Branching random walk with step size coming from a power law

Ayan Bhattacharya
E-mail: ayanbhattacharya.isi@gmail.com
Statistics and Mathematics Unit, Indian Statistical Institute, 203 B. T. Road, Kolkata 700108.

Rajat Subhra Hazra
E-mail: rajatmaths@gmail.com
Statistics and Mathematics Unit, Indian Statistical Institute, 203 B. T. Road, Kolkata 700108.

Parthanil Roy
E-mail: parthanil.roy@gmail.com
Statistics and Mathematics Unit, Indian Statistical Institute, 203 B. T. Road, Kolkata 700108.

Abstract. In their seminal work [15], Brunet and Derrida made predictions on the random point configurations associated with branching random walks. We shall discuss the limiting behavior of such point configurations when the displacement random variables come from a power law. In particular, we establish that two prediction of [15] remains valid in this setup and investigate various other issues mentioned in their paper.

1. Introduction
Branching random walk is a collection of particles that starts from a single particle situated at the origin, and branch and disperse independently of each other. It can be applied to model how a growing population (of bacteria, animals, particles, etc.) occupies an environment and hence is useful in biology, ecology, statistical physics, etc. In this article, we shall assume that the particles will branch in a supercritical fashion satisfying the Kesten-Stigum condition (see (2.1) below) and their displacements will follow a power law (see (2.2) below).

Branching random walk is of interest starting from [19], [20], [8]. Recently, its extreme value theory has gained prominence due to the connection to tree indexed random walk and Gaussian free field; see [12], [18], [4], [3], [22], [9], [10], [11]. See also [14], [13], [21], [6] for related results on extremes of branching Brownian motion. [15] predicted various properties of the limiting point configurations of branching random walks and branching Brownian motion. These predictions have been verified mathematically by [1], [5] and [2] for branching Brownian motion, and by [22] for branching random walks with light-tailed displacements. In this paper, we go beyond the usual setup of light-tailed displacements and announce the results when the step sizes follow a power law. For detailed discussions, technical background and proofs of the stated results in a slightly general setup, the readers are referred to [7].
2. Notation and Model
We first give the description of a Galton-Watson tree which will form a building block for our model. Such a tree is obtained by the following random mechanism. Start with one particle and call it the root of the tree (denoted by o). This particle produces a random number of offsprings according to a discrete probability distribution and dies immediately. These particles form the first generation. Now each of these particle undergoes the same mechanism independently of each other and the past. The new particles form the second generation and this goes on. Let $Z_i$ be the number of particles in the $i^{th}$ generation. We assume that $Z_1 \geq 1$, $\mu:=\langle Z_1 \rangle \in (1, \infty)$ and the Kesten-Stigum condition holds, i.e.,

$$\langle Z_1 \log Z_1 \rangle < \infty. \quad (2.1)$$

It is well-known that under the above assumptions $Z_n/\mu^n$ is a martingale sequence that converges almost surely to a positive random variable $W$.

We denote this Galton-Watson tree by $\mathbb{T} = (V, E)$, where $V$ and $E$ denote the collection of vertices and edges, respectively. After obtaining this infinite tree, we assign independent and identically distributed edge random variables $\{X_e : e \in E\}$ (that are also independent of the Galton-Watson process $\{Z_i\}_{i \geq 0}$) on the edges satisfying the power law

$$P(X_e > x) \sim cx^{-\alpha} \quad (2.2)$$

as $x \to \infty$ for some constant $c > 0$ and $\alpha > 0$. To each vertex $v \in V$, we attach a displacement random variable $S_v$, which is the sum of all edge random variables on the unique geodesic path from the root $o$ to the vertex $v$. The size of this path is called the generation of $v$ and is denoted by $|v|$. Our goal is to find the limit of the random point configuration $\{b_n^{-1}S_v : |v| = n\}$ denoted by $N_n$, where $b_n = c^{1/\alpha}/\mu^{n/\alpha}$ and verify that this limit honours the prediction of [15].

It is important to note that $\mu^{n/\alpha}$ is the right scaling for the weak convergence of the random point configuration of the $n^{th}$ generation displacement random variables because according to [17], it is the correct scaling for the maxima. If we scale by a sequence $b_n \gg \mu^{n/\alpha}$, then all the points will be killed by this normalization. On the other hand, if we scale by $b_n \gg \mu^{n/\alpha}$, then many of the points will diverge to infinity.

3. Main Results
Let $\{T_i\}_{i \geq 1}$ be a sequence of independently and identically distributed random variables with probability mass function

$$\gamma(y) = r \sum_{i=0}^{\infty} \frac{1}{\mu^i} P(Z_i = y) \quad (3.1)$$

with $r = \mu/(\mu - 1)$, for $y \in \mathbb{N}$ independent of $W$. Let $\{jt\}_{t \geq 1}$ be the collection of all Poisson points of the Poisson process on $(0, \infty]$ with intensity $\nu_{\alpha}(dx) = \alpha x^{-\alpha-1}dx$ independent of $\{T_i\}_{i \geq 1}$ and $W$. The point 0 is removed from the state space of the Poisson process because the intensity blows up near 0. Our main result says the limiting random point configuration is a Cox cluster process. The clusters appear in the limit because of the strong dependence among the $n^{th}$ generation displacement random variables.

**Theorem 3.1.** $N_n$ converges in distribution to a random point configuration $N_*$ with $\{(rW)^{1/\alpha}jt\}_{t \geq 1}$ as the collection of points with the $l^{th}$ point repeating $T_1$ many times.

Clearly, the points of the limiting random point configuration arise as a combined effect of two kinds of randomness: one randomness is due to the displacement random variables and the other is coming from the underlying branching process. The clusters appear here due to the strong dependence structure of the displacement random variables $\{S_v : |v| = n\}$. The randomness in
the intensity measure arises from the martingale limit \( W \) in contrast to the light-tailed case, where similar randomness comes from the derivative martingale limit. Note also that a \( W \)-mixture was already present in Theorem 1 of [17].

Recall that according to [15], the limiting random point configuration is a “decorated Poisson point process” and satisfies “superposability” property. In our setup, we can show that the limiting random point configuration is a \textit{randomly scaled scale-decorated Poisson point process} and it satisfies the an analogue of superposability.

First we would like to explain what is meant by a scale-decorated Poisson process. Let \( Q \) (will be referred as scale-decoration ) denotes a random point configuration on \([-∞, ∞] \setminus \{0\} \). Let \( \Lambda \) (will be referred as underlying Poisson process) be a Poisson process on \((0, ∞)\) with intensity measure \( m(dx) \) and \( \{λ_i\}_{i≥1} \) be the collection of Poisson points of \( Λ \). Consider independent copies \( \{Q_i\}_{i≥1} \) of \( Q \) such that the entire sequence is independent of \( Λ \). We multiply each point of the random point configuration \( \Lambda \cup \{Q_i\}_{i≥1} \) by \( λ_i \) and denote it by \( S_{λ_i} Q_i \). The superposed point configuration \( \Lambda \cup S_{λ_i} Q_i \) is called a scale-decorated Poisson point process and we denote it by \( \text{ScDPPP}(m, Q) \). We further multiply each point of this superposed point configuration by a positive random variable \( U \) (will be referred as scaling random variable) independent of both \( Λ \) and \( \{Q_i\}_{i≥1} \). The resultant random point configuration \( \Lambda \cup S_{λ_i} Q_i \) is called a randomly scaled scale-decorated Poisson point process and is denoted by \( \text{SScDPPP}(m, Q, U) \). The next result and the subsequent remark confirm that the predictions of [15] remains valid in our setup.

**Theorem 3.2.** The limiting random point configuration \( N_α \) is a randomly scaled scale-decorated Poisson point process. The scale-decoration is the random point configuration consisting of \( T \) repeating \( T \) many times, where \( T \) has probability mass function \( (3.1) \). The intensity measure of the underlying Poisson process is \( ν_α(dx) \). The scaling random variable is \( (rW)^{1/α} \), where \( r = μ/(μ - 1) \) and \( W \) is the martingale limit of the underlying branching process.

**Remark 3.3.** If \( Z_1 \equiv d \), then the underlying tree is \( d \)-regular. In this case, we denote the limiting point configuration by \( N_α^{(d)} \). Consider two independent copies \( N_α^{(d,1)} \) and \( N_α^{(d,2)} \) of \( N_α^{(d)} \). Take two positive real numbers \( a_1 \) and \( a_2 \) satisfying \( a_1^α + a_2^α = 1 \). Multiply each point of \( N_α^{(d,1)} \) by \( a_1 \), each point of \( N_α^{(d,2)} \) by \( a_2 \), and superpose the scaled points. The new random point configuration will have the same distribution as that of \( N_α^{(d)} \). This is the analogue of superposability enjoyed by our limiting random point configuration and it confirms a related prediction of [15] in our setup. In [10], this property has been investigated and referred to as \( \alpha \)-stability of random point configurations.

### 3.1. Order statistics

The asymptotic distribution of order statistics of the displacement random variables attached to the vertices of the \( n^{th} \) generation can be easily derived from Theorem 3.1. In particular, we rediscover a result of [17] on the limiting behaviour of the rightmost particle coming from the \( n^{th} \) generation. Let \( M_n^{(k)} \) denote the \( k^{th} \) upper order statistic coming from the \( n^{th} \) generation, and \( G_n^{(k)} = M_n^{(k)} - M_n^{(k+1)} \) be the \( k^{th} \) gap statistic. In order to study the asymptotic properties of these statistics, we need a few more notations as described below. We denote by \( \pi \) a partition of an integer \( l \) of the form \( l = i_1y_1 + i_2y_2 + \cdots + i_{|\pi|}y_{|\pi|} \), where each \( i_j \) repeats \( y_j \) many times in the partition, and \( i_1 < i_2 < \cdots < i_{|\pi|} \). Here \(|\pi|\) denotes the number of distinct elements in a partition. The set of all such partitions of the integer \( l \) is denoted by \( Π_l \).

- **Property 1** (\( k^{th} \) upper order statistic) For all \( x > 0 \), \( \lim_{n→∞} P \left( M_n^{(k)} ≤ c^{1/α} μ^{α/α} x \right) = \left( e^{−rW x^{−α}} \right)^{k} + \sum_{l=1}^{k-1} \sum_{π∈Π_l} \left( \prod_{i=1}^{|\pi|} \left( (rW^{−α}γ(i_j))^y_j e^{−rW x^{−α}γ(i_j)} \right) \right) \right. \). \( (3.2) \)
In particular,
\[
\lim_{n \to \infty} P \left( M_n^{(1)} \leq c^{1/\alpha} \mu^{n/\alpha} x \right) = \phi(rx^{-\alpha}), \quad x > 0,
\]
where \(\phi\) is the unique (up to a scale-change) completely monotone function on \(\mathbb{R}^+\) satisfying \(\phi(z) = f(\phi(z/\mu))\) with \(f\) being the probability generating function of the branching random variable \(Z_1\).

• **Property 2** (Joint distribution of \(k^{th}\) and \((k+1)^{th}\) upper order statistics) For all \((u, v)\) such that \(0 < u < v\),
\[
\lim_{n \to \infty} P \left( M_n^{(k+1)} \leq c^{1/\alpha} \mu^{n/\alpha} u, M_n^{(k)} \leq c^{1/\alpha} \mu^{n/\alpha} v \right) = \left\langle \xi_{0,(u,\infty)}(W), \xi_{j,(u,v]}(W) \right\rangle + \sum_{j=1}^{k-1} \left\langle \xi_{l,(v,\infty]}(W), \xi_{j,(u,v]}(W) \right\rangle, \quad (3.3)
\]
where for all \(l \geq 0\) and for all \(A \subset [-\infty, \infty] \setminus \{0\}\) such that \(\nu_\alpha(A) < \infty\),
\[
\xi_{l,A}(W) := \begin{cases} 
    e^{-rW\nu_\alpha(A)} & \text{if } l = 0, \\
    \sum_{\pi \in \Pi_l} \prod_{j=1}^{\mid \pi \mid} (rW\nu_\alpha(A)\gamma(i_j))^y_j \frac{1}{y_j!} e^{-rW\nu_\alpha(A)\gamma(i_j)} & \text{if } l \geq 1.
\end{cases}
\]

• **Property 3** \((k^{th}\) gap statistic) Let \(L : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+\) be the map \(L(u, v) = v - u\).
Then \(P(b_n^{-1}G_n^{(k)} \in \cdot) \to \zeta_k \circ L^{-1}\) where \(\zeta_k\) is a probability measure on \(\mathbb{R}^+ \times \mathbb{R}^+\) with joint cumulative distribution function \((3.3)\).

The second term in \((3.2)\) and the last term in \((3.3)\) are both interpreted as zero when \(k = 1\).

4. **Final Remarks**
We have been able to compute the asymptotic distribution of maxima and the random point configuration explicitly. Such explicit representations of the limiting point configuration is missing for light-tailed displacements. Of course, the heavy-tailed nature of the step sizes help us in obtaining such nice and useful results. This is because the extremes of the displacement random variables essentially come from one large value among the edge random variables. This intuition was already available in \[17\] for the rightmost particle coming from the \(n^{th}\) generation. We had to implement this idea at the level of random point configurations and this is, by far, much more nontrivial than the result for the rightmost particle. The proof of Theorem 3.1 is based on a twofold truncation technique using multivariate extreme value theory. For details of the proof, see \[7\].

We have shown that the predictions in \[15\] remain valid in this set up with appropriate modifications. We also predict that even under moderate dependence among the heavy-tailed edge random variables coming out of the same vertex, the limiting random point configuration will satisfy superposability and can be represented as a randomly scaled scale-decorated Poisson point process.

**Acknowledgement**
We thank the anonymous referees for their suggestions.
References
[1] Addario-Berry L and Reed B 2009 Ann. Prob. 37 1044–1079
[2] Aïdékon E, Berestycki J, Brunet É, and Shi Z 2013 Probab. Theory Related Fields 157 405–451
[3] Aïdékon E 2013 Ann. Prob. 41 1362–1426
[4] Arguin L.-P, Bovier A, and Kistler N 2012 Ann. Appl. Probab. 22 1693–1711
[5] Arguin L.-P, Bovier A, and Kistler N 2013 Probab. Theory Related Fields 157 535–574
[6] Arguin L.-P, Bovier A and Kistler N 2011 Comm. Pure Appl. Math. 64 1647–1676
[7] Bhattacharya A, Hazra R. S, and Roy P November 2014 arXiv:1411.5646
[8] Biggins JD 1976 Adv. Appl. Probab. 8 446–459
[9] Biskup M and Louidor O 2013 arXiv:1306.2602
[10] Biskup M and Louidor O 2014 arXiv:1410.4676
[11] Bramson M, Ding J and Zeitouni O 2013 arXiv:1301.6669
[12] Bramson M and Zeitouni O 2012 Comm. Pure Appl. Math. 65 1–20
[13] Bramson M 1983 Mem. Amer. Math. Soc. 285 (Providence: Amer. Math. Soc.)
[14] Bramson M.D 1978 Comm. Pure Appl. Math. 31 531–581
[15] Brunet É and Derrida B 2011 J. Stat. Phys. 143 420–446
[16] Davydov Y, Molchanov I, and Zuyev S 2008 Electron. J. Probab. 13 259–321
[17] Durrett R 1983 Z. Wahrsch. Verw. Gebiete 62 165–170
[18] Hu Y and Shi Z 2009 Ann. Prob. 37 742–789
[19] Hammersley JM 1974 Ann. Prob. 652–680
[20] Kingman J. F. C 1975 Ann. Prob. 790–801
[21] Lalley S and Sellke T 1987 Ann. Prob. 1052–1061
[22] Madaule T 2011 arXiv:1107.2543