Variance Swap Replication: Discrete or Continuous?

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Abstract The popular replication formula to price variance swaps assumes continuity of traded option strikes. In practice, however, there is only a discrete set of option strikes traded on the market. We present here different discrete replication strategies and explain why the continuous replication price is more relevant.

Key Words: variance swap, volatility, derivatives, Carr, finance

1. Variance Swap

We follow here the approach described in (Carr and Lee, 2007). The payoff \( V \) of a variance swap with volatility strike \( K \) and observations at dates \( t \) at maturity \( T \), is:

\[
V(K, 0) = u^2 \sum_{0 < t_i \leq T} \left( \log \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2 - K^2
\]

where \( u \) is a scaling factor. Typically for an annualization of 252, common in practice, \( u = 100 \sqrt{252/N} \).

The main interest of variance swaps is to have a pure exposure to volatility without any delta or gamma risk.

We assume, unless specifically mentioned, no cash dividend on the asset \( S \). Proportional dividends are however acceptable if the variance payoff is adjusted by the dividend value (usually the case for single name equities). A continuous dividend yield is also acceptable.

Let \( F(t, T) \) be the forward price of the asset \( S \) at maturity \( T \). With an interest rate \( r \) and dividend yield \( q \), \( F(t, T) = S(t) e^{\int_t^T (r(\tau) - q(\tau))d\tau} \). The process \( F \) is a martingale and follows:

\[
\frac{dF}{F} = \sigma(t, \ldots) dW
\]

where the volatility \( \sigma \) is an arbitrary function of time and other parameters. We approximate
the discrete variance by the continuous variance to obtain:

\[
V(0, T) \approx u^2 \mathbb{E} \left[ \frac{1}{T} \int_0^T \sigma^2(t, ...) \, dt \right] 
\]

\[
= u^2 \mathbb{E} \left[ \frac{2}{T} \left( \int_0^T \frac{dF}{F} - \log \frac{F(T, T)}{F(0, T)} \right) \right] 
\]

\[
= u^2 \mathbb{E} \left[ - \frac{2}{T} \log \frac{F(T, T)}{F(0, T)} \right] 
\]

An application of the replication formula from Carr and Madan (2001) leads to:

\[
V(K, T) = u^2 DF(T) \left( \int_0^{F(0, T)} \frac{2}{K^2} P(K, T) \, dK + \int_{F(0, T)}^{+\infty} \frac{2}{K^2} C(K, T) \, dK \right) - DF(T)K^2 
\]

where \( C(K, T) \) and \( P(K, T) \) are respectively undiscounted call and put option prices of strike \( K \) and maturity \( T \), and \( DF(T) \) is the discount factor to time \( T \) taken from the relevant discount curve.

2. Continuous replication in practice

We rely on the volatility surface interpolation, for example SVI (Gatheral, 2004), to find the put and call prices for any strike. The integration is truncated before 0 at \( K_{min} \) and before \( +\infty \) at \( K_{max} \). We then use the adaptive Gauss-Lobatto quadrature from (Gander and Gautschi, 2000), to compute the present value. Initial values for \( K_{min} \) and \( K_{max} \) are computed assuming the asset behaves like a geometrical Brownian motion process by:

\[
K_{min} = F(0, T)e^{-N^{-1}(\epsilon)\sigma\sqrt{T}} 
\]

\[
K_{max} = F(0, T)e^{-N^{-1}(\epsilon)\sigma\sqrt{T}} 
\]

where \( N^{-1} \) is the normal inverse cumulative function, \( \epsilon \) is a tolerance (for example \( \epsilon = 10^{-6} \)), and \( \sigma \) the at-the-money implied volatility. We then refine \( K_{max} \) and \( K_{min} \) by integrating step by step between \( (K_{min}, K_{max}) \) and \( (nK_{max}, (n+1)K_{max}) \) until those integrals have lower value than our tolerance criteria.

3. Discrete replication

3.1 Derman’s method

The idea of (Demeterfi et al., 1999) is to approximate the log payoff by a piecewise-linear function, as piecewise-linear functions can be exactly replicated by a stream of call and put options. The replication formula of (Carr and Madan, 2001) is not directly used.

We consider a set of increasing call option strikes \( K_i^c \) and a set of decreasing put option strikes \( K_i^p \) for \( i = 0, ..., n \) with \( K_0^c = K_0^p = K_0 \).

Because the forward \( F(0, T) \) might not fall exactly on a strike, we need to slightly rewrite Equation (5).

\[
\log \frac{F(T, T)}{F(0, T)} = \log \frac{F(T, T)}{K_0} + \log \frac{K_0}{F(0, T)} 
\]
Instead of replicating directly \( \log \frac{F(T,T)}{K_0} \), as (Demeterfi et al., 1999) derivation is slightly different, they prefer to work with the payoff:

\[
f(x) = \frac{x}{K_0} - 1 - \log\left( \frac{x}{K_0} \right)
\]

(10)

Note that the linear term can be omitted, as it can also directly be replicated by shares, but in the end, this would lead to the exact same price, by linearity.

Figure 1(a) shows the payoff and the piecewise-linear approximation.

![Graph showing payoff and piecewise-linear approximation](image)

(a) Derman's method

![Graph showing integrand and trapezoidal approximation](image)

(b) Trapezoidal method

Figure 1.: Different replications using strikes 60 to 140 by increment of 10

The slope of each segment leads to call and put option weights \( w^c(K^c_i) \), \( w^p(K^p_i) \):

\[
w^c(K_0) = \frac{f(K^c_1) - f(K_0)}{K^c_1 - K_0}
\]

(11)

\[
w^c(K^c_i) = \frac{f(K^c_{i+1}) - f(K^c_i)}{K^c_{i+1} - K^c_i} - \sum_{j=0}^{i-1} w^c(K^c_j)
\]

(12)

\[
w^p(K_0) = \frac{f(K^p_1) - f(K_0)}{K^p_1 - K_0}
\]

(13)

\[
w^p(K^p_i) = -\frac{f(K^p_{i+1}) - f(K^p_i)}{K^p_{i+1} - K^p_i} - \sum_{j=0}^{i-1} w^p(K^p_j)
\]

(14)

and the final discrete replication formula is:

\[
V(0,T) = u^2 \frac{2}{T} \left( 1 - \frac{F(0,T)}{K_0} + \log \frac{F(0,T)}{K_0} \right) \\
+ u^2 \frac{2}{T} D(T) \left( \sum_{i=0}^{n} w^c(K^c_i)C(K^c_i, T) + \sum_{i=0}^{n} w^p(K^p_i)P(K^p_i, T) \right)
\]

(15)
3.2 Trapezoidal method

Instead of approximating the payoff by a piecewise-linear function, we can directly integrate Equation (6) by the Trapezoidal method. For simplicity, we will assume in here that the strikes are equidistributed: \( K_{i+1} - K_i = h \). The method can easily be generalized.

\[
V(0, T) = u^2 \frac{2}{T} \left( \log \frac{F(0, T)}{K_0} \right) + u^2 \frac{2h}{T} DF(T) \left( \frac{C(K_0, T)}{2K_0^2} + \frac{C(K_1^c, T)}{(K_1^c)^2} - \frac{C(K_n^c, T)}{2(K_n^c)^2} \right) \tag{16}
\]

\[
+ \frac{u^2}{T} \frac{2h}{3} DF(T) \left( \sum_{i=0}^{n-1} w_i C(K_i^c, T) \frac{1}{(K_i^c)^2} + \frac{1}{2} \sum_{i=0}^{n-1} w_i P(K_i^p, T) \frac{1}{(K_i^p)^2} \right) \tag{17}
\]

While both approximations involve a piecewise-linear approximation, a different function is approximated in each case. Figure 1(b) shows the function being approximated with the Trapezoidal method while Figure 1(a) shows the function being approximated with Derman’s method.

3.3 Simpson

We can go further and use a more precise quadrature, for example Simpson’s quadrature if we assume that the strikes are equidistributed with width \( h \) and \( n \) is even.

\[
V(0, T) = u^2 \frac{2}{T} \left( \log \frac{F(0, T)}{K_0} \right) + u^2 \frac{2h}{3} DF(T) \left( \sum_{i=0}^{n-1} w_i C(K_i^c, T) \frac{1}{(K_i^c)^2} + \frac{1}{2} \sum_{i=0}^{n-1} w_i P(K_i^p, T) \frac{1}{(K_i^p)^2} \right) \tag{18}
\]

with \( w_0 = w_n = 1 \) and \( w_{2i+1} = 4, w_{2i} = 2 \).

Similarly, we could also change Derman’s method by shifting the segments to the midpoints.

4. Numerical examples

4.1 Replication comparison

In order to compare the various discrete replications, we consider a simple use case with strikes ranging from 60 to 140 by increment of 10, the asset spot at 100, a maturity of 1 year and no interest rate and no dividend rate, that is the at-the-money option is included in the replication. Including interest rates or shifting the spot would not change the conclusion.

We first consider the case of constant low volatility, so that our strike range captures well the distribution of the asset. The replication weights given by the Trapezoidal method are close to the weights from Derman’s method, but the weights from Simpson’s method are very different (Table 1).

Simpson’s method results in a price much closer to the continuous integration (Table 2). If we increased the number of strikes, we would see Simpson’s method converging faster to the Continuous price than Derman or Trapezoidal methods. We see that with just 9 options, it is possible to replicate a flat surface quite well in theory, if we go beyond a simple trapezoidal approximation.
Table 1.: Variance swap replication weights.

| Option | Strike | Derman | Trapezoidal | Simpson |
|--------|--------|--------|-------------|---------|
| PUT    | 60     | 0      | 27.78       | 18.52   |
| PUT    | 70     | 41.24  | 40.82       | 54.42   |
| PUT    | 80     | 31.50  | 31.25       | 20.83   |
| PUT    | 90     | 24.85  | 24.69       | 32.92   |
| PUT    | 100    | 10.72  | 10          | 6.67    |
| CALL   | 100    | 9.38   | 10          | 6.67    |
| CALL   | 110    | 16.60  | 16.53       | 22.04   |
| CALL   | 120    | 13.94  | 13.89       | 9.26    |
| CALL   | 130    | 11.87  | 11.83       | 15.78   |
| CALL   | 140    | 0      | 5.10        | 3.40    |

Table 2.: Variance swap replication under low volatility (10%).

| Method     | Price in vol |
|------------|--------------|
| Continuous | 10.0000      |
| Derman     | 10.8264      |
| Trapezoidal| 10.7986      |
| Simpson    | 10.0055      |

Under higher volatility, for example 40%, the range of strikes becomes too narrow to replicate properly the log payoff as much of the distribution is cut-off as mentioned in (Demeterfi et al., 1999, p.27). As a result, none of the discrete replications give a correct price on a simple flat surface (Table 3). Derman’s method provides a intuitive explanation: a linear approximation is used in the wings, while the payoff $f$ is very far from being linear leading to a potentially large error. Effectively, with the discrete replication we are pricing a corridor variance swap with bounds at the first and last strike instead of a true variance swap (Carr and Lewis, 2004).

Table 3.: Variance swap replication under high volatility (40%).

| Method    | Price |
|-----------|-------|
| Continuous| 40.00 |
| Truncated | 37.18 |
| Derman    | 36.51 |
| Trapezoidal| 37.32 |
| Simpson   | 37.18 |

Instead of pushing the volatility up, we could also have just pushed the time to maturity forward and would have obtained the same effect.

With the approximate replication method of (Carr and Lee, 2007), truncating the domain of integration has a much lower effect on a newly issued volatility swap, as its equivalent payoff is much more linear in the wings. As an illustration, we price a volatility swap by replication, truncating the integration between (40, 160) (see Table 4).
4.2 Jumps effect

Beside the problem of replicating the log contract with vanilla options in practice, another important issue of the variance swap is its difference compared to the log contract assuming jumps (cubic returns cannot be ignored anymore). This is a popular explanation for the single name variance swap market collapse in 2008, as jumps are more pronounced in single name stocks. In the same period, the volatility swap market has increased.

We will see why with the example of the Bates model (Bates, 1996) as it includes jumps in the asset along with stochastic volatility. We consider the stochastic volatility parameters $\kappa = 1.15, \theta = 0.04, \sigma = 0.39, \rho = -0.64, v_0 = 0.04$ and the jump parameters $\lambda = 0.6, \nu = -0.12, \delta = 0.15$ for 1-year variance swap and volatility swap contract with zero interest and dividend rates. $\lambda$ represents the jump intensity, when set to 0, we recover the Heston model, $\nu$ corresponds to the jump mean, a higher value will translate to larger jumps, and $\delta$ is the jump volatility.

We compare the value of the variance swap under the Bates model using a Monte-Carlo simulation, along with the value of the same variance swap under the Local Volatility model, where the volatility surface is the one produced by the Bates model. In theory, when there are no jumps, the prices should be the same as the variance swap payoff can be perfectly replicated by a continuous stream of vanilla options.

Table 5.: Variance swap prices with various jump parameters.

| Parameters | $\delta = 0$ | $\delta = 0.6$ | $\delta = 0.6$ | $\delta = 0.6$ |
|------------|--------------|----------------|----------------|----------------|
| Bates      | 399.0        | 655.3          | 1033           | 3205           |
| Local volatility | 398.3       | 626.1          | 943.3          | 2577           |
| Relative difference | -0.18%      | -4.46%         | -8.68%         | -19.59%        |

We can see that when $\nu$ increases, the difference in the variance swap prices computed with Bates and Local volatility increases by the same scale: the variance swap contract is in deed very dependent on jumps (Table 5).

Table 6.: Volatility swap prices with various jump parameters.

| Parameters | $\delta = 0$ | $\delta = 0.6$ | $\delta = 0.6$ | $\delta = 0.6$ |
|------------|--------------|----------------|----------------|----------------|
| Bates      | 18.69        | 23.37          | 28.28          | 45.7           |
| Local volatility | 19.38       | 24.48          | 29.94          | 48.7           |
| Relative difference | 3.69%       | 4.75%          | 5.87%          | 6.56%          |

Volatility swaps are less influenced by jumps, interestingly the effect of jumps is opposite: assuming jumps in the model decreases the price of a volatility swap (Table 6). Note that without jump, the price under local volatility and the price under Bates are different, as contrary to the variance swap, the volatility swap can not be statically replicated.
5. Conclusion

Discrete replication can be useful to obtain an estimate of how much a hedge of a variance swap would cost. However, we have seen that there are different possible strategies, and those can lead to relatively different hedging prices. More importantly, discrete replication significantly underestimates tail events. This leads to an artificially low variance swap price and invalidate the simple discrete hedging strategy effectiveness in practice. Finally, we have shown through the example of the Bates model that this can be compounded with the effect of jumps: discrete or continuous replication assumes no jumps, and will again underestimate the price in cases of jumps. In contrast, the volatility swap appears less sensitive to tail events and jumps.

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