COMPOUND POISSON APPROXIMATION FOR RANDOM FIELDS WITH APPLICATION TO SEQUENCE ALIGNMENT

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The article presents the theory of stationary regularly varying random fields. In this context, we prove a new compound Poisson approximation theorem under appropriate dependence conditions, and demonstrate a couple of effective methods for checking its conditions. The theory is based on an extension of some standard point processes limiting theory to general Polish spaces. Finally, we apply these theoretical results to shed entirely new light on the classical problem of evaluating local alignments of biological sequences.

1. Introduction. Recent developments in the theory of regularly varying sequences clarified our understanding of several key time series models, see e.g. [BS09, JS14, MW16, BPS18] and references therein. In this article, we extend that theory to regularly varying random fields, and apply this to shed entirely new light on the classical problem of evaluating local alignments of biological sequences. In particular, we give a novel geometric interpretation for asymptotic behavior of scores in local alignments of i.i.d. sequences. In the course of this project, we also adapt and generalize some standard limiting results in point processes theory to general Polish spaces.

1.1. Regularly varying fields. Consider first a stationary regularly varying field \( (X_i : i \in \mathbb{Z}^d) \), i.e. a field whose all finite dimensional distributions satisfy regular variation condition, see (3.1). This property is equivalent to the existence of the so-called tail field \( Y = (Y_i : i \in \mathbb{Z}^d) \) which as \( u \to \infty \), appears as the limit in

\[
\left( u^{-1}X_i : i \in \mathbb{Z}^d \right) \quad |X_0| > u \xrightarrow{d} (Y_i),
\]

using the product topology on the space \( \mathbb{R}^{\mathbb{Z}^d} \), see Section 3 below. By regular variation property, there also exists a sequence of real numbers \( (a_n) \) such that \( \lim_{n \to \infty} n^d \mathbb{P}(|X_0| > a_n) = 1 \). Denote \( J_n = \{1, \ldots, n\}^d \subseteq \mathbb{Z}^d \) for each \( n \in \mathbb{N} \). One of our main goals is to describe the limit in distribution of the point processes of the form \( \sum_{j \in J_n} \delta(j/n, X_j/a_n) \), \( n \in \mathbb{N} \).

In the case when \( X_j \)'s are i.i.d., it is well known (cf. [Res87]) that, for \( n \to \infty \), point processes above tend to a rather simple Poisson point process. On the other hand, for a general dependent regularly varying random field for which extreme values tend to appear in clusters, the limit, if it exists, will usually not be Poissonian. Still, one can give workable sufficient conditions on the extremal dependence structure of the random field \( (X_i) \) under which the point processes \( \sum_{j \in J_n} \delta(j/n, X_j/a_n) \) converge in distribution to a Poisson cluster process.

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Theorem 1.1. Under dependence assumptions given in Theorem 3.6 below,
\[ \sum_{j \in J_n} \delta(j/n, X_j/a_n) \overset{d}{\longrightarrow} \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{Z}^d} \delta(T_i, P_j, Q_j) \]
in the space of point measures on \([0,1]^d \times (\mathbb{R} \setminus \{0\})\), where the limiting Poisson cluster process has the following components

(i) \( \sum_{i \in \mathbb{N}} \delta(T_i, P_i) \) is a Poisson point process on \([0,1]^d \times (0, \infty)\) with intensity measure \( \text{Leb} \times d(-\theta y^{-\alpha}) \) with constant \( \theta > 0 \) depending only on the distribution of the tail field \( Y \) in (1.1);

(ii) \( (Q_{ij})_{j \in \mathbb{Z}^d}, i \in \mathbb{N} \) is a sequence of i.i.d. random fields over \( \mathbb{Z}^d \) independent of \( \sum_{i \in \mathbb{N}} \delta(T_i, P_i) \) and with common distribution again depending only on the tail field \( Y \).

As usual, the topology used for this convergence is the standard vague topology but with bounded sets in \([0,1]^d \times (\mathbb{R} \setminus \{0\})\) being those which are bounded away from the set \([0,1]^d \times \{0\}\), see Section 2.1. Characterization of the constant \( \theta \) and the distribution of \( (Q_{ij})_{j \in \mathbb{Z}^d} \) can be found in Section 3, see expressions (3.9) and (3.11) in particular.

1.2. Local sequence alignment. The local alignment problem was studied extensively in probabilistic setting, see for instance [AGG89, DKZ94b, Han06] and references therein. Since it represents one of the main motivations for our study, we explain here its key ingredients and our main result in that context.

Let \( (A_i)_{i \in \mathbb{N}} \) and \( (B_i)_{i \in \mathbb{N}} \) be two independent i.i.d. sequences taking values in a finite alphabet \( E \). Also, let \( A \) and \( B \) be independent random variables distributed as \( A_1 \) and \( B_1 \), respectively. For a fixed score function \( s : E \times E \to \mathbb{R} \) and for all \( i, j \in \mathbb{N} \) and \( m = 0, 1, \ldots, i \wedge j \) (where \( i \wedge j := \min\{i, j\} \)), let
\[ S_{i,j}^m = \sum_{k=0}^{m-1} s(A_{i-k}, B_{j-k}) \]
be the score of aligning segments \( A_{i-m+1}, \ldots, A_i \) and \( B_{j-m+1}, \ldots, B_j \). Further, for all \( i, j \in \mathbb{N} \) define
\[ S_{i,j} = \max\{S_{i,j}^m : 0 \leq m \leq i \wedge j\}. \]

From biological perspective it is essential to understand the extremal distributional properties of the random matrix \( (S_{i,j} : 1 \leq i, j \leq n) \) as \( n \to \infty \). The following simple assumption is standard in this context, cf. Dembo et al. [DKZ94b].

Assumption 1.2. The distribution of \( s(A, B) \) is nonlattice, i.e. \( \mathbb{P}(s(A, B) \in \delta \mathbb{Z}) < 1 \) for all \( \delta > 0 \), and satisfying
\[ \mathbb{E}[s(A, B)] < 0 \quad \text{and} \quad \mathbb{P}(s(A, B) > 0) > 0. \]

The lattice case is excluded for simplicity in the sequel. It is known to be conceptually similar, although technically more involved. Note further that, like [DKZ94b] and [Han06], we consider only gapless local alignments.
Denote by $\mu_A$ and $\mu_B$ the distributions of $A$ and $B$, respectively and assume for simplicity that $\mu_A(e), \mu_B(e) > 0$ for each letter $e$ in the alphabet $E$. By Assumption 1.2 there exists the unique strictly positive solution $\alpha^*$ of the Lundberg equation

\begin{equation}
(1.4) \quad m(\alpha^*) := \mathbb{E}[e^{\alpha^*s(A,B)}] = 1.
\end{equation}

Let $\mu^*$ be the (exponentially tilted) probability measure on $E \times E$ given by

\begin{equation}
(1.5) \quad \mu^*(a, b) = e^{\alpha^*s(a,b)} \mu_A(a) \mu_B(b), \quad a, b \in E.
\end{equation}

For two probability measures $\mu$ and $\nu$ on a finite set $F$, denote by $H(\nu|\mu)$ the relative entropy of $\nu$ with respect to $\mu$, i.e.

\begin{equation}
H(\nu|\mu) = \sum_{x \in F} \nu(x) \log \frac{\nu(x)}{\mu(x)}.
\end{equation}

Dembo et al. [DKZ94] introduce one final condition on the tilted probability measure $\mu^*$. It essentially restricts extremal dependence within the field $(S_{i,j})$ in a way which seems biologically meaningful and exactly suits their, as well as our, asymptotic analysis of the field.

**ASSUMPTION 1.3 (Condition (E') in [DKZ94b]).** It holds that

\begin{equation}
(1.6) \quad H(\mu^*|\mu_A \times \mu_B) > 2 \{H(\mu^*_A|\mu_A) \vee H(\mu^*_B|\mu_B)\},
\end{equation}

where $\mu^*_A$ and $\mu^*_B$ denote the marginals of $\mu^*$.

Note that (1.6) holds automatically if $\mu_A = \mu_B$ and if the score function $s$ is symmetric (i.e. $s(a, b) = s(b, a)$) and not of the form $s(a, b) = s(a) + s(b)$, see [DKZ94a, Section 3].

Under Assumptions 1.2 and 1.3, Dembo et al. [DKZ94b] (see also Hansen [Han06]) showed that the distribution of the maximal local alignment score $M_n = \max_{1 \leq i, j \leq n} S_{i,j}$, asymptotically follows a Gumbel distribution. More precisely, as $n \to \infty$, for a certain constant $K^* > 0$,

\begin{equation}
(1.7) \quad \mathbb{P}\left(M_n - \frac{2 \log(n)}{\alpha^*} \leq x\right) \to e^{-K^* e^{-\alpha^* x}}, \quad x \in \mathbb{R}.
\end{equation}

Observe that the field $(S_{i,j})$ consists of dependent random variables. For instance, simple arguments can be given (cf. (1.9) below) showing that any extreme score, i.e. score exceeding a given large threshold, will be followed by a run of extreme scores along the diagonal. This phenomena is illustrated in Figure 1.2 for both real life and simulated sequences. The approach of [DKZ94b] is based on showing that the number of such extreme clusters, as both sample size and the threshold tend to infinity, becomes asymptotically Poisson distributed.

In the sequel, we show that one can give much more detailed information about the structure within extreme clusters. In particular, following the method below one can deduce the asymptotic distribution of arbitrary functionals of upper order statistics of the field $(S_{i,j})$.

Observe that for each $i, j \in \mathbb{N}$, $S_{i,j}$ can be seen as the maximum of a truncated random walk $(S^m_{i,j})_{m=0,\ldots,i\wedge j}$ which by (1.3) has negative drift. It can be rigorously shown, see Remark 4.1, that in all our asymptotic considerations this truncation and related edge effects can be
Fig 1. Heatmap of large local scores for alignments of two simulated sequences (left) and two regions of human and fruit-fly genome (right). Each sequence is 1000 nucleotides long.

ignored. Therefore we assume throughout that sequences \((A_i)\) and \((B_i)\) extend over all integers \(i \in \mathbb{Z}\). This makes scores \(S_{i,j}^m\) well defined for all \(i, j \in \mathbb{Z}\) and \(m \geq 0\), and consequently we update the original field of scores \((S_{i,j})\) as follows

\[
S_{i,j} = \sup\{S_{i,j}^m : m \geq 0\}, \quad i, j \in \mathbb{Z}.
\]

By construction, the field \((S_{i,j})\) is stationary. Moreover, it satisfies the following simple (Lindley) recursion along any diagonal, namely

\[
S_{i,j} = (S_{i-1,j-1} + \varepsilon_{i,j})_+ ,
\]

where random variables \(\varepsilon_{i,j} = s(A_i, B_j)\) have negative mean.

By the classical Cramér-Lundberg theory, Assumption 1.2 implies that the tail of \(S_{i,j}\) is asymptotically exponential, or more precisely

\[
P(S_{i,j} > u) \sim Ce^{-\alpha^* u}, \quad \text{as } u \to \infty,
\]

for some \(C > 0\). Note that, in the language of extreme value theory, marginal distribution of the field \((S_{i,j})\), belongs to the maximum domain of attraction of the Gumbel distribution. In this light, the limiting result (1.7) may not be very surprising, but its proof remains quite involved due to the clustering of extremal scores of the field \((S_{i,j})\).

Our main result in this context, strengthens (1.7) to convergence in distribution of point processes based on the \(S_{i,j}\)'s. It turns out that, under Assumptions 1.2 and 1.3, the field \((e^{S_{i,j}})\) satisfies regular variation condition, or more precisely in this setting, as \(u \to \infty\),

\[
(e^{S_{i,j}}/e^{S_{0,0}} : (i,j) \in \mathbb{Z}^2) \mid S_{0,0} > u \overset{d}{\to} (\Theta_{i,j}),
\]

for some field \((\Theta_{i,j})_{(i,j) \in \mathbb{Z}^2}\) with \(\Theta_{i,j} = 0\) for all \(i \neq j\).

Distribution of \(\Theta_{i,j}\)'s can be described in detail using two auxiliary independent i.i.d. sequences \((\varepsilon_i)_{i \geq 1}\) and \((\varepsilon_{i,j}^*)_{i \geq j \geq 1}\) whose distributions correspond to the distributions of \(s(A, B)\) under the
product measure $\mu_A \times \mu_B$ and under the tilted measure $\mu^*$ from (1.5), respectively. Set $S_0^e = 0$ and

\begin{equation}
(1.12) \quad S_m^e = \sum_{i=1}^m \varepsilon_i, \quad \text{for } m \geq 1 \quad \text{and } \quad S_m^e = -\sum_{i=1}^{-m} \varepsilon_i^*, \quad \text{for } m \leq -1,
\end{equation}

then $\Theta_{m,m} = e^{S_m^e}$, for $m \in \mathbb{Z}$. Clearly, $(S_m^e)$ is an asymmetric double sided random walk.

**Theorem 1.4.** Under Assumptions 1.2 and 1.3,

\[ n \sum_{i,j} \delta \left( \left( S_{i,j}^e - \frac{2 \log(n)}{\alpha^*} \right) \right) \xrightarrow{d} \sum_{k \in \mathbb{N}} \sum_{m \in \mathbb{Z}} \delta \left( T_k, \bar{P}_k + \tilde{Q}_m^k \right) \]

in the space of point measures on $[0,1]^2 \times \mathbb{R}$, where the Poisson cluster process in the limit has the following components

(i) $\sum_{k \in \mathbb{N}} \delta \left( T_k, \bar{P}_k \right)$ is a Poisson point process on $[0,1]^2 \times \mathbb{R}$ with intensity measure $\text{Leb} \times d(-\vartheta C e^{-\alpha^* u})$ where, for an exponential random variable $\Gamma$ with parameter $\alpha^*$ independent of $(S_m^e)_{m \in \mathbb{Z}}$,

\[ \vartheta = \mathbb{P} \left( \sup_{m \geq 1} S_m^e + \Gamma \leq 0 \right); \]

(ii) $(\tilde{Q}_m^k)_{m \in \mathbb{Z}}$, $k \in \mathbb{N}$ are i.i.d. two-sided $\mathbb{R}$-valued sequences, independent of $\sum_{k \in \mathbb{N}} \delta \left( T_k, \bar{P}_k \right)$ and with common distribution equal to the distribution of the random walk $(S_m^e)_{m \in \mathbb{Z}}$ conditioned on staying negative for $m < 0$ and non positive for $m > 0$.

For the convergence in distribution above, bounded sets in $[0,1]^2 \times \mathbb{R}$ are those which are contained in $[0,1]^2 \times (x, \infty)$ for some $x \in \mathbb{R}$, again, see Section 2.1. Observe that application of the theorem yields (1.7) at once with the following new expression for the key constant therein

\[ K^* = \vartheta C. \]

Note further that $\vartheta$ can be viewed as the so-called extremal index in this setting, see de Haan et al. [dHRRdV89, Section 3] where the same expression appears together with a suggested algorithm for its numerical computation. The constant $C$ arising from (1.10) is frequently encountered in the literature, for various expressions of $C$ we refer to [Asm03, Part C, XIII.5].

The distribution of random walks conditioned to stay negative (or positive) is discussed in detail by Tanaka [Tan89] and Biggins [Big03]. Consequently, one can apply those results to simulate and precisely describe the distribution of $\tilde{Q}_m^k$’s. Putting all these ingredients together, one can use Theorem 1.4 to give a probabilistic and geometric interpretation of the plots in Figure 1.2. Note that this type of limiting behavior was conjectured already by Metzler et al. [MGW02] who suggested a marked Poissonian model of local alignments with essentially same features.
1.3. Organization of the paper. The rest of the article is organized as follows — in Section 2, building on the classical theorems of Grigelionis (see [Kal83]), as well as Banys [Ban80] and Arratia et al. [AGG89], we present a general type of Poissonian approximation theorem, which allows one to study point processes constructed from general random fields with values in a Polish space under appropriate dependence assumption. We also find sufficient conditions for such a dependence assumption to hold. Section 3 presents the theory of stationary regularly varying random fields indexed over \( \mathbb{Z}^d \), complementing and extending the theory from the case \( d = 1 \). In particular, we introduce the corresponding notions of extremal index and the tail field/process and point out at subtleties of this extension arising from the fact that there is no unique natural ordering of the points in \( d \)-dimensional lattice, for \( d \geq 2 \). Special attention is dedicated to the notion of anchoring of the cluster of extremes, i.e. choosing a reference point in the cluster of large exceedances. Here we show that the choice of the anchoring point (or the choice of group order in \( \mathbb{Z}^d \)) does not influence our main results. Section 4 is entirely dedicated to the alignment problem and the proof of Theorem 1.4. In Section 5 we study stationary fields which are approximable by \( m \)-dependent regularly varying fields. Finally, in Section 6 we give proofs of the main theorems from Section 3 as well as several auxiliary results used in Section 4.

2. On (compound) Poisson approximation in general Polish spaces. Let \((I_n)\) be a sequence of finite index sets such that \( \lim_{n \to \infty} |I_n| = \infty \), where \( |I_n| \) denotes the number of elements in \( I_n \). For each \( n \in \mathbb{N} \), let \((X_{n,i} : i \in I_n)\) be a family of random elements in a separable and complete metric space \((X', d)\). Further, denote by \( \delta_x \) the Dirac measure concentrated at \( x \in X' \) and consider a subset \( X \) of \( X' \) which remains Polish under the relative topology, e.g. \( X = X' \setminus C \) with \( C \) being a closed subset of \( X' \).

The goal of this section is to provide sufficient conditions under which point processes

\[
N_n = \sum_{i \in I_n} \delta_{X_{n,i}} , \quad n \in \mathbb{N},
\]

restricted to the space \( X \), converge in distribution on the space of point measures on \( X \) with respect to a suitable topology. The standard choice of the topology in this setting and the one used in this paper, is the so called vague topology as described in Kallenberg [Kal17, Chapter 4]. However, as proposed in [BP18], we use a slight modification of the definition of vague convergence.

2.1. Basic setup and the notion of vague convergence. For a given family of subsets \( \mathcal{B}' \) in \( X \), called bounded sets, denote by \( \mathcal{B}_b \) the family of bounded Borel measurable sets in \( X \). Denote further by \( \mathcal{M}(X) \) the space of Borel measures on \( X \) which are finite on all sets in \( \mathcal{B}_b \). We say that a measure \( \mu \in \mathcal{M}(X) \) is a point measure if \( \mu = \sum_{K=1}^{K} \delta_{x_k} \) for some \( K \in \{0, 1, 2, \ldots \} \cup \{+\infty\} \) and \( x_1, \ldots, x_K \in X \). The subspace of all point measures is denoted by \( \mathcal{M}_p(X) \).

For measures \( \mu, \mu_1, \mu_2, \ldots \in \mathcal{M}(X) \), we say that \( \mu_n \) converge vaguely to \( \mu \) and denote this \( \mu_n \xrightarrow{v} \mu \), if

\[
\mu_n(f) = \int f(x) d\mu_n(x) \to \int f(x) d\mu(x) = \mu(f), \quad \text{as } n \to \infty,
\]

for all bounded and continuous functions \( f \) on \( X \) with bounded support, i.e. with support being an element of \( \mathcal{B}_b \). Denote this family of functions by \( CB_0(X) \) and by \( CB_0^+(X) \subseteq CB_0(X) \) the subfamily of all non negative functions.
For example, if \( B' \) consists of all subsets of \( X \), the space \( \mathcal{M}(X) \) is the space of all finite Borel measures on \( X \) and the corresponding notion of vague convergence coincides with the usual notion of weak convergence. More interesting example comes from the theory of regularly varying random variables and processes. In this case, for some fixed closed set \( C \subseteq X' \), one chooses \( B' \) as the family of all sets \( B \subseteq X = X' \setminus C \) which are bounded away from \( C \) with respect to the original metric \( d \). More precisely, \( B \in B' \) if \( B \subseteq \{ x \in X : d(x, C) > \epsilon \} \) for some \( \epsilon > 0 \), where \( d(x, C) = \inf \{ d(x, z) : z \in C \} \). The resulting notion of vague convergence is precisely the notion of the so called \( M_\infty \)-convergence (where \( \emptyset = X \)) discussed in Lindskog et al. [LRR14]. See also Hult and Lindskog [HL06].

We say that a subfamily \( C' \subseteq B' \) is a basis of \( B' \) if a set \( B \subseteq X \) is in \( B' \) if and only if \( B \subseteq C \) for some \( C \in C' \). Further, denote by \( \overline{B} \) the closure of the set \( B \) in the space \( X \).

In both of the previous examples it is easy to see that there exists a countable basis \( (K_m)_{m \in \mathbb{N}} \) of \( B' \) where \( K_m \)'s are bounded open subsets of \( X \) which cover \( X \) and satisfy

\[
(2.2) \quad \overline{K}_m \subseteq K_{m+1}, \text{ for all } m \in \mathbb{N}.
\]

Indeed, in the case of weak convergence, it suffices to take \( K_m = X \) for all \( m \in \mathbb{N} \), and in the case of \( M_\infty \)-convergence, one can take \( K_m = \{ x \in X : d(x, C) > 1/m \} \) for all \( m \in \mathbb{N} \).

Interestingly, Hu [Hu66] shows that for any family \( B' \) satisfying the above property, there exists a metric on \( X \) which generates the topology and such that the set \( B \subseteq X \) is in \( B' \) if and only if it is metrically bounded, i.e. contained in some open ball, with respect to this metric. Since this is exactly the framework of [Kal17], the theory developed therein applies to our setting. In particular, by [Kal17, Theorem 4.2], the topology on \( \mathcal{M}(X) \) inducing the notion of vague convergence, called the vague topology, is again Polish, see also [BP18, Section 3].

In the rest of the paper, we will always assume existence of such a sequence \((K_m)\) characterizing the family of bounded sets \( B' \). Furthermore, the space of point measures \( \mathcal{M}_p(X) \) is always endowed with the vague topology and by a point process on \( X \) we mean a random element of the space \( \mathcal{M}_p(X) \) with respect to the Borel \( \sigma \)-algebra. We denote convergence in distribution by \( \xrightarrow{d} \). Recall, for point processes \( N, N_1, N_2, \ldots \), convergence of Laplace functionals \( \mathbb{E}[e^{-N_n(f)}] \to \mathbb{E}[e^{-N(f)}] \) for all \( f \in CB^+_b(X) \) is equivalent to \( N_n \xrightarrow{d} N \) in \( \mathcal{M}_p(X) \), see [Kal17, Theorem 4.11]. It is often useful if one can restrict to a smaller family of functions.

**Definition 2.1.** We say that a family \( F \subseteq CB^+_b(X) \) is (point process) convergence determining if, for point processes \( N, N_1, N_2, \ldots \), convergence \( \mathbb{E}[e^{-N_n(f)}] \to \mathbb{E}[e^{-N(f)}] \) for all \( f \in F \) implies that \( N_n \xrightarrow{d} N \) in \( \mathcal{M}_p(X) \).

For example, one can take the subfamily \( F \subseteq CB^+_b(X) \) of functions which are Lipschitz continuous with respect to a suitable metric. More precisely, this is true for any metric \( \rho \) on \( X \) which generates the topology and is such that for any \( B \in B' \), there exists an \( \epsilon > 0 \) such that \( B^c = \{ x \in X : \rho(x, B) < \epsilon \} \) is also bounded, see [BP18, Proposition 4.6]. Denote the corresponding family of Lipschitz continuous functions by \( LB^+_b(X, \rho) \).

2.2. General Poisson approximation. Recall, for each \( n \in \mathbb{N} \), a family of random elements \((X_{n,i} : i \in I_n)\) in \( X' \) is given. Further, we will assume that \( X_{n,i} \)'s are such that \( \sup_{i \in I_n} \mathbb{P}(X_{n,i} \in \cdot) \)
$B) \to 0$ as $n \to \infty$ for all $B \in \mathcal{B}_b$ and that the intensity measures of point processes $N_n = \sum_{i \in I_n} \delta_{X_{n,i}}$ on $\mathbb{X}$ vaguely converge to a measure $\lambda$ in $\mathcal{M}(\mathbb{X})$, i.e. that $\mathbb{E}[N_n(\cdot)] = \sum_{i \in I_n} \mathbb{P}(X_{n,i} \in \cdot) \xrightarrow{v} \lambda$.

If in addition, for each $n \in \mathbb{N}$, the family $(X_{n,i} : i \in I_n)$ were independent, the famous result of Grigelionis (see [Kal83, Corollary 4.25]) applied to the null array $(\delta_{X_{n,i}} : n \in \mathbb{N}, i \in I_n)$ would imply that point processes $N_n$ converge in distribution to a Poisson point process with intensity measure $\lambda$. In what follows the distribution of such process will be denoted by $\text{PPP}(\lambda)$.

In general, one can still obtain the same Poisson limit if the asymptotic distributional behavior of $N_n$’s is indistinguishable from its independent version. More precisely, let for each $n \in \mathbb{N}$, $(X_{n,i}^* : i \in I_n)$ be independent random elements such that for all $i \in I_n$, $X_{n,i}^*$ is distributed as $X_{n,i}$, and denote by $N_n^* = \sum_{i \in I_n} \delta_{X_{n,i}^*}$ the corresponding point processes on $\mathbb{X}$. For a class of measurable and non negative functions $\mathcal{F}$ on the space $\mathbb{X}$, we say that the family $(X_{n,i} : n \in \mathbb{N}, i \in I_n)$ is asymptotically $\mathcal{F}$–independent (AI($\mathcal{F}$)) if

$$\left| \mathbb{E}[e^{-N_0(f)}] - \mathbb{E}[e^{-N_n^*(f)}] \right| = \left| \mathbb{E}[e^{-\sum_{i \in I_n} f(X_{n,i})}] - \prod_{i \in I_n} \mathbb{E}[e^{-f(X_{n,i})}] \right| \to 0, \text{ as } n \to \infty,$$

for all $f \in \mathcal{F}$, where we set $f(x) = 0$ for all $x \in \mathbb{X}' \setminus \mathbb{X}$. The following result is now immediate.

**Theorem 2.2.** Assume that there exists non zero measure $\lambda \in \mathcal{M}(\mathbb{X})$ such that

(i) For all $m \in \mathbb{N}$

$$\lim_{n \to \infty} \sup_{i \in I_n} \mathbb{P}(X_{n,i} \in K_m) = 0 \ .$$

(ii) As $n \to \infty$,

$$\sum_{i \in I_n} \mathbb{P}(X_{n,i} \in \cdot) \xrightarrow{v} \lambda \ .$$

Then for any convergence determining family $\mathcal{F}$, $N_n \overset{d}{\to} N \overset{d}{=} \text{PPP}(\lambda)$ in $\mathcal{M}_p(\mathbb{X})$ if and only if $(X_{n,i} : n \in \mathbb{N}, i \in I_n)$ is AI($\mathcal{F}$).

**Remark 2.1.** Observe, one can allow $\mathcal{F}$ to contain functions $f$ which are not necessarily continuous but are such that $N(\text{disc}(f)) = 0$ almost surely, where $\text{disc}(f)$ denotes the set of all discontinuity points of $f$. In particular, $\mathcal{F}$ can consist of non negative simple functions with bounded support, see [Kal17, Theorem 4.11].

Consider now the space $[0,1]^d \times \mathbb{X}$ with respect to the product topology and with bounded sets being those which are contained in $K'_m = [0,1]^d \times K_m$ for some $m \in \mathbb{N}$. Let $\text{Leb}$ denote the Lebesgue measure on $[0,1]^d$. As a simple consequence of the previous theorem, one obtains an important special case, cf. [Res87, Proposition 3.21].

**Corollary 2.3.** Assume that $I_n = \{1, 2, \ldots, k_n\}^d \subseteq \mathbb{Z}^d$ for some $d \in \mathbb{N}$ with $k_n \to \infty$ and that $(X_{n,i} : i \in I_n)$ are identically distributed for every $n \in \mathbb{N}$. If there exists a non zero measure $\nu \in \mathcal{M}(\mathbb{X})$ such that as $n \to \infty$,

$$k_n^d \mathbb{P}(X_{n,1} \in \cdot) \xrightarrow{v} \nu \ ,$$

...
then for any convergence determining family $\mathcal{F}'$ on $[0,1]^d \times \mathbb{X}$,

$$N'_n = \sum_{i \in I_n} \delta(i/k_n, X_{n,i}) \xrightarrow{d} N' \overset{d}{=} \text{PPP}(\text{Leb} \times \nu)$$

in $\mathcal{M}_p([0,1]^d \times \mathbb{X})$ if and only if $(i/k_n, X_{n,i}) : n \in \mathbb{N}, i \in I_n)$ is $AI(\mathcal{F}')$.

**Proof.** We simply apply Theorem 2.2 to the random elements $X'_{n,i} := (i/k_n, X_{n,i})$, $n \in \mathbb{N}, i \in I_n$. By [Kal17, Lemma 4.1(iv)], (2.5) implies that for each $m \in \mathbb{N}$

$$\lim_{n \to \infty} \sum_{i \in I_n} \mathbb{P}(X'_{n,i} \in K'_m) = \lim_{n \to \infty} k_n d \mathbb{P}(X_{n,1} \in K_m) \leq \nu(K_m) < +\infty.$$ 

Hence, (2.3) holds since $k_n \to \infty$. Further, note that for arbitrary $a = (a_1, \ldots, a_d)$ and $b = (b_1, \ldots, b_d)$ in $[0,1]^d$ such that $a_j \leq b_j$ for all $j = 1, \ldots, d$ and a set $B \in \mathcal{B}_b$ such that $\nu(\partial B) = 0$, (2.5) implies that as $n \to \infty$,

$$\sum_{i \in I_n} \mathbb{P}(X'_{n,i} \in (a, b] \times B) = \frac{1}{k_n} \prod_{j=1}^d \mathbb{P}([k_n(b_j - a_j)] \cdot k_n d \mathbb{P}(X_{n,1} \in B) \to \prod_{j=1}^d (b_j - a_j) \cdot \nu(B)$$

By [Kal17, Lemma 4.1], this implies that $\sum_{i \in I_n} \mathbb{P}(X'_{n,i} \in \cdot) \xrightarrow{w} \text{Leb} \times \nu$ in $\mathcal{M}([0,1]^d \times \mathbb{X})$, i.e. (2.4) holds with $\lambda = \text{Leb} \times \nu$. \hfill \qed

### 2.3. Sufficient condition for asymptotic $\mathcal{F}$-independence

For each $i \in I_n$, choose a subset of the index set $B_n(i) \subseteq I_n$ containing $i$, and call it the **neighborhood of dependence of $i$**. Intuitively, it will be useful to choose $B_n(i)$ as small as possible but such that $X_{n,i}$ is (nearly) independent of all $X_{n,j}$ for $j \notin B_n(i)$.

Select an arbitrary ordering of the elements in $I_n$. Without loss of generality, we will assume that $I_n = \{1, 2, \ldots, m_n\}$ where $m_n \to \infty$ as $n \to \infty$. For all $i \in I_n$ denote $B_n(i) := \{j \in B_n(i) : j > i\}$ and $B_n^c(i) := \{j \notin B_n(i) : j > i\}.$

For a given neighborhood structure $(B_n(i) : n \in \mathbb{N}, i \in I_n)$ and for all $m, n \in \mathbb{N}$ define

$$b^m_{n,1} = \sum_{i \in I_n} \sum_{j \in B_n(i)} \mathbb{P}(X_{n,i} \in K_m) \cdot \mathbb{P}(X_{n,j} \in K_m),$$

$$b^m_{n,2} = \sum_{i \in I_n} \sum_{j \in B_n(i)} \mathbb{P}(X_{n,i} \in K_m, X_{n,j} \in K_m).$$

Furthermore, for all $n \in \mathbb{N}$ and an arbitrary non negative measurable function $f$ on $\mathbb{X}$ define

$$b^m_{n,3}(f) = \sum_{i \in I_n} \left| \mathbb{E}[e^{-f(X_{n,i})} \prod_{j \in B_n^c(i)} e^{-f(X_{n,j})}] - \mathbb{E}[e^{-f(X_{n,i})}] \cdot \mathbb{E}[\prod_{j \in B_n^c(i)} e^{-f(X_{n,j})}] \right|.$$

**Proposition 2.4.** Let $f$ be a non negative measurable function on $\mathbb{X}$ with bounded support. If $m \in \mathbb{N}$ is such that the support of $f$ is contained in $K_m$, then

$$\left| \mathbb{E} \left[ e^{-\sum_{i \in I_n} f(X_{n,i})} \right] - \prod_{i \in I_n} \mathbb{E} \left[ e^{-f(X_{n,i})} \right] \right| \leq b^m_{n,1} + b^m_{n,2} + b^m_{n,3}(f)$$
for all \( n \in \mathbb{N} \). In particular, if there exists a neighborhood structure \((B_n(i) : n \in \mathbb{N}, i \in I_n)\) such that for all \( m \in \mathbb{N} \) and every \( f \in \mathcal{F} \)

\[
\lim_{n \to \infty} b_{n,1}^m = \lim_{n \to \infty} b_{n,2}^m = \lim_{n \to \infty} b_{n,3}(f) = 0 ,
\]

then the family \((X_n,i : n \in \mathbb{N}, i \in I_n)\) is AF(\(\mathcal{F}\)).

**Proof.** The proof is an adaptation of argument in Nakhapetyan [Nak88, Lemma 3], though the main idea goes back to [Ban80, Theorem 4]. Since \(e^{-f}\) is positive and bounded by 1 it follows that

\[
\left| \mathbb{E} \left[ e^{-\sum_{i \in I_n} f(X_{n,i})} \right] - \prod_{i \in I_n} \mathbb{E} \left[ e^{-f(X_{n,i})} \right] \right| \\
\leq \sum_{i=1}^{m_n-1} \left| \mathbb{E} \left[ e^{-f(X_{n,i})} \prod_{j=i+1}^{m_n} e^{-f(X_{n,j})} \right] - \mathbb{E} \left[ e^{-f(X_{n,i})} \right] \cdot \mathbb{E} \left[ \prod_{j=i+1}^{m_n} e^{-f(X_{n,j})} \right] \right| : \sum_{i=1}^{m_n-1} \varepsilon_i .
\]

Fix now an arbitrary \( i \in \{1, \ldots, m_n - 1\} \). After writing

\[
\prod_{j=i+1}^{m_n} e^{-f(X_{n,j})} = \prod_{j \in B_n(i)} e^{-f(X_{n,j})} \prod_{j \in B_n(i)^C} e^{-f(X_{n,j})} ,
\]

one can easily check that

\[
\varepsilon_i \leq \left| \mathbb{E} \left[ e^{-f(X_{n,i})} \cdot \left( \prod_{j \in B_n(i)} e^{-f(X_{n,j})} - 1 \right) \prod_{j \in B_n(i)^C} e^{-f(X_{n,j})} \right] \right| \\
- \mathbb{E} \left[ e^{-f(X_{n,i})} \right] \cdot \left| \mathbb{E} \left[ \left( \prod_{j \in B_n(i)} e^{-f(X_{n,j})} - 1 \right) \prod_{j \in B_n(i)^C} e^{-f(X_{n,j})} \right] \right| \\
+ \left| \mathbb{E} \left[ e^{-f(X_{n,i})} \prod_{j \in B_n(i)^C} e^{-f(X_{n,j})} \right] - \mathbb{E} \left[ e^{-f(X_{n,i})} \right] \cdot \mathbb{E} \left[ \prod_{j \in B_n(i)^C} e^{-f(X_{n,j})} \right] \right| .
\]

Note that the first summand on the right hand side of the previous inequality equals

\[
\left| \mathbb{E} \left[ \left( e^{-f(X_{n,i})} - 1 \right) \cdot \left( \prod_{j \in B_n(i)} e^{-f(X_{n,j})} - 1 \right) \prod_{j \in B_n(i)^C} e^{-f(X_{n,j})} \right] \right| \\
- \left| \mathbb{E} \left[ \left( e^{-f(X_{n,i})} - 1 \right) \cdot \left( \prod_{j \in B_n(i)} e^{-f(X_{n,j})} - 1 \right) \prod_{j \in B_n(i)^C} e^{-f(X_{n,j})} \right] \right| ,
\]

and since \(e^{-\sum_k f(x_k)} - 1 \neq 0\) implies that \(f(x_k) > 0\), and hence \(x_k \in K_m\), for at least one \( k \), we obtain that

\[
\varepsilon_i \leq \mathbb{P}(X_{n,i} \in K_m, \bigcup_{j \in B_n(i)} \{X_{n,j} \in K_m\}) + \mathbb{P}(X_{n,i} \in K_m) \cdot \mathbb{P}(\bigcup_{j \in B_n(i)} \{X_{n,j} \in K_m\}) \\
+ \left| \mathbb{E} \left[ e^{-f(X_{n,i})} \prod_{j \in B_n(i)^C} e^{-f(X_{n,j})} \right] - \mathbb{E} \left[ e^{-f(X_{n,i})} \right] \cdot \mathbb{E} \left[ \prod_{j \in B_n(i)^C} e^{-f(X_{n,j})} \right] \right| .
\]
Hence,
\[ |E\left[e^{-\sum_{i\in I_n} f(X_{n,i})} \right] - \prod_{i\in I_n} E\left[e^{-f(X_{n,i})} \right] | \leq \sum_{i=1}^{m_n-1} \varepsilon_i \leq b_{n,1}^m + b_{n,2}^m + b_{n,3}(f). \]

\[\square\]

**Remark 2.2.** Recall, \((X_{n,i}^* : i \in I_n)\) are independent random elements such that for all \(i \in I_n, X_{n,i}^*\) is distributed as \(X_{n,i}\). Further, let \((X_{n,i}^* : i \in I_n)\) and \((X_{n,i} : i \in I_n)\) be defined on the same probability space and independent. We can then bound \(b_{n,3}(f)\) by
\[ b_{n,3}(f) \leq \sum_{i\in I_n} E\left|E\left[e^{-f(X_{n,i}^*)} - e^{-f(X_{n,i}^*)} \mid \sigma(X_{n,j} : j \in \tilde{B}_n^c(j)) \right]\right| \]
\[= \sum_{i\in I_n} E\left|E\left[e^{-f(X_{n,i})} \mid \sigma(X_{n,j} : j \in \tilde{B}_n^c(i)) \right] - E\left[e^{-f(X_{n,i})} \right]\right|. \]

Since for any \(f \in CB_b^+(\mathbb{X})\) the function \(1 - e^{-f}\) is also an element \(CB_b^+(\mathbb{X})\) and further bounded by 1, it follows that
\[ \sum_{i\in I_n} E\left|E\left[f(X_{n,i}) \mid \sigma(X_{n,j} : j \in \tilde{B}_n^c(i)) \right] - E\left[f(X_{n,i}) \right]\right| \to 0 \]
for all \(f \in CB_b^+(\mathbb{X})\) which are bounded by 1 implies that \(b_{n,3}(f) \to 0\) for all \(f \in CB_b^+(\mathbb{X})\).

**Remark 2.3.** The concept of neighborhoods implicitly appears already in Banys [Ban80, Theorem 4]. There, essentially the same sufficient conditions for convergence of \(N_n\) to a Poisson point process are given but with, in our notation, neighborhoods of the form \(\tilde{B}_n(i) = \{i+1, \ldots, i+r_n\}\) and \(\tilde{B}_n^c(i) = \{i+r_n+1, \ldots, m_n\}\) for all \(i \in I_n\) where \((r_n)_{n\in\mathbb{N}}\) is a sequence of non negative integers. The proof is similar to ours and even though it is stated only for the case when \(\mathbb{X}\) is locally compact, it transfers directly to the case of a general Polish space.

**Remark 2.4.** Similar results were also obtained by Schuhmacher [Sch05, Theorem 2.1], but with a completely different approach, using the Chen-Stein method. As a consequence, Schuhmacher even provides bounds on the convergence in the so called Barbour-Brown distance \(d_2\). However, this result does not directly imply our results, see [Sch05, Remark 2.4(b)] for the comparison to the result of Banys [Ban80] which is also relevant to our case.

**Example 2.1.** For Bernoulli random variables \(X_{n,i}\) such that \(\lim_{n\to\infty} \sup_{i\in I_n} P(X_{n,i} = 1) = 0\) and \(\lim_{n\to\infty} \sum_{i\in I_n} P(X_{n,i} = 1) = \lambda \in (0, \infty)\), one can set \(\mathbb{X}' = \{0, 1\}\) and \(\mathbb{X} = K_0 = \{1\}\) for all \(m \in \mathbb{N}\). Using Theorem 2.2 together with Proposition 2.4 and Remark 2.2, we recover the result of Arratia et al. [AGG89, Theorem 1] on convergence in distribution of \(\sum_{i\in I_n} X_{n,i}\) to a Poisson random variable with intensity \(\lambda\), but without the bound on the distance in total variation.

Our main motivation for studying Poisson approximation in a general, possibly infinite-dimensional, state space, actually comes from the problem of obtaining a compound Poisson or Poisson cluster limit for a large class of stationary random fields. This is the content of the following section.
3. Regularly varying random fields. Consider a strictly stationary random field \( X = (X_i : i \in \mathbb{Z}^d) \) of random variables indexed over the \( d \)-dimensional integer lattice. We say that \( X \) is \( (jointly) \) regularly varying with tail index \( \alpha > 0 \) if for every non empty and finite subset of indices \( I \subseteq \mathbb{Z}^d \), the finite-dimensional random vector \( X_I := (X_i : i \in I) \) is multivariate regularly varying with index \( \alpha \). As argued in [BS09], this holds if and only if the random variable \( |X_0| \) is regularly varying with index \( \alpha \) and for every non empty and finite \( I \subseteq \mathbb{Z}^d \) there exists a non zero measure \( \mu_I \in \mathcal{M}(\mathbb{R}^{|I|} \setminus \{0\}) \), with bounded sets in \( \mathbb{R}^{|I|} \setminus \{0\} \) being those which are bounded away from \( 0 \), such that

\[
\frac{\mathbb{P}(u^{-1}X_I \in \cdot)}{\mathbb{P}(|X_0| > u)} \xrightarrow{u} \mu_I, \text{ as } u \to \infty.
\]

For stationary regularly varying time series, i.e. when \( d = 1 \), [BS09] introduced the notion of the tail process, which proved to be very useful in the analysis of asymptotic extremal behavior of such time series. In this section we aim to extend this notion and corresponding results to the case \( d \geq 2 \). The main difference is that in this case there is no obvious notion of the past and the future of the process.

To deal with this, we will consider the lexicographic order on \( \mathbb{Z}^d \). More precisely, for indices \( i = (i_1, \ldots, i_d), j = (j_1, \ldots, j_d) \in \mathbb{Z}^d \), we write \( i < j \) if \( i_k < j_k \) for the first \( k \) where \( i_k \) and \( j_k \) differ. Moreover, if \( i < j \) or \( i = j \), we write \( i \preceq j \).

Note that the lexicographic order \( \preceq \) makes \( \mathbb{Z}^d \) into a totally ordered space for every \( d \in \mathbb{N} \). In particular, any finite subset of \( \mathbb{Z}^d \) has a unique minimal and maximal element with respect to \( \preceq \). Moreover, \( \preceq \) is translation–invariant, i.e. for all \( i, j, k \in \mathbb{Z}^d \), \( i \preceq j \) implies \( i + k \preceq j + k \).

We note that the theory below does not actually depend on an order on \( \mathbb{Z}^d \) as long as it is a total order which is translation–invariant.

Further, we denote by \( \preceq \) the component–wise order, thus for \( i = (i_1, \ldots, i_d), j = (j_1, \ldots, j_d) \in \mathbb{Z}^d \) we write \( i \preceq j \) if \( i_k \leq j_k \) for all \( k = 1, \ldots, d \).

3.1. The (spectral) tail field. Set \( \mathbb{Z}_\preceq^d = \{i \in \mathbb{Z}^d : i \succeq 0\} \), \( X_\preceq = X_{\mathbb{Z}_\preceq^d} \) and consider spaces \( \mathbb{R}^{\mathbb{Z}_\preceq^d} \) and \( \mathbb{R}^{\mathbb{Z}^d} \) with their respective product topologies. The following key result corresponds to [BS09, Theorem 2.1]. The proof is given in Section 6.1.1.

**Theorem 3.1.** For a stationary random field \( X \) and \( \alpha > 0 \), the following three statements are equivalent:

1. \( X \) is regularly varying with tail index \( \alpha > 0 \).
2. There exists a random field \( Y_\preceq = (Y_i : i \succeq 0) \) of random variables with \( \mathbb{P}(|Y_0| \geq y) = y^{-\alpha} \) for \( y \geq 1 \) such that, as \( u \to \infty \),

\[
u^{-1}X_\preceq \mid |X_0| > u \xrightarrow{d} Y_\preceq \quad \text{in } \mathbb{R}^{\mathbb{Z}_\preceq^d}.
\]

3. There exists a random field \( Y = (Y_i : i \in \mathbb{Z}^d) \) of random variables with \( \mathbb{P}(|Y_0| \geq y) = y^{-\alpha} \) for \( y \geq 1 \) such that, as \( u \to \infty \),

\[
u^{-1}X \mid |X_0| > u \xrightarrow{d} Y \quad \text{in } \mathbb{R}^{\mathbb{Z}^d}.
\]
The random field $Y$ is called the tail field or the tail process of $X$. As a consequence of regular variation of $X$, the spectral tail field $\Theta = (\Theta_i : i \in \mathbb{Z}^d)$ of $X$, defined by $\Theta_i = Y_i / |Y_0|$, $i \in \mathbb{Z}^d$ is independent of $|Y_0|$ (see [BS09, Theorem 3.1(i)]), and by continuous mapping theorem,

$$|X_0|^{-1} X \mid |X_0| > u \xrightarrow{d} \Theta \text{ in } \mathbb{R}^{\mathbb{Z}^d}.$$ 

**Remark 3.1.** While writing the paper, we learned of parallel study by Wu and Samorodnitsky [WS18] who also consider regularly varying fields with emphasis on various notions of extremal index in this context. They show by an example that for $d \geq 2$ existence of the limit of $(u^{-1} X_i : i \in J) \mid |X_0| > u$ for $J$ being an orthant in $\mathbb{Z}^d$ is not sufficient for regular variation of $X$ and existence of the tail field. Observe, however, that the proof of Theorem 3.1 shows that it is sufficient to take any $J \subseteq \mathbb{Z}^d$ such that for every non empty and finite $I \subseteq \mathbb{Z}^d$ there exists at least one $i^* \in I$ such that $I - i^* \subseteq J$.

Even though the tail field is typically not stationary, regular variation and stationarity of the underlying random field $X$ yield specific distributional properties of $\Theta$ (and hence of $Y$) summarized by the so-called time-change formula: for every integrable (in the sense that one of the expectations below exists) or non negative measurable function $h : \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}$ and all $j \in \mathbb{Z}^d$,

$$\mathbb{E}[h((\Theta_i)_{i \in \mathbb{Z}^d}) 1\{\Theta_{-j} \neq 0\}] = \mathbb{E}[h((\Theta_{i+j})_{i \in \mathbb{Z}^d}) |\Theta_{j}|^\alpha 1\{\Theta_j \neq 0\}] .$$

In the case of time series, (3.2) appears in [BS09] and the proof is easily extended to the case of random fields. Alternatively, one can arrive at (3.2) following the approach of [PS18] who use the so-called tail measure of $X$ introduced in [SO12], see also [DHS17]. In particular, a special, but very useful, case of [PS18, Lemma 2.1] is

$$\mathbb{E}[h((Y_i)_{i \in \mathbb{Z}^d}) 1\{|Y_{-j}| > 1\}] = \mathbb{E}[h((Y_{i+j})_{i \in \mathbb{Z}^d}) 1\{|Y_j| > 1\}] .$$

for all $j \in \mathbb{Z}^d$ and all $h$ as above.

**Remark 3.2.** To verify that $X$ is a regularly varying field with index $\alpha > 0$, it is enough to show that $|X_0|$ is regularly varying random variable with the same index and that there exists a field $\Theta_\geq = (\Theta_i : i \geq 0)$ such that $|X_0|^{-1} X_\geq \mid |X_0| > u \xrightarrow{d} \Theta_\geq$ in $\mathbb{R}^{\mathbb{Z}^d}$, see [BS09, Corollary 3.2]. In that case, the distribution of the whole spectral process $\Theta$ is determined by (3.2) and the tail field of $X$ is given by $Y = Y \Theta$ where $Y$ is independent of $\Theta$ and satisfies $\mathbb{P}(Y \geq y) = y^{-\alpha}$ for $y \geq 1$.

### 3.2. The space for blocks

Denote $J_n = \{1, \ldots, n\}^d \subseteq \mathbb{Z}^d$ for each $n \in \mathbb{N}$, and let $(a_n)$ be a positive sequence of real numbers such that

$$\lim_{n \to \infty} n^d \mathbb{P}(|X_0| > a_n) = 1 .$$

Our goal is to study convergence in distribution of the point processes $\sum_{j \in J_n} \delta_{(j/n, X_j/a_n)}$ in the space $\mathcal{M}_b([0,1]^d \times (\mathbb{R} \setminus \{0\})$ with the sets of interest, i.e. bounded sets, being those which are bounded away from $[0,1]^d \times \{0\}$.
The main idea is not entirely surprising: we break the dependent field \( (X_j : j \in J_n) \) into smaller blocks whose size still tends to \( \infty \), and then consider them as elements of the space of real-valued fields on the \( d \)-dimensional integer lattice vanishing to zero in all directions. This enlargement of the state space will enable us to transform the problem of obtaining a Poisson cluster limit into obtaining a suitable Poisson approximation by an application of Corollary 2.3 from the previous section.

To make these ideas precise, take a sequence of positive integers \( (r_n) \) such that \( \lim_{n \to \infty} r_n = \lim_{n \to \infty} n/r_n = \infty \), let \( k_n = \lfloor n/r_n \rfloor \) and for every \( i \in I_n := \{1, \ldots, k_n\}^d \subseteq \mathbb{Z}^d \) define the block of indices of size \( r_n^d \),

\[
\tag{3.5} J_{n,i} := (j \in \mathbb{Z}^d : (i-1) \cdot r_n + 1 \leq j \leq i \cdot r_n).
\]

Following [BPS18], consider blocks

\[

\tag{3.6} X_{n,i} := X_{J_{n,i}}, \ i \in I_n,
\]

as elements of the space

\[
l_0 = \{ x = (x_i)_{i \in \mathbb{Z}^d} : \lim_{|i| \to \infty} |x_i| = 0 \}
\]

where \(|i| = \max_{k=1,\ldots,d} |i_k| \) for \( i = (i_1, \ldots, i_d) \in \mathbb{Z}^d \). This embedding boils down to concatenating infinitely many zeros around the block \( X_{n,i} \). To develop the asymptotic theory, we actually consider \( X_{n,i} \)'s as elements of the quotient space \( \tilde{l}_0 \) of elements of \( l_0 \) which are shift-equivalent (elements \( (x_i)_i, (y_i)_i \in l_0 \) are shift-equivalent if for some \( j \in \mathbb{Z}^d \), \( (x_{i+j})_i = (y_i)_i \) ), see [BPS18, Section 2].

We equip the space \( l_0 \) with the sup–norm \( \|(x_i)_i\| := \sup_{i \in \mathbb{Z}^d} |x_i| \) and with the corresponding Borel \( \sigma \)-field. Further, it is not difficult to show that the function \( d : \tilde{l}_0 \times \tilde{l}_0 \to [0, \infty) \) defined by

\[
\tag{3.7} d(x, y) = \inf_{j \in \mathbb{Z}^d} \|(x_{i+j})_i - (y_i)_i\|, \ x, y \in \tilde{l}_0
\]

where \((x_i)_i, (y_i)_i \in \tilde{l}_0 \) are arbitrary representatives of the classes \( x \) and \( y \), respectively, is a proper metric on \( \tilde{l}_0 \) and furthermore makes it a separable and complete metric space (cf. [BPS18, Lemma 2.1]).

Now it remains to apply Corollary 2.3 to elements \( X_{n,i}/a_n \) in the space \( \tilde{l}_{0,0} := \tilde{l}_0 \setminus \{0\} \) with bounded sets being those which are bounded away from 0. To show (2.5), one can often apply the following assumption on the random field \( X \). Intuitively speaking, it implies that clustering of extremes is relatively localized and that in a block of size \( r_n^d \), one can find (asymptotically) at most one extreme cluster.

**Assumption 3.2.** There exists a sequence of positive integers \((r_n)_n\) such that \( \lim_{n \to \infty} r_n = \lim_{n \to \infty} n/r_n = \infty \) and for every \( u > 0 \),

\[
\lim_{m \to \infty} \limsup_{n \to \infty} \mathbb{P} \left( \max_{m < |i| \leq r_n} |X_i| > a_n u \mid |X_0| > a_n u \right) = 0.
\]

For each \( n \in \mathbb{N} \), denote by \( J_{r_n} := \{1, \ldots, r_n\}^d = J_{n,1} \) and let \( X_{r_n} := X_{J_{r_n}} = X_{n,1} \) represent the common distribution of blocks \( X_{n,i}, \ i \in I_n \).
Proposition 3.3. Let \((r_n)\) be a sequence of positive integers satisfying Assumption 3.2. Then, as \(n \to \infty\),

\[
k_n^dP(a_n^{-1}X_{r_n} \in \cdot) \xrightarrow{v} \nu(\cdot) = \vartheta \int_0^\infty P(yQ \in \cdot)\alpha y^{-\alpha-1}dy
\]

in \(\mathcal{M}(\hat{l}_0,0)\) with \(\vartheta \in (0,1]\) and random element \(Q\) in \(\hat{l}_0\) being determined by the tail process \(Y\), see (3.9) and (3.10) below.

The previous result is an extension of the case \(d = 1\) shown in [BPS18, Lemma 3.3], and one can prove it in a similar way. However, here we use a slightly different approach.

First of all, following [BPS18], we introduce a very convenient convergence determining family of functions on \(\hat{l}_0,0\). For an element \(x \in \hat{l}_0\) and any \(\delta > 0\) denote by \(x^\delta \in \hat{l}_0\) the equivalence class of the sequence \((x_i1_{\{x_i > \delta\}})_i\), where \((x_i)_i \in l_0\) is an arbitrary representative of \(x\). Let \(\mathcal{F}\) be the family of all functions \(f \in CB_0^+(\hat{l}_0,0)\) such that for some \(\delta > 0\), \(f(x) = f(x^\delta)\) for all \(x \in \hat{l}_0\), where we set \(f(0) = 0\), i.e. \(f\) depends only on coordinates greater than \(\delta\) in absolute value.

It can be shown that such a class \(\mathcal{F}\) is convergence determining in the sense of Definition 2.1 even for general random measures, i.e. random elements in \(\mathcal{M}(\hat{l}_0,0)\), cf. [BP18, Example 4.6]. In particular, \(\mu_n(f) \to \mu(f)\) for all \(f \in \mathcal{F}\) implies that \(\mu_n \to \mu\) in \(\mathcal{M}(\hat{l}_0,0)\).

To shed additional light on (3.8), we note that it is equivalent to \(k_n^dP(\|X_{r_n}\| > a_nu) \to \vartheta u^{-\alpha}\)

with cluster convergence

\[
\|X_{r_n}\|^{-1}X_{r_n} \to Q \quad \text{in} \quad \hat{l}_0,
\]

for all \(u > 0\), see [PS18, Lemma 4.2].

In order to describe \(\vartheta\) and \(Q\), and to prove the previous proposition, we need a couple of auxiliary results, and this is the content of the following subsection.

3.3. Anchoring the tail process. It follows easily (cf. [BS09, Proposition 4.2]) that Assumption 3.2 implies that the tail field \(P(Y \in l_0) = 1\), i.e. that \(P(\lim_{|i| \to \infty}|Y_i| = 0) = 1\). Recall that \(|Y_0| > 1\) so in particular \(\|Y\| > 1\). The theory presented below is inspired by ideas from [PS18].

We say that a measurable function \(A : \{x \in l_0 : \|x\| > 1\} \to \mathbb{Z}^d\) is an anchoring function if

(i) \(A((x_i)_i \in \mathbb{Z}^d) = j\) for some \(j \in \mathbb{Z}^d\) implies that \(|x_j| > 1\);
(ii) For each \(j \in \mathbb{Z}^d\), \(A((x_i)_i) = A((x_{i-j})_i) - j\).

In words, \(A\) picks one of finitely many \(x_i\)'s which are larger than one in absolute value in a way which is insensitive to translations. For example, valid anchoring functions which are important for our studies are:

- first exceedance: \(A^{fe}((x_i)_i) = \min\{j \in \mathbb{Z}^d : |x_j| > 1\}\),
- last exceedance: \(A^{le}((x_i)_i) = \max\{j \in \mathbb{Z}^d : |x_j| > 1\}\),
- first maximum: \(A^{fm}((x_i)_i) = \min\{j \in \mathbb{Z}^d : |x_j| = \|(x_i)_i\|\},\)
where min and max on $\mathbb{Z}^d$ are taken with respect to the lexicographic order $\preceq$. Note, here one can replace $\preceq$ with an arbitrary total order on $\mathbb{Z}^d$ which is translation-invariant.

**Lemma 3.4.** Assume that $\mathbb{P}(Y \in l_0) = 1$. Then for every anchoring function $A$
\[ \mathbb{P}(A(Y) = 0) > 0. \]

**Proof.** Assume that $\mathbb{P}(A(Y) = 0) = 0$. Applying (3.3) we obtain
\[
1 = \sum_{j \in \mathbb{Z}^d} \mathbb{P}(A(Y) = j) = \sum_{j \in \mathbb{Z}^d} \mathbb{P}(A(Y) = j, |Y_j| > 1)
\]
\[
= \sum_{j \in \mathbb{Z}^d} \mathbb{P}(A((Y_{i-j})_i) = j, |Y_{-j}| > 1) = \sum_{j \in \mathbb{Z}^d} \mathbb{P}(A(Y) = 0, |Y_{-j}| > 1) = 0.
\]

Hence, $\mathbb{P}(A(Y) = 0) > 0$. □

Whenever $\mathbb{P}(A(Y) = 0) > 0$, one can consider the anchored tail process $Z^A = (Z^A_i : i \in \mathbb{Z}^d)$ which has the same distribution as $Y = (Y_i)_i$ but conditionally on $A(Y) = 0$. Also, define $Q^A = (Q^A_i : i \in \mathbb{Z}^d)$ by $Q^A_i = Z^A_i / \|Z^A\|.$

**Lemma 3.5.** Assume that $\mathbb{P}(A(Y) = 0) > 0$ and let $A, A'$ be two anchoring functions. Then
\[ \mathbb{P}(A(Y) = 0) = \mathbb{P}(A'(Y) = 0), \]
and for every measurable and bounded function $h : l_0 \to \mathbb{R}$ which is shift-invariant, i.e. $h((x_{i-j})_i) = h((x_i)_i)$ for all $j \in \mathbb{Z}^d$,
\[ \mathbb{E}[h(Z^A)] = \mathbb{E}[h(Z^{A'})]. \]

**Proof.** Using (3.3) and shift-invariance of $h$ we obtain
\[
\mathbb{E}[h(Y)1 \{A(Y) = 0\}] = \sum_{j \in \mathbb{Z}^d} \mathbb{E}[h(Y)1 \{A(Y) = 0, A'(Y) = j, |Y_j| > 1\}]
\]
\[
= \sum_{j \in \mathbb{Z}^d} \mathbb{E}[h(Y)1 \{A(Y) = -j, A'(Y) = 0\}]
\]
\[
= \mathbb{E}[h(Y)1 \{A'(Y) = 0\}].
\]

Taking $h \equiv 1$ yields the first statement, and then the second one follows immediately. □

Define the common value of $\mathbb{P}(A(Y) = 0)$ by $\vartheta$. In particular,
\[
\vartheta = \mathbb{P}(\sup_{j < 0} |Y_j| \leq 1) = \mathbb{P}(\sup_{j > 0} |Y_j| \leq 1) = \mathbb{P}(A^{fm}(Y) = 0).
\]

(3.9)
Furthermore, since every continuous and bounded function $h : \tilde{l}_0 \to \mathbb{R}$ induces a continuous and bounded function on $l_0$ which is shift–invariant, the previous result implies that distribution of $Z^A$ (and hence of $Q^A$), when viewed as an element in $\tilde{l}_0$, does not depend on the anchoring function $A$. Therefore, we can denote by $Z$ and $Q$ random elements in $\tilde{l}_0$ so that

\[(3.10)\quad Z \overset{d}{=} Z^A, \quad \text{and} \quad Q \overset{d}{=} Q^A,\]

for an arbitrary anchoring function $A$.

Note that, conditionally on $A^m(Y) = 0$, $\|Y\|$ is equal to $|Y_0|$ and therefore $Y/\|Y\| = \Theta$. Consequently, $\mathbb{P}(\|Z\| \geq y) = y^{-\alpha}$ for all $y \geq 1$, $\|Z\|$ is independent of $Q$ in $\tilde{l}_0$, and since $A^m(Y) = 0$ if and only if $A^m(\Theta) = 0$,

\[(3.11)\quad Q \overset{d}{=} \Theta \mid A^m(\Theta) = 0 \quad \text{in } \tilde{l}_0,\]

cf. the discussion after [PS18, Proposition 3.6]. Observe here that the function $A^m$ remains well defined on the whole set $l_0$ without $0$.

**Example 3.1.** Let $(\xi_i : i \in \mathbb{Z}^d)$ be i.i.d. random variables with regularly varying distribution with index $\alpha > 0$. Consider the infinite order moving average process $X = (X_i : i \in \mathbb{Z}^d)$ defined by

\[X_i = \sum_{j \in \mathbb{Z}^d} c_j \xi_{i-j},\]

where $(c_j : j \in \mathbb{Z}^d)$ is a field of real numbers satisfying $0 < \sum_{j \in \mathbb{Z}^d} |c_j|^\delta < \infty$, for some $\delta > 0$ such that $\delta < \alpha$ and $\delta \leq 1$. It is easily shown (see e.g. [Res87, Section 4.5]) that this condition ensures that the series above is absolutely convergent. Note also that $\sum_{j \in \mathbb{Z}^d} |c_j|^\alpha < \infty$.

Furthermore, it can be proved as in [Res87, Lemma 4.24] that

\[(3.12)\quad \lim_{u \to \infty} \frac{\mathbb{P}(|X_0| > u)}{\mathbb{P}(|\xi_0| > u)} = \sum_{j \in \mathbb{Z}^d} |c_j|^\alpha.\]

Moreover, one can show that the stationary field $X$ is jointly regularly varying with index $\alpha$ and spectral tail field given by

\[(\Theta_i)_{i \in \mathbb{Z}^d} \overset{d}{=} (Kc_i + J/|c_i|)_{i \in \mathbb{Z}^d}\]

where $K$ is a $\{-1,1\}$–valued random variable with distribution equal to the spectral measure of $\xi_0$ and $J$ an $\mathbb{Z}^d$–valued random variable, independent of $K$, such that $\mathbb{P}(J = j) = |c_j|^\alpha / \sum_{i \in \mathbb{Z}^d} |c_i|^\alpha$ for all $j \in \mathbb{Z}^d$, cf. Meinguet and Segers [MS10, Example 9.2]. In particular, choosing $A^m$ as the anchoring function (see (3.9) and (3.11)), it follows immediately that

\[(3.13)\quad \vartheta = \mathbb{P}(A^m(\Theta) = 0) = \frac{\max_{j \in \mathbb{Z}^d} |c_j|^\alpha}{\sum_{j \in \mathbb{Z}^d} |c_j|^\alpha}, \quad Q \overset{d}{=} \left(\frac{Kc_j}{\max_{i \in \mathbb{Z}^d} |c_i|}\right)_{j \in \mathbb{Z}^d} \quad \text{in } \tilde{l}_0.\]

We are now in position to prove Proposition 3.3 and for readability, the proof is deferred to Section 6.1.2.
3.4. Convergence to a compound Poisson process. Let $\mathcal{F}'$ be the family of all functions $f \in CB_0^+([0,1]^d \times \tilde{l}_0,0)$ such that for some $\delta > 0$, $f(t,x) = f(t,x\delta)$ for all $t \in [0,1]^d$ and $x \in \tilde{l}_0$, where we set $f(t,0) = 0$; see the definition of the family $\mathcal{F}$ given after Proposition 3.3. Again, $\mathcal{F}'$ is convergence determining in the sense of Definition 2.1 (cf. [BP18, Example 4.6]).

Building on the last two subsections one can rephrase Corollary 2.3 and generalize [BPS18, Theorem 3.6] to $d$–dimensional setting. The proof is essentially the same, and therefore omitted.

**Theorem 3.6.** Let $X$ be a stationary regularly varying random field with tail index $\alpha > 0$ such that for some sequence $(r_n)$, Assumption 3.2 holds and the family $\{(i/k_n,X_{n,i}/a_n) : n \in \mathbb{N}, i \in I_n \}$ is $AI(\mathcal{F}')$.

Then

\[(3.14) \quad N_n' = \sum_{i \in I_n} \delta(i/k_n,X_{n,i}/a_n) \xrightarrow{d} N' = \sum_{i \in \mathbb{N}} \delta(T_i,P_i(Q_i^j)) \]

in $\mathcal{M}_p([0,1]^d \times \tilde{l}_0,0)$, where $N' \overset{d}{=} \text{PPP}(\text{Leb} \times \nu)$ and

(i) $\sum_{i \in \mathbb{N}} \delta(T_i,P_i)$ is a Poisson point process on $[0,1]^d \times (0,\infty)$ with intensity measure $\text{Leb} \times d(-\vartheta y^{-\alpha})$;

(ii) $(Q_i^j)_{j \in \mathbb{Z}^d}, i \in \mathbb{N}$ is a sequence of i.i.d. elements in $\tilde{l}_0$, independent of $\sum_{i \in \mathbb{N}} \delta(T_i,P_i)$ and with common distribution equal to the distribution of $Q$.

By applying continuous mapping theorem to the convergence in (3.14) and using similar arguments as in [Kri10, Proposition 1.34], one obtains Theorem 1.1 which is a generalization of [BT16, Theorem 3.1] to $d$–dimensions; the details are omitted.

**Corollary 3.7.** Under notation of Theorem 3.6, if there exists a sequence $r_n \to \infty, r_n/n \to 0$ for which (3.14) holds, then, with $J_n = \{1,\ldots,n\}^d$,

\[\sum_{j \in J_n} \delta(j/n,X_{j}/a_n) \xrightarrow{d} \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{Z}^d} \delta(T_i,P_i,Q_i^j)\]

in $\mathcal{M}_p([0,1]^d \times (\mathbb{R} \setminus \{0\})$) with bounded sets being those which are bounded away from $[0,1]^d \times \{0\}$.

Observe, this is a convergence result for point processes on a simpler state space and in this way one loses the information about the structure of the cluster in the limit, see [BPS18] for more details.

The assumptions of Theorem 3.6 are straightforward to check in the case of $m$–dependent stationary fields, see Section 6.1.2 for the definition of $m$–dependence. In general, however, checking these assumptions is not trivial. To assure the asymptotic independence condition, one can impose some form of mixing on the field $X$ from the literature, see [BP18] for the case of time series. However, such an approach proves to be too restrictive in the case of random fields, for $d > 1$ that is, cf. Theorem 2.1 in [Bra05], so we will not pursue such an approach here.
Still, one can extend convergence in (3.14) to fields which can be approximated by m-dependent fields, such as spatial infinite order moving average processes from Example 3.1, see Section 5. But first, we show that Theorem 3.6 can be applied to the random field of scores arising in the local sequence alignment problem. In particular, this is an example of a field with a non-trivial dependence structure, but for which the asymptotic independence property holds. To show this, we apply Proposition 2.4 and for convenience, we rephrase it in this setting and in the form suitable for our needs.

**Corollary 3.8.** Let for each \( n \in \mathbb{N} \), \( (X_{n,i} : i \in I_n) \) be identically distributed random elements in \( \tilde{I}_0 \) and such that for all \( \epsilon > 0 \),

\[
\limsup_{n \to \infty} k_n^d \mathbb{P}(\|X_{n,1}\| > a_n \epsilon) < \infty. \tag{3.15}
\]

If there exists a neighborhood structure \( (B_n(i) : n \in \mathbb{N}, i \in I_n) \) such that, denoting \( \|B_n\| = \max_{i \in I_n} |B_n(i)| \),

(i) As \( n \to \infty \), \( \|B_n\|/k_n^d \to 0 \) and for all \( \epsilon > 0 \),

\[
k_n^d \|B_n\| \max_{i \in I_n} \mathbb{P}(\|X_{n,i}\| > a_n \epsilon, \|X_{n,j}\| > a_n \epsilon) \to 0; \tag{3.16}
\]

(ii) For \( n \) big enough, \( X_{n,i} \) is independent of \( \sigma(X_{n,j} : j \notin B_n(i)) \) for each \( i \in I_n \).

Then the family \( ((i/k_n, X_{n,i}/a_n) : n \in \mathbb{N}, i \in I_n) \) is \( AI(F') \).

**Proof.** First, observe that for any sequence \( \epsilon_m \downarrow 0 \) sets \( K'_m = [0,1]^d \times \{x \in \tilde{I}_{0,0} : \|x\| > \epsilon_m\} \), \( m \in \mathbb{N} \), form a base for the family of bounded sets of \( [0,1]^d \times \tilde{I}_{0,0} \). Next, regardless of ordering of \( I_n = \{1, \ldots, k_n\}^d, |B_n(i)| \leq |B_n(\tilde{i})| \) for all \( i \in I_n \). Since \( X_{n,i} \)'s are identically distributed, \( b^m_{n,1} = \sum_{i \in I_n} \sum_{j \in B_n(i)} \mathbb{P}((i/k_n, X_{n,i}/a_n) \in K'_m) \cdot \mathbb{P}((j/k_n, X_{n,j}/a_n) \in K'_m) \leq \sum_{n=1}^{k_n^d} \mathbb{P}(\|X_{n,1}\| > a_n \epsilon_m)^2 \).

In view of (3.15), \( \limsup_{n \to \infty} b^m_{n,1} \leq \text{(const.)} \limsup_{n \to \infty} \|B_n\|/k_n^d = 0 \) for all \( m \in \mathbb{N} \). Similarly, (3.16) implies that \( \limsup_{n \to \infty} b^m_{n,2} = 0 \) for all \( m \in \mathbb{N} \), and by (ii), \( b_{n,3}(f) = 0 \) for every measurable function \( f \geq 0 \) on \( [0,1]^d \times \tilde{I}_{0,0} \) and \( n \) big enough. Applying Proposition 2.4 finishes the proof. \( \square \)

**4. Sequence alignment problem.** In this section we use the notation from Section 1.2 and assume throughout that Assumptions 1.2 and 1.3 hold. Recall, \( (A_i)_{i \in \mathbb{Z}} \) and \( (B_i)_{i \in \mathbb{Z}} \) are independent i.i.d. sequences, \( S^m_{i,j} = \sum_{k=0}^{m-1} s(A_{i-k}, B_{j-k}) \) for \( i, j \in \mathbb{Z} \) and \( m \geq 0 \), and \( S_{i,j} = \sup\{S^m_{i,j} : m \geq 0\} \) for \( i, j \in \mathbb{Z} \).

For some of the key technical results in our analysis we are indebted to Hansen [Han06] who even allows sequences \( (A_i) \) and \( (B_i) \) to be Markov chains. In the i.i.d. setting coresponding proofs, which rely on change of measure arguments, are much less involved. For an alternative approach based on combinatorial arguments see Dembo et. al. [DKZ94b].
4.1. The tail field. Consider a positive stationary field \( X = (X_{i,j} : i, j \in \mathbb{Z}) \) defined by

\[
X_{i,j} = e^{S_{i,j}}, \; i, j \in \mathbb{Z}.
\]

Observe that by (1.10), for \( \alpha^* > 0 \) satisfying \( E[e^{\alpha^*s(A,B)}] = 1, \)

\[
P(X_{i,j} > u) \sim C u^{-\alpha^*}, \; \text{as} \; u \to \infty,
\]

i.e. marginal distributions of \( X \) are regularly varying. More importantly, the transformed field \( X \) fits into the framework of Section 3.

**Proposition 4.1.** The field \( X \) is regularly varying with tail index \( \alpha^* \) and with the spectral tail field \( \Theta = (\Theta_{i,j} : i, j \in \mathbb{Z}) \) satisfying

(i) \( \Theta_{i,j} = 0 \) for \( i, j \in \mathbb{Z}, \; i \neq j \).

(ii) \( \Theta_{m,m} = e^{S_m^e} \) for \( m \in \mathbb{Z} \), where \( S_0^e = 0 \) and

\[
S_m^e = \sum_{i=1}^{m} \varepsilon_i, \; \text{for} \; m \geq 1 \quad \text{and} \quad S_m^e = -\sum_{i=1}^{-m} \varepsilon_i^*, \; \text{for} \; m \leq -1,
\]

for independent i.i.d. sequences \( (\varepsilon_i)_{i \geq 1} \) and \( (\varepsilon_i^*)_{i \geq 1} \) whose distributions correspond to the distributions of \( s(A,B) \) under the product measure \( \mu_A \times \mu_B \) and under the tilted measure \( \mu^* \) from (1.5), respectively.

The tail field \( Y = (Y_{i,j})_{i,j \in \mathbb{Z}} \) of \( X \) is therefore given by \( Y_{i,j} = Y \Theta_{i,j} \) where \( Y \) satisfies \( P(Y \geq y) = y^{-\alpha^*} \) for \( y \geq 1 \) and is independent from \( \Theta \). Observe, \( E[\varepsilon_1] = E[s(A,B)] < 0 \) and since the moment generating function \( m(\alpha) = E[e^{\alpha s(A,B)}] \) is strictly convex and \( m(0) = m(\alpha^*) = 1, \)

\[E[\varepsilon^*_1] = E[s(A,B)e^{\alpha^* s(A,B)}] = \frac{dm}{d\alpha}(\alpha^*) > 0.\]

This implies that \( P(\lim_{|m| \to \infty} S_m^e = -\infty) = 1 \) so \( \Theta \) and \( Y \) are elements of \( l_0 \) almost surely. In particular, by (3.9),

\[
0 < \vartheta = P\left( \sup_{(i,j) \neq (0,0)} Y_{i,j} \leq 1 \right) = P(\sup_{m \geq 1} \Theta_{m,m} \leq 1) = P(\log Y + \sup_{m \geq 1} S_m^e \leq 0),
\]

where, notice, \( \log Y \) is a standard exponential random variable with index \( \alpha^* \). Also, by (3.11), the distribution of the random element \( Q = (Q_{i,j})_{i,j \in \mathbb{Z}} \) in \( l_0 \) is determined by:

\[
Q_{i,j} = 0 \; \text{for} \; i \neq j, \; (Q_{m,m})_{m \in \mathbb{Z}} \overset{d}{=} \left( e^{S_m^e}, \; m \in \mathbb{Z} \mid \sup_{m \leq -1} S_m^e < 0, \; \sup_{m \geq 1} S_m^e \leq 0 \right).
\]

To prove Proposition 4.1 we need two auxiliary lemmas. The first one is a rough estimate using Markov inequality, see Section 6.2 for the proof.

**Lemma 4.2.** There exist a constant \( c_0 > 0 \) such that

\[
\lim_{u \to \infty} e^{2\alpha^*u} P\left( \max_{m > c_0 u} S_{0,0}^m \geq 0 \right) = 0.
\]
Before we state the second lemma, observe first that, using $\mathbb{E}[e^{s(A,B)}] = 1$, for all $u \geq 0$ and any integer $m \geq 0$,

$$
\mathbb{P}(S_{0,0}^m \geq u) = \mathbb{E}[e^{-\alpha^* S_{0,0}^m} e^{\alpha^* S_{0,0}^m} 1\{S_{0,0}^m \geq u\}] \leq e^{-\alpha^* u} \mathbb{P}^* (S_{0,0}^m \geq u) \leq e^{-\alpha^* u},
$$

where the tilted measure $\mathbb{P}^*$ makes pairs $(A_k, B_k)$ for $k = -m + 1, \ldots, 0$, independent and distributed according to the measure $\mu^*$. The following result is proved in [Han06, Lemma 5.11] by change of measure arguments and the Azuma-Hoeffding inequality for martingales. The key fact is that, whenever $\mu^* \neq \mu_A^* \times \mu_B^*$ (which holds under (1.6)),

$$
\mathbb{E}_{\nu_A \times \nu_B} [s(A, B)] < \mathbb{E}_{\mu^*} [s(A, B)]
$$

for all $\nu_A \in \{\mu_A, \mu_A^*\}$ and $\nu_B \in \{\mu_B, \mu_B^*\}$, where $\mathbb{E}_{\nu}$ denotes expectation assuming $(A, B)$ is distributed according to $\nu$, see [DKZ94a, beginning of Section 3].

**Lemma 4.3 ([Han06, Lemma 5.11]).** There exists an $0 < \epsilon_0 < 1$ such that for all $u > 0$,

$$
\sup_{i,j \in \mathbb{Z}, i \neq j, m,l \geq 0} \mathbb{P}(S_{0,0}^m > u, S_{i,j}^l > u) \leq 2e^{-(1+\epsilon_0)\alpha^* u}.
$$

**Proof of Proposition 4.1.** Let $\Theta$ be from the statement of the proposition. We first show that

$$
X_{0,0}^{-1} X_I \mid X_{0,0} > u \xrightarrow{d} \Theta_I,
$$

for all $I \subseteq \mathbb{Z}^2 \setminus \{(m,m) : m \leq -1\}$. Since $X_{0,0}$ is regularly varying with index $\alpha^*$, this will prove the regular variation property of $X$ and show that the spectral tail field $\Theta' = (\Theta_{i,j})_{i,j \in \mathbb{Z}}$ of $X$ satisfies

$$
(\Theta_{i,j} : (i,j) \in \mathbb{Z}^2 \setminus \{(m,m) : m \leq -1\}) \xrightarrow{d} (\Theta_{i,j} : (i,j) \in \mathbb{Z}^2 \setminus \{(m,m) : m \leq -1\}),
$$

see Remark 3.2.

Observe, by (1.9), for each $m \geq 1$,

$$
X_{m,m} = \max\{X_{m-1,m-1} e^{s(A_m,B_m)}, 1\}.
$$

Now since $X_{0,0}$ is regularly varying and independent of the i.i.d. sequence $(e^{s(A_k,B_k)})_{k \geq 1}$, [Seg07, Theorem 2.3] implies that for all $m \geq 0$, as $u \to \infty$,

$$
X_{0,0}^{-1} (X_{0,0}, X_{1,1}, \ldots, X_{m,m}) \mid X_{0,0} > u \xrightarrow{d} (1, e^{s(A_1,B_1)}, \ldots, \prod_{k=1}^m e^{s(A_k,B_k)})
$$

$$
\xrightarrow{d} (\Theta_{0,0}, \Theta_{1,1}, \ldots, \Theta_{m,m}).
$$

Since $\Theta_{i,j} = 0$ for all $i,j \in \mathbb{Z}, i \neq j$, (4.5) will follow if we show that for all such $i,j$,

$$
\mathbb{P}(X_{i,j} > X_{0,0} \eta \mid X_{0,0} > u) \leq \mathbb{P}(X_{i,j} > u \eta \mid X_{0,0} > u)
$$

$$
= \mathbb{P}(S_{i,j} > \log u + \log \eta \mid S_{0,0} > \log u) \to 0, \text{ as } u \to \infty,
$$

for all $\eta \in (0,1)$. 

Fix now $i,j \in \mathbb{Z}$ such that $i \neq j$. Using (1.10) and Lemmas 4.2 and 4.3, for every $M \geq 0$,

$$\limsup_{u \to \infty} \mathbb{P}(S_{i,j} > u - M \mid S_{0,0} > u) = \limsup_{u \to \infty} C^{-1} e^{\alpha u} \mathbb{P}(S_{0,0} > u, S_{i,j} > u - M)$$

$$\leq \limsup_{u \to \infty} C^{-1} e^{\alpha u} \mathbb{P}
\left(\max_{1 \leq m \leq \delta_0 u} S_{0,0}^m > u - M, \max_{1 \leq l \leq \delta_0 u} S_{i,j}^l > u - M\right)$$

$$\leq \limsup_{u \to \infty} 2C^{-1} e^{(1+\epsilon)\alpha M} e^{-\epsilon \alpha u} = 0,$$

hence (4.7) holds.

Finally, we extend (4.6) to equality in distribution on whole $\mathbb{R}^{\mathbb{Z}^2}$. First, fix $m \geq 1$ and note that by (3.2) and $\mathbb{E}[e^{\alpha s(A,B)}] = 1$,

$$\mathbb{P}(\Theta_{-m,-m} > 0) = \mathbb{E}[\Theta_{m,m}^{\alpha}] = 1.$$

Further, for arbitrary bounded measurable function $h : \mathbb{R}^{2m+1} \to \mathbb{R}$, using (3.2) and (4.6),

$$\mathbb{E}[h(\Theta_{-m,-m}^\prime, \ldots, \Theta_{m,m}^\prime)] = \mathbb{E}[h(\Theta_{m,m}^{-1}(\Theta_{0,0}, \ldots, \Theta_{2m,2m})) \Theta_{m,m}^{\alpha}]$$

$$= \mathbb{E}[h(e^{-\sum_{k=1}^m \varepsilon_k}, e^{-\sum_{k=2}^m \varepsilon_k}, \ldots, e^{-\sum_{k=m+1}^m \varepsilon_k}, 1, e^{\varepsilon_{m+1}}, \ldots, e^{\sum_{k=m+1}^{2m} \varepsilon_k}) \prod_{k=1}^m e^{\alpha \varepsilon_k}].$$

By definition of $(\Theta_k)_{k \in \mathbb{Z}}$, this implies that

$$\mathbb{E}[h(\Theta_{-m,-m}^\prime, \ldots, \Theta_{m,m}^\prime)] = \mathbb{E}[h(\Theta_{-m,-m}, \ldots, \Theta_{m,m})].$$

$\square$

4.2. Checking assumptions of Theorem 3.6. In view of (4.1), define the sequence $(a_n)$ by

$$a_n = (Cn^2)^{1/\alpha^*}, \quad n \in \mathbb{N},$$

so that $\lim_{n \to \infty} n^2 \mathbb{P}(X_{0,0} > a_n) = 1$. The proof of the following result is postponed to Section 6.2.2.

**Proposition 4.4.** The random field $X$ satisfies Assumption 3.2 for every sequence of positive integers $(r_n)$ such that $\lim_{n \to \infty} r_n = \infty$ and $\lim_{n \to \infty} r_n/n^\epsilon = 0$ for all $\epsilon > 0$.

Take now sequences of positive integers $(l_n)$ and $(r_n)$ such that

$$\lim_{n \to \infty} \log n/l_n = \lim_{n \to \infty} l_n/r_n = \lim_{n \to \infty} r_n/n^\epsilon = 0$$

for all $\epsilon > 0$ and set $k_n = \lfloor n/r_n \rfloor$. Recall blocks of indices $J_{n,i} \subseteq \{1, \ldots, k_n r_n\}^2$ of size $r_n^2$ from (3.5) and blocks $X_{n,i} := X_{J_{n,i}}$ for $i \in I_n := \{1, \ldots, k_n\}^2$. To show that $X_{n,i}$’s satisfy the asymptotic independence condition from Theorem 3.6, we will apply Corollary 3.8. However, to use it we first need to alter the original blocks.

First, cut off edges of $J_{n,i}$’s by $l_n$, more precisely, define

$$\tilde{J}_{n,i} := \{(i,j) : (i-1) \cdot r_n + 1 \leq (i,j) \leq i \cdot r_n - l_n \cdot 1\}, \quad i \in I_n.$$
Further, for all $i,j \in \mathbb{Z}$ and $m \in \mathbb{N}$ let $\varepsilon_{i,j}^m$ be the empirical measure on $E^2$ of the sequence $(A_{i-k}, B_{j-k}), k = 0, \ldots, m - 1$, i.e.

$$\varepsilon_{i,j}^m = \sum_{k=0}^{m-1} \delta_{(A_{i-k}, B_{j-k})}.$$  

Notice, the score $S_{i,j}^m = \sum_{k=0}^{m-1} s(A_{i-k}, B_{j-k})$ is then equal to the integral of the score function $s$ w.r.t. $\varepsilon_{i,j}^m$. Further, for $\eta > 0$ denote by $B_\eta$ the set of all probability measures $\nu$ on $E^2$ such that $\|\nu - \mu^*\| := \sum_{a,b \in E} |\nu(a, b) - \mu^*(a, b)| < \eta$.

Let $b_n = \log a_n$, then for all $\eta > 0$ and $i, j \in \mathbb{Z}$ define the random variable $\tilde{S}_{i,j} = \tilde{S}_{i,j}(n, \eta)$ by

$$\tilde{S}_{i,j} = \max\{S_{i,j}^m : 1 \leq m \leq c_0 b_n, \varepsilon_{i,j}^m \in B_\eta\}$$

with $c_0 > 0$ from Lemma 4.2 and $\max \emptyset := 0$, and define modified blocks $\tilde{X}_{n,i} = \tilde{X}_{n,i}(\eta)$ in $\tilde{I}_0$ by

$$\tilde{X}_{n,i} = (e^{\tilde{S}_{i,j}} : (i, j) \in \tilde{J}_{n,i}).$$

It turns out that by restricting to the $\tilde{X}_{n,i}$’s one does not lose any relevant information. To understand the role of $\tilde{X}_{n,i}$’s, observe that for any nonnegative and measurable function $f$ on $[0, 1]^2 \times \tilde{I}_0$, we have

$$E \left[ e^{-\sum_{a \in I_0} f(i/k_n, X_{n,i}/a_n)} \right] - \prod_{i \in I_0} E \left[ e^{-f(i/k_n, X_{n,i}/a_n)} \right] \leq \left| E \left[ e^{-\sum_{a \in I_0} f(i/k_n, X_{n,i}/a_n)} \right] - E \left[ e^{-\sum_{a \in I_0} f(i/k_n, \tilde{X}_{n,i}/a_n)} \right] \right| + \left| \prod_{i \in I_0} E \left[ e^{-f(i/k_n, X_{n,i}/a_n)} \right] - \prod_{i \in I_0} E \left[ e^{-f(i/k_n, \tilde{X}_{n,i}/a_n)} \right] \right| + \left| E \left[ e^{-\sum_{a \in I_0} f(i/k_n, \tilde{X}_{n,i}/a_n)} \right] - \prod_{i \in I_0} E \left[ e^{-f(i/k_n, \tilde{X}_{n,i}/a_n)} \right] \right| =: I_1 + I_2 + I_3.$$

Recall now the convergence determining family $\mathcal{F}'$ from Section 3.4. The proof of the following result is in Section 6.2.3.

**Lemma 4.5.** For every $\eta > 0$ and every $f \in \mathcal{F}'$, $I_1 + I_2 \to 0$ as $n \to \infty$.

**Remark 4.1.** In particular, since $I_1 \to 0$ for all $f \in \mathcal{F}'$, point processes $\sum_{i \in I_n} \delta_{(i/k_n, \tilde{X}_{n,i})}$, which are based on $\tilde{S}_{i,j}$’s, converge in distribution if and only if point processes $\sum_{i \in I_n} \delta_{(i/k_n, X_{n,i})}$, which are based on $S_{i,j}$’s from (1.8), do, and in that case their limits coincide. Similarly, one can show that the former (and therefore the latter) convergence is equivalent to convergence of point processes of blocks based on nonstationary scores from (1.2). In particular, point process convergence results given below hold even with $S_{i,j}$’s from (1.8) replaced with the ones from (1.2).

By (4.10) and Lemma 4.5, to show that $(i/k_n, X_{n,i}/a_n)$’s are $AI(\mathcal{F}')$, it is sufficient to find at least one $\eta > 0$ such that $I_3 \to 0$ for all $f \in \mathcal{F}'$, i.e. that $(i/k_n, \tilde{X}_{n,i}/a_n)$’s are $AI(\mathcal{F}')$. For that purpose, we apply Corollary 3.8.
For every \( i = (i_1, i_2) \in I_n \) define its neighborhood \( B_n(i) \) by
\[
B_n(i) = \{ j = (j_1, j_2) \in I_n : i_1 = j_1 \text{ or } i_2 = j_2 \}.
\]
Observe, \( |B_n(i)| = 2kn - 1 \) for all \( i \in I_n \) and hence \( \lim_{n \to \infty} ||B_n||/k^2_n = 0 \). Further, by (4.1) for all \( \epsilon > 0 \),
\[
\limsup_{n \to \infty} k^2_n \mathbb{P}(||\mathbf{X}_{n,1}|| > a_n \epsilon) \leq \limsup_{n \to \infty} k^2_n r^2_n \mathbb{P}(\tilde{S}_{0,0} > a_n \epsilon) \\
\leq \limsup_{n \to \infty} k^2_n r^2_n \mathbb{P}(X_{0,0} > a_n \epsilon) = \epsilon^{-\alpha^*} < \infty.
\]
Next, recall that \( S^m_{i,j} = \sum_{k=0}^{m-1} s(A_{i-k}, B_{j-k}) \) so by (4.8), for every \( n \in \mathbb{N} \),
\[
\tilde{S}_{i,j} \in \sigma(A_{i-[c_0 b_n]+1}, \ldots, A_i, B_{j-[c_0 b_n]+1}, \ldots, B_j).
\]
By construction of \( \tilde{J}_{n,i} \)'s and choice of \( (l_n) \) such that, in particular, \( \lim_{n \to \infty} c_0 b_n/l_n = \lim_{n \to \infty} l_n/r_n = 0 \), this implies that, for \( n \) large enough, \( \mathbf{X}_{n,i} \) and blocks \( (\mathbf{X}_{n,j} : j \notin B_n(i)) \) are constructed from completely different sets of \( A_k \)'s and \( B_k \)'s, and therefore independent.

Further, when \( j \in B_n(i), j \neq i \), arbitrary scores \( S^m_{i,j} \) and \( S^l_{i,j} \) which build blocks \( \mathbf{X}_{n,i} \) and \( \mathbf{X}_{n,j} \) respectively (i.e. \( (i,j) \in \tilde{J}_{n,i}, (i',j') \in \tilde{J}_{n,j} \) and \( 1 \leq m,l \leq c_0 b_n \)), for \( n \) large enough, depend on completely different sets of variables from at least one of the sequences \( (A_k) \) or \( (B_k) \). Thus, the following result, which is [Han06, Corollary 5.4], applies. It follows from [Han06, Lemma 5.3] under condition (12) in [Han06], which, when \( (A_i) \) and \( (B_i) \) are i.i.d. sequences, is equivalent to Assumption 1.3, see [Han06, Remark 3.8].

**Lemma 4.6 ([Han06, Corollary 5.4]).** There exist constants \( \epsilon_2, \eta > 0 \) such that for all \( u > 0 \)
\[
\mathbb{P}(S^m_{0,0} > u, S^l_{i,j} > u, \tilde{S}^m_{0,0}, \tilde{S}^l_{i,j} \in B_\eta) \leq e^{-(3/2+\epsilon_2)\alpha^* u}
\]
uniformly over all \( i,j \in \mathbb{Z} \) and \( m,l \in \mathbb{N} \) such that \( \min\{i,j\} < -m+1 \) or \( \max\{i-l,j-l\} > 0 \).

Take now \( \eta > 0 \) from the previous result and recall the corresponding \( \tilde{\mathbf{X}}_{n,i} \)'s. For \( n \) big enough and every \( \epsilon > 0 \) we get that
\[
k^2_n \mathbb{P}(||\mathbf{X}_{n,i}|| > a_n \epsilon, ||\mathbf{X}_{n,j}|| > a_n \epsilon) \leq k^2_n 2k_n r_n^4 \mathbb{P}(c_0 b_n)^2 e^{-(3/2+\epsilon_2)\alpha^* (b_n+\log \epsilon)} \sim (const.) n^2 r_n b_n^2 n^{-3-2\epsilon_2} \to 0,
\]
as \( n \to \infty \), by the choice of \( (r_n) \) and since \( b_n \sim 2 \log n/\alpha^* \).

Hence by Corollary 3.8, for this \( \eta \), blocks \( \tilde{\mathbf{X}}_{n,i} \), and therefore the original blocks \( \mathbf{X}_{n,i} \), satisfy the asymptotic independence condition.

**Theorem 4.7.** Under Assumptions 1.2 and 1.3, for any sequence of positive integers \( (r_n) \) such that \( \lim_{n \to \infty} r_n = \infty \) and \( \lim_{n \to \infty} r_n/n^\epsilon \to 0 \) for all \( \epsilon > 0 \),
\[
\sum_{i \in I_n} \delta(i/k_n, X_{n,i}/(Cn^2)^{1/\alpha^*}) \rightarrow \mathcal{D} \sum_{k \in \mathbb{N}} \delta(T_k, P_k(Q_k))_{i,j \in \mathbb{Z}}
\]
in $\mathcal{M}_p([0,1]^2 \times \tilde{I}_0,0)$ where the limit is described in Theorem 3.6, with $\vartheta > 0$ given by (4.3) and distribution of $Q$ in $\tilde{I}_0$ determined by (4.4).

In particular, application of Corollary 3.7 yields that

$$\sum_{i,j=1}^{n} \delta((i,j)/n, X_{i,j}/(Cn^2)^{1/\alpha}) \xrightarrow{d} \sum_{k \in \mathbb{N} \cup \mathbb{Z}} \delta(T_k, P_k Q_{i,j})$$

in $\mathcal{M}_p([0,1]^2 \times (0,\infty))$. Consider now space $\mathcal{M}_p([0,1]^2 \times \mathbb{R})$ with a set $B \subseteq [0,1]^2 \times \mathbb{R}$ being bounded if $B \subseteq [0,1]^2 \times (x,\infty)$ for some $x \in \mathbb{R}$. It is easy to see that

$$\sum_{k \in \mathbb{N}} \delta_{(tk,x_k)} \mapsto \sum_{k \in \mathbb{N}} \delta_{(tk, log(x_k C^{1/\alpha}))}$$

is a well defined mapping from $\mathcal{M}_p([0,1]^2 \times (0,\infty))$ to $\mathcal{M}_p([0,1]^2 \times \mathbb{R})$ which is also continuous w.r.t. vague topologies on these spaces (see e.g. [Res87, Proposition 3.18]). Theorem 1.4 now follows easily from (4.11) via continuous mapping theorem and standard Poisson process transformation (see e.g. [Res87, Proposition 3.7]).

5. Fields admitting $m$–dependent approximation. Often, a stationary field admits an approximation by a sequence of regularly varying $m$–dependent fields. Recall, a random field $(X_i : i \in \mathbb{Z}^d)$ is $m$–dependent for some $m \in \mathbb{N}$ if for all $I,J \subseteq \mathbb{Z}^d$ such that $\inf\{|i-j| : i \in I, j \in J\} > m$, $\sigma$–algebras $\sigma(X_i : i \in I)$ and $\sigma(X_j : j \in J)$ are independent.

Let $X = (X_i : i \in \mathbb{Z}^d)$ be a stationary random field and assume that there exists a sequence of stationary regularly varying $m$–dependent fields $X^{(m)} = (X^{(m)}_i : i \in \mathbb{Z}^d)$, $m \in \mathbb{N}$, and a sequence of real numbers $(b_n)$ such that for all $m \in \mathbb{N}$

$$n^d \mathbb{P}(|X^{(m)}_0| > b_n) \xrightarrow{d} d^{(m)} > 0.$$ 

Observe that we keep the same normalizing sequence $(b_n)$ for all the fields. In particular, the tail index of $X^{(m)}$ is the same for all $m$. Denote it by $\alpha > 0$. Further, for each $X^{(m)}$ denote by $\vartheta^{(m)}$ the quantity defined in (3.9) and by $Q^{(m)}$ the random element in $\tilde{I}_0$ defined in (3.10).

Assumption 5.1.

(i) There exists an $\sigma > 0$ and a random element $Q$ in $\tilde{I}_0$ such that, as $m \to \infty$, $\vartheta^{(m)} d^{(m)} \to \sigma$ and $Q^{(m)} \xrightarrow{d} Q$ in $\tilde{I}_0$.

(ii) For any $u > 0$

$$\lim_{m \to \infty} \lim_{n \to \infty} \mathbb{P}(\max_{1 \leq i \leq n} |X^{(m)}_{i}-X_i| > b_n u) = 0.$$ 

Notice that, since $\|Q^{(m)}\| = 1$ a.s. for all $m \in \mathbb{N}$, the same holds for $Q$.

Let $(r_n)$ be any sequence of non negative integers satisfying $r_n \to \infty$ and $k_n := \lfloor n/r_n \rfloor \to \infty$. Recall the blocks $X_{n,i}$, $i \in I_n = \{1,\ldots,k_n\}^d$ defined in (3.6).

Theorem 5.2. Assume that $(b_n)$ is a sequence of real numbers satisfying (5.1) and that Assumption 5.1 holds for some $\sigma > 0$ and random element $Q$ in $\tilde{I}_0$. Then

$$N''_n = \sum_{i \in I_n} \delta_{(i/k_n, X_{n,i}/b_n)} \xrightarrow{d} \sum_{i \in \mathbb{N}} \delta_{(T_i, P_i Q^i)}$$
in \(M_p([0,1]^d \times \tilde{l}_{0,0})\), where \(\sum_{i=1}^{\infty} \delta_{(T_i, P_i)}\) is a Poisson point process on \([0,1]^d \times (0, \infty)\) with intensity measure \(\sigma \text{Leb} \times d(-y^{-\alpha})\), independent of the i.i.d. sequence \((Q^i)_{i \in \mathbb{N}}\) with common distribution equal to the distribution of \(Q\).

**Proof.** For every \(X^{(m)}\) denote by \(X^{(m)}_{n,i}\) the blocks from (3.6) and define the point process \(N^{(m)}_n\) on \([0,1]^d \times \tilde{l}_{0,0}\) by

\[
N^{(m)}_n = \sum_{i \in I_n} \delta_{(i/k_n, X^{(m)}_{n,i}/b_n)}.
\]

Notice, for each \(m \in \mathbb{N}\), by (5.1) and regular variation of \(X^{(m)}_0\), the sequence \(a^{(m)}_n := b_n(d^{(m)})^{\frac{1}{n}}\) satisfies (3.4), i.e. \(\lim_{n \to \infty} n^d \mathbb{P}(|X^{(m)}_0| > a^{(m)}_n) = 1\). Since \(X^{(m)}\) is \(m\)-dependent, we can apply Theorem 3.6 and the continuous mapping theorem to show that for every \(m \in \mathbb{N}\),

\[
N^{(m)}_n \xrightarrow{d} N^{(m)} \equiv \text{PPP}(\text{Leb} \times \nu^{(m)})
\]

in \(M_p([0,1]^d \times \tilde{l}_{0,0})\), where

\[
\nu^{(m)}(\cdot) = \nu^{(m)}d^{(m)} \int_0^\infty \mathbb{P}(yQ^{(m)} \leq \cdot) \alpha y^{-\alpha-1} dy.
\]

Further, by Assumption 5.1 (i) and the dominated convergence theorem, as \(m \to \infty\),

\[
\nu^{(m)} \xrightarrow{v} \nu^{(\infty)}(\cdot) = \sigma \int_0^\infty \mathbb{P}(yQ \leq \cdot) \alpha y^{-\alpha-1} dy,
\]

in \(M(\tilde{l}_{0,0})\). This further implies that

\[
\text{PPP}(\text{Leb} \times \nu^{(m)}) \xrightarrow{d} N^{(m)} \xrightarrow{d} N^{(\infty)} \equiv \text{PPP}(\text{Leb} \times \nu^{(\infty)}).
\]

Since the distribution of \(N^{(\infty)}\) coincides with the distribution of the limit in (5.2), by a Slutsky type argument, it suffices to show that for all \(\eta > 0\)

\[
\lim_{m \to \infty} \limsup_{n \to \infty} \mathbb{P}(|N^{(m)}_n(f) - N^{(m)}_n(f)| > \eta) = 0
\]

for every \(f\) in some family of functions on \([0,1]^d \times \tilde{l}_{0,0}\) which is convergence determining in the sense of Definition 2.1.

Recall the metric \(d\) on \(\tilde{l}_0\) defined in (3.7). Adapting the lines of the proof of [DR85, Theorem 2.4, equation (2.1.1)], it follows that Assumption 5.1 (ii) implies (5.3) for every \(f \in LB^+_b([0,1]^d \times \tilde{l}_{0,0}, d')\) where \(d'\) is a metric on \([0,1]^d \times \tilde{l}_{0,0}\) defined by \(d'(t, x, s, y) = |t - s| \vee d(x, y)\) with \(|\cdot|\) denoting the sup–norm on \([0,1]^d\). Since the family \(LB^+_b([0,1]^d \times \tilde{l}_{0,0}, d')\) is convergence determining (see [BP18, Proposition 4.6]), this proves the result.

\(\square\)

Most notable examples of this form are infinite order moving average processes from Example 3.1 which were already studied by [BPS18, Section 3.3] in the case \(d = 1\).
Proof of Theorem 3.1.

6.1. Regularly varying random fields.

6.2. Postponed proofs.

Let $X_i = \sum_{j \in \mathbb{Z}^d} c_j \xi_{i-j}$ for a field of real numbers $(c_j)$ and field of i.i.d. random variables $(\xi_j)$ which are regularly varying with index $\alpha > 0$.

Instead of checking assumptions of Theorem 3.6 for the field $X$, we show that Theorem 5.2 can be elegantly applied. For each $m \in \mathbb{N}$ and $i \in \mathbb{Z}^d$ define

$$X_i^{(m)} = \sum_{|j| \leq m} c_j \xi_{i-j}.$$  

The process $X^{(m)} = (X_i^{(m)} : i \in \mathbb{Z}^d)$ is in general $(2m + 1)$-dependent rather than $m$-dependent but clearly, Theorem 5.2 still applies. Take a sequence of real numbers $(b_n)$ such that $\lim_{n \to \infty} n^d \mathbb{P}(|\xi_0| > b_n) = 1$. It now follows from (3.12) that (5.1) is satisfied with $d^{(m)} = \sum_{|j| \leq m} |c_j|^{\alpha}$. Further, for each $m \in \mathbb{N}$, special case of (3.13) for $c_j = 0$, $|j| > m$ yields

$$\vartheta^{(m)} = \frac{\max_{|j| \leq m} |c_j|^{\alpha}}{\sum_{|j| \leq m} |c_j|^{\alpha}} \quad Q^{(m)} = \left( \frac{K_{c_j} \mathbb{1}_{\{|j| \leq m\}}}{\max_{|j| \leq m} |c_j|} \right)_{j \in \mathbb{Z}^d} \quad \text{in } \tilde{I}_0.$$  

Observe, since $\lim_{|j| \to \infty} |c_j| = 0$, Assumption 5.1 (i) is satisfied for $\sigma = \max_{j \in \mathbb{Z}^d} |c_j|^{\alpha}$ and $Q$ from (3.13). Finally, (3.12) implies that Assumption 5.1 (ii) holds; see [DR85, Lemma 2.3].

Thus, all the conditions of Theorem 5.2 are met and hence convergence in (5.2) holds. Moreover, the limiting point process can be represented as $\sum_{i=1}^{\infty} \delta_{(T_i, P_i, (c_j)_j)}$ where $\sum_{i=1}^{\infty} \delta_{(T_i, P_i)}$ is a Poisson point process on $[0, 1]^d \times (0, \infty)$ with intensity measure $\text{Leb} \times d(-y^{-\alpha})$ and $(K_i)$ is an i.i.d. sequence of random variables distributed as $K$ and independent of $\sum_{i=1}^{\infty} \delta_{(T_i, P_i)}$.

6. Postponed proofs.

6.1. Regularly varying random fields.

6.1.1. Proof of Theorem 3.1. We only prove (ii)$\Rightarrow$(i) since (i)$\Rightarrow$(iii) follows as in [BS09, Theorem 2.1] and (iii)$\Rightarrow$(ii) is obvious. Observe first that (ii) implies that for all $\epsilon > 0$,

$$(6.1) \quad \lim_{u \to \infty} \frac{\mathbb{P}(|X_0| > u \epsilon)}{\mathbb{P}(|X_0| > u)} = \epsilon^{-\alpha},$$  

and moreover that $X_0$ is regularly varying, see [BS09, Theorem 2.1]. Take now an arbitrary finite $I \subseteq \mathbb{Z}^d$ such that $|I| \geq 2$ and for all $\epsilon > 0$ denote $\mu_u(I) = \mathbb{P}(u^{-1}X_I \in \cdot)/\mathbb{P}(|X_0| > u)$. We prove that $X_I$ is multivariate regularly varying by showing that there exists a non zero measure $\mu_I \in \mathcal{M}(\mathbb{R}^{|I|} \setminus \{0\})$ such that $u \mapsto \mu_u \mu_I$ as $u \to \infty$.

Arguing exactly as in [BS09, Theorem 2.1] it follows that the vague limit of $\mu_u^I$, if it exists, is necessarily non–zero, and further that the set $\{\mu_u^I : u > 0\}$ is relatively compact in the vague topology. Let $i^* \in I$ be the minimal element of $I$ w.r.t. the lexicographic order. By [BS09, Lemma 2.2], to show that measures $\mu_u^I$ vaguely converge as $u \to \infty$, it suffices to prove that

$$\lim_{u \to \infty} \mu_u^I(f) = \lim_{u \to \infty} \mu_u^I(f)$$  

exists for all $f \in F$ where $F = \mathcal{F}_1 \cup \mathcal{F}_2 \subseteq CB_k^+(\mathbb{R}^{|I|} \setminus \{0\})$ with

$$\mathcal{F}_1 = \{f : \text{for some } \epsilon > 0, f((x_i)_{i \in I}) = 0 \text{ if } |x_{i^*}| \leq \epsilon\},$$  

$$\mathcal{F}_2 = \{f : f((x_i)_{i \in I}) \text{ does not depend on } x_{i^*}\}.$$
Note that families $\mathcal{F}_1$ and $\mathcal{F}_2$ depend on $I$ but we omit this in the notation.

Denote $I' := I - i^*$ so in particular, $I' \subseteq \mathbb{Z}_d$. Stationarity, (ii) and (6.1) now imply that for all $f \in \mathcal{F}_1$

$$\mu_u^I(f) \to \epsilon^{-\alpha} \mathbb{E}[f(\epsilon Y_{I'})], \text{ as } u \to \infty.$$ 

Further, every $f \in \mathcal{F}_2$ naturally induces a function $\tilde{f}$ in $CB_0^+(\mathbb{R}^{|I'|-1} \setminus \{0\})$ and by stationarity

$$\mu_u^I(f) = \mathbb{E}[\tilde{f}(u^{-1} X_{I \setminus \{i^*\}})] = \mathbb{E}[\tilde{f}(u^{-1} X_{I \setminus \{i^*\}})](\tilde{f}),$$

so $\lim_{u \to \infty} \mu_u^I(f)$ exists for all $f \in \mathcal{F}_2$ if $X_{I \setminus \{i^*\}}$ is multivariate regularly varying. Hence, for an arbitrary finite $I \subseteq \mathbb{Z}_d$ such that $|I| \geq 2$, $X_I$ is multivariate regularly varying if $X_{I \setminus \{i^*\}}$ is, where $i^* \in I$ is the minimal element of $I$ w.r.t. the lexicographic order. Therefore, (i) now follows by regular variation of $X_0$.

6.1.2. Proof of Proposition 3.3. It suffices to show that $\lim_{n \to \infty} k_n^d \mathbb{E}[f(a_n^{-1} X_{r_n})] = \nu(f)$ for all $f \in F$. Observe, since $k_n^d = [n/r_n]^d \sim (r_n^d \mathbb{P}(|X_0| > a_n))^{-1}$, it is equivalent to prove that

$$\lim_{n \to \infty} \frac{\mathbb{E}[f(a_n^{-1} X_{r_n})]}{r_n^d \mathbb{P}(|X_0| > a_n)} = \nu(f).$$

We essentially follow the steps of the proof of [BS09, Theorem 2.2]. Recall, $X_{r_n} = X_{J_{r_n}}$ where $J_{r_n} = \{1, \ldots, r_n\}^d$. Take now an arbitrary $f \in F$ and choose $\epsilon > 0$ such that $f(z) = f(z')$ for all $z \in \tilde{I}_0$. In particular, $f$ is non negative, bounded and $\|z\| \leq \epsilon$ implies that $f(z) = 0$. Hence,

$$\mathbb{E}[f(a_n^{-1} X_{r_n})] = \mathbb{E}[f(a_n^{-1} X_{r_n}) \mathbb{1}\{|X_{r_n}| > a_n \epsilon\}],$$

and decomposing on the first $J_{r_n}$ (w.r.t. $\preceq$) for which $|X_j| > a_n \epsilon$ we get

$$\mathbb{E}[f(a_n^{-1} X_{r_n})] = \sum_{j \in J_{r_n}} \mathbb{E}[f(a_n^{-1} X_{r_n}) \mathbb{1}\{\max_{j' \in J_{r_n}, j' \prec j} |X_{j'}| \leq a_n \epsilon, |X_j| > a_n \epsilon\}].$$

Fix now an $m \in \mathbb{N}$ and take $n$ big enough so that $r_n \geq 2m + 1$. For every $j \in J_{r_n}$ such that $\{j': |j' - j| \leq m\} \subseteq J_{r_n}$, intuitively, by Assumption 3.2, when $|X_j| > a_n \epsilon$ we can assume that

$$\max\{|X_{j'}| : j' \in J_{r_n}, |j - j'| > m\} \leq a_n \epsilon,$$

and in this case, by the properties of $f$,

$$f(a_n^{-1} X_{r_n}) = f(a_n^{-1} X_{\{j' \in \mathbb{Z}^d : |j' - j| \leq m\}}).$$

More precisely, for all such $j'$s, using stationarity, boundedness of $f$ (assume w.l.o.g. that $0 \leq f \leq 1$) and the fact that $\{j': j' \in J_{r_n}, |j - j'| > m\} \subseteq \{j': m < |j - j'| \leq r_n\}$,

$$\mathbb{E}[f(a_n^{-1} X_{r_n}) \mathbb{1}\{\max_{j' \in J_{r_n}, j' \prec j} |X_{j'}| \leq a_n \epsilon, |X_j| > a_n \epsilon\}]$$

$$- \mathbb{E}[f(a_n^{-1} X_{\{j' \leq m\}}) \mathbb{1}\{\max_{|j'| \leq m, j' \prec 0} |X_{j'}| \leq a_n \epsilon, |X_0| > a_n \epsilon\}]$$

$$\leq \mathbb{P}\left(\max_{m < |j'| \leq r_n} |X_{j'}| > a_n \epsilon, |X_0| > a_n \epsilon\right).$$
For remaining \( j \in J_{r_n} \), simply bound the term on the left hand side above by \( \mathbb{P}(|X_0| > a_n \epsilon) \).

Now, going back to (6.2) we can conclude that

\[
\Delta_{n,m} := \left| \mathbb{E}[f(a_n^{-1}X_{r_n})]/\{r_n^d\mathbb{P}(|X_0| > a_n) \right| \\
\quad - \mathbb{E}[f(a_n^{-1} \{j'| \leq m \})] \{ \max_{|j'| \leq m, j' < 0} |X_{j'}| \leq a_n \epsilon, |X_0| > a_n \epsilon \} / \mathbb{P}(|X_0| > a_n) \}
\]
\[
\leq \frac{\mathbb{P}(|X_0| > a_n \epsilon)}{\mathbb{P}(|X_0| > a_n)} \left\{ \frac{r_n^d}{(r_n - (r_n - 2m))^d} + \mathbb{P} \left( \max_{m < j' \leq r_n} |X_{j'}| > a_n \epsilon \right) / \mathbb{P}(|X_0| > a_n) \right\}.
\]

Observe, by regular variation \( \mathbb{P}(|X_0| > a_n \epsilon) \sim \epsilon^{-\alpha} \mathbb{P}(|X_0| > a_n) \) as \( n \to \infty \), so together with \( r_n \to \infty \) and Assumption 3.2 we get that

\[
(6.3) \quad \lim_{n \to \infty} \limsup_{m \to \infty} \Delta_{n,m} = 0.
\]

Next, since \( f \), when viewed as a function on \( \mathbb{R}^{d(2m+1)} \), is bounded and continuous, and since \( \mathbb{P}(\max_{j' \leq m} j' < 0 | Y_{j'} | = | Y_0 | \max_{j' \leq m} j' < 0 | \Theta_{j'} | = 1) = 0 \), by definition of the tail process \( Y \) and the continuous mapping theorem,

\[
\lim_{n \to \infty} \mathbb{E}[f(a_n^{-1}X_{\{j' \leq m \}})] \{ \max_{|j'| \leq m, j' < 0} |X_{j'}| \leq a_n \epsilon \} / \mathbb{P}(|X_0| > a_n) = \mathbb{E}[f(\epsilon Y_{\{j' \leq m \}})] \{ \max_{|j'| \leq m, j' < 0} |Y_{j'}| \leq \epsilon \}.
\]

Now since \( \mathbb{P}(\lim_{j' \to \infty} |Y_{j'}| = 0) = 1 \), (6.3) and application of the bounded convergence theorem yield

\[
\lim_{n \to \infty} \frac{\mathbb{E}[f(a_n^{-1}X_{r_n})]}{r_n^d \mathbb{P}(|X_0| > a_n)} = \lim_{m \to \infty} \epsilon^{-\alpha} \mathbb{E}[f(\epsilon Y_{\{j' \leq m \}})] \{ \max_{|j'| \leq m, j' < 0} |Y_{j'}| \leq 1 \}
\]
\[
= \epsilon^{-\alpha} \mathbb{E}[f(\epsilon Y)] \{ \max_{j' < 0} |Y_{j'}| \leq 1 \}
\]
\[
= \epsilon^{-\alpha} \partial \mathbb{E}[f(\epsilon Z^{1/\alpha})] = \epsilon^{-\alpha} \partial \mathbb{E}[f(\epsilon Z)],
\]
where the last equality follows since \( f \) is a function on \( \tilde{l}_{0,0} \). By the discussion before (3.11),

\[
\epsilon^{-\alpha} \partial \mathbb{E}[f(\epsilon Z)] = \epsilon^{-\alpha} \partial \int_{1}^{\infty} \mathbb{E}[f(\epsilon y \mathcal{Q})] \alpha y^{-\alpha - 1} \, dy = \partial \int_{\epsilon}^{\infty} \mathbb{E}[f(y \mathcal{Q})] \alpha y^{-\alpha - 1} \, dy.
\]

Now since \( ||Q|| = 1 \) and since \( f(x) = 0 \) whenever \( ||x|| \leq \epsilon \), we finally obtain that

\[
\lim_{n \to \infty} \frac{\mathbb{E}[f(a_n^{-1}X_{r_n})]}{r_n^d \mathbb{P}(|X_0| > a_n)} = \partial \int_{0}^{\infty} \mathbb{E}[f(y \mathcal{Q})] \alpha y^{-\alpha - 1} \, dy = \nu(f).
\]

6.2. Local sequence alignments.

6.2.1. Proof of Lemma 4.2. By Markov inequality, for any \( \lambda \geq 0 \) and all \( u > 0 \)

\[
\mathbb{P} \left( \max_{m \geq c_0 u} S_{0,0}^m \geq 0 \right) \leq \sum_{l=0}^{\infty} \mathbb{P} \left( S_{0,0}^{[c_0 u]+l} \geq 0 \right) \leq \sum_{l=0}^{\infty} \mathbb{E} \left[ e^{\lambda S_{0,0}^{[c_0 u]+l}} \right] \leq \sum_{l=0}^{\infty} m(\lambda)^{[c_0 u]+l},
\]
where \( m(\lambda) = \mathbb{E}[e^{\lambda s(A, B)}] \) is the moment generating function of \( s(A, B) \). Fix any \( 0 < \lambda_0 < \alpha^* \). By strict convexity of \( m \) and \( m(\alpha^*) = 1, 0 < m(\lambda_0) < 1 \) and in particular

\[
\mathbb{P} \left( \max_{m \geq c_0u} \frac{S_{0,0}^m}{n} \geq 0 \right) \leq e^{c_0u \log m(\lambda_0)} \sum_{l=0}^{\infty} m(\lambda_0)^l.
\]

Since the series above is summable, taking \( c_0 \) strictly larger than \(-2\alpha^*/\log m(\lambda_0)\) finishes the proof.

6.2.2. Proof of Proposition 4.4. Let \((r_n)\) be an arbitrary sequence of positive integers satisfying \( r_n \to \infty \) and \( r_n/n^\epsilon \to 0 \) for all \( \epsilon > 0 \). We have to show that for an arbitrary \( u > 0 \)

\[
(6.4) \quad \lim_{m \to \infty} \limsup_{n \to \infty} \mathbb{P} \left( \max_{m < |(i,j)| \leq r_n} X_{i,j} > a_n u \mid X_{0,0} > a_n u \right) = 0.
\]

We deal with the diagonal elements using arguments from [Bas00, Lemma 4.1.4]. First, notice that by (1.9), for each \( k \geq 1 \) we can decompose

\[
X_{k,k} = \max \{ e^{\max_{0 \leq i \leq k} S_{k,k}^i}, X_{0,0} e^{S_{k,k}^k} \},
\]

with \((S_{k,k}^i)_{0 \leq i \leq k}\) being independent of \( X_{0,0} \). Hence, using stationarity,

\[
\mathbb{P} \left( \max_{m < |k| \leq r_n} X_{k,k} > a_n u \mid X_{0,0} > a_n u \right) \leq 2 \sum_{k=m+1}^{r_n} \mathbb{P}(X_{k,k} > a_n u \mid X_{0,0} > a_n u) \leq 2r_n \mathbb{P}(e^{\max_{0 \leq i \leq r_n} S_{0,0}^i} > a_n u) + 2 \sum_{k=m+1}^{r_n} \mathbb{P}(X_{0,0} e^{S_{k,k}^k} > a_n u \mid X_{0,0} > a_n u).
\]

Since \( r_n/n^2 \to 0 \), the choice of \((a_n)\) and (4.1) imply that

\[
2r_n \mathbb{P}(e^{\max_{0 \leq i \leq r_n} S_{0,0}^i} > a_n u) \leq 2r_n \mathbb{P}(X_{0,0} > a_n u) \to 0, \quad \text{as } n \to \infty.
\]

For the second term, take an arbitrary \( 0 < \lambda_0 < \alpha^* \) so in particular \( 0 < m(\lambda_0) = \mathbb{E}[e^{\lambda_0 s(A, B)}] < 1 \) by strict convexity of \( m \). Apply Markov’s inequality and use independence between \( X_{0,0} \) and \( S_{k,k}^k \) to obtain

\[
\sum_{k=m+1}^{r_n} \mathbb{P}(X_{0,0} e^{S_{k,k}^k} > a_n u \mid X_{0,0} > a_n u) \leq \frac{\mathbb{E}[X_{0,0}^{\lambda_0} \mathbb{1}_{\{X_{0,0} > a_n u\}}]}{(a_n u)^{\lambda_0} \mathbb{P}(X_{0,0} > a_n u)} \sum_{k=m+1}^{r_n} m(\lambda_0)^k.
\]

Variant of Karamata’s theorem (see [BDM16, Appendix B.4], also [BGT87, pp. 26–28]) now implies that

\[
\lim_{m \to \infty} \limsup_{n \to \infty} \mathbb{P} \left( \max_{m < |k| \leq r_n} X_{k,k} > a_n u \mid X_{0,0} > a_n u \right) \leq \frac{\alpha^*}{\alpha^* - \lambda_0} \lim_{m \to \infty} \sum_{k=m+1}^{r_n} m(\lambda_0)^k = 0.
\]

It remains to deal with the non diagonal terms. More precisely, in order to obtain (6.4), we will show that, denoting \( b_n = \log a_n \) and \( M = \log y \),

\[
\limsup_{n \to \infty} \mathbb{P} \left( \max_{|i,j| \leq r_n, i \neq j} S_{i,j} > b_n + M \mid S_{0,0} > b_n + M \right)
\]

\[
= C^{-1} e^{\alpha^* M} \limsup_{n \to \infty} e^{\alpha^* b_n} \mathbb{P} \left( \max_{|i,j| \leq r_n, i \neq j} S_{i,j} > b_n + M, S_{0,0} > b_n + M \right) = 0.
\]
Notice that \( e^{\alpha^*b_n} = Cn^2 \). First, since \( r_n/n \to 0 \), stationarity and Lemma 4.2 give
\[
\limsup_{n \to \infty} e^{\alpha^*b_n} P \left( \max_{\| i,j \| \leq r_n} S^k_{i,j} > b_n + M, S_{0,0} > b_n + M \right)
\leq \limsup_{n \to \infty} e^{\alpha^*b_n} P \left( 2r_n + 1 \right) 2 \left( c_0 b_n \right)^2 e^{-\alpha^*b_n}
\leq \limsup_{n \to \infty} \frac{(2r_n + 1)^2}{Cn^2} = 0.
\]

Now by Lemma 4.3 there exist an \( \epsilon_0 > 0 \) such that
\[
\limsup_{n \to \infty} e^{\alpha^*b_n} P \left( \max_{\| i,j \| \leq r_n, i \neq j} S_{i,j} > b_n + M, S_{0,0} > b_n + M \right)
= \limsup_{n \to \infty} e^{\alpha^*b_n} P \left( \max_{\| i,j \| \leq r_n, i \neq j} S_{i,j} > b_n + M, \max_{1 \leq k \leq c_0 b_n} S_{0,0} > b_n + M \right)
\leq \limsup_{n \to \infty} e^{\alpha^*b_n} (2r_n + 1)^2 (c_0 b_n)^2 e^{-\epsilon^*b_n}
= 2\epsilon^* e^{-\epsilon^*} \limsup_{n \to \infty} \left( \frac{2r_n + 1}{n^{\epsilon^*}} \right)^2 \left( \frac{b_n}{n^{\epsilon^*}} \right)^2 = 0,
\]
where the last equality follows by the choice of \( (r_n) \) and since \( b_n \sim \frac{2}{\alpha^*} \log n \).

6.2.3. Proof of Lemma 4.5. First, we need the following simple result proved by a change of measure argument and a large deviation bound for empirical measures, cf. the proof of [Han06, Lemma 5.14, Equation (54)].

**Lemma 6.1.** For all \( \eta > 0 \) there exists an \( \epsilon_1 > 0 \) such that
\[
\lim_{u \to \infty} e^{(1+\epsilon_1) \alpha^* u} \sup_{m \geq 1} P(S_{0,0}^m > u, \epsilon_0^m \notin B_\eta) = 0.
\]

**Proof.** Fix \( \eta > 0 \) and denote \( A_m(u) = \{ S_{0,0}^m > u, \epsilon_0^m \notin B_\eta \} \) for \( m \geq 1 \) and \( u > 0 \). Note that, since \( S_0^m = \sum_{k=0}^{m-1} s(A_{-k}, B_{-k}) \), \( P(A_m(u)) = 0 \) whenever \( m \leq u/\|s\| \), so for fixed \( u > 0 \) we only need to deal with \( P(A_m(u)) \) for \( m > u/\|s\| \).

First, a change of measure yields
\[
P(A_m(u)) = E \left[ \frac{\exp(\alpha^* S_{0,0}^m)}{\exp(\alpha^* S_{0,0}^m) \mathbb{1}_{A_m(u)}} \right] \leq e^{-\alpha^* u} P^* (\epsilon_0^m \notin B_\eta),
\]
where \( P^* \) makes \( (A_{-k}, B_{-k}), k = 0, \ldots, m-1 \), i.i.d. elements of \( E^2 \) with common distribution \( \mu^* \). By Sanov’s theorem (see [DZ10, Theorem 2.1.10])
\[
\limsup_{m \to \infty} \frac{1}{m} \log P^* (\epsilon_0^m \notin B_\eta) \leq - \inf_{\pi \notin B_\eta} H(\pi | \mu^*).
\]
Since, for a sequence of probability measures \( (\pi_n) \) on \( E^2 \), \( H(\pi_n | \mu^*) \to 0 \) implies that \( \| \pi_n - \mu^* \| \to 0 \), for we can find a constant \( c = c(\eta) > 0 \) such that \( \inf_{\pi \notin B_\eta} H(\pi | \mu^*) > c \). Hence, for all \( m > u/\|s\| \) with \( u \) large enough
\[
P^* (\epsilon_0^m \notin B_\eta) \leq e^{-mc} \leq e^{-uc/\|s\|}.
\]
To finish the proof, it suffices to take \( \epsilon_1 := \frac{c}{\|s\| \alpha^*} > 0 \). \( \square \)
Proof of Lemma 4.5. Take an arbitrary $f \in \mathcal{F}' \subseteq CB^+_b([0,1]^2 \times \bar{l}_{0,0})$ and let $\epsilon > 0$ be such that $f(t, (x_{i,j})_{i,j}) = f(t, (x_{i,j}1\{\{x_{i,j}\}>\epsilon\})_{i,j})$ for all $t \in [0,1]^2$ and $(x_{i,j})_{i,j}\in \bar{l}_{0,0}$ with $f(t, 0) = 0$.

By the elementary inequality $|\prod_{i=1}^k a_i - \prod_{i=1}^k b_i| \leq \sum_{i=1}^k |a_i - b_i|$ valid for all $k \geq 1$ and $a_i, b_i \in [0,1]$ (see e.g. [Dur10, Lemma 3.4.3]),

\begin{equation}
E\left[|e^{-\sum_{i\in I_n} f(i/k_n, X_{n,i}/a_n)} - e^{-\sum_{i\in I_n} f(i/k_n, \tilde{X}_{n,i}/a_n)}|\right] \\
+ \prod_{i\in I_n} E\left[|e^{-f(i/k_n, X_{n,i}/a_n)} - e^{-f(i/k_n, \tilde{X}_{n,i}/a_n)}|\right] \\
\leq 2 \sum_{i\in I_n} E\left|e^{-f(i/k_n, X_{n,i}/a_n)} - e^{-f(i/k_n, \tilde{X}_{n,i}/a_n)}\right|.
\end{equation}

Further, denote by $J_{r_n} := \{1, \ldots, r_n\}^2 = J_{n,1}$ and $\tilde{J}_{r_n} := \{1, \ldots, r_n - l_n\}^2 = \tilde{J}_{n,1}$. Using stationarity we get that

\begin{equation}
\sum_{i\in I_n} E\left|e^{-f(i/k_n, X_{n,i}/a_n)} - e^{-f(i/k_n, \tilde{X}_{n,i}/a_n)}\right| \leq k_n^2 (A_1 + A_2 + A_3),
\end{equation}

where

\begin{align*}
A_1 &= P(X_{i,j} > a_n \epsilon \text{ for some } (i, j) \in J_{r_n} \setminus \tilde{J}_{r_n}), \\
A_2 &= P(\max_{m > c_0 b_n} e^{S_{i,j}^m} > a_n \epsilon \text{ for some } (i, j) \in \tilde{J}_{r_n}), \\
A_3 &= P(e^{S_{i,j}^m} > a_n \epsilon \text{ and } \epsilon_{i,j}^m \notin B_\eta \text{ for some } (i, j) \in \tilde{J}_{r_n}, 1 \leq m \leq c_0 b_n).
\end{align*}

Observe, $|J_{r_n} \setminus \tilde{J}_{r_n}| \leq 2 r_n l_n$ and $|\tilde{J}_{r_n}| \leq r_n^2$, and recall that $k_n r_n \sim n$ as $n \to \infty$, so using stationarity and then (4.1), Lemma 4.2 and Lemma 6.1, respectively,

\begin{align*}
\limsup_{n \to \infty} k_n^2 A_1 &\leq \limsup_{n \to \infty} 2 k_n^2 l_n P(X_{0,0} > a_n \epsilon) = (\text{const.}) \limsup_{n \to \infty} l_n/r_n = 0, \\
\limsup_{n \to \infty} k_n^2 A_2 &\leq \limsup_{n \to \infty} k_n^2 r_n^2 P(\max_{m > c_0 b_n} S_{0,0}^m \geq 0) \leq \limsup_{n \to \infty} n^{-2} = 0, \\
\limsup_{n \to \infty} k_n^2 A_3 &\leq \limsup_{n \to \infty} k_n^2 r_n^2 c_0 b_n P(S_{0,0}^m > b_n + \log \epsilon, \epsilon_{i,j}^m \notin B_\eta) \\
&\leq (\text{const.}) \limsup_{n \to \infty} b_n/n^{2\epsilon_1} = 0.
\end{align*}

Therefore, the right hand side, and then also the left hand side, of (6.6) tends to 0 as $n \to \infty$, and by (6.5) this proves the lemma. \qed

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