Law of large numbers for Betti numbers of homogeneous and
spatially independent random simplicial complexes

By Shu Kanazawa

Abstract. The Linial–Meshulam complex model is a natural higher-dimensional analog of the Erdős–Rényi graph model. In recent years, Linial and Peled established a limit theorem for Betti numbers of Linial–Meshulam complexes with an appropriate scaling of the underlying parameter. The present paper aims to extend that result to more-general random simplicial complex models. We introduce a class of homogeneous and spatially independent random simplicial complexes, including the Linial–Meshulam complex model and the random clique complex model as special cases, and we study the asymptotic behavior of their Betti numbers. Moreover, we obtain the convergence of the empirical spectral distributions of their Laplacians. A key element in the argument is the local weak convergence of simplicial complexes. Inspired by the work of Linial and Peled, we establish the local weak limit theorem for homogeneous and spatially independent random simplicial complexes.

1. Introduction

Let $K_n$ be the complete graph on $[n] := \{1, 2, \ldots, n\}$. An Erdős–Rényi graph is a random subgraph of $K_n$ with $n$ vertices, where each edge in $K_n$ appears independently with a probability $p \in [0,1]$. The probability distribution is denoted by $G(n,p)$. The Erdős–Rényi graph model has been extensively studied since the early 1960s ([8], [9], [11]) as a typical random graph model. One of the main themes in the study of the Erdős–Rényi graph model is searching the threshold probability $p$, typically a function of $n$, for some graph property. The behavior of the random graph around the threshold probability should also be studied as a further theme. Erdős and Rényi showed that the threshold probability for the appearance of cycles is $p = \frac{1}{n}$.

Furthermore, they established a limit theorem for the number of connected components around the threshold probability as follows.

Theorem 1.1 (Erdős and Rényi [9, Section 6]). Let $c > 0$ be fixed, and let $G_n \sim G(n,p)$ be an Erdős–Rényi graph with $p = c/n$. Then, for any $\varepsilon > 0$,

$$\lim_{n \to \infty} \Pr \left( \left| \frac{\xi(G_n)}{n} - \frac{1}{c} \sum_{s=1}^{\infty} \frac{s^{s-2}}{s!} (ce^{-c})^s \right| > \varepsilon \right) = 0. \quad (1.1)$$

Here, $\xi(G_n)$ denotes the number of connected components of $G_n$.

Recently, there has been growing interest in studying random simplicial complexes as higher-dimensional analogs of random graphs. The systematic study of random simplicial complexes has its origin in the work of Linial and Meshulam [16]. They introduced the 2-Linial–Meshulam complex model as a higher-dimensional generalization of the Erdős–Rényi graph model, and the $d$-Linial–Meshulam complex model was studied by Meshulam and Wulich [19]. Let $\Delta_n$ denote
the \((n-1)\)-dimensional complete complex on \([n]\). For each \(d \in \mathbb{N}\), a \(d\)-Linial–Meshulam complex is a random subcomplex of \(\Delta_n\) with all the \((d-1)\)-simplices in \(\Delta_n\), where each \(d\)-simplex in \(\Delta_n\) appears independently with a probability \(p \in [0,1]\). The probability distribution is denoted by \(Y_d(n,p)\). Note that a 1-Linial–Meshulam complex can be naturally regarded as an Erdős–Rényi graph.

Since the appearance of cycles in the Erdős–Rényi graph can be described as the nontriviality of the first homology group, it is natural to seek the threshold probability for the appearance of the \(d\)th homology group of the \(d\)-Linial–Meshulam complex. Aronshtam and Linial [2] found an upper bound on the threshold probability for the appearance of the \(d\)th homology group with any field coefficient. Linial and Peled [17] proved that the upper bound is tight as long as the characteristic of the coefficient field is zero. They also studied the asymptotic behavior of the Betti numbers of \(d\)-Linial–Meshulam complexes around the threshold probability. For \(c \geq 0\), letting \(t_{d,c} \in (0,1]\) be the smallest positive root of the equation \(t = \exp(-c(1-t)^d)\), we define \(\log x_d \in (0,1]\) of the equation \((d+1)(1-x) + (1+dx) \log x = 0\), define \(c_d := -(\log x_d)/(1-x_d)^d \) for \(d \geq 2\) and set \(c_1 := 1\) (see Remark 1.2 and Appendix B in [17] in detail). The following theorem is immediately obtained by combining Theorems 1.1 and 1.2 in [17] with the Euler–Poincaré formula.

**Theorem 1.2** (Linial and Peled [17]). Let \(d \in \mathbb{N}\) and \(c > 0\) be fixed, and let \(Y_n \sim Y_d(n,p)\) be a Linial–Meshulam complex with \(p = c/n\). Then, for any \(\varepsilon > 0\),

\[
\lim_{n \to \infty} \mathbb{P}\left( \frac{\beta_{d-1}(Y_n)}{n^d} - \frac{h_{d-1}(c)}{d!} > \varepsilon \right) = 0.
\]

![Figure 1](image_url)  
**Figure 1.** Illustration of Theorem 1.2 for \(d = 2\).

When \(d = 1\), Theorem 1.2 corresponds to Theorem 1.1 and \(h_0(c)\) is identical to the limiting constant in Eq. (1.1). Figure 1 illustrates the behavior of the limiting constant \(h_{d-1}(c)/d!\) with respect to the parameter \(c\) when \(d = 2\). The aim of this paper is to generalize Theorem 1.2 to more-general random simplicial complex models.
Kahle [13] introduced another random simplicial complex model called the random clique complex model. Given a simple undirected graph $G$, its clique complex is defined as the inclusion-wise maximal simplicial complex among other simplicial complexes whose underlying graphs are identical to $G$. The clique complex of an Erdős–Rényi graph that follows $G(n,p)$ is called a random clique complex. The probability distribution is denoted by $C(n,p)$. Whenever $k \geq 1$, the $k$th Betti number of the random clique complex behaves almost unimodally with respect to the parameter $p$, unlike the case of the Linial–Meshulam complex model (see Figure 4 in [14] for a numerical experiment by Afra Zomorodian).

Our first result herein is the counterpart of Theorem 1.2 to the random clique complex model.

**Theorem 1.3.** Let $k \geq 0$ and $c > 0$ be fixed, and let $C_n \sim C(n,p)$ be a random clique complex with $p = (c/n)^{1/(k+1)}$. Then, for any $r \in [1, \infty)$,

$$
\lim_{n \to \infty} \mathbb{E} \left[ \beta_k(C_n) - \frac{c^{k/2}h_k(c)}{(k+1)!} \right] = 0.
$$

Figure 2. Illustration of Theorem 1.3 for $k = 1$.

Whenever $k \geq 1$, the limiting constant $c^{k/2}h_k(c)/(k+1)!$ is unimodal with respect to $c$ as shown in Figure 2. Informally speaking, the unimodality comes from the competitive relationship between the effect of creating $k$-dimensional cycles with some $k$-simplices and that of filling them with some $(k+1)$-simplices with an increasing number of total simplices. These two effects complicate the situation and give rise to the critical difference between the Linial–Meshulam complex model and the random clique complex model as seen in Figures 1 and 2.

Herein, we introduce two properties, namely, homogeneity and spatial independence, that are satisfied with both the Linial–Meshulam complex model and the random clique complex model. As seen in Proposition 3.2, these properties turn out to be closely related to the multi-parameter random simplicial complex model explored in [6], [10]. We then provide a limit theorem for the Betti numbers of homogeneous and spatially independent random simplicial complexes (Theorem 5.1 (1)). This result can be regarded as a law of large numbers for the Betti numbers. Applying the result to the Linial–Meshulam complex model and the random clique complex model, we can obtain Theorems 1.2 and 1.3, respectively, as special cases.

To prove the main theorem, we use the concept of the local weak convergence of simplicial complexes. This concept is a generalization of the Benjamini–Schramm convergence of graphs,
introduced by Benjamini and Schramm [5] and Aldous and Steele [1]. The local weak convergence is critical for estimating the asymptotic behavior of the Betti numbers. This type of approach has been studied in various contexts ([2, 3, 7, 17, 18, 21]). Inspired by those studies, especially the formulation in [18], we establish the local weak limit theorem for homogeneous and spatially independent random simplicial complexes (Theorem 4.1). Consequently, we also obtain the convergence of the empirical spectral distributions of their Laplacians (Theorem 5.1 (2)).

This paper is organized as follows. Section 2 presents some basic concepts related to the cohomology of simplicial complexes and defines the local weak convergence of simplicial complexes. In Section 3, we describe the homogeneity and spatial independence of random simplicial complexes. In Section 4, we discuss the local weak limit theorem for homogeneous and spatially independent random simplicial complexes. Finally, in Section 5 we state the main limit theorem and the empirical spectral distributions of their Laplacians. The proof of the main theorem is presented invoking Section 4.

Notation. We use the Bachmann–Landau big-O/little-o notation and some related notation as $n$ tends to infinity. For non-negative functions $f(n)$ and $g(n)$,

- $f(n) = \Omega(g(n))$ means that $g(n) = O(f(n))$,
- $f(n) = \omega(g(n))$ means that $g(n) = o(f(n))$,
- $f(n) \asymp g(n)$ means that $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$, and
- $f(n) \sim g(n)$ means that $\lim_{n \to \infty} f(n)/g(n) = 1$.

For a random variable $X$ and a probability measure $\nu$, we also use the symbol $\sim$. The notation $X \sim \nu$ indicates that the distribution of $X$ coincides with $\nu$. Given a topological space $S$, we denote by $\mathcal{B}_S$ and $\mathcal{P}_S$ the collection of all Borel sets on $S$ and the set of all Borel probability measures on $S$, respectively. Furthermore, let $C_b(S)$ indicate the set of all bounded continuous real functions on $S$. For $a, b \in \mathbb{R}$, we write $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$.

2. Preliminaries

2.1. Simplicial cohomology

Let $X$ be a collection of finite subsets of a set $V$. $X$ is called a simplicial complex on $V$ if $X$ satisfies the following two conditions: (i) $\{v\} \in X$ for all $v \in V$; (ii) $\sigma \in X$ and $\tau \subset \sigma$ together imply that $\tau \in X$. We often simply say that $X$ is a simplicial complex and let $V(X)$ indicate the vertex set $V$. Note that all simplicial complexes include the empty set. Below, we describe some notation and terminology for a given simplicial complex $X$. Each element $\sigma \in X$ is called a simplex in $X$, and a (strict) subset $\tau$ of $\sigma$ is called a (strict) face of $\sigma$. The dimension of $\sigma \in X$ is defined by $\dim \sigma := |\sigma| - 1$. We call $\sigma \in X$ with $\dim \sigma = k$ a $k$-simplex in $X$. The dimension of $X$, denoted by $\dim X$, is defined as the supremum of the dimensions of the simplices in $X$. For $k \geq -1$, let $F_k(X)$ denote the set of all $k$-simplices in $X$, and set $f_k(X) := |F_k(X)|$. The degree of a $k$-simplex $\tau$ in $X$, denoted by $\deg(X; \tau)$, is defined as the number of $(k + 1)$-simplices in $X$ containing $\tau$. $X$ is said to be locally finite if every nonempty simplex in $X$ has a finite degree. Furthermore, $X$ is said to be finite when $V(X)$ is a finite set. A simplicial complex that is contained in $X$ is called a subcomplex of $X$. Given a simplex $\tau$
in $X$, let $K(\tau)$ denote the subcomplex of $X$ consisting of all the faces of $\tau$. For $k \geq -1$, the $k$-skeleton of $X$ is defined as a subcomplex $X^{(k)} := \bigcup_{i=-1}^{k} F_i(X)$.

Next, we introduce the concepts of the simplicial cohomology. Let $X$ be a simplicial complex on $V$. A sequence $(v_0, v_1, \ldots, v_k) \in V^{k+1}$ is called an ordered $k$-simplex in $X$ if \{v_0, v_1, \ldots, v_k\} $\in F_k(X)$. Let $\Sigma_k(X)$ denote the set of all ordered $k$-simplices in $X$. By convention, we set $\Sigma_{-1}(X) := \{\emptyset\}$. When two ordered simplices can be transformed into each other by an even permutation, they are said to be equivalent. We denote the equivalence class of an ordered simplex $\sigma = (v_0, v_1, \ldots, v_k)$ by $\langle \sigma \rangle$ or $\langle v_0, v_1, \ldots, v_k \rangle$, and call it an oriented $k$-simplex. For $k \geq -1$, a map $\phi: \Sigma_k(X) \to \mathbb{R}$ is called a $k$-cochain of $X$ if $\phi$ is alternating, that is, $\phi((v_{\xi(0)}, v_{\xi(1)}, \ldots, v_{\xi(k)})) = (\text{sgn} \xi) \phi((v_0, v_1, \ldots, v_k))$ for any $(v_0, v_1, \ldots, v_k) \in \Sigma_k(X)$ and permutation $\xi$ on \{0, 1, \ldots, k\}. Let $C^k(X)$ be the $\mathbb{R}$-vector space of all $k$-cochains of $X$. Note that $C^{-1}(X) = \mathbb{R}\{\emptyset\} \cong \mathbb{R}$. For $k \geq -1$, the $k$th coboundary map $d_k: C^k(X) \to C^{k+1}(X)$ is defined as the linear extension of

$$d_k \phi(\sigma) := \sum_{i=0}^{k+1} (-1)^i \phi((v_0, \ldots, \hat{v}_i, \ldots, v_{k+1}))$$  \hspace{1cm} (2.1)$$

for $\phi \in C^k(X)$ and $\sigma = (v_0, v_1, \ldots, v_{k+1}) \in \Sigma_{k+1}(X)$. Here, the hat symbol over $v_i$ indicates that this vertex is deleted from $\sigma$. For $k \geq 0$, define $Z^k(X) := \ker d_k$ and $B^k(X) := \text{Im} d_{k-1}$. A straightforward calculation gives $d_k \circ d_{k-1} = 0$ for all $k \geq 0$, that is, $Z^k(X) \supseteq B^k(X)$. The $k$th cohomology vector space with coefficients in $\mathbb{R}$ is defined as $H^k(X) := Z^k(X)/B^k(X)$. When $X$ is finite, the dimension of $H^k(X)$ is called the $k$th Betti number of $X$, denoted by $\beta_k(X)$.

### 2.2. Rooted spectral measure and empirical spectral distribution

Let $H$ be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$, and let $\| \cdot \|$ be the induced norm. A densely defined symmetric operator $L$ on $H$ is said to be essentially self-adjoint if the closure of $L$ is self-adjoint. Associated with an essentially self-adjoint operator $L$ and $\varphi \in \text{Dom}(L)$ with $\| \varphi \| = 1$ is the spectral measure $\mu_{L, \varphi}$, which is a unique probability measure on $\mathbb{R}$ such that for all $m \in \mathbb{N}$,

$$\langle L^m \varphi, \varphi \rangle = \int_{\mathbb{R}} x^m \, d\mu_{L, \varphi}(x).$$

In the case when $N := \dim H < \infty$, the spectral measure $\mu_{L, \varphi}$ is discrete. Let $\lambda_i$ and $\psi_i$ ($i = 1, 2, \ldots, N$) be the eigenvalues of $L$ and the corresponding orthonormal basis of eigenvectors, respectively. Then, a simple calculation gives

$$\mu_{L, \varphi} = \sum_{i=1}^{N} (\varphi, \psi_i)^2 \delta_{\lambda_i}.$$  \hspace{1cm} (2.2)$$

In such a case, we can also consider the empirical spectral distribution of $L$:

$$\mu_L := \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}.$$  

Now, let $X$ be a locally finite simplicial complex. For $k \geq -1$, we consider the Hilbert space
\[ \ell^2 C^k(X) := \{ \varphi \in C^k(X) \mid \sum_{\sigma \in \Sigma_k(X)} \varphi(\sigma)^2 < \infty \} \] with an inner product

\[ (\varphi, \psi)_k := \frac{1}{(k+1)!} \sum_{\sigma \in \Sigma_k(X)} \varphi(\sigma)\psi(\sigma). \]

By the linear extension of Eq. (2.1), we can consider a densely defined operator \( d^{(2)}_k : \ell^2 C^k(X) \to \ell^2 C^{k+1}(X) \) whose domain includes all \( \varphi \in C^k(X) \) with finite support. The \( k \)th up Laplacian \( L^{up}_k(X) \) on \( \ell^2 C(X) \) is defined by

\[ L^{up}_k(X) := (d^{(2)}_{k+1})^* \circ d^{(2)}_k, \]

where \( (d^{(2)}_{k+1})^* \) is the adjoint operator of \( d^{(2)}_{k+1} \) with respect to the inner products \((\cdot, \cdot)_k\) and \((\cdot, \cdot)_{k+1}\). It is easy to confirm that \( L^{up}_k(X) \) is a densely defined symmetric and non-negative definite operator with respect to the inner product \((\cdot, \cdot)_k\). Now, we consider a pair \((X, \tau)\) of \(X\) and a \(k\)-simplex \(\tau\) in \(X\), namely, a \(k\)-rooted simplicial complex. Let \((e_\tau)_{\tau \in \mathcal{F}_k(X)}\) be a canonical orthonormal basis of \(\ell^2 C^k(X)\). When \(L^{up}_k(X)\) is essentially self-adjoint, the spectral measure associated with \(L^{up}_k(X)\) and \(e_\tau\) is called the rooted spectral measure of \((X, \tau)\), denoted by \(\mu_{(X, \tau)}\). When \(X\) is finite, Eq. (2.2) implies that

\[ \mu_{(X, \tau)} = \sum_{i=1}^{f_k(X)} (e_\tau, \psi_i)_k^2 \delta_{\lambda_i}, \]

where \(\lambda_i\) and \(\psi_i\) \((i = 1, 2, \ldots, f_k(X))\) are the eigenvalues of \(L^{up}_k(X)\) and the corresponding orthonormal basis of eigenvectors, respectively. Therefore, we obtain

\[ \frac{1}{f_k(X)} \sum_{\tau \in \mathcal{F}_k(X)} \mu_{(X, \tau)} = \frac{1}{f_k(X)} \sum_{\tau \in \mathcal{F}_k(X)} \left( \sum_{i=1}^{f_k(X)} (e_\tau, \psi_i)_k^2 \delta_{\lambda_i} \right) = \frac{1}{f_k(X)} \sum_{i=1}^{f_k(X)} \left( \sum_{\tau \in \mathcal{F}_k(X)} (\psi_i, e_\tau)_k^2 \right) \delta_{\lambda_i} = \frac{1}{f_k(X)} \sum_{i=1}^{f_k(X)} \delta_{\lambda_i} = \mu_{L^{up}_k(X)}, \]

which implies that the empirical spectral distribution is the spatial average of all the rooted spectral measures. In other words, the rooted spectral measure can be regarded as the local contribution of the root to the empirical spectral distribution.

### 2.3. Local weak convergence of simplicial complexes

For any \(k\)-rooted simplicial complexes \((X, \tau)\) and \((X', \tau')\), the equivalence \((X, \tau) \simeq (X', \tau')\) means that \((X, \tau)\) and \((X', \tau')\) are root-preserving simplicial isomorphic. The equivalence class of \((X, \tau)\) is denoted by \([X, \tau]\). Given a \(k\)-rooted simplicial complex \((X, \tau)\), we define a nondecreasing sequence \((X_l)_{l=0}^\infty\) of subcomplexes of \(X\) iteratively:

\[ X_0 := K(\tau) \quad \text{and} \quad X_{l+1} := X_l \cup \bigcup_{\sigma \in B_l} K(\sigma) \quad \text{for} \ l \geq 0, \]

where \(B_l\) is the set of all simplices in \(X\) containing at least one \(k\)-simplex in \(X_l\). A simplex \(\sigma\) in \(X\) is said to be of distance \(l\) from \(\tau\) if \(\sigma \in X_l \setminus X_{l-1}\). We additionally set \(X_\infty := \bigcup_{l=0}^\infty X_l\). We
then define $k$-rooted simplicial complexes $(X, \tau)_l := (X_l, \tau)$ for $l \geq 0$, and $X(\tau) = (X_\infty, \tau)$. For the simplicity, let us denote the equivalence class of $(X, \tau)_l$ by $[X, \tau]_l$ for $l \geq 0$, and similarly that of $X(\tau)$ by $X[\tau]$.

Let $\mathcal{S}_k$ denote the set of all equivalence classes $[X, \tau]$ such that $X$ is locally finite and $X(\tau) = (X, \tau)$. We define a metric $d_{\text{loc}}$ on $\mathcal{S}_k$ by letting the distance between $[X_1, \tau_1]$ and $[X_2, \tau_2]$ be $2^{-L}$, where $L$ is the supremum of those $l \in \mathbb{Z}_{\geq 0}$ such that $(X_1, \tau_1)_l \simeq (X_2, \tau_2)_l$. Here, we set $2^{-\infty} = 0$ by convention. This metric is called the local distance, and the convergence with respect to $d_{\text{loc}}$ is called the local convergence. This makes $(\mathcal{S}_k, d_{\text{loc}})$ into a complete and separable metric space. Furthermore, it is easy to verify that $d_{\text{loc}}$ is an ultrametric. We say that a sequence $(\mu_n)_{n=1}^\infty$ in $\mathcal{P}_{\mathcal{S}_k}$ converges weakly to $\mu \in \mathcal{P}_{\mathcal{S}_k}$ if

$$\lim_{n \to \infty} \int_{\mathcal{S}_k} g \, d\mu_n = \int_{\mathcal{S}_k} g \, d\mu$$

for all $g \in C_b(\mathcal{S}_k)$. Since $(\mathcal{S}_k, d_{\text{loc}})$ is a separable ultrametric space, every open set of $\mathcal{S}_k$ is expressed by a disjoint union of a countable number of open balls. Therefore, it is easy to confirm that the weak convergence in $\mathcal{P}_{\mathcal{S}_k}$ is characterized by the convergence of mass on all the open balls as follows.

**Lemma 2.1.** Let $(\mu_n)_{n=1}^\infty$ be a sequence in $\mathcal{P}_{\mathcal{S}_k}$, and let $\mu \in \mathcal{P}_{\mathcal{S}_k}$. Then, $\mu_n$ converges weakly to $\mu$ if and only if, for any $[X, \tau] \in \mathcal{S}_k$ and $r > 0$,

$$\lim_{n \to \infty} \mu_n(B([X, \tau], r)) = \mu(B([X, \tau], r)).$$

Here, $B([X, \tau], r)$ indicates the open ball of radius $r$ centered at $[X, \tau]$.

Furthermore, we are often interested in a sequence of non-rooted finite simplicial complexes. Given a finite simplicial complex $X$ with $\dim X \geq k$, we define a probability measure $\lambda_k(X)$ on $\mathcal{S}_k$ by

$$\lambda_k(X) := \frac{1}{f_k(X)} \sum_{\tau \in F_k(X)} \delta_{X[\tau]}.$$ 

Here, $\delta_{X[\tau]}$ means the Dirac measure at $X[\tau] \in \mathcal{S}_k$. The probability measure $\lambda_k(X) \in \mathcal{P}_{\mathcal{S}_k}$ can be regarded as the distribution of the local structure around a uniformly chosen $k$-simplex in $X$. Using this notation, we define the local weak convergence of finite simplicial complexes.

**Definition 2.2.** Let $(X_n)_{n=1}^\infty$ be a sequence of finite simplicial complexes with $\dim X_n \geq k$, and let $\nu \in \mathcal{P}_{\mathcal{S}_k}$. We say that $X_n$ converges locally weakly to $\nu$ as $n \to \infty$ if $\lambda_k(X_n)$ converges weakly to $\nu$ as $n \to \infty$.

**Remark 2.3.** From Lemma 2.1 and Definition 2.2 is equivalent to stating that for any $[X, \tau] \in \mathcal{S}_k$ and $l \in \mathbb{N}$,

$$\lim_{n \to \infty} \frac{\# \{ \tau \in F_k(X_n) \mid (X_n, \tau)_l \simeq (X, \tau)_l \}}{f_k(X_n)} = \nu(\{ \alpha \in \mathcal{S}_k \mid \alpha_l \simeq (X, \tau)_l \})$$. 


3. Homogeneous and spatially independent random simplicial complex

Throughout this section, let \( n \in \mathbb{N} \) be fixed. Recall that \( \Delta_n = 2^{\lbrack n \rbrack} \) indicates the complete complex on \( \lbrack n \rbrack \). We denote the set of all subcomplexes of \( \Delta_n \) by \( S_n \) and consider \( S_n \)-valued random variables, namely, random subcomplexes of \( \Delta_n \) that are defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\).

3.1. Homogeneity and spatial independence

The permutation group on \( \lbrack n \rbrack \) naturally acts on \( S_n \). Indeed, for any permutation \( g \) on \( \lbrack n \rbrack \) and \( Y = \{\sigma_i\}_i \subseteq S_n \), we define \( gY := \{g(\sigma_i)\}_i \subseteq S_n \), where \( g(\sigma_i) \) denotes the image of \( \sigma_i \subseteq \lbrack n \rbrack \) under \( g \). We say that a random subcomplex \( X \) of \( \Delta_n \) is homogeneous if \( X \) and \( gX \) have the same distribution for any permutation \( g \) on \( \lbrack n \rbrack \). We also say that a random subcomplex \( X \) of \( \Delta_n \) is spatially independent if, for any \( Y_1, Y_2 \in S_n \),

\[
\mathbb{P}(Y_1 \cup Y_2 \subset X)\mathbb{P}(Y_1 \cap Y_2 \subset X) = \mathbb{P}(Y_1 \subset X)\mathbb{P}(Y_2 \subset X). \tag{3.1}
\]

The following lemma gives some characterizations of the spatial independence.

**Lemma 3.1.** For a random subcomplex \( X \) of \( \Delta_n \), the following are equivalent:

1. \( X \) is spatially independent;
2. For any \( Y_1, Y_2 \in S_n \) with \( \mathbb{P}(Y_2 \subset X) > 0 \),
   \[
   \mathbb{P}(Y_1 \subset X \mid Y_2 \subset X) = \mathbb{P}(Y_1 \subset X \mid Y_1 \cap Y_2 \subset X); \tag{3.2}
   \]
3. For any \( Y_1, Y_2 \in S_n \) with \( \mathbb{P}(Y_1 \cap Y_2 \subset X) > 0 \),
   \[
   \mathbb{P}(Y_1 \cup Y_2 \subset X \mid Y_1 \cap Y_2 \subset X) = \mathbb{P}(Y_1 \subset X \mid Y_1 \cap Y_2 \subset X)\mathbb{P}(Y_2 \subset X \mid Y_1 \cap Y_2 \subset X); \tag{3.3}
   \]
4. For any \( Y_1, Y_2 \in S_n \) and \( Z \in S_n \) such that \( Y_1 \cap Y_2 \subset Z \) and \( \mathbb{P}(Z \subset X) > 0 \),
   \[
   \mathbb{P}(Y_1 \cup Y_2 \subset X \mid Z \subset X) = \mathbb{P}(Y_1 \subset X \mid Z \subset X)\mathbb{P}(Y_2 \subset X \mid Z \subset X). \tag{3.4}
   \]

**Proof.** It is easy to verify that (4) \( \Rightarrow \) (3) and that (1), (2), and (3) are equivalent. For (1) \( \Rightarrow \) (4), suppose that \( Y_1, Y_2 \in S_n \) and \( Z \in S_n \) such that \( Y_1 \cap Y_2 \subset Z \) and \( \mathbb{P}(Z \subset X) > 0 \). Since \( Y_1 \cap (Y_2 \cup Z) = Y_1 \cap Z \), the spatial independence of \( X \) implies that

\[
\mathbb{P}(Y_1 \cup (Y_2 \cup Z) \subset X)\mathbb{P}(Y_1 \cap Z \subset X) = \mathbb{P}(Y_1 \subset X)\mathbb{P}(Y_2 \cup Z \subset X).
\]

Furthermore,

\[
\mathbb{P}(Y_1 \cup Z \subset X)\mathbb{P}(Y_1 \cap Z \subset X) = \mathbb{P}(Y_1 \subset X)\mathbb{P}(Z \subset X).
\]

Therefore, noting that \( \mathbb{P}(Y_1 \cap Z \subset X) \geq \mathbb{P}(Z \subset X) > 0 \), we have

\[
\mathbb{P}(Y_1 \cup Y_2 \cup Z \subset X)\mathbb{P}(Z \subset X) = \frac{\mathbb{P}(Y_1 \subset X)\mathbb{P}(Z \subset X)}{\mathbb{P}(Y_1 \cap Z \subset X)}\mathbb{P}(Y_2 \cup Z \subset X) = \mathbb{P}(Y_1 \cap Z \subset X)\mathbb{P}(Y_2 \cup Z \subset X).
\]

The conclusion is obtained by dividing both sides of the above equation by \( \mathbb{P}(Z \subset X)^2 \). \( \square \)
By a simple calculation, Eq. (3.2) implies that for any \( Y \in S_n \) and \( Z_1 \subset Z_2 \subset S_n \) with \( P(Z_2 \subset X) > 0 \),
\[
P(Y \subset X \mid Z_1 \subset X) \leq P(Y \subset X \mid Z_2 \subset X).
\] (3.4)

### 3.2. Multi-parameter random simplicial complex model

The class of homogeneous and spatially independent random subcomplexes of \( \Delta_n \) is closely related to the multi-parameter random simplicial complex model introduced in [6], [10]. Let \( p = (p_0, p_1, \ldots, p_{n-1}) \) be a multi-parameter with \( p_i \in [0, 1] \) for every \( i = 0, 1, \ldots, n - 1 \). We start with vertex set \([n]\) and retain each vertex independently with probability \( p_0 \). Each edge with both end points retained appears independently with probability \( p_1 \). Iteratively, for \( i = 2, 3, \ldots, n-1 \), each \( i \)-simplex in \( \Delta_n \) for which all strict faces were included before the current step appears independently with probability \( p_i \). The resulting random simplicial complex is called a multi-parameter random simplicial complex with multi-parameter \( p = (p_0, p_1, \ldots, p_{n-1}) \). We denote its probability distribution by \( X(n, p) \). Let \( X \sim X(n, p) \) be a multi-parameter random simplicial complex, and let \( Y \in S_n \) be fixed. A nonempty simplex \( \sigma \) in \( \Delta_n \) is called an external simplex in \( Y \) if \( \sigma \not\in Y \) and \( \partial \sigma \subset Y \). Here, \( \partial \sigma \) indicates the simplicial complex consisting of all the strict faces of \( \sigma \). Let \( E_k(Y) \) indicate the set of all external \( k \)-simplices in \( Y \), and set \( e_k(Y) := \#(E_k(Y)) \). Then,
\[
P(X = Y) = \prod_{i=0}^{n-1} P(X^{(i)} = Y^{(i)} \mid X^{(i-1)} = Y^{(i-1)}) = \prod_{i=0}^{n-1} p_i^{f_i(Y)}(1 - p_i)^{e_i(Y)}.
\] (3.5)

The homogeneity of \( X \) follows from this equation. Furthermore, \( X \) is spatially independent. Indeed, noting that
\[
P(Y \subset X) = \prod_{i=0}^{n-1} P(Y^{(i)} \subset X^{(i)} \mid Y^{(i-1)} \subset X^{(i-1)}) = \prod_{i=0}^{n-1} p_i^{f_i(Y)},
\]
we have
\[
P(Y_1 \cup Y_2 \subset X)P(Y_1 \cap Y_2 \subset X) = \prod_{i=0}^{n-1} p_i^{f_i(Y_1 \cup Y_2) + f_i(Y_1 \cap Y_2)}
= \prod_{i=0}^{n-1} p_i^{f_i(Y_1) + f_i(Y_2)} = P(Y_1 \subset X)P(Y_2 \subset X)
\]
for any \( Y_1, Y_2 \in S_n \). In fact, the multi-parameter random simplicial complex model is characterized by the homogeneity and spatial independence as follows.

**Proposition 3.2.** All multi-parameter random simplicial complexes are homogeneous and spatially independent. Moreover, for any homogeneous and spatially independent random subcomplex \( X \) of \( \Delta_n \), there exists \( p = (p_0, p_1, \ldots, p_{n-1}) \) such that \( X \sim X(n, p) \).

**Proof.** The first conclusion is clear from the discussion above. Hence, for the second conclusion, let \( X \) be a homogeneous and spatially independent random subcomplex of \( \Delta_n \). We define
a parameter $p = (p_0, p_1, \ldots, p_{n-1})$ by

$$p_k := \begin{cases} 
  \mathbb{P}([k+1] \in X \mid \partial[k+1] \subset X) & \text{if } \mathbb{P}(\partial[k+1] \subset X) > 0, \\
  0 & \text{otherwise.}
\end{cases}$$

Let $Y \in S_n$ be fixed. Then,

$$\mathbb{P}(X = Y) = \prod_{i=0}^{n-1} \mathbb{P}(X^{(i)} = Y^{(i)} \mid X^{(i-1)} = Y^{(i-1)})$$

$$= \prod_{i=0}^{n-1} \mathbb{P} \left( \bigcap_{\sigma \in F_i(Y)} \{ \sigma \in X \} \cap \bigcap_{\tau \in E_i(Y)} \{ \tau \notin X \} \middle| X^{(i-1)} = Y^{(i-1)} \right). \quad (3.6)$$

Now, let $0 \leq i \leq n-1$ be fixed. Noting that

$$\{X^{(i-1)} = Y^{(i-1)}\} = \{Y^{(i-1)} \subset X\} \cap \bigcap_{\tau \in \Delta_{n}^{(i-1)} \setminus Y} \{\tau \notin X\},$$

we obtain

$$\mathbb{P} \left( \bigcap_{\sigma \in F_i(Y)} \{ \sigma \in X \} \cap \bigcap_{\tau \in E_i(Y)} \{ \tau \notin X \} \cap \{X^{(i-1)} = Y^{(i-1)}\} \right)$$

$$= \mathbb{P} \left( \left\{ Y^{(i-1)} \subset X \right\} \cap \bigcap_{\sigma \in F_i(Y)} \{ \sigma \in X \} \cap \bigcap_{\tau \in E_i(Y) \cup \{ \Delta_n^{(i-1)} \setminus Y \}} \{\tau \notin X\} \right)$$

$$= e_i(Y) \#(\Delta_{n}^{(i-1)} \setminus Y) \sum_{k=0}^{e_i(Y)} \sum_{l=0}^{\#(\Delta_{n}^{(i-1)} \setminus Y)} (-1)^{k+l} \sum_{\#S = k} \sum_{\#T = l} p(S, T). \quad (3.7)$$

Here, $p(S, T)$ is defined by

$$p(S, T) := \mathbb{P} \left( \left\{ Y^{(i-1)} \subset X \right\} \cap \bigcap_{\sigma \in F_i(Y)} \{ \sigma \in X \} \cap \bigcap_{\tau \in S \cup T} \{\tau \in X\} \right)$$

$$= \mathbb{P} \left( \bigcap_{\sigma \in F_i(Y) \cup S} \{ \sigma \in X \} \cap \{Y^{(i-1)} \subset X\} \cap \bigcap_{\tau \in T} \{\tau \in X\} \right)$$

$$= \mathbb{P} \left( \bigcup_{\sigma \in F_i(Y) \cup S} K(\sigma) \cup \left( Y^{(i-1)} \cup \bigcup_{\tau \in T} K(\tau) \right) \subset X \right).$$

Now, we set

$$Z := \bigcup_{\sigma \in F_i(Y) \cup E_i(Y)} \partial \sigma.$$
Then, for all $S \subset E_i(Y)$ and $T \subset \Delta_{n}^{(i-1)} \setminus Y$,

$$
\bigcup_{\sigma \in F_i(Y) \cup S} K(\sigma) \cap \left( Y^{(i-1)} \cup \bigcup_{\tau \in T} K(\tau) \right) \subset Z.
$$

Whenever $P(Z \subset X) > 0$, by applying Eq. (3.3) iteratively,

$$
P \left( \bigcup_{\sigma \in F_i(Y) \cup S} K(\sigma) \cup \left( Y^{(i-1)} \cup \bigcup_{\tau \in T} K(\tau) \right) \subset X \bigg| Z \subset X \right) = \prod_{\sigma \in F_i(Y) \cup S} P(K(\sigma) \subset X \big| Z \subset X) P \left( \left( Y^{(i-1)} \cup \bigcup_{\tau \in T} K(\tau) \right) \subset X \bigg| Z \subset X \right) = \frac{1}{P(Z \subset X)} \prod_{\sigma \in F_i(Y) \cup S} P(K(\sigma) \subset X \big| \partial \sigma \subset X) P \left( \left( Y^{(i-1)} \cup \bigcup_{\tau \in T} K(\tau) \right) \subset X \bigg| Z \subset X \right)
$$

$$
= \frac{p_i^{f_i(Y)} + \#S}{P(Z \subset X)} P \left( \left\{ Y^{(i-1)} \subset X \right\} \cap \bigcap_{\tau \in T} \left\{ \tau \in X \right\} \right).
$$

In the fourth line, we use Eq. (3.2) and $Z \subset Y^{(i-1)}$. Thus,

$$
p(S, T) = p_i^{f_i(Y)} + \#S \frac{P \left( \left\{ Y^{(i-1)} \subset X \right\} \cap \bigcap_{\tau \in T} \left\{ \tau \in X \right\} \right)}{P(Z \subset X)}.
$$

When $P(Z \subset X) = 0$, Eq. (3.8) is easily verified from $Z \subset Y^{(i-1)}$. Combining Eqs. (3.7) and (3.8) gives

$$
P \left( \bigcap_{\sigma \in F_i(Y)} \left\{ \sigma \notin X \right\} \cap \bigcap_{\sigma \in E_i(Y)} \left\{ \sigma \notin X \right\} \cap \left\{ X^{(i-1)} = Y^{(i-1)} \right\} \right)
$$

$$
= \sum_{k=0}^{e_i(Y)} \sum_{l=0}^{\#(\Delta_{n}^{(i-1)} \setminus Y)} (-1)^{k+l} \sum_{S \subset E_i(Y)} \sum_{T \subset \Delta_{n}^{(i-1)} \setminus Y} P^{f_i(Y) + k} \left( \left\{ Y^{(i-1)} \subset X \right\} \cap \bigcap_{\tau \in T} \left\{ \tau \in X \right\} \right)
$$

$$
= p_i^{f_i(Y)} (1 - p_i)^{e_i(Y)} \sum_{l=0}^{\#(\Delta_{n}^{(i-1)} \setminus Y)} (-1)^{l} \sum_{T \subset \Delta_{n}^{(i-1)} \setminus Y} P \left( \left\{ Y^{(i-1)} \subset X \right\} \cap \bigcap_{\tau \in T} \left\{ \tau \notin X \right\} \right)
$$

$$
= p_i^{f_i(Y)} (1 - p_i)^{e_i(Y)} P(X^{(i-1)} = Y^{(i-1)}).
$$
From Eq. (3.6), we obtain
\[ P(X = Y) = \prod_{i=0}^{n-1} p_i^{e(Y)}(1 - p_i)^{e_i(Y)}, \]
which implies that the distribution of $X$ is identical to $X(n, p)$ from Eq. (3.5).

From Proposition 3.2, we can consider various examples of homogeneous and spatially independent random subcomplexes of $\Delta_n$ by choosing the multi-parameter $p$ of the multi-parameter random simplicial complex model.

**Example 3.3 (d-Linial–Meshulam complex).** Let $1 \leq d < n$ and $p \in [0, 1]$ be fixed. We define $p = (p_0, p_1, \ldots, p_{n-1})$ by
\[ p_i := \begin{cases} 1 & (0 \leq i \leq d - 1), \\ p & (i = d), \\ 0 & (d + 1 \leq i \leq n - 1). \end{cases} \]
The corresponding random simplicial complex is a $d$-Linial–Meshulam complex that follows $Y_d(n, p)$. When $d = 1$, this can be regarded as an Erdős–Rényi graph that follows $G(n, p)$.

**Example 3.4 (Random $d$-clique complex).** Let $1 \leq d < n$ and $p \in [0, 1]$ be fixed. We define $p = (p_0, p_1, \ldots, p_{n-1})$ by
\[ p_i := \begin{cases} 1 & (0 \leq i \leq d - 1), \\ p & (i = d), \\ 1 & (d + 1 \leq i \leq n - 1). \end{cases} \]
The corresponding random simplicial complex is called a random $d$-clique complex (see [22] and [12, Example 3.2]). We denote its probability distribution by $C_d(n, p)$. When $d = 1$, we obtain a random clique complex that follows $C(n, p)$.

4. Local weak limit theorem for random simplicial complexes

4.1. Statement of the result

In this section, we consider homogeneous and spatially independent random subcomplexes of $\Delta_n$, and we study their local weak convergence as $n$ tends to infinity. To state the theorem, we describe a higher-dimensional generalization of the Galton–Watson tree with Poisson offspring distribution (see also [3, Section 3] and [18, Section 3]).

For $k \geq 0$, a $k$-rooted simplicial complex $(T, \tau)$ is called a $k$-rooted tree if $T$ can be constructed by the following process: we start with $K(\tau)$; at each step $l = 0, 1, 2, \ldots$, to every $k$-simplex $\tau'$ of distance $l$ from $\tau$, we pick a non-negative number $m(\tau')$ of the new vertices $v_1, v_2, \ldots, v_{m(\tau')}$, and add the $(k + 1)$-simplices $\tau' \cup \{v_1\}, \tau' \cup \{v_2\}, \ldots, \tau' \cup \{v_{m(\tau')}\}$ to the simplicial complex constructed before the current step. A simplicial complex $T$ is called a $(k + 1)$-tree if $(T, \tau)$ is a $k$-rooted tree for some $k$-simplex $\tau$ in $T$.

When we sample each number $m(\tau')$ in the generative process of a $k$-rooted tree from the Poisson distribution $\mathcal{P}_c$ with parameter $c \geq 0$ independently of any others, the resulting object
Here, $U \subset \mathcal{P}$ is an independent random subcomplex of the main result in this section.

Next, we introduce two types of parameters for a given homogeneous and spatially independent random subcomplex $X_n$ of $\Delta_n$. For $k \geq -1$,

$$q_k := \mathbb{P}(\tau \in X_n) \text{ and } r_k := \begin{cases} \mathbb{P}(\sigma \in X_n \mid \tau \in X) & (q_k > 0), \\ 0 & (q_k = 0). \end{cases}$$

(4.1)

Here, $\tau$ and $\sigma$ are arbitrary fixed $k$- and $(k+1)$-simplices in $\Delta_n$, respectively, such that $\tau \subset \sigma$. The parameters $q_k$ and $r_k$ are well-defined from the homogeneity of $X_n$. It is easy to confirm that $q_{k+1} = q_k r_k$ for $k \geq -1$ and that $q_k$ is nonincreasing with respect to $k$. The following local weak limit theorem for homogeneous and spatially independent random subcomplexes of $\Delta_n$ is the main result in this section.

**Theorem 4.1.** Let $k \geq 0$ and $c > 0$ be fixed, and let $X_n$ be a homogeneous and spatially independent random subcomplex of $\Delta_n$. If $n^{k+1} q_k = \omega(1)$ and $n r_k \sim c$, then for any open set $U \subset \mathcal{P}$, such that $\nu_k(c) \in U$,

$$\lim_{n \to \infty} \mathbb{P}(\lambda_k(X_n) \in U \mid \dim X_n \geq k) = 1.$$

In other words, $X_n$ under $\mathbb{P}(\cdot \mid \dim X_n \geq k)$ converges locally weakly to $\nu_k(c)$ in distribution as $n \to \infty$.

We devote the rest of this section to proving Theorem 4.1. The proof of Theorem 4.1 is based on an exploration of a given simplicial complex from a selected simplex. The exploration is a higher-dimensional analog of the breadth-first traversal of a given graph. In what follows in this section, let $k \geq 0$ be fixed, and let $X_n$ be a homogeneous and spatially independent random subcomplex of $\Delta_n$.

### 4.2. Breadth-first traversal of simplicial complexes

Given a subcomplex $X$ of $\Delta_n$ and a $k$-simplex $\tau$ in $X$, we start traversing $X$ layer-wise from $\tau$. More precisely, for each step $i \geq 0$, we build a $(k+1)$-tree $T_i$, a current $k$-simplex $\tau_i$, and a bijective map $\varphi_i$ from a subset $W_i$ of $\mathbb{N}^j := \{\emptyset\} \cup \bigcup_{m=1}^{\infty} \mathbb{N}^m$ to $F_k(T_i)$, iteratively. Here, we equip $\mathbb{N}^j$ with a total order as follows: for two elements $\mathbf{i} = (i_1, i_2, \ldots, i_m)$ and $\mathbf{i}' = (i'_1, i'_2, \ldots, i'_m)$, we set $\mathbf{i} < \mathbf{i}'$ if $m < m'$ or if $m = m'$ and there exists $l = 1, 2, \ldots, m$ such that $(i_1, i_2, \ldots, i_{l-1}) = (i'_1, i'_2, \ldots, i'_{l-1})$ and $i_l < i'_l$. Furthermore, we equip $F_k(\Delta_n)$ and $F_{k+1}(\Delta_n)$ with arbitrary total orders in advance to carry out the following procedure uniquely. We start with $T_0 := K(\tau)$, $\varphi_0(\emptyset) := \tau$ with $W_0 = \{\emptyset\}$, and $\tau_1 := \tau$. For each step $i \geq 1$, we define $M_i := \{\sigma \in F_{k+1}(X) \mid K(\sigma) \cap T_{i-1} = K(\tau_i)\}$. If $m_i := \#M_i = 0$, then set $T_i := T_{i-1}$ and $\varphi_i := \varphi_{i-1}$. Otherwise, arrange $M_i = \{\sigma_1, \sigma_2, \ldots, \sigma_{m_i}\}$ in ascending order and define

$$T_i := T_{i-1} \cup \bigcup_{j=1}^{m_i} K(\sigma_j).$$

Furthermore, for each $1 \leq j \leq m_i$, let $\rho_{j,1}, \rho_{j,2}, \ldots, \rho_{j,k+1}$ be the ascending order of the $k$-dimensional faces of $\sigma_j$, distinct from $\tau_i$. Then, we extend $\varphi_{i-1}$ to $\varphi_i$ such that $\varphi_{i-1}^{-1}(\rho_{j,l}) := \ldots$
then we define
\[
\tau_{i+1} := \min(F_k(T_i) \setminus \{\tau_1, \tau_2, \ldots, \tau_i\}),
\]
where the minimum is taken with respect to the total order in \(F_k(T_i)\) induced from \((W_i, \prec)\) under \(\varphi_i\). Otherwise, this process stops, and we encode \(I := i\).

We extend the sequences \((m_i)_{i=1}^\infty, (T_i)_{i=1}^\infty, (\tau_i)_{i=1}^\infty\) by \(m_i := m_I = 0, T_i := T_I, \) and \(\tau_i := \tau_I\) for \(i > I\), respectively. Moreover, we define a filtration
\[
\mathcal{F}_i := \sigma(T^{(n)}_j; 1 \leq j \leq i).
\]
Clearly, \(X(\tau)\) is a \(k\)-rooted tree if and only if \(c_i = 0\) for \(i \geq 1\). We often denote \(m_i, T_i, \tau_i, I, \) and \(c_i\) by \(m_i(X, \tau), T_i(X, \tau), \tau_i(X, \tau), I(X, \tau), \) and \(c_i(X, \tau)\), respectively, to indicate the explored simplicial complex and the initial \(k\)-simplex.

4.3. Estimates on the breadth-first traversal of random simplicial complexes

For \(\tau \in F_k(\Delta_n)\) with \(\mathbb{P}(\tau \in X_n) > 0\), we define a probability space \((\Omega_\tau, \mathcal{F}_\tau, \mathbb{P}_\tau)\) by
\[
\Omega_\tau := \{\tau \in X_n\}, \quad \mathcal{F}_\tau := \{B \in \mathcal{F} \mid B \subset \Omega_\tau\}, \quad \text{and} \quad \mathbb{P}_\tau(\cdot) := \mathbb{P}(\cdot \mid \Omega_\tau).
\]
The expectation with respect to \(\mathbb{P}_\tau\) is denoted by \(\mathbb{E}_\tau\). In what follows in this subsection, let \(\tau\) be a fixed \(k\)-simplex in \(\Delta_n\) with \(\mathbb{P}(\tau \in X_n) > 0\). We can carry out the breadth-first traversal of \(X_n\) from \(\tau\) given \(\tau\) appearing in \(X_n\), and obtain \(m_i(X_n, \tau), T_i(X_n, \tau), \tau_i(X_n, \tau), I(X_n, \tau), \) and \(c_i(X_n, \tau)\) that are defined on the probability space \((\Omega_\tau, \mathcal{F}_\tau, \mathbb{P}_\tau)\). For the simplicity, we denote \(m_i(Xn, \tau), T_i(X_n, \tau), \tau_i(X_n, \tau), I(X_n, \tau), \) and \(c_i(X_n, \tau)\) by \(m_i^{(n)}, T_i^{(n)}, \tau_i^{(n)}, I(n), \) and \(c_i^{(n)}\), respectively. Moreover, we define a filtration \(\mathcal{F}^{(n)} := (\mathcal{F}_{i+1}^{(n)})_{i=1}^\infty\) on \((\Omega_\tau, \mathcal{F}_\tau, \mathbb{P}_\tau)\) by
\[
\mathcal{F}_{i+1}^{(n)} := \sigma(T_i^{(n)}; 1 \leq j \leq i).
\]
It is easy to confirm that \((m_i^{(n)})_{i=1}^\infty\) is \(\mathcal{F}^{(n)}\)-adapted and that \(I(n)\) is an \(\mathcal{F}^{(n)}\)-stopping time.

To provide some estimates on the breadth-first traversal of \(X_n\), we additionally define
\[
s_k := \begin{cases} 
\mathbb{P}(\tau_1 \cup \tau_2 \in X_n \mid \tau_1, \tau_2 \in X_n) & \text{if } \mathbb{P}(\tau_1, \tau_2 \in X_n) > 0, \\
0 & \text{otherwise.}
\end{cases}
\]
Here, \(\tau_1\) and \(\tau_2\) are arbitrary fixed \(k\)-simplices in \(\Delta_n\) such that \(\dim(\tau_1 \cap \tau_2) = k - 1\). This is also well-defined from the homogeneity of \(X_n\). Whenever \(q_k > 0\), Eq. \(1.3\) implies that \(\mathbb{P}(\tau_1, \tau_2 \in X_n) = q_k^2/q_{k-1} > 0\). Therefore, we have \(s_k = q_{k+1}/(q_k^2/q_{k-1}) = r_k/r_{k-1} - 1\).

Lemma 4.2. Let \(T \subset \Delta_n\) be a \((k+1)\)-tree with \(\mathbb{P}(T \subset X_n) > 0\), and let \(\sigma_1, \sigma_2, \ldots, \sigma_i \in F_{k+1}(\Delta_n) \setminus T\). Then, \(\mathbb{P}(\sigma_j \notin X_n \text{ for all } j = 1, 2, \ldots, i \mid T \subset X_n) \geq 1 - is_k\).

Proof. For each \(1 \leq j \leq i\), there exist two distinct \(k\)-dimensional faces \(\tau_j, \tau_j'\) of \(\sigma_j\) such that \(K(\sigma_j) \cap T \subset K(\tau_j) \cup K(\tau_j')\). Therefore, we obtain
\[
\mathbb{P}(\sigma_j \in X_n \text{ for some } j = 1, 2, \ldots, i \mid T \subset X_n)
\]
Moreover, if \( T \subset \Delta_n \) be a \((k+1)\)-tree such that \( f_k(T) \geq i + 1 \). Provided that \( \Pr(T_i^{(n)} = T) > 0 \) and \( is_k < 1 \), for any \( v \in [n] \) with \( \tau_{i+1}(T, \tau) \cup \{v\} \notin T \),

\[
\Pr(\tau_{i+1}(T, \tau) \cup \{v\} \in X_n \mid T_i^{(n)} = T) \leq \frac{sk}{1 - is_k}. \tag{4.2}
\]

Moreover, if \( v \notin V(T) \), then

\[
(1 - is_k)r_k \leq \Pr(\sigma \in X_n \mid T_i^{(n)} = T) \leq \frac{r_k}{1 - is_k}. \tag{4.3}
\]

Furthermore, the events \( \{(\tau_{i+1}(T, \tau) \cup \{w\} \in X_n)\}_{w \in [n] \setminus V(T)} \) are independent under \( \Pr(\cdot \mid T_i^{(n)} = T) \).

**Proof.** For \( w \in [n] \), define an event \( E_w \) by

\[
E_w := \bigcup_{1 \leq j \leq i} \{\tau_j(T, \tau) \cup \{w\} \in X_n \setminus T\}.
\]

Suppose that \( v \in [n] \) with \( \sigma := \tau_{i+1}(T, \tau) \cup \{v\} \notin T \), and define

\[
E := (E_v)^c \text{ and } F := \bigcap_{w \in [n] \setminus \{v\}} (E_w)^c.
\]

Note that \( \{T_i^{(n)} = T\} = \{T \subset X_n\} \cap E \cap F \). Because the event \( F \) is independent of both \( E \) and \( \{\sigma \in X_n\} \cap E \) under \( \Pr(\cdot \mid T \subset X_n) \) from Proposition 3.2 we obtain

\[
\Pr(\sigma \in X_n \mid T_i^{(n)} = T) = \frac{\Pr((\sigma \in X_n) \cap E \cap F \mid T \subset X_n)}{\Pr(E \cap F \mid T \subset X_n)}
= \frac{\Pr((\sigma \in X_n) \cap E \cap F \mid T \subset X_n)}{\Pr(E \cap F \mid T \subset X_n)}
= \frac{\Pr((\sigma \in X_n) \cap E \mid T \subset X_n)}{\Pr(E \mid T \subset X_n)}. \tag{4.4}
\]

By applying Lemma 4.2 to the numerator and denominator in the last line above, we have

\[
\Pr((\sigma \in X_n) \cap E \mid T \subset X_n) \leq \Pr(\sigma \in X_n \mid T \subset X_n) \leq sk \quad \text{(4.5)}
\]
Provided that $P$, this completes the proof.

respectively. Combining Eqs. (4.4), (4.5), and (4.6), we obtain Eq. (4.2).

Now, suppose $e \notin V(T)$. Then,

$$P(\{\sigma \in X_n \} \cap E \mid T \subset X_n) = \frac{P(E \mid \sigma \in X_n, T \subset X_n)P(\sigma \in X_n \mid T \subset X_n)}{P(E \mid \sigma \in X_n, T \subset X_n)P(\sigma \in X_n \mid T \subset X_n)} \quad \text{(from Eq. (3.2))}$$

$$= \frac{P(E \mid \sigma \in X_n, T \subset X_n)\tau_{i+1}(T, \tau) \in X_n)}{P(E \mid \sigma \in X_n, T \subset X_n)r_k} \in [(1 - is_k), r_k]. \quad \text{(4.7)}$$

In the last line above, we use Lemma 4.2 noting that $K(\sigma) \cup T$ is also a $(k+1)$-tree. Eq. (4.3) follows from Eqs. (4.4), (4.6), and (4.7).

Lastly, we consider the independence of the events $\{\tau_{i+1}(T, \tau) \cup \{w\} \in X_n\}_{w \in [n] \setminus V(T)}$.

Again note that $T^{(n)} = T = \{T \subset X_n \} \cap \bigcap_{x \in [n]}(E_x)^c$. Then, from Proposition 3.2 we obtain

$$\mathbb{P}_T \left( \bigcap_{w \in [n] \setminus V(T)} \{\tau_{i+1}(T, \tau) \cup \{w\} \in X_n \mid T^{(n)} = T \} \right)$$

$$= \prod_{w \in [n] \setminus V(T)} \mathbb{P}_T \left( \left\{ \tau_{i+1}(T, \tau) \cup \{w\} \in X_n \right\} \cap \bigcap_{x \in [n]}(E_x)^c \mid T \subset X_n \right)$$

$$= \prod_{w \in [n] \setminus V(T)} \mathbb{P}_T \left( \left\{ \tau_{i+1}(T, \tau) \cup \{w\} \in X_n \right\} \cap \bigcap_{x \in [n]}(E_x)^c \mid T \subset X_n \right)$$

This completes the proof.

Next, we give some estimates on $m_{i+1}^{(n)}$ and $c_{i+1}^{(n)}$ under $\mathbb{P}_T(\cdot \mid T^{(n)} = T)$. Let $\mu_{i+1}^{(n)}$ denote the distribution of $m_{i+1}^{(n)}$ under $\mathbb{P}_T(\cdot \mid T^{(n)} = T)$.

**Proposition 4.4.** Let $i \in \mathbb{N}$ be fixed, and let $T \subset \triangle_n$ be a $(k+1)$-tree such that $f_k(T) \geq i+1$. Provided that $\mathbb{P}_T(T^{(n)} = T) > 0$ and $is_k < 1$,

$$\mathbb{E}_T \left[ m_{i+1}^{(n)} \mid T^{(n)} = T \right] \leq \frac{nr_k}{1 - is_k}$$
Therefore, 

\[ d_{TV}(\mu^{(n)}_{i+1}, P_{(n-k-1)r_k}) \leq \frac{r_k}{1 - is_k} + (f_0(T) - k - 1)r_k. \]

Here, \(d_{TV}\) indicates the total variation distance between two probability measures on \(\mathbb{Z}_{\geq 0}\). Furthermore,

\[ \mathbb{E}_T [c^{(n)}_{i+1} | T_{i}^{(n)} = T] \leq \frac{f_0(T) - k - 1}{1 - is_k} s_k. \]

**Proof.** For \(v \in [n] \setminus V(T)\), we define \(\xi_v := 1_{\{\tau_{i+1}(T, \tau) \cup \{v\} \in X_n\}}\). From Lemma 4.3 and the homogeneity of \(X_n\), the random variables \((\xi_v)_{v \in [n] \setminus V(T)}\) are independent and identically distributed under \(P_T(\cdot | T_{i}^{(n)} = T)\). Moreover, \(p := \mathbb{E}_T [\xi_v | T_{i}^{(n)} = T] \in [(1 - is_k)r_k, (1 - is_k)^{-1}r_k]\). Therefore, \(m_{i+1}^{(n)}\) follows the binomial distribution under \(P_T(\cdot | T_{i}^{(n)} = T)\):

\[ m_{i+1}^{(n)} = \sum_{v \in [n] \setminus V(T)} \xi_v \sim \text{Bin}(n - f_0(T), p). \]

This immediately implies the first conclusion. Furthermore, from some estimates on the total variation distance (see, e.g., [4, Theorem 1] and [20, Formula (5)])], we have

\[ d_{TV}(\text{Bin}(n - f_0(T), p), P_{(n-f_0(T))p}) \leq p \leq \frac{r_k}{1 - is_k} \]

and

\[ d_{TV}(P_{(n-f_0(T))r_k}, P_{(n-k-1)r_k}) \leq (n - k - 1)r_k - (n - f_0(T))r_k = (f_0(T) - k - 1)r_k. \]

Thus,

\[
\begin{align*}
d_{TV}(\mu^{(n)}_{i+1}, P_{(n-k-1)r_k}) &\leq d_{TV}(\text{Bin}(n - f_0(T), p), P_{(n-f_0(T))p}) + d_{TV}(P_{(n-f_0(T))r_k}, P_{(n-k-1)r_k}) \\
&\leq \frac{r_k}{1 - is_k} + (f_0(T) - k - 1)r_k.
\end{align*}
\]

Lastly, again from Lemma 4.3 we have

\[ \mathbb{E}_T [c^{(n)}_{i+1} | T_{i}^{(n)} = T] = \sum_{v \in V(T) \cap \tau_{i+1}(T, \tau)} P_T(\tau_{i+1}(T, \tau) \cup \{v\} \in X_n \setminus T | T_{i}^{(n)} = T) \\
\leq \frac{f_0(T) - k - 1}{1 - is_k} s_k. \]

These estimates complete the proof. \(\square\)

Let \(i \geq 0\) with \(is_k < 1\). From Proposition 4.4, we have

\[ \mathbb{E}_T [m_{i+1}^{(n)} | F_{i}^{(n)}] = \mathbb{E}_T [m_{i+1}^{(n)} 1_{\{f_k(T_{i}^{(n)} \geq i+1)\}} | F_{i}^{(n)}] = \sum_T \mathbb{E}_T [m_{i+1}^{(n)} | T_{i}^{(n)} = T] 1_{\{T_{i}^{(n)} = T\}} \]

\[ \leq \frac{nr_k}{1 - is_k}. \]
The summation in the second line is taken over all \((k+1)\)-trees \(T\) in \(\triangle_n\) such that \(f_k(T) \geq i + 1\) and \(\mathbb{P}_\tau(T_i^{(n)} = T) > 0\). Therefore,

\[
\mathbb{E}_\tau[m_i^{(n)}] = \mathbb{E}_\tau[\mathbb{E}_\tau[m_i^{(n)} | T_i^{(n)}]] \leq \frac{inr_k}{1 - is_k}.
\]

Since \(f_0(T_i^{(n)}) = k + 1 + \sum_{j=0}^{i-1} m_j^{(n)}\), we obtain

\[
\mathbb{E}_\tau[f_0(T_i^{(n)})] = k + 1 + \sum_{j=0}^{i-1} \mathbb{E}_\tau[m_j^{(n)}] \leq k + 1 + \frac{inr_k}{1 - is_k}.
\] (4.8)

Furthermore,

\[
\mathbb{E}_\tau[c_i^{(n)} | T_i^{(n)}] = \mathbb{E}_\tau[c_i^{(n)} 1_{\{f_k(T_i^{(n)}) \geq i+1\}} | T_i^{(n)}] = \sum_T \mathbb{E}_\tau[c_i^{(n)} | T_i^{(n)} = T] 1_{\{T_i^{(n)} = T\}} \\
\leq \sum_T \frac{f_0(T) - k - 1}{1 - is_k} s_k 1_{\{T_i^{(n)} = T\}} \text{ (from Proposition 4.4)} \\
\leq \frac{\mathbb{E}[f_0(T_i^{(n)})] - k - 1}{1 - is_k} s_k \\
\leq \frac{inr_k s_k}{(1 - is_k)^2} \text{ (from Eq. (4.8)).}
\]

The summations in the second and third lines are also taken over all \((k+1)\)-trees \(T\) in \(\triangle_n\) such that \(f_k(T) \geq i + 1\) and \(\mathbb{P}(T_i^{(n)} = T) > 0\). Therefore,

\[
\mathbb{E}_\tau[c_i^{(n)}] = \mathbb{E}_\tau[\mathbb{E}_\tau[c_i^{(n)} | T_i^{(n)}]] \leq \frac{inr_k s_k}{(1 - is_k)^2}.
\] (4.9)

Now, we define a nondecreasing sequence \((I(n;l))_{l=0}^\infty\) of \(\mathcal{F}^{(n)}\)-stopping times by

\[
I(n;l) := \begin{cases} 0 & (l = 0), \\
\sup\{1 \leq i \leq I(n) | \text{the distance of } \tau_i^{(n)} \text{ from } \tau \text{ is less than } l\} & (l \geq 1). 
\end{cases}
\]

Let us denote \(c_n = (n - k - 1)r_k\) and set \(f_k(\alpha) := f_k(T)\) for any \(k\)-rooted tree \(\alpha = (T, \tau)\).

**Lemma 4.5.** Let \(l \in \mathbb{N}\) be fixed, and let \(\alpha\) be a \(k\)-rooted tree with \(f_k(\alpha) r_k < 1\). Then,

\[
\mathbb{P}_\tau((T_i^{(n)}(l), \tau) \simeq \alpha_l) - \mathbb{P}((\mathcal{P}_i(c_n), \tau_o)_l \simeq \alpha_l) \leq \frac{1 + f_k(\alpha_l)nir_k/2}{1 - f_k(\alpha_l)s_k} f_k(\alpha_l)r_k.
\]

**Proof.** For each \(i \geq 0\), we define a new random \(k\)-rooted tree \((\tilde{T}_i^{(n)}, \tau)\) as follows: we start with \((T_i^{(n)}, \tau)\); for each \(j = i+1, i+2, \ldots, f_k(T_i^{(n)})\), we attach a \(k\)-rooted Poisson tree with parameter \(c_n\) as a branch rooted at \(\tau_j^{(n)}\). Let \(Z_j\) denote the degree of the root in the \(k\)-rooted Poisson tree. We may assume that each \(k\)-rooted Poisson tree is defined on the same probability space \((\Omega, \mathcal{F}, \mathbb{P})\) to be independent of \(X_n\) and other trees. Note that \((\tilde{T}_0^{(n)}, \tau)\) is a \(k\)-rooted Poisson tree.
Furthermore, a simple calculation implies that
\[
\left| \mathbb{P}_\tau \left( (\tilde T_{i+1}^{(n)}, \tau) \simeq \alpha_l \right) - \mathbb{P}_\tau \left( (\tilde T_0^{(n)}, \tau) \simeq \alpha_l \right) \right| \\
= \left| \mathbb{P}_\tau \left( (\tilde T_{i}^{(n)}, \tau) \simeq \alpha_l \right) - \mathbb{P}_\tau \left( (\tilde T_0^{(n)}, \tau) \simeq \alpha_l \right) \right| \\
\leq \sum_{h=0}^{l-1} \sum_{i=I(h)}^{I(h+1)-1} \left| \mathbb{P}_\tau \left( (\tilde T_i^{(n)}, \tau) \simeq \alpha_l \right) - \mathbb{P}_\tau \left( (\tilde T_0^{(n)}, \tau) \simeq \alpha_l \right) \right| \\
\leq \sum_{h=0}^{l-1} \sum_{i=I(h)}^{I(h+1)-1} \left| \mathbb{P}_\tau \left( (\tilde T_i^{(n)}, \tau) \simeq \alpha_{h+1} \right) - \mathbb{P}_\tau \left( (\tilde T_0^{(n)}, \tau) \simeq \alpha_{h+1} \right) \right|. \tag{4.10}
\]

In the last line above, we use the fact that
\[
\mathbb{P}_\tau \left( (\tilde T_{i+1}^{(n)}, \tau) \simeq \alpha_l \mid (\tilde T_i^{(n)}, \tau) \simeq \alpha_{h+1} \right) = \mathbb{P}_\tau \left( (\tilde T_i^{(n)}, \tau) \simeq \alpha_l \mid (\tilde T_i^{(n)}, \tau) \simeq \alpha_{h+1} \right).
\]

Now, we estimate the summand in the last line of Eq. (4.10) for a fixed $0 \leq h \leq l - 1$ and $I(h) \leq i < I(h+1)$.

Let $T_i^{(n)}$ be the set of all $(k+1)$-trees $T$ in $\Delta_n$ such that $\mathbb{P}_\tau(T_i^{(n)}) = T > 0$. For $T \in T_i^{(n)}$ and $(m_1, \ldots, m_{I(h+1)-1}) \in \mathbb{Z}_{\geq 0}^{I(h+1)-1}$, define a new $(k+1)$-tree $T(m_1, \ldots, m_{I(h+1)-1})$ as follows: for each $j = 1, 2, \ldots, I(h+1)-i$, we pick $m_j$ numbers of new vertices $v_{i1}^{(j)}, \ldots, v_{i1}^{(j)}$ and add $(k+1)$-simplices $\tau_{i+j}(T, \tau) \cup \{v_{i1}^{(j)}\}, \ldots, \tau_{i+j}(T, \tau) \cup \{v_{m_j}^{(j)}\}$ to $T$. Furthermore, we define
\[
\mathcal{M}(T, \alpha_{h+1}) := \left\{ (m_1, \ldots, m_{I(h+1)-1}) \in \mathbb{Z}_{\geq 0}^{I(h+1)-1} \mid (T(m_1, \ldots, m_{I(h+1)-1}), \tau) \simeq \alpha_{h+1} \right\}
\]
for $T \in T_i^{(n)}$. Then, we have
\[
\mathbb{P}_\tau \left( (\tilde T_{i}^{(n)}, \tau) \simeq \alpha_{h+1} \right) \\
= \sum_{T \in T_i^{(n)}} \mathbb{P}_\tau \left( (m_{i+1}^{(n)}, \tau_{i+2}^{(n)}, \ldots, \tau_{I(h+1)}^{(n)}) \in \mathcal{M}(T, \alpha_{h+1}) \mid T_i^{(n)} = T \right) \mathbb{P}_\tau \left( T_i^{(n)} = T \right) \tag{4.11}
\]
and
\[
\mathbb{P}_\tau \left( (\tilde T_{i}^{(n)}, \tau) \simeq \alpha_{h+1} \right) \\
= \sum_{T \in T_i^{(n)}} \mathbb{P}_\tau \left( (Z_{i+1}, \tau_{i+2}, \ldots, \tau_{I(h+1)}) \in \mathcal{M}(T, \alpha_{h+1}) \mid T_i^{(n)} = T \right) \mathbb{P}_\tau \left( T_i^{(n)} = T \right) \\
= \sum_{T \in T_i^{(n)}} \mathbb{P}_{\alpha_{h+1}}^{\cup I(h+1)-i}(\mathcal{M}(T, \alpha_{h+1})) \mathbb{P}_\tau \left( T_i^{(n)} = T \right). \tag{4.12}
\]

Furthermore, a simple calculation implies that
\[
\left| \mathbb{P}_\tau \left( (m_{i+1}^{(n)}, \tau_{i+2}, \ldots, \tau_{I(h+1)}) \in \mathcal{M}(T, \alpha_{h+1}) \mid T_i^{(n)} = T \right) - \mathbb{P}_{\alpha_{h+1}}^{\cup I(h+1)-i}(\mathcal{M}(T, \alpha_{h+1})) \right| \\
\leq \sup_{A \in \mathcal{G}_{\geq 2}^{I(h+1)-1}} \left| \mathbb{P}_\tau \left( (m_{i+1}^{(n)}, \tau_{i+2}, \ldots, \tau_{I(h+1)}) \in A \mid T_i^{(n)} = T \right) - \mathbb{P}_{\alpha_{h+1}}^{\cup I(h+1)-i}(A) \right| \\
\leq \sup_{A \in \mathcal{G}_{\geq 2}^{I(h+1)-1}} \left| \mathbb{P}_\tau \left( m_{i+1}^{(n)} \in A \mid T_i^{(n)} = T \right) - \mathbb{P}_{\alpha_{h+1}}(A) \right|.
\]
Combining Eqs. (4.11), (4.12), and (4.13), we have

\[ |\mathbb{P}_\tau((\tilde{T}^{(n)}_{i+1}, \tau)_{h+1} \simeq \alpha_{h+1}) - \mathbb{P}_\tau((\tilde{T}^{(n)}_{i}, \tau)_{h+1} \simeq \alpha_{h+1})| \leq \sum_{T \in T_{i}^{(n)}} ((1 - is_k)^{-1} + f_0(T) - k - 1)r_k \mathbb{P}_\tau(T_{i}^{(n)} = T) \]

\[ = ((1 - is_k)^{-1} + \mathbb{E}_\tau[f_0(T_{i}^{(n)})] - k - 1)r_k \leq \frac{1 + inr_k}{1 - is_k}r_k \quad \text{(from Eq. (4.8)).} \quad (4.14) \]

Thus, from Eqs. (4.10) and (4.14), we obtain

\[ |\mathbb{P}_\tau((T_{I_{(n,l)}}^{(n)}, \tau) \simeq \alpha_l) - \mathbb{P}((PT_k(c_n), \tau_o)_{l} \simeq \alpha_l)| \leq \sum_{h=0}^{l-1} \sum_{i=I(l)}^{I(h+1)-1} \frac{1 + inr_k}{1 - is_k}r_k \]

\[ = \sum_{i=0}^{l-1} \frac{1 + inr_k}{1 - is_k}r_k \]

\[ \leq \frac{1 + I(l)n r_k / 2}{1 - I(l)s_k} I(l)r_k, \]

completing the proof.

PROPOSITION 4.6. Let \( l \in \mathbb{N} \) be fixed, and let \( \alpha \) be a \( k \)-rooted tree with \( f_k(\alpha) s_k < 1 \). Then,

\[ |\mathbb{P}_\tau((X_n, \tau)_{l} \simeq \alpha_l) - \mathbb{P}((PT_k(c_n), \tau_o)_{l} \simeq \alpha_l)| \leq \frac{f_k(\alpha_l)^2 nr_k s_k}{2(1 - f_k(\alpha_l) s_k)^2} + \frac{1 + f_k(\alpha_l) n r_k / 2}{1 - f_k(\alpha_l) s_k} f_k(\alpha_l) r_k. \]

PROOF. We define \( Q_n := \Omega_{\tau} \cap \{ c_i^{(n)} = 0 \text{ for all } i = 1, 2, \ldots, I(n; l) \} \). Note that \((X_n, \tau)_{l} = (T_{I_{(n,l)}}^{(n)}, \tau) \) given the event \( Q_n \). Thus, we have

\[ |\mathbb{P}_\tau((X_n, \tau)_{l} \simeq \alpha_l) - \mathbb{P}_\tau((T_{I_{(n,l)}}^{(n)}, \tau) \simeq \alpha_l)| \leq \mathbb{P}_\tau((X_n, \tau)_{l} \simeq \alpha_l) \setminus Q_n) \cup \mathbb{P}_\tau((T_{I_{(n,l)}}^{(n)}, \tau) \simeq \alpha_l) \setminus Q_n) \]

\[ \leq \mathbb{P}_\tau((I(n; l) \leq f_k(\alpha_l)) \setminus Q_n) \]

\[ \leq \mathbb{P}_\tau\left( \sum_{i=0}^{f_k(\alpha_l)-1} c_i^{(n)} \geq 1 \right) \]

\[ \leq \sum_{i=0}^{f_k(\alpha_l)-1} \mathbb{E}_\tau\left[ c_i^{(n)} \right] \quad \text{(from Chebyshev's inequality)} \]

\[ \leq \sum_{i=0}^{f_k(\alpha_l)-1} \frac{in r_k s_k}{(1 - is_k)^2} \quad \text{(from Eq. (4.9))} \]

\[ \leq \frac{f_k(\alpha_l)^2 n r_k s_k}{2(1 - f_k(\alpha_l) s_k)^2}. \quad (4.15) \]
Combining Eq. (4.15) with Lemma 4.5 we obtain
\[ |P_{\tau}(X_n, \tau) - \alpha| \leq |P_{\tau}(X_n, \tau) - \frac{1}{n} \mathbb{P}(|\mathbb{P}_k(c_n), \tau| \leq \alpha)| \]
\[ \leq \frac{f_k(\alpha)^2 n r_k s_k}{2(1 - f_k(\alpha) s_k)^2} + \frac{1 + f_k(\alpha) n r_k/2}{1 - f_k(\alpha) s_k} f_k(\alpha) r_k. \]

4.4. Proof of Theorem 4.1

The following lemma gives some fundamental relations among \( q_k, r_k, \) and \( s_k. \)

**Lemma 4.7.** The following (1), (2), and (3) hold.

1. \( q_k^{i+1} \geq q_k^{i+1} \) for \( 0 \leq i \leq k. \) In particular, \( n^{k+1} q_k = \omega(1) \) implies that \( n^{i+1} q_i = \omega(1). \)
2. \( r_k \geq q_k s_k^{k+1}. \) In particular, \( n q_0 = \omega(1) \) and \( n r_k \sim 1 \) together imply that \( s_k = o(1). \)
3. \( q_k s_k^{k+1} \geq r_k^{k+1}. \) In particular, \( s_k = o(1) \) and \( n r_k \sim 1 \) together imply that \( n^{k+1} q_k = \omega(1). \)

Furthermore, if \( n r_k \sim 1 \), then the following three conditions are equivalent: \( n^{k+1} q_k = \omega(1); n q_0 = \omega(1); s_k = o(1). \)

**Proof.** When \( q_k = 0, \) the conclusions are trivial because \( q_k = 0 \) implies that \( r_k = s_k = 0. \)

Hence, we may assume \( q_0 \geq q_1 \geq \cdots \geq q_k > 0. \) Since
\[ \frac{(n^{i+1} q_i) (n^i q_{i-1})}{(n^{i+1} q_i)^2} = \frac{r_i}{r_{i-1}} = s_i \leq 1 \quad \text{for } 0 \leq i \leq k, \]
the sequence \( (n^{i+1} q_i)_{-i \leq i \leq k} \) is log-concave. Therefore, noting that \( q_{-1} = 1, \) we have \( n^{i+1} q_i \geq (n^{k+1} q_k)^{(i+1)/(k+1)} \) for \( 0 \leq i \leq k, \) which completes (1). Furthermore, \( r_{-1} = q_0 \) and \( r_k \) is nonincreasing with respect to \( k, \) so
\[ q_k = q_k r_k - 1 = q_k - 2 q_k - 2 q_{k-1} - 3 q_k - 2 q_{k-1} - 4 q_k - 2 q_{k-1} - \cdots - q_k q_{k-2} q_k q_{k-1} \geq (r_k-1)^{k+1}. \]

Thus, we obtain \( q_k s_k^{k+1} \geq (r_{k-1} s_k)^{k+1} = r_k^{k+1}, \) which corresponds with (3). For (2), we additionally define
\[ t_i = \mathbb{P}(\tau_1 \cup \tau_2 \cup \tau_3 \in X_n \mid \tau_1, \tau_2, \tau_3 \in X_n) \quad \text{for } 1 \leq i \leq k. \]

Here, \( \tau_1, \tau_2, \) and \( \tau_3 \) are arbitrary fixed i-simplices in \( \Delta_n \) such that \( \dim(\tau_1 \cap \tau_2) = \dim(\tau_2 \cap \tau_3) = \dim(\tau_3 \cap \tau_1) = i - 1 \) and \( \dim(\tau_1 \cap \tau_2 \cap \tau_3) = i - 2. \) Then, Eq. (3.1) implies that \( \mathbb{P}(\tau_1, \tau_2, \tau_3 \in X_n) = q_1 q_2 q_3^{q_1 q_2 - 1} q_3^{q_1 q_2 - 1} q_3^{q_1 q_2 - 1} \) for \( 1 \leq i \leq k. \) Therefore, we have \( t_i = q_i \mathbb{P}(\tau_i) \mathbb{P}(\tau_{i+1} \setminus \tau_i) = r_i r_{i-1}^{-1} = s_i / s_{i-1}, \) which implies that \( s_i \) is nonincreasing with respect to \( i. \) Thus,
\[ r_k = r_{k-1} s_k = r_{k-2} s_{k-1} s_k = \cdots = q_0 s_0 s_1 \cdots s_k \geq q_0 s_k^{k+1}, \]
which corresponds with (2). The last conclusion of the theorem follows immediately from (1), (2), and (3).

**Proposition 4.8.** Let \( c > 0 \) and \( \tau \in F_k(\Delta_n) \) be fixed. If \( n^{k+1} q_k = \omega(1) \) and \( n r_k \sim c, \) then \( X_n[\tau] \) under \( \mathbb{P}_\tau \) converges to \( \mathbb{P}_k(c), \tau \) in distribution as \( n \to \infty. \)
Proof. Let \([\alpha] \in S_k\) and \(l \in \mathbb{N}\). From Lemma 2.1 it suffices to prove that
\[
\lim_{n \to \infty} P_{\tau}((X_n, \tau)_l \simeq \alpha_l) = P'((PT_k(c), \tau_0)_l \simeq \alpha_l).
\]
We may assume \(s_k f_k(\alpha_l) < 1\) from Lemma 4.7. Then, Proposition 4.6 implies that
\[
|P_{\tau}((X_n, \tau)_l \simeq \alpha_l) - P'((PT_k(c_n), \tau_0)_l \simeq \alpha_l)| \leq \frac{f_k(\alpha_l)^2 n r_k s_k}{2(1 - f_k(\alpha_l)s_k)^2} + \frac{1 + f_k(\alpha_l)n r_k/2}{1 - f_k(\alpha_l)s_k} f_k(\alpha_l)r_k.
\]
Since \(nr_k \sim c\) and \(s_k = o(1)\), the right-hand side converges to zero as \(n \to \infty\). Furthermore, noting that \(\lim_{n \to \infty} c_n = c\), we have
\[
\lim_{n \to \infty} P'((PT_k(c_n), \tau_0)_l \simeq \alpha_l) = P'((PT_k(c), \tau_0)_l \simeq \alpha_l).
\]
These estimates complete the proof.

We can also prove the two-root version of Proposition 4.8 in the same manner. To state the proposition, for \(\tau \neq \tau' \in F_k(\triangle_n)\) such that \(P(\tau, \tau' \in X) > 0\), we define a probability space \((\Omega_{\tau, \tau'}, F_{\tau, \tau'}, P_{\tau, \tau'})\) by
\[
\Omega_{\tau, \tau'} := \{\tau, \tau' \in X\}, F_{\tau, \tau'} := \{B \in F \mid B \subset \Omega_{\tau, \tau'}\}, \text{ and } P_{\tau, \tau'}(\cdot) := P(\cdot \mid \Omega_{\tau, \tau'}).
\]
The expectation with respect to \(P_{\tau, \tau'}\) is denoted by \(E_{\tau, \tau'}\). For two disjoint \(k\)-simplices \(\tau, \tau'\) in \(\triangle_n\), we carry out each breadth-first traversal of \(X_n\) from \(\tau\) and \(\tau'\) alternately, avoiding each other. We can carefully modify the estimates in Section 4.3 for the two-root version of breadth-first traversal, and we can confirm that for any \([\alpha], [\beta] \in S_k\) and \(l, m \in \mathbb{N}\),
\[
\lim_{n \to \infty} P_{\tau, \tau'}((X_n, \tau)_l \simeq \alpha_l, (X_n, \tau')_m \simeq \beta_m) = P'((PT_k(c), \tau_0)_l \simeq \alpha_l)P'((PT_k(c), \tau_0)_m \simeq \beta_m).
\]
The above equation yields the following proposition.

Proposition 4.9. Let \(c > 0\), and let \(\tau, \tau' \in F_k(\triangle_n)\) be fixed to be disjoint. If \(n^{k+1}d_k = \omega(1)\) and \(nr_k \sim c\), then \((X_n[\tau], X_n[\tau'])\) under \(P_{\tau, \tau'}\) converges to \((|PT_k(c, \tau_0)|, |PT_k(c, \tau'_0)|)\) in distribution as \(n \to \infty\). Here, \(|PT_k(c, \tau'_0)|\) is an independent copy of \(|PT_k(c, \tau_0)|\).

We now turn to proving Theorem 4.1. The following lemma states that the number of simplicies in \(X_n\) is concentrated around its mean.

Lemma 4.10. Provided that \(n^{k+1}d_k = \omega(1)\), it holds that for any \(r \in [1, \infty)\),
\[
\lim_{n \to \infty} \mathbb{E} \left[ \frac{f_k(X_n)}{n^{k+1}d_k} - \frac{1}{(k+1)!} \right]^r = 0.
\]
In particular, \(\lim_{n \to \infty} P(f_k(X_n) > 0) = 1\).

Proof. It suffices to prove that for all \(m \in \mathbb{N}\),
\[
\lim_{n \to \infty} \mathbb{E} \left[ \left( \frac{f_k(X_n)}{n^{k+1}d_k} \right)^m \right] = \left( \frac{1}{(k+1)!} \right)^m.
\]
Indeed, if it holds, then for any \( r \in [1, \infty) \),
\[
\mathbb{E}
\left[
\left| f_k(X_n) - \frac{1}{n^{k+1}q_k} \right|^{2r} \right]^{\frac{1}{r}} \leq \mathbb{E}
\left[
\left| f_k(X_n) - \frac{1}{n^{k+1}q_k} \right|^{2r} \right] \quad \text{(from Hölder’s inequality)}
\]
\[
= \sum_{m=0}^{2[r]} \binom{2[r]}{m} \mathbb{E}
\left[
\left| f_k(X_n) \right|^m \right] \left( \frac{1}{(k+1)!} \right)^{2[r]-m}
\xrightarrow{n \to \infty} 0.
\]

Now, for \( m \in \mathbb{N} \) and \( 0 \leq l \leq m \), define
\[
x_{m,l} := \sum_{\tau_1, \ldots, \tau_m} P(\tau_1, \tau_2, \ldots, \tau_m \in X_n),
\]
where the summation is taken over all \( \tau_1, \ldots, \tau_m \in F_k(\triangle_n) \) such that for each \( h = l+1, \ldots, m \), the simplex \( \tau_h \) is disjoint from all others. Clearly, \( x_{m,l} \) is nondecreasing with respect to \( l \). Furthermore, we have
\[
\mathbb{E}
\left[
\left| f_k(X_n) \right|^m \right] = \frac{1}{(n^{k+1}q_k)^m} \mathbb{E}
\left[
\left| \sum_{\tau \in F_k(\triangle_n)} 1(\tau \in X_n) \right|^m \right]
\]
\[
= \frac{1}{(n^{k+1}q_k)^m} \sum_{\tau_1, \ldots, \tau_m \in F_k(\triangle_n)} P(\tau_1, \tau_2, \ldots, \tau_m \in X_n)
\]
\[
= \frac{x_{m,m}}{(n^{k+1}q_k)^m}.
\]
Hence, we prove that for all \( m \in \mathbb{N} \),
\[
\lim_{n \to \infty} \frac{x_{m,m}}{(n^{k+1}q_k)^m} = \left( \frac{1}{(k+1)!} \right)^m. \tag{4.16}
\]
We use an inductive argument on \( m \in \mathbb{N} \). When \( m = 1 \), the conclusion is trivial. Assume that Eq. (4.16) holds up to \( m-1 \) for some \( m \geq 2 \). For each \( 0 \leq l \leq m-1 \), we have
\[
x_{m,l+1} - x_{m,l}
\]
\[
= \sum_{i=1}^{k+1} \sum_{\tau_1, \ldots, \tau_m} P(\tau_1, \tau_2, \ldots, \tau_m \in X_n)
\]
\[
= \sum_{i=1}^{k+1} \sum_{\tau_1, \ldots, \tau_m} P(\tau_1, \tau_2, \ldots, \tau_i \in X_n) P(\tau_{i+1} \in X_n \mid \tau_1, \tau_2, \ldots, \tau_i \in X_n) q_k^{m-l-1}
\]
\[
\leq \sum_{i=1}^{k+1} \sum_{\tau_1, \ldots, \tau_m} P(\tau_1, \tau_2, \ldots, \tau_i \in X_n) (q_k/q_{i-1}) q_k^{m-l-1}
\]
\[
\leq \sum_{i=1}^{k+1} \binom{(k+1)l}{i} \binom{n}{k+1} \binom{n}{k+1} q_k^{m-l-1} \sum_{\tau_1, \ldots, \tau_i \in F_k(\triangle_n)} P(\tau_1, \tau_2, \ldots, \tau_i \in X_n) q_k^{m-l-1} q_{i-1}
\]
\[
\leq ((k+1)l)^{k+1} (n^{k+1}q_k)^{m-l} x_{l,l} \sum_{i=1}^{k+1} \frac{1}{m! q_{i-1}}.
\]
Here, the summations in the second, third, and fourth lines are taken over all \(24\ S\). Kanazawa

Furthermore,

\[
\lim_{n \to \infty} \frac{x_{m.0}}{(n^{k+1}q_k)^m} = \lim_{n \to \infty} \frac{n!}{(n+1)!} \frac{1}{n^m (n+1)^m} = \left( \frac{1}{k+1} \right)^m.
\]

Thus, we obtain

\[
\lim_{n \to \infty} \frac{x_{m,m}}{(n^{k+1}q_k)^m} = \lim_{n \to \infty} \sum_{l=0}^{m-1} \frac{x_{m,l+1} - x_{m,l}}{(n^{k+1}q_k)^m} = \left( \frac{1}{k+1} \right)^m.
\]

Finally, we move on to proving Theorem 4.1. Let us denote \(\nu g = \int_{\mathcal{S}_k} g \, d\nu\) for \(g \in C_b(\mathcal{S}_k)\) and \(\nu \in \mathcal{P}_{\mathcal{S}_k}\) by convention.

**Proof of Theorem 4.1.** It suffices to prove that for any \(g \in C_b(\mathcal{S}_k)\),

\[
\lim_{n \to \infty} \mathbb{E}[\lambda_k(X_n)g \mid \dim X_n \geq k] = \nu_k(c)g
\]

and

\[
\lim_{n \to \infty} \mathbb{E}[(\lambda_k(X_n)g)^2 \mid \dim X_n \geq k] = (\nu_k(c)g)^2.
\]

Indeed, Eqs. (4.17) and (4.18) together imply that

\[
\lim_{n \to \infty} \mathbb{E}[(\lambda_k(X_n)g - \nu_k(c)g)^2 \mid \dim X_n \geq k] = 0.
\]

In particular, \(\lambda_k(X_n)g\) under \(P(- \mid \dim X_n \geq k)\) converges to \(\nu_k(c)g\) in distribution as \(n \to \infty\) for any \(g \in C_b(\mathcal{S}_k)\), which is equivalent to the conclusion (see, e.g., \([15, \text{Theorem 4.11}]\)). Hence, suppose that \(g \in C_b(\mathcal{S}_k)\). We define a finite measure on \(\mathcal{S}_k\) by

\[
\tilde{\lambda}_k(X_n) := \frac{1}{n^{k+1}q_k} \sum_{\rho \in F_k(X_n)} \delta_{X[\rho]}.
\]

We begin by considering Eq. (4.17). Let \(\tau \in F_k(\Delta_n)\) be arbitrarily fixed. Then, for any \(A \in \mathcal{B}_{\mathcal{S}_k}\), we have

\[
(\mathbb{E}\tilde{\lambda}_k(X_n))(A) = \frac{1}{n^{k+1}q_k} \sum_{\rho \in F_k(\Delta_n)} \mathbb{P}(\rho \in X_n, X_n[\rho] \in A) = \mathbb{P}(\tau \in X_n, X_n[\tau] \in A)/q_k
\]

(from the homogeneity of \(X_n\))

which implies that \(\mathbb{E}[\tilde{\lambda}_k(X_n)g] = (\mathbb{E}\tilde{\lambda}_k(X_n))g = \mathbb{E}_\tau[g(X_n[\tau])]\). Therefore, Proposition 4.8
yields
\[
\lim_{n \to \infty} \mathbb{E}[\tilde{\lambda}_k(X_n)g] = \mathbb{E}'[g([\text{PT}_k(c), \tau_0])] = \nu_k(c)g.
\] (4.19)

Furthermore, we have
\[
|\mathbb{E}[\lambda_k(X_n)g \mid \dim X_n \geq k] - \mathbb{E}[\tilde{\lambda}_k(X_n)g]|
= \mathbb{E}\left[\left(-\frac{1}{(n+1)q_k} - \frac{1}{(n+1)q_k}\right) \sum_{\rho \in F_k(X_n)} g(X_n[\rho])\right]
\leq \|g\|_\infty \mathbb{E}\left[\left(-\frac{1}{(n+1)q_k} - \frac{f_k(X_n)}{(n+1)q_k}\right)\right]
\leq \|g\|_\infty \left(\mathbb{E}\left[\left(-\frac{1}{(n+1)q_k} - 1\right)\right] + \mathbb{E}\left[\frac{f_k(X_n)}{(n+1)q_k} - 1\right]\right).
\]

Here, \(\|g\|_\infty\) indicates the supremum norm of \(g\). From Lemma \ref{lem:4.10} the last line above converges to zero as \(n \to \infty\). Thus, combining this estimate with Eq. (4.19) yields Eq. (4.17).

Next, we consider Eq. (4.18). Let \(\tau, \tau' \in F_k(\Delta_n)\) be arbitrarily fixed to be disjoint. From the homogeneity of \(X_n\), we have
\[
\mathbb{E}[(\tilde{\lambda}_k(X_n)g)^2]
= \frac{1}{(n+1)^2 q_k^2} \mathbb{E}\left[\left(\sum_{\rho \in F_k(\Delta_n)} 1_{\{\rho \in X_n\}} g(X_n[\rho])\right)^2\right]
= \frac{1}{(n+1)^2 q_k^2} \sum_{i=0}^{k+1} \sum_{\rho, \rho' \in F_k(\Delta_n) \#(\rho \cap \rho') = i} \mathbb{E}[1_{\{\rho, \rho' \in X_n\}} g(X_n[\rho])g(X_n[\rho'])]
= \frac{(n-k-1)}{(n+1)} E_{\tau, \tau'}[g(X_n[\tau])g(X_n[\tau'])] + \sum_{i=1}^{k+1} \frac{1}{(n+1) q_{i-1}} \sum_{\rho, \rho' \in F_k(\Delta_n) \#(\rho \cap \rho') = i} \mathbb{E}_{\rho, \rho'}[g(X_n[\rho])g(X_n[\rho'])].
\] (4.20)

Proposition \ref{prop:4.19} implies that
\[
\lim_{n \to \infty} E_{\tau, \tau'}[g(X_n[\tau])g(X_n[\tau'])] = \mathbb{E}'[g([\text{PT}_k(c), \tau_0])]^2 = (\nu_k(c)g)^2.
\]

Therefore, the first term of Eq. (4.20) converges to \((\nu_k(c)g)^2\) as \(n \to \infty\). Furthermore, for each \(1 \leq i \leq k+1\),
\[
\frac{1}{(n+1)^2 q_{i-1}} \sum_{\rho, \rho' \in F_k(\Delta_n) \#(\rho \cap \rho') = i} \mathbb{E}_{\rho, \rho'}[g(X_n[\rho])g(X_n[\rho'])] \leq \frac{\binom{k+1}{i} (n-k-1+i)}{(n+1) q_{i-1}} \leq \frac{\binom{k+1}{i} n^{k+1}||g||_2^2}{(n+1)^2 q_{i-1}}.
\]

From Lemma \ref{lem:4.7}(1), the right-hand side of the above equation converges to zero as \(n \to \infty\).
Thus, we obtain
\[
\lim_{n \to \infty} \mathbb{E}[(\lambda_k(X_n))^2] = (\nu_k(c))^2.
\] (4.21)

We also have
\[
|\mathbb{E}[(\lambda_k(X_n))^2 \mid \dim X_n \geq k] - \mathbb{E}[(\tilde{\lambda}_k(X_n))^2]| \\
\leq \|g\|_\infty^2 \mathbb{E}\left[\frac{1}{f_k(X_n)^2} \mathbb{P}(f_k(X_n) > 0) \right] - \mathbb{E}\left[\left(\sum_{\nu \in \mathcal{P}_k(X_n)} g(X_n[\nu])\right)^2\right] \\
\leq \|g\|_\infty^2 \mathbb{E}\left[\left(\sum_{\nu \in \mathcal{P}_k(X_n)} g(X_n[\nu])\right)^2\right] - \mathbb{E}\left[\left(\sum_{\nu \in \mathcal{P}_k(X_n)} g(X_n[\nu])\right)^2\right] \\
\leq \|g\|_\infty^2 \mathbb{E}\left[\left(\sum_{\nu \in \mathcal{P}_k(X_n)} g(X_n[\nu])\right)^2\right] - \mathbb{E}\left[\left(\sum_{\nu \in \mathcal{P}_k(X_n)} g(X_n[\nu])\right)^2\right].
\]

From Lemma 4.10, the last line above converges to zero as \( n \to \infty \). Thus, combining this estimate with Eq. (4.21) yields Eq. (4.18).

5. Convergence of Betti numbers and empirical spectral distributions

5.1. Statement of the result

In this section, we consider homogeneous and spatially independent random subcomplexes of \( \Delta_n \) and study the asymptotic behavior of their Betti numbers and the empirical spectral distributions of their Laplacians as \( n \) tends to infinity. Recall the definitions of the parameters \( q_k \) and \( r_k \) as described in Eq. (1.1). The following theorem is the main result in this section.

**Theorem 5.1.** Let \( k \geq 0 \) and \( c > 0 \) be fixed, and let \( X_n \) be a homogeneous and spatially independent random subcomplex of \( \Delta_n \). If \( n^{k+1} q_k = \omega(1) \) and \( nr_k \sim c \), then the following (1) and (2) hold.

1. For any \( r \in [1, \infty) \),
\[
\lim_{n \to \infty} \mathbb{E}\left[\frac{\beta_k(X_n)}{n^{k+1} q_k} - \frac{h_k(c)}{(k+1)!}\right]^r = 0.
\]

2. There exists a deterministic probability measure \( \mu \in \mathcal{P}_\mathbb{R} \) such that for any open set \( U \subset \mathcal{P}_\mathbb{R} \) such that \( \mu \in U \),
\[
\lim_{n \to \infty} \mathbb{P}(\mu_{L_k^{nv}(X_n)} \in U \mid \dim X_n \geq k) = 1.
\]

In other words, \( \mu_{L_k^{nv}(X_n)} \) under \( \mathbb{P}(\cdot \mid \dim X_n \geq k) \) converges to \( \mu \) in distribution as \( n \to \infty \).

We apply Theorem 5.1 (1) to several typical random simplicial complex models.

**Example 5.2** (d-Linial–Meshulam complex). Let \( d \in \mathbb{N} \) and \( c > 0 \) be fixed. Consider a \( d \)-Linial–Meshulam complex \( Y_n \sim Y_d(n, p) \) with \( p \sim c/n \). Letting \( k = d-1 \), we have \( q_k = 1 \) and \( r_k = p \) for \( n \geq d+1 \) (cf. Example 3.3). Then, we obtain \( n^{k+1} q_k = n^d = \omega(1) \) and \( nr_k = np \sim c \).
Therefore, Theorem 5.1 (1) implies that for any \( r \in [1, \infty) \),
\[
\lim_{n \to \infty} \mathbb{E} \left[ \left| \frac{\beta_{d-1}(Y_n)}{n^d} - \frac{h_{d-1}(c)}{d!} \right|^r \right] = 0.
\]
This implies Theorem 1.2.

**Example 5.3 (Random \( d \)-clique complex).** Let \( d \in \mathbb{N}, k \geq d - 1, \) and \( c > 0 \) be fixed. Consider the random \( d \)-clique complex \( C_n \sim C_d(n, p) \) with \( p \sim (c/n)^{1/(k+1)} \). Note that \( q_k = p^{(k+1)} \) and \( r_k = p^{(k+1)} \) for \( n \geq d + 1 \) (cf. Example 3.4). Here, \( (d_{d+1}) = 0 \) by convention. Then, we obtain
\[
n^{k+1}q_k = n^{k+1}p^{(k+1)} = n^{(k+2)d/(d+1)}c^{(k+1-d)/(d+1)} = \mathcal{O}(1) \quad \text{and} \quad nr_k = np^{(k+1)} \sim c.
\]
Therefore, Theorem 5.1 (1) implies that for any \( r \in [1, \infty) \),
\[
\lim_{n \to \infty} \mathbb{E} \left[ \left| \frac{\beta_k(C_n)}{n^{(k+2)d/(d+1)}} - \frac{c^{(k+1-d)/(d+1)}h_k(c)}{(k+1)!} \right|^r \right] = 0.
\]
When \( d = 1 \), this corresponds to Theorem 1.3.

Theorem 4.1 is critical for the proof of Theorem 5.1. Our approach is essentially according to the idea used in [17, 18] for the proof of Theorem 1.2. In what follows, we always fix \( k \geq 0 \) and \( c > 0 \), and we let \( X_n \) be a homogeneous and spatially independent random simplicial complex of \( \Delta_n \) such that \( n^{k+1}q_k = \mathcal{O}(1) \) and \( nr_k \sim c \).

### 5.2. Proof of Theorem 5.1 (2) and upper estimate of the Betti number

For proving Theorem 5.1 (2), we introduce some additional notation. Let \( \mathcal{S}_k^t \) denote the set of all \( [X, \tau] \in \mathcal{S}_k \) such that \( L^{\nu}(X) \) is an essentially self-adjoint operator. It is a fact that, \( \mathbb{P}' \)-almost surely, \( [\mathbb{P}^t_k(c), \tau_0] \in \mathcal{S}_k^t \) (see, e.g., [17, Claim 3.3]). We define a kernel \( M_k = \mathcal{S}_k^t \times \mathcal{B}_R \to [0, 1] \) by \( M_k([X, \tau], B) := \mu_{[X, \tau]}(B) \) for \( [X, \tau] \in \mathcal{S}_k^t \) and \( B \in \mathcal{B}_R \). In fact, \( \mathcal{S}_k^t \ni [X, \tau] \mapsto M_k([X, \tau], \cdot) \in \mathbb{P}_R \) is continuous because taking the rooted spectral measure is continuous (cf. [17, Lemma 3.2]). The proof of Theorem 5.1 follows from Theorem 4.1 using a map \( \times M_k : \mathbb{P}_{\mathcal{S}_k^t} \to \mathbb{P}_R \) defined by
\[
(\times M_k)(\nu) := \nu M = \int_{\mathcal{S}_k^t} \nu([X, \tau])M([X, \tau], \cdot) \quad \text{for} \ \nu \in \mathbb{P}_{\mathcal{S}_k^t}.
\]
It is easy to confirm that the map \( \times M_k \) is also continuous.

**Proof of Theorem 5.1 (2).** Given the event \( \{\dim X_n \geq k\} \),
\[
\mu_{L^\nu_k(X_n)} = \frac{1}{f_k(X_n)} \sum_{\tau \in \mathcal{P}_k(X_n)} \mu_{[X_n, \tau]} = \frac{1}{f_k(X_n)} \sum_{\tau \in \mathcal{P}_k(X_n)} M_k(X_n[\tau], \cdot) = \lambda_k(X_n) M_k.
\]
The first identity follows from Eq. 2.3. In the second identity, we use \( \mu_{[X_n, \tau]} = \mu_{[X_n, \tau]} \) for any \( \tau \in F_k(X_n) \). From combining Theorem 4.1 and the continuous mapping theorem, \( \lambda_k(X_n) M_k \) under \( \mathbb{P}(\cdot | \dim X_n \geq k) \) converges weakly to \( \nu_k(c) M_k \) in distribution as \( n \to \infty \).

**Remark 5.4.** From the above proof, the deterministic probability measure \( \mu \) in Theorem 5.1
can be expressed using the \( k \)-rooted Poisson tree \((\PT_k(c), \tau_0)\):
\[
\mu = \nu_k(c) M_k = \EE[M_k([\PT_k(c), \tau_0], \cdot)] = \EE' [\mu_{[\PT_k(c), \tau_0]}].
\]
From the recursive structure of the \( k \)-rooted Poisson tree, Linial and Peled [17] provided the following upper estimate of \( \EE' [\mu_{[\PT_k(c), \tau_0]}(\{0\})] \) (see also [18 Section 5]):
\[
\EE' [\mu_{[\PT_k(c), \tau_0]}(\{0\})] \leq \max \left\{ t + ct(1-t)^{k+1} - \frac{c}{k+2} \left( 1 - (1-t)^{k+2} \right) \left| t \in [0,1], t = \exp\left(-c(1-t)^{k+1}\right) \right. \right\} = h_k(c).
\]

The upper estimate of the Betti number follows immediately from Theorem 5.1 (2) and Remark 5.4.

**Proposition 5.5.** Let \( \varepsilon > 0 \) be fixed. Then,
\[
\lim_{n \to \infty} \PP \left( \frac{\beta_k(X_n)}{f_k(X_n)} > h_k(c) + \varepsilon \left| \dim X_n \geq k \right. \right) = 0.
\]

**Proof.** Given the event \( \{\dim X_n \geq k\} \), a simple calculation yields
\[
\mu_{L_k^{up}(X_n)}(\{0\}) = \frac{\dim(\ker L_k^{up}(X_n))}{f_k(X_n)} = \frac{\dim Z_k(X_n)}{f_k(X_n)}.
\]
Therefore, from Theorem 5.1 (2) and Remark 5.4 we obtain
\[
\limsup_{n \to \infty} \PP \left( \frac{\beta_k(X_n)}{f_k(X_n)} \geq h_k(c) + \varepsilon \left| \dim X_n \geq k \right. \right) \leq \limsup_{n \to \infty} \PP \left( \frac{\dim Z_k(X_n)}{f_k(X_n)} \geq \EE' [\mu_{[\PT_k(c), \tau_0]}(\{0\})] + \varepsilon \left| \dim X_n \geq k \right. \right) = \limsup_{n \to \infty} \PP \left( \mu_{L_k^{up}(X_n)}(\{0\}) \geq \EE' [\mu_{[\PT_k(c), \tau_0]}(\{0\})] + \varepsilon \left| \dim X_n \geq k \right. \right) = 0.
\]

In the last line above, we use the fact that the map \( \mathcal{P}_R \ni \mu \mapsto \mu(\{0\}) \in \mathbb{R} \) is upper semi-continuous. Thus, the conclusion follows.

### 5.3. Lower estimate of the Betti number

The following inequality is a simple lower estimate of the Betti number of a given finite simplicial complex.

**Proposition 5.6 (A version of the Morse inequality).** Let \( X \) be a finite simplicial complex. Then, it holds that
\[
\beta_k(X) \geq f_k(X) - f_{k+1}(X) - f_{k-1}(X).
\]

**Proof.** Since \( f_k(X) = \dim Z^k(X) + \dim B^{k+1}(X) \), we have
\[
\beta_k(X) = \dim Z^k(X) - \dim B^k(X) = (f_k(X) - \dim B^{k+1}(X)) - \dim B^k(X)
\]
The following lemma follows from Proposition 5.6.

**Lemma 5.7.** Let $\varepsilon > 0$ be fixed. Then, it holds that
\[
\lim_{n \to \infty} P \left( \beta_k(X_n) < 1 - \frac{c}{k + 2} - \varepsilon \mid \dim X_n \geq k \right) = 0.
\]

**Proof.** From Proposition 5.6, we have
\[
\mathbb{P} \left( \frac{\beta_k(X_n)}{f_k(X_n)} < 1 - \frac{c}{k + 2} - \varepsilon \mid \dim X_n \geq k \right) \leq \mathbb{P} \left( \frac{f_{k+1}(X_n)}{f_k(X_n)} - \frac{c}{k + 2} > \varepsilon/2 \mid \dim X_n \geq k \right) + \mathbb{P} \left( \frac{f_{k-1}(X_n)}{f_k(X_n)} > \varepsilon/2 \mid \dim X_n \geq k \right).
\]

Now, given the event $\{\dim X_n \geq k\}$,
\[
\frac{f_{k+1}(X_n)}{f_k(X_n)} = \frac{f_{k+1}(X_n)}{f_k(X_n)} \left( \frac{n}{k+1} \right) q_k \left( \frac{n}{k+2} \right) r_k
\]
and
\[
\frac{f_{k-1}(X_n)}{f_k(X_n)} = \frac{f_{k-1}(X_n)}{f_k(X_n)} \left( \frac{n}{k} \right) q_{k-1} \left( \frac{n}{k+1} \right) r_k.
\]

Note that $n^{k+2} q_{k+1} = \omega(1)$ because $n^{k+1} q_k = \omega(1)$ and $nr_k \sim c$. Therefore, combining Lemmas 4.7 and 4.10, we obtain
\[
\lim_{n \to \infty} \mathbb{P} \left( \left| \frac{f_{k+1}(X_n)}{f_k(X_n)} - \frac{c}{k + 2} \right| > \varepsilon/2 \mid \dim X_n \geq k \right) = 0 \tag{5.1}
\]
and
\[
\lim_{n \to \infty} \mathbb{P} \left( \frac{f_{k-1}(X_n)}{f_k(X_n)} > \varepsilon/2 \mid \dim X_n \geq k \right) = 0. \tag{5.2}
\]

These complete the proof. \qed

Lemma 5.7 gives a simple and useful lower bound of the asymptotic behavior of the $k$th Betti number of $X_n$. However, there is still room for improving the lower bound. To do that, we use the number of (inclusion-wise) maximal $k$-simplices in $X_n$ after some collapsing procedures. Let $X$ be a simplicial complex. A simplex $\tau$ in $X$ is said to be free if there exists a unique maximal simplex $\sigma_\tau$ in $X$, strictly containing $\tau$. The removal of all the simplices $\eta \subset \sigma_\tau$ is called a collapse. Moreover, when $\dim \sigma_\tau = \dim \tau + 1$, we call the collapse an elementary collapse. We then define a collapsing operator $R_k$ as follows. We first list all the maximal $(k + 1)$-simplices $\sigma$ in $X$ containing at least one free $k$-simplex and remove those $\sigma$’s from $X$ together with an arbitrary chosen free $k$-dimensional face of $\sigma$. We denote the resulting subcomplex of $X$ by $R_k(X)$. Note that $X$ and $R_k(X)$ are homotopy equivalent. We
also define $R_k^0(X) := X$ and $R_k^{l+1}(X) := R_k(R_k^l(X))$ for $l \geq 0$. Furthermore, we define $S_k^l(X)$ by removing all the maximal $k$-simplices from $R_k^l(X)$.

Now, let $l \geq 0$ be fixed. When $X$ is finite, we have $f_k(R_k^l(X)) = f_k(S_k^l(X)) + I_k(R_k^l(X))$, where $I_k(R_k^l(X))$ denotes the number of maximal $k$-simplices of $R_k^l(X)$. In addition,

$$f_k(X) - f_k(R_k^l(X)) = f_{k+1}(X) - f_{k+1}(R_k^l(X)) = f_{k+1}(X) - f_{k+1}(S_k^l(X)).$$

Combining these equations, we obtain

$$I_k(R_k^l(X)) = f_k(X) - f_k(S_k^l(X)) - f_k(S_k^l(X)).$$

Therefore,

$$\beta_k(X) = \beta_k(R_k^l(X))$$

$$= \dim Z^k(R_k^l(X)) - \dim B^k(R_k^l(X))$$

$$\geq \dim Z^k(R_k^l(X)) - f_{k-1}(R_k^l(X))$$

$$= \dim Z^k(R_k^l(X)) - f_{k-1}(X)$$

$$\geq I_k(R_k^l(X)) - f_{k-1}(X)$$

$$= f_k(X) - f_{k+1}(X) - f_{k-1}(X) + f_{k+1}(S_k^l(X)) - f_k(S_k^l(X))$$

$$= f_k(X) - f_{k+1}(X) - f_{k-1}(X) + \sum_{\tau \in F_k(X)} 1\{\tau \in S_k^l(X)\} \left( \frac{\deg(S_k^l(X); \tau)}{k+2} - 1 \right). \quad (5.3)$$

We now define a map $D_k^{(l)} : S_k \to \mathbb{R}$ by

$$D_k^{(l)}([X, \tau]) := 1\{\tau \in S_k^l(X)\} \left( \frac{\deg(S_k^l(X); \tau)}{k+2} - 1 \right).$$

Suppose that $([Y_n, \tau_n])_{n=1}^{\infty}$ is a convergent sequence to $[Y, \tau]$ in $S_k$. By the definition of the local distance, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies that $(Y_n, \tau_n)_{i+1} \simeq (Y, \tau)_{i+1}$. Then, we have $D_k^{(l)}([Y_n, \tau_n]) = D_k^{(l)}([Y, \tau])$ for $n \geq N$ because the map $D_k^{(l)}$ depends on only the simplices of distance at most $l + 1$ from the root. This means that $D_k^{(l)}$ is continuous. The following lower bound of the Betti number follows from Theorem 4.1 using a map $\times D_k^{(l)} : \mathcal{P}_{S_k} \to \mathbb{R} \cup \{\infty\}$ defined by

$$(\times D_k^{(l)})(\nu) := \nu D_k^{(l)} = \int_{S_k} \nu([X, \tau]) D_k^{(l)}([X, \tau]) \quad \text{for } \nu \in \mathcal{P}_{S_k}. $$

It is easy to confirm that the map $\times D_k^{(l)}$ is lower semi-continuous.

**Lemma 5.8.** Let $l \geq 0$ and $\varepsilon > 0$ be fixed. Then, it holds that

$$\lim_{n \to \infty} \mathbb{P} \left( \frac{\beta_k(X_n)}{f_k(X_n)} < 1 - \frac{c}{k+2} + \varepsilon \left[ D_k^{(l)}([PT_k(c), \tau_0]) \right] - \varepsilon \left| \dim X_n \geq k \right. \right) = 0.$$
Proof. From Eq. (5.3), given the event \{\dim X_n \geq k\},

\begin{align*}
\frac{\beta_k(X_n)}{f_k(X_n)} &\geq 1 - \frac{f_{k+1}(X_n)}{f_k(X_n)} + \frac{1}{f_k(X_n)} \frac{1}{D_k} \sum_{\tau \in F_k(X_n)} \left(\frac{\deg(S_k(X_n); \tau)}{k + 2} - 1\right) \\
&= 1 - \frac{f_{k+1}(X_n)}{f_k(X_n)} + \frac{1}{f_k(X_n)} \sum_{\tau \in F_k(X_n)} D_k^{(l)}([X_n, \tau]) \\
&= 1 - \frac{f_{k+1}(X_n)}{f_k(X_n)} + \frac{1}{f_k(X_n)} \sum_{\tau \in F_k(X_n)} D_k^{(l)} + \lambda_k(X_n) D_k^{(l)}.
\end{align*}

(5.4)

From Theorem 4.1 and the lower semi-continuity of the map \(\times D_k^{(l)}\), we have

\begin{align*}
\limsup_{n \to \infty} \mathbb{P}(\lambda_k(X_n) D_k^{(l)} &\leq \mathbb{E}'[D_k^{(l)}(\{\text{PT}_k(c), \tau_o\})] - \varepsilon \mid \dim X_n \geq k) \\
&= \limsup_{n \to \infty} \mathbb{P}(\lambda_k(X_n) D_k^{(l)} \leq \nu_k(c) D_k^{(l)} - \varepsilon \mid \dim X_n \geq k) \\
&= 0. \quad (5.5)
\end{align*}

Thus, the conclusion follows from Eqs. (5.1), (5.2), (5.4), and (5.5) in the same manner as the proof of Lemma 5.7.

We now give an overview of the estimate of \(\mathbb{E}'[D_k^{(l)}(\{\text{PT}_k(c), \tau_o\})]\) as described in [13] (see also [3, Section 3]). To provide the estimate, we introduce the concept of \(k\)-rooted tree pruning. For a \(k\)-rooted tree \((T, \tau)\), we define the pruning \(Q_k((T, \tau))\) as below. Initially, let \(\{\tau_1, \tau_2, \ldots, \tau_m\}\) be the set of all the free \(k\)-simplices in \(T\) that are distinct from \(\tau\), and we take the unique simplex \(\sigma_j \in F_{k+1}(T)\) containing \(\tau_j\). We then define \(Q_k(T)\) as a simplicial complex obtained from \(T\) by removing all the simplices \(\tau_j\) and \(\sigma_j\) \((j = 1, 2, \ldots, m)\). Finally, define \(Q_k((T, \tau))\) as the \(k\)-rooted tree \((Q_k(T))(\tau)\). Furthermore, we define \(Q_k^l((T, \tau)) := (T, \tau)\) and \(Q_k^{l+1}((T, \tau)) := Q_k(Q_k^l((T, \tau)))\) for \(l \geq 0\). A straightforward calculation gives the following lemma.

**Lemma 5.9 ([13, Lemma 3.3]).** Let \((t_{k+1,c}^{(l)})_{l \geq -1}\) be a sequence defined by

\[ t_{k+1,c}^{(-1)} = 0 \quad \text{and} \quad t_{k+1,c}^{(l+1)} = \exp(-c(1 - t_{k+1,c}^{(l)})^{k+1}) \quad \text{for} \quad l \geq -1. \]

Furthermore, set \(\delta_{k+1,c}^{(l)} := \deg(Q_k(Q_k^l((\text{PT}_k(c), \tau_o)))) \mid \tau_o\). Then, \(\delta_{k+1,c}^{(l)}\) follows the Poisson distribution with parameter \(c(1 - t_{k+1,c}^{(l-2)})^{k+1}\) for every \(l \geq 0\).

The following lemma gives a lower estimate of \(\mathbb{E}'[D_k^{(l)}(\{\text{PT}_k(c), \tau_o\})]\) using the values \((t_{k+1,c}^{(l)})_{l \geq -1}\).

**Lemma 5.10 ([13, Section 5.2]).** For \(l \geq 1\),

\[ \mathbb{P}'(\tau_o \in S_k^l(\text{PT}_k(c))) \leq 1 - t_{k+1,c}^{(l-1)} - c(1 - t_{k+1,c}^{(l-2)})^{k+1} t_{k+1,c}^{(l-1)} \]

and

\[ \mathbb{E}'[\deg(S_k^l(\text{PT}_k(c)) ; \tau_o ; \tau_o \in S_k^l(\text{PT}_k(c))] \geq c(1 - t_{k+1,c}^{(l-2)})^{k+1} (1 - t_{k+1,c}^{(l-1)}). \]
In particular,
\[ \mathbb{E}'[D_k^{(l)}([PT_k(c), \tau_o])] \geq \frac{c}{k + 2} \left( 1 - l_{k+1,c}^{(l-1)} \right)^{k+1} \left( 1 - l_{k+1,c}^{(l)} \right) - 1 + l_{k+1,c}^{(l-1)} + c \left( 1 - l_{k+1,c}^{(l-2)} \right)^{k+1} l_{k+1,c}^{(l-1)} \]

**Proof.** Since \( \{ \tau_o \in S_k([PT_k(c)]) \} \subset \{ \delta_{k+1,c}^{(l-1)} \geq 2 \} \), we have
\[ \mathbb{P}'(\tau_o \in S_k([PT_k(c)])) \leq \mathbb{P}(\delta_{k+1,c}^{(l-1)} \geq 2) \]
\[ = 1 - \exp(-c(1 - l_{k+1,c}^{(l-2)})^{k+1}) - c(1 - l_{k+1,c}^{(l-2)})^{k+1} \exp(-c(1 - l_{k+1,c}^{(l-2)})^{k+1}) \]
\[ = 1 - l_{k+1,c}^{(l-2)} - c(1 - l_{k+1,c}^{(l-2)})^{k+1} l_{k+1,c}^{(l-1)}. \]
Furthermore, we note that given the event \( \{ \delta_{k+1,c}^{(l-1)} \geq 2 \} \),
\[ 1_{\{ \tau_o \in S_k([PT_k(c)]) \}} \deg(S_k([PT_k(c)]); \tau_o) = \delta_{k+1,c}^{(l-1)}. \]
Therefore,
\[ \mathbb{E}'[\deg(S_k([PT_k(c)]; \tau_o)); \tau_o \in S_k([PT_k(c)])] \]
\[ \geq \mathbb{E}[1_{\{ \tau_o \in S_k([PT_k(c)]) \}} \deg(S_k([PT_k(c)]; \tau_o)); \delta_{k+1,c}^{(l-1)} \geq 2] \]
\[ = \mathbb{E}[^{\delta_{k+1,c}^{(l-1)}; \delta_{k+1,c}^{(l)}}] \]
\[ = c(1 - l_{k+1,c}^{(l-2)})^{k+1} - c(1 - l_{k+1,c}^{(l-1)})^{k+1} \exp(-c(1 - l_{k+1,c}^{(l-2)})^{k+1}) \]
\[ = c(1 - l_{k+1,c}^{(l-2)})^{k+1} (1 - l_{k+1,c}^{(l-1)}). \]

Now, we define
\[ h_k^{(l)}(c) := \max \left\{ 1 - \frac{c}{k + 2} l_{k+1,c}^{(l-1)} + c l_{k+1,c}^{(l-2)} (1 - l_{k+1,c}^{(l-2)})^{k+1} \left( 1 - l_{k+1,c}^{(l-1)} \right)^{k+1} - \frac{c}{k + 2} (1 - (1 - l_{k+1,c}^{(l-2)})^{k+1} (1 - l_{k+1,c}^{(l-1)})) \right\} \]
for \( l \geq 1 \). Then, Lemma 5.10 implies that
\[ 1 - \frac{c}{k + 2} + (0 \vee \mathbb{E}[D_k^{(l)}([PT_k(c), \tau_o])]) \geq h_k^{(l)}(c) \text{ for } l \geq 1. \]
Therefore, the following proposition follows immediately from Lemmas 5.7 and 5.8.

**Proposition 5.11.** Let \( l \geq 1 \) and \( \epsilon > 0 \) be fixed. Then, it holds that
\[ \lim_{n \to \infty} \mathbb{P} \left( \frac{\delta_k(X_n)}{f_k(X_n)} < h_k^{(l)}(c) - \epsilon \mid \dim X_n \geq k \right) = 0. \]

**5.4. Proof of Theorem 5.1 (1)**
From the upper and lower bounds of the Betti number in Propositions 5.5 and 5.11 respectively, we now prove the main result.

**Proof of Theorem 5.7 (1).** Let \( \epsilon > 0 \) be fixed. We can take \( l \in \mathbb{N} \) such that \( |h_k^{(l)}(c) - \delta_k(c)| < \epsilon/2 \)
because \( \lim_{t \to \infty} I_{k+1,c}^{(l)} = t_{k+1,c} \). Then, from Proposition 5.11, we have

\[
\limsup_{n \to \infty} P \left( \frac{\beta_k(X_n)}{f_k(X_n)} < h_k(c) - \varepsilon \mid \dim X_n \geq k \right) \\
\leq \limsup_{n \to \infty} P \left( \frac{\beta_k(X_n)}{f_k(X_n)} < h_k^{(l)}(c) - \varepsilon/2 \mid \dim X_n \geq k \right) = 0.
\]

Combining this estimate with Proposition 5.5, we obtain

\[
\lim_{n \to \infty} P \left( \left| \frac{\beta_k(X_n)}{f_k(X_n)} - h_k(c) \right| > \varepsilon \mid \dim X_n \geq k \right) = 0. \tag{5.6}
\]

Furthermore, given the event \( \{ \dim X_n \geq k \} \),

\[
\left| \frac{\beta_k(X_n)}{n^{k+1} q_k} - \frac{h_k(c)}{(k+1)!} \right| \leq \left| \frac{\beta_k(X_n)}{n^{k+1} q_k} - \frac{\beta_k(X_n)}{(k+1)! f_k(X_n)} \right| + \left| \frac{\beta_k(X_n)}{(k+1)! f_k(X_n)} - \frac{h_k(c)}{(k+1)!} \right| \\
\leq \left| \frac{f_k(X_n)}{n^{k+1} q_k} - \frac{1}{(k+1)!} \right| + \left| \frac{\beta_k(X_n)}{f_k(X_n)} - \frac{h_k(c)}{f_k(X_n)} \right|.
\] \tag{5.7}

For the second inequality above, we use \( \beta_k(X_n) \leq f_k(X_n) \). Combining Eqs. 5.6 and 5.7 with Lemma 4.10, a simple calculation gives the conclusion.

Acknowledgements

This study was supported by JSPS KAKENHI Grant Number 19J11237. The author expresses gratitude to Professors Yasuaki Hiraoka, Masanori Hino, and Matthew Kahle for their valuable comments.

References

[1] D. Aldous and J. M. Steele, *The Objective Method: Probabilistic Combinatorial Optimization and Local Weak Convergence*, in Probability on Discrete Structures, Encyclopaedia Math. Sci. 110, H. Kesten, ed., Springer, 2004, 1–72.

[2] L. Aronshtam and N. Linial, *When does the top homology of a random simplicial complex vanish?*, Random Structures Algorithms 46 (2015), 26–35.

[3] L. Aronshtam, N. Linial, T. Luczak, and R. Meshulam, *Collapsibility and vanishing of top homology in random simplicial complexes*, Discrete Comput. Geom. 49 (2013), 317–334.

[4] A. D. Barbour and P. Hall, *On the rate of Poisson convergence*, Math. Proc. Cambridge Philos. Soc. 95 (1984), 473–480.

[5] I. Benjamini and O. Schramm, *Recurrence of Distributional Limits of Finite Planar Graphs*, in Selected Works of Oded Schramm, Sel. Works Probab. Stat., I. Benjamini and O. Häggström, eds., Springer, 2001, 533–545.

[6] A. Costa and M. Farber, *Random simplicial complexes*, in Configuration Spaces, Springer INdAM Ser. 14, F. Callegaro, F. Cohen, C. De Concini, E. Feichtner, G. Gaiffi, and M. Salvetti, eds., Springer, 2016, 129–153.

[7] G. Elek, *Betti Numbers are Testable*, in Fete of Combinatorics and Computer Science, Bolyai Soc. Math. Stud. 20, G. O. H. Katona, A. Schrijver, T. Szőnyi, and G. Sági, eds., Springer, 2010, 139–149.

[8] P. Erdős and A. Rényi, *On random graphs*, Publ. Math. Debrecen 6 (1959), 290–297.

[9] P. Erdős and A. Rényi, *On the evolution of random graphs*, Publ. Math. Inst. Hungarian Acad. Sci. 5 (1960), 17–60.

[10] C. F. Fowler, *Homology of Multi-Parameter Random Simplicial Complexes*, Discrete Comput. Geom. 62 (2019), 87–127.

[11] E. N. Gilbert, *Random graphs*, Ann. Math. Statist. 30 (1959), 1141–1144.

[12] M. Hino and S. Kanazawa, *Asymptotic behavior of lifetime sums for random simplicial complex processes*, Math. Proc. Cambridge Philos. Soc. 95 (1984), 473–480.
[13] M. Kahle, Sharp vanishing thresholds for cohomology of random flag complexes, Ann. of Math. (2) 179 (2014), 1085–1107.

[14] M. Kahle, Topology of random simplicial complexes: A survey, in Algebraic Topology: Applications and New Directions, Contemp. Math. 620 (2014), 201–222.

[15] O. Kallenberg, Random Measures, Theory and Applications, Probab. Theory Stoch. Model. 77, Springer, 2017.

[16] N. Linial and R. Meshulam, Homological Connectivity Of Random 2-Complexes, Combinatorica 26 (2006), 475–487.

[17] N. Linial and Y. Peled, On the phase transition in random simplicial complexes, Ann. of Math. (2) 184 (2016), 745–773.

[18] N. Linial and Y. Peled, Random Simplicial Complexes: Around the Phase Transition, in A Journey Through Discrete Mathematics, M. Loebl, J. Nešetřil, and R. Thomas, eds., Springer, 2017, 543–570.

[19] R. Meshulam and N. Wallach, Homological connectivity of random k-dimensional complexes, Random Structures Algorithms 34 (2009), 408–417.

[20] B. Roos, Improvements in the Poisson approximation of mixed Poisson distributions, J. Statist. Plann. Inference 113 (2003), 467–483.

[21] M. Schrödl-Baumann, $\ell^2$-Betti numbers of random rooted simplicial complexes, Manuscripta Math. (2019).

[22] D. Taylan, Topology of random $d$-clique complexes, arXiv:1806.02274, 2018.

Shu Kanazawa
Mathematics Department
Tohoku University
Sendai 980–8578, Japan
E-mail: kanazawa.shu.p5@dc.tohoku.ac.jp