Reflecting space and time via Topological and Temporal cylindric algebras

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Abstract

Let $\alpha$ be an arbitrary ordinal, and $2 < n < \omega$. In [45], accepted for publication in Quaestiones Mathematicae, we studied using algebraic logic, interpolation, amalgamation using $\alpha$ many variables for topological logic with $\alpha$ many variables briefly $\text{TopL}_\alpha$. This is a sequel to [45]; the second part on modal cylindric algebras, where we study algebraically other properties of $\text{TopL}_\alpha$. Modal cylindric algebras are cylindric algebras of infinite dimension expanded with unary modalities inheriting their semantics from a unimodal logic $L$ such as $K5$ or $S4$. Using the methodology of algebraic logic, we study topological (when $L = S4$), in symbols $\text{TCA}_n$. We study completeness and omitting types $\text{OTTs}$ for $\text{TopL}_\omega$ and $\text{TenL}_\omega$, by proving several representability results for locally finite such algebras. Furthermore, we study the notion of atom-canonicity for both $\text{TCA}_n$ and $\text{TenL}_n$, a well known persistence property in modal logic, in connection to $\text{OTT}$ for $\text{TopL}_n$ and $\text{TelLCA}_n$, respectively. We study representability, omitting types, interpolation and complexity issues (such as undecidability) for topological cylindric algebras. In Part 2, we introduce temporal cylindric algebras and point out the way how to amalgamate algebras of space (topological algebras) and algebras of time (temporal algebras) forming topological-temporal cylindric algebras that lend themselves to encompassing spacetime geometries, in a purely algebraic fashion. Having a geometric dimension literally, namely, that of their cylindric reducts, the geometry of such algebras is conceptually distinct from the ‘standard tensor and manifolds’ mathematical model for general relativity taken to be a part of algebraic geometry, rather than algebraic logic, which is the path introduced in the 2nd part of this paper.

1 Introduction and overview

1.1 Universal and algebraic logic

One aim of universal logic is to determine the domain of validity of such and such metatheorem (e.g. the completeness theorem, the Craig interpolation theorem, or the Orey-Henkin omitting types theorem of first order logic) and to give general formulations of metatheorems in broader, or even entirely other contexts. This is also done in algebraic logic, by dealing with modifications and variants of first order logic resulting in a natural way during the process of algebraisation, witness for example the omitting types theorem proved in [40]. This kind of investigation is extremely potent for applications and helps to make the distinction between what is really essential to a particular logic and what is not. During the 20th century, numerous logics have been created, to mention only a few: intuitionistic logic, modal logic, topological logic, topological dynamic logic, spatial logic, dynamic logic, tense logic, temporal logic, many-valued logic, fuzzy logic, relevant logic, para-consistent logic, non monotonic logic, etc. The rapid development
of computer science, since the fifties of the 20th century initiated by work of giants like Gödel, Church and Turing, ultimately brought to the front scene other logics as well, like logics of programs and lambda calculus (the last can be traced back to the work of Church). Universal logic owes its birth as a response to the explosion of new logics. After a while it became noticeable that certain patterns of concepts kept being repeated albeit in different logics. But then the time was ripe to make in retrospect an inevitable abstraction like as the case with the field of abstract model theory (Lindstrom’s theorem is an example here). Abstract algebraic logic on the other hand is a major model theoretic trend of universal algebra that formalizes the intuitive notion of a logical system, including syntax, semantics, and the satisfaction relation between them. and it developed to an important foundational theory. Universal logic addresses different logical systems simultaneously in essentially four ways. Either abstracting common features, or building new bridges between them, or constructing new logics from old ones, or, last but not least, combining logics. In this paper we do all four things. We construct new predicate logics that can be seen as modal expansions of first order logic given an algebraizable (in the standard Blok-Pigozzi sense) formalism. For such logics, in the first part of the paper, we study properties known to hold for $L_{\omega,\omega}$ like the celebrated Orey-Henkin Omitting types Theorem We show that OTT remains to hold for modal expansions of first order logic as long as there are variables existing outside (atomic) formulas, but these results do not generalize any further. Our results cover ordinary predicate first order logic, possibly expanded with modalites. Our results apply to cylindric algebras when $L = S5$, to basic Temporal cylindric algebras defined by Georgescu, and so called Tense algebras, that reflects time a the name might suggest, and to so-called topological cylindric algebras, that expressed spatial properties of space in a modal simple setting obtained when $L = S4$. In the last case, the modalities can be redefined topologically using the interior operator relative to some so-called Alexandrof topology defined on the base of a cylindric set algebra.

**Topological logic:** Topological logic provides a framework for studying the confluence of the topological semantics for $S4$ modalities, based on topological spaces rather than Kripke frames, with the $S4$ modality induced by the interior operator. Motivated by questions like: which spatial structures may be characterized by means of modal logic, what is the logic of space, how to encode in modal logic different geometric relations, topological logics are apt for dealing with logic and space. The overall point is to take a common mathematical model of space (like a topological space) and then to fashion logical tools to work with it. One of the things which blatantly strikes one when studying elementary topology is that notions like open, closed, dense are intuitively very transparent, and their basic properties are absolutely straightforward to prove. However, topology uses second order notions as it reasons with sets and subsets of ‘points’. This might suggest that like second order logic, topology ought to be computationally very complex. This apparent dichotomy between the two paradigms vanishes when one realizes that a large portion of topology can be formulated as a simple modal logic, namely, $S4$. This is for sure an asset for modal logics tend to be much easier to handle than first order logic let alone second order. We can summarize the above discussion in the following neat theorem, that we can and will attribute to McKinsey, Tarski and Kripke; this historically is not very accurate. For a topological space $X$ and $\phi$ an $S4$ formula we write $X \models \phi$, if $\phi$ is valid topologically in $X$ (in either of the senses above). For example, $w \models \Box \phi \iff$ for all $w'$, if $w \leq w'$, then $w' \models \phi$, where $\leq$ is the relation $x \leq y \iff y \in \text{cl}\{x\}$ where cl abbreviate ‘the closure of’.
Theorem 1.1. (McKinsey-Tarski-Kripke) Suppose that $X$ is a dense in itself metric space (every point is a limit point) and $\phi$ is a modal S4 formula. Then the following are equivalent:

1. $\phi \in \text{S4}$.
2. $\models \phi$.
3. $X \models \phi$.
4. $\mathbb{R} \models \phi$.
5. $Y \models \phi$ for every finite topological space $Y$.
6. $Y \models \phi$ for every Alexandrov space $Y$.

One can say that finite topological space or their natural extension to Alexandrov topological spaces reflect faithfully the S4 semantics, and that arbitrary topological spaces generalize S4 frames. On the other hand, every topological space gives rise to a normal modal logic. Indeed S4 is the modal logic of $\mathbb{R}$, or any metric that is separable and dense in itself space, or all topological spaces, as indicated above. Also S4 is the modal logic of the Cantor set, which is known to be Baire isomorphic to $\mathbb{R}$. But, on the other hand, modal logic is too weak to detect interesting properties of $\mathbb{R}$, for example it cannot distinguish between $[0,1]$ and $\mathbb{R}$ despite their topological dissimilarities, the most striking one being compactness; $[0,1]$ is compact, but $\mathbb{R}$ is not. However, when we step into the realm of the predicate topological logic, the expressive power becomes substantially stronger.

OTTs for topological predicate logic: It would seem to be a simple matter to outfit a modal logic with the quantifiers. One would simply add the standard rules for quantifiers to the principles of whichever propositional modal logic one chooses. However, adding quantifiers to modal logic involves a number of difficulties. The main points of disagreement concerning the quantifier rules are about how to handle the domain of quantification. The simplest alternative, the fixed-domain approach, assumes a single domain of quantification that contains all the possible objects. Another interpretation, the world-relative interpretation, assumes that the domain of quantification changes from world to world, and contains only the objects that actually exist in a given world. Each of these two alternatives has its pros and cons. Here we adopt the fixed-domain approach which requires no major adjustments to the classical machinery for the quantifiers. Furthermore, we assume that this world carries an Alexandrov topology, inducing infinitely many modalities whose semantics coincide with S4. The fixed-domain interpretation has advantages of simplicity and familiarity. Other topological interpretations of propositional topological logic were recently extended in a natural way to arbitrary theories of full first order logic by Awodey and Kishida using so-called topological pre-sheaves to interpret domains of quantification [1]. They prove that S4∀ (predicate S4 logic) is complete with respect to such extended topological semantics, using techniques related to recent work in topos theory. Indeed, historically Sheaf semantics was first introduced by topoi theorists for higher order intuitionistic logic, and has been applied to first order modal logic, by both modal and categorical logicians. Here, our syntax is similar to op.cit but completeness and OTT (the former deduced as a byproduct) is different than Sheaf semantics.

Topological, tense and Heyting polyadic algebras: In this paper we investigate the OTT to topological predicate logic with $\alpha$ many variables, briefly TopPL$_\alpha$, where $\alpha$ is
an ordinal. Georgescu \[16, 19, 15, 14, 17\] applied algebraic logic outside the realm of first order logic. He applied the well developed theory of Halmos’ theory of polyadic algebras to intuitionistic, modal, temporal and topological logic. Georgescu studied Chang, modal, topological and tense locally polyadic algebras of infinite dimension, which are essentially equivalent to Tarski’s locally finite cylindric algebras. The work of Georgescu in \[17\] is substantially generalized in \[43\] by relaxing the condition of local finiteness and studying besides representability, various amalgamation properties for Heyting polyadic algebras. The work in this paper, preceded with the results established in \[43\], can be seen as a far reaching generalization of the work in Georgescu’s remaining aforementioned references which dealt only with representation theorems of locally finite algebras. The work in \[43\] can be seen as yet another far reaching generalization of he work of Georgescu in the remaining aforementioned references. While in \[43\], we studied interpolation for intuitionistic fragment of Keisler’s logic topological predicate logic, here we go further by studying various forms of representations for various subclasses of topological and tense cylindric algebras of infinite dimension, reflecting the rich interplay between the syntax and semantics of predicate modal logics of space and time respectively; in the latter case dealing with basic temporal predicate logic. Furthermore, we formulate and prove an omitting types theorem for tense predicate logic. We recover the Henkin-Orey OTT for TopPL\(_\alpha\) when \(\alpha\) is an infinite countable ordinal for countable theories. We prove negative OTT\(_n\)s for TopPL\(_n\) when \(n\) is finite \(> 2\), and the types are required to be omitted with respect to certain generalized semantics. We address the case when a single non-principal type is required to be omitted with respect to so-called \(m\)-square models with \(2 < n < m < \omega\). An \(m\)-square modal is only 'locally square' with \(m\) measuring the degree of squareness; it is only an approximation of an ordinary model which is \(\omega\)-square. The idea is that if we approach this \(m\)-square model using a movable window, then there will become a certain point, determined by \(m\) where we will mistake this \(m\)-square model, for an ordinary genuine (\(\omega\)-square) model. For \(2 < n < l < k\) a \(k\)-square model is \(l\) square, but the converse may fail, both are approximations to ordinary models, with the \(k\)-square one a better (closer) approximation. The larger the degree of squareness, the closer the model is to an ordinary one. Figuratively speaking, the limit of this sequence of infinite locally relativized models is the \(\omega\)-square Tarskian models. These locally relativized models (representations) are invented by Hirsch and Hodkinson in the context of relation algebras \[26\, \text{Chapter 13}\] and is adapted to expansions of cylindric algebras here. Considering such clique-guarded semantics swiftly leads us to rich territory.

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Results on cylindric algebras: Fix finite \(n > 2\). Let CRCA\(_n\) denote the class of completely representable CA\(_n\)s and LCA\(_n\) := EICRCA\(_n\) be the class of algebras satisfying the Lyndon conditions. For a class \(K\) of Boolean algebras with operators, let \(K \cap \text{At}\) denote the class of atomic algebras in \(K\). By modifying the games coding the Lyndon conditions allowing \(\forall\) to reuse the pebble pairs on the board, we will show that LCA\(_n\) = EICRCA\(_n\) = EILNR\(_n\)CA\(_\omega\) \(\cap\) At. Define an \(\mathfrak{A} \in\) CA\(_n\) to be strongly representable \(\iff\) \(\mathfrak{A}\) is atomic and the complex algebra of its atom structure, equivalently its Dedekind-MacNeille completion, in symbols \(\text{CmAt}\mathfrak{A}\) is in RCA\(_n\). This is a strong form of representability; of course \(\mathfrak{A}\) itself will be in RCA\(_n\), because \(\mathfrak{A}\) embeds into \(\text{CmAt}\mathfrak{A}\) and RCA\(_n\) is a variety, a fortiori closed under forming subalgebras. We denote the class of strongly representable atomic algebras of dimension \(n\) by SRCA\(_n\). Nevertheless, there are atomic simple countable algebras that are representable, but not strongly representable. In fact, we shall see that there is a countable simple atomic algebra in RCA\(_n\) such that \(\text{CmAt}\mathfrak{A} \notin\) SNR\(_n\)CA\(_{(n)}\) \(\supset\) RCA\(_n\). So in a way some algebras are more representable than others. In fact, the following inclusions are known to hold:

\[
\text{CRCA}_n \subset \text{LCA}_n \subset \text{SRCA}_n \subset \text{RCA}_n \cap \text{At}.
\]

In this paper we delve into a new notion, that of degrees of representability. Not all algebras are representable in the same way or strength. If \(\mathcal{C} \subseteq \text{Nr}_n\mathcal{D}\), with \(\mathcal{D} \in\) CA\(_m\) for some ordinal (possibly infinite) \(m\), we say that \(\mathcal{D}\) is an \(m\)-dilation of \(\mathcal{C}\) or simply a dilation if \(m\) is clear from context. Using this jargon of 'dilating algebras' we say that \(\mathfrak{A} \in\) RCA\(_n\) is strongly representable up to \(m > n\) \(\iff\) \(\text{CmAt}\mathfrak{A} \in\) SNR\(_n\)CA\(_m\). This means that, though \(\mathfrak{A}\) itself is in RCA\(_n\), the Dedekind-MacNeille completion of \(\mathfrak{A}\) is not representable, but nevertheless it has some neat embedding property; it is 'close' to being representable.
Using this jargon, $\mathfrak{A}$ admits a dilation of a bigger dimension. The bigger the dimension of the dilation of the representable algebra the more representable the algebra is, the closer it is to being strongly representable. Through the unfolding of this paper, we will investigate and make precise the notion of an algebra being more representable than another. It is known that LCA$_n$ is an elementary class, but SRCA$_n$ is not. We shall prove below that Str(ONr$_n$CA$_{n+3}$) = $\mathfrak{S}$ : $\exists m \in$ ONr$_n$CA$_{n+3}$ is not elementary with O $\in$ S$_c$, S$_d$, I as defined in the abstract. We prove that any class K, of CA$_n$ atom structures obtained from K (Kripke frames) such that AtNr$_n$TCA$_{n+3}$ $\subseteq$ K $\subseteq$ AtS$_c$Nr$_n$TCA$_{n+3}$, K is not elementary and lifting from atom structures to atomic algebras, we show that any class K($\subseteq$ RCA$_n$) with S$_d$Nr$_n$CA$_{\omega}$ $\cap$ CRCA$_n$ $\subseteq$ K $\subseteq$ S$_c$Nr$_n$CA$_{n+3}$, K is not elementary either. Here NR$_n$ is the operator of taking n neat reducts defined similarly to cylindric algebras, while recall recall S$_c$(S$_d$) is the operation of forming complete (dense) subalgebras.

2 Predicate topological logic via expansions of cylindric algebras

2.1 Basic notions

Let $\alpha$ be an arbitrary ordinal $> 0$ and $X$ be a set. Then $\mathfrak{B}(X)$ denotes the Boolean set algebra $\langle \wp(X), \cup, \cap, \sim, \emptyset, X \rangle$. Let $U$ be a non-empty set. For $s, t \in ^{\alpha}U$ write $s \equiv_i t$ if $s(j) = t(j)$ for all $j \neq i$. For $X \subseteq ^{\alpha}U$ and $i, j < \alpha$, let

$$c_iX = \{s \in ^{\alpha}U : (\exists t \in X)(t \equiv_i s)\}$$

and

$$d_{ij} = \{s \in ^{\alpha}U : s_i = s_j\}.$$

The algebra $\langle \mathfrak{B}(^{\alpha}U), c_i, d_{ij} \rangle_{i,j < \alpha}$ is called the full cylindric set algebra of dimension $\alpha$ with unit (or greatest or top element) $^{\alpha}U$. $^{\alpha}U$ is called a Cartesian space. Instead of taking ordinary set algebras, as in the case of cylindric algebras, with units of the form $^{\alpha}U$, one may require that the base $U$ is endowed with some topology. This enriches the algebraic structure. For given such an algebra, for each $k < \alpha$, one defines an interior operator on $\wp(^{\alpha}U)$ by

$$I_k(X) = \{s \in ^{\alpha}U : s_k \in \text{int}\{a \in U : s_a \in X\}\}, X \subseteq ^{\alpha}U.$$

Here $s_a^k$ is the sequence that agrees with $s$ except possibly at $k$ where its value is $a$. This gives a topological cylindric set algebra of dimension $\alpha$.

The interior operators, as well as the box operators can also be defined on weak spaces, that is, sets of sequences agreeing co-finitely with a given fixed sequence. This makes a difference only when $\alpha$ is infinite.

**Definition 2.1.** A weak space of dimension $\alpha$ is a set of the form $\{s \in ^{\alpha}U : |\{i \in \alpha : s_i \neq p_i\}| < \omega\}$ for a given fixed in advance $p \in ^{\alpha}U$. Now for $k < \alpha$, define

$$I_k(X) = \{s \in ^{\alpha}U^{(p)} : s_k \in \text{int}\{u \in U : s_u^p \in X\}\}.$$
Theorem 2.4. Let $\mathfrak{A}$ be a set algebra with universe $\varphi(V)$; the top element $V$ is a generalized set algebra of dimension $\alpha$. So let $\mathfrak{A} \in \text{RCA}_\alpha$, and assume that $\mathfrak{A} \cong \mathfrak{B}$ where $\mathfrak{B} \in \text{Gs}_\alpha$ has top element the generalized space $V$. The base of $V$ is the set $U = \bigcup_{s \in V} \text{rings}$. Then one defines the interior operator $I_k$ on $\mathfrak{B}$ by:

$$I_k(X) = \{ s \in V : s_k \in \text{int}\{ a \in U : s^k_a \in X \} \}, X \subseteq V.$$ 

and, for that matter the box operator relative to a Chang system $V : U \rightarrow \varphi(\varphi(U))$ as follows

$$s \in \Box_k(X) \iff \{ a \in U : s^k_a \in X \} \in V(s_k), X \subseteq V.$$ 

The following lemma is very easy to prove, so we omit the proof. Formulated only for set algebras, it also holds for weak set algebras.

**Lemma 2.2.** For any ordinal $\mu > 1$, $\mathfrak{A} \in \text{Cs}_\mu$ and $k < \mu$, let $I_k$ and $\Box_k$ be as defined above. Then if $\mathfrak{A} \in \text{Cs}_\alpha$ has top element $\alpha U$ and $\beta > \alpha$, then the following hold for any $Y \subseteq \alpha U$ and $k < \alpha$:

1. $I_k(Y) \subseteq Y$, $\Box_k(Y) \subseteq Y$,

2. If $f : \varphi(\alpha U) \rightarrow \varphi(\beta U)$ is defined via

$$X \mapsto \{ s \in \beta U : s \upharpoonright \alpha \in X \},$$

then $f(I_kX) = I_k(f(X))$ and $f(\Box_kX) = \Box_k(f(X))$, for any $X \subseteq \alpha U$.

In cylindric algebra theory a subdirect product of set algebras is isomorphic to a generalized set algebra. We show that this phenomena persists when the bases carry topologies; we need to describe the topology on the base of the resulting generalized set algebra in terms of the topologies on the bases of the set algebras involved in the subdirect product.

**Definition 2.3.** Let $\{ X_i : i \in I \}$ be a family of topological spaces indexed by $I$. Let $X = \bigcup X_i$ be the disjoint union of the underlying sets. For each $i \in I$ let $\phi_i : X_i \rightarrow X$ be the canonical injection. The *coproduct* on $X$ is defined as the finest topology on $X$ for which the canonical injections are continuous.

That is a subset $U$ of $X$ is open in the coproduct topology on $X$ if and only if its preimage $\phi_i^{-1}(U)$ is open in $X_i$ for each $i \in I$ if and only if its intersection with $X_i$ is open relative to $X_i$ for each $i \in I$.

**Theorem 2.4.** Let $\mathfrak{B}$ be the $\text{Gs}_\alpha$ with universe $\varphi(V)$, where $U_i \cap U_j = \emptyset$ and base $\bigcup_{i \in I} U_i$, carrying a topology. Assume that $\mathfrak{B}$ has universe $\varphi(V)$. Let $\mathfrak{A}_i$ be the $\text{Cs}_\alpha$ with base $U_i$, $U_i$ having the subspace topology and universe $\varphi(U_i)$. Then $f : \mathfrak{B} \rightarrow \prod_{i \in I} \mathfrak{A}_i$, defined by $X \mapsto (X \cap \alpha U_i : i \in I)$ is an isomorphism of cylindric algebras; furthermore it respects the interior operators stimulated by the topologies on the bases.

$\text{TCs}_\alpha(\text{TGs}_\alpha)$ denotes the class of topological (generalized) set algebras. Now such concrete set algebras lend itself to an abstract formulation aiming to capture the concrete set algebras; or rather the variety generated by them. This consists of expanding
the signature of cylindric algebras by unary operators, or modalities, one for each \( k < \alpha \), satisfying certain identities. The axiomatizations we give are actually simpler than those stipulated by Georgescu in \cite{16, 19}, although locally finite polyadic algebras and locally finite cylindric algebras are equivalent. We use only substitutions corresponding to replacements; in the case of dimension complemented algebras all substitutions corresponding to finite transformations are term definable from these, cf. \cite{22}. This makes axiom (A8) on p.1 of \cite{16} superfluous. We start with the standard definition of cylindric algebras \cite{22, Definition 1.1.1}:

**Definition 2.5.** Let \( \alpha \) be an ordinal. A cylindric algebra of dimension \( \alpha \), a \( \text{CA}_\alpha \) for short, is defined to be an algebra

\[
\mathcal{C} = \langle C, +, \cdot, -, 0, 1, c_i, d_{ij} \rangle_{i, j \in \alpha}
\]

obeying the following axioms for every \( x, y \in C, i, j, k < \alpha \)

1. The equations defining Boolean algebras,
2. \( c_i0 = 0 \),
3. \( x \leq c_ix \),
4. \( c_i(x \cdot c_iy) = c_ix \cdot c_iy \),
5. \( c_ic_jx = c_jc_ix \),
6. \( d_{ii} = 1 \),
7. if \( k \neq i, j \) then \( d_{ij} = c_k(d_{ik} \cdot d_{jk}) \),
8. If \( i \neq j \), then \( c_i(d_{ij} \cdot x) \cdot c_i(d_{ij} \cdot -x) = 0 \).

For a cylindric algebra \( \mathfrak{A} \), we set \( q_i x = -c_i - x \) and \( s^\mathcal{A}_i(x) = c_i(d_{ij} \cdot x) \). Now we want to abstract equationally the prominent features of the concrete interior operators defined on cylindric set and weak set algebras. We expand the signature of \( \text{CA}_\alpha \) by a unary operation \( I_i \) for each \( i \in \alpha \). In what follows \( \oplus \) denotes the operation of symmetric difference, that is, \( a \oplus b = (-a + b) \cdot (-b + a) \). For \( \mathfrak{A} \in \text{CA}_\alpha \) and \( p \in \mathfrak{A} \), \( \Delta p \), the dimension set of \( p \), is defined to be the set \( \{ i \in \alpha : c_ip \neq p \} \). In polyadic terminology \( \Delta p \) is called the support of \( p \), and if \( i \in \Delta p \), then \( i \) is said to support \( p \) \cite{16, 19}.

**Definition 2.6.** \cite{16} A topological cylindric algebra of dimension \( \alpha \), \( \alpha \) an ordinal, is an algebra of the form \( (\mathfrak{A}, I_i)_{i < \alpha} \) where \( \mathfrak{A} \in \text{CA}_\alpha \) and for each \( i < \alpha \), \( I_i \) is a unary operation on \( A \) called an interior operator satisfying the following equations for all \( p, q \in A \) and \( i, j \in \alpha \):

1. \( q_i(p \oplus q) \leq q_i(I_ip \oplus I_ip) \),
2. \( I_ip \leq p \),
3. \( I_ip \cdot I_ip = I_i(p \cdot q) \),
4. \( I_ip \leq I_ip \),
5. \( I_i1 = 1 \),
6. \( c_k I_p = I_p, k \neq i, k \notin \Delta p \),
7. \( s_j^i I_p = I_p s_j^i, j \notin \Delta p \).

The class of all such topological cylindric algebras are denoted by \( TCA_\alpha \).

### 2.2 Basic Lemmas

For \( K \) any class of \( CA \), we write \( TK \) for the corresponding class of topological \( CA \). For example, for any ordinal \( \beta \), (recall that) \( TCA_\beta \) and we write \( TDC_\beta \) to denote the classes of all topological \( CA_\beta \)'s and dimension complemented topological \( CA_\beta \)'s, respectively. Throughout this section, unless otherwise indicated, \( \alpha \) denotes an infinite ordinal. Then \( A \in TDC_\alpha \iff \alpha \neq \Delta x \) for all \( x \in A \iff \alpha \sim \Delta x \) is infinite for all \( x \in A \).

#### Lemma 2.7.
1. Let \( C \in CA_\alpha \) and let \( F \) be a Boolean filter on \( C \). Define the relation \( E \) on \( \alpha \) by \((i, j) \in E \) if and only if \( d_{ij} \in F \). Then \( E \) is an equivalence relation on \( \alpha \).

2. Let \( C \in CA_\alpha \) and \( F \) be a Boolean filter of \( C \). Let \( V = \{ \tau \in ^\alpha \alpha : |\{i \in \alpha : \tau(i) \neq i\}| < \omega \} \). For \( \sigma, \tau \in V \), write

\[
\sigma \equiv_E \tau \iff (\forall i \in \alpha)(\sigma(i), \tau(i)) \in E.
\]

and let

\[
E = \{ (\sigma, \tau) \in 2^V : \sigma \equiv_E \tau \}.
\]

Then \( E \) is an equivalence relation on \( V \). Let \( W = V/E \). For \( h \in W \), write \( h = \tau/E \) for \( \tau \in V \) such that \( \tau(j)/E = h(j) \) for all \( j \in \alpha \). Let \( f(x) = \{ \sigma \in W : s_\tau x \in F \} \). Then \( f \) is well defined. Furthermore, \( W \) can be identified with the weak space \( ^\alpha U/E[^\beta] \) where \( \bar{\rho} = (\rho(i)/E : i < \alpha) \) via \( \tau/E \mapsto [\tau] \), where \( [\tau](i) = \tau(i)/E \). Accordingly, we write \( W = ^\alpha U/E[^\beta] \).

#### Definition 2.8.
Let \( A \) be an algebra having a cylindric reduct of dimension \( \alpha \). A Boolean ultrafilter \( F \) of \( A \) is said to be Henkin if for all \( k < \alpha \), for all \( x \in A \), whenever \( c_k x \in F \), then there exists \( l \notin \Delta x \) such that \( s_l^i x \in F \).

#### Lemma 2.9.
Let everything be as in the previous lemma, and assume that \( F \) is a Henkin ultrafilter. Then \( f \) as defined in the previous lemma is a CA homomorphism.

**Proof.** [33].

#### Definition 2.10.
Let everything be as in the hypothesis of lemma 2.11. For \( s \in W \) and \( k < \alpha \) we write \( s_k^u \) for \( s_k^u/E \). For \( k \in \alpha \), then \( I_k \) is the (interior) operator on \( \varphi(W) \) defined by \( I_k(X) = \{ s \in W : s_k \in \text{int}\{ u \in U : s_k^u \in X \} \} \). Similarly, if \( V : U/E \to \varphi(\varphi(U/E)) \) is a Chang system then \( \square_k \) is defined on \( \varphi(W) \) by \( s \in \square_k(X) \iff \{ u/E \in U/E : s_k^u \in X \} \in V[s(i)/E] \).

From now on, we replace the interior operator \( I_i \) by \( \square_i \).
Lemma 2.11. Assume that \( \mathcal{C} \in \text{TDc}_\alpha \), \( F \) is a Henkin ultrafilter of \( \mathcal{C} \) and \( a \in F \). Then there exist a non-empty set \( U \), \( p \in \alpha U \), a topology on \( U/E \) and a homomorphism \( f : \mathcal{C} \to (\varphi(W), \Box_i; i < \alpha) \) with \( f(a) \neq 0 \), where \( W = \alpha[U/E]^p \), with \( E \) as defined in lemma 2.7 and \( \Box_i \) (\( i < \alpha \)) is the concrete interior operator defined in Definition 2.10.

Proof. Let \( W = \alpha[\alpha/E]^{(f_0)} \). Define, as we did before, \( f : \mathfrak{A} \to \varphi(W) \) via

\[
p \mapsto \{ \tilde{r} \in W : s_rp \in F \}.
\]

For \( i \in \alpha \) and \( p \in \mathfrak{A} \), let

\[
O_{p,i} = \{ k/E \in \alpha/E : s^k_p(I(i))p \in F \}.
\]

Let

\[
B = \{ O_{p,i} : i \in \alpha, p \in A \}.
\]

Then it is easy to check that \( B \) is the base for a topology on \( \alpha/E \). To define the interior operations, we set for each \( i < \alpha \)

\[\square_i : \varphi(W) \to \varphi(W)\]

by

\[ [x] \in J_iX \iff \exists U \in B(x_i/E \in U \subseteq \{ u/E \in \alpha/E : [x]_{u/E} \in X \} ), \]

where \( X \subseteq V \). Note that \( [x]_{u/E} = [x]^i u \). We now check that \( f \) preserves the interior operators \( \Box_i \) (\( i < \alpha \)), too. We need to show

\[
\psi(\Box_i p) = \Box_i(\psi(p)).
\]

The reasoning is like [10]; the difference is that in [10], the constants denoted by \( x_i \) are endomorphisms on \( \mathfrak{A} \); the value \( x_i \) at \( j \) corresponds in our adopted approach to \( s^i_u \) where \( u = x_i(j) \). Let \([x]\) be in \( \psi(\Box_i p) \). Let

\[
\sup(x) = \{ k \in \alpha : x_k \neq k \}.
\]

Then, by definition, \( s_x \Box_i p \in F \). Hence

\[
s^i_{x_i} I_i s^i_{x_1} \ldots s^i_{x_n} p \in F,
\]

where

\[
\sup(x) \sim \{ i \} = \{ j_1, \ldots, j_n \}.
\]

Let

\[
y = [j_1|x_1] \circ \ldots [j_n|x_n].
\]

Then \( x_i/E \in \{ u/E : s^i_u I(i) s^j_y p \in F \} \) \( \in \mathcal{Q} \). But \( I_is_y p \leq s_y p \), hence

\[
U = \{ u/E : s^i_u I_s y p \in F \} \subseteq \{ u/E : s^i_u s_y p \in F \}.
\]

It follows that \( x_i/E \in U \subseteq \{ u/E : x^i_y \in \Psi(p) \} \). Thus \([x] \in \Box_i \psi(p)\). Now we prove the other harder direction. Let \([x] \in \Box_i \Psi(p) \). Let \( U \in \mathfrak{B} \) be such that

\[
x_i/E \in U \subseteq \{ u/E : x^i_u \in \alpha/E : s^i_u s_x p \in F \}.
\]
Assume that $U = O_{r,j}$, where $r \in \mathfrak{A}$ and $j \in \alpha$. Let $u \in \alpha \sim [\Delta p \cup \Delta r \cup \{i, j\}]$. By dimension complementedness such a $u$ exists. Then we have:

$$s_u^j \square_j r \in F \iff s_u^j s_x p \in F,$$

$$s_u^j \square_j r \cdot s_u^j s_x p \in F \iff s_u^j \square_j r \in F.$$

But $s_u^j \square_j r = s_u^j \bigcap_j s^j r$, so we have:

$$s_u^j \square_j r \cdot s_u^j s_x p \oplus s_u^j \square_j s^j r \in F,$$

$$s_u^j [\square_j s^j r \cdot s_x p \oplus \square_j s^j r] \in F,$$

$$q_i [\square_j s^j r \cdot s_x p \oplus \square_j s^j r] \in F,$$

$$s_{x_i} [\square_j s^j r \cdot \square_j s_x p \oplus \square_j s^j r] \in F,$$

$$s_{x_i} I_j r \cdot s_{x_i} \square_j s_x p \oplus s_{x_i} \square_j r \in F.$$

But $s_{x_i} \square_j r \in F$, hence $s_{x_i} \bigcap_j s_x p \in F$, and so $x \in \Psi (\square_j p)$ as required.

We also need the notion of compressing dimensions and, dually, dilating them; expressed by the notion of neat reduces.

**Definition 2.12.** Let $\alpha < \beta$ be ordinals and $\mathfrak{B} \in \text{TC}_\beta \mathfrak{A}$. Then $\text{NT}_\alpha \mathfrak{B}$ is the algebra with universe $\text{Nr}_\alpha \mathfrak{A} = \{a \in \mathfrak{A} : \Delta a \subseteq \alpha\}$ and operations obtained by discarding the operations of $\mathfrak{B}$ indexed by ordinals in $\beta \sim \alpha$. $\text{NT}_\alpha \mathfrak{B}$ is called the neat $\alpha$ reduct of $\mathfrak{B}$. If $\mathfrak{A} \subseteq \text{NT}_\alpha \mathfrak{B}$, with $\mathfrak{B} \in \text{TC}_\beta \mathfrak{A}$, then we say that $\mathfrak{B}$ is a $\beta$ dilation of $\mathfrak{A}$, or simply a dilation of $\mathfrak{A}$ if $\beta$ is clear from context.

### 3 OTTs in case of the presence of infinitely many variables

Now we prove an omitting types theorem for $\text{TDC}_\alpha$ and $\text{TL}_\alpha$ when $\alpha$ is a countable infinite ordinal; also generalizing the result in [16] which addresses only topological locally finite algebras. The proof is similar to the proof of [16, Theorem 3.2.4] having at our disposal lemma 2.1. We omit the parts of the proof that overlap with those in [16]. Given $\mathfrak{A} \in \text{TC}_\alpha \mathfrak{A}$, $X \subseteq \mathfrak{A}$ is called a finitary type, if $X \subseteq \text{NT}_n \mathfrak{A}$ for some $n \in \omega$. It is non-principal if $\prod X = 0$.

**Definition 3.1.** A representation of $\mathfrak{A} \in \text{TC}_\alpha \mathfrak{A}$ is a non-zero homomorphism $f : \mathfrak{A} \to \mathfrak{B}$ where $\mathfrak{B}$ is a weak set algebra. If $\mathfrak{A}$ is simple then $f$ is necessarily one to one. $X \subseteq \mathfrak{A}$ is omitted by $f$ if $\bigcap_{x \in X} f(x) = \emptyset$, otherwise it is realized by $f$.

We define certain cardinals; it is consistent that such cardinal are uncountable. Throughout this paper we do not assume the continuum hypothesis.

**Definition 3.2.** 1. A subset $X \subseteq \mathbb{R}$ is meager if it is a countable union of nowhere dense sets. Let $\text{covK}$ be the least cardinal $\kappa$ such that $\mathbb{R}$ can be covered by $\kappa$ many nowhere dense sets. Let $p$ be the least cardinal $\kappa$ such that there are $\kappa$ many meager sets of $\mathbb{R}$ whose union is not meager.

2. A Polish space is a topological space that is metrizable by a complete separable metric.
Examples of Polish spaces are $\mathbb{R}$, the Cantor set $\omega^2$ and the Baire space $\omega^\omega$. These are called real spaces because they are Baire isomorphic. Any second countable compact Hausdorff space, like the Stone space of a countable Boolean algebra, is a Polish space [20].

Theorem 3.3. 1. The cardinals $\text{covK}$ and $p$ are uncountable cardinals, such that $p \leq \text{covK} \leq 2^\omega$.

2. The cardinal $\text{covK}$ is the least cardinal such that the Baire category theorem for Polish spaces fails, and it is also the largest for which Martin’s axiom for countable Boolean algebras holds.

3. If $X$ is a Polish space, then it cannot be covered by $< \text{covK}$ many meager sets. If $\lambda < p$, and $(A_i : i < \lambda)$ is a family of meager subsets of $X$, then $\bigcup_{i \in \lambda} A_i$ is meager.

Proof. For the definition and required properties of $p$, witness [20, pp. 3, pp. 44-45, corollary 22c]. For properties of $\text{covK}$, witness [40, the remark on. pp. 217].

Both cardinals $\text{covK}$ and $p$ have an extensive literature, witness [20] and the references therein. It is consistent that $\omega < p < \text{covK} \leq 2^\omega$ so that the two cardinals are generally different, but it is also consistent that they are equal; equality holds for example in the Cohen real model of Solovay and Cohen. Martin’s axiom implies that they are both equal to the continuum. Let $\mathfrak{A}$ be any Boolean algebra. The set of ultrafilters of $\mathfrak{A}$ is denoted by $\mathcal{U}(\mathfrak{A})$. The Stone topology makes $\mathcal{U}(\mathfrak{A})$ a compact Hausdorff space. We denote this space by $\mathfrak{A}^*$. Recall that the Stone topology has as its basic open sets the sets $\{N_x : x \in A\}$ where

$$N_x = \{F \in \mathcal{U}(\mathfrak{A}) : x \in F\}.$$  

Let $x \in A$, $Y \subseteq A$ and suppose that $x = \sum Y$. We say that an ultrafilter $F \in \mathcal{U}(\mathfrak{A})$ preserves $Y$ whenever $x \in F$, then $y \in F$ for some $y \in Y$. Now let $\mathfrak{A} \in \mathcal{TLf}_\omega$. For each $i \in \omega$ and $x \in A$ let

$$\mathcal{U}_{i,x} = \{F \in \mathcal{U}(\mathfrak{A}) : F \text{ preserves } \{s^i_j x : j \in \omega\}\}.$$  

Then

$$\mathcal{U}_{i,x} = \{F \in \mathcal{U}(\mathfrak{A}) : c_i x \in F \Rightarrow (\exists j \in \omega)s^i_j x \in F\}$$

$$= N_{-c_i x} \cup \bigcup_{j < \omega} N_{s^i_j x}.$$  

Let

$$\mathcal{H}(\mathfrak{A}) = \bigcap_{i \in \omega, x \in A} \mathcal{U}_{i,x}(\mathfrak{A}) \cap \bigcap_{i \neq j} N_{-d_{ij}}.$$  

It is clear that $\mathcal{H}(\mathfrak{A})$ is a $G_\delta$ set in $\mathfrak{A}^*$. For $F \in \mathcal{U}(\mathfrak{A})$, let

$$rep_F(x) = \{\tau \in \omega^\omega : s^\mathfrak{A}_\tau x \in F\},$$

for all $x \in A$. Here for $\tau \in \omega^\omega$, $s^\mathfrak{A}_\tau x$ by definition is $s^\mathfrak{A}_{\tau \Delta x} x$. The latter is well defined because $|\Delta x| < \omega$. When $a \in F$, then $rep_F$ is a representation of $\mathfrak{A}$ such that $rep_F(a) \neq 0$. The following theorem establishes a one to one correspondence between representations of locally finite cylindric algebras and Henkin ultrafilters. $Cs^\text{reg}_\mathfrak{A}$ denotes the class of
regular set algebras; a set algebra with top element $^αU$ is such, if whenever $f, g \in ^αU$, $f \upharpoonright \Delta x = g \upharpoonright \Delta x$, and $f \in X$ then $g \in X$. This reflects the metalogical property that if two assignments agree on the free variables occurring in a formula then both satisfy the formula or none does.

**Theorem 3.4.** If $F \in \mathcal{H}(\mathcal{A})$, then $rep_F$ is a homomorphism from $\mathcal{A}$ onto an element of $Lf_\omega \cap Cs'_\omega$ with base $\omega$. Conversely, if $h$ is a homomorphism from $\mathcal{A}$ onto an element of $Lf_\omega \cap Cs'_\omega$ with base $\omega$, then there is a unique $F \in \mathcal{H}(\mathcal{A})$ such that $h = rep_F$.

The next theorem is due to Shelah, and will be used to show that in certain cases uncountably many non-principal types can be omitted.

**Theorem 3.5.** Suppose that $T$ is a theory, $|T| = \lambda$, $\lambda$ regular, then there exist models $M_i : i < ^\lambda 2$, each of cardinality $\lambda$, such that if $i(1) \neq i(2) < \chi$, $\bar{a}_{i(l)} \in M_{i(l)}$, $l = 1, 2,$, $\tp(\bar{a}_{i(1)}) = \tp(\bar{a}_{i(2)})$, then there are $p_i \subseteq \tp(\bar{a}_{i(i)})$, $|p_i| < \lambda$ and $p_i \vdash \tp(\bar{a}_{i(i)})$ ($\tp(\bar{a})$ denotes the complete type realized by the tuple $\bar{a}$).

**Proof.** [34] Theorem 5.16, Chapter IV].

We shall use the algebraic counterpart of the following corollary obtained by restricting Shelah’s theorem to the countable case:

**Corollary 3.6.** For any countable theory, there is a family of $< \omega 2$ countable models that overlap only on principal types.

**Theorem 3.7.** 1. Let $\mathcal{A} \in TDc_\omega$ be countable. Assume that $\kappa < \mathfrak{p}$. Let $(\Gamma_i : i \in \kappa)$ be a set of non-principal types in $\mathcal{A}$. Then there is a topological weak set algebra $(\mathcal{B}, \square_i)_{i<\omega},$ that is, $\mathcal{B}$ has top element a weak space, and a homomorphism $f : \mathcal{A} \to (\mathcal{B}, \square_i)_{i<\omega}$, such that for all $i \in \kappa$, $\prod_{x \in X_i} f(x) = \emptyset$, and $f(\bar{a}) \neq 0$. If $\mathcal{A}$ is simple, then $\mathfrak{p}$ can be replaced by $\text{covK}$.

2. If $\mathcal{A} \in \text{TLf}_\omega$, and $(\Gamma_i : i \in \kappa)$ is a family of finitary non-principal types then there is a topological set algebra $(\mathcal{B}, \square_i)_{i<\omega}$, that is, $\mathcal{B}$ has top element a Cartesian square, and $\mathcal{B} \in Cs_\omega^{reg} \cap \text{Lf}_\omega$ together with a homomorphism $f : \mathcal{A} \to (\mathcal{B}, \square_i)_{i<\omega}$ such that $\prod_{x \in X_i} f(x) = \emptyset$, and $f(\bar{a}) \neq 0$. If the family of given types are ultrafilters then $\mathfrak{p}$ can be replaced by $2^\omega$, so that $< 2^\omega$ types can be omitted.

**Proof.** For the first part, we have by [22] 1.11.6 that

$$\forall j < \alpha)(\forall x \in A)(c_j x = \sum_{i \in \alpha \setminus \Delta x} s_i^j x.)$$

(1)

Now let $V$ be the weak space $\omega(1d) = \{s \in \omega : \{|i \in \omega : s_i \neq i| < \omega\}$. For each $\tau \in V$ for each $i < \kappa$, let

$$X_{i, \tau} = \{s_{\tau} x : x \in X_i\}.$$

Here $s_{\tau}$ is the unary operation as defined in [22] 1.11.9. For each $\tau \in V$, $s_{\tau}$ is a complete Boolean endomorphism on $\mathcal{A}$ by [22] 1.11.12(iii)]. It thus follows that

$$\forall \tau \in V)(\forall i < \kappa) \prod_{i \in \kappa} X_{i, \tau} = 0$$

(2)
Let $S$ be the Stone space of the Boolean part of $\mathfrak{A}$, and for $x \in \mathfrak{A}$, let $N_x$ denote the clopen set consisting of all Boolean ultrafilters that contain $x$. Then from [1] it follows that for $x \in \mathfrak{A}$, $\beta < \alpha$, $i < \kappa$ and $\tau \in V$, the sets

$$G_{j,x} = N_{c_j,x} \setminus \bigcup_{i \notin \Delta x} N_{s_i^j,x} \quad \text{and} \quad H_{i,\tau} = \bigcap_{x \in X_i} N_{s_{\tau,x}}$$

are closed nowhere dense sets in $S$. Also each $H_{i,\tau}$ is closed and nowhere dense. Let

$$G = \bigcup_{j \in \beta} \bigcup_{x \in B} G_{j,x} \quad \text{and} \quad H = \bigcup_{i \in \kappa, \tau \in V} H_{i,\tau}.$$

By properties of $p$, $H$ can be reduced to a countable collection of nowhere dense sets. By the Baire Category theorem for compact Hausdorff spaces, we get that $\mathfrak{S}(\mathfrak{A}) = S \sim H \cup G$ is dense in $S$. Let $F$ be an ultrafilter in $N_a \cap X$. By the very choice of $F$, it follows that $a \in F$ and we have the following.

$$(\forall j < \beta)(\forall x \in B)(c_j x \in F \implies (\exists j \notin \Delta x)s_j^j x \in F) \quad (3)$$

and

$$(\forall i < \kappa)(\forall \tau \in V)(\exists x \in X_i)s_{\tau,x} x \notin F. \quad (4)$$

Let $V = \omega_\omega^{(id)}$ and let $W$ be the quotient of $V$ as defined above. That is $W = V/E$ where $\tau E \sigma$ if $d_{\tau(i),\sigma(i)} \in F$ for all $i \in \omega$. Define $f$ as before by

$$f(x) = \{\tau \in W : s_{\tau,x} x \in F\}, \quad \text{for } x \in \mathfrak{A}.$$

and the interior operators for each $i < \alpha$ by

$$\Box_i : \wp(W) \to \wp(W)$$

by

$$[x] \in \Box_i X \iff \exists U \in \mathcal{B}(x_i/E \in U \subseteq \{u/E \in \alpha/E : [x]_{u/E} \in X\}),$$

where $X \subseteq W$; here $W$ and $E$ are as defined in lemmas, [2.7] and $\mathcal{B}$ is the base for the topology on $U/E$ defined as in the proof of theorem [2.11]. Then by lemma [2.11] $f$ is a homomorphism such that $f(a) \neq 0$ and it can be easily checked that $\bigcap f(X_i) = \emptyset$ for all $i \in \kappa$, hence the desired conclusion. If $\mathfrak{A}$ is simple, then by the properties of $\text{cov}K$, $\mathfrak{S}(\mathfrak{A}) = S \sim H \cup G$ is non-empty. Let $F \in H(\mathfrak{A})$ and let $a \in F$. The representation built using such $F$ as above, call it $f$, has $f(a) \neq 0$. By simplicity of $\mathfrak{A}$, $f$ is an injection, because $\text{ker}f = \{0\}$, since $a \notin \text{ker}f$ and by simplicity, either $\text{ker}f = \{0\}$ or $\text{ker}f = \mathfrak{A}$.

2. One proceeds exactly like in the previous item, but using, as indicated above, the fact that the operations $s_{\tau}$ for any $\tau \in \omega_\omega$ which are definable in locally finite algebras, via $s_{\tau,x} = s_{\tau|x_{\Delta x}}$, for any $x \in A$. Furthermore, $s_{\tau} \upharpoonright \mathfrak{N}_n \mathfrak{A}$ is a complete Boolean endomorphism, so that we guarantee that infimums are preserved and the sets $H_{i,\tau} = \bigcap_{x \in X_i} N_{s_{\tau,x}}$ remain no-where dense in the Stone topology. Now for the second part. Let $\mathfrak{A} \in \text{TLf}_{\omega}$, $\lambda < 2\omega$ and $F = (X_i : i < \lambda)$ be a family of maximal non-principal finitary types, so that for each $i < \lambda$, there exists $n \in \omega$ such that $X_i \subseteq \mathfrak{N}_n \mathfrak{A}$, and $\prod X_i = 0$; that is $X_i$ is a Boolean ultrafilter in $\mathfrak{N}_n \mathfrak{A}$. Then by Theorem [3.5] or rather its direct algebraic counterpart, there are $\omega^2$ representations such that if $X$ is
an ultrafilter in $\mathfrak{A}_n$ (some $n \in \omega$) that is realized in two such representations, then $X$ is necessarily principal. That is there exist a family of countable locally finite set algebras, each with countable base, call it $(\mathfrak{B}_{ji} : i < 2^\omega)$, and isomorphisms $f_i : \mathfrak{A} \to \mathfrak{B}_{ji}$ such that if $X$ is an ultrafilter in $\mathfrak{A}_n$, for which there exists distinct $k, l \in 2^\omega$ with $\bigcap f_l(X) \neq \emptyset$ and $\bigcap f_j(X) \neq \emptyset$, then $X$ is principal, so that from Corollary 3.6 such representations overlap only on maximal principal types. By Theorem 3.4, there exists an ultrafilter in $\mathfrak{B}_{ji}$, such that $f_i(h) = \psi(F)$. Then for all $i < 2^\omega$, there exists $F$ such that $F$ is realized in $\mathfrak{B}_{ji}$. Let $\psi : 2^\omega \to \varphi(F)$, be defined by $\psi(i) = \{F : F$ is realized in $\mathfrak{B}_{ji}\}$. Then for all $i < 2^\omega$, $\psi(i) \neq \emptyset$. Furthermore, for $i \neq k$, $\psi(i) \cap \psi(k) = \emptyset$, for if $F \in \psi(i) \cap \psi(k)$ then it will be realized in $\mathfrak{B}_{ji}$ and $\mathfrak{B}_{jk}$, and so it will be principal. This implies that $|F| = 2^\omega$ which is impossible.

$\square$

### 3.1 A positive OTT for $L_n$ theories

Unless otherwise indicated $n$ is a finite ordinal $> 1$. In $L_{\omega,1}$ an atomic model for a countable atomic theory (which exists by VT) omits all non–principal types. The last item in the next theorem is the ‘$L_n$ version (expressed algebraically)’ of this property. The condition of maximality expressed in ‘ultrafilters’ meaning ‘maximal filters’ delineates the edge of an independent statement to a provable one, cf. [40, Theorem 3.2.8] and the fourth item the next theorem.

**Theorem 3.8.** Let $n < \omega$. Let $\mathfrak{A} \in \mathbf{S}_{\omega} \mathfrak{A}_n$ be countable. Let $\lambda < 2^{\aleph_0}$ and let $X = (X_i : i < \lambda)$ be a family of non–principal types of $\mathfrak{A}$. Then the following hold:

1. If $\mathfrak{A} \in \mathfrak{A}_n$ and the $X_i$s are non–principal ultrafilters, then $X$ can be omitted in a $\mathfrak{T} \mathfrak{G}_{S_\nu}$.

2. Every subfamily of $X$ of cardinality $< p$ can be omitted in a $\mathfrak{G}_{S_\nu}$; in particular, every countable subfamily of $X$ can be omitted in a $\mathfrak{G}_{S_\nu}$. If $\mathfrak{A}$ is simple, then every subfamily of $X$ of cardinality $< \text{cov}K$ can be omitted in a $\mathfrak{C}_{S_\nu}$.

3. Every subfamily of $X$ of cardinality $< p$ can be omitted in a $\mathfrak{G}_{S_\nu}$; in particular, every countable

4. It is consistent, but not provable (in ZFC), that $X$ can be omitted in a $\mathfrak{G}_{S_\nu}$.

5. If $\mathfrak{A} \in \mathfrak{A}_n$ and $|X| < p$, then $X$ can be omitted $\iff$ every countable subfamily of $X$ can be omitted. If $\mathfrak{A}$ is simple, we can replace $p$ by $\text{cov}K$.

6. If $\mathfrak{A}$ is atomic, not necessarily countable, but have countably many atoms, then any family of non–principal types can be omitted in an atomic $\mathfrak{G}_{S_\nu}$; in particular, $X$ can be omitted in an atomic $\mathfrak{G}_{S_\nu}$; if $\mathfrak{A}$ is simple, we can replace $\mathfrak{G}_{S_\nu}$ by $\mathfrak{C}_{S_\nu}$.
Proof. To substantially simplify the proof while retaining the gist of ideas used in the more general case for the first item we assume that $\mathfrak{A}$ is countable and simple, that is, to say, has no proper ideal. This means that $\mathfrak{A}$ ‘algebraically represents’ a complete countable theory. We have $\prod^B X_i = 0$ for all $i < \kappa$ because, $\mathfrak{A}$ is a complete subalgebra of $\mathfrak{B}$. To see why, assume that $S \subseteq \mathfrak{A}$ and $\sum^S S = y$, and for contradiction that there exists $d \in \mathfrak{B}$ such that $s \leq d < y$ for all $s \in S$. Then, assuming that $A$ generates $\mathfrak{B}$, we can infer that $d$ uses finitely many dimensions in $\omega \sim n$, $m_1, \ldots, m_n$, say. Now let $t = y \cdot -c_{m_1} \ldots c_{m_n}(-d)$. We claim that $t \in \mathfrak{A} = \mathfrak{N}_n \mathfrak{B}$ and $s \leq t < y$ for all $s \in S$. This contradicts $y = \sum^S S$. The first required follows from the fact that $\Delta y \subseteq n$ and that all indices in $\omega \sim n$ that occur in $d$ are cylindrified. In more detail, put $J = \{m_1, \ldots, m_n\}$ and let $i \in \omega \sim n$, then $c_i t = c_i(-c_{i}(d)) = c_i - c_{i}(d) = c_i - c_{i}(j)(d) = -c_{i}(j)(d) = t$. We have shown that $c_i t = t$ for all $i \in \omega \sim n$, thus $t \in \mathfrak{N}_n \mathfrak{B} = \mathfrak{A}$. If $s \in S$, we show that $s \leq t$. We know that $s \leq y$. Also $s \leq d$, so $s \cdot d = 0$. Hence $0 = c_{m_1} \ldots c_{m_n}(s \cdot d) = s \cdot c_{m_1} \ldots c_{m_n}(-d) = s \cdot c_{m_1} \ldots c_{m_n}(d)$, hence $s \leq t$ as required. We finally check that $t < y$. If not, then $t = y$ so $y < t < c_{i}(j)(d) = c_{i}(j)(d)$, and so $y \cdot c_{i} \ldots c_{m_n}(d) = 0$. But $d \leq c_m \ldots c_{m_n}(-d)$, hence $y \cdot d < y \cdot c_{m} \ldots c_{m_n}(d) = 0$. Hence $y \cdot d = 0$ and this contradicts that $d < y$. We have proved that $\sum^S X = 1$ showing that $\mathfrak{A}$ is indeed a complete subalgebra of $\mathfrak{B}$. Since $\mathfrak{B}$ is a locally finite, we can assume that $\mathfrak{B} = \mathfrak{M}_{T}$ for some countable consistent theory $T$. For each $i < \kappa$, let $\Gamma_i = \{\phi/T : \phi \in X_i\}$. Let $F = (\Gamma_j : j < \kappa)$ be the corresponding set of types in $T$. Then each $\Gamma_j (j < \kappa)$ is a non-principal and complete $n$-type in $T$, because each $X_j$ is a maximal filter in $\mathfrak{A} = \mathfrak{N}_n \mathfrak{B}$. (*) Let $(M_i : i < 2^n)$ be a set of countable models for $T$ that overlap only on principal maximal types; these exist by Theorem 3.5.

The rest is exactly like in the proof of item (2) of Theorem 3.7 resorting to Shelah’s Theorem 3.5. Assume for contradiction that for all $i < 2^n$, there exists $\Gamma \in F$, such that $\Gamma$ is realized in $M_i$. Let $\psi : 2^\omega \rightarrow \wp(F)$, be defined by $\psi(i) = \{F \in F : F$ is realized in $M_i\}$. Then for all $i < 2^n$, $\psi(i) \neq \emptyset$. Furthermore, for $i \neq j$, $\psi(i) \cap \psi(j) = \emptyset$, for if $F \in \psi(i) \cap \psi(j)$, then it will be realized in $M_i$ and $M_j$, and so it will be principal. This implies that $|F| = 2^n$ which is impossible. Hence we obtain a model $M \models T$ omitting $X$ in which $\phi$ is satisfiable. The map $\psi$ defined from $\mathfrak{A} = \mathfrak{M}_{T}$ to $\mathfrak{C}_{n}^M$ (the set algebra based on $M$ [22, 4.3.4]) via $\psi_T \mapsto \phi^M$, where the latter is the set of $n$-ary assignments in $M$ satisfying $\phi$, omits $X$. Injectivity follows from the facts that $f$ is non–zero and $\mathfrak{A}$ is simple. The second item: We can assume that $\mathfrak{A} \subseteq_c \mathfrak{N}_n \mathfrak{B}$, and $A$ generates $\mathfrak{B}$, thus we can assume that $\mathfrak{B} \in L_{\kappa_1}$ is countable. Since $\mathfrak{A} \subseteq_c \mathfrak{N}_n \mathfrak{B} \subseteq_c \mathfrak{B}$, then $\mathfrak{A} \subseteq_c \mathfrak{B}$, so that the non-principal types in $\mathfrak{A}$ remain so in $\mathfrak{B}$. By Theorem 3.7 there exists $f : \mathfrak{B} \rightarrow c$, with $c \in \mathfrak{C}_{n}$, such that $f$ omitting $X$. Then $g = f \upharpoonright \mathfrak{A}$ will omit $X$ in $\mathfrak{N}_n c \in \mathfrak{C}_{n}$. For (2) and (3), we can assume that $\mathfrak{A} \subseteq_c \mathfrak{N}_n \mathfrak{B}, \mathfrak{B} \in L_{\kappa_1}$. We work in $\mathfrak{B}$. Using the notation on p. 216 of proof of Theorem 3.3.4] replacing $\mathfrak{M}_{T}$ by $\mathfrak{B}$, we have $H = \bigcup_{\epsilon \in \lambda} \bigcup_{\tau \in \tau} H_{\epsilon, \tau}$ where $\lambda < p$, and $V$ is the weak space $\omega^{\omega\lambda[\text{ld}]}$, can be written as a countable union of nowhere dense sets, and so can the countable union $G = \bigcup_{\epsilon \in \lambda} \bigcup_{\tau \in \tau} G_{\epsilon, \tau}$. So for any $a \neq 0$, there is an ultrafilter $F \in N_{\kappa_1} \cap (S \setminus (H \cup G))$ by the Baire category theorem. This induces a homomorphism $f_a : \mathfrak{A} \rightarrow \mathfrak{C}_a$, $c_a \in \mathfrak{C}_n$ that omits the given types, such that $f_a(a) \neq 0$. (First one defines $f$ with domain $\mathfrak{B}$ as on p.216, then restricts $f$ to $\mathfrak{A}$ obtaining $f_a$ the obvious way.) The map $g : \mathfrak{A} \rightarrow P_{a \in \mathfrak{A} \setminus \{0\}} \mathfrak{C}_a$ defined via $x \mapsto (g_a(x) : a \in \mathfrak{A} \setminus \{0\}) ((x \in \mathfrak{A})$ is as required. In case $\mathfrak{A}$ is simple, then by properties of $\text{covK}$, $S \setminus (H \cup G)$ is non–empty, so if $F \in S \setminus (H \cup G)$, then $F$ induces a non–zero homomorphism $F$ with domain $\mathfrak{A}$ into a $\mathfrak{C}_n$ omitting the given types. By simplicity of $\mathfrak{A}$, $\phi$ is injective.
To prove independence, it suffices to show that \( \text{cov}K \) many types may not be omitted because it is consistent that \( \text{cov}K < 2^\omega \). Fix \( 2 < n < \omega \). Let \( T \) be a countable theory such that for this given \( n \), in \( S_n(T) \), the Stone space of \( n \)-types, the isolated points are not dense. It is not hard to find such theories. An example, is the theory of random graphs. This condition excludes the existence of a prime model for \( T \) because \( T \) has a prime model \( \iff \) the isolated points in \( S_n(T) \) are dense for all \( n \). A prime model which in this context is an atomic model, omits any family of non-principal types (see the proof of the last item). We do not want this to happen. Using exactly the same argument in [12, Theorem 2.2(2)], one can construct a family \( P \) of non-principal 0-types (having no free variable) of \( T \), such that \( |P| = \text{cov}K \) and \( P \) cannot be omitted. Let \( A = \mathfrak{S}_m \) and for \( p \in P \), let \( X_p = \{ \phi/T : \phi \in p \} \). Then \( X_p \subseteq \mathfrak{N}_nA \), and \( \bigwedge X_p = 0 \), because \( \mathfrak{N}_nA \) is a complete subalgebra of \( A \). Then we claim that for any \( 0 \neq a \), there is no set algebra \( C \) with countable base and \( g : A \rightarrow C \) such that \( g(a) \neq 0 \) and \( \bigcap_{x < X_p} f(x) = \emptyset \). To see why, let \( B = \mathfrak{N}_nA \). Let \( a \neq 0 \). Assume for contradiction, that there exists \( f : B \rightarrow D' \), such that \( f(a) \neq 0 \) and \( \bigcap_{x < X_p} f(x) = \emptyset \). We can assume that \( B \) generates \( A \) and that \( D' = \mathfrak{N}_nB' \), where \( B' \subseteq \text{L}f_\omega \). Let \( g = \mathfrak{S}_p^{A \times B'} \). We show that \( g \) is a homomorphism with \( \text{dom}(g) = A \), \( g(a) \neq 0 \), and \( g \) omits \( P \), and for this, it suffices to show by symmetry that \( g \) is a function with domain \( A \). It obviously has co-domain \( B' \). Setting the domain is easy: \( \text{dom}g = \text{dom}\mathfrak{S}_p^{A \times B'} = \mathfrak{S}_p^{A \times B} = \mathfrak{S}_p^{A \times \mathfrak{N}_nA} = A \). Let \( K = \{ A \in \mathfrak{C}_\omega : A \subseteq \mathfrak{S}_p^{A \times \mathfrak{N}_nA} \}(\subseteq \text{L}f_\omega) \). We show that \( K \) is closed under finite direct products. Assume that \( C, D \in K \), then we have \( \mathfrak{S}_p^{C \times D} \mathfrak{N}_n(C \times D) = \mathfrak{S}_p^{C \times D} \mathfrak{N}_n(C \times \mathfrak{N}_nD) = \mathfrak{S}_p^{C \times \mathfrak{N}_nC} \mathfrak{N}_nD = \mathfrak{C} \times D \). Observe that \((*)\):

\[
(a, b) \in g \Delta [(a, b)] \subseteq n \implies f(a) = b.
\]

(Here \( \Delta[(a, b)] \) is the dimension set of \((a, b)\) defined via \( \{ i \in \omega : c_i(a, b) \neq (a, b) \} \)).

Indeed \((a, b) \in \mathfrak{N}_n(\mathfrak{S}_p^{A \times B} \mathfrak{N}_nA) = \mathfrak{S}_p^{\mathfrak{N}_nA \times \mathfrak{N}_nA} \mathfrak{N}_nA = f \). Now suppose that \((x, y), (x, z) \in g \). We need to show that \( y = z \) proving that \( g \) is a function. Let \( k \in \omega \setminus n \). Let \( \oplus \) denote ‘symmetric difference’. Then \((1)\):

\[
(0, c_k(y \oplus z)) = (c_k(0, c_k(y \oplus z)) = c_k(0, y \oplus z) = c_k((x, y) \oplus (x, z)) \in g.
\]

Also \((2)\),

\[
c_k(0, c_k(y \oplus z)) = (0, c_k(y \oplus z)).
\]

From \((2)\) by observing that \( k \) is arbitrarily chosen in \( \omega \sim n \), we get \((3)\):

\[
\Delta[(0, c_k(y \oplus z))] \subseteq n
\]

From \((1)\) and \((3)\) and \((*)\), we get \( f(0) = c_k(y \oplus z) \) for any \( k \in \omega \setminus n \). Fix \( k \in \omega \sim n \). Then by the above, upon observing that \( f \) is a homomorphism, so that in particular \( f(0) = 0 \), we get \( c_k(y \oplus z) = 0 \). But \( y \oplus z \leq c_k(x \oplus z) \), so \( y \oplus z = 0 \), thus \( y = z \). We have shown that \( g \) is a function, \( g(a) \neq 0 \), \( \text{dom}g = A \) and \( g \) omits \( P \). This contradicts that \( P \), by its construction, cannot be omitted. Assuming Martin’s axiom, we get \( \text{cov}K = p = 2^\omega \). Together with the above arguments this proves \((4)\).

We now prove \((5)\). Let \( A = \mathfrak{N}_nD, D \in \text{L}f_\omega \) is countable. Let \( \lambda < p \). Let \( X = (X_i : i < \lambda) \) be as in the hypothesis. Let \( T \) be the corresponding first order theory, so that \( D \cong \mathfrak{S}_m \). Let \( X' = (\Gamma_i : i < \lambda) \) be the family of non-principal types in \( T \) corresponding to \( X \). If \( X' \) is not omitted, then there is a (countable) realizing tree for \( T \), hence there is a realizing tree for a countable subfamily of \( X' \) in the sense of [12, Definition 3.1],
hence a countable subfamily of $X'$ cannot be omitted. Let $X_\omega \subseteq X$ be the corresponding countable subset of $X$. Assume that $X_\omega$ can be omitted in a $G_{S_n}$, via $f$ say. Then by the same argument used in proving item (4) $f$ can be lifted to $\hat{f}_{MT}$ omitting $X'$, which is a contradiction. We leave the part when $\mathfrak{A}$ is simple to the reader.

For (6): If $\mathfrak{A} \in S_n N_{r_n} \mathcal{C}A_\omega$, is atomic and has countably many atoms, then any complete representation of $\mathfrak{A}$, equivalently, an atomic representation of $\mathfrak{A}$, equivalently, a representation of $\mathfrak{A}$ omitting the set of co–atoms is as required.

\[ \square \]

Using the full power of Theorem 3.5 together with the argument in item (1) of Theorem 3.8, one can replace in the last item of the last corollary $\omega$ by any regular uncountable cardinal $\mu$ as explicitly formulated next,

**Theorem 3.9.** Let $\kappa$ be a regular infinite cardinal and $n < \omega$. Assume that $\mathfrak{A} \in N_{r_n} \mathcal{C}A_\omega$ with $|A| \leq \kappa$, that $\lambda$ is a cardinal $< 2^\kappa$, and that $X = (X_i : i < \lambda)$ is a family of non-principal types of $\mathfrak{A}$. If the $X_i$'s are non–principal ultrafilters of $\mathfrak{A}$, then $X$ can be omitted in a $G_{S_n}$.

We show (algebraically) that the maximality condition cannot be removed when we consider uncountable theories.

**Theorem 3.10.** Let $\kappa$ be an infinite cardinal. Then there exists an atomless $\mathcal{C} \in \mathcal{C}A_\omega$ such that for all $2 < n < \omega$, $|N_{r_n} \mathcal{C}| = 2^\kappa$, $N_{r_n} \mathcal{C} \in L\mathcal{C}A_n (= E\mathcal{C}R\mathcal{C}A_n)$, but $N_{r_n} \mathcal{C}$ is not completely representable. Thus the non–principal type of co–atoms of $N_{r_n} \mathcal{C}$ cannot be omitted. In particular, the condition of maximality in Theorem 3.9 cannot be removed.

**Proof.** We use the following uncountable version of Ramsey’s theorem due to Erdos and Rado: If $r \geq 2$ is finite, $k$ an infinite cardinal, then $exp_r(k) \to (k^+)^2 + 1$, where $exp_0(k) = k$ and inductively $exp_{r+1}(k) = 2^{exp_r(k)}$. The above partition symbol describes the following statement. If $f$ is a coloring of the $r + 1$ element subsets of a set of cardinality $exp_r(k)$ in $k$ many colors, then there is a homogeneous set of cardinality $k^+$ (a set, all whose $r + 1$ element subsets get the same $f$-value). We will construct the required $\mathcal{C} \in \mathcal{C}A_\omega$ from a relation algebra (to be denoted in a while by $\mathfrak{A}$) having an ‘$\omega$-dimensional cylindrical basis.’ To define the relation algebra, we specify its atoms and forbidden triples. Let $\kappa$ be the given cardinal in the hypothesis of the Theorem. The atoms are $Id$, $g_0^i : i < 2^\kappa$ and $r_j : 1 \leq j < \kappa$, all symmetric. The forbidden triples of atoms are all permutations of $(Id, x, y)$ for $x \neq y$, $(r_j, r_j, r_j)$ for $1 \leq j < \kappa$ and $(g_0^i, g_0^j, g_0^s)$ for $i, j, s < 2^\kappa$. Write $g_0$ for $\{g_0^i : i < 2^\kappa\}$ and $r_+$ for $\{r_j : 1 \leq j < \kappa\}$. Call this atom structure $\alpha$. Consider the term algebra $\mathfrak{A}$ defined to be the subalgebra of the complex algebra of this atom structure generated by the atoms. We claim that $\mathfrak{A}$, as a relation algebra, has no complete representation, hence any algebra sharing this atom structure is not completely representable, too. Indeed, it is easy to show that if $\mathfrak{A}$ and $\mathfrak{B}$ are atomic relation algebras sharing the same atom structure, so that $At\mathfrak{A} = At\mathfrak{B}$, then $\mathfrak{A}$ is completely representable $\iff \mathfrak{B}$ is completely representable. Assume for contradiction that $\mathfrak{A}$ has a complete representation with base $M$. Let $x, y$ be points in the representation with $M \models r_1(x, y)$. For each $i < 2^\kappa$, there is a point $z_i \in M$ such that $M \models g_0^i(x, z_i) \land r_1(z_i, y)$. Let $Z = \{z_i : i < 2^\kappa\}$. Within $Z$, each edge is labelled by one of the $\kappa$ atoms in $r_+$.

The Erdos-Rado theorem forces the existence of three points $z^1, z^2, z^3 \in Z$ such that $M \models r_j(z^1, z^2) \land r_j(z^2, z^3) \land r_j(z^3, z_1)$, for some single $j < \kappa$. This contradicts the definition of composition in $\mathfrak{A}$ (since we avoided
monochromatic triangles). Let $S$ be the set of all atomic $A$-networks $N$ with nodes $\omega$ such that \{ $r_i : 1 \leq i < \kappa : r_i$ is the label of an edge in $N$ \} is finite. Then it is straightforward to show $S$ is an amalgamation class, that is for all $M, N \in S$ if $M \equiv_{ij} N$ then there is $L \in S$ with $M \equiv L \equiv_j N$, witness [26], Definition 12.8, for notation. We have $S$ is symmetric, that is, if $N \in S$ and $\theta : \omega \rightarrow \omega$ is a finitary function, in the sense that \{ $i \in \omega : \theta(i) \neq i$ \} is finite, then $N\theta$ is in $S$. It follows that the complex algebra $\mathfrak{Ca}(S) \in \mathrm{QEA}_\omega$. Now let $X$ be the set of finite $A$-networks $N$ with nodes $\subseteq \kappa$ such that:

1. each edge of $N$ is either (a) an atom of $A$ or (b) a cofinite subset of $r_+ = \{ r_j : 1 \leq j < \kappa \}$ or (c) a cofinite subset of $g_0 = \{ g_0^i : i < 2^\kappa \}$ and

2. $N$ is ‘triangle-closed’, i.e. for all $l, m, n \in \mathrm{nodes}(N)$ we have $N(l, n) \leq N(l, m); N(m, n)$. That means if an edge $(l, m)$ is labelled by $Id$ then $N(l, n) = N(m, n)$ and if $N(l, m), N(m, n) \leq g_0$ then $N(l, n) : g_0 = 0$ and if $N(l, m) = N(m, n) = r_j$ (some $1 \leq j < \omega$) then $N(l, n) \cdot r_j = 0$.

For $N \in X$ let $\widehat{N} \in \mathfrak{Ca}(S)$ be defined by

\[
\{ L \in S : L(m, n) \leq N(m, n) \text{ for } m, n \in \mathrm{nodes}(N) \}.
\]

For $i \in \omega$, let $N|_{-i}$ be the subgraph of $N$ obtained by deleting the node $i$. Then if $N \in X$, $i < \omega$ then $c_i\widehat{N} = \widehat{N}|_{-i}$. The inclusion $c_i\widehat{N} \subseteq (\widehat{N}|_{-i})$ is clear. Conversely, let $L \in (\widehat{N}|_{-i})$. We seek $M \equiv L$ with $M \in \widehat{N}$. This will prove that $L \in \widehat{c}_i\widehat{N}$, as required. Since $L \in S$ the set $T = \{ r_i \notin L \}$ is infinite. Let $T$ be the disjoint union of two infinite sets $Y \cup Y'$, say. To define the $\omega$-network $M$ we must define the labels of all edges involving the node $i$ (other labels are given by $M \equiv L$). We define these labels by enumerating the edges and labeling them one at a time. So let $j \neq i < \kappa$. Suppose $j \in \mathrm{nodes}(N)$. We must choose $M(i, j) \leq N(i, j)$. If $N(i, j)$ is an atom then of course $M(i, j) = N(i, j)$. Since $N$ is finite, this defines only finitely many labels of $M$. If $N(i, j)$ is a cofinite subset of $g_0$ then we let $M(i, j)$ be an arbitrary atom in $N(i, j)$. And if $N(i, j)$ is a cofinite subset of $r_+$ then let $M(i, j)$ be an element of $N(i, j) \cap Y'$ which has not been used as the label of any edge of $M$ which has already been chosen (possible, since at each stage only finitely many have been chosen so far). If $j \notin \mathrm{nodes}(N)$ then we can let $M(i, j) = r_k \in Y$ some $1 \leq k < \kappa$ such that no edge of $M$ has already been labelled by $r_k$. It is not hard to check that each triangle of $M$ is consistent (we have avoided all monochromatic triangles) and clearly $M \in \widehat{N}$ and $M \equiv L$. The labeling avoided all but finitely many elements of $Y'$, so $M \in S$. So $(\widehat{N}|_{-i}) \subseteq \widehat{c}_i\widehat{N}$. Now let $\widehat{X} = \{ \widehat{N} : N \in X \} \subseteq \mathfrak{Ca}(S)$. Then we claim that the subalgebra of $\mathfrak{Ca}(S)$ generated by $\widehat{X}$ is simply obtained from $\widehat{X}$ by closing under finite unions. Clearly all these finite unions are generated by $\widehat{X}$. We must show that the set of finite unions of $\widehat{X}$ is closed under all cylindric operations. Closure under unions is given. For $\widehat{N} \in X$ we have $-\widehat{N} = \bigcup_{m, n \in \text{nodes}(N)} \widehat{N}_{mn}$ where $N_{mn}$ is a network with nodes $\{ m, n \}$ and labeling $N_{mn}(m, n) = -\widehat{N}(m, n)$. $N_{mn}$ may not belong to $X$ but it is equivalent to a union of at most finitely many members of $\widehat{X}$. The diagonal $d_{ij} \in \mathfrak{Ca}(S)$ is equal to $\widehat{N}$ where $N$ is a network with nodes $\{ i, j \}$ and labeling $N(i, j) = Id$. Closure under cylindrification is given. Let $\mathfrak{C}$ be the subalgebra of $\mathfrak{Ca}(S)$ generated by $\widehat{X}$. Then $\mathfrak{A} = \mathfrak{RaC}$. To see why, each element of $\mathfrak{A}$ is a union of a finite number of atoms, possibly a co-finite subset of $g_0$ and possibly a co-finite subset of $r_+$. Clearly $\mathfrak{A} \subseteq \mathfrak{RaC}$. Conversely, each element $z \in \mathfrak{RaC}$ is a finite union $\bigcup_{N \in F} \widehat{N}$, for some finite subset $F$ of $X$, satisfying $c_iz = z$, for $i > 1$. Let $i_0, \ldots, i_k$ be an
enumeration of all the nodes, other than 0 and 1, that occur as nodes of networks in \( F \). Then, \( c_{i_0} \ldots c_{i_k} = \bigcup_{N \in F} c_{i_0} \ldots c_{i_k} \hat{N} = \bigcup_{N \in F} \{ \hat{N}_{\{0,1\}} \} \in \mathfrak{A} \). So \( \mathfrak{ReC} \subseteq \mathfrak{A} \). Thus \( \mathfrak{A} \) is the relation algebra reduct of \( \mathfrak{C} \in \mathfrak{CA}_\omega \), but \( \mathfrak{A} \) has no complete representation. Let \( n > 2 \). Let \( \mathfrak{B} = \mathfrak{ReC} \). Then \( \mathfrak{B} \in \mathfrak{N}_{\omega} \mathfrak{CA}_\omega \) is atomic, but has no complete representation for plainly a complete representation of \( \mathfrak{B} \) induces one of \( \mathfrak{A} \). In fact, because \( \mathfrak{B} \) is generated by its two dimensional elements, and its dimension is at least three, its \( \mathbf{Df} \) reduct is not completely representable [?, Proposition 4.10]. We show that \( \mathfrak{B} \) is in \( \mathfrak{ELCRA}_\omega = \mathfrak{LCA}_\omega \). By Lemma [4.5] \( \exists \) has a winning strategy in \( G_\omega (\mathbb{A} \mathfrak{B}) \), hence \( \exists \) has a winning strategy in \( G_k (\mathbb{A} \mathfrak{B}) \) for all \( k < \omega \). Using ultrapowers and an elementary chain argument, we get that there is a countable \( \mathfrak{C} \) such that \( \mathfrak{B} \equiv \mathfrak{C} \), so that \( \mathfrak{C} \) is atomic and \( \exists \) has a winning strategy in \( G_\omega (\mathbb{A} \mathfrak{C}) \). Since \( \mathfrak{C} \) is countable then by [27] Theorem 3.3.3 it is completely representable. It remains to show that the \( \omega \)-dilation \( \mathfrak{C} \) is atomless. For any \( N \in X \), we can add an extra node extending \( N \) to \( M \) such that \( \emptyset \subseteq M' \subseteq N' \), so that \( N' \) cannot be an atom in \( \mathfrak{C} \).

4 Clique guarded semantics

Fix \( 2 < n < \omega \) (locally well–behaved) relativized representations, in analogy to the relation algebra case dealt with in [26, Chapter 13]; such localized representations are called \( m \)-square with \( 2 < n < m \leq \omega \), with \( \omega \)-square representations coinciding with ordinary ones. It will always be the case, unless otherwise explicitly indicated, that \( 1 < n < m < \omega \); \( n \) denotes the dimension. But first we recall certain relativized set algebras. A set \( V (\subseteq nU) \) is diagonalizable if \( s \in V \Rightarrow s \circ [i,j] \in V \). We say that \( V \subseteq nU \) is locally square if whenever \( s \in V \) and \( \tau : n \to n \), then \( s \circ \tau \in V \). Let \( \mathcal{D}_n (G_n) \) be the class of set algebras whose top elements are diagonalizable (locally square) and operations are defined like cylindric set algebra of dimension \( n \) relativized to the top element \( V \).

**Theorem 4.1.** [8]. Fix \( 2 < n < \omega \). Then \( \mathcal{D}_n \) and \( \mathcal{G}_n \) are finitely axiomatizable and have a decidable universal (hence equational) theory.

We identify notationally a set algebra with its universe. Let \( \mathcal{M} \) be a relativized representation of \( \mathfrak{A} \in \mathfrak{CA}_\omega \), that is, there exists an injective homomorphism \( f : \mathfrak{A} \to \varphi (V) \) where \( V \subseteq nM \) and \( \bigcup_{s \in V} \text{rng}(s) = M \). For \( s \in V \) and \( a \in \mathfrak{A} \), we may write \( a(s) \) for \( s \in f(a) \). This notation does not refer to \( f \), but whenever used then either \( f \) will be clear from context, or immaterial in the context. We may also write \( 1^M \) for \( V \). We assume that \( \mathcal{M} \) carries an Alexandrov topology. Let \( \mathcal{L}(\mathfrak{A})^m \) be the first order signature using \( m \) variables and one \( n \)-ary relation symbol for each element of \( \mathfrak{A} \).

An \( n \)-clique, or simply a clique, is a set \( C \subseteq \mathcal{M} \) such that \( (a_0, \ldots, a_{n-1}) \in V = 1^M \) for all distinct \( a_0, \ldots, a_{n-1} \in C \). Let

\[
C^m(M) = \{ s \in {}^mM : \text{rng}(s) \text{ is an } n \text{ clique} \}.
\]

Then \( C^m(M) \) is called the \( n \)-Gaifman hypergraph, or simply Gaifman hypergraph of \( M \), with the \( n \)-hyperedge relation \( 1^M \). The \( n \)-clique–guarded semantics, or simply clique–guarded semantics, \( \models_c \), are defined inductively. Let \( f \) be as above. For an atomic \( n \)-ary formula \( a \in \mathfrak{A}, i \in {}^n m \), and \( s \in {}^mM, M, s \models_c a[x_{i_0}, \ldots, x_{i_{n-1}}] \iff (s_{i_0}, \ldots, s_{i_{n-1}}) \in f(a) \). For equality, given \( i < j < m, M, s \models_c x_i = x_j \iff s_i = s_j \). Boolean connectives, and infinitary disjunctions, are defined as expected. Semantics for existential quantifiers (cylindrifiers) are defined inductively for \( \phi \in \mathcal{L}(A)^{m, \omega} \) as follows: For \( i < m \) and \( s \in {}^mM \),
\( M, s \models_c \exists x \phi \iff \) there is a \( t \in C^m(M) \), \( t \equiv_s s \) such that \( M, t \models_c \phi \). Finally, \( M \models \Box_i \phi \iff \exists t \in C^m(M) : t_k \in \text{int}\{ u \in M : s^u_i \models \phi \}

**Definition 4.2.** Let \( A \) be an algebra having the signature of \( CA_n \), \( M \) a relativized representation of \( A \) carrying an Alexandrov topology and \( L(A)^m \) be as above. Then \( M \) is said to be \( m\)-square, if witnesses for cylindrifiers can be found on \( n \)-cliques. More precisely, for all \( s \in C^m(M), a \in A, i < n \), and for any injective map \( l : n \rightarrow m \), if \( M \models c_i A(s_{l(0)} \ldots s_{l(n-1)}) \), then there exists \( t \in C^m(M) \) with \( t \equiv_i s \) and \( M \models a(t_{l(0)} \ldots t_{l(n-1)}) \).

**Definition 4.3.** An \( n \)-dimensional atomic network on an atomic algebra \( A \in CA_n \) is a map \( N : n \Delta \rightarrow \text{At}A \), where \( \Delta \) is a non-empty finite set of nodes carrying a topology (hence an Alexandrov topology because the underlying set of nodes is finite), denoted by \( \text{nodes}(N) \), satisfying the following consistency conditions for all \( i < j < n \):

1. If \( \bar{x} \in n \text{nodes}(N) \) then \( N(\bar{x}) \leq d_{ij} \iff \bar{x}_i = \bar{x}_j 
2. If \( \bar{x}, \bar{y} \in n \text{nodes}(N) \), \( i < n \) and \( \bar{x} \equiv_i \bar{y} \), then \( N(\bar{x}) \leq c_i N(\bar{y}) \),

The proof of the following lemma can be distilled from its RA analogue [26, Theorem 13.20], by reformulating deep concepts originally introduced by Hirsch and Hodkinson for RAs in the CA context, involving the notions of hypernetworks and hyperbase. In the coming proof, we highlight the main ideas needed to perform such a transfer from RAs to CAs [26 Definitions 12.1, 12.9, 12.10, 12.25, Propositions 12.25, 12.27]. Fix \( 1 < n < \omega \). For \( A \in D_n \) with top element \( V \), the base of \( A \) is \( M = \bigcup_{s \in V \text{rng}(s)} s \), so that \( V \subseteq nM \) and \( M \) is the smallest such set. We let \( D_m^n \) be the variety of \( D_n \)'s whose base carry an Alexandro topolgy and with interior operators defined like in cylindric set algebras. In all cases, the \( m \)-dimensional dilation is a set algebra in \( D_m^n \), as stipulated in the statement of the theorem, will have top element \( C^m(M) \), where \( M \) is the \( m \)-relativized representation of the given algebra, and the operations of the dilation are induced by the \( n \)-clique-guarded semantics. For a class \( K \) of BAOs, \( K \cap \text{At} \) denotes the class of atomic algebras in \( K \).

**Lemma 4.4.** [26, Theorems 13.45, 13.36]. Assume that \( 2 < n < m < \omega \) and let \( A \) have the signature of \( TCA_n \). Then \( A \in S_n D_m^{\text{top}} \iff A \) has an \( m \)-square representation. Furthermore, if \( A \) is atomic, then \( A \) has a complete \( m \)-square representation \( \iff A \in S_n \text{At}n(D_m^{\text{top}} \cap \text{At}) \).

### 4.1 Non atom-canonicity and omitting types

We recall that a class \( K \) of Boolean algebras with operators (BAOs) is atom-canonical if whenever \( A \in K \) is atomic and completely additive, then its Dedekind-MacNeille completion, namely, the complex algebra of its atom structure, namely, \( C\text{mA} \) is also in \( K \). We use in what follows instances of the so-called blow up and blur construction. But first a Lemma:

**Lemma 4.5.** Let \( 2 < n < m \leq \omega \). Let \( A \in CA_n \). If \( \forall \) has winning strategy in \( G_m^n(\text{At}A) \), then \( A \) does not have an \( m \)-square representation.

**Definition 4.6.** A \( TCA_n \) atom structure \( \text{At} \) is weakly representable if there is an atomic \( A \in RTCA_n \) such that \( \text{At} = \text{At}A \); it is strongly representable if \( C\text{mA} \in RTCA_n \).
These two notions are distinct for \( 2 < n < \omega \), cf. [31] for the CA case and the next Theorem.

### 4.2 Blowing up and blurring finite rainbow cylindric algebras

In [4] a single blow up and blur construction was used to prove non-atom–canonicity of RRA and RCA\(_n\) for \( 2 < n < \omega \). To obtain finer results, we use two blow up and blur constructions applied to rainbow algebras. To put things into a unified perspective, we formulate a definition:

**Definition 4.7.** Let \( M \) be a variety of completely additive BAOs.

1. Let \( A \in M \) be a finite algebra. We say that \( D \in M \) is obtained by blowing up and blurring \( A \) if \( D \) is atomic, \( A \) does not embed in \( D \), but \( A \) embeds into \( \mathcal{C}m\text{At}D \).

2. Assume that \( K \subseteq L \subseteq M \), such that \( \mathcal{S}L = L \).
   
   (a) We say that \( K \) is not atom-canonical with respect to \( L \) if there exists an atomic \( D \in K \) such that \( \mathcal{C}m\text{At}D / \in L \). In particular, \( K \) is not atom–canonical \( \iff \) \( K \) not atom-canonical with respect to itself.

   (b) We say that a finite algebra \( A \in M \) detects that \( K \) is not atom–canonical with respect to \( L \), if \( A / \in L \), and there is a(n atomic) \( D \in K \) obtained by blowing up and blurring \( A \).

The next proposition and its proof present the construction in [4] in the framework of definition 4.7.

**Proposition 4.8.** Let \( 2 < n < \omega \). Then for any finite \( j > 0 \), RRA \( \cap \mathcal{R}\text{CA}_{2+j} \) is not atom-canonical with respect to RRA, and RCA\(_n\) \( \cap \mathcal{N}\text{r}_n\text{CA}_{n+j} \) is not atom–canonical with respect to RCA\(_n\).

Till the end of this subsection, fix \( 2 < n < \omega \). The most general exposition of CA rainbow constructions is given in [27, Section 6.2, Definition 3.6.9] in the context of constructing atom structures from classes of models. Our models are just coloured graphs [25]. Let \( G, R \) be two relational structures. Let \( 2 < n < \omega \). Then the colours used are:

- **greens**: \( g_i \) (\( 1 \leq i \leq n - 2 \)), \( g_0^i \), \( i \in G \),
- **whites**: \( w_i : i \leq n - 2 \),
- **reds**: \( r_{ij} \) (\( i, j \in R \)),
- **shades of yellow**: \( y_S : S \) a finite subset of \( \omega \) or \( S = \omega \).

A **coloured graph** is a graph such that each of its edges is labelled by the colours in the above first three items, greens, whites or reds, and some \( n-1 \) hyperedges are also labelled by the shades of yellow. Certain coloured graphs will deserve special attention.

**Definition 4.9.** Let \( i \in G \), and let \( M \) be a coloured graph consisting of \( n \) nodes \( x_0, \ldots, x_{n-2}, z \). We call \( M \) an \( i \)-cone if \( M(x_0, z) = g_0^i \) and for every \( 1 \leq j \leq n - 2 \), \( M(x_j, z) = g_j \), and no other edge of \( M \) is coloured green. \( (x_0, \ldots, x_{n-2}) \) is called the **base of the cone**, \( z \) the **apex of the cone** and \( i \) the **tint of the cone**.

The rainbow algebra depending on \( G \) and \( R \) from the class \( K \) consisting of all coloured graphs \( M \) such that:
1. $M$ is a complete graph and $M$ contains no triangles (called forbidden triples) of the following types:

\begin{align*}
(g_i, g_j, g_k), (g_i, g_i, w_i) & \quad \text{any } 1 \leq i \leq n - 2, \\
(g^j_0, g^k_0, w_0) & \quad \text{any } j, k \in G, \\
(r_{ij}, r_{j'k'}, r^{*r}_{i'k'}) & \quad \text{unless } |\{(j, k), (j', k'), (j^{*}, k^{*})\}| = 3
\end{align*}

and no other triple of atoms is forbidden.

2. If $a_0, \ldots, a_{n-2} \in M$ are distinct, and no edge $(a_i, a_j)$ $i < j < n$ is coloured green, then the sequence $(a_0, \ldots, a_{n-2})$ is coloured a unique shade of yellow. No other $(n - 1)$ tuples are coloured shades of yellow. Finally, if $D = \{d_0, \ldots, d_{n-2}, \delta\} \subset M$ and $M \upharpoonright D$ is an $i$ cone with apex $\delta$, inducing the order $d_0, \ldots, d_{n-2}$ on its base, and the tuple $(d_0, \ldots, d_{n-2})$ is coloured by a unique shade $y_S$ then $i \in S$.

Let $G$ and $R$ be relational structures as above. Take the set $J$ consisting of all surjective maps $a : n \to \Delta$, where $\Delta \in K$ and define an equivalence relation $\sim$ on this set relating two such maps iff they essentially define the same graph [25]; the nodes are possibly different but the graph structure is the same. Let $\text{At}$ be the atom structure with underlying set $J \sim$. We denote the equivalence class of $a$ by $[a]$. Then define, for $i < j < n$, the accessibility relations corresponding to $ij$th–diagonal element, and $i$th–cylindrifier, as follows:

1. $[a] \in E_{ij}$ iff $a(i) = a(j)$,
2. $[a]T_i[b]$ iff $a \upharpoonright n \setminus \{i\} = b \upharpoonright n \setminus \{i\}$.

This, as easily checked, defines a $\text{CA}_n$ atom structure. The complex $\text{CA}_n$ over this atom structure will be denoted by $\mathfrak{A}_{G,R}$. The dimension of $\mathfrak{A}_{G,R}$, always finite and $> 2$, will be clear from context. For rainbow atom structures, there is a one to one correspondence between atomic networks and coloured graphs [25], Lemma 30, so for $2 < n < m \leq \omega$, we use the graph versions of the games $G^m_k$, $k \leq \omega$, and $G^m$ played on rainbow atom structures of dimension $m$ [25] pp.841–842. The the atomic $k$ rounded game game $G^m_k$ where the number of nodes are limited to $n$ to games on coloured graphs [25] lemma 30]. The game $G^m$ lifts to a game on coloured graphs, that is like the graph games $G^m_\omega$ [25], where the number of nodes of graphs played during the $\omega$ rounded game does not exceed $m$, but $\forall$ has the option to re-use nodes. The typical winning strategy for $\forall$ in the graph version of both atomic games is bombarding $\exists$ with cones having a common base and green tints until she runs out of (suitable) reds, that is to say, reds whose indicies do not match [25, 4.3].

**Definition 4.10.** A $\text{CA}_n$ atom structure $\text{At}$ is weakly representable if there is an atomic $\mathfrak{A} \in \text{RCA}_n$ such that $\text{At} = \text{At}\mathfrak{A}$; it is strongly representable if $\text{CmAt} \in \text{RCA}_n$.

These two notions are distinct, cf. [31] and the following Theorem 4.11, see also the forthcoming Theorem 5.7. Let $V \subseteq W$ be varieties of BAOs. $W$ say that $V$ is atom-canonical with respect to $W$ if for any atomic $\mathfrak{A} \in V$, its Dedekind-MacNeille completion, namely, $\text{CmAt}\mathfrak{A}$ is in $W$. Let $\text{R}_{ca}$ denote ‘cylindric reduct’

**Theorem 4.11.** Let $2 < n < \omega$ and $t(n) = n(n+1)/2 + 1$. The variety $\text{TRCA}_n$ is not atom-canonical with respect to $\text{SNr}_n\text{CA}_{t(n)}$. In fact, there is a countable atomic simple
$A \in \text{TRCA}_n$ such that $\text{RCA}_{\omega}$ does not have an $t(n)$-square, a fortiori $t(n)$-flat, representation.

Proof. The proof is long and uses many ideas in [31]. We will highlight only the differences in detail from the proof in [31] needed to make our result work. When parts of the proof coincide we will be more sketchy. The proof is divided into four parts:

1: Blowing up and blurring $\mathfrak{B}_f$ forming a weakly representable atom structure $\text{At}$: Take the finite rainbow $\text{CA}_n$, $\mathfrak{B}_f$ where the reds $R$ is the complete irreflexive graph $n$, and the greens are $\{g_i : 1 \leq i < n-1\} \cup \{g_0 : 1 \leq i \leq n(n-1)/2\}$, endowed with the cylindric operations. We will show $\mathfrak{B}$ detects that $\text{RCA}_n$ is not atom-canonical with respect to $\text{Sn}_{n}\text{CA}_{\omega}(n)$ with $\alpha(t)$ as specified in the statement of the theorem. Denote the finite atom structure of $\mathfrak{B}_f$ by $\text{At}_f$, so that $\text{At}_f = \text{At}(\mathfrak{B}_f)$. One then defines a larger the class of coloured graphs like in [31, Definition 2.5]. Let $2 < n < \omega$. Then the colours used are like above except that each red is ‘split’ into $\rho$ many having ‘copies’ the form $r_{ij}^l$ with $i < j < n$ and $l \in \omega$, with an additional shade of red $\rho$ such that the consistency conditions for the new reds (in addition to the usual rainbow consistency conditions) are as follows:

- $(r_{jk}^l, r_{j'k'}^l, r_{j^*k^*})$ unless $i = i'$ and $|\{(j, k), (j', k'), (j^*, k^*)\}| = 3$
- $(r, \rho, \rho)$ and $(r, r^*, \rho)$, where $r, r^*$ are any reds.

The consistency conditions can be coded in an $L_{\omega, \omega}$ theory $T$ having signature the reds with $\rho$ together with all other colours like in [27, Definition 3.6.9]. The theory $T$ is only a first order theory (not an $L_{\omega_1, \omega}$ theory) because the number of greens is finite which is not the case with [27] where the number of available greens are countably infinite coded by an infinite disjunction. One construct an $n$-homogeneous model $M$ is as a countable limit of finite models of $T$ using a game played between $\exists$ and $\forall$ like in [31, Theorem 2.16]. In the rainbow game $\forall$ challenges $\exists$ with cones having green tints $(g_i^0)$, and $\exists$ wins if she can respond to such moves. This is the only way that $\forall$ can force a win. $\exists$ has to respond by labelling apexes of two successive cones, having the same base played by $\forall$. By the rules of the game, she has to use a red label. She resorts to $\rho$ whenever she is forced a red while using the rainbow reds will lead to an inconsistent triangle of reds; [31, Proposition 2.6, Lemma 2.7]. The number of greens make [31, Lemma 3.10] work with the same proof. using only finitely many green and not infinitely many. The winning strategy is implemented by $\exists$ using the red label $\rho$ that comes to her rescue whenever she runs out of ‘rainbow reds’, so she can always and consistently respond with an extended coloured graph. This proof will is implemented in the framework of an entirely analogous blow up and blur constructions applied to strikingly similar finite rainbow atom structures in [26]. In both cases, the relational structures $G$ and $R$ used satisfy $|G| = |R| + 1$. For $\text{RA}$, $R = 3$ and for $\text{CA}_n$, $R = n$ (the dimension), where the finite ordinals $3$ and $n$ are viewed as complete irreflexive graphs. From Hodkinson’s construction in [31], we know that $\mathcal{M} \mathfrak{B}(\mathfrak{B}_f, r, \omega) \not\in \text{Sn}_m\text{CA}_n$ for some finite $m > n$, where $\mathfrak{B}(\mathfrak{B}_f, r, \omega)$ denotes the result of blowing up $\mathfrak{B}_f$ by splitting each red atom into $\omega$-many ones, to be denoted henceforth by $A$. The (semantical) argument used in [31] does not give any information on the value of such $m$. By truncating the greens to be

\footnote{Worthy of note, is that it is commonly accepted that relation algebras have dimension three being a natural habitat for three variable first order logic. Nevertheless, sometimes it is argued that the dimension should be three and a half in the somewhat loose sense that RAs lie ‘halfway’ between $\text{CA}_3$ and $\text{CA}_4$ manifesting behaviour of each.}
This is needed for representing $A$ with only one non-principle ultrafilter, that can be identified with the shade of red its canonical extension, in a fairly simple step by step manner. The atom structure $\text{At} = Uf(\mathcal{B}(\mathcal{B}_f,r,\omega))$ of $\mathfrak{A}$ consists of principal ultrafilters generated by atoms, together with only one non-principle ultrafilter, that can be identified with the shade of red $\rho$. This is needed for representing $\mathfrak{A}$, but not completely; the atom structure $\text{At}$ is not and cannot be completely representable; it is not even stongly representable. As a matter of fact, it is just weakly representable, with all these notions of representability for atom structures are taken from \[27\].

2. Representing a term algebra (and its completion) as (generalized) set algebras: Having $M$ at hand, one constructs two atomic $n$-dimensional set algebras based on $M$, sharing the same atom structure and having the same top element. The atoms of each will be the set of coloured graphs, seeing as how, quoting Hodkinson \[31\] such coloured graphs are `literally indivisible'. Now $L_n$ and $L_{\infty,\omega}$ are taken in the rainbow signature (without $\rho$). Continuing like in op.cit, deleting the one available red shade, set $W = \{\bar{a} \in {}^nM : M \models (\wedge_{i<j<n} -\rho(x_i,x_j))(\bar{a})\}$, and for $\phi \in L_{\infty,\omega}^n$, let $\phi^W = \{s \in W : M \models \phi[s]\}$. Here $W$ is the set of all $n$-ary assignments in $^nM$, that have no edge labelled by $\rho$. Let $\mathfrak{A}$ be the relativized set algebra with domain $\{\varphi^W : \varphi$ a first-order $L_n -$formula$\}$ and unit $W$, endowed with the usual concrete quasipolyadic operations read off the connectives. Classical semantics for $L_n$ rainbow formulas and their semantics by relativizing to $W$ coincide \[31\] Proposition 3.13] but not with respect to $L_{\infty,\omega}$ rainbow formulas. Hence the set algebra $\mathfrak{A}$ is isomorphic to a cylindric set algebra of dimension $n$ having top element $^nM$, so $\mathfrak{A}$ is simple, in fact its DF reduct is simple. Let $\mathcal{E} = \{\varphi^W : \phi \in L_{\infty,\omega}^n\}$ \[31\] Definition 4.1] with the operations defined like on $\mathfrak{A}$ the usual way. $\mathcal{E}mAt$ is a complete CA$_n$ and, so like in \[31\] Lemma 5.3 we have an isomorphism from $\mathcal{E}mAt$ to $\mathcal{E}$ defined via $X \mapsto \bigcup X$. Since At$\mathfrak{A} = \text{At}\mathcal{M}(\text{At}\mathfrak{A})$, which we refer to only by $\text{At}$, and $\mathcal{E}\text{mAt}\mathfrak{A} \subseteq \mathfrak{A}$, hence $\mathcal{E}\text{mAt}\mathfrak{A} = \text{mAt}\mathfrak{A}$ is representable. The atoms of $\mathfrak{A}$, $\text{mAt}\mathfrak{A}$ and $\mathcal{E}\text{mAt}\mathfrak{A} = \mathcal{E}mAt$ are the coloured graphs whose edges are not labelled by $\rho$. These atoms are uniquely determined by the interpretation in $M$ of so-called MCA formulas in the rainbow signature of $\text{At}$ as in \[31\] Definition 4.3]. Giving $M$ the discrete topology make both algebras topological set algebras, who extra unary modal operators all coincide with the identity operator, which we denote by adding with a sight abuse of notation, denote by the notation used for their cylindric reducts. No confusion is like to ensue. Though the shades of red is outside signature, it was as a label during an $\omega$–rounded game played on labelled finite graphs—which can be seen as finite models in the extended signature having size $\leq n$—in which $\exists$ had a winning strategy, enabling her to construct the required $M$ as a countable limit of the finite graphs played during the game. The construction entails that any subgraph (substructure) of $M$ of size $\leq n$, is independent of its location in $M$; it is uniquely determined by its isomorphism type. A relativized set algebra $\mathfrak{A}$ based on $M$ was constructed by discarding all assignments whose edges are labelled by these shades of reds, getting a set of $n$-ary sequences $W \subseteq {}^nM$. This $W$ is definable in $^nM$ by an $L_{\infty,\omega}$ formula and the semantics with respect to $W$ coincides with classical Tarskian semantics (on $^nM$) for formulas of the signature taken in $L_n$ (but not for formulas taken in $L_{\infty,\omega}$). This was proved in both cases using certain
The heart and soul of the proof: In the set algebra $A$, one replaces the red label by suitable non-red binary relation symbols within an $n$ back-and-forth system, so that one can adjust that the system maps a tuple $\bar{b} \in {}^nM/W$ to a tuple $\bar{c} \in W$ and this will preserve any formula containing the non-red symbols that are ‘moved’ by the system. In fact, all injective maps of size $\leq n$ defined on $M$ modulo an appropriate permutation of the reds will form an $n$ back-and-forth system. This set algebra $\mathfrak{A}$ was further atomic, countable, and simple (with top element $^nM$). The subgraphs of size $\leq n$ of $M$ whose edges are not labelled by any shade of red are the atoms of $\mathfrak{A}$, expressed syntactically by MCA formulas. The Dedekind-MacNeille of $\mathfrak{A}$, in symbols $\mathfrak{CmAt}\mathfrak{A}$, has top element $W$, but it is not in $\mathbb{SNr_nCA}_{t(n)}$ in case of the rainbow construction, let alone representable. In this constructions ‘the shades of red’ – which can be intrinsically identified with non-principal ultrafilters in $\mathfrak{A}$, were used as colours, together with the principal ultrafilters to completely represent $\mathfrak{A}^+$, inducing a representation of $\mathfrak{A}$. Non-representability for Monk like constructions use an uncontrolable Ramsey number determined by Ramsey’s theory. The non neat-embeddability of thr rainbow like algebra $\mathfrak{CmAt}\mathfrak{A}$ in the present more stronger case, we used a finite number of greens that gave us more delicate information on when $\mathfrak{CmAt}\mathfrak{A}$ stops to be representable. The reds, particularly $\rho$ acting as a non-principle ultrafilter had to do with representing $\mathfrak{A}$ using non-atomic networks.

3. Embedding $\mathfrak{A}_{n+1,n}$ into $\mathfrak{Cm}(\mathfrak{At}(\mathfrak{Bb}(\mathfrak{A}_{n+1,n}, r, \omega)))$: Let $\mathfrak{CRG}_f$ be the class of coloured graphs on $\mathfrak{At}_f$ and $\mathfrak{CRG}$ be the class of coloured graph on $\mathfrak{At}$. We can (and will) assume that $\mathfrak{CRG}_f \subseteq \mathfrak{CRG}$. Write $M_a$ for the atom that is the (equivalence class of the) surjection $a : n \to M$, $M \in \mathfrak{CRG}$. Here we identify $a$ with $[a]$; no harm will ensue. We define the (equivalence) relation $\sim$ on $\mathfrak{At}$ by $M_b \sim N_a$, $(M, N \in \mathfrak{CRG})$:

- $a(i) = a(j) \iff b(i) = b(j)$,
- $M_a(a(i), a(j)) = r^l \iff N_b(b(i), b(j)) = r^k$, for some $l, k \in \omega$,
- $M_a(a(i), a(j)) = N_b(b(i), b(j))$, if they are not red,
- $M_a(a(k_0), \ldots, a(k_{n-2})) = N_b(b(k_0), \ldots, b(k_{n-2}))$, whenever defined.

We say that $M_a$ is a copy of $N_b$ if $M_a \sim N_b$ (by symmetry $N_b$ is a copy of $M_a$.) Indeed, the relation ‘copy of’ is an equivalence relation on $\mathfrak{At}$. An atom $M_a$ is called a red atom, if $M_a$ has at least one red edge. Any red atom has $\omega$ many copies, that are cylindrically equivalent, in the sense that, if $N_a \sim M_b$ with one (equivalently both) red, with $a : n \to N$ and $b : n \to M$, then we can assume that $\text{nodes}(N) = \text{nodes}(M)$ and that for all $i < n$, $a \upharpoonright n \sim \{i\} = b \upharpoonright n \sim \{i\}$. In $\mathfrak{CmAt}$, we write $M_a$ for $\{M_a\}$ and we denote suprema taken in $\mathfrak{CmAt}$, possibly finite, by $\sum$. Define the map $\Theta$ from $\mathfrak{A}_{n+1,n} = \mathfrak{CmAt}_f$ to $\mathfrak{CmAt}$, by specifying first its values on $\mathfrak{At}_f$, via $M_a \mapsto \sum_j M_a^{(j)}$ where $M_a^{(j)}$ is a copy of $M_a$. So each atom maps to the suprema of its copies. This map is well-defined because $\mathfrak{CmAt}$ is complete. We check that $\Theta$ is an injective homomorphism. Injectivity is easy. We check preservation of all the $\mathbb{CA}_n$ extra Boolean operations.

$n$ back-and-forth systems, thus $\mathfrak{A}$ is representable classically, in fact it (is isomorphic to a set algebra that) has base $M$. 

The heart and soul of the proof; In the set algebra $A$, one replaces the red label by suitable non-red binary relation symbols within an $n$ back-and-forth system, so that one can adjust that the system maps a tuple $\bar{b} \in {}^nM/W$ to a tuple $\bar{c} \in W$ and this will preserve any formula containing the non-red symbols that are ‘moved’ by the system. In fact, all injective maps of size $\leq n$ defined on $M$ modulo an appropriate permutation of the reds will form an $n$ back-and-forth system. This set algebra $\mathfrak{A}$ was further atomic, countable, and simple (with top element $^nM$). The subgraphs of size $\leq n$ of $M$ whose edges are not labelled by any shade of red are the atoms of $\mathfrak{A}$, expressed syntactically by MCA formulas. The Dedekind-MacNeille of $\mathfrak{A}$, in symbols $\mathfrak{CmAt}\mathfrak{A}$, has top element $W$, but it is not in $\mathbb{SNr_nCA}_{t(n)}$ in case of the rainbow construction, let alone representable. In this constructions ‘the shades of red’ – which can be intrinsically identified with non-principal ultrafilters in $\mathfrak{A}$, were used as colours, together with the principal ultrafilters to completely represent $\mathfrak{A}^+$, inducing a representation of $\mathfrak{A}$. Non-representability for Monk like constructions use an uncontrolable Ramsey number determined by Ramsey’s theory. The non neat-embeddability of thr rainbow like algebra $\mathfrak{CmAt}\mathfrak{A}$ in the present more stronger case, we used a finite number of greens that gave us more delicate information on when $\mathfrak{CmAt}\mathfrak{A}$ stops to be representable. The reds, particularly $\rho$ acting as a non-principle ultrafilter had to do with representing $\mathfrak{A}$ using non-atomic networks.
Lemma 4.5, the required follows.

Then, $B$ is distinct

In more detail, $t \in \mathrm{win}$ this game in $Ehrenfeucht–Fraissé$ forth game to the graph game on 'longer' than $n$ \(\sum_{\mathbf{M} \leq \mathbf{n}}\). Using (any) $t > n$ many pairs of pebbles available on the board $\forall$ can win this game in $n + 1$ many rounds. $\forall$ lifts his winning strategy from the lst private Ehrenfeucht–Fraissé forth game to the graph game on $\mathbf{At_f = At(\mathfrak{B}_f)}$ \[25\] pp. 841 forcing a win using $t(n)$ nodes. One uses the $n(n - 1)/2 + 2$ green relations in the usual way to force a red clique $C$, say with $n(n - 1)/2 + 2$. Pick any point $x \in C$. Then there are $n(n - 1)/2$ points $y$ in $C \{x\}$. There are only $n(n - 1)/2$ red relations. So there must be distinct $y, z \in C \{x\}$ such that $(x, y)$ and $(x, z)$ both have the same red label (it will be some $r^m_{ij}$ for $i < j < n$). But $(y, z)$ is also red, and this contradicts \[31\] Definition 2.5(2), 4th bullet point. In more detail, $\forall$ bombards $\exists$ with cones having common base and distinct green tints until $\exists$ is forced to play an inconsistent red triangle (where indicies of reds do not match). He needs $n - 1$ nodes as the base of cones, plus $|P| + 2$ more nodes, where $P = \{(i, j) : i < j < n\}$ forming a red clique, triangle with two edges satisfying the same $r^m_{ij}$ for $p \in P$. Calculating, we get $t(n) = n - 1 + n(n - 1)/2 + 2 = n(n + 1)/2 + 1$ By Lemma \[14\] $\mathfrak{B}_f \notin \mathbf{SNr_n CA}_{t(n)}$ when $2 < n < \omega$. Since $\mathfrak{B}_f$ is finite, then $\mathfrak{B}_f \notin \mathbf{SNr_n CA}_{t(n)}$, because $\mathfrak{B}_f$ coincides with its canonical extension and for any $\mathfrak{D} \in \mathbf{CA}_\mathbf{n}$, $\mathfrak{D} \in \mathbf{SNr}_n \mathbf{CA}_{2n} \implies \mathfrak{D}^+ \in \mathbf{SNr}_n \mathbf{CA}_{2n}$. But $\mathfrak{B}_f$ embeds into $\mathfrak{N}_\omega \mathfrak{C} \mathfrak{m} \mathfrak{At}_2 \mathfrak{A}^{\text{top}}$, hence $\mathfrak{N}_\omega \mathfrak{C} \mathfrak{m} \mathfrak{A} \mathfrak{t}_2 \mathfrak{A}^{\text{top}}$ is outside the variety $\mathbf{SNr}_n \mathbf{CA}_{t(n)}$, as well. By the second part of Lemma \[14\] the required follows. \[\Box\]
4.3 On non-elementary classes related to the class of completely representable algebras

**Definition 4.12.** For an $n$-dimensional atomic network $N$ on an atomic $\text{CA}_n$ and for $x, y \in \text{nodes}(N)$, set $x \sim y$ if there exists $\bar{z}$ such that $N(x, y, \bar{z}) \leq d_{01}$. Define the equivalence relation $\sim$ over the set of all finite sequences over $\text{nodes}(N)$ by $\bar{x} \sim \bar{y}$ iff $|\bar{x}| = |\bar{y}|$ and $x_i \sim y_i$ for all $i < |\bar{x}|$. (It can be easily checked that this indeed an equivalence relation.) A hypernetwork $N = (N^a, N^h)$ over an atomic $\text{CA}_n$ consists of an $n$-dimensional network $N^a$ together with a labelling function for hyperlabels $N^h : \text{nodes}(N) \to \Lambda$ (some arbitrary set of hyperlabels $\Lambda$) such that for $\bar{x}, \bar{y} \in \text{nodes}(N)$ if $\bar{x} \sim \bar{y}$ then $N^h(\bar{x}) = N^h(\bar{y})$. If $|\bar{x}| = k \in \mathbb{N}$ and $N^h(\bar{x}) = \lambda$, then we say that $\lambda$ is a $k$-ary hyperlabel. $\bar{x}$ is referred to as a $k$-ary hyperedge, or simply a hyperedge. A hyperedge $\bar{x} \in \text{nodes}(N)$ is short, if there are $y_0, \ldots, y_{n-1}$ that are nodes in $N$, such that $N(x_i, y_0, \bar{z}) \leq d_{01}$ for all $i < |\bar{x}|$, for some (equivalently for all) $\bar{z}$. Otherwise, it is called long. This game involves, besides the standard cylindrifier move, two new amalgamation moves. Concerning his moves, this game with $m$ rounds ($m \leq \omega$), call it $H_m$. For can play a cylindrifier move, like before but now played on $\lambda$—neat hypernetworks ($\lambda$ a constant label). Also $\exists$ can play an transformation move by picking a previously played hypernetwork $N$ and a partial, finite surjection $\theta : \omega \to \text{nodes}(N)$, this move is denoted $(N, \theta)$. $\exists$'s response is mandatory. She must respond with $N\theta$. Finally, $\forall$ can play an amalgamation move by picking previously played hypernetworks $M, N$ such that $M\mid_{\text{nodes}(M) \cap \text{nodes}(N)} = N\mid_{\text{nodes}(M) \cap \text{nodes}(N)}$, and $\text{nodes}(M) \cap \text{nodes}(N) \neq \emptyset$. This move is denoted $(M, N)$. To make a legal response, $\exists$ must play a $\lambda_0$–neat hypernetwork $L$ extending $M$ and $N$, where $\text{nodes}(L) = \text{nodes}(M) \cup \text{nodes}(N)$.

The next Lemma will be needed to prove Theorem 4.13 and Corollary 6.14 which are the main results in this section. With Theorem 4.13 they constitute the core of this article.

**Theorem 4.13.** Let $\alpha$ be a countable atom structure. If $\exists$ has a winning strategy in $H_{\omega}(\alpha)$, then there exists a complete $\mathcal{D} \in \text{RCA}_\omega$ such that $\mathcal{C}^\alpha \equiv \mathcal{N}_i, \mathcal{D}$ In particular, $\mathcal{C}^\alpha \equiv \mathcal{N}_i, \text{CA}_\omega$

**Proof.** Fix some $a \in \alpha$. The game $H_\omega(\alpha)$ is designed so that using $\exists$'s winning strategy in the game $H_\omega(\alpha)$ one can define a nested sequence $M_0 \subseteq M_1, \ldots$ of $\lambda$–neat hypernetworks where $M_0$ is $\exists$'s response to the initial $\forall$-move $a$, such that: If $M_r$ is in the sequence and $M_r(x) \leq \epsilon_a$ for an atom $a$ and some $i < n$, then there is $s \geq r$ and $d \in \text{nodes}(M_s)$ such that $M_s(y) = a$, $y_i = d$ and $\bar{y} \equiv_i \bar{x}$. In addition, if $M_r$ is in the sequence and $\theta$ is any partial isomorphism of $M_r$, then there is $s \geq r$ and a partial isomorphism $\theta^+$ of $M_s$ extending $\theta$ such that $\text{rng}(\theta^+) \subseteq \text{nodes}(M_r)$. (This can be done using $\exists$'s responses to amalgamation moves). Now let $M_\omega$ be the limit of this sequence, that is $M_\omega = \bigcup_j M_i$, the labelling of $n - 1$ tuples of nodes by atoms, and hyperedges by hyperlabels done in the obvious way using the fact that the $M_i$s are nested. Let $L$ be the signature with one $n$-ary relation for each $b \in \alpha$, and one $k$-ary predicate symbol for each $k$-ary hyperlabel $\lambda$. Now we work in $L_{\infty, \omega}$. For fixed $f_a \in \text{nodes}(M_\omega)$, let $U_a = \{ f \in \text{nodes}(M_a) : i < \omega : g(i) \neq f_a(i) \}$ is finite. We make $U_a$ into the base of an $L$ relativized structure $M_a$ allowing a clause for infinitary disjunctions. In more detail, for $b \in \alpha$, $l_0, \ldots, l_{n-1}, i_0, \ldots, i_{k-1} < \omega$, $k$-ary hyperlabels $\lambda$, and all $L$-formulas $\phi, \phi_i, \psi,$
and \( f \in U_a: \)

\[
\begin{align*}
M_a, f \models b(x_0, \ldots, x_{n-1}) & \iff M_a(f(l_0), \ldots, f(l_{n-1})) = b, \\
M_a, f \models \lambda(x_{i_0}, \ldots, x_{i_{k-1}}) & \iff M_a(f(i_0), \ldots, f(i_{k-1})) = \lambda, \\
M_a, f \models \phi & \iff M_a, f \not\models \phi, \\
M_a, f \models \bigvee_{i \in I} \phi_i & \iff (\exists i \in I)(M_a, f \models \phi_i), \\
M_a, f \models \exists x \phi & \iff M_a, f[i/m] \models \phi, \text{ some } m \in \text{nodes}(M_a).
\end{align*}
\]

For any such \( L\)-formula \( \phi \), write \( \phi^{M_a} \) for \( \{ f \in \mathcal{U}_a : M_a, f \models \phi \} \). Let \( D_a = \{ \phi^{M_a} : \phi \text{ is an } L\text{-formula} \} \) and \( \mathcal{D}_a \) be the weak set algebra with universe \( D_a \). Let \( \mathcal{D} = \mathcal{P}_{a \in \alpha} \mathcal{D}_a \). Then \( \mathcal{D} \) is a generalized complete weak set algebra [22 Definition 3.1.2 (iv)]. Now we show \( \mathcal{Cm}=\mathfrak{N}_{\mathcal{D}} \mathcal{O} \). Let \( X \subseteq \mathfrak{N}_{\mathcal{D}} \mathcal{O} \). Then by completeness of \( \mathcal{D} \), we get that \( d = \sum\mathcal{D} X \) exists. Assume that \( i \notin n \), then \( c_i d = c_i \sum X = \sum_{x \in X} c_i x = \sum X = d \), because the \( c_i \)s are completely additive and \( c_i x = x \), for all \( i \notin n \), since \( x \in \mathfrak{N}_{\mathcal{D}} \mathcal{O} \). We conclude that \( d \in \mathfrak{N}_{\mathcal{D}} \mathcal{O} \), hence \( d \) is an upper bound of \( X \) in \( \mathfrak{N}_{\mathcal{D}} \mathcal{O} \). Since \( d = \sum\mathcal{D} X \) there can be no \( b \in \mathfrak{N}_{\mathcal{D}} \mathcal{O} (\subseteq \mathcal{D}) \) with \( b < d \) such that \( b \) is an upper bound of \( X \) for else it will be an upper bound of \( X \) in \( \mathcal{D} \). Thus \( \sum\mathcal{D} X = d \) We have shown that \( \mathfrak{N}_{\mathcal{D}} \mathcal{O} \) is complete. Making the legitimate identification \( \mathfrak{N}_{\mathcal{D}} \mathcal{O} \subseteq_d \mathcal{Cm} \) by density, we get that \( \mathfrak{N}_{\mathcal{D}} \mathcal{O} = \mathcal{Cm} \alpha \) (since \( \mathfrak{N}_{\mathcal{D}} \mathcal{O} \) is complete), hence \( \mathcal{Cm} \in \mathfrak{N}_{\mathcal{D}} \mathfrak{C}_{\alpha} \). \( \square \)

**Theorem 4.14.** Let \( \mathcal{O} \in \{ \mathcal{S}_{d}, \mathcal{S}_{d}, \mathcal{I}, \} \), where \( \mathcal{S}_{d} \) denotes the operation of forming dense subalgebras and let \( k \geq 3 \). Then the class of frames \( \mathcal{K}_k = \{ \mathcal{G} : \mathcal{Cm}\mathcal{G} \in \mathfrak{N}_{\mathcal{D}} \mathfrak{C}_{\alpha+n+k} \} \) is not elementary. In particular, the class of extremely representable algebras up to \( n+k \) is not elementary.

**Proof.** (1) **Defining a rainbow-like atom structure \( \alpha \):** We use the algebra in [44 Theorem 5.12]. The algebra \( \mathcal{C}_{\mathcal{Z}_n}(\mathcal{RCA}_n) \) based on \( \mathcal{Z} \) (greens) and \( \mathcal{N} \) (reds) denotes the rainbow-like algebra used in op.cit which is defined as follows: The reds \( \mathcal{R} \) is the set \( \{ r_{ij} : i < j < \omega (= \mathcal{N}) \} \) and the green colours used constitute the set \( \{ g_{i} : 1 \leq i < n - 1 \} \cup \{ g_{0} : i \in \mathcal{Z} \} \). In complete coloured graphs the forbidden triples are like the usual rainbow constructions based on \( \mathcal{Z} \) and \( \mathcal{N} \), with a significant addition: First the colours used are:

- greens: \( g_{i} : 1 \leq i \leq n - 2 \), \( g_{0} : i \in \mathcal{Z} \),
- whites: \( w_{i} : i \leq n - 2 \),
- reds: \( r_{ij} : (i, j) \in \mathcal{N} \),
- shades of yellow: \( y_{S} : S \) a finite subset of \( \omega \) or \( S = \omega \).

The rainbow algebra depending on \( \mathcal{N} \) and \( \mathcal{Z} \) from the class \( \mathcal{K} \) consisting of all coloured graphs \( M \) such that:

1. \( M \) is a complete graph and \( M \) contains no triangles (called forbidden triples) of the following types:

\[
\begin{align*}
(g, g', g^{*}), (g_{i}, g_{i}, w_{i}) & \text{ any } 1 \leq i \leq n - 2, \\
(g_{0}^{j}, g_{0}^{k}, w_{0}) & \text{ any } j, k \in A, \\
(r_{ij}, r_{j'k'}, r_{i'k'}^{*}) & \text{ unless } i = i', j = j' \text{ and } k' = k^{*}
\end{align*}
\]
Observe that this 1.7 is not as item 1.3 in the proof of Theorem 4.11. Here inconsistent triples of reds are defined differently.

2. The triple $(g^i_0, g^i_0, r_{kl})$ is also forbidden if \{\{(i, k), (j, l)\}\} is not an order preserving partial function from $Z \to \mathbb{N}$

It is proved in *op.cit* that $\exists$ has a winning strategy in $G_k(\text{At} \mathfrak{C}_{Z, N})$ for all $k \in \omega$, so that $\mathfrak{C}_{Z, N} \in \text{ELCRCA}_n$. With some more effort it can be proved that $\exists$ has a winning strategy $\sigma_k$ say in $H_k(\text{At} \mathfrak{C}_{Z, N})$ for all $k \in \omega$. Let $\alpha = \text{At} \mathfrak{C}_{Z, N}$.

(2) $\exists$ **has a winning strategy** in $H_\omega(\alpha)$: We describe $\exists$’s strategy in dealing with labelling hyperedges in $\lambda$–neat hypernetworks, where $\lambda$ is a constant label kept on short hyperedges and, not to interrupt the main stream, we defer the rest of the highly technical proof to the appendix. In a play, $\exists$ is required to play $\lambda$–neat hypernetworks, so she has no choice about the the short edges, these are labelled by $\lambda$. In response to a cylindrifier move by $\forall$ extending the current hypernetwork providing a new node $k$, and a previously played coloured hypernetwork $M$ all long hyperedges not incident with $k$ necessarily keep the hyperlabel they had in $M$. All long hyperedges incident with $k$ in $M$ are given unique hyperlabels not occurring as the hyperlabel of any other hyperedge in $M$. In response to an amalgamation move, which involves two hypernetworks required to be amalgamated, say $(M, N)$ all long hyperedges whose range is contained in $\text{nodes}(M)$ have hyperlabel determined by $M$, and those whose range is contained in $\text{nodes}(N)$ have hyperlabels determined by $N$. If $x$ is a long hyperedge of $\exists$’s response $L$ where $\text{rng}(x) \not\subseteq \text{nodes}(M), \text{nodes}(N)$ then $x$ is given a new hyperlabel, not used in any previously played hypernetwork and not used within $L$ as the label of any hyperedge other than $x$. This completes her strategy for labelling hyperedges. In *op.cit* it is shown that $\exists$ has a winning strategy in $G_k(\text{At} \mathfrak{C}_{Z, N})$ where $0 < k < \omega$ is the number of rounds. With some more effort it can be prove that $\exists$ has a winning strategy $\text{in} H_k(\mathfrak{C})$ for each $k < \omega$, call it $\sigma_k$. We can assume that $\sigma_k$ is deterministic. Let $\mathfrak{D}$ be a non–principal ultrapower of $\mathfrak{C}_{Z, N}$. Then $\exists$ has a winning strategy $\sigma$ in $H_\omega(\text{At} \mathfrak{D})$ — essentially she uses $\sigma_k$ in the $k$’th component of the ultraproduct so that at each round of $H_\omega(\text{At} \mathfrak{D})$, $\exists$ is still winning in co-finitely many components, this suffices to show she has still not lost. We can also assume that $\mathfrak{C}_{Z, N}$ is countable by replacing it by the term algebra. Now one can use an elementary chain argument to construct countable elementary subalgebras $\mathfrak{C}_{Z, N} = \mathfrak{A}_0 \preceq \mathfrak{A}_1 \preceq \ldots \preceq \ldots \mathfrak{D}$ in this manner. One defines $\mathfrak{A}_{i+1}$ be a countable elementary subalgebra of $\mathfrak{D}$ containing $\mathfrak{A}_i$ and all elements of $\mathfrak{D}$ that $\sigma$ selects in a play of $G(\text{At} \mathfrak{D})$ in which $\forall$ only chooses elements from $\mathfrak{A}_i$. Now let $\mathfrak{B} = \bigcup_{i<\omega} \mathfrak{A}_i$. This is a countable elementary subalgebra of $\mathfrak{D}$, hence necessarily atomic, and $\exists$ has a winning strategy in $H_\omega(\text{At} \mathfrak{B})$ and $\mathfrak{B} \equiv \mathfrak{C}_{Z, N}$. Thus $\text{At} \mathfrak{B} \in \text{At} \text{NR}_n \text{CA}_\omega$ and $\mathfrak{C}m \text{At} \mathfrak{B} \in \text{NR}_n \text{CA}_\omega$. (This does not imply that $\mathfrak{B} \in \text{NR}_n \text{CA}_\omega$, cf. example 6.4). Since $\mathfrak{B} \subseteq_d \mathfrak{C}m \text{At} \mathfrak{B}$, $\mathfrak{B} \in \text{S}_d \text{NR}_n \text{CA}_\omega$, so $\mathfrak{B} \in \text{S}_c \text{NR}_n \text{CA}_\omega$. Being countable, it follows by *op.cit* Theorem 5.3.6 that $\mathfrak{B} \in \text{CRCA}_n$.

(3) $\forall$ **has a winning strategy** in $G^{n+3}(\alpha)$: We now show that $\forall$ has a winning strategy in $G^{n+3}(\text{At} \mathfrak{C}_{Z, N})$ (denoted in *op.cit* by $F^{n+3}(\text{At} \mathfrak{C}_{Z, N})$), hence by Lemma 4.5 $\mathfrak{C}_{Z, N} \notin \text{S}_c \text{NR}_n \text{CA}_{n+3}$. It can be shown that $\forall$ has a winning strategy in the graph version of the game $G^{n+3}(\mathfrak{C})$ played on coloured graphs [25]. The rough idea here, is that, as is the case with winning strategy’s of $\forall$ in rainbow constructions, $\forall$ bombards $\exists$ with cones having distinct green tints demanding a red label from $\exists$ to apexes of successive cones. The number of nodes are limited but $\forall$ has the option to re-use them, so this process will not end after finitely many rounds. The added order preserving condition relating two greens and a red, forces $\exists$ to choose red labels, one of whose indices form
There is a finite decreasing sequence in $\mathbb{N}$. In $\omega$ many rounds $\forall$ forces a win, so by Lemma 15 $\mathcal{C} \notin S_{cN}CA_{n+3}$. More rigorously, $\forall$ plays as follows: In the initial round $\forall$ plays a graph $M$ with nodes $0, 1, \ldots, n - 1$ such that $M(i, j) = w_0$ for $i < j < n - 1$ and $M(i, n - 1) = g_i$ $(i = 1, \ldots, n - 2)$, $M(0, n - 1) = g_0$ and $M(0, 1, \ldots, n - 2) = y_Z$. This is a 0 cone. In the following move $\forall$ chooses the base of the cone $(0, \ldots, n - 2)$ and demands a node $n$ with $M_2(i, n) = g_i$ $(i = 1, \ldots, n - 2)$, and $M_2(0, n) = g_0^{-1}$. $\exists$ must choose a label for the edge $(n + 1, n)$ of $M_2$. It must be a red atom $r_{mk}$, $m, k \in \mathbb{N}$. Since $-1 < 0$, then by the ‘order preserving’ condition we have $m < k$. In the next move $\forall$ plays the face $(0, 1, \ldots, n - 2)$ and demands a node $n + 1$, with $M_3(i, n) = g_i$ $(i = 1, \ldots, n - 2)$, such that $M_3(0, n + 2) = g_0^{-2}$. Then $M_3(n + 1, n) = g_0$ and $M_3(n + 1, n - 1)$ both being red, the indices must match. $M_3(n + 1, n) = r_{lk}$ and $M_3(n + 1, r - 1) = r_{km}$ with $l < m \in \mathbb{N}$. In the next round $\forall$ plays $(0, 1, \ldots, n - 2)$ and re-uses the node 2 such that $M_4(0, 2) = g_0^{-3}$. This time we have $M_4(n, n - 1) = r_{jl}$ for some $j < l < m \in \mathbb{N}$. Continuing in this manner leads to a decreasing sequence in $\mathbb{N}$. We have proved the required.

(4): proving the required Let $K$ be a class between $S_dNr_nCA_\omega \cap CRCA_n$ and $S_dNr_nCA_{n+3}$. Then $K$ is not elementary, because $\mathcal{E}_{Z,N} \notin S_dNr_nCA_{n+3}(\exists K)$, $\exists \in S_dNr_nCA_\omega \cap CRCA_n(\subseteq K)$, and $\mathcal{E}_{Z,N} \equiv \mathcal{B}$. It clearly suffices to show that $K_k = \{\mathcal{A} \in CA_n \cap At : \mathcal{E}mAt\mathcal{A} \in ONr_nCA_k\}$ is not elementary. $\exists$ has a winning strategy in $H_\omega(\alpha)$ for some countable atom structure $\alpha$, $\mathcal{E}m\mathcal{A} \subseteq \mathcal{E}m\alpha \in Nr_nCA_\omega$ and $\mathcal{E}m\alpha \in CRCA_n$. Since $\mathcal{E}_{Z,N} \notin S_dNr_nCA_{n+3}$, then $\mathcal{E}_{Z,N} = \mathcal{E}mAt\mathcal{E}_{Z,N} \notin K_k$, $\mathcal{E}_{Z,N} \equiv \mathcal{E}m\alpha \in K_k$ because $\mathcal{E}m\alpha \in Nr_nCA_\omega \subseteq S_dNr_nCA_\omega \subseteq S_cNr_nCA_\omega$. We have shown that $\mathcal{E}_{Z,N} \in EIK_k \sim K_k$, proving the required.

Having the necessary tools at our disposal in the previous proof, we obtain the following substantial generalization of Theorem 5.8. Let $S_d$ denote the operation of forming dense subagbras. First observe that from Theorem 5.10 the two classes $CRCA_n$ and $S_dNr_nCA_\omega$ are mutually distinct.

Corollary 4.15. Let $2 < n < \omega$ and $m \geq n + 3$. Then any any class $K$ such that $S_dNr_nCA_\omega \cap CRCA_n \subseteq K \subseteq S_cNr_nCA_m$ is not elementary.

Proof. The rainbow-like atomic algebra algebra $\mathcal{E}_{Z,N}$ having countably many atoms is outside $S_dNr_nCA_{n+3}$. However, the algebra $\mathcal{E}_{Z,N}$ is elementary equivalent to a countable atomic algebra $\mathcal{B}$ such that $\exists$ has a winning strategy in $H_\omega(\exists$), so by Lemma 4.13 $\mathcal{E}mAt\mathcal{B} \in Nr_nCA_\omega$. Thus $\mathcal{B} \in S_dNr_nCA_\omega \subseteq S_cNr_nCA_\omega$. Being countable, by [11, Theorem 5.3.3], we get $\mathcal{B} \in CRCA_n$.

For a variety $V$ of BAOs, $\text{Str}(V) = \{\mathcal{E} : \mathcal{E}m\mathcal{E} \in V\}$. Fix finite $k > 2$. Then $V_k = \text{Str}(SNr_nCA_{n+k})$ is not elementary $\Rightarrow V_k$ is not-atom canonical because (arguing contrapositively) in the case of atom-canonicity, we get that $\text{Str}(SNr_nCA_{n+k}) = \text{At}(SNr_nCA_{n+k})$, and the last class is elementary [26, Theorem 2.84]. However, the converse implication may fail. In particular, we do not know whether $\text{Str}(SNr_nCA_{n+k})$, for a particular finite $k \geq 3$, is elementary or not. Nevertheless, it is easy to show that there has to be a finite $k < \omega$ such that $V_j$ is not elementary for all $j > k$:

Theorem 4.16. There is a finite $k$ such that the class of strongly representable algebras up to $n + k$ is not elementary.
Proof. We show that $\text{Str}(\text{SN}_n \text{CA}_m)$ is not elementary for some finite $m \geq n + 2$. Let $(\mathfrak{A}_i : i \in \omega)$ be a sequence of (strongly) representable $\text{CA}_n$s with $\mathcal{C}m\text{At}\mathfrak{A}_i = \mathfrak{A}_i$ and $\mathfrak{A} = \Pi_{i \in \omega} \mathfrak{A}_i$ is not strongly representable with respect to any non-principal ultrafilter $U$ on $\omega$. Such algebras exist [27]. Hence $\mathcal{C}m\text{At}\mathfrak{A} \not\subseteq \text{SN}_n \text{CA}_\omega = \bigcap_{i \in \omega} \text{SN}_n \text{CA}_{n + i}$, so $\mathcal{C}m\text{At}\mathfrak{A} \not\subseteq \text{SN}_n \text{CA}_l$ for all $l > m$, for some $m \in \omega$, $m \geq n + 2$. But for each such $l$, $\mathfrak{A}_i \in \text{SN}_n \text{CA}_l (\subseteq \text{RCA}_n)$, so $(\mathfrak{A}_i : i \in \omega)$ is a sequence of algebras such that $\mathcal{C}m\text{At}(\mathfrak{A}_i) \not\subseteq \text{SN}_n \text{CA}_l (i \in I)$, but $\mathcal{C}m(\text{At}(\Pi_{i \in \omega} \mathfrak{A}_i)) = \mathcal{C}m\text{At}(\mathfrak{A}) \not\subseteq \text{SN}_n \text{CA}_l$, for all $l \geq m$. 

\section{An application on OTTs, Vaught’s Theorem for TopL$_\alpha$}

\textbf{Definition 5.1.} Let $R$ be an atomic relation algebra. An $n$–dimensional basic matrix, or simply a matrix on $R$, is a map $f : 2^n \to \text{At}R$ satisfying the following two consistency conditions $f(x, x) \leq \text{Id}$ and $f(x, y) \leq f(x, z); f(y, z)$ for all $x, y, z < n$. For any $f, g$ basic matrices and $x, y < n$ we write $f \equiv_x g$ if for all $w, z \in m \setminus \{x, y\}$ we have $f(w, z) = g(w, z)$. We may write $f \equiv g$ instead of $f \equiv_x g$.

\textbf{Definition 5.2.} An $n$–dimensional cylindrical basis for an atomic relational algebra $R$ is a set $\mathcal{M}$ of $n$–dimensional matrices on $R$ with the following properties:

- If $a, b, c \in \text{At}R$ and $a \leq b; c$, then there is an $f \in \mathcal{M}$ with $f(0, 1) = a, f(0, 2) = b$ and $f(2, 1) = c$
- For all $f, g \in \mathcal{M}$ and $x, y < n$, with $f \equiv_x g$, there is $h \in \mathcal{M}$ such that $f \equiv h \equiv y g$.

For the next lemma, we refer the reader to [26] Definition 12.11] for the definition of hyperbasis for relation algebras as well as to [26] Chapter 13, Definitions 13.4, 13.6] for the notions of $n$–flat and $n$–square representations for relation algebras ($n > 2$) to be generalized below to cylindric algebras, cf. Definition [4.2]. For a relation algebra $R$, recall that $R^+$ denotes its canonical extension.

\textbf{Lemma 5.3.} Let $R$ be a relation algebra and $3 < n < \omega$. Then the following hold:

1. $R^+$ has an $n$–dimensional infinite basis $\iff$ $R$ has an infinite $n$–square representation.
2. $R^+$ has an $n$–dimensional infinite hyperbasis $\iff$ $R$ has an infinite $n$–flat representation.

Proof. [26] Theorem 13.46, the equivalence (1) $\iff$ (5) for basis, and the equivalence (7) $\iff$ (11) for hyperbasis].

One can construct a $\text{CA}_n$ in a natural way from an $n$–dimensional cylindrical basis which can be viewed as an atom structure of a $\text{CA}_n$ (like in [26] Definition 12.17] addressing hyperbasis). For an atomic relation algebra $R$ and $l > 3$, we denote by $\text{Mat}_n(\text{At}R)$ the set of all $n$–dimensional basic matrices on $R$. $\text{Mat}_n(\text{At}R)$ is not always an $n$–dimensional cylindrical basis, but sometimes it is, as will be the case described next. On the other hand, $\text{Mat}_3(\text{At}R)$ is always a 3–dimensional cylindrical basis; a result of Maddux’s, so that $\mathcal{C}m\text{Mat}_3(\text{At}R) \in \text{CA}_3$. The following definition to be used in the sequel is taken from [4]:

\textbf{Definition 5.4.} [4] Definition 3.1] Let $R$ be a relation algebra, with non–identity atoms $I$ and $2 < n < \omega$. Assume that $J \subseteq \varphi(I)$ and $E \subseteq ^3 \omega$. 

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1. We say that \((J, E)\) is an \(n\)-blur for \(R\), if \(J\) is a complex \(n\)-blur defined as follows:

   (1) Each element of \(J\) is non-empty,

   (2) \( \bigcup J = I \),

   (3) \((\forall P \in I)(\forall W \in J)(I \subseteq P; W)\),

   (4) \((\forall V_1, \ldots, V_n, W_2, \ldots, W_n \in J)(\exists T \in J)(\forall 2 \leq i \leq n)\) \(\text{safe}(V_i, W_i, T)\), that is there is for \(v \in V_i, w \in W_i\) and \(t \in T\), we have \(v; w \leq t\),

   (5) \((\forall P_2, \ldots, P_n, Q_2, \ldots, Q_n \in I)(\forall W \in J)W \cap P_2; Q_2 \cap \ldots P_n; Q_n \neq \emptyset\).

   and the tenary relation \(E\) is an index blur defined as in item (ii) of [4] Definition 3.1.

2. We say that \((J, E)\) is a strong \(n\)-blur, if it \((J, E)\) is an \(n\)-blur, such that the complex \(n\)-blur satisfies:

   \((\forall V_1, \ldots, V_n, W_2, \ldots, W_n \in J)(\forall T \in J)(\forall 2 \leq i \leq n)\) \(\text{safe}(V_i, W_i, T)\).

The following theorem concisely summarizes the construction in [4] and says some more easy facts.

**Theorem 5.5.** Let \(2 < n \leq l < \omega\). Let \(R\) be a finite relation algebra with an \(l\)-blur \((J, E)\) where \(J\) is the \(l\)-complex blur and \(E\) is the index blur, as in definition 5.4.

1. Then for \(R = \mathcal{B}(R, J, E)\), with atom structure \(\mathcal{A}(R)\) obtained by blowing up and blurring \(R\) (with underlying set is denoted by \(\mathcal{A}(R)\) on [4, p.73]), the set of \(l\) by \(l\)-dimensional matrices \(\mathcal{A}(R)_{ca} = \text{Mat}(\mathcal{A}(R))\) is an \(l\)-dimensional cylindric basis, that is a weakly representable atom structure [4, Theorem 3.2]. The algebra \(\mathcal{B}(R, J, E)\), with last notation as in [4] Top of p. 78] having atom structure \(\mathcal{A}(R)_{ca}\) is in \(\text{RCA}_l\). Furthermore, \(R\) embeds into \(\mathcal{C}(\mathcal{A}(R))\) which embeds into \(\mathcal{R}(\mathcal{A}(R)_{ca})\).

2. For very \(n < l\), there is an \(R\) having a strong \(l\)-blur but no finite representations. Hence \(\mathcal{A}(R)\) obtained by blowing up and blurring \(R\) and the \(\text{CA}_m\) atom structure \(\mathcal{A}(R)_{ca}\) as in the previous item are not strongly representable.

3. Let \(m < \omega\). If \(R\) is as in the hypothesis, \((J, E)\) is a strong \(l\)-blur, and \(R\) has no \(m\)-dimensional hyperbasis, then \(l < m\).

4. If \(n = l < m < \omega\) and \(R\) as above has no infinite \(m\)-dimensional hyperbasis, then \(\mathcal{C}(\mathcal{R}) \mathcal{B}(R, J, E) \notin \text{SN}_n \text{CA}_m\), and the latter class is not atom-canonical.

5. If \(2 < n \leq l < m \leq \omega\), and \((J, E)\) is a strong \(m\)-blur, definition 5.4, then \((J, E)\) is a strong \(l\)-blur, \(\mathcal{B}(R, J, E) \cong \mathcal{R}(\mathcal{B}(R, J, E))\) and \(R \cong \mathcal{R}(\mathcal{B}(R, J, E)) \cong \mathcal{R}(\mathcal{B}(R, J, E))\).

**Proof.** Cf. [4]. For notation, cf. p.73, p.80, and for proofs cf. Lemmata 3.2, 4.2, 4.3]. We start by an outline of (1). Let \(R\) be as in the hypothesis. The idea is to blow up and blur \(R\) in place of the Maddux algebra \(\mathcal{E}_k(2, 3)\) dealt with in [4] Lemma 5.1] (where \(k < \omega\) is the number of non-identity atoms and it depends on \(l\)). Let \(3 < n \leq l\). We blow up and blur \(R\) as in the hypothesis. \(R\) is blown up by splitting all of the atoms each to infinitely many. \(R\) is blurred by using a finite set of blurs (or colours) \(J\). This can be expressed by the product \(\mathcal{A}(R) = \omega \times \mathcal{A}(R) \times J\), which will define an infinite atom structure of a new
relation algebra. (One can view such a product as a ternary matrix with $\omega$ rows, and for each fixed $n \in \omega$, we have the rectangle $AtR \times J$.) Then two partitions are defined on $At$, call them $P_1$ and $P_2$. Composition is re-defined on this new infinite atom structure; it is induced by the composition in $R$, and a ternary relation $E$ on $\omega$, that ‘synchronizes’ which three rectangles sitting on the $i,j,k$ $E$–related rows compose like the original algebra $R$. This relation is definable in the first order structure $(\omega, \prec)$. The first partition $P_1$ is used to show that $R$ embeds in the complex algebra of this new atom structure, namely $\mathcal{C}mAt$. The second partition $P_2$ divides $At$ into finitely many (infinite) rectangles, each with base $W \in J$, and the term algebra denoted in [4] by $\mathbb{B}b(R, J, E)$ over $At$, consists of the sets that intersect co–finitely with every member of this partition. On the level of the term algebra $R$ is blurred, so that the embedding of the small algebra into the complex algebra via taking infinite joins, do not exist in the term algebra for only finite and co–finite joins exist in the term algebra. The algebra $\mathbb{B}b(R, J, E)$ is representable using the finite number of blurs. These correspond to non–principal ultrafilters in the Boolean reduct, which are necessary to represent this term algebra, for the principal ultrafilter alone would give a complete representation, hence a representation of the complex algebra and this is impossible. Thereby, in particular, as stated in theorem 5.3, an atom structure that is weakly representable but not strongly representable is obtained. Because $(J, E)$ is a complex set of $l$–blurs, this atom structure has an $l$–dimensional cylindric basis, namely $At_{ca} = \mathcal{M}at_{l}(At)$. The resulting $l$–dimensional cylindric term algebra $\mathcal{M}at_{l}(At)$, and an algebra $C$ having tatom structure $At_{ca}$ denoted in [4] by $\mathbb{B}b_{l}(R, J, E)$, such that $\mathcal{M}at_{l}(At) \subseteq C \subseteq \mathcal{C} mAt_{l}(At)$ is shown to be representable.

For (2): The Maddux relation algebra $\mathcal{E}_{k}(2, 3)$ in [4, Lemma 5.1], one take $l \geq 2n - 1$, $k \geq (2n - 1)l$, $k \in \omega$, and then take the finite integral relation algebra $\mathcal{E}_{k}(2, 3)$ where $k$ is the number of non-identity atoms in $\mathcal{E}_{k}(2, 3)$, with $k$ depending on $l$ as in [4, Lemma 5.1] is the required $R$ in (2).

We prove (3). Let $(J, E)$ be the strong $l$–blur of $R$. Assume for contradiction that $m \leq l$. Then we get by [4, item (3), p.80], that $\mathfrak{A} = \mathbb{B}b_{n}(R, J, E) \cong \mathfrak{N}_{n}\mathbb{B}b_{l}(R, J, E)$. But the cylindric $l$–dimensional algebra $\mathbb{B}b_{l}(R, J, E)$ is atomic, having atom structure $\mathcal{M}at_{l}(\mathbb{B}b(R, J, E))$, so $\mathfrak{A}$ has an atomic $l$–dilation. Hence $\mathfrak{A} = \mathfrak{N}_{n}\mathfrak{D}$ where $\mathfrak{D} \subseteq \mathcal{C}A_{l}$ is atomic. But $R \subseteq \mathfrak{N}_{n}\mathfrak{D}$, $\mathfrak{D} \subseteq \mathfrak{R}_{n}\mathfrak{D}$. Hence $R$ has a complete $l$–flat representation, hence a complete $m$–flat representation, because $m < l$ and $l \in \omega$. This is a contradiction. We prove (4). Assume that $R$ is as in the hypothesis. Take $\mathfrak{B} = \mathbb{B}b_{n}(R, J, E)$. Then by the above $\mathfrak{B} \in \mathcal{R}CA_{m}$. We claim that $\mathfrak{C} = \mathcal{C}mAt\mathfrak{B} \notin SN_{n}CA_{m}$. To see why, suppose for contradiction that $\mathfrak{C} \subseteq \mathfrak{N}_{n}\mathfrak{D}$, where $\mathfrak{D}$ is atomic, with $\mathfrak{D} \subseteq \mathcal{C}A_{m}$. Then $\mathfrak{C}$ has a (necessarily infinite $m$–flat representation), hence $\mathfrak{R}a\mathfrak{C}$ has an infinite $m$–flat representation as an $\mathcal{R}A$. But $R$ embeds into $\mathcal{C}mAt(\mathbb{B}b(R, J, E))$ which, in turn, embeds into $\mathfrak{R}a\mathfrak{C}$, so $R$ has an infinite $m$–flat representation. By lemma 5.3 $R$ has a $m$–dimensional infinite hyperbases which contradicts the hypothesis.

Now we prove (the last) item (5). For $2 < n \leq l < m < \omega$. If the $m$–blur happens to be strong, in the sense of definition 5.3 and $n \leq l < m$ then we get by [4, item (3) pp. 80], that $\mathbb{B}b_{l}(R, J, E) \cong \mathfrak{N}_{l}\mathbb{B}b_{m}(R, J, E)$. This is proved by defining an embedding $h : \mathfrak{N}_{l}\mathbb{B}b_{m}(R, J, E) \rightarrow \mathbb{B}b_{l}(R, J, E)$ via $x \mapsto \{ M \mid l \times l : M \in x \}$ and showing that $h \mid \mathfrak{N}_{l}\mathbb{B}b_{m}(R, J, E)$ is an isomorphism onto $\mathbb{B}b_{l}(R, J, E)$ [4, p.80]. Surjectiveness uses the condition $(J5)_l$. The resulting $l$–dimensional cylindric term algebra $\mathcal{M}at_{l}(At)$, and an algebra $C$ having tatom structure $At_{ca}$ denoted in [4] by $\mathbb{B}b_{l}(R, J, E)$, such that $\mathcal{M}at_{l}(At) \subseteq C \subseteq \mathcal{C}mAt_{l}(At)$ is shown to be representable. The complex algebra $\mathcal{C}mAt_{l}(At) = \mathcal{C}mAt\mathfrak{C}$ is outside in $SN_{n}CA_{m}$, because $R$ embeds into $\mathcal{C}mAt$ which
embeds into $\mathfrak{Ra}c m \text{Mat}_l(\text{At})$, so if $\mathfrak{c m} \text{Mat}_l(\text{At}) \in \mathfrak{S N r}_n \text{CA}_m$, then $R \in \mathfrak{Ra}S \mathfrak{N r}_n \text{CA}_m \subseteq \mathfrak{S R a} \text{CA}_m$ which is contrary to assumption. \hfill $\square$

Fix $2 < n \leq l < m \leq \omega$. The statement $\Psi(l, m)$ is:

There is an atomic, countable, topological and complete $L_n$ theory $T$, such that the type $\Gamma$ consisting of co-atom is realizable in every $m$-square model, but any formula isolating this type has to contain more than $l$ variables.

By an $m$-square model $M$ of $T$ we understand an $m$-square representation of the algebra $\mathfrak{S m}_T$ with base $M$. Let $\mathcal{V}(l, m)) = \neg \Psi(l, m)$, short for Vaught’s Theorem holds at the parameters $l$ and $m'$ where by definition, we stipulate that $\mathcal{V}(\omega, \omega)$ is just Vaught’s Theorem for $L_{\omega, \omega}$: Countable atomic theories have countable atomic models. For $2 < n \leq l < m \leq \omega$ and $l = m = \omega$, we investigate the likelihood and plausability of the following statement which we abbreviate by (**): $\mathcal{V}(l, m) \iff l = m = \omega$. In the next Theorem several conditions are given implying $\Psi(l, m)$ for various values of $l$ and $m$. $\Psi(l, m)_f$ is the formula obtained from $\Psi(l, m)$ by replacing square by flat. In the first item by no infinite $\omega$-dimensional hyperbasis (basis), we understand no representation on an infinite base. By $\omega$-flat (square) representation, we mean an ordinary representation, and by complete $\omega$-flat (square) representation, we mean a complete representation.

**Theorem 5.6.** Let $2 < n \leq l < m \leq \omega$. Then every item implies the immediately following one.

1. There exists a finite relation algebra $R$ algebra with a strong $l$-blur and no infinite $m$-dimensional hyperbasis,

2. There is a countable atomic $A \in \mathfrak{N r}_n \text{CA}_l \cap \text{RCA}_m$ such that $\mathfrak{c m} \text{At} A$ does not have an $m$-flat representation,

3. There is a countable atomic $A \in \mathfrak{N r}_n \text{CA}_l \cap \text{RCA}_m$ such that $\mathfrak{c m} \text{At} A \not\in \mathfrak{S N r}_n \text{CA}_m$.

4. There is a countable atomic $A \in \mathfrak{N r}_n \text{CA}_l \cap \text{RCA}_m$ such that $A$ has no complete infinity $m$-flat representation,

5. There is a countable atomic $A \in \mathfrak{N r}_n \text{CA}_l \cap \text{RCA}_m$ such that $A \not\in \mathfrak{S c N r}_n \text{CA}_m$,

6. $\Psi(l, m)_f$ is true,

7. $\Psi(l', m')_f$ is true for any $l' \leq l$ and $m' \geq m$.

The same implications hold upon replacing infinite $m$-dimensional hyperbasis by $m$-dimensional relational basis (not necessarily infinite), $m$-flat by $m$-square and $\mathfrak{S N r}_n \text{CA}_m$ by $\mathfrak{S N r}_n \text{D}_m$. Furthermore, in the new chain of implications every item implies the corresponding item in Theorem 5.6. In particular, $\Psi(l, m) \implies \Psi(l, m)_f$.

**Proof.** Let $R$ be as in the hypothesis with strong $l$-blur $(J, E)$. The idea is to ‘blow up and blur’ $R$ in place of the Maddux algebra $\mathfrak{c c}(2, 3)$ dealt with in [4] Lemma 5.1], where $k < \omega$ is the number of non-identity atoms and $l$ depends recursively on $k$. Let $2 < n \leq l < \omega$. The relation algebra $R$ is blown up by splitting all of the atoms each to infinitely many. $R$ is blurred by using a finite set of blurs (or colours) $J$. Then two partitions are defined on $\text{At}$, call them $P_1$ and $P_2$. Composition is re-defined on this new infinite atom structure; it is induced by the composition in $R$, and a ternary relation
$E$ on $\omega$, that ‘synchronizes’ which three rectangles sitting on the $i,j,k \in E$–related rows compose like the original algebra $R$. (This relation is definable in the first order structure $(\omega, \leq)$ [2].) The first partition $P_1$ is used to show that $R$ embeds in the complex algebra of this new atom structure, namely, $\mathcal{CmAt}$. The second partition $P_2$ divides $\mathcal{A}$ into \textit{finitely many} (infinite) rectangles, each with base $W \in J$, and the term algebra denoted in [2] by $\mathbb{B}(R,J,E)$ over $\mathcal{A}$ (where $(J,E)$ is the strong $l$–blur for $R$ assumed to exist by hypothesis) consists of the sets that intersect co–finitely with every member of this partition. One proves that $\mathcal{B}(R,J,E)$ with atom structure $\mathcal{A}$ is representable using the finite number of blurs in $J$. Because $(J,E)$ is a strong $l$–blur, then, by definition, it is a strong $j$–blur for all $n \leq j \leq l$, so the atom structure $\mathcal{A}$ has a $j$–dimensional cylindric basis for all $n \leq j \leq l$, namely, $\mathcal{Mat}_j(\mathcal{A})$. For all such $j$, there is an $\mathcal{RCA}_j$ denoted on [2] Top of p. 78 by $\mathcal{B}_j(R,J,E)$ such that $\mathcal{Mat}_j(\mathcal{A}) \subseteq \mathcal{B}_j(R,J,E) \subseteq \mathcal{CmMat}_j(\mathcal{A})$ and $\mathcal{A} \mathcal{B}_j(R,J,E)$ is a weakly representable atom structure of dimension $j$. Now take $A = \mathcal{B}_n(R,J,E)$. We claim that $A$ endowed with the identity operators as modalities induces by the discrete topology on it as base as required. Since $R$ has a strong $j$–blur $(J,E)$ for all $n \leq j \leq l$, then $A \equiv \mathcal{R}_n \mathcal{B}_j(R,J,E)$, with $\mathcal{B}_j(R,J,E)$ expanded to a $\mathcal{TCA}_j$ the same way for all $n \leq j \leq l$ as proved in [2] item (3) p. 80 for ‘cylindric educts. Identity operators on both sides are obviously preserved. In particular, taking $n = l$, $A \in \mathcal{TCA}_n \cap \mathcal{R}_n \mathcal{CA}_n$. We show that $\mathcal{R}_n \mathcal{CmAtA}$ does not have an $m$–flat representation. Assume for contradiction that $\mathcal{CmAtA}$ does have an $m$–flat representation $M$. Then $M$ is infinite of course. Since $R$ embeds into $\mathcal{B}(R,J,E)$ which in turn embeds into $\mathcal{R}_n \mathcal{CmAtA}$, then $R$ has an $m$–flat representation with base $M$. But since $R$ is finite, $R = R^+$, and consequently $R$ has an infinite $m$–dimensional hyperbasis. This is contrary to our assumption and we are done.

(2) $\implies$ (3): Fix $2 < n < m < \omega$. As above, let $\mathcal{L}(A)^m$ denote the signature that contains an $n$–ary predicate symbol for every $a \in A$. We that the existence of $m$–flat representations, implies the existence of $m$–dilations. Let $M$ be an $m$–flat representation of $A$. We show that $A \subseteq \mathcal{R}_n \mathcal{D}$, for some $\mathcal{D} \in \mathcal{CmAt}$, and that $A$ actually has an infinitary $m$–flat representation. For $\phi \in \mathcal{L}(A)^m$, let $\phi^M = \{\bar{a} \in C^m(M) : M \models \phi(\bar{a})\}$, where $C^m(M)$ is the $n$–Gaifman hypergraph. Let $\mathcal{D}$ be the algebra with universe $\{\phi^M : \phi \in \mathcal{L}(A)^m\}$ and with cylindric operations induced by the $n$–clique–guarded (flat) semantics. For $r \in A$, and $\bar{x} \in C^m(M)$, we identify $r$ with the formula it defines in $\mathcal{L}(A)^m$, and we write $r(\bar{x})^M \iff M, \bar{x} \models \phi$. Then certainly $\mathcal{D}$ is a subalgebra of the $\mathcal{Cr}_m$ (the class of algebras whose units are arbitrary sets of $m$–ary sequences) with domain $\phi(C^m(M))$, so $\mathcal{D} \in \mathcal{Cr}_m$ with unit $1^D = C^m(M)$. Since $M$ is $m$–flat, then cylindrifiers in $\mathcal{D}$ commute, and so $\mathcal{D} \in \mathcal{CmAt}$. Now define $\theta : A \rightarrow \mathcal{D}$, via $r \mapsto r(\bar{x})^M$. Then exactly like in the proof of [2] Theorem 13.20, $\theta$ is a neat embedding, that is, $\theta(A) \subseteq \mathcal{R}_n \mathcal{D}$. It is straightforward to check that $\theta$ is a homomorphism. We show that $\theta$ is injective. Let $r \in A$ be non–zero. Then $M$ is a relativized representation, so there is $\bar{a} \in M$ with $r(\bar{a})$, hence $\bar{a}$ is a clique in $M$, and so $M \models r(\bar{x})(\bar{a})$, and $\bar{a} \in \theta(r)$, proving the required. $M$ itself might not be infinitary $m$–flat, but one can build an infinitary $m$–flat representation of $A$, whose base is an $\omega$–saturated model of the consistent first order theory, stipulating the existence of an $m$–flat representation [2] Proposition 13.17, Theorem 13.46 items (6) and (7)]. This idea (of using saturation) will be given in more detail in the last item.

(3) $\implies$ (4): A complete $m$–flat representation of (any) $\mathcal{B} \in \mathcal{CmAt}$ induces an $m$–flat representation of $\mathcal{CmAtB}$ which implies by Theorem 4.4 that $\mathcal{CmAtB} \in \mathcal{SNR}_m \mathcal{CA}_m$. To see why, assume that $\mathcal{B}$ has an $m$–flat complete representable via $f : \mathcal{B} \rightarrow \mathcal{D}$, where $\mathcal{D} = \phi(V)$ and the base of the representation $M = \bigcup_{s \in V} \text{rng}(s)$ is $m$–flat. Let
\( C = C_{m \text{At} B} \). For \( c \in C \), let \( c \downarrow = \{ a \in \text{At} C : a \leq c \} = \{ a \in \text{At} B : a \leq c \} \); the last equality holds because \( \text{At} B = \text{At} C \). Define, representing \( C \), \( g : C \to D \) by \( g(c) = \sum_{x \in c} f(x) \). The map \( g \) is well defined because \( C \) is complete so arbitrary suprema exist in \( C \). Furthermore, it can be easily checked that \( g \) is a homomorphism into \( \varphi(V) \) having base \( M \) (basically because by assumption \( f \) is a homomorphism).

(4) \( \implies \) (5): if \( A \in S_c \text{Nr}_n \text{CA}_m \) then it has an \( m \)-flat complete representation. Essentially a completeness theorem, this is a ‘truncated version’ of Henkin’s neat embedding theorem: Existence of atomic \( m \)-dimensional dilations \( \implies \) existence of infinitary complete \( m \)-flat representations. One constructs an infinitary \( m \)-flat representation \( M \) of \( A \) Suppose that \( A \subseteq c \text{Nr}_m D \), and \( D \) is atomic. We first show that \( D \) has an \( m \)-dimensional hyperbasis, lifted from relation algebra the obvious way. First, it is not hard to see that for every \( n \leq l \leq m \), \( \text{Nr}_l D \) is atomic. The set of non–atomic labels \( \Lambda \) is the set \( \bigcup_{k<m} \text{Nr}_k D \). Before proceeding we need a piece of notation that is somewhat technical. . Let \( m \) be a finite ordinal \( > 0 \). An \( s \) word is a finite string of substitutions (\( S^1 \)) \( (i, j < m) \), a \( c \) word is a finite string of cylindrifications (\( c_i \)), \( i < m \); an \( sc \) word \( w \), is a finite string of both, namely, of substitutions and cylindrifications. An \( sc \) word induces a partial map \( \hat{w} : m \to m \):

- \( \hat{e} = Id \),
- \( \hat{w}^i_j = \hat{w} \circ [i|j] \),
- \( \hat{w}_c i = \hat{w} \upharpoonright (m \setminus \{i\}) \).

If \( \hat{a} \in \varnothing m \) \( m \) \( m \), we write \( s_a \), or \( s_{a_0 \ldots a_{k-1}} \), where \( k = |\hat{a}| \), for an arbitrary chosen \( sc \) word \( w \) such that \( \hat{w} = \hat{a} \). Such a \( w \) exists by [26, Definition 5.23 Lemma 13.29]. Resuming main stream proof, For each atom \( a \) of \( \hat{D} \), define a labelled hypergraph \( N_a \) as follows: Let \( \hat{b} \in \varnothing m \rightleftharpoons m \). Then if \( |\hat{b}| = n \), so that \( \hat{b} \) has to get a label that is an atom of \( \hat{D} \), one sets \( N_a(\hat{b}) \) to be the unique \( r \in \text{At} \hat{D} \) such that \( a \leq s_b r \); notation here is given as above. If \( n \neq |\hat{b}| \rightleftharpoons m \), \( N_a(\hat{b}) \) is the unique atom \( r \in \text{Nr}_m \hat{D} \) such that \( a \leq s_b r \). Since \( \text{Nr}_m \hat{D} \) is atomic, this is well defined. Note that this label may be a non–atomic one; it might not be an atom of \( \hat{D} \). But by definition it is a permitted label. Now fix \( \lambda \in \Lambda \). The rest of the labelling is defined by \( N_a(\hat{b}) = \lambda \). Then \( N_a \) as an \( m \)-dimensional hypernetwork, for each such chosen \( a \), and \( \{ N_a : a \in \text{At} \hat{D} \} \) is the required \( m \)-dimensional hyperbasis. The rest of the proof consists of a fairly straightforward adaptation of the proof [26 Proposition 13.37], replacing edges by \( n \)–hyperedges.

(5) \( \implies \) (6): By [22 §4.3], we can (and will) assume that \( A = \tilde{A} \text{At}_T \) for a countable, atomic theory \( L_a \) theory \( T \). Let \( \Gamma \) be the \( n \)-type consisting of co–atoms of \( T \). Then \( \Gamma \) is realizable in every \( m \)-flat model, for if \( M \) is an \( m \)-flat model omitting \( \Gamma \), then \( M \) would be the base of a complete \( m \)-flat representation of \( \tilde{A} \), and so \( \tilde{A} \in S_c \text{Nr}_n \text{CA}_m \) which is impossible. Suppose for contradiction that \( \phi \) is an \( l \) witness, so that \( T \models \phi \rightarrow \alpha \), for all \( \alpha \in \Gamma \), where recall that \( \Gamma \) is the set of coatoms. Then since \( \tilde{A} \) is simple, we can assume without loss that \( \tilde{A} \) is a set algebra with base \( M \) say. Let \( M = (M, R_{i})_{i \in \omega} \) be the corresponding model (in a relational signature) to this set algebra in the sense of [22 §4.3]. Let \( \phi^M \) denote the set of all assignments satisfying \( \phi \) in \( M \). We have \( M \models T \) and \( \phi^M \in \tilde{A} \), because \( \tilde{A} \in \text{Nr}_n \text{CA}_{n-1} \). But \( T \models \exists x \phi \), hence \( \phi^M \neq 0 \), from which it follows that \( \phi^M \) must intersect an atom \( \alpha \in \tilde{A} \) (recall that the latter is atomic). Let \( \psi \) be the formula, such that \( \psi^M = \alpha \). Then it cannot be the case that \( T \models \phi \rightarrow \neg \psi \), hence \( \phi \) is not a witness, contradiction and we are done.
(6) \implies (7): follows from the definitions.

For squareness the proofs are essentially the same undergoing the obvious modifications. In the first implication ‘infinite’ in the hypothesis is not needed because any finite relation algebra having an infinite m-dimensional relational basis has a finite one, cf. [26 Theorem 19.18]. On the other hand, there are finite relation algebras having infinite m-dimensional hyperbasis for m \geq 5 but has no finite ones [26 Prop. 19.19].

□

We show that (4) and (5) are actually equivalent. If \( \mathfrak{A} \) has a complete infinitary m-flat representation \( \mathfrak{A} \in S_n \text{Nr}_n \text{CA}_m \). One works in \( L^{m}_{\infty, \omega} \) instead of first order logic. Let \( M \) be the given representation of \( \mathfrak{A} \). In this case, the dilation \( \mathcal{D} \) of \( \mathfrak{A} \) having again top element the Gaifman hypergraph \( C^m(M) \), where \( M \) is the complete infinitary m-flat representation of \( \mathfrak{A} \), will now have (the larger) universe \( \{ \phi^M : \phi \in \mathcal{L}(A)^{m}_{\infty, \omega} \} \) with operations also induced by the n-clique-guarded semantics extended to \( L^{m}_{\infty, \omega} \). Like before \( \mathcal{D} \) will be a \( \text{CA}_m \), but this time, it will be an atomic one. To prove atomicity, let \( \phi^M \) be a non-zero element in \( \mathcal{D} \). Choose \( \bar{a} \in \phi^M \), and consider the following infinitary conjunction (which we did not have before when working in \( L_m \)): \( \tau = \bigwedge \{ \psi \in \mathcal{L}(A)^{m}_{\infty, \omega} : M \models \psi(\bar{a}) \} \). Then \( \tau \in \mathcal{L}(A)^{m}_{\infty, \omega} \), and \( \tau^M \) is an atom below \( \phi^M \). The neat embedding will be an atomic one, hence it will be a complete neat embedding [26 p. 411].

Corollary 5.7. For \( 2 < n < \omega \) and \( n \leq l < \omega \), \( \Psi(n, t(n)) \) and \( \Psi(l, \omega) \) hold.

Proof. The first case, follows from Theorem 4.11 and 5.6 (by taking \( l = n \) and \( m = n + 3 \)). For the second case, it suffices by Theorem 5.6 (by taking \( m = \omega \)) to find a countable algebra \( \mathcal{E} \in \text{Nr}_n \text{CA}_l \cap \text{RCA}_n \) such that \( \mathcal{E} \mathfrak{m} \mathfrak{A} \notin \text{RCA}_n \). This algebra is constructed in [4], cf. item (1) of Theorem 5.6.

Reproving the main results in [25] and [33] in a completely different way using Monk like algebras rather than rainbow ones, we get:

Corollary 5.8. Let \( 2 < n < \omega \)

1. The set of equations using only one variable that holds in each of the varieties \( \text{RCA}_n \) and \( \text{RRA} \), together with any finite first order definable expansion of each, cannot be derived from any finite set of equations valid in the variety \( \text{LCA}_n \). Furthermore, \( \text{LCA}_n \) is not finitely axiomatizable.

2. The classes \( \text{CRCA}_n \) and \( \text{CRRA} \) are not elementary.

Proof. 1. Let \( \mathcal{R}_l = \mathfrak{B}_l(R_l, J_l, E_l) \in \text{RRA} \) where \( \mathcal{R}_l \) is the relation algebra having atom structure denoted \( \mathfrak{A} \) in [4 p. 73] when the blown up and blurred algebra denoted \( \mathcal{R}_l \) happens to be the finite Maddux algebra \( \mathcal{E}_{f(l)}(2, 3) \) and let \( \mathcal{R}_l = \mathfrak{N}_n \mathfrak{B}_l(R_l, J_l, E_l) \in \text{RCA}_n \) as defined in [4 Top of p.80] (with \( R_l = \mathcal{E}_{f(l)}(2, 3) \)). Then \( (\mathfrak{A} \mathcal{R}_l : l \in \omega \sim n) \), and \( (\mathfrak{A} \mathcal{A}_l : l \in \omega \sim n) \) are sequences of weakly representable atom structures that are not strongly representable with a completely representable ultraproduct.

2. The algebra \( \mathfrak{B} \) constructed in Theorem 5.10 satisfies that \( \mathfrak{B} \in \text{Nr}_n \text{CA}_\omega \subseteq \text{LCA}_n = \text{EICRCA}_n \). To see why, \( \mathfrak{B} \in \text{S}_n \text{Nr}_n \text{CA}_\omega \) is atomic, then by Lemma 4.5 \( \exists \) has a winning strategy in \( G^\omega(\mathfrak{A} \mathfrak{B}) \), hence in \( G_\omega(\mathfrak{A} \mathfrak{B}) \), a fortiori, \( \exists \) has a winning strategy in \( G_k(\mathfrak{A} \mathfrak{B}) \) for all \( k < \omega \), so (by definition) \( \mathfrak{B} \in \text{LCA}_n \). We have already dealt with the CA case in

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\(^2\)There are set-theoretic subtleties involved here, that we prefer to ignore.
Theorem 5.6, since the algebra B constructed therein satisfies that B ∈ EICRCAₙ ∼ CRCAₙ. For relation algebras, we use the algebra A constructed in the previous Theorem, too. We have A ∈ RaCAₙ and A has no complete representation. The rest is like the CA case, using the Ra analogue of Lemma 1.5, when the dilation is ω-dimensional, namely, A ∈ SₙRaCAₙ ⇒ ∃ has a winning strategy in Pω with the last notation taken from [23]. The last argument proves that RaCAₙ ⊆ EICRRA.

Remark 5.9. Observe (from the proof of Proposition 3.10) that any atomic algebra in NrₙCAₙ (RaCAₙ) with no complete representation, witnesses that CRCAₙ (CRRA) is not elementary. This cannot be witnessed on algebras having countably many atoms because restricting to such algebras we have SₙNrₙCAₙ ⊆ CRCAₙ (SₙRaCAₙ ⊆ CRRA.)

Nevertheless, if we impose extra conditions on theories and possibly uncountably many non-principal types to be omitted, we get positive results in terms of ω-square Tarskian usual semantics. We prove a positive OTT for Lₙ theories by imposing ‘elimination of quantifiers’ on theories and maximality conditions on non-principal types we wish to omit, such as being complete. If T is a first order theory in a language L, then a set of formulas each using at most m variables) Γ say, is complete, if for any L-formula φ having at most m variables, either T ∪ Γ ⊨ φ or T ∪ Γ ⊨ ¬φ.

Corollary 5.10. Let n be any finite ordinal. Let T be a countable and consistent Lₙ theory and λ be a cardinal < p. Let F = (Γᵢ : i < λ) be a family of non-principal types of T. Suppose that T admits elimination of quantifiers. Then the following hold:

1. If φ is a formula consistent with T, then there is a model M of T that omits F, and φ is satisfiable in M. If T is complete, then we can replace p by covK,

2. If the non-principal types constituting F are maximal, then we can replace p by 2ω.

Proof. Let T be as given in a signature L having n variables. Let A = ℵmₜ, and Gᵢ = {φₜ : φ ∈ Γᵢ}. Then Gᵢ is a a non-principal ultrafilter; maximality follows from the completeness of types considered. By completeness of T, A is simple. Since T admits elimination of quantifiers, then ℵmₜ ∈ NrₙCAₙ. Indeed, let Tₙ be the theory in the same signature L but using ω many variables. Let C = ℵmₜ be the Tarski-Lindenbaum quotient cylindric algebra algebra. Then C ∈ CAₙ (because we have ω many variables); in fact C ∈ ICₙC, and the map Φ defined from A to NrₙC via φ/≡ₜ → φ/≡ₜ is injective and bijective, that is to say, Φ having domain A and codomain NrₙC is in fact onto NrₙC due to quantifier elimination. An application of Theorem 5.6 finishes the proof.

Let n < ω. By observing that if T is a topological theory using n variable admitting quantifier elimination, then ℵmₜ ∈ NrₙTCAₙ, cf [40] Theorem 3.2.10], then we get:

Corollary 5.11. Let T be a topological countable predicate theory in n variables, that admits elimination of quantifiers. Then any family of < 2ⁿ non-principal complete types can be omitted in a countable topological model.

6 Non elementary classes of atom structures (Kripke frames)

We start with an easy lemma to be used in the last item of the next theorem. If B is a Boolean algebra and b ∈ B, then NRbB denotes the Boolean algebra with domain {x ∈ B : x ≤ b}, top element b, and other Boolean operations those of B relativized to b.
Lemma 6.1. In the following \( \mathfrak{A} \) and \( \mathcal{D} \) are Boolean algebras.

1. If \( \mathfrak{A} \) is atomic and \( 0 \neq a \in \mathfrak{A} \), then \( \mathfrak{R}_a \mathfrak{A} \) is also atomic. If \( \mathfrak{A} \subseteq_d \mathcal{D} \), and \( a \in A \), then \( \mathfrak{R}_a \mathfrak{A} \subseteq_d \mathfrak{R}_a \mathcal{D} \).

2. If \( \mathfrak{A} \subseteq_d \mathcal{D} \) then \( \mathfrak{A} \subseteq_d \mathcal{D} \). In particular, for any class \( K \) of BAOs, \( K \subseteq S_d K \subseteq S_c K \).

Proof. (1): Entirely straightforward.

(2): Assume that \( \sum S = 1 \) and for contradiction that there exists \( b' \in \mathcal{D} \), \( b' < 1 \) such that \( s \leq b' \) for all \( s \in S \). Let \( b = 1 - b' \) then \( b \neq 0 \), hence by assumption (density) there exists a non-zero \( a \in \mathfrak{A} \) such that \( a \leq b \), i.e. \( a \leq (1 - b') \). If \( a \cdot s \neq 0 \) for some \( s \in S \), then \( a \) is not less than \( b' \) which is impossible. So \( a \cdot s = 0 \) for every \( s \in S \), implying that \( a = 0 \), contradiction. Now we prove the second part. Assume that \( \mathfrak{A} \subseteq_d \mathcal{D} \) and \( \mathcal{D} \) is atomic. Let \( b \in \mathcal{D} \) be an atom. We show that \( b \in \mathfrak{A} \). By density there is a non-zero \( a' \in \mathfrak{A} \), such that \( a' \leq b \in \mathcal{D} \). Since \( \mathfrak{A} \) is atomic, there is an atom \( a \in \mathfrak{A} \) such that \( a \leq a' \leq b \). But \( b \) is an atom of \( \mathcal{D} \), and \( a \) is non-zero in \( \mathcal{D} \), too, so it must be the case that \( a = b \in \mathfrak{A} \). Thus \( \mathfrak{A} \subseteq_d \mathfrak{A} \) and we are done.

Fix \( 2 < n < \omega \). Call an atomic \( \mathfrak{A} \in \mathfrak{C} \alpha_n \) weakly (strongly) representable \( \iff \mathfrak{A} \) is weakly (strongly) representable. Let \( \mathfrak{W} \mathfrak{R} \mathfrak{A}_n \) (\( \mathfrak{R} \mathfrak{C} \mathfrak{A}_n \)) denote the class of all such \( \mathfrak{C} \alpha_n \)s, respectively. Then the class \( \mathfrak{R} \mathfrak{C} \mathfrak{A}_n \) is not elementary and \( \mathfrak{L} \mathfrak{A}_n \subseteq \mathfrak{R} \mathfrak{A}_n \subseteq \mathfrak{W} \mathfrak{R} \mathfrak{A}_n \) [27]: the strictness of the two inclusions follow from the fact that the classes \( \mathfrak{L} \mathfrak{A}_n \) and \( \mathfrak{W} \mathfrak{R} \mathfrak{A}_n \) are elementary. For an atom structure \( \mathfrak{A} \), let \( \mathfrak{F}(\mathfrak{A}) \) be the subalgebra of \( \mathcal{F}_m \mathfrak{A} \) consisting of all sets of atoms in \( \mathfrak{A} \) of the form \( \{ a \in \mathfrak{A} : \mathfrak{A} \models \phi(a, \bar{b}) \} \in \mathcal{F}_m \mathfrak{A} \), for some first order formula \( \phi(x, \bar{y}) \) of the signature of \( \mathfrak{A} \) and some tuple \( \bar{b} \) of atoms, cf. [26], item (3), p. 456] for the analogous definition for relation algebras. Let \( \mathfrak{F} \mathfrak{C} \mathfrak{A}_n \) be the class of all such \( \mathfrak{C} \alpha_n \)s. Then it can be proved, similarly to the RA case that \( \mathfrak{R} \mathfrak{C} \mathfrak{A}_n \subseteq \mathfrak{F} \mathfrak{C} \mathfrak{A}_n \) and that \( \mathfrak{F} \mathfrak{C} \mathfrak{A}_n \) is elementary, cf. [26] Theorem 14.17], hence the inclusion is strict.

In the following \( \mathfrak{U} \mathfrak{p} \), \( \mathfrak{U} \mathfrak{r} \), \( \mathfrak{P} \) and \( \mathfrak{H} \) denote the operations of forming ultraproducts, ultraroots, products and homomorphic images, respectively.

Theorem 6.2. For \( 2 < n < \omega \) the following hold:

1. For \( n < m \leq \omega \), \( \mathfrak{N}_n \mathfrak{C} \alpha_m \) is a pseudo elementary class that is not elementary; it is closed under \( \mathfrak{P} \), \( \mathfrak{H} \) but not under \( \mathfrak{S}_d \), to fortiori \( \mathfrak{S}_c \) nor \( \mathfrak{S} \), nor \( \mathfrak{U} \mathfrak{r} \). The elementary theory of \( \mathfrak{N}_n \mathfrak{K}_\omega \) is recursively enumerable.

2. For any class \( K \) of frames such that \( \mathfrak{A} \mathfrak{T} \mathfrak{C} \mathfrak{R} \mathfrak{A}_n \subseteq K \subseteq \mathfrak{A} \mathfrak{T} \mathfrak{S} \mathfrak{R} \mathfrak{A}_n \), \( K \) generates \( \mathfrak{R} \mathfrak{C} \mathfrak{A}_n \). If \( K \) is elementary then \( \mathfrak{R} \mathfrak{C} \mathfrak{A}_n \) is canonical, but \( \mathfrak{S}_m \mathfrak{A} \mathfrak{T} \mathfrak{A} \not\subseteq \mathfrak{R} \mathfrak{C} \mathfrak{A}_n \), since the last class is ot atom-canonical.

3. Although \( \mathfrak{S} \mathfrak{t} \mathfrak{R} \mathfrak{C} \mathfrak{A}_n = \{ \mathfrak{F} \in \mathfrak{R} \mathfrak{A}_n : \mathfrak{X} \mathfrak{m} \mathfrak{F} \in \mathfrak{R} \mathfrak{C} \mathfrak{A}_n \} \) is not elementay and \( \mathfrak{S}_m \mathfrak{A} \mathfrak{T} \mathfrak{A} \not\subseteq \mathfrak{R} \mathfrak{C} \mathfrak{A}_n \), there is an elementary class of frames that generates \( \mathfrak{R} \mathfrak{C} \mathfrak{A}_n \), so the last class is canonical.

4. \( \mathfrak{N}_n \mathfrak{C} \alpha_\omega \subseteq \mathfrak{S}_d \mathfrak{N}_n \mathfrak{C} \alpha_\omega \subseteq \mathfrak{S}_c \mathfrak{N}_n \mathfrak{C} \alpha_\omega \subseteq \mathfrak{S}_m \mathfrak{N}_n \mathfrak{C} \alpha_\omega \not\subseteq \mathfrak{R} \mathfrak{C} \mathfrak{A}_n \). Furthermore, the strictness of inclusions are witnessed by atomic algebras.

Proof. The case when \( m \) is finite is easy we need only a tw sorted defining theory. To show that \( \mathfrak{N}_n \mathfrak{C} \alpha_\omega \) is pseudo-elementary, we use a three sorted defining theory, with one
sort for a cylindric algebra of dimension \( n \) (\( c \)), the second sort for the Boolean reduct of a cylindric algebra (\( b \)) and the third sort for a set of dimensions (\( \delta \)); the argument is analogous to that of Hirsch used for relation algebra reducts [23, Theorem 21]. We use superscripts \( n, b, \delta \) for variables and functions to indicate that the variable, or the returned value of the function, is of the sort of the cylindric algebra of dimension \( n \), the Boolean part of the cylindric algebra or the dimension set, respectively. The signature includes dimension sort constants \( i^b \) for each \( i < \omega \) to represent the dimensions. The defining theory for \( \text{Nr}_n \text{CA}_\omega \) includes sentences stipulating that the constants \( i^b \) for \( i < \omega \) are distinct and that the last two sorts define a cylindric algebra of dimension \( \omega \). For example the sentence

\[
\forall x^\delta, y^\delta, z^\delta (d^b(x^\delta, y^\delta) = c^b(z^\delta, d^b(x^\delta, z^\delta), d^b(z^\delta, y^\delta)))
\]

represents the cylindric algebra axiom \( d_{ij} = c_b(d_{ik}, d_{kj}) \) for all \( i, j, k < \omega \). We have have a function \( I^b \) from sort \( c \) to sort \( b \) and sentences requiring that \( I^b \) be injective and to respect the \( n \) dimensional cylindric operations as follows: for all \( x^r \)

\[
I^b(d_{ij}) = d^b(i^\delta, j^\delta)
\]

\[
I^b(c_{r}; x^r) = c^b(I^b(x^r)).
\]

Finally we require that \( I^b \) maps onto the set of \( n \) dimensional elements

\[
\forall y^b((\forall z^\delta (z^\delta \neq 0^b, \ldots, (n-1)^\delta \rightarrow c^b(z^\delta, y^b) = y^b)) \leftrightarrow \exists x^r(y^b = I^b(x^r))).
\]

In all cases, it is clear that any algebra of the right type is the first sort of a model of this theory. Conversely, a model for this theory will consist of an \( n \) dimensional cylindric algebra type (sort \( c \)), and a cylindric algebra whose dimension is the cardinality of the \( \delta \)-sorted elements, which is at least \( |m| \). Thus the three sorted theory defines the class of neat reduct, furthermore, it is clearly recursive. Finally, if \( K \) be a pseudo elementary class, that is \( K = \{ M^a | L : M \models U \} \) of \( L \) structures, and \( L, L', U \) are recursive. Then there a set of first order recursive theory \( T \) in \( L \), so that for any \( \mathfrak{A} \) an \( L \) structure, we have \( \mathfrak{A} \models T \) iff there is a \( \mathfrak{B} \in K \) with \( \mathfrak{A} \equiv \mathfrak{B}. \) In other words, \( T \) axiomatizes the closure of \( K \) under elementary equivalence, see [26, Theorem 9.37] for unexplained notation and proof. Closure under \( \mathbf{P} \) follows from that and that \( \text{PNr}_n \text{CA}_\omega = \text{Nr}_n \text{CA}_\omega \). Let \( \langle A_i : i \in I \rangle \) be a system of \( \text{CA}_m \)'s indexed by \( \mathfrak{A} \) non empty set \( I \). Then, in fact, \( \mathbf{P}_i \models \text{Nr}_n \mathfrak{A}_i = \text{Nr}_n \mathfrak{P}_i \models \mathfrak{A}_i \). Closure under \( \mathbf{U} \mathbf{p} \) follows from that \( \text{Nr}_n \text{CA}_m \) for any \( n < m \leq \omega \) is pseudo-elementary, cf. [23, Theorem 21] and [26, §9.3] for similar cases. Hence it is cde under ultraproducts. Using the system of algebras as above with \( F \) a non principal ultrafilter on \( I \) (more explicitly), we have \( \mathbf{P}_i \models \text{Nr}_n \mathfrak{A}_i / F \cong \text{Nr}_n \mathbf{P}_i \in T \mathfrak{A}_i / F \) Closure of \( \text{Nr}_n \text{CA}_m \) under \( \mathbf{H} \) is proved in [46]. Th last is not closed under \( \mathbf{S}_d \) by example [5.4], and not under \( \mathbf{U}_r \), because it is pseudo-elementary, but not elementary. Being closed under ultrapoducts, the Keisler-Shelah Ultrapower Theorem finishes the proof, showing that \( \text{Nr}_n \text{CA}_m \) is not closed under ultraroots, that is to say, that here is an algebra \( \mathfrak{A} \notin \text{Nr}_n \text{CA}_m \) but \( \mathfrak{A}^F / F \in \text{Nr}_n \text{CA}_m \).

(2): Given an algebra \( \mathfrak{A} \) having \( \text{CA}_n \) signature, then \( \mathfrak{A} \in \text{RCA}_n \iff \) the canonical extension \( \mathfrak{A}^+ \) of \( \mathfrak{A} \), based on the ultrafilter frame in symbols \( U\mathfrak{A} \) or Stone space of \( \mathfrak{A} \), whose underlying set consists of all Boolean ultrafilters of \( \mathfrak{A} \), namely, the complex algebra of \( U \), in symbols, \( C^n U\mathfrak{A} \), is completely representable. Therefore \( \text{RCA}_n \) is atomically generated by \( \text{CRCA}_n \) in the (strong sense sense), that is to say, \( \text{SCRCA}_n = \text{RCA}_n \) (without the help of \( \mathbf{HP}. \) ) For \( t(n) = n(n + 1)/2 + 1 \), \( \text{SNr}_n \text{CA}_m \) is not Sahlqvist axiomatizable by
because it is not atom-canonical a fortiori not closed under Dedekind-MacNeille completions.

(3) The variety $RCA_n$ is not atom-canonical $\iff$ $S\emptyset mAtRCA_n \not\subseteq RCA_n$. Since by Theorem 4.11 there exist $n < m < \omega$, namely, $m = n(n + 1)/2 + 1$ such that $SNr_nCA_m \supset RCA_n$ and $SNr_nCA_m$ is not atom-canonical with respect to $RCA_n$, it follows that $S\emptyset mAt\not\subseteq RCA_n$, though $S\emptyset mStrCA_n \subseteq RCA_n$. Although $StrRCA_n$ is not elementary, then for any elementary class $K$ such that $LCA_n \subseteq K \subseteq RCA_n$, where recall that $LCA_n = ElCRCA_n$ is elementary by definition, consisting only of atomic algebras, we would have $AtK$ elementary generating $RCA_n$ in the strong sense, meaning that $S\emptyset mAtK = RCA_n$ hence $RCA_n$ is canonical.

(4): The algebra $\mathcal{B}$ used in example 6.3 witnesses that $Nr_nCA_\omega \subseteq S_dNr_nCA_\omega$, because, as proved in [40], $\mathcal{B} \notin ElNr_nCA_\omega(\supset Nr_nCA_\omega)$ and $\mathcal{E} \in_d \mathcal{A}$ where $\mathcal{A} \in Nr_nCA_\omega$ is the full $Cs_n$ with top element $nQ$ (and universe $\varphi(nQ)$). Let $\mathcal{A} \in CA_n$ be the algebra constructed in Theorem 4.11. We know that $\mathcal{A} \in RCA_n \cap At$, but $\mathcal{A} \notin LCA_n$, because $At\mathcal{A}$ does not satisfy the Lyndon conditions, lest $\emptyset mAt\mathcal{A} \in LCA_n(\subseteq RCA_n)$. We conclude that $\mathcal{A} \notin ElNr_nCA_\omega$ proving the strictness of the last inclusion. Since $\mathcal{E}, \mathcal{C}$ and $\mathcal{A}$ are all atomic, we are done. To show that $S_dNr_nCA_\omega \subseteq S_cNr_nCA_\omega$, we slightly modify the construction in [11] Lemma 5.1.3, Theorem 5.1.4 as done below. The algebra denoted by $\mathcal{B}$ in op. cit. witnesses the strictness of the inclusion.

\begin{theorem}
Let $2 < n < \omega$. Then the following hold:

1. $S_dNr_nCA_\omega \cap \text{Count} = CRCA_n \cap \text{Count},$

2. $CRCA_n \subseteq S_cNr_n(\text{CA}_\omega \cap \text{At}) \cap \text{At} \subseteq S_cNr_nCA_\omega \cap \text{At}$. At least two of the above three classes are distinct but they coincide on algebras having countably many atoms. Non of all these classes is elementary.

3. $El(S_cNr_nCA_\omega \cap \text{At}) = ElS_cNr_nCA_\omega \cap \text{At} = LCA_n,$

4. $SNr_nCA_\omega \cap \text{At} = WRCA_n,$

5. $PElS_cNr_nCA_\omega \cap \text{At} \subseteq SRCA_n,$ and $ElPElS_cNr_nCA_\omega \cap \text{At} \subseteq FCA_n.$

\end{theorem}

\begin{proof}
The first required is already dealt with. For the second required, we know show that $CRCA_n \subseteq S_dNr_n(\text{CA}_\omega \cap \text{At}) \cap \text{At}$. Let $\mathcal{A} \in CRCA_n$. Assume that $M$ is the base of a complete representation of $\mathcal{A}$, whose unit is a generalized cartesian space, that is, $1^M = \bigcup U_i$, where $U_i \cap U_j = \emptyset$ for distinct $i$ and $j$, in some index set $I$, that is, we have an isomorphism $t : \mathcal{B} \rightarrow \mathcal{C}$, where $\mathcal{C} \in Gs_n$ has unit $1^M$, and $t$ preserves arbitrary meets carrying them to set-theoretic intersections. For $i \in I$, let $E_i = U_i$ and let $W_i = \{ f \in \omega U_i : |\{ k \in \omega : f(k) \notin f_i(k) \}| < \omega \}. \text{ Let } \mathcal{C}_i = \varphi(W_i).$ Then $\mathcal{C}_i \in Ws_\omega$ and is atomic; indeed the atoms are the singletons. Let $x \in \mathfrak{M}_nC_i$, that is $c_i x = x$ for all $n \leq i < \omega$. Now if $f \in x$ and $g \in W_i$ satisfy $g(k) = f(k)$ for all $k < n$, then $g \in x$. Hence $\mathfrak{M}_nC_i$ is atomic; its atoms are $\{ g \in W_i : \forall i < n \subseteq U_i \}$. Define $h_i : \mathcal{A} \rightarrow \mathfrak{M}_nC_i$ by $h_i(x) = \{ f \in W_i : \exists a \in At\mathcal{A}, a' \leq a, (f : i < n) \in t(a') \}$. Let $\mathcal{D} = \mathcal{P}_iC_i$. Let $\pi_i : \mathcal{D} \rightarrow \mathcal{C}_i$ be the $i$th projection map. Now clearly $\mathcal{D}$ is atomic, because it is a product of atomic algebras, and its atoms are $\{ \pi_i(\beta) : \beta \in At(\mathcal{C}_i) \}$. Now $\mathcal{A}$ embeds into $\mathfrak{M}_nD$ via $J : a \mapsto (\pi_i(a) : i \in I)$. If $x \in \mathfrak{M}_n\mathcal{D}$, then for each $i$, we have $\pi_i(x) \in \mathfrak{M}_nC_i$, and if $x$ is non–zero, then $\pi_i(x) \neq 0$. By atomicity of $\mathcal{C}_i$, there is an $n$–ary

\end{proof}
tuple $y$, such that $\{g \in W_i : g(k) = y_k\} \subseteq \pi_i(x)$. It follows that there is an atom of $b \in \mathfrak{A}$, such that $x \cdot J(b) \neq 0$, and so the embedding is atomic, hence complete. We have shown that $\mathfrak{A} \in S_n \text{satisf} \cap \text{At}$, and since $\mathfrak{A}$ is atomic because $\mathfrak{A} \in \text{CRCA}_n$ we are done with the first inclusion. The construction of an atomic $\mathfrak{A} \in \text{satisf} (\text{having uncountably many atoms})$ that lacks a complete representation in Theorem 3.10 shows that the first and last classes are distinct. We show that $\text{LCA}_n = \text{El} \cap \text{At}$.

Assume that $\mathfrak{A} \in \text{LCA}_n$. Then, by definition, for all $k < \omega$, $\exists$ a winning strategy in $G_k(\text{At}\mathfrak{A})$. Using ultrapowers followed by an elementary chain argument like in [27, Theorem 3.3.5], $\exists$ has a winning strategy in $G_\omega(\text{At}\mathfrak{B})$ for some countable $\mathfrak{B} \equiv \mathfrak{A}$, and so by [27, Theorem 3.3.3] $\mathfrak{B}$ is completely representable. Thus $\mathfrak{A} \in \text{El} \cap \text{At}$. If $\mathfrak{A} \in S_n \text{satisf} \cap \text{At}$, then by Lemma 4.5, $\exists$ has a winning strategy in $G_\omega(\text{At}\mathfrak{A})$, hence in $G_k(\text{At}\mathfrak{A})$, a fortiori, $\exists$ has a winning strategy in $G_k(\text{At}\mathfrak{A})$ for all $k < \omega$, so (by definition) $\mathfrak{A} \in \text{LCA}_n$ so (since $\text{LCA}_n$ is elementary) $\exists \in (S_n \text{satisf} \cap \text{At}) \subseteq \text{LCA}_n$. So $\text{LCA}_n = \text{El} \cap \text{At}$, and so (by definition) taking into account that $\text{RCA}_n = \text{satisf} \in \text{At}$.

Assume that $\mathfrak{D} \in \text{CRCA}_n$ and $\mathfrak{A} \subseteq \mathfrak{D}$. Identifying set algebras with their domain let $f : \mathfrak{D} \to \wp(V)$ be a complete representation of $\mathfrak{D}$, where $V$ is a $\text{Gs}_n$ unit. We claim that $g = f \mid \mathfrak{A}$ is a complete representation of $\mathfrak{A}$. Let $X \subseteq \mathfrak{A}$ be such that $\sum X = 1$. Then by $\mathfrak{A} \subseteq \mathfrak{D}$, we have $\sum X = 1$. Furthermore, for all $x \in X(\subseteq \mathfrak{A})$ we have $f(x) = g(x)$, so that $\sum_{x \in X} f(x) = \sum_{x \in X} g(x) = V$, since $f$ is a complete representation, and we are done. Let $\mathfrak{C}$ be any of the two remaining classes. Closure under $S_n$ follows from that $S_n \subseteq \text{satisf} \cap \text{At}$.

Non-closure under $H$ is trivial for a subalgebra of an atomic algebra may not be atomic. We prove non-closure under $H$ for all three classes in one blow. Take a family $(U_i : i \in \mathbb{N})$ of pairwise disjoint non-empty sets. Let $V_i = \text{a}^* U_i(i \in \mathbb{N})$. Take the full $\text{Gs}_n$, $\mathfrak{A}$ with universe $\wp(V)$, where $V = \bigcup_{i \in \mathbb{N}} V_i$. Then $\mathfrak{A} \in \text{CRCA}_n \subseteq \mathfrak{C}$. Let $I$ be the ideal consisting of elements of $\mathfrak{A}$ that intersect only finitely many of the $V_i$’s. Then $\mathfrak{A}/I$ is not atomic, so $\mathfrak{A}/I$ is outside all three classes. Now we approach closure under $\text{Ur}$. Let $\mathfrak{C} \in \text{CRCA}_n$ be atomic having countably many atoms and elementary equivalent to a $\mathfrak{B} \in \text{CRCA}_n$. Such algebras exist, cf. [25, Theorem 5.12]. Then $\mathfrak{C} \equiv \mathfrak{B}$, $\mathfrak{C}$ will be outside all three classes (since they coincide on atomic algebras having countably many atoms), while $\mathfrak{B}$ will be inside them all proving that non of the three is elementary, so being closed under $\text{Ur}$, since they are pseudo-elementary classes (cf. [23 Theorem 21] and [26, §9.3 for analogous cases), by the Keisler-Shelah ultrapower Theorem they are not closed under $\text{Ur}$.

Item (4) follows from that $\text{LCA}_n \subseteq \text{RCA}_n$, that (it is easy to check that) $\text{SRCA}_n$ is closed under $\mathfrak{P}$, that $\text{SRCA}_n \subseteq \text{FCA}_n$ and finally that the last class is elementary.
where \(t(n) = n(n + 1)/2 + 1\) as shown in Theorem 4.11, but cannot happen for \(\text{CRCA}_n\).

**Example 6.4.** Assume that \(1 < n < \omega\). Let \(V = {}^\omega \mathbb{Q}\) and let \(\mathfrak{A} \in \mathbb{C}_\omega\) have universe \(v(V)\). Then \(\mathfrak{A} \in \text{Nr}_n \text{CA}_\omega\). Let \(y = \{s \in V : s_0 + 1 = \sum_{i > 0} s_i\}\) and \(\mathfrak{C} = G_\mathfrak{A}^{\mathfrak{A}}(\{y\} \cup X)\), where \(X = \{\{s\} : s \in V\}\). Now \(\mathfrak{C}\) and \(\mathfrak{A}\) having same top element \(V\), share the same atom structure, namely, the singletons, so \(\mathfrak{C} \cap \mathfrak{A} = \mathfrak{A}\). Thus \(\mathfrak{A} \cap \text{At} \mathfrak{C} = \mathfrak{A}\). Thus \(\mathfrak{A} \in \text{At} \text{Nr}_n \text{CA}_\omega\) and \(\mathfrak{A} = \mathfrak{C} \cap \mathfrak{A} \in \text{Nr}_n \text{CA}_\omega\). Since \(\mathfrak{C} \subseteq \mathfrak{A}\), \(\mathfrak{C} \in S_\omega \text{Nr}_n \text{CA}_\omega \subseteq S_\omega \text{Nr}_n \text{CA}_\omega\), but as proved in [46] \(\mathfrak{C} \notin \text{EL} \text{Nr}_n \text{CA}_{n+1} \subseteq \text{Nr}_n \text{CA}_{n+1} \supseteq \text{Nr}_n \text{CA}_\omega\). This can be generalized as follows: Let \(\alpha\) be an ordinal \(> 1\); could be infinite. Let \(\mathfrak{A}\) is field of characteristic 0.

\[V = \{s \in {}^\alpha \mathfrak{A} : \left\{ i : \alpha : s_i \neq 0 \right\} < \omega\},\]

\[\mathfrak{C} = (v(V), \cup, \cap, \cdot, 0, V, c_i, d_i, j_{i,j} \in \alpha)\]

Then clearly \(v(V) \in \text{Nr}_\alpha \text{CA}_{\alpha + \omega}\). Indeed let \(W = \alpha^{\omega} \mathfrak{A}^{(0)}\). Then \(\psi : (V) \rightarrow \text{Nr}_\alpha v(W)\)
defined via

\[X \mapsto \{s \in W : s \alpha \subset X\}\]

is an isomorphism from \(v(V)\) to \(\text{Nr}_\alpha v(W)\). We shall construct an algebra \(\mathfrak{B}\), \(\mathfrak{B} \notin \text{Nr}_\alpha \text{CA}_{\alpha + 1}\). Let \(y\) denote the following \(\alpha\)-ary relation:

\[y = \{s \in V : s_0 + 1 = \sum_{i > 0} s_i\}\]

Let \(y_s\) be the singleton containing \(s\), i.e. \(y_s = \{s\}\). Define as before \(\mathfrak{B} \in \text{CA}_\alpha\) as follows:

\[\mathfrak{B} = G_\mathfrak{C}^{\mathfrak{C}}(y, y_s : s \in y)\]

The first order sentence that codes the idea of the proof says that \(\mathfrak{B}\) is neither an elementary nor complete subalgebra of \(v(V)\). Let \(\text{At}(x)\) be the first order formula asserting that \(x\) is an atom. Let

\[\tau(x, y) = c_1(c_0 x \cdot s_1 y \cdot c_1 y) \cdot c_1 x \cdot c_0 y\]

Let

\[\text{At}(x) := c_0 x \cap c_1 x = x\]

\[\phi := \forall x (x \neq 0 \rightarrow \exists y (\text{At}(y) \land y \leq x)) \land \forall x (\text{At}(x) \rightarrow \text{At}(x))\]

\[\alpha(x, y) := \text{At}(x) \land x \leq y\]

and \(\psi(y_0, y_1)\) be the following first order formula

\[\forall z \forall x (\alpha(x, y_0) \rightarrow x \leq z) \land y_0 \leq z) \land \forall x (\text{At}(x) \rightarrow \text{At}(c_0 x \cap y_0) \land \text{At}(c_1 x \cap y_0))\]

\[\rightarrow [\forall x_1 \forall x_2 (\alpha(x_1, y_0) \land \alpha(x_2, y_0) \rightarrow \tau(x_1, x_2) \leq y_1)\]

\[\land \forall z (\forall x_1 \forall x_2 (\alpha(x_1, y_0) \land \alpha(x_2, y_0) \rightarrow \tau(x_1, x_2) \leq z) \rightarrow y_1 \leq z)]\]

Then

\[\text{Nr}_\alpha \text{CA}_\beta \models \phi \rightarrow \forall y_0 \exists y_1 \psi(y_0, y_1)\]

But this formula does not hold in \(\mathfrak{B}\). We have \(\mathfrak{B} \models \phi\) and not \(\mathfrak{B} \models \forall y_0 \exists y_1 \psi(y_0, y_1)\). In words: we have a set \(X = \{y_s : s \in V\}\) of atoms such that \(\sum X = y\), and \(\mathfrak{B}\) models \(\phi\) in the sense that below any non zero element there is a rectangular atom, namely a singleton. Let \(Y = \{\tau(y_0, y_1), r, s \in V\}\), then \(Y \subseteq \mathfrak{B}\), but it has no supremum in \(\mathfrak{B}\), but it does have one in any full neat reduct \(\mathfrak{A}\) containing \(\mathfrak{B}\), and this is \(\tau(x, y)\), where \(\tau(x, y) = c_0(s_1 c_0 x \cdot s_0 c_0 y)\). In \(v(V)\) this last is \(w = \{s \in {}^\omega \mathfrak{A}^{(0)} : s_0 + 2 = s_1 + 2 \sum_{i > 1} s_i\}\), and \(w \notin \mathfrak{B}\). For \(y_0 = y\), there is no \(y_1 \in \mathfrak{B}\) satisfying \(\psi(y_0, y_1)\).
If $K$ is class of BAOs, and $\mathfrak{A} \in K$ is atomic, then plainly $\text{At}\mathfrak{A} \in \text{At}K$. But the converse might not be true. But, conversely, if $\mathfrak{A} \in \text{CRCA}_n$ has atom structure $\text{At}$, then if $\mathfrak{A} \in \text{CA}_n$ and $\text{At}\mathfrak{A} = \text{At}$, then $\mathfrak{A} \in \text{CRCA}_n$. This motivates:

**Definition 6.5.**  
(1) The class $K$ is gripped by its atom structures, or simply gripped, if for $\mathfrak{A} \in \text{CA}_n$, whenever $\text{At}\mathfrak{A} \in \text{At}K$, then $\mathfrak{A} \in K$.

(2) An $\omega$-rounded game $H$ grips $K$, if whenever $\mathfrak{A} \in \text{CA}_n$ is atomic with countably many atoms and $\exists$ has a winning strategy in $H(\mathfrak{A})$, then $\mathfrak{A} \in K$. The game $H$ weakly grips $K$, if whenever $\mathfrak{A} \in \text{CA}_n$ is atomic with countably many atoms and $\exists$ has a winning strategy in $H(\text{At}\mathfrak{A})$, then $\text{At}\mathfrak{A} \in K$. The game $H$ densely grips $K$, if whenever $\mathfrak{A} \in \text{CA}_n$ is atomic with countably many atoms and $\exists$ has a winning strategy in $H(\text{At}\mathfrak{A})$, then $\text{At}\mathfrak{A} \in K$ and $\text{CmAt}\mathfrak{A} \in K$.

**Example 6.6.** Let $2 < n < m \leq \omega$.

- The classes $\text{RCA}_n$ and $\text{Nr}_n\text{CA}_m$ are not gripped, by [44, 41] and example 6.4. In fact, by [44], if $m \geq n + 3$, then $\text{SNr}_n\text{CA}_m$ is not gripped. For any $n < \omega$, the class $\text{CRCA}_n$, and its elementary closure, namely, $\text{LCA}_n$ (the class of algebras satisfying the Lyndon conditions) are gripped.
- The class $\text{ScNr}_n\text{CA}_m$ is gripped.
- The usual atomic game $G$ weakly grips, densely grips and grips $\text{CRCA}_n$.

Now we define the game $H$, mentioned above outline of Theorem ??, that densely grips, hence weakly grips, $\text{Nr}_n\text{CA}_\omega$, but we it can be shown that $H$ does not grip $\text{Nr}_n\text{CA}_\omega$. Accordingly $H$ will be used to prove a slightly weaker result as stated in the idea of proof.

The following definition, inspired by the result in Theorem 4.11 stresses the fact that some algebras are ‘more representable’ than others.

**Definition 6.7.** Let $m \leq \omega$ and $\mathfrak{A} \in \text{RCA}_n$ be atomic.

1. Then $\mathfrak{A}$ is strongly representable up to $m$ if if $\text{CmAt}\mathfrak{A} \in \text{SNr}_n\text{CA}_{n+m}$.
2. We say that $\mathfrak{A}$ is extremely representable up to $m$ if if $\text{CmAt}\mathfrak{A} \in \text{ScNr}_n\text{CA}_{n+m}$.

**Example 6.8.** The atomic algebra $\mathfrak{C} = \mathfrak{C}_{\mathbb{Z},\mathbb{N}}$ used in [44], to be recalled below is $\text{Cm(At}\mathfrak{C}) \in \text{RCA}_n$, hence $\mathfrak{C}$ strongly representable up to $m$ for any $m \leq \omega$, but $\mathfrak{C}$ is not extremely representable up to $\omega$ since it lacks a complete representation to be proved in a while upon observing that it has only countably many atoms. Worthy of note is that in fact $\mathfrak{C} \in \text{LCA}_n(\subseteq \text{SRCA}_n)$. However, if $\mathfrak{A} \in \text{RCA}_n$ is finite and $m \leq \omega$, then $\mathfrak{A}$ is strongly representable up to $m$ if $\mathfrak{A}$ is extremely representable up to $m$.

**Theorem 6.9.** An atomic algebra $\mathfrak{A} \in \text{RCA}_n$ is strongly representable up to $\omega$ if $\mathfrak{A}$ is extremely representable up to $\omega$. If $\mathfrak{A}$ is atomic with countably many atoms, then $\mathfrak{A}$ extremely representable up to $\omega$ if $\mathfrak{A}$ is in $\text{CRCA}_n$.

**Proof.** The first part follows by observing that $\text{RCA}_n = \text{SNr}_n\text{CA}_\omega$. For the second part. Assume that $\mathfrak{A}$ has countably many atoms and $\mathfrak{A}$ is extremely representable up to $\omega$. Then $\text{CmAt}\mathfrak{A} \in \text{ScNr}_n\text{CA}_\omega$. Since $\mathfrak{A} \subseteq \mathfrak{C}$, $\text{CmAt}\mathfrak{A}$, then $\mathfrak{A} \subseteq \mathfrak{C}$, $\text{CmAt}\mathfrak{A}$, and since the operator $\mathfrak{S}_\mathfrak{C}$ is obviously idempotent ($\mathfrak{S}_\mathfrak{C}\mathfrak{S}_\mathfrak{C} = \mathfrak{S}_\mathfrak{C}$), we get that $\mathfrak{A} \in \text{ScNr}_n\text{CA}_\omega$. Having countably many atoms we get $\mathfrak{A} \in \text{CRCA}_n$. Conversely, if $\mathfrak{A} \in \text{CRCA}_n$ is atomic having countably many atoms, then $\text{CmAt}\mathfrak{A} \in \text{CRCA}_n$, $\mathfrak{A} \in \text{ScNr}_n\text{CA}_\omega$. □
Lemma 6.10. Let \( \alpha \) be a countable atom structure. If \( \exists \) has a winning strategy in \( H_\omega(\alpha) \), then there exists a complete \( D \in \text{RCA}_\omega \) such that \( \alpha \cong \text{AtNr}_nD \).

Proof. We show that \( \alpha \cong \text{AtNr}_nD \) with \( D \) as in Theorem 4.13 Let \( x \in D \). Then \( x = (x_a : a \in \alpha) \), where \( x_a \in D_a \). For \( b \in \alpha \) let \( \pi_b : D \to D_b \) be the projection map defined by \( \pi_b(x_a : a \in \alpha) = x_b \). Conversely, let \( \iota_a : D_a \to D \) be the embedding defined by \( \iota_a(y) = (x_b : b \in \alpha) \), where \( x_a = y \) and \( x_b = 0 \) for \( b \neq a \). Suppose \( x \in \text{Nr}_nD \setminus \{0\} \). Since \( x \neq 0 \), then it has a non-zero component \( \pi_a(x) \in D_a \), for some \( a \in \alpha \). Assume that \( \emptyset \neq \phi(x_{i_0}, \ldots, x_{i_{k-1}})^{D_a} = \pi_a(x) \), for some L-formula \( \phi(x_{i_0}, \ldots, x_{i_{k-1}}) \). We have \( \phi(x_{i_0}, \ldots, x_{i_{k-1}})^{D_a} \in \text{Nr}_nD_a \). Pick \( f \in \phi(x_{i_0}, \ldots, x_{i_{k-1}})^{D_a} \) and assume that \( M_a, f \models b(x_0, \ldots, x_{n-1}) \) for some \( b \in \alpha \). We show that \( b(x_0, x_1, \ldots, x_{n-1})^{D_a} \subseteq \phi(x_{i_0}, \ldots, x_{i_{k-1}})^{D_a} \).

Take any \( g \in b(x_0, x_1, \ldots, x_{n-1})^{D_a} \), so that \( M_a, g \models b(x_0, \ldots, x_{n-1}) \). The map \( \{(f(i), g(i)) : i < n\} \) is a partial isomorphism of \( M_a \). Here that short hyperedges are constantly labelled by \( \lambda \) is used. This map extends to a finite partial isomorphism \( \theta \) of \( M_a \) whose domain includes \( f(i_0), \ldots, f(i_{k-1}) \). Let \( g' \in M_a \) be defined by

\[
g'(i) = \begin{cases} 
\theta(i) & \text{if } i \in \text{dom}(\theta) \\
g(i) & \text{otherwise}
\end{cases}
\]

We have \( M_a, g' \models \phi(x_{i_0}, \ldots, x_{i_{k-1}}) \). But \( g'(0) = \theta(0) = g(0) \) and similarly \( g'(n-1) = g(n-1) \), so \( g \) is identical to \( g' \) over \( n \) and it differs from \( g \) on only a finite set. Since \( \phi(x_{i_0}, \ldots, x_{i_{k-1}})^{D_a} \in \text{Nr}_nD_a \), we get that \( M_a, g \models \phi(x_{i_0}, \ldots, x_{i_k}) \), so \( g \in \phi(x_{i_0}, \ldots, x_{i_{k-1}})^{D_a} \) (this can be proved by induction on quantifier depth of formulas).

This proves that

\[
b(x_0, x_1 \ldots x_{n-1})^{D_a} \subseteq \phi(x_{i_0}, \ldots, x_{i_k})^{D_a} = \pi_a(x),
\]

and so

\[
\iota_a(b(x_0, x_1 \ldots x_{n-1})^{D_a}) \leq \iota_a(\phi(x_{i_0}, \ldots, x_{i_{k-1}})^{D_a}) \leq x \in D_a \setminus \{0\}.
\]

Now every non-zero element \( x \) of \( \text{Nr}_nD_a \) is above a non-zero element of the following form \( \iota_a(b(x_0, x_1 \ldots x_{n-1})^{D_a}) \) (some \( a, b \in \alpha \)) and these are the atoms of \( \text{Nr}_nD_a \). The map defined via \( b \mapsto (b(x_0, x_1, \ldots, x_{n-1})^{D_a} : a \in \alpha) \) is an isomorphism of atom structures, so that \( \alpha \in \text{AtNr}_n\text{CA}_\omega \).

Having Lemma 6.10 at our hand, if we go down to the level of atom structures (of atomic algebras), then we can remove the \( S_d \) appearing in Theorem 4.14. Going from atomic algebras to atom structures this is feasible, but the process is not reversible as shown in example 6.4. More succinctly the implication Corollary 6.11 \( \implies \) Theorem 4.14 is not valid, because precisely \( \text{Nr}_n\text{CA}_m \) for any \( 2 < n < m \leq m \) is not gripped.

Corollary 6.11. Any class of frames \( K \), such \( \text{At} (\text{Nr}_n\text{CA}_\omega \cap \text{CRCA}_n) \subseteq K \subseteq \text{At}_CA_n \) is not elementary

Proof. Using the technique in Theorem 4.14 together with Lemmata 4.13 and 6.10.

Theorem 6.12. Any class \( K \) such that \( \text{Nr}_n\text{CA}_\omega \subseteq K \subseteq \text{Sr}_n\text{C}A_{n+1} \), \( K \) is not elementary.

Proof. We slightly modify the construction in [4] Lemma 5.1.3, Theorem 5.1.4]. Using the same notation, the algebras \( \mathcal{A} \) and \( \mathcal{B} \) constructed in op.cit satisfy \( \mathcal{A} \in \text{Nr}_n\text{CA}_\omega \), \( \mathcal{B} \notin \text{Nr}_n\text{CA}_{n+1} \) and \( \mathcal{A} \equiv \mathcal{B} \). As they stand, \( \mathcal{A} \) and \( \mathcal{B} \) are not atomic, but it can be
fixed that they are atomic, giving the same result with the rest of the proof unaltered. This is done by interpreting the uncountably many tenary relations in the signature of \( M \) defined in \cite[Lemma 5.1.3]{41}, which is the base of \( \mathfrak{A} \) and \( \mathfrak{B} \) to be disjoint in \( M \), not just distinct. The construction is presented this way in \cite{38}, where (the equivalent of) \( M \) is built in a more basic step-by-step fashion. We work with \( 2 < n < \omega \) instead of only \( n = 3 \). The proof presented in \textit{op.cit} lift verbatim to any such \( n \). Let \( u = ^n n \). Write \( 1_u \) for \( \chi_u^{M} \) (denoted by \( 1_u \) for \( n = 3 \) in \cite[Theorem 5.1.4]{41}). We denote by \( \mathfrak{A}_u \) the Boolean algebra \( \mathcal{R}L_1 \mathfrak{A} = \{ x \in \mathfrak{A} : x \leq 1_u \} \) and similarly for \( \mathfrak{B} \), writing \( \mathfrak{B}_u \) short hand for the Boolean algebra \( \mathcal{R}L_1 \mathfrak{B} = \{ x \in \mathfrak{B} : x \leq 1_u \} \). Then exactly like in \cite{41}, it can be proved that \( \mathfrak{A} \equiv \mathfrak{B} \). Using that \( M \) has quantifier elimination we get, using the same argument in \textit{op.cit} that \( \mathfrak{A} \in \mathcal{R}l_n \mathfrak{C}_\omega \). The property that \( \mathfrak{B} \notin \mathcal{R}l_n \mathfrak{C}_n+1 \) is also still maintained. To see why, consider the substitution operator \( s(0, 1) \) (using one spare dimension) as defined in the proof of \cite[Theorem 5.1.4]{41}. Assume for contradiction that \( \mathfrak{B} = \mathcal{R}l_n \mathfrak{C}_u \), with \( \mathfrak{C} \in \mathfrak{C}_n+1 \). Let \( u = (1, 0, 2, \ldots, n - 1) \). Then \( \mathfrak{A}_u = \mathfrak{B}_u \) and so \( |\mathfrak{B}_u| > \omega \). The term \( n_s(0, 1) \) acts like a substitution operator corresponding to the transposition \( 0, 1 \); it ‘swaps’ the first two co-ordinates. Now one can show that \( n_s(0, 1) \mathfrak{B}_u \subseteq \mathfrak{B}_{u,0} = \mathfrak{B}_{1d} \), so \( |n_s(0, 1)\mathfrak{B}_u| \) is countable because \( \mathfrak{B}_{1d} \) was forced by construction to be countable. But \( n_s(0, 1) \) is a Boolean automorphism with inverse \( s(1, 0) \), so that \( |\mathfrak{B}_u| = |n_s(0, 1)\mathfrak{B}_u| > \omega \), contradiction. Since \( \mathfrak{A} \equiv \mathfrak{B} \) and \( \mathfrak{A} \in \mathfrak{N}_\kappa \mathfrak{C}_\omega \cap \mathfrak{C}_\omega \mathfrak{A} \), it suffices to show (since \( \mathfrak{B} \) is atomic) that \( \mathfrak{A} \) is in fact outside \( \mathfrak{S}_d\mathfrak{N}_\kappa \mathfrak{C}_{n+1} \cap \mathfrak{A} \). Take \( \kappa \) the signature of \( M \); more specifically, the number of \( n \)-ary relation symbols to be \( 2^{2^c} \), and assume for contradiction that \( \mathfrak{B} \in \mathfrak{S}_d\mathfrak{N}_\kappa \mathfrak{C}_{n+1} \cap \mathfrak{A} \). Then \( \mathfrak{B} \subseteq \mathfrak{N}_\kappa \mathfrak{D} \), for some \( \mathfrak{D} \in \mathfrak{C}_{n+3} \) and \( \mathfrak{N}_\kappa \mathfrak{D} \) is atomic. For brevity, let \( \mathfrak{E} = \mathfrak{N}_\kappa \mathfrak{D} \). Then by item (1) of Lemma \cite[9]{6.1} \( \mathfrak{N}_\kappa \mathfrak{D} \subseteq \mathfrak{S}_d\mathfrak{N}_\kappa \mathfrak{C}_\omega \). Since \( \mathfrak{E} \) is atomic, then by item (1) of the same Lemma \( \mathfrak{S}_d\mathfrak{N}_\kappa \mathfrak{C}_\omega \) is also atomic. Using the same reasoning as above, we get that \( |\mathfrak{S}_d\mathfrak{N}_\kappa \mathfrak{C}_\omega| > 2^{2^c} \) (since \( \mathfrak{E} \in \mathfrak{S}_d\mathfrak{N}_\kappa \mathfrak{C}_{n+1} \)). By the choice of \( \kappa \), we get that \( |\mathfrak{A} \mathfrak{S}_d\mathfrak{N}_\kappa \mathfrak{C}_\omega| > \omega \). By density, we get from item (2) of Lemma \cite[6.1]{} that \( \mathfrak{A} \mathfrak{S}_d\mathfrak{N}_\kappa \mathfrak{C}_\omega \subseteq \mathfrak{S}_d\mathfrak{N}_\kappa \mathfrak{C}_\omega \). Hence \( |\mathfrak{A} \mathfrak{S}_d\mathfrak{N}_\kappa \mathfrak{C}_\omega| \geq |\mathfrak{A} \mathfrak{S}_d\mathfrak{N}_\kappa \mathfrak{C}_\omega| > \omega \). But by the construction of \( \mathfrak{B} \), \( |\mathfrak{S}_d\mathfrak{N}_\kappa \mathfrak{C}_\omega| = |\mathfrak{A} \mathfrak{S}_d\mathfrak{N}_\kappa \mathfrak{C}_\omega| = \omega \), which is a contradiction and we are done. But \( \mathfrak{B} \) is completely representable, thus \( \mathfrak{B} \in \mathfrak{S}_d \mathfrak{N}_\kappa \mathfrak{C}_\omega \). Thus \( \mathfrak{B} \in \mathfrak{E}(\mathfrak{N}_\kappa \mathfrak{C}_\omega \cap \mathfrak{C}_\omega \mathfrak{A}) \sim \mathfrak{S}_d \mathfrak{N}_\kappa \mathfrak{C}_\omega \). \qed

It can be shown that neither of the two classes \( \mathfrak{S}_d \mathfrak{N}_\kappa \mathfrak{C}_{n+1} \) and \( \mathfrak{S}_d \mathfrak{N}_\kappa \mathfrak{C}_{n+3} \) are included in the other. The algebra \( \mathfrak{B} \) in example \cite[6.4]{} is in the last, because it is completely representable, but not the first, while in \cite[6.1]{} it is shown that for any \( k \geq 1 \), there is a finite \( \mathfrak{D}_k \in \mathfrak{C}_n \) such that \( \mathfrak{D}_k \in \mathcal{S}_\kappa \mathfrak{C}_{n+k+1} \sim \mathfrak{N}_\kappa \mathfrak{C}_{n+k} \). Since for any such algebra \( \mathfrak{D}_k = \mathfrak{D}_k^\kappa \), and for any \( \mathfrak{E} \in \mathfrak{C}_n \) and \( m > n \), \( \mathfrak{E} \in \mathcal{S}_\kappa \mathfrak{C}_m \iff \mathfrak{E} \in \mathfrak{S}_d \mathfrak{N}_\kappa \mathfrak{C}_m \). From this we immediately get that \( \mathfrak{D}_2 \in \mathfrak{S}_d \mathfrak{N}_\kappa \mathfrak{C}_{n+3} \sim \mathfrak{S}_d \mathfrak{N}_\kappa \mathfrak{C}_{n+1} \). Excluding elementary classes between \( \mathfrak{N}_\kappa \mathfrak{C}_\omega \) and \( \mathfrak{S}_d \mathfrak{N}_\kappa \mathfrak{C}_\omega \) and excluding ones between \( \mathfrak{S}_d \mathfrak{N}_\kappa \mathfrak{C}_\omega \) and \( \mathfrak{S}_d \mathfrak{N}_\kappa \mathfrak{C}_{n+3} \), this prompts the following: Is there an elementary class \( K \) between \( \mathfrak{N}_\kappa \mathfrak{C}_\omega \) and \( \mathfrak{S}_d \mathfrak{N}_\kappa \mathfrak{C}_{n+3} \)? Observe that if \( K \lneq \mathfrak{S}_d \mathfrak{N}_\kappa \mathfrak{C}_\omega \), then \( K \cap \mathfrak{S}_d \mathfrak{N}_\kappa \mathfrak{C}_\omega \neq \emptyset \).

**Conjecture.** For \( 2 < n < \omega \), any class \( K \) such that \( \mathfrak{N}_\kappa \mathfrak{C}_\omega \cap \mathfrak{C}_\omega \mathfrak{A} \subseteq K \subseteq \mathfrak{S}_d \mathfrak{N}_\kappa \mathfrak{C}_{n+3} \), \( K \) is not elementary.

**Summary of results on (non-) first order definability of classes of algebras involving the operators \( \mathcal{R}L \mathfrak{A} \) and \( \mathfrak{N}_\kappa \) \((2 < n < \omega)\):** In the next table we summarize the results obtained on non first order definability proved in theorems ?, ?, ?, ?. The last column in the second row remains unsettled for RAs.
7 Representability Theory

7.1 Notions of representability and neat embeddings

Corollary 7.1. Let $2 < n < \omega$. Then the following hold:

1. $\text{Nr}_n \text{CA}_\omega \cap \text{At} \subseteq S_n \text{Nr}_n \text{CA}_\omega \cap \text{At} \subseteq S_n \text{Nr}_n \text{CA}_\omega \cap \text{At} \subseteq \text{EILS}_n \text{Sr}_n \text{CA}_\omega \cap \text{At} = \text{EIL}(S_n \text{Nr}_n \text{CA}_\omega \cap \text{At}) = \text{LCA}_n \subseteq \text{SRC} \subseteq \text{EISRCA}_n \subseteq \text{FC} \subseteq \text{WRCA}_n$.

2. $\text{CRCA}_n \subseteq S_n \text{Sr}_n \text{CA}_\omega \cap \text{At} \subseteq \text{EILCRCA}_n = \text{EILS}_n \text{Sr}_n \text{CA}_\omega \cap \text{At} = \text{LCA}_n$.

3. For any class $K(\subseteq \text{RCA}_n \cap \text{At})$ occurring in the previous two items, $\text{EIK}$ is not finitely axiomatizable, and $SK = \text{RCA}_n$.

Proof. The algebra $\mathfrak{B}$ used in the last item of Theorem 7.1 is in $S_n \text{Sr}_n \text{CA}_\omega \cap \text{At} \sim S_n \text{Sr}_n \text{CA}_\omega$. For the strictness of the last inclusion in the first item, we refer the reader to [26, Theorem 14.17] for the relation algebra analogue. Item (2) is already dealt with. Now we approach item (3): From [22, Construction 3.2.76, p. 94], it can be easily distilled that the elementary closure of any class $K$, such that $\text{Nr}_n \text{CA}_\omega \cap \text{At} \subseteq K \subseteq \text{RCA}_n$, $K$ is not finitely axiomatizable. In the aforementioned construction, non–representable finite (Monk) algebras outside $\text{RCA}_n$ are constructed, such that any (atomic) non–trivial ultraproduct of such algebras is in $\text{Nr}_n \text{CA}_\omega \cap \text{At}$. For proving non–finite axiomatizability one uses [22, Construction 3.2.76, pp. 94]. In op.cit non–representable finite Monk algebras outside $\text{RCA}_n \supseteq \text{EISRCA}_n \supseteq \text{LCA}_n$ are constructed, such that any (atomic) non–trivial ultraproduct of such algebras is in $\text{Nr}_n \text{CA}_\omega \cap \text{At} \subseteq \text{EILN}_n \text{CA}_\omega \cap \text{At} \subseteq \text{EILS}_n \text{Sr}_n \text{CA}_\omega \cap \text{At} = \text{LCA}_n \subseteq \text{EISRCA}_n$. We give the details. Fix $2 < n < \omega$, and let $K$ be any elementary class between $\text{Nr}_n \text{CA}_\omega$ and $\text{RCA}_n$. For $3 \leq n$, $i < \omega$, with $n - 1 \leq i$, $\mathfrak{C}_{n,i}$ denotes the finite $\text{CA}_n$ associated with the cylindric atom structure as defined on [22, p. 95]. Then by [22, Theorem 3.2.79] for $3 \leq n$, and $j < \omega$, $\mathfrak{R}_d \mathfrak{C}_{n,n+j}$ can be neatly embedded in a $\text{CA}_3+j+1$ (1). By [22, Theorem 3.2.84], we have for every $j \in \omega$, there is an $3 \leq n$ such that $\mathfrak{R}_d \mathfrak{R}_d \mathfrak{C}_{n,n+j}$ is a non–representable $\text{DF}_3$ (2). Now suppose that $m \in \omega$. By (2), there is a $j \in \omega \sim 3$ so that $\mathfrak{R}_d \mathfrak{R}_d \mathfrak{R}_d \mathfrak{C}_{j,j+m+n-4}$ is not a representable $\text{DF}_3$.

By (1) we have $\mathfrak{R}_d \mathfrak{R}_d \mathfrak{R}_d \mathfrak{C}_{j,j+m+n-4} \subseteq \mathfrak{R}_d \mathfrak{B}_m$, for some $\mathfrak{B}_m \in \text{CA}_{n+m}$. We can assume that $\mathfrak{R}_d \mathfrak{C}_{j,j+m+n-4}$ generates $\mathfrak{B}_m$, so that $\mathfrak{B}_m$ is finite. Put $\mathfrak{A}_m = \mathfrak{R}_n \mathfrak{B}_m$, then $\mathfrak{A}_m$ is finite, too, and $\mathfrak{R}_d \mathfrak{R}_d \mathfrak{A}_m$ is not representable, a fortiori $\mathfrak{A}_m \not\in \text{RCA}_n$. Therefore $\mathfrak{A}_m \not\in \text{EILN}_n \text{CA}_\omega$. Let $\mathfrak{C}_m$ be an algebra similar to $\text{CA}_\omega$'s such that $\mathfrak{B}_m = \mathfrak{R}_n \mathfrak{C}_m$. Then $\mathfrak{A}_m = \mathfrak{R}_n \mathfrak{C}_m$. (Note that $\mathfrak{C}_m$ cannot belong to $\text{CA}_\omega$, for else $\mathfrak{A}_m$ will be representable). If $F$ is a non–trivial ultralinear on $\omega$, we have $\Pi_{m \in \omega} \mathfrak{A}_m/F = \Pi_{m \in \omega} (\mathfrak{R}_n \mathfrak{C}_m)/F = \mathfrak{R}_n (\Pi_{m \in \omega} \mathfrak{C}_m/F)$. But $\Pi_{m \in \omega} \mathfrak{C}_m/F \in \text{CA}_\omega$, we conclude that $\Pi_{m \in \omega} \mathfrak{A}_m \sim \text{EIK}$ is not closed under ultraproducts, because $\mathfrak{A}_m \not\in \text{RCA}_n \supseteq \text{EILN}_n \text{CA}_\omega$ and $\Pi_{m \in \omega} \mathfrak{A}_m/F \in \text{Nr}_n \text{CA}_\omega \subseteq \text{EIK}$.

We prove the last part of item (3): Let $\mathfrak{A} \in \text{RCA}_n$. Then $\mathfrak{A} \cong \mathfrak{B}$, $\mathfrak{B} \in \text{GS}_n$ with top element $V$ say. Let $\mathfrak{C}$ be the full $\text{GS}_n$ with top element $V$ (and universe $\phi(V)$). Then
$C \in \text{Nr}_n \text{CA}_\omega \cap \text{At}$ and $B \subseteq C$. Thus $A \in S(\text{Nr}_n \text{CA}_\omega \cap \text{At})$. For classes in the last item one has another option; one can take canonical extensions instead of generalized full set algebras upon observing that $A \in \text{RCA}_n \iff A^+ \in \text{CRCA}_n$.

Using the notation and proof of [27, Theorem 3.7.4] dealing with inclusions and first order definability of atom structures, we get:  

**Theorem 7.2.** Let $2 < n < \omega$. Then the following hold with elementary classes of atom structures or algebras underlined:

1. $\text{At} \text{Nr}_n \text{CA}_\omega \nsubseteq \text{CRAS}_n$ and $\text{At} \text{Nr}_n \text{CA}_\omega \subseteq \text{At} \text{S}_d \text{Nr}_n \text{CA}_\omega \subseteq \text{At} \text{S}_c \text{Nr}_n \text{CA}_\omega \subseteq \text{At} \text{EL} \text{Sc} \text{Nr}_n \text{CA}_\omega = \text{EL} \text{Sc} \text{Nr}_n \text{CA}_\omega$.

2. $\text{CRAS}_n \nsubseteq \text{At} \text{S}_c \text{Nr}_n \text{CA}_\omega \nsubseteq \text{At} \text{EL} \text{Sc} \text{Nr}_n \text{CA}_\omega = \text{LCAS}_n \subseteq \text{SRAS}_n \subseteq \text{WRAS}_n$.

3. $\text{At} \text{Nr}_n \text{CA}_\omega \nsubseteq \text{CRAS}_n$.

We start by a neat embedding theorem, a NET for short, formulated for TCA$s$ and TeCA$s$, lifting Henkin’s famous neat embedding theorem to the topological and temporal context, respectively.

**Theorem 7.3.** An algebra $A \in \text{TCA}_\alpha$ is representable if and only if $A \in \text{SNr}_\alpha \text{TCA}_{\alpha+\omega}$.

**Proof.** First for any pair of ordinals $\alpha < \beta$, $\text{SNr}_\alpha \text{TCA}_\beta$ is a variety exactly like the CA case. To show that $\text{RTCA}_\alpha \subseteq \text{SNr}_\alpha \text{TCA}_{\alpha+\omega}$, it suffices to consider algebras in $\text{TWs}_\alpha$ the weak set algebras as defined in [22] whose base is endowed with an Alexandrov topology. Since $\text{SNr}_\alpha \text{TCA}_{\alpha+\omega}$ is closed under SP. Let $A \in \text{TWs}_\alpha$ and assume that $A$ has top element $\alpha U(p)$. Let $B = \alpha + \omega$ and let $p^* \in \beta U$ be a fixed sequence such that $p^* \upharpoonright \alpha = p$. Let $C$ be the TCA$\beta$ with top element $\beta U(p^*)$; cylindrifiers and diagonal elements are defined the usual way and the interior operators induced by the topology on $U$. Define $\psi : A \to C$ via $X \mapsto \{ s \in \beta U(p^*) : s \upharpoonright \alpha \in X \}$. Then $\psi$ is a homomorphism, further it is injective, and as easily checked, $\psi$ is a neat embedding that is $\psi(A) \subseteq \text{Nr}_\alpha C$. Maybe the hardest part is to show that if $A \in \text{SNr}_\alpha \text{TCA}_{\alpha+\omega}$ then it is representable. But this follows from the fact that we can assume that $A \subseteq \text{Nr}_\alpha B$, where $B \in \text{TDc}_{\alpha+\omega}$. By the representability of dimension complemented algebras, baring in mind that a neat reduct of a representable algebra is representable, we get the required result. 

Unless otherwise indicated $\alpha$ is an arbitrary ordinal and we sometimes write $n$ instead of $\alpha$ if the last is finite.

### 7.2 Non-finite axiomatizability and other complexity issues for the variety $\text{RTCA}_\alpha$

Now we prove quite intricate and sharp non-finite axiomatizability results using their cylindric version. We address both finite and infinite dimensions. But we start with a very simple fact that allows us to recursively associate with every CA of any dimension both a TCA and a TeCA of the same dimension, such that the last two algebras are representable if and only if the original CA is. This mechanical procedure will be the main technique we use to obtain negative results for both TCAs and TeCAs by bouncing them back to their cylindric counterpart.
Definition 7.4. Let $\mathfrak{A} \in \mathcal{C}A_\alpha$. Then the algebra $\mathfrak{A^{\text{top}}} \in \text{TCA}_\alpha$ is a topologizing of $\mathfrak{A}$ if $\mathfrak{M}_\text{ca}\mathfrak{A^{\text{top}}} = \mathfrak{A}$. The discrete topologizing of $\mathfrak{A}$ is the TCA$_\alpha$ obtained from $\mathfrak{A}$ by expanding $\mathfrak{A}$ with $\alpha$ many identity operators.

Obersee that if $\mathfrak{A}$ is a generalized set algebra, then this can be done by giving all of its subsbasis the discrete Alexandrov topology.

Theorem 7.5. The discrete topologizing of $\mathfrak{A} \in \mathcal{C}A_\alpha$ is unique up to isomorphism. Furthermore, if $\mathfrak{A^{\text{top}}}$ is the discrete topologizing of $\mathfrak{A}$, then $\mathfrak{A}$ is representable if and only if $\mathfrak{A^{\text{top}}}$ is representable.

Proof. The first part is trivial. The second part is also very easy. If $\mathfrak{A^{\text{top}}}$ is representable then obviously $\mathfrak{A} = \mathfrak{M}_\text{ca}\mathfrak{A^{\text{top}}}$ is representable. For the last part if $\mathfrak{A}$ is representable with base $U$ then $\mathfrak{A^{\text{top}}}$ have the same universe of $\mathfrak{A}$, hence it is representable by endowing $U$ with the discrete topology, which induces the identity interior operators. 

For a BAO, $\mathfrak{A}$, say, and $a \in \mathfrak{A}$, we write $\mathfrak{M}_a\mathfrak{A}$ for the algebra with universe $\{x \in \mathfrak{A} : x \leq a\}$, top element $a$ and operations relativized to $\mathfrak{A}$. If $\mathfrak{A} \in \mathcal{C}A_\alpha$; it is not always the case that $\mathfrak{M}_a\mathfrak{A}$ is a $\mathcal{C}A_\alpha$, too. We show that for any ordinal $\alpha > 2$, for any $r \in \omega$, and for any $k \geq 1$, there exists $\mathfrak{B}r \in \text{SNe}_r\text{TeCA}_\alpha+k \sim \text{SNe}_r\text{TeCA}_\alpha+k+1$ such that $\Pi_{r/U}\mathfrak{B}r \in \text{TeCA}_\alpha$, for any non-principal ultrafilter on $\omega$. We will use quite sophisticated constructions of Hirsch and Hodkinson for relation and cylindric algebras reported in [26]. Assume that $3 \leq m \leq n < \omega$. For $r \in \omega$, let $\mathfrak{C}_r = \mathfrak{C}a(\text{H}_{m}^{n+1}(\mathfrak{A}(n, r), \omega))$ as defined in [26] definition 15.3. We denote $\mathfrak{C}_r$ by $\mathfrak{C}(m, n, r)$. Then the following hold:

Lemma 7.6. (1) For any $r \in \omega$ and $3 \leq m \leq n < \omega$, we have $\mathfrak{C}(m, n, r) \in \text{N}_m\mathcal{C}A_n$, $\mathfrak{C}(m, n, r) \notin \text{SNe}_m\text{CA}_n$ and $\Pi_{r/U}\mathfrak{C}(m, n, r) \in \text{RCA}_m$. Furthermore, for any $k \in \omega$, $\mathfrak{C}(m, m + k, r) \cong \mathfrak{M}_m\mathfrak{C}(m + k, m + k, r)$.

(2) If $3 \leq m < n, k \geq 1$ is finite, and $r \in \omega$, there exists $x_n \in \mathfrak{C}(n, n + k, r)$ such that $\mathfrak{C}(m, m + k, r) \cong \mathfrak{M}_x\mathfrak{C}(n, n + k, r)$ and $c_i x_n \cdot c_j x_n = x_n$ for all $i, j < m$.

Proof. 1. Assume that $3 \leq m \leq n < \omega$, and let

$$\mathfrak{C}(m, n, r) = \mathfrak{C}a(\text{H}_{m}^{n+1}(\mathfrak{A}(n, r), \omega)),$$

be as defined in [26] Definition 15.4. Here $\mathfrak{A}(n, r)$ is a finite Monk-like relation algebra

[26] Definition 15.2] which has an $n + 1$-wide $m$-dimensional hyperbasis $\text{H}_{m}^{n+1}(\mathfrak{A}(n, r), \omega)$ consisting of all $n + 1$-wide $m$-dimensional wide $\omega$ hypernetworks [26] Definition 12.21.

For any $r$ and $3 \leq m \leq n < \omega$, we have $\mathfrak{C}(m, n, r) \in \text{N}_m\mathcal{C}A_n$. Indeed, let $H = \text{H}_{m}^{n+1}(\mathfrak{A}(n, r), \omega)$. Then $H$ is an $n + 1$-wide $m$ dimensional $\omega$ hyperbasis, so $\mathfrak{C}aH \in \mathcal{C}A_n$. But, using the notation in [26] Definition 12.21 (5), we have $\text{H}_{m}^{n+1}(\mathfrak{A}(n, r), \omega) = \text{H}_{m}^{n+1}$. Thus

$$\mathfrak{C}(m, n, r) = \mathfrak{C}a(\text{H}_{m}^{n+1}(\mathfrak{A}(n, r), \omega)) = \mathfrak{C}a(\text{H}_{m}^{n+1}) \cong \mathfrak{M}_m\mathfrak{C}aH.$$

The second part is proved in [26] Corollary 15.10], and the third in [26] exercise 2, p. 484).

2. Let $3 \leq m < n$. Take

$$x_n = \{ f : \leq n+k+1 \to \text{At}\mathfrak{A}(n + k, r) \cup \omega : m \leq j < n \to \exists i < m, f(i, j) = 1d \}.$$

Then $x_n \in \mathfrak{C}(n, n + k, r)$ and $c_i x_n \cdot c_j x_n = x_n$ for distinct $i, j < m$. Furthermore

$$I_n : \mathfrak{C}(m, m + k, r) \cong \mathfrak{M}_{x_n}\mathfrak{M}_m\mathfrak{C}(n, n + k, r),$$

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via the map, defined for $S \subseteq H_{m+k+1}^n(\mathfrak{A}(m+k,r), \omega)$, by

$$I_n(S) = \{f : \leq n+k+1 \rightarrow \mathfrak{A}(n+k,r) \cup \omega : f \downarrow \leq m+k+1 \in S, \forall j (m \leq j < n \rightarrow \exists i < m, f(i,j) = \text{Id})\}.$$ 

The essential argument used in the next proof is basically a lifting argument initiated by Monk [22, Theorem 3.2.67].

**Theorem 7.7.** Let $\alpha > 2$ be an ordinal. Then for any $r \in \omega$, for any finite $k \geq 1$, for any $l \geq k + 1$ (possibly infinite), there exist $\mathcal{B}^r \in \text{SN}_\alpha \text{TCA}_{\alpha+1}$, $\mathfrak{R}_\alpha \mathcal{B} \not\sim \text{SN}_\alpha \text{TCA}_{\alpha+k+1}$ such $\Pi_{r/\in} \mathcal{B}^r \in \text{SN}_\alpha \text{TCA}_{\alpha+k}$. Also $\text{TCA}_\alpha$ cannot be axiomatized with a set of universal formulas having only finitely many variables. Same holds for TCAs.

**Proof.** We use the algebras $\mathfrak{C}(m, n, r)$ in Theorem 7.6 in the signature of $\text{TCA}_m$, by static temporalization for both algebras, the $\mathfrak{C}(m, m+k, r)$ and its dilation by defining $G = H = \text{Id}$ as the identity function, $T = \{t\}$ and $<$ be the empty set. Using the same notation for the expanded algebras, we still obviously have $\mathfrak{C}(m, m+k, r) \cong \mathfrak{R}_\alpha \mathfrak{C}(m, m+k, r)$ for any $k \in \omega$. Fix $r \in \omega$. Let $I = \{\Gamma : \Gamma \subseteq \alpha, |\Gamma| < \omega\}$. For each $\Gamma \in I$, let $M_\Gamma = \{\Delta \in I : \Gamma \subseteq \Delta\}$, and let $F$ be an ultrafilter on $I$ such that $\forall \Gamma \in I$, $M_\Gamma \in F$. For each $\Gamma \in I$, let $\rho_\Gamma$ be a one to one function from $|\Gamma|$ onto $\Gamma$. Let $\mathcal{C}_\Gamma^r$ be an algebra similar to $\text{TeCA}_\alpha$ such that

$$\mathfrak{R}_\alpha \mathcal{C}_\Gamma^r = \mathfrak{C}(|\Gamma|, |\Gamma| + k, r).$$

Let

$$\mathcal{B}^r = \Pi_{r/\in} \mathcal{C}_\Gamma^r.$$ 

Then it can be proved

1. $\mathcal{B}^r \in \text{SN}_\alpha \text{TCA}_{\alpha+k}$,
2. $\mathfrak{R}_\alpha \mathcal{B} \not\sim \text{SN}_\alpha \text{CA}_{\alpha+k+1}$,
3. $\Pi_{r/\in} \mathcal{B}^r \in \text{RTCA}_\alpha$.

For the first part, for each $\Gamma \in I$ we know that $\mathfrak{C}(|\Gamma| + k, |\Gamma| + k, r) \in \mathfrak{K}_{|\Gamma|+k}$ and $\mathfrak{R}(|\Gamma|) \mathfrak{C}(|\Gamma| + k, |\Gamma| + k, r) \cong \mathfrak{C}(|\Gamma|, |\Gamma| + k, r)$. Let $\sigma_\Gamma$ be a one to one function $(|\Gamma| + k) \rightarrow (\alpha + k)$ such that $\rho_\Gamma \subseteq \sigma_\Gamma$ and $\sigma_\Gamma(|\Gamma| + i) = \alpha + i$ for every $i < k$. Let $\mathcal{A}_\Gamma$ be an algebra similar to a $\mathfrak{CA}_{\alpha+k}$ such that $\mathfrak{R}_\alpha \mathcal{A}_\Gamma = \mathfrak{C}(|\Gamma| + k, |\Gamma| + k, r)$. We claim that $\Pi_{r/\in} \mathcal{A}_\Gamma \in \text{TeCA}_{\alpha+k}$. For this it suffices to prove that each of the defining axioms for $\text{TeCA}_{\alpha+k}$ hold for $\Pi_{r/\in} \mathcal{A}_\Gamma$. Let $\sigma = \tau$ be one of the defining equations for $\text{TeCA}_{\alpha+k}$, and we assume to simplify notation that the number of dimension variables is one. Let $i \in \alpha + k$, we must prove that $\Pi_{r/\in} \mathcal{A}_\Gamma \models \sigma(i) = \tau(i)$. If $i \in \text{rng}(\rho_\Gamma)$, say $i = \rho_\Gamma(i_0)$, then $\mathfrak{R}_\alpha \mathcal{A}_\Gamma \models \sigma(i_0) = \tau(i_0)$, since $\mathfrak{R}_\alpha \mathcal{A}_\Gamma \in \mathfrak{CA}_{|\Gamma|+k}$, so $\mathcal{A}_\Gamma \models \sigma(i) = \tau(i)$. Hence $\{\Gamma \in I : \mathcal{A}_\Gamma \models \sigma(i) = \tau(i)\} \supseteq \{\Gamma \in I : i \in \text{rng}(\rho_\Gamma)\} \in F$, hence $\Pi_{r/\in} \mathcal{A}_\Gamma \models \sigma(i) = \tau(i)$. Thus, as claimed, we have $\Pi_{r/\in} \mathcal{A}_\Gamma \in \mathfrak{CA}_{\alpha+k}$. We prove that $\mathcal{B}^r \subseteq \text{SN}_\alpha \Pi_{r/\in} \mathcal{A}_\Gamma$. Recall that $\mathcal{B}^r = \Pi_{r/\in} \mathcal{C}_\Gamma^r$ and note that $\mathcal{C}_\Gamma^r \subseteq \mathcal{A}_\Gamma$ (the universe of $\mathcal{C}_\Gamma^r$ is $C(|\Gamma|, |\Gamma| + k, r)$, the
universe of $\mathfrak{A}_\Gamma$ is $C(|\Gamma| + k, |\Gamma| + k, r)$). So, for each $\Gamma \in I$,

$$\mathfrak{A}^{\omega^\rho} C^r_\Gamma = \mathfrak{C}((|\Gamma|, |\Gamma| + k, r)$$

$$\cong \mathfrak{N}_{|\Gamma|} \mathfrak{A} \mathfrak{C}((|\Gamma| + k, |\Gamma| + k, r)$$

$$= \mathfrak{N}_{|\Gamma|} \mathfrak{A} \mathfrak{C}(\mathfrak{A}_\Gamma)$$

$$= \mathfrak{A}^{\omega^\rho} \mathfrak{N}_{|\Gamma|} \mathfrak{A}_\Gamma$$

$$= \mathfrak{A}^{\omega^\rho} \mathfrak{A}_\Gamma$$

Thus (using a standard Los argument) we have: $\Pi_{\Gamma/F} C^r_\Gamma \cong \Pi_{\Gamma/F} \mathfrak{A} \mathfrak{C}(\mathfrak{A}_\Gamma) = \mathfrak{N}_{|\Gamma|} \Pi_{\Gamma/F} \mathfrak{A}_\Gamma$, proving (1). The above isomorphism $\cong$ follows from the following reasoning. Let $\mathfrak{B}_\Gamma = \mathfrak{N}_{|\Gamma|} \mathfrak{A}_\Gamma$. Then universe of the $\Pi_{\Gamma/F} C^r_\Gamma$ is identical to that of $\Pi_{\Gamma/F} \mathfrak{A}^{\omega^\rho} C^r_\Gamma$ which is identical to the universe of $\Pi_{\Gamma/F} \mathfrak{B}_\Gamma$. Each operator $\alpha$ of $\mathfrak{A}_\alpha$ is the same for both ultraproducts because $\{\Gamma \in I : \dim(\alpha) \subseteq \mbox{rng}(\rho_{|\Gamma|})\} \in F$.

Now we prove (2). For this assume, seeking a contradiction, that $\mathfrak{N}_{|\Gamma|} \mathfrak{B}^r \in SNr_{|\Gamma|} \mathfrak{A}_{|\Gamma|+1}$, $\mathfrak{B}^r \subseteq \mathfrak{N}_{|\Gamma|} \mathfrak{C}$, where $\mathfrak{C} \in \mathfrak{A}_{|\Gamma|+1}$. Let $3 \leq m < \omega$ and $\lambda : m + k + 1 \rightarrow \alpha + k + 1$ be the function defined by $\lambda(i) = i$ for $i < m$ and $\lambda(m + i) = \alpha + i$ for $i < k + 1$. Then $\mathfrak{N}^{\omega^\lambda} \mathfrak{C} \in \mathfrak{A}_{m+k+1}$ and $\mathfrak{N}_{|\Gamma|} \mathfrak{B}^r \subseteq \mathfrak{N}_{|\Gamma|} \mathfrak{N}^{\omega^\lambda} \mathfrak{C}$). For each $\Gamma \in I$, let $I_{|\Gamma|}$ be an isomorphism

$$\mathfrak{C}(m, m + k, r) \cong \mathfrak{N}_{|\Gamma|} \mathfrak{N}_{\omega^\lambda} \mathfrak{C}(|\Gamma|, |\Gamma| + k, r).$$

Exists by item (2) of lemma 7.10. Let $x = (x_{|\Gamma|} : \Gamma)/F$ and let $i(b) = (I_{|\Gamma|} b : \Gamma)/F$ for $b \in \mathfrak{C}(m, m + k, r)$. Then $i$ is an isomorphism from $\mathfrak{C}(m, m + k, r)$ into $\mathfrak{N}_{|\Gamma|} \mathfrak{B}^r$. Then by [?, theorem 2.6.38] we have $\mathfrak{N}_{|\Gamma|} \mathfrak{B}^r \in SN_{|\Gamma|} \mathfrak{A}_{|\Gamma|+1}$ which is a contradiction and we are done.

Now we prove the third part of the theorem, putting the superscript $r$ to use. The first two items are as before. Now we prove (3) putting the superscript $r$ to use. Recall that $\mathfrak{B}^r = \Pi_{\Gamma/F} C^r_\Gamma$, where $C^r_\Gamma$ has the type of $\mathcal{TCA}_\alpha$ and $\mathfrak{A}^{\omega^\rho} C^r_\Gamma = \mathfrak{C}((|\Gamma|, |\Gamma| + k, r)$. We know from item (1) of lemma 7.10 that $\Pi_{r/U} \mathfrak{A}^{\omega^\rho} C^r_\Gamma = \Pi_{r/U} \mathfrak{C}(|\Gamma|, |\Gamma| + k, r) \subseteq \mathfrak{N}_{\mbox{TeCA}_{|\Gamma|+\omega}}$ for some $\mathfrak{A}_\Gamma \in \mathfrak{TeCA}_{|\Gamma|+\omega}$. Let $\lambda_\Gamma : |\Gamma| + k + 1 \rightarrow \alpha + k + 1$ extend $\rho_{|\Gamma|} : |\Gamma| \rightarrow (\subseteq \alpha)$ and satisfy

$$\lambda_\Gamma((|\Gamma| + i) = \alpha + i$$

for $i < k + 1$. Let $k + 1 \leq l \leq \omega$. Let $\mathfrak{F}_\Gamma$ be a $\mathcal{TCA}_{|\Gamma|+\omega}$ type algebra such that $\mathfrak{N}^{\omega^\lambda} \mathfrak{F}_\Gamma = \mathfrak{N}_{\omega^\lambda} \mathfrak{A}_\Gamma$. As before, $\Pi_{\Gamma/F} \mathfrak{F}_\Gamma \in \mathfrak{TCA}_{|\Gamma|+\omega}$. And

$$\Pi_{r/U} \mathfrak{B}^r = \Pi_{r/U} \Pi_{\Gamma/F} C^r_\Gamma$$

$$= \Pi_{\Gamma/F} \Pi_{r/U} C^r_\Gamma$$

$$\subseteq \Pi_{\Gamma/F} \mathfrak{N}_{|\Gamma|} \mathfrak{A}_\Gamma$$

$$= \Pi_{\Gamma/F} \mathfrak{N}_{|\Gamma|} \mathfrak{A}_\Gamma$$

$$\subseteq \mathfrak{N}_{|\Gamma|} \mathfrak{A}_\Gamma$$

Hence, we get by the neat embedding theorem that $\Pi_{r/U} \mathfrak{B}^r \in SNr_{|\Gamma|} \mathfrak{TCA}_{|\Gamma|+\omega}$ and we are done.

Let $k$ be as before; $k$ is finite and $> 0$ and let $l$ be as in the hypothesis of the theorem, that is, $l \geq k + 1$, and we can assume without loss that $l \leq \omega$. Recall that $\mathfrak{B}^r = \Pi_{\Gamma/F} C^r_\Gamma$, where $C^r_\Gamma$ has the type of $\mathfrak{C}_\alpha$ and $\mathfrak{A}^{\omega^\rho} C^r_\Gamma = \mathfrak{C}((|\Gamma|, |\Gamma| + k, r)$. We know (this is the main novelty here) from item (2) that $\Pi_{r/U} \mathfrak{A}^{\omega^\rho} C^r_\Gamma = \Pi_{r/U} \mathfrak{C}((|\Gamma|, |\Gamma| + k, r) \subseteq \mathfrak{N}_{|\Gamma|} \mathfrak{A}_\Gamma$, for some $\mathfrak{A}_\Gamma \in \mathfrak{C}_{|\Gamma|+\omega}$. If $\rho : \omega \rightarrow \alpha$ is an injection, then $\rho$ extends recursively to a function
\( \rho^+ \) from \( \text{CA}_0 \) terms to \( \text{CA}_\alpha \) terms. On variables \( \rho^+(v_k) = v_k \), and for compound terms like \( c_k \tau \), where \( \tau \) is a \( \text{CA}_0 \) term, and \( k < \omega \), \( \rho^+(c_k \tau) = c_{\rho(k)} \rho^+(\tau) \). For an equation \( e \) of the form \( \sigma = \tau \) in the language of \( \text{CA}_\alpha \), \( \rho^+(e) \) is the equation \( \rho^+(\tau) = \rho^+(\sigma) \) in the language of \( \text{CA}_\alpha \). This last equation, namely, \( \rho^+(e) \) is called an \( \alpha \) instance of \( e \) obtained by applying the injection \( \rho \). Let \( k \geq 1 \) and \( l \geq k + 1 \). Assume for contradiction that \( SN_{\alpha,TCA_{\alpha+l}} \) is axiomatizable by a finite schema over \( SN_{\alpha,TCA_{\alpha+k}} \). We can assume that there is only one equation, such that all its \( \alpha \) instances, axiomatize \( SN_{\alpha,TCA_{\alpha+l}} \) over \( SN_{\alpha,TCA_{\alpha+k}} \). So let \( \sigma \) be such an equation in the signature of \( \text{TC}_{\alpha+1} \), and let \( \mathcal{E} \) be its \( \alpha \) instances; so that for any \( \mathcal{A} \in SN_{\alpha,TCA_{\alpha+k}} \) we have \( \mathcal{A} \in SN_{\alpha,TCA_{\alpha+l}} \iff \mathcal{A} \models \mathcal{E} \). Then for all \( r \in \omega \), there is an instance of \( \sigma, \sigma_r \), say, such that \( \mathcal{B}^r \) does not model \( \sigma_r \). \( \sigma_r \) is obtained from \( \sigma \) by some injective map \( \mu_r : \omega \to \alpha \). For \( r \in \omega \), let \( v_r \in \text{n} \alpha \), be an injection such that \( \mu_r(i) = v_r(i) \) for \( i \in \text{ind}(\sigma_r) \), and let \( \mathcal{A}_r = \text{Rd}^{v_r} \mathcal{B}^r \). Now \( \Pi_{r\in\omega} \mathcal{A}_r \models \sigma \). But then
\[
\{ r \in \omega : \mathcal{A}_r \models \sigma \} = \{ r \in \omega : \mathcal{B}^r \models \sigma_r \} \in U,
\]
contradicting that \( \mathcal{B}^r \) does not model \( \sigma_r \) for all \( r \in \omega \).

We note that all of Andréka’s complexity results and other results showing the resilient robust undecidability of ‘three variable first order logic (and more finitely many variables \( n \) say)’ expressed in classes between \( \text{CA}_3 \) and \( \text{RCA}_3 \) (between \( \text{RCA}_n \) and \( \text{CA}_n \)) proved in [2] and [32, 29] respectively, with possibly no exception lift to the topological contexts by forming discrete topologizing. Let \( 2 < n < \omega \). We approach the modal version of \( L_n \) without equality, namely, \( S5^n \). The corresponding class of modal algebras is the variety \( \text{Rdf}_n \) of diagonal free \( \text{RCA}_n \)'s [22, 29]. Let \( \text{Rd}_{df} \) denote ‘diagonal free reduct’.

**Lemma 7.8.** Let \( 2 < n < \omega \). If \( \mathcal{A} \in \text{CA}_n \) is such that \( \text{Rd}_{df} \mathcal{A} \in \text{Rdf}_n \), and \( \mathcal{A} \) is the smallest subalgebra of itself containing \( J = \{ x \in \mathcal{A} : \Delta x \neq n \} \) and closed under complementation, infinite intersection, cylindrifications and diagonal elements, then \( \mathcal{A} \in \text{RCA}_n \).

**Proof.** Easily follows from [22, Lemma 5.1.50, Theorem 5.1.51]. Assume that \( \mathcal{A} \in \text{CA}_n \), \( \text{Rd}_{df} \mathcal{A} \) is a set algebra (of dimension \( n \)) with base \( U \), and \( R \subseteq U \times U \) as are in the hypothesis of [22, Theorem 5.1.49]. Let \( E = \{ x \in A : (\forall y \in nU)(x,y) \models (x < y \iff (x \in X \iff y \in X)) \} \). Then \( \{ x \in \mathcal{A} : \Delta x \neq n \} \subseteq E \) and \( E \in \text{CA}_n \) is closed under infinite intersections. The required follows.

**Theorem 7.9.** Let \( 2 < n < \omega \).

1. Any universal axiomatization of \( \text{RTCA}_\alpha \) must contain infinitely many variables and infinitely many diagonal constants.

2. It is undecidable whether a finite \( \text{TCA}_n \) is representable or not. In particular, for any such \( n \), the variety of representable algebras in the two cases cannot be axiomatized in \( m \)th order logic for any finite \( m \).

3. The class of topological Kripke frames of the form \( \{ \mathfrak{F} \in \text{AtTCA}_n : \mathfrak{C}m\mathfrak{F} \in \text{RTCA}_n \} \) is not elementary.

4. Any axiomatization of any model logic between \( S5^n \) and modal topological logics with \( n \) variables, viewed as an \( n \)-dimensional multi-modal logic has to contain genuinely
second order formulas on Kripke frames, and furthermore cannot be axiomatized by first order formulas a fortiori Sahlqvist, nor canonical ones.

5. Though canonical, any equational axiomatization of $\text{TRCA}_n$ must contain infinitely many non-canonical equations.

Proof. 1. Let $2 < n < \omega$, and $k \in \omega$. Then in $[2]$ a non-representable $\text{CA}_n$ is constructed (by splitting an atom in a $\text{CS}_n$-a cylindric set algebra of dimension $n$- into $k+1$ parts) such that its $k$ generated subalgebras, that is, subalgebras generated by $\leq k$ many elements, are representable. One expands $\mathfrak{A}$ with the interior identity operators, it remains non-representable of course, but its $k$ generated subalgebras are representable in the expanded signature, the representation of the newly added identity $\text{S4}$ modalities induced by the discrete topology on the base.

2. In $[29]$, it is proved that it is undecidable to tell whether a finite frame is a frame for $\mathcal{L}$, and this gives the non-finite axiomatizability result required as indicated in op.cit, and obviously implies undecidability. The rest follows by transferring the required results holding for $S5^n [10] [29]$ to $\mathcal{L}$ since $S5^n$ is finitely axiomatizable over $\mathcal{L}$, and any axiomatization of $\text{Rdf}_n$ must contain infinitely many non-canonical equations. We prove the second item for $\text{TCA}_n$. The proof uses the main result in $[29]$ using the recursive procedure of static topologizing. That is if the problem is decidable for $\text{TCA}_n$, then this implies its decidability for $\text{CA}_n$ for any finite $n > 2$. Indeed take any finite $\mathfrak{A} \in \text{CA}_n$. By discrete topologizing we obtain $\mathfrak{A}^{\text{top}}$, apply the available algorithm for finite $\text{TCA}_n$s, the answer is the correct one for $\mathfrak{A}$, since $\mathfrak{A}$ is representable $\iff \mathfrak{A}^{\text{top}}$ is. The last part concerning non-finite axiomatizability is proved for both relation and cylindric algebras (having an analogous undecidability result) in $[26] [27] [29]$. The idea is the existence of any such finite axiomatization in $m$th order logic for any positive $m$ gives a decision procedure for telling whether a finite algebra is representable or not.

3. Let $\mathcal{L}$ be the class of square frames for $S5^n$. Then $\mathcal{L}(\mathcal{L}) = S5^n [32]$ p.192]. But the class of frames $\mathfrak{F}$ valid in $\mathcal{L}(\mathcal{L})$ coincides with the class of strongly representable $\text{Df}_n$ atom structures which is not elementary as proved in $[10]$. This gives the required result for $S5^n$. With Lemma 7.8 at our disposal, a slightly different proof can be easily distilled from the construction addressing $\text{CA}_n$s in $[27]$ or $[29]$. We adopt the construction in the former reference, using the Monk-like $\text{CA}_n$s $\mathcal{M}(\Gamma), \Gamma$ a graph, as defined in $[27]$ Top of p.78]. For a graph $\mathfrak{G}$, let $\chi(\mathfrak{G})$ denote it chromatic number. Then it is proved in op.cit that for any graph $\Gamma$, $\mathcal{M}(\Gamma) \in \text{RCA}_n$ $\iff \chi(\Gamma) = \infty$. By Lemma 7.8 and discretely topologizing $\mathcal{M}(\Gamma)$, getting the expansion $\mathcal{M}(\Gamma)^{\text{top}}$, say, we have $\mathcal{M}(\Gamma)^{\text{top}} \in \text{RTCA}_n$ $\iff \mathfrak{M}_{\text{df}}\mathcal{M}(\Gamma) \in \text{Rdf}_n$ $\iff \chi(\Gamma) = \infty$, because $\mathcal{M}(\Gamma)$ is generated by the set $\{x \in \mathcal{M}(\Gamma) : \Delta x \neq n\}$ using infinite unions and $\mathfrak{A} \in \text{RCA}_n$ $\iff \mathfrak{A}^{\text{top}} \in \text{RTCA}_n$. Now we adopt the argument in $[27]$. Using Erdos’ probabilistic graphs $[18]$, for each finite $\kappa$, there is a finite graph $G_\kappa$ with $\chi(G_\kappa) > \kappa$ and with no cycles of length $< \kappa$. Let $\Gamma_\kappa$ be the disjoint union of the $G_l$ for $l > \kappa$. Then $\chi(\Gamma_\kappa) = \infty$, and so $\mathcal{M}(\Gamma_\kappa)^{\text{top}} \in \text{RTCA}_n$.

Now let $\Gamma$ be a non-principal ultraproduct $\Pi_{\mathcal{D}}\Gamma_\kappa$ for the $\Gamma_\kappa$s. For $\kappa < \omega$, let $\sigma_\kappa$ be a first-order sentence of the signature of the graphs stating that there are no cycles of length less than $\kappa$. Then $\Gamma_l \models \sigma_\kappa$ for all $l > \kappa$. By Loś’s Theorem, $\Gamma \models \sigma_\kappa$ for all $\kappa$. So $\Gamma$ has no cycles, and hence by $\chi(\Gamma) \leq 2$. Thus $\mathfrak{M}_{\text{df}}\mathcal{M}(\Gamma)$ is not representable. (Observe that the the term algebra $\text{ImAt}(\mathcal{M}(\Gamma))$ is representable (as a $\text{CA}_n$), because the class of weakly representable atom structures is elementary $[26]$ Theorem 2.84.)

4. The first part follows from that the class of strongly representable $\text{Df}_n$ atom structures is not elementary proved in the previous item and in $[10]$. Since Sahlqvist
formulas have first order correspondents, then $S5^n$ is not Sahlqvist.

5. From the construction in [10] using discrete topologizing. □

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