Contraction in $L^1$ and large time behavior for a system arising in chemical reactions and molecular motors

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Abstract

We prove a contraction in $L^1$ property for the solutions of a nonlinear reaction–diffusion system whose special cases include intercellular transport as well as reversible chemical reactions. Assuming the existence of stationary solutions we show that the solutions stabilize as $t$ tends to infinity. Moreover, in the special case of linear reaction terms, we prove the existence and the uniqueness (up to a multiplicative constant) of the stationary solution.

Key words: weakly coupled system, molecular motor, transport, parabolic systems, contraction property.

AMS subject classification: 34D23, 35K45, 35K50, 35K55, 35K57, 92C37, 92C45.

1 Introduction

We start with two specific reaction-diffusion systems. The first one describes a reversible reaction and the other one a molecular motor. We first consider the reversible chemical reaction (see also Bothe [4], Bothe and Hilhorst [5], Desvillettes and Fellner [10] and Érdi and Tóth [11]). It involves a reaction-diffusion system of the form

\begin{align}
  u_t &= d_1 \Delta u - \alpha k (r_A(u) - r_B(v)) \quad \text{in } \Omega \times (0,T), \quad \Omega \subset \mathbb{R}^d, \\
  v_t &= d_2 \Delta v + \beta k (r_A(u) - r_B(v)) \quad \text{in } \Omega \times (0,T), \quad \Omega \subset \mathbb{R}^d, 
\end{align}

(1.1)
together with the homogeneous Neumann boundary conditions, where \( d_1, d_2, \alpha, \beta, k \) and \( T \) are positive constants and where \( \Omega \) is a bounded subset of \( \mathbb{R}^d \) with smooth boundary. Such systems describe, with a suitable choice of the functions \( r_A \) and \( r_B \), chemical reactions for two mobile species. For example, functions \( r_A(u) = u^k \), \( r_B(v) = v^m \) correspond to a reversible reaction \( kA \rightleftharpoons mB \). Reactions of the type \( q_1 A_1 + \ldots q_k A_k \rightleftharpoons q_1 B_1 + \ldots q_m B_m \) can also be described by similar systems with more complicated reactions terms.

Another model problem is a system in \( d = 1 \) space dimension and \( n \) unknown variables \( u_1, \ldots, u_n \), \( n > 1 \), for intercellular transport, namely

\[
\frac{\partial u_i}{\partial t} = \frac{\partial}{\partial x} \left( \sigma \frac{\partial u_i}{\partial x} + u_i \psi'_i \right) + \sum_{j=1}^{n} a_{ij} u_j \quad \text{in} \quad Q_T = [0, 1] \times (0, T)
\]

\[
\sigma \frac{\partial u_i}{\partial x} + u_i \psi'_i = 0 \quad \text{on} \quad \partial Q_T = \{0, 1\} \times (0, T),
\]

where

\[
a_{ii} \leq 0, \quad a_{ij} \geq 0 \quad \text{for all} \quad i \in \{1, \ldots, n\}, i \neq j,
\]

\[
\sum_{i=1}^{n} a_{ij} = 0 \quad \text{for all} \quad i, j \in \{1, \ldots, n\}. \tag{1.2}
\]

It models transport via motor proteins in the eukaryotic cell where chemical energy is transduced into directed motion. A derivation of the system from a mass transport viewpoint is given in [7]. For an analysis of the steady state solutions and for further references we refer to [6], [12], [13], and [20].

In this paper we study the corresponding system in higher space dimension, namely

\[
\frac{\partial u_i}{\partial t} = \text{div} \left( \sigma_i \nabla u_i + u_i \nabla \psi_i \right)
+ \alpha_i \left( \sum_{j=1}^{n} \lambda_{ij} r_j(u_j(x,t), x) \right) \quad \text{in} \quad Q_T, \tag{1.3a}
\]

where \( i \in \{1, \ldots, n\} \), and \( u_i(x,t) : Q_T \rightarrow \mathbb{R}^+ \), with \( Q_T = \Omega \times (0, T) \), \( \Omega \) an open bounded subset of \( \mathbb{R}^d \) with smooth boundary, and \( T \) some positive constant. We supplement this system with the Robin (no-flux) boundary conditions

\[
\sigma_i \frac{\partial u_i}{\partial \nu} + u_i \frac{\partial \psi_i}{\partial \nu} = 0, \quad i \in \{1, \ldots, n\}, \quad \text{on} \quad \partial \Omega \times (0, T), \tag{1.3b}
\]
where $\nu$ is the outward normal vector to $\partial \Omega$, and the initial conditions

$$u_1(x,0) = u_{0,1}(x), \ldots, u_n(x,0) = u_{0,n}(x), \quad x \in \Omega. \quad (1.3c)$$

We assume that the following hypotheses hold

1. The constants $\sigma_i$ and $\alpha_i \in \mathbb{R}$, where $i \in \{1, \ldots, n\}$, are strictly positive;
2. For $i, j \in \{1, \ldots, n\}$, $\lambda_{ii} \leq 0$, $\lambda_{ij} \geq 0$ if $i \neq j$, $\sum_{k=1}^{n} \lambda_{kj} = 0$;
3. for all $i \in \{1, \ldots, n\}$, the smooth functions $r_i$ are nondecreasing with respect to the first variable; $r_i(0,x) = 0$ and we assume that the functions $\psi_i$ are smooth as well;
4. $u_i(.,0) = u_{0i} \in C(\bar{\Omega})$, $u_{0i} \geq 0$.

In the linear case of the molecular motors, it amounts to choosing

$$r_i(s,x) = s, \quad \lambda_{ij} = a_{ij} \text{ and } \alpha_i = 1 \text{ for all } i, j \in \{1, \ldots, n\}. \quad (1.4)$$

We denote by Problem (P) the system (1.3a) together with the boundary and initial conditions (1.3b), (1.3c), and admit without proof that Problem (P) possesses a unique smooth and bounded solution on each time interval $(0,T]$. An essential idea for proving the existence of a solution would be to apply the Comparison principle Theorem 2.2 below to deduce that any solution of Problem (P) has to be nonnegative and bounded from above by a stationary solution.

Finally, we note that because of the boundary conditions (1.3b) the quantity

$$\sum_{i=1}^{n} \frac{1}{\alpha_i} \int_{\Omega} u_i(x,t) \, dx \quad (1.5)$$

is conserved in time.

The organization of this paper is as follows. In Section 2 we prove a comparison principle for Problem (P). The main idea, which permits to show that Problem (P) is cooperative, is a change of functions which transforms the Robin boundary conditions into the homogeneous Neumann boundary conditions. In Section 3 we establish a contraction in $L^1$ property for the corresponding semigroup solution. Let us point out the similarity with an old result due to Crandall and Tartar [8] where they proved in a scalar case that in the presence of a conservation of the integral property such as (1.5), a comparison principle such as Theorem 2.2 is equivalent to a contraction in $L^1$ property such as the inequality (3.4) below. As far as we know such an abstract result is not known in the case of systems.

Section 4 deals with the large time behavior of the solutions. Supposing the existence of a stationary solution, we construct a continuum of
stationary solutions and prove that the solutions stabilize as \( t \) tends to infinity. Let us mention a result by Perthame [19] who proved the stabilization in the case of the two component one-dimensional molecular motor problem. Finally in Section 5, show the existence and uniqueness (up to a multiplicative constant) of the stationary solution of the molecular motor problem.

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2 Comparison principle

First, we remark that the system of equations (1.3a) is cooperative. However, since nothing is known about the sign of the coefficients \( \frac{\partial \psi_i}{\partial \nu} \) in the Robin boundary conditions (1.3b), we cannot decide whether the Problem (P) is cooperative. This leads us to perform a change of variables which transforms the Robin boundary conditions into the homogeneous Neumann boundary conditions.

2.1 The change of unknown functions

Performing the change of variables

\[
\begin{align*}
\bar{w}_i(x,t) &= u_i(x,t) e^{\psi_i(x)/\sigma_i}, \quad i \in \{1, \ldots, n\},
\end{align*}
\]

we deduce from (1.3) that \( \bar{w} := (w_1, \ldots, w_n) \) satisfies the parabolic problem

\[
\begin{align*}
\frac{\partial \bar{w}_i}{\partial t} &= \sigma_i e^{\psi_i(x)/\sigma_i} \operatorname{div} \left( e^{-\psi_i(x)/\sigma_i} \nabla \bar{w}_i \right) \\
&+ \alpha_i e^{\psi_i(x)/\sigma_i} \left( \sum_{j=1}^{n} \lambda_{ij} r_j \left( w_j(x,t) e^{-\psi_j(x)/\sigma_j}, x \right) \right) \quad \text{in } Q_T,
\end{align*}
\]

and the initial conditions

\[
\begin{align*}
\bar{w}_i(x,0) &= u_{0,i}(x) e^{\psi_i(x)/\sigma_i}, \quad i \in \{1, \ldots, n\}, \quad x \in \Omega.
\end{align*}
\]
In the following, we denote by Problem $P_N$ — the problem (2.2), (2.3), (2.4). To begin with we define the operators

\[ L_i (w_i) = \frac{\partial w_i}{\partial t} - \sigma_i e^\psi_i / \sigma_i \text{div} \left( e^{-\psi_i / \sigma_i} \nabla w_i \right) - \alpha_i e^\psi_i / \sigma_i \left( \sum_{j=1}^n \lambda_{ij} r_j (w_j (x, t)) e^{-\psi_j / \sigma_j (x)} \right) \] in $Q_T$. (2.5)

We say that $(w_1, \ldots, w_n)$ is a subsolution of Problem $P_N$ if

\[ L_i (w_i) \leq 0 \quad \text{in} \quad Q_T, \]
\[ \frac{\partial w_i}{\partial \nu} \leq 0 \quad \text{on} \quad \partial \Omega \times (0, T), \]
\[ w_i (x, 0) \leq w_i (x, 0), \quad x \in \Omega \] for all $i \in \{1, \ldots, n\}$. We define similarly a supersolution $(\overline{w}_1, \ldots, \overline{w}_n)$ of Problem $P_N$ by the inequalities

\[ L_i (\overline{w}_i) \geq 0 \quad \text{in} \quad Q_T, \]
\[ \frac{\partial \overline{w}_i}{\partial \nu} \geq 0 \quad \text{on} \quad \partial \Omega \times (0, T), \]
\[ \overline{w}_i (x, 0) \geq w_i (x, 0), \quad x \in \Omega. \] (2.6)

The following comparison theorem holds ([2], [21]).

**Theorem 2.1.** Let $(w_1, \ldots, w_n)$ and $(\overline{w}_1, \ldots, \overline{w}_n)$, be a sub- and a super-solution, respectively, for the operators $L_j$ defined by (2.5) with $j \in \{1, \ldots, n\}$, which means that (2.6) and (2.7) hold for $i \in \{1, \ldots, n\}$. Then $w_i \leq \overline{w}_i$ in $Q_T$. Moreover, for all $i \in \{1, \ldots, n\}$ such that $w_i \equiv \overline{w}_i$ and $w_i \neq \overline{w}_i$ on $\{t = 0\} \times \Omega$ then $w_i < \overline{w}_i$ in $Q_T$. ■

This comparison theorem immediately translates into a comparison theorem for solutions of the original Problem (P). For all $i \in \{1, \ldots, n\}$, we define the operators

\[ L_i (u_i) = (u_i)_t - \text{div} (\sigma_i \nabla u_i + u_i \nabla \psi_i) \]
\[ - \alpha_i \left( \sum_{j=1}^n \lambda_{ij} r_j (u_j (x)) \right) \] in $Q_T$. (2.8)

The following result holds.

**Theorem 2.2.** Let $(u_1, \ldots, u_n)$ and $(\overline{u}_1, \ldots, \overline{u}_n)$, be a sub- and a super-solution, respectively, for the operators $L_j$, defined by (2.8) with $j \in \{1, \ldots, n\}$. Then $u_i \leq \overline{u}_i$ in $Q_T$. Moreover, for all $i \in \{1, \ldots, n\}$ such that $u_i \equiv \overline{u}_i$ and $u_i \neq \overline{u}_i$ on $\{t = 0\} \times \Omega$ then $u_i < \overline{u}_i$ in $Q_T$. ■
Next we state two immediate corollaries of Theorem 2.2.

**Corollary 2.3.** (uniqueness) If \((u_1^1, \ldots, u_n^1)\) and \((u_1^2, \ldots, u_n^2)\) are solutions of Problem \((P)\) with the same initial condition \((u_{0,1}, \ldots, u_{0,n}) \in (C(\Omega))^n\), then for all \(i \in \{1, \ldots, n\}\), \(u_i^1 = u_i^2\). ■

**Corollary 2.4.** (positivity) If \((u_1, \ldots, u_n)\) is the solution of Problem \((P)\) with the nonnegative initial condition \((u_{0,1}, \ldots, u_{0,n}) \in (C(\Omega))^n\), then for all \(i \in \{1, \ldots, n\}\), such that \(u_{0,i} \geq 0\) and \(u_{0,i} \neq 0\), \(u_i > 0\) in \(\Omega\). ■

### 3 Contraction property

The purpose of this section is to show a contraction in \((L^1(\Omega))^n\) property for the solutions of Problem \((P)\) with the initial conditions belonging to \((L^\infty(\Omega))^n\). The main steps of the proof rely upon arguments due to [3] and [18].

We first introduce some notation. We suppose that the functions \((u_1^1, \ldots, u_n^1)\) and \((u_1^2, \ldots, u_n^2)\) are the solutions of Problem \((P)\) with the initial conditions \((u_{0,1}, \ldots, u_{0,n})\) and \((u_{0,1}^2, \ldots, u_{0,n}^2)\), respectively. Define

\[
(U_1, \ldots, U_n) := (u_1^1 - u_1^2, \ldots, u_n^1 - u_n^2). \tag{3.1}
\]

Then

\[
(U_i)_t = \text{div}(\sigma_i \nabla U_i + U_i \nabla \psi_i) \\
+ \alpha_i \sum_{j=1}^n \lambda_{ij} (r_j(u_j^1(x,t), x) - r_j(u_j^2(x,t), x)) \quad \text{in} \quad Q_T, \tag{3.2}
\]

\[
\sigma_i \frac{\partial U_i}{\partial \nu} + U_i \frac{\partial \psi_i}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega \times (0,T),
\]

\[
U_i(x,0) = U_{0,i}(x) \quad \text{for} \quad x \in \Omega,
\]

together with

\[
U_{0,i} = u_{0,i}^1 - u_{0,i}^2, \tag{3.3}
\]

for each \(i \in \{1, \ldots, n\}\).

Next we prove the following contraction in \(L^1\) property.

**Theorem 3.1.** For all \(t > 0\),

\[
\frac{1}{\alpha_1} \|U_1(\cdot,t)\|_{L^1(\Omega)} + \ldots + \frac{1}{\alpha_n} \|U_n(\cdot,t)\|_{L^1(\Omega)} \leq \frac{1}{\alpha_1} \|U_{0,1}(\cdot)\|_{L^1(\Omega)} + \ldots + \frac{1}{\alpha_n} \|U_{0,n}(\cdot)\|_{L^1(\Omega)}, \tag{3.4}
\]


where $U_i$ and $U_{0,i}$, $i \in \{1, \ldots, n\}$, are defined by (3.1) and (3.3), respectively.

**Proof** Dividing each partial differential equation of (3.2) by $\alpha_i$ and summing them up, we obtain

$$\frac{d}{dt} \left( \sum_{i=1}^{n} \frac{1}{\alpha_i} U_i \right) = \sum_{i=1}^{n} \frac{1}{\alpha_i} \text{div} (\sigma_i \nabla U_i + U_i \nabla \psi_i)$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{ij} \left( r_j(u_j^1(x,t), x) - r_j(u_j^2(x,t), x) \right)$$

$$= \sum_{i=1}^{n} \frac{1}{\alpha_i} \text{div} (\sigma_i \nabla U_i + U_i \nabla \psi_i)$$

$$+ \sum_{i=1}^{n} \left\{ \left( r_j(u_j^1(x,t), x) - r_j(u_j^2(x,t), x) \right) \sum_{i=1}^{n} \lambda_{ij} \right\}$$

$$= \sum_{i=1}^{n} \frac{1}{\alpha_i} \text{div} (\sigma_i \nabla U_i + U_i \nabla \psi_i),$$

where we have used Hypothesis 2.

This, together with the boundary conditions (1.3b), implies the conservation in time of the quantity

$$\frac{d}{dt} \sum_{i=1}^{n} \frac{1}{\alpha_i} \int_{\Omega} U_i(x,t) \, dx = 0. \quad (3.5)$$

Let us look closer at the nonlinear term in (3.2). We can write, for fixed index $i$

$$\sum_{j=1}^{n} \lambda_{ij} \left( r_j(u_j^1(x,t), x) - r_j(u_j^2(x,t), x) \right)$$

$$= \sum_{j=1}^{n} \lambda_{ij} \int_{0}^{1} \frac{\partial}{\partial u} r_j(\theta u_j^1 + (1-\theta)u_j^2, x) \, d\theta = \sum_{j=1}^{n} A_{ij} U_j.$$

Freezing the functions $u^k_i$ for $i \in \{1, \ldots, n\}$, $k \in \{1, 2\}$, we deduce that the functions $U_1, \ldots, U_n$ satisfy a system of the form

$$(U_i)_t = \text{div} \left( \sigma_i \nabla U_i + U_i \nabla \psi_i \right) + \sum_{j=1}^{n} A_{ij} U_j \quad \text{in} \quad Q_T, \quad (3.6)$$

with the boundary and initial conditions

$$\sigma_i \frac{\partial U_i}{\partial \nu} + U_i \frac{\partial \psi_i}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega \times (0,T),$$

$$U_i(x,0) = U_{0,i}(x), \quad x \in \Omega. \quad (3.7)$$
for \( i \in \{1, \ldots, n\} \), where \( A_{ij} \) are functions of space and time.

In order to make the notation more concise, we write

\[
\vec{U}_0 = (U_{0,1}, \ldots, U_{0,n}),
\]

\[
\vec{U} = (U_1, \ldots, U_n),
\]

\[
\vec{U}_0^\pm = (U_{0,1}^\pm, \ldots, U_{0,n}^\pm),
\]

\[
\vec{U}^\pm = (U_1^\pm, \ldots, U_n^\pm),
\]

where \( s^+ = \max\{s, 0\} \), \( s^- = \max\{-s, 0\} \). By (3.6), (3.7) and Corollary 2.3 we can write \( \vec{U} \) in the form

\[
(\vec{U})(x,t) = S(t)\vec{U}_0(x) = (S_1(t)\vec{U}_0, \ldots, S_n(t)\vec{U}_0)(x)
\]

with some operator \( S(t) \). We set

\[
(W_1, \ldots, W_n) = -(U_1 e^{\psi_1(x)/\sigma_1}, \ldots, U_n e^{\psi_n(x)/\sigma_n}),
\]

and \( \tilde{A}_{ij} = A_{ij} e^{\psi_i(x)/\sigma_i} e^{-\psi_j(x)/\sigma_j} \). Then, the system of equations (3.6) can be expressed in the form

\[
(W_i)_t = \sigma_i e^{\psi_i(x)/\sigma_i} \text{div}\left(e^{-\psi_i(x)/\sigma_i} \nabla W_i\right) + \sum_{j=1}^n \tilde{A}_{ij} W_j \leq 0 \quad \text{in } Q_T, \quad (3.8)
\]

with the boundary and initial conditions

\[
\frac{\partial W_i}{\partial \nu} = 0 \quad \text{on } \partial \Omega \times (0, T), \quad (3.9)
\]

\[
W_i(x,0) = -U_{0,i} e^{\psi_i(x)/\sigma_i}, \quad x \in \Omega, \quad (3.10)
\]

for \( i \in \{1, \ldots, n\} \).

Next we show that the solutions \( W_i \) of the problem (3.8) – (3.10) with nonpositive initial conditions are nonpositive in \( \Omega \) for all \( t \in (0, T) \). To that purpose we consider the auxiliary problem

\[
(W_i)_t - \vartheta_i(x)\text{div}\left(\zeta_i(x)\nabla W_i\right) - \sum_{j=1}^n \gamma_{ij} W_j \leq 0 \quad \text{in } Q_T, \quad (3.11)
\]

\[
\frac{\partial W_i}{\partial \nu} \leq 0 \quad \text{on } \partial \Omega \times (0, T), \quad (3.12)
\]

\[
W_i(x,0) = W_{0,i}(x) \leq 0 \quad x \in \Omega, \quad (3.13)
\]

for \( i \in \{1, \ldots, n\} \). We assume that \( \vartheta_i(x) \) and \( \zeta_i(x) \) are nonnegative in \( \Omega \) and that the coefficients \( \gamma_{ij} \) satisfy the same assumptions as the coefficients \( \lambda_{ij} \) in Problem (P). The following result holds.
Lemma 3.2. Let \((W_1, \ldots, W_n)\) be a smooth and bounded solution of the problem (3.11) – (3.13) with nonpositive initial conditions \(W_{0,i}\) on a time interval \([0,T]\). Then \(W_i(x,t) \leq 0\) in \(\Omega \times (0,T)\). Moreover, for each \(i \in \{1, \ldots, n\}\) such that \(W_{0,i} \leq 0\) and \(W_{0,i} \neq 0\), \(W_i < 0\) in \(\Omega \times (0,T)\).

Proof. The result of Lemma 3.2 follows from the fact that the system (3.11), (3.12), (3.13), with the inequalities \(\leq\) replaced by the equalities \(=\), is a cooperative system. However, for the sake of completeness, we present a proof below. We first remark that, in view of [21, Remark (i), p. 191], one can always satisfy the condition

\[
\sum_{j=1}^{n} \gamma_{ij} \leq 0 \text{ for all } i \in \{1, \ldots, n\},
\tag{3.14}
\]

for the matrix of coefficients \(\left(\gamma_{ij}\right)_{i,j=1}^{n}\) by performing the change of variables \(W_i = W_i e^{-ct}\) for all \(i \in \{1, \ldots, n\}\) and \(c > 0\) large enough. Thanks to the regularity of each \(W_i\), we can apply Theorem 15, p. 191 from [21] to conclude that \(W_i - M \leq 0\) in \(\Omega \times (0,T)\) for some \(M > 0\) and all \(i \in \{1, \ldots, n\}\). In fact, we can deduce that \(W_i - M < 0\) in \(\Omega \times (0,T)\).

Indeed, if for some \(k \in \{1, \ldots, n\}\), \(W_k = M\) in an interior point \((\tilde{x}, \tilde{t}) \in \Omega \times (0,T)\), then Theorem 15, p. 191 in [21] implies that \(W_k \equiv M\) for all \(0 < t < \tilde{t}\), which is impossible since \(W_k(x,0) \leq 0\). If the maximum of \(W_k\) is attained at a boundary point \(P \in \partial \Omega \times (0,T)\) then either there exists an open ball \(K \subset \Omega \times (0,T)\) such that \(P \in \partial K\) and \(W_k - M < 0\) in \(K\), and the last part of Theorem 15, p. 191 in [21] contradicts the boundary inequality (3.12), or for all open balls \(K \subset \Omega \times (0,T)\) such that \(P \in \partial K\) there exists a point \((\hat{x}, \hat{t}) \in K\) such that \(W_i(\hat{x}, \hat{t}) = M\), and we proceed as in the case before.

Hence, there exists \(\hat{M} > 0\), such that \(W_i \leq \hat{M} < M\) in \(\Omega \times [0,T]\) for all \(i \in \{1, \ldots, n\}\). Then we can repeat the reasoning for all \(M > 0\) until \(M = 0\). Indeed, if this would not be the case, we find the least real number \(\overline{M} > 0\), with \(W_i \leq \overline{M} \leq \hat{M}\) in \(\Omega \times [0,T]\), which leads again to the existence of a real number \(0 \leq \hat{M} < \overline{M}\) with the same property. This contradicts the fact that \(\overline{M}\) was defined as the least such real number. ■

Since the functions \(u^1_i, u^2_i\) are bounded on \(\overline{\Omega} \times [0,T]\), it follows that the functions \(W_i\) are bounded on \(\overline{\Omega} \times [0,T]\) for all \(i \in \{1, \ldots, n\}\). Then we are in a position to apply Lemma 3.2 with \(\vartheta_i(x) = e^{\psi_i / u_i}, \zeta_i(x) = \sigma_i e^{-\psi_i / u_i}\) and \(\gamma_{ij} = A_{ij}\) for \(i, j \in \{1, \ldots, n\}\). We deduce that the solutions \(W_i\) of the problem (3.8) – (3.10) with nonpositive initial conditions are nonpositive in \(\overline{\Omega}\) for all \(t \in (0,T)\).
Next we remark that the above reasoning can be applied either with $\vec{U}_0$ replaced by $U_0^+$ or with $\vec{U}_0$ replaced by $U_0^-$. This permits to show that $S_i(t)\vec{U}_0^+, S_i(t)\vec{U}_0^- \geq 0$ and that

$$S_i(t)\vec{U}_0^\pm > 0 \text{ if } \vec{U}_0^\pm \neq 0. \quad (3.15)$$

We easily compute

$$\sum_{i=1}^{n} \frac{1}{\alpha_i} \|U_i(\cdot, t)\|_{L^1(\Omega)} - \sum_{i=1}^{n} \frac{1}{\alpha_i} \|U_{0,i}(\cdot)\|_{L^1(\Omega)}$$

$$= \sum_{i=1}^{n} \frac{1}{\alpha_i} \|S_i(t)\vec{U}_0^+ - S_i(t)\vec{U}_0^-\|_{L^1(\Omega)} - \sum_{i=1}^{n} \frac{1}{\alpha_i} \|U_{0,i}(\cdot)\|_{L^1(\Omega)}$$

$$= \sum_{i=1}^{n} \int_{\Omega} \frac{1}{\alpha_i} \left\{ \max \{S_i(t)\vec{U}_0^+, S_i(t)\vec{U}_0^-\} \right\} \, dx$$

$$- \frac{1}{\alpha_i} \min \{S_i(t)\vec{U}_0^+, S_i(t)\vec{U}_0^-\} \right\} \, dx - \sum_{i=1}^{n} \frac{1}{\alpha_i} \int_{\Omega} \{U_{i,0}^+, U_{i,0}^-\} \, dx$$

$$= \sum_{i=1}^{n} \int_{\Omega} \frac{1}{\alpha_i} \left( S_i(t)\vec{U}_0^+ + S_i(t)\vec{U}_0^- \right) \, dx - \sum_{i=1}^{n} \frac{1}{\alpha_i} \int_{\Omega} \{U_{i,0}^+, U_{i,0}^-\} \, dx$$

$$- 2 \sum_{i=1}^{n} \int_{\Omega} \frac{1}{\alpha_i} \min \{S_i(t)\vec{U}_0^+, S_i(t)\vec{U}_0^-\} \, dx$$

$$= - 2 \sum_{i=1}^{n} \int_{\Omega} \frac{1}{\alpha_i} \min \{S_i(t)\vec{U}_0^+, S_i(t)\vec{U}_0^-\} \, dx \leq 0,$$

(3.17)

which completes the proof of (3.14). □

**Corollary 3.3.** Let $(u_{0,1}^1, \ldots, u_{0,n}^1), (u_{0,1}^2, \ldots, u_{0,n}^2) \in (C(\overline{\Omega}))^n$ be as in Theorem 3.1. Moreover, let us assume that for at least one index $k \in \{1, \ldots, n\}$ the difference $u_{0,k}^1 - u_{0,k}^2$ changes the sign. Then, the inequality (3.4) is strict for all $t > 0$, so that solution satisfies a strict contraction property.
4 Large time behavior of solutions

In this section we assume the existence and uniqueness of a positive solution \( \vec{v} = (v_1, \ldots, v_n) \in (C(\overline{\Omega}) \cap C^2(\Omega))^n \) of the elliptic problem

\[
\text{div}(\sigma_i \nabla v_i + v_i \nabla \psi_i) + \alpha_i \left( \sum_{j=1}^{n} \lambda_{ij} r_j(v_j(x), x) \right) = 0 \quad \text{in} \quad \Omega, \quad (4.1)
\]

\[
s_i \frac{\partial v_i}{\partial \nu} + v_i \frac{\partial \psi_i}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega, \quad (4.2)
\]

\[
\sum_{i=1}^{n} \frac{1}{\alpha_i} \int_{\Omega} v_i(x) \, dx = 1, \quad (4.3)
\]

for \( i \in \{1, \ldots, n\} \).

**Definition 4.1.** We say that a vector function \( \vec{v} = (v_1, \ldots, v_n) \in (C(\overline{\Omega}))^n \) is nonnegative (resp. positive) if \( v_i(x) \geq 0 \) (resp. \( v_i(x) > 0 \)) for all \( x \in \overline{\Omega} \) and all \( i \in \{1, \ldots, n\} \).

Next we introduce the semigroup notation for the unique solution of Problem (P), namely

\[
\vec{u}(t) = T(t) \vec{u}_0 = \left( T_1(t) \vec{u}_0, \ldots, T_n(t) \vec{u}_0 \right),
\]

with the initial data \( \vec{u}_0 \in (C(\overline{\Omega}))^n \). The method of the proof is based upon an idea of Osher and Ralston [18]. It mainly exploits the contraction properties for the nonlinear semigroup \( T(t) \) given by Theorem 3.1 and Corollary 3.3. A similar reasoning was developed in other contexts by Bertsch and Hilhorst [3], Hilhorst and Hulshof [14] and Hilhorst and Peletier [15].

We suppose there exists a set \( \mathcal{H} \subset (C(\overline{\Omega}) \cap C^2(\Omega))^n \) of positive stationary solutions with the following property which we denote by \( \mathcal{I} \):

For each \( \vec{f} = (f_1, \ldots, f_n) \in (C(\overline{\Omega}) \cap C^2(\Omega))^n \) either \( \vec{f} \in \mathcal{H} \) or there exists \( (\xi_1, \ldots, \xi_n) \in \mathcal{H} \), such that \( f_i - \xi_i \) changes the sign for at least one index \( i \in \{1, \ldots, n\} \).

One can prove that a set \( \mathcal{H} \) satisfying Property \( \mathcal{I} \) exists in at least two cases:

i) In the case of the system (1.1) where the Robin boundary conditions reduce to the homogeneous Neumann boundary conditions,
the set $\mathcal{H}$ is given by
\[
\mathcal{H} = \left\{ (a, b) : a > 0, \ b = r_B^{-1}(r_A(a)) \right\}
\]
and \( \frac{a}{\alpha} + \frac{b}{\beta} = \int_{\Omega} \left( \frac{u}{\alpha} + \frac{v}{\beta} \right) dx \).

For more details we refer to [5].

ii) In the case of the molecular motor with a linear \( n \)-component system the set $\mathcal{H}$ is given by
\[
\mathcal{H} = \left\{ c\vec{v} : c \in \mathbb{R}^+ \right\},
\]
where \( \vec{v} \) is a unique solution of the elliptic problem (4.1) – (4.3).

**Proposition 4.2.** The continuum $\mathcal{H}$ is such that for each 
\[
\vec{f} = (f_1, \ldots, f_n) \in (C(\overline{\Omega}) \cap C^2(\Omega))^n
\]
either \( \vec{f} \in \mathcal{H} \), or there exists \( (\xi_1, \ldots, \xi_n) \in \mathcal{H} \) such that \( f_i - \xi_i \) changes the sign for at least one index \( i \in \{1, \ldots, n\} \).

**Proof**

i) In the case of system (1.1) the proof is rather obvious since the continuum $\mathcal{H}$ is composed of constant pairs.

ii) In the case of the molecular motor, let us assume that \( \vec{f} \notin \mathcal{H} \). Then there does not exist any positive constant \( c \) such that \( c \vec{v} = \vec{f} \). In particular, there exists an index \( i \in \{1, \ldots, n\} \) such that \( v_i \) is not proportional to \( f_i \), or in other words \( cv_i \neq f_i \) for all \( c > 0 \). Without loss of generality we can assume that the first coordinate has this property. Let \( x_0 \in \Omega \) be arbitrary. Since \( v_1 \) is strictly positive in \( \overline{\Omega} \), we can define
\[
c_0 = \frac{f_1(x_0)}{v_1(x_0)},
\]
so that
\[
(f_1 - c_0 v_1)(x_0) = 0.
\]
Let \( Z = \{ x \in \overline{\Omega} : (f_1 - c_0 v_1)(x) = 0 \} \). From the continuity of \( f_1 \) and \( v_1 \), \( Z \) is closed as a subset of \( \Omega \). If there exist \( x_1, x_2 \in \mathbb{R} \), such that \( (f_1 - c_0 v_1)(x_1) \) and \( (f_1 - c_0 v_1)(x_2) \) are of different signs, then the proof is complete. Now suppose that \( (f_1 - c_0 v_1)(x) \) is positive for all \( x \in \mathbb{R} \). In particular
\[
(f_1 - c_0 v_1)(\hat{x}) = d > 0
\]
for some fixed $\tilde{x} \in \mathcal{Z}^c$. Then choosing $\varepsilon = \frac{d}{2v_1(\tilde{x})}$ we see that

$$(f_1 - (c_0 + \varepsilon)v_1)(\tilde{x}) = \frac{d}{2} > 0.$$ 

However

$$(f_1 - (c_0 + \varepsilon)v_1)(x_0) < 0.$$ 

We proceed similarly when $(f_1 - c_0v_1)(x)$ is negative for all $x \in \mathcal{Z}^c$. ■

In the sequel we suppose that the initial data $\vec{u}_0 = (u_0,1,\ldots,u_0,n)$ from $(C(\overline{\Omega}))^n$ also satisfy the following property:

There exists $\vec{h} \in \mathcal{H}$ such that $0 \leq \vec{u}_0 \leq \vec{h}$ in $\overline{\Omega}$, (4.4)

and remark that this property is satisfied in both the cases (i) and (ii).

**Proposition 4.3.** Let $\vec{u}_0 = (u_{0,1},\ldots,u_{0,n}) \in (C(\overline{\Omega}))^n$ satisfy the property \[4.4\]. Then the solution $(u_1,\ldots,u_n)$ of Problem (P) is such that $0 \leq \vec{u}(t) \leq \vec{h}$ for all $t > 0$.

**Proof** We remark that $\vec{0}$ is a subsolution of Problem (P) and that $\vec{h}$ is a supersolution, and apply Theorem 2.2. ■

Next we prove the main result of this section. To that purpose we first define the norm $\| \cdot \|_1$ by

$$\| \vec{f} \|_1 = \sum_{i=1}^{n} \frac{1}{\alpha_i} \| f_i \|_{L^1(\Omega)}.$$ 

Note that this norm is equivalent to the usual product norm in the space $(L^1(\Omega))^n$.

**Theorem 4.4.** For all nonnegative $\vec{u}_0 = (u_{0,1},\ldots,u_{0,n}) \in (C(\overline{\Omega}))^n$ there exists $\vec{f} = (f_1,\ldots,f_n) \in \mathcal{H}$, such that

$$\lim_{t \to \infty} \| T(t) \vec{u} - \vec{f} \|_1 = 0.$$ 

**Proof** The proof consists of several steps. To begin with we define the $\omega$-limit set

$$\omega(\vec{u}_0) = \left\{ \vec{g} \in (L^1(\Omega))^n : \text{there exists a sequence } t_k \to \infty \right.$$ 

$$\text{as } k \to \infty, \text{ such that } \lim_{k \to \infty} \| T(t_k) \vec{u}_0 - \vec{g} \|_1 = 0 \right\}, \quad (4.5)$$

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The organization of the proof is as follows. First we show that \( \omega(\vec{u}_0) \) is not empty. In the second step we define the Lyapunov functional
\[
\mathcal{V}(\vec{\xi}) = \|\vec{\xi} - \vec{w}\|_1,
\]
where \( \vec{w} \) is a stationary solution and check that it is constant on \( \omega(\vec{u}_0) \). We then deduce that \( \omega(\vec{u}_0) \subset \mathcal{H} \), and finally prove that \( \omega(\vec{u}_0) \) consists of exactly one function.

**Step 1.** \( \omega(\vec{u}_0) \) is not empty.
Let \( \varepsilon > 0 \) be arbitrary. Suppose that \( \Omega' \subset\subset \Omega \) satisfy
\[
|\Omega \setminus \Omega'| \leq \frac{\varepsilon}{2K}.
\]
and set
\[
K = \sum_{i=1}^{n} \frac{2}{\alpha_i} \|h_i\|_{C(\Omega)},
\]
where \( \vec{h} \) has been introduced in (4.4). We have already proved in Proposition 4.3 that \( T(t) \vec{u}_0 \) is bounded in \( (L^\infty(\Omega))^n \). Therefore there exist a vector function \( \vec{g} \in (L^\infty(\Omega))^n \) and a sequence \( \{\vec{u}(t_k)\} \) such that
\[
\vec{u}(t_k) \rightharpoonup \vec{g} \quad \text{weakly in} \quad (L^2(\Omega))^n, \tag{4.7}
\]
as \( t_k \to \infty \). Next we deduce from [10, Chap. III, Theorem 10.1] that there exists a positive constant \( C \) such that
\[
|u_i(x_1,t) - u_i(x_2,t)| \leq C|x_1 - x_2|^\alpha
\]
for all \( x_1, x_2 \in \Omega' \) and all \( t > 0 \). Therefore, it follows from the Ascoli-Arzela Theorem (see, e.g., [1, Theorem 1.33]) that \( \vec{u}(t_k) \to \vec{g} \) as \( t_k \to \infty \), uniformly in \( \Omega' \). We choose \( t_0 \) large enough such that for all \( t_k \geq t_0 \)
\[
\|\vec{u}(\cdot,t_k) - \vec{g}(\cdot)\|_{1,\Omega'} \leq \frac{\varepsilon}{2}, \tag{4.8}
\]
where \( \|\cdot\|_{1,\Omega'} \) corresponds to the \( L^1 \) norm in \( \Omega' \). We deduce that, in view of (4.6) and (4.7) that
\[
\|\vec{u}(\cdot,t_k) - \vec{g}(\cdot)\|_{1,\Omega \setminus \Omega'} \leq K|\Omega \setminus \Omega'| \leq \frac{\varepsilon}{2},
\]
which together with (4.8) yields
\[
\|\vec{u}(\cdot,t_k) - \vec{g}(\cdot)\|_1 \leq \varepsilon.
\]

**Step 2.** \( \omega(\vec{u}_0) \subset \mathcal{H} \).
Indeed, let \( \vec{g} \in \omega(\vec{u}_0) \) and suppose \( \vec{g} \notin \mathcal{H} \). According to Proposition
we can find a steady state solution \( \vec{w} \in \mathcal{H} \), such that at least one component of \( \vec{w} - \vec{g} \) changes the sign. Without loss of generality we can assume that it happens for the first component, namely that \( f_1 - w_1 \) changes the sign. We remark that, by the contraction property in Theorem 3.1, the functional

\[
\mathcal{V}(\vec{\xi}) = \|\vec{\xi} - \vec{w}\|_1
\]

is a Lyapunov functional for Problem (P), where \( \vec{\xi} \in (L^1(\Omega))^n \). Next we describe some of its properties.

**Property (a)** The functional \( \mathcal{V} \) is constant on \( \omega(\vec{u}_0) \).

Since \( \mathcal{T}(t) \vec{w} = \vec{w} \) and \( \mathcal{T}(t) \) has the contraction property (3.4), the functional \( \mathcal{V} \) is nonincreasing in time along the trajectory \( \mathcal{T}(t) \vec{u}_0 \), which yields

\[
\mathcal{V}(\mathcal{T}(t) \vec{u}_0) = \|\mathcal{T}(t) \vec{u}_0 - \vec{w}\|_1 \leq \|\vec{u}_0 - \vec{w}\|_1 < \infty.
\]

Thus there exists a finite limit \( \mathcal{V}^* \) of \( \mathcal{V}(\mathcal{T}(t) \vec{u}_0) \) as \( t \to \infty \). Let \( \vec{h}_1, \vec{h}_2 \in \omega(\vec{u}_0) \). We can find a sequence \( t_k \to \infty \) as \( k \to \infty \), such that

\[
\|\mathcal{T}(t_{2k}) \vec{u}_0 - \vec{h}_1\|_1 \to 0 \quad \text{and} \quad \|\mathcal{T}(t_{2k+1}) \vec{u}_0 - \vec{h}_2\|_1 \to 0,
\]

as \( k \) tends to \( \infty \). It follows that \( \mathcal{V}(\vec{h}_1) = \mathcal{V}(\vec{h}_2) = \mathcal{V}^* \).

**Property (b)** The \( \omega \)-limit set \( \omega(\vec{u}_0) \) is invariant with respect to the semigroup \( \mathcal{T}(t) \), namely if \( \vec{h} \in \omega(\vec{u}_0) \), then for all \( t > 0 \) also \( \mathcal{T}(t) \vec{h} \in \omega(\vec{u}_0) \).

Let the sequence \( t_k \to \infty \) as \( k \to \infty \) be such that \( \|\mathcal{T}(t_k) \vec{u}_0 - \vec{h}\|_1 \to 0 \).

From the contraction property (3.4)

\[
\|\mathcal{T}(t_k + t) \vec{u}_0 - \mathcal{T}(t) \vec{h}\|_1 = \|\mathcal{T}(t) \mathcal{T}(t_k) \vec{u}_0 - \mathcal{T}(t) \vec{h}\|_1 \leq \|\mathcal{T}(t_k) \vec{u}_0 - \vec{h}\|_1.
\]

Since the last term above tends to \( 0 \) as \( k \) tends to \( \infty \) this shows that \( \mathcal{T}(t) \vec{h} \in \omega(\vec{u}_0) \).

Now, remember that \( \vec{g} \in \omega(\vec{u}_0) \) is such that \( \vec{g} \notin \mathcal{H} \) and \( \vec{w} \in \mathcal{H} \) is such that the first component of \( \vec{w} - \vec{g} \) changes the sign in \( \Omega \). Then, Corollary 3.3 yields

\[
\mathcal{V}(\mathcal{T}(t) \vec{g}) = \|\mathcal{T}(t) \vec{g} - \vec{w}\|_1 = \|\mathcal{T}(t) \vec{g} - \mathcal{T}(t) \vec{w}\|_1 < \|\vec{g} - \vec{w}\|_1 = \mathcal{V}(\vec{g}),
\]

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Step 3. The set $\omega(\vec{u}_0)$ contains only one element.
Suppose that $\vec{g}_1, \vec{g}_2 \in \omega(\vec{u}_0)$. Then we can find two sequences $t_k, s_k$
tending to $\infty$ as $k \to \infty$, such that $s_k \leq t_k$ and $\|T(t_k) \vec{u}_0 - \vec{g}_1\|_1, \|T(s_k) \vec{u}_0 - \vec{g}_2\|_1 \to 0$ as $t_k \to \infty$. Since $\omega(\vec{u}_0) \subset \mathcal{H}$, it follows that

$$
\|\vec{g}_1 - \vec{g}_2\|_1 \leq \|T(t_k) \vec{u}_0 - \vec{g}_1\|_1 + \|T(t_k) \vec{u}_0 - \vec{g}_2\|_1
$$

$$
= \|T(t_k) \vec{u}_0 - \vec{g}_1\|_1 + \|T(t_k - s_k) \vec{u}_0 - T(s_k) \vec{u}_0 - \vec{g}_2\|_1
$$

$$
\leq \|T(t_k) \vec{u}_0 - \vec{g}_1\|_1 + \|T(s_k) \vec{u}_0 - \vec{g}_2\|_1,
$$

which tends to 0 as $k \to \infty$. 

5 Stationary solutions for the linear molecular motor problem

In this section we show the existence and the uniqueness (up to a multiplicative constant) of the classical stationary solution of the problem for the molecular motor. We suppose that $\Omega$ is an open bounded subset of $\mathbb{R}^d$ with smooth boundary $\partial \Omega$.

We consider the linear system

$$
\text{div} (\sigma_i \nabla v_i(x) + v_i(x) \nabla \psi_i(x)) + \sum_{j=1}^{n} a_{ij} v_j(x) = 0 \quad \text{in} \quad \Omega, \quad (5.1)
$$

where $i \in \{1, \ldots, n\}$, $n > 1$. The system (5.1) is supplemented with the Robin boundary conditions

$$
\sigma_i \frac{\partial v_i}{\partial \nu} + v_i \frac{\partial \psi_i}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega, \quad (5.2)
$$

where $i \in \{1, \ldots, n\}$. Thus, the problem can be written as

$$
\mathcal{A} \vec{v} = 0,
$$

with a linear operator $\mathcal{A}$ in a suitable Banach space $\mathcal{X}$ of functions on $\Omega$, to be made precise later. Moreover, we impose the integral constraint

$$
\sum_{i=1}^{n} \int_{\Omega} v_i(x) \, dx = 1. \quad (5.3)
$$

The adjoint problem $\mathcal{A}^* \vec{\varphi} = 0$ to (5.1), in a dual space $\mathcal{X}^*$, is now

$$
\sigma_i \Delta \varphi_i - \nabla \psi_i \cdot \nabla \varphi_i + \sum_{j=1}^{n} a_{ij} \varphi_j = 0, \quad \text{in} \quad \Omega, \quad (5.4)
$$
with the Neumann boundary conditions for each \( i = 1, \ldots, n \)
\[
\frac{\partial \psi_i}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega. \tag{5.5}
\]
Since \( \sum_{j=1}^{n} a_{ji} = 0 \), the problem (5.4) has the obvious solution
\[
\vec{\varphi} = (\varphi_1, \ldots, \varphi_n) = (1, \ldots, 1). \tag{5.6}
\]
We are going to apply the Krein-Rutman theorem on the first eigenvalues and eigenvectors of positive operators, and this will permit us to conclude that the problem (5.1)–(5.2) has a one-dimensional space of solutions. Therefore, under the additional constraint (5.3), the original problem (5.1)–(5.2) has a unique solution.

Perthame and Souganidis sketched this argument for \( n > 1 \) and \( d = 1 \) in [20].

**Theorem 5.1.** Under the assumption \( \sum_{j=1}^{n} a_{ji} = 0 \), there exists a unique smooth solution \( \vec{v} \) of the system (5.1)–(5.3).

Before proving Theorem 5.1 we recall some basic definitions as well as the Krein-Rutman theorem from [9, Ch. VIII, p. 188–191].

**Definition 5.2** (Reproducing cone). We say that a closed set \( K \) in \( \mathcal{X} \) is a cone, if it possesses the following properties:
1. \( 0 \in K \),
2. \( u, v \in K \implies \alpha u + \beta v \in K \), for all \( \alpha, \beta \geq 0 \),
3. \( v \in K \) and \( -v \in K \implies v = 0 \).

A cone \( K \subset \mathcal{X} \) is said to be reproducing if \( \mathcal{X} = K - K \equiv \{ k_1 - k_2 : k_1, k_2 \in K \} \).

**Definition 5.3** (Dual cone). If \( K \) is a cone in \( \mathcal{X} \), then the set \( K^* \subset \mathcal{X}^* \) is said to be a dual cone if
\[
\langle f^*, v \rangle \geq 0,
\]
for every \( v \in K \).

**Definition 5.4** (Strict positivity). Let \( \mathcal{B} \) be a linear operator on \( \mathcal{X} \).
Then \( \mathcal{B} \) is said to be strongly positive if \( \mathcal{B} v \in K^0 \) for all \( v \in K \) such that \( v \neq 0 \).

**Theorem 5.5.** Let \( K \) be a reproducing cone in a Banach space \( \mathcal{X} \), with nonempty interior \( K^0 \neq \emptyset \), and let \( \mathcal{B} \) be a strongly positive compact operator on \( K \) in a sense of Definition 5.4. Then the spectral radius of \( \mathcal{B} \), \( r(\mathcal{B}) \), is a simple eigenvalue of \( \mathcal{B} \) and \( \mathcal{B}^* \), and their associated eigenvectors belong to \( K^0 \) and \((K^*)^0\). More precisely, there exists a unique associated eigenvector in \( K^0 \) (resp. \((K^*)^0\)) of norm 1. Furthermore, all other eigenvalues are strictly less in absolute value than \( r(\mathcal{B}) \).
Proof We will apply Theorem 5.5 to the space $\mathcal{X} = (C(\overline{\Omega}))^n \subset (L^1(\Omega))^n$ endowed with the usual supremum norm, and the operators

$$
B = (\lambda I - A)^{-1} : \mathcal{X} \to \mathcal{X},
$$

$$
B^* = (\lambda I - A^*)^{-1} : \mathcal{X}^* \to \mathcal{X}^*,
$$

where $\lambda > 0$ is a strictly positive real number to be fixed later.

Let

$$
K = \{ \vec{u} \in \mathcal{X} : u_i(x) \geq 0 \text{ for each } x \in \overline{\Omega}, \ i = 1, \ldots, n \}.
$$

We remark that $K$ is a reproducing cone, with nonempty interior

$$
K^o = \{ \vec{u} \in \mathcal{X} : \inf_{x \in \Omega} u_i(x) > 0, \ i = 1, \ldots, n \}.
$$

From the standard theory [17, Theorem 2.1 and Theorem 3.1, Ch. 7] for elliptic partial differential linear systems, the boundary value problem

$$
\sigma_i \Delta \varphi_i - \nabla \psi_i \cdot \nabla \varphi_i + \sum_{i=1}^n a_{ji} \varphi_j - \lambda \varphi_i = f_i \text{ in } \Omega, \tag{5.7}
$$

with the homogeneous Neumann conditions (5.5) on $\partial \Omega$, for $\lambda = \tilde{\lambda} > 0$ sufficiently large, has a solution $\vec{\varphi} = (\varphi_1, \ldots, \varphi_n) \in \mathcal{X}$ for each $\vec{f} = (f_1, \ldots, f_n) \in \mathcal{X}$. Moreover, if $f_i(x) \geq 0$ for each $i = 1, \ldots, n$, and $x \in \overline{\Omega}$, then $\varphi_i(x) \geq 0$ (in fact, $\varphi_i(x) > 0$ in $\Omega$), which is a consequence of the maximum principle (cf. also Example 3 on p. 196–197 in [9]). Thus, the operator $B^* = (\tilde{\lambda} I - A^*)^{-1}$ is a strongly positive and compact operator, and by Theorem 5.5, the largest eigenvalue $\mu$ of $B$ and $B^*$ is simple.

Since

$$
-\sigma_i \Delta \varphi_i + \nabla \psi_i \cdot \nabla \varphi_i - \sum_{j=1}^n a_{ji} \varphi_j + \lambda \varphi_i = \tilde{\lambda} \varphi_i \text{ in } \Omega
$$

$$
\frac{\partial \varphi_i}{\partial \nu} = 0 \text{ on } \partial \Omega,
$$

for all $i \in \{1, \ldots, n\}$, with $\vec{\varphi} = (\varphi_1, \ldots, \varphi_n) = (1, \ldots, 1)$, and since $(1, \ldots, 1) \in (K^*)^o$, it follows that $\frac{1}{\lambda} = r((\tilde{\lambda} I - A^*)^{-1})$ is a simple eigenvalue of the operator $(\tilde{\lambda} I - A^*)^{-1}$. Applying again Theorem 5.5, we deduce that $\frac{1}{\lambda}$ is the largest eigenvalue of the operator $(\tilde{\lambda} I - A)^{-1}$ and that it is simple, and that there exists $\vec{v} \in K^o \subset \mathcal{X}$ such that

$$
(\tilde{\lambda} I - A)^{-1} \vec{v} = \frac{1}{\lambda} \vec{v},
$$

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which is equivalent to
\[ A\vec{v} = 0. \]
This proves the existence of the solution of the problem (5.1)–(5.3).

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