Oscillations of relativistic axisymmetric tori and implications for modelling kHz-QPOs in neutron star X-ray binaries

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ABSTRACT
We perform a global linear perturbative analysis and investigate oscillation properties of relativistic, non-self-gravitating tori orbiting around neutron stars in the slow rotation limit approximation. Extending the work done in Schwarzschild and Kerr backgrounds, we consider the axisymmetric oscillations of vertically integrated tori in the Hartle–Thorne space–time. The equilibrium models are constructed by selecting a number of different non-Keplerian distributions of specific angular momentum, allowing for disc sizes \(L \sim 0.5–600\) gravitational radii. Our results, obtained after solving a global eigenvalue problem to compute the axisymmetric \(p\)-modes, indicate that such oscillation modes could account for most observed lower (\(v_L\)) and upper (\(v_U\)) high-frequency quasi-periodic oscillations for Sco X-1, and for some \(Z\) and Atoll sources with \(v_L \gtrsim 500\) Hz. However, when \(v_L \lesssim 500\) Hz, \(p\)-mode oscillations do not account for the linear relation \(v_U = A v_L + B, B \neq 0\), between the upper and lower high-frequency quasi-periodic oscillations that are observed in neutron star low-mass X-ray binaries.

Key words: stars: neutron – X-rays: binaries.

1 INTRODUCTION
Quasi-periodic oscillations (QPOs) observed in the X-ray spectra of binary systems are transient phenomena associated with nonthermal states and state transitions. QPOs observed at high frequencies, in the range of \(\sim 200–1200\) Hz (van der Klis 2005), are referred to as high-frequency QPOs (kHz-QPOs). An important feature of such kHz-QPOs is that they usually appear in couples consisting of a lower and an upper kHz-QPO. The entire set of kHz-QPOs detected in black hole binaries (BHBs), i.e. binaries having a black hole as an accretor, contains only seven sources,\(^1\) four of which\(^2\) show both the upper and lower kHz-QPOs. On the other hand, there are approximately 20 neutron star binaries (NSBs), i.e. binaries having a neutron star as the accretor, that show the kHz-QPO phenomenology (see Miller 2010 for a recent review of QPOs in NSBs), including both \(Z\) and Atoll type\(^3\) (Lewin & van der Klis 2006). Interestingly, these QPO frequencies correspond to the frequencies of orbits a few gravitational radii away from a stellar-mass compact object, which explains why kHz-QPOs have been considered a promising tool to investigate the gravitational forces in the strong field regime. It should also be noted, however, that in a recent work, Sanna et al. (2010) have shown that NSB J1701–462 provides an example of a source that, during the same outburst, presents spectral and timing characteristic of both \(Z\) and Atoll sources. Because the kHz-QPOs detected are remarkably different in the two spectral states, and because such differences cannot be attributed to changes in the gravitational field of the central compact object, the authors conclude that the coherence and rms amplitude of the kHz-QPOs cannot be used to deduce the existence of the innermost stable circular orbit around a neutron star.

Although it is still not clear whether there is a unique physical mechanism responsible for the generation of kHz-QPOs in both BHBs and NSBs, a few pieces of observational evidence have emerged over the years which may indicate the existence of two distinct mechanisms.

(i) First of all, the kHz-QPO peak separation in NSBs is typically within 20 per cent of the neutron star spin frequency \(v_{\text{spin}}\), or half of that (van der Klis 2005). In particular, sources with \(v_{\text{spin}} \lesssim 400\) Hz have \(\Delta v \sim v_{\text{spin}}\), while sources with \(v_{\text{spin}} > 400\) Hz have \(\Delta v \sim v_{\text{spin}}/2\) (Muno et al. 2001). However, Strohmayer, Markwardt & Kuulkers (2008) have found that the low-mass NSB 4U 614+091 does not show a strong connection between the kHz-QPO frequency difference and the neutron star spin frequency (\(v_{\text{spin}} = 414\) Hz, while \(\Delta v \sim 320\) Hz).
(ii) Secondly, while in NSBs the frequencies observed may vary by a factor of 2 in association with changes of the luminosity, the frequencies in BHBs are much more stable and vary at most by 15 per cent, even when the luminosity changes by a factor of 3 (Remillard & McClintock 2006). This effect is clearly shown in fig. 1 of Belloni, Méndez & Homan (2005) or in fig. 3 of Török et al. (2006), where the upper frequencies, \(v_U\), are plotted versus the lower frequencies, \(v_L\), showing that while kHz-QPOs from a single BHB are represented essentially as a point, kHz-QPOs from a single NSB are scattered along straight lines, the so-called ‘Bursa lines’ (Török et al. 2006; Abramowicz et al. 2007).

(iii) An additional peculiar property of twin kHz-QPOs in the four BHBs, where they have been simultaneously observed, is that such frequencies appear in couples obeying the ratio 3:2 to a high degree of accuracy (see fig. 1 of Török et al. 2005). Over the last few years, there has been an ongoing debate about the existence of the same phenomenon among kHz-QPOs in NSBs. Observations, in fact, clearly indicate that Bursa lines \(v_U = A v_L + B\) of kHz-QPOs in NSBs are not compatible with a constant 3:2 ratio \((A = 1.5, B = 0)\), but it remains controversial whether the peak at 1.5 in the distribution of the observed \(v_U/v_L\) ratios is physical or not (Belloni et al. 2005; Török et al. 2006; Belloni, Méndez & Homan 2007; Török et al. 2008; Boutelier et al. 2010).

Several models have been proposed over the years to explain the physical mechanism responsible for the generation of kHz-QPOs in the X-ray spectra of binary systems (Miller, Lamb & Psaltis 1998; Stella & Vietri 1999; Abramowicz et al. 2003; Lamb & Miller 2003; Rezzolla et al. 2003; Zhang 2004; Arras, Blaes & Turner 2006). In particular, the model proposed by Rezzolla et al. (2003) for kHz-QPOs in BHBs suggests the existence of an oscillating torus around the black hole, and identifies the time variability in the X-ray spectra with inertial-acoustic modes (\(p\)-modes) of the relativistic thick accretion torus which acts as a resonant cavity for the \(p\)-mode axisymmetric oscillations. The frequencies of the fundamental and the first-overtone modes appear approximately in a 3:2 ratio, and within the range of kHz-QPOs depending on the spin of the black hole and the size of the torus. Moreover, the model by Rezzolla et al. (2003) also accounts for the \(M^{-1}\) scaling of the frequency, where \(M\) is the mass of the black hole.

The main properties of the axisymmetric \(p\)-mode oscillations of relativistic, non-self-gravitating tori orbiting black holes have been investigated in a series of papers either using a linear perturbative approach by Rezzolla, Yoshida & Zanotti (2003), Montero, Rezzolla & Yoshida (2004) (hereafter Papers I and II, respectively) or through non-linear general relativistic hydrodynamic simulations of both, non-self-gravitating tori (Zanotti, Rezzolla & Font 2003; Zanotti et al. 2005; Montero et al. 2007) and self-gravitating tori (Montero, Font & Shibata 2008, 2010). The linear perturbative approach, computationally less demanding, allows for a more detailed investigation of the parameter space. Overall, these perturbative analyses confirmed the results of non-linear hydrodynamic simulations which revealed that the lowest-order eigenfrequencies appear in a sequence of small integers 2:3:4:..., for a wide range of models. It is also worth noting that some of these modes have been related in the analysis performed by Blaes, Arras & Fragile (2006) to surface gravity waves, at least in the slender torus limit.

However, it remains unclear whether a kHz-QPO model based on axisymmetric inertial-acoustic oscillations is flexible enough to explain the rich phenomenology of kHz-QPOs in NSBs, and in particular to account for the distribution of the upper and lower kHz-QPOs along the ‘Bursa lines’ observed in NSBs. To address this question we perform a global linear perturbative analysis of equilibrium thick discs around neutron stars, whose external metric is described in the slow limit approximation. The set of perturbed relativistic equations is reduced to the solution of an eigenvalue problem, following the approach described in Papers I and II. In order to explore exhaustively the deviations from the 3:2 ratio that axisymmetric inertial-acoustic oscillations display, we consider a much wider parameter space than that studied in Papers I and II, particularly in terms of the rotation law and the extension of the accretion disc.

The plan of this paper is as follows. In Section 2 we introduce the basic assumptions and equations employed in the definition of our general relativistic, vertically integrated tori. In Section 3 we derive the perturbation equations, and in Section 4 we list the properties of the non-self-gravitating equilibrium models studied. In Section 5, on the other hand, we present the results of the global analysis, while Section 6 is devoted to a discussion of the implications for explaining kHz-QPOs in NSBs. Finally, Section 7 contains our conclusions.

In the following, we will assume a signature \(\{−, +, +, +\}\) for the space–time metric and we will use Greek letters \(\mu, \nu, \lambda, \ldots\) (running from 0 to 3) for four-dimensional space–time tensor components, while Latin letters \(i, j, k, \ldots\) (running from 1 to 3) will be employed for three-dimensional spatial tensor components. We also adopt a geometrized system of units by setting \(c = G = 1\).

### 2 Equilibrium Tori in the Hartle–Thorne Space–Time

Equilibrium thick discs around a rotating neutron star are constructed assuming that their self-gravity can be neglected and that the background space–time takes the form of the Hartle–Thorne metric (Hartle & Thorne 1968), which describes the metric around a slowly rotating neutron star. Since we are interested in the region of the space–time around the equatorial plane, we use cylindrical coordinates \((t, \sigma, \phi, z)\), and consider only the zeroth-order terms in the ratio \(z/\sigma\) (Wilson 1972; Novikov & Thorne 1973), where \(\sigma\) is the cylindrical radial coordinate (see equation 1 in Paper II for this form of the Kerr line element). Then, the Hartle–Thorne metric in cylindrical coordinates is derived from the Kerr metric, in Boyer–Lindquist coordinates, by retaining only terms that are first order in the ratio \(a/\sigma\), where \(a\) is the Kerr parameter, and by replacing the spin of the black hole with the angular momentum of the neutron star. In this way, we obtain

\[
\Delta = \sigma^2 - 2M \sigma a^2 + a^2 \left(1 - \frac{2M}{\sigma a^2} + \frac{\sigma^2}{a^2}\right) 
\]

\[
\simeq \sigma^2 \left(1 - \frac{2M}{\sigma a^2}\right), \quad (1)
\]

\[
A = \sigma^4 + \sigma^2 a^2 + 2M \sigma a^2 - \sigma^4 \left(1 + \frac{a^2}{\sigma^2} + \frac{2M}{\sigma^3 a^2}\right) 
\]

\[
\simeq \sigma^4,\quad (2)
\]

\[
\omega = \frac{2Ma}{A} \simeq \frac{2J}{\sigma^3}, \quad (3)
\]

where \(M\) is the neutron star mass and \(J\) is the angular momentum of the neutron star (Rezzolla, Ahmedov & Miller 2001), which we assume to be constant. Then, the line element of the Hartle–Thorne
metric becomes
\[
\frac{1}{s^2} = \left(1 - \frac{2M}{c^2} \right)dt^2 + \left(1 - \frac{2M}{\sigma c^2} \right)^{-1}d\sigma^2 - 2\omega\sigma^2 d\phi dt + \sigma^2 d\phi^2 + dz^2. \tag{4}\]

In order to construct hydrostatic equilibrium models of rotating thick discs we solve the continuity equation \(\nabla \cdot (\rho u\sigma) = 0\) and the conservation of energy momentum, \(\nabla \cdot T^\rho = 0\), where the symbol \(\nabla\) refers to a covariant derivative with respect to the metric (4). Here, \(T^\rho = (\varepsilon + p)u^\rho u^\sigma + pg^\rho\sigma\) are the components of the stress–energy tensor of a perfect fluid, with \(u^\rho\) being the components of the four-velocity, \(\rho\) the rest-mass density, \(\varepsilon\) the energy density and \(p\) the pressure.

It is also useful to introduce an orthonormal tetrad carried by the local stationary observer and defined by the one-forms with components
\[
\omega^\rho = \sigma \sqrt{A} \tilde{A} dt, \quad \omega^\phi = \sqrt{A} (d\phi - \omega dt)/\sigma, \quad \omega^z = dz, \quad \omega^\sigma = \sigma / \sqrt{A} d\sigma. \tag{5}\]

In this frame, the components of the four-velocity of the fluid are denoted by \(u^\rho\) and the three-velocity components are defined as
\[
v^i = \omega^i \omega^\rho u^\rho, \quad i = \sigma, z, \phi. \tag{6}\]

We consider a perfect fluid that follows a polytropic equation of state (EOS) \(p = k\rho^{\gamma}\), where \(k\) and \(\gamma \equiv d\ln p/d\ln \rho\) are the polytropic constant and the adiabatic index, respectively. Following Papers I and II, we introduce a vertically integrated pressure
\[
P(\sigma) \equiv \int_H^z p dz, \tag{7}\]
and a vertically integrated rest-mass density
\[
\Sigma(\sigma) \equiv \int_H^z \rho dz, \tag{8}\]
where \(H = H(\sigma)\) is the local ‘thickness’ of the torus. We further assume that \(P\) and \(\Sigma\) obey an ‘effective’ polytropic EOS,
\[
P = K\Sigma^\Gamma, \tag{9}\]
so that \(K\) and \(\Gamma \equiv d\ln P/d\ln \Sigma\) play the role of the polytropic constant and the adiabatic index, respectively.

After the vertical integration, we enforce the conditions of hydrostatic equilibrium and axisymmetry (i.e. assume \(\partial_\sigma = 0 = \partial_\phi\)) and simplify the equation of energy–momentum conservation to a Bernoulli-type form (Kozlowski, Jaroszynski & Abramowicz 1978)
\[
\frac{\partial_t P}{E + P} = - \left(\partial_\sigma \ln u^\sigma - \frac{\ell}{1 - \Omega \ell} \partial_\Omega \Omega\right), \tag{10}\]
where \(\ell \equiv -\dot{\sigma}_t \dot{\sigma}_t\) is the specific angular momentum, \(\Omega = u^t/\ell\) is the angular velocity and
\[
(u^\sigma)^{-2} = - (\dot{\sigma}_t + 2\Omega \dot{\sigma}_\phi + \Omega^2 \dot{\sigma}_{\phi\phi}). \tag{11}\]

After simple manipulations, equation (10) can be rewritten as
\[
\partial_\sigma \ln u^\sigma - \frac{\ell}{1 - \Omega \ell} \partial_\Omega \Omega = \frac{(u^\sigma)^2}{2} \left(\partial_\sigma \dot{\sigma}_t + 2\Omega \partial_t \dot{\sigma}_\phi + \Omega^2 \partial_{\phi} \dot{\sigma}_{\phi\phi}\right). \tag{12}\]

By using the metric terms of equation (4) into equation (10), we derive the following force balance equation for a non-self-gravitating disc in the Hartle–Thorne space-time:
\[
\frac{1}{E + P} \frac{dP}{d\sigma} = - M/\sigma^2 - \Omega \sigma (\omega + \Omega) \left(\varepsilon + p\right) / (1 - 2M/\sigma c^2 + \Omega^2 (2\omega - \Omega)^2). \tag{13}\]

where \(E\) is the vertically integrated energy density, defined in complete analogy to equations (7) and (8).

3 Perturbation Equations

We next perturb the hydrodynamical equations introducing Eulerian perturbations of the hydrodynamical variables with a harmonic time-dependence of the type
\[
\delta V^\rho = \delta \tilde{V}^\rho, \delta \Omega = \delta \tilde{W}, \delta V^\phi = \delta \tilde{W}^\phi, \delta V^z = \delta \tilde{W}^z, \delta V^\sigma = \delta \tilde{W}^\sigma, \tag{14}\]
where \(\delta \tilde{W}\) is the vertically averaged velocity perturbations respectively as
\[
\delta \tilde{W} = \frac{1}{2H} \int_{H}^z \delta v^\rho dz, \delta \tilde{W}^\phi = \frac{1}{2H} \int_{H}^z \delta \tilde{W}^\rho dz, \delta \tilde{W}^z = \frac{1}{2H} \int_{H}^z \delta \tilde{W}^\sigma dz, \tag{15}\]
We assume that the Eulerian perturbations in the metric functions can be neglected, i.e. \(\delta g_{\alpha\beta} = 0\) (Cowling approximation; Cowling 1941). While this condition does not hold in general, it represents a very good approximation in the case of non-self-gravitating tori.

To eliminate the imaginary part from the system of equations, we introduce the following quantities:
\[
\delta U \equiv i\delta \tilde{W}, \quad \delta W \equiv \delta \tilde{W}^\phi, \tag{16}\]
and after a bit of straightforward algebra, we derive the following set of ordinary differential equations:
\[
\sigma^2 \frac{\Delta}{A} \delta U = \alpha \frac{\Delta}{\sigma^2} \delta Q + \left(\frac{\Delta^{3/2}}{A}\right)^{\prime} \left(\frac{A}{\sigma^2}\right)^{\prime} \Omega - \frac{\Delta^{3/2}}{\sigma^2} (\omega - \Omega) \left(\partial_\tau \delta W\right), \tag{17}\]
\[
\sigma^2 \frac{\Delta}{A} \delta W = \left(\frac{\partial_\sigma \partial_\phi}{\sigma^2 \Delta} \right)^{\prime} \left(\omega - \Omega\right) + \left(\frac{A}{\sigma^2 \Delta}\right)^{\prime} \left(\omega - \Omega\right) - \frac{A^2}{\sigma^2 \Delta} \left(\frac{A}{\sigma^2}\right)^{\prime} \left(\omega - \Omega\right) \delta Q = 0, \tag{18}\]
\[
\sigma^2 \frac{\Delta}{A} \delta Q + \frac{\Delta}{\sqrt{A}} \delta U = \left(\frac{\Delta}{\sqrt{A}}\right)^{\prime} \left(\frac{P}{E + P}\right), \tag{19}\]
where \(\alpha \equiv 1/(u^\sigma)^2\), \(\tilde{\Gamma} \equiv \Gamma P/E + P\), and the index \(\prime\) indicates the derivative with respect to \(\sigma\).

Equations (17)–(19) are the \(\sigma\) - and \(\phi\)-components of the perturbed relativistic Euler equations and the perturbed contiguity equation, respectively. They can be solved numerically for the eigenfrequencies and for the eigenfunctions of \(m\)-mode oscillations of an oscillating vertically integrated thick disc in the Hartle–Thorne space–time. In practice, we solve the system of equations (17)–(19) as an eigenvalue problem using a ‘shooting’ method (Press et al. 1986) in which, once the appropriate boundary conditions are provided, two trial solutions are found, starting from the inner and outer edges of the disc respectively, and these are then matched at
an intermediate point where the Wronskian of the two solutions is evaluated. This procedure is iterated until a zero of the Wronskian is found, thus providing a value for \( \sigma \) and a solution for \( \delta Q, \delta U \) and \( \delta W \). The numerical method employed here to solve the eigenvalue problem is the same as that discussed in Papers I and II, where a more detailed discussion can be found.

### 4 Equilibrium Models

First of all, we need to fix the angular momentum of the star which is defined as \( J = I \Omega_1 \), where \( \Omega_1 = 2 \pi / P \), is the angular velocity of the star. In geometric units, we have

\[
J = \frac{I_{\text{as}}}{P_1} \left( \frac{M_\odot}{M} \right)^2 7.136 \times 10^{-4},
\]

(20)

where \( I_{\text{as}} \) is the moment of inertia of the star in units of \( 10^{45} \text{ g cm}^2 \), which we take as \( I_{\text{as}} = 1 \), while \( P_1 \) is the period of rotation in units of second. For instance, the typical Atoll source 4U 1608–52 (van Straaten, van der Klis & Méndez 2003) has a mass \( M = 1.7 M_\odot \), a spin period of \( P = 1.61 \text{ ms} \), and the angular momentum, computed from equation (20), is therefore \( J = 0.15 \). Even assuming a binary system with an accreting neutron star that rotates as fast as the fastest known millisecond pulsar PSR J1748–2446ad, namely with \( P = 1.39 \text{ ms} \) and with a canonical mass \( M = 1.4 M_\odot \), would yield to \( J = 0.26 \).

In Sections 5 and 6, we report results obtained after assuming \( J = 0.1 \). However, we have also solved the eigenvalue problem for the case \( J = 0.2 \), without finding any significant difference in the results, so our conclusions remain unchanged.

Next, we define the distribution of the specific angular momentum \( \ell = \ell(\sigma) \) within the disc. We consider tori with distributions of specific angular momentum that are constant in space, i.e. \( \ell(\sigma) = \text{const.} \), and also tori with non-constant distributions of the specific angular momentum.\(^4\) We note that \( \ell(\sigma) = \text{const.} \) is a useful mathematical case which leads to analytic initial data, while non-constant distributions of the specific angular momentum are a more realistic assumption. In the case of \( \ell(\sigma) = \text{const.} \), the value of the specific angular momentum must satisfy the condition \( \ell_{\text{min}} < \ell < \ell_{\text{max}} \), where \( \ell_{\text{min}} \) and \( \ell_{\text{max}} \) are the specific angular momenta of the marginally stable and marginally bound orbit in the Hartle–Thorne space–time (see e.g. Abramowicz et al. 2003). On the other hand, in the case of tori with non-constant distributions of the specific angular momentum, we consider a power-law distribution of the type

\[
\ell = \ell_c \sigma^q,
\]

(21)

where both \( \ell_c \) and \( q \) are positive constants. The power-law angular momentum distributions are chosen such that the position of the cusp is always located between the marginally bound and marginally stable orbits. The position of the cusp, as well as the position of the maximum rest-mass density \( \sigma_{\text{max}} \) in the torus, is obtained by imposing that the specific angular momentum at these two points coincides with the Keplerian value (Kozlowski et al. 1978). The inner edge of the torus \( \sigma_{\text{in}} \) is determined by fixing the potential gap

\[
\Delta W_{\text{in}} = W_{\text{in}} - W_{\text{cusp}},
\]

defined as

\[
\Delta W_{\text{in}} = \ln([-u_{\text{in}}]) - \ln([-u_{\text{cusp}}]) - \int_{u_{\text{cusp}}}^{u_{\text{in}}} \frac{\Omega df}{1 - \Omega \ell}. \tag{22}
\]

\(^4\) Note that Qian et al. (2009) considered an alternative ansatz for the distribution of the specific angular momentum.

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**Table 1.** Main properties of the equilibrium models studied. From left to right the columns report: the name of the model, the type of the specific angular momentum distribution, the constant coefficient \( \ell_c \), the power-law index \( q \) (cf. equation 21) or its range of values, the minimum radial size for the corresponding sequence of tori, and the maximum radial size for the sequence of tori. The angular momentum of the neutron star is set to be \( J = 0.1 \) in all of the models.

| Model | \( \ell(\sigma) \) | \( \ell_c \) | \( q \) | \( L_{\text{min}} \) | \( L_{\text{max}} \) |
|-------|-----------------|----------|------|---------------|---------------|
| A1    | const.          | 3.60     | 0.0  | 0.4           | 1.9           |
| A2    | const.          | 3.65     | 0.0  | 0.5           | 5.7           |
| A3    | const.          | 3.70     | 0.0  | 0.5           | 10.0          |
| A4    | const.          | 3.75     | 0.0  | 0.4           | 16.8          |
| A5    | const.          | 3.80     | 0.0  | 0.8           | 29.9          |
| A6    | const.          | 3.85     | 0.0  | 0.7           | 71.4          |
| B1    | power law       | 3.0      | [0.1, 0.15] | 2.5  | 374.3         |
| B2    | power law       | 3.1      | [0.08, 0.15] | 2.1  | 348.6         |
| B3    | power law       | 3.2      | [0.07, 0.15] | 1.9  | 279.6         |
| B4    | power law       | 3.3      | [0.05, 0.15] | 0.8  | 318.4         |
| B5    | power law       | 3.4      | [0.04, 0.15] | 2.5  | 363.3         |
| B6    | power law       | 3.5      | [0.02, 0.15] | 1.7  | 265.3         |
| B7    | power law       | 3.6      | [0.006, 0.15] | 0.4  | 327.1         |
| B8    | power law       | 3.7      | [0.001, 0.15] | 1.9  | 276.7         |
| B9    | power law       | 3.8      | [0.001, 0.15] | 2.6  | 299.1         |
| B10   | power law       | 3.9      | [0.001, 0.15] | 2.7  | 281.6         |
| C1    | power law       | 2.59     | 0.2  | 3.3           | 160.8         |
| C2    | power law       | 2.15     | 0.3  | 5.3           | 548.8         |
| C3    | power law       | 2.19     | 0.3  | 7.6           | 575.1         |
| C4    | power law       | 2.29     | 0.3  | 1.4           | 550.9         |
| C5    | power law       | 2.35     | 0.3  | 1.9           | 545.1         |
| C6    | power law       | 2.39     | 0.3  | 0.7           | 156.6         |
| C8    | power law       | 1.79     | 0.4  | 51.5          | 593.9         |

On the other hand, the outer edge of the torus \( \sigma_{\text{out}} \) is defined as the position at which \( P = 0 \) and it is obtained by integration of the hydrostatic balance equation (13). Then, for a given distribution of specific angular momentum, sequences of tori having the same \( \sigma_{\text{max}} \) but different radial extents can be constructed by varying the potential gap \( \Delta W_{\text{in}} \).

In order to investigate how the axisymmetric oscillations depend on the parameters of the discs, we have constructed sequences of models, having different radial extents and distributions of the specific angular momentum. The main properties of the various models considered are listed in Table 1. Models of class A (henceforth models A) are sequences of equilibrium tori with a constant distribution of the specific angular momentum, while models B and C have a specific angular momentum increasing outwards according to equation (21). Unlike models A and C, which correspond to sequences of discs with different radial sizes for a given pair of values of \( \ell_c \) and \( q \), models B refer to different disc sequences which not only have different radial sizes but also different values of the power-law index \( q \) for each of the constant coefficients \( \ell_c \); that is for each set of models from B1 to B10, we have constructed discs with values of the power-law index \( q \) varying between the minimum and maximum values listed in the fourth column of Table 1 at intervals of \( \Delta q = 0.005 \). The last sequence C8 corresponds to discs with a distribution of angular momentum having a power-law index \( q \approx 0.4 \), close to the Keplerian value \( q_{\text{Kep}} = 0.5 \). Therefore, all these models allow for an extensive investigation of the parameter space in terms of disc sizes and distributions of specific angular momentum, varying from constant to almost Keplerian.

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5 RESULTS OF THE GLOBAL ANALYSIS

The main properties of axisymmetric oscillations of tori in a Hartle–Thorne background are analogous to those found for tori in Schwarzschild and Kerr space–times (Papers I and II). Overall, a fundamental mode of oscillation and a sequence of overtones are found (collectively referred to as p-mode oscillations) which depend on the position of the rest-mass density maximum, on the radial size of the disc, on the distribution of angular momentum, while they are rather independent of the EOS (Rezzolla et al. 2003; Montero et al. 2004; Zanotti et al. 2003, 2005; Montero et al. 2007). These properties can be summarized as follows.

(i) The eigenfrequencies of p-mode oscillations increase as the radial size of the disc decreases.

(ii) The fundamental mode tends to the values of the radial epicyclic frequency at the position of the rest-mass density maximum as the radial size of the tori tends to zero.

(iii) For any radial extent, the model with the largest fundamental mode eigenfrequency has its rest-mass density maximum located at the position at which the epicyclic frequency has a maximum.

(iv) The ratio between the frequency of the fundamental mode (f) and its first overtone (o1) for tori with constant distributions of specific angular momentum, ℓ(σ) = const., appears approximately in a 2:3 ratio. As the size of the tori tends to zero, the ratio o1/f tends to o1/f ∼ 1.52.

(v) The ratio between the frequency of the fundamental mode (f) and its first overtone (o1) for tori with non-constant distributions of specific angular momentum, ℓ(σ) = ℓ(σ) const, can deviate significantly from o1/f ∼ 3/2 for very small discs. As the size of the disc increases the o1/f ratio tends to o1/f ∼ 1.44.

In Fig. 1, we show the eigenfrequencies (in units of Hz and scaled for a neutron star mass M = 1.6 M⊙) for the fundamental mode corresponding to some representative tori of our sample (i.e. A6, B9 with q = 0.03, B1 with q = 0.135, C2 and C7) and reported as a function of the radial size of the disc expressed in units of the gravitational radii r_g = GM/c². Each line corresponds to a sequence of tori having the same σ_max but different radial extents L and the solid circles correspond to the values of the Keplerian radial epicyclic frequency at σ_max in the Hartle–Thorne metric (Abramowicz et al. 2003). As expected, for modes behaving effectively as sound waves trapped in the disc, the eigenfrequencies decrease like L⁻¹ as the radial extent of the torus increases. Note also that, as shown in Papers I and II for tori orbiting around a Schwarzschild and a Kerr black hole, the eigenfrequencies of the fundamental mode tend to the values of the radial epicyclic frequency at σ_max as the radial dimension of the discs tends to zero. As their size diminishes, the role of pressure gradients inside the disc becomes negligible and the discs effectively behave as rings of particles in circular orbits, oscillating with the epicyclic frequency at the maximum rest-mass density point.

A key feature of the axisymmetric p-modes oscillations of tori around black holes is that the eigenfrequencies of the fundamental mode and the first overtone appear in an approximately 2:3 harmonic sequence, although deviations are possible, in particular for the non-constant specific angular momentum case. This feature is also present in the case of sub-Keplerian discs in the Hartle–Thorne space–time. In Fig. 2 we show the ratio o1/f as a function of the radial size of the disc for models with a constant distribution of specific angular momentum, i.e. models A1–A6. As the size of the disc decreases, the ratio o1/f decreases and tends to a value of ~1.52, independently of the constant distribution of specific angular momentum. On the other hand, the behaviour of the o1/f ratio, as the disc size decreases, is more complex for a non-constant angular momentum disc. In Fig. 3 we plot the ratio o1/f as a function of the radial size of the disc for some representative models with a constant distribution of specific angular momentum. These models belong to the sequences B and C, and the values of the constant coefficient ℓ_c and of the power-law index q are also showed in Fig. 3. We observe that for small discs, the ratio o1/f decreases as the power-law index q increases for a given value of the constant coefficient ℓ_c, i.e. there exists a variation of about 30 per cent in the o1/f ratio for small discs. In particular, o1/f has an upper limit
of $\sim 1.52$ (for discs with an almost constant distribution of specific angular momentum), and a lower limit of $\sim 1.15$ for models with a power-law index $q$ close to the Keplerian value $q_{Kepler} = 0.5$. On the other hand, for large-size discs the $a_1/f$ ratio tends to $a_1/f \approx 1.44$.

6 IMPLICATIONS FOR KHZ-QPOs IN NEUTRON STAR LOW-MASS X-RAY BINARIES

Based on these properties of the axisymmetric $p$-mode oscillations of thick discs, Rezzolla et al. (2003) proposed a model of kHz-QPOs in BHBs that explains the observed frequencies in terms of $p$-mode oscillations of a small accretion thick disc orbiting close to the black hole (Schnittman & Rezzolla 2006). This model accounts very well for the $M^{-1}$ scaling of the observed frequencies, for the observed variations in the relative strength of the peaks, which are interpreted as due to variations in the perturbations that the torus is experiencing, and for the fact that twin kHz-QPOs in the four BHBs show frequencies obeying the ratio 3:2 to a high degree of accuracy. As discussed in the introduction, on the other hand, the phenomenology of kHz-QPOs in NSBs presents peculiar features that distinguish them from those detected in BHBs. In particular, the upper and lower kHz-QPO frequencies $v_l$ and $v_U$ can vary by hundreds of Hertz along straight lines $v_l / v_U = A v_l + B$, with $B \neq 0$. For convenience, we have listed in Table 2 the best-fitting linear parameters obtained by Belloni et al. (2005, 2007) for the Atoll sources, Z sources, Sco X-1 and Cir X-1.

Table 2. Best-fitting linear parameters obtained by Belloni et al. (2005, 2007) for the Atoll sources, Z sources, Sco X-1 and Cir X-1.

| Source       | A      | B      |
|--------------|--------|--------|
| Atoll sources| 0.94 ± 0.02 | 350 ± 15 |
| Z sources    | 0.85 ± 0.01 | 383 ± 8  |
| Sco X-1     | 0.73 ± 0.01 | 469 ± 7  |
| Cir X-1     | 2.34 ± 0.47 | 104 ± 58 |

The possibility that the observed $v_l$ and $v_U$ kHz-QPO frequencies correspond to the fundamental frequency and to the second overtone of an oscillating torus encounters similar difficulties (see the right-hand panel of Fig. 4). Although the area covered by the computed $p$-mode oscillations matches some of the observed kHz-QPO frequencies, particularly those of Z sources with $v_l$ in the range of $150–500$ Hz, the observations of most of the Atoll sources, Cir X-1 and several Z sources remain unexplained.

Overall, the properties of axisymmetric $p$-mode oscillations of vertically integrated thick discs are such that, in a plot of $v_l$ versus $v_U$, the first overtone and the fundamental mode frequency follow a straight line for which $A$ may depart from an exact 3:2 ratio by 30 per cent (in the case of discs with a non-constant distribution of the specific angular momentum), but for which $B \approx 0$.

7 CONCLUSIONS

We have performed a detailed analysis of the oscillation properties of a thick disc (torus) around a slowly rotating neutron star. Our approach extends previous investigations of Rezzolla et al. (2003) and Montero et al. (2004) by considering a much wider parameter space and by solving the linear perturbative eigenvalue problem in the Hartle–Thorne metric. In particular, the rotation law of the

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torus spans the whole range between a constant distribution of the specific angular momentum and an almost Keplerian rotation. We have computed a fundamental mode of oscillations and a sequence of overtones which can in principle be all excited depending on the perturbation acted upon the torus.

We showed that there are discs (B and C models) with fundamental mode frequency $f > 500$ Hz and a first overtone frequency which can be in agreement with most observed $v_2$ and $v_1$ kHz-QPO frequencies for Sco X-1, and for some Z and Atoll sources.

However, when these results are used for explaining kHz-QPOs in neutron star low-mass X-ray binaries with twin QPOs with $v_1 \lesssim 500$ Hz, a major difficulty arises. In fact, unlike kHz-QPOs in black hole binaries, the upper and lower kHz-QPOs in neutron star binaries obey a linear relation $v_1 \approx A v_2 + B$, with $A$ significantly different from 1.5 (e.g. $A \approx 0.94$ for Atoll sources, $A \approx 0.85$ for Z sources and $A \approx 0.73$ for Sco X-1) and $B \neq 0$. In contrast, the computed axisymmetric $p$-modes, either in the ratio $o_1f$ or $o_2f$, follow a straight line with $0.8 \lesssim A \lesssim 1.5$ and with $B \approx 0$ for $o_1f$, and with $A \gtrsim 1.5$ and $B \approx 0$ for $o_2f$.

Therefore, with the assumptions adopted in this paper, axisymmetric $p$-mode oscillations of a thick disc around a neutron star do not provide an explanation for the observed twin QPOs in neutron star X-ray binaries with $v_1 \lesssim 500$ Hz. Nevertheless, additional physics should be taken into account to have a better understanding of the differences between the Z and Atoll sources. For instance, the thickening of the disc due to radiation pressure, the interaction of the accreting thick disc with the magnetosphere of the neutron star, the presence of a magnetic field (Balbus & Hawley 1991) or non-axisymmetric instabilities (Papaloizou & Pringle 1984; Kiuchi et al. 2011) may play an important role.

It is known that the luminosities of the Z sources are typically close to the Eddington luminosity ($L \sim L_\text{Edd}$), while the luminosities of the Atoll sources appear in a lower and broader range ($L \sim 0.001$–$0.5 L_\text{Edd}$). Moreover, Hasinger & van der Klis (1989) first suggested that the differences between the two types of sources may be due to differences in the mass accretion rates, i.e. low and high accretion rates show the source as an Atoll or a Z one. Lin, Remillard & Homan (2009) found that the source XTE J1701–462 evolved from super-Eddington luminosity to quiescence, displaying an evolution with features of Cyg-like Z, Sco-like Z and Atoll sources, supporting the idea that changes in the accretion rate are responsible for this secular evolution. In addition, Lin et al. (2009) pointed out that the Atoll stage may be characterized by a constant inner disc radius, while the Z stage may have a luminosity-dependent location of the inner disc. Similar results have recently been obtained by Ding et al. (2011), who focused on the interaction between the NS magnetosphere and the radiation–pressure-dominated accretion disc. As the disc is thickened by radiation pressure, the disc gas pressure reduces and the magnetosphere expands, thus making the inner disc radius increase.

Overall, there is growing evidence that the different source stages are due to different disc structures, and that the hot regions of the flow where the QPO oscillations are generated could be qualitatively different. Interestingly, the oscillation properties of relativistic, non-self-gravitating vertically integrated equilibrium tori orbiting NS agree better with the data for the Atoll (low accretion rate) stage and the Sco-like stage than for the Z (high accretion rate) stage. This seems to indicate that both the NS magnetosphere interaction and the thickening of the disc due to radiation pressure may play a crucial role, and will be considered in our future research for a more realistic interpretation of kHz-QPOs from neutron star binaries.

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