On Mixed Iterated Revisions

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Abstract

Several forms of iterable belief change exist, differing in the kind of change and its strength: some operators introduce formulae, others remove them; some add formulae unconditionally, others only as additions to the previous beliefs; some only relative to the current situation, others in all possible cases. A sequence of changes may involve several of them: for example, the first step is a revision, the second a contraction and the third a refinement of the previous beliefs. The ten operators considered in this article are shown to be all reducible to three: lexicographic revision, refinement and severe withdrawal. In turn, these three can be expressed in terms of lexicographic revision at the cost of restructuring the sequence. This restructuring needs not to be done explicitly: an algorithm that works on the original sequence is shown. The complexity of mixed sequences of belief change operators is also analyzed. Most of them require only a polynomial number of calls to a satisfiability checker, some are even easier.

1 Introduction

New information come in different forms. At the one end of the spectrum, it is believed to be always true (lexicographic revision [68, 52]); at the far opposite, it is known to be false (contraction [1, 65]). Middle cases exist: it may be believed only as long as the current situation is concerned (natural revision [10, 41]), or it may be believed only as long as it does not contradict the previous beliefs (refinement [55, 9]). Sequence of changes are hardly all of the same form, like someone who embraces every single new theory with all of his hearth or instead so skeptical to refuse every piece of information that contradicts what known.

Example 1 An example against natural revision [10, 41] is that of the red bird [38]: an animal looks like a bird (b) and upon coming close turned out to be red (r); finding out not to be a bird (¬b) makes natural revision discard it being red in spite of no evidence of the contrary.

This is a situation where natural revision is not to be used. Yet, a small variant of the conditions turns the very same formulae, with the very same meaning of variables, into a case for it.

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A hunter meets a peasant friend in the countryside, who told him having seen a strange bird in the thicket a couple of miles away (b). Lured by the unique trophy he could make out of it, and by bragging about its hunting at the village fête that evening, the hunter rushes to there.

Midway, he encounters the village postman. In a hurry, the hunter explains he could not stop and why. The postman answers he understands, as he actually saw something red in the thicket (r). This would explain why the peasant friend told the bird was strange, because no red bird has ever been seen around there.

Arrived at the thicket, the hunter checks for anything red but the bushes and trees are too thick to see anything inside. Entering is out of discussion, as any bird would fly away upon hearing the noise. Keeping ready with his rifle, the hunter throws a stone in the thicket, but nothing happens. A second and third stone confirm that no bird is there. The peasant friend and the postman made fun of him by having him run with no reason. No bird is there (¬b), nothing indicates something red (r is no longer believed).

While the sequence of formulae is exactly the same (first b, then r, finally ¬b), with the same meaning of the variables (b=bird, r=red), natural revision gives exactly the expected result: since the first piece of information was made up, there is no reason to believe the second.

Should hunters always use natural revisions?

Of course not.

Seeing the hunter throwing stones for no clear reason, the village policeman approaches the hunter to ask why. Concealing how he was made a fool is useless, as the peasant friend and the postman will tell the story to everyone at the village fête that evening; everyone will know even before the parade. The policeman laughs, but to the hunter’s surprise it’s not about the practical joke. A red animal would be unique in the area. Even if one was there, and the hunter managed to shot it, he would have been fined and confiscated the trophy. He could not have shown it in his living room, and certainly not brag about it at the village fête. If it is red, it cannot be hunted (r → ¬h).

As it comes from a police officer while on duty, this information is totally reliable. It is also a general rule, not specific to the current situation. No matter if it was a bird (b or ¬b), if it is red it cannot be hunted (r → ¬h). The situation where a red bird that can be hunted was there (b, r and h) is less likely than one where it can (b, r, ¬h), and the same if it is red but not a bird (¬b, r and h is less likely than ¬b, r and ¬h). That hunting it is forbidden is more likely in every situation, even if a red animal that is not a bird later turn out to be there.

The hunter is better revising lexicographically (by ¬h) this time if he wants to avoid being fined, after previously having revised naturally (by ¬b). A mixed sequence of revisions is the best course of actions.

A related question is why to apply a sequence of revisions in the first place. Why not just coming back to where the hunter started, with no information about the thicket, rather than revising by b, then r and then by ¬b? Once the bird is nowhere to be found, everything could be just canceled. The epistemic state is the same as the beginning.

While talking with the policeman, the hunter hears a sound of feathers from the thicket. Feathers mean bird (b). Maybe one that does not fly, or is wounded and unable to fly. The peasant and the postman might have been truthful, after all. Nothing
indicates they joked any longer. None of them, including the postman. The bird might be red \( r \) after all. This would not be the case if the hunter just forgot everything was told.

Natural revision, then lexicographic revision. Not two lexicographic revisions, not two natural revisions. A mixed sequence of revisions is what to do in this example. As Booth and Meyer [7] put it: "it is clear that an agent need not, and in most cases, ought not to stick to the same revision operator every time that it has to perform a revision.”

Other forms of revisions exist. They may mix in any order. A lexicographic revision may be followed by a natural revision, a restrained revision and a radical revision. Such a sequence is used as a running example:

\[
\emptyset \text{lex}(y) \text{nat}(\neg x) \text{res}(x \land z) \text{rad}(\neg z)
\]

The sequence begins with \( \emptyset \), the complete lack of knowledge. The first information acquired is \( y \), and is considered valid in all possible situations; it is a lexicographic revision. The next is \( \neg x \), but is deemed valid only in the current conditions; this is a natural revision. The next is \( x \land z \), but is only accepted as long as it does not contradict what is currently known; it is a refinement. Finally, \( \neg z \) is so firmly believed that all cases where it does not hold are excluded; this is a radical revision.

No semantics is better than the others. Natural revision has its place. As well as lexicographic revision. As well as refinement, radical revision, severe antiremovals and severe revision. None of them is the best. Each is right in the appropriate conditions and wrong in the others. For example, a sequence of severe revisions is problematic because it coarsens the plausibility degree of different scenarios [25, 59]; yet, it is a common experience that sometimes new information makes unlikely conditions as likely as others. A truck full of parrots to be sold as pets crushed nearby, freeing thousands of exotic animals; a red bird is now as likely to be stumbled upon as a local wild animal. New information may raise the plausibility of a situation to the level of another, making them equally likely. This is a consequence of the new information, not a fault of the revision operator. The problem comes when using severe revisions only, without other revisions that separate the likeness of different conditions [25, 59].

The solution is not to search for a new framework that encompasses all possible cases, but to deal with mixes of different revision kinds. How to decide which type of revision to use at each time depends on how the information has been obtained and on the information itself. This is a separate problem [8, 48], not considered here.

The problem considered here is to determine the outcome of a sequence of mixed revisions. Semantically, each modifies the order of plausibility of the models in a different way. For example, natural revision “promotes” some models to the maximal plausibility; lexicographic revision makes some models being more plausible than others. Keeping in memory a complete description of these orderings is unfeasible, even in the propositional case: the number of models is exponential in the number of variables. Several operators such as lexicographic revision, refinement and restrained revision can generate orderings that compare equal no two models, making exponential every representation that is linear in the number of equivalence classes of the ordering, such as the ordered sequence of formulae by Rott [60] and the partitions by Meyer, Ghose and Chopra [51]. Many
distance-based one-step revisions suffer from a similar problem: the result of even a single revision may be exponential in the size of the involved formulae [12]. Iterated revisions typically do not employ distances, and the problem can be overcome:

- the ten belief change operators considered in this article (lexicographic revision, refinement, severe withdrawal, natural revision, restrained revision, plain severe revision, severe revision, moderate severe revision, very radical revision) can be reduced to three: lexicographic revision, refinement and severe withdrawal; these reductions are local: they replace an operator without changing the rest of the sequence before and after it;

- refinement and severe withdrawal can be reduced to lexicographic revision; this however requires structuring the sequence of belief change operators; however, the result is a sequence that behave like the original on subsequent changes;

- this restructuring needs not to be done explicitly; an algorithm that works on the original sequence is shown; it does not change the sequence, but behaves as if it were restructured; apart from the calls to a satisfiability checker, the running time is polynomial.

This mechanism determines the result of an arbitrary sequence of revisions from an initial ordering that reflects a total lack of information. This is not a limitation, as an arbitrary ordering can be created by a sequence of lexicographic revisions [8].

During its execution, the algorithm calculates some partial results called unformulae, which can be used when the sequence is extended by further revisions. The need for a satisfiability checker is unavoidable, given that belief change operates on propositional formulae. However, efficient solvers have been developed [4, 2]. Restricting to a less expressive language [14] such as Horn logic may also reduce the complexity of the problem, as it is generally the case for one-step revisions [19, 54, 50, 45], since satisfiability in this case can be solved efficiently.

Some complexity results are proved: some imply the ones announced without proofs in a previous article [47], but extend them to the case of mixed sequences of revisions. Entailment from a sequence of lexicographic, natural, restrained, very radical and severe revisions, refinements and severe antiwithdrawals is in the complexity class \( \Delta^p_2 \), and is \( \Delta^p_2 \)-hard even if the sequence contains only lexicographic revisions and refinements. Two groups of belief change operators are relevant to complexity. The first is called lexicographic-finding and comprises the ones that behave like lexicographic revision on consistent sequences of formulae; lexicographic and moderate severe revisions are in this group. The second is called bottom-refining as it includes the revisions that separate the most likely scenarios when some are consistent with the new information; natural revision, restrained revision and severe revision are in this group. Entailment from a sequence of operators all of the first kind or all of the second is \( \Delta^p_2 \)-complete. Three revision operators require a separate analysis. Entailment from a sequence of very radical revision is \( \Delta^p_2[\log n] \)-complete. The same complexity comes from sequences of plain severe and full meet revisions only.

The rest of this article is organized as follows: Section 2 introduces the main concepts of total preorder, lexicographic revision, refinement and severe withdrawal; Section 3...
shows how to reduce the other change operators to these three; Section 4 shows an algorithm for computing the result of a sequence of revisions; Section 5 presents the computational complexity results; Section 6 discusses the results in this article, compares them with other work in the literature and presents the open problems.

2 Preliminaries

A propositional language over a finite alphabet is assumed. Given a formula \( F \), its set of models is denoted \( \text{Mod}(F) \), while a formula having \( S \) as its set of models is denoted \( \text{Form}(S) \). The symbol \( \top \) denotes a tautology, a formula satisfied by all models over the given alphabet.

A base is a propositional formula denoting what is believed in certain moment. Historically, revision was defined as an operator that modifies a base in front of new information; an ordering was employed to take choices when this integration may be done in multiple ways, which is usually the case. Assuming this ordering as fixed or depending on the base only is the AGM model or revision \([1, 28]\). Iterated revision is problematic using this approach; the solution is to reverse the role of the base and the ordering. Instead of being a supporting element, the ordering becomes the protagonist. The base derives from it as the set of most plausible formulae \([16, 43]\). Such plausibility information can be formalized in several equivalent ways: epistemic entrenchments \([29, 23]\), systems of spheres \([32, 30]\), rankings \([71, 69, 40]\), and KM preorders \([39, 56]\).

2.1 Total preorders

Katsuno and Mendelzon \([39]\) proved that AGM revision can be reformulated in terms of a total preorder over the set of models, where the models of the base are exactly the minimal ones according to the ordering. Iterated revision can be defined by demoting the base from primary information to derived one. Instead of revising a base using the ordering as a guide, the ordering itself is modified. The base is taken to be just the set of formulae implied by all most plausible models.

**Definition 1** A total preorder \( C \) is a partition of the models into a finite sequence of classes \([C(0), C(1), C(2), \ldots, C(m)]\).

Such an ordering can be depicted as a stack, the top boxes containing the most plausible models. This is equivalent to a reflexive, transitive and total relation, but makes for simpler definitions and proofs about iterated revisions.
A KM total preorder is the same as a partition by Mayer, Ghose and Chopra [51], who use a formula for each class in place of its set of models. In turns, such a partition is similar to the system for expressing such orderings in possibilistic logic [3], and correspond to a sequence of formulae by Rott [60] \( \mathcal{h}_1 \prec \cdots \prec \mathcal{h}_m \) via \( C(0) \cup \cdots \cup C(i) = \text{Mod}(\mathcal{h}_{i+1} \land \cdots \land \mathcal{h}_m) \) and to an epistemic entrenchment [1].

Being a partition, \( \mathcal{C} = [\mathcal{C}(0), \ldots, \mathcal{C}(m)] \) contains all models. As a result, every model is in a class. No model is “inaccessible”, or excluded from consideration when performing revision. Revisions producing such models could still be formalized by giving a special status to the last class \( \mathcal{C}(m) \), as the set of such inaccessible models, but they are not studied in this article. Their analysis is left as an open problem.

Classes are allowed to be empty, even class zero \( \mathcal{C}(0) \). The base represented by a total preorder \( \mathcal{C} = [\mathcal{C}(0), \ldots, \mathcal{C}(m)] \) cannot therefore being defined as \( \text{Form}(\mathcal{C}(0)) \) but as the minimal models according to \( \mathcal{C} \), denoted by \( \text{Mod}(\mathcal{C}, \top) \).

More generally, given a formula \( P \) the notation \( \text{min}(\mathcal{C}, P) \) indicates the set of minimal models of \( P \) according to the ordering \( \mathcal{C} \). Formally, if \( i \) is the lowest index such that \( \mathcal{C}(i) \cap \text{Mod}(P) \) is not empty, then \( \text{min}(\mathcal{C}, P) = \mathcal{C}(i) \cap \text{Mod}(P) \). Several iterated revision depends on such an index \( i \) and its corresponding set of models \( \mathcal{C}(i) \cap \text{Mod}(P) \).

Another consequence of allowing empty classes is that two total preorder may be different yet comparing models in the same way. For example, \([\text{Mod}(T)]\) and \( [\emptyset, \text{Mod}(T)]\) both place all models in the same class, which is class zero for the former and class one for the latter. They are in this sense equivalent. They coincide when removing the empty classes. The minimal models of every formula are the same [55].

**Definition 2** Two total preorder \( \mathcal{C} \) and \( \mathcal{C}' \) are equivalent if \( \text{min}(\mathcal{C}, P) = \text{min}(\mathcal{C}, P') \) for every formula \( P \).

Revising by the same formula modifies equivalent orderings into equivalent orderings. This holds for all revision semantics considered in this article.

The amount of information an ordering carries can be informally identified with its ability of telling the relative plausibility of two models. Ideally, an ordering should have a single minimal model, representing what is believed to be the state of the world, and a single model in each class, allowing to unambiguously decide which among two possible states of the world is the most likely. Most revision indeed refine the ordering by splitting its classes. At the other end of the spectrum, the total order \( \emptyset = [\text{Mod}(T)] \) carries no information: not only its base comprises all models and is therefore tautological, but all models are also considered equally plausible. Studies on the practical use of revision [47] assume an initial empty ordering that is then revised to obtain a more discriminating one. Equivalently, an ordering can be expressed as a suitable sequence of revisions applied to the empty total preorder.

Not all operators considered in this article are revisions, only the ones that produce an ordering whose base implies the revising formula. Some other operators just split classes (like refinement) or merge them (like severe withdrawal). The result of an operator ope modifying a total preorder \( C \) by a formula \( P \) is defined by the infix notation \( \text{Cope}(P) \). This is a new total preorder whose base entails \( P \) if ope is a revision operator. More specifically, AGM revisions produce a base out of the minimal models of \( P \) in \( C \):

\[
\text{min}(\text{Cope}(P), \top) = \text{min}(C, P)
\]
2.2 Iterated revisions

Several iterated belief revision operators are considered. These can be all expressed in terms of three of them: lexicographic revision, refinement, and severe withdrawal. Intuitively, this is because each of these three includes a basic operation that can be performed over an ordering: moving, splitting and merging classes. The correspondence is not exact, as the lexicographic revision perform both moving and splitting, but can be made to move a single class from a position of the sequence to another.

These three operators are defined in this section. The others will be then introduced in the next, and immediately proved to be reducible to these three. This allows to concentrate on the computational aspects only on the three basic ones.

2.2.1 Lexicographic revision

Lexicographic revision is one of the two earliest iterated belief revision operator [68]. While its authors initially rejected it, later research have reconsidered it [52, 47, 53]. The tenant of this operator can be summarized as: revising by \( P \) means that \( P \) is true no matter of everything else. Technically, all models satisfying \( P \) are more plausible that every other one.

**Definition 3** The lexicographic revision of a total preorder \( C \) by a formula \( P \) is defined as the following total preorder, where \( i \) and \( j \) are respectively the indexes of the minimal and maximal classes of \( C \) containing models of \( P \):

\[
C_{\text{lex}}(P)(k) = \begin{cases} 
C(k + i) \cap \text{Mod}(P) & \text{if } k \leq j - i \\
C(k - j + i - 1) \setminus \text{Mod}(P) & \text{otherwise}
\end{cases}
\]

Alternatively, a formula directly based on sequences can be taken as the definition of lexicographic revision:

\[
[C(0), \ldots, C(m)]_{\text{lex}}(P) = [C(0) \cap \text{Mod}(P), \ldots, C(m) \cap \text{Mod}(P), C(0) \setminus \text{Mod}(P), \ldots, C(m) \setminus \text{Mod}(P)]
\]

This definition does not exactly coincide with the previous one because of some empty classes, which means that the two produce equivalent total preorders. A graphical representation of revising a total preorder by a formula \( P \) is the following one:
In words, the models of $P$ are “cut out” from the ordering and shifted together to the top. Their relative ordering is not changed, but they are made more plausible than every model of $\neg P$. By construction, $\min(C\lex(P), \top)$ is equal to $\min(C, P)$, making this operator a revision.

2.2.2 Refinement

Contrary to lexicographic revision, refinement [55, 9] is not a revision. It is still a basic form of belief revision in which belief in a formula $P$ is strengthened, but never so much to contradict previous information. Technically, the models of every class are split depending on whether they satisfy $P$ or not. This way, two models are separated only if they were previously considered equally plausible, and only if one satisfies $P$ and the other does not.

**Definition 4** The refinement of a total preorder $C$ by formula $P$ is the following total preorder:

$$C_{ref}(P)(k) = \begin{cases} 
C(0) \cap \text{Mod}(P) & \text{if } k = 0 \text{ and } C(0) \cap \text{Mod}(P) \neq \emptyset \\
C(0) \setminus \text{Mod}(P) & \text{if } k = 0 \text{ and } C(0) \cap \text{Mod}(P) = \emptyset \\
C(k/2) \cap \text{Mod}(P) & \text{if } k > 0 \text{ even} \\
C(k/2) \setminus C_{ref}(P)(k - 1)) & \text{if } k > 0 \text{ odd}
\end{cases}$$

Alternatively, refinement can be defined directly on partitions:

$$[C(0), \ldots, C(m)]_{ref}(P) = 
[C(0) \cap \text{Mod}(P), C(0) \setminus \text{Mod}(P), \ldots, C(m) \cap \text{Mod}(P), C(m) \setminus \text{Mod}(P)]$$

Some of these classes may be empty, and can therefore be removed respecting preorder equivalence. Graphically, refining a total preorder $C$ by a formula $P$ can be seen as follows:
2.2.3 Severe antiwithdrawal

While this operator was defined \cite{62,24} as a form of contraction, it is technically cleaner to use it in reverse, with the negated formula. Removing a formula $\neg P$ is the same of creating the consistency with $P$, but the second definition has been advocated has a most direct formalization of the actual process of belief change \cite{31}.

In the specific case of severe antiwithdrawal, creating consistency with $P$ is obtained by merging all classes of the ordering that are in the same class or one of lower index with the minimal models of $P$. This is motivated by the principle of equal-treated-equally when applied to the plausibility of models: in order to make $P$ consistent some models of $P$ have to become minimal; but the models of $\neg P$ that are in lower classes have the same plausibility or greater, so they should not be excluded.

**Definition 5** The severe anticontraction of the total preorder $C$ by formula $P$ is the following total preorder, where $i$ is the minimal index such that $C(i) \cap \text{Mod}(P) \neq \emptyset$:

$$
\begin{align*}
C_{\text{sev}}(P)(k) &= \begin{cases} 
C(0) \cup \cdots \cup C(i) & \text{if } k = 0 \\
C(k + i) & \text{if } k > 0
\end{cases}
\end{align*}
$$

Lexicographic antiwithdrawal can also be defined in terms of sequences. If $i$ is the lowest index such that $C(i) \cap \text{Mod}(P) \neq \emptyset$, then:

$$
[C(0), \ldots, C(i), C(i + 1), \ldots, C(m)]_{\text{sev}}(P) = [C(0) \cup \cdots \cup C(i), C(i + 1), \ldots, C(m)]
$$

Graphically, severe antiwithdrawal merges all classes of index lower or equal to the minimal class intersecting $\text{Mod}(P)$:
This way, the base of the revised preorder $\text{min}(\text{Csev}(P), \top)$ is guaranteed to contain some models of $P$, which means that it has been made consistent with $P$. At the same time, the relative plausibility of two models is never reversed: a model that is more plausible than another according to $C$ is never made less plausible than that according to $\text{Csev}(P)$.

3 Reductions

Many belief change operators exist. Many of them are expressible in terms of the three presented in the previous section: lexicographic revision, refinement and severe anti-withdrawal. The reductions do not affect what is before or after then replaced operator applications, which is not the case for the transformations shown in the next section.

Example 2 The following sequence of revisions is used as a running example. The following sections show how to make it into a sequence that only contains lex, ref and sev.

$\emptyset \text{lex}(y) \text{nat}(\neg x) \text{res}(x \land z) \text{rad}(\neg z)$

3.1 Natural revision

This revision was first considered and discarded by Spohn [68], and later independently reintroduced by Boutilier [10]. Among revisions, it can be considered at further opposite to lexicographic revision, in that a formula $P$ is made true by a minimal change to the ordering. This amounts to making the minimal models of $P$ the new class zero of the ordering, and changing nothing else.

Formally, if $\text{min}(C, P) = C(i) \cap \text{Mod}(P)$ then:

$[C(0), \ldots, C(m)]\text{nat}(P) = \\
[C(i) \cap \text{Mod}(P), C(0), \ldots C(i - 1), C(i) \backslash \text{Mod}(P), C(i + 1), \ldots, C(m)]$

Graphically, $\text{min}(C, P)$ is “cut out” from the total preorder $C$ and moved to the beginning of the sequence, making it the new class zero $\text{Cnat}(P)(0)$. Since by definition $\text{min}(C, P)$ is not empty, it holds $\text{min}(\text{Cnat}(P), \top) = \text{Cnat}(P)(0) = \text{min}(C, P)$, meaning that it is an AGM revision operator.
Theorem 1 For every total preorder $C$ and formula $P$, it holds $C_{nat}(P) \equiv C_{lex}(K)$ where $K = Form(\min(C, P))$.

Proof. Let $K = Form(\min(C, P))$ and $i$ the index such that $\min(C, P) = C(i) \cap Mod(P)$. By the properties of set difference, it holds $C(i) \setminus Mod(P) = C(i) \setminus (C(i) \cap Mod(P)) = C(i) \setminus Mod(K)$. Since classes do not share variable and $Mod(K) \subseteq C(i)$, it holds $C(j) = C(j) \setminus Mod(K)$ for every $j \neq i$. Natural revision can therefore be recast as:

\[
[C(0), \ldots, C(m)]_{nat}(P) = \\
\equiv [C(i) \cap Mod(M), C(0), \ldots C(i - 1), C(i) \setminus Mod(P), C(i + 1), \ldots, C(m)] \\
\equiv [Mod(K), C(0) \setminus Mod(K), \ldots, C(i - 1) \setminus Mod(K), C(i) \setminus Mod(K), C(i + 1) \setminus Mod(K), \ldots, C(m) \setminus Mod(K)] \\
\equiv [Mod(K), \emptyset, \ldots, \emptyset, C(0) \setminus Mod(K), \ldots, C(m) \setminus Mod(K)] \\
\equiv [C(0) \cap Mod(K), C(1) \cap Mod(K), \ldots C(m) \cap Mod(K), C(0) \setminus Mod(K), \ldots, C(m) \setminus Mod(K)]
\]

The equivalences are correct because: first, $C(i) \cap Mod(K) = Mod(K)$ since $Mod(K) \subseteq C(i)$; second, empty classes can be introduced at every point of every ordering, and $C(j) \cap Mod(K) = \emptyset$ for every $j \neq i$. The resulting total preorder is $C_{lex}(K)$. □

This transformation does not just tell how to compute the propositional result of natural revision, that is, the base $Form(C_{nat}(P)(0))$ of the revised ordering. To the contrary, it requires it, as $Mod(K) = \min(C, P) = C_{nat}(P)(0)$. After $K$ has been calculated, $lex(K)$ produces the same exact preorder as $nat(P)$ when applied to the same preorder, not just two preorders having the same base. This means that all subsequent revisions are unaffected by the replacement. In other words, for every initial preorder $C$ and every sequence of previous and future belief changes, it holds:

\[
C_{ope_1}(P_1) \ldots \text{ope}_{n-1}(P_{n-1})\text{nat}(P)\text{ope}_{n+1}(P_{n+1}) \ldots \text{ope}_{n'}(P_{n'}) \equiv \\
C_{ope_1}(P_1) \ldots \text{ope}_{n-1}(P_{n-1})\text{lex}(K)\text{ope}_{n+1}(P_{n+1}) \ldots \text{ope}_{n'}(P_{n'})
\]

The other reductions in this section have all this property, that an operator application in whichever position of a sequence can be replaced without affecting the final ordering. Natural revision requires the minimal models of $P$ to be calculated, some other operators do not. Natural revision is replaced by a single lexicographic revision, the others may require some lexicographic revisions, refinements and severe antithrowdraws.

Example 3 The second operation in the running example is a natural revision.

\[
\emptyset\text{lex}(y)\text{nat}(\neg x)\text{res}(x \land z)\text{rad}(\neg z)
\]
Since $\emptyset \text{lex}(y)$ is $[\text{Mod}(y), \text{Mod}(\neg y)]$, and class zero of this ordering contains models of $\neg x$, then $\min(\emptyset, \neg x) = \text{Mod}(\neg x \land y)$. As a result, the sequence can be simplified into:

$$\emptyset \text{lex}(y) \text{lex}(\neg x \land y) \text{res}(x \land z) \text{rad}(\neg z)$$

Some other operators are reduced to natural revision, which can in turn be reduced to lexicographic revision. For example, restrained revision is a refinement followed by natural revision (or vice versa). The above theorem shows that it can be further reformulated as a refinement and a lexicographic revision.

### 3.2 Restrained revision

Restrained revision [7] can be seen as a minimal modification of refinement to turn it into a form of revision. Indeed, refining a total preorder by a formula $P$ does not generally makes $P$ entailed by the refined total preorder. This is indeed the case only if $\min(C, \top)$ contains some models of $P$.

Restrained revision can be seen as an intermediate form of revision: while natural revision changes the preorder in a minimal way to make the revising formula entailed and lexicographic revision makes the formula to be preferred in all possible cases, restrained revision makes it to be preferred only when this is consistent with previous beliefs, and makes it entailed by a minimal change in the ordering.

Restrained revision is defined as follows, where $\min(C, P) = C(i) \cap \text{Mod}(P)$.

\[
[C(0), \ldots, C(m)]\text{res}(P) = \\
\quad = [C(i) \cap \text{Mod}(P), C(0) \cap \text{Mod}(P), C(0) \setminus \text{Mod}(P), \ldots, C(i - 1) \cap \text{Mod}(P), C(i - 1) \setminus \text{Mod}(P), C(i) \setminus \text{Mod}(P), C(i + 1) \cap \text{Mod}(P), C(i + 1) \setminus \text{Mod}(P), \ldots, C(m) \cap \text{Mod}(P), C(m) \setminus \text{Mod}(P)]
\]

The following quite obvious theorem is proved only for the sake of completeness, its statement being almost a direct consequence of the definition.

**Theorem 2** For every total preorder $C$ and formula $P$, it holds $C\text{res}(P) \equiv C\text{ref}(P)\text{nat}(P)$

**Proof.** Let $\min(C, P) = C(i) \cap \text{Mod}(P)$. The ordering $C\text{ref}(P)\text{nat}(P)$ is:

\[
[C(0), \ldots, C(m)]\text{ref}(P)\text{nat}(P) = \\
\quad = [C(0) \cap \text{Mod}(P), C(0) \setminus \text{Mod}(P), \ldots, C(m) \cap \text{Mod}(P), C(m) \setminus \text{Mod}(P)]\text{nat}(P) = \\
\quad = [C(i) \cap \text{Mod}(P), C(0) \cap \text{Mod}(P), C(0) \setminus \text{Mod}(P), \ldots, C(i - 1) \cap \text{Mod}(P), C(i - 1) \setminus \text{Mod}(P), \emptyset, C(i) \setminus \text{Mod}(P), C(i + 1) \cap \text{Mod}(P), C(i + 1) \setminus \text{Mod}(P), \ldots, C(m) \cap \text{Mod}(P), C(m) \setminus \text{Mod}(P)]
\]

12
By assumption, the minimal class of $C$ containing models of $P$ is $C(i)$. As a result, the minimal class of $C_{ref}(P)$ containing models of $P$ is $C(i) \cap \text{Mod}(P)$. As a result, $\min(C_{ref}(P), P) = \min(C, P)$. The total preorder above is therefore equivalent to $C_{ref}(P)$, since empty classes do not affect equivalence.

This reduction is applied to the running example.

**Example 4** Restrained revision can be replaced by a refinement followed by natural revision.

$$\emptyset \text{lex}(y) \text{lex}(\neg x \land y) \text{res}(x \land z) \text{rad}(\neg z)$$

This operation results into the following sequence:

$$\emptyset \text{lex}(y) \text{lex}(\neg x \land y) \text{ref}(x \land z) \text{nat}(x \land z) \text{rad}(\neg z)$$

The resulting natural revision can be then replaced by lexicographic revision. It can be seen that the minimal models of $x \land z$ in the ordering just before the natural revision are these of $x \land y \land z$:

$$\emptyset \text{lex}(y) \text{lex}(\neg x \land y) \text{ref}(x \land z) \text{lex}(x \land y \land z) \text{rad}(\neg z)$$

### 3.3 Very radical revision

Irrevocable revision [67] formalizes hypothetical reasoning by excluding from consideration all models that do not satisfy the assumption. Formally, these models are made inaccessible to revision, which cannot therefore recover them (hence the name). The scope of this article is limited to revisions that consider all models. While irrevocable revision excludes some model, the very radical revision variant by Rott [60] does not. Formally, it is defined as follows.

$$[C(0), \ldots, C(m)] \text{rad}(P) = [C(0) \cap \text{Mod}(P), \ldots, C(m) \cap \text{Mod}(P), (C(0) \cup \cdots \cup C(m)) \mod(P)]$$

The original definition has the first part $C(0) \cap \text{Mod}(P), \ldots, C(m) \cap \text{Mod}(P)$ only for the classes that intersect $\text{Mod}(P)$. The difference is inessential since the other classes are empty and empty classes are irrelevant.

Very radical revision can be expressed in terms of a sequence of a lexicographic revision, a severe antiwithdrawal and a second lexicographic revision. Intuitively, this is because very radical revision merges the classes not satisfying $P$, which is equivalent to making them minimal by a lexicographic revision by $\neg P$ and then by a severe antiwithdrawal by $P$; a further lexicographic revision is needed to restore the correct ordering.
Proof. By definition, the Levy identity only specifies the base of the revised ordering, the set of its minimal can be applied to severe withdrawal, leading to the definition of severe revision. However, The Levi identity \[33\] allows constructing a revision operator from a contraction. This

3.4 Severe revisions

drawback.

Example 5 The previous replacements turned the sequence of revisions of the running example into the following.

\[\emptyset \text{ lex}(y) \text{ lex}(-x \land y) \text{ ref}(x \land z) \text{ lex}(x \land y \land z) \text{ rad}(-z)\]

The last revision of the sequence \(\text{rad}(-z)\) is replaced by \(\text{lex}(-z)\text{sev}(-z)\text{lex}(-z)\):

\[\emptyset \text{ lex}(y) \text{ lex}(-x \land y) \text{ ref}(x \land z) \text{ lex}(x \land y \land z) \text{ lex}(z) \text{ sev}(-z) \text{ lex}(-z)\]

This sequence contains only lexicographic revisions, a refinement and a severe anti-withdrawal.

3.4 Severe revisions

The Levy identity \(\mathbb{L}\) allows constructing a revision operator from a contraction. This can be applied to severe withdrawal, leading to the definition of severe revision. However, the Levy identity only specifies the base of the revised ordering, the set of its minimal
models. The rest of the ordering can be obtained in at least three different ways, leading to different revision operators [60].

The first definition is called just “severe revision”. Since the symbol sev is already taken for severe antiwithdrawal, sevr is used for this revision.

**Definition 6** If \( \min(C, P) = C(i) \cap \text{Mod}(P) \), the severe revision \( \text{sevr} \) revises the total preorder \( C \) by formula \( P \) as follows.

\[
[C(0), \ldots , C(m)]_{\text{sevr}(P)} = \left[ C(i) \cap \text{Mod}(P), (C(0) \cup \cdots \cup C(i)) \setminus \text{Mod}(P), C(i + 1), \ldots , C(m) \right]
\]

This operator can be shown to be reducible to a severe antiwithdrawal followed by a natural revision, the latter being reducible to lexicographic revision as proved above.

**Theorem 4** For every total preorder \( C \) and formula \( P \), it holds \( C_{\text{sevr}(P)} \equiv \text{sev}(P)_{\text{nat}(P)} \).

**Proof.** Let \( C = [C(0), \ldots , C(m)] \) and \( i \) be the minimal index such that \( C(i) \cap \text{Mod}(P) \neq \emptyset \). Revising \( C \) by \( \text{sev}(P) \) and then \( \text{nat}(P) \) produces:

\[
[C(0), \ldots , C(m)]_{\text{sev}(P)_{\text{nat}(P)}} = \left[ C(0) \cup \cdots \cup C(i), C(i + 1), \ldots , C(m) \right]_{\text{nat}(P)}
\]

Since class zero of this ordering is \( C(0) \cup \cdots \cup C(i) \) and \( C(i) \) intersects \( \text{Mod}(P) \), it follows that the minimal index of a class of this ordering interesting \( \text{Mod}(P) \) is zero. As a result, natural revision produces:

\[
[C(0) \cup \cdots \cup C(i), C(i + 1), \ldots , C(m)]_{\text{nat}(P)} = \left[ (C(0) \cup \cdots \cup C(i)) \cap \text{Mod}(P), (C(0) \cup \cdots \cup C(i)) \setminus \text{Mod}(P), C(i + 1), \ldots , C(m) \right]_{\text{nat}(P)}
\]

The last step follows from \( C(i) \) being the minimal-index class of \( C \) intersecting \( \text{Mod}(P) \), which implies \( C(j) \cap \text{Mod}(P) = \emptyset \) for every \( j < i \). The last total preorder is \( C_{\text{sevr}(P)} \).

Moderate severe revision mixes a severe withdrawal with the changes lexicographic revision makes to a preorder. It will indeed be proved to be equivalent as a sequence of a severe antiwithdrawal and a lexicographic revision.

**Definition 7** If \( \min(C, P) = C(i) \cap \text{Mod}(P) \), the moderate severe revision \( \text{msev} \) revises the total preorder \( C \) by formula \( P \) as follows.

\[
[C(0), \ldots , C(m)]_{\text{msev}(P)} = \left[ C(i) \cap \text{Mod}(P), (C(0) \cup \cdots \cup C(i)) \setminus \text{Mod}(P), C(i + 1) \setminus \text{Mod}(P), \ldots , C(m) \setminus \text{Mod}(P) \right]
\]
Moderate severe revision can be proved to be equivalent to a severe antiwithdrawal followed by a lexicographic revision.

**Theorem 5** For every total preorder $C$ and formula $P$, it holds $C_{\text{msev}}(P) \equiv \text{sev}(P)\text{lex}(P)$.

**Proof.** Let $C = [C(0), \ldots, C(m)]$ and $i$ be the index such that $\min(C, P) = C(i) \cap \text{Mod}(P)$. Revising $C$ by $\text{sev}(P)$ and then $\text{lex}(P)$ produces:

$$[C(0), \ldots, C(m)]_{\text{sev}(P)\text{lex}(P)} =$$

$$= [C(0) \cup \cdots \cup C(i), C(i+1), \ldots, C(m)]_{\text{lex}(P)}$$

$$= [(C(0) \cup \cdots \cup C(i)) \cap \text{Mod}(P), C(i+1) \cap \text{Mod}(P), \ldots, C(m) \cap \text{Mod}(P),$$

$$\quad (C(0) \cup \cdots \cup C(i)) \setminus \text{Mod}(P), C(i+1) \setminus \text{Mod}(P), \ldots, C(m) \setminus \text{Mod}(P)]$$

$$= [C(i) \cap \text{Mod}(P), C(i+1) \cap \text{Mod}(P), \ldots, C(m) \cap \text{Mod}(P),$$

$$\quad (C(0) \cup \cdots \cup C(i)) \setminus \text{Mod}(P), C(i+1) \setminus \text{Mod}(P), \ldots, C(m) \setminus \text{Mod}(P)]$$

Equivalence $(C(0) \cup \cdots \cup C(i)) \cap \text{Mod}(P) = C(i) \cap \text{Mod}(P)$ holds because $C(i)$ is by assumption the lowest-index class intersecting $\text{Mod}(P)$. For the same reason, $C(0) \cap \text{Mod}(P), \ldots, C(i-1) \cap \text{Mod}(P)$ are all empty; therefore, their introduction leads to an equivalent preorder. The preorder obtained this way is $C_{\text{msev}}(P)$.

The last variant of severe revision is plain severe revision. Let $\min(C, P) = C(i) \cap \text{Mod}(P)$ and $j$ be the index such that $i < j$ and $C(j) \neq \emptyset$ if any, otherwise $j = i + 1$. Plain severe revision is defined as follows.

**Definition 8** If $\min(C, P) = C(i) \cap \text{Mod}(P)$, the plain severe revision $\text{psev}$ revises the total preorder $C$ by formula $P$ as follows.

$$[C(0), \ldots, C(m)]_{\text{psev}(P)} =$$

$$= [C(i) \cap \text{Mod}(P), C(0) \cup \cdots \cup (C(i) \setminus \text{Mod}(P)) \cup C(j), C(j+1), \ldots, C(m)]$$

Plain severe revision can be reformulated in terms of severe antiwithdrawal and lexicographic revision.

**Theorem 6** For every total preorder $C$ and formula $P$, it holds $C_{\text{psev}}(P) \equiv C_{\text{sev}}(\neg K')\text{lex}(K)$ where $K = \text{Form}(\min(C, P))$ and $K' = \text{Form}(\min(C_{\text{sev}}(P), \top))$.

$$[C(0), \ldots, C(i), C(i+1), \ldots, C(m)]_{\text{sev}(P)} =$$

$$[C(0) \cup \cdots \cup C(i), C(i+1), \ldots, C(m)]$$
Proof. Let $i$ and $j$ be the indexes as in the definition of plain severe revision. The models of $K'$ are the first non-empty class of the ordering $\text{Csev}(P)$, where $C = [C(0), \ldots, C(m)]$:

$$[C(0), \ldots, C(i), C(i+1), \ldots, C(m)]\text{sev}(P) = \begin{cases} [C(0) \cup \cdots \cup C(i), C(i+1), \ldots, C(m)] \\ [C(0) \cap \cdots \cap C(i), C(i+1), \ldots, C(m)] \end{cases}$$

By definition $i$ is such that $C(i) \cap \text{Mod}(P) \neq \emptyset$. As a result, $C(0) \cup \cdots \cup C(i)$ is not empty and therefore defines the set of models of $K$. The models of $\neg K'$ are the other ones:

$$\text{Mod}(\neg K') = C(i+1) \cup \cdots \cup C(m)$$

The ordering $\text{Csev}(\neg K')\text{lex}(K)$ can now be determined. By construction, none of the classes $C(0), \ldots, C(i)$ intersect $\text{Mod}(\neg K')$. The next class $C(i+1)$ may, but only if it is not empty. In particular, the lowest index class intersecting $\text{Mod}(K')$ is $C(j)$.

$$[C(0), \ldots, C(m)]\text{sev}(\neg K')\text{lex}(K) = \begin{cases} [C(0) \cup \cdots \cup C(i) \cup \cdots \cup C(j), C(j+1), \ldots, C(m)]\text{lex}(K) \\ [C(i) \cap \text{Mod}(P), C(0) \cup \cdots \cup (C(i) \setminus \text{Mod}(P)) \cup \cdots \cup C(j), C(j+1), \ldots, C(m)] \\ [C(i) \cap \text{Mod}(P), C(0) \cup \cdots \cup (C(i) \setminus \text{Mod}(P)) \cup C(j), C(j+1), \ldots, C(m)] \end{cases}$$

The last simplification can be done because all classes between $C(i)$ and $C(j)$ are by definition empty. What results coincides with the definition of $\text{Cpsev}(P)$. □

The following theorem shows that plain severe revision is not able to increase the number of levels over two. It also links it with full meet revision, to be defined in the next section.

**Theorem 7** If $C$ has at most two non-empty classes, then $\text{Cpsev}(P) \equiv [C(i) \cap \text{Mod}(P), \text{Mod}(\top) \setminus (C(i) \cup \text{Mod}(P))]$ holds for every formula $P$, where $\min(C, P) = C(i) \cap \text{Mod}(P)$.

Proof. Since $C$ has at most two non-empty classes, and removing empty classes produces an equivalent preorder, it can be assumed $C = [C(0), C(1)]$ where $C(0) \neq \emptyset$ while $C(1)$ may be empty. The definition of the plain severe revision depends on the minimal class $i$ intersecting $\text{Mod}(P)$ and the minimal non-empty class of index greater than $i$. Since $C$ has only two classes, $i$ can only be 0 or 1. In the first case, $j$ can only be 1, regardless of whether $C(1)$ is empty or not. As a result:

$$[C(0), C(1)]\text{psev}(P) = \begin{cases} [C(0) \cap \text{Mod}(P), (C(0) \setminus \text{Mod}(P)) \cup C(1)] \\ [C(0) \cap \text{Mod}(P), \text{Mod}(\top) \setminus (C(0) \cap \text{Mod}(P))] \\ [C(0) \cap \text{Mod}(P), \text{Mod}(\top) \setminus (C(i) \cap \text{Mod}(P))] \end{cases}$$
If $C(0) \cap Mod(P) = \emptyset$, then $i = 1$ and $j = i + 1 = 2$ since no class of index greater than $i$ exists, therefore none is different from the empty set. As a result:

\[
[C(0), C(1)]_{psev}(P) = \\
= [C(1) \cap Mod(P), C(0) \cup (C(1) \setminus Mod(P)) \cup C(2)] \\
= [C(1) \cap Mod(P), Mod(\top) \setminus (C(1) \cap Mod(P))] \\
= [C(1) \cap Mod(P), Mod(\top) \setminus (C(i) \cap Mod(P))]
\]

Since $C$ has only two classes, $C(2)$ is empty and can be removed. Since $C(0) \cap Mod(P) = \emptyset$, the set $C(0) \cup (C(1) \setminus Mod(P))$ is equal to $(C(0) \cup C(1)) \setminus Mod(P)$, in turn equal to $Mod(\top) \setminus (C(1) \cap Mod(P))$. \hfill \Box

### 3.5 Full meet revision

Full meet revision was initially defined in the one-step revision case [1, 28]. In particular, it was the result of disjoining all possible ways of minimally revising a propositional theory, formalizing both the assumption of minimal change and that of a complete lack of knowledge about the plausibility of the various choices. The initial plausibility ordering is not used other than for its set of minimal models. The resulting ordering only distinguish models in two classes: the base and the others.

**Definition 9** The full meet revision $\text{full}$ revises an ordering $C$ by a formula $P$ as follows, where $\text{min}(C, P) = C(i) \cap Mod(P)$.

\[
[C(0), \ldots, C(m)]_{\text{full}}(P) = \\
= [C(i) \cap Mod(P), Mod(\top) \setminus (C(i) \cap Mod(P))]
\]

Due to its simplicity, full meet revision can be expressed in a number of ways in terms of the other operators. For example, it is equivalent to a sequence made of a lexicographic revision followed by a severe antiwithdrawal and another lexicographic revision.

**Theorem 8** For every total preorder $C$ and formula $P$, it holds $C_{\text{full}}(P) \equiv C_{\text{lex}(\neg K)_{\text{sev}(K)}_{\text{lex}(K)}}$, where $K = \text{Form}(\text{min}(C, P))$.

**Proof.** Let $C = [C(0), \ldots, C(m)]$ and $i$ be the lowest index such that $C(i) \cap Mod(P) \neq \emptyset$. By definition, $Mod(K) = C(i) \cap Mod(P)$. Since $Mod(K) \subseteq C(i)$, it holds $Mod(K) \cap C(j) = \emptyset$ for every $j \neq i$.

\[
[C(0), \ldots, C(m)]_{\text{lex}(\neg K)_{\text{sev}(K)}_{\text{lex}(K)}} = \\
= [C(0) \cap Mod(\neg K), \ldots, C(m) \cap Mod(\neg K), \\
C(0) \setminus Mod(\neg K), \ldots, C(m) \setminus Mod(\neg K)]_{\text{sev}(K)_{\text{lex}(K)}} \\
= [C(0) \setminus Mod(K), \ldots, C(m) \setminus Mod(K)],
\]
\[ C(0) \cap \text{Mod}(K), \ldots, C(m) \cap \text{Mod}(K) \] \text{sev}(K) \text{lex}(K) \]

\[ \equiv [\text{Mod}(\top), \emptyset] \text{lex}(K) \]

\[ \equiv [\text{Mod}(\top)] \text{lex}(K) \]

\[ = [\text{Mod}(K), \text{Mod}(\neg K)] \]

Since \( \text{Mod}(K) = C(i) \cap \text{Mod}(P) \), the final total preorder is \( C_{\text{full}}(P) \).

An alternative reduction is \( C_{\text{full}}(P) = C_{\text{lex}}(\neg \text{Form}(\{I\})) \text{sev}(\text{Form}(\{I\})) \text{lex}(K) \), where \( I \) is an arbitrary propositional interpretation. Indeed, the proof relies on \( C_{\text{lex}}(\neg F) \text{sev}(F) = [\text{Mod}(\top)] \), which holds for every formula \( F \) such that \( \text{Mod}(F) \) is contained in a single class of \( C \). This is the case for \( \text{min}(C, P) \), but also for every formula having only one model.

Yet another reduction is \( C_{\text{full}}(P) = \emptyset \text{lex}(K) \) dove \( K = \text{Form}(\text{min}(C, P)) \). This is however not used because, contrarily to the other reductions it affects the previous sequence of revisions. For example, \( C_{\text{nat}}(N)_{\text{full}}(P) \) is turned into \( \emptyset \text{lex}(K) \), therefore making the initial natural revision disappear.

The previous revisions are instead preserved by the reduction \( C_{\text{full}}(P) \equiv C_{\text{rad}}(K) \) where \( K = \text{Form}(\text{min}(C, P)) \), which however requires the calculation of the minimal models of \( P \) in the total preorder \( C \) before being applied. The very radical revision can be then expressed in terms of two lexicographic revisions and a severe antithreaddrawal.

Starting from an empty ordering, full meet revision and plain severe revision behave in exactly the same way. Formally, for every sequence of formulae \( P_1, \ldots, P_n \) it holds:

\[ \emptyset_{\text{full}}(P_1) \ldots_{\text{full}}(P_n) \equiv \emptyset_{\text{psev}}(P_1) \ldots_{\text{psev}}(P_n) \]

This means that a sequence of mixed plain severe and full meet revisions can be turned into one containing only one type of revisions. This fact is a consequence of how they change an ordering comprising one or two classes: they both produce an ordering containing the class \( \text{min}(C, P) \) and the class containing all other models. For full meet revision, this is the definition and holds in all cases. For plain severe revision this is proved in Theorem 7.

4 The algorithm

The previous section shows that every considered belief change operator can be reduced to a sequence of lexicographic revisions, refinements and severe antithreaddrawals. As a result, every sequence of operators can be turned into one made of these three only. This section presents an algorithm for computing the base of the ordering at every time step of such a sequence.

This is done by first proving that refinements and severe antithreaddrawals can be removed by suitably modifying the sequence. This is done differently than the reductions in the previous section, which only modify the sequence locally: nothing is changed
before or after the operator that is replaced. Removing refinements instead requires introducing lexicographic revisions in other points of the sequence, and removing severe antiwithdrawals requires changing the previous lexicographic revisions.

The algorithm that computes the bases of a sequence of lexicographic revisions is then modified to work on the original sequence. The detour to the sequence of lexicographic revisions is necessary to prove that the final algorithm works. In particular, it is shown to do the same as the original algorithm on the simplified sequence.

All sequences are assumed to start with the empty ordering \( \emptyset \). Every other ordering \( C = [C(0), \ldots, C(m)] \) is the result of a sequence of lexicographic revision applied to the empty ordering: \( \emptyset \text{lex}(\text{Form}(C(m))) \ldots \text{lex}(\text{Form}(C(0))) \).

### 4.1 Simplification

A sequence of lexicographic revisions ending in either a refinement or a severe antiwithdrawal can be turned into a sequence of lexicographic revisions that has exactly the same final ordering when applied to the same original ordering. As a result, a sequence containing every of these three operators can be scanned from the beginning until the first operator that is not a lexicographic revising is found. The initial part of the sequence is then turned into a sequence containing only lexicographic revisions, and the process restarted.

Removal of the refinements is done thanks to the following theorems, which proves that a refinement can be moved at the beginning of a sequence of lexicographic revisions, and then turned into a lexicographic revision itself.

**Theorem 9** For every ordering \( C \) and two formulae \( R \) and \( L \), it holds \( C \text{lex}(L)\text{ref}(R) = C\text{ref}(R)\text{lex}(L) \).

**Proof.** According to the definitions of lex and ref, the ordering \( C \text{lex}(L)\text{ref}(R) \) is:

\[
[C(0), \ldots, C(m)]\text{lex}(L)\text{ref}(R) = \\
= [C(0) \cap \text{Mod}(L), \ldots, C(m) \cap \text{Mod}(L), C(0) \setminus \text{Mod}(L), \ldots, C(m) \setminus \text{Mod}(L)]\text{ref}(R) \\
= [C(0) \cap \text{Mod}(L) \cap \text{Mod}(R), C(0) \cap \text{Mod}(L) \setminus \text{Mod}(R) \\
\ldots C(m) \cap \text{Mod}(L) \cap \text{Mod}(R), C(m) \cap \text{Mod}(L) \setminus \text{Mod}(R), \\
C(0) \setminus \text{Mod}(L) \cap \text{Mod}(R), C(0) \setminus \text{Mod}(L) \setminus \text{Mod}(R) \\
\ldots C(m) \setminus \text{Mod}(L) \cap \text{Mod}(R) C(m) \setminus \text{Mod}(L) \setminus \text{Mod}(R)]
\]

The ordering resulting from the opposite application \( C\text{ref}(R)\text{lex}(L) \) is:

\[
[C(0), \ldots, C(m)]\text{ref}(R)\text{lex}(L) = \\
= [C(0) \cap \text{Mod}(R), C(0) \setminus \text{Mod}(R) \ldots C(m) \cap \text{Mod}(R), C(m) \setminus \text{Mod}(R)]\text{lex}(L) \\
= [C(0) \cap \text{Mod}(R) \cap \text{Mod}(L), C(0) \cap \text{Mod}(R) \setminus \text{Mod}(L) \\
\ldots C(m) \cap \text{Mod}(R) \cap \text{Mod}(L), C(m) \cap \text{Mod}(R) \setminus \text{Mod}(L), \\
C(0) \cap \text{Mod}(R) \setminus \text{Mod}(L), C(0) \setminus \text{Mod}(R) \setminus \text{Mod}(L) \\
\ldots C(m) \cap \text{Mod}(R) \setminus \text{Mod}(L), C(m) \setminus \text{Mod}(R) \setminus \text{Mod}(L)]
\]
Since ∩ and \ commute, these two sequences are the same. □

This proves that \( \emptyset \text{lex}(L_1) \ldots \text{lex}(L_{n-1}) \text{ref}(R) \) is equal to \( \emptyset \text{lex}(L_1) \ldots \text{lex}(L_{n-1}) \text{ref}(R) \text{lex}(L_n) \). Iteratively applying commutativity produces \( \emptyset \text{ref}(R) \text{lex}(L_1) \ldots \text{lex}(L_n) \). Since \( \emptyset = [\text{Mod}(T)] \), by definition \( \emptyset \text{ref}(R) = [\text{Mod}(R), \text{Mod}(\neg R)] \), and this is also the total preorder \( \emptyset \text{lex}(R) \). As a result, the whole sequence is equivalent to \( \emptyset \text{lex}(R) \text{lex}(L_1) \ldots \text{lex}(L_n) \).

**Corollary 1** For every formulae \( L_1, \ldots, L_n \) and \( R \), it holds:

\[
\emptyset \text{lex}(L_1) \ldots \text{lex}(L_{n-1}) \text{ref}(R) \equiv \emptyset \text{lex}(R) \text{lex}(L_1) \ldots \text{lex}(L_{n-1}) \text{lex}(L_n)
\]

If a sequence contains lex and ref, every ref\((R)\) in order can be moved to the beginning of the sequence and then replaced by lex\((L)\). What results is a sequence containing only lexicographic revisions.

**Example 6** The sequence in the running example was changed to comprise lexicographic revisions, refinements and severe antiwithdrawals only:

\[
\emptyset \text{lex}(y) \text{lex}(\neg x \land y) \text{ref}(x \land z) \text{lex}(x \land y \land z) \text{lex}(z) \text{sev}(\neg z) \text{lex}(\neg z)
\]

The corollary above shows that ref\((x \land z)\) can be moved to the beginning of the sequence and there turned into a lexicographic revision:

\[
\emptyset \text{lex}(x \land z) \text{lex}(y) \text{lex}(\neg x \land y) \text{lex}(x \land y \land z) \text{lex}(z) \text{sev}(\neg z) \text{lex}(\neg z)
\]

This transformation can be used to prove the folklore theorem linking lexicographic revision with maxsets:

\[
\text{maxset}(F) \equiv F
\]
\[
\text{maxset}(F_1, F_2) \equiv \begin{cases} F_1 \land F_2 & \text{if consistent} \\ F_1 & \text{otherwise} \end{cases}
\]
\[
\text{maxset}(F_1, \ldots, F_n) \equiv \text{maxset}(\text{maxset}(F_1, \ldots, F_{n-1}), F_n)
\]

The theorem establishes that maxset can be used to determine the minimal models of a formula in the ordering resulting from a sequence of lexicographic revisions. The proof is included here for the sake of completeness.

**Theorem 10** For every formula \( P \) and sequence of formulae \( L_1, \ldots, L_n \), it holds:

\[
\min(\emptyset \text{lex}(L_1) \ldots \text{lex}(L_n), P) = \text{Mod}(\text{maxset}(P, L_n, \ldots, L_1))
\]

**Proof.** Proved by induction on the length of the sequence. With \( n = 0 \), \( \text{maxset}(P) = P \) and \( \min(\emptyset, P) = \text{Mod}(P) \). The claim therefore holds.

If the claim holds for \( n - 1 \) formulae, then \( \min(\emptyset \text{lex}(L_2) \ldots \text{lex}(L_n), P) = \text{Mod}(\text{maxset}(P, L_n, \ldots, L_2)) \). The same has to be proved with a formula \( L_1 \) more.
Let $C = [C(0), \ldots, C(m)] = \emptyset \text{lex}(L_2) \ldots \text{lex}(L_n)$. By the above theorem, 
$\emptyset \text{lex}(L_1) \text{lex}(L_2) \ldots \text{lex}(L_n) = \emptyset \text{lex}(L_2) \ldots \text{lex}(L_n) \text{ref}(L_1) = C \text{ref}(L_1)$.

Let $i$ be the index such that $\text{min}(C, P) = C(i) \cap \text{Mod}(P)$. This implies that 
$C(0), \ldots, C(i-1)$ do not intersect $\text{Mod}(P)$. By definition, $C \text{ref}(L_1)$ is:

$$[C(0), \ldots, C(m)] \text{ref}(L_1) =$$

$$= [C(0) \cap \text{Mod}(L_1), C(0) \setminus \text{Mod}(L_1), \ldots, C(i-1) \cap \text{Mod}(L_1), C(i-1) \setminus \text{Mod}(L_1),$$

$$C(i) \cap \text{Mod}(L_1), C(i) \setminus \text{Mod}(L_1), \ldots, C(m) \cap \text{Mod}(L_1), C(m) \setminus \text{Mod}(L_1)]$$

Since $C(0), \ldots, C(i-1)$ do not intersect $\text{Mod}(P)$, the minimal class of $C \text{ref}(P)$ doing 
that is $C(i) \cap \text{Mod}(L_1)$ if not empty and $C(i) \setminus \text{Mod}(L_1)$ otherwise. In the second case, 
$C(i) \setminus \text{Mod}(L_1) = C(i)$ since $C(i) \cap \text{Mod}(L_1) = \emptyset$. Therefore, $\text{min}(C \text{ref}(L_1), P)$ is $C(i) \cap \text{Mod}(L_1) \cap \text{Mod}(P)$ if not empty and $C(i) \cap \text{Mod}(P)$ otherwise.

By the inductive assumption, $\text{min}(C, P) = \text{Mod}(\text{maxset}(P, L_1, \ldots, L_2))$, and by definition of $i$ it holds $C(i) \cap \text{Mod}(P) = \text{min}(C, P)$. Therefore, $\text{min}(C \text{ref}(L_1), P)$ is $\text{Mod}(\text{maxset}(P, L_1, \ldots, L_2)) \cap \text{Mod}(L_1)$ if not empty, and $\text{Mod}(\text{maxset}(P, L_1, \ldots, L_2))$ otherwise. In terms of formulae, $\text{min}(C \text{ref}(P), P)$ is 
the set of models of $\text{maxset}(P, L_1, \ldots, L_2) \land L_2$ if this formula is consistent and 
of $\text{maxset}(P, L_1, \ldots, L_2)$ otherwise. By the recursive definition of maxset, 
$\text{min}(C \text{ref}(L_1), P) = \text{Mod}(\text{maxset}(P, L_1, \ldots, L_2, L_1))$. \square

In a sequence of lexicographic revisions, refinements and severe antiwithdrawals, if 
the first operator of the sequence that is not a lexicographic revision is a refinement it can 
be turned into a lexicographic revision and moved to the beginning of the sequence. If it 
is a severe antiwithdrawal, a more complex change needs to be applied to the sequence.

As the previous refinements can be turned into lexicographic revisions, the previous 
belief change operators can be all assumed to be lexicographic revisions. In other 
words, the considered sequence has all lexicographic revisions but the last operator, 
which is a severe antiwithdrawal. Such a sequence can be modified as follows, where 
$B = \text{under}(S; L_1, \ldots, L_1)$ is a formula defined below.

$$\emptyset \text{lex}(L_1) \ldots \text{lex}(L_n) \text{sev}(S) = \emptyset \text{lex}(L_1 \lor B) \ldots \text{lex}(L_n \lor B) \text{lex}(B)$$

Intuitively, $B$ is constructed so that it collects all models that are in the same class 
of the minimal ones of $S$ or in lower classes. Disjoining every revising formula with $B$ 
ensures that these models remain in class zero over each revision. The claim therefore 
requires two proofs: first, that $B$ actually comprises these models; second, that modifying 
the lexicographic sequence this way does not change the resulting total preorder.

**Definition 10** The underformula of a sequence of formulae is:

$$\text{under}(S; \epsilon) = \top$$
$$\text{under}(S; L_n, L_{n-1}, \ldots, L_1) =$$

$$= \begin{cases} 
L_n \land \text{under}(S \land L_n; L_{n-1}, \ldots, L_1) & \text{if } S \land L_n \text{ is consistent} \\
L_n \lor \text{under}(S; L_{n-1}, \ldots, L_1) & \text{otherwise}
\end{cases}$$

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Informally, this construction includes as alternatives the formulae that are excluded from \( \text{maxset}(P, L_n, \ldots, L_1) \) because they are inconsistent with the partially built maxset \( M \). Starting from \( M = S \), the procedure of maxset construction adds \( L_i \) to \( M \) if \( M \land L_i \) is consistent. Otherwise, \( L_i \) is skipped. This procedure results in the minimal models of \( S \). If \( M \land L_i \) is inconsistent only because of \( S \), its models are in lower classes than all models of \( S \) in the final total preorder. Disjoining \( M \) with \( L_i \) gathers all such models. This is obtained in the last case of the definition: \( L_i \) is disjoined with the underformula but not added to \( M \). This intuition is formalized by the following theorem.

**Lemma 1** If \( C = [C(0), \ldots, C(m)] = \emptyset \text{lex}(L_1) \ldots \text{lex}(L_n) \) and \( i \) is the minimal index such that \( C(i) \cap \text{Mod}(S) \neq \emptyset \), then:

\[
C(0) \cup \cdots \cup C(i) = \text{Mod}(\text{under}(S; L_n, \ldots, L_1))
\]

**Proof.** The class of a model \( I \) is lower or equal than all classes of \( \text{Mod}(S) \) if and only if \( I \in \min(C, S \lor \text{Form}(I)) \). As a result, \( C(0) \cup \cdots \cup C(i) \) is defined by \( \{ I \mid I \models \text{maxset}(S \lor \text{Form}(I), L_n, \ldots, L_1) \} \). This set can be inductively proved to be equal to \( \text{Mod}(\text{under}(S; L_n, \ldots, L_1)) \). By induction on \( n \), the following is proved:

\[
I \models \text{maxset}(S \lor \text{Form}(I), L_n, \ldots, L_1) \iff I \models \text{under}(S; L_n, \ldots, L_1)
\]

With \( m = 0 \), by definition \( \text{under}(S; \epsilon) = \top \), which contains all models; \( \text{maxset}(S \lor \text{Form}(I)) = S \lor \text{Form}(I) \), which always contains \( I \). The base case is therefore proved.

The induction step assumes:

\[
I \models \text{maxset}(S \lor \text{Form}(I), L_{n-1}, \ldots, L_1) \iff I \models \text{under}(S; L_{n-1}, \ldots, L_1)
\]

The claim is the same with \( L_n \) added. The underformula of the claim is by definition:

\[
\text{under}(S; L_n, L_{n-1}, \ldots, L_1) = \begin{cases} 
L_n \land \text{under}(S \land L_n; L_{n-1}, \ldots, L_1) & \text{if } S \land L_1 \text{ is consistent} \\
L_n \lor \text{under}(S; L_{n-1}, \ldots, L_1) & \text{otherwise}
\end{cases}
\]

The maxset is instead:

\[
\text{maxset}(S \lor \text{Form}(I), L_n, L_{n-1}, \ldots, L_1) = \begin{cases} 
\text{maxset}((S \lor \text{Form}(I)) \land L_n, L_{n-1}, \ldots, L_1) & \text{if } (S \lor \text{Form}(I)) \land L_n \text{ is consistent} \\
\text{maxset}(S \lor \text{Form}(I), L_{n-1}, \ldots, L_1) & \text{otherwise}
\end{cases}
\]

The claim is proved if \( I \models \text{maxset}(S \lor \text{Form}(I), L_n, L_{n-1}, \ldots, L_1) \) is shown to be the same as \( I \models \text{under}(S; L_n, L_{n-1}, \ldots, L_1) \). This is done by reformulating the first condition. Since \( (S \lor \text{Form}(I)) \land L_n \) is consistent if and only if \( S \land L_n \) is consistent or \( I \models L_n \), the first case in the definition of maxset can be divided into three:
maxset\((S \lor \text{Form}(I)), L_n, L_{n-1}, \ldots, L_1\) =
\[
\begin{cases}
\text{maxset}((S \lor \text{Form}(I)) \land L_n, L_{n-1}, \ldots, L_1) & \text{if } S \land L_n \text{ is consistent and } I \models L_n \\
\text{maxset}((S \lor \text{Form}(I)) \land L_n, L_{n-1}, \ldots, L_1) & \text{if } S \land L_n \text{ is consistent and } I \not\models L_n \\
\text{maxset}((S \lor \text{Form}(I)) \land L_n, L_{n-1}, \ldots, L_1) & \text{if } S \land L_n \text{ is inconsistent and } I \models L_n \\
\text{maxset}(S \lor \text{Form}(I), L_{n-1}, \ldots, L_1) & \text{if } S \land L_n \text{ is inconsistent and } I \not\models L_n 
\end{cases}
\]

In the first case, since \(I \models L_n\) it follows that \(I \models \text{maxset}((S \lor \text{Form}(I)) \land L_n, L_{n-1}, \ldots, L_1)\) is equivalent to \(I \models L_n \lor \text{maxset}((S \lor \text{Form}(I)) \land L_n, L_{n-1}, \ldots, L_1)\). The first argument \((S \lor \text{Form}(I)) \land L_n\) can be rewritten \((S \land L_n) \lor \text{Form}(I)\) since \(I \models L_n\). By the induction assumption, \(I \models L_n \land \text{maxset}((S \land L_n) \lor \text{Form}(I), L_{n-1}, \ldots, L_1)\) is the same as \(I \models L_n \land \text{under}(S \land L_n; L_{n-1}, \ldots, L_1)\). Since \(S \land L_1\) is consistent, the latter is equivalent to \(I \models \text{under}(S; L_n, \ldots, L_1)\)

In the second case, the claim is proved by showing that \(I\) satisfies neither \(\text{maxset}((S \lor \text{Form}(I)) \land L_n, L_{n-1}, \ldots, L_1)\) nor \(\text{under}(S; L_n, L_{n-1}, \ldots, L_1)\). Since \(S \land L_n\) is consistent, \((S \lor \text{Form}(I)) \land L_n\) is also consistent. As a result, \(\text{maxset}((S \lor \text{Form}(I)), L_n, L_{n-1}, \ldots, L_1)\) implies \((S \lor \text{Form}(I)) \land L_n\), which implies \(L_n\). Since \(I \not\models L_n\), it follows \(I \not\models \text{maxset}((S \lor \text{Form}(I)), L_n, L_{n-1}, \ldots, L_1)\). This model does not satisfy \(\text{under}(S; L_n, L_{n-1}, \ldots, L_1)\) either, because this formula is equal to \(L_n \land \text{under}(S \land L_n; L_{n-1}, \ldots, L_1)\). Since \(S \land L_1\) is consistent, \(L_n \land \text{under}(S \land L_n; L_{n-1}, \ldots, L_1)\) when \(S \land L_n\) is consistent.

In the third case, \((S \lor \text{Form}(I)) \land L_n\equiv \text{Form}(I)\) since \(S \land L_n\) is inconsistent and \(I \models L_n\). As a result, \(I\) is the only model of \(\text{maxset}(S \lor \text{Form}(I), L_n, L_{n-1}, \ldots, L_1)\).

Together with the fourth case, this means that if \(S \land L_n\) is inconsistent then \(I \models \text{maxset}(S \lor \text{Form}(I), L_n, L_{n-1}, \ldots, L_1)\) if and only if \(I \models L_n\) or \(I \models \text{maxset}(S \lor \text{Form}(I), L_n, L_{n-1}, \ldots, L_1)\). By the induction assumption, the latter is equivalent to \(I \models \text{under}(S; L_n, L_{n-1}, \ldots, L_1)\). Therefore, the condition can be rewritten as \(I \models L_n \lor \text{under}(S; L_n, L_{n-1}, \ldots, L_1)\). Since \(S \land L_n\) is inconsistent, \(L_n \lor \text{under}(S; L_n, L_{n-1}, \ldots, L_1)\) is inconsistent.

It is now shown that \(\text{under}(S; L_n, \ldots, L_1)\) allows rewriting the sequence without affecting the final total preorder.

**Theorem 11** If \(B = \text{under}(S; L_n, \ldots, L_1)\), then:

\[\emptyset \text{lex}(L_1) \ldots \text{lex}(L_n)\text{sev}(S) \equiv \emptyset \text{lex}(L_1 \lor B) \ldots \text{lex}(L_n \lor B)\text{lex}(B)\]

**Proof.** The claim is proved by showing that for every consistent formula \(P\), its minimal models according to the two total preorders are the same.

By the previous theorem, if \(B = \text{under}(S; L_n, \ldots, L_1)\) then \(\text{Mod}(B) = C(0) \cup \cdots \cup C(i)\), where \(C = [C(0), \ldots, C(m)] = \emptyset \text{lex}(L_1) \ldots \text{lex}(L_n)\) and \(i\) is the minimal index such that \(C(i) \cap \text{Mod}(S) \neq \emptyset\). Since \(\text{Csev}(S) = [C(0) \cup \cdots \cup C(i), C(i+1), \ldots, C(m)] = [\text{Mod}(B), C(i+1), \ldots, C(m)]\), it follows that:

\[\min(P, C\text{sev}(S)) = \begin{cases}
\text{Mod}(P) \cap \text{Mod}(B) & \text{if not empty} \\
\text{Mod}(P) \cap C(j) & \text{otherwise, for } j\text{ minimal index such that this set is not empty}
\end{cases}\]
Since $C = \emptyset \text{lex}(L_1) \ldots \text{lex}(L_n)$, $\text{Mod}(P) \cap C(j)$ for the minimal $j$ for which this set is not empty is the set of models of $\text{maxset}(P, L_n, \ldots, L_1)$. As a result, $\min(P, C \text{sev}(S))$ can be rewritten as:

$$\min(P, C \text{sev}(S)) = \begin{cases} \text{Mod}(P \land B) & \text{if consistent} \\ \text{Mod}(\text{maxset}(P, L_n, \ldots, L_1)) & \text{otherwise} \end{cases}$$

Let $C' = \emptyset \text{lex}(L_1 \lor B) \ldots \text{lex}(L_n \lor B) \text{lex}(B)$. It is now shown that $\min(C \text{sev}(S), P) = \min(C', P)$. Since $C'$ results from applying a number of lexicographic revisions to an empty total preorder, it holds:

$$\min(P, C') = \text{maxset}(P, B, L_n \lor B, \ldots, L_1 \lor B)$$

If $P \land B$ is consistent, then it is consistent with all formulae $L_i \lor B$ but also entails all of them. As a result:

$$\begin{align*}
\min(P, C') & = \text{maxset}(P, B, L_n \lor B, \ldots, L_1 \lor B) \\
& = \text{maxset}(P \land B, L_n \lor B, \ldots, L_1 \lor B) \\
& = P \land B \land (L_n \lor B) \land \cdots \land (L_1 \lor B) \\
& = P \land B
\end{align*}$$

If $P \land B$ is inconsistent, then $P \models \neg B$; therefore:

$$\begin{align*}
\min(P, C') & = \text{maxset}(P, B, L_n \lor B, \ldots, L_1 \lor B) \\
& = \text{maxset}(P, L_n \lor B, \ldots, L_1 \lor B)
\end{align*}$$

Since $P \models \neg B$, then $P \models \neg B$. As a result, $P \land (L_n \lor B)$ is equivalent to $P \land L_i$. This implies that $\text{maxset}(P, L_n \lor B, \ldots, L_1 \lor B)$ is $\text{maxset}(P \land L_n, L_{n-1} \ldots, L_1 \lor B)$ if $P \land L_n$ is consistent, otherwise it is $\text{maxset}(P, L_{m-1} \ldots, L_1 \lor B)$. This argument can be repeated for every $P \land \land L$ with $L \subseteq \{L_n, \ldots, L_i\}$, proving that the result is $\text{maxset}(P, L_n, \ldots, L_1)$.

These theorems tell how to modify a sequence into one that only contains lexicographic revisions: starting from the beginning, the first operator that is not lexic can be ref($R$) or sev($S$); in the first case, it is turned into lex($R$) and moved at the beginning of the sequence; in the second case, the underformula $B$ of the previous revisions (which are all lexicographic by assumption) is used to replace sev($S$) with lex($B$) and is disjoined to all previous revisions. This part of the sequence now contains only lexicographic revisions, and the process can therefore be repeated for the next ref or sev operator. The final result is a sequence of lexicographic revisions applied to the empty preorder.

Such a sequence not only has the correct value $\min(C, P)$ at each step, but also the same final preorder of the original sequence. This implies that it is equivalent to it even regarding subsequent revisions.
Example 7  The sequence in the running example has been shown to be equivalent to the following one, which only contains lexicographic revisions and a severe antiwithdrawal.

$$\emptyset \text{lex}(x \land z) \text{lex}(\neg x \land y) \text{lex}(x \land y \land z) \text{lex}(z) \text{sev}(\neg z) \text{lex}(\neg z)$$

The severe antiwithdrawal can be turned into a lexicographic revision by first calculating its underformula:

\[
\text{under}(\neg z; z, x \land y \land z, \neg x \land y, y, x \land z) = \\
= z \lor \text{under}(\neg z; x \land y \land z, \neg x \land y, y, x \land z) \\
= z \lor (x \land y \land z) \lor \text{under}(\neg z; \neg x \land y, y, x \land z) \\
= z \lor (x \land y \land z) \lor (\neg x \land y \land \text{under}(\neg z \land \neg x \land y, y, x \land z)) \\
= z \lor (x \land y \land z) \lor (\neg x \land y \land ((x \land z) \lor \text{under}(\neg z \land \neg x \land y \land y))) \\
= z \lor (x \land y \land z) \lor (\neg x \land y \land ((x \land z) \lor \top))
\]

This formula is equivalent to $B = z \lor (\neg x \land y)$. The sequence is therefore turned into:

$$\emptyset \text{lex}(x \land z) \text{lex}(y) \text{lex}(\neg x \land y) \text{lex}(x \land y \land z) \text{lex}(z) \text{sev}(\neg z) \text{lex}(\neg z)$$

Some simplifications can be then applied. For example, $(x \land z) \lor B = (x \land z) \lor (z \lor (\neg x \land y)) \equiv z \lor (\neg x \land y)$ and $y \lor (z \lor (\neg x \land y)) \equiv y \lor z$.

In this particular case, only one severe antiwithdrawal occurs. More generally, they are transformed into lexicographic revisions starting from the first.

The only apparent drawback of this procedure is that every $\text{sev}(S)$ requires the underformula $B$ to be disjoined to all previous formulae. This makes $B$ to be included in the underformula of the next $\text{sev}(S')$. This problem is solved by leaving the sequence as it is and processing it as if the transformation has been done.

4.2 Algorithm

A sequence contains only lexicographic revisions and refinements applied to the empty ordering can be turned into a sequence of lexicographic revisions by moving all refinements to the beginning. After this change, the minimal models of a formula $P$ can be calculated using the maxset construction. Since the refinements are moved to the start of the sequence in the order in which they are encountered, they end up there in reverse order. As a result, the maxset can be calculated from the original sequence following the order that would result from the simplification:
1. start with $M = P$;

2. proceeding from the end to the start of the sequence, for every $\text{lex}(L_i)$ turn $M$ into $M \land L_i$ if this formula is consistent;

3. from the start to the end of the sequence, for every $\text{ref}(R_i)$ turn $M$ into $M \land R_i$ if this formula is consistent.

**The back and forth algorithm.**

The following figure shows how the algorithm proceeds when computing a formula equivalent to the set of the minimal models of $P$. Every formula encountered following the arrows is conjoined with it if that does not result in contradiction.

The back and forth algorithm works because it builds the maxset starting from $P$ and proceeding in the same order as if the refinements were moved to the start of the sequence. Its correctness is therefore proved by Theorem 1. For the same reason, a similar mechanism can be used to determine an underformula instead of a maxset.

This is important because a sequence may contain lexicographic revisions, refinements and severe antiwithdrawals. Assuming that the underformulae for the latter have all been determined, at each severe antiwithdrawal encountered while going back, if $B$ is consistent with the current maxset $M$ then $M$ is turned into $M \land B$ and the procedure “bounces” back in the forward direction. This is because if $M \land B$ is consistent then the previous $\text{lex}(L)$ in the original sequence would be turned into $\text{lex}(L \lor B)$ in the modified sequence. As a result, $M$ is consistent with all of them, but their addition is irrelevant because $M$ is already conjoined with $B$. 
1. Start at the end of the sequence with $M = P$ and go back;

2. for every $\text{lex}(L_i)$ turn $M$ into $M \land L_i$ if this formula is consistent; regardless, continue going backwards;

3. for every $\text{sev}(S)$, if $M$ is consistent with its underformula $B$ then turn $M$ into $M \land B$ and bounce forward, toward the end of the sequence; otherwise, continue going backwards;

4. at the start of the sequence, bounce forward, toward the end of the sequence;

5. when proceeding forward, for every $\text{ref}(R_i)$ turn $M$ into $M \land R_i$ if this formula is consistent; regardless, continue going forward.

**The back, bounce and forth algorithm**

The fourth point can be omitted by placing $\text{sev}(\top)$ at the very beginning of the sequence. This marker signals the algorithm to bounce forward without the need to verify whether the sequence is at the start. At the end $M$ has models $\min(C, P)$ where $C$ is the final ordering.

The algorithm works because it builds a formula that is the same that would have been produced when creating the maxset of the modified sequence that only contains lexicographic revisions.

The following figure shows how the algorithm moves in a segment of the sequence. When it reaches $\text{sev}(F)$, if the formula that is currently being built is consistent with the underformula of this severe antiwithdrawal then the algorithm bounces forward to $\text{ref}(H)$, otherwise it keeps going back, to $\text{lex}(D)$.

```
ref(C)  lex(D)  ref(E)  sev(F)  lex(G)  ref(H)
```

This construction produces a maxset. The underformula of each severe antiwithdrawal is built similarly. The effect of a severe anticontraction $\text{sev}(S')$ with underformula $B'$ encountered during the construction of another underformula may have only two possible effects: $B'$ is ignored; or, $B'$ is added but makes the algorithm bounce forward. This proves that an underformula $B$ may at most contain a single previous underformula $B'$ and some formulae in between. As a result, the underformulae do not exponentially blow up.

**Example 8** The algorithm is applied to the sequence of the running example. It first determines the underformula of the severe antiwithdrawal, then the base at the end of the sequence.

```
\emptyset \text{lex}(y) \text{lex}(\neg x \land y) \text{ref}(x \land z) \text{lex}(x \land y \land z) \text{lex}(z) \text{sev}(\neg z) \text{lex}(\neg z)
```
The first step is to determine the underformula of the first severe revision in the sequence. This is done by following the back and forth procedure: first go back through the lexicographic revisions, then come forth through the refinements.

\[
0 \ \text{lex}(y) \ \text{lex}(\neg x \land y) \ \text{ref}(x \land z) \ \text{lex}(x \land y \land z) \ \text{lex}(z) \ \text{sev}(\neg z) \ \text{lex}(\neg z)
\]

Following the numbers, the formulae are in the sequence \(\neg z, z, x \land y \land z, \neg x \land y, y, x \land z\). As a result, the underformula of the severe antiwithdrawal is calculated on this sequence:

\[
\text{under}(\neg z; z, x \land y \land z, \neg x \land y, y, x \land z)
\]

This has been previously shown to be equivalent to \(B = z \lor (\neg x \land y)\). It allows determining the final base by following the arrows as in the back, bounce and forth algorithm.

\[
0 \ \text{lex}(y) \ \text{lex}(\neg x \land y) \ \text{ref}(x \land z) \ \text{lex}(x \land y \land z) \ \text{lex}(z) \ \text{sev}(\neg z) \ \text{lex}(\neg z)
\]

The choice of keeping going back or bouncing forth depends on the consistency of the formula under construction with the underformula of \(\text{sev}(\neg z)\). In this case, the formula is \(\neg z\) and the underformula \(z \lor (\neg x \land y)\). Since their conjunction \(\neg z \land \neg x \land y\) is consistent, the algorithm bounces. Since there are no refinement after the severe antiwithdrawal, the resulting base is this formula.

The algorithm can be adapted to work with the other considered revisions without replacing them with \text{lex}, \text{ref} and \text{sev}, since each of them can be “locally” replaced with a sequence of these three. As a result, while going forward or backwards, sufficies to behave in the same way as if the replacement has been done.

5 Complexity

The reductions shown in the previous sections prove that each considered belief change operators can be turned into a lexicographic revision, possibly by first calculating a maxset or an underformula. Both operations can be done by a polynomial number of calls to a propositional satisfiability solver. Therefore, the complexity of a sequence of arbitrary and mixed belief change operators is in the complexity class \(\Delta_2^p\), which contains all problems that can be solved by a polynomial number of calls to an NP oracle. The problem is also easily shown to be hard for the same class even if all operators are lexicographic revisions. This was previously published without proof [47].

\textbf{Theorem 12} The problems of establishing whether the base resulting from a sequence of lexicographic, natural, restrained, very radical and severe revisions, refinements and severe antiwithdrawals applied to the empty ordering implies a formula is in \(\Delta_2^p\), and is \(\Delta_2^p\)-hard even if the sequence comprises lexicographic revisions only or refinements only.
The back, bounce and forth algorithm shown in the previous section calculates a formula equivalent to the set of the minimal models of a formula in polynomial time, not counting that needed to determine propositional satisfiability. This proves that the problem is in $\Delta_2^p$ for sequence comprising only lexicographic revisions, refinements and severe antiwithdrawal. Since all other considered operators can be reduced to these three, the problem is in $\Delta_2^p$ for all of them.

Hardness is proved by reduction from the problem of establishing whether the maximal lexicographic model of a formula $F$ over the alphabet $\{x_1, \ldots, x_n\}$ satisfies $x_n$, which is $\Delta_2^p$-complete [44]. A simple reduction translates this problem into the similar one where the formula is satisfiable: a possibly unsatisfiable formula $F$ is turned into the satisfiable formula $(\neg x_0 \land \neg x_1 \land \cdots \land \neg x_n) \lor F$, where $x_0$ is a new variable. A further reduction shows that the maximal model of a satisfiable formula $F$ satisfies $x_n$ if and only if the base of $\emptyset \text{lex}(x_n) \ldots \text{lex}(x_1) \text{lex}(F)$ implies $x_n$. This proves that entailment for a sequence of lexicographic revisions is $\Delta_2^p$-hard. The sequence is equivalent to $\emptyset \text{ref}(F) \text{ref}(x_1) \ldots \text{ref}(x_n)$, proving the $\Delta_2^p$-hardness for refinements only. More generally, it is hard for every alternation of these two belief change operators.

Since severe antiwithdrawal turns an empty total preorder into an empty total preorder, a sequence comprising only this operator has low complexity: entailment is equal to validity, coNP complete.

Sequences of mixed belief change operators are now considered. Two classes can be shown to be $\Delta_2^p$-hard: operators that can produce a lexicographically maximal model, and operators that can refine the lowest class of an ordering. In both cases, the alternation of operators does not matter, as long as they have the given behavior.

### 5.1 Lexicographic-finding revisions

Entailment from a sequence of lexicographic revisions is $\Delta_2^p$-hard by Theorem 12. Some other belief change operators can be intermixed without changing complexity. These are the ones that produce the same results when applied after a sequence of revisions whose formulae are consistent.

**Theorem 13** If $\text{rev}$ is any of a class of revision operators such that $\emptyset \text{rev}(S_1) \ldots \text{rev}(S_n) \equiv \emptyset \text{lex}(S_1) \ldots \text{lex}(S_n)$ whenever $S_1 \land \cdots \land S_n$ is consistent, then entailment for $\text{rev}$ is $\Delta_2^p$-hard.

**Proof.** Checking whether the lexicographically maximal model of a formula $F$ satisfies $x_n$ is $\Delta_2^p$-hard [44]. This model is also the only element of class zero of $\emptyset \text{lex}(x_n) \ldots \text{lex}(x_1) \text{lex}(F)$. Since $x_1, \ldots, x_n$ is consistent, $\emptyset \text{lex}(x_n) \ldots \text{lex}(x_1)$ is equivalent to $\emptyset \text{rev}(x_n) \ldots \text{rev}(x_1)$ by assumption. Therefore, entailment from these two sequences is the same. \qed

Moderate severe revision satisfies the premise of this theorem: it coincides with lexicographic revision on all consistent sequences.

**Theorem 14** If $S_1 \land \cdots \land S_n$ is consistent, then:

$$\emptyset \text{msev}(S_1) \ldots \text{msev}(S_n) = \emptyset \text{lex}(S_1) \ldots \text{lex}(S_n)$$
Proof. It is inductively proved that $\emptyset \text{msev}(S_1) \ldots \text{msev}(S_n) = \emptyset \text{lex}(S_1) \ldots \text{lex}(S_n)$ and that $\emptyset \text{msev}(S_1) \ldots \text{msev}(S_n)(0) = \text{Mod}(S_1 \land \cdots \land S_n)$.

The base case is with $n = 0$, where the claim holds because $\emptyset = \emptyset$ and the conjunction of an empty sequence is $\top$.

Assuming that $\emptyset \text{msev}(S_1) \ldots \text{msev}(S_{n-1}) = \emptyset \text{lex}(S_1) \ldots \text{lex}(S_{n-1})$, and that $\emptyset \text{msev}(S_1) \ldots \text{msev}(S_{n-1})(0) = \text{Mod}(S_1 \land \cdots \land S_{n-1})$, the same are proved with the addition of $S_n$.

Let $C = [C(0), \ldots, C(m)]$ be $\emptyset \text{msev}(S_1) \ldots \text{msev}(S_{n-1})$, and $M = S_1 \land \cdots \land S_{n-1}$. By the induction assumptions, $C = \emptyset \text{lex}(S_1) \ldots \text{lex}(S_{n-1})$, and $C(0) = \text{Mod}(M)$.

The definition of $C\text{msev}(S_n)$ depends on $\min(C, S_n) = C(0) \cap \text{Mod}(S_n)$. Since $C(0) = \text{Mod}(M)$ and $M \land S_n$ is by assumption consistent, $i = 0$. The definition of moderate severe revision specializes to $i = 0$ as:

$$[C(0), \ldots, C(m)]\text{msev}(S_n) =$$

$$= [C(0) \cap \text{Mod}(S_n), \ldots, C(m) \cap \text{Mod}(S_n),$$

$$(C(0) \cup \cdots \cup C(0)) \setminus \text{Mod}(S_n), C(1) \setminus \text{Mod}(S_n), \ldots, C(m) \setminus \text{Mod}(S_n))]$$

$$= [C(0) \cap \text{Mod}(S_n), \ldots, C(m) \cap \text{Mod}(S_n),$$

$$C(0) \setminus \text{Mod}(S_n), C(1) \setminus \text{Mod}(S_n), \ldots, C(m) \setminus \text{Mod}(S_n)]$$

$$= [C(0), \ldots, C(m)]\text{lex}(S_n)$$

The total preorder $[C(0), \ldots, C(m)]\text{lex}(S_n)$ is $\emptyset \text{lex}(S_1) \ldots \text{lex}(S_{n-1}) \text{lex}(S_n)$ because of the induction assumption $C = \emptyset \text{lex}(S_1) \ldots \text{lex}(S_{n-1})$. Since $C(0) = \text{Mod}(S_1 \land \cdots \land S_{n-1})$, it follows that the class zero of this ordering is $C\text{msev}(S_n)(0) = C(0) \cap \text{Mod}(S_n) = \text{Mod}(S_1 \land \cdots \land S_{n-1} \land S_n)$, which concludes the proof of the induction claim.

A consequence of the two theorems above is that entailment from a sequence of moderate severe revision is $\Delta^P_2$-hard. The same holds even if lexicographic and moderate severe revisions are mixed.

**Corollary 2** Entailment from a sequence of moderate severe revision is $\Delta^P_2$-hard.

### 5.2 Bottom-refining revisions

A revision operator is bottom-refining if it “refines” the lowest-index class of the ordering that has models of the revising formula.

**Definition 11** An operator $\text{rev}$ is a revision if $\min(C\text{rev}(P), T) = \min(C, P)$ and is a bottom-revising revision if $C\text{rev}(P)(1) = C(0) \setminus \text{Mod}(P)$ also holds whenever $C(0) \cap \text{Mod}(P) \neq \emptyset$.

A revision makes minimal the models that are the minimal models satisfying the revising formula. Removing empty classes, class zero of $C\text{rev}(P)$ is the non-empty intersection $C(i) \cap P$ of minimal $i$. If $\text{rev}$ is also bottom-refining, this is $C(0) \cap \text{Mod}(P)$ if this intersection is not empty. In this case, revising $C$ by $P$ splits $C(0)$ based on $P$, as shown in the following example.
The name bottom-refining derives from the way the “bottom” class \( C(0) \) is partitioned (refined) into the part satisfying \( P \) and the part not satisfy \( P \). How the other classes are changed is not constrained.

**Theorem 15** Natural revision, restrained revision and severe revision are bottom-refining revisions.

**Proof.** All three operators are revisions, since they make the non-empty intersection \( C(i) \cap \text{Mod}(P) \) of minimal \( i \) the new class zero.

The condition of bottom-refining only concerns the case where \( C(0) \cap \text{Mod}(P) \neq \emptyset \), where this index \( i \) is 0. The definition of natural revisions specializes as follows:

\[
[C(0), \ldots, C(m)]_{\text{nat}}(P) = \\
[C(i) \cap \text{Mod}(M), C(0), \ldots C(i-1), C(i) \backslash \text{Mod}(P), C(i+1), \ldots, C(m)] \\
[C(0) \cap \text{Mod}(M), C(0) \backslash \text{Mod}(P), C(1), \ldots, C(m)]
\]

The definition of restrained revisions specializes as follows:

\[
[C(0), \ldots, C(m)]_{\text{res}}(P) = \\
= [C(i) \cap \text{Mod}(P), \\
\quad C(0) \cap \text{Mod}(P), C(0) \backslash \text{Mod}(P), \ldots, C(i-1) \cap \text{Mod}(P), C(i-1) \backslash \text{Mod}(P), \\
\quad C(i) \backslash \text{Mod}(P), \\
\quad C(i+1) \cap \text{Mod}(P), C(i+1) \backslash \text{Mod}(P), \ldots, C(m) \cap \text{Mod}(P), C(m) \backslash \text{Mod}(P)] \\
= [C(0) \cap \text{Mod}(P), C(0) \backslash \text{Mod}(P), C(1) \cap \text{Mod}(P), C(1) \backslash \text{Mod}(P), \\
\quad \ldots C(m) \cap \text{Mod}(P), C(m) \backslash \text{Mod}(P)]
\]

The definition of severe revisions specializes as follows:

\[
[C(0), \ldots, C(m)]_{\text{sevr}}(P) = \\
= [C(i) \cap \text{Mod}(P), (C(0) \cup \cdots \cup C(i)) \backslash \text{Mod}(P), C(i+1), \ldots, C(m)] \\
= [C(0) \cap \text{Mod}(P), C(0) \backslash \text{Mod}(P), C(1), \ldots, C(m)]
\]
All three revisions makes $C(0) \cap \text{Mod}(P)$ the new class zero and $C(0) \setminus \text{Mod}(P)$ the new class one.

The complexity of arbitrary sequences of bottom-refining revisions is established by the following theorem.

**Theorem 16** Inference from a sequence of bottom-refining revisions is $\Delta^p_2$-hard.

**Proof.** The claim is proved by reduction from the problem of establishing whether the lexicographically maximal model of a consistent formula $F$ over alphabet $\{x_1, \ldots, x_n\}$ satisfies $x_n$. The corresponding sequence of bottom refining revisions rev is the following, where $y_1, \ldots, y_n$ are fresh variables in bijective correspondence with $x_1, \ldots, x_n$:

$$\emptyset \text{rev}(F) \text{rev}(y_1) \text{rev}(y_1 \rightarrow x_1) \ldots \text{rev}(y_n) \text{rev}(y_n \rightarrow x_n)$$

The reduction first introduces the models of $F$ as the class zero, then refines it by $x_1$ if consistent, then by $x_2$ if consistent, etc. A single bottom-refining revision for each variable cannot do that: if $x_1$ is inconsistent with $F$ the effect of $F \text{rev}(x_1)$ is to move the models of $x_1$ in a class lower than that of $F$. This is why the new variable $y_1$ is introduced first.

$$\emptyset \text{rev}(F) \text{rev}(y_1) \text{rev}(y_1 \rightarrow x_1) \ldots \text{rev}(y_n) \text{rev}(y_n \rightarrow x_n)$$

The empty ordering has all models in class zero. A revision operator cuts Mod($F$) out from it to make the new class zero when revising by a consistent formula $F$. The class Mod($\neg F$) is created only if $F$ is not tautological, but this is irrelevant.

The new class zero Mod($F$) is refined into Mod($F \land y_1$) and Mod($F \land \neg y_1$) because of the bottom-refining condition: since $F$ is consistent and does not contain $y_1$, the conjunctions $F \land y_1$ and $F \land \neg y_1$ are consistent.

Revising this ordering by $y_1 \rightarrow x_1$ depends on the consistency of $F \land x_1$. If $F$ is consistent with $x_1$ then $F \land y_1 \land (y_1 \rightarrow x_1)$ is consistent; therefore, class zero Mod($F \land y_1$) contains some models of $y_1 \rightarrow x_1$. The resulting class zero comprises them: Mod($F \land y_1 \land (y_1 \rightarrow x_1)$) = Mod($F \land x_1 \land y_1$). If $F$ is inconsistent with $x_1$, then $F \land y_1 \land (y_1 \rightarrow x_1)$ is inconsistent; therefore, Mod($F \land y_1$) does not contain any model of $y_1 \rightarrow x_1$. Instead, Mod($F \land \neg y_1$) does, since Mod($F \land \neg y_1 \land (y_1 \rightarrow x_1)$) is the same as Mod($F \land \neg y_1$), which is consistent because $F$ is consistent and does not mention $y_1$. The class zero resulting from revising by $y_1 \rightarrow x_1$ is Mod($F \land \neg y_1 \land (y_1 \rightarrow x_1)$) = Mod($F \land \neg y_1$) since rev is a revision.

This proves that the result of revising first by $y_1$ and then by $y_1 \rightarrow x_1$ is an ordering that has Mod($F \land x_1 \land y_1$) as its class zero if $F \land x_1$ is consistent and Mod($F \land \neg y_1$) otherwise. Apart from $y_1$, which is unlinked to the rest of the formula and the other variables $y_i$, these are the models of $F \land x_1$ if consistent and the models of $F$ otherwise.

Iterating the procedure on the remaining variables $x_2, \ldots, x_n$ produces an ordering whose class zero comprises the lexicographically maximally model of $F$ only. Checking whether it entails $x_n$ is the final step of the translation.
This intuition is made a formal proof by induction. For every \( i \), class zero of \( \emptyset \text{rev}(F) \text{rev}(y_1) \text{rev}(y_1 \rightarrow x_1) \ldots \text{rev}(y_i) \text{rev}(y_i \rightarrow x_i) \) is \( \text{Mod}(\bigwedge Y' \land \text{maxset}(F, x_1, \ldots, x_i)) \) for some consistent \( Y' \subseteq \{y_1, \neg y_1, \ldots, y_i, \neg y_i\} \). This is the lexicographically maximal partial model over variables \( x_1, \ldots, x_i \), apart from some of the variables \( y_1, \ldots, y_i \). Assuming that this condition is true for \( i \), it is shown to remain true after revising by \( y_{i+1} \) and \( y_{i+1} \rightarrow x_{i+1} \).

Let \( C = \emptyset \text{rev}(F) \text{rev}(y_1) \text{rev}(y_1 \rightarrow x_1) \ldots \text{rev}(y_i) \text{rev}(y_i \rightarrow x_i) \) and \( M = \bigwedge Y' \land \text{maxset}(F, x_1, \ldots, x_i) \). The inductive assumption is \( C(0) = \text{Mod}(M) \). Since \( M \) does not mention \( y_{i+1} \) and is consistent, \( M \land y_{i+1} \) is consistent. As a result, \( C(0) \cap \text{Mod}(y_{i+1}) \) is not empty. A bottom-refining revision splits the class zero in two:

\[
\text{Crev}(y_{i+1})(0) = \text{Mod}(M \land y_{i+1}), \\
\text{Crev}(y_{i+1})(1) = \text{Mod}(M \land \neg y_{i+1})
\]

This ordering is further revised by \( y_{i+1} \rightarrow x_{i+1} \). The resulting class zero of \( \text{Crev}(y_{i+1}) \text{rev}(y_{i+1} \rightarrow x_{i+1}) \) is the first non-empty of the following two sets, since \( \text{rev} \) is by assumption a revision operator and at least the second is not empty since \( M \) is consistent and does not contain \( y_{i+1} \).

\[
\text{Crev}(y_{i+1})(0) \cap \text{Mod}(y_{i+1} \rightarrow x_{i+1}) = \text{Mod}(M \land y_{i+1} \land (y_{i+1} \rightarrow x_{i+1})) = \text{Mod}(M \land y_{i+1} \land x_{i+1}) \\
\text{Crev}(y_{i+1})(1) \cap \text{Mod}(y_{i+1} \rightarrow x_{i+1}) = \text{Mod}(M \land \neg y_{i+1} \land (y_{i+1} \rightarrow x_{i+1})) = \text{Mod}(M \land \neg y_{i+1})
\]

Since \( M = \bigwedge Y' \land \text{maxset}(F, x_1, \ldots, x_i) \) and \( F \) is a formula over variables \( \{x_1, \ldots, x_n\} \) only, \( M \land y_{i+1} \land x_{i+1} \) is consistent if and only if \( \text{maxset}(F, x_1, \ldots, x_i) \land x_{i+1} \) is consistent. Depending on this condition:

- \( \text{maxset}(F, x_1, \ldots, x_i) \land x_{i+1} \) is **consistent**: \( M \land y_{i+1} \land x_{i+1} \) is also consistent; therefore, the models of this formula are the new class zero; \( \bigwedge Y' \land \text{maxset}(F, x_1, \ldots, x_i) \land y_{i+1} \land x_{i+1} \) is the same as \( \bigwedge Y'' \land \text{maxset}(F, x_1, \ldots, x_i) \) where \( Y'' = Y' \cup \{y_{i+1}\} \) since by assumption \( \text{maxset}(F, x_1, \ldots, x_i) \land x_{i+1} \) is consistent;

- \( \text{maxset}(F, x_1, \ldots, x_i) \land x_{i+1} \) is **inconsistent**: since \( M \land y_{i+1} \land x_{i+1} \) is inconsistent, the first non-empty of the two sets above is \( \text{Mod}(M \land \neg y_{i+1}) \); replacing \( M \) with its definition, \( M \land y_{i+1} \) becomes \( \bigwedge Y' \land \text{maxset}(F, x_1, \ldots, x_i) \land \neg y_{i+1} \) and this is equal to \( \bigwedge Y'' \land \text{maxset}(F, x_1, \ldots, x_i, x_{i+1}) \) where \( Y'' = Y' \cup \{\neg y_i\} \). Indeed, \( \text{maxset}(F, x_1, \ldots, x_i, x_{i+1}) = \text{maxset}(F, x_1, \ldots, x_i) \) since by assumption \( \text{maxset}(F, x_1, \ldots, x_i) \land x_{i+1} \) is inconsistent.
5.3 Very radical revision

Very radical revision is neither lexicographic-finding nor bottom-refining. It is indeed easier, as the classes of $\emptyset \text{rad}(R_1) \ldots \text{rad}(R_n)$ are relatively easy to determine.

**Theorem 17** For every formulae $R_1, \ldots, R_n$, the total preorder $\emptyset \text{rad}(R_1) \ldots \text{rad}(R_n)$ is equivalent to the following preorder $C$.

\[
\begin{align*}
C(0) &= \text{Mod}(\neg \bot \land R_1 \land R_2 \land R_3 \land \cdots \land R_n) \\
C(1) &= \text{Mod}(\neg R_1 \land R_2 \land R_3 \land \cdots \land R_n) \\
C(2) &= \text{Mod}(\neg R_2 \land R_3 \land \cdots \land R_n) \\
\vdots \\
C(n-1) &= \text{Mod}(\neg R_{n-1} \land R_n) \\
C(n) &= \text{Mod}(\neg R_n)
\end{align*}
\]

**Proof.** The proof is by induction on the length of the sequence. For $n = 1$, the total preorder $C = \emptyset \text{rad}(R_1)$ splits the single class $\emptyset(0) = \text{Mod}(\top)$ into the two classes $C(0) = \text{Mod}(\top) \cap \text{Mod}(R_1) = \text{Mod}(\neg \bot \land R_1)$ and $C(1) = \text{Mod}(\top) \setminus \text{Mod}(R_1) = \text{Mod}(\neg R_1)$. The claim therefore holds.

Assuming that the claim holds for the preorder $C = [C(0), \ldots, C(n-1)] = \emptyset \text{rad}(R_1) \ldots \text{rad}(R_{n-1})$, it is proved for $C \text{rad}(R_n)$. From the definition of rad:

\[
[C(0), \ldots, C(m)]\text{rad}(R_n) = \\
= [C(0) \cap \text{Mod}(R_n), \ldots, C(n-1) \cap \text{Mod}(R_n), (C(0) \cup \cdots \cup C(n-1)) \setminus \text{Mod}(R_n)] \\
= [\text{Mod}(\neg \bot \land R_1 \land \cdots \land R_{n-1}) \cap \text{Mod}(R_n), \\
\quad \ldots, \text{Mod}(\neg R_{n-1} \land R_n) \cap \text{Mod}(R_n)] \\
= [\text{Mod}(\neg \bot \land R_1 \land \cdots \land R_{n-1} \land R_n), \ldots, \text{Mod}(\neg R_{n-1} \land R_n), \text{Mod}(\neg R_n)]
\]

The second equality holds because by definition $C$ includes all models; therefore, $C(0) \cup \cdots \cup C(n) = \text{Mod}(\top)$. The third holds because $\text{Mod}(\top) \setminus \text{Mod}(R_n)$ is the set of all models but the ones of $R_n$. \qed

This theorem tells how to determine $\min(\emptyset \text{rad}(R_1) \ldots \text{rad}(R_n), \top)$: by conjoining $R_n$ with $R_{n-1}$, then with $R_{n-2}$ and so on until consistent.

**Definition 12** The longest consistent conjunction of a sequence of formulae $\text{longest}(L_1, \ldots, L_n)$ is $L_1 \land \cdots \land L_i$ such that either $i = n$ or $L_1 \land \cdots \land L_i \land L_{i+1}$ is inconsistent.

A longest sequence is a simplified form of maxset: the maxsets conjoin formulae in order skipping every one that would create an inconsistency; the longest sequences stop altogether at the first. A sequence of very radical revisions from the empty ordering can be reformulated in terms of this definition.
Theorem 18 For every formulae $R_1, \ldots, R_n$, formula $\text{Form}(\min(\forall \text{rad}(R_1) \ldots \text{rad}(R_n), \top))$ is equivalent to $\text{longest}(R_n, \ldots, R_1)$.

Proof. The previous theorem shows that the classes of $\forall \text{rad}(R_1) \ldots \text{rad}(R_n)$ are the models of the following formulae:

\[-\bot \land R_1 \land R_2 \land R_3 \land \cdots \land R_n\]
\[-R_1 \land R_2 \land R_3 \land \cdots \land R_n\]
\[-R_2 \land R_3 \land \cdots \land R_n\]
\[\vdots\]
\[-R_{n-1} \land R_n\]
\[-R_n\]

As a result, $\min(\forall \text{rad}(R_1) \ldots \text{rad}(R_n), \top)$ is the set of models of the first consistent formula in the list. This formula may be the first or any of the others. The first is $\neg \bot \land R_1 \land R_2 \land R_3 \land \cdots \land R_n$, which is equivalent to $R_1 \land \cdots \land R_n$. If it is consistent, then $R_n \land \cdots \land R_i$ is consistent for $i = 1$, and is therefore the same as $\text{longest}(R_n, \ldots, R_1)$.

The other case is that the first consistent formula of the list is $\neg R_{i-1} \land R_i \land \cdots \land R_n$ for some index $i$. Since it is consistent, its subformula $R_i \land \cdots \land R_n$ is consistent too. This is the first part of the definition of the longest consistent conjunction, the second being the inconsistency of $R_{i-1} \land R_i \land \cdots \land R_n$.

To the contrary, let $M$ be a model of $R_{i-1} \land R_i \land \cdots \land R_n$. If $M$ satisfies all formulae $R_1, \ldots, R_{i-2}$ then $R_1 \land \cdots \land R_n \land \neg \bot$ is consistent, contrary to assumption. Therefore, $M$ falsifies some formula among $R_1, \ldots, R_{i-2}$. Let $j \leq i - 2$ be the highest index such that $M$ falsifies $R_j$. Since $M$ falsifies this formula, it satisfies its negation $\neg R_j$. Because of the highest index, $M$ satisfies all formulae $R_{j+1}, \ldots, R_{i-2}$ if any. As a result, $M$ satisfies $\neg R_j \land R_{j+1} \land \cdots \land R_{i-2}$. Since it also satisfies $R_{i-1} \land R_i \land \cdots \land R_n$, it satisfies $\neg R_j \land R_{j+1} \land \cdots \land R_1$. The consistency of this sequence with $j \leq i - 2$ contradicts the assumption that $i$ is the lowest index such that $\neg R_{i-1} \land R_i \land \cdots \land R_1$ is consistent.

This proves that $R_{i-1} \land R_i \land \cdots \land R_n$ is inconsistent. Since $R_i \land \cdots \land R_n$ is consistent, this is longest($R_n, \ldots, R_1$). \qed

By this theorem, the complexity of $\text{longest}(L_1, \ldots, L_n) \models Q$ is the same as inference from a sequence of very radical revisions from an empty sequence. This problem is investigated under the condition that each formula $L_i$ is consistent.

Theorem 19 For every sequence of consistent formulae $L_1, \ldots, L_n$ and formula $Q$, checking whether $\text{longest}(L_1, \ldots, L_n) \models Q$ is $\Delta^p_2[\log n]$-complete, and $\text{BH}_{2n-1}$-complete if $n$ is a constant.

Proof. Entailment $\text{longest}(L_1, \ldots, L_n) \models Q$ holds if $L_1 \land \cdots \land L_i$ is consistent and entails $Q$ for some $i \in \{1, \ldots, n\}$. The check for the inconsistency of $L_1 \land \cdots \land L_i \land L_{i+1}$ is not necessary: if it does not hold, then $L_{i+1}$ is added to the conjunction $L_1 \land \cdots \land L_i \models Q$, and the result still entails $Q$. 

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These consistency and entailment tests can be done in parallel; if they succeed for the same index $i$, then $\text{longest}(L_1, \ldots, L_n)$ entails $Q$. The problem is therefore in $\Delta_2^p[\log n]$. If $n$ is a constant, an exact computation of the tests to be performed is needed; these are:

1. $\models Q$; or
2. $L_1$ is consistent and $L_1 \models Q$; or
3. $L_1 \land L_2$ is consistent and $L_1 \land L_2 \models Q$; or

$\vdots$

$n + 1$ $L_1 \land \cdots \land L_n$ is consistent and entails $Q$.

Under the assumption that every single $L_i$ is consistent, the first two ones can be simplified. Indeed, the second one “$L_1$ is consistent and $L_1 \models Q$” is the same as $L_1 \models Q$. This condition is entailed by $\models Q$, which becomes unnecessary. The conditions can therefore be rewritten as:

1. $L_1 \models Q$; or
2. $L_1 \land L_2$ is consistent and $L_1 \land L_2 \models Q$; or

$\vdots$

$n$ $L_1 \land \cdots \land L_n$ is consistent and entails $Q$.

The first test is in coNP, the other $n - 1$ ones are in $D^p$. By the definition of the Boolean hierarchy [70], the problem is in BH$_{2n-1}$.

Hardness for $\Delta_2^p[\log n]$ and BH$_{2n-1}$ is proved for unbounded and constant $n$ by a reduction from the following problem: given $F_1, \ldots, F_{2n-1}$ with $F_i$ consistent implying $F_{i-1}$ consistent, check whether the number of consistent $F_i$ is even [22]. This problem is reduced to checking longest$(L_1, \ldots, L_n) \models Q$. This implies that entailment from the longest sequence is both $\Delta_2^p[\log n]$-hard in the general case and BH$_{2n-1}$-hard if $n$ is constant.

The first step of the reduction is rewriting of each formula $F_i$ on a private alphabet. This is assumed already done, and does not change consistency.

New variables are introduced, one for each formula: $\{y_1, \ldots, y_m\}$. The query $Q$ is $\neg y_1 \lor \neg y_3 \lor \cdots$, the disjunction of all literals $\neg y_i$ with $i$ odd.

The first formula $L_1$ is $y_1 \rightarrow F_1$, which is consistent because it is satisfied by setting $y_1$ to false. If $F_1$ is inconsistent, $\neg y_1$ is entailed. This is correct since the number of consistent formulae is zero, which is even.
The second formula $L_2$ contains $y_1$ and an unrelated variable. The conjunction of the first two formulae contains $(y_1 \rightarrow F_1) \land y_1 \equiv F_1 \land y_1$. If $F_1$ is consistent, this formula does not entail $\neg y_1$ and is consistent. The construction of the longest consistent sequence continues.

Overall, if $F_1$ is inconsistent then $\neg y_1$ is entailed. Otherwise, $\neg y_1$ is not entailed and the construction of the longest consistent sequence continues.

The rest of the sequence works similarly: the construction stops at the first inconsistent formula $F_i$; if $i$ is odd then $\neg y_i$ is entailed, which is correct since the number of consistent formulae is $i - 1$, even.

The other formulae are as follows, where $i$ is odd.

Assuming that the construction of the longest consistent sequence includes $L_{i-1}$, it proceeds as follows.

Three cases are possible: $F_{i-1}$ is inconsistent; it is consistent but $F_i$ is not; they are both consistent.

If $F_{i-1}$ is inconsistent, $L_i$ is not added to the sequence because $L_{i-1} \land L_i$ contains $y_{i-1} \land (y_{i-1} \rightarrow F_i)$, which implies the inconsistent formula $F_{i-1}$. Therefore, the longest consistent sequence does not contain $y_i$, and it therefore does not entail $\neg y_i$. This is correct since the number of consistent formulae is $i - 2$, odd.

If $F_{i-1}$ is consistent, then $L_i$ is added to the sequence because the final part of its conjunction is $y_{i-1} \land (y_{i-1} \rightarrow F_{i-1}) \land (y_i \rightarrow F_i)$, which is equivalent to $y_{i-1} \land F_{i-1} \land (y_i \rightarrow F_i)$ and is therefore consistent. If $F_i$ is inconsistent this formula entails $\neg y_i$, which is correct because the number of consistent formulae is $i - 1$, even.

The final case is that $F_i$ is consistent. Not only $y_{i-1} \land F_{i-1} \land (y_i \rightarrow F_i)$ is consistent, but is also consistent with $L_{i+1}$, which is $y_i$ plus an unrelated variable. Therefore, the construction of the longest consistent sequence continues.

Technically, the formulae $L_i, L_{i+1}$ and $Q$ are as follows, where $i$ is odd:
\[ L_1 = y_1 \rightarrow F_1 \]
\[ L_2 = y_1 \land y_2 \]
\[ \vdots \]
\[ L_i = (y_{i-1} \rightarrow F_{i-1}) \land (y_i \rightarrow F_i) \]
\[ L_{i+1} = y_i \land y_{i+1} \]
\[ Q = \neg y_1 \lor \neg y_3 \lor \cdots \]

Each formula is consistent by itself: the formulae of odd index are satisfied by the model that assigns false to all variables, the formulae of even index by that assigning true to all variables.

The claim is that \( \text{longest}(L_1, \ldots, L_n) \) entails \( Q \) if and only if the number of consistent formulae \( F_i \) is even.

The entailment \( \text{longest}(L_1, \ldots, L_m) \models Q \) simplifies because \( \text{longest}(L_1, \ldots, L_m) \) is a conjunction of formulae \( L_i \), where the variables \( y_i \) occur in separate subformulae. This means that \( \text{longest}(L_1, \ldots, L_m) \) entails \( Q \) if and only if it entails some variables \( \neg y_i \) with \( i \) odd.

The number of consistent formulae \( F_i \) is even is the same as the consistency of \( F_1 \land \cdots \land F_{i-1} \) and the inconsistency of \( F_1 \land \cdots \land F_{i-1} \land F_i \) with \( i \) odd because of the separation of the variables and the consistency of all formulae preceding a consistent one.

The conjunction \( L_1 \land \cdots \land L_i \) with \( i \) odd is \( y_1 \land (y_1 \rightarrow F_1) \land \cdots \land y_{i-1} \land (y_{i-1} \rightarrow F_{i-1}) \land (y_i \rightarrow F_i) \). This formula is equivalent to \( y_1 \land F_1 \land \cdots \land y_{i-1} \land F_{i-1} \land (y_i \rightarrow F_i) \).

The conjunction with \( i \) even is obtained by taking \( L_1 \land \cdots \land L_{i+1} \) in the definition above and decreasing \( i \) by 1. It is \( y_1 \land (y_1 \rightarrow F_1) \land \cdots \land y_{i-1} \land (y_{i-1} \rightarrow F_{i-1}) \land y_i \), which is equivalent to \( y_1 \land F_1 \land \cdots \land y_{i-1} \land F_{i-1} \land y_i \).

The claim can now be proved in each direction.

If \( F_1 \land \cdots \land F_{i-1} \) is consistent and \( F_1 \land \cdots \land F_{i-1} \land F_i \) is not with \( i \) odd then \( L_1 \land \cdots \land L_i = y_1 \land F_1 \land \cdots \land y_{i-1} \land F_{i-1} \land (y_i \rightarrow F_i) \) is consistent and entails \( \neg y_i \). The longest consistent conjunction contains it and therefore entails \( \neg y_i \) as well.

To prove the other direction, the longest consistent conjunction \( L_1 \land \cdots \land L_i \) is assumed to entail a literal \( \neg y_j \). Since this conjunction is consistent, it does not contain \( y_j \). Since it contains all variables \( y_j \) with \( j < i \) regardless of whether \( i \) is even or odd, the only possible \( j \) is \( i \). If \( i \) is even then \( L_1 \land \cdots \land L_i \) contains \( y_i \). Therefore, \( i \) is odd. \( \square \)

Since a sequence of very radical revisions from the empty ordering is exactly the same as the longest consistent conjunction, and all formulae are consistent by assumption, the problem of entailment for very radical revision is \( \Delta_2^p[\log n] \)-complete in the general case and \( \text{BH}_{2n-1} \)-complete for constant \( n \).

### 5.4 Plain severe revision and full meet revision

On total preorders comprising at most two classes, plain severe and full meet revision coincide, and always generate an ordering of at most two classes. As a result, when the initial ordering is empty, sequence of plain severe and full meet revision coincide:
\[ \emptyset \text{psev}(P_1) \ldots \text{psev}(P_n) \equiv \emptyset \text{full}(P_1) \ldots \text{full}(P_n) \]

More generally, mixed sequences of plain severe and full meet revisions applied to an ordering comprising at most two classes are equivalent to sequence of full meet revisions only and to sequences of plain severe revisions only.

These two revisions are neither lexicographic-finding nor bottom-refining. A lexicographic-finding sequence of revisions \( x_1 \land x_3, \neg x_1, x_2 \) produces \( \neg x_1 \land x_2 \land x_3 \), but the same sequence of full meet revisions instead produces to \( \neg x_1 \land x_2 \). A sequence of bottom-refining revisions \( x_1, x_2 \) produces an ordering with a class one equal to \( x_1 \land \neg x_2 \), but the same sequence of full meet revisions instead gives \( \neg x_1 \lor \neg x_2 \). They are also different from very radical revision, as seen from the sequence of revisions \( x_1, x_2, \neg x_1 \), where very radical revision produces \( x_2 \land \neg x_1 \) while full meet produces \( \neg x_1 \).

**Theorem 20** Inference from a sequence of full meet and plain severe revisions applied to the empty ordering is \( \Delta^p_2[\log n] \)-complete.

**Proof.** The \( \Delta^p_2[\log n] \) class includes all problems that can be solved by a polynomial number of nonadaptive calls to an NP-oracle. Nonadaptive means that no call depends on the others. Equivalently, these calls are in parallel [35, 11].

A sequence of full meet revisions \( \emptyset \text{full}(S_1) \ldots \text{full}(S_n) \) requires establishing the satisfiability of the following quadratic number of formulae.

\[
S_1 \\
S_1 \land S_2 \\
\vdots \\
S_1 \land S_2 \land \cdots \land S_n \\
S_2 \\
S_2 \land S_3 \\
\vdots \\
S_2 \land S_3 \land \cdots \land S_n \\
\vdots \\
S_n
\]

The first group of formulae starts with \( S_1 \) and adds a formula at time until the last. The second starts with \( S_2 \) and does the same. This is repeated for all subsequent formulae \( S_3, \ldots, S_n \). All these conjunctions are checked for satisfiability regardless of the satisfiability of the others.

Given the result of these tests, the result of full meet revision is calculated in polynomial time. First, the longest continuous conjunction \( F_1 \land \cdots \land F_i \) is determined. If \( i = n \), it is the final result. Otherwise, since \( i < n \) then \( i + 1 \) is less than or equal to \( n \). Therefore, \( F_{i+1} \) is a formula of the sequence. The longest continuous conjunction
$F_{i+1} \land \cdots \land F_j$ is again determined. If $j = n$, it is the final result. Otherwise, the process continues with $F_{j+1}$. This is repeated until $F_n$ is in the conjunction.

Hardness was announced for full meet revision in a previous article, but without proof [47]. The proof provided here is by reduction from the problem of deciding $\text{longest}(L_1, \ldots, L_n) \models Q$. Given the consistent formulae $L_1, \ldots, L_n$, the reduction builds the following sequence.

\begin{align*}
F_n &= a_n \rightarrow (L_1 \land \cdots \land L_n) \\
F_{n-1} &= a_n \land (a_{n-1} \rightarrow (L_1 \land \cdots \land L_{n-1})) \\
F_{n-2} &= a_{n-1} \land (a_{n-2} \rightarrow (L_1 \land \cdots \land L_{n-2})) \\
&\vdots \\
F_2 &= a_3 \land (a_2 \rightarrow (L_1 \land L_2)) \\
F_1 &= a_2 \land L_1
\end{align*}

The sequence of revisions $F_n, \ldots, F_1$ applied to the empty order entails $Q$ if and only if $\text{longest}(L_1, \ldots, L_n)$ does. This is the case because an inconsistent conjunction $L_1 \land \cdots \land L_i$ makes $F_i$ inconsistent with the following formulae $F_{i+1}$, which therefore takes its place. Otherwise, they are conjoined and the process continues.

Technically, if $L_1 \land \cdots \land L_i$ is the longest consistent conjunction then $L_1 \land \cdots \land L_{i+1}$ is inconsistent. Since $F_{i+1}$ contains $a_{i+1} \rightarrow (L_1 \land \cdots \land L_{i+1})$, it is inconsistent with $a_{i+1}$, which is contained in $F_i$. As a result, full meet revision produces $F_i$. The subsequent formulae $F_{i-1}, \ldots, F_1$ are consistent with $F_i$. Indeed, $F_i \land F_{i+1}$ implies $L_1 \land \cdots \land L_i$, which is consistent and entails all implications $a_j \rightarrow (L_1 \land \cdots \land L_j)$ in the following formulae. What remains after their removal is only a number of positive literals $a_j$, which are therefore consistent.

This proves that the result of the sequence of revision is $F_i \land \cdots \land F_1$, which is equivalent to the longest consistent conjunction $L_1 \land \cdots \land L_i$ apart from some unrelated variables $a_j$. Entailment of $Q$ is therefore the same.

\[\square\]

6 Conclusions

This article advocates and studies mixed sequences of belief change operators, in which revisions, refinements and withdrawals may occur. With some exceptions [46, 42, 27, 17, 36, 6], the semantics for iterated belief revision mostly work on objects that are equivalent to total preorders, which lets using different kinds of changes at different times. Even the memoryless operators such as full meet revision [1] and the distance-based revision [15, 57] can be embedded in this framework: they produce a plausibility order which does not depend at all on the previous one except for their zero class.

The main technical result of this article is a method for computing the result of a mixed sequence of revisions. It directly works on sequences of lexicographic revisions, refinements and severe antiwithdrawals, which may result from translating an arbitrary sequence of lexicographic revisions, refinements, severe withdrawal, natural, severe, plain severe, moderate severe and very radical revisions, alternating in every possible way. 41
The requirement of being able to solve propositional satisfiability problems is not too demanding, given the current efficiency of SAT algorithms \cite{4, 2} and given that belief revision cannot be easier than its underlying logical language \cite{19, 50}. The polynomial running time (not counting the satisfiability tests) implies that the required amount of memory is also polynomial, as well as the resulting knowledge bases at each step. This was not obvious, as some belief change operators may produce orderings comprising an exponential number of classes, which forbids storing them explicitly in practice.

**Example 9** The running example can be solved by an explicit representation of the pre-orders.

\[
\emptyset \text{lex}(y) \text{nat}(\neg x) \text{ref}(x \land z) \text{rad}(\neg z)
\]

The initial preorder is empty: \( \emptyset = [\text{Mod}(\top)] \). The revisions change it as follows, where Mod() is omitted from the classes for simplicity.

\[
\begin{array}{cccccc}
\emptyset & \text{lex}(y) & \text{nat}(\neg x) & \text{res}(x \land z) & \text{rad}(\neg z) \\
T & y & \neg x \land y & x \land y & x \land y \land z & \neg x \land y \land z \\
 & \neg y & x \land y & \neg x \land y & \neg x \land y \land z & \neg y \land \neg z \\
 & & \neg y \land x \land z & \neg y \land (\neg x \lor \neg z) & &
\end{array}
\]

The final result is the same as obtained by the algorithm: the base of the last preorder is \( \neg x \land y \land \neg z \). However, explicitly storing the preorder means representing all its classes, which in this example increased in number up to five. In general, with \( n \) variables there may be as many as \( 2^n \) models, and therefore as many as \( 2^n \) nonempty classes. In this case, the bound \( 2^3 = 8 \) was almost reached after \( \text{res}(x \land z) \).

A side result is that the resulting knowledge base only takes polynomial space since it is generated by an algorithm that works in polynomial space. This is not the case for several one-step revisions \cite{12}. A stricter characterization can also be given: if none of the original operators is of a kind that is translated using severe antiwithdrawal, the result is the conjunction of some formulae in the sequence. Otherwise, it may also contain an underformula, which by definition may include disjunctions. The ability of generating results that contain both conjunctions and disjunction can be seen as informal evidence that mixed sequences of revisions have superior expressive power \cite{37} than sequences of a single kind of revisions.

Other articles explored the translations from different belief change operators into a single formalism. Rott has shown that severe withdrawal, irrevocable and irrefutable revision can be expressed in terms of revision by comparison \cite{59}, natural and lexicographic in terms of bounded revision \cite{61}. Several single-step revisions can be recast in some forms of circumscription \cite{49}.

Several computational complexity results about belief revision are known. Eiter and Gottlob \cite{20} proved that most distance-based and syntax approaches are \( \Pi^p_2 \)-complete in the single-step case. In a further article \cite{21}, the same authors proved (among other results) that the same applies to positive right-nested counterfactuals, which are equivalent to a form of iterated revision. Nebel \cite{54} proved a number of results, the most
relevant to the present article being the that one-step syntactic-lexicographic revision is $\Delta^p_2$-complete. This operator can encode lexicographic revision as defined in the iterated case by placing each formula in a separate priority class. Other iterated revisions have a similar degree of complexity [17].

A number of problems are left open. The algorithm requires a SAT solver, which is unavoidable given that the underlying language is propositional logic and SAT expresses its basic problems of satisfiability, mutual consistency and entailment. However, some restricted languages such as Horn and Krom require only polynomial time for checking satisfiability [66]. As a result, it makes sense to investigate their computational properties on iterated change. The analysis would not be obvious because underformulae include both disjunction and conjunction, which may result in a non-Horn and non-Krom formula. The Horn restriction has been studied in single-step revisions by Eiter and Gottlob [18], and has recently been considered as a contributor to the semantics of revision [14].

Some iterated belief change operators such as radical revision (as opposed to very radical revision, considered in this article) consider some models “inaccessible” [67, 26]. In terms of total preorders, this amounts to shifting from a partition into ordered classes into a sequence of non-overlapping subsets; the models that are not in any of them are the inaccessible one. Alternatively, the highest-level class is given the special status of inaccessible model container. These operators have not been considered in this article, but the analysis could be extended to them.

Other operators not considered in this article include the ones based on numerical rankings [68, 71, 38, 64] and bidimensional ones [13, 25, 61]. They allow for specifying the strength of a revision either by a number or indirectly by referring to that of another formula. Either way, revision is by a pair of a formula and an expression of its strength. A preliminary analysis suggests that at least a form of bidimensional change, revision by comparison, can be recast in terms of lexicographic and severe antiwithdrawal, at the cost of first determining an underformula and a maxset of the previous lexicographic revisions. Other two-place operators may be amenable to such reductions. Other recent work include iterated contraction [43, 5, 65] and operators where conditions on the result are specified, rather than mandating a mechanism for obtaining them [34].

Memoryless revision operators [15, 63] may be treated as if they had memory: this is the case of full meet revision, which is indeed oblivious to the previous history of revision. The ordering it generates is always $[\text{Mod}(K), \text{Mod}(\neg K)]$. In spite of its simplicity, it is still useful to characterize a tabula rasa step of reasoning, forgetting all previously acquired data to start over from a single simple information.

Operators with full memory [16, 12, 27, 17] require a different analysis, since they work from the complete history of revisions rather than from a total preorder that is modified at each step. The same applies to operators working from structure more complex than total preorders over models [36, 6].

Finally, given that a revision may be performed using different operators, a question is how to decide which. This is related to merging and non-prioritized revision. An answer may be to use the history of previous revision to find out the credibility of a source [18], which affects the kind of incorporation. For example, trustworthy sources produce lexicographic revisions, plausible but not very reliable sources produce natural
revisions, the others refinements. Still better, sources providing information that turned out to be valid after all subsequent changes are better treated by lexicographic revisions; source providing information that turned out to be specific to the current case are formalized by natural revision. As an alternative, every new information may be initially treated as a natural revision; if observations suggest its generality, they are promoted to lexicographic.

Example 10 (cont.) Sound of feathers. A bird, after all?
The hunter and the policeman turn their head, eager to find out. What comes out from the bushes is a drag queen in red feathers, who stopped by the thicket for the obvious reason while coming for the parade at the village fête. Not a bird (¬b) but red (r), not to be hunted anyway (¬h).

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