A-phase origin in B20 helimagnets

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A-phase origin in cubic helimagnets (MnSi, FeGe etc) is explained. It is shown that its upper bound is a result of the spin-wave instability at \( H_{\perp} > \sqrt{8/3}\Delta = H_{A2} \) where \( H_{\perp} \) are the magnetic field perpendicular to the helix axis \( k \) and \( \Delta \) the spin-wave gap respectively. The last appears due to the spin-wave interaction if one takes into account that the Dzyaloshinskii-Moriya interaction acts between different spins. The infra-red divergences (IRD) in the \( 1/S \) series for the magnetic energy at \( H_{\perp} \rightarrow H_{A2} \) are responsible for the lower A-phase boundary \( H_{A1} \). It is shown that the A-phase exists at all \( T < T_C \) but if \( T \ll T_C \) it is very narrow and can not be observed. However in the critical region just below \( T_C \) its width increasing strongly. Preliminary estimations demonstrate semi-quantitative agreement with the existing experimental data.

The existence of the spin-wave gap is a crucial point of our consideration. In Appendix A we present its derivation in more transparent form than in previous publication.

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I. INTRODUCTION

Unusual properties of noncentrosymmetric cubic B20 helimagnets (MnSi, FeGe and related compounds) with Dzyaloshinskii-Moriya interaction (DMI) attracted a lot of attention during more than thirty years (see for example\cite{1,2,10,11} and references therein). Renascence in this field began with a discovery of a quantum phase transition to a disordered (partially ordered) state in MnSi at high pressure\cite{5,6}. Then this transition was observed in FeGe also\cite{7}.

Apparently complicated behavior of the helix wave-vector \( k \) (the helix axis) in magnetic field \( H \) is one of the most striking phenomenon observed in these compounds. Indeed suppressing the weak cubic anisotropy the field aligns the helix axis along itself and gives rise a conical magnetic structure which transforms to the ferromagnetic one at critical field \( H_C \). This simple behavior holds almost in whole region of the \((H,T)\) phase diagram. However just below the transition \( T_C \) to paramagnetic state the helix vector \( k \) suddenly rotates perpendicular to the field. This so-called A-phase discovered by B. Leech in FeGe\cite{10} exists in rather narrow field range \( H_{A1} < H < H_{A2} < H_C \) and above \( H_{C2} \) the helix axis returns to the field direction again\cite{11,12,13,14}.

Further small angle neutron scattering experiments revealed unexpected feature. If the neutron beam is along the field the small-angle magnetic scattering appears in the A-Phase only and the magnetic Bragg reflections form the six-fold structure which dependent weakly on the field orientation relative to the crystal axes. This phenomenon was observed in MnSi\cite{12}, FeGe\cite{13} and several B20 compounds\cite{15}. It could be described as superposition of three helices with wave vectors \( k_1 + k_2 + k_3 = 0 \) where \( |k_1| = k \) which are perpendicular to the field \( H \).

Meanwhile it was claimed that this structure is a new skyrmion lattice state\cite{14}. However the higher-order reflections which has to be in this case were not observed. Hence a nature of this six-fold structure inside the A has not been understood yet.

Moreover the very existence of the A-phase with perpendicular \( k \) orientation has to be explained. Indeed in zero field the multi-domain state is realized with \( k \)-axes are along \((111)\) or \((001)\) depending on a sign of very weak cubic anisotropy\cite{13}. Then with the field increasing the vectors \( k \) rotate to the field and conical helix structure develops with the cone angle determined by \( \sin \alpha = -H/H_C\)\cite{14,15}. This behavior is the same as in antiferromagnets above the spin-flop transition where we have not any hint to the A-phase state. It should be noted also that the classical magnetic energy depends on \( H_\parallel \) the field component along the helix vector \( k \) only and \( H_{\perp} \) dependence appears as a quantum phenomenon only\cite{13}.

Behavior of the cubic helimagnets in the field is related with more important multiferroic problem. Indeed in RMnO\(_4\) materials multiferroic properties are connected directly to spin helices mediated by DMI\cite{16,17} and a lot of transitions in magnetic field were observed (see\cite{15} and references therein). However we have not now any satisfied explanation of these transitions. Hence understanding of the helices behavior in magnetic field may be considered as an urgent problem.

In this paper we explain the A-phase origin developing theory of the helix behavior in the field which takes into account the interaction between spin-waves. This interaction gives rise the infra-red divergences (IRD) in perturbation \( 1/S \) series for the field-depending part of the magnetic energy at \( H_{\perp}^2 - H_{A2}^2 \rightarrow 0 \), where \( H_{\perp} \) is the field component perpendicular to the helix vector \( k \) and \( H_{A2} \) is the upper A-phase boundary respectively. We demonstrate also that
FIG. 1: Two parts of the magnetic energy $E_\parallel$ and $E_\perp$ with the helix vector $k$ along and perpendicular to the field respectively. Between $H_{A1}$ and $H_{A2}$ in the A-phase region present approach is applicable in the multi-domain case only.

$H_{A2} = \sqrt{8/3} \Delta$ where $\Delta$ is the spin-wave gap. In the Hartree-Fock approximation it is given by\textsuperscript{14}

$$\Delta_{HF}^2 = \frac{(Ak^2)^2}{4S} \sum D_q D_0$$

where $S$ is the unit-cell spin.

We demonstrate that the magnetic energy is a sum of two parts $E_M = E_\parallel + E_\perp$ differently depending on $H_\parallel$ and $H_\perp$ as shown in Fig.1. This anisotropy is a consequence of an unusual form of the spin-wave energy at $H_\parallel \ll H_C$\textsuperscript{14,20}

$$\epsilon_q = \sqrt{(Ak^2)(q_\parallel^2 + 3q_\perp^4/8k^2) + \Delta^2 - 3H_\perp^2/8, \ q \leq k; \ Aq\sqrt{q^2 + k^2}, \ q \gg k,$$

where $A$ is the spin-wave stiffness at $q \gg k$. This $q$ anisotropy is a result of the DMI which mixes spin-waves with $q$ and $q \pm k$. These umklapps produce satellite spin-wave structure with $q \pm nk$ where $n = 1, 2, ...$ observed in\textsuperscript{22}.

It should be noted that in\textsuperscript{21} was claimed that in B20 helimagnets the spin-waves are the gapless Goldstones due to translation invariance along the helix vector $k$. Really this statement is correct in the linear theory only and is broken in higher approximations if one takes into account that the DMI acts between different spins\textsuperscript{14,23–25}. The essence of the problem is following. Conventional macroscopic expression for the DMI energy is given by $D(M(r) \cdot [\nabla \times M(r)])$ is ill-defined as both $M$ operators act in the same $r$ point. It is unimportant in the cases of the classical groundstate energy and the linear spin-wave theory but becomes crucial if one is interested in the spin-wave interaction. Indeed if both $M$ operators act in the same point they non-commute and we have not a gap. Otherwise they commute and the gap appears in the Hartree-Fock approximation\textsuperscript{14,22} (see also line below Eq.(15) and Appendix A). In this case the DMI feels the lattice structure the above mentioned translation invariance is broken. The sum in Eq.(1) for $\Delta^2$ is saturated at $q \sim 1/a$ where the macroscopic approach is not applicable. Moreover in\textsuperscript{22} was shown that the magneto-elastic interaction mixes $k = 0$ magnons with the $2k$ phonons. As a result in the second order on this interaction a negative contribution to $\Delta^2$ appears and we have

$$\Delta^2 = \Delta_{HF}^2 + \Delta_{ME}^2,$$

It was suggested\textsuperscript{24,25} that the quantum phase transition at pressure observed in\textsuperscript{5,6,9} is a result of a competition between these two terms and holds when $\Delta^2 = 0$.

From Eq.(2) follows also that uniform spin-waves are unstable if

$$H_\perp > H_{A2} = \Delta\sqrt{8/3},$$
and the perpendicular state is impossible. Hence $H_{A2}$ is the upper A-phase bound. Below in agreement with experiment we assume that $H_{A2}^2 < H_{T_C}^2$, and demonstrate that the lower bound $H_{A1}$ is a result of the infra-red divergences (IRD) which appear in the $1/S$ perturbation expansion for the magnetic energy if $H \downarrow H_{A2}$. At low $T$ this boundary is very close to $H_{A2}$ and the A-phase cannot be seen. However just below the transition $H_{A1}$ decreases strongly due to critical slowing down (decreasing of the spin-wave stiffness $A$ as $T \rightarrow T_C$) and the A-phase becomes visible. Qualitative agreement with experimental data of Ref. was demonstrated. For detailed comparison with experiment one has to have more precise experimental data for the $T$ dependence of $H_{A1,2}$ in the critical region.

II. SPIN-WAVE INTERACTION

At the beginning we have to summarize briefly principal theoretical results which will be explored below. We use the Bak-Jensen model adding the Zeeman energy and omitting weak cubic anisotropy which is unimportant. Corresponding Hamiltonian is given by

$$ H = \sum (-J_q(S_q \cdot S_{-q})/2 + iD_q(q \cdot [S_q \times S_{-q}]) + \sqrt{N}(H \cdot S_q)), $$

where the first second and third terms are the ferromagnetic exchange interaction, DMI and the Zeeman energy respectively. In the case of the helical structure for $S_q$ we have

$$ S_q = S^c_q + S^A_q \mathbf{A} + S^A^*_q \mathbf{A}^*, $$

$$ S^c_q = S^c_q \sin \alpha + S^A_q \cos \alpha, S^A_q = S^c_{q-k} \cos \alpha - S^c_{q-k} \sin \alpha + iS^m_{q-k}, S^A^*_q = S^c_{q-k} \cos \alpha - S^c_{q-k} \sin \alpha - iS^m_{q+k}, $$

where $\mathbf{A} = (\hat{a} - i\hat{b})/2$, unit vectors $\hat{a}, \hat{b}, \hat{c}$ form the right-handed orthogonal frame and the spin operators in the Dyson-Maleev representation are given by

$$ S^c_q = N^{1/2}S_0\hat{q} - (a^+ a)q; S^m_q = -i\sqrt{S/2}[aq - a^+ - (a^+ a^2)/2S]; S^c = \sqrt{S/2}[aq + a^+ - (a^+ a^2)/2S], $$

where $a_q$ and $a^+_q$ are conventional Bose operators.

In the classical approximation we have $k = SD_0\hat{c}/A$ where $A = S(J_0 - J_k)/k^2$ is the spin-wave stiffness at $q \gg k$, $\sin \alpha = -H_{H}/H_C$ where $H_{H}$ is the field component along $k$ and $H_C = Ak^{213,14}$. Corresponding part of the magnetic energy have the form $E(H_{H}) = -H^2/2H_C$.

We are interested below low-field region $H < H_C$ and for simplicity put $\alpha = 0$. Conventional spin-wave Hamiltonian is given by

$$ H_2 = \sum |E_qa^+_qa_q + B_q(a_qa_{-q} + a^+_qa^+_q)|/2, $$

where at $H_{H} < H_C$ we have $E_q = S(M_{q,k} - M_{q,k}) + B_q \simeq A(q^2 + k^2/2), B_q = S(M_{q,k} - J_k)/2 \simeq Ak^2/2, M_{q,k} = (J_{q+k} + J_{q-k})/2 + 2D_q(k \cdot \hat{c}),$ (9)

where approximate equalities hold at $q \ll 1/a$ and $a$ is the lattice spacing. The equilibrium condition is given by $D_q(k \cdot \hat{c}) = Ak^2/S$.

The second term in Eq.(5) mixes excitations with $q$ and $q \pm k$ and gives rise the $q$ anisotropy in Eq.(2) at $q \leq k^{13,23}$. For following we have to consider the spin-wave interaction in more details in comparison with $^{14}$ From Eqs.(5-7) at small $\alpha$ for the interaction energy we have $V_I = V_4 + V_6$ and

$$ V_4 = -(1/2)\sum (M_{1-3,k} - M_{1,k})a^+_qa^+_qa^+_qa^+_q + (1/4)\sum (J_1 - M_{1,k})(a_1^+ a_1^+)(a^+_q a^+_q), $$

$$ V_6 = -(1/16S)\sum (J_1 - M_{1,k})(a^+_q a^+_q)(a^+_q a^+_q) - (Ak^2/16S^2)\sum (a^+_q a^+_q)(a^+_q a^+_q), $$

where $1, 2, 3$ label corresponding momenta. The first term in Eq.(10) is a generalization of the Dyson interaction for ferromagnets. We do not consider here the $V_6$ interaction as it gives further terms in $1/S$ expansion.

In the Hartree-Fock (HF) approximation Eq.(10) gives rise non-Hermitian spin-wave Hamiltonian

$$ H_{SW} = \sum |E_qa^+_qa_q + (B_qa_qa_{-q} + B_qa^+_q a^+_q)|/2, $$

(12)
where $E_q = E_q + \Sigma$, $\tilde{B}_q^{(+)} = B_q + \Pi^{(+)}$ and corresponding diagram is shown in Fig.2a. Corresponding expressions at $q = 0$ are given by

$$\Sigma = (1/2) \sum [(J_1 - M_{1,k})(n_1 + f_1) + (J_0 - M_{0k})(n_1 + f_1/2)];$$  \hspace{1cm} (13)

$$\Pi = \sum [(M_{0,k} - M_{1,k}) + J_0/M_{0k})]f_1;$$

$$\Pi^+ = (1/2) \sum [(J_1 - M_{1,k})(n_1 + f_1) + 2(J_0 - M_{0k})n_1].$$

where

$$n_1 = a^+_q a_q = [(2N_1 + 1)E_1 - \epsilon_1]/2\epsilon_1, \hspace{1cm} f_1 = -B_1(2N_1 + 1)/2\epsilon_1$$  \hspace{1cm} (14)

where $N_1$ is the Plank function. From these expressions using definitions (9) for the spin-wave gap we obtain (for discussion see Appendix A)

$$\Delta_{HF}^2 = 2E_0\Sigma - B_0(\Pi + \Pi^+) = \frac{(Ak^2)^2}{4S} \sum D_k, \hspace{1cm} (15)$$

where $E_0 = B_0 = Ak^2/2$. We have $\Delta_{HF}^2 \neq 0$ due to $-1/2$ term in expression for $n_1$. Forgetting that the DMI acts always between different spins we must replace $n_1 \rightarrow n_1 + 1/2$ and $\Delta_{HF}^2 \rightarrow 0$. It should be noted also that the sum in Eq.(13) is saturated at $q \sim 1/4$ and may be considered as $T$-independent at $T \leq T_C$.

One has to note also that Fig.2a diagram is of order of $S^0$ and $\Delta^2 \sim S$ whereas the main part of $\epsilon_q^2 \sim S^2$.

### III. PERPENDICULAR FIELD ($H \perp k$)

From Eqs.(5-7) for interaction with the perpendicular field at $\alpha \ll 1$ we have4

$$V_\perp = (H \cdot A)[\sqrt{S/2}(a_{-k} - a_k^+)] - \sum a_{q-k}^+ a_q + h.c. = V_L + V_{UM}, \hspace{1cm} (16)$$

where the terms linear in the $a_{\pm_k}^{(+)}$ operators lead to the spin-wave Bose condensation at $q = \pm k$. Other terms mix excitations with $q$ and $q \pm k$.

In Appendix B we obtain a system of equations for $a_0^{(+)}$ and $a_{\pm k}^{(+)}$ considering them as classical variables. From their solution we have26

$$a_k = \sqrt{S/2}(H \cdot A)/Ak^2; a_{-k} = -\sqrt{S/2}(H \cdot A^*)/Ak^2.$$  \hspace{1cm} (17)

and the magnetic energy is given by26

$$E_{M0} = \frac{SH^2}{2H_C} - \frac{SH^2}{4H_C}. \hspace{1cm} (18)$$

where the factor 1/4 is very transparent: $1/4 = 1/2 < \cos^2 \varphi >$ where $\varphi$ is the angle between the helical spin and $H_1$. According to this equation the helix axis $k$ has to be along the field as was observed in all $H, T$ region of the phase diagram except small A-phase pocket just below $T_C$.12,13,14,15

Taking into account the spin-wave BC we get from $V_4$ additional contribution to the spin-wave Hamiltonian (12)

$$H_{BC} = -(H^2 + 16Ak^2) \sum (a_q^+ a_{-q} + a_{-q}^+ a_q + 2a_q a_{-q}). \hspace{1cm} (19)$$

As a result we obtained $-3H^2 / 8$ term in Eq.(1) instead of $-H^2 / 2$ in13,14,15. This part of the spin-wave Hamiltonian has been taken into account in Appendix B for the Green function evaluating and produce non-symmetric $H^2\perp$ terms in $F_{q}$ and $F^+ \perp$ Green functions (see below).

These umklapps gives rise to two infinite systems of conjugated linear equations for the Green functions. At $H_\perp \ll H_C$ they can be truncated and we have two systems of the linear equations considered in Appendix. Their solution may be divided on two parts: direct and umklapp Green functions. For the first one we have

$$G_q(\omega) = \bar{G}_q(-\omega) = \frac{i\omega + E_q + \Sigma - 7H^2 / 16 Ak^2}{\omega^2 + \epsilon_q^2}; \hspace{1cm} F_q = \frac{B_q + \Pi}{\omega^2 + \epsilon_q^2}; \hspace{1cm} F^+ = \frac{B_q + \Pi^+ - H^2 / 8Ak^2}{\omega^2 + \epsilon_q^2}. \hspace{1cm} (20)$$
FIG. 2: Diagrammatic expansion for the magnetic energy. Lines are Green functions $G = \leftarrow$, $F = \rightarrow\leftarrow$ and $F^+ = \leftarrow\rightarrow$. At top is the sum of the HF loop and the BC part Eq.(19) used in the diagrams. The IRD first time appears in Fig.2d diagram.

The umklapp functions are given by

$$G_+ = G_- = -F_+^+ = -F_- = -\frac{(H \cdot A)}{2(\omega_n^2 + \epsilon_q^2)}; \quad G_- = G_+ = -F_-^+ = -F_+ = -\frac{(H \cdot A^*)}{2(\omega_n^2 + \epsilon_q^2)},$$

where $\epsilon_q^2$ is given by Eq.(2) and in numerators we neglected some terms proportional to $q^2_\perp$ and $q^4_\perp$ which are unimportant to us (see 14). Expressions (14) for $n_q$ and $f_q$ are direct consequences of these equations.

IV. MAGNETIC ENERGY

We demonstrate below that the A-phase lower bound is a result of the infrared divergence (IRD) at $H_{A2} - H_\perp \to 0$ which appear in the $1/S$ perturbation series for the interaction energy. Using Eqs.(12, 18,19) for the total magnetic energy we have

$$E = E_{MO} + < H_{SW} + H_{BC} > + < V - V_{HF} >,$$

where the first term us given by Eq.(18). For the second term we have very transparent expression

$$E_{SW} = \sum [\epsilon_q n_q + (\epsilon_q - E_q)/2],$$

where off-diagonal terms in Eq.(16) are responsible for correct $-3H^2_\perp/8$ field dependence of $\epsilon^2_q$. This expression includes the energy of thermally excited magnons and zero-point motion with energy given by Eq.(2). The last term of Eq.(22) does not contain the HF part Fig.2b as it was used in Eq.(12) and the $1/S$ expansion begins with Fig.2b diagram.

In the principal $1/S$ order left and right vertexes in Fig.2 IRD diagrams may be expressed using Eqs.(13)

$$\Gamma = [\Sigma - (\Pi + \Pi^+)/2][a_+^qa_q + (1/2)(\Pi a_+ + \Pi^+a_-)](a_q + a_-^q),$$

where according to Eqs.(19,B5) we have to add to $\Sigma, \Pi^{(+)}$ the BC parts (first line in Fig.2). As a result we get

$$\Gamma = (\Delta^2/4k^2)[R_1 a_+^q a_q - (R_2 a_+^q + R_3 a_-^q)(a_q + a_-^q)/2]$$

where $R_1 = 1 + H^2_{\perp}/3H^2_{A2} \simeq 4/3, \quad R_2 = 1 + H^2_{\perp}/6H^2_{A2} \simeq 7/6, \text{ and } R_3 = H^2_{\perp}/3H^2_{A2} \simeq 1/3$. Approximate equalities hold et $H_\perp \to H_{A2}$ and will be used below.

Using Eq.(25) and taking into account direct (20) and umklapp (21) Green functions for the first IRD diagram Fig.2d we obtain

$$E_d = -\frac{H^4_{A2} T}{8} \sum_{q,\omega_n} \frac{1 + H^2_{\perp}/32k^2}{(\omega_n^2 + \epsilon_q^2)^2}.$$
where we neglected in the numerator terms bilinear in \((H_1^2, \Delta^2)\) and in following will omit the \(H_1^2\) term.

We are interested in \(T \gg Ak^2\) region and the IRD displaces the \(\omega_n = 0\) term only and we obtain

\[
E_d = -\frac{T(ak)^3}{18\pi(Ak^2)^2} \left( \frac{H_1^2}{H_1^2 - H_1^2} + 1 \right) \approx E_d \frac{H_1^2}{H_1^2}.
\]

where \(a\) is the lattice constant. and instead of Eq.(18) we have

\[
E_M = -\frac{SH_1^2}{2HC} - \frac{SH_1^2}{4HC} \left( 1 - \frac{4HC E_d}{SH_1^2} \right)
\]

Neglecting higher order IRD terms (see below) we obtain that the A-phase is energetically profitable if the expression in the brackets becomes larger than two. Hence at the S-phase lower boundary we have

\[
E_d = -4SH_1^2/H_C
\]

and using Eq.(27) we get

\[
H_{A1} = H_{A2}[1 - 2\sqrt{3/8(ka)^3}TH_C/9\pi S(Ak^2)^2]^{1/2}
\]

Using well known low-T parameters for \(MnSi\): \(T_C \approx 29.5K, H_C \approx Ak^2 \approx 0.6T \approx 0.8K, a \approx 0.46nm, k \approx 0.38nm^{-1}\) and \(S = 1.6\) (see for example[14] and references therein) we obtain \(H_{A1}/H_{A2} = (1 - 0.0053T/T_C)^{1/2}\). Hence at \(T \ll T_C\) the A-phase exists but it is so narrow that can be hardly observable. Moreover at \(T = T_C\) the ratio \(r = H_{A1}/H_{A2} = 0.9975\).

We demonstrate now that there is strong decreasing of the ratio \(r = H_{A1}/H_{A2}\) just below \(T_C\). It is connected mainly with the spin-waves critical slowing down. Unfortunately there are not any theoretical predictions for it and our discussion is restricted by analysis of Eq.(30) where all parameters may be measured independently.

First of all the low-T condition \(H_C \approx Ak^2\) is violated. Indeed \(H_C(T_C) \approx H_C(0)/\tau^2, k \approx \text{almost constant}^{15}\) whereas the spin-wave stiffness \(A\) is renormalized strongly. Indeed it is equal to 0.52meV\(\cdot\)nm \(2\) and \(0.24meV\cdot\)nm \(2\) at \(T = 5K\) and \(2\)\(K\) respectively\(21,22\), while \(H_C\) remains almost unchanged\(15\). One may await that near \(T_C\) \(A = A_0\tau^2\) where \(\tau = (T_C - T)/T_C\).

The unit cell spin \(S = S_0\tau^{0.22,23,24}\). According to\(24\) this scaling behavior begins at \(T \approx 25K\) (\(\tau \approx 0.14\)) and \(S_0 \approx 1.2\). The upper A-phase bound is almost \(\tau\) independent\(21,24\) and from Eqs.(1) and (4) we have \(A \sim S^{1/2}\) and \(A = A_0\tau\). Putting \(A_0 = 0.24meV\cdot\)nm \(2\) we obtain

\[
H_{A1} = H_{A2}(1 - 0.033h/\tau^{0.44})^{1/2},
\]

where \(h = H_C(T)/H_C(0)\).

The most detailed available data for the A-phase in \(MnSi\) are shown in Fig.1 of Ref.\(2\) and we compare our results with them. i. \(T = 25K, \tau = 0.14, h \approx 1\) \(r = H_{A1}/H_{A2} = 0.96\) and A-phase was not seen. ii. \(T \approx 28.5K, \tau \approx 0.03h \approx 0.8; r = 0.94\) whereas the observed ratio is 0.6 - 0.7. Hence the theory explains the A-phase phenomenon at least qualitatively. It has to mention also that all parameters including values of \(\tau\) used above are known with very low accuracy. For example at \(\tau = 0.01\) we have \(r = 0.89\) and replacing in Eq.(31) 0.033 \(\rightarrow\) 0.1 we get \(r = 0.63\) We mention also that according to\(21,24\)

\[
H_{A2} \approx 0.22T; \text{ and } \Delta \approx 0.13T = 15\mu eV.
\]

V. 1/\(S\) Corrections

The next IRD term is represented by Fig.2e diagram and in general form is given by

\[
E_e = \langle \Gamma V_4 \Gamma \rangle.
\]

Interaction \(V_4\) given by Eq.(10) consists of two parts. The first one gives the central vertex in Fig.2e diagram which disappears at zero momenta and leads to rather weak IRD singularity which may be neglected. The second remains constant and gives main correction to the above results.

Using Eqs.(2),(20) and (A3) for the most singular e2 correction we obtain

\[
E_e = -\frac{3Ak^2}{S} \left( \frac{\Delta^2R_1}{Ak^4} \right)^2 \sum \frac{B_1(F_1^+ - F_1)B_2^2}{Z_1Z_2^2} \sum \frac{T}{\epsilon_1^2}.
\]
where \( Z_l = \omega_0^2 + \epsilon_l^2 \) and \( l = 1, 2 \). This expression is proportional to \( (H_{A2}^2 - H_0^2)^{-2} \). Using Eq.(26) we obtain \( \sum(T/\epsilon^4) = -8E_d/H_{A2}^4 \) and

\[
E_c = -\frac{15E_d^2AK^2H_2^2}{4SH_{A2}^4}
\]

Considering this expression as a small correction to \( E_d \) in Eq.(29) we must replace

\[
E_d \rightarrow E_d(1 + 15AK^2/16HC). \tag{36}
\]

At low \( T \) this correction of order of unity and one has to examine all \( 2/S \) series. However near \( T_C \) where \( A \rightarrow A_0T^{0.11} \) it may be neglected. In any case deep into the A-phase and near its upper boundary the full series examining is unavoidable if one assume conventional helical structure \( (c\hat{A}) \).

VI. CONCLUSIONS

A-phase origin in cubic helimagnets \( (MnSi, FeGe \text{ etc}) \) is examined. It is shown that its upper bound is a result of the spin-wave instability at \( H_\perp > \sqrt{3}\Delta = H_{A2} \).

The infra-red divergences (IRD) in the \( 1/S \) series for the magnetic energy at \( H_\perp \rightarrow H_{A2} \) are responsible for the lower A-phase boundary \( H_{A1} \). It is shown that these IRD are anomalously strong due to softens of the spin-wave spectrum at \( q < k \). As a result the low-momenta fluctuations behave as in 2D systems.

We demonstrate that the A-phase exists at all \( T < T_C \) but if \( T \ll T_C \) it is very narrow and can not be observed. However in the critical region just below \( T_C \) its width increasing strongly. Preliminary estimations show semi-quantitative agreement with the existing experimental data. However the problem demands further theoretical and experimental studies. We can formulate following unresolved problems. i. Examination of the full \( 1/S \) expansion at low \( T \) as the second correction to the \( H_{A1} \) boundary is not small. In this respect we wish to point out to the hint to the A-phase observed in \( MnSi \) at \( T = 10K \) (Fig.3b in Ref.), ii. Better understanding the temperature dependence of the spin-wave energy at \( q < k \). iii. More precise measurements of the A-phase boundaries then it was done in previous studies. In this respect we wish to note that there are the first order transitions at both a-phase boundaries which has to be accompanied by the specific heat jumps. The first time the specific heat anomalies in this region were observed in but more precise measurements would be important.

In this discussion we avoided the nature of the A-phase itself. There are two possibilities: exotic "skyrmion" state proposed in or the tree domain structure. However at present we have not any real theory of the skyrmion state as well as an explanation of the three domain state.

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Appendix A

As very existence of the gap is crucial for us we repeat here some details of corresponding calculations which were presented in in a rather cumbersome form. In Eq.(15) terms proportional to \( J_0 - M_{q,k} \) cancel. For remained parts of \( \Sigma, \Pi(+) \) using Eqs.(9,13,14) we obtain

\[
\Sigma = \Pi^+; \quad \Pi = \Sigma - \sum(M_{1,k} - J_1)/4 \tag{A1}
\]

\[
\sum(M_{1,k} - J_1)/4 = (1/2) \sum D_q(k \cdot \hat{c}) \tag{A2}
\]

as \( \sum J_{1,k} = \sum J_q = 0 \) due to condition that the exchange interaction acts between different spins. Using equality \( D_0(k \cdot \hat{c}) = Ak^2/S \) we obtain Eq.(1).

It may be shown that expression \( (A2) \) is much larger than other terms in Eqs.(13) as they have additional small factors which vanish if \( k \rightarrow 0 \). Hence we may put

\[
\Pi = -2\Delta^2/Ak^2, \quad \Sigma = \Pi^+ = 0. \tag{A3}
\]
From Eq.(A2) follows that $\Delta^2 \neq 0$ if $\sum D_q \neq 0$. We demonstrate now that this condition is fulfilled. In the single lattice point

$$V_{DM} = (1/2) \sum D_{R,R'} (\nabla - \nabla') [S_R \times S_{R'}] = i \sum D_q (q \cdot [S_q \times S_{-q}]),$$  \tag{A4}$$

where $D_{R,R'} = D_{R',R}$, the condition $R \neq R'$ holds and $\sum D_q = 0$.

We present now simple example where the truncated system is given by

$$\text{Eqs.(17).}$$

At $H_\perp \ll H_C$ truncated equations for the Bose condensed magnons are given by (cf.

$$E_0 a_0^+ + B^+ a_0 - h a_k^+ - f a_k^- = 0, \quad \text{B1}$$

$$B a_0^+ + E a_0 - h a_{-k}^+ - f a_k = 0,$$

$$-f a_0^+ + E_1 a_k^+ + B a_{-k} = \sqrt{S/2} f,$$

$$-f a_0 + B a_k^+ + E_1 a_{-k} = -\sqrt{S/2} f,$$

$$-h a_0^+ + E_1 a_k^+ + B a_{-k} = -\sqrt{S/2} h,$$

$$-h a_0 + B a_{-k}^+ + E_1 a_k = \sqrt{S/2} h,$$

and we obtain Eqs.(17).

We have two sets of the Green functions determined as follow

$$G_q = -<Ta_q, a_q^+>; F_q^+ = -<Ta_q, a_q^->; G_{\pm} = -<Ta_{q\pm k}, a_q^+>; F_{\pm} = -<Ta_{-q\mp k}, a_q^+>, \quad \text{B2}$$

$$\bar{G}_q = -<Ta_q^+, a_q>; F_q = -<Ta_{-q}, a_q>; \bar{G}_{\pm} = -<Ta_{q\pm k}^+, a_q>; F_{\pm} = -<Ta_{-q\mp k}^+, a_q>, \quad \text{B3}$$

They are solutions of two conjugated systems of linear equations. We consider here the second set only. Corresponding truncated system is given by

$$(i \omega + \bar{E}) \bar{G} + \tilde{B}^+ F - h \bar{G} - f \bar{G}_{-} = -1, \quad \text{B4}$$

where $h = f^* = (A \cdot H)$, $E_1 \simeq E_{q \perp k}$ if $q \ll k$ and according to Eqs.(12,13) and the BC contribution (19) we have

$$\bar{E} = E_0 + \Sigma - H^2 / 16Ak^2; \; \tilde{B} = B + \Pi - H^2 / 8Ak^2; \; \tilde{B}^+ = B + \Pi^+ - H^2 / 4Ak^2, \quad \text{B5}$$

Solution of these equations as well as the conjugated one are given by Eqs.(20) and (21).

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In Eq.(2) we have $-3H^{2\perp}/8$ instead of $-H^{2\perp}/2$ in [14] where the spin-wave Bose condensation in the spin-wave Hamiltonian was not be considered. See Eq.(19) below.