GAUSSIAN FLUCTUATIONS OF SPATIAL AVERAGES OF A SYSTEM OF STOCHASTIC HEAT EQUATIONS

Abstract. We consider a system of $d$ non-linear stochastic heat equations driven by an $m$-dimensional space-time white noise on $\mathbb{R}_+ \times \mathbb{R}$. In this paper we study the asymptotic behavior of spatial averages over large intervals $[-R, R]$. We establish a rate of convergence to a multivariate normal distribution in the Wasserstein distance and a functional central limit theorem.

1. Introduction

In this paper we are interested in the following system of stochastic heat equations

\begin{equation}
\frac{\partial u_i}{\partial t}(t, x) = \frac{\partial^2 u_i}{\partial x^2}(t, x) + \sum_{j=1}^{m} \sigma_{ij}(u(t, x)) \dot{W}_j(t, x),
\end{equation}

where $u = (u_1, \ldots, u_d)$ is a $d$-dimensional random vector depending on the space and time parameters $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$. We assume a constant initial condition $u_i(0, x) = 1$, $1 \leq i \leq d$. For all $1 \leq i \leq d$ and $1 \leq j \leq m$, $\sigma_{ij} : \mathbb{R}^d \to \mathbb{R}$ are Lipschitz functions. The noise $\dot{W} := (\dot{W}_1, \ldots, \dot{W}_m)$ is a vector of $m$ independent Gaussian space-time white noises on $\mathbb{R}_+ \times \mathbb{R}$.

It is well-known (see, for instance, [8]) that equation (1.1) has a unique mild solution $u$ such that for each $i = 1, \ldots, d$, it satisfies the equation

\begin{equation}
u_i(t, x) = 1 + \sum_{j=1}^{m} \int_{0}^{t} \int_{\mathbb{R}} p_{t-s}(x-y) \sigma_{ij}(u(s, y)) W_j(ds, dy),
\end{equation}

where in the right-hand side, the stochastic integral is interpreted in the Itô-Walsh sense, and $p_t(x) = (2\pi t)^{-\frac{d}{2}} e^{-x^2/2t}$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, is the heat kernel. This type of systems of stochastic heat equations have been extensively studied and, for instance, we refer to [2] for a detailed analysis of hitting probabilities.

Fix a time $t > 0$. The goal of this paper is to study the asymptotic behaviour, as $R$ tends to infinity, of the the spatial averages $F^R(t) := (F^R_1(t), \ldots, F^R_d(t))$, where, for $i = 1, \ldots, d$,

\begin{equation}
F^R_i(t) := \frac{1}{\sqrt{R}} \left( \int_{-R}^{R} u_i(t, x) dx - 2R \right).
\end{equation}

In the case $d = m = 1$ functional and quantitative central limit theorems have been established in [3] using the techniques of Malliavin calculus combined with Stein’s method for normal approximations. Our work is an extension of that paper to a system of equations. In this case, the limit in law of the family of $d$-dimensional random processes \{$F^R(t), t \in [0, T]$\} is a $d$-dimensional Gaussian martingale. For the rate of convergence, instead of the total variation distance considered in [3], we will deal with the Wasserstein distance, because it is more appropriate in the multidimensional case, and we will establish a rate of convergence of the order $R^{-1/2}$, assuming a suitable non-degeneracy condition.

Key words and phrases. Stochastic heat equation, Malliavin calculus, central limit theorem.

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The paper is organized as follows. Section 2 contains the main results of this paper. In Section 3 we introduce some preliminaries and in Section 4 we present some important results, which we use later in the proof of the main theorems. Finally, the proof of the main results is given in Section 5.

2. Main results

The spatial integral \( \int_{-R}^{R} u_i(t, x) dx \) behaves like a sum of i.i.d random variables and it will be natural to expect a central limit theorem to hold in this situation. Along the paper, we will denote by \( C(t) \) the symmetric and nonnegative definite \( d \times d \) matrix given by

\[
C_{ij}(t) := 2 \sum_{k=1}^{m} \int_{0}^{t} \eta_{ij}^{(k)}(r) dr, \quad 1 \leq i, j \leq d,
\]

where

\[
\eta_{ij}^{(k)}(r) := E \left[ \sigma_{ik}(u(r, x))\sigma_{jk}(u(r, x)) \right].
\]

Notice that \( \eta_{ij}^{(k)}(r) \) does not depend on \( x \) because for any fixed \( t > 0 \), the process \( \{u(t, x), x \in \mathbb{R}\} \) is stationary. We will show that for each \( 1 \leq i, j \leq d \)

\[
\lim_{R \to \infty} E \left( F_i^R(t) F_j^R(t) \right) = C_{ij}(t).
\]

We will make use of the following non-degeneracy condition:

\[(H1)\] The vector space spanned by the family of vectors \( \{\sigma_{1k}(\mathbb{I}), \ldots, \sigma_{dk}(\mathbb{I})\}, k = 1, \ldots, m \) has dimension \( d \), where \( \mathbb{I} = (1, \ldots, 1) \).

Our first result is the following quantitative central limit theorem.

**Theorem 2.1.** Suppose that \( u(t, x) = (u_1(t, x), \ldots, u_d(t, x)) \) is the mild solution to equation (1.1) and let \( F^R(t) \) be given as in (1.3). Assume condition \((H1)\). Let \( d_W \) denote the Wasserstein distance defined in (3.6) below and let \( N(t) \) be a \( d \)-dimensional random vector with the multivariate normal distribution \( \mathcal{N}(0, C(t)) \) with mean zero and covariance matrix \( C(t) \) defined by (2.1). Then, there is a constant \( c > 0 \) depending on \( t \), such that for any \( R \geq 1 \),

\[
d_W(F^R(t), N(t)) \leq \frac{c}{\sqrt{R}}.
\]

Our second result is a functional central limit theorem.

**Theorem 2.2.** Let \( u(t, x) = (u_1(t, x), \ldots, u_d(t, x)) \) be the mild solution to equation (1.1) and let \( \eta_{ij}^{(k)}(r) \) be as in (2.2). Then for each \( T > 0 \)

\[
\left( \frac{1}{\sqrt{R}} \int_{-R}^{R} u(t, x) dx - 2R \right)_{t \in [0, T]} \rightarrow (M(t))_{t \in [0, T]},
\]

as \( R \) tends to infinity, where \( (M(t))_{t \in [0, T]} \) is a \( d \)-dimensional centered Gaussian martingale which satisfies

\[
E(M_i(t)M_j(s)) = \sum_{k=1}^{m} \int_{0}^{t \wedge s} 2\eta_{ij}^{(k)}(r) dr,
\]

for each \( s, t \in [0, T] \) and all \( 1 \leq i, j \leq d \).
3. Preliminaries

Let us first introduce the white noise on $\mathbb{R}_+ \times \mathbb{R}$. We denote by $\mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R})$ the collection of Borel sets $A \subseteq \mathbb{R}_+ \times \mathbb{R}$ with finite Lebesgue measure, denoted by $|A|$. Consider a centered Gaussian family of random variables $W = \{W_j(A), A \in \mathcal{B}_b, 1 \leq j \leq m\}$, defined in a complete probability space $(\Omega, \mathcal{F}, P)$, with covariance

$$E[W_j(A)W_k(B)] = 1_{(j=k)}|A \cap B|.$$  

We assume that the $\sigma$-algebra $\mathcal{F}$ is generated by $W$ and the $P$-null sets. For any $t \geq 0$, we denote by $\mathcal{F}_t$ the $\sigma$-algebra generated by the random variables

$$\{W_j([0, s] \times A) : 1 \leq j \leq m, 0 \leq s \leq t, A \in \mathcal{B}_b(\mathbb{R})\}.$$  

As proved in [8], for any $m$-dimensional adapted random field $\{X(s, y), (s, y) \in \mathbb{R}_+ \times \mathbb{R}\}$ that is jointly measurable and

$$\int_0^\infty \int_{\mathbb{R}} E[|X(s, y)|^2]dyds < \infty,$$

the following stochastic integral

$$\sum_{j=1}^m \int_0^\infty \int_{\mathbb{R}} X_j(s, y)W_j(ds, dy)$$

is well defined. Next we will introduce the basic elements of Malliavin calculus which are required to prove our results.

3.1. Malliavin Calculus. In this section we will discuss some basic facts about the Malliavin calculus associated with $W$. We refer the reader to [5] for a detailed account on the Malliavin calculus with respect to a Gaussian process.

Consider the Hilbert space $\mathcal{H} = L^2(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R}^m)$. The Wiener integral

$$W(h) = \sum_{j=1}^m \int_0^\infty \int_{\mathbb{R}} h_j(t, x)W^j(dt, dx)$$

provides an isometry between the Hilbert space $\mathcal{H}$ and $L^2(\Omega)$. In this sense $\{W(h), h \in \mathcal{H}\}$ is an isonormal Gaussian process.

We denote by $C^\infty_p(\mathbb{R}^n)$ the space of infinitely differentiable functions with all their partial derivatives having at most polynomial growth at infinity. Let $S$ be the space of simple and smooth random variables of the form

$$F = f(W(h^{(1)}), \ldots, W(h^{(n)}))$$

for $f \in C^\infty_p(\mathbb{R}^n)$ and $h^{(i)} \in \mathcal{H}, 1 \leq i \leq n$. Then, the Malliavin derivative $DF$ is the $\mathcal{H}$-valued random variable defined by

$$D^k_{s, y}F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(h^{(1)}), \ldots, W(h^{(n)}))h^{(i)}_k(s, y),$$

where $1 \leq k \leq m$ and $(s, y) \in \mathbb{R}_+ \times \mathbb{R}$. The derivative operator $D$ is a closable operator with values in $L^p(\Omega; \mathcal{H})$ for any $p \geq 1$. For any $p \geq 1$, let $\mathbb{D}^{1,p}$ be the completion of $S$ with respect to the norm

$$||F||_{1,p} = (E|F|^p + E||DF||^p_{\mathcal{H}})^{\frac{1}{p}}.$$  

We denote by $\delta$ the adjoint of the derivative operator given by the duality formula

$$E(\delta(u)F) = E(\langle u, DF \rangle_{\mathcal{H}})$$
for any $F \in \mathbb{D}^{1,2}$, and any $u \in L^2(\Omega; \mathcal{H})$ in the domain of $\delta$, denoted by $\text{Dom } \delta$. The operator $\delta$ is also called Skorohod integral and in the Brownian motion case it coincides with an extension of the Itô integral introduced by Skorohod (see [7]). More generally, in the context of the space-time white noise $W$, any $m$-dimensional adapted random field $X$ which is jointly measurable and satisfies (3.1) belongs to the domain of $\delta$ and $\delta(X)$ coincides with the Itô-Walsh integral

$$\delta(X) = \sum_{j=1}^{m} \int_{0}^{\infty} \int_{\mathbb{R}} X_j(s, y) W_j(ds, dy).$$

As a consequence, the mild equation to equation (1.1) can be written as

$$u_i(t, x) = 1 + \delta(1_{[0, t]}(\bullet)p_{t-\bullet}(x - s)\sigma_{ij}(u(\bullet, s)),

where $(\bullet, s)$ denotes a variable in $\mathbb{R}_+ \times \mathbb{R}$ and $\dagger \in \{1, \ldots, m\}$. It is known that for any $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ and each $i = 1, \ldots, d$, $u_i(t, x)$ belongs to $\mathbb{D}^{1,p}$ for any $p \geq 2$ and the derivative satisfies the following linear equation for $t \geq s$ and $1 \leq \ell \leq m$,

$$D_{s,y}^\ell u_i(t, x) = p_{t-s}(x - y)\sigma_{i\ell}(u(s, y)) + \sum_{j=1}^{m} \sum_{k=1}^{d} \int_{s}^{t} \int_{\mathbb{R}} p_{t-z}(x - z)\Sigma_{ij}^{(k)}(r, z)D_{s,y}^\ell u_k(r, z)W_j(dr, dz),

where $\Sigma_{ij}^{(k)}(r, z)$ is an adapted process, bounded by the Lipschitz constant of $\sigma_{ij}$. If $\sigma_{ij}$ is continuously differentiable, then $\Sigma_{ij}^{(k)}(r, z) = \partial_{x_j} \sigma_{ij}(u(r, z))$. This result is proved in Proposition 2.4.4 of [5] in the case of Dirichlet boundary condition on $[0,1]$ and with $d = m = 1$. The proof can be easily extended to the multidimensional case and with the space variable on $\mathbb{R}$. We also refer to [1, 6], where this result is used when the coefficient is continuously differentiable.

We use Stein method, which is a probabilistic technique which allows us to measure the distance between a probability distribution and a normal distribution, to prove our first theorem. With this aim, we next define the Wasserstein distance.

3.2. Wasserstein Distance. Let $F$ and $G$ denote two integrable $d$-dimensional random vectors defined on the probability space $(\Omega, \mathcal{F}, P)$. Then the Wasserstein distance is defined as

$$d_W(F, G) = \sup_{h \in \text{Lip}(1)} |E[h(F)] - E[h(G)]|.$$

Here, Lip($K$) stands for the set of functions $h : \mathbb{R}^d \to \mathbb{R}$ that are Lipschitz with constant $K > 0$, that is, satisfying $|h(x) - h(y)| \leq K|x - y|$ for all $x, y \in \mathbb{R}^d$.

4. Basic results

The next two results provide upper bounds for the Wasserstein distance between a $d$-dimensional random vector whose components can be expressed as divergences and a random vector with a $d$-dimensional Gaussian distribution.

**Proposition 4.1.** Let $F = (F_1, \ldots, F_d)$, with $F_i := \delta(v_i)$, where $v_i \in \text{Dom}(\delta), i = 1, \ldots, d$. Suppose that $N$ is a $d$-dimensional Gaussian centered vector with an invertible covariance
matrix $A$. Then

$$d_W(F, N) \leq \sqrt{d} \|A^{-1}\|_{\text{op}} \frac{1}{\delta_p} \left( \sum_{i,j=1}^d E(\|A_{ij} - \langle v_i, D F_j \rangle_H \|^2) \right),$$

where $\| \cdot \|_{\text{op}}$ denotes the operator norm of a matrix.

Proof. Using Theorem 4.41 in [4, page 85], we have

$$d_W(F, N) \leq \sup_{f \in \mathcal{F}^d_W(A)} \left| E \left[ \langle A, \text{Hess} f(F) \rangle_{HS} \right] - E \left[ \langle F, \nabla f(F) \rangle_{\mathbb{R}^d} \right] \right|,$$

where $\mathcal{F}^d_W(A)$ denotes the class of twice continuously differentiable functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$ such that

$$\sup_{x \in \mathbb{R}^d} \|\text{Hess} f(x)\|_{HS} \leq \sqrt{d} \|A^{-1}\|_{\text{op}} \frac{1}{\delta_p}.$$

Consider the expression

$$\Phi := E \left[ \langle A, \text{Hess} f(F) \rangle_{HS} \right] - E \left[ \langle F, \nabla f(F) \rangle_{\mathbb{R}^d} \right]$$

$$= \sum_{i,j=1}^d E \left[ A_{ij} \partial_i \partial_j f(F) \right] - \sum_{i=1}^d E \left[ F_i \partial_i f(F) \right],$$

where $\partial_i$ is a short notation for $\frac{\partial}{\partial x_i}$. Using the duality formula in the second term, we can write

$$\Phi = \sum_{i,j=1}^d E \left[ A_{ij} \partial_i \partial_j f(F) \right] - \sum_{i=1}^d E \left[ \langle v_i, D(\partial_i f(F))_H \rangle \right]$$

$$= \sum_{i,j=1}^d E \left[ A_{ij} \partial_i \partial_j f(F) \right] - \sum_{i=1}^d E \left[ \langle v_i, \sum_{j=1}^d \partial_j \partial_i f(F) D F_j \rangle_H \right]$$

$$= \sum_{i,j=1}^d E \left[ (A_{ij} - \langle v_i, D F_j \rangle_H) \partial_j \partial_i f(F) \right].$$

Finally, this clearly implies using the Cauchy-Schwartz inequality

$$|\Phi| \leq \left( \sum_{i,j=1}^d E(\|\partial_j \partial_i f(F)\|^2) \sum_{i,j=1}^d E(\|A_{ij} - \langle v_i, D F_j \rangle_H \|^2) \right)^{\frac{1}{2}},$$

which completes the proof. \hfill \Box

In the next proposition we will apply Proposition 4.1 to the random vector $F^R(t)$ defined in (1.3). Notice that from (3.4) we have the representation $F_i^R(t) = \delta(v_i^R(t))$ for $i = 1, \ldots, d$, where

$$v_i^R(t)(s, y) = 1_{[0,t]}(s) \frac{1}{\sqrt{R}} \sigma_{lk}(u(s, y)) \int_{-R}^R p_{t-s}(x-y) dx, \quad 1 \leq k \leq m.$$

**Proposition 4.2.** Let $F^R(t)$ be as defined in (1.3), and let $v^R(t) = (v_1^R(t), \ldots, v_d^R(t))$ be as defined in (4.2) for $i = 1, \ldots, d$. Suppose that $N^R(t)$ is a $d$-dimensional centered Gaussian
random variable with covariance matrix $C^R(t)$ such that $C^R_{ij}(t) = E[F^R_i(t)F^R_j(t)]$ for all $i, j = 1, \ldots, d$. Then,

\begin{equation}
(4.3) \quad d_W(F^R(t), N^R(t)) \leq \sqrt{d} \|(C^R(t))^{-1}\|_{op} \|C^R(t)\|_{op}^{\frac{1}{2}} \sum_{i,j=1}^{d} \text{Var}(\langle v^R_i(t), DF^R_j(t) \rangle_{\mathcal{H}}).
\end{equation}

Proof. By the duality formula (3.3) we can write

$$C^R_{ij}(t) = E(F^R_i(t)F^R_j(t)) = E(\delta(v^R_i(t))F^R_j(t)) = E[\langle v^R_i(t), DF^R_j(t) \rangle_{\mathcal{H}}].$$

As a consequence, from (4.1) we have

$$d_W(F^R(t), N^R(t)) \leq \sqrt{d} \|(C^R(t))^{-1}\|_{op} \|C^R(t)\|_{op}^{\frac{1}{2}} \sum_{i,j=1}^{d} \text{Var}(\langle v^R_i(t), DF^R_j(t) \rangle_{\mathcal{H}}).$$

This completes the proof of the proposition. \qed

Next we compute the entries of the asymptotic covariance matrix of the process $\{F^R(t), t \in [0, T]\}$.

**Proposition 4.3.** Let $G^R_i(t) = \int_{-R}^{R} u_i(t, x)dx - 2R$ for any $i = 1, \ldots, d$. Then

$$\lim_{R \to \infty} \frac{1}{R} E[G^R_i(t)G^R_j(s)] = 2 \sum_{k=1}^{m} \int_{0}^{t \wedge s} \eta^{(k)}_{ij}(r)dr,$$

where $\eta^{(k)}_{ij}(r) = E[\sigma_{ik}(u(r, x))\sigma_{jk}(u(r, x))]$ has been defined in (2.2).

Proof. The proof is similar to the one done in Proposition 3.1 of [3]. For the sake of completion we will give some details below. We can write

$$E[u_i(t, x)u_j(s, x')] = 1 + \sum_{k=1}^{m} \int_{0}^{t \wedge s} \int_{R}^{R} \sigma_{ik}(x-y)p_{t-r}(x'-y)\sigma_{jk}(u(r, x))dydr$$

$$= 1 + \sum_{k=1}^{m} \int_{0}^{t \wedge s} \eta^{(k)}_{ij}(r)p_{t-r}(x'-y)dydr$$

$$= 1 + \sum_{k=1}^{m} \int_{0}^{t \wedge s} \eta^{(k)}_{ij}(r)p_{t+s-2r}(x-x')dr.$$
As a consequence,
\[
\lim_{R \to \infty} \frac{1}{R} \text{Cov} \left( G_i^R(t), G_j^R(s) \right) = \lim_{R \to \infty} 2 \sum_{k=1}^{m} \int_{0}^{t} \eta_{ij}^{(k)}(r) \int_{0}^{2R} p_{t+s-2r}(z)(2 - \frac{z}{R})dzdr = 2 \sum_{k=1}^{m} \int_{0}^{t} \eta_{ij}^{(k)}(r)dr,
\]
which completes the proof of the proposition.

\[\square\]

5. PROOF OF THE MAIN RESULTS

In this section we will show Theorems 2.1 and 2.2.

5.1. Proof of Theorem 2.1.

Proof. Let us recall that we used \( C^R(t) \) to denote the covariance matrix of \( F^R(t) \) i.e. \( C^R(t) = E(F_i^R(t)F_j^R(t)) \). In the proof of Proposition 4.3 we obtained that
\[
C^R_ij(t) = 2 \sum_{k=1}^{m} \int_{0}^{t} \eta_{ij}^{(k)}(r) \int_{0}^{2R} p_{t+s-2r}(z)(2 - \frac{z}{R})dzdr.
\]

Let \( N^R(t) \) and \( N(t) \) denote two \( d \)-dimensional Gaussian random vectors with covariance matrices \( C^R(t) \) and \( C(t) \), where \( C(t) \) has been defined in (2.1). Applying the triangle inequality for the Wasserstein distance and can write
\[
d_W(F^R(t), N) \leq d_W(F^R(t), N^R(t)) + d_W(N^R(t), N(t)),
\]
The proof of Theorem 2.1 will be done in two steps.

Step 1. We claim that for any \( R \geq 1 \),
\[
(5.1) \quad d_W(F^R(t), N_R(t)) \leq \frac{c}{\sqrt{R}},
\]
where \( c \) is a constant depending on \( t \). To show (5.1) we make use of the following bound proved in Proposition 4.2:
\[
(5.2) \quad d_W(F^R(t), N_R(t)) \leq \sqrt{d\|C^R(t)\|_\infty^{-1}\|C^R(t)\|_\infty^{\frac{3}{2}}} \sqrt{\sum_{i,j=1}^{d} \text{Var}(\langle v_i^R(t), DF_j^R(t) \rangle_{\mathcal{H}})}.
\]

In order to compute \( \text{Var}(\langle v_i^R(t), DF_j^R(t) \rangle_{\mathcal{H}}) \), applying Fubini’s theorem we can write
\[
F_i^R(t) = \frac{1}{\sqrt{R}} \left( \int_{-R}^{R} u_i(t, x)dx - 2R \right)
= \sum_{k=1}^{m} \frac{1}{\sqrt{R}} \left( \int_{-R}^{R} \int_{0}^{t} \int_{R}^{2R} p_{t+s-2r}(x-y)\sigma_{ik}(u(s, y))W_k(ds, dy)dx \right)
= \sum_{k=1}^{m} \int_{0}^{t} \left( \frac{1}{\sqrt{R}} \int_{-R}^{R} p_{t+s-2r}(x-y)\sigma_{ik}(u(s, y))dx \right) W_k(ds, dy).
\]
We recall that for any fixed $t \geq 0$, $F_i^R(t) = \delta(v_i^R(t))$, where $v_i^R(t)$ is defined in (4.2). Moreover,

$$D_{s,y}^\ell F_i^R(t) = \mathbf{1}_{[0,t]}(s) \frac{1}{\sqrt{R}} \int_{-R}^R D_{s,y}^\ell u_i(t,x) dx.$$

From (3.5), we can write for $\ell = 1, \ldots, m,$

$$D_{s,y}^\ell u_i(t,x) = p_{t-s}(x-y)\sigma_{ie}(u(s,y))$$

$$+ \sum_{j=1}^m \sum_{k=1}^d \int_s^t \int_{-R}^R p_{t-r}(x-z)\Sigma_{ij}^{(k)}(r,z)D_{s,y}^\ell u_k(r,z)W_j(dr,dz),$$

where $\Sigma_{ij}^{(k)}(r,z)$ is the adapted process defined there. Therefore,

$$(DF_i^R(t), v_j^R(t))_H$$

$$= \sum_{\ell=1}^m \frac{1}{R} \int_0^t \int_{-R}^R \left( \int_{-R}^R p_{t-s}(x-y)dx \right)^2 \sigma_{ie}(u(s,y)\sigma_{j\ell}(u(s,y))dyds$$

$$+ \frac{1}{R} \sum_{\ell=1}^m \sum_{q=1}^m \sum_{k=1}^d \int_0^t \int_{-R}^R \int_{-R}^R \int_{-R}^R \int_s^t p_{t-r}(x-y)\sigma_{j\ell}(u(s,y))$$

$$\times \left( \int_{-R}^R \int_{-R}^R p_{t-r}(x-z)\Sigma_{iq}^{(k)}(r,z)D_{s,y}^\ell u_k(r,z)W_q(dr,dz) \right) dx dx'dyds.$$

Now using arguments analogous to those in the proof of Theorem 1.1 in [3], we deduce the bound

(5.3) $\text{Var} \left( (DF_i^R(t), v_j^R(t))_H \right) \leq cR^{-1}$,

where $c$ is a constant depending on $t$.

On the other hand, we claim that for a fixed $t > 0$,

(5.4) $\sup_{R>0} \|C^R(t)\|_{op} < \infty$

and

(5.5) $\sup_{R \geq 1} \| (C^R(t))^{-1} \|_{op} < \infty$.

Then the estimate (5.1) will be a consequence of the bounds (5.3), (5.4) and (5.5).

It is easy to show the estimate (5.4) and we omit the details. In order to show the claim (5.5) it suffices to get a lower bound, uniformly in $R \geq 1$, for the determinant of the matrix $C^R(t)$. We have

$$\det C^R(t) \geq \left( \inf_{|\xi| = 1} \xi^T C^R(t) \xi \right)^d.$$

We can obtain a lower bound for the quadratic form $\xi^T C^R(t) \xi$ as follows

$$\xi^T C^R(t) \xi = 2 \sum_{i,j,k=1}^d \int_0^t \eta_{ij}^{(k)}(r)\xi_i\xi_j \int_0^{2R} p_{2t-2r}(z)(2 - \frac{z}{R})dzdr$$

$$= 2 \sum_{k=1}^d \int_0^t E \left( \sum_{i=1}^d \xi_i \sigma_{ik}(u(r,0)) \right)^2 \int_0^{2R} p_{2t-2r}(z)(2 - \frac{z}{R})dzdr.$$
We have
\[ \int_0^{2R} p_{2t-2r}(z)(2 - \frac{z}{R})dz \geq \int_0^R p_{2t-2r}(z)dz \geq \int_0^1 p_{2t-2r}(z)dz. \]
Set
\[ \varphi_\xi(r) = 2 \sum_{k=1}^m E \left( \left| \sum_{i=1}^d \xi_i \sigma_{ik}(u(r, 0)) \right|^2 \right). \]
Then, we can write
\[ \xi^t C^R(t) \xi \geq \int_0^t \varphi_\xi(r) \int_0^1 p_{2t-2r}(z)dzdr \geq \int_0^1 p_t(z)dz \int_0^{t/2} \varphi_\xi(r)dr. \]
Our assumption (H1) implies that for any unit vector \( \xi \),
\[ \varphi_\xi(0) = 2 \sum_{k=1}^m \left( \left\| \sum_{i=1}^d \xi_i \sigma_{ik}(T) \right\|^2 \right) = 0. \]
This implies that \( \xi^t C^R(t) \xi > 0 \) and, by continuity, \( \inf_{|\xi|=1} \xi^t C^R(t) \xi > 0 \), which completes the proof of claim (5.5).

Step 2. We claim that for \( r \geq 1 \),
\[ d_W(N^R(t), N(t)) \leq cR^{-1}, \]
for some constant \( c > 0 \) depending on \( t \). The proof this estimate we make use of the following bound on the Wasserstein distance between two multidimensional Gaussian vectors (see, for instance, Exercise 4.5.3 in [4]):
\[ d_W(N^R(t), N(t)) \leq Q(C^R(t), C(t))\|C^R(t) - C(t)\|_{HS}, \]
where
\[ Q(C^R(t), C(t)) = \sqrt{d} \min\{\|(C^R(t))^{-1}\|_{op}, \|C(R)(t)\|_{op}^{1/2}, \|C(t)^{-1}\|_{op}, \|C(t)\|_{op}^{1/2}\}. \]
The estimates (5.4) and (5.5) imply that \( Q(C^R(t), C(t)) \) is uniformly bounded by a constant depending on \( t \) for any \( R \geq 1 \). Moreover,
\[ \|C^R(t) - C(t)\|_{HS} \leq \left[ \sum_{i,j=1}^d |C^R_{ij}(t) - C_{ij}(t)|^2 \right]^{1/2} \]
\[ \leq \left[ \sum_{i,j=1}^d 2 \sum_{k=1}^m \int_0^t \sum_{i,j=1}^d \eta_{ij}^{(k)}(r) \left( 1 - \int_0^{2R} p_{2t-2r}(2 - \frac{z}{R})dz \right) dr \right]^{1/2} \]
\[ \leq c \int_0^t \left| 1 - \int_0^{2R} p_{2t-2r}(2 - \frac{z}{R})dz \right| dr \]
\[ \leq c \int_0^t \left| 1 - 2 \int_0^{2R} p_{2t-2r}(z)dz \right| dr + c \int_0^t \int_0^{2R} p_{2t-2r}(z)dzdr \]
\[ \leq \frac{c}{R^2}. \]
This completes the proof of the estimate (5.6). \( \square \)
5.2. Proof of Theorem 2.2.

Proof. The proof is similar to that of Theorem 1.2 in [3] and we skip the details. It suffices to show the weak convergence of finite-dimensional distributions and the tightness property. Tightness follows from the next lemma, whose proof is analogous to that of Proposition 4.1 in [3].

Lemma 5.1. Let $u(t, x) = (u_1(t, x), \ldots, u_d(t, x))$ be the solution to equation (1.1). Then for each $i$, $1 \leq i \leq d$ and any $0 \leq s < t \leq T$ and any $p \geq 1$ there exists a constant $K = K(p, T)$ such that

$$
E \left( \left| \int_{-R}^{R} u_i(t, x)dx - \int_{-R}^{R} u_i(s, x)dx \right|^p \right) \leq KR^p(t - s)^{p/2}.
$$

In order to show the convergence in law of the finite-dimensional distributions, we fix points $0 \leq t_1 \leq \cdots \leq t_M \leq T$. Recall that for each $i_0, 1 \leq i_0 \leq d$

$$
F^{R}_{i_0}(t_i) = \frac{1}{\sqrt{R}} \left( \int_{-R}^{R} u_{i_0}(t_i, x)dx - 2R \right)
$$

and set, for $1 \leq i, j \leq M$ and $1 \leq p, q \leq d$,

$$
A^{ij}_{pq} := 2 \sum_{k=1}^{m} \int_{t_i \land t_j}^{t_i \land t_j} \eta^{(k)}_{pq}(r)dr, \quad 1 \leq i, j \leq M,
$$

where $\eta^{(k)}_{pq}(r)$ is defined in (2.2). With arguments similar to those the proof of Theorem 1.2 in [3], we can show that for all $1 \leq i, j \leq M$ and $1 \leq p, q \leq d$,

$$
\lim_{R \to \infty} E \left[ (A^{ij}_{pq} - \langle DF^{R}_{p}(t_i), v^{R}_{q}(t_j) \rangle_{H})^2 \right] = 0.
$$

We can then complete the proof by the methodology used in the proof of Theorem 1.2 in [3]. □

REFERENCES

[1] Chen, L., Hu, Y. and Nualart, D.: Regularity and strict positivity of densities for the nonlinear stochastic heat equation. Mem. Amer. Math. Soc. 273 (2021).
[2] Dalang, R. C., Khoshnevisan, D. and Nualart, E.: Hitting probabilities for a system of non-linear stochastic heat equations with multiplicative noise. Probab. Theory Rel. Fields 144 (2009) 371-427.
[3] Huang, J., Nualart, D. and Vitsaari, L.: A central limit theorem for the stochastic heat equation. Stochastic Process. Appl. 149 (2022) 7170-7184.
[4] Nourdin, I. and Peccati, G.: Normal approximation with Malliavin Calculus. From Stein’s method to universality. Cambridge University Press, Cambridge, 2012.
[5] Nualart, D.: The Malliavin Calculus and related topics. Springer, 2006.
[6] Nualart, D. and Quer-Sardanyons, L.: Existence and smoothness of the density for spatially homogeneous SPDEs. Potential Anal. 27 (2007) 281-299.
[7] Skorohod, A. V.: On a generalization of a stochastic integral. Theory Probab. Appl. 20 (1975) 219-233.
[8] Walsh, J. B.: An introduction to stochastic partial differential equations. In: École d’Été de Probabilités de Saint-Flour XIV-1984. Lecture notes in Math. 1180 265-439, Springer, Berlin, 1986.

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