A Lyapunov Approach for the Exponential Stability of a Damped Timoshenko Beam

Andrea Mattioni ⊗, Member, IEEE, Yongxin Wu ⊗, Member, IEEE, and Yann Le Gorrec ⊗, Senior Member, IEEE

Abstract—In this technical note, we consider the stability properties of a viscously damped Timoshenko beam equations with spatially varying parameters. With the help of the port-Hamiltonian framework, we first prove the existence of solutions and show, by the use of an appropriate Lyapunov function, that the system is exponentially stable and has an explicit decay rate. The explicit exponential bound is computed for an illustrative example for which we provide some numerical simulations.

Index Terms—Distributed parameter systems, exponential stability, port-Hamiltonian systems, viscous damping.

I. INTRODUCTION

The Timoshenko beam theory is often used in engineering applications to represent the propagation of vibrations in mechanical systems, such as buildings, aircraft structures, flexible robots, and microgrippers [1], [2]. In this technical note, we consider the Timoshenko beam described by partial differential equations (PDEs) with space-varying parameters and viscous damping. In the case of constant parameters, the system has already been proven to be exponentially stable in [3] using the Gearhart–Herbst-Prüss–Huang spectral method [4]. In [3], the authors prove that there exist $M > 0$ and $\alpha > 0$ such that $\| T(t)z_0 \| \leq Me^{-\alpha t}$ for all $z_0 \in \mathbb{R}$, but do not provide any estimation of these two quantities. The same result with space varying parameters has been proven in [5] using the same techniques. In [6], the authors constructed a Lyapunov function to prove exponential stability in case of constant parameters, but without making explicit the state's norm decay rate. Furthermore, different studies have focused on the stabilization problem in the case of the presence of damping in only one beam dynamics, e.g., vertical or rotational dynamics. In particular, in [7], the authors used a Lyapunov function to show that the system is exponentially stable if and only if the wave propagation speeds of the two dynamics are identical. A technical extension to linear and nonlinear operator equations using Lyapunov techniques can be found in [8] and [9].

Over the last 20 years, the port-Hamiltonian (PH) framework has proved to be a useful tool for the stability analysis and control design for PDEs. It has been used to design static [10], linear dynamic [2], and nonlinear dynamic [11] PDEs boundary controllers able to exponentially stabilize the origin of the closed-loop system. Existing results using the PH framework have been obtained without considering internal dissipation (e.g., viscous damping for flexible beams). Finding the exponential bound parameters becomes challenging due to the lack of internal dissipation, resulting in only assessing exponential decay without explicit decay of the norm [12].

In this technical note, inspired by [9] and [7], we propose a Lyapunov function with crossing terms in order to prove the exponential stability of a Timoshenko beam in the case of spatially varying parameters and viscous damping in both the vertical and rotational dynamics. The proposed Lyapunov function allows us to compute the parameters $M$, $\alpha$ of the exponential bound $\| T(t)z_0 \| \leq Me^{-\alpha t}$. This work relies on the PH framework [13], [14] for the result on existence and uniqueness of solutions, and on [15] for the state variable selection.

The rest of this article is organized as follows. In Section II, we recall some technical preliminaries that will be useful for the stability proof. In Section III, is stated the main result of the article, i.e., exponential stability with an explicit formulation of the decay rate of the solution’s norm. Then, a numerical example is presented to validate the theoretical results. This technical note ends with some conclusions in Section IV.

II. PRELIMINARIES

A. Useful Inequalities

Throughout the article, we make use of some standard inequalities that are very often used in the literature on control of PDE. We recall two classical inequalities, that hold for all functions $f$, $g : \Omega \to \mathbb{R}$ with $\Omega \subseteq \mathbb{R}^N$, $N \in \mathbb{N}_2$:

Young’s inequality

$$fg \leq \frac{1}{2\alpha} \left( |f|^2 + \frac{\alpha}{2} |g|^2 \right)$$

(1)

for all $\alpha > 0$.

Cauchy–Schwarz inequality

$$\int_0^L f(\xi)g(\xi) d\xi \leq \left( \int_0^L f(\xi)^2 d\xi \right)^{\frac{1}{2}} \left( \int_0^L g(\xi)^2 d\xi \right)^{\frac{1}{2}}.$$  

(2)

In the next lemma we introduce the Poincaré-type inequality that can be derived from [16, Th. 256], changing the integration interval from $[0,1]$ to $[0,L]$.

Lemma 2.1 (Variation of the Wirtinger’s inequality): For any absolutely continuous function $f$ such that $f(0) = 0$,

$$\int_0^L f(\xi)^2 d\xi \leq \left( \frac{2L}{\pi} \right)^2 \int_0^L \left( \frac{d}{d\xi} f(\xi) \right)^2 d\xi.$$  

(3)
B. Lyapunov Stability Theory

Let $z$ belong to a Hilbert space $Z$ and consider the linear differential equation
\[ \dot{z} = Az, \quad z(0) = z_0 \tag{4} \]
where we assume that the operator $A$ with domain $D(A)$ is the infinitesimal generator of a $C_0$-semigroup $T(t)$ on the state space $Z$. In the following, we denote the solution of (4) with initial condition $z_0$ as $z(t, z_0) = T(t)z_0$. Now, we introduce the concept of Lyapunov function for (4).

**Definition 2.2:** A continuous functional $V : Z \to [0, \infty)$ is a Lyapunov functional for (4) on $Z$ if $V(z(t, z_0))$ is Dini differentiable at $t = 0$ for all $z_0 \in X$ and the following inequality holds:
\[ \dot{V}(z_0) := \limsup_{t \to 0} \frac{V(z(t, z_0)) - V(z_0)}{t} \leq 0. \tag{5} \]

Since in most practical cases the limit (5) is not easy to compute, we rely on [17, Lemma 11.2.5] to establish the relation between the Dini time derivative (see [17, Definition A.5.43]) and the Fréchet derivative (see [17, Definition A.5.31]). In fact, if $V$ is Fréchet differentiable, then for $z \in D(A)$, $V(z(t, z_0))$ is Dini differentiable and
\[ \dot{V}(z_0) := dV(z_0)Az_0 \tag{6} \]
where $dV$ is the Fréchet derivative of $V$. In the following, we cite a part of [17, Th. 11.2.7], that will be instrumental to prove the exponential stability.

**Theorem 2.3:** Suppose that $V$ is a Lyapunov functional for (4) with $V(0) = 0$. If there exist two positive constants $\kappa_1, \kappa_2 > 0$ such that $V(z) \geq \kappa_1 ||z||^2$ and $\dot{V}(z) \leq -\kappa_2 V(z)$ for all $z \in Z$, then the origin is globally exponentially stable, and
\[ ||z(t, z_0)|| \leq \sqrt{\frac{V(z_0)}{\kappa_1}} e^{-\frac{\kappa_2}{2} t}. \tag{7} \]

III. MAIN RESULT

A. Port Hamiltonian Formulation of the Timoshenko’s Beam With Viscous Damping

We consider the equations of motion of a clamped Timoshenko beam with viscous damping
\begin{align*}
\rho \frac{\partial^2 w}{\partial t^2} &= \frac{\partial}{\partial \xi} \left( K \left( \frac{\partial w}{\partial \xi} - \phi \right) \right) - \gamma \frac{\partial w}{\partial t} \\
I_{\rho} \frac{\partial^2 \phi}{\partial t^2} &= \frac{\partial}{\partial \xi} \left( EI \frac{\partial \phi}{\partial \xi} + \left( \frac{\partial w}{\partial \xi} - \phi \right) \right) - \frac{\partial \phi}{\partial t} \\
w(0, t) &= \phi(0, t) = 0 \\
K(L) \frac{\partial w}{\partial \xi}(L, t) - \phi(L, t) &= \gamma(L) \frac{\partial w}{\partial t}(L, t) \\
EI(L, t) \frac{\partial \phi}{\partial \xi}(L, t) &= \delta(L) \frac{\partial \phi}{\partial t}(L, t). \tag{8}
\end{align*}

Throughout the article, the terms $\xi, s \in [0, L]$ identify the spatial coordinate, while $w(\xi, t)$ and $\phi(\xi, t)$ represent the deflection and the relative rotation of a beam cross-section in the rotating frame at position $\xi$ and time $t$, respectively. $E(\xi), I(\xi)$ are the spatially dependent Young’s modulus and moment of inertia of the beam’s cross-section, respectively. $\rho(\xi), I_{\rho}(\xi)$ are the spatially dependent density and mass moment of inertia of the beam’s cross-section, respectively. $K(\xi)$ is the cross-sectional area. $\gamma(\xi)$ and $\delta(\xi)$ represent the spatially translating and the rotating components of the viscous damping, respectively. Throughout this article, all physical parameters and their reciprocals are assumed to be absolutely continuous, positive definite, and belong to $L_\infty([0, L])$. Following [15], we define the energy variables as
\[ z_1 = \rho \frac{\partial w}{\partial t}, \quad z_2 = I_{\rho} \frac{\partial \phi}{\partial t}, \quad z_3 = \frac{\partial w}{\partial \xi} - \phi, \quad z_4 = \frac{\partial \phi}{\partial \xi} \tag{9} \]
such to write the port-Hamiltonian representation of (8) with the state variable $z = [z_1 z_2 z_3 z_4]^T$
\[ \dot{z} = P_1 \frac{\partial}{\partial \xi}(H z) + (P_0 - G_0)(H z) \tag{10} \]
where $H z$ are the coenergy variables and
\[ P_1 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}, \quad P_0 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad G_0 = \begin{bmatrix}
\gamma & 0 & 0 & 0 \\
0 & \delta & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}. \tag{11} \]

We define the state space $Z = L_2([0, L], \mathbb{R}^4)$ and we equip it with the energy inner product
\[ \langle z_1, z_2 \rangle_Z = \langle z_1, H z_2 \rangle_{L_2} = \int_0^L z_1^T H z_2 d\xi. \tag{12} \]

The energy of the beam is defined by
\[ E = \frac{1}{2} \langle z, z \rangle_Z. \tag{13} \]

Following [18], we define the boundary flow and effort as a composition of the coenergy variables at the boundary of the spatial domain
\[ f_0(t) \begin{bmatrix}
f_0(t) \\
e_0(t)
\end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix}
P_1 & -I_1 \\
-I_1 & I_1
\end{bmatrix} \begin{bmatrix}
(H z)(0, t) \\
(H z)(L, t)
\end{bmatrix}. \tag{14} \]

As shown in the following, the boundary flow and effort are instrumental to define the boundary operators such to obtain a well-posed (in the Hadamard sense) set of PDE
\begin{align*}
\mathcal{B}_1 z(t) &= W_{B1} \begin{bmatrix}
f_0(t) \\
e_0(t)
\end{bmatrix} = \begin{bmatrix}
\frac{1}{\sqrt{2}} z_1(0, t) \\
\frac{1}{\sqrt{2}} z_2(0, t)
\end{bmatrix} \\
\mathcal{B}_2 z(t) &= W_{B2} \begin{bmatrix}
f_0(t) \\
e_0(t)
\end{bmatrix} = \begin{bmatrix}
-K(L) z_3(L, t) \\
KEI(0) z_4(0, t)
\end{bmatrix} \\
\mathcal{C}_1 z(t) &= W_{C1} \begin{bmatrix}
f_0(t) \\
e_0(t)
\end{bmatrix} = \begin{bmatrix}
K(0) z_3(0, t) \\
KEI(0) z_4(0, t)
\end{bmatrix} \\
\mathcal{C}_2 z(t) &= W_{C2} \begin{bmatrix}
f_0(t) \\
e_0(t)
\end{bmatrix} = \begin{bmatrix}
\frac{1}{\sqrt{2}} z_1(L, t) \\
\frac{1}{\sqrt{2}} z_2(L, t)
\end{bmatrix}. \tag{15}
\end{align*}

with
\begin{align*}
W_{B1} &= -\frac{1}{\sqrt{2}} \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \\
W_{B2} &= \frac{1}{\sqrt{2}} \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\end{align*}
We can now define the operator

\[
\mathcal{J} z = P_1 \frac{d}{dx}(\mathcal{H} z) + (P_0 - G_0)(\mathcal{H} z)
\]

with domain

\[
D(\mathcal{J}) = \{ z \in Z \mid \mathcal{H} z \in H^1, \mathcal{B}_1 z = 0, \mathcal{B}_2 z = -S(L)\mathcal{E}_2 z \}
\]

and \( S = \text{diag}\{\gamma, \delta\} \). In the following proposition, we show that the operator \( \mathcal{J} \) with domain \( D(\mathcal{J}) \) generates a contraction \( C_0\)-semigroup, or equivalently that the dynamical system (10) is well posed.

**Proposition 3.1:** The operator \( \mathcal{J} \) in (17) with domain (18) generates a contraction \( C_0\)-semigroup on the state space \( Z \). Moreover,

\[
\dot{E}_+ = \langle \mathcal{J} z, z \rangle Z
\]

\[
= -\int_0^L \left[ \frac{\gamma}{\rho} z_z^2 + \frac{\delta}{\rho} z_z^2 \right] d\xi - (\mathcal{E}_2 z)^T S(L)\mathcal{E}_2 z. \tag{19}
\]

**Proof:** For the generation result, it is sufficient to use [13, Th. 6.9].

For the energy time derivative, we compute

\[
\dot{E}_+ (z) = dE_+(\mathcal{J} z) = \langle \mathcal{J} z, z \rangle Z
\]

\[
= \int_0^L \left( P_1 \frac{d}{dx}(\mathcal{H} z) + (P_0 - G_0)(\mathcal{H} z) \right)^T \mathcal{H} z d\xi
\]

\[
= -\int_0^L (\mathcal{H} z)^T G_0(\mathcal{H} z) d\xi
\]

\[
+ \int_0^L \left( P_1 \frac{d}{dx}(\mathcal{H} z) + P_0(\mathcal{H} z) \right)^T (\mathcal{H} z) d\xi. \tag{20}
\]

The first term of the last equation corresponds to the first term in (19), while the second term, after integration by parts, makes appear the second term in (19).

Next, we present two inequalities that will be useful for the stability analysis we will propose considering Lyapunov arguments.

**Lemma 3.2:** For any function \( z_3, z_4 \in L_2([0, L], \mathbb{R}) \), the following inequalities hold:

\[
\int_0^L \left( \int_0^L K z_3 d\xi \right)^2 d\xi \leq k_1 \int_0^L K z_3^2 d\xi \tag{21}
\]

\[
\int_0^L \left( \int_0^L E I z_4 d\xi \right)^2 d\xi \leq k_2 \int_0^L E I z_4^2 d\xi \tag{22}
\]

with \( k_1 = (\frac{4L}{\pi})^2 K \) and \( k_2 = (\frac{4L}{\pi})^2 E I \), where \( K = \text{ess sup}_{[0, L]} K(\xi) \) and \( E I = \text{ess sup}_{[0, L]} E I(\xi) \).

**Proof:** To obtain the first inequality, we apply Wirtinger’s inequality of Lemma 2.1

\[
\int_0^L \left( \int_0^L K z_3 d\xi \right)^2 d\xi \leq \left( \frac{2L}{\pi} \right)^2 \int_0^L (K z_3)^2 d\xi
\]

\[
\leq \left( \frac{2L}{\pi} \right)^2 K \int_0^L (z_3)^2 d\xi. \tag{23}
\]

The second inequality is obtained in the same manner.

**B. Stability Analysis**

The aim of this section is to propose an appropriate Lyapunov function allowing to show the exponential stability of the system and to explicit its decay rate. The proposed Lyapunov function is composed of the natural energy of the system together with two cross-coupling terms:

\[
V = n_0 E + n_1 F_1 + n_2 F_2 \tag{24}
\]

with \( n_0, n_1, n_2 > 0 \) and \( F_1, F_2 \) defined as

\[
F_1 = \int_0^L z_1 \left( \int_0^\xi K z_3 d\xi \right) d\xi, \quad F_2 = \int_0^L z_2 \left( \int_0^\xi E I z_4 d\xi \right) d\xi. \tag{25}
\]

**Lemma 3.3:** For any state \( z \in Z \) the Lyapunov function (24) is well defined, i.e., it is finite in all the state space \( Z \).

**Proof:** The energy term \( E \) in (24) is bounded as soon as \( z \in Z \). By using the first Young’s inequality and Lemma 3.2, we get

\[
\int_0^L z_1 \left( \int_0^\xi K z_3 d\xi \right) d\xi \leq \frac{1}{2} \int_0^L \left( \int_0^\xi K z_3^2 d\xi \right) d\xi
\]

\[
+ \frac{1}{2} \int_0^L z_1^2 d\xi \leq \frac{1}{2} k_1 \int_0^L K z_3^2 d\xi + \frac{1}{2} \int_0^L z_1^2 d\xi. \tag{26}
\]

that shows that \( F_1 \) is bounded as soon as \( z \in Z \). The term \( F_2 \) can be shown to be bounded in a very similar manner.

Since the objective of this Lyapunov study is to obtain an inequality of the type \( \dot{V} \leq -n V \), the choice of \( F_1 \) and \( F_2 \) cross terms is justified by the need of making appear the missing negative square terms in the time derivative of the Lyapunov functional. Similarly, as in [7], the general idea comes from the fact that for \( i \in \{1, 2, 3, 4\} \)

\[
\int_0^L \frac{\partial z_i}{\partial t} \left( \int_0^\xi z_i d\xi \right) d\xi = \left[ z_i \int_0^\xi z_i d\xi \right]_0^L - \int_0^L z_i^2 d\xi. \tag{27}
\]

In the next proposition, we show that the functional \( V \) is positive definite and bounded proportionally to the energy if the constants \( n_0, n_1, n_2 \) are chosen appropriately.

**Proposition 3.4:** For all \( n_0, n_1, n_2 > 0 \), the Lyapunov function \( V \) in (24) is such that

i) \( V(z) \geq k_1 ||z||^2 \) for all \( z \in Z \), with \( k_1 = \min\{\left(\frac{n_0}{\rho} - \frac{n_1}{2}\rho\right), \left(\frac{n_0}{2} - \frac{n_2}{2}\right)\} \), \( \rho = \text{ess sup}_{[0, L]}(\mathcal{E}_1(\xi)) \), and \( \mathcal{E}_1 = \text{ess sup}_{[0, L]} \mathcal{E}_1(\xi) \).

ii) \( V(z) \leq n E \) for all \( z \in Z \), with \( \eta = \max\{n_0 + n_1\mathcal{E}_1, n_2\mathcal{E}_1\} \).

**Proof:** i) We apply Young’s inequality (with \( \alpha = 1 \) and \( f \) replaced by \( -f \)) to get

\[
V \geq \int_0^L \left\{ \left(\frac{n_0}{2} - \frac{n_1}{2}\rho\right) z_1^2 + \left(\frac{n_0}{2} - \frac{n_2}{2}\right) z_1^2 \right\} d\xi
\]

\[
+ \frac{n_0}{2} K z_3^2 + \frac{n_0}{2} E I z_4^2 - \frac{n_1}{2} \left( \int_0^\xi K z_3 d\xi \right)^2
\]

\[
- \frac{n_2}{2} \left( \int_0^\xi E I z_4 d\xi \right)^2 \tag{28}
\]

\[
\geq \int_0^L \left( \frac{n_0}{2} - \frac{n_1}{2}\rho\right) z_1^2 + \left(\frac{n_0}{2} - \frac{n_2}{2}\right) z_1^2 \mathcal{E}_1(\xi) d\xi.
\]
where Lemma 3.2 has been applied to obtain the second inequality. Defining \( \kappa_1 = \min\{a_1, a_2, a_3, a_4\} \), we obtain the inequality of item i).

ii) We apply Cauchy–Schwarz and Young’s Inequalities with \( \alpha = 1 \) to get

\[
V \leq \int_0^L \left\{ \left( \frac{n_0}{2} \right)^2 + \frac{n_0 \rho}{2} + \left( \frac{n_0}{2} + n_2 I_\rho \right) \frac{z_2^2}{I_\rho} \right\} ds
\]

\[
+ \frac{n_2}{2} K z_3^2 + \frac{n_2}{2} E I z_3^2 + \frac{n_2}{2} \left( \int_0^\xi K z_3 ds \right)^2 d\xi
\]

\[
+ \frac{n_2}{2} \left( \int_0^\xi E I z_3 ds \right)^2 d\xi
\]

\[
\leq \frac{1}{2} \int_0^L \left( \frac{b_1}{n_0 + n_1 \beta} \right)^2 + \frac{b_2}{n_0 + n_2 I_\rho} \right\} ds
\]

\[
\left( n_0 + n_1 k_1 \right) K z_3^2 + \left( n_0 + n_2 k_2 \right) E I z_3^2 d\xi
\]

(29)

where Lemma 3.2 has been applied to obtain the second inequality. We define the constant \( \eta = \max\{b_1, b_2, b_3, b_4\} \) to obtain the inequality of item ii).

In the following theorem, we present the main result of this article, i.e., we show the exponential stability of the Timoshenko beam model with viscous damping making use of the Lyapunov function (24).

**Theorem 3.5:** Consider the Timoshenko’s beam equation with space-varying parameters (10) and the Lyapunov functional \( V \) (24) with \( n_0, n_1, n_2 \) selected such to satisfy points 1)–4) after (38) and to render \( \kappa_1 > 0 \) of point i) of Proposition 3.4. The norm of the \( C_0 \)-semigroup generated by the operator (17)–(18) can be bounded by

\[
\|z(t, z_0)\| \leq \sqrt{V(z_0) \kappa_1} e^{-\frac{\gamma}{\rho} t}
\]

(30)

where \( \kappa_2 = \frac{\beta}{\eta} > 0 \) with \( \eta \) defined in point ii) of Proposition 3.4 and \( \beta = \min\{b_1, b_2, b_3, b_4\} > 0 \) with \( c_i \) defined in (38) and \( c_i = \text{ess inf}_{\xi \in [0, T]} c_i(\xi) \).

**Proof:** We start by computing the estimates of the Dini’s time derivative of the functionals \( F_1, F_2 \) composing the Lyapunov functional (24)

\[
\dot{F}_{1,+} = \int_0^L \left\{ \left( \frac{\partial}{\partial z_3} (K z_3) - \frac{\gamma}{\rho} z_1 \right) \left( \int_0^\xi K z_3 ds \right) \right.
\]

\[
+ z_1 \left( \int_0^\xi K \left( \frac{\partial}{\partial s} \frac{z_1}{\rho} \right) - z_2 \right) ds \left\} d\xi
\]

\[
+ \frac{\alpha_3}{2} \left( \frac{z_2}{\rho} \right)^2 + \frac{1}{2a_1} \left( \frac{2L}{\pi} \right)^2 \left( K z_3 \right)^2 d\xi
\]

\[
\left. + \frac{K z_3^2 + \frac{1}{2} \left( \int_0^\xi z_2^2 ds \right)^2 \right\} \left( \int_0^L \left( \frac{K z_3}{I_\rho} \right)^2 d\xi \right) \right\} \left( \int_0^\xi \frac{2L}{\pi} \right)^2 \frac{z_2^2}{\rho}
\]

\[
\left. + \frac{1}{2a_1} \left( \frac{2L}{\pi} \right)^2 \frac{z_2^2}{\rho} \right\} \left( \int_0^\xi \frac{2L}{\pi} \right)^2 \frac{z_2^2}{\rho}
\]

\[
\left. + \frac{1}{2a_1} \left( \frac{2L}{\pi} \right)^2 \frac{z_2^2}{\rho} \right\} \left( \int_0^\xi \frac{2L}{\pi} \right)^2 \frac{z_2^2}{\rho}
\]

\[
+ \frac{L}{2} (K z_3 (L, t))^2.
\]

With a very similar procedure as for \( F_1 \) we bound the \( F_2 \) time derivative with

\[
\dot{F}_{2,+} = \int_0^L \left( \frac{\partial}{\partial z_4} (E I z_4) - K z_3 - \frac{\delta}{I_\rho} z_2 \right) \left( \int_0^\xi E I z_4 ds \right) \right.
\]

\[
+ z_2 \left( \int_0^\xi E I \frac{\partial}{\partial s} \frac{1}{I_\rho} \right) \left( \int_0^\xi \frac{2L}{\pi} \right)^2 \frac{z_2^2}{\rho}
\]

\[
\leq \frac{L}{2} (E I (L)) (E I z_4) + \int_0^L \left( \frac{1}{2} (E I z_3)^2 - (E I z_4)^2 \right)
\]

\[
+ \frac{\alpha_2}{2} (K z_3)^2 + \frac{2L^2}{2a_3 \pi^4} \left( \int_0^\xi \right)^2 + \frac{\alpha_3}{2} \frac{\delta}{I_\rho} \right) \left( \int_0^\xi \frac{2L}{\pi} \right)^2 \frac{z_2^2}{\rho}
\]

\[
+ \frac{2L^2}{2a_3 \pi^4} \left( \int_0^\xi \right)^2 + \frac{\alpha_3}{2} \frac{\delta}{I_\rho} \right) \left( \int_0^\xi \frac{2L}{\pi} \right)^2 \frac{z_2^2}{\rho}
\]
\[ + \frac{1}{2} \left( \frac{2L}{\pi} \right)^2 \left( \frac{EI_d}{\rho} \right)^2 \] 
\[ - \int_0^L \left\{ \left( \alpha_3 + \frac{2EI}{2L} + \frac{1}{21\rho} \right) \left( \frac{2LEI_d}{\pi} \right)^2 \right\} \, d\xi \] 
\[ + \frac{\alpha_2 K_z^2}{K_z} - \left( \frac{EI}{2} - \frac{EI(2L)^2}{2\alpha_3\pi^2} - \frac{EI(2L)^2}{2\alpha_3\pi^2} \right) \cdot EI_z^2 \right\} \, d\xi + \frac{L}{2} \left( EI(L)z_z(L, t) \right)^2 \] 
(35)

where \( EI_d = \frac{dEI}{d\theta} \) and \( \alpha_1, \alpha_2, \alpha_3 > 0 \) are constants to be determined later. We replace (34), (35), and (19) in the Lyapunov function’s time derivative

\[ \dot{V}_+ = n_0 \dot{E}_+ + n_1 \dot{F}_{1,+} + n_2 \dot{F}_{2,+} \] 
(36)

and considering \( \partial_2 z = 0 \), \( \partial_2 z = -S(L)\rho_z \), we obtain

\[ \dot{V}_+ \leq - \int_0^L \left\{ \left( c_1 \frac{z_1^2}{\rho} + c_2 \frac{z_2^2}{P_1} + c_3 \alpha K_z^2 + c_4 EI z_4^2 \right) \right\} \, d\xi \] 
\[ - c_5 \left( \frac{z_1(L,t)}{\rho(L)} \right)^2 - c_6 \left( \frac{z_2(L,t)}{P_1(L)} \right)^2 \] 
(37)

with functions

\[ c_1 = \frac{n_0\gamma}{\rho^2} - n_1 \alpha_1^2 - n_1 K - n_1 \rho - \frac{1}{2} \left( \frac{2LK_d}{\pi} \right)^2 \] 
\[ c_2 = \frac{n_0\delta}{P_1} - \frac{n_1}{2L} - \frac{K}{2} - n_1 \rho - \frac{1}{2} \left( \frac{2L}{\pi} \right)^2 \] 
\[ c_3 = \frac{n_1 K}{2\alpha_1} - n_1 \alpha_1^2 - n_2 \alpha_2 K - \frac{L_n\gamma(L)^2}{2} \] 
\[ c_4 = \frac{n_0 EI}{2} - n_2 EI(2L)^2 - \frac{L_n\gamma(L)^2}{2} \] 
\[ c_5 = \frac{n_0}{\rho} - \frac{L_n\gamma(L)^2}{2} \] 
(38)

Then, the constants \( n_0, n_1, n_2 \) and \( \alpha_1, \alpha_2, \alpha_3 \) could be chosen as following:

1) Fix an arbitrary \( n_2 > 0 \).
2) Select \( \alpha_2, \alpha_3 \) sufficiently large to obtain \( c_4 > 0 \) \( \forall \xi \in [0, L] \).
3) Select \( \alpha_1 \) and \( n_1 \) sufficiently large such that \( c_4 > 0 \) \( \forall \xi \in [0, L] \).
4) The constant \( n_0 \) is selected sufficiently large such that \( c_1, c_2, c_5, c_6 > 0 \) \( \forall \xi \in [0, L] \) and \( \kappa_i \) of point \( i \) of Proposition 3.4 is strictly positive \( \kappa_i > 0 \).

Therefore, we have

\[ \dot{V}_+ \leq -\beta E \] 
(39)

with \( \beta \) defined in the Theorem’s statement. Using point \( ii \) of Proposition 3.4, we obtain

\[ \dot{V}_+ \leq -\kappa_2 V \] 
(40)

with \( \kappa_2 = \frac{\beta}{\rho^2} \). Hence, using Theorem 2.3, we can conclude that the origin is an exponentially stable equilibrium, and the trajectories of system (10) fulfil the estimation (30).

**Remark 1:** The boundary conditions at \( \xi = 0 \) and \( \xi = L \) can be interchanged without changing the result of Theorem 3.5. In fact, \( \xi \) could be replaced by \( L - \xi \) in (12) and, using the linearity and the derivative sign change, return to the same problem.

**Remark 2:** In case of constant parameters \( \rho, I, K, EI \), it is possible to prove that the Dini time derivative of the cross-term functions in (25) becomes

\[ \dot{F}_{1,+} \leq \int_0^L \left\{ \left( \frac{\alpha_1 \gamma^2}{\rho} + K + \rho \right) \frac{z_1^2}{\rho} - \frac{1}{21\rho} \left( \frac{2LK}{\pi} \right)^2 \right\} \, d\xi \] 
\[ - \left( \frac{K}{2} - \frac{K}{2\alpha_1} \right) \frac{K_z^2}{K_z} \right\} \, d\xi \] 
\[ + \frac{L}{2} \left( K z_z(L,t) \right)^2 \] 
(41)

\[ \dot{F}_{2,+} \leq \int_0^L \left\{ \left( \frac{\alpha_3 \delta^2}{2\rho} + 3 \frac{EI}{L} \right) \frac{z_2^2}{\rho} + \frac{\alpha_2 K z_4^2}{2} \right\} \, d\xi \] 
\[ - \left( \frac{EI}{2} - \frac{EI(2L)^2}{2\alpha_3\pi^2} - \frac{EI(2L)^2}{2\alpha_3\pi^2} \right) \frac{EI z_4^2}{L} \right\} \, d\xi \] 
\[ + \frac{L}{2} \left( EI z_4(L,t) \right)^2 \] 
(42)

Therefore, the Dini time derivative of the Lyapunov function takes the same form as in (37), but with constant coefficients

\[ c_1 = n_0 \gamma - \frac{n_1 \alpha_1^2}{2\rho} - n_1 K - n_1 \rho \] 
\[ c_2 = n_0 \delta \frac{1}{P_1} - \frac{n_1}{2L} - \frac{1}{21\rho} \frac{2K}{\pi} \] 
\[ c_3 = \frac{n_1 K}{2\alpha_1} - \frac{n_1 \alpha_1^2}{2\rho} - \frac{n_2 \alpha_2 K}{2} \] 
\[ c_4 = \frac{n_0 EI}{2} - \frac{n_2 EI(2L)^2}{2\alpha_3\pi^2} - \frac{n_2 EI(2L)^2}{2\alpha_3\pi^2} \] 
\[ c_5 = \frac{n_0}{\rho} - \frac{L_n\gamma(L)^2}{2} \] 
\[ c_6 = \frac{L_n\gamma(L)^2}{2} \] 
(43)

![Fig. 1. \( w(\xi,t) \) and \( w(\xi,t) \) evolution along time.](image-url)
exponential bound is conservative. This is because the proposed Lyapunov function depends on $\eta$.

Example 1: Assume that the Timoshenko’s beam equation in (8) have a length $L = 1$ and the parameters $\rho, I_g, K, E, I, \gamma, \delta$ have the following value:

$$\phi = 0.4 + 0.01 \sin(2\pi \xi + \phi(t))$$ (44)

with

$$\phi_0 = \frac{\pi}{4}, \phi_\nu = \frac{3\pi}{4}, \phi_K = \frac{\pi}{6}, \phi_E1 = \frac{2\pi}{3}, \phi_\gamma = 0, \phi_\delta = \frac{\pi}{2}.$$

Consider the Lyapunov function in (24) with constants $n_0 = 37$, $n_1 = 67$, $n_2 = 39$, and $\alpha_1 = 5$, $\alpha_2 = 1$, $\alpha_3 = 6$. Therefore, according to Theorem 3.5, we can compute the exponential bound (30) coefficients $k_1 = 4.77$ and $k_2 = \frac{\pi}{n_1} \frac{\beta_1}{64} = 0.0622$.

In order to show the exponential bound on the system’s state norm, we perform the numerical simulations using the MATLAB environment and the “ode23tb” time integration algorithm. To do that, a pH structure-preserving finite element spatial discretization as described in [19, Sec. 2.2] has been carried on (10) to obtain a finite dimensional linear time invariant (LTI) pH approximation. In this specific example, the system has been divided into 50 discretizing elements; therefore, the LTI system has 200 states. To perform the numerical simulations, we impose the initial conditions $z_1(\xi, 0) = 0$, $z_2(\xi, 0) = 0$, and $z_3 = \frac{1}{2}(1 - \cos(2\pi \xi))$, $z_4 = 1 - \cos(2\pi \xi)$. Fig. 1 shows the trajectory time evolution of the beam deformation $w(\xi, t)$ and its velocity $\dot{w}(\xi, t)$, while Fig. 2 shows the state’s norm evolution together with the computed exponential bound (30). We remark that the computed exponential bound is conservative. This is because the proposed Lyapunov parameters are not optimal with respect to the maximum decay rate.

IV. CONCLUSION

In this technical note, the exponential stability problem of Timoshenko’s beam equations with space-varying parameters and with viscous damping in both the vertical and rotational dynamics has been considered. After recalling some basic inequalities, Timoshenko’s equations have been rewritten in the port-Hamiltonian framework and the existence and uniqueness of solutions have been proven. The exponential bound of the state norm has been obtained using Lyapunov arguments. The defined Lyapunov function is composed of the internal energy and two crossing terms and has been proven to be finite in all the state space. Therefore, the time derivative of the Lyapunov function along the system trajectories has been computed, and the exponential stability has been proven. In an illustrative example, the exponential bound coefficients are computed for Timoshenko’s beam equations with space-varying parameters.

The future work will focus on the stabilization problem in case the viscously damped flexible beam is part of a larger mechanism. For this purpose, the Lyapunov function proposed in this technical note can be used, in composition with other terms, to prove exponential stability.

REFERENCES

[1] A. Mattioni, Y. Wu, and Y. L. Gorrec, “Infinite dimensional model of a double flexible-link manipulator: The port-Hamiltonian approach,” Appl. Math. Model., vol. 83, pp. 59–75, 2020.
[2] H. Ramírez, Y. L. Gorrec, A. Macchelli, and H. Zwart, “Exponential stabilization of boundary controlled port-Hamiltonian systems with dynamic feedback,” IEEE Trans. Autom. Control, vol. 59, no. 10, pp. 2849–2855, Oct. 2014.
[3] C. A. Raposo, J. Ferreira, M. L. Santos, and N. N. O. Castro, “Exponential stability for the Timoshenko system with two weak damping,” Appl. Math. Lett., vol. 18, pp. 535–541, 2004.
[4] F. Huang, “Characteristic conditions for exponential stability of linear dynamical systems in hilbert spaces,” Ann. Differ. Equ., vol. 1, pp. 43–55, 1985.
[5] D. Shi and D. Feng, “Exponential decay of Timoshenko beam with locally distributed feedback,” IMA J. Math. Control Inf., vol. 18, no. 3, pp. 395–403, 2001.
[6] J. U. Kim and Y. Renardy, “Boundary control of the Timoshenko beam,” SIAM J. Control Optim., vol. 25, no. 6, pp. 1417–1429, 1987.
[7] D. S. A. Júnior, M. L. Santos, and J. E. M. Rivera, “Stability to weakly dissipative Timoshenko systems,” Math. Methods Appl. Sci., vol. 36, no. 14, pp. 1965–1976, 2013.
[8] A. Haraux and E. Zuazua, “Decay estimates for some semilinear hyperbolic problems,” Arch. Rational Mech. Anal., vol. 100, pp. 191–206, 1988.
[9] E. Zuazua, “Exponential decay for the semilinear wave equation with locally distributed damping,” Commun. Partial Differ. Equ., vol. 15, no. 2, pp. 205–235, 1990.
[10] J. A. Villegas, H. Zwart, Y. Le Gorrec, B. Maschke, and A. J. V. D. Schaft, “Stability and stabilization of a class of boundary control systems,” in Proc. IEEE 44th Conf. Decis. Control, 2005, pp. 3850–3855.
[11] B. Augner, “Well-posedness and stability of infinite-dimensional linear port-Hamiltonian systems with nonlinear boundary feedback,” SIAM J. Control Optim., vol. 57, no. 3, 2019, Art. no. 1818.
[12] B. Augner, “Stabilisation of infinite-dimensional port-Hamiltonian systems via dissipative boundary feedback,” Ph.D. Dissertation, Univ. Wuppertal, Wuppertal, Germany, 2016.
[13] J. A. Villegas, “A port-hamiltonian approach to distributed parameter systems,” Ph.D. Dissertation, Univ. Twente, Enschede, The Netherlands, 2007.
[14] B. Jacob and H. Zwart, Linear Port-Hamiltonian Systems on Infinite-Dimensional Spaces, Number 223 in Operator Theory: Advances and Applications. Berlin, Germany: Springer, 2012.
[15] A. Macchelli and C. Melchiorri, “Modeling and control of the Timoshenko beam: The distributed port Hamiltonian approach,” SIAM J. Control Optim., vol. 43, no. 2, pp. 743–767, 2004.
[16] G. H. Hardy, J. E. Littlewood, and G. Polya, Inequalities, 2nd ed. New York, NY, USA: Cambridge Univ. Press, 1959.
[17] R. Curtain and H. Zwart, Introduction to Infinite-Dimensional Linear Systems Theory, a State Space Approach, 1st ed. Berlin, Germany: Springer, 2020.
[18] Y. L. Gorrec, H. Zwart, and B. Maschke, “Dirac structures and boundary control systems associated with skew-symmetric differential operators,” SIAM J. Control Optim., vol. 44, pp. 1864–1892, 2005.
[19] A. Mattioni, “Modelling and stability analysis of flexible robots: A distributed parameter port-Hamiltonian approach,” Ph.D. Dissertation, Université de Bourgogne Franche-Comté, Besançon, France, 2021.