Gradient expansion for anisotropic hydrodynamics

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We compute the gradient expansion for anisotropic hydrodynamics. The results are compared with the corresponding expansion of the underlying kinetic-theory model with the collision term treated in the relaxation time approximation. We find that a recent formulation of anisotropic hydrodynamics based on an anisotropic matching principle yields the first three terms of the gradient expansion in agreement with those obtained for the kinetic theory. This gives further support for this particular hydrodynamic model as a good approximation of the kinetic-theory approach. We further find that the gradient expansion of anisotropic hydrodynamics is an asymptotic series, and the singularities of the analytic continuation of its Borel transform indicate the presence of non-hydrodynamic modes.

I. INTRODUCTION

Theories of relativistic hydrodynamics have enjoyed a remarkable decade of advances both in the theoretical aspects of their formulation and in their numerical implementation. The driving force behind these developments was the successful application of this type of description to the physics of the quark-gluon plasma (QGP). During this time a number of different theories of relativistic hydrodynamics have been formulated, and one of the aims of this paper is to clarify how they are related. We adopt the point of view that computing gradient expansions of the energy-momentum tensor provides a natural means of making such comparisons. It is also a direct way to compare the effective, hydrodynamic description with an underlying microscopic theory or model.

Our main focus of interest here are theories of anisotropic hydrodynamics (AHYDRO) [1, 2], whose form looks quite different from other approaches. One of our goals is to use the gradient expansion as a way to clarify similarities and differences which may not be obvious at first glance. Another major motivation of this work is to compare the leading terms in the gradient expansion of the energy-momentum tensor of AHYDRO with the corresponding terms calculated recently directly at the level of kinetic theory [3]. To this end, we calculate the gradient expansion for anisotropic hydrodynamics in the special case of boost-invariant and transversely homogeneous systems that have been the focus of many studies in recent years in the context of early thermalisation and hydrodynamization of matter produced in heavy-ion collisions [4–11].

We analyse two different formulations of anisotropic hydrodynamics [12, 13], as well as Müller-Israel-Stewart viscous hydrodynamics (MIS) [14–16], and especially its modern version [17], which we will refer to as BRSSS. We compare the corresponding gradient expansions with the expansion obtained for the kinetic-theory model with the collision term treated in the relaxation time approximation (RTA) [3].

A comparison of the gradient expansions provides an indication of how successful the various hydrodynamic approaches are in reproducing close-to-equilibrium dynamics governed by the underlying microscopic theory. We find that the formulation of anisotropic hydrodynamics based on the anisotropic matching principle [13] yields the first three terms in the gradient expansion which agree exactly with the terms obtained for the kinetic theory. This gives support for this particular anisotropic hydrodynamics description as a good approximation for the underlying kinetic-theory model.

From a wider perspective, this suggests that effective hydrodynamic descriptions tailored to a specific microscopic theory may provide a better picture for a given system than a general framework such as the BRSSS theory. The latter provides a universal set of equations valid for all relativistic systems sufficiently close to equilibrium. Different microscopic theories, such as Quantum Chromodynamics (QCD), N=4 supersymmetric Yang-Mills theory, or kinetic-theory models, are distinguished by different sets of transport coefficients. However, the gradient expansions are guaranteed to match only up to second order, beyond which they differ. In contrast, anisotropic hydrodynamics, as understood here, aims to provide a structure more closely attuned to the kinetic theory in the RTA, which is the reason why it can provide a better description.

We also examine the large order behaviour of the gradient series generated by the anisotropic hydrodynamic theories and find that they are asymptotic, with the coefficients $g_n$ growing as $n!$. The Borel transform technique applied to this series indicates the presence of purely damped non-hydrodynamic modes. This parallels earlier findings in the BRSSS theory [18].

The paper is organised as follows: In Sec. II we discuss the basic theoretical structures of perfect-fluid and viscous relativistic hydrodynamics. We distinguish between the notion of hydrodynamic expansion that is used to construct hydrodynamic equations from the underlying kinetic theory [19–21] and the gradient expansion which is a formal, infinite, expansion of the energy-momentum tensor of a given theory around the perfect fluid form. In Sec. III we introduce a reorganised hydrodynamic expansion that leads to the concept of anisotropic hydrodynamics. The constraints resulting from imposing longitudinal boost invariance and transverse
homogeneity are implemented in Sec. IV. The kinetic theory model considered in this paper (based on the relaxation time approximation) is shortly described in Sec. V. In Sec. VI we present two different versions of equations of the anisotropic hydrodynamics. In the first case one uses moments of the Boltzmann equation [12], while in the second case one uses the anisotropic matching principle combined with the exact treatment of the dynamical equations for the pressure corrections [13]. The latter method introduces an infinite set of the coupled equations that is, eventually, truncated at the leading order. The Müller-Israel-Stewart approach to hydrodynamics is discussed in Sec. VII. Our main results on the gradient expansion for anisotropic hydrodynamics are presented in Secs. VIII and IX. We summarize and conclude in Sec. X.

Throughout the paper we use natural units with \( c = \hbar = 1 \). The metric tensor is \( g_{\mu\nu} = \text{diag}(1, -1, -1, -1) \).

II. THEORIES OF RELATIVISTIC HYDRODYNAMICS

A. Dynamical variables and evolution equations

By a hydrodynamic description one means a theoretical framework that uses a small set of fluid variables. For a perfect fluid these can be chosen as the local energy density \( \varepsilon(x) \) and local hydrodynamic flow vector \( U^\mu(x) \), which is normalized as \( U^\mu U_\mu = 1 \). To set the stage for the developments described in the remainder of the paper, it is worthwhile to briefly review the basic conceptual structures used to formulate theories of relativistic hydrodynamics.

The point of departure is the assumption that we are dealing with a system which reaches global thermodynamic equilibrium at late times. At this stage one can be agnostic about the fundamental physics governing this system: it could be composed of well defined quasiparticles, but it need not be.

The equilibrium energy-momentum tensor in the rest-frame is given by

\[
T^\mu_\nu_{\text{EQ}} = \text{diag}(\varepsilon_{\text{EQ}}, P(\varepsilon_{\text{EQ}}), P(\varepsilon_{\text{EQ}}), P(\varepsilon_{\text{EQ}})),
\]

where we assume that the equation of state is known, so that the pressure \( P \) is a given function of the energy density \( \varepsilon_{\text{EQ}} \). It is worth stressing, that \( T^\mu_\nu_{\text{EQ}} \) is a classical object which introduces the bulk viscous pressure \( \Pi \) (the trace part of \( \Pi^{\mu\nu} \)) and the shear stress tensor \( \sigma^{\mu\nu} \) which is symmetric, \( \Pi^{\mu\nu} = \Pi^{\nu\mu} \), traceless, \( \Pi^{\mu\mu} = 0 \), and orthogonal to \( U^\mu \), \( \Pi^{\mu\nu} U_\nu = 0 \).

Equation (7) encodes ten independent components of \( T^{\mu\nu} \) in terms of the effective temperature, three independent components of \( U^\mu \), five independent components of \( \Pi^{\mu\nu} \), and the bulk viscous pressure \( \Pi \). We note that the latter vanishes for conformal systems, for which the entire energy-momentum tensor is traceless.

We still have the four conservation equations at our disposal, but to obtain a closed system of equations one needs additional information. The most straightforward option is to express \( \Pi^{\mu\nu} \) in terms of the hydrodynamic variables and their gradients. Since the perfect-fluid energy-momentum tensor contains no gradients, it is natural to try to build up the theory as a series of corrections in gradients. The simplest possibility is to include terms with only a single gradient, which leads to the relativistic Navier-Stokes theory [23], in which the bulk pressure and shear stress tensor are given by the gradients of the flow vector

\[
\Pi = -\zeta \partial_\mu U^\mu, \quad \pi^{\mu\nu} = 2\eta \sigma^{\mu\nu}.
\]

Here \( \zeta \) and \( \eta \) are the bulk and shear viscosity coefficients, respectively, and \( \sigma^{\mu\nu} \) is the shear flow tensor defined as

\[
\sigma^{\mu\nu} = \Delta^{\mu\nu}_\alpha \partial_\alpha U^\beta,
\]
where the projection operator $\Delta^{\alpha\beta}_{\alpha\beta}$ has the form

$$\Delta^{\alpha\beta}_{\alpha\beta} = \frac{1}{2} (\Delta^{\mu\nu}_{\mu\nu} + 3 \Delta^{\rho\sigma}_{\rho\sigma} \Delta^{\mu\nu}_{\mu\nu}) - \frac{1}{3} \Delta^{\mu\nu}_{\mu\nu} \Delta^{\rho\sigma}_{\rho\sigma}. \quad (12)$$

While the evolution equations obtained in this way are covariant, they have solutions which propagate with arbitrarily high velocities, leading to causality violation and instabilities in numerical simulations [24, 25].

Adding extra terms with higher gradients on the right-hand side of (10) does not help to solve the problem with causality. The only known way to avoid it is to relax the assumption that II and $\pi^{\mu\nu}$ are expressed locally in terms of the hydrodynamic variables $T, U^\mu$, and (a finite number of) their space-time derivatives [14]. This means, however, that the conservation equations alone are no longer enough to determine the dynamics of $T^{\mu\nu}$ and one needs to postulate additional dynamic equations, or derive them (possibly by some heuristic means). The outcome, a closed set of hydrodynamic equations, will clearly involve additional degrees of freedom beyond those already present in the theory of the perfect fluid. To write down such equations will require additional assumptions, or additional information beyond what is embodied by conservation laws.

### B. Approaches to finding evolution equations

A well-known and widely applied approach to the task of positing a set of closed equations for the hydrodynamic fields is the Müller-Israel-Stewart theory [14–16], in its modern incarnation described in Ref. [17] (BRSSS). This approach (discussed further in Sec. VII) basically parameterizes dominant contributions classified by symmetries and the number of gradients. It does not make any special assumptions about the microscopic dynamics, which accounts for its generality. A major advantage of this approach is that the resulting equations are causal at least for some domain in the space of transport coefficients.

If we commit to a specific microscopic model, we gain the option of deriving (at least in a heuristic way) a set of hydrodynamic equations which can provide a better physical picture than a generic approach such as the BRSSS theory. An important testbed for this idea is provided by kinetic theory with an idealized collision kernel, further discussed below in Sec. V. In this case a number of complementary approaches exist, which we now briefly review.

One important approach makes use of the hydrodynamic expansion: this is the process of constructing the dynamical equations order by order in the Knudsen and inverse Reynolds numbers [19–21]. The hydrodynamic expansion is performed around the local equilibrium state that corresponds to the perfect-fluid limit $T^{\mu\nu}_{eq}$. The Knudsen number is the ratio of the molecular mean free path length to a representative physical length scale. On the other hand, the inverse Reynolds number describes deviations of the energy-momentum components from their local equilibrium values — they are typically expressed by the ratios $\sqrt{\frac{\pi^{\mu\nu} \pi^{\rho\sigma}}{P}}$ and $\Pi/P$. The hydrodynamic expansion serves as a tool to systematically derive hydrodynamic equations from kinetic theory.

Another approach to the task of formulating a closed set of hydrodynamic equations for models of kinetic theory is known under the name of anisotropic hydrodynamics [1, 2] (for a recent review see [26]). This name originated in the desire of finding hydrodynamic equations suited to describing early stages of evolution of the quark-gluon plasma (QGP) produced in heavy-ion collisions. Equations of anisotropic hydrodynamics were formulated in such a way as to capture some features of highly anisotropic initial states, but also to ensure that at late times their predictions should be consistent with BRSSS. In modern formulations, the equations of anisotropic hydrodynamics are suitable for studying arbitrary flows. Further details of this approach (which is central to the present article) are given in Sec. III.

### C. Gradient expansions

The gradient expansion is an effective way to quantify the approach to equilibrium as well as to gather information about the non-hydrodynamic sector. This latter information is attainable by considering perturbations around the hydrodynamic solution, or through the study of large order behavior of the gradient series, with the former being encoded in the leading terms.

As briefly reviewed in Sec. II A, once dissipative effects are incorporated within a hydrodynamic framework, one loses the universality of perfect fluid theory, and many different sets of hydrodynamic equations are possible. This raises the question of how they are to be compared, and how can they be reconciled with computations carried out directly in the microscopic theory.

One measuring stick that can be used to survey this realm of possibilities is the gradient expansion: a formal, infinite, expansion of the energy-momentum tensor $T^{\mu\nu}$ around the perfect fluid form $T^{\mu\nu}_{eq}$ in powers of gradients of the fluid variables $T(x)$ and $U^\mu(x)$:

$$T^{\mu\nu} = T^{\mu\nu}_{eq} + \text{powers of gradients of } T \text{ and } U^\mu. \quad (13)$$

Such an expression can arise in calculations based on microscopic models (such as the AdS/CFT representation of $N=4$ supersymmetric Yang-Mills theory [27] or kinetic theory [3]). Crucially, it also arises from any set of hydrodynamic equations as a generic late-time solution. Given the set of evolution equations for the shear stress tensor, we can always find such a solution by writing down the most general gradient series consistent with Lorentz symmetry (and any other constraints, such as perhaps conformal invariance), and determine the scalar coefficient functions by using the evolution equation order by order in gradients.

We wish to stress the need to make a clear distinction between the gradient expansion discussed here and the hydrodynamic expansions discussed in the previous subsection as a means of deriving hydrodynamic equations from kinetic theory. As discussed there, various methods exist for writing down hydrodynamic equations; once some set of hydrodynamic equations is found, one can look for a formal solution in the form of a gradient expansion of the energy-momentum tensor as in Eq. (13).

By comparing such formal solutions, one may quantify differences between different hydrodynamic theories, as well as compare them to a given microscopic model. This process also determines any phenomenological parameters appearing at the level of hydrodynamics in terms of parameters appearing in the fundamental theory.
### III. ANISOTROPIC HYDRODYNAMICS – REORGANIZED HYDRODYNAMIC EXPANSION

As discussed above, in the standard approach to viscous hydrodynamics, the energy-momentum tensor components $T^{\mu \nu}$ may be treated as functions of $T, U^\mu, \pi^{\mu \nu}$, and $\Pi$, which we write schematically as

$$
T^{\mu \nu} = T^{\mu \nu}(T, U^\mu, \pi^{\mu \nu}, \Pi)

= T^{\mu \nu}_{\text{eq}}(T, U) + \pi^{\mu \nu} - \Pi \Delta^{\mu \nu}

\equiv T^{\mu \nu}_{\text{eq}} + \delta T^{\mu \nu}.
$$

(14)

The hydrodynamic equations that determine the dynamics of $T^{\mu \nu}$ contain various terms that may be characterised by the power of space-time gradients and/or the power of the dissipative terms they include. For example, in the first order of viscous hydrodynamics one deals with dissipative terms they include. In the second order, the products of $\pi^{\mu \nu}$ and $\Pi$ appear, as well as the gradients of $\pi^{\mu \nu}$ and $\Pi$. Such approach may be continued to higher orders but, in practical applications, one stops at the third order (for example, see Ref. [28]).

Anisotropic hydrodynamics can be treated as a method to reorganize this kind of expansion within the framework of the kinetic theory. Thus, from now on we use the concepts of the phase space distribution function $f(x, p)$ and express different physical quantities as the moments of $f(x, p)$ (in the three-momentum space).

Within AHYDRO one separates the description of viscous effects into two parts. The first part is characterised by the new fluid variables $\xi^{\mu \nu}$ and $\phi$, see Refs. [12, 13, 29–32]. They may account for large possible values of the shear stress tensor and bulk viscosity and should be treated in a non-perturbative manner, similarly to $T$ and $U^\mu$. The second part is characterised by the tensor $\tilde{\pi}^{\mu \nu}$ and $\tilde{\Pi}$ that are treated similarly as $\pi^{\mu \nu}$ and $\Pi$ in the standard case [33–36]. Thus, we write

$$
T^{\mu \nu} = T^{\mu \nu}(T, U^\mu, \xi^{\mu \nu}, \phi, \tilde{\pi}^{\mu \nu}, \tilde{\Pi})

= T^{\mu \nu}_{\text{eq}}(T, U, \xi^{\mu \nu}, \phi) + \tilde{\pi}^{\mu \nu} - \tilde{\Pi} \Delta^{\mu \nu}

\equiv T^{\mu \nu}_{\text{eq}} + \delta T^{\mu \nu}.
$$

(15)

Several comments are in order now:

i) Introducing the fluid variables $\xi^{\mu \nu}$ and $\phi$ together with $\tilde{\pi}^{\mu \nu}$ and $\tilde{\Pi}$ means that the “viscous” degrees of freedom may be doubled and we need more than typical ten hydrodynamic equations to determine dynamics of the energy-momentum tensor. This can be easily achieved within kinetic theory approach by including, for example, a sufficient number of the moments of the kinetic equation. The selection of the moments is, however, not well defined. One considers usually the lowest possible moments [33], since they are most sensitive to the low momentum sector which is expected to be well described by hydrodynamics-like models. We come back to the discussion of this ambiguity below in the point vii).

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1 Strictly speaking one considers the powers of the ratios $\pi^{\mu \nu}/P$ and $\Pi/P$ that are known as the inverse Reynolds numbers.

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ii) The tensor $\xi^{\mu \nu}$ has similar geometric properties as $\pi^{\mu \nu}$, namely, it is symmetric, transverse to $U^\mu$ and traceless [13, 31, 32]. This means that it has in general five independent components. However, in practical applications one often uses a simplified versions of $\xi^{\mu \nu}$ that contains one or two independent parameters. In such cases only these degrees of freedom may be “doubled”. We note that the use of simplified forms of $\xi^{\mu \nu}$ is very often a consequence of the system’s symmetries such as boost invariance, homogeneity in the transverse plane or cylindrical symmetry.

iii) The tensor $\tilde{\pi}^{\mu \nu}$ is also symmetric, transverse to $U^\mu$ and traceless. Consequently, using the Landau frame, one can determine the effective temperature of the system $T$ and the flow $U^\mu$ by the equation

$$
eq_T(T(x)) = T^{\mu \nu}_{\text{eq}}(x) U_\nu(x).
$$

(16)

iv) Since the substantial part of the viscous effects is included with the help of the variables $\xi^{\mu \nu}$ and $\phi$, one expects that the terms $\tilde{\pi}^{\mu \nu}$ and $\tilde{\Pi}$ are small compared to the equilibrium pressure $P$. The expansion in the ratios $\sqrt{\pi^{\mu \nu}}, \pi^{\mu \nu}/P$ and $\Pi/P$ is discussed in this context as an expansion in the modified inverse Reynolds numbers [33–36].

v) Using the kinetic theory approach, Eq. (15) is reproduced with the distribution function that has a structure

$$
f(x, p) = f_a(x, p) + \delta f(x, p).
$$

(17)

Here $f_a(x, p)$ is the anisotropic distribution function in the momentum space. It can be regarded as an extension of the equilibrium distribution $f_{\text{eq}}(x, p)$, which depends not only on $T$ and $U^\mu$ but also on $\xi^{\mu \nu}$ and $\phi$. In the limit $\xi^{\mu \nu}, \phi \to 0$ one finds that $f_a(x, p) \to f_{\text{eq}}(x, p).$ Two special forms of $f_a(x, p)$ will be discussed below, see Eqs. (40) and (41).

vi) In the leading order of anisotropic hydrodynamics, we neglect the corrections $\delta T^{\mu \nu}$ in (15) and $\delta \tilde{T}^{\mu \nu}$ in (17). The complete energy-momentum tensor has the form

$$
T^{\mu \nu} = T^{\mu \nu}_{\text{eq}} + k \int \frac{d^3p}{(2\pi)^3 P^0} p^\mu p^\nu f_a(x, p).
$$

(18)

where $k$ is a degeneracy factor. In this case, the components of $T^{\mu \nu}_{\text{eq}}$ depend, in general, on ten independent parameters contained in the set: $T, U^\mu, \xi^{\mu \nu}$ and $\phi$. The equations of anisotropic hydrodynamics specify the dynamics of $T^{\mu \nu}_{\text{eq}}$. They include four equations that follow directly from the energy-momentum conservation law and additional six equations that should be derived from some other (microscopic) theory.

vii) Doubling of the viscous degrees of freedom can be avoided if the use of a certain parameter in the set ($\xi^{\mu \nu}, \phi$) is accompanied with the elimination of some parameter in the set ($\pi^{\mu \nu}, \Pi$). For example, using the bulk parameter $\phi$ we can set $\Pi = 0$. The extreme strategy in this context is to assume that the parameters $\xi^{\mu \nu}$ and $\phi$ are chosen in such a way that

$$
T^{\mu \nu} = T^{\mu \nu}_{\text{eq}}.
$$

(19)

This formula represents the anisotropic matching principle introduced by Tinti [13]. We note that (19) is...
formally equivalent to (18), however, it is obtained with different assumptions: instead of neglecting the term \( \delta f(x,p) \) in (17) one assumes that \( \delta f(x,p) \) might be finite but it does not contribute to \( T^{\mu \nu} \).

We can now define the gradient expansion for the leading-order anisotropic hydrodynamics. Given \( T(x) \) and \( U^\mu(x) \) we define \( f_{eq}(x,p) \) and \( T_{eq}^{\mu \nu} \) and write

\[
T^{\mu \nu} = T_{eq}^{\mu \nu} + \delta T^{\mu \nu} = T_{eq}^{\mu \nu} + (T_\parallel^{\mu \nu} - T_{eq}^{\mu \nu}) \quad (20)
\]

This formula suggests to use the gradient expansion of anisotropic hydrodynamics in the form

\[
T^{\mu \nu} = T_{eq}^{\mu \nu} + \text{powers of gradients of } T, U^\mu, \xi^{\mu \nu} \text{ and } \phi. \quad (21)
\]

Compared to (13), the expansion (21) includes also the gradients of \( \xi^{\mu \nu} \) and \( \phi \). On the other hand, similarly to (13), the expansion (21) should be done around the perfect-fluid solution that is determined solely by the conservation law (6), see Sec. IV B.

IV. BOOST-INvariant AND TRANSVERSELY-HOMOGENEOUS SYSTEMS

A. Tensor decompositions

In this work we consider transversely-homogeneous and boost-invariant systems. In this case, following Refs. [12, 29, 37], it is convenient to introduce a basis of four-vectors that can be used to construct all tensor structures necessary in our analysis:

\[
U^\mu(t,0,0,z/\tau) , \quad Z^\mu(z,\tau,0,0) , \quad X^\mu(0,1,0,0) , \quad Y^\mu(0,0,0,1) .
\]

Here \( t \) and \( z \) are time and space coordinates, and \( \tau = \sqrt{t^2 - z^2} \) is the (longitudinal) proper time. The time-like four-vector \( U^\mu \) describes the longitudinal Bjorken flow of matter [38]. The four-vectors \( X, Y, \) and \( Z \) are space-like and orthogonal to \( U \).

We find that

\[
\Delta^{\mu \nu} = g^{\mu \nu} - U^\mu U^\nu = -X^\mu X^\nu - Y^\mu Y^\nu - Z^\mu Z^\nu \quad (26)
\]

and

\[
\pi^{\mu \nu} = \pi_\perp (X^\mu X^\nu + Y^\mu Y^\nu) + \pi_\parallel Z^\mu Z^\nu , \quad (27)
\]

where \( 2\pi_\perp + \pi_\parallel = 0 \) due to the fact that \( \pi^{\mu \nu}_\mu = 0 \). This leads to the following decomposition of the energy-momentum tensor for local equilibrium

\[
T_{eq}^{\mu \nu} = \varepsilon U^\mu U^\nu + P (X^\mu X^\nu + Y^\mu Y^\nu + Z^\mu Z^\nu) \quad (28)
\]

and for locally anisotropic state

\[
T^{\mu \nu} = T_\parallel^{\mu \nu} = \varepsilon U^\mu U^\nu + P_\perp (X^\mu X^\nu + Y^\mu Y^\nu) + P_\parallel Z^\mu Z^\nu . \quad (29)
\]

Here we have introduced the transverse and longitudinal pressure components

\[
P_\perp = P + \Pi + \frac{1}{2} \pi_\parallel , \quad P_\parallel = P + \Pi - \pi_\parallel . \quad (30)
\]

Equations (20), (28), and (29) indicate that the gradient expansion in our case is defined by the gradient expansion of the bulk viscosity \( \Pi \) and the difference of the longitudinal and transverse pressure

\[
\Delta P = P_\parallel - P_\perp = -\frac{3}{2} \pi_\parallel . \quad (31)
\]

For boost-invariant systems, the gradients of scalar quantities such as \( P_\parallel \) or \( P_\perp \) are expressed by the derivatives with respect to the proper time \( \tau \). Hence, our task is twofold: we need to find the dependence of the fluid variables \( T, \xi^{\mu \nu} \) and \( \phi \) on the proper time \( \tau \) and, then, to express the corrections to the energy-momentum tensor, \( \Pi \) and \( \Delta P \), in terms of \( T, \xi^{\mu \nu} \) and \( \phi \). In the conformal limit it is enough to study \( \Delta P \), since \( \Pi = \phi = 0 \) in this case. In Sec. V B we argue that our symmetry constraints allow us to use only one independent scalar anisotropy parameter \( \xi \) instead of the full five-component tensor \( \xi^{\mu \nu} \).

B. Bjorken perfect-fluid model

In the case where the dissipative terms are neglected we deal with Eqs. (6) only. For boost-invariant and transversely homogeneous systems, with the flow vector given by Eq. (22), they reduce to a single equation of the form [38]

\[
\frac{d\varepsilon}{d\tau} = -\varepsilon + P \tau , \quad (32)
\]

For conformal systems studied here \( \varepsilon = 3P \), hence, Eq. (32) has a solution

\[
\varepsilon = \varepsilon_0 \left( \frac{\tau_0}{\tau} \right)^{1/3} . \quad (33)
\]

where \( \varepsilon_0 \) is the energy density at the initial proper time \( \tau_0 \), \( \varepsilon_0 = \varepsilon(\tau_0) \).

Since \( \varepsilon \) and \( P \) describe the system in local equilibrium, we may use the thermodynamic relations \( d\varepsilon = T s d\tau \) and \( dP = s dT \), where \( s \) is the entropy density. This leads to a scaling solution for the entropy density

\[
s = \frac{s_0 \tau_0}{\tau} \quad (34)
\]

while for the temperature one obtains

\[
T = T_0 \left( \frac{\tau_0}{\tau} \right)^{1/3} . \quad (35)
\]

Here \( s_0 \) and \( T_0 \) are the initial values of the entropy density and temperature, respectively. Dissipative effects introduce corrections to the above equations which will be analysed below with the help of the gradient expansion.

V. BOLTZMANN KINETIC EQUATION IN THE RELAXATION TIME APPROXIMATION

A. Kinetic equation

The kinetic equation in the relaxation time approximation has the form [39]

\[
p^{\mu} \partial_{\mu} f(x,p) = p \cdot U \frac{f_{eq}(x,p) - f(x,p)}{\tau_{eq}} , \quad (36)
\]
where \( f(x,p) \) is the one-particle phase-space distribution function depending on particle space-time coordinates \( x \) and momenta \( p \), and \( f_{\text{eq}}(x,p) \) is the background equilibrium distribution function. In this work we neglect the effects coming from quantum statistics and set particle masses equal to zero. Thus, in local equilibrium we deal with the Boltzmann distribution function \( f_{\text{eq}} \) that may be rewritten with the help of the four-vectors (23)–(25) as

\[
f_{\text{eq}} = \exp \left( -\frac{1}{T} \sqrt{ (p \cdot X)^2 + (p \cdot Y)^2 + (p \cdot Z)^2 } \right),
\]

where \( T \) is the system’s (effective) temperature which is determined by the condition that the energy densities obtained with the distributions \( f \) and \( f_{\text{eq}} \) are equal, see Eq. (5).

The quantity \( \tau_{\text{eq}} \) in (36) is the relaxation time. In this work we consider conformal systems where \( \tau_{\text{eq}} \) depends on the inverse of the effective temperature, namely, we use the formula

\[
\tau_{\text{eq}} = \frac{c}{T},
\]

where \( c \) is a numerical constant. The calculation of the shear viscosity coefficient \( \eta \) done with Eq. (36) leads to the result [40–42]

\[
\tau_{\text{eq}} = \frac{5\eta}{T},
\]

where \( \eta \) is the ratio of the shear viscosity to the entropy density, \( \eta = \eta/s \). This means that the coefficient \( c \) may be related directly to the shear viscosity of the system.

We note that for boost invariant and transversely-homogeneous systems, Eq. (36) can be solved exactly [43–46] and its solutions can be used to assess the agreement between the kinetic-theory approach and hydrodynamic approaches that have been constructed with its help [45, 46]. Our present study, based on the gradient expansion, is complementary to this type of study.

### B. Anisotropic distribution functions

Within the original formulation of anisotropic hydrodynamics [1, 2], one assumes that the non-equilibrium distribution function \( f \) is well approximated by the spheroidal Romatschke-Strickland form [47],

\[
f_a = \exp \left( -\frac{1}{\Lambda} \sqrt{ (p \cdot U)^2 + \xi (p \cdot Z)^2 } \right)
\]

\[
= \exp \left( -\frac{1}{\Lambda} \sqrt{ (p \cdot X)^2 + (p \cdot Y)^2 + (1 + \xi)(p \cdot Z)^2 } \right).
\]

The parameter \( \Lambda \) in (40) defines a typical transverse-momentum scale in the system, while \( \xi \) is a scalar anisotropy parameter. In local equilibrium \( \xi \to 0 \) and \( \Lambda \) can be identified with the temperature \( T \).

In Ref. [12] a generalized ellipsoidal parameterization of the anisotropic distribution function was proposed as a good approximation for \( f \), namely

\[
f_a = \exp \left( -\frac{E}{\Lambda} \right),
\]

where

\[
E^2 = (1 + \xi X)(p \cdot X)^2 + (1 + \xi Y)(p \cdot Y)^2 + (1 + \xi Z)(p \cdot Z)^2.
\]

The form (41) becomes important in the case where radial expansion of the system is present [12], since in such cases the pressure anisotropies in the \( X \) and \( Y \) directions are generally different, which is not included in the original formulation of HYDRO based on (40).

In this work we consider a boost-invariant and transversely-homogeneous system in which case the two formulations (40) and (41) are completely equivalent. We present (41) for completeness as several results presented below are obtained within the framework defined in [12]. We note that the anisotropy parameters \( \xi_i \) in (41) satisfy the condition [12]

\[
\sum_i \xi_i = \xi_X + \xi_Y + \xi_Z = 0.
\]

Consequently, the parameterizations (40) and (41) are connected through the following set of simple transformations

\[
\xi_X = \xi_Y = \xi_\perp = -\frac{\xi/3}{1 + \xi/3},
\]

\[
\xi_Z = \xi_\parallel = \frac{2\xi/3}{1 + \xi/3},
\]

\[
\lambda = \Lambda (1 + \xi/3)^{-1/2}.
\]

The form of the anisotropic distribution can be made even more general if we use the expression

\[
f_a = \exp \left( -\frac{1}{\Lambda} \sqrt{p_\mu(U_\mu U_\nu + \xi^{\mu\nu})p_\nu} \right),
\]

where the anisotropy tensor \( \xi^{\mu\nu} \) appears. The five parameters in \( \xi^{\mu\nu} \) together with three independent parameters defining the flow vector \( U_\mu \) and \( \lambda \) should be taken in such a way that one reproduces nine independent components of the conformal energy-momentum tensor, see Eq. (19) that represents the anisotropic matching principle introduced by Tinti in [13]. The latter is a generalisation of the Landau matching condition which demands that the parameters of the distribution function reproduce only the energy density and energy flow vector, see Eq. (16). For a boost-invariant and transversely-homogeneous system, the distribution (45) agrees with (41), since in this case we have

\[
\xi^{\mu\nu} = \xi_\perp (X^\mu X^\nu + Y^\mu Y^\nu) + \xi_\parallel Z^\mu Z^\nu.
\]

### VI. ANISOTROPIC HYDRODYNAMICS EQUATIONS

Our last considerations show that with our symmetry constraints we may use the original Romatschke-Strickland form of the distribution function with a single anisotropy parameter. We have introduced, however, the forms (41) and (45) because they serve as the starting points for two different formulations of anisotropic hydrodynamics — the first one [12] uses the moments of the Boltzmann equation, while the second one [13] uses the anisotropic matching principle combined with the exact treatment of the dynamical equations for the pressure corrections. The latter introduces an infinite
set of the coupled equations that is finally truncated by the assumption that $f = f_s$.

In the next two sections we present the form of anisotropic hydrodynamics equations derived in [12] and [13], respectively. Our discussion above suggests that in the two considered cases the hydrodynamic equations can be written as two coupled ordinary differential equations for the functions $T(\tau)$ and $\xi(\tau)$.

A. First option

For purely longitudinal and boost invariant expansion, the equations of anisotropic hydrodynamics derived in Ref. [12] can be cast into the following form

$$T^4 = R(\xi)\Lambda^4,$$  \hspace{1cm} (47)

$$\frac{4R(\xi)}{\Lambda} \frac{d\Lambda}{d\tau} + R'(\xi) \frac{d\xi}{d\tau} = -\frac{1}{\tau} \left( R(\xi) + \frac{1}{3} R_{||}(\xi) \right)$$  \hspace{1cm} (48)

and

$$-\frac{1}{1 + \xi} \frac{d\xi}{d\tau} + \frac{2}{\tau} = \frac{\xi}{\tau_{eq}} \left( \frac{T}{\Lambda} \right)^5 (1 + \xi)^{1/2}.$$  \hspace{1cm} (49)

These are three equations for three functions of the proper time: the effective temperature, $T(\tau)$, the transverse-momentum scale, $\Lambda(\tau)$, and the anisotropy parameter, $\xi(\tau)$.

Equations (47) and (48) follow from the first moment of the kinetic equation (36), i.e., from the energy-momentum conservation. Equation (47) expresses the condition that the energy density obtained from the anisotropic distribution (40) characterised by $\Lambda$ and $\xi$ is equal to the energy density obtained from the reference equilibrium distribution (37) characterised by the temperature $T$. The functions $R$ and $R_{||}$ are defined by the formulas [2]

$$R(\xi) = \frac{1}{2} \left( \frac{1}{1 + \xi} + \frac{\tanh^{-1}(\sqrt{\xi})}{\sqrt{\xi}} \right)$$  \hspace{1cm} (50)

and

$$R_{||}(\xi) = \frac{3}{\xi} \left( R(\xi) - \frac{1}{1 + \xi} \right).$$  \hspace{1cm} (51)

Equation (49) is obtained from the second moment of the kinetic equation (37). Its form is determined by the condition that it agrees with the BRSSS theory for systems that are close to equilibrium, for more details see derivation of Eqs. (43), (48) and (57) presented in [48].

Using Eqs. (38) and (47) in Eqs. (48) and (49) we obtain two coupled ordinary differential equations for $T$ and $\xi$ only,

$$4 \frac{R(\xi)}{T} \frac{dT}{d\tau} = -\frac{1}{\tau} \left( R(\xi) + \frac{R_{||}(\xi)}{3} \right)$$  \hspace{1cm} (52)

and

$$-\frac{d\xi}{d\tau} + \frac{2(1 + \xi)}{\tau} = \frac{\xi T R(\xi)^{5/4}}{c} (1 + \xi)^{3/2}.$$  \hspace{1cm} (53)

B. Second option

As the second option for the anisotropic hydrodynamics equations we choose the form derived recently in [13]. This form follows from the anisotropic matching principle. One can check that this matching is consistent with Eq. (52). On the other hand, Eq. (53) should be replaced by Eq. (82) from [13]. In the conformal limit, the latter has the form

$$\frac{d\Delta P}{d\tau} = -\frac{T \Delta P}{c} - \frac{F}{\tau},$$  \hspace{1cm} (54)

where $\Delta P$ is the difference of the longitudinal and transverse pressures, see Eq. (31). Using definitions given in [13] one finds that $\Delta P$ can be expressed as

$$\Delta P = -\frac{6kT^4}{\xi} \left( \frac{\xi + 3}{\xi + 1} + \frac{(\xi - 3) \tan^{-1}(\sqrt{\xi})}{\sqrt{\xi}} \right).$$  \hspace{1cm} (55)

Similarly, one finds the form of the function $F$ appearing on the right-hand side of (54), namely

$$F = -2(1 + \xi) \frac{\partial \Delta}{\partial \xi}.$$  \hspace{1cm} (56)

Using Eq. (47) to express $\Lambda$ in terms of $T$ and $\xi$ in (31) and (56), and substituting (31) and (56) into (54) we find an alternative for (53).

To summarise the last two Sections, we state that we have two options for anisotropic hydrodynamics equations, these are either Eqs. (52) and (53), denoted below as AHYDRO I, or Eqs. (52) and (54), denoted below as AHYDRO II. In both cases these are two ordinary coupled differential equations for the two functions of the proper time, $T(\tau)$ and $\xi(\tau)$.

VII. MÜLLER-ISRAEL-STEWART VISCOUS HYDRODYNAMICS

The hydrodynamic equations introduced in the previous subsections were tailored to a specific microscopic theory: kinetic theory in the relaxation time approximation. In particular, they did not introduce any free parameters beyond the single parameter, $c$, present already in the microscopic model. In this Section we briefly review the complementary approach, which assumes essentially nothing beyond thermodynamic equilibrium in the far future. In particular, no quasiparticle picture or specific microscopic model is adopted, yielding a very powerful, general theory. This generality, as we shall see, comes at a price: we cannot expect this theory to provide as fine a picture as that provided by hydrodynamic equations developed to mimic a specific microscopic model.

As described in Sec. II, to formulate a theory of dissipative relativistic hydrodynamics in the conformal cases one needs to provide five equations in addition to the four conservation equations (6). The MIS theory postulates that these should take the form of relaxation equations [14-16]

$$\Delta_{\mu\nu} U^\gamma \partial_\gamma \pi^{\mu\nu} = -\frac{1}{\tau_\pi} \left( \pi^{\alpha\beta} - 2\rho \sigma^{\alpha\beta} \right) + \ldots$$  \hspace{1cm} (57)

where $\tau_\pi$ is the relaxation time. This guarantees that the shear stress tensor approaches the Navier-Stokes form (10)
at late times. It is easy to see that Eq. (57) can be solved iteratively obtaining
\[ \pi^{\alpha \beta} = 2 \eta g^{\alpha \beta} + \ldots, \]
where the ellipsis contains terms of second and higher order in gradients. It is clear that all orders in gradients will appear here.

The formulation of BRSSS theory in Ref. [17] rests on the observation that by a judicious choice of terms on the RHS of Eq. (57) one can generate all possible terms up to second order in gradients with coefficients which can be chosen at will. In this way, by a suitable choice of these coefficients one can match a calculation of the energy-momentum tensor expectation value in any microscopic theory or model up to second order in gradients. Of course, all higher orders will typically not be matched, but sufficiently close to equilibrium this is not an issue. It is instructive to recall at this point that the formulation of the BRSSS equations was prompted by a calculation of the gradient expansion of the energy-momentum tensor performed in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory [27] using the AdS/CFT correspondence. It turned out that the original MIS theory did not have all the terms necessary to match the result of the microscopic calculation.

The complete set of BRSSS equations for conformal hydrodynamics given in Ref. [17] involves 5 second order transport coefficients $\tau_1, \lambda_1, \lambda_2, \lambda_3, \kappa$. If boost invariance is imposed, the resulting BRSSS equations involve only $\tau_1, \lambda_1$, and the shear viscosity:
\[ \tau_1 = \frac{4}{3} \varepsilon + \phi, \]
\[ \tau_{1i} = 4\eta \frac{\lambda_1}{2\eta} - \frac{3\tau_1}{\tau} \phi - \phi, \]
where the dot denotes a proper time derivative and $\phi \equiv -\pi_1$ is the single independent component of the shear stress tensor.

For comparison, we also analyse below: i) the hydrodynamic equations derived in Ref. [19], which for the case of the Bjorken flow differ from (59) by the second line, which reads
\[ \tau_{1i} = 4\eta \frac{\lambda_1}{2\eta} - \frac{3\tau_1}{\tau} \phi - \phi, \]
and ii) a truncated version of the second line in (59)
\[ \tau_{1i} = 4\eta \frac{\lambda_1}{2\eta} - \frac{3\tau_1}{\tau} \phi - \phi, \]
which can be connected with one of the early versions of the Israel-Stewart theory. The results obtained with (60) and (61) will be denoted by the labels DNMR and MIS, respectively. In the following Section, it will be clarified in what sense Eqs. (59), (60), and (61) are consistent with each other.

**VIII. DIMENSIONLESS GRADIENT EXPANSION**

In this Section we study the gradient expansion for the Bjorken flow, which amounts to calculating gradient corrections to the perfect-fluid Bjorken solution [38]. Thus, it is convenient, as seen in a number of recent studies, to consider a dimensionless function $g$ of a dimensionless variable $w$, namely
\[ g = \frac{1}{T} \frac{dT}{dw}, \quad w = \tau T. \]
This quantity is related in a trivial way to the dimensionless pressure anisotropy
\[ \Delta = \frac{\Delta P}{P} = 3 \frac{P_0 - P_T}{\varepsilon} = 12 \left( g - \frac{2}{3} \right). \]
The gradient expansion for boost-invariant flow takes the form of an expansion
\[ g(w) = \sum_{n=0}^{\infty} g_n w^{-n}, \]
where $g_0 = 2/3$, which corresponds to the perfect-fluid behavior $T \sim \tau^{-1/4}$ [38]. The fact that only integer powers appear in Eq. (64) is a very convenient feature.

In Ref. [18], where the BRSSS theory was considered, it was possible to find a closed first order ordinary differential equation satisfied by the function $g(w)$. The gradient expansion was then calculated by looking for a solution in the form of a series in $1/w$. In the present case, it appears difficult to find a closed equation for $g(w)$, so we first calculate the expansions of $T$ and $\xi$ in powers of $\tau$ and then determine the coefficients $g_n$ by solving Eq. (62) as an expansion of the form (64).

It is convenient to write the series for $T(\tau)$ and $\xi(\tau)$ in the forms
\[ T = T_0 \left( \frac{\tau}{\tau_0} \right)^{1/3} \left( 1 + \sum_{n=1}^{\infty} \left( \frac{c_{nT}}{T_0} \right)^n \tau_n \left( \frac{\tau}{\tau_0} \right)^{2n/3} \right), \]
\[ \xi(\tau) = \sum_{n=1}^{\infty} \left( \frac{2c_{nT}}{T_0} \right)^n \xi_n \left( \frac{\tau}{\tau_0} \right)^{2n/3}, \]
where $T_0$ is the initial temperature at some initial proper time $\tau_0$ for the Bjorken solution [38]. We insert (65) and (66) into either Eqs. (52) and (53) or Eqs. (52) and (54). These two pairs of equations correspond to two different options of constructing anisotropic hydrodynamics, see Sections (VI A) and (VI B), respectively. The form of these equations is cumbersome, but they are first order differential equations which can be solved order by order in powers of $\tau^{2/3}$ to determine the values of the coefficients $\xi_n$ and $\tau_n$.

For the first option of anisotropic hydrodynamics based on Eqs. (52) and (53), one finds
\[ (\xi_n) = \left( \frac{1}{15}, \frac{1}{15}, \frac{62}{1575}, \frac{893}{10125}, \ldots \right), \quad n = 1, 2, \ldots \]
and
\[ (\tau_n) = \left( \frac{1}{15}, \frac{2}{315}, \frac{268}{14175}, \ldots \right), \quad n = 0, 1, \ldots, \]
while for the second option based on Eqs. (52) and (54) we get
\[ (\xi_n) = \left( \frac{1}{15}, \frac{82}{105}, \frac{782}{1575}, \frac{8325224}{38201625}, \ldots \right), \quad n = 1, 2, \ldots \]

\[ \text{In [18] this function was called } f. \]
and
\[(t_n) = \left(1, -\frac{2}{15}, -\frac{4}{315}, 1208/99225, \ldots\right), \quad n = 0, 1, \ldots\]  
Equation (60)

The results (67) and (68) imply the following coefficients in Eq. (64)
\[(g_n) = \left(\frac{2}{3}, \frac{8c^2}{45}, \frac{184c^3}{4725}, \ldots\right), \quad n = 0, 1, \ldots\]  
Equation (71)

Similarly, the series (69) and (70) imply
\[(g_n) = \left(\frac{2}{3}, \frac{16c^2}{45}, \frac{176c^3}{6615}, \ldots\right), \quad n = 0, 1, \ldots\]  
Equation (72)

The leading term is the correct perfect fluid value. The first subleading term represents the viscous correction. The most general form of the energy momentum tensor in conformal hydrodynamics [49] implies that
\[\frac{\eta}{s} = \frac{9}{4} g_1,\]  
Equation (73)

so in the theory considered here \(\eta/s = c/5\). This value is already known from other considerations [40–42]. It is also confirmed by comparing our results with the results of a direct computation of the gradient expansion of kinetic theory in the RTA [3].

It is interesting to compare the gradient expansion of AHYDRO to the underlying kinetic theory, as well as to the BRSSS theory. The leading terms of the gradient expansions are listed in Table I (we have set the coefficient \(c = 1\) for readability; it can be restored using the fact that \(g_k \sim c^k\)). For Bjorken flow BRSSS has effectively three parameters \(\tau, \eta, \lambda_1\) so one can adjust them to reproduce the exact computation in the RTA model up to second order.

The only free parameter of AHYDRO for conformal systems is a constant \(c\) fixing the viscosity to entropy density ratio. It appears in the same way in AHYDRO as in the underlying kinetic equation through Eq. (38) that fixes the relaxation time. The agreement of the first order terms in the gradient expansion for AHYDRO and the kinetic theory shows that AHYDRO (in the two considered versions) properly includes the effects of shear viscosity.

In the second order, the first version of AHYDRO misses the RTA results by a factor of two yielding \(8c^2/945\) instead of \(16c^2/945\). Very interestingly, the second version of AHYDRO reproduces exactly the RTA result. This gives further support for anisotropic hydrodynamics based on the anisotropic matching principle [13].

At third order however both MIS and AHYDRO depart from the kinetic theory results, but AHYDRO is significantly closer. Numerically BRSSS gives 30% of the RTA result, AHYDRO I gives 88%, while AHYDRO II gives 60%.

In Table II we show the leading coefficients of gradient expansions for the kinetic theory model and various hydrodynamic approaches (RTA, BRSSS, DNMR and MIS, respectively). It is interesting to observe that BRSSS and DNMR agree up to the second order. This is expected, since both BRSSS and DNMR have been constructed as consistent expansions. They both agree with RTA: the first because the parameters \(\tau, \eta, \lambda_1\) of BRSSS have been adjusted to reproduce the RTA result, the second because the kinetic coefficients used in (60) have been obtained directly from the RTA kinetic equation [19].

The MIS results differ from those obtained for RTA, BRSSS and DNMR already at the second order. Note that the old MIS theory has \(g_3\) of the wrong sign. This points to the importance of a nonzero value of the \(\lambda_1\) term in (59). The poorer performance of MIS compared to DNMR in the gradient expansion is similar to the situation described in [46]. In this work solutions of (61) and (60) were compared with the exact solutions of the kinetic theory indicating that the DNMR approach better reproduces the kinetic-theory results.

### IX. LARGE ORDER BEHAVIOR

It has recently become clear that large order behavior of gradient expansions contains important information about the non-hydrodynamic sector of the theory. This is the case both at the level of microscopic theories [3, 50] and at the level of hydrodynamics [18, 51]. Since hydrodynamics can be treated as an effective description of microscopic systems, one may aim not only to match the low orders of the gradient expansion, but also the large order behavior, which is tantamount to matching the non-hydrodynamic sectors. Of course, one may choose to refrain from using the hydrodynamic description when dependence on the non-hydrodynamic sector (the “regulator sector” in the language of Ref. [52]) is nontrivial, in which case only matching of the low orders would be important. In this section, however, we will examine the large order behavior of the formal gradient expansions of anisotropic hydrodynamics to determine what kind of non-hydro modes these theories contain.

Using the methods described in the previous section one may, with relatively modest effort, determine the coefficients \(g_n\) to order 140 in both models of anisotropic hydrodynamics considered in this paper. Examination of these coefficients shows that the series has vanishing radius of convergence, with \(g_n \sim n!\). For the series (72), this is illustrated in Fig. 1 below; the result for the series (71) is analogous.

As in [18, 51, 53], we will study the singularities of the analytic continuation of the Borel transform as a means to learn

| \(n\) | RTA | BRSSS | AHYDRO I | AHYDRO II |
|-------|-----|-------|----------|-----------|
| 0     | 2/3 | 2/3   | 2/3      | 2/3       |
| 1     | 4/45| 4/45  | 4/45     | 4/45      |
| 2     | 16/945| 16/945| 8/945    | 16/945    |
| 3     | −208/4725 | −1712/99225 | −184/4725 | −176/6615 |

**TABLE I.** Leading coefficients of gradient expansions for RTA, BRSSS, AHYDRO I and AHYDRO II.

| \(n\) | RTA | BRSSS | DNMR | MIS |
|-------|-----|-------|------|-----|
| 0     | 2/3 | 2/3   | 2/3  | 2/3 |
| 1     | 4/45| 4/45  | 4/45 | 4/45|
| 2     | 16/945| 16/945| 8/135| 8/135|
| 3     | −208/4725 | −1712/99225 | −304/33075 | 112/2025 |

**TABLE II.** Leading coefficients of gradient expansions for RTA, BRSSS, DNMR and MIS.
FIG. 1. The coefficients in Eq. (72) grow as $n!$

FIG. 2. Poles of the symmetric Pade approximant to the Borel transform of the series $\{g_n\}$ in Eq. (72).

about non-hydrodynamic modes of the theory. The Borel transform of $g$ is given by

$$g_B(\xi) = \sum_{n=0}^{\infty} \frac{g_n}{n!} \xi^n,$$  \hspace{1cm} (74)

and represents a series which has a finite radius of convergence. The analytic continuation of series (74), denoted by $\tilde{g}_B(\xi)$, would be needed to invert the Borel transform via the formula

$$g_R(w) = w \int_C \frac{d\xi}{2\pi i} e^{-w\xi} \tilde{g}_B(\xi),$$  \hspace{1cm} (75)

where $C$ denotes a contour in the complex plane connecting 0 and $\infty$. We perform the analytic continuation using diagonal Padé approximants of order 70. This function has a sequence of poles along the positive real axis, starting at $\xi_0 = 1.500$, which signals the presence of a cut originating at that point. A consequence of this is a “nonperturbative” ambiguity of the same kind as that seen in [18]. The implication of this is that the hydrodynamic series must be regarded as the lowest order element of a transseries. This line of reasoning can be continued as in Ref. [18] (see also [51, 53]). The main conclusion for our present purposes is however that the cut along the real axis indicates the presence of a single non-hydrodynamic mode, which is purely decaying, as in the BRSSS theory. The location of the start of the cut determines the rate of exponential decay to be $3/2c$.

X. SUMMARY

We have examined the gradient series solutions in two formulations of anisotropic hydrodynamics and compared them to similar results in BRSSS theory, as well as the gradient series for the model of kinetic theory in the relaxation time approximation which is the underlying microscopic theory for the anisotropic hydrodynamics considered here.

The gradient expansions in hydrodynamic theories are divergent, so their usefulness, apart from formal comparisons, lies in the fact that keeping only a few leading terms gives a reasonable approximation at late times. The theory of asymptotic series provides the concept of optimal truncation, which in the cases considered is of the order of a few terms. When comparing with a microscopic model, such as the kinetic theory under consideration here, it is interesting to ask how many terms in such an expansion should a hydrodynamic description aim to capture. A conservative point of view would be to assume that the first two orders are the most relevant, so that one should determine the coefficients which enter the BRSSS equations and use these. However, experience from the numerical study of Ref. [46] suggests that one can do better, and the evidence provided by the gradient expansions studied in this paper suggests that matching higher orders is perhaps indicative of better numerical performance. One should however keep in mind that a given initial condition typically involves non-hydrodynamic modes along with hydrodynamic ones, and while the decay of the former is exponential, they will affect early time evolution. Thus conclusions based on low orders of the gradient expansion are relevant only for sufficiently late times.

We have also studied the large-order behavior of the gradient series of anisotropic hydrodynamics, establishing that the series is divergent, as in the two cases studied previously [18, 51]. The singularities of the Borel transform indicate that this theory involves a single, purely decaying non-hydrodynamic mode, very much like what is found in MIS theory. This suggests that the pattern of attaining the hydrodynamic attractor should be similar in both cases.

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