Making Nonlinear Systems Negative Imaginary via State Feedback

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Abstract—This paper provides a state feedback stabilization approach for nonlinear systems of relative degree less than or equal to two by rendering them nonlinear negative imaginary (NI) systems. Conditions are provided under which a nonlinear system can be made a nonlinear NI system or a nonlinear output strictly negative imaginary (OSNI) system. Roughly speaking, an affine nonlinear system that has a normal form with relative degree less than or equal to two, after possible output transformation, can be rendered nonlinear NI and nonlinear OSNI. In addition, if the internal dynamics of the normal form are input-to-state stable, then there exists a state feedback input that stabilizes the system. This stabilization result is then extended to achieve stability for systems with a nonlinear NI uncertainty.

Index Terms—nonlinear negative imaginary systems, nonlinear output strictly negative imaginary systems, state feedback stabilization, robust control.

I. INTRODUCTION

Negative imaginary (NI) systems theory, which was introduced by Lanzon and Petersen in [1] and [2], provides an approach to the robust control of flexible structures [3]–[5]. As negative velocity feedback control [6] may not be suitable for some highly resonant systems, NI systems theory provides an alternative approach that uses positive feedback control. An NI system can be regarded as a positive real (PR) system cascaded with an integrator. Typical mechanical NI systems are systems with colocated force actuators and position sensors. Roughly speaking, a square real-rational proper transfer matrix $F(s)$ is said to be NI if $jF( j\omega) - F(j\omega^*) \geq 0$ for all $\omega \geq 0$. The fundamental stability results in NI systems theory are intuitive yet useful.

Under mild assumptions, the positive feedback interconnection of an NI system $F(s)$ and a strictly negative imaginary (SNI) system $G(s)$ is internally stable if and only if the DC loop gain has all its eigenvalues strictly less than unity; i.e., $\lambda_{\text{max}}(F(0)G(0)) < 1$. Since it was introduced in 2008 [1], NI systems theory has attracted attention from many control theorists [7]–[11] and has been applied in many fields including nano-positioning control [12]–[15] and the control of lightly damped structures [16]–[18], etc.

NI systems theory was extended to nonlinear systems in [19]–[21] through the notion of counterclockwise input-output dynamics [22]. Roughly speaking, a system is said to be nonlinear NI if it has a positive definite storage function and is dissipative with respect to the supply rate $u^T y$, where $u$ and $y$ are the input and output of the system, respectively. This definition is generalized in [23], to only require positive semidefiniteness of the storage function, in order to allow for systems with poles at the origin. Also introduced in [20] and [23] is the notion of nonlinear output strictly negative imaginary (OSNI) systems (see [11] and [24] for the definition of linear OSNI systems). Under the control of suitable nonlinear OSNI controllers, nonlinear NI systems can be asymptotically stabilized under mild assumptions. An advantage of linear and nonlinear NI systems theory is that NI systems can have relative degree of zero, one and two, in comparison to PR and passive systems whose relative degree can only be zero or one. With this advantage, NI systems theory can provide stability for systems of relative degree less than or equal to two, which cannot be dealt with by passivity or PR systems theory. One such example arises in the problem of state feedback stabilization.

Passivity and PR systems theory is applied in many papers to achieve state feedback stabilization [25]–[31]. The general idea applied in these papers is to render part of a nonlinear system PR or passive using state feedback. Then stability can be obtained using the passivity or PR properties of the resulting system. Such state feedback passivity results are significant not only because they provide a generalization to the feedback linearization method, but also because feedback analysis design for passive systems is comparatively simple and intuitive [27]. However, due to the nature of passive systems, a common assumption made in these papers is that the systems in question must have relative degree one. This restriction rules out a wide variety of systems which have output entries of relative degree two.

Since NI systems theory can deal with systems with relative degree zero, one and two, it is useful as a complement to passivity and PR systems theory in addressing the state feedback stabilization problem. In [32], conditions are given for linear time-invariant (LTI) systems with relative degree one and relative degree two to be rendered NI or strongly strictly NI (SSNI) using state feedback control. This result is then generalized in the paper [33], which gives necessary and sufficient conditions under which an LTI system is state feedback equivalent to an NI, OSNI or SSNI system. In [33], the system is allowed to have mixed relative degree one and two. Stabilization results are also provided in [32] and [33] for systems with SNI uncertainties.

Considering the nonlinear nature of most control systems, this paper investigates the problem of making affine nonlinear systems nonlinear NI using state feedback, in order to provide a method of stabilization for nonlinear systems.
of relative degree less than or equal to two. This paper provides conditions under which a system can be rendered nonlinear NI or OSNI, as well as corresponding formulas for the control inputs. Roughly speaking, if an affine nonlinear system of relative degree less than or equal to two can be transformed into a normal form (see [28], [34] for a description of the normal form), then there exists state feedback control such that the resulting system is NI or OSNI. If in addition, the internal dynamics in the normal form are input-to-state stable (ISS), then there exists a state feedback control that stabilizes the system. Furthermore, such a system with a nonlinear NI plant uncertainty can also be stabilized if in addition there exists a storage function for this system, which is positive definite with respect to a specific subset of the state variables.

From the technical point of view, the contribution of this work is providing an alternative approach to the previous passivity-based state feedback stabilization results (e.g., [27]) to overcome their limitations by allowing systems with output entries of relative degree two. More importantly, it broadens the class of systems to which nonlinear NI systems theory is applicable.

This paper is organized as follows: Section II reviews the essential nonlinear NI systems definitions. Section III provides conditions to render a system nonlinear NI or OSNI, locally and globally, as shown in Theorems 1 and 2. Using the nonlinear NI properties, Theorems 3 and 4 provide conditions and formulas for state feedback stabilization, locally and globally. In Section IV, Theorems 5 and 6 provide conditions and formulas for the state feedback stabilization of a system with a nonlinear NI uncertainty. A conclusion is given in Section V. A full archive version of this paper including proofs of the results, some remarks and a numerical example can be found in [35].

Notation: The notation in this paper is standard. $\mathbb{R}$ denotes the fields of real numbers. $\mathbb{R}^{m \times n}$ denotes the space of real matrices of dimension $m \times n$. $A^T$ denotes transpose of a matrix $A$. $A^{-T}$ denotes transpose of the inverse of $A$; i.e., $A^{-T} = (A^{-1})^T = (A^T)^{-1}$. $\lambda_{\max}(A)$ denotes the largest eigenvalue of a matrix $A$ with real spectrum. $\| \cdot \|$ denotes the standard Euclidean norm. $C^k$ represents the class of $k$-time continuously differentiable functions. Given a scalar function $h(x)$ and a vector field $f(x)$, $L_fh(x)$ denotes the Lie derivative of $h(x)$ with respect to $f(x)$; i.e., $L_fh(x) := \frac{\partial h(x)}{\partial x} f(x)$. For two vector fields $f$ and $g$ on $D \subset \mathbb{R}^n$, the Lie bracket $[f, g]$ is a third vector field defined by

$$[f, g](x) = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x),$$

where $\frac{\partial g}{\partial x}$ and $\frac{\partial f}{\partial x}$ are Jacobian matrices. Repeated bracketing of $g$ with $f$ can be represented using the following adjoint representation for simplicity:

$$ad^0_f g(x) = g(x),$$
$$ad^1_f g(x) = [f, g](x),$$
$$ad^k_f g(x) = [f, ad^{k-1}_f g](x), \quad k \geq 1.$$
III. STABILIZATION OF A SYSTEM USING NONLINEAR NI SYSTEMS THEORY

Let us consider a nonlinear system of the form (1). We aim to stabilize this system by rendering it a nonlinear OSNI system as per Definition 2 in the case that the system has relative degree less than or equal to two. We now provide a formal definition of systems of relative degree less than or equal to two.

Definition 7: A system of the form (1) is said to have relative degree less than or equal to two if it has a vector relative degree $r = \{r_1, \cdots, r_p\}$, where $1 \leq r_i \leq 2$ for all $i = 1, \cdots, p$. Without loss of generality, assume the components of the output vector are sorted such that the components in the vector relative degree are in nondecreasing order; i.e., $r_i = 1$ for $i = 1, 2, \cdots, p_1$ and $r_i = 2$ for $i = p_1 + 1, p_1 + 2, \cdots, p$, where $p_1$ is the number of ones in the vector relative degree $r$.

The paper [28] provides conditions for a system with a vector relative degree to have local and global normal forms. Here, we focus on the specific case that the system has relative degree less than or equal to two.

Lemma 1: (see also [28]) Suppose the system (1) has relative degree less than or equal to two at $x = 0$. If the distribution

$G = \text{span}\{g^1, g^2, \cdots, g^p\}$

is involutive, then the system (1) can be described locally around $x = 0$ by the following normal form

$\Sigma:\begin{align}
\dot{z} &= f^*(z, \xi), \\
\dot{\xi}_1 &= a_1(z, \xi) + b_1(z, \xi)u, \\
\dot{\xi}_2 &= \xi_3, \\
\dot{\xi}_3 &= a_2(z, \xi) + b_2(z, \xi)u, \\
y &= [\xi_1]_{L^2} \\
\end{align}$

(5a)-(5e)

where $u$ and $y$ are the input and output of the system. The vector $[z^T \xi^T]^T$ is the new state of the system, where $z \in \mathbb{R}^m (m \geq 0)$ and $\xi = [\xi_1, \xi_2, \xi_3]^T$. The vector $\xi_1 \in \mathbb{R}^n$ contains the state vector entries corresponding to the ones in the vector relative degree. The vector $\xi_2 \in \mathbb{R}^{p_2}$ ($p_2 := p - p_1$) contains the state vector entries corresponding to the twos in the vector relative degree and $\xi_3 \in \mathbb{R}^{p_2}$ is defined as the derivative of $\xi_2$. Here, $f^*, a_1, b_1, a_2, b_2$ are functions of suitable dimensions. Also,

$a_1(z, \xi) = \begin{bmatrix} f^* h_1(x) \\ \vdots \\ f^* h_{p_1}(x) \end{bmatrix}$,

and

$b_1(z, \xi) = \begin{bmatrix} g^1 h_1(x) \\ \vdots \\ g^1 h_{p_1}(x) \end{bmatrix}.$

Hence,

$\begin{bmatrix} b_1(z, \xi) \\ b_2(z, \xi) \end{bmatrix} = A(x)$

as in (4) and is nonsingular for $(z, \xi)$ at $(0, 0)$.

In the normal form (5), the dynamics described by (5a) are called the internal dynamics of the system. When the output is identically zero, the internal dynamics are called the zero dynamics [28], [34], [36]. In the case of system (5), $y$ being identically zero implies $\xi = 0$. Therefore, the zero dynamics are described by

$\dot{z} = f^*(z, 0).$

Lemma 2: (see [28]) The system (1) is globally diffeomorphic to a system having the normal form (5) if:

H1: the system has uniform relative degree less than or equal to two;

H2: the vector fields

$X^k = ad^{-1}_{\xi} \hat{g}_i, \quad 1 \leq i \leq p, \quad 1 \leq k \leq r_i$

are complete;

H3: $[X^1, X^j] = 0$ for all $1 \leq i, j \leq p$.

Here,

$\hat{f} = f - gA^{-1}(x) \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_p \end{bmatrix}$,

and

$\hat{g} = gA^{-1}(x).$

Lemma 3: Consider the system (5) where $[b_1(z, \xi), b_2(z, \xi)]$ is nonsingular. Then it can be rendered a nonlinear NI system as in Definition 1 using the state feedback control law

$u = \begin{bmatrix} b_1(z, \xi) \\ b_2(z, \xi) \end{bmatrix}^{-1} \left(v - \begin{bmatrix} a_1(z, \xi) + (\partial_1 \xi_1 \xi_1^T) \\ a_2(z, \xi) + (\partial_2 \xi_1 \xi_1^T) + \lambda \xi_3 \end{bmatrix} \right),$ (6)

where $v \in \mathbb{R}^p$ is the new input, $V_1(\xi_1)$ and $V_2(\xi_2)$ can be any positive semi-definite functions, and $\lambda \geq 0$ is a scalar. Moreover, if $\lambda > 0$, then the resulting system is a nonlinear OSNI system as per Definition 2 with degree of output strictness $\varepsilon = \min\{1, \lambda\}$. The storage function of the nonlinear NI (OSNI) system is

$V(z, \xi) = \hat{V}(\xi) = V_1(\xi_1) + V_2(\xi_2) + \frac{1}{2} \xi_3^T \xi_3.$ (7)

Theorem 1: Suppose the system (1) has relative degree less than or equal to two at $x = 0$ and the distribution

$G = \text{span}\{g^1, g^2, \cdots, g^p\}$

is involutive. Then the system (1) can be rendered a nonlinear NI (OSNI) system locally around $x = 0$ using the state feedback control

$u = A(x)^{-1} \left(v - \begin{bmatrix} L_f^1 h_1(x) \\ \vdots \\ L_f^p h_p(x) \end{bmatrix} - \begin{bmatrix} (\partial_1 \xi_1 \xi_1^T) \\ (\partial_2 \xi_1 \xi_1^T) + \lambda \xi_3 \end{bmatrix} \right),$ (8)

where $A(x)$ is defined as in (4), $v \in \mathbb{R}^p$ is the new input, $V_1(\xi_1)$ and $V_2(\xi_2)$ can be any positive semi-definite functions, and $\lambda \geq 0 \ (\lambda > 0)$ is a scalar. Also, the function (7) is a storage function for the resulting nonlinear NI (OSNI) system.
**Theorem 2:** Suppose the system (1) satisfies H1, H2 and H3. Then the system (1) can be globally rendered a nonlinear NI (OSNI) system using the state feedback control (8). Also, the function $V(z, \xi)$ defined in (7) is a storage function for the resulting nonlinear NI (OSNI) system.

Considering the restriction in condition (ii) of Definition 3, some systems do not have vector relative degrees. However, for a system that does not have a vector relative degree, sometimes there exists an output transformation that transforms it into a system with a vector relative degree. We generalize Theorem 1 by showing that the result is invariant to a nonsingular output transformation.

**Lemma 4:** A system with output $y$ and input $u$ is a nonlinear NI (OSNI) system if and only if the system with output $\tilde{y} = T_y y$ and input $\tilde{u} = T_y^T u$ is a nonlinear NI (OSNI) system. Here $T_y$ is a nonsingular constant matrix.

**Lemma 5:** Consider a system of the form (1) and an output transformation $\tilde{y} = T_y y$ where $T_y \in \mathbb{R}^{p \times p}$ is a nonsingular constant matrix. If there exists a state feedback control law

$$u = k_x(x)x + k_u(x)v,$$

under which the system with the new input $v \in \mathbb{R}^p$ is a nonlinear NI (OSNI) system, then the output transformed system; i.e., the system with input $u$ and output $\tilde{y} = T_y y$, can also be rendered a nonlinear NI (OSNI) system using the state feedback control law

$$u = k_x(x)x + k_u(x)T_y^T \tilde{v},$$

where $\tilde{v} \in \mathbb{R}^p$ is the new input.

**Corollary 1:** Suppose the system (1) can be output transformed into a system with relative degree less than or equal to two at $x = 0$ using the output transformation

$$\tilde{y} = T_y y,$$

where $T_y \in \mathbb{R}^{p \times p}$ is a nonsingular constant matrix. Also, suppose the distribution

$$G = \text{span}(g^1, g^2, \ldots, g^p)$$

is involutive. Then the system (1) can be rendered a nonlinear NI (OSNI) system locally around $x = 0$ using state feedback control.

**Corollary 2:** Suppose the system (1) can be output transformed into a system satisfying H1, H2 and H3 using the output transformation (9). Then the system (1) can be globally rendered a nonlinear NI (OSNI) system using state feedback control.

If the system (1) is rendered a nonlinear OSNI system with a storage function which is positive definite with respect to $\xi$, then according to the dissipation inequality (3), giving zero input to the system will result the boundedness of the state $\xi$. Indeed, as will be shown later, the state $\xi$ will converge to zero under zero input. Given the stability of the state $\xi$, we consider the question of what additional conditions are needed in order to make the state $z$ also stable. First, we need to provide several definitions regarding to the internal dynamics in the normal form of a nonlinear system.

**Definition 8 ((Globally) Minimum Phase):** [27], [34] A nonlinear system which has a normal form is said to be (globally) minimum phase if its zero dynamics have a (globally) asymptotically stable equilibrium at the origin.

In the case of system (5), the minimum phase property guarantees that when $\xi = 0$, if $z$ is finite, it will also converge to zero. In other words, if we view the state $\xi$ as the input to the internal dynamics $\dot{z} = f^*(z, \xi)$, then (global) minimum phase property implies that the internal dynamics is (globally) asymptotically stable with zero input, namely 0-AS (0-GAS) for short. However, examples in [36]–[38] show that for systems that are 0-AS (0-GAS), its state may diverge under a bounded input that converges to zero. This phenomena motivated the concept of input-to-state stability (ISS) [37], [38]. As is discussed in [39], the asymptotic stability of the zero dynamics is sometimes insufficient for control design purposes until it is combined with the ISS property of the internal dynamics. This is a common requirement (see for example [40]). Let us now recall the definitions of ISS and locally ISS (LISS) systems.

To avoid introducing new system models, let us consider the system of the form (5a). Let us rewrite it in the following as a sepaerate system:

$$\dot{z} = f^*(z, \xi),$$

where $\xi$ acts as the input to this system.

**Definition 9 (Input-to-State Stability):** [34], [36]–[38], [41] The system (10) is said to be input-to-state stable (ISS) if there exist a class $\mathcal{K}$ function $\beta$ and a class $\mathcal{K}$ function $\gamma$ such that for any initial state $z(t_0)$ and any bounded input $\xi(t)$, the solution $z(t)$ exists for all $t \geq t_0$ and satisfies

$$\|z(t)\| \leq \beta(\|z(t_0)\|, t - t_0) + \gamma \left( \sup_{t_0 \leq \tau < t} \|\xi(\tau)\| \right).$$

**Definition 10 (Locally Input-to-State Stability):** [42] The system (10) is said to be locally input-to-state stable (LISS) if there exist a class $\mathcal{K}$ function $\beta$ and a class $\mathcal{K}$ function $\gamma$ and constants $\rho_z, \rho_\xi > 0$ such that for any initial state $z(t_0)$ with $\|z(t_0)\| \leq \rho_z$ and any bounded input $\xi(t)$ with $\sup_{t_0 \leq \tau < t} \|\xi(\tau)\| \leq \rho_\xi$ for all $t \geq t_0$, the solution $z(t)$ exists and satisfies (11) for all $t \geq t_0$.

**Lemma 6:** [36], [41] Suppose the system (10) is ISS. If $\xi(t) \rightarrow 0$ as $t \rightarrow \infty$, so does $z(t)$.

**Lemma 7:** [36], [41] If the system (10) is LISS, then there exist constants $\tilde{\rho}_z, \tilde{\rho}_\xi > 0$ such that for any initial state $z(t_0)$ with $\|z(t_0)\| \leq \tilde{\rho}_z$ and any bounded input $\xi(t)$ with $\sup_{t_0 \leq \tau < \infty} \|\xi(\tau)\| \leq \tilde{\rho}_\xi$ and $\xi(t) \rightarrow 0$ as $t \rightarrow \infty$, we have $z(t) \rightarrow 0$ as $t \rightarrow \infty$.

As is proved in [42], ISS implies 0-GAS and LISS implies 0-AS. We now introduce a new version of the (global) minimum phase property in the following, in which the 0-AS (0-GAS) requirement is replaced by an LISS (ISS) requirement. Hence, the new definition is stricter than Definition 8.

**Definition 11 ((Globally) Strictly Minimum Phase):** A system is said to be (Globally) strictly minimum phase if its internal dynamics of the form

$$\dot{z} = f^*(z, \xi)$$

is (Globally) minimum phase in the following, in which the 0-AS (0-GAS) requirement is replaced by an LISS (ISS) requirement.
are LISS (ISS) with respect to \( \xi \).

**Theorem 3:** After possible output transformation (9), suppose the system (1) satisfies the following:

(i). it has relative degree less than or equal to two at \( x = 0 \);
(ii). the distribution
\[
G = \text{span}\{g^1, g^2, \ldots, g^p\}
\]
is involutive;
(iii). the system (1) is strictly minimum phase around \( x = 0 \). Then the system (1) can be locally asymptotically stabilized using the state feedback control law
\[
u = -A(x)^{-1}
\begin{pmatrix}
\left[ L_f^T h_1(x) \right]
\vdots
\left[ L_f^T h_p(x) \right]
\end{pmatrix}
+ \left[ \frac{\partial \mathcal{V}_1(\xi_1)}{\partial \xi_1} \right]_T + \lambda \xi_3,
\]
where \( A(x) \) is defined in (4), \( \mathcal{V}_1(\xi_1) \) and \( \mathcal{V}_2(\xi_2) \) can be any positive definite functions, and \( \lambda > 0 \) is a scalar.

**Theorem 4:** After possible output transformation (9), suppose the system (1) satisfies H1, H2 and H3. Also, suppose the system (1) is globally strictly minimum phase. Then the system (1) can be globally asymptotically stabilized using the state feedback (12).

**IV. CONTROLLER SYNTHESIS FOR A SYSTEM WITH NONLINEAR NI UNCERTAINTY**

Suppose a system of the form (1) has uncertainty that can be modelled as a nonlinear NI system. Denote the uncertainty as \( H_c \). The system model of \( H_c \) is
\[
H_c : \begin{align*}
x_c &= f_c(x_c, u_c), \\
y_c &= h_c(x_c),
\end{align*}
\]
where \( x_c \in \mathbb{R}^{n_c} \) is the state, \( u_c \in \mathbb{R}^p \) is the input, and \( y_c \in \mathbb{R}^p \) is the output. \( f_c : \mathbb{R}^{n_c} \times \mathbb{R}^p \rightarrow \mathbb{R}^{n_c} \) is a Lipschitz continuous function and \( h_c : \mathbb{R}^{n_c} \rightarrow \mathbb{R}^p \) is a class \( C^1 \) function. Suppose the system has at least one equilibrium. Then without loss of generality, we can assume \( f_c(0, 0) = 0 \) and \( h_c(0) = 0 \) after a possible coordinate shift.

When full state information is available, we aim to stabilize the uncertain system using a state feedback controller as shown in the left-hand side (LHS) of Fig. 1.

The interconnection can be described by the following equations:
\[
\begin{align*}
\dot{x} &= f(x) + g(x)(u + w), \\
y &= h(x), \\
\dot{x}_c &= f_c(x_c, u_c), \\
y_c &= h_c(x_c), \\
w &= y_c, \\
u_c &= y.
\end{align*}
\]

**Theorem 5:** Suppose the nominal plant \( \Sigma \) of the form (1) is strictly minimal phase and has relative degree less than or equal to two around \( x = 0 \) and the distribution \( G = \text{span}\{g^1, g^2, \ldots, g^p\} \) is involutive. Let \( \xi_1 = [y_1^T, \ldots, y_{p+1}^T]^T \) and \( \xi_2 = [y_{p+2}^T, \ldots, y_p^T]^T \) denote the vectors containing the output entries corresponding to the ones and twos in the vector relative degree, respectively. Let \( \xi_3 = \xi_2 \). Suppose that the systems (13) is nonlinear NI with storage function \( V_c(x_c) \). If there exist positive definite functions \( \mathcal{V}_1(\xi_1) \) and \( V_c(x_c) \) such that the function defined as
\[
W(\xi, x_c) = V_c(x_c) - h_c(x_c) \left( \xi_1 \right) \xi_2
\]
is positive definite, then the system (14) is locally asymptotically stabilized by the state feedback control law
\[
u = A(x)^{-1}
\begin{pmatrix}
I - \left[ L_g L_f^{-1} h_1(x) \right]
\vdots
\left[ L_g L_f^{-1} h_p(x) \right]
\end{pmatrix} w
+ \left[ \frac{\partial \mathcal{V}_1(\xi_1)}{\partial \xi_1} \right]_T + \lambda \xi_3 + \left[ \frac{\partial \mathcal{V}_2(\xi_1)}{\partial \xi_2} \right]_T + \lambda \xi_3
\]
where \( A(x) \) is defined in (4), \( w \in \mathbb{R}^p \) is the output of the uncertainty \( H_c \), and \( \lambda > 0 \) is a scalar.

**Theorem 6:** Suppose the nominal plant \( \Sigma \) of the form (1) is globally strictly minimal phase and satisfies H1, H2 and H3. Let \( \xi_1 = [y_1^T, \ldots, y_{p+1}^T]^T \) and \( \xi_2 = [y_{p+2}^T, \ldots, y_p^T]^T \) denote the vectors containing the output entries corresponding to the ones and twos in the vector relative degree, respectively. Let \( \xi_3 = \xi_2 \). Suppose that the systems (13) is nonlinear NI with storage function \( V_c(x_c) \). If there exist positive definite functions \( \mathcal{V}_1(\xi_1) \) and \( V_c(x_c) \) such that the function (15) is positive definite, then the system (14) is globally robustly stabilized by the state feedback control law (16).

**V. CONCLUSION**

This paper investigates a state feedback stabilization problem using nonlinear NI systems theory for affine nonlinear systems with relative degree less than or equal to two. For such a system that also has a normal form, we provide a state feedback control law that makes it nonlinear NI or
OSNI. In this case, if the internal dynamics of the system are ISS, then there exists state feedback control that stabilizes the system. In the case that the system has a nonlinear NI uncertainty, there exists a state feedback control law that stabilizes the system if a positive definiteness-like assumption is satisfied for the storage function of the closed-loop system.

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