0. Introduction. The following observation, due to E. Trubowitz [7], illustrates an intimate relationship between spectral theory and Hamiltonian mechanics in the presence of constraints. Let \( q(s) \) be a real periodic function such that Hill’s operator,

\[
L = \left( \frac{d}{ds} \right)^2 - q(s),
\]

has only a finite number \( g \) of simple eigenvalues. There exist \( g + 1 \) periodic eigenfunctions \( x_0, \ldots, x_g \) and corresponding eigenvalues \( a_0, \ldots, a_g \) of \( L \) such that

\[
1 = \sum_{r=0}^{g} x_r^2 \quad \text{and} \quad q = -\sum_{r=0}^{g} (a_r x_r^2 + y_r^2),
\]

where \( y_r = dx_r/ds \). The equations \( L x_r = a_r x_r \) \( (r = 0, \ldots, g) \) are equivalent to the classical Neumann system [7].

H. Flaschka [3] obtained similar results from a different point of view. His approach is based on the articles [2 and 5] of I. V. Cherednik and I. M. Krichever. The familiar Lax pairs, the constants of motion and the quadrics of the Neumann system emerge as consequences of the Riemann-Roch Theorem.

The purpose of our work is to apply Flaschka’s techniques to operators of order \( n \geq 2 \). We will be defining higher Neumann systems whose theory is closely tied to the spectral theory of linear differential operators of order \( n \).

C. Tomei [9], using scattering theory, obtained some of our \( n = 3 \) formulas.
assume that the genus $g_R$ is related to $m, n$ and $l$ by the following important formula, $g_R = \frac{1}{2}(n-1)(2(m+1)-nN-(l+1))$. It is known that two rational functions on a Riemann surface satisfy a polynomial equation. Since that equation, it turns out, follows from the Baker function theory below, we need not discuss the existence of Riemann surfaces with the properties above.

Since $n$ and $l$ are relatively prime, there exist $r_j, s_j \in \mathbb{Z}$ such that $\lambda^j z^{s_j}$ has a pole of order $j$ at $\infty$. Let $t = (t_j | j \in W)$ be a vector of $g_R$ complex “time” parameters. Let $\theta = \sum_{j \in W} t_j \lambda^j z^{s_j}$.

(1.3) BAKER FUNCTIONS. Let $\delta$ be a positive nonspecial divisor of degree $g_R$ that does not meet $\infty$ and satisfies $L(\delta - \infty) = \{0\}$. It is known that there exists a unique function $\psi = \psi_{\delta}(t, p)$, called the Baker function of $\delta$, with the following two properties. $\psi$ is meromorphic in $R - \infty$ and any pole of $\psi$ lies in $\delta$. Near $\infty$, $\psi$ is given by $\psi e^{-\theta} = 1 + \xi_1(t) \kappa^{-1} + \xi_2(t) \kappa^{-2} + \cdots$, where the $\xi_j$ are functions analytic on an open subset of $\mathbb{C}^g$ containing $t = 0$.

(1.4) DUAL BAKER FUNCTION. By the Riemann-Roch Theorem there exists a unique abelian differential $\Omega$ and a positive nonspecial divisor $\delta'$ of degree $g_R$ such that $(\Omega) = \delta + \delta' - 2\infty$ and $\Omega = -\kappa^2(1 + O(\kappa^{-2})) d\kappa^{-1}$ at $\infty$. Let $\phi = \psi_{\delta'}(-t, p)$. We will refer to $\phi$ as the Baker function dual to $\psi$ and $\delta'$ will be called the dual divisor [2].

(1.5) NEUMANN SYSTEMS. There exists a linear differential operator $L$ of order $n$ in $d/dt_1$ and, for each $j \in W$, a linear differential operator $L_j$ of order $j$ in $d/dt_1$ such that

\begin{equation}
L(t) \psi(t, p) = \lambda(p) \psi(t, p) \quad \text{and} \quad \tilde{L}_j(t) \psi(t, p) = \frac{\partial \psi}{\partial t_j}(t, p).
\end{equation}

Let $L^*$ be the formal real adjoint of $L$ (for instance, $(qD^j)^* = (-1)^j D^j q$). The article [2] contains a clever proof of the following formulas:

\begin{equation}
L(t)^* \phi(t, p) = \lambda(p) \phi(t, p) \quad \text{and} \quad \tilde{L}_j(t)^* \phi(t, p) = \frac{\partial \phi}{\partial t_j}(t, p).
\end{equation}

We are now in position to define the main object of our analysis. Let $\rho_r = \text{Res}_{(r)}(z^r \Omega)$ and choose constants $\alpha_r, \beta_r \in \mathbb{C}^*$ such that $\rho_r = \alpha_r \beta_r$. We evaluate the Baker functions $\psi$ and $\phi$ over the poles of $z$ to make the following definitions:

\begin{equation}
x^r_1(t) = \alpha_r \psi(t, r) \quad \text{and} \quad u^r_n(t) = \beta_r \phi(t, r), \quad r = 0, \ldots, m.
\end{equation}

Let $m \in \mathbb{C}^{2n(m+1)}$ be the point whose coordinates are $x^r_1, u^r_n$ and their first $n - 1$ derivatives with respect to $t_1$. We are concerned with the equations obtained from (1.5.1) by setting $p = (r), r = 0, \ldots, m$.

(1.6) SOLITON EQUATIONS. The integrability condition of the simultaneous linear equations (1.5.1) is the partial differential equation

\begin{equation}
\frac{\partial L}{\partial t_j} = [\tilde{L}_j, L], \quad j \in W.
\end{equation}

The Lax equation usually suggests that certain spectral data associated to $L$ are preserved in time. In the present setup it is the Riemann surface $R$ that is preserved. Two of the equations $(\ast)$ are important in their applications to soliton mathematics. If $n = 2$ and $j = 3$, $(\ast)$ is the Korteweg-de Vries
equation. If \( n = 3 \) and \( j = 2 \), (*) is the Boussinesq equation in the form of a system of equations.

**Results.**

(2.1) SYMPLECTIC MANIFOLD AND TRACE FORMULAS. The differential
\[ \eta = \psi_j^{(i)} \phi_j^{(i)} \Omega \]
is meromorphic because the exponents of \( \psi \) and \( \phi \) at \( \infty \)
cancel. The meromorphic differential \( \eta = \lambda^k z \eta \) has simple poles in \( (z)_{\infty} \) and it may have a pole at \( \infty \). Let \( C_\eta = \sum_{p \in \mathbb{R}} \text{Res}_p(\eta) \). The classical formula
\[ \sum_{p \in \mathbb{R} - \infty} \text{Res}_p(\eta) = -\text{Res}_\infty(\eta) \]
expresses \( \text{Res}_\infty(\eta) \) in terms of \( m \). If \( \text{Res}_\infty(\eta) \) is constant (in \( t \)) the equation \( C_\eta = 0 \) defines a hypersurface in \( C^{2n(m+1)} \).

The functions \( C_\eta \) with \( \text{Res}_\infty(\eta) \) constant are called constraints.

(2.1.1) THEOREM. The algebraic subset \( M \) of \( C^{2n(m+1)} \) defined in terms
of the quadratic constraints \( C_\eta = 0 \) is a symplectic manifold. The dimension
of \( M \) is given by \( \dim(M) = 2g_R + 2(m + 1) \).

(2.1.2) THEOREM. The coefficients of \( L \) are expressible in terms of the
point \( m \) associated with the Baker function and the poles of \( z \).

It follows then that the equations (1.5.1) with \( p = (r), r = 0, \ldots, m \), define
\( g_R \) autonomous vector fields \( X_j^*, j \in W, \) on \( M \). The \( (n = 2) \) vector field \( X_1^* \)
is a generalization of the Neumann system [3 and 4].

(2.2) LAX EQUATIONS. One of the nicest results of Flaschka’s work is a
systematic derivation of the well-known Neumann-Lax pairs. The best expla­
nation for the existence of the Neumann-Lax pairs comes from Krichever’s
theory of commutative rings of matrix differential operators. The divisor
\( \Delta' = \delta' + (z)_{\infty} \) is nonspecial and its degree is \( g_R + m \). Following [4] we
call \( \Delta' \) the augmented dual divisor. According to [8], there exists a vector
function \( \Phi = (\Phi^0, \ldots, \Phi^m)^T \) with the following two properties. \( \Phi \) is meromorphic in \( R - (z)_{\infty} \) and any pole in \( \Phi \) lies in \( \Delta' \). Near \( (r) \), \( \Phi^s \) is given by
\[ \Phi^s e^{-\theta} = \alpha_r \delta_r, s + O(z^{-1}) \]
where \( (; ; \) is the bilinear form associated to \( L \) by the Lagrange identity,
\[ d(f; g)/dt_1 = Lf \cdot g - f \cdot L^* g. \]

H. Flaschka discovered the \( n = 2 \) version of the very beautiful formula,

\[ (2.2.1) \quad \Phi^r(t, p) = (x_1^*(t); \phi(t, p)) \quad \frac{z^{-1}(p)}{\lambda(p)} e^{\theta(t, p)} \]

According to Krichever there exists an \( (m + 1) \times (m + 1) \) matrix \( B_j \) that
depends polynomially on \( z \) such that \( \Phi_{t_j} = B_j \Phi \). Using Flaschka’s formula
(2.2.1) we are able to express \( B_j \) in terms of \( m \). The function \( \lambda z^n \)
belongs to the ring \( H^0(R - (z)_{\infty}, O_R) \). Thus according to Krichever there exists
an \( (m + 1) \times (m + 1) \) matrix \( L \) that depends polynomially on \( z \) such that
\( L\Phi = \lambda z^n \Phi \). The Lax equation \( L_{t_j} = [B_j, L] \) is immediate. Our explicit
formulas show that \( L \) is a rank \( n \) perturbation of the diagonal matrix \( az^n \)
in that the range of \( L - az^n \) is spanned by \( x_1, \ldots, x_n \). The \( (n = 2) \) and \( B_1 \)
generalize the Neumann-Lax pairs in [1, 3 and 4].

We have \( \Delta' - (\phi)_{\infty} \geq 0 \) and therefore \( \phi e^{\theta} \) belongs to the linear space
of Baker functions spanned by the components of \( \Phi \). This observation led
Flaschka to the $n = 2$ version of the following formula:

$$\phi(t, p)e^\theta = \sum_{r=0}^{m} u_r^\ast(t) \Phi_r(t, p) = \langle u_n(t), \Phi(t, p) \rangle.$$  

The formula has two applications. We use (2.2.2) to obtain explicit formulas for the operators $\tilde{L}_j$. Such formulas were one of Cherednik’s objectives [2]. When $\Phi$ is eliminated from (2.2.2) by use of (2.2.1) we obtain the following result.

**Theorem.** There exists an $n \times n$ matrix $Z = Z(m, \lambda)$, rational in $\lambda$, whose spectrum is independent of $t$. The algebraic relationship (1.2) between $\lambda$ and $z$ is given by the characterization polynomial $\det(Z - zI) = 0$.

**Complete integrability.** The $m + 1$ Hamiltonians $(x_r^\ast; u_r^\ast)$, $r = 0, \ldots, m$, are rather trivial involutive constants of motion. A reduction of $M$ by these Hamiltonians defines a symplectic manifold which, by (2.1.1), has dimension $2g_R$. We use the fact that the eigenvalues of $L$ and $Z$ are constants of the motion to construct a Hamiltonian $\tilde{H}_j^\ast$ for each vector field $X_j^\ast$, $j \in W$.

**Theorem.** The $g_R$ Neumann vector fields $X_j^\ast$ of (2.1.4) form a completely integrable Hamiltonian system.

It is known that the level surface $M_C \triangleq \{ m^\ast \in M | H_j(m) = c_j \}$ of a completely integrable system, if real and compact, is a torus. Our last result is concerned with the structure of these energy level sets.

**Theorem.** The level surface of the reduced manifold is locally isomorphic to the Zariski-open subset, Jacobian-(theta divisor) of the Jacobian variety of the algebraic curve given by $\det(Z(A) - zI)$.

The idea in the proofs of (2.3.1,2) is an algebro-geometrical version of the solitonic inverse scattering transform. Let $M$ be one of the symplectic manifolds of Theorem (2.1.1). We assign to each point $m \in M$ an algebraic curve $C$ and a divisor $\delta = \delta_m$ on $C$. The isomorphism of Theorem (2.3.2), called the divisor map, is given by

$$m \in M \rightarrow (C, \delta) \rightarrow (\text{Jac}(C), A(\delta))$$

where $A$ is the Abel map. It contains a method for linearizing the equations of motion. The important ideas can be found in [1 and 5]. We apply McKean’s pole conditions [6, p. 624] to make certain results, especially the description of $\delta$, more explicit.

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