ROUNDING NEAR-OPTIMAL QUANTUM STRATEGIES FOR NONLOCAL GAMES TO STRATEGIES USING A MAXIMALLY ENTANGLED STATE

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ABSTRACT. We establish approximate rigidity results for several well-known families of nonlocal games. In particular, we show that near-perfect quantum strategies for boolean constraint system (BCS) games are approximate representations of the corresponding BCS algebra. Likewise, for the class of XOR nonlocal games, we show that near-optimal quantum strategies are approximate representations of the corresponding ∗-algebra associated with optimal quantum values for the game. In both cases, the approximate representations are with respect to the normalized Hilbert-Schmidt norm and independent of the Hilbert space or quantum state employed in the strategy.

We also show that approximate representation of the BCS (resp. XOR-algebra) yields measurement operators for near-perfect (resp. near-optimal) quantum strategies where the players employ a maximally entangled state in the game. As a corollary, every near-perfect BCS (resp. near-optimal XOR) quantum strategy is close to a near-perfect (resp. near-optimal) quantum strategy using a maximally entangled state. Lastly, we establish that every synchronous algebra is ∗-isomorphic to a certain BCS algebra called the SynchBCS algebra. This allows us to apply our BCS rigidity results to the class of synchronous games as well.

1. INTRODUCTION

A two-player nonlocal game is a scenario involving two players, commonly referred to as Alice and Bob, and a referee. In the game, the referee sends each player a question, and each player replies with an answer. The players are unable to communicate once the game begins. However, they may share a bipartite quantum state and perform measurements on the state as part of their strategy. The players win if their answers satisfy the rule predicate, otherwise they lose. The rule predicate is known to the players beforehand allowing them to predetermine their strategy.

It is well-known that there are nonlocal games where by using entanglement the players can win with a higher probability than if they had only classical resources. There are even examples of nonlocal games where the players can win perfectly (with probability one) using an entangled strategy, while any classical strategy for the game has a nonzero losing probability. However, determining if a nonlocal game admits some quantum advantage is not easy. Not only can it be hard to find the optimal winning probability amongst classical strategies, a quantity known as the classical value, it was recently established that deciding if the optimal winning probability for a nonlocal game amongst all entangled strategies, a quantity known as the entangled value, is 1 or greater than 1/2 is equivalent to the halting problem [JNV+22].

Despite the computational hardness surrounding and entangled value; for several classes of nonlocal games, the existence of optimal and or perfect quantum strategies can be characterized in purely algebraic terms. This notion is often referred to as...
the “rigidity property” of the optimal strategies for a nonlocal game. Abstractly, the rigidity relations amongst the observables in an optimal strategy can be viewed as generators and relations of a finitely presented $*$-algebra associated with the nonlocal game. By construction, the finite-dimensional representations of these nonlocal game algebras yield quantum strategies that obtain the quantum (or commuting operator) value for such games.

The first instance of this correspondence is in the context of XOR nonlocal games through the algebraic characterization of specific quantum correlations in two-output Bell scenarios and is attributed to Tsirelson [Tsi85]. Tsirelson’s result implies that the optimal quantum strategies for XOR games are representations of a certain finitely presented $*$-algebra with a finite-dimensional tracial state [Tsi87,Slo11]. There are now several families of nonlocal games for which a correspondence between optimal quantum strategies and representations of the nonlocal game algebra has been established. This includes the class of $\mathbb{Z}_2$-linear constraint systems (LCS) games [CM14,CLS17], the class of synchronous games [PSS+16], boolean constraint system (BCS) games [CM14,JI13], and the general class of imitation games [LMP+20].

A particularly useful application of this correspondence is in providing lower bounds on the amount of entanglement required to achieve the quantum value of the game, for example, [Slo11,Slo18]. Another application of this correspondence is enabling an observer to verify information about the quantum measurements and states employed by the players when they achieve the optimal winning probability in a game. This concept is more formally known as self-testing and is an important ingredient in device-independent cryptography, see for example [MY04, WBMS16, BˇSCA18a, BˇSCA18b, Kan20].

With these applications in mind, it is natural to wonder how the correspondence between optimal quantum strategies and representations is affected by noise. More precisely, we say a quantum strategy is $\epsilon$-optimal if the probability of winning is at most $\epsilon$-away from the entangled value. In the case where the entangled value is one, we say that a quantum strategy is $\epsilon$-perfect if it wins with probability at least $1 - \epsilon$. In this work, we focus on the case where $\epsilon$ is significantly less than the smallest joint question probability.[1] In this regime, we will see that $\epsilon$-optimal strategies correspond to approximate representations. Informally, an approximate representation or $\epsilon$-representation of a finitely presented $*$-algebra is a function from the generators to matrices where the defining relations hold approximately. The parameter $\epsilon > 0$ measures how far, according to some metric, the relations are from being satisfied.

There are already several previous results about approximate rigidity in the literature. In [Slo11], Slofstra showed that the correspondence between optimal quantum strategies and representations of the XOR-algebra is robust, in the sense that $\epsilon$-optimal strategies are $O(\epsilon^{1/8}d^{2/3})$-representations of the XOR-algebra, where $d$ is the dimension of the local strategy Hilbert space $H_A$ (or equivalently $H_B$) supporting the quantum strategy. In the case of $\mathbb{Z}_2$-linear constraint systems (LCS) nonlocal games, Slofstra and Vidick established that $\epsilon$-perfect quantum strategies correspond to unitary $O(\epsilon^{1/4})$-representations of the associated solution group [SV18]. Unlike in the XOR game case, for LCS games the quality of the approximate representation does not depend on the

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[1] When this is not the case, the noise from the state is indistinguishable from a strategy employing losing answers and other issues arise which we do not explore here.
Hilbert space or the state in the quantum strategy. This independence is a much-desired property in the context of device independence.

Our main result is that the correspondence between perfect (or optimal in the XOR case) quantum strategies and representations is robust and Hilbert space independent for the class of boolean constraint system (BCS), synchronous, and XOR nonlocal games.

**Theorem 1.1.**

1. Each $\epsilon$-perfect strategy for a BCS nonlocal game corresponds to an $O(\epsilon^{1/4})$-representation of the BCS algebra.
2. Each $\epsilon$-perfect strategy for a synchronous nonlocal game corresponds to an $O(\epsilon^{1/8})$-representation of the synchronous algebra.
3. Each $\epsilon$-optimal strategy for an XOR nonlocal game corresponds to an $O(\epsilon^{1/8})$-representation of the XOR algebra.

The precise definitions of a $\epsilon$-perfect and optimal strategies are given in Section 4, along with the definitions of the algebras $\mathcal{B}(\mathcal{G})$, and $\mathcal{C}(\mathcal{G})$, while approximate representation are formally defined in Section 3 Definition 3.2. The more precise statements of Theorem 1.1 are stated in Proposition 4.11, Proposition 4.26, and Proposition 4.33. All of the approximate representations are measured with respect to the little Frobenius norm $\| \cdot \|_f$. In particular, Theorem 1.1(1) can be seen as a generalization of the result in [SV18] to the class of more general BCS nonlocal games. Theorem 1.1(2) provides an alternative proof of the result in [Vid22] in the case of games. While Theorem 1.1(3) can be seen as an improvement of the result in [Slo11], as it removed the Hilbert space dependence in the approximate representation.

The proof of the first theorem consists of two parts. First, we establish that every near-perfect (resp. near-optimal) strategy is an approximate representation of the BCS (resp. XOR) algebra with respect to a particular state-dependent semi-norm. This state-dependent semi-norm is determined by the quantum state employed as part of the quantum strategy used by the players. The second step involves showing that each state-dependent approximate representation can be “rounded” to an approximate representation in the little Frobenius norm. This removes the state/dimension dependence in the approximate representation. The rounding of the state-dependent approximate representations from near-optimal quantum strategies to an approximate representation in the little Frobenius norm is achieved through Lemma 3.26, which builds on the techniques developed in [SV18] in the group setting.

Although our results do not depend on the dimension of the approximate representation, the approximate representations do depend on the properties of the game algebra. In particular, this means that approximate representations may depend (exponentially in some cases) on the size of the question and answer sets from the nonlocal game. This means that although the techniques apply to fixed games, they do not apply to families of games with these parameters. We leave the problem of tightly analyzing this dependence for future work.

Another contribution of this work is furthering the connection between synchronous games and boolean constraint system (BCS) games. In Proposition 4.22, we establish that the synchronous algebra is isomorphic to the BCS algebra of a certain BCS nonlocal game we call the SynchBCS game. This isomorphism allows us to extend several of our results to the class of synchronous algebras. Connections between synchronous
and BCS games have been previously noted, but have focussed exclusively on the case where the constraints are all linear, for example [KPS18, Goi21, Fri20].

Our second main result is a converse to Theorem 1.1. In particular, we show that approximate representations for certain game algebras are close to near-perfect (near-optimal in the XOR case) quantum strategies employing a maximally-entangled state.

**Theorem 1.2.**

1. Each $\epsilon$-representation of the BCS algebra is close to an $O(\epsilon^2)$-perfect strategy for the corresponding BCS nonlocal game employing a maximally-entangled state.
2. Each $\epsilon$-representation of the synchronous algebra is close to an $O(\epsilon^2)$-perfect strategy for the synchronous nonlocal game where the players employ a maximally-entangled state.
3. Each $\epsilon$-representation of the XOR algebra is close to an $O(\epsilon^2)$-optimal strategy for the XOR nonlocal game where the players employ a maximally-entangled state.

By *close* we mean that each measurement operator in the strategy is at most $O(\epsilon)$-away from the generators in the approximate representation. Because all of the above algebras are quotients of group algebras, we assume that the operator norm of each generator in the approximate representation is at most one. In this case the $O(\epsilon)$ is entirely independent of the Hilbert space and depends only on the presentation of the nonlocal game algebra. The formal statements for Theorem 1.2 are found in Proposition 4.12, Proposition 4.27, and Proposition 4.34. As a consequence of the proofs of Theorem 1.1 and Theorem 1.2 we obtain the following corollary:

**Corollary 1.3.**

1. Each $\epsilon$-perfect quantum strategy for a BCS nonlocal game is $O(\epsilon^{1/4})$-away from an $O(\epsilon^{1/2})$-perfect quantum strategy using a maximally entangled state.
2. Each $\epsilon$-perfect quantum strategy $S$ for a synchronous nonlocal game is $O(\epsilon^{1/8})$-away from an $O(\epsilon^{1/4})$-perfect quantum strategy using a maximally entangled state.
3. Each $\epsilon$-optimal quantum strategy $S$ for an XOR nonlocal game is $O(\epsilon^{1/8})$-away from an $O(\epsilon^{1/4})$-optimal quantum strategy using a maximally entangled state.

Notably, Corollary 1.3 shows that you can reduce the analysis of near-perfect (near-optimal in the XOR case) strategies with an arbitrary state to the analysis of near-perfect (near-optimal in the XOR case) strategies with a maximally-entangled state without amplifying the error too much.

Recently, independent but similar results to Theorem 1.1(2) for approximate synchronous correlations were established in [Vid22]. Both results are based on techniques in [SV18] but we emphasize that this work takes a more algebraic perspective and focuses on extending the techniques to arbitrary BCS games. One advantage of their result is that it applies to the more general case of correlations (not just strategies). However, none of their results apply to the case of XOR games as they are far from synchronous.

The remainder of the paper is outlined as follows: Section 2 covers some mathematical preliminaries, Section 3 covers the relevant concepts and results from Approximate Representation Theory we use in the work, including the key rounding result
(Lemma 3.20), and Section 4 defines the nonlocal game algebras associated with BCS, synchronous, and XOR games, while also examining the connection between approximate representations and near-optimal strategies for these games.

2. Preliminaries

Let $S$ be a finite set, we let $\mathbb{C}^*(S)$ denote the free (complex) $\ast$-algebra generated by $S$. Let $R \subset \mathbb{C}^*(X)$ be a finite collection of elements (noncommutative $\ast$-polynomials) from $\mathbb{C}^*(S)$. The \textbf{finitely-presented} $\ast$-\textit{algebra} $\mathcal{A} = \mathbb{C}^*(S : R)$ is the quotient of $\mathbb{C}^*(S)$ by $\langle \langle R \rangle \rangle$, where $\langle \langle R \rangle \rangle$ is the smallest (two-sided) $\ast$-ideal containing $R$. We call the pair $(S, R)$ a finite \textbf{presentation} of the $\ast$-algebra $\mathcal{A}$. We write $1$ to represent the unit in a $\ast$-algebra. A priori, elements of $\mathcal{A} = \mathbb{C}^*(S : R)$ are not bounded in representations. To address this, let $\mathcal{A}_{\ast aa} = \{ a \in \mathcal{A} : a = a^\ast \}$ be the $\ast$-subalgebra of self-adjoint elements. Define the subset of \textbf{sums-of-squares (SOS)} to be the elements $\mathcal{A}_+ = \{ a \in \mathcal{A} : a = \sum_{k \in K} b_k^2 b_k \}$. In the language of [Oza13], the subset $\mathcal{A}_+ \subset \mathcal{A}$ is a $\ast$-\textbf{positive cone} for $\mathcal{A}$. The $\ast$-positive cone induces a partial order on the self-adjoint elements $\mathcal{A}_{\ast aa}$, where $a \leq b$ if $a - b \in \mathcal{A}_+$. The $\ast$-subalgebra of bounded elements is defined as $\mathcal{A}_{\ast bdd} = \{ a \in \mathcal{A} : \exists \lambda > 0 \text{ such that } a^\ast a \leq \lambda 1 \}$. A finitely presented $\ast$-algebra $\mathcal{A} = \mathbb{C}^*(S : R)$ is said to be \textbf{archimedean} if $\mathcal{A} = \mathcal{A}_{\ast bdd}$. In this case, we say that the relations $R$ are archimedean. For a finitely presented $\ast$-algebra $\mathbb{C}^*(a_1, \ldots, a_n : r_1, \ldots, r_m)$ being archimedean is equivalent to the ideal generated by $(r_1, \ldots, r_m)$ contains the relation $n v^2 1 - \sum_{i=1}^n a_i a_i^\ast$ for some $v > 0$. Additionally, whenever $\mathcal{A} = \mathcal{A}_{\ast bdd}$ we say that $\mathcal{A}$ is a \textbf{semi-pre-$\mathbb{C}^*$-algebra}. A \textbf{representation} of $\mathcal{A}$ is a $\ast$-homomorphism $\psi : \mathcal{A} \to B(H)$, where $B(H)$ are the bounded operators on a Hilbert space $H$. Note that a finitely-presented $\ast$-algebra $\mathcal{A} = \mathbb{C}^*(S : R)$ is archimedean if $S \subset \mathcal{A}_{\ast bdd}$. Remark that the relation $n v^2 1 - \sum_{i=1}^n a_i a_i^\ast$ implies that in any $\ast$-representation, the image of each generator has operator norm at most $v$. Given a finitely presented $\ast$-algebra $\mathcal{A}$ we let $\vartheta_{\mathcal{A}} = \inf_\psi \{ n v^2 1 - \sum_{i=1}^n a_i a_i^\ast \in \langle r_1, \ldots, r_m \rangle \}$ denote the \textbf{bounded radius} of $\mathcal{A}$.

For $A \in M_d(\mathbb{C})$, $\|A\|_{\text{op}}$ denotes the \textbf{operator norm} of $A$, while $\|A\|_F$ denotes the \textbf{Frobenius (or Hilbert-Schmidt) norm} of $A$. We write $\|\cdot\|$ when the matrix norm is left unspecified. For a finite-dimensional Hilbert space $H$, we denote by $\mathcal{L}(H)$ the set of linear operators from $H$ to $H$. Whenever $H \cong \mathbb{C}^d$, we define $d = \dim(H)$ and we have that $M_d(\mathbb{C}) \cong \mathcal{L}(H)$ is a Hilbert space with the \textbf{Frobenius (Hilbert-Schmidt) inner-product} $\langle A, B \rangle_F := \text{tr}(A^\ast B)$, for $A, B \in \mathcal{L}(H)$. We also use the \textbf{little Frobenius (or normalized Hilbert-Schmidt) norm}, denoted by $\|A\|_F^2 := \text{tr}(A^\ast A) = \frac{1}{d} \|A\|_F^2$, for $A \in M_d(\mathbb{C})$. The normalization in the little Frobenius norm ensures that $\|1\|_F = 1$, in contrast to $\|1\|_F = \sqrt{d}$. It’s worth noting that unlike its unnormalized version, the Frobenius norm $\|\cdot\|_F$, the little Frobenius norm $\|\cdot\|_F$ is not submultiplicative. Nonetheless, if $A, B, C \in M_d(\mathbb{C})$ we do have the bimodule property $\|ABC\|_F \leq \|A\|_{\text{op}} \|B\|_F \|C\|_{\text{op}}$.

A (pure) \textbf{quantum state} $|\psi\rangle$ is a unit vector in a Hilbert space $H$. Each quantum state $|\psi\rangle$ gives rise to a positive-semidefinite matrix with trace one, called a \textbf{density matrix} (or sometimes just a state) $\rho$ through the identification $|\psi\rangle \mapsto |\psi\rangle \langle \psi| := \rho$. Density matrices can also represent ensembles or mixtures of pure states. The \textbf{state-induced semi-norm (or $\rho$-norm)} for a density operators $\rho \in \mathcal{L}(H)$ is given by $\|T\|_\rho := \|T\rho^{1/2}\|_F$ for all $T \in \mathcal{L}(H)$. The failure of positive definiteness in the $\rho$-norm is the result of $\rho$ having a 0-eigenvalue. In the case where $\rho = 1/d$, the $\rho$-norm
A bipartite quantum state is a unit vector $|\psi\rangle$ in the tensor product of Hilbert spaces $H_A \otimes H_B$. A state $|\psi\rangle \in H_A \otimes H_B$ is said to be **maximally entangled** if its reduced density matrix $\text{tr}_{H_A}(\rho) = \rho_B$ on $H_B$ (or $\rho_A = \text{tr}_{H_B}(\rho)$ on $H_A$) is $1/\dim(H_B)$ (or $1/\dim(H_A)$). Thus, starting from a maximally entangled state the induced $\rho$-norm on $H_B$ (or $H_A$) is the little Frobenius norm $\| \cdot \|_f$. Every bipartite vector state has a **Schmidt decomposition** $|\psi\rangle = \sum_{i \in I} \alpha_i |u_i\rangle \otimes |v_i\rangle$, where $\{|u_i\rangle\}_{i \in I}$ and $\{|v_i\rangle\}_{i \in I}$ are orthonormal subsets of $H_A$ and $H_B$ respectively, and $\alpha_i > 0$ for all $i \in I$. The **support** of a bipartite vector state $|\psi\rangle$ is the image of $H_A \otimes H_B$ under the projection $\Pi = \sum_{i \in I} |u_i\rangle \langle u_i| \otimes |v_i\rangle \langle v_i|$. The support on $H_A$ or $H_B$ are the images under the local projections $\Pi_A = \sum_{i \in I} |u_i\rangle \langle u_i|$ and $\Pi_B = \sum_{i \in I} |v_i\rangle \langle v_i|$ respectively. The support Hilbert space is defined by $\text{Im}(\Pi) := \Pi H$. Note that for the maximally entangled state, the local support projections are $\text{Im}(\Pi) = H_A$ and $\text{Im}(\Pi) = H_B$. We denote the maximally entangled state by $|\tau\rangle = \sum_{i \in I} |u_i\rangle \otimes |v_i\rangle$. For a self-adjoint matrix $A^* = A$ we observe that $A \otimes 1 |\tau\rangle = 1 \otimes A^T |\tau\rangle = 1 \otimes \overline{A} |\tau\rangle$, where the transpose is taken with respect to the Schmidt basis of $|\tau\rangle$, since $A^T = (A^*)^T = \overline{A}$. Moreover, this identification shows that there is a correspondence between the norms $\|A \otimes 1 |\psi\rangle\| = \|A\|_\rho$ where $|\psi\rangle \in H \otimes H$ is a **purification** of $\rho \in \mathcal{L}(H)$.

For positive real functions $f, g : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ as $x \rightarrow 0$ we write $f(x) = O(g(x))$, if there exists constants $C, K > 0$ such that for all $x \in (0, C)$ we have that $f(x) \leq Kg(x)$. This is read as “$f$ is big-Oh of $g$”, and means for small $x$ the behaviour of $f$ is dominated by a constant times the function $g$.

The **unitary part** of a $d \times d$ complex matrix $A$ we mean any unitary $U$ satisfying the **polar decomposition** equation $A = U|A|$ for $|A| = \sqrt{A^*A}$. Every matrix has a unitary part, the simplest construction comes from the the singular value decomposition. Moreover, when $|A|$ is invertible the unitary part of $A$ is unique and given by $U = A|A|^{-1} = A(A^*A)^{-1/2}$. If $A$ is self-adjoint then $\text{sgn}(A)$ is a (self-adjoint) unitary which satisfies $A = \text{sgn}(A)|A|A$.

### 3. Approximate representation theory

In the first part of this section, we present the key definitions and concepts from approximate representation theory. The second part of this section contains the proof of the main technical lemma. All of the Hilbert spaces in this section are complex and finite-dimensional unless stated otherwise. Moreover, we assume every finitely-presented *-algebra is a $\mathbb{C}$-vector space.

**Definition 3.1.** Let $\mathcal{A} = \mathbb{C}^*[S : R]$ and $\mathcal{B} = \mathbb{C}^*[T : U]$ be finitely presented *-algebras. If $\psi : \mathcal{A} \rightarrow \mathcal{B}$ is a *-homomorphism, then the **lift** of $\psi$ is the unique *-homomorphism $\tilde{\psi} : \mathbb{C}^*[S] \rightarrow \mathbb{C}^*[T]$ such that $\tilde{\psi}(r) = 0$ for all $r \in R$. Equivalently, we say that $\psi$ **descends** to the *-homomorphism $\psi$.

Of particular interest is when $\mathcal{B}$ is the *-algebra of linear operators on $H$ with the usual antilinear involution. When $H$ is finite-dimensional, this is matrix algebra by the standard identification $\mathcal{L}(H) \cong M_d(\mathbb{C})$ where $H \cong \mathbb{C}^d$. In this case, the *-homomorphisms (or representations) $\mathcal{A} \rightarrow \mathcal{L}(H)$ are in one-to-one correspondence with the **lifts** $\tilde{\psi} : \mathbb{C}^*[S] \rightarrow \mathcal{L}(H)$. This point of view is essential for motivating the definition of an approximate representation.
Definition 3.2. Let $\mathcal{A} = C^*(S : R)$ be a finitely-presented $*$-algebra. If $H$ is a Hilbert space, $\rho$ a state (i.e. density operator) in $L(H)$, a state-dependent $\epsilon$-representation of $\mathcal{A}$ or $(\epsilon, \rho)$-representation is a $*$-homomorphism 
$$
\phi : C^*(S) \to L(H) \text{ such that } \|\phi(r)\|_\rho \leq \epsilon, \text{ for all } r \in R.
$$

Remark 3.3. If the $\rho$-norm is non-degenerate then every $(0, \rho)$-representation descends to a $*$-homomorphism $\mathcal{A} \to L(H)$. Moreover, if $T \subset R$ is a subset of relations and $\phi$ is a non-degenerate $(\epsilon, \rho)$-representation of $\mathcal{A} = C^*(S : R)$ where $\|\phi(r)\|_\rho = 0$ for all $r \in T$, then $\phi$ satisfies the relations $T \subset R$. In particular, this means that $\phi$ descends to a representation of $C^*(S : T)$.

The universal notions identified in Definition 3.2 illustrate why $\epsilon$-representation $\phi$ of $\mathcal{A}$ are formally $*$-homomorphisms of the free $*$-algebra $C^*(S)$ such that $\|\phi(R)\| \approx_\epsilon 0$. However, they also indicate some degree of flexibility in the notion of an approximate representation. In particular, there are multiple ways to quantify $\|\phi(R)\| \approx_\epsilon 0$. As such, other notions of approximate representations exist in the literature, see for example [GH17, Tho18]. Returning to Definition 3.2 we highlight the important case where the state $\rho$ is the maximally-mixed state $\rho = 1/d$, with $\dim(H) = d$. In this case, the semi-norm or $\rho$-norm coincides with the little Frobenius (a.k.a. normalized Hilbert-Schmidt) norm $\| \cdot \|_{1/d} = \| \cdot \|_f$. This norm has many nice qualities and is a popular norm for studying approximate representations. We make the following definition.

Definition 3.4. Let $H \cong \mathbb{C}^d$. An $(\epsilon, \rho)$-representation of $\mathcal{A} = C^*(S : R)$ where $\rho = 1/d$ is called a state-independent $\epsilon$-representation or simply an $\epsilon$-representation. In this case, it is clear that the relations hold approximately in the little Frobenius norm $\| \cdot \|_f$ on $L(H)$.

A keen reader may be aware that $*$-representations of $C^*(X)$ on arbitrary (possibly infinite-dimensional) Hilbert spaces are not bounded. This means that there is no universal bound on the operator norm of an element in an $(\epsilon, \rho)$-representation on an arbitrary Hilbert space. That being said, in any given $(\epsilon, \rho)$-representation every $\|\phi(x)\|_\rho$ is finite when $H$ is finite-dimensional. Indeed, since $X$ is a finite set for any given $(\epsilon, \rho)$-representation $\max\{\|\phi(x)\|_\rho : x \in X\}$ bounds the norm of every $\phi(x)$. Hence, by letting $\kappa_\phi > 0$ be the largest singular value among generators of an $(\epsilon, \rho)$-representation we can explicitly track how results about $(\epsilon, \rho)$-representations depend on this quantity. On the other hand, we recall that if a finitely-presented $*$-algebra $\mathcal{A} = C^*(S : R)$ is archimedean then in every $*$-representation $\psi : C^*(S) \to L(H)$ of $\mathcal{A}$, the largest singular value of each $\psi(s)$ is bounded by the radius $\vartheta_\mathcal{A} > 0$ of $\mathcal{A}$, which depend on the presentation of $\mathcal{A}$ and not on $H$. Hence, it is not unreasonable to expect that approximate representations have the same or similar bounded property, especially since as $\epsilon \to 0$ we expect for $\kappa_\phi$ to coincide with $\vartheta_\mathcal{A}$. To keep things simple, we can also restrict the domain of our approximate representations when $\mathcal{A}$ is archimedean.

Definition 3.5. Let $\mathcal{A} = C^*(S : R)$ be an archimedean finitely-presented $*$-algebra, and let $\vartheta_\mathcal{A} > 0$ be the bounded radius. An $(\epsilon, \rho)$-representation $\phi : C^*(S) \to L(H)$ is a bounded approximate representation if each $\phi(s)$ has singular value at most $\vartheta_\mathcal{A}$, for all $s \in S$.

\[\text{In particular, the definition in Definition 3.2 can be viewed as a worst case notion of approximate representation. This is contrasted with an average case notion, where } \epsilon \text{ represents the average error over all the relations according to a measure on } R, \text{ see for instance [CVY23].}\]
Although the results of this section are for general approximate representations, we note that by restricting to the class of **bounded approximate representations** some result can be strengthened, in particular when the bounded radius of \( \mathcal{A} \) is 1 and we take our approximate representations to be bounded as well.

3.1. **Stability and replacement.** One of the central questions in approximate representation theory is when are approximate representation close to exact representations? The answer to this question is captured by the notion of stability for finitely presented \(*\)-algebras. Intuitively, the more stable the algebra the more closely approximate representations correspond to genuine representations.

**Definition 3.6.** Let \( g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) be a non-negative function. A finitely presented \(*\)-algebra \( \mathcal{A} = C^*(S : R) \) is \((g, C)\)-**stable** if for every non-degenerate \((\epsilon, \rho)\)-representation of \( \mathcal{A} \phi : C^*(S) \to \mathcal{L}(H) \) with \( \epsilon \leq C \), there is a \(*\)-homomorphism \( \psi : C^*(S) \to \mathcal{L}(H) \) of \( \mathcal{A} \) such that

\[
\| \phi(s) - \psi(s) \|_\rho \leq g(\epsilon),
\]

for all \( s \in S \). Alternatively, we say that \( \mathcal{A} \) is \( g\)-**stable** if it is \((g, C)\)-stable for all \( \epsilon \geq 0 \), and **stable** if \( g(\epsilon) = O(\epsilon) \).

The stability function \( g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) describes the behaviour of how exact representations relate to approximate representations. The asymptotics of \( g \) gives us an idea of how much we need to perturb or shift \( \phi \) to obtain a genuine representation.

We make two remarks: firstly, the notion of stability should be Hilbert space free in the sense that it should not depend on the dimension of \( H \). Secondly, although the stability of a finitely presented algebra is sensitive to the choice of presentation, the following result shows that for state-independent approximate representations changing the presentation will not affect the stability asymptotically. Despite our earlier emphasis on state-dependent approximate representations, several facts about stability are significantly harder to establish in this regime because the \( \rho\)-norm generally fails to have the bimodule property with respect to the operator norm. So we proceed in the state-independent case and mention when a result holds for the state-dependent case.

**Proposition 3.7.** Let \( \mathcal{A} = C^*(S : R) \) and \( \mathcal{B} = C^*(T : U) \) be finitely presented \(*\)-algebras and \( H \) a Hilbert space. If \( \phi : C^*(T) \to \mathcal{L}(H) \) is an \( \epsilon\)-representation and \( \psi : \mathcal{A} \to \mathcal{B} \) a \(*\)-homomorphism, then there exists a constant \( C > 0 \) so that \( \phi \circ \psi \) is an \( C\epsilon\)-representation of \( \mathcal{A} \), where \( \psi \) is the lift of \( \psi \).

**Proof.** Let \( \varphi : C^*(S) \to \mathcal{A} \) and \( \eta : C^*(T) \to \mathcal{B} \) be the quotient maps induced by the (two-sided) \(*\)-ideals \( \langle \langle R \rangle \rangle \) and \( \langle \langle U \rangle \rangle \) respectively. Furthermore, let \( \tilde{\psi} \) be the lift of the \(*\)-homomorphism \( \psi : \mathcal{A} \to \mathcal{B} \). Since \( \varphi \circ \psi(r) = \eta \circ \tilde{\psi}(r) = 0 \) for all \( r \in R \), we conclude that \( \tilde{\psi}(r) \in \langle \langle U \rangle \rangle \) for all \( r \in R \). Consider a single \( r \in R \), and note that \( \tilde{\psi}(r) \in \langle \langle U \rangle \rangle \) if and only if there exists a collection \( i \in I \subseteq U \), coefficients \( \gamma_i \in \mathbb{C} \), monomials \( \{w_i, v_i\} \in C^*(T) \), and relations \( \{u_i\} \in U \), so that \( \psi(r) = \sum_{i \in I} \gamma_i w_i u_i v_i \). Then, if \( \phi : C^*(T) \to \mathcal{L}(H) \) is an \( \epsilon\)-representation of \( \mathcal{B} \), we see that each \( r \) is bounded by

\[
\| \phi \circ \tilde{\psi}(r) \|_f \leq \sum_{i \in I} |\gamma_i| \| \phi(w_i) \| \| \phi(v_i) \| \leq \sum_{i \in I} C_0 \| \phi(u_i) \| \| \phi(v_i) \| \leq \sum_{i \in I} C_0 \| \phi(u_i) \| \leq |I|C_0 \epsilon,
\]

where \( C_0 \) is a constant depending on the presentation which bound the coefficients \( \gamma_i \), and the operator norms of the monomials \( \{v_i\}_{i \in I} \) and \( \{u_i\}_{i \in I} \) respectively. That is, we
let \( C_0 = \max \{ |\gamma_i| \| \phi(w_i) \|_{op} \| \phi(v_i) \|_{op} \} \). Furthermore, if \( \kappa_\phi = \max_{t \in T} \{ \| \phi(t) \|_{op} \} \), then \( \| \phi(w_i) \|_{op} \) (resp. \( \| \phi(v_i) \|_{op} \)) are bounded by \( \kappa_\phi^l \) where \( l = \max_i \{ \text{len}(w_i), \text{len}(v_i) \} \) is the longest monomial in \( \{ v_i, w_i : i \in I \} \), which is finite but not given explicitly from the presentation. Now, we define the constant \( C_r = C_0 |I| \), and the result follows by taking the largest \( C_r \) among the relations in \( R \), that is \( C = \max_{r \in R} \{ C_r \} \).

In particular, if \( \mathbb{C}^* \langle S | R \rangle \) and \( \mathbb{C}^* \langle T | W \rangle \) are both presentations of a finitely presented \(*\)-algebra \( \mathcal{A} \) then \( \mathbb{C}^* \langle S | R \rangle \cong_{\psi} \mathbb{C}^* \langle T | W \rangle \) and if \( \mathbb{C}^* \langle S | R \rangle \) is stable with \( g(\epsilon) = C \epsilon \), then there exists a constant \( C' > 0 \) such that \( \mathbb{C}^* \langle T | W \rangle \) is stable with \( g(\epsilon) = C' \epsilon \) and vice versa. We do note that the constant in Proposition 3.7 does depend on \( \kappa_\phi \) the operator norm of the approximate representation, and so one should be cautious in applying this result in a case where \( \phi \) is not bounded and or \( \mathcal{A} \) is not archimedean. Fortunately, for our applications this is not an issue.

Up until now the discussion of stability has been quite abstract. In reality stability for matrices is quite a concrete notion. For example, the \(*\)-algebra of self-adjoint matrices

\[
\mathbb{C}^* \langle X_1, \ldots, X_n : X_i^* - X_i, \text{ for all } 1 \leq i \leq n \rangle
\]

is stable with \( g(\epsilon) = \epsilon / 2 \). We can see this by remarking that for any \( X_i \) with \( \| X_i - X_i^* \|_f \leq \epsilon \), setting \( Y_i = \frac{1}{2} (X_i + X_i^*) \) the following two conditions holds: (i) each \( Y_i \) is self-adjoint, and (ii) each \( Y_i \) is close to \( X_i \), since \( \| X_i - Y_i \|_f \leq \frac{1}{2} \| X_i - X_i^* \|_f \leq \epsilon / 2 \).

We can repeat the construction for all \( 1 \leq i \leq n \) to get a collection of self-adjoints \( Y_{1}, \ldots, Y_{m} \), and we see the self-adjoint relations are \( \epsilon / 2 \)-stable.

This description of stability in terms of stable relations is intentional, and it motivates the following case. Suppose the self-adjoint relations are a subset of the defining relations of some finitely presented \(*\)-algebra. Using the fact that the algebra is stable, we could replace an approximate representation on \( \mathbb{C}^* \langle X \rangle \) with the one that satisfies the self-adjoint relations. In this case, we would like to know the extent to which this affects the remaining relations. The next result called the replacement lemma, gives an upper bound on the quality of the approximate representation obtained in this way. Specifically, by replacing the approximate representation with another approximate representation, whose distance on the generators is known.

**Lemma 3.8.** Let \( \mathcal{A} = \mathbb{C}^* \langle S : R \rangle \) and let \( R' \subset R \) be a subset of the relations so that \( \mathcal{A}' = \mathbb{C}^* \langle S : R' \rangle \) is archimedean. There exists constants \( K > 0 \), such that if \( \phi : \mathbb{C}^* \langle S \rangle \to \mathcal{L}(H) \) is an \( \epsilon \)-representation of \( \mathcal{A} \) on a finite-dimensional Hilbert space \( H \) with and \( \psi : \mathbb{C}^* \langle S \rangle \to \mathcal{L}(H) \) is a representation of the quotient \( \mathbb{C}^* \langle S : R' \rangle \) with

\[
\| \phi(s) - \psi(s) \|_f \leq \delta,
\]

for all \( s \in S \), then \( \psi \) is a \((K\delta + \epsilon)\)-representation of \( \mathcal{A} \).

**Proof.** Our proof proceeds in two steps. First, we claim that there exists a constant \( M_L > 0 \) such that for any monomial \( \alpha \in \mathbb{C}^* \langle S \rangle \) of length \( L \) we have that \( \| \psi(\alpha) - \phi(\alpha) \|_f \leq M_L \delta. \) To begin, we observe that

\[
(3.1) \quad \| \phi(\alpha) - \psi(\alpha) \|_f = \| \phi(s_{i_1}) \cdots \phi(s_{i_L}) - \psi(s_{i_1}) \cdots \psi(s_{i_L}) \|_f \leq \sum_{k=0}^{L-1} C_0^{k} C_1^{L-(k+1)} \delta.
\]

where \( C_1 = \max_{s \in S} \{ \| \phi(s) \|_{op} \} = \kappa_\phi \) and \( C_0 = \max_{s \in S} \{ \| \psi(s) \|_{op} \} = \nu_{\mathcal{A}'} \) are the largest singular values amongst all the generators in the image of \( \phi \) and \( \psi \) respectively. Noting
that
\[
\sum_{k=0}^{L-1} C_0^k C_1^{L-(k+1)} = \begin{cases} 
\frac{C_0 - C_0^L}{C_1 - L C_0} & \text{when } C_0 \neq C_0, \\
\frac{C_1 - C_0}{L C_0^{L-1}} & \text{when } C_0 = C_1,
\end{cases}
\]
we can take \( C = \max\{C_0, C_1\} \) and conclude that the Eq. (3.2) is bounded by \( M_L := L C_0^{L-1} \). Since each relation \( r \in R \) is a finite sum of monomials in the generators \( s \in S \) with complex coefficients, each \( r \in R \) can be written as a sum over monomials \( \alpha \) of increasing lengths
\[
r = \sum_{\alpha} c_{\alpha} \alpha = \sum_{\ell=0}^{N} \left( \sum_{\alpha: \text{len}(\alpha) = \ell} c_{\alpha} \alpha \right).
\]
Therefore if \( \phi \) is an \( \epsilon \)-representation, for any \( r \in R \), we see that
\[
\|\psi(r)\|_f \leq \|\psi(r) - \phi(r)\|_f + \epsilon
\]
\[
\leq \sum_{\ell=0}^{N} \left( \sum_{\alpha: \text{len}(\alpha) = \ell} |c_{\alpha}| \|\psi(\alpha) - \phi(\alpha)\|_f \right) + \epsilon
\]
\[
\leq \sum_{\ell=0}^{N} \left( \sum_{\alpha: \text{len}(\alpha) = \ell} |c_{\alpha}| \right) M_r \delta + \epsilon
\]
\[
\leq \sum_{\ell=0}^{N} \max_{\alpha} \{|c_{\alpha}| : \text{len}(\alpha) = \ell\} \|S\|^\ell M_r \delta + \epsilon
\]
the result follows by setting \( K = N \max_{\ell} \{c_\ell |S|^\ell M_r\} \), where \( c_\ell = \max_{\alpha} \{|c_{\alpha}| : \text{len}(\alpha) = \ell\} \). Since \( r \) was chosen arbitrarily the result follows. \( \square \)

If we perform replacement on the stable relations we have that \( \delta = O(\epsilon) \) and we obtain the following important corollary.

**Corollary 3.9.** If a finitely presented algebra \( \mathbb{C}^*(S : R) \) contains a subset of relations \( W \subset R \) for which the quotient algebra \( \mathbb{C}^*(S : W) \) is stable and archimedean, then replacing the \( \epsilon \)-representation \( \phi : \mathbb{C}^*(S) \to \mathcal{L}(H) \) by an approximate representation \( \psi : \mathbb{C}^*(S) \to \mathcal{L}(H) \) that descends to a \( * \)-homomorphism \( \tilde{\psi} : \mathbb{C}^*(S : W) \to \mathcal{L}(H) \) will be an \( O(\epsilon) \)-representation on \( \mathbb{C}^*(S : R) \). In particular, on the relations \( r \in R \setminus W \).

**Remark 3.10.** In the proof of Lemma 3.8 if the approximate representation \( \phi \) is bounded then we have \( C_1 = C_0 = \vartheta_{\mathcal{L}} \). In this case, \( L \vartheta_{\mathcal{L}}^{-1} \) bounds the term in Equation (3.1) for each \( L > 1 \) and depends only on the presentation of \( \mathcal{L} \). Moreover, unlike Proposition 3.7, the bound in Lemma 3.8 is explicit due to the fact that the \( * \)-polynomials \( r \in R \) are explicitly listed in the presentation so the lengths of the monomials are known explicitly.

Another consequence of the replacement lemma is in the case where the operator norms of the generators in the approximate representations are at most one.

**Proposition 3.11.** If a finitely presented algebra \( \mathcal{L} = \mathbb{C}^*(S : R) \) is archimedean with \( \vartheta_{\mathcal{L}} = 1 \) and \( R \) contains a subset of relations \( W \) for which the algebra \( \mathbb{C}^*(S : W) \) is stable and archimedean, then replacing the bounded \( \epsilon \)-representation \( \phi : \mathbb{C}^*(S) \to \mathcal{L}(H) \) by an approximate representation \( \psi : \mathbb{C}^*(S) \to \mathcal{L}(H) \) that descends to a \( * \)-homomorphism...
\[ \tilde{\psi} : C^* \langle S : W \rangle \to \mathcal{L}(H), \] 
will be an \( O(\epsilon) \)-representation on \( C^* \langle S : R \rangle \) and the constant depends only on the presentation of \( \mathcal{A} \).

**Proof.** The proof is identical to the proof of Lemma 3.8. In particular, since \( \theta_{\mathcal{A}} \leq 1 \) we see that Eq. (3.1) is bounded by \( L_0 \) (see Remark 3.10). Following through the rest of the proof, we see that the resulting constant \( K \) depends only on the presentation of \( \mathcal{A} \). \( \square \)

Another important class of \(*\)-algebras/reations we consider in this work are the **unitary relations** \( \{ s^*s - 1, s^*s - 1 : \) for all \( s \in S \} \subset R \). Suppose that \( C^* \langle S : R \rangle \) is a finitely presented \(*\)-algebra and \( R \) contains the unitary relations. If \( \phi \) is an \( \epsilon \)-representation of \( C^* \langle S : R \rangle \) and \( \phi \) satisfies the unitary relations, then we call \( \phi \) a **unitary \( \epsilon \)-representation.** Remark that every unitary approximate representation is bounded and in particular \( \kappa_{\phi} \leq 1 \).

**Proposition 3.12.** For any \( d \times d \) matrix \( X \) with \( \|X\|_{op} \leq 1 \), there is a unitary \( W \) such that
\[ \|X - W\|_f \leq \|X^*X - 1\|_f. \]

**Proof.** Consider the singular value decomposition \( X = UV \) so that \( \Sigma \) is a diagonal matrix with non-negative singular values \( \sigma_j \in [0, 1] \) for all \( 1 \leq j \leq d \). Let \( W = UV \) and observe that
\[ \|X - UV\|_f = \|U(\Sigma - 1)V\|_f = \|\Sigma - 1\|_f \]
\[ \leq \|((\Sigma - 1)(\Sigma + 1))\|_f = \|\Sigma^2 - 1\|_f = \|V^*(\Sigma U^*U \Sigma - 1)V\|_f \]
\[ = \|V^*\Sigma^*U^*U \Sigma V - V^*V\|_f = \|X^*X - 1\|_f, \]
since \( 1 - \sigma_j^2 = (1 - \sigma_j)(1 + \sigma_j) \geq 1 - \sigma_j \) for all \( \sigma_j \in [0, 1] \). \( \square \)

On their own, we see that the unitary relations are \( \epsilon \)-stable provided the initial approximate unitary has singular values at most 1 (i.e. it is bounded). If the largest singular value of \( X \) is greater than 1, one could naively take the normalization \( \tilde{X} \) with respect to \( \|\cdot\|_{op} \), so that \( \tilde{X} \) has singular value 1. However, in this case the distance \( \|X - \tilde{W}\|_f \) to \( W \) will depend on \( \|X\|_{op} \), where \( W \) is the unitary in the singular value decomposition of \( \tilde{X} \). In the worst case, without any prior bound on \( \|X\|_{op} \), we have that \( \|X\|_{op} \leq \sqrt{d}\|X\|_f \), but the resulting stability would not be Hilbert space free (as it would depend on \( d \)) even if \( \|X\|_f \) is explicitly bounded.

### 3.2. Stability for self-adjoint unitaries and PVM algebras

For our applications to nonlocal games in Section 4, we focus our attention on two important finitely presented algebras: the algebra of self-adjoint unitaries, and the algebra of projective measurement operators (PVMs). Collections of self-adjoint unitaries arise in the context of quantum strategies for boolean constraint system nonlocal games, where the measurement operators can be taken without loss of generality to be boolean (\( \pm 1 \)-valued) observables. With this in mind, we define the finitely presented \(*\)-algebra of self-adjoint unitaries, denoted by

\[ \mathcal{U}_n = C^*(x_1, \ldots, x_n : x_i^2 - 1, x_i^*x_i - 1, x_i x_i^* - 1, x_i^* x_i^2 - 1 \text{ for all } 1 \leq i \leq n). \]

We remark that \( \mathcal{U}_n \) is archimedean with radius \( \delta_{\mathcal{U}_n} = 1. \)
Lemma 3.13. (\textit{MSZ23} [Lemma 3.10]) If $A$ is a $d \times d$ matrix which satisfies (i) $\|A^2 - 1\|_f \leq \epsilon$, (ii) $\|A^* A - 1\|_f \leq \epsilon$, (iii) $\|A A^* - 1\|_\rho \leq \epsilon$, (iv) $\|A^{*2} - A\|_f \leq \epsilon$, and (v) $\|A - A^*\|_f \leq \epsilon$, then there exists an self-adjoint unitary $\tilde{A}$ such that $\|A - \tilde{A}\|_f \leq 2\epsilon$.

The idea in the proof is to pick the unitary $\tilde{A} = \text{sgn}(A^{*+} A)$. The result in \textit{MSZ23} show that the above holds in the state-dependent case as well. We refer the reader to the proof in \textit{MSZ23}, Lemma 3.13 gives the following immediate result.

Corollary 3.14. The $*$-algebra of self-adjoint unitaries $\mathbb{C}^* \langle x_1, \ldots, x_n : x_i^2 - 1, x_i^* x_i - 1, x_i x_i^* - 1, x_i^* - 1, x_i - x_i^* \rangle$ for all $1 \leq i \leq n$ is stable with respect to matrices and $\| \cdot \|_f$.

We note that the stability of $\mathcal{U}_n$ is Hilbert space free in the sense that there is no dependence on $d$, additionally it does not depend on $\kappa_\phi$! Another stability result we require concerns the stability of the group algebra $\mathbb{C}Z_2^n$, which is equivalent to the $*$-algebra of self-adjoint unitaries $\mathcal{U}_n$ modulo the $*$-ideal generated by the commutators $[x_i, x_j] = x_i x_j - x_j x_i$ for all $1 \leq i \neq j \leq n$.

Lemma 3.15. (\textit{Slo19} [Lemma 24]) There exists a constant $C > 0$, such that if $\phi$ is an $\epsilon$-representation of the group algebra $\mathbb{C}Z_2^n$ in $M_d(\mathbb{C})$ then there is a representation $\psi$ of $\mathbb{C}Z_2^n$ in $M_d(\mathbb{C})$ such that $\|\psi(s_i) - \phi(s_i)\|_f \leq C \epsilon$, for all $1 \leq i \leq n$. In particular, $\mathbb{C}Z_2^n$ is stable with respect to $M_d(\mathbb{C})$ and $\| \cdot \|_f$.

We refer the reader to the proof in \textit{Slo19}. We remark that although they consider unitary approximate representations (which are bounded), by our result Corollary 3.14 stability in that case is sufficient. Since by Lemma 3.8 we can first obtain an $O(\epsilon)$-representation that is self-adjoint and unitary. However, we note that the constant $C$ could in this case depend on $n$ and $\kappa_\phi$, however, since we treat $n$ as a fixed parameter it does not affect the stability asymptotically. Furthermore, if we assume $\phi$ is a bounded approximate representation, then the quality of the resulting approximate representation depends only on the presentation $\mathcal{A}$ by Proposition 3.11.

The other important $*$-algebra comes from projective quantum measurements.

**Definition 3.16.** The PVM algebra $\mathcal{A}^{(\mathcal{T}, \mathcal{O})}_{PV, M}$ is the $*$-algebra:

$$
\mathbb{C}^* \langle \{p_a\}_{a \in \mathcal{O}, i \in \mathcal{T}} : p_a^2 - p_a = 0, p_a^* p_a = p_a p_a^* = p_a, p_a^* - p_a = p_a - p_a^*, p_a^* i - p_a i^* = 0 \rangle
$$

Satisfying the additional relations:

(i) $p_a p_b$ for all $a \neq b \in \mathcal{O}$ (mutual orthogonality), and

(ii) $1 - \sum_{a \in \mathcal{O}} p_a^2$ for each $i \in \mathcal{T}$ (completeness).

Like the algebra of self-adjoint unitaries, this algebra is also archimedean with $\vartheta_{\mathcal{A}^{(\mathcal{T}, \mathcal{O})}_{PV, M}} = 1$. This follows from noting it is a quotient of the $*$-algebra of positive contractions. We claim that this $*$-algebra is stable with respect to matrices and $\| \cdot \|_f$. We first collect some results, which are almost certainly known to experts.

**Lemma 3.17.** If $A$ is a $d \times d$ matrix which satisfies (i) $\|A^2 - A\|_f \leq \epsilon$, (ii) $\|A^* A - A\|_f \leq \epsilon$, (iii) $\|A A^* - A\|_f \leq \epsilon$, (iv) $\|A^{*2} - A\|_f \leq \epsilon$, then there exists an orthogonal projection $\tilde{A}$ such that $\|A - \tilde{A}\|_f \leq 2(\sqrt{2} + 1) \epsilon$.

Before we prove Lemma 3.17 we establish several intermediate claims.
Proposition 3.18. ([KPS18](Lemma 3.4)) If $C$ is $d \times d$ positive contraction, then there exists a matrix $P$, such that $P^2 = P$ and $P^* = P$, and moreover $\|C - P\|_f \leq 2\sqrt{2}\|C^2 - C\|_f$.

For a positive contraction $C$, we call the orthogonal projection $P$ in Proposition 3.18 the projective part of $C$ and denote it going forward as $C_{(0,1)}$.

Proposition 3.19. If $B$ is a $d \times d$ positive (semidefinite) matrix then there exists a positive contraction $D$ with the property that $\|B - D\|_f \leq \|B^2 - B\|_f$.

Proof. Let $\{\lambda_1, \ldots, \lambda_d\}$ be the eigenvalues of $B$, and let $V \subseteq \mathbb{C}^d$ be the image of the joint spectral projections $\{\Pi_{\lambda_i} : \lambda_i \in [0,1]\}$ of $B$ whose corresponding eigenvalues $\lambda_i$ are contained in the interval $[0,1]$ for $1 \leq i \leq d$. We define $D$ as the operator which, when restricted to $V$ is equal to $B$ (i.e. $D|_V = B|_V$). The space orthogonal to $V$ is the image of all spectral projections $\{\Pi_{\lambda_i} : \lambda_i > 1\}$ of $B$ for which the corresponding eigenvalues are strictly greater than 1. On $V^\perp$, we define $D$ to be equal to this projection with eigenvalue 1 (i.e. $D|_{V^\perp}$ is the identity matrix). By construction $D$ a positive contraction. Moreover, the operator $B - D$ has eigenvalues

$$\mu_i = \begin{cases} 0, & \text{if } 0 \leq \lambda_i \leq 1 \\ \lambda_i - 1, & \text{if } \lambda_i > 1, \end{cases}$$

for $1 \leq i \leq d$. Now, if $\lambda_i \in \mathbb{R}$ satisfies $\lambda_i > 1$ then we observe that

$$\lambda_i^2 - \lambda_i = \lambda_i(\lambda_i - 1) > (\lambda_i - 1).$$

On the other hand, the operator $B^2 - B$ has spectrum consisting of the eigenvalues $\lambda_i^2 - \lambda_i$ for $1 \leq i \leq d$. The result follows from the calculation

$$\|B - D\|_f^2 = \frac{1}{d} \sum_{i=1}^d \mu_i^2 = \frac{1}{d} \sum_{\lambda_i > 1} (\lambda_i - 1)^2 \leq \frac{1}{d} \left( \sum_{\lambda_i > 1} (\lambda_i^2 - \lambda_i)^2 + \sum_{\lambda_i \leq 1} (\lambda_i^2 - \lambda_i)^2 \right) = \|B^2 - B\|_f^2,$$

where there is equality if $\lambda_i \in \{0,1\}$ for all $1 \leq i \leq d$. 

We call the matrix $D$ in Proposition 3.19 the contractive part of the positive matrix $B$, and denote it by $B_{[0,1]}$ going forward. We are now ready to prove Lemma 3.17.

Proof of Lemma 3.17. To begin, it is clear that the matrix $A_+ = \left(\frac{A^+ + A}{2}\right)^2$ is positive (semi-definite). Furthermore, we observe that $A_+$ is close to $A$ since

$$\|A - \left(\frac{A^+ + A}{2}\right)^2\|_f \leq \frac{1}{4} \left( \|A - A^2\|_f + \|A - A^*A\|_f + \|A - AA^*\|_f + \|A - A^+\|_f \right) \leq \epsilon,$$

by using properties (i)-(iv). Next, we consider the contractive part of $A_+$, which is defined in Proposition 3.19 and we denote by $A_{[0,1]}$. If we let $\{\lambda_1, \ldots, \lambda_d\}$ be the eigenvalues of $A_+$, then we observe that

$$\|A_{[0,1]}^2 - A_{[0,1]}\|_f^2 = \frac{1}{d} \sum_{\lambda_i \in [0,1]} (\lambda_i^2 - \lambda_i)^2 \leq \frac{1}{d} \left( \sum_{\lambda_i \in [0,1]} (\lambda_i^2 - \lambda_i)^2 + \sum_{\lambda_i > 0} (\lambda_i^2 - \lambda_i)^2 \right) = \|A_+^2 - A_+\|_f^2.$$
Next, we see that
\[
\|A^2_+ - A_+\|_f \leq \frac{1}{4} \left( \|A^2 - A^*\|_f + \|AA^* - A^*\|_f + \|AA^* - A\|_f + \|A^2 - A\|_f \right) \leq \epsilon,
\]
again using (i)-(iv) and the fact that \(\|A^2 - A^*\|_f = \|A^2 - A\|_f\) and \(\|AA^* - A^*\|_f = \|AA^* - A\|_f\). For the final step of the proof we let \(\tilde{A}\) be the projective part \(A_{(0,1)}\) of the positive contraction \(A_{(0,1)}\), then by the triangle inequality along with Proposition 3.18 and equations Eq. (3.4), Eq. (3.5), and Eq. (3.6) we see that
\[
\|\tilde{A} - A\|_f \leq \|A_{(0,1)} - A_{(0,1)}\|_f + \|A_{(0,1)} - A_+\|_f + \|A_+ - A\|_f \\
\leq 2\sqrt{2}\|A^2_{(0,1)} - A_{(0,1)}\|_f + \|A^2_+ - A_+\|_f + \|A_+ - A\|_f \\
\leq 2\sqrt{2}\epsilon + 2\epsilon \\
= 2(\sqrt{2} + 1)\epsilon,
\]
as desired. \(\Box\)

**Corollary 3.20.** The \(\ast\)-algebra of orthogonal projections \(\mathbb{C}^\ast\langle p_1, \ldots, p_m : p_i^2 - p_i, p_ip_j - p_i, p_ip_i - p_i, \text{ for all } 1 \leq i \leq m \rangle\) is stable with respect to matrices and \(\|\cdot\|_f\).

Again we note that this stability is Hilbert space free in that there is no dependence of \(d\) nor on the operator norms of the elements in the approximate representations.

**Remark 3.21.** If \(s\) is the largest singular value of \(d \times d\) matrix \(A\), and \(A\) satisfies \(\|A^2 - A^*\|_f \leq \delta\) and \(\|A - A^*\|_f \leq \delta\) for some \(\delta > 0\), then properties (i), (ii), (ii), and (iv) in Lemma 3.17 all hold with \(\epsilon = (4s + 1)\delta\). This suggests that the relations \(p_i^2 - p_i\) and \(p_ip_i - p_i\) in the orthogonal projection algebra are fundamental in obtaining a stability result that is independent of the operator norm.

The stability of the PVM algebra comes from the following lemma.

**Lemma 3.22.** ([PP23][Lemma 2.47] & [Har24][Remark 2.8]) There exists a constant \(C > 0\), such that if \(\epsilon > 0\), and \(A_1, \ldots, A_n\) be positive contractions in \(M_d(\mathbb{C})\) with the property that (i) \(\sum_{i=1}^{m} \|A_i^2 - A_i\|_f \leq \epsilon\), (ii) \(\sum_{1 \leq i < j \leq n} \|A_iA_j\|_f \leq \epsilon\), (iii) \(\sum_{i=1}^{m} A_i - 1\|_f \leq \epsilon\), then there exists a collection of orthogonal projections \(P_1, \ldots, P_n\) such that \(P_iP_j = 0\) for all \(1 \leq i \neq j \leq m\), \(\sum_{i=1}^{m} P_i = 1\), and \(\|A_i - P_i\|_f \leq C\) for all \(1 \leq i \leq n\).

We refer to the proof in [Har24][Lemma 2.7]. Both proofs are based on techniques presented in [KPS18][Lemma 3.5]. We note that the constant \(C\) in Lemma 3.22 depends exponentially on \(m\), however, in our case, we treat \(m\) as a fixed parameter so this is not an issue in this work.

**Corollary 3.23.** \(A_{PV_M}^{(\mathcal{L})}\) is stable with respect to \(M_d(\mathbb{C})\) and \(\|\cdot\|_f\).

### 3.3. Rounding and the approximate tracial property

We now move on to establishing the key technical result in this work. Before we state the result we define and review a property of state-dependent approximate representations called the approximate tracial property.

**Definition 3.24.** Let \(\rho \in \mathcal{L}(H)\) be a density matrix and \(\mathcal{A} = \mathbb{C}^\ast\langle S : R \rangle\) a finitely presented \(\ast\)-algebra. An \((\epsilon, \rho)\)-representation \(\phi : \mathbb{C}^\ast\langle S \rangle \to \mathcal{L}(H)\) is \(\delta\)-tracial if
\[
\|\phi(s)\sqrt{\rho} - \sqrt{\rho}\phi(s)\|_F \leq \delta,
\]
for all \(s \in S\).
We make a few remarks: first, unlike our earlier properties, this property is defined in terms of the (unnormalized) Frobenius norm $\| \cdot \|_F$. Secondly, if an $(\epsilon, \rho)$-representation $\phi$ is $(0, \rho)$-tracial, then the linear functional $\text{tr}(\phi(x)\rho)$ has the tracial (cyclic) property $\text{tr}(\phi(x)\phi(y)\rho) = \text{tr}(\phi(y)\phi(x)\rho)$ for any $x, y \in X$. Trace linear functionals, or \textit{tracial states}, play an important role in the representation theory of finite-dimensional $C^*$-algebras and so it is not surprising to have something resembling these in the approximate case. The approximately tracial property is a requirement to state our main rounding lemma.

**Definition 3.25.** A finitely-presented $*$-algebra $\mathbb{C}^*\langle S : R \rangle$ is \textbf{generated by self-adjoint unitaries} if $R$ contains the relations $W = \{s^2 - 1, s^*s - 1, ss^* - 1, s^2 - 1, s^* - s\}$ for all $s \in S$. Moreover, we say that $\phi : \mathbb{C}^*\langle S \rangle \to \mathcal{L}(H)$ is a \textbf{self-adjoint unitary approximate representation} if $\phi(r) = 0$ for all $r \in W$. In particular, every finitely presented $*$-algebra generated by self-adjoint unitaries is a quotient of the algebra of self-adjoint unitaries $U_S$. Such algebras are archimedean with bounded radius at most 1. These algebras are the subject of the following rounding lemma.

**Lemma 3.26.** Let $\mathcal{G} = \mathbb{C}^*\langle S : R \rangle$ be a finitely-presented $*$-algebra generated by self-adjoint unitaries. If $H$ is a finite-dimensional Hilbert space, $\varphi : \mathbb{C}^*\langle S \rangle \to \mathcal{L}(H)$ a self-adjoint unitary $(\epsilon, \rho)$-representation of $\mathcal{G}$, and $\varphi$ is $\epsilon$-tracial, then there exists a non-zero subspace $\tilde{H}$ of $H$ and a state-independent $O(\epsilon^{1/2})$-representation $\phi : \mathbb{C}^*\langle S \rangle \to \mathcal{L}(\tilde{H})$. Moreover, we can choose $\phi$ to be a self-adjoint unitary approximate representation of $\mathcal{G}$ on $\tilde{H}$.

Notably, there is no dependence on the dimension of $H$ nor of $\tilde{H}$ in any of the resulting approximate representations. Lemma 3.26 can be seen as an extension of the proof of Theorem 5.1 in [SV18] from certain groups to certain $*$-algebras. Before we prove Lemma 3.26, we collect some facts and definitions that are required for the proof. Let $\chi_I$ be the indicator function for the real interval $I \subseteq \mathbb{R}$. Then for a self-adjoint operator $T \in \mathcal{L}(H)$ and measurable subset $I \subseteq \mathbb{R}$, the operator $\chi_I(T)$ is the spectral projection onto $I \cap \text{spec}(T)$, where $\text{spec}(T)$ is the spectrum of $T$. For $\alpha \in \mathbb{R}$, we let $(\geq \alpha)$ denote the interval $[\alpha, +\infty)$ so that we can express the spectral projection onto $\text{spec}(T) \cap [\alpha, +\infty)$ as the operator $\chi_{\geq \alpha}(T)$. The following result is a finite-dimensional version of the “Connes’s joint distribution trick” [Con76].

**Proposition 3.27.** If $\lambda$ and $\lambda'$ are positive semi-definite operators on a finite-dimensional Hilbert space $H$, then

\begin{equation}
\int_0^{+\infty} \|\chi_{\geq \sqrt{\alpha}}(\lambda) - \chi_{\geq \sqrt{\alpha}}(\lambda')\|_F^2 \, d\alpha \leq \|\lambda - \lambda'\|_F \|\lambda + \lambda'\|_F.
\end{equation}

Rather than giving the proof we direct the reader to the concise proof in [SV18] Lemma 5.5. Readers wishing to see the more general case can consult the seminal work [Con76] Lemma 1.2.6. Along with the above technical result, we require the following simple result.

**Proposition 3.28.** Let $B$ be a positive semi-definite operator on a finite-dimensional Hilbert space

\[ \int_0^{+\infty} \chi_{\geq \sqrt{\alpha}}(B) \, d\alpha = B^2. \]
Again, a short proof can be found in [SV18][Lemma 5.6]. In particular, if \( \lambda = \sqrt{\rho} \) for a density matrix \( \rho \), then we see that

\[
\int_0^{+\infty} \text{tr} \left( \chi_{\geq \sqrt{\pi}(\lambda)} \right) d\alpha = 1.
\]

In the preliminaries, we defined a unitary part of a matrix. However for Lemma 3.26 we need to show that a particular unitary part or a self-adjoint matrix exists. In particular, there exists a unitary part that when restricted to a certain subspace is a unitary matrix on that subspace).

**Lemma 3.29.** Let \( X \) be \( d \times d \) self-adjoint matrix and \( P \) be a \( d \times d \) orthogonal projection. There exists a unitary part of \( PXP \) which restricts to a unitary on \( \text{Im}(P) \).

The proof can be found in the appendix.

**Proof of Lemma 3.26.** Suppose \( \mathcal{A} = \mathbb{C}^*(S : R) \) is a finitely presented \( \ast \)-algebra and \( \varphi : \mathbb{C}^*(S) \to \mathcal{L}(H) \); sending \( s_i \to X_j \), is a self-adjoint unitary \((\epsilon, \rho)\)-representation that is \( \epsilon \)-tracial. Let \( \varphi(r) \) represent the image of the relations \( r \in R \) in the self-adjoint unitary representatives \( \{X_1, \ldots, X_n\} \). We begin by showing that there is a non-zero orthogonal projection \( P \) on \( H \) for which:

\[
\int_0^{+\infty} \left\| X_j P - PX_j \right\|_F^2 d\alpha = O(\epsilon^{1/2}) \text{tr}(P)^{1/2} \quad \text{for } 1 \leq j \leq n, \quad \text{and}
\]

\[
\int_0^{+\infty} \left\| \varphi(r) P \right\|_F^2 d\alpha = O(\epsilon^{1/2}) \text{tr}(P)^{1/2} \quad \text{for all } r \in R.
\]

Both claims follow from Proposition 3.27. In particular, for each \( 1 \leq j \leq n \), we see that

\[
\int_0^{+\infty} \left\| X_j \chi_{\geq \sqrt{\alpha}(\lambda)} - X_j \chi_{\geq \sqrt{\alpha}(\lambda)} \right\|_F^2 d\alpha
\]

\[
= \int_0^{+\infty} \left\| \chi_{\geq \sqrt{\alpha}(\lambda)} - X_j^* \chi_{\geq \sqrt{\alpha}(\lambda)} X_j \right\|_F^2 d\alpha
\]

\[
\leq \left\| \lambda - X_j^* \lambda X_j \right\|_F \left\| \lambda + X_j^* \lambda X_j \right\|_F
\]

\[
= \left\| X_j \lambda - \lambda X_j \right\|_F \left\| X_j \lambda + \lambda X_j \right\|_F
\]

\[
\leq 2 \left\| X_j \lambda - \lambda X_j \right\|_F
\]

\[
= O(\epsilon),
\]

by the \( \epsilon \)-tracial property of the approximate representation \( \varphi \). Additionally, by Proposition 3.28 we have

\[
\int_0^{+\infty} \left\| \varphi(r) \chi_{\geq \sqrt{\alpha}(\lambda)} \right\|_F^2 d\alpha = \left\| \varphi(r) \right\|_{\rho}^2 = O(\epsilon^2),
\]

for each of the relations \( r \in R \). Recall that for \( \epsilon \leq 1 \), we have \( \epsilon^2 \leq \epsilon \). Hence, we have that \( O(\epsilon^2) = O(\epsilon) \). Therefore, summing over all relations \( r \in R \) we see that

\[
\int_0^{+\infty} \left( \sum_{j=1}^{n} \left\| \chi_{\geq \sqrt{\pi}(\lambda)} - X_j^* \chi_{\geq \sqrt{\pi}(\lambda)} X_j \right\|_F^2 + \sum_{r \in R} \left\| \varphi(r) \chi_{\geq \sqrt{\pi}(\lambda)} \right\|_F^2 \right) d\alpha
\]

\[
\leq O(\epsilon) \int_0^{+\infty} \text{tr} \left( \chi_{\geq \sqrt{\pi}(\lambda)} \right) d\alpha,
\]
holds for any \( \alpha \geq 0 \). Moreover, each integrand is zero if \( \alpha > \| \lambda \|_{op}^2 \), by the definition of \( \chi_{\geq \sqrt{\alpha}}(\lambda) \). So there exists a value \( 0 < c_0 \leq \| \lambda \|_{op}^2 \) such that if we set \( P := \chi_{\geq \sqrt{c_0}}(\lambda) \), then \( P \) is a non-zero projection and

\[
(3.12) \quad \sum_{j=1}^{n} \| X_j P - PX_j \|_F^2 + \sum_{r \in R} \| \varphi(r) P \|_F^2 = O(\epsilon) \text{tr}(P).
\]

This in turn, bounds each summand on the left hand side of equation Eq. (3.12) by \( O(\epsilon) \text{tr}(P) \) as all the terms are positive, establishing that both Eq. (3.8) and Eq. (3.9) hold. Let \( \tilde{X}_j \) be the unitary part of \( PX_j P \) that restricts to a unitary on \( Im(P) \). We note that such unitaries exists by Lemma 3.29. We now show that the following holds:

(a) \( \| \tilde{X}_j - X_j P \|_F = O(\epsilon^{1/2}) \text{tr}(P) \) for all \( 1 \leq j \leq n \), and

(b) if \( X_{j_1} \cdots X_{j_k} \) for \( 1 \leq j_1, \cdots, j_k \leq n \) is a word of length \( k \), then

\[
(3.13) \quad \| X_{j_1} \cdots X_{j_k} P - \tilde{X}_{j_1} \cdots \tilde{X}_{j_k} \|_F = O(\epsilon^{1/2}) \text{tr}(P)
\]

where the constant depends only on \( k \), for each \( 1 \leq j \leq n \).

We begin by proving (a), since \( (PX_j P)^* = PX_j P \), we have that

\[
(3.14) \quad \| (PX_j P)^2 - P \|_F = \| PX_j PX_j P - P^3 \|_F \leq \| PX_j PX_j - P \|_F = \| PX_j - X_j P \|_F,
\]

is \( O(\epsilon^{1/2}) \text{tr}(P)^{1/2} \) using the unitary invariance of \( \| \cdot \|_F \) along with the fact that \( P = P^* P \leq 1 \). Now, observe that \( \| PX_j P \|_{op} \leq \| X_j \|_{op} \leq 1 \), hence using Proposition 3.12 we can conclude that

\[
(3.15) \quad \| \tilde{X}_j - X_j P \|_F = O(\epsilon^{1/2}) \text{tr}(P).
\]

The matrix \( \tilde{X}_j \) is a unitary by recalling its a unitary part of \( PX_j P \). Before continuing, observe that we can use Eq. (3.15) to show that

\[
(3.16) \quad \| PX_j P - X_j P \|_F = \| PX_j P - X_j P^2 \|_F \leq \| PX_j - X_j P \|_F \| P \|_{op} = O(\epsilon^{1/2}) \text{tr}(P)^{1/2},
\]

and by the triangle inequality:

\[
(3.17) \quad \| \tilde{X}_j - X_j P \|_F \leq \| \tilde{X}_j - PX_j P \|_F + \| PX_j P - X_j P \|_F = O(\epsilon^{1/2}) \text{tr}(P)^{1/2}.
\]

The result follows since \( (PX_j P)^* (PX_j P) \) is almost the identity on \( Im(P) \).

For (b) we note that \( \tilde{X}_j = P \tilde{X}_j \) for any \( 1 \leq j \leq n \), and therefore

\[
\| X_{j_1} \cdots X_{j_k} P - \tilde{X}_{j_1} \cdots \tilde{X}_{j_k} \|_F \leq O(\epsilon^{1/2}) \text{tr}(P)^{1/2}
\]

\[
+ \| X_{j_1} \cdots X_{j_{k-1}} P \tilde{X}_{j_k} - \tilde{X}_{j_1} \cdots \tilde{X}_{j_{k-1}} \|_F.
\]

The result follows by induction on \( k \in \mathbb{N} \).

We now conclude the proof by showing that the map \( \phi : \mathbb{C}^* \langle S \rangle \to \mathcal{L}(Im(P)) \) sending \( s_j \mapsto \tilde{X}_j \) is an \( O(\epsilon^{1/2}) \)-representation on \( Im(P) \subset H \) with respect to \( \| \cdot \|_f \). By construction, we know that \( \tilde{X}_j \) is a unitary on \( Im(P) \). To see that each \( \tilde{X}_j \) is close to an involution on \( Im(P) \), we verify

\[
\| \tilde{X}_j^2 - P \|_f = \frac{1}{\text{tr}(P)^{1/2}} \| \tilde{X}_j^2 - P \|_F
\]

\[
\leq \frac{1}{\text{tr}(P)^{1/2}} \left( \| \tilde{X}_j^2 - (PX_j P)^2 \|_F + \| (PX_j P)^2 - P \|_F \right)
\]
are already self-adjoint and unitary.

Results because the approximate representations that come from quantum strategies depend on the case by the result of [MSZ23]. We note that this does not affect our main assumption since the stability of the self-adjoint unitary algebra holds in the state-dependent case by Lemma 3.26. To do this, one would need a state-dependent version of the replacement assumptions from the state-dependent approximate representation in the hypothesis of Lemma 3.26. However, in the application of strategies for nonlocal games, we always obtain an approximate representation where both \( \gamma \) and \( \delta \) are determined by the winning probability of the game, so this technicality does not arise.

For the remaining relations, we recall the image of each relation under the approximate representation \( \varphi : \mathbb{C}^n(S) \to \mathcal{L}(\mathcal{H}) \) is a *-polynomial \( \phi(r) = \sum_{M \subseteq [n]} \gamma_M \prod_{j \in M} \varphi(s_j) \) where \( \gamma_M \) are coefficients. By the triangle inequality, it suffices to bound each of the monomials. However, each monomial is bounded by the proofs of claims (a) and (b) earlier. Hence, for any relation \( r \in R \) of \( \mathcal{G} \) we conclude that

\[
\| \phi(r) \|_f = \frac{1}{\text{tr}(P)^{1/2}} \| \phi(r) \|_F \leq \frac{1}{\text{tr}(P)^{1/2}} (\| \phi(r) - \varphi(r)P \|_F + \| \varphi(r)P \|_F )
\]

\[
\leq \frac{1}{\text{tr}(P)^{1/2}} \sum_{M \subseteq [n]} \gamma_M \| \prod_{j \in M} X_j - \prod_{j \in M} X_j P \|_F + O(\epsilon^{1/2})
\]

\[
\leq \frac{1}{\text{tr}(P)^{1/2}} \sum_{M \subseteq [n]} |\gamma_M| \sum_{j \in M} \| \tilde{X}_j - X_j P \|_F + O(\epsilon^{1/2})
\]

\[
\leq \sum_{M \subseteq [n]} |\gamma_M| \sum_{j \in M} O(\epsilon^{1/2}) + O(\epsilon^{1/2})
\]

is \( O(\epsilon^{1/2}) \) completing the proof. \( \square \)

The assumption on the size of \( \epsilon \) in Lemma 3.26 depends on the relationship between the quantities in Eq. (3.10). In particular, for the regime of interest with \( \epsilon \to 0 \), we take the larger of the two quantities. One could imagine a version of Lemma 3.26 with an \((\epsilon, \rho)\)-representation that is \( \delta \)-tracial. The resulting approximate representation would ultimately depend on both \( \epsilon \) and \( \delta \). However, in the application of strategies for nonlocal games, we always obtain an approximate representation where both \( \delta \) and \( \epsilon \) are determined by the winning probability of the game, so this technicality does not arise.

An interesting open question is whether you can remove the self-adjoint unitary assumptions from the state-dependent approximate representation in the hypothesis of Lemma 3.26. To do this, one would need a state-dependent version of the replacement lemma. If something like this holds, then one would expect that it could remove this assumption since the stability of the self-adjoint unitary algebra holds in the state-dependent case by the result of MSZ23. We note that this does not affect our main results because the approximate representations that come from quantum strategies are already self-adjoint and unitary.
4. Near-optimal quantum strategies and nonlocal game algebras

A two-player nonlocal game is a scenario involving two players, Alice and Bob, and a referee. In the game, Alice (resp. Bob) receives questions \( i \in \mathcal{I}_A \) (resp. \( j \in \mathcal{I}_B \)) from the referee according to a probability distribution \( \pi : \mathcal{I}_A \times \mathcal{I}_B \rightarrow \mathbb{R}_{\geq 0} \). Alice (resp. Bob) responds to each question with answers \( a \in \mathcal{O}_A \) (resp. \( b \in \mathcal{O}_B \)). However, once they receive their questions they are not permitted to communicate with each other. The goal of the players is to satisfy the rule predicate \( V : \mathcal{O}_A \times \mathcal{O}_B \times \mathcal{I}_A \times \mathcal{I}_B \rightarrow \{0, 1\} \), a function such that \( V(a, b|i, j) = 0 \) indicates a loss and \( V(a, b|i, j) = 1 \) a win. The description of the game and the predicate is known to the players before the game begins. The goal of the players is to maximize their winning probability.

Although communication between the players is not permitted once the game begins, the players can share a bipartite quantum state. With this resource, the players can make local quantum measurements on their subsystem to obtain their answers. This allows the players to employ quantum correlations in their strategy to win the game. Strategies that employ these quantum correlations are called quantum (or entangled) strategies, while strategies that use only classical correlations are called classical strategies.

**Definition 4.1.** A quantum strategy \( S \) for a nonlocal game \( G \) consists of:

(i) finite-dimensional Hilbert space \( H_A \) and \( H_B \),
(ii) collections of orthogonal projections \( \{P^a_i\}_{a \in \mathcal{O}_A : i \in \mathcal{I}_A} \) acting on \( H_A \), such that \( \sum_a P^a_i = \mathbb{1}_{H_A} \), for all \( i \in \mathcal{I}_A \), and a collection of orthogonal projections \( \{Q^b_j\}_{b \in \mathcal{O}_B : j \in \mathcal{I}_B} \) acting on \( H_B \), such that \( \sum_b Q^b_j = \mathbb{1}_{H_B} \), for all \( j \in \mathcal{I}_B \), and
(iii) a quantum state \( |\psi\rangle \in H_A \otimes H_B \).

These collections of projective-valued measures (PVMs) and a bipartite state correspond to a quantum strategy \( S \) in the sense that they model the correlations

\[
p(a, b|i, j) = \langle \psi | P^a_i \otimes Q^b_j | \psi \rangle,
\]

for all outcomes \( a \in \mathcal{O}_A, b \in \mathcal{O}_B, \) and inputs \( i \in \mathcal{I}_A, \) and \( j \in \mathcal{I}_B, \) that can occur in the game. More generally, quantum correlations can be modelled by positive-operator value-measures (or POVMs) and mixed states (i.e. density operators). However, in this work, we restrict to the class of PVM strategies with pure states. This is justified by the fact that Naimark’s dilation theorem tells us that any correlation achieved by a POVM on a finite-dimensional Hilbert space can be achieved by a projective-value-measure (PVM) on a larger (but still) finite-dimensional Hilbert space. Similarly, a standard purification argument shows that any correlation achieved with a mixed state can be achieved with a pure state\(^3\). Players employing a quantum strategy \( S \) for a nonlocal game \( G \) win with probability

\[
\omega(G; S) = \sum_{i, j \in \mathcal{I}_A \times \mathcal{I}_B} \pi(i, j) \sum_{a, b \in \mathcal{O}_A \times \mathcal{O}_B} V(a, b|i, j) \langle \psi | P^a_i \otimes Q^b_j | \psi \rangle.
\]

The probability \( \omega(G; S) \) is often referred to as the value of \( G \) under \( S \). The optimal winning probability, called the quantum (or entangled) value of the game, is the supremum over all quantum strategies and is denoted by \( \omega^*(G) = \sup_S \omega(G; S) \).

\(^3\)In fact, one can even find a pure state on the same dimensional Hilbert space as the mixed state \( |\psi\rangle \) [SVW16].
Definition 4.2. A strategy \( S \) for a nonlocal game \( \mathcal{G} \) is perfect if \( \omega(S; \mathcal{G}) = 1 \). More generally, a strategy is optimal if \( \omega(\mathcal{G}; S) = \omega^*(\mathcal{G}) \). The notion of near-optimal and near-perfect quantum strategies are natural extensions of the ideal case. That is, a quantum strategy \( S \) is \( \epsilon \)-optimal if \( |\omega(\mathcal{G}; S) - \omega^*(\mathcal{G})| \leq \epsilon \) and is \( \epsilon \)-perfect\(^4\) if \( \omega(\mathcal{G}; S) \geq 1 - \epsilon \).

For convenience, we let \( p_{ij}(S) \) denote the probability of winning with strategy \( S \) given question pair \( (i, j) \in I_A \times I_B \). It is not hard to see that a quantum strategy \( S \) is perfect if and only if \( p_{ij}(S) = 1 \) for all \( (i, j) \). Moreover, any strategy for which \( p_{ij}(S) \geq 1 - \epsilon \) for all questions \( (i, j) \) will be \( \epsilon \)-perfect. However, we note that the converse is not true. Consider the case where the distribution of questions is uniform. In this case, a strategy that is \( \epsilon \)-perfect implies that \( p_{ij}(S) \geq 1 - |O_A||O_B|\epsilon \), since a strategy losing on some question with probability \( |O_A||O_B|\epsilon \) could still win the overall game with probability \( 1 - \epsilon \). Nonetheless, when the questions are asked uniformly the property of the strategy \( S \) being \( \epsilon \)-perfect and the property that \( p_{ij}(S) \geq 1 - \epsilon \) for all \( (i, j) \) are equivalent up to a constant that depends on \( \mathcal{G} \). Hence, we restrict to the uniform case in this work and leave the non-uniform case for future work.

4.1. BCS nonlocal games. We now focus our attention on the study of Boolean constraint system (BCS) nonlocal games. BCS games have previously been called “binary constraint” nonlocal games, for instance in [CM14]; however, we prefer the term “boolean constraint” as it avoids any confusion with the subclass of “2-ary” constraints. Before we define the BCS nonlocal game, we review the formal concept of a boolean constraint system.

We adopt the multiplicative convention and associate \(-1\) with the boolean TRUE value and \(1\) with the boolean FALSE. A \( k \)-ary boolean relation \( \mathcal{R} \) is a subset of \( \{ \pm 1 \}^k \) for \( k > 0 \). Given a set of boolean variables \( V = \{ x_1, \ldots, x_n \} \), a constraint \( \mathcal{C} \) is a pair \( (\mathcal{U}, \mathcal{R}) \) where the the context \( \mathcal{U} \) is the subset of variables \( V \) that make up the constraint \( \mathcal{C} \), and \( \mathcal{R} \) is a \( k \)-ary boolean relation. A satisfying assignment to a constraint \( \mathcal{C} \) is a function \( \phi : V \to \{ \pm 1 \} \) such that \( \phi(\mathcal{U}) \in \mathcal{R} \). A boolean constraint system \( \mathcal{B} \) is a pair \( (V, \{ \mathcal{C}_i \}_{i=1}^m) \), where \( V \) is a set of variables and \( \{ \mathcal{C}_i \}_{i=1}^m \) is a collection of constraints. An assignment to a BCS \( \mathcal{B} \) is a function \( \phi : V \to \{ \pm 1 \} \). The function \( \phi \) is a satisfying assignment if \( \phi(\mathcal{U}_i) \in \mathcal{R}_i \) for all \( 1 \leq i \leq m \). A BCS is satisfiable if it has a satisfying assignment. For a single constraint \( \mathcal{C}_i \) we denote the set of satisfying assignments by \( sat(\mathcal{C}_i) = \{ \phi : \mathcal{U}_i \to \{ \pm 1 \} : \phi(\mathcal{U}_i) \in \mathcal{R}_i \} \).

For \( z \in \{ \pm 1 \}^k \), and for each \( k \)-ary relation \( \mathcal{R} \), we can associate the indicator function \( f_{\mathcal{R}} : \{ \pm 1 \}^k \to \{ \pm 1 \} \) that evaluates to \( f(z) = -1 \) whenever \( z \in \mathcal{R} \) and \( 1 \) otherwise. Given an indicator function \( f_{\mathcal{R}} \) for a \( k \)-ary relation \( \mathcal{R} \) we define the indicator polynomial

\[
(4.2) \quad F_{\mathcal{R}}(\mathcal{U}) = \sum_{z \in \{ \pm 1 \}^k} f_{\mathcal{R}}(z) \prod_{j \in \mathcal{U}} \frac{1 + z_j x_j}{2} = \sum_{M \subseteq \mathcal{U}_i} \gamma_M \prod_{j \in M} x_j,
\]

for some coefficients \( \gamma_M \in \mathbb{R} \). In other words, the indicator polynomial for a constraint \( \mathcal{C} \) is a real multilinear polynomial in the context \( \mathcal{U} \subset V \).

For a constraint \( \mathcal{C} \), the indicator polynomial \( F_{\mathcal{R}}(\mathcal{U}) \) takes the value \(-1\) whenever it is evaluated on a satisfying assignment. Hence, any propositional formula over the

\[^4\text{We note that our definition of \( \epsilon \)-perfect differs from the one used in [SV13].}\]
quantum strategies in terms of these. It follows that $\beta$ and observe that employing the strategy where they both use $\phi$ of questions is uniform on the variables and constraints. A quantum strategy $BCS$ game consists of PVMs $B_i$ satisfying assignment for $I_i$. That is Alice has a projective measurement system over $(\forall \phi \in \{\pm 1\})$ the bias of a strategy $S$ is the probability of winning minus the probability of losing. Let $\beta_{ij}(S) = |\langle \psi | Y_i \otimes X_j | \psi \rangle|$ be the bias on question $(i, j)$ with quantum strategy $S$ and observe that

$$|\langle \psi | Y_i \otimes X_j | \psi \rangle| = \sum_{a \in \text{sat}({\mathcal C}_i)} \sum_{b \in \mathbb{Z}_2} a \cdot b |\langle \psi | P_a^i \otimes Q_b^j | \psi \rangle|$$

$$= 2 \left[ \sum_{b \in \mathbb{Z}_2} \sum_{a \in \text{sat}({\mathcal C}_i)} p(a, b | i, j) \right] - 1$$

$$= 2p_{ij}(S) - 1.$$
Proposition 4.4. Let $\mathcal{S}$ be an $\epsilon$-perfect quantum strategy for a BCS game $\mathcal{G}(\mathcal{B})$ with vector state $|\psi\rangle \in H_A \otimes H_B$. Then there is a collection of $\pm 1$-valued observables $\{Y_{ij}\}_{i,j \in [m] \times [n]}$ in $\mathcal{L}(H_A)$ and $\{X_{ij}\}_{j=1}^n$ in $\mathcal{L}(H_B)$ such that $[Y_{ij}, Y_{ik}] = 0$ for all $j, k$, and $\langle \psi | Y_{ij} \otimes X_j | \psi \rangle \geq 1 - O(\epsilon)$, whenever $\mathcal{V}(a, b, i, j) = 1$, for all $i, j \in [m] \times [n]$.

Proof. Take $Y_{ij}$ and $X_j$ to be the operators defined in Eq. (4.3) for all $1 \leq i \leq m$ and $1 \leq j \leq n$. By construction each $Y_{ij}$ and $X_j$ is a self-adjoint unitary. Moreover, the commutation relations $[Y_{ij}, Y_{ik}] = 0$ hold, since each $Y_{ij}$ and $Y_{ik}$ is a $\mathbb{Z}_2$-linear combinations of orthogonal projections $P_{ij}^a$ for all $1 \leq i \leq m$. Lastly, if there is an $\epsilon$-perfect strategy then $p_{ij}(\mathcal{S}) \geq 1 - \epsilon$ and

$$\langle \psi | Y_{ij} \otimes X_j | \psi \rangle = 2p_{ij}(\mathcal{S}) - 1 \geq 1 - 2\epsilon,$$

whenever $\mathcal{V}(a, b, i, j) = 1$, for all $i, j \in [m] \times [n]$ as desired. $\square$

Cleve and Mittal observed that the collection of mutually commuting $\pm 1$-valued observables $X_j$ derived from a perfect quantum strategy for a BCS nonlocal game satisfy the multilinear polynomials in Eq. (4.2), see [CM14 Theorem 1]. Hence, perfect quantum strategies for a BCS game correspond to a matrix-valued satisfying assignments to the BCS.

Definition 4.5. Given a BCS $\mathcal{S}$ a matrix-valued satisfying assignment for $\mathcal{B}$ is a collection of mutually commuting $\pm 1$-valued observables $\{X_1, \ldots, X_n\}$ such that

$$\mathcal{F}_{\mathcal{S}}(X_1, \ldots, X_{jk}) = -1,$$

for all constraints $(\mathcal{U}_i, \mathcal{R}_i)$, where $j_1, \ldots, j_k \in \mathcal{U}_i$ is an abuse of notation to indicate the index of the variables appearing in $\mathcal{U}_i$, for $1 \leq i \leq m$. Moreover, it is not hard to see that we can write the constraint polynomial as a difference of projections

$$\mathcal{F}_{\mathcal{S}}(\mathcal{U}_i) = \sum_{\phi(\mathcal{U}_i) \in \mathcal{R}_i} \prod_{j \in \mathcal{U}_i} \frac{1 + \phi(x_j)X_j}{2} - \sum_{\phi(\mathcal{U}_i) \in \mathcal{R}_i} \prod_{j \in \mathcal{U}_i} \frac{1 + \phi(x_j)X_j}{2}$$

$$= 2 \sum_{\phi(\mathcal{U}_i) \in \mathcal{R}_i} \prod_{j \in \mathcal{U}_i} \frac{1 + \phi(x_j)X_j}{2} - 1.$$

For convenience, we will denote the orthogonal projections

$$\Pi_{\phi,i}(X_j) = \prod_{j \in \mathcal{U}_i} \frac{1 + \phi(x_j)X_j}{2},$$

where $\phi : \mathcal{U}_i \to \{\pm 1\}$ and $1 \leq i \leq m$.

It is not hard to see that the converse is also true. Specifically, given a matrix satisfying assignment of $\pm 1$-valued observables $\{X_1, \ldots, X_n\}$ of dimension $d$, the BCS nonlocal game can be played perfectly using a strategy where Bob employs the observables $1 \otimes X_j$. Alice employs the observables $Y_{ij} \otimes 1 = X_j^\dagger \otimes 1$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$, and they share a maximally entangled state $|\tau\rangle = \sum_{i=1}^d |i\rangle \otimes |i\rangle$. We refer the reader to the details in [CM14][Ji13].

The BCS algebra is based on the observation that polynomial constraint relations abstractly define perfect strategies for BCS nonlocal games.

Definition 4.6. The BCS algebra $\mathcal{B}$ of a boolean constraint system $\mathcal{B}$ is the self-adjoint unitary algebra $\mathcal{U}_n$ (defined in Eq. (3.3)) subject to the additional relations:
(1) \( \mathcal{F}_{\mathcal{A}_i}(\mathcal{U}_i) = -1 \) for all \( 1 \leq i \leq m \), and
(2) \( x_j x_\ell = x_\ell x_j \) whenever \( x_j, x_\ell \in \mathcal{U}_i \), for all \( 1 \leq i \leq m \).

Here, each \( \mathcal{F}_{\mathcal{A}_i}(V) = \mathcal{F}_{\mathcal{A}_i}(\mathcal{U}_i) \) is the multilinear indicator polynomial for the constraint \( \mathcal{C}_i \) with context \( \mathcal{U}_i \).

In a perfect strategy for BCS nonlocal games, one can show that Alice and Bob’s measurement operators must be the same on the support of the state (up to transposition in the Schmidt basis of the state vector). For self-adjoint operators, the transpose is equivalent to the conjugate (in that basis). The following lemma shows that a similar result holds in the approximate case.

**Lemma 4.7.** Let \( X \) and \( Y \) be self-adjoint unitary operators on a finite-dimensional Hilbert space \( H \) and \( |\psi\rangle \in H \otimes H \). Then,

\[
\langle \psi | X \otimes Y | \psi \rangle \geq 1 - O(\epsilon),
\]

if and only if

\[
\| Y \lambda - X \lambda \|_F \leq O(\epsilon^{1/2}),
\]

where \( \lambda \) is the square root of the reduced density matrix \( \rho \) for the state \( |\psi\rangle \). Moreover, in any case where the above holds we also have that

(i) \( \| Y \lambda - Y \rho \|_F \leq O(\epsilon^{1/2}) \), and

(ii) \( \| X \lambda - X \rho \|_F \leq O(\epsilon^{1/2}) \).

The proof can be found in [SV18] [Proposition 5.4]. By combining this result with Proposition 4.4 we obtain a simple corollary.

**Corollary 4.8.** Let \( \mathcal{S} \) be a quantum strategy for a BCS game \( \mathcal{G}(\mathcal{B}) \) presented in terms of \( \pm 1 \)-valued observables \( \{ Y_{ij} \}_{i,j=1}^{m,n} \) and \( \{ X_{ij} \}_{j=1}^{n} \), along with a maximally entangled state \( |\tau\rangle \in H_A \otimes H_B \). If

\[
e^2 \geq \| Y_{ij}^\top - X_j \|_F^2 = 2(1 - \langle \tau | Y_{ij} \otimes X_j | \tau \rangle)
\]

for all \( 1 \leq i \leq m \), and \( 1 \leq j \leq n \), then \( p_{ij}(\mathcal{S}) \geq 1 - \frac{e^2}{4} \) for all question pairs \((i, j)\) and \( \mathcal{S} \) is \( e^2/4 \)-perfect.

We now show that Bob’s (or analogously Alice’s) operators in any \( \epsilon \)-perfect strategy to the BCS game is a state-dependant approximate representation of the BCS algebra \( \mathcal{B}(\mathcal{G}) \). In the remainder of the section, we assume that the state \( |\psi\rangle \) is fully supported on the spaces \( H_A \) and \( H_B \). In this case, the \( \rho \)-seminorm is a norm on the space of linear operators on \( H_B \) (or \( H_A \)).

**Proposition 4.9.** Let \( \rho = \lambda^* \lambda \) on \( H_B \) be the reduced density matrix of the state \( |\psi\rangle \in H_A \otimes H_B \) on \( H_B \). If \( \{ Y_{ij} \}_{i,j=1}^{m,n}, \{ X_{ij} \}_{j=1}^{n}, |\psi\rangle \in H_A \otimes H_B \) is an \( \epsilon \)-perfect strategy for the BCS game \( \mathcal{G} \), then \( \{ X_{ij} \}_{j=1}^{n} \) is an \( (O(\epsilon^{1/2}), \rho) \)-representation of the BCS algebra \( \mathcal{B}(\mathcal{G}) \). Moreover, the approximate representation is \( O(\epsilon^{1/2}) \)-tracial.

**Proof.** A priori each \( X_j \) is a self-adjoint unitary, so all that remains is to establish that

(a) \( \| \mathcal{F}_{\mathcal{A}_i}(V) + I \|_\rho \leq O(\epsilon^{1/2}) \) for all \( 1 \leq i \leq m \), and

(b) \( \| X_\ell X_j - X_j X_\ell \|_\rho \leq O(\epsilon^{1/2}) \) whenever \( X_\ell, X_k \in \mathcal{U}_i \), for all \( 1 \leq i \leq m \).
For (a), abusing some notation we let \( \mathcal{W}_i \) be the variables that appear in the \( i \)th constraint polynomial \( \mathcal{F}_{i} \). For convenience, we define \( Z_{ij} := \mathcal{Y}_{ij} \) for all \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \). Then, it is not too hard to show that

\[
\left\| \prod_{j \in \mathcal{W}_i} X_j \lambda - \lambda \prod_{j \in \mathcal{W}_i} Z_{ij} \right\|_F \leq \sum_{j \in \mathcal{W}_i} \|X_j \lambda - \lambda Z_{ij}\|_F,
\]

for any \( M \subset V \).

Since Alice’s observables are a matrix-valued satisfying assignment, we have that \( \mathcal{F}_{i}(Z_{i_1}, \ldots, Z_{i_k}) = -1 \). Thus, since \( \|X_j \lambda - \lambda Z_{ij}\|_F \leq O(\epsilon^{1/2}) \) for each \( j \in M \) by Lemma 4.7, we can deduce that

\[
\left\| \mathcal{F}_{i}(\mathcal{W}_i) + \mathbf{1} \right\|_\rho \leq \sum_{S \subset \mathcal{W}_i} \gamma_M \left\| \prod_{j \in M} X_j \lambda - \lambda (-1) \prod_{j \in M} Z_{ij} \right\|_F \leq \sum_{S \subset \mathcal{W}_i} \|\gamma_M \left\| \prod_{j \in M} X_j \lambda - \lambda \prod_{j \in M} Z_{ij} \right\|_F = O(\epsilon^{1/2}),
\]
as desired. To see that (b) holds, we observe that

\[
\|X_k X_j - X_j X_k\|_\rho \leq \|X_k X_j \lambda - \lambda Z_{ij} Z_{ik}\|_F + \|X_j X_k \lambda - \lambda Z_{jk} Z_{ik}\|_F,
\]
since the variables \( \{Z_j : j \in \mathcal{W}_i\} \) all commute by virtue of Alice employing a valid quantum strategy. Now we apply the inequality from Eq. (4.13) to Eq. (4.14) and conclude that (b) holds by Lemma 4.7. Lastly, the approximate tracial property \( \|X_j \lambda - \lambda X_j\|_F^2 = O(\epsilon) \) for all \( 1 \leq j \leq n \) follows directly from the second statement of Lemma 4.7.

We immediately obtain the following corollary.

**Corollary 4.10.** If the state \( |\psi\rangle \in H_A \otimes H_B \) in the \( \epsilon \)-perfect strategy \( S \) for a BCS game \( G \) is maximally entangled, then the operators \( \{X_j\}_{j=1}^n \) are a state-independent \( O(\epsilon^{1/4}) \)-representation of the BCS algebra \( B \) on \( H_B \).

For arbitrary states we employ the rounding lemma (Lemma 3.26) to obtain our main result.

**Proposition 4.11.** If \( S \) is an \( \epsilon \)-perfect strategy for a BCS nonlocal game \( G \), then restricted to a non-zero subspace of \( H_B \), Bob’s measurement operators are state-independent \( O(\epsilon^{1/4}) \)-representation of the BCS algebra \( B \).

**Proof.** It follows from Proposition 4.9 that any \( \epsilon \)-perfect strategy for a BCS game \( G \) with reduced density matrix \( \rho \), gives a state-dependent \( O(\epsilon^{1/2}) \)-representation of the BCS algebra \( B \) that is \( O(\epsilon^{1/2}) \)-tracial. Since the operators in any strategy are \( \pm 1 \)-valued observables the approximate representation already satisfies the self-adjoint unitary relations exactly. Hence, the result follows by applying Lemma 3.26 which results in an \( O(\epsilon^{1/4}) \)-representation of the BCS algebra in the \( \| \cdot \|_F \)-norm.

To prove Theorem 1.2(1) we show that each \( \epsilon \)-representations of the BCS algebra can be used to obtain near-perfect strategies for the corresponding BCS nonlocal game provided the players share a maximally entangled state. For this result, we appeal to the stability of the group algebra \( \mathbb{C} \mathbb{Z}_2^k \).
Proposition 4.12. If $\phi$ is a bounded $\epsilon$-representation of the BCS algebra $\mathcal{B}$ on a finite-dimensional Hilbert space $H_B$, then there is a $O(\epsilon^2)$-perfect strategy to the BCS game $\mathcal{G}$ using the maximally entangled state $|\tau\rangle \in H_B \otimes H_B$.

Proof. Let $\varphi$ be an $\epsilon$-representation of $\mathcal{B}(\mathcal{G})$ on $H_B$. For a fixed constraint $\mathcal{C}_i$, consider the $\epsilon$-representation restricted to the context $\mathcal{U}_i$. On these variables $\varphi$ is an $\epsilon$-representation of $\mathcal{C}Z_{2^{|\mathcal{U}_i|}}$. By Lemma 3.15 the algebra $\mathcal{C}Z_{2^{|\mathcal{U}_i|}}$ is stable. In particular, there exists an exact representation $\psi_i$ of $\mathcal{C}Z_{2^{|\mathcal{U}_i|}}$ such that $\|\varphi(x_j) - \psi_i(x_j)\|_f \leq C \epsilon$ for all $j \in \mathcal{U}_i$. Define the $\pm 1$-observables $W_{ij} = \psi_i(x_j)$ for all $j \in \mathcal{U}_i$. The collection of observables $\{W_{ij}\}_{j \in \mathcal{U}_i}$ correspond to measuring an assignment $\phi : \mathcal{U}_i \to \{\pm 1\}$, however, it may not be a satisfying assignment for $\mathcal{C}_i$. In particular, if $\Pi_{\phi,i}(W_{ij}) = \prod_{j \in \mathcal{U}_i} \frac{1}{2}(1 + \phi(x_j)W_{ij})$ then the the projection onto unsatisfying assignments $\sum_{\phi(\mathcal{U}_i) \notin \mathcal{U}_i} \Pi_{\phi,i}(W_{ij})$ may be non-zero. However, this happens if and only if an irreducible block of $W_{ij}$ with corresponding weight $\nu(x_j) \in \{\pm 1\}$ does not agree with $\phi(x_j)$ (i.e. if $\phi(x_j) = -\nu(x_j)$).

Hence, multiplying each unsatisfying irreducible block in each $W_{ij}$ by $-1$, leaving the other blocks untouched, we obtain a new $\pm 1$-observable $Y_{ij}$. Repeating this for each $j \in \mathcal{U}_i$ gives us a new collection of observables for which $\sum_{\phi(\mathcal{U}_i) \notin \mathcal{U}_i} \Pi_{\phi,i}(Y_{ij}) = 0$. More precisely, define the new collection of observables

\[
Y_{ij} = \sum_{\phi(\mathcal{U}_i) \notin \mathcal{U}_i} (1 - \Pi_{\phi,i}(W_{ij}))W_{ij} - \sum_{\phi(\mathcal{U}_i) \notin \mathcal{U}_i} \Pi_{\phi,i}(W_{ij})W_{ij},
\]

for all $j \in \mathcal{U}_i$. In addition to these observables being a satisfying assignment for $\mathcal{C}_i$, we have that

\[
\|W_{ij} - Y_{ij}\|_f = \|W_{ij} - \sum_{\phi(\mathcal{U}_i) \notin \mathcal{U}_i} (1 - \Pi_{\phi,i}(W_{ij}))W_{ij} + \Pi_{\phi,i}(W_{ij})W_{ij}\|_f \\
= \|1 - \sum_{\phi(\mathcal{U}_i) \notin \mathcal{U}_i} (1 - \Pi_{\phi,i}(W_{ij})) + \Pi_{\phi,i}(W_{ij})\|_f \\
= \|2 \sum_{\phi(\mathcal{U}_i) \notin \mathcal{U}_i} \Pi_{\phi,i}(W_{ij})\|_f \\
\leq 2\| \sum_{\phi(\mathcal{U}_i) \notin \mathcal{U}_i} \Pi_{\phi,i}(W_{ij}) - \sum_{\phi(\mathcal{U}_i) \notin \mathcal{U}_i} \Pi_{\phi,i}(X_j)\|_f + 2\| \sum_{\phi(\mathcal{U}_i) \notin \mathcal{U}_i} \Pi_{\phi,i}(X_j)\|_f \\
\leq 2 \sum_{\phi(\mathcal{U}_i) \notin \mathcal{U}_i} \|\Pi_{\phi,i}(W_{ij}) - \Pi_{\phi,i}(X_j)\|_f + \epsilon \\
\leq 2 \sum_{j \in \mathcal{U}_i} \|W_{ij} - X_j\|_f + \epsilon \\
\leq 2 \sum_{j \in \mathcal{U}_i} \|\psi_i(x_j) - \varphi(x_j)\|_f + \epsilon \\
= (2|\mathcal{U}_i|C + 1)\epsilon,
\]

for all $j \in \mathcal{U}_i$. This means that the number of assignments in $W_{ij}$ that we had to correct (by multiplying by $-1$) was relatively small, provided we have $\epsilon$ small. Now, consider the strategy consisting of observables $\{X_j\}_{j=1}^n, \{Y_{ij}\}_{i=1,j=1}^{m,n}$ and a maximally entangled state $|\tau\rangle \in H_B \otimes H_B$. By Corollary 4.8 the strategy is $(Cn + 1)\epsilon^2$-perfect, since $\|Y_{ij} - X_j\|_f \leq \|Y_{ij} - W_{ij}\|_f + \|W_{ij} - X_j\|_f = 2(Cn + 1)\epsilon$, for all $1 \leq j \leq n$, and $1 \leq i \leq m$. \qed
Remark 4.13. The only caveat in the case where $\phi$ is not bounded a priori is that the resulting approximate representations could depend on $\kappa_\phi$. If this quantity is independent of $d$ then the result will still be Hilbert space free in this case as well.

As a corollary, we obtain the final result of this section.

Corollary 4.14. For any $\epsilon$-perfect quantum strategy $S$ for a BCS nonlocal game there is an $O(\epsilon^{1/2})$-perfect quantum strategy $\tilde{S}$ using a maximally entangled state $|\tilde{\psi}\rangle$, such that each measurement in $\tilde{S}$ is at most $O(\epsilon^{1/4})$-away from the measurement in $S$ with respect to $\| \cdot \|_f$ on the local support of $|\tilde{\psi}\rangle$ on $H_B$.

4.2. **Synchronous nonlocal games.** Synchronous games are class of nonlocal games introduced in [PSS+16]. In a synchronous nonlocal game $G$, the set of questions and answers are the same for each player, i.e. $I_A = I_B = I$ and $O_A = O_B = O$. Additionally, in a synchronous game, the players lose whenever they give different answers to the same question. This latter property is called the **synchronous condition**. Like BCS nonlocal games there is a finitely presented $*$-algebra we can associate with each synchronous game. Unlike BCS algebras, the associated game algebra is typically presented in terms of collections of orthogonal projections:

**Definition 4.15.** The synchronous algebra $A(G)$ of a synchronous game $G$ is a quotient of the PVM algebra $A_{PVM}(I,O)$ (see Definition 3.16) satisfying the additional rule relations: for all $(a,b,i,j) \in O^2 \times I^2$ if $V(a,b,i,j) = 0$, then $p_a^i p_b^j = 0$.

The class of synchronous algebras is well studied in the literature, and typically in the context of the more general commuting-operator strategies, see for instance [PT15,DP16,HMPS19]. It is not hard to see that any representation of the synchronous on a finite-dimensional Hilbert space $H$, along with a maximally entangled state gives a perfect quantum strategy for the associated synchronous case. The other direction is less straightforward, but also true as a result of [PT15, Theorem 5.5]. Hence, a synchronous nonlocal game has a perfect quantum strategy $S$, if and only if Bob’s PVM measurement operators $\{Q^i_a \mid a \in O : i \in I\}$ restricted to the support of the reduced density matrix $\rho_B$ forms a representation of the synchronous algebra associated with $G$. To analyze near-perfect strategies for synchronous games, we start by mentioning the following simple lemma.

**Lemma 4.16.** (See [Slo11, Lemma 4.1]) If $E$ and $F$ are self-adjoint operators on Hilbert spaces $H_A$ and $H_B$ respectively, and $\lambda = \rho^{1/2}$ is the reduced density matrix of a pure state $|\psi\rangle \in H_A \otimes H_B$ on $H_B$, then

$$\| (E \otimes 1 - 1 \otimes F)|\psi\rangle \| = \| \lambda E - F \lambda \|_F,$$

where $E$ is the entry-wise conjugate taken with respect to the basis of eigenvectors for $\lambda \in L(H_B)$. Therefore if $\| (E \otimes 1 - 1 \otimes F)|\psi\rangle \|_{H_A \otimes H_B} \leq \epsilon$ then $\| E \lambda - \lambda F \|_F \leq \epsilon$.

**Proposition 4.17.** Suppose $G$ is a synchronous nonlocal game with a uniform distribution on questions. If $\{E^i_a \mid a \in O : i \in I\}$ and $\{F^i_a \mid a \in O : i \in I\}$ are Alice and Bob PVM’s from an $\epsilon$-perfect strategy for $G$, using the state $|\psi\rangle$, then $\| E^i_a \lambda - \lambda F^i_a \|_F \leq O(\epsilon^{1/2})$ for all $i \in I$, and $a \in O$.

**Proof.** If $S$ is an $\epsilon$-perfect strategy then for any pair of questions $(i,j)$ we have

$$\sum_{a,b \in O \times O} V(a,b|i,j) p(a,b|i,j) \geq 1 - n^2 \epsilon.$$
In particular $\mathcal{V}(a, b, i, j) = 0$ whenever $a \neq b$, so for $\mathcal{S}$ to be $\epsilon$-perfect it must be that 
\[
\sum_{a \neq b} \langle \psi | E_a^i \otimes F_b^i | \psi \rangle \leq n^2 \epsilon
\]
holds, for all $1 \leq i \leq m$. Hence, we see that
\[
(4.16) \quad \| (E_a^i \otimes 1 - 1 \otimes F_a^i) | \psi \rangle \|^2 = \sum_{b \neq a} \langle \psi | E_a^i \otimes F_b^i | \psi \rangle + \sum_{a' \neq a} \langle \psi | E_{a'}^i \otimes F_a^i | \psi \rangle \leq 2n^2 \epsilon,
\]
and the result follows from Lemma 4.16. 

A synchronous nonlocal game is **symmetric** if $\mathcal{V}(b, a | j, i) = 0$ whenever $\mathcal{V}(a, b | i, j) = 0$, for all $a, b \in \mathcal{O}$, $i, j \in \mathcal{I}$. Although not every synchronous predicate is symmetric, it is not hard to show that every perfect quantum strategy is symmetric. That is, if $\mathcal{S}$ is a perfect quantum strategy, then $p(a, b | i, j) = p(b, a | j, i) = 0$, see [HMPS19] Corollary 2.2.

**Proposition 4.18.** Let $\mathcal{S} = (\{E_a^i\}_{a \in \mathcal{O}} : i \in \mathcal{I}), (\{F_a^i\}_{a \in \mathcal{O}} : i \in \mathcal{I}), (| \psi \rangle)$ be a $\epsilon$-perfect strategy for a synchronous nonlocal game $\mathcal{G}$, where $\rho = \lambda^2$ is the reduced density matrix on $H_B$. If $\mathcal{V}(a, b, i, j) = 0$ for $a, b \in \mathcal{O}$, $i, j \in \mathcal{I}$, then $p(a, b | i, j) = p(b, a | j, i) + O(\epsilon^{1/2})$.

**Proof.** We observe that
\[
p(a, b | i, j) = \langle \psi | E_a^i \otimes F_b^i | \psi \rangle \\
= \langle \psi | (E_a^i \otimes 1)(1 \otimes F_b^i) | \psi \rangle \\
= \langle \psi | (E_a^i \otimes 1 - 1 \otimes F_a^i)(1 \otimes F_a^i - E_a^i \otimes 1)| \psi \rangle + \langle \psi | E_a^i \otimes F_b^i | \psi \rangle \\
+ \langle \psi | (1 \otimes F_b^i)(1 \otimes F_a^i - E_a^i \otimes 1)| \psi \rangle + \langle \psi | (E_a^i \otimes 1 - 1 \otimes F_a^i)(E_a^i \otimes 1)| \psi \rangle \\
\leq \| E_a^i \lambda - \lambda F_b^i \|_F \| \lambda E_a^i \|_F + \| E_a^i \lambda - \lambda F_b^i \|_F + \| \lambda E_a^i - F_b^i \|_F \\
+ \rho(p(a, b | i, j),
\]
the result follows from Proposition 4.17. 

**Proposition 4.19.** If $\mathcal{S} = (\{E_a^i\}_{a \in \mathcal{O}} : i \in \mathcal{I}), (\{F_a^i\}_{a \in \mathcal{O}} : i \in \mathcal{I}), (| \psi \rangle)$ is an $\epsilon$-perfect strategy for a synchronous game $\mathcal{G}$, where $\rho = \lambda^2$ is the reduced density matrix of $| \psi \rangle$ on $H_B$, then $\{F_a^i\}_{a \in \mathcal{O}} : i \in \mathcal{I}$ is an $O(\epsilon^{1/4})$-representation of the synchronous game algebra in $\mathcal{L}(H_B)$ with respect to the state-induced semi-norm $\| \cdot \|_\rho$. Moreover, the $O(\epsilon^{1/4})$-representation is $(O(\epsilon^{1/2}), \lambda)$-tracial.

**Proof.** Since $\mathcal{S}$ is $\epsilon$-perfect, whenever $\mathcal{V}(a, b | i, j) = 0$ we have that
\[
p(a, b | i, j) = \langle \psi | E_a^i \otimes F_b^i | \psi \rangle = \text{tr}(E_a^i \lambda F_b^i \lambda) \leq n^2 \epsilon
\]
hence, by Proposition 4.18 we have $p(b, a | j, i) = \text{tr}(E_b^i \lambda F_a^i \lambda) = O(\epsilon^{1/2})$. From Cauchy-Schwarz we deduce that
\[
\| F_a^i F_b^i \|_\rho^2 = \| F_a^i F_b^i \|_F^2 \\
= \text{tr}(\lambda F_a^i F_b^i \lambda) \\
= \text{tr}(\lambda F_a^i F_a^i (F_b^i \lambda - \lambda F_b^i)) + \text{tr}(\lambda F_a^i F_a^i \lambda E_b^i) \\
\leq \| F_a^i F_b^i \|_F \| F_b^i \lambda - \lambda E_b^i \|_F + \text{tr}(\lambda F_a^i \lambda E_b^i F_b^i) \\
= \| F_a^i F_b^i \|_F \| F_b^i \lambda - \lambda E_b^i \|_F + \text{tr}(\lambda F_a^i \lambda E_b^i F_b^i) \\
+ \| F_b^i \lambda F_b^i \|_F^2 \\
= \| F_a^i F_b^i \|_F \| F_b^i \lambda - \lambda E_b^i \|_F + \text{tr}(\lambda F_a^i \lambda E_b^i (\lambda F_b^i - E_b^i \lambda)) + \| F_b^i \lambda F_b^i \|_F^2,
\]
for all $i$. Proposition 4.17. Since $\{\{F_a^i\}_{a \in O} : i \in I\}$ is a PVM, each $F_a^i$ is an orthogonal projection and $\sum_{a \in O} F_a^i = 1_{H_B}$ for each $i \in I$, and so the remaining relations in Definition 4.15 hold automatically. The fact that the approximate representation is $O(\epsilon^{1/2}, \lambda)$-tracial follow from the second part of Lemma 4.17, by letting $X_a^i = 1 - 2E_a^i$ and $Y_a^i = 1 - 2F_a^i$ we obtain self-adjoint unitaries satisfying $\|Y_a^i - \lambda X_a^i\|_F \leq O(\epsilon^{1/2})$ for all $i \in I$ and $a \in O$, completing the proof.

We leave it as an open problem whether this bound on the state-dependent approximate representation is tight in the degree of $\epsilon$. It seems plausible that Proposition 4.18 could be improved to $O(\epsilon)$, which would lead to an $O(\epsilon^{1/2})$-representation in Proposition 4.19. Despite this issue, we obtain the immediate corollary when the state in the strategy is maximally entangled.

**Corollary 4.20.** Let $S$ be an $\epsilon$-perfect strategy for a synchronous nonlocal game $G$. If the state $\ket{\psi}$, from the strategy $S$, is maximally entangled with Schmidt rank equal to $\text{dim}(H_B)$, then Bob’s measurement operators $\{\{F_a^i\}_{a \in O} : i \in I\}$ are an $O(\epsilon^{1/4})$-representation of the synchronous algebra on $L(H_B)$ with respect to $\| \cdot \|_F$.

Unlike in the BCS case, we cannot directly apply Lemma 3.26 to Proposition 4.19 and obtain a state-independent approximate representation. This is because the presentation of the synchronous algebra is in terms of orthogonal projections and not self-adjoint unitaries. In the next subsection, we show that there is an alternative presentation of the synchronous algebras as the BCS algebra of a certain BCS built from the original synchronous game.

### 4.3. The SynchBCS algebra of a synchronous nonlocal game.

Although synchronous and BCS games may initially appear different they are essentially the same. First off, there is a synchronous version of any BCS game by considering the game where Alice and Bob each receive a constraint $\mathcal{C}_i$ and $\mathcal{C}_j$ and must reply with satisfying assignments. In this “constraint-constraint” version of the BCS game, the players win perfectly if and only if their assignment to all variables in the intersection of the contexts match. This synchronous version of a BCS game is well-known. In particular, the authors of [KPS18] employ this idea to construct a synchronous nonlocal game for which there is a $*$-homomorphism from the synchronous algebra of a linear BCS game to the corresponding $C^*$-algebra of the corresponding solution group of the linear system.

In this work, we focus on the other direction. We consider a “constraint-variable” version of a synchronous nonlocal game which we call the SynchBCS game. Given a synchronous nonlocal game $G$ the SynchBCS game associated with $G$ is the BCS game where:

- for each question $i \in I$ and answer $a \in O$, we add a $\{\pm 1\}$-valued variable $z_a^i$, and
- in the synchronous game, whenever $\mathcal{V}(a,b|i,j) = 0$ we add the constraint $\overline{\text{AND}}(z_a^i, z_b^j) = 1$, and
- to ensure that each \( z_a^i \) comes from a single measurement, we add the constraint
\[
\text{XOR}_{a \in \mathcal{O}}(z_a^i) = -1
\]
for each question \( i \in \mathcal{I} \).

This last constraint prevents two different \(-1\)'s from each question while ensuring at least one \(-1\) output is given for each input \( i \). In this SynchBCS game, the players can receive any of these constraints and they must reply with a satisfying assignment to the variables in the context. The distribution on these constraints is informed by the distribution of questions in the original synchronous game. Since this is a BCS game, we can consider the corresponding BCS algebra associated with any synchronous game \( G \) through this transformation.

**Definition 4.21.** The SynchBCS algebra \( B(G) \) of the synchronous nonlocal game \( G \) is a quotient of the self-adjoint unitary algebra \( U_{\mathcal{I} \times \mathcal{O}} \) (see Eq. (3.3)), with self-adjoint unitary generators \( \{ z_a^i : (i, a) \in \mathcal{I} \times \mathcal{O} \} \), satisfying the additional relations:

1. \( \text{AND}(z_a^i, z_b^j) = 1 \), whenever \( V(a, b|i, j) = 0 \) for all \( i, j \in \mathcal{I} \) and \( a, b \in \mathcal{O} \),
2. \( \prod_{a \in \mathcal{O}} z_a^i = -1 \), each \( i \in \mathcal{I} \),
3. \( z_a^i z_{a'}^{j'} = z_a^{i'} z_{a'}^{j} \) for all pairs \( a, a' \in \mathcal{O} \), and each \( i \in \mathcal{I} \).

By construction, the finite-dimensional representations of the SynchBCS algebras give quantum satisfying assignment to the associated SynchBCS nonlocal game. Hence, there is a BCS nonlocal game for each synchronous nonlocal game. In \[Fri20, Gol21\], the authors showed that the synchronous LCS game algebra is isomorphic to the synchronous algebra of projections. We establish the following complementary result.

**Proposition 4.22.** The synchronous game algebra \( A(G) \) is *-isomorphic to the SynchBCS algebra \( B(G) \).

**Proof.** We begin by describing the *-homomorphism \( \phi : A(G) \to B(G) \). Define the function on the generators \( p_a^i \to (1-z_a^i)/2 \). This function extends to a *-homomorphism \( \phi \) on \( A(G) \). We now check that it descends to a *-homomorphism to \( B(G) \). First, we note that each \( \phi(p_a^i) \) is an orthogonal projection since \( z_a^i = z_a^{i*} = z_a^{j} \), and \( z_a^{i+1} = 1 \), for all \( a \in \mathcal{O} \). The AND relation together with the relation \( z_a^{i} z_{a'}^{j} = z_{a'}^{j} z_a^{i} \) for all \( a, a' \in \mathcal{O} \), implies that \( 1 - z_a^i - z_b^j + z_a^{i} z_{b}^{j} = 0 \) whenever \( V(a, b|i, j) = 0 \), and thus
\[
\phi(p_a^i)\phi(p_b^j) = \frac{(1 - z_a^{i})}{2} \cdot \frac{(1 - z_b^{j})}{2} = 0
\]
is satisfied whenever \( V(a, b|i, j) = 0 \).

For each \( i \in \mathcal{I} \) with \( |\mathcal{O}| = n \), observe that the unit 1 can be expanded as the sum of indicator polynomials in the variables \( z_a^i \), giving us
\[
1 = \sum_{(e_1, \ldots, e_n) \in \{\pm 1\}^n} \prod_{a \in \mathcal{O}} \frac{(1 + e_a z_a^i)}{2}.
\]
However, upon enforcing the orthogonality relations we note that \( \prod_{a \in \mathcal{O}} \frac{(1 + e_a z_a^i)}{2} = 0 \), whenever there is a pair \( a, a' \in \mathcal{O} \) with \( e_a = e_a' = -1 \). Thus, there are only two cases we need to consider.

\[\text{For any } i \in \mathcal{I} \text{ the subset of } z_a^i \text{'s are jointly measurable, and exactly one of them outputs a } -1.\]

\[\text{As mentioned, one of the directions was also established in } [KPS18]. \text{ We also note that although they use “BCS” in their title, their results are only for BCS games with linear constraints.}\]
The first is when \( e_a = 1 \) for all \( a \in \mathcal{O} \). In this case, we have the term
\[
\prod_{a \in \mathcal{O}} \left( 1 + \frac{z_a^i}{2} \right) = \frac{1}{2^n} \left( \sum_{S \subseteq [n]} \prod_{a \in S} z_a^i \right).
\]

Recalling that the rule predicate relation shows \( \prod_{a \in \mathcal{O}} z_a^i = -1 \), we observe that
\[
\prod_{a \in S} z_a^i + \prod_{a \in [n] \setminus S} z_a^i = 0,
\]
for any \( S \subseteq [n] \), by recalling that each \( z_a^2 = 1 \) by the self-adjoint unitary relations. It follows that Eq. (4.18) is 0 because each subset \( S \subseteq [n] \) is in bijection with its complementary subset \( S^c = [n] \setminus S \), and so by equation Eq. (4.19) each term with an \( S \) product cancels out with the term for \( S^c \) product.

In the other case, the remaining terms are those with exactly one \( a \in \mathcal{O} \) with \( e_a = -1 \). For this case, let \( f_\alpha^i = (1 - z_a^i)/2 \) and \( f_{\alpha'}^i = (1 - z_a^j)/2 \) and observe that \( f_\alpha^i \) and \( f_{\alpha'}^i \) are self-adjoint orthogonal projections with \( f_\alpha^i f_{\alpha'}^i = 0 \), and therefore \( f_\alpha^i (1 - f_{\alpha'}^i) = f_\alpha^i - f_\alpha^i f_{\alpha'}^i = f_\alpha^i \). Then, noting \( 1 - f_{\alpha'}^i = (1 + z_a^j)/2 \), it follows that \( f_\alpha^i \left[ \prod_{\alpha' \neq a} (1 - f_{\alpha'}^i) \right] = f_\alpha^i \). These being the only remaining terms in Eq. (4.19), we see that
\[
1 = \sum_{a \in \mathcal{O}} \frac{1 - z_a^i}{2} \prod_{\alpha' \neq a} \frac{1 + z_a^j}{2} = \sum_{a \in \mathcal{O}} \frac{1 - z_a^i}{2} = \sum_{a \in \mathcal{O}} \phi(p_a^i),
\]
for all \( i \in \mathcal{I} \), as desired.

On the other hand, consider the \(*\)-homomorphism \( \varphi : \mathcal{B}(\mathcal{G}) \rightarrow \mathcal{A}(\mathcal{G}) \) defined by extending the function \( z_a^i \mapsto (1_A - 2p_a^i) \). Recalling that \( p_a^i p_b^j = 0 \) for all \( a \neq b \), we see that
\[
\prod_{a \in \mathcal{O}} \varphi(z_a^i) = \prod_{a \in \mathcal{O}} (1 - 2p_a^i) = \sum_{S \subseteq \mathcal{O}} (-2)^{|S|} \prod_{a \in S} p_a^i = 1_A + (-2) \sum_{a \in \mathcal{O}} p_a^i = -1_A,
\]
by recalling the completeness relation in \( \mathcal{A}(\mathcal{G}) \). Now, if \( V(a, b|i, j) = 0 \) then we have that \( p_a^i p_b^j = 0 \), hence
\[
\overline{\text{AND}}(\varphi(z_a^i), \varphi(z_b^j)) = \frac{1}{2} (1_A + (1 + 2p_a^i)(1_A + 2p_b^j))
- (1_A + 2p_a^i)(1_A + 2p_b^j))
= \frac{1}{2} \left( 1_A + 2p_a^i + 2p_b^j - 2p_a^i - 2p_b^j - 4p_a^i p_b^j \right)
= 1_A.
\]
Lastly, since \( V(a, b|i, i) = 0 \), it follows that \( \varphi(z_a^i) \varphi(z_b^j) = (1_A + 2p_a^i)(1_A + 2p_b^j) = 1_A + 2p_a^i + 2p_b^j = \varphi(z_b^j) \varphi(z_a^i) \) for all \( a \neq b \) as desired.

It remains to show that \( \varphi \) and \( \phi \) are mutual inverses. Observe that,
\[
\varphi(\phi(p_a^i)) = \varphi \left( \frac{1 - z_a^i}{2} \right) = \frac{1}{2} \left( \varphi(1) - \varphi(z_a^i) \right) = \frac{1}{2} \left( 1_A - (1_A - 2p_a^i) \right) = p_a^i.
\]
Similarly,
\[
\phi(\varphi(z_a^i)) = \phi(1_A - 2p_a^i) = \phi(1_A) - 2\phi(p_a^i) = 1 - 2\left( \frac{1 - z_a^i}{2} \right) = z_a^i,
\]
thus \( \varphi \circ \phi = id_{\mathcal{A}(\mathcal{G})} \) and \( \phi \circ \varphi = id_{\mathcal{B}(\mathcal{G})} \), and the result follows. \( \square \)
Recall from an $\epsilon$-perfect strategy for a BCS nonlocal game with an arbitrary state, our rounding result will give us an $O(\epsilon^{1/4})$-representation of the BCS algebra in the $\| \cdot \|_f$-norm. To apply our rounding result in the synchronous algebra case we need to ensure that $\epsilon$-representations of the synchronous algebra $\mathcal{A}(\mathcal{G})$ gives us $O(\epsilon)$-representation of the SynchBCS algebra in the $\rho$-norm under the isomorphism in Proposition 4.22. We remark that we cannot directly apply Proposition 3.7 here because the approximate representation is not state-independent!

**Proposition 4.23.** If $\psi$ is an $(\epsilon, \rho)$-representation of $\mathcal{A}(\mathcal{G})$, then $\phi \circ \psi$ is a $(O(\epsilon), \rho)$-representation of $\mathcal{B}(\mathcal{G})$, where the $\ast$-homomorphism $\phi : \mathcal{A}(\mathcal{G}) \to \mathcal{B}(\mathcal{G})$ is described in Proposition 4.22.

**Proof of Proposition 4.23.** It is straightforward to show that if $\| F^i_{a}^{\ast} - F^i_{\bar{a}}^{\ast} \|_\rho \leq \epsilon$ and $\| F^i_{a} - F^i_{\bar{a}} \|_\rho \leq \epsilon$, then $\| X^i_{a}^{\ast} - 1 \|_\rho \leq 4\epsilon$, and similarly $\| X^i_{a} - X^i_{\bar{a}} \|_\rho \leq 2\epsilon$. So all that remains to show that relations (1), (2), and (3) hold approximately as well. If $\mathcal{V}(a, b| i, j) = 0$ then $\| F^i_{a} F^j_{b} \|_\rho \leq \epsilon$, hence, we set $X^i_{a} = 1 - 2F^i_{a}$. Note that $\| \tilde{\text{AND}}(z^i_{a}, z^j_{\bar{a}}) - 1 \|_\rho = \| z^i_{a} - z^j_{\bar{a}} + z^i_{a}z^j_{\bar{a}} \|_\rho$. Then

\begin{equation}
(4.20) \quad \| 1 - X^i_{a} - X^j_{\bar{a}} + X^i_{a}X^j_{\bar{a}} \|_\rho = 4 \left\| \frac{(1 - X^i_{a})}{2} \frac{(1 - X^j_{\bar{a}})}{2} \right\|_\rho = 4 \| F^i_{a} F^j_{\bar{a}} \|_\rho \leq 4\epsilon.
\end{equation}

Lastly,

\begin{align*}
\| \prod_{a \in \mathcal{A}} X^i_{a} + 1 \|_\rho &= \left\| \prod_{a \in \mathcal{A}} (1 - 2F^i_{a}) + 1 \right\|_\rho \\
&= \left\| \sum_{|\alpha| = 1} (-2)^{|\alpha|} \prod_{a \in \alpha} F^i_{a} + 1 \right\|_\rho \\
&\leq 2 \| 1 - \sum_{a} F^i_{a} \|_\rho + \sum_{|\alpha| > 1} 2^{|\alpha|} \| \prod_{a \in \alpha} F^i_{a} \|_\rho \\
&\leq 2\epsilon + \sum_{|\alpha| > 1} 2^{|\alpha|} \| \prod_{a \in \alpha \setminus \{a, a'\}} F^i_{a} \|_\rho \\
&\leq 2\epsilon + 22^{|A|} C^{|A| - 1} \| F^i_{a} F^i_{a'} \|_\rho \\
&= O(\epsilon),
\end{align*}

where $C$ is the constant that bounds the operator norm of each $F^i_{a}$. Now, we ensure that the commutation relation holds for all $a \neq a', i \in \mathcal{I}$

\begin{align*}
\| X^i_{a'} X^i_{a} - X^i_{a} X^i_{a'} \|_\rho &= \| (1 - 2F^i_{a})(1 - 2F^i_{a'}) - (1 - 2F^i_{a'})(1 - 2F^i_{a}) \|_\rho \\
&\leq 4 \| F^i_{a} F^i_{a'} \|_\rho + \| F^i_{a'} F^i_{a} \|_\rho.
\end{align*}
\[ = O(\epsilon), \]

as desired. \qed

**Remark 4.24.** Although Proposition 4.23 works for arbitrary \((\epsilon, \rho)\)-representations, for our applications the approximate representation of \(\mathcal{A}(\mathcal{G})\) in the \(\rho\)-norm already satisfy exactly many of the relations in the synchronous algebra Definition 4.15. This is because they begin as projective measurements over satisfying assignments for some quantum strategy. Moreover, if \(X^i_a = 1 - 2F^i_a\) is the \(\pm 1\)-valued observable assigned to the orthogonal projection onto outcome \((i, a)\) then under the isomorphism in Proposition 4.22 the collection of observables \(\{X^i_a\}_{(i, a) \in I \times O}\) generate a self-adjoint unitary \(O(\epsilon^{1/2})\)-representation of the SynchBCS algebra \(\mathcal{B}(\mathcal{G})\). It is clear that the relations self-adjoint unitary relations, as well as relations (2), and (3) in Definition 4.21 hold exactly in this approximate representation due to the fact \(\mathcal{S}\) is a representation of the PVM algebra. Therefore it only remains to check that relation (1) in Definition 4.21 holds approximately, which follows from Eq. (4.20). Lastly, we see that since the projections satisfy the property in Proposition 4.17, their corresponding observables satisfy the hypothesis of Lemma 4.7.

**Corollary 4.25.** If \(\mathcal{S}\) is an \(\epsilon\)-perfect synchronous strategy for a synchronous nonlocal game \(\mathcal{G}\), then the corresponding \((O(\epsilon^{1/4}), \rho)\)-representation of the SynchBCS algebra is \(O(\epsilon^{1/2})\)-tracial.

Corollary 4.25 allows us to apply our rounding result to synchronous algebras by considering the approximate representations of the SynchBCS algebra. In particular, the representation coming from a near-perfect strategy for the synchronous game yields a self-adjoint unitary state-dependent approximate representation of the SynchBCS algebra. Hence, we can apply Lemma 3.26 to the SynchBCS algebra to obtain a self-adjoint unitary state-independent approximate representation of the SynchBCS algebra. Then, by the isomorphism, we can return the state-independent approximate representation of the SynchBCS algebra to obtain a state-independent approximate representation of the synchronous algebra. We summarize the result in the following proposition.

**Proposition 4.26.** If \(\mathcal{S}\) is an \(\epsilon\)-perfect strategy for a synchronous nonlocal game \(\mathcal{G}\), then restricted to a non-zero subspace of \(H_B\), Bob’s measurement operators are an \(O(\epsilon^{1/8})\)-representation of the synchronous algebra \(\mathcal{A}(\mathcal{G})\).

**Proof.** The synchronous case is similar to the BCS case. By Proposition 4.19 any \(\epsilon\)-perfect strategy for a synchronous game \(\mathcal{G}\) with reduced density matrix \(\rho\) gives a state-dependent \((O(\epsilon^{1/4}), \rho)\)-representation of the synchronous algebra \(\mathcal{A}(\mathcal{G})\) that is \(O(\epsilon^{1/2})\)-tracial. By Corollary 4.25 this state-dependent approximate representation is a \(O(\epsilon^{1/2})\)-tracial \((O(\epsilon^{1/4}), \rho)\)-representation of the synchBCS algebra \(\mathcal{B}(\mathcal{G})\) that is exact on the self-adjoint unitary relations. By applying Lemma 3.26 we obtain a \(O(\epsilon^{1/8})\)-representation in the \(\|\cdot\|_f\) norm of the synchBCS algebra on a subspace of \(H_B\). Finally, by the \(\ast\)-isomorphism described in Proposition 4.22 combined with Proposition 3.7 we obtain a state-independent \(O(\epsilon^{1/8})\)-representation of the synchronous algebra. \qed

Our final task in this subsection is to show that approximate representations of the synchronous algebra is close to near-perfect quantum strategies.
Proposition 4.27. If \( \phi \) is a bounded \( \epsilon \)-representation of the synchronous algebra \( \mathcal{A}(\mathcal{G}) \) on a Hilbert space \( H \), then there is \( O(\epsilon^2) \)-perfect synchronous strategy using a maximally entangled state.

Proof. Suppose that \( \phi \) is an \( \epsilon \)-representation of the synchronous algebra on a Hilbert space \( H \). By the stability of the PVM algebra \( \mathcal{A}^{(T,O)} \) Corollary 3.23, there exists a constant \( C_0 \geq 0 \) and orthogonal projections \( \{ \Pi_a^i : a \in A \} : i \in I \), such that 
\[
\sum_a \Pi_a^i = 1 \quad \text{for all } i \in I, \quad \Pi_a^i \Pi_b^i = 0 \quad \text{for } a \neq b, \quad \text{for all } i \in I, \quad \text{and } \| \Pi_a^i - \phi(p_a^i) \|_f = C_0 \epsilon, \quad \text{for all } i \in I, \quad a \in A.
\]
Moreover, by replacing the almost-PVMs with the genuine PVMs the replacement lemma shows that new PVMs still approximately satisfy the rule relations.

More precisely, by Lemma 3.8 there exists a constant \( C_1 > 0 \) such that \( \| \Pi_a^i \Pi_b^j \|_f = C_1 \epsilon \) whenever \( \mathcal{V}(a,b|i,j) = 0 \). Now, consider the quantum strategy where Alice employs the PVMs \( \{ P_a^i = \Pi_a^i : a \in A \} : i \in I \), Bob employs PVMs \( \{ Q_b^j = \Pi_b^j : b \in A \} : j \in I \), and they use a shared maximally entangled state \( |\tau\rangle \in H \otimes H \). Given this strategy, the probability of losing on question pair \((i,j)\) is at most
\[
\sum_{a,b} \mathcal{V}(a,b|i,j) = \sum_{a,b} |\langle \phi | P_a^i \otimes Q_b^j |\phi\rangle |^2 = \sum_{a,b} \| P_a^i \Pi_b^j \|_f^2 \\
\leq |O|^2 C_1^2 \epsilon^2.
\]

It follows that the strategy is \( O(\epsilon^2) \)-perfect. \( \square \)

Corollary 4.28. For any \( \epsilon \)-perfect quantum strategy \( S \) for a synchronous nonlocal game, there is an \( O(\epsilon^{1/4}) \)-perfect quantum strategy \( \tilde{S} \) using a maximally entangled state \( |\tilde{\psi}\rangle \), such that each measurement in \( \tilde{S} \) is at most \( O(\epsilon^{1/8}) \)-away from the measurement in \( S \) with respect to \( \| \cdot \|_f \) on the local support of \( |\tilde{\psi}\rangle \) on \( H_B \).

4.4. XOR nonlocal games. Unlike the case of BCS and synchronous games, XOR nonlocal games do not admit non-classical perfect quantum strategies [CHTW10]. However, in many cases, there are quantum strategies for XOR games that can achieve higher winning probabilities than the best classical strategies [Slo11]. Nevertheless, XOR games have an affiliated finitely presented \( \ast \)-algebra \( \mathcal{C}(\mathcal{G}) \), called the XOR algebra, for which optimal quantum strategies correspond to representations.

An XOR game is a nonlocal game where Alice and Bob are given questions \( i \in \mathcal{I}_A = [m] \) and \( j \in \mathcal{I}_B = [n] \) according to a probability distribution \( \pi(i,j) \), and they respond with 1-bit answers \( a \in \mathcal{O}_A = \{0,1\} \) and \( b \in \mathcal{O}_B = \{0,1\} \). The rule predicate for the game is determined by the XOR of the answer bits. For any XOR game, we can describe the predicate by an \( m \times n \) \{0,1\}-matrix \( T \) with entries \( (T)_{i,j} = t_{ij} \), so that
\[
\mathcal{V}(a,b|i,j) = \left\{ \begin{array}{ll}
1, & \text{if } a \oplus b = t_{ij}, \\
0, & \text{otherwise}
\end{array} \right.
\]
(4.21)

Letting \( w_{ij} = (-1)^{t_{ij}} \pi(i,j) \), we obtain the cost matrix \( W \) of an XOR game. Given the cost matrix of a game, one can conveniently express the bias of a strategy \( \mathcal{S} \) consisting of \( \pm 1 \)-valued observables \( \{ Y_1, \ldots, Y_m \} \) for Alice, \( \{ X_1, \ldots, X_n \} \) for Bob, and a vector
state \( |\psi\rangle \in H_A \otimes H_B \), as

\[
\beta(G; S) = \sum_{i=1, j=1}^{m,n} w_{ij} \langle Y_i \otimes X_j | \psi \rangle.
\]

The supremum over all quantum strategies \( S \) gives the optimal bias denoted by \( \beta_q(G) \) for the XOR game \( G \).

A result of Tsirelson implies that the quantum value of an XOR game can be computed using a semi-definite program \([\text{CHTW}10, \text{Weh}06, \text{CSUU}08]\). Unlike for BCS games, this characterization often makes computing the entangled value of an XOR game computationally tractable. Also, using the formulation of Tsirelson it was shown in \([\text{Slo}11]\) that if a strategy \( S \) for an XOR game is optimal, then the measurement observables satisfy the following relation

\[
\sum_{j=1}^{n} w_{ij} X_j \lambda = r_i \lambda Y_i,
\]

for all \( i \in [m] \). Here, the collection \( \{r_i\}_{i \in [m]} \) are the marginal row biases associated with the questions, and \( \lambda \) is the square root of the reduced density matrix of the strategy quantum state \( |\psi\rangle \) on \( H_B \). From this relation, one can define the XOR-algebra associated to an XOR nonlocal game \( G \) in terms of abstract relations resembling equation Eq. (4.23) along with the \( \pm 1 \)-valued observable relations, see \([\text{Slo}11]\).

**Definition 4.29.** Let \( G \) be an XOR game with an \( m \times n \) cost matrix \( W \), and marginal row biases \( \{r_i \in \mathbb{R} : i \in [m]\} \). The XOR algebra \( C(G) \) is a quotient of the self-adjoint unitary algebra \( U_n \) (see Eq. (3.3)), subject to the additional relations:

\[
\left( \sum_{j=1}^{n} w_{ij} x_j \right)^2 = r_i^2 \cdot 1 \quad \text{for all } 1 \leq i \leq m
\]

The characterization of optimal strategies in terms of the semi-definite program was also applicable in the approximate setting, in particular, it implies the following approximate rigidity result.

**Theorem 4.30** (Theorem 3.1 in \([\text{Slo}11]\)). For every XOR game \( G \) there exists a collection of constants \( r_i \geq 0 \) such that if \( S = (\{Y_i\}_{i=1}^{m}, \{X_j\}_{j=1}^{n}, |\psi\rangle) \) is an \( \epsilon \)-optimal strategy of \( \pm 1 \)-valued observables, and \( 0 \leq \epsilon \leq \frac{1}{4(m+n)} \), then

\[
\left\| \sum_{j=1}^{n} w_{ij} (1 \otimes X_j) - r_i (Y_i \otimes 1) \right\|_{|\psi\rangle} = O(\epsilon^{1/4}),
\]

for all \( 1 \leq i \leq m \), and the constants hidden in the \( O(\epsilon^{1/4}) \) depends only on the size of the question sets \( m \) and \( n \).

In other words, Theorem 4.30 establishes that near-optimal strategies are state-dependent approximate representation of the XOR algebra (given in Definition 4.29). With this fact, we can establish the following result:

**Proposition 4.31.** Let \( S = (\{Y_i\}_{i=1}^{m}, \{X_j\}_{j=1}^{n}, |\psi\rangle) \in H_A \otimes H_B \) be an \( \epsilon \)-optimal strategy to an XOR game \( G \) where \( |\psi\rangle \) has reduced density matrix \( \rho \in \mathcal{L}(H_B) \), then
the observables \(\{X_1, \ldots, X_n\}\) are an \((O(\epsilon^{1/4}), \rho)\)-representation of the solution algebra \(\mathcal{C}(\mathcal{G})\). Additionally, the \((O(\epsilon^{1/4}), \rho)\)-representation is \((O(\epsilon^{1/4}), \rho)\)-tracial.

Proof. If \(S\) is an \(\epsilon\)-optimal strategy for the XOR game \(\mathcal{G}\) and \(\rho = \lambda^* \lambda\) is the reduced density matrix of the state on \(H_B\), since each \(X_j\) is a self-adjoint unitary, it only remains to show that the remaining relations in Definition 4.29 hold approximately. In particular, we claim that

\[
\left\| r_i^2 \mathbb{1} - \left( \sum_{j=1}^{n} w_{ij} X_j \right)^2 \right\|_\rho = O(\epsilon^{1/4}),
\]

for all \(1 \leq i \leq m\). This holds by Theorem 4.30 and Lemma 4.7 as they show that in general

\[
(4.26) \quad \left\| \sum_{j=1}^{n} w_{ij} X_j \lambda - \lambda r_i \mathbb{1} \right\|_F = O(\epsilon^{1/4}).
\]

From this, it follows that

\[
\left\| r_i \mathbb{1} - \left( \sum_{j=1}^{n} w_{ij} X_j \right) \right\|_F = \left\| r_i^2 \mathbb{1} - \left( \sum_{j=1}^{n} w_{ij} X_j \right)^2 \right\|_F \\
\leq \left\| r_i^2 \mathbb{1} - \sum_{j=1}^{n} w_{ij} X_j r_i \lambda \mathbb{1} \right\|_F + \left\| \sum_{j=1}^{n} w_{ij} X_j r_i \lambda \mathbb{1} - \left( \sum_{j=1}^{n} w_{ij} X_j \right) \left( \sum_{\ell=1}^{n} w_{i\ell} X_{\ell} \right) \lambda \right\|_F \\
\leq \left\| r_i \lambda \mathbb{1} - \sum_{j=1}^{n} w_{ij} X_j r_i \lambda \right\|_F + \sum_{j=1}^{n} \left\| w_{ij} \right\| \left\| X_j \left( r_i \lambda \mathbb{1} - \sum_{\ell=1}^{n} w_{i\ell} X_{\ell} \lambda \right) \right\|_F \\
\leq \left\| r_i \lambda \mathbb{1} - \sum_{j=1}^{n} w_{ij} X_j \lambda \right\|_F + \sum_{j=1}^{n} \left\| w_{ij} \right\| \left\| r_i \lambda \mathbb{1} - \sum_{\ell=1}^{n} w_{i\ell} X_{\ell} \lambda \right\|_F
\]

is \(O(\epsilon^{1/4})\) by equation Eq. (4.26). That this approximate representation is \((O(\epsilon^{1/4}), \lambda)\)-tracial follows easily from Lemma 4.7 and equation Eq. (4.26).

\[\square\]

Corollary 4.32. If the state \(|\psi\rangle\) in the \(\epsilon\)-optimal strategy \(S\) for the XOR nonlocal game \(\mathcal{G}\) is maximally entangled then the observables \(\{X_1, \ldots, X_n\}\) are a state-independent \(O(\epsilon^{1/4})\)-representation of \(\mathcal{C}(\mathcal{G})\).

To obtain the result in the general case, we apply our Lemma 4.26 to obtain the following result.

Proposition 4.33. If \(S\) is an \(\epsilon\)-optimal strategy for an XOR nonlocal game \(\mathcal{G}\), then restricted to a subspace of \(H_B\), Bob’s measurement observables are a state-independent \(O(\epsilon^{1/8})\)-representation of the XOR algebra \(\mathcal{C}(\mathcal{G})\).
Proof. For the XOR case, Proposition 3.31 establishes that any \( \epsilon \)-perfect strategy for an XOR game \( G \) results in an \( O(\epsilon^{1/4}) \)-tracial \((O(\epsilon^{1/4}), \rho)\)-representation of the XOR algebra \( C(G) \). Again, because the operators are \( \pm 1 \)-valued observables, the approximate representation is exact for the self-adjoint unitary relations. Hence, by Lemma 3.26 we obtain a state-independent \( O(\epsilon^{1/8}) \)-representation of the XOR algebra. \( \square \)

Our last task is to determine the optimality of strategies arising from \( \epsilon \)-representations in the \( \| \cdot \|_f \)-norm of the XOR algebra.

**Proposition 4.34.** If \( \phi \) is a bounded \( \epsilon \)-representations of the solution algebra \( C(G) \) on a Hilbert space \( H \), then there is an \( O(\epsilon^2) \)-optimal strategy for the corresponding XOR game using a maximally entangled state.

**Proof.** Let \( \phi \) be an \( \epsilon \)-representation of \( C(G) \). Start by defining the operators \( \phi(x_j) \) for all \( 1 \leq j \leq n \) on the Hilbert space \( H_B \). We note that these observables may not be \( \pm 1 \)-valued observables, but they are close. In particular, by Lemma 3.13 we can find a nearby \( \pm 1 \)-valued observables \( X_j \) for all \( 1 \leq j \leq n \) such that each \( \| X_j - \phi(s_j) \|_f \leq 2\epsilon \). Let these self-adjoint unitaries \( \{ X_1, \ldots, X_j \} \) be the observables in Bob’s quantum strategy. To start building Alice’s strategy we first define

\[
Z_i = \frac{1}{r_i} \sum_j w_{ij} X_j^\top,
\]

for each \( 1 \leq i \leq m \). By the Lemma 3.38 there exists a constant \( K_0 \) such that \( \| Z_i^2 - 1 \|_f \leq K_0 \epsilon \). Moreover, by noting that each \( Z_i \) is self-adjoint by construction, we have that \( \| Z_i Z_i^* - 1 \|_f, \| Z_i^* Z_i - 1 \|_f, \text{ and } \| Z_i - Z_i^* \|_f, \) are all at most \( K_0 \epsilon \) as well. Hence, applying Lemma 3.13 again we obtain self-adjoint unitaries \( Y_i \) such that \( Y_i^2 = 1 \), \( Y_i^* = Y_i \) and \( \| Z_i^\top - Y_i \|_f \leq 2K_0 \epsilon \) for all \( 1 \leq i \leq n \). Then, if Alice’s strategy consists of the operators \( \{ Y_1, \ldots, Y_m \} \), as defined for each \( 1 \leq i \leq m \), and they share a maximally entangled state \( |\tau\rangle \in H_B \otimes H_B \), we observe that

\[
|\beta_\delta(G) - \beta(\mathcal{S}; G)| = \left| \sum_i r_i - \sum_{ij} w_{ij} \langle \psi | Y_i \otimes X_j | \psi \rangle \right|
\leq \sum_i r_i \left| 1 - \langle \psi | Y_i \otimes Z_i^\top | \psi \rangle \right|
\leq \sum_i \frac{r_i}{2} \left| \text{tr}(21 - 2Y_i^\top Z_i) \right|
\leq \sum_i \frac{r_i}{2} \| Y_i^\top - Z_i \|_f^2
\leq 2n \max_i \{ r_i \} K_0 \epsilon^2 = O(\epsilon^2),
\]

as desired. \( \square \)

It remains to show that the measurement operators in the rounded strategy using the maximally entangled state are not too far from the operators in the original \( \epsilon \)-optimal strategy. Towards this point, in the proof of Lemma 3.26 we see that on the support of the projection \( P \), each unitary \( \tilde{X}_j \) is close to the starting self-adjoint unitary \( X_j \) (see Eq. (3.17)). In particular, this distance depends on the initial approximate representation. In the XOR case, we see that measurements obtained from Lemma 3.26
are at most $O(\epsilon^{1/8})$-away in the $\| \cdot \|_f$-norm on the subspace $\tilde{H}$. Lastly, in the proof of Proposition 4.34 it follows from stable replacement that whenever we obtain a strategy for the corresponding nonlocal game from an $O(\epsilon')$-representation, the measurement operators are never more than $O(\epsilon')$-away from the measurement operators in the initial strategy.

**Corollary 4.35.** For any $\epsilon$-optimal quantum strategy $S$ for an XOR nonlocal game, there is an $O(\epsilon^{1/4})$-optimal quantum strategy $\tilde{S}$ using a maximally entangled state $|\tilde{\psi}\rangle$, such that each measurement in $\tilde{S}$ is at most $O(\epsilon^{1/8})$-away from the measurement in $S$ with respect to $\| \cdot \|_f$ on the local support of $|\tilde{\psi}\rangle$ on $H_B$.

**Acknowledgements**

The author would like to thank William Slofstra and Arthur Mehta for several helpful discussions. They would also like to thank Yuming Zhao, Eric Culf, Taro Spirig, Denis Rochette and the anonymous referees for valuable feedback on earlier drafts of this work. This work was primarily completed while the author was at the University of Waterloo and the Institute for Quantum Computing.

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Proof of Lemma 3.22. To begin let \( Y = P X P \) and for now suppose that \( P \) has rank \( k \) and that \( P \) is diagonal in the standard basis i.e. \( P = \mathbb{1}_k \oplus 0_{d-k} \). Since \( Y \) is supported on \( \text{Im}(P) \) we can write \( Y = \begin{pmatrix} \hat{Y} & 0_{k \times d-k} \\ 0_{d-k,k} & 0_{d-k,d-k} \end{pmatrix} \) for some self-adjoint matrix \( \hat{Y} \in M_k(\mathbb{C}) \).

Now take \( \tilde{U} \) to be a unitary part of \( \tilde{C} \) on \( \mathbb{C}^k \). To extend \( \tilde{U} \) to matrix on \( \mathbb{C}^d \) we
let $U = \tilde{U} \oplus \mathbb{1}_{d-k}$. In addition to $U$ being a unitary on $\mathbb{C}^d$, we observe that $U$ restricts to a unitary on $\text{Im}(P)$. Since $\tilde{Y} = \tilde{U}|Y|$, all that remains is to verify is if $|Y|$ equals \[
abla
\begin{pmatrix}
|\tilde{Y}| & 0_{k \times d-k} \\
0_{d-k,k} & 0_{d-k,d-k}
\end{pmatrix}.
\]
With this in mind, let $\tilde{V}$ be the unitary in $M_k(\mathbb{C})$ that diagonalizes $\tilde{Y}$ (i.e. $\tilde{Y} = \tilde{V}\tilde{D}\tilde{V}^*$ for a diagonal matrix $\tilde{D} \in M_k(\mathbb{C})$). We observe that
\[
|C| = \begin{pmatrix}
|\tilde{Y}| & 0_{k \times d-k} \\
0_{d-k,k} & 0_{d-k,d-k}
\end{pmatrix} = \begin{pmatrix}
\tilde{V} & 0_{k \times d-k} \\
0_{d-k,k} & \mathbb{1}_{d-k}
\end{pmatrix}\begin{pmatrix}
\tilde{D} & 0_{k \times d-k} \\
0_{d-k,k} & 0_{d-k,d-k}
\end{pmatrix}\begin{pmatrix}
\tilde{V}^* & 0_{k \times d-k} \\
0_{d-k,k} & \mathbb{1}_{d-k}
\end{pmatrix} = \begin{pmatrix}
|\tilde{D}| & 0_{k \times d-k} \\
0_{d-k,k} & 0_{d-k,d-k}
\end{pmatrix} = \begin{pmatrix}
\tilde{V}^* & 0_{k \times d-k} \\
0_{d-k,k} & \mathbb{1}_{d-k}
\end{pmatrix}.
\]
as required. Next, we verify that
\[
U|C| = \begin{pmatrix}
\tilde{U} & 0_{k \times d-k} \\
0_{d-k,k} & \mathbb{1}_{d-k}
\end{pmatrix}\begin{pmatrix}
|\tilde{Y}| & 0_{k \times d-k} \\
0_{d-k,k} & 0_{d-k,d-k}
\end{pmatrix} = \begin{pmatrix}
\tilde{U}|\tilde{Y}| & 0_{k \times d-k} \\
0_{d-k,k} & 0_{d-k,d-k}
\end{pmatrix} = C.
\]

In the case that $P$ is not diagonal in the standard basis there exists a unitary $W \in M_d(\mathbb{C})$ so that $W^*P$ is diagonal. In this case, we repeat the steps above with $WYW^* = \begin{pmatrix}
\tilde{Y} & 0_{k \times d-k} \\
0_{d-k,k} & 0_{d-k,d-k}
\end{pmatrix}$. Doing so provides us with a unitary $U$ satisfying $U|WYW^*| = WYW^*$ which is equivalent to $W^*UW|Y| = Y$. In this case, we take the unitary part of $Y$ to be $U' = W^*UW$. All that remains is to show that $U'$ restricts to a unitary on $\text{Im}(P)$. Recall that the non-zero eigenvectors $\{|v_1\rangle, \ldots, |v_k\rangle\}$ of $P$ are an orthonormal basis for $\text{Im}(P)$. Then, for any $1 \leq i \leq k$ we have
\[
U'|v_i\rangle = U'|W^*|e_i\rangle = W^*U|e_i\rangle = \sum_{j=1}^{k} \gamma_j W|e_j\rangle = \sum_{j=1}^{k} \gamma_j |v_j\rangle \in \text{Im}(P),
\]
where $\sum_{i=1}^{k} |\gamma_j|^2 = 1$ for all $1 \leq i \leq k$. Moreover, the columns of $U'$ restricted to this basis are orthonormal, completing the proof. \hfill \square

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