K3 SURFACES WITH INTERESTING
GROUPS OF AUTOMORPHISMS

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ABSTRACT. By the fundamental result of I.I. Piatetsky-Shapiro and I.R. Shafarevich (1971), the automorphism group Aut(X) of a K3 surface X over \( \mathbb{C} \) and its action on the Picard lattice \( S_X \) are prescribed by the Picard lattice \( S_X \). We use this result and our method (1980) to show finiteness of the set of Picard lattices \( S_X \) of rank \( \geq 3 \) such that the automorphism group Aut(X) of the K3 surface X has a non-trivial invariant sublattice \( S_0 \) in \( S_X \) where the group Aut(X) acts as a finite group. For hyperbolic and parabolic lattices \( S_0 \) it has been proved by the author before (1980, 1995). Thus we extend this results to negative sublattices \( S_0 \).

We give several examples of Picard lattices \( S_X \) with parabolic and negative \( S_0 \). We also formulate the corresponding finiteness result for reflective hyperbolic lattices of hyperbolic type over purely real algebraic number fields.

These results are important for the theory of Lorentzian Kac–Moody algebras and Mirror Symmetry.

0. Introduction

Here we want to extend theory of 2-reflective and reflective hyperbolic lattices to so called hyperbolic type. For elliptic type this theory was developed by the author [N2]—[N14] and É. B. Vinberg (see e.g. [V1]—[V3]). For parabolic type it was developed in [N11].

This theory is important for Lorentzian Kac–Moody algebras and Mirror symmetry. See [B1]—[B6], [GH], [GN1]—[GN7], [N1]—[N14].

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1. AUTOMORPHISM GROUPS OF K3 SURFACES AND
2-REFLECTIVE LATTICES OF HYPERBOLIC TYPE

First we recall some general results about automorphism groups of K3 surfaces. All of them one can find in the fundamental work by I.I. Piatetsky-Shapiro and I.R. Shafarevich [P-SSh] where, in particular, automorphism groups of K3 surfaces over algebraically closed fields of characteristic 0 were described using the group of 2-reflections of their Picard lattices.

Let \( X \) be a projective algebraic K3 surface over an algebraically closed field \( k \). Remind that it means: \( X \) is a non-singular projective algebraic surface with the canonical class \( K_X = 0 \) and \( H^1(X, \mathcal{O}_X) = 0 \). Let \( S = S_X \) be the Picard lattice

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of $X$. The lattice $S$ is hyperbolic (has signature $(1, k)$). Let $V^+(S)$ be the half-cone of the light cone $V(S) = \{ x \in S \otimes \mathbb{R} \mid x^2 > 0 \}$ of $S$ which contains a class of hyperplane section (i.e. a polarization) of $X$ and $\mathcal{L}(S) = V^+(S)/\mathbb{R}_+$ the corresponding hyperbolic space with the distance $\cosh \rho(\mathbb{R}_+ x, \mathbb{R}_+ y) = (x, y)/\sqrt{x^2y^2}$, $x, y \in V^+(S)$. (Then the curvature is equal to $-1$.) The automorphism group $O^+(S) = \{ \phi \in O(S) \mid \phi(V(S)) = V(S) \}$ of the lattice $S$ is discrete in $\mathcal{L}(S)$ and has a fundamental domain of finite volume. We denote by $W(2)(S)$ its subgroup generated by reflections in all elements with square $-2$ of $S$. Let $\text{NEF}(X)$ be the nef cone of $X$. The convex polyhedron $\mathcal{M} = \text{NEF}(X)/\mathbb{R}_{++} \subset \mathcal{L}(S)$ is a fundamental chamber for the group $W(2)(S)$, and the set $P(\mathcal{M})$ of orthogonal vectors with square $-2$ to $\mathcal{M}$ coincides with classes of all exceptional curves on $X$ (all of them are irreducible non-singular rational). We have the semi-direct product $O^+(S) = W(2)(S) \rtimes A(\mathcal{M})$ where $A(\mathcal{M}) = \{ \phi \in O^+(S) \mid \phi(\mathcal{M}) = \mathcal{M} \}$ is the group of symmetries of $\mathcal{M}$. The group $O(S)$ is an arithmetic group. From the general theory of arithmetic groups (e.g., see [R]), it follows that if $\text{rk} S \geq 3$, then one can find a fundamental domain $\mathcal{M}_0 \subset \mathcal{M}$ for $A(\mathcal{M})$ acting in $\mathcal{M}$ which has geometrically finite volume in $\mathcal{L}(S)$ (it means that it is a convex envelope of a finite set of points in $\mathcal{L}(S)$, possibly with some of them at infinity).

The group $A(\mathcal{M})$ is extremely important in geometry of the nef cone $\text{NEF}(X)$ (and its dual Mori cone) of a K3 surface and geometry of the K3 surface $X$ itself. For example, it is known that the kernel of the natural homomorphism $\pi : \text{Aut}(X) \to A(\mathcal{M})$ is finite. Thus, if one has an information about the group $A(\mathcal{M})$, he gets some information about the group $\text{Aut}(X)$. For example, if the group $A(\mathcal{M})$ is small, then the group $\text{Aut}(X)$ is also small.

One of the fundamental results of [P-SSh] is that over the basic field $k$ of characteristic $0$ the homomorphism $\pi$ has also finite cokernel. In other words, the groups $\text{Aut}(X)$ and $A(\mathcal{M})$ coincide up to finite groups. Thus, in many cases one can replace the automorphism group $\text{Aut}(X)$ by the group $A(\mathcal{M})$. More exact results about $\pi$ one can find in [N1], [N4]. We mention some of them. The kernel $G_0$ of $\pi$ is a cyclic group $\mathbb{Z}/n\mathbb{Z}$. If a prime $p\mid n$, then the lattice $S$ is $p$-elementary: $S^*/S$ is $(\mathbb{Z}/p\mathbb{Z})^a$. In particular, if $n$ is divided by two different primes, the lattice $S$ is unimodular. For a general K3 surface $X$ with the Picard lattice $S$, the image of $\pi$ is equal to $A(\mathcal{M})_0 = \{ g \in A(\mathcal{M}) \mid g|_{S^*/S} = \text{id} \}$. See more exact statements in [N4, §10].

All questions we consider below don’t change if one replaces the group $A(\mathcal{M})$ by its subgroup of finite index. Below we work with the group $A(\mathcal{M})$, but instead one can always consider the automorphism group $\text{Aut}(X)$ and its action on the Picard lattice $S$.

Further we work with an arbitrary integral hyperbolic lattice $S$ and its group of 2-reflections $W(2)(S)$ and the fundamental polyhedron $\mathcal{M}$ for $W(2)(S)$ acting in $\mathcal{L}(S)$. We leave to a reader a “difficult problem” of reformulation of our results below replacing the group $A(\mathcal{M})$ by the automorphism group $\text{Aut}(X)$ in cases when the hyperbolic lattice $S$ is isomorphic to the Picard lattice of a K3 surface $X$. Over the field $k$ of characteristic $0$ a hyperbolic lattice $S$ is isomorphic to a Picard lattice of a K3 surface if and only if there exists a primitive embedding $S \subset L_{K3}$ where $L_{K3}$ is the even unimodular lattices of signature $(3, 19)$ (in general, it follows from epimorphism of the Torelli map proved by Vic.S. Kulikov in [Ku]). In [N2] simple necessary and sufficient conditions on $S$ are given to have this primitive embedding.
embedding. For example, any even hyperbolic lattice \( S \) of rank \( \text{rk} \ S \leq 11 \) has a primitive embedding to \( L_{K3} \).

**Definition 1.** A hyperbolic lattice \( S \) is called 2-reflective if there exists a subgroup \( G \subset A(M) \) of finite index and an \( G \)-invariant sublattice \( S_0 \subset S \) such that \( S_0 \neq \{0\} \), and the group \( G|_{S_0} \) is finite. There are three cases:

1. The group \( A(M) \) has elliptic type if the lattice \( S_0 \) is hyperbolic (has one positive square). Obviously, this is equivalent for the group \( A(M) \) to be finite. Then the maximal \( S_0 \), one can take, is equal to \( S_0 = S \).

2. The group \( A(M) \) has parabolic type if \( A(M) \) is infinite and \( S_0 \) has a one dimensional kernel \( \mathbb{Z} c \) where \( c^2 = 0 \) and \( c \neq 0 \). One can prove that then the element \( c \) is fixed by the full group \( A(M) \) and \( \mathbb{R}_{++} c \in M \) (see [N13]). The full group \( A(M) \) is trivial on the sublattice \( S_0 = [c] \). The maximal sublattice \( S_0 \) is equal to the primitive sublattice in \( (c)^{\perp}_S \) generated by \( c \) and all elements with square \( -2 \) in \( (c)^{\perp}_S \).

3. The group \( A(M) \) has hyperbolic type if \( A(M) \) has not elliptic or parabolic type and the lattice \( S_0 \) is negative. Replacing \( S_0 \) by a sublattice generated by the finite set of negative sublattices \( g(S_0), g \in A(M) \), we can always suppose that the full group \( A(M) \) is finite on a non-zero negative sublattice \( S_0 \). Obviously, there exists a maximal negative sublattice \( S_0 \) in \( S \) with this property.

According to the type of the group \( A(M) \), the lattice \( S \) is called 2-reflective of elliptic, parabolic or hyperbolic type respectively.

We mention that in Definition 1 of a 2-reflective hyperbolic lattice, replacing \( S_0 \) by \( (S_0)^{\perp}_S \), one can always drop the condition that \( G \) is finite on \( S_0 \).

If \( \text{rk} \ S = 1 \), then \( S \) is obviously 2-reflective. If \( \text{rk} \ S = 2 \), then the \( S \) is 2-reflective if and only if \( S \) has a non-zero element with square \( -2 \) or \( 0 \). It was proved in [N3]—[N9] that number of 2-reflective hyperbolic lattices \( S \) of elliptic type is finite, and all of them were classified. For \( \text{rk} \ S = 4 \) it was done by É.B. Vinberg (see [N8]). Recently in [N11] finiteness of the set of reflective hyperbolic lattices of parabolic type also was proved (there we considered the general case of reflective (not only 2-reflective) lattices). Here we want to extend these our results for hyperbolic type using the same method we had applied for elliptic and parabolic type. Moreover, we can prove this finiteness for more general class of lattices \( S \).

**Definition 2.** A hyperbolic lattice \( S \) is 2-reflective of hyperbolic type if there exists a subgroup \( G \subset A(M) \) of finite index and a non-zero real negative subspace \( K \subset S \otimes \mathbb{R} \) such that \( g(K) = K \) for any \( g \in G \). Equivalently, the group \( G \) preserves a proper subspace \( L_0 \subset L(S) \) which is orthogonal to \( K \).

It seems, Definition 2 is more general than Definition 1. Obviously, if \( S \) is 2-reflective of hyperbolic type in the sense of Definition 1, then \( S \) is 2-reflective of hyperbolic type by Definition 2. Any hyperbolic lattice \( S \) of the rank \( \text{rk} \ S \leq 2 \) is 2-reflective in the sense of Definition 2.

We prove

**Theorem 1.** For a fixed rank \( \text{rk} \ S \geq 3 \) number of 2-reflective hyperbolic lattices \( S \) of the hyperbolic type in the sense of Definition 2 is finite.

**Proof.** Theorem 1 follows from Lemma 1 below. The same Lemma is valid for elliptic and parabolic type with the better constant 14 instead of 10 + \( 4 \sqrt{2} \).
Lemma 1. Let $S$ be a 2-reflective hyperbolic lattice of hyperbolic type and $n = \text{rk } S \geq 3$. Then there are $\{\delta_1, ..., \delta_n\} \subset P(\mathcal{M})$ such that

1) $\text{rk } [\delta_1, ..., \delta_n] = n$

and

2) $-2 < (\delta_i, \delta_j) \leq 10 + 4\sqrt{5} \approx 18.94427191$, $1 \leq i < j \leq n$.

Theorem 1 follows from Lemma 1 since number of lattices $[\delta_1, ..., \delta_n]$ generated by $\{\delta_1, ..., \delta_n\}$ is obviously finite (because $\delta_i^2 = -2$ for any $\delta_i \in P(\mathcal{M})$) and $[\delta_1, ..., \delta_n] \subset S \subset [\delta_1, ..., \delta_n]^*$.

**Proof of Lemma 1.** First, suppose that $n \geq 4$. Equivalently, $\text{dim } \mathcal{L}(S) \geq 3$. We consider two cases.

Case 1. Suppose that there exists a hyperplane $\mathcal{H}_\delta$, $\delta \in P(\mathcal{M})$, of a face (of the highest dimension) of $\mathcal{M}$ which does not intersect $\mathcal{L}_0$ including infinite points of $\mathcal{L}_0$.

Let $p : \mathcal{L}(S) \rightarrow \mathcal{L}_0$ be the orthogonal projection into $\mathcal{L}_0$. The set $p(\mathcal{H}_\delta) \subset \mathcal{L}_0$ is compact. We consider an open ball $D$ in $\mathcal{L}_0$ which contains this set, and the orthogonal cylinder $C_{\mathcal{M}}$ over $D$. The intersection $C_{\mathcal{M}} \cap \mathcal{M}$ has finite volume in $\mathcal{L}(S)$, and $\mathcal{M}$ is geometrically finite in $C_{\mathcal{M}}$ (i.e. $\mathcal{M} \cap C_{\mathcal{M}}$ is equal to intersection of $C_{\mathcal{M}}$ with a convex polyhedron in $\mathcal{L}(S)$ of geometrically finite volume). It follows that the face $\gamma$ of $\mathcal{M}$ containing in $\mathcal{H}_\delta$ has geometrically finite volume (in $\mathcal{H}_\delta$).

Since $\text{dim } \mathcal{L}(S) \geq 3$, there exists a 2-dimensional face $\gamma_2 \subset \gamma$ of geometrically finite volume. By Geometrical Lemma 3.2.1 in [N5], the plane polygon $\gamma_2$ has a narrow part which is given by four consecutive vertices $A_1A_2A_3A_4$ of $\gamma_2$ ($A_1 = A_4$ if $\gamma_2$ is a triangle) such that the distance $\rho((A_1A_2),(A_3A_4)) < \text{arc cosh}(7)$. We denote by $(AB)$ the line containing two different points $A, B$. (In fact, below we will prove a similar and even more complicated statement when we consider $n = 3$.)

We consider the subset $\Delta \subset P(\mathcal{M})$ which consists of all $\delta \in P(\mathcal{M})$ such that the hyperplane $\mathcal{H}_\delta$ contains one of lines $(A_1A_2)$, $(A_2A_3)$ or $(A_3A_4)$. Obviously, then the distance $\rho(\mathcal{H}_\delta, \mathcal{H}_{\delta'}) \leq \text{arc cosh}(7)$ for any $\delta, \delta' \in \Delta$. It follows that $(\delta, \delta') = 2 \cosh \rho(\mathcal{H}_\delta, \mathcal{H}_{\delta'}) < 14$. The set $\Delta$ has both properties 1) and 2) of Lemma 1. The property 1) will be satisfied since all hyperplanes $\mathcal{H}_\delta$, $\delta \in \Delta$, don’t have a common point and are not orthogonal to a hyperplane in $\mathcal{L}(S)$. Deleting unnecessary elements from $\Delta$, we get Lemma 1 for this case.

Case 2. Suppose that all hyperplanes $\mathcal{H}_\delta$, $\delta \in P(\mathcal{M})$, intersect the subspace $\mathcal{L}_0$. The polyhedron $\mathcal{M}$ has geometrically finite volume in the orthogonal cylinder $C_k$ over any compact subset $K \subset \mathcal{L}_0$. It follows that the polyhedron $\mathcal{M}$ has a vertex $v$ which is not contained in $\mathcal{L}_0$ (see considerations in [N13, Lemma 1.2.2] where it is also proved that $\mathcal{M}$ contains a face $\gamma \subset \mathcal{M}$ of geometrically finite volume which is not contained in $\mathcal{L}_0$ and $\text{dim } \gamma \geq \text{dim } \mathcal{L} - \text{dim } \mathcal{L}_0$). Consider a line $l$ which contains the vertex $v$ and is orthogonal to $\mathcal{L}_0$. Let $P = l \cap \mathcal{L}_0$. Consider a compact ball $D \subset \mathcal{L}_0$ with the center $P$ and the orthogonal cylinder $C_D$ over $D$. Since $\mathcal{M}$ has geometrically finite volume in $C_D$, there exists a hyperplane $\mathcal{H}_\delta$, $\delta \in P(\mathcal{M})$, which intersects the line $l$ in a point $B$ which is different from $A$ and $\mathcal{H}_\delta$ does not contain $A$. Consider any hyperplane $\mathcal{H}_e$, $e \in P(\mathcal{M})$, which contains the vertex $v$. Since both hyperplanes $\mathcal{H}_e$ and $\mathcal{H}_\delta$ intersect the subspace $\mathcal{L}_0$, it follows that $\rho(\mathcal{H}_e, \mathcal{H}_\delta) \leq \text{arc cosh} 3$. It follows $(e, \delta) = 2 \cosh \rho(\mathcal{H}_e, \mathcal{H}_\delta) \leq 6$. We consider the subset $\Delta \subset P(\mathcal{M})$ which consists of the $\delta$ and all $e \in P(\mathcal{M})$ such that the hyperplane $\mathcal{H}_e$ contains $v$. Again the set $\Delta$ obviously has both properties 1) and 2). It finishes the proof for $n > 4$. 

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Suppose that \( n = 3 \) and the group \( G \) is infinite. The \( \mathcal{L}(S) \) is a hyperbolic plane and \( \mathcal{L}_0 \) is a line.

The orthogonal projection of \( \mathcal{M} \) into \( \mathcal{L}_0 \) is an interval \([A, B]\) preserving by the group \( G \). If one of points \( A \) or \( B \) is finite, then the group \( G \) is finite. It follows that \( \mathcal{L}_0 \subset \mathcal{M} \). Let \( e \) be an orthogonal vector to \( \mathcal{L}_0 \) with \( e^2 = -2 \). All lines \( \mathcal{H}_\delta, \delta \in \mathcal{P}(\mathcal{M}) \), of sides of \( \mathcal{M} \) (or the corresponding \( \delta \in \mathcal{P}(\mathcal{M}) \)) are divided on two types depending on in what half-plane bounded by the line \( \mathcal{L}_0 \) they are contained. It is defined by the sign \((e, \delta)\). Since all lines \( \mathcal{H}_\delta, \delta \in \mathcal{P}(\mathcal{M}) \), may intersect the line \( \mathcal{L}_0 \) only at infinity, it follows that \(|(e, \delta)| \geq 2 \) for any \( \delta \in \mathcal{P}(\mathcal{M}) \). There exists also a possibility (which may really happen) that the line \( \mathcal{L}_0 \) is one of lines \( \mathcal{H}_\delta, \delta \in \mathcal{P}(\mathcal{M}) \), defining a side of \( \mathcal{M} \). Then we assume that \( e \in \mathcal{P}(\mathcal{M}) \).

We consider the invariant of \( \mathcal{M} \)

\[
a = \max\{|(e, \delta)| \mid \delta \in \mathcal{P}(\mathcal{M})\}.
\]

Geometrically \( a = 2 \cosh \rho \) where \( \rho \) is the maximal distance of lines of sides of the polygon \( \mathcal{M} \) from the line \( \mathcal{L}_0 \). Since the group \( G \) has finite index in \( \mathcal{A}(\mathcal{M}) \) and the fundamental domain for \( G \) in \( \mathcal{M} \) is obviously equal to the orthogonal cylinder with the base over the fundamental interval of \( G \) acting on the line \( \mathcal{L}_0 \), it follows that the maximum \( a \) always exists.

We fix \( f_2 \in \mathcal{P}(\mathcal{M}) \) such that \(|(f_2, e)| = a\). Replacing \( e \) by \(-e\), we can suppose that \( a = (e, f_2) > 0 \). Since the line \( \mathcal{H}_{f_2} \) may intersect the line \( \mathcal{L}_0 \) only at infinity, it follows that \( a = (e, f_2) \geq 2 \). Suppose that \( a = (e, f_2) = 2 \). Then all lines \( \mathcal{H}_\delta, \delta \in \mathcal{P}(\mathcal{M}) \), have \(|(\delta, e)| \leq 2 \). It then follows that they all intersect the line \( \mathcal{L}_0 \) in one of its infinite points. Then number of these lines is finite (it is not greater than 4), and the group \( G \) is finite. Thus, the invariant \( a > 2 \), and the line \( \mathcal{H}_{f_2} \) does not intersect the line \( \mathcal{L}_0 \). Consider the lines \( \mathcal{H}_{f_1}, \mathcal{H}_{f_3}, f_1, f_3 \in \mathcal{P}(\mathcal{M}) \), which define the neighboring sides to the side \( \mathcal{H}_{f_2} \) of \( \mathcal{M} \). Thus, \( \mathcal{H}_{f_1}, \mathcal{H}_{f_2} \) and \( \mathcal{H}_{f_3} \) are lines of three consecutive sides of the fundamental polygon \( \mathcal{M} \). Lemma 2 below is the analog of the statement above we used for a plane polygon \( \gamma_2 \) of geometrically finite volume.

**Lemma 2.** We have: \((f_1, f_2) \leq 2, (f_2, f_3) \leq 2 \) and

\[-2 < (f_1, f_3) \leq 14 + \frac{64}{a^2 - 4}.
\]

**Proof.** Since lines \( \mathcal{H}_{f_1} \) and \( \mathcal{H}_{f_2} \) intersect each other, \((f_1, f_2) \leq 2 \). Similarly, \((f_2, f_3) \leq 2 \). We have \((f_1, f_3) = 2 \cosh \rho(\mathcal{H}_{f_1}, \mathcal{H}_{f_3}) \). Thus, we should estimate the distance \( \rho(\mathcal{H}_{f_1}, \mathcal{H}_{f_3}) \). Consider the terminals \( P \) of the line \( \mathcal{H}_{f_1} \) and \( Q \) of the line \( \mathcal{H}_{f_3} \) on the absolute (i.e. the infinity of \( \mathcal{L}(S) \)), such that the line \( \mathcal{H}_{f_2} \) is between lines \( (PQ) \) and \( \mathcal{L}_0 \). Clearly, \( \rho((PQ), \mathcal{L}_0) \geq \rho(\mathcal{H}_{f_2}, \mathcal{L}_0) \). Let \( f \) be the orthogonal vector with square \( f^2 = -2 \) to the line \( (PQ) \), and \((f, e) > 0 \). Then \( b = (f, e) \geq (f_2, e) = a \). Thus, it is sufficient to prove Lemma replacing the element \( f_2 \) by \( f \) and \( a \) by \( b \) respectively. Let us consider the perpendiculars \( PP_1 \) and \( QQ_1 \) to the line \( \mathcal{L}_0 \) where \( P_1, Q_1 \in \mathcal{L}_0 \). Consider the line \( (P'P) \) which is obtained from the line \( (PQ) \) by the symmetry with respect to the line \( (PP_1) \). Similarly, consider the line \( (QQ') \) which is obtained from the line \( (PQ) \) by the symmetry with respect to the line \( (QQ_1) \). The lines \( (P'P) \) and \( (QQ') \) have the same distance from \( \mathcal{L}_0 \) as the line \( (PP_1) \). Thus, they are further from the line \( \mathcal{L}_0 \) than the lines \( \mathcal{H}_{f_2} \) and \( \mathcal{H}_{f_3} \).
because \((f_1, e) \leq a \leq b\) and \((f_3, e) \leq a \leq b\). It then follows that any interval with terminals on the lines \((P^P)\) and \(QQ'\) intersects both lines \(H_{f_1}\) and \(H_{f_3}\). Thus, \(\rho((P^P), (QQ')) \geq \rho(H_{f_1}, H_{f_3})\). Let \(f'_1\) and \(f'_3\) be orthogonal vectors with square \((f'_1)^2 = (f'_3)^2 = -2\) to the lines \((P^P)\) and \((QQ')\) respectively, and \((f'_1, e) > 0\), \((f'_3, e) > 0\). Then \((f'_1, f'_3) \geq (f_1, f_3)\), and it is sufficient to estimate \((f'_1, f'_3)\). Let \(g\) and \(h\) are orthogonal vectors with square \(-2\) to lines \(PP_1\) and \(QQ_1\) and directed outwards from the quadrangle \(P_1QQ_1\). Then the vectors \(f'_1\) and \(f'_3\) are given by the reflections in \(g\) and \(h\) respectively. We have \(f'_1 = f + 2g\) and \(f'_3 = f + 2h\). Thus, \((f'_1, f'_3) = f^2 + 2(f, h) + 2(f, g) + 4(g, h) = 6 + 4x\) where \(x = (g, h)\). The Gram matrix of the four vectors \(e, g, f, h\) is equal to

\[
\Gamma = \begin{pmatrix}
-2 & 0 & b & 0 \\
0 & -2 & 2 & x \\
b & 2 & -2 & 2 \\
0 & x & 2 & -2
\end{pmatrix}.
\]

We have \(\det(\Gamma) = x^2(b^2 - 4) - 16x - 4b^2 - 16 = 0\) and \(x = 2 + 16/(b^2 - 4)\). Thus \((f_1, f_3) \leq (f'_1, f'_3) = 14 + 64/(b^2 - 4) \leq 14 + 64/(a^2 - 4)\). It proves Lemma 2.

We denote by \(a_0 = 2\sqrt{2} + \sqrt{5} = 4.116342\ldots\) the root of the equation

\[
x^4 - 16x^2 - 16 = 0.
\]

We consider three cases.

Case 1. Suppose that \(a \geq a_0\).

Then elements \(f_1, f_2, f_3 \in P(M)\) of Lemma 2 give elements for the Lemma 1 we are looking for.

Case 2. Suppose that \(a \leq a_0\) and the line \(L_0\) is one of lines of sides of \(M\). Equivalently, \(e \in P(M)\).

Consider a vertex \(v\) of \(M\). We take two lines \(H_{f_1}\) and \(H_{f_2}\), \(f_1, f_2 \in P(M)\), of the two consecutive sides of \(M\) which contain the vertex \(v\). Then elements \(f_1, f_2, f_3 = e\) give elements of Lemma 1 we are looking for. Really, \(-2 < (f_1, f_2) \leq 2\) because lines \(H_{f_1}\) and \(H_{f_2}\) intersect each other. We have \((f_1, f_3) \leq a \leq a_0 \leq 18\), \((f_2, f_3) \leq a \leq a_0 \leq 18\) by definition of the invariant \(a\).

Case 3. Suppose that \(a \leq a_0\) and the line \(L_0\) is not a line of a side of \(M\).

Like for the case 2, we take elements \(f_1, f_2 \in P(M)\) such that \(H_{f_1}\) and \(H_{f_2}\) contain a vertex \(v\) of the polyhedron \(M\). We can suppose that \((f_1, e) > 0\) and \((f_2, e) > 0\). Consider a perpendicular \((vP)\) to the line \(L_0\). The line \((vP)\) intersects another line \(H_{f_3}\), \(f_3 \in P(M)\), of a side of \(M\). This line is contained in the second half-plane bounded by \(L_0\). Thus, the line \(L_0\) is between lines \(H_{f_1}\), \(H_{f_3}\), and \(H_{f_2}\), \(H_{f_3}\), as well. Suppose that the line \((vP)\) is equal to \((RS)\) where \(R, S\) are points on the absolute and \(v\) is between \(R\) and \(P\). Consider lines \(l_1\) and \(l_2\) with orthogonal vectors with square \(-2\) \(w_1\) and \(w_2\) respectively and such that \(l_1\) contains \(R\), \(l_2\) contains \(S\), lines \(l_1, l_2\) are contained in different half-planes bounded by \((RS)\) and in different half-planes bounded by \(L_0\), and \((w_1, e) = a\), \((w_2, e) = -a\). Then \((w_1, w_2) \geq (f_1, f_3)\) and \((w_1, w_2) \geq (f_2, f_3)\) because \(0 < (f_1, e) \leq a\), \(0 < (f_2, e) \leq a\), and \(-a < (f_3, e) < 0\) by definition of the invariant \(a\). (Geometrically, lines \(l_1\) and \(l_2\) are further from each other than the lines \(H_{f_1}\), \(H_{f_3}\) and lines \(H_{f_2}\), \(H_{f_3}\) as well. One should draw the corresponding picture to make sure.) Lines \(l_1\) and \(l_2\) are symmetric with respect to the point of intersection of lines \(L_0\) and \((RS)\). If \(a\) is the orthogonal
vector with $g^2 = -2$ to the line $(RS)$ and $(g, w_1) = 2$, then $w_2$ is obtained from $w_1$ by the composition of symmetries in $g$ and $e$. It follows that $w_2 = w_1 + 2g + ae$ and $(w_1, w_2) = 2 + a^2$. Thus, we have proved that $(f_3, f_1) \leq 2 + a^2$ and $(f_3, f_2) \leq 2 + a^2$.

Since $a \leq a_0$, elements $f_1, f_2, f_3$ give elements of Lemma 1 we are looking for.

The equation for $a_0$, we have introduced above, is the equation $2 + x^2 = 14 + 64/(x^2 - 4)$ when Lemma 2 and considerations of Case 3 give the same result.

It finishes the proof of Lemma 1 and Theorem 1.

2. Reflective hyperbolic lattices

Let $K$ be a purely real algebraic number field of finite degree $[K : \mathbb{Q}]$ and $\mathcal{O}$ its ring of integers. A lattice $S$ over $K$ is a projective module over $\mathcal{O}$ of a finite rank equipped with a non-degenerate symmetric bilinear form with values in $\mathcal{O}$. A lattice $S$ is called hyperbolic if there exists an embedding $\sigma^{(+)}: K \to \mathbb{R}$ (it is called the geometric embedding) such that the real symmetric bilinear form $S \otimes \mathbb{R}$ is hyperbolic (i.e. it has exactly 1 positive square), and for all other embeddings $\sigma \neq \sigma^{(+)}$ the real form $S \otimes \mathbb{R}$ is negative definite. Like for integral hyperbolic lattices, using the geometric embedding, one can define the hyperbolic space $L(S)$, the groups $O^+(S)$, the group $W(S)$ generated by all reflections of the lattice $S$ (i.e. automorphisms from $O^+(S)$ which act as reflections with respect to hyperplanes in $L(S)$), fundamental polyhedron $M \subset L(S)$ for $W(S)$, set $P(M) \subset S$ of orthogonal vectors to faces of $M$, and the group of symmetries $A(M)$ of $M$.

Applying Definitions 1 and 2 to this situation, we get definition of reflective hyperbolic lattices of elliptic, parabolic and hyperbolic types. Here a sublattice $S_0 \subset S$ (over $\mathcal{O}$) is called hyperbolic, parabolic or negative if it generates a real subspace of this type for the geometric embedding. For parabolic type, one should consider $\mathcal{O}$ instead of $\mathbb{Z}$; it then follows that $K = \mathbb{Q}$.

Using combination of considerations here for the proof of Theorem 1 and considerations in [N5] and [N6], we get

**Theorem 2.** i) For a fixed $N$, the set of fields $K$ such that $[K : \mathbb{Q}] = N$ and there exists a hyperbolic reflective lattice $S$ over $K$ of rank $rk S \geq 3$ (of any type: elliptic, parabolic or hyperbolic) is finite.

ii) For a fixed rank $n \geq 3$ and a fixed purely real algebraic number field $K$, the set of reflective hyperbolic lattices (of any type: elliptic, parabolic or hyperbolic) $S$ over $K$ and of the rank $n$ is finite up to multiplication of the form of $S$ by $m \in K$.

For reflective hyperbolic lattices of elliptic type it was proved in [N5] and [N6]. For parabolic type it is proved in [N11]. Thus, Theorem 2 generalizes that results for hyperbolic type. Like for elliptic and parabolic type, the proof is based on

**Lemma 3.** For any reflective hyperbolic lattice $S$ of rank $rk S = n \geq 3$ there are $\delta_1, ..., \delta_n \in P(M)$ such that

1) $rk [\delta_1, ..., \delta_n] = n$;

2) the Gram diagram of $\delta_1, ..., \delta_n$ is connected (i.e. one cannot divide the set $\{\delta_1, ..., \delta_n\}$ on two non-empty subsets which are orthogonal to one another);

3) for the geometric embedding,

$$\frac{4(\delta_i, \delta_j)^2}{\epsilon_i \epsilon_j} < 200^2, \quad 1 \leq i < j \leq n.$$
Proof. For elliptic type Lemma 3 was proved in [N5], for parabolic type in [N11] with the constant 62 instead of 200. For hyperbolic type, the proof is similar and uses combination of considerations in [N5] and considerations here for the proof of Lemmas 1 and 2 and Theorem 1. In fact, like in [N11] and [N1], one can introduce a class of convex locally finite polyhedra \( \mathcal{M} \) of restricted hyperbolic type and prove Lemma 3 for these polyhedra like we did it for polyhedra of elliptic and restricted parabolic type. Lemma 3 for this class of polyhedra is important as itself: for example, for Mori polyhedra (e.g. see [N13]) and for Borcherds type automorphic products (see [B5], [N11] and [GN5]). We hope to present details of the proof somewhere.

We mention that Theorem 2 partly gives an answer to our question in [N9, Sect.3]: to generalize results about crystallographic reflection groups in hyperbolic spaces for so called generalized crystallographic reflection groups in hyperbolic spaces.

3. SOME EXAMPLES OF 2-REFLECTIVE HYPERBOLIC LATTICES

Here we present some examples of 2-reflective hyperbolic lattices over \( \mathbb{Z} \).

3.1. 2-ELEMENTARY 2-REFLECTIVE HYPERBOLIC LATTICES AND K3 SURFACES WITH INVOLUTION. This example had been first considered in [N4] (and, in fact, in [N2], [N3] before, see also related results in [N7], [N9]). Let \( S \) be a 2-elementary even hyperbolic lattice and \( S \subset L_{K3} \). We remind that \( S \) is called 2-elementary if \( S^*/S \cong (\mathbb{Z}/2\mathbb{Z})^a \). By the Global Torelli Theorem [P-SSH], there exists a K3 surfaces \( X \) over \( \mathbb{C} \) with \( S = S_X \) and with the canonical involution \( \sigma \) which acts trivially on \( S_X \) and as multiplication by \(-1\) on the transcendental lattice \( T_X = S_{X}^1 \). Obviously, the \( \text{Aut}(X) \) normalizes the involution \( \sigma \). In particular, \( \text{Aut}(X) \) preserves the set \( X^\sigma \) of points of \( X \) fixed by this involution. The set \( X^\sigma \) is a non-singular curve whose components generate the sublattice \( S_0 \subset S \) which is obviously invariant with respect to \( \text{Aut}(X) \), and the group \( \text{Aut}(X) \) is finite on \( S_0 \). Thus, if \( X^\sigma \neq \emptyset \), the lattice \( S \) is 2-reflective.

In [N4] (actually, it had been done in [N2] and [N3]) the set \( X^\sigma \) and the lattice \( S_0 \) were described explicitly using invariants of the lattice \( S \), and all 2-elementary lattices \( S \) having a primitive embedding to \( L_{K3} \) also were described. Let \( r = \text{rk} \, S \). For \( S \cong U(2) \oplus E_8(2) \) the set \( X^\sigma = \emptyset \) and the lattice \( S \) is not 2-reflective (because it does not have elements with the square \(-2\)). (Here we denote by \( K(t) \) a lattice \( K \) with the form multiplied by \( t \in \mathbb{Q} \). The lattice \( U \) is the standard even unimodular lattice of signature \((1,1)\). The lattice \( E_8 \) is the standard even unimodular lattice of signature \((0,8)\).) For \( S \cong U \oplus E_8(2) \), the set \( X^\sigma = C_1^{(1)} \prod C_i^{(2)} \) and the lattice \( S \) is 2-reflective of parabolic type. If \( S \) is different from the two lattices above, then \( X^\sigma = C_0 \prod C_0^{(1)} \prod \cdots \prod C_0^{(k)} \) where \( g = (22 - r - a)/2, \, k = (r - a)/2 \). Here the below index denotes the genus \( g \) (or \( 0 \)) of the corresponding smooth irreducible curve, equivalently \( C_0^g = 2g - 2 \) and \( (C_0^t)^2 = -2, \, 1 \leq t \leq k \). Obviously, \( g + k \leq 11 \) and excluding some exceptions for \( g + k = 10 \) and \( g + k = 11 \), all other cases \( g, k \) satisfying \( g \geq 0, \, k \geq 0 \) and \( g + k \leq 11 \) correspond to the K3 surfaces with involution.

In particular, if \( g > 1 \), the \( S \) has elliptic type. If \( g = 1 \), the lattice \( S \) has elliptic or parabolic type. It was shown in [N4] that for \( g = 1 \) only the lattice \( C_0 \) has elliptic or parabolic type. Finally, if \( g = 0 \), the lattice \( S \) has parabolic type, and \( S_{X}^1 = E_8(1) \). (Here \( E_8(1) \) is the standard even unimodular lattice of signature \((8,8)\).)
$S = U \oplus E_8 \oplus E_8 \oplus \langle -2 \rangle$ has elliptic type. We remark that for $g = 1$ the K3 surface $X$ has the canonical elliptic fibration $|C_1|$ which is preserved by the group Aut($X$).

If $g = 0$, then $S$ is 2-reflective of hyperbolic or parabolic type.

Thus, we get a lot of examples of 2-reflective hyperbolic lattices $S$ of elliptic, parabolic and hyperbolic type of all possible rank $rk \leq 20$. All 2-elementary hyperbolic lattices $S$ having a primitive embedding to $L_{K3}$ are 2-reflective except the lattice $U(2) \oplus E_8(2)$.

3.2. A general result about 2-reflective hyperbolic lattices $S$ of $rk \geq 6$.

In [N4] we proved that for any K3 surface $X$ with $rk \geq 6$ the $X$ always has an elliptic fibration with infinite automorphism group if the full automorphism group $Aut(X)$ is infinite. It is interesting that this is true for arbitrary characteristic of the ground field. It then follows that for parabolic and hyperbolic types of Definition 1 the canonical lattice $S_0$ is equal to intersection of all primitive sublattices in $S$ generated by components of fibers of elliptic fibrations on $X$ having an infinite automorphism group. (In [N4] we also claimed that $S_0$ is generated up to finite index by irreducible curves in $X$, but this statement is might be wrong.) If the lattice $S_0 \neq 0$, then $S$ is 2-reflective of parabolic type if $S_0$ is parabolic, and $S$ is 2-reflective of hyperbolic type if $S_0$ is negative.

It shows geometrical sense of $S_0$ when $X$ has elliptic fibrations with infinite automorphism groups. The same statement is valid for all hyperbolic lattices $S$ of $rk \geq 6$ if one replaces the group $Aut(X)$ by $A(M)$ and elliptic fibrations by automorphism groups of infinite vertices of $M$.

We mention that using this idea one can prove that 2-reflective (and reflective) hyperbolic lattices over $\mathbb{Z}$ of hyperbolic type do not exist in high dimension. This is known for elliptic type [N3], [N4], [N6] and [V2] and parabolic type [N11].

3.3. 2-reflective hyperbolic lattices of rank 3.

We consider hyperbolic lattices $S_k = U \oplus \langle -2k \rangle$, $k \in \mathbb{N}$. We consider the standard basis $e_1, e_2$ of the lattice $U$ and the standard bases $e_3$ of $\langle -2k \rangle$, i.e. with the Gram matrix

$$
(e_i, e_j) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2k \end{pmatrix}.
$$

Consider $k = 2$. For the fundamental polyhedron $M_0$ of the full reflection group $W(S_2)$ we have (using Vinberg’s algorithm [V1])

$$P(M_0)_{pr} = \{\delta_{0,1} = (0, 0, 1), \delta_{0,2} = (2, 0, -1), \delta_1 = (-1, 1, 0)\}
$$

with the Gram matrix

$$G(P(M_0)_{pr}) = \begin{pmatrix} -4 & 4 & 0 \\ 4 & -4 & 2 \\ 0 & 2 & -2 \end{pmatrix}.
$$

Here the first index $i$ in $\delta_{i,j}$ shows appearing of this element on the $i$-th step of the Vinberg’s algorithm with the center $\rho = (1, 0, 0)$. Index pr shows that we consider primitive elements. We denote by $\Delta(2)(S)$ the set of all elements of $S$ with square $-2$.

All intermediate lattices $S_2 \supset S = S_{2,1} \supset [\Delta(2)(S_2)]$ of index $l = [S_2 : S]$ are equal to

$$S_2 = S_{2,1} = [\delta_{0,1}, \delta_{0,2}, \delta_1, (\delta_{0,1} + \delta_{0,2})/2] \supset S_{2,2} = [\delta_{0,1}, \delta_{0,2}, \delta_1]
\supset S_{2,3} = [2\delta_{0,1}, \delta_{0,2}, \delta_1] = [\Delta(2)(S_2)]$$

for $l = 2, 3, 4$. Let $h_{2,1}$ be the highest root of $S_2 = S_{2,1} = U(2) \oplus E_8(2)$. We consider the standard basis $e_1, e_2$ of the lattice $U$ and the standard bases $e_3$ of $\langle -2 \rangle$, i.e. with the Gram matrix

$$G(U(2)) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$
and for all these lattices
\[ P(M_0) = \{ \delta_{0,1} = (0, 0, 1), \delta_{0,2} = (2, 0, -1), \delta_1 = (-1, 1, 0) \}. \]

One should multiply that elements by some constants from \( \mathbb{N} \) to get elements from \( S_{2,l} \). For all these lattices
\[ P(M) = A(M)(\delta_1), \quad A(M) = [s_{\delta_{0,1}}, s_{\delta_{0,2}}] \cong D_\infty. \]

We denote by \( D_\infty \) the group on a line generated by two different reflections. Here \( s_\delta \) denote the reflection in \( \delta \). Thus, the group \( A(M) \) and the corresponding group \( Aut(X) \) of a K3 surfaces \( X \) with \( S_X \cong S_{2,l} \) are isomorphic to \( \mathbb{Z} \). It follows that all lattices \( S_{2,l} \) are 2-reflective of parabolic type because the group \( A(M) \) fixes the parabolic sublattice \( \mathbb{Z}c \) where \( c = \delta_{0,1} + \delta_{0,2} \) with \( c^2 = 0 \). The linear system \( |c| \) defines the unique elliptic fibration on the K3 surface \( X \). Moreover, this elliptic fibration has infinite automorphism group (it is \( \mathbb{Z} \)). Thus, we get three examples of K3 surfaces with the parabolic automorphism group \( Aut(X) \cong \mathbb{Z} \) and with very nice geometry.

One can check that
\[ S_2 \cong U \oplus \langle -4 \rangle; \quad S_{2,2} \cong \langle 4 \rangle \oplus \langle -2 \rangle^2; \quad S_{2,4} \cong \langle 16 \rangle \oplus \langle -2 \rangle^2. \]

We get similar examples for \( k = 3, 5, 7, 13 \). The case \( k = 3 \) gives three lattices \( S_3 \cong U \oplus \langle -6 \rangle; \quad S_{3,3} \cong \langle 18 \rangle \oplus A_2; \quad S_{3,6} \cong \langle 72 \rangle \oplus A_2 \). The case \( k = 5 \) gives the only lattice \( S_5 \cong U \oplus \langle -10 \rangle \). The case \( k = 7 \) gives two lattices \( S_7 = U \oplus \langle -14 \rangle \) and \( S_{7,2} = [\Delta(2)(S_7)] \). The case \( k = 13 \) gives the only lattice \( U \oplus \langle -26 \rangle \). Here one should use calculations in Sect. 4 below.

Using the general method described above for the proof of Theorem 1, we can prove

**Theorem 3.** Lattices \( S_{k,l}, k = 2, 3, 5, 7, 13 \), defined above, are the only Picard lattices of rank 3 of K3 surfaces \( X \) such that \( X \) has the only one elliptic fibration, moreover this fibration has an infinite automorphism group (it is then isomorphic to \( \mathbb{Z} \)).

Now we can look for all other 2-reflective lattices in the series \( S_k = U \oplus \langle -2k \rangle \) and their intermediate lattices \( S_k \supset S_{k,l} \supset [\Delta(2)(S_k)] \). We get the following and the only following 2-reflective lattices: The lattice \( S_1 = U \oplus \langle -2 \rangle \) is 2-reflective of elliptic type. All other lattices will be 2-reflective of parabolic type. They are: the lattice \( S_4 \) and its intermediate sublattices \( S_{4,1}, S_{4,2} \), \( S_{4,4} \), \( S_{4,8} = [\Delta(2)(S_4)] \). The lattice \( S_9 \) and its intermediate sublattices \( S_{9,3} = [\Delta(2)(S_9)] \). The lattice \( S_{25}. \) (Here one should use calculations in Sect. 4 below.) Lattices \( S_4, S_9 \) and \( S_{25} \) and their intermediate sublattices give Picard lattices or rank 3 of K3 surfaces having the unique elliptic fibration with infinite automorphism group and some other elliptic fibrations with finite automorphism group. One need to add very few cases to get the full list of K3 surfaces of this type.

Consider the lattice \( S = U(11) \oplus \langle -2 \rangle \) with the standard bases. The fundamental polyhedron \( M_2 \) for \( W(S) \) has the set of primitive orthogonal vectors \( \delta_1, \ldots, \delta_{12} \).
with coordinates given by lines of the matrix \( P(\mathcal{M}_0) \) and with the Gram matrix \( G(P(\mathcal{M}_0)) \) given below

\[
P(\mathcal{M}_0) = \begin{pmatrix}
1 & 0 & -1 \\
0 & 0 & 1 \\
-1 & 1 & 0 \\
3 & 3 & -10 \\
8 & 6 & -23 \\
5 & 2 & -11
\end{pmatrix}, \quad G(P(\mathcal{M}_0)) = \begin{pmatrix}
-2 & 2 & 11 & 13 & 20 & 0 \\
2 & -2 & 0 & 20 & 46 & 22 \\
11 & 0 & -22 & 0 & 22 & 33 \\
13 & 20 & 0 & -2 & 46 & 11 \\
20 & 46 & 22 & 2 & -2 & 0 \\
0 & 22 & 33 & 11 & 0 & -22
\end{pmatrix}.
\]

It follows that the fundamental polyhedron \( \mathcal{M} \) for \( W^{(2)}(S) \) has

\[
P(\mathcal{M}) = A(\mathcal{M})(\{\delta_1, \delta_2, \delta_4, \delta_5\}), \quad A(\mathcal{M}) = [s_{\delta_3}, s_{\delta_6}] \cong D_\infty.
\]

The lattice \( S \) is 2-reflective of hyperbolic type since \( A(\mathcal{M}) \) keeps the sublattice \([\delta_3, \delta_6] \) generated by \( w = (2, 2, -7) \) with \( w^2 = -10 \). Thus, the lattice \( S \) is 2-reflective of hyperbolic type. A K3 surface \( X \) over \( \mathbb{C} \) with the Picard lattice \( S \cong U(11) \oplus \langle -2 \rangle \) has the automorphism group \( \text{Aut}(X) \cong \mathbb{Z} \) but this group is not the automorphism group of any elliptic fibration on \( X \). All elliptic fibrations on \( X \) have finite automorphism groups. We remark that the lattice \( S = U(11) \oplus \langle -2 \rangle \) is 11-dual to the lattice \( S_{11} = U \oplus \langle -22 \rangle \) which we will consider in Sect. 4 below.

Similarly one can check that the lattices \( U(15) \oplus \langle -2 \rangle \) and \( U(24) \oplus \langle -2 \rangle \) are 2-reflective of hyperbolic type. K3 surfaces over \( \mathbb{C} \) with these Picard lattices have the group \( \text{Aut}(X) \cong \mathbb{Z} \) but this group is not a group of automorphisms of any elliptic fibration on \( X \). All elliptic fibrations on \( X \) have finite automorphism groups. We remark that \( U(15) \oplus \langle -2 \rangle \) is 15-dual to the lattice \( S_{15} = U \oplus \langle -30 \rangle \), and the lattice \( U(24) \oplus \langle -2 \rangle \) is dual to the lattice \( S_{24} = U \oplus \langle -48 \rangle \) which we consider below.

We hope to classify all 2-reflective hyperbolic lattices of rank 3 later. For elliptic type of rank 3 this had been done in [N8].

4. Reflective hyperbolic lattices of rank 3 representing 0

Here we consider the fundamental series \( S_k = U \oplus \langle -2k \rangle \), \( k \in \mathbb{N} \), of hyperbolic lattices of rank 3 representing 0.

Below we give results of our calculations using Vinberg’s algorithm [V1] of the fundamental polyhedron \( \mathcal{M} \) for the full group \( W(S_k) \) generated by reflections, if \( k \leq 60 \). We use the standard bases \( e_1, e_2, e_3 \) introduced above. The first matrix (or the matrices \( e, f \) for the hyperbolic type) gives the set \( P(\mathcal{M}) \) of primitive orthogonal vectors to \( \mathcal{M} \). The second matrix (or \( G(e), G(f) \) for the hyperbolic type) gives their Gram matrix. For the hyperbolic type the vector \( w \) generates a negative sublattice \( S_0 \) in \( S_k \) which is invariant with respect to the group \( A(\mathcal{M}) \) which is given by the set of generators \( C_i \). For hyperbolic type the set \( P(\mathcal{M}) = A(\mathcal{M})(e \cup f) \) or \( P(\mathcal{M}) = A(\mathcal{M})(e) \) (if we don’t give the set \( f \)).

\[
n = 1 : \quad \begin{pmatrix}
1 & 0 & -1 \\
0 & 0 & 1 \\
-1 & 1 & 0
\end{pmatrix}, \quad \begin{pmatrix}
-2 & 2 & 1 \\
2 & -2 & 0 \\
1 & 0 & -2
\end{pmatrix}.
\]

\[
n = 2 : \quad \begin{pmatrix}
2 & 0 & -1 \\
0 & 0 & 1 \\
-1 & 1 & 0
\end{pmatrix}, \quad \begin{pmatrix}
-4 & 4 & 2 \\
4 & -4 & 0 \\
2 & 0 & -2
\end{pmatrix}.
\]
\[
\begin{align*}
n &= 3: \\
&\quad \begin{pmatrix} 3 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} -6 & 6 & 3 \\ 6 & -6 & 0 \\ 3 & 0 & -2 \end{pmatrix}.
\end{align*}
\]

\[
\begin{align*}
n &= 4: \\
&\quad \begin{pmatrix} 4 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} -8 & 8 & 4 \\ 8 & -8 & 0 \\ 4 & 0 & -2 \end{pmatrix}.
\end{align*}
\]

\[
\begin{align*}
n &= 5: \\
&\quad \begin{pmatrix} 5 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \\ 2 & 2 & -1 \end{pmatrix}, \quad \begin{pmatrix} -10 & 10 & 5 & 0 \\ 10 & -10 & 0 & 10 \\ 5 & 0 & -2 & 0 \\ 0 & 10 & 0 & -2 \end{pmatrix}.
\end{align*}
\]

\[
\begin{align*}
n &= 6: \\
&\quad \begin{pmatrix} 6 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \\ 2 & 2 & -1 \end{pmatrix}, \quad \begin{pmatrix} -12 & 12 & 6 & 0 \\ 12 & -12 & 0 & 12 \\ 6 & 0 & -2 & 0 \\ 0 & 12 & 0 & -4 \end{pmatrix}.
\end{align*}
\]

\[
\begin{align*}
n &= 7: \\
&\quad \begin{pmatrix} 7 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \\ 3 & 2 & -1 \end{pmatrix}, \quad \begin{pmatrix} -14 & 14 & 7 & 0 \\ 14 & -14 & 0 & 14 \\ 7 & 0 & -2 & 1 \\ 0 & 14 & 1 & -2 \end{pmatrix}.
\end{align*}
\]

\[
\begin{align*}
n &= 8: \\
&\quad \begin{pmatrix} 8 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \\ 8 & 8 & -3 \end{pmatrix}, \quad \begin{pmatrix} -16 & 16 & 8 & 16 \\ 16 & -16 & 0 & 48 \\ 8 & 0 & -2 & 0 \\ 16 & 48 & 0 & -16 \end{pmatrix}.
\end{align*}
\]

\[
\begin{align*}
n &= 9: \\
&\quad \begin{pmatrix} 9 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \\ 4 & 2 & -1 \end{pmatrix}, \quad \begin{pmatrix} -18 & 18 & 9 & 0 \\ 18 & -18 & 0 & 18 \\ 9 & 0 & -2 & 2 \\ 0 & 18 & 2 & -2 \end{pmatrix}.
\end{align*}
\]

\[
\begin{align*}
n &= 10: \\
&\quad \begin{pmatrix} 10 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \\ 4 & 2 & -1 \end{pmatrix}, \quad \begin{pmatrix} -20 & 20 & 10 & 0 \\ 20 & -20 & 0 & 20 \\ 10 & 0 & -2 & 2 \\ 0 & 20 & 2 & -4 \end{pmatrix}.
\end{align*}
\]

\[
\begin{align*}
n &= 11: \\
&\quad \begin{pmatrix} 11 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \\ 33 & 33 & -10 \end{pmatrix}, \quad \begin{pmatrix} -22 & 22 & 11 & 143 & 220 & 0 \\ 22 & -22 & 0 & 220 & 506 & 22 \\ 11 & 0 & -2 & 0 & 22 & 3 \\ 143 & 220 & 0 & -22 & 22 & 11 \end{pmatrix}.
\end{align*}
\]

\[
\begin{align*}
n &= 12: \\
&\quad \begin{pmatrix} 12 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \\ 3 & 3 & -1 \end{pmatrix}, \quad \begin{pmatrix} -24 & 24 & 12 & 12 \\ 24 & -24 & 0 & 24 \\ 12 & 0 & -2 & 0 \\ 12 & 24 & 0 & -6 \end{pmatrix}.
\end{align*}
\]

\[
\begin{align*}
n &= 13: \\
&\quad \begin{pmatrix} 13 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \\ 4 & 3 & -1 \end{pmatrix}, \quad \begin{pmatrix} -26 & 26 & 13 & 13 & 0 \\ 26 & -26 & 0 & 26 & 26 \\ 13 & 0 & -2 & 1 & 4 \\ 13 & 26 & 1 & -2 & 0 \\ 0 & 26 & 4 & 0 & 2 \end{pmatrix}.
\end{align*}
\]
\[
\begin{bmatrix}
14 & 0 & -1 \\
0 & 0 & 1 \\
-1 & 1 & 0 \\
7 & 7 & -2 \\
6 & 2 & -1
\end{bmatrix}
\begin{bmatrix}
-28 & 28 & 14 & 42 & 0 \\
28 & -28 & 0 & 56 & 28 \\
14 & 0 & -2 & 0 & 4 \\
42 & 56 & 0 & -14 & 0 \\
0 & 28 & 4 & 0 & -4
\end{bmatrix}
\]

\[
\begin{bmatrix}
15 & 0 & -1 \\
0 & 0 & 1 \\
-1 & 1 & 0 \\
15 & 15 & -4 \\
60 & 30 & -11 \\
7 & 2 & -1
\end{bmatrix}
\begin{bmatrix}
-30 & 30 & 15 & 105 & 120 & 0 \\
30 & -30 & 0 & 120 & 330 & 30 \\
15 & 0 & -2 & 0 & 30 & 5 \\
105 & 120 & 0 & -30 & 30 & 15 \\
120 & 330 & 30 & 30 & -30 & 0 \\
0 & 30 & 5 & 15 & 0 & -2
\end{bmatrix}
\]

\[
\begin{bmatrix}
16 & 0 & -1 \\
0 & 0 & 1 \\
-1 & 1 & 0 \\
5 & 3 & -1 \\
48 & 16 & -7
\end{bmatrix}
\begin{bmatrix}
-32 & 32 & 16 & 16 & 32 \\
32 & -32 & 0 & 32 & 224 \\
16 & 0 & -2 & 2 & 32 \\
16 & 32 & 2 & -2 & 0 \\
32 & 224 & 32 & 0 & -32
\end{bmatrix}
\]

\[
\begin{bmatrix}
17 & 0 & -1 \\
0 & 0 & 1 \\
-1 & 1 & 0 \\
4 & 4 & -1 \\
85 & 51 & -16 \\
204 & 102 & -35 \\
19 & 8 & -3 \\
8 & 2 & -1
\end{bmatrix}
\begin{bmatrix}
-34 & 34 & 17 & 34 & 323 & 544 & 34 & 0 \\
34 & -34 & 0 & 34 & 544 & 1190 & 102 & 34 \\
17 & 0 & -2 & 0 & 34 & 102 & 11 & 6 \\
34 & 34 & 0 & -2 & 0 & 34 & 6 & 6 \\
323 & 544 & 34 & 0 & -34 & 34 & 17 & 34 \\
544 & 1190 & 102 & 34 & 34 & -34 & 0 & 34 \\
34 & 102 & 11 & 6 & 17 & 0 & -2 & 0 \\
0 & 34 & 6 & 6 & 34 & 34 & 0 & -2
\end{bmatrix}
\]

\[
\begin{bmatrix}
18 & 0 & -1 \\
0 & 0 & 1 \\
-1 & 1 & 0 \\
4 & 4 & -1 \\
8 & 2 & -1
\end{bmatrix}
\begin{bmatrix}
-36 & 36 & 18 & 36 & 0 \\
36 & -36 & 0 & 36 & 36 \\
18 & 0 & -2 & 0 & 6 \\
36 & 36 & 0 & -4 & 4 \\
0 & 36 & 6 & 4 & -4
\end{bmatrix}
\]

\[
\begin{bmatrix}
19 & 0 & -1 \\
0 & 0 & 1 \\
-1 & 1 & 0 \\
31 & 30 & -7 \\
171 & 152 & -37 \\
190 & 152 & -39 \\
6 & 3 & -1 \\
9 & 2 & -1
\end{bmatrix}
\begin{bmatrix}
-38 & 38 & 19 & 304 & 1482 & 1406 & 19 & 0 \\
38 & -38 & 0 & 266 & 1406 & 1482 & 38 & 38 \\
19 & 0 & -2 & 1 & 19 & 38 & 3 & 7 \\
304 & 266 & 1 & -2 & 0 & 38 & 7 & 66 \\
1482 & 1406 & 19 & 0 & -38 & 38 & 19 & 304 \\
1406 & 1482 & 38 & 38 & -38 & 38 & 0 & 266 \\
19 & 38 & 3 & 7 & 19 & 0 & -2 & 1 \\
0 & 38 & 7 & 66 & 304 & 266 & 1 & -2
\end{bmatrix}
\]

\[
\begin{bmatrix}
20 & 0 & -1 \\
0 & 0 & 1 \\
-1 & 1 & 0 \\
4 & 4 & -1 \\
15 & 5 & -2
\end{bmatrix}
\begin{bmatrix}
-40 & 40 & 20 & 40 & 20 \\
40 & -40 & 0 & 40 & 80 \\
20 & 0 & -2 & 0 & 10 \\
40 & 40 & 0 & -8 & 0 \\
20 & 80 & 10 & 0 & -10
\end{bmatrix}
\]

\[
\begin{bmatrix}
21 & 0 & -1 \\
0 & 0 & 1 \\
-1 & 1 & 0 \\
6 & 3 & -1 \\
10 & 2 & -1
\end{bmatrix}
\begin{bmatrix}
-42 & 42 & 21 & 21 & 0 \\
42 & -42 & 0 & 42 & 42 \\
21 & 0 & -2 & 3 & 8 \\
21 & 42 & 3 & -6 & 0 \\
0 & 42 & 0 & 0 & 2
\end{bmatrix}
\]
The translation $n = 22$:

$$n = 23$: Hyperbolic type with $(w, w) = -138$, the symmetry group generated by the translation $C$, and

$$e = \begin{pmatrix} 23 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}, \quad G(e) = \begin{pmatrix} -46 & 46 & 23 \\ 46 & -46 & 0 \\ 23 & 0 & -2 \end{pmatrix}, \quad G(f) = \begin{pmatrix} -46 & 46 \end{pmatrix}, \quad w = \begin{pmatrix} -23 \\ 7 \end{pmatrix}.$$

$n = 24$:

$$n = 25$:

$$n = 26$:

$n = 27$: Not reflective.
$n = 28$:

$$
\begin{pmatrix}
28 & 0 & -1 \\
0 & 0 & 1 \\
-1 & 1 & 0 \\
21 & 21 & -4 \\
84 & 56 & -13 \\
112 & 56 & -15 \\
9 & 3 & -1 \\
35 & 7 & -3
\end{pmatrix} \quad 
\begin{pmatrix}
-56 & 56 & 28 & 364 & 840 & 728 & 28 & 28 \\
56 & -56 & 0 & 224 & 728 & 840 & 56 & 168 \\
28 & 0 & -2 & 0 & 28 & 56 & 6 & 28 \\
364 & 224 & 0 & -14 & 28 & 168 & 28 & 210 \\
840 & 728 & 28 & 28 & -56 & 56 & 28 & 364 \\
728 & 840 & 56 & 168 & 56 & -56 & 0 & 224 \\
28 & 56 & 6 & 28 & 28 & 0 & -2 & 0 \\
28 & 168 & 28 & 210 & 364 & 224 & 0 & -14
\end{pmatrix}
$$

$n = 29$: Hyperbolic type with $(w, w) = -348$, the symmetry group generated by the translation $C$, and

$$
e = \begin{pmatrix} 29 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \\ 70 & 70 & -13 \\ 957 & 928 & -175 \end{pmatrix}, \quad G(e) = \begin{pmatrix} -58 & 58 & 29 & 1276 & 16762 \\ 58 & -58 & 0 & 754 & 10150 \\ 29 & 0 & -2 & 0 & 29 \\ 1276 & 754 & 0 & -2 & 0 \\ 16762 & 10150 & 29 & 0 & -58 \end{pmatrix},$$

$$f = \begin{pmatrix} 26 & 10 & -3 \\ 7 & 4 & -1 \\ 348 & 290 & -59 \end{pmatrix}, \quad G(f) = \begin{pmatrix} -58 & 58 & 58 & 116 & 26854 \\ 58 & -58 & 0 & 29 & 10150 \\ 26854 & 10150 & 754 & 0 & -58 \end{pmatrix},$$

$$C = \begin{pmatrix} 121 & 464 & 2552 \\ 116 & 441 & 2436 \\ -22 & -84 & -463 \end{pmatrix}, \quad w = \begin{pmatrix} -29 \\ 8 \end{pmatrix}. $$

$n = 30$:

$$
\begin{pmatrix}
30 & 0 & -1 \\
0 & 0 & 1 \\
-1 & 1 & 0 \\
5 & 5 & -1 \\
9 & 3 & -1 \\
14 & 2 & -1
\end{pmatrix} \quad 
\begin{pmatrix}
-60 & 60 & 30 & 90 & 30 & 0 \\
60 & -60 & 0 & 60 & 60 & 0 \\
30 & 0 & -2 & 0 & 6 & 12 \\
90 & 60 & 0 & -10 & 0 & 20 \\
30 & 60 & 6 & 0 & -6 & 0 \\
0 & 60 & 12 & 20 & 0 & -4
\end{pmatrix}
$$

$n = 31$: Hyperbolic type with $(w, w) = -434$, the symmetry group generated by the translation $C$, and

$$
e = \begin{pmatrix} 31 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \\ 6 & 5 & -1 \\ 2728 & 1581 & -373 \end{pmatrix}, \quad G(e) = \begin{pmatrix} -62 & 62 & 31 & 93 & 25885 \\ 62 & -62 & 0 & 62 & 23126 \\ 31 & 0 & -2 & 0 & 1147 \\ 93 & 62 & 1 & -2 & 0 \\ 25885 & 23126 & 1147 & 0 & -62 \end{pmatrix},$$

$$f = \begin{pmatrix} 10 & 3 & -1 \\ 33 & 15 & -4 \\ 496 & 248 & -63 \end{pmatrix}, \quad G(f) = \begin{pmatrix} -62 & 62 & 62 & 5208 & 103726 \\ -62 & -62 & 0 & 1147 & 23126 \\ 62 & 0 & -2 & 1 & 62 \\ 5208 & 1147 & 1 & -2 & 0 \\ 103726 & 23126 & 62 & 0 & -62 \end{pmatrix},$$

$$C = \begin{pmatrix} 484 & 2511 & 12276 \\ 744 & 155 & -61 \\ 33 & 15 & -4 \\ 496 & 248 & -63 \end{pmatrix}, \quad w = \begin{pmatrix} -93 \\ -31 \end{pmatrix}. $$

$n = 32$: Not reflective.

$n = 33$:
\[
\left( \begin{array}{ccc}
33 & 0 & -1 \\
0 & 0 & 1 \\
-1 & 1 & 0 \\
11 & 11 & -2 \\
8 & 4 & -1 \\
99 & 33 & -10 \\
264 & 66 & -23 \\
37 & 8 & -3 \\
121 & 22 & -9 \\
16 & 2 & -1 \\
\end{array} \right) \left( \begin{array}{ccc}
36 & 66 & 33 \\
0 & -66 & 0 \\
-2 & 0 & 4 \\
132 & 0 & 132 \\
-22 & 0 & 66 \\
66 & 66 & 0 \\
-22 & 0 & 0 \\
-2 & 0 & 14 \\
-22 & 0 & -66 \\
-2 & 0 & 14 \\
\end{array} \right) =
\left( \begin{array}{ccc}
16 & V.V. NIKLUN
\end{array} \right)
\]

\[
C =
\left( \begin{array}{ccc}
34 & 0 & -1 \\
0 & 0 & 1 \\
-1 & 1 & 0 \\
35 & 144 & 840 \\
9 & 35 & 210 \\
-3 & -12 & -71 \\
\end{array} \right) \left( \begin{array}{ccc}
36 & 0 & -1 \\
0 & 0 & 1 \\
-1 & 1 & 0 \\
8 & 4 & -1 \\
40 & 8 & -3 \\
63 & 9 & -4 \\
\end{array} \right) =
\left( \begin{array}{ccc}
-68 & 68 & 34 \\
68 & -68 & 0 \\
34 & 0 & -2 \\
17 & 17 & -3 \\
8 & 4 & -1 \\
11 & 3 & -1 \\
16 & 2 & -1 \\
\end{array} \right) \left( \begin{array}{ccc}
68 & -68 & 204 \\
-68 & 68 & 0 \\
204 & 68 & 68 \\
34 & 0 & -2 \\
34 & 8 & 34 \\
34 & 0 & -2 \\
0 & 68 & 14 \\
\end{array} \right) =
\left( \begin{array}{ccc}
374 & 68 & 34 \\
204 & 68 & 68 \\
-34 & 0 & 34 \\
102 & 12 & 2 \\
12 & 2 & -4 \\
\end{array} \right).
\]

For \( n = 34 \): Hyperbolic type with \((w, w) = -30\), the symmetry group generated by the central symmetry \( C_1 \) and the translation \( C_2 \), and
\[
e =
\left( \begin{array}{ccc}
35 & 0 & -1 \\
0 & 0 & 1 \\
-1 & 1 & 0 \\
35 & 144 & 840 \\
9 & 35 & 210 \\
-3 & -12 & -71 \\
\end{array} \right) \quad G(e) =
\left( \begin{array}{ccc}
-70 & 70 & 35 \\
70 & -70 & 0 \\
35 & 0 & -2 \\
805 & 420 & 0 \\
5 & 28 & 140 \\
-1 & -6 & -29 \\
\end{array} \right) \quad e(n) =
\left( \begin{array}{ccc}
-72 & 72 & 36 \\
72 & -72 & 0 \\
36 & 0 & -2 \\
72 & 72 & 4 \\
72 & 72 & 8 \\
36 & 288 & 54 \\
\end{array} \right).
\]

For \( n = 36 \): Hyperbolic type with \((w, w) = -66\), the symmetry group generated by the translation \( C \), and
\[
e =
\left( \begin{array}{ccc}
37 & 0 & -1 \\
0 & 0 & 1 \\
-1 & 1 & 0 \\
6 & 6 & -1 \\
259 & 185 & -36 \\
\end{array} \right) \quad G(e) =
\left( \begin{array}{ccc}
-74 & 74 & 37 \\
74 & -74 & 0 \\
37 & 0 & -2 \\
148 & 74 & 0 \\
4181 & 2664 & 74 \\
\end{array} \right) \quad e(n) =
\left( \begin{array}{ccc}
-74 & 74 & 37 \\
74 & -74 & 0 \\
37 & 0 & -2 \\
148 & 74 & 0 \\
4181 & 2664 & 74 \\
\end{array} \right).
\]

For \( n = 37 \): Hyperbolic type with \((w, w) = -532\), the symmetry group generated by the central symmetry \( C_1 \) and the translation \( C_2 \), and
\[
e =
\left( \begin{array}{ccc}
37 & 0 & -1 \\
0 & 0 & 1 \\
-1 & 1 & 0 \\
6 & 6 & -1 \\
259 & 185 & -36 \\
\end{array} \right) \quad G(e) =
\left( \begin{array}{ccc}
-74 & 74 & 37 \\
74 & -74 & 0 \\
37 & 0 & -2 \\
148 & 74 & 0 \\
4181 & 2664 & 74 \\
\end{array} \right) \quad e(n) =
\left( \begin{array}{ccc}
-74 & 74 & 37 \\
74 & -74 & 0 \\
37 & 0 & -2 \\
148 & 74 & 0 \\
4181 & 2664 & 74 \\
\end{array} \right).
\]
$n = 40$: Hyperbolic type with $(w, w) = -120$, the symmetry group generated by the skew-symmetry $C$, and

$$
C = \begin{pmatrix}
40 & 0 & -1 \\
0 & 0 & 1 \\
120 & 120 & -19 \\
280 & 240 & -41
\end{pmatrix}
$$

$$
w = \begin{pmatrix}
-30 \\
-10 \\
3
\end{pmatrix}.
$$

$n = 41$: Not reflective.

$n = 42$: Hyperbolic type with $(w, w) = -94$, the symmetry group generated by the skew-symmetry $C$, and

$$
C = \begin{pmatrix}
21 & 2 & -1 \\
43 & 0 & -1 \\
0 & 0 & 1 \\
172 & 189 & 2924 \\
7 & 6 & -1 \\
25 & 43 & 430 \\
-10 & -17 & -171
\end{pmatrix}
$$

$$
w = \begin{pmatrix}
-2 \\
-10 \\
3
\end{pmatrix}.
$$

$n = 43$: Hyperbolic type with $(w, w) = -66$, the symmetry group generated by the central symmetry $C$, and the translation $C_2$, and

$$
C = \begin{pmatrix}
-76 & 76 & 38 & 152 & 4256 \\
76 & -76 & 0 & 76 & 2964 \\
32 & 171 & 912 \\
19 & 98 & 532 \\
-4 & -21 & -113
\end{pmatrix}
$$

$$
w = \begin{pmatrix}
-114 \\
-38 \\
11
\end{pmatrix}.
$$

$n = 44$: Hyperbolic type with $(w, w) = -66$, the symmetry group generated by the central symmetry $C$, and the translation $C_2$, and...
\[
\begin{align*}
  n &= 46: \text{Hyperbolic type with } (w, w) = -184, \text{ the symmetry group generated by} \\
  &\quad \text{the central symmetry } C_1 \text{ and by the translation } C_2, \text{ and} \\
  e &= \begin{pmatrix} 46 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}, \quad G(e) = \begin{pmatrix} -92 & 92 & 46 \\ 92 & -92 & 0 \\ 46 & 0 & -2 \end{pmatrix}.
  \\
  C_1 &= \begin{pmatrix} 529 & 529 & -78 \\ 158 & 154 & -23 \\ 966 & 920 & -139 \end{pmatrix} \\
  C_2 &= \begin{pmatrix} 49 & 184 & 1288 \\ 46 & 169 & 1196 \\ -7 & -26 & -183 \end{pmatrix}.
\end{align*}
\]

\[
\begin{align*}
  n &= 47: \text{Not reflective.}
\end{align*}
\]
$n = 50$: Not reflective.

$n = 52$: Hyperbolic type with $(w, w) = -1040$, the symmetry group generated by the skew-symmetry $C$, and

$$G(e) = \begin{pmatrix}
-2 & 0 & 130 & 676 & 2496 & 430 \\
0 & -8 & 0 & 104 & 520 & 96 \\
130 & 0 & -26 & 52 & 624 & 130 \\
676 & 104 & 52 & -104 & 104 & 52 \\
2496 & 520 & 624 & 104 & -104 & 0 \\
430 & 96 & 130 & 52 & 0 & -2
\end{pmatrix}$$

$n = 53$: Not reflective.

$n = 54$: Not reflective.

$n = 55$: Not reflective.

$n = 56$: Hyperbolic type with $(w, w) = -56$, the symmetry group generated by the central symmetry $C$, and
where this series was considered for the model B. We can see that there are a lot of cases of hyperbolic type.

\[ C = \begin{pmatrix} 56 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}, \quad G(e) = \begin{pmatrix} -112 & 112 & 56 & 280 & 168 & 1680 & 1456 \\ 112 & -112 & 0 & 112 & 112 & 1456 & 1680 \\ 56 & 0 & -2 & 0 & 6 & 112 & 168 \\ 280 & 112 & 0 & -14 & 0 & 112 & 280 \\ 168 & 112 & 6 & 0 & -2 & 0 & 56 \\ 1680 & 1456 & 112 & 112 & 0 & -112 & 112 \\ 1456 & 1680 & 168 & 280 & 56 & 112 & -112 \end{pmatrix} \]

\[ C_1 = \begin{pmatrix} 1183 & 10952 & 53872 \\ -52 & 1183 & 5824 \\ 9 & 6 & -1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 32 & 343 & 1568 \\ 7 & 72 & 336 \\ -2 & -21 & -97 \end{pmatrix}, \quad w = \begin{pmatrix} -98 \\ -14 \\ 5 \end{pmatrix}. \]

\[ n = 57: \text{Hyperbolic type with } (w, w) = -494, \text{ the symmetry group generated by the skew-symmetry } C, \text{ and} \]

\[ C = \begin{pmatrix} 321 & 30 & -13 \\ 28 & 2 & -1 \\ 57 & 0 & -1 \end{pmatrix}, \quad G(e) = \begin{pmatrix} -6 & 0 & 228 & 1482 & 291 & 714 \\ 0 & -2 & 0 & 114 & 26 & 72 \\ 228 & 0 & -114 & 114 & 57 & 228 \\ 1482 & 114 & 114 & -114 & 0 & 114 \\ 291 & 26 & 57 & 0 & -2 & 3 \\ 714 & 72 & 228 & 114 & 3 & -6 \end{pmatrix}. \]

\[ n = 58: \text{Not reflective.} \]

\[ n = 59: \text{Not reflective.} \]

\[ n = 60: \text{Hyperbolic type with } (w, w) = -80, \text{ the symmetry group generated by the central symmetry } C_1 \text{ and the translation } C_2, \text{ and} \]

\[ C_1 = \begin{pmatrix} 12615 & 110224 & 577680 \\ 1444 & 12615 & 66120 \\ -551 & -4814 & -25231 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 12 & 125 & 600 \\ 5 & 48 & 240 \\ -1 & -10 & -49 \end{pmatrix}, \quad w = \begin{pmatrix} -50 \\ -10 \\ 3 \end{pmatrix}. \]

As a result we get that cases

\[ k = 1 \rightarrow 22, 24 \rightarrow 26, 28, 30, 33, 34, 36, 39, 42, 45, 49, 50, 55 \]

are reflective of elliptic type; cases

\[ k = 23, 29, 31, 35, 37, 38, 40, 43, 44, 46, 48, 52, 56, 60 \]

are reflective of hyperbolic type, and cases

\[ k = 27, 32, 41, 47, 51, 53, 54, 58, 59 \]

are not reflective. We can see that there are a lot of cases of hyperbolic type.

These calculations are mirror symmetric to results in [G], [GH], [GN1]—[GN6] where this series was considered for the model B.
GROUPS OF AUTOMORPHISMS OF K3 SURFACES

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