Finite abelian subgroups of the mapping class group

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Abstract

The problem of enumeration of conjugacy classes of finite abelian subgroups of the mapping class group \(M_\sigma\) of a compact closed surface \(X\) of genus \(\sigma\) is considered. A complete method of enumeration is achieved for finite elementary abelian subgroups and steps are taken toward enumeration of finite abelian subgroups.

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1 Introduction

Let \(S\) be a closed, smooth, orientable surface of genus \(\sigma \geq 2\). The mapping class group \(M_\sigma\) of \(S\) (or MCG) is the group of isotopy classes of homeomorphisms of \(S\). In this paper we shall investigate the conjugacy classes of finite abelian subgroups of \(M_\sigma\). While the general finite subgroup classification is important we focus on the elementary abelian case as a tractable case where complete classification methods by standard linear algebra is possible and the abelian case where positive steps can be made toward such a classification. The main result of our work is to describe methods which may be employed to completely classify all elementary abelian actions in a given genus and steps toward a classification of abelian actions. The classification is complex since it involves understanding the representation theory of certain subgroups of symmetric groups, but for a fixed low genus since the symmetric groups are small, we are able to produce very explicit results.

Our main results are presented in three parts. Theorem 8 gives a decomposition of an abelian subgroup \(G\) into an unramified part (hyperbolic) and a ramified part (elliptic). For elementary abelian subgroups, the classification of the unramified part is completely described in Section 3 and methods allowing
one to classify the ramified part, as well as some explicit examples, are in Section 4. In Section 5 steps are taken to classify general unramified abelian actions. Though complete results are not obtained, enough information is gathered to classify unramified abelian actions on surfaces up to genus 65 (see Example 18) and the general method of how to classify abelian actions is described through other examples.

We should note that our methods fail for non-abelian groups. There is a general classification theory which describes the classes of subgroups as a finite sequence of quotients of finite sets by the action of infinite groups. In the abelian case the general method simplifies greatly and we get a nice splitting into ramified and unramified cases and the subcases are tractable. For the general case however, these simplifications do not occur and calculations become complicated very quickly and are only really possible through computer calculation. For low genus, some partial results are known, and many cases of infinite families, as mentioned further in this introduction.

The importance of a good understanding of the finite subgroups of $M_\sigma$ was made apparent when Maclachlan showed that $M_\sigma$ is generated by torsion elements in [13]. Following this, a number of different people have found sets of torsion generators for $M_\sigma$, see [1] and [2]. An understanding of how such elements generate $M_\sigma$ may be possible through an analysis of the conjugacy classes of finite subgroups of $M_\sigma$. Another motivating reason for our work is that each of the conjugacy classes of finite subgroups of $M_\sigma$ are in 1-1 correspondence to finite groups of orientation preserving groups of homeomorphisms of $S$ up to topological equivalence. See [5] for a detailed discussion and the complete classification in genus 2 and 3 (with one omission [6]). A brief summary of the equivalence is given in Subsection 2.1 below. Another motivating reason is that the finite subgroups describe the singularity structure of moduli space with implications about the structure of the cohomology of the mapping class group. The moduli space $\mathcal{M}_\sigma$ of surfaces of genus $\sigma$ is the space of conformal equivalence classes of surfaces of genus $\sigma$. The moduli space may be obtained as a quotient of the Teichmüller space $\mathcal{M}_\sigma = \mathcal{T}_\sigma / \mathcal{M}_\sigma$, where $\mathcal{T}_\sigma$ is homeomorphic to an open ball in $\mathbb{C}^{3\sigma-3}$. The singular points of $\mathcal{M}_\sigma$ are caused by fixed points of finite subgroups of $M_\sigma$ which in turn controls the torsion cohomology of the mapping class group. For more details and applications see [4].

A starting point for our work are the papers [10] and [12] discussing cyclic and abelian groups of surface automorphisms. We adapt and extend the methods of [11] especially in the discussion of unramified actions using the correspondence between groups of conformal automorphisms and conjugacy classes of the MCG given by the Nielsen Realization Problem, see [14]. The advantage of considering $G \leq M_\sigma$ in this way is that there has been tremendous progress in classification results of automorphism groups of compact Riemann surfaces. A review of some results are given in the paper [5] and the monograph [3]. Some additional calculations are available in preprint form [8] and [9].
dition, there is much literature dedicated to infinite families of surfaces whose automorphism groups share certain properties, see for example [10] or [15]. For reference, a comprehensive study in this area is given in Breuer’s monograph. We note that in Breuer’s monograph, group actions are considered equivalent if their representations on the first homology group are equivalent - a coarser classification than the classification up to conjugacy in the mapping class group.

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2 Reduction of classification to finite group calculations

It is standard to study surface groups, Teichmüller space, and the moduli space in terms of Fuchsian groups so we present our problem in that context.

2.1 Finite group actions and finite subgroups of $\mathcal{M}_\sigma$

Let $G$ be a finite group. The group $G$ is said to act (in an orientation preserving manner) on surface a $S$ of genus $\sigma \geq 2$ if there is an injection $\epsilon : G \hookrightarrow \text{Homeo}^+(S)$ into the group of orientation preserving homeomorphisms. Two actions $\epsilon_1, \epsilon_2$ are said to be topologically equivalent if there is a homeomorphism $h$ of $S$ and an automorphism $\omega$ of $G$ such that

$$\epsilon_2(\omega(g)) = h \circ \epsilon_1(g) \circ h^{-1}.$$ 

This is equivalent to saying that the images $\epsilon_1(G)$ and $\epsilon_2(G)$ are conjugate in $\text{Homeo}^+(S)$. The following are well known for $\sigma \geq 2$:

- The map to the mapping class group $\mathcal{M}_\sigma = \mathcal{M}(S)$ is injective
  $$G \xrightarrow{\sim} \text{Homeo}^+(S) \xrightarrow{\iota} \mathcal{M}_\sigma \quad (1)$$
  because $G$ acts faithfully on $H_1(S)$.

- Given a finite subgroup $H \subset \mathcal{M}_\sigma$ there is a finite group $G \subset \text{Homeo}^+(S)$ such that $H = \iota(G)$ and at least one conformal structure on $S$ such that $G$ is a group of conformal automorphisms with respect to this conformal structure (Nielsen Realization problem [14]).

Thus we have the following:

**Proposition 1** The map induced by (1) is a 1-1 correspondence between topological equivalence classes of $G$-actions and conjugacy classes of the mapping class group.


2.2 Covering Fuchsian group and generating vectors

Now suppose that $G$ acts conformally on $S$. The quotient $T = S/G$ is a surface of some genus $\rho$ and the quotient map $S \to S/G$ is branched over $r$ points $Q_1, \ldots, Q_r$ with periods (or branching orders) $m_1, \ldots, m_r$. We say that $S = (\rho : m_1, \ldots, m_r)$ is the signature or branching data of $G$ acting on $S$ and that $S$ admits a $G$-$(\rho : m_1, \ldots, m_r)$ action. Since $G$ acts conformally there is a uniformizing Fuchsian group $\Gamma$ for the $G$-action and the signature of $\Gamma$ is $(\rho : m_1, \ldots, m_r)$. More precisely we have a map

$$\Pi \hookrightarrow \Gamma \twoheadrightarrow G,$$

where $\Pi$ is a torsion free surface group acting on $\mathbb{H}$ so that $S \approx \mathbb{H}/\Pi$, and the $G$-action on $S$ is induced by the isomorphism $\eta : \Gamma \twoheadrightarrow G$ and the natural action of $\Gamma/\Pi$ on $\mathbb{H}/\Pi$.

The epimorphisms $\Gamma \twoheadrightarrow G$ can be neatly summarized in the context of finite groups by generating vectors (see [4] and [5]). For the conformal action of $G$ on a surface $S$ with signature $S = (\rho : m_1, \ldots, m_r)$, the group $\Gamma$ has signature $(\rho : m_1, \ldots, m_r)$ and a presentation of the following type:

$$\Gamma \cong \Gamma(\rho : m_1, \ldots, m_r) \quad \text{(2)}$$

$$\cong \langle \alpha_i, \beta_i, \gamma_j : 1 \leq i \leq \rho, 1 \leq j \leq r : \prod_{i=1}^{\rho} [\alpha_i, \beta_i] \prod_{j=1}^{r} \gamma_j = \gamma_1^{m_1} = \cdots = \gamma_r^{m_r} = 1 \rangle,$$

and the genus $\sigma$ of $S$ is given by the Riemann-Hurwitz equation

$$\frac{(2\sigma - 2)}{|G|} = (2\rho - 2 + r) - \sum_{j=1}^{r} \frac{1}{m_j}. \quad \text{(3)}$$

Let $a_i, b_i, c_j$ be images of the generators $\alpha_i, \beta_i, \gamma_j$ under the epimorphism $\eta : \Gamma \twoheadrightarrow G$, i.e.,

$$\eta : \alpha_i \to a_i, \eta : \beta_i \to b_i, \eta : \gamma_j \to c_j. \quad \text{(4)}$$

Then the set $\{a_i, b_i, c_j : 1 \leq i \leq \rho, 1 \leq j \leq r\}$ is a generating set for $G$ satisfying the following properties:

$$\prod_{i=1}^{\rho} [a_i, b_i] \prod_{j=1}^{r} c_j = 1, \quad \text{(5)}$$

and

$$o(c_j) = m_j. \quad \text{(6)}$$

Each such generating set gives us a $(2\rho + r)$-tuple $(a_1, \ldots, a_\rho, b_1, \ldots, b_\rho, c_1, \ldots, c_r)$ satisfying (3) and (4) and is called a $(\rho : m_1, \ldots, m_r)$-generating vector. There is a 1-1 correspondence between the set of $(\rho : m_1, \ldots, m_r)$-generating vectors of $G$ and $Epi(\Gamma, G)$, epimorphisms $\Gamma \to G$ preserving the orders of the $\gamma_j$, once a generating set $G = \{\alpha_i, \beta_i, \gamma_j : 1 \leq i \leq \rho, 1 \leq j \leq r\}$ of $\Gamma$ has been fixed.
Let $X^\circ(G,S) = X^\circ(G,\rho : m_1,\ldots,m_r)$ denote the set of $(\rho : m_1,\ldots,m_r)$-generating vectors of $G$ acting on $S$.

There is a natural action of $\text{Aut}(G) \times \text{Aut}(\Gamma)$ on $\text{Epi}(\Gamma,G)$ by

$$\eta \longrightarrow \omega \circ \eta \circ \xi^{-1}$$

where $(\omega, \xi) \in \text{Aut}(G) \times \text{Aut}(\Gamma)$. The action transfers to $X^\circ(G,S)$ in a natural way. The following is well known and the justification provided in some detail in [5].

**Proposition 2** Let notation be as above. Then we have:

1. Each finite subgroup of the mapping class group has a uniquely determined signature which may be recovered from the homology representation.

2. The conjugacy classes of finite subgroups of the mapping class group are in 1-1 correspondence to the $\text{Aut}(G) \times \text{Aut}(\Gamma)$ orbits on $X^\circ(G,S)$.

3. The group $\text{Aut}(\Gamma)$ may be constructed geometrically from the mapping class group $M_B$ of $T$ preserving the branch point set and the branching orders.

**Remark 3** Let’s give a bit more detail on the last point of the proposition. Let $B = \{Q_1,\ldots,Q_r\}$ denote the branch point set, and $T^o = T - B$ and $Q_0 \in T^o$. Every homeomorphism of $T^o$ may be identified with a homeomorphism of the pair $(T,B)$ and vice versa. The fundamental group, $\pi_1(T^o,Q_0)$ has a presentation of the form:

$$\langle \tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\gamma}_j, 1 \leq i \leq r, 1 \leq j \leq r : \prod_{i=1}^{r}[\tilde{\alpha}_i, \tilde{\beta}_i] \prod_{j=1}^{r} \tilde{\gamma}_j = 1 \rangle$$

where the $\tilde{\alpha}_i, \tilde{\beta}_i$ are the canonical pair around the $i$'th handle and $\tilde{\gamma}_j$ encircles the $j$'th puncture. There is an epimorphism $\pi_1(T^o,Q_0) \to \Gamma$ given by

$$\tilde{\alpha}_i \to \alpha_i, \tilde{\beta}_i \to \beta_i, \tilde{\gamma}_j \to \gamma_j.$$ 

Let $M(T^o,S)$ denote the subgroup of the mapping class group of $M(T^o)$ that preserves periods of the branch points. Then the canonical map $M(T^o,S) \to \text{Out}(\Gamma)$ is an isomorphism, see [16].

### 2.3 The $\text{Aut}(G) \times \text{Aut}(\Gamma)$ actions for abelian groups.

It is clear how $\text{Aut}(G)$ acts on $X^\circ(G,S)$ namely $\omega \in \text{Aut}(G)$ acts via:

$$(a_1,\ldots,a_p,b_1,\ldots,b_p,c_1,\ldots,c_p) \to (\omega a_1,\ldots,\omega a_p,\omega b_1,\ldots,\omega b_p,\omega c_1,\ldots,\omega c_r).$$

In order to compute the action of $\text{Aut}(\Gamma)$ on $X^\circ(G,S)$ we need a generating set for $\text{Aut}(\Gamma)$. This can be constructed as follows. In [14] Birman constructs
a set of geometrically defined generators of $\mathcal{M}(T, S)$ using Dehn twists and spin maps. The action of these geometric generators on $\pi_1(T, Q_0)$ can be written as substitution formulas in the generators $(\alpha_i, \beta_i, \gamma_j, 1 \leq i \leq \rho, 1 \leq j \leq r)$ using the presentation $(7)$ and then writing $(\alpha_i, \beta_i, \gamma_j, 1 \leq i \leq \rho, 1 \leq j \leq r)$ as substitution formulas in the generators using the homomorphism $(8)$ and the presentation $(2)$. Any substitution formula can be tested against the presentation to see if it really is an automorphism. Here are some examples, there are many more but we will not record them here since we are only going to use the much simpler transformations for Abelian groups.

### Action of automorphisms $\mathfrak{A}_i, \mathfrak{B}_i, \mathfrak{C}_i, \mathfrak{R}_i, \mathfrak{S}_i, \mathfrak{T}_i$

|          | $\alpha_i$ | $\beta_i$ | $\alpha_{i+1}$ | $\beta_{i+1}$ |
|----------|------------|------------|----------------|----------------|
| $\mathfrak{A}_i$ | $\alpha_i$ | $\beta_i\alpha_i$ | $\alpha_{i+1}$ | $\beta_{i+1}$ |
| $\mathfrak{B}_i$ | $\alpha_i\beta_i$ | $\beta_i$ | $\alpha_{i+1}$ | $\beta_{i+1}$ |
| $\mathfrak{R}_i$ | $\alpha_i\beta_i\alpha_i^{-1}$ | $\alpha_i^{-1}$ | $\alpha_{i+1}$ | $\beta_{i+1}$ |
| $\mathfrak{S}_i$ | $\beta\alpha_i\delta^{-1}$ | $\alpha_i^{-1}\delta^{-1}$ | $\alpha_i$ | $\beta_i$ |
| $\mathfrak{T}_i$ | $\alpha_i\alpha_{i+1}$ | $\alpha_i\beta_i\alpha_{i+1}$ | $\epsilon\alpha_{i+1}$ | $\beta_{i+1}\epsilon^{-1}$ |

$\delta = [\alpha_i, \beta_i], \epsilon = [\alpha_{i+1}^{-1}, \beta_i]$ 

### Action of $\mathfrak{T}_j$

| $\mathfrak{T}_j$ | $\gamma_j$ | $\gamma_{j+1}$ | notes |
|-----------------|------------|----------------|-------|
| $\mathfrak{T}_j$ | $\gamma_j\gamma_{j+1}\gamma_j^{-1}$ | $\gamma_j$ | $o(\gamma_j) = o(\gamma_{j+1})$ |
| $\mathfrak{T}_{j+1}^{-1}$ | $\gamma_{j+1}$ | $\gamma_{j+1}^{-1}\gamma_j\gamma_{j+1}$ | $o(\gamma_j) = o(\gamma_{j+1})$ |

### Action of $\mathfrak{U}_{i,j}$, $\mathfrak{V}_{i,j}$

| $\mathfrak{U}_{i,j}$ | $\alpha_i\beta_i\gamma_j^{-1}$ | $\gamma_j$ | notes |
|---------------------|-----------------------------|------------|-------|
| $\mathfrak{U}_{i,j}$ | $\alpha_i\beta_i\gamma_j^{-1}$ | $\gamma_j$ | $x = \alpha_i^{-1}\beta_i^{-1}w_2, y = x^{-1}\alpha_i^{-1}x$ |
| $\mathfrak{V}_{i,j}$ | $\alpha_i\beta_i\gamma_j^{-1}$ | $y\gamma_j^{-1}$ | $u = \beta_i\alpha_i^{-1}\beta_i^{-1}w_2, u = v = u^{-1}\beta_iu$ |

where 

$$w_2 = \prod_{k=i+1}^{\rho} [\alpha_k, \beta_k] \prod_{k=1}^{j-1} \gamma_k$$

When the group $G$ is an additively written abelian group the formulas are much simpler since all commutators will disappear and elements are equal to their conjugates. To avoid confusing the additive formulas with the multiplicative formulas in the general case we will use uppercase letters for a generating vector viz., $(A_1, A_2, A_3, B_1, B_2, B_3)$. In this case an arbitrary assignment of elements of $G$ to the generators $\eta : \alpha_i \to A_i, \beta_i \to B_i$ will define an element of $\text{Hom}(\Gamma, G)$ since the commutation relation is trivially satisfied. Our
transformations above may be written
\[ \mathfrak{A}_i^k : B_i \rightarrow B_i + kA_i \]  
\[ \mathfrak{B}_i^k : A_i \rightarrow A_i + kB_i \]  
\[ \mathfrak{Z}_i : A_i \rightarrow A_i + kA_{i+1}, B_i \rightarrow B_{i+1} - kB_i \]  
\[ \mathfrak{R}_i : A_i \rightarrow B_i, B_i \rightarrow -A_i \]  
\[ \mathfrak{S}_i : A_i \rightarrow A_{i+1}, B_i \rightarrow B_{i+1} - kB_i \]
and trivial on all other generators. If we include branch points then we must also add an assignment \( \eta : \gamma_j \rightarrow C_j \), with
\[ o(C_j) = m_j, \]  
and
\[ C_1 + C_2 + \cdots + C_r = 0. \]
We can then add to our table of transformations
\[ \mathfrak{X}_i : C_j \rightarrow C_{j+1}, C_{j+1} \rightarrow C_j, \quad m_j = m_{j+1} \]  
\[ \mathfrak{U}^k_{i,j} : B_i \rightarrow B_i + kC_j \]  
\[ \mathfrak{V}^k_{i,j} : A_i \rightarrow A_i + kC_j \]
Obviously, any permutation of the \( C_j \) preserving order is permissible.

When \( G \) is abelian then the induced action of \( \text{Aut}(\Gamma) \) on a generating vector can be written in matrix vector form. Suppose that \( \xi^{-1} \in \text{Aut}(\Gamma) \) is induced by a homeomorphism of \( T^\circ \) preserving branch order. Then
\[ (A_1, \ldots A_\rho, B_1, \ldots B_\rho, C_1, \ldots C_r) \rightarrow (A_1, \ldots A_\rho, B_1, \ldots B_\rho, C_1, \ldots C_r)M \]
where \( M \) is a matrix of the form
\[ M = \begin{bmatrix} Sp & Z \\ 0 & P \end{bmatrix} \]  
where \( Sp \) is a \( 2\rho \times 2\rho \) integral symplectic matrix, \( P \) is an \( r \times r \) permutation matrix, and \( Z \) is arbitrary integer valued matrix. The matrix \( Sp \) is the induced symplectic automorphism on \( H_1(T; \mathbb{Z}) \). Perhaps the easiest way to see this is that each of the generators listed above has the given matrix decomposition.

### 2.4 Hyperbolic - elliptic decomposition, abelian case

In the group \( \Gamma \) the generators \( (\alpha_i, \beta_i : 1 \leq i \leq \rho) \) are hyperbolic elements and the \( \{ \gamma_j : 1 \leq j \leq r \} \) are elliptic elements. Accordingly we define the hyperbolic (unramified) and elliptic (ramified) parts of \( G : G^h = \langle A_i, B_i : 1 \leq i \leq \rho \rangle \), \( G^e = \langle C_j : 1 \leq j \leq r \rangle \). We are interested in the action of \( \text{Aut}(\Gamma) \) on the splitting of \( G \). According define for \( \xi \in \text{Aut}(\Gamma) \)
\[ G^h,\xi = \langle \eta \circ \xi^{-1}(\alpha_i), \eta \circ \xi^{-1}(\beta_i) : 1 \leq i \leq \rho \rangle, \quad G^e,\xi = \langle \eta \circ \xi^{-1}(\gamma_j) : 1 \leq j \leq r \rangle. \]
We have the following:
Proposition 4 Let \( \eta : \alpha_i \to A_i, \beta_i \to B_i, \gamma_j \to C_j \), be an arbitrary assignment. Then \( \eta \) extends to an epimorphism \( \eta : \Gamma \to G \) if and only if the relations (11) and (12) hold and \( G = \langle G^\eta, G^\xi \rangle \).

Proposition 5 Let \( \eta : \Gamma \to G \) be an epimorphism then \( G^\xi = G^\eta \xi \) for all \( \xi \in \text{Aut}(\Gamma) \).

Proof. Let \( \Gamma^e = \langle \gamma_j : 1 \leq j \leq r \rangle \). If \( N(\Gamma^e) \) is the normal closure of \( \Gamma^e \) then \( G^e = \eta(N(\Gamma^e)) \). It is well known that in \( \Gamma \) every elliptic element is conjugate to one of the \( \gamma_j \), and hence \( \langle \eta \circ \xi^{-1}(C_j) : 1 \leq j \leq r \rangle = \eta(\langle \xi^{-1}(N(\Gamma^e)) \rangle) = \eta(N(\Gamma^e)) = G^e \).

Proposition 6 The elliptic components of two abelian generating vectors \( C_j : 1 \leq j \leq r, C'_j : 1 \leq j \leq r \), are \( \mathcal{M}(\Gamma) \)-equivalent if and only if one is a permutation of the other: \( C'_j = C_{\pi j} \) for some permutation \( \pi \).

Proof. The automorphism \( \xi \) is induced by a homeomorphism \( h \) of \( (T^o, Q_0) \) which must permute the punctures \( Q_1, \ldots, Q_r \). Since \( h \) is orientation-preserving then \( h^{-1}(\tilde{\gamma}_j) \) is a conjugate of some \( \tilde{\gamma}_{\pi j} \) for some permutation \( \pi \).

Next let \( G^e \) be a subgroup of \( G \) and consider all possible subgroups \( H \) such that \( G = \langle H, G^e \rangle \). Let \( H_0 \) be one of smallest order. Then we may also assume that \( G^h = H_0 \).

Proposition 7 Let \( \eta : \Gamma \to G \) be an action and let \( H_0 \) be the subgroup of smallest order such that \( G = \langle H_0, G^e \rangle \). Then there is \( \xi \in \mathcal{M}(\Gamma) \) such that \( G^{h\xi} = H_0 \) and \( G^{e\xi} = G^e \).

Proof. The equality \( G^{e\xi} = G^e \) is automatic. Each \( A_i \) and \( B_i \) may be written

\[
A_i = A'_i + a_{i,1}C_1 + \cdots + a_{i,r}C_r, \quad A'_i \in H_0
\]

\[
B_i = B'_i + b_{i,1}C_1 + \cdots + b_{i,r}C_r, \quad B'_i \in H_0
\]

By applying \( \Psi^{-a_{i,j}} \) we can remove the component \( a_{i,j}C_j \) from the expression for \( A_i \) and affect no other terms. Thus all the \( A_i \) can be reduced to \( A'_i \). Similarly the \( B_i \) can be reduced to \( B'_i \). Now \( \langle A'_i, B'_i : 1 \leq i \leq \rho \rangle \subseteq H_0 \) and \( G = \langle \langle A'_i, B'_i : 1 \leq i \leq \rho \rangle, G^e \rangle \) by construction, so \( H_0 = \langle A'_i, B'_i : 1 \leq i \leq \rho \rangle \) by minimality.

2.5 The elementary abelian case

Finally let us assume that \( G = \mathbb{F}^e_p \), an elementary abelian group considered as a vector space over \( \mathbb{F}_p \). The generators \( A_i, B_i, C_j \) may then be thought of as vectors over \( \mathbb{F}_p \). Now let \( X_{AB} \) be the matrix \( \left[ A_1 \cdots A_\rho \ B_1 \cdots B_\rho \right] \) and \( X_C \) be the matrix \( \left[ C_1 \cdots C_r \right] \). Then by equation (13) the action of \( (\omega, \xi) \) on \( X = \left[ X_{AB} \ X_C \right] \) is:

\[
\left[ X_{AB} \ X_C \right] \to M_\omega \left[ X_{AB} \ X_C \right] M_\xi^{-1}
\]
where $M_\omega \in GL_w(p)$ and

$$M_\xi^{-1} = \begin{bmatrix} Sp & Z \\ 0 & P \end{bmatrix}$$

Next by Proposition [4] we assume that $\eta$ is chosen so that $F^w_p = G^h \oplus G^e$.
Now write $F^w_p = F^u_p \oplus F^v_p$ and find an $N \in GL_w(p)$ such $NG^h = F^u_p \oplus 0$ and $NG^e = 0 \oplus F^v_p$. It then follows that

$$N \begin{bmatrix} X_{AB} & X_C \end{bmatrix} = \begin{bmatrix} Y_{AB} & 0 \\ 0 & Y_C \end{bmatrix} \quad (14)$$

where $Y_{AB}$ is a $u \times 2\rho$ matrix of rank $u$ and $Y_C$ is a $v \times r$ matrix of rank $v$. Thus we may assume the generating vector $X$ of our initial $\eta$ has the form of the right hand side of equation (14). Now suppose that we look at a transformed generating vector $M_\omega XM_\xi^{-1}$ that has the same form i.e.,

$$\begin{bmatrix} W_{AB} & 0 \\ 0 & W_C \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} Y_{AB} & 0 \\ 0 & Y_C \end{bmatrix} \begin{bmatrix} Sp & Z \\ 0 & P \end{bmatrix}$$

$$= \begin{bmatrix} M_{11}Y_{AB}Sp & M_{11}Y_{AB}Z + M_{12}Y_CP \\ M_{21}Y_{AB}Sp & M_{22}Y_{AB}Z + M_{22}Y_CP \end{bmatrix}$$

Since $M_{21}Y_{AB}Sp = 0$ and $Y_{AB}Sp$ has rank $u$ then $M_{21} = 0$, and $M_{11}$ and $M_{22}$ are invertible. Since $M_{11}Y_{AB}Z + M_{12}Y_CP$ is also assumed zero, it follows that

$$\begin{bmatrix} W_{AB} & 0 \\ 0 & W_C \end{bmatrix} = \begin{bmatrix} M_{11}Y_{AB}Sp & 0 \\ 0 & M_{22}Y_CP \end{bmatrix}$$

$$= \begin{bmatrix} M_{11} \\ M_{22} \end{bmatrix} \begin{bmatrix} Y_{AB} & 0 \\ 0 & Y_C \end{bmatrix} \begin{bmatrix} Sp & Z \\ 0 & P \end{bmatrix}$$

i.e., we can just find the equivalence classes of the unramified and the ramified components independently. We summarize the results in a theorem.

**Theorem 8** Suppose that the group $G = F^w_p$ acts on a surface $S$ with signature $(\rho : p^r)$, and genus $1 + p^w(\rho - 1) + p^{w-1}(p-1)$ determined by an epimorphism $\eta : \Gamma \to G$ (note that $r = 1$ is infeasible). Then there are unique integers $u, v$ such that $w = u + v$, $0 \leq u \leq 2\rho$, $1 \leq v < r$, a $u \times 2\rho$ matrix $W_{AB}$ of rank $u$, and a $v \times r$ matrix $W_C$ of rank $v$ such that the matrix $\begin{bmatrix} W_{AB} & 0 \\ 0 & W_C \end{bmatrix}$ is a representative of the topological equivalence class of $G$-actions on $S$. Moreover $\begin{bmatrix} W'_{AB} & 0 \\ 0 & W'_C \end{bmatrix}$ is another representative satisfying similar rank conditions for $W'_{AB}$ and $W'_C$ then $W_{AB}$ and $W'_{AB}$ define equivalent unramified actions with signature $(\rho : -)$ and $W_C$ and $W'_C$ define equivalent purely ramified actions with signature $(0 : p^r)$. 

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Corollary 9 For $0 \leq u \leq 2\rho$, let $h_u$ be the number of equivalence classes of unramified actions of $\mathbb{F}_p^u$ on a surface of genus $1 + p^u(\rho - 1)$ with signature $(\rho : -)$ and $h_u = 0$ otherwise. For $1 \leq v < r$ let $e_v$ be the number of equivalence classes of purely ramified actions of $\mathbb{F}_p^v$ on a surface $S$ of genus $1 + p^{v-1}(p-1) - p^v$ with signature $(0 : p^r)$ and set $e_v = 0$ otherwise. Then the number of inequivalent actions of $\mathbb{F}_p^w$ on a surface of genus $1 + p^w(\rho - 1) + p^{w-1}(p-1)$ with signature $(\rho : p^r)$ is given by

$$\#\text{actions} = \sum_{u=0}^{n} h_u e_{w-u}.$$ 

2.6 Cohomological invariants

For abelian covers we may use cohomology to concoct an $\mathcal{M}$-invariant to distinguish classes. As noted above, for an abelian group $G$, $\text{Hom}(\Gamma, G)$ classifies covers $T$ by subgroups of $G$. We have the following sequence of equivalences

$$\text{Hom}(\Gamma, G) \cong \text{Hom}(H^1(T; \mathbb{Z}), G) \cong H^1(T; G).$$

Given two elements of $\eta_1, \eta_2 \in H^1(T; G)$ we may consider the cup product $\eta_1 \cup \eta_2 \in H^1(T; G \otimes G)$. If $h$ is an orientation-preserving homeomorphism then

$$h^*\eta_1 \cup h^*\eta_2 = h^*(\eta_1 \cup \eta_2) = \eta_1 \cup \eta_2.$$ 

In particular if $\eta_1 = \eta_2 = \eta$ then

$$h^*\eta \cup h^*\eta = \eta \cup \eta$$

since $h$ is orientation preserving. It is not hard to show that if $\eta : \alpha_i \to A_i$, $\eta : \beta_i \to B_i$, then

$$\eta \cup \eta = \sum_{i=1}^{\rho} (A_i \otimes B_i - B_i \otimes A_i)$$

(15)

Observe that the invariance of $\eta \cup \eta$ under the transforms in Subsection 2.3 may also be proven by direct computation.

Example 10 Let $G = C_n \oplus C_m$ with $x$ generating $C_n$ and $y$ generating $C_m$ additively (cyclic groups of order $m$ and $n$). Consider the following epimorphisms

$$\eta_1 : (\alpha_1, \beta_1, \alpha_2, \beta_2) \to (x, y, 0, 0)$$

$$\eta_2 : (\alpha_1, \beta_1, \alpha_2, \beta_2) \to (x, 0, y, 0)$$

Then

$$\eta_1 \cup \eta_1 = x \otimes y - y \otimes x$$

$$\eta_2 \cup \eta_2 = 0$$

and so the epimorphisms are different under the action of Aut($\Gamma(2; -)$).
3 The unramified abelian case

In this section, we consider the case when $G$ is abelian and $\Gamma = \Pi_\rho$ is torsion free. For the special case when $G$ is elementary abelian, complete results were derived in [7]. We shall apply the ideas from this case to the more general abelian case. Due to the complexity of the problem, we shall only produce partial results, though the results we derive will be sufficient to produce explicit results for genus up to 65. We shall also illustrate how in principle one could use the results to classify all fixed point abelian actions for arbitrary genus. It should be noted that we do not consider the abelian case for general $\Gamma$ because Theorem 8 no longer holds so there could be overlap between $G^h$ and $G^e$ making the problem much more difficult.

3.1 The unramified elementary abelian case

The following was proved in [7] and completely classifies all unramified elementary abelian actions up to topological equivalence (by $p$-rank of an abelian group $G$, we mean the number of invariant factors of $G$). The corollary immediately follows.

**Theorem 3.1** Suppose $\Gamma = \Pi_\rho$ is a surface group of orbit genus $\rho$ and genera-tors $\alpha_1, \ldots, \alpha_\rho, \beta_1, \ldots, \beta_\rho$, and $G$ is an elementary abelian group of $p$-rank $r \leq 2\rho$ with generators $\omega_1, \ldots, \omega_r$. Then there exists an integer $r/2 \leq K \leq \min(\rho, r)$ such that any epimorphism from $\Gamma$ onto $G$ is $\text{Aut}(G) \times \text{Aut}(\Gamma)$-equivalent to one of those below:

$$
\eta := \begin{cases} 
\eta(\alpha_i) = \omega_i & i \leq K \\
\eta(\alpha_i) = 0 & i > K \\
\eta(\beta_i) = \omega_i + K & i \leq r - K \\
\eta(\beta_i) = 0 & i > r - K 
\end{cases}
$$

**Corollary 11** Suppose $G$ is an elementary abelian group of $p$-rank $r$, let $\mathcal{M}_\sigma$ denote the mapping class group of a closed surface of genus $\sigma$ and let $\rho = (\sigma - 1 + p')/p'$. If $\rho$ is not an integer, there are no conjugacy classes of subgroups of $\mathcal{M}_\sigma$ isomorphic to $G$ with fixed point free action. Else, the number of conjugacy classes of subgroups of $\mathcal{M}_\sigma$ isomorphic to $G$ with fixed point free action is calculated as follows: if $r \leq \rho$, there are $r/2 + 1$ different classes of epimorphisms for $r$ even and $(r + 1)/2$ epimorphisms if $r$ is odd, if $r = \rho + i$ with $0 < i < \rho$ there are $(\rho - i)/2$ classes of epimorphisms if $\rho - i$ is even and $(\rho - i + 1)/2$ if $\rho - i$ is odd, and if $r = 2\rho$ or $r = 1$, there is just one class.

We illustrate with an example.

**Example 12** Suppose $\Gamma$ has genus 2 and $G$ is an elementary abelian $p$-group of $p$-rank 2. If $\eta: \Gamma \to G$ is a surface kernel epimorphism, then using the Riemann-Hurwitz formula, the genus $\sigma$ of the kernel will be $\sigma = p^2 + 1$. In fact, by simple
application of the Riemann Hurwitz formula, it can be shown that for \( p > 5 \), \( \Gamma \) is the only Fuchsian group which admits an elementary abelian quotient of order \( p^2 \) with kernel of orbit genus \( p^2 + 1 \) (for \( p = 5 \), see Example 37). This means for \( p > 5 \), in \( M_{p^2+1} \), the conjugacy classes of elementary abelian subgroups of order \( p^2 \) will be in \( 1-1 \) correspondence with the classes of epimorphisms from \( \Gamma \) onto \( G \). Applying Corollary 17, there are two such classes.

3.2 The general unramified abelian case

We now consider partial results for the general abelian case. Before we start, we introduce some notation and terminology.

Notation 13 For the rest of this section, \( G \) will denote an additively written abelian group of \( p \)-rank \( r \) with invariant factors \( n_1 \geq n_2 \geq \cdots \geq n_r \) where \( n_{i+1} | n_i \) and \( \omega_1, \ldots, \omega_r \) are a fixed set of generators of orders \( n_1, \ldots, n_r \) respectively. Also, for an integer \( n \), \( C_n \) denotes the cyclic group of order \( n \).

The following two steps can be taken to classify all fixed point free \( G \)-actions on a surface - first determine a set of epimorphisms with the property that every epimorphism is equivalent to one in this set, and then reduce this set so no two epimorphisms are equivalent. In the following, we shall consider the first step. Though in general we shall not consider the problem of distinguishing between classes, we shall present explicit examples showing how in principle one could tackle this problem. Note that \( r \leq 2\rho \), so we only need consider epimorphisms from \( \Gamma \) to \( G \) with \( p \)-rank at most \( 2\rho \).

Proposition 14 Any epimorphism \( \eta: \Gamma \rightarrow G \) is equivalent to one of those described below:

1. \( r \leq \rho \).

\[
\eta := \begin{cases} 
\eta(\alpha_i) = \omega_i & i \leq r \\
\eta(\alpha_i) = 0 & i > r \\
\eta(\beta_i) = \sum_{j=i+1}^{r} N_{\beta_i, \omega_j} \omega_j & i \leq r \\
\eta(\beta_i) = 0 & i > r 
\end{cases}
\]

where

- \( (n_{i+1}, N_{\beta_i, \omega_{i+1}}) > 1 \) or \( N_{\beta_i, \omega_{i+1}} = 1 \) for each \( i \leq r - 1 \)
- for each \( i \), if \( j > i + 1 \) is the largest integer such that \( N_{\beta_i, \omega_j} \neq 0 \), then \( \langle N_{\beta_i, \omega_j} \rangle \leq \langle N_{\beta_{i+1}, \omega_j} \rangle \).

2. \( r > \rho \).
Proof. We shall use induction on the $p$-rank of the group $G$. For $p$-rank $r = 1$, we are done by Theorem 14 of [11]. Assuming the result holds for $p$-rank $r$, we shall prove it holds for $p$-rank $r+1$. The proof falls into two different cases depending upon whether $1 < r < p$ or $r = p$.

Let $\eta: \Gamma \to G = C_{n_1} \times \cdots \times C_{n_r}$ denote the epimorphism onto $G$ and $\Phi: G \to C_{n_1} \times \cdots \times C_{n_{r-1}}$ the projection map onto the quotient group $G/C_{n_r}$. (By abuse of notation, we identify $C_{n_1} \times \cdots \times C_{n_{r-1}}$ with the corresponding subgroup of $G$.) Since $\Gamma$ is torsion free, all subgroups will be torsion free. In particular, the map $\Phi \circ \eta$ will be a surface kernel epimorphism from $\Gamma$ onto $C_{n_1} \times \cdots \times C_{n_{r-1}}$, so by induction, will be equivalent to one as given in the statement of the proposition. Lifting to $G$ and composing with appropriate automorphisms of $G$, it follows that $\eta$ is equivalent to an epimorphism of the following form:

$$
\eta := \begin{cases} 
\eta(\alpha_i) = \omega_i & 1 \leq i \leq \rho \\
\eta(\beta_i) = \sum_{j=i+1}^{\rho} N_{\beta_i,\omega_j} \omega_j, & 1 \leq i \leq 2 \rho - r \\
\eta(\beta_i) = \omega_{2\rho-i+1} + \sum_{j=i+1}^{2\rho-i} N_{\beta_i,\omega_j} \omega_j & 2\rho - r < i < \rho \\
\eta(\beta_{\rho}) = \omega_{\rho+1} & i = \rho
\end{cases}
$$

where:

- for $i > 2\rho - r$ either the order of $N_{\beta_i,\omega_j}$ does not divide $n_r$ or $N_{\beta_i,\omega_j} = 0$.
- for $i \leq 2\rho - r$ if $j > i + 1$ is the largest integer such that $N_{\beta_i,\omega_j} \neq 0$, then $\langle N_{\beta_i,\omega_j} \rangle \preceq \langle N_{\beta_{i+1},\omega_j} \rangle$.
- for $i \leq 2\rho - r - 1$, $(n_i, N_{\beta_i,\omega_{i+1}}) > 1$ or $N_{\beta_i,\omega_i+1} = 1$.

**Proof.** We shall use induction on the $p$-rank of the group $G$. For $p$-rank $r = 1$, we are done by Theorem 14 of [11]. Assuming the result holds for $p$-rank $r$, we shall prove it holds for $p$-rank $r+1$. The proof falls into two different cases depending upon whether $1 < r < p$ or $r = p$. First suppose that $r \leq p$.

Let $\eta: \Gamma \to G = C_{n_1} \times \cdots \times C_{n_r}$ denote the epimorphism onto $G$ and $\Phi: G \to C_{n_1} \times \cdots \times C_{n_{r-1}}$ the projection map onto the quotient group $G/C_{n_r}$. (By abuse of notation, we identify $C_{n_1} \times \cdots \times C_{n_{r-1}}$ with the corresponding subgroup of $G$.) Since $\Gamma$ is torsion free, all subgroups will be torsion free. In particular, the map $\Phi \circ \eta$ will be a surface kernel epimorphism from $\Gamma$ onto $C_{n_1} \times \cdots \times C_{n_{r-1}}$, so by induction, will be equivalent to one as given in the statement of the proposition. Lifting to $G$ and composing with appropriate automorphisms of $G$, it follows that $\eta$ is equivalent to an epimorphism of the following form:

$$
\eta := \begin{cases} 
\eta(\alpha_i) = \omega_i & 1 \leq i \leq \rho \\
\eta(\alpha_i) = m_i \omega_r & i \geq \rho \\
\eta(\beta_i) = M_i \omega_r + \sum_{j=i+1}^{r-1} N_{\beta_i,\omega_j} \omega_j & 1 \leq r \\
\eta(\beta_i) = M_i \omega_r & i > r
\end{cases}
$$

for integers $m_i$, $r \leq i \leq \rho$ and $M_j$, $1 \leq j \leq \rho$. Similar to the elementary abelian case in [11], we now reduce using the automorphisms from Aut($\Gamma$) developed in Section 2.3 and the automorphisms of the abelian group $G$. In most instances, since the reduction steps are between pairs of pairs of generators and constant on all others, we shall just consider the pairs which are changed. To start, let $k$ be the smallest integer such that $(M_2 - kM_1)\omega_r$ generates the subgroup generated by $M_1 \omega_r$ and $M_2 \omega_r$ and let $\Phi \in \text{Aut}(G)$ where $\Phi(\omega_1) = \omega_1 + k \omega_2$ and the identity map on all other generators. Then the map $\Phi^k$...
Finally, we eliminate the image of $\beta_i$ for some integer $i \geq 3$. Specifically, after renaming the coefficients, we get:

$$(\alpha_1,\beta_1,\alpha_2,\beta_2) \rightarrow (\omega_1, M_1\omega_r + \sum_{j=2}^{r-1} N_{\beta_1,\omega_j} \omega_j, \omega_2, M_2\omega_r + \sum_{j=3}^{r} N_{\beta_2,\omega_j} \omega_j)$$

$$\rightarrow (\omega_1, \sum_{j=2}^{r-1} N_{\beta_1,\omega_j} \omega_1, \omega_2, \sum_{j=3}^{r} N_{\beta_2,\omega_j} \omega_1)$$

where $\langle N_{\beta_1,r}\omega_r \rangle \leq \langle N_{\beta_2,r}\omega_r \rangle$. Applying similar transformations to all proceeding pairs of pairs up to the pair $(\alpha_r, \beta_r)$ we get:

$$\eta := \begin{cases} 
\eta(\alpha_i) = \omega_i & i < r \\
\eta(\alpha_i) = m_i\omega_r & i \geq r \\
\eta(\beta_i) = \sum_{j=i+1}^{r} N_{\beta_i,\omega_j} \omega_j & i \leq r \\
\eta(\beta_i) = M_i\omega_r & i > r 
\end{cases}$$

where $\langle N_{\beta_1,r}\omega_r \rangle \leq \langle N_{\beta_2,r+1}\omega_{r+1} \rangle$ for all $i$. In particular, this implies

$$\langle N_{\beta_1,r}, N_{\beta_2,r}, \ldots, N_{\beta_{r-1},r}\omega_r \rangle \leq \langle N_{\beta_r,r}\omega_r \rangle.$$

Next, we reduce the pairs $(\alpha_i, \beta_i)$ with $i \geq r$. First, to any such pair, we apply $\mathfrak{A}_n^k$ where $k$ is the smallest integer such that $(m_i + kM_i)\omega_r$ generates the subgroup generated by $m_i\omega_r$ and $M_i\omega_r$ (for brevity, by abuse of notation, we rename $(m_i + kM_i)$ by $m_i$). Then for a given $i$, since $\langle M_i\omega_r \rangle \leq \langle m_i\omega_r \rangle$, there exists $n$ such that $\langle M_i + nm_i \rangle \omega_r = 0$. Applying the automorphism $\mathfrak{A}_n^i$, we get $(\alpha_i, \beta_i) \rightarrow (0, M_i\omega_r)$. Following this, for the pairs, $(\alpha_{p-1}, \beta_{p-1}, \alpha_p, \beta_p)$, we apply $\mathfrak{A}_n^{p-1}$ where $k$ is the smallest integer such that $(M_p - kM_{p-1})\omega_r$ generates the subgroup generated by $M_p\omega_r$ and $M_{p-1}\omega_r$. Renaming $(M_p - kM_{p-1})\omega_r$ by $M_p$, we get $(\alpha_{p-1}, \beta_{p-1}, \alpha_p, \beta_p) \rightarrow (M_{p-1}\omega_r, 0, M_p\omega_r, 0)$ where $\langle M_{p-1}\omega_r \rangle \leq \langle M_p\omega_r \rangle$. Finally, we eliminate the image of $\beta_{p-1}$ by applying $\mathfrak{A}_n^{p-1}$ where $n$ is the smallest integer such that $\langle M_{p-1} - nM_p \rangle\omega_r = 0$ and then apply $\mathfrak{G}_{p-1}$ and $\mathfrak{R}_{p-1}$ giving $(\alpha_{p-1}, \beta_{p-1}, \alpha_p, \beta_p) \rightarrow (0, 0, M_{p-1}\omega_r, 0)$ (after renaming coefficients). Applying similar transformations to pairs $(\alpha_i, \beta_i, \alpha_{i+1}, \beta_{i+1})$ with $i > r$, we get:

$$\eta := \begin{cases} 
\eta(\alpha_i) = \omega_i & i < r \\
\eta(\alpha_i) = M_i\omega_r & i = r \\
\eta(\alpha_i) = 0 & i > r \\
\eta(\beta_i) = \sum_{j=i+1}^{r} N_{\beta_i,\omega_j} \omega_j & i \leq r \\
\eta(\beta_i) = 0 & i > r 
\end{cases}$$

for some integer $M$. Finally, we apply $\mathfrak{A}_r^k$ where $k$ is the smallest integer such that $(M - kN_{\beta_{r-1},\omega_r}, n_r) = 1$ (note that such an integer exists since $M\omega_r$ and $N_{\beta_{r-1},\omega_r}$ generate $\langle \omega_r \rangle$). Finally, by applying $\mathfrak{G}_r$ and an appropriate automorphism of $G$, after renaming the coefficients, we get:
For the last reduction step, if \( \langle N_{\beta_{r-1}, \omega_r}, n_r \rangle = 1 \), we first apply \( \Phi \in \text{Aut}(G) \) such that \( \Phi(N_{\beta_{r-1}, \omega_r}, \omega_r) = \omega_r \) and identity on all other generators. Then, assuming \( b \) is the integer with \( \Phi(\omega_r) = b\omega_r \), we perform \( \Theta_r \circ 3_{r-1}^{i+1} \circ \Theta_r \circ \Phi \) giving \( \langle \alpha_{r-1}, \beta_{r-1}, \alpha_r, \beta_r \rangle \rightarrow \langle \omega_{r-1}, \omega_r, \omega_r, 0 \rangle \). Note that for \( l < r \), none of the transformations used during this proof change the fact that \( n_i, N_{\beta_{r-1}, \omega_r} \rangle > 1 \) or \( N_{\beta_{r-1}, \omega_r} \rangle = 1 \). Observe also that the reduction methods used imply that if \( j > i + 1 \) is the largest integer such that \( N_{\beta, \omega_j} \neq 0 \), then \( \langle N_{\beta_i, \omega_j} \rangle \leq \langle N_{\beta_{i+1}, \omega_j} \rangle \).

Now suppose that \( r > p \). Since the arguments regarding transformations are similar to those used for the previous case, we shall skip many steps. Assuming the result holds when the \( p \)-rank of the group is \( r-1 \geq p \), induction implies after composing with appropriate automorphisms of \( G \), \( \eta \) has the following form:

\[
\eta := \begin{cases} 
\eta(\alpha_i) = \omega_i & 1 \leq i \leq \rho \\
\eta(\beta_i) = M_i \omega_r + \sum_{j=i+1}^{r-1} N_{\beta_i, \omega_j} \omega_j, & 1 \leq i \leq 2\rho - r + 1 \\
\eta(\beta_i) = \omega_{2\rho - i + 1} + \sum_{j=i+1}^{2\rho - r} N_{\beta, \omega_j} \omega_j, & 2\rho - r + 1 < i < \rho \\
\eta(\beta_r) = \omega_{\rho + 1} & \end{cases}
\]

As with the previous case, for each \( i < r \), we can use the transformations \( 3_i, 2_i \) and appropriate automorphisms of \( G \) to move a generator for \( \langle \omega_r \rangle \) to the image of \( \beta_{2\rho - r + 1} \) giving \( \eta(\beta_{2\rho - r + 1}) = \omega_{2\rho - r + 1} + \sum_{j=r+1}^{2\rho - r} N_{\beta_i, \omega_j} \omega_j \). Applying these transformations, we get

\[
\eta := \begin{cases} 
\eta(\alpha_i) = \omega_i & 1 \leq i \leq \rho \\
\eta(\beta_i) = \sum_{j=i+1}^{r-1} N_{\beta_i, \omega_j} \omega_j, & 1 \leq i \leq 2\rho - r \\
\eta(\beta_i) = \omega_{2\rho - i + 1} + \sum_{j=i+1}^{2\rho - r} N_{\beta_i, \omega_j} \omega_j, & 2\rho - r + 1 \leq i < \rho \\
\eta(\beta_r) = \omega_{\rho + 1} & \end{cases}
\]

where \( \langle N_{\beta_i, \omega_j} \rangle \leq \langle N_{\beta_{2\rho - r + 1}, \omega_j} \rangle \leq \cdots \leq \langle N_{\beta_{2\rho - r + 1}, \omega_j} \rangle \).

For the last reduction step, we apply an appropriate automorphism of \( G \) that acts trivially on all generators except \( \omega_r \) and eliminates all elements from the sum \( \sum_{j=r+1}^{r-1} N_{\beta_i, \omega_j} \omega_j \) with the order of \( N_{\beta_i, \omega_j} \omega_j \) dividing \( n_r \). Then we get:
where the order of \( N_{\beta_i, \omega_j} \) does not divide \( n_r \) for \( i > 2\rho - r \). As with the last case, if \( i \leq 2\rho - r \), the reduction methods used implies that if \( j > i + 1 \) is the largest integer such that \( N_{\beta_i, \omega_j} \neq 0 \), then \( \langle N_{\beta_i, \omega_j} \rangle \leq \langle N_{\beta_{i+1}, \omega_j} \rangle \). □

We note that though more refined sets of epimorphisms can be obtained (in particular, the elementary abelian case), our goal was to provide a set of epimorphisms independent of the genus and invariant factors. We finish with an explicit example showing how to use Proposition 14 to determine all classes of fixed point free actions.

**Example 15** Suppose that \( \Gamma \) has genus 2 and \( G \) has \( p \)-rank 3 with invariant factors 4, 4, 2. Proposition 14 implies there are up to six classes of epimorphisms of the form \((\alpha_1, \beta_1, \alpha_2, \beta_2) \rightarrow (\omega_1, a\omega_2 + b\omega_3, \omega_2, \omega_3)\) where \( a = 0, 1, 2 \) and \( b = 0, 1 \). Using elements of \( \text{Aut}(G) \times \text{Aut}(\Gamma) \) it is fairly straightforward to reduce each of these epimorphisms to one of the following three:

- \( \eta_1 := (\alpha_1, \beta_1, \alpha_2, \beta_2) \rightarrow (\omega_1, \omega_2, \omega_3, 0) \)
- \( \eta_2 := (\alpha_1, \beta_1, \alpha_2, \beta_2) \rightarrow (\omega_1, \omega_3, \omega_2, 0) \)
- \( \eta_3 := (\alpha_1, \beta_1, \alpha_2, \beta_2) \rightarrow (\omega_1, \omega_2, 2\omega_2, \omega_3) \)

We shall show that these three epimorphisms are distinct.

First note that if \( \eta_i \) is equivalent to \( \eta_j \), then there exists \( \theta \in \text{Aut}(G) \) and \( \zeta \in \text{Aut}(\Gamma) \) such that \( \theta \circ \eta_i \circ \zeta = \eta_j \). Since \( \cup \) is an \( \text{Aut}(\Gamma) \) invariant, it follows that \( (\theta \circ \eta_i) \cup (\theta \circ \eta_i) = \eta_j \cup \eta_j \). Therefore, in order to show distinctness, it suffices to show that for \( i \neq j \), there does not exist \( \theta \in \text{Aut}(G) \) such that \( (\theta \circ \eta_i) \cup (\theta \circ \eta_i) = \eta_j \cup \eta_j \). Since \( \text{Aut}(G) \) is finite, this can be checked directly.

We illustrate by showing that \( \eta_1 \) and \( \eta_2 \) are not equivalent.

First, taking the cup product we get

\[
\eta_1 \cup \eta_1 = \omega_1 \otimes \omega_2 - \omega_2 \otimes \omega_1
\]

and

\[
\theta \circ \eta_1 \cup \theta \circ \eta_2 = \theta(\omega_1) \otimes \theta(\omega_3) - \theta(\omega_3) \otimes \theta(\omega_1).
\]

Assuming that \( \rho(\omega_1) = A\omega_1 + B\omega_2 + C\omega_3 \) and \( \rho(\omega_3) = D\omega_1 + E\omega_2 + F\omega_3 \), by simplifying the tensor product and combining like terms, in order for these products to be equal, we must have \((AE - DB) = 1 \mod (4)\). However, this is impossible since both \( D \) and \( E \) must be divisible by 2 (since \( \omega_3 \) has order 2).

Thus \( \eta_1 \) and \( \eta_2 \) are not equivalent. All other cases follow similar arguments and so it follows that there are three topological equivalence classes of fixed point \( G \)-actions on a surface of genus 33.
3.3 Classification for Genera \( \leq 65 \)

Due to the ad-hoc style arguments which seem required to distinguish classes, we fall short of a general classification for abelian groups. However, Proposition 14 coupled with the following results will allow a classification up to genus 65 (and in fact is enough to classify certain infinite families).

**Lemma 16** If \( G \) has \( p \)-rank 2 and \( n_2 \) is square free, then for each divisor \( N \) of \( n_2 \) with \( 1 \leq N \leq n_2 \), any epimorphism \( \eta: \Gamma \rightarrow G \) is equivalent to one of those described below:

\[
\eta := \begin{cases} 
\eta(\alpha_1) = \omega_1 \\
\eta(\alpha_2) = \omega_2 \\
\eta(\alpha_i) = 0 & i > 2 \\
\eta(\beta_1) = N\omega_2 \\
\eta(\beta_i) = 0 & i > 1 
\end{cases}
\]

Moreover, no two such epimorphisms are equivalent under the action of \( \text{Aut} (G) \times \text{Aut} (\Gamma) \), so in particular, there are \( d(n_2) \) epimorphisms up to the action of \( \text{Aut} (G) \times \text{Aut} (\Gamma) \) (where \( d(n_2) \) denotes the number of divisors of \( n_2 \)).

**Proof.** By Proposition 14 any epimorphism from \( \Gamma \) onto a \( p \)-rank 2 abelian group will be equivalent to one of the form \( \eta_N: (\alpha_1, \beta_1, \alpha_2, \beta_2) \rightarrow (\omega_1, N\omega_2, \omega_2, 0) \) with the image of all other generators being trivial. Therefore, we just need to determine when \( \eta_N \) and \( \eta_M \) are equivalent for \( M \neq N \). We start by showing that if \( \gcd(n, N) \neq \gcd(n, M) \), then \( \eta_N \) and \( \eta_M \) are not equivalent.

First, using the arguments from Example 15 it suffices to show that there does not exist \( \theta \in \text{Aut} (G) \) such that \( (\theta \circ \eta_N) \cup (\theta \circ \eta_N) = \eta_M \cup \eta_M \). In general, if \( \theta \in \text{Aut} (G) \), then \( \theta \) can be identified with a \( 2 \times 2 \) matrix with integer coefficients with the action of \( \theta \) on the generators \( \omega_1 \) and \( \omega_2 \) defined as follows:

\[
\begin{pmatrix} A & B \\
C & D \end{pmatrix} \begin{pmatrix} \omega_1 \\
\omega_2 \end{pmatrix} = \begin{pmatrix} A\omega_1 + B\omega_2 \\
C\omega_1 + D\omega_2 \end{pmatrix}
\]

Under such a realization, since this map must restrict to an automorphism of the subgroup \( C_{n_2} \times C_{n_2} \), the determinant \( \det (\rho) = AD - BC \) must be a unit in \( C_{n_2} \). Assuming \( \theta \) has this form, calculating the two cup products and simplifying, we get

\[
(\theta \circ \eta_N) \cup (\theta \circ \eta_N) = N \det (\theta)(\omega_1 \otimes \omega_2 - \omega_2 \otimes \omega_1))
\]

and

\[
\eta_M \cup \eta_M = M(\omega_1 \otimes \omega_2 - \omega_2 \otimes \omega_1). 
\]

Since \( \det (\theta) \) is a unit in \( C_{n_2} \), if \( \gcd(n, N) \neq \gcd(n, M) \), then we can never have \( N \det (\theta) = M \) (else they would be associates in the ring \( C_{n_2} \) and thus we would have \( \gcd(n, N) = \gcd(n, M) \)). Thus, if \( \gcd(n, N) \neq \gcd(n, M) \) then \( \eta_N \) and \( \eta_M \) define inequivalent classes. To finish, we need to show if \( \gcd(n, N) = \gcd(n, M) \), then \( \eta_N \) and \( \eta_M \) are equivalent.
Suppose $n = p_1 \ldots p_s q_1 \ldots q_r$ where the $p_i$ and $q_i$ are all distinct primes and $d = p_1 \ldots p_s = \gcd(n, N)$. Observe that by applying $\mathcal{S}_2$ followed by $3_k^N$ for any integer $k$ to $\eta_N$, we have the epimorphism $(\alpha_1, \beta_1, \alpha_2, \beta_2) \rightarrow (\omega_1, N\omega_2, 0, (1-kN)\omega_2)$. Moreover, if $(1-kN, n_2) = 1$, then there exists an integer $a$ which induces an automorphism of $C_{n_2}$ via multiplication, such that $a(1-kN) = 1 \mod (n_2)$. Applying this automorphism and reapplying $\mathcal{S}_2$, we see that $\eta_N$ is equivalent to $\eta_{aN}$ for any such $a$ and $(n_2, aN) = d$ (since $a$ is a unit in $C_{n_2}$).

Assume $((1-kN, n_2) = 1$ and $(1-cN, n_2) = 1$, let $a = (1-kN)^{-1}$ and $b = (1-cN)^{-1}$ (all taken $\mod (n_2)$). In order to avoid redundancy, we may assume $0 \leq k, c \leq q_1 \ldots q_r$. Now if $aN = bN \mod (n_2)$, then $(1-kN) = (1-cN)$ and consequently $q_1 \ldots q_r | (k-c)$. However, this implies $k = c + aq_1 \ldots q_r$ which contradicts that $0 \leq k, c \leq q_1 \ldots q_r$ unless $a = 0$. Thus $\eta_{aN} = \eta_{bN}$ only when $a = b$. Therefore, since there are precisely $\phi(n_2/d) = (q_1 - 1)(q_2 - 1) \ldots (q_r - 1)$ different integers $M$ with $\langle n_2, M \rangle = d$, it suffices to show that there are precisely $(q_1 - 1)(q_2 - 1) \ldots (q_r - 1)$ choices for $k$ such that $(1-kN, n_2) = 1$.

Observe that $p_i$ does not divide $1-kN$ for any value of $i$ or $k$, so it follows that $(1-kN, n_2) = 1$ provided $T = q_1 \ldots q_i$, does not divide $1-kN$ for any divisor $T$ of $q_1 \ldots q_r$. Suppose $T = q_1 \ldots q_i$ is a divisor of $q_1 \ldots q_r$. Then $(T, N) = 1$ (since $\gcd(n_2, N) = d$), so there exists $k$ and $c$ such that $cT + kN = 1$, or $cT = 1-kN$. Moreover, if $c_1 T + k_1 N = 1$ for any other $c_1$ and $k_1$, we must have $c_1 = c + vN$ for some integer $v$ and for every integer $v$, if we define $c_1 = c + vN$ and $k_1 = k -Tv$, then $c_1 T = 1 - k_1 N$. This implies there will be precisely $q_1 \ldots q_r/T$ values of $k$ for which $T$ divides $1-kN$. Since there are $q_1 \ldots q_r$ total choices for $k$ and $q_1 \ldots q_r/T$ choices which are divisible by $T$ for each divisor $T$ of $q_1 \ldots q_r$, we can form a weighted sum over all the divisors of $q_1 \ldots q_r$ to determine precisely how many are not divisible by any such $T$. Specifically, the number of values of $k$ with $0 \leq k < q_1 \ldots q_r$ such that $(1-kN, q_1 \ldots q_r) = 1$ is

$$\sum_{n=0}^{r} \left[ (-1)^n \sum_{r \geq i_1 > i_2 > \ldots > i_n \geq 1} \frac{q_1 \ldots q_r}{q_{i_1} \ldots q_{i_n}} \right] = (q_1 - 1)(q_2 - 1) \ldots (q_r - 1).$$

The result follows. \qed

In fact Lemma 16 can be generalized further so for $p = 2$ we don’t have $16|n_2$ (though the proof is more technical). The next result is a generalization of the elementary abelian case and the proof is identical.

**Lemma 17** If $n_2 = p$ a prime, then there exists $K$ with $r/2 \leq K \leq \min(\rho, v)$ such that any epimorphism $\eta: \Gamma \rightarrow G$ is equivalent to one and only one of those described below.

$$\eta := \begin{cases} 
\eta(\alpha_i) = \omega_i & i \leq K \\
\eta(\alpha_i) = 0 & i > K \\
\eta(\beta_i) = \omega_{i+K} & i \leq r - K \\
\eta(\beta_i) = 0 & i > r - K
\end{cases}$$
We now have enough information to complete the classification of fixed point free abelian actions up to genus 65.

**Example 18** First we eliminate all cases which can be classified by our results. Suppose $G$ has invariant factors $n_1, n_2, \ldots, n_r$. By Theorem 3.1 we may assume that $n_1 \neq p$ for $p$ a prime and by Lemma 17 if the $p$-rank is greater than 1, we may assume that $n_2 \neq p$ for $p$ a prime either. Consequently, if $G$ has $p$-rank at least 3 and $n_3 = a$, then $n_2 = ab$ and $n_1 = abc$ where if $a$ is prime, then $b \neq 1$. Thus we would have $|G| = a^3b^2c \geq 64$ unless $a = 2$, $b = 2$ and $c = 1$. However, in this case the invariant factors are 4, 4, and 2 and we classified all such epimorphisms in Example 15. Thus we may assume $G$ has $p$-rank 2.

For $p$-rank 2, Lemma 17 implies we may assume there exists a prime $p$ such that $p^2 | n_2$ (and consequently $p^2 | n_1$). If $p \geq 3$, then $|G| \geq 3^3$ and so the genus of the resulting surface will be at least 82, so we may assume $p = 2$. This means $n_1 = 4k$ for some integer $k$. If $k > 3$, the resulting surface will have genus greater than 65, so we only need $n_1 = n_2 = 4$, $n_1 = 8$ and $n_2 = 4$, and $n_1 = 12$ and $n_2 = 4$. However, in all these cases, there is a unique value $N$ for which $(N, n_2) > 1$, so using the cohomology arguments of Lemma 10 and the general form given in Proposition 14, it is easy to check that there are just three classes:

$$
\begin{align*}
\eta_1 &: (\alpha_1, \beta_1, \alpha_2, \beta_2) \rightarrow (\omega_1, \omega_2, 0, 0) \\
\eta_2 &: (\alpha_1, \beta_1, \alpha_2, \beta_2) \rightarrow (\omega_1, 2\omega_2, \omega_2, 0) \\
\eta_3 &: (\alpha_1, \beta_1, \alpha_2, \beta_2) \rightarrow (\omega_1, 0, \omega_2, 0)
\end{align*}
$$

4 The totally ramified case

Assume that $G = \mathbb{F}_p^r$ and that our action is purely ramified, namely $S/G$ has genus 0, and that there are $r$ branch points. We are going to describe a method for computing the number of equivalence classes of actions of $G$. In each case there will be several small exceptional primes and a finite number of infinite families for which the number of actions is expressed as a polynomial in the prime. The infinite families are defined by the existence or non-existence of certain roots of unity and so each family is typically the set of primes in an arithmetic progression. Unfortunately, our method depends on the enumeration of the finite subgroups of the symmetric group on $r$ points, so exact general results are impossible for all $p$-ranks and numbers of branch points. Thus, we will describe the general method, but only derive the precise details for 3 and 4 branch points (Table 5.1).

We may associate a generating set $\{C_j : 1 \leq j \leq r\}$ with a matrix $X$

$$
\{C_j : 1 \leq j \leq r\} \leftrightarrow \begin{bmatrix} X_1 & X_2 & \cdots & X_r \end{bmatrix} = X
$$

where

$$
X \text{ is a } v \times r \text{ matrix of rank } v \quad (16)
$$

such that

$$
X_1 + X_2 + \cdots + X_r = 0. \quad (17)
$$
Note that $1 \leq v < r$ and $r \geq 2$. Let us explicitly define the action as we shall need it in our calculations. Let $g \in GL(v, \mathbb{F}_p)$ be a $v \times v$ invertible matrix over $\mathbb{F}_p$, and $\alpha \in \Sigma_r$. Let $\pi_\alpha$ be the standard permutation matrix defined by

$$
\pi_\alpha = \begin{bmatrix}
E_\alpha(1) \\
E_\alpha(2) \\
\vdots \\
E_\alpha(r)
\end{bmatrix}
$$

where $E_1, \ldots, E_n$ are the rows of the identity matrix. Then $\pi_{\alpha\beta} = \pi_\alpha \pi_\beta$ and the action $\text{Aut}(G) \times \text{Aut}(\Gamma)$ on the set of matrices $X$ is the action of $GL(v, \mathbb{F}_p) \times \Sigma_r$ given by

$$(g, \alpha) \cdot X = gX\pi_\alpha^T.$$

i.e.,

$$gX\pi_\alpha^T = \begin{bmatrix} gX_{\alpha(1)} & gX_{\alpha(2)} & \cdots & gX_{\alpha(r)} \end{bmatrix}$$

The problem of computing the number of equivalence classes is the same as computing the number of orbits under the given action. Before we start setting up the machinery for the computation of the orbits, let us prove a trivial case for all $p$-ranks.

**Proposition 19** Suppose that the number of branch points is one more than the $p$-rank of $G$. Then there is a unique action of $G$.

**Proof.** Let $g = \begin{bmatrix} X_1 & X_2 & \cdots & X_v \end{bmatrix}$ be the square matrix obtained by dropping the last column of $X$. Since $X_1 + X_2 + \cdots + X_r = 0$ then the column space of $g$ is the same as the column space of $X$. It follows that $g$ is full rank and hence invertible. Then $g^{-1}X = \begin{bmatrix} I_v & -E_v \end{bmatrix}$ where $I_v$ is the $v \times v$ identity matrix and $E_v$ a column of ones. \[\blacksquare\]

### 4.1 Orbit spaces via equisymmetry

Let $\Omega$ be a finite set upon which the finite group $G$ (different from our $G$ above) acts. Consider the following standard constructs. Given a subset $Y \subseteq \Omega$ or $H \subseteq G$ let

$$G_Y = \{ g \in G : gx = x, \forall x \in Y \} \quad \text{and} \quad G_x = G_{\{x\}}$$

and

$$\Omega^H = \{ x \in \Omega : gx = x, \forall g \in H \} \quad \text{and} \quad \Omega^g = \Omega_{\{g\}}$$

For $H \subseteq G$ let $\overline{H}$ denote the action closure of $H$, the largest group fixing all the points fixed by all elements of $H$,

$$\overline{H} = G_{\Omega^H} = \bigcap_{x \in \Omega^H} G_x.$$
Also \( Y \) denotes the \textit{action closure} of a subset \( Y \subseteq \Omega \), the set of all points fixed by all elements of \( G \) fixing all of \( Y \),

\[
Y = \Omega^G_Y = \bigcap_{g \in G_Y} \Omega^g.
\]

Next for a subgroup \( H \subseteq G \) let \( \langle H \rangle \) denote the conjugacy class of subgroups of \( G \) determined by \( H \). The set of conjugacy classes has a partial order \( \langle H_1 \rangle \prec \langle H_2 \rangle \) if and only if \( H_1 \subseteq gH_2g^{-1} \) for some \( g \in G \). We make the following additional definitions.

\textbf{Definition 20} Two points \( x, y \in \Omega \) are called \textit{equi-isotropic} if \( G_x = G_y \) and \textit{equisymmetric} if \( G_x \) and \( G_y \) are conjugate subgroups. The group \( G_x \) is called the \textit{isotropy type} of \( x \) and \( \langle G_x \rangle \) is called the \textit{symmetry type} of \( x \). If \( H = G_x \) for some \( x \) then \( \hat{\Omega}^H \), the \( H \)-isotropic stratum or the isotropic stratum of \( x \) is the set of all points with isotropy type \( H \).

\[
\hat{\Omega}^H = \{ y \in \Omega : G_y = H \}
\]

Likewise if \( H = G_x \) for some \( x \) then \( \hat{\langle \Omega \rangle}^H \), the \( H \)-equisymmetric stratum or the equisymmetric stratum of \( x \), is the set of all points with symmetry type \( \langle H \rangle \).

\[
\hat{\langle \Omega \rangle}^H = \{ y \in \Omega : G_y \in \langle H \rangle \}
\]

The action closure \( \Omega^H \) of \( \hat{\langle \Omega \rangle}^H \) is the set of points with \( H \)-isotropy or greater, i.e., \( \Omega^H = \{ x \in \Omega : H \subseteq G_x \} \). It is easily seen that

\[
\Omega^H = \hat{\Omega}^H \cup \bigcup_K \hat{\Omega}^K
\]

(19)

a disjoint union for a set of closed groups \( K \) strictly containing \( K \supset H \). We can actually take the union over all subgroups \( K \supset H \) fixing any point, though some of the \( \hat{\Omega}^K \) may be empty. We get a similar union

\[
\langle \Omega^H \rangle = \langle \hat{\Omega}^H \rangle \cup \bigcup_K \langle \hat{\Omega}^K \rangle,
\]

(20)

for a set of closed point stabilizers \( K \) such that \( \langle K \rangle \supset \langle H \rangle \). The equations (19) and (20) may be rewritten to iteratively compute

\[
|\Omega^H| = |\Omega^H| - \sum_K |\hat{\Omega}^K|
\]

(21)

\[
|\langle \Omega^H \rangle| = |\langle \Omega^H \rangle| - \sum_K |\langle \hat{\Omega}^K \rangle|
\]

(22)

where the sums are over the same set of subgroups as above.

21
If \(x\) and \(y\) have the same symmetry type, i.e., \(G_x\) and \(G_y\) are conjugate, then the orbits \(Gx\) and \(Gy\) have the same size \(|G|/|G_x|\). Since \(G_gx = gG_xg^{-1}\), then \(g\hat{\Omega}^H = \hat{\Omega}^{gHg^{-1}}\) for a point stabilizer \(H\) and so the set \(\hat{\Omega}^H\) is \(G\)-invariant. Indeed we have a disjoint union

\[
\langle \hat{\Omega}^H \rangle = \bigcup_{K \in \langle H \rangle} \hat{\Omega}^K = \bigcup_{g \in G/N_G(H)} g\hat{\Omega}^H
\]

from which we get \(|\langle \hat{\Omega}^H \rangle| = |\langle H \rangle| \cdot |\hat{\Omega}^H|\). Since \(\langle \hat{\Omega}^H \rangle \to \langle \hat{\Omega}^H \rangle /G\) is a \(|G|/|H|\) to 1 map

\[
|\langle \hat{\Omega}^H \rangle /G| = \frac{|H|}{|G|} \cdot |\langle \hat{\Omega}^H \rangle| = \frac{|H|}{|N_G(H)|} \cdot |\hat{\Omega}^H|\]

(23)

and it follows that

\[
|\Omega /G| = \sum_H |\langle \hat{\Omega}^H \rangle /G| = \sum_H \frac{|H|}{|N_G(H)|} \cdot |\hat{\Omega}^H|,
\]

(24)

where \(\langle H \rangle\) runs over all conjugacy classes of point stabilizers. So computing \(|\Omega /G|\) is just a matter of determining the point stabilizers \(H\) and the equisymmetric or the equi-isotropic strata. The cardinalities of the closed subsets \(\Omega^H\) are generally easier to calculate directly which is why we use the above formulas.

### 4.2 Matrix formulation of Möbius inversion

The iterative calculation of \(|\Omega /G|\) via the formulas (24) and (24), which amounts to Möbius inversion, can be put into a very succinct matrix format. Consider the set of subgroups \(H \subseteq G\) such that \(\Omega^H\) is non-empty. For each conjugacy class of subgroups in this set select a representative and then list the subgroups \(\langle id \rangle = H_1, H_2, \ldots, H_s\), in such a way that \(\langle H_i \rangle \subseteq \langle H_j \rangle \implies i \leq j\). This can always be done, though generally not in a unique way. From equation (19) we have

\[
|\Omega^{H_i}| = \langle \hat{\Omega}^{H_i} \rangle + \sum_{K \supset H_i} |\hat{\Omega}^K|
\]

(25)

\[
= \langle \hat{\Omega}^{H_i} \rangle + \sum_{j > i} \left( \sum_{K \in \langle H_j \rangle, H_i \subseteq K} |\hat{\Omega}^K| \right)
\]

For \(H, K \subseteq G\), let

\[
U(H, K) = \{gKg^{-1} : g \in G, H \subseteq gKg^{-1}\}
\]

(26)

\[
D(H, K) = \{g^{-1}Hg : g \in G, g^{-1}Hg \subseteq K\}
\]

(27)

and set \(u_{i,j} = |U(H_i, H_j)|, \, d_{i,j} = |D(H_i, H_j)|\). Observe that \(u_{i,i} = d_{i,i} = 1\) and \(u_{i,j} = d_{i,j} = 0\) if \(i > j\) and from Lemma 24 below we have

\[
u_{i,j} = d_{i,j} \frac{|\langle H_j \rangle|}{|\langle H_i \rangle|}.
\]

(28)
Then equation (25) can be rewritten
\[
\Omega^H_i = |\Omega^H_i| + \sum_{i<j} u_{i,j} |\Omega^H_j| \tag{29}
\]
and
\[
\langle \Omega^H_i \rangle = |\Omega^H_i| \langle H_i \rangle \tag{30}
\]
Now let us convert to matrix notation. Define
\[
U = [u_{i,j}], \quad D = [d_{i,j}]
\]
\[
S = \text{diag}(|H_1|, |H_2|, \ldots, |H_s|)
\]
\[
N = \text{diag}(|N_G(H_1)|, |N_G(H_2)|, \ldots, |N_G(H_s)|)
\]
\[
T = \text{diag}(|H_1|, |H_2|, \ldots, |H_s|).
\]
The matrices \(U = [u_{i,j}]\) and \(D = [d_{i,j}]\) are upper triangular with 1’s on the diagonal. It follows from equation 28 that \(U = S^{-1} DS\). Let \(L, L^o, E^o, O^o\) be the vectors
\[
L = \begin{bmatrix}
|\Omega^H_1| & |\Omega^H_2| & \cdots & |\Omega^H_s|
\end{bmatrix}^T
\]
\[
L^o = \begin{bmatrix}
|\Omega^H_1| & |\Omega^H_2| & \cdots & |\Omega^H_s|
\end{bmatrix}^T
\]
\[
E^o = \begin{bmatrix}
|\langle \Omega^H_1 \rangle| & |\langle \Omega^H_2 \rangle| & \cdots & |\langle \Omega^H_s \rangle|
\end{bmatrix}^T
\]
\[
O^o = \begin{bmatrix}
|\langle \Omega^H_1 \rangle / G| & |\langle \Omega^H_2 \rangle / G| & \cdots & |\langle \Omega^H_s \rangle / G|
\end{bmatrix}^T
\]
The vectors \(E^o\) and \(O^o\) are the cardinalities of the various equisymmetric strata and corresponding orbits, \(|\Omega/G|\) is simply the sum of the entries of \(O^o\). The equations (25) to (30) can be translated into the following matrix equations:
\[
L = U L^o = S^{-1} D S L^o
\]
\[
L^o = U^{-1} L = S^{-1} D^{-1} S L
\]
\[
E^o = S L^o = S S^{-1} D^{-1} S L = D^{-1} S L
\]
\[
O^o = \frac{1}{|G|} T E^o = \frac{1}{|G|} T D^{-1} S L.
\]
Finally we have the matrix equation \(N^{-1} = \frac{1}{|G|} S\) relating normalizers and conjugacy classes resulting in our final equation
\[
O^o = T D^{-1} N^{-1} L. \tag{31}
\]
The following lemma relates the \(u_{i,j}\) and \(d_{i,j}\).
Lemma 21 Let $H \subseteq K$ be subgroups of a finite group $G$. Let $U(H, K)$ and $D(H, K)$ be the sets defined in equations (26) and (27). Then

$$U(H, K) |\langle H \rangle| = |D(H, K)| |\langle K \rangle|.$$  

(32)

**Proof.** Let $P = \{(S, T) \in \langle H \rangle \times \langle K \rangle : S \subseteq T \}$. By considering the projections $P \to \langle H \rangle$, $P \to \langle K \rangle$ we see that $|U(H, K)| |\langle H \rangle| = |P| = |D(H, K)| |\langle K \rangle|$.

**Remark 22** The entries of $D$ can be found by enumerating the subgroups of $H_j$ and then determining to which $H_i$ they are conjugate. In our calculation we will just work with subgroups of small symmetric groups.

4.3 Structure of stabilizers

Let $\Omega$ be the set of matrices $X$ satisfying conditions (16) and (17), and $G = GL(v, \mathbb{F}_p) \times \Sigma_r$ with the action given in equation (18). There are two canonical representations $\rho_1 : G \to GL(v, \mathbb{F}_p)$, $(g, \alpha) \to g$ and $\rho_2 : G \to \Sigma_r \to GL(r, \mathbb{F}_p)$, $(g, \alpha) \to \pi_\alpha$. We are going to describe the structure of stabilizers $H \subseteq G$ in terms of the two representations. Let $X_0 \in \Omega$ and suppose that $H$ is a subgroup fixing $X_0$, (the subgroup need not be action closed). If $X$ is any matrix stabilized by $H$ then $\rho_1(h)X\rho_2(h) = X$, $h \in H$ or

$$\rho_1(h)X = X \rho_2(h), h \in H,$$  

(33)

i.e., $X \in \text{Hom}_H(\mathbb{F}_p^v, \mathbb{F}_p^r)$ is an intertwining operator. Now, the restricted map $p_2 : H \to \Sigma_r$, is 1-1, for if $(g, \alpha) \in \ker(p_2)$ then $\alpha = id$ and $X_0 = gX_0$. Since $X_0$ has rank $v$ then $g = I$. Let $H' = p_2(H)$ denote the isomorphic image of $H$ in $\Sigma_r$. Reversing the process, we can construct candidate stabilizers by selecting a subgroup $H' \subseteq \Sigma_r$, and a suitable representation $q = \rho_1 \circ p_2^{-1} : H' \to GL(v, \mathbb{F}_p)$ and then construct $H$ by

$$H = \{(q(\alpha), \alpha) : \alpha \in H'\}$$  

(34)

It is not clear that an arbitrary $H$ so constructed fixes any matrices in $\Omega$, in fact the representation $q$ must satisfy certain properties which we proceed to discuss.

For the remainder of our discussion we are going to assume that $p \nmid |H|$, so that we may use the theory of reducible representations to aid our computations. Since $|H|$ divides $r!$ we are only excluding the small primes $p \leq r$. Both $H$-modules $\mathbb{F}_p^v$, $\mathbb{F}_p^r$ can be rewritten as a direct sum of $\mathbb{F}_p$-irreducibles, $\mathbb{F}_p^v = V_1 \oplus V_2 \oplus \cdots \oplus V_m$, $\mathbb{F}_p^r = W_1 \oplus W_2 \oplus \cdots \oplus W_n$. By a change in coordinates, we may
and hence we may consider $Y$ acting on the space $W$ in the algebraic closure of $Y$. Then $Y$ then $Y$.

Definition 23

For a subgroup $H$ of $\Sigma_r$ acting naturally on the space $F_p^*$, write

$$F_p^* = W \oplus \left( \begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix}^T \right)$$

as a direct sum of subrepresentations. If $\pi$ denotes the natural representation of $H'$, we call $\pi_W$, the natural representation restricted to $W$, the reduced natural representation of $H'$ and denote it by $\overline{\pi}$. If $p \nmid r$ then may take $W = \{(x_1, \ldots, x_r) \in F_p^r : x_1 + \cdots + x_r = 0 \}$.

Remark 24

Though a bit pedantic, we remind ourselves on how to construct $A$ and $B$ since the details will be used in a later proof. The matrix $A$ may be constructed by choosing column vectors that span $V_1$ then adding vectors that span $V_2$ and so on to get $A = [A_1 A_2 \cdots A_m]$, where $A_i$ is a $v \times v_i$ matrix. Construct $B = [B_1 B_2 \cdots B_n]$, similarly. Now write

$$A^{-1} = \left[ (A_1^*)^T (A_2^*)^T \cdots (A_m^*)^T \right]^T$$

as a column of matrices with $A_i^*$ of
size $v_i \times v$. Define the $B_j^*$ similarly. We have the following transformation formula

$$Y_{i,j} = A_i^* XB_j, \quad X = \sum_{i,j} A_{i,j} B_j^* \quad (38)$$

The next definition helps us characterize the representations we are looking for in the defining equation (34).

**Definition 25** Let $H$ be a finite group and $k$ a field whose characteristic is coprime to $|H|$. Let $\rho_1, \rho_2$ be two representations of finite degree on $k$ vector spaces $V, W$. We say that $\rho_2$ dominates $\rho_1$ ($\rho_1 \preceq \rho_2$) if and only if for each $k$-irreducible representation $\phi$ of $H$, $\langle \phi, \rho_1 \rangle \leq \langle \phi, \rho_2 \rangle$ where $\langle \phi, \rho \rangle$ is the number of times the representation $\phi$ is contained in a representation $\rho$ of $H$.

**Proposition 26** Let $H, k$ and $\rho_1, \rho_2$ be as in Definition 25. Then $\rho_1 \preceq \rho_2$ if and only if the space of intertwining operators $\text{Hom}_H(W, V)$ contains a surjective map. Similarly $\rho_2 \preceq \rho_1$ if and only if $\text{Hom}_H(W, V)$ contains an injective map.

When restated in terms $H$-modules over $\mathbb{F}_p$ the proposition is self-evident. In proof of Proposition 28, which characterizes the representations defining subgroups that fix a point, we will see that the dominance condition $\rho_1 \preceq \rho_2$ is equivalent to the existence of an intertwining operator $X$ with linearly independent rows.

**Remark 27** Suppose the action of $H$ has several orbits on $\{1, 2, \ldots, r\}$ and that by selecting a suitable conjugate of $H'$ each orbit is an interval of $t$ integers $\{s, s+1, \ldots, s+t-1\}$. We may write

$$X = \begin{bmatrix} Z_1 & Z_2 & \cdots & Z_l \end{bmatrix} \quad (39)$$

as a block matrix where each block is defined by an orbit and hence invariant under $H'$. Then $\rho_1$ must satisfy

$$\rho_1(h)Z_k = Z_k \rho_{2,k}(h), k = 1, \ldots, l \quad (40)$$

where $\rho_{2,k}(h)$ is the induced permutation matrix on the orbit defining $Z_k$. We use these representations to characterize when a pair $H' \subseteq \Sigma_r, q : H' \to GL(v, \mathbb{F}_p)$ fixes a point in $\Omega$.

**Proposition 28** Assume that $v > 1$ if $p = 2$. Let $H' \subseteq \Sigma_r$ such that $p \nmid |H|$, let $q : H' \to GL(v, \mathbb{F}_p)$ be an arbitrary representation, and let $H = \{(q(\alpha), \alpha) : \alpha \in H'\}$. Let $\rho, \overline{\rho}$ be the standard and reduced representations of $H'$ afforded by $H' \subseteq \Sigma_r$, and $\rho_i = \rho_{2,i} \circ \rho_2^{-1}$ be the representations of $H'$ determined by the orbits of $H'$ on $\{1, 2, \ldots, r\}$. Then $H$ fixes a point in $\Omega$ if and only if all the following conditions hold.

- $q$ is dominated by $\overline{\rho}$, and
- $q$ and $\rho_k$ have a common irreducible for $k = 1, \ldots, l$.  

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Proof. By taking a conjugate of $H'$ we may assume that the orbits occur in intervals as Remark 24. Let $\rho_1$ and $\rho_2$, $\rho_{2,k}$, $Z_k$ be as defined previously so that $\rho_1 = q \circ p_2$, and, $\rho_2 = \rho \circ p_2$, $\rho_{2,k} = \rho_i \circ p_2$, $\rho_{2,i} = \rho_i \circ p_2$. To show that $H$ fixes an element of $\Omega$ we need to find a $v \times r$ matrix $X$ such that

- $\rho_1(h)X = X\rho_2(h), h \in H$,
- $X_1 + X_2 + \cdots + X_r = 0$,
- $X$ is surjective, i.e., has linearly independent rows,
- each column $X_i$ is non-zero.

We leave it to the reader to show that the conditions are necessary, we show how to construct an $X$ if the conditions hold. Our proof depends on carefully setting up the diagonalizing matrices $A$ and $B$, and then carefully selecting the $Y_{i,j}$. Because $(g, \alpha)$ in $H$ acts on a column $X_j$ by $X_j \rightarrow gX_{\alpha(j)}$, then a $Z_k$ defined by an $H'$ orbit will be non-zero if and only if all the columns of the given $Z_k$ are non-zero. Thus we merely need to construct $X$ satisfying the first three bullets and all $Z_i$ non-zero. The diagonalizing matrix $A$ may be constructed according to any decomposition, we need more care with $B$. Decompose $\rho_{2,1}$ into $H$-irreducibles and then place the corresponding basis vectors into $B$ as columns as described in Remark 27. Next decompose $\rho_{2,2}$ and add the basis vectors as columns, and continue on to the last orbit. Each orbit determines a unique trivial subrepresentation of the $\rho_{2,k}$. A spanning vector for this subrepresentation is the vector with 1's in the locations corresponding to the orbit and zeros elsewhere. By construction, for each of these vectors, $B$ contains a column which is a scalar multiple of this vector. We assume that the scalar is 1 and that the last column of $B$ is the vector corresponding to the orbit defining $\rho_{2,1}$. Now the sum of these columns is $[1 \ 1 \ \cdots \ 1]^T$. If we replace the last column of $B$ with the vector $[1 \ 1 \ \cdots \ 1]^T$ we obtain a matrix $C$ with the same column span as $B$ and hence $C$ is invertible. Moreover as we exchanged one invariant vector for another $B^{-1}\rho_2(h)B = C^{-1}\rho_2(h)C$. Thus we can assume that $B$ has $B_n = [1 \ 1 \ \cdots \ 1]^T$ as its last column.

Now we are going to further modify $A$ and $B$ as follows. We assume that we have selected our representations $\phi_i$ and $\psi_j$ so that if any two are equivalent then they are equal. This can be achieved by modifying the columns of $A$ or $B$ corresponding to the $H$-invariant subspace corresponding to a $\phi_i$ or $\psi_j$. Note that we do not need to alter the last column of $B$. In this circumstance we define the matrix components of $Y$ in equation (36)

$$Y_{i,j} = y_{i,j}I_v, \text{ if } \phi_i = \psi_j \quad (41)$$

$$Y_{i,j} = 0, \text{ otherwise.}$$

By equation (37) the above equations always define an intertwining operator constructed via equation (58). Now $X_1 + X_2 + \cdots + X_r = XB_n$, where $B_n = [1 \ 1 \ \cdots \ 1]^T$. But $XB_n$ equals the last column of $XB = AY$. It follows
then that $X_1 + X_2 + \cdots + X_r = 0$ if and only if the last column of $Y$ is zero, i.e., $Y_{i,n} = 0$ for all $i$. Next we modify $Y$ so that $X$ is surjective. From the hypotheses $\overline{p_2}$ dominates $\rho_1$, and hence for each irreducible $\phi_i$ there can be chosen a $\psi_j(i)$ with $\phi_i = \psi_j(i), j(i) < n$, and such that for distinct $i_1, i_2$ we have $j(i_1) \neq j(i_2)$. If we select $y_{i,j(i)} \neq 0$ and all other $y_{i,j} = 0$ then the resulting matrix $Y$, and consequently $X$, has linearly independent rows. (This argument shows surjectivity in Proposition 20.)

Finally, we show that the $Y_{i,j}$ can be chosen so that each $Z_k$ in equation (39) is non-zero. By equation (39) $Y = A^{-1}XB = \begin{bmatrix} A^{-1}Z_1B & A^{-1}Z_2B & \cdots & A^{-1}Z_kB \end{bmatrix}$ so if some $Z_k = 0$ then $Y$ has a block column equal to zero and by construction there is a set $J_0$ of $j$’s such that $Y_{i,j} = 0, 1 \leq i \leq m, j \in J_0$. The $J_0$ consists of all $j$ such that $\psi_j$ comes from the subspace determined by the orbit corresponding to $Z_k$. By assumption $\rho_1$ and $\rho_2,k$ have a common irreducible. Then we can set some $Y_{i,j} = y_{i,j}I_n$, for $i$ and a $j \in J_0$, and any $y_{i,j} \neq 0$. The modified $Y$ will still define an intertwining operator, and $Y$ will remain full rank since we are modifying columns that were initially zero. Thus we can guarantee that all $Z_k$ are non-zero, and still leave $X$ a surjective intertwining operator. The potential problem is if we were forced to change something in the last column of $Y$. This can only happen if the trivial representation is the only common irreducible between $\rho_2,l$ and $\overline{p_2}$. Now suppose that there is another $\rho_2,v$ such that $\overline{p_2}$ and $\rho_2,v$ have a common non-trivial irreducible. Then by taking a conjugate of $H'$ we can switch $\rho_2,l$ and $\rho_2,v$ so that the $Y_{i,j}$ can be adjusted without affecting the last column of $Y$. Now suppose that even switching in not possible. Then it follows that $q$ is trivial and that the number of orbits is greater than than $v$. Define $X$ as follows.

$$X = \begin{bmatrix} E_{r_1} & 0 & \cdots & 0 & r_1y_1E_{r_v+1} & \cdots & r_1y_{v-1}E_{r_1} \\ 0 & E_{r_2} & \cdots & 0 & r_2y_1E_{r_v+1} & \cdots & r_2y_{v-1}E_{r_1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & E_{r_v} & r_vy_1E_{r_v+1} & \cdots & r_vy_{v-1}E_{r_1} \end{bmatrix}$$

where $E_{n}$ is a row matrix of 1’s, $r_s$ is the size of the $s$’th orbit and

$$y_1r_{v+1} + \cdots + y_{v-1}r_1 = -1, \ y_i \neq 0, \ r_i \neq 0. \quad (42)$$

Assuming that the $y_j$’s satisfy the given constraints and at least one of the $r_i$’s is non-zero then $X$ meets all the requirements. Since $p$ is coprime to $|H|$, none of the $r_i$’s are zero. Set $s = t - v > 0$ and assume that $p \neq 2$. According to Proposition 20. in the next section, only $\frac{(p-1)^s-(-1)^s}{p}$ of the $(p - 1)^s$ selections of $y_i$ sum to zero. Thus at least one of the sums in equation (42) is non-zero and it may be scaled to equal $-1$. If $p = 2$ and $v > 1$ then choose $X$ of the following form

$$X = \begin{bmatrix} E_{r_1} & yE_{r_1} & \cdots & 0 & E_{r_v+1} & E_{r_v+2} & \cdots & E_{r_l} \\ 0 & E_{r_1} & \cdots & 0 & E_{r_v+1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & E_{r_v} & E_{r_v+1} & 0 & \cdots & 0 \end{bmatrix}$$

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Since all the \( r_i \) are odd then \( y \) may be chosen so that all row sums are zero and that no column is zero. ■

### 4.4 Normalizers and conjugates

Let us describe how to compute normalizers of subgroups fixing a point. Suppose that \((g, \alpha)\) normalizes \(H\). Then for \((q(\beta), \beta) \in H\), \((g, \alpha)(q(\beta), \beta)(g^{-1}, \alpha^{-1}) = (gq(\beta)g^{-1}, \alpha\beta\alpha^{-1}) = (q(\alpha\beta\alpha^{-1}), \alpha\beta\alpha^{-1}) \in H\) so we must have

\[
\alpha\beta\alpha^{-1} \in H' \\
gq(\beta)g^{-1} = q(\alpha\beta\alpha^{-1})
\]

(43)

Thus we must already have \( \alpha \in N_{\Sigma_n}(H') = N' \). The normalizer \( N' \) permutes the representations of \( H' \) by the formula

\[
q^\alpha(\beta) = q(\alpha\beta\alpha^{-1}),
\]

so there is \((g, \alpha)\) normalizing \(H\) if and only if \(q\) and \(q^\alpha\) are equivalent over \(\mathbb{F}_p\). In terms of characters, \(\alpha\) must fix the character of \(q\). Let \(N''\) be the subgroup satisfying \(H' \subseteq N'' \subseteq N'\) and fixing the character of \(q\). If \((g_1, \alpha), (g_2, \alpha)\) both belong to the normalizer of \(H\) then equation (43) implies \(g_1q(\beta)g_1^{-1} = q(\alpha\beta\alpha^{-1}) = g_2q(\beta)g_2^{-1}\) or \(q(\beta) = (g_1^{-1}g_2)(g_1^{-1}g_2)^{-1}\) for all \(\beta \in H'\). Thus, \(h = g_1^{-1}g_2 \in Z = Z_{GL(v, \mathbb{F}_p)}(q(H'))\). It follows that the normalizer of \(H\) contains \(Z\) as a subgroup with quotient \(N''\) and so the normalizer has size \(|Z| |N''|\).

Two subgroups fixing a point \(H_1, H_2\) are conjugate if and only \(H_2 = \alpha H_2 \alpha^{-1}\) and the two representations \(q'\) and \(q\) satisfy

\[
q'(\alpha\beta\alpha^{-1}) = gq(\beta)g^{-1}
\]

for some \(g \in GL(v, \mathbb{F}_p)\).

### 4.5 Fixed point subsets and strata

There are three ways we shall consider to calculate the size of fixed point subsets. We illustrate the first two by example with \(v = 2, r = 4, H' = ((1, 2)(3, 4))\) and \(q((1, 2)(3, 4)) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\). The first method is by brute force and can easily be implemented by computer.

**Example 29** A typical \(X = \begin{bmatrix} x & x & x & x \\ y_1 & y_2 & y_3 & y_4 \end{bmatrix}\) must satisfy the equations

\[
x_1 + x_2 + x_3 + x_4 = y_1 + y_2 + y_3 + y_4 = 0 \\
x_1 - x_2 = x_3 + x_4 = y_1 - y_2 = y_3 + y_4 = 0
\]

and so \(X = \begin{bmatrix} x & x & -x & -x \\ y & -y & z & -z \end{bmatrix}\). Clearly \(x \neq 0\) and at least one of the six \(2 \times 2\) minors must be non-zero, i.e., one of \(xz, x(y - z), x(y + z), xy\) must be
It follows that our conditions are $x \neq 0$ and $(y, z) \neq (0, 0)$ and the number of valid $X$’s is $(p - 1)(p^2 - 1)$.

A drawback to this method is that the representation must be constructed. However, we note that it works if $p$ divides the order of a stabilizer. The second method uses the “diagonalized” form of the intertwining operator $Y = A^{-1}XB$, and can usually be determined by inspection of the characters of $q$ and $\overline{\rho}$.

Example 30 Observe that the character of $q$ is $\chi_0 + \chi_1$ and $\overline{\rho} = \chi_0 + 2\chi_1$, where $\chi_0$ and $\chi_1$ are the trivial and non-trivial characters of $H'$ respectively. The matrices $A, B,$ and $Y$ can be chosen as:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} x & 0 & 0 & 0 \\ 0 & y & z & 0 \end{bmatrix}.$$ 

We need $x \neq 0$ and $(y, z) \neq (0, 0)$ giving the same result as Example 29. Note that we don’t need to know $A$ and $B$ explicitly to determine $Y$, just the characters.

The two previous methods are useful when the fixed point set has small dimension and the non-vanishing polynomials are simple. At the other extreme we need to calculate the fixed point set for the trivial point stabilizer. As a starting point we need to calculate the size of $\Omega$. We can calculate the number of vectors by inclusion-exclusion followed by a specialized Möbius inversion. Let $V$ be an arbitrary vector space of dimension $v$ over $\mathbb{F}_p$ and let

$$\overline{\Omega} (V) = \{ X = (X_1, \ldots, X_r) \in V^r : X_i \neq 0, i = 1, \ldots, r, X_1 + X_2 + \cdots + X_r = 0 \}$$

$$\Omega^r(V) = \{ X \in \overline{\Omega} (V) : V = \langle X_1, \ldots, X_r \rangle \}$$

and set $\overline{\Omega}^{v,r} = \overline{\Omega} (\mathbb{F}_p^v)$, $\Omega^{v,r} = \Omega^r (\mathbb{F}_p^v)$. Clearly the cardinalities of the sets depend only on $r$ and $v = \dim V$.

Proposition 31 Let $\overline{\Omega} (V)$ and $\Omega^r(V)$ be defined as above and set

$$\overline{\omega}(v, r) = \left| \overline{\Omega} (V) \right|, \quad \omega(v, r) = |\Omega^r(V)|.$$ 

Then,

$$\overline{\omega}(v, r) = \frac{(p^v - 1)^r - (-1)^r}{p^v} + (-1)^r \quad (44)$$

$$\omega(1, r) = \overline{\omega}(1, r)$$

$$\omega(v, r) = \overline{\omega}(v, r) - \sum_{l=1}^{v-1} n_l \omega(l, r)$$
where
\[ n_{v,l} = \frac{\prod_{j = l+1}^{v} (p^j - 1)}{\prod_{j=1}^{v-1} (p^j - 1)} \]

is the number of subspaces \( L \) of dimension \( l \) in \( V \).

**Proof.**

Let
\[ \Omega_0 = \{ (X_1, \ldots, X_r) \in V^r : X_1 + X_2 + \cdots + X_r = 0 \} \]
\[ \Omega_i = \{ (X_1, \ldots, X_r) \in V^r : X_i = 0, X_1 + X_2 + \cdots + X_r = 0 \}, i = 1, \ldots, r \]
\[ \Omega = \{ (X_1, \ldots, X_r) \in V^r : X_i \neq 0, i = 1, \ldots, r, X_1 + X_2 + \cdots + X_r = 0 \} \]

so that \( \Omega^r(V) = \Omega = \Omega_0 - \bigcup_i \Omega_i \). Then, by inclusion-exclusion:
\[ \omega(v, r) = |\Omega| \]
\[ = |\Omega_0| - \sum_i |\Omega_i| + \sum_{i < j} |\Omega_i \cap \Omega_j| - \sum_{i < j < k} |\Omega_i \cap \Omega_j \cap \Omega_k| + \cdots \]
\[ + (-1)^{r-1} |\Omega_i \cap \cdots \cap \Omega_r| \]
\[ = p^{v(r-1)} - \binom{r}{1} p^{v(r-2)} + \binom{r}{2} p^{v(r-3)} - \binom{r}{3} p^{v(r-3)} + \cdots \]
\[ + (-1)^{r-1} \binom{r}{r-1} + (-1)^r \]
\[ = \frac{(p^v - 1)^r - (-1)^r}{p^v} + (-1)^r \]

Next let \( L \) denote an arbitrary proper subspace of \( V \) of dimension \( l \). Then
\[ \Omega^r(L) = \Omega^r(V) \cup \bigcup_{0 \subset L \subset V} \Omega^r(L) \]

It follows that
\[ \omega(v, r) = |\Omega^r(L)| = |\Omega^r(V)| + \sum_{l=1}^{v-1} \sum_{\dim L = l} \Omega^r(L) \]
\[ = |\Omega^r(F_p^v)| + \sum_{l=1}^{v-1} n_{v,l} |\Omega^r(F_p^l)| \]
\[ = \omega(v, r) + \sum_{l=1}^{v-1} n_{v,l} \omega(l, r) \]

The remaining formulas in equations (44) follow immediately. Finally, by noting that \( |GL_v(F_p)| = \prod_{j=1}^{v} (p^j - 1) p^{(n^2 - n)/2} \), by a homogeneous space argument,
letting $GL_v(F_p)$ act on the subspaces, we have

$$n_{v,1} = \frac{|GL_v(F_p)|}{|GL_1(F_p)||GL_{v-1}(F_p)|p^{(v-1)}} = \frac{\prod_{j=l+1}^{v} (p^j - 1)}{\prod_{j=1}^{v-l} (p^j - 1)}.$$

Example 32 For later work we will need the following values $|\Omega^r(F_p^v)|$.

| $v$ | $r$ | $|\Omega^r(F_p^v)|$ |
|-----|-----|------------------|
| 1   | 2   | $p - 1$         |
| 1   | 3   | $(p - 1) (p - 2)$ |
| 1   | 4   | $(p - 1) (p^2 - 3p + 3)$ |
| 1   | 5   | $(p - 1) (p^2 - 4p^2 + 6p - 4)$ |
| 2   | 3   | $p (p - 1) (p^2 - 1)$ |
| 2   | 4   | $p (p - 1) (p^2 - 1) (p^2 + p - 3)$ |
| 2   | 5   | $p (p - 1) (p^2 - 1) (p^2 + p - 3 - 6p + 6)$ |
| 3   | 4   | $p^2 (p - 1) (p^2 - 1) (p^3 - 1)$ |

Note that, since $GL_v(F_p)$ acts freely by left multiplication on $\Omega^r(F_p^v)$ then $|\Omega^r(F_p^v)|$ is always evenly divided by $|GL_v(F_p)|$. The $|\Omega^r(F_p^v)|$ have been written to show this fact (some can be further factored).

Example 33 Assume that $q : H' \to GL_v(F_p)$ is trivial. Then $H$ orbits determine a partition of $r$, $r = r_1 + r_2 + \cdots + r_s$ with $1 \leq r_1 \leq r_2 \leq \cdots \leq r_s$. Such a partition of $r$ determines $(r_1, \ldots, r_s)$ partitions of $\{1, \ldots, r\}$, with the canonical one being

$$\{\{1, \ldots, r_1\}, \{r_1 + 1, \ldots, r_1 + r_2\}, \ldots, \{r_1 + \cdots + r_{s-1} + 1, \ldots, r\}\}$$

A typical element of the stratum is obtained by selecting distinct non-zero $X_1, \ldots, X_s$ satisfying

$$r_1 X_1 + \cdots + r_s X_s = 0$$

$$\text{rank} \left[ \begin{array}{c} X_1 \\ \vdots \\ X_s \end{array} \right] = v$$

and then taking in order $r_1$ of $X_1, r_2$ of $X_2, \ldots, r_s$ of $X_s$. To count the points fixed by $H$ we need the following generalization

Proposition 34 Suppose that $a_1, \ldots, a_s$ are non-zero scalars in $F_p$. Then the cardinality of the set

$$\left\{ X = \left[ \begin{array}{c} X_1 \\ \vdots \\ X_s \end{array} \right] \in F_p^{\nu s} : X_i \neq 0, i = 1, \ldots, s, \quad a_1 X_1 + \cdots + a_s X_s = 0, \text{ rank} X = v \right\}$$

is $|\Omega^{\nu s}|$.

Proof. The map $\left[ \begin{array}{c} X_1 \\ \vdots \\ X_s \end{array} \right] \leftrightarrow \left[ \begin{array}{c} a_1 X_1 \\ \vdots \\ a_s X_s \end{array} \right]$ is a bijection between the given set and $\Omega^{\nu r}$. ■
4.6 Singular Primes and Galois fusion

The forgoing representation theory depends on certain divisibility properties of primes. The first problem occurs when a prime \( p \) divides the order of \( H' \). Let us call such a prime a singular prime. The singular primes must satisfy \( p \leq r \). The main problem is that complete reducibility of the representations \( q \) and \( \rho \) fails. Moreover there is new twist in that the sum over an orbit may be zero if the size of the orbit is divisible by the prime. In this case one needs to directly compute the representations and then compute the normalizers and fixed point set by brute force.

Another problem that can occur is that \( \mathbb{F}_p \) may not be a splitting field for the representations \( H' \). In trying to describe the representations completely in terms of characters will have to be take characters together in Galois equivalent groups in order that the representations \( q \) be defined over the base field \( \mathbb{F}_p \). We will call this Galois fusion. Furthermore, the formulas for the number of fixed points will have different polynomials in the primes. To help in describing this theory we use the following notation. For a prime \( p \) and an integer \( n \) let \( U_n(\mathbb{F}_p) \) be the set of primitive \( n \)'th roots of 1 in \( \mathbb{F}_p \) and let \( \beta_n(p) = |U_n(\mathbb{F}_p)| \). Again, instead of developing the full theory here we will illustrate the ideas with the sample calculations in the next subsection.

4.7 Rank 2 actions with 4 branch points

Our general approach to compute the number of orbits with given \( p \)-rank and number of branch points is as follows.

1. Determine all subgroups fixing at least one point, one representative for each conjugacy class. List them as \( H_1, H_2, \ldots, H_s \) as described in Subsection 4.2. The sequence of groups will depend upon the prime.

2. Compute \( |\Omega^H| \) and \( |N_G(H)| \) for each subgroup.

3. Compute the matrix \( D = [d_{i,j}] \) for the sequence of groups.

4. Compute \( \left| \langle \Omega^H \rangle / G \right| \) for each subgroup using formula 31 and the add up the results.

We will illustrate the steps by giving complete details for \( p \)-rank 2 and 4 branch points. For Step 1 we use a computer algebra system such as Magma or GAP to do the following:

- Determine a representative \( H' \) of each conjugacy class of \( \Sigma_4 \).
- Determine the character table of \( H' \).
- Decompose the reduced natural representation of \( H' \) into irreducibles over \( \mathbb{C} \). Determine all compatible representations of \( H' \) and how they reduce over \( \mathbb{F}_p \).
Remark 35 For each modular representation of an arbitrary group $G$ defined over $p$ there is complex valued Brauer character defined on the $p$-regular elements of $G$ (order coprime to $p$). In the case $p 
mid |G|$ each Brauer character is an ordinary character and the modular representations of $G$ can be completely described by the irreducible characters. Thus in the discussion below we describe the modular characters by sums of irreducible ordinary characters from the character table of $G$.

Step 1. There are 11 conjugacy classes of subgroups of $\Sigma_4$. Using Magma, the orbits, and the decomposition of the reduced permutation (in terms of the characters of the subgroup), and the degrees of the representation are computed. The column giving the reduced permutation representation is written in terms of the characters of the subgroup using the order in the character table produced by Magma. The list of the degrees is in the order given by the character table. The following notation is used: $\Sigma_k$, $A_k$ denote respectively the symmetric and alternating groups on $k$ points, $C_{k_1,\ldots,k_s}$ the cyclic group with cycle structure $(k_1,\ldots,k_s)$, $D_k$ dihedral group on $k$ points and $V_4$ denotes the Klein 4 group.

| $H'$   | $|H'|$ | orbits             | red perm rep | degrees |
|--------|-------|--------------------|--------------|---------|
| $\Sigma_4^2$ | 1     | $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ | $3\chi_1$   | 1       |
| $\Sigma_2 \times \Sigma_1^3$ | 2     | $\{1,2\}$, $\{3\}$, $\{4\}$ | $2\chi_1 + \chi_2$ | 1,1     |
| $C_2 \times C_2$ | 2     | $\{1,2\}$, $\{3,4\}$ | $\chi_1 + 2\chi_2$ | 1,1     |
| $A_3 \times \Sigma_1$ | 3     | $\{1,2,3\}$, $\{4\}$ | $\chi_1 + \chi_2 + \chi_3$ | 1,1,1   |
| $\Sigma_2 \times \Sigma_2$ | 4     | $\{1,2\}$, $\{3,4\}$ | $\chi_1 + \chi_3 + \chi_4$ | 1,1,1,1 |
| $V_4$ | 4     | $\{1,2,3,4\}$ | $\chi_2 + \chi_3 + \chi_4$ | 1,1,1,1 |
| $C_4$ | 4     | $\{1,2,3,4\}$ | $\chi_2 + \chi_3 + \chi_4$ | 1,1,1,1 |
| $\Sigma_3 \times \Sigma_1$ | 6     | $\{1,2,3\}$, $\{4\}$ | $\chi_1 + \chi_3$ | 1,1,2   |
| $D_4$ | 8     | $\{1,2,3,4\}$ | $\chi_3 + \chi_5$ | 1,1,1,1,2 |
| $A_4$ | 12    | $\{1,2,3,4\}$ | $\chi_4$ | 1,1,1,3  |
| $\Sigma_4$ | 24    | $\{1,2,3,4\}$ | $\chi_4$ | 1,1,2,3,3 |

Next we determine representatives of conjugacy classes of subgroups fixing a point, given in Table 4.2. Using Proposition 38 a stabilizer can be specified by a compatible character if one exists. Note that here we need to take away a one-dimensional character from the reduced representation to form $q$. This automatically eliminates $\Sigma_4$, $A_4$ and $\Sigma_3 \times \Sigma_1$. If there is a fixed point of $H'$ then $q$ must contain the trivial representation $\chi_1$. This eliminates $\Sigma_3 \times \Sigma_1$ and taking $\chi_1$ away from the reduced representation of $A_3 \times \Sigma_1$. Also in the $A_3 \times \Sigma_1$ case, since we can take only one $\chi_2$ or $\chi_3$ then $\mathbb{F}_p$ must contain primitive cube roots of 1. Different characters may lead to conjugate stabilizers. By the discussion in Subsection 4.4 if $\alpha$ normalizes $H'$, and $g \in GL(2,\mathbb{F}_p)$ then the subgroup $H'$ and $g \alpha g^{-1}$ determines a conjugate subgroup. The $N_{H'} \chi$-orbits column in Table 4.2 lists the non-trivial orbits of characters under conjugation. If two linear
combinations of characters are equivalent under the normalizer then conjugate stabilizers are determined and so we need only write one down. Furthermore, depending on the number of roots of unity in $\mathbb{F}_p$ some representations may not reduce completely. In that case the characters have to be taken together as a group, namely the set of all Galois conjugate characters. The non-trivial Galois orbits are noted in the *Galois fusion* column. Cases 11 and 11a are the same subgroup but are distinguished because $|\Omega^{H_i}|$, Table 4.3, depends on the value of $\beta_4(p)$.

| Case | $H'$ | $q$ | $N_{H'}$ $\chi$-orbits | Galois Fusion | restrictions |
|------|------|----|-------------------------|---------------|--------------|
| 1    | $\Sigma_4^1$ | $2\chi_1$ | $\chi_1 + \chi_2$ | $\chi_1 + \chi_2$ | $\chi_1 + \chi_2$ |
| 2    | $\Sigma_2 \times \Sigma_2$ | $2\chi_1$ | $\chi_1 + \chi_2$ | $\chi_1 + \chi_2$ | $\chi_1 + \chi_2$ |
| 3    | $\Sigma_2 \times \Sigma_2$ | $\chi_1 + \chi_2$ | $\chi_1 + \chi_2$ | $\chi_1 + \chi_2$ | $\chi_1 + \chi_2$ |
| 4    | $C_2.2$ | $\chi_1 + \chi_2$ | $\chi_1 + \chi_2$ | $\chi_1 + \chi_2$ | $\chi_1 + \chi_2$ |
| 5    | $C_2.2$ | $2\chi_2$ | $\chi_1 + \chi_2$ | $\chi_1 + \chi_2$ | $\chi_1 + \chi_2$ |
| 6    | $A_3 \times \Sigma_1$ | $\chi_1 + \chi_2$ | $\chi_1 + \chi_2$ | $\chi_1 + \chi_2$ | $\chi_1 + \chi_2$ |
| 7    | $\Sigma_2 \times \Sigma_2$ | $\chi_1 + \chi_2$ | $\chi_1 + \chi_2$ | $\chi_1 + \chi_2$ | $\chi_1 + \chi_2$ |
| 8    | $\Sigma_2 \times \Sigma_2$ | $\chi_1 + \chi_2$ | $\chi_1 + \chi_2$ | $\chi_1 + \chi_2$ | $\chi_1 + \chi_2$ |
| 9    | $K_4$ | $\chi_1 + \chi_2$ | $\chi_1 + \chi_2$ | $\chi_1 + \chi_2$ | $\chi_1 + \chi_2$ |
| 10   | $C_4$ | $\chi_1 + \chi_2$ | $\chi_1 + \chi_2$ | $\chi_1 + \chi_2$ | $\chi_1 + \chi_2$ |
| 11   | $C_4$ | $\chi_1 + \chi_2$ | $\chi_1 + \chi_2$ | $\chi_1 + \chi_2$ | $\chi_1 + \chi_2$ |
| 11a  | $C_4$ | $\chi_1 + \chi_2$ | $\chi_1 + \chi_2$ | $\chi_1 + \chi_2$ | $\chi_1 + \chi_2$ |

**Step 2.** Next we list $|N_G(H_i)|$ and the $|\Omega^{H_i}|$. Each calculation of $|\Omega^{H_i}|$ was confirmed by using the brute force method using Maple. We have the following table:
Table 4.3 fixed point data for $\Sigma_4$, $p \neq 2, 3$

| Case | $|H'|$ | $N_G(H_i)$ | $|\Omega^H_i|$ |
|------|-------|------------|------------|
| 1    | 1     | $24p(p-1)(p^2-1)$ | $p(p-1)(p^2-1)(p^2+p-3)$ |
| 2    | 2     | $4p(p-1)(p^2-1)$ | $p(p-1)(p^2-1)$ |
| 3    | 2     | $4(p-1)^2$ | $(p-1)(p^2-1)$ |
| 4    | 2     | $8(p-1)^2$ | $(p-1)(p^2-1)$ |
| 5    | 2     | $8p(p-1)(p^2-1)$ | $p(p-1)(p^2-1)$ |
| 6    | 3     | $3(p-1)^2$ | $(p-1)^4$ |
| 7    | 4     | $4(p-1)^2$ | $(p-1)^4$ |
| 8    | 4     | $8(p-1)^2$ | $(p-1)^4$ |
| 9    | 4     | $8(p-1)^2$ | $(p-1)^4$ |
| 10   | 4     | $4(p-1)^2$ | $(p-1)^4$ |
| 11   | 4     | $8(p-1)^2$ | $p^2-1$ |
| 11a  | 4     | $8(p^2-1)$ | $(p-1)^4$ |
| 12   | 10    | $8(p-1)$ | $p-1$ |

**Step 3.** To compute the $d_{i,j}$’s, let us first look at the case where all roots of unity are available $\beta_3(p) = \beta_4(p) = 2$. The other cases may be derived from this case. We present the $d_{i,j}$’s, in matrix form with this ordering of the groups.

Table 4.4 Subgroup ordering

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 |
|-----|---|---|---|---|---|---|
| Case | 1 | 2 | 3 | 4 | 5 | 6 |
| $H'_i$ | $\Sigma_4$ | $\Sigma_2 \times \Sigma_3$ | $\Sigma_2 \times \Sigma_3^2$ | $C_{2,2}$ | $C_{2,2}$ | $A_3 \times \Sigma_1$ |
| $q_i$ | $2\lambda_1$ | $2\lambda_1$ | $\chi_1 + \chi_2$ | $\chi_1 + \chi_2$ | $2\lambda_2$ | $\lambda_1 + \chi_2$ |

| $i$ | 7 | 8 | 9 | 10 | 11 | 12 |
|-----|---|---|---|----|----|----|
| Case | 7 | 8 | 9 | 10 | 11a | 12 |
| $H'_i$ | $\Sigma_2 \times \Sigma_2$ | $\Sigma_2 \times \Sigma_2$ | $K_4$ | $C_4$ | $C_4$ | $D_4$ |
| $q_i$ | $\chi_1 + \chi_3$ | $\chi_3 + \chi_4$ | $\chi_2 + \chi_3$ | $\chi_2 + \chi_3$ | $\chi_3 + \chi_4$ | $\chi_5$ |

The values of $d_{i,j} = |\text{D}(H_i, H_j)|$ can be computed as follows. Using Magma or GAP, find all the subgroup classes of $H'_j$. For each subgroup class $\langle K \rangle$, $K \subseteq H'_j$ find the corresponding $H_i$’s such that $H'_i$ is conjugate to $K$. Transport the character $q_j$ of $H_j$ to $H'_i$ to determine the subgroup $H_i$. The entry for $d_{i,j}$ is then $|\langle K \rangle|$ (computed in $H'_j$).
Example 36  Here is the matrix $D$ for the set of subgroups above.

$$
\begin{bmatrix}
H_1 & H_2 & H_3 & H_4 & H_5 & H_6 & H_7 & H_8 & H_9 & H_{10} & H_{11} & H_{12} \\
H_1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
H_2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
H_3 & 1 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 2 & 0 \\
H_4 & 1 & 0 & 0 & 1 & 0 & 2 & 1 & 0 & 2 & 0 & 0 \\
H_5 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
H_6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
H_7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
H_8 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
H_9 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
H_{10} & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
H_{11} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
H_{12} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
$$

Step 4. Finally we compute the number of orbits according to formula \([31]\). We have

$$
O^\circ = TD^{-1}N^{-1}L
$$

where $T$ and $N$ are the diagonal matrices formed from columns 2 and 3 of Table 4.3, respectively, and $L$ is the vector formed from column 4 of Table 4.3. Accordingly we obtain the vector of orbits and the total number of orbits

$$
O^\circ = \begin{bmatrix} \frac{p^2 - 8p + 7}{24} & 0 & \frac{p - 3}{2} & \frac{p^2 - 5}{4} & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}^T
$$

$$
|\Omega/G| = \frac{p^2 + 10p + 37}{24}
$$

We may check that all the numbers are integers by substituting in $p = 12k + 1$, since 12 divides $p - 1$.

There are a total of 4 families of primes which depend upon whether $\beta_4(p) = 0, 2$ and $\beta_3(p) = 0, 2$. For $p > 3$ these families depend on the congruence class of $p \mod 12$. For each family certain subgroup cases from Table 4.1 are excluded. The corresponding matrices $D$ are obtained from the sample matrix above by deleting the row(s) and column(s) corresponding to the excluded subgroups. Depending on the whether $\beta_4(p) = 0, 2$ one chooses the same subgroup $H_{10}$ but with a different normalizer and fixed point data given by cases 11 and 11a.

| $\beta_3(p)$ | $\beta_4(p)$ | $p \mod 12$ | excluded subgroups | $|\Omega/G|$ |
|---|---|---|---|---|
| 0 | 0 | 11 | $H_6, H_{10}$ | $\frac{1}{24}(p^2 + 6p + 9)$ |
| 2 | 0 | 7 | $H_{10}$ | $\frac{1}{24}(p^2 + 6p + 25)$ |
| 0 | 2 | 5 | $H_6$ | $\frac{1}{24}(p^2 + 6p + 31)$ |
| 2 | 2 | 1 | none | $\frac{1}{24}(p^2 + 6p + 37)$ |

The complete results for 3 and 4 branch points are given in Table 5.1.
5 Low genus examples

To finish, we illustrate our results through a number of interesting examples. To emphasize the explicit results which can be obtained, the first example we consider is very specific. Following this, we shall present some general examples which hold independent of the prime \( p \).

**Example 37** Let \( G \) be an elementary abelian group of order 25. Using the Riemann-Hurwitz formula, it can be shown that there are exactly two different signatures for \( \Gamma \) for a surface of genus 26 which give rise to subgroups of \( \mathcal{M}_{26} \) isomorphic to \( G \) - the signatures \((2, -)\) and \((1; 5, 5, 5)\). To find the total number of conjugacy classes of subgroups of \( \mathcal{M}_{26} \) isomorphic to \( G \), we need to find the number of classes induced by each of these signatures.

For the signature \((2, -)\), a direct application of Corollary 11 gives two different classes of subgroups. For the signature \((1; 5, 5, 5)\), there are two possibilities for the group \( G_{h} \), either trivial or cyclic of order 5. If \( G_{h} \) is trivial, \( G_{e} = G \), so Proposition 19 implies there is just one epimorphism from \( \Gamma \) onto \( G \). Else, the image is cyclic of order 5. In this case, we apply Corollary 18. For the hyperbolic part, observe that Corollary 11 tells us there will be a single epimorphism arising from \( \Gamma \) with signature \((1; -)\) onto \( G^{h} \). For the elliptic part, Example 38 below shows that all epimorphisms from \( \Gamma \) with signature \((0; 5, 5, 5)\) onto \( G^{e} \) are equivalent. Thus there is just one epimorphism arising from the elliptic part. Therefore, in total, there are \( 2 + 1 + 1 \times 1 = 4 \) conjugacy classes of subgroups isomorphic to \( G \) in \( \mathcal{M}_{26} \).

**Example 38** We can use our results to enumerate the equivalence classes of totally ramified actions for 3 or 4 branch points. Complete results for these cases are given in the following table:

| \( r \) | \( v \) | \# branch points | \# equiv classes |
|---|---|---|---|
| 3 | 1 | 2 | 0 |
| 3 | 1 | 3 | \( (p+1) \) |
| 3 | 1 | \( \beta_4(p) = 0 \) | \( p = 5 \mod 6 \) |
| 3 | 1 | \( \beta_4(p) = 2 \) | \( p = 1 \mod 6 \) |
| 3 | 2 | all | 1 |
| 4 | 1 | 2, 3 | 1 |
| 4 | 1 | \( \beta_4(p) = 0 \) | \( p = 7, 11 \mod 12 \) |
| 4 | 1 | \( \beta_4(p) = 2 \) | \( p = 1, 5 \mod 12 \) |
| 4 | 2 | 3 | 2 |
| 4 | 2 | \( \beta_4(p) = 0, \beta_4(p) = 0 \) | \( p = 11 \mod 12 \) |
| 4 | 2 | \( \beta_4(p) = 2, \beta_4(p) = 0 \) | \( p = 7 \mod 12 \) |
| 4 | 2 | \( \beta_4(p) = 0, \beta_4(p) = 2 \) | \( p = 5 \mod 12 \) |
| 4 | 2 | \( \beta_4(p) = 2, \beta_4(p) = 2 \) | \( p = 1 \mod 12 \) |
| 4 | 3 | all | 1 |
The techniques and results developed in the previous examples allow us to develop some further results for general families of signatures and groups.

Example 39 Suppose $\Gamma$ has signature $(g_1; p, p, p)$ with $p > 3$ and let $G$ denote an elementary abelian group of order $p^2$. In this case, $G^h$ is either trivial or has order $p$. We first consider the case when $G^h$ has order $p$. By Example 38, there will be $(p + 1 + 2\beta_3(p))/6$ totally ramified epimorphisms onto $C_p$ and Corollary 11 implies there is a unique hyperbolic epimorphism onto $C_p$ giving a total of $1 + (p + 1 + 2\beta_3(p))/6$. If $G^h$ is trivial, then Proposition 14 implies there exists a unique elliptic epimorphism. Thus there is a total of

$$\frac{p + 7 + 2\beta_3(p)}{6}$$

conjugacy classes of groups induced by this signature.

The results we have obtained for elementary abelian subgroups can in certain special cases be extended to provide information about other groups. We illustrate with the following two examples.

Example 40 By Proposition 14, there is a unique conjugacy class of elementary abelian subgroups of order $p^2$ in $M_{(p-1)(p-2)/2}$ with signature $(0; p, p, p)$. In fact, this can be derived explicitly by showing that any two epimorphisms $\eta_1, \eta_2 : \Gamma \rightarrow G$ from $\Gamma$ with signature $(0; p, p, p)$ differ by an automorphism $\alpha \in \text{Aut}(G)$, $\eta_1 = \alpha \circ \eta_2$. In [15], it is shown that the kernel $\text{Ker}(\eta_1)$ is in fact a uniformizing surface group for the $p$th Fermat curve with defining equation $x^p + y^p = 1$. The full automorphism group of the $p$th Fermat curve is isomorphic to the semi-direct product $S_3 \rtimes (\mathbb{F}_p \times \mathbb{F}_p) \cong \Gamma_1/\text{Ker}(\eta_1)$ where $\Gamma_1$ has signature $(0; 2, 3, 2p)$. By the uniqueness of $\text{Ker}(\eta_1)$ and the uniqueness of the elementary abelian subgroup of $S_3 \rtimes (\mathbb{F}_p \times \mathbb{F}_p)$, it follows that any two epimorphisms $\eta_1, \eta_2 : \Gamma_1 \rightarrow S_3 \rtimes (\mathbb{F}_p \times \mathbb{F}_p)$ must differ by an automorphism $\alpha \in \text{Aut}(S_3 \rtimes (\mathbb{F}_p \times \mathbb{F}_p))$. In particular, there will be a unique conjugacy class of subgroups of $M_{(p-1)(p-2)/2}$ isomorphic to $S_3 \rtimes (\mathbb{F}_p \times \mathbb{F}_p)$ induced by $\Gamma$ with signature $(0; 2, 3, 2p)$.

Example 41 Let $G$ be an elementary abelian group of order $p^2$ where $p \geq 5$ and suppose that $\Gamma$ has signature $(0; p, p, p, p)$. In [15] it is shown that there exists a unique epimorphism $\eta_1$ from $\Gamma_1$ with signature $(0; 2, 2p, 4)$ onto $H = D_4 \rtimes (C_p \times C_p)$ (up to the action of $\text{Aut}(H)$) which restricts to an epimorphism $\eta_1|_{\Gamma_1} : \Gamma \rightarrow C_p \times C_p$. In particular, this implies there exists a unique class of subgroups of $M_{(p-1)2}$ isomorphic to $H$ with signature $(0; 2, 2p, 4)$. Alternatively, this can also be seen on the level of generating vectors of the restriction $\eta_1|_{\Gamma}$. Specifically, it can be shown that if $x$ and $y$ generate $G$, then all generating vectors for epimorphisms which extend to epimorphisms from $\Gamma_1$ to $H$ are $\text{Aut}(G) \times \text{Aut}(\Gamma)'$ equivalent to $(x, x^{-1}, y, y^{-1})$. This implies there exists a unique class of groups isomorphic to $G$ in $M_{(p-1)2}$ contained in a class of subgroups isomorphic to $H$.

Then, by the uniqueness of $G \leq H$, it follows that there exists a unique class of groups isomorphic to $H$ with signature $(0; 2, 2p, 4)$ in $M_{(p-1)2}$.
Observe that the generating vector in Example 41 is highly symmetric and this symmetry is reflected by the fact that $G$ is contained in a larger subgroup of $\mathcal{M}_{(p-1)^2}$. It seems plausible that if a generating vector is highly symmetric, then the corresponding conjugacy class of subgroups of the MCG will be contained in a class of larger finite subgroups of the MCG. This could lead to a new way to determine and enumerate classes of subgroups of the MCG which are not abelian using the methods we have developed for abelian subgroups.

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