Convergence of a Multi-Agent Projected Stochastic Gradient Algorithm for Non-Convex Optimization

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Abstract— We introduce a new framework for the convergence analysis of a class of distributed constrained non-convex optimization algorithms in multi-agent systems. The aim is to search for local minimizers of a non-convex objective function which is supposed to be a sum of local utility functions of the agents. The algorithm under study consists of two steps: a local stochastic gradient descent at each agent and a gossip step that drives the network of agents to a consensus. Under the assumption of decreasing stepsize, it is proved that consensus is asymptotically achieved in the network and that the algorithm converges to the set of Karush-Kuhn-Tucker points. As an important feature, the algorithm does not require the double-stochasticity of the gossip matrices. It is in particular suitable for use in a natural broadcast scenario for which no feedback messages between agents are required. It is proved that our results also holds if the number of communications in the network per unit of time vanishes at moderate speed as time increases, allowing potential savings of the network’s energy. Applications to power allocation in wireless ad-hoc networks are discussed. Finally, we provide numerical results which sustain our claims.

I. INTRODUCTION

Stochastic gradient descent is a widely used procedure for finding critical points of an unknown function $f$ [33]. Formally, it can be summarized as an iterative scheme of the form $\theta_{n+1} = \theta_n + \gamma_{n+1}(-\nabla f(\theta_n) + \xi_{n+1})$ where $\nabla$ is the gradient operator and where $\xi_{n+1}$ represents a random perturbation. Relevant selection of the step size $\gamma_n$ ensures that, for a well behaved function $f$, sequence $(\theta_n)_{n \geq 0}$ will eventually converge to a critical point.

In this paper, we investigate a distributed optimization problem which is of practical interest in many multi-agent contexts such as parallel computing [8], statistical estimation [36], [35], [4], [29], robotics [13] or wireless networks [30]. Consider a network of $N$ agents. To each agent $i = 1, \ldots, N$, we associate a possibly non-convex continuously differentiable utility function $f_i : \mathbb{R}^d \to \mathbb{R}$ where $d \geq 1$. Let $G \subset \mathbb{R}^d$ be a nonempty compact convex subset. We address the the following optimization problem:

$$
\min_{\theta \in G} \sum_{i=1}^{N} f_i(\theta).
$$

(1)

The set $G$ is assumed to be known by all agents. However, a given agent $i$ does not know the utility functions $f_j$’s of other agents $j \neq i$. Cooperation between agents is therefore needed to find minimizers of (1). Moreover, any utility function $f_i$ may be unperfectly observed by agent $i$ itself, due to the presence of random observation noise. We thus address the framework of distributed stochastic approximation.

The literature contains at least two different cooperation approaches for solving (1). The so-called incremental approach is used by [27], [24], [28], [18], [31]: a message containing an estimate of the desired minimizer iteratively travels all over the network. At any instant, the agent which is in possession of the message updates its own estimate and adds its own contribution, based on its local observation. Incremental algorithms generally require the message to go through a Hamiltonian cycle in the network. Finding such a path is known to be a NP complete problem. Relaxations of the Hamiltonian cycle requirement have been proposed: for instance, [24] only requires that an agent communicates with another agent randomly selected in the network (not necessarily in its neighborhood) according to the uniform distribution. However, substantial routing is still needed. In [21], problem (1) is solved using a different approach, assuming that agents perfectly observe their utility functions and know also the utility functions of their neighbors.

This paper focuses on another cooperation approach based on average consensus techniques, see references [15], [40]. In this context, each agent maintains its own estimate. Agents separately run local gradient algorithms and simultaneously communicate in order to eventually reach an agreement over the whole network on the value of the minimizer. Communicating agents combine their local estimates in a linear fashion: a receiver computes a weighted average between its own estimate and the ones which have been transmitted by its neighbors. Such combining techniques are often referred to as gossip methods.

The idea underlying the algorithm of interest in this paper is not new. Its roots can be found in [40], [41] where a network of processors seeks to optimize some objective function known by all agents (possibly up to some additive noise). More recently, numerous works extended this kind of algorithm to more involved multi-agent scenarios, see [25], [26], [32], [23], [19], [38] for a non exhaustive list. Multi-agent systems are indeed more difficult to deal with, because individual agents does not know the global objective function to be minimized. Reference [25] addresses the problem of unconstrained optimization, assuming convex but not necessarily differentiable utility functions. Convergence to a global minimizer is established assuming that utility functions have bounded (sub)gradients. Let us also mention [38] which focuses on the case of quadratic objective functions. Unconstrained optimization is also investigated in [9] assuming differentiable but non neces-
sarily convex utility functions and relaxing boundedness conditions on the gradients. Convergence to a critical point of the objective function is proved and the asymptotic performance is evaluated under the form of a central limit theorem. In [26], the objective function is proved and the asymptotic performance is sarily convex utility functions and relaxing boundedness con-

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• Our assumptions encompass the case of non-doubly stochastic matrices $W_n$ and, as a particular case, the natural broadcast scheme of reference [3]. Our proofs reveal that, loosely speaking, the relaxation of column stochasticity brings a “noise-like” term in the algorithm dynamics, but which is not powerful enough to prevent convergence to the KKT points.

• We show that our convergence result still holds in case the number of communications in the network per unit of time vanishes at moderate speed as time increases.

As an illustration, we apply our results to the problem of power allocation in the wireless interference channel.

The paper is organized as follows. Section II introduces the distributed algorithm and the main assumptions on the network and the observation model. The main result is stated in Section III. Section IV is devoted to its proof. We discuss applications to power allocation in Section V. Section VI describes some standard communication schemes in more details, and provides numerical results.

II. THE DISTRIBUTED ALGORITHM

A. Description of the Algorithm

After an [Initialization step] where each agent starts at a given $\theta_{0,i}$, each node $i$ generates a stochastic process $(\theta_{n,i})_{n\geq 1}$ in $\mathbb{R}^d$ using a two-step iterative algorithm:

[Local step] Node $i$ generates at time $n$ a temporary estimate $\hat{\theta}_{n,i}$ given by

$$
\hat{\theta}_{n,i} = P_G [\theta_{n-1,i} + \gamma_n Y_{n,i}],
$$

where $\gamma_n$ is a deterministic positive step size, $Y_{n,i}$ is a random variable, and $P_G$ represents the projection operator onto the set $G$. Random variable $Y_{n,i}$ is to be interpreted as a perturbed version of the negative gradient of $f_i$ at point $\theta_{n-1,i}$. As will be made clear by Assumption 1(e) below, it is convenient to think of $Y_{n,i}$ as $Y_{n,i} = -\nabla f_i(\theta_{n-1,i}) + \delta M_{n,i}$ where $\delta M_{n,i}$ is a martingale difference noise which stands for the random perturbation.

[Gossip step] Node $i$ is able to observe the values $\hat{\theta}_{n,j}$ of some other $j$’s and computes the weighted average:

$$
\theta_{n,i} = \sum_{j=1}^{N} w_n(i,j) \hat{\theta}_{n,j},
$$

where for any $i$, $\sum_{j=1}^{N} w_n(i,j) = 1$. In the sequel, we define the $N \times N$ matrix $W_n := [w_n(i,j)]_{i,j=1,...,N}$.

Define the random vectors $\theta_n$ and $Y_n$ as $\theta_n := (\theta_{n,1}, \ldots, \theta_{n,N})^T$ and $Y_n := (Y_{n,1}, \ldots, Y_{n,N})^T$. The algorithm reduces to:

$$
\theta_n = (W_n \otimes I_d) P_G N [\theta_{n-1} + \gamma_n Y_n],
$$

where $\otimes$ denotes the Kronecker product, $I_d$ is the $d \times d$ identity matrix and $P_G N$ is the projector onto the $N$th order product set $G_N := G \times \cdots \times G$. 

In each of these works, the gossip communication scheme can be represented by a sequence of matrices $(W_n)_{n\geq 1}$ of size $N \times N$, where the $(i,j)$th component of $W_n$ is the weight given by agent $i$ to the message received from $j$ at time $n$, and is equal to zero in case agent $i$ receives no message from $j$. In most works (see for instance [25], [26], [32], [9]), matrices $W_n$ are assumed doubly stochastic, meaning that $W_n^T 1 = W_n 1 = 1$ where $1$ the $N \times 1$ vector whose components are all equal to one and where $T$ denotes transposition. Although row-stochasticity ($W_n 1 = 1$) is rather easy to ensure in practice, column-stochasticity ($W_n^T 1 = 1$) implies more stringent restrictions on the communication protocol. For instance, in [12], each one-way transmission from an agent $i$ to another agent $j$ requires at the same time a feedback link from $j$ to $i$. Double stochasticity prevents one from using natural broadcast schemes, in which a given agent may transmit its local estimate to all its neighbors without expecting any immediate feedback [3]. Very recently, [23] made a major step forward, getting rid of the column stochasticity condition, and thus opening the road to road broadcast-based constrained distributed optimization algorithms. It is worth noting however that the algorithm of [23] is such that only receiving agents update their estimates. Otherwise stated, an agent deletes its local observations as long as it is not the recipient of a message. Moreover, except perhaps in some special network topologies, the algorithm of [23] strongly relies on a specific choice of the stepsize. In particular, a necessary condition for the convergence to the desired consensus is that the stepsize vanishes at speed $1/n$. However, in practice, it is often desirable to have a leeway on the choice of the stepsize to avoid slow convergence issues.

Contributions

In this paper, we address the optimization problem [1] using a distributed projected stochastic gradient algorithm involving random gossip between agents and decreasing stepsize.

• Unlike previous works, utility functions are allowed to be non-convex. We introduce a new framework for the analysis of a general class of distributed optimization algorithm, which does not rely on convexity properties of the utility functions. Instead, our approach relies on recent results of reference [7] about perturbed differential inclusions. Under a set of assumptions made clear in the next section, we establish that, almost surely, the sequence of estimates of any agent shadows the behavior of a differential variational inequality, and eventually converges to the set of Karush-Kuhn-Tucker (KKT) points of [1].
B. Observation and Network Models

Random processes $(Y_n, W_n)_{n \geq 1}$ are defined on a measurable space equipped with a probability $\mathbb{P}$. Notation $\mathbb{E}$ represents the corresponding expectation. For any $n \geq 1$, we introduce the $\sigma$-field $\mathcal{F}_n = \sigma(\theta_0, Y_1, \ldots, Y_n, W_1, \ldots, W_n)$. The distribution of the random vector $Y_{n+1}$ conditioned on $\mathcal{F}_n$ is assumed to be such that:

$$\mathbb{P}(Y_{n+1} \in A \mid \mathcal{F}_n) = \mu_{\theta_n}(A)$$

(4)

for any measurable set $A$, where $\mu_{\theta_n} \in \mathbb{R}^{dN}$ is a given family of probability measures on $\mathbb{R}^{dN}$. We denote by $[x]$ the Euclidean norm of a vector $x$. We denote by $E^c$ the complementary set of any set $E$. Notation $\times \cap \gamma$ stands for $\max(x, \gamma)$.

**Assumption 1:** The following conditions hold:

a) $(W_n)_{n \geq 1}$ is a sequence of matrix-valued random variables with non-negative components s.t.
   - $W_n$ is row stochastic for any $n$:
     - $W_n 1 = 1$,
   - $\mathbb{E}(W_n)$ is column stochastic for any $n$:
     - $1^T \mathbb{E}(W_n) = 1$.

b) The spectral norm $\rho_n$ of matrix $\mathbb{E}(W_n^T(I_N-11^T/N)W_n)$ satisfies:

$$\lim_{n \to \infty} n(1-\rho_n) = +\infty$$

(5)

c) Conditionally to $\mathcal{F}_n$, $W_{i,n+1}$ and $Y_{n+1}$ are independent. Moreover, the probability distribution of $Y_{n+1}$ conditionally to $\mathcal{F}_n$ depends on $\theta_n$ only. It is denoted $\mu_{\theta_n}$.

d) For any $i=1,\ldots,N$, $f_i$ is continuously differentiable.

e) For any $n \geq 1$,

$$\mathbb{E}[Y_{n+1} | \mathcal{F}_n] = -\nabla f_1(\theta_{n,1})^T, \ldots, \nabla f_N(\theta_{n,N})^T$$

f) $\sup_{\theta \in G} \int |y|^2 d\mu_\theta(y) < \infty$.

We now discuss the above Assumption. Conditions (a) and (b) summarize our assumptions on matrices $W_n$ that is, on the communication scheme used in the network. Following the work of reference [12], random gossip is assumed in this paper. Each matrix $W_n$ must be row stochastic, this means that each agent $i = 1, \ldots, N$ must compute a weighted average $\sum_j w_{n,i,j} 1 = 1$. Note that a quite classical condition in the literature is to further assume that $W_n$ is column-stochastic for any $n$ [25], [26], [32], [9]. Column stochasticity inevitably goes with some restrictions on the communication protocol as discussed in [12]. Here, our assumption is weaker. We only require that $W_n$ is column stochastic on average. This is for instance the case in the natural broadcast scheme of reference [3] which will be discussed in the Section II-C.

Assumption (b) is a connectivity condition of the underlying network graph which will be discussed in more details in Section II-C.

Assumptions (c-e) are related to the observation model. Assumption (c) states three different things. First, random variables $W_{n+1}$ and $Y_{n+1}$ are independent conditionally on the past. Second, $(W_n)_{n \geq 1}$ form an independent sequence (not necessarily identically distributed). Finally, the distribution of $Y_{n+1}$ conditionally on the past is as in (4). Assumption (e) means that each $Y_{n,i}$ can be interpreted as a noisy version of $-\nabla f_i(\theta_{n-1,i})$. The distribution of the random additive perturbation $Y_{n,i} - (\nabla f_i(\theta_{n-1,i})$ is likely to depend on the past through the value of $\theta_{n-1}$, but has a zero mean for any given value of $\theta_{n-1}$. By Markov’s inequality, Assumption (e) implies that $\mu(\theta)_{\theta \in G}$ is tight i.e., for any $\epsilon > 0$, there exists a compact set $K \subset \mathbb{R}^{dN}$ such that $\sup_{\theta \in G} \mu_\theta(K^c) < \epsilon$.

**Assumption 2:**

a) The deterministic sequence $(\gamma_n)_{n \geq 1}$ is positive and such that $\sum \gamma_n = +\infty$.

b) There exists $\alpha > 1/2$ such that:

$$\lim_{n \to \infty} n^{\alpha} \gamma_n = 0$$

(6)

$$\liminf_{n \to \infty} \frac{1 - \rho_n}{n^\alpha \gamma_n} > 0$$

(7)

Note that (6) implies (but is not equivalent to) $\sum \gamma_n < \infty$, which is a rather usual assumption in the framework of decreasing step size stochastic algorithms [20]. In order to have some insights on (7), first consider the case where the matrices $(W_n)_{n \geq 1}$ form an i.i.d. sequence i.e., the spectral radius $\rho := \rho_n$ does not depend on $n$. Then both conditions (5) and (7) are satisfied if and only if:

$$\rho < 1$$

(8)

Nevertheless, matrices $(W_n)_{n \geq 1}$ do not need to be i.i.d. An interesting example is when matrix $W_n$ is likely to be equal to identity with a probability that tends to one as $n \to \infty$. From a communication point of view, this means that the exchange of information between agents becomes rare as $n \to \infty$. This context is especially interesting in case of wireless networks, where it is often required to limit as much as possible the communication overhead. Let us emphasize that if $W_n$ is assumed i.i.d with $\rho < 1$, then the usual assumptions $\sum \gamma_n = +\infty$ and $\sum \gamma_n^2 < \infty$ can be substituted to Assumption 2. Assumption 2 is designed to take into account the non-stationnarity of $W_n$.

Consider for instance the case where $1 - \rho_n = a/n^\gamma$ and $\gamma_n = \gamma_0/n^\xi$ for some constants $a, \gamma_0 > 0$. Then, a sufficient condition for Assumption 2 is:

$$0 \leq \eta < \xi - 1/2 \leq 1/2$$

In particular, $\xi \in (1/2, 1]$ and $\eta \in [0, 1/2)$.

C. Illustration: Some Examples of Gossip schemes

Here, we focus on two standard communication schemes and give the corresponding sequence $(W_n)_{n \geq 1}$ for each of them. We refer the reader to reference [14] for a more complete picture and for more general gossip strategies. We introduce what we shall refer to as the pairwise and the broadcast schemes. The first one can be found in the paper of Boyd et al. [12] on average consensus while the second is inspired from the broadcast scheme depicted in [3]. The network of agents is represented as a nondirected graph $(V, E)$ where $V$ is the set of $N$ nodes and $E$ corresponds to the set edges between nodes.

1) Pairwise Gossip: At time $n$, a single node $i$ wakes up (node $i$ is chosen at random, uniformly within the set of nodes and independently from the past). Node $i$ randomly selects a node $j$ among its neighbors in the graph. Node $i$ and $j$ exchange their temporary estimates $\theta_{n,i}$ and $\theta_{n,j}$ and compute
the weighted average \( \theta_{n,i} = \theta_{n,j} = \beta \theta_{n,i} + (1 - \beta) \theta_{n,j} \) where \( 0 < \beta < 1 \). Other nodes \( k \notin \{i, j\} \) simply set \( \theta_{n,k} = \theta_{n,k} \). Set \( \beta = 1/2 \) for simplicity. In this case, the corresponding matrix \( W_n \) is given by \( W_n = I_N - (e_i - e_j)(e_i - e_j)^T / 2 \) where \( e_i \) denotes the \( i \)th vector of the canonical basis in \( \mathbb{R}^N \). Note that for each \( n \), \( W_n \) forms an i.i.d. sequence of doubly stochastic matrices. Assumption 1(a) is obviously satisfied. Moreover, the spectral radius \( \rho \) of matrix \( \mathbb{E}(W_n^T (I_N - 11T^T / N) W_n) \) satisfies \( \rho \leq 1 \) if and only if \((\mathcal{V}, \mathcal{E})\) is a connected graph (see [12]).

2) Broadcast Gossip: At time \( n \), a random node \( i \) wakes up. The latter node is supposed to be chosen at random w.r.t. the uniform distribution on the set of vertices. Node \( i \) broadcasts its temporary update to all its neighbors. Any neighbor \( j \), computes the weighted average \( \theta_{n,j} = \beta \theta_{n,i} + (1 - \beta) \theta_{n,j} \). On the other hand, any node \( k \) which does not belong to the neighborhood \( \mathcal{N}_i \) of \( i \) (this includes \( i \) itself) simply sets \( \theta_{n,k} = \theta_{n,k} \). Note that, as opposed to the pairwise scheme, the transmitter node \( i \) does not expect any feedback from its neighbors. It is straightforward to show that the \((k, \ell)\)th component of matrix \( W_n \) corresponding to such a scheme writes:

\[
W_n(k, \ell) = \begin{cases} 
1 & \text{if } k \notin \mathcal{N}_i \text{ and } k = \ell \\
\beta & \text{if } k \in \mathcal{N}_i \text{ and } \ell = i \\
1 - \beta & \text{if } k \in \mathcal{N}_i \text{ and } k = \ell \\
0 & \text{otherwise.}
\end{cases}
\]

As a matter of fact, the above matrix \( W_n \) is not doubly stochastic since \( 1^T W_n \neq 1^T \). Nevertheless, it is straightforward to check that \( 1^T \mathbb{E}(W_n) = 1^T \) (see for instance [3]). Thus, the sequence of matrices \( (W_n)_{n \geq 1} \) satisfies the Assumption 1(a). Once again, straightforward derivations which can be found in [3] show that the spectral radius \( \rho \) satisfies \( \rho \leq 1 \) if and only if \((\mathcal{V}, \mathcal{E})\) is a connected graph.

### III. CONVERGENCE W.P.1

#### A. Framework and Assumptions

We study the case where for any \( i = 1, \ldots, N \), the set \( G \) is determined by a finite set of \( p \) inequality constraints \( (1 \leq p < \infty) \):

\[
G := \{ \theta \in \mathbb{R}^d : \forall j = 1, \ldots, p, \; q_j(\theta) \leq 0 \} \tag{9}
\]

for some functions \( q_1, \ldots, q_p \) which satisfy the following conditions. For any \( \theta \in \mathbb{R}^d \), we denote by \( A(\theta) \subset \{1, \ldots, p\} \) the active set i.e., \( A(\theta) = \{j : q_j(\theta) = 0, \; \theta \in G\} \). Denote by \( \partial G \) the boundary of \( G \).

Assumption 3: a) The set \( G \) defined by (9) is nonempty and compact.
b) For any \( j = 1, \ldots, p, \; q_j : \mathbb{R}^d \rightarrow \mathbb{R} \) is a convex function, continuously differentiable in a neighborhood of \( \partial G \).
c) For any \( \theta \in \partial G, \; (\nabla q_j(\theta) : j \in A(\theta)) \) is a linearly independent collection of vectors.

In other terms, some regularity is imposed on \( G \). Moreover, (9,c) is a simple qualification assumption [34]: note that the same set \( G \) can be expressed using different constraints; for instance, one can always duplicate constraints arbitrarily. The qualification assumption says that it is up to the user to remove redundant constraints.

#### B. Notations

Recall that \( |x| \) represents the Euclidean norm of a vector \( x \). Denote by \( \nabla \) the gradient operator. We denote by

\[
J := (11^T / N) \otimes I_d, \quad J_\perp := I_{dN} - J, \tag{10}
\]

resp. the projector onto the consensus subspace \( \{1 \otimes \theta : \theta \in \mathbb{R}^d \} \) and the projector onto the orthogonal subspace. For any vector \( x \in \mathbb{R}^{dN} \), define the vector of \( \mathbb{R}^d \)

\[
\langle x \rangle := \frac{1}{N} (1^T \otimes I_d) x. \tag{11}
\]

Note that \( \langle x \rangle = (x_1 + \cdots + x_N) / N \) in case we write \( x = (x_1^T, \ldots, x_N^T) \) for some \( x_1, \ldots, x_N \in \mathbb{R}^d \). We denote by \( x_\perp := J_\perp x \) the projection of \( x \) on the orthogonal to the consensus space. Remark that \( x = 1 \otimes \langle x \rangle + x_\perp \). In particular, we set \( \theta_\perp, n := J_\perp \theta_\perp, n \) where \( J_\perp \) is given by (10) and refer to \( \theta_\perp, n \) as the disagreement vector. Denote by

\[
f := \frac{1}{N} \sum_{i=1}^N f_i
\]

the average of utility functions. Define the set of KKT points of \( f \) on \( G \) (also called the set of stationary points) as:

\[
\mathcal{L} := \{ \theta \in \mathbb{R}^d : \; -\nabla f(\theta) \in N_G(\theta) \} \tag{12}
\]

where \( N_G(\theta) \) is the normal cone i.e., \( N_G(\theta) := \{ v \in \mathbb{R}^d : \forall \theta' \in G, \; v^T (\theta' - \theta) \geq 0 \} \). Define \( d(x, A) := \inf \{ |x - a| : a \in A \} \) for any \( \theta \in \mathbb{R}^{dN} \) and any set \( A \). We say that a random sequence \( (x_n)_{n \geq 1} \) converges almost surely (a.s.) to a set \( A \) if \( d(x_n, A) \) converges a.s. to zero as \( n \rightarrow \infty \). Please note that convergence to \( A \) does not imply convergence to some point of \( A \).

#### C. Main result

**Theorem 1:** Assume that \( f(\mathcal{L}) \) has an empty interior. Under Assumptions 1, 2, 3, the sequence \((\theta_n)_{n \geq 1}\) converges a.s. to the set \( \{1 \otimes \theta : \theta \in \mathcal{L}\} \). Moreover, \((\langle \theta_n \rangle)_{n \geq 1}\) converges a.s. to a connected component of \( \mathcal{L} \).

Theorem 1 establishes two points. First, a consensus is achieved as \( n \) tends to infinity, meaning that \( \max_{i,j} |\theta_{n,i} - \theta_{n,j}| \) converges a.s. to zero. Second, the average estimate \( \langle \theta_n \rangle \) converges to the set \( \mathcal{L} \) of KKT points. As a consequence, if \( \mathcal{L} \) contains only isolated points, sequence \( \langle \theta_n \rangle \) converges a.s. to one of these points.

In particular, when \( f \) is convex, \( \langle \theta_n \rangle \) converges to the set of global solutions to the minimization problem (1). However, as already remarked, our result is more general and does not rely on the convexity of \( f \). If \( f \) is not convex, sequence \( \langle \theta_n \rangle \) does not necessarily converge to a global solution. Nevertheless, it is well known that the KKT conditions are satisfied by any local minimizer [11].

The condition that \( f(\mathcal{L}) \) has an empty interior is satisfied in most practical cases. From Sard’s theorem, it holds as soon as \( f \) is \( d \) times continuously differentiable.
IV. PROOF OF THEOREM 1

A. Sketch of Proof

We first provide some insights on the proof (all statements are made rigorous in the next subsections).

The proof is decomposed in the following steps.

1) First, establish convergence to consensus with probability 1 (see Lemma 1). In this step we show that iterating matrices \( W_n \) yields to consensus whatever the behavior of sequence \( \{\theta_n\}_{n \geq 1} \). See subsection IV-C.

2) Show then that \( \{\theta_n\} \) is ruled by the following discrete time dynamical system (see Proposition 1 in subsection IV-D):

\[
\theta_n = \theta_{n-1} - \gamma_n \nabla f(\theta_{n-1}) + \gamma_n g_n(\theta_{n-1}) + \gamma_n \xi_n + \gamma_n r_n, \tag{13}
\]

In order to give some insight, assume just for a moment that \( g \) is identically 0. In that case, one could write:

\[
\frac{\langle \theta_n \rangle - \langle \theta_{n-1} \rangle}{\gamma_n} = -\nabla f(\theta_{n-1}) + \xi_n + r_n,
\]

and view \( \{\theta_n\} \) as a noisy discrete approximation of the well studied Ordinary Differential Equation (ODE):

\[
\dot{x} = -\nabla f(x),
\]

where \( \dot{x} \) stands for the derivative of function \( t \mapsto x(t) \). See, for instance, [17].

This line of reasoning is the so-called “ODE” method (see, for instance, [10]). If function \( g \) was regular enough, one could still use the ODE method and study an ODE of the form \( \dot{x} = -\nabla f(x) + h(x) \) where \( h \) is a function derived from \( g \) (we skip the details, which are not important here). Unfortunately, in our case, function \( g \) is not regular enough. In that case, the standard ODE method fails for two reasons: i) such an ODE might have several solutions, ii) it does no longer ensure that discretized versions of the ODE stay close to suitable solutions of the original ODE.

3) The framework of Differential Inclusions (DI) addresses these two issues. The ODE is replaced by the DI

\[
\dot{x} \in F(x),
\]

(see (14) in what follows), where, at any time instant \( t \), \( F(x(t)) \) is a set of vectors instead of the single vector as in an ODE. An important property is that the set-valued function \( F \) should now be upper semi-continuous. It remains to prove two assertions to finish the proof:

a) Show that the noisy discretized dynamical system (13) asymptotically behaves “the same way” as a solution to the DI. To formalize this, we shall rely on the notion of perturbed differential inclusions of [7]. We prove in Section IV-E that the continuous-time process obtained from a suitable interpolation of (13) is in fact a perturbed solution to the DI. Using the results of [7] which we recall in Section IV-B the limit set of the interpolated process can be characterized by simply studying the behavior of the solutions \( x(t) \) to the DI for large \( t \).

b) The asymptotic behavior of the DI is addressed in Section IV-F where it is shown that \( F \) is a Lyapunov function for the set \( \mathcal{L} \) of KKT points under the dynamics induced by the DI.

B. Preliminaries: Useful Facts about Set-Valued Dynamical Systems

Before providing the details of the proof, we recall some useful facts about perturbed differential inclusions. All definitions and statements made in this paragraph can be found in reference [7]. However, for the sake of readability and completeness, it is worth recalling some facts.

Consider an arbitrary set-valued function \( F \) which maps each point \( \theta \in \mathbb{R}^d \) to a set \( F(\theta) \subset \mathbb{R}^d \). Assume that \( F \) satisfies the following conditions:

**Condition 1:** The following holds:

- \( F \) is a closed set-valued map \( i.e., \{ (\theta, y) : y \in F(\theta) \} \) is a closed subset of \( \mathbb{R}^d \times \mathbb{R}^d \).
- For any \( \theta \in \mathbb{R}^d \), \( F(\theta) \) is a nonempty compact convex subset.
- There exists \( c > 0 \) such that for any \( \theta \in \mathbb{R}^d \), \( \sup_{z \in F(\theta)} |z| < c(1 + |\theta|) \).
- A function \( x : \mathbb{R} \rightarrow \mathbb{R}^d \) is called a solution to the differential inclusion

\[
\frac{dx}{dt} \in F(x), \tag{14}
\]

if it is absolutely continuous and if \( \frac{dx}{dt}(t) \in F(x(t)) \) for almost every \( t \in \mathbb{R} \). Let \( \Lambda \) be a compact set in \( \mathbb{R}^d \). A continuous function \( V : \mathbb{R}^d \rightarrow \mathbb{R} \) is called a Lyapunov function for \( \Lambda \) if, for any solution \( x \) to (14) and for any \( t > 0 \),

\[
V(x(t)) \leq V(x(0))
\]

and where the inequality is strict whenever \( x(0) \notin \Lambda \).

Finally, a function \( y : [0, \infty) \rightarrow \mathbb{R}^d \) is called a perturbed solution to (14) if it is absolutely continuous and if there exists a locally integrable function \( t \mapsto U(t) \) such that:

- For any \( T > 0 \), \( \lim_{t \to \infty} \sup_{s \in [0, T]} |\int_t^{t+\delta} U(s)ds| = 0 \).
- There exists a function \( \delta : [0, \infty) \rightarrow [0, \infty) \) such that \( \lim_{t \to \infty} \delta(t) = 0 \) and such that for almost every \( t > 0 \),

\[
\frac{dy(t)}{dt} - U(t) \in F^\delta(y(t)), \text{ where for any } \delta > 0:
\]

\[
F^\delta(\theta) := \{ z \in \mathbb{R}^d : \exists \theta', |\theta - \theta'| < \delta, d(z, F(\theta')) < \delta \}.
\tag{15}
\]

**Remark 1:** Note that there is no guarantee that a perturbed solution remains close to a solution from a given time (for instance, \( y(t) = \log(1 + t) \) is a perturbed solution to \( \dot{x} = 0 \)). However, loosely speaking, \( y \) is a perturbed solution if the behavior of \( y \) on the interval \( [t, t+T] \) shadows the differential inclusion for \( t \) large enough, for each fixed width \( T \).

The following result, found in reference [7], will be used in our proofs. Denote by \( \mathcal{S} \) the closure of a set \( \mathcal{S} \).

**Theorem 2 ([7]):** Let \( V \) be a Lyapunov function for \( \Lambda \). Assume that \( V(\Lambda) \) has an empty interior. Let \( y \) be a bounded perturbed solution to (14). Then,

\[
\bigcap_{t \geq 0} y([t, \infty)) \subset \Lambda.
\]
Proof: The result is a consequence of Theorem 4.2, Theorem 4.3 and Proposition 3.27 in [7].

\section{C. Agreement between Agents}

\textbf{Lemma 1 (Agreement):} Assume that \( G \) is a compact convex set. Under Assumptions 2 and 3, \( \sum_{n \geq 1} \mathbb{E}[|\theta_{\perp,n}|^2] < \infty \). As a consequence, \( \theta_{\perp,n} \) converges to zero a.s.

Proof: We rewrite (14) as \( \theta_n = (W_n \otimes I_d)(\theta_{n-1} + \gamma_n Z_n) \) where

\[
Z_n := \frac{P_G^N[\theta_{n-1} + \gamma_n Y_n] - \theta_{n-1}}{\gamma_n} .
\] (16)

Before going into the details of the proof of Lemma 1, it is worth noting that \( |Z_n|^2 \leq |Y_n|^2 \) (just note that \( \theta_{n-1} = P_{G^N}[\theta_{n-1}] \) and use the fact that \( G^N \) is convex). By Assumption 3, the sequence \( \mathbb{E}[|Z_n|^2] \) is therefore bounded.

We now study \( \theta_{\perp,n} \). As \( W_n = 1 \), it is straightforward to show that \( J_\perp(W_n \otimes I_d) = J_\perp(W_n \otimes I_d)J_\perp \). As a consequence, \( \theta_{\perp,n} = J_\perp(W_n \otimes I_d)(\theta_{n-1} + \gamma_n Z_n) \). Using the so-called "mixed product rule", \( ([A \otimes B] \cdot [C \otimes D] = AC \otimes BD) \), we expand the square Euclidean norm of the latter vector:

\[
|\theta_{\perp,n}|^2 = (\theta_{\perp,n-1} + \gamma_n Z_n)^TH_n(\theta_{\perp,n-1} + \gamma_n Z_n) .
\]

where \( W_n := (W_n^T \otimes I_d)((I_N - 11^T/N) \otimes I_d)^2(W_n \otimes I_d) \). Note that \( W_n = (W_n^T(I_N - 11^T/N)W_n \otimes I_d) \). Integrate both sides of the above equation w.r.t. the random variable \( W_n \):

\[
\mathbb{E}[|\theta_{\perp,n}^2|] \leq \rho_n|\theta_{\perp,n-1} + \gamma_n Z_n|^2 .
\]

Expand the right-hand side and take the expectation. Using the fact that \( \rho_n < 1 \) for \( n \) large enough,

\[
\mathbb{E}[|\theta_{\perp,n}^2|] \leq \rho_n \mathbb{E}[|\theta_{\perp,n-1}^2| + 2\gamma_n \mathbb{E}[|\theta_{\perp,n-1}| Z_n]] + \gamma_n^2 \mathbb{E}[|Z_n|^2] .
\]

As \( \mathbb{E}[|Z_n|^2] \) is uniformly bounded, we obtain from Cauchy-Schwarz's inequality:

\[
\mathbb{E}[|\theta_{\perp,n}^2|] \leq \rho_n \mathbb{E}[|\theta_{\perp,n-1}|^2] + \gamma_n \sqrt{C \mathbb{E}[|\theta_{\perp,n-1}|^2]} + \gamma_n^2 C
\]

for some constant \( C \).

Let us denote \( v_n := \mathbb{E}[Z_n^2] \). Using \( \gamma_n = \sqrt{C} \gamma_n \) (which also fulfills Assumption 3), we get:

\[
v_n \leq \rho_n v_{n-1} + \gamma_n \sqrt{v_{n-1}} + \gamma_n^2 .
\] (17)

Let \( u_n := n^{2\alpha}v_n \) for some \( \alpha > 1/2 \) satisfying (9) and (7). Then,

\[
u_n \leq \left( 1 + \frac{1}{n-1} \right)^{2\alpha} \rho_n u_{n-1} + n^{2\alpha}\gamma_n^2 + \left( 1 + \frac{1}{n-1} \right)^{\alpha} \sqrt{u_{n-1}} + n^{2\alpha}\gamma_n^2 .
\] (18)

This implies in turn:

\[
u_n - u_{n-1} \leq \left( -a_n u_n + b_n \sqrt{u_n} + c_n \right) n^{2\alpha}\gamma_n^2 ,
\]

where \( b_n = (1 + \frac{1}{n-1})^{2\alpha} a_n = \frac{1}{n^{\alpha}\gamma_n} \), and \( c_n = n^{\alpha}\gamma_n \).

A straightforward analysis of function \( \phi_n : u \mapsto -a_n u + b_n \sqrt{u} + c_n \) shows that \( u > \bar{t}_n \) implies \( \phi_n(u) < 0 \) where \( \bar{t}_n := (b_n/a_n + c_n/b_n)^2 \). Note that, using Assumption 3b), \( \bar{u} \sim \frac{1-\phi_n}{n^{\alpha}\gamma_n} \) and using Assumption 3, \( t_n \) is bounded above, say by a constant \( K > 0 \). Moreover, when \( u \leq t_n, \phi_n(u) \leq \phi_n(b_n/2a_n) = c_n + b_n^2/4a_n \). Notice again that \( \phi_n(b_n/2a_n) \) is bounded above, say by a constant \( L > 0 \). We have proved that if \( u_{n-1} \leq K \) then \( u_n \leq K + L \) and if \( u_{n-1} > K, u_n \leq u_{n-1} \). This implies that \( u_n \leq \max(K + L, u_0) \). Hence \( \sum v_n < \infty \).

\section{D. Average Estimate}

For any \( \gamma > 0 \), \( \theta \in G^N \), define \( g_{\gamma}(\theta) := \gamma^{-1} \langle T \theta \otimes I_d \rangle \int (P_{G^N}[\theta + \gamma y] - \theta - \gamma y) d\mu(\theta \cdot y) \). Under Assumption 3, this means that:

\[
g_{\gamma_n}(\theta_{n-1}) = \frac{\langle E[P_{G^N}[\theta_{n-1}+\gamma_n Y_n] - \theta_{n-1} - \gamma_n Y_n | F_{n-1}] \rangle}{\gamma_n} .
\]

\section{Proposition 1: Under Assumptions 1, 2, 3 there exists two stochastic processes \( (\xi_n)_{n \geq 1}, (r_n)_{n \geq 1} \) such that for any \( n \geq 1 \):

\[
\langle \theta_n \rangle = \langle \theta_{n-1} \rangle - \gamma_n \nabla f((\theta_{n-1})) + \gamma_n g_{\gamma_n}(\theta_{n-1}) + \gamma_n \xi_n + \gamma_n r_n
\]

and satisfying w.p.1:

\[
\lim_{n \rightarrow \infty} \sup_{k \geq n} \sum_{\ell=1}^{k} \gamma_\ell \xi_\ell = 0
\]

\[
\lim_{n \rightarrow \infty} r_n = 0 .
\]

The proof is provided in Appendix 4.

Note that the third term in the right-hand side of (19) is zero whenever \( \theta_{n-1} + \gamma_n Y_n \) lies in \( G^N \) i.e., when the projector is inoperant. In order to have some insights, assume just for a moment that this holds for any \( n \) after a certain value. In this case, equation (19) simply becomes

\[
\langle \theta_n \rangle = \langle \theta_{n-1} \rangle - \gamma_n \nabla f((\theta_{n-1})) + \gamma_n \xi_n + \gamma_n r_n
\]

(21)

In this case, by the continuity of \( \nabla f \) and using the above conditions on the sequences \( \xi_n \) and \( r_n \), the asymptotic behavior of sequence \( \langle \theta_n \rangle \) can be directly characterized using classical stochastic approximation results (see [20], [1], [6], [16]). Indeed, a sequence \( \langle \theta_n \rangle \) satisfying (21) converges to the set of critical points of \( f \). Nevertheless, the projector \( P_{G^N} \) is generally active in practice, so that the term \( g_{\gamma_n}(\theta_{n-1}) \) may be nonzero infinitely often. This additional term raises at least two problems. First, it depends on the whole vector \( \theta_n \) and not only on the average \( \langle \theta_n \rangle \); equation (19) looks thus nothing like a usual iteration of a stochastic approximation algorithm. Second, \( g_{\gamma}(\theta) \) is not a continuous function of \( \theta \), whereas standard approaches often assume the continuity of the mean field of the stochastic approximation algorithm.
E. Interpolated Process

Define \( \mu := \sup_{\theta \in G} \int |y| d\mu_\theta(y) \). Define the following set-valued function \( F \) on \( \mathbb{R}^d \) which maps any \( \theta \) to the set:

\[
F(\theta) := \{ -\nabla f(\theta) - z : z \in N_G(\theta), |z| \leq 3\mu \}.
\]

Using the fact that \( f \) is continuously differentiable and that \( G \) is closed and convex, it can be shown that \( F \) satisfies Condition 1. Recall notation \( F^0(\theta) \) in (15). Consider stochastic processes \( (\xi_n, r_n)_{n \geq 1} \) as in Proposition 1.

Proposition 2: Under Assumptions 1, 2, 3, there exists a sequence of random variables \( (\delta_n)_{n \geq 1} \) converging a.s. to zero and an integer \( n_0 \) such that for any \( n \geq n_0 \),

\[
\frac{\langle \theta_n \rangle - \langle \theta_{n-1} \rangle}{\gamma_n} - \xi_n - r_n \in F^0_n(\langle \theta_{n-1} \rangle).
\]

Proof: From the definition of \( g_j \), it is straightforward to show that \( |g_j(\theta)| \leq 2\mu \). Note that for any \( \theta \in G \) and \( y \in \mathbb{R}^d \), the vector \( \theta + \gamma y - P_G[\theta + \gamma y] \) belongs to the normal cone \( N_G(P_G[\theta + \gamma y]) \) at point \( P_G[\theta + \gamma y] \). Otherwise stated, \( P_G[\theta + \gamma y] - \theta - \gamma y \) can be written as a linear combination of the gradient vectors associated with the active constraints, where the coefficients of the linear combination are nonnegative (see for instance, [34], theorem 6.14). The latter linear combination is moreover unique due to the qualification constraint given by Assumption 3(c). More precisely, if \( A(P_G[\theta + \gamma y]) \) represents the active set at point \( P_G[\theta + \gamma y] \) for any \( \theta, \gamma, y \), there exists a unique collection of nonnegative coefficients \( (\lambda_j(\theta, \gamma, y) : j \in A(P_G[\theta + \gamma y])) \) such that:

\[
-\frac{P_G[\theta + \gamma y] - \theta - \gamma y}{\gamma} = \sum_{j \in A(P_G[\theta + \gamma y])} \lambda_j(\theta, \gamma, y) \nabla q_j(P_G[\theta + \gamma y]).
\]

Throughout the paper, we use the convention that \( \lambda_j(\theta, \gamma, y) = 0 \) in case \( j \notin A(P_G[\theta + \gamma y]) \). The following technical lemma is proved in Appendix IV.

Lemma 2: Under Assumptions 1, 2, and 3, there exists a constant \( C > 0 \) and a function \( \gamma \mapsto \epsilon(\gamma) \) on \([0, +\infty)\) satisfying \( \lim_{\gamma \to 0} \epsilon(\gamma) = 0 \) such that the following holds. For any \( \theta \in G^N \) and any \( \gamma > 0 \), there exists \( (\alpha_1, \cdots, \alpha_p) \in [0, C]^p \) such that

\[
-\gamma_n^{\phi(\theta)} = \sum_{j \in A(\theta, \gamma)} \alpha_j \nabla q_j(\langle \theta \rangle) \leq \epsilon(\gamma \vee \langle \theta \vee \rangle).
\]

The proof is provided in Appendix III. The sum in the left-hand side of (27) is a (nonnegative) linear combination of the gradient vectors of the constraints at point \( \langle \theta \rangle \). However, this does not necessarily imply that this term belongs to the normal cone \( N_G(\langle \theta \rangle) \) because, for a fixed \( \epsilon > 0 \), the set \( A(\theta, \epsilon) \) is in general larger than the active set \( A(\theta) \). Nevertheless, the following lemma states that \( A(\theta, \epsilon) \) is no larger than a certain active set \( A(\theta') \) for some \( \theta' \) in a neighborhood of \( \theta \).

Lemma 3: Under Assumption 3, there exists a constant \( C > 0 \) and a function \( \gamma \mapsto \delta(\epsilon) \) on \([0, +\infty)\) satisfying \( \lim_{\gamma \to 0} \delta(\epsilon) = 0 \) and there exists \( \epsilon_0 > 0 \) such that for any \( \epsilon < \epsilon_0 \) and any \( \theta \in G \), there exists \( \theta' \in G \) s.t.

\[
|\theta - \theta'| < \delta(\epsilon) \quad \text{and} \quad A(\theta, \epsilon) \subset A(\theta').
\]

The proof is given in Appendix IV. We put all pieces together. Consider constant \( C \) and functions \( \epsilon(,) \) and \( \delta(,) \) as in Lemma 3 and 4 respectively. Define \( \epsilon_n := \epsilon(\gamma_n \vee \langle \theta_{n-1} \rangle) \) and \( \delta_n := \max(\epsilon_n + C\phi(\epsilon_n), \delta(\epsilon_n)) \). Clearly, \( \epsilon_n \) (and consequently \( \delta_n \)) converges to zero a.s. due to Lemma 4 and to the fact that \( \gamma_n \to 0 \). In particular, there exists an integer \( n_0 \) s.t. \( \epsilon_n < \epsilon_0 \) for any \( n \geq n_0 \). By Lemma 4, for any \( n \geq n_0 \), there exists \( \theta'_n \in G \) satisfying \( |\theta'_n - \langle \theta_{n-1} \rangle| < \delta(\epsilon_n) \) and \( A(\theta_{n-1}, \epsilon_n) \subset A(\theta'_n) \). Thus, by Lemma 3 there exists \( (\alpha_1, \cdots, \alpha_p) \in [0, C]^p \) such that

\[
-\gamma_n^{\phi(\theta_{n-1})} - \sum_{j \in A(\theta'_n)} \alpha_j \nabla q_j(\langle \theta_{n-1} \rangle) \leq \epsilon_n.
\]

Define \( z_n := \sum_{j \in A(\theta'_n)} \alpha_j \nabla q_j(\theta'_n) \). Clearly, \( z_n \in N_G(\theta'_n) \). Using inequality (29),

\[
-\gamma_n^{\phi(\theta_{n-1})} - z_n \leq \epsilon_n + C \sum_{j \in A(\theta'_n)} |\nabla q_j(\theta'_n) - \nabla q_j(\langle \theta_{n-1} \rangle)| \leq \epsilon_n + C\phi(\delta(\epsilon_n)) \leq \delta_n.
\]

As \( |g_j(\theta)| \leq 2\mu \), this moreover implies that \( |z_n| \leq 3\mu \) provided that \( \delta_n \) is small enough. Thus,

\[
d(-\gamma_n^{\phi(\theta_{n-1})}, N_G(\theta'_n) \cap \{z : |z| \leq 3\mu\}) \leq \delta_n.
\]
for all but a finite number of $n$’s. The proof of Proposition 2 is completed by using (19).

Define $\tau_0 = 0$ and $\tau_n := \sum_{i=k}^{n} \gamma_k$ for any $n \geq 1$. Define the continuous time process $\Theta : [0, +\infty) \to \mathbb{R}^{d}$ as:

$$\Theta(\tau_{n-1} + t) := (\theta_n) + t \frac{(\theta_n - \theta_{n-1})}{\tau_n - \tau_{n-1}}$$

for any $t \in [0, \gamma_n)$ and any $n \geq 1$.

**Proposition 3:** Under Assumptions 1, 2, 3 the interpolated process $\Theta$ is a perturbed solution to (14) w.p.1.

**Proof:** The proof follows more or less the same idea as the proof of Proposition 1.3 in reference [7]. There exists an event $\Omega_0$ of probability one such that $\delta_n \to 0$ and $r_n \to 0$ for any sample point $\omega \in \Omega_0$. From now on, we fix such an $\omega$ and we study function $\Theta$ for this fixed sample point. For any $n \geq 1$ and $\tau_{n-1} < t < \tau_n$, $d\Theta(t)/dt \in (\theta_n - (\theta_{n-1})/\gamma_n$. By Proposition 2

$$d\Theta(t)/dt \in \xi_n + r_n + F^\delta_n((\theta_n - \theta_{n-1})) \quad (\tau_{n-1} < t < \tau_n).$$

The following property is easy to check. For any set-valued function $F$, any $r \in \mathbb{R}^{d}$, $\delta > 0$,

$$\forall (\theta, \theta') \in \mathbb{R}^{d} \times \mathbb{R}^{d}, \quad r + F^\delta(\theta) \subset F^{\delta + |r| + |\theta - \theta'|}(\theta') .$$

Now, for any $n$ and any $\tau_{n-1} < t < \tau_n$, define $\eta(t) := \delta_n + |r_n| + |\Theta(t) - (\theta_{n-1})|$ and $U(t) = \xi_n$. We obtain:

$$d\Theta(t)/dt - U(t) \in F^{\gamma}(\Theta(t)) .$$

It is straightforward to show from (20) that for any $T > 0$, $\sup_{0 \leq s \leq T} |\int_{t}^{t+s} U(s)ds|$ tends to zero as $t \to \infty$ (we refer the reader to Proposition 1.3 in reference [7] for details). We now prove that $\eta(t)$ tends to zero as $t \to \infty$. To this end, note that for any $\tau_{n-1} < t < \tau_n$,

$$|\eta(t)| \leq \delta_n + |r_n| + |\Theta(t) - (\theta_{n-1})| ,$$

thus,

$$|\eta(t)| \leq \delta_n + (1 + \gamma_n)|r_n| + \gamma_n \sup_{\theta \in G} \nabla f(\theta) + \gamma_n|\theta_{n-1}| + \gamma_n|\xi_n| .$$

The first three terms of the right-hand side of the above inequality converge to zero as $\gamma_n \to 0$, $r_n \to 0$ and $\delta_n \to 0$. The fourth term tends to zero as well because $|g_i(\theta)|$ is uniformly bounded in $(\gamma, \theta)$, as remarked in the proof of Proposition 2. Finally, $\gamma_n|\xi_n$ tends to zero by (20). Thus $\eta(t)$ tends to zero as $t \to \infty$. This completes the proof of Proposition 3.

**F. Study of the Differential Inclusion Dynamics**

**Proposition 4:** [2] Any solution $x$ of the differential inequality $dx/dt \in F(x)$ with $F$ defined by eq. (22) such that $\forall t \in \mathbb{R}$, $x(t) \in G$ satisfies for almost all $t \in \mathbb{R}$:

$$dx/dt (t) = P_{T_G(x(t))}(-\nabla f(x(t)))$$

where $P_{T_G(x)}$ stands for the projection onto the tangent cone $T_G(x)$ at point $x$.

**Proof:** For the sake of completeness, we reproduce here the proof of reference [2, pp. 266]. Let $x(t)$ be a solution of eq. (14); it is differentiable for almost every $t$ by definition. At such $t$ one has $dx/dt(t) = \lim_{n \to 0} x(t + \epsilon) - x(t) \in T_G(x(t))$ since $x(t) \in G$ for all $t$. For the same reason (using $e < 0$), one has $-dx/dt(t) \in T_G(x(t))$. By convexity of $G$, $T_G(\theta)$ is the dual cone of $N_G(\theta)$ for every $\theta$. Hence $\forall t \in [0, \infty)$:

$$dx/dt(t), v \in 0 \Rightarrow \langle \frac{dx/dt(t), v} > 0. \text{ Now, considering that } x \text{ is a solution of eq. (14) with } F \text{ defined by eq. (22): } dx/dt(t) = -\nabla f(x(t)) \text{ for some } v \in N_G(x(t)).$$

To conclude, $-\nabla f(x(t))$ can be written $v + dx/dt(t)$ with $\langle \frac{dx/dt(t), v} > 0$ and it is a classical fact from convex analysis (see, for instance, reference [34]) that, $N_G(x(t))$ and $T_G(x(t))$ are dual cones: $v = P_{T} (-\nabla f(x(t)))$.

The following proposition is straightforward to prove but has an important role.

**Proposition 5:** Any solution $x$ of eq. (30) admits $f$ as a Lyapunov function for the set $L$ of KKT points defined by eq. (12).

**Proof:** One has $\frac{dx/dt}{dt} f(x(t)) = \langle \nabla f(x(t)), \frac{dx/dt}{dt} \rangle$. From Proposition 4 we deduce:

$$\frac{dx/dt}{dt} f(x(t)) = -\langle \nabla f(x(t)), P_{T_G(x(t))}(-\nabla f(x(t))) \rangle .$$

When $T$ and $N$ are dual cones, the decomposition $v = P_{T}(v) + P_{N}(v)$, $\langle P_{T}(v), P_{N}(v) \rangle = 0$ holds. Using this decomposition with $v = -\nabla f(x(t))$, $T = T_G(x(t))$ and $N = N_G(x(t))$, one deduces

$$\frac{dx/dt}{dt} f(x(t)) = -|P_{T_G(x(t))}(-\nabla f(x(t)))|^2 .$$

This gives the sought result.

We are now in a position to apply Theorem 2 with Lyapunov function $f$ itself and $\Lambda$ the set of KKT points of our optimization program (see Proposition 5).

**V. APPLICATION: POWER ALLOCATION IN AD-HOC WIRELESS NETWORKS**

The context of power allocation for wireless networks has recently raised a great deal of attention in the field of distributed optimization and game theory [37], [5], [22]. Application of distributed optimization to power allocation has been previously investigated in [30].

Consider an $ad$-hoc network composed of $N$ source-destination pairs. We focus on the so-called interference channel i.e., the signal received by the destination of a given pair $i = 1, \ldots, N$ is corrupted both by an additive white Gaussian noise of variance $\sigma_i^2$ and by the interference produced by other sources $j \neq i$. Denote by $P_i^t \geq 0$ the transmission power of source $i$. The power of the useful received signal at the destination $i$ is proportional to $P_i^{\delta}\delta_{i}$ where $\delta_{i}^{\delta}$ represents the channel gain between the $i$th source and the corresponding destination. On the other hand, the level of interference endured by the destination $i$ is proportional to $\sum_{j \neq i} p_j^{\delta}\delta_{j,i}$ where $\delta_{j,i}^{\delta}$ is the (positive) channel gain between source $j$ and destination $i$. As will be explained below, the reliability of the
link between the $i$th source and its destination is usually expressed as an increasing function of the signal to interference-plus-noise ratio, defined as $A^{i,j}p_i/(\sigma_i^2 + \sum_{j \neq i} A^{j,i}p_j)$.

We assume that there is no Channel State Information at the Transmitter (no CSIT) i.e., all channel gains are unknown at all transmitters. However, we assume that the destination associated with the $i$th source-destination pair

- knows the set of channel gains $A^i := (A^{i,1}, \ldots, A^{i,N})^T$,
- ignores all other channel gains $A^j$ for $j \neq i$.

Figure [1] below illustrates the interference channel with $N = 2$ transmit-destination pairs. We assume that $0 \leq p^i \leq P_i$.

![Example of a 2 x 2 interference channel](image)

where $P_i$ is the maximum allowed power for user $i$. Define $\theta = (p_1, \ldots, p_N)^T$ as the vector of all powers of all users. The aim is to select a relevant value for parameter $\theta$. We assume that destinations are able to communicate according to an underlying connected graph. The proposed algorithm works as follows.

1) In a first step, the set of destination nodes cooperate and jointly search for a relevant global power allocation $\theta$. The desired vector $\theta$ corresponds to a local minimizer of an optimization problem which will be made clear below.

2) Once an agreement is found on the power allocation vector $\theta$, each destination $i$ possesses its own source with the corresponding power $p^i$ using a dedicated channel.

Consider fixed deterministic channels. As a performance metric, consider the error probability observed at each destination. Assuming for instance that each transmitter uses a 4-QAM modulation, the error probability at the $i$th destination is given by [39, Section 3.1]:

$$P_{e,i}(\theta, A^i) := Q \left( \frac{A^{i,i}p_i}{\sigma_i^2 + \sum_{j \neq i} A^{j,i}p_j} \right).$$  \hspace{1cm} (31)

where $Q(x) = (\sqrt{2\pi})^{-1} \int_x^{\infty} e^{-t^2/2}dt$. We investigate the following minimization problem:

$$\min_{\theta \in G} \sum_{i=1}^{N} \beta_i P_{e,i}(\theta, A^i)$$  \hspace{1cm} (32)

where $\beta_i$ is an arbitrary positive deterministic weight known only by agent $i$ and where $G := \{ (p_1, \ldots, p_N) \in \mathbb{R}^N : \forall i = 1, \ldots, N, 0 \leq p^i \leq P^i \}$. The above optimization problem is non-convex. Note that, utility functions [31] can of course be replaced by any other continuously differentiable functions of the signal-to-interference-plus-noise ratio without changing the results of this section. Section II suggests the following deterministic distributed gradient algorithm. Each user $i$ has an estimate $\hat{\theta}_{n,i}$ of the whole vector $\theta$ at the $n$th iteration. Here, we stress the fact that a given user has not only an estimate of its own power allocation $p^i$, but also an estimate of what should be the power allocation of other users $j \neq i$. Denote by $\theta_n = (\theta_{n,1}^T, \ldots, \theta_{n,N}^T)^T$ the vector of size $N^2$ which gathers all local estimates. Denote by $A := ((A^1)^T, \ldots, (A^N)^T)^T$ the vector which gathers all $N^2$ channel gains. The distributed algorithm writes:

$$\theta_n = (W_n \otimes I_d)P_{GN} [\theta_{n-1} + \gamma_n \nabla \mathcal{Y} (\theta_{n-1}; A)]$$  \hspace{1cm} (33)

where for any $\theta = (\theta_1^T, \ldots, \theta_N^T)^T \in \mathbb{R}^{N^2}$, we set

$$\nabla \mathcal{Y} (\theta; A) = (\beta_1 \nabla \theta P_{e,1}(\theta; A^1)^T, \ldots, \beta_N \nabla \theta P_{e,N}(\theta; A^N)^T)^T$$

and where $\nabla \theta$ is the gradient operator with respect to the first argument $\theta$ of $P_{e,i}(\theta, A^i)$.

**Corollary 1:** Under the stated assumptions on the sequences $(\gamma_n)_{n \geq 1}$ and $(W_n)_{n \geq 1}$, the algorithm (33) is such that sequence $(\theta_n)_{n \geq 1}$ converges to the set of KKT points of (32).

**Remark:** In many situations, the channel gains are random and rapidly time-varying. In this case, it is more realistic to assume that each destination $i$ observes a sequence of random i.i.d. channel gains $(A_n^i)_{n \geq 1}$. The algorithm (33) can be extended to this context without difficulty. This yields an algorithm which for solving the following optimization problem:

$$\min_{\theta \in G} \sum_{i=1}^{N} \beta_i E[P_{e,i}(\theta, A^i)].$$  \hspace{1cm} (34)

**VI. NUMERICAL RESULTS**

**A. Scenario #1**

As a benchmark, we address the convex optimization scenario formulated in [23]. Define $G \subset \mathbb{R}^2$ as the unit disk in $\mathbb{R}^2$ centered at the origin. Consider the minimization of $\sum_{i=1}^{N} f_i(\theta)$ w.r.t. $\theta \in G$, where for any $i = 1, \ldots, N$, $f_i(\theta) := E[(R_i - s_i^2 \theta)^2]$. Here, $(R_1, \ldots, R_N)$ is a collection of i.i.d. real Gaussian distributed random variables with mean 0.5 and unit variance, and $(s_1, \ldots, s_N)$ is a collection of deterministic elements of $\mathbb{R}^2$. The number of agents is set as $N = 10$ or $N = 50$. We used different graphs: the complete graph where any agent is connected to all other agents, and the cycle. We evaluate the performance of both pairwise and broadcast algorithms described in Section II-C. The weighting coefficient $\beta$ used to compute the average is set to $\beta = 0.5$. As for comparison, we also evaluate the performance of the broadcast-based algorithm of [23]. The common point between the algorithm of [23] and the broadcast algorithm described in Section II-C is that they both rely on the broadcast gossip scheme of [3] but the core of the algorithms is rather different as explained in Section IV. In order to distinguish both broadcast algorithms, we will designate the algorithm of [23] as the broadcast algorithm with sleeping phases, referring to the fact that each agent does not update its estimates as long as it is not the recipient of a message. On the other hand, we refer to the broadcast algorithm of Section II-C as the broadcast algorithm without sleeping phases.
It is worth remarking that a fair comparison between different stochastic approximation algorithms is generally a delicate task, because the behavior of each particular algorithm is sensitive to the choice of the stepsize. In this paragraph, we simply set $\gamma_n = \gamma_0/n^k$ for all $n$, where $\gamma_0 > 0$ and $0.5 < \xi \leq 1$ are parameters chosen in an ad-hoc fashion. More degrees of freedom are of course possible when choosing $\gamma_n$, but a complete discussion would be out the scope of this paper. Recall that the algorithm of [23] requires a more specific choice of the stepsize which solely depends on the initial step. We shall denote by $\gamma^0_0$ the latter initial stepsize used with the algorithm [23], where the upper superscript $s$ stands for sleeping phases.

For each algorithm, we evaluate the deviation of the estimates from the global minimizer $\theta^*$:

$$\Delta_n := \left(\frac{1}{N} \sum_{i=1}^{N} |\theta_{n,i} - \theta^*|^2\right)^{1/2}.$$  

Note that $\Delta_n$ depends on the parameters $(s_1, \cdots, s_N)$. We consider 50 Monte-Carlo runs, each of them consisting of 10000 iterations of each algorithm. For each run, we randomly select the parameters $(s_1, \cdots, s_N)$ according to the uniform distribution on the unit disk $G$. The $k$th Monte-Carlo run yields a sequence $(\Delta_n^{(k)} : 1 \leq n \leq 10000)$ for each algorithm.

Figure 2 represents the average deviation $(1/50) \sum_{k=1}^{50} \Delta_n^{(k)}$ as a function of the number $n$ of iterations. In Figure 2(a), we set $N = 50$ and the graph is a cycle. In Figure 2(b), we set $N = 10$ and the graph is a complete graph. It is worth noting that the pairwise gossip algorithm behaved at least as well as both broadcast based algorithms in our experiments. This fact might seem surprising at first glance. Indeed, in the framework of average consensus i.e., when the aim is not to optimize an objective function but simply to compute an average in a distributed fashion [12], the broadcast gossip algorithm of [3] is known to i) reach a consensus faster than the pairwise algorithm of [12] and ii) fail to converge to the desired value. In the context of distributed optimization, a different phenomenon happens: our theoretical analysis showed that broadcast based optimizers do converge to the desired value. However, in this example, there is no clear gain in using a broadcast-based algorithm. Convergence has been established in Theorem 1. Appendix A reveals that part of the perturbation (denoted $\xi_n$) is due to the fact that $1^T W_n \neq 1^T$ (see the term $\zeta_n^{(1)}$ at equation (35)). This part of the perturbation is clearly zero when $W_n$ is doubly stochastic. This is the case for the pairwise algorithm, but not for the broadcast algorithm.

As a conclusion, there should be interesting comparisons to make between pairwise optimizer and the broadcast ones.

B. Scenario #2

Consider the distributed power allocation algorithm of Section V. In order to validate the proposed algorithm, we study the $2 \times 2$ interference channels shown in Figure 4. As a toy but revealing example, first assume fixed channel gains chosen as $A^{1,1} = A^{2,2} = 2, A^{1,2} = A^{2,1} = 1$. The noise variance is equal to $\sigma^2_1 = \sigma^2_2 = 0.1$. The powers $p^1$ and $p^2$ of the users must not exceed a maximum power of $P_1 = P_2 = 10$. The aim is to minimize the weighted sum of the error probabilities as in (22) where $\beta_1 = 2/3, \beta_2 = 1/3$. Strictly speaking, we actually implement a distributed gradient descent w.r.t. to the parameter vector $\theta$ in log-scale in order to avoid slow convergence. Figure 3(a) represents the objective function $J(\theta)$ w.r.t. $(p^1, p^2)$ in dB (the $x$-axis and $y$-axis are $10 \log_{10} P^1$ and $10 \log_{10} P^2$ respectively). On this example, there exists a unique minimum achieved at point $(p^1, p^2) = (10, 5.4)$. Figure 3(b) represents, on a single run, the trajectory of the estimates $\theta_{n,1} = (p^1_{n,1}, p^2_{n,2})$ of the first agent as a function of the number of iterations. We compare the pairwise and the broadcast gossip schemes. Note that we only plot the result for the broadcast scheme without sleeping phase, as we observed slow convergence of the algorithm of [23] on this particular example. The two upper curves represent the estimate of power $p_1$ (using a pairwise and a broadcast scheme respectively) while the two lower curves represent the estimate of power $p_2$. Each algorithm converges to the desired value.
of the powers $p^1$ and $p^2$ in dB - Fixed Deterministic channels - $A^1 = A^{1,2} = 2$, $A^{1,1} = 1 - \beta_1 = 2/3$, $\beta_1 = 1/3$ - $\sigma_1^2 = 0.1$. $\mathcal{P}_1 = \mathcal{P}_2 = 10$. The minimum is achieved at point $(p^1, p^2) = (10, 5.4)$. (b) First agent’s estimates of $p^1$ and $p^2$ as a function of the number of iterations - $\gamma_n = 200/n^{0.7}$ for $n \leq 3000$ - $\gamma_n = 30/n^{0.7}$ for $n > 3000$.

Fig. 3. (a) Weighted sum of error probabilities for $N = 2$ as a function of the powers $p^1$ and $p^2$ in dB - Fixed Deterministic channels - $A^1 = A^{1,2} = 2$, $A^{1,1} = 1 - \beta_1 = 2/3$, $\beta_1 = 1/3$ - $\sigma_1^2 = 0.1$. $\mathcal{P}_1 = \mathcal{P}_2 = 10$. The minimum is achieved at point $(p^1, p^2) = (10, 5.4)$. (b) First agent’s estimates of $p^1$ and $p^2$ as a function of the number of iterations - $\gamma_n = 200/n^{0.7}$ for $n \leq 3000$ - $\gamma_n = 30/n^{0.7}$ for $n > 3000$.

Fig. 4. Powers $p^1$ and $p^2$ as a function of the number of iterations, averaged w.r.t. 50 Monte-Carlo runs - $\gamma_n = 200/n^{0.7}$ for $n \leq 3000$ - $\gamma_n = 30/n^{0.7}$ for $n > 3000$.

VII. CONCLUSION

We introduced a new framework for the analysis of a class of constrained optimization algorithms for multi-agent systems. The methodology uses recent powerful results about dynamical systems which do not rely on the convexity of the objective function, thus addressing a wider range of practical distributed optimization problems. Also, the proposed framework allows to alleviate the common assumption of double-stochasticity of the gossip matrices, and therefore encompasses the natural broadcast gossip scheme. The algorithm has been proved to converge to a consensus. The interpolated process of average estimates is proved to be a perturbed solution to a differential variational inequality, w.p.1. As a consequence, the average estimate converges almost surely to the set of KKT points.

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APPENDIX I

PROOF OF PROPOSITION

From (1) and Assumption (e), it is straightforward to show that the decomposition (19) holds if one sets:

$$r_n = \nabla f(\langle \theta_{n-1} \rangle) - \frac{1}{N} \sum_{i=1}^{N} \nabla f_i(\theta_{n-1,i})$$

and $\xi_n = \xi_n^{(1)} + \xi_n^{(2)}$ where:

$$\xi_n^{(1)} := \frac{1}{\gamma_n} \left( \frac{1}{N} W_n - \frac{1}{N} I_d \right) P_{G^n} [\theta_{n-1} + \gamma_n Y_n]$$

and

$$\xi_n^{(2)} := (Z_n) - \mathbb{E}(Z_n) | \mathcal{F}_{n-1}$$

where $\gamma_n = \gamma_n^{(1)} + \gamma_n^{(2)}$.
where $Z_n$ is given by (16). We first prove that $r_n$ tends to zero. Remark that:

$$|r_n| \leq \frac{1}{N} \sum_{i=1}^{N} |\nabla f_i((\theta_{n-1})_i) - \nabla f_i((\theta_{n-1},i))|.$$ 

Each gradient $\nabla f_i$ is continuous, and thus uniformly continuous on the compact set $G$. By Lemma [1] $|\theta_{n-1} - \theta_{n-1,i}|$ converges to zero a.s. for any $i$. Therefore, $r_n$ converges a.s. to zero. To prove Proposition [1], it is thus sufficient to show that $\sup_{k \geq n} \gamma_k \xi_k^{(1)}$ tends to zero when $n \to \infty$ for $j = 1, 2$.

First, consider $\xi_k^{(1)}$. Recalling that $W_n$ is row-stochastic, it follows that $\{(1^T W_n - I) \otimes I_d\} J = 0$. Thus, one may write:

$$\gamma_k \xi_k^{(1)} = \left(\frac{1}{N} \sum_{i=1}^{N} (1^T W_n - I) \otimes I_d\right) (\theta_{1,n-1} + \gamma_n Z_n),$$

where the random vector $Z_n$ is given by (16). Define $M_n := \sum_{k=1}^{n} \gamma_k \xi_k^{(1)}$. It is straightforward to show that $M_n$ is a martingale adapted to $(\mathcal{F}_n)_{n \geq 1}$. Indeed, by Assumption [1(c)], $W_n$ and $Z_n$ are independent conditioned on $\mathcal{F}_{n-1}$. Therefore:

$$\mathbb{E}[\gamma_k \xi_k^{(1)} | \mathcal{F}_{n-1}] = \left(\frac{1}{N} \sum_{i=1}^{N} (1^T W_n - I) \otimes I_d\right) (\theta_{1,n-1} + \gamma_n \mathbb{E}[Z_n | \mathcal{F}_{n-1}]) = 0$$

because $1^T \mathbb{E}(W_n) = 1^T$ by Assumption [1(a)]. We derive:

$$\mathbb{E}[M_n^2] = \sum_{k=1}^{n} \mathbb{E} \left[ \left(\frac{1}{N} \sum_{i=1}^{N} (1^T W_n - I) \otimes I_d\right) (\theta_{1,k-1} + \gamma_k Z_k) \right]^2 \leq \sum_{k=1}^{n} \mathbb{E} \left[ (\theta_{1,k-1} + \gamma_k Z_k)^2 \mathbb{V}(\theta_{1,k-1} + \gamma_k Z_k) \right]$$

where $\mathbb{V}_k \equiv \left(\frac{(1^T W_k - I)^T (1^T W_k - 1^T)}{N} \otimes I_d\right)$ and where the last equality is obtained by integrating the inner terms w.r.t. $W_k$ only. Remark that $\mathbb{E}(W_k^T - I - 1) (1^T W_k - 1^T) = \mathbb{E}(W_k^T - 1^T W_k - 1^T)$. As the spectral radius of matrix $W_k^T - 1^T W_k$ is uniformly bounded, there exists a constant $C' > 0$ such that:

$$\mathbb{E}[M_n^2] \leq C' \sum_{k=1}^{\infty} \mathbb{E}[|\theta_{1,k-1} + \gamma_k Z_k|^2] \leq 2C' \sum_{k=1}^{\infty} \mathbb{E}[|\theta_{1,k-1}|^2] + \sum_{k=1}^{\infty} \gamma_k^2 \mathbb{E}[|Z_k|^2].$$

By Lemma [1] the first term in the right-hand side of the above inequality is finite. Recalling that $|Z_k| \leq |Y_k|$, we deduce from Assumption [1(f)] that $\mathbb{E}[Z_k^2]$ is uniformly bounded. As $\sum_{k=1}^{\infty} \gamma_k^2 < \infty$, we conclude that $\sup_n \mathbb{E}[M_n^2] < \infty$. This implies that the martingale converges a.s. to a finite random variable $M_\infty$. Thus, for any $k \geq n$,

$$\left| \sum_{k=n}^{k=\infty} \gamma_k \xi_k^{(1)} \right| = \left| (M_k - M_\infty) - (M_{n-1} - M_\infty) \right|.$$

Therefore, $\sup_{k \geq n} \left| \sum_{k=n}^{\infty} \gamma_k \xi_k^{(1)} \right|$ tends a.s. to zero as $n \to \infty$.

We now study $\xi_k^{(2)}$. Clearly, $\xi_k^{(2)}$ is a martingale difference noise sequence. Therefore,

$$\mathbb{E}\left[ \left( \sum_{k=1}^{n} \gamma_k \xi_k^{(2)} \right)^2 \right] = \sum_{k=1}^{\infty} \gamma_k^2 \mathbb{E}\left[ (\mathbb{E}[Z_k | \mathcal{F}_{k-1}])^2 \right] \leq \sum_{k=1}^{\infty} \gamma_k^2 \mathbb{E}[|Z_k|^2] \leq \sup_{n \geq 1} \mathbb{E}[|Y_n|^2] \sum_{k=1}^{\infty} \gamma_k^2,$$

where we used the fact that $|Y_k| \leq |Z_k|$ for any $k$. Note that $\mathbb{E}[|Y_n|^2 | \mathcal{F}_{n-1}]$ is uniformly bounded with $\sup_\theta \int |y|^2 d\mu_\theta$ by Assumption [1(c)]. The latter constant is finite by Assumption [1(f)]. Therefore, $\sup_{n \geq 1} \mathbb{E}[|Y_n|^2] \to 0$. Since $\sum_{k=1}^{\infty} \gamma_k^2 < \infty$, we conclude that $\frac{1}{\gamma} \sum_{k=1}^{\infty} \gamma_k^2$ tends to zero using the same arguments. This completes the proof of Proposition [1].

**APPENDIX II**

**PROOF OF LEMMA [2]**

Let us define $\mathbb{Q}(\theta)$ as the matrix $\mathbb{Q}(\theta) := [\nabla q_j(\theta)]_{j \in A(\theta)}$. Denote by $\Lambda_1(\theta)$ the smallest eigenvalue of $\mathbb{Q}(\theta)^T \mathbb{Q}(\theta)$. We first show that $\Lambda_1(\theta)$ is lower semicontinuous, i.e., for a sequence $\theta_n \in G$ converging to $\theta \in G$:

$$\Lambda_1(\theta_n) \leq \liminf_{n \to \infty} \Lambda_1(\theta_n). \quad (36)$$

Continuity of all functions $q_j$ ensures that $A(\theta)$ is upper semicontinuous, i.e., for any $\theta$ in a neighborhood of $\theta_0$, $A(\theta) \subset A(\theta_0)$. Hence, for $n$ large enough $A(\theta_n) \subset A(\theta)$. Denote by $Q(\theta_n)$ the matrix $d \times p$:

$$\tilde{\mathbb{Q}}(\theta_n) = [\nabla q_j(\theta_n) \mathbb{I}_{A(\theta_n)}(j)]_{j=1}^{d}$$

where $\mathbb{I}_A$ stands for the indicator function of set $A$. There exists a sequence of $p \times 1$ vectors $\tilde{v}_n$ with unit norm such that $|\tilde{Q}(\theta_n) \tilde{v}_n|^2 = \Lambda_1(\theta_n)$ and $\tilde{v}_n(j) = 0$ if $j \notin A(\theta_n)$. Since $\tilde{v}_n$ has unit norm, one can extract a converging subsequence $\tilde{v}(\phi_n)$ towards a unit norm $p \times 1$ vector $\tilde{v}_n$ such that $|\tilde{Q}(\theta_n) \tilde{v}(\phi_n)|^2$ converges to $\liminf \Lambda_1(\theta_n)$. Using the inclusion $A(\theta_n) \subset A(\theta)$, one has $\tilde{v}_n(j) = 0$ when $j \notin A(\theta)$. Moreover, under Assumption [3](b), functions $\nabla q_j$ are continuous, which implies that $\tilde{Q}(\theta_n)$ converges to $\tilde{Q}(\theta)$. Hence vector $\tilde{v}_n$ satisfies $|\tilde{Q}(\theta_n) \tilde{v}_n|^2 = \liminf |\tilde{Q}(\theta_n) \tilde{v}(\phi_n)|^2$. Since $\tilde{v}_n(j) = 0$ when $j \notin A(\theta)$, there exists a vector $\tilde{v}_n$ such that $|\tilde{Q}(\theta_n) \tilde{v}_n|^2 \leq \liminf |\tilde{Q}(\theta_n) \tilde{v}(\phi_n)|^2$. Hence

$$\Lambda_1(\theta_n) \leq |\tilde{Q}(\theta_n) \tilde{v}_n|^2 = |\tilde{Q}(\theta) \tilde{v}_n|^2 \leq \liminf \Lambda_1(\theta_n).$$

This proves (36). Under Assumption [3](a) $G$ is a compact set, so lower semicontinuity of $\Lambda_1(\theta)$ ensures that $\Lambda_1$ reaches its minimum $m > 0$ ($m = 0$ would contradict Assumption [3](c)). Now, let us denote by $\lambda := (\lambda_2(\theta,\gamma,y))^T$ and $\nu := \frac{1}{\gamma}(\theta + \gamma - P_G(\theta + \gamma))$. One has

$$\lambda = \left(\mathbb{Q}(P_G(\theta + \gamma)) \mathbb{Q}(P_G(\theta + \gamma))^T\right)^{-1} \mathbb{Q}(P_G(\theta + \gamma)) \nu.$$

Hence $|\lambda| \leq \Lambda_1^{-1}(P_G(\theta + \gamma))|\mathbb{Q}(P_G(\theta + \gamma)) \nu|$. Continuity of $\nabla q_j$ and compactness of $G$ ensure the existence of $L > 0$.
such that: \(|Q(P_G(\theta + \gamma y))^2 v| \leq L|v|\). To conclude, remark that \(|v| \leq |y|\) so \(|\lambda| \leq \frac{L}{m}|y|\). Hence,

\[
\int \lambda_j(\theta_i, \gamma, y_i)^2 d\mu(y_i, \ldots, y_N) \leq \left( \frac{L}{m} \right)^2 \int |y_i|^2 d\mu(y_i, \ldots, y_N) < \infty.
\]

**Appendix III**

**Proof of Lemma 3**

Define constant \(M_1\) as the supremum in equation (24): \(0 \leq M_1 < \infty\) by Lemma 2. We set \(M_2 = \sup_{\theta \in G, j=1,\ldots,p} |\nabla q_j(\theta)|\). Define for any \(x > 0\), \(M(x) = \sup_{\theta \in G,N} \int 1_{\{|y| > x^{-1/2}\}} d\mu(y)\) and:

\[
\epsilon(x) = \sqrt{x} + 2p\sqrt{M_1}\phi(\sqrt{x} + 2p\sqrt{M_1}M_2(M(x)))^{1/2}
\]

where we recall the definition (25) of \(\phi\). Using the fact that \(\phi(x)\) tends to zero as \(x \downarrow 0\) and using the tightness of the family \((\mu(\theta)_{\theta \in G})\) (which is a consequence of Assumption (I)), it is straightforward to show that \(\epsilon(x)\) tends to zero as \(x \downarrow 0\). Set \(y_i : N = y_1, \ldots, y_N\). We decompose \(-g_\gamma(\theta)\) as \(-g_\gamma(\theta) = s_\gamma(\theta) + t_\gamma(\theta) + u_\gamma(\theta)\) where:

\[
s_\gamma(\theta) = \frac{1}{N} \sum_{i=1}^N \sum_{j \in A(P_G[\theta_i + \gamma y_i])} \lambda_j(\theta_i, \gamma, y_i) 1_{|y_i| \leq 1/\sqrt{\gamma}} \nabla q_j((\theta_i)) d\mu(y_i; N)
\]

\[
t_\gamma(\theta) = \frac{1}{N} \sum_{i=1}^N \sum_{j \in A(P_G[\theta_i + \gamma y_i])} \lambda_j(\theta_i, \gamma, y_i) 1_{|y_i| > 1/\sqrt{\gamma}} \nabla q_j(P_G[\theta_i + \gamma y_i]) d\mu(y_i; N)
\]

\[
u_\gamma(\theta) = \frac{1}{N} \sum_{i=1}^N \sum_{j \in A(P_G[\theta_i + \gamma y_i])} \lambda_j(\theta_i, \gamma, y_i) 1_{|y_i| \leq 1/\sqrt{\gamma}} \nabla q_j(P_G[\theta_i + \gamma y_i]) d\mu(y_i; N)
\]

Consider first \(s_\gamma(\theta)\). When the indicator \(1_{|y_i| \leq 1/\sqrt{\gamma}}\) is active (equal to one), inequality \(|y_i| \leq 1/\sqrt{\gamma}\) holds true. In this case,

\[
|P_G[\theta_i + \gamma y_i] - (\theta_i)| \leq |P_G[\theta_i + \gamma y_i] - \theta_i| + |\theta_i - (\theta_i)| \leq |\gamma y_i| + |\theta_i| \leq \sqrt{\gamma} + |\theta_i| \leq \epsilon(\sqrt{\gamma} + |\theta_i|)
\]

Therefore, as soon as \(|y_i| \leq 1/\sqrt{\gamma}\), \(A(P_G[\theta_i + \gamma y_i])\) is included in the set \(A((\theta_i), \epsilon(\sqrt{\gamma} + |\theta_i|))\) where \(A\) is defined by (26). As a consequence,

\[
s_\gamma(\theta) = \sum_{j \in A((\theta_i), \epsilon(\sqrt{\gamma} + |\theta_i|))} \alpha_j \nabla q_j((\theta_i))
\]

where \(\alpha_j := \frac{1}{N} \sum_{i=1}^N \lambda_j(\theta_i, \gamma, y_i) 1_{|y_i| \leq 1/\sqrt{\gamma}} d\mu(y_i; N)\). By Jensen’s inequality, \(0 \leq \alpha_j \leq \sqrt{M_1}\) for any \(j\).

Consider the second term \(t_\gamma(\theta)\). It is straightforward to show from triangle and Cauchy-Schwartz’s inequalities that:

\[
|t_\gamma(\theta)| \leq \frac{M_2}{\sqrt{N}} \sum_{i=1}^N \sum_{j=1}^p \left( \int \lambda_j(\theta_i, \gamma, y_i)^2 d\mu(y_i; N) \right)^{1/2} \int \left( 1_{|y_i| > 1/\sqrt{\gamma}} d\mu(y_i; N) \right)^{1/2} \leq p\sqrt{M_1}M_2(M(\gamma))^{1/2} \leq 0.5 \epsilon(\sqrt{\gamma} + |\theta_i|)
\]

Finally, consider \(u_\gamma(\theta)\):

\[
|u_\gamma(\theta)| \leq \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^p \int \lambda_j(\theta_i, \gamma, y_i) 1_{|y_i| \leq 1/\sqrt{\gamma}} \left| \nabla q_j(P_G[\theta_i + \gamma y_i]) - \nabla q_j((\theta_i)) \right| d\mu(y_i; N)
\]

Note that \(|\nabla q_j(P_G[\theta_i + \gamma y_i]) - \nabla q_j((\theta_i))| \leq \phi(|P_G[\theta_i + \gamma y_i] - (\theta_i)| \leq \sqrt{\gamma} + |\theta_i|\). As \(\phi\) is non decreasing, it is clear that \(1_{|y_i| \leq 1/\sqrt{\gamma}} \phi(|P_G[\theta_i + \gamma y_i] - (\theta_i)| \leq \phi(\sqrt{\gamma} + |\theta_i|)\). Therefore,

\[
|u_\gamma(\theta)| \leq p\sqrt{M_1} \phi(\sqrt{\gamma} + |\theta_i|) \leq 0.5 \epsilon(\sqrt{\gamma} + |\theta_i|)
\]

This completes the proof of Lemma 3.

**Appendix IV**

**Proof of Lemma 4**

For any \(\epsilon \geq 0\), \(j = 1, \ldots, p\), define \(\partial G_j := \{\theta \in G : \exists \theta' \in q_j^{-1}(\{0\}), |\theta' - \theta| \leq \epsilon\}\). It is useful to remark that \(\partial G_j = q_j^{-1}(\{0\}) \cap G\) is the set of points in \(G\) for which the \(j\)th constraint is active. In particular, that \(\partial G_j \subset \partial G_j^0\) for any \(\epsilon \geq 0\). Denote by \(d_H\) the Hausdorff distance between sets. Define:

\[
\delta(\epsilon) = \max_{E \subset \{1, \ldots, p\}} d_H \left( \bigcap_{j \in E} \partial G_j, \bigcap_{j \in E} \partial G_j^0 \right)
\]

The key point is to show that \(\lim_{\epsilon \downarrow 0} \delta(\epsilon) = 0\). By contradiction, assume that this is not the case. Then there exists a constant \(c > 0\) and a sequence \(\epsilon_n \downarrow 0\) such that \(\delta(\epsilon_n) > c\) for each \(n\). As there is a finite number of subsets of \(\{1, \ldots, p\}\), it is straightforward to show that there exists a certain subset \(E \subset \{1, \ldots, p\}\) such that for any \(n \geq 1\),

\[
d_H \left( \bigcap_{j \in E} \partial G_j, \bigcap_{j \in E} \partial G_j^0 \right) > c
\]

First note that \(\bigcap_{j \in E} \partial G_j^0\) is nonempty. Indeed, if it was empty, \(\bigcap_{j \in E} \partial G_j^0\) would be empty as well, so that the Hausdorff distance in the lefthand side of (37) would be \(d_H(0,0) = 0 < c\). Thus, for any \(n \geq 1\), there exists \(\theta_n \in \bigcap_{j \in E} \partial G_j^0\) such that

\[
d_H (\theta_n, \bigcap_{j \in E} \partial G_j) > c
\]

The sequence \((\theta_n)_{n \geq 1}\) lies in the compact set \(G\). Thus, there exists a subsequence which converges to some point
θₙ ∈ G. Without loss of generality, we shall still denote this subsequence by (θₙ)ₙ≥1 in order to simplify the notations. We thus consider that \lim_{n→∞} θₙ = θ*. We shall now prove that θₙ ∈ ∩ₖ∈E ΩGₖ. For any n ≥ 1, θₙ ∈ ∩ₖ∈E ΩGₖ. Thus, there exists θ⁽j⁾ ∈ G such that qⱼ(θ⁽j⁾) = 0 and ∥θ⁽j⁾ − θ⁽j+1⁾∥ ≤ εₙ. As qⱼ is convex, it is also Lipschitz on the compact set G. Denote by Kⱼ its Lipschitz constant on G:

\[ |qⱼ(θₙ) − qⱼ(θ⁽j⁾)| ≤ Kⱼεₙ. \]

Since qⱼ is continuous and εₙ ↓ 0, it follows that qⱼ(θₙ) = 0. Thus θₙ ∈ ∩ₖ∈E ΩGₖ. Therefore, by (58), ∥θₙ − θ*∥ > εₙ > 0. This contradicts the fact that (θₙ)ₙ≥1 converges to θ*. This proves that δ(ε) tends to zero as ε ↓ 0.

It is useful to remark that, as a by-product of the above proof, we also obtained the following result. Consider any set E ⊂ {1, · · · , p} and assume that there exists a sequence εₙ ↓ 0 s.t. for any n ≥ 1 there exists θₙ ∈ ∩ₖ∈E ΩGₖ. Due to the above arguments, any limit point of such a sequence (θₙ)ₙ≥1 belongs to the set ∩ₖ∈E ΩGₖ which is thus nonempty. Let us state this result the other way around: for any E such that ∩ₖ∈E ΩGₖ = ∅, there exists E₀ > 0 such that for any ε < E₀, ∩ₖ∈E ΩGₖ = ∅. We set ε₀ = min{E₀ : ∩ₖ∈E ΩGₖ = ∅}. It remains to prove (58). Let 0 < ε < ε₀ and θ ∈ G. Trivially,

θ ∈ ∩ₖ∈E ΩGₖ.

As ε < ε₀, the set ∩ₖ∈E(A(θ, ε) ∩ Gₖ) is nonempty. There exists θ' in the latter set such that |θ − θ'| ≤ δ(ε). By definition of θ', qⱼ(θ') = 0 for any j ∈ A(θ, ε). This proves that A(θ, ε) ⊂ A(θ').

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