Plasma-sheath effects on the Debye screening problem

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Abstract

The classical Debye-Hückel screening effect of the electrostatic field generated by isolated charged particles immersed in a plasma is reviewed. The validity of the underlying mathematical model, and particularly of the weak-field approximation, are analyzed. It is shown that the presence of the plasma sheath around test particles and the resulting effect of charge screening are essential for the description of plasmas which are strongly coupled.

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I. INTRODUCTION

In this work we intend to analyze the consistency of the traditional mathematical model for the so-called Debye screening problem (DSP) originally formulated by Debye and Hückel (DH model). In particular, we intend to prove that the 3D Poisson equation involved in the DH model does not admit physically acceptable solutions, i.e., solutions which are provided by ordinary functions and are at least continuous in the domain of existence, i.e., are so-called classical (or strong) solutions. For this purpose a modified model is proposed which takes into account the effect of local plasma sheath (i.e., the local domain near test particles where the plasma must be considered discrete). Basic consequences of the model are discussed, which concern the asymptotic properties of the solutions determined both for weakly and strongly-coupled plasmas. As an application the charge screening effect in strongly-coupled plasmas is investigated and an explicit expression of the effective charge for the asymptotic DH potential is determined.

An ubiquitous property of plasmas, either quasi-neutral or non-neutral and weakly or strongly-coupled, is the so-called Debye shielding of the electrostatic potential. This is generated by a single charged particle (test particle) of the plasma, located at position \( \mathbf{r}(t) \) and belonging to the N-body system (with \( N \gg 1 \)) of charged particles forming a plasma, or more generally by a finite collection of test particles (with positions \( \mathbf{r}_i(t), i = 1, N_t \), possibly with \( N_t \ll N \)). The test particle and the plasma are both assumed non-relativistic when referred to a suitable Galileian reference frame, i.e., defined in such a way that \( |V|/c \ll 1, |\mathbf{r}(t)|/c \ll 1, V \) and \( \mathbf{r}(t) \) being respectively the fluid velocity of the plasma and the test particle velocity \( \mathbf{r}(t) \), both to be assumed generally non-vanishing. It is well-known that this phenomenon occurs, in particular, if the plasma is assumed suitably close to kinetic equilibrium, namely if each particle species \( s \) is described by a Maxwellian kinetic distribution function \( f_{Ms} \), carrying a finite number density \( N_s \), temperature \( T_s \) and flow velocity \( \mathbf{V} \). Here, \( T_s, \mathbf{V} \) and \( N_s \) (the latter only in the absence of test particles) must result constant or at least, in a suitable sense slowly dependent with respect to position \( (\mathbf{r}) \) and time \( (t) \). As a consequence, sufficiently far from each test particle, in a sense to be defined, the Coulomb electrostatic potential results effectively screened and decays exponentially on a characteristic scale specified by the Debye length (\( \lambda_D \); see below). Of course, this phenomenology applies only provided local perturbations of the kinetic distribution
function are suitably small. In such a case, due to spherical symmetry of the test particle, the screened potential must result manifestly function only of the distance $\rho = |\mathbf{r} - \mathbf{r}(t)|$ between the point of measurement $\mathbf{r}$ and the position of the test particle $\mathbf{r}(t)$. In addition, if both the test particle and the plasma are non-relativistic, the potential must be also independent of the velocity of the test particle $\mathbf{r}(t)$. This is the subject of the so-called Debye screening problem, which regards the determination of the electrostatic potential generated by test particles in a quasi-neutral plasma. The mathematical model of DSP in its original formulation \[1\] (recalled in detail below, see Sec.2) concerns point-like test particles and is based both on the neglect of the local plasma sheath around each test particle (or the finite size of the test particles, for example in the case of large-size dust particles) and on the so-called weak-field approximation. The first assumption, concerns the neglect of the screening effect produced by the presence of discrete interactions close to the test particles (local plasma sheath). This set is generally species-dependent and is defined as the spherical domain $R_{\{\rho_{os}\}} = \{\rho : \rho_p \leq \rho < \rho_{os}, \rho \equiv |\mathbf{r} - \mathbf{r}(t)| \in \mathbb{R}^+\}$, $\rho_p$ being the radius of a spherically-symmetric test particle and $\rho_{os}$ the characteristic radius of the plasma sheath produced by the $s$–th particle species. The characteristic radius $\rho_{os}$ can manifestly be identified with the mean interparticle distance in the plasma for the species $s$, i.e., $\rho_{os} = \left(\frac{3}{4\pi N_{os}}\right)^{1/3}$, $N_{os}$ denoting the $s$–th species plasma number density in the absence of test particles. This set is centered at the position of the test particle, $\mathbf{r}(t)$, in which the species $s$ of the plasma must be treated as a discrete system; instead, for any distance $\rho \geq \rho_{os}$ the same species is assumed to be described in terms of a continuous kinetic distribution function. The second approximation, which permits the linearization of the Poisson equation for the electrostatic potential, is based on the requirement that, for arbitrary $\rho \equiv |\mathbf{r} - \mathbf{r}(t)| \in \mathbb{R}^+$, the following asymptotic ordering holds

\[ \hat{\Phi}(\rho) \sim O(\varepsilon) \ll 1, \]  

$\varepsilon$ being an infinitesimal and $\hat{\Phi} = |e\Phi| / T_e$ the normalized electrostatic potential and $T_e$ the electron temperature measured in energy units. As a result, Debye and Hückel obtained, for the electrostatic potential generated by a point-particle carrying the electric charge $q$, the asymptotic solution

\[ \Phi(\rho) \cong \Phi_o(\rho) \equiv \frac{\hat{c}}{4\pi |\mathbf{r} - \mathbf{r}(t)|} e^{-|\mathbf{r} - \mathbf{r}(t)| / \lambda_D} \]
which is the so-called DH potential, also known as the Yukawa potential. Here $\hat{c}$ is the DH effective charge to be approximated

$$\hat{c} \cong q,$$

and $\lambda_D$ is the Debye length

$$\frac{1}{\lambda_D^2} = \sum_j \frac{1}{\lambda_{Dj}^2},$$

where the summation is carried out over all plasma species $j$. Furthermore, $\lambda_{Dj}^{-1} = \sqrt{\frac{4\pi Z_j^2 e^2 N_{oj}}{T_j}}$ is the $j$-th species Debye length and $N_{oj}$ the $j$-th species plasma density defined in the absence of test particles (i.e., letting $\hat{\Phi}(\rho) \equiv 0$).

Despite previous attempts to construct approximate or exact solutions to the DH model \[8, 9, 10\], the related mathematical model appears incomplete and can be shown to be physically unacceptable, due to the neglect of the local plasma sheath. In fact, it is obvious that sufficiently close to the point-particle the weak-field approximation (1) is violated making the DH model invalid. In the past \[3\] it was pointed out that in such a case the test particle does not produce any electric field, but only complete charge neutralization by the plasma, thus producing a Debye length which effectively vanishes. Other objections concerned the asserted indeterminacy of the solution for $x = 0$ due to its divergence in the same point \[6\]. These issues were later addressed in a more general context \[4\], including the 2D case where complete neutralization cannot be achieved. To recover the correct physical picture the effect of local plasma sheath must be included. Nevertheless, for suitably dense plasmas or in the case of plasma species characterized by very high electric charges (high-Z), such as dusty plasmas, the weak-field approximation may be locally violated. This circumstance, when the effect of finite local plasma sheath is included, occurs if the normalized electrostatic potential $\hat{\Phi}(\rho)$ results of order unit or larger on the boundary of the plasma sheath (produced by at least one of the $s$ plasma species), namely for $\rho = \rho_{os}$. Such a condition can be expressed in terms of the Coulomb coupling parameter $\Gamma_{ks}$, i.e., by the requirement that for a test particle of species $k$ there results at least for a species $s$

$$\Gamma_{ks} \equiv \frac{\beta_k}{x_{os}} = \frac{Z_k e^2}{T_e \rho_{os}} \sim 1$$

(strong coupling condition). Here the notation is standard. Thus, $\beta_k = \frac{Z_k e^2}{T_e \lambda_D}$, $\rho_{os} = \left(\frac{3}{4\pi N_{os}}\right)^{1/3}$ and $x_{os} = \rho_{os}/\lambda_D$, denote respectively the dimensionless electric charge carried
by a test particle of species $k$ and the radius of the $s-th$ species plasma sheath surrounding
the test particle. We stress that, ordinarily, in plasmas the asymptotic condition $x_{os} \ll 1$ has
also to be imposed on each particle species $s$. For example, in a space dusty plasma typical
values for dust grain charges, plasma temperature and plasma density are $Z_d \sim 10^4 \div 10^5$,
$T \sim 1 \text{ eV}, N_o \sim 10^9 \text{ cm}^{-3}$. In this case the Coulomb parameter for a negatively-charged dust
particle in the presence of the plasma sheath produced by hydrogen ions may result typically
$\Gamma_{di} \simeq 0.91 \times 10^{-6} \frac{Z_d (N_o \text{ cm}^{-3})^{1/3}}{(T \text{ eV})^{1/3}} \simeq 4\pi \times (15 \div 150)$, while the dimensionless radius of the ion
plasma sheath can be estimated $x_{oi} \simeq 0.83 \times 10^{-3} \frac{(N_o \text{ cm}^{-3})^{1/6}}{\sqrt{T \text{ eV}}} \simeq 0.03$. However, in principle,
in a dusty plasma the plasma Debye length can also become significantly smaller than in the
 corresponding ”clean” plasma (i.e., before injection of the dusty species), which may influence,
i.e., increasing, the values of $\Gamma_{di}$ and $x_{oi}$. Further important aspects of DSP concern the
investigation of possible charge screening effects, namely the reduction to the effective charge
c with respect to the asymptotic value characterizing the weak-field approximation \([3]\) of
the DH potential \([2]\). These effects, produced by non-linear contributions in the Poisson
equation, while usually negligible for weakly coupled plasmas, are known to be significant in
strongly-coupled plasmas \([2]\), such as dusty plasmas \([11, 15]\). Indeed the investigation of the
effective interactions characterizing high-Z grains in plasmas has attracted interest in recent
years especially for numerical simulations (see, for example, \([12, 14, 16]\)). However, analytic
estimates of the effective charge characterizing the DH potential in strongly-coupled plasmas
are still not available.

Goal of this work is the analysis of DSP and the definition of a suitably modified math-
ematical model to take into account the effect of local plasma sheaths in quasi-neutral
plasmas. In particular, in Sec. 2 the traditional DH model is recalled, while the new model
is presented in Sec.3. In the same section the basic mathematical results are given which
concern the asymptotic properties of the solutions obtained in the limit of vanishing sheath
radii ($\rho_{os} \to 0^+$). We intend to prove that the Poisson equation characterizing the DH
model (here denoted Debye-Poisson equation) can be considered as a limit equation ob-
tained from a suitably modified formulation of DSP. The latter is obtained by including the
effect of finite local plasma sheaths (i.e., requiring $x_{os} > 0$, for all species). Basic feature of
the present approach is the representation of the Poisson equation in integral form. This
permits to analyze the asymptotic properties of the solutions of the modified problem in the
limit $x_{os} \to 0^+$. It is found, that the limit solution of the modified DSP for $x_{os} \to 0^+$ is a
distribution which vanishes identically for all $\rho > 0$ and is discontinuous in $\rho = 0$. It is found (see THM.1 in Sec.3) that in this limit the solution of the modified DSP results uniquely determined as a function of the normalized charge carried by the test particle ($\beta$) and is represented by a well-defined distribution. Precisely it follows that (see THM.1 in Sec.3)

$$\lim_{x_o \to 0^+} \sinh \hat{\Phi}_{x_o}(x) = \frac{\beta}{x^2} \delta(x),$$

while one can prove that the limit function $\lim_{x_o \to 0^+} \hat{\Phi}_{x_o}(x) \equiv \hat{\Phi}(x)$ reads (see THM.3 in Sec.4)

$$\lim_{x_o \to 0^+} \hat{\Phi}_{x_o}(x) = \lim_{x_o \to 0^+} \hat{\Phi}_{x_o}(x_o) \exp \{y^{(int)}(x, x_o)\},$$

where $y^{(int)}(x, x_o)$ is a suitable smooth real function to be defined [see Sec.4, Eq.(60)] so that there results $y(x_o, x_o) = 0$ while for $x \neq x_o$ the following limit holds

$$\lim_{x_o \to 0^+} y^{(int)}(x, x_o) = -\infty.$$ 

Hence, obviously, the limit function $\hat{\Phi}(x)$ is not a strong solution of the DP equation. This is therefore a characteristic property of the DH model. In particular, as a basic consequence, the effective charge of the DH asymptotic solution $c$ vanishes identically in such a limit and results independent of the charge of the test particle.

As a further development, we intend to investigate strongly-coupled plasmas (Sec.4 and 5), for which there results for a test particle of species $k$ (for example to be identified with a dusty grain)

$$\Gamma_{ks} \sim \frac{1}{O(\delta)} \gg 1,$$

$$x_{os} \sim O(\delta^k),$$

(\textit{strong coupling ordering}) where $k = 0, 1$ and $\delta$ is an infinitesimal. The asymptotic solutions of the modified DSP are here determined explicitly for the external and internal asymptotic solutions. The former, which are valid outside the Debye sphere (i.e., in the domain $x > 1$, or $x \gg 1$) coincides with the customary DH potential. The latter instead, which occurs inside the Debye sphere near the boundary of the local plasma sheath, describes the nonlinear effects due to the screening generated by plasma species having opposite charge with respect to that of the test particle. In addition, in Sec.4 also the asymptotic solution of the modified DSP close to the boundary of the local plasma sheath (i.e., for $\Delta x = x - x_{os}$
infinitesimal) is constructed. It is proven that the asymptotic solution can be determined uniquely, together with its internal boundary conditions (defined on the boundary of the plasma sheath, i.e., for \( x = x_{\text{os}} \)). Finally, by comparing the internal and external asymptotic solutions for a test particle in a strongly-coupled plasma, an "asymptotic" upper bound is obtained (Sec.5) for the effective charge characterizing the DH potential. As a result, it is found that in strongly-coupled plasmas the effective charge of highly-charged test particles, which characterizes the DH asymptotic solution, results strongly reduced with respect the value of the isolated test particle (charge screening effect). The result appears potentially relevant especially for strongly-coupled dusty plasmas, since the charge screening effect can dramatically reduce the DH potential, particularly for highly charged test particles.

II. THE DEBYE-HÜCKEL MODEL AND THE DSP PROBLEM

The traditional formulation of the DSP, based on the Debye-Hückel model [1], regards the test particles as point-like and having a spherically-symmetric charge distribution while ignoring the effect of local plasma sheath. This implies, from the physical standpoint, to neglect the discrete nature of the plasma. As a consequence the background plasma is treated as a continuum medium formed potentially by several particle species, each described by a Maxwellian kinetic distribution, so that its total charge density results as

\[
\rho_p = \sum_s Z_s e N_{os} \exp \left\{ \frac{-Z_s e \Phi}{2 T_s} \right\},
\]

where the constraint \( \sum_s Z_s e N_{os} = 0 \) must be imposed for quasi-neutral plasma. Here the summation is performed on all particle species of the plasma, while \( N_{os} \) and \( T_s \) (species density coefficient and temperature) are both assumed constant.

Furthermore, the charge density of the test particles is manifestly \( \rho_n = \sum_i q_i \delta(\rho - \rho_i(t)) \), where \( q_i \equiv Z_q e \) and \( \rho_i(t) \) are the electric charge and position vector of the \( i \)-th test particle and the summation is carried out on the collection of all test particles. In the remainder we shall assume, without loss of generality, for all test charges \( q_i > 0 \). Hence, in terms of dimensionless electrostatic potential \( \hat{\Phi} = \frac{4 e \Phi}{T} \), one obtains, for example, \( \rho_p = 2 e N_o \sinh \hat{\Phi} \), for a quasi-neutral electron-proton plasma. In a case of a single point charge, in terms of the Debye length \( \lambda_D = \sqrt{\frac{T}{8\pi e^2 N_o}} \) and the dimensionless distance \( x = \rho/\lambda_D \) (where \( \rho = |\mathbf{r} - \mathbf{r}(t)| \in \mathbb{C} \)), \( \hat{\Phi}(x) \) must satisfy the equation:

\[
\nabla_x^2 \hat{\Phi}(x) = -\frac{\beta \delta(x)}{x^2} + \sinh \hat{\Phi}(x), \quad (11)
\]
(DP equation), where \( \nabla_x^2 \equiv \frac{1}{x^2} \frac{d}{dx} \left( x^2 \frac{d}{dx} \right) \) (the Laplacian expressed in dimensionless variables). We require that \( \hat{\Phi}(x) \) obeys the boundary conditions, i.e.,

\[
\lim_{x \to \infty} \hat{\Phi}(x) = 0, \quad \lim_{x \to 0} x^2 \frac{d}{dx} \hat{\Phi}(x) = -\beta.
\]

Equation (11), together with (12), (13), define the so-called Debye screening problem (DSP).

We notice that, in contrast to what stated earlier [6], both boundary conditions (12), (13) can be satisfied by \( \hat{\Phi}(x) \). In fact, it is immediate to prove that in the domain \( x > 0 \), and imposing (12), (13), the DP equation can be cast in the integral form

\[
\hat{\Phi}(x) = \frac{\beta}{x} - \left[ \frac{1}{x} \int_0^x dx' x'^2 + \int_x^\infty dx' x' \right] \sinh \hat{\Phi}(x').
\] (14)

A strong solution of (11), (12), (13) [or, equivalent, of Eq.(14)] is a solution belonging to the functional class \( \hat{C}^{(2)}(\mathbb{R}^+) \equiv \{ \hat{\Phi}(x) : C^{(0)} ([0, \infty]), C^{(2)} ([0, \infty]) \} \). In the traditional approach [1] the DSP problem is solved requiring the validity for all \( x \in \mathbb{R}^+ \) of the weak electric field ordering (1) and assuming that \( \hat{\Phi}(x) = \hat{\Phi}_o(x) + O(\varepsilon) \) (weak field approximation), \( \hat{\Phi}_o(x) \) being the solution of the linearized DP equation:

\[
\nabla_x^2 \hat{\Phi}_o(x) = -\frac{\beta}{x^2} \delta(x) + \hat{\Phi}_o(x),
\] (15)

with \( \hat{\Phi}_o(x) \), by assumption, belonging to the same functional class of \( \hat{\Phi}(x) \). This equation admits the exact solution \( \hat{\Phi}_o(x) = \frac{\varepsilon}{x} e^{-x} \), i.e., Eq.(2), defined in terms of the DH effective charge \( c = \beta \). Finally, we stress that, assuming the validity of the weak electric field ordering, due to the linearity of Eq.(15) a similar result is implied also for a collection of point charge sources. In this case, in fact, the corresponding solution of (15) is just a sum of DH potentials, one associated with each source.

III. CONSISTENCY OF THE MATHEMATICAL MODEL - THE MODIFIED DSP

In this section we define the modified Debye screening problem, based on the introduction of a simple model of local plasma sheath. In the sequel we shall consider for simplicity of notation the case of a two species-plasma, formed by electrons and Hydrogen ions, having an unique plasma sheath. However, the generalization to multispecies plasma is straightforward.
Thus, we shall assume that the test particle is represented by a spherically symmetric charge of radius \( \rho_p \). For a particle in which \( \rho_p < \rho_o \) the plasma sheath is represented by the spherical shell centered at the position (center) of the test particle for which \( \rho_p \leq \rho < \rho_o \), in which the plasma charge density (except for the presence of the test particle) results negligible. In the sequel we can also let in particular \( \rho_p = 0 \) (point-like test particle) or \( \rho_p = \rho_o \) (finite-size test particle).

The customary DH model is obviously recovered letting \( \rho_p = 0 \) and taking the limit \( \rho_o \rightarrow 0 \) (or in dimensionless variables, requiring \( x_p \equiv \rho_p/\lambda_D = 0 \) and \( x_o \equiv \rho_o/\lambda_D \rightarrow 0 \)). Denoting \( \hat{\Phi}_{x_o}(x) \) the solution of the Poisson equation, here we intend to determine its asymptotic properties in the limit \( x_o \rightarrow 0^+ \), while also letting \( x_p = 0 \) (see THM.1). As a consequence and in agreement with [3, 4], in such a case it follows that the limit function

\[
\lim_{x_o \rightarrow 0^+} \hat{\Phi}_{x_o}(x) = 0.
\]

In addition, in the same set we intend to prove the identity

\[
\beta - \lim_{x_o \rightarrow 0^+} \int_{x_o}^{x} dx' x'^2 \sinh \hat{\Phi}_{x_o}(x') \hat{\Theta}(x' - x_o) = 0,
\]

which implies necessarily Eq. (16), \( \hat{\Theta}(x - x_o) \) being the weak Heaviside function

\[
\hat{\Theta}(x - x_o) = \begin{cases} 
0 & x < x_o \\
1 & x \geq x_o.
\end{cases}
\]

In detail the relevant equations valid in each subdomain for the normalized electrostatic potential \( \hat{\Phi}_{x_o}(x) \) are as follows. In the internal domain \( 0 \leq x < x_p \) the electrostatic potential is assumed constant

\[
\hat{\Phi}_{x_o}(x) = \hat{\Phi}_{x_o}(x_p).
\]

In the plasma sheath \( x_p \leq x < x_o \), \( \hat{\Phi}_{x_o}(x) \) satisfies the customary Poisson equation in the presence of the charge density produced by a finite-size spherically-symmetric charge

\[
\nabla_x^2 \hat{\Phi}_{x_o} = -\frac{\beta}{x^2} \delta(x - x_p).
\]

Finally, in the external domain \( x > x_o \) there holds the Poisson equation in the presence of the plasma charge density:

\[
\nabla_x^2 \hat{\Phi}_{x_o} = \hat{\Theta}(x - x_o) \sinh \hat{\Phi}_{x_o}.
\]
The boundary conditions analogous to (12), (13), imposed respectively at infinity and at the boundary of the plasma sheath, are specified as follows

\[ \lim_{x \to \infty} \hat{\Phi}_{x_0}(x) = 0, \]  
\[ x^2 \frac{d}{dx} \hat{\Phi}_{x_0}(x) \bigg|_{x=x_0} = -\beta. \]

We notice that, if \( x_p < x_o \) (for example, \( x_p = 0 \)), \( \hat{\Phi}_{x_0}(x) \) results by assumption at least of class \( C^1(\mathbb{R}\{x_p\}) \), where \( \mathbb{R}\{x_p\} \equiv [x_p, \infty[ \) . Here \( x_o, \beta \) are both assumed constant and strictly positive real numbers. The problem defined by (21), (20), together with the boundary conditions (22), (23), will be here denoted as modified DSP. From the physical standpoint Eqs. (21), (20) may be viewed as the Poisson equation for a spherical ideally conducting charge, or for a point particle in the presence of a plasma sheath, of radius \( r_o \) (i.e., \( x_o = r_o/\lambda_D \) in non-dimensional variables) which is in electrostatic equilibrium and is immersed in a spatially uniform quasi-neutral and Maxwellian plasma. As for the previous DP equation, it follows that, for solutions satisfying the boundary conditions (22), (23), in the domain \( x \in \mathbb{R}\{x_o\} \) Eq. (21) can be cast in the integral form

\[ \hat{\Phi}_{x_0}(x) = \frac{\beta \hat{\Theta}(x - x_0)}{x} - \left[ \frac{1}{x} \int_{x_o}^{x} dx' x'^2 + \int_{x}^{\infty} dx' x' \right] \sinh \hat{\Phi}_{x_0}(x') \hat{\Theta}(x' - x_o). \]  

In particular, thanks to continuity at \( x = x_o \) of \( \hat{\Phi}_{x_0}(x) \), one obtains the constraint

\[ \hat{\Phi}_{x_0}(x_o) = \Gamma - \int_{x_o}^{\infty} dx' x' \sinh \hat{\Phi}_{x_0}(x') \hat{\Theta}(x' - x_o), \]

with \( \Gamma \equiv \frac{\beta}{x_o} \) denoting the Coulomb coupling parameter. It is immediate to establish, the existence and uniqueness of \( \hat{\Phi}_{x_0}(x) \) in the functional class \( \hat{C}^\infty(\mathbb{R}\{x_o\}) \), together with its continuous dependence on initial data, in particular the continuity with respect to the parameter \( x_o \in \{0\} \) . Moreover, assuming that the weak-fields approximation (11) applies (this condition is manifestly fulfilled identically in the weak-coupling ordering, \( 0 < \Gamma \sim O(\varepsilon) \ll 1 \)), and is satisfied at least for \( x \gg 1 \) suitably large, it is immediate to prove that in this subset an asymptotic solution of the modified DSP is provided by the external asymptotic solution

\[ \hat{\Phi}_{x_0}(x) \approx \hat{\Phi}_{x_0}^{(ext)}(x) = \frac{c}{x} e^{-x + x_o}. \]
Here denoted as of the modified DSP and \( c = c(x_o, \Gamma) \) is the \textit{effective dimensionless charge}. Hence, \( \hat{\Phi}_{x_o}^{(ext)}(x) \) reduces formally to the previous DH potential \((2)\) when \( x \gg x_o \). In the weak-coupling ordering it follows \( c(x_o, \Gamma) = \frac{\beta}{1 + x_o} \), while for strongly-coupled plasmas a lower value is expected. Furthermore, it is obvious that the limit function \( \lim_{x_o \to 0^+} \hat{\Phi}_{x_o}(x) \) coincides with the solution of DSP, i.e.,

\[
\lim_{x_o \to 0^+} \hat{\Phi}_{x_o}(x) = \Phi(x). \quad (28)
\]

Let us now investigate the asymptotic properties of the exact solution \( \hat{\Phi}_{x_o}(x) \) in the limit \( x_o \to 0^+ \).

\textbf{A. THEOREM 1 - Asymptotic properties of} \( \hat{\Phi}_{x_o}(x) \)

For any strong solution of the modified DSP, \( \hat{\Phi}_{x_o}(x) \) obtained letting \( x_p = 0 \), the limit function \( \lim_{x_o \to 0^+} \hat{\Phi}_{x_o}(x) \) has the following properties:

1) \[
\lim_{x_o \to 0^+} \hat{\Phi}_{x_o}(x_o) = +\infty; \quad (29)
\]

2) for any \( x > 0 \) Eq.\((16)\) holds identically.

3) for any \( x > 0 \) the integral limit \((17)\) is satisfied by \( \hat{\Phi}_{x_o}(x) \). This implies that the limit function \( \hat{\Phi}(x) = \lim_{x_o \to 0^+} \hat{\Phi}_{x_o}(x) \) results such that Eq.\((6)\) is satisfied identically. Moreover, for any \( x > 0, x \in \{0\} \)

\[
\hat{\Phi}(x) = \lim_{x_o \to 0^+} \hat{\Phi}_{x_o}(x) = 0. \quad (30)
\]

4) the following limit is satisfied by the boundary value \( \hat{\Phi}_{x_o}(x_o) \)

\[
\lim_{x_o \to 0^+} x_o \hat{\Phi}_{x_o}(x_o) = 0. \quad (31)
\]

5) the limit value of the effective dimensionless charge \( c(x_o, \Gamma) \) for \( x_o \to 0^+ \), obtained keeping \( \Gamma \) finite, is

\[
\lim_{x_o \to 0^+} c(x_o, \Gamma) = 0. \quad (32)
\]

\textbf{PROOF}

1) In fact, as a consequence of the integral equation \((26)\) and the continuous dependence of \( \hat{\Phi}_{x_o}(x) \) on the initial data, it follows

\[
\lim_{x_o \to 0^+} x_o \hat{\Phi}_{x_o}(x_o) = \beta - \lim_{x_o \to 0^+} x_o \int_{x_o}^{\infty} dx' x' \sinh \hat{\Phi}_{x_o}(x') \Theta(x' - x_o), \quad (33)
\]
which implies

\[
\lim_{x_o \to 0^+} \hat{\Phi}_{x_o}(x_o) = \infty, \tag{34}
\]

\[
\lim_{x_o \to 0^+} \int_x^\infty dx' x' \sinh \hat{\Phi}_{x_o}(x') \hat{\Theta}(x' - x_o) = \infty, \tag{35}
\]

i.e., the limit function \(\lim_{x_o \to 0^+} \hat{\Phi}_{x_o}(x)\) diverges in \(x = x_o\). Therefore, due to the continuity of \(\hat{\Phi}_{x_o}(x)\) with respect to \(x \in [0, \infty[\) it follows that infinitesimally close to \(x, x_o = 0\), and when \(x, x_o\) are infinitesimal of the same order, \(\hat{\Phi}_{x_o}(x)\) must diverge logarithmically as

\[
\hat{\Phi}_{x_o}(x) \sim \ln \left\{ \frac{1}{x^3} \right\}. \tag{36}
\]

2) Let us now consider the implications of the integral equation (24) for the limit function \(\lim_{x_o \to 0^+} \hat{\Phi}_{x_o}(x)\) for arbitrary \(x \in (x_o)\). There follows

\[
\lim_{x_o \to 0^+} \hat{\Phi}_{x_o}(x) = \beta \frac{1}{x} \lim_{x_o \to 0^+} \frac{1}{x} \int_{x_o}^x dx' x'^2 \sinh \hat{\Phi}_{x_o}(x') \hat{\Theta}(x' - x_o) -
\]

\[
- \int_x^\infty dx' x' \lim_{x_o \to 0^+} \sinh \hat{\Phi}_{x_o}(x') \hat{\Theta}(x' - x_o), \tag{37}
\]

where, due to the asymptotic estimate (36), the second term on the r.h.s. necessarily diverges

\[
\lim_{x_o \to 0^+} \frac{1}{x} \int_{x_o}^x dx' x'^2 \sinh \hat{\Phi}_{x_o}(x') \hat{\Theta}(x' - x_o) = \infty \tag{38}
\]

unless there results for any \(x \neq x_o, x \in (x_o)\)

\[
\lim_{x_o \to 0^+} \hat{\Phi}_{x_o}(x) = 0. \tag{39}
\]

3) As a consequence of Eq.(39), from the integral equation (24) it follows necessarily that for all \(x > 0:\)

\[
\lim_{x_o \to 0^+} \frac{\beta \hat{\Theta}(x - x_o)}{x} = \lim_{x_o \to 0^+} \frac{1}{x} \int_{x_o}^x dx' x'^2 \sinh \hat{\Phi}_{x_o}(x') \hat{\Theta}(x' - x_o), \tag{40}
\]

which proves the limit (17). As a consequence it must result necessarily that the limit \(\lim_{x_o \to 0^+} \sinh \hat{\Phi}_{x_o}(x)\) is a Dirac delta, in accordance with Eqs. (16) and (4).

4) The limit (31) follows immediately from the boundary condition (26).

5) Finally, Eq.(39) implies manifestly the limit (32).

As an immediate consequence of THM.1 it follows that the DP equation of the DH model does not admit strong solutions.
B. THEOREM 2 - Non-existence of classical solutions of DSP

In the functional class $\hat{C}^{(2)}(0)$ the DSP problem defined by Eqs. (11), (12) and (13) has no strong solution.

PROOF

In fact, first, we notice that the limit function

$$\lim_{x_o \to 0^+} \hat{\Phi}_{x_o}(x) \equiv \hat{\Phi}(x),$$

is manifestly a solution of the DP equation which satisfies the required boundary conditions (12), (13). Other hand, due to THM.1, this solution is discontinuous in $x = x_o$ and results a distribution. Hence it is not a strong (classical) solution of the modified DSP problem.

The basic implication of THM.1 and 2 is that the DP equation (11), provided by the DH model, must be regarded as physically unacceptable, since it does not admit strong solutions. In this regard it should be noted that, as a basic principle, physically acceptable of solutions of ordinary (or partial) differential equations characterizing the classical theory of fields must be suitably smooth strong solutions. The modified Debye screening problem here defined, instead, exhibits smooth strong solutions and therefore appears, from this viewpoint, consistent.

IV. INTERNAL ASYMPTOTIC SOLUTION OF THE MODIFIED DSP

In this section we intend to construct explicitly the internal asymptotic solution of the modified DSP problem, defined by (21), (20), (22), (23), in the case of strongly-coupled plasmas, for which the asymptotic ordering defined by the inequalities (9), (10) applies. In particular, we shall consider a suitable neighborhood of the boundary of the plasma sheath $(x_o)$, characterized by an infinitesimal amplitude $\Delta x = x - x_o$ defined so that

$$0 \leq \Delta x \ll \frac{x_o w_o}{\Gamma} \sqrt{\ln w_o},$$

where by hypothesis $w_o \equiv \ln \left( \frac{2r^2}{x_o^2} \right) \gg 1$. Under such assumptions, we intend to construct approximate solutions of the modified DSP which are accurate to leading order in the infinitesimal dimensionless parameter $\delta$ and with respect to $\Delta x$. These will be denoted as internal asymptotic solution of the modified DSP. We intend to prove, as a consequence,
that in the neighborhood of $x_o$ defined by (42) the internal asymptotic solution can be uniquely determined, to leading order in the relevant asymptotic parameter. In fact, the following theorem holds.

A. THEOREM 3 - Internal asymptotic solution for strongly-coupled plasma

In validity of the asymptotic ordering of strongly-coupled plasma [see inequalities (9), (10)], requiring $x_p = 0$ (or at least $x_p \lesssim x_o$) and in the neighborhood of $x_o$ defined by the inequalities (42), the asymptotic solution of the modified DSP [(21), (20), (22), (23)] reads

$$\hat{\Phi}_{x_o}(x) \approx \hat{\Phi}^{(int)}_{x_o}(x) = \hat{\Phi}_o \exp \{ y^{(int)}(x, x_o) \}$$

(43)

where $\hat{\Phi}_o \equiv \hat{\Phi}_{x_o}(x_o)$ and

$$y^{(int)}(x, x_o) \equiv y_o(x', x_o) \equiv -\alpha (x - x_o)$$

(44)

(internal asymptotic solution), where $\alpha = \frac{r}{x_o^2 \Phi_o}$ and the integration constant $\hat{\Phi}_o$ results

$$\hat{\Phi}_o \approx \ln \left( \frac{2\Gamma^2}{x_o^2} \right).$$

(45)

PROOF

In order to prove these results let us represent $\hat{\Phi}_{x_o}(x)$ in the form

$$\hat{\Phi}_{x_o}(x) = \hat{\Phi}_o \exp \{ y(x, x_o) \},$$

(46)

It follows that in the domain $x > x_o$, $y(x, x_o)$ satisfies the ordinary differential equation

$$y'' + \left( y' \right)^2 + \frac{2}{x} y' = \frac{1}{2\Phi_o} \exp \left\{ \hat{\Phi}_o \exp \{ y \} - y(x, x_o) \right\} -$$

$$- \frac{1}{2\Phi_o} \exp \left\{ -\hat{\Phi}_o \exp \{ y \} - y(x, x_o) \right\} \equiv S(y(x, x_o)),$$

(47)

(48)

where primes denote differentiation with respect to $x$. Consistent with (22), (23), there follow the boundary conditions

$$y(x_o, x_o) = 0,$$

(49)

$$\lim_{x_o \to 0^+} y(x, x_o) = -\infty,$$

(50)
\[
\lim_{x \to \infty} y(x, x_o) = -\infty, \quad (51)
\]
\[
y'(x, x_o) \big|_{x = x_o} = -\frac{\Gamma}{x_o \Phi_o}. \quad (52)
\]
As a consequence, it is immediate to prove that \(y(x, x_o)\) for arbitrary \(x, x_o > 0 \ (x \geq x_o)\) with admits the exact integral representation
\[
y(x, x_o) = y_o(x, x_o) + F(x, x_o), \quad (53)
\]
\[
y_o(x', x_o) = -\alpha(x - x_o). \quad (54)
\]
Here \(F(x, x_o)\) is the solution of the integral equation
\[
F(x, x_o) = \int_{x_o}^{x} dx'(x - x') \left\{ S(y(x', x_o)) - y(x', x_o)^2 - \frac{2}{x} y'(x', x_o) \right\}, \quad (55)
\]
while \(S(y(x', x_o))\) is defined by the r.h.s. of Eq. (47). Furthermore, a direct evaluation of the explicit integral in Eq. (55) permits to cast \(F(x, x_o)\) in the form
\[
F(x, x_o) = F_o(x, x_o) + F_1(x, x_o), \quad (56)
\]
\[
F_1(x, x_o) = \int_{x_o}^{x} dx'(x - x') G_1(x', x_o), \quad (57)
\]
where \(F_o(x, x_o)\) and \(G_1(x', x_o)\) are respectively
\[
F_o(x, x_o) = \int_{x_o}^{x} dx'(x - x') \left\{ S(y_o(x', x_o)) - y_o(x', x_o)^2 - \frac{2}{x} y'_o(x', x_o) \right\}, \quad (58)
\]
\[
G_1(x', x_o) = S(y(x', x_o)) - S(y_o(x', x_o)) - F_1(x', x_o)^2 - 2F_1(x', x_o)y_o(x', x_o) - \frac{2}{x} F'_1(x', x_o). \quad (59)
\]

Using Eqs. (56), (58) and (59) it is immediate to prove that \(y(x, x_o)\) satisfies also the boundary conditions (49), (51) and (52) and in particular that the limit (50) is actually satisfied. The proof follows by noting that, in the strong-coupling ordering (9), (10) and for \(0 < (x - x_o) \sim O(\delta^0)\), the dominant contribution in \(y(x, x_o)\) is negative and is provided by the second term on the r.h.s. of Eq. (58). This implies that an asymptotic approximation for \(y(x, x_o)\) is provided in this case by:
\[
y(x, x_o) \cong y^{(int)}(x, x_o) = -\alpha(x - x_o) + F_o(x, x_o), \quad (60)
\]
and consequently also Eq.(44). Manifestly \( y^{(int)}(x, x_o) \) satisfies identically all the boundary conditions (49), (50), (51) and (52). Finally, to determine the initial condition \( \hat{\Phi}_{x_o}(x_o) \equiv \hat{\Phi}_o \), it is sufficient to notice that in asymptotic orderings (9),(10) and (42) the following identity must be satisfied, correct to leading order in \( \delta \),

\[
\beta \simeq \frac{1}{2} \int_{x_o}^{x} dx' x'^2 \exp \left\{ \hat{\Phi}_o \exp \{ y(x', x_o) \} \right\}.
\] (61)

The integral can be estimated asymptotically to yield

\[
\hat{\Phi}_{x_o}(x_o) \simeq \ln \frac{2\Gamma}{x_o^2} \left[ \Gamma - \hat{\Phi}_{x_o}(x_o) \right],
\] (62)

which to leading order in \( \delta \) implies Eq.(45). To briefly comment these results, it is important to remark, that, by assumption, the internal asymptotic solution here obtained, (60), is valid if the parameter \( x_o \) results of order \( O(1) \) or is infinitesimal in \( \delta \) [see the ordering (10)], while the parameter \( \Gamma \) satisfies (9). As a consequence, in the limit \( x_o \to 0^+ \) Eq.(61) results consistent with Eq.(7), thus proving also the validity of Eq.(6). Finally, we notice that the sub-domain (42) in which the internal asymptotic solution (60) is valid, allows \( \hat{\Phi}_{x_o}(x) \ll 1 \). This is also consistent with the weak-field approximation (1).

V. CONCLUSIONS: ASYMPTOTIC UPPER BOUND FOR THE EFFECTIVE CHARGE

As an important application, the previous analytic results can be used to obtain an asymptotic estimate for the effective dimensionless charge \( c(x_o, \Gamma) \) carried by the DH potential in strongly-coupled plasmas. We intend to show that test particles which have a high charge undergo a strong charge screening effect, close to the local plasma sheath, produced by non-linear effects in the Poisson equation. As a consequence, outside the Debye sphere (i.e., for \( x > 1 \)) the DH potential generated by highly charged test particles in strongly-coupled plasmas results strongly reduced with respect to the theoretical value observed in the corresponding weakly-coupled systems.

The effective charge of the test particle can be estimated by comparing the internal and external asymptotic solutions in a suitable neighborhood of the boundary of the plasma sheath (i.e., near \( x = x_o \)), belonging to the sub-domain defined by (42). For this purpose we impose the matching condition

\[
\hat{\Phi}^{(int)}_{x_o}(x_c) \simeq \hat{\Phi}^{(ext)}_{x_o}(x_C),
\] (63)
where, by assumption, \( x_C \) is defined so that there results \( \Phi(x_c) \ll 1 \) and moreover

\[
\frac{\Gamma}{\Phi(x_o)} \Delta x_c \sim O(\delta) \ll 1.
\]

(64)

being \( \Delta x_c \equiv x_c - x_o \) and where \( \frac{\Gamma}{\Phi(x_o)} \gg 1 \), consistent with the assumption of strongly-coupled plasma (9),(10). The matching condition can be used to obtain an upper estimate for \( c(x_o, \Gamma) \).

In fact, in validity of the ordering Eq.(64), \( \Phi(x_c) \) can be approximated by

\[
\Phi(x_c) \approx \Phi(x_o) \exp \left\{ -\frac{\Gamma}{\Phi(x_o)} (x - x_o) \right\}.
\]

Then, ignoring higher order corrections in \( \delta \), the following inequality manifestly holds:

\[
c(x_o, \Gamma) \lesssim c^{(a)}(x_o, \Gamma) \equiv x_o \ln \frac{2\Gamma^2}{x_o^2},
\]

(65)

which delivers therefore an asymptotic upper bound for the effective charge \( c(x_o, \Gamma) \). In particular, it follows that \( c^{(a)}(x_o, \Gamma) \) depends logarithmically on the normalized charge of the isolated test particle \( \beta \).

Thus, in general for strongly-coupled plasmas the effective dimensionless charge appears much smaller than in weakly-coupled plasmas. The result can be significant for the investigation of dusty plasmas, particularly to describe the charge screening effect of highly charged dusty grains.

Comparisons with numerical simulations, based on the numerical solution of modified DSP [defined by Eqs. (21), (20) and the boundary conditions (22), (23)], indicate that the asymptotic estimate \( c^{(a)}(x_o, \Gamma) \) holds already for \( \beta \gtrsim 1 \) and \( x_o \lesssim 0.3 \). For numerical calculations, the effective charge \( c(x_o, \Gamma) \) has been determined numerically as the limit for \( x \gg 1 \) (i.e., \( x \to \infty \)) of the the function

\[
f_c(x) = \frac{\Phi(x_o)x e^{-x_o}}{\beta},
\]

(66)

where \( \Phi(x_o) \) denotes the numerical solution of the modified DSP. As previously discovered \[9\], it is found that for strongly-coupled plasmas (i.e., assuming \( \Gamma > 1 \)) \( f_c(x) \) approaches rapidly the asymptotic value \( c(x_o, \Gamma) \), even at a ”distance” \( x \) smaller than the Debye Length (i.e., for \( x < 1 \)). For reference in figs 1-3 the numerical estimates of \( f_c(x) \), \( c(x_o, \Gamma) \) and \( c^{(a)}(x_o, \Gamma) \) have been reported for the cases \( \beta = 1, 5, 10 \) and \( x_o = 0.05 \div 0.3 \). Its is found that the upper bound \( c^{(a)}(x_o, \Gamma) \) provided by the majorization \[65\] can actually be used to obtain an approximate estimate of the effective charge \( c(x_o, \Gamma) \).

Finally, it is worthwhile pointing out that the asymptotic estimate provided by \( c^{(a)}(x_o, \Gamma) \) satisfies also the correct limit set by THM.1. In fact the it results

\[
\lim_{x_o \to 0^+} c(x_o, \Gamma) = 0
\]

(67)
and this limit is satisfied even in the case in which $\Gamma \sim 1/\delta \sim 1/x_o$.

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[1] P. Debye and E. Hückel, Phys. Zeit. 24(9), 186 (1923); see also The collected papers of Peter J.W. Debye (Ox Bow Press, Conn., USA,1988), p.217.
[2] E.E. Salpeter, Australian J.Phys. 7, 353 (1954).
[3] M.A. Lampert and R.S. Crandall, Phys.Rev. A 21, 362 (1980).
[4] A.J. Garrett, Phys. Rev.A 37, 4354 (1988).
[5] A. Gregoratto, C. Lo Surdo, M. Tessarotto and R. Zorat, Proceedings of the 2nd Symposium on Plasma Dynamics: Theory and Applications, edited by M. Tessarotto (C.D.C., Udine, Italy, 1992), p.39.
[6] C. Lo Surdo, A. Nocentini, Proceedings of the 1994 International Conference of Plasma Physics, edited by. P.H. Sakanaka, E. Del Bosco and M.V. Alves (INPE, Sao Jose’ dos Campos, Brazil, 1994), p.163.
[7] R. Clemente and P. Martin, J.Phys. Soc.J. 61(6), 1969 (1992).
[8] R.J. Perez and P. Martin, Astr.Space Sc. 256, 263 (1998).
[9] P. Martin and G.A. Baker Jr., J.Math.Phys. 32, 1490 (1991).
[10] Chang Lin, Jin-bao Zhao and Xiu-Iian Zhang, Physica Scripta 62, 405 (2000).
[11] P. Ricci, G. Lapenta, U de Angelis and V.N. Tsytovich, Phy.Plasma 8, 769 (2001).
[12] M.O.Robbins, K.Kremer, G.S. Grest, J.Chem. Phys. 88, 3286 (1988).
[13] G. Dupont, Mol. Phys.79, 453 (1996).
[14] E.Allayrov, H.Löwen, S.Trigger, Phys. Rev.E 57, 5518 (1998).
[15] O. Bysternko and A. Zagorodny, Phys.Lett. A 255, 325 (1999).
[16] O. Bysternko, T. Bysternko and A. Zagorodny, Cond.Matter Physics 6 (3), 425 (2003).
Figure captions

**Figure 1** - Comparison between $f_c(x), c(x_o, \Gamma)$ and $c^{(a)} (x_o, \Gamma)$. The data are normalized with respect to $\beta$, the normalized charge of the isolated test particle. The figure concerns the case with $\beta = 1$ and $x_o = 0.05$, yielding $\Gamma = 20$. The horizontal straight line represents the asymptotic estimate $c^{(a)} (x_o, \Gamma)$, while the curve below it is the graph of $f_c(x)$. It follows $f_c(x_o) \approx 0.564$, while the asymptotic value $c(x_o, \Gamma) \approx 0.493$ is reached approximately at $x \approx 0.3$, and the upper bound for the normalized effective charge is $c^{(a)} (x_o, \Gamma) \approx 0.589$.

**Figure 2** - Comparison between $f_c(x), c(x_o, \Gamma)$ and $c^{(a)} (x_o, \Gamma)$ for $\beta = 5$ and $x_o = 0.2$ (with $\Gamma = 25$). In this case $f_c(x_o) \approx 0.369$, while the asymptotic value $c(x_o, \Gamma) \approx 0.28$ is reached approximately at $x \approx 0.4$, and $c^{(a)} (x_o, \Gamma) \approx 0.38$.

**Figure 3** - Comparison between $f_c(x), c(x_o, \Gamma)$ and $c^{(a)} (x_o, \Gamma)$ for $\beta = 10$ and $x_o = 0.3$ (with $\Gamma \approx 33$). In this case it is found $f_c(x_o) \approx 0.273$, while the asymptotic value $c(x_o, \Gamma) \approx 0.188$ is reached approximately at $x \approx 0.5$, and $c^{(a)} (x_o, \Gamma) \approx 0.303$. 
