Autonomous Operator and Differential Systems of order 1 over Integral Domains

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January 8, 2019

Abstract

In this paper we introduce the autonomous operator which is a nonlinear map that sends sequences to sequences in an integral domain $R$, in particular sequences over the ring of exponential generating functions. Properties of this operator are studied: its relationship with the derivative of the ring of sequences, its characterization by groups of $k$-homogeneity, conditions of injectivity and surjectivity and its restriction to a linear operator. Autonomous differential dynamical systems over integral domain are defined. It is also shown that the flow of such differential equations is the exponential generating function of the sequence obtained via the action of autonomous operator acting over the Hurwitz expansion of the vectorial field of such equation. It is shown that the flow is a torsion-free cyclic $R$-module. The $R\times$-modules associated to the groups of $k$-homogeneity are constructed. We finish showing some examples of autonomous differential equations and we find its $R\times$-modules when $R$ is one of the rings: $\mathbb{Z}$, $\mathbb{C}$, $\mathbb{Z}[i]$, $\mathbb{Z} [\omega]$ and $\mathbb{Z}[\sqrt{2}]$.

Keywords: autonomous operator, Hurwitz expansion, homogeneity groups, dynamical systems
Mathematics Subject Classification: 47H99, 16W99, 34A34, 11B83

1 Introduction

An important issue in the theory of autonomous differential systems is to find a general analytic solution for the equation $x' = f(x)$ where $f$ is an arbitrary vector field $f : E \subset \mathbb{R}^k \to \mathbb{R}^k$. One of the main approaches to the solution of this equation is the use of formal series of rooted trees, introduced by Cayley [3] and developed by Merson and Butcher [5]. Butcher is best known for showing the existence of a group structure in the Runge-Kutta methods for ordinary differential equations when those methods are expanded in power series. All this methods implicitly use the Faá di Bruno formula for the composition of functions. Finally, Foissy[4] studied the combinatorial form of the Dyson-Schwinger equation using Hopf-Faá di Bruno algebras in which the coefficients of the solution use the Bell polynomials. In [6] an analytical solution to the autonomous equation of dimension $k$ is found by mean using Bell polynomials. In this paper a nonlinear operator defined by means of the Bell polynomials is studied and applied to the study of one-dimensional dynamical systems over integral domain.

Bell polynomials are a very useful tool in mathematics to represent the $nth$ derivative of the composite of two functions [2]. In effect, let $f$ and $g$ be two analytical functions with power series representations $\sum a_n x^n/n!$ and $\sum b_n x^n/n!$ respectively, with $a_n, b_n \in \mathbb{C}$. Then

$$f(g(x)) = f(b_0) + \sum_{n=1}^{\infty} Y_n(b_1, ..., b_n; a_1, ..., a_n) \frac{x^n}{n!},$$

where $Y_n$ is the $nth$ Bell polynomial. For instance it is well known that

\begin{align*}
Y_1(b_1; a_1) &= a_1 b_1, \\
Y_2(b_1, b_2; a_1, a_2) &= a_1 b_2 + a_2 b_1^2, \\
Y_3(b_1, b_2, b_3; a_1, a_2, a_3) &= a_1 b_3 + a_2 (3b_1 b_2) + a_3 b_1^3.
\end{align*}
Bell polynomials can be expressed implicitly using the Faà di Bruno formula [2],

$$ Y_n(b_1, ..., b_n; a_1, ..., a_n) = \sum_{k=1}^{n} B_{n,k}a_k, $$

where

$$ B_{n,k} = \sum_{|p(n)|=k} \frac{n!}{j_1!j_2! \cdots j_n!} \left[ \frac{b_1}{1!} \right]^{j_1} \left[ \frac{b_2}{2!} \right]^{j_2} \cdots \left[ \frac{b_n}{n!} \right]^{j_n}, $$

where the sum runs over all the partitions \( p(n) \) of \( n \), i.e., \( n = j_1 + 2j_2 + \cdots + nj_n, \) \( j_h \) denotes the number of parts of size \( h \) and \( |p(n)| = j_1 + j_2 + \cdots + j_n \) is the length of partition \( p(n) \).

Let \( f \in C^\infty(S) \), where \( S \subset \mathbb{R} \) and let

\[
\begin{align*}
\begin{cases}
u' = f(u) \\
u(0) = x
\end{cases}
\end{align*}
\]

the autonomous ordinary differential equation with initial value problem.

In [6] the author introduced the autonomous polynomials, which are the coefficients of the analytical solution of the equation (4) expressed as a function of the initial value in the next way:

$$ \begin{align*}
A_1(f(x)) &= f(x), \\
A_{n+1}(f(x), f'(x), ..., f^{(n)}(x)) &= Y_n(A_1(f(x)), ..., A_n(f(x)); f'(x), ..., f^{(n)}(x))
\end{align*} $$

From [6] we can see that every \((n + 1)\)-th autonomous polynomial is defined over \( f(x), f'(x), ..., f^{(n)}(x) \) and in general every autonomous polynomial \( A_n \) is defined on the sequence \((f^{(n)}(x))_{n \geq 0}\) of derivatives of \( f(x) \). We define an operator by means of autonomous polynomials defined over the sequences of derivatives of exponential generating functions with coefficients in an integral domain. In this paper \( R \) or \( \text{Frac}(R) \), the fractions field of \( R \), contains to \( \mathbb{Q} \). Also, \( \mathbb{N} \) will be the monoid of nonnegative integer and \( \mathbb{N}_1 \) will be the semigroup of positive integers.

This paper is organized as follows. In Section 2, the ring of exponential generating functions and the ring of sequences over this are introduced. Section 3 introduces the autonomous operator \( \mathfrak{A} \). In Section 4, the properties of \( \mathfrak{A} \) are studied. In Section 5, the results of the previous section are applied to dynamical systems over integral domain. The \( G \)-modules of flows of dynamical systems are constructed. Finally, in Section 6 examples of dynamical systems are showed.

## 2 The \( R \)-algebra \( \Delta \text{Exp}_R[X] \)

Let \( R \) be an integral domain with identity and consider the power series of the form \( \sum_{n=0}^{\infty} a_n \frac{X^n}{n!} \) with \( a_n \in R \), for all \( n \geq 0 \) and let \( \text{Exp}_R[X] \) denote the set of all this series. Is is clear that \( (\text{Exp}_R[X], +, \cdot) \) is a ring with componentwise addition and ordinary product of series, i.e.

\[
\begin{align*}
f(X) + g(X) &= \sum_{n=0}^{\infty} (a_n + b_n) \frac{X^n}{n!}, \\
f(X)g(X) &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \frac{X^n}{n!},
\end{align*}
\]

with \( f(X) = \sum_{n=0}^{\infty} a_n \frac{X^n}{n!}, \) \( g(X) = \sum_{n=0}^{\infty} b_n \frac{X^n}{n!} \in \text{Exp}_R[X]. \)

Let \( \delta \) denote the formal derivative over the ring \( \text{Exp}_R[X] \). Then the ring of constant of \( \text{Exp}_R[X] \), i.e. the kernel \( \text{Ker} \delta \), is all the ring \( R \).
Definition 1. Let \( f(X) \in \text{Exp}_R[X] \). The sequence of derivatives of \( f(X) \), denoted \( \Delta f(X) \), is the sequence \( (\delta^n f(X))_{n \in \mathbb{N}} \).

Proposition 1. Take \( f \) and \( g \) of \( \text{Exp}_R[X] \). Then
1. \( \Delta(f + g) = \Delta f + \Delta g \).
2. \( \Delta(cf) = c\Delta f \), with \( c \in R \).
3. \( \Delta(fg) = \left( \sum_{k=0}^{n} \binom{n}{k} \delta^k f \delta^{n-k} g \right)_{n \in \mathbb{N}} \).
4. \( \Delta(f \circ g) = (Y_n(\delta g, \delta^2 g, ..., \delta^n g; (\delta f) \circ g, (\delta^2 f) \circ g, ..., (\delta^n f) \circ g))_{n \in \mathbb{N}} \), where \( Y_n \) is the \( n \)th Bell polynomial and \( Y_0 = f \circ g \).

Proof. 1 and 2 are obtained directly of properties of lineality of \( \delta \). 3 is obtained from the Leibniz formula. And 4 is obtained from definition of Bell polynomials.

Let \( \Delta \text{Exp}_R[X] \) denote the set of all sequences \( \Delta f \) for all \( f \in \text{Exp}_R[X] \). From the previous proposition it is followed that \( \Delta \text{Exp}_R[X] \) is an abelian group because \( \Delta f + \Delta g = \Delta(f + g) \in \Delta \text{Exp}_R[X] \). Let * denote the convolution product in \( \Delta \text{Exp}_R[X] \) given by

\[
\Delta f(X) * \Delta g(X) = \left( \sum_{k=0}^{n} \binom{n}{k} \delta^k f(X) \delta^{n-k} g(X) \right).
\]

Then from 3 in previous proposition it is followed that

\[
\Delta f(X) * \Delta g(X) = \Delta(f(X)g(X)) \in \Delta \text{Exp}_R[X].
\]

As \( c\Delta f = \Delta(cf) \in \Delta \text{Exp}_R[X] \), then \( \Delta \text{Exp}_R[X] \) is an \( R \)-algebra with identity \( e_\Delta = \Delta(1) = (1,0,...) \) and zero element \( 0 = (0,0,...) \). Then \( \Delta : \text{Exp}_R[X] \to \Delta \text{Exp}_R[X] \) is an \( R \)-isomorphism of algebras with \( \text{Ker} \Delta = \{0\} \). The ring \( \Delta \text{Exp}_R[X] \) is a subring of the ring \( H \text{Exp}_R[X] \) of series of Hurwitz with coefficients in \( \text{Exp}_R[X] \) and \( \Delta f(X) \) is known as Hurwitz expansion of \( f(X) \). (See [1] for more details).

3 Autonomous Operator \( \mathfrak{A} \)

Definition 2. Let \( \text{Exp}_R[X]^N \) be the multiplicative algebra of sequences over \( \text{Exp}_R[X] \) with multiplication defined term to term. We define autonomous operator \( \mathfrak{A} \) as the nonlinear map \( \mathfrak{A} : \Delta \text{Exp}_R[X] \to \text{Exp}_R[X]^N \), defined by \( \mathfrak{A}(\Delta f) = (A_n([\delta^{n-1} f]))_{n \geq 1} \), with \( [\delta^n f] = (f, \delta f, ..., \delta^n f) \), in which the \( A_n \) are recursively defined in the next way:

\[
\begin{align*}
A_1([\delta^0 f]) &= A_1([f]) = f, \\
A_{n+1}([\delta^n f]) &= Y_n(A_1([\delta^0 f]), A_2([\delta^1 f]), ..., A_n([\delta^{n-1} f]); \delta^1 f, ..., \delta^n f),
\end{align*}
\]

\( n \geq 1 \).

The polynomials \( A_n \) in the indeterminates \( \delta^0 f(X), \delta^1 f(X), ..., \delta^{n-1} f(X) \) will be called autonomous polynomial.

Some polynomials \( A_n \) are

\[
\begin{align*}
A_1([\delta^0 f]) &= f, \\
A_2([\delta^1 f]) &= A_1([\delta^0 f])\delta^1 f = f \delta f, \\
A_3([\delta^2 f]) &= A_2([\delta^1 f])\delta f + A_1^2([\delta^0 f])\delta^2 f, \\
&= f(\delta f)^2 + f^2 \delta^2 f, \\
A_4([\delta^3 f]) &= A_3([\delta^2 f])\delta^3 f + 3A_1([\delta^0 f])A_2([\delta^1 f])\delta^2 f + A_1^3([\delta^0 f])\delta^3 f, \\
&= f(\delta f)^3 + 4f^2 \delta^1 f \delta^2 f + f^3 \delta^3 f.
\end{align*}
\]
Let $R^\mathbb{N}$ denote the ring of sequences over the ring $R$ with componentwise addition and product and indentity $e = (1, 1, \ldots)$. The map $\gamma : R^\mathbb{N} \to \text{Exp}_R[X]$, with $\gamma(a) = f_a(X) = \sum_{n=0}^{\infty} a_n X^n$ is obviously an isomorphism of ring. Then the ring $\text{Exp}_R[X]$ is the ring of exponential generating functions of sequences in $R^\mathbb{N}$. Let $\mathcal{E}_a : \text{Exp}_R[X] \to R$ denote the evaluation map in the element $a$ of $R$ and extend this map to $\text{Exp}_R[X]^\mathbb{N}$ in this sense

$$\mathcal{E}_a \Delta f(X) = (\mathcal{E}_a \delta^n f(X))_{n \in \mathbb{N}_0} = (\delta^n f(a))_{n \in \mathbb{N}}.$$

Then

$$\mathcal{E}_0 \Delta f_a(X) = (f_a(0), \delta f_a(0), \delta^2 f_a(0), \ldots) = (a_0, a_1, a_2, \ldots) = a \quad (7)$$

and $R^\mathbb{N} \subset \Delta \text{Exp}_R[X]$. Also

$$\mathcal{E}_0 \mathfrak{A}(\Delta f_a(X)) = (f_a(0), f_a(0) \delta f_a(0), f_a(0) (\delta f_a(0))^2 + f_a^2(0) \delta^2 f_a(0), \ldots)$$

$$= (a_0, a_0 a_1, a_0 a_1^2 + a_0^2 a_2, \ldots)$$

Thus $\mathfrak{A}$ act over any sequence in $R^\mathbb{N}$ although $R$ is not necessarily a differential ring. On the other hand, is not easy, at least to the author, to find closed formulas for the images of $\mathfrak{A}$, especially with of the sum, product or composition of functions in $\text{Exp}_R[X]$. For this reason, we calculate the action $\mathfrak{A}$ over $\alpha \Delta f(X)$, $\exp(X)f(X)$ and $\exp(f(X))$.

**Proposition 2.** Let $f \in \text{Exp}_R[X]$ and $\alpha \in R$. Then

$$\mathfrak{A}(\alpha \Delta f(X)) = (\alpha^n)_{n \in \mathbb{N}} \mathfrak{A}(\Delta f(X)), \quad (8)$$

**Proof.** We shall prove that $A_n((\alpha \delta^{n-1} f)) = \alpha^n A_n([\delta^{n-1} f])$. The proof is by induction. This is true for $n = 1$, since $A_1((\alpha \delta^0 f)) = a f = \alpha A_1([f])$. Suppose that the statement it holds for all natural number smaller than $n$ and let’s consider the case $n + 1$. Then

$$A_{n+1}((\alpha \delta^n f)) = \sum_{k=1}^{n} \sum_{|p(n)|=k} \frac{n!}{j_1! \cdots j_n!} \left[ \alpha A_1 \right]^{j_1} \cdots \left[ \alpha^n A_n \right]^{j_n} \frac{\alpha^n A_n}{n!} \delta^k f$$

$$= \sum_{k=1}^{n} \sum_{|p(n)|=k} \frac{n!}{j_1! \cdots j_n!} \left[ A_1 \right]^{j_1} \cdots \left[ A_n \right]^{j_n} \frac{1}{n!} \delta^k f$$

$$= \alpha^{n+1} \sum_{k=1}^{n} \sum_{n=1}^{k} \frac{n!}{j_1! \cdots j_n!} \left[ A_1 \right]^{j_1} \cdots \left[ A_n \right]^{j_n} \frac{1}{n!} \delta^k f$$

$$= \alpha^{n+1} A_{n+1}([\delta^n f])$$

and the statement follows. \hfill \Box

**Proposition 3.** Let $f(X) \in \text{Exp}_R[X]$ and let $\alpha \in R$, then

$$\mathfrak{A}(\Delta(\exp(\alpha X)f(X))) = (\exp(\alpha n X))_{n \in \mathbb{N}} \mathfrak{A}(F(X)) \quad (9)$$

where

$$F(X) = (F_{n,\alpha}(f(X)))_{n \geq 0}$$

with

$$F_{n,\alpha}(f(X)) = \sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} \delta^k f(X). \quad (10)$$

**Proof.** In the previous proposition was shown that $\mathfrak{A}(\alpha f(X)) = (\alpha^n)_{n \geq 1} \mathfrak{A}(f(X))$ for all $f(X)$ in $\text{Exp}_R[X]$. We use this fact together with

$$\delta^n(\exp(\alpha X)f(X)) = \exp(\alpha X) F_{n,\alpha}(f(X))$$
Theorem 1. Take $\delta$.
In this section properties of $A$.

In the next theorem we will see the relationship between the operators $A$ and the derivative $\delta$ of ring $\text{Exp}_R[X]$. Also we characterize $\mathfrak{A}$ by groups of $k$-homogeneity in $R^N$. We study condition for their injectivity and surjectivity. Their action over $R[X]$ and its restriction to a linear operator.

Proposition 4. Let $f(X) \in \text{Exp}_R[X]$. Then

$$\mathfrak{A}(\Delta \text{Exp}(f(X))) = \mathfrak{A}(\text{Exp}(nf(X)))_{n \in \mathbb{N}} \mathfrak{A}(Y(X)),$$

where

$$Y(X) = \left( Y_n(f^{(1)}(X), f^{(2)}(X), ..., f^{(n)}(X)) \right)_{n \in \mathbb{N}}$$

with $Y_n$ the $n$-th Bell’s polynomial.

Proof. From (2),

$$Y_n = \sum_{k=1}^{n} B_{n,k} \delta^k \text{Exp}(f(X)) = \text{Exp}(f(X)) \sum_{k=1}^{n} B_{n,k}.$$

where

$$B_{n,k} = \sum_{|p(n)|=k} \frac{n!}{j_1! j_2! ... j_n!} \left[ \frac{f^{(1)}(0)}{1!} \right]^{j_1} \left[ \frac{f^{(2)}(0)}{2!} \right]^{j_2} ... \left[ \frac{f^{(n)}(0)}{n!} \right]^{j_n}.$$

4 The structure of $\mathfrak{A}(\Delta \text{Exp}_R[X])$

In this section properties of $\mathfrak{A}$ are studied. In particular the relationship between $\mathfrak{A}$ and the derivative $\delta$ of ring $\text{Exp}_R[X]$. Also we characterize $\mathfrak{A}$ by groups of $k$-homogeneity in $R^N$. We study condition for their injectivity and surjectivity. Their action over $R[X]$ and its restriction to a linear operator.

4.1 Relationship between $\delta$ and $A_n$

In the next theorem we will see the relationship between the operators $\mathfrak{A}$ and $\delta$.

Theorem 1. Take $f(X) \in \text{Exp}_R[X]$. Then

$$A_{n+1}(\delta^n f(X)) = f(X)\delta(A_n(\delta^{n-1} f(X))),$$

with $A_0 \equiv 1$.

Proof. By definition $A_2(\delta f) = f \delta(f)$. Suppose true $A_n(\delta^{n-1} f) = f \delta(A_{n-1}(\delta^{n-2} f))$. Set $A_n \equiv A_n(\delta^{n-1} f)$. On the one hand we have

$$A_{n+1} = \delta f A_n + \delta^2 f \{ A_1 A_{n-1} + A_2 A_{n-2} + \cdots \} + \cdots$$

$$= \delta(f) \delta(A_{n-1}) + \delta^2(f) \{ f \delta(A_{n-2}) + \delta(A_1) A_{n-2} + \cdots \} + \cdots$$

$$= f \delta(f) \delta(A_{n-1}) + \delta^2(f) \{ f \delta(A_{n-2}) + \delta(A_1) A_{n-2} + \cdots \} + \cdots$$

And on the other hand,

$$f \delta(A_n) = f \delta(f) A_{n-1} + \delta(f) \{ A_1 A_{n-2} + A_2 A_{n-3} + \cdots \} + \cdots$$

$$= f \delta(f) \delta(A_{n-1}) + \delta^2(f) A_{n-1} + \delta^2(f) \delta(A_1) A_{n-2} + \delta^2(f) A_1 \delta(A_{n-2}) + \cdots$$

$$= f \delta(f) \delta(A_{n-1}) + \delta^2(f) \{ f \delta(A_{n-2}) + \delta(A_1) A_{n-2} + \cdots \} + \cdots$$

We can see that $A_{n+1} = f \delta(A_n)$ for comparing both results.
Corollary 1. Take \( f(X) \in \text{Exp}_R[X] \). Then
\[
A_n([\delta^{n-1}f(X)]) = \overbrace{f(X)\delta(\cdots f(X)\delta(f(X)))}^{n}.
\] (14)

Proof. From above theorem, \( A_2([\delta^1f]) = f(\delta(A_1([\delta^0f])) = f(\delta(f)) \). Inductively the claim is followed. \(\square\)

4.2 Characterization by groups of \( k \)-homogeneity

Let \( R^N \Delta \text{Exp}_R[X] \) denote the set
\[
\{a \Delta \text{Exp}_R[X] : a \in R^N\}
\] (15)
where the product \( a \Delta f(X) \) is componentwise. \( R^N \Delta \text{Exp}_R[X] \) is an \( R^N \)-module since

1. \( a(\Delta f(X) + \Delta g(X)) = a\Delta f(X) + a\Delta g(X), \)
2. \( (a + b)\Delta f(X) = a\Delta f(X) + b\Delta f(X), \)
3. \( (ab)\Delta f(X) = a(b\Delta f(X)), \)
4. \( e\Delta f(X) = \Delta f(X). \)

If we define product in \( R^N \Delta \text{Exp}_R[X] \) as
\[
a\Delta f(X) \ast b\Delta g(X) = (ab)(\Delta f(X) \ast \Delta g(X))
\] (16)
then \( R^N \Delta \text{Exp}_R[X] \) becomes an \( R^N \)-algebra.

We wish to study the action of \( \mathfrak{A} \) over \( R^N \Delta \text{Exp}_R[X] \). Let \( h_\alpha = (\alpha^n)_{n \geq 0} \in R^N, \alpha \neq 0 \). We extend \( h_\alpha \) for all \( k \in \mathbb{N} \) by means of to define
\[
h_\alpha^k = \begin{cases} (\alpha^k)^n_{n \geq 0}, & \text{if } k \neq 0, \\ e = (1)^n_{n \geq 0}, & \text{if } k = 0. \end{cases}
\] (17)

Then \( h_\alpha^k = h_\alpha^k \) and \( h_\alpha^k h_\beta^l = h_{\alpha^k \beta^l} \). Let \( \mathcal{M}_1 \) denote the commutative multiplicative monoid
\[
\mathcal{M}_1 = \{h_\alpha \in R^N : \alpha \neq 0\}.
\] (18)

If all \( \alpha \) are taken to be units in \( R \), then \( \mathcal{M}_1 \) is a group and denote this by \( \mathcal{H}_1 \).

In the following theorem we characterize \( \mathfrak{A} \) by the monoid \( \mathcal{M}_1 \)

Theorem 2. \( \mathfrak{A} \) is homogeneous under the action of \( \mathcal{M}_1 \), that is,
\[
\mathfrak{A}(\mathcal{M}_1 \Delta \text{Exp}_R[X]) = \mathcal{M}_1 \mathfrak{A}(\Delta \text{Exp}_R[X]).
\] (19)

Proof. Take \( h_\alpha \in \mathcal{M}_1 \). We wish to show that
\[
A_n([\alpha^{(n-1)}\delta^{n-1}f(X)]) = \alpha^{(n-1)}A_n([\delta^{n-1}f(X)])
\]
The proof is by induction on \( n \). The case \( n = 1 \) is trivial, since \( A_1([\alpha^0f(X)]) = A_1([f(X)]) = f(X) = \alpha^0f(X) = \alpha^0A_1([f(X)]) \). Suppose true for all natural number smaller than \( n \) and we prove for \( n + 1 \)
\[
A_{n+1}([\alpha^n\delta^n f(X)]) = \sum_{k=1}^{n} \sum_{[p(n)] = k} \frac{n!}{j_1! \cdots j_n!} \left[ \frac{\alpha^0A_1}{1!} \right]^{j_1} \cdots \left[ \frac{\alpha^{(n-1)}A_n}{n!} \right]^{j_n} \alpha^k f(X)
\]
\[
= \sum_{k=1}^{n} \sum_{[p(n)] = k} \frac{n!\alpha^{(0j_1+2j_2+\cdots+(n-1)j_{n+1})}}{j_1! \cdots j_n!} \left[ \frac{A_1}{1!} \right]^{j_1} \cdots \left[ \frac{A_n}{n!} \right]^{j_n} \delta^k f(X)
\]
\[
= \alpha^n \sum_{k=1}^{n} \sum_{[p(n)] = k} \frac{n!}{j_1! \cdots j_n!} \left[ \frac{A_1}{1!} \right]^{j_1} \cdots \left[ \frac{A_n}{n!} \right]^{j_n} \delta^k f(X)
\]
\[
= \alpha^n A_{n+1}([\delta^n f(X)]).
\]
From which it follows that $\mathfrak{A}(h_0 \Delta f(X)) = h_0 \mathfrak{A}(\Delta f(X))$ for all nonzero $\Delta f(X) \in \Delta \text{Exp}_R[X]$ and all $h_0 \in M_1$. □

If $\alpha$ is a unity, then $\mathfrak{A}$ is homogeneous under the action the group $\mathcal{H}_1$. The subscripts in $M_1$ and $\mathcal{H}_1$ are justified by the order of homogeneity of $h_0$. We call $M_1$ and $\mathcal{H}_1$ monoid and group of 1-homogeneity, respectively. The previous theorem tells us that the monoid $M_1$ defines a partition in $R^N \Delta \text{Exp}_R[X]$ in homogeneity class $M_1 \Delta f(X)$, for all $f(X) \in \text{Exp}_R[X]$. Then the action of $\mathfrak{A}$ over $R^N$ as an action of class of 1-homogeneity. Take also $g(X)$ in $\text{Exp}_R[X]$. If $f(X) = g(\alpha X)$, then $\Delta f(X) = h_0 \Delta g(X) \circ (\alpha X)_{\alpha \in \mathbb{N}}$ implies that $\mathfrak{A}(\Delta f(X)) = h_0 \mathfrak{A}(\Delta g(X)) \circ (\alpha X)_{\alpha \in \mathbb{N}}$, where the composition is componentwise.

Now we study the groups of $k$-homogeneity in $R^N \Delta \text{Exp}_R[X]$ for $k > 1$.

**Definition 3.** For all $k \geq 2$ define

$$\mathcal{H}_k = \{ h \in R^N : \mathfrak{A}(h \Delta f(X)) = h^k \mathfrak{A}(\Delta f(X)), \ 0 \neq \Delta f(X) \in \Delta \text{Exp}_R[X], \ h \neq 0 \}$$

as the $k$-homogeneity set of $\mathfrak{A}$ and we will say that $\mathfrak{A}$ is $\mathcal{H}_k$-homogeneous of degree $k$.

**Theorem 3.** Let $\mathcal{H}_k$ be the $k$-homogeneity set of $\mathfrak{A}$. All element $h$ of $\mathcal{H}_k$ has the next form

$$h = \{ a^{1-n} b^n \}_{n \geq 0}$$

where $a^{k-1} = 1$ y $b^{k-1} = a$, $k > 1$.

**Proof.** Take $h = \{ h_n \}_{n \geq 0}$ in $\mathcal{H}_k$. For $n = 1$, we have that $A_1([h_0 x_0]) = h_0^0 A_1([x_0])$ implies that $h_0 = h_0^0$ and therefore $h_0^{k-1} = 1$. Now for $n = 2$, we have that $A_2([h_1 x_1]) = h_0^k A_2([x_1])$ implies $h_0 h_1 = h_1^k$ and therefore $h_1^{k-1} = h_0$ and thus we have the desired relations. On the other hand, it is easy to see that $h_0 = h_0^0 h_1^0$ and $h_1 = h_0^{-(1-1)} h_1^1$. Since $h \in \mathcal{H}_k$, then $A_{n+1}([h_n x_n]) = h_0^k A_{n+1}([x_n])$ implies that

$$h_0^k h_1^1 h_2^1 \cdots h_{n+1-r}^1 x_0^1 x_1^1 x_2 \cdots x_{n+1-r} = h_0^k x_0^1 x_1^1 x_2 \cdots x_{n+1-r}$$

with $l_1 + l_2 + \cdots + l_{n-r} = n$. Then $h_0 h_1^n = h_0^0 h_1^0$ and $h_n = h_0^{-(n-1)} h_1^n$. Hence it is followed that

$$h_n^k = (h_0^{-(n-1)})^k (h_1^n)^k$$

$$= (h_0^k)^{-(n-1)} (h_1^k)^n$$

$$= h_0^{-(n-1)} (h_0 h_1)^n$$

$$= h_0^{-(n-1)} h_0^n h_1^n$$

$$= h_0 h_1^n.$$

We want to show that $h_0^k h_1^1 h_2^1 \cdots h_{n+1-r}^1 = h_0 h_1^n$ for all $r$. We have

$$h_0^k h_1^1 h_2^1 \cdots h_{n+1-r}^1 = h_0^k h_0^{1-l_1} h_1^1 h_0^{1-l_2} h_1^2 \cdots h_0^{1-l_{n+1-r}} h_1^{l_{n+1-r}}$$

$$= h_0^k h_0^{1-l_1} h_0^{1-l_2} \cdots h_0^{1-l_{n+1-r}} h_1^{l_{n+1-r}} h_1^{l_1 l_2 + \cdots + l_{n+1-r}}$$

$$= h_0^{n+1-(l_1 + l_2 + \cdots + l_{n+1-r})} h_1^{l_1 l_2 + \cdots + l_{n+1-r}}$$

$$= h_0 h_1^n.$$ 

The preceding result does not depend on the length $n$, so we can deduce that $h_n = h_0^{-(n-1)} h_1^n$ for all $n$. On the other hand, suponga que $h_n = h_0^{-(n-1)} h_1^n$. Then

$$A_{n+1}([h_n x_n]) = A_{n+1}([h_0^{-(n-1)} h_1^n x_n])$$

$$= h_0 h_1^n A_{n+1}([x_n])$$

$$= h_0 h_1^n A_{n+1}([x_n])$$

and

$$h = \{ h_0^{-(n-1)} h_1^n \}_{n \geq 0} \in \mathcal{H}_k.$$ □
**Theorem 4.** \( \mathcal{H}_k \subset R^N \) is a torsion group of exponent \((k-1)^2\).

**Proof.** Take \( f, g \in \mathcal{H}_k \). Then

\[
\mathfrak{A}(fX) = \mathfrak{A}(f(gX)) = f^k \mathfrak{A}(gX) = f^{k-1} \mathfrak{A}(aX) = (fg)^k aX (\Delta f(X)).
\]

Hence \( fg \in \mathcal{H}_k \). Obviously \( e \in \mathcal{H}_k \). From the previous theorem we know that all element in \( \mathcal{H}_k \) is invertible because if \( (a^{-n}b^n)_{n \in \mathbb{N}} \) exist in \( \mathcal{H}_k \), then \( (a^{-n}b^n)_{n \in \mathbb{N}} \) is its inverse and thus \( \mathcal{H}_k \) is a group. From \( (a^{-n}b^n)^{(k-1)^2} = (a^{k-1}b)^n = 1 \), is followed that \( f^{k-1} = e \) for all \( f \in \mathcal{H}_k \) and it follow that \( \mathcal{H}_k \) is a torsion group of exponent \((k-1)^2\).

As \( \mathfrak{A}(\mathcal{H}_k \Delta \text{Exp}_R[X]) = \mathcal{H}_k \mathfrak{A}(\Delta \text{Exp}_R[X]) \), then the operator \( \mathfrak{A} \) is an action over class of \( k \)-homogeneity in \( R^N \Delta \text{Exp}_R[X] \). Also let \( f(X) = ag(a^{-1}bX) \in \text{Exp}_R[X] \) with \( a^{k-1} = 1 \) and \( b^{k-1} = a \). Then \( \Delta f(X) = h\Delta g(X) \circ (a^{-1}bX)_{n \in \mathbb{N}} \) and \( \mathfrak{A}(\Delta f(X)) = \mathfrak{A}(h\Delta g(X) \circ (a^{-1}bX)_{n \in \mathbb{N}}) = h^k \mathfrak{A}(\Delta g(X)) \circ (a^{-1}bX)_{n \in \mathbb{N}} \).

As the exponential generating functions of elements in \( \mathcal{H}_k \) are exponential functions we call to the groups \( \mathcal{H}_k \) exponential groups. We finish this section showing the exponential groups \( \mathcal{H}_k \) for some rings.

### 4.2.1 Homogeneity group in \( \mathbb{Z}^N \)

The units of \( \mathbb{Z} \) are \{−1, 1\}. Then \( \mathcal{H}_1 = \{ e, (1, -1, 1, -1, \ldots) \} \) is a binary group and \( \mathcal{H}_k = \{ e \} \) for \( k > 1 \).

### 4.2.2 Homogeneity group in \( \mathbb{R}^N \)

As \( \mathbb{R} \) is a field then \( \mathcal{H}_1 \cong \mathbb{R}^\times \). Also \( \mathcal{H}_k = \{ e \} \) for all \( k > 1 \).

### 4.2.3 Homogeneity group in \( \mathbb{C}^N \)

Since \( a^{k-1} = 1, b^{k-1} = a \), then \( a \) is a \((k-1)th\) root of the unity, i.e., \( a \in \mathbb{Z}_{k-1} \) and \( b \in k\sqrt[k]{a} \mathbb{Z}_{k-1} \). Then

1. \( \mathcal{H}_1 \cong \mathbb{C}^\times \)
2. \( \mathcal{H}_2 = \{ e \} \)
3. \( \mathcal{H}_3 = \{((-1)^n)_{n \in \mathbb{N}}, (-i^n)_{n \in \mathbb{N}}, (-(-i)^n)_{n \in \mathbb{N}}\} \cong \mathbb{Z}_4 \)
4. \( \mathcal{H}_4 = \left( \{\omega^n, \omega^{2n} \}_{n \in \mathbb{N}}, \left( \omega^{\frac{3+4n}{4}}, \omega^{\frac{3-4n}{4}} \right)_{n \in \mathbb{N}}, \left( \omega^{\frac{5+4n}{4}}, \omega^{\frac{5-4n}{4}} \right)_{n \in \mathbb{N}} \right) \)

where \( \omega = e^{2\pi i/3} \).

In general

\[
\mathcal{H}_k = \left\{ e^{2\pi i(j(1-n)+ln)(k-1)+jn}/(k-1)^2 \right\}_{n \in \mathbb{N}} : j = 0, 1, \ldots, k - 2, l = 0, 1, \ldots, k - 2 \quad (22)
\]

\[ \]
4.2.4 Homogeneity group in \( \mathbb{Z}[w_d]^N \)

Let \( d \) be a square-free integer. Then \( \mathbb{Z}[w_d] = \{a + bw_d : a, b \in \mathbb{Z}\} \) is the ring known as the quadratic ring, where

\[
w_d = \begin{cases} (1 + \sqrt{d})/2 & \text{if } d \equiv 1 \pmod{4} \\ \sqrt{d} & \text{if } d \not\equiv 1 \pmod{4} \end{cases}
\]

We find the groups of homogeneity in \( \mathbb{Z}[w_d]^N \). We start with the cases \( d = -1, -3 \), which correspond to the gaussian and eisenstein rings, respectively.

1. \( \mathbb{Z}[i] \).

\[
\mathcal{H}_1 = \{e, ((-1)^n)_{n \in \mathbb{N}}, (i^n)_{n \in \mathbb{N}}, ((-i)^n)_{n \in \mathbb{N}}\} \cong \mathbb{Z}_4,
\]

\[
\mathcal{H}_3 = \{(a^{1-n}b^n)_{n \in \mathbb{N}} : (a, b) \in \{(1, 1), (1, -1), (-1, i), (-1, -i)\}\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2
\]

\[
\mathcal{H}_5 = \{(a^{1-n}b^n)_{n \in \mathbb{N}} : (a, b) \in \{(1, 1), (1, -1), (1, i), (1, -i)\}\} = \mathcal{H}_1 \cong \mathbb{Z}_4
\]

\[
\mathcal{H}_{2l+1} = \mathcal{H}_1, l \geq 3,
\]

\[
\mathcal{H}_q = \{e\}, q \neq 2^l + 1.
\]

Then only there are two non-isomorphs nontrivial exponential groups in \( \mathbb{Z}[i]^N \): \( \mathcal{H}_1 \) and \( \mathcal{H}_3 \).

2. \( \mathbb{Z}[\omega], \omega = -\frac{1+\sqrt{-3}}{2} \).

\[
\mathcal{H}_1 = \{e, ((-1)^n)_{n \in \mathbb{N}}, (\omega^n)_{n \in \mathbb{N}}, ((-1)^n\omega^n)_{n \in \mathbb{N}}, (\omega^{2n})_{n \in \mathbb{N}}, ((-1)^n\omega^{2n})_{n \in \mathbb{N}}\}
\]

\[\cong \mathbb{Z}_6\]

\[
\mathcal{H}_3 = \{(a^{1-n}b^n)_{n \in \mathbb{N}} : (a, b) \in \{(1, 1), (1, -1)\}\} \cong \mathbb{Z}_2
\]

\[
\mathcal{H}_4 = \{(a^{1-n}b^n)_{n \in \mathbb{N}} : (a, b) \in \{(1, 1), (1, \omega), (1, \omega^2)\}\} \cong \mathbb{Z}_3
\]

\[
\mathcal{H}_7 = \{(a^{1-n}b^n)_{n \in \mathbb{N}} : (a, b) \in \{(1, \pm 1), (1, \pm \omega), (1, \pm \omega^2)\}\} = \mathcal{H}_1 \cong \mathbb{Z}_6
\]

\[
\mathcal{H}_{3 \cdot \sqrt{3} + 1} = \mathcal{H}_1, l \geq 2
\]

\[
\mathcal{H}_{2l+1} = \mathcal{H}_3, l \geq 2,
\]

\[
\mathcal{H}_{3m \cdot \sqrt{3} + 2l+1} = \mathcal{H}_4, m \geq 2, l \geq 0,
\]

\[
\mathcal{H}_q = \{e\}, q \neq 2^l + 1, q \neq 3^m \cdot 2^l + 1, q \neq 3 \cdot 2^l + 1.
\]

Then only there are two non-isomorphs nontrivial exponential groups in \( \mathbb{Z}[\omega]^N \): \( \mathcal{H}_1, \mathcal{H}_3 \) and \( \mathcal{H}_4 \).

The case \( d > 1 \), correspond to the real quadratic rings. For this ring there are not nontrivial homogeneity groups \( \mathcal{H}_k, k > 1 \).

1. \( \mathbb{Z}[\sqrt{2}] \).

\[
\mathcal{H}_1 = \{((\pm(1 + \sqrt{2})^m)_{n \in \mathbb{N}} : m \in \mathbb{Z}\}.
\]

2. \( \mathbb{Z}[\sqrt{3}] \).

\[
\mathcal{H}_1 = \{((\pm(2 + \sqrt{3})^m)_{n \in \mathbb{N}} : m \in \mathbb{Z}\}.
\]

3. \( \mathbb{Z}[\varphi] \), \( \varphi \) the golden radio. \( \mathcal{H}_1 = \{((\pm\varphi^m)_{n \in \mathbb{N}} : m \in \mathbb{Z}\}.
\]

4. In general for all \( d > 1 \), \( \mathcal{H}_1 = \{((\pm e^m)_{n \in \mathbb{N}} : m \in \mathbb{Z}\} \), where \( e \) is a fundamental unity of \( \mathbb{Z}[^d] \).

4.3 Injectivity and Surjectivity of \( \mathfrak{a} \)

Now we study the injectivity and the surjectivity of \( \mathfrak{a} \) when this is restricted to subset of \( R^N \).
Theorem 5. The set of zeros of $\mathfrak{A}$ is the ideal

$$R^N_0 = \{(x_n)_{n \geq 0} \in R^N : x_0 = 0\} \subset R^N,$$

i.e., $\text{Null}(\mathfrak{A}) = R^N_0$.

Proof. Take $f(X) \in \text{Exp}_R[X]$ and let $x^*$ be a zero of $f(X)$. From (13) if $A_1([f(x^*)]) = f(x^*) = 0$, then $A_n(\delta^{n-1} f(x^*)) = f(x^*) \delta(A_n([\delta^{(n-2)} f(x^*)])) = 0$ for all $n > 1$. Hence, all the $\Delta f(x^*) = (\delta^n f(x^*))_{n \in \mathbb{N}}$ with $f(x^*) = 0$ are in $\text{Null}(\mathfrak{A})$, for all $f(X) \in \text{Exp}_R[X]$. On the other hand, let $y \in \text{Null}(\mathfrak{A})$. Then

$$\mathfrak{A}(y) = (A_1([y_0]), A_2([y_1]), A_3([y_2])...) = (0, 0, 0,...).$$

Since $A_1([y_0]) = 0$ implies $y_0 = 0$, then $y \in R^N_0$ and $\text{Null}(\mathfrak{A}) = R^N_0$.

Consequently, $\mathfrak{A}$ is a map from quotient ring $R^N/R^N_0$ to $R^N$.

Theorem 6. $\mathfrak{A} : R^N/R^N_0 \rightarrow R^N$ is injective.

Proof. Take $x = (x_n)_{n \geq 0}$ and $y = (y_n)_{n \geq 0}$ in $R^N$ with exponential generating functions $f_X(X)$ and $f_Y(Y)$, respectively, such that $x_n \neq 0$ and $y_0 = 0$ and suppose that $A_n([x_{n-1}]) = A_n([y_{n-1}])$ for all $n \geq 1$. Then $x_n = y_n$. From (13), we have that $f_X(0) \delta(A_{n-1}([\delta^{(n-2)} f_X(0)])) = f_Y(0) \delta(A_{n-1}([\delta^{(n-2)} f_Y(0)]))$. By induction $x_n = y_n$ for all $n \geq 0$ and therefore $x = y$.

Then, we show that the operator $\mathfrak{A}$ is bijective if it is restricted to those sequences $R^N$ having first element invertible. Thus

Theorem 7. Let $R^N_U$ denote the set $R^x \times R^N_1$. Then $\mathfrak{A} : R^N_U \rightarrow R^N_U$ is bijective.

Proof. Indeed we note that $\mathfrak{A}(x)$ falls into $R^N_U$ when $x = (x_n) \in R^N_U$. Take $y \in R^N$ and let $x \in R^N$ with $\mathfrak{A}(x) = y$. Then from the above theorem $\mathfrak{A}$ restricted to $R^N_U$ is injective. Let $x \in R^N$ such that $\mathfrak{A}(x) = y$. Then $x_0 = y_0$, $x_1 = y_0^{-1} y_1$, $x_2 = y_0^{-2} y_2 - y_0^{-1} y_1^2$ and as $y_0$ is an unity in $R$, therefore all element $x_n$ is expressed in terms of $y_0^{-1}, y_1, y_2, ...$. Thus there is an only $x \in R^N_U$ such that $\mathfrak{A}(x) = y$. Hence $\mathfrak{A}$ is surjective when it is restricted to $R^N_U$.

In general $\mathfrak{A} : R^N \rightarrow R^N$ is not surjective. For instance, the set $R^N \setminus \{0\}$ is never the image the some subset of $R^N$ under the action of $\mathfrak{A}$. This is noticed because if we take $x \in R^N$ with $A_1([x_0]) = x_0 = 0$, then $A_n([x_{n-1}]) = 0$ for all $n > 1$. Then $\mathfrak{A}(x) = 0$ and $\mathfrak{A}(x) \notin R^N \setminus \{0\}$.

Take $a$ in $R^N$ and let $f_a(X)$ its exponential generating function in $\text{Exp}_R[X]$. Let $f_a(X) \text{Exp}_R[X]^N$ denote the subring of $\text{Exp}_R[X]^N$ of sequences in $\text{Exp}_R[X]$ multiplied by a factor $f_a(X)$. Then

Proposition 5. $\mathfrak{A}(\Delta(\text{Exp}_R[X])) \subset \bigcup_{f_a(X) \in \text{Exp}_R[X]} f_a(X) \text{Exp}_R[X]^N.$

Proof. From (14) we have

$$\mathfrak{A}(\Delta(f_a(X))) = f_a(X)(1, \delta(A_{n-1}([\delta^{(n-2)} f_a(X)])))_{n \geq 2} \in f_a(X) \text{Exp}_R[X]^N.$$

Corollary 2. $\mathfrak{A}(R^N) \subset \bigcup_{x \in R} \langle x \rangle^N.$

Proof. Follows from above proposition and from to evaluate $f_a(X)$ in $X = 0$.

Proposition 6. Let $\langle a \rangle$ be a principal ideal of $R$. Then $\mathfrak{A}(\langle a \rangle^N) \subset M_1 \langle a \rangle^N.$

Proof. Take $ax \in \langle a \rangle^N$. From proposition 2, $\mathfrak{A}(ax) = ah_a\mathfrak{A}(x)$. As $\langle a \rangle^N$ is an ideal in $R^N$, then $h_a(\mathfrak{A}(ax)) \in M_1 \langle a \rangle^N$. 

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Let $J = \{I_i\}_{i \in \mathbb{N}}$ be a sequences of ideals in $R$ with the $I_i$ not necessarily different. It is clear that $J$ is an ideal of $R^N$. Define

$$\mathfrak{F}_J(X) := \{f_n(X) : a \in J\} \quad (24)$$

as the class of exponential generating functions in $J$ and

$$\Delta(\mathfrak{F}_J(X)) := \{\Delta(f_n(X)) : f_n(X) \in \mathfrak{F}_J(X)\} \quad (25)$$

as the class of Hurwitz expansion of $\mathfrak{F}_J(X)$. Then

$$\mathfrak{A}(\Delta(\mathfrak{F}_J(X))) = (A_n([\delta^{n-1}\mathfrak{F}_J(X)])_{n \in \mathbb{N}}. \quad (26)$$

As we have seen, not every element in $R^N$ is image of $\mathfrak{A}$. However, if $F = \text{Frac}(R)$, then from the theorem $7\mathfrak{A} : F^N \to F^N$ is bijective.

### 4.4 $\mathfrak{A}$ acting over $R[x]$

**Definition 4.** For each $k \geq 1$ we define the ring

$$R_k^N = \{x \in R^N : x = (x_0, x_1, \ldots, x_{k-1}, 0, 0, \ldots)\} \quad (27)$$

such that $R_k^N \cap R_0^N = \{0\}$.

The condition $R_k^N \cap R_0^N = \{0\}$ implies that $x_0 \neq 0$ for all $x = (x_0, x_1, \ldots, x_{k-1}, 0, 0, \ldots)$ in $R_k^N$ and $k \geq 2$, then we have the series of inclusions $\{0\} \subset R_1^N \subset R_2^N \subset \cdots \subset R_k^N$.

The elements in the ring $R_k^N$ correspond to the polynomials $\sum_{n=0}^{k-1} a_n \frac{x^n}{n!}$ of grade $k - 1$ in $\text{Exp}_R[X]$.

#### 4.4.1 Structure of $\mathfrak{A}(R_1^N)$

The ring $R_1^N$ is isomorphic to $R$ and is the simpler ring from which act $\mathfrak{A}$ since $\mathfrak{A}(R_1^N) = R_1^N$. Over $R_1^N$ the operator $\mathfrak{A}$ fulfil $\mathfrak{A}(x + y) = \mathfrak{A}(x) + \mathfrak{A}(y)$ and $\mathfrak{A}(xy) = \mathfrak{A}(x)\mathfrak{A}(y)$ for all $x, y$ in $R_1^N$.

#### 4.4.2 Estructura de $\mathfrak{A}(R_2^N)$

We have that $R_2^N = (R_2^N \setminus R_1^N) \cup R_1^N$. If $x = (a, b, 0, 0, \ldots) \in R_2^N$, then

$$\mathfrak{A}(x) = \begin{cases} x, & \text{if } x \in R_1^N \\ ah_b, & \text{if } x \in R_2^N \setminus R_1^N. \end{cases} \quad (28)$$

Now take $x, y$ en $R_2^N \setminus R_1^N$. Then we can say that $\mathfrak{A}(xy) = \mathfrak{A}(x)\mathfrak{A}(y)$ and that $\mathfrak{A}(R_2^N \setminus R_1^N)$ is a multiplicative group. Also if $x = (a, b, 0, 0, \ldots), y = (c, d, 0, 0, \ldots)$, then $\mathfrak{A}(x+y) = (a + c)h_{b+d}$ and $\mathfrak{A}$ is no linear in $R_2^N \setminus R_1^N$.

### 4.5 Linear part of $\mathfrak{A}$

**Definition 5.** Let $\alpha \in K$. The linear part of the operator $\mathfrak{A}$ is the set

$$\text{Lin}\mathfrak{A} = \{x \in R^N : \mathfrak{A}(x) \in R_1^N\} \quad (29)$$

The set $\text{Lin}\mathfrak{A}$ is a sub-algebra of $R^N$ and the restriction of $\mathfrak{A}$ to $\text{Lin}\mathfrak{A}$ is a linear operator, i.e., if $x, y \in \text{Lin}\mathfrak{A}$ and if $a, b \in R$, then $\mathfrak{A}(ax + by) = a\mathfrak{A}(x) + b\mathfrak{A}(y)$. It is easy to note that $\text{Lin}\mathfrak{A} = R_1^N$. 

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5 Dynamical system associated with $\mathfrak{A}(\Delta f(x))$

We know from [6] that the solution $\phi$ of [4] is given by

$$\phi(t, x) = x + \sum_{n=1}^{\infty} A_n([f^{(n-1)}(x)]) \frac{t^n}{n!}. \quad (30)$$

This solution is a dynamical system defined on the sequence $f(x), f^{(1)}(x), f^{(2)}(x), \ldots$. Also, it is well known that the solution $\phi(t, x)$ is a flow, i.e., satisfy

1. $\phi(0, x) = x$
2. $\phi(t, \phi(s, x)) = \phi(t + s, x)$

The property 2) tell us that $\phi(t, x)$ for all $t \in \mathbb{R}$ is an oneparametric group, with identity $\phi(0, x)$ and composition product $\circ$. From this we can note that the operator $\mathfrak{A}$ lead the sequence of derivative of $f(x)$ the group $\{\phi(t, x)\}_{t \in \mathbb{R}}$. Then, we will define a dynamical system over the integral domain $\Delta \text{Exp}_{\mathbb{R}}[X]$.

**Definition 6.** Take $\Delta f(x)$ in $\Delta \text{Exp}_{\mathbb{R}}[X]$. We define the dynamical system $\Phi : \mathbb{R} \times \mathbb{R} \times \Delta \text{Exp}_{\mathbb{R}}[X] \to \text{Exp}_{\mathbb{S}}[T], \ S = \text{Exp}_{\mathbb{R}}[X]^N$, related to $\Delta f(x)$ as

$$\Phi(t, x, \Delta f(x)) = x + \sum_{n=1}^{\infty} A_n([\delta^{(n-1)} f(x)]) \frac{t^n}{n!}. \quad (31)$$

From previous definition is concluded that the flow associate to Hurwitz expansion $\Delta f(x)$ is the exponential generating function of $\mathfrak{A}(\Delta f(X))$. From the theorem [7] is deduced that if $a \in \mathbb{R}^N_U$, then there is a function $f(x)$ in $\text{Exp}_{\mathbb{R}}[X]$ such that $\phi_0 \mathfrak{A}(\Delta f(x)) = a$, i.e, all sequence in $\mathbb{R}^N_U$ is mapped to a funcion generadora exponencial $\phi_0 \Phi(t, x, \Delta f(X))$ in $\text{Exp}_{\mathbb{S}}[T]$ for some function $f(x) \in \text{Exp}_{\mathbb{R}}[X]$. In particular, the previous it holds for all sequence in $\text{Frac}(\mathbb{R})^N$. For instance, the sequences $(1, 1, 2, 5, 14, 42, \ldots)$ and $(1, 1, 2, 3, 5, 8, 13, \ldots)$ have associates the exponential generating functions $\phi_0 \Phi(t, x, \Delta f(X))$ and $\phi_0 \Phi(t, x, \Delta g(X))$ for some functions $f$ and $g$ in $\text{Exp}_{\mathbb{R}}[X]$. An important issue is how to obtain such functions, if possible, but this will not be studied in this paper. We will deal with from now on with basic results from the theory of dynamical systems and we will apply the previous results in this paper.

**Theorem 8.** $\Phi(t, x, \Delta f(x))$ is a flow.

**Proof.** When $t = 0$ we have from (31) that $\Phi(0, x, \Delta f(x)) = x$. Now the property 2) will be proved. On one hand, the Taylor expansion of $\Phi(s + t, x, \Delta f(x))$ is $\sum_{n=0}^{\infty} \delta^n \Phi(s, x, \Delta f(x)) \frac{t^n}{n!}$. On the other hand, making $\Phi_s = \Phi(s, x, \Delta f(x))$ we have

$$\Phi(t, \Phi(s, x, \Delta f(x)), \Delta f(x)) = \Phi_s + \sum_{n=1}^{\infty} A_n([\delta^{n-1} f(\Phi_s)]) \frac{t^n}{n!}$$

$$= \Phi_s + \sum_{n=1}^{\infty} \delta^n \Phi(s, x, \Delta f(x)) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \delta^n \Phi(s, x, \Delta f(x)) \frac{t^n}{n!}$$

$$= \Phi(s + t, x, \Delta f(x)).$$

\[ \square \]
On the other hand, from \([13]\) we know that the operators \(\mathfrak{A}\) and \(\delta\) are related by equation \(A_{n+1}(\delta^n f(x)) = f(x)\delta(A_n(\delta^{n-1} f(x)))\). Write \(\delta_x = \delta\) and suppose that \(\delta_x\) extend as derivative to \(\Phi\), i.e., \(\delta_x\) derives formally to the dynamical system \(\Phi(t, x, \Delta f(x))\) with respect to the variable \(x\). Also define \(\delta_t\) the formal derivative of \(\Phi\) with respect to variable \(t\). In the following theorem we will show how are related the partial derivatives \(\delta_x\) and \(\delta_t\) of \(\Phi\).

**Theorem 9.** Take \(\Delta f(x) \in R^\mathbb{N}\). Then

\[
f(x)\delta_x \Phi(t, x, \Delta f(x)) = \delta_t \Phi(t, x, \Delta f(x)) = f(\Phi) \tag{32}
\]

**Proof.** From \([31]\), we have

\[
f(x)\delta_x \Phi(t, x, \Delta f(x)) = f(x)\delta_x \left(x + \sum_{n=1}^{\infty} A_n(\delta^{n-1} f(x))^\frac{t^n}{n!}\right)
\]

\[
= f(x) + \sum_{n=1}^{\infty} f(x)\delta_x (A_n(\delta^{n-1} f(x)))^\frac{t^n}{n!}
\]

\[
= A_1([f(x)]) + \sum_{n=1}^{\infty} A_{n+1}(\delta^n f(x))^\frac{t^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} A_{n+1}(\delta^n f(x))^\frac{t^n}{n!}
\]

\[
= \delta_t \Phi(t, x, \Delta f(x)).
\]

\(\square\)

A result of great importance is that we can construct an \(R\)-module from the flow \(\Phi(t, x, r\Delta f(x))\) for a fixed \(r \in R\). First we prove the following theorem who says that the flows of \(\delta_x \Phi = f(\Phi)\) and \(\delta_t \Phi = rf(\Phi)\) have matching trajectories.

**Theorem 10.** Fix an \(r \in R\). Then \(\Phi(t, x, r\Delta f(x)) = \Phi(rt, x, \Delta f(x))\) for all \(t \in R\).

**Proof.** From proposition 2 we know that \(\mathfrak{A}(r \Delta f(x)) = (r^n)_{n \geq 1} \mathfrak{A}(\Delta f(x))\). Then

\[
\Phi(t, x, r\Delta f(x)) = x + \sum_{n=0}^{\infty} r^n A_n([\delta^{n-1} f(x)])^\frac{(rt)^n}{n!}
\]

\[
= x + \sum_{n=0}^{\infty} A_n([\delta^{n-1} f(x)])^\frac{(rt)^n}{n!}
\]

\[
= \Phi(rt, x, \Delta f(x)),
\]

as we wanted to show. \(\square\)

Now we use the previous results for to construct an \(R\)-module over the flow \(\Phi\).

**Theorem 11.** Fix an \(r\) and an \(x\) in \(R\) and let \(M_R(x) = \{\Phi_{t,r}(x)\}_{t \in R}\) be an abelian group, where \(\Phi_{t,r}(x) = \Phi(x, r\Delta f(x))\) and the action of ring \(R\) over \(M_R(x)\) is given by the map \(*: \{\Phi_{t,r}(x)\}_{t \in R} \times R \to \{\Phi_{t,s}(x)\}_{t \in R}\) defined by \(\Phi_{t,s}(x) = \Phi_{t,s}(x)\). Then \(M_R(x)\) is a right \(R\)-module.

**Proof.** From above theorem \(\Phi_{t,r}(x) = \Phi_{rt,1}(x) = \Phi_{r,t}(x) = \Phi_{1,rt}(x)\). Then we have

1. \((\Phi_{t,r}(x) \circ \Phi_{s,r}(x)) \star v = \Phi_{t+s,r}(x) \star v = \Phi_{tv+sv, r}(x) = \Phi_{tv, r}(x) \circ \Phi_{sv, r}(x) = (\Phi_{t,r}(x) \star v) \circ (\Phi_{s,r}(x) \star v),\)
2. \( \Phi_{t,r}(x) \ast (v+w) = \Phi_{t(v+w),r}(x) = \Phi_{tv+w,r}(x) = \Phi_{tv,r}(x) \circ \Phi_{w,r}(x) = (\Phi_{t,r}(x) \ast v) \circ (\Phi_{t,r}(x) \ast w) \),

3. \( \Phi_{t,r}(x) \ast (vw) = \Phi_{tvw,r}(x) = \Phi_{tv,r}(x) \ast w = (\Phi_{t,r}(x) \ast v) \ast w \),

4. \( \Phi_{t,r}(x) \ast 1_R = \Phi_{t,r}(x) \),

where \( v, w \in R \). \( \square \)

For any \( x \) in \( R \) defines the map \( \sigma : R \to M_\Phi(x) \) by \( \sigma(t) = \Phi_{t,r}(x) \). Then \( \sigma(t+s) = \Phi_{t+s,r}(x) = \sigma(t) \circ \sigma(s) \), \( \sigma(st) = \Phi_{st,r}(x) = \sigma(s) \ast t = \sigma(t) \ast s \). As the ring \( R \) is an \( R \)-module, then the map \( \sigma \) becomes an homomorphism of \( R \)-modules.

Let

\[
\Gamma_x := \{ \epsilon_x \Phi(t,y, \Delta f(y)) : t \in R \}
\]  

(33)

denote the orbit or trajectory of \( x \). Then, all sequence in \( R^n \) has to the trajectory \( \Gamma_0 \) of the flow \( \Phi(t, x, \Delta f(x)) \) as exponential generating function la trayectoria. If \( \Gamma_x = \{ x \} \), then \( x \) is an equilibrium point for \( \Phi \). The equilibrium points are obtained when \( A_n([\delta^{n-1} f(x)]) = 0 \) for all \( n \), i.e., when \( f(x) = 0 \) for some \( x \in R \). Thus a solution of equilibrium of \( \delta_t \Phi = f(\Phi) \) correspond with a sequence in the ideal \( R^n_0 \). If \( x \) is not an equilibrium point, then this will be called a regular point of \( \Phi \).

Let

\[
\text{Ker } \sigma = \{ t \in R : \sigma(t) = 1_{M_\Phi} \} = \{ t \in R : \Phi(t, x, \Delta f(x)) = x \}
\]

be the kernel \( \sigma \). If \( \pi \) is an equilibrium point, \( \sigma : R \to \{ \pi \} \) and \( \text{Ker } \sigma = R \). If \( x \) is regular point, \( \text{Ker } \sigma = \{ 0 \} \) and \( \sigma \) is injective. As \( \sigma \) is surjective, then \( \sigma \) is an isomorphisms of \( R \)-modules. We shall say that the module \( M_\Phi(x) \) is a trivial module if \( x \) is an equilibrium point. Otherwise, will be called nontrivial module.

**Theorem 12.** A non-trivial \( R \)-module \( M_\Phi(x) \) is a torsion-free cyclic module.

*Proof.* Suppose \( x \) is a regular point. To show that \( M_\Phi(x) \) is a cyclic module, is enough to take an unit \( u \) in \( R \). Then \( \Phi_{a,r}(x) \ast R = \Phi_{aR,r}(x) = M_\Phi(x) \) and thus \( M_\Phi(x) \) is cyclic. On the one hand, there is an ideal \( I \subset R \) such that \( M_\Phi(x) \) is isomorph to \( R/I \). Yet we are already that \( R \) and \( M_\Phi(x) \) are isomorph. Thus \( I \) is the zero ideal. On the other hand, \( M_\Phi(x) \) is an cyclic \( R \)-module is equivalent to saying that the multiplication homomorphism \( \tau_s : R \to M_\Phi(x) \), \( \tau_s(a) = \Phi_{s,r}(x) \ast a \), is a surjective \( R \)-module homomorphism.

Write \( I = \text{Ker } (\tau_s) \). By the first isomorphism theorem, \( \tau_s \) is an isomorphism from \( R/I \) to \( M_\Phi(x) \). Since \( \text{Ker } (\tau_s) \) is the annihilator \( \text{Ann}(\Phi_{s,r}(x)) \) of \( \Phi_{s,r}(x) \), then \( \text{Ann}(\Phi_{s,r}(x)) = I = 0 \) for any \( \Phi_{s,r}(x) \in M_\Phi(x) \) and 0 is only torsion element in \( M_\Phi(x) \). \( \square \)

### 5.1 The flow \( \Phi \) as an \( R^\infty \)-module

In this section we study the existing relationship between the exponential groups \( H_k \) and the flow \( \Phi \) of a differential equation \( \delta_t \Phi = f(\Phi) \). Actually, we note that each group \( H_k \) defines an action over \( \Phi \) as \( G \)-module. First, we study the \( G \)-module induced by \( H_1 \) and then the \( G \)-modules induced by \( H_k \) for \( k \geq 2 \). We define a right \( R \)-algebra via composition in this way

1. \((\Delta f(x) + \Delta g(x)) \circ (ax)_{n\in\mathbb{N}} = \Delta f(x) \circ (ax)_{n\in\mathbb{N}} + \Delta g(x) \circ (ax)_{n\in\mathbb{N}}\)
2. \(\Delta f(x) \circ (x)_{n\in\mathbb{N}} = \Delta f(x)\)
3. \(\Delta f(x) \circ (abx)_{n\in\mathbb{N}} = (\Delta f(x) \circ (ax)_{n\in\mathbb{N}}) \circ (bx)_{n\in\mathbb{N}}\).
4. \((\Delta f(x) \ast \Delta g(x)) \circ (ax)_{n \in \mathbb{N}} = (\Delta f(x) \circ (ax)_{n \in \mathbb{N}}) \ast (\Delta g(x) \circ (ax)_{n \in \mathbb{N}})\).

The composition \(\circ\) extend to \(\text{Exp}_S[T]\) by to do
\[
\sum_{n=0}^{\infty} \delta^n f(x) \circ (ax) \frac{t^n}{n!} = \left( \sum_{n=0}^{\infty} \delta^n f(x) \frac{t^n}{n!} \right) \circ (ax).
\]

This fact is used to show the existing relationship between the trajectory \(\Gamma_{\text{ax}}\) of the flow associated to \(\Delta f(x)\) and the flow associated to \(\Delta f(ax)\).

**Theorem 13.** Let \(a\) be non-zero in \(R\) and let \(\mathcal{H}_1\) be the group of 1-homogeneity of \(\mathfrak{A}\) and \(g_n = (a^n)_{n \in \mathbb{N}_0} \in \mathcal{H}_1\). Then for all sequence \(\Delta f(x)\) in \(\Delta \text{Exp}_R[X]\) we have
\[
a\Phi(t, x, \Delta f(ax)) = a\Phi(t, x, g_n \Delta f(x) \circ (ax)_{n \in \mathbb{N}_0}) = \Phi(at, x, \Delta f(x) \circ (ax)) \tag{34}
\]
where the composition is in the \(x\) variable.

**Proof.** If \(g_n \in \mathcal{H}_1\), we have that
\[
a\Phi(t, x, \Delta f(ax)) = a\Phi(t, x, g_n \Delta f(x) \circ (ax)_{n \in \mathbb{N}_0}) \nonumber
= ax + a \sum_{n=1}^{\infty} A_n([a^{n-1}\delta^{n-1} f(x) \circ (ax)]) \frac{t^n}{n!} \nonumber
= ax + a \sum_{n=1}^{\infty} a^{n-1} A_n([\delta^{n-1} f(x)]) \circ (ax) \frac{t^n}{n!} \nonumber
= ax + \sum_{n=1}^{\infty} A_n([\delta^{n-1} f(x)]) \circ (ax) \frac{(at)^n}{n!} \nonumber
= \Phi(at, x, \Delta f(x)) \circ (ax).
\]

With the previous result we obtain the first \(G\)-module induced by the homogeneity group \(\mathcal{H}_1\).

**Theorem 14.** Denote \(\mathcal{M}_1(\Phi)\) be the set
\[
\{u\Phi(t, x, \Delta f(u)) : u \in R^x\} \tag{35}
\]
and let \(\circ : \mathcal{M}_1(\Phi) \times R^x \rightarrow \mathcal{M}_1(\Phi)\) be a right group action given by
\[
\Phi(t, x, \Delta f(x)) \circ u = \Phi(t, x, u \Delta f(x)) \circ (ux) \tag{36}
\]
Then \(\mathcal{M}_1(\Phi)\) is a \(R^x\)-module.

**Proof.** From previous theorem, \(\Phi(t, x, u \Delta f(x)) \circ (ux) = u\Phi(t, x, \Delta f(ux))\). Then
1. \(\Phi(t, x, \Delta f(x)) \circ u \in \mathcal{M}_1(\Phi)\),
2. \(\Phi(t, x, \Delta f(x)) \circ 1_{R^x} = \Phi(t, x, \Delta f(x))\),
3. \(\Phi(t, x, \Delta f(x)) \circ uv = (\Phi(t, x, \Delta f(x)) \circ u) \circ v\),
4. \(\Phi(t + s, x, \Delta f(x)) \circ u = (\Phi(t, x, \Delta f(x)) \circ u) \circ (\Phi(s, x, \Delta f(x)) \circ u)\).

**Definition 7.** The \(R^x\)-module \(\mathcal{M}_1(\Phi)\) will be called \(R^x\)-module of homogeneity of order 1.
Equally, we will show the existing relationship between the trajectory $\Gamma_{a^{-1}bx}$ of the flow associated to $\Delta f(x)$ and the flow associated to $a\Delta f(a^{-1}bx)$, where $a$ and $b$ hold $a^{k-1} = 1$ and $b^{k-1} = a$, $k \geq 2$.

**Theorem 15.** Let $\mathcal{H}_k$ be $k$-homogeneity group of $\mathfrak{A}$ with $k > 1$ in $R^N$ and let $h = (a^{-1}nb^n)_{n \in \mathbb{N}_0} \in \mathcal{H}_k$ Then for all sequence $\Delta f(x) = (\delta^n f(x))_{n \in \mathbb{N}_0}$ we have

$$a^{-1}b\Phi(t, x, a\Delta f(a^{-1}bx)) = a^{-1}b\Phi(t, x, h\Delta f(x) \circ (a^{-1}bx)_{n \in \mathbb{N}_0} = \Phi(a^{-1}bt, x, \Delta f(x)) \circ (a^{-1}bx).$$

(37)

**Proof.** Igual que el caso anterior. □

Ahora obtendremos los $G$-module asociados a los grupos $\mathcal{H}_k$, $k \geq 2$.

**Theorem 16.** Denote $\mathcal{M}_k(\Phi)$, $k \geq 2$, be the set

$$\{a^{-1}b\Phi(t, x, a\Delta f(a^{-1}bx)) : a^{k-1} = 1, b^{k-1} = a, a, b \in R^x\}$$

(38)

and let $\bullet : \mathcal{M}_k(\Phi) \times R^x \rightarrow \mathcal{M}_k(\Phi)$ be a right group action given by

$$\Phi(t, x, \Delta f(x)) \bullet (a^{-1}b) = \Phi(t, x, a^{-1}b\Delta f(x)) \circ (a^{-1}bx).$$

(39)

Then $\mathcal{M}_k(\Phi)$ is a $R^x$-module.

**Proof.** From previous theorem, $\Phi(t, x, a^{-1}b\Delta f(x)) \circ (a^{-1}bx) = a^{-1}b\Phi(t, x, a\Delta f(a^{-1}bx))$. Then

1. $\Phi(t, x, \Delta f(x)) \bullet (a^{-1}b) \in \mathcal{M}_k(\Phi)$,
2. $\Phi(t, x, \Delta f(x)) \bullet 1_{R^x} = \Phi(t, x, \Delta f(x))$,
3. $\Phi(t, x, \Delta f(x)) \bullet (a^{-1}bc^{-1}d) = (\Phi(t, x, \Delta f(x)) \bullet (a^{-1}b)) \bullet (c^{-1}d)$,
4. $\Phi(t + s, x, \Delta f(x)) \bullet (a^{-1}b) = (\Phi(t, x, \Delta f(x)) \bullet (a^{-1}b)) \circ (\Phi(s, x, \Delta f(x)) \bullet (a^{-1}b))$.

□

**Definition 8.** The $R^x$-module $\mathcal{M}_k(\Phi)$ will be called $R^x$-module of homogeneity of order $k$, $k \geq 2$.

Using the exponential groups $\mathcal{H}_k$ found in the section 4.2 we have the following theorem.

**Theorem 17.** From the rings $\mathbb{Z}$, $\mathbb{C}$, $\mathbb{Z}[\omega]$ and $\mathbb{Z}[\epsilon]$ we have the following $G$-modules

1. $\mathbb{Z}_2$-module

$$\mathcal{M}_1(\Phi) = \{\pm \Phi(t, x, \Delta f(\pm x))\}.$$

2. (a) $\mathbb{C}^x$-module

$$\mathcal{M}_1(\Phi) = \{c\Phi(t, x, \Delta f(cx)) : c \in \mathbb{C}^x\}.$$

(b) $\mathbb{Z}_{(k-1)^2}$-module

$$\mathcal{M}_k(\Phi) = \{\zeta \Phi(t, x, \Delta f(\zeta x)) : \zeta^{(k-1)^2} = 1\}.$$

3. (a) $\mathbb{Z}_6$-module

$$\mathcal{M}_1(\Phi) = \{\pm \Phi(t, x, \Delta f(\pm x)), \pm \omega \Phi(t, x, \Delta f(\pm \omega x)), \pm \omega^2 \Phi(t, x, \Delta f(\pm \omega^2 x))\}.$$

(b) $\mathbb{Z}_2$-module

$$\mathcal{M}_1(\Phi) = \{\pm \Phi(t, x, \Delta f(\pm x))\}.$$
4. $\mathbb{Z}[e]^\times$-module

$$\mathcal{M}_1(\Phi) = \{ \pm e^m \Phi(t, x, \Delta f(\pm e^m x)) : m \in \mathbb{Z} \}.$$ 

**Theorem 18.** From the gaussian ring $\mathbb{Z}[i]$ all the $G$-modules $\mathcal{M}_k(\Phi)$ are $\mathbb{Z}_4$-modules.

**Proof.** From section 4.2.4, the $G$-modules $\mathcal{M}_1(\Phi)$ and $\mathcal{M}_3(\Phi)$ are $\mathbb{Z}_4$-modules and $\mathbb{Z}_2 \times \mathbb{Z}_2$-module, respectively. From exponential group $H_3$ we have that $a^{-1}b \in \mathbb{Z}_4$. Then $\mathcal{M}_1(\Phi) = \mathcal{M}_3(\Phi)$.

Let $x^*$ be an equilibrium point of $\Phi$. If $G$ is some of the above groups, then each $g^{-1}x^*$ is an equilibrium point of $\Phi(t, x, \Delta f(gx))$ for all $g \in G$. Then, all equilibrium point $x^*$ of $\Phi$ is invariant under the action of the $G$-module.

6 Examples of one-dimensional dynamical systems over integral domain

We finish this paper showing the flos and the $R^\times$-modules for the equations $\delta_t \Phi = a$, $\delta_t \Phi = a+b\Phi$ and $\delta_t \Phi = a \exp(\Phi)$ defined over the rings $\mathbb{Z}$, $\mathbb{C}$, $\mathbb{Z}[i]$, $\mathbb{Z}[\omega]$ and $\mathbb{Z}[\sqrt{2}]$. Also some trajectories for the $G$-modules are showed.

6.1 Equation $\delta_t \Phi = a$

We have $f(x) = a$. Then

$$\Delta f(x) = (a, 0, 0, 0, \ldots)$$

and

$$A(\Delta f(x)) = \Delta f(x) = (a, 0, 0, \ldots).$$

Thus the solution is

$$\Phi(t, x, \Delta f(x)) = x + at.$$  

6.1.1 $R = \mathbb{Z}$

$\mathbb{Z}_2$-module

$$\mathcal{M}_1(\Phi) = \{ \pm (x + at) \}.$$ 

Figure 1: $\mathbb{Z}_2$-module para $\delta_t \Phi = 2$
6.1.2 \( R = \mathbb{C} \)
1. \( \mathbb{C}^\times \)-module
   \[ \mathcal{M}_1(\Phi) = \{ c(x + at) : c \in \mathbb{C}^\times \} \]
2. \( \mathbb{Z}_{(k-1)^2} \)-module
   \[ \mathcal{M}_k(\Phi) = \{ \zeta(x + at) : \zeta^{(k-1)^2} = 1 \} \].

6.1.3 \( R = \mathbb{Z}[i] \)
\( \mathbb{Z}_4 \)-module
   \[ \mathcal{M}_1(\Phi) = \{ \pm(x + at), \pm i(x - at) \} \].

Figure 2: \( \mathbb{Z}_4 \)-module for \( \delta_t \Phi = 2 \) and orbit \( \Gamma_2 \)

6.1.4 \( R = \mathbb{Z}[\omega] \)
1. \( \mathbb{Z}_6 \)-module
   \[ \mathcal{M}_1(\Phi) = \{ \pm(x + at), \pm \omega(x + at), \pm \omega^2(x + at) \} \].
2. \( \mathbb{Z}_2 \)-module
   \[ \mathcal{M}_3(\Phi) = \{ \pm(x + at) \} \].
3. \( \mathbb{Z}_3 \)-module
   \[ \mathcal{M}_4(\Phi) = \{ x + at, \omega(x + at), \omega^2(x + at) \} \].

Figure 3: \( \mathbb{Z}_3 \)-module for \( \delta_t \Phi = 2 \) and orbit \( \Gamma_1 \)

6.1.5 \( R = \mathbb{Z}[\sqrt{2}] \)
\( \mathbb{Z}[\sqrt{2}]^\times \)-module
   \[ \mathcal{M}_1(\Phi) = \{ \pm(1 + \sqrt{2})^m(x + at) : m \in \mathbb{Z} \} \].

Figure 4: \( \mathbb{Z}[\sqrt{2}]^\times \)-module for \( \delta_t \Phi = 2 \) and orbit \( \Gamma_2 \)
6.2 Equation $\delta_t \Phi = a + b\Phi$

We have $f(x) = a + bx$. Then

$$\Delta f(x) = (a + bx, b, 0, 0, \ldots) \quad (40)$$

and

$$\mathcal{A}(\Delta f(x)) = ([a + bx]^n)_{n\in\mathbb{N}_0}. \quad (41)$$

Thus the solution is

$$\Phi(t, x, \Delta f(x)) = x + \left(\frac{a}{b} + x\right) (\exp(bt) - 1). \quad (42)$$

When $a \equiv 0$, then (42) is reduced to

$$\Phi(t, x, \Delta f(x)) = x \exp(bt) \quad (43)$$

which is a solution to the knowed linear equation $\delta_t \Phi(t, x) = b\Phi$.

6.2.1 $R = \mathbb{Z}$

$\mathbb{Z}_2$-module

$$\mathcal{M}_1(\Phi) = \left\{ \pm x + \left(\frac{a}{b} \pm x\right) (\exp(\pm bt) - 1) \right\}. \quad (44)$$

Figure 5: $\mathbb{Z}_2$-module para $\delta_t \Phi = 1 + \Phi$

6.2.2 $R = \mathbb{C}$

1. $\mathbb{C}^\times$-module

$$\mathcal{M}_1(\Phi) = \left\{ cx + \left(\frac{a}{b} + cx\right) (\exp(bct) - 1) : c \in \mathbb{C}^\times \right\}. \quad (45)$$

2. $\mathbb{Z}_{(k-1)^2}$-module

$$\mathcal{M}_k(\Phi) = \left\{ \zeta x + \left(\frac{a}{b} + \zeta x\right) (\exp(b\zeta t) - 1) : \zeta^{(k-1)^2} = 1 \right\}. \quad (46)$$

6.2.3 $R = \mathbb{Z}[i]$

$\mathbb{Z}_4$-module

$$\mathcal{M}_1(\Phi) = \left\{ \pm x + \left(\frac{a}{b} \pm x\right) (\exp(\pm bt) - 1), \pm ix + \left(\frac{a}{b} \pm ix\right) (\exp(\pm ibt) - 1) \right\}. \quad (47)$$

Figure 6: $\mathbb{Z}_4$-module for $\delta_t \Phi = 1 + \Phi$ and orbit $\Gamma_2$
6.2.4 \( R = \mathbb{Z}[\omega] \)

1. \( \mathbb{Z}_6 \)-module
   \[
   \mathcal{M}_1(\Phi) = \left\{ \pm x + \left( \frac{a}{b} \pm x \right) (\exp(\pm bt) - 1), \pm \omega x + \left( \frac{a}{b} \pm \omega x \right) (\exp(\pm \omega bt) - 1), \right. \\
   \left. \pm \omega^2 x + \left( \frac{a}{b} \pm \omega^2 x \right) (\exp(\pm \omega^2 bt) - 1) \right\}.
   \]

2. \( \mathbb{Z}_2 \)-module
   \[
   \mathcal{M}_3(\Phi) = \left\{ \pm x + \left( \frac{a}{b} \pm x \right) (\exp(\pm bt) - 1) \right\}.
   \]

3. \( \mathbb{Z}_3 \)-module
   \[
   \mathcal{M}_4(\Phi) = \left\{ x + \left( \frac{a}{b} x \right) (\exp(b t) - 1), \omega x + \left( \frac{a}{b} \omega x \right) (\exp(\omega bt) - 1), \right. \\
   \left. \omega^2 x + \left( \frac{a}{b} \omega^2 x \right) (\exp(\omega^2 bt) - 1) \right\}.
   \]

Figure 7: \( \mathbb{Z}_3 \)-module for \( \delta \Phi = 1 + \Phi \) and orbit \( \Gamma_2 \)

6.2.5 \( R = \mathbb{Z}[\sqrt{2}] \)

\( \mathbb{Z}[\sqrt{2}] \)-module

\[
\mathcal{M}_1(\Phi) = \left\{ \pm (1 + \sqrt{2})^m x + \left( \frac{a}{b} \pm (1 + \sqrt{2})^m x \right) \left( \exp(\pm (1 + \sqrt{2})^m bt) - 1 \right) : m \in \mathbb{Z} \right\}.
\]

Figure 8: \( \mathbb{Z}[\sqrt{2}] \times \)-module for \( \delta \Phi = 1 + \Phi \) and orbit \( \Gamma_2 \)

6.3 Equation \( \delta \Phi = a \exp(\Phi) \)

In this case \( f(x) = a \exp(x) \) and \( \Delta f = (a \exp(x))_{n \in \mathbb{N}_0} \). Then

\[
\mathcal{A}(\Delta f) = (a^n \exp(nx))_{n \in \mathbb{N}}((n - 1)!)_{n \in \mathbb{N}}.
\]

Thus the solution is

\[
\Phi(t, x, \Delta F) = x - \ln (1 - a \exp(x)t)
\]
6.3.1 \( R = \mathbb{Z} \)

\( \mathbb{Z}_2 \)-module

\[ \mathcal{M}_1(\Phi) = \{ \pm x - \ln (1 \mp a \exp(\pm x)t) \}. \]

Figure 9: \( \mathbb{Z}_2 \)-module for \( \delta_t \Phi = \exp(\Phi) \) and orbit \( \Gamma_2 \)

6.3.2 \( R = \mathbb{C} \)

1. \( \mathbb{C}^\times \)-module

\[ \mathcal{M}_1(\Phi) = \{ cx - \ln (1 - ca \exp(cx)t) : c \in \mathbb{C}^\times \}. \]

2. \( \mathbb{Z}_{(k-1)^2} \)-module

\[ \mathcal{M}_k(\Phi) = \{ \zeta x - \ln (1 - a\zeta \exp(\zeta x)t) : \zeta^{(k-1)^2} = 1 \}. \]

6.3.3 \( R = \mathbb{Z}[i] \)

\( \mathbb{Z}_4 \)-module

\[ \mathcal{M}_1(\Phi) = \{ \pm x - \ln (1 \mp a \exp(\pm x)t), \pm ix - \ln (1 \mp ia \exp(\pm ix)t) \}. \]

Figure 10: \( \mathbb{Z}_4 \)-module for \( \delta_t \Phi = \exp(\Phi) \) and orbit \( \Gamma_2 \)

6.3.4 \( R = \mathbb{Z}[\omega] \)

1. \( \mathbb{Z}_6 \)-module

\[ \mathcal{M}_1(\Phi) = \{ \pm x - \ln (1 \mp a \exp(\pm x)t), \pm \omega x - \ln (1 \mp \omega a \exp(\pm \omega x)t), \pm \omega^2 x - \ln (1 \mp \omega^2 a \exp(\pm \omega^2 x)t) \}. \]

2. \( \mathbb{Z}_2 \)-module

\[ \mathcal{M}_3(\Phi) = \{ \pm x - \ln (1 \mp a \exp(\pm x)t) \}. \]

3. \( \mathbb{Z}_3 \)-module

\[ \mathcal{M}_4(\Phi) = \{ x - \ln (1 - a \exp(x)t), \omega x - \ln (1 - \omega a \exp(\omega x)t), \omega^2 x - \ln (1 - \omega^2 a \exp(\omega^2 x)t) \}. \]
6.3.5 $R = \mathbb{Z}[\sqrt{2}]$

$\mathbb{Z}[\sqrt{2}]$-module

$$\mathfrak{M}_1(\Phi) = \left\{ \pm (1 + \sqrt{2})^m x - \ln \left( 1 \mp (1 + \sqrt{2})^m \exp(\pm(1 + \sqrt{2})^m x) \right) : m \in \mathbb{Z} \right\}.$$  

7 Conclusions and Further research

In this paper the operator $\mathfrak{A}$ and its properties were studied. A goal was to link sequences in an integral domain with the flow of a dynamical system. Another result of great importance was to show that $\mathfrak{A}$ has an inverse. Thus to all sequence with first element in the group of units of the domain is obtained via the action of $\mathfrak{A}$. Other very important result was to show that the flow of an onedimensional dynamical system over an integral domain is a torsion-free cyclic module. This will be very useful as a part of the construction of a qualitative theory for this system. Also we construct an structure of $G$-module for the flows and were classified for some rings.

Some potential issues for further research are:

1. To find a closed formula for $\mathfrak{A}(\Delta f(X) * \Delta g(X))$.

2. Is $\mathfrak{A}(\Delta \text{Exp}_R[X])$ a ring? or a module or an algebra?

3. To find a closed formula for the inverse of the operator $\mathfrak{A}$ and to study its properties.

4. To classify equilibrium point for dynamical system over integral domains. Also to study other concepts of the qualitative theory of dynamical systems.

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