INTERSECTION COHOMOLOGY OF PURE SHEAF SPACES USING KIRWAN’S DESINGULARIZATION

KIRYONG CHUNG AND YOUNGHO YOON

Abstract. Let $M_n$ be the Simpson compactification of twisted ideal sheaves $\mathcal{I}_{L,Q}(1)$ where $Q$ is a rank 4 quadric hypersurface in $\mathbb{P}^n$ and $L$ is a linear subspace of dimension $n-2$. This paper calculates the intersection Poincaré polynomial of $M_n$ using Kirwan’s desingularization method. We obtain the intersection Poincaré polynomial of the moduli space for one-dimensional sheaves on del Pezzo surfaces of degree $\geq 8$ by considering wall-crossings of stable pairs and complexes.

1. Introduction

1.1. Kronecker quiver and related works. Let $M_n$ be the space parameterizing semi-stable sheaves $F$ on the projective space $\mathbb{P}^n$ with a linear free resolution

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \oplus \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n} \to F \to 0.$$ (1.1)

By a deformation theoretic argument of sheaves, $M_n$ is an irreducible normal variety with dimension $4n-3$ and generic moduli points of $M_n$ parameterize twisted ideal sheaves $\mathcal{I}_{L,Q}(1)$, where $Q$ is a rank 4 quadric hypersurface in $\mathbb{P}^n$ and $L$ is a linear subspace of dimension $n-2$. The space $M_n$ (as a quiver representation space) has been studied in several areas, including homological mirror symmetry, birational geometry, and curve counting theory. King ([Kin94]) showed that a general quiver representation space is projective under suitable conditions, and this space has been used in several areas of algebraic geometry. For our purpose in this paper, let $K_n(a,b)$ be the moduli space of Kronecker quiver representations with dimension vector $(a,b)$ and $(n+1)$-simple arrows (see Section 2.3 for the complete definition). Hosono and Takagi ([HT16]) used Kronecker modules space $K_n(2,2)$ (called the double symmetroid) as the starting point to find a pair of derived equivalent but not birationally equivalent Calabi-Yau threefolds.

In terms of birational geometry, Kontsevich’s moduli space $M_0(\text{Gr}(d, n+1), d)$ of degree $d$ and genus zero stable maps to $\text{Gr}(d, n+1)$ is birationally equivalent to $K_n(2, d)$ ([CM17, Proposition 4.8]). This paper focuses on cases for $d = 2$, since $M_n \cong K_n(2, 2)$ is a minimal birational contraction of $M_0(\text{Gr}(2, n+1), 2)$ ([CM17]).

On the other hand, the moduli space $M_S(c, \chi)$ of semi-stable pure sheaves $F$ with $c_1(F) = c$ and $\chi(F) = \chi$ on a del Pezzo surface $S$ has been studied in the virtual curve counting theory, where the Gopakumar-Vafa (GV) invariant of the local surface (i.e., the total space Tot$(K_S)$ of the canonical line bundle $K_S$ on $S$) is conjectured to be the topological Euler number of the moduli space $M_S(c, \chi)$ whenever it is smooth ([Kat08]). One method to compute the Euler number is to use Bridgeland wall-crossings of $M_S(c, \chi)$ ([BMW14, CHW14, CC15]). The moduli space is

2010 Mathematics Subject Classification. 14B05, 14F43, 14N3, 32S35, 32S60, 55N33.

Key words and phrases. Partial desingularization, Intersection Poincaré polynomial, Open cone of a link.
regarded as the moduli space of semi-stable objects on the derived category of \( \text{Coh}(S) \). The Kronecker modules space \( K_n(a, b) \) (or a projective bundle over \( K_n(a, b) \)) naturally arises as the final model of Bridgeland wall-crossings of \( M_S(c, \chi) \).

This paper calculates the intersection cohomology (of middle perversity) of \( M_n \) using the geometric invariant theoretic (GIT) quotient description for \( M_n \cong K_n(2, 2) \) combined with Kirwan’s method ([Kir86a, Kir86b]). Subsequently, we use the result to compute the intersection cohomology group for \( M_S(c, \chi) \) on del Pezzo surfaces \( S \) of degree \( \geq 8 \).

1.2. Main result and application. The main result of this paper is the following.

**Theorem 1.1.** For each integer \( n \geq 2 \), the intersection Poincaré polynomial of \( M_n \) is

\[
\text{IP}(M_n) = \frac{(1 - t^{4n+4})(1 - t^{4\lfloor \frac{n}{2} \rfloor})(1 - t^{4\lfloor \frac{n+1}{2} \rfloor})}{(1 - t^2)(1 - t^4)^2},
\]

where \( \lfloor x \rfloor \) is the largest integer \( \leq x \).

The key ingredient of the proof of Theorem 1.1 is that a partial desingularization of \( M_n \) is isomorphic to the moduli space \( M_0(\text{Gr}(2, n+1), 2) \) of degree two stable maps to Grassmannian variety \( \text{Gr}(2, n+1) \) ([CM17, Theorem 5.1]). We calculate the intersection Poincaré polynomial of \( M_n \) ([Kir86a, Kir86b]) considering the variation of intersection Betti numbers of intermediate moduli spaces. One key issue is to check that each term is pure and balanced Hodge type (see (5) of Remark 2.11 for the definition). Thus Theorem 1.1 is recovered in the level of the intersection E-polynomial by letting \( t^2 := uv \).

As corollaries of Theorem 1.1, relating \( M_n \) and \( M_S(\beta, \chi) \) using wall-crossings of pairs and complexes, we obtain topological invariants of moduli space \( M_S(\beta, \chi) \) on del Pezzo surfaces such as the Hirzebruch surface \( \mathbb{F}_k = \mathbb{P}(O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-k)) \) for \( k = 0, 1 \) and the projective plane \( \mathbb{P}^2 \). More precisely,

**Corollary 1.2.** Let \( M_S(\beta, \chi) \) be the moduli space of semi-stable sheaves \( F \) on a del Pezzo surface \( S \) with \( c_1(F) = \beta \in \text{H}_2(S, \mathbb{Z}) \) and \( \chi(F) = 2 \). Then the intersection Euler numbers of the moduli space can be expressed as

| \( S \) | \( \beta \in \text{H}_2(S, \mathbb{Z}) \) | Intersection Euler number of \( M_S(\beta, \chi) \) |
|---|---|---|
| \( \mathbb{F}_0 \) | \( c_1(O_{\mathbb{F}_0}(2, 2)) \) | 36 |
| \( \mathbb{F}_1 \) | \( c_1(O_{\mathbb{F}_1}(4, 2)) \) | 110 |
| \( \mathbb{P}^2 \) | \( c_1(O_{\mathbb{P}^2}(4)) \) | 192 |

More generally, we calculate virtual intersection Poincaré polynomials of moduli spaces in Corollary 4.1, 4.12, and 4.13. A key idea for the proof of these Corollaries is the following. The difference between the E-polynomials and the intersection E-polynomials of a quasi-projective variety \( Y \) is completely measured by geometric information from the analytic neighborhood of the singular locus of \( Y \) ([CMS08]). Since the space \( M_n \) has the same singularity type as the moduli space \( M_S(c, \chi) \) (cf. Remark 2.1), we can obtain the intersection Poincaré polynomial of \( M_S(c, \chi) \) from the singularity type of \( M_n \).
Remark 1.3. Upon replacing $\chi(F) = 2$ by $\chi(F) = 1$ in Corollary 1.2, the (intersection) Euler numbers of the moduli spaces (GV-numbers) do not change ([BS14, Proposition 12], [CvGKT18, Proposition 4.9], and [CC17, Corollary 5.2]). This is interesting, since the moduli spaces may not be isomorphic to each other for different $\chi$. For example, if $S = \mathbb{P}^2$, $M_{\mathbb{P}^2}(d, \chi)$ is isomorphic to $M_{\mathbb{P}^2}(d', \chi')$ if and only if $d = d'$ and $\chi \equiv \pm \chi' \pmod{d}$ ([Woo13, Theorem 8.1]).

Remark 1.4. The fact that the moduli space of sheaves on the del Pezzo surface $\geq 8$ is birationally equivalent to the space $M_n$ gives us a chance to compute the E-polynomial of the moduli spaces.

1.3. Structure of this paper. Section 2 reviews geometric properties of several moduli spaces that are subsequently used to calculate the intersection Poincaré polynomial of $M_n$. We also recall some basic notions and properties related to the (intersection) E-polynomial of a quasi-projective variety. In section 3 we prove Theorem 1.1 using Kirwan’s method ([Kir86b, (2,1) and (2,28)]) and obtain a numerical relationship with open cones from the singular loci of $M_n$ (Corollary 3.4). In section 4 we calculate the (virtual) intersection Poincaré polynomial for moduli space $M_S(\beta, \chi)$ on del Pezzo surfaces (Corollaries 4.1, 4.12, and 4.13) using explicit birational morphisms and wall-crossings among related spaces.

Acknowledgement. Some parts of this work were completed while K. Chung attended the Mixed Hodge Modules and Birational Geometry summer course at Johannes Gutenberg University Mainz in July, 2018, and he thanks the organizers for their invitation and hospitality. The authors gratefully acknowledge many helpful suggestions from Seung-Jo Jung, Joonyeong Won, and Sang-Bum Yoo during the preparation of the paper. We also thank the anonymous reviewer for valuable comments and suggestions to improve the quality of the paper.

2. Preliminaries

This section reviews several properties for moduli spaces of our interest and the E-polynomial of a quasi-projective variety which we will use.

2.1. Moduli of stable maps to Grassmannian. Let us recall the definition and geometric properties for spaces of stable maps. Let $X$ be a projective variety with a fixed embedding in $\mathbb{P}^n$, and $C$ be a projective connected reduced curve. A map $f : C \to X$ is called stable if $C$ has at worst nodal singularities and $|\text{Aut}(f)| < \infty$. Let $M_g(X, d)$ be the moduli space of stable maps with arithmetic genus $g(C) = g$ and degree $\text{deg}(f) = d$. If $X$ is a convex variety and $g = 0$, then moduli space $M_0(X, d)$ is a projective variety with at most finite group quotient singularity ([FP97, Theorem 2]). This paper focuses on the case $X = \text{Gr}(2, n + 1)$ and $d = 2$, denoted as

$$K_n := M_0(\text{Gr}(2, n + 1), 2).$$

2.2. Moduli space of semi-stable sheaves. Let $X$ be a smooth projective variety with a fixed polarization $L$. For a coherent sheaf $F$ on $X$, the Hilbert polynomial $P(F)(m)$ is defined as $\chi(F \otimes L^m)$. If the support of $F$ has dimension $d$, $P(F)(m)$ has degree $d$ and can be expressed as

$$P(F)(m) = \sum_{i=0}^{d} a_i \frac{m^i}{i!},$$
where \( r(F) := a_d \) is called the multiplicity of \( F \), and the reduced Hilbert polynomial is \( p(F)(m) := P(F)(m)/r(F) \). A pure sheaf \( F \) is semi-stable if for every nonzero proper subsheaf

\[
F' \subset F, \quad p(F')(m) \leq p(F)(m)
\]

for \( m \gg 0 \). We say \( F \) is stable if the inequality is strict. For each semi-stable sheaf \( F \), there is a filtration (Jordan-Hölder filtration) \( 0 = F_0 \subset F_1 \subset \cdots \subset F_n = F \) such that \( \text{gr}_i(F) := F_i/F_{i-1} \) is stable and \( p(F)(m) = p(\text{gr}_i(F))(m) \) for all \( i \). Finally, two semi-stable sheaves \( F \) and \( G \) are \( S \)-equivalent if \( \text{gr}(F) \cong \text{gr}(G) \), where \( \text{gr}(F) := \oplus_i \text{gr}_i(F) \). Simpson [Sim94] proved there is a projective coarse moduli space \( M_L(X, P(m)) \) of \( S \)-equivalent classes of semi-stable sheaves for a fixed Hilbert polynomial \( P(m) \). This moduli space has several connected components with respect to \( \beta = c_1(F) \in H_2(X, \mathbb{Z}) \), that is,

\[
M_L(X, P(m)) = \bigsqcup_{\beta \in H_2(X, \mathbb{Z})} M_L(X, \beta, P(m)).
\]

For brevity, we denote \( M_L(X, P(m)) \) by \( M_X(P(m)) \) and \( M_L(X, \beta, P(m)) \) by \( M_X(\beta, P(m)) \).

Let \( M_{\mathbb{P}^n}(P(m)) \) be the moduli space of semi-stable sheaves \( F \) with Hilbert polynomial \( P(m) = 2\chi(O_{\mathbb{P}^n-1}(m)) - 2\chi(O_{\mathbb{P}^n-1}(m-1)) \). Let \( M^0 \) be the space of twisted ideal sheaves \( \mathcal{I}_{L,Q}(1) \) such that \( L \in |O_{\mathbb{P}^n}(1)| \) and \( Q \in |O_{\mathbb{P}^n}(2)| \) with \( \text{rank}(Q) = 4 \). Then we summarize the results of Section 4.2 in [CM17] as follows.

1. The closure \( M_n \) of \( M^0 \) in \( M_{\mathbb{P}^n}(P(m)) \) is an irreducible normal variety of dimension \( 4n - 3 \).
2. \( M_n \) is a connected component of \( M_{\mathbb{P}^n}(P(m)) \).
3. Each semi-stable sheaf \( F \) parameterized by \( M_n \) has a free resolution as in (1.1).
4. The singular locus of \( M_n \) is isomorphic to \( \text{Sing}(M_n) \cong \text{Sym}^2(\mathbb{P}^n^*) \) parameterizing pure sheaves of the form \( O_H \oplus O_{H'} \) for hyperplanes \( H \) and \( H' \) in \( \mathbb{P}^n \).

**Remark 2.1.** From the resolution (1.1) of \( F \), \( \text{Ext}^2(F, F) = 0 \), which implies that the Quot scheme arising in the GIT construction of \( M_n \) ([HL10]) is smooth. Thus, from Luna’s étale slice theorem, the analytic normal neighborhood of \( \text{Sing}(M_n) \) in \( M_n \) is the same as that of the moduli space of vector bundles over a smooth projective curve ([Las96] and [Kir86a]).

2.3. **Resolution of \( M_n \) using Kirwan’s method.** The birational relationship between \( M_n \) and Kontsevich’s map space \( K_n \) was explicitly studied in [CM17, Section 5] using Kirwan’s desingularization method. One critical point is the interpretation of \( M_n \) as a Kronecker quiver representation space. For convenience, we recall the results in detail.

Fix two positive integers \( a, b \) and let \( V^* \) be a vector space of \( \dim V^* = n + 1 \). A Kronecker \( V^* \)-module is a quiver representation of an \( n \)-Kronecker quiver

\[
\bullet \underset{i}{\longleftarrow} \overset{i}{\longrightarrow} \bullet
\]

with dimension vector \( (a, b) \). Two Kronecker \( V^* \)-modules \( \phi = (\phi_i) \) and \( \psi = (\psi_i) \) are equivalent if there are \( A \in \text{SL}_a \) and \( B \in \text{SL}_b \), such that \( \phi = B \circ \psi \circ A \). We may regard the GIT quotient

\[
K_n(a, b) := \mathbb{P}\text{Hom}(V^* \otimes \mathbb{C}^a, \mathbb{C}^b)\big/\text{SL}_a \times \text{SL}_b
\]

as the moduli space of semi-stable Kronecker \( V^* \)-modules. We are interested in the case \( a = b = 2 \) and \( G := \text{SL}_2 \times \text{SL}_2 \).
Proposition 2.3: Let $M = \mathbb{P} \text{Hom}(V^* \otimes \mathbb{C}^2, \mathbb{C}^2) \cong \mathbb{P}(V^* \otimes \mathfrak{g}_2) := X$. If $M \in X^{ss} \setminus X^s$, then $M$ is equivalent to
\[
\begin{bmatrix}
g & 0 \\
0 & h
\end{bmatrix}
\]
for some $g, h \in V^* \setminus \{0\}$ where

1. $\text{Stab } M \cong \text{SL}_2 \times \mathbb{Z}_2$ if $g$ is proportional to $h$ and
2. $\text{Stab } M \cong \mathbb{C}^* \times \mathbb{Z}_2$ otherwise.

Lemma 2.2. Let $Y \in X^{ss}$ be the locus of matrices equivalent to (1) of Lemma 2.2. At each point $M = g \cdot \text{Id} \in Y$, the normal bundle $N_{Y_0/X_M^s}|_M$ is isomorphic to $H \otimes \mathfrak{sl}_2$ where $H \cong V^*/\langle g \rangle$. From Luna’s slice theorem, there is a normal neighborhood of $\overline{M} \in X//G$ that is isomorphic to $H \otimes \text{sl}_2//\text{Stab } M \cong H \otimes \text{sl}_2//\text{SL}_2$,

where $\text{SL}_2$ acts on $\text{sl}_2$ in the standard way and $H$ in a trivial way. Also, $\mathbb{Z}_2$ acts trivially. Thus, from [Kir85, Lemma 3.11],

Proposition 2.4. Let $\pi_1: X^1 \rightarrow X^{ss} := X^0$ be the blow-up of $X^0$ along $Y_0$. Then GIT quotient $X^1//G$ is the blow-up of $X//G$ along $Y_0//G \cong \mathbb{P}^n$.

Consider Kirwan’s partial desingularization of $K_n(2, 2) = X//G$ along the loci described in Lemma 2.2. Let $Y_0 \subset X^{ss}$ be the locus of matrices equivalent to (1) of Lemma 2.2. At each point $M = g \cdot \text{Id} \in Y_0$, the normal bundle $N_{Y_0/X_M^s}|_M$ is isomorphic to $H \otimes \mathfrak{sl}_2$ where $H \cong V^*/\langle g \rangle$. From Luna’s slice theorem, there is a normal neighborhood of $\overline{M} \in X//G$ that is isomorphic to $H \otimes \text{sl}_2//\text{Stab } M \cong H \otimes \text{sl}_2//\text{SL}_2$.

where $\text{SL}_2$ acts on $\text{sl}_2$ in the standard way and $H$ in a trivial way. Also, $\mathbb{Z}_2$ acts trivially. Thus, from [Kir85, Lemma 3.11],

Proposition 2.5. Let $\pi_2: X^2 \rightarrow (X^1)^{ss}$ be the blow-up of $(X^1)^{ss}$ along $(\overline{Y_1})^{ss}$. Then GIT quotient $X^2//G$ is the blow-up of $X^1//G$ along $\overline{Y_1}/G \cong \text{bl}_\Delta (\mathbb{P}^n \times \mathbb{P}^n//\mathbb{Z}_2)$.

Let $\pi_i: X^i//G \rightarrow X^{i-1}//G$ be the induced quotient map of $\pi_i$ for $i = 1, 2$.

1. For $M \in Y_0$,

$$\pi_1^{-1}(\overline{M}) \cong \mathbb{P}(H \otimes \text{sl}_2)//\text{SL}_2.$$ 

The second blow-up $\pi_2$ also provides partial a desingularization of $\mathbb{P}(H \otimes \text{sl}_2)//\text{SL}_2$ which is isomorphic to moduli space $M_0(\mathbb{P}H, 2)$ ([Kie07, Theorem 4.1]).

2. For $M \in Y_1 \setminus Y_0$,

$$\pi_2^{-1}(\overline{M}) \cong \mathbb{P}^{n-2} \times \mathbb{P}^{n-2}.$$
Thus, we have the following theorem.

**Theorem 2.6** ([CM17, Theorem 5.1]). The partial desingularization of $M_n$ is the second blown-up space $X^2 // G \cong K_n = \mathcal{M}_0(Gr(2, n + 1), 2)$:

$$X // G \cong M_n \xrightarrow{\pi_2} X^2 // G := M'_{n} \xrightarrow{\pi_2} X^2 // G = K_n.$$ 

2.4. **(Intersection) E-polynomial of a variety.** We review several polynomial invariants related to a quasi-projective variety $X$. All polynomial invariants defined in this section are related to the mixed Hodge structure on the compactly supported cohomology $H^*_c(X, \mathbb{C})$, the cohomology $H^*(X, \mathbb{C})$, the compactly supported intersection cohomology $IH^*_c(X, \mathbb{C})$, and intersection cohomology $IH^*(X, \mathbb{C})$. We use the notation $h^{p,q,i}_{c}$ for $\dim_{\mathbb{C}} Gr^p F^{p+q} Gr^W_{p+q} H^i_c(X, \mathbb{C})$, where $F^\bullet$ and $W^\bullet$ are Hodge and weight filtration respectively.

**Definition 2.7.** Let $X$ be a quasi-projective variety of dimension $\dim_{\mathbb{C}} X = n$. The compactly supported Poincaré-Deligne polynomial of $X$ can be expressed as

$$PD^c_X(u, v, t) := \sum_{p,q=0}^{n} 2^n \sum_{i=0}^{2n} h^{p,q,i}_{c} u^p v^q t^i;$$

the compactly supported E-polynomial (or Serre polynomial) of $X$ as

$$E^c_X(u, v) := PD^c_X(u, v, -1) = \sum_{p,q=0}^{n} 2^n \sum_{i=0}^{2n} (-1)^i h^{p,q,i}_{c} u^p v^q;$$

the compactly supported Poincaré polynomial of $X$ as

$$P^c_X(t) := PD^c_X(1, 1, t) = \sum_{p,q=0}^{n} 2^n \sum_{i=0}^{2n} h^{p,q,i}_{c} t^i = \sum_{i=0}^{2n} \dim_{\mathbb{C}} H^i_c(X, \mathbb{C}) t^i;$$

and the virtual Poincaré polynomial of $X$ as

$$P^v_X(t) := PD^c_X(-t, -t, -1) = \sum_{p,q=0}^{n} 2^n \sum_{i=0}^{2n} (-1)^i h^{p,q,i}_{c} (-t)^{p+q} = \sum_{m=0}^{2n} 2^n \sum_{i=0}^{2n} (-1)^{i+m} \dim_{\mathbb{C}} Gr^W_{m} H^i_c(X, \mathbb{C}) t^m.$$ 

The compactly supported E-polynomial $E^c_X(u, v)$ of $X$ has a special property from the Grothendieck group of varieties $K_0(var)$ (or $K_0(var/pt)$ in a relative version).

**Definition 2.8.** The Grothendieck group $K_0(var)$ of complex algebraic varieties is a free abelian group of isomorphism classes with an equivalence relation

$$[X] = [Z] + [X \setminus Z],$$

where $Z$ is a Zariski closed subvariety in a variety $X$. It also has a ring structure with multiplication structure

$$[X] \cdot [Y] = [X \times Y].$$

For a quasi-projective variety $X$ of dimension $n$, the compactly supported cohomology group $H^i_c(X, \mathbb{Q})$ carries a mixed Hodge structure, which induces the class

$$[H^*_c(X)] := \sum_{i=0}^{2n} (-1)^i [H^i_c(X, \mathbb{Q})]$$

and
in the Grothendieck group $K_0(HS)$ of pure Hodge structures. There is a ring homomorphism

$$[H^*_c] : K_0[\text{var}] \to K_0(HS)$$

defined by $[H^*_c](\{X\}) = [H^*_c(X)]$. Also, there is another ring homomorphism (the Hodge-Euler polynomial)

$$E_{Hdg} : K_0(HS) \to \mathbb{Z}[u^\pm, v^\pm], \quad E_{Hdg}([H]) = \sum_{p,q} (\dim_{\mathbb{C}} \text{Gr}_F^p \text{Gr}_W^{W_{p+q}} H_c) u^p v^q.$$ 

Hence the compactly supported E-polynomial $E^c_X(u, v)$ of variety $X$ is nothing but $(E_{Hdg} \circ [H^*_c])(\{X\})$.

**Notation 2.9.** $E_c(X) := E^c_X(u, v) \in \mathbb{Z}[u^\pm, v^\pm]$.

From the ring homomorphism $E_{Hdg} \circ [H^*_c]$,

**Proposition 2.10.**
1. $E_c(\mathbb{C}^n) = (uv)^n$.
2. $E_c(X) = E_c(Z) + E_c(X \setminus Z)$ for any closed subset $Z \subset X$.
3. $E_c(X) = E_c(F) \cdot E_c(B)$ for the Zariski (resp. étale) locally trivial fibration $X \to B$ with constant fiber $F$ (resp. $\text{Gr}(k, n)$) ([BJ12, Lemma 3.1]).

In particular, the virtual Poincaré polynomial $P^{vir}(t) = E_c(X)(-t, -t)$ has similar properties to (2) and (3), called motivic properties. We can define $PD_X(u, v, t)$, $E_X(u, v)$ and $P_X(t)$ from cohomology groups after replacing $h^{p,q,i}_c$ by $h^{p,q,i} := \dim_{\mathbb{C}} \text{Gr}_F^p \text{Gr}_W^{W_{p+q}} H^{i}(X, \mathbb{C})$, but the map $[H^*_c] : K_0[\text{var}] \to K_0(HS)$ is not homomorphism. Thus, motivic properties do not hold for any of the polynomial invariants from cohomology groups. However, there are a number of useful identities for calculation of these invariants. Let us shortly denote $E_X(u, v)$ by $E(X)$.

**Remark 2.11.**
1. If $X$ is smooth and connected, then $E(X)(u, v) = u^n v^n E_c(X)(u^{-1}, v^{-1})$ from the Poincaré duality.
2. If $X$ is a compactification of an algebraic variety $U$, then $E_c(U) = E(X) - E(X \setminus U)$ ([PS08, Section 5.5.2]).
3. If $\pi : (\tilde{X}, E) \to (X, D)$ is a proper modification with discriminant $D$, then $E(X) = E(\tilde{X}) - E(E) + E(D)$ ([PS08, Theorem 5.37]).
4. If $X$ is projective, then $E_c(X) = E(X)$.
5. If $X$ is pure (i.e., $h^{p,q,i}_c = 0$ for $p + q \neq i$) and balanced (i.e., $h^{p,q,i}_c = 0$ for $p \neq q$) type, then $P^c_X(t) = P^{vir}_X(t)$.

**Definition 2.12.** The (compactly supported) intersection cohomology of a complex $n$-dimensional variety $X$ is defined by the hypercohomology

$$\text{III}^i_X(\mathbb{C}, \mathbb{Q}) := \mathbb{H}^i_X(\mathbb{Q}, \text{IC}_X[-n]),$$

where $\text{IC}_X$ is the intersection complex on $X$ of the middle perversity.

Since $\text{III}^i_X(\mathbb{C}, \mathbb{Q})$ carries a mixed Hodge structure, we define the (compactly supported) intersection cohomology E-polynomial

$$E_c(X) = E_{Hdg} \circ [\text{III}^*_c(X)].$$
where the map $[IH^*_c]$ is defined by $X \mapsto [IH^*_c(X)] := \sum_{i=0}^{2n} (-1)^i [IH^*_c(X, \mathbb{Q})] \in K_0[HS]$ for a quasi-projective variety $X$. Unfortunately, $[IH^*_c]$ does not provide group homomorphism from $K_0(var)$ in general.

The (compactly supported) cohomology $H^i_c(X, \mathbb{Q})$ and the (compactly supported) intersection cohomology $IH^i_c(X, \mathbb{Q})$ of variety $X$ are not isomorphic in general when $X$ is a singular variety. However, there is an isomorphism for some special cases.

**Proposition 2.13.** ([Max18, Theorem 6.6.3] and [Bri99, Proposition A.1]) If a variety $X$ has at most finite quotient singularities (more generally, rationally smooth manifold), then $IC_X$ is quasi-isomorphic to $Q_X[\dim X]$. In particular,

$$IH^*_c(X, \mathbb{Q}) = H^*_c(X, \mathbb{Q}).$$

### 2.5. Comparison via algebraic stratification.

We give a relationship between polynomial invariants of $X$ from $H^i_c(X, \mathbb{Q})$ and $IH^i_c(X, \mathbb{Q})$. We give a more general statement ahead. Let $f : X \to Y$ be a proper morphism of complex algebraic varieties. We fix a complex algebraic Whitney stratification $\mathcal{V}$ of $f$ such that all strata of $X$ and $Y$ are smooth and $f$ is a stratified submersion. This satisfies the frontier condition: if $W \cap V \neq \emptyset$, then $W \subset V$. We have partial order in $\mathcal{V}$ by $W \leq V$ if $W \subset V$ and $W < V$ for $\dim(W) < \dim(V)$. Let $Y^0$ be a dense open stratum in $Y$, then $Y^0$ is the maximal element in $\mathcal{V}$. For each order pair $W < V$ and $w \in W$, consider a local analytic embedding $(V, w) \hookrightarrow (\mathbb{C}^n, 0)$ of neighborhood $(V, w)$ of $w$. Let $N$ be a smooth, normal neighborhood of $w$ transversally meeting with $W$ only at $w$ and $\dim N = \text{codim}_{\mathbb{C}^n} W$. Let $L_{w, V} := V \cap N \cap \partial B_\delta(w)$, where $B_\delta(w)$ is an open ball in $\mathbb{C}^n$ with radius $0 < \delta \ll 1$ centered at $w$. Then the stratification satisfies

- the open cone $c^0 L_{w, V} := (L_{w, V} \times [0, 1]) / (L_{w, V} \times \{0\})$ is homeomorphic to $V \cap N \cap B_\delta(w)$, and

- the homeomorphic type of $L_{w, V}$ does not depend on the choice of $w \in W$.

**Notation 2.14.** We call $L_{w, V}$ the link $L_{W, V}$ of $W$ in $V$ and the open cone of $L_{W, V}$ is denoted as $c^0 L_{W, V}$. We denote $c^0 L_{W, V}$ by $c^0 L_{W, V}$ unless stated otherwise.

Since the open cone has canonical mixed Hodge structure, we can define the E-polynomials $E(c^0 L_{W, V})$ and $IE(c^0 L_{W, V})$. The difference between $E(X)$ and $IE(X)$ for a variety $X$ can be obtained from [CMS08].

**Proposition 2.15.** Let $f : X \to Y$ be a proper morphism between algebraic varieties. Fix an algebraic stratification of $Y$ satisfying the above conditions, and assume that $f$ induces a trivial fibration on each stratum. Then,

$$[IH^*_c(X)] = [IH^*_c(Y)] \cdot [H^*(F)] + \sum_{V < Y^0} [IH^*_c(V)] \cdot ([H^*(F_V)] - [H^*(F)] \cdot [IH^*(c^0 L_{V, Y})]),$$

where $[IH^*_c(V)]$ is inductively defined by

$$[IH^*_c(V)] := [IH^*_c(V)] - \sum_{W < V} [IH^*_c(W)] \cdot [IH^*(c^0 L_{W, V})],$$

for $V \in \mathcal{V}$. The difference between $E(X)$ and $IE(X)$ for a variety $X$ can be obtained from [CMS08].
and $F$ (resp. $F_V$) is the fiber over $Y^\circ$ (resp. $V \in \mathcal{V}$).

**Proof.** Let $M = \mathbb{Q} \mathbb{H}^2_X$ and apply $k'_s$ ($k'_t$) for compactly supported cases to the identity of Corollary 3.4 in [CMS08], where $k' : Y \to pt$. Classes $[H^*(F)]$, $[H^*(F_V)]$, $[IH^*(c^o L_{V,Y})]$, and $[IH^*(c^o L_{W,V})]$ are in the Grothendieck group $K_0(MHM(pt))$ of mixed Hodge modules on a point. Since $k'_s$ ($k'_t$) is a $K_0(MHM(pt))$-linear map, we obtain the result. \hfill $\Box$

We drop subscript $c$ for compactly supported cohomologies in the remainder of this paper. However, $\text{IE}(c^o L_{A,B})$ of any open cone $c^o L_{A,B}$ always refers to the intersection cohomology, rather than the compactly supported intersection cohomology.

**Corollary 2.16.** Under assumptions of Proposition 2.15,

$$E(X) = \text{IE}(Y) \cdot E(F) + \sum_{V < Y^\circ} \text{IE}(V) \cdot (E(F_V) - E(F) \cdot \text{IE}(c^o L_{V,Y})), $$

where $\text{IE}(V)$ is inductively defined by

$$\text{IE}(V) := \text{IE}(V) - \sum_{W < V} \text{IE}(W) \cdot \text{IE}(c^o L_{W,V}), $$

and $F$ (resp. $F_V$) is the fiber over $Y^\circ$ (resp. $V \in \mathcal{V}$).

**Proof.** The identity of the claim is obtained by applying ring isomorphism $E_{Hdg}$ in (2.2) and (2.3) respectively. \hfill $\Box$

**Corollary 2.17.**

$$E(Y) = \text{IE}(Y) + \sum_{V < Y^\circ} \text{IE}(V) \cdot (1 - \text{IE}(c^o L_{V,Y})), $$

for any stratification $\mathcal{V}$.

**Proof.** The result follows immediately by letting $X = Y$ and $f = \text{id}$ in Corollary 2.16. \hfill $\Box$

**Example 2.18.** Let $X$ be a smooth projective variety. The symmetric product $\text{Sym}^2(X)$ has the $\mathbb{Z}_2$-quotient singularity along the diagonal $\Delta(\cong X) \subset \text{Sym}^2(X)$, and hence $\text{E}(\text{Sym}^2(X)) = \text{IE}(\text{Sym}^2(X))$ from Proposition 2.13. Applying Corollary 2.17 for the stratification $\mathcal{V} = \{\Delta, V := \text{Sym}^2(X) \setminus \Delta\}$,

$$\text{IE}(c^o L_{\Delta,V}) = 1.$$

**Proposition 2.19.** Let $Q := \{xy - zw = 0\} \subset \mathbb{C}^4$ be the quadric cone in $\mathbb{C}^4$. Then the IE-polynomial of the open cone of the link $Q$ at the origin $(0,0,0,0) \in \mathbb{C}^4$ is given by

$$\text{IE}(c^o L_{\{0\},Q}) = uv + 1.$$ 

**Proof.** Let $\bar{Q} = \{xy - zw = 0\} \subset \mathbb{P}^4$ be the closure of $Q$ under standard embedding $\mathbb{C}^4 \subset \mathbb{P}^4, (x,y,z,w) \mapsto [x : y : z : w : 1]$. Let $P = [0 : 0 : 0 : 1]$ be the singular point of $Q$. There are two resolutions $\hat{Q}_1$ and $\hat{Q}_2$ for the singular point $P$. The exceptional divisor of the resolution $\hat{Q}_1 \to \hat{Q}$ is $\mathbb{P}^1 \times \mathbb{P}^1$ and $\hat{Q}_1$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1 \times \mathbb{P}^1$. Thus,

$$E(\hat{Q}) = E(\hat{Q}_1) - E(\mathbb{P}^1 \times \mathbb{P}^1) + E(P) = (uv)^3 + 2(uv)^2 + (uv) + 1.$$
The resolution $\tilde{Q}_2 \to Q$ is small with the exceptional locus $\mathbb{P}^1$, hence
\[
\text{IE}(\tilde{Q}) = \text{E}(\tilde{Q}_2) - \text{E}(P) + \text{E}(\mathbb{P}^1) = (uv)^3 + 2(uv)^2 + 2(uv) + 1;
\]
and from Corollary 2.16,
\[
\text{IE}(c^0 L_{(0),Q}) = uv + 1. \tag*{$\square$}
\]

We use the notation $P(X) := \text{E}(X)(-1, -1) = P_X^{\text{vir}}(t)$ and $\text{IP}(X) := \text{IE}(X)(-1, -1)$, where $\text{IP}(c^0 L_{A,B})$ comes from the usual intersection cohomology of the open cone $c^0 L_{A,B}$.

3. PROOF OF MAIN RESULT

This section proves Theorem 1.1. The intersection Poincaré polynomial of $M_n$ is exactly the same as the virtual one for $M_n$ because all spaces arising in the computation are pure and balanced Hodge types. We propose a numerical relationship between the virtual intersection Poincaré polynomials of open cones of singular loci in $M_n$.

3.1. Intersection cohomology of $M_n$ using Kirwan’s resolution. For a pure dimensional variety $X$, let us write the truncated intersection Poincaré polynomial of $X$ as
\[
\text{IP}(X)_{<k} := \sum_{i=0}^{k-1} \dim \text{IH}^i(X, Q)t^i.
\]

Lemma 3.1. The intersection Poincaré polynomial of the GIT-quotient space $\mathbb{P}(\text{Sym}^2 C^2 \otimes C^n)/\text{SL}(2)$ is
\[
\frac{(1 - t^{2n})(1 - t^{2n+2})(1 - t^{2n-2}) - t^2(1 - t^{14} - 1)(1 - t^{4} - 1)}{(1 - t^2)^2(1 - t^4)},
\]
where $\lfloor x \rfloor$ is the largest integer $\leq x$.

Proof. Since the blow-up of the stable maps space $M_0(\mathbb{P}^{n-1}, 2)$ along a $\mathbb{P}^2$-bundle over $\text{Gr}(2, n)$ is isomorphic to the blow-up of the Hilbert scheme $H(\mathbb{P}^{n-1})$ of conics along a $\mathbb{P}^2$-bundle over $\text{Gr}(3, n)$ (see [Kie07, Section 4] and [CHK12, Section 3] for an explicit geometric description),
\[
P(M_0(\mathbb{P}^{n-1}, 2)) = P(H(\mathbb{P}^{n-1})) - P(\mathbb{P}^2)P(\text{Gr}(3, n)) + P(\mathbb{P}^2)P(\text{Gr}(2, n)).
\]

Kirwan’s partial desingularization of $\mathbb{P}(\text{Sym}^2 C^2 \otimes C^n)/\text{SL}(2)$ along the strictly semi-stable locus $\mathbb{P}(\text{Sym}^2 C^2) \times \mathbb{P}(C^n)/\text{SL}(2) \cong \mathbb{P}^{n-1}$ is isomorphic to the moduli space $M_0(\mathbb{P}^{n-1}, 2)$, where the exceptional locus is a $\text{Sym}^2 \mathbb{P}^{n-2}$-fibration over $\mathbb{P}^{n-1}$ ([Kie07, Theorem 4.1]). Since $\pi_1(\mathbb{P}^{n-1}) = 0$, we can apply [Kir86b, 2.28],
\[
\text{IP}(\mathbb{P}(\text{Sym}^2 C^2 \otimes C^n)/\text{SL}(2)) = P(M_0(\mathbb{P}^{n-1}, 2)) - \text{IP}(\mathbb{P}^{n-1}) \cdot (t^2 Q(t) + t^{4n-8} Q(\frac{1}{t})),
\]
where $Q(t) := \text{IP}(\text{Sym}^2 \mathbb{P}^{n-2})_{<2n-4}$. Note that $P(M_0(\mathbb{P}^{n-1}, 2)) = \text{IP}(M_0(\mathbb{P}^{n-1}, 2))$ from Proposition 2.13. The Hilbert scheme $H(\mathbb{P}^{n-1})$ is also isomorphic to a $\mathbb{P}^5$-bundle over $\text{Gr}(3, n)$ and $\text{IP}(\text{Sym}^2 \mathbb{P}^{n-2}) = \frac{1}{2}(1 - t^{2n-2})^2 + \frac{1 - t^{4n-4}}{1 - t^2}$ ([MOVG09, Lemma 2.6]).

Thus the result follows from (3.1) and (3.2). \tag*{$\square$}
Martin proved the following proposition using torus localization method ([LM14, Theorem 3.1]). We use an alternative birational geometric proof to confirm that the related moduli spaces are pure and balanced types.

**Proposition 3.2.** The Poincaré polynomial of $K_n = M_0(Gr(2, n + 1), 2)$ is

$$
\frac{[(1 + t^{2n+2})(1 + t^6) - t^2(1 + t^2)(t^4 + t^{2n-2})] \cdot [1 - t^{2n+2})(1 - t^{2n})(1 - t^{2n-2})]}{(1 - t^2)^3(1 - t^4)^2}.
$$

**Proof.** Let $H(Gr(2, n + 1))$ be the Hilbert scheme of conics in $Gr(2, n + 1)$. Let $Gr(2, U)$ be the Grassmannian bundle over the universal sub-bundle $U$ of $Gr(4, n + 1)$. Consider the relative Hilbert scheme $H(Gr(2, U))$ of conics over $Gr(4, n + 1)$. Following the method used to prove Proposition 4.2 in [CHL18], the natural forgetful map $H(Gr(2, U)) \rightarrow H(Gr(2, n + 1))$ is a blow-up map along a $\mathbb{P}^5$-bundle over $Gr(3, n + 1)$.

On the other hand, the space $K_n$ is a blow-up of space $H(Gr(2, n + 1))$ along a $\mathbb{P}^{n-2}$-bundle over $Gr(1, 3, n + 1)$ followed by blowing-down along a $\mathbb{P}^2$-bundle over $Gr(1, 3, n + 1)$ ([CHK12, Corollary 5.3]). Therefore, $P(K_n) = P(H(Gr(2, U))) - P(\mathbb{P}^5)P(Gr(3, n + 1))(P(\mathbb{P}^{n-3}) - 1) + P(Gr(1, 3, n + 1))(P(\mathbb{P}^2) - P(\mathbb{P}^{n-2}))$, since $H(Gr(2, 4))$ is the blow-up of $Gr(3, 6)$ along two copies of disjoint $\mathbb{P}^5$. Note that from Proposition 2.10 (3) and [MOVG09, Lemma 2.1], the relevant spaces are pure and balanced type. $\square$

**Proof of Theorem 1.1.** Applying [Kir86b, (2.28)] to the first blow-up of Proposition 2.4,

$$
\text{IP}(M_n) = \text{IP}(M'_n) - P(\mathbb{P}^5)[t^2 R(t) + t^{n-6} R_1(t)] - ih^{3n-5}(\mathbb{P}(\text{Sym}^2 \mathbb{C}^2 \otimes \mathbb{C}^n) / SL(2)) t^{4n-2},
$$

where

$$
R(t) = \text{IP}(\mathbb{P}(\text{Sym}^2 \mathbb{C}^2 \otimes \mathbb{C}^n) / SL(2))_{3n-4} \text{ and }
$$

$$
ih^{3n-5}(\mathbb{P}(\text{Sym}^2 \mathbb{C}^2 \otimes \mathbb{C}^n) / SL(2)) = \dim \text{IH}^{3n-5}(\mathbb{P}(\text{Sym}^2 \mathbb{C}^2 \otimes \mathbb{C}^n) / SL(2), \mathbb{Q}).
$$

Applying [Kir86b, (2.1)] to the second blow-up of Proposition 2.5 ([Kir86a, Section 5], [MOVG09, Lemma 2.6] and [LMN13, Remark 2.7]),

$$
\text{IP}(M'_n) = \text{IP}(K_n) - \frac{1}{2}[P(\mathbb{P}^5)^2 + \frac{1 - (-t^2)^{2n+2}}{1 - t^4} + 2 \cdot P(\mathbb{P}^5) \cdot \frac{t^{2n} - t^2}{t^2 - 1}] \cdot \sum_{j=1}^{2n-4} \left\lfloor \frac{\min\{j + 1, 2n - 2 - j\}}{2} \right\rfloor t^{2j}
$$

$$
- \frac{1}{2}[P(\mathbb{P}^5)^2 - \frac{1 - (-t^2)^{2n+2}}{1 - t^4}] \cdot \sum_{j=2}^{2n-5} \left\lfloor \frac{\min\{j, 2n - 3 - j\}}{2} \right\rfloor t^{2j},
$$

where $\lfloor x \rfloor$ is the largest integer $\leq x$.

The proof follows by combining these two relation with Lemma 3.1 and Proposition 3.2. $\square$

**Remark 3.3.** $M_2 \cong K_2(2, 2) \cong \mathbb{P}^5$ ([LP93, Corollary 4.3]) and hence $\text{IP}(M_2) = 1 + t^2 + t^4 + t^6 + t^8 + t^{10}$, which is consistent with Theorem 1.1.

The following proposition is used subsequently in the paper. Recall that $\Delta := Y_0 / G = \mathbb{P}^n$ and $S_n := Y_1 / G \cong \text{Sym}^2(\mathbb{P}^n)$ are the blow-up centers of $M_n$. 

Corollary 3.4. For the stratification $\mathcal{V} = \{\Delta, S_n \setminus \Delta, M_n \setminus S_n\}$ of $M_n$, the intersection Poincaré polynomials of open cones are related by

$$(t^2 + 1)\text{IP}(c^0 L_{\Delta,M_n}) + t^4 \frac{1 - t^{2n}}{1 - t^2} \text{IP}(c^0 L_{S_n \setminus \Delta,M_n})$$

$$(3.3) = \frac{(t^{2n-2} - 1)(2t^{2n+4} - t^4 - 1) + \frac{(-1)^n+1}{2}(t^{2n-2} + t^{4n})(1 - t^2)^2}{(t^2 - 1)(t^4 - 1)}.$$  

Proof. The result follows from Corollary 2.17, Theorem 1.1, Example 2.18, and Proposition 7.2 of [CM17].

□

Example 3.5. When $n = 3$, the intersection cohomology for each open cone can be calculated using (3.3) and Proposition 2.19. From Luna’s slice theorem, the space $M_n$ at $[E] = [\mathcal{O}_H \oplus \mathcal{O}_{H'}]$, $H \neq H'$ is locally isomorphic to

$$\text{Ext}^1(E,E)//\text{Aut}(E) \cong (\text{Ext}^1(\mathcal{O}_H,\mathcal{O}_H) \oplus \text{Ext}^1(\mathcal{O}_{H'},\mathcal{O}_{H'})) \times Y \subset (\mathbb{C}^n \oplus \mathbb{C}^n) \times \mathbb{C}^{(n-1)^2},$$

where $S_n$ at $[E]$ corresponds to the affine space $(\text{Ext}^1(\mathcal{O}_H,\mathcal{O}_H) \oplus \text{Ext}^1(\mathcal{O}_{H'},\mathcal{O}_{H'})) \times \{0\}$ at the origin $\{(0 \oplus 0) \times \{0\}\}$, and $Y$ is isomorphic to the affine cone of the Segre variety $\mathbb{P}^{n-2} \times \mathbb{P}^{n-2} \subset \mathbb{P}^{n^2-2n}$ ((2.1) and [Dré04, Proposition 7.16]).

For the case $n = 3$, let us choose the normal slice $N = \{(0 \oplus 0)\} \times \mathbb{C}^4$ at $[E]$. Then,

$$c^0 L_{S_3,M_3} \cong c^0 L_{\{0\},Q}.$$  

Substituting $\text{IP}(c^0 L_{S_3,M_3}) = 1 + t^2$ into (3.3), $\text{IP}(c^0 L_{\Delta,M_3}) = 1$.

4. APPLICATION TO LOCAL SURFACES

This section calculates the intersection Poincaré polynomial of the moduli space of pure one-dimensional sheaves on del Pezzo surfaces ($\mathbb{F}_0$, $\mathbb{F}_1$, and $\mathbb{P}^2$) using the intersection Poincaré polynomial of the space $M_n$ ($3 \leq n \leq 5$).

Recall that $M_S(c,\chi)$ is the moduli space of semi-stable sheaves $F$ with $c_1(F) = c$ and $\chi(F) = \chi$ on a del Pezzo surface $S$. From the Serre duality and the semi-stability of $F$, $\text{Ext}^2_S(F,F) \cong \text{Ext}^0_S(F,F \otimes K_S) = 0$, and hence the Quot scheme arising in the GIT-construction of $M_S(c,\chi)$ is smooth. Therefore, the analytic neighborhood of the singular locus in $M_S(c,\chi)$ is isomorphic to that of vector bundles case (cf. Remark 2.1).

For $n = 3$ and $5$, we can use the explicit birational maps between spaces $M_S(c,\chi)$ and $M_n$ (see [CM16, Theorem 5.7] and [CM17, Proposition 7.4]). However, we use Corollary 3.4 for $n = 4$ since we do not know any explicit birational relation between $M_S(c,\chi)$ and $M_n$. Lastly, we conjecture that the intersection Poincaré polynomial of the moduli space $M_S(c,\chi)$ does not depend on the Euler characteristic $\chi$.

---

1There is a small error on page 648 of [CM17]. In (7.1), the term $P((P^{n-2})^2) - 1$ $(\frac{1}{2} \left(P(P^n)^2 + \frac{1-2^{n+2}}{1-2^{n+3}}\right) - P(P^n))$ must be replaced by $P(\text{Sym}^2(P^n \times P^{n-2})) - P(P^n) \cdot P(\text{Sym}^2 P^{n-2}) - (P(\text{Sym}^2 P^n) - P(P^n))$ ((2.1) and [LMN13, Proposition 2.6]).
4.1. Cases \( n = 3 \) and 5. Let \( M_{F_0}(2, 2) = M_{O_{F_0}(1, 1)}(\mathbb{P}_0, (2, 2), 4m + 2) \) be the moduli space of semi-stable sheaves \( F \) with \( c_1(F) = (2, 2) \in H_2(\mathbb{P}_0, \mathbb{Z}) \) with Hilbert polynomial \( P(F)(m) = 4m + 2 \).

**Corollary 4.1.** Let \( M_{F_0}(2, 2) \) be the moduli space of pure sheaves on \( \mathbb{P}_0 \). Then the intersection Poincaré polynomial of \( M_{F_0}(2, 2) \) is

\[
1 + 3t^2 + 4t^4 + 4t^6 + 4t^8 + 4t^{10} + 4t^{12} + 4t^{14} + 3t^{16} + t^{18}.
\]

**Proof.** From Theorem 5.7 of [CM16], there exists a birational morphism

\[
M_{F_0}(2, 2) \to M_3
\]

that is a smooth blow-up at two distinct smooth points. From the blow-up formula for cohomology groups,

\[
\text{IP}(M_{F_0}(2, 2)) = \text{IP}(M_3) + 2 \cdot \text{P}(\{\text{pt}\})(\text{P}(\mathbb{P}^n \text{dim } M_3) - 1),
\]

and \( \text{P}(\mathbb{P}^n) = \frac{1-t^{2n+2}}{1-t^2} \) the result follows from Theorem 1.1. \( \square \)

**Remark 4.2.** The space \( M_{F_0}(2, 2, 1) \) is isomorphic to the relative Hilbert scheme of one point over the complete linear system \( |O_{\mathbb{P}_0}(2, 2)| \) ([BS14, Proposition 12]), where the latter space is isomorphic to a \( \mathbb{P}^3 \)-bundle over \( \mathbb{P}_0 \). Thus, from Proposition 2.10, \( \text{IE}(M_{F_0}(2, 2, 1)) = \text{IE}(\mathbb{P}^3 \times \mathbb{P}^3) \), which is \( \text{IE}(M_{F_0}(2, 2, 2)) \).

**Corollary 4.3.** Let \( M_{\mathbb{P}_2}(4, 2) \) be the moduli space of semi-stable sheaves on \( \mathbb{P}_2 \) with the Hilbert polynomial \( 4m + 2 \). Then the intersection Poincaré polynomial of \( M_{\mathbb{P}_2}(4, 2) \) is

\[
1 + 2t^2 + 6t^4 + 10t^6 + 14t^8 + 15t^{10} + 16t^{12} + 16t^{14} + 16t^{16}
+ 16t^{18} + 16t^{20} + 16t^{22} + 15t^{24} + 14t^{26} + 10t^{28} + 6t^{30} + 2t^{32} + t^{34}.
\]

**Proof.** The spaces \( M_{\mathbb{P}_2}(4, 2) \) and \( M_5 \) are related by Bridgeland wall-crossings on \( \mathbb{P}_2 \) ([BMW14, Section 6] and [CM17, Proposition 7.4]) with wall-crossing loci given in Table 1. Since \( \text{dim Ext}^1(F, F) = 17 = \text{dim } M_{\mathbb{P}_2}(4, 2) \) for \( F, F' \in W_1 \) or \( W_2 \), the wall-crossing loci are contained in the smooth part of moduli spaces. The result follows by comparing intersection cohomology groups. \( \square \)

**Remark 4.4.** The intersection Poincaré polynomial of \( M_{\mathbb{P}_2}(4, 2) \) is exactly the same as that of \( M_{\mathbb{P}_2}(4, 1) \) ([CC17, Corollary 5.2]).

| First wall (\( W_1 \)) | Second wall (\( W_2 \)) |
|-------------------------|-------------------------|
| 0 \to \mathcal{O}_{\mathbb{P}_2}(1) \to F \to \mathcal{O}_{\mathbb{P}_2}(-3)[1] \to 0 | 0 \to I_p(1) \to F \to I_q'(\mathcal{O}_{\mathbb{P}_2}(-3)[1] \to 0 \text{ for } p \text{ and } q \in \mathbb{P}^2 |
| 0 \to \mathcal{O}_{\mathbb{P}_2}(-3)[1] \to F' \to \mathcal{O}_{\mathbb{P}_2}(1) \to 0 | 0 \to I_q'(\mathcal{O}_{\mathbb{P}_2}(-3)[1] \to F' \to I_p(1) \to 0 \text{ for } p \text{ and } q \in \mathbb{P}^2 |

**Table 1.** Bridgeland wall-crossings between \( M_{\mathbb{P}_2}(4, 2) \) and \( M_5 \)
4.2. Case \( n = 4 \). Let \( F_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \). Consider the blow-up map \( F_1 \to \mathbb{P}^2 \) at a point. \( H_2(F_1, \mathbb{Z}) \cong \mathbb{Z} \cdot h \oplus \mathbb{Z} \cdot e \), where \( h \) is the hyperplane class and \( e \) is the exceptional divisor class. The canonical divisor of \( F_1 \) is \( K_{F_1} = -3h + e \) and the arithmetic genus of curve \( C \) in \( F_1 \) with \( c_1(\mathcal{O}_C) = dh - ne \) is

\[
 p_0(C) = \frac{(d - 1)(d - 2)}{2} - \frac{n(n - 1)}{2}
\]

from the adjunction formula.

Let \( M_{F_1}(F_1, 2) = M_{K_{F_1}}((4, 2), 10m + 2) \) be the moduli space of semi-stable sheaves \( F \) with \( c_1(F) = 4h - 2e \in H_2(F_1, \mathbb{Z}) \) and the Hilbert polynomial \( P(F)(m) = 10m + 2 \). To obtain the Poincaré polynomial of \( M_{F_1}(4, 2) \), we use the wall-crossings of the moduli space of \( \alpha \)-stable pairs. For the ample line bundle \( L = -K_{F_1} \), let \( P(F)(m) = \chi(F \otimes L^m) \) be the Hilbert polynomial of a coherent sheaf \( F \) on \( F_1 \). A pair \((s, F)\) consists of a coherent sheaf \( F \) on \( F_1 \) and a nonzero section \( \mathcal{O}_{F_1} \to F \). The pair is \( \alpha \)-semi-stable if \( F \) is pure and, for any subsheaf \( F' \subset F \)

\[
\frac{P(F')(m) + \delta \cdot \alpha}{r(F')} \leq \frac{P(F)(m) + \alpha}{r(F)}
\]

holds for \( m \gg 0 \), where \( r(F) = -K_{F_1} \cdot c_1(F) \) and \( \delta = 1 \) if the section \( s \) factors through \( F' \) and \( \delta = 0 \) otherwise. When the strict inequality holds, \((s, F)\) is called an \( \alpha \)-stable pair.

There exists a projective scheme \( M_{F_1}^0(F_1, P(m)) \) parameterizing \( S \)-equivalence classes of \( \alpha \)-semi-stable pairs with the Hilbert polynomial \( P(m) \) ([He98, Theorem 2.6]). We also have a decomposition of the moduli space

\[
 M_{F_1}^0(F_1, P(m)) = \bigsqcup_{\beta \in H_2(F_1, \mathbb{Z})} M_{F_1}^0(F_1, \beta, P(m)).
\]

**Notation 4.5.** We denote \( M_{F_1}^0(F_1, \beta, P(m)) \) by \( M_{F_1}^0(\beta, P(0)) \). If \( \alpha \) is sufficiently large (resp. small), we denote \( \alpha = \infty \) (resp. \( \alpha = 0^+ \)).

Wall-crossing phenomena of moduli spaces \( M_{F_1}^0((4, 2), 2) \) can be analyzed by the following propositions.

**Proposition 4.6 ([BJRR10]).**

Let \( h^i(m, n) := \dim H^i(F_1, \mathcal{O}_{F_1}(mh - ne)) \). Then,

\( 1 \) \( h^0(m, n) = \binom{m + 2}{2} - \binom{n + 1}{2} \)

\( 2 \) \( h^1(m, n) = \begin{cases} 
\binom{-n}{2} - \binom{-m - 1}{2} & \text{if } m \geq n \text{ and } -2 \geq n, \\
\binom{n + 1}{2} - \binom{m + 2}{2} & \text{if } m - n \leq -2 \text{ and } 1 \leq n, \\
0 & \text{otherwise.}
\end{cases} \)

\( 3 \) \( h^2(m, n) = \begin{cases} 
\binom{-m - 1}{2} - \binom{-n}{2} & \text{if } m \leq 0 \text{ and } n \leq 0, \\
0 & \text{otherwise,}
\end{cases} \)

where \( \binom{r}{2} = 0 \) for \( r < 2 \).
\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
$\alpha$ & $(s, ((4, 2), 2)) = (s, (d_1 h - n_1 e, \chi_1)) \oplus (0, (d_2 h - n_2 e, \chi_2))$ \\
\hline
$\frac{1}{2}$ & $(s, ((2, 0), 1)) \oplus (0, ((2, 2), 1))$ \\
\hline
$\frac{4}{5}$ & $(s, ((2, 2), 0)) \oplus (0, ((2, 0), 2))$ \\
\hline
$\frac{4}{3}$ & $(s, ((3, 2), 1)) \oplus (0, ((1, 0), 1))$ \\
\hline
3 & $(s, ((3, 1), 1)) \oplus (0, ((1, 1), 1))$ \\
\hline
8 & $(s, ((3, 1), 0)) \oplus (0, ((1, 1), 2))$ \\
\hline
$8'$ & $(s, ((4, 3), 1)) \oplus (0, ((0, -1), 1))$ \\
\hline
13 & $(s, ((3, 1), -1)) \oplus (0, ((1, 1), 3))$ \\
\hline
\end{tabular}
\caption{Numerical walls of $M^0_{F_1}((4, 2), 2)$}
\end{table}

**Proposition 4.7** ([He98, Corollary 1.6]). Let $\Lambda = (s, F)$ and $\Lambda' = (s', F')$ be pairs on a smooth projective variety $X$. Then, there exists a long exact sequence

$$0 \to \text{Hom}(\Lambda, \Lambda') \to \text{Hom}(F, F') \to \text{Hom}(s, H^0(F')/s')$$

$$\to \text{Ext}^1(\Lambda, \Lambda') \to \text{Ext}^1(F, F') \to \text{Hom}(s, H^1(F'))$$

$$\to \text{Ext}^2(\Lambda, \Lambda') \to \text{Ext}^2(F, F') \to \text{Hom}(s, H^2(F')) \to \cdots .$$

**Proposition 4.8.** The $\infty$-stable pairs space $M^0_{F_1}((4, 2), 2)$ is a $\mathbb{P}^8$-bundle over the Hilbert scheme of three points on $F_1$.

**Proof.** From Proposition 4.6, $h^0(O_{F_1}(4h - 2e)) = 12$ and $h^1(O_{F_1}(4h - 2e)) = 0$. The line bundle $O_{F_1}(4h - 2e)$ is also 2-very ample from [DR96]. Thus, the result follows by applying the same argument as Lemma 2.3 in [CC17].

During wall-crossings, a pair $(s, F)$ is lying in the wall at $\alpha$ if

$$(s, F) = (s, F_1) + (0, F_2)$$

and

$$\frac{\chi(F) + \alpha}{K_{F_1} : c_1(F)} = \frac{\chi(F_1) + \alpha}{K_{F_1} : c_1(F_1)} = \frac{\chi(F_2)}{K_{F_1} : c_1(F_2)} .$$

The numerical walls of $M^0_{F_1}((4, 2), 2)$ are listed in Table 2 by a direct calculation.

**Lemma 4.9.** The walls at $\alpha = \frac{1}{2}, \frac{4}{3},$ and $13$ (Table 2) are empty.

**Proof.** The case $\alpha = \frac{1}{2}$ cannot occur. Let $F_2$ be a stable sheaf with $c_1(F_2) = 2h - 2e$ and $\chi(F_2) = 1$. Support of $F$ should be the fiber of the projection map $p : F_1 = \mathbb{P}(O_{F_1} \oplus O_{F_1}(-1)) \to \mathbb{P}^1$. From $\chi(F_2) = 1$, we have the canonical map $s : O_{F_1} \to F_2$. Hence the image of $s$ is of the form $\text{im}(s) = O_C$, where $C$ is supported on the fiber of the map $p$. Therefore, the possible classes for $C$ are only $c_1(O_C) = h - e$ or $2h - 2e$, but both classes violate the stability of $F_2$.

For $\alpha = \frac{4}{3}$, let $(s, F_1)$ be a stable pair with $c_1(F_1) = 2h - 2e$ and $\chi(F_1) = 0$. Then the image of the section map $s : O_{F_1} \to F_1$ is $\text{im}(s) = O_C$, such that $c_1(O_C) = h - e$ or $2h - 2e$, which is a contradiction to the stability of $(s, F_1)$. 

For $\alpha = 13$, let $(s, F_1)$ be a stable pair with $c_1(F_1) = 2h - 2e$ and $\chi(F_1) = 0$. Then the image of the section map $s : O_{F_1} \to F_1$ is $\text{im}(s) = O_C$, such that $c_1(O_C) = h - e$ or $2h - 2e$, which is a contradiction to the stability of $(s, F_1)$. 

...
| \(\alpha\) | Blow-up center at \(\alpha + \epsilon\) | Blow-up center at \(\alpha - \epsilon\) |
|---|---|---|
| \(\frac{4}{3}\) | a \(\mathbb{P}^1\)-bundle over \(\mathbb{P}^2 \times \mathbb{P}^6\) | a \(\mathbb{P}^2\)-bundle over \(\mathbb{P}^2 \times \mathbb{P}^6\) |
| 3 | a \(\mathbb{P}^1\)-bundle over \((a \mathbb{P}^1\text{-bundle over } \mathbb{P}_1) \times \mathbb{P}^7\) | a \(\mathbb{P}^1\)-bundle over \((a \mathbb{P}^1\text{-bundle over } \mathbb{P}_1) \times \mathbb{P}^7\) |
| \(8\) | a \(\mathbb{P}^3\)-bundle over \(\mathbb{P}^8 \times \mathbb{P}^1\) | a \(\mathbb{P}^1\)-bundle over \(\mathbb{P}^8 \times \mathbb{P}^1\) |
| \(8'\) | a \(\mathbb{P}^3\)-bundle over \(\mathbb{P}^8\) | a \(\mathbb{P}^2\)-bundle over \(\mathbb{P}^8\) |

Table 3. Blow-up centers of \(M_{F_1}((4, 2), 2)\)

Finally, for \(\alpha = 13\), let \((s, F_1)\) be the stable pair with \(c_1(F_1) = 3h - e\) and \(\chi(F_1) = -1\). The dual \(F_1^D := \text{Ext}^1(F_1, \omega_{F_1})\) of \(F_1\) fits into the unique non-split extension

\[
0 \to \mathcal{O}_C \to F_1^D \to \mathbb{C}_p \to 0
\]

for \(c_1(\mathcal{O}_C) = 3h - e\), \(p \in C\), hence \(F_1 \cong F_1^D = I_{p,C}\) (cf. [CvGKT18, Proposition 4.4]). Since \(h^0(F_1) = 0\), the wall is empty. \(\square\)

**Proposition 4.10.** There exist wall-crossings among moduli spaces \(M_{F_1}^\alpha((4, 2), 2)\) of \(\alpha\)-stable pairs on \(F_1\):

\[
M_{F_1}^\infty((4, 2), 2) \leftrightarrow M_{F_1}^+(4, 2), 2)
\]

where the blow-up centers at each wall are listed in Table 3.

**Proof.** For wall \(\alpha = \frac{4}{3}\), let \((s, F_1)\) be the stable pair with \(c_1(F_1) = 3h - 2e\) and \(\chi(F_1) = 1\). From (4.1), the section map \(s : \mathcal{O}_C \to F_1\) must be an isomorphism and hence the pairs \((s, F_1)\) are parameterized by \(|\mathcal{O}_{F_1}(3h - 2e)| \cong \mathbb{P}^6\) (Proposition 4.6). By the same argument, the locus for pairs \((0, F_2)\) with \(c_1(F_2) = h\) and \(\chi(F_2) = 1\) is parameterized by \(|\mathcal{O}_{F_1}(h)| \cong \mathbb{P}^2\). For these pairs, the wall crossing locus at \(\alpha = \frac{4}{3} + \epsilon\) parameterizes the non-split extensions

\[
0 \to (0, F_2) \to (s, F') \to (s, F_1) \to 0.
\]

On the other hand, the wall locus at \(\alpha = \frac{4}{3} - \epsilon\) parameterizes the non-split extensions

\[
0 \to (s, F_1) \to (s, F'') \to (0, F_2) \to 0.
\]

The results in Table 3 follow since \(\text{Ext}^1((s, F_1), (0, F_2)) = \mathbb{C}^4\) and \(\text{Ext}^1((0, F_2), (s, F_1)) = \mathbb{C}^3\) (Proposition 4.6 and Proposition 4.7).

The other cases can be derived by the same method and we omit the detail. \(\square\)

We compare spaces \(M_{F_1}^+(4, 2), 2)\) and \(M_{F_1}^+(4, 2), 2)\). For the polystable sheaf \(F \in M_{F_1}((4, 2), 2) \setminus M_{F_1}^+(4, 2), 2)\), \(F \cong \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2}\) for some curves \(C_1\) and \(C_2\) with \(c_1(\mathcal{O}_{C_i}) = 2h - e\) for \(i = 1, 2\). From Proposition 4.6, the space \(\text{Sym}^2\mathbb{P}^4\) parametrize such sheaves.

**Proposition 4.11.** Let \(\phi : M_{F_1}^+(4, 2), 2) \to M_{F_1}((4, 2), 2)\) be the forgetful map \((s, F') \mapsto F\).

1. \(\phi\) is a \(\mathbb{P}^1\)-fibration over stable locus \(M_{F_1}^+(4, 2), 2)\).
2. Let \(\Delta \subset \text{Sym}^2\mathbb{P}^4\) be the diagonal.
   a. For \([F] \in \text{Sym}^2\mathbb{P}^4 \setminus \Delta \subset M_{F_1}((4, 2), 2) \setminus M_{F_1}^+(4, 2), 2)\), the fiber \(\phi^{-1}([F]) = (s, F)\)
   parameterizes the non-split extension class

\[
0 \to (0, \mathcal{O}_{C_2}) \to (s, F) \to (s, \mathcal{O}_{C_1}) \to 0,
\]
which is a \((\mathbb{P}^3 - \{\text{pt}\})\)-bundle over \(\mathbb{P}^4 \times \mathbb{P}^4 \setminus \mathbb{P}^4\).

(b) For \([F] \in \Delta\), the fiber \(\phi^{-1}([F])\) parametrizes the unique pair \((s, \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2})\) such that \(C_1 \neq C_2\).

(3) Over \(\Delta \cong \mathbb{P}^4 = |\mathcal{O}_{F_1}(2h - e)| \subset \mathbf{M}_{\mathbb{F}_1}((4, 2), 2) \setminus \mathbf{M}_{\mathbb{F}_1}^0((4, 2), 2)\), \(\phi\) is a \(\mathbb{P}^3\)-fibration over its base space \(\Delta\).

Proof. Since \(\chi(F) = 2\) for each \(F \in \mathbf{M}_{\mathbb{F}_1}((4, 2), 2), h^0(F) \geq 2\). If \(h^0(F) \geq 3\), then \(h^0(FD) \geq 1\) from the Serre duality. Hence there is a non-zero homomorphism \(\mathcal{O}_C \to F^D\) for \(\text{Supp}(FD) = C\), \(c_1(\mathcal{O}_C) = 4h - 2e\), which violates the semi-stability of \(FD\). Thus, \(h^0(F) = 2\) which implies item (1).

The remaining proof of the claim follows [CM16, Proposition 3.6] by changing the extension groups into

\[
\text{Ext}^1_{\mathbb{F}_1}(\mathcal{O}_{C_1}, \mathcal{O}_{C_2}) = \begin{cases} 
\mathbb{C}^3, & \text{for } C_1 \neq C_2 \in |\mathcal{O}_{F_1}(2h - e)| \\
\mathbb{C}^4, & \text{for } C_1 = C_2 \in |\mathcal{O}_{F_1}(2h - e)|,
\end{cases}
\]

and we omit the detail. \(\square\)

**Corollary 4.12.** The virtual Poincaré polynomial of \(\mathbf{M}_{\mathbb{F}_1}((4, 2), 2)\) is

\[
P(\mathbf{M}_{\mathbb{F}_1}((4, 2), 2)) = 1 + 3t^2 + 6t^4 + 8t^6 + 7t^8 + 7t^{10} + 6t^{12} + 8t^{14} + 8t^{16} + 10t^{18} + 9t^{20} + 8t^{22} + 3t^{24} + t^{26}.
\]

**Proof.** From Proposition 4.10,

\[
P(\mathbf{M}_{\mathbb{F}_1}^+(((4, 2), 2)) = P(\mathbf{M}_{\mathbb{F}_1}^\infty((4, 2), 2)) + (P(\mathbb{P}^2) - P(\mathbb{P}^3))E(\mathbb{P}^8) + (P(\mathbb{P}^1) - P(\mathbb{P}^3))P(\mathbb{P}^8)P(\mathbb{P}^1) + P(\mathbf{F}_1)P(\mathbb{P}^7)P(\mathbb{P}^1)(P(\mathbb{P}^1) - P(\mathbb{P}^2)) + P(\mathbb{P}^0)P(\mathbb{P}^2)(P(\mathbb{P}^2) - P(\mathbb{P}^3)),
\]

and from Proposition 4.8 \(P(\text{Hilb}^3(\mathbf{F}_1)) = t^{12} + 3t^{10} + 9t^8 + 14t^6 + 9t^4 + 3t^2 + 1\) \([\text{GS93}]\) and \(P(\mathbf{M}_{\mathbb{F}_1}^\infty((4, 2), 2)) = P(\text{Hilb}^3(\mathbf{F}_1)) \cdot P(\mathbb{P}^8)\). Thus,

\[
P(\mathbf{M}_{\mathbb{F}_1}^+(((4, 2), 2))) = t^{28} + 4t^{26} + 11t^{24} + 18t^{22} + 23t^{20} + 24t^{18} + 24t^{16} + 24t^{14} + 24t^{12} + 24t^{10} + 23t^8 + 18t^6 + 11t^4 + 4t^2 + 1.
\]

On the other hand, from Propositions 4.11 and 2.10,

\[
P(\mathbf{M}_{\mathbb{F}_1}^+(((4, 2), 2))) = P(\mathbb{P}^4)P(\mathbf{M}_{\mathbb{F}_1}^0((4, 2), 2)) + (P(\mathbb{P}^3) - 1)(P(\mathbb{P}^4 \times \mathbb{P}^4) - P(\mathbb{P}^4)) + (P(\text{Sym}^2\mathbb{P}^4) - P(\mathbb{P}^4)) + P(\mathbb{P}^3) \cdot P(\mathbb{P}^4)
\]

and thus we obtain \(P(\mathbf{M}_{\mathbb{F}_1}^+(((4, 2), 2)))\). Finally, \(P(\mathbf{M}_{\mathbb{F}_1}((4, 2), 2)) = P(\mathbf{M}_{\mathbb{F}_1}^+((4, 2), 2)) + P(\text{Sym}^2\mathbb{P}^4)\). \(\square\)

**Corollary 4.13.** The virtual intersection Poincaré polynomial of \(\mathbf{M}_{\mathbb{F}_1}((4, 2), 2)\) is

\[
\text{IP}(\mathbf{M}_{\mathbb{F}_1}((4, 2), 2)) = 1 + 3t^2 + 8t^4 + 10t^6 + 11t^8 + 11t^{10} + 11t^{12} + 11t^{14} + 11t^{16} + 11t^{18} + 10t^{20} + 8t^{22} + 3t^{24} + t^{26}.
\]

**Proof.** From Corollaries 2.17 and 4.12, it is sufficient to calculate intersection cohomology for the open cones of singular locus in \(\mathbf{M}_{\mathbb{F}_1}((4, 2), 2)\). Since the analytic neighborhoods of the strictly semi-stable loci of \(\mathbf{M}_4\) and \(\mathbf{M}_{\mathbb{F}_1}((4, 2), 2)\) are isomorphic to each other (cf. Remark 2.1 and Section 4, paragraph 1), the result follows from the result of Corollary 3.4. \(\square\)
Remark 4.14. We use the intersection cohomology of the open cones of singular loci of the moduli spaces for the case $F_1$ since we do not know any explicit birational relation among the relevant moduli spaces, unlike the cases $F_0$ and $P^2$.

Remark 4.15. The virtual intersection Poincaré polynomial of $M_{F_1}((4, 2), 2)$ in Proposition 4.13 is exactly the same as that of $M_{F_1}((4, 2), 1)$ ([CvGKT18, Proposition 4.9]).

From Remarks 4.2, 4.4, and 4.15,

Conjecture 4.16. The (virtual) intersection Poincaré polynomial of the space $M_S(c, \chi)$ depends only on the first Chern class $c$.

References

[BJ12] B. Bakker and A. Jorza. Higher rank stable pairs on k3 surfaces. Commun. Number Theory Phys., 6(4):805–847, 2012.

[BJRR10] Ralph Blumenhagen, Benjamin Jurke, Thorsten Rahn, and Helmut Roschy. Cohomology of line bundles: A computational algorithm. Journal of Mathematical Physics, 51(10):103525, 2010.

[BMW14] Aaron Bertram, Cristian Martinez, and Jie Wang. The birational geometry of moduli spaces of sheaves on the projective plane. Geom. Dedicata, 173:37–64, 2014.

[Bri99] M. Brion. Rational smoothness and fixed points of torus actions. Transformation Groups, 4(2):127–156, Jun 1999.

[BS14] Edoardo Ballico and Huh Sukmoon. Stable sheaves on a smooth quadric surface with linear hilbert biquadric, The Scientific World Journal, 2014, 2014.

[CC15] Jinwon Choi and Kiryong Chung. The geometry of the moduli space of one-dimensional sheaves. Sci. China Math., 58(3):487–500, 2015.

[CC17] Jinwon Choi and Kiryong Chung. Moduli spaces of $\alpha$-stable pairs and wall-crossing on $\mathbb{P}^2$. J. Math. Soc. Jap., 68(2):685–789, 2017.

[CHK12] Kiryong Chung, Jaehyun Hong, and Young-Hoon Kiem. Compactified moduli spaces of rational curves in projective homogeneous varieties. J. Math. Soc. Japan, 64(4):1211–1248, 2012.

[CHL18] Kiryong Chung, Jaehyun Hong, and SangHyun Lee. Geometry of moduli spaces of rational curves in linear sections of grassmannian $gr(2, 5)$. Journal of Pure and Applied Algebra, 222(4):868–888, 2018.

[CHW14] Izzet Coskun, Jack Huizenga, and Matthew Woolf. The effective cone of the moduli space of sheaves on the plane. arXiv:1401.1613, 2014.

[CMP97] W. Fulton and R. Pandharipande. Notes on stable maps and quantum cohomology. In Algebraic geometry—Santa Cruz 1995, volume 62 of Proc. Sympos. Pure Math., pages 45–96. Amer. Math. Soc., Providence, RI, 1997.

[GS93] Lothar Göttsche and Wolfgang Soergel. Perverse sheaves and the cohomology of hilbert schemes of smooth algebraic surface. Mathematische Annalen, 296(2):235–246, 1993.

[He98] Min He. Espaces de modules de systèmes cohérents. Internat. J. Math., 9(5):545–598, 1998.
[HL10] Daniel Huybrechts and Manfred Lehn. *The geometry of moduli spaces of sheaves*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2010.

[HT16] Shinobu Hosono and Hiromichi Takagi. Double quintic symmetroids, reye congruences, and their derived equivalence. *J. Differential Geom.*, 104(3):443–497, 2016.

[Kat08] Sheldon Katz. Genus zero Gopakumar-Vafa invariants of contractible curves. *J. Differential Geom.*, 79(2):185–195, 2008.

[Kie07] Young-Hoon Kiem. Hecke correspondence, stable maps, and the Kirwan desingularization. *Duke Math. J.*, 136(3):585–618, 2007.

[Kin94] A. D. King. Moduli of representations of finite-dimensional algebras. *Quart. J. Math. Oxford Ser. (2)*, 45(180):515–530, 1994.

[Kir85] Frances Clare Kirwan. Partial desingularisations of quotients of nonsingular varieties and their Betti numbers. *Ann. of Math.* (2), 122(1):41–85, 1985.

[Kir86a] Frances Clare Kirwan. On the homology of compactifications of moduli spaces of vector bundles over a rie mann surface. *Proc. London Math. Soc.*, 53:237–266, 1986.

[Kir86b] Frances Clare Kirwan. Rational intersection cohomology of quotient varieties. *Invent. Math.*, 86(3):471–505, 1986.

[Las96] Yves Laszlo. Local structure of the moduli space of vector bundles over curves. *Commentarii Mathematici Helvetici*, 71(1):373–401, Dec 1996.

[LM14] Alberto López Martín. Poincaré polynomials of stable map spaces to Grassmannians. *Rend. Semin. Mat. Univ. Padova*, 131:193–208, 2014.

[LMN13] M. Logares, Vicente Muñoz, and P. E. Newstead. Hodge polynomials of SL(2, C)-character varieties for curves of small genus. *Rev. Mat. Complut.*, 26(3):635–703, 2013.

[LP93] J. Le Potier. Faisceaux semi-stables de dimension 1 sur le plan projectif. *Rev. Roumaine Math. Pures Appl.*, 38(7-8):635–678, 1993.

[Max18] Laurentiu G. Maxim. Intersection homology and perverse sheaves, with applications to singularities. *Book Project*, 2018.

[MOVG09] Vicente Muñoz, Daniel Ortega, and Maria-Jesús Vázquez-Gallo. Hodge polynomials of the moduli spaces of triples of rank (2, 2). *Q. J. Math.*, 60(2):235–272, 2009.

[PS08] Chris A. M. Peters and Joseph H. M. Steenbrink. *Mixed Hodge Structures*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Springer Berlin Heidelberg, 2008.

[Sim94] Carlos T. Simpson. Moduli of representations of the fundamental group of a smooth projective variety. I. *Inst. Hautes Études Sci. Publ. Math.*, (79):47–129, 1994.

[Woo13] Matthew Woolf. Nef and effective cones on the moduli space of torsion sheaves on the projective plane. arXiv:1305.1465, 2013.