The Gromov-Lawson codimension 2 obstruction to positive scalar curvature and the $C^*$-index

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Abstract

Gromov and Lawson developed a codimension 2 index obstruction to positive scalar curvature for a closed spin manifold $M$, later refined by Hanke, Pape and Schick. Kubota has shown that also this obstruction can be obtained from the Rosenberg index of the ambient manifold $M$ which takes values in the K-theory of the maximal $C^*$-algebra of the fundamental group of $M$, using relative index constructions.

In this note, we give a slightly simplified account of Kubota’s work and remark that it also applies to the signature operator, thus recovering the homotopy invariance of higher signatures of codimension 2 submanifolds of Higson, Schick, Xie.

1 Introduction

Let $M$ be a closed spin manifold and $N \subset M$ a submanifold of codimension 2 with trivial normal bundle. Much current research is devoted to the question when such a manifold $M$ does admit a Riemannian metric of positive scalar curvature. The submanifold $N$ can provide an obstruction to this, as was first explored in rather special situations by Gromov and Lawson [2]. The core of the argument lead to the following version [3]:

1.1 Theorem. Let $M$ be a closed manifold, $N \subset M$ a connected submanifold codimension 2 with trivial normal bundle. Assume that the induced map $\pi_1(N) \to \pi_1(M)$ is injective and $\pi_2(N) \to \pi_2(M)$ is surjective.

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If $M$ is spin and the Rosenberg index of $N$ doesn’t vanish, $0 \neq \alpha(N) \in K_*(C_{\text{max}}^*\pi_1(N))$, then $M$ does not admit a Riemannian metric with positive scalar curvature.

The proof uses index theory of the Dirac operator, but not on $M$ itself. It is a common theme that $\alpha(M)$ should be the most powerful index theoretic obstruction to positive scalar curvature on $M$, formulated as a conjecture in [9]. However, for the codimension 2 obstruction this remained elusive. Some partial results in this direction have been carried out in [1] and [7], but they were of topological nature and required the validity of the strong Novikov conjecture for $\pi_1(M)$ to answer the question.

Higson, Schick, and Xie proved in [5] a companion result: the unexpected homotopy invariance of higher signatures of submanifolds of codimension 2 in the situation of Theorem 1.1, with the spin condition replaced by orientability.

Recently, in [6, Section 3.3] the first author proved the desired result about the Rosenberg index of $M$:

1.2 Theorem. In the situation of Theorem 1.1, the Rosenberg index of $M$ itself is non-zero: $0 \neq \alpha(M) \in K_*(C_{\text{max}}^*\pi_1(M))$.

More precisely, the situation determines a homomorphism (induced from a KK-element)

$$\rho_{\;M,N} : K_*(C_{\text{max}}^*\pi_1(M)) \to K_{*-2}(C_{\text{max}}^*\pi_1(N)),$$

mapping the Rosenberg index of $M$ to the one of $N$.

The proof uses a relative higher index setup, using more or less the interplay of $\pi_1(N \times S^1) \subset \pi_1(W)$ for the manifold $W := M \setminus U(N)$ with boundary constructed from $M$ by removing a tubular neighborhood $U(N) \cong N \times D^2$ of $N$.

In this note, we give an account of the proof in [6], together with a slight simplification avoiding relative higher index theory. Moreover, we observe that this method also gives a simpler proof of the main result of [5], even of the following strengthening. Here we write $\text{Sgn}(M; V_M)$ for the $C^*$-algebraic higher signatures, i.e. the Mishchenko-Fomenko index of the signature operator twisted with the Hilbert $C^*$-module bundle $V_M := \tilde{M} \times_{\pi_1(M)} C_{\text{max}}^*\pi_1(N)$.

1.3 Theorem. Let $M$ with submanifold $N$ be as in Theorem 1.1. If $M$ is oriented then the homomorphism $\rho_{\;M,N}$ of Theorem 1.2 maps the $C^*$-algebraic higher signature of $M$ to twice the one of $N$. More generally, let $f : M_1 \to M$ be a map between closed oriented manifolds of degree 1. Assume $N$ is transversal to $f$ and set $N_1 := f^{-1}(N) \subset M_1$. Then

$$\rho_{\;M,N}(\text{Sgn}(M_1; f^*V_M)) = 2\text{Sgn}(N_1; f^*V_N) \in K_*(C_{\text{max}}^*\pi_1(N)).$$

(1.4)

In particular, if $f$ is a homotopy equivalence then $f^*V_M = V_{M_1}$ and by homotopy invariance of the $C^*$-algebraic signature $\text{Sgn}(M_1; V_{M_1}) = \text{Sgn}(M; V_M)$ we also get

$$2(\text{Sgn}(N_1; f^*V_N) - \text{Sgn}(N; V_N)) = 0,$$

recovering the main theorem of [5].
2 Proof of the theorem

We set \( \pi := \pi_1(N) \), \( \Gamma := \pi_1(M) \), and \( \Pi := \pi \times \mathbb{Z} \). Moreover, we set \( m := \dim(M) \). Note that \( n := \dim(N) \equiv m \pmod{2} \). Until the last section, all group \( C^* \)-algebras are maximal group \( C^* \)-algebras and we simply write e.g. \( C^* \pi \) for the maximal group \( C^* \)-algebra of \( \pi \).

The first step in the proof of the main theorem, used in all approaches, is a well known product formula for the Mishchenko index of the Dirac, as well as the signature operator of \( N \times S^1 \):

2.1 Proposition. Let \( N \) be a closed manifold with fundamental group \( \pi \) and identify \( \pi_1(N \times S^1) \) with \( \Pi \). The Künneth formula provides canonical homomorphisms

\[
K_* (C^* \pi) \to K_{*-1} (C^* \Pi) \xrightarrow{\beta} K_* (C^* \pi)
\]

(Indeed induced by KK-elements) with composition the identity. If \( N \) is spin these homomorphisms map the Mishchenko-Fomenko index of \( N \) and of \( N \times S^1 \) to each other.

If \( N \) is oriented and \( W \) is a bundle of finitely generated projective \( C^* \pi \)-modules on \( N \), \( \beta \) maps the analytic higher signature class \( \text{Sgn}(N \times S^1; W \boxtimes V_{S^1}) \) (where \( W \boxtimes V_{S^1} \) denotes the exterior tensor product bundle) to \( 2^\epsilon \text{Sgn}(N; W) \) with

\[
\epsilon = \begin{cases} 
0; & m \equiv 0 \pmod{2}, \\
1; & m \equiv 1 \pmod{2}.
\end{cases}
\]

This is well known and follows from the principle “boundary of Dirac is Dirac” (compare e.g. [8]) and “boundary of signature is \( 2^\epsilon \) times signature”, this factor of 2 for the signature operator is explained e.g. in [5, 2.13].

It therefore suffices to relate the Mishchenko-Fomenko index of Dirac and signature operator on \( N \times S^1 \), considered as the boundary of the tubular neighborhood of \( N \) in \( M \) to the one of \( M \).

For this, we make the following well known observations.

(1) As a purely topological fact, we have a Mayer-Vietoris sequence in generalized homology for the decomposition \( M = W \cup_{N \times S^1} N \times D^2 \) where we identify a closed tubular neighborhood of \( N \) in \( M \) with \( N \times D^2 \) and let \( W \) be the complement of its interior.

We apply this to K-homology and get the boundary map

\[
\partial_{MV}: K_*(M) \to K_{*-1}(N \times S^1).
\]

A spin structure on \( M \) defines a fundamental K-homology class \([M] \in K_{m}(M)\), and

\[
\partial_{MV}( [M] ) = [N \times S^1] \in K_{m-1}(N \times S^1)
\]

is the fundamental class of \( N \times S^1 \), another implementation of “boundary of Dirac is Dirac”. For the signature operator of an orientation of \( M \), we similarly get

\[
\partial_{MV}( f_*[M]_{\text{Sgn}} ) = 2^{\epsilon+1} f_*[N \times S^1]_{\text{Sgn}} \in K_{m-1}(N \times S^1).
\]
Here $[M]_{S^m}$ is the K-homology class of the signature operator and as before $\epsilon = \begin{cases} 0; & m \equiv 0 \pmod{2} \\ 1; & m \equiv 1 \pmod{2} \end{cases}$.

(2) The Mishchenko bundle

$\mathcal{V}_{N \times S^1} = \widetilde{N} \times S^1 \times \pi_{x \pi} C^*\Pi$

of $N \times S^1$ defines a class in K-theory with coefficients in $C^*\Pi$, namely

$[\mathcal{V}_{N \times S^1}] \in K^0(N \times S^1; C^*\Pi) := K_0(C(N \times S^1) \otimes C^*\Pi)$. Moreover, the Mishchenko-Fomenko index of the Dirac operator is simply obtained as the pairing of the fundamental K-homology class of $N \times S^1$ with this K-theory class of $N \times S^1$ as

$\alpha(N \times S^1) = ([N \times S^1], [\mathcal{V}_{N \times S^1}] \in K_{m-1}(C^*(\pi \times \mathbb{Z}))$. The corresponding statement holds for the signature and for $M$, in particular $\alpha(M) = ([M], [\mathcal{V}_M]) \in K_m(C^*\Gamma)$.

(3) The pairing between K-homology and K-theory is compatible with the Mayer-Vietoris boundary map:

$\langle [N \times S^1], [\mathcal{V}_{N \times S^1}] \rangle \in K_m(C^*\Pi) = \langle \partial_{MV}([M]), [\mathcal{V}_{N \times S^1}] \rangle_{K_*C^*\Pi}$

in $K_m(C^*\Pi)$. Here $\delta_{MV} : K^0(N \times S^1; C^*\Pi) \to K^1(M; C^*\Pi)$ is the boundary map of the Mayer-Vietoris sequence in K-theory for the above decomposition of $M$, where we use fixed coefficients in $C^*\Pi$.

All of this is standard and relatively easy to derive. For the proof of the main theorem it therefore remains “only” to relate $([M], [\mathcal{V}_M]) \in K_m(C^*\Gamma)$ to $([M], \delta_{MV}[\mathcal{V}_{N \times S^1}]) \in K_m(C^*\Pi)$, and for this the obvious strategy is to relate $[\mathcal{V}_M] \in K^0(M; C^*\Gamma)$ to $\delta_{MV}[\mathcal{V}_{N \times S^1}] \in K^1(M; C^*\Pi)$. This latter task, however, is not obvious at all and is achieved in [6, Section 3.3], stated in different terms there.

We retrace these steps here, using our slightly simpler setup. First, construct the standard Hilbert $C^*\Pi$-module $\mathcal{H}_{C^*\Pi} := \ell^2N \otimes C^*\Pi$ with its $C^*$-algebras of compact and bounded adjointable operators $\mathcal{K}_{C^*\Pi} := \mathcal{K}_{C^*\Pi}(\mathcal{H}_{C^*\Pi})$ and $\mathcal{B}_{C^*\Pi} := \mathcal{B}_{C^*\Pi}(\mathcal{H}_{C^*\Pi})$, giving rise to the short exact sequence of $C^*$-algebras

$0 \to \mathcal{K}_{C^*\Pi} \to \mathcal{B}_{C^*\Pi} \to Q_{C^*\Pi} := \mathcal{B}_{C^*\Pi}/\mathcal{K}_{C^*\Pi} \to 0$.

The associated long exact sequence in K-theory yields the boundary isomorphism (induced by a KK-element)

$\delta_Q : K_*(Q_{C^*\Pi}) \to K_{*+1}(C^*\Pi)$ (2.3)

because $K_* (\mathcal{B}_{C^*\Pi}) = 0$ and by stability of K-theory we have the canonical isomorphism $K_*(C^*\Pi) \cong K_*(\mathcal{K}_{C^*\Pi})$.

The standard construction of the boundary homomorphism as in (2.3) allows us here to explicitly find a representative of the class in $K_1(Q_{C^*\Pi})$ corresponding to $[\mathcal{V}_{N \times S^1}]$:...
2.4 Lemma. Consider the stabilized bundle $V_{N \times S^1} \oplus H_{C^*\Pi}$ of Hilbert $C^*$-modules. Its structure group, the unitary group $U(H_{C^*\Pi}(C^*\Pi \oplus H_{C^*\Pi}))$ with the norm topology, is contractible by the appropriate version of Kuiper’s theorem. Therefore we have a trivialization (unique up to homotopy) $H_{C^*\Pi} \rightarrow V_{N \times S^1} \oplus H_{C^*\Pi}$. Composed with the obvious projection $V_{N \times S^1} \oplus H_{C^*\Pi} \rightarrow H_{C^*\Pi}$ we obtain a norm continuous map $\Psi: N \times S^1 \rightarrow B_{C^*\Pi}$ with values epimorphisms with finitely generated projective kernel. Therefore, composed with the projection to the Calkin algebra $Q_{C^*\Pi}$ the map takes values in unitaries and gives $p \circ \Psi: N \times S^1 \rightarrow U(Q_{C^*\Pi})$, thus representing a class

$$[p \circ \Psi] \in K^1(N \times S^1; Q_{C^*\Pi}).$$

Its image under the map induced by $\delta_Q$ of (2.3) is precisely $[V_{N \times S^1}] \in K^0(N \times S^1; C^*\Pi)$.

Proof. There is a standard way to compute the boundary map: Lift $p \circ \Psi: N \times S^1 \rightarrow U(Q_{C^*\Pi})$ to $\Psi$. The kernel is obviously isomorphic to the continuous section space $C(N \times S^1; V_{N \times S^1})$ and the cokernel is trivial. Then the kernel represents the image under the boundary map, which corresponds to the class of the bundle $[V_{N \times S^1}]$ in $K^0(N \times S^1; C^*\Pi)$ in the bundle description of $K^0$. \square

Next, there is a standard description of the Mayer-Vietoris boundary map

$$\delta_{MV}: K^1(N \times S^1; Q_{C^*\Pi}) \rightarrow K^0(M; Q_{C^*\Pi}).$$

2.6 Lemma. The class $\delta_{MV}([p \circ \Psi]) \in K^0(M; Q_{C^*\Pi})$ is represented by the Hilbert $Q_{C^*\Pi}$-module bundle $V$ obtained by gluing the trivial free $Q_{C^*\Pi}$-module bundles of rank one over $W$ and over $N \times D^2$ along their common boundary using the isomorphism $[p \circ \Psi]: N \times S^1 \rightarrow U(Q_{C^*\Pi})$. Here, we glue via left multiplication whereas the right module structure of the fibers is given by right multiplication.

Finally, taking the Mayer-Vietoris sequence is compatible with the K-theory sequence induced from an extension of coefficient $C^*$-algebras, meaning

$$\delta_{Q^{-1}}(\delta_{MV}[V_{N \times S^1}]) = \delta_{MV}(\delta_{Q^{-1}}[V_{N \times S^1}]) = \delta_{MV}([p \circ \Psi]) = [V].$$

Again, these are standard constructions. The main point now is the following crucial theorem, which in a different form is the main idea of [6, Section 3.3].

2.8 Theorem. There is a unitary representation $\rho: \Gamma \rightarrow U(Q_{C^*\Pi})$ where we consider $Q_{C^*\Pi}$ as the $Q_{C^*\Pi}$-endomorphisms of the free $Q_{C^*\Pi}$-module of rank one such that the associated Hilbert $Q_{C^*\Pi}$-bundle $U$ (which is flat) is isomorphic to the bundle $V$ of Lemma 2.6.

With the other preparations, this is the heart of the proof of the main theorems:

2.9 Corollary. The class $\delta_{Q^{-1}}(\alpha(N \times S^1)) \in K_m(Q_{C^*\Pi})$ is the image under $\rho_*: K_m(C^*\Gamma) \rightarrow K_m(Q_{C^*\Pi})$ of $\alpha(M)$ where $\rho_*: C^*\Gamma \rightarrow Q_{C^*\Pi}$ is the homomorphism induced by the representation $\rho$ of Theorem 2.8 via the universal property of $C^*\Gamma$. 
Consequently, \( \alpha(N) \) is the image of \( \alpha(M) \) under the composition

\[
\rho_{M,N} := \beta \circ \delta_Q \circ \rho_* : K_* (C^* \Gamma) \to K_{*-2}(C^* \pi)
\]

(with the Künneth map \( \beta \) of Proposition 2.7).

In the Situation of Theorem 1.3, for the \( C^* \)-algebraic signature we get

\[
\rho_{M,N}(\text{Sgn}(M_1; f^* V_M)) = 2 \text{Sgn}(N_1; f^* V_N).
\]

**Proof.** For any \( \xi \in K_0(M) \) we have

\[
\langle \xi, [U] \rangle = \langle \xi, [V] \rangle = \langle \xi, \delta_{MV} \delta_{Q}^{-1}([V_\times S^1]) \rangle = \delta_{Q}^{-1}(\langle \partial_{MV} \xi, [V_\times S^1] \rangle).
\]

Applying \( \beta \circ \delta_Q \) to both sides, we get

\[
\beta \circ \delta_Q(\langle \xi, [U] \rangle) = \beta(\langle \partial_{MV} \xi, [V_\times S^1] \rangle) = (2.10)
\]

In the case that \( \xi \) is the Dirac fundamental class \([M]\), (2.10) implies

\[
\rho_{M,N}(\alpha(M)) = \beta(\alpha(N \times S^1)) = \alpha(N)
\]

since \( \rho_*(\alpha(M)) = \langle [M], [U] \rangle \in K_m(S_{[\cdot]} \Gamma) \) by [1] Lemma 3.1. Similarly, applying (2.10) to the pushed signature class \( \xi = f_* \rho_1[M_1] \text{Sgn} \) we get

\[
\rho_{M,N} \text{Sgn}(M_1; f^* V_M) = (\beta \circ \delta_Q)(\langle f_* \rho_1[M_1] \text{Sgn}, [U] \rangle)
\]

\[
= 2 \beta(\langle \partial_{MV} f_* \rho_1[M_1] \text{Sgn}, [V_\times S^1] \rangle)
\]

\[
= 2 \beta(\langle f_* [N_1 \times S^1] \text{Sgn}, [V_\times S^1] \rangle)
\]

\[
= 2 \text{Sgn}(N_1 \times S^1, f^* V_{N_1}^*)
\]

where \( \epsilon \) is as in Theorem 1.3. For the last equality, we remark that there is an isomorphism \( f^* V_{N_1} \cong f^* V_{N_1} \boxtimes S^1 \). This is because \( f|_{N \times S^1} \) is identified with \( f|_{N} \times 1 \) under the trivialization of the normal bundle of \( N_1 \) pulled back from that of the normal bundle of \( N \).

It remains to prove Theorem 2.8. For this, following [6] Section 3.3 we construct the flat Hilbert \( QC_{\pi} \)-module bundle \( U \) ad hoc, corresponding automatically to a representation \( \rho \), and then check by hand that \( U \) and \( V \) are isomorphic Hilbert \( QC_{\pi} \)-module bundles.

For this, first we choose a basepoint \( * \) in \( N \times S^1 \subset M \) and identify the fundamental groups as \( \Gamma = \pi_1(M,*) \), \( \pi = \pi_1(N \times D^2,*) \) and \( \Pi = \pi_1(N \times S^1,*) \). Let \( q: (M_{\pi},*) \to (M,*) \) be the covering projection with \( \pi_1(M_{\pi},*) = \pi \subset \Gamma \) and let \( N \times D^2 \subset M_\pi \) be the corresponding lift of the embedded \( N \times D^2 \subset M \), namely the connected component of \( q^{-1}(N \times D^2) \) containing the basepoint. Define \( W_\infty := M_{\pi} \backslash (N \times D^2) \). It is a crucial consequence of the conditions on \( \pi_2 \) derived in [3] Theorem 4.3, that the inclusion of the boundary \( N \times S^1 \to W_\infty \) induces a split injection \( \Pi = \pi \times \mathbb{Z} \to \pi_1(W_\infty,*) \). Let \( W := q^{-1}(W) \to W \) be the restriction of the covering \( M_{\pi} \to W \), a subset of \( W_\infty \). Set \( G := \pi_1(W,*) \) and
$H := \pi_1(W_\pi, \ast)$. The covering and inclusion maps give a commutative diagram of spaces, inducing the one of fundamental groups

$$
\begin{array}{ccc}
N \times S^1 & \xrightarrow{C} & N \times D^2 \\
\cap & & \cap \\
W_\pi & \xrightarrow{C} & W_\infty \\
\cap & & \cap \\
W & \xrightarrow{C} & M
\end{array}
\quad
\begin{array}{ccc}
\Pi = \pi \times \mathbb{Z} & \xrightarrow{\pi} & \pi \\
\downarrow & & \downarrow \\
H & \xrightarrow{\pi_1(W_\infty, \ast)} & G
\end{array}
$$

(2.11)

The horizontal maps of groups are surjective, the vertical ones injective. The inclusion of $H \rightarrow \pi \times \mathbb{Z}$ induces $\pi \times \mathbb{Z} \rightarrow H \rightarrow G$ (the first a split of $H \rightarrow \pi \times \mathbb{Z}$ of (2.11)) and by the van Kampen theorem the normal subgroup $\Lambda := \langle \langle \mathbb{Z} \rangle \rangle$ generated by $\mathbb{Z}$ is the kernel of both epimorphisms $H \rightarrow \pi$ and $G \rightarrow \Gamma$. Through the epimorphism $H \rightarrow \Pi$ we get an induced map $H \rightarrow U(\mathbb{B}_{C^*\Pi}(C^*\Pi))$ (acting by left multiplication). The associated Hilbert $C^*\Pi$-module bundle on $W_\pi$ is the restriction of the Mishchenko bundle of $W_\infty$ to $W_\pi$ (associated to the canonical map $\Pi \rightarrow \mathbb{B}_{C^*\Pi}(C^*\Pi)$).

Inducing the representation $H \rightarrow \Pi \rightarrow U(C^*\Pi)$ up from $H$ to $G$ we obtain the unitary representation of $G$ on the space of square-summable sections of the induced $C^*\Pi$-bundle

$$X := G \times_H C^*\Pi \cong \Pi_{g \in G/H} C^*\Pi$$
on the discrete space $G/H \cong \Gamma/\Pi$, where the action is by left multiplication. We denote it by

$$\rho_G : G \rightarrow U(\mathbb{B}_{C^*\Pi}(\ell^2(G/H, X))).$$

The above isomorphism $X \cong \Pi_{g \in G/H} C^*\Pi$ means that $\ell^2(G/H, X)$ is the completion of an (infinite) algebraic direct sum of free Hilbert $C^*\Pi$-modules of rank one, on which $G$ acts via a combination of permutations and left $\Pi$-multiplication. The corresponding flat Hilbert $C^*\Pi$-module bundle

$$\mathcal{H}_W := W \times_G \ell^2(G/H, X)$$
on $W$ is the pushdown of the bundle $W \times_H C^*\Pi$ on $W_\pi$: the fiber over $x \in W$ is the completed direct sum of all the fibers in the inverse image of $x$ in $W_\pi$.

2.12 Lemma. Restricted to $\pi \times \mathbb{Z} = \Pi$, the representation $\rho_\Pi := \rho_G|_\Pi$ decomposes as a direct sum

$$\rho_\Pi = \lambda_\Pi \oplus \rho_{\text{rest}} : \Pi \rightarrow U(\mathbb{B}_{C^*\Pi}(C^*\Pi) \oplus \mathbb{B}_{C^*\Pi}(\ell^2((G \setminus H)/H, X)))$$

(2.13)

where the map $\lambda_\Pi$ to the first summand comes from left multiplication. Moreover, the map $\rho_{\text{rest}}$ to the second summand factors through the projection $pr_\pi : \Pi = \pi \times \mathbb{Z} \rightarrow \pi$, i.e. is written as $\rho_{\pi} \circ pr_\pi$ with

$$\rho_{\pi} := \rho_G|_\pi : \pi \rightarrow U(\mathbb{B}_{C^*\Pi}(\ell^2((G \setminus H)/H, X))).$$

Correspondingly, the restriction of $\mathcal{H}_W$ to $N \times S^1$ is the direct sum of flat bundles $\mathcal{V}_{N \times S^1}$ and

$$\mathcal{H}_{\text{rest}} := N \times S^1 \times_H \ell^2((G \setminus H)/H, X),$$

where $\mathcal{H}_{\text{rest}}$ extends to a flat bundle over $N \times D^2$. 
Define the left multiplication by $\rho H$ on the summand $H \times_H C^*\Pi = C^*\Pi$ is given by left multiplication.

We write $t$ for the generator of $\mathbb{Z} \subseteq H \subset G$. Then $\rho H(t)$ preserves each rank one direct summand $gH \subseteq (G \setminus H)/H$, since $t \cdot gH = g \cdot g^{-1}tgH$ and $g^{-1}tg \in H$. This observation also shows that $\rho H(t)$ stabilizes any $gH \subseteq (G \setminus H)/H$. We show that it is contractible in $W_{\infty}$ of the inverse image of $\mathcal{Q}(C^*\Pi)$ and hence also $\rho(g^{-1}tg) = e$ for any $g \in G \setminus H$, which concludes that $\rho H(t)$ acts on $H_{\text{rest}}$ trivially.

The element $g^{-1}tg \in H = \pi_1(W_\pi)$ is represented by the concatenation of the lift of the loop $g$ to a (non-closed, as $g \notin H$) path $\gamma$ in $W_\pi$ from the base point $*$, the corresponding lift of the loop $t \in S^1$ and the inverse of the path $\gamma$. We have to show that it is contractible in $W_{\infty}$. However, $\gamma$ ends in a different component of the inverse image of $N \times S^1$ under the covering projection $W_\pi \rightarrow W \subset W_{\infty}$. In $W_{\infty}$, this component is the boundary of a covering of $N \times D^2$ and therefore the lift of $t$ is contractible in $W_{\infty}$ and hence also $\rho(g^{-1}tg) = e = \pi_1(W_{\infty}) = \Pi$. 

**2.14 Definition.** Define $\rho \colon \Gamma = G/\langle \langle \mathbb{Z} \rangle \rangle \rightarrow \mathcal{U}(\mathcal{Q}(C^*\Pi))$ of Theorem 2.8 as induced by the composition

$$
\bar{\rho}_G \colon G \xrightarrow{\rho_G} \mathbb{B}_{C^*\Pi}(\ell^2(G/H,\mathcal{X})) \rightarrow \mathcal{Q}(C^*\Pi)(\ell^2(G/H,\mathcal{X})) \cong \mathcal{Q}(C^*\Pi),
$$

using that the kernel $\Lambda$ of $G \rightarrow \Gamma$, normally generated by $\mathbb{Z}$, acts by Lemma 2.12 as the identity on $\ell^2((G \setminus H)/H,\mathcal{X})$ and therefore as the identity in $\mathcal{Q}(C^*\Pi)$.

**2.16 Remark.** For the proof of Theorem 2.8 we remark a relation between $\mathcal{H}_{C^*\Pi}$-bundles and $\mathbb{B}_{C^*\Pi}$-bundles. Let $\mathcal{H}$ be a locally trivial bundle of Hilbert $C^*$-modules whose typical fiber is $\mathcal{H}_{C^*\Pi}$ and with structure group $\mathcal{U}(\mathcal{H}_{C^*\Pi})$ with norm topology. We write $\mathcal{U}(\mathcal{H})$ for corresponding right principal $\mathcal{U}(\mathcal{H}_{C^*\Pi})$-bundle, with $\mathcal{U}(\mathcal{H})_x = \mathcal{U}(\mathcal{H}_{C^*\Pi},\mathcal{H}_x)$ for $x \in X$ (note that, in the usual convention of the product of endomorphisms, the products $(a, x) \mapsto ax$ and $(x, b) \mapsto xb$ induce a left $\mathbb{B}_{C^*\Pi}(\mathcal{H})$-action and a right $\mathbb{B}_{C^*\Pi}(\mathcal{H})$-action on $\mathbb{B}_{C^*\Pi}(\mathcal{H})$). We have the associated bundles $B(\mathcal{H}) := \mathcal{U}(\mathcal{H}) \times_{\mathcal{U}(\mathcal{H}_{C^*\Pi})} \mathbb{B}_{C^*\Pi}$ and $\mathcal{Q}(\mathcal{H}) := \mathcal{U}(\mathcal{H}) \times_{\mathcal{U}(\mathcal{H}_{C^*\Pi})} \mathcal{Q}(C^*\Pi)$. Then the following are easily verified:

1. For the group $G = \pi_1(X)$ and its unitary representation $\rho \colon G \rightarrow \mathbb{B}_{C^*\Pi}$, the associated bundle $X \times_G \mathbb{B}_{C^*\Pi}$ is isomorphic to $B(X \times G) \mathcal{H}_{C^*\Pi})$.

2. The fiber of the bundle $B(\mathcal{H})$ at $x \in X$ is $\mathbb{B}_{C^*\Pi}(\mathcal{H}_{C^*\Pi},\mathcal{H}_x)$. Hence, a bounded bundle map $T \colon \mathcal{H} \rightarrow \mathcal{K}$ induces $B(\mathcal{H}) \rightarrow B(\mathcal{K})$, fiberwise given by postcomposition. Similarly, $T$ also induces $\mathcal{Q}(\mathcal{H}) \rightarrow \mathcal{Q}(\mathcal{K})$.

3. If $\mathcal{H}$ and $\mathcal{K}$ are trivial, $T \colon X \times \mathcal{H}_{C^*\Pi} \rightarrow X \times \mathcal{K}_{C^*\Pi}$ is identified with a continuous function $X \rightarrow \mathbb{B}_{C^*\Pi}$. The bundle map on $B(X \times \mathcal{H}_{C^*\Pi}) \cong X \times \mathbb{B}_{C^*\Pi}$ (or $\mathcal{Q}(X \times \mathcal{H}_{C^*\Pi}) \cong X \times \mathcal{Q}(C^*\Pi)$, respectively) induced from $T$ is the multiplication of $T$ from the left.

**Proof of Theorem 2.8** We now prove that $V$ is isomorphic to $U$ by showing that the restrictions of $U$ to $W$ and to $N \times D^2$ both can be trivialized, and that the change of trivialization over $N \times S^1$ is precisely the gluing map which produces $V$.

For this, note that the restriction of $U$ to $W$ is the flat bundle associated to the representation $\rho G$ of (2.13). As such, it is identified with $\mathcal{Q}(\mathcal{H}_W)$ by
Remark 2.16 (1). We consider the flat bundle $Q(\mathcal{H}_{\text{rest}}) \cong N \times D^2 \times \mathcal{H}_{\text{rest}} \otimes \mathcal{C}^*\Pi$ over $N \times D^2$ and glue them by the bundle map $Q(\mathcal{H}_W)|_{N \times S^1} \to Q(\mathcal{H}_{\text{rest}})|_{N \times S^1}$ induced from the second projection

$$w: \mathcal{H}_W|_{N \times S^1} \cong V_{N \times S^1} \oplus \mathcal{H}_{\text{rest}} \to \mathcal{H}_{\text{rest}}$$

as is discussed in Remark 2.16 (2), to get the bundle $U'$. Since $w$ is a lift of the $\Pi$-invariant projection $\ell^2(G/H, \mathcal{X}) \to \ell^2((G \setminus H)/\mathcal{X})$ to the associated bundles, the flat connection on $U|_W$ extends to $U'$ and its monodromy representation coincides with $\rho_G$. This means that $U \cong U'$.

By Kuiper’s theorem, there are trivializations $v_W: W \times \mathcal{H}_{C^*\Pi} \to \mathcal{H}_W$ and $v_{N \times D^2}: N \times D^2 \times \mathcal{H}_{C^*\Pi} \to \mathcal{H}_{\text{rest}}$. Then $U'$ is obtained by gluing two trivial $\mathcal{C}^*\Pi$-bundles along $N \times S^1$ by the bundle map induced from

$$v_{N \times D^2}^*wv_W: N \times S^1 \times \mathcal{H}_{C^*\Pi} \to N \times S^1 \times \mathcal{H}_{C^*\Pi}.$$ 

This is a bundle map whose kernel bundle is precisely $V_{N \times S^1}$ and with trivial cokernels. By Remark 2.16 (3), this is precisely the gluing map for $V$, consequently $U \cong U' \cong V$ as Hilbert $\mathcal{C}^*\Pi$-module bundles. This finishes the proof of Theorem 2.8 and therefore, in view of Corollary 2.9 also our two main results, Theorem 1.2 and Theorem 1.3.

2.17 Remark. The Mayer-Vietoris boundary map $\delta_M: K^1(N \times S^1; \mathcal{C}^*\Pi) \to K^0(M; \mathcal{C}^*\Pi)$ of (2.5) is not injective. In particular, $[V] = \delta_M([\mathcal{V}_{N \times S^1} - [\mathcal{C}^*\Pi]])$ is also associated to of the Mishchenko bundle over $N \times S^1$ and the trivial rank 1 Hilbert $\mathcal{C}^*\Pi$-module bundle over $N \times S^1$. The latter is used in some calculations of [6, Section 3.3].

2.18 Proposition. We have the following strengthening of Theorem 1.2. In the situation of Theorem 1.2, the Rosenberg index $\alpha(M)$ is not contained in the image of the map $K_*(C_{\text{max}}\pi_1(N)) \to K_*(C_{\text{max}}\pi_1(M))$. This follows from Theorem 2.8. Indeed, the composition $\rho: C^*\pi \to C^*\Gamma \to \mathcal{C}^*\Pi$ induces the zero map in K-theory since the diagram

$$\begin{array}{ccc}
C^*\pi & \longrightarrow & B_{C^*\Pi} \\
\downarrow & & \downarrow \\
C^*\Gamma & \longrightarrow & \mathcal{C}^*\Pi
\end{array}$$

commutes and $K_*(B_{C^*\Pi}) = 0$. Note that this strengthening is also a consequence of [6, Theorem 3.7].

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