Linear collective collocation and Galerkin approximations for parametric and stochastic elliptic PDEs

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Abstract

Consider the parametric elliptic problem

\[- \text{div} (a(y)(x)\nabla u(y)(x)) = f(x) \quad x \in D, \quad y \in I^\infty, \quad u|_{\partial D} = 0,\]

where \(D \subset \mathbb{R}^m\) is a bounded Lipschitz domain, \(I^\infty := [-1, 1]^\infty\), \(f \in L_2(D)\), and the diffusions \(a\) satisfy the uniform ellipticity assumption and are affinely dependent with respect to \(y\). The parametric variable \(y\) may be deterministic or random. In the present paper, a central question to be studied is as follows. Assume that we have an approximation property that there is a sequence of finite element approximations with a certain error convergence rate in energy norm of the space \(V := H_0^1(D)\) for the nonparametric problem \(- \text{div} (a(y_0)(x)\nabla u(y_0)(x)) = f(x)\) at every point \(y_0 \in I^\infty\). Then under what assumptions does this sequence induce a sequence of finite element approximations with the same error convergence rate for the parametric elliptic problem in the norm of the Bochner spaces \(L_\infty(I^\infty, V)\) or \(L_2(I^\infty, V)\)?

We solved this question by linear collective Taylor, collocation and Galerkin methods, based on Taylor expansions, Lagrange polynomial interpolations and Legendre polynomial expansions, respectively, on the parametric domain \(I^\infty\). Under very light conditions, we show that all these approximation methods give the same error convergence rate as that by the sequence of finite element approximations for the nonparametric elliptic problem. Hence the curse of dimensionality is broken by linear methods.

Keywords and Phrases: high-dimensional problems, parametric and stochastic elliptic PDEs, linear collective Taylor and collocation approximations, affine dependence of the diffusion coefficients.

Mathematics Subject Classifications (2010): 65N35, 65N30, 65N15, 65L10, 65D05, 65C30.

1 Introduction

In the recent decades, various approaches and methods have been proposed for the numerical solving of parametric partial differential equations of the form

\[\mathcal{D}(u, y) = 0,\]  \hfill (1.1)
where $u \mapsto D(u, y)$ is a partial differential operator that depends on $d$ parameters represented as the vector $y = (y_1, ..., y_d) \in \Omega \subset \mathbb{R}^d$. If we assume that the problem (1.1) is well-posed in a Banach space $X$, then the solution map $y \mapsto u(y)$ is defined from the parametric domain $\Omega$ to the solution space $X$. We refer the reader to [12, 21, 29] for surveys and bibliography on different aspects in study of approximation and numerical methods for the problem (1.1).

Depending on the nature of the object modeled by the equation (1.1), the parameter $y$ may be either deterministic or random variable. The main challenge in numerical computation is to approximate the entire solution map $y \mapsto u(y)$ up to a prescribed accuracy with acceptable cost. This problem becomes actually difficult when $d$ may be very large. Here we suffer the so-called curse of dimensionality coined by Bellman: the computational cost grows exponentially in the dimension $d$ of the parametric space. Moreover, in some models the number of parameters may be even countably infinite. In the present paper, a central question to be considered is: Under what assumptions does a sequence of finite element approximations with a certain error convergence rate for the nonparametric problem $D(u, y_0) = 0$ at every point $y_0 \in \Omega$ induce a sequence of finite element approximations with the same error convergence rate for the parametric problem (1.1)?

We will solve it for a model parametric elliptic equation by linear collective methods, and therefore, show that the curse of dimensionality is broken by them. However, we believe that our approach and methods can be extended to more general equations of the form (1.1).

Let $D \subset \mathbb{R}^m$ be a bounded domain with a Lipschitz boundary $\partial D$ and $\mathbb{I}^\infty := [-1, 1]^\infty$. Consider the parametric elliptic problem

$$-\text{div}(a(y)\nabla u(y)) = f \text{ in } D, \quad u|_{\partial D} = 0, \quad y \in \mathbb{I}^\infty,$$

where the gradient operator $\nabla$ is taken with respect to $x$, the diffusions $a(y)(x) := a(x, y)$ are functions of $x = (x_1, ..., x_m) \in D$ and of parameters $y = (y_1, y_2, ...) \in \mathbb{I}^\infty$ on $D \times \mathbb{I}^\infty$, and the function $f(x)$ is functions of $x = (x_1, ..., x_m) \in D$. Throughout the present paper we preliminarily assume that $f \in L_2(D)$ and the diffusions $a$ satisfy the uniform ellipticity assumption

$$0 < r < a(y)(x) = a(x, y) \leq R < \infty, \quad x \in D, \quad y \in \mathbb{I}^\infty,$$

and are affinely dependent with respect to $y$, or more precisely,

$$a(y)(x) = \varpi(x) + \sum_{j=1}^{\infty} y_j \psi_j(x), \quad x \in D, \quad y \in \mathbb{I}^\infty, \quad \varpi, \psi_j \in W^1_{\infty}(D),$$

where $W^1_{\infty}(D)$ is the space of functions $v$ on $D$, equipped with the semi-norm and norm

$$|v|_{W^1_{\infty}(D)} := \max_{1 \leq i \leq m} \|\partial_{x_i} v\|_{L_{\infty}(D)}, \quad \|v\|_{W^1_{\infty}(D)} := \|v\|_{L_{\infty}(D)} + |v|_{W^1_{\infty}(D)}.$$

Based on finite element approximations with respect to the spatial variable $x$ and polynomial approximations with respect to the parametric variable $y$, there have been proposed several numerical methods for solving (1.2). Many works have been devoted to the development of the parametric Galerkin and collocation techniques for the numerical solving of (1.2). As shown in [16], these methods are promising since they can use the possible regularity of the solution $u(y)$ with respect to the parameters $y$ to achieve faster convergence than sampling methods like Monte Carlo. A parametric Galerkin method is a projection technique over a set of orthogonal polynomials with respect to an appropriate probability measure $\varpi \psi_j \in W^1_{\infty}(D)$. A collocation method is an approximation by a sum of Lagrangian interpolants based on the data of particular solution instances $u(y^{(i)})$ for some chosen values $y^{(1)}, ..., y^{(k)}$. In the case of problems with affine parameter dependence such as (1.2), adaptive methods based on Taylor expansions have been investigated in [14, 22].
In [10]–[14], [23] based on the \( \ell_p \)-assumption \( (\|v_j\|_{W^1_p(D)})_{j \in \mathbb{N}} \in \ell_p(\mathbb{N}) \) for some \( 0 < p < 1 \) on the affine expansion \([1,4]\), the authors proposed nonlinear \( n \)-term approximation methods in energy norm by establishing \textit{a priori} the set of the \( n \) most useful infinite dimensional polynomials in Taylor expansion, Legendre polynomials expansion and Lagrange interpolation. The obtained \( n \)-term approximands then are approximated by finite element methods. It is worth to emphasize that the \( \ell_p \)-assumption crucially influences the convergence rate of the approximation error due to involving Stechkin’s lemma. The results of \([13,14]\) have been improved \([4,5]\) and extended to a class of parametric semi-linear elliptic PDEs \([22,9]\) and to parametric nonlinear PDEs \([9,12]\). The reader can find a survey and bibliography on this direction in \([12]\).

In the recent papers \([16,17]\), we have considered a particular case of the equation \([1,2]\) where \( D = [0,1]^m \), with an \textit{a priori} assumption that the solution possesses higher order mixed smoothness of Sobolev-Korobov type or of Sobolev-analytic type simultaneously on spatial variable \( x \) and parametric variable \( y \). Applying results on hyperbolic cross approximation in infinite dimension, we constructed linear collective Galerkin methods on both variables \( x \) and \( y \) for approximation of the solution which give the convergences rate in energy norm as the same as that of approximation by Galerkin methods for solving the corresponding nonparametric elliptic problem the domain \([0,1]^m \). Moreover, the infinite-variate parametric part of the problem completely disappeared from the cost of complexities and influences only the constants.

Let \( V := H_0^1(D) \) and denote by \( W \) the subspace of \( V \) equipped with the semi-norm and norm
\[
|v|_W := \|\Delta v\|_{L^2(D)}, \quad \|v\|_W := \|v\|_V + |v|_W.
\]
Assume that we have the following approximation property on the spatial domain \( D \): There are a nested sequence of subspaces \((V_n)_{n \in \mathbb{N}}\) in \( V \), a sequence of linear bounded operators \((P_n)_{n \in \mathbb{N}}\) from \( V \) into \( V_n \), and a number \( 0 < \alpha \leq 1/m \) such that \( \dim V_n \leq n \) and
\[
\|v - P_n(v)\|_V \leq C_D n^{-\alpha} \|v\|_W, \quad \forall v \in W.
\]
In the present paper, we propose collective Taylor, collocation and Galerkin approximations in the Bochner spaces \( L_\infty(\mathbb{R}^\infty, V) \) and \( L_2(\mathbb{R}^\infty, V) \) for solving \([1,2]\), based on this approximation property and Taylor expansions, Lagrange polynomial interpolations and Legendre polynomials expansions, respectively, on the parametric domain \( \mathbb{R}^\infty \). All the methods are linear and constructive. The Taylor and Galerkin approximations are based on hyperbolic crosses, while the collocation method on sparse grids. Moreover, they are collective with regard to spatial variable \( x \) and parametric variable \( y \). This means that in constructing these methods, the \( m \)-variate spatial part and the infinite-variate parametric part are not separately but collectively treated.

We put a light restriction on the diffusions \( a(y) \): the inclusion
\[
(\|\psi_j\|_{W^1_\infty(D)})_{j \in \mathbb{N}} \in \ell_\infty(\mathbb{N})
\]
with \( p(\alpha) = \frac{1}{1+\alpha} \) for the collective Taylor and collocation approximations, and with \( p(\alpha) = \frac{2}{1+2\alpha} \) for the collective Galerkin approximation. Under these conditions on the diffusions \( a(y) \), we show that our methods give the \textit{same convergence rate} \( n^{-\alpha} \) of the error of the approximation of the solution of the nonparametric elliptic problem using the approximation property \([15]\) (see \([2.1]\) and \([2.5]\) in Subsection \(2.1\)). All the conditions on the diffusions \( a(y) \) in particular, the \( \ell_{\infty}(\mathbb{N}) \)-assumption do not affect the convergence rate of the approximation error, completely disappear from it and influence only the constant. Finally, notice also that the construction of linear collective approximations in the present paper is completely different from the construction of finite element approximations in \([11,13,14]\), and from the construction of linear collective approximations in \([16,17]\).

The outline of the present paper is the following. In Section 2, as a preliminary we investigate a general collective approximation in the space \( L_\infty(\mathbb{R}^\infty, V) \). Section 3 is devoted to the construction and error
estimation of collective Taylor methods for solving (1.2). Section 4 is devoted to the construction and error estimation of collective collocation methods for solving (1.2). In Section ??, we extend the construction and methods in Section 3 to the construction and error estimation of collective Legendre and Galerkin methods for solving (1.2). Section 7 is devoted to some concluding remarks.

2 A general collective approximation

2.1 Nonparametric elliptic problem

Let us preliminarily consider the nonparametric complex-valued situation when we have only one equation:

\[-\text{div}(a \nabla u) = f \quad \text{in} \quad D, \quad u|_{\partial D} = 0,\] (2.1)

where \(f, a\) are complex-valued functions on \(D\), \(f \in L_2(D)\) and \(a\) satisfies the ellipticity assumption

\[0 < r < \Re[a(x)] \leq |a(x)| \leq R < \infty, \quad x \in D.\]

By the well-known Lax-Milgram lemma, there exists a unique solution \(u \in V\) in weak form which satisfies the variational equation

\[\int_D a(x) \nabla u(x) \cdot \nabla v(x) \, dx = \int_D f(x) v(x) \, dx, \quad \forall v \in V.\]

Moreover, this solution satisfies the inequality

\[\|u\|_V \leq \frac{\|f\|_{V^*}}{r},\] (2.2)

where \(V^* = H^{-1}(D)\) denotes the dual of \(V\). Observe that there holds the embedding \(L_2(D) \hookrightarrow V^*\) and the inequality \(\|f\|_{V^*} \leq \|f\|_{L_2(D)}\).

If we assume that \(a \in W_1^\infty(D)\), then the solution \(u\) of (2.1) is in \(W\). Moreover, \(u\) satisfies the estimates

\[|u|_W \leq \frac{1}{r} \left(1 + \frac{|a|_{W_1^\infty(D)}}{r}\right) \|f\|_{L_2(D)},\]

and

\[\|u\|_W \leq \frac{1}{r} \left[1 + \left(1 + \frac{|a|_{W_1^\infty(D)}}{r}\right)\right] \|f\|_{L_2(D)}.\] (2.3)

Suppose that we have an approximation property in the following assumption.

**Assumption (i):** There are a nested sequence of subspaces \((V_n)_{n \in \mathbb{N}}\) in \(V\), a sequence of linear bounded operators \((P_n)_{n \in \mathbb{N}}\) from \(V\) into \(V_n\), and a number \(0 < \alpha \leq 1/m\) such that \(\dim V_n \leq n\) and

\[\|v - P_n(v)\|_V \leq C_D n^{-\alpha} \|v\|_W, \quad \forall v \in W,\] (2.4)

where \(C_D\) is a constant which may depend on the domain \(D\).

For example, classical error estimates \[7\] yield that the convergence rate in (2.4) with \(\alpha = 1/m\) can be achieved by using Lagrange finite elements on quasi-uniform partitions. Throughout the remainder of the present paper, \(\alpha\) is fixed and used only for denoting the convergence rate in Assumption (i).
We also use the notation \(|X|\) for a semi-norm.

Let Assumption (i) hold. Let \(v \in L_\infty(\mathbb{I}^\infty, V)\) be represented as the series
\[
v(y)(x) = \sum_{s \in \mathcal{F}} g_s(x) \varphi_s(y)
\]
converging unconditionally in \(L_\infty(\mathbb{I}^\infty, V)\) where \(g_s \in W\) and \((\|g_s\|_W)_{s \in \mathcal{F}}\) belongs to \(\ell_1(\mathbb{F})\), and \(\varphi_s \in L_\infty(\mathbb{I}^\infty)\) with \(\|\varphi_s\|_{L_\infty(\mathbb{I}^\infty)} = 1\). Then \(v(y)\) can be represented as the series
\[
v(y)(x) = \sum_{(k, s) \in \mathbb{Z}_+ \times \mathbb{F}} \delta_k(g_s)(x) \varphi_s(y), \quad y \in \mathbb{I}^\infty,
\]
converging unconditionally in \(L_\infty(\mathbb{I}^\infty, V)\).
Proof. Let us first prove the convergence of the series (2.8) for a sequence of special form \((G_N^*)_{N \in \mathbb{N}}\) with
\[
G_N^* = ((k, s) \in \mathbb{Z}_+ \times \mathbb{F} : 0 \leq k \leq N, \ s \in A_N),
\]
where \((A_N)_{N \in \mathbb{N}}\) is any sequence of finite subsets in \(\mathbb{F}\) which exhausts \(\mathbb{F}\).

We have for every \(y \in \ell_1\),
\[
\left\| v(y) - \sum_{(k, s) \in G_N^*} \delta_k(g_s)\varphi_s(y) \right\|_V \leq \left\| v(y) - \sum_{s \in A_N} g_s\varphi_s(y) \right\|_V + \left\| \sum_{s \in A_N} g_s\varphi_s(y) - \sum_{(k, s) \in G_N^*} \delta_k(g_s)\varphi_s(y) \right\|_V.
\]
Hence, due to the unconditional convergence (2.7) it is sufficient to show that
\[
\lim_{N \to \infty} \left\| \sum_{s \in A_N} g_s\varphi_s - \sum_{(k, s) \in G_N^*} \delta_k(g_s)\varphi_s \right\|_{L_1(\ell_1, V)} = 0. \tag{2.9}
\]
Using the assumptions of the lemma gives for every \(y \in \ell_1\),
\[
\left\| \sum_{s \in A_N} g_s\varphi_s(y) - \sum_{(k, s) \in G_N^*} \delta_k(g_s)\varphi_s(y) \right\|_V = \left\| \sum_{s \in A_N} g_s\varphi_s(y) - \sum_{s \in A_N} \sum_{k=0}^N \delta_k(g_s)\varphi_s(y) \right\|_V
\]
\[
= \left\| \sum_{s \in A_N} \left[ g_s - P_{2^N}(g_s) \right] \varphi_s(y) \right\|_V \leq \sum_{s \in A_N} \left\| g_s - P_{2^N}(g_s) \right\|_V
\]
\[
\leq \sum_{s \in A_N} C_D 2^{-\alpha N} \left\| g_s \right\|_W \leq C_D 2^{-\alpha N} \left\| \left( \left\| g_s \right\|_W \right) \right\|_{\ell_1(\mathbb{F})}
\]
which proves (2.9).

Let \((G_N)_{N \in \mathbb{N}}\) be any sequence of finite subsets in \(\mathbb{Z}_+ \times \mathbb{F}\) which exhausts \(\mathbb{Z}_+ \times \mathbb{F}\). For any \(\varepsilon > 0\), there exists \(M = M(\varepsilon)\) such that
\[
\left\| v - \sum_{(k, s) \in G_M^*} \delta_k(g_s)\varphi_s \right\|_{L_1(\ell_1, V)} \leq \frac{\varepsilon}{2}.
\]
We have by (2.6) that
\[
\sum_{(k, s) \notin G_M^*} \left\| \delta_k(g_s) \right\|_V \leq (2^\alpha + 1) C_D 2^{-\alpha M} \sum_{(k, s) \notin G_M^*} \left\| g_s \right\|_W \leq (2^\alpha + 1) C_D 2^{-\alpha M} \left( \left\| \left( \left\| g_s \right\|_W \right) \right\|_{\ell_1(\mathbb{F})} \right).
\]
Consequently, we may also assume that
\[
\sum_{(k, s) \notin G_M^*} \left\| \delta_k(g_s) \right\|_V \leq \frac{\varepsilon}{2}.
\]
Since \((G_N)_{N \in \mathbb{N}}\) exhausts \(\mathbb{Z}_+ \times \mathbb{F}\), there exists \(N^*\) such that \(G_M^* \subset G_N\) for all \(N \geq N^*\). Hence we derive that
\[
\left\| v - \sum_{(k, s) \in G_N} \delta_k(g_s)\varphi_s \right\|_{L_1(\ell_1, V)} \leq \left\| v - \sum_{(k, s) \in G_M^*} \delta_k(g_s)\varphi_s \right\|_{L_1(\ell_1, V)} + \sum_{(k, s) \notin G_M^*} \left\| \delta_k(g_s) \right\|_V \leq \varepsilon.
\]
The proof is complete. \(\square\)

Let \(v \in L_1(\ell_1, V)\) be represented as the series (2.7) converging unconditionally in \(L_1(\ell_1, V)\) where \(g_s \in W\) and \(\left( \left\| g_s \right\|_W \right)_{s \in \mathbb{F}}\) belongs to \(\ell_1(\mathbb{F})\), and \(\varphi_s \in L_1(\ell_1)\) with \(\left\| \varphi_s \right\|_{L_1(\ell_1)} = 1\). We know that if in
addition Assumption (i) holds, \( v \) can be represented by the series (2.8). We are interested in approximation of \( v \) by its partial sums. To this end, for a finite subset \( G \) in \( \mathbb{Z}_+ \times \mathbb{F} \), we define the function

\[
S_Gv(y)(x) := \sum_{(k,s) \in G} \delta_k(g_s)(x) \varphi_s(y).
\]

Notice that the function \( S_Gv(y)(x) \) is defined collectively with regards to the spatial variables \( x \) and parameter variables \( y \), i.e., \( x \) and \( y \) are not separated in constructing it.

Let \( 0 < p < \infty \) and \( \sigma := (\sigma_s)_{s \in \mathbb{F}} \) be a positive sequence. For \( T > 0 \), define the following subset in \( \mathbb{Z}_+ \times \mathbb{F} \)

\[
G(T) = G_{p,\sigma}(T) := \{ (k,s) \in \mathbb{Z}_+ \times \mathbb{F} : 2^k \sigma_s^p \leq T \}.
\]

Clearly, \( G(T) \) is a finite set for every \( T > 0 \). We will approximate \( v \) by the partial sums \( S_{G(T)}v \) in the norm of \( L_\infty(\mathbb{F}^\infty, V) \). To estimate the error of this approximation with regard to the parameter \( T \), we will need the unconditional convergence of the series (2.8) and a lemma on estimation of sums over the complement of the sets \( G(T) \).

If \( \alpha > 0 \), \( 0 < p < 1 \), we use the notation: \( \alpha^* := \alpha \) for \( \alpha \leq 1/p - 1 \), and \( \alpha^* := \alpha - 1/p + 1 \) for \( \alpha > 1/p - 1 \).

**Lemma 2.2** Let \( \alpha > 0 \), \( 0 < p < 1 \) and \( \sigma := (\sigma_s)_{s \in \mathbb{F}} \) be a positive sequence such that the sequence \( (\sigma_s^{-1})_{s \in \mathbb{F}} \) belongs to \( \ell_p(\mathbb{F}) \). Then we have for every \( T > 0 \),

\[
\sum_{(k,s) \notin G(T)} 2^{-ak} \sigma_s^{-1} \leq C T^{-\min(1/p-1,\alpha)},
\]

where

\[
C := \frac{1}{2^{\alpha^*} - 1} \left\| (\sigma_s^{-1}) \right\|_{\ell_p(\mathbb{F})}^p.
\]

**Proof.** We first consider the case \( \alpha \leq 1/p - 1 \). We have for every \( N \in \mathbb{N} \),

\[
\sum_{(k,s) \notin G(T)} 2^{-ak} \sigma_s^{-1} \leq \sum_{s \in \mathbb{F}} \sigma_s^{-1} \sum_{k \geq 2^k > T \sigma_s^{-p}} 2^{-ak} \leq \sum_{s \in \mathbb{F}} \frac{1}{2^{\alpha^*} - 1} \sigma_s^{-1} \left( T \sigma_s^{-p} \right)^{-\alpha} = \frac{T^{-\alpha}}{2^{\alpha^*} - 1} \sum_{s \in \mathbb{F}} \sigma_s^{-(1/p\alpha)} \leq C T^{-\alpha}.
\]

In the last step we used the inequality \( 1 - p\alpha \geq p \).

We next consider the case \( \alpha > 1/p - 1 \). We have for every \( N \in \mathbb{N} \),

\[
\sum_{(k,s) \notin G(T)} 2^{-ak} \sigma_s^{-1} \leq \sum_{k \geq 2^k \geq (T \sigma_s^{-p})^{1/p}} 2^{-ak} \sigma_s^{-1} \leq \sum_{k \geq 0} 2^{-ak} \sum_{\sigma_s \geq (T \sigma_s^{-p})^{1/p}} \sigma_s^{-(1/p)} \sigma_s^{-p} = T^{-1/p-1} \sum_{k \geq 0} 2^{-(\alpha-1/p+1)k} \sum_{\sigma_s \in \mathbb{F}} \sigma_s^{-p} \leq C T^{-(1/p-1)}.
\]

In the last step we used the inequality \( \alpha - 1/p + 1 > 0 \). \( \square \)

The following theorem gives a upper bound of the approximation of \( v \) by the approximant \( S_{G(T)}v \).
Theorem 2.1 Let Assumption (i) hold. Let \( v \in L_\infty(\mathbb{I}^\infty, V) \) be represented as the series
\[
v(y)(x) = \sum_{s \in \mathbb{G}} g_s(x) \varphi_s(y)
\]
converging unconditionally in \( L_\infty(\mathbb{I}^\infty, V) \) where \( \varphi_s \in L_\infty(\mathbb{I}^\infty) \) with \( \| \varphi_s \|_{L_\infty(\mathbb{I}^\infty)} = 1 \). Let the sequence \( (\sigma_s^{-1})_{s \in \mathbb{F}} \) belong to \( \ell_p(\mathbb{F}) \) for some \( 0 < p < 1 \) and
\[
\| g_s \|_W \leq C' \sigma_s^{-1}, \ \forall s \in \mathbb{F}. \tag{2.11}
\]
Then we have for every \( T > 0 \),
\[
\left\| v - S_{G(T)} v \right\|_{L_\infty(\mathbb{I}^\infty, V)} \leq C T^{-\min(1/p^{-1}, \alpha)},
\]
where
\[
C := C' C_D \frac{2^\alpha + 1}{2^\alpha - 1} \| (\sigma_s^{-1}) \|_{\ell_p(\mathbb{F})}^p.
\]

Proof. By Lemma 2.1 the series (2.8) converging unconditionally in \( L_\infty(\mathbb{I}^\infty, V) \) to \( v \), and therefore, we can write for every \( y \in \mathbb{I}^\infty \),
\[
\left\| v(y) - S_{G(T)} v(y) \right\|_V = \left\| \sum_{(k,s) \notin G(T)} \delta_k(g_s)(y) \right\|_V \leq \sum_{(k,s) \notin G(T)} \| \delta_k(g_s) \|_V.
\]
From (2.6) we derive that
\[
\| \delta_k(g_s) \|_V \leq (2^\alpha + 1) C_D 2^{-\alpha k} \| g_s \|_W, \ \forall (k, s) \in \mathbb{Z}_+ \times \mathbb{F}.
\]
Hence, by (2.11) we have that
\[
\left\| v - S_{G(T)} v \right\|_{L_\infty(\mathbb{I}^\infty, V)} \leq (2^\alpha + 1) C_D C' \sum_{(k,s) \notin G(T)} 2^{-\alpha k} \sigma_s^{-1}.
\]
By applying Lemma 2.2 we conclude the proof.

Our approximation strategy is as follows. In the remainder of this paper, based on Assumption (i) and Taylor expansion and Lagrange polynomial interpolation on the parametric domain \( \mathbb{I}^\infty \), we will construct collective expansions of the form (2.8) for linear collective Taylor and collocation. In the next step, for each particular approximation, we will construct the linear approximation operators \( S_{G(T)} v(y)(x) \) where \( G(T) = G_p,\sigma(T) \) with properly chosen sequence \( \sigma = (\sigma_s)_{s \in \mathbb{F}} \) such that there hold the assumptions of Theorem 2.1 in particular, the inequalities (2.11) for the spatial components \( g_s \). There may be many ways to construct a sequence \( \sigma = (\sigma_s)_{s \in \mathbb{F}} \) satisfying (2.11). Here, we suggest the way based on a direct estimate for \( \| g_s \|_W \) derived from smoothness properties of the solution \( u \). See [15] for another way based on analytic regularities of solutions.

3 Collective Taylor approximation

We return now to the parametric equation (1.2). Let us extend the definition of the solution \( u(y) \) to \( u(z) \) for \( z \) belonging to the unit polydics
\[
\mathbb{U}^\infty := \{ z = (z_1, z_2, ...) \in C^\infty : |z_j| \leq 1, \ j \in \mathbb{N} \}.
\]
To this end, based on the affine expansion (1.4) we extend $a(y)$ to $C^\infty$ by
\[ a(z)(x) = a(x, z) := \overline{a}(x) + \sum_{j=1}^\infty z_j \psi_j(x), \quad x \in D, \ z \in C^\infty, \ \psi_j \in L_\infty(D). \] (3.1)

We then consider the complex expansion of (1.2):
\[ -\text{div}(a(z)\nabla u(z)) = f \quad \text{in} \quad D, \quad u|_{\partial D} = 0, \quad z \in U^\infty. \]

Observe that the uniform ellipticity (1.3) implies the complex uniform ellipticity
\[ 0 < r < \Re[a(x, z)] \leq |a(x, z)| \leq 2R < \infty, \quad x \in D, \ z \in U^\infty. \] (3.2)

Clearly, in the case where $f(x)$ and $a(x, z)$ are real valued, the restriction of $u(z)$ to $I^\infty$ coincides with $u(y)$.

Since the complex uniform ellipticity (3.2) holds, the map $z \mapsto u(z)$ is a $V$-valued and bounded analytic function in certain domains are larger than $U^\infty$. Following (14), for $0 < \delta < r$, let us define
\[ A^\infty_\delta := \{ z \in C^\infty : \delta \leq \Re[a(x, z)] \leq |a(x, z)| \leq 2R \}. \]

Observe that $U^\infty$ is contained in $A^\infty_\delta$.

Due to the Lax-Migram lemma in complex form, if $f \in L_2(D)$ is given, then for all $z \in A^\infty_\delta$ there exists a unique solution $u(z) \in V$ in weak form which satisfies the variational equation
\[ \int_D a(x, z)\nabla u(x, z) \cdot \nabla v(x) \, dx = \int_D f(x) v(x) \, dx, \quad \forall v \in V. \] (3.3)

Here and throughout we use the convention: $u(x, z) := u(z)(x)$. This solution also satisfies the inequality
\[ \|u(z)\|_V \leq \frac{1}{\delta} \|f\|_{V'}. \]

For $0 < \delta < 2R$ and $B > 0$, let us define
\[ A^\infty_\delta,B := \{ z \in C^\infty : \delta \leq \Re[a(x, z)] \leq |a(x, z)| \leq 2R, \ |a(z)|_{W^1_\infty(D)} \leq B, \ \forall x \in D \}. \]

We have seen in (2.3) that under the complex uniform ellipticity assumption (3.2), for $0 < \delta < r$ and sufficiently large $B$ the set $A^\infty_\delta,B$ is nonempty, $u(z) \in W$ for every $z \in A^\infty_\delta,B$, and moreover,
\[ \|u(z)\|_W \leq C_{\delta,B} := \frac{1}{\delta} \left[ \frac{1}{\delta} \left( 1 + \left( \frac{B}{\delta} \right) \right) \right] \|f\|_{L_2(D)}, \quad z \in A^\infty_\delta,B. \]

We consider the Taylor expansion of the solution $u(z)$ with respect to the parametric variable $z$. For $s \in F$ with $\text{supp}(s) \subset \{1, 2, \ldots, J\}$, we define the partial derivative
\[ \partial_z^s u := \frac{\partial^{\text{sup}} u}{\partial^{s_1} z_1 \cdots \partial^{s_J} z_J}, \]
where $|s| := \sum_{j=1}^J |s_j|$. We will need a condition for unconditional convergence towards $u(z)$ of the Taylor series
\[ u(z) = \sum_{s \in F} t_s z^s, \]
where the Taylor coefficients $t_s$ are defined by

$$t_s(x) := \frac{1}{s!} \partial_s^x u(0)(x)$$

with $s! := \prod_{j=1}^J s_j!$ and $z^s := \prod_{j=1}^J z_j^{s_j}$ (we use the convention $z_0^0 := 1$ for $z_j \in \mathbb{C}$).

It was proven in [14, Lemma 2.2] that at any $z \in A_s^\infty$, the function $z \mapsto u(z)$ admits a complex derivative $\partial_z u(z) \in V$ with respect to each variable $z_j$. This derivative is the weak solution of the problem: for $z \in A_s^\infty$, find $\partial_z u(z) \in V$ such that

$$\int_D a(x, z) \nabla \partial_z u(x, z) \cdot \nabla v(x) \, dx = -\int_D \psi_j(x) \nabla u(x, z) \cdot \nabla v(x) \, dx, \quad \forall v \in V.$$  

Hence, starting with $t_0 := u(0)$ we can recursively find all $t_s$ as the unique solution of the variational equation

$$\int_D \bar{a}(x) \nabla t_s(x) \cdot \nabla v(x) \, dx = -\sum_{j: s_j \neq 0} \int_{D} \psi_j(x) \nabla t_{s-e^j}(x) \cdot \nabla v(x) \, dx, \quad \forall v \in V,$$

where $e^j \in F$ denotes the vector with value 1 at position $j$ and 0 otherwise.

**Lemma 3.1** Assume that there exist a sequence $\sigma = (\sigma_s)_{s \in F}$ and a constant $C$ such that the sequence $(\sigma^{-1}_s)_{s \in F}$ belongs to $\ell_1(F)$ and

$$\|t_s\|_V \leq C \sigma^{-1}_s, \quad s \in F.$$ 

Then $(\|t_s\|_V)_{s \in F}$ belongs to $\ell_1(F)$ and

$$u(y)(x) = \sum_{s \in F} t_s(x) y^s, \quad x \in D, \quad y \in \mathbb{F}, \quad (3.4)$$

converging unconditionally in $L_\infty(\mathbb{F}^\infty, V)$.

**Proof.** This lemma can be proven in a similar way to the proof of [14, Theorem 1.3] which states that if $(\|\psi_j\|_{L_\infty(D)})_{j \in \mathbb{N}} \in \ell_p(\mathbb{N})$ for some $0 < p < 1$, then $(\|t_s\|_V)_{s \in F}$ belongs to $\ell_p(F)$, and there holds the expansion (3.4) converging unconditionally in $L_\infty(\mathbb{F}^\infty, V)$.

We derive the unconditional convergence of a collective expansion based on the approximation property on the spatial domain $D$ in Assumption (i) and the Taylor expansion on the parametric domain $\mathbb{F}^\infty$.

**Lemma 3.2** Let Assumption (i) hold. Assume that there exist a sequence $\sigma = (\sigma_s)_{s \in F}$ and a constant $C$ such that the sequence $(\sigma^{-1}_s)_{s \in F}$ belongs to $\ell_1(F)$ and

$$\|t_s\|_W \leq C \sigma^{-1}_s, \quad s \in F.$$ 

Then $(\|t_s\|_W)_{s \in F}$ belongs to $\ell_1(F)$ and $u(y)$ can be represented as the series

$$u(y)(x) = \sum_{(k, s) \in \mathbb{Z}_+ \times F} \delta_k(t_s)(x) y^s, \quad y \in \mathbb{F}^\infty, \quad (3.5)$$

converging unconditionally in $L_\infty(\mathbb{F}^\infty, V)$. 

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Proof. This lemma follows from Lemmas 3.1 and 2.1 by putting \( v(y)(x) = u(y)(x), \) \( g_s(x) = t_s(x) \) and \( \varphi_s(y) = y^s. \)

Based on the collective expansion (3.5) of the solution \( u \), the approximation property (2.10) and an estimate for \( ||t_s||_W \) of the form \( ||t_s||_W \leq M \sigma_s^{-1}, \) \( s \in F, \) for which there holds the \( \ell_p \)-summability of the sequence \( \sigma = (\sigma_s)_{s \in F} \) for some \( 0 < p < 1, \) we now construct linear collective Taylor approximations of the solution \( u \) and estimate the approximation error in the norm of \( \ell\infty(\mathbb{R}^\infty, V) \), by applying the general theory established in Subsection 2.2. Let us formulate the exact condition on the sequences \( (||t_s||_W)_{s \in F} \) as an assumption.

Assumption (ii): There exist \( 0 < p < 1, \) a sequence \( \sigma = (\sigma_s)_{s \in F} \) and a constant \( M \) such that the sequence \( (\sigma_s^{-1})_{s \in F} \) belongs to \( \ell_p(F) \) and

\[
\|t_s\|_W \leq M \sigma_s^{-1}, \quad s \in F.
\]

For a finite subset \( G \) in \( \mathbb{Z}_+ \times F, \) denote by \( V^T(G) \) the subspace in \( \ell\infty(\mathbb{R}^\infty, V) \) of all functions \( v \) of the form

\[
v(y)(x) = v(x, y) = \sum_{(k,s) \in G} v_k(x) y^s, \quad y \in \mathbb{R}^\infty, \quad v_k \in V_2^x,
\]

and define the linear operator \( S^T_G : \ell\infty(\mathbb{R}^\infty, V) \to V^T(G) \) by

\[
S^T_G u(y)(x) = S^T_G u(x, y) := \sum_{(k,s) \in G} \delta_k(t_s)(x) y^s.
\]

Theorem 3.1 Let Assumptions (i) and (ii) hold. For \( T > 0, \) consider the set \( G(T) = G_{p, \sigma}(T) \) as in (2.10). Then we have for every \( T > 0, \)

\[
\left\| u(y) - S^T_{G(T)} u(y) \right\|_{\ell\infty(\mathbb{R}^\infty, V)} \leq C T^{-\min\left(1/p-1, \alpha\right)},
\]

where

\[
C := M C_D \frac{2^\alpha + 1}{2^\alpha - 1} \left\| (\sigma_s^{-1}) \right\|_{\ell_p(F)}^p.
\]

Proof. Again, put \( v(y)(x) = u(y)(x), \) \( g_s(x) = t_s(x) \) and \( \varphi_s(y) = y^s. \) By Lemma 3.2 the series (2.8) converges unconditionally in \( \ell\infty(\mathbb{R}^\infty, V) \) to \( v. \) By applying Theorem 2.1 we prove the theorem.

We show that under the assumptions of Theorem 3.1 for a given \( n \in \mathbb{N}, \) the respective operator \( S^T_{G(T_n)} \) with properly chosen \( T_n \) is a bounded linear operator in \( \ell\infty(\mathbb{R}^\infty, V) \) of rank \( n \) which gives the convergence rate of the approximation to \( u(y) \) as \( n^{-\alpha}. \) For any \( n \in \mathbb{N}, \) let \( T_n \) be the number defined by the inequalities

\[
2 \left\| (\sigma_s^{-1}) \right\|_{\ell_p(F)}^p T_n \leq n < 4 \left\| (\sigma_s^{-1}) \right\|_{\ell_p(F)}^p T_n.
\]

Lemma 3.3 Let \( 0 < p < \infty, \) let \( \sigma = (\sigma_s)_{s \in F} \) be a positive sequence such that the sequence \( (\sigma_s^{-1})_{s \in F} \) belongs to \( \ell_p(F). \) Then we have for \( G(T) := G_{p, \sigma}(T) \) and for every \( T > 0, \)

\[
\sum_{(k,s) \in G(T)} 2^k \leq C T,
\]

where

\[
C := 2 \left\| (\sigma_s^{-1}) \right\|_{\ell_p(F)}^p.
\]
Proof. The lemma is trivial for $T < 1$ since in this case the set $G(T)$ is empty. Let us prove it for $T \geq 1$. Note that, for every $(k,s) \in G(T)$, it follows from the definition of $G(T)$ that $\sigma_s^p \leq T$. Hence, we have

$$\sum_{(k,s) \in G(T)} 2^k \leq \sum_{\sigma_s^p \leq T} 2^k \leq \sum_{\sigma_s^p \leq T} 2 \sum_{s \in \mathbb{B}} \sigma_s^p \leq 2T \sum_{s \in \mathbb{B}} \sigma_s^p \leq CT.$$  

\[ \square \]

**Theorem 3.2** Let the assumptions and notations of Theorem 3.1 hold. For any $n \in \mathbb{N}$, let $T_n$ be the number defined as in (3.6) and put $\mathcal{V}_n^T := \mathcal{V}^T(G(T_n))$, $\mathcal{P}_n := S_{G(T_n)}^T$. Then there holds the following.

- $\{\mathcal{V}_n^T\}_{n \in \mathbb{N}}$ is a nested sequence of subspaces in $L_\infty(\mathbb{P}, V)$ and $\dim \mathcal{V}_n^T \leq n$;
- $\{\mathcal{P}_n\}_{n \in \mathbb{N}}$ is a sequence of linear bounded operators from $L_\infty(\mathbb{P}, V)$ into $\mathcal{V}_n^T$; and
- For every $n \in \mathbb{N}$,
  $$\|u - \mathcal{P}_n u\|_{L_\infty(\mathbb{P}, V)} \leq C n^{-\min(1/p - 1, \alpha)},$$
  with the same $\alpha$ as in the convergence rate of the approximation in Assumption (i) and $p$ as in Assumption (ii), where
  $$C := MC_D 4^\alpha 2^{\alpha + 1} (\alpha - 1) \|\sigma_s^{-1}\|_{\ell_p(F)}^{\alpha}.$$  
  (3.7)  
  Moreover, if in addition, $p = \frac{1}{1 + \alpha}$ in Assumption (ii), then we have that
  $$\|u - \mathcal{P}_n u\|_{L_\infty(\mathbb{P}, V)} \leq C n^{-\alpha},$$

Proof. From Assumption (i) we have

$$\dim \mathcal{V}^T(G(T_n)) \leq \sum_{(k,s) \in G(T_n)} \dim V_{2^k} \leq \sum_{(k,s) \in G(T_n)} 2^k.$$  

Hence, by Lemma 3.3 and (3.6) we derive that

$$\dim \mathcal{V}^T(G(T_n)) \leq 2 \|\sigma_s^{-1}\|_{\ell_p(F)}^{\alpha} T_n \leq n.$$

(3.8)  

On the other hand, by (3.6),

$$T_n^{-\min(1/p - 1, \alpha)} \leq 4^\alpha \|\sigma_s^{-1}\|_{\ell_p(F)}^{\alpha} n^{-\min(1/p - 1, \alpha)}$$

which together with Theorem 3.1 and (5.12) completes the proof of the theorem.  

\[ \square \]

Observe that as in (4.12) the approximation methods $\mathcal{P}_n$ of $u(y)$ give the same convergence rate as that by the approximation methods $P_n$ in Assumption (i) for solving the corresponding nonparametric elliptic problem in the domain $D$. The parametric infinite-variate part as well as Assumption (ii) do not affect the convergence rate, completely disappears from it and influence only the constant $C$ given in (3.7).

From Theorem 3.1 we see that under Assumption (i) the problem of construction of a linear collective Taylor approximation is reduced to find a number $0 < p < 1$, a constant $M$ and a sequence $\sigma = (\sigma_s)_{s \in \mathbb{E}}$ satisfying Assumption (ii). We present a way based an estimate for $\|\partial_n^p u\|_{L_\infty(\mathbb{P}, W)}$ (see [13, Theorem 8.2] for a similar estimate). We define the following constant $K$ and sequence $b$ as follows.

$$K := \frac{1}{r} \left[ 1 + \left( 1 + \frac{|a| L_\infty(1, W_\alpha^1(D))}{r} \right) \|f\|_{L^2(D)} \right];$$  

(3.9)  

$$b = (b_j)_{j \in \mathbb{N}}, \quad b_j := \frac{1}{r} \left( \left( \frac{|a| L_\infty(1, W_\alpha^1(D))}{r} + 2 \right) \|\psi_j\|_{L_\infty(D)} + |\psi_j|_{W_\alpha^1(D)} \right).$$  

(3.10)
Lemma 3.4 Assume that \( a \in L_\infty(\mathbb{I}^\infty, W_\infty^1(D)) \). Then we have
\[
\| \partial_y^a u \|_{L_\infty(\mathbb{I}^\infty, W)} \leq K |s|! b^s, \quad s \in \mathbb{F}.
\]

Proof. Let us prove the lemma by induction on \(|s|\). For \( s = 0 \), from (2.3) we derive that
\[
\| u \|_{L_\infty(\mathbb{I}^\infty, W)} \leq \frac{1}{r} \left[ 1 + \left( 1 + \frac{|a| L_\infty(\mathbb{I}^\infty, W_\infty^1(D))}{r} \right) \| f \|_L(D) \right] = K.
\]
Suppose that the lemma holds true for all \( \nu \in \mathbb{F} \) with \(|\nu| < |s|\). We will prove it for \( s \). Let a \( k \in \mathbb{Z}_+^m \) with \(|k| \leq \nu - 2\) be given. Taking differentiation both sides of the equation By (1.2) we get
\[
-a \Delta u = f + \nabla a \cdot \nabla u,
\]
we obtain
\[
- a \partial_y^s (a \Delta u) = \partial_y^s (\nabla a \cdot \nabla u).
\]
Applying the Leibniz rule of multivariate differentiation to the both sides we obtain
\[
- \sum_{0 \leq \nu \leq s} \left( \binom{k}{\nu} \partial_y^a \partial_y^{s-\nu} (\Delta u) = \sum_{0 \leq \nu \leq s} \left( \binom{s}{\nu} \partial_y^a (\nabla a) \cdot \partial_y^{s-\nu} (\nabla u) \right).
\]
Hence, due to (3.1) we get
\[
-a \Delta (\partial_y^s u) = \sum_{j: s_j \neq 0} s_j \psi_j \Delta (\partial_y^{s-e_j} u) + \nabla a \cdot \nabla (\partial_y^s u) + \sum_{j: s_j \neq 0} s_j \nabla \psi_j \cdot \nabla (\partial_y^{s-e_j} u)
\]
which implies that
\[
r \| \partial_y^s u \|_{L_\infty(\mathbb{I}^\infty, W)} \leq |a| L_\infty(\mathbb{I}^\infty, W_\infty^1(D)) \| \partial_y^s u \|_{L_\infty(\mathbb{I}^\infty, W)}
\]
\[
+ \sum_{j: s_j \neq 0} s_j \| \psi_j \|_{L_\infty(D)} \| \partial_y^{s-e_j} u \|_{L_\infty(\mathbb{I}^\infty, W)} + \sum_{j: s_j \neq 0} s_j \| \psi_j \|_{W_\infty^1(D)} \| \partial_y^{s-e_j} u \|_{L_\infty(\mathbb{I}^\infty, W)}
\]
It has been proven in [13 (4.11)] that
\[
\| \partial_y^s u \|_{L_\infty(\mathbb{I}^\infty, W)} \leq \sum_{j: s_j \neq 0} s_j \| \psi_j \|_{L_\infty(D)} \| \partial_y^{s-e_j} u \|_{L_\infty(\mathbb{I}^\infty, W)}.\]

All these together with the induction assumption give
\[
\| \partial_y^s u \|_{L_\infty(\mathbb{I}^\infty, W)} \leq \| \partial_y^s u \|_{L_\infty(\mathbb{I}^\infty, W)} + |\partial_y^s u|_{L_\infty(\mathbb{I}^\infty, W)} \leq \sum_{j: s_j \neq 0} s_j b_j \| \partial_y^{s-e_j} u \|_{L_\infty(\mathbb{I}^\infty, W)}
\]
\[
\leq K \sum_{j: s_j \neq 0} s_j b_j (|s| - 1)! b^{s-e_j} \leq K \sum_{j: s_j \neq 0} s_j (|s| - 1)! b^s = K |s|! b^s.
\]

Lemma 3.5 Let \( 0 < p < \infty \), \( c = (c_j)_{j \in \mathbb{N}} \) be a positive sequence. Then we have the following.
\[
\left( \frac{|s|!}{s!} e^s \right) \in \ell_p(\mathbb{F}) \iff \begin{cases} \| c \|_{\ell_1(N)} < 1, \quad c \in \ell_p(N), & \text{for } p \leq 1; \\ \| c \|_{\ell_1(N)} \leq 1, & \text{for } p > 1. \end{cases}
\]

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This lemma was proved in [13, Theorem 7.2] for \( p \leq 1 \) and in [17, Theorem 5.2] for \( p > 1 \). From it and Lemma 3.4 we obtain

**Corollary 3.1** Let the function \( a \) belong to \( L_\infty(\mathbb{I}^\infty, W_\infty^1(D)) \). Assume that there exists \( 0 < p < 1 \) such that the sequence \( (\|\psi_j\|_{W_\infty^1(D)})_{j \in \mathbb{N}} \) belong to \( \ell_p(\mathbb{N}) \) and that \( \|b\|_{\ell_1(\mathbb{N})} < 1 \). Then there holds Assumption (ii) for \( p, M = K \) and the sequence

\[ \sigma := (\sigma_s)_{s \in \mathbb{F}}, \quad \sigma_s^{-1} := \frac{|s|^p}{s!}b^s. \]

### 4 Collective collocation approximation

#### 4.1 Tensorisation

Our collective collocation method of polynomial interpolation is based on the approximation property in Assumption (i) and on the standard principle of tensorisation of difference of successive one-dimensional interpolation operators introduced in [11]. Let us recall it as well some auxiliary results from there.

Let \( (\xi_j)_{j \in \mathbb{Z}_+} \) be a sequence of mutually distinct points in \( \mathbb{I} \). Then the univariate Lagrange interpolation operator \( I_k \) associated with the section \( \{\xi_0, ..., \xi_k\} \), is defined by

\[ I_k v := \sum_{j=0}^{k} v(\xi_j) \ell_j^k, \quad \ell_j^k(y) := \prod_{i=0}^{k} \frac{y - \xi_i}{\xi_j - \xi_i} \]

for a function \( v \) defined on \( \mathbb{I} \). For \( s \in \mathbb{Z}_+ \), let us introduce the difference operator

\[ \Delta_s := I_s - I_{s-1} \]

with the convention \( I_{-1}(v) = 0 \). If \( s \in \mathbb{F} \), we define

\[ \xi_s := (\xi_s^j)_{j \in \mathbb{Z}_+} \in \mathbb{I}^\infty, \]

and the tensor product difference operator

\[ \Delta_s := \bigotimes_{j \in \mathbb{N}} \Delta_{s^j}. \]

For a finite set \( \Lambda \subset \mathbb{F} \), we introduce the interpolation operator

\[ I_\Lambda := \sum_{s \in \Lambda} \Delta_s, \]

the space of polynomials

\[ V_\Lambda := \text{span}\{y^s : s \in \Lambda\} \]

and the grid

\[ \Gamma_\Lambda := \{\xi_s : s \in \Lambda\}. \]

A set \( \Lambda \subset \mathbb{F} \) is called lower if \( s \in \Lambda \), then \( s - e^j \in \Lambda \) for every \( j \) such that \( s_j > 0 \). For every lower set \( \Lambda \), the generalization of the interpolation operator \( I_\Lambda \) to the \( V \)-valued setting is straightforward: \( I_\Lambda u \) is the unique solution in \( V_\Lambda \) that coincides with \( u \) at the points \( \xi_s \) for \( s \in \Lambda \).
Denote by $L_\infty(I^\infty)$ the space of bounded complex-valued functions $v$ on $I^\infty$ equipped with the sup norm
\[ \|v\|_{L_\infty(I^\infty)} := \sup_{y \in I^\infty} |v(y)|. \]

Then the Lebesgue constant of the interpolation operator $I_\Lambda$ is defined as
\[ \mathcal{L}_\Lambda := \sup_{\|v\|_{L_\infty(I^\infty)} \leq 1} \|I_\Lambda v\|_{L_\infty(I^\infty)}. \]

We are interested in selecting sequences $(\xi_j)_{j \in \mathbb{Z}_+}$ so that the univariate Lebesgue constants
\[ \lambda_k := \sup_{\|v\|_{L_\infty(I^1)} \leq 1} \|I_k v\|_{L_\infty(I^1)}, \]
associated with the univariate operator $I_k$ are increasing moderately. (Note that $\lambda_0 = 0$ for any choice of $(\xi_j)_{j \in \mathbb{Z}_+}$.) Such a sequence is the projection of a Leja sequence on the complex disk $U$ to $I$ which is defined inductively by fixing a point $z_0 \in U$ and defining
\[ z_k := \text{Argmax}_{z \in U} \prod_{j=0}^{k-1} |z - z_j|. \]

The following lemma has been proven in [11].

**Lemma 4.1** Let the Lebesgue constants $\lambda_k$ satisfy $\lambda_k \leq (k+1)^\theta$, $k \in \mathbb{Z}_+$, for some $\theta \geq 1$. Then $\mathcal{L}_\Lambda \leq |\Lambda|^\theta$ for every lower set $\Lambda$.

Upper bounds of the form $\lambda_k \leq (k+1)^\theta$ can be derived for some $\theta > 0$ from the fact that $\lambda_k = \mathcal{O}(k^\gamma)$ for some $\gamma > 0$. For the sequence $(\xi_j)_{j \in \mathbb{Z}_+}$ given by the projection of the Leja sequence on the complex disk $U$ to $I$ with $z_0 = 1$, it has been proven in [8] that
\[ \lambda_k \leq 3(k+1)^2 \log(k+1) = \mathcal{O}(k^{2+\varepsilon}) \]
for arbitrary fixed $\varepsilon > 0$.

For $k \in \mathbb{Z}_+$, we introduce the univariate polynomials $h_k$ of degree $k$ associated with the sequence $(\xi_j)_{j \in \mathbb{Z}_+}$ by
\[ h_0(y) := 1, \quad h_k(y) := \prod_{j=0}^{k-1} \frac{y - \xi_j}{\xi_k - \xi_j}, \quad k \in \mathbb{N}. \]

If $s \in F$, we define the tensor product function
\[ h_s(y) := \prod_{j \in \mathbb{N}} h_{s_j}(y_j). \]

Let $(\Lambda_n)_{n \in \mathbb{N}}$ be a nested sequence of sets with $n = |\Lambda_n|$. Then the grids $(\Gamma_{\Lambda_n})_{n \in \mathbb{N}}$ are also nested. Note that each set $\Lambda_n$ can be seen as the section $\{s^1, \ldots, s^n\}$ of a sequence $(s^k)_{k \in \mathbb{N}}$. This allows us to construct an algorithm for the computation $I_{\Lambda_n} v$ from $I_{\Lambda_{n-1}} v$. Namely, the polynomials $I_{\Lambda_n} v$ can be given by
\[ I_{\Lambda_n} v = \sum_{s \in \Lambda_n} v_s h_s = \sum_{k=1}^n v_{s^k} h_{s^k}, \]
where $v_{s^k}$ are defined recursively by
\[ v_{s^1} := v(\xi_0) \]
\[ v_{s^{k+1}} := v(\xi_{s^k+1}) - I_{\Lambda_k} v(\xi_{s^k+1}) = v(\xi_{s^k+1}) - \sum_{j=1}^k v_{s^j} h_{s^j}(\xi_{s^k+1}). \]
4.2 An estimate of Taylor coefficients

Following [14], for \( \rho := (\rho_j) \) be a sequence of positive numbers and \( \delta, B > 0 \), we say that \( \rho \) is \((\delta, B)\)-admissible, if

\[
\sum_{j=1}^{\infty} \rho_j |\psi_j(x)| \leq \Re[\bar{a}(x)] - \delta, \quad x \in D,
\]

and

\[
\sum_{j=1}^{\infty} \rho_j \max_{1 \leq i \leq m} |\partial_{x_i} \psi_j(x)| \leq B - |\bar{a}|_{W^1_\infty(D)}, \quad x \in D.
\]

For a proof of the following lemma, see [14, Lemma 5.4].

**Lemma 4.2** Let \( \rho := (\rho_j) \) be \((\delta, B)\)-admissible for \( 0 < \delta < r \) and sufficiently large \( B \). Then we have

\[
\|t_s\|_{W} \leq C_{\delta, B} \rho^{-s}, \quad s \in \mathcal{F}.
\]

Assume that the sequence \( (\|\psi_j\|_{W^1_\infty(D)})_{j \in \mathbb{N}} \) belongs to \( \ell_1(\mathbb{N}) \). For a given number \( q > 1 \), let us give an estimate for \( p_s \|t_s\|_{W}, \quad s \in \mathcal{F} \), where

\[
p_s := \prod_{j \in \mathbb{N}} (s_j + 1)^q.
\]

By the assumptions we may choose \( \lambda > 1 \) and \( j_0 \) so that

\[
(\lambda - 1) \sum_{j \in E} \|\psi_j\|_{W^1_\infty(D)} \leq \frac{r}{6} \sum_{j > j_0} \|\psi_j\|_{W^1_\infty(D)} \leq \frac{r}{12 e^q}.
\]

We split \( \mathbb{N} \) into the two sets \( E := \{1, ..., j_0\} \) and \( F := \{j_0, j_0 + 1, ...\} \), and for each \( s \in \mathcal{F} \) define the sequence \( \rho = \rho(s) \) by

\[
\rho_j := \begin{cases} 
\lambda, & j \in E, \\
1, & j \in F, s_j = 0, \\
e^q + \frac{r}{4 s_j \|\psi_j\|_{W^1_\infty(D)}}, & j \in F, s_j \neq 0.
\end{cases}
\]

Let us show that \( \rho \) is \((r/2, B)\)-admissible, where

\[
B := \|\bar{a}\|_{W^1_\infty(D)} + \sum_{j \in E} \|\psi_j\|_{W^1_\infty(D)} + \frac{r}{2}.
\]

We verify, for instance the condition (4.2), the condition (4.1) can be verified in a similar way. Indeed, we have for every \( x \in D \),

\[
\sum_{j=1}^{\infty} \rho_j \max_{1 \leq i \leq m} |\partial_{x_i} \psi_j(x)| + |\bar{a}|_{W^1_\infty(D)} \leq \lambda \sum_{j \in E} |\psi_j|_{W^1_\infty(D)} + e^q \sum_{j \in F} |\psi_j|_{W^1_\infty(D)}
\]

\[
+ \frac{r}{4} \sum_{j \in F} \frac{s_j}{\|\psi_j\|_{W^1_\infty(D)}} |\psi_j|_{W^1_\infty(D)} + |\bar{a}|_{W^1_\infty(D)}
\]

\[
\leq \sum_{j \in E} \|\psi_j\|_{W^1_\infty(D)} + \frac{r}{6} + \frac{r}{12} + \frac{r}{4} + \|\bar{a}\|_{W^1_\infty(D)} = B.
\]
By applying Lemma 4.2 we have that
\[ \|t_s\|_W \leq C r/2, \rho^{-s}, \quad s \in \mathbb{F}. \]
Hence, we derive that
\[ p_s \|t_s\|_W \leq C_{j_0, \lambda, q} C r/2, \sigma_s^{-1}, \quad s \in \mathbb{F}, \quad (4.3) \]
where
\[ \sigma_s := \left( \prod_{j \in E} \left( \frac{2\lambda}{\lambda + 1} \right)^{s_j} \right) \left( \prod_{j \in F} \rho_j^{s_j}(s_j + 1)^q \right). \quad (4.4) \]
In a way similar to (4.23)–(4.26) in the proof of [11, Theorem 4.3] we can prove the estimate
\[ \sigma_s^{-1} \leq \tilde{\sigma}_s^{-1}, \quad s \in \mathbb{F}, \]
where
\[ \tilde{\sigma}_s := \left( \prod_{j \in E} \left( \frac{2\lambda}{\lambda + 1} \right)^{s_j} \right) \left( \prod_{j \in F} \left( \frac{|s_F| d_j}{s_j} \right)^{-s_j} \right), \quad s \in \mathbb{F}, \]
and
\[ d_j := \frac{4e^q}{r} \|\psi_j\|_{W^1(D)}. \]
By the construction we have also that
\[ \sum_{j \in F} d_j \leq \frac{1}{3}. \]
Hence, by [13, Lemma 7.1] we can conclude that
\[ \left( \|\psi_j\|_{W^1(D)} \right)_{j \in \mathbb{N}} \in \ell_p(\mathbb{N}), \quad 0 < p \leq 1, \quad \Rightarrow \quad (\tilde{\sigma}_s^{-1})_{s \in \mathbb{F}}, \quad (\sigma_s^{-1})_{s \in \mathbb{F}} \in \ell_p(\mathbb{F}). \quad (4.5) \]

4.3 Linear collective collocation approximation

For a finite lower subset \( G \) in \( \mathbb{Z}_+ \times \mathbb{F} \), we define the linear operator
\[ I_G := \sum_{(k,s) \in G} \delta_k \Delta_s, \]
which is a mapping from \( L_\infty(\mathbb{I}^\infty, V) \) to the subspace \( \mathcal{V}^T(G) \). We want to approximate \( u(y) \) by \( I_{G(T)}u(y) \) in the norm of \( L_\infty(\mathbb{I}^\infty, V) \).

**Theorem 4.1** Let Assumption (i) hold. Assume that there exists \( 0 < p < 1 \) such that the sequence \( (\|\psi_j\|_{W^1(D)} \) \( j \in \mathbb{N} \) belong to \( \ell_p(\mathbb{N}) \). Let the sequence \( (\xi_j)_{j \in \mathbb{Z}_+} \) be chosen so that \( \lambda_j \leq (j + 1)^{q-1} \) for some \( q > 1 \). Let \( \sigma := (\sigma_s)_{s \in \mathbb{F}} \) be the sequence defined by (4.4). For \( T > 0 \), consider the set \( G(T) = G_{p,\sigma}(T) \) as in (2.10). Then we have for every \( T > 0 \),
\[ \left\| u - I_{G(T)}u \right\|_{L_\infty(\mathbb{I}^\infty, V)} \leq C T^{-\min(1/p - 1, \alpha)}, \]
where
\[ C := (C_D(2^\alpha + 1) + C_{j_0, \lambda, q}) \frac{C r/2, B}{2^{\alpha^2 - 1}} \left\| (\sigma_s^{-1}) \right\|_{\ell_p(\mathbb{F})}. \]

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Hence, we have

\[ \Lambda_k := \{ s \in \mathbb{F} : (k, s) \in G(T) \} = \{ s \in \mathbb{F} : \sigma^2_s \leq 2^{-k}T \}. \tag{4.6} \]

Observe that \( \Lambda_k = \emptyset \) for all \( k > k^* := \lfloor \log_2 T \rfloor \), and consequently, we have that

\[ \mathcal{I}_{G(T)} u = \sum_{k=0}^{k^*} \delta_k \left( \sum_{s \in \Lambda_k} \Delta_s \right) u = \sum_{k=0}^{k^*} \delta_k \Lambda_k u. \tag{4.7} \]

Moreover, by the construction \( (\sigma_s)_{s \in \mathbb{F}} \) is an increasing sequence and, consequently, \( \Lambda_k \) are lower sets. This yields that the sequence \( \{ \Lambda_k \}_{k=0}^{k^*} \) is nested in the inverse order, i.e., \( \Lambda_{k'} \subset \Lambda_k \) if \( k' > k \), and \( \Lambda_0 \) is the largest and \( \Lambda_{k^*} = \{0\} \). Observe that \( I_{\Lambda_k} y^s = y^s \) for every \( s \in \Lambda_k \) and \( \Delta_s y^{s'} = 0 \) for every \( s \not\in s' \). By \( (4.5) \) and Lemma 3.4, the Taylor series unconditionally converges. Hence, we can write

\[ I_{\Lambda_k} u(y) = I_{\Lambda_k} \left( \sum_{s \in \mathbb{F}} t_s y^s \right) = \sum_{s \in \mathbb{F}} t_s I_{\Lambda_k} y^s = \sum_{s \in \Lambda_k} t_s y^s + \sum_{s \not\in \Lambda_k} t_s I_{\Lambda_k \cap R_s} y^s. \]

Therefore, from \( (4.7) \) we derive that

\[ \mathcal{I}_{G(T)} u(y) = \sum_{k=0}^{k^*} \sum_{s \in \Lambda_k} \delta_k(t_s) y^s + \sum_{k=0}^{k^*} \sum_{s \not\in \Lambda_k} \delta_k(t_s) I_{\Lambda_k \cap R_s} y^s \]

\[ = \mathcal{S}^T_{G(T)} u(y) + \sum_{(k, s) \not\in G(T)} \delta_k(t_s) I_{\Lambda_k \cap R_s} y^s. \]

This together with \( (3.5) \) implies that

\[ u(y) - \mathcal{I}_{G(T)} u(y) = u(y) - \mathcal{S}^T_{G(T)} u(y) - \sum_{(k, s) \not\in G(T)} \delta_k(t_s) I_{\Lambda_k \cap R_s} y^s. \]

Hence, we have

\[ \| u - \mathcal{I}_{G(T)} u \|_{L^\infty(I^\infty, V)} \leq \| u - \mathcal{S}^T_{G(T)} u \|_{L^\infty(I^\infty, V)} + \sum_{(k, s) \not\in G(T)} \| \delta_k(t_s) \|_V \| I_{\Lambda_k \cap R_s} y^s \|_{L^\infty(I^\infty)}. \tag{4.8} \]

From \( (4.3) \) and \( (4.5) \) it follows that there holds Assumption (ii) for the number \( p \), the sequence \( \sigma := (\sigma_s)_{s \in \mathbb{F}} \) defined in \( (4.4) \) and \( M = C_{j_0, \lambda, q} C_{r/2, B} \). Therefore, by Theorem 3.1 we obtain for every \( T > 0 \),

\[ \| u(y) - \mathcal{S}^T_{G(T)} u(y) \|_{L^\infty(I^\infty, V)} \leq C' T^{-\min(1/p-1, \alpha)}, \tag{4.9} \]

where

\[ C' := C_{j_0, \lambda, q} C_{r/2, B} C_{D} \frac{2^\alpha + 1}{2^{\alpha^*} - 1} \left\| (\sigma_s^{-1}) \right\|_{l^p_{(r)}}. \]

For the second sum in \( (4.8) \) we have the estimate

\[ \sum_{(k, s) \not\in G(T)} \| \delta_k(t_s) \|_V \| I_{\Lambda_k \cap R_s} y^s \|_{L^\infty(I^\infty)} \leq \sum_{(k, s) \not\in G(T)} 2^{-\alpha k} \| t_s \|_W \mathcal{L}_{\Lambda_k \cap R_s}. \tag{4.10} \]

Lemma 4.1 yields that for every \( s \in \mathbb{F} \),

\[ \mathcal{L}_{\Lambda_k \cap R_s} \leq |\Lambda_k \cap R_s|^q \leq |R_s|^q = \prod_{j \in \mathbb{N}} (1 + s_j)^q = p_s \]
which together with (4.3) and (4.10) gives
\[
\sum_{(k,s) \notin G(T)} \| \delta_k(t_s) \| V \| I_{\Lambda_k \cap R_s} y^s \|_{L_\infty(1^\infty)} \leq \sum_{(k,s) \notin G(T)} 2^{-\alpha k} p_s \| t_s \| W \leq C_{j_0,\lambda,q} C_{r/2} \sum_{(k,s) \notin G(T)} 2^{-\alpha k} \sigma_s^{-1}.
\]

(4.11)

From Assumption (ii) we know that \((\sigma_s^{-1}) \in \ell_1(F)\). Hence, by applying Lemma 2.2 to the sum in the right-hand side of (4.11) we obtain
\[
\sum_{(k,s) \notin G(T)} \| \delta_k(t_s) \| V \| I_{\Lambda_k \cap R_s} y^s \|_{L_\infty(1^\infty)} \leq C_{j_0,\lambda,q} C_{r/2} \frac{1}{2^{\alpha - 1}} \| (\sigma_s^{-1}) \|_{\ell_p(F)} T^{-\min(1/p-1,\alpha)}.\]

Combining the last estimate, (4.8) and (4.9) proves the theorem. \(\square\)

Similarly to the proof of Theorem 3.2 from Theorem 4.1 and Lemma 3.3 we derive the following

**Theorem 4.2** Let the assumptions and notation of Theorem 4.1 hold. For any \(n \in \mathbb{N}\), let \(T_n\) be the number defined as in (3.6) and put \(V_n^T := \mathcal{V}^T(G(T_n))\), \(I_n := I_{G(T_n)}\). Then there holds the following.

- \(\{V_n^T\}_{n \in \mathbb{Z}_+}\) is a nested sequence of subspaces in \(L_\infty(1^\infty, V)\) and \(\dim V_n^T \leq n\);
- \(\{I_n\}_{n \in \mathbb{Z}_+}\) is a sequence of linear bounded operators from \(L_\infty(1^\infty, V)\) into \(V_n^T\); and
- For every \(n \in \mathbb{N}\),
  \[
  \| u - I_n u \|_{L_\infty(1^\infty, V)} \leq C n^{-\min(1/p-1, \alpha)},
  \]
  with the same \(\alpha\) as in the convergence rate of the approximation in Assumption (i), where
  \[
  C := 4^\alpha (C_D(2^\alpha + 1) + C_{j_0,\lambda,q}) \frac{C_{r/2} B}{2^{\alpha - 1}} \| (\sigma_s^{-1}) \|_{\ell_p(F)}^{p \alpha}.
  \]

Moreover, if in addition, \(p = \frac{1}{1 + \alpha}\), then we have that
\[
\| u - I_n u \|_{L_\infty(1^\infty, V)} \leq C n^{-\alpha},
\]

(4.12)

Let us show that the collective polynomial interpolation method \(I_{G(T)}\) is a collocation method and how to construct it. From the proof of Theorem 4.1 we know that for the sets \(\Lambda_k\) introduced in (4.10), \(\Lambda_k = \emptyset\) for all \(k > k^* := \lfloor \log_2 T \rfloor\), and therefore,
\[
I_{G(T)} u = \sum_{k=0}^{k^*} \delta_k \left( \sum_{\Lambda_k} \Delta s \right) u = \sum_{k=0}^{k^*} \delta_k I_{\Lambda_k} u.
\]

Moreover, \(\Lambda_k\) are lower sets nested in the inverse order, i.e., \(\Lambda_0 \supset \Lambda_1 \cdots \supset \Lambda_{k^*}\) and
\[
\Lambda_0 = \{ s \in F : \sigma_s^p \leq T \}, \quad \Lambda_{k^*} = \{ 0 \}.\]

As mentioned above, \(\Lambda_k\) can be seen as the section \(\{ s^0, \ldots, s^{j_k} \}\) of a sequence \((s^j)_{j \in \mathbb{N}}\). Consequently,
\[
I_{\Lambda_k} u(y) := \sum_{s \in \Lambda_k} u_s h_s(y) = \sum_{j=0}^{j_k} u_{s^j} h_{s^j}(y),
\]
where \( u_{s,j} \) are recursively constructed by the algorithm
\[
\begin{align*}
  u_{s,0} & := u(\xi_0), \\
  u_{s,j+1} & := u(\xi_{s,j+1}) - \sum_{j'=1}^{j} u_{s,j'} h_{s,j'}(\xi_{s,j+1}).
\end{align*}
\] (4.13)

Observe that \( I_{\Lambda_k} \) can be constructed from \( I_{\Lambda_{k+1}} \) starting with \( I_{\Lambda_j}^* := u(\xi_0 F) \). Hence, \( I_G(T) u \) is a collocation method based on the particular solutions \( u(\xi_s), s \in \Lambda_0 \), of the forms
\[
I_G(T) u = \sum_{k=0}^{k^*} \delta_k I_{\Lambda_k} u = \sum_{k=0}^{k^*} \sum_{j=0}^{j_k} \delta_k (u_{s,j}) h_{s,j}.
\] (4.14)

Finally, we give an analysis on the computational cost of the approximation \( I_G(T) u \) to the solution \( u \). If we take any point \( y \in \mathbb{I}^\infty \) and use the operator \( P_l \) in Assumption (i) to approximate the particular solution \( u(y) \), then \( l \) can be considered as the computational cost of this approximation. This yields that the computational cost \( N \) of the operator \( \delta_k (u(y)) \) does not exceed \( 2^k \). Hence, the computational cost of the term \( \delta_k I_{\Lambda_k} u \) does not exceed \( 2^k |\Lambda_k| \), and consequently by the formulas (4.13–4.14) and Lemma 3.3 the computational cost \( N \) of the approximation \( I_G(T) u \) does not exceed
\[
N \leq \sum_{k=0}^{k^*} 2^k |\Lambda_k| = \sum_{(k,s) \in G(T)} 2^k \leq CT,
\]

where \( C := 2 \left\| (\sigma_s^{-1}) \right\|_{L_p(\mathbb{I})}^p \). Thus, if the assumption of Theorem 4.1 holds for \( p = \frac{1}{1+\alpha} \), then we can conclude that with the computational cost \( N \) we achieve the approximation error
\[
\left\| u - I_G(T) u \right\|_{L^\infty(\mathbb{I}^\infty, V)} \leq C N^{-\alpha}
\]
with an absolute positive constant \( C \).

5 Galerkin approximation

Let us define a probability measure \( \mu \) on \( \mathbb{I}^\infty \) as the infinite tensor product measure of the univariate uniform probability measures on the one-dimensional \( \mathbb{I} \):
\[
d\mu(y) = \bigotimes_{j \in \mathbb{Z}} \frac{1}{2} dy_j.
\]
The sigma algebra \( \Sigma \) for \( \mu \) is generated by the finite rectangles \( \prod_{j \in \mathbb{N}} I_j \) where only a finite number of the \( I_j \) are different from \( \mathbb{I} \). Then \( (\mathbb{I}^\infty, \Sigma, \mu) \) is a probability space. Let \( L_2(\mathbb{I}^\infty, \mu) \) denote the Hilbert space of functions on \( \mathbb{I}^\infty \) equipped with the inner product
\[
(f, g) := \int_{\mathbb{I}^\infty} f(y) \overline{g(y)} \, d\mu(y).
\]

Consider two types of Legendre univariate polynomials expansions different only in their normalization for basis. The univariate Legendre basis \( (P_n)_{n \in \mathbb{N}} \) is defined with \( L_\infty(\mathbb{I}) \)-normalization: \( \| P_n \|_{L_\infty(\mathbb{I})} = 1 \). The
Similarly, assume that

Moreover, from the identity

For a Hilbert space

the unique expansion

with the change to ess sup norm when

converging in the Hilbert space

We also use the notation

for a semi-norm

in

to

for which the following norm is finite

Denote by

the subset in

of all

such that

is the set of all

such that

are defined by

with the change to ess sup norm when

For simplicity we identify

with

For a Hilbert space

and

the space of all mappings

from

to

for which the following norm is finite

We also use the notation

for a semi-norm

in

for any. The probability measure

induces the Bochner space

of

-measurable mappings

from

to

which are

-p-summable. The norm in

is defined by

with the change to ess sup norm when

For simplicity we identify

with

For a Hilbert space

and

the Bochner space

coincides with the tensor product

Due to

there hold the inclusions

Hence it follows that

admits the unique expansion

converging in the Hilbert space

where the Legendre coefficients

are defined by

Moreover, from the identity

it follows Parseval’s identity

Similarly, assume that

then by

we have the inclusions

and therefore, the convergence of the Legendre expansion

in the Hilbert space

and Parseval’s identity

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For \( s \in \mathbb{F} \) with \( \text{supp}(s) \subset \{1, 2, \ldots, J\} \), we define the partial derivative

\[
\partial_y^s u := \frac{\partial^{|s|} u}{\partial x_1^{s_1} \cdots \partial x_J^{s_J}},
\]

where \(|s| := \sum_{j=1}^J |s_j| \).

It is known \([13]\) that at any \( y \in \mathbb{F}^\infty \), the function \( y \mapsto u(y) \) admits a partial derivative \( \partial_y^s u \). Moreover, starting with \( u(y) \) which is the unique solution in \( V \) of the variational equation \((5.3)\), we can recursively find all \( \partial_y^s u(y) \) as the unique solution of the variational equation

\[
\int_D a(y)(x) \nabla \partial_y^s u(y)(x) \cdot \nabla v(x) \, dx = - \sum_{j: s_j \neq 0} s_j \int_D \psi_j(x) \nabla \partial_y^{s-e^j} u(y)(x) \cdot \nabla v(x) \, dx. \quad \forall v \in V. \quad (5.6)
\]

By use of \((5.1)\) we derive from \((5.6)\) by inductive integration by parts in the variables \( y_j \) the formulas for the Legendre coefficients

\[
v_s = \frac{1}{s!} \prod_{j: s_j \neq 0} \left( \frac{2s_j + 1}{2s_j} \right)^{1/2} \int_\mathbb{F}^{\infty} \partial_y^s u(y) \prod_{j: s_j \neq 0} (1 - y_j^2)^{s_j} \, d\mu(y), \quad (5.7)
\]

where \( s! := \prod_{j=1}^J s_j! \).

Since \( u \in L_2(\mathbb{F}^\infty, V, \mu) \), it can be defined as the unique solution of the variational problem: Find \( u \in L_2(\mathbb{F}^\infty, V, \mu) \) such that

\[
B(u, v) = F(v) \quad \forall v \in L_2(\mathbb{F}^\infty, V, \mu),
\]

where

\[
B(u, v) := \int_{\mathbb{F}^\infty} \int_D a(x, y) \nabla u(x, y) \cdot \nabla v(x, y) \, dx \, d\mu(y),
\]

\[
F(v) := \int_{\mathbb{F}^\infty} \int_D f(x) v(x, y) \, dx \, d\mu(y).
\]

For a subset \( G \) in \( \mathbb{Z}_+ \times \mathbb{F} \), denote by \( \mathcal{V}^L(G) \) the subspace in \( L_\infty(\mathbb{F}^\infty, V) \) of all functions \( v \) of the form

\[
v(y)(x) = \sum_{(k, s) \in G} v_k(x) L_s(y), \quad y \in \mathbb{F}^\infty, \quad v_k \in V_{2^k},
\]

and define the linear operator \( \mathcal{S}_G^L : L_\infty(\mathbb{F}^\infty, V) \to \mathcal{V}^L(G) \) by

\[
\mathcal{S}_G^L u(y)(x) := \sum_{(k, s) \in G} \delta_k(v_s)(x) L_s(y) = \sum_{(k, s) \in G} \delta_k(u_s)(x) P_s(y).
\]

If \( G \) is a finite set, we define the Galerkin approximation \( u_G \) to \( u \) as the unique solution to the problem: Find \( u_G \in \mathcal{V}^L(G) \) such that

\[
B(u_G, v) = F(v) \quad \forall v \in \mathcal{V}^L(G).
\]

By Céa’s lemma we have the estimate

\[
\|u - u_G\|_{L_2(\mathbb{F}^\infty, V, \mu)} \leq \sqrt{\frac{R}{r}} \inf_{v \in \mathcal{V}^L(G)} \|u - v\|_{L_2(\mathbb{F}^\infty, V, \mu)},
\]

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and consequently,
\[ \|u - u_G\|_{L_2(\mathbb{I}^\infty, V, \mu)} \leq \sqrt{\frac{R}{r}} \|u - \mathcal{S}^L_{G_L} u\|_{L_2(\mathbb{I}^\infty, V, \mu)}. \] (5.8)

For linear collective Galerkin approximations we need the following assumption.

**Assumption (iii):** There exist a sequence \( \sigma = (\sigma_s)_{s \in \mathbb{F}} \) and a constant \( M \) such that the sequence \( (\sigma^{-1}_s)_{s \in \mathbb{F}} \) belongs to \( \ell_{p(\alpha)}(\mathbb{F}) \) for \( p(\alpha) = \frac{2}{1 + 2\alpha} \) and
\[ \|v_s\|_W \leq M \sigma^{-1}_s, \quad s \in \mathbb{F}. \]

**Theorem 5.1** Let Assumptions (i) and (iii) hold and \( a \in L_\infty(\mathbb{I}^\infty, W^1_\infty(D)) \). For \( T > 0 \), consider the set \( G(T) = G_{p, \sigma}(T) \) as in (2.10) for \( p = \frac{2}{1 + 2\alpha} \). Then we have for every \( T > 0 \),
\[ \|u - u_{G(T)}\|_{L_2(\mathbb{I}^\infty, V, \mu)} \leq \sqrt{\frac{R}{r}} \|u - \mathcal{S}^L_{G(T)} u\|_{L_2(\mathbb{I}^\infty, V, \mu)} \leq C \sqrt{\frac{R}{r}} T^{-\alpha}, \]
where
\[ C := MCD \frac{2^\alpha + 1}{2^\alpha - 1} \|\sigma^{-1}\|_{\ell_{p(\alpha)}}. \]

**Proof.** We preliminarily show that
\[ \lim_{N \to \infty} \|u - \mathcal{S}^L_{G_N} u\|_{L_2(\mathbb{I}^\infty, V, \mu)} = 0, \] (5.9)
where \( G_N := \{(k, s) \in \mathbb{Z}_+ \times \mathbb{F} : 0 \leq k \leq N\} \). Obviously, by the definition,
\[ \mathcal{S}^L_{G_N} u = \sum_{s \in \mathbb{F}} \sum_{k=0}^N \delta_k(v_s) L_s = \sum_{s \in \mathbb{F}} P_{2^N}(v_s) L_s. \]
By the assumptions we have the inclusion \( u \in L_2(\mathbb{I}^\infty, W, \mu) \subset L_2(\mathbb{I}^\infty, V, \mu) \). From the uniform boundedness of the operators \( P_{2^N} \) and (5.4)
\[ \|\mathcal{S}^L_{G_N} u\|^2_{L_2(\mathbb{I}^\infty, V, \mu)} = \sum_{s \in \mathbb{F}} \|P_{2^N}(v_s)\|^2_V \leq C_D^2 \sum_{s \in \mathbb{F}} \|v_s\|^2_V = C_D^2 \|u\|^2_{L_2(\mathbb{I}^\infty, V, \mu)}. \]
This means that \( \mathcal{S}^L_{G_N} u \in L_2(\mathbb{I}^\infty, V, \mu) \). Hence, by (5.4), Assumption (i) and (5.5) we deduce that
\[ \|u - \mathcal{S}^L_{G_N} u\|^2_{L_2(\mathbb{I}^\infty, V, \mu)} = \sum_{s \in \mathbb{F}} \|v_s - P_{2^N}(v_s)\|^2_V \leq C_D^2 2^{-2\alpha N} \sum_{s \in \mathbb{F}} \|v_s\|^2_V \leq C_D^2 2^{-2\alpha N} \|u\|^2_{L_2(\mathbb{I}^\infty, W, \mu)} \]
which prove (5.9).

Let \( T \) be given and \( \varepsilon \) arbitrary positive number. Then since \( G(T) \) is finite from the definition of \( G_N \) and (5.9) there exists \( N = N(T, \varepsilon) \) such that \( G(T) \subset G_N \) and
\[ \|u - \mathcal{S}^L_{G_N} u\|_{L_2(\mathbb{I}^\infty, V, \mu)} \leq \varepsilon. \] (5.10)
By the triangle inequality,
\[ \|u - \mathcal{S}^L_{G(T)} u\|_{L_2(\mathbb{I}^\infty, V, \mu)} \leq \|u - \mathcal{S}^L_{G_N} u\|_{L_2(\mathbb{I}^\infty, V, \mu)} + \|\mathcal{S}^L_{G_N} u - \mathcal{S}^L_{G(T)} u\|_{L_2(\mathbb{I}^\infty, V, \mu)}. \] (5.11)
We have by (5.4) and (2.6) that
\[
\|S_{G,N}^L(u) - S_{G(T)}^L u\|_{L_2(\mathbb{I}^\infty, V, \mu)}^2 = \left\| \sum_{s \in \mathbb{F}} \sum_{k=0}^N \delta_k(v_s) L_s - \sum_{s \in \mathbb{F}} \sum_{2^k > T \sigma_s^p} \delta_k(v_s) L_s \right\|_{L_2(\mathbb{I}^\infty, V)}^2
\]
\[
= \sum_{s \in \mathbb{F}} \sum_{2^k > T \sigma_s^p, 2^k < N} \delta_k(v_s) L_s \left\|_V \right. \left. \right\|^2
\]
\[
\leq \sum_{s \in \mathbb{F}} \left( \sum_{2^k > T \sigma_s^p, 2^k < N} \|\delta_k(v_s)\|_V \right)^2
\]
\[
\leq \sum_{s \in \mathbb{F}} \left( \sum_{2^k > T \sigma_s^p, 2^k < N} (2^{\alpha} + 1)C_D 2^{-\alpha k}\|v_s\|_W \right)^2
\]
\[
\leq (2^{\alpha} + 1)^2 C_D^2 \sum_{s \in \mathbb{F}} \|v_s\|_W^2 \left( \sum_{2^k > T \sigma_s^p} 2^{-\alpha k} \right)^2.
\]
Hence, by Assumption (iii) and the equation $2(1 - p\alpha) = p$ we derive that
\[
\|S_{G,N}^L(u) - S_{G(T)}^L u\|_{L_2(\mathbb{I}^\infty, V, \mu)}^2 \leq (2^{\alpha} + 1)^2 C_D^2 \sum_{s \in \mathbb{F}} \sigma_s^{-2} \left( \sum_{2^k > T \sigma_s^p} 2^{-\alpha k} \right)^2
\]
\[
\leq T^{-2\alpha} M^2 C_D^2 \left( \frac{2^{\alpha} + 1}{2^{\alpha} - 1} \right)^2 \sum_{s \in \mathbb{F}} \sigma_s^{-2(1 - p\alpha)}
\]
\[
= T^{-2\alpha} M^2 C_D^2 \left( \frac{2^{\alpha} + 1}{2^{\alpha} - 1} \right)^2 \sum_{s \in \mathbb{F}} \sigma_s^{-p}
\]
\[
= C^2 T^{-2\alpha}.
\]
which in combining with (5.10) and (5.11) gives
\[
\|u - S_{G(T)}^L u\|_{L_2(\mathbb{I}^\infty, V, \mu)} \leq \varepsilon + CT^{-\alpha}
\]
for arbitrary positive number $\varepsilon$. Hence,
\[
\|u - S_{G(T)}^L u\|_{L_2(\mathbb{I}^\infty, V, \mu)} \leq C T^{-\alpha}
\]
which together with (5.8) proves the theorem.

We show that under the assumptions of Theorem (5.1) for a given $n \in \mathbb{N}$, the respective operator $S_{G(T_n)}^L$ with properly chosen $T_n$ is a bounded linear operator in $L_\infty(\mathbb{I}^\infty, V)$ of rank $\leq n$ which gives the convergence rate of the approximation to $u(y)$ as $n^{-\alpha}$.

**Theorem 5.2** Let the assumptions and notation of Theorem (5.1) hold. For any $n \in \mathbb{N}$, let $T_n$ be the number defined as in (3.6) and put $V_n := V^L(G(T_n))$, $P_n := S_{G(T_n)}^L$, $u_n := u_{G(T_n)}$. Then
• \( \{ V_n^k \}_{n \in \mathbb{Z}_+} \) is a nested sequence of subspaces in \( L_2(\mathbb{I}^\infty, V, \mu) \) and \( \dim V_n^k \leq n \);

• \( \{ P_n \}_{n \in \mathbb{Z}_+} \) is a sequence of linear bounded operators from \( L_2(\mathbb{I}^\infty, V, \mu) \) into \( V_n^k \); and

• for every \( n \in \mathbb{N} \),

\[
\| u - u_n \|_{L_2(\mathbb{I}^\infty, V, \mu)} \leq \sqrt{R} \| u - P_n u \|_{L_2(\mathbb{I}^\infty, V, \mu)} \leq C \sqrt{R} n^{-\alpha},
\]

with the same \( \alpha \) as in the convergence rate of the approximation in Assumption (i), where

\[
C := MCD4^\alpha \frac{2^\alpha + 1}{\sqrt{2^\alpha - 1}} \| (\sigma_s^{-1}) \|_{\ell_p(\mathbb{F})}^{p/2},
\]

Proof. We have that

\[
\dim V^L(G(T)) \leq \sum_{(k,s) \in G(T)} \dim V_{2^k} \leq \sum_{(k,s) \in G(T)} 2^k \\
\leq \sum_{\sigma_s^k \leq T} \sum_{2^k \leq T \sigma_s^{-p}} 2^k \leq 2 \sum_{\sigma_s^k \leq T} T \sigma_s^{-p} \\
\leq 2T \sum_{s \in \mathbb{F}} \sigma_s^{-p} \leq 2 \| (\sigma_s^{-1}) \|_{\ell_p(\mathbb{F})}^p T.
\]

Hence, by (3.6) we derive that

\[
\dim V^L(G(T_n)) \leq 2 \| (\sigma_s^{-1}) \|_{\ell_p(\mathbb{F})}^p T_n \leq n.
\] (5.12)

On the other hand, by (3.6),

\[
T_n^{-\alpha} \leq 4^\alpha \| (\sigma_s^{-1}) \|_{\ell_p(\mathbb{F})}^p n^{-\alpha}
\]

which together with Theorem 5.1 and (5.12) completes the proof of the theorem. \( \square \)

Lemma 5.1 Assume that \( a \in L_\infty(\mathbb{I}^\infty, W^1_{\infty}(D)) \). Let the constant \( K \) be as in (3.9) and the sequence \( b \) as in (3.10). Define the sequence

\[
d = (d_j)_{j \in \mathbb{N}}, \quad d_j := b_j / \sqrt{3}.
\] (5.13)

Then we have

\[
\| v_s \|_W \leq K \frac{|s|!}{s!} d^s, \quad s \in \mathbb{F}.
\]

Proof. From (5.7) we derive that

\[
\| v_s \|_W \leq \frac{3^{-|s|/2}}{|s|!} \| \partial_y^n u \|_{L_\infty(\mathbb{I}^\infty, W)}
\]

which combining with Lemma 3.4 prove the lemma. \( \square \)

Corollary 5.1 Let the function \( a \) belong to \( L_\infty(\mathbb{I}^\infty, W^1_{\infty}(D)) \), \( p(\alpha) = \frac{2}{1+2\alpha} \) and the sequence \( d = (d_j)_{j \in \mathbb{N}} \) defined in (5.13) satisfy the condition

\[
\begin{cases}
\| d \|_{\ell_p(\mathbb{N})} < 1, & d \in \ell_p(\mathbb{N}), \quad \text{for } \alpha \geq 1/2; \\
\| d \|_{\ell_1(\mathbb{N})} \leq 1, & \text{for } \alpha < 1/2.
\end{cases}
\]

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Then there holds Assumption (iii) for $M = K$ and the sequence

$$\sigma := (\sigma_s)_{s \in \mathbb{F}}, \quad \sigma_s^{-1} := \frac{|s|!}{s!}d^s.$$  

**Proof.** By definition we have that $0 < p(\alpha) \leq 1$ for $\alpha \geq 1/2$, and $1 < p(\alpha) < \infty$ for $\alpha < 1/2$. Hence, by Lemma 3.5

$$\left(\frac{|s|!}{s!}d^s\right) \in \ell_{p(\alpha)}(\mathbb{F}) \iff \begin{cases} 
\|d\|_{\ell_\alpha} < 1, & d \in \ell_{p(\alpha)}(\mathbb{N}), \quad \text{for } \alpha \geq 1/2; \\
\|d\|_{\ell_\alpha} \leq 1, & \text{for } \alpha < 1/2
\end{cases}$$

which together with Lemma 5.1 proves the corollary. 

Notice that according to Assumption (i) $0 < \alpha \leq 1/m$, where $m$ is the dimension of the spatial domain $D$. Hence the inequality $\alpha \geq 1/2$ may hold only for $m = 1, 2$, and $\alpha < 1/2$ for all $m > 2$. This means that in Assumption (i) the inequality $p(\alpha) \leq 1$ may hold only in the case when $m = 1, 2$, and except this case we always have $p(\alpha) > 1$.

### 6 Legendre approximation

The collective Legendre approximation is constructed on the basis of a representation of the solution $u$ by a series converging unconditionally in $L_\infty(I_\infty, V)$ as in the following lemma.

**Lemma 6.1** Let Assumption (i) hold and let the sequence $(\|\psi_j\|_{W_\infty^1(D)})_{j \in \mathbb{N}}$ belong to $\ell_1(\mathbb{N})$. Then $(\|u_s\|_W)_{s \in \mathbb{F}}$ belongs to $\ell_1(\mathbb{F})$ and $u(y)$ can be represented as the series

$$u(y) = \sum_{(k,s) \in \mathbb{Z} \times \mathbb{F}} \delta_k(u_s) P_s(y), \quad y \in I_\infty,$$

converging unconditionally in $L_\infty(I_\infty, V)$.

**Proof.** This theorem can be proven in a similar way to the proof of Lemma 3.2.

For the linear collective Legendre approximation of the solution $u$ we need the following assumption.

**Assumption (iv):** There exist $0 < p < 1$, a sequence $\sigma = (\sigma_s)_{s \in \mathbb{F}}$ and a constant $M$ such that the sequence $(\sigma_s^{-1})_{s \in \mathbb{F}}$ belongs to $\ell_p(\mathbb{F})$ and

$$\|u_s\|_W \leq M \sigma_s^{-1}, \quad s \in \mathbb{F}.$$

The following two theorems can be proven in a similar way to the proofs of Theorems 3.1 and 3.2 respectively.

**Theorem 6.1** Let Assumptions (i) and (iv) hold. For $T > 0$, consider the set $G(T) = G_{p,\sigma}(T)$ as in (2.10). Then we have for every $T > 0$,

$$\left\| u - S^L_{G(T)} u \right\|_{L_\infty(I_\infty, V)} \leq CT^{-\min(1/p-1, \alpha)},$$

where

$$C := MC_D \frac{2^{\alpha^*} + 1}{2^{\alpha^*} - 1} \left\| (\sigma_s^{-1}) \right\|_{\ell_p(\mathbb{F})},$$

$$\alpha^* := \alpha \text{ for } \alpha \leq 1/p - 1, \text{ and } \alpha^* := \alpha - 1/p + 1 \text{ for } \alpha > 1/p - 1.$$
**Theorem 6.2** Let the assumptions and notation of Theorem 6.1 hold. For any \( n \in \mathbb{N} \), let \( T_n \) be the number defined as in (3.6) and put \( \mathcal{V}_n := \mathcal{V}^L(G(T_n)), \ P_n := S^L_{G(T_n)} \). Then

- \( \{\mathcal{V}_n\}_{n \in \mathbb{Z}^+} \) is a nested sequence of subspaces in \( L_\infty(\mathbb{R}^N, V) \) and \( \dim \mathcal{V}_n \leq n \);
- \( \{P_n\}_{n \in \mathbb{Z}^+} \) is a sequence of linear bounded operators from \( L_\infty(\mathbb{R}^N, V) \) into \( \mathcal{V}_n \); and
- for every \( n \in \mathbb{N} \),
  \[
  \|u - P_n u\|_{L_\infty(\mathbb{R}^N, V)} \leq C n^{-\min(1/p-1, \alpha)},
  \]
  with the same \( \alpha \) as in the convergence rate of the approximation in Assumption (i) and \( p \) as in Assumption (iv), where

\[
C := MC_D 4^\alpha \frac{2^\alpha + 1}{2^\alpha - 1} \| (\sigma_s^{-1}) \|_{L_p(\mathbb{F})}.
\]

Moreover, if in addition, \( p = \frac{1}{1+\alpha} \) in Assumption (iv), then we have that

\[
\|u - P_n u\|_{L_\infty(\mathbb{R}^N, V)} \leq C n^{-\alpha}.
\]

From Theorems 6.1 and 6.2 we see that the problem of construction of a linear collective Legendre approximation is reduced to the construction of a sequence \( \sigma = (\sigma_s)_{s \in F} \) satisfying Assumption (iv).

**Corollary 6.1** Let the constant \( K \) be as in (3.9) and the sequence \( b \) as in (3.10). Assume that the function \( a \in L_\infty(\mathbb{R}^N, W^{1,\infty}_\infty(D)) \), there exists \( 0 < p < 1 \) such that the sequence \( \{\|\psi_j\|_{W^{1,\infty}_\infty(D)}\}_{j \in \mathbb{N}} \) belongs to \( L_p(\mathbb{N}) \) and \( \|b\|_{\ell_1(\mathbb{N})} < 1 \). Then there holds Assumption (iv) for \( p, M = K \) and the sequence

\[
\sigma := (\sigma_s)_{s \in F}, \quad \sigma_s^{-1} := \frac{|s|!}{s!} b^s.
\]

**Proof.** By using of (5.7) and Lemma 3.4 we derive that

\[
\|u_s\|_W \leq \frac{1}{s!} \|\partial_y^s u\|_{L_\infty(\mathbb{R}^N, W)} \leq K \frac{|s|!}{s!} b^s = K \sigma_s^{-1}.
\]

On the other hand, from the assumptions we have that \( b \in L_p(\mathbb{N}) \) and \( \|b\|_{\ell_1(\mathbb{N})} < 1 \). Hence by Lemma 3.5 the sequence \( (\sigma_s^{-1})_{s \in \mathbb{F}} \) belongs to \( L_p(\mathbb{F}) \). This proves the corollary.

### 7 Concluding remarks

- We have constructed linear collective methods for Taylor, collocation, Galerkin and Legendre approximations for parametric elliptic PDEs (1.2) with affine parametric dependence of diffusion coefficients on the basis of a sequence of approximations to one nonparametric elliptic PDEs with a certain error convergence rate.
- These methods are "optimal" in the sense that they give the same error convergence rate of the inducing approximations for nonparametric elliptic PDEs.
- All the conditions on the parametric part disappear in the convergence rate and only influence the constant which can be explicitly estimated.

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In constructing these methods, the spatial variables and the parametric variables are not split, but treated collectively.

The curse of dimensionality is broken by linear methods.

In the present paper, the parameter $\alpha$ defining the convergence rate of the approximation error in Assumption (i) is restricted by the condition $0 < \alpha \leq 1/m$ caused by the restriction of the regularity of the diffusion coefficients $a(y)$, the function $f$ and the domain $D$. However, we can extend our results to the case where $\alpha$ may be arbitrarily large if we require a proper regularity of $a(y)$, $f$ and $D$.

Hopefully, the approach and methods which have been considered in this paper can be extended to more general problems. In a forthcoming paper, we extend them to the parametric elliptic PDEs \[12\] with the diffusion coefficients $a(y)$ not necessarily affinely dependent with respect to $y$, as well to a semi-linear extension and to parametric and stochastic parabolic PDEs.

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