The Hörmander index in the finite-dimensional case

Yuting ZHOU\textsuperscript{1}, Li WU\textsuperscript{2}, Chaofeng ZHU\textsuperscript{1}

1 Chern Institute of Mathematics and LPMC, Nankai University, Tianjin 300071, China
2 Department of Mathematics, Shandong University, Jinan, 250100, China

© Higher Education Press and Springer-Verlag Berlin Heidelberg 2017

Abstract In this paper, we calculate Hörmander index in the finite-dimensional case. Then we use the result to give some iteration inequalities, and prove almost existence of mean indices for given complete autonomous Hamiltonian system on compact symplectic manifold with symplectic trivial tangent bundle and given autonomous Hamiltonian system on regular compact energy hypersurface of symplectic manifold with symplectic trivial tangent bundle.

Keywords Maslov index, Hörmander index, Maslov-type index, symplectic reduction

MSC 53D12, 58J30

1 Introduction

Let \((V, \omega)\) be a symplectic vector space. Let \(\lambda_1, \lambda_2, \mu_1, \mu_2\) be four Lagrangian subspaces of \(V\). The Hörmander index \(s(\lambda_1, \lambda_2; \mu_1, \mu_2)\) has been introduced by L. Hörmander [8, Sect. 3.3] in the finite-dimensional case when \(H = \lambda_j \oplus \mu_k\) hold for \(j, k = 1, 2\), who also gave the explicit formula to calculate it. The notion was generalized by B. Booss and K. Furutani [1, Proposition 2.1] in the finite-dimensional case and [1, Definition 5.2] in the strong symplectic Hilbert case when \((\lambda_1, \mu_1), (\lambda_2, \mu_1)\) are two Fredholm pairs of Lagrangian subspaces of \(V\) and \(\mu_1/(\mu_1 \cap \mu_2)\) is finite-dimensional. Recently, B. Booss and the third author [2, Definition 3.4.4] generalized the notion to the symplectic Banach case.

The splitting number [16, Definition 9.1.4] is a special case of the Hörmander index. It turns out that the study of the Hörmander index in the full generality is very important in the study of Hamiltonian systems (see [16] for the applications of the splitting numbers and [17, 14] for the study of multiplicity of the
brake orbits).

In [4], M. de Gosson gave a very elegant definition of the Hörmander index in the finite-dimensional case in great generality. His definition differs slightly from ours. By admitting half-integer indices, it yields more simple proofs, but may be more difficult to be used in concrete applications in Morse theory.

We calculate the Hörmander index in the finite-dimensional case and get the following main result.

Theorem 1.1. Let $(V, \omega)$ be a complex symplectic vector space of dimension $2n$. Let $\lambda_1, \lambda_2, \mu_1, \mu_2$ be four Lagrangian subspaces of $V$. Denote by $i(\cdot, \cdot, \cdot)$ the triple index defined by [5, (2.16)] (see Corollary 3.12 below). Then we have

$$s(\lambda_1, \lambda_2; \mu_1, \mu_2) = i(\lambda_1, \lambda_2, \mu_2) - i(\lambda_1, \lambda_2, \mu_1)$$

$$= i(\lambda_1, \mu_1, \mu_2) - i(\lambda_2, \mu_1, \mu_2).$$

Our main result does not require the transversal conditions $V = \lambda_j \oplus \mu_k, j, k = 1, 2$. We use the result to get some new iteration inequalities of Maslov-type index. Then we use the inequalities to prove almost existence of mean indices for given complete autonomous Hamiltonian system on compact symplectic manifold with symplectic trivial tangent bundle and given autonomous Hamiltonian system on regular compact energy hypersurface of symplectic manifold with symplectic trivial tangent bundle.

The paper is organized as follows. In §1 we review the historical literatures and introduce our main result. In §2 we review the notions of Maslov index and Maslov-type index in the finite-dimensional case. In §3 we study the Hörmander index in the finite-dimensional case and prove Theorem 1.1. In §4 we prove some iteration inequalities of Maslov-type index. In §5 we prove almost existence of mean indices for given complete autonomous Hamiltonian system on compact symplectic manifold with symplectic trivial tangent bundle and given autonomous Hamiltonian system on regular compact energy hypersurface of symplectic manifold with symplectic trivial tangent bundle.

In this paper we denote the sets of natural, integral, real, complex numbers, the unit circle in the complex plane, the set of all linear operators on $V$, the set of all invertible linear transformations on $V$ and the set of all self-adjoint operators on Hilbert space $V$, by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, $S^1$, $\text{End}(V)$, $\text{GL}(V)$ and $\mathcal{B}^{sa}(V)$ respectively. We denote by $\text{dim} V$ the complex dimension of a complex linear space $V$. We denote by $\text{Hom}(V_1, V_2)$ the set of all linear maps between vector spaces $V_1$ and $V_2$. For a linear operator $A \in \text{Hom}(V_1, V_2)$, we denote by $\text{Gr}(A)$ the graph of $A$. For two maps $f : X \to Y$ and $g : Y \to Z$, we denote by $g \circ f : X \to Z$ the composite map defined by $(g \circ f)(x) = g(f(x))$ for each $x \in X$. Without further explanation, the coefficient field is $\mathbb{C}$ in the rest of this paper.
2 The Maslov index and the Maslov-type index

In this section we review the definition of the Maslov index and the Maslov-type index in the finite-dimensional case. Firstly we recall the basic concepts and properties of symplectic vector space.

Definition 2.1. Let \( V \) be a complex vector space.

(a) A mapping \( \omega : V \times V \to \mathbb{C} \)
is called a symplectic form on \( V \), if it is a non-degenerate skew-Hermitian form. Then we call \((V, \omega)\) a complex symplectic vector space.

(b) Let \((V, \omega_1)\) and \((V, \omega_2)\) be two finite-dimensional symplectic vector spaces. A linear map \( L \in \text{Hom}(V_1, V_2) \) is called symplectic, if \( L \) is invertible and \( \omega_2(Lx, Ly) = \omega_1(x, y) \) for each \( x, y \in V_1 \). We denote by \( \text{Sp}((V_1, \omega_1), (V_2, \omega_2)) \) the set of all such symplectic linear maps \( L \), and \( \text{Sp}(V, \omega) := \text{Sp}((V, \omega), (V, \omega)) \) for the finite-dimensional symplectic vector space \((V, \omega)\). We denote by \( \text{Sp}(V_1, V_2) = \text{Sp}((V_1, \omega_1), (V_2, \omega_2)) \) and \( \text{Sp}(V) = \text{Sp}(V, \omega) \) if there is no confusion.

(c) Let \( \lambda \) be a linear subspace of \( V \). The annihilator \( \lambda^\omega \) of \( \lambda \) is defined by
\[
\lambda^\omega := \{ x \in V; \omega(x, y) = 0 \text{ for all } y \in \lambda \}.
\]

(d) A linear subspace \( \lambda \) of \( V \) is called symplectic, isotropic, co-isotropic, or Lagrangian if
\[
\lambda \cap \lambda^\omega = \{0\}, \quad \lambda \subset \lambda^\omega, \quad \lambda \supset \lambda^\omega, \quad \lambda = \lambda^\omega,
\]
respectively.

(e) The Lagrangian Grassmannian \( \mathcal{L}(V, \omega) \) consists of all Lagrangian subspaces of \((V, \omega)\). We write \( \mathcal{L}(V) := \mathcal{L}(V, \omega) \) if there is no confusion.

If \( \dim V < +\infty \), the space \( \mathcal{L}(V) \) is path-connected. It is nonempty if and only if the signature \( \text{sign}(i\omega) = 0 \).

The following definition of the Maslov index is taken from \[2, \S 2.2\].

Let \((V, \omega)\) be a \( 2n \)-dimensional complex vector space. Let \( \langle \cdot, \cdot \rangle \) be an inner product on \( V \). Then there is an operator \( J \in \text{GL}(V) \) such that \( iJ \) is self-adjoint and \( \omega(x, y) = \langle Jx, y \rangle \) for all \( x, y \in V \).

Denote by \( V^\pm \) the positive (negative) eigenspace of \( iJ \). Given a path \((\lambda(s), \mu(s)), s \in [a, b]\) of pairs of Lagrangian subspaces of \((V, \omega)\), let \( U(s), V(s) : V^- \to V^+ \) be generators for \((\lambda(s), \mu(s))\), i.e., \( \lambda(s) = \text{Gr}(U(s)) \) and \( \mu(s) = \text{Gr}(V(s)) \) (see \[3, \text{Proposition 2}\]). Then the family \( \{U(s)V(s)^{-1}\}_{s \in [a, b]} \) is a continuous family of unitary operators on Hilbert space \((V^+, -i\omega|_{V^+})\).

Note that the eigenvalues of \( U(s)V(s)^{-1} \) are on the unit circle \( S^1 \). Recall that each map in \( C([a, b], S^1) \) can be lifted to a map in \( C([a, b], \mathbb{R}) \). By \[9\]
Theorem II.5.2], there are \(n\) continuous functions \(\theta_1, \ldots, \theta_n \in C([a, b], \mathbb{R})\) such that the eigenvalues of the operator \(U(s)V(s)^{-1}\) for each \(s \in [a, b]\) (counting algebraic multiplicity) have the form
\[
e^{i\theta_j(s)}, \quad j = 1, \ldots, n.
\]

Denote by \([a]\) the integer part of \(a \in \mathbb{R}\) and \(\{a\} := a - [a]\). Define
\[
E(a) := \begin{cases} a, & a \in \mathbb{Z}; \\ [a] + 1, & a \notin \mathbb{Z}. \end{cases}
\]

**Definition 2.2.** We define the Maslov index of the path \((\lambda, \mu)\) by
\[
\text{Mas}\{\lambda, \mu\} = \text{Mas}_+\{\lambda, \mu\} = \sum_{j=1}^{n} \left( E\left(\frac{\theta_j(b)}{2\pi}\right) - E\left(\frac{\theta_j(a)}{2\pi}\right) \right),
\]
\[
\text{Mas}_-\{\lambda, \mu\} = \sum_{j=1}^{n} \left( \frac{\theta_j(b)}{2\pi} - \frac{\theta_j(a)}{2\pi} \right).
\]

By definition, \(\text{Mas}_\pm\{\lambda(s), \mu(s); s \in [a, b]\}\) is an integer which does not depend on the choices of the arguments \(\theta_j(s)\). By [3, Proposition 6], it does not depend on the particular choice of the inner product.

**Remark 2.3.** Let \(V\) be a real symplectic \(2n\)-dimensional symplectic space. Let \(\alpha\) be in \(\mathcal{L}(V)\) and \(\lambda(s), s \in [a, b]\) be a path in \(\mathcal{L}(V)\) such that \(\lambda(a) \cap \alpha = \lambda(b) \cap \alpha = \{0\}\). Denote by \([\lambda : \alpha]\) the Maslov-Arnol’d index defined by [5, (2.8)]. By [2, Proposition 3.27], we have \(\text{Mas}\{\lambda, \alpha\} = - \text{Mas}\{\alpha, \lambda\} = -[\lambda : \alpha]\).

We recall some concepts in [5, 19] for the calculation of the Maslov index.

Let \((V, \omega)\) be a finite-dimensional symplectic vector space. Let \(\alpha(s), s \in (-\varepsilon, \varepsilon)\) be a path in \(\mathcal{L}(V)\) differentiable at \(s = 0\). Define the form \(Q(\alpha, 0)\) on \(\alpha(0)\) by
\[
Q(\alpha, 0)(x, y) := \frac{d}{ds}|_{s=0} \omega(x, y(s)),
\]
where \(x, y \in \alpha(0), y(s) \in \alpha(s)\) and \(y(s) - y \in \beta\). It is well-known that the Hermitian form \(Q(\alpha, 0)\) is independent on the choice of \(\beta \in \mathcal{L}(V)\) with \(V = \alpha(0) \oplus \beta\).

**Definition 2.4.** Let \((V, \omega)\) be a finite-dimensional symplectic vector space. Let \((\lambda(s), \mu(s)), s \in [a, b]\) be a \(C^1\) curve of pairs of Lagrangian subspaces of \((V, \omega)\). For \(t \in [a, b]\), the crossing form \(\Gamma(\lambda, \mu, t)\) on \(\lambda(t) \cap \mu(t)\) is defined by
\[
\Gamma(\lambda, \mu, t)(u, v) := Q(\lambda(t))(u, v) - Q(\mu, t)(u, v),
\]
where \(u, v \in \lambda(t) \cap \mu(t)\). A crossing is a time \(t \in [a, b]\) such that \(\lambda(t) \cap \mu(t) \neq \{0\}\). A crossing \(t\) is called *regular* if \(\Gamma(\lambda, \mu, t)\) is non-degenerate.
Proposition 2.5. ([3] Lemma 2.5, [17] Proposition 4.1) Let \((V, \omega)\) be a finite-dimensional symplectic vector space. Let \((\lambda(s), \mu(s)), s \in [a, b]\) be a \(C^1\) curve of pairs of Lagrangian subspaces of \((V, \omega)\) with only regular crossings. Then the crossings are finite, and we have

\[
\text{Mas}(\lambda, \mu) = m^+(\Gamma(\lambda, \mu, a)) - m^-(\Gamma(\lambda, \mu, b)) + \sum_{a < s < b} \text{sign}(\Gamma(\lambda, \mu, s)),
\]

where we denote by \(m^*(Q), * = +, 0, -\) the Morse positive index, the nullity, and the Morse negative index of an Hermitian form \(Q\) respectively.

For each \(\tau > 0\), we define

\[
\mathcal{P}_\tau(V) := \{ \gamma \in C([0, \tau], \text{Sp}(V)); \gamma(0) = I_V \}.
\]

Definition 2.6. (cf. [19] Definition 4.6) Let \((V_i, \omega_l), l = 1, 2\) be two finite-dimensional symplectic vector space. Then \((V = V_1 \oplus V_2, (-\omega_1) \oplus \omega_2)\) is a symplectic vector space. Let \(W \in \mathcal{L}(V)\). Let \(\gamma(t), 0 \leq t \leq \tau\) be a path in \(\text{Sp}(V_1, V_2)\). The Maslov-type index \(i_W(\gamma)\) is defined to be \(\text{Mas}(\text{Gr} \circ \gamma, W)\). If \(P \in \text{Sp}(V_1, V_2)\), we define \(i_P(\gamma) := i_{\text{Gr}(P)}(\gamma)\). If \((V_1, \omega_1) = (V_2, \omega_2)\) and \(\gamma \in \mathcal{P}_\tau(V_1)\), we denote by \(i_1(\gamma) := i_{I_{V_1}}(\gamma), \nu_1(\gamma) := \dim(\ker(\gamma(\tau) - I_{V_1})), \nu_W(\gamma) := \dim(\text{Gr}(\gamma(\tau)) \cap W)\) and \(\nu_P(\gamma) := \dim(\ker(\gamma(\tau) - P))\).

The following lemma gives the tangent vector of given symplectic path.

Lemma 2.7. (cf. [3] Lemma 3.1) Let \((V_i, \omega_l), l = 1, 2\) be two finite-dimensional symplectic vector space. Then \((V = V_1 \oplus V_2, (-\omega_1) \oplus \omega_2)\) is a symplectic vector space. Let \(\gamma(t), t \in (-\varepsilon, \varepsilon)\) be a path in \(\text{Sp}(V_1, V_2)\). Then for each \(u, v \in V_1\), we have

\[
Q(\text{Gr} \circ \gamma, 0)((u, \gamma(0)u), (v, \gamma(0)v)) = \omega_1(-\gamma(0)^{-1}\dot{\gamma}(0)u, v),
\]

where we denote by \(\dot{\gamma} := \frac{d}{dt}\gamma\).

The Maslov-type indices have the following properties.

Lemma 2.8. (cf. [16] Theorem 6.1.8) Let \((V, \omega)\) be a finite-dimensional symplectic vector space. Let \(W \in \mathcal{L}(V \oplus V, (-\omega) \oplus \omega)\). Let \(\gamma(t) \in \mathcal{P}_\tau(V)\) be a symplectic path. Then there exists a \(C^0\) neighborhood \(\mathcal{N}\) of \(\gamma\) in \(\mathcal{P}_\tau(V)\) such that, for each \(\tilde{\gamma} \in \mathcal{N}\), there holds that

\[
i_W(\gamma) \leq i_W(\tilde{\gamma}) \leq i_W(\gamma) + \nu_W(\gamma).
\]

Proof. Let \(\mathcal{N}\) be a convex neighborhood of 0 in \(\text{sp}(V)\), the Lie algebra of \(\text{Sp}(V)\) such that the map \(\exp : \mathcal{N} \to \exp(\mathcal{N})\) is a diffeomorphism. Set

\[
\mathcal{N} := \{ \tilde{\gamma} \in \mathcal{P}_\tau(V); \tilde{\gamma}(t) \in \gamma(t) \exp(N), \text{ for each } t \in [0, \tau] \}.
\]

Then for each \(\tilde{\gamma} \in \mathcal{N}\), there exists \(A \in C([0, \tau], \mathcal{N})\) and \(A(0) = 0\), such that \(\dot{\gamma}(t) = \gamma(t) \exp(A(t)), t \in [0, \tau]\). Thus we have a homotopy \(\phi(s, t) :
Remark 3.1. Assume that $\omega$ the path $\gamma(\tau) \exp(sA(\tau))$, $s \in [0,1]$. By the definition of the Maslov-type index we can choose $N$ small enough such that $0 \leq i_W(\gamma) \leq i_W(\gamma)$ for each $\gamma \in \mathcal{N}$. By the homotopy invariance and path additivity of the Maslov-type index we have $i_W(\gamma) = i_W(\gamma) + i_W(\gamma)$. The inequality (11) then follows.

Lemma 2.9. (cf. [19, Lemma 4.4]) Let $(V, \omega_l), l = 1, 2, 3, 4$ be finite-dimensional symplectic vector spaces. Let $W$ be a Lagrangian subspace of $(V_1 \oplus V_4, (-\omega_l) \oplus \omega_4)$. Let $\gamma_l \in C([0,1], Sp(V, V_{i+1}))$, $l = 1, 2, 3$ be symplectic paths. Then we have

$$i_W(\gamma_3 \gamma_2 \gamma_1) = i_W(\gamma_2) + i_W(\gamma_3 \gamma_2(0) \gamma_1),$$

(12)

where $W' = \text{diag}(\gamma_1(1), \gamma_3(1)^{-1})W$.

### 3 Calculation of the Hörmander index

In this section we study the form $Q(\cdot, \cdot; \cdot)$ and the triple index $i(\cdot, \cdot, \cdot)$. Then we express the Hörmander index via the triple index.

#### 3.1 The form $Q(\cdot, \cdot; \cdot)$

Let $(V, \omega)$ be a complex symplectic vector space with three isotropic subspaces $\alpha, \beta, \gamma$. Define $Q := Q(\alpha, \beta; \gamma)$ on $\alpha \cap (\beta + \gamma)$ by

$$Q(x_1, x_2) := \omega(x_1, y_2) = \omega(z_1, y_2) = \omega(x_1, z_2)$$

(13)

for all $x_j = -y_j + z_j \in \alpha \cap (\beta + \gamma)$, where $y_j \in \beta$, $z_j \in \gamma$, $j = 1, 2$.

Remark 3.1. Assume that $\alpha, \beta, \gamma$ are Lagrangian subspaces of $V$ and $V = \alpha \oplus \beta = \beta \oplus \gamma$. Then the form $Q(\alpha, \beta; \gamma)$ defined by [15 (2.3)] is $-Q(\alpha, \beta; \gamma)$ here.

With the above notions, we have $\omega(x_1 + y_1, x_2 + y_2) = 0$, $\omega(x_1, x_2) = 0$ and $\omega(y_1, y_2) = 0$. It follows that $\omega(x_1, y_2) = -\omega(y_1, x_2) = \omega(x_2, y_1)$. So $Q(x_1, x_2)$ does not depend on the choices of $y_1$ and $y_2$ and the form $Q$ is a well-defined Hermitian form on $\alpha \cap (\beta + \gamma)$. Moreover, the last two equalities in [13] hold, i.e., $\omega(x_1, y_2) = \omega(z_1, y_2) = \omega(x_1, z_2)$, and we have

$$Q(\alpha, \beta; \gamma) = -Q(\alpha, \gamma; \beta).$$

(14)

The following lemma is well-known in the non-degenerate case.

Lemma 3.2. Let $(V, \omega)$ be a complex symplectic vector space with three isotropic subspaces $\alpha, \beta, \gamma$. Denote by $Q_1 := Q(\alpha, \beta; \gamma)$, $Q_2 := Q(\beta, \gamma; \alpha)$ and $Q_3 = Q(\gamma, \alpha; \beta)$.

(a) Let $z_j = x_j + y_j \in \gamma \cap (\alpha + \beta)$, where $x_j \in \alpha$, $y_j \in \beta$, $j = 1, 2$. Then we have $Q_1(x_1, x_2) = Q_2(y_1, y_2) = Q_3(z_1, z_2)$.

(b) We have $m^\pm(Q_1) = m^\pm(Q_2) = m^\pm(Q_3)$. 

Proof. (a) Note that $x_j = z_j - y_j$. So we have
\[ Q_1(x_1, x_2) = \omega(x_1, z_2) = \omega(y_1, -z_2) = Q_2(y_1, y_2). \]
Similarly we have $Q_1(x_1, x_2) = Q_3(z_1, z_2)$.

(b) Let $X$ be a linear subspace of $\alpha \cap (\beta + \gamma)$ such that $Q_1|_X < 0$ with a base $\{x_1, \ldots, x_k\}$. Let $y_j \in \beta$, $j = 1, \ldots, k$ be such that $x_j + y_j \in \gamma$. Then $y_j \in \beta \cap (\alpha + \gamma)$. Let $Y$ be the $\mathbb{C}$-linear span of $\{y_1, \ldots, y_k\}$. By (a), for each $(a_1, \ldots, a_k) \in \mathbb{C}^k \setminus \{0\}$ we have $Q_2(y, y) = Q_1(x, x) < 0$, where $x := \sum_{j=1}^k a_j x_j$ and $y := \sum_{j=1}^k a_j y_j$. In this case $y \neq 0$. So $y_1, \ldots, y_k$ are linearly independent, and $\dim Y = k$. Hence $m^-(Q_2) \geq \dim Y = \dim X$. Since $X$ is arbitrarily chosen, we have $m^-(Q_2) \geq m^-(Q_1)$. Similarly we have $m^+(Q_1) \geq m^+(Q_2)$. It follows that $m^-(Q_1) = m^+(Q_2)$. Similarly we have $m^+(Q_1) = m^+(Q_2)$ and $m^+(Q_2) = m^+(Q_3)$. \qed

Here we give the kernel of the form $Q(\alpha, \beta; \gamma)$.

Lemma 3.3. Let $(V, \omega)$ be a complex symplectic vector space with three isotropic subspaces $\alpha, \beta, \gamma$. Then we have
\[ \alpha \cap (\beta \cap (\alpha + \gamma))' = \alpha \cap (\gamma \cap (\alpha + \beta))', \quad \text{and} \quad \ker Q(\alpha, \beta; \gamma) = \alpha \cap (\beta + \gamma \cap (\beta \cap (\alpha + \gamma))'). \]
Similarly we have
\[ \alpha \cap (\beta \cap (\alpha + \gamma))' = \alpha \cap (\alpha \cap (\alpha + \beta))' = \alpha \cap (\gamma \cap (\alpha + \beta))'. \]

In particular, $\ker Q(\alpha, \beta; \gamma) = \alpha \cap \beta + \alpha \cap \gamma$ holds if $(\alpha \cap (\beta + \gamma))' = \alpha + \beta \cap \gamma$, or $(\beta \cap (\alpha + \gamma))' = \beta \cap \alpha \cap \gamma$, or $(\gamma \cap (\alpha + \beta))' = \gamma \cap \alpha \cap \beta$.

Proof. Since $\beta \cap (\alpha + \gamma) \subset \alpha + \gamma \cap (\alpha + \beta)$, we have
\[ \alpha \cap (\beta \cap (\alpha + \gamma))' \supset \alpha \cap (\alpha \cap (\alpha + \beta))' = \alpha \cap (\gamma \cap (\alpha + \beta))'. \]
Similarly we have $\alpha \cap (\beta \cap (\alpha + \gamma))' \subset \alpha \cap (\gamma \cap (\alpha + \beta))'$. So (15) holds.

Since $\beta \subset \beta' \subset (\beta \cap (\alpha + \gamma))'$, by (14) we have
\[ \ker Q(\alpha, \beta; \gamma) = \alpha \cap (\beta + \gamma \cap (\beta \cap (\alpha + \gamma))') = \alpha \cap (\beta + \gamma \cap (\beta \cap (\alpha + \gamma))'). \]

So (16) follows. Similarly we get (18). By (15) and (16) we get (17).

If $(\alpha \cap (\beta + \gamma))' = \alpha + \beta \cap \gamma$, by (17) we have
\[ \ker Q(\alpha, \beta; \gamma) = \alpha \cap (\beta + \gamma \cap (\alpha + \beta \cap \gamma)) = \alpha \cap (\beta + \alpha \cap \beta \cap \gamma) = \alpha \cap (\beta + \alpha \cap \gamma) = \alpha \cap \beta + \alpha \cap \gamma. \]

Similarly, by (16) and (18), $\ker Q(\alpha, \beta; \gamma) = \alpha \cap \beta + \alpha \cap \gamma$ holds if $(\beta \cap (\alpha + \gamma))' = \beta + \alpha \cap \gamma$, or $(\gamma \cap (\alpha + \beta))' = \gamma + \alpha \cap \beta$. \qed
Proof. By the symmetry of the statement, we only need to prove that (i)⇒(ii).

Assume that (i) holds. Clearly we have \( \alpha \cap \beta + \beta \cap \gamma \subseteq (\alpha + \gamma) \cap \beta \). Let \( y \in (\alpha + \gamma) \cap \beta \). Then there exist \( x \in \alpha \) and \( z \in \gamma \) such that \( y = x + z \). So \( x = y - z \in (\beta + \gamma) \cap \alpha \). By (i), there exist \( y_1 \in \alpha \cap \beta \) and \( z_1 \in \alpha \cap \gamma \) such that \( x = y_1 + z_1 \). So \( z_1 + z = y - y_1 \in \beta \cap \gamma \) and \( y = y_1 + (z_1 + z) \in \alpha \cap \beta + \beta \cap \gamma \). \( \square \)

Lemma 3.6. Let \( V \) be a vector space with three finite-dimensional linear subspaces \( \alpha, \beta, \gamma \). Then we have

\[
\dim(\alpha \cap \beta) + \dim(\alpha \cap \gamma) + \dim(\beta \cap \gamma) \leq \dim \alpha + \dim \beta + \dim \gamma + \dim(\alpha \cap \beta \cap \gamma) - \dim(\alpha + \beta + \gamma).
\]  
(19)

The equality in (19) holds if and only if \( \alpha \cap \beta + \alpha \cap \gamma = \alpha \cap (\beta + \gamma) \).

Proof. We have \( \dim(\beta \cap \gamma) = \dim \beta + \dim \gamma - \dim(\beta + \gamma) \). Since \( \alpha \cap \beta + \alpha \cap \gamma \subseteq \alpha \cap (\beta + \gamma) \), we have

\[
\dim(\alpha \cap \beta) + \dim(\alpha \cap \gamma) = \dim(\alpha \cap \beta + \alpha \cap \gamma) + \dim(\alpha \cap \beta \cap \gamma)
\leq \dim(\alpha \cap (\beta + \gamma)) + \dim(\alpha \cap \beta \cap \gamma)
= \dim \alpha + \dim(\beta + \gamma) - \dim(\alpha + \beta + \gamma) + \dim(\alpha \cap \beta \cap \gamma).
\]

So (19) holds. The equality in (19) holds if and only if \( \dim(\alpha \cap \beta + \alpha \cap \gamma) = \dim(\alpha \cap (\beta + \gamma)) \), if and only if \( \alpha \cap \beta + \alpha \cap \gamma = \alpha \cap (\beta + \gamma) \). \( \square \)

Corollary 3.7. Let \( (V, \omega) \) be a complex symplectic vector space of dimension \( 2n \) with three Lagrangian subspaces \( \alpha, \beta, \gamma \). Then we have

\[
\dim(\alpha \cap \beta) + \dim(\alpha \cap \gamma) + \dim(\beta \cap \gamma) \leq n + 2 \dim(\alpha \cap \beta \cap \gamma).
\]  
(20)

The equality in (20) holds if and only if \( \alpha \cap \beta + \alpha \cap \gamma = \alpha \cap (\beta + \gamma) \), if and only if \( Q(\alpha, \beta; \gamma) = 0 \).
Proof. Since $\alpha, \beta, \gamma$ are Lagrangian subspaces of $V$ and $\dim V = 2n$, we have $\dim \alpha = \dim \beta = \dim \gamma = n$ and

$$\dim(\alpha \cap \beta \cap \gamma) = \dim((\alpha + \beta + \gamma)') = 2n - \dim(\alpha + \beta + \gamma).$$

By Lemma 3.6, the inequality (20) holds, and the equality in (20) holds if and only if $\alpha \cap \beta + \alpha \cap \gamma = \alpha \cap (\beta + \gamma)$. By Corollary 3.3, $\alpha \cap \beta + \alpha \cap \gamma = \alpha \cap (\beta + \gamma)$ holds if and only if $Q(\alpha, \beta; \gamma) = 0$.

If $\epsilon$ is an isotropic subspace of $(V, \omega)$ such that $\epsilon = \epsilon' / \epsilon$, $\omega$ defines a symplectic form $\tilde{\omega}$ on $\epsilon' / \epsilon$. Moreover, for each isotropic subspace $\delta$, the image $\pi_\epsilon(\delta)$ of $\delta \cap \epsilon'$ under the canonical homomorphism: $\pi_\epsilon : \epsilon' / \epsilon \rightarrow \epsilon' / \epsilon$ is an isotropic subspace of $(\epsilon' / \epsilon, \tilde{\omega})$.

Let $\alpha, \beta, \gamma$ be three isotropic subspaces of $V$. Assume that $\epsilon \subset \beta \subset \epsilon'$. Then we have $\beta \cap \epsilon' + \epsilon = \beta$, and

$$(\alpha \cap \epsilon' + \epsilon) \cap (\beta + \gamma \cap \epsilon' + \epsilon) = (\alpha + \epsilon) \cap \epsilon' \cap (\beta + \gamma) \cap \epsilon'
= \alpha \cap (\beta + \gamma) \cap \epsilon' + \epsilon.
So we have $\pi(\alpha \cap (\beta + \gamma)) = (\pi\alpha) \cap (\pi\beta + \pi\gamma)$, and (cf. [5] (2.11)])

$$Q(\alpha, \beta; \gamma)(x_1, x_2) = Q(\pi\alpha, \pi\beta; \pi\gamma)(\pi x_1, \pi x_2),$$

(21)

here $x_1, x_2 \in \alpha \cap (\beta + \gamma) \cap \epsilon'$ and $\pi = \pi_\epsilon$.

Lemma 3.8. Let $(V, \omega)$ be a finite-dimensional complex symplectic vector space. Let $\alpha(s)$, $s \in (-\epsilon, \epsilon)$ be a path in $L(V)$ differentiable at $s = 0$. Let $\beta$ be a Lagrangian subspace of $V$. Assume that $Q(\alpha, 0)$ is positive definite. Then there exists an $\epsilon_1 \in (0, \epsilon)$ such that for $s \in (0, \epsilon_1)$, we have $V = \alpha(\pm s) + \beta$, and

$$m^\tau(Q(\alpha(0), \beta; \alpha(\pm s))) = m^\tau(Q(\beta, \alpha(\pm s); \alpha(0))) = 0.$$ (22)

Proof. By Proposition 2.5, there exists an $\epsilon_2 \in (0, \epsilon)$ such that $V = \alpha(\pm s) + \beta$ holds for each $s \in (0, \epsilon_2)$. Set $\epsilon := \alpha(0) \cap \beta$ and $\pi := \pi_\epsilon$. By the proof of [24, Corollary 1.3.4], there is a $\tilde{\beta} \in L(V)$ such that $V = \alpha(0) + \beta$ and $\pi\beta = \pi\tilde{\beta}$. Since $Q(\alpha, 0)$ is positive definite, there exists an $\epsilon_1 \in (0, \epsilon_2)$ such that for $s \in (0, \epsilon_1)$, the form $Q(\alpha(0), \tilde{\beta}; \alpha(\pm s))$ is positive (negative) definite. Let $s$ be in $(0, \epsilon_1)$. By [24], the form $Q(\pi\alpha(0), \pi\tilde{\beta}; \pi\alpha(\pm s))$ is positively (negatively) definite. Note that $\alpha(0) \subset \epsilon'$. By [24] again, the form $Q(\alpha(0), \beta; \alpha(\pm s))$ is positive (negative) definite. By Lemma 3.2, (22) holds.

3.2 The triple index and the Hörmander index

Let $(V, \omega)$ be a complex symplectic vector space of dimension $2n$. Let $\lambda_1, \lambda_2, \mu_1, \mu_2$ be four Lagrangian subspaces of $(V, \omega)$. 

The Hörmander index in the finite-dimensional case
Definition 3.11. ([2] Definition 3.4.4) Assume that there are continuous paths \(\lambda(s)\) and \(\mu(s)\), \(s \in [a, b]\) of Lagrangian subspaces of \((V, \omega)\) such that \(\lambda(a) = \lambda_1\), \(\lambda(b) = \lambda_2\), \(\mu(a) = \mu_1\), \(\mu(b) = \mu_2\). Then the Hörmander index \(s(\lambda_1, \lambda_2; \mu_1, \mu_2)\) is defined by

\[
s(\lambda_1, \lambda_2; \mu_1, \mu_2) = \text{Mas}\{\lambda, \mu_2\} - \text{Mas}\{\lambda, \mu_1\}
\]

and

\[
s(\lambda_2, \mu_1) - \text{Mas}\{\lambda_1, \mu\}.
\]

Lemma 3.10. Let \(\lambda(s)\), \(s \in [a, b]\) be a Lagrangian path of complex symplectic vector space \((V, \omega)\). Let \(\alpha \in \mathcal{L}(V)\) be such that \(\alpha \cap \lambda(a) = \alpha \cap \lambda(b) = \emptyset\). Then we have

\[
s(\alpha, \lambda(a); \lambda, \lambda(b)) = m^{-}(Q(\lambda(a), \alpha; \lambda(b))),
\]

\[
\text{Mas}\{\lambda(a), \lambda\} = \text{Mas}\{\alpha, \lambda\} + m^{-}(Q(\lambda(a), \alpha; \lambda(b))).
\]

Proof. Let \(A(s) \in \text{Hom}(\lambda(a), \alpha)\), \(s \in [a, b]\) be a path of linear maps such that, the form \(\omega(x, A(s)y)\), \(x, y \in \lambda(a)\) is Hermitian for each \(s \in [a, b]\), \(A(a) = 0\), \(\lambda(b) = \text{Gr}(A(b))\). Consider a special path \(\lambda(s) = \text{Gr}(A(s))\). By [2] Lemma 2.3.2, we have

\[
s(\alpha, \lambda(a); \lambda, \lambda(b)) = \text{Mas}\{\lambda(a), \lambda\} - \text{Mas}\{\alpha, \lambda\}
\]

\[
= -\text{Mas}\{\lambda, \lambda(a)\} + \text{Mas}\{\lambda, \alpha\}
\]

\[
= m^{-}(Q(\lambda(a), \alpha; \lambda(b))).
\]

The equality (29) follows from (28).

Corollary 3.11. ([5] (3.3.5),(3.3.7)), ([6] (2.10),(2.13)) Let \(\lambda_1, \lambda_2, \mu_1, \mu_2\) be Lagrangian subspaces of complex symplectic vector space \((V, \omega)\). Assume that \(\lambda_i \cap \mu_j = 0\), \(i, j = 1, 2\). Then we have

\[
s(\lambda_1, \lambda_2; \mu_1, \mu_2) = -s(\mu_1, \mu_2; \lambda_1, \lambda_2)
\]

\[
= m^{-}(Q(\lambda_1, \lambda_2; \mu_1)) - m^{-}(Q(\lambda_1, \lambda_2; \mu_2)).
\]

Proof. By [27], [25], [26], Lemma 5.10, Corollary 3.4 and [14] we have

\[
s(\lambda_1, \lambda_2; \mu_1, \mu_2) = -s(\mu_1, \mu_2; \lambda_1, \lambda_2)
\]

\[
= s(\mu_2, \lambda_1; \lambda_1, \lambda_2) - s(\mu_1, \lambda_1; \lambda_1, \lambda_2)
\]

\[
= m^{-}(Q(\lambda_1, \mu_2; \lambda_2)) - m^{-}(Q(\lambda_1, \mu_1; \lambda_2))
\]

\[
= m^{-}(Q(\lambda_1, \lambda_2; \mu_1)) - m^{-}(Q(\lambda_1, \lambda_2; \mu_2)).
\]
The following corollary was proved by J. J. Duistermaat \[5\, (2.16)\].

**Corollary 3.12.** Let $\alpha, \beta, \gamma$ be Lagrangian subspaces of complex symplectic vector space $V$. Define the triple index of $\alpha, \beta, \gamma$ by

$$i(\alpha, \beta, \gamma) = m^- (Q(\alpha, \delta; \beta)) + m^- (Q(\beta, \delta; \gamma)) - m^- (Q(\alpha, \delta; \gamma)). \quad (32)$$

where $\delta \in \mathcal{L}(V)$ be such that $\delta \cap \alpha = \delta \cap \beta = \delta \cap \gamma = \{0\}$. Then the triple index is well-defined.

Now we calculate the triple index $i(\alpha, \beta, \gamma)$.

**Lemma 3.13.** Let $\alpha, \beta, \gamma$ be three Lagrangian subspaces of $V$. Then we have

$$i(\alpha, \beta, \gamma) = m^+ (Q(\alpha, \beta; \gamma)) + \dim(\alpha \cap \gamma) - \dim(\alpha \cap \beta \cap \gamma) \quad (33)$$

$$\leq n - \dim(\alpha \cap \beta) - \dim(\beta \cap \gamma) + \dim(\alpha \cap \beta \cap \gamma). \quad (34)$$

**Proof.** Denote by $\epsilon := \alpha \cap \beta + \beta \cap \gamma$ and $\pi := \pi_\epsilon$. Recall that $\pi \alpha = (\alpha + \epsilon) \cap \epsilon^\perp / \epsilon$. Note that $\epsilon \subset \beta$,

$$\epsilon^\perp = (\alpha + \beta) \cap (\beta + \gamma), \text{ and}$$

$$\alpha + \epsilon = \alpha + \alpha \cap \beta + \beta \cap \gamma = \alpha + \beta \cap \gamma.$$ It follows that $(\pi \alpha) \cap (\pi \beta) = \{0\}$, and

$$(\pi \alpha) \cap (\pi \gamma) = ((\alpha + \epsilon) \cap (\alpha + \beta) \cap (\beta + \gamma) \cap (\gamma + \epsilon)) / \epsilon$$

$$= ((\alpha + \epsilon) \cap (\gamma + \epsilon)) / \epsilon$$

$$= ((\alpha + \epsilon) \cap (\epsilon + \gamma)) / \epsilon$$

$$= ((\alpha + \beta \cap \gamma) \cap (\gamma + \epsilon)) / \epsilon$$

$$= ((\alpha \cap \gamma + \beta \cap \gamma + \alpha \cap \beta + \beta \cap \gamma) / (\alpha \cap \beta + \beta \cap \gamma)$$

$$\cong (\alpha \cap \gamma) / (\alpha \cap \beta + \beta \cap \gamma).$$

Then by Lemma 3.3 we have

$$m^0 (Q(\pi \alpha, \pi \beta; \pi \gamma)) = \dim((\pi \alpha) \cap (\pi \beta)) + \dim((\pi \alpha) \cap (\pi \gamma))$$

$$= \dim(\alpha \cap \gamma) - \dim(\alpha \cap \beta \cap \gamma).$$

By Lemma 3.2 [12], Remark 3.1 and 3 [Lemma 2.4], we have

$$i(\alpha, \beta, \gamma) = (m^+ + m^0)(Q(\pi \alpha, \pi \beta; \pi \gamma))$$

$$= m^+ (Q(\alpha, \beta; \gamma)) + \dim(\alpha \cap \gamma) - \dim(\alpha \cap \beta \cap \gamma)$$

$$\leq \dim(\pi \alpha) = n - \dim(\alpha \cap \beta \cap \gamma)$$

$$= n - \dim(\alpha \cap \beta) - \dim(\beta \cap \gamma) + \dim(\alpha \cap \beta \cap \gamma).$$

$\square$
Let $\alpha, \beta$ be two Lagrangian subspaces of $V$. We define
\[
L_0(\alpha, \beta) := \{\gamma \in L(V); \alpha \cap \gamma + \beta \cap \gamma = (\alpha + \beta) \cap \gamma\}.
\] (35)

**Corollary 3.14.** Let $\alpha, \beta, \gamma$ be three Lagrangian subspaces of $V$. Assume that $\gamma \in L_0(\alpha, \beta)$. Let $(\alpha_1, \beta_1, \gamma_1)$ be a permutation of $(\alpha, \beta, \gamma)$. Then we have
\[
i(\alpha_1, \beta_1, \gamma_1) = \dim(\alpha_1 \cap \gamma_1) - \dim(\alpha_1 \cap \beta_1 \cap \gamma_1).
\] (36)

In particular, we have
\[
i(\alpha, \alpha, \beta) = i(\beta, \alpha, \alpha) = 0, \quad i(\alpha, \beta, \alpha) = n - \dim(\alpha \cap \beta).\] (37)

**Proof.** By Corollary 3.14 and Lemma 3.13, we have $Q(\alpha_1, \beta_1; \gamma_1) = 0$. By Lemma 3.13, (36) holds. Since $\beta \in L_0(\alpha, \alpha)$, (37) holds.

Denote by $J \in \text{End}(V)$ with $\omega(x, y) = \langle Jx, y \rangle$ for each $x, y \in V$.

By Lemma 3.8 there is an $\epsilon > 0$ such that, for each $s \in (0, \epsilon)$, we have $(e^{Js}\gamma) \cap \alpha = (e^{Js}\gamma) \cap \beta = (e^{Js}\gamma) \cap \gamma = \{0\}$, and
\[
m^-(Q(\beta, e^{Js}\gamma; \gamma)) = m^-(Q(\alpha, e^{Js}\gamma; \gamma)) = 0.
\]

By Corollary 3.12 we have
\[
i(\alpha, \beta, \gamma) = m^-(Q(\alpha, e^{Js}\gamma; \beta)).\] (38)

**Lemma 3.15.** Let $\lambda, \mu : [a, b] \rightarrow L(V)$ be two paths of Lagrangian subspaces of $V$. Then there is an $\epsilon > 0$ such that for each $s \in (0, \epsilon)$, we have
\[
s(\lambda(a), \lambda(b); e^{Js}(\mu(a), e^{Js}(\mu(b)) = s(\lambda(a), \lambda(b); \mu(a), \mu(b)).\] (39)

**Proof.** Let $s > 0$ be sufficiently small. According to Proposition 2.5 we have $\text{Mas}\{\lambda(s_1), e^{Js}\mu(s_2); t \in [0, s]\} = 0$, for $s_1, s_2 \in \{a, b\}$. So by the definition of Hörmander index [24], we have $s(\lambda(a), \lambda(b); \mu(s_1), e^{Js}\mu(s_1)) = 0$, for $s_1 = a, b$. Meanwhile by (25) and (26), (39) holds.

**Proof of Theorem 1.1.** Let $s > 0$ be sufficiently small. By Lemma 3.15 the proof of Corollary 3.11 and (38) we have
\[
s(\lambda_1, \lambda_2; \mu_1, \mu_2) = s(\lambda_1, \lambda_2; e^{Js}\mu_1, e^{Js}\mu_2)
= m^-(Q(\lambda_1, e^{Js}\mu_2; \lambda_2)) - m^-(Q(\lambda_1, e^{Js}\mu_1; \lambda_2))
= i(\lambda_1, \lambda_2, \mu_2) - i(\lambda_1, \lambda_2, \mu_1).
\]
By (27), Lemma 3.13 and Lemma 3.2, we have
\[
\begin{align*}
s(\lambda_1, \lambda_2; \mu_1, \mu_2) &= -s(\mu_1, \mu_2; \lambda_1, \lambda_2) + \sum_{j,k \in \{1,2\}} (-1)^{j+k+1} \dim(\lambda_j \cap \mu_k) \\
&= i(\mu_1, \mu_2, \lambda_1) - i(\mu_1, \mu_2, \lambda_2) + \sum_{j,k \in \{1,2\}} (-1)^{j+k+1} \dim(\lambda_j \cap \mu_k) \\
&= m^+(Q(\mu_1, \mu_2; \lambda_1)) + \dim(\lambda_1 \cap \mu_1) - \dim(\lambda_1 \cap \mu_1 \cap \mu_2) \\
&\quad - m^+(Q(\mu_1, \mu_2; \lambda_2)) - \dim(\lambda_2 \cap \mu_1) + \dim(\lambda_2 \cap \mu_1 \cap \mu_2) \\
&\quad + \sum_{j,k \in \{1,2\}} (-1)^{j+k+1} \dim(\lambda_j \cap \mu_k) \\
&= m^+(Q(\lambda_1, \mu_1; \mu_2)) + \dim(\lambda_1 \cap \mu_2) - \dim(\lambda_1 \cap \mu_1 \cap \mu_2) \\
&\quad - m^+(Q(\lambda_2, \mu_1; \mu_2)) - \dim(\lambda_2 \cap \mu_2) + \dim(\lambda_2 \cap \mu_1 \cap \mu_2) \\
&= i(\lambda_1, \mu_1, \mu_2) - i(\lambda_2, \mu_1, \mu_2).
\end{align*}
\]

□

Corollary 3.16. Let \((V, \omega)\) be a complex symplectic vector space of dimension \(2n\). Let \(\lambda \in \mathcal{C}([a, b], \mathcal{L}(V))\) be a Lagrangian path. Then for each \(\mu \in \mathcal{L}(V)\), we have
\[
\begin{align*}
s(\lambda(a), \lambda(b); \lambda(a), \mu) &= -i(\lambda(b), \lambda(a), \mu) \tag{40} \\
&\leq \dim(\lambda(a) \cap \lambda(b) \cap \mu) - \dim(\lambda(b) \cap \mu) \leq 0, \tag{41} \\
s(\lambda(a), \lambda(b); \lambda(a), \mu) &\geq \dim(\lambda(a) \cap \lambda(b)) + \dim(\lambda(a) \cap \mu) - \dim(\lambda(a) \cap \lambda(b) \cap \mu) - n, \tag{42} \\
&\quad - \dim(\lambda(a) \cap \lambda(b) \cap \mu) - n, \\
s(\lambda(a), \lambda(b); \lambda(b), \mu) &= i(\lambda(a), \lambda(b), \mu) \tag{43} \\
&\leq \dim(\lambda(a) \cap \lambda(b) \cap \mu) - \dim(\lambda(b) \cap \mu) \leq 0, \\
s(\lambda(a), \lambda(b); \lambda(a), \mu) &\geq \dim(\lambda(a) \cap \lambda(b)) + \dim(\lambda(a) \cap \mu) - \dim(\lambda(a) \cap \lambda(b) \cap \mu) - n, \\
&\quad - \dim(\lambda(a) \cap \lambda(b) \cap \mu) - n, \\
s(\lambda(a), \lambda(b); \lambda(b), \mu) &= i(\lambda(a), \lambda(b), \mu) - i(\lambda(a), \lambda(b), \lambda(b)) \\
&= i(\lambda(a), \lambda(b), \mu) \geq 0. \tag{44}
\end{align*}
\]

Proof. By Theorem 1.1, Lemma 3.13 and (37), we have
\[
\begin{align*}
s(\lambda(a), \lambda(b); \lambda(a), \mu) &= i(\lambda(a), \lambda(a), \mu) - i(\lambda(b), \lambda(a), \mu) \\
&= -i(\lambda(b), \lambda(a), \mu) \\
&\leq \dim(\lambda(a) \cap \lambda(b) \cap \mu) - \dim(\lambda(b) \cap \mu) \leq 0, \\
s(\lambda(a), \lambda(b); \lambda(a), \mu) &\geq \dim(\lambda(a) \cap \lambda(b)) + \dim(\lambda(a) \cap \mu) - \dim(\lambda(a) \cap \lambda(b) \cap \mu) - n, \\
&\quad - \dim(\lambda(a) \cap \lambda(b) \cap \mu) - n, \\
s(\lambda(a), \lambda(b); \lambda(b), \mu) &= i(\lambda(a), \lambda(b), \mu) - i(\lambda(a), \lambda(b), \lambda(b)) \\
&= i(\lambda(a), \lambda(b), \mu) \geq 0.
\end{align*}
\]

By Definition 3.9 we obtain (44). □
4 Iteration inequalities of Maslov-type index

4.1 Iteration inequalities with periodical boundary condition

Let \( \gamma_l \in \mathcal{P}_{\tau_l}(V) \), \( l = 1, \ldots, k \) be \( k \) symplectic paths starting from the identity \( I_V \), where \( \tau_l \geq 0 \). For each \( l = 1, \ldots, k \), we set

\[
M_l := \gamma_l(\tau_l), \quad T_l := \sum_{j=1}^{l} \tau_j, \quad \tilde{M}_l = \prod_{j=1}^{l} M_{l+1-j}.
\] (45)

The iteration of \( \gamma_1, \ldots, \gamma_k \) is a symplectic path \( \tilde{\gamma} \in \mathcal{P}_{T_k}(V) \) defined by

\[
\tilde{\gamma}(t) = \gamma_l(t - T_{l-1}) \tilde{M}_{l-1}, \quad T_{l-1} \leq t \leq T_l, \; l = 1, \ldots, k.
\] (46)

Let \( \tau > 0 \) be a positive number and \( k \in \mathbb{N} \) be a positive integer. For each symplectic path \( \gamma \in \mathcal{P}_\tau(V) \), we define its \( k \)-th iteration \( (\gamma,k) \) to be the iteration path of \( k \) copies of \( \gamma \).

If \( \gamma \) is a real symplectic path, the iteration inequalities for \( (\gamma,k) \) with periodical boundary condition was obtained by C. Liu and Y. Long in 1997 (cf. \[16, Theorem 10.1.3\]).

Our iteration inequalities with periodic boundary condition read as follows.

**Theorem 4.1.** Let \((V,\omega)\) be a symplectic vector space of dimension \( m \) and \( k \geq 2 \) be an integer. Let \( \gamma_l \in \mathcal{P}_{\tau_l}(V), \; l = 1, \ldots, k \) be \( k \) symplectic paths starting from the identity \( I_V \), where \( \tau_l \geq 0 \). Let \( \tilde{\gamma} \in \mathcal{P}_{T_k}(\gamma) \) be the iteration of \( \gamma_1, \ldots, \gamma_k \).

Let \( M_l \) and \( \tilde{M}_l \) be defined by \[(45)\]. Denote by \( \nu_1(M) := \dim \ker(M - I_V) \) and \( \mathcal{N}_1(M) := \ker(M - I_V) \) for \( M \in \text{Sp}(V) \). For \( \mathcal{M} := (M_1, M_2, \ldots, M_k) \), we define

\[
A(\mathcal{M}) := \dim \left( \bigcap_{l=1}^{k} \mathcal{N}_1(M_l) \right),
\] (47)

\[
B(\mathcal{M}) := \sum_{l=2}^{k} \dim(\mathcal{N}_1(M_l) \cap \mathcal{N}_1(\tilde{M}_{l-1})) - \sum_{l=2}^{k-1} \nu_1(\tilde{M}_l).
\] (48)

Then we have

\[
\sum_{l=1}^{k} \nu_1(M_l) - B(\mathcal{M}) - m(k-1) \leq i_1(\tilde{\gamma}) - \sum_{l=1}^{k} i_1(\gamma_l)
\leq B(\mathcal{M}) - \nu_1(\tilde{M}_k),
\] (49)

\[
B(\mathcal{M}) \leq A(\mathcal{M}).
\] (50)

**Proof.** We divide the proof into four steps.

**Step 1.** The case that \( k = 2 \).
In this case we have \( A(\mathcal{M}) = B(\mathcal{M}) \). By Lemma 2.9, the definition of the Maslov-type index and the Hörmander index we have
\[
i_1(\tilde{\gamma}) - i_1(\gamma_1) - i_1(\gamma_2) = i_1(\gamma_2 M_1) - i_1(\gamma_2) \\
= \text{Mas}\{\gamma_2, \text{Gr}(M_1^{-1})\} - \text{Mas}\{\gamma_2, \text{Gr}(I_V)\} \\
= s(\text{Gr}(I_V), \text{Gr}(M_2); \text{Gr}(I_V), \text{Gr}(M_1^{-1})).
\]
By Corollary 3.16 we have
\[
\nu_1(M_1) + \nu_1(M_2) - \dim(\ker(M_2 - I_V) \cap \ker(M_1 - I_V)) - m \\
\leq i_1(\tilde{\gamma}) - i_1(\gamma_1) - i_1(\gamma_2) \\
= -i(\text{Gr}(M_2), \text{Gr}(I_V), \text{Gr}(M_1^{-1})) \\
\leq -\nu_1(M_1 M_2) + \dim(\ker(M_2 - I_V) \cap \ker(M_1 - I_V)).
\]
So the case that \( k = 2 \) follows.

**Step 2.** The inequalities \((49)\) and \((50)\) hold. By Step 1, for \( l = 2, \ldots, k \) we have
\[
\nu_1(\tilde{M}_{l-1}) + \nu_1(M_l) - \dim(\mathcal{R}_l(M_l) \cap \mathcal{R}_l(\tilde{M}_{l-1})) - m \\
\leq i_1(\tilde{\gamma}|_{[0, T_l]} - i_1(\tilde{\gamma}|_{[0, T_{l-1}]} - i_1(\gamma_l) \\
\leq -\nu_1(M_l) + \dim(\mathcal{R}_l(M_l) \cap \mathcal{R}_l(\tilde{M}_{l-1})).
\]
Add up \((53)\) for \( l = 2, \ldots, k \), we obtain \((49)\) and \((50)\).

**Step 3.** Let \( V_1, V_2, V_3 \) be linear subspaces of a vector space \( V \). Assume that \( V_1 \subset V_2 \) and \( \dim V_2 < +\infty \). Then we have
\[
\frac{V_1}{V_1 \cap V_3} \approx \frac{V_1 + V_3}{V_3} \subset \frac{V_2 + V_3}{V_3} \approx \frac{V_2}{V_2 \cap V_3}.
\]
So we obtain
\[
\dim \frac{V_1}{V_1 \cap V_3} \leq \dim \frac{V_2}{V_2 \cap V_3}.
\]

**Step 4.** The inequality \((51)\) holds. For \( l = 2, \ldots, k \) we have \( \bigcap_{j=1}^{l-1} \mathcal{R}(M_j) \subset \mathcal{R}(\tilde{M}_{l-1}) \). By Step 3 we have
\[
\dim \bigcap_{j=1}^{l-1} \mathcal{R}(M_j) \leq \dim \frac{\mathcal{R}(\tilde{M}_{l-1})}{\mathcal{R}(\tilde{M}_{l-1}) \cap \mathcal{R}(M_l)}.
\]
Add up \((55)\) for \( l = 2, \ldots, k \), we obtain
\[
\nu_1(M_1) - A(\mathcal{M}) \leq \nu_1(M_1) - B(\mathcal{M}).
\]
So \((51)\) holds.

Now we generalize [16, Theorem 10.2.2] to the complex case. Our method gives another proof of [16, Theorem 10.2.2].

Proof. For each \( R \lambda \) by a similar proof in \([16, \text{Theorem 1.3.1, Lemma 1.3.2, Theorem 1.7.3}]\), \( (V, R) \) \( \lambda \) notation. Y. Long’s book \([16]\) deals with standard real symplectic space \((\mathbb{R}^{2n}, \omega)\), the real symplectic matrix \( M \in \text{Sp}(2n, \mathbb{R}) := \{M \in \mathbb{R}^{2n \times 2n} \mid M^TJM = J\} \), where \( M^T \) is the transpose of matrix \( M \) and \( J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \). The set of real symplectic paths from the identity is defined by \([16, 2.0.1]:\)

\[ \mathcal{P}_r(2n, \mathbb{R}) := \{\gamma \in C([0, \tau], \text{Sp}(2n, \mathbb{R})); \gamma(0) = I_{2n}\}. \]

Let \((\mathbb{C}^{2n}, \omega)\) be the standard complex symplectic space, where \( \omega(x, y) = \langle Jx, y \rangle \), \( \forall x, y \in \mathbb{C}^{2n} \). Since \( \mathbb{C}^{2n} = \mathbb{R}^{2n} \otimes \mathbb{C} = \mathbb{R}^{2n} \oplus \sqrt{-1} \mathbb{R}^{2n}, \) \( \text{Sp}(2n, \mathbb{R}) \subset \text{Sp}(\mathbb{C}^{2n}) \). Thus if \( \gamma \in \mathcal{P}_r(2n, \mathbb{R}) \), then \( \gamma \in \mathcal{P}_r(\mathbb{C}^{2n}) \), so \( i_1(\gamma) \) has been defined in Definition 2.6.

We have these facts: if a path \( \gamma \in \mathcal{P}_r(2n, \mathbb{R}) \), then

\[ i_1(\gamma) = i_1^L(\gamma) \text{ (in } [16, \text{Chapter 5}]) + n, \]

\[ \nu_1(M) = \nu_1^I(M) \text{ (in } [16, \text{Definition 5.1}]). \]

For \( M \in \text{Sp}(V) \), we define the elliptic height \( e(M) \) of \( M \) by the total algebraic multiplicity of all eigenvalues of \( M \) on the unit circle \( S^1 \).

**Theorem 4.3.** (cf. \([16, \text{Theorem 10.2.2}]\)) Let \((V, \omega)\) be a symplectic vector space of dimension \( m \). For any \( \gamma \in \mathcal{P}_r(V) \), set \( M := \gamma(\tau) \). Then for any \( k_1, k_2 \in \mathbb{N} \), we have

\[
\nu_1(M^{k_1}) + \nu_1(M^{k_2}) - \nu_1(M^{(k_1,k_2)}) - \frac{e(M) + m}{2} \\
\leq i_1(\gamma, k_1 + k_2) - i_1(\gamma, k_1) - i_1(\gamma, k_2) \\
\leq \nu_1(M^{(k_1,k_2)}) - \nu_1(M^{k_1 + k_2}) + \frac{e(M) - m}{2},
\]

where \((k_1, k_2)\) is the greatest common divisor of \( k_1 \) and \( k_2 \).

**Proof.** For each \( \lambda \in \mathbb{C} \), we denote by \( E_\lambda(M) \) the root vector space of \( M \) belonging to the eigenvalue \( \lambda \). Set

\[
V_1 := \bigoplus_{|\lambda|=1} E_\lambda(M), \quad V_2 := \bigoplus_{|\lambda| \neq 1} E_\lambda(M), \\
M_1 := M|_{V_1}, \quad M_2 := M|_{V_2}, \\
\alpha := \bigoplus_{|\lambda| > 1} E_\lambda(M), \quad \beta := \bigoplus_{|\lambda| < 1} E_\lambda(M), \\
A := M|_{\alpha}, \quad B := M|_{\beta}.
\]

By a similar proof in \([16, \text{Theorem 1.3.1, Lemma 1.3.2, Theorem 1.7.3}]\), \( V_1, V_2 \) are symplectic subspaces of \( V, V_1 = V_2^\perp, M_l \in \text{Sp}(V_l), l = 1, 2, V = V_1 \oplus V_2, M = M_1 \oplus M_2, e(M_1) = \dim V_1, e(M_2) = 0, \alpha, \beta \) are Lagrangian subspaces of \( V_2, V_2 = \alpha \oplus \beta, \) and \( M_2 = A \oplus B \). We denote by \( m_l := \dim V_l, l = 1, 2 \). Then we have \( m_2 = 2 \dim \alpha \). Clearly for each \( k \in \mathbb{N} \) we have

\[
\nu_1(M^k) = \nu_1(M_1^k), \quad \nu_1(M_2^k) = 0, \\
e(M) = e(M_1), \quad e(M_2) = 0.
\]
Note that \( \text{ker}(M_1^{k_1} - I_{V_1}) \cap \text{ker}(M_1^{k_2} - I_{V_1}) = \text{ker}(M_1^{(k_1,k_2)} - I_{V_1}) \). By (52) we have

\[
\nu_1(M_1^{k_1}) + \nu_1(M_1^{k_2}) - \nu_1(M_1^{(k_1,k_2)}) - m_1 \\
\leq -i(\text{Gr}(M_1^{k_2}), \text{Gr}(I_{V_1}), \text{Gr}(M_1^{-k_1})) \\
\leq \nu_1(M_1^{(k_1,k_2)}) - \nu_1(M_1^{k_1+k_2}).
\]

(57)

Since \( e(M_2) = 0 \), we have

\[
\text{Gr}(M_2^{k_2}) \cap \text{Gr}(I_{V_2}) = \text{Gr}(M_2^{-k_1}) \cap \text{Gr}(I_{V_2}) = \text{Gr}(M_2^{k_2}) \cap \text{Gr}(M_2^{-k_1}) = \{0\}.
\]

So the form \( Q := Q(\text{Gr}(M_2^{k_2}), \text{Gr}(I_{V_2}); \text{Gr}(M_2^{-k_1})) \) is a non-degenerate form on \( \text{Gr}(M_2^{k_2}) \). Since \( \alpha \times \alpha \in \mathcal{L}(V_2 \times V_2) \) and \( M_2 \) is in quasi-diagonal form, that is, with respect a suitable basis for \( V_2 = \alpha \oplus \beta \), \( M_2 = \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix} \), \( A \in \text{GL}(\mathbb{C}^{\nu}) \), we have

\[
Q|_{\text{Gr}(M_2^{k_2}) \cap (\alpha \times \alpha)} = Q|_{\text{Gr}(M_2^{k_2}) \cap (\beta \times \beta)} = 0,
\]

\[
\text{Gr}(M_2^{k_2}) = (\text{Gr}(M_2^{k_2}) \cap (\alpha \times \alpha)) \oplus (\text{Gr}(M_2^{k_2}) \cap (\beta \times \beta)).
\]

So we have

\[
i(\text{Gr}(M_2^{k_2}), \text{Gr}(I_{V_2}), \text{Gr}(M_2^{-k_1})) = m^+(Q) = \text{dim } \alpha = \frac{m_2}{2}.
\]

(58)

Clearly we have

\[
i(\text{Gr}(M_2^{k_2}), \text{Gr}(I_{V}), \text{Gr}(M^{-k_1})) = i(\text{Gr}(M_1^{k_2}), \text{Gr}(I_{V_1}), \text{Gr}(M_1^{-k_1})) \\
+ i(\text{Gr}(M_2^{k_2}), \text{Gr}(I_{V_2}), \text{Gr}(M_2^{-k_1})).
\]

Our result then follows from (52), (57) and (58).

\[\square\]

### 4.2 Iteration inequalities with non-periodical boundary conditions

In this subsection we derive some iteration inequalities with non-periodical boundary conditions from Theorem 1.1.

Firstly we give the following useful lemma in the study of Maslov-type index with brake symmetry.

Let \( (V, \omega) \) be a complex symplectic vector space of dimension \( m \). Denote by \( (\bar{V}, \bar{\omega}) := (V \times V, (-\omega) \times \omega) \). Define \( f : \bar{V} \to \bar{V} \) by \( f(x, y) = (y, x) \) for each \( x, y \in V \). For each \( \gamma \in C([0, \tau], \text{Sp}(V)) \), we define \( R(\gamma) \in C([0, \tau], \text{Sp}(V)) \) by \( R(\gamma)(t) = \gamma(\tau - t) \) for \( t \in [0, \tau] \).

**Lemma 4.4.** Let \( f \) and \( R \) be the maps defined above. For each \( \gamma \in C([0, \tau], \text{Sp}(V)) \) and \( W \in \mathcal{L}(V) \), we have

\[
i_W(\gamma(0)^{-1} R(\gamma)(\tau - t)^{-1}; -\omega) = i_f(W)(\gamma).
\]

(59)
Proof. Define $T(\gamma) \in C([0, 2\tau], \text{Sp}(V))$ by
\[
T(\gamma)(t) = \begin{cases} 
\gamma(t) & \text{if } t \in [0, \tau], \\
(\gamma(t) - \tau)^{-1}\gamma(0) & \text{if } t \in [\tau, 2\tau].
\end{cases}
\]
Then $T(\gamma)$ is a contractible loop with a contraction $T(\gamma(s))$ for $s \in [0, 1]$. So we have
\[
i_{f(W)}(T(\gamma)) = i_{f(W)}(\gamma) + i_{f(W)}(\gamma(\tau)^{-1}\gamma(0)) = 0.
\]
By the definition of Maslov-type index and $f \in \text{Sp}((\hat{V}, -\tilde{\omega}),(\hat{V}, \tilde{\omega}))$ we have
\[
i_{W}(\gamma(0)^{-1}R(\gamma)\gamma^{-1}; -\omega) = i_{f(W)}(\gamma(\tau)^{-1}\gamma(0)) = -i_{f(W)}(\gamma(\tau)^{-1}\gamma(0)).
\]
So \([50]\) holds.

Let $\dim V = 2n$, $\alpha_1, \alpha_2$ be two Lagrangian subspaces of $(V, \omega)$ with $V = \alpha_1 \oplus \alpha_2$. Set $\tilde{\alpha}_j := \alpha_j \times \alpha_j$ for $j = 1, 2$. Then we have $\tilde{\alpha}_j \in \mathcal{L}(\hat{V})$. Set $N = (-I_{\alpha_1}) \oplus I_{\alpha_2}$. Then $N^2 = IV$ and $\omega(Nx, Ny) = -\omega(x, y)$ for each $x, y \in V$.

Corollary 4.5. Let $f$ and $R$ be the maps defined above. For each $\gamma \in C([0, \tau], \text{Sp}(V))$ and $W \in \mathcal{L}(V)$, we have
\[
i_{W}(N\gamma(0)^{-1}R(\gamma)\gamma^{-1}N) = i_{f(\text{diag}(N, N)W)}(\gamma).
\]
Proof. By Lemma [29] and $N \in \text{Sp}((V, \omega), (V, -\omega))$ we have
\[
i_{W}(N\gamma(0)^{-1}R(\gamma)\gamma^{-1}N) = i_{\text{diag}(N, N)W}(\gamma(0)^{-1}R(\gamma)\gamma^{-1}; -\omega) = i_{f(\text{diag}(N, N)W)}(\gamma).
\]

Note that we have $\text{Gr}(I_V) \in \mathcal{L}_0(\tilde{\alpha}_1, \tilde{\alpha}_2)$, where $\mathcal{L}_0(\cdot, \cdot)$ is defined by \([35]\).

We denote by $p_j$ the projection of $\hat{V}$ onto $\tilde{\alpha}_j$ induced by $\hat{V} = \tilde{\alpha}_1 \oplus \tilde{\alpha}_2$ for $j = 1, 2$. Then for each $\mu \in \mathcal{L}_0(\tilde{\alpha}_1, \tilde{\alpha}_2)$, we have
\[
p_j(\mu) = \tilde{\alpha}_j \cap \mu, \quad \text{Gr}(I_V) \cap \mu = S_1(\mu) \oplus S_2(\mu),
\]
where we denote by $S_j(\mu) := p_j(\mu) \cap \text{Gr}(I_{\alpha_j})$. Moreover, the map $p_j$ induces a bijection between $\mathcal{L}_0(\tilde{\alpha}_1, \tilde{\alpha}_2)$ and the Grassmannian of $\tilde{\alpha}_j$ for each $j = 1, 2$.

The following proposition is useful in comparing Maslov-type indices with a class of Lagrangian boundary conditions.

Proposition 4.6. Let $\lambda, \mu_1, \mu_2$ be three Lagrangian subspaces of $\hat{V}$. Assume that $\mu_1, \mu_2 \in \mathcal{L}_0(\tilde{\alpha}_1, \tilde{\alpha}_2)$ and $p_2(\mu_1) \subset p_2(\mu_2)$. Then we have
\[
s(\text{Gr}(I_V), \lambda; \mu_1, \mu_2) = \dim S_2(\mu_2) - \dim S_2(\mu_1) - i(\lambda, \mu_1, \mu_2) = i(\lambda, \mu_2, \mu_1) - \dim S_1(\mu_1) + \dim S_1(\mu_2) \geq \dim \frac{\lambda \cap \mu_1}{\lambda \cap \mu_1 \cap \mu_2} - \dim S_1(\mu_1) + \dim S_1(\mu_2).
\]
Hence there holds that $\text{Gr}(I_V) \cap (\mu_1 + \mu_2) = S_1(\mu_1) \oplus S_2(\mu_2)$
$$= \text{Gr}(I_V) \cap \mu_1 + \text{Gr}(I_V) \cap \mu_2,$$
$$\text{Gr}(I_V) \cap (\mu_1 \cap \mu_2) = S_1(\mu_2) \oplus S_2(\mu_1).$$

Hence there holds that $\text{Gr}(I_V) \in L_0(\mu_1, \mu_2)$. By Corollary 3.14 we have
$$i(\text{Gr}(I_V), \mu_1, \mu_2) = \dim(\text{Gr}(I_V) \cap \mu_2) - \dim(\text{Gr}(I_V) \cap \mu_1 \cap \mu_2)$$
$$= (\dim S_1(\mu_2) + \dim S_2(\mu_2)) - (\dim S_1(\mu_2) + \dim S_2(\mu_1))$$
$$= \dim S_2(\mu_2) - \dim S_2(\mu_1),$$
and similarly $i(\text{Gr}(I_V), \mu_2, \mu_1) = \dim S_1(\mu_1) - \dim S_1(\mu_2)$. By Theorem 1.1 and Lemma 3.13 we have
$$s(\text{Gr}(I_V), \lambda; \mu_1, \mu_2) = i(\text{Gr}(I_V), \mu_1, \mu_2) - i(\lambda, \mu_1, \mu_2)$$
$$= \dim S_2(\mu_2) - \dim S_2(\mu_1) - i(\lambda, \mu_1, \mu_2)$$
$$= -s(\text{Gr}(I_V), \lambda; \mu_2, \mu_1) \quad \text{(by 26)}$$
$$= i(\lambda, \mu_2, \mu_1) - i(\text{Gr}(I_V), \mu_2, \mu_1)$$
$$= i(\lambda, \mu_2, \mu_1) - \dim S_1(\mu_1) + \dim S_1(\mu_2)$$
$$\geq \dim \frac{\lambda \cap \mu_1}{\lambda \cap \mu_1 \cap \mu_2} - \dim S_1(\mu_1) + \dim S_1(\mu_2).$$

\[\square\]

**Proposition 4.7.** Let $\lambda$ be a Lagrangian subspace of $\tilde{V}$ and $\beta$ be a Lagrangian subspace of $V$. Then we have
$$s(\text{Gr}(I_V), \lambda; \alpha_1, \beta \times \alpha_1) \geq \dim \frac{\lambda \cap \alpha_1}{\lambda \cap ((\beta \times \alpha_1) \times \alpha_1)} + \dim(\beta \cap \alpha_1) - n. \quad (65)$$

**Proof.** We have $\text{Gr}(I_V) \in L_0(\beta \times \alpha_1, \tilde{\alpha}_1)$. In fact, we have
$$\text{Gr}(I_V) \cap (\beta \times \alpha_1 + \tilde{\alpha}_1) = \text{Gr}(I_V) \cap \tilde{\alpha}_1 = \text{Gr}(I_V) \cap (\beta \times \alpha_1) + \text{Gr}(I_V) \cap \tilde{\alpha}_1.$$ 

By Corollary 5.14 we have
$$i(\text{Gr}(I_V), \beta \times \alpha_1, \tilde{\alpha}_1) = \dim(\text{Gr}(I_V) \cap \tilde{\alpha}_1)$$
$$- \dim(\text{Gr}(I_V) \cap (\beta \times \alpha_1) \cap \tilde{\alpha}_1)$$
$$= n - \dim(\beta \cap \alpha_1).$$

By (26), Theorem 1.1 and Lemma 3.13 we have
$$s(\text{Gr}(I_V), \lambda; \tilde{\alpha}_1, \beta \times \alpha_1) = i(\lambda, \beta \times \alpha_1, \tilde{\alpha}_1) - i(\text{Gr}(I_V), \beta \times \alpha_1, \tilde{\alpha}_1)$$
$$\geq \dim \frac{\lambda \cap \tilde{\alpha}_1}{\lambda \cap ((\beta \times \alpha_1) \times \alpha_1)} - (n - \dim(\beta \cap \alpha_1))$$
$$= \dim \frac{\lambda \cap \tilde{\alpha}_1}{\lambda \cap ((\beta \times \alpha_1) \times \alpha_1)} + \dim(\beta \cap \alpha_1) - n.$$
The iteration of a symplectic path \( \gamma \in \mathcal{P}_\tau(V) \) for the brake symmetry is defined as follows.

**Definition 4.8.** (cf. [13 (4.3),(4.4)]) Given a \( \tau > 0 \), a positive integer \( k \), and a path \( \gamma \in \mathcal{P}_\tau(V) \), we define

\[
\gamma^{(k)}(t) = \begin{cases} 
\gamma(t - 2j\tau)(\gamma(2\tau))^j, & t \in [2j\tau,(2j+1)\tau], j \in [0,\frac{k-1}{2}], j \in \mathbb{Z}, \\
N\gamma(2j\tau - t)N(\gamma(2\tau))^j, & t \in [(2j-1)\tau,2j\tau], j \in [1,\frac{k}{2}], j \in \mathbb{Z},
\end{cases}
\]

(66)

where \( \gamma(2\tau) = N\gamma(\tau)^{-1}N\gamma(\tau) \). We call \( \gamma^{(k)} \) the \( k \)-th \( N \)-brake iteration of \( \gamma \). The map \( \gamma(2\tau) \) is called the Poincaré map of \( \gamma \) at \( 2\tau \).

**Theorem 4.9.** Let \( \tilde{\gamma} \in \mathcal{P}_{\tau_1+\tau_2}(V) \) be the iteration of \( \gamma_1 \in \mathcal{P}_{\gamma_1}(V) \) and \( \gamma_2 \in \mathcal{P}_{\tau_2}(V) \). Then we have

\[
i_{\tilde{\alpha}_2}(\tilde{\gamma}) - i_{\tilde{\alpha}_2}(\gamma_1) - i_1(\gamma_2) \in [\nu_1(\gamma_2) - 2n,0].
\]

(67)

**Proof.** Set \( M_j := \gamma_j(\tau_j) \) for \( j = 1,2 \). By Definition 2.6, Lemma 2.9, Definition 3.9 and Corollary 3.16 we have

\[
i_{\tilde{\alpha}_2}(\tilde{\gamma}) - i_{\tilde{\alpha}_2}(\gamma_1) - i_1(\gamma_2) = i_{\tilde{\alpha}_2}(\gamma_2 M_1) - i_1(\gamma_2) \\
= i_{(M_1\alpha_2)\times\alpha_2}(\gamma_2) - i_1(\gamma_2) \\
= s(Gr(I_V),Gr(M_2);Gr(I_V),(M_1\alpha_2)\times\alpha_2) \\
= -i(Gr(M_2),Gr(I_V),(M_1\alpha_2)\times\alpha_2) \in [\nu_1(\gamma_2) - 2n,0].
\]

(68)

Then we get the following iteration inequality with brake symmetry (cf. [12 Theorem 2.4. 1°]).

**Corollary 4.10.** For each \( \gamma \in \mathcal{P}_\tau(V) \) and \( k \in \mathbb{N} \), we have

\[
i_{\tilde{\alpha}_2}(\gamma^{(k)}) \in i_{\tilde{\alpha}_2}(\gamma^{(k-2(\frac{k-1}{2}))}) + \left[ \frac{k-1}{2} \right] i_1(\gamma^{(2)}) \\
+ \left[ \frac{k-1}{2} \right] (\nu_1(\gamma^{(2)}) - 2n,\nu_1(\gamma^{(2)}) - \nu_1(\gamma^{(2)}\frac{k-1}{2})).
\]

(69)

**Proof.** By Corollary 4.5 and Lemma 2.9 we have

\[
i_1(NR(\gamma)\gamma(\tau)^{-1}N) = i_1(\gamma),
\]

(69)

\[
i_1(NR(\gamma)\gamma(\tau)^{-1}N \gamma(\tau)) = i_1(\gamma(\tau)NR(\gamma)\gamma(\tau)^{-1}N) = i_1(\gamma N \gamma(\tau)^{-1}N).
\]

(70)

By (69), (70) and path additivity of Maslov-type index, we have

\[
i_1((\gamma^{(3)})|_{\tau,3\tau})\gamma(\tau)^{-1} = i_1(\gamma^{(2)}).
\]

(71)
Note that $\gamma^{(\lfloor \frac{k-1}{2} \rfloor)}$ is the $[k-1]/2$-th iteration of $\gamma^{(2)}$. By (71), Theorem 4.9 and Theorem 4.11 we have
\[
i_{\tilde{\alpha}_2}^{(\gamma)}(\gamma) = i_{\tilde{\alpha}_2}^{(\gamma)}(\gamma) + i_1^{(\gamma)}[(\gamma^{(2)}(\lfloor \frac{k-1}{2} \rfloor)) + [\nu(\gamma^{(2)}(\lfloor \frac{k-1}{2} \rfloor))] - 2n, 0]
\subset i_{\tilde{\alpha}_2}^{(\gamma)}(\gamma) + \left[\frac{k-1}{2}\right]i_1^{(\gamma)}(2\gamma)\left[\nu(\gamma) - 2n, \nu(\gamma) - \nu(\gamma^{(2)}(\lfloor \frac{k-1}{2} \rfloor))\right].
\]

□

We have the following iteration inequality with focal-type boundary value condition.

**Theorem 4.11.** Let $\alpha$ be a Lagrangian subspace of $V$. Let $\gamma_1 \in \mathcal{P}_{1+\tau_2}(V)$ be the iteration of $\gamma_1 \in \mathcal{P}_{1}(V)$ and $\gamma_2 \in \mathcal{P}_{\tau_2}(V)$. Set $M_j := \gamma_j(\tau_j)$ for $j = 1, 2$. Then we have
\[
i_{\alpha \times 1}^{(\gamma)}(\gamma) - i_{\alpha \times 1}^{(\gamma)}(\gamma) - i_{\tilde{\alpha}_1}^{(\gamma)}(\gamma)
\geq \dim \frac{\text{Gr}(M_2) \cap \tilde{\alpha}_1}{\text{Gr}(M_2) \cap ((\alpha_1 \cap (M_1 \alpha)) \times \alpha_1)} + \dim((M_1 \alpha) \cap \alpha_1) - n.
\]

**Proof.** By Definition 2.6, Lemma 2.9, Definition 3.9 and Proposition 4.7 we have
\[
i_{\alpha \times 1}^{(\gamma)}(\gamma) - i_{\alpha \times 1}^{(\gamma)}(\gamma) - i_{\tilde{\alpha}_1}^{(\gamma)}(\gamma) = i_{\alpha \times 1}^{(\gamma)}(\gamma_2 M_1) - i_{\tilde{\alpha}_1}^{(\gamma)}(\gamma_2)
\geq \dim \frac{\text{Gr}(M_2) \cap \tilde{\alpha}_1}{\text{Gr}(M_2) \cap ((\alpha_1 \cap (M_1 \alpha)) \times \alpha_1)} + \dim((M_1 \alpha) \cap \alpha_1) - n.
\]

□

The above theorem has the following two important corollaries.

The following corollary generalizes and strengthens [16, Theorem 13.5.4]. Note that the inequality $i_{\tilde{\alpha}_1}^{(\gamma)}(\gamma) \geq n$ holds there.

**Corollary 4.12.** For each $k \in \mathbb{N}$, we have
\[
i_{\tilde{\alpha}_2}^{(\gamma)}(\gamma) \geq k(i_{\tilde{\alpha}_1}^{(\gamma)} + \nu(\gamma) - n).
\]

**Proof.** Set $M := \gamma(\tau)$. If $x, M x \in \alpha_1$, we have $NM^{-1}NM x = x$. So we have $\nu(i_{\tilde{\alpha}_1}^{(\gamma)}) \geq \nu(i_{\tilde{\alpha}_1}^{(\gamma)})$ for each $j \in \mathbb{N}$. Let the map $R$ be defined in Lemma 4.4. By Corollary 4.3 we have
\[
i_{\tilde{\alpha}_1}^{(\gamma)}(NR(\gamma)\gamma(\tau)^{-1}N) = i_{\tilde{\alpha}_1}^{(\gamma)}.
\]

(74)
By Proposition 4.6 and Theorem 4.11, we have
\[ i\bar{\alpha}_2(\gamma^k) \geq i\bar{\alpha}_1(\gamma^k) + \nu\bar{\alpha}_1(\gamma^k) - n \]
\[ \geq ki\bar{\alpha}_1(\gamma) + \sum_{j=1}^{k}(\nu\bar{\alpha}_1(\gamma^j) - n) \]
\[ \geq k(i\bar{\alpha}_1(\gamma) + \nu\bar{\alpha}_1(\gamma) - n). \]

\[ \square \]

The following corollary is different from [15, Theorem 3.7]. It is enough to prove [15, Theorem 4.2]. Note that the inequality
\[ i\bar{\alpha}_1(\gamma^{(2)}) \geq i\bar{\alpha}_1(\gamma) \geq n \]
holds there.

**Corollary 4.13.** For each \( k \in \mathbb{N} \), we have
\[ i_{\alpha_2 \times \alpha_1}(\gamma^{(2k+1)}) \geq i\bar{\alpha}_1(\gamma^{(2k+1)}) - n \]
\[ \geq ki\bar{\alpha}_1(\gamma^{(2)}) + \nu\bar{\alpha}_1(\gamma^{(2j)}) - n + i\bar{\alpha}_1(\gamma) - n \]
\[ \geq k(i\bar{\alpha}_1(\gamma^{(2)}) + \nu\bar{\alpha}_1(\gamma^{(2)}) - n) + i\bar{\alpha}_1(\gamma) - n. \]

**5 The almost existence of mean indices**

In this section we follow the lines of [15, §1.8] and prove almost existence of mean indices for given complete autonomous Hamiltonian system on compact symplectic manifold with symplectic trivial tangent bundle and given autonomous Hamiltonian system on regular compact energy hypersurface of symplectic manifold with symplectic trivial tangent bundle. Our main tools will be Kingman’s subadditive ergodic theorem and Theorem 4.1.

Firstly we review Kingman’s subadditive ergodic theorem.

**Definition 5.1.** A family \( x = (x_{kl}; k, l \in \mathbb{N}, k < l) \) of random variables satisfying the following three conditions is called subadditive process.

\( (S_1) \) Whenever \( j < k < l \), \( x_{j,l} \leq x_{j,k} + x_{k,l} \).

\[ \nu\bar{\alpha}_1(\gamma^{(2)}) \geq \nu\alpha_1 \times \alpha_2(\gamma). (76) \]
(S2) The process \((x_{k+1,l+1})\) has the same joint distributions as the process \((x_{k,l})\).

(S3) The expectation \(g_l = E(x_{0,l})\) exists and satisfies \(g_l \geq -Al\) for some constant \(A\) and all \(l > 1\).

**Theorem 5.2** (Kingman’s subadditive ergodic theorem (cf. [10])). If \(x\) is a subadditive process, then the finite limit

\[
F = \lim_{l \to \infty} \frac{x_{0,l}}{l}
\]

exists with probability one and in \(L^1\), and

\[
E(F) = \inf_{l \geq 1} \frac{g_l}{l}.
\]

We need the following preparations that are known to the experts. For the sake of completeness, we include a proof here.

Let \((V, \omega)\) be a symplectic Hilbert space of dimension \(m\), and \(J \in \text{GL}(V)\) be such that \(\omega(x,y) = \langle Jx, y \rangle\) for each \(x, y \in V\). The following lemma is used to compare the Maslov-type indices.

**Lemma 5.3.** Let \(B_0, B_1 \in C([0, \tau], \mathcal{B}^{sa}(V))\) be two paths of self-adjoint operators on \(V\), where \(\tau > 0\). Let \(\gamma_s(t), t \in [0, \tau]\) be the solution of

\[
\dot{\gamma}_s(t) = -J^{-1} B_s(t) \gamma_s(t), \quad \gamma_s(0) = I_V,
\]

for \(s = 0, 1\). Assume that \(B_0(t) \leq B_1(t)\) for each \(t \in [0, \tau]\). Then we have

\[
i_1(\gamma_0) \leq i_1(\gamma_1).
\]

**Proof.** Set \(\Delta B(t) := B_1(t) - B_0(t)\) and \(B_s(t) := B_0(t) + s\Delta B(t)\). Let \(\gamma_s(t), 0 \leq s \leq 1\) be the solution of

\[
\begin{cases}
\dot{\gamma}_s(t) = -J^{-1} B_s(t) \gamma_s(t), \\
\gamma_s(0) = I_V.
\end{cases}
\]

We claim that \(Q(\text{Gr}(\gamma_s(\tau)), s)\) is semi-positive for each \(s \in [0, 1]\). The idea comes from [5, Proposition 4.1]. We solve the variational equation

\[
\begin{cases}
\frac{\partial}{\partial t} \frac{\partial \gamma_s(t)}{\partial s} = -J^{-1} \Delta B(t) \gamma_s(t) - J^{-1} B_s(t) \frac{\partial \gamma_s(t)}{\partial s}, \\
\frac{\partial \gamma_s(0)}{\partial s} = 0.
\end{cases}
\]

Then we get

\[
\frac{\partial \gamma_s(t)}{\partial s} = -\int_0^t \gamma_s(t) \gamma_s(\hat{t})^{-1} J^{-1} \Delta B(\hat{t}) \gamma_s(\hat{t}) d\hat{t}.
\]
For each \( u, v \in V \), set
\[
Q_s(\tau)(u, v) := Q(\text{Gr}(\gamma_s(\tau)), s)((u, \gamma_s(\tau)u), (v, \gamma_s(\tau)v)).
\]
By Lemma 2.7, we have
\[
Q_s(\tau)(u, v) = \langle - J_{\gamma_s(\tau)}^{-1} \frac{\partial \gamma_s(\tau)}{\partial s} u, v \rangle
= \langle J \int_0^\tau \gamma_s(\tilde{t})^{-1} \Delta B(\tilde{t}) \gamma_s(\tilde{t}) d\tilde{t}(u), v \rangle
= \int_0^\tau \langle \Delta B(\tilde{t}) \gamma_s(\tilde{t}) u, \gamma_s(\tilde{t}) v \rangle d\tilde{t}.
\]
So we have \( Q_s(\tau) \geq 0 \). By [2, Proposition 3.2.11] we have \( i_1(\gamma_s(\tau); s \in [0, 1]) \geq 0 \).

By homotopy invariance property and path additivity property of Maslov-type index, we have
\[
\begin{align*}
\forall \gamma_1 \in P \tau (V), 
& i_1(\gamma_1) = i_1(\gamma_0) + i_1(\gamma_s(\tau); s \in [0, 1]) \geq i_1(\gamma_0). 
\end{align*}
\]

The following lemma gives Maslov-type indices for a kind of symplectic paths.

**Lemma 5.4.** Let \( c \in \mathbb{R} \) and \( \tau \geq 0 \). Set \( J_1 := (- J_2)^{-\delta} J \). Define \( \gamma \in \mathbb{P}_\tau (V) \) by \( \gamma(t) := e^{cJ_1t} \). Then we have
\[
i_1(\gamma) = m E(\frac{ct}{2\pi}).
\]  

**Proof.** We have \( i_1(\gamma) = 0 \) if \( ct = 0 \). Since \( J_1^2 = -I_V \), \( \ker(\gamma(t) - I_V) \neq \emptyset \) holds if and only if \( \frac{ct}{2\pi} \in \mathbb{Z} \). If \( \frac{ct}{2\pi} \in \mathbb{Z} \), we have \( \gamma(t) = I_V \). By Lemma 2.7, \( Q(\text{Gr} \circ \gamma, t) \) is positive (negative) for each \( t \in [0, \tau] \) if \( c > 0 \) \((c < 0)\) and \( \tau > 0 \). By Proposition 2.5, we have
\[
i_1(\gamma) = \sum_{\frac{ct}{2\pi} \in \mathbb{Z}, t \in [0, \tau]} m = m E(\frac{ct}{2\pi})
\]
if \( c > 0 \) and \( \tau > 0 \), and
\[
i_1(\gamma) = - \sum_{\frac{ct}{2\pi} \in \mathbb{Z}, t \in (0, \tau]} m = -m E(\frac{ct}{2\pi}) = m E(\frac{ct}{2\pi})
\]
if \( c < 0 \) and \( \tau > 0 \).

We come to the almost existence of the mean indices.

**Assumption 5.5.** We make the following assumptions.
(i) Let \((V, \omega)\) be a symplectic Hilbert space of dimension \(m\), and \(J \in \text{GL}(V)\) be such that \(\omega(x, y) = \langle Jx, y \rangle\) for each \(x, y \in V\).

(ii) Let \(M\) be a compact Hausdorff space with a flow \(\varphi : \mathbb{R} \times M \to M\) and a \(\varphi\)-invariant measure \(\mu\) such that \(\mu(M) \in (0, +\infty)\).

(iii) Let \(B \in C(M, B^{sa}(V))\) be a continuous map.

**Definition 5.6.** Assume that Assumption 5.5 holds. For each \(\xi \in M\), we denote by \(\gamma_{\xi}\) the fundamental solution of \(\dot{x} = -J^{-1}B(\varphi(t, \xi))x\). For each \(\xi \in M\) and \(\tau > 0\), the Maslov-type index \(i_{\tau}(\xi)\) is defined by

\[
i_{\tau}(\xi) := i_{1}(\gamma_{\xi}|_{[0, \tau]}).
\]

The main result of this section is the following.

**Theorem 5.7.** Assume that Assumption 5.5 holds. Then there is a Borelian function \(F : M \to \mathbb{R}\) such that

\[
F \circ \varphi(t, \cdot) = F \quad \forall t \geq 0,
\]

\[
\frac{i_{\tau}}{\tau} \to F \text{ when } \tau \to +\infty,
\]

the convergence being \(L^{1}\) and almost everywhere.

Before proving the above theorem, we give the following lemma.

**Lemma 5.8.** Define the process \(x := (x_{k,l}; k, l \in \mathbb{N}, k < l)\) by \(x_{0,k} = i_{k}(\xi), k \in \mathbb{N}\) and \(x_{k+1,l+1}(\xi) = x_{k,l}(\varphi(1, \xi))\), \(k, l \in \mathbb{N}, k < l\). Then the process \(x\) is subadditive.

**Proof.** We claim that for fixed \(\tau > 0\), \(i_{\tau}(\cdot) : M \to \mathbb{R}\) is a measurable function.

In fact, by Lemma 2.8, for each \(a \in \mathbb{R}\), the set \(A_{a} := \{\xi | i_{\tau}(\xi) \geq a\}\) is open and hence measurable. So \(i_{\tau}(\cdot) : M \to \mathbb{R}\) is a measurable function.

Now we check that the conditions \((S_{1})-(S_{3})\) in Definition 5.1 of subadditive process hold.

\((S_{1})\): By the definition of \(x\), for \(j < k < l, j, k, l \in \mathbb{N}\), we have

\[
x_{j,k}(\xi) = x_{j-1,k-1}(\varphi(1, \xi)) = \cdots = x_{0,k-j}(\varphi(j, \xi)) = i_{k-j}(\varphi(j, \xi)).
\]

Similarly, we have

\[
x_{k,l}(\xi) = x_{j-1,l-k}(\varphi(k, \xi)) = i_{l-k}(\varphi(k, \xi))
\]

\[
x_{j,l}(\xi) = x_{j-1,l-j}(\varphi(j, \xi)) = i_{l-j}(\varphi(j, \xi)).
\]

Since \(\varphi\) is a flow, we have

\[
\varphi(k, \xi) = \varphi(k - j, \varphi(j, \xi)).
\]
By the above equalities and (52), we get \((S_1)\).

\((S_2)\): \(S_2\) follows from the definition of \(x\) and the measure-preserving property of \(\varphi(\cdot, \cdot)\).

\((S_3)\): Since \(M\) is compact, \(i_\tau(\cdot)\) is integrable on \(M\), and there is a constant \(c \geq 0\) such that \(B(\xi) \geq -c(-J^2)^{\frac{1}{2}}\) for each \(\xi \in M\). We then have the probability space \((M, \frac{\mu}{\mu(M)})\). By Lemma 5.3 and 5.4, we have \(i_\tau(\xi) \geq -m\left[\frac{\tau}{2\pi}\right],\) and

\[
g_l = \int_M x_0(\xi) \frac{d\mu}{\mu(M)} = \int_M i_l(\xi) \frac{d\mu}{\mu(M)} \geq -\frac{clm}{2\pi},
\]

for all \(l > 1\). So \((S_3)\) is verified.

Finally, we get the subadditive process \(x\) on the space \((M, \frac{\mu}{\mu(M)})\).

Now we can apply Theorem 5.2 to prove Theorem 5.7.

**Proof of Theorem 5.7.**

By Theorem 5.2, \(F := \lim_{l \to \infty} l_{\frac{\mu}{\mu(M)}} i_\tau \) exists almost everywhere and in \(L^1\), and \(F(\varphi(\tau, \xi)) = F(\xi),\) a.e.

For each nonnegative real number \(\tau\), by (52), we have

\[
-m + i_{\tau}([\tau], \xi) \leq i_\tau(\xi) \leq i_{\tau}([\tau], \xi) + i_{\{\tau\}}(\varphi([\tau], \xi)).
\]

(85)

Since \(M\) is compact, there is a constant \(c \geq 0\) such that \(-c(-J^2)^{\frac{1}{2}} \leq B(\xi) \leq c(-J^2)^{\frac{1}{2}}\) for each \(\xi \in M\). By Lemma 5.3 and (82), for \(t \geq 0\) we have

\[
-m\left[\frac{ct}{2\pi}\right] \leq i_t(\xi) \leq mE\left(\frac{ct}{2\pi}\right).
\]

(86)

By (85) and (86) we have

\[
\frac{i_{\tau}([\tau], \xi)}{[\tau]} - m\left[\frac{\tau}{2\pi}\right] + m \leq \frac{i_\tau(\xi)}{\tau} \leq \frac{i_{\tau}([\tau], \xi)}{[\tau]} + mE\left(\frac{\tau}{2\pi}\right).
\]

(87)

Since \(\frac{i_{\tau}(\xi)}{[\tau]}\) converges almost everywhere and in \(L^1\) as \(\tau \to +\infty\), by (87) there holds that \(\frac{i_\tau(\xi)}{\tau}\) converges almost everywhere and in \(L^1\) as \(\tau \to +\infty\).

Moreover, we claim that

\[
F(\xi) = F(\varphi(t, \xi)) \quad \forall t \geq 0, \xi \in M.
\]

(88)

In fact,

\[
F(\xi) = \lim_{\tau \to +\infty} \frac{i_\tau(\xi)}{\tau},
\]

\[
F(\varphi(t, \xi)) = \lim_{\tau \to +\infty} \frac{i_\tau(\varphi(t, \xi))}{\tau}.
\]
By (52) we have
\[-m \leq i_{\tau + t}(\xi) - i_{\tau}(\xi) - i_{\tau}(\varphi(\tau, \xi)) \leq 0,\]
\[-m \leq i_{\tau + t}(\xi) - i_{\tau}(\xi) - i_{\tau}(\varphi(\tau, \xi)) \leq 0.\]

So we have
\[-m \leq i_{\tau}(\varphi(t, \xi)) + i_{\tau}(\xi) - i_{\tau}(\xi) - i_{\tau}(\varphi(\tau, \xi)) \leq m. \tag{89}\]

By (86) and (89) we have
\[-\frac{2m[\frac{d}{\tau}] + m}{\tau} \leq \frac{i_{\tau}(\varphi(t, \xi))}{\tau} - \frac{i_{\tau}(\xi)}{\tau} \leq \frac{2mE(\frac{d}{\tau}) + m}{\tau}. \tag{90}\]

By letting \(\tau \to +\infty\), we get (88).

Let \((M, \omega)\) be a \(C^2\) compact symplectic manifold of dimension 2\(n\) with \(C^2\) boundary. Let \(H : M \to \mathbb{R}\) be a \(C^2\) function, called Hamiltonian function, which induces a Hamiltonian vector field \(X_H : M \to TM\) defined by
\[\omega(X_H(\xi), Y) = -d\xi H(\xi)Y \quad \text{for} \quad Y \in T_{\xi}M.\]

Denote by \(\varphi(t, \xi)\) the Hamiltonian flow on \(M\) generated by the vector filed \(X_H\), that is,
\[\frac{d}{dt}\varphi(t, \xi) = X_H(\varphi(t, \xi)), \quad \varphi(0, \xi) = \xi. \tag{91}\]

Assume that \(X_H\) is complete, i.e., \(\varphi(t, \xi)\) is well-defined for all \((t, \xi) \in \mathbb{R} \times M\). Denote by \(\mu_M := \frac{\omega}{m}\) the Liouville form of \((M, \omega)\). By Cartan’s formula \(L_X\omega = X_\omega(d\omega) + d(X_\omega\omega)\), where \(L_X\) denotes the Lie derivative of vector field \(X\) and \(X_\omega\) denotes the interior multiplication with \(X\), we have \(\varphi(t, \cdot)^*\omega = \omega\) and then \(\mu_M\) is \(\varphi\)-invariant.

Set \(V := \mathbb{R}^{2n}\) with the standard symplectic form \(\omega_0\) on \(\mathbb{R}^{2n}\), then \(M \times V\) is a trivial symplectic vector bundle. We set
\[\omega_0(x, y) = (J_0 x, y) \quad \text{for all} \quad x, y \in \mathbb{R}^{2n},\]
where \((x, y) = y^T x, x, y \in \mathbb{R}^{2n}\), \(y^T\) is the transpose of vector \(y\), and
\[J_0 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.\]

In [18], the Maslov index was defined for non-degenerate periodic solutions of Hamiltonian systems which are contractible loops on a symplectic manifold \(M\) provided that the first Chern class \(c_1(TM)\) of the tangent bundle vanishes over the 2-dimensional homotopy group \(\pi_2(M)\).
In our paper, we assume that there is a $C^1$ symplectic trivialization $f : TM \to M \times V$ of the tangent bundle $TM$ of $M$, that is, $f$ is a $C^1$ bundle isomorphism and $f^*\omega_0 = \omega$, such that the following diagram commutes:

$$\begin{align*}
TM & \xrightarrow{f} M \times V \\
\pi & \downarrow \quad \quad \quad \quad \quad \quad \downarrow p_1 \\
M
\end{align*}$$

where $\pi : TM \to M$ is the natural projection map, $p_1$ is the projection on the first factor.

Now we use the $C^1$ symplectic trivialization $f$ to define a linear isomorphism $\Phi(\xi) \in \text{Hom}(V, T\xi M)$ for each $\xi \in M$ by $f(\Phi(\xi)x) = (\xi, x)$ for each $x \in V$. Then we have $\Phi(\xi)^*\omega = \omega_0$ for each $\xi \in M$. As in [11], we will often commit the usual mild sin of identifying $T\xi M$ with its image under the canonical injection $X \in T\xi M \mapsto (\xi, X) \in TM$, and will use any of the notations $(\xi, X)$, $X_\xi$, and $X$ for a tangent vector in $T\xi M$, depending on how much emphasis we wish to give the point $\xi \in M$.

Consider the linearized flow along the Hamiltonian flow $\varphi(t, \cdot) : M \to M$ of $(91)$, the differential with respect to $\xi \in M$, $d\xi\varphi(t, \xi) : T\xi M \to T_{\varphi(t, \xi)M} M$, $t \in \mathbb{R}$, we can define a map $\gamma(t, \xi) : T\xi M \to T\xi M$ by

$$\gamma(t, \xi) := \Phi(\xi)^{-1} \varphi(t, \xi) \Phi(\xi)^{-1} d\xi\varphi(t, \xi),$$

where $\Phi(\xi)^{-1} : T\xi M \to V$ is the inverse of the linear isomorphism $\Phi(\xi)$, for any $\xi \in M$, then we have $\gamma(t, \xi)^*\omega = \omega$. We can use the symplectic trivialization to identity $T\xi M$ with $V$, for each $\xi \in M$. Then for any fixed $\xi \in M$, we get a path $\gamma_0(t, \xi)$, $t \in \mathbb{R}$ in $\text{Sp}(V_0, \omega_0)$ by

$$\gamma_0(t, \xi) := \Phi(\xi)^{-1} \gamma(t, \xi) \Phi(\xi) = \Phi(\varphi(t, \xi))^{-1} d\xi\varphi(t, \xi) \Phi(\xi).$$

(92)

For any $x \in V$,

$$f(\varphi(t, \xi), d\xi\varphi(t, \xi)\Phi(\xi)x) = f(\varphi(t, \xi), \Phi(\varphi(t, \xi))\gamma_0(t, \xi)x) = (\varphi(t, \xi), \gamma_0(t, \xi)x).$$

For any fixed $\xi_0 \in M$, choose a chart $(U_0, \psi)$ for $M$ with coordinate functions $(\xi^1, ..., \xi^{2n})$, such that $\xi_0 \in U_0$. Define a map $\tilde{\psi} : \pi^{-1}(U_0) \to \psi(U_0) \times \mathbb{R}^{2n}$ by

$$\tilde{\psi}(\sum_{k=1}^{2n} v^k \frac{\partial}{\partial \xi^k}) = (\xi^1(\xi), ..., \xi^{2n}(\xi), v^1, ..., v^{2n}),$$

where $\frac{\partial}{\partial \xi^k} \in T\xi M$ are the coordinate vectors associated with the given chart and $(\frac{\partial}{\partial \xi^1}, ..., \frac{\partial}{\partial \xi^{2n}})$ form a $C^1$ local frame for $TM$ over $U_0$. The $C^1$ map $\tilde{\psi}$ is a bijection onto its image, obviously linear on fibers and satisfies $p_1 \circ \tilde{\psi} = \psi$. [11]
Denote by $g$ and by the chain rule for total derivatives, we have

$$\pi. \text{ The coordinates } (\xi^1, ..., \xi^{2n}, v^1, ..., v^{2n}) \text{ are called standard coordinates for } TM. \text{ Using the local coordinate representation, the Cauchy problem of the Hamiltonian system (91) reduces to an initial value problem of a first order } C^1 \text{ system of ordinary differential equations. Then the solution } \varphi(t, \xi) \text{ of (91) is } C^1 \text{ on } G_0 := \{(t, \xi) : t \in (-\delta_{g_0}, \delta_{g_0}), \xi \in U_{g_0}\} \tag{93}
$$

for some positive number $\delta_{g_0}$ and open set $U_{g_0} \subseteq U_0$ such that $\varphi(G_0) \subset U_0$. Denote by $g(t, \xi) := (\varphi(t, \xi), d_\xi \varphi(t, \xi) \Phi(\xi)x) \in TM$ for $t \in \mathbb{R}$, $\xi \in M$, $x \in V$. Then for fixed $\xi \in M$, $g(t, \xi)$ is a curve in $C^1$ manifold $TM$. We claim that $\frac{d}{dt}g(t, \xi)$ exists and is continuous on $G_0$. Thus for each $\xi \in U_{g_0}$, $\frac{d}{dt}g(t, \xi)$ is the velocity vector field of the curve $g(t, \xi), t \in (-\delta_{g_0}, \delta_{g_0})$. Locally, using the standard coordinates chart $(\pi^{-1}(U_0), \tilde{\psi})$ for $TM$,

$$\tilde{\psi}(\varphi(t, \xi), d_\xi \varphi(t, \xi) \Phi(\xi)x) = \tilde{\psi} \circ d_\xi \varphi(t, \xi) \circ \varphi^{-1} \circ \tilde{\psi}(\Phi(\xi)x)$$

$$= (\psi(\varphi(t, \xi)), (\frac{\partial \psi^j}{\partial \xi^k}(t, \xi))_{j,k=1}^{2n} \tilde{\psi}(\Phi(\xi)x),$$

where the matrix $(\frac{\partial \psi^j}{\partial \xi^k})_{j,k=1}^{2n}$ of partial derivatives is the Jacobian matrix of $\psi \circ \varphi \circ \psi^{-1}$. In fact, although $\varphi$ may not be $C^2$, by the standard coordinates $(\xi^1, ..., \xi^{2n}, v^1, ..., v^{2n})$ representation and the $C^1$ system (91), $(\frac{\partial \psi^j}{\partial \xi^k}(t, \xi))_{j,k=1}^{2n} \tilde{\psi}(\Phi(\xi)x)$ is continuous on $G_0$ and $\psi(t, \xi), d_\xi \varphi(t, \xi)$ exist. Then we have $\frac{d}{dt}(\frac{\partial \psi^j}{\partial \xi^k}(t, \xi))_{j,k=1}^{2n}$ exists and $\frac{d}{dt}(\frac{\partial \psi^j}{\partial \xi^k}(t, \xi))_{j,k=1}^{2n} = (\frac{\partial \psi^j}{\partial \xi^k}(t, \xi))_{j,k=1}^{2n}$, since every term in the matrices is just the second-order mixed partial derivative. So

$$\frac{d}{dt}(\frac{\partial \psi^j}{\partial \xi^k}(t, \xi))_{j,k=1}^{2n} \tilde{\psi}(\Phi(\xi)x) = \frac{d}{dt}(\frac{\partial \psi^j}{\partial \xi^k}(t, \xi))_{j,k=1}^{2n} \tilde{\psi}(\Phi(\xi)x)$$

$$= (\frac{\partial \psi^j}{\partial \xi^k}(t, \xi))_{j,k=1}^{2n} \tilde{\psi}(\Phi(\xi)x),$$

thus $\frac{d}{dt}g(t, \xi)$ exists and is continuous on $G_0$. Since $f$ is $C^1$, using the chain rule for total derivatives to the composite curve, we obtain that $\gamma_0(t, \xi)$ is continuous on $G_0$. Since $M$ is compact, $M$ can be covered by a finite number of such neighborhood $U_{g_0}$. Let $\delta_0$ denote the smallest of the corresponding positive number $\delta_{g_0}$. For any $\xi \in M$, since $\varphi(0, \xi) = \xi$, we have $\gamma_0(0, \xi) = I_V.$ For any $\xi \in M$, $s, t \geq 0$, since

$$\varphi(s + t, \xi) = \varphi(s, \varphi(t, \xi))$$

and by the chain rule for total derivatives, we have

$$d_\xi \varphi(s + t, \xi) = d_\xi \varphi(s, \xi)|_{\xi=\varphi(t,\xi)}d_\xi \varphi(t, \xi).$$
By (92) and (95), we have
\[ \gamma_0(s + t, \xi) = \gamma_0(s, \varphi(t, \xi))\gamma_0(t, \xi). \] (96)

We claim that \( \varphi(t, \xi) \) is \( C^1 \) on \( \mathbb{R} \times M \). In fact, for any \( t \in \mathbb{R} \), we can choose an \( N_0 \in \mathbb{N} \), such that \( |t|_0 < \delta_0 \), then by (91) and (93), we have this claim.

So by (90), \( \gamma_0(t, \xi) \) exists and is continuous on \( \mathbb{R} \times M \). Since simple computations show that \( -J_0\gamma_0(t, \xi)\gamma_0(t, \xi)^{-1} \) are symmetric matrices. Define \( B : M \to \mathcal{B}^{\omega\alpha}(V) \) by \( B(\xi) = -J_0\gamma_0(t, \xi)|_{t=0} \), then we get a \( B \in C(M, \mathcal{B}^{\omega\alpha}(V)) \). Altogether, we have for fixed \( \xi \), \( \gamma_0(t, \xi) \) is the fundamental solution of \( \dot{x} = -J_0^{-1}B(\varphi(t, \xi))x \). By Theorem 5.7 we have

Corollary 5.9. Under the above assumptions, the results of Theorem 5.7 hold.

Let \((M, \omega)\) be a \( C^3 \) compact symplectic manifold of dimension \( 2n \). Set \( V := \mathbb{R}^{2n} \). Assume that there is a \( C^1 \) symplectic trivialization \( f : TM \to M \times V \) of the tangent bundle \( TM \) of \( M \). Let \( \Sigma \) be an orientable compact \( C^3 \) hypersurface of \( M \). By [4, §4.2, p. 114], there exists a function \( H \in C^2(U, \mathbb{R}) \) defined on an open neighborhood \( U \) of \( \Sigma \) representing \( \Sigma = \{ x \in U; H(x) = 0 \} \) and satisfying \( dH \neq 0 \). Moreover, there exists an \( \epsilon > 0 \) and a diffeomorphism
\[ \psi : \Sigma \times I \to U \subset M \]
such \( U = H^{-1}(I) \), and
\[ \psi(x, 0) = x, \ H(\psi(x, t)) = t, \ \text{for} \ (x, t) \in \Sigma \times I, \]
where \( I = (-\epsilon, \epsilon) \).

\( H \) induces a Hamiltonian vector field \( X_H \) on \( U \). Denote by \( \varphi(t, \xi) \) the Hamiltonian flow generated by \( X_H \), then \( X_H \) is complete on \( U \), and \( \Sigma \) is an invariant subset of the flow \( \varphi \). Define \( B : M \to \mathcal{B}^{\omega\alpha}(V) \) as in Corollary 5.9. Denote by \( \mu_M := \omega^\alpha \) the Liouville form of \((M, \omega)\).

Lemma 5.10. Under the above assumptions, there exists a \((2n-1)\)-form \( \mu_\Sigma \) on \( \Sigma \) such that the form \( \mu_\Sigma \) is \( \varphi \)-invariant.

Proof. Let \( x = (x_1, x_2, ..., x_{2n}) \in U' \subset \mathbb{R}^{2n} \) be a local coordinate of \( \xi \in U \). Then there exists a positive function \( a \) on \( U' \) such that \( \mu_M = adx_1 \wedge dx_2 \wedge ... \wedge dx_{2n} \).

Let \( \mu = \sum_{i=1}^{2n} (-1)^{i-1}a_idx_1 \wedge ... \wedge \hat{dx}_i ... \wedge dx_{2n} \) be a \((2n-1)\)-form defined in the chart such that
\[ \mu_M = dH \wedge \mu, \] (97)
where the hat indicates omitted argument. Denote by \( H_{x_j} := \frac{\partial H}{\partial x_j} \). Then we have
\[ \sum_{j=1}^{2n} H_{x_j}a_j = a. \]
Denote by \( \iota : \Sigma \hookrightarrow M \) the inclusion map. We claim that, if there are two \((2n - 1)\)-forms \( \mu_1, \mu_2 \) which satisfy (97), then \( \iota^* \mu_1 = \iota^* \mu_2 \). In fact, \( H(x) = 0 \) holds for \( x \in \Sigma \), so we have \( \sum_{j=1}^{2n} H_{x_j} dx_j = 0 \). Since \( H'(x) \neq 0 \), without generality, we assume \( H_{x_1} \neq 0 \), then we have \( \iota^* \mu = \frac{\partial}{\partial x_1} dx_2 \wedge \ldots \wedge dx_{2n} |_{\Sigma} \), which is unique determined by \( H \). The argument also shows the existence of \( \mu \) in chart.

By patching the local defined \( \iota^* \mu \), we obtain a globally defined \((2n - 1)\)-form \( \mu_\Sigma := \iota^* \mu \) on \( \Sigma \).

We have \( \varphi(t, \cdot)^* \mu_M = \mu_M \) by Liouville theorem and \( \varphi(t, \cdot)^* dH = dH \) since \( H \circ \varphi(t, \cdot) = H \). So we have \( \varphi(t, \cdot)^* \mu_M = (\varphi(t, \cdot)^* dH) \wedge (\varphi(t, \cdot)^* \mu) \), and

\[
\mu_M = dH \wedge (\varphi(t, \cdot)^* \mu) \tag{98}
\]

By the uniqueness of \( \iota^* \mu \) we have \( \iota^* \varphi(t, \cdot)^* \mu = \iota^* \mu \). Since \( \varphi(t, \cdot) \circ \iota = \iota \circ \varphi(t, \cdot) \) on \( \Sigma \), we finally get

\[
\varphi(t, \cdot)^* \iota^* \mu = \iota^* \mu. \tag*{□}
\]

By Theorem 5.7 we have

**Corollary 5.11.** Under the above assumptions of Lemma 5.10, there is a Borelian function \( F : \Sigma \rightarrow \mathbb{R} \) such that

\[
F \circ \varphi(t, \cdot) = F \quad \forall t \geq 0, \tag{99}
\]

\[
\frac{i_\tau}{\tau} \rightarrow F \quad \text{when} \quad \tau \rightarrow +\infty, \tag{100}
\]

the convergence being \( L^1 \) and almost everywhere. Here the flow \( \varphi \) and the function \( i_\tau \) are defined on \( \Sigma \).

**Acknowledgements** We would like to thank the referees of this paper for their critical reading and very helpful comments and suggestions. The authors were partially supported by NSFC (No.11221091 and No.11471169), LPMC of MOE of China.

**References**

1. Booss-Bavnbek B, Furutani K. Symplectic functional analysis and spectral invariants. In: Geometric aspects of partial differential equations (Roskilde, 1998). Contemp Math, Vol 242. Providence, RI: Amer Math Soc, 1999, 53–83
2. Booß-Bavnbek B, Zhu C. Maslov index in symplectic Banach spaces. Mem Amer Math Soc, to appear. [arXiv:1406.1569v4]
3. Booß-Bavnbek B, Zhu C. The Maslov index in weak symplectic functional analysis. Ann Global Anal Geom, 2013, 44: 283–318
4. De Gosson M. On the usefulness of an index due to Leray for studying the intersections of Lagrangian and symplectic paths. J Math Pures Appl (9), 2009, 91(6): 598–613
5. Duistermaat J J. On the Morse index in variational calculus. Advances in Math, 1976, 21(2): 173–195
6. Ekeland I. Convexity methods in Hamiltonian mechanics. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], Vol 19. Berlin: Springer-Verlag, 1990

7. Hofer H, Zehnder, E. Symplectic invariants and Hamiltonian dynamics. Modern Birkhäuser Classics. Basel: Birkhäuser Verlag, 2011, Reprint of the 1994 edition

8. Hörmander L. Fourier integral operators. I. Acta Math, 1971, 127(1-2): 79-183

9. Kato T. Perturbation theory for linear operators. Berlin: Springer, 1995

10. Kingman J F C. The ergodic theory of subadditive stochastic processes. J Roy Statist Soc Ser B, 1968, 30: 499–510

11. Lee John M. Introduction to smooth manifolds. Graduate Texts in Mathematics, Vol 218. New York: Springer-Verlag, 2003

12. Liu C. Minimal period estimates for brake orbits of nonlinear symmetric Hamiltonian systems. Discrete and Continuous Dynamical Systems Series A, 2010, 27(1): 337–355

13. Liu C, Zhang D. Iteration theory of L-index and multiplicity of brake orbits. J Differential Equations, 2014, 257(4): 1194–1245

14. Liu C, Zhang D. Seifert conjecture in the even convex case. Communications on Pure and Applied Mathematics, 2014, 67(10): 1563–1604

15. Long Y. The minimal period problem of classical Hamiltonian systems with even potentials. Ann Inst H Poincaré Anal Non Linéaire, 1993, 10(6): 605–626

16. Long Y. Index theory for symplectic paths with applications. Progress in Mathematics, Vol 27. Basel: Birkhäuser Verlag, 2002

17. Long Y, Zhang D, Zhu C. Multiple brake orbits in bounded convex symmetric domains. Advances in Mathematics, 2006, 203(2): 568–635

18. Salamon D, Zehnder E. Morse theory for periodic solutions of Hamiltonian systems and the Maslov index. Comm Pure Appl Math, 1992, 45(10): 1303–1360

19. Zhu C. A generalized Morse index theorem. In: Analysis, geometry and topology of elliptic operators. Hackensack, NJ: World Sci Publ, 2006, 493–540