Hyperbolic Manifolds, Harmonic Forms, and Seiberg-Witten Invariants

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Abstract

New estimates are derived concerning the behavior of self-dual harmonic 2-forms on a compact Riemannian 4-manifold with non-trivial Seiberg-Witten invariants. Applications include a vanishing theorem for certain Seiberg-Witten invariants on compact 4-manifolds of constant negative sectional curvature.

1 Introduction

Seiberg-Witten theory gives rise to a powerful interplay between the geometry and topology of smooth 4-manifolds. For example, a remarkable theorem of Taubes [12] asserts that any symplectic 4-manifold with $b^+ \geq 2$ has a non-zero Seiberg-Witten invariant. On the other hand, if a 4-manifold with $b^+ \geq 2$ admits a metric of positive scalar curvature, Witten [14] observed that its Seiberg-Witten invariants must all vanish. The existence of a metric satisfying a suitable curvature condition may thus be sufficient to rule out the existence of a symplectic structure on a given smooth, compact 4-manifold.

Despite this, there are many “naturally occurring” classes of 4-manifolds for which we do not yet know whether the Seiberg-Witten invariants all vanish. In particular, the following still appears to be open:

**Conjecture 1.1** Let $M^4 = \mathcal{H}^4/\Gamma$ be a compact hyperbolic 4-manifold. Then all the Seiberg-Witten invariants of $M$ vanish. In particular, $M$ does not admit symplectic structures.

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By contrast, the complex-hyperbolic 4-manifolds, which by definition are compact quotients of the unit ball in $\mathbb{C}^2$, are all Kähler manifolds, and so carry symplectic structures compatible with their standard orientations. One might expect, however, for the situation to be quite different regarding the non-standard orientation of such a manifold:

**Conjecture 1.2** Let $M^4 = \overline{\mathcal{CH}_2}/\Gamma$ be a reverse-oriented compact complex-hyperbolic 4-manifold. Then all the Seiberg-Witten invariants of $M$ vanish for the fixed orientation. In particular, there is no symplectic structure on $M$ compatible with the non-complex orientation.

In fact, both of these speculations may be interpreted as special cases of a more general conjecture. Recall that the 2-forms on an oriented 4-manifold decompose as

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-,$$

where $\Lambda^\pm$ is the $(\pm 1)$ eigenspace of Hodge star operator $\star$. Thinking of the curvature tensor $\mathcal{R}$ as a linear map $\Lambda^2 \to \Lambda^2$, we thus get a decomposition

$$\mathcal{R} = \begin{pmatrix}
W_+ + \frac{\mathbf{s}}{12} & \mathbf{r} \\
\mathbf{r}^* & W_- + \frac{\mathbf{s}}{12}
\end{pmatrix}$$

into irreducible pieces. Here the self-dual and anti-self-dual Weyl curvatures $W_\pm$ are the trace-free pieces of the appropriate blocks. The scalar curvature $s$ is understood to act by scalar multiplication, whereas the trace-free part $\mathbf{r} = r - \frac{s}{4}g$ of the Ricci curvature acts on 2-forms by

$$\psi_{ab} \mapsto \mathbf{r}_{ac} \psi^c_b - \mathbf{r}_{bc} \psi^c_a.$$

An oriented 4-manifold is said to be self-dual if $W_- \equiv 0$, or anti-self-dual if $W_+ \equiv 0$. An orientable real-hyperbolic 4-manifold is both self-dual and anti-self-dual, which is just another way of saying that any such manifold is locally conformally flat. On the other hand, a complex-hyperbolic 4-manifold is self-dual with respect to the orientation determined by the complex structure; thus, since reversing the orientation of a 4-manifold interchanges $W_+$ and
$W_\omega$, a complex-hyperbolic 4-manifold is anti-self-dual with respect to its non-complex orientation. Also notice that both real- and complex-hyperbolic 4-manifolds are Einstein --- i.e. they satisfy $\ddot{r} \equiv 0$. Thus the above conjectures might be viewed as simply special cases of the following:

**Conjecture 1.3** Let $(M^4, g)$ be a compact anti-self-dual Einstein manifold with negative scalar curvature. Then all the Seiberg-Witten invariants vanish for the fixed orientation of $M$.

This paper will present some tantalizing, albeit inconclusive, evidence in favor of these conjectures. To this end, let us first draw the reader’s attention to a beautiful recent result of Armstrong [1] asserting that anti-self-dual Einstein spaces of negative scalar curvature never admit non-trivial self-dual harmonic 2-forms of constant length. On the other hand, this article will prove that if such a space has a non-trivial Seiberg-Witten invariant, it necessarily admits a self-dual harmonic 2-form whose length is “nearly constant,” by two different quantitative measures. A quantitative sharpening of Armstrong’s result might therefore provide exactly the tool needed to prove some version of the above conjectures.

## 2 Harmonic 2-Forms

In this section, we introduce two different invariants which offer quantitative obstructions to the existence of non-trivial self-dual harmonic 2-forms of constant length on a given 4-manifold.

Our first invariant is simply the minimal angle between the point-wise norm of the form and the constant 1, considered as vectors in the Hilbert space $L^2$:

**Definition 2.1** Let $(M, g)$ be a compact, oriented Riemannian 4-manifold with $b^+(M) \geq 1$. Let $H_g^+ = \{ \phi \in \mathcal{E}^2(M) \mid \phi = \ast \phi, \ d\phi = 0 \}$ be the space of self-dual harmonic 2-forms on $(M, g)$, so that $\dim H_g^+ = b^+(M) > 0$. We define

$$\theta(M, g) = \min_{\phi \in (H_g^+ - 0)} \cos^{-1} \left( \frac{\int_M |\phi| d\mu_g}{V^{1/2} \left( \int_M |\phi|^2 d\mu \right)^{1/2}} \right)$$
where \( V = \int_M 1 \, d\mu \) is the total volume of \((M, g)\).

Our second invariant is rather more subtle, and is best understood in the context of the following observation:

**Proposition 2.2** Let \((M, g)\) be any compact oriented Riemannian 4-manifold, and let \( \phi \) be any self-dual harmonic 2-form. Then the function \( f = \sqrt{|\phi|} \) belongs to the Sobolev space \( L^2 \), and satisfies

\[
\int_M |df|^2 d\mu \leq \int_M \left( \sqrt{\frac{2}{3}} |W_+| - \frac{s}{6} \right) |\phi| \, d\mu,
\]

(1)

**Proof.** Since \( \phi \) is smooth by elliptic regularity, \( f \) is certainly continuous, and thus belongs to \( L^2 \). We therefore just need to show is that \( |df| \) belongs to \( L^2 \), and satisfies (1).

We may assume henceforth that \( \phi \not\equiv 0 \), since otherwise there is nothing to prove. It then follows that the nodal set where \( \phi \) vanishes is of measure zero; indeed \[^4\], its Hausdorff dimension is \( \leq 2 \). The function \( |df|^2 \) is smooth outside this nodal set, and our objective is just to show that its integral over the complement of the nodal set is finite.

To this end, recall that the harmonicity of \( \phi \) implies the Weitzenböck formula \[^4\]

\[
0 = \frac{1}{2} \Delta|\phi|^2 + |\nabla \phi|^2 - 2W_+(\phi, \phi) + \frac{s}{3} |\phi|^2
\]

\[
\geq \frac{1}{2} \Delta|\phi|^2 + |\nabla \phi|^2 + \left( \frac{s}{3} - 2\sqrt{\frac{2}{3}} |W_+| \right) |\phi|^2
\]

On the open set defined by \( \phi \not\equiv 0 \), we therefore have

\[
0 \geq \frac{\Delta|\phi|^2}{2|\phi|} + \frac{|\nabla \phi|^2}{|\phi|} + \left( \frac{s}{3} - 2\sqrt{\frac{2}{3}} |W_+| \right) |\phi|
\]

\[
= \Delta|\phi| - \left( \frac{d|\phi|}{|\phi|} \right)^2 + \left( \frac{s}{3} - 2\sqrt{\frac{2}{3}} |W_+| \right) |\phi|
\]

But since \( \phi \) is harmonic, we therefore have the refined Kato inequality \[^5\]

\[
|\nabla \phi|^2 \geq \frac{3}{2} \left( \frac{d|\phi|}{|\phi|} \right)^2,
\]

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so the above yields

\[ 0 \geq \Delta |\phi| + \frac{1}{2} \frac{|d\phi|}{|\phi|}^2 + \left( \frac{s}{3} - 2\sqrt{\frac{2}{3}|W_+|} \right) |\phi|, \]

and we thus have

\[ (2\sqrt{\frac{2}{3}|W_+| - \frac{s}{3})|\phi| \geq \Delta |\phi| + 2|df|^2 \]

wherever \( \phi \neq 0 \).

Now let \( F = |\phi| = f^2 \), and let \( \varepsilon^2 \) be any positive regular value of the smooth function \( F^2 \). Let \( M_\varepsilon \) be the set where \( F \leq \varepsilon \), and observe that, since \( \Delta \) is the \emph{positive} Laplacian, Stokes’ theorem tells us that

\[
\int_{M-M_\varepsilon} \Delta |\phi| \, d\mu = - \int_{M-M_\varepsilon} d \star dF = \int_{\partial M_\varepsilon} \star dF > 0.
\]

Thus

\[
\int_{M-M_\varepsilon} (2\sqrt{\frac{2}{3}|W_+| \frac{s}{3})|\phi| \, d\mu \geq \int_{M-M_\varepsilon} \Delta |\phi| \, d\mu + 2 \int_{M-M_\varepsilon} |df|^2 \, d\mu \geq 2 \int_{M-M_\varepsilon} |df|^2 \, d\mu
\]

On the other hand, \( \bigcap_{\varepsilon > 0} M_\varepsilon \) is the nodal set, which has measure zero in \((M,d\mu)\). Thus

\[
\int_M |df|^2 \, d\mu = \limsup_{\varepsilon \downarrow 0} \int_{M-M_\varepsilon} |df|^2 \, d\mu \\
\leq \limsup_{\varepsilon \downarrow 0} \int_{M-M_\varepsilon} (\sqrt{\frac{2}{3}|W_+| \frac{s}{6})|\phi| \, d\mu \\
= \int_M (\sqrt{\frac{2}{3}|W_+| \frac{s}{6})|\phi| \, d\mu < \infty,
\]

so that \( |df| \) is an \( L^2 \) function, and satisfies the promised estimate. \( \square \)

In particular, the following invariant is \emph{a priori} finite:
Definition 2.3 Let \((M,g)\) be a compact oriented Riemannian 4-manifold with \(b^+(M) \geq 1\). We then define

\[
\nu(M,g) := \min_{\phi \in (\mathcal{H}_g^+ - 0)} \frac{\int \sqrt{|\phi|} d\mu_g}{\int |\phi| d\mu_g}
\]

One might do well to compare this definition with that of the spectral invariant

\[
\lambda_1(M,g) = \inf \left\{ \frac{\int |df|^2 d\mu_g}{\int |f|^2 d\mu_g} \middle| f \in L^2(M,g), f \not\equiv 0, \int fd\mu = 0 \right\}.
\]

By contrast,

\[
\nu(M,g) = \inf \left\{ \frac{\int |df|^2 d\mu_g}{\int |f|^2 d\mu_g} \middle| \exists \phi \in (\mathcal{H}_g^+ - 0) \text{ s.t. } f = \sqrt{|\phi|} \right\}.
\]

Despite the analogy, however, these invariants would seem to have little to say about each other. On one hand, \(\nu\) is defined in terms of a finite-dimensional family of functions \(f\); on the other hand, these functions are not orthogonal to the constants!

The invariant \(\nu\) is not scale-invariant; if the metric \(g\) is replaced by \(cg\), where \(c\) is a positive constant, \(\nu\) gets replaced by \(c^{-1} \nu\); thus \(\nu\) rescales in analogy to the scalar curvature \(s\) or the first eigenvalue \(\lambda_1\) of the Laplacian.

As we are primarily concerned here with manifolds of constant negative scalar curvature, we shall simplify the statements of many of our results by assuming the metric in question satisfies \(s \equiv -12\), as does the standard \(K = -1\) metric on a real-hyperbolic 4-manifold. For example:

**Proposition 2.4** Let \((M,g)\) be an compact anti-self-dual Einstein manifold with \(s = -12\). Then \(\theta(M,g) \neq 0\), and \(\nu(M,g) \in (0, 2]\).

**Proof.** Since \(s \equiv -12\) and \(W_+ \equiv 0\) by assumption, the inequality (1) tells us that

\[
\int |d\sqrt{\phi}|^2 d\mu \leq 2 \int |\phi| d\mu
\]

for any \(\phi \in \mathcal{H}^+ - 0\). This shows that \(\nu(M,g) \leq 2\), as claimed.

If, on the other hand, either \(\theta(M,g)\) or \(\nu(M,g)\) were zero, \((M,g)\) would admit a self-dual harmonic 2-form \(\phi\) of constant, non-zero length.
Multiplying this form by a constant would then give us a harmonic self-dual 2-form $\omega$ of point-wise norm $\equiv \sqrt{2}$. This symplectic form would then be expressible as $\omega(\cdot, \cdot) = g(J\cdot, \cdot)$ for a unique almost-complex structure $J$ on $M$, making $(M, g, J, \omega)$ an almost-Kähler manifold. However, Armstrong [1] has shown that compact anti-self-dual almost-Kähler Einstein manifolds with $s < 0$ do not exist. By contradiction, $\nu(M, g)$ and $\theta(M, g)$ must therefore be positive.

By contrast, $\nu = 0$ for any 4-dimensional Kähler manifold, since the Kähler form is a self-dual harmonic 2-form of constant length. Nonetheless, Kähler manifolds give us an instructive set of examples when we examine the harmonic forms which are orthogonal to the Kähler form:

**Proposition 2.5** Let $(M^4, g)$ be any compact Kähler manifold with $s = -12$, and let $\phi \not\equiv 0$ be any self-dual harmonic 2-form which is $L^2$-orthogonal to the Kähler form $\omega$. Then

$$\frac{\int_M |d\sqrt{|\phi|}^2 d\mu}{\int_M |\phi| d\mu} = \frac{3}{2}.$$ 

**Proof.** On a Kähler manifold of real dimension 4,

$$\Lambda^+ \otimes \mathbb{C} = \mathbb{C}\omega \oplus \Lambda^{2,0} \oplus \Lambda^{0,2},$$

and any real self-dual harmonic 2-form can be uniquely written as

$$\phi = a\omega + \varphi + \bar{\varphi},$$

where $a$ is a real constant and $\varphi$ is a holomorphic $(2,0)$-form. If $\phi$ is $L^2$-orthogonal to $\omega$, $a = 0$, and $|\phi|^2 = 2|\varphi|^2$. But the Ricci form $\rho$ of $(M, \omega)$ is given by

$$\rho = i\partial \bar{\partial} \log |\varphi|^2 = i\partial \bar{\partial} \log |\phi|^2$$

away from the nodal set of $\phi$, and taking the trace against $\omega$ yields

$$s = -\Delta \log |\phi|^2 = -4\Delta \log f,$$

where $f = \sqrt{|\phi|}$. Thus

$$-\frac{s}{4} f^2 = f \Delta f + |df|^2.$$
away from the nodal set. Again setting $F = |\phi| = f^2$, and letting $M_\varepsilon$ be the set where $F < \varepsilon$}, for $\varepsilon > 0$ any regular value of $F$, we have

$$ - \int_{M-M_\varepsilon} \frac{s}{4} f^2 d\mu = 2 \int_{M-M_\varepsilon} |df|^2 d\mu + \frac{1}{2} \int_{\partial M_\varepsilon} \ast dF. $$

On the other hand

$$ \left| \int_{\partial M_\varepsilon} \ast dF \right| < C\varepsilon $$

because $|dF| = |d|\phi|| \leq |\nabla \phi|$ by the Kato inequality, and the 3-dimensional volume of the hypersurface $\partial M_\varepsilon$ is $< \text{const} \cdot \varepsilon$ by the Weierstrass preparation theorem for holomorphic functions. Taking the limit as $\varepsilon \searrow 0$, we thus obtain

$$ - \int_M \frac{s}{4} f^2 d\mu = 2 \int_M |df|^2 d\mu. $$

Setting $s = -12$, this tells us that

$$ \frac{\int_M |df|^2 d\mu}{\int_M f^2 d\mu} = -\frac{s}{8} = -\frac{3}{2}, $$

as claimed.

Thus, it does not seem unreasonable to hope that the value of $\nu$ may turn out to be of the order of 1 for many manifolds with $s = -12$. With this in mind, we now state our main result:

**Theorem A** Let $(M, g)$ be a compact anti-self-dual Einstein manifold with $s = -12$ and $b^+(M) \geq 2$. If $\nu(M, g) \geq 2 - \sqrt{3} \approx 0.268$, all the Seiberg-Witten invariants of the oriented 4-manifold $M$ must vanish.

### 3 Seiberg-Witten Estimates

In this section, we will derive a new set of Seiberg-Witten estimates by combining ideas previously used in [8] and [9]. While these estimates suffice to imply the vanishing results contained in the last section of the paper, they also have an intrinsic interest of their own, as well as other ramifications which would seem to be worthy of exploration.
Let $M$ be a smooth compact oriented connected 4-manifold with $b^+(M) \geq 2$, and let $c$ be any spin$^c$ structure on $M$. For any Riemannian metric $g$ on $M$, we then have rank-2 complex Hermitian vector bundles $V_\pm \to M$ which formally satisfy
\[ V_\pm = S_\pm \otimes L^{1/2}, \]
where $S_\pm$ are the locally-defined left- and right-handed spinor bundles of $g$, and $L = \wedge^2 V_\pm$ is a globally defined Hermitian line bundle. As a convenient abuse, we will use $c_1$ to denote the image of $c_1(L) = c_1(V_\pm) \in H^2(M, \mathbb{Z})$ in the real cohomology $H^2(M, \mathbb{R})$, and refer $c_1$ as the first Chern class of $c$. Now the Hodge theorem tells us that $H^2(M, \mathbb{R})$ can be identified with the space of harmonic 2-forms $\mathcal{H}_g^2$ on $(M, g)$; and the latter splits as the direct sum
\[ \mathcal{H}_g^2 = \mathcal{H}_g^+ \oplus \mathcal{H}_g^- \]
of the self-dual and anti-self-dual harmonic forms. This allows us to uniquely write
\[ c_1 = c_1^+ + c_1^-, \]
where the cohomology classes $c_1^\pm \in H^2(M, \mathbb{R})$ have harmonic representatives in $\mathcal{H}_g^\pm$, respectively.

For any given self-dual form $\eta$ on $(M, g)$, the corresponding perturbed Seiberg-Witten equations \cite{12} read
\begin{align*}
D_A \Phi &= 0 \tag{2} \\
i F_A^+ + \sigma(\Phi) &= \eta, \tag{3}
\end{align*}
where the unknowns are a Hermitian connection $A$ on the line bundle $L$ associated with $c$, and a section $\Phi$ of the twisted spinor bundle $V_+$. Here $D_A : \Gamma(V_+) \to \Gamma(V_-)$ is the Dirac operator determined by $A$, and $F_A^+$ is the self-dual part of the curvature of $A$, whereas $\sigma : V_+ \to \Lambda^+$ is the natural real-quadratic map induced by the isomorphism $\Lambda^+ \otimes \mathbb{C} = \odot^2 S_+$, with the conventional normalization that $|\sigma(\Phi)|^2 = |\Phi|^4/8$.

Let $\mathcal{U}$ denote the affine space of differentiable unitary connections on $L$, and let $\mathcal{V}$ denote the vector space of differentiable sections of $V_+$. The solution space
\[ \mathcal{S}_{c,g,\eta} = \{ (A, \Phi) \mid (2) \text{ and (3) are satisfied} \} \]
of the perturbed Seiberg-Witten equations is thus a subset of $U \times V$. Moreover, as long as the harmonic part of $\eta$ is different from $2\pi c_1^+$, any solution of (2–3) will be irreducible, in the sense that $\Phi \not\equiv 0$; thus, for generic $\eta$,

$$S_{c,g,\eta} \subset U \times (V - 0).$$

Now the gauge group $G = \{ u : M \to S^1 \}$ acts on $U \times (V - 0)$ by $(A, \Phi) \mapsto (A - 2d\log u, u\Phi)$, and this action preserves the solution space $S_{c,g,\eta}$. The quotient space

$$M_{c,g,\eta} = S_{c,g,\eta} / G$$

is called the Seiberg-Witten moduli space associated with the given spin$^c$ structure, and is tautologically a subset of the configuration space

$$\mathcal{B} = [U \times (V - 0)] / G,$$

which is homotopy equivalent to $T_b(M) \times \mathbb{CP}_\infty$. For generic $\eta$, the moduli space is a smooth, compact manifold of dimension

$$\ell = \frac{c_2^2 - (2\chi + 3\tau)}{4},$$

where $\chi$ and $\tau$ respectively denote the Euler characteristic and signature of $M$; in particular, if this integer is negative, the moduli space is empty for generic $\eta$. Moreover, an orientation of the vector space $H^1(M, \mathbb{R}) \oplus H^+_g$ determines an orientation of the moduli space. Thus the homology class of $M_{c,g,\eta} \subset \mathcal{B}$ gives us an element of $H_\ell(\mathcal{B}, \mathbb{Z}) \cong H_\ell(T_b(M) \times \mathbb{CP}_\infty, \mathbb{Z})$ which turns out to be independent of the metric $g$ and the generic self-dual form $\eta$, and which is called the (generalized) Seiberg-Witten invariant of $(M, c)$. For our purposes, the only important point is that when this invariant is non-zero, the equations

$$D_A \Phi = 0 \quad \text{(4)}$$
$$-i F^+_A = \sigma(\Phi) - t\phi, \quad \text{(5)}$$

must have a solution for any metric $g$, any self-dual harmonic 2-form $\phi$, and any real constant $t$. Moreover, provided $\phi \not\equiv 0$, this solution will be irreducible for any sufficiently large $t$.

**Lemma 3.1** Let $(M, g)$ be a compact oriented Riemannian 4-manifold, and let $\phi$ be a self-dual harmonic 2-form on $(M, g)$. If, for a given spin$^c$ structure
the perturbed Seiberg-Witten equations (4–5) have a solution for a given real number $t$, then

$$V^{1/3} \left( \int_M \frac{2}{3} s + 2w - t2\sqrt{2} |\phi| \right)^{2/3} \geq 8t^2 [\phi]^2 - 32\pi tc_1 \cdot [\phi],$$

where $V$ denotes the total volume of $(M, g)$, $w : M \to (-\infty, 0]$ is the lowest eigenvalue of the self-dual Weyl curvature $W_+ : \Lambda^+ \to \Lambda^+$ of $g$, $d\mu$ is the volume form of $g$, $\cdot$ is the point-wise norm determined by $g$, and $c_1 = c_1(L)$ is the first Chern class of the spin$^c$ structure $c$.

**Proof.** The Dirac equation $D_A \Phi = 0$ implies the Weitzenböck formula

$$0 = \frac{1}{2} \Delta |\Phi|^2 + |\nabla \Phi|^2 + \frac{s}{4} |\Phi|^2 + 2\langle -iF_A^+, \sigma(\Phi) \rangle.$$

For a solution of the perturbed Seiberg-Witten equations (4–5), we also have $-iF_A^+ = \sigma(\Phi) - t\phi$, so it follows that

$$0 \geq 2\Delta |\Phi|^2 + 4|\nabla \Phi|^2 + s|\Phi|^2 + |\Phi|^4 - 8t\langle \phi, \sigma(\Phi) \rangle \geq 2\Delta |\Phi|^2 + 4|\nabla \Phi|^2 + s|\Phi|^2 + |\Phi|^4 - t2\sqrt{2} |\phi| |\Phi|^2. \quad (6)$$

Multiplying by $|\Phi|^2$, we thus have

$$0 \geq 2|\Phi|^2 \Delta |\Phi|^2 + 4|\Phi|^2 |\nabla \Phi|^2 + (s - t2\sqrt{2} |\phi|) |\Phi|^4 + |\Phi|^6,$$

and, upon integrating, we obtain

$$0 \geq \int_M \left[ 2 |d|\Phi|^2|^2 + 4|\Phi|^2 |\nabla \Phi|^2 + (s - t2\sqrt{2} |\phi|) |\Phi|^4 + |\Phi|^6 \right] d\mu,$$

so that

$$\int \left[ (-s)|\Phi|^4 - 4|\Phi|^2 |\nabla \Phi|^2 \right] d\mu \geq \int \left[ |\Phi|^6 - t2\sqrt{2} |\phi| |\Phi|^4 \right] d\mu. \quad (7)$$

Now recall that any self-dual 2-form $\psi$ on any oriented 4-manifold satisfies the Weitzenböck formula

$$(d + d^*)^2 \psi = \nabla^* \nabla \psi - 2W_+(\psi, \cdot) + \frac{s}{3} \psi,$$
where $W_+$ is the self-dual Weyl tensor. It follows that

$$
\int_M [-2W_+(\psi, \psi)]d\mu \geq \int_M \left(- \frac{8}{3}\right)|\psi|^2 d\mu - \int_M |\nabla \psi|^2 d\mu,
$$

so that

$$
- \int_M 2w|\psi|^2 d\mu \geq \int_M \left(- \frac{8}{3}\right)|\psi|^2 d\mu - \int_M |\nabla \psi|^2 d\mu,
$$

and hence

$$
- \int_M \left(\frac{2}{3}s + 2w\right)|\psi|^2 d\mu \geq \int_M (-s)|\psi|^2 d\mu - \int_M |\nabla \psi|^2 d\mu.
$$

On the other hand, the particular self-dual 2-form $\varphi = \sigma(\Phi)$ satisfies

$$
|\varphi|^2 = \frac{1}{8}|\Phi|^4,
$$

$$
|\nabla \varphi|^2 \leq \frac{1}{2}|\Phi|^2|\nabla \Phi|^2.
$$

Setting $\psi = \varphi$, we thus have

$$
- \int_M \left(\frac{2}{3}s + 2w\right)|\Phi|^4 d\mu \geq \int_M \left[(-s)|\Phi|^4 - 4|\Phi|^2|\nabla \Phi|^2\right] d\mu.
$$

Combining this with (7), we thus obtain

$$
- \int_M \left(\frac{2}{3}s + 2w\right)|\Phi|^4 d\mu \geq \int_M \left[|\Phi|^6 - t\sqrt{2}|\phi||\Phi|^4\right] d\mu,
$$

and hence

$$
- \int_M \left(\frac{2}{3}s + 2w - t\sqrt{2}|\phi|\right)|\Phi|^4 d\mu \geq \int_M |\Phi|^6. \tag{8}
$$

By the Hölder inequality, this implies

$$
\left(\int \left|\frac{2}{3}s + 2w - t\sqrt{2}|\phi|\right|^3 d\mu\right)^{1/3} \left(\int |\Phi|^6 d\mu\right)^{2/3} \geq \int |\Phi|^6 d\mu,
$$

and hence that

$$
\int \left|\frac{2}{3}s + 2w - t\sqrt{2}|\phi|\right|^3 d\mu \geq \int |\Phi|^6 d\mu.
$$
But the Hölder inequality also tells us that
\[
V^{1/3} \left( \int |\Phi|^4 \, d\mu \right)^{2/3} \geq \int |\Phi|^4 d\mu,
\]
so we now have
\[
V^{1/3} \left( \int_M \left( \frac{2}{3} s + 2w - t2\sqrt{2}|\phi| \right)^3 \, d\mu \right)^{2/3} \geq \int |\Phi|^4 d\mu
\]
\[
= 8 \int |\sigma(\Phi)|^2 d\mu
\]
\[
= 8 \int |-iF_A^+ + t\phi|^2 d\mu
\]
\[
= 8 \int (t^2|\phi|^2 - 2t(iF_A^+, \phi) + |iF_A^+|^2) \, d\mu
\]
\[
= 8 \left( t^2|\phi|^2 - 2t(2\pi c_1) \cdot [\phi] + \int |iF_A^+|^2 d\mu \right)
\]
\[
\geq 8t^2|\phi|^2 - 32\pi tc_1 \cdot [\phi],
\]
as claimed. \[\square\]

**Lemma 3.2** Let \( \gamma = [g] \) be a smooth conformal class on a smooth compact oriented 4-manifold \( M \), and let \( \phi \not\equiv 0 \) be a closed 2-form which is self-dual with respect to \( \gamma \). Suppose, moreover, that for a fixed spin\(^c\) structure \( \mathfrak{c} \) and a fixed positive real number \( t \) that the perturbed Seiberg-Witten equations (4–5) have an irreducible solution for every metric in the conformal class \( \gamma \). Then, for any metric \( g \in \gamma \), the scalar curvature \( s \) and Weyl curvature \( W \) satisfy the inequality
\[
\int_M \left( \frac{2}{3} s + 2w - t2\sqrt{2}|\phi| \right)^2 \, d\mu \geq 8t^2|\phi|^2 - 32\pi tc_1 \cdot [\phi].
\]

**Proof.** The key step in the argument is a conformal rescaling trick, the general idea of which is due to Gursky [6]. We begin by observing that there is a \( C^2 \) metric \( g_\gamma \in \gamma \) for which the function \( \mathcal{G} = s + 3w - t3\sqrt{2}|\phi| \) is constant. Indeed, if \( \hat{g} = u^2 g \), the corresponding curvature quantity is given by
\[
\mathcal{G}_{\hat{g}} = u^{-3} (6\Delta_g u + \mathcal{G}_g u)
\]
because \( w_\tilde{g} = u^{-2}w_g \) and \( |\phi|_\tilde{g} = u^{-2}|\phi|_g \). The metric \( g_\gamma \) may therefore be constructed by minimizing the functional

\[
\mathcal{F}(g) = \frac{\int_M \mathfrak{G}_g d\mu_g}{\sqrt{\int_M d\mu_g}}
\]

among metrics in the conformal class \( \gamma \). The infimum of this functional is negative because the Weitzenböck formula (6) shows that every metric in \( \gamma \) has \( s - t2\sqrt{2}|\phi| \leq 0 \) somewhere, and \( \mathfrak{G} \leq s - t2\sqrt{2}|\phi| \) everywhere, with strict inequality at any point where \( \phi \neq 0 \). Trudinger’s approach to the Yamabe problem thus produces a minimizer \( g_\gamma \) of regularity \( C^{2,\alpha} \) for any \( \alpha < 1 \).

Since \( \mathfrak{G}_{g_\gamma} \) is automatically a negative constant by the Euler-Lagrange equations, we have

\[
\int_M \left( \frac{2}{3} s_{g_\gamma} + 2w_{g_\gamma} - t2\sqrt{2}|\phi|_{g_\gamma} \right)^2 d\mu_{g_\gamma} = V_{g_\gamma}^{1/3} \left( \int_M \left( \frac{2}{3} s_{g_\gamma} + 2w_{g_\gamma} - t2\sqrt{2}|\phi|_{g_\gamma} \right)^3 d\mu_{g_\gamma} \right)^{2/3},
\]

so that

\[
\int_M \left( \frac{2}{3} s_{g_\gamma} + 2w_{g_\gamma} - t2\sqrt{2}|\phi|_{g_\gamma} \right)^2 d\mu_{g_\gamma} \geq 8t^2[\phi]^2 - 32\pi tc_1 \cdot [\phi]
\]

by the previous lemma. Thus the desired inequality at least holds for the particular metric \( g_\gamma \in \gamma \).

We now compare the left-hand side with analogous expression for the given metric \( g \), following an idea of [3]. To do so, we express \( g \) in the form \( g = u^2g_\gamma \), where \( u \) is a positive \( C^2 \) function, and observe that

\[
\int_M \mathfrak{G}_g u^2 d\mu_{g_\gamma} = \int u^{-3} (6\Delta_g u + \mathfrak{G}_g u) u^2 d\mu_{g_\gamma} \\
= \int (-6u^{-2}|du|^2 + \mathfrak{G}_{g_\gamma}) d\mu_{g_\gamma} \\
\leq \int \mathfrak{G}_{g_\gamma} d\mu_{g_\gamma}.
\]

Applying Cauchy-Schwarz, we thus have

\[
-V_{g_\gamma}^{1/2} \left[ \int \mathfrak{G}_g^2 d\mu_g \right]^{1/2} = -V_{g_\gamma}^{1/2} \left( \int \mathfrak{G}_g^2 u^4 d\mu_{g_\gamma} \right)^{1/2}
\]
\[ \leq \int_M \mathcal{S}_g u^2 d\mu_g, \]
\[ \leq \int_M \mathcal{S}_g d\mu_g, \]
\[ = -V_g^{1/2} \left[ \int_M \mathcal{S}_g^2 d\mu_g \right]^{1/2}. \]

Thus
\[ \int_M \left( \frac{2}{3}s_g + 2w_g - t2\sqrt{2} |\phi|_g \right)^2 d\mu_g = \frac{4}{9} \int_M \mathcal{S}_g^2 d\mu_g \]
\[ \geq \frac{4}{9} \int_M \mathcal{S}_g^2 d\mu_g, \]
\[ = \int_M \left( \frac{2}{3}s_g + 2w_g - t2\sqrt{2} |\phi|_g \right)^2 d\mu_g, \]
\[ \geq 8t^2 |\phi|^2 - 32\pi tc_1 \cdot [\phi], \]

as claimed.

**Theorem 3.3** Let \( M^4 \) be a smooth compact oriented 4-manifold with \( b^+ \geq 2 \), and suppose that \( \mathfrak{c} \) is a spin\(^c\) structure with non-trivial Seiberg-Witten invariant. Let \( g \) be any Riemannian metric on \( M \), and let \( \phi \) be a \( g \)-self-dual harmonic 2-form with de Rham class \([\phi] \in H^2(M, \mathbb{R})\). Then the scalar curvature \( s \) and lowest eigenvalue \( w \) of the self-dual Weyl curvature \( W_+ \) of \( g \) satisfy
\[ \int \left( \frac{2}{3}s + 2w \right) |\phi| \sqrt{2} d\mu \leq 4\pi c_1 \cdot [\phi]. \]
Here \( d\mu \) and \(| \cdot |\) are respectively the volume form and point-wise norm determined by the metric \( g \), while \( c_1 = c_1(V_+) \) is the first Chern class of the spin\(^c\) structure \( \mathfrak{c} \).

**Proof.** By Lemma 3.2, we have
\[ \int_M \left( \frac{2}{3}s + 2w - t2\sqrt{2} |\phi| \right)^2 d\mu \geq 8t^2 |\phi|^2 - 32\pi tc_1 \cdot [\phi] \]
for all $t > 0$. Thus

$$8t^2[\phi]^2 - 4\sqrt{2}t \int_M \left( \frac{2}{3}s + 2w \right) |\phi| d\mu + \int_M \frac{2}{3}s + 2w d\mu \geq 8t^2[\phi]^2 - 32\pi tc_1 \cdot [\phi],$$

and hence

$$\int_M \left( \frac{2}{3}s + 2w \right) \frac{|\phi|}{\sqrt{2}} d\mu - \frac{8}{t} \int_M \frac{2}{3}s + 2w d\mu \leq 4\pi c_1 \cdot [\phi].$$

Taking the limit as $t \to \infty$ then yields the desired result.

**Corollary 3.4** Let $M^4$ be a smooth compact oriented 4-manifold with $b^+ \geq 2$, and suppose that $c$ is a spin$^c$ structure with non-trivial Seiberg-Witten invariant. Let $g$ be any Riemannian metric on $M$, and let $\phi$ be a $g$-self-dual harmonic 2-form with de Rham class $[\phi] \in H^2(M, \mathbb{R})$. Then the scalar curvature $s$ and Weyl curvature $W_+$ of $g$ satisfy

$$\int \left( s - \sqrt{6}|W_+| \right) |\phi| d\mu \leq 6\sqrt{2}\pi c_1 \cdot [\phi].$$

**Proof.** Because $W_+$ is a trace-free endomorphism of $\Lambda^+$,

$$-\sqrt{\frac{2}{3}}|W_+| \leq w.$$

Substituting this into Theorem 3.3 and multiplying by $3/\sqrt{2}$, we thus obtain the desired result.

Since Theorem 3.3 applies to every metric conformal to a given $g$, we can improve it, as follows:

**Theorem 3.5** Let $M^4$ be a smooth compact oriented 4-manifold with $b^+ \geq 2$, and suppose that $c$ is a spin$^c$ structure with non-trivial Seiberg-Witten invariant. Let $g$ be any Riemannian metric on $M$, and let $\phi$ be a $g$-self-dual harmonic 2-form with de Rham class $[\phi] \in H^2(M, \mathbb{R})$. Then the function $f = \sqrt{|\phi|}$ satisfies

$$\int_M \left( \frac{2}{3}s_g + 2w_g \right) |\phi|_g d\mu_g + 4 \int_M |df|_g^2 d\mu_g \leq (4\pi \sqrt{2})c_1 \cdot [\phi].$$
Proof. We may obviously assume that $\phi \neq 0$, since otherwise there is nothing to prove.

Now observe that, for any smooth positive function $u$ on $M$, the metric $\hat{g} = u^2 g$ satisfies

$$(\frac{2}{3} s_g + 2 w_g) |\phi|_{\hat{g}} d\mu_{\hat{g}} = u^{-1} |\phi|_g \left[ 4 \Delta_g u + \left( \frac{2}{3} s_g + 2 w_g \right) u \right] d\mu_g,$$

and, since $|\phi|$ has bounded derivative, we may integrate by parts to obtain

$$\int_M \left( \frac{2}{3} s_g + 2 w_g \right) |\phi|_{\hat{g}} d\mu_{\hat{g}} = \int_M \left( \frac{2}{3} s_g + 2 w_g \right) |\phi|_g d\mu_g + 4 \int_M \langle d(u^{-1}|\phi|), du \rangle_g d\mu_g.$$

Applying Theorem 3.3 to $\hat{g}$ thus gives us

$$\int_M \left( \frac{2}{3} s_g + 2 w_g \right) |\phi|_{\hat{g}} d\mu_{\hat{g}} + 4 \int_M \langle d(u^{-1}|\phi|), du \rangle_g d\mu_g \leq (4\pi \sqrt{2}) c_1 \cdot [\phi]. \tag{9}$$

Now, for some $\epsilon > 0$, let us take $u = u_\epsilon$ to be given by

$$u_\epsilon = \sqrt{\alpha_{\epsilon}(|\phi|)},$$

where $\alpha_{\epsilon} : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function with

$$\alpha_{\epsilon}(x) = x \ \forall x \in [\epsilon, \infty),$$

$$\alpha_{\epsilon}'(x) = 0 \ \forall x \in (-\infty, \frac{\epsilon}{2}], \text{ and}$$

$$\alpha_{\epsilon}''(x) \geq 0 \ \forall x \in \mathbb{R}.$$

Then, since $\alpha_{\epsilon}'(x) \leq 1$ for all $x$,

$$\langle d(u_\epsilon^{-1}|\phi|), du_\epsilon \rangle_g = \frac{\alpha_{\epsilon}'(|\phi|)}{2\alpha_{\epsilon}(|\phi|)} \left( 1 - \frac{|\phi|\alpha_{\epsilon}'(|\phi|)}{2\alpha_{\epsilon}(|\phi|)} \right) |d|\phi| |^2_g$$

$$\geq \frac{1}{4 \alpha_{\epsilon}(|\phi|)} |d|\phi| |^2_g,$$

with equality when $|\phi| \geq \epsilon$. Hence $\langle d(u_\epsilon^{-1}|\phi|), du_\epsilon \rangle \geq 0$ everywhere, and $\langle d(u_\epsilon^{-1}|\phi|), du_\epsilon \rangle = |d\phi|^2$ on the set $M - M_\epsilon$ where $|\phi| \geq \epsilon$. Thus

$$\int_M \langle d(u^{-1}|\phi|), du \rangle d\mu > \int_{M - M_\epsilon} |d\phi|^2 d\mu,$$

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and (9) therefore implies
\[ \int_M \left( \frac{2}{3}s + 2w \right) |\phi| d\mu + 4 \int_{M-M_\varepsilon} |df|^2 d\mu < (4\pi \sqrt{2}) c_1 \cdot [\phi]. \]

However, \( \varepsilon > 0 \) is arbitrary, and \( \cap_\varepsilon M_\varepsilon \) is the nodal set of \( \phi \), which has measure zero. Taking the supremum of the left-hand side over \( \varepsilon > 0 \) thus gives us the desired inequality.

\[ \square \]

4 Vanishing Theorems

One of Witten’s most elegant observations is the fact that a given 4-manifold can only have finitely many spin\(^c\) structures for which the Seiberg-Witten invariant is non-zero; this follows from the fact that there is an \textit{a priori} upper bound on \((c_1^+)^2\) in terms of scalar curvature \([14]\). When the 4-manifold admits a hyperbolic metric, however, we will now see that the invariant must also vanish for most of the spin\(^c\) structures which slip under the bar of Witten’s upper bound.

**Theorem B** Let \((M, g)\) be a compact, anti-self-dual Einstein manifold with scalar curvature \( s = -12 \). Suppose, moreover, that \( b^+(M) \geq 2 \), and that \( c_1 \) is the first Chern class of a spin\(^c\) structure on \( M \) for which the (generalized) Seiberg-Witten invariant is non-zero; let \( c_1^+ \) denote its self-dual part of \( c_1 \) with respect to \( g \). Then

\[
\frac{(c_1^+)^2}{2\chi + 3\tau} \leq \frac{(2 - \nu)^2 \cos^2 \theta}{3},
\]

where \( \chi \) and \( \tau \) respectively denote the Euler characteristic and signature of \( M \), and where the invariants \( \theta \) and \( \nu \) of \((M, g)\) are as in Definitions \( 2.1 \) and \( 2.3 \).

**Proof.** For any self-dual harmonic 2-form on \((M, g)\), Theorem \( 3.5 \) tell us that

\[
\int_M \frac{2}{3}s_g |\phi_g| d\mu_g + 4 \int_M |d\sqrt{|\phi|}_g|^2 d\mu_g \leq (4\pi \sqrt{2}) c_1 \cdot [\phi]
\]
because the anti-self-duality of $g$ guarantees that $w_g \equiv 0$. Since $s$ is constant, the definition of $\nu(M, g)$ thus tells us that

$$\frac{2}{3} s + 4\nu \leq \frac{\int_M \frac{2}{3} s|\phi| d\mu_g + 4 \int_M |d\sqrt{|\phi|}|^2 d\mu_g}{\int |\phi| d\mu_g} \leq (4\pi \sqrt{2}) c_1 \cdot [\phi].$$

Taking $[\phi] = -c_1^+$, we therefore have

$$-s - 6\nu \geq 6\pi \sqrt{2} \frac{(c_1^+)^2}{\int |\phi| d\mu_g} = 6\pi \sqrt{2} \frac{(\int |\phi|^2 d\mu_g)^{1/2}}{\int |\phi| d\mu_g} \sqrt{(c_1^+)^2} \geq 6\pi \sqrt{2} \frac{V^{-1/2}}{\cos \theta} q \sqrt{2\chi + 3\tau}$$

where

$$q = \sqrt{\frac{(c_1^+)^2}{2\chi + 3\tau}}.$$

However, we have the Gauss-Bonnet-like formula [7]

$$2\chi + 3\tau = \frac{1}{4\pi^2} \int_M \left( 2|W_+|^2 + \frac{s^2}{24} - \frac{|\phi|^2}{2} \right) d\mu,$$

for any metric on $M$, and for our anti-self-dual Einstein metric $g$ this simplifies to become

$$2\chi + 3\tau = \frac{s^2 V}{96\pi^2}.$$

Thus, since $s < 0$,

$$|s| - 6\nu \geq \sqrt{72\pi^2 V^{-1/2}} \frac{s^2 V}{96\pi^2} \frac{q \sqrt{\frac{s^2 V}{96\pi^2}}}{\cos \theta} = \sqrt{\frac{3}{2}} |s| q \frac{\cos \theta}{\cos \theta},$$

and

$$\frac{1}{\sqrt{3}} (2 - \frac{12}{|s|}) \nu \cos \theta \geq q.$$

With the normalization $s = -12$, this then gives us

$$\frac{1}{3} (2 - \nu)^2 \cos^2 \theta \geq q^2 = \frac{(c_1^+)^2}{2\chi + 3\tau},$$

as claimed.
Corollary 4.1 Let $(M, g)$ be a compact, anti-self-dual Einstein manifold with scalar curvature $s = -12$ and $b^+(M) \geq 2$. Then the Seiberg-Witten invariant vanishes for any spin$^c$ structure for which

$$\text{(c}_1^+\text{)}^2 > \frac{(2 - \nu)^2 \cos^2 \theta}{3} (2\chi + 3\tau)$$

or for which

$$\text{|(c}_1^-\text{)}^2| > \frac{(2 - \nu)^2 \cos^2 \theta - 3}{3} (2\chi + 3\tau)$$

Proof. If the Seiberg-Witten invariant were non-zero, we would necessarily have

$$(\text{c}_1^+\text{)}^2 - |(c}_1^-\text{)^2| = c^2_1 \geq 2\chi + 3\tau$$

because the virtual dimension of the moduli space must be non-negative. Theorem 3 thus guarantees that

$$\left(\frac{(2 - \nu)^2 \cos^2 \theta}{3} - 1\right) (2\chi + 3\tau) \geq (\text{c}_1^+\text{)}^2 - (2\chi + 3\tau) \geq |(c}_1^-\text{)^2|.$$ 

This proves the corollary by contraposition. \hfill \Box

In particular, since $|(c}_1^-\text{)^2| \geq 0$ for all spin$^c$ structures, we obtain

Corollary 4.2 Let $(M, g)$ be a compact, anti-self-dual Einstein manifold with scalar curvature $s = -12$ and $b^+(M) \geq 2$. If

$$\nu > 2 - \sqrt{3} \sec \theta,$$

then all the Seiberg-Witten invariants of $M$ must vanish for the given orientation.

Theorem 3 now follows immediately, since $\theta \neq 0$ for the spaces in question. But it remains to be seen, of course, whether this result is actually non-vacuous! Can one at least show that $\nu \geq 2 - \sqrt{3}$ for some real-hyperbolic 4-manifolds?

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