar{g}(\sigma F) \Delta_B \Gamma(m_2 - i\hat{p}_2) \bar{g}(\sigma F) \Delta_B] \rangle_A. \quad (77)
\]
one readily obtains for \(\Gamma = \gamma_5\)
\[
\mathcal{M} \equiv \text{tr}[\gamma_5(m_1 - i\hat{p}_1)(\sigma_1 F)\gamma_5(m_2 - i\hat{p}_2)(\sigma_2 F)] = \text{tr}[(m_1 - i\hat{p}_1)(\sigma_1 F)(m_2 - i\hat{p}_2)(\sigma_2 F)]^T, \quad (78)
\]
\[
(m_1 - i\hat{p}_1)(\sigma F_1) = \begin{pmatrix} (m_1 + \omega_1)\sigma H_1 - i\sigma p \sigma E_1, & (m_1 + \omega_1)\sigma E_1 - i\sigma p \sigma H_1 \\ i\sigma p \sigma H_1 + (m_1 - \omega_1)\sigma E_1, & i\sigma p \sigma E_1 + (m_1 - \omega_1)\sigma H_1 \end{pmatrix}. \quad (79)
\]
\[
((m_2-i\hat{p}_2)(\sigma F_2))^T = \begin{pmatrix}
i\sigma p\sigma E_2 + (m_2 + \omega_2)\sigma H_2, & i\sigma p\sigma H_2 + (m_2 + \omega_2)\sigma E_2 \\
(m_2 - \omega_2)\sigma E_2 - i\sigma p\sigma H_2, & (m_2 - \omega_2)\sigma H_2 - i\sigma p\sigma E_2
\end{pmatrix}.
\]

Combining (79) and (80), one obtains \((H_i \equiv H(x_i), i = 1, 2)\).

\[
\mathcal{M} = tr_\sigma \{\sigma H_1\sigma H_2(2m_1m_2 + 2\omega_1\omega_2) - 2(\sigma H_1)\sigma p(\sigma H_2)\sigma p) +
\]

\[\sigma E_1\sigma E_2(2m_1m_2 - 2\omega_1\omega_2) + 2(\sigma E_1)\sigma p(\sigma E_2)\sigma p) +
\]

\[\sigma E_1\sigma H_2(-2i\sigma p(\omega_1 + \omega_2)) + \sigma H_1\sigma E_2(2i\sigma p(\omega_1 + \omega_2))\} \tag{81}
\]

In what follows we disregard first the terms, containing \((\sigma p)\) or \((\sigma p)(\sigma p)\).

In the nonrelativistic limit, \(|p| \to 0, \omega_i \to m_i\), one has

\[
\langle \mathcal{M} \rangle_A = 8m_1m_2\langle H_i(x_1)H_i(x_2) \rangle_A. \tag{82}
\]

Field correlators are expressed via two scalar correlators \(D(x_1 - x_2)\) and \(D_1(x_1 - x_2)\) as in (35),

Comparing with the standard definition for the nonrelativistic hyperfine (hf) term, one has

\[
V_{hf} = \frac{\sigma(1)\sigma(2)}{12m_1m_2}V_4^{(H)}(r), \quad V_4^{(H)}(r) = \int_{-\infty}^{\infty} dv \frac{g^2}{N_c} \langle H_i(x)H_i(y) \rangle, \quad \nu \equiv x_4 - y_4, \tag{83}
\]

where

\[
u = x_\mu - y_\mu, \quad \mu = 1, 2, 3, 4; \quad r = |x - y| = |u|, \quad \nu \equiv u_4. \tag{84}
\]

One can separate perturbative part in \(D_1\)

\[
D_1(x) = D_1^{pert}(x) + D_1^{up}(x), \quad D_1^{pert}(x) = \frac{16\alpha_s}{3\pi x^4} + O(\alpha_s^2), \tag{85}
\]

and define the potentials

\[
V_4^{(H)}(r) = \int_{-\infty}^{\infty} dv \frac{g^2}{N_c} \langle H_i(x)H_i(y) \rangle =
\]

\[
= \int_{-\infty}^{\infty} dv \left( 3D(r, \nu) + 3D_1(r, \nu) + 2r^2 \frac{\partial^2 D_1(r, \nu)}{\partial r^2} \right) = V_4^{(D)}(r) + V_4^{(1)}(r), \tag{86}
\]

\[17\]
\[ V_4^{(E)}(r) = \int_{-\infty}^{\infty} d\nu \left( 3D(r, \nu) + 3D_1(r, \nu) + (3\nu^2 + r^2) \frac{\partial D_1(r, \nu)}{\partial r^2} \right) = V_4^{(D)} - V_4^{(1)}(r). \]  

(87)

As it is known [21], the nonperturbative part of \( D_1 \) and \( D \) yield much lower input in \( V_4^{(E,H)} \), the leading part is due to \( D_1^{\text{pert}} \), i.e. \( V_4^{(1)} \). Inserting this, one obtains

\[ V_4(H) \equiv V_4^{(H)\text{pert}}(r) = \frac{32\pi \alpha_s}{3} \delta(3)(r), \]  

(88)

and the result (83), (88) coincides with the known nonrelativistic limit.

Now we turn to the relativistic case, \( \omega_i \gg m_i \). First of all we note, that

\[ V_4(E) \equiv V_4^{(E)\text{pert}}(r) = -\frac{32\pi \alpha_s}{3} \delta(3)(r). \]  

(89)

Looking at (81), one can see that in the relativistic case, when \( \omega_i \gg m_i \), there is a cancellation in the spin-spin interaction, in the combination

\[ 2(m_1m_2 + \omega_1\omega_2)V_4^{(H)\text{pert}} + 2(m_1m_2 - \omega_1\omega_2)V_4^{(E)\text{pert}} = 2\omega_1\omega_2 \cdot 2V_4^{(1)} + 2m_1m_22V_4^{(D)}, \]  

(90)

and multiplying this result with \( \Delta_B, \bar{\Delta}_B \) as in (77), in the \( B = 0 \) case we obtain for \( \Delta_B \approx \frac{1}{2\Omega_1^2}, \bar{\Delta}_B = \frac{1}{2\Omega_2^2} \).

\[ V_{\text{hf}} = \frac{\sigma^{(1)}\sigma^{(2)}}{12\omega_1\omega_2} \left\{ V_4^{(H)}(r) \left( 1 + \frac{p^2}{3\omega^2} \right) + \frac{m^2}{\omega^2}V_4^{(D)}(r) \right\}. \]  

(91)

Here \( \bar{\omega}_i = \frac{\Omega_i^2}{\omega_i} \), and \( \Omega_i \) is defined in (53) one can derive that \( \bar{\omega}_i \geq \omega_i^{(0)} \), e.g. in the nonrelativistic limit \( \omega_i^{(0)} \rightarrow m_i \), \( \Omega_i \rightarrow m_i \) and also \( \bar{\omega}_i \rightarrow m_i \). The same happens in relativistic case, when \( eB \gg \sigma \). In what follows for \( \bar{\omega}_1 \neq \bar{\omega}_2 \) it is implied, e.g., that \( \bar{\omega}_1^{(+-)} = \bar{\omega}_1^{(+--)} \).

We now turn to the case of nonzero \( B \) and now take into account the noncommutative (2 × 2) terms in \( H_\sigma \) and in \( V_{\text{hf}} \), which we write in the total mass as

\[ M = \bar{M} - \mu_1\sigma_1z + \mu_2\sigma_2z + a\sigma_1\sigma_2, \]  

(92)

where \( \mu_i = \frac{eB}{2\omega}, a = \frac{1}{12\omega^2} \langle V_4^H \rangle \).

For \( \pi^0, \rho^0(s_z = 0) \) states one obtains a standard mixing of \( \langle ++ | \) and \( \langle -+ | \) states of \( \langle \sigma_1z, \sigma_2z \rangle \), with \( V_{\text{hf}} \), where now for \( B \neq 0 \) we distinguish

\[ a_{11} = \langle ++ | a\sigma_1\sigma_2 | ++ \rangle = \frac{1}{12\bar{\omega}_+^2} \langle V_4^{(H)} \rangle, \]  

(93)
\[ a_{22} = \langle - + |a\sigma_1\sigma_2| - + \rangle = \frac{1}{12\bar{\omega}_{++}^2} \langle V_4^{(H)} \rangle, \tag{94} \]

\[ 2a_{12} = 2a_{21} = \langle - + |a\sigma_1\sigma_2| + - \rangle = \frac{2}{12\bar{\omega}_{+-}\bar{\omega}_{-+}} \langle V_4^H \rangle, \tag{95} \]

We also define \( M_{11} \), \( M_{22} \) as follows

\[ M_{11} = (\bar{M} - (\mu_1 + \mu_2) - a_{11}))\omega_{+-}; \quad M_{22} = (\bar{M} + (\mu_1 + \mu_2) - a_{22}))\omega_{-+}. \tag{96} \]

Finally, from \( \text{det}(M - E) = 0 \), we obtain the eigenvalues of \( M \),

\[ E_{1,2} = \frac{1}{2}(M_{11} + M_{22}) \pm \sqrt{\left( \frac{M_{22} - M_{11}}{2} \right)^2 + 4a_{12}a_{21}}. \tag{97} \]

Here \( \omega_{+-} \) and \( \omega_{-+} \) correspond to the diagonal states of \( \Delta_B\bar{\Delta}_B \), i.e. for neutral mesons to the stationary values of \( \omega^{(0)} \) in the states with spin projections \( \langle + - | \) and \( \langle - + | \) respectively. It is clear, that at large \( eB \), the values of \( \omega \) behave differently, i.e. \( \omega_{+-} \sim \text{const} \), while \( \omega_{-+} \sim \sqrt{eB} \), and \( M_{22} \sim \sqrt{eB} \), hence at large \( eB \) the nondiagonal part of \( M \) in (97) is decreasing, and \( M \) tends to its diagonal eigenvalues.

\[ E_1(eB \to \infty) \to M_{11}, \quad E_2(eB \to \infty) \to M_{22}. \tag{98} \]

One can notice, that for the \( \langle + - | \) state \( M_{11} \) contains at large \( eB \) the fast decreasing part, \( -a_{11} \sim -\psi^2(0) \), and the latter is large in modulus, \( \psi^2(0) \sim \sim eB \), hence leading to the negative mass \( E_1 \).

This happens already for only colormagnetic contribution \( \langle H_i H_k \rangle \) to \( V_4^{(H)} \), while our conclusion in section 2 was, that eigenvalues of \( \langle m_i^2 - \hat{D}_i^2 \rangle \) are positive, together with the total mass eigenvalues of the operator \( \left( \frac{1}{m_1^2 - D_1^2} \frac{1}{m_2^2 - D_2^2} \right) \). Hence we conclude, that the perturbation theory in \( V_{hf} \) breaks down at large \( eB \) and one has to replace it with some modified form. However, as was understood already in [2, 3] even at \( B = 0 \) the perturbation theory with the potential \( V_{ss}^{(0)}(r) \sim c\delta^{(3)}(r) \) is diverging since \( V_{ss}^{(0)} \) for any \( c < 0 \) ensures infinite number of bound states, which are physically irrelevant. Therefore one should in any case take into account the relativistic smearing of the hf interaction, which appears due to the time integration in (38), which is taken in (38) along the straight line, instead of the complicated relativistic trajectory of the quark with time fluctuations, see [15]. The resulting smearing length
\[ \lambda > \lambda_{\text{conf}}, \text{where } \lambda_{\text{conf}} \text{ is the scale of } D(x), \text{connected to the gluelump mass} \]

\[ \lambda_{\text{conf}} \approx 0.1 \div 0.15 \text{ fm}. \text{ On the lattice } \lambda \geq a, a \text{ is the lattice unit,} \]

\[ a \approx 0.1 \div 0.24 \text{ fm}. \text{ therefore we replace } V^{(0)}_{ss}(r) \text{ by a smeared out version, e.g.} \]

\[ \tilde{\delta}^{(3)}(r) = \left( \frac{\lambda}{\sqrt{\pi}} \right)^3 e^{-\mu^2 r^2}; \quad \tilde{V}_{ss}(r) = c \tilde{\delta}^{(3)}(r), \quad \mu = \frac{1}{\lambda} \approx (1 \div 2) \text{ GeV}. \quad (99) \]

Using the wave function \( \tilde{\psi}(\eta) \) from (54) one obtains for \( \langle \tilde{V}_{ss} \rangle \), in the PS meson

\[ \langle \tilde{V}_{ss} \rangle = \frac{c \mu^3}{\pi^{3/2} \sqrt{1 + \mu^2 r_\perp^2 (1 + \mu^2 r_\perp^2)}}; \quad c = -\frac{8 \pi \alpha_s}{3 \omega_1 \omega_2}. \quad (100) \]

For \( \mu \to \infty \) one regains the original answer, \( \langle \tilde{V}_{ss} \rangle \to c \psi^2(0) \).

Since \( r_\perp^2 = \frac{2}{eB} \left( 1 + \frac{(\omega_2)^2}{(eB)^2} \right)^{-1/2} \), \( r_\perp (eB \to \infty) \to 0 \), and \( \langle \tilde{V}_{ss} \rangle \) tends to a constant limit at large \( eB \), preventing in this way breakdown of the vacuum due to vanishing of the meson mass.

5 The tensor forces in magnetic field

As was established in the previous section, the hf interaction, has the form (91) and in the m.f. the coefficients \( \frac{1}{\omega_1} \) transform into \( \frac{1}{\omega_1 \omega_{i'k'}} \) where \( (ik), (i'k') = (\sigma_{1z}, \sigma_{2z}) = (+-) \) or \((-+)\), for \( S_z = 0 \) see Eqs. (93)-(95). We now turn to the tensor forces and discuss, how they are transformed in the m.f. As was found in \[16\] \[19\] the total spin-dependent forces without m.f. can be written as

\[ V^ss_3 + V^ss_4 \equiv \frac{\sigma^{(1)} \sigma^{(2)}}{12 \omega_1 \omega_2} V_4(r) + \frac{1}{12 \omega_1 \omega_2} (3(\sigma^{(1)} n)(\sigma^{(2)} n) - \sigma^{(1)} \sigma^{(2)}) V_5(r) \quad (101) \]

It is clear from (101), that for the spherically symmetric \( q\bar{q} \) states the term \( V_5(r) \) is irrelevant, and therefore requires a nonzero angular momentum.

The situation is drastically changing in the m.f., since in this case the form of the wave function is distorted as in the elongated ellipsoid.

This fact implies the appearance of tensor forces in m.f. even in the ground \( q\bar{q} \) state with the zero angular momentum. This situation was studied for the hydrogen atom case in m.f. in \[34\] in the nonrelativistic treatment of the tensor forces in the \( q\bar{q} \) system.
We are using again the correlator technic and the perturbative correlator $D_1^{\text{pert}}$, in terms of which the tensor interaction can be written as in (37).

The expectation value of the tensor term in the ground state with the wave function $\tilde{\psi}(\rho, z)$ (54) can be written as

$$\langle V_{ss}^3 \rangle = \frac{8\alpha_s \sigma^{(1)}_k \sigma^{(2)}_l}{9\pi \omega_1 \omega_2} \int_{-\infty}^{\infty} d\nu d\rho dz d\varphi (3n, n, k - \delta_{ik}) \psi^2(\rho, z) (-r^2) \frac{1}{\partial r^2 (r^2 + \nu^2)^2}$$

(102)

Using the relations

$$\frac{1}{(r^2 + \nu^2)^3} = \frac{1}{2} \frac{\partial^2}{\partial \alpha^2} \bigg|_{\alpha=0} \frac{1}{r^2 + \nu^2 + \alpha}; \frac{1}{r^2 + \nu^2 + \alpha} = \int_0^{\infty} d\beta e^{-\beta(r^2 + \nu^2 + \alpha)}.$$

(103)

and the $q\bar{q}$ wave function in the zeroth approximation

$$\psi^2(\rho, z) = Ne^{-\frac{r_1^2}{4} - \frac{r_3^2}{4}}; \quad N = \frac{1}{\pi^{3/2} r_1^2 r_3^2},$$

(104)

one can calculate the sum

$$\langle V_{3ss}^3 + \langle V_{4ss}^4 \rangle = \frac{8\pi \alpha_s}{9\omega_1 \omega_2} N \left\{ \sigma_1 \sigma_2 + \frac{K}{4} \left( \frac{1}{r_1^2} - \frac{1}{r_3^2} \right) (-\sigma_2^{(1)} \sigma_2^{(2)} - \sigma_1^{(1)} \sigma_1^{(2)} + 2\sigma_2^{(1)} \sigma_2^{(2)}) \right\},$$

(105)

Here $K$ is

$$K = r_1^2 \int_0^{\infty} \frac{2u^4 du}{(u^2 + 1)^2(u^2 + \frac{r_2^2}{r_3^2})^{3/2}}.$$

(106)

Note, that we have omitted here for simplicity in $\langle V_{4ss}^4 \rangle \sim \sigma_1 \sigma_2$, the additional terms, which appear in (91). The integral $K$ in (106) can be done explicitly, using the relation

$$\int_0^{\infty} \frac{du}{(u^2 + p^2)(u^2 + q^2)^{1/2}} = \frac{1}{2p\sqrt{p^2 - q^2}} \ln \left( \frac{p + \sqrt{p^2 - q^2}}{p - \sqrt{p^2 - q^2}} \right).$$

(107)

It is important, that $K \leq r_1^2$, therefore since $r_1^2 \to 0$ for $eB \to \infty$, see eq. (55), the relative role of the tensor term in (105) is diminishing with growing $eB$, and the absolute value of $\langle V_{3ss}^3 \rangle$ depends only on the values of $\omega_1, \omega_2$. To
find these values, we again as in (92) write the total mass, but now with the addition of the tensor term and for any quark charges

\[ M = \bar{M} + a\sigma_1\sigma_2 + c^{(1)}\sigma_{1z} + c^{(2)}\sigma_{2z} + b(-\sigma_x^{(1)}\sigma_x^{(2)} - \sigma_y^{(1)}\sigma_y^{(2)} + 2\sigma_z^{(1)}\sigma_z^{(2)}) \equiv \bar{M} + h, \]

where

\[ a = \frac{8\pi\alpha_s}{9\bar{\omega}_1\bar{\omega}_2}N, \quad c^{(1)} = \frac{-e_1B}{2\omega_1}, \quad c^{(2)} = \frac{-e_2B}{2\omega_2}, \]

\[ b = \frac{8\pi\alpha_s}{9\bar{\omega}_1\bar{\omega}_2}N\frac{K}{4}(\frac{1}{r_1^2} - \frac{1}{r_2^2}). \]

Moreover, one must distinguish the values of coefficients \(a, b, c\) in different spin projection states \((\sigma_1, \sigma_2)\) namely the values of \(\bar{\omega}_i\) in \(a, b, c\) in different \((\sigma_x^{(1)}, \sigma_x^{(2)})\) states. E.g. for \(S_z = +1(-1)\) one must write \(\bar{\omega}_1\bar{\omega}_2 \rightarrow \bar{\omega}_2^2(\bar{\omega}_2^2)\), while for \(S_z = 0\), one has the matrix elements as in (93)-(95). In a similar way for our general case with tensor forces in (105), one can write

\[
\begin{align*}
\langle + | h | + \rangle &= -a_{11} - 2b_{11} + c^{(1)}_{11} - c^{(2)}_{11}, \\
\langle - | h | - \rangle &= -a_{22} - 2b_{22} + c^{(1)}_{22} - c^{(2)}_{22}, \\
\langle + | h | - \rangle &= \langle - | h | + \rangle = 2(a_{12} - b_{12}) = 2(a_{21} - b_{21}),
\end{align*}
\]

and all diagonal terms have the corresponding \(\bar{\omega}_i\), e.g. \(\bar{\omega}_1 = \bar{\omega}_2 = \bar{\omega}_\pm\) for \(a_{11}, b_{11}, c^{(i)}_{11}\), and in nondiagonal matrix elements \(\bar{\omega}_1\bar{\omega}_2 = \bar{\omega}_+\bar{\omega}_-\).

From \(\det(h - E) = 0\) with elements in (111) one obtains two eigenvalues of the spin-dependent part \(E_1, E_2\).

The resulting expressions for \(E_1, E_2\) coincide with those in (97), when one makes the following replacements

\[ a_{11} \rightarrow a_{11} + 2b_{11}, \quad a_{22} \rightarrow a_{22} + 2b_{22}, \quad a_{12} \rightarrow a_{12} - b_{12}, \]

and \(\mu_1 \rightarrow -c^{(1)}\).

Eqs. (51), (52) contain a prescription for the values of \(\omega_{ij}, i, j = +, -\), entering in \(M_{11}, M_{22}\), which is valid also in the case of nonzero tensor forces.

### 6 Conclusions and prospectives

We have obtained explicit relativistic expressions for the spin-spin interaction terms in the \(q\bar{q}\) system in the arbitrarily strong m.f. As a by-product we also
obtained in the same formalism expressions for the np self-energy corrections in m.f. These formulas are generalizations of the previously found expressions for the $q\bar{q}$ mesons in absence of m.f., see e.g. [16], and we also found corrections to those expressions, see Eq. (91), where the terms $\frac{p^2}{2\omega^2} V_4^{(H)}(r)$ and $\frac{m^2}{\omega^2} V_4^{(D)}(r)$ are new.

It is remarkable, how m.f. changes the spin-spin forces. First of all, the matrix element of the hf term $V_4^{(H)}(r) \sim \delta^{(3)}(r)$, in the strong m.f. is proportional to the $\psi^2(0)$ – the probability of coming together of $q$ and $\bar{q}$, which in strong m.f. grows as $eB$

This effect is known in nonrelativistic case, where the hf term is $\langle V_{hf} \rangle \sim \frac{\psi^2(0)}{M_1 M_2}$ and was discussed recently in [34] for the hydrogen case. However, in this case $M_2 = M_p$ is very large and $\langle V_{hf}(\text{hydr}) \rangle$ is small and this growth of $\langle V_{hf} \rangle$ was thought to cease in relativistic limit, $eB \sim m^2_c$.

However, as we have shown here in the paper, the growth of $\langle V_{hf} \rangle$ in the relativistic case for $eB \to \infty$ is possible whenever $\omega^2$ in the denominator of $\langle V_{hf} \rangle \sim \frac{\psi^2(0)}{\omega^2}$ does not grow with $eB$, which occurs for the $q\bar{q}$ states, where magnetic moment terms compensate the growth of the rest part of mass, e.g. for the $< + - |$ states of the neutral mesons, like $\pi^0$. Then hf terms yield negative ($-3\langle V_{hf} \rangle$) contribution to the mass, linearly growing with $eB$ in modulus, thus giving the absurd negative mass result.

To disprove this result, we have given in Section 2 the proof, that the term $V_4^{(H)}$, which causes this problem, cannot generate negative mass in any m.f., and we are coming to the conclusion, that this discrepancy occurs due to the use of the perturbation theory for the hf term, proportional to the delta-function. Therefore we have suggested in section 4 the smearing procedure of the hf term, which takes into account the relativistic Zitterbewegung of quarks and should be supplemented with a rigorous procedure of the summation or estimation of the whole perturbative series. We stress, that this situation occurs solely due to m.f., which cannot, as shown in section 2, provide the pair creation and vacuum reconstruction, in contrast to the real Minkowskian electric fields, which are capable of the pair creation (as in the Schwinger famous formula [24]) and of the vacuum reconstruction with emission of positrons (as in superheavy atoms with $Z > Z_{crit}$).

Another interesting new effect is the appearance of the tensor force effects at nonzero m.f., which can be tested both in atoms and mesons. However, as we have shown above, at large m.f. the tensor force contribution does not grow with $eB$, unlike hf term, and is always smaller than the latter.
In this way we have completed the main part of the strong dynamics of mesons in magnetic field for the zero angular momentum, started in [15].

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