BPX preconditioner for nonstandard finite element methods for diffusion problems *

Binjie Li†  Xiaoping Xie ‡
School of Mathematics, Sichuan University, Chengdu 610064, China

Abstract
This paper proposes and analyzes an optimal preconditioner for a general linear symmetric
positive definite (SPD) system by following the basic idea of the well-known BPX framework.
The SPD system arises from a large number of nonstandard finite element methods for dif-
fusion problems, including the well-known hybridized Raviart-Thomas and Brezzi-Douglas-
Marini mixed element methods, the hybridized discontinuous Galerkin method, the Weak
Galerkin method, and the nonconforming Crouzeix-Raviart element method. We prove that
the presented preconditioner is optimal, in the sense that the condition number of the precon-
ditioned system is independent of the mesh size. Numerical experiments provided confirm the
theoretical result.

Keywords. BPX, preconditioner, hybridized discontinuous Galerkin method, Weak Galerkin
method, nonconforming Crouzeix-Raviart element

1 Introduction
This paper is to design an efficient preconditioner for a large class of nonstandard finite element
methods for solving the diffusion model

\[
\begin{align*}
-\text{div}(A \nabla u) &= f \quad \text{in } \Omega, \\
\quad u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

(1.1)

where \( \Omega \subset \mathbb{R}^d \) (\( d = 2, 3 \)) is a bounded polygonal domain, the diffusion tensor \( A : \Omega \to \mathbb{R}^{d \times d} \) is a
matrix function that is assumed to be symmetric and uniformly positive definite, and \( f \in L^2(\Omega) \).
The choice of homogeneous boundary condition is made for ease of presentation, since similar
results are valid for other boundary conditions.

*This work was supported by National Natural Science Foundation of China (11171239), Major Research Plan of
National Natural Science Foundation of China (91430105) and Open Fund of Key Laboratory of Mountain Hazards
and Earth Surface Processes, CAS.
†Email: jieruiscu@gmail.com
‡Corresponding author. Email: xpxie@scu.edu.cn
Let $\mathcal{T}_h$ be a triangulation of $\Omega$, and $\mathcal{F}_h$ be the set of all faces of $\mathcal{T}_h$. We introduce a finite dimensional space

$$M_{h,k}^0 := \{ \mu_h \in L^2(\mathcal{F}_h) : \mu_h|_F \in P_k(F), \forall F \in \mathcal{F}_h, \mu_h|_{\partial \Omega} = 0 \},$$

with $P_k(F)$ denoting the set of polynomials with order no greater than $k$ on $F$, and consider the following general symmetric and positive definite (SPD) system for equation (1.1): Seek $\lambda_h \in M_{h,k}^0$ such that

$$d_h(\lambda_h, \mu_h) = b(\mu_h), \forall \mu_h \in M_{h,k}^0.$$  \hfill (1.3)

Here $d_h(\cdot, \cdot) : M_{h,k}^0 \times M_{h,k}^0 \to \mathbb{R}$ is an inner-product on $M_{h,k}^0$ and $b_h(\cdot) : M_{h,k}^0 \to \mathbb{R}$ is a linear functional on $M_{h,k}^0$.

The first class of nonstandard finite element methods that fall into the framework (1.3) are hybrid or hybridized finite element methods ([5, 33, 36, 37, 4, 20, 19, 30]). Due to the relaxation of the constraint of continuity at the inter-element boundaries by introducing some Lagrange multipliers, the corresponding hybrid method allows for piecewise-independent approximation to the potential or flux solution. Thus, after local elimination of unknowns defined in the interior of elements, the method leads to a SPD discrete system of the form (1.3), where the unknowns are only the globally coupled degrees of freedom describing the Lagrange multiplier. In [3, 14], the Raviart-Thomas (RT) [35] and Brezzi-Douglas-Marini (BDM) mixed methods were shown to have equivalent hybridized versions. A new characterization of the approximate solution of hybridized mixed methods was developed and applied in [16] to obtain an explicit formula for the entries of the matrix equation for the Lagrange multiplier unknowns. An overview of some new hybridization techniques was presented in [17]. In [4] a unifying framework for hybridization of finite element methods was developed. Error estimates of some hybridized discontinuous Galerkin (HDG) methods were derived in [18, 19, 30].

The weak Galerkin (WG) method [40, 32, 31] is the second class of nonstandard approach that applies to the framework (1.3). The WG method is designed by using a weakly defined gradient operator over functions with discontinuity, and allows the use of totally discontinuous functions in the finite element procedure. The concept of weak gradients provides a systematic framework for dealing with discontinuous functions defined on elements and their boundaries in a near classical sense [40]. Similar to the hybrid methods, the WG scheme can be reduced to the form (1.3) after local elimination of unknowns defined in the interior of elements. We note that when $A$ in (1.1) is a piecewise-constant matrix, the WG method is, by introducing the discrete weak gradient as an independent variable, equivalent to the hybridized version of the RT or BDM mixed methods. For the discretization of the diffusion model (1.1) on simplicial 2D or 3D meshes, we refer to [29] for a multigrid WG algorithm, and to [15] for an auxiliary space multigrid preconditioner for the WG method as well as a reduced system of the weak Galerkin method involving only the degrees of freedom on edges/faces.
Besides, some nonconforming methods, e.g. the nonconforming Crouzeix-Raviart element method [22], can also lead to a SPD discrete system of the form (1.3). To this end, one needs to introduce a special projection of the flux solution to the element boundaries as the trace approximation. We refer to [12, 6, 13, 11, 28, 33, 27, 38, 46] for multigrid algorithms or preconditioning for the CR or CR-related nonconforming finite element methods. In particular, in [13], an optimal-order multigrid method was proposed and analyzed for the lowest-order Raviart-Thomas mixed element based on the equivalence between Raviart-Thomas mixed methods and certain nonconforming methods.

As far as we know, the first preconditioner for the system (1.3) was developed in [24], where a Schwarz preconditioner was designed for the hybridized RT and BDM mixed element methods. In [25] a convergent V-cycle multigrid method was proposed for the hybridized mixed methods for Poisson problems. By following the idea of [25], a non-nested multigrid V-cycle algorithm, with a single smoothing step per level, was analyzed in [21] for the system (1.3) arising from one type of HDG method.

It is well known that the BPX multigrid framework, developed by Bramble, Pasciak and Xu [10], is widely used in the analysis of multigrid and domain decomposition methods. We refer to [7, 8, 9, 11, 23, 26, 30, 41, 42, 43, 45] for the development and applications of the BPX framework. In [41] an abstract framework of auxiliary space method was proposed and an optimal multigrid technique was developed for general unstructured grids. Especially, in [42] an overview of multilevel methods, such as V-cycle multigrid and BPX preconditioner, was given for solving various partial differential equations on quasi-uniform meshes, and the methods were extended to graded meshes and completely unstructured grids.

In this paper, we shall follow the basic ideas of [10], [41], [42] to construct a BPX preconditioner for the system (1.3), which is, due to the definition of the discrete space $M^0_{h,k}$, corresponding to nonnested multilevel finite element spaces. We will show the proposed preconditioner is optimal.

We arrange the rest of the paper as follows. Section 2 introduces some notations and preliminaries. Section 3 constructs the BPX preconditioner and derives the condition number estimation of the preconditioned system. Section 4 shows some applications of the proposed preconditioner. Finally, Section 5 provides some numerical results.

2 Notations and preliminaries

Throughout this paper, we use the standard definitions of Sobolev spaces and their norms and semi-norms (cf. [2]), namely for an arbitrary open set $D \subset \mathbb{R}^d$ and any nonnegative integer $s$,

$$H^s(D) := \{ v \in L^2(D) : \partial^\alpha v \in L^2(D), \forall |\alpha| \leq s \},$$

$$\|v\|_{s,D} := \left( \sum_{|\alpha| \leq s} \int_D |\partial^\alpha v|^2 \right)^{\frac{1}{2}}, \quad |v|_{s,D} := \left( \sum_{|\alpha| = s} \int_D |\partial^\alpha v|^2 \right)^{\frac{1}{2}}.$$
We denote respectively by \( (\cdot, \cdot)_D \) and \( (\cdot, \cdot)_{\partial D} \) the \( L^2 \) inner products on \( L^2(D) \) and \( L^2(\partial D) \), and respectively by \( \|\cdot\|_D \) and \( \|\cdot\|_{\partial D} \) the \( L^2 \)-norms on \( L^2(D) \) and \( L^2(\partial D) \). In particular, \( (\cdot, \cdot) \) and \( \|\cdot\| \) abbreviate \( (\cdot, \cdot)_{\Omega} \) and \( \|\cdot\|_{\Omega} \), respectively.

Let \( \Omega \subset \mathbb{R}^d \) be a polygonal, and let \( T_h \) be a conforming shape-regular triangulation of \( \Omega \). For any \( T \in T_h \), \( h_T \) denotes the diameter of \( T \), and we set \( h := \max_{T \in T_h} h_T \). We define the mesh-dependent inner product \( (\cdot, \cdot)_h : \mathbb{M}_{h,k}^0 \times \mathbb{M}_{h,k}^0 \to \mathbb{R} \) and the norm \( \|\cdot\|_h : \mathbb{M}_{h,k}^0 \to \mathbb{R} \) as follows: for any \( \lambda_h, \mu_h \in \mathbb{M}_{h,k}^0 \),

\[
(\lambda_h, \mu_h)_h := \sum_{T \in T_h} h_T \int_{\partial T} \lambda_h \mu_h, \quad \|\mu_h\|_h := (\mu_h, \mu_h)_h^{1/2}. \tag{2.1}
\]

We also need the following notations: for any \( \mu \in L^2(\partial T) \),

\[
\|\mu\|_{h,\partial T} := h_T^{1/2} \|\mu\|_{\partial T}, \\
\|\mu\|_{h,\partial T} := h_T^{-1} \|\mu - m_T(\mu)\|_{\partial T}, \quad \text{with} \quad m_T(\mu) := \frac{1}{|\partial T|} \int_{\partial T} \mu, \\
\|\mu\|_h := (\sum_{T \in T_h} \|\mu\|_{h,\partial T}^2)^{1/2}. \tag{2.2}
\]

In the context, we use \( x \lesssim y \) to denote \( x \leq cy \), where \( c \) is a positive constant independent of \( h \) which may be different at its each occurrence. The notation \( x \sim y \) abbreviates \( x \lesssim y \lesssim x \). For the bilinear form \( d_h(\cdot, \cdot) \) in the system (1.3), we give the following abstract assumption.

**Assumption 2.1.** For any given \( \mu_h \in \mathbb{M}_{h,k}^0 \), it holds

\[
d_h(\mu_h, \mu_h) \sim \|\mu_h\|_h^2. \tag{2.3}
\]

**Remark 2.1.** This assumption is valid for many nonstandard finite element methods, as will be shown in Section 4. We note that the Schwarz preconditioner constructed in (2.4) can also be extended to the system (1.3) under Assumption 2.4.

Basing on Assumption 2.1 we are ready to present an estimate that character the conditioning of the system (1.3).

**Theorem 2.1.** Suppose \( T_h \) to be quasi-uniform. Under Assumption 2.4 it holds

\[
\|\mu_h\|_h^2 \lesssim d_h(\mu_h, \mu_h) \lesssim h^{-2} \|\mu_h\|_h^2, \quad \forall \mu_h \in \mathbb{M}_{h,k}^0. \tag{2.4}
\]

**Proof.** See the proof of Theorem 2.3 in [24]. \( \square \)

We introduce the operator \( D_h : \mathbb{M}_{h,k}^0 \to \mathbb{M}_{h,k}^0 \) with

\[
\langle D_h \mu_h, \eta_h \rangle_h = d_h(\mu_h, \eta_h), \quad \forall \mu_h, \eta_h \in \mathbb{M}_{h,k}^0. \tag{2.5}
\]

Obviously, \( D_h \) is an SPD operator, and from Theorem 2.4 it follows the condition number estimate

\[
\kappa(D_h) \lesssim h^{-2}, \tag{2.6}
\]

where \( \kappa(D_h) := \frac{\lambda_{\max}(D_h)}{\lambda_{\min}(D_h)} \) and \( \lambda_{\max}(D_h), \lambda_{\min}(D_h) \) denote the maximum and minimum eigenvalues of \( D_h \) respectively. In fact, some further analysis can show that \( \kappa(D_h) \sim h^{-2} \) (cf. [30]).
3 BPX preconditioner

3.1 Preconditioner construction

Suppose we are given a coarse quasi-uniform triangulation $T_0$. Then we obtain a nested sequence of triangulations $\{T_j : 0 \leq j \leq J\}$ through a successive refinement process, i.e., $T_j$ is the uniform refinement of $T_{j-1}$ for $j > 0$. We denote by $h_j$ the mesh size of $T_j$, i.e., the maximum diameter of the simplex in $T_j$. We set $T_h = T_J$ and $h = h_J$. Associated with each triangulation $T_j$, we define $V_j$ by

$$V_j := \{ v \in H_0^1(\Omega) : v|_T \in P_1(T), \forall T \in T_j \}. \quad (3.1)$$

For each $V_j$, we denote by $\{\varphi_{j,i} : i = 0, 1, \ldots, N_j\}$ the standard nodal basis of $V_j$, where $N_j$ is the dimension of $V_j$. We set $\{\eta_i : i = 1, 2, \ldots, M\}$ to be the standard nodal basis of $M_{0,h,k}^0$.

Let $\Pi_h : V_J \to M_{0,h,k}^0$ be a linear operator given by

$$\Pi_h v_h|_F := \frac{1}{|F|} \int_F v_h, \forall F \in F_h, \forall v_h \in V_J, \quad (3.2)$$

where $|F|$ denotes the $d-1$ dimensional Hausdorff measure of $F$. Define the adjoint operator $\Pi_h^*: M_{0,h,k}^0 \to V_J$ by

$$(\Pi_h^* \mu_h, v_h) = (\mu_h, \Pi_h v_h)_h, \forall \mu_h \in M_{0,h,k}^0, \forall v_h \in V_J. \quad (3.3)$$

With the operators $\Pi_h$, $\Pi_h^*$, the nodal basis, $\{\varphi_{j,i} : i = 0, 1, \ldots, N_j\}$, of $V_j$, and the nodal basis, $\{\eta_i : i = 1, 2, \ldots, M\}$, of $M_{0,h,k}^0$, we define the BPX preconditioner (in operator form) for the operator $D_h$ given in (2.4) as follows:

$$B_h \mu_h = h^{2-d} \sum_{i=1}^M (\mu_h, \eta_i)_h \eta_i + \sum_{(j,i) \in \Lambda} h_j^{2-d} (\Pi_h^* \mu_h, \varphi_{j,i}) \Pi_h \varphi_{j,i}, \forall \mu_h \in M_{0,h,k}^0, \quad (3.4)$$

where $\Lambda := \{(j,i) : 0 \leq j \leq J, 0 \leq i \leq N_j\}$. It’s trivial to verify that $B_h$ is a SPD operator with respect to $\langle \cdot, \cdot \rangle_h$.

3.2 Estimation of condition number

We shall follow the abstract framework of [41] to analyze the condition number of the preconditioned system for (1.3).

For the sake of convenience, in this subsection we assume

$$\mu_i \in \text{span}\{\eta_i\} \quad \text{for} \quad i = 1, \ldots, M,$$

$$v_{j,i} \in \text{span}\{\varphi_{j,i}\} \quad \text{for} \quad (j,i) \in \Lambda.$$

At first, we give a characterization of $B_h^{-1}$ in the lemma below.
Lemma 3.1. For any $\mu_h \in M_{h,k}^0$, it holds

$$
\langle B_h^{-1} \mu_h, \mu_h \rangle_h = \inf_{\sum_{i=1}^M \mu_i + \sum_{(j,i) \in A} \Pi_h v_{j,i} = \mu_h} \left\{ \sum_{i=1}^M \frac{h^{d-2}}{\| \eta_i \|_h^2} \| \mu_i \|^2_h + \sum_{(j,i) \in A} \frac{h^{d-2}}{\| \phi_{j,i} \|^2} \| v_{j,i} \|^2 \right\}. \tag{3.5}
$$

Proof. The proof is a trivial modification of the proof of the Lemma 2.4 ([44]). Set

$$
\tilde{\mu}_i := h^2 - d B_h^{-1} \mu_h, \eta_i, \quad \tilde{\nu}_{j,i} := h^2 - d (\Pi_h B_h^{-1} \mu_h, \phi_{j,i}) \phi_{j,i}.
$$

We easily have

$$
\sum_{i=1}^M \tilde{\mu}_i + \Pi_h \left( \sum_{(j,i) \in A} \tilde{\nu}_{j,i} \right) = \mu_h,
$$

$$
\langle B_h^{-1} \mu_h, \mu_h \rangle_h = \sum_{i=1}^M \frac{h^{d-2}}{\| \eta_i \|_h^2} \| \tilde{\mu}_i \|^2_h + \sum_{(j,i) \in A} \frac{h^{d-2}}{\| \phi_{j,i} \|^2} \| \tilde{\nu}_{j,i} \|^2. \tag{3.6}
$$

For any $\sum_{i=1}^M \mu_i + \Pi_h (\sum_{(j,i) \in A} v_{j,i}) = 0$, it follows from

$$
\frac{h^{d-2}}{\| \eta_i \|_h^2} \langle \tilde{\mu}_i, \mu_i \rangle_h = \langle B_h^{-1} \mu_h, \mu_i \rangle_h, \tag{3.7}
$$

$$
\frac{h^{d-2}}{\| \phi_{j,i} \|} \langle \tilde{\nu}_{j,i}, v_{j,i} \rangle_h = \langle B_h^{-1} \mu_h, \Pi_h v_{j,i} \rangle_h \tag{3.8}
$$

that

$$
\sum_{i=1}^M \frac{h^{d-2}}{\| \eta_i \|_h^2} \langle \tilde{\mu}_i, \mu_i \rangle_h + \sum_{(j,i) \in A} \frac{h^{d-2}}{\| \phi_{j,i} \|} \langle \tilde{\nu}_{j,i}, v_{j,i} \rangle_h = 0. \tag{3.9}
$$

Combining (3.6) with (3.9), we obtain

$$
\sum_{i=1}^M \frac{h^{d-2}}{\| \eta_i \|_h^2} \| \tilde{\mu}_i \|^2_h + \sum_{(j,i) \in A} \frac{h^{d-2}}{\| \phi_{j,i} \|} \| \tilde{\nu}_{j,i} + v_{j,i} \|^2
$$

$$
= \sum_{i=1}^M \frac{h^{d-2}}{\| \eta_i \|_h^2} \| \tilde{\mu}_i \|^2_h + \sum_{(j,i) \in A} \frac{h^{d-2}}{\| \phi_{j,i} \|} \| \tilde{\nu}_{j,i} \|^2 + \sum_{i=1}^M \frac{h^{d-2}}{\| \eta_i \|_h^2} \| \mu_i \|^2 + \sum_{(j,i) \in A} \frac{h^{d-2}}{\| \phi_{j,i} \|} \| v_{j,i} \|^2
$$

$$
= \langle B_h^{-1} \mu_h, \mu_h \rangle_h + \sum_{i=1}^M \frac{h^{d-2}}{\| \eta_i \|_h^2} \| \mu_i \|^2 + \sum_{(j,i) \in A} \frac{h^{d-2}}{\| \phi_{j,i} \|} \| v_{j,i} \|^2,
$$

which yields (3.5) immediately.

Through standard scaling arguments, it’s easy to derive the following lemma.

Lemma 3.2. It holds

$$
\| \Pi_h v_h \|_h \sim \| v_h \|_{1, \Omega}, \quad \forall v_h \in V_J, \tag{3.10}
$$

$$
\sum_i \frac{h^{d-2}}{\| \eta_i \|_h^2} \| \mu_i \|^2_h \sim \frac{h^2}{2} \sum_i \| \mu_i \|^2_h. \tag{3.11}
$$
By Lemma 3, Theorem 4 and Theorem 5 in [42], we have the estimate
\[
\inf_{\sum_{(j,i) \in A} v_{j,i} = v_h} \sum_{(j,i) \in A} \frac{h_{j}^{d-2}}{\|\phi_{j,i}\|^2} \|v_{j,i}\|^2 \sim \|v_h\|_{1,\Omega}, \quad \forall v_h \in V_J.
\] (3.12)

In light of Lemma 3.1, Lemma 3.2 and (3.12), we show a new characterization of $B_h^{-1}$ below.

Lemma 3.3. For any $\mu_h \in M_{h,k}^0$, it holds
\[
\langle B_h^{-1} \mu_h, \mu_h \rangle \sim \inf_{\eta_h + \Pi_h \nu_h = \mu_h} h^{-2} \|\eta_h\|_h^2 + \|\Pi_h \nu_h\|_h^2.
\] (3.13)

where $\eta_h \in M_{h,k}^0$ and $v_h \in V_J$.

Next, we define a linear operator $P_h : M_{h,k}^0 \to V_h$ as follows: For each node $a$ of $T_J$,
\[
P_h \mu_h(a) := \begin{cases} \frac{\sum_{T \in \omega_a} m_T(\mu_h)}{\sum_{T \in \omega_a} 1}, & \text{if } a \text{ is an interior node}, \\ 0, & \text{if } a \in \partial \Omega, \end{cases}
\] (3.14)

where $\omega_a$ denotes the set of simplexes that share the node $a$.

We have the following estimates for $P_h$.

Lemma 3.4. For any $\mu_h \in M_{h,k}^0$, it holds
\[
\|P_h \mu_h\|_{1,\Omega} \lesssim \|\mu_h\|_h, \quad \|I - \Pi_h P_h \mu_h\|_h \lesssim h \|\mu_h\|_h.
\] (3.15) (3.16)

Proof. For each $T \in T_h$, we use $N(T)$ to denote the set of vertexes of $T$ and $\omega_T$ the set $\{T' \in T_h : T' \text{ and } T \text{ share a same vertex}\}$.

We have
\[
\|P_h \mu_h\|_{1,\Omega}^2 = \|m_T(\mu_h) - P_h \mu_h\|_{1,T}^2 \\
\lesssim h_T^{d-2} \sum_{a \in N(T)} |m_T(\mu_h) - (P_h \mu_h)(a)|^2 \\
\lesssim h_T^{d-2} \sum_{a \in N(T)} \sum_{T_1, T_2 \in \omega_a} |m_{T_1}(\mu_h) - m_{T_2}(\mu_h)|^2 \\
\lesssim h_T^{-1} \sum_{T' \in \omega_T} \|\mu_h - m_{T'}(\mu_h)\|_{\partial T'}^2 \\
\lesssim \sum_{T' \in \omega_T} \|\mu_h\|_{h,\partial T'}^2,
\]

which implies
\[
\|P_h \mu_h\|_{1,\Omega}^2 = \sum_{T \in T_h} \|P_h \mu_h\|_{1,T}^2 \lesssim \|\mu_h\|_h^2,
\]
i.e. the estimate (3.15) holds.
On the other hand, since
\[ \|m_T(\mu_h) - \Pi_h P_h \mu_h \|^2_{\partial T} \lesssim h_T d^{-1} \sum_{a \in \mathcal{N}(T)} \sum_{T_1, T_2 \in \omega_a, T_1, T_2 \text{ share a same face}} |m_{T_1}(\mu_h) - m_{T_2}(\mu_h)|^2 \]
we get
\[ \|\mu_h - \Pi_h P_h \mu_h\|^2_{\partial T} \lesssim h_T \|\mu_h\|^2_{h, \partial T} + \|m_T(\mu_h) - \Pi_h P_h \mu_h\|^2_{\partial T} \]
Therefore, it holds
\[ \| (I - \Pi_h P_h) \mu_h \|_h^2 = \sum_{T \in T_h} h_T \| (I - \Pi_h P_h) \mu_h \|^2_{\partial T} \lesssim h^2 \|\mu_h\|_h^2. \]
This completes the proof.

Basing on the lemmas above, we are ready to estimate the condition number of the operator $B_h D_h$. Lemmas 3.5-3.6 present the estimation of the lower and upper bounds of the eigenvalues of $B_h D_h$, respectively.

**Lemma 3.5.** Under Assumption 2.1, for any $\mu_h \in \mathcal{M}_{h,k}^0$, it holds
\[ \langle B^{-1}_h \mu_h, \mu_h \rangle_h \lesssim d_h(\mu_h, \mu_h), \] (3.17)
which implies
\[ 1 \lesssim \lambda_{\text{min}}(B_h D_h), \] (3.18)
where $\lambda_{\text{min}}(B_h D_h)$ denotes the minimum eigenvalue of $B_h D_h$.

**Proof.** By Lemma 3.3, Lemma 3.2 and Lemma 3.4, we have
\[ \langle B^{-1}_h \mu_h, \mu_h \rangle_h \lesssim h^{-2} \| (I - \Pi_h P_h) \mu_h \|_h^2 + \| \Pi_h P_h \mu_h \|_h^2 \]
\[ \lesssim h^{-2} \| (I - \Pi_h P_h) \mu_h \|_h^2 + | P_h \mu_h |_{1, \Omega}^2 \]
\[ \lesssim \| \mu_h \|_h^2. \]
Then (3.17) follows from Assumption 2.1 immediately, and (3.18) is a direct conclusion from (3.17).

**Lemma 3.6.** Under Assumption 2.1, for any $\mu_h \in \mathcal{M}_{h,k}^0$, it holds
\[ d_h(\mu_h, \mu_h) \lesssim \langle B^{-1}_h \mu_h, \mu_h \rangle_h, \] (3.19)
which implies
\[ \lambda_{\text{max}}(B_h D_h) \lesssim 1, \] (3.20)
where $\lambda_{\text{max}}(B_h D_h)$ denotes the maximum eigenvalue of $B_h D_h$. 

8
Proof. For any $v_h \in V_J$, an inverse estimate indicates

$$
\|\mu_h\|_h^2 \lesssim \|\mu_h - \Pi_h v_h\|_h^2 + \|\Pi_h v_h\|_h^2 \lesssim h^{-2} \|\mu_h - \Pi_h v_h\|_h^2 + \|\Pi_h v_h\|_h^2,
$$

which, together with Lemma 3.3, implies

$$
\|\mu_h\|_h^2 \lesssim \langle B^{-1}_h \mu_h, \mu_h \rangle_h.
$$

Then (3.19) follows from Assumption 2.1, and (3.20) is just a trivial conclusion from (3.19). \qed

From Lemmas 3.5-3.6, we obtain the main result of our paper for the estimation of the condition number of $B_h D_h$:

**Theorem 3.1.** Under Assumption 2.1 it holds

$$
\kappa(B_h D_h) \lesssim 1, \quad (3.21)
$$

where

$$
\kappa(B_h D_h) = \frac{\lambda_{\text{max}}(B_h D_h)}{\lambda_{\text{min}}(B_h D_h)},
$$

and $D_h, B_h$ are defined by (2.4), (3.4), respectively.

### 3.3 Implementation

We recall that $\{\eta_i : 1 \leq i \leq M\}$ is the standard nodal basis of $M_{h,k}^0$ and $\{\phi_{j,i} : i = 0, 1, \ldots, N_j\}$ is the standard nodal basis of $V_j$ for $j = 0, 1, \ldots, J$. For each $\mu_h \in M_{h,k}^0$, we use $\bar{\mu}_h \in \mathbb{R}^M$ to denote the vector of coefficients of $\mu_h$ with respect to the basis $\{\eta_1, \eta_2, \ldots, \eta_M\}$. Let $D_h \in \mathbb{R}^{M \times M}$ be the stiffness matrix with respective to the operator $D_h$ defined in (2.4) with

$$
\lambda^2_h D_h \bar{\mu}_h = \langle D_h \mu_h, \eta_h \rangle_h, \quad \forall \lambda_h, \mu_h \in M_{h,k}^0.
$$

Then it follows from Theorem 2.1 or the estimate (2.5), that

$$
\kappa(D_h) \lesssim h^{-2}.
$$

By the definition, (3.2), of $\Pi_h$, there exists a matrix $I_j \in \mathbb{R}^{M \times N_j}$ for $j = 0, 1, \ldots, J$, such that

$$
\Pi_h(\phi_{j,1}, \phi_{j,2}, \ldots, \phi_{j,N_j}) = (\eta_1, \eta_2, \ldots, \eta_M) I_j. \quad (3.22)
$$

We set $I_h \in \mathbb{R}^{M \times M}$ to be the identity matrix. From the definition, (3.4), of $B_h$, it follows, for any $\mu_h \in M_{h,k}^0$,

$$
B_h D_h \mu_h = h^{2-d} \sum_{i=1}^M (D_h \mu_h, \eta_i) h \eta_i + \sum_{(j,i) \in \Lambda} h^{2-d}(\Pi^*_h D_h \mu_h, \phi_{j,i}) \Pi_h \phi_{j,i}
$$

$$
= h^{2-d} \sum_{i=1}^M (D_h \mu_h, \eta_i) h \eta_i + \sum_{(j,i) \in \Lambda} h^{2-d}(D_h \mu_h, \Pi_h \phi_{j,i}) h \Pi_h \phi_{j,i}.
$$


Thus, in view of (3.22), we have
\[ \tilde{B}_h \tilde{D}_h \mu_h = B_h D_h \tilde{\mu}_h, \quad \forall \mu_h \in M_{h,k}^0, \quad (3.23) \]
where \( B_h \) is the matrix representation of the operator \( B_h \) given by
\[ B_h = h^{2-d} I + \sum_{k=0}^{J} h^{2-d} I_j I_j^t, \quad (3.24) \]

From Theorem 3.1 it follows
\[ \kappa(B_h D_h) \lesssim 1. \quad (3.25) \]
This means that the matrix \( B_h \) is an optimal preconditioner for the stiffness matrix \( D_h \).

**Remark 3.1.** From the definition (3.24), it’s easy to see that the preconditioner \( B_h \) preserves the advantage of the well-known BPX preconditioner, i.e., it’s optimal and perfect for parallel computation.

## 4 Applications

Firstly, let \( V(T) \) and \( W(T) \) be two local finite dimensional spaces for \( T \in T_h \). Define
\[
V_h := \{ v \in L^2(\Omega) : v_T |_{T} \in V(T), \quad \forall T \in T_h \},
\]
\[
W_h := \{ \tau \in [L^2(\Omega)]^d : \tau_T |_{T} \in W(T), \quad \forall T \in T_h \}.
\]

Then we introduce another two local spaces,
\[
M(F) := P_k(F), \quad \forall F \in \mathcal{F}_h,
\]
\[
M(\partial T) := \{ \mu \in L^2(\partial T) : \mu|_F \in P_k(F), \quad \forall \text{ face } F \text{ of } T \},
\]
and define a local projection operator \( P^0_T : H^1(T) \to M(\partial T) \) by
\[
(P^0_T v, \mu)_{\partial T} = \langle v, \mu \rangle_{\partial T}, \quad \forall v \in H^1(T), \forall \mu \in M(\partial T).
\]
We recall
\[ M_{h,k}^0 := \{ \mu_h \in L^2(\mathcal{F}_h) : \mu_h|_F \in M(F), \quad \forall F \in \mathcal{F}_h, \mu_h|_{\partial \Omega} = 0 \}. \quad (4.1) \]

### 4.1 Hybridized discontinuous Galerkin method

The general framework of HDG method for the problem (1.1) reads as follows ([4]): Seek \((u_h, \lambda_h, \sigma_h) \in V_h \times M_{h,k}^0 \times W_h\), such that
\[ (C \sigma_h, \tau_h) + (u_h, \text{div}_h \tau_h) - \sum_{T \in \mathcal{T}_h} \langle \lambda_h, \tau_h \cdot n \rangle_{\partial T} = 0, \quad (4.2a) \]
\[ -(v_h, \text{div}_h \sigma_h) + \sum_{T \in \mathcal{T}_h} \langle \alpha_T (P^0_T u_h - \lambda_h), v_h \rangle_{\partial T} = (f, v_h), \quad (4.2b) \]
\[ \sum_{T \in \mathcal{T}_h} \langle \sigma_h \cdot n - \alpha_T (P^0_T u_h - \lambda_h), \mu_h \rangle_{\partial T} = 0, \quad (4.2c) \]
hold for all \((v_h, \mu_h, \tau_h) \in V_h \times M^0_{h,k} \times W_h\), where \(C = A^{-1}\), \(\text{div}_h\) is the broken \(\text{div}\) operator defined by
\[
\text{div}_h \tau_h|T := \text{div}(\tau_h|T), \quad \forall \tau_h \in W_h, T \in T_h,
\]
and \(\alpha_T\) denotes a nonnegative penalty function defined on \(\partial T\).

For any \(T \in T_h\), we introduce two local problems as follows.

**Local problem 1:** For any given \(\lambda_h \in M(\partial T)\), seek \((u_{\lambda_h}, \sigma_{\lambda_h}) \in V(T) \times W(T)\) such that
\[
(C_{\lambda_h} \sigma_{\lambda_h}, \tau)|T + (u_{\lambda_h}, \text{div}\tau)|T = \langle \lambda_h, \tau \cdot n \rangle_{\partial T}, \quad (4.3a)
\]
\[
-(v, \text{div}\sigma_{\lambda_h})|T + \langle \alpha_T P^0_T u_{\lambda_h}, v \rangle_{\partial T} = \langle \alpha_T \lambda_h, v \rangle_{\partial T}, \quad (4.3b)
\]
hold for all \((v, \tau) \in V(T) \times W(T)\).

**Local problem 2:** For any given \(f \in L^2(T)\), seek \((u_f, \sigma_f) \in V(T) \times W(T)\) such that
\[
(C_{\lambda_h} \sigma_{\lambda_h}, \tau)|T + (u_f, \text{div}\tau)|T = 0, \quad (4.4a)
\]
\[
-(v, \text{div}\sigma_f)|T + \langle \alpha_T P^0_T u_f, v \rangle_{\partial T} = \langle f, v \rangle_{T}, \quad (4.4b)
\]
hold for all \((v, \tau) \in V(T) \times W(T)\).

**Theorem 4.1.** Suppose \((u_h, \lambda_h, \sigma_h) \in V_h \times M^0_{h,k} \times W_h\) to be the solution of the system
\[(1.3)\], and suppose, for any \(T \in T_h\), \((u_{\lambda_h}, \sigma_{\lambda_h})|T \in V(T) \times W(T)\) and \((u_f, \sigma_f)|T \in V(T) \times W(T)\) to be the solutions of the local problems \(4.3\) and \(4.4\), respectively. Then it holds
\[
\sigma_h|T = \sigma_{\lambda_h} + \sigma_f, \quad (4.5)
\]
\[
u_h|T = u_{\lambda_h} + u_f, \quad (4.6)
\]
and \(\lambda_h \in M^0_{h,k}\) is the solution of the system \(1.3\), i.e.
\[
d_h(\lambda_h, \mu_h) = b_h(\mu_h), \quad \forall \mu_h \in M^0_{h,k},
\]
where
\[
d_h(\lambda_h, \mu_h) := (C_{\lambda_h} \sigma_{\lambda_h}, \sigma_{\mu_h}) + \sum_{T \in T_h} \langle \alpha_T (P^0_T u_{\lambda_h} - \lambda_h), P^0_T u_{\mu_h} - \mu_h \rangle_{\partial T}, \quad (4.7)
\]
\[
b_h(\mu_h) := (f, u_{\mu_h}), \quad (4.8)
\]
and for any \(T \in T_h\), \((u_{\mu_h}, \sigma_{\mu_h})|T \in V(T) \times W(T)\) denotes the solution of the local problem \(4.3\) by replacing \(\lambda_h\) with \(\mu_h\).

We list in what follows several types of HDG methods for which Assumption \(2.1\) holds.

**Type 1.** \(V(T) = P_h(T), W(T) = \left[P_h(T)\right]^d + P_h(T)\mathbf{x}\) and \(\alpha_T = 0\). The corresponding HDG scheme \(4.2\) turns out to be the well-known hybridized RT mixed element method \(3\).
**Type 2.** \( V(T) = P_{k-1}(T) \) \((k \geq 1)\), \( W(T) = [P_k(T)]^d \) and \( \alpha_T = 0 \). The corresponding HDG method turns out to be the well-known hybridized BDM mixed element method ([14]).

For Types 1-2 HDG methods, it was shown in [24] that Assumption 2.1 holds.

**Type 3.** \( V(T) = P_k(T) \), \( W(T) = [P_k(T)]^d \) and \( \alpha_T = O(1) \). The corresponding HDG method was proposed in [4] and analyzed in [19]. It was shown in [21] that Assumption 2.1 holds.

**Type 4.** \( V(T) = P_{k+1}(T) \), \( W(T) = [P_k(T)]^d \) and \( \alpha_T = O(h_T^{-1}) \). The corresponding HDG method was proposed and analyzed in [30], where Assumption 2.1 was shown to hold.

**Remark 4.1.** It has been shown in [16, 17] that, when \( A \) is a piecewise constant matrix and \( k \geq 1 \), the stiffness matrices of the bilinear form \( d_h(\cdot, \cdot) \) arising from the hybridized RT mixed element method, i.e. the Type 1 HDG method, and from the corresponding hybridized BDM mixed element method, i.e. the Type 2 HDG method, are the same. Then any preconditioner for the Type 1 HDG method is also a preconditioner for the Type 2 HDG method, and vice versa.

### 4.2 Weak Galerkin method

We follow [10] to introduce the discrete weak gradients. Here we make a little modification, just for the sake of convenience.

At first, for \( T \in \mathcal{T}_h \) we define \( \nabla_i^w : V(T) \rightarrow W(T) \) by

\[
(\nabla_i^w v, q)_T = -(v, \text{div} q)_T, \; \forall q \in W(T).
\]

(4.9)

Secondly, we define \( \nabla^0_w : M(\partial T) \rightarrow W(T) \) by

\[
(\nabla^0_w \mu, q)_T = \langle \mu, q \cdot n \rangle_{\partial T}, \; \forall \mu \in M(\partial T), \; \forall q \in W(T).
\]

(4.10)

Then we define \( \nabla_w : V(T) \times M(\partial T) \rightarrow W(T) \) by

\[
\nabla_w (v, \mu) = \nabla^i_w v + \nabla^0_w \mu, \; \forall (v, \mu) \in V(T) \times M(\partial T).
\]

(4.11)

The WG method reads as follows: Seek \( (u_h, \lambda_h) \in V_h \times M^0_{h,h} \) such that

\[
\begin{align*}
(A(\nabla^i_w u_h + \nabla^0_w \lambda_h), \nabla_w v_h) + \sum_{T \in \mathcal{T}_h} \langle \alpha_T (P^0_T u_h - \lambda_h), v_h \rangle_{\partial T} &= (f, v_h), \\
(A(\nabla^i_w u_h + \nabla^0_w \lambda_h), \nabla_w \mu_h) - \sum_{T \in \mathcal{T}_h} \langle \alpha_T (P^0_T u_h - \lambda_h), \mu_h \rangle_{\partial T} &= 0,
\end{align*}
\]

(4.12a)

(4.12b)

hold for \( (v_h, \mu_h) \in V_h \times M^0_{h,h} \), where \( \alpha_T \) denotes a nonnegative penalty function defined on \( \partial T \).

We shall follow the same routine as in the previous subsection. We begin with defining two local problems as follows.

**Local problem 1:** For any given \( f \in L^2(T) \), seek \( u_f \in V(T) \) such that

\[
(A \nabla_w u_f, \nabla_w v)_T + \langle \alpha_T P^0_T u_f, P^0_T v \rangle_{\partial T} = (f, v)_T, \; \forall v \in V(T).
\]

(4.13)
Local problem 2: For any $\lambda h \in M(\partial T)$, seek $u_{\lambda h} \in V(T)$ such that

$$
(A \nabla w u_{\lambda h}, \nabla w^i v)_T + \langle \alpha_T P_T^0 u_{\lambda h}, P_T^0 v \rangle_{\partial T} = -(A \nabla w \lambda h, \nabla w^i v)_T + \langle \alpha_T \lambda h, P_T^0 v \rangle_{\partial T}, \forall v \in V(T).
$$

(4.14)

Similar to Theorem 4.1, the following conclusion holds.

**Theorem 4.2.** Suppose $(u_h, \lambda h) \in V_h \times M_{h,k}^0$ to be the solution of the system (4.12), and suppose, for any $T \in T_h$, $u_f$ and $u_{\lambda h}$ to be the solutions of the local problems (4.13) and (4.14), respectively.

Then it holds

$$
u_h = u_{\lambda h} + u_f,
$$

(4.15)

and $\lambda h \in M_{h,k}^0$ is the solution of the system (1.3), i.e.

$$d_h(\lambda h, \mu h) = b_h(\mu h), \forall \mu h \in M_{h,k}^0,
$$

where

$$
d_h(\lambda h, \mu h) := (A \nabla w u_{\lambda h, \lambda h}, \nabla w(u_{\mu h, \mu h})) + \sum_{T \in T_h} (\alpha_T (P_T^0 u_{\lambda h} - \lambda h), P_T^0 u_{\mu h} - \mu h)_{\partial T},
$$

(4.16)

$$b_h(\mu h) := (A \nabla w^i u_{\mu h, \mu h}, \nabla w^i u_f).
$$

(4.17)

We consider two basic cases of the WG method (4.12) (10):

- $V(T) = P_h(T)$, $W(T) = [P_k(T)]^d + P_K(T)x$ and $\alpha_T = 0$;
- $V(T) = P_{k-1}(T)$ ($k \geq 1$), $W(T) = [P_k(T)]^d$ and $\alpha_T = 0$.

In both cases, we can prove that Assumption 2.1 holds by using a similar technique used in [24].

**Remark 4.2.** We note that the reduced system (1.3) is nothing but the Schur complement system of the WG method.

**Remark 4.3.** If $A$ is a piecewise-constant matrix, the two WG methods are equivalent to the hybridized RT mixed element method and the hybridized BDM mixed element method, respectively.

We refer to (Remark 2.1, [29]) for the details.

### 4.3 Nonconforming finite element method

In this section we take Crouzeix-Raviart element method [22] as an example to show that the theory in Section 3 also applies to nonconforming methods.

At first, we introduce the Crouzeix-Raviart finite element space $L_{1}^{CR}(T_h)$ as follows.

$$
L_{1}^{CR}(T_h) := \{v_h \in L^2(\Omega) : v_h|_T \in P_1(T), \forall T \in T_h, v_h \text{ is continuous at the gravity point of each interior face of } T_h \text{ and vanishes at the gravity point of each face of } T_h \text{ that lies on } \partial \Omega\}.
$$

(4.18)
As we know, the standard discretization of CR element method reads as follows: Seek \( u_h \in L^1_c(T_h) \) such that
\[
(\mathbf{A} \nabla_h u_h, \nabla_h v_h) = (f, v_h), \quad \forall v_h \in L^1_c(T_h),
\]
where \( \nabla_h v_h \) is given by
\[
\nabla_h v_h |_T := \nabla(v_h |_T), \quad \forall T \in T_h.
\]
We define an operator \( \tilde{\Pi}_h : L^1_c(T_h) \to M^0_{h,0} \) by
\[
\tilde{\Pi}_h v_h |_F := \frac{1}{|F|} \int_F v_h = v_h(c_F), \quad \forall F \in F_h.
\]
where \( c_F \) denotes the gravity point of \( F \). Obviously, \( \tilde{\Pi}_h \) is a bijective map, and its inverse map \( \Pi^{-1}_h : M^0_{h,0} \to L^1_c(T_h) \) satisfies
\[
\int_F \tilde{\Pi}^{-1}_h \mu_h = \int_F \mu_h, \quad \forall F \in F_h.
\]
By denoting \( \mu_h := \tilde{\Pi}_h v_h, \lambda_h := \Pi u_h \), the system (4.19) is equivalent to the system (1.3), i.e. \( \lambda_h \in M^0_{h,k} \) satisfies
\[
d_h(\lambda_h, \mu_h) = b_h(\mu_h), \quad \forall \mu_h \in M^0_{h,k},
\]
where
\[
d_h(\lambda_h, \mu_h) := (\mathbf{A} \nabla_h \tilde{\Pi}^{-1}_h \lambda_h, \nabla_h \tilde{\Pi}^{-1}_h \mu_h), \quad b_h(\mu_h) := (f, \tilde{\Pi}^{-1}_h \mu_h).
\]
By similar estimates as in Lemma 3.2, it is easy to verify Assumption 2.1 in this case.

**Remark 4.4.** When \( \mathbf{A} \) is a piecewise constant matrix, we can show that the Type 4 HDG method described in Subsection 4.1 with \( k = 0 \) is of the same stiffness matrix as the CR method (4.19). In fact, in this case we have, for any \( T \in T_h \),
\[
M(\partial T) := \{ \mu \in L^2(\partial T) : \mu |_F \in P_0(F), \forall \text{ face } F \text{ of } T \},
\]
\[
V(T) = P_1(T), \quad W(T) = [P_0(T)]^d, \quad \alpha_T = O(h_T^{-1}).
\]
From the relation (4.3b) it follows
\[
(\alpha_T (P^0_T(u_\lambda - \lambda), P^0_T v - \mu)_{\partial T} = 0, \quad \forall (v, \mu) \in V(T) \times M(\partial T).
\]
Thus, in view of (4.7), it holds
\[
d_h(\lambda_h, \mu_h) = (C \sigma_{\lambda_h}, \sigma_{\mu_h}).
\]
On the other hand, the relation (4.3a), together with (4.21) and integration by parts, yield
\[
(C \sigma_{\mu_h}, \tau)_T = (\mu_h, \tau \cdot n)_{\partial T} = (\tilde{\Pi}^{-1}_h \mu_h, \tau \cdot n)_{\partial T} = (\nabla \tilde{\Pi}^{-1}_h \mu_h, \tau)_T
\]
for $\tau \in W(T)$. Since $C = A^{-1}$ is a constant matrix on $T$, the above equality means
\[ \nabla_h \tilde{\Pi}_h^{-1} \mu_h = C\sigma_h. \]

This relation, together with (4.24) and (4.22), shows the Type 4 HDG method and the nonconforming CR element method results in the same stiffness matrix of $d_h(\cdot, \cdot)$.

**Remark 4.5.** As shown in [16, 17], when $A$ is a piecewise constant matrix, the stiffness matrix of $d_h(\cdot, \cdot)$ arising from the lowest order hybridized RT mixed finite element method, i.e. the Type 1 HDG method in Subsection 4.1 with $k = 0$, is the same as the one arising from the CR element method.

**Remark 4.6.** From Remarks 4.3-4.5, we know that when $A$ is a piecewise constant matrix and $k = 0$, the four methods, namely the Type 1 and Type 4 HDG methods in Subsection 4.1, the first WZ method Subsection 4.2, and the nonconforming CR element method, lead to the same stiffness matrix of $d_h(\cdot, \cdot)$, and then share the same optimal preconditioner.

## 5 Numerical experiments

In this section, we report some numerical experiments in two-space dimensions to verify the theoretical result of Theorem 3.1.

We consider two types of domains: a square domain $\Omega = (0, 1) \times (0, 1)$ (Figure 1) and an $L$-type domain $\Omega = (-1, 1) \times (0, 1) \cup (0, 1) \times (-1, 0)$ (Figure 2). Given a coarse triangulation $T_0$ of $\Omega$ as in Figure 1 or Figure 2, we produce a sequence of triangulations $\{T_j : j = 0, 1, \ldots, 6\}$ by bisection, i.e. connecting the midpoints of three edges of each simplex. Suppose $D_h$ to be the stiffness matrix with respect to the operator $D_h$ defined by (2.4), and $B_h$ to be the preconditioner defined by (3.24).

For the diffusion tensor $A$, we consider two cases:

**Case 1.** We set $A = I$, with $I$ the identity matrix.

In this case we compute the condition numbers of the system (1.3) and its preconditioned one arising from the Types 3-4 HDG methods with $k = 0$:

- $V(T) = P_0(T)$, $M(F) = P_0(F)$, $W(T) = [P_0(T)]^2$ and $\alpha_T = 1$;
- $V(T) = P_1(T)$, $M(F) = P_0(F)$, $W(T) = [P_0(T)]^2$ and $\alpha_T|F = \frac{1}{2}$ for each face $F$ of $T$.

We list the numerical results of $\kappa(D_h)$ and $\kappa(B_hD_h)$ in Tables 12 for the square domain and $L$-type domain cases, respectively.

**Case 2.** We set
\[ A = \begin{cases} 
1 + \sin(\pi x)^2 \sin(\pi y)^2 & 0 \\
0 & 1 + \sin(\pi x)^2 \sin(\pi y)^2 
\end{cases} \]  
(5.1)

15
We compute the condition numbers of the system (1.3) and its preconditioned one arising from the nonconforming CR element method, and list the corresponding numerical results in Table 3.

From Tables 1-3 we can see that, for all cases, $\kappa(D_h)$ is of $O(h^{-2})$ and $\kappa(B_hD_h)$ is of $O(1)$, which show that our proposed preconditioner is optimal.

![Figure 1: $T_0$ (left) and $T_1$ (right) on square domain](image1)

![Figure 2: $T_0$ (left) and $T_1$ (right) on L-type domain](image2)

Table 1: Condition numbers for types 3-4 HDG methods with $k = 0$: square domain

| Method    | h      | 1/2 | 1/4 | 1/8 | 1/16 | 1/32 |
|-----------|--------|-----|-----|-----|------|------|
| Type 3    | $\kappa(D_h)$ | 15.5 | 59.0 | 233.8 | 933.6 | 3733.7 |
|           | $\kappa(B_hD_h)$ | 6.8 | 10.5 | 12.4 | 13.6 | 14.3 |
| Type 4    | $\kappa(D_h)$ | 15.9 | 59.4 | 234.5 | 934.8 | 3735.0 |
|           | $\kappa(B_hD_h)$ | 7.2 | 10.9 | 12.6 | 13.7 | 14.4 |

16
Table 2: Condition numbers for types 3-4 HDG methods with \( k = 0 \): \( L \)-type domain

| Method  | \( h \) | 1/2 | 1/4 | 1/8 | 1/16 | 1/32 |
|---------|--------|-----|-----|-----|------|------|
| Type 3  | \( \kappa(D_h) \) | 31.4 | 120.9 | 478.8 | 1911.0 |
|         | \( \kappa(B_hD_h) \) | 13.2 | 16.2 | 18.5 | 20.1 |      |
| Type 4  | \( \kappa(D_h) \) | 32.4 | 122.8 | 482.8 | 1919.2 |
|         | \( \kappa(B_hD_h) \) | 13.9 | 16.6 | 18.7 | 20.3 |      |

Table 3: Condition numbers for C-R element method

| Domain   | \( h \) | 1/2 | 1/4 | 1/8 | 1/16 | 1/32 |
|----------|--------|-----|-----|-----|------|------|
| square   | \( \kappa(D_h) \) | 21.2 | 91.0 | 382.6 | 1572.4 | 6384.0 |
|          | \( \kappa(B_hD_h) \) | 9.8  | 16.3 | 20.4 | 23.5 | 25.6 |
| \( L \)-type | \( \kappa(D_h) \) | 40.6 | 186.1 | 788.3 | 3230.0 | -    |
|          | \( \kappa(B_hD_h) \) | 18.1 | 25.2 | 30.7 | 34.9 | -    |

References

[1] T. ARBOGAST, Z. CHEN, On the implementation of mixed methods as nonconforming methods for second-order elliptic problems, Math. Comp., 64 (1995), 943-972.
[2] R. A. ADAMS, J. J. F. FOURNIER, Sobolev Spaces, Academic Press, 2nd ed., 2003.
[3] D. N. ARNOLD, F. BREZZI, Mixed and non-conforming finite element methods: implementation, post-processing and error estimates, Modél. Math. Anal. Numér., 19 (1985), 7-35.
[4] D. N. ARNOLD, F. BREZZI, B. COCKBURN, L. D. MARINI, Unified analysis of discontinuous Galerkin methods for elliptic problems, SIAM J. Numer. Anal., 39 (2002), 1749-1779.
[5] I. BABUSKA, J. ODEN, J. LEE, Mixed-hybrid finite element approximations of second-order elliptic boundary-value problems, Comput. Methods Appl. Mech. Engrg., 11 (1977), 175-206.
[6] D. BRAESS, R. VERFÜRTH, Multigrid methods for nonconforming finite element methods, SIAM J. Numer. Anal., 27(1990), 979-986.
[7] J. H. BRAMBLE, Multigrid Methods, Pitman Research Notes in Mathematics, V. 294, John Wiley and Sons, 1993.
[8] J. H. BRAMBLE, D.Y. KWAK, J. E. PASCIAK, Uniform convergence of multigrid V-cycle iterations for indefinite and nonsymmetric problems, SIAM J. Numer. Anal., 31 (1994), 1746-1763.
[9] J. H. BRAMBLE, J. E. PASCIAK, New estimates for multilevel algorithms including the Vcycle, Math. Comp., 60 (1993) 447-471.
[10] J. H. BRAMBLE, J. E. PASCIAK, J. XU, Parallel multilevel preconditioners, Math. Comp., 55 (1990), 1-22.

[11] J. H. BRAMBLE, X. ZHANG, Uniform convergence of the multigrid V-cycle for an anisotropic problem, Math. Comp., 70 (2001), 979-986.

[12] S. C. BRENNER, An optimal-order multigrid method for P1 nonconforming finite elements, Math. Comp., 52 (1989), 1-16.

[13] S. C. BRENNER, A multigrid algorithm for the lowest-order Raviart-Thomas mixed triangular finite element method, SIAM J. Numer. Anal., 29 (1992), 647-678.

[14] F. BREZZI, J. DOUGLAS, JR., L. D. MARINI, Two families of mixed finite elements for second order elliptic problems, Numer. Math., 47 (1985), 217-235.

[15] L. CHEN, J. WANG, Y. WANG, X. YE, An auxiliary space multigrid preconditioner for the weak Galerkin method, arXiv preprint [arXiv:1410.1012], 2014.

[16] B. COCKBURN, J. GOPALAKRISHNAN, A characterization of hybridized mixed methods for second order elliptic problems, SIAM J. Numer. Anal., 42 (2004), 283-301.

[17] B. COCKBURN, J. GOPALAKRISHNAN, New hybridization techniques, GAMM-Mitt., 2 (2005), 154-183.

[18] B. COCKBURN, B. DONG, J. GUZMÁN, A superconvergent LDG-hybridizable Galerkin method for second-order elliptic problems, Math. Comp., 77 (2008), 1887-1916.

[19] B. COCKBURN, J. GOPALAKRISHNAN, F. J. SAYAS, A projection-based error analysis of HDG methods, Math. Comp., 79 (2010), 1351-1367.

[20] B. COCKBURN, J. GOPALAKRISHNAN, R. LAZAROV, Unified hybridization of discontinuous Galerkin, mixed, and conforming Galerkin methods for second order elliptic problems, SIAM J. Numer. Anal., 47 (2009), 1319-1365.

[21] B. COCKBURN, O. DUBOIS, J. GOPALAKRISHNAN, Multigrid for an HDG Method. IMA J. Numer. Anal. 2013, doi: 10.1093/imanum/drt024.

[22] M. CROUZEIX, P. A. RAVIART, Conforming and nonconforming finite element methods for solving the stationary stokes equations, RAIRO Modél. Math. Anal. Numér., 7 (1973), 33-75.

[23] H. DUAN, S. GAO, R. TAN, S. ZHANG, A generalized BPX multigrid framework covering nonnested V-cycle methods, Math. Comp., 76 (2007), 137-152

[24] J. GOPALAKRISHNAN, A Schwarz preconditioner for a hybridized mixed method, Comput. Meth. Appl. Math., 3 (2003), 116-134.

[25] J. GOPALAKRISHNAN, A convergent multigrid cycle for the hybridized mixed method, Numer. Linear Algebra Appl., 16 (2009), 689-714.
[26] J. GOPALAKRISHNAN, G. KANSCHAT, A multilevel discontinuous Galerkin method, Numer. Math., 95 (2003) 527-550.

[27] J. KRAUS, S. MARGENOV, J. SYNKA, On the multilevel preconditioning of Crouzeix-Raviart elliptic problems, Numer. Linear Algebra Appl., 15(2008), 395-416.

[28] R.D. LAZAROV, S.D. MARGENOV, On a two-level parallel MIC (0) preconditioning of Crouzeix-Raviart non-conforming FEM systems, Numer. Meth. App., Lecture Notes in Computer Science, 2542(2003), 192-201.

[29] B. LI, X. XIE, Multigrid weak Galerkin finite element method for diffusion problems, arXiv preprint arXiv:1405.7506, 2014.

[30] B. LI, X. XIE, Analysis of a family of HDG methods for second order elliptic problems, arXiv preprint arXiv:1408.5545, 2014.

[31] L. MU, J. WANG, Y. WANG, X. YE, A computational study of the weak Galerkin method for second-order elliptic equations, arXiv:1111.0618v1, 2011, Numerical Algorithms, 2012, DOI:10.1007/s11075-012-9651-1.

[32] L. MU, J. WANG, X. YE, A weak Galerkin finite element methods with polynomial reduction, arXiv:1304.6481, 2013.

[33] J. ODEN, J. LEE, Dual-Mixed Hybrid finite element method for second-order elliptic problems, Lecture Notes in Mathematics 606 (1977), 275-291.

[34] T. RAHMAN, X. XU, R. HOPPE, Additive Schwarz methods for the Crouzeix-Raviart mortar finite element for elliptic problems with discontinuous coefficients Numer. Math., 101(2005), 551-572.

[35] P.-A. RAVIART, J. M. THOMAS, A mixed finite element method for second order elliptic problems, Mathematical Aspects of Finite Element Method (I. Galligani and E. Magenes, eds.), Lecture Notes in Math. 606, Springer-Verlag, New York, 1977, 292-315.

[36] P. A. RAVIART, J. M. THOMAS, Primal hybrid finite element methods for 2nd order elliptic equations[J], Mathematics of computation, 31 (1977), 391-413.

[37] P. A. RAVIART, J. M. THOMAS, Dual finite element models for second order elliptic problems[J]. Energy methods in finite element analysis.(A 79-53076 24-39) Chichester, Sussex, England, Wiley-Interscience, 1979, 175-191.

[38] F. WANG, J. CHEN, P. HUANG, A multilevel preconditioner for the C-R FEM for elliptic problems with discontinuous coefficients, Sci. China Math., 55(2012),1513-1526.

[39] J. WANG, Convergence analysis of multigrid algorithms for nonselfadjoint and indefinite elliptic problems, SIAM J. Numer. Anal., 30 (1993), 275-285.

[40] J. WANG, X. YE, A weak Galerkin finite element method for second-order elliptic problems, J. Comp. and Appl. Math, 241 (2013), 103-115.
[41] J. XU, The auxiliary space method and optimal multigrid preconditioning techniques for unstructured meshes. Computing, 56 (1996), 215-235.

[42] J. XU, L. CHEN, R. H. NOCHETTO, Optimal multilevel methods for $H(\text{grad})$, $H(\text{curl})$, and $H(\text{div})$ systems on graded and unstructured grids, Multiscale, Nonlinear and Adaptive Approximation, 2009, 599-659.

[43] J. XU, Y. ZHU, Uniform convergent multigrid methods for elliptic problems with strongly discontinuous coefficients. Math. Models Methods Appl. Sci., 18 (2008), 77-105.

[44] J. XU, L. ZIKATANOV, The method of alternating projections and the method of subspace corrections in Hilbert space, J. Amer. Math. Soc., 15 (2002), 573-597.

[45] X. XU, J. CHEN, Multigrid for the mortar element method for $P_1$ nonconforming element, Numer. Math., 88 (2001), 381-398.

[46] Y. ZHU, Analysis of a multigrid preconditioner for Crouzeix-Raviart discretization of elliptic PDE with jump coefficient, Numer. Linear Algebra Appl., 21 (2014), 2438.