Pathwise Uniqueness and Non-explosion Property of Skorohod SDEs with a Class of Non-Lipschitz Coefficients and Non-smooth Domains

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Abstract
Here, we study stochastic differential equations with a reflecting boundary condition. We provide sufficient conditions for pathwise uniqueness and non-explosion property of solutions in a framework admitting non-Lipschitz continuous coefficients and non-smooth domains.

Keywords Skorohod SDE · Non-Lipschitz coefficient · Pathwise uniqueness · Non-explosion property

Mathematics Subject Classification 60H10

1 Introduction
Let \( w = \{w(t)\}_{t \geq 0} \) be a one-dimensional Brownian motion on \( \mathbb{R} \) starting in \([0, \infty)\). A reflecting Brownian motion \( \xi = \{\xi(t)\}_{t \geq 0} \) on \([0, \infty)\) is characterized by the solution

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of the following (pathwise) equation:

\[
\begin{cases}
\xi = w + \phi, \\
\phi \text{ is non-decreasing on } [0, \infty), \phi(0) = 0, \text{ and} \\
\phi(t) = \int_0^t 1_{[0]}(\xi(s)) \, d\phi(s), \quad t \geq 0.
\end{cases}
\]  

Equation (1.1) for a continuous function \( w = \{w(t)\}_{t \geq 0} \) with a nonnegative initial value is called the Skorohod problem for \((\mathbb{R}^+, w)\). This equation has a unique solution described as

\[
\xi(t) = \begin{cases} 
  w(t), & 0 \leq t \leq \tau, \\
  w(t) - \inf\{w(s) \mid \tau \leq s \leq t\}, & t > \tau,
\end{cases}
\]

where \( \tau = \inf\{s > 0 \mid w(s) < 0\} \). Given a multidimensional domain \( D \subset \mathbb{R}^d \) and an \( \mathbb{R}^d \)-valued continuous function \( w \) on \([0, \infty)\), the Skorohod problem for \((D, w)\) can be considered similarly to (1.1) (see [10] for a precise formulation). Tanaka [11, Theorem 2.1] showed that the Skorohod problem has a unique solution if \( D \) is a convex domain. Saisho [10, Theorem 4.1] extended this result to more general domains satisfying conditions (A) and (B), which are defined in Sect. 2. The class of domains \( D \) satisfying these conditions includes all convex domains and domains with a bounded \( C^2 \)-boundary and admits some non-smoothness.

The Skorohod problem is generalized to a stochastic differential equation (SDE) as follows. Let \((\Omega, F, P)\) be a probability space. Let \( D \) be a domain of \( \mathbb{R}^d \) and denote its closure by \( \overline{D} \). Given an \( \mathbb{R}^d \)-valued function \( b \) and a \( d \times d \) matrix-valued function \( \sigma \) on \([0, \infty) \times \Omega \times \overline{D} \), we are concerned with the following SDE:

\[
\begin{cases}
dX(t) = \sigma(t, \cdot, X(t)) \, dB(t) + b(t, \cdot, X(t)) \, dt + d\Phi_X(t), & t \geq 0, \\
X(0) \in \overline{D}.
\end{cases}
\]  

Here, \( \{B(t)\}_{t \geq 0} \) denotes a \( d \)-dimensional Brownian motion, and \( \Phi_X \) is a reflection term, which is an unknown continuous function of bounded variation with properties (2.2), (2.3), and (2.4), which are presented below. Equation (1.2) is called a Skorohod SDE, which is a natural generalization of the Skorohod problem. If \( b = 0 \) and \( \sigma \) is the identity matrix, the solution to (1.2) is simply a reflecting Brownian motion on \( \overline{D} \). If \( D \) satisfies conditions (A) and (B), and if coefficients \( \sigma \) and \( b \) depend on only \( x \in \overline{D} \) and are bounded continuous functions, then a solution exists for (1.2) (see [10, Remark 5.1] and also Remark 2 below). If \( \sigma \) and \( b \) depend on only \( x \in \overline{D} \) and are Lipschitz continuous on \( \overline{D} \), a standard argument by the Gronwall inequality leads us to pathwise uniqueness of (1.2) ([10, Lemma 5.6]). We are therefore interested in the case where \( \sigma \) and \( b \) are not necessarily Lipschitz continuous. Since [14] obtained a satisfactory sufficient condition for pathwise uniqueness in the one-dimensional case, we consider general dimensions.

Note that for usual SDEs without reflection terms, pioneering works by Yamada and Watanabe [12,13] have already treated non-Lipschitz coefficients. Although a number
of related studies have been done since then, here we cite only a couple of works directly related to this paper [4,7]. The arguments in [4] were adapted in [3] for study of the Skorohod SDE (1.2) and to obtain sufficient conditions for pathwise uniqueness when \( D \) is admissible, that is, when \( D \) is roughly described as

\[
D = \{ x \in \mathbb{R}^d \mid \varphi(x) > 0 \} \quad \text{and} \quad \partial D = \{ x \in \mathbb{R}^d \mid \varphi(x) = 0 \}
\]

for some nice \( C^2 \)-function \( \varphi \). In particular, \( D \) has a smooth boundary. Their arguments depend on this smoothness.

In this paper, we consider the Skorohod SDE (1.2) for domains \( D \) satisfying only condition (A) (and (B) for some claims). First, we prove pathwise uniqueness under certain conditions that allow non-Lipschitz coefficients \( \sigma \) and \( b \) (Theorem 1). Our arguments are based on those of [7, Theorem 2] and inherit the assumptions in that work. Note that condition (A) does not necessarily hold even when the boundary of \( D \) is in \( C^1 \). However, this is consistent with the fact that the pathwise uniqueness of a reflecting Brownian motion on \( \overline{D} \) can fail for \( C^1 \) domains. See [2, Theorem 4.1] for such an example.

Since the solution to (1.2) may explode in general, we provide two sufficient conditions for the solution to be not explosive (Theorems 2 and 3). In Theorem 2, the main conditions are described using a Lyapunov-type function. This result corresponds to [7, Theorem 1], where the non-explosion property of SDEs without reflection terms was discussed. As seen in Example 1, these conditions fit for convex domains and domains whose boundaries are (globally) described by smooth functions. In both cases, the norms of coefficients \( \sigma \) and \( b \) in (1.2) can grow as \( |x|(\log |x|)^{1/2} \) in \( x \) as \( |x| \to \infty \).

In Theorem 3, we discuss the non-explosion property for more general domains in a sense, but with more restrictive conditions on coefficients. The following are typical examples where Theorem 3 can be applied:

1. The coefficients grow by at most \( |x|(\log |x|)^{1/2} \) in \( x \) as \( |x| \to \infty \) and are bounded near the boundary (Example 2(1));
2. The coefficients have at most sub-linear growth of order \( (1/2) - \varepsilon \) in \( x \) for some \( \varepsilon > 0 \) as \( |x| \to \infty \) (Example 2(2)).

The proof of Theorem 3 is quite different from that of Theorem 2; the key idea is to prove that solutions of (1.2) approach the boundary of \( D \) only a finite number of times in a sense almost surely on each finite time interval (Lemma 4).

The remainder of this paper is organized as follows: In Sect. 2, we set up our framework, state its main theorems (Theorems 1, 2, and 3), and present some typical examples. In Sect. 3, we use the arguments in [7] to prove Theorems 1 and 2. As an application of Theorem 1, we give a sufficient condition for (1.2) to have a strong solution (Corollary 1). In Sect. 4, we prove Theorem 3.

**Notation** The following symbols are used in the paper.

- \( \mathbb{R}_+ \) is the half-line \([0, \infty)\).
- \( \mathbb{R}^d \otimes \mathbb{R}^d \) denotes the set of all real square matrices of size \( d \), and \( A^* \) denotes the transpose of \( A \in \mathbb{R}^d \otimes \mathbb{R}^d \).
- \( \langle \cdot, \cdot \rangle \) and \( | \cdot | \) denote the standard inner product and norm of \( \mathbb{R}^n \), respectively.
\| \cdot \| \) denotes the Hilbert–Schmidt norm of $\mathbb{R}^d \otimes \mathbb{R}^d$.
- For $x \in \mathbb{R}^d$ and $r > 0$, $B(x, r)$ (resp. $\overline{B}(x, r)$) denotes the open (resp. closed) ball in $\mathbb{R}^d$ with center $x$ and radius $r$. We write $B(r)$ for $B(0, r)$.
- For $a, b \in [-\infty, \infty]$, we write $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$.
- $\#I$ is the cardinality (number of elements) of a set $I$.
- $\inf \emptyset = \infty$ by convention.

## 2 Main Results

Let $D$ be a domain of $\mathbb{R}^d$. Let $\overline{D}_\Delta$ denote $\overline{D}$ or $\overline{D} \cup \{\Delta\}$ (the one-point compactification of $\overline{D}$) according whether $\overline{D}$ is compact or non-compact, respectively. We define

$$\mathcal{W}(\overline{D}_\Delta) = \left\{ w : \mathbb{R}_+ \to \overline{D}_\Delta \mid w \text{ is continuous, } w(0) \in \overline{D}, \text{ and if } w(t) = \Delta \text{ for some } t \geq 0, \text{ then } w(s) = \Delta \text{ for any } s \geq t. \right\}$$

For each $w \in \mathcal{W}(\overline{D}_\Delta)$, we define the lifetime $\zeta(w)$ of $w$ as

$$\zeta(w) = \inf\{t > 0 \mid w(t) = \Delta\}.$$ 

For $x \in \partial D (= \overline{D} \setminus D)$ and $r \in (0, \infty)$, we define $\mathcal{N}_{x,r}$ and $\mathcal{N}_x$ as

$$\mathcal{N}_{x,r} = \{ \mathbf{n} \in \mathbb{R}^d \mid |\mathbf{n}| = 1, \ B(x - r\mathbf{n}, r) \cap D = \emptyset \}, \quad \mathcal{N}_x = \bigcup_{r \in (0, \infty)} \mathcal{N}_{x,r}.$$ 

An element in $\mathcal{N}_x$ is called an inward normal unit vector at $x$.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a filtered probability space with the usual condition. The expectation with respect to $P$ is denoted by $E[\cdot]$. We denote by $E[\cdot : A]$ the expectation on event $A \in \mathcal{F}$. Let $\sigma : \mathbb{R}_+ \times \Omega \times \mathcal{D} \to \mathbb{R}^d \otimes \mathbb{R}^d$ and $b : \mathbb{R}_+ \times \Omega \times \mathcal{D} \to \mathbb{R}^d$ be measurable functions. Throughout this paper, we assume the following:

- $\mathbb{R}_+ \times \Omega \ni (t, \omega) \mapsto \sigma(t, \omega, x)$ and $\mathbb{R}_+ \times \Omega \ni (t, \omega) \mapsto b(t, \omega, x)$ are progressively measurable for any fixed $x \in \mathcal{D}$.

We now provide an explicit definition of the Skorohod SDE.

**Definition 1** A pair of $\{\mathcal{F}_t\}_{t \geq 0}$-adapted processes $(X, \Phi_X)$ is called a solution of the Skorohod SDE

$$\begin{cases} 
    dX(t) = \sigma(t, \cdot, X(t)) \, dB(t) + b(t, \cdot, X(t)) \, dt + d\Phi_X(t), & t \geq 0, \\
    X(0) \in \overline{D} 
\end{cases} \quad (2.1)$$

if the following conditions are satisfied:

- For $P$-a.s., $X = \{X(t)\}_{t \geq 0}$ belongs to $\mathcal{W}(\overline{D}_\Delta)$.
For $P$-a.s., $\Phi_X = \{\Phi_X(t)\}_{t \geq 0}$ is an $\mathbb{R}^d$-valued continuous function on $[0, \zeta_X)$ of bounded variation on each compact interval, where we define $\zeta_X(\omega) = \zeta(X(\omega))$, $\omega \in \Omega$. Furthermore, for $P$-a.s.,

$$\Phi_X(0) = 0, \quad |\Phi_X|_t = \int_0^t \mathbf{1}_{\partial D}(X(s)) \, d|\Phi_X|_s, \quad t < \zeta_X,$$

and

$$\Phi_X(t) = \int_0^t \mathbf{n}(s) \, d|\Phi_X|_s, \quad t < \zeta_X.$$  \hspace{1cm} (2.4)

Here, $|\Phi_X|_t$ denotes the total variation process of $\Phi_X$, and $\mathbf{n}(s) \in \mathcal{N}_X(s)$ if $X(s) \in \partial D$.

For $P$-a.s.,

$$\int_0^t \{\|\sigma(s, \cdot, X(s))\|^2 + |b(s, \cdot, X(s))|\} \, ds < \infty, \quad t \in [0, \xi_X).$$  \hspace{1cm} (2.5)

For $P$-a.s.,

$$X(t) = X(0) + \int_0^t \sigma(s, \cdot, X(s)) \, dB(s) + \int_0^t b(s, \cdot, X(s)) \, ds + \Phi_X(t),$$

$t \in [0, \zeta_X)$,

where $\{B(t)\}_{t \geq 0}$ is a $d$-dimensional $\mathcal{F}_{t}$-Brownian motion.

We often say that $X$ is a solution of (2.1) without referring to $\Phi_X$.

Following [8,10], we introduce conditions (A) and (B) on $D$ as follows:

(A) There exists $r_0 \in (0, \infty)$ such that for any $x \in \partial D$,

$$\mathcal{N}_x = \mathcal{N}_{x, r_0} \neq \emptyset.$$  \hspace{1cm} (A)

(B) There exist $\delta \in (0, \infty)$ and $\beta \in [1, \infty)$ with the requirement that for any $x \in \partial D$ there exists a unit vector $\mathbf{1}_x$ such that

$$\langle \mathbf{1}_x, \mathbf{n} \rangle \geq 1/\beta$$

for any $\mathbf{n} \in \bigcup_{y \in B(x, \delta) \cap \partial D} \mathcal{N}_y$.

Remark 1 ([10, Remark 1.1]) Let $x \in \partial D$, $r > 0$, and let $\mathbf{n} \in \mathbb{R}^d$ be a unit vector. Then, the following conditions are equivalent:

- $\mathbf{n} \in \mathcal{N}_{x, r}$.
- For any $y \in D$, $\langle y-x, \mathbf{n} \rangle + (2r)^{-1} |y-x|^2 \geq 0$. 

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Indeed, \( n \in \mathcal{N}_{x,r} \) if and only if \( |y - (x - r n)| \geq r \) for any \( y \in \overline{D} \), that is,
\[
|y - x|^2 + 2(y - x, r n) + r^2 \geq r^2 \quad \text{for any } y \in \overline{D}.
\]

**Remark 2**

1. Our definition of the Skorohod SDE is more general than that in [10], in that the lifetime is considered.
2. By the same argument as in [10, Theorem 5.1], Eq. (2.1) possesses a (not necessarily strong) solution with infinite lifetime if the following are satisfied (see also [10, Remark 5.1]):
   - \( D \) satisfies conditions (A) and (B).
   - For \( P \)-a.s., \( \sigma \) and \( b \) are continuous in \((t, x) \in \mathbb{R}_+ \times \overline{D} \).
   - For any \( T > 0 \),
   \[
   \operatorname{ess} \sup_{\omega \in \Omega} \sup_{t \in [0, T], x \in \overline{D}} \left\{ \| \sigma(t, \omega, x) \| \lor |b(t, \omega, x)| \right\} < \infty.
   \]
3. In [9, Theorem 2.2], the authors provided sufficient conditions for Eq. (2.1) to have a solution when \( \sigma \) and \( b \) are only measurable.

Even if \( \sigma \) and \( b \) are locally bounded, we can obtain local solutions for (2.1) by Remark 2 (2). The solutions are strong once we prove the pathwise uniqueness of solutions for (2.1). We can then obtain a strong solution to (2.1) from a standard localization argument. Such discussions are rigorously presented in Corollary 1, below. Therefore, we first study the pathwise uniqueness of the Skorohod equation (2.1). The definition is as follows:

**Definition 2**

We say that pathwise uniqueness of solutions for (2.1) holds if for any two solutions \((X, \Phi_X)\) and \((Y, \Phi_Y)\) of (2.1) that are defined on the same filtered probability space with the same \( d \)-dimensional Brownian motion \( \{B(t)\}_{t \geq 0} \) such that \( X(0) = Y(0) \) \( P \)-a.s., we have \( X(t) = Y(t) \) for all \( t \geq 0 \) \( P \)-a.s.

**Remark 3**

By the Yamada–Watanabe theorem [13, Corollary 3], the existence and pathwise uniqueness of solutions to (2.1) ensure the existence of a strong solution to (2.1). Note that the Yamada–Watanabe theorem was proved for SDEs without reflection terms. However, the proof is also valid for (2.1).

To describe a sufficient condition for pathwise uniqueness, we introduce the following condition for a nonnegative and Borel measurable function \( \Lambda \) on \([0, 1] \):

1. There exists some \( \varepsilon_0 \in (0, 1) \) such that \( \Lambda \) is continuous and non-decreasing on \([0, \varepsilon_0] \), and
\[
\int_0^{\varepsilon_0} \frac{1}{\Lambda(s)} \, ds = \infty. \tag{2.6}
\]

Note that (2.6) implies
\[
\int_0^\varepsilon \frac{1}{\Lambda(s)} \, ds = \infty, \quad \varepsilon \in (0, \varepsilon_0]. \tag{2.7}
\]
For example, \( \Lambda(s) = s \), \( \Lambda(s) = s \log(1/s) \), and \( \Lambda(s) = s \log(1/s) \times \log \log(1/s) \) satisfy the above conditions.

Let \( g = \{g(t, \cdot)\}_{t \geq 0} \) be a nonnegative progressively measurable process such that, for any \( T \geq 0 \),
\[
\int_0^T g(s, \cdot) \, ds < \infty \quad P\text{-a.s.} \quad (2.8)
\]

A sufficient condition for pathwise uniqueness is given as follows:

**Theorem 1** Assume condition \((A)\) and that for each \( R > 0 \) there exists a Borel measurable function \( \Lambda_R : [0, 1) \to \mathbb{R}_+ \) satisfying \((L)\) and for \( P\)-a.s. \( \omega \),
\[
\|\sigma(t, \omega, x) - \sigma(t, \omega, y)\|^2 + 2\langle x - y, b(t, \omega, x) - b(t, \omega, y) \rangle
\leq g(t, \omega) \Lambda_R(|x - y|^2) \quad (2.9)
\]
for any \( t \geq 0 \) and \( x, y \in \overline{D} \cap B(R) \) with \( |x - y| < 1 \). Then, the pathwise uniqueness of solutions for (2.1) holds.

Combining Theorem 1 and Remarks 2(2) and 3, we obtain the following sufficient condition for (2.1) to have a strong solution:

**Corollary 1** Assume conditions \((A)\) and \((B)\) and the following:

1. For each \( R > 0 \) there exists a Borel measurable function \( \Lambda_R : [0, 1) \to \mathbb{R}_+ \) satisfying \((L)\) and for \( P\)-a.s. \( \omega \),
\[
\|\sigma(t, \omega, x) - \sigma(t, \omega, y)\|^2 + 2\langle x - y, b(t, \omega, x) - b(t, \omega, y) \rangle
\leq g(t, \omega) \Lambda_R(|x - y|^2) \]
for any \( t \geq 0 \) and \( x, y \in \overline{D} \cap B(R) \) with \( |x - y| < 1 \).

2. For \( P\)-a.s. \( \omega \in \Omega \), the maps \( \mathbb{R}_+ \times \overline{D} \ni (t, x) \mapsto \sigma(t, \omega, x) \) and \( \mathbb{R}_+ \times \overline{D} \ni (t, x) \mapsto b(t, \omega, x) \) are continuous.

3. For any \( T > 0 \) and \( R > 0 \),
\[
\text{ess sup}_{\omega \in \Omega} \sup_{t \in [0, T], \, x \in \overline{D} \cap B(R)} \{ \|\sigma(t, \omega, x)\| \vee |b(t, \omega, x)| \} < \infty.
\]

Then, (2.1) possesses a strong solution.

Next, we discuss the non-explosion property of the solution. Let \( \gamma : \mathbb{R}_+ \to [1, \infty) \) be a continuous and non-decreasing function such that \( \lim_{s \to \infty} \gamma(s) = \infty \) and
\[
\int_0^\infty \frac{1}{\gamma(s)} \, ds = \infty. \quad (2.10)
\]

Functions \( \gamma(s) = s + 1 \), \( \gamma(s) = s \log(s + 1) + 1 \) are typical examples satisfying the above conditions.
Theorem 2 Assume condition (A), and that there exists a nonnegative function $V \in C^{1,2}([0, \infty) \times \mathbb{R}^d)$ with the following conditions:

(V.1) For any $t > 0$,
\[
\lim_{R \to \infty} \inf_{s \in [0,t], x \in \bar{D} \setminus B(R)} V(s, x) = \infty.
\]

(V.2) For any $x \in \partial D$, $t \geq 0$, and $\mathbf{n} \in \mathcal{N}_x$, $((\nabla V)(t, x), \mathbf{n}) \leq 0$.

(V.3) For $P$-a.s. $\omega$,
\[
\|\sigma(t, \omega, x)\|^2(\Delta V)(t, x) + 2\langle b(t, \omega, x), (\nabla V)(t, x) \rangle + 2\frac{\partial V}{\partial t}(t, x) \leq g(t, \omega) \gamma(V(t, x))
\]
(2.11)
for any $t \geq 0$ and $x \in \bar{D}$.

Then, the solutions to (2.1) are non-explosive, that is, $P(\xi_X = \infty) = 1$.

The following describes another sufficient condition for non-explosion. For $x \in \bar{D}$, $\delta > 0$, and $T > 0$, we set
\[
M(x, \delta, T) = \text{ess sup}_{\omega \in \Omega} \sup_{t \in [0, T], z \in B(x, \delta) \cap \mathcal{D}} \{\|\sigma(t, \omega, z)\|^2 + |b(t, \omega, z)|^2\}.
\]

Theorem 3 Assume conditions (A) and (B), and the following:

(1) For $P$-a.s. $\omega$,
\[
\|\sigma(t, \omega, x)\|^2 + |b(t, \omega, x)|^2 \leq g(t, \omega) \gamma(|x|^2)
\]
(2.12)
for any $t \geq 0$ and $x \in \bar{D}$.

(2) For each $T > 0$, there exist constants $C > 0$, $\nu \in [0, 1)$, $\delta > 0$, $\beta \in (0, 1)$, points $\{x_n\}_{n=1}^\infty \subset \partial D$, and positive numbers $\{\delta_n\}_{n=1}^\infty \subset [\delta, \infty)$ such that $\partial D \subset \bigcup_{n=1}^\infty B(x_n, \beta \delta_n)$ and
\[
M(x_n, \delta_n, T) \leq C \delta_n^\nu \text{ for any } n \in \mathbb{N}.
\]
(2.13)

Then, the solutions of (2.1) are non-explosive.

Example 1 The following are some examples meeting the assumptions in Theorem 2:

(1) If $D$ is an unbounded convex domain, it satisfies conditions (A) and (B). We take $x_0 \in D$ and set
\[
V(t, x) = |x - x_0|^2, \quad t \in [0, \infty), \quad x \in \mathbb{R}^d.
\]
We write \( V(x) \) for \( V(t, x) \) since \( V(t, x) \) does not depend on \( t \). \( V \) satisfies (V.1). Because \( D \) is convex, it follows that for any \( x \in \partial D \) and \( n \in \mathcal{N}_x \)

\[
\langle \nabla V(x), n \rangle = 2(x - x_0, n) \leq 0,
\]

proving (V.2). If there exists \( C > 0 \) such that

\[
\|\sigma(t, \omega, x)\| \vee |b(t, \omega, x)| \leq C[|x|(|\log(|x|) + 1)|^{1/2} + 1]
\]

for any \( (t, \omega, x) \in \mathbb{R}_+ \times \Omega \times \overline{D} \), we then see that (V.3) holds with \( \gamma(s) = s \log(s + 1) + 1 (s \geq 0) \) and a sufficiently large constant function \( g \).

(2) Let \( H : [-1, \infty) \to \mathbb{R} \) be a smooth function such that \( H(-1) = 0 \) and \( H(s) > 0 \) for any \( s > -1 \). We also assume that there exist \( m > 0 \) and \( M > -1 \) such that \( H'(s) \vee H''(s) \leq m \) for any \( s > -1 \), and

\[
\int_{-1}^{s} H(u) \, du \geq \frac{s^2}{m} \tag{2.14}
\]

for any \( s > M \). We define a domain \( D \subset \mathbb{R}^d \) as

\[
D = \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d | x_1 > -1, \, x_2^2 + \cdots + x_d^2 < H(x_1)^2 \}.
\]

Since \( H''(s) \) is bounded above, \( D \) satisfies condition (A). Moreover, \( \mathcal{N}_x \) is a singleton for any \( x = (x_1, x_2, \ldots, x_d) \in \partial D \) with \( x_1 > -1 \). For each \( x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \), we write \( \tilde{x} = (x_2, \ldots, x_d) \) and denote the length of \( \tilde{x} \) as \( |\tilde{x}| \) with an abuse of notation. Then, for any \( x = (x_1, \tilde{x}) \in \partial D \) with \( x_1 > -1 \), the inward unit vector \( n \) at \( x \) is

\[
n = \frac{1}{\sqrt{H'(x_1)^2 + 1}} \left( H'(x_1), -\frac{\tilde{x}}{|\tilde{x}|} \right).
\]

We define a nonnegative function \( V : [0, \infty) \times D \to \mathbb{R} \) as

\[
V(t, x) = \int_{-1}^{x_1} H(s) \, ds + \frac{m}{2} |\tilde{x}|^2, \quad t \in [0, \infty), \, x = (x_1, \tilde{x}) \in D.
\]

We write \( V(x) \) for \( V(t, x) \). Then, \( V(x) \) extends to a smooth function on \( \mathbb{R}^d \). We see that \( V \) satisfies conditions (V.1) and (V.2). Assume that there exists \( C_1 > 0 \) such that

\[
\|\sigma(t, \omega, x)\| \vee |b(t, \omega, x)| \leq C_1[|x|(|\log(|x|) + 1)|^{1/2} + 1]
\]

for any \( (t, \omega, x) \in [0, \infty) \times \Omega \times \overline{D} \). For any \( x \in \overline{D} \), we then have

\[
(\Delta V)(x) \leq md, \quad |(\nabla V)(x)|^2 \leq m^2(x_1 + 1)^2 + m^2|\tilde{x}|^2.
\]
We see that the left-hand side of (2.11) is less than or equal to
\[ C_2[|x|^2 \log(|x| + 1) + 1], \quad x = (x_1, \tilde{x}) \in \bar{D} \]
for some $C_2 > 0$. It follows from (2.14) that there exists $C_3 > 0$ such that $V(x) \geq C_3|x|^2$ for any $x \in \bar{D}$. Therefore, (V.3) holds with $\gamma(s) = s \log(s + 1) + 1 \ (s \geq 0)$ and a sufficiently large constant function $g$.

**Example 2** The following examples can apply Theorem 3. Assume that $D$ satisfies (A) and (B) in both cases.

1. Suppose assumption (1) in Theorem 3. Moreover, for each $T > 0$, suppose that there exists $\delta > 0$ such that
   \[ \sup_{\omega \in \Omega} \sup_{t \in [0,T], x \in D(\delta)} \|\sigma(t, \omega, x)\| \wedge |b(t, \omega, x)| < \infty, \]
   where $D(\delta) = \bigcup_{y \in \partial D} B(y, \delta) \cap \bar{D}$. Then, assumption (2) holds with $\nu = 0$, $\beta = 1/2$, $\delta_n = \delta$ for $n \in \mathbb{N}$, and $\{x_n\}_{n=1}^{\infty} \subset \partial D$ being taken so that $\partial D \subset \bigcup_{n \in \mathbb{N}} B(x_n, \delta/2)$.

2. Suppose that there exist $C > 0$ and $\varepsilon \in (0, 1/2)$ such that
   \[ \|\sigma(t, \omega, x)\| \wedge |b(t, \omega, x)| \leq C(|x|^{1/2-\varepsilon} + 1) \]
   for any $(t, \omega, x) \in [0, \infty) \times \Omega \times \bar{D}$. Then, assumptions (1) and (2) in Theorem 3 hold with $\nu = 1 - 2\varepsilon$, $\beta = 1/2$, $\{x_n\}_{n=1}^{\infty} \subset \partial D$ such that $\partial D \subset \bigcup_{n \in \mathbb{N}} B(x_n, (|x_n| + 1)/2)$, and $\delta_n = |x_n| + 1$ for $n \in \mathbb{N}$.

3 **Proofs of Theorems 1, 2, Corollary 1**

We introduce the following lemma for later use:

**Lemma 1** (1) Let $f$ be a nonnegative and non-decreasing function on an interval $(0, \varepsilon]$ such that $\int_0^{\varepsilon} f(t)^{-1} \, dt = \infty$. Then, $\int_0^\varepsilon (f(t) \wedge t)^{-1} \, dt = \infty$.

(2) Let $h$ be a nonnegative and non-decreasing function on an interval $[r, \infty)$ such that $h \geq 1$ on $[r, \infty)$ and $\int_r^\infty h(t)^{-1} \, dt = \infty$. Then, $\int_r^\infty (h(t) \wedge t)^{-1} \, dt = \infty$.

**Proof** (1) Let $A = \{ t \in (0, \varepsilon] \mid f(t) < 1/t \}$. If $A = \emptyset$, then the assertion is obvious. Suppose $A \neq \emptyset$ and let $a = \inf A$. If $a > 0$, then
   \[ \int_0^\varepsilon \frac{1}{f(t) \wedge t} \, dt \geq \int_0^a \frac{1}{f(t)} \, dt = \infty. \]
   Otherwise, we can take a decreasing sequence $\{t_n\}_{n=1}^{\infty}$ converging to 0 such that $t_n \in A$ and $t_{n+1} < t_n/2$ for every $n$. Then, because
   \[ f(t) \wedge t \leq f(t_n) \wedge t_n = t_n, \quad t \in [t_{n+1}, t_n], \]
   for some $C_2 > 0$. It follows from (2.14) that there exists $C_3 > 0$ such that $V(x) \geq C_3|x|^2$ for any $x \in \bar{D}$. Therefore, (V.3) holds with $\gamma(s) = s \log(s + 1) + 1 \ (s \geq 0)$ and a sufficiently large constant function $g$. **Example 2** The following examples can apply Theorem 3. Assume that $D$ satisfies (A) and (B) in both cases.

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2. Suppose that there exist $C > 0$ and $\varepsilon \in (0, 1/2)$ such that
   \[ \|\sigma(t, \omega, x)\| \wedge |b(t, \omega, x)| \leq C(|x|^{1/2-\varepsilon} + 1) \]
   for any $(t, \omega, x) \in [0, \infty) \times \Omega \times \bar{D}$. Then, assumptions (1) and (2) in Theorem 3 hold with $\nu = 1 - 2\varepsilon$, $\beta = 1/2$, $\{x_n\}_{n=1}^{\infty} \subset \partial D$ such that $\partial D \subset \bigcup_{n \in \mathbb{N}} B(x_n, (|x_n| + 1)/2)$, and $\delta_n = |x_n| + 1$ for $n \in \mathbb{N}$.

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   \[ \int_0^\varepsilon \frac{1}{f(t) \wedge t} \, dt \geq \int_0^a \frac{1}{f(t)} \, dt = \infty. \]
   Otherwise, we can take a decreasing sequence $\{t_n\}_{n=1}^{\infty}$ converging to 0 such that $t_n \in A$ and $t_{n+1} < t_n/2$ for every $n$. Then, because
we obtain that
\[
\int_0^\varepsilon \frac{1}{f(t) \vee t} \, dt \geq \sum_{n=1}^\infty \int_{t_{n+1}}^{t_n} \frac{1}{f(t) \vee t} \, dt \\
\geq \sum_{n=1}^\infty \int_{t_{n+1}}^{t_n} \frac{1}{t_n} \, dt \geq \sum_{n=1}^\infty \frac{t_n}{2} \cdot \frac{1}{t_n} = \infty.
\]

(2) The proof is similar to that of (1). Let \( A = \{ t \in [r, \infty] \mid h(t) < t \} \). If \( A = \emptyset \), then the assertion is obvious. Suppose \( A \neq \emptyset \) and let \( a = \sup A \). If \( a < \infty \), then
\[
\int_a^\infty \frac{1}{h(t) \vee t} \, dt \geq \int_a^\infty \frac{1}{h(t)} \, dt = \infty.
\]
Otherwise, we can take an increasing sequence \( \{t_n\}_{n=1}^\infty \) diverging to \( \infty \) such that \( t_n \in A \) and \( t_{n+1} > 2t_n \) for every \( n \). Then, since
\[
h(t) \vee t \leq h(t_{n+1}) \vee t_{n+1} = t_{n+1}, \quad t \in [t_n, t_{n+1}],
\]
we obtain that
\[
\int_r^\infty \frac{1}{h(t) \vee t} \, dt \geq \sum_{n=1}^\infty \int_{t_n}^{t_{n+1}} \frac{1}{h(t) \vee t} \, dt \\
\geq \sum_{n=1}^\infty \int_{t_n}^{t_{n+1}} \frac{1}{t_{n+1}} \, dt \geq \sum_{n=1}^\infty \frac{t_{n+1}}{2} \cdot \frac{1}{t_{n+1}} = \infty.
\]

\( \square \)

**Proof of Theorem 1** Let \((X, \Phi_X)\) and \((Y, \Phi_Y)\) be two solutions to (2.1) that are defined on the same filtered probability space with the same \(d\)-dimensional Brownian motion \(\{B(t)\}_{t \geq 0}\) such that \(X(0) = Y(0)\) \(P\)-a.s. We define
\[
\eta(t) = X(t) - Y(t), \\
\xi(t) = |\eta(t)|^2, \\
e(t) = \sigma(t, \cdot, X(t)) - \sigma(t, \cdot, Y(t)), \\
f(t) = b(t, \cdot, X(t)) - b(t, \cdot, Y(t)).
\]

From the Itô formula,
\[
d\xi(t) = \langle 2e^*(t)\eta(t), dB(t) \rangle + \langle 2\eta(t), f(t) \rangle \, dt \\
+ \langle 2\eta(t), d\Phi_X(t) - d\Phi_Y(t) \rangle + \|e(t)\|^2 \, dt. \quad (3.1)
\]
Fix $R > 0$. By Lemma 1 (1), $\Lambda_R(s) \lor s$ ($s \in [0, 1]$) also satisfies the conditions imposed on $\Lambda_R$. Thus, we may assume

$$\Lambda_R(s) \geq s \quad \text{for } s \in [0, 1)$$

(3.2)

without loss of generality. For each $r > 0$, we define $\phi_r : [0, 1] \to \mathbb{R}_+$ by

$$\phi_r(s) = \int_0^s \frac{1}{\Lambda_R(u) + r} \, du.$$  

Then, for any $s \in (0, \epsilon_0)$, we have

$$\phi_r(s) \nearrow \int_0^s \frac{1}{\Lambda_R(s)} \, du = \infty \quad \text{as } r \to 0 \quad (3.3)$$

and

$$\phi'_r(s) = \frac{\partial \phi_r}{\partial s}(s) = \frac{1}{\Lambda_R(s) + r} \geq 0. \quad (3.4)$$

Fix $r > 0$. We take a concave function $\bar{\phi}_r : \mathbb{R} \to \mathbb{R}$ such that $\bar{\phi}_r(s) = \phi_r(s)$ for $s \in [0, \epsilon_0)$. For $M > 0$, we define

$$\tau_R = \inf \{ t > 0 \mid |X(t)| \lor |Y(t)| \geq R \},$$

$$\chi_{M, R} = \tau_R \wedge \inf \left\{ t > 0 \mid \int_0^t \|e(s)\|^2 \, ds \lor |\Phi_X|_{t \wedge \tau_R} \lor |\Phi_Y|_{t \wedge \tau_R} \lor \int_0^t g(s, \cdot) \, ds \geq M \right\}.$$  

For each $\epsilon \in (0, \epsilon_0)$, we define $U_\epsilon$ by

$$U_\epsilon = \inf \{ t > 0 \mid |\xi(t)| \geq \epsilon \}.$$  

Fix $M > 0$ and $\epsilon \in (0, \epsilon_0)$. To simplify the notation, we write

$$\rho := U_\epsilon \wedge \chi_{M, R}.$$  

Applying the Itô–Tanaka formula to $\bar{\phi}_r \circ \xi$ and using (3.1), we obtain

$$\bar{\phi}_r(\xi(\rho)) = \bar{\phi}_r(\xi(0)) + 2 \int_0^\rho \bar{\phi}'_r(\xi(s)) \langle e^a(s) \eta(s), dB(s) \rangle$$

$$+ 2 \int_0^\rho \bar{\phi}'_r(\xi(s)) \langle e(s), f(s) \rangle \, ds + \int_0^\rho \bar{\phi}''_r(\xi(s)) \|e(s)\|^2 \, ds$$

$$+ 2 \int_0^\rho \bar{\phi}'_r(\xi(s)) \langle \eta(s), d\Phi_X(s) - d\Phi_Y(s) \rangle + \frac{1}{2} \int_\mathbb{R} \bar{\phi}''(da), \quad (3.5)$$

$$\square$$ Springer
where \( \{L_t^a\}_{t \geq 0} \) denotes the local time at \( a \) of the semimartingale \( \{\xi(t)\}_{t \geq 0} \), and \( \bar{\phi}_r'' \) the second derivative of \( \bar{\phi}_r \) in the sense of distribution. Because \( \bar{\phi}_r \) is a concave function, the last term of (3.5) is non-positive. For \( P \)-a.s., \( \xi(0) = 0 \) and \( \bar{\phi}_r(\xi(0)) = 0 \). For any \( s \in [0, U_\varepsilon] \), we see from (2.9) that

\[
2\langle \eta(s), f(s) \rangle + \|e(s)\|^2 \leq g(s, \cdot) \Lambda_R(\xi(s)).
\]

Therefore, the sum of the third and the fourth terms of (3.5) is dominated by

\[
\int_0^\rho \frac{g(s, \cdot) \Lambda_R(\xi(s))}{\Lambda_R(\xi(s)) + r} \, ds \leq \int_0^\rho g(s, \cdot) \, ds \leq M.
\]

By Remark 1, it holds that

\[
\frac{1}{2r_0} |Y(s) - X(s)|^2 \geq \langle X(s) - Y(s), n(s) \rangle
\]

if \( X(s) \in \partial D \). From (2.4) and (2.3), we have

\[
\langle X(s) - Y(s), d\Phi_X(s) \rangle = \langle X(s) - Y(s), n(s) \rangle \, d|\Phi_X|_s
\]

\[
\leq \frac{1}{2r_0} |Y(s) - X(s)|^2 \, d|\Phi_X|_s.
\]

(3.6)

By exchanging the roles of \( X \) and \( Y \), we have

\[
\langle Y(s) - X(s), d\Phi_Y(s) \rangle \leq \frac{1}{2r_0} |X(s) - Y(s)|^2 \, d|\Phi_Y|_s.
\]

(3.7)

From (3.6), (3.7), (3.4), and (3.2), it follows that for any \( t \geq 0 \),

\[
\int_0^\rho \bar{\phi}_r'(\xi(s)) \langle \eta(s), d\Phi_X(s) - d\Phi_Y(s) \rangle \\
\leq \frac{1}{2r_0} \int_0^\rho \bar{\phi}_r'(\xi(s)) |X(s) - Y(s)|^2 \, d(|\Phi_X|_s + |\Phi_Y|_s) \\
= \frac{1}{2r_0} \int_0^\rho \frac{\xi(s)}{\Lambda_R(\xi(s)) + r} \, d(|\Phi_X|_s + |\Phi_Y|_s) \\
\leq \frac{1}{2r_0} (|\Phi_X|_{\rho} + |\Phi_Y|_{\rho}) \leq M/r_0.
\]

Combining these estimates, we get

\[
\bar{\phi}_r(\xi(\rho)) \leq 2 \int_0^\rho \bar{\phi}_r'(\xi(s)) (e^*\eta(s), dB(s)) + M + \frac{2M}{r_0}.
\]

(3.8)
For any $s \in [0, \rho]$, we have

$$|\tilde{\phi}'_r(\xi(s))e^s(s)\eta(s)|^2 \leq |\tilde{\phi}'_r(\xi(s))|^2\|e(s)\|^2|\xi(s)| \leq \frac{1}{\rho}\|e(s)\|^2.$$ 

Therefore, we see that \(\int_{0}^{t\wedge\rho} \tilde{\phi}'_r(\xi(s))\langle e^s(s)\eta(s), dB(s)\rangle\) is a martingale. Then, taking the expectations of both sides of (3.8), we obtain

$$E[\tilde{\phi}_r(\xi(\rho))] \leq M + \frac{2M}{r_0}.$$ 

From the monotone convergence theorem,

$$E\left[\int_{0}^{\xi(\rho)} \frac{1}{\Lambda_R(s)} ds\right] = \lim_{r \to 0} E[\tilde{\phi}_r(\xi(\rho))] \leq M + \frac{2M}{r_0} < \infty.$$ 

In view of (2.7) with $\Lambda$ replaced by $\Lambda_R$, we obtain that $\xi(\rho) = 0$ $P$-a.s. Therefore,

$$0 = E[\xi(\rho)] \geq E[\xi(U_\varepsilon \wedge \chi_{M,R}) : U_\varepsilon < \chi_{M,R}]$$

$$= \varepsilon P(U_\varepsilon < \chi_{M,R}),$$

which implies that

$$U_\varepsilon \geq \chi_{M,R} \quad P\text{-a.s.}$$

By letting $\varepsilon \downarrow 0$, then $M \to \infty$, we see from (2.5) and (2.8) that

$$X(t) = Y(t) \text{ for } t < \tau_R, \quad P\text{-a.s.}$$

for any $R > 0$. This, in particular, implies that the lifetimes of $X$ and $Y$ are the same $P$-a.s. Accordingly, we have $P(X(t) = Y(t), t \geq 0) = 1$. 

\textit{Proof of Corollary 1} We fix $n \in \mathbb{N}$. Define 1-Lipschitz functions $u_n : \overline{D} \to \mathbb{R}$ and $v_n : \mathbb{R}_+ \to \mathbb{R}$ as

$$u_n(x) = 0 \lor (n + 1 - |x|) \land 1, \quad x \in \overline{D},$$

$$v_n(t) = 0 \lor (n + 1 - t) \land 1, \quad t \in \mathbb{R}_+.$$ 

We define functions $\sigma_n$ and $b_n$ on $\mathbb{R}_+ \times \Omega \times \overline{D}$ as

$$\sigma_n(t, \omega, x) = \sigma(t, \omega, x)u_n(x)v_n(t),$$

$$b_n(t, \omega, x) = b(t, \omega, x)u_n(x)v_n(t).$$
For \( P \)-a.s. \( \omega \in \Omega \), the maps \( \mathbb{R}_+ \times \overline{D} \ni (t, x) \mapsto \sigma_n(t, \omega, x) \) and \( \mathbb{R}_+ \times \overline{D} \ni (t, x) \mapsto b_n(t, \omega, x) \) are bounded continuous. We fix \( R > 0 \) and set

\[
K_{n, R} = \text{ess sup}_{\omega \in \Omega} \sup_{t \in [0, n+1], z \in \overline{D} \cap B(R)} \{|\sigma(t, \omega, z)| \vee |b(t, \omega, z)|\} < \infty.
\]

It follows that for each \( (t, \omega) \in \mathbb{R}_+ \times \Omega \) and \( x, y \in \overline{D} \cap B(R) \),

\[
\|\sigma_n(t, \omega, x) - \sigma_n(t, \omega, y)\| \leq u_n(x)v_n(t)|\sigma(t, \omega, x) - \sigma(t, \omega, y)| + \|(u_n(x) - u_n(y))v_n(t)\sigma(t, \omega, y)\|
\]

\[
\leq \|\sigma(t, \omega, x) - \sigma(t, \omega, y)\| + K_{n, R}|x - y|
\]

and

\[
\langle x - y, b_n(t, \omega, x) - b_n(t, \omega, y) \rangle = u_n(x)v_n(t)\langle x - y, b(t, \omega, x) - b(t, \omega, y) \rangle + \langle x - y, v_n(t)b(t, \omega, y)(u_n(x) - u_n(y)) \rangle
\]

\[
\leq u_n(x)v_n(t)\langle x - y, b(t, \omega, x) - b(t, \omega, y) \rangle + K_{n, R}|x - y|^2.
\]

From the assumption, there exists a Borel measurable function \( A_R : [0, 1) \to \mathbb{R}_+ \) satisfying (L) such that for \( P \)-a.s. \( \omega \),

\[
\|\sigma(t, \omega, x) - \sigma(t, \omega, y)\|^2 \leq g(t, \omega)A_R(|x - y|^2)
\]

and

\[
\|\sigma(t, \omega, x) - \sigma(t, \omega, y)\|^2 + 2\langle x - y, b(t, \omega, x) - b(t, \omega, y) \rangle
\]

\[
\leq g(t, \omega)A_R(|x - y|^2)
\]

for any \( t \geq 0 \) and \( x, y \in \overline{D} \cap B(R) \) with \( |x - y| < 1 \). From these estimates, we see that for \( P \)-a.s. \( \omega \) and such \( t, x, y \),

\[
\|\sigma_n(t, \omega, x) - \sigma_n(t, \omega, y)\|^2 + 2\langle x - y, b_n(t, \omega, x) - b_n(t, \omega, y) \rangle
\]

\[
\leq 2\|\sigma(t, \omega, x) - \sigma(t, \omega, y)\|^2 + 2K_{n, R}^2|x - y|^2
\]

\[
+ 2u_n(x)v_n(t)(x - y, b(t, \omega, x) - b(t, \omega, y)) + K_{n, R}|x - y|^2
\]

\[
\leq (2 - u_n(x)v_n(t))g(t, \omega)A_R(|x - y|^2) + u_n(x)v_n(t)g(t, \omega)A_R(|x - y|^2)
\]

\[
+ (2K_{n, R}^2 + 2K_n)|x - y|^2
\]

\[
= 2g(t, \omega)A_R(|x - y|^2) + (2K_{n, R}^2 + 2K_n)|x - y|^2.
\]
By Lemma 1(1), we may assume that \( \Lambda s \geq s \) for any \( s \in [0, 1] \). Therefore, we see that \( \sigma_n \) and \( b_n \) satisfy (2.9). Then, by Theorem 1 and Remarks 3 and 2(2), the equation

\[
X_n(t) = X(0) + \int_0^t \sigma_n(s, \cdot, X_n(s)) \, dB(s) + \int_0^t b_n(s, \cdot, X_n(s)) \, ds \\
+ \Phi X_n(t), \quad t \geq 0
\]  

(3.9)

has a strong solution. That is, (3.9) has a solution for a given \( d \)-dimensional Brownian motion. We define \( \tau_n := \inf \{ t > 0 \mid X_n(t) \notin \overline{D} \cap B(n) \} \). Then, by the definition of \( \sigma_n \) and \( b_n \), we see that \( X_n = \{ X_n(t) \}_{t \geq 0} \) solves (2.1) up to \( n \) and \( \tau_n \). Then, we can define \( X = \{ X(t) \}_{t \geq 0} \) as a solution of (2.1) and \( \zeta_X = \lim_{n \to \infty} n \wedge \tau_n \), by letting \( X(t) = \Delta \) for \( t \in [\zeta_X, \infty) \).

\( \square \)

**Proof of Theorem 2** We define \( \hat{\xi} = \{ \hat{\xi}(s) \}_{s \geq 0} \) by

\[
\hat{\xi}(s) = V(s, X(s)), \quad s \geq 0.
\]

It follows from the Itô formula that

\[
d\hat{\xi}(s) = \langle \sigma^*(s, \cdot, X(s))(\nabla V)(s, X(s)), dB(s) \rangle \\
+ \langle (\nabla V)(s, X(s)), b(s, \cdot, X(s)) \rangle \, ds + \langle (\nabla V)(s, X(s)), d\Phi X(s) \rangle \\
+ \frac{1}{2} \| \sigma(s, \cdot, X(s)) \|^2 (\Delta V)(s, X(s)) \, ds + \frac{\partial V}{\partial s}(s, X(s)) \, ds.
\]  

(3.10)

where \( \nabla \) and \( \Delta \) are differentiations with respect to the second variable. For \( R > 0 \) and \( M > 0 \), we set

\[
\tau_R = \inf \{ s > 0 \mid |X(s)| \geq R \},
\]

\[
\chi_M = \inf \left\{ s > 0 \mid \int_0^s g(u, \cdot) \, du \geq M \right\},
\]

and define \( \psi : \mathbb{R}_+ \to \mathbb{R}_+ \) as

\[
\psi(s) = \int_0^s \frac{1}{\gamma(u)} \, du, \quad s \in \mathbb{R}_+.
\]

For now, we fix \( t > 0, R > 0, \) and \( M > 0, \) and write

\[
\rho = t \wedge \tau_R \wedge \chi_M.
\]

Since \( \gamma \) is a non-decreasing function on \( \mathbb{R}_+ \), \( \psi \) can extend to a concave function on \( \mathbb{R} \), which is denoted as the same symbol. By applying the Itô–Tanaka formula to \( \psi \circ \hat{\xi} \)
and using (3.10), we have

\[
\begin{align*}
\psi(\hat{\xi}(\rho)) &= \psi(\hat{\xi}(0)) + \int_0^\rho \psi'(\hat{\xi}(s)) \, d\hat{\xi}(s) + \frac{1}{2} \int_{\mathbb{R}} \hat{L}_\rho^a \psi''(da) \\
&= \psi(V(0, X(0))) + \int_0^\rho \psi'(\hat{\xi}(s)) \langle \sigma^*(s, \cdot, X(s))(\nabla V)(s, X(s)), dB(s) \rangle \\
&\quad + \int_0^\rho \psi'(\hat{\xi}(s)) \langle (\nabla V)(s, X(s)), b(s, \cdot, X(s)) \rangle \, ds \\
&\quad + \frac{1}{2} \int_0^\rho \psi'(\hat{\xi}(s)) \| \sigma(s, \cdot, X(s)) \|^2 (\Delta V)(s, X(s)) \, ds \\
&\quad + \frac{1}{2} \int_0^\rho \psi'(\hat{\xi}(s)) \frac{\partial V}{\partial s}(s, X(s)) \, ds + \frac{1}{2} \int_{\mathbb{R}} \hat{L}_\rho^a \psi''(da). \tag{3.11}
\end{align*}
\]

Here, \( \{ \hat{L}_s^a \}_{s \geq 0} \) denotes the local time at \( a \) of the semimartingale \( \{ \hat{\xi}(s) \}_{s \geq 0} \), and \( \psi'' \) the second derivative of \( \psi \) in the sense of distribution. Because \( \psi \) is a concave function, it follows that

\[
\int_{\mathbb{R}} \hat{L}_\rho^a \psi''(da) \leq 0.
\]

From (V.3), the sum of the third, the fifth, and the sixth terms of (3.11) is dominated by

\[
\int_0^\rho g(s, \omega) \frac{\gamma(\hat{\xi}(s))}{2\gamma'(\hat{\xi}(s))} \, ds \leq \frac{M}{2}.
\]

Since \( d\Phi_X(s) = n(s) \, d|\Phi_X|_s \), where \( n(s) \in \mathcal{N}_X(s) \) if \( X(s) \in \partial D \), it follows from (V.2) that the fourth term of (3.11) is equal to

\[
\int_0^\rho \psi'(\hat{\xi}(s)) \langle (\nabla V)(s, X(s)), n(s) \rangle \, d|\Phi_X|(s) \leq 0.
\]

By using these estimates, we obtain

\[
\psi(\hat{\xi}(\rho)) \leq \psi(V(0, X(0))) + \int_0^\rho \psi'(\hat{\xi}(s)) \langle \sigma^*(s, \cdot, X(s))(\nabla V)(s, X(s)), dB(s) \rangle + \frac{M}{2}. \tag{3.12}
\]

We define a local martingale \( S = \{ S(t) \}_{t \geq 0} \) as

\[
S(t) = \int_0^{t \wedge t_R \wedge t_M} \psi'(\hat{\xi}(s)) \langle \sigma^*(s, \cdot, X(s))(\nabla V)(s, X(s)), dB(s) \rangle, \quad t \geq 0.
\]
There exists an increasing sequence of stopping times \( \{ \theta_n \}_{n=1}^{\infty} \) such that \( \lim_{n \to \infty} \theta_n = \infty \) \( P \)-a.s. and \( \{ S(t \wedge \theta_n) \}_{t \geq 0} \) is a martingale for each \( n \in \mathbb{N} \). Notice that (3.12) is valid if we replace \( \rho \) by \( \rho \wedge \theta_n \), \( n \in \mathbb{N} \). Then, by Fatou’s lemma, for any \( t \geq 0, r > 0, R > 0, \) and \( M > 0 \),

\[
E[\psi(\hat{\xi}(t \wedge \tau_R \wedge \chi_M)) : |X(0)| < r] \leq \lim_{n \to \infty} E[\psi(\hat{\xi}(t \wedge \tau_R \wedge \chi_M \wedge \theta_n)) : |X(0)| < r] \\
\leq E[\psi(V(0, X(0))) : |X(0)| < r] + \lim_{n \to \infty} E[S(t \wedge \theta_n) : |X(0)| < r] + M/2 \\
\leq \sup_{x \in B(r)} V(0, x) + 0 + M/2.
\]

Therefore, we have

\[
E[\int_0^{\hat{\xi}(\tau_R)} \frac{1}{\gamma(s)} \, ds : \xi_X \leq t \wedge \chi_M, \ |X(0)| < r] < \infty. \tag{3.13}
\]

From (V1),

\[
\lim_{R \to \infty} \hat{\xi}(\tau_R) = \infty \quad P\text{-a.s. on } \{ \xi_X < \infty \}.
\]

Therefore, (3.13) implies that

\[
P(\xi_X \leq t \wedge \chi_M, \ |X(0)| < r) = 0
\]

for any \( t \geq 0, M > 0, \) and \( r > 0 \). In view of (2.8) and the fact that \( X(0) \in \overline{D} \ P\text{-a.s.}, \) we arrive at the conclusion. \( \square \)

### 4 Proof of Theorem 3

For a continuous functions \( w : \mathbb{R}^d \to \mathbb{R}, s, t \in \mathbb{R}_+ \) with \( s < t \), and for \( \theta \in (0, 1] \), we define

\[
\Delta_{s,t}(w) = \sup_{s \leq t_1 < t_2 \leq t} |w(t_2) - w(t_1)|,
\]

\[
\|w\|_{H_{[s,t],\theta}} = \sup_{s \leq t_1 < t_2 \leq t} \frac{|w(t_2) - w(t_1)|}{|t_2 - t_1|^\theta},
\]

\[
|w|^2_{\Pi} = \sup_{\Pi} \sum_{k=1}^{N} |w(t_k) - w(t_{k-1})|,
\]

where \( \Pi = \{ s = t_0 < t_1 < \cdots < t_N = t \} \) is a partition of the interval \( [s, t] \).
Let \( \{X(t)\}_{t \geq 0} \) be a solution of (2.1) with a Brownian motion \( \{B(t)\}_{t \geq 0} \). Define \( \{W(t)\}_{0 \leq t < \xi_X} \) as

\[
W(t) = X(0) + \int_0^t \sigma(s, \cdot, X(s)) \, dB(s) + \int_0^t b(s, \cdot, X(s)) \, ds, \quad t < \xi_X.
\]

The following lemma is a slight modification of [1, Lemma 2.3], originally due to [10, Proposition 3.1]. The proof is the same as that of [1, Lemma 2.3].

**Lemma 2** Let \( T > 0 \) and \( \theta \in (0, 1] \). There exist positive constants \( C_1, C_2 \) depending only on \( \theta \), and \( r_0, \beta, \delta \) in assumptions (A) and (B) such that, for \( P \)-a.s.,

\[
|X|^t_s \leq C_1 \left( 1 + (t-s) \|W\|_H^{1/\theta} \right) e^{C_2 \Delta_{s,t}(W)} \Delta_{s,t}(W), \quad 0 \leq s < t < T \land \xi_X.
\]

**Remark 4** Following the proof of [1, Lemma 2.3], we can take

\[
C_1 = 24\beta(1+\beta)\left\{ (4\delta^{-1}(\beta+2))^{1/\theta} + 1 \right\} \exp\{\beta\delta(1+\delta^{-1})r_0^{-1}\}
\]

and

\[
C_2 = (1+\delta^{-1})\beta r_0^{-1}.
\]

However, we do not use such specific quantities below.

In what follows, we suppose that the assumptions in Theorem 3 are satisfied. We introduce some stopping times and random integers. We fix \( T, M \in (0, \infty) \) and set

\[
\kappa = \inf \left\{ t > 0 \left| \int_0^t g(s, \cdot) \, ds \geq M \right. \right\} \land T.
\]

For \( R \in (0, \infty) \), we set

\[
\kappa_R = \inf \{ t > 0 \left| |X(t)| \geq R \right. \} \land \kappa
\]

and define

\[
\kappa_\infty = \lim_{R \to \infty} \kappa_R \left( = \xi_X \land \kappa \right).
\]

We further define subsets \( \{U_n\}_{n=0}^\infty \) and \( \{V_n\}_{n=0}^\infty \) of \( \overline{D} \) as

\[
U_n = B(x_n, \hat{\beta}\delta_n), \quad V_n = B(x_n, \delta_n), \quad n \geq 1,
\]

and

\[
U_0 = \overline{D} \setminus \bigcup_{n=1}^{\infty} \overline{B}(x_n, \hat{\beta}\delta_n/2), \quad V_0 = \overline{D} \setminus \bigcup_{n=1}^{\infty} \overline{B}(x_n, \hat{\beta}\delta_n/3).
\]
Note that $\overline{D} \subset \bigcup_{n=0}^{\infty} U_n$. For $R \in \mathbb{N} \cup \{\infty\}$, we define stopping times $\{\tau_k^{(R)}\}_{k=0}^{\infty}$ and random sequences $\{n_k^{(R)}\}_{k=0}^{\infty}$ as

$$\tau_0^{(R)} = 0, \quad n_0^{(R)} = \inf\{n \geq 0 \mid X(\tau_0^{(R)}) \in U_n\},$$

and for $k \geq 0$,

$$\tau_{k+1}^{(R)} = \inf\{t > \tau_k^{(R)} \mid X(t) \notin V_{n_k^{(R)}} \} \wedge \kappa_R,$$

$$n_{k+1}^{(R)} = \begin{cases} 
\inf\{n \geq 0 \mid X(\tau_{k+1}^{(R)}) \in U_n\} & \text{if } \tau_{k+1}^{(R)} < \kappa_R, \\
\infty & \text{if } \tau_{k+1}^{(R)} = \kappa_R.
\end{cases}$$

Let

$$\Gamma_0 = \{k \geq 0 \mid n_k^{(\infty)} = 0\} = \{k \geq 0 \mid n_k^{(\infty)} = 0 \text{ and } \tau_k^{(\infty)} < \kappa_\infty\},$$

$$\Gamma_1 = \{k \geq 0 \mid n_k^{(\infty)} \in \mathbb{N}\} = \{k \geq 0 \mid n_k^{(\infty)} \in \mathbb{N} \text{ and } \tau_k^{(\infty)} < \kappa_\infty\},$$

and

$$\Sigma = \{k \geq 0 \mid n_k^{(\infty)} \in \mathbb{N} \text{ and } \tau_k^{(\infty)} < \kappa_\infty\}.$$

**Lemma 3** If $\#\Sigma$ is finite $P$-a.s., then both $\#\Gamma_0$ and $\#\Gamma_1$ are finite $P$-a.s.

**Proof** Because $\tau_k^{(\infty)}$ is non-decreasing in $k$, there is at most one $k$ (depending on $\omega \in \Omega$) such that $\tau_k^{(\infty)} < \kappa_\infty$ and $\tau_{k+1}^{(\infty)} = \kappa_\infty$. Therefore, $\#\Gamma_1 \leq \#\Sigma + 1$ $P$-a.s. Moreover, if $n_k^{(\infty)} = 0$, then $n_{k+1}^{(\infty)} \in \mathbb{N} \cup \{\infty\}$ from the definition of $n_k^{(\infty)}$. Because there is at most one $k$ such that $n_k^{(\infty)} = 0$ and $n_{k+1}^{(\infty)} = \infty$, we have $\#\Gamma_0 \leq \#\Gamma_1 + 1$ $P$-a.s. This completes the proof. \qed

**Lemma 4** $\#\Sigma$ is finite $P$-a.s.

**Proof** We define a sequence of random numbers $\{	ilde{l}_j\}_{j=1}^{\infty} \subset \mathbb{N} \cup \{0, \infty\}$ as

$$\tilde{l}_1 = \inf\{k \geq 0 \mid n_k^{(\infty)} \in \mathbb{N}\} \wedge T,$$

$$\tilde{l}_{j+1} = \inf\{k > l_j \mid n_k^{(\infty)} \in \mathbb{N}\} \wedge T, \quad j \in \mathbb{N}.$$

Also, let

$$l_j = \begin{cases} 
\tilde{l}_j & \text{if } \tau_{\tilde{l}_j+1}^{(\infty)} < \kappa_\infty \\
T & \text{if } \tau_{\tilde{l}_j+1}^{(\infty)} = \kappa_\infty,
\end{cases} \quad j \in \mathbb{N}.$$
Hereafter, we omit the superscript \((\infty)\) and write \(n_k\) and \(\tau_k\) for \(n_k^{(\infty)}\) and \(\tau_k^{(\infty)}\), respectively. For \(n \in \mathbb{N}\), we write \(M_n = M(x_n, \delta_n, T)\). We set

\[
B_j = \{l_j < T\}, \quad B_{j,n} = B_j \cap \{n_j = n\},
\]

\[
\tilde{B}_j = \{\tilde{l}_j < T\}, \quad \tilde{B}_{j,n} = \tilde{B}_j \cap \{n_j = n\}, \quad j \in \mathbb{N}, \ n \in \mathbb{N}.
\]

Note that \(\{B_j\}_{j=1}^{\infty}\) is a decreasing sequence and \(\bigcap_{j=1}^{\infty} B_j = \{\# \Sigma = \infty\}\). On each \(B_j\), \(X(\tau_{l_j}) \in \tilde{B}(x_{n_j}, \beta \delta_{n_j})\) and \(X(\tau_{l_j+1}) \notin B(x_{n_j}, \delta_{n_j})\), implying that

\[
|X(\tau_{l_j+1}) - X(\tau_{l_j})| \geq (1 - \hat{\beta})\delta_{n_{l_j}}. \tag{4.1}
\]

We fix \(j \in \mathbb{N}\) and write \(\tau = \tau_{l_j}, \tilde{\tau} = (\tau_{l_j} + 1/j) \wedge \tau_{l_j+1}, x = x_{n_{l_j}}, \) and \(\delta = \delta_{n_{l_j}}\). Both \(\tau\) and \(\tilde{\tau}\) are \(\{\mathcal{F}_t\}_{t \geq 0}\)-stopping times. In the following, \(c\) denotes an unimportant positive constant that may vary line by line. Take \(p\) such that \(p > (1 - \nu)^{-1}\). Fix \(n \in \mathbb{N}\). For \(s, t\) with \(0 \leq s < t\), we have

\[
E[|W((\tau + t) \wedge \tilde{\tau}) - W((\tau + s) \wedge \tilde{\tau})|^2p : B_{j,n}] \leq \frac{E[|W((\tau + t) \wedge \tilde{\tau}) - W((\tau + s) \wedge \tilde{\tau})|^2p : \tilde{B}_{j,n}]}{E[|W((\tilde{\tau} + t) \wedge \tilde{\tau}) - W((\tilde{\tau} + s) \wedge \tilde{\tau})|^2p]}, \tag{4.2}
\]

where

\[
\tilde{\tau} = \begin{cases} \tau & \text{on } \tilde{B}_{j,n}, \\ T & \text{on } \Omega \setminus \tilde{B}_{j,n} \end{cases}
\]

Since \(\tilde{B}_j \in \mathcal{F}_\tau\) and

\[
\{n_{l_j} = n\} = \{\tau < \kappa_{\infty}\} \cap \left\{X_t \in U_n \setminus \bigcup_{k=0}^{n-1} U_k \right\} \in \mathcal{F}_\tau,
\]

\(\tilde{B}_{j,n} = \tilde{B}_j \cap \{n_{l_j} = n\} \in \mathcal{F}_\tau\) and \(\tilde{\tau}\) is an \(\{\mathcal{F}_t\}_{t \geq 0}\)-stopping time. Therefore, from the Burkholder–Davis–Gundy inequality, the last term of (4.2) is dominated by

\[
\begin{align*}
c E\left[ \left( \int_{(\tau + s) \wedge \tilde{\tau}}^{(\tau + t) \wedge \tilde{\tau}} \|\sigma(s, \cdot, X(s))\|^2 \, ds \right)^p + \left( \int_{(\tau + s) \wedge \tilde{\tau}}^{(\tilde{\tau} + t) \wedge \tilde{\tau}} \|b(s, \cdot, X(s))\| \, ds \right)^{2p} \right] \\
\leq c \left( (t - s) M_n \right)^p P(\tilde{B}_{j,n}) + c \left( (t - s) M_n^{1/2} \right)^{2p} P(\tilde{B}_{j,n}) \\
\leq c M_n^p ((t - s)^p + (t - s)^{2p}) P(\tilde{B}_{j,n}).
\end{align*}
\]
For $\alpha \in (1, \infty)$ and $\lambda \in (1/\alpha, 1)$, the Garsia–Rodemich–Rumsey inequality [5, Corollary A.2] (see also [6, Lemma 1.1] for the original inequality) implies

$$E \left[ \frac{\|W((\tau + t) \wedge \hat{\tau}) - W((\tau + s) \wedge \hat{\tau})\|^\alpha}{(t - s)^{\lambda - (1/\alpha)}} : B_{j,n} \right] \leq c \int_0^1 \int_0^1 E \left[ \|W((\tau + t) \wedge \hat{\tau}) - W((\tau + s) \wedge \hat{\tau})\|^\alpha : B_{j,n} \right] \, ds \, dt. \quad (4.3)$$

Then, letting $\alpha = 2p$ and $\lambda = 1/\alpha + v/2$ ($< 1$) in (4.3), we obtain that

$$E \left[ \|W\|_{\mathcal{H}([\tau, \hat{\tau}], v/2)}^{2p} : B_{j,n} \right] \leq c M_n^{p} \int_0^1 \int_0^1 \frac{|t - s|^{2p} + |t - s|^p}{|t - s|^{2 + pv}} \, ds \, dt P(\tilde{B}_{j,n}) \leq c M_n^{p} \int_0^1 \int_0^1 |t - s|^{-2 + p(1-v)} \, ds \, dt P(\tilde{B}_{j,n}) \leq c M_n^{p} j^{-p(1-v)} P(\tilde{B}_{j,n}). \quad (4.4)$$

Now, let $Z_j = B_j \cap \{\|W\|_{\mathcal{H}([\tau, \hat{\tau}], v/2) \geq M_n^{1/2} \}$. Then $Z_j = \bigcup_{n=1}^\infty Z_{j,n}$, where

$$Z_{j,n} = B_{j,n} \cap \{\|W\|_{\mathcal{H}([\tau, \hat{\tau}], v/2) \geq M_n^{1/2} \}.$$ 

The Markov inequality and (4.4) lead us to

$$P(Z_{j,n}) \leq M_n^{-p} E[\|W\|_{\mathcal{H}([\tau, \hat{\tau}], v/2)}^{2p} : B_{j,n}] \leq c j^{-p(1-v)} P(\tilde{B}_{j,n}).$$

Thus, we have

$$P(Z_j) = \sum_{n=1}^\infty P(Z_{j,n}) \leq c j^{-p(1-v)} P(\tilde{B}_j) \leq c j^{-p(1-v)}.$$

Since $p(1 - v) > 1$, we have $\sum_{n=1}^\infty P(Z_j) < \infty$. From Borel–Cantelli’s lemma, for $P$-a.s. $\omega$ there exists $j_0 = j_0(\omega)$ such that for all $j \geq j_0$,

$$\|W\|_{\mathcal{H}([\tau, \hat{\tau}], v/2) \leq M_n^{1/2} \text{ if } \omega \in B_j. \quad (4.5)$$

Fix such $\omega$ in $\bigcap_{j_0}^\infty B_j = \{\#(\Sigma) = \infty \}$. Let $j \in \mathbb{N}$, let $N$ be the smallest integer such that $N \geq \delta_{n_{j_0}}$, and write

$$t_k = \left( \tau_{j} + \frac{k}{jN} \right) \wedge \tau_{j+1}, \quad k = 0, 1, \ldots, N.$$ 

For an integer $k$ with $0 \leq k < N$ and $s, t$ with $t_k \leq s < t \leq t_{k+1}$, we have from (4.5)

$$|W(t) - W(s)| \leq M_{n_{j_0}}^{1/2}(t - s)^{v/2}.$$
This, together with (2.13), implies that
\[
\Delta_{tk,tk+1}(W) \leq M_{n_j}^{1/2} (jN)^{-v/2} \leq M_{n_j}^{1/2} (j\delta_{n_j})^{-v/2} \leq cj^{-v/2}.
\] (4.6)

Applying Lemma 2 and using (4.6), (4.5), and (2.13), we obtain that
\[
|X|_{tk} \leq C_1 \left(1 + \frac{1}{jN} (M_{n_j}^{1/2})^{2/v}\right) e^{C_2 cj^{-v/2}} c j^{-v/2} \leq cj^{-v/2}
\]
for any integer \(k\) with \(0 \leq k < N\). Thus, we arrive at
\[
|X|_{(\tau_{lj} + 1/j) \wedge \tau_{lj} + 1} \leq \sum_{k=0}^{N-1} |X|_{tk+1} \leq Nc j^{-v/2}
\]
\[
\leq c(1 + \hat{\delta}^{-1})\delta_{n_j} j^{-v/2},
\]
where \(\hat{\delta}\) is a constant in the assumption of Theorem 3. For sufficiently large \(j\) (say, greater than or equal to \(j_1 = j_1(\omega) \geq j_0(\omega)\)), \(c(1 + \hat{\delta}^{-1}) j^{-v/2} < 1 - \beta\). In view of (4.1), it holds for such \(j\) that \(\tau_{lj} + 1/j < \tau_{lj} + 1\). Then, for \(P\)-a.s. \(\omega\) in \(\bigcap_{j=1}^{\infty} B_j = \{\#\Sigma = \infty\}\),
\[
T \geq \sum_{j=j_1(\omega)}^{\infty} (\tau_{lj+1} - \tau_{lj}) \geq \sum_{j=j_1(\omega)}^{\infty} \frac{1}{j} = \infty,
\]
which is absurd. Therefore, \(P(\#\Sigma = \infty) = 0\). \(\square\)

**Lemma 5** For \(k \geq 0\),
\[
\sup_{t_k^{(\infty)} \leq t < t_{k+1}^{(\infty)}} |X(t)| < \infty \quad P\text{-a.s. on } \{n_k^{(\infty)} = 0\}.
\]

**Proof** For \(k \geq 0\) and \(R \in \mathbb{N} \cup \{\infty\}\), we set
\[
A_k^{(R)} = \{\omega \in \Omega \mid n_k^{(R)}(\omega) = 0\},
\]
which is \(\mathcal{F}_{t_k^{(\infty)}}\)-measurable. For the moment, we fix \(k \geq 0\) and \(R \in \mathbb{N}\), and suppress the superscript \((R)\) from the notation. We define
\[
\xi_k(t) = |X(t \wedge \tau_{k+1}) - X(t \wedge \tau_k)|^2, \quad t \geq 0.
\]
For \(t, u \geq 0\), we define
\[
\varphi(t, u) = \int_{0}^{\min(t,u)} \frac{1}{\gamma(2s+2u)} \, ds.
\]
For each $u \geq 0$, $\mathbb{R}_+ \ni t \mapsto \varphi(t, u) \in \mathbb{R}$ is a concave function since $\gamma$ is a non-decreasing function on $\mathbb{R}_+$, so it can extend to a concave function on $\mathbb{R}$. This extension is still denoted by $\varphi$. We denote by $\varphi'$ the derivative of $\varphi(t, u)$ in $t$. Applying the Itô–Tanaka formula to $\varphi(\cdot, u) \circ \xi_k$, we obtain for $u \geq 0$ that

$$
\varphi(\xi_k(t_{k+1}), u) = \int_{t_k}^{t_{k+1}} \varphi'(\xi_k(s), u) \, d\xi_k(s) + \frac{1}{2} \int_{\mathbb{R}} (L^a_{t_{k+1}} - L^a_{t_k}) \varphi''(da, u)
=: I_1 + I_2.
$$

(4.7)

Here, $\varphi''(da, u)$ denotes the second derivative of $t \mapsto \varphi(t, u)$ in the sense of distribution. Then, $I_2 \leq 0$. We also have

$$
I_1 = 2 \int_{t_k}^{t_{k+1}} \varphi'(\xi_k(s), u) \langle \sigma^*(s, \cdot, X(s))(X(s) - X(t_k)), dB(s) \rangle
+ \int_{t_k}^{t_{k+1}} \varphi'(\xi_k(s), u) \| \sigma(s, \cdot, X(s)) \|^2 \, ds
+ 2 \int_{t_k}^{t_{k+1}} \varphi'(\xi_k(s), u)(X(s) - X(t_k), b(s, \cdot, X(s))) \, ds
+ 2 \int_{t_k}^{t_{k+1}} \varphi'(\xi_k(s), u)(X(s) - X(\tau_k), d\Phi_X(s)).
$$

(4.8)

On event $A_k$, we have $d\Phi_X = 0$ on $[\tau_k, \tau_{k+1}]$ since $X(t) \notin \partial D$ for any $t \in [\tau_k, \tau_{k+1}]$. The first term of (4.8) is expressed as

$$
2 \int_0^{t_{k+1}} \varphi'(\xi_k(s), u) \langle \sigma^*(s, \cdot, X(s))(X(s) \wedge \tau_{k+1}) - X(s \wedge \tau_k), dB(s) \rangle.
$$

By (2.12) and the fact that $\tau_{k+1} \leq \kappa_R$, we see that

$$
\left\{ \int_0^t \varphi'(\xi_k(s), |X(\tau_k)|^2) \langle \sigma^*(s, \cdot, X(s))(X(s \wedge \tau_{k+1}) - X(s \wedge \tau_k), dB(s) \rangle \right\}_{t \in [0, \infty]}
$$

is a martingale by confirming that its quadratic variation is integrable at $t = \infty$. Therefore, by letting $u = |X(\tau_k)|^2$ in (4.7) and taking the expectation of both sides of (4.7) on $A_k$, we obtain

$$
E[\varphi(\xi_k(t_{k+1}), |X(\tau_k)|^2) : A_k]
\leq E \left[ \int_{t_k}^{t_{k+1}} \varphi'(\xi_k(s), |X(\tau_k)|^2) \| \sigma(s, \cdot, X(s)) \|^2 \, ds : A_k \right]
+ 2 \left[ \int_{t_k}^{t_{k+1}} \varphi'(\xi_k(s), |X(\tau_k)|^2) \langle X(s) - X(\tau_k), b(s, \cdot, X(s)) \rangle \, ds : A_k \right]
=: J_1 + J_2.
$$
Recall that $\gamma(s): \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing. By Lemma 1(2), we may assume $\gamma(s) \geq s$ for $s \geq 0$ without loss of generality. Then, we see from (2.12) that

$$J_1 \leq E \left[ \int_{\tau_k}^{\tau_{k+1}} \frac{g(s, \cdot)\gamma(|X(s)|^2)}{\gamma(2|X(s) - X(\tau_k)|^2 + 2|X(\tau_k)|^2)} \, ds : A_k \right]$$

and

$$J_2 \leq E \left[ \int_{\tau_k}^{\tau_{k+1}} \frac{|X(s) - X(\tau_k)|^2 + |b(s, \cdot, X(s))|^2}{\gamma(2|X(s) - X(\tau_k)|^2 + 2|X(\tau_k)|^2)} \, ds : A_k \right] \leq T + M.$$

Combining these estimates, we obtain

$$E \left[ \varphi \left( |X(\tau_{k+1})^R) - X(\tau_k)^R)|^2, |X(\tau_k)^R)|^2 \right) : A_k^{(R)} \right] \leq T + 2M. \quad (4.9)$$

It clearly holds that $\lim_{R \to \infty} A_k^{(R)} = \bigcup_{R \in \mathbb{N}} A_k^{(R)} = A_k^{(\infty)}$. By letting $R \to \infty$ in (4.9) and using (2.10), we obtain the conclusion.

**Proof of Theorem 3** By Lemmas 3 and 4, for $P$-a.s. $\omega$, the number of $k \geq 0$ such that $n_k^{(\infty)}(\omega) < \infty$ is finite. Let $\hat{k} = \hat{k}(\omega)$ be the largest integer $k$ such that $n_k^{(\infty)}(\omega) < \infty$. If $n_k^{(\infty)}(\omega) = 0$, then Lemma 5 implies that

$$\zeta_X(\omega) \geq \tau_{k+1}^{(\infty)}(\omega) = \kappa_{\infty}(\omega) = \kappa(\omega). \quad (4.10)$$

If $n_k^{(\infty)}(\omega) \in \mathbb{N}$, (4.10) clearly holds. Thus, it holds that $\zeta_X \geq \kappa$ $P$-a.s. Because $T$ and $M$ are arbitrarily chosen, we complete the proof from (2.8).

**Compliance with Ethical Standards**

**Conflict of interest** The authors declare that they have no conflict of interest.

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