CRYSTAL BASES AND CATEGORIFICATIONS

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ABSTRACT. This is a survey of the theory of crystal bases, global bases and cluster algebra structure on the quantum coordinate rings.

1. Crystal bases

The notion of quantum groups (or quantized universal enveloping algebras) was introduced by Drinfeld [7] and Jimbo [9] around 1985 in order to explain trigonometric $R$-matrices in 2-dimensional solvable models in statistical mechanics. Since then, the quantum group has been one of the important tools to describe new symmetries in representation theory and other fields.

The quantum group $U_q(\mathfrak{g})$ is an algebra over $\mathbb{C}(q)$, in which $q$ is a parameter of temperature in the 2-dimensional solvable model, and $q = 0$ corresponds to the absolute temperature zero. The notion of crystal bases was motivated by the belief that something extraordinary should happen at the absolute temperature zero. In fact, it turns out that the representations of $U_q(\mathfrak{g})$ have good bases at $q = 0$, which we call crystal bases ([19, 20]). Here a basis at $q = 0$ of a $\mathbb{C}(q)$-vector space $V$ means a pair $(L, B)$ consisting of a free module $L$ of $V$ over the ring $A_0 := \{f \in \mathbb{C}(q) \mid f$ is regular at $q = 0\}$ and a basis $B$ of the $\mathbb{C}$-vector space $L/qL$ together with an isomorphism $\mathbb{C}(q) \otimes_{A_0} L \cong V$.

The crystal bases have nice properties such as uniqueness, stability by tensor products, etc. Moreover the modified actions of the simple root vectors induce a combinatorial structure on the crystal basis, called crystal graph. This permits us to reduce many problems in the representation theory to problems of the combinatorics.

For example, the combinatorics describing the representation theory of $\mathfrak{gl}_n$ can be well understood with crystal bases. The irreducible representation over $\mathfrak{gl}_n$ is parameterized by the highest weight, which corresponds to a Young diagram in the combinatorial language. Then Young tableaux with the given Young diagram label the crystal basis of the corresponding irreducible representation. The Littlewood-Richardson rule,
describing the decomposition of the tensor product of a pair of representations into irreducible components, may be also clearly explained by means of crystal bases.

The combinatorial description of crystal bases by Young tableaux is generalized to other types of simple Lie algebras in [25]. Littelmann gave a completely different combinatorial description of crystal bases (path model) in [34]. Lusztig gave also bases at \( q = 0 \) from the PBW bases in the ADE case ([35]).

See [21, 22] for surveys of crystal bases.

2. Global bases

A crystal basis is a basis at \( q = 0 \). However, we can extend this basis to the whole \( q \)-space in a unique way to obtain a true basis of the representation, which we call the lower global basis ([20]). More precisely, the representation \( V \) is a \( \mathbb{C}(q) \)-vector space and it is equipped a bar-involution \( c: V \xrightarrow{\sim} V \), i.e., a \( \mathbb{C} \)-linear involution of \( V \) satisfying \( c(qv) = q^{-1}c(v) \) for any \( v \in V \). Moreover \( V \) has a certain \( \mathbb{C}[q^{\pm 1}] \)-form, i.e., a free \( \mathbb{C}[q^{\pm 1}] \)-submodule \( V_{\mathbb{C}[q^{\pm 1}]} \) of \( V \) such that \( \mathbb{C}(q) \otimes_{\mathbb{C}[q^{\pm 1}]} V_{\mathbb{C}[q^{\pm 1}]} \xrightarrow{\sim} V \). Then we can prove that the map \( L \cap c(L) \cap V_{\mathbb{C}[q^{\pm 1}]} \to L/qL \) is an isomorphism and the inverse image \( B \subset L \cap c(L) \cap V_{\mathbb{Q}} \) of \( B \) is a basis of the \( \mathbb{C}(q) \)-vector space \( V \) and also a basis of the \( A_0 \)-module \( L \). We call \( B \) the lower global basis of \( V \).

The negative half \( U_q^- (\mathfrak{g}) \) of \( U_q (\mathfrak{g}) \) has also a crystal basis \( B(U_q^- (\mathfrak{g})) \), and it lifts to the lower global basis \( B^{low} \).

It is first introduced by Lusztig under the name of canonical basis in the ADE cases inspired by the work of Ringel ([39]) describing \( U_q^- (\mathfrak{g}) \) as the Hall algebras associated with quivers (see [35, 36]). It is shown that the canonical basis and the global basis coincide (Grojnowski-Lusztig [10]).

The dual basis of the lower global basis is called the upper global basis. The lower global basis \( B^{low} \) of \( U_q^- (\mathfrak{g}) \) is a basis of the integral form \( U_{\mathbb{Z}[q^{\pm 1}]}^- (\mathfrak{g}) \) of the negative half of \( U_q (\mathfrak{g}) \). The upper global basis \( B^{up} \) is the basis of \( A_q (n) \), the dual form of the integral form \( U_{\mathbb{Z}[q^{\pm 1}]}^- (\mathfrak{g}) \) ([23]). With a canonical symmetric bilinear form on \( U_q^- (\mathfrak{g}) \), the dual form \( A_q (n) \) may be regarded as a subalgebra of \( U_q^- (\mathfrak{g}) \). However, at \( q = 1 \), the integral form \( U_{\mathbb{Z}[q^{\pm 1}]}^- (\mathfrak{g}) \) becomes the universal enveloping algebra \( U(n) \) of the negative half \( n \) of \( \mathfrak{g} \), while \( A_q (n) \) becomes the coordinate ring \( \mathbb{C}[n] \) of \( n \).

3. Quiver Hecke algebras

The notion of quiver Hecke algebras (sometimes called the KLR algebras) is introduced independently by Rouquier ([40, 41]) and Khovanov-Lauda ([26, 27]).

It is a family of \( \mathbb{Z} \)-graded algebras which categorifies the negative half \( U_q^- (\mathfrak{g}) \) of a quantum group. More precisely, there exists a family of algebras \( \{ R(n) \}_{n \in \mathbb{Z}_{\geq 0}} \) (quiver Hecke algebras) such that the (split) Grothendieck group \( K(R\text{-gproj}) \) of the direct
sum $R$-gproj := $\bigoplus_{n \in \mathbb{Z}_{\geq 0}} R(n)$-gproj of the categories of finitely generated projective graded $R(n)$-modules is isomorphic to the integral form $U_{\mathbb{Z}[q^{\pm 1}]}(\mathfrak{g})$ of the negative half of $U_q(\mathfrak{g})$ as a $\mathbb{Z}[q^{\pm 1}]$-algebra. Note that the Grothendieck group $K(R$-gproj) has a natural structure of $\mathbb{Z}[q^{\pm 1}]$-algebra, where the multiplication is induced by the monoidal category structure of $R$-gproj given by the convolution product and the action of $q$ is induced by the grading shift functor.

Remark. (i) The quiver Hecke algebra $\{R(n)\}_{n \in \mathbb{Z}_{\geq 0}}$ depends on a base ring $k$, an index set $I$ (the index of simple roots) and a family of polynomials $Q_{ij}(u, v) \in k[u, v]$ ($i \neq j \in I$) such that $Q_{ij}(u, v) = Q_{ji}(v, u)$. It is related with $U_q(\mathfrak{g})$ as follows. Let $A = \{a_{ij}\}_{i,j \in I}$ be a generalized Cartan matrix for $\mathfrak{g}$. Then $Q_{ij}(u, v)$ is a polynomial in $u$ (with coefficients in $k[v]$) of degree $-a_{ij}$ with an element of $k^\times$ as the top coefficient. (For the precise definition, see e.g., [40, 18].)
(ii) For a given generalized Cartan matrix $A$, the representation theory of $R$ depends on a choice of $\{Q_{ij}\}$ (see [24]).
(iii) The quiver Hecke algebra is called symmetric if $Q_{ij}(u, v) = (u - v)^{-a_{ij}}$ up to a constant multiple.

The Grothendieck group $K(R$-gproj) has the basis consisting of the isomorphism classes of indecomposable project modules. If $R$ is symmetric (and the base field is characteristic 0), then the lower global basis corresponds to this basis of $K(R$-gproj) (Varagnolo-Vasserot [42]).

There is a dual statement. The Grothendieck group $K(R$-gmod) of the direct sum $R$-gmod := $\bigoplus_{n \in \mathbb{Z}_{\geq 0}} R(n)$-gmod of the categories of finite-dimensional graded $R(n)$-modules is isomorphic to the dual of $K(R$-gproj). Hence $K(R$-gmod) is isomorphic to $A_q(n)$, the dual form of the integral form $U_{\mathbb{Z}[q^{\pm 1}]}(\mathfrak{g})$. Moreover, if $R$ is symmetric (and the base field is characteristic 0), then the upper global basis corresponds to the set of the isomorphism classes of simple modules.

As a consequence, the coefficients $C_{b_1, b_2}^b$ appearing in the multiplication

$$b_1 \cdot b_2 = \sum_b C_{b_1, b_2}^b b \quad \text{where } b_1, b_2, b \in B^{\text{low}} \text{ or } B^{\text{up}}$$

belong to $\mathbb{Z}_{\geq 0}[q, q^{-1}]$ when the generalized Cartan matrix is symmetric. Here, $\mathbb{Z}_{\geq 0}[q, q^{-1}]$ is the set of Laurent polynomials in $q$ with non-negative integers as coefficients. In general, the positivity fails (first observed in the $G_2$ case by S. Yamane ([43]).

Note that Lauda-Vazirani ([31]) proved that the set of the isomorphism classes of simple modules in $R$-gmod is canonically isomorphic to $B(U_q^-(\mathfrak{g}))$ for an arbitrary quiver Hecke algebra.

The cyclotomic quotient $R^\Lambda(n)$ of $R(n)$ provides a categorification of the integrable highest weight module $V(\Lambda)$ of $U_q(\mathfrak{g})$ ([16]) with a dominant integral weight $\Lambda$ as a highest weight.
One of the motivations of these categorification theorems originated from the so-called LLT-Ariki theory. In 1996, Lascoux-Leclerc-Thibon ([30]) conjectured that the irreducible representations of Hecke algebras of type $A$ are controlled by the upper global basis of the basic representation of the quantum affine algebra $U_q(A^{(1)}_{N-1})$. Soon after, Ariki ([1]) proved this conjecture by showing that the cyclotomic quotients of affine Hecke algebras categorify the irreducible highest weight modules over $U(A^{(1)}_{N-1})$, the universal enveloping algebra of affine Kac-Moody algebra of type $A^{(1)}_{N-1}$. In [4, 40], Brundan-Kleshchev ([4]) and Rouquier ([41]) showed that the affine Hecke algebra of type $A$ is isomorphic to the quiver Hecke algebra of type $A^{(1)}_{N-1}$ or of type $A_\infty$ up to a specialization and a localization. Thus the quiver Hecke algebras can be understood as a graded version of the affine Hecke algebras of type $A$, and Kang-Kashiwara’s cyclotomic categorification theorem ([16]) is a generalization of Ariki’s theorem on $A^{(1)}_{N-1}$ and $A_\infty$ to all symmetrizable Cartan datum.

4. Cluster algebras

As one can imagine from the fact that $A_q(n)$ is a commutative algebra at $q = 1$, the upper global basis of $A_q(n)$ has an interesting multiplicative property.

Berenstein and Zelevinsky (cf. [2]) conjectured that, when the generalized Cartan matrix is of finite type, there exists a family $\mathcal{F}$ of finite subsets of the upper global basis $B^{\text{up}}$ of $A_q(n)$ satisfying the following properties:

(a) Any pair $(x, y)$ of elements of $C \in \mathcal{F}$ is $q$-commutative (i.e., $xy = q^n yx$ for some $n \in \mathbb{Z}$),
(b) For any $C \in \mathcal{F}$, any $C$-monomial, i.e., an element of the form $x_1 \cdots x_\ell$ with $x_1, \ldots, x_\ell \in C$, belongs to $q^Z B^{\text{up}} := \{q^n b \mid n \in \mathbb{Z}, b \in B^{\text{up}}\}$.
(c) $B^{\text{up}}$ is the union of $C$-monomials (up to a constant multiple) where $C$ ranges over $\mathcal{F}$.

Then Leclerc ([32]) gave a counterexample to this conjecture. He called an element $b \in B^{\text{up}}$ a real vector if $b^2 \in q^Z B^{\text{up}}$. Otherwise, $b$ is called imaginary. If Conjecture were true, then any $b \in B^{\text{up}}$ would be a real vector. He gave examples of imaginary vectors for types $A_n$ ($n \geq 5$), $B_n$ ($n \geq 3$), $C_n$ ($n \geq 3$), $D_n$ ($n \geq 4$), and all exceptional types.

Although their conjecture failed, their idea survives and it was one of the motivations of the introduction of cluster algebras by Fomin and Zelevinsky ([8]). They replaced condition (c) with a weaker condition

$$(c)' \quad U_\mathfrak{g}^- (\mathfrak{g}) \text{ is generated by } \bigcup_{C \in \mathcal{F}} C \text{ as a } \mathbb{C}(q)\text{-algebra.}$$

They call $C \in \mathcal{F}$ a cluster, and reformulated the conjecture in the language of cluster algebras.
A cluster algebra is a \( \mathbb{Z} \)-subalgebra of a rational function field given by a set of generators, called the cluster variables. These generators are grouped into overlapping subsets, called clusters. The clusters are defined inductively by a procedure called mutation from the initial cluster \( \{ X_i \}_{1 \leq i \leq r} \). The mutation is controlled by an exchange matrix \( \tilde{B} = (b_{ij})_{ij} \) as follows. By the mutation at \( k \) (1 \( \leq k \leq r \)), a new cluster is created from the old cluster by replacing the \( k \)-th variable \( X_k \) with
\[
X'_k = \frac{\prod_{i: b_{ik} > 0} X_i^{b_{ik}} + \prod_{i: b_{ik} < 0} X_i^{-b_{ik}}}{X_k}.
\]

The exchange matrix \( \tilde{B} = (b_{ij})_{ij} \) is also mutated to the new exchange matrix \( \mu_k(\tilde{B}) = (b'_{ij})_{ij} \) given by
\[
b'_{ij} = \begin{cases} 
-b_{ij} & \text{if } i = k \text{ or } j = k, \\
b_{ij} + (-1)^{\delta(b_{ik} < 0)} \max(b_{ik} b_{kj}, 0) & \text{otherwise}.
\end{cases}
\]

A cluster monomial is a monomial of cluster variables in one cluster. In the conjecture above, every \( C \in \mathcal{F} \) is a cluster. Moreover, any two members of \( \mathcal{F} \) are connected by successive mutations. We do not assume (c) but we assume that the algebra is generated (as an algebra) by the cluster monomials.

Fomin and Zelevinsky proved that every cluster variable is a Laurent polynomial of the initial cluster \( \{ X_i \}_{1 \leq i \leq r} \). They conjectured that this Laurent polynomial has positive coefficients ([8]). This positivity conjecture was proved by Lee and Schiffler in the skew-symmetric cluster algebra case in [33]. The linearly independence conjecture on cluster monomials was proved in the skew-symmetric cluster algebra case in [5].

The notion of quantum cluster algebras, introduced by Berenstein and Zelevinsky in [3], can be considered as a \( q \)-analogue of cluster algebras. It is an algebra over \( \mathbb{Z}[q^{\pm 1/2}] \). The cluster variables in a cluster \( q \)-commute with each other. As in the cluster algebra case, every cluster variable belongs to \( \mathbb{Z}[q^{\pm 1/2}][X_i^{\pm 1}]_{1 \leq i \leq r} \) for the initial cluster \( \{ X_i \}_{1 \leq i \leq r} \) ([3]), and is expected to be an element of \( \mathbb{Z}_{\geq 0}[q^{\pm 1/2}][X_i^{\pm 1}]_{1 \leq i \leq r} \), which is referred to as the quantum positivity conjecture (cf. [6, Conjecture 4.7]). In [29], Kimura and Qin proved the quantum positivity conjecture for quantum cluster algebras containing acyclic seed and specific coefficients.

Assume that the generalized Cartan matrix is symmetric. In a series of papers [15, 14, 13], Geiß, Leclerc and Schröer showed that the quantum unipotent coordinate algebra \( A_q(\mathfrak{n}(w)) \) has a skew-symmetric quantum cluster algebra structure whose initial cluster consists of quantum minors. Here, \( A_q(\mathfrak{n}(w)) \) is a \( \mathbb{Z}[q^{\pm 1}] \)-subalgebra of \( A_q(\mathfrak{n}) \) associated with a Weyl group element \( w \). In [28], Kimura proved that \( A_q(\mathfrak{n}(w)) \) is compatible with the upper global basis \( \mathbf{B}^{up} \) of \( A_q(\mathfrak{n}) \); i.e., the set \( \mathbf{B}^{up}(w) := A_q(\mathfrak{n}(w)) \cap \mathbf{B}^{up} \) is a basis of \( A_q(\mathfrak{n}(w)) \). Thus, one can expect that every cluster monomial of
\(A_q(n(w))\) is contained in the upper global basis \(B^{up}(w)\), which is named the quantization conjecture by Kimura ([28]).

5. Monoidal categorification

This conjecture (in the symmetric generalized Cartan matrix case) is proved affirmatively by Kang-Kashiwara-Kim-Oh([18]), using the monoidal categorification of \(A_q(n(w))\) by a subcategory of the module category \(R\text{-gmod}\) over the quiver Hecke algebras.

In [11], Hernandez and Leclerc introduced the notion of a monoidal categorification of a cluster algebra. Let \((\mathcal{C}, \otimes)\) be a monoidal category. We say that a simple object \(S\) of \(\mathcal{C}\) is real if \(S \otimes S\) is simple. We say that a simple object \(S\) is prime if there exists no non-trivial factorization \(S \simeq S_1 \otimes S_2\). They say that \(\mathcal{C}\) is a monoidal categorification of a cluster algebra \(A\) if the Grothendieck ring of \(\mathcal{C}\) is isomorphic to \(A\) and if

(M1) any cluster monomial of \(A\) corresponds to the class of a real simple object of \(\mathcal{C}\),

(M2) any cluster variable of \(A\) corresponds to the class of a real simple prime object of \(\mathcal{C}\).

(Note that the above version is weaker than the original definition of the monoidal categorification in [11].) They proved that certain categories of modules over symmetric quantum affine algebras \(U'_q(g)\) give monoidal categorifications of cluster algebras. Nakajima extended it to the cases of the cluster algebras of type \(A, D, E\) ([37]) (see also [12]).

Once a cluster algebra \(A\) has a monoidal categorification, the positivity of cluster variables of \(A\) and the linear independence of cluster monomials of \(A\) follow.

In order to give a monoidal categorification of \(A_q(n(w))\), we use a monoidal cluster, a categorification of a cluster. It is a finite set of real simple objects \(\{M_i\}_{1 \leq i \leq r}\) in the monoidal category \((R\text{-gmod}, \otimes)\) of the finite-dimensional graded modules over the quiver Hecke algebras, which satisfies the condition: \(M_i \otimes M_j \simeq M_j \otimes M_i\) up to a grading shift. Then the mutation at \(k\) \((1 \leq k \leq r)\) creates a new monoidal cluster from the old monoidal cluster by replacing the \(k\)-th object \(M_k\) with a real simple object \(M'_k\). Here, the mutated object \(M'_k\) is related with the original monoidal cluster by the following exact sequences (up to grading shifts) instead of the relation (1):

\[
\begin{align*}
0 & \rightarrow \bigotimes_{i; b_{ik} < 0} M_i ^{\otimes (-b_{ik})} \rightarrow M_k \otimes M'_k \rightarrow \bigotimes_{i; b_{ik} > 0} M_i ^{\otimes b_{ik}} \rightarrow 0, \\
0 & \rightarrow \bigotimes_{i; b_{ik} > 0} M_i ^{\otimes b_{ik}} \rightarrow M'_k \otimes M_k \rightarrow \bigotimes_{i; b_{ik} < 0} M_i ^{\otimes (-b_{ik})} \rightarrow 0.
\end{align*}
\]

Note that if one passes to the Grothendieck group level, then (1) is a consequence of (2).
In [18], it is proved that if the first step mutations starting from the initial monoidal cluster are possible, then every successive mutations are possible. As its consequence, the quantum unipotent coordinate algebra $A_q(n(w))$ has a monoidal categorification, and the quantization conjecture follows. Note that F. Qin also provided a proof of the conjecture for a large class with a condition on the Weyl group element $w$ in a completely different method ([38]).

Note that the converse of (M1) is conjectured by several experts (e.g., see [15, 28]): (M1)' any real element of $B(A_q(n(w)))$ is a cluster monomial.

It is still open.

6. Real objects

By the monoidal categorification, a real element of the upper global basis corresponds to a real object of the monoidal category $R$-gmod of finite-dimensional graded modules over the quiver Hecke algebra. Real simple objects have remarkable properties. Let $(\mathcal{C}, \otimes)$ be a monoidal category which is either the monoidal category $R$-gmod of finite-dimensional graded modules over the symmetric quiver Hecke algebra, or the monoidal category of finite-dimensional modules over an affine quantum group. Let $k$ be the base field of $\mathcal{C}$. Then we have the following propositions.

**Proposition 1** ([17, 18]). Let $M$ and $N$ be simple objects of $\mathcal{C}$. We assume that either $M$ or $N$ is real. Then we have (forgetting grading shifts)

(i) $\text{Hom}(M \otimes N, M \otimes N) = k\text{id}_{M \otimes N},$

(ii) there exists a non-zero morphism $r: M \otimes N \to N \otimes M$ such that $\text{Hom}(M \otimes N, N \otimes M) = kr,$

(iii) $\text{Im}(r)$ is simple and it coincides with the head of $M \otimes N$ and also with the socle of $N \otimes M$.

Conversely, a simple object $M$ is real as soon as $\text{End}(M \otimes M) = k\text{id}_{M \otimes M}$.

We denote by $M \nabla N$ the simple head of $M \otimes N$.

**Proposition 2** ([17]). Let $M$ be a real simple object of $\mathcal{C}$ and let $N, N'$ be simple objects. If $M \nabla N \simeq M \nabla N'$, then $N \simeq N'$.

**Remark.** (i) Note that the Grothendieck ring $K(\mathcal{C})$ is commutative if $\mathcal{C}$ is the module category over an affine quantum group. Similarly, the Grothendieck ring $K(R$-gmod) is commutative if we forget the grading shifts.

(ii) In (2), we have

$$M_k \nabla M'_k \simeq \bigotimes_{i; b_{ik} > 0} M_i^{\otimes b_{ik}}$$

and

$$M'_k \nabla M_k \simeq \bigotimes_{i; b_{ik} < 0} M_i^{\otimes -b_{ik}}.$$
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