Automorphisms of the category of finitely generated free groups of the some subvariety of the variety of all groups

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September 16, 2019

Abstract

In universal algebraic geometry the category $\Theta_0$ of the finite generated free algebras of some fixed variety $\Theta$ of algebras and the quotient group $A/Y$ are very important. Here $A$ is a group of all automorphisms of the category $\Theta_0$ and $Y$ is a group of all inner automorphisms of this category.

In the varieties of all the groups, all the abelian groups [18], all the nilpotent groups of the class no more than $n$ ($n \geq 2$) [22] the group $A/Y$ is trivial. B. Plotkin posed a question: ”Is there a subvariety of the variety of all the groups, such that the group $A/Y$ in this subvariety is not trivial?” A. Tsurkov hypothesized that exist some varieties of periodic groups, such that the groups $A/Y$ in these varieties is not trivial. In this paper we give an example of one subvariety of this kind.

Keywords: Universal algebraic geometry, category theory, automorphic equivalence, nilpotent groups, periodic groups.

Mathematics Subject Classification 2010: 08A99, 08B20, 18A99, 20F18, 20F50.
1 Introduction

The paper is devoted to some aspects of universal algebraic geometry, i.e., geometry over universal algebras (for definition of universal algebra see, for example [12, Chapter 3, 1. 3]). In fact, universal algebra is the set with the same list (signature) of operations. We will say shortly "algebra" instead "universal algebra".

All definitions of the basic notions of the universal algebraic geometry can be found, for example, in [13, 14, 15] and [16]. Also, there are fundamental papers [2, 10] and [3, 4].

One of the natural question of universal algebraic geometry is as follows:

Problem 1.1 When do two algebras $H_1$ and $H_2$ from the some variety of algebras $\Theta$ have the same algebraic geometry?

Under the sameness of geometries over $H_1$ and $H_2$ we mean an isomorphism of the categories of algebraic sets over $H_1$ and $H_2$, respectively. So, Problem 1.1 is ultimately related to the following one:

Problem 1.2 What are the conditions which provide an isomorphism of the categories of algebraic sets over the algebras $H_1$ and $H_2$?

Notions of geometric and automorphic equivalences of algebras play here a crucial role.

In universal algebraic geometry we consider some variety $\Theta$ of universal algebras of the signature $\Omega$. We denote by $X_0$ an infinite countable set of symbols. By $\mathcal{F}(X_0)$ we denote the set of all finite subsets of $X_0$. We will consider the category $\Theta^0$, whose objects are all free algebras $F(X)$ of the variety $\Theta$ generated by finite subsets $X \in \mathcal{F}(X_0)$. Morphisms of the category $\Theta^0$ are homomorphisms of such algebras. We will occasionally denote $F(X) = F(x_1, x_2, \ldots, x_n)$ if $X = \{x_1, x_2, \ldots, x_n\}$.

We consider a system of equations $T \subseteq F \times F$, where $F \in \text{Ob} \Theta^0$, and we solve these equations in arbitrary algebra $H \in \Theta$.

The set $\text{Hom}(F, H)$ serves as an affine space over the algebra $H$: the solution of the system $T$ is a homomorphism $\mu \in \text{Hom}(F, H)$ such that $\mu(t_1) = \mu(t_2)$ holds for every $(t_1, t_2) \in T$ or $T \subseteq \text{ker} \mu$. $T_H = \{\mu \in \text{Hom}(F, H) \mid T \subseteq \text{ker} \mu\}$
will be the set of all the solutions of the system \( T \). We call these sets \textit{algebraic}, as in the classical algebraic geometry.

For every set of points \( R \subseteq \text{Hom}(F,H) \) we consider a congruence of equations defined in this way: \( R'_H = \bigcap_{\mu \in R} \ker \mu \). This is a maximal system of equations which has the set of solutions \( R \). For every set of equations \( T \) we consider its algebraic closure \( T''_H = \bigcap_{\mu \in T} \ker \mu \) with respect to the algebra \( H \). A set \( T \subseteq F \times F \) is called \( H \)-closed if \( T = T''_H \). An \( H \)-closed set is always a congruence. We denote the family of all \( H \)-closed congruences in \( F \) by \( \text{Cl}_H(F) \).

**Definition 1.1** Algebras \( H_1, H_2 \in \Theta \) are \textit{geometrically equivalent} if and only if for every \( F \in \text{Ob}\Theta^0 \) and every \( T \subseteq F \times F \) the equality \( T''_{H_1} = T''_{H_2} \) is fulfilled.

By this definition, algebras \( H_1, H_2 \in \Theta \) are geometrically equivalent if and only if the families \( \text{Cl}_{H_1}(F) \) and \( \text{Cl}_{H_2}(F) \) coincide for every \( F \in \text{Ob}\Theta^0 \).

**Definition 1.2** \cite{[15]} We say that algebras \( H_1, H_2 \in \Theta \) are \textit{automorphically equivalent} if there exist an automorphism \( \Phi : \Theta^0 \to \Theta^0 \) and the bijections

\[ \alpha(\Phi)_F : \text{Cl}_{H_1}(F) \to \text{Cl}_{H_2}(\Phi(F)) \]

for every \( F \in \text{Ob}\Theta^0 \), coordinated in the following sense: if \( F_1, F_2 \in \text{Ob}\Theta^0 \), \( \mu_1, \mu_2 \in \text{Hom}(F_1, F_2) \), \( T \in \text{Cl}_{H_1}(F_2) \) then

\[ \tau \mu_1 = \tau \mu_2, \]

if and only if

\[ \overline{\tau} \Phi (\mu_1) = \overline{\tau} \Phi (\mu_2), \]

where \( \tau : F_2 \to F_2/T, \overline{\tau} : \Phi(F_2) \to \Phi(F_2)/\alpha(\Phi)_{F_2}(T) \) are the natural epimorphisms.

The definition of the automorphic equivalence in the language of the category of coordinate algebras was considered in \cite{[15]} and \cite{[24]}. Intuitively we can say that algebras \( H_1, H_2 \in \Theta \) are automorphically equivalent if and only if the families \( \text{Cl}_{H_1}(F) \) and \( \text{Cl}_{H_2}(\Phi(F)) \) coincide up to a changing of coordinates. This changing is defined by the automorphism \( \Phi \).

**Definition 1.3** An automorphism \( \Upsilon \) of an arbitrary category \( \mathcal{R} \) is \textit{inner}, if it is isomorphic as a functor to the identity automorphism of the category \( \mathcal{R} \).

It means that for every \( F \in \text{Ob}\mathcal{R} \) there exists an isomorphism \( \sigma^\Upsilon_F : F \to \Upsilon(F) \) such that for every \( \mu \in \text{Mor}_{\mathcal{R}}(F_1, F_2) \)

\[ \Upsilon(\mu) = \sigma^\Upsilon_{F_2} \mu \sigma^\Upsilon_{F_1}^{-1} \]

holds. It is clear that the set \( \mathcal{Y} \) of all inner automorphisms of an arbitrary category \( \mathcal{R} \) is a normal subgroup of the group \( \mathfrak{A} \) of all automorphisms of this category.
If an inner automorphism \( \Upsilon \) provides the automorphic equivalence of the algebras \( H_1 \) and \( H_2 \), where \( H_1, H_2 \in \Theta \), then \( H_1 \) and \( H_2 \) are geometrically equivalent (see [15, Proposition 9]). Therefore the quotient group \( \mathfrak{A}/\mathfrak{Y} \) measures the possible difference between the geometric equivalence and automorphic equivalence of algebras from the variety \( \Theta \): if the group \( \mathfrak{A}/\mathfrak{Y} \) is trivial, then the geometric equivalence and automorphic equivalence coincide in the variety \( \Theta \). The converse is not true. For example, in the variety of the all linear spaces over some fixed field \( k \) of characteristic 0 we have that \( \mathfrak{A}/\mathfrak{Y} \cong \text{Aut} k \), where \( \text{Aut} k \) is the group of all the automorphisms of the field \( k \). The proof of this fact can be achieved by the method of [23]. But all linear spaces over every fixed field \( k \) are geometrically equivalent. This fact is a simple conclusion from [17, Theorem 3].

In the varieties of all the groups, all the abelian groups [18], all the nilpotent groups of the class no more then \( n \) (\( n \geq 2 \)) [22] the group \( \mathfrak{A}/\mathfrak{Y} \) is trivial, so the geometric equivalence and the automorphic equivalence coincide in these varieties. B. Plotkin posed a question: "Is there a subvariety of the variety of all the groups, such that the group \( \mathfrak{A}/\mathfrak{Y} \) in this subvariety is not trivial?" A. Tsurkov hypothesized that exist some varieties of periodic groups, such that the groups \( \mathfrak{A}/\mathfrak{Y} \) in these varieties is not trivial. In this article, we confirm this hypothesis.

We consider a subvariety \( \Theta \) of the variety of all groups. Our subvariety is defined by identities

\[
x^4 = 1, \quad (1.1)
\]

\[
((x_1, x_2), (x_3, x_4)) = 1, \quad (1.2)
\]

and

\[
(((x_1, x_2), x_3), x_4), x_5) = 1, \quad (1.3)
\]

in other words, this is a variety of all nilpotent class no more then 4, metabelian and Sanov [19] groups. We will use the method of the verbal operations elaborated in [18] for the calculation of the quotient group \( \mathfrak{A}/\mathfrak{Y} \) for the variety \( \Theta \). In the next Section we will explain this method.

## 2 Method of verbal operations

In this section we will explain the method of the verbal operations for the computing of the quotient group \( \mathfrak{A}/\mathfrak{Y} \) in the case of arbitrary variety \( \Theta \) of universal algebras of the signature \( \Omega \). The reader also can see the explanation and application of this method in [18, 21, 22, 24 and 25].

### 2.1 First definitions and basic facts

This method we can apply only if the following condition holds in the variety \( \Theta \):

**Condition 2.1** [18] \( \Phi (F(x)) \cong F(x) \) for every automorphism \( \Phi \) of the category \( \Theta^0 \) for every \( x \in X_0 \).
In this case, by [24, Theorem 2.1], for every $\Phi \in \mathfrak{A}$ there exists a system of bijections

$$S = \{ s_F : F \to \Phi (F) \mid F \in \text{Ob}\Theta^0 \},$$

such that for every $\psi \in \text{Mor}_{\Theta^0} (A, B)$ the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{s_A} & \Phi(A) \\
\downarrow\psi & & \downarrow\Phi(\psi) \\
B & \xrightarrow{s_B} & \Phi(B)
\end{array}
$$

is commutative. It means that $\Phi$ acts on the morphisms $\psi : A \to B$ of $\Theta^0$ as follows:

$$\Phi(\psi) = s_B \psi s_A^{-1}. \quad (2.2)$$

**Definition 2.1** We say that the system of bijections (2.1) is a system of bijections associated with the automorphism $\Phi \in \mathfrak{A}$ if this system fulfills the condition (2.2).

One automorphism of the category $\Theta^0$ in general can be associated with various systems of bijections and some system of bijections can be associated with various automorphisms.

In [18] the notion of the strongly stable automorphism of the category $\Theta^0$ was defined:

**Definition 2.2** An automorphism $\Phi$ of the category $\Theta^0$ is called strongly stable if it satisfies the conditions:

1. $\Phi$ preserves all objects of $\Theta^0$,
2. there exists one system of bijections associated with the automorphism $\Phi$ such that

$$s_F | X = id_X \quad (2.3)$$

holds for every $F(X) \in \text{Ob}\Theta^0$.

In other words, we can say that an automorphism of the category $\Theta^0$ is called strongly stable if it preserves all objects of $\Theta^0$ and there is some system of bijections associated with this automorphism such that all the bijections of this system preserve all generators of domains.

It is clear that the set $\mathfrak{S}$ of all strongly stable automorphisms of the category $\Theta^0$ is a subgroup of the group $\mathfrak{A}$ of all automorphisms of this category. By [24, Theorem 2.3], $\mathfrak{A} = \mathfrak{Y} \mathfrak{S}$ holds if in the category $\Theta^0$ fulfills the Condition 2.1. In this case we have that $\mathfrak{A}/\mathfrak{Y} \cong \mathfrak{S}/(\mathfrak{S} \cap \mathfrak{Y})$. So to study $\mathfrak{A}/\mathfrak{Y}$ we must compute the groups $\mathfrak{S}$ and $\mathfrak{S} \cap \mathfrak{Y}$. 

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2.2 Strongly stable automorphism and strongly stable system of bijections

We consider the strongly stable automorphism $\Phi \in \mathcal{S}$. There exists a system of bijections associated with this automorphism which is a subject of Definition 2.2. This system of bijections is uniquely defined by the automorphism $\Phi$, because the equality $s_A(a) = \Phi(\alpha)(x)$ holds for every $A \in \text{Ob}\Theta^0$ and every $a \in A$, where $\alpha : F(x) \to A$ is a homomorphism defined by $\alpha(x) = a$ (see [24, Proposition 3.1]). We denote this system of bijections by $S_{\Phi}$, and its bijections we denote by $s_{\Phi}^F$ for every $F \in \text{Ob}\Theta^0$.

**Definition 2.3** The system of bijections $S = \{s_F : F \to F \mid F \in \text{Ob}\Theta^0\}$ is called strongly stable if for every $A, B \in \text{Ob}\Theta^0$ and every $\mu \in \text{Mor}_{\Theta^0}(A, B)$ the mappings $s_B \mu s_A^{-1}, s_B^{-1} \mu s_A : A \to B$ are homomorphisms and the condition (2.3) are fulfilled.

The set of all the strongly stable system of bijections we denote by $\text{SSSB}$. It is clear that system of bijections $S_{\Phi}$ is strongly stable. Hence the mapping $A : \mathcal{S} \to \text{SSSB}$ such that $A(\Phi) = S_{\Phi}$ is well defined by [24, Proposition 3.1].

This mapping is one to one and onto by [24, Proposition 3.2]. If $\Phi_1, \Phi_2 \in \mathcal{S}$ then there are strongly stable systems of bijections

$$A(\Phi_1) = S_{\Phi_1} = \{s_{F}^{\Phi_1} : F \to F \mid F \in \text{Ob}\Theta^0\}$$

and

$$A(\Phi_2) = S_{\Phi_2} = \{s_{F}^{\Phi_2} : F \to F \mid F \in \text{Ob}\Theta^0\}. $$

For every $\psi \in \text{Mor}_{\Theta^0}(F_1, F_2)$ the equality $\Phi_2 \Phi_1(\psi) = s_{F_2}^{\Phi_2} s_{F_1}^{\Phi_1} \psi \left(s_{F_1}^{\Phi_1}\right)^{-1} \left(s_{F_2}^{\Phi_2}\right)^{-1}$.

It means that the system of bijections

$$\{s_{F}^{\Phi_2} s_{F}^{\Phi_1} : F \to F \mid F \in \text{Ob}\Theta^0\}$$

is associated with the automorphism $\Phi_2 \Phi_1$. But it is clear that this system is strongly stable, so this system of bijections is uniquely defined strongly stable system of bijections corresponds to the strongly stable automorphism $\Phi_2 \Phi_1$, in other words,

$$A(\Phi_2 \Phi_1) = \{s_{F}^{\Phi_2} s_{F}^{\Phi_1} : F \to F \mid F \in \text{Ob}\Theta^0\}. $$

2.3 Strongly stable system of bijections and applicable systems of words

We consider the algebra $F = F(x_1, \ldots, x_n) \in \text{Ob}\Theta^0$ and take a word (element) $w = w(x_1, \ldots, x_n) \in F(x_1, \ldots, x_n)$.

**Definition 2.4** The operation $\omega^* : \omega^*(h_1, \ldots, h_n) = w(h_1, \ldots, h_n)$ is called verbal operation defined on the algebra $H$ by the word $w$, where $h_i \in H, 1 \leq i \leq n$, and $H \in \Theta$ is an arbitrary algebra of the variety $\Theta$. 
The reader can compare this definition with the definition of word maps, \cite{20,11} and references therein.

Denote the signature of our variety \( \Theta \) by \( \Omega \). For every \( \omega \in \Omega \) which has an arity \( \rho_\omega \) we consider the algebra \( F_\omega = F(x_1,\ldots,x_{\rho_\omega}) \in \text{Ob}\Theta^0 \). Having a system of words \( W = \{w_\omega \mid \omega \in \Omega \} \) where \( w_\omega \in F_\omega \), denote by \( H^*_W \) the algebra which coincides with \( H \) as a set, but instead of the original operations \( \{\omega \mid \omega \in \Omega \} \) it possesses the system of the operations \( \{\omega^* \mid \omega \in \Omega \} \) where \( \omega^* \) is a verbal operation defined by word \( w_\omega \).

We can consider the algebras \( H \) and \( H^*_W \) as algebras with the same signature \( \Omega \): the realization of the operation \( \omega \in \Omega \) in the algebra \( H \) is the operation \( \omega \) and the realization of the operation \( \omega \in \Omega \) in the algebra \( H^*_W \) is the operation \( \omega^* \). So, if \( A \) and \( B \) are algebras with the original operations \( \{\omega \mid \omega \in \Omega \} \), \( A^*_W \) and \( B^*_W \) are algebras with the operations \( \{\omega^* \mid \omega \in \Omega \} \), we can consider the homomorphisms from \( A \) to \( B^*_W \), from \( A^*_W \) to \( B \) and so on.

**Definition 2.5** The system of words \( W = \{w_\omega \mid \omega \in \Omega \} \) is called applicable if \( w_\omega(x_1,\ldots,x_{\rho_\omega}) \in F_\omega \) and for every \( F = F(X) \in \text{Ob}\Theta^0 \) there exists an isomorphism \( s_F : F \to F^*_W \) such that \( s_F|_X = id_X \).

The set of all the applicable systems of words we denote by \( ASW \). This set is never empty. The trivial example of the applicable system of words, which always exists, give as the system \( W = \{w_\omega \mid \omega \in \Omega \} \), such that \( w_\omega = \omega \) for every \( \omega \in \Omega \).

We suppose that \( W = \{w_\omega \mid \omega \in \Omega \} \) is an applicable system of words and consider the system of isomorphisms \( S = \{s_F : F \to F^*_W \mid F \in \text{Ob}\Theta^0 \} \) mentioned in Definition \( 2.3 \). The isomorphism \( s_F \) as mapping from algebra \( F \in \text{Ob}\Theta^0 \) to itself is only a bijection, which fulfill conditions \( 2.3 \). The mappings \( s_B \mu s_A^{-1} : A \to B \) are homomorphisms by \cite{24} Corollary 2 from Proposition 3.4 for every \( A,B \in \text{Ob}\Theta^0 \) and every \( \mu \in \text{Mor}_{\Theta^0}(A,B) \). So \( S = \{s_F : F \to F \mid F \in \text{Ob}\Theta^0 \} \) is a strongly stable system of bijections. From \cite{24} Proposition 3.5 we conclude that the isomorphisms \( s_F : F \to F^*_W \) such that \( 2.3 \) holds are uniquely defined by the system of words \( W \). So the system of bijections \( S \) is uniquely defined by \( W \). We denote this system by \( S_W \). Therefore the mapping \( B : ASW \to SSSB \) such that \( B(W) = S_W \) is well defined. This mapping is one to one and onto by \cite{24} Proposition 3.6. In particular, if system of bijections \( S = \{s_F : F \to F \mid F \in \text{Ob}\Theta^0 \} \) is a strongly stable system of bijections, then a word \( w_\omega \) from the applicable system of words \( W = B^{-1}(S) \) we can obtain by the formula

\[
w_\omega(x_1,\ldots,x_{\rho_\omega}) = s_{F_\omega}(\omega(x_1,\ldots,x_{\rho_\omega})) \in F_\omega,
\]

where \( \omega \in \Omega \) (see \cite{18} Susection 2.4, \cite{24} Equation (3.1)]).

Now we can conclude \cite{24} Theorem 3.1 that there is one to one and onto correspondence \( C = B^{-1}A : \mathcal{S} \to ASW \). We denote \( C(\Phi) \) by \( W_\Phi \). The systems of words \( W_\Phi \) is defined by formula \( 2.4 \) where bijections \( s_{F_\omega} = s_{F_\omega}^\Phi \) are the corresponding bijections of the system \( A(\Phi) = S_\Phi \).
Therefore we can calculate the group \( \mathfrak{G} \) if we are able to find all applicable system of words.

If \( \Phi_1, \Phi_2 \in \mathfrak{G} \) and

\[
\mathcal{A}(\Phi_1) = S_{\Phi_1} = \left\{ s_{F_1}^\Phi : F \to F \mid F \in \text{Ob}\Theta^0 \right\},
\]

\[
\mathcal{A}(\Phi_2) = S_{\Phi_2} = \left\{ s_{F_2}^\Phi : F \to F \mid F \in \text{Ob}\Theta^0 \right\}
\]

are strongly stable systems of bijections correspond to automorphisms \( \Phi_1 \) and \( \Phi_2 \), then as we saw in the previous section, the strongly stable system of bijections

\[
\mathcal{A}(\Phi_2 \Phi_1) = S = \left\{ s_{F_2}^\Phi s_{F_1}^\Phi : F \to F \mid F \in \text{Ob}\Theta^0 \right\}
\]

corresponds to the strongly stable automorphism \( \Phi_2 \Phi_1 \). Hence, by (2.4), the applicable systems of words \( B^{-1}(S) = C(\Phi_2 \Phi_1) \) we can obtain by formula

\[
w_\omega(x_1, \ldots, x_{\rho_w}) = s_{F_2}^\Phi s_{F_1}^\Phi(\omega(x_1, \ldots, x_{\rho_w})),
\]

(2.5)

where \( \omega \in \Omega \).

### 2.4 Automorphisms, which are strongly stable and inner

For calculation of the group \( \mathfrak{G} \cap \mathbb{Q} \) we also have the following

**Criterion 2.1** [13] **Lemma 3**/The strongly stable automorphism \( \Phi \) of the category \( \Theta^0 \), such that \( C(\Phi) = W_\Phi = W \), is inner if and only if for every \( F \in \text{Ob}\Theta^0 \) there exists an isomorphism \( c_F : F \to F_W^\ast \) such that

\[
c_B\psi = \psi c_A
\]

(2.6)

is fulfilled for every \( A, B \in \text{Ob}\Theta^0 \) and every \( \psi \in \text{Mor}_{\Theta^0}(A, B) \).

Also we have

**Proposition 2.1** [6] **Proposition 23**/The system of functions \( \{c_A : A \to A \mid A \in \text{Ob}\Theta^0\} \) fulfills the equality (2.6) for every \( A, B \in \text{Ob}\Theta^0 \) and every \( \psi \in \text{Mor}_{\Theta^0}(A, B) \) if and only if there exists \( c(x) \in F(x) \) such that

\[
c_A(a) = c(a),
\]

(2.7)

for every \( A \in \text{Ob}\Theta^0 \) and every \( a \in A \).

**Proof.** We consider \( c(x) \in F(x) \) and define the system of functions \( \{c_A : A \to A \mid A \in \text{Ob}\Theta^0\} \) by (2.7). We have for every \( \psi \in \text{Mor}_{\Theta^0}(A, B) \) and every \( a \in A \) that the equality \( \psi c_A(a) = c(\psi(a)) = c(\psi(a)) = c_{B\psi}(a) \) holds, because \( \psi \in \text{Mor}_{\Theta^0}(A, B) \).

We suppose that exists a system of functions \( \{c_A : A \to A \mid A \in \text{Ob}\Theta^0\} \) which fulfills equality (2.6) for every \( A, B \in \text{Ob}\Theta^0 \) and every \( \psi \in \text{Mor}_{\Theta^0}(A, B) \). We consider the algebra \( F = F(x) \in \text{Ob}\Theta^0 \). There exists \( c(x) = c_F(x) \in F(x) \). For every \( A \in \text{Ob}\Theta^0 \) and every \( a \in A \) we can consider the homomorphism \( \alpha_a : F(x) \to A \), such that \( \alpha_a(x) = a \). Therefore \( c_A(a) = c_A(\alpha_a(x)) = \alpha_a(c_F(x)) = \alpha_a(c(x)) = c(a) \).
3 Application of the method of verbal operations

We consider every group as universal algebra with signature which has 3 operations:

\[ \Omega = \{1, -1, \cdot\} \]

where the 0-ary operation 1 gives us an unit of a group, the 1-ary operation \(-1\) give us for an arbitrary element \(g\) of a group \(G\) the inverse element \(g^{-1}\) and the 2-ary operation \(\cdot\) give us for two elements of a group \(G\) its product.

The IBN (invariant basis number) property or invariant dimension property was defined initially in the theory of rings and modules, see, for example, [8, Definition 2.8]. But then this concept was generalized to arbitrary varieties of algebras:

Definition 3.1 We say that the variety \(\Theta\) has an IBN property if for every \(F_{\Theta}(X), F_{\Theta}(Y) \in \text{Ob}\Theta^0\) the \(F_{\Theta}(X) \cong F_{\Theta}(Y)\) holds if and only if \(|X| = |Y|\).

By [5] our variety \(\Theta\) has an IBN property. It is easy to conclude from this fact that in the variety \(\Theta\) the Condition 2.1 fulfills. So, the method of verbal operations is valid in our variety.

Thus the strategy of our research is clear. First of all we will compute the 2-generated free group of our variety \(F_{\Theta}(x,y)\).

After that we will find all applicable system of words

\[ W = \{w_1, w_{-1}(x), w.(x,y)\} \]

where \(w_1\) is a constant which correspond to the 0-ary operation 1, \(w_{-1}(x) \in F_{\Theta}(x)\) is a word which correspond to the 1-ary operation \(-1\) and \(w.(x,y) \in F_{\Theta}(x,y)\) is a word which correspond to the 2-ary operation \(\cdot\). We will use the Definition 2.5 for the finding of the applicable system of words. The necessary conditions for the system of words to be applicable we will conclude from the fact that the isomorphism \(s_F : F \to F^*_W\), which exists for every \(F \in \text{Ob}\Theta^0\), provide the fulfilling of all identities of the variety \(\Theta\) in the groups \(F^*_W\). It will give us 4 systems of words of the form (3.1), which can be applicable.

In the next step of our research we will prove that all these systems of words are applicable. We will prove that for all these system \(W\) all identities of the variety \(\Theta\) really fulfill in the groups \(F^*_W\) for every \(F \in \text{Ob}\Theta^0\). This will allow us to construct the homomorphism \(s = s_{F(X)} : F(X) \to (F(X))^*_W\), such that \(s|_X = id_X\) for every \(F(X) \in \text{Ob}\Theta^0\). After that we will find the inverse maps for every \(s_{F(X)}\). It allow to conclude that all homomorphisms \(s_{F(X)}\) are isomorphisms and all 4 considered systems of words are applicable and provide the strongly stable automorphisms of the category \(\Theta^0\).

And we will finish our research when we will compute for the category \(\Theta^0\) the group \(\frak{G} \cap \frak{S}\) by Criterion 2.1 and Proposition 2.1. We will see in the end of our research that the group \(\frak{A}/\frak{G}\) of the category \(\Theta^0\) contains 2 elements.
Some properties of the varieties $\mathcal{N}_4$ and $\Theta$

In this paper $\mathcal{N}_4$ is the variety of nilpotent groups of class no more than 4. The free groups of this variety, generated by generators $x_1, \ldots, x_n$ we will denote by $N_4(x_1, \ldots, x_n)$.

We will denote $(((\ldots((x, y), z), \ldots), t))$ as $(x, y, z, \ldots, t)$. Also we will denote for every group $G$ the $\gamma_1(G) = G$ and $\gamma_i(G) = (\gamma_{i-1}(G), G)$. And we will denote by $Z(G)$ the center of the group $G$.

In our computation we will frequently use the identities

\[(xy, z) = (x, z)(y, z), \quad (x, y)z = (x, y)(x, z), \quad (x^{-1}, y) = (y, x)(x^{-1}, y),\]

which fulfill in every group (see [7, (10.2.1.2) and (10.2.1.3)] and [9, p. 20, (3)]).

From these identities we can conclude these facts about arbitrary group $G \in \mathcal{N}_4$:

1. for every $g_1, g_2 \in G$ and every $l_1, l_2 \in \gamma_4(G)$
   \[(g_1l_1, g_2l_2) = (g_1, g_2),\]

2. for every $g \in G$ and every $l_1, l_2 \in \gamma_2(G)$
   \[(l_1l_2, g) = (l_1, g)(l_2, g), \quad (g, l_1l_2) = (g, l_2)(g, l_1),\]

3. for every $g_1, g_2, g_3 \in G$ and every $l_1, l_2, l_3 \in \gamma_3(G)$
   \[(g_1l_1, g_2l_2, g_3l_3) = (g_1, g_2, g_3),\]

4. for every $g_1, g_2, g_3, g_4 \in G$ and every $l_1, l_2, l_3, l_4 \in \gamma_2(G)$
   \[(g_1l_1, g_2l_2, g_3l_3, g_4l_4) = (g_1, g_2, g_3, g_4),\]

5. every commutator of the length 4 is a multiplicative function by all its 4 arguments:
   \[w(g_1, \ldots, g_4) = w(g_1, \ldots, g_i, \ldots, g_4)w(g_1, \ldots, l_i, \ldots, g_4), \quad 1 \leq i \leq 4,\]

where $w(x_1, \ldots, x_4) \in \gamma_4(N_4(x_1, \ldots, x_4))$, $1 \leq i \leq 4$, holds for every $g_1, \ldots, g_4, l_i \in G$.

For every $G \in \mathcal{N}_4$ we have that $\gamma_4(G) \subseteq Z(G)$ and $\gamma_5(G) = \{1\}$. For every $G \in \Theta$ the group $\gamma_2(G)$ is an abelian group. We will use these facts later in our computations without special reminder. Also we use the identity $yx = xy(y, x)$, which fulfills in every group, and the identity \[(1.1) \quad (x, y)z = (x, y)(x, z),\]

which fulfills in an every group of the variety $\Theta$ without special reminder.
In this subsection we will describe the free group of our variety \( \Theta \) generated by 2 generators. This group is a quotient group \( N_4(x, y)/T \), where \( T \) is a normal subgroup of the identities with two variables of the subvariety \( \Theta \) in the variety \( M_4 \).

By \([9, \text{Theorem 17.2.2}]\), if \( G \) is finitely generated nilpotent group then there exist central (in particular, normal) series:

\[
G = G_1 > G_2 > \ldots > G_s > G_{s+1} = \{1\}
\]

such that \( G_i/G_{i+1} = \langle a_i G_{i+1} \rangle \) \( \iff G_i = \langle a_i, G_{i+1} \rangle \), \( a_i \in G_i \).

\( \langle a_i G_{i+1} \rangle \cong \mathbb{Z}_n \) (\( n \geq 2 \)), or \( \langle a_i G_{i+1} \rangle \cong \mathbb{Z} \). Therefore every \( g \in G \) can be uniquely represented in the form

\[
\prod a_{\alpha_i} (x_i G_{i+1}) = \prod a_{\alpha_i} \mathbb{Z}_n \quad \text{or} \quad \prod a_{\alpha_i} \mathbb{Z}.
\]

Therefore every \( g \in G \) can be uniquely represented in the form

\[
\prod a_{\alpha_i} (x_i G_{i+1}) = \prod a_{\alpha_i} \mathbb{Z}_n \quad \text{or} \quad \prod a_{\alpha_i} \mathbb{Z}.
\]

### Definition 4.1

We say that the set \( \{a_1, a_2, \ldots, a_s\} \) is a base of the group \( G \) and numbers \( \alpha_1, \alpha_2, \ldots, \alpha_s \) are coordinates of the element \( g \) in this base.

The base of \( N_4(x, y) \) we can denote by

\[
C_1 = x, C_2 = y, C_3 = (y, x), C_4 = (y, x, y), C_5 = (y, x, x),
\]

\[
C_6 = (y, x, x, x), C_7 = (y, x, y, y), C_8 = (y, x, y, x).
\]

This is a base of Shirshov, which we can compute by the algorithm explained in \([1, 2.3.5]\).

In particular, if we substitute in \([7, 10.2.1.4]\) \( (y, x) \) instead \( x \) and \( x \) instead \( z \), we obtain

\[
((y, x), y^{-1}, x)^y (y, x^{-1}, (y, x))^x (x, (y, x)^{-1}, y)^{(y, x)} = 1.
\]

So, by \([7, 10.2.1.4]\), we can conclude that

\[
((y, x), y, x)^{-1} (y, x, (y, x))^{-1} (x, (y, x), y)^{-1} = 1.
\]

We have that

\[
(y, x, (y, x)) = ((y, x), (y, x)) = 1,
\]

and

\[
(x, (y, x), y) = ((x, (y, x)), y) = ((y, x), x)^{-1} = (y, x, x, y)^{-1},
\]

hence

\[
(y, x, x, y)^{-1} (y, x, x, y) = 1
\]

and

\[
(y, x, y, x) = (y, x, y, x) = C_8.
\]

### Proposition 4.1

The identity

\[
(xy)^4 = x^4 y^4 (y, x)^6 (y, x, y)^{14} (y, x, y)^{11} (y, x, x, y)^{11} (y, x, x, x),
\]

fulfills in the variety \( M_4 \).
Proof. We will consider the group $G \in \mathfrak{N}_4$ and $x, y \in G$.
Initially we go to compute $(xy)^2$. We have that

$$(xy)^2 = xyxy = x^2 y(y, x) y = x^2 y^2 (y, x, x, y).$$

After this we compute $(xy)^3$ by same method:

$$(xy)^3 = (xy)^2 (xy) = x^2 y^2 (y, x, x, y) x y = x^2 y^2 x (y, x) (y, x, x, x) (y, x, y, x, y) y.$$

Now we will compute $y^2 x$:

$$y^2 x = y(y, x) = x y x y (y, x, y) = (4.12)$$

because elements of $\gamma_2 (G)$ commute with elements of $\gamma_3 (G)$ in every $G \in \mathfrak{N}_4$.

Hence, by (4.12) and (4.10), we have the equality

$$(xy)^3 = x^3 y^2 (y, x, x, y) (y, x, x, x) (y, x, y, x, y) y =$$

$$x^3 y^3 (y, x, y)^3 (y, x, y) (y, x, y)^2 (y, x, x, x) (y, x, y) (y, x, y, x) y = (4.13)$$

$$x^3 y^3 (y, x, y)^3 (y, x, x, x) (y, x, y) (y, x, y)^2 (y, x, y, x)^2.$$

Now we will compute $(xy)^4$. By (4.13) we have that

$$(xy)^4 = xy(xy)^3 = xy x^3 y^3 (y, x, x, y)^3 (y, x, y)^3 (y, x, y)^3 (y, x, y)^2 (y, x, y, x)^2 (4.14)$$

After this we can compute that

$$y^2 x = y(y, x) = x y x y (y, x, y) =$$

$$y^2 x = y y x^3 y^3 (y, x, x, x) (y, x, x, y)^3 (y, x, x, x)^3 (y, x, x, x, y)^3 y^3. (4.15)$$

We have that

$$(y, x, x, x) y^3 = y^3 (y, x, x, x). (4.16)$$

Also we can compute that

$$(y, x, x, y)^3 y^3 = y (y, x, x, y)^3 (y, x, x, y) y^2 =$$

$$(y, x, x, y)^3 y^2 (y, x, x, y)^3 (y, x, x, y)^3 (y, x, x, y)^3 (y, x, x, y)^3 y^3 (4.17)$$

and

$$(y, x, y)^3 y^3 = y (y, x, y)^3 (y, x, y) y^2 =$$

$$(y, x, x, y)^3 (y, x, y)^3 (y, x, y)^3 (y, x, y)^3 (y, x, y)^3 y =$$
\[
y^2(y, x)^3 (y, x, y)^6 (y, x, y, y)^3 y =  (4.18)
\]

\[
y^3(y, x)^3 (y, x, y)^3 (y, x, y, y)^6 (y, x, y, y)^3 =
\]

\[
y^3(y, x)^3 (y, x, y)^9 (y, x, y, y)^9.
\]

Therefore, by (4.15), (4.16), (4.17) and (4.18),

\[
xy^3 y^3 = x^4 y^4 (y, x, y)^3 (y, x, y, y)^9 (y, x, x)^9 (y, x, x, x).
\]

After this, we have, by (4.14) and (4.19), that

\[
(xy)^4 = x^4 y^4 (y, x)^3 (y, x, y)^9 (y, x, y, y)^9 (y, x, x)^9 (y, x, x, x).
\]

By (4.10) we have the

**Corollary 1** The identity

\[
1 = (y, x)^2 (y, x, y)^2 (y, x, x, x) (y, x, y, y)^{-1} (y, x, y, x)^{-1}.
\]

fulfills in the variety \( \Theta \).

We denote the images of elements of the base \( \{C_1, \ldots, C_8\} \) by the natural homomorphism \( N_4(x, y) \to N_4(x, y) / T = F_\Theta(x, y) \) by same notation: \( \{C_1, \ldots, C_8\} \).

**Proposition 4.2** The relations:

\[
C_i^2 = 1, \quad (4 \leq i \leq 8)
\]

\[
C_2^2 C_6 C_7 C_8 = 1
\]

in \( F_\Theta(x, y) \) are conclusions from the identities of \( \Theta \).

**Proof.** The (4.20) is an identity in \( \Theta \), so in (4.20) we can substitute \( x \) instead of \( y \) and vice versa. Therefore

\[
1 = (x, y)^2 (x, y, x)^2 (x, y, y, y) (x, y, x, x)^{-1} (x, y, x)^{-1} =
\]

\[
(y, x)^2 (y, x, x)^2 (y, x, y, y) (y, x, x, x)^{-1} (y, x, x)^{-1}.
\]

By (4.13), (4.17), (4.18) and (4.10) we have that \( (y, x)^2 = (x, y)^2, \quad (y, x, x, x)^{-1} = (x, y, x, x)^{-1}, \quad (y, x, y, y) = (y, y, x, y)^{-1}, \quad (y, x, x) = (y, y, x, y)^{-1} \). Therefore we conclude from (4.23) that

\[
(x, y)^2 (y, x, x) = (y, x, x)^2 (x, y, x, x)^{-1}.
\]
Also we have by (4.3) that
\[(x, y, x) = (y, x, x)^{-1} (x, (y, x), (x, y)) = (y, x, x)^{-1} = C_5^{-1}.\]
Therefore \((x, y, x)^2 = C_5^{-2} = C_5^2\). Now we conclude from (4.24) that
\[C_4^2 = C_5^2 C_8^2.\]  
(4.25)

Now we substitute in (4.20) \((y, x)\) instead \(x\) and \(x\) instead \(y\):
\[1 = (x, (y, x))^2 (x, (y, x), (y, x), (y, x)), (y, x)) - (x, (y, x), x, (y, x))^{-1} = (x, (y, x))^2 (x, (y, x), x) = (y, x)^{-2} (y, x, x)^{-2} \]
So the relation
\[C_5^2 C_8^2 = 1\]  
(4.26) holds.

Analogously we substitute in (4.20) \(y\) instead \(x\) and \((y, x)\) instead \(y\) and conclude that
\[1 = C_4^2.\]  
(4.27)

Now by (4.20) and (4.26) we have that
\[C_5^2 = C_6^2 = C_8^2.\]  
(4.28)

Also, when we substitute in (4.20) \((y, x, x)\) instead \(y\), we obtain that
\[1 = C_6^2.\]  
(4.29)

And when we substitute in (4.20) \((y, x, y)\) instead \(x\), we conclude
\[1 = C_7^{-2} = C_7^2.\]  
(4.30)

Therefore, we conclude (4.21) from (4.27), (4.28), (4.29), (4.30). And after this the (4.20) has form
\[1 = C_3^2 C_4^2 C_5 C_7^{-1} C_8^{-1} = C_3^2 C_6 C_7 C_8.\]

Now we consider in the group \(N_4(x, y)\) the minimal normal subgroup \(R\) which contains elements \(x^4, y^4\) and the left parts of the relations (4.21) and (4.22). Here we consider the elements \(x = C_1, y = C_2, C_3, \ldots, C_8\) as elements of \(N_4(x, y)\). The images of the elements \(C_1, \ldots, C_8\) by the natural epimorphism \(N_4(x, y) \rightarrow N_4(x, y) / R\) we also denote by \(C_1, \ldots, C_8\). We see from the Proposition (4.4) that the base of the group \(N_4(x, y) / R\) is \(\{C_1, C_2, \ldots, C_7\}\) and, if \(1 \leq i \leq 3\), then \(|C_i| = 4\), if \(4 \leq i \leq 7\), then \(|C_i| = 2\).

Our goal is to prove that \(N_4(x, y) / R = F_{60}(x, y)\). For this we must study the group \(N_4(x, y) / R\) and prove some lemmas about it’s properties. These lemmas we will use in the proof of the Theorem (6.1) and in other computations.
5 Some lemmas about the group $N_4(x, y) / R$

In this section we will denote the group $N_4(x, y) / R$ by $G$.

**Lemma 5.1** $\gamma_2(G)$ is a commutative group.

**Proof.** We have that $\gamma_3(N_4(x, y)) \leq Z(\gamma_2(N_4(x, y)))$ and quotient group $\gamma_2(N_4(x, y)) / \gamma_3(N_4(x, y)) = ((y, x) \gamma_3(N_4(x, y)))$ is a cyclic group. Therefore $\gamma_2(N_4(x, y))$ is a commutative group. $G$ is a homomorphic image of $N_4(x, y)$, so, $\gamma_2(G)$ is a commutative group. ■

**Lemma 5.2** The group $\gamma_3(G)$ is a group of exponent 2.

**Proof.** We have that $\gamma_3(G) = \langle C_4, \ldots, C_7 \rangle$. Lemma 5.1 and the consideration of relations (4.21) completes the proof. ■

**Lemma 5.3** For every $h \in \gamma_2(G)$ the inclusion $h^2 \in \gamma_4(G)$ holds.

**Proof.** We have that $\gamma_2(G) = \langle C_4, \ldots, C_7 \rangle$. Lemma 5.1 and the consideration of relations (4.21) and (4.22) completes the proof. ■

**Lemma 5.4** For every $a, b, c \in G$ the following equalities holds:

\[
(ab, c)^2 = (a, c)^2(b, c)^2, \quad (a, bc)^2 = (a, c)^2(a, b)^2 \\
(a^{-1}, b)^2 = (a, b)^2 \\
(a, b^{-1})^2 = (a, b)^2.
\]

**Proof.** We have that

\[
(ab, c)^2 = ((a, c)^b(b, c))^2 = ((a, c)^2)^b(b, c)^2
\]

by (4.1) and by Lemma 5.1. And now by Lemma 5.3 we conclude (5.1). By similar computation we can conclude (5.1) from (4.2) and Lemma 5.3.

By Lemmas 5.3 and 5.2 $\gamma_2(G)$ is a group of exponent 4. Therefore, by (4.3) and Lemma 5.3 we have that

\[
(a^{-1}, b)^2 = ((b, a)^2)^2 = ((b, a)^2)^{a^{-1}} = (b, a)^2 = (a, b)^{-2} = (a, b)^2.
\]

By similar computation we can conclude (5.3). ■

**Lemma 5.5** If $g \in G$, $h \in \gamma_2(G)$, then the $(gh)^4 = g^4$.

**Proof.** We know that the identity (4.11) holds in the variety $\mathfrak{N}_4$. So this identity holds in $G$. Hence we have that

\[
(gh)^4 = g^4h^4(h, g)^6(h, g, h)^{14}(h, g, h)^{14}(h, g, g, h)^{11}(h, g, g, g)^{11}(h, g, g, g).
\]

In our case $(h, g, h), (h, g, h, h), (h, g, g, h), (h, g, g, g) \in \gamma_5(G)$. By Lemmas 5.3 and 5.2 we have that $h^4 = (h, g)^6 = (h, g)^4 = 1$. Therefore $(gh)^4 = g^4$. ■

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6 Computation of the group $F_{\Theta} (x, y)$

**Theorem 6.1** $N_4 (x, y) / R = F_{\Theta} (x, y)$.

**Proof.** In this proof we also denote the group $N_4 (x, y) / R$ by $G$.

By Proposition 4.2, the relations $r = 1$, where $r \in R$, are conclusions from the identities which define the variety $\Theta$. So we only must prove that $G \in \Theta$.

It is clear that the group $G$ is a nilpotent groups of class 4.

As we said in the proof of the Lemma 5.3, $G$ is a metabelian group.

Now we will prove that the group $G$ fulfills the identity (1.1). By Lemma 5.2, it remains for us to prove that for every $0 \leq \alpha_1, \alpha_2 \leq 3$ the

$$(x^{\alpha_1} y^{\alpha_2})^4 = 1$$

holds in $G$. We substitute in (4.11) $x^{\alpha_1}$ instead $x$ and $y^{\alpha_2}$ instead $y$. $(x^{\alpha_1})^4 = (y^{\alpha_2})^4 = 1$ holds in $G$. Therefore we must only prove that

$$(y^{\alpha_2}, x^{\alpha_1})^6 (y^{\alpha_2}, x^{\alpha_1}, y^{\alpha_2})^{14} (y^{\alpha_2}, x^{\alpha_1}, y^{\alpha_2}, y^{\alpha_2})^{11} (y^{\alpha_2}, x^{\alpha_1}, x^{\alpha_1})^4$$

holds in $G$. By Lemmas 5.3 and 5.2 we have that

$$(y^{\alpha_2}, x^{\alpha_1})^6 = (y^{\alpha_2}, x^{\alpha_1})^2,$$

$$(y^{\alpha_2}, x^{\alpha_1}, y^{\alpha_2})^{14} = 1,$$

$$(y^{\alpha_2}, x^{\alpha_1}, y^{\alpha_2}, y^{\alpha_2})^{11} = (y^{\alpha_2}, x^{\alpha_1}, y^{\alpha_2}, y^{\alpha_2}),$$

$$(y^{\alpha_2}, x^{\alpha_1}, x^{\alpha_1})^4 = 1,$$

$$(y^{\alpha_2}, x^{\alpha_1}, x^{\alpha_1}, y^{\alpha_2})^{11} = (y^{\alpha_2}, x^{\alpha_1}, x^{\alpha_1}, y^{\alpha_2}).$$

We denote

$$v (\alpha_1, \alpha_2) = (y^{\alpha_2}, x^{\alpha_1})^2 (y^{\alpha_2}, x^{\alpha_1}, y^{\alpha_2}, y^{\alpha_2}) (y^{\alpha_2}, x^{\alpha_1}, x^{\alpha_1}) (y^{\alpha_2}, x^{\alpha_1}, x^{\alpha_1}, x^{\alpha_1}) (y^{\alpha_2}, x^{\alpha_1}, x^{\alpha_1}, x^{\alpha_1})$$

So it remains for us to prove that

$$v (\alpha_1, \alpha_2) = 1 \quad (6.1)$$

holds in $G$ for every $0 \leq \alpha_1, \alpha_2 \leq 3$.

It is clear that (6.1) holds in $G$ if $\alpha_1 = 0$ or $\alpha_2 = 0$. If $\alpha_1 = \alpha_2 = 1$, than by (4.11), we have that

$$v (1, 1) = (y, x)^2 (y, x, y) (y, x, y) (y, x, x, y) (y, x, x, x) = C_3^2 C_7 C_8 C_6 = 1.$$

Now we will prove (6.1) by induction on $\alpha_1$ and $\alpha_2$. We suppose that (6.1) holds for all $\alpha_1, \alpha_2$ such that $\alpha_1 \leq \beta_1$, $\alpha_2 \leq \beta_2$ and $0 \leq \alpha_1, \alpha_2$. We have by (5.2) that

$$(y^{\beta_2}, x^{\beta_1 + 1})^2 = (y^{\beta_2}, x)^2 (y^{\beta_2}, x^{\beta_1})^2. \quad (6.2)$$
We have by (4.8) that
\[(y^{\beta_2}, x^{\beta_1+1}, y^{\beta_2}, y^{\beta_2}) = (y^{\beta_2}, x^{\beta_1}, y^{\beta_2}, y^{\beta_2}) (y^{\beta_2}, x, y^{\beta_2}, y^{\beta_2}), \] (6.3)
\[(y^{\beta_2}, x^{\beta_1+1}, x^{\beta_1+1}, y^{\beta_2}) = (y^{\beta_2}, x, x, y^{\beta_2}) (\beta_1+1)^2 =
(y^{\beta_2}, x, x, y^{\beta_2}) (\beta_1+1)^2 (y^{\beta_2}, x, x, y^{\beta_2}) \]
and
\[(y^{\beta_2}, x^{\beta_1+1}, x^{\beta_1+1}, x^{\beta_1+1}) = (y^{\beta_2}, x, x, x) (\beta_1+1)^3 =
(y^{\beta_2}, x, x, x)^{\beta_1} (y^{\beta_2}, x, x, x)^{\beta_1} (\beta_1+1) (y^{\beta_2}, x, x, x). \]
By Lemma 5.2 we have that
\[(y^{\beta_2}, x, x, y^{\beta_2})^{\beta_1} = (y^{\beta_2}, x, x, x)^{\beta_1 (\beta_1+1)} = 1, \]
because \(\beta_1 (\beta_1+1)\) is an even number. Hence
\[(y^{\beta_2}, x^{\beta_1+1}, x^{\beta_1+1}, y^{\beta_2}) = (y^{\beta_2}, x, x, y^{\beta_2})^{\beta_1} (y^{\beta_2}, x, x, y^{\beta_2}) = \] (6.4)
\[(y^{\beta_2}, x^{\beta_1}, x^{\beta_1}, y^{\beta_2}) (y^{\beta_2}, x, x, y^{\beta_2}) \]
and
\[(y^{\beta_2}, x^{\beta_1+1}, x^{\beta_1+1}, x^{\beta_1+1}) = (y^{\beta_2}, x, x, x)^{\beta_1} (y^{\beta_2}, x, x, x) = \] (6.5)
\[(y^{\beta_2}, x^{\beta_1}, x^{\beta_1}, x^{\beta_1}) (y^{\beta_2}, x, x, x). \]
Therefore, by (6.2), (6.3), (6.4), (6.5) and by our hypothesis about \(v(\beta_1, \beta_2)\)
and \(v(1, \beta_2)\), we have that
\[v(\beta_1+1, \beta_2) = v(\beta_1, \beta_2) v(1, \beta_2) = 1. \]
By (5.1) we have that
\[(y^{\beta_2+1}, x^{\beta_1})^2 = (y^{\beta_2}, x^{\beta_1})^2 (y, x^{\beta_1})^2, \] (6.6)
And now, similarly to the previous arguments, we conclude that
\[(y^{\beta_2+1}, x^{\beta_1}, x^{\beta_1}, x^{\beta_1}) = (y^{\beta_2}, x^{\beta_1}, x^{\beta_1}, x^{\beta_1}) (y, x^{\beta_1}, x^{\beta_1}, x^{\beta_1}), \] (6.7)
\[(y^{\beta_2+1}, x^{\beta_1}, x^{\beta_1}, x^{\beta_1}) = (y^{\beta_2}, x^{\beta_1}, x^{\beta_1}, y^{\beta_2}) (y, x^{\beta_1}, x^{\beta_1}, y^{\beta_2}), \] (6.8)
and
\[(y^{\beta_2+1}, x^{\beta_1}, y^{\beta_2+1}, y^{\beta_2+1}) = (y^{\beta_2}, x^{\beta_1}, y^{\beta_2}, y^{\beta_2}) (y, x^{\beta_1}, y, y). \] (6.9)
Hence, by (6.6), (6.7), (6.8), (6.9) and by our hypothesis about \(v(\beta_1, \beta_2)\)
and \(v(\beta_1, 1)\),
\[v(\beta_1+1, \beta_2) = v(\beta_1, \beta_2) v(\beta_1, 1) = 1. \]
Therefore we prove that (6.1) holds in \(G\) for every \(0 \leq \alpha_1, \alpha_2 \leq 3\). This completes our proof. □
Now, when we know that \(N_4(x,y) / R = F_{\alpha}(x,y)\), we can prove the
Corollary 1 The Lemmas 5.1, 5.2, 5.3 and 5.4 hold when we consider as group \( G \) an arbitrary group of the variety \( \Theta \).

Proof. Lemma 5.1 is fulfilled by definition of the variety \( \Theta \).

Now we will prove that every \( G \in \Theta \) fulfills the conclusion of the Lemma 5.3. Every \( b \in \gamma_2 (G) \) is generated by commutators \((a,b)\), such that \( a,b \in G \). There exists the homomorphism \( \varphi : F_\Theta (x,y) \to G \) such that \( \varphi (x) = b, \varphi (y) = a \). We apply \( \varphi \) to (4.22) and conclude that \((a,b)^2 \in \gamma_4 (G)\).

Also every \( G \in \Theta \) fulfills the conclusion of the Lemma 5.2 because the group \( \gamma_3 (G) \) is generated by the commutators \((a,b)\), where \( a \in G, b \in \gamma_2 (G) \). We also consider the homomorphism \( \varphi : F_\Theta (x,y) \to G \) from the previous part of the proof, apply it to (4.22) and now, because \( b \in \gamma_2 (G) \), conclude that \((a,b)^2 \in \gamma_5 (G)\).

The proof of the fact that every \( G \in \Theta \) fulfills the conclusion of the Lemma 5.4 coincides with the proof of the Lemma 5.4 for the group \( N_4 (x,y) / R \). □

7 Applicable systems of words. Necessary conditions

Proposition 7.1 If \( W \) (see (7.1)) is applicable system of words in our variety \( \Theta \) then always \( w_1 = 1 \), \( w_{-1} (x) = x^{-1} \).

Proof. We suppose that \( W \) is an applicable system of words. \( w_1 \in F_\Theta (\emptyset) = \{1\} \), so \( w_1 = 1 \).

\( w_{-1} (x) \in F_\Theta (x) \cong \mathbb{Z}_4 \). We denote \( F_\Theta (x) \) by \( F \). Because \( W \) is applicable system of words, by Definition 2.5 there exists the isomorphism \( s_F : F \to F_W \), such that \( s_F (x) = x \). We have that \( s_F (x^{-1}) = w_{-1} (s_F (x)) = w_{-1} (x) \). If \( w_{-1} (x) = 1 \), then \( s_F (x^{-1}) = 1 \), but \( s_F (1) = w_1 = 1 \), and this contradicts the assumption that \( s_F \) is an injective mapping.

If \( w_{-1} (x) = x \), then \( s_F (x^{-1}) = x = s_F (x) \) which gives the same contradiction.

If \( w_{-1} (x) = x^2 \), then \( x = s_F (x) = s_F ((x^{-1})^{-1}) = w_{-1} (w_{-1} (x)) = w_{-1} (x^2) = (x^2)^2 = x^4 = 1 \). It also gives us a contradiction.

Therefore, there is only one possibility: \( w_{-1} (x) = x^3 = x^{-1} \). □

For studying of words \( w : (x,y) \) we need to consider the group \( F_\Theta (x,y) \). We denote this group by \( G \).

Because \( W \) is applicable system of words, by Definition 2.5 there exists the isomorphism \( s_G : G \to G_W \), which fix \( x \) and \( y \).

Proposition 7.2 If \( W \) (see (7.1)) is applicable system of words in our variety \( \Theta \) then always

\[ w : (x,y) = xyC_3^{\alpha_3}C_4^{\alpha_4}C_5^{\alpha_5}C_6^{\alpha_6}C_7^{\alpha_7}, \]  \hspace{1cm} (7.1)

where \( 0 \leq \alpha_3 < 4, \alpha_i \in \{0,1\}, \) when \( 4 \leq i \leq 7 \).
Proof. We use the considerations of [22, Proposition 2.1]. \( w(x,y) \in G \), so \( w(x,y) = x^{\alpha_1}y^{\alpha_2}g_2(x,y) \), where \( g_2(x,y) \in \gamma_2(G), 0 \leq \alpha_1, \alpha_2 < 4 \).

We have that \( x = s_G(x \cdot 1) = w(s_G(x), s_G(1)) = w(x, w1) = w(x, 1) = x^{\alpha_1}g_2(x, 1) = x^{\alpha_1} \) holds, because \( g_2(x, 1) \) is the result of substitution of 1 instead \( y \) in \( g_2(x, y) \). Therefore, \( \alpha_1 = 1 \). We obtain by similar computations that \( \alpha_2 = 1 \).

In the next Proposition we will get the stronger result about word \( w(x,y) \) from applicable system of words \( W \).

Proposition 7.3 If \( W \) (see (3.7)) is applicable system of words in our variety \( \Theta \) then always \( w(x,y) = xyC_3^{\alpha_3}, \) where \( \alpha_3 = 0,1,2,3 \).

Proof. The equalities \( x(xy) = (xx)y \) and \( x(yy) = (xy)y \) hold in \( G = F_{\Theta}(x,y) \).

We apply the isomorphism \( s_G : G \to G_W \) to the both hands of the first equality and have that
\[
s_G(x(xy)) = w(s_G(x), s_G(xy)) =
\]
\[
w(s_G(x), w(s_G(x), s_G(y))) = w(x, w(x, y))
\]
and
\[
s_G((xx)y) = w(s_G(xx), s_G(y)) =
\]
\[
w(s_G(x), s_G(x)), s_G(y)) = w(x, x), y).
\]
Therefore
\[
w(x, w(x, y)) = w(x, x), y).
\]
We conclude be similar computations from the second equality that
\[
w(x, w(y, y)) = w(x, x), y)
\]
holds. If we will denote the operation defined by the word \( (7.1) \) by symbol \( \circ \), then we can rewrite these equalities in the form
\[
x \circ (x \circ y) = (x \circ x) \circ y \quad (7.2)
\]
and
\[
x \circ (y \circ y) = (x \circ y) \circ y. \quad (7.3)
\]

Now we will compute the left hand of \( (7.2) \). We have that
\[
x \circ (x \circ y) = x \circ xyC_3^{\alpha_3}C_4^{\alpha_4}C_5^{\alpha_5}C_6^{\alpha_6}C_7^{\alpha_7}, =
\]
\[
xyC_3^{\alpha_3}C_4^{\alpha_4}C_5^{\alpha_5}C_6^{\alpha_6}C_7^{\alpha_7}, \quad L_3 = L_4 \quad L_4 \quad L_6^{\alpha_6}L_7^{\alpha_7}, \quad (7.4)
\]
where
\[
L_3 = (xyC_3^{\alpha_3}C_4^{\alpha_4}C_5^{\alpha_5}C_6^{\alpha_6}C_7^{\alpha_7}, x) = (xyC_3^{\alpha_3}C_4^{\alpha_4}C_5^{\alpha_5}, x) \quad (7.5)
\]
by \( (6.4) \).

\[
L_4 = (xyC_3^{\alpha_3}C_4^{\alpha_4}C_5^{\alpha_5}C_6^{\alpha_6}C_7^{\alpha_7}, x, xyC_3^{\alpha_3}C_4^{\alpha_4}C_5^{\alpha_5}C_6^{\alpha_6}C_7^{\alpha_7}, x) = (xyC_3^{\alpha_3}, x, xyC_3^{\alpha_3}) \quad (7.6)
\]

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by (4.6),
\[ L_5 = (xyC_3^3C_4^4C_5^5C_6^6C_7^7, x, x) = (xyC_3^3, x, x) \]  
(7.7)

by (4.6),
\[ L_6 = (xyC_3^3C_4^4C_5^5C_6^6C_7^7, x, x, x) = (xy, x, x, x) \]  
(7.8)

by (4.7),
\[ L_7 = (xyC_3^3C_4^4C_5^5C_6^6C_7^7, x, xyC_3^3C_4^4C_5^5C_6^6C_7^7, xyC_3^3C_4^4C_5^5C_6^6C_7^7) = (xy, x, x, y) \]  
(7.9)

by (4.7).

By (7.5) and (4.1) we have that
\[ L_3 = (xy, x)(xy, x, C_3^3C_4^4C_5^5) (xyC_3^3C_4^4C_5^5, x) \].  
(7.10)

By (4.1) the equality
\[ (xy, x) = (x, x)(x, y)(y, x) = (y, x) \]  
(7.11)

holds. By (4.6) and (7.11) we conclude that
\[ (xy, x, C_3^3C_4^4C_5^5) = ((y, x), (y, x)^{α_3}) = 1. \]  
(7.12)

By (4.5) we have that
\[ (C_3^3C_4^4C_5^5, x) = (C_3, x)^{α_3} (C_4, x)^{α_4} (C_5, x)^{α_5} = C_3^αC_4^αC_5^α. \]  
(7.13)

Hence, by (7.10), (7.11), (7.12) and (7.13) we have that
\[ L_3 = C_3C_4C_5C_6 \].  
(7.14)

By (4.1), (4.5), (7.11) and (7.12) we conclude that
\[ (xyC_3^3, x) = (xy, x)(x, x, C_3^3) (C_3^3, x) = C_3C_3^3. \]  
(7.15)

By (4.7), (7.15), (4.5), (4.2), (4.7) and (4.3), we have that
\[ L_4 = (C_3C_5^5, xyC_3^3) = (C_3, xyC_3^3) (C_3, xyC_3^3)^{α_3} = (C_3, C_3^3)(C_3, xy)(C_3, xy, C_3^3) (C_5, x)^{α_3} (C_5, y)^{α_3}. \]  
(7.16)

By (4.2) and (4.10) the equalities
\[ (C_3, xy) = (C_3, y)(C_3, x) (C_3, x, y) = C_4C_5C_8 \]  
(7.17)

fulfills. Therefore, by (7.10), (7.17), (4.10) and because \( (C_3, xy, C_3^α) \in γ_5 (G) \) the
\[ L_4 = C_4C_5C_8C_6^αC_8^α = C_4C_5C_6^αC_8^α+1 \]  
(7.18)

holds.
By (7.7), (7.15) and (4.5) we have that
\[ L_5 = (C_5 C_{\alpha^3}, x) = (C_5, x)(C_{\alpha^3} x) = C_5 C_{\alpha^3}. \] (7.19)

By (7.8), (7.9) and (4.8) we can conclude that
\[ L_6 = (y, x, x, x) = C_6 \] (7.20)
and
\[ L_7 = (y, x, x, x)(y, x, x, y)(y, x, x, y)(y, x, y, y) = C_7 C_8, \] (7.21)
because, by (4.10) and (4.21), \((y, x, y, x)(y, x, x, y) = C_2 = 1.\)
Therefore, by (7.24), (7.25), (7.26), (7.27), (7.28), (7.29), we have that
the left hand of (7.2) is equal to
\[ x \circ (x \circ y) = x^2 y C_3^3 C_4^3 C_5^3 C_6^3 C_7^3. \]
\[ C_3^3 C_5^3 C_8^3 C_6^3 C_7^3 C_9^3 C_7^3 \cdot C_4^3 C_5^3 C_6^3 C_7^3 C_8^3 C_9^3 C_7^3 C_8^3 = \]
\[ x^2 y C_3^3 C_5^3 C_8^{3 + \alpha_4} C_9^{3 + \alpha_4} C_7^{3 + \alpha_4} = \]
\[ (x \circ x) \circ y = x^2 \circ y = x^2 y S_3^3 S_4^3 S_5^3 S_6^3 S_7^3 \] (7.23)
where
\[ S_3 = (y, x^2), \] (7.24)
\[ S_4 = (y, x^2, y), \] (7.25)
\[ S_5 = (y, x^2, x^2), \] (7.26)
\[ S_6 = (y, x^2, x^2, x^2), \] (7.27)
\[ S_7 = (y, x^2, y, y). \] (7.28)

By (7.24) and (7.2) we have that
\[ S_3 = (y, x)(y, x)(y, x) = C_3^3 C_5. \] (7.29)

By Lemma (5.3) \(C_3^3 \in \gamma_4(G),\) hence, by (7.25), (4.4) and (4.10),
\[ S_4 = (S_3, y) = (C_5, y) = C_8. \] (7.30)

By (7.26), (7.27), (4.4), (4.8) and (4.21) we have that
\[ S_5 = (S_3, x^2) = (C_5, x^2) = C_9^2 = 1. \] (7.31)

Also by (7.28) and (7.31)
\[ S_6 = (S_5, x^2) = 1. \] (7.32)
By (7.28) and (7.29)
\[ S_7 = (S_3, y, y) = 1 \] (7.33)
because \( S_3 \in \gamma_3(G) \). Therefore, by (7.23), (7.29), (7.30), (7.31), (7.32) and (7.33) we have that
\[ (x \circ x) \circ y = x^2 y C_3^{2 \alpha_3} C_5^{\alpha_5} C_6^{\alpha_6} \] (7.34)

By (7.2) we conclude from (7.22) and (7.34) that
\[ x^2 y C_3^{2 \alpha_3} C_5^{\alpha_5 + \alpha_4} C_6^{\alpha_3 + \alpha_5} C_8^{\alpha_4} = x^2 y C_3^{2 \alpha_3} C_5^{\alpha_5} C_6^{\alpha_6}. \] (7.35)

We compare the exponents of the basic elements \( C_5 \) and \( C_6 \) in both sides of this equality and deduce these two congruences:
\[ \alpha_3^2 + \alpha_4 \equiv \alpha_3 \pmod{2}, \]
\[ \alpha_3 \alpha_4 + \alpha_5 \equiv 0 \pmod{2}. \]

When \( \alpha_3 \equiv 0 \pmod{2} \) both when \( \alpha_3 \equiv 1 \pmod{2} \), we conclude from these congruences that \( \alpha_4 \equiv 0 \pmod{2} \) and \( \alpha_7 \equiv 0 \pmod{2} \). Therefore, the word \( w(x, y) \) in the applicable system of words necessary has a form
\[ w(x, y) = x y C_3^{\alpha_3} C_5^{\alpha_5} C_6^{\alpha_6}, \] (7.35)
where \( 0 \leq \alpha_3 < 4, \alpha_5, \alpha_6 \in \{0, 1\} \).

Now we will compute the right hand of (7.3) when \( \circ \) is the verbal operation defined by the word (7.35):
\[ (x \circ y) \circ y = w(x y C_3^{\alpha_3} C_5^{\alpha_5} C_6^{\alpha_6}, y) = x y C_3^{\alpha_3} C_5^{\alpha_5} C_6^{\alpha_6} y Q_3^{\alpha_3} Q_5^{\alpha_5} Q_6^{\alpha_6} = x^2 y C_3^{\alpha_3} (C_3^{\alpha_3}, y) C_5^{\alpha_5} (C_5^{\alpha_5}, y) C_6^{\alpha_6} Q_3^{\alpha_3} Q_5^{\alpha_5} Q_6^{\alpha_6}, \] (7.36)

because \( C_6^{\alpha_6} \in \gamma_4(G) \). Here
\[ Q_3 = (y, x y C_3^{\alpha_3} C_5^{\alpha_5} C_6^{\alpha_6}) = (y, x y C_3^{\alpha_3} C_5^{\alpha_5}), \] (7.37)
by (4.4); 
\[ Q_5 = (y, x y C_3^{\alpha_3} C_5^{\alpha_5} C_6^{\alpha_6}, x y C_3^{\alpha_3} C_5^{\alpha_5} C_6^{\alpha_6}) = (y, x y C_3^{\alpha_3}, x y C_3^{\alpha_3}), \] (7.38)
by (4.6); and 
\[ Q_6 = (y, x y C_3^{\alpha_3} C_5^{\alpha_5} C_6^{\alpha_6}, x y C_3^{\alpha_3} C_5^{\alpha_5} C_6^{\alpha_6}, x y C_3^{\alpha_3} C_5^{\alpha_5} C_6^{\alpha_6}) = (y, x y, x y) \] (7.39)
by (4.7).

By (4.5) we have that
\[ (C_3^{\alpha_3}, y) = (C_3, y)^{\alpha_3} = (y, x, y)^{\alpha_3} = C_4^{\alpha_3}, \] (7.40)
and

\[(C_5^{\alpha}, y) = (C_5, y)^{\alpha_5} = (y, x, x, y)^{\alpha_5} = C_8^{\alpha_5} \quad (7.41)\]

by (4.10).

We obtain the next equality from (4.2), (4.5), (4.21) and Lemma 5.1:

\[(y, xyC_3^{\alpha_3}) = (y, C_3^{\alpha_3})(y, xy, C_3^{\alpha_3}) = (y, C_3^{\alpha_3} (y, y)(y, x, x, y)^{\alpha_5} = C_8^{\alpha_5} \quad (7.42)\]

After this we conclude from (7.37), (4.2), (7.41), (7.42), (4.10) and (4.21) that

\[Q_3 = (y, xyC_3^{\alpha_3} C_5^{\alpha_5}) = (y, C_3^{\alpha_3})(y, xyC_3^{\alpha_3}, C_5^{\alpha_5}) = C_3 C_4^{\alpha_3 + 1} C_5^{\alpha_5} \quad (7.43)\]

By (7.38), (7.42), (4.5), (4.2), (4.8), (4.21), (4.10) and because \((C_3, xy, C_4^{\alpha_3}) \in \gamma_5(G)\) we have that

\[Q_5 = (C_4 C_3^{\alpha_3 + 1}, xyC_3^{\alpha_3}) = (C_3, xyC_4^{\alpha_3}) (C_4, xyC_3^{\alpha_3})^{\alpha_3 + 1} = (C_3, xy) \quad (7.44)\]

From (4.8), (4.10) and (4.21) we conclude that

\[Q_6 = (y, xy, x, y) = (y, x, x, y, y) \quad (7.45)\]

Therefore, by (7.36), (7.40), (7.41), (7.43), (7.44), (7.45), (4.21) and by Lemma 5.1

\[(x \circ y) \circ y = x \circ (y \circ y) = xy \quad (7.46)\]

Now we will compute the left side of (7.46). As above \(\circ\) is the verbal operation defined by the word (7.35). We have that

\[x \circ (y \circ y) = w. \quad (x, y^2) = xy^2 U_3^6 U_5^5 U_6^6, \quad (7.47)\]

where

\[U_3 = (y^2, x), \quad (7.48)\]

\[U_5 = (y^2, x, x), \quad (7.49)\]

\[U_6 = (y^2, x, x, x). \quad (7.50)\]
By (7.48), (4.1) and by Lemma 5.1 we conclude that
\[ U_3 = (y,x)(y,x,y)(y,x) = C_3^2 C_4. \]  

(7.51)

By (7.49), (7.51), (4.5) and, because, by Lemma 5.3, 
\[ (C_3^2, x) \in \gamma_5(G), \]
we deduce that
\[ U_5 = (U_3, x) = (C_3^2, x)(C_4, x) = (C_4, x) = C_8. \]  

(7.52)

Also by (7.50), (4.8) and (4.21) we have that
\[ U_6 = (y,x,x,x)^2 = C_2^6 = 1. \]  

(7.53)

Therefore, by (7.47), (7.51), (7.52), (7.53) and by Lemma 5.1 we obtain
\[ x \circ (y \circ y) = xy^2 C_3^{2\alpha_3} C_4^{\alpha_3} C_8^{\alpha_5}. \]  

(7.54)

By (7.3) we conclude from (7.46) and (7.54) that
\[ xy^2 C_3^{2\alpha_3} C_4^{\alpha_3} C_7^{\alpha_3 \alpha_5 + \alpha_5 + \alpha_6} C_8^{\alpha_5} = xy^2 C_3^{2\alpha_3} C_4^{\alpha_3} C_8^{\alpha_5}. \]

We compare the exponents of the basic elements \( C_4 \) and \( C_7 \) in both sides of this equality and deduce these two congruences:
\[ \alpha_3^2 + \alpha_5 \equiv \alpha_3 \pmod{2}, \]
\[ \alpha_3 \alpha_5 + \alpha_5 + \alpha_6 \equiv 0 \pmod{2}. \]

When \( \alpha_3 \equiv 0 \pmod{2} \) both when \( \alpha_3 \equiv 1 \pmod{2} \), we conclude from these congruences that \( \alpha_5 \equiv 0 \pmod{2} \) and \( \alpha_6 \equiv 0 \pmod{2} \). Therefore, the word \( w. (x,y) \) in the applicable system of words necessary has a form
\[ w. (x,y) = xyC_3^{\alpha_3}, \]  

(7.55)

where \( 0 \leq \alpha_3 < 4 \).

From Propositions 7.4 and 7.3 we conclude that in the variety \( \Theta \) the applicable system of words can have only these four forms
\[ W_{\alpha} = \{ w_1, w_{-1} (x) = x^{-1}, w. (x,y) = xyC_3^{\alpha} \}, \]  

(7.56)

where \( 0 \leq \alpha < 4 \).

8 Applicable systems of words. Sufficient conditions

We will prove in this section that all the systems of words mentioned in (7.56) are applicable.

It is obvious that the system of words \( W_0 \) is applicable (see Subsection 2.3).

In the begging of this section we will prove that the systems of words \( W_1 \) and \( W_2 \) are applicable and after this we will conclude that the systems of words \( W_3 \) is applicable.
8.1 System of words $W_1$

Now we consider the system of words $W_1$. In this system of words $w_1(x, y) = xy$, $w_1(y, x) = yx$. We denote by $\circ$ the verbal operation defined by the word $w_1(x, y) = yx$. We will prove that for every $G \in \Theta$ the universal algebra $G\ast W_1$ is also a group of the variety $\Theta$.

It is clear that for every $G \in \Theta$ and every $x \in G$ the identities

\[ x \circ 1 = 1 \circ x = x \]

hold.

**Proposition 8.1** The operation $\circ$ is an associative operation.

**Proof.** For every $G \in \Theta$ and every $x, y, z \in G$ we have that \((x \circ y) \circ z = z \circ (yx) = (zy) \circ x = x \circ (y \circ z)\). \(\blacksquare\)

We denote for every $m \in \mathbb{Z}$ by $x^m$ the degree $m$ defined system of words $W_1$ of the element $x \in G$, where $G \in \Theta$. It is clear that $x^m = x^m$, so for every $G \in \Theta$ and every $x \in G$ the identities

\[ x \circ x^{m-1} = x^{m-1} \circ x = 1 \]

and

\[ x^4 = 1 \]

hold.

For every $G \in \Theta$ and every $x, y \in G$ we will denote $(x, y)_1 = x^{-1} \circ y^{-1} \circ x \circ y$.

**Proposition 8.2** For every $G \in \Theta$ and every $x_1, x_2, x_3, x_4 \in G$ the identity

\[ ((x_1, x_2)_1, (x_3, x_4)_1)_1 = 1 \]

holds.

**Proof.** We have that

\[ (x, y)_1 = y^{-1} x^{-1} \circ y x = y x y^{-1} x^{-1} = (y^{-1}, x^{-1}) = (x^{-1}, y^{-1})^{-1}. \] (8.1)

Therefore

\[
((x_1, x_2)_1, (x_3, x_4)_1)_1 = ((x_1^{-1}, x_2^{-1}), (x_4^{-1}, x_3^{-1}))_1 = \\
((x_4^{-1}, x_3^{-1})^{-1}, (x_2^{-1}, x_1^{-1})^{-1}) = ((x_3^{-1}, x_4^{-1}), (x_1^{-1}, x_2^{-1})) = 1.
\]

\(\blacksquare\)
Proposition 8.3 For every $G \in \Theta$ and every $x_1, x_2, x_3, x_4, x_5 \in G$ the identity

$$\left(\left(\left(\left((x_1, x_2)_1, x_3\right)_1, x_4\right)_1, x_5\right)_1 = 1 \right)$$

holds.

Proof. By (8.1) we have that

$$\left(\left(\left(\left((x_1, x_2)_1, \ldots, x_n\right)_1, x_{n+1}\right)_1, x_n\right)_1 = \left((\left((x_1, x_2)_1, \ldots, x_n\right)^{-1}, x_{n+1}^{-1}\right)^{-1} = \left(\left(\left((x_1, x_2)_1, \ldots, x_n\right), x_{n+1}\right)^{-1}, x_n^{-1}\right)^{-1}. \right)$$

(8.2)

holds when $n = 2$. We suppose that (8.2) holds. So, by (8.1), we have that

$$\left(\left(\left(\left(\left((x_1, x_2)_1, \ldots, x_n\right)_1, x_{n+1}\right)_1, x_n\right)_1 = \left((\left((x_1, x_2)_1, \ldots, x_n\right)^{-1}, x_{n+1}^{-1}\right)^{-1} = \left(\left(\left((x_1, x_2)_1, \ldots, x_n\right), x_{n+1}\right)^{-1}, x_n^{-1}\right)^{-1}. \right)$$

Therefore, we proved (8.2) for every $n \geq 2$. In particular, for every $G \in \Theta$ and every $x_1, x_2, x_3, x_4, x_5 \in G$ we have that

$$\left(\left(\left(\left((x_1, x_2)_1, x_3\right)_1, x_4\right)_1, x_5\right)_1 = (x_1^{-1}, \ldots, x_5^{-1})^{-1} = 1. \right)$$

Therefore, we proved that for every $G \in \Theta$ the universal algebra $G^*_W$ is also a group of the variety $\Theta$.

In particular, we have that $F^*_W \in \Theta$ for every $F \in \text{Ob}\Theta^0$. So, for every $F = F_\Theta(X) \in \text{Ob}\Theta^0$ there exists a homomorphism $s_F^{(1)} : F \to F^*_W$ such that $s_{F|X}^{(1)} = \text{id}_X$.

Proposition 8.4 The system of words $W_1$ is an applicable system of words.

Proof. For every $F = F_\Theta(X) \in \text{Ob}\Theta^0$ and every $a, b \in F$ we have that

$$\left(s_F^{(1)} \right)^2(ab) = s_F^{(1)} \left(s_F^{(1)}(a) \circ s_F^{(1)}(b) \right) = s_F^{(1)} \left(s_F^{(1)}(b) \circ s_F^{(1)}(a) \right) =$$

$$\left(s_F^{(1)} \right)^2(b) \circ \left(s_F^{(1)} \right)^2(a) = \left(s_F^{(1)} \right)^2(a) \left(s_F^{(1)} \right)^2(b).$$

So, $\left(s_F^{(1)} \right)^2 : F \to F$ is a homomorphism. The equality $\left(s_F^{(1)} \right)^2 = \text{id}_F$ holds, hence $\left(s_F^{(1)} \right)^2 = \text{id}_F$. Therefore, $s_F^{(1)}$ is a bijection. It means that $s_F^{(1)}$ is an isomorphism. Hence $W_1$ is a subject of Definition 2.5. 

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8.2 System of words \(W_2\)

We will prove in this subsection that the system of words \(W_2\) is an applicable system of words. \(w \cdot (x, y) = xy(y, x)^2\) in this system of words. As above we denote by \(\circ\) the verbal operation defined by the word \(w \cdot (x, y) = xy(y, x)^2\). We will prove that for every \(G \in \Theta\) the universal algebra \(G_{W_2}^*\) is also a group of the variety \(\Theta\).

It is clear that for every \(G \in \Theta\) and every \(x \in G\) the identities

\[ x \circ 1 = 1 \circ x = x \]

hold.

**Proposition 8.5** The operation \(\circ\) is an associative operation.

**Proof.** For every \(G \in \Theta\) and every \(x, y, z \in G\) we have that

\[
\left( x \circ y \right) \circ z = xy(y, x)^2 \circ z = xy(y, x)^2 \circ z = x \left( xy(y, x)^2 \right) \circ z = x \left( xy(y, x)^2 \right) \circ z = xy(y, x)^2 (z, xy(y, x)^2) = x z, xy(y, x)^2 (z, xy(y, x)^2) = x z, xy(y, x)^2 (z, xy(y, x)^2).
\]

In this computation we use Corollary 1 from Theorem 6.1, Lemma 5.3 and (5.2). By similar computation we conclude that

\[
x \circ \left( y \circ z \right) = x \circ yz(z, y)^2 = x \circ yz(z, y)^2 = x \circ yz(z, y)^2 = x \circ yz(z, y)^2.
\]

\[
x \circ \left( x \circ y \right) = x \circ yz(y, x)^2 = x \circ yz(y, x)^2 = x \circ yz(y, x)^2 = x \circ yz(y, x)^2.
\]

\[
x \circ \left( x \circ y \right) = x \circ yz(y, x)^2 = x \circ yz(y, x)^2 = x \circ yz(y, x)^2 = x \circ yz(y, x)^2.
\]

\[
x \circ \left( x \circ y \right) = x \circ yz(y, x)^2 = x \circ yz(y, x)^2 = x \circ yz(y, x)^2 = x \circ yz(y, x)^2.
\]

As above we denote for every \(m \in \mathbb{Z}\) by \(x_{\circ m}\) the degree \(m\) defined system of words \(W_2\) of the element \(x \in G\), where \(G \in \Theta\). And just as before, it is clear that \(x_{\circ m} = x^m\), so for every \(G \in \Theta\) and every \(x \in G\) the identities

\[ x \circ x_{\circ -1} = x_{\circ -1} \circ x = 1 \]

and

\[ x_{\circ 4} = 1 \]

hold.

For every \(G \in \Theta\) and every \(x, y \in G\) we will denote \((x, y)_{\circ 2} = x^{-1} \circ y^{-1} \circ x \circ y\).

**Proposition 8.6** For every \(G \in \Theta\) and every \(x, y \in G\) the equality

\[
(x, y)_{\circ 2} = (x, y)
\]

holds.
Corollary 1 For every $G \in \Theta$ and every $x_1, x_2, x_3, x_4, x_5 \in G$ the identities
\[(x_1, x_2)_2, (x_3, x_4)_2 = 1\]
and
\[(((x_1, x_2)_2, x_3)_2, x_4)_2, x_5_2 = 1\]
hold.

Therefore, we proved that for every $G \in \Theta$ the universal algebra $G_{W_2}^{\ast}$ is also a group of the variety $\Theta$.

In particular, we have that $F_{W_2} \in \Theta$ for every $F \in \text{Ob}\Theta^0$. So, for every $F = F\Theta (X) \in \text{Ob}\Theta^0$ there exists a homomorphism $s^{(2)}_{F} : F \to F_{W_2}^{\ast}$ such that $s^{(2)}_{F|X} = \text{id}_X$.

Proposition 8.7 The system of words $W_2$ is an applicable system of words.

Proof. It is clear, that for every $G \in \Theta$, every $a \in G$ and every $b \in \gamma_4 (G)$ the equality $a \circ b = ab$ holds. Therefore, for every $F = F\Theta (X) \in \text{Ob}\Theta^0$ and every $a, b \in F$ we have by Proposition 8.6 by Corollary 1 from Theorem 6.1 and Lemma 5.3 and by (6.1) that
\[
\left(s^{(2)}_F \right)^2 (ab) = s^{(2)}_F \left(s^{(2)}_F (a) \circ s^{(2)}_F (b) \right) =
\]
\[
s^{(2)}_F \left(s^{(2)}_F (a) \circ s^{(2)}_F (b) \right) \left(s^{(2)}_F (b), s^{(2)}_F (a) \right)^2 =
\]
\[
\left(s^{(2)}_F \right)^2 (a) \circ \left(s^{(2)}_F \right)^2 (b) \left(s^{(2)}_F \right)^2 (b), \left(s^{(2)}_F \right)^2 (a) \right)^2 =
\]
\[
\left(s^{(2)}_F \right)^2 (a) \left(s^{(2)}_F \right)^2 (b) \left(s^{(2)}_F \right)^2 (b), \left(s^{(2)}_F \right)^2 (a) \right)^2 =
\]
\[
\left(s^{(2)}_F \right)^2 (a) \left(s^{(2)}_F \right)^2 (b) \left(s^{(2)}_F \right)^2 (b), \left(s^{(2)}_F \right)^2 (a) \right)^4 = \left(s^{(2)}_F \right)^2 (a) \left(s^{(2)}_F \right)^2 (b).
\]
So, $s^{(2)}_F : F \to F$ is a homomorphism. And, as in Proposition 8.4 this completes the proof. ■
8.3 System of words $W_3$

We proved that the systems of words $W_1$ and $W_2$ are applicable. So, there exist $C^{-1}(W_1) = \Phi_1, C^{-1}(W_2) = \Phi_2 \in \mathcal{S}$. Hence, the applicable systems of words $C(\Phi_2 \Phi_1)$ we can obtain by (2.3), where $s_{F_\omega} = s_{F_\omega}^{(1)}, s_{F_\omega} = s_{F_\omega}^{(2)}, \omega \in \Omega = \{1, -1, \}$. 

**Proposition 8.8** The equality $C(\Phi_2 \Phi_1) = W_3$ holds.

**Proof.** $s_{F_\omega}^{(1)}$ and $s_{F_\omega}^{(2)}$ fix the words $w_1 = 1$ and $w_{-1}(x) = x^{-1}$. So, it’s enough to compute $s_{G_x}^{(2)}(s_{G_x}^{(1)}(xy))$, where $G = F_\omega(x, y)$. This word will be $w(x, y)$ in the applicable system of words $C(\Phi_2 \Phi_1)$. $s_{G_x}^{(1)} : G \rightarrow G_{W_1}$ and $s_{G_x}^{(2)} : G \rightarrow G_{W_2}$ are isomorphisms and they fix the generators $x$ and $y$. Therefore, by (1.10),

$$s_{G_x}^{(2)}(s_{G_x}^{(1)}(xy)) = s_{G_x}^{(2)}(x \circ y) = s_{G_x}^{(2)}(yx) = y \circ x = yx(xy)^2 = xy(x, y)^2 = xy(x, y) = xy(y, x)^3.$$

We conclude from this proposition that $W_3$ is an applicable systems of words.

9 Group $\mathcal{S} \cap \mathcal{Q}$ and group $\mathcal{A}/\mathcal{Q}$

We conclude from Section 8 that group $\mathcal{S}$ contains 4 elements: automorphisms $\Phi_\omega = C^{-1}(W_\omega)$, where $0 \leq \omega < 4$.

**Theorem 9.1** The $\mathcal{S} \cap \mathcal{Q} = \{\Phi_0, \Phi_1\}$ and $|\mathcal{A}/\mathcal{Q}| = 2$ hold.

**Proof.** By Criterion [2.1] the automorphism $\Phi_\omega$ is inner if and only if for every $F \in \text{Ob}\Theta^0$ there exists an isomorphism $c_F^{(\omega)} : F \rightarrow F_{W_\omega}$, which fulfills condition [2.6] for every $A, B \in \text{Ob}\Theta^0$ and every $\psi \in \text{Mor}_{\Theta^0}(A, B)$. By Proposition [2.1] it means, in particular, that there exists $c(x) \in F_\Theta(x)$ such that the equality [2.7] holds.

On the other hand isomorphisms $c_F^{(\omega)}$, where $F \in \text{Ob}\Theta^0$, must be bijections. The group $F_\Theta(x)$ contains only 4 elements: $c_1(x) = x^3$, where $0 \leq i < 4$. For every $F \in \text{Ob}\Theta^0$ we consider mappings $(c_i)_F : F \rightarrow F$ defined for every $f \in F$ by formula $(c_i)_F(f) = c_i(f) = f^i$. It is easy to check that $\text{im}(c_0)_{F_\Theta(x)} = \{1\} \neq F_\Theta(x)$ and $\text{im}(c_2)_{F_\Theta(x)} = \{1, x^2\} \neq F_\Theta(x)$. When $i = 1$ or $i = 3$ then the mappings $(c_i)_F : F \rightarrow F$, such that for every $f \in F$ the $(c_1)_F(f) = f$ and $(c_3)_F(f) = f^3 = f^{-1}$ holds, are bijections, because for every $F \in \text{Ob}\Theta^0$ we have that $(c_1)_F = \text{id}_F$ and $((c_3)_F)^2 = \text{id}_F$.

We will denote $(c_1)_F = c_F^{(0)}$ and $(c_3)_F = c_F^{(1)}$ for every $F \in \text{Ob}\Theta^0$. It is clear that for every $F \in \text{Ob}\Theta^0$ the mapping $c_F^{(0)} = \text{id}_F : F \rightarrow F_{W_0}$ is an isomorphism, because $F = F_{W_0}$.

Also for every $F \in \text{Ob}\Theta^0$ and every $a, b \in F$ the equality

$$c_F^{(1)}(a) \circ c_F^{(1)}(b) = a^{-1} \circ b^{-1} = b^{-1}a^{-1} = (ab)^{-1} = c_F^{(1)}(ab)$$

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holds. Therefore, $c^{(1)}_F : F \to F^*_W$ is an isomorphism. By Proposition 2.1 we have that condition (2.6) holds for isomorphisms $c^{(0)}_F : F \to F^*_W$ and isomorphisms $c^{(1)}_F : F \to F^*_W$ ($F \in \text{Ob} \Theta^0$). It proves that $\Phi_0, \Phi_1 \in \mathcal{S} \cap \mathcal{Y}$.

We will denote $F_\Theta(x, y) = G$. We have that

$c^{(0)}_G(x) \circ c^{(0)}_G(y) = x \circ y = xy(x, x)^2 \neq c^{(0)}_G(xy) = xy$

and

$c^{(1)}_G(x) \circ c^{(1)}_G(y) = x^{-1} \circ y^{-1} = x^{-1}y^{-1}(y^{-1}, x^{-1})^2 \neq c^{(1)}_G(xy) = (xy)^{-1} = y^{-1}x^{-1}$

because

$xy^{-1}y^{-1} (y^{-1}, x^{-1})^2 = (x^{-1}, y^{-1}) (y^{-1}, x^{-1})^2 = (y^{-1}, x^{-1}) \neq 1$.

Therefore, neither mapping $c^{(0)}_G$ nor mapping $c^{(1)}_G$ are isomorphisms $F \to F^*_W$, so the automorphism $\Phi_2 \notin \mathcal{S} \cap \mathcal{Y}$. The Lagrange Theorem argument completes the proof.

10 Open problem

As we said in the Section 1, we can’t conclude from fact that the group $\mathfrak{A}/\mathfrak{Y}$ is not trivial that in our variety $\Theta$ the difference between geometric and automorphic equivalences exists. We must construct a specific example of the two groups from the variety $\Theta$, such that they are automorphically equivalent but are not geometrically equivalent. This construction is yet the open problem.

11 Acknowledgement

The first author acknowledge the support of Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - CAPES (Coordination for the Improvement of Higher Education Personnel, Brazil).

We are thankful to Prof. E. Aladova for her important remarks, which helped a lot in writing this article.

We acknowledge Prof. A. I. Reznokov from St.-Petersburg State University, which provide to the authors the copy of [19].

References

[1] Yu. A. Bahturin. Identical Relations in Lie Algebras (VNU Science Press, Utrecht, 1987).

[2] G. Baumslag, A. Myasnikov and V. Remeslennikov. Algebraic geometry over groups I: Algebraic sets and ideal theory. Journal of Algebra. 219:1 (1999), 16-79. DOI: 10.1006/jabr.1999.7881.

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[3] E. Daniyarova, A. Myasnikov, V. Remeslennikov. Algebraic geometry over algebraic structures II: Foundations. *J. Math. Sci.* **185**:3 (2012), 389 – 416. DOI: 10.1007/s10958-012-0923-z.

[4] E. Daniyarova, A. Myasnikov and V. Remeslennikov, Algebraic geometry over algebraic structures V. The case of arbitrary signature, *Algebra Logic.* **51**:1 (2012), 28 – 40. DOI: 10.1007/s10469-012-9168-7.

[5] T. Fujiwara. Note on the Isomorphism Problem for Free Algebraic Systems. *Proc. Japan Acad.*, **31**:3 (1955), 135-136.

[6] M. M. Gomes de Araújo. Strongly Stable Automorphisms of the Categories of finitely generated Free Algebras of the varieties of all Linear Nilpotent Algebras of degree 5. M.Sc. thesis. Federal University of Rio Grande do Norte (2017).

[7] M. Hall. *The Theory of Groups* (The Macmillan Company, 1959).

[8] Th. W. Hungerford. *Algebra* (Springer-Verlag, 1974).

[9] M.I. Kargapolov and Ju.I. Merzljakov. *Fundamentals of the Theory of Groups* (Springer-Verlag, 1979).

[10] A. Myasnikov and V. Remeslennikov. Algebraic geometry over groups II: Logical Foundations. *Journal of Algebra.* **234**:1 (2000), 225–276.

[11] A. Kanel-Belov, B. Kunyavskii and E. Plotkin. Word Equations In Simple Groups And Polynomial Equations In Simple Algebras. *Vestnik St. Petersburg University: Mathematics.* **46**(1) (2013), 3 – 13.

[12] A. G. Kurosh. *Lectures in general algebra* (Pergamon Press, 1965).

[13] B. Plotkin. Varieties of algebras and algebraic varieties. Categories of algebraic varieties. *Siberian Advanced Mathematics.* **7**(2) (1997), 64–97.

[14] B. Plotkin. Some notions of algebraic geometry in universal algebra. *St. Petersburg Math. J.* **9**(4) (1998), 859–879.

[15] B. Plotkin. Algebras with the same (algebraic) geometry. *Proceedings of the Steklov Institute of Mathematics.* **242** (2003), 17–207. DOI: 10.1134/S0081543812070048.

[16] B. Plotkin and E. Plotkin. Multi-sorted logic and logical geometry: some problems. *Demonstratio Mathematica.* **XLVIII**(4) (2015), 578–618. DOI: 10.1515/dema-2015-0042.

[17] B. Plotkin, E. Plotkin and A. Tsurkov. Geometrical equivalence of groups. *Communications in Algebra.* **27**:8 (1999), 4015-4025.

[18] B. Plotkin and G. Zhitomirski. On automorphisms of categories of free algebras of some varieties. *Journal of Algebra.* **306**:2 (2006), 344–367. DOI: 10.1016/j.jalgebra.2006.07.028.
[19] I. Sanov. Solution of Burnside’s problem for exponent 4. Leningrad State Univ. Ann. 10 (1940), 166-170. (In Russian.)

[20] D. Segal. Words: notes on verbal width in groups. London Math. Soc. Lecture Notes Ser. vol. 361 (Cambridge Univ. Press, 2009).

[21] A. Tsurkov. Automorphic equivalence of algebras. International Journal of Algebra and Computation. 17(5/6) (2007), 1263–1271. DOI: 10.1142/S0218196707004128.

[22] A. Tsurkov. Automorphisms of the category of the free nilpotent groups of the fixed class of nilpotency. International Journal of Algebra and Computation. 17(5/6) (2007), DOI: 10.1142/S021819670700413X.1273–1281.

[23] A. Tsurkov. Automorphic Equivalence of Linear Algebras. Journal of Algebra and Its Applications. 13:7, DOI: 10.1142/S0219498814500261.

[24] A. Tsurkov. Automorphic Equivalence of Many-Sorted Algebras. Applied Categorical Structures. 24:3 (2016), 209-240. DOI: 10.1007/s10485-015-9394-y.

[25] A. Tsurkov. Automorphic Equivalence in the Classical Varieties of Linear Algebras. International Journal of Algebra and Computations. 27:8 (2017), 973–999. DOI: 10.1142/S021819671750045X