Self-full ceers and the uniform join operator

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Abstract

A computably enumerable equivalence relation (ceer) $X$ is called self-full if whenever $f$ is a reduction of $X$ to $X$, then the range of $f$ intersects all $X$-equivalence classes. It is known that the infinite self-full ceers properly contain the dark ceers, i.e., the infinite ceers which do not admit an infinite computably enumerable transversal. Unlike the collection of dark ceers, which are closed under the operation of uniform join, we answer a question from [4] by showing that there are self-full ceers $X$ and $Y$ so that their uniform join $X \oplus Y$ is non-self-full. We then define and examine the hereditarily self-full ceers, which are the self-full ceers $X$ so that for any self-full $Y$, $X \oplus Y$ is also self-full: we show that they are closed under uniform join and that every non-universal degree in $\text{Ceers}^\infty$ have infinitely many incomparable hereditarily self-full strong minimal covers. In particular, every non-universal ceer is bounded by a hereditarily self-full ceer. Thus, the hereditarily self-full ceers form a properly intermediate class in between the dark ceers and the infinite self-full ceers, which is closed under $\oplus$.

Keywords: Computably enumerable equivalence relation, self-full equivalence relation, computable reducibility on equivalence relations

1 Introduction

We investigate the notion of self-fullness, which has turned out to be quite important in the study of computably enumerable equivalence relations (called ceers) under computable reducibility. We recall that if $R,S$ are equivalence relations on the set of natural numbers $\omega$, then $R$ is computably reducible (or, simply, reducible) to $S$ (notation: $R \leq S$) if there is a computable total function $f$ such that

$$\forall x, y [x R y \iff f(x) S f(y)].$$

In this case, we write $f : R \to S$. This reducibility (due to Ershov [6], see also [5]) has been widely exploited in recent years as a convenient tool for measuring the computational complexity of classification problems in computable mathematics. For instance (see [8]), the isomorphism relation for various familiar classes of computable groups is $\Sigma^0_1$-complete under $\leq$. Restricted to ceers, this reducibility has been used to study familiar equivalence relations from logic, such as the provable equivalence relation of sufficiently expressive formal systems (see [1] for a survey), word problems

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and isomorphism problems of finitely presented groups ([12, 13]), c.e. presentations of structures (see e.g. [9, 11]).

As a reducibility, ≤ gives rise in the usual way to a degree structure. Due to the importance of ceers within general equivalence relations on ω, a considerable amount of attention has been given to its substructure, called Ceers, consisting of the degrees of ceers. The systematic study of Ceers, a poset with a greatest element (usually called the universal element), was initiated by Gao and Gerdes [10]. Its algebraic structure, in particular its structure under joins and meets, was thoroughly investigated by Andrews and Sorbi [4], who have proposed a partition of the ceers into the three following classes: the finite ceers, i.e. the ceers with only finitely many equivalence classes; the light ceers, i.e. the ceers following classes: the finite ceers, i.e. the ceers with only finitely many equivalence classes; the dark ceers, i.e. the ceers which are neither finite nor light. These classes have been extensively investigated in relation to the existence or non-existence of joins and meets in the poset Ceers. For instance, no pair of incomparable degrees of dark ceers has join or meet. The classes of degrees corresponding to the classes of the above partition are first-order definable in Ceers in the language of posets.

Another class of ceers which has emerged in [4] consists of the self-full ceers, where a ceer R is self-full if and only if whenever a computable function f provides a reduction f : R → R, then range(f) intersects all R-equivalence classes. Equivalently, R is self-full if and only if f ⊕ Id ≤ R, where Id is the identity ceer; the dark ceers, i.e. the ceers which are neither finite nor light. These classes have been extensively investigated in relation to the existence or non-existence of joins and meets in the poset Ceers. For instance, no pair of incomparable degrees of dark ceers has join or meet. The classes of degrees corresponding to the classes of the above partition are first-order definable in Ceers in the language of posets.

The self-full ceers and their degrees play an important role in the theory of ceers. The following are few examples of the prominent role the self-full ceers have played in the theory of ceers. The existence in Ceers/I of infinitely many self-full strong minimal covers above any non-universal degree has been exploited in the recent paper [3] to show, among other things, that the first order theory of the degrees of light ceers is isomorphic to true first-order arithmetic. The self-full degrees form an automorphism base of the continuum many automorphisms of the poset Ceers [4,Corollary 11.5]. The degrees of self-full ceers are first-order definable in Ceers in the language of posets. They coincide exactly with the non-universal meet-irreducible degrees [4,Theorem 7.8].

The three classes of the partition of ceers (finite, light and dark) introduced in [4] are closed under the operation of uniform join. Closure under ⊕ is an important issue for a class of ceers, not only as regards the investigation of the existence of infima or suprema of the degrees of the ceers in the class but also because uniform joins correspond to coproducts in the category of equivalence relations. If R, S are equivalence relations, then a morphism μ : R → S, from R to S, is a function mapping R-equivalence classes to S-equivalence classes, for which there is a computable function f such that for every x, μ maps the R-equivalence class of x to the S-equivalence class of f(x). So a class of ceers is closed under ⊕ if and only if the corresponding full subcategory of ceers is closed under coproducts of any pair of objects. The category-theoretic approach to numberings and equivalence
relations is due to Ershov, see in particular [7]. Since reductions correspond to monomorphisms, by [4, Lemma 1.1] the category-theoretic jargon allows for yet another characterization of the self-full ceers, namely a ceer $R$ is self-full if and only if every monomorphism $\mu : R \to R$ is an isomorphism.

A natural question is therefore whether or not the self-full ceers are closed under $\oplus$ (see [4, Question 1]). We answer this question (Theorem 2.1) by showing that there are self-full ceers $X, Y$ such that $X \oplus Y$ is not self-full. Notice that both $X$ and $Y$ must be infinite since $F \oplus X$ is self-full whenever $F$ is finite and $X$ is self-full [4, Corollary 4.3].

Motivated by the fact that the uniform join of two self-full ceers need not be self-full, in the last section of the paper, we introduce the notion of a ceer $X$ being hereditarily self-full, i.e. $X \oplus Y$ is self-full whenever $Y$ is self-full. Let $\text{HSF}$ be the class of hereditarily self-full ceers. It follows from [4, Corollary 4.3] that $\text{HSF}$ contains the finite ceers. On the other hand, we show in Corollary 3.9 that all dark ceers are hereditarily self-full.

In this section, we show that the collection of self-full ceers is not closed under uniform join operator $\oplus$.

**THEOREM 2.1**

There are self-full ceers $X$ and $Y$ so that $X \oplus Y$ is non-self-full.

**PROOF.** We construct a ceer $Z_s$, which will be $X \oplus Y$ for ceers $X$ and $Y$. We let $X$ refer to $Z \upharpoonright \text{Evens}$ (i.e. $x \in X$ if and only if there are $Z_1 \leq Z_2$ with $Z_1 \oplus Z_2 = Y$) and $Y$ refer to $Z \upharpoonright \text{Odds}$ (i.e. $x \in Y$ if and only if there are $Z_1 \leq Z_2$ with $(2x+1) \leq Z_2$). We enumerate the ceer $Z$ in stages. At stage $s$, we define a ceer $Z_s$ so that $Z = \bigcup_s Z_s$ is our desired final ceer. We build $Z_{s+1}$ extending $Z_s$ by the collapsing technique described at the end of Section 1. Finally, we let $Z_{s+1} = Z_s \upharpoonright \text{Evens}$, and similarly $Y_{s+1} = Z_s \upharpoonright \text{Odds}$. $X$-collapsing $k, k'$ at any stage means that we $Z$-collapse $2k$ and $2k'$. Similarly, $Y$-collapsing $k$ and $k'$ means that we $Z$-collapse $2k + 1$ and $2k' + 1$.

## 2 Self-fullness is not closed under uniform join

In this section, we show that the collection of self-full ceers is not closed under $\oplus$.

**THEOREM 2.1**

There are self-full ceers $X$ and $Y$ so that $X \oplus Y$ is non-self-full.
We also construct a reduction function \( f : Z \rightarrow Z \). In addition to ensuring that \( f \) is a reduction and that its image omits the \( Z \)-class of 0, we have the following requirements:

\[
SF_{f,k}X : \begin{align*}
\text{if } \varphi_j \text{ is a reduction of } X \text{ to } X', \text{ then } [k]_X \text{ intersects the range of } \varphi_j. \\
\text{if } \varphi_j \text{ is a reduction of } Y \text{ to } Y', \text{ then } [k]_Y \text{ intersects the range of } \varphi_j.
\end{align*}
\]

We will often say that a number \( z \) is in \( X \) if \( z \) is even and \( z \) is in \( Y \) if \( z \) is odd. We will also say that some numbers will be \( X \)-bound and other numbers may be \( Y \)-bound. Namely, when we say that a number \( u \) is \( X \)-bound, this means that if \( l \) is the greatest number such that \( f^{(l)}(u) \) has been already defined then as we choose further \( f \)-images of the number, i.e. we choose \( f^{(m)}(u) \) for \( m > l \), we will choose from \( X \) (in other words, the image will be an even number). When we say that \( u \) is \( Y \)-bound, then we will choose from \( Y \) (in other words, the image will be an odd number).

We describe a module called \text{DiagonalizationModule}(x)\) that we will use to ensure that \( \varphi_j \) is not a reduction of \( X \) to \( X \). We call this module (at some stage \( s \)) when we have an element \( x \) such that \( 2x \) is \( Y \)-bound and we have \( \varphi_j(x) \downarrow \) and \( \varphi_j(x) \not \rightarrow Xs \). We may also use this module to ensure that \( \varphi_j \) is not a reduction of \( Y \) to \( Y \) if we have an element \( y \) such that \( 2y + 1 \) is \( X \)-bound and we have \( \varphi_j(y) \downarrow \) and \( \varphi_j(y) \not \rightarrow Y \). Though we discuss the diagonalization module for \( X \), and the strategy for \( SF_{f,k}X \), everything is symmetric for \( Y \).

**Remark 2.2**

When dealing with \( SF_{f,k}X \)-requirements, we let \( \hat{\varphi}(u) = \frac{f(2u)}{2} \), if \( f(2u) \in X \). We define \( \hat{f}^{(n)}(u) = v \) if and only if \( f^{(n)}(2u) = 2v \). Notice that we abuse the notation in that \( \hat{f}^{(n)} \) may properly contain the \( n \)th iterate of the function \( \hat{\varphi} \). If \( \hat{f}^{(n)}(u) \) and \( \hat{f}^{(n)}(v) \) are both defined, then it is easy to see, using that \( f : X \oplus Y \rightarrow X \oplus Y \) is a reduction, that \( u \equiv X \equiv v \) if and only if \( \hat{f}^{(n)}(u) \equiv X \equiv \hat{f}^{(n)}(v) \). Symmetrically, when dealing with \( SF_{f,k}Y \)-requirements, we let \( \hat{\varphi}(u) = \frac{f(2u+1)}{2} \). In this case \( \hat{f}^{(n)}(u) = v \) if and only if \( f^{(n)}(2u + 1) = 2v + 1 \).

**Informal description of DiagonalizationModule(\( x \))**

**Goal 1.** The first goal of the module is to find some element \( z \) (which might not be \( x \)) so \( 2z \) is \( Y \)-bound and a stage \( t > s \) so that for every \( n \) such that \( f^{(n)}(z) \) is defined, \( \varphi_j(z) \not \rightarrow f^{(n)}(z) \). More precisely, let \( S \) be the set of \( n \) for which, by stage \( s \), \( f^{(n)}(2x) \) is determined and in \( X \). Notice that no other \( f^{(m)}(2x) \) is in \( X \) (since \( 2x \) is \( Y \)-bound so all later choices of \( f^{(m)}(2x) \) will be odd). If \( \varphi_j(z) \not \rightarrow f^{(n)}(x) \) for any \( n \in S \), then we let \( z = x \), and we move to Goal 2.

Otherwise, wait for a stage \( t > s \) such that at this stage \( \varphi_j(\hat{f}^{(n)}(x)) \) converges for each \( n \in S \). As we will argue in Lemma 2.11, either we see at this point that \( \varphi_j \) is not a reduction on the elements \( \{\hat{f}^{(n)}(x) : n \in S\} \), or we see that there is some \( m \in S \), \( m > 0 \), such that, taking \( z = \hat{f}^{(m)}(x) \), we have that \( \varphi_j(z) \not \rightarrow f^{(n)}(z) \), for every \( n \) so that \( f^{(n)}(z) \) is defined. After this we move to Goal 2.

**Goal 2.** We are given a \( z \) so that \( 2z \) is \( Y \)-bound and so that \( \varphi_j(z) \) converges by stage \( t \) and is not \( X \)-equivalent to any \( \hat{f}^{(n)}(z) \) which is already defined. We consider a new element \( w \) and wait for a stage \( s' > t \) at which \( \varphi_j(w) \) converges. In the meantime, we ensure that \( f^{(n)}(2w) \) is in \( X \) if and only if \( f^{(n)}(2z) \) is in \( X \). The fact that \( w \) is new will ensure that the class of \( 2w \) does not intersect the range of \( f \). Thus, we may later \( X \)-collapse \( w \) with \( z \) (by \( Z \)-collapsing \( 2w \) with \( 2z \)) if we so wish. Once \( \varphi_j(w) \) converges at \( s' \), if \( \varphi_j(w) \not \rightarrow \varphi_j(z) \), then we \( X \)-collapse \( w \) with \( z \) (i.e. we \( Z \)-collapse \( 2w \) with \( 2z \)). Since \( \varphi_j(z) \) is not \( X \)-equivalent to any element \( \hat{f}^{(n)}(z) \) and the class of \( 2w \) does not intersect the image.
of $f$, this does not cause $\psi_j(z)$ to $X$-collapse with $\psi_j(w)$. Indeed, the $X$-collapse of $w$ with $z$ entails only (as $f$ is a reduction from $Z$ to $Z$) the $Z$-collapse of each $f^{(n)}(2z)$ with $f^{(n)}(2w)$, i.e. the $X$-collapse of $f^{(n)}(z)$ with $f^{(n)}(w)$. Therefore, $\psi_j(z)$ is not $X$-collapsed to any element as a consequence of the execution of Goal 2.

We have thus diagonalized to ensure that $\psi_j$ is not a reduction of $X$ to $X$.

We now describe the strategy to satisfy $SF^X_{j,k}$. The strategy to satisfy $SF^Y_{j,k}$ is symmetric.

**Informal description of the $SF^X_{j,k}$-strategy**  
First, suppose that $2k$ is either not bound by a higher-priority requirement or is $Y$-bound by a higher-priority requirement. If it is not bound by a higher-priority requirement, then we make $2k$ $Y$-bound. When the strategy is called at stage $s$, we may have already defined $f(2k), \ldots, f^{(l)}(2k)$ for some $l$. This means that we will choose $f^{(m)}(2k)$ to be in $Y$ (i.e. odd) for $m > l$. Next, we wait for $\psi_j(k)$ to converge, at, say, stage $s$. If $\psi_j(k)$ is in $X$, we make $2k$ $X$-bound. Next, we wait for $\psi_j(k')$ to converge, at, say, stage $s$. If $\psi_j(k')$ converges and is already $X$-equivalent to $k'$, then we $X$-collapse $k'$ with $k$ and we are done. Otherwise, $\psi_j(k')$ converges and is not $X$-equivalent to $k'$. In this case, we determine that $2k'$ is $Y$-bound and call DiagonalizationModule($k'$) to ensure that $\psi_j$ is not a reduction of $X$ to $X$.

Next, we consider the case where $2k$ is $X$-bound by a higher-priority requirement, thus we cannot use the previous strategy. We begin by choosing a new element $k'$, and we ensure that if we have defined $f(2k), \ldots, f^{(l)}(2k)$ already, then we define $f^{(m)}(2k')$ for $m \leq l$ to be in $X$ if and only if $f^{(m)}(2k)$ is in $X$, and we make $2k'$ $X$-bound. Next, we wait for $\psi_j(k')$ to converge, at, say, stage $s$. If $\psi_j(k')$ converges and is already $X$-equivalent to $k'$, then we $X$-collapse $k'$ with $k$ and we are done. Otherwise, $\psi_j(k')$ converges and is not $X$-equivalent to $k'$. In this case, we determine that $2k'$ is $Y$-bound and call DiagonalizationModule($k'$) to ensure that $\psi_j$ is not a reduction of $X$ to $X$.

**Construction:** As anticipated at the beginning of the proof, we enumerate our desired ceer $Z$ in stages, so that at stage $s$, $Z_s$ will be a ceer, and we let $X_s$ be $Z_s \mid$ Evens and let $Y_s$ be $Z_s \mid$ Odds.

We say that a requirement $SF^X_{j,k}$ or $SF^Y_{j,k}$ requires attention at stage $s$ if it has not acted since it was last initialized, or if it sees some computations $\psi_j(u)$ converge for some numbers $u$, and it had been waiting for these computations to converge. Requirements may determine that numbers are $X$-bound or $Y$-bound. When we write that a strategy makes a number $k$ $X$-bound or $Y$-bound, it simultaneously makes $f^{(n)}(k)$ $X$-bound (or $Y$-bound) for every $n$ so that $f^{(n)}(k)$ is already defined. If no active strategy is making a number $X$-bound or $Y$-bound, then we say that the number is free. A number is new at stage $s + 1$, if it is bigger than $s + 1$ and none of its $X_{s-}$, $Y_{s-}$ or $Z_s$-equivalence classes contains any number so far used in the construction.

We initialize a requirement $SF^X_{j,k}$ or $SF^Y_{j,k}$ by reverting it back to the beginning of its strategy, i.e. to the initial distinction between Case 1 and Case 2 described below.

Stage 0. All requirements are initialized. Let $Z_0 = \text{Id}$.

Stage $s + 1$. Let $SF^X_{j,k}$ (or $SF^Y_{j,k}$, for which we act symmetrically, by just replacing $X$ with $Y$ and the even numbers with the odd numbers) be the highest-priority requirement which requires attention at stage $s + 1$. Since almost all requirements are initialized, there is such a requirement. Then, we re-initialize all requirements of lower-priority and act as follows:

- The requirement requires attention because it is initialized. We distinguish the following two cases:

  **Case 1.** The number $2k$ is free or $Y$-bound. If $2k$ is currently free, then we make $2k$ $Y$-bound. Wait for a stage $t > s + 1$ where $\psi_j(k)$ converges and $\psi_j(k) \not\in X_k$. We say that the strategy has entered the **Case 1-waiting outcome**.

  **Case 2** The number $2k$ is $X$-bound.
Step 0 of Case 2. Let \( k' \) be a new element. Let \( n \) be greatest so that \( f^{(n)}(2k) \) is already defined. Define \( f^{(m)}(2k') \) to be new elements for each \( m \leq n \) ensuring that \( f^{(m)}(2k') \) is in \( X \) if and only if \( f^{(m)}(2k) \) is in \( X \). Make \( 2k' \) \( X \)-bound. Wait for \( \varphi_j(k') \) to converge. We say that the strategy has entered the Case 2 Step 0-waiting outcome.

The requirement requires attention because it was in the Case 1-waiting outcome or in the Case 2 Step 0-waiting outcome, and now the awaited computations have converged. If it was in the Case 1-waiting outcome, then we will return to Step 1 of DiagonalizationModule(x) (as described below) with \( x := k \). If it was in the Case 2 Step 0-waiting outcome, then it will return to Case 2 Step 1.

Step 1 of Case 2. If \( \varphi_j(k') \) \( X_s \ k' \), then \( X_{s+1} \)-collapse \( k' \) with \( k \). Declare the requirement satisfied. Otherwise, make \( 2k' \) \( Y \)-bound and go to Step 1 of DiagonalizationModule(x) (as described below) with \( x := k' \).

If the requirement requires attention because it was in the DiagonalizationModule(x) Step 1-waiting outcome, and now the awaited computation has converged, then it will return to Step 2 of DiagonalizationModule(x):

If the requirement requires attention because it was in the DiagonalizationModule(x) Step 3(z)-waiting outcome, and now the awaited computation has converged then it will return to Step 4(z, w) of DiagonalizationModule(x).

\( \text{DiagonalizationModule}(x) \) (Called at \( s + 1 \) for a \( Y \)-bound 2x, with \( \varphi_j(x) \downarrow \) and \( \varphi_j(x) \not\in X_s \))

Step 1. We define \( S \) the set of \( n \) so that \( f^{(n)}(2x) \) is defined by stage \( s \) and is in \( X \). Notice that since \( 2x \) is \( Y \)-bound, then we will not define any \( f^{(m)}(2x) \) to be in \( X \) at a later stage. If \( \varphi_j(x) \not\in X^{f^{(n)}(x)} \) for each \( n \in S \), then we go to Step 3(x). Otherwise, wait for \( \varphi_j(f^{(n)}(x)) \) to converge for every \( n \in S \). We say the module has entered the DiagonalizationModule Step 1-waiting outcome.

Step 2. If the wait for the computations \( \varphi_j(f^{(n)}(x)) \) in Step 1 to converge is over, then pick some \( z = f^{(n)}(x) \) for \( n \in S \) so that \( \varphi_j(z) \not\in X^{f^{(m)}(z)} \) for every \( m \) such that \( f^{(m)}(2z) \) is determined before stage \( s \) and is in \( X \). Then, go to Step 3(z). We will argue below (in the proof of Lemma 2.11) that either the requirement is satisfied (i.e. we see some \( n \) so that \( \varphi_j(n) X k \), or \( \varphi_j \) is not a reduction of \( X \) to \( X \)) or that such a \( z \) must exist. Note that since \( 2z = f^{(n)}(2x) \), \( 2z \) is also \( Y \)-bound.

Step 3(z). Take a new element \( w \). For every \( m \) for which \( f^{(m)}(2z) \) is defined, define \( f^{(m)}(2w) \) to be a new number so that \( f^{(m)}(2w) \) is in \( X \) if and only if \( f^{(m)}(2z) \) is in \( X \). Make \( 2w \) be \( Y \)-bound (since \( 2z \) is \( Y \)-bound this preserves our ability to \( X \)-collapse \( w \) with \( z \)). Wait for \( \varphi_j(w) \) to converge. We say the module has entered the DiagonalizationModule Step 3(z)-waiting outcome.

Step 4(z, w). If the wait for the computation \( \varphi_j(w) \) in Step 3(z) to converge is over, then we have \( \varphi_j(z) \) and \( \varphi_j(w) \) both converged by stage \( s \). If \( \varphi_j(z) \not\in X_s \varphi_j(w) \), then we do nothing and declare the requirement satisfied. Otherwise, we collapse \( z \) \( X_{s+1} \) \( w \) and declare the requirement satisfied.

We choose the first number \( v \) on which we have not defined \( f(v) \). If \( v \) is \( X \)-bound, we take a new number \( x \) in \( X \) and define \( f(v) = x \). Otherwise, we take a new number \( y \) in \( Y \) and define \( f(v) = y \). If \( v \) was \( X \)-bound (or \( Y \)-bound), then we also make \( f(v) \) be \( X \)-bound (or \( Y \)-bound).

Let \( Z_{s+1}^0 \) be the equivalence relation generated by \( Z_s \) along with any pairs that we have decided to \( Z \)-collapse during this stage.

Lastly, we do further collapses in order to ensure \( f \) is a reduction. That is, we let \( Z_{s+1} \) be the result of closing \( Z_{s+1}^0 \) under the implication: If \( z \in Z_{s+1} \) \( w \) and \( f(z) \) and \( f(w) \) are defined, then \( f(z) Z_{s+1} f(w) \).
Note that since \( f \) is only defined on finitely many values, this causes finitely much further collapse. We then stop the stage and go to stage \( s + 2 \).

**Verification:** We will adopt the following notation throughout the verification: given a parameter \( x \) chosen by an \( SF \)-strategy \( R \), we will write \( \tilde{x} = 2x \) if \( R = SF_{j,k}X \) for some \( j, k \) and \( \tilde{x} = 2x + 1 \) if \( R = SF_{j,k}Y \) for some \( j, k \).

**Lemma 2.3**
Each requirement is re-initialized only finitely often.

**Proof.** Each requirement can act to re-initialize lower-priority requirements, without being re-initialized itself, at most finitely often. \( \square \)

In the following definition, an \( SF \)-strategy or a DiagonalizationModule module is called active for a parameter \( x \) if the module has already chosen this parameter and is waiting for a computation involving this parameter to converge.

**Definition 2.4**
A time during the construction is either the beginning of a stage or immediately follows any \( Z \)-collapse or the assignment of any new parameter or unassignment (via initialization) of any parameters.

If \( s \) is a stage, we abuse the notation letting \( s \) also represent the time that is the beginning of stage \( s + 1 \). Note that this agrees with the previous definition.

At each time \( t \) in the construction, let \( D_t \) be the set of elements \( \tilde{w} \) which refer to parameters \( w \) for an active DiagonalizationModule which is waiting in Step 3 (\( z \)).

For each time \( t \), let \( E_t \) be the set of elements \( \tilde{k}' \) which refer to parameters \( k' \) for active \( SF \)-strategies in Case 2 Step 0.

For each time \( t \), let \( I_t \) be the set of elements which are in the image of \( f \) at time \( t \).

For a time \( t \), we let \( Z_t \) be the ceer \( Z \) as it appears at time \( t \). That is, if \( t \) is during stage \( s + 1 \), \( Z_t \) is the equivalence relation generated by \( Z_s \) and any \( Z \)-collapse so far done during stage \( s + 1 \). Note that if \( s \) is a stage, this definition of \( Z_s \) agrees with the previous definition.

We say that a number \( x \) is new at time \( t \) (a time during stage \( s + 1 \)) if \( x > s + 1 \) and none of its \( X_s \)-, \( Y_s \)- or \( Z_s \)-equivalence classes contain any number so far used in the construction.

**Observation 2.5**
At each time \( t \), the sets \( D_t, E_t \) and \( I_t \) are disjoint.

**Proof.** Each of the three types of parameters: \( w \) for an active DiagonalizationModule which is in Step 3 (\( z \)), \( k' \) for active \( SF \)-strategies in Case 2 Step 0, and elements \( f(n) \), are chosen to be new. Thus, it is impossible that the same number is in two of these sets. \( \square \)

**Lemma 2.6**
For every time \( t \), if \( x, y \in D_t \cup E_t \cup I_t \) and \( x Z_t y \), then \( x = f(n) \) and \( y = f(m) \) and \( n Z_t m \) for some \( n, m \) where \( f(n) \) and \( f(m) \) are defined at time \( t \).

For any times \( t \leq t' \) (i.e. \( t \) occurs before \( t' \) in the construction): if \( x \) is a number which is not new at time \( t \) and \( x \) is not \( Z_t \)-equivalent to any member of \( D_t \cup E_t \cup I_t \), then \( x \) is not \( Z_{t'} \)-equivalent to any member of \( D_{t'} \cup E_{t'} \cup I_{t'} \).
PROOF. We prove both claims by simultaneous induction on times $s$. We assume the claims hold for all times $t, t' \leq s$. Let $s'$ be the next time (i.e. exactly one action happens between $s$ and $s'$). We verify that they still hold where $t, t'$ are assumed to be times $\leq s'$. The claims clearly hold for $s$ being the beginning of the construction (i.e. $Z_s = \text{Id}$ and no parameters have been chosen). Exactly one action takes place between times $s$ and $s'$. There are three possibilities for this action:

- We may remove parameters via re-initialization or choose a new parameter which enters $D_s' \cup E_s' \cup I_s'$.
- We can cause $Z$-collapse to satisfy a requirement. This happens either between $\tilde{k}$ and $\tilde{k}'$ in Case 2 Step 1 of a $SF^X_{j,k}$- or $SF^Y_{j,k}$-strategy or between $\tilde{w}$ and $\tilde{z}$ in Step 4($z, w$) of a DiagonalizationModule.
- We may $Z$-collapse $f(z)$ with $f(w)$ at the end of the stage because we have already $Z$-collapsed $z$ with $w$.

We first consider removing or choosing new parameters. Removing parameters clearly maintain both claims since there are strictly fewer active numbers for the claims to hold for. Since all new parameters are chosen to be new numbers, their classes are singletons. Thus, both claims are maintained by assigning new parameters.

We next consider $Z$-collapse between $\tilde{k}$ and $\tilde{k}'$ in Case 2 Step 1 of a $SF^X_{j,k}$- or $SF^Y_{j,k}$-strategy. By the first claim of the inductive hypothesis, before this collapse $\tilde{k}'$ was the only member of its $Z_s$-class which was in $D_s \cup E_s \cup I_s$. Since we make this collapse and simultaneously make the requirement satisfied (i.e. inactive), $\tilde{k}'$ is not in $D_s' \cup E_s' \cup I_s'$. After this action, thus, to the class of $\tilde{k}$, we have added no member of $D_s' \cup E_s' \cup I_s'$. We also note that the collection of numbers which are now equivalent to active numbers is a subset of the numbers that were equivalent to active numbers before the collapse. Thus, this action did not make either of the claims false.

Next, we consider $Z$-collapse between $\tilde{w}$ and $\tilde{z}$ in Step 4($z, w$) of a DiagonalizationModule. Once again, $\tilde{w}$ is no longer active after this collapse since the strategy is now satisfied and no longer active. By the first claim of the inductive hypothesis, before this collapse $\tilde{w}$ was the only member of its $Z_s$-class which was in $D_s \cup E_s \cup I_s$. As in the previous case, this maintains both claims.

Finally, we consider what happens when we collapse $f(z)$ with $f(w)$ once we have already collapsed $z$ with $w$. Suppose this caused $Z_s$-collapse between $x$ and $y$ for $x, y \in D_s' \cup E_s' \cup I_s'$. Note that this action does not change the values of these sets, so $x, y \in D_s \cup E_s \cup I_s$. By possibly re-naming $x$ and $y$, we may assume $x Z_s f(z)$ and $y Z_s f(w)$. By inductive hypothesis, this is only possible if $x = f(n)$ and $y = f(m)$ where $n Z_s z$ and $m Z_s w$. So we see that $n Z_s z$ and $m Z_s w$ and the first claim is preserved. Similarly, the second claim is preserved since we did not change the $Z$-closure of the set $D_s' \cup E_s' \cup I_s' = D_s \cup E_s \cup I_s$.

\[\square\]

**Lemma 2.7**

If $x$ is even and $y$ is odd, then we never $Z$-collapse $x$ with $y$. That is, $Z = X \oplus Y$.

**Proof.** We cause collapse in two cases: the first is $Z$-collapsing $\tilde{k}'$ with $\tilde{k}$ in Case 2 Step 1. In this case, since we defined $f$ on $\tilde{k}'$ so that $f^{(m)}(\tilde{k}')$ is in $X$ if and only if $f^{(m)}(\tilde{k})$ is in $X$, and both $\tilde{k}$ and $\tilde{k}'$ were $X$-bound, or both $Y$-bound, when we $Z$-collapse $\tilde{k}$ with $\tilde{k}'$ (even after closing under making $f$ a reduction), we do not cause any $Z$-collapses between even and odd numbers.

The second case is $Z$-collapsing $\tilde{z}$ with $\tilde{w}$ in Step 4($z, w$). In this case, we have made $\tilde{z}$ $Z$-collapse with $\tilde{w}$ where we have ensured that $f^{(m)}(\tilde{w})$ is in $X$ if and only if $f^{(m)}(\tilde{z})$ is in $X$ and then we made both $\tilde{z}$ and $\tilde{w}$ be $X$-bound or both be $Y$-bound. Thus, we see that we do not cause any $Z$-collapse between even and odd numbers. \[\square\]
**Lemma 2.8**

\( f \) is a reduction of \( X \oplus Y \) to \( X \oplus Y \) and \([0]_{X \oplus Y} \) is not in the range of \( f \).

**Proof.** Since whenever we have \( zZ_t w \), we also cause \( f(z)Z_t f(w) \), it remains to see that if \( f(z)Z_t f(w) \), then we also have \( zZ_t w \). This follows from the first claim of Lemma 2.6.

By the second claim in Lemma 2.6 and the fact that 0 is not new even at stage 0, 0 never becomes \( Z \)-equivalent to a number in the image of \( f \). \( \square \)

**Lemma 2.9**

Suppose \( x \) and \( y \) are mentioned by a strategy \( R \) at stage \( s \) and \( xZ_t y \). Suppose that \( t > s \) is a stage so that the strategy \( R \) has not been re-initialized or acted at any stage between \( s \) and \( t \). Then \( xZ_t y \).

**Proof.** It suffices to show that no lower-priority requirement can cause a collapse which will make \( x \) \( Z \)-collapse with \( y \). We begin by showing that any number which is not new at stage \( s \) (such as \( x \) or \( y \)) does not become \( Z_t \)-equivalent to parameters \( \tilde{k}' \) for a strategy of lower-priority than \( R \) in Case 2 Step 0 or parameters \( \tilde{w} \) for a DiagonalizationModule for a strategy of lower-priority than \( R \) which is waiting in Step 3(\( z \)) for any time \( s < t' \leq t \). Similarly, if \( a \) is not new at stage \( s \) and is not \( Z_t \)-equivalent to a member of \( I_s \), then \( a \) is not \( Z_t \)-equivalent to a member of \( I_t \) for any time \( s < t' \leq t \).

Let \( t' \) be a time so that \( s < t' \leq t \), and let \( a \) be not new at stage \( s \). At stage \( s \), all lower-priority requirements are re-initialized. Thus, either \( a \) is not \( Z_t \)-equivalent to a member of \( D_s \cup E_s \cup I_s \), so the second claim of Lemma 2.6 ensures that it is not \( Z_t \)-equivalent to a member of \( D_t \cup E_t \cup I_t \), or it is \( Z_t \)-equivalent to a member of \( D_s \cup E_s \cup I_s \).

If \( a \) is \( Z_t \)-equivalent to a member \( u \in D_s \cup E_s \), then \( u \) is a parameter for a requirement of priority at least as high as \( R \), which does not act or get re-initialized before stage \( s \), so \( u \in D_t \cup E_t \). Similarly, if \( u \in I_s \), then \( u \in I_t \), since \( I_s \subseteq I_t \). The first claim of Lemma 2.6 shows that \( u \) is not \( Z_t \)-equivalent to parameters \( \tilde{k}' \) for a lower-priority requirement in Case 2 Step 0, or parameters \( \tilde{w} \) for a lower-priority DiagonalizationModule which is waiting in Step 3(\( z \)). And if \( u \in D_s \cup E_s \), the first claim of Lemma 2.6 also shows that \( u \) is not \( Z_t \)-equivalent to a member of \( I_t \). We conclude that any number \( a \) which is not new at stage \( s \) is never equivalent to a parameter \( \tilde{k}' \) for a lower-priority requirement in Case 2 Step 0 or parameters \( \tilde{w} \) for a lower-priority DiagonalizationModule which is waiting in Step 3(\( z \)) at any \( t' \) between \( s \) and \( t \). Further, if \( a \) is not \( Z_t \)-equivalent to a member of \( I_s \), then \( a \) is not \( Z_t \)-equivalent to a member of \( I_t \).

We define a unique finite sequence of classes \([a_0]_{Z_t}, \ldots, [a_n]_{Z_t}\) as follows. We let \( a_0 = x \). Having defined a class \([a_i]_{Z_t}\), we let the class \([a_{i+1}]_{Z_t}\) be the (unique by the first claim in Lemma 2.6) class of a number \( u \) so that \( f(u) \in [a_i]_{Z_t} \). If no such class exists, we simply end the finite sequence. We similarly define a sequence of classes \([b_1]_{Z_t}, \ldots, [b_m]_{Z_t}\) beginning with \( b_0 = y \).

**Claim 2.10** Let \( t' \) be a time so \( s < t' \leq t \). Suppose that \( c \) is a number so that \( f^{(k)}(c) \) is defined at time \( t' \) and is \( Z_{t'} \)-equivalent to \( x \). Then \( c \) \( Z_t \) \( a_i \) for some \( i \). Similarly for \( y \) and the \( b_j \).

**Proof.** We distinguish between two cases. Case 1: \( k > n \). Then by the first claim of Lemma 2.6 applied \( n \) times, \( f^{k-n}(c) Z_t a_n \). But \( a_n \) is not new at stage \( s \) and is not \( Z_t \)-equivalent to a member of \( I_s \), so it is not \( Z_{t'} \)-equivalent to a member of \( I_{t'} \). This is a contradiction. Case 2: \( k \leq n \). Then applying the first claim of Lemma 2.6 \( k \) times, we see that \( c \) \( Z_{t'} \) \( a_k \). \( \square \)

Since no member of \([a_i]_{Z_t}\) is new at stage \( s \) for any \( i \leq n \), we know that they cannot be \( Z_{t'} \)-equivalent to a parameter \( \tilde{k}' \) for a lower-priority requirement in Case 2 Step 0 or parameters \( \tilde{w} \) for a lower-priority DiagonalizationModule which is waiting in Step 3(\( z \)) at any \( t' \) between \( s \) and \( t \).
Similarly for the \([b_j]_\mathcal{Z}\). This, along with the previous claim, shows that whenever a lower-priority requirement acts, it cannot collapse the classes of any of the higher-priority requirements. In particular, even after we ensure that \(f\) is a reduction (i.e. we collapse \(f(z)\) with \(f(w)\) if we have already collapsed \(z\) with \(w\)), we will not collapse \(x\) with \(y\).

**Lemma 2.11**
Each requirement is satisfied, so both \(X\) and \(Y\) are self-full.

**Proof.** Let us consider a requirement \(S\mathcal{E}^Xf^Y_{j,k}\). By Lemma 2.3, we can fix a stage \(s\) to be the last time this strategy will ever be re-initialized. After stage \(s\), when it next chooses parameters \(k', z\) and \(w\), these choices are permanent. If we are in Case 1, then we either have \(\varphi_j(k)\) diverges, or it converges and \(\varphi_j(k) X k\), or we go to DiagonalizationModule\((k)\). In the first two cases, we have satisfied the requirement because either \(\varphi_j\) is not total or \([k]_\mathcal{X}\) intersects the image of \(\varphi_j\). We consider the last case below. Similarly, if we are in Case 2, then we either have \(\varphi_j(k')\) diverge or \(\varphi_j(k') X s k\) at the next stage \(s\) when the requirement acts, in which case we cause \(k' X k\), or we enter DiagonalizationModule\((k')\).

In the first two cases, the requirement is satisfied because again either \(\varphi_j\) is not total or \([k]_\mathcal{X}\) intersects the image of \(\varphi_j\).

So, we can suppose that a DiagonalizationModule begun. If we wait in Step 1 forever, this guarantees that \(\varphi_j\) is not total and the requirement is satisfied. Otherwise, at some stage \(s + 1\) we begin Step 2. We claim that in Step 2, either the requirement is satisfied or we must succeed in picking some \(z\). Consider the finite set of \(X\)-classes \(F = \{[\hat{f}^{(n)}(x)]_\mathcal{X} \mid n \in S\}\). If \(\varphi_j\) is a reduction of \(X\) to \(\mathcal{X}\), then it must be injective on the classes in \(\mathcal{X}\). So, it either sends some class in \(\mathcal{X}\) to a class outside of \(F\) or it is surjective on \(\mathcal{X}\). In the latter case, \([x]_\mathcal{X} = [\hat{f}^{(0)}(x)]_\mathcal{X}\) intersects the range of \(\varphi_j\) and the requirement is satisfied. In the former case, there is some \(n \in S\) so that \(\varphi_j(\hat{f}^{(n)}(x)) X \hat{f}^{(m)}(x)\) for every \(m\). Thus, we will find the needed \(z = \hat{f}^{(n)}(x)\). Thus, we begin Step 3. If we are stuck in Step 3, then \(\varphi_j\) is not total. If we get to Step 4(\(z, w\)) at say stage \(s' + 1\), we have two cases to consider: if \(\varphi_j(z) X \varphi_j(w)\), then we do not \(X\)-collapse \(z\) with \(w\). This guarantees that \(z X t w\) for all \(t > s'\) by Lemma 2.9, therefore \(\varphi_j\) is not a reduction. On the other hand, if we have \(\varphi_j(z) X \varphi_j(w)\), then we \(X\)-collapse \(z X s' + 1 w\), which does not cause \(\varphi_j(z) X s' + 1 \varphi_j(w)\). By Lemma 2.9, we have \(\varphi_j(z) X \varphi_j(w)\), and \(\varphi_j\) is not a reduction.

The proof is now complete. \(\square\)

### 3 The hereditarily self-full ceers

We introduce the hereditarily self-full ceers and show that they properly contain the dark ceers.

**Definition 3.1**
A ceer \(X\) is hereditarily self-full if whenever \(Y\) is self-full, \(X \oplus Y\) is self-full.

Notice that by [4,Corollary 4.3] every finite ceer is hereditarily self-full.

The next definition highlights a property, which, when accompanied by self-fullness, is sufficient to guarantee hereditary self-fullness as proved in Theorem 3.3 below.

**Definition 3.2**
We say that a ceer \(X\) is co-ceer-resistant if whenever \(C\) is a \(\Pi^0_1\)-equivalence relation with infinitely many classes, then there are \(x, y\) so that \(x X y\) and \(x \not\in C\).
**Theorem 3.3**

Every self-full co-ceer-resistant cer is hereditarily self-full.

**Proof.** Given any reduction \( f : X \oplus Y \to X \oplus Y \), define \( n \sim f m \) if for every \( k \in \omega \), \( f^{(k)}(2n) \) is even if and only if \( f^{(k)}(2m) \) is even. Note that \( \sim f \) is a \( \Pi^0_1 \)-equivalence relation and \( X \) is a refinement of \( \sim f \) (i.e. if \( n \sim f m \) then \( n \sim f m \)). We represent a \( \sim f \)-class by a sequence in \( \{X, Y\}^\omega \). Namely, we define \( t^f_n \in \{X, Y\}^\omega \) by \( t^f_n(k) = X \) if \( f^{(k)}(2n) \) is even (or \( f^{(k)}(2n) \in X \), as we will sometimes write), and \( t^f_n(k) = Y \) if \( f^{(k)}(2n) \) is odd (or \( f^{(k)}(2n) \in Y \), as we will sometimes write). It is immediate to see that \( n \sim f m \) if and only if \( t^f_n = t^f_m \), thus in fact we can identify the \( \sim f \)-equivalence class of \( n \) with \( t^f_n \).

A sequence \( \tau \in \{X, Y\}^\omega \) is eventually periodic if there exist finite sequences \( \rho, \sigma \in \{X, Y\}^\omega \) so that \( \tau = \rho \check{\sigma} \omega \), where the symbol \( \check{\cdot} \) denotes concatenation and \( \sigma^\omega = \sigma \check{\sigma} \check{\cdots} \check{\sigma} \cdots \) is the infinite string obtained by concatenating infinitely many times \( \sigma \) with itself. Such a string \( \sigma \) is a period of \( \tau \).

We say that a \( \sim f \)-class is periodic of period \( k \) if the infinite string \( \tau \) which represents \( \sim f \) is periodic with a period of length \( k \).

Let now \( X \) be a self-full co-ceer-resistant cer and suppose \( Y \) is self-full. We must show that \( X \oplus Y \) is self-full. Suppose that \( f : X \oplus Y \to X \oplus Y \) is a reduction. We must show that the range of \( f \) intersects all the equivalence classes of \( X \oplus Y \). Since \( X \subseteq \sim f \) and \( X \) is co-ceer resistant, it follows that \( \sim f \) has only finitely many classes.

**Lemma 3.4** Each of the finitely many \( \sim f \)-classes is represented by an eventually periodic sequence \( \tau \in \{X, Y\}^\omega \).

**Proof.** Let \( \tau \) be a sequence which represents a \( \sim f \)-class, say \( \tau = t^f_n \). If \( \tau = \rho \check{\sigma} \omega \) for some finite \( \rho \) (where, to be more precise, \( Y^\omega = \langle Y \rangle^\omega \), where \( \langle Y \rangle \) denotes the string of length one consisting of the sole bit \( Y \) then \( \tau \) is eventually periodic. Otherwise, let \( \{k_i\}_{i \in \omega} \) be the sequence in ascending order so that \( k_i = X \) (notice that \( k_0 = 0 \)). For every \( i \), let \( \tau_{\geq k_i} \) be the tail of \( \tau \) beginning from the bit \( k_i \) of \( \tau \), i.e. \( \tau_{\geq k_i} = \tau_{k_i + k} \). Clearly for every \( i \) there exists \( n_i \) so that \( \tau_{\geq k_i} = t^f_{n_i} \). Just take \( n_i = \frac{f^{(k_i)}(2n)}{2} \). If it were \( \tau_{\geq k_i} \neq \tau_{\geq k_j} \) for every \( i \), and \( j < i \), then there would be infinitely many \( \sim f \)-equivalence classes, contrary to our previous conclusion. Therefore, there exist a least \( j \), and for this \( j \) a least \( i > j \), so that \( \tau_{\geq k_i} = \tau_{\geq k_j} \). But then we see that there is a period, namely the finite string \( \sigma \) such that \( \tau_{\geq k_j} = \sigma \check{\tau}_{\geq k_j} \).

Let \( N_1 \) be a common multiple of all periods of \( \sim f \)-classes. We replace \( f \) by \( f_1 = f^{(N_1)} \), which is still a reduction of \( X \oplus Y \) to \( X \oplus Y \), and now all \( \sim f_1 \)-classes have period \( 1 \). It follows that all the \( \tau \) which represent \( \sim f_1 \)-classes are of the form \( \rho \check{\sigma} \check{\omega} \) or \( \rho \check{\sigma} \omega \), for some \( \rho \). Let \( N_2 \) be a number greater than the length of \( \rho \) for each \( \sim f_1 \)-class. Once again, we replace \( f_1 \) by \( f_2 = f_1^{(N_2)} \), which ensures that each \( \sim f_2 \)-class is either represented by \( X^\omega \) or \( Y^\omega \). Thus, we have at most two \( \sim f_2 \)-classes. Let \( C^f_X = \{2n : t^f_n = X^\omega \} \) and \( C^f_Y = \{2n : t^f_n = Y^\omega \} \).

**Lemma 3.5**

The range of \( f_2 \mid C^f_X \) intersects all the \( X \oplus Y \)-classes of \( C^f_X \), and there are no odd numbers in \( f_2^{-1}(C^f_X) \).

**Proof.** First of all, notice that \( C^f_X, C^f_Y \) partition \( 2\omega \), they are both \( X \oplus Y \)-closed, and they are decidable, being two \( \Pi^0_1 \)-sets that partition a decidable set. Therefore, we can define a reduction.
For every $m$ be such that $g$ have 2 since $X$ is self-full, it follows that the range of $g$ restricted to $\{n : 2n \in C_{f_2}^f\}$ intersects all the $X$-classes of $\{n : 2n \in C_{f_2}^f\}$, and thus the range of $f_2 \upharpoonright C_{f_2}^f$ intersects all the $X \oplus Y$-classes of $C_{f_2}^f$. Thus, there cannot be any odd $2m + 1$ such that $f_2(2m + 1) \in C_{f_2}^f$. Otherwise, let $f_2(2m + 1) = 2n \in C_{f_2}^f$, and let $2n' \in C_{f_2}^f$ be such that $f_2(2n') X \oplus Y 2n$; as $f_2(2m + 1) X \oplus Y f_2(2n')$, it would follow $2m + 1 X \oplus Y 2n'$, a contradiction.

**Lemma 3.6**

For every $m$, either there is an odd $y$ so that $2m + 1 X \oplus Y f_2^2(y)$ or there is an odd $y$ so that $2m + 1 X \oplus Y f_2^2(y)$.

**Proof.** Define the function

$$g(n) = \begin{cases} \frac{f_2(2n + 1) - 1}{2}, & \text{if } f_2(2n + 1) \text{ is odd,} \\ \frac{f_2^2(2n + 1) - 1}{2}, & \text{otherwise.} \end{cases}$$

We first observe that $g(n)$ is always an integer. If $f_2(2n + 1)$ is even, then $f_2(2n + 1)$ must be in $C_{f_2}^f$, since $f_2^{-1}(C_{f_2}^f)$ cannot contain $2n + 1$ by the previous Lemma. Thus, $f_2^2(2n + 1)$ is odd. Suppose that $n \not\equiv m$. Since $f_2$ is a reduction of $X \oplus Y$ to $X \oplus Y$, we see that $f_2(2n + 1) X \oplus Y f_2(2m + 1)$, thus they have the same parity. Thus, either $g(n) = \frac{f_2(2n + 1) - 1}{2} Y f_2(2m + 1) - 1\frac{2}{2} = g(m)$ or $g(n) = \frac{f_2^2(2n + 1) - 1}{2} Y f_2^2(2m + 1) - 1\frac{2}{2} = g(m)$. Now, suppose that $g(n) Y g(m)$. If $f_2(2n + 1)$ has the same parity as $f_2(2m + 1)$, then we can use the fact that $f_2$ is a reduction of $X \oplus Y$ to $X \oplus Y$ to conclude that $n \not\equiv m$. If they have different parities, then we have without loss of generality: $g(n) = \frac{f_2(2n + 1) - 1}{2} Y f_2^2(2m + 1) - 1\frac{2}{2} = g(m)$.

But then we have $f_2(2n + 1) X \oplus Y f_2^2(2m + 1)$. But since $f_2$ is a reduction of $X \oplus Y$ to $X \oplus Y$, we have $2n + 1 X \oplus Y f_2(2m + 1)$, but the latter is even. This is a contradiction. Thus, we have shown that $g$ is a reduction of $Y$ to $Y$. Since $Y$ is self-full, we conclude that every class of $Y$ intersects the range of $g$. That is, $m \not\equiv g(k)$ for some $k$. If $f_2(2k + 1)$ is odd, then $2m + 1 X \oplus Y f_2(2k + 1)$. If $f_2(2k + 1)$ is even, then $2m + 1 X \oplus Y f_2^2(2k + 1)$.

**Lemma 3.7**

For every $k$, $[k]_{X \oplus Y}$ intersects the range of $f_2$.

**Proof.** If $k = 2m + 1$ then Lemma 3.6 ensures that $[k]_{X \oplus Y}$ intersects the range of $f_2$.

Consider now $k = 2m$. If $2m \in C_{f_2}^f$, then we have shown in Lemma 3.5 that $[k]_{X \oplus Y}$ intersects the range of $f_2$. So we suppose that $f_2(2m) = 2n + 1$. But we have shown in Lemma 3.6 that $2n + 1$ is $X \oplus Y$-equivalent to $f_2(y)$ for an odd $y$ or $f_2^2(y)$ for an odd $y$. In the former case, we have...
LEMMA 3.8
Every dark ceer $X$ is co-ceer resistant.

Proof. Suppose $C$ is a $\Pi^0_1$-equivalence relation and has infinitely many classes. Then $Id \leq C$ because $C$ is $\Pi^0_1$. Indeed, if $C = W^e_1$ and $W^e$ is the $e$-th c.e. set, then a reduction $f : Id \to C$ can be defined as follows: to define $f(n)$ search for the least $(x, s)$ so that $\{f(i), x \mid i < n\} \subseteq W^e_s$ (such a number exists since $C$ has infinitely many classes) and let $f(n) = x$. It follows that if $X$ is a ceer such that $Id \not\leq X$, then there are $n, m$ so that $nXm$ and $n \not\subseteq m$.

COROLLARY 3.9
Every dark ceer is hereditarily self-full.

Proof. Every dark ceer is self-full by [4] and is co-ceer resistant by Lemma 3.8. By Theorem 3.3, dark ceers are hereditarily self-full.

The next theorem shows that every non-universal degree in $Ceers/\mathcal{I}$ have infinitely many incomparable hereditarily self-full strong minimal covers. This is akin to [4, Theorem 4.10], which proves analogous result for the class of self-full ceers. Throughout the theorem and its proof, we employ the following notation. For every $k \geq 1$, we let $Id_k$ denote a fixed ceer with exactly $k$ classes.

THEOREM 3.10
Let $A$ be any non-universal ceer. Then, there are infinitely many incomparable hereditarily self-full ceers $(E_i)_{i \in \omega}$ so that for every $n, l \in \omega$ and ceer $X$, $A \oplus Id_n \not\leq E_i$ and

$$X < E_i \Rightarrow (\exists k)[X \not\leq A \oplus Id_k].$$

Proof. We construct ceers $E_i$ with the property that the function $x \mapsto 2x$ is a reduction $A \leq E_i$, satisfying the following requirements:

$CCR^k_i$: if $V_j$ is a co-c.e. equivalence relation with at least two distinct equivalence classes, then there are $x, y$ so that $xE_jy$ and $x \not\subseteq y$. (Here, $V_j = W^e_j$, i.e. the complement of the c.e. set $W^e_j$.)

$SF^{k,l}_i$: if $W_i$ intersects infinitely many $E_l$-classes which do not contain an even number, then $W_i$ intersects $[k]_X$.

$D^l_j$: $\phi_j$ is not a reduction of $E_l$ to $E_l$.

We first argue that any sequence of ceers $(E_i)_{i \in \omega}$ constructed with the above properties will be incomparable, will be hereditarily self-full and will have the properties that $A \oplus Id_n \not\leq E_i$ and $X < E_i \Rightarrow (\exists k)[X \not\leq A \oplus Id_k]$.

The $E_i$ are clearly incomparable by the $D$-requirements. By the fact that we will make $x \mapsto 2x$ be a reduction of $A$ to $E_i$, it is immediate that $A \leq E_i$ for each $i$. Thus, if there is some pair $n, l$ so that $A \oplus Id_n \not\leq E_i$, then $E_i \equiv A \oplus Id_k$ for some $k \leq n$ by [4, Lemma 2.5]. Let $k$ be minimal so that there is some $E_l$ with $E_l \equiv A \oplus Id_k$. Then $E_l \leq E_{l'}$ for every other $l'$, contradicting incomparability. So, $A \oplus Id_n \leq E_i$ for every pair $n, l$. 

$f_2(2m) X \oplus Y f_2(y)$, which would imply that $2m X \oplus Y y$ which is impossible since $y$ is odd. Thus, we have $f_2(2m) X \oplus Y f_2(2m) y$, thus $2m X \oplus Y f_2(y)$.

This shows that the range of $f_2$ intersects every $X \oplus Y$-class, so the range of $f$ must also intersect every $X \oplus Y$-class. Thus, $X \oplus Y$ is self-full.
Suppose that $\varphi$ is a reduction witnessing $X \subseteq E_I$ for some ceer $X$. There are two cases to consider. Either $\varphi$ intersects only finitely many, say, $k$, classes which contain no even number, in which case $X \subseteq A \oplus \text{Id}_k$, or by the SF-requirements applied to the c.e. image of $\varphi$, $\varphi$ is onto the classes of $E_I$. In the latter case, $X \equiv E_I$ [4, Lemma 1.1]. This shows that $X < E_I \Rightarrow ( \exists k ) [ X \subseteq A \oplus \text{Id}_k ]$. Applying the same argument to any reduction $\varphi$ of $E_I$ to itself shows that $\varphi$ must be onto the classes of $E_I$. Thus, $E_I$ is self-full. Finally, by the CCR-requirements, $E_I$ is also co-ceer resistant. It follows by Theorem 3.3 that the $E_I$ are each hereditarily self-full.

The strategies. We now give the strategies for each requirement:

To achieve that $f(x) = 2x$ is a reduction of $A$ to $X$, we guarantee that throughout the construction, we $E_I$-collapse a pair of distinct even numbers $2x, 2y$ if and only if $x, y$ is

For the CCR$^\varphi$-requirement, we wait until we see a pair $\langle x, y \rangle$ so that $\langle x, y \rangle$ appears in $W_j$. At this point, we pick a new odd $z$, we wait for either $\langle z, x \rangle$ or $\langle y, z \rangle$ to appear in $W_j$. While waiting, restrain the equivalence class of $z$ from being $E_I$-collapsed to other classes due to the action of lower-priority requirements. If and when the wait is over, drop the restraint. If at that point we see $\langle x, z \rangle \in W_j$, then we $E_I$-collapse $x$ and $z$. If we see $\langle y, z \rangle \in W_j$ but not as yet $(x, z) \in W_j$, then we $E_I$-collapse $y$ and $z$. Note that if $V_j$ is an equivalence relation with at least two equivalence classes then eventually such a pair $\langle x, y \rangle$ appears in $W_j$. Then, for every $z$ either $xW_jz$ or $yW_jz$. So whatever $z$ we pick, eventually we see $\langle x, z \rangle \in W_j$ or $\langle y, z \rangle \in W_j$.

For the SF$^{\varphi}_{k,j}$-requirement, we wait until $W_i$ enumerates an odd number $x$ which is not currently $E_I$-equivalent to any even number or lies in $E_I$-class currently restrained by a higher-priority requirement. We then $E_I$-collapse this $x$ with $k$.

For the $D_I^{k,j}$-requirement: we would like to employ the natural direct diagonalization strategy, i.e. take two new numbers in $E_I$ and choose to collapse them if and only if we do not choose to collapse their $\varphi_j$-image in $E_I$. The problem is that the $\varphi_j$-images in $E_I$ may be $E_I$-equivalent to even numbers (in addition to the usual problem of them being equivalent to the finitely many numbers restrained by higher-priority strategies). So, we cannot control whether or not they collapse because this is determined by $A$. The solution is to use the non-universality of $A$. In particular, as long as it appears that $\varphi_j$ is giving a reduction of $E_I$ into the classes of even numbers in $E_I$ (i.e. into $A$), we will encode a universal ceer into $E_I$. This will have one of two possible outcomes: either $\varphi_j$ will eventually have elements in its range which are not $E_I$-equivalent to even numbers or we will permanently witness a diagonalization. In either case, our threat to encode a universal ceer into $E_I$ will have a finite outcome and we will not actually make $E_I$ universal.

More explicitly, the $D_I^{k,j}$-strategy is as follows: we fix a universal ceer $T$. As we proceed, we will choose a sequence of parameters $a_0, a_1, \ldots$ which are not $E_I$-equivalent to any even numbers and we will ensure $a_i : E_i a_k$ if and only if $i + k < t$. At every stage, we will have only chosen finitely many parameters, thus will have encoded only a finite fragment of $T$. When we see that for every pair $i, k$ for which we have chosen parameters, $a_i : E_i a_k$ if and only if $\varphi_j(a_i) \ E_I \varphi(a_k)$, we will choose a new parameter $a_m$ for the least $m$ where we had not yet chosen $a_m$. In particular, we take one more step towards making $i \mapsto a_i$ be a reduction of $T$ to $E_I$. We stop this strategy if we see two numbers $i, k$ so that $\varphi_j(a_i), \varphi(a_k)$ are neither $E_I$-equivalent to each other nor to an even number nor a number restrained for a higher-priority requirement. If this happens, we are ready to perform a direct diagonalization. We $E_I$-collapse $a_i$ and $a_k$, and we restrain the pair $\varphi_j(a_i), \varphi_j(a_k)$ from any future collapse in $E_I$. This strategy either ends with the direct diagonalization as described, or by $\varphi_j$ being partial or by there being a pair $i, k$ so that $a_i : E_i a_k$ if and only if $\varphi_j(a_i) \ E_I \varphi(a_k)$. In any case, $\varphi_j$ will not be a reduction of $E_I$ to $E_I$. 


Building the sequence \((E_i)_{i \in \omega}\) is carried out via a standard finite injury priority construction.

**The construction:** We build \((E_i)_{i \in \omega}\) in stages by using the collapsing technique as described at the end of Section 1.

To initialize a \(D\)-requirement or a \(CCR\)-requirement at a stage means to cancel its parameters, if any, and to drop its restraint, if any. Further, a \(D\)-requirement which was satisfied is no longer considered satisfied after initialization. To initialize an \(SF\)-requirement means to do nothing.

Stage 0. Let \(X_0 = \text{Id}\). All requirements are initialized.

Stage \(s+1\). We begin the stage by collapsing even numbers in every \(E_i\) so that 2\(x\ E_i\ x\ y\) whenever \(x\ A_y\ y\).

If \(R = \text{CCR}_j\), we say that \(R\) is already satisfied at \(s+1\) if there is \((w, z) \in W_{j,s}\) so that \(w\ E_i\ z\). If \(R = \text{SF}_{i}^{k,l}\), we say that \(R\) is already satisfied at \(s+1\) if there is \(x \in W_{i,s}\) so that \(x\ E_i\ k\).

A new number \(x\) at stage \(s+1\) is an odd number bigger than any number so far mentioned in the construction. In particular, if \(x\) is new at stage \(s+1\), then \([x]_{E_i,s} = [x]\) for each \(l\).

Scan the requirements \(R\) in decreasing order of priority. Suppose you have scanned all \(R' < R\) and distinguish the following cases:

\[ R = \text{CCR}_j \]

If \(R\) is already satisfied then it drops its restraint if any, and we move to the next requirement.

If the requirement is not already satisfied at \(s+1\) and it is initialized (i.e. has no chosen parameters), but we now see numbers \(x, y\) with \((x, y) \in W_{j,s}\), then take a new odd number \(z\) which becomes the current witness of \(R\). Restrain the equivalence class of \(z\) from being \(E_j\)-collapsed to other classes due to lower-priority requirements. After this, if we see \((x, z) \in W_{j,s}\) or \((y, z) \in W_{j,s}\), then drop the restraint, \(E_j\)-collapse \(x\) and \(z\) or \(y\) and \(z\) according to the case. Whether \(\text{CCR}_j\) chooses \(z\) or we cause \(E_j\)-collapse, we say that \(\text{CCR}_j\) has acted at stage \(s+1\). After this action, stop the stage and initialize all lower-priority requirements. If we neither choose \(z\) nor cause \(E_j\)-collapse, then proceed to the next requirement.

\[ R = \text{SF}_{i}^{k,l} \]

Suppose the requirement is not already satisfied at \(s+1\), but we now see an odd number \(x \in W_{i,s}\) not in the equivalence class of any number currently restrained by some higher priority \(R'\), nor \(E_{i,s}\)-equivalent to any even number. We view this \(x\) as the once-and-for-all witness of the requirement. Then, \(E_j\)-collapse \(x\) with \(k\). We say that \(\text{SF}_{i}^{k,l}\) has acted at stage \(s+1\). After the action, stop the stage and initialize all requirements \(R'\) having lower priority than \(R\). If no such \(x\) is found, then proceed to the next requirement.

\[ R = \text{D}_j^{l,u} \]

If \(R\) is currently initialized and not yet satisfied, then choose two new odd numbers \(a_0, a_1\) which become the current witnesses of \(R\). Restrain the classes of the two witnesses from \(E_i\)-collapses due to lower-priority requirements. After choosing \(a_0\) and \(a_1\) and placing restraint, stop the stage and initialize all requirements \(R'\) having lower priority than \(R\).

If \(R\) already has witnesses, let \(m\) be the least so that \(a_m\) is not yet chosen. For \(i, k < m\), \(E_i\)-collapse \(a_i\) with \(a_k\) if and only if \(iT_k\ k\) (recall that \(T\) is our fixed universal ceeer). If this causes any new collapses, then we say we have caused \(E_j\)-collapse. We distinguish the following three cases:

Case 1: there are \(i, k < m\) so that \(\varphi_j(a_i), \varphi_j(a_k)\) both converge by stage \(s\) to numbers which are not \(E_{l,s}\)-equivalent to an even number nor a number restrained by higher-priority requirements and \(\varphi_j(a_i) \neq \varphi_j(a_k)\). We \(E_j\)-collapse \(a_i\) with \(a_k\) and place a permanent restraint to not allow lower-priority requirements to cause \(\varphi_j(a_i)\) to \(E_j\) collapse with \(\varphi_j(a_k)\). We let \(\varphi_j(a_i)\) and \(\varphi_j(a_k)\) permanently
be parameters $c$ and $d$ of the requirement. We now say that the requirement is satisfied (this is permanent unless the requirement is re-initialized). After this action, stop the stage and initialize all lower-priority requirements.

Case 2: not Case 1 and $\phi_i(a_i)$ converges by stage $s$ for every $i < m$ and for each $i, k < m$, $a_i E_{l,s} a_k$ if and only if $\phi_j(a_i) E_{l,s} \phi_j(a_k)$. In this case, we define $a_m$ to be a new odd number. After this action, stop the stage and initialize all lower-priority requirements.

Case 3: not Case 1 or Case 2: if we have caused $E_l$-collapse, then stop the stage and initialize all lower-priority requirements. Otherwise, move to the next requirement.

Notice that eventually we are sure to move on to stage $s + 2$ since the action of any initialized $D$-requirement stops the stage, and there are co-finitely many initialized requirements at any stage.

The verification: We split the verification into the following lemmata. Notice that at each stage, exactly one requirement acts.

Let us say that a witness for a requirement $R$ is $E_l$-active if it is a parameter $z$ for a $CCR^l_j$-requirement or is an $a_i$ parameter for a $D^l'_{j'}$-requirement or is a $c$ or $d$ parameter for a $D^l_d$-requirement. That is, it is a parameter for which some requirement restrains $E_l$-collapse.

**Lemma 3.11**

At every stage $s$, if $x$ and $y$ are $E_l$-active witnesses for different requirements, then $x E_{l,s} y$.

At every stage $s$, if $x, y$ are $E_l$-active witnesses and $x E_{l,s} y$, then $x = a_i$ and $y = a_j$ for a $D$-requirement and $i T_j s j$.

At every stage $s$, if $x$ is $E_l$-active, then $x$ is not $E_{l,s}$-equivalent to any even number.

**Proof.** We prove all three claims by simultaneous induction on $s$. The lemma is certainly true at stage 0, as there are no $E_l$-active witnesses. If the claim is true at $s$, then at $s + 1$ at most one $R$ can effect $E_l$. (i) $E_{l,s+1}$-collapses the class of its witness with some other class, or (ii) $R$ is a $D$-requirement)

If $R$ is a $CCR$-requirement, then $R$’s witness $z$ ceases to be active. So, it has collapsed $[z]_{E_{l,s}}$ to some other class $[w]_{E_{l,s}}$. By inductive hypothesis, the only member of $[w]_{E_{l,s}}$ which was either $E_l$-active or even was $z$ itself. Since $z$ is no longer active at stage $s + 1$, we have added neither an $E_l$-active number nor an even number to $[w]_{E_{l,s+1}}$.

If $R$ is a $D$-requirement, it collapses $a_i$ with $a_j$ only if $i T_j s j$. Since neither $a_i$ nor $a_j$ were $E_{l,s}$-equivalent to another requirement’s $E_l$-active witness or an even number by inductive hypothesis, the combined class maintains this property.

If $R$ is an $SF^L_{k,l}$-requirement, then it collapses its once-and-for-all witness $x$ to $k$. This $x$ is not $E_{l,s}$-equivalent to any $E_{l,s+1}$-active number since it is not $E_{l,s}$-equivalent to a number restrained by a higher-priority requirement (and this action initializes all lower-priority requirements) nor an even number, and it initializes all lower-priority requirements. Thus, it has added neither an even number nor an $E_l$-active number to $[k]_{E_{l,s+1}}$.

Finally, if $R$ appoints new $E_l$-active witnesses, these witnesses are either new, so their classes are currently disjoint singletons, or are the $c$ and $d$ for a $D$-requirement, which are appointed only if they are in distinct $E_l$-classes from every other $E_l$-active or even number.

**Lemma 3.12**

The map $i \mapsto 2i$ is a reduction witnessing $A \leq E_l$ for each $l$.

**Proof.** Since no active parameter is ever equivalent to an even number by Lemma 3.11, we never cause collapse of two even numbers aside from the beginning of each stage. At that point, we collapse.
2x \( E_1 x 2y \) if and only if \( x A s y \). Since we never cause any other collapse among even numbers, we have 2x \( E_1 2y \) if and only if \( x A y \).

**Lemma 3.13**
Every requirement is initialized finitely often, eventually stops acting and sets up only a finite restraint.

**Proof.** The proof is by induction on the priority ranking of the requirement \( R \), since every requirement acts only finitely often after each initialization. This is clear for each requirement except \( D \)-requirements. Clearly, they can act via Case 1 at most once after last initialization. Suppose towards a contradiction that it acts infinitely often via Case 2 after last initialization. Then \( \psi : i \mapsto a_i \) is a reduction of \( T \) to \( E_{r_i} \). Further, the range of \( \psi \) is contained in the classes which contain even numbers and the finitely many (by inductive hypothesis) classes restrained for higher priority strategies. Thus, using the fact that \( E_{r_i} \) restricted to the even numbers is equivalent to \( A \) by Lemma 3.12, we can alter \( \psi \) to give a reduction of \( T \) to \( A \oplus \text{Id}_k \) for some \( k \). But since \( A \) is assumed to be non-universal, the universal degree is uniform join-irreducible [2, Proposition 2.6], \( A \oplus \text{Id}_k \) is non-universal. This is a contradiction. Finally, since the requirement acts only finitely often in Case 2, it defines only finitely many parameters \( a_i \). Thus, it can only cause \( E_{r_i} \)-collapse finitely often and can only initialize lower-priority requirements in Case 3 finitely often. So, \( D \)-requirements also act finitely often, place finite restraint and initialize lower-priority requirements finitely often after their last initialization.

**Lemma 3.14**
Every requirement is eventually satisfied.

**Proof.** Notice that \( CCR \)- or \( SF \)-requirements can never be injured after they have acted. If they collapse, then the collapse permanently satisfies the requirement. Suppose that every \( R' < R \) is eventually satisfied.

We now distinguish the various possibilities for \( R \).

\( R = SF_{i,j}^l \). We must only worry if \( W_i \) intersects infinitely many \( E_{r_i} \)-classes not containing even numbers. In this case, eventually some odd number \( x \) appears in \( W_i \) not as yet \( E_{r_i} \)-equivalent to any even number and avoiding the finite restraint imposed by the higher-priority requirements. So \( R \) acts by picking such an \( x \) (the once-and-for-all witness of the requirement), and \( x \) is \( E_{r_i} \)-collapsed to \( k \), which makes \( R \) permanently satisfied.

\( R = CCR_{i,j}^l \). By the fact that lower-priority \( SF \)-requirements will choose their once-and-for-all witnesses avoiding higher-priority restraints, and thus not interfering with \( CCR_{i,j}^l \), the only possible injury to \( CCR_{i,j}^l \) can be made, thanks to Lemma 3.11, by higher-priority requirements. By Lemma 3.13 let \( s_0 \) be the least stage such that these higher-priority requirements do not act at any \( s \geq s_0 \). If \( V_j \) is an equivalence relation with at least two distinct equivalence classes, then eventually we either see at some point that \( R \) is already satisfied without having acted (and so the requirement is permanently satisfied) or we see at some point a pair \( \langle x, y \rangle \) to appear in \( W_i \). At that point, we pick the final witness \( z \), a new odd number. At some later stage, we collapse \( z \) with \( x \) or \( y \), which makes \( R \) permanently satisfied.

\( R = D_{i,j}^l \). By the fact that lower-priority \( SF \)-requirements will choose their once-and-for-all witnesses avoiding higher-priority restraint, and thus not interfering with \( D_{i,j} \), the only possible injury
to $D_l$ can be made, thanks to Lemma 3.11, by higher-priority requirements. By Lemma 3.13 let $s_0$ be the least stage such that these higher-priority requirements do not act at any $s \geq s_0$. Requirement $D^{l'}_j$ is never re-initialized after $s_0$. After stage $s_0$, we have several possible outcomes for $D^{l'}_j$. We have already shown that it cannot take Case 2 infinitely often. If it ever takes Case 1, then we have two numbers $a_i, a_k$ so that $a_i \equiv E(a_i)\equiv \varphi_j(a_i) \equiv a_k$. The remaining possibility is that it takes Case 3 co-finitely often. In this case, we see that either $\varphi_j$ is not total or is not a reduction (either case is witnessed on the finite set $\{a_i \mid i < m\}$).

The proof of the theorem is now complete.

Note the following corollary which follows from letting $A$ be $\text{Id}$:

**Corollary 3.15**

There are light ceers which are hereditarily self-full.

We conclude by showing that the property of being hereditarily self-full is itself hereditary in one sense and not in another.

**Observation 3.16**

If $Z$ is hereditarily self-full, then there is a self-full cer $X$ so that $Z \oplus X$ is not hereditarily self-full.

**Proof.** Let $Z$ be hereditarily self-full, and let $X$ and $Y$ be self-full so that $X \oplus Y$ is non-self-full. Then $Z \oplus X$ is not hereditarily self-full because $Z \oplus X \oplus Y$ is non-self-full. $\square$

**Observation 3.17**

If $X$ is hereditarily self-full and $Y$ is hereditarily self-full, then $X \oplus Y$ is hereditarily self-full.

**Proof.** Let $E$ be any self-full cer. Then since $Y$ is hereditarily self-full, we see that $Y \oplus E$ is self-full. Thus, since $X$ is hereditarily self-full, we see that $X \oplus (Y \oplus E)$ is self-full. We conclude that whenever $E$ is self-full, $(X \oplus Y) \oplus E$ is self-full, showing that $X \oplus Y$ is hereditarily self-full. $\square$

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