Inverse proximity effect in an s-wave superconductor coupled to a topological insulator

Henning G. Hugdal, Morten Amundsen, Jacob Linder, and Asle Sudbø

Center for Quantum Spintronics, Department of Physics, NTNU,
Norwegian University of Science and Technology, NO-7491 Trondheim, Norway

Using a field-theoretical approach, we study the inverse proximity effect in a bilayer consisting of a thin superconductor (S) and a topological insulator (TI) in the weak coupling limit. Integrating out the topological fermions of the TI, we find that spin-orbit coupling is induced in the S, which leads to spin-triplet p-wave correlations in the anomalous S Green’s function. Solving the self-consistency equation for the superconducting order parameter, we find that the inverse proximity effect is negligible except in a limited region of parameter space. We show, however, that by tuning the chemical potentials of the S and TI, it is in principle possible to suppress superconductivity in the interfacial layer entirely. This happens when the Fermi momenta of the TI and S are similar, leading to a large degree of hopping between the layers. For parameters relevant for conventional s-wave superconductors such as Nb and a TI such as HgTe, we find that the inverse proximity effect is negligible, as has thus far been assumed in most works studying the proximity effect in S-TI structures. In superconductors with low Fermi-energies, such as high-Tc cuprate and heavy-fermion superconductors, the situation may be drastically altered.

I. INTRODUCTION

Topological insulators are insulating in the bulk, but host metallic surface states protected by the topology of the material.1–3 For three-dimensional topological insulators, the two-dimensional (2D) surface states can be described by a massless analog to the relativistic Dirac equation, having linear dispersions and spin-momentum locking. Many interesting phenomena are predicted to occur by coupling the TI to a superconductor, thus inducing a superconducting gap in the TI.4 For instance, such systems have been predicted to host Majorana bound states,5 which could be used for topological quantum computing. Moreover, the Dirac-like Hamiltonian \( \sigma \cdot \mathbf{k} \) has consequences for the response to exchange fields, allowing the phase difference in a Josephson junction to be tuned by an in-plane magnetization to values other than 0 and \( \pi \), and inducing vortexes by an in-plane magnetic field.6–8

Numerous papers have studied the interesting phenomena that have been discovered in topological insulators with proximity-induced superconductivity.9–21 To our knowledge, however, much less attention has been paid to the inverse superconducting, or topological,22 proximity effect, i.e. the effect that the topological insulator has on the superconductor order-parameter. There have been indications that superconductivity might be suppressed.17 One recent study demonstrated that the proximity to the TI induces spin-orbit coupling in the S, possibly making a Fulde-Ferrel23 superconducting state energetically more favourable near the interface of a magnetically doped TI.23 Another study showed that the TI surface states can leak into the superconductor, resulting in a Dirac cone in the density of states.25 In this paper, we focus on the superconducting gap itself and study under what circumstances the inverse proximity effect is negligible, as is often assumed in theoretical works.

Using a field-theoretical approach, we study an atomically thin Bardeen-Cooper-Schrieffer (BCS) s-wave superconductor coupled to a TI. While this is an approximation for conventional superconductors such as Nb and Al, superconductivity has been observed in single-layer NbSe2.26 Integrating out the TI fermions, we obtain an effective action for the S electrons.

Due to the induced spin-orbit coupling, spin-triplet p-wave correlations are induced in the S. Solving the mean-field equations, we find that the inverse proximity effect is negligible, except when the system parameters are chosen such that the separation between the Fermi momenta in the TI and S is small, allowing a large degree of hopping between the layers. However, by tuning the system parameters, superconductivity can be suppressed entirely at the S-TI interface, resulting in a normal state density of states near the Fermi level. Using parameter values for Nb as S and HgTe as TI, we find that the inverse proximity effect is weak, which is in agreement with experiments.19

The remainder of the article is organized as follows: The model system is presented in Sec. II, and the effective action for the S fermions and order parameter is derived in Sec. III. In Sec. IV we derive the mean field gap equations for the order parameter in the weak coupling limit. Numerical results for the superconducting gap and density of states are presented and discussed in Sec. V, and summarized in Sec. VI. Further details on the calculation of the density of states and the Nambu space field integral are presented in the Appendix.

II. MODEL

We model the bilayer consisting of a thin s-wave superconductor (S) coupled to a TI by the Lagrangian density

\[
\mathcal{L} = \mathcal{L}_S + \mathcal{L}_{TI} + \mathcal{L}_t.
\]

(1)

In imaginary time and real space, the superconductor is described by

\[
\mathcal{L}_S = c_\uparrow^\dagger(r) \left( \partial_\tau - \frac{\nabla^2}{2m} - \mu \right) c_\uparrow(r) - gc_\uparrow^\dagger(r)c_\downarrow^\dagger(r)c_\downarrow(r)c_\uparrow(r),
\]

(2)

where \( c_\uparrow(r) = [c_\uparrow(r) \ c_\downarrow(r)]^T \) with \( c_\uparrow(r) \) denoting the annihilation operator for spin-up (spin-down) electrons, \( m \) is the electron mass, \( \mu \) is the chemical potential in the S, and \( g \) is the BCS coupling constant. In Fourier space, the coupling constant is assumed to be non-zero only when \( |k^2/2m - \mu| < \omega_D \),
where $k$ is the wavenumber, and $\omega_D$ is the Debye frequency. We will set $\hbar = 1$ throughout the paper. For the TI we use the Dirac Lagrangian
\[ \mathcal{L}_{TI} = \Psi^\dagger(r)(\partial_r - iv_F \nabla \cdot \sigma - \mu_{TI})\Psi(r), \]
where $\Psi(r) = [\psi^\dagger(r) \psi(r)]^T$ describes the TI fermions, $v_F$ is the Fermi velocity, and $\mu_{TI}$ is the chemical potential in the TI. The S and TI layers are coupled by a hopping term $t$.

In Matsubara frequency and momentum representation, the term in the S action, $\mathcal{S}_0$, in Eq. (8) leads to an induced spin-orbit coupling $\sim k \cdot \sigma$ in the S, in agreement with Ref. 24.

Performing a Hubbard-Stratonovich decoupling, the 4-fermion term in the S action can be rewritten in terms of bosonic fields $\varphi(q)$ and $\varphi^\dagger(q)$,
\[ -\frac{g}{\beta V^3} \sum_{k,k',q} c^\dagger_k(k') c^\dagger_{-k'}(-q) c_{-k}(-q) c_{k}(k) \]
\[ = -\frac{1}{\beta V} \sum_{k,q} \mathcal{C}(q) \mathcal{C}^\dagger(-q) c^\dagger_{k}(-q) + \text{h.c.}. \]
(11)

This also leads to an additional term in the total system action
\[ S^0_\varphi = \frac{\beta V}{g} \sum_q \varphi^\dagger(q)\varphi(q), \]
(12)

and a functional integration of the bosonic fields in the partition function. Note that the decoupling is performed such that the bosonic fields have units of energy.

Defining the Nambu spinor
\[ \mathcal{C}(k) = [c_{-k}(-q) c_{k'}(-k') \mathcal{C}(k') \mathcal{G}^{-1}(k,k') \mathcal{C}(k'), \]
(13)

the effective S action can be written
\[ S_{S} = \frac{1}{2\beta V} \sum_{k,k'} \mathcal{C}(k) \mathcal{G}^{-1}(k,k') \mathcal{C}(k'), \]
(14)

where
\[ \mathcal{G}^{-1}(k,k') = \left( \begin{array}{cc} G_0^{-1}(k) \delta_{k,k'} & \varphi(k) \varphi(-k')i \sigma_y \\ -\varphi^\dagger(-k+k')i \sigma_y & -[G_0^{-1}(-k)]^T \delta_{k,k'} \end{array} \right). \]
(15)

Performing the functional integration over the fermionic fields, we arrive at the effective action for the bosonic fields
\[ S_\varphi = \frac{\beta V}{g} \sum_q \varphi^\dagger(q)\varphi(q) - \frac{1}{2} \text{Tr} \ln(-\mathcal{G}^{-1}). \]
(16)

The additional factor 1/2 in front of the trace is due to the change in integration measure when changing to the Nambu spinor notation, see the appendix and e.g. Ref. 30 for details.

### IV. MEAN FIELD THEORY

Since $G_0^{-1}(\omega_n, k)$ is still inversion symmetric in the diagonal basis (see the Appendix), we assume that the order parameter is spatially homogeneous as in the regular BCS case. However, a recent study has shown that introducing in-plane magnetic fields in the TI breaks this symmetry and can make a Fulde-Ferrell order parameter energetically more favourable. Calculating the matrix $G(k)$ assuming a spatially homogeneous order parameter $\phi(q) = \delta_{q,0} \Delta$, we get
\[ G(k) = \left( \begin{array}{cc} G(k) & F(k) \\ F^\dagger(k) & -G^T(-k) \end{array} \right), \]
(17)

where to leading order in $t$
\[
G(k) = -\frac{\epsilon_k + i\omega_n}{\xi_k + \omega_n^2} - i^2 \left(\frac{\epsilon_k + i\omega_n}{\xi_k + \omega_n^2}\right)^2 \left[\epsilon_k^2 \omega_n^2 - v_F k^2 - (\omega_n + \mu_{TI})^2\right] - i^2 \left[\Delta^2 \left[\epsilon_k^2 \omega_n^2 - v_F k^2 - (\omega_n + \mu_{TI})^2\right]\right],
\]
\[
F(k) = -\frac{\Delta}{\xi_k + \omega_n^2} \left[1 + 2i^2 \left(\epsilon_k - \mu\right) \left(v_F^2 k^2 - \mu_{TI}^2 - \omega_n^2\right) \mu_{TI} - \omega_n^2 (v_F^2 k^2 + \mu_{TI}^2 + \omega_n^2)\right] + 2i^2 \left(\epsilon_k - \mu\right) (v_F^2 k^2 - \mu_{TI}^2 + \omega_n^2) - 2\omega_n^2 \mu_{TI}\right] \left(v_F k \cdot \sigma\right) i\gamma_y,
\]

with \(\epsilon_k = k^2/2m - \mu\) and \(\xi_k = \sqrt{\epsilon_k^2 + |\Delta|^2}\). As mentioned above, the proximity-induced spin-orbit coupling leads to nondiagonal terms in \(G(k)\). Moreover, \(F(k)\) now has diagonal terms \(\propto k \cdot \sigma \sigma_y\), signalling that \(p\)-wave triplet superconducting correlations are induced in the superconductor. This has been shown to be the case when the spin-degeneracy is lifted by spin-orbit coupling.\(^{31}\) A similar expression was found for the anomalous Green’s function on the TI side of an S/TI bilayer in Ref. 32.

While the above Green’s function contains information about the correlations in the superconductor, the superconducting gap must be determined self-consistently. The gap equation for \(\Delta\) is found by requiring \(\delta S_\Delta/\delta \Delta = 0\),\(^{29}\) which yields
\[
\Delta^+ = \frac{g}{2\beta V} \sum_k F^+(k)
\]
Inserting Eq. (19) and performing the sum over Matsubara frequencies, we get the gap equation to leading order in \(t\),
\[
1 = \frac{g}{2V} \sum_k \tanh(\beta\xi_k/2) + \frac{g^2}{2V} \sum_k \left\{ \frac{\beta \xi_k (\epsilon_k^2 - \mu_{TI}^2 - v_F^2 k^2 - \beta \epsilon_k \mu_{TI} (\epsilon_k^2 - \mu_{TI}^2 + v_F^2 k^2))}{2 \cosh^2(\beta\xi_k/2) \xi_k^2 (\epsilon_k^2 - v_F^2 k^2)} - \frac{v_F^2 k^2 (\epsilon_k^2 - 10 \mu_{TI}^2 \epsilon_k^2 + \mu_{TI}^2)}{\xi_k^2 (\epsilon_k^2 - v_F^2 k^2)} - \frac{v_F^2 k^2 (\epsilon_k^2 + \mu_{TI}^2 + v_F^2 k^2)}{\xi_k^2 (\epsilon_k^2 - v_F^2 k^2)} + \frac{\epsilon_k + v_F k - \mu_{TI}}{\xi_k^2 (\epsilon_k^2 - v_F^2 k^2)} \tanh(\beta\xi_k/2) / 2} + \frac{\epsilon_k + v_F k - \mu_{TI}}{\xi_k^2 (\epsilon_k^2 - v_F^2 k^2)} \tanh(\beta\xi_k/2) / 2}
\]
Setting \(t = 0\) simply yields the regular BCS gap equation, which results in a gap \(\Delta_0 = 2\omega_D e^{-1/\lambda}\),\(^{31}\) where \(\lambda = g D_0/V\) is a dimensionless coupling constant, and \(D_0\) is the density of states at the Fermi level. For \(t \neq 0\), the above equation can be expressed in terms of an energy integral over \(\epsilon_k\) using \(v_F |k| = 2mv_F^2 \sqrt{\epsilon_k + \mu}\).

\section{RESULTS AND DISCUSSION}

Using numerical values \(\mu \sim 5\text{ eV}, h\omega_D \sim 25\text{ meV}\), \(k^2/2m \sim 40\text{ meV} \cdot \text{nm}^2\), \(h\nu_F \sim 300\text{ meV} \cdot \text{nm}^{-1}\),\(^{19,35}\) and \(\lambda = 0.2\), we solve the above gap equation in Eq. (21) for different values of \(t\) and \(\mu_{TI}\) at \(T = 0\). The results in Fig. 1(a) show that the absolute value of the gap is not changed significantly due to the inverse proximity effect, except in a small region in parameter space. Both for \(\mu_{TI} > 0\) and below this region, the inverse proximity effect is small, signifying that the disappearing gap in the region where the inverse proximity effect is strong cannot be simply related to the increasing density of states in the TI.

The disappearance of the order parameter can, however, be understood from a quite simple kinematic argument: The particles that are free to interact are located in a thin shell around the Fermi level, with momenta \(k_T = \sqrt{2m\mu}\) in the S, and \(k_{TI}^2 = |\mu_{TI}|/v_F\) in the TI, stemming from the positive \(\mu_{TI} > 0\) or negative \(\mu_{TI} < 0\) chiral band. This means that there is a finite separation between the Fermi momenta of the two layers, and therefore a separation \(\delta\epsilon\) between the bands at the S Fermi momentum, as illustrated in Fig. 2. Since the coupling term, Eq. (4) is diagonal in \(k\), the particles in the S must change their energy by \(\delta\epsilon\) when hopping between the layers. This process is suppressed unless \(|\delta\epsilon| \lesssim t\). Hence we find that the inverse proximity effect is strong when
\[
\sqrt{2mv_F^2 \mu - t} \lesssim |\mu_{TI}| \lesssim \sqrt{2mv_F^2 \mu + t},
\]
which corresponds to a small separation between the respective Fermi momenta. The limits of this region is indicated in
which the gap disappears, and the system becomes non-superconducting. The figure shows that, depending on the two chemical potentials, films. This is also evident between the Fermi level and $\mu_{\text{TI}}$, Eq. (4), is diagonal in $\delta k = k_{\text{S}}^F - k_{\text{TI}}^F$ between the Fermi momenta of the two bands, and a difference $\delta \epsilon$ between the energies of the bands at the two Fermi momenta. Since the hopping between the S and TI, Eq. (4), is diagonal in $k$, hopping between the layers is only significant when $\delta \epsilon \lesssim t$.

For typical parameter values in TIs and $s$-wave superconductors, the region in parameter space where superconductivity vanishes is inaccessible, tuning $\mu_{\text{TI}}$ by several eV would place the Fermi level inside the bulk bands of the TI, where our model is no longer valid. The fact that an unrealistically high chemical potential is needed in the TI to observe the vanishing of superconductivity, might explain why the superconductor seems to be unaffected by the coupling to a TI in e.g. Ref. 19, where the Fermi level was placed inside the gap. Since conventional $s$-wave superconductors have high Fermi energies, it might not be possible to reach the parameter regions where superconductivity vanishes, unless the chemical potential in the S can be lowered significantly, or the Fermi velocity of the TI is lowered by renormalization, as was proposed in Ref. 25. However, similar effects might be present also for unconventional, high-$T_c$ superconductors, for which the Fermi energy is lower. Examples of such superconductors would be the high-$T_c$ cuprates and the heavy-fermion superconductors. 38

VI. SUMMARY

We have theoretically studied the inverse superconducting proximity effect between an $s$-wave superconductor and a topological insulator. Using a field-theoretical approach, we have found that the inverse proximity effect is weak for experimentally relevant parameter values, specifically using numerical values for Nb and HgTe. However, we find regions in parameter space where superconductivity vanishes entirely, even at $T = 0$. Using a simple kinematic argument, we give a criterion for when the inverse proximity effect cannot be neglected. Namely, the chemical potentials of the layers must be tuned such that the separation of the Fermi momenta of the S and TI is small. For conventional $s$-wave superconductors and the
where The eigenenergies of the superconductor are now given by variation of the order parameter in a superconductor with finite thickness. We also find that the spin-triplet $p$-wave superconducting correlations are induced in the S due to the proximity-induced spin-orbit coupling. These correlations are not visible in the density of states normal to the z-axis could reveal signatures of $p$-wave pairing.

Possible further work could include studying the spatial variation of the order parameter in a superconductor with finite thickness. Furthermore, adding a process which allows a change in momentum, such as electron-phonon scattering, might enhance the inverse proximity effect, as this would allow hopping between the layers which is non-diagonal in $k$. 

**ACKNOWLEDGMENTS**

J. L. and A. S. acknowledge funding from the Research Council of Norway Center of Excellence Grant Number 262633, Center for Quantum Spintronics. A. S. and H. G. H. also acknowledge funding from the Research Council of Norway Grant Number 250985. J. L. acknowledges funding from Research Council of Norway Grant No. 240806. J. L. and M. A. also acknowledge funding from the NV-faculty at the Norwegian University of Science and Technology. H. G. H. thanks F. N. Krohg for useful discussions.

**Appendix A: Calculation of density of states**

Since the density of states is given by the trace of the normal part of the Green’s function, $^{36}$

$$D(\omega) = -\frac{1}{\pi} \sum_k \text{Im tr} G(\omega_n, k) \bigg|_{i\omega_n \to \omega + i\eta}, \quad (A1)$$

where $\eta = 0^+$ is a positive infinitesimal, we can change to the basis which diagonalizes the non-superconducting normal inverse Green’s function $G^{-1}_0$. We find $G^{-1}_0(k) = P(k)G^{-1}_{d,0}(k)P^\dagger(k)$, where $G^{-1}_{d,0}(k) = \text{diag}(G^{-1}_{+0}(k), G^{-1}_{-0}(k))$, with

$$G^{-1}_{\pm,0}(k) = i\omega_n - \epsilon_k - \frac{t^2}{i\omega_n + \mu_{TI} \mp t \nu_F |k|} \quad (A2)$$

and

$$P(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & e^{-i\phi_k} \\ 1 & e^{i\phi_k} \end{pmatrix}. \quad (A3)$$

Here $\phi_k$ is the angle of $k$ relative the $k_x$ axis. $+(-)$ here denotes the Green’s function for positive (negative) chirality states. Inverting $G^{-1}_{d,0}$ we find the Green’s functions

$$G_{\pm,0}(k) = \frac{i\omega_n + t \nu_F |k| + \mu_{TI}}{|i\omega_n - \epsilon_+^\pm(k)|} \quad (A4)$$

where

$$\epsilon_\alpha^\pm(k) = \frac{1}{2} \left[ \epsilon_k + \alpha \nu_F |k| - \mu_{TI} ^\pm \right] \quad (A5)$$

with $\alpha, \gamma = \pm 1$.

We next transform the entire inverse Green’s function $\mathcal{G}$ using $\mathcal{G}^{-1}(k) = \mathcal{P}(k)\mathcal{G}^{-1}(k)\mathcal{P}^\dagger(k)$, where

$$\mathcal{P}(k) = \begin{pmatrix} P(k) & 0 \\ 0 & P^*(-k) \end{pmatrix}, \quad (A6)$$

which yields

$$\mathcal{G}^{-1}_d(k) = \begin{pmatrix} G^{-1}_{d,0}(k) & -\Delta e^{-i\phi_k} \sigma_z \\ -\Delta e^{i\phi_k} \sigma_z & G^{-1}_{d,0}(-k) \end{pmatrix}. \quad (A7)$$

Hence the full Green’s function matrix for the superconductor is

$$\mathcal{G}_d(k) = \begin{pmatrix} G_d(k) & F_d(k) \\ F_d^\dagger(k) & -G_d(-k) \end{pmatrix}, \quad (A8)$$

where we have defined the $2 \times 2$ matrices $G_d(k) = \text{diag}(G_+(k), G_-(k))$ and $F_d(k) = \text{diag}(F_+(k), F_-(k))$, and Green’s functions

$$G_\pm(k) = \frac{[i\omega_n + \epsilon_+^\pm(k)][i\omega_n + \epsilon_-^\pm(k)][i\omega_n \mp t \nu_F |k| + \mu_{TI}]}{[i\omega_n - \epsilon_+^\pm(k)][i\omega_n + \epsilon_+^\pm(k)][i\omega_n - \epsilon_-^\pm(k)][i\omega_n + \epsilon_-^\pm(k)]} \quad (A9a)$$

and

$$F_\pm(k) = \pm \frac{\Delta e^{-i\phi_k} |i\omega_n|^2 - (t \nu_F |k| - \mu_{TI})^2} {[i\omega_n - \epsilon_+^\pm(k)][i\omega_n + \epsilon_-^\pm(k)][i\omega_n - \epsilon_-^\pm(k)][i\omega_n + \epsilon_+^\pm(k)]} \quad (A9b)$$

The eigenenergies of the superconductor are now given by

$$\xi_{\alpha}^\pm(k) = \frac{1}{\sqrt{2}} \left\{ \epsilon_k^2 + (\alpha \nu_F |k| - \mu_{TI})^2 + 2t^2 + \gamma \sqrt{[\xi_k^2 - (\alpha \nu_F |k| - \mu_{TI})^2]^2 + 4t^2(\epsilon_k + \alpha \nu_F |k| - \mu_{TI})^2 + |\Delta|^2} \right\}^{1/2}. \quad (A10)$$
In terms of the diagonalized Green's function, the density of states can be written

\[
D(\omega) = -\frac{1}{\pi} \sum_{k} \text{Im}[G_+(-i\omega + \eta, k) + G_-(i\omega + \eta, k)]
\]

\[
= \sum_{k} \sum_{a} \frac{(\omega + \epsilon^+_a(k))(\omega + \epsilon^-_a(k))(\omega - \alpha v_F |k| + \mu \tau)}{4\xi^+_a(k)\xi^-_a(k)}
\times \left\{ [\xi^+_a(k) - \xi^-_a(k)]^{-1} [\delta(\omega - \xi^+_a(k)) - \delta(\omega + \xi^+_a(k)) - \delta(\omega - \xi^-_a(k)) + \delta(\omega + \xi^-_a(k))] 
- [\xi^+_a(k) + \xi^-_a(k)]^{-1} [\delta(\omega - \xi^+_a(k)) + \delta(\omega + \xi^+_a(k)) - \delta(\omega - \xi^-_a(k)) - \delta(\omega + \xi^-_a(k))] \right\},
\]

where we have used \(\text{Im}[x + iy]^{-1} = -\pi \delta(x)\). Using \(D(\epsilon) = \sum_{k} \delta(\epsilon - \epsilon_k)\) and \(|k| = \sqrt{2m(\epsilon_k + \mu)}\), we can express the above in terms of an integral over the energy \(-\mu < \epsilon < \infty\). Numerically the density of states can be solved by integrating Eq. (A11) directly, or by approximating the \(\delta\)-functions by a Gaussian distribution centered at the origin, before integrating Eq. (A12).

### Appendix B: Functional integral in Nambu spinor notation

We begin by considering the Gaussian integral over Grassmann variables,

\[
I = \left( \prod_i \int da_i \right) e^{-\frac{1}{2} \sum_{i,j} a_i M_{ij} a_j}
\]

Combining these two expressions, we get

\[
S^\text{eff}_S = -\frac{1}{\beta V} \sum_{k,k'} C^T(-k) \left( \begin{array}{cc}
\varphi(k' - k)^2 - i\sigma_y \\
\sigma_x + i\sigma_y
\end{array} \right) \varphi(k' - k) \sigma_x + i\sigma_y \right) C(k') \equiv -\frac{1}{\beta V} \sum_{k,k'} C^T(-k) A(k,k') C(k')
\]

\[
= -\frac{1}{\beta V} \sum_{k,k'} C^T(k) \left( -[G^{-1}_0(k)]^T \delta_{k,k'} - \varphi(k' - k) \sigma_x - i\sigma_y \varphi(k' - k) \right) C(-k') = -\frac{1}{\beta V} \sum_{k,k'} C^T(k)[-A(k', k)]^T C(-k').
\]

Combining these two expressions, we get

\[
S^\text{eff}_S = -\frac{1}{2\beta V} \sum_{k,k'} C^T(-k) \left( -[G^{-1}_0(-k)]^T \delta_{k,k'} - \varphi(k' - k) \sigma_x - i\sigma_y \varphi(k' - k) \right) C(k') \equiv -\frac{1}{2\beta V} \sum_{k,k'} C^T(-k) A^A(k,k') C(k')
\]

where \(A^A(k,k')\) denotes the anti-symmetric part of \(A\). Hence, the functional integral results in

\[
Z = \int Dc \, e^{-S^\text{eff}_S} = \sqrt{\det[-A^A]},
\]

where \(\text{Pf}(M - M^T)\) is the Pfaffian of the antisymmetric part of \(M\), where \(\text{Pf}(A)^2 = \det(A)\). As an example we consider only two variables, \(a_1\) and \(a_2\). In this case, terms containing \(M_{ij}\) disappear, since \(\sigma^2_2 = 0\), as do second order terms in \(M\). For the above integral we therefore get

\[
I = \int da_1 da_2 \frac{1}{2} (-a_1 M_{12} a_2 - a_2 M_{21} a_1) = \frac{M_{12} - M_{21}}{2},
\]

Here, \(M^A\) is the anti-symmetric part of \(M\).

Applying this to the problem of integrating \(\exp(-S^\text{eff}_S)\), we first write the action in terms of the Nambu spinor \(c\):

\[
S^\text{eff}_S = -\frac{1}{\beta V} \sum_{k,k'} C^T(-k) \left( \begin{array}{cc}
\varphi(k' - k)^2 - i\sigma_y \\
\sigma_x + i\sigma_y
\end{array} \right) \varphi(k' - k) \sigma_x + i\sigma_y \right) C(k') \equiv -\frac{1}{\beta V} \sum_{k,k'} C^T(-k) A(k,k') C(k')
\]

\[
= -\frac{1}{\beta V} \sum_{k,k'} C^T(k) \left( -[G^{-1}_0(k)]^T \delta_{k,k'} - \varphi(k' - k) \sigma_x - i\sigma_y \varphi(k' - k) \right) C(-k') = -\frac{1}{\beta V} \sum_{k,k'} C^T(k)[-A(k', k)]^T C(-k').
\]
where we have neglected various numerical constants. By interchanging an even number of rows, it can be shown that $2A^A(k, k') \rightarrow G^{-1}(k, k')$, and since the determinant is invariant under an even number interchanges, we find
\[
Z = e^{\frac{1}{2} \text{Tr} \ln(-v^{-1})}.
\]