Ashtekar Variables and Matter Coupling

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Abstract

It has been shown for low-spin fields that the use of only the self-dual part of the connection as basic variable does not lead to spurious equations or inconsistencies. We slightly generalize the form of the chiral Lagrangian of half-integer spin fields and express its imaginary part in a simple form. If the imaginary part is non-vanishing, it will lead to spurious equations. As an example, for (Majorana) Rarita-Schwinger fields the equations of motion of the torsion is solved and it is shown that it vanishes owing to the Fierz identity.

1. Introduction

In the mid-1980s Ashtekar has presented a new formulation of general relativity from a non-perturbative point of view, in terms of which all the constraints of the gravity become simple polynomials of the canonical variables [1]–[4]. This formulation can be extended to include matter sources. In particular, in the cases of spin-1/2 fields [4, 5] and N = 1 supergravity [3, 6], the constraints are again polynomials of the canonical variables. On the other hand, the Ashtekar formulation of N = 1 supergravity was reformulated in the method of the two-form gravity [7]. The progress of the Ashtekar formalism can be traced from the references compiled in [8].

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Here the complex chiral action using the self-dual connection and the tetrads $e^i_\mu$ is

$$S^{(+)} = \int d^4 x \, e R^{(+)} + \text{[matter terms]},$$

(1.1)

where $e = \det(e^i_\mu)$ and

$$R^{(+)} := \frac{1}{2} \left( R - \frac{i}{2} \varepsilon^{ijkl} R_{\mu
u} e^\mu_i e^\nu_j \right).$$

(1.2)

Greek letters $\mu, \nu, \cdots$ are space-time indices, and Latin letters $i, j, \cdots$ are local Lorentz indices. Using the Bianchi identity, the chiral action (1.1) is reexpressed as

$$S^{(+)} = \frac{1}{2} \int d^4 x \, e R(e) + \text{[quadratic terms of torsion]} + \text{[matter terms]}.$$  

(1.3)

The first term of (1.3) is the Einstein-Hilbert action. Since the torsion equals zero in the source-free case, the complex chiral action is equivalent to the Einstein-Hilbert action. In this paper we shall take the form of matter terms to be slightly general, and analyze the consistency of the field equations.

In Sec.2 we will contemplate the half-integer spin fields minimally coupled to gravity, and introduce a slightly general form of the complex chiral Lagrangian of matter fields. In Sec.3 we will argue if the field equations are influenced by the imaginary part of the chiral Lagrangian. In the final section our result is summarized.

### 2. Lagrangian of Matter Fields

In the source-free case, although the chiral gravitational Lagrangian is complex, the imaginary part of the Lagrangian turns out to be simply the Bianchi identity. Since the Lagrangian of integer spin fields (spin-0,1) does not contain the Lorentz connection, the situation remains unaltered. Therefore, let us consider half-integer spin fields ($\psi$ and $\psi^i_\mu$ for spin-1/2 and spin-3/2 fields, respectively) as matter fields.

Firstly, let us suppose that the matter Lagrangian is obtained by the minimal prescription; namely, by replacing ordinary derivatives by covariant derivatives,

$$\partial_i \rightarrow e^i_\mu D_\mu$$

(2.1)
with
\[ D_\mu = \partial_\mu + \frac{i}{2} A_{ij\mu} S^{ij}. \] (2.2)

Here \( e^\mu_i \) is a tetrad field, \( A_{ij\mu} \) means the Lorentz connection, and \( S_{ij} \) stands for the \( \text{SL}(2,\mathbb{C}) \) generator. The Lorentz connection \( A_{ij\mu} \) is divided into the Ricci rotation coefficients \( A_{ij\mu}(e) \) and contorsion tensor \( K_{ij\mu} \),
\[ A_{ij\mu} = A_{ij\mu}(e) + K_{ij\mu}. \] (2.3)

Secondly, we suppose that the chiral Lagrangian of matter fields is described by using the self-dual part of the Lorentz connection. According to the equation
\[ A^{(+)}_{ij\mu} S^{ij} = A_{ij\mu} S^{ij} \frac{1 + \gamma_5}{2}, \] (2.4)
this demand that only the terms expressed by \( D_\mu \psi_R \) and \( \bar{\psi}_L D_\mu \psi \) should appear in the matter Lagrangian. \( \psi_R (\psi_L) \) is the right (left)-handed spinor field;
\[
\begin{align*}
\psi_R &= \frac{1 + \gamma_5}{2} \psi, \\
\psi_L &= \frac{1 - \gamma_5}{2} \psi.
\end{align*}
\] (2.5)

Let us consider a (Majorana) Rarita-Schwinger field. (Spin-1/2 fields can be considered in the similar manner.) Its Lagrangian in Minkowski space is
\[ L_{RS} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_5 \gamma_\rho \partial_\sigma \psi_\nu. \] (2.6)

Applying the minimal prescription to (2.6), the Lagrangian density of a Rarita-Schwinger field becomes
\[ \mathcal{L}_{RS} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_5 \gamma_\rho D_\sigma \psi_\nu, \] (2.7)
which is used in \( N = 1 \) supergravity [9]-[11].

* In our convention \( S_{ij} = \frac{1}{4} \{ \gamma_i, \gamma_j \} \) and \( \{ \gamma_i, \gamma_j \} = -2 \eta_{ij} \). We denote the Minkowski metric by \( \eta_{ij} = \text{diag}(-1, +1, +1, +1) \). \( \epsilon_{\mu\nu\rho\sigma} \) is the totally antisymmetric tensor normalized as \( \epsilon_{0123} = +1 \).
On the other hand, the Lagrangian (2.6) can be rewritten as

\[ L_{RS} = -\epsilon^{\mu\nu\rho\sigma} \overline{\psi}_{R\mu} \gamma_\rho \partial_\sigma \psi_{R\nu} + [\text{tot. div. term}]. \]  

(2.8)

The minimal prescription applied to (2.8) leads to the chiral Lagrangian density,

\[ L_{RS}^{(+)} = -e \epsilon^{\mu\nu\rho\sigma} \overline{\psi}_{R\mu} \gamma_\rho D_\sigma \psi_{R\nu}. \]  

(2.9)

Here the total divergence term in (2.8) is discarded. (The chiral Lagrangian density for the antiself-dual connection can be similarly defined by using the left-handed spinor field.)

The Lagrangian densities (2.7) and (2.9) can respectively be reexpressed as

\[ \mathcal{L}_{RS} = \mathcal{L}_{RS}(e) + \frac{i}{4} e \epsilon^{\mu\nu\rho\sigma} \overline{\psi}_\mu \gamma_5 \gamma_\rho K_{ij} S^{ij} \psi_\nu, \]  

(2.10)

\[ \mathcal{L}_{RS}^{(+)} = \mathcal{L}_{RS}(e) + \frac{i}{2} e \epsilon^{\mu\nu\rho\sigma} \overline{\psi}_\mu \gamma_5 \gamma_\rho K_{ij}^{(+)} S^{ij} \psi_\nu, \]  

(2.11)

where \( K_{ij}^{(+)} \) is the self-dual part of contorsion tensor \( K_{ij} \). In (2.11) a total divergence term has been omitted. We notice that the factor of the second term in (2.11) is twice that in (2.10).

Let us consider matter fields of which the real Lagrangian density is written as

\[ \mathcal{L}_{M} = \mathcal{L}_{M}(e) + e X_{ijk} K_{ijk}, \]  

(2.12)

where \( X_{ijk} = X_{[ijk]} \) is a real tensor made of matter fields. Based on the case for spin-3/2 (and spin-1/2) fields, we assume that the complex chiral Lagrangian density of those matter fields is expressed as follows;

\[ \mathcal{L}_{M}^{(+)} = \mathcal{L}_{M}(e) + 2e X_{ijk} K_{ijk}^{(+)} \]  

(2.13)

It is noted that only the self-dual part \( X_{ijk}^{(+)} \) contributes in (2.13).

In order to analyze the Lagrangian densities (2.12) and (2.13), it is convenient to decompose \( K_{ijk} \) into irreducible parts [12]:

\[ K_{ijk} = u_{ijk} - \frac{2}{3} \eta_{k[i} v_{j]} + \frac{1}{2} \epsilon_{ijk} a^l, \]  

(2.14)

\[ \dagger \] The \( u_{ijk} \) of (2.14) is related to \( t_{ijk} \) of Ref. [13] by \( u_{k(ij)} = t_{ijk} \).
where the tensor $u_{ijk} = u_{[ijk]}$ is traceless and

$$
\varepsilon^{ijkl}u_{jkl} = 0. \tag{2.15}
$$

Here the vector $v_i$ is

$$
v_i = K_{ij}^j, \tag{2.16}
$$

and the axial vector $a_i$ is

$$
a_i = \frac{1}{3}\varepsilon_{ijkl}K^{jkl}. \tag{2.17}
$$

In the same way, $K_{ijk}^{(+)}$ is decomposed as

$$
K_{ijk}^{(+)} = u_{ijk}^{(+)} - \frac{2}{3}\eta_{[i}^{(+)}v_{j]}^{(+)} + \frac{1}{2}\varepsilon_{ijkl}a_{l}^{(+)}, \tag{2.18}
$$

where $u_{ijk}^{(+)}$ is self-dual part of $u_{ijk}$, and

$$
a_i^{(+)} = \frac{2}{3}iv_i^{(+)} = \frac{1}{2}(a_i + \frac{2}{3}iv_i). \tag{2.19}
$$

Substituting (2.14) and (2.18) into (2.12) and (2.13), respectively, the matter Lagrangian densities can be represented as

$$
\mathcal{L}_M = \mathcal{L}_M(e) + e(B_{ijk}u_{ijk}^{(+)} + C_i v_i^{(+)} + D_i a_i^{(+)}), \tag{2.20}
$$

$$
\mathcal{L}_M^{(+)} = \mathcal{L}_M(e) + 2e(B_{ijk}u_{ijk}^{(+)} + C_i v_i^{(+)} + D_i a_i^{(+)}), \tag{2.21}
$$

where $B_{ijk} = B_{[ijk]}, C_i$ and $D_i$ are related to $X_{ijk}$. We notice that only the self-dual part $B_{ijk}^{(+)}$ contributes in (2.21).

### 3. Consistency of the field equations

Let us write the quadratic terms of the torsion in (1.3) explicitly. The gravitational Lagrangian density $\mathcal{L}_G = eR$ is represented as follows;

$$
\mathcal{L}_G = \mathcal{L}_G(e) + \frac{e}{2} \left( \frac{1}{2} u^2 - \frac{2}{3} v^2 + \frac{3}{2} a^2 \right), \tag{3.1}
$$
where \( u^2 = u_{ijk} u_{ijk} \), etc. We define the total Lagrangian density as the sum of \( (3.1) \) and \( (2.20) \);

\[
\mathcal{L} = \mathcal{L}(e) + \frac{e}{2} \left( \frac{1}{2} u^2 - \frac{2}{3} v^2 + \frac{3}{2} a^2 \right) + e (B_{ijk} u_{ijk} + C_i v^i + D_i a^i). \quad (3.2)
\]

On the other hand, the chiral gravitational Lagrangian \( \mathcal{L}_G^{(+)} = eR^{(+)} \) of \( (1.1) \) becomes

\[
\mathcal{L}_G^{(+)} = \mathcal{L}_G(e) + e \left( \frac{1}{2} u^{(+)}_{}^2 - \frac{2}{3} v^{(+)}^2 + \frac{3}{2} a^{(+)}^2 \right), \quad (3.3)
\]

and the total chiral Lagrangian density is

\[
\mathcal{L}^{(+)} = \mathcal{L}^{(+)}(e) + e \left( \frac{1}{2} u^{(+)}_{}^2 - \frac{2}{3} v^{(+)}^2 + \frac{3}{2} a^{(+)}^2 \right) + 2e (B_{ijk} u^{(+)}_{ijk} + C_i v^{(+)}^i + D_i a^{(+)}^i), \quad (3.4)
\]

which is the sum of \( (3.3) \) and \( (2.21) \).

To begin with, let us consider the equations of motion for \( u_{ijk}, v_i \) and \( a_i \). The equations of motion derived from \( (3.2) \) are

\[
\begin{cases}
  u_{ijk} = -2B_{ijk}, \\
  v_i = \frac{3}{2} C_i, \\
  a_i = -\frac{2}{3} D_i.
\end{cases} \quad (3.5)
\]

On the other hand, regarding \( u_{ijk}^{(+)} \) and \( v_i^{(+)} \) as independent variables in \( (3.4) \), we obtain the equations of motion as

\[
\begin{cases}
  u_{ijk}^{(+)} = -2B_{ijk}^{(+)}, \\
  v_i^{(+)} = \frac{3}{4} \left( C_i + \frac{2}{3} i D_i \right),
\end{cases} \quad (3.6)
\]

where \( B_{ijk}^{(+)} \) is self-dual part of \( B_{ijk} \). It can be shown that \( u_{ijk}, v_i \) and \( a_i \) derived from \( (3.6) \) coincide with \( (3.5) \). If \( u_{ijk}, v_i \) and \( a_i \) are regarded as independent variables in \( \mathcal{L}^{(+)} \) of \( (3.4) \), both \( \text{Re}\mathcal{L}^{(+)} \) and \( \text{Im}\mathcal{L}^{(+)} \) give the same result as \( (3.3) \); namely, although \( \mathcal{L}^{(+)} \) is complex, extra conditions do not appear.

Next, let us turn to investigate the detailed form of \( \text{Im}\mathcal{L}^{(+)} \). Using the solution \( (3.5) \) in \( (3.4) \), \( \text{Im}\mathcal{L}^{(+)} \) can be written in terms of \( u_{ijk}, v_i \) and \( a_i \) as

\[
\text{Im}\mathcal{L}^{(+)} = \frac{e}{8} (\epsilon_{ijmn} u_{ijk} u^{mn}_k - 8v_i a^i). \quad (3.7)
\]
After a little calculation, we get
\[ \text{Im} \mathcal{L}^{(+)} = \frac{e}{8} \epsilon_{ijmn} T^{kij} T^{mn}_k, \] (3.8)

where \( T_{ijk} = K_{ijk} - K_{ikj} \) is the torsion tensor.

In the case of spin-1/2 fields,
\[
\begin{align*}
  u_{ijk} &= 0, \\
  v_i &= 0, \\
  a_i &= \bar{\psi} \gamma^5 \gamma_i \psi.
\end{align*}
\] (3.9)

Substituting (3.9) into (3.7), we obtain \( \text{Im} \mathcal{L}^{(+)} = 0 \). For a (Majorana) Rarita-Schwinger field, we have
\[
\begin{align*}
  u_{ijk} &= -i \frac{4}{3} \epsilon_{ij} \epsilon_{mnkr} \bar{\psi}^m \gamma_s \psi^n, \\
  v_i &= \frac{i}{2} \bar{\psi}_i \gamma^j \psi_j, \\
  a_i &= -i \frac{12}{3} \epsilon_{ijkl} \bar{\psi}^j \gamma^l \psi^k.
\end{align*}
\] (3.10)

Although \( u_{ijk} \) are considerably complicated, it can be shown after a little calculation that the torsion takes a simple form
\[ T_{kij} = -\frac{i}{2} \bar{\psi}_i \gamma_k \psi_j, \] (3.11)

which is the same result as \( N = 1 \) supergravity. With the help of the following identity
\[ \epsilon^{ijkl} (\bar{\psi}_k \gamma_m \psi_l) \gamma^m \psi_j = 0, \] (3.12)

which is valid because of the Fierz identity \([13]\), substituting (3.11) into (3.8) shows that \( \text{Im} \mathcal{L}^{(+)} \) vanishes.

4. Summary
Based on the considerations of spin-1/2 fields and (Majorana) Rarita-Schwinger fields, we have generalized the chiral Lagrangian of half-integer spin fields a little. This generalized chiral Lagrangian of matter fields holds also in the case of $N = 2$ supergravity and (Dirac) Rarita-Schwinger fields, etc. For $N$-(Majorana) Rarita-Schwinger fields, we may merely replace $\psi_\mu$ in this paper by $\psi^I_\mu$ where $I$ runs from 1 to $N$. We have seen that the field equations for the self-dual connection derived from the complex chiral Lagrangian is compatible with those derived from the real Lagrangian. Furthermore, we have shown that the imaginary part of the chiral Lagrangian takes a simple form in terms of the torsion tensor $T_{ijk}$.

For a (Majorana) Rarita-Schwinger field, $\text{Im}\mathcal{L}^{(+)}$ vanishes owing to the identity (3.12). However, there is a possibility that $\text{Im}\mathcal{L}^{(+)} \neq 0$ for $N = 2$ supergravity, because (3.12) does not hold for this case. If this is the case, $\text{Im}\mathcal{L}^{(+)}$ will lead to spurious equations. (Dirac) Rarita-Schwinger fields are also under study.
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[13] See, for example, eq.(7) in p.365 of Ref.[11].