COMPLEX AND QUATERNIONIC OPTIMIZATION

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Abstract. We introduce and suggest to research a special class of optimization problems, wherein an objective function is a real-valued complex variables function under constraints, comprising complex-valued complex variables functions: “Complex Optimization”. We demonstrate multiple examples to show a rich variety of problems, describing Complex Optimization as an optimization subclass as well as a Mixed Integer-Real-Complex Optimization.

Next, we introduce more general concept: “Quaternionic Optimization” for optimization over quaternion subsets.

1. Introduction. It is well-known that an optimization problem can be represented in the following way: given a function $f : G \rightarrow \mathbb{R}$ from some set $G$ to the real numbers; sought: an element $x_0 \in G$ such that $f(x_0) \leq f(x)$ for all $x \in G$ (“minimization”), or such that $f(x_0) \geq f(x)$ for all $x \in G$ (“maximization”).

Typically, $G$ is some subset of the Euclidean space $\mathbb{R}^n$, specified by a set of constraints and the function $f$ is called an objective function or target function.

The case, when $G$ is some subset of two-dimensional complex plane and target function $f : \mathbb{C} \rightarrow \mathbb{R}$ is real-valued complex variable function is very poor investigated yet.

In [9, 15], the mathematical formulation of the ptychographic phase retrieval problem is examined and the corresponding minimization of intensity Gaussian error metric (for the corresponding discrete Fourier operator and probes: scanning positions) over potentially complex vector is investigated (Least Squares Problem). Phase retrieval is the process of algorithmically finding solutions to the phase problem. In physics, the phase problem is a problem of loss of information concerning the phase that can occur when making a physical measurement.

In [12, 13, 14] methods to solve unconstrained nonlinear optimization problems of real-valued complex functions in several complex variables are developed in order to overcome the fact that due to Cauchy-Riemann conditions, the real-valued functions in complex variables are necessarily nonanalytic.

As we see, currently optimization of real-valued complex variable functions over the complex plane is considered and investigated just for some specific cases, no general models, comprising wide variety of targets and constraints are considered and investigated yet.

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The purpose of this paper is to introduce and describe wide variety of optimization problems of $f : \mathbb{C} \to \mathbb{R}$ and $f : \mathbb{C}^n \to \mathbb{R}$, specified by the constraints, comprising complex functions of one and several complex variables, and further, we introduce more general concept: “Quaternionic Optimization”, when $G$ is some quaternion subset.

2. Complex Optimization. Basic definitions, methods and algorithms for modern complex analysis is considered, e.g., in [10] and [11].

Complex analysis proves a powerful tool for solving wide variety of problems in fundamental science and engineering: the analysis of electrical circuits, hydro- and aerodynamics, and so on.

In [1, 3, 6, 8] formulas in various engineering applications and science are considered: e.g., for electrical impedance, electromechanical circuits, electromagnetic fields, hydro- and aerodynamics. Imposing the corresponding constraints on complex input parameters we could obtain various complex optimization problems.

Let $|z|$ be the absolute value of a complex number $z = \text{Re}(z) + \text{Im}(z)i = a + ib, a, b \in \mathbb{R}, i^2 = -1$ and $\arg(z)$ the argument of $z$ : the principal value. (See, e.g., [10, 11]).

Let us introduce and demonstrate various optimization problems, defined in terms of complex numbers and functions, contained in various targets and constraints.

2.1. Single complex variable. In this section we describe a rich variety of optimization problems, containing complex functions of a single complex variable in the objective functions (targets) and constraints, including very simple cases and more sophisticated. This section consists of subsections, each one having a title, describing its specific features.

2.1.1 Simple absolute value-comprising constraint

$\text{cop211} = \{ \min |z|, \text{ subject to } |z| \geq 1 \}$,

$\text{argmin}(\text{cop211}) = \{ z: |z| = 1 \}$.

2.1.2. Imaginary part-comprising target.

$\text{cop212} = \{ \min -\text{Im}(z), \text{ subject to } |z| \leq 1 \}$,

$\text{argmin}(\text{cop212}) = i$.

2.1.3. Real part-comprising target.

$\text{cop213} = \{ \min \text{Re}(z), \text{ subject to } |z| \leq 1 \}$,

$\text{argmin}(\text{cop213}) = -1$.

2.1.4. Real and Imaginary part-comprising constraints.

$\text{cop214} = \{ \max |z|, \text{ subject to } 0 \leq \text{Re}(z) \leq 1, 0 \leq \text{Im}(z) \leq 1 \}$,

$\text{argmax}(\text{cop214}) = 1+i$.

2.1.5. Real and Imaginary part-comprising target and constraints.

$\text{cop215} = \{ \max \text{Re}(z) + \text{Im}(z), \text{ subject to } 0 \leq \text{Re}(z) \leq 1, 0 \leq \text{Im}(z) \leq 1 \}$,

$\text{argmax}(\text{cop215}) = 1+i$.

2.1.6. Argument-comprising constraint.

$\text{cop216} = \{ \max \text{Re}(z) + \text{Im}(z), \text{ subject to } 0 \leq \text{Re}(z) \leq 1, 0 \leq \text{Im}(z) \leq 1, \arg(z) = 0 \}$,
2.1.7. **Real and Imaginary part-comprising target and constraints.**

cop217 = {max Im(z), subject to 0 ≤ Re(z) ≤ 1, 0 ≤ Im(z) ≤ 1, Im(z) ≤ Re(z)},
argmax(cop217) = 1+i.

2.1.8. **Absolute value and Real and Imaginary part-comprising target and constraints.**

cop218 = {max |z|, subject to Im(z) ≥ Re^2(z), Re(z) ≥ Im^2(z)},
argmax(cop218) = 1+i.

2.1.9. **Polynomial Complex Optimization.**

cop219 = {max |c_n z^n + ... + c_1 z| subject to
|a_1 z^n + ... + a_1 z| ≤ b_1, ..., |a_m z^n + ... + a_m z| ≤ b_m,
z ∈ C, a_j ∈ C, b_i ∈ R, c_j ∈ C, 1 ≤ i ≤ m, 1 ≤ j ≤ n, n ∈ N, m ∈ N}.

By introducing the slack variables w_1 ≥ 0, ..., w_m ≥ 0, the above inequalities can be converted into the following equations:

\[ |a_1 z^n + ... + a_1 z| + w_1 = b_1, ..., |a_m z^n + ... + a_m z| + w_m = b_m. \]

(More sophisticated examples would contain rational meromorphic complex functions).

2.1.10. **Non-Linear Complex Optimization.**

cop2110 = {max |e^z + \sin(\pi z)|, subject to |\cos(\pi z)| ≤ a, 0 ≤ Re(z) ≤ 1, 0 ≤ Im(z) ≤ 1, z ∈ C, a ∈ R}.

2.2. **Several complex variables.** In this section we describe a rich variety of optimization problems, containing complex functions of several complex variables in the objective functions (targets) and constraints, including very simple cases and more sophisticated as well. This section consists of subsections, each one having a title, describing its specific features.

2.2.1. **Absolute values-comprising target and constraints.**

cop221 = {max |z_1 + z_2| subject to |z_1| ≤ 1, |z_2| ≤ 1}.

2.2.2. **Argument-comprising constraint.**

cop222 = {max |z_1 + z_2| subject to |z_1| ≤ 1, |z_2| ≤ 1, arg(z_1 z_2) ≤ \frac{\pi}{4}}.

2.2.3. **Linear Complex Optimization.**

cop223a = {max |c_1 z_1 + ... + c_n z_n| subject to
|a_1 z_1 + ... + a_1 z_n| ≤ b_1, ..., |a_m z_1 + ... + a_m z_n| ≤ b_m,
z_j ∈ C, a_j ∈ C, b_i ∈ R, c_j ∈ C, 1 ≤ i ≤ m, 1 ≤ j ≤ n, n ∈ N, m ∈ N}.

By introducing the slack variables w_1 ≥ 0, ..., w_m ≥ 0 the above inequalities can be converted into the following equations:

\[ |a_1 z_1 + ... + a_1 z_n| + w_1 = b_1, ..., |a_m z_1 + ... + a_m z_n| + w_m = b_m. \]

cop223b = {max |c_1 z_1 + ... + c_n z_n| subject to
a_1 z_1 + ... + a_1 z_n = b_1, ..., a_m z_1 + ... + a_m z_n = b_m,
z_j ∈ C, a_j ∈ C, b_i ∈ C, c_j ∈ C, (Az = b), 1 ≤ i ≤ m, 1 ≤ j ≤ n, n ∈ N, m ∈ N}. 
2.2.4. Real and Imaginary part-comprising constraints.

cop224 = \{ \text{max} \ |z_1 + \ldots + z_n| \ \text{subject to} \n \begin{align*}
\text{Re}(a_{11}z_1 + \ldots + a_{1n}z_n) & \leq b_1, \ldots, \ \text{Re}(a_{m1}z_1 + \ldots + a_{mn}z_n) \leq b_m, \\
\text{Im}(a_{11}z_1 + \ldots + a_{1n}z_n) & \leq c_1, \ldots, \ \text{Im}(a_{m1}z_1 + \ldots + a_{mn}z_n) \leq c_m, \\
z_j \in \mathbb{C}, \ a_{ij} \in \mathbb{C}, b_i \in \mathbb{R}, c_i \in \mathbb{R}, 1 \leq i \leq m, 1 \leq j \leq n, n \in \mathbb{N}, m \in \mathbb{N}. 
\end{align*} \}

2.2.5. Argument-comprising constraints.

cop225 = \{ \text{max} \ |z_1 + \ldots + z_n| \ \text{subject to} \n \begin{align*}
\arg(a_{11}z_1 + \ldots + a_{1n}z_n) & \leq b_1, \ldots, \ \arg(a_{m1}z_1 + \ldots + a_{mn}z_n) \leq b_m, \\
\text{Im}(a_{11}z_1 + \ldots + a_{1n}z_n) & \leq c_1, \ldots, \ \text{Im}(a_{m1}z_1 + \ldots + a_{mn}z_n) \leq c_m, \\
z_j \in \mathbb{C}, \ a_{ij} \in \mathbb{C}, b_i \in \mathbb{R}, c_i \in \mathbb{R}, 1 \leq i \leq m, 1 \leq j \leq n, n \in \mathbb{N}, m \in \mathbb{N}. 
\end{align*} \}

2.2.6. Argument-comprising target.

cop226 = \{ \text{max} \ \arg(z_1 \ldots z_n) \ \text{subject to} \n \begin{align*}
\text{Re}(a_{11}z_1 + \ldots + a_{1n}z_n) & \leq b_1, \ldots, \ \text{Re}(a_{m1}z_1 + \ldots + a_{mn}z_n) \leq b_m, \\
\text{Im}(a_{11}z_1 + \ldots + a_{1n}z_n) & \leq c_1, \ldots, \ \text{Im}(a_{m1}z_1 + \ldots + a_{mn}z_n) \leq c_m, \\
\arg(z_j) & \leq d_j, z_j \in \mathbb{C}, a_{ij} \in \mathbb{C}, b_i \in \mathbb{R}, c_i \in \mathbb{R}, d_j \in \mathbb{R}, \\
1 \leq i \leq m, 1 \leq j \leq n, n \in \mathbb{N}, m \in \mathbb{N}. 
\end{align*} \}

2.2.7. Quadratic Complex Optimization.

cop227 = \{ \text{max} \ |z_1^2 + \ldots + z_n^2 - i_z1z_2| \ \text{subject to} \n \begin{align*}
|a_{11}z_1 + \ldots + a_{1n}z_n| & \leq b_1, \ldots, \ |a_{m1}z_1 + \ldots + a_{mn}z_n| \leq b_m, \\
z_j \in \mathbb{C}, \ a_{ij} \in \mathbb{C}, b_i \in \mathbb{R}, 1 \leq i \leq m, 1 \leq j \leq n, n \in \mathbb{N}, m \in \mathbb{N}. 
\end{align*} \}

2.2.8. Integer Complex Optimization. Similar to the very well known in Mathematical Optimization, Integer Optimization(see, e.g., [5]): optimization over integer points, we introduce here Complex Optimization over the subsets of the Gaussian Integers: Integer Complex Optimization (ICOP).

Its well-known in number theory a complex number whose real and imaginary parts are both integers: Gaussian Integer. The Gaussian integers are the set: \( \mathbb{Z}[i] := \{a + bi, a, b \in \mathbb{Z}\} \), where \( i^2 = -1 \). Gaussian integers are closed under addition and multiplication and form commutative ring, which is a subring of the field of complex numbers. When considered within the complex plane the Gaussian integers constitute the 2-dimensional integer lattice. The Gaussian integers form unique factorization domain: it is irreducible if and only if it is a prime(Gaussian primes).

The field of Gaussian rationals consists of the complex numbers whose real and imaginary part are both rational(see, e.g., [7]).

The norm of a Gaussian integer is its product with its conjugate:

\[ N(a + bi) = (a + bi)(a - bi) = a^2 + b^2. \]

The norm is multiplicative, that is, one has:

\[ N(zw) = N(z)N(w), \quad z, w \in \mathbb{Z}[i]. \]

\[ \text{cop228} = \{ \text{max} \ |z_1^4 + \ldots + z_n^4| \ \text{subject to} \n \begin{align*}
b_1 \leq |a_{11}z_1 + \ldots + a_{1n}z_n| \leq c_1, \ldots, \ b_m \leq |a_{m1}z_1 + \ldots + a_{mn}z_n| \leq c_m, \\
z_j \in \mathbb{Z}[i], \ a_{ij} \in \mathbb{C}, b_i \in \mathbb{R}, c_i \in \mathbb{R}, b_i > 0, \\
1 \leq i \leq m, 1 \leq j \leq n, n \in \mathbb{N}, m \in \mathbb{N}. 
\end{align*} \]
triangular lattice in the complex plane, in contrast with Gaussian integers, which form a square lattice in the complex plane. The Eisenstein integers form a commutative ring as well and similar to Gaussian integers form a Euclidean domain, which supposes unique factorization of Eisenstein integers into Eisenstein primes.

2.2.9. Mixed Real-Complex Optimization (MRCOP).

\[
\text{cop229} = \left\{ \max x^3y|z^2 - c^2| - x^2 + y^3 \quad \text{subject to} \right.
\]
\[
|z| \leq a, \ b \leq x \leq c, \ d \leq y \leq e, \ z \in \mathbb{C}, \ x \in \mathbb{R}, \ y \in \mathbb{R}, \ a, b, c, d, e \in \mathbb{R} \right\}.
\]

2.2.10. Mixed Integer-Real-Complex Optimization (MIRCOP).

\[
\text{cop2210} = \left\{ \min |kz_1^2 - z_2^2| - x^2 + y^3t^2 \quad \text{subject to} \right.
\]
\[
x \geq N, \ a_1 \leq |z_1| \leq b_1, \ a_2 \leq |z_2| \leq b_2, \ a_3 \leq x \leq b_3, \ a_4 \leq y \leq b_4, \ a_5 \leq t \leq b_5, \ z_1 \in \mathbb{C}, \ z_2 \in \mathbb{Z}[i], \ x \in \mathbb{Z}, \ y \in \mathbb{Z}, \ t \in \mathbb{R}, \ a_i, b_i \in \mathbb{R}, \ N \in \mathbb{N}, \ a_i \geq 0, 1 \leq i \leq 5 \right\}.
\]

Note that in addition, each such example may comprise complex conjugations as well.

3. Quaternionic Optimization. Quaternions are generally represented in the form: \( q = a + bi + cj + dk \), where \( a \in \mathbb{R} \), \( b \in \mathbb{R} \), \( c \in \mathbb{R} \), \( d \in \mathbb{R} \), and \( i, j \) and \( k \) are the fundamental quaternion units and are a number system that extends the complex numbers (see, e.g., [4]). Quaternions find uses in both pure and applied mathematics: in three-dimensional computer graphics, computer vision, robotics, control theory, signal processing, attitude control, physics, bioinformatics, molecular dynamics, computer simulations, orbital mechanics, crystallographic texture analysis. In quantum mechanics, the spin of an electron and other matter particles can be described using quaternions. In 1999 it was shown that Einstein equations of general relativity could be formulated using quaternions.

The set of all quaternions \( \mathbb{H} \) is a normed algebra, where the norm is multiplicative: \( ||pq|| = ||p|| ||q||, \ p \in \mathbb{H}, q \in \mathbb{H}, ||q||^2 = a^2 + b^2 + c^2 + d^2 \).

This norm makes it possible to define the distance \( d(p, q) = ||p - q|| \) which makes \( \mathbb{H} \) into a metric space.

Let us introduce Quaternionic Optimization - optimization of real-valued quaternionic functions over quaternionic subsets.

3.1. Linear Quaternionic Optimization.

\[
\text{qop31a} = \left\{ \max \ ||c_1q_1 + \ldots + c_nq_n|| \quad \text{subject to} \right.
\]
\[
||a_{11}q_1 + \ldots + a_{1n}q_n|| \leq b_1, \ \ldots, \ ||a_{m1}q_1 + \ldots + a_{mn}q_n|| \leq b_m, \ q_j \in \mathbb{H}, \ a_{ij} \in \mathbb{H}, \ b_i \in \mathbb{R}, \ c_j \in \mathbb{H}, 1 \leq i \leq m, 1 \leq j \leq n, n \in \mathbb{N}, m \in \mathbb{N} \right\}.
\]

By introducing the slack variables \( w_1 \geq 0, \ldots, w_m \geq 0 \) the above inequalities can be converted into the following equations:

\[
||a_{11}q_1 + \ldots + a_{1n}q_n|| + w_1 = b_1, \ \ldots, \ ||a_{m1}q_1 + \ldots + a_{mn}q_n|| + w_m = b_m.
\]

\[
\text{qop31b} = \left\{ \max \ ||c_1q_1 + \ldots + c_nq_n|| \quad \text{subject to} \right.
\]
\[
a_{11}q_1 + \ldots + a_{1n}q_n = b_1, \ \ldots, \ a_{m1}q_1 + \ldots + a_{mn}q_n = b_m, \ q_j \in \mathbb{H}, \ a_{ij} \in \mathbb{H}, \ b_i \in \mathbb{R}, \ c_j \in \mathbb{H}, (Aq = b), \ 1 \leq i \leq m, 1 \leq j \leq n, n \in \mathbb{N}, m \in \mathbb{N} \right\}.
\]

3.2. Polynomial Quaternionic Optimization.

\[
\text{qop32} = \left\{ \max \ ||c_nq^n + \ldots + c_1q|| \quad \text{subject to} \right.
\]
\[ \|a_{11}q^n + \cdots + a_{11}q\| \leq b_1, \quad \cdots, \quad \|a_{mn}q^n + \cdots + a_{mn}q\| \leq b_m, \]
\[ q \in \mathbb{H}, \; a_{ij} \in \mathbb{H}, \; b_i \in \mathbb{R}, \; c_j \in \mathbb{H}, \; 1 \leq i \leq m, 1 \leq j \leq n, n \in \mathbb{N}, m \in \mathbb{N}. \]

By introducing the slack variables \( w_1 \geq 0, \ldots, w_m \geq 0 \) the above inequalities can be converted into the following equations:
\[ \|a_{11}q^n + \cdots + a_{11}q\| + w_1 = b_1, \quad \cdots, \quad \|a_{mn}q^n + \cdots + a_{mn}q\| + w_m = b_m. \]

### 3.3. Mixed Integer-Real-Complex-Quaternionic Optimization (MIRCQOP).

Similar to Gaussian Integers in Complex Analysis, let us consider a set of integer points for quaternions: \( \mathbb{L} := \{ q : q = a + bi + cj + dk \mid a \in \mathbb{Z}, b \in \mathbb{Z}, c \in \mathbb{Z}, d \in \mathbb{Z} \} - \) Lipschitz quaternions. Similarly for Hurwitz Integers.

**qop33** = \( \{ \min \| p - q\|/(z^2 - z^2) - x^2 + y^2t^2 \quad \text{subject to} \]
\[ xy \geq N, \; a_1 \leq \|p\| \leq b_1, a_2 \leq \|q\| \leq b_2, \; a_3 \leq \|z_1\| \leq b_3, \; a_4 \leq \|z_2\| \leq b_4, \]
\[ a_5 \leq x \leq b_5, \; a_6 \leq y \leq b_6, \; a_7 \leq t \leq b_7, \; p \in \mathbb{H}, q \in \mathbb{L}, z_1 \in \mathbb{C}, \; z_2 \in \mathbb{Z}[i], \]
\[ x \in \mathbb{Z}, \; y \in \mathbb{Z}, \; t \in \mathbb{R}, \; a_i, b_i \in \mathbb{R}, \; N \in \mathbb{N}, \; a_i \geq 0, 1 \leq i \leq 7 \}. \]

### 3.4. Non-Linear Quaternionic Optimization.

**qop34** = \( \{ \max \| \exp(p) - \ln(q)\| \quad \text{subject to} \]
\[ a_1 \leq \|p\| \leq b_1, a_2 \leq \|q\| \leq b_2, \]
\[ p \in \mathbb{H}, q \in \mathbb{H}, \; a_i, b_i \in \mathbb{R}, \; a_i \geq 0, i = 1, 2 \}. \]

### 4. Open Problems.

Despite such optimization problems actually could be translated and considered in terms of optimization problems over the Euclidean space, it may be not always so “easy” task (complexity problems, etc.).

That is why, it would be preferable to develop specific, “direct” methods for Complex and Quaternionic Optimization problems using Complex and Quaternionic Analysis.

The corresponding complexity evaluations for the Complex and Quaternionic Optimization problems would be developed as well: for example in binary encoded length of the coefficients (see, e.g., \( [5, 2] \)), and, in particular, finding conditions for the polynomial-time optimization.

Complex and Quaternionic Optimization ideas may be further extended for octonions and other hypercomplex systems, forming Hypercomplex Optimization, as well as useful for similar approaches in other subfields of the Optimization Theory, e.g., in Optimal Control Theory.

### 5. Conclusions.

We described a rich variety of optimization problems, comprising complex numbers and complex functions in their targets and constraints: “Complex Optimization” and quaternion variables and functions: “Quaternionic Optimization” and the corresponding open problems: complexity, extension for octonions and other hypercomplex systems, forming Hypercomplex Optimization. It would stimulate researchers to develop the corresponding new methods and algorithms.

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