The Landscape of Matrix Factorization Revisited

Hossein Valavi, Sulin Liu, and Peter J. Ramadge
Department of Electrical Engineering, Princeton University
Princeton, NJ 08544
hvalavi, liu, ramadge @princeton.edu

Abstract

We revisit the landscape of the simple matrix factorization problem. For low-rank matrix factorization, prior work has shown that there exist infinitely many critical points all of which are either global minima or strict saddles. At a strict saddle the minimum eigenvalue of the Hessian is negative. Of interest is whether this minimum eigenvalue is uniformly bounded below zero over all strict saddles. To answer this we consider orbits of critical points under the general linear group. For each orbit we identify a representative point, called a canonical point. If a canonical point is a strict saddle, so is every point on its orbit. We derive an expression for the minimum eigenvalue of the Hessian at each canonical strict saddle and use this to show that the minimum eigenvalue of the Hessian over the set of strict saddles is not uniformly bounded below zero. We also show that a known invariance property of gradient flow ensures the solution of gradient flow only encounters critical points on an invariant manifold \( M_C \) determined by the initial condition. We show that, in contrast to the general situation, the minimum eigenvalue of strict saddles in \( M_0 \) is uniformly bounded below zero. We obtain an expression for this bound in terms of the singular values of the matrix being factorized. This bound depends on the size of the nonzero singular values and on the separation between distinct nonzero singular values of the matrix.

1 Introduction

Factor analysis is a well known problem in machine learning and many methods have been introduced for posing and solving problems of this form. Some of the best known examples include Principal Component Analysis (PCA) [1] [2] [3], Canonical Correlation Analysis (CCA) [4], Independent Component Analysis (ICA) [5], Positive Matrix Factorization (PMF) [6], and Dictionary Learning [7]. All of these problems share the characteristic of being non-convex optimization problems over matrix arguments. While some of the problems can be solved using well established tools (e.g. SVD), others require the application of iterative solution methods such as gradient descent. This motivates developing a better understanding the landscape of these problems. This has gained additional momentum recently because of a connection to representation learning in deep networks.

In the late 1980’s, Baldi and Hornik [8] examined the landscape question in the context of training a one hidden layer low-rank linear neural network (a linear auto-encoder). Their paper provides a characterization of the associated critical points, proves that these are either global minima or (strict) saddle points, and shows that the global minimum value corresponds to the residual of the projection of the training data onto the subspace generated by the first principal vectors of a covariance matrix associated with the training patterns. This connection to PCA had been previously established by Bourlard in [9]. The results in [8] are proved by vectorizing the relevant matrix differentials and seeking small perturbations around a critical point to make the objective smaller.

Recently, there has been a surge of interest in characterizing the global landscape of the objective functions used in these types of problems. Chen et al. [10] study the landscape of the generalized
This paper revisits the landscape of the simple matrix factorization (MF) problem. Prior results indicate that the minimum eigenvalue of the Hessian is not uniformly bounded below zero over all strict saddles, where the objective has a negative curvature, the second contains neighborhoods of all global minima, and the third is the complement of the first two regions. Mohammadi et al. [12] examine the equilibrium points of the best rank-one approximation under the gradient flow. Ge et al. [13] study the landscape of the matrix sensing, matrix completion and robust PCA problems, showing that for these problems all local minima are global minima. Sun et al. [14] study the phase retrieval problems and dictionary recovery. Boumal [16] studies phase synchronization. Much effort has also recently been devoted to studying the related problem of the optimization landscape of deep neural networks under simplified assumptions [17, 18, 19, 20].

Several papers have investigated provable guarantees for the local and global convergence properties of optimization on non-convex problems under gradient flow and gradient descent. A common theme in these papers is the concept of a strict saddle function introduced in [21]. The main required properties are that the functions are twice-differentiable and that the minimum eigenvalue of the Hessian is positive at all local minima and negative at all other critical points. For such functions, with high probability, stochastic gradient descent (SGD) converges to a local minimum in a polynomial number of iterations. Several subsequent papers show that a wide class of non-convex functions are indeed strict saddle functions. For instance, orthogonal tensor decompositions [21], deep linear neural networks [17], deep linear residual neural networks [19], matrix completion [22], generalized phase retrieval problem [14], complete dictionary recovery over the sphere [15], low-rank matrix recovery [23], are all examples of problems that satisfy the strict saddle property. Lee et al. [24] use the stable manifold theorem to show that for a twice continuously differentiable function with the strict saddle property, almost surely, gradient descent with random initialization converges to a local minimum or negative infinity. Jin et al. [25] show that for $l$-smooth and $\rho$-Hessian Lipschitz functions with the strict saddle property, with high probability, perturbed gradient descent (PGD) converges to a local minimum in a poly-logarithmic number of iterations.

Other papers have investigated the implicit constraints imposed by gradient flow in training over-parameterized models such as deep neural networks [26, 27]. Arora et al. [26] considers over-parameterized multi-layer linear neural networks and shows that gradient flow implicitly balances the underlying factors. Du et al. [27] extends this result to fully-connected and convolutional linear sections of multi-layer neural networks.

This paper revisits the landscape of the simple matrix factorization (MF) problem. Prior results indicate that critical points are either global minima or strict saddles [8]. Our analysis is based on orbits of factorizations under the general linear group. After defining these orbits, we construct a representative point (called a canonical point) on each orbit of critical points. This construction exploits a known connection to PCA/SVD. We then obtain an expression for the minimum eigenvalue of the Hessian at each canonical strict saddle. This bound can be moved along a curve in the orbit to show that there is no uniform negative bound on the minimum eigenvalue of the Hessian over the orbit. This implies that the minimum eigenvalue of the Hessian is not uniformly bounded below zero over all strict saddles. We then analyze how an invariance property of gradient flow impacts the strict saddles that can be encountered. This natural invariant depends on the initial condition for gradient flow. We focus on one interesting form of the invariant. When the initial conditions start in this invariant manifold, the flow restricts the symmetry of the problem to the orthogonal group and this allows us to to give a uniform negative bound on the minimum eigenvalue of the Hessian map at all strict saddles on the invariant manifold. Moreover, we provide exact expressions for the minimum eigenvalue of the Hessian at each strict saddle. Our results are applicable to the situations $k < r$ and $k > r$, where $k$ denotes the rank of the factorization.

## 2 Preliminaries

For positive integers $m, n$, let $\mathbb{R}^{m \times n}$ denote the set of $m \times n$ real matrices, $\text{GL}_m \subset \mathbb{R}^{n \times n}$ denote the general linear group of invertible $n \times n$ matrices, $\mathcal{O}_n \subset \text{GL}_n$ denote the group of orthogonal $n \times n$ matrices, and for $k \leq m$, $\text{St}_{m,k}$ denote the subset of $m \times k$ real matrices with orthonormal columns. For $A, B \in \mathbb{R}^{m \times n}$, $A_{:,k}$ (resp. $A_{k,:}$) denotes the $k$-th column (resp. row) of $A$. $\langle A, B \rangle$ denotes the standard inner product of $A$ and $B$, and $\|A\|_F$ denotes the Frobenius norm of $A$. 


Let \( f : \mathbb{R}^{m \times n} \to \mathbb{R} \) be a twice continuously differentiable function. The derivative of \( f \) with respect to \( X \) evaluated at a given point \( X_0 \) is a linear map from \( \mathbb{R}^{m \times n} \) to \( \mathbb{R} \). Its action on \( H \in \mathbb{R}^{m \times n} \), denoted by \( D_X f(X_0)[H] \), satisfies

\[
D_X f(X_0)[H] = \lim_{\alpha \to 0} \frac{f(X_0 + \alpha H) - f(X_0)}{\alpha}.
\]

The gradient of \( f \) at \( X_0 \), denoted \( \nabla_X f(X_0) \), is the unique point in \( \mathbb{R}^{m \times n} \) such that

\[
D_X f(X_0)[H] = \langle \nabla_X f(X_0), H \rangle.
\]

When no confusion is possible we simply write \( Df(X_0) \) and \( \nabla f(X_0) \). The gradient can also be regarded as a function \( X \mapsto \nabla f(X) \). The derivative of this function at \( X_0 \) is a linear map \( \nabla_X^2 f(X_0) : \mathbb{R}^{m \times n} \to \mathbb{R}^{m \times n} \). Its action on \( H \in \mathbb{R}^{m \times n} \) is denoted by \( \nabla_X^2 f(X_0)[H] \). When no confusion is possible we simply write \( \nabla^2 f(X_0) \). It is natural to call the linear map \( \nabla^2 f(X_0) \) the Hessian map. The second derivative of \( f \) is then defined by

\[
D^2_X f(X_0)[H, K] = \langle \nabla_X^2 f(X_0)[K], H \rangle.
\]

We will always evaluate the second derivative with \( K = H \). In this case, we simply write \( D^2 f(X_0)[H] \). The second derivative is then a scalar valued function from \( \mathbb{R}^{m \times n} \) to \( \mathbb{R} \). The linear function \( \nabla^2 f(X_0)[H] \) embedded in the second derivative brings in eigenvalues and eigenvectors associated with the second derivative.

\( X_0 \) is a critical point of \( f \) if \( \nabla J(X_0) = 0 \). The second order Taylor series of \( f \) about a critical point \( X_0 \) in the direction of \( H \) is

\[
f(X_0 + tH) = f(X_0) + \frac{1}{2} t^2 D^2 f(X_0)[H].
\]

Together with eigenvalues of \( \nabla^2 f(X_0) \), this can be used to partially classify critical points. However, this fails if there is nonzero \( H \) with \( D^2 f(X_0)[H] = 0 \). In this event, \( X_0 \) is termed a degenerate critical point. Motivated by the observation that many non-convex optimization problems have degenerate critical points, the concepts of a strict saddle and a strict saddle function have been introduced \[21\]. A strict saddle is a critical point for which \( \nabla^2 f(X_0) \) has a least one negative eigenvalue (this includes local maxima).

### 2.1 Basic Matrix Factorization

In unconstrained matrix factorization, given \( X \in \mathbb{R}^{m \times n} \) with rank(\( X \)) = \( r \), and a factorization dimension \( k \), we seek a solution of

\[
\min_{(W, S) \in \mathcal{X}} J(W, S) = \frac{1}{2} \| X - WS \|_F^2.
\]

Here \( \mathcal{X} \equiv \mathbb{R}^{m \times k} \times \mathbb{R}^{k \times n} \) with inner product \( \langle (G, H), (G', H') \rangle = \langle G, G' \rangle + \langle H, H' \rangle \) and associated norm \( \| (G, H) \|_F^2 = \| G \|_F^2 + \| H \|_F^2 \).

When \( k \leq r \), problem \(3\) has a well known solution derived from any compact SVD \( X = U \Sigma V^T \). Let \( U_k \) (resp. \( V_k \)) consist of the first \( k \) columns of \( U \) (resp. \( V \)) and \( \Sigma_k \) be the top left \( k \times k \) submatrix of \( \Sigma \). Then \( (W, S) = (U_k \sqrt{\Sigma_k}, \sqrt{\Sigma_k} V_k^T) \) is a global minimum of \(3\). However, rather than solving \(3\) using the SVD, we are interested in finding a solution using gradient descent methods. This motivates an interest in the global landscape of \( J \).

We will work extensively with the gradient, Hessian, and second derivative of \( J \) defined in \(3\). Setting \( E \equiv WS - S \), we list the easily obtained equations for these functions below.

\[
\nabla J(W, S) = (EST^T, WT^TE)
\]

\[
\nabla^2 J(W, S)[(G, H)] = (GSS^T + WHS^T + EHT, WTWH + W^T GS + G^T E),
\]

\[
D^2 J(W, S)[(G, H)] = \| GS \|_F^2 + \| WH \|_F^2 + 2 \text{trace}(HT^T GS + H^T G^T E),
\]

A point \((W, S)\) is a critical point of \( J \) if \( \nabla J(W, S) = 0 \). By \(4\) this is equivalent to

\[
EST = (WS - X)S^T = 0 \quad \text{and} \quad WT^E = W^T(WS - X) = 0.
\]
Because $J$ has a continuous symmetry group that leaves its value invariant (see §3), every nonzero critical point of $J$ is degenerate. To see this, let $(W, S) \neq 0$ be a critical point, and $K \in \mathbb{R}^{k \times k}$ be any matrix with $(WK, -KS)$ nonzero. Then using (5) and (7) we see that

$$\nabla^2 J(W, S)[(WK, -KS)] = (-ES^T K, K^T W^T E) = 0.$$  
Thus $(WK, -KS)$ is an eigenvector of $\nabla^2 J(W, S)$ with eigenvalue 0. To address this degeneracy, we employ the concept of a strict saddle [21]. A critical point $(W, S)$ is a strict saddle if $\lambda_{\min}(\nabla^2 J(W, S)) < 0$. The corresponding eigenvector provides an escape direction from the neighborhood of the critical point. Of particular interest is determining a negative upper bound on $\lambda_{\min}(\nabla^2 J(W, S))$ for the strict saddles of $J$.

### 2.1.1 Orbits and Basic Orbit Properties

For $A \in GL_k$, let $L_A : X \rightarrow X$ denote the linear map $L_A : (G, H) \mapsto (GA, A^{-1}H)$. Then for given $(W, S) \in X$ let

$$\Theta(W, S) \triangleq \{ L_A(W, S) : A \in GL_k \} \quad \text{and} \quad O(W, S) \triangleq \{ L_Q(W, S) : Q \in O_k \}. \quad (8)$$

We call $\Theta(W, S)$ the orbit of $(W, S)$ under $GL_k$, and $O(W, S)$ the suborbit of $(W, S)$ under $O_k$. The value $J(W, S)$ is constant on each orbit. Hence if any point on an orbit is a global minimum, all points on the orbit are global minima. Moreover, the gradient $\nabla J(W, S)$ must be orthogonal to $\Theta(W, S)$ at $(W, S)$. To prove this, take the derivative of $(WA, A^{-1}S)$ with respect to $A$, and then set $A = I_k$. This yields the set of tangents to $\Theta(W, S)$ at $(W, S)$:

$$T_{W,S} = \{(WK, -KS) : K \in \mathbb{R}^{k \times k}\}. \quad (9)$$

Orthogonality is then verified by taking the inner product of any element in (9) with (4).

Qualitative properties of point neighborhoods are also “preserved” along the orbit. To see this let $(G, H) \in X$ and $t > 0$. Then for any $A \in GL_k$,

$$J(W + tG, S + tH) = J(WA + t(GA), A^{-1}S + t(A^{-1}H)).$$

So the values of $J$ traced out moving from $(W, S)$ in a line in the direction of $(G, H)$ are the same as those traced out moving from $(WA, A^{-1}S)$ along a line in the direction of $(GA, A^{-1}H)$. The following lemma (proved in §A) transfers these observations to the derivatives of interest.

**Lemma 2.1.** For all $(W, S) \in X$ and $A \in GL_k$:

(a) $\nabla J(L_A(W, S)) = L_{A^{-1}} \nabla J(W, S)$,

(b) $DJ(L_A(W, S))[(G, H)] = D_J(W, S)[L_{A^{-1}}(G, H)]$,

(c) $\nabla^2 J(L_A(W, S))[(G, H)] = L_{A^{-1}}(\nabla^2 J(W, S))[L_{A^{-1}}(G, H)]$,

(d) $D^2 J(L_A(W, S))[(G, H)] = D^2 J(W, S)[L_{A^{-1}}(G, H)]$.

For a critical point $(W, S)$, the inertia of $\nabla^2 J(W, S)$ is the triple $(i_+, i_-, i_0)$, consisting of the number of its positive, negative, and zero eigenvalues, respectively. We have already noted that if any point on an orbit is a global minimum, so are all points on the orbit. The same holds for strict saddles. In addition, the inertia of $\nabla^2 J$ is an invariant of an orbit. The following theorem (proved in §A) lists these and other orbit properties.

**Theorem 2.1.** Let $(W, S) \in X$ and $A \in GL_k$. Then

(a) If $(W, S)$ is a critical point (global minimum, strict saddle), so are all points in $\Theta(W, S)$.

(b) $\nabla^2 J$ has the same inertia at all points in $\Theta(W, S)$.

(c) $\nabla^2 J$ has the same eigenvalues at all points in $O(W, S)$.

(d) For each $A \in GL_k$, $\lambda_{\min}(\nabla^2 J(L_A(W, S))) \leq \frac{\lambda_{\min}(\nabla^2 J(W, S))}{\max\{\lambda_{\max}(AA^T), \lambda_{\min}(AA^T)\}}$.

Note that $\max\{\lambda_{\max}(AA^T), \lambda_{\min}^{-1}(AA^T)\} \geq 1$ with equality when $A \in O_k$. For an orthogonal matrix $A$, $\lambda_{\min}(\nabla^2 J(L_A(W, S))) = \lambda_{\min}(\nabla^2 J(W, S))$. In this case, the bound in (d) is an equality.

In summary, if an orbit contains a critical point, all points in the orbit are critical points; if it contains a strict saddle, all points in the orbit are strict saddles, and if it contains a global minimum, all points
on the orbit are global minima. In addition, the known result that all critical points are either global minima and strict saddles [8], we see that there are three kinds of orbits: orbits of global minima, orbits of strict saddles, and orbits of noncritical points.

3 The Landscape of Matrix Factorization

Our approach is to first construct a representative point, called a canonical point, on each orbit of critical points. We will use each canonical point to reason about all of the points on its orbit. In particular, we examine $\lambda_{\text{min}}(\nabla^2 J(W, S))$ over orbits of strict saddles. As a by-product we recover the known result that every critical point of $J$ is either a global minimum or a strict saddle [8].

The construction below exploits the known connection between matrix factorization and an SVD of the data matrix $X$. We will use an SVD of $X$ as a theoretical tool in our definitions and proofs. However, we do not intend to numerically compute an SVD of $X$. For clarity we assume $m < n$. This is without loss of generality since symmetric arguments apply when $n < m$. Since $m < n$, the $m \times m$ matrix $XX^T$ is of interest. This has $r$ positive eigenvalues and $m - r$ zero eigenvalues.

Denote these by $\sigma_1^2 \geq \sigma_2^2 \geq \cdots \geq \sigma_r^2 > 0$ and $\sigma_{r+1}^2 = \cdots = \sigma_m^2 = 0$. Let $u_1, \ldots, u_m$ denote a set of $m$ corresponding orthonormal eigenvectors with $XX^T u_i = \sigma_i^2 u_i$ for $i \in [1:r]$, and $XX^T u_i = 0$ for $i \in [r+1:m]$. These orthonormal eigenvectors may not be unique (the nonzero eigenvalues may be repeated). Place the eigenvectors in the columns of $U \in \mathbb{R}^{m \times m}$ and form a compatible full SVD $X = U \Sigma V^T = \sum_{i=1}^{m} \sigma_i u_i v_i^T$.

The singular values of $X$ are unique, but in general $U$ and $V$ are not. It does not matter for our purposes which SVD of $X$ is selected as long as it is used consistently.

3.1 Orbit Representation: Canonical Points

Fix a factorization dimension $k$, and for $q \in \{1: \min\{k, m\}\}$, place $q$ selected singular values of $X$, denoted by $\lambda_1^2 \geq \lambda_2^2 \geq \cdots \geq \lambda_q^2$, in decreasing order in a diagonal matrix $\Lambda^2 \in \mathbb{R}^{q \times q}$, and a corresponding set of $q$ left singular vectors of $X$ in the columns of $\tilde{U} \in \mathbb{R}^{m \times q}$. So $XX^T \tilde{U} = \tilde{U} \Lambda^2$.

Then $\tilde{U}^T X = \tilde{U}^T U \Sigma V^T = \Lambda \tilde{V}^T$, where $\tilde{V}$ denotes the submatrix of $V$ in (10) corresponding to the columns of $\tilde{U}$. Let $V_0 = \{v_{r+1}, \ldots, v_n\}$ and $C_0 \in \mathbb{R}^{(n-r) \times (k-q)}$. Now form

$$
(W_c, S_c) = \left( \begin{bmatrix} \tilde{U} & 0_{m \times (k-q)} \end{bmatrix}, \begin{bmatrix} \Lambda \tilde{V}^T \\ C_0 \end{bmatrix} \right). 
$$

Any point of the form (11) will be called a canonical point.

**Theorem 3.1.** For $1 \leq q \leq \min\{k, m\}$, the following hold:

(a) $(W_c, S_c)$ in (11) is a critical point of $J$ with $J(W_c, S_c) = 1/2 \left( \sum_{i=1}^{r} \sigma_i^2 - \sum_{j=1}^{q} \lambda_j^2 \right)$.

(b) If $(W, S)$ is a critical point of $J$ with rank$(W) = q$, then there exists $(W_c, S_c)$ of the form (11) and $A \in \text{GL}_k$ such that $(W, S) = L_A(W_c, S_c)$.

**Proof.** (a) We show that $(W_c, S_c)$ satisfies (7). $(W_c, S_c - X)S_c^T = (\tilde{U} \Lambda \tilde{V}^T - U \Sigma V^T)[\tilde{V}^T, V_0 C_0] = 0$, and $W_c^T(W_c S_c - X) = \left[ \begin{bmatrix} -1 \sum_{i=1}^{q} \sigma_i^2 - \sum_{j=1}^{q} \lambda_j^2 \end{bmatrix} \right]$. In addition, $J(W_c, S_c) = 1/2 \sum_{i=1}^{r} \sigma_i^2 - \sum_{j=1}^{q} \lambda_j^2$.

(b) Step (i). Since rank$(W) = q$, there is a permutation matrix $P \in \text{GL}_k$ such that the first $q$ columns of $WP$ are linearly independent. If $\hat{W}$ denotes the matrix of these first $q$ columns, then

$$
W = \begin{bmatrix} \hat{W} & 0_{m \times (k-q)} \end{bmatrix} = \begin{bmatrix} I_q & F \\ 0_{(k-q) \times q} & I_{(k-q)} \end{bmatrix} P^T,
$$

where $F$ is determined by the last $k - q$ columns of $WP$.

Step (ii). Let $U \Sigma V^T$ be a compact SVD of $\hat{W}$. Noting that $\hat{U} \in \text{St}_{m, q}$ and $\hat{V}, \hat{\Sigma} \in \text{GL}_q$, we have

$$
W = \begin{bmatrix} \hat{U} & 0_{m \times (k-q)} \end{bmatrix} \begin{bmatrix} \hat{\Sigma} \tilde{V}^T \\ 0_{(k-q) \times q} \\ I_{(k-q)} \end{bmatrix} P^T.
$$

5
Let $C$ denote the product of the two rightmost matrices in (13). Since $\hat{\Sigma}, \hat{V} \in \text{GL}_q, \hat{C} \in \text{GL}_k$. Let $(W_1, S_1) = L_{C^{-1}}((W, S)$). By (13), $W_1 = [\hat{U} \ 0_{n \times (k-q)}]$, and by Theorem 2.1 $(W_1, S_1)$ is a critical point. Write $S_1^T = [S_a^T \ S_b^T]$ with $S_a \in \mathbb{R}^{q \times n}$ and $S_b \in \mathbb{R}^{(k-q) \times n}$. Then by (7),

$$(\hat{U}S_a - X) [S_a^T \ S_b^T] = 0 \quad \text{and} \quad \begin{bmatrix} \hat{U}^T & 0_{(k-q) \times m} \end{bmatrix} [\hat{U}S_a - X] = 0.$$ 

The second condition implies $S_a = \hat{U}^T X$. Then the first implies (c-i) $(\hat{U}\hat{U}^T - I)XX^T \hat{U} = 0$ and (c-ii) $(\hat{U}\hat{U}^T - I)XS_b^T = 0$.

Step (iii). Condition (c-i) implies that range of $\hat{U}$ is invariant under $XX^T$, i.e., $XX^T \mathcal{R}(\hat{U}) \subseteq \mathcal{R}(\hat{U})$. Since $\mathcal{R}(\hat{U})$ has dimension $q$, there are $q$ orthonormal eigenvectors of $XX^T$ that form a basis for $\mathcal{R}(\hat{U})$. Let $\hat{U}$ be the matrix with this basis as its columns arranged in decreasing order of the corresponding eigenvalue. Since every column of $\hat{U}$ has a representation in this basis, for some $Q \in \text{GL}_q$, $\hat{U} = \hat{U}Q$. In addition, $I_k = \hat{U}^T \hat{U} = Q^T Q$. Hence $Q \in \mathcal{O}_k$. Now rewrite (13) as

$$W = \begin{bmatrix} \hat{U} & 0_{m \times (k-q)} \end{bmatrix} \begin{bmatrix} Q & 0_{(k-q) \times q} \ 0_{(k-q) \times q} & I_{(k-q)} \end{bmatrix} \begin{bmatrix} \hat{\Sigma} \hat{V}^T & \hat{\Sigma} \hat{V}^T F \end{bmatrix} P^T, \quad (14)$$

Let $D$ denote the matrix in (14) containing $Q$, and let $(W_2, S_2) = L_{(DC)^{-1}}((W, S)$. Then $(W_2, S_2)$ is a critical point with $W_2 = W(DC)^{-1} = [\hat{U} \ 0_{m \times (k-q)}]$, and $S_2 = (DC)S = D[\begin{bmatrix} S_a^T \ S_b^T \end{bmatrix} = [Q^{\hat{U}^T X} S_a^T \ S_b^T = [\hat{\Lambda}^{\hat{U}^T X} S_a^T \ S_b^T]$. By construction, $\hat{\Lambda}^2$ a diagonal matrix with eigenvalues of $XX^T$ arranged in decreasing order down its diagonal, the columns of $\hat{U}$ are corresponding eigenvectors of $XX^T$ with $XX^T \hat{U} = \hat{U} \hat{\Lambda}^2$, and by (c-ii) $(\hat{U}\hat{U}^T - I)XS_b^T = 0$.

Step (iv). Lemma 2.2 shows that $S_b^T = \hat{V} \hat{C} + V_0 C_0$ where $V_0 = [v_{t+1}, \ldots, v_n]$, $\hat{C} \in \mathbb{R}^{q \times (k-q)}$, and $C_0 \in \mathbb{R}^{(n-r) \times (k-q)}$. Hence

$$(W_2, S_2) = \begin{bmatrix} \hat{U} & 0_{m \times (k-q)} \end{bmatrix} \begin{bmatrix} \hat{C}^T \hat{V}^T + C_0^T V_0^T \end{bmatrix}, \quad (15)$$

Step (v). $(W_c, S_c) = \begin{bmatrix} \hat{U} & 0_{m \times (k-q)} \end{bmatrix} \begin{bmatrix} \hat{C}^T \hat{V}^T \end{bmatrix}$ has the canonical form (11). Moreover, if $E = [\begin{bmatrix} I_q \ 0_{(k-q) \times q} \end{bmatrix} \ 0_{(k-q) \times q}] \in \text{GL}_q$ with $E^{-1} = [\begin{bmatrix} I_q \ 0_{(k-q) \times q} \end{bmatrix} \ 0_{(k-q) \times q}]$, then $[\hat{U} \ 0] E = [\hat{U} \ 0]$. Hence $L_E(W_c, S_c) = (W_2, S_2) = L_{(DC)^{-1}}((W, S)$. Thus $(W, S) = L_{(DC)E}(W_c, S_c)$. 

By Theorem 3.1 every critical point with $W$ of positive rank lies on the orbit of a canonical point. This is depicted schematically in Figure 1. Note that there is also a family of critical points of the form $W = 0_{n \times k}$ and $S^T = [C_0^T V_0^T]$ with $C_0 \in \mathbb{R}^{(n-r) \times k}$. These are addressed separately below.

The results and proof of Theorem 3.1 simplify when $q = k$. In this case, there is no need for the term $S_b$ and condition (c-ii). In contrast, when $q \leq \min\{k, m\}, W$ has a nontrivial null space and $S$ can be decomposed into the sum of a term $S_a \in \mathcal{N}(W)^\perp$ and a term $S_o \in \mathcal{N}(W)$. The term $S_o$ is redundant since it does not impact the value of $J$, but we need to account for its possible presence.
3.2 The Critical Points \((0, C_0^T V_0^T)\)

We first examine the family of critical points associated with the origin. These take the form
\[
(W, S) = (0, C_0^T V_0^T),
\]
where \(V_0 = [v_{r+1} \ldots v_n] \in \mathbb{R}^{n \times (n-r)}\) and \(C_0 \in \mathbb{R}^{(n-r) \times k}\). This analysis is a warm-up for the corresponding analysis of the canonical points \((15)\).

Throughout this section \((W, S)\) is a point of the form \((16)\). To verify that \((W, S)\) is a critical point, note that \(WS = 0\), \(W^T E = 0\) and \(EST = VX_0C_0 = 0\). Hence \((7)\) is satisfied. Since \(SS^T = C_0^T C_0\) is symmetric positive semidefinite, there exists \(Z \in \mathbb{R}^k\) and a diagonal matrix \(\Omega \in \mathbb{R}^{k \times k}\) with diagonal \(\omega_1 \geq \cdots \geq \omega_k > 0\), such that \(C_0^T C_0 = Z \Omega Z^T\). We use the columns of \(Z\) to define the following matrices. For \(j \in [1:k]\), let
\[
G_{0,j} = u_j z_j^T, \quad i \in [1:m], \quad H_{j,i}^0 = z_j v_i^T, \quad i \in [1:n].
\]
These matrices will identify the eigenvectors and eigenvalues of the Hessian at \((W, S)\).

**Theorem 3.2.** (a) For \(i \in [1:r]\) and \(j \in [1:k]\), there exist \(\delta_{i,j}, \delta'_{i,j} \in \mathbb{R}\) with \(\delta_{i,j} \delta'_{i,j} = -1\) such that 
\((G_{0,j}, \delta H_{j,i}^0)\) and \((G_{i,j}, \delta' H_{j,i}^0)\) are orthogonal eigenvectors of \(\nabla^2 J(W, S)\) with corresponding eigenvalues 
\[
\rho_{i,j} = \frac{a}{c} - \frac{\sqrt{\sigma_i^2 + \left(\frac{\omega_j}{\sigma_i}\right)^2}}{\sigma_i}, \quad \text{and} \quad \rho'_{i,j} = \frac{a}{c} + \frac{\sqrt{\sigma_i^2 + \left(\frac{\omega_j}{\sigma_i}\right)^2}}{\sigma_i} > 0.
\]
(b) \((G_{0,j}, 0)\) and \((0, H_{j,i}^0)\), \(i \in [r+1:m]\), \(j \in [1:k]\), are orthogonal eigenvectors of \(\nabla^2 J(W, S)\) with eigenvalues \(\omega_j\), and \(0\) respectively.
(c) \((0, H_{j,i}^0)\), \(i \in [m+1:n]\), \(j \in [1:k]\), is an eigenvector of \(\nabla^2 J(W, S)\) with eigenvalue \(0\).
(d) \(\lambda_{\min}(\nabla^2 J(W, S)) = \frac{a}{c} - \sqrt{\sigma_i^2 + \left(\frac{\omega_j}{\sigma_i}\right)^2} < 0\).

**Proof.** From \((5)\) and the above definitions, \(\nabla^2 J(W, S)((G, H)) = (G Z \Omega Z^T - X H^T, -G^T X)\).

(a) \(\nabla^2 J(W, S)((G_{0,j}, \delta H_{j,i}^0)) = (u_i z_j^T Z \Omega Z^T - \delta X v_i z_j^T, -z_j u_i^T X) = (\omega_j - \delta \sigma_i) G_{i,j}, -\sigma_i H_{j,i}^0\).

This has the form \(\rho(G_{i,j}, \delta H_{j,i}^0)\) if and only if
\[
\rho = \omega_j - \delta \sigma_i = -\sigma_i / \delta.
\]
Since \(\sigma_i > 0\), this equivalent to \(\delta^2 - \frac{\omega_j}{\sigma_i} \delta - 1 = 0\), yielding real solutions \(\delta_{i,j} = \frac{\omega_j}{\sigma_i} + \frac{1}{2 \sigma_i} \sqrt{\left(\frac{\omega_j}{\sigma_i}\right)^2 + \frac{4}{\sigma_i^2}}\).

(b) For \(i \in [r+1:m]\), \(\sigma_i = 0\). Hence \(u_i^T X = 0\) and \(X v_i = 0\). Then \(\nabla^2 J(W, S)((G_{0,j}, 0)) = (u_i z_j^T Z \Omega Z^T, -z_j u_i^T X) = \omega_j (G_{0,j}, 0)\), and \(\nabla^2 J(W, S)((0, H_{j,i}^0)) = (0, H_{j,i}^0)\).

(c) For \(i \in [m+1:n]\), \(j \in [1:k]\), \(\nabla^2 J(W, S)((0, H_{j,i}^0)) = (0, H_{j,i}^0)\), and \(\nabla^2 J(W, S)((0, H_{j,i}^0)) = (0, H_{j,i}^0)\).

(d) Each negative eigenvalue in \((17)\) has the form \(\rho = -\sqrt{x^2 + y^2} + y\) with \(y > 0\). Taking the derivative w.r.t. \(y\) yields, \(1 - y / \sqrt{x^2 + y^2} > 0\), i.e., \(\rho\) is monotonically increasing in \(y\). Hence the minimum eigenvalue is attained with \(i = 1\) and \(j = k\).

Theorem 3.3 part (a) confirms (as expected from \((8)\) that the critical points \((16)\) are all strict saddles. The new contribution is the explicit formulas for the eigenvalues of the Hessian and the insight provided by part (d) into how the least eigenvalue changes over this family of critical points. Specifically, \(\lambda_{\min}(\nabla^2 J(W, S)) = -\sigma_i\) when \(\omega_k = 0\), and is monotone increasing as \(\omega_k\) increases, asymptotically to zero as \(\omega_k \rightarrow \infty\). So without a constraint on the size of \(\omega_k\), there is no \(\gamma < 0\) such that \(\lambda_{\min}(\nabla^2 J(W, S)) < \gamma\) over this family of strict saddles.
3.3 The Canonical Points

We now examine the landscape around the canonical points specified by (11). The analysis builds on that given in §2. Complete details are given in Appendix B. Here we summarize and discuss the main results.

We say that a canonical point is maximal if \( \lambda_j = \sigma_j, \ j \in [1:q] \). This holds if and only if the \( q \) columns of \( \bar{U} \) are left singular vectors of \( X \) for a set of its \( q \) largest singular values. If this does not hold, then we say that is \( (W, S) \) not maximal. In this case, there exists a least integer \( p \in [1:q] \) such that \( \lambda_p < \sigma_p \).

**Theorem 3.3.** Let \( 1 \leq q \leq \min\{k, m\} \) and \( (W_c, S_c) \) be a canonical point of the form (11). If \((W_c, S_c)\) is not maximal, let \( p \in [1:q] \) denote the least integer with \( \lambda_p < \sigma_p \).

(a) If \( q = m \), or \( q = k < m \) and \((W_c, S_c)\) is maximal, then \((W_c, S_c)\) is a global minimum of \( J \).

(b) If \( q = k < m \) and \((W_c, S_c)\) is not maximal, then \((W_c, S_c)\) is a strict saddle with

\[
\lambda_{\min}(\nabla^2 J(W_c, S_c)) = -\frac{\sigma^2 - \lambda^2}{\frac{q^2}{2} + \sqrt{\left(\frac{q^2}{2}\right)^2 + \lambda^2}},
\]

(c) If \( q < \min\{k, m\} \), then \((W_c, S_c)\) is a strict saddle with \( \lambda_{\min}(\nabla^2 J(W_c, S_c)) \) given by

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{\omega_{k+q}}{2} - \sqrt{\sigma^2_{q+1} + \left(\frac{\omega_{k+q}}{2}\right)^2}, \\
\min \left\{ \frac{\omega_{k+q}}{2} - \sqrt{\sigma^2_p + \left(\frac{\omega_{k+q}}{2}\right)^2}, \frac{1}{2} \left(\lambda^2_q + 1 - \sqrt{(\lambda^2_q - 1)^2 + 4\sigma^2_p}\right) \right\}
\end{array} \right. \\
(W_c, S_c) \text{ maximal;}
\end{align*}
\]

\( (W_c, S_c) \) maximal; otherwise.

**Proof.** (a) If \( q = m \), then by Theorem 3.1 \( J(W_c, S_c) = \sum_{i=1}^r \sigma_i^2 - \sum_{j=1}^m \lambda_j^2 = -\sum_{r+1}^m \sigma_i^2 = 0 \).

If \( q = k < m \) and \((W_c, S_c)\) be maximal, then \( \{\lambda_j\}_{j=1}^k \) is a set of \( k \) largest singular values of \( X \).

Hence \( J(W_c, S_c) = \sum_{i=1}^m \sigma_i^2 - \sum_{j=1}^k \lambda_j^2 = \sum_{i=k+1}^m \sigma_i^2 \) achieves its lower bound.

(b) A detailed analysis of this situation is given in [B.1] See Corollary [B.1] with \( a = 1 \).

(c) A detailed analysis of this situation is given in [B.2] See equations (39) and (40). \( \square \)

Theorem 3.3 confirms that the canonical points (11) are either global minima or strict saddles. By Theorem 2.1 the same holds for all critical points. This is the main result in [3]. The new contributions are the explicit formulas for the eigenvalues of the Hessian (see Appendix B for full details) and for the minimum eigenvalue of the Hessian at a canonical strict saddle (parts (b) and (c)). The latter expressions give insight into how the least eigenvalue of the Hessian changes over the family of canonical points. For example, if the rank of \( W_c \) is \( \min\{k, m\} \) (no zero columns in \( W_c \)), the minimum eigenvalue of the Hessian depends on the separation of the squared values of two distinct nonzero singular values of \( X, \sigma_p \) and \( \lambda_k \). In the best case, \( \sigma_p \) is large and \( \lambda_k \) is very small and the minimum eigenvalue is approximately \(-\sigma_p\). This minimum value increases as \( \lambda_k \) increases. For typical data, this suggests that the worst case is when \( k \) is large, \( \sigma_p = \sigma_k \) and \( \lambda_k = \sigma_{k+1} \).

In contrast, when the rank of \( W_c \) is \( q < \min\{k, m\} \) (there are zero columns in \( W_c \)), and \( (W_c, S_c) \) is maximal, the minimum eigenvalue depends only the size of a nonzero singular value of \( X \) and a singular value of the matrix \( C_d^T C_0 \). This is given by the first line in (20). Notice the similarity of this equation to the first equation in (17). The same caveats given there, also apply here. When \( (W_c, S_c) \) is not maximal, there are two ways that negative eigenvalues can arise and this results in two expressions competing to provide the least eigenvalue. This is displayed the second line of (20). When \( \omega_{k-q} = 0 \), the first term reduces to \(-\sigma_p\) and this is the minimum of the two terms. As the value \( \omega_{k-q} \) increases so does the first term. Eventually, the second term is the least and this is a constant that does not depend on \( \omega_{k-q} \).

Using Theorem 2.1 we can give a negative upper bound on \( \lambda_{\min}(\nabla^2 J) \) for any strict saddle \((W, S)\).

**Corollary 3.1.** Let \((W_c, S_c)\) be a canonical strict saddle and \( A \in \text{GL}_k \). Then

\[
\lambda_{\min}(\nabla^2 J(L_A(W_c, S_c))) \leq \frac{\lambda_{\min}(\nabla^2 J(W_c, S_c))}{\max\{\lambda_{\max}(AA^T), \lambda_{\min}^{-1}(AA^T)\}} < 0.
\]
Proof. By Theorem 2.1, dividing the negative upper bound on $\lambda_{\min} \nabla^2 J(W_c, S_c)$ in Theorem 3.3 by $\|L_A\|^2$ gives a negative upper bound on $\lambda_{\min}(\nabla^2 J(L_A(W_c, S_c)))$. □

4 Negative Upper Bound on $\lambda_{\min}(\nabla^2 J)$ Over Strict Saddles

For a maximal canonical point $(W_c, S_c)$ with $q < \min\{k, m\}$, $\lambda_{\min}(\nabla^2 J(W_c, S_c))$ given in the first line of (20) depends on a singular value of $C_0$. Since there is no a priori bound on the singular values of $C_0$, there is no uniform negative upper bound for $\lambda_{\min}(\nabla^2 J)$ over all canonical strict saddles. By this we mean that for no $\gamma < 0$ is it the case that $\lambda_{\min}(\nabla^2 J) < \gamma$ for all canonical strict saddles. Even if $C_0 = 0$, there is a second issue. Since $\|A\|^2$ is unbounded over $GL_k$, the bound in Corollary 5.1 can be arbitrarily close to 0. That does not say, however, that a uniform bound does not exist. This section examines this issue.

We first prove the negative result that even when $C_0 = 0$, there is no uniform negative upper bound for $\lambda_{\min}(\nabla^2 J(W, S))$ over all strict saddles.

Theorem 4.1. For given $k < m$, and any canonical strict saddle $(W_c, S_c)$ with $\text{rank}(W_c) = k$, there is no $\gamma < 0$ such that $\lambda_{\min}(\nabla^2 J(W, S)) \leq \gamma$ for all $(W, S) \in \Theta(W_c, S_c)$.

Proof. Consider the curve of strict saddles $\{(W, a, a^{-1} S_c), a > 0\} \in \Theta(W_c, S_c)$. Corollary B.1 gives an expression for $\lambda_{\min}(\nabla^2 J(W, a, a^{-1} S_c))$. The expression indicates that $\lambda_{\min}(\nabla^2 J(W, a, a^{-1} S_c))$ can be made arbitrary close to 0 by making $a$ sufficiently small or sufficiently large. □

4.1 A Uniform Negative Bound for $\lambda_{\min}(\nabla^2 J)$ Over Strict Saddles in $\mathcal{M}_0$

We now restrict attention to an interesting subset of $\mathcal{X}$ and show that there is a uniform negative upper bound for the minimum eigenvalue of the Hessian at all critical points in this subset.

Let $\mathcal{M}_C \triangleq \{(W, S) : WW^T - SS^T = C\}$, where $C \in \mathbb{R}^{k \times k}$ is a symmetric matrix. $\mathcal{M}_C$ is of interest for several reasons. First, the factorization problem (3) permits imbalance between $W$ and $S$ in the sense that $A \in GL_k$ can make $WA$ very large (resp. small) while making $A^{-1} S$ very small (resp. large) without changing the value of the objective. However, if $(W, S) \in \mathcal{M}_C$, the difference between the norms of $W$ and $S$ is bounded: $\|W\|^2_F - \|S\|^2_F = \text{trace}(C)$. In particular, if $C = 0$, $\|W\|^2_F = \|S\|^2_F$. This is referred to as a balance condition [26][27]. Second, the term $C_0^T V_0^T$ in (11) is redundant and we have no a priori bound on its value. We show that for critical points in $\mathcal{M}_0$, $C_0 = 0$. Third, it is known that the $\mathcal{M}_C$ is invariant under gradient flow. An initial value for $(W, S)$ specifies $C$, and the gradient flow o.d.e. $(W_t, S_t) = -\nabla J(W_t, S_t))$ ensures $(W_t, S_t) \in \mathcal{M}_C$ for $t \geq 0$ [26]. Theorem 1]. Lemma[4] gives a self-contained proof of this result.

Motivated by the above, we now focus on critical points in $\mathcal{M}_0$. Each critical point in $\mathcal{M}_0$ must be in the orbit of some canonical point. We show below that the orbit of a canonical point intersects $\mathcal{M}_0$ if and only if $A$ is invertible and $C_0 = 0$.

Theorem 4.2. For a canonical point $(W_c, S_c)$ of the form (11),

(a) There exists $A \in GL_k$ such that $L_A(W_c, S_c) \in \mathcal{M}_0$ if and only if $A$ is invertible and $C_0 = 0$.

(b) If $L_A(W_c, S_c) \in \mathcal{M}_0$, then for each $Q \in \mathcal{O}_k$, $L_{AQ}(W_c, S_c) \in \mathcal{M}_0$.

Proof. Let $(W_c, S_c)$ be a canonical point and $A \in GL_k$. $L_A(W_c, S_c) \in \mathcal{M}_0$ if and only if $A^T W_c^T W_c A = A^{-1} S_c S_c^T A^{-T}$. Since $A \in GL_k$, this is equivalent to $AA^T(W_c^T W_c)AA^T = S_c S_c^T$. (22)

(If) Assume $A \in GL_k$ and $C_0 = 0$. For $A = \begin{bmatrix} \Lambda & 0 \\ 0 & I_{k-q} \end{bmatrix} \in GL_k$, $AA^T(W_c^T W_c)AA^T = \begin{bmatrix} \Lambda^2 & 0 \\ 0 & I_{k-q} \end{bmatrix} = S_c S_c^T$. Hence $L_A(W_c, S_c) \in \mathcal{M}_0$.

(Only If) There exists $A \in GL_k$, with $L_A(W_c, S_c) \in \mathcal{M}_0$. So $A$ satisfies (22). In general, $W_c^T W_c = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}$ and $S_c S_c^T = \begin{bmatrix} \Lambda^2 & 0 \\ 0 & 0 \end{bmatrix} = S_c S_c^T$. Let $R = AA^T > 0$ and write $R = \begin{bmatrix} R_1 & R_3 \\ R_3^T & R_3 \end{bmatrix}$. By (22), $R$ satisfies $R \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} R = \begin{bmatrix} R_1 & R_3 \\ R_3^T & R_3 \end{bmatrix} = \begin{bmatrix} \Lambda^2 & 0 \\ 0 & 0 \end{bmatrix}$. It follows that $R_1^2 = \Lambda^2$, $R_1 R_3 = 0$, and $R_3^2 R_3 = 0$.
Let $A \in \text{GL}_k$ satisfy (22) and $Q \in O_k$, then $(AQ)(AQ)^T = AA^T$. So $AQ$ satisfies (22).

For the rest of this section we only consider canonical points with $C_0 = 0$ and $\Lambda \in \text{GL}_q$. The second condition requires $q \leq \min\{k, r\}$. This allows for factorization with $k \leq r$, and with $k > r$. Under these assumptions we can apply $L_{\Lambda/2}$ to map the canonical point in (11) into $\mathcal{M}_0$. This yields

$$(W_0, S_0) = \left( \left[ \tilde{U} \Lambda^{1/2} 0_{m \times (k-q)} \right], \left[ \Lambda^{1/2} \tilde{V}^T \right] \right).$$

(23)

**Corollary 4.1.** The set of critical points in $\mathcal{M}_0$ is $\{ O(W_0, S_0): (W_0, S_0) has the form (23) \}$. 

**Proof.** This follows from (23) and part (b) of Theorem 4.2.

By Corollary 4.1 and Theorem 3.1 to determine the landscape around a critical point on $\mathcal{M}_0$ we need only examine the landscape around the point $(W_0, S_0)$ in (23). Each such $(W_0, S_0)$ lies in the orbit of a companion canonical point $(W_c, S_c)$ sharing the same $U$ and $\Lambda$.

**Theorem 4.3.** Consider the critical point $(W_0, S_0) \in \mathcal{M}_0$ given by (23). If $(W_0, S_0)$ is not maximal, let $p \in [1:q]$ denote the least integer with $\lambda_p < \sigma_p$.

(a) If $q = \min\{k, r\}$ and $(W_0, S_0)$ is maximal, then it is a global minimum.

(b) If $q = \min\{k, r\}$ and $(W_0, S_0)$ is not maximal, then $\lambda_{\min}(\nabla^2 J(W_0, S_0)) = -(\sigma_p - \lambda_k)$.

(c) If $q < \min\{k, r\}$, then $\lambda_{\min}(\nabla^2 J(W_0, S_0)) = \begin{cases} -\sigma_{q+1} & \text{if } (W_0, S_0) \text{ is maximal;} \\ -\sigma_p & \text{if } (W_0, S_0) \text{ not maximal.} \end{cases}$

**Proof.** Let $(W_0, S_0)$ be in the orbit of the canonical point $(W_c, S_c)$.

(a) Since $(W_0, S_0)$ is maximal so is $(W_c, S_c)$. Then by Theorem 4.2, $(W_c, S_c)$ is a global minimum. Hence $(W_0, S_0)$ is a global minimum.

(b) $q = \min\{k, r\}$ and $(W_0, S_0)$ not maximal implies $q = k < r$. By Lemma 4.4, for each $i \in [1:r]$ and $j \in [1:k]$, with $u_i^T \tilde{U} = 0$ and $\lambda_j < \sigma_i$, $\nabla^2 J(W_c, S_c)$ has a negative eigenvalue. For each such pair $i, j$ we show that $\nabla^2 J(W_0, S_0)$ also has a negative eigenvalue. Let $G_{i,j} = u_i e_j^T$ and $H_{i,j} = e_i u_j^T$. Since $U^T u_i = 0$ and $V^T v_i = 0$, we have $H_{i,j} S_0 = 0$ and $W_0^T G_{i,j} = 0$. Using (6) and $W_0^T W_0 = S_0 S_0^T = \Lambda$ we have $\nabla^2 J(W_0, S_0) (G_{i,j}, H_{i,j}) = (u_i e_j^T \Lambda + (W_0 S_0 - X) v_i e_j^T, A e_i u_j^T + e_i u_j (W_0 S_0 - X)) = (\lambda_j - \sigma_i) (G_{i,j}, H_{i,j})$. Thus $-(\sigma_i - \lambda_j)$ is a negative eigenvalue of $\nabla^2 J(W_0, S_0)$. By Theorem 2.1 part (b), these are the only negative eigenvalues of $\nabla^2 J(W_0, S_0)$. Hence $\lambda_{\min}(\nabla^2 J(W_0, S_0)) = -(\sigma_p - \lambda_k)$.

(c) Let $q < \min\{k, r\}$ and $(W_c, S_c)$ be maximal. Lemma B.8 shows that for $i \in [1:r], j \in [q+1:k]$ with $u_i^T \tilde{U} = 0,(G_{i,j}, H_{i,j})$ is an eigenvector of $\nabla^2 J(W_c, S_c)$ with eigenvalue $-\sigma_i$. Moreover, these are the only negative eigenvalues. The same proof for the same pairs $i, j$, shows that $(G_{i,j}, H_{i,j})$ is also an eigenvector of $\nabla^2 J(W_0, S_0)$ with eigenvalue $-\sigma_i$. By Theorem 2.1 part (b) these are the only negative eigenvalues of $\nabla^2 J(W_0, S_0)$. Hence in this case, $\lambda_{\min}(\nabla^2 J(W_0, S_0)) = -\sigma_{q+1}$.

Now suppose $(W_c, S_c)$ is not maximal. Then $\nabla^2 J(W_c, S_c)$ has two groups of negative eigenvalues and these are the only negative eigenvalues. Group 1: By Lemma 4.4 and Lemma 4.7, for each $i \in [1:r]$ and $j \in [1:q]$, with $u_i^T \tilde{U} = 0$ and $\sigma_i < \lambda_j$, $\nabla^2 J(W_c, S_c)$ has a negative eigenvalue. Group 2: By Lemma 4.4, for each $i \in [1:r]$ and $j \in [q+1:k]$, with $u_i^T \tilde{U} = 0$, $\nabla^2 J(W_c, S_c)$ has a negative eigenvalue. We show that for each pair of indices in each group, $\nabla^2 J(W_0, S_0)$ has a negative eigenvalue. First, by adapting the result in part (b) to the current situation, we see that $-(\sigma_j - \lambda_j)$ is a negative eigenvalue of $\nabla^2 J(W_0, S_0)$ for all pairs of indices $i, j$ in group 1. Second, the first result in part (c) shows that $-\sigma_i$ is a negative eigenvalue of $\nabla^2 J(W_0, S_0)$ for all pairs indices $i, j$ in group 2. By Theorem 2.1 part (b), these are all of negative eigenvalues of $\nabla^2 J(W_0, S_0)$. The least eigenvalue is attained in the second group by selecting $i = p$ and $j = q + 1$. Hence in this case, $\lambda_{\min}(\nabla^2 J(W_0, S_0)) = -\sigma_p$. 

\(\Box\)
We believe that this yields greater clarity and insight. For example, it permits us to determine eigenvalues. By definition, the latter expression allows us to show that the minimal eigenvalue was not uniformly bounded below zero over all strict saddles. There are two reasons for this. First, the matrix $C_0$ that appears, for example, in the family of critical points $(0, C_0, V_0^T)$ can push the minimum eigenvalue of the Hessian towards 0. Second, moving a strict saddle along its orbit under $GL_k$ can also push the minimum eigenvalue of the Hessian towards 0. However, we show that constraining attention to a particular manifold $\mathcal{M}_0$ ensures the least eigenvalue of the Hessian at strict saddles is uniformly bounded below zero. We prove this by characterizing the critical points on $\mathcal{M}_0$ and using this to obtain an explicit expression for $\lambda_{\min}(\nabla^2 J(W, S))$ for each strict saddle $(W, S)$ in $\mathcal{M}_0$.

The manifold $\mathcal{M}_0$ is special in that points on the manifold satisfy the so-called balance condition $\|W\|^2_2 = \|S\|^2_2$. Moreover, it is known that $\mathcal{M}_0$ is invariant under gradient flow \cite{26,27}. However, despite its special characteristics, continuity ensures that our results above degrade gracefully as one deviates from this particular manifold.

Finally, our development has used the natural setting of the problem, made no assumptions of symmetry or artificially created symmetry, and has avoided vectorization of the relevant differentials. We believe that this yields greater clarity and insight. For example, it permits us determine eigenvectors with a simple interpretable structure at every canonical critical point. We also posit that this approach is more amenable to generalization to related problems.

A Auxiliary Lemmas and Proofs

Proof of Lemma 2.1 (a) $\nabla J(L_A(W, S)) = (E S^T A^{-T}, A^T W T E) = L_A^{-T} \nabla J(W, S)$.
(b) $D^2 J(L_A(W, S))[(G, H)] = \langle L_A^{-T}(\nabla J(W, S)), (G, H) \rangle = \langle (\nabla J(W, S)), L_A^{-1}(G, H) \rangle$.
(c) $\nabla^2 J(L_A(W, S))[(G, J)] = (G A^{-1}) S^T A^{-T} + W (AH) S^T A^{-T} + E H^T, A^T W T W (AH) + A^T W T (G A^{-1}) S + G T E) = L_A^{-T} (\nabla^2 J(W, S)[L_A^{-1}(G, H)])$.
(d) By definition, $D^2 J(L_A(W, S))[(G, H)] = \langle \nabla^2 (L_A(W, S)), (G, H) \rangle$. Using part (c), the RHS of the previous equation can be written as $\langle L_A^{-T} (\nabla^2 (W, S)[L_A^{-1}(G, H)]), (G, H) \rangle = \langle \nabla^2 (W, S)[L_A^{-1}(G, H)], L_A^{-1}(G, H) \rangle = D^2 J(W, S)[L_A^{-1}(G, H)]$. □

Proof of Theorem 2.1 (a) (i) If $\nabla J(W, S) = 0$, then by part (a) of Lemma 2.1 and the linearity of $L_A^{-T}$, $\nabla J(L_A(W, S)) = 0$. (ii) Let $\nabla^2 J(W, S) = \lambda (G, H)$ with $\lambda < 0$. Then $D^2 J(W, S) = \langle \nabla^2 (W, S), (G, H) \rangle = \lambda (G, H) < 0$. By part (d) of Lemma 2.1 $D^2 J(L_A(W, S)) = D^2 J(W, S) < 0$. Thus $\nabla^2 J(L_A(W, S))$ also has a negative eigenvalue.
(b) Fix an orthonormal basis \( \{(G_i, H_i)\}_{i=1}^{k(m+n)} \) for \( \mathcal{X} \). With respect to this basis, each \((G, H) \in \mathcal{X}\) has a unique coordinate vector \( g = \phi(G, H) \in \mathbb{R}^{k(m+n)} \), and each linear map \( L_A : \mathcal{X} \to \mathcal{X} \), with \( A \in \text{GL}_k \), has a unique matrix representation \( M(L_A) \in \mathbb{R}^{k(m+n)} \). We then have \( g_A = M(L_A)g \) where \( g_A = \phi(L_A(G, H)) \). Inner products are preserved using coordinates: \( \langle (G, H), (G', H') \rangle = \langle (G', H'), (G, H) \rangle \). We then have

\[
\begin{align*}
\langle L_A^T (G, H), (G', H') \rangle &= \langle M(L_A^T) g, g' \rangle = g^T M(L_A^T) g', \\
\langle (G, H), L_A(G', H') \rangle &= \langle g, M(L_A) g' \rangle = g^T M(L_A) g'.
\end{align*}
\]

These expressions must be equal for all \( g, g' \). Thus \( M(L_A)^T = M(L_A) \). Since \( \nabla^2 J(W, S) \) is a linear map on \( \mathcal{X} \), it has a matrix \( P \) in the given basis. Similarly, \( \nabla^2 J(L_A(W, S)) \) has a matrix \( Q \). Since \( P \) and \( Q \) represent Hessian maps, both are symmetric matrices. By part (c) of Lemma 2.1, \( \nabla^2 J(L_A(W, S))[((G, H)], L_A^{-1}((G, H))] \) = \( L_A^{-1}(\nabla^2 J(W, S)[L_A^{-1}(G, H]]) \). Letting \( g = \phi(G, H) \), and writing this equation using coordinates yields \( Qg = M(L_{A^{-1}})^T PM(A^{-1})g \). Since this must hold for all \( g \), we conclude that \( Q = M(L_{A^{-1}})^T PM(L_{A^{-1}}) \). Thus the matrices \( P \) and \( Q \) are congruent. The result then follows by Sylvester’s theorem of inertia [28, Theorem 4.5.8].

(c) Let \((G, H)\) be an eigenvector of \( \nabla^2 J(W, S) \) with eigenvalue \( \lambda \). By Lemma 2.1 part (c),

\[
\nabla^2 J(L_A(W, S))[L_A(G, H)] = \lambda L_A^{-T}((G, H)) = \lambda(GA^{-T}, A^T H).
\]

If \( A \in \mathcal{O}_k \), then \( A^{-1} = A^T \). Hence \( \nabla^2 J(W, S)[L_A(G, H)] = \lambda L_A(G, H) \). Thus \( \lambda \) is also an eigenvalue of \( \nabla^2 J(W, A^{-1} S) \). A symmetric argument proves the converse result.

(d) Let \( \lambda = \lambda_{\text{min}}(\nabla^2 J(W, S)) \) have eigenvector \((G, H)\). By Lemma 2.1 part (c),

\[
\begin{align*}
D^2 J(L_A(W, S))[L_A(G, H)] &= \lambda \langle (GA^{-T}, A^T H), (GA, A^{-1} H) \rangle = \lambda \text{trace}(G^T G + HH^T).
\end{align*}
\]

Dividing the above equation by the squared norm of \((GA, A^{-1} H)\) yields

\[
\lambda_{\text{min}}(\nabla^2 J(L_A(W, S))) \leq \frac{\lambda \text{trace}(G^T G + HH^T)}{\text{trace}(A^T G^T A + A^{-1} H H^T A^{-T})} \leq \lambda/\|L_A\|^2,
\]

where \( \|L_A\| \) denotes the induced norm of \( L_A \). The result then follows by Lemma A.1.

**Lemma A.1.** \( \|L_A\| = \max\{\lambda^{1/2}_{\text{max}}(AA^T), \lambda^{-1/2}_{\text{min}}(AA^T)\} \).

**Proof.** \( \|L_A\| = \max_{\|G\|_F = 1} \|G A^{-1} H\|_F \). Let \( u \) (resp. \( v \)) be a unit norm eigenvector of \( AA^T \) (resp. \( (AA^T)^{-1} \)) with eigenvalue \( \lambda_{\text{max}} \triangleq \lambda_{\text{max}}(AA^T) \) (resp. \( \mu_{\text{max}} \triangleq \lambda_{\text{min}}^{-1}(AA^T) \)). Let \((G, H)\) satisfy \( \|G, H\|_F = 1 \) and maximize

\[
\|G, H\|_F^2 = \text{trace}(GAA^T G^T) + \text{trace}(H^T (AA^{-1})^{-1} H).
\]

If a nonzero row of \( G \) is replaced by a scaled version of \( u^T \) with the same norm, the constraint remains satisfied and the objective can increase. The same holds if a nonzero column of \( H \) is replaced by a suitably scaled version of \( v \). Hence there is an optimal \((G, H)\) of the form \( G = \sum \alpha_i e_i u^T \) and \( H = \sum \beta_j e_j v^T \), with \( \sum \alpha_i^2 = 1 \) and \( \sum \beta_j^2 = 1 \), with optimal value \( (\sum \alpha_i^2) \lambda_{\text{max}} + (\sum \beta_j^2) \mu_{\text{max}} \). This value is achieved by \((G^*, H^*) = (\alpha e u^T, \beta e v^T)\) with \( \alpha^2 + \beta^2 = 1 \). Thus the optimal value of (24) is \( \min_{\alpha^2 + \beta^2 = 1} \alpha^2 \lambda_{\text{max}} + \beta^2 \mu_{\text{max}} = \max \{\lambda_{\text{max}}, \mu_{\text{max}}\} \). So \( \|L_A\| = \max \{\sqrt{\lambda_{\text{max}}}, \sqrt{\mu_{\text{max}}}\} \).

The following lemma concerns \( S_0 \in \mathbb{R}^{(k-q) \times n} \) introduced in the proof of Theorem 3.1.

**Lemma A.2.** Let the columns of \( \bar{U} \) be a set of \( q \) left singular vectors of \( X \), the columns of \( \bar{V} \) be a matching set of right singular vectors of \( X \). Let \( \Lambda \) be the diagonal matrix with the singular values corresponding to \( \bar{U} \) on the diagonal, and \( V_0 \triangleq [v_{r+1} \ldots v_n] \). Then

(a) \( \Sigma^T V_0 = 0 \) and \( \Lambda \bar{V}^T V_0 = 0 \).

(b) \( (\bar{U} \bar{U}^T - I) X S_0^T = 0 \equiv S_0^T = \bar{V} \bar{C} + V_0 C_0 \) for some \( \bar{C} \in \mathbb{R}^{q \times (k-q)} \) and \( C_0 \in \mathbb{R}^{(n-r) \times (k-q)} \).
B Hessians

Hessians and Eigenvectors

Let \( \{ z_j \}_{j=1}^k \) denote an orthonormal basis in \( \mathbb{R}^k \), \( G_{i,j} = u_i z_j^T \), \( i \in [1:m] \), and \( H_{j,i} = z_j u_i^T \), \( i \in [1:n] \). Then let \( \mathcal{V} = \{(G_{i,j}, 0) : j \in [1:k], i \in [1:m]\} \cup \{(0, H_{j,i}) : j \in [1:k], i \in [1:n]\} \).

Lemma B.1. \( \mathcal{V} \) is a set of \( k(m+n) \) orthonormal vectors in \( \mathcal{X} \).

Proof. It is clear that \( \mathcal{V} \) contains \( k(m+n) \) vectors and that every vector in the first subset is orthogonal to every vector in the second subset. Each of the two subsets forming \( \mathcal{V} \) is orthonormal. For example, \( \langle (G_{i,j}, 0), (G_{i,j'}, 0) \rangle = \text{trace}(z_j^T z_j u_i^T u_i) = 1 \) if \( i = i', j = j' \), and is 0 otherwise. A similar equation proves the same holds for the second subset.

Lemma B.2. Let \( \alpha, \alpha' \in \mathbb{R} \) with \( \alpha \alpha' = -1 \). Then any pair of vectors \((G_{i,j}, 0)\) and \((0, H_{s,t})\) in \( \mathcal{V} \) can be replaced by the pair of vectors \((G_{i,j}, \alpha H_{s,t})\) and \((G_{i,j}, \alpha' H_{s,t})\) without changing the orthogonality of the elements in the modified set \( \mathcal{V}' \).

Proof. The new vectors are orthogonal: \( \langle (G_{i,j}, \alpha H_{s,t}), (G_{i,j}, \alpha' H_{s,t}) \rangle = \text{trace}(z_j^T z_j u_i^T u_i) + \alpha \alpha' \text{trace}(u_i^T z_j u_i^T z_j) = 0 \). So span \((G_{i,j}, \alpha H_{s,t}), (G_{i,j}, \alpha' H_{s,t})\) = span \((G_{i,j}, 0), (0, H_{s,t})\). Hence \((G_{i,j}, \alpha H_{s,t}), (G_{i,j}, \alpha' H_{s,t})\) are orthogonal to all other elements of \( \mathcal{V} \).

B.1 A Full Rank Canonical Point

We now determine the eigenvalues and eigenvectors of the Hessian at a full rank canonical point \((W_c, S_c)\). Here \( W_c \in \mathbb{R}^{m \times k} \) has rank \( k \). Hence \( k < m \). It will be convenient to derive a slightly more general result by considering the curve \( \{ (W, a, a^{-1} S_c) : a \in \mathbb{R}, a \neq 0 \} \subset \Theta(W_c, S_c) \). Each point \((W, S)\) on this curve is a critical point. The following three lemmas obtain expressions for the \( k(m+n) \) eigenvalues and corresponding orthogonal eigenvectors of \( \nabla^2 J(W, S) \). To simplify the exposition we will assume \( r \leq m \leq n \). Symmetric arguments cover the case \( n < m \). For \( j \in [1:k] \), we set \( G_{i,j} = u_i e_j^T \), \( i \in [1:m] \), and \( H_{j,i} = e_j v_i^T \), \( i \in [1:n] \).

Lemma B.3. For \( i \in [m+1:n] \) and \( j \in [1:k] \), \((0, H_{j,i})\) is an eigenvector of \( \nabla^2 J(a W_c, a^{-1} S_c) \) with eigenvalue \( \rho_0 = a^2 \).

Proof. \[ \nabla^2 J(a W_c, a^{-1} S_c)[(0, H_{j,i})] = (W_c H_{j,i} S_c^T + (W_c S_c - X)(H_{j,i})^T, a^2 W_c^T W_c H_{j,i}) = (W_c e_j v_i^T \tilde{V} A + (W_c S_c - X) e_j e_j^T, a^2 e_j v_i^T) = a^2 (0, H_{j,i}) \].

Now consider vectors constructed from \( G_{i,j} \) and \( H_{j,i} \) for \( i \in [1:m] \) with \( u_i \) not a column of \( \tilde{U} \). The following lemma separates this into two parts; first \( i \in [1:r] \), then \( i \in [r+1:m] \).

Lemma B.4. For \( j \in [1:k] \), and \( i \in [1:m] \) with \( u_i \) not a column in \( \tilde{U} \), the following hold: (a) If \( i \in [1:r] \), there exist \( \alpha_{i,j}, \alpha'_{i,j} \in \mathbb{R} \) with \( \alpha_{i,j} \alpha'_{i,j} = -1 \) such that \((G_{i,j}, \alpha_{i,j} H_{j,i})\) and \((G_{i,j}, \alpha'_{i,j} H_{j,i})\) are eigenvectors of \( \nabla^2 J(W_c, a^{-1} S_c) \) with corresponding eigenvalues

\[
\rho_{i,j} = \frac{1}{2} \left[ \frac{\lambda_i^2 + a^4}{a^2} - \sqrt{\left( \frac{\lambda_i^2 - a^4}{a^2} \right)^2 + 4a^2} \right] \begin{cases} 
> 0, & \text{if } \lambda_j > \sigma_i \\
geq 0, & \text{if } \lambda_j = \sigma_i \\
< 0, & \text{if } \lambda_j < \sigma_i.
\end{cases}
\]
\[
\rho'_{ij} = \frac{1}{2} \left( \frac{\lambda_i^2 + a^4}{a^2} + \sqrt{\left( \frac{\lambda_i^2 - a^4}{a^2} \right)^2 + 4\sigma_i^2} \right) > 0. \tag{26}
\]

(b) Alternatively, if \( i \in [r+1:m] \), the pair of vectors \((G_{i,j}, 0)\) and \((0, H_{j,i})\), are eigenvectors of \(\nabla^2 J(aW_c, a^{-1}S_c)\) with eigenvalues \(\rho_j = \lambda_j^2/a^2 \geq 0\), and \(\rho_0 = a^2 > 0\), respectively.

**Proof.** (a) Under the stated assumptions, \(\sigma_i > 0\, u_i^T \bar{U} = 0\) and \(v_i^T \bar{V} = 0\). Let \((G, H) = (u_i e_j^T, \alpha e_j e_i^T)\) with \(\alpha \in \mathbb{R}\). For \((W, S) = (W_c, a^{-1}S_c)\) we seek \(\alpha, \rho \in \mathbb{R}\) such that \((G, H)\) is an eigenvector of \(\nabla^2 J\) with eigenvalue \(\rho\). Using \((\ref{eq:diff_eigenvalue})\), this is equivalent to

\[
G S S^T + W H S T^T + (W S - X) H T = \rho G, \quad W T W H + W T G S + G T (W S - X) = \rho H. \tag{27}
\]

In the present context these equations become \(u_i e_j^T (a^{-2} \lambda^2) - \sigma_i^2 v_i e_j^T = (a^{-2} \lambda^2 - \alpha \sigma_i) u_i e_j^T = \rho u_i e_j^T\), and \(a \sigma_i^2 v_i e_j^T - \sigma_j^2 u_i e_j^T = \alpha (a^2 - \sigma_i) e_j e_i^T = \rho e_j e_i^T\). Solving for \(\rho\) we obtain

\[
\rho = a^{-2} \lambda_j^2 - \alpha \sigma_i \quad \text{and} \quad \rho = a^2 - \sigma_i/\alpha.
\]

Thus \(\nabla^2 J(W_c, a^{-1}S_c) ([G, H]) = \rho \, (G, H)\) if and only if \(\alpha\) is a real root of the equation

\[
\sigma_i^2 a^2 - \lambda_j^2 a^4 = \alpha - \sigma_i = 0. \tag{29}
\]

This equation has two real roots \(\alpha^+\) and \(\alpha^-\) with \(\alpha^+ = \frac{1}{2\sigma_i} \left( \frac{\lambda_j^2 - a^4}{\lambda_j^2 - a^4} \pm \sqrt{\left( \frac{\lambda_j^2 - a^4}{\lambda_j^2 - a^4} \right)^2 + 4\sigma_i^2} \right)\). Using the first equation for \(\rho\) above and this result we find

\[
\rho(\alpha^+) = \frac{1}{2} \left( \frac{\lambda_j^2 + a^4}{a^2} + \sqrt{\left( \frac{\lambda_j^2 - a^4}{a^2} \right)^2 + 4\sigma_i^2} \right), \quad \text{eigenvector:} \quad (u_i e_j, \alpha^+ e_j e_i^T),
\]

\[
\rho(\alpha^-) = \frac{1}{2} \left( \frac{\lambda_j^2 + a^4}{a^2} + \sqrt{\left( \frac{\lambda_j^2 - a^4}{a^2} \right)^2 + 4\sigma_i^2} \right), \quad \text{eigenvector:} \quad (u_i e_j, \alpha^- e_j e_i^T). \tag{31}
\]

It is readily checked from \((\ref{eq:diff_eigenvalues})\) that \(\alpha^+\alpha^- = -1\). Hence these eigenvectors have the form \((G_{i,j}, \alpha_{i,j} H_{i,j})\) and \((G_{i,j}, \alpha_{i,j} H_{i,j})\) with \(\alpha_{i,j} = \alpha^+, \alpha_{i,j} = \alpha^-\) and \(\alpha_{i,j} \alpha_{i,j} = -1\). Let

\[
c = \frac{\lambda_j^2 + a^4}{a^2}, \quad \text{and} \quad d = \sqrt{\left( \frac{\lambda_j^2 - a^4}{a^2} \right)^2 + 4\sigma_i^2}.
\]

Then \(c, d > 0, \rho(\alpha^+) = \frac{1}{2}(c - d)\), and \(\rho(\alpha^-) = \frac{1}{2}(c + d)\). Clearly \(\rho(\alpha^-) > 0\). Simple algebra verifies that \(c^2 - d^2 = 4(\sigma_i^2 - \lambda_j^2)\). This gives the sign classifications of \(\rho_{i,j}\) in \((\ref{eq:egenvectors})\).

(b) In this case, \(\sigma_i = 0\). Hence \(u_i^T X = 0\) and \(X v_i = 0\). In addition, \(u_i^T \bar{U} = 0\) and \(v_i^T \bar{V} = 0\). Hence \(u_i^T W S = u_i^T \bar{U} A \bar{V}^T = 0\). Thus \(\nabla^2 J(aW_c, a^{-1}S_c) ([G_{i,j}, 0]) = (u_i e_j^T a^{-2} \lambda^2, 0) = a^{-2} \lambda_j^2 (G_{i,j}, 0), \) and \(\nabla^2 J(aW_c, a^{-1}S_c) ([0, H_{j,i}]) = (0, a^2 e_j e_i^T) = a^2 (0, H_{j,i}).\)

Lemma \([\ref{lem:diff_eigenvalues}]\) and Lemma \([\ref{lem:diff_eigenvalues2}]\) have identified \((m + n)k - 2k^2\) eigenvectors. The remaining \(2k^2\) eigenvectors are found by considering indices for which \(u_i\) is a column of \(\bar{U}\).

For each column \(\bar{u}_j\) of \(\bar{U}, j \in [1:k]\), there exists \(i \in [1:m]\) such that \(\bar{u}_j = u_i, \bar{v}_j = v_i, \) and \(\lambda_j = \sigma_i\). If \(i \in [1:r]\), then \(\lambda_j = \sigma_i > 0\); otherwise \(i \in [r+1:m]\) and \(\lambda_j = \sigma_i = 0\). We can partition the index set \([1:k]\) accordingly into \(\mathcal{S}\), with \(s \in \mathcal{S}\) if \(\lambda_s > 0\), and \(\mathcal{T}\), with \(t \in \mathcal{T}\) if \(\lambda_t = 0\). We assume \(\mathcal{S}\) and \(\mathcal{T}\) are nonempty. But the case when one of \(\mathcal{S}\) or \(\mathcal{T}\) is empty, is also covered by the result below.

**Lemma B.5.** (a) For each \( j \in [1:k] \) and \( s \in \mathcal{S} \), there exist \(\beta_{j,s}, \beta'_{j,s} \in \mathbb{R}\) with \(\beta_{j,s} \beta'_{j,s} = -1\), such that \((\bar{u}_j e_s^T, \beta_{j,s} e_s \bar{v}_s^T)\) and \((\bar{u}_j e_s^T, \beta'_{j,s} e_s \bar{v}_s^T)\) are eigenvectors of \(\nabla^2 J(W_c, a^{-1}S_c)\) with corresponding eigenvalues \(\rho_{j,s} = 0\) and \(\rho'_{j,s} = \lambda_j^2/a^2 + a^2 > 0\), respectively.

(b) For each \( j \in [1:k] \) and \( t \in \mathcal{T} \), \((\bar{u}_j e_s, 0)\), and \((0, e_j \bar{v}_t^T)\), are eigenvectors of \(\nabla^2 J(aW_c, a^{-1}S_c)\) with eigenvalues \(0\) and \(\rho_0 = a^2 > 0\), respectively.
Proof. Recall that $W_c = \bar{U}, S_c = \Lambda \bar{V}^T, W_c^T W_c = I \kappa$, and $S_c S_c^T = \Lambda^2$. In addition, $\bar{u}_j^T X = \lambda_j \bar{v}_j$, $X \bar{v}_j = \lambda_j u_j$, $\bar{u}_j^T (W_c S_c - X) = 0$, and $(W_c S_c - X) \bar{v}_j = 0, j \in [1:k]$. 

(a) Consider $(G,H) = (\bar{u}_j e_j^T, \beta e_j \bar{v}_j^T)$, for $j \in [1:k], s \in \mathcal{S}$, and $\beta \neq 0$. By (5), $(G,H)$ is an eigenvector of $\nabla^2 J(W_c a, a^{-1} S_c)$ with eigenvalue $\rho$ if and only if 

$$\bar{u}_j e_j^T (a^2 \Lambda^2) + \beta \bar{U} e_j \bar{v}_j^T \bar{V} \Lambda + (\bar{U} \Lambda \bar{V}^T - X) \bar{v}_j e_j^T = (a^{-2} \lambda_j^2 + \beta \lambda_j) \bar{u}_j e_j^T = \rho \bar{u}_j e_j^T, \beta a e_j \bar{v}_j^T + \bar{U}^T \bar{u}_j e_j^T \Lambda \bar{V}^T + e_j \bar{u}_j^T (\bar{U} \Lambda \bar{V} - X) = (\beta a^2 + \lambda_j) e_j \bar{v}_j^T = \rho e_j \bar{v}_j^T.$$ 

Solving these equations for $\rho$ gives $\rho = a^{-2} \lambda_j^2 + \beta \lambda_j = a^2 + \lambda_j / \beta$. These are the equations in (28) except that $\sigma_i$ has been replaced by $-\lambda_j$ and $\alpha$ by $\beta$. After these adjustments to (29) we find two real roots $\beta^+$ and $\beta^-$ with, 

$$\beta^\pm = -\frac{1}{2\lambda_j} \left( \frac{\lambda_j^2 + \alpha^2}{a^2} \pm \sqrt{ \left( \frac{\lambda_j^2 - \alpha^2}{a^2} \right)^2 + 4 \lambda_j^2} \right).$$ (33) 

Note that $s \in \mathcal{S}$ and hence $\lambda_j > 0$. Let $d'$ denote the square root term in (33). Then $d' = (\lambda_j^2 - 2a^2 \lambda_j^2 + 4 \lambda_j^2 a^4)^{1/2} = \lambda_j^2 + a^2$. Hence $\beta^+ = -\lambda_j / a^2$, and $\beta^- = a^2 / \lambda_j$. Note that $\beta^+ \beta^- = -1$. The expressions for $\beta^\pm$ yield the following eigenvalues and corresponding eigenvectors 

$$\rho(\beta^+) = 0,$$ 

$$\rho(\beta^-) = \lambda_j^2 / a^2 + a^2 > 0,$$ 

with eigenvector: $(\bar{u}_j e_j^T, -\lambda_j / a^2 e_j \bar{v}_j^T)$; 

$$\rho(\beta^-) = \lambda_j^2 / a^2 + a^2 > 0,$$ 

with eigenvector: $(\bar{u}_j e_j^T, -\lambda_j / a^2 e_j \bar{v}_j^T)$.

(b) For $(\bar{u}_j e_j, 0)$, and $(0, e_j \bar{v}_j^T), j \in [1:k], t \in \mathcal{T}$, we have $\nabla^2 J(W_c a, a^{-1} S_c)((U, e_j, 0)) = (\bar{u}_j e_j^T (a^{-2} \Lambda^2), \bar{U}^T \bar{u}_j e_j^T \Lambda \bar{V}^T) = (\bar{u}_j e_j^T (a^{-2} \Lambda^2), e_j \lambda_j \bar{v}_j^T)$. Noting that $\lambda_i = 0$, this simplifies to $(0, e_j \bar{v}_j^T, 0)$. Similarly, $\nabla^2 J(W_c a, a^{-1} S_c)((0, e_j \bar{v}_j^T)) = (\bar{U} e_j \bar{v}_j^T \bar{V} \Lambda, a^2 e_j \bar{v}_j^T) = (\bar{u}_j e_j^T \lambda_i, a^2 e_j \bar{v}_j^T) = a^2 (0, e_j \bar{v}_j^T)$. Thus $\{(\bar{u}_j e_j, 0); j \in [1:k], t \in \mathcal{T}\}$ is a set of $k|\mathcal{T}|$ eigenvectors of $\nabla^2 J(W_c a, a^{-1} S_c)$ with eigenvalue 0, and $\{(0, e_j \bar{v}_j^T); j \in [1:k], t \in \mathcal{T}\}$ is a set of $k|\mathcal{T}|$ eigenvectors with eigenvalue $a^2 > 0$.

The above three lemmas have displayed $k(n+m)$ eigenvalues of $\nabla^2 J(W_c a, a^{-1} S_c)$. As expected, as $\alpha$ varies over the nonzero reals, the positive eigenvalues remain positive and the negative eigenvalues remain negative. We are particularly interested in the negative eigenvalues.

Lemma B.6. $\lambda_{\min}(\nabla^2 J(W_c a, a^{-1} S_c)) < 0$ if and only if there exists $p \in [1:k]$ with $\lambda_p < \sigma_p$.

Proof. (Only If) Assume $\nabla^2 J(W_c a, a^{-1} S_c)$ has a negative eigenvalue. Then by Lemma [3.4] there exists $i \in [1:r]$ such that $u_i$ not a column in $\bar{U}$, and $j \in [1:k]$ such that $\lambda_j < \sigma_i$. For some $s \in [1:m], \bar{u}_j = u_s$ and hence $\lambda_j = \sigma_j < \sigma_i$. Thus $i < s$. So $u_s$ has been omitted from $\bar{U}$ and $\sigma_i$ from the diagonal of $\Lambda$. Yet $\sigma_i < \sigma_i$ is included in the diagonal of $\Lambda$. It follows that the diagonal of $\Lambda$ does not contain a set of $k$ largest singular values of $X$. Hence there is a least $p \in [1:k]$ such that $\lambda_p < \sigma_p$. 

(If) Assume that for some $j \in [1:k], \lambda_j < \sigma_j$. Then there exists a least $p$ with $\lambda_p < \sigma_p$. Thus for $j \in [1:p-1], \lambda_j = \sigma_j$ and a corresponding left singular vector $u_j$ occupies column $j$ of $\bar{U}$. But $\bar{u}_p$ is a left singular vector for a singular value $\lambda_p < \sigma_p$. Hence a corresponding left singular vector $u_p$ for $\sigma_p$ has been omitted from $\bar{U}$. Thus there exists a left singular vector $u_p$, such that $u_p$ is not a column of $\bar{U} = 0$, and an integer $j = p \in [1:k]$ such that $\lambda_j < \sigma_p$. 

By Lemma [B.6] $(W_c, S_c)$ is a strict saddle if and only if it is not maximal. We now determine the least eigenvalue of $\nabla^2 J(W_c a, a^{-1} S_c)$ when $(W_c, S_c)$ is a strict saddle.

Theorem B.1. Assume $(W_c, S_c)$ is not maximal and $p$ is the least integer for which $\lambda_p < \sigma_p$. Then 

$$\lambda_{\min}(\nabla^2 J(W_c a, a^{-1} S_c)) = \frac{1}{2} \left( \frac{\lambda_j^2 + \alpha^2}{a^2} - \sqrt{ \left( \frac{\lambda_j^2 - \alpha^2}{a^2} \right)^2 + 4 \sigma_p^2} \right) < 0$$ (36)
Proof. Since there exists \( j \in [1:k] \) with \( \lambda_j < \sigma_j \), Lemma \[B.6\] implies \( \nabla^2 J(Wa, a^{-1}, S) \) has a negative eigenvalue \( \rho \), and Lemma \[B.4\] implies
\[
2|\rho| = \sqrt{\left(\frac{\lambda^2 - a^4}{\sigma^2}\right)^2 + 4\sigma^2 - \frac{\lambda^2 + a^4}{\sigma^2}}
\]
where \( \lambda \in \{ \lambda_j : j \in [1:k] \} \), \( \sigma \in \{ \sigma_i : \sigma_i > \lambda_j, u_i^T \bar{U} = 0 \} \). We have
\[
2 \frac{d|\rho|}{d\sigma} = \frac{\left(\frac{\lambda^2 - a^4}{\sigma^2}\right)}{\sqrt{\left(\frac{\lambda^2 - a^4}{\sigma^2}\right)^2 + 4\sigma^2}} - \frac{\lambda}{\sigma} = \frac{\lambda}{\sigma} \left( - \frac{\lambda}{\sigma} \right) - 1 = \frac{\lambda}{\sigma} \left( - \frac{\lambda}{\sigma} - 1 \right).
\]
The terms \( c \) and \( d \), defined in \[52\], satisfy \( c < d \). Hence \( \frac{d|\rho|}{d\sigma} \) is negative and \( |\rho| \) is monotone decreasing in \( \lambda \). It is clear from \[57\] that \( |\rho| \) is monotonically increasing in \( \sigma \). So we let \( p \) be the least index such that \( \sigma_p > \lambda_p \). Then \( \sigma_p > \lambda_p \geq \lambda_k \). So we select \( \lambda = \lambda_k \). This gives \[56\].

\[\Box\]

Corollary B.1. Under the assumptions of Theorem \[B.1\]
\[
\lambda_{\min}(\nabla^2 J(Wa, a^{-1}, S_c)) = \frac{-(\lambda^2 - \lambda_k^2)}{2\sigma^2 + \left(\frac{\lambda_k^2 + a^4}{\sigma^2}\right)^2} + \frac{\sigma^2}{\sigma_p - \lambda_k^2}
\]

Proof. By Theorem \[B.1\] \( \lambda_{\min}(\nabla^2 J(Wa, a^{-1}, S_c)) \) is given by \[56\]. Let \( c = \frac{\lambda^2 + a^4}{\sigma^2} \) and \( d = \sqrt{\left(\frac{\lambda^2 - a^4}{\sigma^2}\right)^2 + 4\sigma^2} \). Then \( \lambda_{\min}(\nabla^2 J(Wa, a^{-1}, S_c)) = \frac{1}{2}(c - d) \). Use \( c^2 - d^2 = (c - d)(c + d) \) to write \( \lambda_{\min}(\nabla^2 J(Wa, a^{-1}, S_c)) = \frac{1}{2} \frac{c^2 - d^2}{\sigma^2} \). Noting that \( c^2 - d^2 = -4(\sigma^2 - \lambda_k^2) \) and evaluating \( 1/2(c + d) \) using the definitions of \( c \) and \( d \) yields \[58\].

\[\Box\]

B.2 A Canonical Point with \( q < k \)

We now determine the eigenvalues and eigenvectors of the Hessian at a canonical point \((W_c, S_c)\) with \( W_c = [W \ 0_{k-q}] \), \( S_c = [S^T \ V_0 C_0]^T \), \( k-q > 0 \), and \( \text{rank}(W) = q \). Noting that \((W, S)\) is a full rank canonical point for dimension \( q \), we first “lift” the \( q(m+n) \) eigenvalues and eigenvectors of \( \nabla^2 J(W, S) \) to eigenvalues and eigenvectors of \( \nabla^2 J(W_c, S_c) \).

Lemma B.7. If \((G, H)\) is an eigenvector of \( \nabla^2 J(W, S) \) with eigenvalue \( \rho \) then \([(G \ 0_{k-q})\], \([H^T \ 0_{k-q}]^T\)) is an eigenvector of \( \nabla^2 J(W_c, S_c) \) with eigenvalue \( \rho \).

Proof. Simple algebra shows that \((W_c, S_c)\), \([(G \ 0_{k-q})\], \([H^T \ 0_{k-q}]^T\)) and \( \rho \) satisfy \[27\].

\[\Box\]

We obtain \((k-q)(n-m)\) additional eigenvalues from Lemma \[B.3\]. Specifically, for \( i \in [m+1:k] \) and \( j \in [q+1:k] \), \((0, H_{i,j})\) is an eigenvector of \( \nabla^2 J(W_c, S_c) \) with eigenvalue \( \rho_0 = 1 \).

There are \( 2(k-q)m - 2(k-q)^2 \) remaining eigenvalues. Since \( C_0^T C_0 \in \mathbb{R}^{(k-q) \times (k-q)} \) is symmetric positive semidefinite, \( C_0^T C_0 = Z \Omega Z^T \) where \( Z \in \mathcal{O}_{k-q} \), and \( \Omega \) is diagonal with the eigenvalues \( \omega_j \) of \( C_0^T C_0 \) listed in decreasing order on the diagonal. Let \( \bar{z}_j = [0, z_j^T]^T \). Then for \( i \in [1:m] \) with \( u_i \) not in \( \bar{U} \), and \( j \in [q+1:k] \), set \( \bar{G}_{i,j} = u_i \bar{z}_j^T \) and \( \bar{H}_{i,j} = \bar{z}_j u_i^T \).

Lemma B.8. For \( i \in [1:m] \) with \( u_i \) not a column in \( \bar{U} \), and \( j \in [q+1:k] \), the following hold:
(a) If \( i \in [1:r] \), there exist \( \delta_{i,j}, \delta'_{i,j} \in \mathbb{R} \) with \( \delta_{i,j} \delta'_{i,j} = -1 \) such that \((\bar{G}_{i,j}, \delta_{i,j} \bar{H}_{i,j})\) and \((\bar{G}_{i,j}, \delta'_{i,j} \bar{H}_{i,j})\) are eigenvectors of \( \nabla^2 J(W_c, S_c) \) with eigenvalues \( \rho_{i,j} \) and \( \rho'_{i,j} \) given in \[17\].
(b) If \( i \in [r+1:m] \), the pair of vectors \((\bar{G}_{i,j}, 0)\) and \((0, \bar{H}_{i,j})\), are eigenvectors of \( \nabla^2 J(W_c, S_c) \) with eigenvalues \( \omega_j \geq 0 \), and \( 0 \), respectively.

Proof. (a) Write \( W = \bar{U} \) and \( S = \Lambda \bar{V}^T \). Then \( W_c, S_c = WS = \bar{U} \Lambda \bar{V}^T \). In addition, \( W_c^T W_c = \begin{bmatrix} I_r \ 0 \ 0 \end{bmatrix} \) and using Lemma \[A.2\] part (a), \( S_c S_c^T = \begin{bmatrix} \Lambda^2 \ 0 \ 0 \ C_0^2 C_0 \end{bmatrix} \). From \[5\], the first component of
The expression above simplifies when
\[ C \nabla (b) \]
and the analysis there yields the result.

(b) The first component of \( \nabla^2 J(W_c, S_c) (\tilde{G}_{i,j}, 0) \) is \( u_i \left[ 0 \ z^T_j \right] = \omega_j u_i z^T_j \); the second is \( \left[ \begin{array}{cc} \tilde{U} & 0 \\ 0 & \tilde{U} \end{array} \right] u_i z^T_j s_c + \tilde{z}_j u^T \tilde{U} A V^T - X) v_i z^T_j = \omega_j (e_j, 0). \) The first component of \( \nabla^2 J(W_c, S_c) (0, \tilde{H}_{i,j}) \) is \( u_i \left[ 0 \ z^T_j \right] = \omega_j u_i z^T_j = 0. \) Thus \( \nabla^2 J(W_c, S_c) (0, \tilde{H}_{i,j}) = 0 \). \( \square \)

It remains to find \( 2(k-q)^2 \) eigenvalues. To do so, consider indices \( i \in [1:m] \) for which \( u_i \) is a column of \( \tilde{U} \). For each column \( \tilde{u}_j \) of \( \tilde{U} \), \( j \in [1:q] \), there exists \( i \in [1:m] \) such that \( \tilde{u}_j = u_i, \tilde{v}_j = v_i, \) and \( \lambda_j = \sigma_i \). We partition the index set \([1:q] \) into \( S \), with \( s \in S \) if \( \lambda_s > 0 \), and \( T \), with \( t \in T \) if \( \lambda_t = 0 \). One can verify that the remaining eigenvalues and eigenvectors of the Hessian have the form given in Lemma B.5 for when \( k = q \) and \( a = 1 \). This determines all the eigenvalues.

To find the least eigenvalue of \( \nabla^2 J(W_c, S_c) \) we consider two situations. First, using the notation defined above, if \( (W, S) = (U, A V^T) \) is maximal, the eigenvalues of \( \nabla^2 J(W, S) \) are non-negative. Hence the only negative eigenvalues of \( \nabla^2 J(W_c, S_c) \) are given in Lemma B.8. The least among these is clearly

\[
\lambda_{\min}(\nabla^2 J(W_c, S_c)) = \frac{\omega_k - q}{2} - \sqrt{\frac{\sigma_2^2}{\sigma_1^2} + \frac{(\omega_k - q)^2}{\sigma_1^2}}.
\]

Second, if \( (W, S) \) is not maximal, then there is a least \( p \in [1:q] \) with \( \sigma_p > \lambda_p \). In this case, negative eigenvalues emerge from two sources. First, by “lifting” the eigenvectors of \( \nabla^2 J(W, S) \) (Lemma B.7 and Theorem B.1). Second, negative eigenvalues arise because of the existence of zero columns in \( W_c \); Lemma B.8. The least eigenvalue of \( \nabla^2 J(W, S) \) is then the minimum eigenvalue from these two sources. Using Lemma B.8 and Theorem B.1 this can be expressed as

\[
\lambda_{\min}(\nabla^2 J(W_c, S_c)) = \min \left\{ \frac{\omega_k - q}{2} - \sqrt{\frac{\sigma_2^2}{\sigma_1^2} + \frac{(\omega_k - q)^2}{\sigma_1^2}}, \right. \]

\[
\left\{ \frac{1}{2} \left( \lambda_2^2 + 1 - \sqrt{\left( \lambda_2^2 - 1 \right)^2 + 4 \sigma_2^2} \right) \right\}.
\]

The expression above simplifies when \( C_0 = 0 \).

**Theorem B.2.** For a canonical point \((W_c, S_c)\) with \( q < k \) and \( C_0 = 0 \),

\[
\lambda_{\min}(\nabla^2 J(W_c, S_c)) = \begin{cases} -\sigma_{q+1}, & \text{if } (W_c, S_c) \text{ is maximal;} \\ -\sigma_p, & \text{if } (W_c, S_c) \text{ is not maximal;} \end{cases}
\]

where in the second case, \( p \in [1:q] \) is the least index with \( \lambda_p < \sigma_p \).

**Proof.** When \( C_0 = 0, \omega_j = 0 \) for \( j \in [1:k-q] \). When \( (W_c, S_c) \) is maximal, using [39], the least eigenvalue is \( -\sigma_{q+1} \). Otherwise, there are two possibilities. We show that \(-\sigma_p \) is the least eigenvalue. Let \( -\sigma_p < \frac{\lambda_2^2 - 1}{2} - \sqrt{\left( \frac{\lambda_2^2 - 1}{2} \right)^2 + \sigma_2^2} \). This is equivalent to \( \sigma_p + \frac{\lambda_2^2 - 1}{2} > \sqrt{\left( \frac{\lambda_2^2 - 1}{2} \right)^2 + \sigma_2^2} \). Squaring both sides, expanding, and eliminating common terms confirms the claimed ordering. \( \square \)

### C The Manifold

\[ W^T W - S S^T = C \]

**Lemma C.1** (26 Theorem 1). Along every solution of the gradient flow o.d.e., \( W^T W_t - S_t S^T_t \) is a constant symmetric \( k \times k \) matrix and \( \|W_t\|^2_{F} - \|S_t\|^2_{F} \) is a constant.

**Proof.** Given an initial condition \((W_0, S_0) \in X\), the gradient flow o.d.e. \( \frac{d}{dt} (W_t, S_t) = -\nabla J(W_t, S_t) \) defines a curve \((W_t, S_t), t \geq 0, \) in \( X \). Taking the inner product of both sides of the o.d.e with \( T_{W,S}(H) \) in [9], and using \((T_{W,S}(H), \nabla J(W,S)) = 0\), yields \( H^T W^T W_t - S^T_t H^T S_t = \)
\[ \langle H, W^T W_t - S_t S^T_t \rangle = 0, \forall H \in \mathbb{R}^{k \times k}. \] Thus \( W^T_t W_t - S_t S^T_t = 0 \). Adding this to its transpose gives \( \frac{d}{dt}(W^T_t W_t - S_t S^T_t) = 0 \). Thus \( W^T_t W_t - S_t S^T_t = W^T_0 W_0 - S_0 S^T_0 \). Now note that \( \|W_t\|_F^2 - \|S_t\|_F^2 = \text{trace}(W^T_t W_t) - \text{trace}(S^T_t S_t) = \text{trace}(W^T_t W_t - S_t S^T_t) \), and \( \text{trace}(W^T_t W_t - S_t S^T_t) \) is a constant.

### References

[1] Harold Hotelling. Analysis of a complex of statistical variables into principal components. *Journal of educational psychology*, 24(6):417, 1933.

[2] Harold Hotelling. Relations between two sets of variates. *Biometrika*, 28(3/4):321–377, 1936.

[3] Ian Jolliffe. *Principal component analysis*. Springer, 2011.

[4] David R. Hardoon, Sandor Szedmak, and John Shawe-Taylor. Canonical correlation analysis: An overview with application to learning methods. *Neural computation*, 16(12):2639–2664, 2004.

[5] Aapo Hyvärinen and Erkki Oja. Independent component analysis: algorithms and applications. *Neural networks*, 13(4-5):411–430, 2000.

[6] Pentti Paatero and Unto Tapper. Positive matrix factorization: A non-negative factor model with optimal utilization of error estimates of data values. *Environmetrics*, 5(2):111–126, 1994.

[7] Julien Mairal, Francis Bach, Jean Ponce, and Guillermo Sapiro. Online dictionary learning for sparse coding. In *Proceedings of the 26th annual international conference on machine learning*, pages 689–696. ACM, 2009.

[8] Pierre Baldi and Kurt Hornik. Neural networks and principal component analysis: Learning from examples without local minima. *Neural networks*, 2(1):53–58, 1989.

[9] Y. Bourlard, H. and Kamp. Auto-association by multilayer perceptrons and singular value decomposition. *Biological Cybernetics*, 59(4):291–294, Sep 1988.

[10] Zhehui Chen, Xingguo Li, Lin F. Yang, Jarvis Haupt, and Tuo Zhao. On landscape of lagrangian functions and stochastic search for constrained nonconvex optimization. *arXiv preprint arXiv:1806.05151*, 2018.

[11] Xingguo Li, Junwei Lu, Raman Arora, Jarvis Haupt, Han Liu, Zhaoran Wang, and Tuo Zhao. Symmetry, saddle points, and global optimization landscape of nonconvex matrix factorization. *IEEE Transactions on Information Theory*, 2019.

[12] Hesameddin Mohammadi, Meisam Razaviyayn, and Mihailo R Jovanović. On the stability of gradient flow dynamics for a rank-one matrix approximation problem. In *2018 Annual American Control Conference (ACC)*, pages 4533–4538. IEEE, 2018.

[13] Rong Ge, Chi Jin, and Yi Zheng. No spurious local minima in nonconvex low rank problems: A unified geometric analysis. In *Proceedings of the 34th International Conference on Machine Learning-Volume 70*, pages 1233–1242. JMLR. org, 2017.

[14] Ju Sun, Qing Qu, and John Wright. A geometric analysis of phase retrieval. *Foundations of Computational Mathematics*, 18(5):1131–1198, 2018.

[15] Ju Sun, Qing Qu, and John Wright. Complete dictionary recovery over the sphere. In *2015 International Conference on Sampling Theory and Applications (SampTA)*, pages 407–410. IEEE, 2015.

[16] Nicolas Boumal. Nonconvex phase synchronization. *SIAM Journal on Optimization*, 26(4):2355–2377, 2016.

[17] Kenji Kawaguchi. Deep learning without poor local minima. In *Advances in Neural Information Processing Systems*, pages 586–594, 2016.

[18] Quynh Nguyen and Matthias Hein. The loss surface of deep and wide neural networks. In *Proceedings of the 34th International Conference on Machine Learning-Volume 70*, pages 2603–2612. JMLR. org, 2017.

[19] Moritz Hardt and Tengyu Ma. Identity matters in deep learning. *arXiv preprint arXiv:1611.04231*, 2016.
[20] Rong Ge, Jason D. Lee, and Tengyu Ma. Learning one-hidden-layer neural networks with landscape design. arXiv preprint arXiv:1711.00501, 2017.

[21] Rong Ge, Furong Huang, Chi Jin, and Yang Yuan. Escaping from saddle points? online stochastic gradient for tensor decomposition. In Conference on Learning Theory, pages 797–842, 2015.

[22] Rong Ge, Jason D Lee, and Tengyu Ma. Matrix completion has no spurious local minimum. In Advances in Neural Information Processing Systems, pages 2973–2981, 2016.

[23] Srinadh Bhojanapalli, Behnam Neyshabur, and Nati Srebro. Global optimality of local search for low rank matrix recovery. In Advances in Neural Information Processing Systems, pages 3873–3881, 2016.

[24] Jason D Lee, Max Simchowitz, Michael I Jordan, and Benjamin Recht. Gradient descent only converges to minimizers. In Conference on Learning Theory, pages 1246–1257, 2016.

[25] Chi Jin, Rong Ge, Praneeth Netrapalli, Sham M Kakade, and Michael I Jordan. How to escape saddle points efficiently. In Proceedings of the 34th International Conference on Machine Learning-V o lume 70, pages 1724–1732. JMLR. org, 2017.

[26] S. Arora, N. Cohen, and E. Hazan. On the optimization of deep networks: Implicit acceleration by overparameterization. In Proc. of the 35th International Conference on Machine Learning, pages 244–253. PMLR, 10–15 Jul 2018.

[27] Simon S. Du, Wei Hu, and Jason D. Lee. Algorithmic regularization in learning deep homogeneous models: Layers are automatically balanced. In S. Bengio, H. Wallach, H. Larochelle, K. Grauman, N. Cesa-Bianchi, and R. Garnett, editors, Advances in Neural Information Processing Systems 31, pages 384–395. Curran Associates, Inc., 2018.

[28] Roger A. Horn and Charles R. Johnson. Matrix Analysis. Cambridge University Press, 2nd edition, 2013.