LOW MACH NUMBER LIMIT FOR THE COMPRESSIBLE MAGNETOHYDRODYNAMIC EQUATIONS IN A PERIODIC DOMAIN

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Abstract. This paper studies the convergence of the compressible isentropic magnetohydrodynamic equations to the corresponding incompressible magnetohydrodynamic equations with ill-prepared initial data in a periodic domain. We prove that the solution to the compressible isentropic magnetohydrodynamic equations with small Mach number exists uniformly in the time interval as long as that to the incompressible one does. Furthermore, we obtain the convergence result for the solutions filtered by the group of acoustics.

1. Introduction. In this paper we study the following isentropic compressible magnetohydrodynamic (MHD) equations (see [20, 21]):

\[
\begin{align*}
\partial_t \rho^\epsilon + \text{div} (\rho^\epsilon u^\epsilon) &= 0, \\
\partial_t (\rho^\epsilon u^\epsilon) + \text{div} (\rho^\epsilon u^\epsilon \otimes u^\epsilon) + \nabla P^\epsilon &= H^\epsilon \cdot \nabla H^\epsilon - \frac{1}{2} |H^\epsilon|^2 + \mu \Delta u^\epsilon + (\mu + \lambda) \nabla \text{div} u^\epsilon, \\
\partial_t H^\epsilon + (\text{div} u^\epsilon) H^\epsilon + u^\epsilon \cdot \nabla H^\epsilon - H^\epsilon \cdot \nabla u^\epsilon &= \nu \Delta H^\epsilon, \quad \text{div} H^\epsilon = 0, \\
(\rho^\epsilon, u^\epsilon, H^\epsilon)|_{t=0} &= (\rho_0^\epsilon, u_0^\epsilon, H_0^\epsilon).
\end{align*}
\]

Here the unknowns \( \rho^\epsilon = \rho^\epsilon(t, x) \in \mathbb{R}^+ \) denotes the dimensionless density of the fluid, \( u^\epsilon = u^\epsilon(t, x) \in \mathbb{R}^N \) the fluid velocity field, and \( H^\epsilon = H^\epsilon(t, x) \in \mathbb{R}^N \) the magnetic field, respectively. The spatial variable \( x \in T^N_a (N = 2, 3) \), where \( T^N_a \) denotes the torus with period \( 2\pi a_i \) in the \( i \)th component, and \( a = (a_1, \ldots, a_N) \) with \( a_i > 0 (i = 1, \ldots, N) \). We assume that the (rescaled) pressure \( P^\epsilon = P(\rho^\epsilon) \) is a smooth function of \( \rho^\epsilon \). The constants \( \mu > 0 \) and \( \lambda \) denote the shear and bulk viscosity coefficients of the flow, respectively. Physically, for the Newtonian flow, \( \mu \) and \( \lambda \) satisfy the condition \( 2\mu + N\lambda > 0 \).

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In [21], cooperated with D. Wang, we proved that the compressible MHD equations (1)-(4) converge to the following incompressible MHD equations when $\epsilon$ tends to zero in the whole space $\mathbb{R}^N$ with ill-prepared initial data:

$$
\begin{align*}
\partial_t v + v \cdot \nabla v - \mu \Delta v + \nabla \pi &= B \cdot \nabla B - \frac{1}{2} \nabla (|B|^2), \quad \text{div } v = 0, \\
\partial_t B + v \cdot \nabla B - B \cdot \nabla v &= \nu \Delta B, \quad \text{div } B = 0, \\
(v, B)|_{t=0} &= (v_0, B_0).
\end{align*}
$$

In the present paper, we study the same limit process in the periodic domain $T_a^N$ with ill-prepared initial data. The main difference between the periodic case and the whole space case is that, for the whole space case, the oscillation is dispersive hence disappears when $\epsilon$ goes to zero, while for the periodic case, the oscillation will survive for ever. Thus, some new ideas must be introduced to deal with the oscillations. (For the well-prepared initial data case, there is no essential difference between the whole space case and the periodic case, for example, see [22, 18].)

Without loss of generality, we assume that $\rho' \sim 1$ and $P'(1) = 1$. We assume further that the initial data $(\rho'_0 = 1 + \epsilon b'_0, u'_0, H'_0)$ are uniformly bounded (in some functional space). Thus, we can set $\rho := 1 + \epsilon b'$ and introduce the viscosity operator $A := \mu \Delta + (\lambda + \mu) \nabla \text{div}$ and the function $I(z) := \frac{1}{1+z}$, and denote

$$
\frac{\rho'(1+z)}{1+z} = 1 + \kappa z + z\tilde{K}(z) \quad \text{with} \quad \tilde{K}(0) = 0.
$$

With these notations, the problem (1)-(4) can be rewritten as follows:

$$
\begin{align*}
&\partial_t b' + \frac{\text{div } u'}{\epsilon} = -\text{div } (b' u'), \\
&\partial_t u' - Au' + \frac{(H' \cdot \nabla H' - \frac{1}{2} \nabla \cdot |H'|^2)}{1 + \epsilon b'} - u' \cdot \nabla u' - \kappa b' \cdot \nabla b' - \tilde{K}(\epsilon b')b' \cdot \nabla b' - I(\epsilon b')Au', \\
&\partial_t H' + (\text{div } u')H' + u' \cdot \nabla H' - H' \cdot \nabla u' = \nu \Delta H', \quad \text{div } H' = 0, \\
&(b', u', H')|_{t=0} = (b'_0, u'_0, H'_0), \quad \text{div } H'_0 = 0.
\end{align*}
$$

For presentation simplicity, we shall assume that the initial data $(b'_0, u'_0, H'_0)$ do not depend on $\epsilon$ and will be merely denoted by $(b_0, u_0, H_0)$ satisfying $v_0 = \mathcal{P} u_0$ and $B_0 = H_0$. Here $\mathcal{P}$ stands for the Leray projector on solenoidal vector fields and is defined by $\mathcal{P} := I - Q$ with $Q := \Delta^{-1} \nabla \text{div}$. As for the general case, since $\mathcal{P} u'_0$ tends to $v_0$ and $H'_0$ tends to $B_0$ when $\epsilon$ goes to 0, it can be treated in a similar way.

Let us mention some previous mathematical works on the low Mach number limit to the isentropic MHD equations (1)-(3). Hu and Wang [16] proved the convergence of the weak solutions of the compressible MHD equations (1)-(3) to a weak solution of the viscous incompressible MHD equations (5)-(7) in the whole space, the torus, or the bounded domain. Jiang, Ju, and Li obtained the convergence of the weak solutions of the compressible MHD equations (1)-(3) to the strong solution of the ideal incompressible MHD equations (i.e. (5)-(7) with $\mu = \nu = 0$) in the whole space [19] or the viscous incompressible MHD equations in torus [17] with general initial data. Later, Feireisl, Novotný, and Sun [13] extended and improved the results in [19] to the unbounded domain case. Li [22] studied the inviscid, incompressible limit of the viscous isentropic compressible MHD equations for local solutions with well-prepared initial data. Dou, Jiang, and Ju [11] studied the low Mach number limit for the compressible MHD equations in a 2D bounded domain with perfectly
conducting boundary. Fan, Li, and Nakamura [12] studied the low Mach number limit for the compressible magnetohydrodynamic equations (1)-(3) in a 3D bounded domain with small initial data. It should be pointed out that all of the above results were carried out in the framework of Sobolev spaces.

As for the framework of Besov space, we first recall some related results on the isentropic Navier-Stokes system (namely, $H = 0$ in the equations (1)-(3)). In [5], Danchin proved the global well-posedness of isentropic Navier-Stokes equations in the critical Besov space when the initial data is a small perturbation around some given constant state. Later, the results of [5] were extended to more general Besov space in [2, 4, 15]. In [6, 7, 8], Danchin obtained the local well-posedness of solutions the isentropic Navier-Stokes equations with large initial data. In [9, 10], Danchin studied the low Mach number limit of the isentropic Navier-Stokes equations in the whole space $\mathbb{R}^N$ with ill-prepared data, respectively. In [24], the second author considered the low Mach number limit for the compressible MHD equations (1)-(3) in the whole space $\mathbb{R}^N$ with small initial data in the critical Besov spaces. Cooperated with D. Wang [21], we established the convergence of compressible MHD equations to the incompressible MHD equations in the whole space $\mathbb{R}^N$. Here we consider the low Mach number limit to the system (1)-(3) in the torus $T^N_a$ with ill-prepared initial data.

As in [25], we introduce the following skew-symmetric operator $L$ defined on $\mathcal{S}'(T^N) \times \mathcal{S}'(T^N)$ by

$$L \begin{pmatrix} b \\ u \end{pmatrix} := \begin{pmatrix} \text{div } u \\ \nabla b \end{pmatrix}.$$ 

It is easy to see that $\text{Ker} L$ is the set of couples $(b, u)$ with $b$ constant and $\text{div } u = 0$, and

$$\text{Ker} L^\perp = \left\{ (b, \nabla \varphi) \mid b \in \mathcal{S}'(T^N), \int_{T^N} b(x) \, dx = 0 \text{ and } \varphi \in \mathcal{S}'(T^N) \right\}.$$

As pointed out in [10], the incompressible part of the velocity $\mathcal{P} u^\epsilon$ isn’t affected by the penalization. However, the compressible parts $(b^\epsilon, Q u^\epsilon)$ will experience high oscillations. In whole space $\mathbb{R}^N$, $(b^\epsilon, Q u^\epsilon)$ converge strongly to $(0,0)$ since the penalization operator has enough dispersive properties. However, in the torus $T^N_a$, we need to filter $(b^\epsilon, Q u^\epsilon)$ with the aid of the group $L(\tau) := e^{-\tau L}$ generated by $L$. The filter method has been used by many authors in the study of incompressible limit, for example, see [25, 14, 23, 17].

Denoting by $L^1(\tau)$ the first component of $L(\tau)$, and by $L^2(\tau)$ the last $N$ components. Letting $\eta := \lambda + 2\mu > 0$ and defining $V^\epsilon := \mathcal{L}(\frac{-t}{\epsilon})^T (b^\epsilon, Q u^\epsilon)$. We then obtain the following equations of $V^\epsilon$:

$$\begin{align*}
\partial_t V^\epsilon + \mathcal{Q}_1(\mathcal{P} u^\epsilon, V^\epsilon) + \mathcal{Q}_2(V^\epsilon, V^\epsilon) - \eta \mathcal{A}_2(D) V^\epsilon &= \mathcal{L} \left( -\frac{t}{\epsilon} \right) \\
0 &= \mathcal{Q} \left( \frac{1}{1 + \epsilon^2}(H^\epsilon \cdot \nabla H^\epsilon - \frac{1}{2} \nabla |H^\epsilon|^2) - \mathcal{P} u^\epsilon \cdot \nabla \mathcal{P} u^\epsilon - I(\epsilon b^\epsilon) \mathcal{A} u^\epsilon - \tilde{K}(\epsilon b^\epsilon) b^\epsilon \cdot \nabla b^\epsilon \right)
\end{align*}$$

with

$$\mathcal{A}_2(D) B := \mathcal{L} \left( -\frac{t}{\epsilon} \right) \left( \Delta \left( \mathcal{L}^2 \left( \frac{t}{\epsilon} \right) B \right) \right).$$

Formally, using the non-stationary phase arguments, the equations (12) tends to the following limit system

\[
\begin{aligned}
\partial_t V + Q_1(v, V) + Q_2(V, V) - \frac{\eta}{2} \Delta V &= 0, \\
V|_{t=0} &= V_0 := T(b_0, Q_{u_0}).
\end{aligned}
\]  

In the above system, \(v\) stands for the solution to the incompressible MHD equations (5)-(7), and the term \(Q_i(i = 1, 2)\) is bilinear.

Before stating our result, we introduce the following function spaces

\[
\begin{aligned}
F_T^a &:= \left\{ (u, H) \in \left( \tilde{C}_T(H^{s-1}) \right)^{N+1} \mid (u, H) \in \left( \tilde{L}_T^1(H^{s+1}) \right)^{N+1} \right\}, \\
G_T^a &:= \left( \tilde{L}^2(0, T; H^s) \right) \cap \tilde{C}_T(H^{s-1})^N, \\
E_{T, \sigma}^a &:= \left\{ (b, u, H) \in \tilde{C}_T(H_{s, \infty}^\sigma) \times (H^{s-1})^{N+1} \mid b \in \tilde{L}_T^1(H_{s, 1}^\sigma) \right. \\
&\quad \left. \text{and } (u, H) \in \left( \tilde{L}_T^1(H^{s+1}) \right)^{N+1} \right\}
\end{aligned}
\]

endowed with the norm

\[
\| (b, u, H) \|_{E_{T, \sigma}^a} := \|b\|_{\tilde{L}_T^1(H_{s, 1}^\sigma)} + \|b\|_{\tilde{L}^2_T(H_{s, \infty}^\sigma)} + \|(u, H)\|_{\tilde{L}_T^1(H^{s+1})}.
\]

For more details on the meanings of the above notations, see Appendix A blow.

Now, we state our main result as follows.

\textbf{Theorem 1.1.} Let \(\delta > 0\) and \(T_0 \in (0, +\infty]\). Assume that \(b_0 \in H_{s+1}^\delta\) and \((u_0, H_0) \in H_{s-1+\delta}^\delta\). Suppose that the incompressible MHD equations (5)-(8) with initial data \((P u_0, H_0)\) has a solution \((v, B) \in F_{T_0}^{s+\delta}\) on the time interval \([0, T_0]\) (in \(\mathbb{R}^+\) if \(T_0 = +\infty\). Then the limit system (13)-(14) has a unique solution \(V \in F_{T_0}^{s+\delta}\). Moreover, for a positive constant \(\epsilon_0\) depending only on \(\delta, N, \lambda, \mu, \nu, p, b_0, u_0, H_0, v, B\), and for all \(0 < \epsilon \leq \epsilon_0\), the system (8)-(11) has a unique solution \((b^\epsilon, u^\epsilon, H^\epsilon) \in E_{T_0, \epsilon\eta}^{s+\delta}\) uniformly in \(\epsilon\) satisfying that

\[
(P u^\epsilon - v, H^\epsilon - B) = O(\epsilon^{\frac{s}{s+1}}) \quad \text{in} \quad F_{T_0}^{s+\delta},
\]

and \(V^\epsilon\) tends to \(V\) in \(G_{T_0}^{s+\delta'}\) with \(\delta' < \delta\).

\textbf{Remark 1.1.} When \(H = 0\), the isentropic MHD equations (8)-(11) is reduced to the isentropic Navier-Stokes system. We point out that the results in Theorem 1.1 coincide with the ones obtained by Danchin \[10\] on the isentropic Navier-Stokes system. Thus, our results can be regarded as an extension of Danchin’s results to the isentropic MHD equations.

To prove Theorem 1.1, besides the difficulties mentioned in \[10\], we encounter a few new difficulties. For instance, in order to obtain the priori estimates for
the solution (see Proposition 4.1 below), we must find new appropriate reduced system analogous to the incompressible MHD equations (see (69)-(71) below). In this process, we need to do refined analyses to the nonlinear terms of the velocity and the magnetic field. Because of the strong coupling of the velocity and the magnetic field, the estimates for (69)-(71) are not as concise as the case of the isentropic Navier-Stokes equations. On the other hand, the coupling terms are analyzed in detail in each step of the proof of Theorem 1.1, see Sections 2–5 below. In our analysis, we shall make full use of the special structure of the isentropic compressible MHD equations.

Our paper is organized as follows. In Section 2, we prove that \( V^\epsilon \) tends to \( V \). In Section 3, we study the convergence \( (P\mu^\epsilon, H^\epsilon) \) to \( (v, B) \). In Section 4, we obtain a priori estimates for \( (b^\epsilon, u^\epsilon, H^\epsilon) \). In Section 5, based on the results obtained in Sections 2–4, we complete the proof of Theorem 1.1. Finally, we present an appendix to give the definition of the Littlewood-Paley decomposition, and recall some results in harmonic analysis.

**Notation.** Throughout the paper, the letter \( C \) stands for a generic constant independent of \( \epsilon \), and we use the notation \( A \lesssim B \) as an equivalent to \( A \leq CB \). The notation \( A \approx B \) means that both \( A \lesssim B \) and \( B \lesssim A \) hold simultaneously.

### 2. Convergence of the oscillating part of the system

Throughout the proof, below we shall use the following notations. Let \( s := \frac{N}{2} + \delta \). For \( T \in (0, +\infty) \), define:

\[
\begin{align*}
\zeta^\epsilon &:= V^\epsilon - V, \quad \omega_1 := P\mu^\epsilon - v, \quad \omega_2 := H^\epsilon - B, \\
X^\epsilon(T) &:= \|(b^\epsilon, Q\mu^\epsilon)\|_F, \quad X(T) := \|V\|_F, \\
P^\epsilon(T) &:= \|(P\mu^\epsilon, H^\epsilon)\|_F, \quad P(T) := \|(v, B)\|_F, \\
W^\epsilon(T) &:= \|P\mu^\epsilon - v\|_F, \quad Z^\epsilon(T) := \|V^\epsilon - V\|_F.
\end{align*}
\]

\[
X_0 := \|u_0\|_{L^\infty_T} + \|Q u_0\|_{H^s \Delta}, \quad P_0 := \|P u_0\|_{H^s \Delta}, \quad H_0 := \|H_0\|_{H^s \Delta}.
\]

\[
Y^\epsilon(T) := \|(1 + \epsilon b^\epsilon)^{-1}\|_{L^\infty_T (L^\infty)} + \|1 + \epsilon b^\epsilon\|_{L^\infty_T (L^\infty)}.
\]

Clearly, by means of Lemma A.3 and the definition of the space \( E^\epsilon_{T, \sigma} \), the following inequality holds:

\[
\|V^\epsilon\|_{L^\infty_T (H^{s \Delta - 1 \frac{3}{2}})} \leq X^\epsilon(T) \quad \text{for} \quad 2 \leq r \leq +\infty. \tag{15}
\]

In this section, we mainly conclude the following proposition.

**Proposition 2.1.** Let \( T \in (0, +\infty) \), \( (b^\epsilon, u^\epsilon, H^\epsilon) \in E^\epsilon_{T, \epsilon} \) be a solution to the problem (8)-(11), and \( (v, B) \in F^\epsilon_T \) (resp. \( V \in F^\epsilon_T \)) be the solution to the problem (5)-(7) (resp. (13)-(14)). Then the following estimate holds:

\[
Z^\epsilon \leq C(Y^\epsilon) e^{C(X^2 + P^2)} \left\{(Z^\epsilon)^2 + W^\epsilon (P + P^\epsilon + X^\epsilon + W^\epsilon)
+ \tau(\epsilon) \left(X_0 + X_0^2 + P_0^2 + X^2 + X^3 + P^2 + P^3 + X^\epsilon + (X^\epsilon)^2 + (X^\epsilon)^3
+ (P^\epsilon)^2 + (P^\epsilon)^3\right) + \frac{\epsilon^2}{2} X^\epsilon \left(X^\epsilon + P^\epsilon + (P^\epsilon)^2\right)\right\}, \tag{16}
\]

where \( C \) is a constant and \( \tau \) tends to zero when \( \epsilon \) goes to zero.
Since the proof of this proposition is similar to the case of the isentropic Navier-Stokes equations given by Proposition 3.1 in [10], we only give the sketch of the proof for brevity.

As in [10], we introduce the Hilbert basis $(\Phi^\alpha_k)_{\alpha \in -1, 1, k \in \mathbb{Z}^N \setminus \{0\}}$ of the space $\text{Ker} L^\perp$,

$$\Phi_k^\alpha (X) := c_N \left( - \alpha \text{sg}(k) k / |k| \right) e^{i k \cdot x} \quad \text{and} \quad c_N := (2|\mathbb{T}|^N)^{-\frac{1}{2}},$$

where $\text{sg}(k)$ denotes a sign function on $\mathbb{R}^N \setminus \{0\}$ and its value is 1 if and only if the first nonzero component of $k$ is positive, $-1$ elsewhere.

Clearly, we have $L \Phi_k^\alpha = -i \alpha \text{sg}(k) |k| \Phi_k^\alpha$. For any function $A := \sum_{\alpha, k} \hat{A}_k^\alpha \Phi_k^\alpha \in \text{Ker} L^\perp$ and $\mathcal{L}(\tau) = e^{-\tau L}$, one obtains that

$$\mathcal{L}(\tau) A = \sum_{\alpha, k} \hat{A}_k^\alpha \Phi_k^\alpha e^{i \alpha \text{sg}(k) |k| \tau}.$$

Meanwhile, by letting $v = \sum_k \hat{v}_k e^{i k \cdot x}$, then $Q'_1(u, B)$ and $Q'_2(A, B)$ can be written as

$$Q'_1(v, B) = \frac{i}{2} \sum_{m, \gamma} \left\{ \sum_{k, l = m} \hat{B}_k^\alpha \cdot \hat{v}_l \left( 1 + \frac{\alpha \gamma \text{sg}(k) \text{sg}(m)}{|k| |m|} (l + m) \cdot k \right) \right.\left. \times e^{i \frac{\chi}{2} (\alpha \text{sg}(k) |k| - \gamma \text{sg}(m)|m|)} \right\} \Phi_m^\gamma,$$

$$Q'_2(A, B) = -\frac{i c_N}{2} \sum_{m, \gamma} \left\{ \sum_{\alpha, \beta} \text{sg}(m)|m| \left( \frac{\hat{A}_k^\alpha \hat{B}_l^\beta + \hat{B}_k^\alpha \hat{A}_l^\beta}{2} \right) \right.\left. \times \left( \beta \text{sg}(l) \text{sg}(m) \frac{l \cdot m}{|m| |l|} + \frac{\gamma \kappa}{2} + \frac{\alpha \beta \gamma}{2} \text{sg}(k) \text{sg}(l) \frac{k \cdot l}{|k| |l|} \right) \right.\left. \times e^{i \frac{\chi}{2} (\alpha \text{sg}(k) |k| + \beta \text{sg}(l)|l| - \gamma \text{sg}(m)|m|)} \right\} \Phi_m^\gamma.$$

We define $Q_1(u, B)$ and $Q_2(A, B)$ as follows:

$$Q_1(u, B) = \frac{i}{2} \sum_{\gamma, m} \left\{ \sum_{k, l = m} \hat{B}_k^\alpha \cdot \hat{u}_l \frac{k \cdot m}{|k| |m|} \right\} \Phi_m^\gamma,$$

$$Q_2(A, B) = -i c_N \left( \frac{k + 3}{4} \right) \sum_{\gamma, m} \text{sg}(m)|m| \left\{ \sum_{k, l = m} \hat{A}_k^\alpha \hat{B}_l^\gamma \right\} \Phi_m^\gamma.$$

When $\epsilon$ goes to zero, it is easy to check that $Q'_i \to Q_i$ for $i = 1, 2$ in the sense of distributions. Finally, we have the convergence

$$\mathcal{A}_i^\epsilon(D)V = -\frac{1}{2} \sum_{\alpha, \gamma, k} \alpha \gamma |k|^2 \partial V \cdot \epsilon^{i \frac{\chi}{2} (\alpha - \gamma) \text{sg}(k)|k|} \Phi_m^\gamma \to \frac{1}{2} \Delta V \quad \text{as} \quad \epsilon \to 0.$$

In the proof of Proposition 2.1, we need the following Lemma. For the heat equation of the following type

$$\partial_t V + \hat{Q}(V) + \hat{Q}(V) - \mu \Delta V = F + G,$$

$$V_{t=0} = V_0,$$

we have
Lemma 2.1 ([10]). Let $s \in \mathbb{R}$ and $\dot{Q}$ (resp. $\dot{Q}$) be a time dependent nonlinear operator on $H^s$ (resp. $H^{s-1}$) valued in $H^{s-1}$ (resp. $H^{s-2}$). Let $V \in L^2(0; H^s) \cap L^\infty(0; T; H^{s-1})$ be a solution to the problem (17)-(18) with $F \in L^1(0; H^{s-1}), G \in L^2(0; T; H^{s-2})$ and $V_0 \in H^{s-1}$. Assume that for all $(A, B) \in H^s \times H^{s-1}$ and for a.e. $t \in (0, T)$,

$$
\|\dot{Q}(B)(t)\|_{H^s} \leq \tilde{K}(t)\|B\|_{H^{s-1}}, \quad \|\dot{Q}(A)(t)\|_{H^{s-1}} \leq \tilde{\tilde{K}}(t)\|A\|_{H^s}
$$

with $\tilde{K}, \tilde{\tilde{K}} \in L^2(0, T)$. Then there exists a universal constant $c > 0$ such that, for all $t \in [0, T]$,

$$
\|V\|_{L^2_1(H^{s-1})}^2 + c\mu\|V\|_{L^2_1(H^s)}^2 \leq 4e^\frac{c}{\mu} t_0^s (\tilde{K}^2(t) + \tilde{\tilde{K}}^2(t))d\tau \times \left\{ \|V_0\|_{H^{s-1}}^2 + \|F\|_{L^1_1(H^{s-2})}^2 + \frac{\|G\|_{L^2_1(H^{s-2})}^2}{c\mu} \right\}.
$$

By applying similar arguments to Lemma 4.2 in [10], we have

Lemma 2.2. Let $T \in (0, +\infty), (v', u', H') \in E^s_{0, e_n}$ be a solution to the problem (8)-(11), $(v, B) \in F^s_0$ (resp. $V \in F^s_0$) be the solution to the problem (5)-(7) (resp. (13)-(14)). Then, $\partial_x v, \partial_x B, \partial_x V$ and $\partial_x V'$ belong to the space $L^1_T(H^{s-1})$. Moreover, the following inequalities hold

$$
\|\partial_t(v, B)\|_{L^2_1(H^{s-1})} \leq C(P + P^2),
$$

$$
\|\partial_x V\|_{L^2_1(H^{s-1})} \leq CX(1 + P + X),
$$

$$
\|\partial_x V'\|_{L^2_1(H^{s-1})} \leq C(Y') \left\{ X' + (X')^2 + (P')^2 + c_2 X' (X' + P' + (P')^2) \right\}.
$$

Meanwhile, the following Lemma is needed in this section and next section.

Lemma 2.3 ([10]). The following estimates hold for $0 < s < 1 + \frac{N}{2}$ :

$$
\|Q_1^*(v, B)\|_{H^{s-1}} \lesssim \|Fv\|_{H^{s-1}(\tilde{\Omega}(2\pi N))} \|B\|_{H^s} + \|F B\|_{H^{s-1}(\tilde{\Omega}(2\pi N))} \|v\|_{H^s},
$$

$$
\|Q_1^*(v, B)\|_{H^{s-2}} \lesssim \min(\|v\|_{H^{s-2}}, \|B\|_{H^s} \|v\|_{H^{s-1}}), \quad (s > 2 - \frac{N}{2}),
$$

$$
\|Q_2^*(v, B)\|_{H^{s-2}} \lesssim \min(\|v\|_{H^{s-2}}, \|B\|_{H^s} \|v\|_{H^{s-1}}).
$$

Now, we begin to prove Proposition 2.1.

Proof of Proposition 2.1. The equation for $z'$ reads

$$
\partial_t z' + Q'_1(v, z') + 2Q'_2(V, z') + Q'_2(z', z') - \frac{n}{2} \Delta z' = F' + R^{1, z'} + R^{2, z'} + R^{3, z'} + S'
$$

with

$$
F' = -\mathcal{L} \left( -\frac{t}{\epsilon} \right) \left( \mathcal{Q}(v \cdot \nabla \omega' + \omega'_1 \cdot \nabla P u' + I(\epsilon b'), Au' + b' \tilde{K}(\epsilon b') \nabla b') \right) + \mathcal{L} \left( -\frac{t}{\epsilon} \right)
$$
In order to apply Lemma 2.1, we first need to decompose $Q_1$, then we need to make the change of function $\varphi$. If we denote $\tilde{Q}$, then $R^{1,\epsilon}$ and $S^\epsilon$ can be rewritten as $R^{1,\epsilon} = \mathcal{L} \left( -\frac{t}{\epsilon} \right) \left( Q(B) \cdot \nabla B - \frac{1}{2} \nabla |B|^2 - Q(v) \cdot \nabla v \right)$, $R^{2,\epsilon} = (Q_1 - \tilde{Q}) (v, \gamma)$, $R^{3,\epsilon} = (Q_2 - \tilde{Q}_2) (V, V) = \eta (A^\epsilon_2 (D) - \Delta/2) V^\epsilon$. If we denote $Q(B) \cdot \nabla B - \frac{1}{2} \nabla |B|^2 = \sum_{\alpha,k} \hat{B}_k^\alpha \Phi_k^\alpha$, and $Q(v) \cdot \nabla v = \sum_{\alpha,k} \hat{B}_k^\alpha \Phi_k^\alpha$, then $R^{1,\epsilon}$ and $S^\epsilon$ can be rewritten as $R^{1,\epsilon} = \sum_{\alpha,k} \left( \hat{B}_k^\alpha - \hat{B}_{\alpha k} \right) e^{i \frac{\epsilon}{2} \text{sgn}(k) |k|} \Phi_k^\alpha$, $S^\epsilon = \frac{1}{2} \sum_{\alpha,k} |k|^2 \hat{B}_k^\alpha e^{i \frac{\epsilon}{2} \text{sgn}(k) |k|} \Phi_k^\alpha$. Meanwhile, for $\gamma \in \{-1, 1\}$ and $m \in \hat{Z}^N \setminus \{0\}$, we denote $B^{1,\gamma}_m = \{ (\alpha, k, l) \in \{-1, 1\} \times \hat{Z}^N \times \hat{Z}^N \mid \text{sgn}(k) |k| \neq \gamma \text{sgn}(m) |m| \}$ and $k + l = m \}$, $B^{2,\gamma}_m = \{ (\alpha, \beta, k, l) \in \{-1, 1\} \times \hat{Z}^N \times \hat{Z}^N \mid \text{sgn}(k) |k| + \beta \text{sgn}(l) |l| \neq \gamma \text{sgn}(m) |m| \}$ and $k + l = m \}$. Then $R^{2,\epsilon}$ and $R^{3,\epsilon}$ read $R^{2,\epsilon} = -\frac{i}{2} \sum_{m, \gamma} \sum_{(\alpha, k, l) \in B^{1,\gamma}_m} \hat{V}^\alpha_k \cdot \hat{v}_l \left( 1 + \frac{\alpha \text{sgn}(k) \text{sgn}(m)}{|k||m|} \right) \left( l + m \cdot k \right) \times e^{i \frac{\epsilon}{2} \text{sgn}(k) |k| - \gamma \text{sgn}(m) |m|} \Phi_m^\gamma$, $R^{3,\epsilon} = \frac{i \gamma \epsilon}{2} \sum_{m, \gamma} \sum_{(\alpha, \beta, k, l) \in B^{2,\gamma}_m} \hat{V}^\alpha_k \hat{V}^\beta_l \text{sgn}(m) |m| \times \text{sgn}(k) \text{sgn}(l) \frac{l \cdot m}{|m||l|} + \frac{\alpha \gamma}{2} \frac{\text{sgn}(k) \text{sgn}(l) k \cdot l}{|k||l|} \times e^{i \frac{\epsilon}{2} \text{sgn}(k) |k| + \beta \text{sgn}(l) |l| - \gamma \text{sgn}(m) |m|} \Phi_m^\gamma$. In order to apply Lemma 2.1, we first need to decompose $\mathcal{Q}_1 (v, B)$ into $\mathcal{Q}_1 (v, B) = \tilde{Q}_1 (v, B) + \mathcal{Q}_1 (v, B)$ with $\tilde{Q}_1 (v, B) = \mathcal{Q}_1 (v, B) - \mathcal{Q}_1 (v, B)$, $\mathcal{Q}_1 (v, B) = \mathcal{L} \left( -\frac{t}{\epsilon} \right) \left( \text{div} \left( \frac{1}{\epsilon} \mathcal{L}^{\frac{1}{2}} B \right) \right)$. Then we need to make the change of function $\varphi = x^\epsilon - \epsilon \tilde{R}_M$ with $\tilde{R}_M = R^{1,\epsilon} + R^{2,\epsilon} + R^{3,\epsilon} + S^\epsilon$, and $S^\epsilon = \frac{i}{2} \sum_{|k| \leq M} \text{sgn}(k) |k| \hat{V}^\alpha_k \cdot e^{i \frac{\epsilon}{2} \text{sgn}(k) |k|} \Phi_k^{-\alpha}$.
According to (23) and (24), we obtain that
\[ \tilde{R}^{1,\epsilon}_{M} := i \sum_{\alpha,|\alpha| \leq M} \alpha_{\sg(k)}|k|^{-1} (\tilde{\beta}_{\k} - \tilde{\nu}_{\k}) e^{-i \frac{\alpha}{2} \sg(k)|k|} \Phi_{\k}^{\alpha}, \]
\[ \tilde{R}^{2,\epsilon}_{M} := -\frac{1}{2} \sum_{m,\gamma} \frac{V_{\k}^{\alpha} k \cdot \dot{\nu}_{l}}{\sg(k||k| - \sg(m)||m|} \left( 1 + \frac{\alpha_{\sg(k)}sg(m)}{|k||m|} (l + m) \cdot k \right) \times e^{i \frac{\alpha}{2} \sg(k)|k| - \sg(m)|m|} \Phi_{m}^{\gamma}, \]
\[ \tilde{R}^{3,\epsilon}_{M} := \frac{c_{N}}{2} \sum_{m,\gamma} \frac{V_{\k}^{\alpha} k \cdot \dot{\nu}_{l}}{\sg(k||k| + \sg(l)||l| - \sg(m)||m|} \left( \frac{\beta_{\sg(l)}sg(m)}{||m||} \right) \times e^{i \frac{\alpha}{2} \sg(k)|k| + \sg(l)|l| - \sg(m)|m|} \Phi_{m}^{\gamma}. \]

Here,
\[ B_{m,M}^{1,\gamma} := \{(\alpha, k, l) \in B_{m}^{1,\gamma} | |k| \leq M, |l| \leq M\}, \]
\[ B_{m,M}^{2,\gamma} := \{(\alpha, \beta, k, l) \in B_{m}^{2,\gamma} | |k| \leq M, |l| \leq M\}. \]

Further, we denote by \( R^\epsilon_{M} \) “the low frequency part” of \( R^\epsilon := R^{1,\epsilon} + R^{2,\epsilon} + R^{3,\epsilon} + S^\epsilon \) obtained by keeping only the indices \((k, l)\) such that \(|k|, |l| \leq M\) in the summations, and \( R^{\epsilon, M} := R^\epsilon - R^\epsilon_{M} \). Clearly, one has
\[ \epsilon \partial_{t} R_{M}^{\epsilon} = R_{M}^{\epsilon} + \epsilon \tilde{R}_{M}^{\epsilon} \]
with \( \tilde{R}_{M}^{\epsilon} := \tilde{R}_{M}^{1,\epsilon} + \tilde{R}_{M}^{2,\epsilon} + \tilde{R}_{M}^{3,\epsilon} + \tilde{S}_{M}^{\epsilon} \) and
\[ \tilde{S}_{M}^{\epsilon} := -\frac{i}{4} \sum_{|\k| \leq M} \sg(k)|k| \partial_{t} \hat{V}_{\k}^{\alpha,\epsilon} e^{2 \int_{0}^{t} \sg(k)|k| - \alpha}, \]
\[ \tilde{R}_{M}^{1,\epsilon} := i \sum_{|\k| \leq M} \sg(k)|k|^{-1} (\partial_{t} \hat{B}_{\k}^{\epsilon} - \partial_{l} \hat{V}_{\k}^{\beta}) e^{-i \frac{\alpha}{2} \sg(k)|k|} \Phi_{\k}^{\alpha}, \]
\[ \tilde{R}_{M}^{2,\epsilon} := -\frac{1}{2} \sum_{m,\gamma} \frac{\partial_{t} (\hat{V}_{\k}^{\alpha} k \cdot \dot{\nu}_{l})}{\sg(k)|k| - \sg(m)||m|} \left( 1 + \frac{\alpha_{\sg(k)}sg(m)}{|k||m|} (l + m) \cdot k \right) \times e^{i \frac{\alpha}{2} \sg(k)|k| - \sg(m)|m|} \Phi_{m}^{\gamma}, \]
\[ \tilde{R}_{M}^{3,\epsilon} := \frac{c_{N}}{2} \sum_{m,\gamma} \frac{\partial_{t} (\hat{V}_{\k}^{\alpha} \hat{V}_{\k}^{\beta})}{\sg(k||k| + \sg(l)||l| - \sg(m)||m|} \left( \frac{\beta_{\sg(l)}sg(m)}{||m||} \right) \times e^{i \frac{\alpha}{2} \sg(k)|k| + \sg(l)|l| - \sg(m)|m|} \Phi_{m}^{\gamma}. \]

According to (23) and (24), we obtain that
\[ \partial_{t} \varphi_{M}^{\epsilon} + \tilde{Q}_{1}(v, \varphi_{M}) + \tilde{Q}_{2}(v, \varphi_{M}) + 2 \varphi_{M} - \frac{\eta}{2} \Delta \varphi_{M}^{\epsilon}. \]
Applying Lemma 2.1 to the equation (25), we gather that

\[
\|\varphi_M\|_{L^p_T(H^{\frac{N}{2} - 1})} + \|\varphi_M\|_{L^p_T(H^{\frac{N}{2} - 2})} \leq C \|v\|_{L^2_T(H^{\sigma + \gamma})} + \|\varphi_M\|_{L^p_T(H^{\frac{N}{2} - 1})} + \|\varphi_M\|_{L^p_T(H^{\frac{N}{2} - 2})} + \|F^\varepsilon\|_{L^p_T(H^{\frac{N}{2} - 1})} + \|\varphi_M\|_{L^p_T(H^{\frac{N}{2} - 1})} + \|\varphi_M\|_{L^p_T(H^{\frac{N}{2} - 2})} + \|\varphi_M\|_{L^p_T(H^{\frac{N}{2} - 2})} \leq C \|v\|_{L^2_T(H^{\sigma + \gamma})} + \|F^\varepsilon\|_{L^p_T(H^{\frac{N}{2} - 1})} + \|\varphi_M\|_{L^p_T(H^{\frac{N}{2} - 1})} + \|\varphi_M\|_{L^p_T(H^{\frac{N}{2} - 2})} + \|\varphi_M\|_{L^p_T(H^{\frac{N}{2} - 2})}.
\]

With regard to the estimates in (26), we shall use frequently the following facts: For any $v \in S'(\mathbb{T}^N)$, $r \in [1, +\infty)$, $\sigma \in \mathbb{R}$, and $\eta > 0$, one gets

\[
\|v_M\|_{L^p_T(H^{\sigma + \gamma})} \leq CM^\eta \|v\|_{L^p_T(H^{\sigma + \gamma})}, \\
\|v_M\|_{L^p_T(H^{\sigma})} \leq CM^{-\eta} \|v\|_{L^p_T(H^{\sigma + \gamma})}, \\
\|F^\varepsilon\|_{L^p_T(H^{\frac{N}{2} + \gamma})} \leq CM^{-\eta} \|v\|_{H^{\sigma + \gamma}}.
\]

with $v_M := \sum_{|k| \leq M} \hat{v}_k e^{ikx}$, and $v_M := \sum_{|k| > M} \hat{v}_k e^{ikx}$.

Firstly, we claim that

\[
\|\hat{R}_M\|_{L^p_T(H^{\frac{N}{2} - 1})} + \|\hat{R}_M\|_{L^p_T(H^{\frac{N}{2} - 2})} \leq CM^\eta (X^\varepsilon + X^2 + P^2), \\
\|\hat{R}_M(0)\|_{H^{\frac{N}{2} - 1}} \leq CM^\eta (X^\varepsilon + X^2 + P^2).
\]

Clearly, the following inequalities hold

\[
\|\varphi_M^\varepsilon - \zeta^\varepsilon\|_{L^p_T(H^{\frac{N}{2} - 1})} + \|\varphi_M^\varepsilon - \zeta^\varepsilon\|_{L^p_T(H^{\frac{N}{2} - 2})} \leq CM^\eta (X^\varepsilon + X^2 + P^2), \\
\|\varphi_M(0)\|_{H^{\frac{N}{2} - 1}} \leq CM^\eta (X^\varepsilon + X^2 + P^2).
\]

Indeed, we denote \(\hat{R}_M^{1,\varepsilon} := \hat{R}_{M,1}^{1,\varepsilon} - \hat{R}_{M,2}^{1,\varepsilon}\) with

\[
\hat{R}_{M,1}^{1,\varepsilon} := i \sum_{|k| \leq M} \text{asg}(k) |k|^{-1} e^{-i\frac{\text{asg}(k)}{2} |k|^2} |\phi_k^\varepsilon|, \\
\hat{R}_{M,2}^{1,\varepsilon} := i \sum_{|k| \leq M} \text{asg}(k) |k|^{-1} e^{-i\frac{\text{asg}(k)}{2} |k|^2} |\phi_k^\varepsilon|.
\]

By the facts (27)-(29) and Lemma A.3, one has

\[
\|\hat{R}_{M,1}^{1,\varepsilon}\|_{L^p_T(H^{\frac{N}{2} - 1})} \leq CM^{1-\delta} \|\hat{R}_{1}^{1,\varepsilon}\|_{L^p_T(H^{\sigma - 3})} \leq CM^{1-\delta} \|Q(B \cdot \nabla B - \frac{1}{2} \nabla |B|^2)\|_{L^p_T(H^{\sigma - 3})} \leq CM^{1-\delta} P^2, \\
\|\hat{R}_{M,1}^{1,\varepsilon}\|_{L^p_T(H^{\frac{N}{2} - 2})} \leq CM^{1-\delta} \|\hat{R}_{1}^{1,\varepsilon}\|_{L^p_T(H^{\sigma - 2})} \leq CM^{1-\delta} \|Q(B \cdot \nabla B - \frac{1}{2} \nabla |B|^2)\|_{L^p_T(H^{\sigma - 2})} \leq CM^{1-\delta} P^2.
\]

Similarly,

\[
\|\hat{R}_{M,2}^{1,\varepsilon}\|_{L^p_T(H^{\frac{N}{2} - 1})} \leq CM^{1-\delta} \|Q(v \cdot \nabla v)\|_{L^p_T(H^{\sigma - 3})} \leq CM^{1-\delta} P^2.
\]
\[ \| \tilde{R}_{M,2}^{\epsilon} \|_{L^p_{\theta}(H^{\frac{N}{2}})} \leq CM^{1-\delta} \| Q(v \cdot \nabla v) \|_{L^p_{\theta}(H^{s-2})} \leq CM^{1-\delta} P^2. \]

Therefore, we have
\[ \| \tilde{R}_{M}^{\epsilon} \|_{L^p_{\theta}(H^{\frac{N}{2}-1})} + \| \tilde{R}_{M}^{\epsilon} \|_{L^p_{\theta}(H^{\frac{N}{2}})} \leq CM^{1-\delta} P^2. \]

As for \( \tilde{R}_{M}^{2,\epsilon}, \tilde{R}_{M}^{3,\epsilon} \) and \( \tilde{S}_{M}^{\epsilon} \), they can be treated similarly to that in [10]. Here, we only give their bounds as follows:
\[ \| \tilde{S}_{M}^{\epsilon} \|_{L^p_{\theta}(H^{\frac{N}{2}-1})} + \| \tilde{S}_{M}^{\epsilon} \|_{L^p_{\theta}(H^{\frac{N}{2}})} \leq CM^{1-\delta} X^{\epsilon}, \]
\[ \| \tilde{R}_{M}^{2,\epsilon} \|_{L^p_{\theta}(H^{\frac{N}{2}-1})} + \| \tilde{R}_{M}^{2,\epsilon} \|_{L^p_{\theta}(H^{\frac{N}{2}})} \leq C_{M}^1 M^2 X P, \]
\[ \| \tilde{R}_{M}^{3,\epsilon} \|_{L^p_{\theta}(H^{\frac{N}{2}-1})} + \| \tilde{R}_{M}^{3,\epsilon} \|_{L^p_{\theta}(H^{\frac{N}{2}})} \leq C_{M}^2 M^2 X^2 \]
with
\[ C_{M}^1 := \max_{\gamma \leq 1, \gamma \in \{1,1\}} \max_{(a,b) \in B_{\gamma,\gamma,0}^M} |\alpha(s)k| - \gamma s(\gamma)|m|^{-1}, \]
\[ C_{M}^2 := \max_{\gamma \leq 1, \gamma \in \{1,1\}} \max_{(a,b) \in B_{\gamma,\gamma,0}^M} |\beta(s)k| + \beta(s)l| - \gamma s(\gamma)|m|^{-1}. \]

Define
\[ C_{M} := C \max \left\{ \left( C_{M}^1 + C_{M}^2 \right) M^2, M^{1-\delta}, M^{2-2\delta} \right\}. \]

Collecting all the above estimates, then we conclude that (30) holds. It is easy to prove (31), we omit it here.

Next, we shall estimate the remaining terms on the right hand side of (26).

**Estimates for \( R_{M}^{\epsilon,\epsilon} \):** By (27)-(29) and Lemma A.3, one has
\[ \| R_{M}^{\epsilon,\epsilon} \|_{L^p_{\theta}(H^{\frac{N}{2}-1})} \leq CM^{-\delta} \| R_{M}^{\epsilon,\epsilon} \|_{L^p_{\theta}(H^{\frac{N}{2}+\epsilon-1})} \leq CM^{-\delta} \| Q(B \cdot \nabla B - \frac{1}{2} \nabla |B|^2 - \nabla v) \|_{L^p_{\theta}(H^{\epsilon-1})} \leq CM^{-\delta} P^2. \]

We give the estimates for \( R_{M}^{2,\epsilon,\epsilon}, \tilde{R}_{M}^{\epsilon,\epsilon} \) and \( \tilde{S}_{M}^{\epsilon,\epsilon} \) below and refer to [10] for more detail:
\[ \| R_{M}^{2,\epsilon,\epsilon} \|_{L^p_{\theta}(H^{\frac{N}{2}-1})} \leq CM^{-\delta} X \]
\[ \| R_{M}^{3,\epsilon,\epsilon} \|_{L^p_{\theta}(H^{\frac{N}{2}-2})} \leq CM^{-\delta} X^2 \]
\[ \| S_{M}^{\epsilon,\epsilon} \|_{L^p_{\theta}(H^{\frac{N}{2}-2})} \leq CM^{-\delta} X^2. \]

Thus, we have
\[ \| R_{M}^{\epsilon,\epsilon} \|_{L^p_{\theta}(H^{\frac{N}{2}-1}) + L^p_{\theta}(H^{\frac{N}{2}-2})} \leq CM^{-\delta}(X^{\epsilon} + X^2 + P^2). \] (34)

**Estimates for \( F^{\epsilon} \):** With the help of Lemma A.3, the following estimate holds:
\[ \| F^{\epsilon} \|_{L^p_{\theta}(H^{\frac{N}{2}-1})} \leq \| v \cdot \nabla \omega_{1} \|_{L^p_{\theta}(H^{\frac{N}{2}-1})} + \| \omega_{1} \cdot \nabla P_{\theta} \|_{L^p_{\theta}(H^{\frac{N}{2}-1})} + \| I(e_{0})A_{\theta}^{\epsilon} \|_{L^p_{\theta}(H^{\frac{N}{2}-1})} + \| B_{\theta}^{\epsilon} K(e_{0}) \nabla b_{\theta}^{\epsilon} \|_{L^p_{\theta}(H^{\frac{N}{2}-1})} + \| Q_{1}(\omega_{1}^{\epsilon}, V_{1}^{\epsilon}) \|_{L^p_{\theta}(H^{\frac{N}{2}-1})} + \| \nabla (\omega_{2}^{\epsilon} \cdot \nabla H^{\epsilon}) \|_{L^p_{\theta}(H^{\frac{N}{2}-1})} + \| B_{\theta} \cdot \nabla \omega_{2}^{\epsilon} \|_{L^p_{\theta}(H^{\frac{N}{2}-1})} + \| \nabla (\omega_{2}^{\epsilon} \cdot \nabla H^{\epsilon}) \|_{L^p_{\theta}(H^{\frac{N}{2}-1})} + \| \nabla \omega_{2}^{\epsilon} \cdot B \|_{L^p_{\theta}(H^{\frac{N}{2}-1})} + \| I(e_{0})^{\epsilon}(H^{\epsilon} \cdot \nabla H^{\epsilon} - \frac{1}{2} \nabla |H^{\epsilon}|^2) \|_{L^p_{\theta}(H^{\frac{N}{2}-1})}. \] (35)
For some terms on the right hand side of (35), we have the following estimates:
\[
\|\omega^2 \cdot \nabla H^e\|_{L^2_t(H^m \frac{H}{2} - \frac{1}{2})} \leq C\|\omega^2\|_{L^2_t(H^m \frac{H}{2} - \frac{1}{2})} \|\nabla H^e\|_{L^2_t(H^m \frac{H}{2} - \frac{1}{2})} \leq C\epsilon^r P^e,
\]
\[
\|B \cdot \nabla \omega^2\|_{L^2_t(H^m \frac{H}{2} - \frac{1}{2})} \leq C\|B\|_{L^2_t(H^m \frac{H}{2} - \frac{1}{2})} \|\nabla \omega^2\|_{L^2_t(H^m \frac{H}{2} - \frac{1}{2})} \leq CP^e,
\]
\[
\|\nabla \omega^2\|^2 \|_{L^2_t(H^m \frac{H}{2} - \frac{1}{2})} \leq C\|\omega^2\|^2_{L^2_t(H^m \frac{H}{2} - \frac{1}{2})} \leq C(W^r)^2,
\]
\[
\|\nabla (\omega^2 \cdot B)\|_{L^2_t(H^m \frac{H}{2} - 1)} \leq C\|\omega^2\|_{L^2_t(H^m \frac{H}{2} - \frac{1}{2})} \|B\|_{L^2_t(H^m \frac{H}{2} - 1)} \leq CP^e,
\]
\[
\|I(e^r)\|_{L^2_t(H^m \frac{H}{2} - 1)} \leq C\|I(e^r)\|_{L^\infty_t(L^\infty_{\infty})(H^m \frac{H}{2} - \frac{1}{2})} \|H^e \cdot \nabla H^e - \frac{1}{2}\nabla |H^e|^2\|_{L^2_t(H^m \frac{H}{2} - 1)}
\leq C\epsilon^r X^e(P^e)^2.
\]

The other terms on the right hand side of (35) are as same as these in [10] and we just list their estimates as follows:
\[
\|\omega^e \cdot \nabla \omega^e\|_{L^2_t(H^m \frac{H}{2} - \frac{1}{2})} \leq CP^e,
\]
\[
\|\omega^e \cdot \nabla P^e\|_{L^2_t(H^m \frac{H}{2} - \frac{1}{2})} \leq CP^e W^e,
\]
\[
\|I(e^r)\cdot A u^e\|_{L^2_t(H^m \frac{H}{2} - 1)} \leq C\epsilon^r X^e(X^e + P^e),
\]
\[
\|\nabla \cdot \nabla \omega^e\|_{L^2_t(H^m \frac{H}{2} - 1)} \leq C\epsilon^r X^e 3,
\]
\[
\|\nabla (\omega^e \cdot V^e)\|_{L^2_t(H^m \frac{H}{2} - 1)} \leq CP^e X^e.
\]

Thus, \(\|F^e\|_{L^2_t(H^m \frac{H}{2} - 1)}\) can be controlled by
\[
\|F^e\|_{L^2_t(H^m \frac{H}{2} - 1)} \leq CP^e (P + P^e + X^e + W^e)
+ C\epsilon^r X^e \left\{ P^e + X^e + (X^e)^2 + (P^e)^2 \right\}.
\]  

**Estimates for \(\tilde{R}^t_{M}^{i,e}\):** According to the expressions of \(\tilde{S}^{t,e}_{M}\) and \(\tilde{R}^{i,t,e}_{M}\) (1 \(\leq i \leq 3\)), we have
\[
\|\tilde{S}^{t,e}_{M}\|_{L^2_t(H^m \frac{H}{2} - 1)} \leq CM^1 - \delta \|\partial_t V^e\|_{L^2_t(H^m \frac{H}{2} - 1)},
\]
\[
\|\tilde{R}^{t,t,e}_{M}\|_{L^2_t(H^m \frac{H}{2} - 1)} \leq CM^2 \left\{ \|\partial_t V\|_{L^2_t(H^m \frac{H}{2} - 1)} \|V\|_{L^2_t(H^m \frac{H}{2} - 1)}
+ \|\partial_t B\|_{L^2_t(H^m \frac{H}{2} - 1)} \|B\|_{L^2_t(H^m \frac{H}{2} - 1)} \right\},
\]
\[
\|\tilde{R}^{t,t,e}_{M}\|_{L^2_t(H^m \frac{H}{2} - 1)} \leq C M^2 \left\{ \|\partial_t V\|_{L^2_t(H^m \frac{H}{2} - 1)} \|V\|_{L^2_t(H^m \frac{H}{2} - 1)}
+ \|\partial_t V\|_{L^2_t(H^m \frac{H}{2} - 1)} \|V\|_{L^2_t(H^m \frac{H}{2} - 1)} \right\}.
\]

Using Lemma 2.2, we eventually gather that
\[
\epsilon\|\tilde{R}^{t,e}_{M}\|_{L^2_t(H^m \frac{H}{2} - 1)} \leq C\epsilon C M \left\{ P^2 + P^3 + X^2 + X^3 + (X^e)^2 + (P^e)^2
+ \epsilon^r X^e (X^e + P^e + (P^e)^2) \right\}.
\]
The remaining terms on the right hand side of (26) have the following bounds, and we can refer to [10] for more details:

\[
\|Q_2(z^\epsilon, z^\epsilon)\|_{L^2_T(H^{s-2})} \leq C(Z^\epsilon)^2, \\
\|\tilde{Q}_1(v, R_M^\epsilon)\|_{L^1_T(H^{s-1})} + \|\tilde{Q}_1(v, R_M^\epsilon)\|_{L^2_T(H^{s-2})} + \|Q_2(V, R_M^\epsilon)\|_{L^2_T(H^{s-2})} \\
\leq CC_M(X + P)(X^\epsilon + P^2 + X^2).
\] (39)

Choosing \(M = \chi^{-1}(\epsilon^{-1})\) where \(\chi^{-1}\) is the inverse function of \(\chi\) with \(\chi(M) := C_M M^\delta\) and plugging (30)-(34) and (36)-(39) into (26), we conclude that (16) holds with \(\tau(\epsilon) := (\chi^{-1}(\epsilon^{-1}))^{-\delta}\).

3. Convergence of the incompressible part of the flow. This section is devoted to show the following proposition.

**Proposition 3.1.** Let \(T \in (0, +\infty), (b^\epsilon, u^\epsilon, H^\epsilon) \in E_T^\epsilon\) be a solution to the problem (8)-(11), \((v,B) \in F_T^\epsilon\) be the solution to the problem (5)-(7). Then the following estimate holds on \([0,T]\):

\[
W^\epsilon \leq C(\lambda^\epsilon)e^{C(X^\epsilon + P)} \left\{ (W^\epsilon)^2 + \epsilon \tau^\epsilon \left\{ X_0 P_0 + X^\epsilon \left( P + P^2 + P^\epsilon + (P^\epsilon)^2 \right) + (X^\epsilon)^2 \right\} + (P + P^\epsilon) \left( X^\epsilon + (X^\epsilon)^2 + (P^\epsilon)^2 \right) \right\},
\] (40)

where \(C > 0\) is a positive constant.

First, according to the equations in (8)-(11) and (5)-(7), we deduce the following system for \((\omega^\epsilon_1, \omega^\epsilon_2)\):

\[
\partial_t \omega^\epsilon_1 + \mathcal{P}(A^\epsilon \cdot \nabla \omega^\epsilon_1 + \omega^\epsilon_1 \cdot \nabla A^\epsilon) - \mathcal{P}(B \cdot \nabla \omega^\epsilon_2 + \omega^\epsilon_2 \cdot \nabla B) - \mu \Delta \omega^\epsilon_1 \\
= -\mathcal{P}(Qu^\epsilon \cdot \nabla v + v \cdot \nabla Qu^\epsilon + \omega^\epsilon_1 \cdot \nabla \omega^\epsilon_1) - \mathcal{P}(I(\epsilon b^\epsilon)u^\epsilon) \\
- \mathcal{P}(I(\epsilon b^\epsilon)(H^\epsilon \cdot \nabla H^\epsilon - \frac{1}{2} \nabla |H^\epsilon|^2)) + \mathcal{P}(\omega^\epsilon_2 \cdot \nabla \omega^\epsilon_2),
\] (41)

\[
\partial_t \omega^\epsilon_2 + \mathcal{P}(A^\epsilon \cdot \nabla \omega^\epsilon_2 - \omega^\epsilon_2 \cdot \nabla A^\epsilon) + \mathcal{P}(\omega^\epsilon_1 \cdot \nabla B - B \cdot \nabla \omega^\epsilon_1) - \nu \Delta \omega^\epsilon_2 \\
= \mathcal{P}(B \cdot \nabla Qu^\epsilon - Qu^\epsilon \cdot \nabla B + \omega^\epsilon_2 \cdot \nabla \omega^\epsilon_1 - \omega^\epsilon_1 \cdot \nabla \omega^\epsilon_2 - (\nabla u^\epsilon)H^\epsilon)
\] (42)

with \(A^\epsilon := Qu^\epsilon + v\). Here we have used the fact that \(\mathcal{P}(Qu^\epsilon \cdot \nabla Qu^\epsilon) = 0\).

Before we give the proof of Proposition 3.1, we study the estimate of the heat system of the following type:

\[
\partial_t Z + \mathcal{P}(A \cdot \nabla Z + Z \cdot \nabla A) - \mathcal{P}(E \cdot \nabla B + B \cdot \nabla E) - \mu \Delta Z = f, \\
\partial_t E + \mathcal{P}(A \cdot \nabla E - E \cdot \nabla A) + \mathcal{P}(Z \cdot \nabla B - B \cdot \nabla Z) - \nu \Delta E = g, \\
\text{div } f = \text{div } g = \text{div } Z = \text{div } E = 0, \\
(Z, E)|_{t=0} = (Z_0, E_0), \quad \text{div } Z_0 = \text{div } E_0 = 0.
\] (46)

We have

**Proposition 3.2.** Let \(s \in (1 - \frac{N}{2}, 1 + \frac{N}{2}), \mu := \min\{\mu, \nu\}\) and \((Z, E)\) be a solution to the problem (43)-(46). Then, there exists a constant \(C = C(N,s) > 0\) and a universal constant \(c > 0\) such that for all positive \(T\) (possibly infinite),

\[
\|(Z, E)\|_{L^\infty_T(H^{s+1})} + cT \|(Z, E)\|_{L^2_T(H^{s+1})}
\]
With regard to each term in the above equality, using Lemma 7.5 in [9], we have

$$
\leq \exp \left\{ C \int_{0}^{T} \left( \| A, B \|_{H^{\frac{3}{2}+\gamma} \cap \text{Lip}} + \| \nabla A \nabla B \|_{H^{\frac{3}{2}} \cap L^\infty} \right) dt \right\}
\times \left\{ \|(Z(0), E(0))\|_{H^{-1}} + \| (f, g) \|_{L^1_{T}(H^{-1})} \right\}.
$$

(47)

**Proof.** For brevity, we first study the following simplified model:

$$
\partial_t Z + \mathcal{P}(T_A, \partial_j Z - T_B, \partial_j E) - \mu \Delta Z = F,
$$

(48)

$$
\partial_t E + \mathcal{P}(T_A, \partial_j E - T_B, \partial_j Z) - \nu \Delta E = G.
$$

(49)

Localizing the system (48)-(49) by means of the Littlewood-Paley decomposition, and taking energy integration, finally, their summation, we get

$$
\frac{1}{2} \frac{d}{dt} \left( \| \Delta_q Z \|_{L^2}^2 + \| \Delta_q E \|_{L^2}^2 \right) + \mu \| \nabla \Delta_q Z \|_{L^2}^2 + \nu \| \nabla \Delta_q E \|_{L^2}^2
$$

$$
- \frac{1}{2} \int_{T_N} S_{q-1} \text{div} A(\| \Delta_q E \|^2 + |\Delta_q Z|^2) \, dx + \int_{T_N} S_{q-1} \text{div} B \Delta_q Z \Delta_q E \, dx
$$

$$
= (\Delta_q F, \Delta_q Z) + (\Delta_q G, \Delta_q E)
$$

$$
+ (S_{q-1} A^j \Delta_q \partial_j Z - \Delta_q T_A, \partial_j Z, \Delta_q Z) + (S_{q-1} A^j \Delta_q \partial_j E - \Delta_q T_A, \partial_j E, \Delta_q E)
$$

$$
- (S_{q-1} B^j \Delta_q \partial_j E - \Delta_q T_B, \partial_j E, \Delta_q Z) - (S_{q-1} B^j \Delta_q \partial_j Z - \Delta_q T_B, \partial_j Z, \Delta_q E).
$$

With regard to each term in the above equality, using Lemma 7.5 in [9], we have

$$
\int_{T_N} S_{q-1} \text{div} A(\| \Delta_q E \|^2 + |\Delta_q Z|^2) \, dx \leq C \| S_{q-1} \text{div} A \|_{L^\infty} \| (\Delta_q E, \Delta_q Z) \|_{L^2}^2
$$

$$
\leq C \| \nabla A \|_{L^\infty} \left( \| \Delta_q Z \|_{L^2}^2 + \| \Delta_q E \|_{L^2}^2 \right),
$$

$$
\int_{T_N} S_{q-1} \text{div} B \Delta_q Z \Delta_q E \, dx \leq C \| S_{q-1} \text{div} B \|_{L^\infty} \| \Delta_q Z \|_{L^2} \| \Delta_q E \|_{L^2}
$$

$$
\leq C \| \nabla B \|_{L^\infty} \left( \| \Delta_q Z \|_{L^2} + \| \Delta_q E \|_{L^2} \right),
$$

$$
(S_{q-1} A^j \Delta_q \partial_j Z - \Delta_q T_A, \partial_j Z, \Delta_q Z) \leq \| S_{q-1} A^j \Delta_q \partial_j Z - \Delta_q T_A, \partial_j Z \|_{L^2} \| \Delta_q Z \|_{L^2}
$$

$$
\leq C \| \nabla A \|_{L^\infty} \sum_{|q'-q| \leq 3} \| \Delta_{q'} Z \|_{L^2} \| \Delta_q Z \|_{L^2},
$$

$$
(S_{q-1} A^j \Delta_q \partial_j E - \Delta_q T_A, \partial_j E, \Delta_q E) \leq \| S_{q-1} A^j \Delta_q \partial_j E - \Delta_q T_A, \partial_j E \|_{L^2} \| \Delta_q E \|_{L^2}
$$

$$
\leq C \| \nabla A \|_{L^\infty} \sum_{|q'-q| \leq 3} \| \Delta_{q'} E \|_{L^2} \| \Delta_q E \|_{L^2},
$$

$$
(S_{q-1} B^j \Delta_q \partial_j E - \Delta_q T_B, \partial_j E, \Delta_q Z) \leq \| S_{q-1} B^j \Delta_q \partial_j E - \Delta_q T_B, \partial_j E \|_{L^2} \| \Delta_q Z \|_{L^2}
$$

$$
\leq C \| \nabla B \|_{L^\infty} \sum_{|q'-q| \leq 3} \| \Delta_{q'} E \|_{L^2} \| \Delta_q Z \|_{L^2},
$$

$$
(S_{q-1} B^j \Delta_q \partial_j Z - \Delta_q T_B, \partial_j Z, \Delta_q E) \leq \| S_{q-1} B^j \Delta_q \partial_j Z - \Delta_q T_B, \partial_j Z \|_{L^2} \| \Delta_q E \|_{L^2}
$$

$$
\leq C \| \nabla B \|_{L^\infty} \sum_{|q'-q| \leq 3} \| \Delta_{q'} Z \|_{L^2} \| \Delta_q E \|_{L^2}.
$$

Therefore, we gather that

$$
\frac{1}{2} \frac{d}{dt} \left( \| \Delta_q Z \|_{L^2}^2 + \| \Delta_q E \|_{L^2}^2 \right) + \nu \left( \| \nabla \Delta_q Z \|_{L^2}^2 + \| \nabla \Delta_q E \|_{L^2}^2 \right)
$$

$$
\leq C \left( \| \Delta_q Z \|_{L^2} + \| \Delta_q E \|_{L^2} \right) \left\{ \| \Delta_q F \|_{L^2} + \| \Delta_q G \|_{L^2} \right\}
$$
With the aids of Proposition A.4, it follows that

\[
\|\Delta_q Z\|_{L^2} + \|\Delta_q E\|_{L^2} + C 2^q (\|\Delta_q Z\|_{L^1} + \|\Delta_q E\|_{L^1}) \\
\leq \|\Delta_q Z(0)\|_{L^2} + \|\Delta_q E(0)\|_{L^2} + \|\Delta_q F\|_{L^1} + \|\Delta_q G\|_{L^1} \\
+ C \sum_{|q' - q| \leq 3} \int_0^t (\|\nabla A\|_{L^\infty} + \|\nabla B\|_{L^\infty}) (\|\Delta_q Z\|_{L^2} + \|\Delta_q E\|_{L^2}) \, dt. \tag{50}
\]

Taking the time integration and using the Bernstein's inequality in the above inequality, we obtain that

\[
\|\Delta_q Z\|_{L^\infty} + \|\Delta_q E\|_{L^\infty} + C 2^q (\|\Delta_q Z\|_{L^1} + \|\Delta_q E\|_{L^1}) \\
\leq \|\Delta_q Z(0)\|_{L^2} + \|\Delta_q E(0)\|_{L^2} + \|\Delta_q F\|_{L^1} + \|\Delta_q G\|_{L^1} \\
+ C \sum_{|q' - q| \leq 3} \int_0^t (\|\nabla A\|_{L^\infty} + \|\nabla B\|_{L^\infty}) (\|\Delta_q Z\|_{L^2} + \|\Delta_q E\|_{L^2}) \, dt. \tag{51}
\]

Here we have used Minkowski's inequality to bound the last term.

Since \(\mathcal{P}(T_{A, \partial_t Z}, \mathcal{P}(T_{B, \partial_t E}) \neq 0\) and \(\partial_t E_0 = 0\), it is straightforward to arrive at

\[
\|\hat{Z}_0\|_{L^2} \leq \|\hat{Z}_0(0)\|_{L^2} + \|\hat{F}_0\|_{L^1} + \|\hat{G}_0\|_{L^1}.
\]

Adding the above inequalities into (51), we end up with

\[
\|\hat{Z}\|_{L^\infty} + \|\hat{E}\|_{L^\infty} + C 2^q (\|\hat{Z}\|_{L^1} + \|\hat{E}\|_{L^1}) \\
\leq \|\hat{Z}(0)\|_{H^{s-1}} + \|\hat{E}(0)\|_{H^{s-1}} + \|\hat{F}\|_{L^1} + \|\hat{G}\|_{L^1} \\
+ \int_0^t (\|\nabla A\|_{L^\infty} + \|\nabla B\|_{L^\infty}) (\|\hat{Z}\|_{H^{s-1}} + \|\hat{E}\|_{H^{s-1}}) \, dt. \tag{52}
\]

Now, we return to the system (43)-(46). Let

\[
F := f - \mathcal{P}(T_{A, \partial_t Z} A) + \mathcal{P}(T_{B, \partial_t E} B) + \mathcal{P}(E \cdot \nabla B) - \mathcal{P}(Z \cdot \nabla A),
\]

\[
G := g - \mathcal{P}(T_{A, \partial_t E} A) + \mathcal{P}(T_{B, \partial_t Z} B) + \mathcal{P}(E \cdot \nabla A) - \mathcal{P}(Z \cdot \nabla B).
\]

With the aids of Proposition A.4, it follows that

\[
\|F - f\|_{L^1} \\
\leq C \int_0^t \left(\|T_{A, \partial_t Z} A\|_{H^{s-1}} + \|T_{B, \partial_t E} B\|_{H^{s-1}} + \|Z \cdot \nabla A\|_{H^{s-1}} + \|E \cdot \nabla B\|_{H^{s-1}}\right) \, dt \\
\leq C \int_0^t (\|A, B\|_{H^{s-1}} + \|\nabla A, \nabla B\|_{H^{s-1} \cap L^\infty}) (\|Z(\tau), E(\tau)\|_{H^{s-1}}) \, dt.
\]

Similarly,

\[
\|G - g\|_{L^1} \\
\leq C \int_0^t (\|A, B\|_{H^{s-1}} + \|\nabla A, \nabla B\|_{H^{s-1} \cap L^\infty}) (\|Z(\tau), E(\tau)\|_{H^{s-1}}) \, dt.
\]
Therefore, (52) can be rewritten as
\[
\|Z\|_{L^\infty_t(H^{2s-1})} + \|E\|_{L^\infty_t(H^{2s-1})} + c_2\left(\|Z\|_{L^1_t(H^{2s-1})} + \|E\|_{L^1_t(H^{2s-1})}\right)
\leq \|Z(0)\|_{H^{2s-1}} + \|E(0)\|_{H^{2s-1}} + \|f\|_{L^1_t(H^{2s-1})} + \|g\|_{L^1_t(H^{2s-1})} + \int_0^t (\|(A, B)\|_{H^{2s-1}} + \|(\nabla A, \nabla B)\|_{H^{2s-1}})\left(\|Z\|_{H^{s-1}} + \|E\|_{H^{s-1}}\right)\mathrm{d}t.
\]

By applying Gronwall’s inequality, we complete the proof. \(\square\)

In order to show Proposition 3.1, we first make a brief analysis. Since \(Qu^e\) does not tend to zero in any space \(L^r_\tau(H^s)\) as \(\epsilon\) goes to zero, \(Qu^e \cdot v, Qu^e \cdot B, v \cdot \nabla Qu^e, B \cdot \nabla Qu^e\) are not likely to be small in \(L^r_\tau(H^{s-1})\). Therefore, we can’t apply Proposition 41 to the system (41)-(42) directly. But we notice that \(V^e = L(-\frac{1}{\tau})^{-1}(b^e, Qu^e)\) converge strongly to some limit \(V\).

We make the following change of function as in Section 2:
\[
\psi_M^e = \omega^e + c^p \tilde{R}_M^e
\]
with
\[
\omega^e := \begin{pmatrix} \omega_1^e \\ \omega_2^e \end{pmatrix}, \quad \tilde{R}_M^e := \begin{pmatrix} \tilde{R}_{M, 1}^e \\ \tilde{R}_{M, 2}^e \end{pmatrix},
\]
\[
\tilde{R}_{M, 1}^e := -c_N \sum \left\{ \sum_{\alpha, k, l = m} \left( \frac{1}{|k|} \hat{\theta}_k + \frac{1}{|l|} \hat{\theta}_l \right) \hat{\varphi}^e_{\lambda, \epsilon} e^{i \lambda \frac{|\sigma|}{2} |l|} \right\} e^{i \sigma},
\]
\[
\tilde{R}_{M, 2}^e := -c_N \sum \left\{ \sum_{\alpha, k, l = m} \left( \frac{1}{|k|} \hat{\theta}_k - \frac{1}{|l|} \hat{\theta}_l \right) + \tau \hat{\varphi}^e_{\lambda, \epsilon} e^{i \lambda \frac{|\sigma|}{2} |l|} \right\} e^{i \sigma}.
\]

Indeed, denoting
\[
v := \sum_k \hat{\theta}_k e^{ikx}, \quad B := \sum_k \hat{B}_k e^{ikx}, \quad H^e := \sum_k \hat{H}^e_k e^{ikx},
\]
\[
Qu^e := L^2(\frac{L}{\epsilon})V^e = -i c_N \sum_{\alpha, l} \alpha \sigma \left( \frac{1}{|l|} \right) \hat{\varphi}^e_{\lambda, \epsilon} e^{i \lambda \frac{|\sigma|}{2} |l|} e^{i \sigma},
\]
we have
\[
Qu^e \cdot v = -i c_N \sum_{\alpha, k, l = m} \left\{ \sum \alpha \sigma \left( \frac{1}{|l|} \right) \hat{\varphi}^e_{\lambda, \epsilon} e^{i \lambda \frac{|\sigma|}{2} |l|} \right\} e^{i \sigma},
\]
\[
v \cdot \nabla Qu^e = -i c_N \sum_{\alpha, k, l = m} \left\{ \sum \alpha \sigma \left( \frac{1}{|l|} \right) \hat{\varphi}^e_{\lambda, \epsilon} e^{i \lambda \frac{|\sigma|}{2} |l|} \right\} e^{i \sigma},
\]
\[
Qu^e \cdot B = -i c_N \sum_{\alpha, k, l = m} \left\{ \sum \alpha \sigma \left( \frac{1}{|l|} \right) \hat{B}^e_k \hat{\varphi}^e_{\lambda, \epsilon} e^{i \lambda \frac{|\sigma|}{2} |l|} \right\} e^{i \sigma},
\]
\[
B \cdot \nabla Qu^e = -i c_N \sum_{\alpha, k, l = m} \left\{ \sum \alpha \sigma \left( \frac{1}{|l|} \right) \hat{B}^e_k \hat{\varphi}^e_{\lambda, \epsilon} e^{i \lambda \frac{|\sigma|}{2} |l|} \right\} e^{i \sigma},
\]
\[
\text{div} u^e H^e = \text{div} Qu^e H^e = -i c_N \sum_{\alpha, k, l = m} \left\{ \sum \alpha \sigma \left( \frac{1}{|l|} \right) \hat{H}^e_k \hat{\varphi}^e_{\lambda, \epsilon} e^{i \lambda \frac{|\sigma|}{2} |l|} \right\} e^{i \sigma}.
\]
Therefore, one gets

$$\epsilon \partial_t \tilde{R}_M^e = R_M^e + \epsilon \tilde{R}_{M}^{e,\epsilon},$$

where $R_M^e$ is the low frequency part of $R^e$ obtained by keeping only the indices $l$ and $k$ such that $|k|, |l| \leq M$, with

$$R^e := \left( Q u^e \cdot \nabla v + v \cdot \nabla Q u^e \right)$$

and

$$\tilde{R}_{M}^{e,\epsilon} := \left( \tilde{R}_{M,1}^{e,\epsilon}, \tilde{R}_{M,2}^{e,\epsilon} \right)$$

with

$$\tilde{R}_{M,1}^{e,\epsilon} = -c_N \sum_m \left\{ \sum_{\alpha, k, l = m} \partial_i \left( \left( \frac{l \cdot k}{|l|^2} \right) \hat{V}_i^{\alpha,\epsilon} e^{i \alpha_m} \right) \right\} e^{i m \cdot x},$$

$$\tilde{R}_{M,2}^{e,\epsilon} = -c_N \sum_m \left\{ \sum_{\alpha, k, l = m} \partial_i \left( \left( \frac{l \cdot k}{|l|^2} \right) \hat{V}_i^{\alpha,\epsilon} e^{i \alpha_m} \right) \right\} e^{i m \cdot x}.$$

Let $R_M^{e,M} := R^e - R_M^e$, it follows that the equations for $\psi_M^e$ reads

$$\begin{aligned}
\partial_t \psi_{M,1}^e + \mathcal{P} \left( A^e \cdot \nabla \psi_{M,1}^e + \psi_{M,1}^e \cdot \nabla A^e \right) - \mathcal{P} \left( \psi_{M,2}^e \cdot \nabla B + B \cdot \nabla \psi_{M,2}^e \right) - \mu \Delta \psi_{M,1}^e \\
= -\mathcal{P} R_1^{e,M} + \epsilon \mathcal{P} \tilde{R}_{M,1}^{e,\epsilon} - \mathcal{P} \left\{ \omega_1^e \cdot \nabla \omega_1^e + I(e b^e) A u^e \right\} \\
- \mathcal{P} \left\{ I(e b^e) \left( H^e \cdot \nabla H^e - \frac{1}{2} \nabla |H^e|^2 \right) - \omega_2^e \cdot \nabla \omega_2^e \right\} - \epsilon \mu \mathcal{P} \Delta \tilde{R}_M^e,
\end{aligned}$$

$$\begin{aligned}
\partial_t \psi_{M,2}^e + \mathcal{P} \left( A^e \cdot \nabla \psi_{M,2}^e - \psi_{M,2}^e \cdot \nabla A^e \right) - \mathcal{P} \left( \psi_{M,1}^e \cdot \nabla B - B \cdot \nabla \psi_{M,1}^e \right) - \nu \Delta \psi_{M,2}^e \\
= -\mathcal{P} R_2^{e,M} + \epsilon \mathcal{P} \tilde{R}_{M,2}^{e,\epsilon} - \mathcal{P} \left\{ \omega_2^e \cdot \nabla \omega_1^e - \omega_1^e \cdot \nabla \omega_2^e \right\} - \epsilon \nu \mathcal{P} \Delta \tilde{R}_M^{e,\epsilon},
\end{aligned}$$

with initial data $\psi_M^e(0) = \epsilon \tilde{R}_M^e(0)$.

Applying Proposition 3.2 to the system (53)-(54) yields

$$\begin{aligned}
\| \psi_{M}^e \|_{L^2_{t}(H^{s-1})} + \| \psi_{M}^e \|_{L^1_{t}(H^{s+1})} \lesssim e^{\int_{0}^{t} \| (A^e(\tau), B(\tau)) \|_{H^{s-1}}} + d \tau \\
\times \left\{ \| R^{e,M} \|_{L^2_{t}(H^{s-1})} + \sum_{i,j=1}^2 \| \omega_{i}^e \cdot \nabla \omega_{j}^e \|_{L^1_{t}(H^{s+1})} + \| I(e b^e) A u^e \|_{L^1_{t}(H^{s-1})} + \| I(e b^e) (H^e \cdot \nabla H^e - \frac{1}{2} \nabla |H^e|^2) \|_{L^1_{t}(H^{s-1})} + \| R_1^{e,M} \|_{L^2_{t}(H^{s-1})} + \| R_2^{e,M} \|_{L^2_{t}(H^{s-1})}
\end{aligned}$$
In order to simply the presentation of the estimates below, we first introduce some notations,
\[
\tilde{V}^\epsilon(t,x) := -c_N \sum_{\alpha,k} \tilde{V}_k^{\alpha,\epsilon} e^{i \tilde{\omega} x} e^{i \tilde{\omega} s\{k\} |x| e^{ikx}}, \\
\hat{V}^\epsilon(t,x) := ic_N \sum_{\alpha,k} \tilde{\omega} s\{k\} \tilde{V}_k^{\alpha,\epsilon} e^{i \tilde{\omega} x} e^{i \tilde{\omega} s\{k\} |x| e^{ikx}}, \\
\tilde{v}^{\epsilon}(t,x) := -c_N \sum_{\alpha,k} \partial_k \tilde{V}_k^{\alpha,\epsilon} e^{i \tilde{\omega} x} e^{i \tilde{\omega} s\{k\} |x| e^{ikx}}.
\]

According to Lemma A.2 and the definitions of Sobolev spaces, we deduce that for any \( r \in [1, +\infty], \sigma \in \mathbb{R} \),
\[
\|\tilde{V}^\epsilon\|_{L^r_x(H^\sigma)} + \|\hat{V}^\epsilon\|_{L^r_x(H^\sigma)} \approx \|V^\epsilon\|_{L^r_x(H^\sigma)}, \|\tilde{v}^{\epsilon}\|_{L^r_x(H^\sigma)} \approx \|\partial_t V^\epsilon\|_{L^r_x(H^\sigma)}. \tag{56}
\]

Meanwhile, \( \tilde{R}^\epsilon_M \) can be rewritten as
\[
\tilde{R}^\epsilon_M = \left( \hat{R}^\epsilon_M,1 \right) \left( \hat{R}^\epsilon_M,2 \right) = \left( v_M \cdot \nabla \nabla^{-1} \tilde{V}^\epsilon_M + \nabla^2 \Delta^{-1} \tilde{V}^\epsilon_M \cdot v_M, \nabla B \cdot \nabla \Delta^{-1} \tilde{V}^\epsilon_M + \nabla^2 \Delta^{-1} \tilde{V}^\epsilon_M \cdot B_M + \tilde{V}^\epsilon_M \cdot H^\epsilon_M \right). \tag{57}
\]

**Estimates for \( \tilde{R}^\epsilon_M \)**: We claim
\[
\|\tilde{R}^\epsilon_M\|_{L^\infty_x(H^{r-1})} + \|\tilde{R}^\epsilon_M\|_{L^1_x(H^{r+1})} \leq CM (P + P^\epsilon) X^\epsilon, \tag{58}
\]
\[
\|\tilde{R}^\epsilon_M(0)\|_{H^{r-1}} \leq CM P_0 X_0. \tag{59}
\]

According to Proposition A.2, and (56), we gather that
\[
\|\nabla B_M \cdot \nabla \Delta^{-1} \tilde{V}^\epsilon_M\|_{L^1_x(H^{r+1})} \\
\leq C \left\{ \|\nabla B_M\|_{L^1_x(H^r)} \|\tilde{V}^\epsilon_M\|_{L^1_x(H^{r+1})} + \|B_M\|_{L^1_x(H^{r+1})} \|\tilde{V}^\epsilon_M\|_{L^1_x(H^{r+1})} \right\} \\
\leq CM \left\{ \|B_M\|_{L^1_x(H^{r+1})} \|\tilde{V}^\epsilon_M\|_{L^1_x(H^{r+1})} + \|B\|_{L^1_x(H^{r+1})} \|\tilde{V}^\epsilon_M\|_{L^1_x(H^{r+1})} \right\} \\
\leq CM \left\{ \|B\|_{L^1_x(H^{r+1})} \|\tilde{V}^\epsilon_M\|_{L^1_x(H^{r+1})} + \|B\|_{L^1_x(H^r)} \|\tilde{V}^\epsilon_M\|_{L^1_x(H^r)} \right\} \\
\|B^\epsilon_M \cdot \tilde{V}^\epsilon_M\|_{L^1_x(H^{r+1})} \\
\leq C \left\{ \|B\|_{L^1_x(H^{r+1})} \|\tilde{V}^\epsilon_M\|_{L^1_x(H^r)} + \|B\|_{L^1_x(H^r)} \|\tilde{V}^\epsilon_M\|_{L^1_x(H^r)} \right\} \\
\leq CM \left\{ \|B\|_{L^1_x(H^r)} \|\tilde{V}^\epsilon_M\|_{L^1_x(H^r)} + \|B\|_{L^1_x(H^r)} \|\tilde{V}^\epsilon_M\|_{L^1_x(H^r)} \right\} \\
\leq CM \left\{ \|B\|_{L^1_x(H^r)} \|\tilde{V}^\epsilon_M\|_{L^1_x(H^r)} + \|B\|_{L^1_x(H^r)} \|\tilde{V}^\epsilon_M\|_{L^1_x(H^r)} \right\}
\]

Therefore, we have
\[
\|\tilde{R}^\epsilon_M\|_{L^1_x(H^{r+1})} \leq CM (P + P^\epsilon) X^\epsilon.
\]

Similarly,
\[
\|\nabla B_M \cdot \nabla \Delta^{-1} \tilde{V}^\epsilon_M\|_{L^1_x(H^{r-1})} \leq C \|\nabla B_M\|_{L^1_x(H^{r-1})} \|\nabla \Delta^{-1} \tilde{V}^\epsilon_M\|_{L^1_x(H^{r-1})} \|L^\infty_x(H^r) \cap L^\infty_x \}
\]
Therefore, we also have
\[ \| \tilde{R}_{M,2}^\varepsilon \|_{L_T^\infty (H^{s-1})} \leq CM (P + P^\varepsilon) X^\varepsilon. \]
Similar arguments can be applied to \( \tilde{R}_{M,1}^\varepsilon \) and \( \tilde{R}_M^\varepsilon (0) \), and we eventually arrive at
\[ \| \tilde{R}_{M,1}^\varepsilon \|_{L_T^\infty (H^{s-1})} + \| \tilde{R}_{M,1}^\varepsilon \|_{L_T^1 (H^{s+1})} \leq CM P X^\varepsilon, \]
\[ \| \tilde{R}_M^\varepsilon (0) \|_{H^{s-1}} \leq CM P_0 X_0, \]
which complete the proof of (58) and (59).
Next we deal with the right-hand side of (55).

**Estimates for \( \tilde{R}_{M,2}^\varepsilon \):** We claim that
\[
\| \tilde{R}_{M,2}^\varepsilon \|_{L_T^1 (H^{s-1})} \leq C (X^\varepsilon) \left\{ X^\varepsilon \left( P + P^2 + P^\varepsilon (1 + X^\varepsilon + P^\varepsilon) \right) \right. \\
+ (P + P^\varepsilon) \left( X^\varepsilon + (X^\varepsilon)^2 + (P^\varepsilon)^2 + \varepsilon^2 X^\varepsilon (X^\varepsilon + P^\varepsilon + (P^\varepsilon)^2) \right) \}.
\]
(60)
As in [10], \( \| \tilde{R}_{M,2}^\varepsilon \|_{L_T^1 (H^{s-1})} \) can be controlled by
\[
\| \tilde{R}_{M,2}^\varepsilon \|_{L_T^1 (H^{s-1})} \leq M X^\varepsilon \| \partial_t V^\varepsilon \|_{L_T^1 (H^{s-1})} + MP \| \partial_t V^\varepsilon \|_{L_T^1 (H^{s-1})}.
\]
(61)
Next, we deal with \( \| \tilde{R}_{M,2}^\varepsilon \|_{L_T^1 (H^{s-1})} \). We rewrite \( \tilde{R}_{M,2}^\varepsilon \) as
\[
\tilde{R}_{M,2}^\varepsilon = \nabla \partial_t B_M \cdot \nabla \Delta^{-1} \tilde{V}_M^\varepsilon + \nabla^2 \Delta^{-1} \tilde{V}_M^\varepsilon \cdot \partial_t B_M + \nabla B_M \cdot \nabla \Delta^{-1} \tilde{V}_M^\varepsilon \\
+ \nabla^2 \Delta^{-1} \tilde{V}_M^\varepsilon \cdot \partial_t B_M + \partial_t H_M \tilde{V}_M^\varepsilon + H_M \tilde{V}_M^\varepsilon.
\]
With the help of the spectral localization of \( \tilde{R}_{M,2}^\varepsilon \) and Remark A.2, we have
\[
\| \nabla \partial_t B_M \cdot \nabla \Delta^{-1} \tilde{V}_M^\varepsilon \|_{L_T^1 (H^{s-1})} \leq C \| \nabla \partial_t B_M \|_{L_T^1 (H^{s-1})} \| \nabla \Delta^{-1} \tilde{V}_M^\varepsilon \|_{L_T^\infty (H^{s-1})} \\
\leq CM \| \partial_t B_M \|_{L_T^1 (H^{s-1})} \| \tilde{V}_M^\varepsilon \|_{L_T^\infty (H^{s-1})} \leq CM X^\varepsilon \| \partial_t B_M \|_{L_T^1 (H^{s-1})},
\]
\[
\| \partial_t H_M \tilde{V}_M^\varepsilon \|_{L_T^1 (H^{s-1})} \leq C \| \partial_t H_M \|_{L_T^1 (H^{s-1})} \| \tilde{V}_M^\varepsilon \|_{L_T^\infty (H^{s-1})} \leq CM X^\varepsilon \| \partial_t H_M \|_{L_T^1 (H^{s-1})},
\]
\[
\| H_M \tilde{V}_M^\varepsilon \|_{L_T^1 (H^{s-1})} \leq C \| H_M \|_{L_T^\infty (H^{s-1})} \| \tilde{V}_M^\varepsilon \|_{L_T^\infty (H^{s-1})} \leq CM X^\varepsilon \| H_M \|_{L_T^1 (H^{s-1})}.
\]
\[ \leq CM \|H^e\|_{L^\infty_T(H^{-s-1})} \|\partial_t x^e\|_{L^1_T(H^{s+1})} \leq C M P^e \|\partial_t x^e\|_{L^1_T(H^{s+1})}. \]

Collecting all the above estimates and combining with (61), we deduce that
\[
\|\tilde{R}_1^{e,M}\|_{L^1_T(H^{s+1})} \lesssim M X^e \left\{ \|\partial_t (v,B)\|_{L^1_T(H^{s+1})} + \|\partial_t H^e\|_{L^1_T(H^{s+1})} \right\}
+ M (P + P^e) \|\partial_t x^e\|_{L^1_T(H^{s+1})},
\]

On the other hand, using (10), it is straightforward to obtain
\[
\|\partial_t H^e\|_{L^1_T(H^{s+1})} \leq C P^e (1 + X^e + P^e).
\]

Finally, combining it with Lemma 2.2, we easily obtain (60).

**Estimates for \( \tilde{R}_1^{e,M} \):** We claim that
\[
\|\tilde{R}_1^{e,M}\|_{L^1_T(H^{s+1})} \leq C M^{-\delta} (P + P^e) X^e. \tag{62}
\]

As for \( \tilde{R}_2^{e,M} \), it has already estimated in [10], we just list the result here:
\[
\|\tilde{R}_2^{e,M}\|_{L^1_T(H^{s+1})} \leq C M^{-\delta} P X^e. \tag{63}
\]

As for \( \tilde{R}_2^{e,M} \), we first rewrite it as
\[
\tilde{R}_2^{e,M} = \nabla B^M : \nabla |D|^{-1} \tilde{V}^e + \nabla B^M : \nabla |D|^{-1} \tilde{V}^e M^e - \nabla^2 |D|^{-1} \tilde{V}^e \cdot B^M
- \nabla^2 |D|^{-1} \tilde{V}^e M^e \cdot B^M + |D| \tilde{V}^e M^e \cdot H^e
\]
with \( D = \sqrt{-\Delta} \). By making use of Propositions A.1 and A.2, we have
\[
\|\nabla B^M : \nabla |D|^{-1} \tilde{V}^e\|_{L^1_T(H^{s+1})} + \|B^M : \nabla^2 |D|^{-1} \tilde{V}^e\|_{L^1_T(H^{s+1})}
\lesssim \|B^M\|_{L^p_T(H^{\frac{s}{2}+1})} \|\tilde{V}^e\|_{L^1_T(H^{s+1})} + \|B^M\|_{L^p_T(H^{\frac{s}{2}+1})} \|\tilde{V}^e\|_{L^1_T(H^{s+1})}
\lesssim M^{-\delta} \left\{ \|B\|_{L^p_T(H^{s+1})} \|\tilde{V}^e\|_{L^1_T(H^{s+1})} + \|B\|_{L^p_T(H^{s+1})} \|\tilde{V}^e\|_{L^1_T(H^{s+1})} \right\}
\lesssim M^{-\delta} P X^e,
\]
\[
\|\nabla B^M : \nabla |D|^{-1} \tilde{V}^e M^e\|_{L^1_T(H^{s+1})} + \|B^M : \nabla^2 |D|^{-1} \tilde{V}^e M^e\|_{L^1_T(H^{s+1})}
\lesssim \|B^M\|_{L^p_T(H^{s+1})} \|\tilde{V}^e M^e\|_{L^1_T(H^{s+1})} + \|B^M\|_{L^p_T(H^{s+1})} \|\tilde{V}^e M^e\|_{L^1_T(H^{s+1})}
\lesssim M^{-\delta} \left\{ \|B\|_{L^p_T(H^{s+1})} \|\tilde{V}^e M^e\|_{L^1_T(H^{s+1})} + \|B\|_{L^p_T(H^{s+1})} \|\tilde{V}^e M^e\|_{L^1_T(H^{s+1})} \right\}
\lesssim M^{-\delta} P X^e,
\]
\[
\|D \tilde{V}^e \cdot H^e M^e\|_{L^1_T(H^{s+1})}
\lesssim \|H^e M^e\|_{L^p_T(H^{\frac{s}{2}+1})} \|\tilde{V}^e\|_{L^1_T(H^{s+1})} + \|H^e M^e\|_{L^p_T(H^{\frac{s}{2}+1})} \|\tilde{V}^e\|_{L^1_T(H^{s+1})}
\lesssim M^{-\delta} \left\{ \|H^e\|_{L^p_T(H^{s+1})} \|\tilde{V}^e\|_{L^1_T(H^{s+1})} + \|H^e\|_{L^p_T(H^{s+1})} \|\tilde{V}^e\|_{L^1_T(H^{s+1})} \right\}
\lesssim M^{-\delta} P X^e,
\]
\[
\|D \tilde{V}^e M^e \cdot H^e\|_{L^1_T(H^{s+1})}
\lesssim \|H^e M^e\|_{L^p_T(H^{\frac{s}{2}+1})} \|\tilde{V}^e M^e\|_{L^1_T(H^{s+1})} + \|H^e M^e\|_{L^p_T(H^{\frac{s}{2}+1})} \|\tilde{V}^e M^e\|_{L^1_T(H^{s+1})}
\lesssim M^{-\delta} \left\{ \|H^e\|_{L^p_T(H^{s+1})} \|\tilde{V}^e M^e\|_{L^1_T(H^{s+1})} + \|H^e\|_{L^p_T(H^{s+1})} \|\tilde{V}^e M^e\|_{L^1_T(H^{s+1})} \right\}
\lesssim M^{-\delta} P X^e.
\]
Collecting all the above estimates and (63), we deduce that (62) holds. For the remaining terms on the right hand side of (55), their estimates is easy, and we only give their bounds as follows:

\[ \|\omega_\epsilon \cdot \omega_\epsilon\|_{L^1_t(H_{s-1})} \leq C\|\omega_\epsilon\|_{L^P_t(H_{s-1})} \|\omega_\epsilon\|_{L^2_t(H_{s+1})} \leq C(W_\epsilon)^2, \]

\[ \|I(\epsilon b')Au\|_{L^1_t(H_{s-1})} \leq C(\epsilon^2)\epsilon^\frac{s}{r} X^\epsilon(X^\epsilon + P^\epsilon), \]

\[ \|I(\epsilon b')(H^\epsilon \cdot \nabla H^\epsilon - \frac{1}{2}\nabla|H^\epsilon|^2)\|_{L^1_t(H_{s-1})} \]

\[ \leq C(\epsilon^2)\epsilon\|b'\|_{L^P_t(H_{s-1})} \|H^\epsilon \cdot \nabla H^\epsilon - \frac{1}{2}\nabla|H^\epsilon|^2\|_{L^1_t(H_{s-1})} \]

\[ \leq C(\epsilon^2)\epsilon\|b'\|_{L^P_t(H_{s-1})} \|H^\epsilon\|_{L^P_t(H_{s-1})} \|H^\epsilon\|_{L^1_t(H_{s+1})} \]

\[ \leq C(\epsilon^2)\epsilon\|\epsilon b'\|_{L^P_t(H_{s-1})} \|H^\epsilon\|_{L^1_t(H_{s+1})} \]

Choosing \( M = \epsilon^{-\frac{1}{4}} \) and plugging (58)-(58), (60), (62), (64)-(67) into (55), and meanwhile noticing

\[ \|\psi_M - \omega_M\|_{L^P_t(H_{s-1})} + \|\psi_M - \omega_M\|_{L^1_t(H_{s+1})} \leq C\epsilon^{\frac{s}{r+2}} X^\epsilon(P + P^\epsilon), \]

we finally conclude that Proposition 3.1 holds.

4. A priori estimates for the solution. In this section, we mainly obtain a priori estimates of \((b', u', H')\) in the space \(E^s_{T, \epsilon, q}\). Namely, we shall prove the following proposition.

**Proposition 4.1.** Let \( \epsilon > 0, s > 1, 1 \leq p, r \leq 2 \) and \((b', u', H') \in E^s_{T, \epsilon, q}\) be a solution to the problem (8)-(11). Then, for any \( t \in [0, T] \), there exists a nondecreasing function \( C \) depending on \( s, N, \lambda, \mu, \nu, p, r \) and on the pressure function, but independent of the ratios \( a_2/a_1, \cdots, a_N/a_1 \), such that

\[ \|b'\|_{L^\infty_t(H_{s+1})} + \|u'\|_{L^\infty_t(H_{s-1})} + \|H'\|_{L^1_t(H_{s+1})} \leq C(\epsilon^2)\epsilon^{\frac{s}{r+2}} \]

\[ \leq C(\epsilon^2)\epsilon^{\frac{s}{r+2}} \times \left\{ \|b'\|_{L^\infty_t(H_{s+1})} + \|u'\|_{L^\infty_t(H_{s-1})} + \|H'\|_{L^1_t(H_{s+1})} \right\} \]

\[ + \epsilon^{\frac{s}{r+2}} \||u'|_{L^1_t(H_{s+1})} + \epsilon^{\frac{s}{r+2}} \|H'\|_{L^1_t(H_{s+1})} + \epsilon^{\frac{s}{r+2}} \|b'\|_{L^1_t(H_{s+1})} \|

\[ \text{with} \ V_{\epsilon}^{p, r}(t) := \int_0^t \left( \|u'(\tau)\|_{L^p_t} + \|H'(\tau)\|_{L^\infty_t} + \|b'(\tau)\|_{L^p_t} \right) d\tau. \]

Here \( \|u'(\tau), H'(\tau)\|_{L^\infty_t} \) stands for \( \|u'(\tau), H'(\tau)\|_{L^\infty_t} \) if \( p = 1 \).

For brevity, we begin to study the following reduced system:

\[ \partial_t c + \partial_j T_j c + \text{div} v = F, \]

\[ \partial_t v + T_j v - A \partial_j c = G, \]

(69) \hspace{1cm} (70)
\[ \partial_t B + T_{z_j} \partial_j B - \nu \Delta B = L. \]  

(71)

For the above system, we claim that

**Proposition 4.2.** Let \( s \in \mathbb{R}, 1 \leq p, r < +\infty \) and \((c, v, B)\) be a solution to the problem (69)-(71). Denote \( \tilde{v} = \min\{\mu, \eta, \nu\} \) and \( \tilde{v} = \eta \max\{1, \eta/\mu, \eta/\nu\} \). Then the following estimate holds with a constant \( C \) depending only on \( N, p, r \) and \( s \),

\[
\|c\|_{L^p_t(\tilde{R}^\infty_s)} + \|(v, B)\|_{L^p_t(H^{r-1}_s)} + \tilde{v}\|c\|_{L^1_t(\tilde{R}^\infty_s)} + \tilde{v}\|(v, B)\|_{L^1_t(H^{r+1}_s)} \leq Ce^{C\hat{V}^{P,r}(t)} \times \left\{ \|c_0\|_{\tilde{R}^\infty_s} + \|(v_0, B_0)\|_{H^{r-1}_s} + \|F\|_{L^1_t(\tilde{R}^\infty_s)} + \|(G, L)\|_{L^1_t(H^{r+1}_s)} \right\}. \]  

(72)

Here

\[
\hat{V}^{P,r}(t) := \int_0^t \left( \tilde{v}\|\nabla z(\tau)\|_{L^2}^p + \tilde{v}^{r-1}\|\nabla z(\tau)\|_{L^\infty} \right) d\tau.
\]

**Proof.** Since we need to estimate the hybrid norms, the following quantity \( k_q \) must be controlled,

\[
k_q = \begin{cases} \sqrt{\|\Delta_q c\|_{L^2_s}^2 + \|\Delta_q (v, B)\|_{L^2_s}^2} & \text{if } q \leq q_0 - 1, \\ \sqrt{\|\eta\nabla \Delta_q c\|_{L^2_s}^2 + \|\Delta_q (v, B)\|_{L^2_s}^2} & \text{if } q \geq q_0 - 1. \end{cases}
\]

with \( q_0 := 1 - \lfloor \log_2 \eta \rfloor \). If we apply standard energy method to the localized system (69)-(71), it is difficult to obtain suitable estimates for \( k_q \). For this purpose, we shall find some \( f_q \), which is equivalent to \( k_q \), namely,

\[
f_q := \begin{cases} \sqrt{\|\Delta_q c\|_{L^2_s}^2 + \|\Delta_q (v, B)\|_{L^2_s}^2 + \frac{1}{4}(\eta \nabla \Delta_q c | \Delta_q v)} & \text{if } q \leq q_0 - 1, \\ \sqrt{\|\eta \nabla \Delta_q c\|_{L^2_s}^2 + 2 \|\Delta_q (v, B)\|_{L^2_s}^2 + 2(\eta \nabla \Delta_q c | \Delta_q v)} & \text{if } q \geq q_0 - 1. \end{cases}
\]

Clearly, according to (69)-(71), it is easy to check that the following five equalities hold:

\[
\frac{1}{2} \frac{d}{dt} \|\Delta_q c\|_{L^2_s}^2 + (\Delta_q \text{div } v | \Delta_q c) + (\Delta_q \partial_j T_{z_j} c | \Delta_q c) = (\Delta_q F | \Delta_q c),
\]

(73)

\[
\frac{1}{2} \frac{d}{dt} \|\eta \nabla \Delta_q c\|_{L^2_s}^2 + (\eta \nabla \Delta_q v | \eta \nabla \Delta_q c) + (\eta \nabla \Delta_q \partial_j T_{z_j} c | \eta \nabla \Delta_q c) = (\eta \nabla \Delta_q F | \eta \nabla \Delta_q c),
\]

(74)

\[
\frac{1}{2} \frac{d}{dt} \|\Delta_q v\|_{L^2_s}^2 + (\Delta_q T_{z_j} \partial_j v | \Delta_q v) + \mu \|\nabla \Delta_q P v\|_{L^2_s}^2 + \eta \|\nabla \Delta_q Q v\|_{L^2_s}^2 - (\Delta_q \text{div } v | \Delta_q c) = (\Delta_q G | \Delta_q v),
\]

(75)

\[
\frac{1}{2} \frac{d}{dt} \|\Delta_q B\|_{L^2_s}^2 + (\Delta_q T_{z_j} \partial_j B | \Delta_q B) + \nu \|\nabla \Delta_q B\|_{L^2_s}^2 = (\Delta_q L | \Delta_q B),
\]

(76)

\[
\frac{d}{dt} [\eta \nabla \Delta_q c | \Delta_q v) + \eta \|\nabla \Delta_q c\|_{L^2_s}^2 - \eta \|\nabla \Delta_q Q v\|_{L^2_s}^2 - \eta (\Delta_q Q v | \eta \nabla \Delta_q c) ]
\]

\[
= (\eta \nabla \Delta_q F | \Delta_q v) + (\eta \Delta_q G | \nabla \Delta_q c) - (\eta \nabla \Delta_q T_{z_j} \partial_j c | \Delta_q v)
\]

\[
- \eta (\Delta_q T_{z_j} \partial_j v | \nabla \Delta_q c) - \eta (\nabla \Delta_q T_{\text{div } z} c | \Delta_q v),
\]

(77)

where \((a|b)\) stands for the scalar product in \(L^2(T^N_\alpha)\).

To begin with, we deal with the case for \( q \leq q_0 - 1 \). By means of (73), (75)-(77), it follows that

\[
\frac{1}{2} \frac{d}{dt} f_q^2 + \mu \|\nabla \Delta_q P v\|_{L^2_s}^2 + \frac{7}{8} \eta \|\nabla \Delta_q Q v\|_{L^2_s}^2 + \frac{1}{8} \|\nabla \Delta_q c\|_{L^2_s}^2
\]
\[ \begin{aligned}
&+ \nu \| \nabla \Delta_q B \|_{L^2}^2 - \frac{1}{8} \eta (\Delta \eta \Delta \Delta_q Q v | \eta \nabla \Delta_q c)
=(\Delta_q F | \Delta_q c) + (\Delta_q G | \Delta_q v) + (\Delta_q L | \Delta_q B) + \frac{1}{8} \eta (\nabla \Delta_q F | \Delta_q v)
+ \frac{1}{8} \eta (\Delta_q G | \nabla \Delta_q c) - (\Delta_q \partial_T z_j \partial c | \Delta_q c) - (\Delta_q T_j \partial \partial_j v | \Delta_q v) - (\Delta_q T_j \partial \partial_j B | \Delta_q B)
- \frac{1}{8} \eta (\| \nabla \Delta_q T_j \partial \partial_j c | \Delta_q v \| + (\Delta_q T_j \partial \partial_j v | \nabla \Delta_q c)) \right) - \frac{1}{8} \eta (\nabla \Delta_q T \text{div } z | \Delta_q v). \quad (78)
\end{aligned} \]

Noticing that \( 2^q \eta \leq 2 \) and \( F(\Delta \Delta_q Q v) \) is supported in \( \mathbb{C}^{(0, \frac{5}{6} 2^q, \frac{12}{5} 2^q)} \), we then obtain that
\[
|((\Delta \Delta_q Q v | \eta \nabla \Delta_q c)| \leq \frac{48}{7} \| \nabla \Delta_q Q v \|_{L^2}^2 + \frac{21}{25} \| \nabla \Delta_q c \|_{L^2}^2. \quad (79)
\]

For the term \( (\Delta_q T_j \partial \partial_j v | \Delta_q v) \), we can rewrite it as
\[
(\Delta_q T_j \partial \partial_j v | \Delta_q v) = (\Delta_q T_j \partial \partial_j v - S_{-1} \partial \partial_j \Delta \Delta_q v | \Delta_q v) + (S_{-1} \partial \partial_j \Delta \Delta_q v | \Delta_q v).
\]

With the help of Lemma 7.5 in [9], we arrive at
\[
|((\Delta_q T_j \partial \partial_j v - S_{-1} \partial \partial_j \Delta \Delta_q v | \Delta_q v)| \leq 2^q |(2-\frac{5}{7}) \| \nabla v \|_{L^2} \left( \sum_{|q-q'| \leq 3} \| \Delta_q v \|_{L^2} \right) \| \Delta_q v \|_{L^2}.
\]

Applying integration by parts to \( (S_{-1} \partial \partial_j \Delta \Delta_q v | \Delta_q v) \), we have
\[
(S_{-1} \partial \partial_j \Delta \Delta_q v | \Delta_q v) = -\frac{1}{2} \int S_{-1} \text{div } z \| \Delta_q v \|_{L^2} dx \leq \| S_{-1} \text{div } z \|_{L^\infty} \| \Delta_q v \|_{L^2}^2 \leq C 2^q |(2-\frac{5}{7}) \| \nabla v \|_{L^2} \| \Delta_q v \|_{L^2}.
\]

Therefore, we get
\[
|((\Delta_q T_j \partial \partial_j v | \Delta_q v)| \leq C 2^q |(2-\frac{5}{7}) \| \nabla v \|_{L^2} \left( \sum_{|q-q'| \leq 3} \| \Delta_q v \|_{L^2} \right) \| \Delta_q v \|_{L^2}. \quad (80)
\]

Similarly, it holds
\[
|((\Delta_q T_j \partial \partial_j B | \Delta_q B)| \leq C 2^q |(2-\frac{5}{7}) \| \nabla v \|_{L^2} \left( \sum_{|q-q'| \leq 3} \| \Delta_q B \|_{L^2} \right) \| \Delta_q B \|_{L^2}. \quad (81)
\]

For the term \( (\Delta_q \partial \partial_j T_j \partial c | \Delta \Delta_q c) \), it can be rewritten as
\[
(\Delta_q \partial \partial_j T_j \partial c | \Delta \Delta_q c) = (\Delta_q \partial \partial_j T_j \partial c | \Delta \Delta_q c) + (\Delta_q T \text{div } z c | \Delta \Delta_q c).
\]

According to the definition of the paraproduct and the facts (109), the equality \( \Delta_q T \text{div } z c = \sum_{|q-q'| \leq 3} \Delta_q (S_{-1} \text{div } z \Delta \Delta_q c) \) holds. It is straightforward to obtain that
\[
|((\Delta_q T \text{div } z c | \Delta \Delta_q c)| \leq C 2^q |(2-\frac{5}{7}) \| \nabla v \|_{L^2} \left( \sum_{|q-q'| \leq 3} \| \Delta_q c \|_{L^2} \right) \| \Delta \Delta_q c \|_{L^2}.
\]

Therefore, it is easy to get
\[
|((\partial \partial_j T_j \partial c | \Delta \Delta_q c)| \leq C 2^q |(2-\frac{5}{7}) \| \nabla v \|_{L^2} \left( \sum_{|q-q'| \leq 3} \| \Delta_q c \|_{L^2} \right) \| \Delta \Delta_q c \|_{L^2}, \quad (82)
\]
\[
|\eta (\nabla \Delta_q T \text{div } z c | \Delta_q v)| \leq C 2^q |(2-\frac{5}{7}) \| \text{div } z \|_{L^2} \left( \sum_{|q-q'| \leq 3} \| \Delta_q \nabla c \|_{L^2} \right) \| \Delta_q v \|_{L^2}

\leq C 2^q |(2-\frac{5}{7}) 2^q \eta \| \text{div } z \|_{L^2} \left( \sum_{|q-q'| \leq 3} \| \Delta_q c \|_{L^2} \right) \| \Delta_q v \|_{L^2}
\]

\]
\[
\leq C 2^{g(2-\frac{1}{p})} \| \nabla z \|_{\frac{p}{2}-2} \left( \sum_{|q-q'| \leq 3} \| \Delta_q c \|_{L^2} \right) \| \Delta_q v \|_{L^2}. \quad (83)
\]

Finally, in order to estimate the term \( \eta(\nabla \Delta_q T_z \partial_j c | \Delta_q v) + \eta(\Delta_q T_z \partial_j v | \nabla \Delta_q c) \), we can rewrite it as
\[
\eta(\nabla \Delta_q T_z \partial_j c | \Delta_q v) + \eta(\Delta_q T_z \partial_j v | \nabla \Delta_q c) \\
= \eta(\nabla \Delta_q T_z \partial_j c - S_{q-1} z_j \partial_j \Delta_q \nabla c | \Delta_q v) + \eta(\Delta_q T_z \partial_j v - S_{q-1} z_j \partial_j \Delta_q v | \nabla \Delta_q c) \\
+ \eta(S_{q-1} z_j \partial_j \Delta_q c | \Delta_q v) + \eta(S_{q-1} z_j \partial_j \Delta_q v | \nabla \Delta_q c).
\]

Clearly, we have
\[
|\eta(\nabla \Delta_q T_z \partial_j c - S_{q-1} z_j \partial_j \Delta_q \nabla c | \Delta_q v)| \\
\leq C 2^{g(2-\frac{1}{p})} \| \nabla z \|_{\frac{p}{2}-2} \left( \sum_{|q-q'| \leq 3} \| \Delta_q c \|_{L^2} \right) \| \Delta_q v \|_{L^2}, \quad (84)
\]
\[
\eta(\Delta_q T_z \partial_j v - S_{q-1} z_j \partial_j \Delta_q v | \nabla \Delta_q c) \\
\leq C 2^{g(2-\frac{1}{p})} \| \nabla z \|_{\frac{p}{2}-2} \left( \sum_{|q-q'| \leq 3} \| \Delta_q v \|_{L^2} \right) \| \Delta_q c \|_{L^2}. \quad (85)
\]

And for \( \eta(S_{q-1} z_j \partial_j \Delta_q \nabla c | \Delta_q v) + \eta(S_{q-1} z_j \partial_j \Delta_q v | \nabla \Delta_q c) \), applying integration by parts, we get
\[
\eta(S_{q-1} z_j \partial_j \Delta_q \nabla c | \Delta_q v) + \eta(S_{q-1} z_j \partial_j \Delta_q v | \nabla \Delta_q c) = \eta \int S_{q-1} z_j \partial_j (\Delta_q \nabla c \cdot \Delta_q v) dx \\
= - \int S_{q-1} \text{div}(z(\Delta_q \nabla c \cdot \Delta_q v)) dx \leq C \|S_{q-1} \text{div} z\|_{L^{\infty}} \|\Delta_q \nabla c\|_{L^2} \|\Delta_q v\|_{L^2} \\
\leq C 2^{g(2-\frac{1}{p})} \| \nabla z \|_{\frac{p}{2}-2} \| \Delta_q c \|_{L^p} \| \Delta_q v \|_{L^2}. \quad (86)
\]

Now, plugging (79)-(86) into (78), we arrive at
\[
\frac{1}{2} \frac{d}{dt} f_q^2 + \mu \| \nabla \Delta_q P v \|_{L^2}^2 + \frac{1}{50} \eta \| \nabla \Delta_q Q v \|_{L^2}^2 + \frac{1}{50} \eta \| \nabla \Delta_q c \|_{L^2}^2 + \eta \| \nabla \Delta_q B \|_{L^2}^2 \\
\leq C f_q \left\{ \| \Delta_q F \|_{L^2} + \| \Delta_q G \|_{L^2} + 2^{g(1-\frac{1}{p})} \| \nabla z \|_{\frac{p}{2}-2} \sum_{|q-q'| \leq 3} k_q \right\}. \quad (87)
\]

Since \( f_q \approx k_q \), the time integration of (87) gives
\[
\| \Delta_q c \|_{L^p_c (L^2)} + \| \Delta_q (v, B) \|_{L^p_c (L^2)} \\
+ \tilde{p} 2^q \left\{ \| \Delta_q c \|_{L^1_c (L^2)} + \| \Delta_q v \|_{L^1_c (L^2)} + \| \Delta_q B \|_{L^1_c (L^2)} \right\} \\
\lesssim \| \Delta_q v_0 \|_{L^2} + \| \Delta_q (v_0, B_0) \|_{L^2} + \| \Delta_q F \|_{L^1_c (L^2)} + \| \Delta_q G \|_{L^1_c (L^2)} + \| \Delta_q L \|_{L^1_c (L^2)} \\
+ A^{p-1} \tilde{p}^{1-p} \sum_{|q-q'| \leq 3} \int_0^t \| \nabla z \|_{\frac{p}{2}-2} \| \Delta_q' c(\tau) \|_{L^2} + \| \Delta_q' v(\tau) \|_{L^2} + \| \Delta_q' B(\tau) \|_{L^2} d\tau \\
+ A^{-1} \tilde{p} \sum_{|q-q'| \leq 3} 2^{q} \int_0^t \| \Delta_q' c(\tau) \|_{L^2} + \| \Delta_q' v(\tau) \|_{L^2} + \| \Delta_q' B(\tau) \|_{L^2} d\tau \quad (88)
\]

Here we have used the following inequality:
\[
2^{2q(1-\frac{1}{p})} \| \nabla z \|_{\frac{p}{2}-2} \leq \frac{1}{p} \left( \frac{A}{\tilde{p}} \right)^{p-1} \| \nabla z \|_{\frac{p}{2}-2} + \frac{(p-1)\tilde{p}}{pA} 2^{2q}.
\]
Next, we deal with the case: \( q \geq q_0 \). From (74)-(77), it follows that
\[
\frac{1}{2} \frac{d}{dt} f_q^2 + 2\mu \| \nabla \Delta_q P v \|^2 + \eta \| \nabla \Delta_q Q v \|^2 + 2\nu \| \nabla \Delta_q B \|^2 + \eta \| \nabla \Delta_q c \|^2
\]
\[
+ 2(\Delta_q Q v | \nabla \Delta_q c) + \eta(\nabla \Delta_q F \eta \nabla \Delta_q c) + 2(\Delta_q G | \Delta_q c) + 2(\Delta_q L | \Delta_q B)
\]
\[
+ \eta(\nabla \Delta_q F | \Delta_q v) + \eta(\Delta_q G | \nabla \Delta_q c) - \eta(\nabla \Delta_q T_z \partial_j c | \eta \nabla \Delta_q c)
\]
\[
- 2(\Delta_q T_z \partial_j c | \Delta_q v) - 2(\Delta_q T_z \partial_j B | \Delta_q B) - \eta(\nabla \Delta_q T_{div} z | \eta \nabla \Delta_q c)
\]
\[
- \eta(\nabla \Delta_q T_{div} z | \Delta_q v) - \eta \left\{ (\nabla \Delta_q T_z \partial_j c | \Delta_q v) + (\Delta_q T_z \partial_j c | \nabla \Delta_q c) \right\}. \quad (89)
\]
Noticing that \( 2^4 \eta \geq 2 \) and that \( \mathcal{F}(\Delta_q v) \) is supported away from \( B(0, \frac{2}{3}) \). Thus, we have
\[
\| \eta \nabla \Delta_q P v \|^2 \leq \frac{25}{9} \| \Delta_q P v \|_{L^2}^2,
\]
\[
\| \eta \nabla \Delta_q B \|^2 \leq \frac{25}{9} \| \Delta_q B \|_{L^2}^2,
\]
\[
\| \eta \nabla \Delta_q Q v \|^2 - 2(\eta \nabla \Delta_q c \Delta_q Q v) \geq \frac{7}{9} \| \Delta_q Q v \|^2 - \frac{1}{2} \| \eta \nabla \Delta_q c \|^2. \quad (90)
\]
As for the terms on the right hand side of (89), their estimates are similar to the corresponding terms on the right hand side of (78), thus we omit the details and give only their bounds as follows:
\[
| (\Delta_q T_z \partial_j c | \Delta_q v) | + | (\Delta_q T_z \partial_j B | \Delta_q B) | \leq C \| \nabla z \|_{L^\infty}
\]
\[
\times \left\{ \| \Delta_q v \|_{L^2} \sum_{|q-q'| \leq 3} \| \Delta_q' v \|_{L^2} + \| \Delta_q B \|_{L^2} \sum_{|q-q'| \leq 3} \| \Delta_q' B \|_{L^2} \right\}, \quad (91)
\]
\[
| \eta(\nabla \Delta_q T_z \partial_j c | \eta \nabla \Delta_q c) | \leq C \| \nabla z \|_{L^\infty} \| \eta \nabla \Delta_q c \|_{L^2} \sum_{|q-q'| \leq 3} \| \eta \nabla \Delta_q c \|_{L^2}, \quad (92)
\]
\[
| \eta(\nabla \Delta_q T_{div} z c | \Delta_q v) | + | \eta(\nabla \Delta_q T_{div} z c | \eta \nabla \Delta_q c) | \leq C \| \nabla z \|_{L^\infty}
\]
\[
\times \left\{ \| \eta \nabla \Delta_q c \|_{L^2} \sum_{|q-q'| \leq 3} \| \Delta_q' v \|_{L^2} + \| \Delta_q v \|_{L^2} \sum_{|q-q'| \leq 3} \| \eta \nabla \Delta_q c \|_{L^2} \right\}. \quad (94)
\]
\[
\text{Plugging (90)-(94) into (89), we gather that}
\]
\[
\frac{1}{2} \frac{d}{dt} f_q^2 + \frac{50}{9} \frac{\mu}{\eta^2} \| \nabla \Delta_q P v \|^2_{L^2}
\]
\[
+ \frac{1}{\eta} \left\{ \frac{7}{9} \| \nabla \Delta_q Q v \|^2_{L^2} + \| \eta \nabla \Delta_q c \|^2_{L^2} \right\} + \frac{50}{9} \frac{\nu}{\eta^2} \| \nabla \Delta_q B \|^2_{L^2}
\]
\[
\leq C f_q \left\{ \| \eta \nabla \Delta_q F \|_{L^2} + \| (\Delta_q G, \Delta_q L) \|_{L^2} + \| \nabla z \|_{L^\infty} \sum_{|q-q'| \leq 3} k_{q'} \right\}. \quad (95)
\]
As \( f_q \approx k_q \), the time integration of (95) gives
\[
\| \eta \nabla \Delta_q c \|_{L^\infty(L^2)} + \| \Delta_q(v, B) \|_{L^\infty(L^2)}
\]
\[
+ \tilde{\nu}^{-1} \left\{ \| \eta \nabla \Delta_q c \|_{L^2(L^2)} + \| \Delta_q v \|_{L^2(L^2)} + \| \Delta_q B \|_{L^2(L^2)} \right\}
\]
\[
\lesssim \| \eta \nabla \Delta_q c_0 \|_{L^2} + \| \Delta_q(v_0, B_0) \|_{L^2} + \| \eta \nabla \Delta_q F \|_{L^2(L^2)} + \| \Delta_q(G, L) \|_{L^2(L^2)}
\]
+ (A\nu)^{-1} \sum_{|q-q'| \leq 3} \int_0^t \| \nabla z \|_{L^\infty} \left\{ \| \eta \nabla \Delta_{q'} c(\tau) \|_{L^2} + \| \Delta_{q'} v(\tau) \|_{L^2} + \| \Delta_{q'} B(\tau) \|_{L^2} \right\} d\tau \\
+ (A\nu)^{-1} \sum_{|q-q'| \leq 3} \int_0^t \left\{ \| \eta \nabla \Delta_{q'} c(\tau) \|_{L^2} + \| \Delta_{q'} v(\tau) \|_{L^2} + \| \Delta_{q'} B(\tau) \|_{L^2} \right\} d\tau. \tag{96}

Here we have used the following inequality:

\[ \| \nabla z \|_{L^\infty} \leq \frac{1}{r} \left( \frac{A}{\nu} \right)^{r-1} \| \nabla z \|_{L^\infty}^{r-1} + \frac{r-1}{rA\nu}. \]

Multiplying (88) by \( 2^{q(s-1)} \), squaring it and summing over \( q \leq q_0 - 1 \), likewise, multiplying (96) by \( 2^{q(s-1)} \), squaring it and summing over \( q \geq q_0 \), summarizing the obtained inequalities and taking the square root of it, we eventually arrive at

\[
\| c \|_{L^\infty_t(H^{s-1})} + \| (v, B) \|_{L^\infty_t(H^{s-1})} + \tilde{\nu} \| c \|_{L^1_t(H^{s-1})} \\
+ \tilde{\nu} \left\{ \sum_{q \in \mathbb{Z}} \left( 2^{2^{q} s^{-1}} \min(2^{2^{q}}, \eta^{-2}) \| (\Delta_q v, \Delta_q B) \|_{L^1_t(L^2)} \right) ^2 \right\} ^{\frac{1}{2}} \\
\lesssim \| c_0 \|_{H^{s-1}} + \| (v_0, B_0) \|_{H^{s-1}} + \| F \|_{L^1_t(H^{s-1})} + \| (G, L) \|_{L^1_t(H^{s-1})} \\
+ \left\{ \sum_{q \in \mathbb{Z}} \left( 2^{2^{q} s^{-1}} \int_0^t \| (A\nu) \|^{-1} \| \nabla z \|_{L^\infty} + (A/\nu)^{p-1} \| \nabla z \|_{L^p}^{p-2} \right) \left( \| c \|_{H^{s-1}} + \| (v, B) \|_{H^{s-1}} \right) d\tau \right\} ^{\frac{1}{2}} \\
+ A^{-1} \tilde{\nu} \left\{ \| c \|_{L^1_t(H^{s-1})} + \left( \sum_{q \in \mathbb{Z}} \left( 2^{2^{q} s^{-1}} \min(2^{2^{q}}, \eta^{-2}) \| (\Delta_q v, \Delta_q B) \|_{L^1_t(L^2)} \right) ^2 \right\} ^{\frac{1}{2}}. 
\]

Choosing \( A \) large enough, using the Minkowski’s inequality and introducing the low frequency cut-off \( (u^{BF}, B^{BF}) := (\chi^{BF}(D)v, \chi^{BF}(D)B) := (S_{q_0} v, S_{q_0} B) \), then denoting the corresponding high frequency part \( \chi^{HF}(D) = I - S_{q_0} \), we finally conclude that

\[
\| c \|_{L^\infty_t(H^{s-1})} + \| (v, B) \|_{L^\infty_t(H^{s-1})} + \tilde{\nu} \| c \|_{L^1_t(H^{s-1})} + \tilde{\nu} \| (u^{BF}, B^{BF}) \|_{L^1_t(H^{s-1})} \\
\lesssim \| c_0 \|_{H^{s-1}} + \| (v_0, B_0) \|_{H^{s-1}} + \| F \|_{L^1_t(H^{s-1})} + \| (G, L) \|_{L^1_t(H^{s-1})} \\
+ \int_0^t \left( \tilde{\nu}^{r-1} \| \nabla z \|_{L^\infty} + \tilde{\nu}^{1-p} \| \nabla z \|_{L^p}^{p-2} \right) \left( \| c \|_{H^{s-1}} + \| (v, B) \|_{H^{s-1}} \right) d\tau. \tag{97}
\]

Next, we treat \( (v^{HF}, B^{HF}) \). In fact, from (70) and (71), we obtain that

\[
\frac{1}{2} \frac{d}{dt} \| \Delta_q v \|_{L^2} + \mu \| \Delta_q P v \|_{L^2} + \eta \| \nabla \Delta_q Q v \|_{L^2} \\
\leq \| \Delta_q v \|_{L^2} \left( \| \Delta_q G \|_{L^2} + \| \nabla \Delta_q c \|_{L^2} \right) + (\| \Delta_q T_{z,j} \partial_j v | \Delta_q v |) + (\| \Delta_q T_{z,j} \partial_j B | \Delta_q B |). 
\]

Integrating in time for the above two inequalities and applying the previous estimates, we get, for a universal constant \( \kappa \),

\[
\| \Delta_q (v, B) \|_{L^t_t(L^2)} + \kappa \tilde{\nu} \| \Delta_q (v, B) \|_{L^t_t(L^2)} \leq \| \Delta_q (v_0, B_0) \|_{L^2} + \| \Delta_q (G, L) \|_{L^t_t(L^2)} \\
+ C \| \Delta_q c \|_{L^t_t(L^2)} + C \tilde{\nu} A^{-1} \sum_{|q' - q| \leq 3} 2^{2^q} \| \Delta_q (v, B) \|_{L^1_t(L^2)} 
\]
For the last term

\[ + C\tilde{p}^{1-p}A^{p-1}\sum_{q\leq q_0} \int_0^t \|\nabla z(\tau)\|^p \frac{p}{p-2} \left( \|\Delta_q v(\tau)\|_{L^2} + \|\Delta_q B(\tau)\|_{L^2} \right) d\tau. \]

Multiplying the above inequality by \(2^{q(s-1)}\), squaring the result, summing over \(q \geq q_0\), choosing a suitable large \(A\), and finally using the Minkowski’s inequality, we end up with

\[ \|v_{HF}\|_{L^\infty_t L^{1,\infty}_x} + \|B_{HF}\|_{L^\infty_t L^{1,\infty}_x} \lesssim \|(u_0, B_0)\|_{H^{s-1}} + \|F\|_{L^1_t L^{1,\infty}_x} + \|G\|_{L^1_t L^{1,\infty}_x} + \int_0^t \|\nabla z(\tau)\|^p \frac{p}{p-2} \|\nabla z(\tau)\|^p \frac{p}{p-2} \left( \|c\|_{L^\infty_t L^{1,\infty}_x} + \|(u, B)\|_{H^{s-1}} \right) d\tau. \]  

(98)

Applying the Gronwall inequality to (98) completes the proof of Proposition 4.2. \(\square\)

Next, we want to deduce the case \(c = 1\) of Proposition 68 from Proposition 69. To this end, we first consider the following reduced system

\[ \partial_t b + \partial_j T_{uj} b + \text{div} u = -\tilde{F} - \chi D^F(D) (\partial_j T_{uj} u_j), \]
\[ \partial_t u + T_{uj} \partial_j u - Au + \nabla b = -T'_{uj} u_j + K(b) \nabla b - T'_{Auj} I(b) + \tilde{G}, \]
\[ \partial_t H + T_{uj} \partial_j H - \nu \Delta H = \tilde{L} - T'_{uj} H u_j \]

with \(K(b) = (\kappa + \tilde{K}(b))b\). Applying Proposition 69 to the above system, we get

\[ \|b\|_{L^\infty_t L^{1,\infty}_x} + \|(u, H)\|_{L^\infty_t L^{1,\infty}_x} \lesssim \|b_0\|_{L^\infty_t L^{1,\infty}_x} + \|(u_0, H_0)\|_{L^{1,\infty}_x} \]
\[ + \|T'_{Auj} I(b)\|_{L^1_t L^{1,\infty}_x} + \|K(b) \nabla b\|_{L^1_t L^{1,\infty}_x} + \|T'_{uj} H u_j\|_{L^1_t L^{1,\infty}_x}. \]  

(99)

For the last term \(\|T'_{uj} H u_j\|_{L^1_t L^{1,\infty}_x}\), the following estimate holds

\[ \|T'_{uj} H u_j\|_{L^1_t L^{1,\infty}_x} \lesssim \sum_{q \in \mathbb{Z}^N} \left\{ \int_0^t 2^{q(s-1)} \|S_{q+1} \partial_j H(\tau) \Delta_q u_j(\tau)\|_{L^2} d\tau \right\}^2 \]
\[ \lesssim \sum_{q \in \mathbb{Z}^N} \left\{ \int_0^t \|\nabla H\| \frac{p}{p-2} \left( 2^{q(s-1)} \|\Delta_q u\|_{L^2} \right) \frac{p}{2} \cdot \left( 2^{q(s+1)} \|\Delta_q u\|_{L^2} \right)^{1-\frac{1}{p}} d\tau \right\}^2 \]
\[ \lesssim \sum_{q \in \mathbb{Z}^N} \left\{ \int_0^t \|\nabla H\| \frac{p}{p-2} \left( 2^{q(s-1)} \|\Delta_q u\|_{L^2} \right)^2 \right\} \cdot \left\{ \int_0^t 2^{q(s+1)} \|\Delta_q u\|_{L^2} d\tau \right\}^{2-\frac{2}{p}}. \]
\[
\leq \sum_{g \in \mathbb{Z}^N} \left\{ \int_0^t \| \nabla H \|_{L^\infty}^p \left( 2^{g(s-1)} \| \Delta_g u \|_{L^2} \right) \right\}^2 \cdot 2^{2(p-1)} A^{2(p-1)}
\]
\[
+ \sum_{g \in \mathbb{Z}^N} \left\{ \int_0^t 2^{g(s+1)} \| \Delta_g u \|_{L^2} \right\}^2 A^{-2}
\]

with \( A > 0 \). Therefore,

\[
\| T_{b,t}^u u \|_{L^1_t(H^{s-1})} \lesssim A^{-1} \| u \|_{L^1_t(H^{s+1})} + A^{p-1} \int_0^t \| \nabla H \|_{L^\infty}^p \| u(\tau) \|_{H^{s-1}} \, d\tau. \tag{100}
\]

Meanwhile, we have the following result.

**Lemma 4.1** ([10]). For all \( A > 0 \), the following estimates hold true:

\[
\| T_{b,t}^u u \|_{L^1_t(H^{s-1})} \lesssim A^{-1} \| u \|_{L^1_t(H^{s+1})} + A \int_0^t \| b(\tau) \|_{L^\infty}^2 \| u(\tau) \|_{H^{s-1}} \, d\tau,
\]

\[
\| T_{b,t}^u u \|_{L^1_t(H^{s-1})} \lesssim A^{-1} \| u \|_{L^1_t(H^{s+1})} + A^{p-1} \int_0^t \| \nabla u \|_{L^\infty}^p \| u(\tau) \|_{H^{s-1}} \, d\tau,
\]

\[
\| K(b) \nabla b \|_{L^1_t(H^{s-1})} \lesssim C(\mathcal{Y}(t)) \left( A^{-1} \| b \|_{L^1_t(\tilde{H}^{s-1})} + A \int_0^t \| b(\tau) \|_{L^\infty}^2 \| \nabla(\tau) \|_{L^\infty} \, d\tau \right),
\]

\[
\| T_{b,\tau} I(b) \|_{L^1_t(H^{s-1})} \lesssim C(\mathcal{Y}(t)) \left( A^{-1} \| b \|_{L^1_t(\tilde{H}^{s-1})} + A \int_0^t \| b(\tau) \|_{L^\infty}^2 \| \nabla(\tau) \|_{L^\infty} \, d\tau \right).
\]

Plugging (100) and the inequalities in Lemma 4.1 into (99), choosing large constant \( A \), and then applying the Gronwall inequality again, we obtain

\[
\| b \|_{L^\infty_t(\tilde{H}^{s-1})} + \| (u, H) \|_{L^\infty_t(H^{s-1})} + \| b \|_{L^1_t(\tilde{H}^{s-1})} + \| (u, H) \|_{L^1_t(H^{s+1})} \leq C(\mathcal{Y}(t)) e^C \| (u_0, H_0) \|_{H^{s-1}} + \| F \|_{L^1_t(\tilde{H}^{s-1})} + \| (\tilde{G}, \tilde{L}) \|_{L^1_t(H^{s-1})} \tag{101}
\]

Denote

\[
\tilde{F} = -\chi^{HF}(D)(\partial_j T_{b,t}^u), \quad \tilde{L} = H \cdot \nabla u - (\text{div } u) H,
\]

\[
\tilde{G} = -T_{I(b),\tau} u - I(b)(H \cdot \nabla H - \frac{1}{2} \Delta |H|^2) + H \cdot \nabla H - \frac{1}{2} \nabla |H|^2.
\]

We shall get the estimate for \( \epsilon = 1 \) of Proposition 68. Thanks to Remark A.2, it follows that

\[
\| \chi^{HF}(D)(\partial_j T_{b,t}^u) \|_{L^1_t(\tilde{H}^{s-1})} \leq C \| b \|_{L^\infty_t(\tilde{L}^{\infty})} \| u \|_{L^1_t(H^{s+1})},
\]

\[
\| T_{I(b),\tau} u \|_{L^1_t(H^{s-1})} \leq C(\mathcal{Y}(t)) \| b \|_{L^\infty_t(\tilde{L}^{\infty})} \| u \|_{L^1_t(H^{s+1})},
\]

\[
\| I(b)(H \cdot \nabla H - \frac{1}{2} \Delta |H|^2) \|_{L^1_t(H^{s-1})} \leq C \| b \|_{L^\infty_t(\tilde{L}^{s-1})} \| H \|_{L^1_t(H^{s+1})} \| H \|_{L^1_t(H^{s+1})},
\]

\[
\| H \cdot \nabla H - \frac{1}{2} \Delta |H|^2 \|_{L^1_t(H^{s-1})} \leq C \| H \|_{L^1_t(H^{s+1})} \| H \|_{L^1_t(H^{s+1})},
\]

\[
\| H \cdot \nabla u - (\text{div } u) H \|_{L^1_t(H^{s-1})} \leq C \| H \|_{L^1_t(H^{s+1})} \| u \|_{L^1_t(H^{s+1})}.
\]
Therefore, we arrive at
\begin{equation}
\|b\|_{L^\infty(T_{\tau_0}\rightarrow T)} + \|(u, H)\|_{L^\infty(H^{s-1})} + \|b\|_{L^1_t(H^{s+1})} + \|(u, H)\|_{L^1_t(H^{s+1})} \\
\leq C(\lambda^1(t))\epsilon \\
\times \left\{ \|b_0\|_{H^{s-1}} + \|(u_0, H_0)\|_{H^{s-1}} + \|b\|_{L^\infty(L^\infty)}\|u\|_{L^1_t(H^{s+1})} \right\} \\
+ \|H\|_{L^\infty(H^{s-1})}\left\{ \|u\|_{L^1_t(H^{s+1})} + \|H\|_{L^1_t(H^{s+1})} \right\} \\
+ \|b\|_{L^\infty(T_{\tau_0}\rightarrow T)}\|H\|_{L^1_t(H^{s-1})}\|H\|_{L^1_t(H^{s+1})} \right\}.
\end{equation}

Finally, we deduce the general case from (102). In fact, we take the following change of function,
\begin{align*}
\tilde{b}(t, x) &= \epsilon b'(\epsilon^2 t, c x), \quad \tilde{u}(t, x) = \epsilon u'(\epsilon^2 t, c x), \quad \tilde{H}(t, x) = \epsilon H'(\epsilon^2 t, c x) \\
\tilde{b}_0(x) &= \epsilon b'(c x), \quad \tilde{u}_0(x) = \epsilon u'(c x), \quad \tilde{H}_0(x) = \epsilon H'(c x),
\end{align*}
Clearly, if \((b', u', H')\) solves the problem (8)-(11), then \((\tilde{b}, \tilde{u}, \tilde{H})\) must be a solution to the problem (8) - (11) for \(\epsilon = 1\). Using Proposition A.1, straightforward computations yield the desired inequality for \((b', u', H')\).

5. **Proof of Theorem 1.1.** In this section, we mainly prove Theorem 1.1. We shall follow and adopt some ideas developed in [10]. Briefly, we shall divided the proof into six steps:

Step 1. The estimate of \(V^\epsilon - V\) in the space \(G^N_T\).
Step 2. The estimate of \((Pu^\epsilon - v, H^\epsilon - B)\) in the space \(F^N_T\).
Step 3. The estimate of \((b', u', H')\) in the space \(E^N_T, c\). For \(V\) in \(F^N_T\), we prove that the system (8)-(11) has a solution in \(E^N_T, c\).
Step 4. By the first three steps and bootstrap argument, we give the uniform bound of \((b', u', H')\) in \(E^N_{T, c}\) and the convergence of \(V^\epsilon - V, Pu^\epsilon - v, H^\epsilon - B\) in the corresponding spaces, respectively.
Step 5. Global existence for the limit system (13)-(14).
Step 6. We complete the proof of Theorem 1.1 with the help of continuation argument.

In the first four steps of the following proof, we assume that
\begin{itemize}
\item **Assumption (H).** The system (5)-(7) (resp., (13)-(14)) has a solution \((v, B)\) (resp. \(V\)) in \(F^N_T\) for a \(T \in (0, +\infty]\). The system (8)-(11) has a solution in \(E^N_{T, c}\) which satisfies \(\|b'\|_{L^\infty(0, T \times \mathbb{T}^N)} \leq \frac{\epsilon}{2}\).
\end{itemize}

**Step 1.** Convergence of \(V^\epsilon\) to \(V\). Under the assumption (H) and \(\epsilon \leq 1\), the inequality (16) in Section 2 reads
\begin{align}
Z^\epsilon &\leq C(\lambda^c)\epsilon C(\chi^2 + p^3) \left\{ (Z^\epsilon)^2 + W^\epsilon (P + P^\epsilon + X^\epsilon + W^\epsilon) \\
+ \tau(\epsilon) \left[ X_0 + X_0^2 + P_0^2 + X^2 + X^3 + P^2 + P^3 + X^\epsilon + (X^\epsilon)^2 \\
+ (X^\epsilon)^3 + (P^\epsilon)^2 + (P^\epsilon)^3 \right] \right\}.
\end{align}
\textbf{Step 2.} Convergence of $Pu^\epsilon$ to $u$. Under the assumption $(\mathcal{H})$ and $\epsilon \leq 1$, the inequality (40) in Section 3 reads

$$W^\epsilon \leq C(Y^\epsilon)e^{C(X^\epsilon + P)} \left\{ (W^\epsilon)^2 + \epsilon \frac{\delta}{1+2\epsilon} \left( X_0 P_0 + X^\epsilon (P + P^2 + P^\epsilon + P^\epsilon)^2 + X^\epsilon + (X^\epsilon)^2 + (P + P^\epsilon) \left( X^\epsilon + (X^\epsilon)^2 + (P^\epsilon)^2 + X^\epsilon (P^\epsilon)^2 \right) \right) \right\}. \quad (104)$$

\textbf{Step 3.} A priori estimates in $E^\epsilon_{T,x\eta}$ for $(b^\epsilon, u^\epsilon, H^\epsilon)$. Under the assumption $(\mathcal{H})$ and applying Proposition 4.1 with $s = \frac{N}{2} + \delta$, $p = 2$, $r = 1 + \frac{\delta}{2}$, we arrive at

$$\begin{aligned}
&\|b^\epsilon\|_{L^\infty_t(\tilde{H}^{s\infty}_t)} + \| (u^\epsilon, H^\epsilon) \|_{L^\infty_t(\tilde{H}^{s-1}_t)} + \| b^\epsilon \|_{L^1_t(\tilde{H}^{s+1}_t)} + \| (u^\epsilon, H^\epsilon) \|_{L^1_t(\tilde{H}^{s+1}_t)} \\
&\leq C e^{CV(t)} \times \left\{ \| b_0 \|_{\tilde{H}^{s\infty}_0} + \| (u_0, H_0) \|_{\tilde{H}^{s-1}_0} \\
&+ \epsilon \| b^\epsilon \|_{L^\infty_t(\tilde{H}^{s\infty}_t)} \| u^\epsilon \|_{L^1_t(\tilde{H}^{s+1}_t)} + \epsilon^\delta \| (u^\epsilon, H^\epsilon) \|_{L^1_t(\tilde{H}^{s+1}_t)} \| H^\epsilon \|_{L^1_t(\tilde{H}^{s+1}_t)} \\
&+ \epsilon^{2\delta} \| b^\epsilon \|_{L^\infty_t(\tilde{H}^{s\infty}_t)} \| H^\epsilon \|_{L^\infty_t(\tilde{H}^{s+1}_t)} \| H^\epsilon \|_{L^1_t(\tilde{H}^{s+1}_t)} \right\} \quad (105)
\end{aligned}$$

with

$$V^\epsilon(t) := \int_0^t \left( \| (b^\epsilon(\tau), u^\epsilon(\tau), H^\epsilon(\tau)) \|_{\tilde{L}^\infty}^2 + \epsilon^\delta \| \nabla u^\epsilon(\tau) \|_{L^2}^{1+\frac{s}{2}} \right) d\tau.$$

For $V^\epsilon(t)$, the following estimates hold:

$$\begin{aligned}
\| \nabla u^\epsilon \|_{L^1_t(\tilde{H}^{s+\frac{1}{2}})} &\leq \| \nabla u^\epsilon \|_{L^1_t(\tilde{H}^{s+\frac{1}{2}})} + \| \nabla u^\epsilon \|_{L^2_t(\tilde{H}^{s+\frac{1}{2}+\delta})} \\
\| (b^\epsilon, u^\epsilon, H^\epsilon) \|_{L^2_t(\tilde{H}^{s+\frac{1}{2}})} &\leq \| (P u^\epsilon, H^\epsilon) \|_{L^1_t(\tilde{H}^{s+\frac{1}{2}+\delta})} + \| (b^\epsilon, Qu^\epsilon) \|_{L^1_t(\tilde{H}^{s+\frac{1}{2}})} \\
&\leq P + W^\epsilon + \frac{\| \mathcal{L}(\frac{t}{\epsilon}) V \|_{L^1_t(\tilde{H}^{s+\frac{1}{2}})}}{2} + \frac{\| \mathcal{L}(\frac{t}{\epsilon}) z^\epsilon \|_{L^2_t(\tilde{H}^{s+\frac{1}{2}})}}{2} \\
&\leq P + W^\epsilon + X + (Z^\epsilon)^{\frac{1}{2}} (X + X^\epsilon)^{\frac{1}{2}}.
\end{aligned}$$

Thus, we can rewrite (105) as follows:

$$X^\epsilon + P^\epsilon \leq Ce^{C(P^2 + X^2 + (W^\epsilon)^2 + Z^\epsilon (X + X^\epsilon) + \epsilon^\delta (X^\epsilon + P^\epsilon)^{1+\frac{s}{2}})} \times \left\{ X_0 + P_0 \\
+ \epsilon \| b^\epsilon \|_{L^\infty_t(\tilde{L}^\infty)} (X^\epsilon + P^\epsilon) + \epsilon^\delta P^\epsilon (X^\epsilon + P^\epsilon) + \epsilon^{2\delta} X^\epsilon (P^\epsilon)^2 \right\}. \quad (106)$$

\textbf{Step 4.} Bootstrap argument. In this step, we mainly close the estimates (103), (104), and (106). Namely, we have

\textbf{Proposition 5.1.} Under the assumption $(\mathcal{H})$, there exists an $\epsilon_0 > 0$ depending only on $\delta, N, (P(T), X(T))$ and on the initial data, and such that the following inequalities hold true for some constant $C = C(N, \mu, \lambda, \nu, p, \delta)$ when $\epsilon \leq \epsilon_0$,

$$\begin{aligned}
X^\epsilon(T) + P^\epsilon(T) &\leq U_M := 2Ce^{C(P^2(T) + X^2(T))} (X_0 + P_0), \\
W^\epsilon(T) &\leq \epsilon \frac{\delta}{1+2\epsilon} W_M, \\
Z^\epsilon(T) &\leq \max \left\{ \epsilon \frac{1+2\epsilon}{\delta}, \tau(\epsilon) \right\} Z_M
\end{aligned}$$

with

$$W_M := 2Ce^{C(P + U_M)} \left\{ X_0 P_0 + U_M \left( P + P^2 + U_M + U_M^2 \right) \\
+ (P + U_M) \left( U_M + U_M^2 + U_M^3 \right) \right\},$$
Thanks to the inequalities (103), (104) and (106), we get, when
By the continuity of
Suppose that
Proposition 5.2. Continuation argument. We state the following proposition:
Section 8 in [10]. Thus we omit the details here.
Step 5. Global existence for the system (13)-(14). The proof is similar to that of
Section 8 in [10]. Thus we omit the details here.
Step 6. Continuation argument. We state the following proposition:
Proposition 5.2. Suppose that N = 2 or 3, δ > 0, b_0 \in H^{\frac{N}{2}+\delta} with \|b_0\|_{L^\infty} \leq \frac{1}{2}, (u_0, H_0) \in H^{\frac{N}{2}-1+\delta}. Then there exists a positive time T^* such that the problem (8)-(11) has a unique solution (b^*, u^*, H^*) \in E^{\frac{N}{2}+\delta}_{T^*, c_0} with \|b^*\|_{L^\infty_{T^*, (L^\infty)}} \leq \frac{3}{4}. Moreover, (T^*)^2 can be bounded by below by
\sup \left\{ T \left| \begin{array}{c} T \leq \frac{c}{\zeta} \\
\sum_{q \in \mathbb{Z}^N} 2^q (\frac{N}{2}+\delta-1) \left( \frac{1-e^{-c2^q}}{c} \right)^2 \| (\Delta_q u_0, \Delta_q H_0) \|_{L^2}^2 \leq \frac{c}{\zeta^2} \end{array} \right. \right\}.\]
where \( c = c(\lambda, \mu, \nu) \) and \( \zeta \) is a positive function of \( \|b_0\|_{H^{\frac{\nu}{2}+\delta}}, \|(u_0, H_0)\|_{H^{\frac{\nu}{2}+\delta-1}}, \) and \( \epsilon \|b_0\|_{L^\infty} \) depending also on \( \lambda, \mu, \nu, p, \epsilon, \delta, N. \)

The above statement may be obtained by going along the lines of the proof of Theorem 1.1 and Remark 2.2 in \([10]\). However, it has been assumed that \( \epsilon = 1, b_0 \in B_{2,1}^{\frac{\nu}{2}+\delta}, (u_0, H_0) \in B_{2,1}^{\frac{\nu}{2}+\delta-1}. \) Using appropriate change of variable (see Section 4), we can get the case \( \epsilon > 0 \) from \( \epsilon = 1. \) Moreover, Theorem 1.1 and Remark 2.2 in \([21]\) still hold in the Sobolev spaces setting. Collecting the above argument and Proposition 5.1, one can prove that the solution \((v', u'), (H')\) may be continued until the time \( T_0 \) or on \( \mathbb{R}^+ \) if \( T_0 = +\infty \) when \( \epsilon \) is suitably small. The reader can find more details in \([21]\). Therefore, \((H)\) is fulfilled on \([0, T_0]\) and meanwhile, Proposition 5.1 holds on \([0, T_0]\). Finally, according to \( V^\epsilon \) tends to \( V \) in \( G_{T_0}^{\frac{\nu}{2}+\delta} \) and \( V^\epsilon \) is uniformly bounded in \( G_{T_0}^{\frac{\nu}{2}+\delta} \), we can conclude that \( V^\epsilon \) tends to \( V \) in \( G_{T_0}^{\frac{\nu}{2}+\delta'} (\delta' < \delta) \) and in \( C_{T_0} (H^{\frac{\nu}{2}-1+\delta}) \). Thus, the proof of Theorem 1.1 is completed.

Appendix A. Some results in harmonic analysis. Let us first recall the definition and some basic properties of Sobolev space. Most of the materials stated below can be found in \([10]\). We shall use the letter \( C \) to denote the positive constant which may change from line to line.

We shall denote by \( \mathcal{F} z = (\hat{z}_k)_{k \in \mathbb{Z}^N} \) the Fourier series of a distribution \( z \in S'(\mathbb{T}^N) \) such that

\[
  z = \sum_{k \in \mathbb{Z}^N} \hat{z}_k e^{ik \cdot x}, \quad (107)
\]

where \( \hat{Z}^N := \mathbb{Z}/a_1 \times \cdots \times \mathbb{Z}/a_N \) and \( |\mathbb{T}^N| \) stands for the measure of \( \mathbb{T}^N \). We first introduce a \( C^\infty \) symmetric function \( \varphi \) of one variable supported in \( \{ r \in \mathbb{R}, 5/6 \leq |r| \leq 12/5 \} \) such that

\[
  \sum_{q \in \mathbb{Z}} \varphi(2^{-q}r) = 1 \quad \text{for} \quad r \neq 0.
\]

We then define the following dyadic blocks:

\[
  \Delta_q u := \sum_{k \in \mathbb{Z}^N} \varphi(2^{-q|k|}) \hat{u}_k e^{ik \cdot x},
\]

and the following low-frequency cut-off:

\[
  S_q u := \hat{u}_0 + \sum_{p \leq q^{-1}} \Delta_p u.
\]

Therefore, we have the following simple propositions:

\[
  \Delta_q u = 0 \quad \text{for negative enough} \quad q, \quad (108)
\]

\[
  \Delta_k \Delta_q u \equiv 0 \quad \text{if} \quad |k - q| \geq 2 \quad \text{and} \quad \Delta_k (S_{q-1} u \Delta_q u) \equiv 0 \quad \text{if} \quad |k - q| \geq 4. \quad (109)
\]

Next, using the Littlewood-Paley decomposition, we define the Sobolev and Besov space and their corresponding Hybrid spaces as follows:

\[
  H^\alpha(\mathbb{T}^N) := \left\{ u \in S'(\mathbb{T}^N) \mid \|u\|_{H^\alpha} = \left( |\hat{u}_0|^2 + \sum_{q \in \mathbb{Z}} 2^{2\alpha q} \|\Delta_q u\|_{L^2}^2 \right)^{\frac{1}{2}} < +\infty \right\},
\]

\[
  B^\alpha(\mathbb{T}^N) := \left\{ u \in S'(\mathbb{T}^N) \mid \|u\|_{B^\alpha} = |\hat{u}_0| + \sum_{q \in \mathbb{Z}} 2^{\alpha q} \|\Delta_q u\|_{L^2} < +\infty \right\},
\]
Then, we state several propositions as follows:

\[
\hat{H}^s_r(T^N) := \left\{ u \in S'(T^N) \mid \| u \|_{\hat{H}^s_r} = \left( |\hat{u}_0|^2 + \sum_{q \in \mathbb{Z}} (2^{sq} \max (\sigma, 2^{-q}) 1 - \frac{3}{2} \| \Delta_q u \|_{L^2} )^2 \right)^{\frac{1}{2}} < +\infty \right\},
\]

\[
\tilde{B}^s_r(T^N) := \left\{ u \in S'(T^N) \mid \| u \|_{\tilde{B}^s_r} = |\tilde{u}_0| + \sum_{q \in \mathbb{Z}} 2^{sq} \max (\sigma, 2^{-q}) 1 - \frac{3}{2} \| \Delta_q u \|_{L^2} < +\infty \right\}.
\]

We shall use the notation \( u = u - |T^N|^{-1} \int_{T^N} u \, dx \) for any \( u \in S'(T^N) \), and \( H^s, B^s \) stand for the subset of distribution \( u \) of \( \hat{H}^s, \tilde{B}^s \) with zero average (i.e. \( \hat{u}_0 = 0 \)). Then, we state several propositions as follows:

**Proposition A.1** ([10]). The following properties hold:

1. **Sobolev embeddings:**
   
   (a) For all \( s \in \mathbb{R} \), we have
   
   \[ B^s \hookrightarrow H^s \hookrightarrow C^{s - \frac{N}{2}} \]
   
   where \( \hookrightarrow \) stands for “continuous embedding”, and \( C^s \) for the Hölder space
   
   \[ C^s := \left\{ u \in S'(T^N) \mid \| u \|_{C^s} := \max \left( |\hat{u}_0|, \max_{q \in \mathbb{Z}} 2^{sq} \| \Delta_q u \|_{L^\infty} \right) < +\infty \right\}. \]

   (b) For any \( 0 < \delta' < \delta \), we have
   
   \[ \| u \|_{L^\infty} \leq C \| u \|_{H^{\delta'-1}_{\frac{N}{2}+\delta'}} \leq C_1 \| u \|_{\hat{B}^{\delta'-1}_{\frac{N}{2}+\delta'}}, \]
   
   and when \( \delta = 0 \), the above inequality reads
   
   \[ \| u \|_{L^\infty} \leq C \| u \|_{B^{\frac{N}{2}}_{\frac{N}{2}}} \leq C_1 \| u \|_{\hat{B}^{\frac{N}{2}}_{\frac{N}{2}}}. \]

2. **Scaling properties:**
   
   (a) For all \( \lambda > 0 \) and \( u \in X^s \) (here \( X^s \) stands for \( B^s \) or \( H^s \)), we have
   
   \[ \| u(\lambda) \|_{X^s(T^N_{\lambda^{-1}})} \approx \lambda^{s - \frac{N}{2}} \| u \|_{X^s(T^N)}. \]

   (b) For \( \sigma > 0, s \in \mathbb{R}, r \in [1, +\infty) \) and \( X = B \) or \( X = H \), we have
   
   \[ \| u(\lambda) \|_{X^s_r(T^N_{\lambda^{-1}})} \approx \lambda^{s - \frac{N}{2} + \frac{3}{2} - 1} \| u \|_{X^{s,r}_r(T^N_{\lambda^{-1}})}. \]

**Proposition A.2** ([10]). (1) If \( u \in H^{s_1} \) and \( v \in H^{s_2} \) with \( s_1 < \frac{N}{2}, s_2 < \frac{N}{2} \) and \( s_1 + s_2 > 0 \), then \( uv \in H^{s_1+s_2-\frac{N}{2}} \) and

\[
\| uv \|_{H^{s_1+s_2-\frac{N}{2}}} \leq C \| u \|_{H^{s_1}} \| v \|_{H^{s_2}}. \tag{110}
\]

If \( s_1 = \frac{N}{2} \), then \( \| u \|_{H^{s_1}} \) must be replaced with \( \| u \|_{H^{s_1} \cap L^\infty} \).

(2) If \( s \geq 0 \) and \( u, v \in H^s \), then we have

\[
\| uv \|_{H^s} \leq C \| \mathcal{F} u \|_{L^1(\mathbb{Z}^N)} \| v \|_{H^s} + C \| \mathcal{F} v \|_{L^1(\mathbb{Z}^N)} \| u \|_{H^s}. \tag{111}
\]

(3) If \( u \in H^{s_1} \cap H^{s_2} \) and \( v \in H^{t_1} \cap H^{t_2} \) with \( s_1 < \frac{N}{2}, t_1 < \frac{N}{2} \) and \( s_1 + t_1 > s_2 + t_2 > 0 \), then \( uv \in H^{s_1+t_2-\frac{N}{2}} \) and

\[
\| uv \|_{H^{s_1+t_2-\frac{N}{2}}} \leq C \| u \|_{H^{s_1}} \| v \|_{H^{t_2}} + C \| u \|_{H^{s_2}} \| v \|_{H^{t_1}}. \tag{112}
\]
(4) Let \( s \geq 0, u \in H^s \cap L^\infty \), and \( F \in W^{[s]+2,\infty}_{loc}(\mathbb{R}) \) be such that \( F(0) = 0 \). Then \( F(u) \in H^s \cap L^\infty \) and there exists a constant \( C = C(s, N, F, \|u\|_{L^\infty}) \) such that
\[
\|F(u)\|_{H^s} \leq C\|u\|_{H^s}.
\] (113)

**Remark A.1** ([10]). Similar properties stated in Proposition A.2 still hold in Besov spaces \( B^s \). Actually, (110) holds for \( s_1 = \frac{N}{2} \), or \( s_2 = \frac{N}{2} \), and (111) and (113) are true for \( s > 0 \) only.

**Lemma A.1** ([10]). Let \( s \in \mathbb{R}, u \in H^s \) (resp. \( u \in B^s \)) and \( \psi \in L^\infty(\mathbb{R}^N) \) be supported in the annulus \( C(0, R_1, R_2) \). There exists a sequence \((c_q)_{q \in \mathbb{Z}}\) such that \( \sum_q c_q^2 \leq 1 \) (resp. \( \sum_q c_q \leq 1 \)) and for all \( q \in \mathbb{Z} \),
\[
\|\psi(2^{-q}D)\|_{L^2} \lesssim q2^{-qs}\|u\|_{H^s}.
\]

Conversely, suppose that \( u = \sum_q u_q \) in \( S'(\mathbb{T}^N) \) with \( \supp \hat{u}_q \subset 2^q C(0, R_1, R_2) \) (If \( s > 0, \supp \hat{u}_q \subset 2^q B(0, R) \) suffices) and that
\[
\left( \sum_{q \in \mathbb{Z}} 2^{qs}\|u_q\|_{L^2}^2 \right)^{1/2} = K < +\infty.
\]

Then \( u \in H^s \) (resp. \( u \in B^s \)) and \( \|u\|_{H^s} \lesssim K \) (resp. \( \|u\|_{B^s} \lesssim K \)).

**Lemma A.2** ([10]). Let \( A \) be a function on \( \mathbb{Z}^N \times \mathbb{Z}^N \) such that for some \( \alpha \in \mathbb{R}, \beta, \gamma \geq 0 \), we have
\[
|A(k, m)| \leq K|m|^\alpha |k|\beta |m-k|^\gamma.
\]

Let
\[
Q(u, v) = \sum_{m,k} A(k, m) \bar{u}_k \bar{v}_{m-k} e^{imx}.
\]

Then the following inequality holds true for \( \sigma \geq 0 \),
\[
\|Q(u, v)\|_{H^{s-\sigma}} \lesssim K \|F\|_{H^{s+\sigma}} + \|F\|_{H^{s+\sigma}} \|u\|_{H^{s+\sigma}}.
\]

Next, we introduce the following time-space Sobolev spaces, initiated by [3];
\[
\tilde{L}_{T}^{s}(H^s) := \left\{ u \in S'(0, T \times \mathbb{T}^N) \mid \sup_{T} 2^{qs}\|\Delta u\|_{L^2(\mathbb{T}^N)}^2 < +\infty \right\}.
\]

Comparing the space \( \tilde{L}_{T}^{s}(H^s) \) with the spaces \( L^r(0, T; H^s) \), we have
\[
\begin{cases}
L^r(0, T; H^s) = \tilde{L}_{T}^{s}(H^s) & \text{if } r = 2, \\
\tilde{L}_{T}^{s}(H^s) \hookrightarrow L^r(0, T; H^s) & \text{if } r > 2,
\end{cases}
\] (114)

For any \( s' > s \) and \( r \in [1, +\infty] \), we have
\[
\tilde{L}_{T}^{s'}(H^{s'}) \hookrightarrow L^r(0, T; H^s) \quad \text{and} \quad L^r(0, T; H^{s'}) \hookrightarrow \tilde{L}_{T}^{s}(H^s).
\] (115)

Meanwhile, we have the following interpolation inequalities:
\[
\|u\|_{\tilde{L}_{T}^{s}(H^s)} \leq \|u\|_{L_{T}^{\frac{2s}{s'}}(H^{s'})}^{1-\theta} \|u\|_{L_{T}^{s}(H^s)}^{\theta}.
\] (116)

\[
\|u\|_{\tilde{L}_{T}^{s}(H^s)} \leq \|u\|_{L_{T}^{\frac{s}{s'}}(H^{s'})}^{1-\theta} \|u\|_{L_{T}^{s}(H^s)}^{\theta}.
\] (117)

where \( \theta \in [0, 1], \frac{1}{r_1} = \theta \frac{1}{r_1} + \left(1 - \theta\right) \frac{1}{r_2} \) and \( s = \theta s_1 + (1 - \theta)s_2 \).
Remark A.2 ([10]). The product and composition laws in $\tilde{L}^r_T(H^s)$ or $\tilde{L}^r_T(B^s)$ are the same as in Sobolev or Besov spaces. For example, instead of (112), we have
\[
\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r_3} + \frac{1}{r_4} = \frac{1}{r}.
\]

We need estimates for the composition of functions in hybrid spaces.

Proposition A.3 ([10]). Let $s > \max\{0, 1 - \frac{2}{r}\}, \sigma > 0, r \in [1, +\infty], T \in (0, +\infty)$ and $u \in \tilde{X}^{s,r}_\sigma \cap L^\infty (\text{resp. } u \in \tilde{L}^r_T(X^{s,r}_\sigma) \cap L^\infty (0, T \times \mathbb{T}^N))$ with $X = H$ or $B$.

Let $\gamma := [s + \max\{0, \frac{2}{r} - 1\}] + 1$ and $F \in W^{\gamma+1,\infty}(-M, M)$ with $M > \|u\|_{L^\infty}$ (resp. $M > \|u\|_{L^\infty(0, T \times \mathbb{T}^N)}$) such that $F(0) = 0$. Then $F(u) \in \tilde{X}^{s,r}_\sigma \cap L^\infty (\text{resp. } F(u) \in \tilde{L}^r_T(X^{s,r}_\sigma) \cap L^\infty (0, T \times \mathbb{T}^N))$ and
\[
\|F(u)\|_{\tilde{X}^{s,r}_\sigma} \leq C\|u\|_{\tilde{X}^{s,r}_\sigma} \quad (\text{resp. } \|F(u)\|_{\tilde{L}^r_T(X^{s,r}_\sigma)} \leq C\|u\|_{\tilde{L}^r_T(X^{s,r}_\sigma)})
\]
for a constant $C = C(s, r, N, F, \|u\|_{L^\infty})$ (resp. $C = C(s, r, N, F, \|u\|_{L^\infty(0, T \times \mathbb{T}^N)})$).

We recall the definition of the paraproduct introduced by Bony [1]. The paraproduct between $u$ and $v$ is defined by
\[
T_u v := \sum_{q \in \mathbb{Z}} S_{q-1} u \Delta_q v.
\]

We thus have the following formal decomposition:
\[
uv := T_u v + T_v u + R(u, v),
\]
with
\[
R(u, v) := \sum_{q \in \mathbb{Z}} \Delta_q u \tilde{\Delta}_q v \quad \tilde{\Delta}_q = \Delta_{q-1} + \Delta_q + \Delta_{q+1}.
\]

We will use the notation $T'_u v := T_u v + R(u, v)$.

Let us state some estimates for the paraproduct in $H^s$ and $\tilde{L}^r_T(H^s)$ spaces.

Proposition A.4 ([10]). Let $1 \leq r, r_1, r_2 \leq +\infty$ such that $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$, then for all $t \in \mathbb{R}$ and $s < \frac{N}{2}$, we have
\[
\|T_u v\|_{H^{s+t} - \frac{N}{2}} \lesssim \|u\|_{H^s} \|v\|_{H^t},
\]
\[
\|T_u v\|_{\tilde{L}^r_T(H^{s+t} - \frac{N}{2})} \lesssim \|u\|_{\tilde{L}^r_T(H^s)} \|v\|_{\tilde{L}^{r'_2}(H^t)}.
\]

If $s = \frac{N}{2}$, the above inequalities hold true provided $\|u\|_{H^s}$ (resp. $\|u\|_{\tilde{L}^r_T(H^s)}$) has been replaced with $L^\infty$ (resp. $L^r(0, T; L^\infty)$). If $(s, t) \in \mathbb{R}^2$ satisfies $s + t > 0$, then
\[
\|R(u, v)\|_{H^{s+t} - \frac{N}{2}} \lesssim \|u\|_{H^s} \|v\|_{H^t},
\]
\[
\|R(u, v)\|_{\tilde{L}^r_T(H^{s+t} - \frac{N}{2})} \lesssim \|u\|_{\tilde{L}^r_T(H^s)} \|v\|_{\tilde{L}^{r'_2}(H^t)}.
\]

Finally, for the operator $L(\tau)$, we have the following result:

Lemma A.3 ([10]). For all $\alpha \in \mathbb{R}$, the operator
\[
\begin{cases}
S'(0, T \times \mathbb{T}^N) \to S'(0, T \times \mathbb{T}^N) \\
u \to L(\alpha t) u
\end{cases}
\]
is an isomorphism on $L^r(0, T; H^s), \tilde{L}^r_T(H^s), L^r(0, T; B^s)$ and $\tilde{L}^r_T(B^s)$. 

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