Markov Lemma for Countable Alphabets

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Abstract—Strong typicality and the Markov lemma have been used in the proofs of several multiterminal source coding theorems. Since these two tools can be applied to finite alphabets only, the results proved by them are subject to the same limitation. Recently, a new notion of typicality, namely unified typicality, has been defined. It can be applied to both finite or countably infinite alphabets, and it retains the asymptotic equipartition property and the structural properties of strong typicality. In this paper, unified typicality is used to derive a version of the Markov lemma which works on both finite or countably infinite alphabets so that many results in multiterminal source coding can readily be extended. Furthermore, a simple way to verify whether some sequences are jointly typical is shown.

I. INTRODUCTION

The Markov lemma was first used by Berger [1] to extend multiterminal source coding theory. It has been used in the achievability part of the coding theorems in source coding with side information [2, Section 15.8], rate distortion with side information [2, Section 15.9], channel coding with side information [3, Section 6.2], a large class of multiterminal noiseless source coding problems [4], etc. The different versions of the Markov lemma given in [1]–[4] have the same limitation that all of them cannot be applied to countably infinite alphabets because they are based on strong typicality [1][5]. Note that the Markov lemma for Gaussian sources has been shown in [6].

Recently, Ho and Yeung have defined a new notion of typical sequences, called unified typicality, which works for countable alphabet [7]. Unified typicality retains the asymptotic equipartition property and the structural properties of strong typicality [8]. We will further show in this paper that unified typicality can give a version of the Markov lemma for countable alphabets, which can be used to extend the achievability parts of the aforementioned coding problems. Also, the new Markov lemma further supports that unified typicality is a right notion for generalizing strong typicality to countable alphabets.

In order to show that some sequences are jointly weakly typical, we need to show $2^k - 1$ nonnegative quantities in [2, (15.24)] sufficiently small for a problem with $k$ random variables. It seems that unified typicality suffers the same trouble. In this paper, we will demonstrate a simple method which requires to show only two nonnegative quantities sufficiently small in order to show jointly unified typical.

In the next section, we introduce unified typicality and some notations. In Section III-A, the Markov lemma which works on both finite or countably infinite alphabet is shown, and its consequences are discussed. Then some useful lemmas and the trick to ease the verification of jointly unified typical sequences are shown in Section III-B before the new Markov lemma is proved in Section III-C. In this paper, the base of the logarithm is 2.

II. UNIFIED TYPICALITY

Consider some countable alphabets $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$. For any sequences $\mathbf{y} = (y_1, \ldots, y_n) \in \mathcal{Y}^n$, we say that a sequence of random variables $\mathbf{X} = (X_1, X_2, \ldots, X_n) \in \mathcal{X}^n$ is drawn $\sim \prod_i p(x_i|y_i)$ if $X_i$ are independent and

$$\Pr(\mathbf{X} = \mathbf{x}) = \prod_{i=1}^{n} p(x_i|y_i),$$

where $\mathbf{x} = (x_1, \ldots, x_n) \in \mathcal{X}^n$. Let $\mathbf{z} = (z_1, \ldots, z_n) \in \mathcal{Z}^n$. We call $Q_{XYZ} = \{q(xyz)\}$ the empirical distribution of the sequences $(\mathbf{X}, \mathbf{y}, \mathbf{z})$, where $q(xyz) = n^{-1}N(x, y, z; \mathbf{X}, \mathbf{y}, \mathbf{z})$ and $N(x, y, z; \mathbf{X}, \mathbf{y}, \mathbf{z})$ is the number of occurrences of $(x, y, z)$ in the sequences $(\mathbf{X}, \mathbf{y}, \mathbf{z})$. Note that $Q_{XYZ}$ is also called the type of $(\mathbf{X}, \mathbf{y}, \mathbf{z})$ [9] and $Q_{XYZ}$ is a random variable as $\mathbf{X}$ is random. The marginal distribution $\{q(xy)\}$ is denoted by $Q_{XY}$ and the other marginal distributions of $Q_{XYZ}$ and $P_{XYZ} = \{p(xyz)\}$ are defined in a similar fashion. We use $X - Y - Z$ to denote a Markov chain with respect to $P_{XYZ}$, i.e., $p(xyz) = p(x|y)p(yz)$ for all $x$, $y$ and $z$. Now, we use the Kullback-Leibler divergence $D(\cdot \mid \cdot)$ and entropy $H(\cdot)$ (see e.g., [2][5]) to define unified typicality [7]. We always assume $H(P_{XYZ}) < \infty$.

Definition 1: The unified jointly typical set $U_{\{X_{XY}, Y, Z\}}^n$ with respect to $P_{XYZ}$ is the set of sequences $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in \mathcal{X}^n \times \mathcal{Y}^n \times \mathcal{Z}^n$ such that

$$D(Q'_{XY}||P_{XYZ}) + |H(Q'_{XY}) - H(P_{XY})| + |H(Q'_{XZ}) - H(P_{XZ})| + |H(Q'_{Y}) - H(P_{Y})| + |H(Q'_{Z}) - H(P_{Z})| \leq \gamma,$$

where $Q'_{XYZ} = \{q'(xyz)\}$ is the empirical distribution of $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ with $q'(xyz) = n^{-1}N(x, y, z; \mathbf{x}, \mathbf{y}, \mathbf{z})$.

The definition of $U_{\{X_{XY}, Y, Z\}}^n$ is similar to $U_{\{X_{XY}, Y\}}^n$ with $D(Q_{XY}||P_{XYZ})$ replaced by $D(Q'_{XY}||P_{XYZ})$, and all the absolute values involving $X$ being dropped.
III. MAIN RESULTS

A. The Markov Lemma

The Markov lemma for countable alphabets is given in Theorem 1 and its proof will be deferred to Section III-C. In this paper, we consider only those $P_{XYZ}$ satisfying $H(P_{XYZ}) < \infty$ and

$$\sum_x p(x|y) (\log p(x|y))^2 < C$$

for $y \in \mathcal{Y}$, where $C$ is finite. These assumptions enable us to simplify the proofs by using Chebyshev’s inequality.

**Theorem 1**: Consider $P_{XYZ}$ with $H(P_{XYZ}) < \infty$. Assume that (3) is satisfied and $X - Y - Z$. If for any $\gamma > 0$ and any given $(y, z) \in U^n_{[Y,Z]_\gamma}$, $X$ is drawn $\sim \prod_i p(x_i|y_i)$, then

$$\Pr \{ (X, y, z) \in U^n_{[X,Y,Z]_\gamma} \} \geq 1 - \gamma$$

for $n$ sufficiently large and $\gamma$ sufficiently small.

**Remarks**: 

i) This is a generalization of [2, Lemma 15.8.1]. Since unified typicality retains the asymptotic equipartition property and the structural properties of strong typicality [7][8], it is readily to generalize the achievability parts of Theorem 15.8.1 and Theorem 15.9.1 in [2] with $X$ and $Y$ taking values from countable alphabets.

ii) A result similar to [3, (1.27)] with strong typicality replaced by unified typicality can be easily shown from Theorem 1

iii) Theorem 1 can easily generalize the version of the Markov lemma in [1] to countably infinite alphabet as follows.

**Corollary 2**: Consider $P_{XYZ}$ with $H(P_{XYZ}) < \infty$. Assume that (3) is satisfied and $X - Y - Z$. If for any $\gamma > 0$ and any given $z \in U^n_{[Z]_\gamma}$, $(X, Y)$ is generated according to

$$\Pr \{ (X, Y) = (x, y) \} = \prod_i p(x_i|y_i),$$

then

$$\Pr \{ (X, z) \in U^n_{[X,Z]_\gamma} \} \geq 1 - \gamma,$$

for $n$ sufficiently large and $\gamma$ sufficiently small.

**Proof**: If $(X, Y, z) \in U^n_{[X,Y,Z]_\gamma}$, then $(X, z) \in U^n_{[X,Z]_\gamma}$ from the consistency theorem in [7, Theorem 5]. Therefore,

$$\Pr \{ (X, z) \in U^n_{[X,Z]_\gamma} \} \geq \Pr \{ (X, Y, z) \in U^n_{[X,Y,Z]_\gamma} \} \geq \sum_{y: \sum_i p(x_i|y_i)} \Pr \{ Y = y | (X, z) \in U^n_{[Y,Z]_\gamma} \} \geq 1 - \gamma,$$

where (2) follows from Theorem 1.

B. Some Lemmas

In order to prove Theorem 1 we have to first establish the results in this subsection. Let $E_i$ be an event for all $i$. In this paper, we will frequently use the following lemma and the fact that if $E_1$ implies $E_2$, then $\Pr \{ E_1 \} \leq \Pr \{ E_2 \}$.

**Lemma 1**: If $\Pr \{ E_i \} \geq 1 - \delta_i$, then

$$\Pr \{ \cap_i E_i \} \geq 1 - \sum_i \delta_i.$$  

**Proof**: By the union bound,

$$\Pr \{ \cap_i E_i \} = 1 - \Pr \{ \cup_i E_i^c \} \geq 1 - \sum_i \Pr \{ E_i^c \} \geq 1 - \sum_i \delta_i.$$  

In the following lemma, we consider the variational distance (see e.g., [5]) between $Q_{XYZ}$ and $P_{XYZ}$ which is defined as

$$V(Q_{XYZ}, P_{XYZ}) = \sum_{xyz} |q(xyz) - p(xyz)|.$$  

**Lemma 2**: Assume $X - Y - Z$. If for any $\epsilon > 0$ and any given $(y, z) \in U^n_{[Y,Z]_\gamma}$, $X$ is drawn $\sim \prod_i p(x_i|y_i)$, then

$$\Pr \{ V(Q_{XYZ}, P_{XYZ}) \leq \epsilon \} \geq 1 - \epsilon,$$

for $n$ sufficiently large and $\eta$ sufficiently small.

**Proof**: The proof is similar to the proof of [1, Lemma 4.1] except that $P_{XYZ}$ is defined on countable alphabets here. Fix any $(x, y, z) \in X \times Y \times Z$. For $1 \leq i \leq n$, let $B_i$ be binary and independently distributed. If $(y, z) = (y_i, z_i)$, let

$$B_i = \begin{cases} 0 & \text{with probability } 1 - p(x|y) \\ 1 & \text{with probability } p(x|y) \end{cases}$$

If $(y, z) \neq (y_i, z_i)$, let $B_i = 0$. Then $N(x, y, z; X, y, z)$ and $\sum_{i=1}^n B_i$ have the same distribution on the set of integers. So

$$E[N(x, y, z; X, y, z)] = \sum_{i=1}^n E[B_i] = p(x|y)N(y, z; y, z).$$

Since $B_i$ are binary and independent, the variance of $N(x, y, z; X, y, z)$ is

$$\text{Var}[N(x, y, z; X, y, z)] = \sum_{i=1}^n \text{Var}[B_i] \leq n.$$

For any $\delta > 0$, Chebyshev’s inequality [2, (3.32)] can be applied to show

$$\Pr \{ |N(x, y, z; X, y, z) - p(x|y)N(y, z; y, z)| \geq n\delta \} \leq \frac{\text{Var}[N(x, y, z; X, y, z)]}{(n\delta)^2} \leq \frac{1}{n\delta^2} \leq \delta,$$

where the last inequality holds for sufficiently large $n$. Since $q(xyz) = n^{-1}N(x, y, z; X, y, z)$ and $q(yz) = n^{-1}N(y, z; y, z)$, (16) is equivalent to

$$\Pr \{ |q(xyz) - p(x|y)q(yz)| \leq \delta \} \geq 1 - \delta.$$
Now for any $\epsilon > 0$, let
\[ \eta = \frac{\epsilon^2}{32}. \] (18)

Since $(y, z) \in U^n_{Y \mid Z}$, $D(Q_{Y \mid Z} \mid P_{Y \mid Z}) \leq \eta = \frac{\epsilon^2}{32}$. By Pinsker’s inequality [2] and the fact that $\ln 2 < 1$,
\[ \frac{\epsilon}{4} \geq \sum_{y^i z^i} |q(y^i z^i) - p(y^i z^i)| \]
(19)
\[ = \sum_{x^i y^i z^i} p(x^i | y^i) |q(y^i) - p(y^i)| \]
(20)
\[ = \sum_{x^i y^i z^i} |p(x^i | y^i)q(y^i) - p(x^i z^i)|, \]
(21)
where (21) follows from that $X - Y - Z$. Let $M = |S|$ where $S \subseteq X \times Y \times Z$ is a finite subset such that
\[ \sum_{(x, y, z) \in S} p(xyz) \geq 1 - \frac{\epsilon}{8}. \] (22)
Here, the left side of (22) goes to 1 as $M \rightarrow \infty$, so that such $S$ must exist. Let $E_{xyz} = 1 \{ |q(xyz) - p(xyz)| \leq \frac{\epsilon}{8} \}$ and suppose $E_{xyz} = 1$ for all $(x, y, z) \in S$. Then
\[ \sum_{(x, y, z) \in S} |q(xyz) - p(xyz)| \leq \frac{\epsilon}{8}. \] (23)
Together with (21), we have
\[ \sum_{(x, y, z) \in S} |q(xyz) - p(xyz)| \leq \frac{3\epsilon}{8}. \] (24)
and hence,
\[ \sum_{(x, y, z) \in S} q(xyz) \geq \sum_{(x, y, z) \in S} p(xyz) - \frac{3\epsilon}{8} \geq 1 - \frac{\epsilon}{2}. \] (25)
where the last inequality follows from (22). Thus,
\[ \sum_{x^i y^i z^i} |q(x^i y^i z^i) - p(x^i y^i z^i)| \]
\[ \leq \sum_{(x, y, z) \in S} \frac{1}{\sum_{x^i y^i z^i} q(x^i y^i z^i)} \left( 1 - \sum_{(x, y, z) \in S} q(x^i y^i z^i) \right) \]
\[ = \sum_{x^i y^i z^i} \left( 1 - \sum_{(x, y, z) \in S} p(x^i y^i z^i) \right) \]
\[ = \frac{\epsilon}{8} + \frac{\epsilon}{8} \]
\[ = \epsilon, \] (26)
where (26) follows from (22), (24) and (25). Therefore, if $E_{xyz} = 1$ for all $(x, y, z) \in S$, then $V(Q_{XYZ}, P_{XYZ}) \leq \epsilon$. So we can put $\delta = \frac{\epsilon}{8\eta M}$ into (17) and apply Lemma 1 to show that when $n$ is sufficiently large,
\[ \Pr \{ V(Q_{XYZ}, P_{XYZ}) \leq \epsilon \} \geq \Pr \{ \cap_{(x, y, z) \in S} E_{xyz} \} \]
\[ \geq 1 - \frac{\epsilon}{8} \geq 1 - \epsilon. \] (30)

We now establish a result regarding the Kullback-Leibler divergence and entropy difference between $P_{X \mid Y \mid Z}$ and $Q_{X \mid Y \mid Z}$. In the following lemma, $(y^n, z^n)$ is not necessarily jointly typical. Also, $Q_{X \mid Y = y, Z = z}$ and $P_{X \mid Y = y, Z = z}$ are the probability distributions of $X$ when $Y = y$ and $Z = z$ are given. Recall that we consider only those $P_{X \mid Y \mid Z}$ satisfying (3) and $H(P_{X \mid Y \mid Z}) < \infty$.

**Lemma 3**: Assume $X - Y - Z$. If for any $\epsilon > 0$ and any given $(y^n, z^n)$, $X$ is drawn $\sim \prod_i p(x_i | y_i)$, then
\[ \Pr \left\{ \sum_{y} q(y) (D(Q_{X \mid Y = y, Z = z} \mid P_{X \mid Y = y, Z = z}) + H(Q_{X \mid Y = y, Z = z}) - H(P_{X \mid Y = y, Z = z})) \right\} \leq \epsilon \right\} \geq 1 - \epsilon \] (31)
for $n$ sufficiently large.

**Proof**: For $1 \leq i \leq n$, let $A_i = \log p(x_i | y_i)$. Since $X_i$ are independent, $A_i$ are also independent. Together with (3), the upper bound on the variance of $\sum_{i=1}^n A_i$ is given by
\[ \Var \left\{ \sum_{i=1}^n A_i \right\} = \sum_{i=1}^n \Var[A_i] \leq \sum_{i=1}^n \mathbb{E}[A_i^2] \leq nC, \] (32)
By Chebychev’s inequality,
\[ \Pr \left\{ \frac{1}{n} \left| \sum_{i=1}^n A_i - \mathbb{E} \left[ \sum_{i=1}^n A_i \right] \right| \geq \epsilon n \right\} \leq \frac{\Var \left\{ \sum_{i=1}^n A_i \right\}}{(n\epsilon)^2} \leq \frac{C}{n\epsilon^2} \leq \epsilon \] (33)
when $n$ is sufficiently large. Then
\[ \Pr \left\{ \left| \sum_{i=1}^n A_i - \mathbb{E} \left[ \sum_{i=1}^n A_i \right] \right| \leq \epsilon \right\} \geq 1 - \epsilon, \] (34)
where the left sides of (31) and (33) are equal because
\[ n^{-1} \mathbb{E} \left[ \sum_{i=1}^n A_i \right] - n^{-1} \sum_{i=1}^n A_i \]
\[ = n^{-1} \sum_{i=1}^n \sum_x p(x | y_i) \log p(x | y_i) - n^{-1} \sum_{i=1}^n \log p(X_i | y_i) \]
\[ = n^{-1} \sum_{i=1}^n N(y; y) \sum_x p(x | y) \log p(x | y) \]
\[ - n^{-1} \sum_{x, y} N(x, y; X, y) \log p(x | y) \]
\[ = \sum_{x, y} p(x | y)q(y) \log p(x | y) - \sum_{x, y} q(x, y) \log p(x | y) \]
\[ = \sum_{x, y} q(y) \sum_x (p(x | y) - q(x | y)) \log p(x | y) \]
\[ = \sum_{x, y} q(y) \sum_x \left( \frac{q(x | y) \log \frac{q(x | y)}{p(x | y)} - q(x | y) \log q(x | y) + p(x | y) \log p(x | y)}{p(x | y)} \right), \] (40)
where (39) follows from that $X - Y - Z$. 

If $(y, z) \in U^n_{[Y Z]\eta}$, the following lemma simplifies (31).

**Lemma 4:** For any $\epsilon > 0$, there exists $\eta > 0$ such that if $(y, z) \in U^n_{[Y Z]\eta}$, then

$$\sum_{yz} (q(yz) - p(yz)) H(P_{X|Y=y, Z=z}) \leq \epsilon,$$

where $\epsilon \to 0$ as $\eta \to 0$.

**Proof:** Since

$$\sum_x p(x|y) (\log p(x|y))^2 \geq \sum_{x:p(x|y) > 0.5} p(x|y) (\log p(x|y))^2$$

$$\geq \sum_{x:p(x|y) \leq 0.5} p(x|y) (\log p(x|y)),$$

it is easily shown that $H(P_{X|Y=y}) \leq 0.5 + C$ from (3). Since $p(x|y) = p(x|y)$ for all $(x,y,z)$ as $X - Y - Z$,

$$\sum_{yz} (q(yz) - p(yz)) H(P_{X|Y=y, Z=z})$$

$$= \sum_{yz} (q(yz) - p(yz)) H(P_{X|Y=y})$$

$$\leq \sum_{yz:q(yz) \geq p(yz)} (q(yz) - p(yz))(0.5 + C) +$$

$$\sum_{yz:q(yz) < p(yz)} (p(yz) - q(yz))(0.5 + C)$$

$$= (0.5 + C) \sum_{yz} |p(yz) - q(yz)|,$$

$$\leq (0.5 + C) \sqrt{2\eta \ln 2},$$

where (46) follows from $(y, z) \in U^n_{[Y Z]\eta}$ and Pinsker's inequality. By letting $\eta = \frac{\epsilon^2}{(0.5 + C)^2 \ln 2}$, the lemma is proved. $\blacksquare$

Now we use Lemma 4 to simplify (31) in the following lemma, which uses the conditional Kullback-Leibler divergence $D(Q_X|Y Z||P_X|Y Z)$ [10].

**Lemma 5:** Assume $X - Y - Z$. If for any $\epsilon > 0$ and any given $(y, z) \in U^n_{[Y Z]\eta}$, $X$ is drawn $\sim \prod_i p(x_i|y_i)$, then

$$\Pr \left\{ |D(Q_X|Y Z||P_X|Y Z)(Q Y Z) + H(Q_X|Y Z) - H(P_X|Y Z)| \leq \epsilon \right\} \geq 1 - \epsilon$$

for $n$ sufficiently large and $\eta$ sufficiently small.

**Proof:** For any $\epsilon > 0$, there exists a sufficiently small $\eta$ such that

$$\sum_{yz} (q(yz) - p(yz)) H(P_{X|Y=y, Z=z}) \leq \frac{\epsilon}{2}$$

from Lemma 4. Now, suppose

$$\sum_{yz} q(yz)(D(Q_X|Y=y, Z=z)||P_X|Y=y, Z=z) +$$

$$H(Q_X|Y=y, Z=z) - H(P_X|Y=y, Z=z)) \leq \frac{\epsilon}{2},$$

Adding (48) and (49) gives

$$|D(Q_X|Y Z||P_X|Y Z)(Q Y Z) + H(Q_X|Y Z) - H(P_X|Y Z)| \leq \epsilon.$$
then \( H(Q_{X|YZ}) - H(P_{X|YZ}) \geq -\epsilon \). On the other hand, if (50) is satisfied, then \( \epsilon \geq H(Q_{X|YZ}) - H(P_{X|YZ}) \). Therefore, if both (50) and (55) are satisfied, then \( |H(Q_{X|YZ}) - H(P_{X|YZ})| \leq \epsilon \). When \( n \) is sufficiently large and \( \eta \) is sufficiently small, Lemma 2 shows that

\[
Pr \left\{ V(Q_{XYZ}, P_{XYZ}) \leq \min \left\{ \delta, \frac{\epsilon}{2} \right\} \right\} \geq 1 - \frac{\epsilon}{2}. \tag{59}
\]

Also, Lemma 3 shows that (50) is true with probability larger than \( 1 - \frac{\epsilon}{2} \). Therefore, (57) can be shown from Lemma 1. Similarly, (56) can be verified by Lemma 1, Lemma 3, and together with (57).

Due to the following theorem, we just need to bound two instead of eight quantities in (2) in order to verify that \((x, y, z) \in U_{n|XYZ}^n\).

**Theorem 3:** Assume \( H(P_{AB}) \) is finite. If \( \lim_{m \to \infty} V(P_{A_m B_m}, P_{AB}) = 0 \) and \( \lim_{m \to \infty} |H(P_{A_m B_m}) - H(P_{AB})| = 0 \), then

\[
\lim_{m \to \infty} |H(P_{A_m}) - H(P_A)| = 0. \tag{60}
\]

**Proof:**

\[
\lim_{m \to \infty} H(P_{A_m}) = \lim_{m \to \infty} H(P_{A_m B_m}) - H(P_{B_m|A_m}) \tag{61}
\]

\[
= H(P_{AB}) - \lim_{m \to \infty} H(P_{B_m|A_m}) \tag{62}
\]

\[
\leq H(P_{AB}) - H(P_{B|A}) \tag{63}
\]

\[
= H(P_A), \tag{64}
\]

where (63) follows from Lemma 6. On the other hand, \( \lim_{m \to \infty} H(P_{A_m}) \geq H(P_A) \) because entropy is lower semi-continuous [11]. Therefore, the theorem is proved.

Suppose \( |H(Q_{XYZ}) - H(P_{XYZ})| \) and \( D(Q_{XYZ}||P_{XYZ}) \) are sufficiently small. In this case, \( V(Q_{XYZ}, P_{XYZ}) \) is small from Pinsker’s inequality and Theorem 3 tells that all the nonnegative quantities in (2) are also small.

**C. Proof of Theorem 1**

We first show that for any \( \epsilon > 0 \),

\[
Pr\{ |H(Q_{XYZ}) - H(P_{XYZ})| \leq \epsilon \} \geq 1 - \frac{\epsilon}{2}. \tag{65}
\]

and

\[
Pr\{ D(Q_{XYZ}||P_{XYZ}) \leq \epsilon \} \geq 1 - \frac{\epsilon}{2} \tag{66}
\]

when \( n \) is sufficiently large and \( \eta \) is sufficiently small.

Let \( \eta = \frac{\epsilon}{2} \) so that \( |H(Q_{Y|Z}) - H(P_{Y|Z})| \leq \frac{\epsilon}{2} \). If \( |H(Q_{X|YZ}) - H(P_{X|YZ})| \leq \frac{\epsilon}{2} \), then

\[
\epsilon \geq |H(Q_{XYZ}) - H(P_{XYZ})| + |H(Q_{Y|Z}) - H(P_{Y|Z})| \geq |H(Q_{XYZ}) - H(P_{XYZ})|. \tag{67}
\]

Together with Lemma 7 (65) follows from

\[
Pr\{ |H(Q_{XYZ}) - H(P_{XYZ})| \leq \epsilon \} \geq Pr\{ |H(Q_{XYZ}) - H(P_{XYZ})| \leq \frac{\epsilon}{2} \} \geq 1 - \frac{\epsilon}{2}. \tag{68}
\]

Since \( \eta = \frac{\epsilon}{2} \), \( D(Q_{YZ}||P_{YZ}) \leq \frac{\epsilon}{2} \) as \((y, z) \in U_{n|XYZ}^n\). If \( D(Q_{X|YZ}||P_{X|YZ}Q_{YZ}) \leq \frac{\epsilon}{2} \), then

\[
\epsilon \geq D(Q_{X|YZ}||P_{X|YZ}Q_{YZ}) + D(Q_{YZ}||P_{YZ}) \tag{69}
\]

\[
= D(Q_{XYZ}||P_{XYZ}). \tag{70}
\]

Together with Lemma 7 (65) follows from

\[
Pr\{ D(Q_{XYZ}||P_{XYZ}) \leq \epsilon \} \geq Pr\{ D(Q_{XYZ}||P_{XYZ}) \leq \frac{\epsilon}{2} \} \geq 1 - \frac{\epsilon}{2}. \tag{71}
\]

For any \( \gamma > 0 \), there exists a sufficiently small \( \epsilon \leq \frac{\gamma}{2} \) from Theorem 5 such that if (67) and (70) are satisfied, then all the absolute values in (2) are less than \( \frac{\gamma}{2} \), and hence, (2) is satisfied. Therefore, by (65) and (66).

\[
Pr\{ (x, y, z) \in U_{n|XYZ}^n \} \geq Pr\{ |H(Q_{XYZ}) - H(P_{XYZ})| \leq \epsilon \} \tag{72}
\]

\[
\geq 1 - \epsilon \tag{73}
\]

\[
\geq 1 - \gamma. \tag{74}
\]

\[
\geq 1 - \gamma. \tag{75}
\]

**IV. Conclusion**

A version of the Markov lemma which works on both finite or countably infinite alphabets has been proved. We have also demonstrated a method to ease the verification of jointly unified typical sequences. These results can readily generalize the achievability parts in some existing coding theorems to countably infinite alphabet and are potentially useful for proving coding theorems that apply to both finite and infinite alphabets.

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