Delayed singularity formation for three-dimensional compressible Euler equations with non-zero vorticity

Fei Hou¹ | Huicheng Yin²

¹Department of Mathematics, Nanjing University, Nanjing, China
²School of Mathematical Sciences, Nanjing Normal University, Xianlin, Nanjing, China

Abstract
For the 3D compressible isentropic Euler equations with an initial perturbation of size $\varepsilon$ of a rest state, if the initial vorticity is of size $\delta$ with $0 < \delta \leq \varepsilon$ and $\varepsilon$ is small, we establish that the lifespan of the smooth solutions is $T^P_\delta = O(\min\{e^{\varepsilon^2}, \frac{1}{\delta}\})$ for the polytropic gases, and $T^C_\delta = O(\frac{1}{\delta})$ for the Chaplygin gases. Our result illustrates that the time of existence of smooth solutions depends crucially on the size of the vorticity of the initial data, as long as the initial data are sufficiently close to a constant state. The main ingredients in the paper are as follows: Through looking for the new good unknown instead of the velocity and introducing some suitably weighted energies, we derive the pointwise space-time decay estimates of solutions and establish the required weighted estimates on the vorticity.

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1 | INTRODUCTION

1.1 Setting of the problem and statement of the main result

In this paper, we are concerned with the long time existence of smooth solutions to the 3D compressible Euler equations

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla p &= 0, \\
\rho(0, x) &= \bar{\rho} + \rho^0(x), \quad u(0, x) = u^0(x),
\end{align*}
\]

(1.1)

where \((t, x) = (t, x_1, x_2, x_3) \in \mathbb{R}_+^{1+3} := [0, \infty) \times \mathbb{R}^3, \nabla = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})^T, \) and \(u = (u_1, u_2, u_3)^T, \rho, \ p \) stand for the velocity, density, pressure, respectively. In addition, \(\bar{\rho} > 0\) is a constant, \(\rho(0, x) > 0, u^0(x) = (u^0_1(x), u^0_2(x), u^0_3(x))^T,\) and \((\rho^0(x), u^0(x)) \in C^\infty(\mathbb{R}^3)\). Assume that the pressure \(p = p(\rho)\) is smooth on its argument \(\rho\).

For the polytropic gases (see [13]),

\[ p(\rho) = A\rho^\gamma, \]

(1.2)

where \(A\) and \(\gamma (1 < \gamma < 3)\) are some positive constants.

For the Chaplygin gases (see [13] or [15]),

\[ p(\rho) = P_0 - \frac{B}{\rho}, \]

(1.3)

where \(P_0 > 0\) and \(B > 0\) are constants.

If \((\rho, u) \in C^1\) and \(\rho > 0\), then (1.1) can be reduced to

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t u + u \cdot \nabla u + \frac{c^2(\rho)}{\rho} \nabla \rho &= 0,
\end{align*}
\]

(1.4)

where \(c(\rho) = \sqrt{p'(\rho)}\) is the local sound speed. If we let \((\rho, u)(t, x) = (\hat{\rho}, \hat{u})(t, s)\) with \(s = x \cdot \xi, \xi = (\xi_1, \xi_2, \xi_3)^T \in S^2,\) then system (1.4) becomes

\[
\begin{align*}
\partial_t \hat{\rho} + (\xi_1 \hat{u}_1 + \xi_2 \hat{u}_2 + \xi_3 \hat{u}_3) \partial_s \hat{\rho} + \hat{\rho}(\xi_1 \partial_s \hat{u}_1 + \xi_2 \partial_s \hat{u}_2 + \xi_3 \partial_s \hat{u}_3) &= 0, \\
\partial_t \hat{u}_1 + (\xi_1 \hat{u}_1 + \xi_2 \hat{u}_2 + \xi_3 \hat{u}_3) \partial_s \hat{u}_1 + \frac{\xi_1 c^2(\hat{\rho})}{\hat{\rho}} \partial_s \hat{\rho} &= 0, \\
\partial_t \hat{u}_2 + (\xi_1 \hat{u}_1 + \xi_2 \hat{u}_2 + \xi_3 \hat{u}_3) \partial_s \hat{u}_2 + \frac{\xi_2 c^2(\hat{\rho})}{\hat{\rho}} \partial_s \hat{\rho} &= 0, \\
\partial_t \hat{u}_3 + (\xi_1 \hat{u}_1 + \xi_2 \hat{u}_2 + \xi_3 \hat{u}_3) \partial_s \hat{u}_3 + \frac{\xi_3 c^2(\hat{\rho})}{\hat{\rho}} \partial_s \hat{\rho} &= 0.
\end{align*}
\]

(1.5)
It follows from direct computation that (1.5) has four eigenvalues
\[
\hat{\lambda}_1 = \xi_1 \hat{u}_1 + \xi_2 \hat{u}_2 + \xi_3 \hat{u}_3 - c(\hat{\varrho}), \quad \hat{\lambda}_{2,3} = \xi_1 \hat{u}_1 + \xi_2 \hat{u}_2 + \xi_3 \hat{u}_3, \quad \hat{\lambda}_4 = \xi_1 \hat{u}_1 + \xi_2 \hat{u}_2 + \xi_3 \hat{u}_3 + c(\hat{\varrho}),
\]
and the corresponding four right eigenvectors are
\[
\hat{r}_1 = (\hat{\varrho}, c(\hat{\varrho})\xi_1, c(\hat{\varrho})\xi_2, c(\hat{\varrho})\xi_3)^T, \\
\hat{r}_2 = (0, -\xi_2, \xi_1, 0)^T, \\
\hat{r}_3 = (0, -\xi_3, 0, \xi_1)^T, \\
\hat{r}_4 = (\hat{\varrho}, c(\hat{\varrho})\xi_1, c(\hat{\varrho})\xi_2, c(\hat{\varrho})\xi_3)^T.
\]

It is easy to verify that for each \(\xi \in S^2\),
\[
\nabla_{\hat{\varrho}, \hat{u}} \hat{\lambda}_i \cdot \hat{r}_i \equiv 0, \quad i = 2, 3.
\]
For the Chaplygin gases, then
\[
\nabla_{\hat{\varrho}, \hat{u}} \hat{\lambda}_i \cdot \hat{r}_i \equiv 0, \quad i = 1, 4.
\]
For the polytropic gases, then
\[
\nabla_{\hat{\varrho}, \hat{u}} \hat{\lambda}_i \cdot \hat{r}_i = \hat{\varrho}c'(\hat{\varrho}) + c(\hat{\varrho}) > 0 \quad \text{for } \hat{\varrho} > 0, \quad i = 1, 4.
\]
By the definition on page 89 of [28], (1.4) is totally linearly degenerate for the Chaplygin gases, and (1.4) is genuinely nonlinear with respect to the first eigenvalue \(\lambda_1\) and the fourth eigenvalue \(\lambda_4\) for the polytropic gases when \(\varrho > 0\). Generally speaking, for the nonlinear hyperbolic conservation laws with small initial data or small perturbed initial data, the genuinely nonlinear condition can arise the blowup of smooth solutions in finite time and corresponds to the formation of shock (see [2, 3, 7–9, 11, 12, 14, 17, 25, 27] and [32]); for 1D nonlinear hyperbolic conservation laws, the totally linearly degenerate condition can produce the global smooth small data solutions (see [26]). For the multidimensional case of nonlinear hyperbolic conservation laws with totally linearly degenerate condition, A. Majda posed the following conjecture on page 89 of [28]:

**Conjecture.** If the \(d (d \geq 2)\)-dimensional nonlinear symmetric system is totally linearly degenerate, then it typically has smooth global solutions when the initial data are in \(H^s(\mathbb{R}^d)\) with \(s > \frac{d}{2} + 1\) unless the solution itself blows up in finite time. In particular, the shock wave formation never happens for any smooth initial data.

By our knowledge, so far this conjecture has not been solved yet even for the small initial data. As illustrated on page 89 of [28], the above conjecture is mainly of mathematical interest but its resolution would elucidate both the nonlinear nature of the conditions requiring linear degeneracy of each wave field and also might isolate the fashion in which the shock wave formation arises in quasilinear hyperbolic systems. In fact, with respect to the small perturbed problem of 3D compressible Euler equations of Chaplygin gases with general non-zero vorticity (as the typical model
of multidimensional nonlinear symmetric system with totally linearly degenerate condition).

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla p &= 0, \\
\rho(0, x) &= \bar{\rho} + \varepsilon \tilde{\rho}^0(x), \quad u(0, x) = \varepsilon \tilde{u}^0(x),
\end{align*}
\] (1.6)

it is still completely unknown whether the global smooth solution \((\rho, u)\) of (1.6) exists or not.

Next we give some illustrations on the irrotational case of (1.1) with the small perturbed initial data \((\rho, u)(0, x) = (\bar{\rho} + \varepsilon \tilde{\rho}^0(x), \varepsilon \tilde{u}^0(x))\), where \((\tilde{\rho}^0(x), \tilde{u}^0(x)) \in C^\infty_0(\mathbb{R}^3)\) are supported in the ball \(B(0, R)\). Without loss of generality, \(\bar{\rho} = c(\bar{\rho}) = 1\) is assumed. Set the initial vorticity

\[
\text{curl} \, \tilde{u}^0 := (\partial_2 \tilde{u}^0_3 - \partial_3 \tilde{u}^0_2, \partial_3 \tilde{u}^0_1 - \partial_1 \tilde{u}^0_3, \partial_1 \tilde{u}^0_2 - \partial_2 \tilde{u}^0_1)^T \equiv 0.
\] (1.7)

Then \(\text{curl} \, u(t, x) \equiv 0\) always holds as long as \((\rho, u) \in C^1\). This means that there exists a potential function \(\phi\) such that \(u = \nabla \phi\). The Bernoulli’s law shows \(\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + h(\rho) = 0\) with \(h'(\rho) = \frac{c^2(\rho)}{\rho}\) and \(h(\bar{\rho}) = 0\). One can easily check that \(h(\rho) = \frac{c^2(\rho)}{\gamma - 1}\), where \(1 < \gamma < 3\) for the polytropic gases in (1.2) and \(\gamma = -1\) for the Chaplygin gases in (1.3). From this, the density \(\rho\) can be expressed as

\[
\rho = \left[1 - (\gamma - 1)(\partial_t \phi + \frac{1}{2} |\nabla \phi|^2)\right]^{\frac{1}{\gamma - 1}}.
\]

Substituting this into the first equation of (1.1) yields

\[
\partial_t^2 \phi - \Delta \phi + 2 \sum_{k=1}^3 \partial_k \phi \partial_k \phi + (\gamma - 1) \partial_t \phi \Delta \phi + \sum_{i,j=1}^3 \partial_i \phi \partial_j \partial_{ij} \phi + \frac{\gamma - 1}{2} |\nabla \phi|^2 \Delta \phi = 0,
\] (1.8)

where the Laplace operator \(\Delta := \sum_{i=1}^3 \partial_i^2\). In addition, the initial data of \(\phi\) are

\[
\begin{align*}
\phi(0, x) &= \varepsilon \int_{-R}^{x_1} \tilde{u}^0(\eta, x_2, x_3) d\eta, \\
\partial_t \phi(0, x) &= -\varepsilon \tilde{\rho}^0(x) + \varepsilon^2 \nu(x, \varepsilon),
\end{align*}
\] (1.9)

where \(\nu(x, \varepsilon) = -\frac{1}{2} \sum_{k=1}^3 (\tilde{u}^0_k)^2(x) - (\tilde{\rho}^0)^2(x) \int_0^1 \left(\frac{c^2(\rho)}{\rho}\right)' |_{\rho = 1 + \eta \varepsilon \tilde{\rho}^0(x)(1 - \eta)} d\eta\).

For Equation (1.8) with (1.9), when \(\varepsilon > 0\) is small, it follows from Theorem 6.5.3 in [17] that the lifespan \(T_\varepsilon\) of smooth solution \(\phi\) satisfies \(T_\varepsilon \geq e^{C \varepsilon}\) with some positive constant \(C\). On the other hand, for the polytropic gases (1.2) with \(1 < \gamma < 3\), it has been known that the lifespan \(T_\varepsilon = O(e^{\frac{C}{\varepsilon}})\) is optimal (see [5] and [23]) and meanwhile the shock can be formed (see [12, 16, 32]); for the Chaplygin gases (1.3) with \(\gamma = -1\), the lifespan \(T_\varepsilon = +\infty\) holds since the corresponding null condition holds (see [10] and [24]). In the present paper, we are concerned with such an interesting question:

**When \(\text{curl} \, u^0(x) \neq 0\) and \(\text{curl} \, u^0(x) = o(\varepsilon)\), what is the lifespan \(T_\varepsilon\) of the classical solution \((\rho, u)\) to the 3D compressible Euler equations with (1.2) or (1.3)?**
So far, by the author’s knowledge, only a few results on the above problem have been obtained for the 2D or 3D compressible Euler equations (1.1) with non-zero vorticity. For examples, with respect to the 2D compressible Euler equations of polytropic gases with the following rotationally invariant initial data

\[
\rho(0, x) = \bar{\rho} + \varepsilon \rho^0(r), \quad u(0, x) = \varepsilon u^0(r) \frac{x}{r} + \varepsilon u^0_\perp \frac{x^\perp}{r},
\]  

where \( r = |x| \) and \( x^\perp = (-x^2, x^1)^T \), [1] has shown that the lifespan \( T_\varepsilon \) is of order \( O\left(\frac{1}{\varepsilon^2}\right) \). The global existence of smooth solution to 2D compressible Euler equations (1.1) of Chaplygin gases with the initial data (1.10) was established in [18, 19]. Without the assumption on the rotationally invariant initial data, if \( \text{curl} u^0(x) = O(\varepsilon^{1+\alpha}) \) with the constant \( \alpha \geq 0 \), it follows from Theorem 1 and Theorem 2 of [30] that the lifespan \( T_\varepsilon \) of 2D compressible Euler equations (1.1) with (1.2) fulfills \( T_\varepsilon = O(\varepsilon^{\min\{1+\alpha, 2\}}) \). In addition, for the 3D compressible Euler equations (1.4) of polytropic gases with the small initial perturbed density of order \( O(\varepsilon) \), when the divergence of initial velocity is of \( O(\varepsilon) \), and the vorticity of initial velocity is of order \( O(\varepsilon^\mu) \) with \( 1 < \mu < \frac{6}{5} \), the authors in [29] proved that the lifespan of smooth solution \( (\rho, u) \) is of \( O\left(\frac{1}{\varepsilon^\mu}\right) \).

We now investigate the influence of the vorticity on the existence time of smooth solutions to 3D compressible Euler equations with (1.2) or (1.3). To this end, we introduce the following two quantities that capture the size of the perturbed initial data \((\rho^0, u^0)\) and initial vorticity \(\text{curl} u^0\):

\[
\varepsilon := \sum_{k \leq N} \|\langle x \rangle \nabla^k (\rho^0(x), u^0(x))\|_{L^2},
\]

\[
\delta := \sum_{k \leq N-1} \|\langle x \rangle \nabla^k \text{curl} u^0(x)\|_{L^2},
\]

where \( N \geq 8 \), and \( \langle x \rangle = \sqrt{1 + |x|^2} \). The main result in the paper is as follows:

**Theorem 1.1.** For the numbers \( \varepsilon \) and \( \delta \) defined in (1.11), then there exist three constants \( \varepsilon_0, \delta_0, \kappa_0 > 0 \) such that when \( \varepsilon \leq \varepsilon_0 \) and \( \delta \leq \delta_0 \),

(i) (1.1) with the state equation (1.2) of polytropic gases has a solution \((\rho - \bar{\rho}, u) \in C([0, T^P], H^N(\mathbb{R}^3)) \cap C^1([0, T^P], H^{N-1}(\mathbb{R}^3))\), where \( T^P = \min\{\varepsilon_0^{-\frac{1}{2}}, 1, \frac{\kappa_0}{\delta}\} \).

(ii) (1.1) with the state equation (1.3) of Chaplygin gases has a solution \((\rho - \bar{\rho}, u) \in C([0, T^C], H^N(\mathbb{R}^3)) \cap C^1([0, T^C], H^{N-1}(\mathbb{R}^3))\), where \( T^C = \frac{\kappa_0}{\delta} \).

### 1.2 Remarks and sketch of proof

**Remark 1.** When \( \delta = e^{-\frac{1}{\varepsilon \ell}} \) with \( \ell \geq 1 \) or \( e^{-e \frac{1}{\varepsilon \ell}} \) with \( \rho > 0 \) are chosen in Theorem 1.1, one knows that the existence time of smooth solution \((\rho, u)\) to (1.1) for the Chaplygin gases is larger than \( \kappa_0 e^{-\frac{1}{\varepsilon \ell}} \) or \( \kappa_0 e^{-e \frac{1}{\varepsilon \ell}} \), which means that the order of lifespan \( T^C_\delta \) of small perturbed solution \((\rho, u)\) is only essentially influenced by the size of the initial vorticity. On the other hand, the optimality of the obtained lifespans in Theorem 1.1 will be further studied.
Remark 2. In [29], through decomposing the solution \((\rho, u)\) of 3D compressible Euler equations (1.4) as the sum of the irrotational part, the incompressible part and the remainder and by applying the direct energy method, the authors establish that the lifespan of smooth solution \((\rho, u)\) is of order \(O(\frac{1}{\epsilon^\mu})\) \((1 < \mu < \frac{6}{5})\) when the initial vorticity is of \(O(\epsilon^\mu)\) and the perturbed density is of \(O(\epsilon)\). Instead of the time-decay estimates \((1 + t)^{-1}\) of the irrotational part in [29], one of our ingredients is to derive the more precise pointwise space-time decay estimates \((1 + t)^{-1} (|x| - t)^{-\alpha}\) for suitable \(\alpha > 0\).

Remark 3. In our paper [21], for the 2D compressible isentropic Euler equations (1.1) of Chaplygin gases, when the initial data are a perturbation of size \(\epsilon\), and the initial vorticity is of any size \(\delta\) with \(0 < \delta \leq \epsilon\), we have established the lifespan \(T^C_\delta = O(\frac{1}{\delta})\). The main methods in [21] are: establishing a new class of weighted space-time \(L^\infty-L^\infty\) estimates for the solution itself and its gradients of 2D linear wave equations, introducing some suitably weighted energies and taking the \(L^p\) \((1 < p < \infty)\) estimates on the vorticity due to the requirements of Sobolev embedding theorem. Since the space-time decay rates of smooth solutions to 3D and 2D wave equations are different, moreover, the vorticity equations in 3D and 2D Euler equations have also some differences (for example, the vorticity \(\text{curl} u\) in 2D Euler equations is a scalar function and there exists the conservation form \((\partial_t + u \cdot \nabla)(\text{curl} u) \equiv 0\); however, for the 3D Euler equations, the vorticity \(\text{curl} u\) is a 3D vector and satisfies a nonlinear system \((\partial_t + u \cdot \nabla)\text{curl} u = \text{curl} u \cdot \nabla u - \text{curl} u \cdot \nabla u\)), then there are a few different techniques and methods between the present paper and [21] in order to derive the lifespan of smooth solutions.

Remark 4. For the 3D Euler–Maxwell system, when the initial data are of order \(O(\epsilon)\) and the initial vorticity is of order \(O(\delta)\) with \(0 < \delta \leq \epsilon\), the authors in [22] have proved that the existent time is larger than \(C_\delta\), where \(C\) is some positive constant.

Now we give some illustrations on the proof of Theorem 1.1. At first, we introduce the perturbed sound speed \(\sigma = \frac{c(\rho)^{-1}}{\lambda}\) as the new unknown to rewrite (1.1) as

\[
\begin{align*}
\partial_t \sigma + \text{div} u &= Q_1 := -\lambda \sigma \text{div} u - u \cdot \nabla \sigma, \\
\partial_t u + \nabla \sigma &= Q_2 := -\lambda \sigma \nabla \sigma - u \cdot \nabla u,
\end{align*}
\]

where \(\lambda = \frac{\gamma - 1}{2}\), and the \(i\)-th component of the vector \(Q_2\) is \(Q_{2i} = -\lambda \sigma \partial_i \sigma - u \cdot \nabla u_i\). In order to derive the different estimates on the irrotational part and the incompressible part of the velocity, the Helmholtz decomposition will be applied. Second, as in [21], the new good unknown \(g\) is introduced in the region \(|x| > 0\),

\[
g := (g_1, g_2, g_3)^T = u - \omega \sigma \quad \text{with} \quad g_i = u_i - \sigma \omega_i, \quad i = 1, 2, 3,
\]

where \(\omega = (\omega_1, \omega_2, \omega_3)^T = (\frac{x_1}{|x|}, \frac{x_2}{|x|}, \frac{x_3}{|x|})^T \in \mathbb{S}^2\). Note that the introduction of \(g\) is motivated by the second-order quasilinear wave equation (1.8) although (1.1) admits the non-zero vorticity and cannot be transformed a wave equation directly: In (1.8), due to \(u_i = \partial_i \phi\) and \(\sigma = -\partial_i \phi + \) higher order error terms of \(\partial \phi\), then \(g_i = (\partial_i + \omega_i \partial_t) \phi + \) higher order error terms of \(\partial \phi\). It is well known that \((\partial_i + \omega_i \partial_t) \phi\) is the good derivative in the study of the nonlinear wave equation (see [6]) since \((\partial_i + \omega_i \partial_t) \phi\) will admit more rapid space-time decay rates. By some ideas and methods
dealing with the null condition structures in [18–20] for the non-compactly supported solutions of 2D quasilinear wave equations, we can obtain better $L^\infty$ space-time decay rates of $g$. On the other hand, since the optimal time-decay rate of solutions to the 3D free wave equation is merely $(1 + t)^{-1}$ near the forward light cone surface, which is far to derive the existence time $T_C^\delta = \frac{\epsilon}{\delta}$ in Theorem 1.1 (ii). In fact, when such a $\delta = \epsilon - \epsilon^2$ is chosen, then the integral $\int_0^{T_C^\delta} \frac{dt}{1 + t} = O(e^{c\epsilon})$ is sufficiently large as $\epsilon \to 0$, which leads to that the usual energy $E(t)$ cannot be controlled well by the corresponding inequality $E'(t) \leq E(0) + \frac{c\epsilon}{1 + t}E(t)$. To overcome this difficulty, some better space-time weighted $L^\infty-L^\infty$ estimates of $g$ and $(\sigma, u)$ are obtained by introducing some auxiliary energies (including the space-time weighted energies of $\text{div} u$ and $\text{curl} u$) and looking for the null condition in the system (1.1) for the Chaplygin gases, meanwhile the suitable energy estimates of the vorticity are also derived. Based on these key estimates, Theorem 1.1 can be proved.

This paper is organized as follows. In Section 2, we will introduce the basic bootstrap assumptions, Helmholtz decomposition and some pointwise estimates. The estimates of the auxiliary energies and the good unknown $g$ are established in Section 3. Collecting the pointwise space-time estimates in Section 2 and 3, the Hardy inequality and the ghost weight method in [4], the related energy estimates are derived in Section 4. In Section 5, based on the previous energy inequalities and Gronwall’s inequalities, the proof of Theorem 1.1 is finished by the continuity argument.

2  \ SOME PRELIMINARIES

2.1  \ The vector fields and bootstrap assumptions

Define the spatial rotation vector fields

$$\Omega := x \wedge \nabla = (\Omega_{23}, \Omega_{31}, \Omega_{12})^T, \quad \Omega_{ij} := x_i \partial_j - x_j \partial_i.$$  

For a 3D vector-valued function $U$, denote

$$\tilde{\Omega}_{ij} U := \Omega_{ij} U + e_i \otimes e_j U - e_j \otimes e_i U,$$

where $e_i := (0, \ldots, 1, \ldots, 0)^T$. Let $\tilde{\Omega}_{ij} U_k = (\tilde{\Omega}_{ij} U)_k$ be the $k$-component of $\tilde{\Omega}_{ij} U$ rather than the operator $\tilde{\Omega}_{ij}$ acts on the component $U_k$.

According to the definitions of $\Omega$ and $\tilde{\Omega}$, it is easy to check that for the scalar function $f$ and the 3D vector-valued functions $U, V$,

$$\begin{align*}
\Omega_{ij} \text{div} U &= \text{div} \tilde{\Omega}_{ij} U, \quad \tilde{\Omega}_{ij} \text{curl} U = \text{curl} \tilde{\Omega}_{ij} U, \\
\tilde{\Omega}_{ij} \nabla f &= \nabla \Omega_{ij} f, \quad \tilde{\Omega}_{ij} (U \cdot \nabla V) = U \cdot \nabla (\tilde{\Omega}_{ij} V) + (\tilde{\Omega}_{ij} U) \cdot \nabla V.
\end{align*} \tag{2.1}$$

The spatial derivatives can be decomposed into the radial and angular components for $r = |x| \neq 0$,

$$\nabla = \omega \partial_r - \frac{1}{|x|} \omega \wedge \Omega.$$
For convenience, we schematically denote this decomposition as

\[ \partial_i = \omega_i \partial_r + \frac{1}{|x|} \Omega. \]  

(2.2)

For the multi-index \( a \), let

\[ \Gamma^a = S^a_z Z^a_z, \quad Z \in \{ \partial_i, \nabla, \Omega \}, \quad \tilde{\Gamma}^a = S^a_z \tilde{Z}^a_z, \quad \tilde{Z} \in \{ \partial_i, \nabla, \tilde{\Omega} \}. \]  

(2.3)

As a consequence, by acting \((S + 1)^a_z Z^a_z\) on the first equation and \((S + 1)^a_z \tilde{Z}^a_z\) on the second equation in (1.12), one can find the equations of \((\Gamma^a \sigma, \tilde{\Gamma}^a u)\) as follows:

\[ \begin{aligned}
\partial_t \Gamma^a \sigma + \text{div} \ \tilde{\Gamma}^a u &= Q^a_1 := \sum_{b+c=a} C^a_{bc} Q^{bc}_1, \\
\partial_t \tilde{\Gamma}^a u + \nabla \Gamma^a \sigma &= Q^a_2 := \sum_{b+c=a} C^a_{bc} Q^{bc}_2,
\end{aligned} \]  

(2.4)

where \( C^a_{bc} \) are some suitable constants \((C^a_{a0} = C^a_{0a} = 1)\) and

\[ \begin{aligned}
Q^{bc}_1 &:= -\lambda \Gamma^b \sigma \text{div} \ \Gamma^c u - \Gamma^b u \cdot \nabla \Gamma^c \sigma, \\
Q^{bc}_2 &:= -\lambda \Gamma^b \sigma \nabla \Gamma^c \sigma - \Gamma^b u \cdot \nabla \tilde{\Gamma}^c u.
\end{aligned} \]  

(2.5)

For integers \( m, m_1, m_2 \in \mathbb{N} \) with \( m_1 \geq 1 \) and \( m_2 \geq 2 \), set

\[ \begin{aligned}
E_m(t) &:= \sum_{|a| \leq m} \| (\tilde{\Gamma}^a u, \Gamma^a \sigma)(t, x) \|_{L^2_x}, \\
\mathcal{X}_{m_1}(t) &:= \sum_{|a| \leq m_1 - 1} \| (|x| - t) (\text{div} \ \tilde{\Gamma}^a u, \partial_i \tilde{\Gamma}^a u, \nabla \Gamma^a \sigma, \partial_i \Gamma^a \sigma)(t, x) \|_{L^2_x}, \\
\mathcal{Y}_{m_2}(t) &:= \sum_{|a| \leq m_2 - 2} \| (t) \nabla \text{div} \ \tilde{\Gamma}^a u, \nabla^2 \Gamma^a \sigma)(t, x) \|_{L^2(|x| \leq (t)/2)}, \\
\mathcal{W}_m(t) &:= \sum_{|a| \leq m} \| (|x|) \text{curl} \ \Gamma^a u(t, x) \|_{L^2_x}.
\end{aligned} \]  

(2.6)

Choose the integer \( N_1 \) such that \( N_1 + 3 \leq N \leq 2N_1 - 2 \). Throughout the whole paper, we make the following bootstrap assumptions: For \( t \delta \leq \kappa_0 \),

\[ \begin{aligned}
E_N(t) + \mathcal{X}_N(t) &\leq M \varepsilon, \\
\mathcal{Y}_{N_1}(t) &\leq M \varepsilon + M \delta (1 + t)^{1 + M' \varepsilon}, \\
\mathcal{W}_N(t) &\leq M \delta, \\
\mathcal{W}_{N-1}(t) &\leq M \delta (1 + t)^{M' \varepsilon},
\end{aligned} \]  

(2.7)

\[ \delta \leq \varepsilon, \quad 0 < M' \varepsilon \leq \frac{1}{8}, \quad M(\varepsilon + \kappa_0) \leq 1, \]

where the constants \( M \geq 1, M' > 0 \) and \( \kappa_0 > 0 \) will be chosen later on. In Section 5, we will prove that the constant \( M \) on the right hands of the first two lines in (2.7) can be improved to \( \frac{1}{2} M \).
2.2 The Helmholtz decomposition and commutators

For the 3D rapidly decaying vector function $U = (U_1, U_2, U_3)$, we divide it into the curl-free part $P_1U$ (irrotational) and the divergence-free part $P_2U$ (solenoidal), which is called Helmholtz decomposition

$$U = P_1U + P_2U := -\nabla(-\Delta)^{-1}\text{div}U + (-\Delta)^{-1}\text{curl}^2U. \quad (2.8)$$

In addition, it is noted that

$$\text{curl}^2U = \text{curl}^2P_2U = -\Delta P_2U + \nabla\text{div}P_2U = -\Delta P_2U. \quad (2.9)$$

We now give some useful inequalities.

**Lemma 2.1.** *For the 3D vector function $U$, it holds that*

$$\|\nabla U\|_{L^2_x} \lesssim \|\text{div}U\|_{L^2_x} + \|\text{curl}U\|_{L^2_x}, \quad (2.10)$$

$$\|\langle|x|-t\rangle\nabla U\|_{L^2_x} \lesssim \|\langle|x|-t\rangle\text{div}U\|_{L^2_x} + \langle t \rangle\|\langle|x|\rangle\text{curl}U\|_{L^2_x} + \|U\|_{L^2_x},$$

*where and below $A \lesssim B$ means $A \leq CB$ for a generic positive constant $C$ which is independent of $\varepsilon, \delta, \kappa_0, M, M'$ and $t$.*

*Proof.* Since the proof of (2.10) follows from the integration by parts directly, we omit it here. \hfill \Box

**Lemma 2.2.** *For the vector fields $\tilde{\Gamma}$ defined in (2.3), we have the commutators $[\tilde{\Gamma}, P_1] := \tilde{\Gamma}P_1 - P_1\tilde{\Gamma} = 0$ and $[\tilde{\Gamma}, P_2] = 0$.*

*Proof.* It suffices to prove $\tilde{\Gamma}P_2U = P_2(\tilde{\Gamma}U)$ since it is easy to know $[\tilde{\Gamma}, P_1] = [\tilde{\Gamma}, \text{Id} - P_2] = 0$ if $[\tilde{\Gamma}, P_2] = 0$.

For $\tilde{\Gamma} \in \{\partial_t, V\}$, then $[\tilde{\Gamma}, P_2] = 0$ is obvious. We now focus on the case of $\tilde{\Gamma} \in \{\tilde{\Omega}, S\}$.

According to (2.1) and (2.9), it holds that

$$-\Delta P_2(\tilde{\Omega}_{ij}U) = \text{curl}^2 \tilde{\Omega}_{ij}U = \tilde{\Omega}_{ij}\text{curl}^2U = \tilde{\Omega}_{ij}(-\Delta)P_2U = -\Delta(\tilde{\Omega}_{ij}P_2U).$$

This, together with the uniqueness of the solution to the equation $\Delta w = 0$ when $w$ suitably decays, yields $P_2(\tilde{\Omega}_{ij}U) = \tilde{\Omega}_{ij}P_2U$.

Analogously, $P_2(SU) = SP_2U$ comes from

$$-\Delta P_2(SU) = \text{curl}^2 SU = (S + 2)\text{curl}^2U = (S + 2)(-\Delta)P_2U = -\Delta(SP_2U). \quad \Box$$

2.3 The pointwise estimates

**Lemma 2.3.** *For the multi-indices $a, b$ with $|a| \leq N - 2$ and $b \leq N - 3$, it holds that*

$$\langle|x|\rangle^2 \langle \text{curl} \Gamma^a u(t, x) \rangle \lesssim W_{|a|+2}(t), \quad (2.11)$$
\[
\langle |x| - t \rangle \frac{1}{2} (|\Gamma^a u(t,x)| + |\Gamma^a \sigma(t,x)|) \lesssim E_{|a|+2}(t) + \mathcal{X}_{|a|+2}(t) + \langle t \rangle \mathcal{W}_{|a|+1}(t),
\]
and
\[
\langle |x| - t \rangle \frac{1}{2} (|\nabla \Gamma^b u(t,x)| + |\nabla \Gamma^b \sigma(t,x)|) \lesssim E_{|b|+3}(t) + \mathcal{X}_{|b|+3}(t) + \langle t \rangle \mathcal{W}_{|b|+2}(t).
\]

**Proof.** The proofs of (2.11) and (2.12) are motivated by Lemma 3.3 in [31]. Note that the main difference between (2.12) and Proposition 3.3 in [31] lies in the appearance of the vorticity on the right-hand side.

Recall the Sobolev-type inequalities (3.14b), (3.14c) and (3.14d) in [31] that for any \( |x| > 0 \),
\[
\langle |x| - t \rangle \frac{1}{2} |U(t,x)| \lesssim \| \partial_r \tilde{\Omega}^{\leq 1} U(t,y) \|_{L^2(|y| > |x|)} + \| \tilde{\Omega}^{\leq 2} U(t,y) \|_{L^2(|y| > |x|)},
\]
\[
\langle |x| - t \rangle \frac{1}{2} |V(t,x)| \lesssim \| \partial_r \tilde{\Omega}^{\leq 1} V(t,y) \|_{L^2(|y| > |x|)} + \| \tilde{\Omega}^{\leq 2} V(t,y) \|_{L^2(|y| > |x|)},
\]
where \( \tilde{\Omega}^{\leq m} := \sum_{0 \leq |a| \leq m} \tilde{\Omega}^a \) and \( W, U, V \) can be 3D vectors or scalar functions (for scalar functions, \( \tilde{\Omega} \) in (2.13) is replaced by \( \Omega \)).

At first, we deal with (2.11) in the region \( |x| \geq 1/4 \). Let \( W = \tilde{\Gamma}^a \text{curl} u \) in the first inequality of (2.13). Then
\[
|\langle |x| - t \rangle | \text{curl} \tilde{\Gamma}^a u(t,x)| \lesssim \| \partial_r \tilde{\Omega}^{\leq 1} \text{curl} \tilde{\Gamma}^a u(t,y) \|_{L^2(|y| > |x|)} + \| \partial_r \tilde{\Omega}^{\leq 2} \text{curl} \tilde{\Gamma}^a u(t,y) \|_{L^2(|y| > |x|)}
\]
\[
\lesssim \mathcal{W}_{|a|+2}(t),
\]
which derives (2.11) for \( |x| \geq 1/4 \). On the other hand, in the domain of \( |x| \leq 1/4 \), (2.11) follows from the standard Sobolev embedding theorem directly.

The proof of (2.12) in the region \( |x| \geq 1/4 \) follows from the choices of \( U = \tilde{\Gamma}^a u, \Gamma^a \sigma, V = \nabla \tilde{\Gamma}^a u, \nabla \Gamma^a \sigma \) in (2.13) and the weighted inequality (2.10).

Finally, we deal with (2.12) in the domain of \( |x| \leq 1/4 \). Choosing the cut-off function \( \chi(s) \in C^\infty \) such that
\[
0 \leq \chi(s) \leq 1, \quad \chi(s) = \begin{cases} 1, & s \leq 1/4, \\ 0, & s \geq 1/2. \end{cases}
\]
Applying the standard Sobolev embedding theorem to \( \chi(|x|)\Gamma^a u(t,x) \), one has
\[
|\chi(|x|)\Gamma^a u(t,x)| \lesssim \| \Gamma^a u(t,x) \|_{L^2(|x| \leq 1)} + \langle t \rangle^{-1} \| |x| - t \nabla \nabla^{\leq 1} \Gamma^a u(t,x) \|_{L^2(|x| \leq 1)}.
\]
By using (2.13) to the first term on the right-hand side of (2.15), we arrive at
\[
\langle t \rangle^{\frac{1}{2}} \| \Gamma^a u(t,x) \|_{L^2(|x| \leq 1)} \lesssim \left\| \langle |x| - t \rangle \frac{1}{2} \Gamma^a u(t,x) \right\|_{L^2(|x| \leq 1)} \leq \| \langle |x| - t \rangle \partial_r \tilde{\Omega}^{\leq 1} \Gamma^a u(t,y) \|_{L^2} + \| \tilde{\Omega}^{\leq 2} \Gamma^a u(t,y) \|_{L^2}.
\]
Substituting (2.16) into (2.15) yields (2.12) for $\Gamma^a u$. On the other hand, for $\Gamma^a \sigma, \nabla \Gamma^a u$ and $\nabla \Gamma^a \sigma$, (2.12) can be analogously proved.

**Lemma 2.4** (Sharp time decay of $P_1 u$ away from the conic surface $|x| = \langle t \rangle$). For $|a| \leq N - 2$ and $|x| \leq \langle t \rangle/4$, it holds that

$$
\langle t \rangle^\frac{3}{2} (|P_1 \Gamma^a u(t, x)| + |\Gamma^a \sigma(t, x)|) \leq E_{|a|+2}(t) + \mathcal{X}_{|a|+2}(t) + \mathcal{Y}_{|a|+2}(t) + \langle t \rangle \mathcal{W}_{|a|}(t). 
$$

(2.17)

**Proof.** At first, we consider the general function $\chi(\frac{|x|}{\langle t \rangle}) U(t, x)$, where the cut-off function $\chi$ is defined in (2.14). Let $x = \langle t \rangle y$ and then it follows from the standard Sobolev embedding theorem $H_x^2(\mathbb{R}^3) \hookrightarrow L_x^\infty(\mathbb{R}^3)$ that

$$
\|\chi(\frac{|x|}{\langle t \rangle}) U(t, x)\|_{L_x^\infty} = \|\chi(|y|) U(t, \langle t \rangle y)\|_{L_y^\infty}
\lesssim \|U(t, \langle t \rangle y)\|_{L_y^2} + \|\langle t \rangle (\nabla U(t, \langle t \rangle y))\|_{L_y^{2(|y| \leq 1/2)}} + \|\langle t \rangle^2 \chi(|y|) (\nabla_x U)(t, \langle t \rangle y)\|_{L_y^2}
\lesssim \langle t \rangle^{-\frac{3}{2}} \left( \|U(t, x)\|_{L_x^2} + \|\langle t \rangle \nabla U(t, x)\|_{L_x^{2(|x| \leq \langle t \rangle/2)}} + \langle t \rangle^2 \|\chi(\frac{|x|}{\langle t \rangle}) \nabla^2 U(t, x)\|_{L_x^2} \right). 
$$

(2.18)

Here we point out that $U(t, x) = P_1 \Gamma^a u(t, x)$ or $U(t, x) = \Gamma^a \sigma(t, x)$ will be chosen in (2.18). Note that the second term in the last line of (2.18) can be estimated as follows:

$$
\|\langle t \rangle \nabla P_1 \Gamma^a u\|_{L_x^{2(|x| \leq \langle t \rangle/2)}} \lesssim \|\langle |x| - t \rangle \nabla \Gamma^a u\|_{L_x^2} + \|\langle t \rangle \nabla P_2 \Gamma^a u\|_{L_x^2}
\lesssim \mathcal{X}_{|a|+1}(t) + \langle t \rangle \mathcal{W}_{|a|}(t) + E_{|a|}(t) + \langle t \rangle \|\text{curl} \ P_2 \Gamma^a u\|_{L_x^2}
\lesssim E_{|a|+2}(t) + \mathcal{X}_{|a|+2}(t) + \langle t \rangle \mathcal{W}_{|a|}(t),
$$

(2.19)

where the second inequality in (2.10) is used. We next deal with the third term in the last line of (2.18) with $U(t, x) = P_1 \Gamma^a u(t, x)$.

$$
\left\|\chi(\frac{|x|}{\langle t \rangle}) \nabla^2 P_1 \Gamma^a u(t, x)\right\|_{L_x^2} \lesssim \left\|\nabla \left( \chi(\frac{|x|}{\langle t \rangle}) P_1 \nabla \Gamma^a u(t, x) \right)\right\|_{L_x^2} + \langle t \rangle^{-1} \|\nabla P_1 \Gamma^a u\|_{L_x^{2(|x| \leq \langle t \rangle/2)}}. \quad (2.20)
$$

Applying the first inequality in (2.10) to $\chi(\frac{|x|}{\langle t \rangle}) P_1 \nabla \Gamma^a u$ yields

$$
\left\|\nabla \left( \chi(\frac{|x|}{\langle t \rangle}) P_1 \nabla \Gamma^a u(t, x) \right)\right\|_{L_x^2} \lesssim \left\|\chi(\frac{|x|}{\langle t \rangle}) \nabla \text{div} \ \Gamma^a u(t, x)\right\|_{L_x^2} + \langle t \rangle^{-1} \|\nabla P_1 \Gamma^a u\|_{L_x^{2(|x| \leq \langle t \rangle/2)}}. \quad (2.21)
$$

Substituting (2.19) and (2.21) into (2.20) derives

$$
\langle t \rangle^2 \left\|\chi(\frac{|x|}{\langle t \rangle}) \nabla^2 P_1 \Gamma^a u(t, x)\right\|_{L_x^2} \lesssim E_{|a|+2}(t) + \mathcal{X}_{|a|+2}(t) + \mathcal{Y}_{|a|+2}(t) + \langle t \rangle \mathcal{W}_{|a|}(t).
$$

(2.22)

Then (2.17) is achieved by plugging (2.19) and (2.22) into (2.18) with $U = P_1 \Gamma^a u$ or $U = \Gamma^a \sigma$. ☐
Lemma 2.5. For the multi-index $|a| \leq N - 2$, it holds that
\[
\|P_2 \tilde{\Gamma}^a u(t, x)\|_{L_x^\infty} \lesssim \mathcal{W}_{|a|+1}(t).
\] (2.23)

Proof. It concludes from the standard Sobolev embedding theorems $W^{1,6}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$, $H^1(\mathbb{R}^3) \cap C_0(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ and (2.10) that
\[
\|P_2 U(t, x)\|_{L_x^\infty} \lesssim \|P_2 U(t, x)\|_{L_x^6} + \|\nabla P_2 U(t, x)\|_{L_x^6}
\lesssim \|\nabla P_2 U(t, x)\|_{L_x^2} + \|\nabla^2 P_2 U(t, x)\|_{L_x^2}
\lesssim \|\text{curl} U(t, x)\|_{L_x^2} + \|\nabla \text{curl} U(t, x)\|_{L_x^2}.
\]
Thus, (2.23) is proved by choosing $U = \tilde{\Gamma}^a u$. □

Combining Lemmas 2.4 and 2.5 implies the following corollary.

Corollary 2.6. For $|a| \leq N - 2$ and $|x| \leq \langle \tau \rangle /4$, it holds that
\[
|\tilde{\Gamma}^a u(t, x)| + |\Gamma^a \sigma(t, x)| \lesssim \langle \tau \rangle^{-\frac{3}{2}} [E_{|a|+2}(t) + \mathcal{X}_{|a|+2}(t) + \mathcal{Y}_{|a|+2}(t)] + \mathcal{W}_{|a|+1}(t).
\] (2.24)

3 ESTIMATES OF THE AUXILIARY ENERGIES AND THE GOOD UNKNOWN $g$

3.1 Estimates of the auxiliary energies $\mathcal{X}_N(t)$ and $\mathcal{Y}_N(t)$

Lemma 3.1 (Weighted $\dot{H}_x^1$ estimate). Under bootstrap assumptions (2.7), for the integer $m$ with $1 \leq m \leq N$, it holds that
\[
\mathcal{X}_m(t) \lesssim E_m(t) + \mathcal{W}_{m-1}(t).
\] (3.1)

Proof. For the multi-index $|a| \leq m - 1$, it follows from the equations in (2.4) and direct computations that
\[
(|x|^2 - t^2)\partial_j \tilde{\Gamma}^a u_i = |x|^2 (Q_{2i}^a - \partial_i \Gamma^a \sigma) - t ST \Gamma^a u_i + tx_j \partial_j \tilde{\Gamma}^a u_i
\]
\[
= |x|^2 Q_{2i}^a - x_j (x_j \partial_i - x_i \partial_j) \Gamma^a \sigma - x_i ST \Gamma^a \sigma + tx_j \partial_i \Gamma^a \sigma - t ST \Gamma^a u_i
\]
\[
+ tx_j (\partial_j \tilde{\Gamma}^a u_i - \partial_i \tilde{\Gamma}^a u_j) + t(x_j \partial_i - x_i \partial_j) \tilde{\Gamma}^a u_j + tx_i \text{div} \tilde{\Gamma}^a u
\]
\[
= |x|^2 Q_{2i}^a - x_j \Omega_{ji} \Gamma^a \sigma - x_i ST \Gamma^a \sigma + tx_i Q_{1i}^a - t ST \Gamma^a u_i
\]
\[
+ tx_j \varepsilon_{ijk} \text{curl} \tilde{\Gamma}^a u_k + t \Omega_{ji} (\tilde{\Gamma}^a u_j),
\] (3.2)
where we have used the Einstein summation, and the fact of $\partial_j U_i - \partial_i U_j = \varepsilon_{ijk} \text{curl} U_k$ with the volume form $\varepsilon_{ijk}$ being the sign of the arrangement $\{i,j,k\}$. Note that the main difference between (3.2) and similar equality of $\partial_i P_1 \Gamma^a u_i$ in [30] lies in the presence of the vorticity $\text{curl} \tilde{\Gamma}^a u$ in (3.2).
Analogously, one can get
\[
(|x|^2 - t^2) \partial_t \Gamma^a \sigma = |x|^2 Q^a_x - x_j \Omega_{ji}(\Gamma^a u_i) - x_i S \Gamma^a u_i - t S \Gamma^a \sigma + t x_i Q^a_{2i},
\]
\[
(|x|^2 - t^2) \partial_i \Gamma^a \sigma = x_j \Omega_{ji} \Gamma^a \sigma + x_i S \Gamma^a \sigma - t x_i Q^a_1 - t^2 Q^a_{2i} + t S \Gamma^a \sigma
\]
\[
- t x_i \epsilon_{ijk} \text{curl} \hat{\Gamma}^a u_k - t \Omega_{ji}(\hat{\Gamma}^a u_j),
\]
\[
(|x|^2 - t^2) \text{div} \Gamma^a u = x_j \Omega_{ji}(\Gamma^a u_i) + x_i S \Gamma^a u_i - t x_i Q^a_1 - t^2 Q^a_{2i} + t S \Gamma^a \sigma.
\]

Due to \(||x| - t| \leq 1 + ||x| - t|\), by dividing \(|x| + t\) and then taking \(L^2_x\) norm on the both sides of (3.2) and (3.3), we arrive at
\[
\mathcal{X}_m(t) \leq E_m(t) + W_{m-1}(t) + \sum_{|b| + |c| \leq m-1, |c| \leq N_1-2} ||(|x| + t)(|Q^1_{bc}| + |Q^2_{bc}|)||_{L^2_x},
\]
where \(Q^1_{bc}, Q^2_{bc}\) are defined in (2.5).

Next, we estimate the term \(||(|x| + t)(|Q^1_{bc}| + |Q^2_{bc}|)||_{L^2_x}\) on the right-hand side of (3.4).

Since \(|b| + |c| \leq N - 1 \leq 2N_1 - 3\), then \(|b| \leq N_1 - 1\) or \(|c| \leq N_1 - 2\) holds. For the case of \(|c| \leq N_1 - 2\), applying (2.12) to \(\nabla \Gamma^c \sigma, \nabla \hat{\Gamma}^c u\) directly yields
\[
\sum_{|b| + |c| \leq m-1, |b| \leq N_1-1} ||(|x| + t)(|Q^1_{bc}| + |Q^2_{bc}|)||_{L^2_x} \leq E_m(t) [E_{N_1+1}(t) + \mathcal{X}_{N_1+1}(t) + \langle t \rangle W_{N_1}(t)] \leq E_m(t),
\]
where the assumptions (2.7) are used.

When \(|b| \leq N_1 - 1\), the related integral domain will be divided into two parts of \(|x| \geq \langle t \rangle / 8\) and \(|x| \leq \langle t \rangle / 8\).

In the region of \(|x| \geq \langle t \rangle / 8\), by using (2.12) to \(\Gamma^b \sigma, \hat{\Gamma}^b u\), one has
\[
\sum_{|b| + |c| \leq m-1, |b| \leq N_1-1} ||(|x| + t)(|Q^1_{bc}| + |Q^2_{bc}|)||_{L^2_x(|x| \geq \langle t \rangle / 8)} \leq E_m(t) [E_{N_1+1}(t) + \mathcal{X}_{N_1+1}(t) + \langle t \rangle W_{N_1}(t)] \leq E_m(t).
\]

In the region of \(|x| \leq \langle t \rangle / 8\), it concludes from (2.10) and (2.24) that
\[
||\langle |x| + t\rangle(|Q^1_{bc}| + |Q^2_{bc}|)||_{L^2_x(|x| \leq \langle t \rangle / 8)} \leq \|\hat{\Gamma}^b u, \Gamma^b \sigma\|_{L^\infty(|x| \leq \langle t \rangle / 8)} \left\{ ||(|x| - t) \nabla \Gamma^c \sigma||_{L^2_x} + \left\langle |||x| - t|| \nabla \Gamma^c u||_{L^2_x} \right\rangle \right. \]
\[
\leq \left\langle (t)^{-\frac{3}{2}} [E_{|b|+2}(t) + \mathcal{X}_{|b|+2}(t) + \mathcal{Y}_{|b|+2}(t)] + \mathcal{W}_{|b|+1}(t) \right\rangle \left\{ E_m(t) + \mathcal{X}_m(t) + \langle t \rangle W_{m-1}(t) \right\}. \]

Substituting the assumptions (2.7) into (3.7) yields
\[
\sum_{|b| + |c| \leq m-1, |b| \leq N_1-1} ||(|x| + t)(|Q^1_{bc}| + |Q^2_{bc}|)||_{L^2_x(|x| \leq \langle t \rangle / 8)} \leq E_m(t) + M \varepsilon \mathcal{X}_m(t) + \mathcal{W}_{m-1}(t).
\]

Collecting (3.4)–(3.6), (3.8) with the smallness of \(M \varepsilon\), we have achieved (3.1).
Lemma 3.2 (Weighted $\dot{H}^2_x$ estimate away from the conic surface $|x| = \langle t \rangle$). Under bootstrap assumptions (2.7), for each integer $m$ with $2 \leq m \leq N$, it holds that

$$\mathcal{Y}_m(t) \lesssim E_m(t) + \langle t \rangle \mathcal{W}_{m-1}(t).$$

(3.9)

Proof. For $|a| \leq m - 2$, replacing $\tilde{\Gamma}^a, \Gamma^a, Q^a_1, Q^a_2$ by $\partial_x \tilde{\Gamma}^a, \partial_x \Gamma^a, \partial_x Q^a_1, \partial_x Q^a_2$ in the last two identities of (3.3), respectively, one has that

$$\begin{align*}
(|x|^2 - t^2) \partial_x \Gamma^a \sigma &= x_J \Omega_{ji} \partial_x \Gamma^a \sigma + x_i S \partial_x \Gamma^a \sigma - t x_i \partial_x Q^a_1 - t^2 \partial_x Q^a_2 \\
+ t S \partial_x \tilde{\Gamma}^a u_i - t x_j \varepsilon_{jik} \partial_x \tilde{\Gamma}^a u_k - t \Omega_{ji} (\partial_x \tilde{\Gamma}^a u_j),
\end{align*}$$

$$\begin{align*}
\langle |x|^2 - t^2 \rangle \partial_x \Gamma^a u &= x_J \Omega_{ji} (\partial_x \Gamma^a u_i) + x_i S \partial_x \Gamma^a u_i - t x_i \partial_x Q^a_2 - t^2 \partial_x Q^a_1 \\
+ t S \partial_x \Gamma^a \sigma.
\end{align*}$$

(3.10)

Taking $L^2(|x| \leq \langle t \rangle / 2)$ norm on the both sides of (3.10) yields

$$\begin{align*}
\mathcal{Y}_m(t) &\lesssim \mathcal{X}_m(t) + \langle t \rangle \mathcal{W}_{m-1}(t) + \sum_{|a| \leq m-2} \|\langle |x|- t \rangle \nabla \Gamma^a u\|_{L^2} \\
+ \sum_{|a| \leq m-2} \|\langle t \rangle^2 (|\partial_x Q^a_1| + |\partial_x Q^a_2|)\|_{L^2(|x| \leq \langle t \rangle / 2)} \\
&\lesssim E_m(t) + \mathcal{X}_m(t) + \langle t \rangle \mathcal{W}_{m-1}(t) + \sum_{|a| \leq m-2} \|\langle |x|- t \rangle \nabla \Gamma^a u\|_{L^2} \|
\n+ \sum_{|a| \leq m-2} \|\langle |x|- t \rangle \nabla \Gamma^a u\|_{L^2} \|
\n\lesssim E_m(t) + \mathcal{X}_m(t) + \langle t \rangle \mathcal{W}_{m-1}(t) + \sum_{|a| \leq m-2} \|\langle |x|- t \rangle \nabla \Gamma^a u\|_{L^2} \|

(3.11)

where the inequality (2.10) is used. By the definitions of $Q^a_1, Q^a_2$ in (2.4), we have

$$\begin{align*}
\partial_x Q^a_1 &= - \sum_{b+c=a} C^a_{bc} (\lambda \Gamma^b \sigma \partial_x \nabla \Gamma^c u + \tilde{\Gamma}^b u \cdot \nabla \partial_x \Gamma^c \sigma + \lambda \partial_x \tilde{\Gamma}^b u \cdot \nabla \nabla \Gamma^c \sigma), \\
\partial_x Q^a_2 &= - \sum_{b+c=a} C^a_{bc} (\lambda \Gamma^b \nabla \partial_x \Gamma^c u + \Gamma^b u \cdot \nabla \partial_x \Gamma^c u + \lambda \partial_x \Gamma^b u \cdot \nabla \Gamma^c u). \\
\end{align*}$$

(3.12)

It suffices to deal with $\partial_x \tilde{\Gamma}^b u \cdot \nabla \Gamma^c u$ and $\Gamma^b u \cdot \nabla \partial_x \Gamma^c u$ in (3.12) since the treatments on the other left terms are similar.

In view of $|b| + |c| \leq N - 2 \leq 2N_1 - 4$, then $|b| \leq N_1 - 2$ or $|c| \leq N_1 - 2$ holds. Using (2.12) to $\partial_x \Gamma^b u$ or $\nabla \Gamma^c u$ directly leads to

$$\begin{align*}
\langle t \rangle \sum_{b+c=a} \|\partial_x \tilde{\Gamma}^b u \cdot \nabla \Gamma^c u\|_{L^2(|x| \leq \langle t \rangle / 2)} \\
\lesssim \sum_{|b| \leq N_1 - 2} \|\partial_x \tilde{\Gamma}^b u\|_{L^\infty} \|\langle |x|- t \rangle \nabla \Gamma^c u\|_{L^2} + \sum_{|c| \leq N_1 - 2} \|\langle |x|- t \rangle \partial_x \tilde{\Gamma}^b u\|_{L^2} \|\nabla \Gamma^c u\|_{L^\infty} \\
\lesssim \langle t \rangle^{-1} [E_{m-1}(t) + \mathcal{X}_{m-1}(t) + \langle t \rangle \mathcal{W}_{m-1}(t)] [E_{N_1+1}(t) + \mathcal{X}_{N_1+1}(t) + \langle t \rangle \mathcal{W}_{N_1}(t)] \\
\lesssim \langle t \rangle^{-1} [E_{m-1}(t) + \mathcal{X}_{m-1}(t) + \langle t \rangle \mathcal{W}_{m-1}(t)],
\end{align*}$$

(3.13)

where assumptions (2.7) and inequality (2.10) are used.
To treat $\bar{\Gamma}^b u \cdot \nabla \bar{\partial}_x \bar{\Gamma}^c u$, we divide the integral region $|x| \leq \langle t \rangle / 2$ into two parts of $\langle t \rangle / 4 \leq |x| \leq \langle t \rangle / 2$ and $|x| \leq \langle t \rangle / 4$.

In the region $\langle t \rangle / 4 \leq |x| \leq \langle t \rangle / 2$, it follows from (2.7), (2.10) and (2.12) that

$$\|\bar{\Gamma}^b u \cdot \nabla \bar{\partial}_x \bar{\Gamma}^c u\|_{L^2(\langle t \rangle / 4 \leq |x| \leq \langle t \rangle / 2)} \lesssim \langle t \rangle^{-1} \|\bar{\Gamma}^b u\|_{L^\infty(\langle t \rangle / 4 \leq |x| \leq \langle t \rangle / 2)} \|\langle |x| \rangle - t \rangle \nabla \bar{\partial}_x \bar{\Gamma}^c u\|_{L^2}$$

$$\lesssim \langle t \rangle^{-\frac{5}{2}} [E_m(t) + \mathcal{X}_m(t) + \langle t \rangle \mathcal{W}_{m-1}(t)][E_{|b|+2}(t) + \mathcal{X}_{|b|+2}(t) + \langle t \rangle \mathcal{W}_{|b|+1}(t)]$$

$$\lesssim \langle t \rangle^{-2} [E_m(t) + \mathcal{X}_m(t) + \langle t \rangle \mathcal{W}_{m-1}(t)].$$

In the region $|x| \leq \langle t \rangle / 4$, it concludes from (2.10), (2.22) and (2.24) that

$$\langle t \rangle^2 \|\bar{\Gamma}^b u \cdot \nabla \bar{\partial}_x \bar{\Gamma}^c u\|_{L^2(|x| \leq \langle t \rangle / 4)}$$

$$\lesssim \langle t \rangle^2 \|\bar{\partial}_x \bar{\Gamma}^c u\|_{L^\infty(|x| \leq \langle t \rangle / 4)} \left\{ \|\nabla \bar{\partial}_x P_2 \Gamma^c u\|_{L^2} + \|\chi\langle |x| \rangle \nabla \bar{\partial}_x P_1 \bar{\Gamma}^c u\|_{L^2} \right\}$$

$$\lesssim \left\{ \langle t \rangle^{-\frac{3}{2}} [E_m(t) + \mathcal{X}_m(t) + \mathcal{Y}_m(t)] + \mathcal{W}_{|b|+1}(t) \right\}$$

$$\times \left\{ \langle t \rangle^2 \mathcal{W}_{|c|+1}(t) + E_m(t) + \mathcal{X}_m(t) + \mathcal{Y}_m(t) \right\}$$

$$\lesssim \langle t \rangle^2 \mathcal{W}_{|c|+1}(t) \{M(\varepsilon\langle t \rangle)^{-\frac{3}{2}} + M\delta(t)^{M\varepsilon^{-\frac{1}{2}}} + \mathcal{W}_{|b|+1}(t)\} + \mathcal{X}_m(t) + \mathcal{W}_{m-1}(t) + M\varepsilon \mathcal{Y}_m(t),$$

where the assumptions (2.7) are applied.

Since $|b| \leq N_1 - 1$ or $|c| \leq N_1 - 1$ always holds, by using assumptions (2.7) again, we have

$$\langle t \rangle^2 \mathcal{W}_{|b|+1}(t) \mathcal{W}_{|c|+1}(t) \lesssim \langle t \rangle \mathcal{W}_{m-1}.$$ (3.16)

Therefore, combining (3.11)–(3.16) with the smallness of $M\varepsilon$ derives (3.9). □

### 3.2 Estimates of the good unknown $g$

**Lemma 3.3.** Under bootstrap assumptions (2.7), for the good unknown $g$ defined by (1.13) and $m \leq N - 1$, it holds that

$$\sum_{|a| \leq m} \|(\langle |x| \rangle \nabla \Gamma^a g\|_{L^2(|x| \geq \langle t \rangle / 8)} \lesssim E_{m+1}(t) + \mathcal{W}_m(t).$$ (3.17)

**Proof.** According to (2.2), it only needs to deal with $r \partial_r \bar{\Gamma}^a g_i = x \partial_j \bar{\Gamma}^a(u - \sigma \omega)_i$ in order to estimate $(|x|) V\Gamma^a g$ on the left-hand side of (3.17). It is deduced from direct computation that there exist some bounded smooth functions $f_i^{a,b}(x)$ and $f_{ij}^{a,b}(x)$ in $|x| \geq 1 / 8$ such that

$$\bar{\Gamma}^a(\sigma \omega)_i = \omega_i \Gamma^a \sigma + \frac{1}{|x|} \sum_{b+c \leq a} f_i^{a,b}(x) \Gamma^c \sigma,$$

$$\partial_j \bar{\Gamma}^a(\sigma \omega)_i = \omega_i \partial_j \Gamma^a \sigma + \frac{1}{|x|} \sum_{b+c \leq a} [f_i^{a,b}(x) \partial_j \Gamma^c \sigma + f_{ij}^{a,b}(x) \Gamma^c \sigma].$$ (3.18)
Thereafter, we arrive at
\[ r \partial_r \tilde{\Gamma}^a g_{ij} + \omega_j \sum_{b+c \leq a} \left[ f^{a,b}_i(x) \partial_j \Gamma^c \sigma + f^{a,b}_{ij}(x) \Gamma^c \sigma \right] \]
\[ = x_j \partial_j \tilde{\Gamma}^a u_i - \omega_i x_j \partial_i \tilde{\Gamma}^a \sigma \]
\[ = x_j (\partial_j \tilde{\Gamma}^a u_i - \partial_i \tilde{\Gamma}^a u_j) + (x_j \partial_i - x_i \partial_j) \tilde{\Gamma}^a u_j + x_i \text{div} \tilde{\Gamma}^a u - \omega_i S \tilde{\Gamma}^a \sigma + \omega_i t \partial_i \tilde{\Gamma}^a \sigma \]
\[ = x_j \varepsilon_{ijk} \text{curl} \tilde{\Gamma}^a u_k + \Omega_{ji}(\tilde{\Gamma}^a u_j) + x_i Q^a_i + \omega_i (t - |x|) \partial_i \tilde{\Gamma}^a \sigma - \omega_i S \tilde{\Gamma}^a \sigma. \]

By taking the \( L^2(\{|x| \geq \langle t \rangle/8\}) \) norm on the both sides of (3.19) and then substituting (3.1), (3.5) and (3.6) into the resulted inequality, one has
\[
\sum_{|a| \leq m} \|\langle |x| \rangle \partial_r \tilde{\Gamma}^a g\|_{L^2(\{|x| \geq \langle t \rangle/8\})} \lesssim E_{m+1}(t) + \mathcal{X}_m(t) + \mathcal{W}_m(t) + \sum_{|b|+|c| \leq |a|} \|\langle |x| \rangle Q^{bc}_1\|_{L^2(\{|x| \geq \langle t \rangle/8\})}
\]
\[
\lesssim E_{m+1}(t) + \mathcal{W}_m(t).
\]
This together with (2.2) yields
\[
\sum_{|a| \leq m} \|\langle |x| \rangle \nabla \tilde{\Gamma}^a g\|_{L^2(\{|x| \geq \langle t \rangle/8\})} \lesssim \sum_{|a| \leq m} \|\langle |x| \rangle \partial_r \tilde{\Gamma}^a g\|_{L^2(\{|x| \geq \langle t \rangle/8\})} + E_{m+1}(t).
\]
Thus (3.17) is proved.

**Lemma 3.4.** Under bootstrap assumptions (2.7), for \(|a| \leq N-2\) and \(|x| \geq \langle t \rangle/8\), it holds that
\[ \langle |x| + t \rangle^{\frac{3}{2}} |\tilde{\Gamma}^a g(t, x)| \lesssim E_{|a|+2}(t) + \mathcal{W}_{|a|+1}(t). \] (3.20)

**Proof.** Recall (3.19) of [31] that
\[
\left( |x|^4 |R(|x|)|^2 \int_{S^2} |U(t, |x|\omega)|^2 d\omega \right)^{\frac{1}{2}} \lesssim \left( \int_{|y| \geq |x|} \left[ |R(|y|)|^2 \partial_t U(t, y) \right]^2 + |R'(|y|)|^2 |U(t, y)|^2 \right)^{\frac{1}{2}} \left( \int_{|y| \geq |x|} |\tilde{\Omega}^{<1} U(t, y)|^2 dy \right)^{\frac{1}{2}}.
\] (3.21)
By choosing \( R(|x|) = \langle |x| \rangle \) and \( U(t, x) = \tilde{\Omega}^{<1} \tilde{\Gamma}^a g(t, x) \) in (3.21), one can deduce from \( \mathcal{W}^{1,4}(S^2) \hookrightarrow L^\infty(S^2) \) that for \(|x| \geq \langle t \rangle/8\),
\[
\langle |x| + t \rangle^{\frac{3}{2}} \langle |x^a| \rangle \lesssim \left( |x|^4 \langle |x| \rangle^2 \int_{S^2} \tilde{\Omega}^{<1} \tilde{\Gamma}^a g(t, |x|\omega)|^4 d\omega \right)^{\frac{1}{2}} \lesssim \sum_{|b| < |a|+2} \|\tilde{\Gamma}^b g(t, y)\|_{L^2(\{|y| \geq \langle t \rangle/8\})} + \sum_{|b| < |a|+1} \|\langle |y| \rangle \partial_r \tilde{\Gamma}^b g(t, y)\|_{L^2(\{|y| \geq \langle t \rangle/8\})} \lesssim E_{|a|+2}(t) + \mathcal{W}_{|a|+1}(t),
\]
where (3.17) is used in the last inequality. This completes the proof of Lemma 3.4.
4 | ENERGY ESTIMATES

Substituting Lemmas 3.1 and 3.2 into Lemmas 2.3, 2.4 and Corollary 2.6 yields the following:

**Lemma 4.1.** Under bootstrap assumptions (2.7), for multi-indices $a, b$ with $|a| \leq N - 2$ and $|b| \leq N - 3$, it holds that

\[
\langle |x| \rangle \langle |x| - t \rangle \frac{1}{2} (|\Gamma^a u(t, x)| + |\Gamma^a \sigma(t, x)|) \lesssim E_{|a|+2}(t) + \langle t \rangle \mathcal{W}_{|a|+1}(t),
\]

(4.1)

\[
\langle |x| \rangle \langle |x| - t \rangle \left(|\nabla \Gamma^b u(t, x)| + |\nabla \Gamma^b \sigma(t, x)|\right) \lesssim E_{|b|+3}(t) + \langle t \rangle \mathcal{W}_{|b|+2}(t),
\]

(4.2)

and for $|x| \leq \langle t \rangle / 4$,

\[
|\Gamma^a u(t, x)| + |\Gamma^a \sigma(t, x)| \lesssim \langle t \rangle^{-\frac{3}{2}} E_{|a|+2}(t) + \mathcal{W}_{|a|+1}(t),
\]

(4.3)

\[
|P_1 \Gamma^a u(t, x)| \lesssim \langle t \rangle^{-\frac{3}{2}} E_{|a|+2}(t) + \langle t \rangle^{-\frac{1}{2}} \mathcal{W}_{|a|+1}(t).
\]

(4.4)

4.1 | Elementary energy estimates

**Lemma 4.2.** Under bootstrap assumptions (2.7), it holds that if $1 < \gamma < 3$,

\[
E^2_N(t') \lesssim E^2_N(0) + \int_0^{t'} (M\varepsilon(t)^{-1} + M\delta) E^2_N(t) dt;
\]

(4.5)

if $\gamma = -1$,

\[
E^2_N(t') \lesssim E^2_N(0) + \int_0^{t'} \left\{ (M\varepsilon(t)^{-\frac{3}{2}} + M\delta) E^2_N(t) + M^2 \delta^2 \langle t \rangle M^\varepsilon^{-1} E_N(t) \right\} dt.
\]

(4.6)

**Remark 5.** Although the estimates of $g$ in Lemmas 3.3 and 3.4 are not used in proving (4.5), these estimates still hold true for the polytropic gases corresponding to $1 < \gamma < 3$ of (1.2).

**Proof.** For $|a| \leq N$, multiplying the first equation by $e^q \Gamma^a \sigma$ and the second equation by $e^q \Gamma^a u$ in (2.4), where the ghost weight $e^q = e^q(|x| - |t|) = e^{\arctan(|x| - |t|)}$ (such a ghost weight is chosen in [4] to establish the global small data solutions of 2D quasilinear wave equations with the first null and the second null conditions), and subsequently adding them, we have

\[
\frac{1}{2} \partial_t [e^q (|\Gamma^a \sigma|^2 + |\Gamma^a u|^2)] + \text{div} [e^q (1 + \lambda \sigma) \Gamma^a \sigma \Gamma^a u] + \frac{1}{2} \text{div} [e^q u (|\Gamma^a \sigma|^2 + |\Gamma^a u|^2)]
\]

\[
+ e^q \frac{1}{2(|x| - t)^2} \sum_{i=1}^3 \left\{ |\Gamma^a u_i - \omega_i \Gamma^a \sigma|^2 - u_i \omega_i (|\Gamma^a \sigma|^2 + |\Gamma^a u|^2) - 2\lambda \sigma \omega_i \Gamma^a \sigma \Gamma^a u_i \right\}
\]

\[
= \frac{1}{2} e^q (|\Gamma^a \sigma|^2 + |\Gamma^a u|^2) \text{div} u + \lambda e^q \Gamma^a \sigma \Gamma^a u \cdot \nabla \sigma + \sum_{b+c=a, \; c < a} e^q C^a_{bc}(Q^b_c \Gamma^a \sigma + Q^b_c \cdot \Gamma^a u).
\]

(4.7)
Integrating the above equality over $[0, t'] \times \mathbb{R}^3$ yields

$$E_{|u|}^2(t') + \sum_{i=1}^{3} \int_0^{t'} \int \frac{1}{|x| - t} |\bar{\Gamma}^a u_i - \omega_i \Gamma^a \sigma|^2 dx dt$$

$$\lesssim E_{|u|}^2(0) + \int_0^{t'} \int \left\{ |I^a| + \sum_{b+c=a, \quad \omega_c < \omega_a} \left( |Q_i^{bc} \Gamma^a \sigma| + |Q_2^{bc} \cdot \bar{\Gamma}^a u| \right) \right\} dx dt,$$

where

$$I^a := (|\Gamma^a \sigma|^2 + |\Gamma^a u|^2) \text{div} u + 2\lambda \Gamma^a \sigma \Gamma^a u \cdot \nabla \sigma$$

$$+ \frac{1}{|x| - t} \sum_{i=1}^{3} \left\{ u_i \omega_i (|\Gamma^a \sigma|^2 + |\Gamma^a u|^2) + 2\lambda \omega_i \Gamma^a \sigma \Gamma^a u_i \right\}.$$ (4.9)

To treat the integral of (4.9), the related integral domain will be divided into two parts of $|x| \leq \langle t \rangle/8$ and $|x| \geq \langle t \rangle/8$.

Since $|b| + |c| \leq N \leq 2N_1 - 2$, then $|b| \leq N_1 - 1$ or $|c| \leq N_1 - 2$ holds. In the region $|x| \leq \langle t \rangle/8$, by using (4.3) to $I^a, Q_1^{bc}, Q_2^{bc}$ with assumptions (2.7), we obtain

$$\int_{|x| \leq \langle t \rangle/8} |I^a| dx \lesssim E_N^2(t) \left\{ M \varepsilon \langle t \rangle^{-\frac{3}{2}} + M \delta \right\}$$

and

$$\int_{|x| \leq \langle t \rangle/8} \left( |Q_1^{bc} \Gamma^a \sigma| + |Q_2^{bc} \cdot \bar{\Gamma}^a u| \right) dx$$

$$\lesssim E_N^2(t) \left\{ \sum_{|b| \leq N_1 - 1} \left[ \langle t \rangle^{-\frac{3}{2}} E_{|b|+2}(t) + \mathcal{W}_{|b|+1}(t) \right] \right\} + \sum_{|c| \leq N_1 - 2} \left[ \langle t \rangle^{-\frac{3}{2}} E_{|c|+3}(t) + \mathcal{W}_{|c|+2}(t) \right]$$

$$\lesssim E_N^2(t) \left\{ M \varepsilon \langle t \rangle^{-\frac{3}{2}} + M \delta \right\}.$$ (4.11)

For the polytropic gases of $1 < \gamma < 3$, applying (4.1) and (4.2) to $I^a, Q_1^{bc}, Q_2^{bc}$ in the region $|x| \geq \langle t \rangle/8$ yields

$$\int_{|x| \geq \langle t \rangle/8} \left\{ |I^a| + \sum_{b+c=a, \quad \omega_c < \omega_a} \left( |Q_i^{bc} \Gamma^a \sigma| + |Q_2^{bc} \cdot \bar{\Gamma}^a u| \right) \right\} dx \lesssim E_N^2(t) \left\{ M \varepsilon \langle t \rangle^{-1} + M \delta \right\}.$$ (4.12)

Substituting (4.10)–(4.12) into (4.8) implies (4.5).

Next, we turn to the proof of (4.6). In this case, $\gamma = -1$ and $\lambda = -\frac{\gamma - 1}{2} = -1$ hold. We point out that in the region $|x| \geq \langle t \rangle/8$, the null condition structures of nonlinearities and the estimates in Subsection 3.2 will play a crucial role. According to the definition of good unknown (1.13) and
identities (3.18), one easily gets that
\[
\hat{\Gamma}^b u_i = \Gamma^b \gamma_i + \omega_i \Gamma^b \sigma + \frac{1}{|x|} \sum_{b_1 + b_2 \leq b} f_{i, b_1}^{b, b_2}(x) \Gamma^b \sigma
\]
\[
\partial_j \hat{\Gamma}^c u_i = \partial_j \hat{\Gamma}^c g_i + \omega_j \partial_j \Gamma^c \sigma + \frac{1}{|x|} \sum_{c_1 + c_2 \leq c} [f_{i, c_1}^{c, c_2}(x) \partial_j \Gamma^c \sigma + f_{i j}^{c, c_2}(x) \Gamma^c \sigma]
\]  
(4.13)

At first, we deal with \( I^a \) defined by (4.9). It concludes from the definition of good unknown \( g \) that
\[
I^a = \sum_{i=1}^3 \left\{ \text{div } u |\hat{\Gamma}^a u_i - \omega_i \Gamma^a \sigma|^2 + 2 \Gamma^a \sigma \hat{\Gamma}^a u_i (\omega_i \text{div } u - \partial_i \sigma) \right\}
\]
\[
+ \frac{1}{(|x| - t)^2} \sum_{i, j=1}^3 \left\{ u_i \omega_i |\hat{\Gamma}^a u_j - \omega_j \Gamma^a \sigma|^2 + 2 \omega_i \omega_j \Gamma^a \sigma \hat{\Gamma}^a u_j \right\}
\]
(4.14)

It follows from (2.2) and the second equality of (4.13) that
\[
\omega_i \text{div } u - \partial_i \sigma = \omega_i \left\{ \partial_j g_j + \frac{1}{|x|} \left[ f_{j}^{0,0}(x) \partial_j \sigma + f_{j j}^{0,0}(x) \sigma \right] \right\} + \frac{1}{|x|} \Omega \sigma
\]
(4.15)

Applying (3.20), (4.1) and (4.2) to (4.14) and (4.15) derives
\[
\int_{|x| \geq (t/8)} |I^a| dx \lesssim \sum_{i=1}^3 \int \frac{M \varepsilon}{(|x| - t)^2} |\hat{\Gamma}^a u_i - \omega_i \Gamma^a \sigma|^2 dx + E_2^2(t)\{M \varepsilon(t)^{-\frac{1}{2}} + M \delta\},
\]
(4.16)

where the Young’s inequality is used.

Next, we focus on the treatments of \( Q_{21}^{bc} \) and \( Q_{22}^{bc} \) defined by (2.5) with \( \gamma = -1 \).

For \( |c| \leq N_1 - 2 \), by using the second equality of (4.13), one arrives at
\[
Q_{21}^{bc} = -\Gamma^b u_j \partial_j \hat{\Gamma}^c u_i + \Gamma^b \sigma \partial_j \Gamma^c \sigma
\]
\[
= -\partial_j \hat{\Gamma}^c u_i (\Gamma^b u_j - \omega_j \Gamma^b \sigma) + \Gamma^b \sigma (\partial_j \Gamma^c \sigma - \omega_j \partial_j \hat{\Gamma}^c u_i)
\]
\[
= -\partial_j \hat{\Gamma}^c u_i (\Gamma^b u_j - \omega_j \Gamma^b \sigma) + \frac{1}{|x|} \Gamma^b \sigma \Omega \Gamma^c \sigma
\]
\[
- \omega_j \Gamma^b \sigma \left\{ \partial_j \hat{\Gamma}^c g_j + \frac{1}{|x|} \sum_{c_1 + c_2 \leq c} [f_{i, c_1}^{c, c_2}(x) \partial_j \Gamma^c \sigma + f_{i j}^{c, c_2}(x) \Gamma^c \sigma] \right\},
\]
and
\[
Q_{11}^{bc} = \Gamma^b \sigma \left\{ \partial_i \hat{\Gamma}^c g_i + \frac{1}{|x|} \sum_{c_1 + c_2 \leq c} [f_{i, c_1}^{c, c_2}(x) \partial_i \Gamma^c \sigma + f_{i i}^{c, c_2}(x) \Gamma^c \sigma] \right\}
\]
\[
- \partial_i \Gamma^c \sigma (\Gamma^b u_i - \omega_i \Gamma^b \sigma).
\]
(4.17)
Applying (4.2) to \( \nabla \Gamma^c u, \nabla \Gamma^c \sigma \) and then using the Young’s inequality to the resulted inequality yield

\[
\sum_{|c| \leq N_1 - 2} \int_{|x| \geq \langle t \rangle / 8} |\nabla \tilde{\Gamma}^c u (\tilde{\Gamma}^b u_j - \omega_j \Gamma^b \sigma)| dx \lesssim \int \frac{M \varepsilon}{\langle |x| - t \rangle^2} |\Gamma^b u_j - \omega_j \Gamma^b \sigma|^2 dx + E_N^2(t) \{ M \varepsilon(t)^{-2} + M \delta \}.
\] (4.19)

Similarly to the proof of (4.19), it concludes from (3.20), (4.1) and (4.2) with (4.17) and (4.18) that

\[
\sum_{b+c=a, |c| \leq N_1 - 2} \int_{|x| \geq \langle t \rangle / 8} ([Q_{1}^{bc} \Gamma^a \sigma] + [Q_{2}^{bc} \cdot \tilde{\Gamma}^a u]) dx \lesssim \sum_{b \leq a} \sum_{i=1}^{3} \int \frac{M \varepsilon}{\langle |x| - t \rangle^2} |\Gamma^b u_i - \omega_i \Gamma^b \sigma|^2 dx + E_N^2(t) \{ M \varepsilon(t)^{-\frac{3}{2}} + M \delta \}.
\] (4.20)

For \( |b| \leq N_1 - 1 \), substituting the first equality of (4.13) into (4.17) and (4.18) yields

\[
Q_{2i}^{bc} = -\partial_j \tilde{\Gamma}^c u_i \left\{ \Gamma^b g_j + \frac{1}{|x|} \sum_{b_1 + b_2 \leq b} f_j^{b_1 b_2} (x) \Gamma^b \sigma \right\} + \frac{1}{|x|} \Gamma^b \sigma \Omega^c \sigma \\
- \omega_j \Gamma^b \sigma \left\{ \tilde{\partial}_j \Gamma^c g_i + \frac{1}{|x|} \sum_{c_1 + c_2 \leq c} \left[ f_{j i}^{c_1 c_2} (x) \tilde{\partial}_j \Gamma^c \sigma + f_{j i}^{c_2 c_1} (x) \Gamma^c \sigma \right] \right\},
\] (4.21)

and

\[
Q_{1}^{bc} = \Gamma^b \sigma \left\{ \tilde{\partial}_i \Gamma^c g_i + \frac{1}{|x|} \sum_{c_1 + c_2 \leq c} \left[ f_{i j}^{c_1 c_2} (x) \tilde{\partial}_j \Gamma^c \sigma + f_{i i}^{c_2 c_1} (x) \Gamma^c \sigma \right] \right\} \\
- \partial_i \Gamma^c \sigma \left\{ \Gamma^b g_i + \frac{1}{|x|} \sum_{b_1 + b_2 \leq b} f_{i j}^{b_1 b_2} (x) \Gamma^b \sigma \right\}.
\] (4.22)

It is deduced from (3.17), (3.20), (4.1) and (4.2) that

\[
\sum_{b+c=a, c < a, |b| \leq N_1 - 1} \int_{|x| \geq \langle t \rangle / 8} ([Q_{1}^{bc} \Gamma^a \sigma] + [Q_{2}^{bc} \cdot \tilde{\Gamma}^a u]) dx \lesssim E_N^2(t) \{ M \varepsilon(t)^{-\frac{3}{2}} + M \delta \} + M^2 \delta^2 \langle t \rangle M \varepsilon^{-1} E_N(t).
\] (4.23)

Substituting (4.10), (4.11), (4.16), (4.20) and (4.23) into (4.8) derives (4.6). \(\square\)

### 4.2 Energy estimates of the vorticity

**Lemma 4.3.** Under bootstrap assumptions (2.7), it holds that

\[
\mathcal{W}_{N_1}^{2}(t') \lesssim \mathcal{W}_{N_1}^{2}(0) + \int_0^{t'} \left( M \varepsilon(t)^{-\frac{2}{3}} + M \delta \right) \mathcal{W}_{N_1}^{2}(t) dt,
\] (4.24)
\[
W_{N-1}^2(t') \lesssim W_{N-1}^2(0) + \int_0^{t'} (M\varepsilon(t)^{-1} + M\delta)W_{N-1}^2(t)dt.
\] (4.25)

**Proof.** It is easy to find the equation of vorticity as follows:

\[
(\partial_t + u \cdot \nabla) \text{curl } u = \text{curl } u \cdot \nabla u - \text{curl } u \text{ div } u.
\] (4.26)

By acting \((S + 1)^a Z^a\) on Equation (4.26), one can find the equation of \(\text{curl } \tilde{\Gamma}^au\):

\[
(\partial_t + u \cdot \nabla) \text{curl } \tilde{\Gamma}^a u = \sum_{b+c=a, \ c<\alpha} J^b_1 + \sum_{b+c=a} J^b_2,
\] (4.27)

where

\[
J^b_1 := \tilde{\Gamma}^b u \cdot \nabla \text{curl } \tilde{\Gamma}^c u,
\]

\[
J^b_2 := \text{curl } \tilde{\Gamma}^c u \cdot \nabla \tilde{\Gamma}^b u - \text{curl } \tilde{\Gamma}^c u \text{ div } \tilde{\Gamma}^b u.
\] (4.28)

Multiplying (4.27) by \(\langle|\cdot|\rangle^2 e^{q|\cdot|-t}\) curl \(\tilde{\Gamma}^a u\) and then integrating the resulted equality over \([0, t'] \times \mathbb{R}^3\) yield

\[
W_{|a|}^2(t') + \int_0^{t'} \int \frac{|\langle|\cdot|\rangle| \text{curl } \tilde{\Gamma}^a u|^2}{\langle|\cdot|\rangle - t} \, dx \, dt
\lesssim W_{|a|}^2(0) + \int_0^{t'} \int \frac{\langle|\cdot|\rangle \text{curl } \tilde{\Gamma}^a u|^2 \{ |\text{div } u| + \frac{|u|}{\langle|\cdot|\rangle} + \frac{|u|}{\langle|\cdot|\rangle - t} \}}{\langle|\cdot|\rangle - t} \, dx \, dt
\] (4.29)

\[
+ \int_0^{t'} \int \langle|\cdot|\rangle^2 |\text{curl } \tilde{\Gamma}^a u| \left\{ \sum_{b+c=a, \ c<\alpha} |J^b_1| + \sum_{b+c=a} |J^b_2| \right\} \, dx \, dt,
\]

where we have used integration by parts with respect to the space variables.

For the integrand in the second line of (4.29), it concludes from (4.1), (4.2), (4.3) and Young’s inequality that

\[
|\text{div } u| + \frac{|u|}{\langle|\cdot|\rangle} + \frac{|u|}{\langle|\cdot|\rangle - t} \lesssim M\varepsilon(t)^{-\frac{3}{2}} + \frac{M\varepsilon}{\langle|\cdot|\rangle - t}^2 + M\delta, \quad |\cdot| \geq \langle t/4,\n\]

\[
|\text{div } u| + |u| \lesssim M\varepsilon(t)^{-\frac{3}{2}} + M\delta, \quad |\cdot| \leq \langle t/4.
\] (4.30)

Next, we deal with \(J^b_1\) and \(J^b_2\) in the third line of (4.29). Similarly to the estimates in the former subsections, the related integral domain is also divided into two parts of \(|\cdot| \geq \langle t/4\) and \(|\cdot| \leq \langle t/4.\) \(J^b_1\) with \(|b| \leq |a| - 1 \leq N - 2\) in the region \(|\cdot| \leq \langle t/4: Due to \(|b| + |c| \leq N - 1 \leq 2N_1 - 3, then \(|b| \leq N_1 - 1\) or \(|c| \leq N_1 - 1\) holds. Therefore, it follows from (4.3) directly that

\[
\sum_{c<\alpha, |b|<|a|-1} \|\langle|\cdot|\rangle J^b_1\|_{L^2(|\cdot| \leq \langle t/4)} \lesssim \sum_{c<\alpha, |b|<|a|-1} W_{|c|+1}(t)\|\tilde{\Gamma}^b u\|_{L^\infty(|\cdot| \leq \langle t/4)}
\lesssim \sum_{c<\alpha, |b|<|a|-1} W_{|c|+1}(t)\langle t\rangle^{-\frac{3}{2}} E_{|b|+2}(t) + W_{|b|+1}(t)\}
\] (4.31)

\[
\lesssim W_{|a|}(t)M\varepsilon(t)^{-\frac{3}{2}} + M\delta.
\]
$J_{1}^{bc}$ with $|b| \leq |a| - 1$ in the region $|x| \geq \langle t \rangle / 4$; By using (4.1) to $\hat{\Gamma}^{b}u$ and taking the Young's inequality, one easily gets that for $|b| \leq |a| - 1$,

$$
\begin{align*}
|\hat{\Gamma}^{b}u(t,x)| & \lesssim \langle t \rangle^{-1}(|x| - t)^{-\frac{1}{2}}\{E_{|b|+2}(t) + \langle t \rangle\mathcal{W}_{|b|+1}(t)\} \\
& \lesssim M\varepsilon(t)^{-\frac{1}{2}} + \mathcal{W}_{|b|+1}(t) \\
& \lesssim M\varepsilon(t)^{-\frac{4}{3}} + M\varepsilon(|x| - t)^{-2} + \mathcal{W}_{|b|+1}(t).
\end{align*}
$$

(4.32)

For $J_{2}^{bc}$, note that $|b| \leq N_{1} - 2$ or $|c| \leq N_{1}$ holds. Subsequently, similarly to (4.31) and (4.32), we have that for $|b| \leq |a| - 2 \leq N - 3$,

$$
\begin{align*}
\sum_{|b| \leq |a|-2} \|\langle|x|\rangle J_{2}^{bc} \|_{L^{2}(|x| \leq \langle t \rangle / 4)} & \lesssim \sum_{|b| \leq |a|-2} \mathcal{W}_{|c|}(t)\|\nabla\hat{\Gamma}^{b}u\|_{L^{\infty}(|x| \leq \langle t \rangle / 4)} \\
& \lesssim \sum_{|b| \leq |a|-2} \mathcal{W}_{|c|}(t)\{\langle t \rangle^{-\frac{3}{2}}E_{|b|+3}(t) + \mathcal{W}_{|b|+2}(t)\} \\
& \lesssim \mathcal{W}_{|a|}(t)\{M\varepsilon(t)^{-\frac{3}{2}} + M\delta\},
\end{align*}
$$

(4.33)

and for $|x| \geq \langle t \rangle / 4$,

$$
\begin{align*}
|\nabla\hat{\Gamma}^{b}u(t,x)| & \lesssim \langle t \rangle^{-1}(|x| - t)^{-1}\{E_{|b|+2}(t) + \langle t \rangle\mathcal{W}_{|b|+2}(t)\} \\
& \lesssim M\varepsilon(t)^{-2} + M\varepsilon(|x| - t)^{-2} + \mathcal{W}_{|b|+2}(t).
\end{align*}
$$

(4.34)

By (4.31)–(4.34) and the fact of $|a| \leq N_{1} \leq N - 3$, in order to achieve the lower order energy estimate (4.24), it remains to control $J_{1}^{bc}$ with $b = a$ and $J_{2}^{bc}$ with $|b| \geq |a| - 1$ in the region $|x| \leq \langle t \rangle / 4$. For this purpose, by the Helmholtz decomposition (2.8), we have

$$
\begin{align*}
\sum_{b=a,c<a} \|\langle|x|\rangle J_{1}^{bc} \|_{L^{2}(|x| \leq \langle t \rangle / 4)} & \lesssim \sum_{b=a,c<a} \mathcal{W}_{|c|+1}(t)\|P_{1}\hat{\Gamma}^{b}u\|_{L^{\infty}(|x| \leq \langle t \rangle / 4)} \\
& \lesssim \mathcal{W}_{|a|}(t)\{M\varepsilon(t)^{-\frac{3}{2}} + M\delta\},
\end{align*}
$$

(4.35)

Applying the Hardy's inequality and (2.10) to the last term in (4.35) yield

$$
\|\langle|x|\rangle^{-1}P_{2}\hat{\Gamma}^{b}u\|_{L^{2}} \lesssim \|\nabla P_{2}\hat{\Gamma}^{b}u\|_{L^{2}} \lesssim \|\text{curl} P_{2}\hat{\Gamma}^{b}u\|_{L^{2}} \lesssim \mathcal{W}_{|a|}(t).
$$

(4.36)

Thereafter, by plugging (2.11), (4.4) and (4.36) into (4.35), one obtains

$$
\begin{align*}
\sum_{b=a,c<a} \|\langle|x|\rangle J_{1}^{bc} \|_{L^{2}(|x| \leq \langle t \rangle / 4)} & \lesssim \mathcal{W}_{|a|}(t)\{M\varepsilon(t)^{-\frac{3}{2}} + M\delta\} + \mathcal{W}_{3}(t)\mathcal{W}_{|a|}(t) \\
& \lesssim \mathcal{W}_{N_{1}}(t)\{M\varepsilon(t)^{-\frac{3}{2}} + M\delta\},
\end{align*}
$$

(4.37)
where we have used the bootstrap assumptions (2.7) with \( N_1 \geq 5 \) in the last line of (4.37). Analogously, one can get the following estimate of \( J_{2}^{bc} \),

\[
\sum_{\substack{b+c=a, \\
|b|>\mu |a|}} \|\langle |x| \rangle J_{2}^{bc} \|_{L^2(|x|\leq \langle t \rangle/4)} \lesssim \sum_{|b|\leq |a|, |c|<1} \left\{ W_{[c]}(t) \|P_1 \nabla \Gamma^b u\|_{L^\infty(|x|\leq \langle t \rangle/4)} + \|\langle |x| \rangle \text{curl} \Gamma^c u\|_{L^\infty} \|P_2 \nabla \Gamma^b u\|_{L^2} \right\} \]

(4.38)

\[
\lesssim W_{N_1}(t) \{ M\varepsilon(t)^{-\frac{3}{2}} + M\delta \}.
\]

Collecting (4.29)–(4.34), (4.37), (4.38) together with all \(|a| \leq N_1\), we eventually achieve

\[
W_{N_1}^2(t') + \sum_{|a|\leq N_1} \int_0^{t'} \left\| \langle |x| \rangle \frac{\text{curl} \Gamma^a u}{\langle |x| - t \rangle} \right\|^2_{L^2} dt \lesssim \left\{ \begin{array}{ll}
W_{N_1}^2(0) + \int_0^{t'} W_{N_1}^2(t) [M\varepsilon(t)^{-\frac{4}{3}} + M\delta] dt \\
+ M\varepsilon \sum_{|b|\leq N_1} \int_0^{t'} \left\| \langle |x| \rangle \frac{\text{curl} \Gamma^b u}{\langle |x| - t \rangle} \right\|^2_{L^2} dt
\end{array} \right.
\]

This together with the smallness of \( M\varepsilon \) implies (4.24).

At last, the proof of (4.25) is reduced to the case of \( N_1 + 1 \leq |a| \leq N - 1 \). In view of (4.31)–(4.34), we only need to treat \( J_1^{bc} \) and \( J_2^{bc} \) for \(|b|>\mu |a| - 1\). Note that \(|b| \leq |a| \leq N - 1 \) and \(|c| \leq 1 \leq N_1 - 3 \leq N - 4 \) hold. In the region \(|x| \geq \langle t \rangle/4\), applying (2.7) and (2.11) directly leads to

\[
\|\langle |x| \rangle J_1^{bc} \|_{L^2(|x|\geq \langle t \rangle/4)} \lesssim \langle t \rangle^{-1} E_{|b|}(t) \|\langle |x| \rangle^2 \nabla \Gamma^c u\|_{L^\infty} \lesssim \langle t \rangle^{-1} E_{|b|}(t) W_{[c]+3}(t) \lesssim M\varepsilon(t)^{-1} W_{N-1}(t).
\]

(4.39)

In the region \(|x| \leq \langle t \rangle/4\), it concludes form the Hardy inequality, (2.7), (2.10) and (3.1) that

\[
\|\langle |x| \rangle J_1^{bc} \|_{L^2(|x|\leq \langle t \rangle/4)} \lesssim \langle t \rangle^{-1} \left\| \langle |x| - t \rangle \frac{\nabla \Gamma^b u}{\langle |x| \rangle} \right\|_{L^2} \|\langle |x| \rangle^2 \nabla \Gamma^c u\|_{L^\infty} \lesssim \langle t \rangle^{-1} W_{[c]+3}(t) \{ E_{|b|}(t) + X_{|b|+1}(t) + \langle t \rangle W_{|b|}(t) \} \]

(4.40)

\[
\lesssim \langle t \rangle^{-1} W_{N-1}(t) E_{N}(t) + W_{N_1}(t) W_{N-1}(t) \lesssim W_{N-1}(t) [M\varepsilon(t)^{-1} + M\delta].
\]

Similarly, one can get the following estimate of \( J_{2}^{bc} \),

\[
\|\langle |x| \rangle J_{2}^{bc} \|_{L^2} \lesssim \|\langle |x| \rangle \frac{\text{curl} \Gamma^c u}{\langle |x| \rangle} \|_{L^\infty} \|\langle |x| - t \rangle \nabla \Gamma^b u\|_{L^2} \lesssim \langle t \rangle^{-1} W_{[c]+2}(t) \{ E_{|b|+1}(t) + \langle t \rangle W_{|b|}(t) \} \]

(4.41)

\[
\lesssim W_{N-1}(t) [M\varepsilon(t)^{-1} + M\delta(t)].
\]
For all $|a| \leq N - 1$, based on the estimate (4.24), substituting (4.30)–(4.34) and (4.39)–(4.41) into (4.29) yields

$$
\mathcal{W}_{N-1}^2(t') + \sum_{|a| \leq N-1} \int_0^{t'} \left\| \langle |x| \rangle \text{curl} \Gamma^a u \right\|_{L_x^2}^2 dt
\leq \mathcal{W}_{N-1}^2(0) + \int_0^{t'} \mathcal{W}_{N-1}^2(t) \{ M\varepsilon(t)^{-1} + M\delta \} dt
+ M\varepsilon \sum_{|b| \leq N-1} \int_0^{t'} \left\| \langle |x| \rangle \text{curl} \Gamma^b u \right\|_{L_x^2}^2 dt.
$$

Then (4.25) is proved.

\[\square\]

5 | PROOF OF THEOREM 1.1

Proof of Theorem 1.1.

(i) Applying the Growall’s inequality to (4.5), (4.24), (4.25) and then combining the resulted inequalities with (1.11), (3.1) and (3.9), we know that there exist two positive constants $C_1, C_2 \geq 1$ such that

$$
E_N(t) + \mathcal{X}_N(t) \leq C_1 \varepsilon (1 + t)^{C_2 M\varepsilon}, \quad \mathcal{Y}_N(t) \leq C_1 \varepsilon (1 + t)^{C_2 M\varepsilon} + C_1 \delta (1 + t)^{1+C_2 M\varepsilon},
$$

$$
\mathcal{W}_{N-1}(t) \leq C_1 \delta (1 + t)^{C_2 M\varepsilon}, \quad \mathcal{W}_{N-1}(t) \leq C_1 \delta.
$$

Choosing $M = 2eC_1, M' = 2eC_1C_2, \kappa_0 = \frac{1}{4eC_1C_2}$ and $\varepsilon_0 = \delta_0 = \frac{1}{4eC_1}$, one then obtains that for $t \leq T^p = \min \{ e^{\frac{\kappa_0}{\varepsilon}} - 1, \frac{\kappa_0}{\delta} \}$,

$$
E_N(t) + \mathcal{X}_N(t) \leq \frac{1}{2} M\varepsilon, \quad \mathcal{Y}_N(t) \leq \frac{1}{2} M\varepsilon + \frac{1}{2} M\delta (1 + t)^{1+M'\varepsilon},
$$

$$
\mathcal{W}_{N-1}(t) \leq \frac{1}{2} M\delta (1 + t)^{M'\varepsilon}, \quad \mathcal{W}_{N-1}(t) \leq \frac{1}{2} M\delta.
$$

This, together with the local existence of classical solution to (1.12) (see Chapter 2 of [28]), implies that (1.12) with (1.2) admits a unique solution $(\sigma, u) \in C([0, T^p], H^N(\mathbb{R}^3)) \cap C^1([0, T^p], H^{N-1}(\mathbb{R}^3))$. Hence, the proof of Theorem 1.1 (i) is completed.

(ii) Let $\tilde{E}_N(t) := \sup_{0 \leq s \leq t} E_N(s)$ and for convenience we still denote $\tilde{E}_N(t)$ as $E_N(t)$. Then it follows from (4.6) that

$$
E_N(t') \leq E_N(0) + \frac{M^2 \delta^2(t)^{M'\varepsilon}}{M'\varepsilon} + \int_0^{t'} (M\varepsilon(t)^{-\frac{3}{2}} + M\delta) E_N(t) dt.
$$

If $\delta \leq O(\varepsilon^{\frac{8}{7}})$, then for $t\delta \leq \kappa_0$, one has $\delta(t)^{M'\varepsilon} \leq \delta(t)^{\frac{1}{7}} \leq \delta^\frac{7}{8} \leq \varepsilon$. Plugging this inequality into (5.2) and utilizing the Growall’s inequality to the resulted inequality with (4.24) and (4.25), similarly to
the proof of (5.1), we know that there exist two positive constants $C_3, C_4 \geq 1$ such that

$$E_N(t) + \mathcal{X}_N(t) \leq C_3 \varepsilon (1 + \frac{M^2}{M'})^t, \quad \mathcal{Y}_N(t) \leq C_3 \varepsilon (1 + \frac{M^2}{M'}) + C_3 \delta (1 + t)^{1+ C_4 M \varepsilon},$$

$$\mathcal{W}_{N-1}(t) \leq C_3 \delta (1 + t)^C_4 M \varepsilon, \quad \mathcal{W}_{N_1}(t) \leq C_3 \delta.$$

Let $M = 4C_3$, $M' = \max\{16C_3^2, 4C_3 C_4\}$, $\kappa_0 = \frac{1}{8C_3}$ and $\varepsilon_0 = \delta_0 = \frac{1}{8M'}$, then for $t \leq T^C = \frac{\kappa_0}{\delta}$, we achieve

$$E_N(t) + \mathcal{X}_N(t) \leq \frac{1}{2} M \varepsilon, \quad \mathcal{Y}_N(t) \leq \frac{1}{2} M \varepsilon + \frac{1}{2} M \delta (1 + t)^{1+ M' \varepsilon},$$

$$\mathcal{W}_{N-1}(t) \leq \frac{1}{2} M \delta (1 + t)^{M' \varepsilon}, \quad \mathcal{W}_{N_1}(t) \leq \frac{1}{2} M \delta.$$  \hspace{1cm} (5.3)

If $\delta = O(\varepsilon^{1+\alpha})$ with $0 < \alpha < \frac{1}{7}$ and $M' \varepsilon_0 \leq \frac{\alpha}{1+\alpha} \leq \frac{1}{8}$, one finds $\delta (t)^{M' \varepsilon} \lesssim (1-M' \varepsilon_0) \lesssim \varepsilon$. Analogously, choosing $M, M', \kappa_0$ as before and $\varepsilon_0 = \delta_0 = \frac{\alpha}{M'(1+\alpha)}$, one can get (5.3). Therefore, (1.12) with (1.3) will admit a unique solution $(\sigma, u) \in C\left(\left[0, T^C\right], H^N(\mathbb{R}^3)\right) \cap C^1\left(\left[0, T^C\right], H^{N-1}(\mathbb{R}^3)\right)$, which completes the proof of Theorem 1.1 (ii) by the continuity argument. \hfill \Box

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