Timescales of Kozai-Lidov oscillations at quadrupole and octupole order in the test particle limit

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ABSTRACT

Kozai-Lidov (KL) oscillations in hierarchical triple systems have found application to many astrophysical contexts, including planet formation, type Ia supernovae, and supermassive black hole dynamics. The period of these oscillations is known at the order-of-magnitude level, but dependences on the initial mutual inclination or inner eccentricity are not typically included. In this work I calculate the period of KL oscillations ($t_{KL}$) exactly in the test particle limit at quadrupole order (TPQ). I explore the parameter space of all hierarchical triples at TPQ and show that except for triples on the boundary between libration and rotation, the period of KL oscillations does not vary by more than a factor of a few. The exact period may be approximated to better than 2 per cent for triples with mutual inclinations between 60° and 120° and initial eccentricities less than $\sim$0.3. In addition, I derive an analytic expression for the period of octupole-order oscillations due to the ‘eccentric KL mechanism’ (EKM). I show that the timescale for EKM oscillations is proportional to $\epsilon_{oct}^{-1/2}$, where $\epsilon_{oct}$ measures the strength of octupole perturbations relative to quadrupole perturbations.

1 INTRODUCTION

Triple systems are common in the Galaxy, comprising $\sim$10% of systems in which the primary is $\sim$1 $M_\odot$ (Duquennoy & Mayor 1991; Raghavan et al. 2010; Tokovinin 2014; Riddle et al. 2015). All observed triples are ‘hierarchical,’ in that the relative distance between two components of the triple is much smaller than the relative distance between them and the third. Such systems are stable if they are sufficiently hierarchical (Mardling & Aarseth 1999, 2001).\footnote{While stable non-hierarchical triple systems are possible (e.g., Chenciner & Montgomery 2000; Suvakov & Dmitrasinovic 2013), they may require fine tuning to form and have never been observed in nature.}

In general, if the tertiary is highly inclined relative to the inner binary, the eccentricity of the inner binary will undergo oscillations, known as Kozai-Lidov (KL) oscillations (Lidov 1962; Kozai 1962). KL oscillations have been invoked in many contexts to explain a wide variety of phenomena such as the formation of hot Jupiters (Wu & Murray 2003; Wu et al. 2007; Fabrycky & Tremaine 2007; Naoz et al. 2011, 2012; Petrovich 2013), the formation of blue stragglers (Perets & Fabrycky 2009; Naoz & Fabrycky 2014), the merger of WD-WD binaries (Thompson 2011; Katz et al. 2011), the merger of supermassive and intermediate-mass black holes (Miller & Hamilton 2002b; Blaes et al. 2002; Wen 2003), the distribution of dark matter around supermassive black hole binaries (Naoz & Silk 2014), and as a source of unique gravitational wave signals (Miller & Hamilton 2002a; Gould 2011; Seto 2013; Antonini et al. 2014; Antognini et al. 2014; Bode & Wegg 2014).

KL oscillations are a secular phenomenon, occurring on timescales much longer than the orbital periods. It is therefore possible to average the motions of the individual stars over their orbits and study only the secular changes to the orbital elements. If there is a large mass ratio in the inner binary then on even longer timescales the strength of the KL oscillations (i.e., the maximum eccentricity reached) will vary (Ford et al. 2000; Katz et al. 2011; Lithwick & Naoz 2011; Naoz et al. 2013a). These variations have been termed the ‘eccentric KL mechanism’ (EKM), and in some cases can cause the inner binary to pass through an inclination of 90° with respect to the outer binary in a ‘flip’ from prograde to retrograde or vice versa. During a flip the eccentricity of the inner binary can be driven to extremely large values because the strength of KL oscillations is very sensitive to the mutual inclination, with arbitrarily strong oscillations occurring as the inclination approaches 90° exactly in the test particle limit. Although EKM oscillations do not occur when the two stars of the inner binary are of equal mass, mass loss from one of the stars in the course of stellar evolution can induce EKM oscillations (Shappee & Thompson 2013; Michaely & Perets 2014). EKM oscillations and flips have
generally been studied in the context of hierarchical triples, but flips occur over a wider range of parameter space in both 2+2 quadruples (Pejcha et al. 2013) and 3+1 quadruples (Hamers et al. 2014).

The period of KL oscillations, \( t_{KL} \), is a particularly useful quantity because it determines not only on what timescale the oscillations occur, but whether other drivers of precession of the inner orbit (e.g., relativistic effects or tides) will dominate the dynamics of the triple and suppress KL oscillations. In this paper we use the action angle formalism to derive the period of KL oscillations exactly in the test particle limit.

Because the extreme eccentricity oscillations that occur during a flip can affect the evolution of the objects in the inner binary, the timescale for EKM oscillations is another important quantity. Yet no derivation of the timescale for EKM oscillations has appeared in the literature, although several have asserted that \( t_{EKM} \sim t_{KL}/e_{opt} \) is a plausible timescale (e.g. Katz et al. 2011, Naoz et al. 2013b, Li et al. 2013b), where \( e_{opt} \) measures the strength of the octupole order term relative to the quadrupole order term of the Hamiltonian (see equation 50 for a definition). I show that \( t_{EKM} \sim t_{KL}/\sqrt{e_{opt}} \).

In Section 2 I present the basic parameters and equations that govern a hierarchical three-body system. In Section 3 I then derive the period of KL oscillations. In Section 4 I explore how the period varies over the parameter space and in Section 5 I provide an approximation to the exact period. In Section 6 I treat the period of EKM oscillations and derive the corrected timescale. I conclude in Section 7.

To perform the calculations in this paper I wrote the kozai Python module. This module can evolve hierarchical triple systems in the secular approximation up to hexadecapole order using either the Delaunay orbital elements or the eccentricity and angular momentum vectors. I have released this code under the MIT license and it is available at https://github.com/joe-antognini/kozai.

2 BASIC EQUATIONS

2.1 Notation

Throughout this paper orbital properties referring to the inner and outer binary are labelled with a subscript 1 and 2, respectively. The masses of the two components of the inner binary are \( m_1 \) and \( m_2 \), and the mass of the tertiary is \( m_3 \).

We will often refer to the orbital parameters using Delaunay’s elements: the mean anomalies, \( t_x \); the arguments of periapsis, \( g_x \); and the longitudes of ascending nodes, \( h_x \), where \( x = 1 \) or 2 and refers to the inner or outer binary, respectively. Their conjugate momenta are

\[
L_1 = \frac{m_1 m_2}{m_1 + m_2} \sqrt{G(m_1 + m_2)a_1},
\]

(1)

\[
L_2 = \frac{m_3(m_1 + m_2)}{m_1 + m_2 + m_3} \sqrt{G(m_1 + m_2 + m_3)a_2},
\]

(2)

\[
G_x = L_x \sqrt{1 - e_x^2},
\]

(3)

and

\[
H_x = G_x \cos i_x.
\]

(4)

Delaunay’s elements form a set of canonical variables. Note that \( G_1 \) and \( G_2 \) are the angular momenta of the inner and outer binaries, respectively. We furthermore define the convenient parameter

\[
\epsilon_x^2 \equiv 1 - e_x^2.
\]

(5)

The angular momentum of an orbit may thus be written \( G_x = L_x \epsilon_x \).

2.2 The Hamiltonian

If a three-body system is sufficiently hierarchical, its Hamiltonian may be considered to be that of two isolated binaries (the inner binary, consisting of the two closest bodies, and the outer binary, consisting of the distant body plus the inner binary taken as a point mass) plus a perturbative interaction term:

\[
H = \frac{G m_1 m_2}{2a_1} + \frac{G(m_1 + m_2)m_3}{2a_2} + H_{pert}.
\]

(6)

This interaction term captures the change in the orbital motion of each binary in the tidal field of the other. Because we are assuming that the triple is hierarchical, the semi-major axis ratio, \( \alpha = a_1/a_2 \), is a small parameter that we can use to expand the perturbative component of the Hamiltonian in a multipole expansion (Harrington 1968).

\[
H_{pert} = \frac{G}{a_2} \sum_{j=2}^{\infty} \alpha^j \mathcal{M}_j \left( \frac{r_2}{a_1} \right) j+1 \right) P_j(\cos \Phi),
\]

(7)

where \( P_j \) is the \( j \)th Legendre polynomial, \( r_x \) is the distance between the two components of the \( x \)th binary, \( \Phi \) is the angle between \( r_2 \) and \( r_1 \) and \( \mathcal{M}_j \) is a mass parameter defined by

\[
\mathcal{M}_j = m_1 m_2 m_3 \frac{m_1^{-1} - (m_2)^{-j-1}}{(m_1 + m_2)^j}.
\]

(8)

If we are only interested in changes to the orbital elements that occur on timescales much longer than the orbital periods (so-called ‘secular’ changes), we must average the Hamiltonian over both mean anomalies. To do this while maintaining the canonical structure of the Hamiltonian requires a technique known as von Zeipel averaging. The general case for three massive bodies is quite complicated even at quadrupole order as one must be careful to include the longitudes of ascending nodes and eliminate them from the Hamiltonian. The resulting double-averaged Hamiltonian at quadrupole order in the test particle limit is

\[
H_n = C_2 \left[ (2 + 3e_1^2) (1 - 3\theta^2) - 15e_1^2 (1 - \theta^2) \cos 2\theta \right],
\]

(9)

where \( C_2 \) is a constant parameterizing the strength of the quadrupole term given by

\[
C_2 = \frac{G m_1 m_2 m_3}{16(m_1 + m_2)a_2(1 - e_2^2)^{3/2}} \alpha^2.
\]

(10)

The semi-major axes and \( e_2 \) do not change at quadrupole order.
order, so \( C_2 \) is also constant. We will henceforth refer to the dimensionless Hamiltonian,
\[
\hat{H}_q = \frac{H_q}{C_2}.
\]

### 2.3 Integrals of motion

There are no dissipative forces in the problem, so the total energy, \( \hat{H}_q \), remains constant. Moreover, because no energy is transferred between the two binaries, each term of the Hamiltonian is conserved separately, so \( \hat{H}_q \) remains constant as well.

The total angular momentum is also conserved and may be expressed in the form of the geometrical relation,
\[
\theta \equiv \cos i = \frac{G_{\text{tot}}^2 - G_1^2 - G_2^2}{2G_1G_2}.
\]

This relation is valid in the general case of three massive bodies. In the test particle limit the geometrical relation may be approximated by
\[
G_{\text{tot}} \simeq G_2 + G_1\theta.
\]

Since \( G_{\text{tot}} \) and \( G_2 \) are constant, we must have that \( G_1\theta \) is constant as well. Furthermore, \( G_1 = L_1\epsilon_1 \), and \( L_1 \) is also constant, so this requires that \( \epsilon_1\theta \) be constant as well. We note this constant of motion as
\[
\Theta \equiv (\epsilon_1\theta)^2.
\]

In this form, the constant of motion is known as ‘Kozai’s integral’ [Holman et al. 1993]. Kozai’s integral implies that the component of angular momentum perpendicular to the plane of the outer binary is conserved. However, the test particle assumption is crucial to its derivation. In the general case of three massive bodies this component of angular momentum is not conserved, although other, more complicated, integrals of motion may be derived from the geometrical relation instead (e.g., [Wen 2003]).

Because \( \hat{H}_q \) only depends on \( \epsilon_1 \), \( \theta \), and \( g_1 \), and there are two integrals of motion, \( \hat{H}_q \) and \( \Theta \), there is only one degree of freedom and so the system is integrable. Moreover, because these variables are all bounded, the motion is periodic (with the exception of a locus of stationary points of measure zero). The Hamiltonian to quadrupole order may be expressed as
\[
\hat{H}_q = \frac{1}{\epsilon_1^2} \left[ (5 - 3\epsilon_1^2)(\epsilon_1^2 - 3\Theta) - 15(1 - \epsilon_1^2)(\epsilon_1^2 - \Theta) \cos 2g_1 \right]
\]

in terms of \( \epsilon_1 \) and \( \Theta \).

### 2.4 Equations of motion

We are interested in the time evolution of the variables \( \epsilon_1 \), \( \theta \), and \( g_1 \). Of these, only \( g_1 \) is a canonical variable so its time evolution follows directly from Hamilton’s equations:
\[
\frac{dg_1}{dt} = \frac{\partial \hat{H}_q}{\partial \epsilon_1} = \frac{C_2}{L_1} \frac{\partial \hat{H}_q}{\partial \epsilon_1}
\]

Carrying out the differentiation of equation [15] we find
\[
\frac{dg_1}{dt} = \frac{6C_2}{L_1} \frac{1}{\epsilon_1} \left[ 5(\Theta - \epsilon_1^4) (1 - \cos 2g_1) + 4\epsilon_1^4 \right]
\]

The variable \( \epsilon_1 \) is related to a canonical variable, \( G_1 \), by a constant, so we find its time evolution to be
\[
\frac{d\epsilon_1}{dt} = \frac{1}{L_1} \frac{\partial \hat{H}_q}{\partial g_1} = \frac{C_2}{L_1} \frac{\partial \hat{H}_q}{\partial g_1}
\]

Again carrying out the differentiation of equation [15] we find
\[
\frac{d\epsilon_1}{dt} = \frac{30C_2}{L_1} \frac{1}{\epsilon_1^2} \left( 1 - \epsilon_1^2 \right) \left( \epsilon_1^2 - \Theta \right) \sin 2g_1.
\]

The time evolution of the inclination can be approximated by
\[
\Theta \equiv (\epsilon_1\theta)^2.
\]

### 2.5 Libration vs. rotation

During a KL oscillation, the argument of periapsis of the inner binary may either rotate or librate. This is to say, \( g_1 \) may sweep through the full range of angles from 0 to \( 2\pi \) (rotation) or it may be restricted to just a subset of them (libration). In the case of libration, the set of librating trajectories must librate about a fixed point of \( g_1 \) and \( \epsilon_1 \). Inspection of equation [19] reveals that \( \epsilon_1 \) is stationary only when \( g_1 \) takes half- or whole-integer multiples of \( \pi \) (recall that \( \Theta < \epsilon_1^2 \)).

Now, inspection of equation [17] reveals that \( \Theta \) cannot be stationary at integer multiples of \( \pi \). This implies that trajectories can only librate about half-integer multiples of \( \pi \), so \( g_1,\Theta = \pm \pi/2 \) and \( \epsilon_1^2 = \sqrt{5}/3 \).

To determine whether a particular system (i.e., a given \( \hat{H}_q \) and \( \Theta \)) librates or rotates we must see whether there exists a physical solution of equation [15] for \( \epsilon_1 \) when \( g_1 = 0 \). Setting \( g_1 = 0 \) in equation [15] and solving for \( \epsilon_1^2 \) we find
\[
\epsilon_1^2 = \frac{1}{12} (10 + \hat{H}_q + 6\Theta).
\]

The critical system on the boundary between libration and rotation will have a solution for \( \epsilon_1 \) exactly equal to unity and libration will occur if the only solution for \( \epsilon_1 \) exceeds unity. Defining the libration constant as
\[
C_{KL} \equiv \frac{1}{12} \left( 2 - \hat{H}_q - \Theta \right),
\]

we will have libration if \( C_{KL} < 0 \) and rotation if \( C_{KL} > 0 \). This constant was first presented in [Katz et al. 2011] and may be calculated equivalently by
\[
C_{KL} = \epsilon_1^2 \left( 1 - \frac{5}{2} \sin^2 i \sin^2 2g_1 \right).
\]

Note that the condition for rotation then becomes
\[
\sin g_1 \leq \sqrt{\frac{2}{5}} \frac{1}{\sin i}.
\]

Because \( C_{KL} \) naturally parameterizes a dynamical property of the triple, it is often convenient to work with it instead of \( \hat{H}_q \) where possible.

### 3 DERIVATION OF THE PERIOD OF KL OSCILLATIONS

Because the Hamiltonian at quadrupole order is integrable, we may use action angle variables to determine the period
of KL oscillations exactly. In general, the period, $P$, of a canonical variable $q$ is given by

$$\frac{1}{P} = \frac{\partial H}{\partial J},$$

(24)

where $J$ is the action variable defined by

$$J = \oint p \, dq,$$

(25)

with $p$ being the conjugate momentum of $q$. Switching the coordinates and the conjugate momenta is a valid canonical transformation, so we also have

$$J = \oint q \, dp.$$

(26)

In this problem the canonical variable is $g_1$ and the conjugate momentum is $G_1$, so we have

$$J = \oint g_1 \, dG_1.$$

(27)

In principle, we would have to carry out this integral analytically, solve the resulting equation for $H$ in terms of $J$, and then differentiate with respect to $J$ to determine the period. However, because our system is physical we may assume that it is mathematically well behaved enough that the period of a KL oscillation, $t_{KL}$, is given by

$$t_{KL} = \frac{1}{C_2} \oint \frac{\partial g_1}{\partial H_q} \, dG_1.$$

(28)

Furthermore, because $H_q$ is a constant parameter for any given system, we may differentiate under the integral, yielding

$$t_{KL} = \frac{L_1}{C_2} \oint \frac{\partial g_1}{\partial H_q} \, dG_1.$$

(29)

Factoring out the constant part of $G_1$, we have

$$t_{KL} = \frac{L_1}{C_2} \oint \frac{\partial g_1}{\partial H_q} \, d\theta.$$

(30)

We solve equation (15) for $g_1$ to find

$$g_1 = \frac{1}{2} \arccos \left( \frac{(3 - 3\xi^2)(3\xi^2 - 3\Theta) - H_q\xi^2}{15(1 - \xi^2)(1 - \Theta)} \right).$$

(31)

Differentiating and substituting into the integral, we have

$$t_{KL} = \frac{L_1}{30C_2} \oint \frac{\xi^2}{(1 - \xi^2)(1 - \Theta)} \times \left[ 1 - \left( \frac{3\xi^2 + \xi^2(9\Theta - 90 - 15\Theta)}{15(1 - \xi^2)(1 - \Theta)} \right)^2 \right] \, d\theta.$$

(32)

The integral equation (32) proceeds from the maximum value of $\xi$ to the minimum value of $\xi$ and back again to the maximum value of $\xi$, so we may instead integrate from $\xi_{\text{min}}$ to $\xi_{\text{max}}$ and multiply by two. Eliminating $H_q$ in favor

$^2$ Strictly speaking this transformation introduces a negative sign, but because the period is necessarily positive, we absorb the negative sign into the direction of the loop integral.

of $C_{KL}$ by making use of equation (21), and rearranging, we have

$$t_{KL} = \frac{L_1}{15C_2} \int_{\xi_{\text{min}}}^{\xi_{\text{max}}} \frac{1}{(1 - \xi^2)} \right] \times \left[ 1 - \left( \frac{(\Theta)^2}{\xi^2} - \left( \frac{1}{5} - \Theta \right) + \frac{4}{3} \frac{C_{KL}}{1 - \xi^2} \right) \right]^{-\frac{1}{2}} \, d\xi. $$

(33)

Eccentricity maxima ($\xi_{\text{max}}$) occur for $g_1 = \pm \pi/2$. Eccentricity minima ($\xi_{\text{min}}$) also occur at $g_1 = \pm \pi/2$ in the case of libration but occur at $g_1 = 0$ or $\pi$ in the case of rotation. We may therefore solve for $\xi_{\text{min}}$ and $\xi_{\text{max}}$ by substituting the appropriate values of $g_1$ into equation (15) and solving for $\xi$. We therefore have

$$\xi_{\text{min}} = \frac{1}{\sqrt{6}} \left( \zeta - \sqrt{\zeta^2 - 60\Theta} \right)$$

(34)

$$\xi_{\text{max}} = \frac{1}{\sqrt{6}} \left( \zeta + \sqrt{\zeta^2 - 60\Theta} \right), \quad C_{KL} < 0$$

(35)

$$\xi_{\text{max}} = \sqrt{1 - C_{KL}} \quad C_{KL} > 0.$$  

(36)

where we have defined

$$\zeta = 3 + 5\Theta + 2C_{KL}.$$  

(37)

For convenience, we define the integral in equation (33) to be $f(C_{KL}, \Theta)$ such that

$$f(C_{KL}, \Theta) = \frac{15t_{KL}C_2}{L_1}.$$  

(38)

Having calculated the limits of integration, we can now use equation (33) to calculate the period of KL oscillations to quadrupole order in the test particle limit for any hierarchical triple.

**4 A BRIEF SURVEY OF PARAMETER SPACE**

We now turn to a brief exploration of the range of values that the integral in equation (33) may take. The overall timescale for KL oscillations is determined by the coefficient before the integral, which we present in more detail in Section 5.1. The integral, however, depends on only two parameters describing the triple: $H_q$ and $\Theta$, or equivalently, $C_{KL}$ and $\Theta$. Thus, once the timescale of KL oscillations has been set, only two degrees of freedom remain.

What values may $H_q$, $C_{KL}$, and $\Theta$ take? It is easy to see from equation (14) that $0 \leq \Theta \leq 1$ since both $\epsilon$ and $\cos i$ are bounded by 0 and 1. Moreover, it is clear from equation (22) that

$$-\frac{3}{2} \leq C_{KL} \leq 1$$

(39)

since all the terms are bounded by 0 and 1. From the bounds on $\Theta$ and $C_{KL}$, we can conclude from equation (21) that the bounds on $H_q$ are

$$-10 \leq H_q \leq 20.$$  

(40)

However, the limits on $H_q$ and $\Theta$ are not independent. In the case of $g_1 = 0$, the requirement that $\Theta \leq \epsilon$ implies that

$$-10 + 6\Theta \leq H_q \leq 20.$$  

(41)
This, in turn, translates to the requirement in $C_{KL}$ that
\[ C_{KL} \leq 1 - \Theta. \] (42)

In order for equation (33) to have a solution, the square roots in equations (34), (35), and (36) must exist. The existence of the inner square root in equation (34) requires
\[ C_{KL} \geq -\frac{1}{2} \left( 5\Theta - 2\sqrt{15\Theta} + 3 \right). \] (43)

This requirement is always satisfied in the case of rotation ($C_{KL} > 0$). In the case of libration ($C_{KL} < 0$), this requirement may instead be written in terms of $C_{KL}$ as
\[ \Theta \leq \frac{1}{5} \left( 3 - 2\sqrt{-6C_{KL} - 2C_{KL}} \right), \quad C_{KL} \leq 0. \] (44)

In the case of libration, the square root in equation (35) exists everywhere that the square root in equation (34) does, so the existence of this square root adds no new constraints. In the case of rotation, the requirement that the square root in equation (36) exist is satisfied by the same condition set in equation (42).

Equation (44) implies that there is a critical inclination, below which librating KL oscillations do not occur. Taking $C_{KL} = 0$ we recover the usual critical inclination of $\Theta \geq \sqrt{3/5}$. Although we do not provide an explicit derivation, we note that the criterion in equation (44) can also be arrived at by requiring that
\[ \frac{d^2\epsilon_1}{dt^2} < 0 \] (45)
when $\epsilon_1$ is at a maximum. In other words, KL oscillations occur when the minimum eccentricity is an unstable equilibrium.

Knowing now the region of parameter space in which KL oscillations occur, we can numerically integrate the integral in equation (33) over the entire parameter space. The results of this procedure are presented in Figure 1. Except for a narrow strip of parameter space centered around the boundary between rotation and libration ($C_{KL} = 0$) the integral only varies by a factor of a few. Near the rotation-libration boundary the integral diverges and KL oscillations have arbitrarily large periods. Figure 1 also indicates that the period of KL oscillations depends most strongly on $C_{KL}$ and only weakly on $\Theta$.

5 APPROXIMATIONS

5.1 The timescale of KL oscillations

So long as the integral in equation (33) is of order unity, the period of KL oscillations will be given by the coefficient before the integral to within an order of magnitude:
\[ t_{KL} \simeq \frac{L_1}{15C_{KL}^2}. \] (46)

Substituting equations (1) and (10) and noting that we are working in the test particle limit so $m_2 \to 0$, we have the timescale in terms of the semi-major axes, masses, and eccentricities:
\[ t_{KL} \simeq \frac{16}{15} \left( \frac{a_1^3}{a_1^{3/2}} \right) \sqrt{\frac{m_1}{Gm_3}} \left( 1 - e_2^2 \right)^{3/2}. \] (47)

This timescale may be expressed more elegantly in terms of the periods of the inner and outer orbits, $P_{in}$ and $P_{out}$, respectively, by making use of Kepler’s law:
\[ t_{KL} \simeq \frac{8}{15\pi} \left( 1 + \frac{m_1}{m_3} \right) \frac{P_{out}^2}{P_{in}} (1 - e_2^2)^{3/2}. \] (48)

This is the form of the KL period that typically appears in the literature, but with an additional mass term and numerical coefficient. The mass term implies that KL oscillations lengthen indefinitely as the tertiary approaches zero mass. In the case of a massive tertiary but a test particle primary and secondary (e.g., a WD-WD binary orbiting a SMBH), the period of KL oscillations approaches a constant value. Note that in the case of an equal mass primary and tertiary, neglecting the numerical coefficient will lead to an overestimate of the period of KL oscillations by a factor of nearly three.

5.2 High inclination, low eccentricity triples

In most cases of interest in astronomy, the inner binary of a hierarchical triple starts with a low to moderate eccentricity. Moreover, KL oscillations are strongest (and therefore most interesting) when the tertiary is at high inclination. It is therefore worth finding an approximation to $t_{KL}$ in the high inclination, low initial eccentricity limit. In this limit, we have $\Theta \to 0$ and $C_{KL} \to 0$ and equation (33) may be solved exactly:
\[ f(C_{KL}, \Theta) \simeq \frac{5}{4\sqrt{6}} \ln \left( \frac{1 + \epsilon_1}{1 - \epsilon_1} \right)^{\epsilon_{max}/\epsilon_{min}}. \] (49)
We also have in this limit that $\epsilon_{\text{min}} \ll 1$ and $1 - \epsilon_{\text{max}} \ll 1$ so that

$$f(C_{\text{KL}}, \Theta) \approx \frac{5}{4\sqrt{6}} \ln \left( \frac{2}{1 - \epsilon_{\text{max}}} \right).$$

where

$$\epsilon_{\text{max}} \approx 1 + \frac{C_{\text{KL}}}{3}$$

(51)

for libration ($C_{\text{KL}} < 0$), and

$$\epsilon_{\text{max}} \approx 1 - \frac{C_{\text{KL}}}{2},$$

(52)

for rotation ($C_{\text{KL}} > 0$).

The dependence of $f(C_{\text{KL}}, \Theta)$ on $\Theta$ is non-trivial to approximate from first principles. After experimenting with several functional forms, we found that $f(C_{\text{KL}}, \Theta)$ varies most closely with $(1 - \Theta)$. If there is a $\Theta$ dependence both inside and outside the logarithm, we then expect $f(C_{\text{KL}}, \Theta)$ to take the form

$$f(C_{\text{KL}}, \Theta) \approx \frac{5}{4\sqrt{6}} \ln \left( \frac{a(1 - \Theta)^{b}}{C_{\text{KL}}} \right)(1 - \Theta)^{c},$$

(53)

where $a$, $b$, and $c$ are fitting parameters. We fit numerical integrations of $f(C_{\text{KL}}, \Theta)$ to this form over the range $0 < \Theta < 0.25, -0.1 < C_{\text{KL}} < 0.1$ and find the remarkably good fit,

$$t_{\text{KL}} \approx \frac{1}{3\sqrt{2}} \left[ 3 + \frac{m_{1}}{m_{3}} \right] \frac{P_{\text{out}}^{2}}{P_{\text{in}}} \left( 1 - \epsilon_{2}^{2} \right)^{3/2} \times \ln \left( \frac{9.42(1 - \Theta)^{2.36}}{C_{\text{KL}}} \right)(1 - \Theta)^{-1.53}.$$

(54)

We attempted to add several auxiliary parameters but found that they did not substantially improve the fit.

The approximation provided in equation (51) fits the true value of $f(C_{\text{KL}}, \Theta)$ to within 2% over the range sampled, and over the vast majority of the range sampled the residuals are less than 0.3%. This is therefore an appropriate formula to use for triples in which the inner binary has an eccentricity $\epsilon_{1} \leq 0.3$ and an inclination $60^\circ \geq i \geq 120^\circ$. (Note that the triple need only have high inclination and low eccentricity at some point in the KL cycle for the approximation to be valid.) Contours of the residuals of equation (51) are shown in Figure 2.

6 THE ECCENTRIC KL MECHANISM

If the two masses of the inner binary are not equal and the outer orbit has non-zero eccentricity, the next term in the expansion of the Hamiltonian, the octupole order term, becomes dynamically significant. This term leads to changes to the orbital parameters of the outer orbit that are slow relative to individual KL oscillations. These long-term changes can cause the inner orbit to eventually pass through an inclination of $90^\circ$. During these orbital flips, the very large inclination leads to strong KL oscillations which drive the inner binary to extremely large eccentricities. For this reason, the dynamical effect of the octupole term has been called the ‘eccentric KL mechanism’ (EKM) (e.g., Lithwick & Naoz [2011]).

The introduction of the octupole term breaks the integrability of the Hamiltonian. Consequently, neither $C_{\text{KL}}$ or $\Theta$ remain constants of the motion. Furthermore, in the test particle limit at quadrupole order it is possible to eliminate the longitude of ascending node, $\Omega$, from the Hamiltonian. At octupole order either this parameter or $g_{2}$ necessarily enters into the equations of motion. In this work we follow the analysis of Katz et al. (2011) and work in terms of the longitude of ascending node of the eccentricity vector, $\Omega_{e}$, defined such that $e = e(\sin i_{e}, \cos \Omega_{e}, \sin i_{c}, \cos i_{c})$, and $e$ points toward periapsis of the inner binary.

In the case of rotation ($C_{\text{KL}} > 0$), the parameters $\Omega_{e}$, $C_{\text{KL}}$, and $\Theta$ all change on a timescale which is long compared to individual KL cycles. It is therefore possible to assume that $\Omega_{e}$, $C_{\text{KL}}$, and $\Theta$ are all approximately constant over individual KL oscillations and only examine the long-term changes to these parameters. In this approximation the system remains integrable with new integrals of motion. Due to the integrability of the system the variations in $C_{\text{KL}}$, $\Theta$, and $\Omega_{e}$ are all strictly periodic. In this section we derive the period of these EKM oscillations.

6.1 Equations of motion and integrals of motion

Since energy is conserved, the quadrupole order term of the Hamiltonian, $H_{q}$, is also conserved in the time-averaged behavior of the system. This implies that the relationship between $C_{\text{KL}}$, $\Theta$, and $H_{q}$ in equation (21) remains valid and that the quantity

$$\phi_{q} \equiv C_{\text{KL}} + \frac{1}{2} \Theta$$

(55)

is a constant of motion.

It is convenient to work with the parameter $\epsilon_{\text{oct}}$, which measures the relative size of the octupole order term of the
Hamiltonian to the quadrupole order term. The parameter $\epsilon_{\text{oct}}$ is conventionally defined as

$$\epsilon_{\text{oct}} \equiv \frac{e_2}{1 - e_2^2} \frac{a_1}{a_2}.$$  (56)

Some authors have added a mass term (e.g., Naoz et al. 2013b) to capture the fact that the octupole term is zero and EKM oscillations do not occur for an equal mass inner binary. However, because we are working exclusively in the test particle limit we do not do so here.

Following Katz et al. (2011), the long-term evolution in $\Omega_e$ and $\Theta$ are given by

$$\frac{d\Omega_e}{d\tau} = -\Theta \left( \frac{6E(x) - 3K(x)}{4K(x)} \right),$$  (57)

$$\frac{d\Theta}{d\tau} = -\frac{15\pi \epsilon_{\text{oct}}}{64\sqrt{10}} \left[ \frac{11C_{\text{KL}} - \sqrt{6 + 4C_{\text{KL}}}}{K(x)} \right] \sin \Omega_e,$$  (58)

where $K(x)$ and $E(x)$ are complete elliptic functions of the first and second kind, respectively,

$$x(C_{\text{KL}}) = \frac{3(1 - C_{\text{KL}})}{3 + 2C_{\text{KL}}},$$  (59)

and the time coordinate has been scaled to the KL period during a flip:

$$\tau = \frac{t}{t_{\text{KL}} = 90}. $$  (60)

Katz et al. (2011) also derive another integral of motion,

$$\chi \equiv F(C_{\text{KL}}) - \epsilon_{\text{oct}} \cos \Omega_e,$$  (61)

where the function $F(C_{\text{KL}})$ is defined to be

$$F(C_{\text{KL}}) = \frac{32\sqrt{2}}{\pi} \frac{1}{x(C_{\text{KL}})} \left[ K(\eta) - 2E(\eta) \right] \left( \frac{41\eta - 21}{\sqrt{2} \eta + 3} \right)^{-1},$$  (62)

Although there are two integrals of motion, $\phi_q$ and $\chi$, they are not sufficient to completely describe the dynamical behavior of the triple. This is because $\epsilon_{\text{oct}}$ carries dynamical information as well, most importantly whether or not flips are possible. The dynamical significance of $\epsilon_{\text{oct}}$ can be seen from the fact that $\epsilon_{\text{oct}}$ enters into the definition of $\chi$. Thus, in the octupole case there are three independent parameters describing the system as opposed to the case of quadrupole-order KL oscillations in which there are only two.

### 6.2 The period of EKM oscillations

In the case of EKM oscillations it is easier to derive their period directly from the equations of motion rather than from action angle variables. We have from equation (58) that

$$\frac{dC_{\text{KL}}}{dt} = \frac{1}{2} \frac{d\Theta}{d\tau},$$  (63)

so the period may be written

$$\tau_{\text{EKM}} = \int \frac{d\tau}{dC_{\text{KL}}} \left( \frac{dC_{\text{KL}}}{d\tau} \right).$$  (64)

Substituting equation (58) we find

$$\tau_{\text{EKM}} = \int \frac{128\sqrt{10}}{15\pi \epsilon_{\text{oct}}} \frac{K(x)}{\sqrt{2(\phi_q - C_{\text{KL}}) \sin \Omega_e}} \frac{1}{(4 - 11C_{\text{KL}})^2 + 4C_{\text{KL}}^2} dC_{\text{KL}}.$$  (65)

To write $\Omega_e$ in terms of $C_{\text{KL}}$, we note that equation (61) implies that

$$\sin \Omega_e = \sqrt{1 - \left( \frac{\chi - F(C_{\text{KL}})}{\epsilon_{\text{oct}}} \right)^2}.$$  (66)

Substituting equation (60) into equation (65) and explicitly writing the limits of the integral yields

$$\tau_{\text{EKM}} = \frac{256\sqrt{10}}{15\pi \epsilon_{\text{oct}}} \int_{C_{\text{KL}}}^{C_{\text{KL,max}}} \frac{K(x)}{\sqrt{2(\phi_q - C_{\text{KL}})(4 - 11C_{\text{KL}})}} \times \left[ 1 - \left( \frac{\chi - F(C_{\text{KL}})}{\epsilon_{\text{oct}}} \right)^2 \right]^{-\frac{1}{2}} dC_{\text{KL}}.$$  (67)

The upper limit of the integral can be deduced by noting that $C_{\text{KL}}$ is maximized when $\Theta$ is minimized and that $\Theta = 0$ during a flip. We therefore have

$$C_{\text{KL,max}} = \phi_q.$$  (68)

The lower limit is more subtle. It is clear from equation (61) that $\Theta$ is maximized when $\sin \Omega_e = 0$. This implies from equation (61) that

$$F(C_{\text{KL,min}}) = \chi \pm \epsilon_{\text{oct}}.$$  (69)

To decide whether to take the plus or minus sign, we must solve both for $C_{\text{KL}}$ and then take the value of $\phi_q$.

$$F(C_{\text{KL,min}}) = \chi \pm \epsilon_{\text{oct}}.$$  (70)

As in the quadrupole case we first explore over what region of parameter space EKM oscillations with flips may occur. We then determine the variation in $\epsilon_{\text{oct}}$ over this range of parameter space. Unfortunately, the parameter space cannot be mapped quite as easily as in the case of quadrupole KL oscillations because there are now three parameters describing the system instead of two: $\phi_q$, $\chi$, and $\epsilon_{\text{oct}}$. As such, we explore parameter space for two choices of $\epsilon_{\text{oct}}$: $\epsilon_{\text{oct}} = 10^{-3}$ and $\epsilon_{\text{oct}} = 10^{-2}$. Strong octupole-order effects occur in many triple systems with $\epsilon_{\text{oct}} = 10^{-2}$, but these effects are much weaker for most triples when $\epsilon_{\text{oct}} = 10^{-3}$ (e.g., Lithwick & Naoz 2011).

To determine the boundaries of the parameter space of spin flips we first recall that $0 \leq \Theta \leq 1$, and for rotation $0 \leq C_{\text{KL}} \leq 1$ (which is the only case we are considering to octupole order). The occurrence of a spin flip is equivalent to having $\Theta = 0$, and hence during a flip $C_{\text{KL}} = \phi_q$. Since $\cos \Omega_e$ is bounded by $\pm 1$, we then have the following constraint:

$$F(\phi_q) - \epsilon_{\text{oct}} \leq \chi \leq F(\phi_q) + \epsilon_{\text{oct}}.$$  (70)

The parameter space can be divided into two regions based on the maximum of the function $F(\phi_q)$. This maximum can be found by solving $K(x_{\text{crit}}) = 2E(x_{\text{crit}})$ for $x_{\text{crit}}$, which yields $x_{\text{crit}} \approx 0.826$, and then calculating

$$\phi_q_{\text{crit}} = \frac{3(1 - x_{\text{crit}})}{3 + 2x_{\text{crit}}} \approx 0.112.$$  (71)
Now, φ_q cannot be arbitrarily large because F(φ_q) diverges at φ_q = 4/11. Thus we have

$$\phi_q < \frac{4}{11}. \quad (72)$$

Since, for φ_q < φ_q,crit, F(φ_q) cannot be less than zero, this then implies a constraint on χ:

$$\chi > \epsilon_{oct} \quad (\phi_q < \phi_q,crit). \quad (73)$$

Finally, the above relation implies that

$$F(\phi_{q,min}) = \epsilon_{oct}. \quad (74)$$

Taken together, these relations bound the parameter space over which flips are possible. The resulting maps of parameter space for ε_{oct} = 10^{-3} and ε_{oct} = 10^{-2} are shown in Figure 3. Because the parameter space over which flips occur is somewhat narrow and the dependence of τ_{EKM} on φ_q is fairly complicated we do not show contours as we did at quadrupole order in Fig. 1. Instead, we show τ_{EKM} as a function of φ_q with the choice of χ = F(φ_q) and χ = F(φ_q) ± ε_{oct}/2 in Fig. 4. The timescale for EKM oscillations depends most sensitively on φ_q. The timescale has two singularities: one at the maximum value of φ_q of 4/11, and another which is dependent on the choice of χ, but is near φ_q,crit. Except very close to these singularities, the period of EKM oscillations does not vary by more than a factor of a few. Thus, over a broad range of parameter space EKM oscillations have similar timescales.

### 6.4 The dependence on ε_{oct}

If the constants φ_q and χ are held fixed and ε_{oct} is varied, how does the period of EKM oscillations vary? Equation (67) exhibits a ε_{oct}^{-1} dependence in the coefficient before the integral, so it is tempting to conclude that the timescale for EKM oscillations scales as ε_{oct}^{-1}. This conclusion has been asserted in several studies in the literature, but we show here that it is incorrect. The integral in equation (67) in fact exhibits a ε_{oct}^{-1/2} dependence.

To determine this dependence we first note that for EKM oscillations to occur, in general C_{KL} \ll 1. This then implies that x is very close to unity; so we may write x = 1 - ε, where ε \ll 1. For values of x very close to unity, the complete elliptic integral of the first kind may be approximated

$$K(1 - \varepsilon) \simeq -\frac{1}{2} \ln \varepsilon, \quad (75)$$

and the complete elliptic integral of the second kind is approximated by E(1 - ε) \simeq 1. We note that the coefficient in equation (75) is off by several tens of percent for realistic values of ε, but the important feature of this approximation is that it carries the correct dependence on ε. The function F(C_{KL}) can then be approximated as

$$F(C_{KL}) \simeq -\frac{8}{5\pi} \frac{3}{5} \int_0^\varepsilon \left( \frac{1}{2} \ln \left( \varepsilon' \right) + 2 \right) d\varepsilon'. \quad (76)$$

Now, because we are integrating over a small range, the integral can then be approximated as

$$F(C_{KL}) \simeq -\frac{8}{5\pi} \frac{3}{5} \left( \frac{1}{2} \ln \left( \frac{\varepsilon'}{2} \right) + 2 \right) \varepsilon \quad (77)$$

Furthermore, we note that

$$\varepsilon \simeq \frac{2}{3} C_{KL} \quad (78)$$

so we finally have

$$F(C_{KL}) \simeq -\frac{16}{15\pi} \sqrt{\frac{3}{5}} C_{KL} \left( \frac{1}{2} \ln \left( \frac{C_{KL}}{3} \right) + 2 \right). \quad (79)$$

Let us now consider the lower limit of the integral in equation (67). For simplicity, let us for the time being restrict ourselves to the locus χ = F(φ_q) since here flips occur for arbitrarily small values of ε_{oct}. We then have

$$F(C_{KL,min}) = F(\phi_q) - \epsilon_{oct}. \quad (80)$$

Now, the approximation in equation (79) may be written more simply as F(C_{KL}) \sim kC_{KL}, where k is a parameter that has only a sub-linear dependence on C_{KL}. For small C_{KL}, then, the function F is nearly linear in C_{KL}. This then implies that for points on the locus we are considering

$$\phi_q - C_{KL,min} \sim \frac{\epsilon_{oct}}{k}. \quad (81)$$

This then means that the width over which we are integrating, ΔC_{KL} is proportional to ε_{oct} since

$$\Delta C_{KL} \equiv C_{KL,max} - C_{KL,min} = \phi_q - C_{KL,min} \sim \epsilon_{oct}. \quad (82)$$

Let us now consider the various terms of the integrand of equation (67). We have already seen that because x is close to unity, K(x) \sim \ln(C_{KL}/3). This term is sublinear so we ignore it. The (4 - 11C_{KL}) term reduces to 4, and similarly the \sqrt{1 + 4C_{KL}} term reduces to \sqrt{6}. The term \sqrt{2(\phi_q - C_{KL})} reduces by equation (81) to \sim \sqrt{2\epsilon_{oct}}. This leaves only the sinΩ term. Now, if \phi_q - C_{KL} \sim \epsilon_{oct} and F is approximately linear in this limit, it must be the case that

$$F(\phi_q) - C_{KL} \equiv \chi - C_{KL} \sim \epsilon_{oct}. \quad (83)$$

Comparing this to equation (67), we find that to lowest order, sinΩ does not exhibit any dependence on ε_{oct}. It is straightforward to verify this claim numerically.

Putting these results together, we find that the only dependencies on ε_{oct} in the integral come from the width of integration (which yields a dependence of ε_{oct}), and from the term 1/\sqrt{2(\phi_q - C_{KL})} (which yields a dependence of ε_{oct}^{1/2}). Since the integral has a coefficient of ε_{oct}^{-1}, this then implies that the overall dependence of the period of the EKM is

$$\tau_{EKM} \sim \frac{1}{\sqrt{\epsilon_{oct}}}. \quad (84)$$

We demonstrate this dependence explicitly in Fig. 5 by numerically calculating the period using equation (67) for fixed values of φ_q and χ but over a range of ε_{oct}. We have compared these values with the periods obtained by integrating the secular equations of motion directly and find excellent agreement.

By combining this result with the numerical coefficient of equation (67) we find that

$$t_{EKM} \sim \frac{256\sqrt{10}}{15\pi \sqrt{\epsilon_{oct}}} t_{KL,i=90^\circ}. \quad (85)$$

During a single EKM cycle the inner binary will undergo two flips, so the flip timescale is half this value. The flip timescale can then be obtained by substituting for t_{KL,i=90^\circ}, taking
Figure 3. Parameter space where EKM oscillations with flips are possible for two choices of $\epsilon_{\text{oct}}$. We only explore the parameter space where individual KL cycles are rotating instead of librating (i.e., $C_{\text{KL}} > 0$), as librating cycles cannot be correctly analyzed using this technique of averaging over individual KL oscillations. At smaller values of $\epsilon_{\text{oct}}$ the area of parameter space where rotating flips are possible shrinks about the line $\chi = F(\phi)$.

Figure 4. The period of EKM oscillations with flips as a function of $\phi_q$ for three choices of $\chi$. The solid line is given by the choice $\chi = F(\phi_q)$, the dashed line by $\chi = F(\phi_q) - \epsilon_{\text{oct}}/2$, and the dotted line by $\chi = F(\phi_q) + \epsilon_{\text{oct}}/2$. Except very near the two singularities, the period of EKM oscillations does not vary by more than a factor of a few. Over a broad range of parameter space EKM oscillations have similar timescales.

Over most of parameter space this expression is valid to within a factor of a few. For extremely large values of $\epsilon_{\text{oct}}$ ($\epsilon_{\text{oct}} \sim 0.1$) our numerical experiments demonstrate that the dependence of $t_{\text{EKM}}$ on $\epsilon_{\text{oct}}$ steepens and this expression overpredicts the timescale for flips, but in this limit non-secular effects become important, so the above analysis does not apply (e.g., Katz & Dong 2012; Seto 2013; Antonini et al. 2014; Bode & Wegg 2014; Antognini et al. 2014).
not occur, the timescale for EKM oscillations does not vary
that apart from near two singularities where spin flips do
rameter space over which spin flips occur (Fig. 3) and show
KL cycles to calculate the period of EKM oscillations, and
(in equation 54).

between $t$ and $\chi$.

rotation and libration ($\epsilon$).

In this regime the timescale for flips steepens as a function o f
value of $\epsilon$. Note that $\epsilon < 1$.

The period of the EKM relative to the period of KL oscil-
$\epsilon^{-1/2}$ and there is excellent agreement between the
and analytic calculations. We show this relationship for an arbitrary choice of $\phi_0 = 0.015$ and two choices of $\chi = F(\phi_0)$
black line and points), and $\chi = F(\phi_0) + 9 \times 10^{-4}$ (gray dotted
line and points). For $\chi = F(\phi_0)$ flips are possible at arbitrarily
small values of $\epsilon$, whereas for $\chi = F(\phi_0) + 9 \times 10^{-4}$ flips are
only possible for values of $\epsilon > 9 \times 10^{-4}$. The relationship
between $t_{\text{EKM}}$ and $\epsilon$ becomes slightly shallower near this critical
value of $\epsilon$. Note that $\epsilon$ cannot exceed $\chi$. Although flips oc-
cur at larger values of $\epsilon$, the evolution is no longer integrable
because the inner binary switches between rotation and libration.
In this regime the timescale for flips steepens as a function of $\epsilon$, although there is no longer a simple relationship between the two
because the evolution becomes essentially chaotic.

7 CONCLUSIONS

Using action angle variables we have derived the period of KL oscillations at quadrupole order and in the test particle
limit (equation [33]). From the exact period we have derived the timescale for KL oscillations. We have explored the full
range of parameter space over which KL oscillations are possible and found that except very near the boundary between
rotation and libration ($|C_{\text{KL}}| \ll 1$) the period of KL oscillations
does not vary by more than a factor of a few from the derived timescale (Fig. 1). By employing several approxima-
tions in the high-inclination, low eccentricity limit we have found a function that matches the true KL period
within 2% for triples for which $e_1 \lesssim 0.3$ and $i \gtrsim 60^\circ$
(equation [33]).

The strength of KL oscillations varies due to the oct-
tuple term of the Hamiltonian. We average over individual
KL cycles to calculate the period of EKM oscillations, and
hence, the timescale for spin flips to occur. We map the pa-
rameter space over which spin flips occur (Fig. 3) and show
that apart from near two singularities where spin flips do
not occur, the timescale for EKM oscillations does not vary
by more than a factor of a few (Fig. 4). Finally, we show numerically and analytically that the dependence of $\epsilon_{\text{oct}}$
on the timescale for EKM oscillations is $\epsilon_{\text{oct}}^{-1/2}$ (Fig. 5) in con-
trast to previous studies. We provide the EKM timescale in
equation (35) and the timescale for flips in equation (36).

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Figure 5. The period of the EKM relative to the period of KL
oscillations as a function of $\epsilon_{\text{oct}}$ calculated analytically using equation [34] (lines) and by integrating the secular equations of motion (points). The timescale for the EKM is almost exactly pro-
portional to $\epsilon_{\text{oct}}^{-1/2}$ and there is excellent agreement between the
secular and analytic calculations. We show this relationship for an arbitrary choice of $\phi_0 = 0.015$ and two choices of $\chi = F(\phi_0)$
(black line and points), and $\chi = F(\phi_0) + 9 \times 10^{-4}$ (gray dotted
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