PARAMETRIC INSTABILITY OF STEADY FLUID FLOW IN A CORRUGATED PIPE

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ABSTRACT

In this paper, we show that the quasi-one-dimensional flow of an ideal inviscid fluid in a corrugated pipe is parametrically unstable in certain frequency bands. First-order perturbation theory is used to analyze the stability of the flow, and shows that parametric instability occurs even if the velocity of the base flow is zero. Stability diagrams of the system as a function of the amplitude and period of the corrugation, for different velocities of the base flow, are obtained using parallel numerical computation. The analysis shows that the higher-order bands of instability are strongly dependent on the base flow velocity.

Keywords inviscid instabilities · inviscid flows · parametric instabilities · steady flow · quasi-one-dimensional flow

1 Introduction

Corrugated pipes are widely known for their use in laminar-turbulent transition studies [1, 2, 3, 4] as well as for their interesting acoustic properties [5, 6]. For example, it is known that fluid flow through a pipe with transversally corrugated walls generates tonal sounds [7]. Corrugated pipes also have many practical applications because of their large-scale flexibility combined with local rigidity. As a rule, flow studies on corrugated pipes involve three-dimensional or axisymmetric modeling of the local effects in the corrugation cavities [6, 8]. On the other hand, corrugated pipes with a relatively large corrugation period (in comparison with the pipe diameter) can be described by a quasi-one-dimensional model. In these quasi-one-dimensional fluid dynamics equations the cross-section of the pipe provides spatially dependent coefficients, meaning that the system can be excited parametrically. The aim of this study is to show that the flow of an ideal fluid in a corrugated pipe has parametric instability at certain frequency bands, even in the absence of a base flow through the pipe. The first analytical study of this problem, with a linear approximation relative to flow velocity and amplitude of corrugation, was performed in [9]. In the present paper a numerical study of the instability is performed in order to obtain the stability diagrams of the system without this simplification.

2 Theoretical description of quasi-one-dimensional flow in a corrugated pipe

We consider a semi-infinite corrugated pipe with a flow inlet placed at spatial coordinate \( x = 0 \). We describe the fluid flow inside the pipe by the following quasi-one-dimensional equations

\[
\frac{\partial}{\partial t} (\rho S) + \frac{\partial}{\partial x} (\rho v S) = 0,
\]

\[
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0.
\]

Here \( \rho(x, t), v(x, t) \), and \( p(x, t) \) are the density, velocity, and pressure of the fluid, respectively, and \( S(x) \) is the cross-sectional area of the pipe. A linear (first order) perturbation of the flow can be expressed as follows:

\[
\rho(x, t) = \rho^{(0)} + \varepsilon \rho^{(1)}(x, t),
\]
where \( \gamma \) is the complex conjugate) and \( \rho(0) = \rho(x,0) = \rho(0) \), where \( c \) is the sound speed. The parameters of the incompressible steady base flow are denoted with subscript \((0)\) and the perturbations are denoted with subscript \((1)\). After substitution of Eqs. (3)- (5) into Eqs. (1)-(2) and further simplification, the following equations for the flow perturbations can be obtained:

\[
\frac{\partial}{\partial t} \left( \frac{\rho^{(1)}}{\rho^{(0)}} \right) + \frac{\partial \rho^{(1)}}{\partial x} \frac{v^{(0)}}{S(x)} + \frac{\partial \rho^{(1)}}{\partial x} + S(x) \frac{v^{(1)}}{S(x)} = 0,
\]

\[
\frac{\partial v^{(1)}}{\partial t} + v^{(0)} \frac{\partial v^{(1)}}{\partial x} + c^2 \frac{\partial}{\partial x} \left( \frac{\rho^{(1)}}{\rho^{(0)}} \right) + v^{(0)}(x) v^{(1)}(x) v^{(0)}(x) \frac{\rho^{(1)}}{\rho^{(0)}} = 0,
\]

while the base flow velocity and pressure are governed by the equations

\[
v^{(0)}(x) = \frac{A}{S(x)},
\]

\[
p^{(0)}(x) = p^{(0)}(0) - \frac{1}{2} \rho^{(0)} \left( v^{(0)}(x) \right)^2 = p^{(0)}(0) - \rho^{(0)} A^2 / 2S^2(x).
\]

Here \( A = v^{(0)}(0) S(0) \) is a constant value for the given inlet velocity and geometry of the pipe.

For the harmonic wave the desired solution can be expressed as \( \rho^{(1)}(x,t)/\rho^{(0)} = r(x)e^{-i\omega t}/2 + c.c. \), (where c.c. denotes the complex conjugate) and \( v^{(1)}(x,t)/c = u(x)e^{-i\omega t}/2 + c.c. \), and Eqs. (6) and (7) are written

\[
u' + M(x) r' + S'(x) u - ikr = 0,
\]

\[
r' + M(x) u' + M(x) M' r = (M(x) - ik) u = 0,
\]

where \( k = \omega/c \) and \( M(x) = v^{(0)}(x)/c \) is the Mach number of the base flow. Introducing a periodic cross-section for the pipe

\[
S(x) = S_m (1 + \mu \sin 2\beta x),
\]

where \( S_m \) is its mean cross-sectional area, \( \mu \) is the corrugation amplitude and \( \beta/\pi \) is the corrugation period, one can obtain

\[
\frac{S'(x)}{S(x)} = \frac{2\beta \mu \cos 2\beta x}{1 + \mu \sin 2\beta x}.
\]

Denoting \( z = \beta x \) we obtain the following dimensionless equations:

\[
u' + M(z) r' + S'(z) u(z) - i\gamma r(z) = 0,
\]

\[
r' + M(z) u' + M(z) M' r(z) + (M'(z) - i\gamma) u(z) = 0.
\]

It was shown in previous work [2] that linearization of Eqs. (12) and (13) with respect to \( \mu \), with the additional assumption that \( M \ll 1 \), leads to a relatively simple Hill equation with the first region of instability approximately described by the inequality

\[
1 - \frac{\mu}{2} < \frac{\gamma}{1 - M^2} < 1 + \frac{\mu}{2},
\]

where \( \gamma = k/\beta \) is the dimensionless propagation constant.

It should also be mentioned that if the velocity of the base flow is zero, Eqs. (12) and (13) lead to a second-order differential equation

\[
r'' + S'(z) r' - \gamma^2 r = 0,
\]

which can be simplified by introducing a new variable \( y(z) = r(z) \sqrt{S_0(z)} \) to obtain a Hill equation:

\[
y'' + \left( \gamma^2 + f(z) \right) y(z) = 0,
\]

where \( f(z) \) is a periodic function depending on \( S(z) \):

\[
f(z) = \frac{S'(z)}{2S(z)} + \frac{S''(z)}{2S(z)}.
\]
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For the function $S(z)$ following Eq. (10), this leads to the following expression for $f(z)$:

$$f(z) = -\mu \frac{2 \sin 2z - \mu (1 - 3 \sin^2 2z)}{(1 + \mu \sin 2z)^2}.$$  (18)

The linear part of $f(z)$ with respect to $\mu$ gives $f(z) = -2\mu \sin 2z + O(\mu^2)$, which leads to a classical form of Mathieu equation

$$y''(z) + (\gamma^2 - 2\mu \sin 2z)y(z) = 0,$$  (19)

for which the presence of parametric resonance is well known [10].

3 Numerical results and discussion

We investigate the regions of stability of the flow numerically by solving Eqs. (12) and (13) with initial conditions $u(0) = u_0$ and $r(0) = r_0$ for different $\mu$ and $\gamma$ on each point of a $(\mu, \gamma)$ grid with dimension 1000×1000. The condition for which the solution is treated as unstable is that the amplitude $u(z)$ (or $r(z)$) at $z = 1000$ is more than 10 times greater than $u_0$ (or $r_0$). To speed up the process a parallel computation was performed. By solving this problem for several flow velocities $M_0 = v(0)/c$, we obtain the stability diagrams shown in Fig. 1. The instability bands are colored in black, i.e., for every and inside these bands the solution of Eqs. (12) and (13) is unstable.

Figure 1 shows the first four instability bands, with approximate central frequencies $\omega_n \approx n\beta c$, $n = 1, 2, 3, 4$. The first and largest instability band (with central propagation constant $\gamma 1$) is almost symmetric and is relatively accurately described by Eq. (14), especially for low $M_0$ numbers. The higher-order instability bands have a threshold for the corrugation amplitude $\mu$, and the higher the order of the band the greater this threshold is. The whole picture for higher-order bands is strongly dependent on $M_0$. It is notable that for $M_0 = 0.3$ [Fig. 1d], a region appears for which the instability is unconditional with respect to $\gamma$ (where $\mu > 0.7$, approximately). Despite this interesting behavior of the system at $M_0 = 0.3$, there is no point in investigating larger values of $M_0$ due to the main assumption that the flow is incompressible.

4 Conclusion

This study has shown that for a quasi-one-dimensional description of ideal inviscid fluid flow in corrugated pipes there is a parametric instability of linear perturbations. To find the exact regions of this instability we conducted a numerical investigation of the solution with different flow velocities and corrugation parameters (amplitude and period.
of corrugation). The borders of the first band can be approximately described analytically. In general, the borders of the higher-order instability bands are sensitive to the flow velocity.

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