INTRODUCTION

A possible quantum probability increase of the cylindrical gravitational field

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Abstract

As known, the cylindrical gravitational wave have been canonically quantized and its wave function, as the quantum one, interpreted in probability terms. We show in this work, using quantum Zeno methods, that this probability may be substantially increased and even approach unity. For that we first show, in detailed manner, that the cylindrical gravitational wave may be discussed in the commutation number representation. We also discuss this field in the transverse-traceless (TT) gauge and calculate the related trapped surface.

Keywords: Cylindrical gravitational wave, Quantization, Zeno effect

Pacs numbers: 03.70.+k, 04.30.-w, 03.65.Xp

1 INTRODUCTION

The problem of quantizing the general gravitational wave (GW) [1] [2] have, theoretically, been discussed by different persons and methods beginning from the earlier works of Rosenfeld [3], Bergmann [4] and Schwinger [5] to the canonical methods of Dewitt [6], Arnowitt-Desser-Misner (ADM) [7] and Dirac [8]. Among the supposed quantum characteristics of the GW is the probability interpretation [9] [10] which is extended from the quantum regime so that one interprets [11] the GW function as a probability amplitude [9] [10]. This problem of
quantizing and interpreting, in probability terms, the GW entails in turn a second interesting
problem of how and in which way to increase the probability of some specific GW. Taking
the cylindrical GW as an example and assuming that one starts from a hypersurface with a
cylindrical geometry one may ask how to increase its probability so that this same cylindrical
geometry may persist upon this hypersurface.

This problem is also found, under different terms and terminology, in other disciplines of
general relativity. Thus, a central issue of quantum gravity [12] is the problem of substantially
increasing the probability of some quantum foam [13], with a typical order of magnitude of
the Planck-Wheeler length \( \left( \frac{G \hbar}{c^3} \right) \) [14, 15], so that it may be realized in full grown human scale
with a unity probability (as phrased in [16] regarding a Planck-Wheeler sized wormhole [17]).

As known, the mentioned first problem of quantizing the GW finds its solution when
one restricts the discussion to limited dynamical regions of geometrodynamics such as the
minisuperspace and miniphasespace discussed by Dewitt [6], Misner [18] and, especially, by
Kuchar [11] which applied it for quantizing the cylindrical GW. Kuchar did it by extending
the extrinsic time idea, which was first introduced [7, 19, 20] for the linearized theory [14, 15]
of general relativity, to the nonlinear theory [21] and, especially, to the cylindrical spacetime.
This extrinsic time variable, which is of the Tomonaga-Schwinger many-fingered time kind
[22, 23], is canonically conjugate to momentum and not to energy as is the intrinsic time
which serves more as a label to distinguish between spacelike hypersurfaces [14] in a one-
parameter family of them.

The mentioned property of being fully quantized makes the gravitational cylindrical min-
isuperspace a starting point for a possible general quantum gravity theory [24, 25, 26].
Moreover, it has even triggered a discussion [27] about cylindrical blackholes and trapped
cylinders.

Using the extrinsic time variable one may obtain [11], as known from [7], a formalism
which is identical to that used for discussing the parametrized [11] cylindrical massless scalar
field in a flat Minkowskian background. This suggests, as noted in [11], that all the results
obtained from the later case may, theoretically, be applied also for the cylindrical GW in a
curved spacetime. These applications include also the canonical quantization of the cylindri-
cal GW and the derivation \[11\] of a functional time-independent Schroedinger-type equation
for it. Moreover, Kuchar has succeeded \[11\] to apply for the cylindrical GW not only quan-
tum ideas such as the inner product of states \[9, 10\] or path independence of the evolution
of them but also their probability interpretation \[9, 10\].

In this work we use this successful quantization of the cylindrical GW as a basis for dis-
cussing the mentioned second problem of finding the conditions under which the mentioned
probability increases and even approaches unity. In the quantum regime, where states change
with both time and space \[9, 10\], there exists the known Zeno effect \[28, 29, 30, 31\] which
causes these changing states to become constant and fixed and, therefore, to cause their
probability, as in classical physics, to become unity. This effect, which were experimentally
validated \[32, 33\], has three different versions:

1. Repeating a large number of times, in a finite total time, the same experiment of checking
the present state of some quantum system that has been prepared in an initial specific state
so that in the limit of a very large number of repetitions during the same total time the
initial state is preserved in time \[28, 32\].

2. Performing many sets of experiments where each set is composed of a large number of
slightly different experiments so that the overall effect is as if one advances through many
different possible paths of states (Feynman paths \[34, 35\]). Thus, taking the limit of doing
these experiments in a “dense” manner one may “realize” \[29, 30\] any specific path of states
in the sense that the probability to proceed along all its constituent states tends to unity.

3. Simultaneously performing the same experiment in a large number of non-overlapping
spatial subregions all included in a finite total region so that when these subregions infinites-
imally shrink, keeping the total region fixed, one avoid any spatial shifting of the state \[31\].
This may be explained by the example of trying to locate a very small particle-like object
in the finite total spatial region which become easier when this region is divided into several
equal parts each occupying the same small object and the searching for it is done in each of these smaller regions. It is obvious that the smaller become these regions in the finite total region the probability to locate the small object in each of them grows so that in the limit in which they infinitesimally shrink this probability tends to unity.

The first two cases (1)-(2) are, respectively, known \[29, 30\] as the static and dynamic Zeno effects while the third (3) is the space Zeno effect \[31\].

We note that whereas the quantum states are related to the ordinary intrinsic time the gravitational cylindrical states are characterized by \[11\] an extrinsic time variable which is related (and actually borrows \[11\] its name) to the extrinsic curvature. That is, the gravitational state, represented by the probability that the related hypersurface has cylindrical geometry, changes in spacetime with respect to extrinsic time. Thus, since this extrinsic time, like the ordinary spatial variable, is (see Sec VII in \[11\]) canonically conjugate to momentum (which is connected to extrinsic curvature) the corresponding gravitational Zeno effect should also have spatial characteristics as in the mentioned space Zeno effect.

We directly show in this work, using space Zeno terms \[31\], that one may avoid any space shifting of the cylindrical GW thereby fixing its cylindrical geometry and causing its probability to approach unity. In the following we precede this Zeno demonstration with a discussion which shows that the cylindrical GW may be represented as a large number of constituent parts in some finite region of space so that it may be discussed in spatial Zeno terms \[31\]. The appropriate representation which enables one to do so is the occupation number one \[9, 10\]. Thus, we discuss here, in detail, this representation \[9, 10\] in relation to the cylindrical GW. We note in this respect that although this representation in the context of cylindrical GW is mentioned and calculated in the literature (under the name of Fock quantization see \[11, 24, 25, 36\]) we proceed here in a detailed manner along the methods used in \[9\] for the harmonic oscillator.

As known from quantum mechanics in the occupation number representation \[9, 10, 37\], one may, theoretically, prepare any quantum state by merely applying the relevant creation
and destruction operators [9, 10] any required number of times upon some initial basis state. One also knows from the canonical formalism [7, 14, 15] of general relativity that it is possible to theoretically prepare the geometry of some spacetime hypersurface by controlling the form of the lapse and shift functions [7, 14, 15]. That is, any specific theoretical evolution of spacetime should be preceded by determining beforehand these functions so that one can be sure (with a unity probability) that the related spacetime is developed along the specified route. This operation, characterized by the determination of the lapse and shift functions, should be related, when discussing the quantum properties of the GW, to the corresponding operation of the mentioned quantum creation and destruction operators upon some initial basis state. We derive here in detail the appropriate expressions which accordingly relate the cylindrical lapse and shift functions to the creation and destruction operators.

When the mentioned probability of the cylindrical GW approach unity (also in consequence of the spatial Zeno effect) the related GW produce certain effects upon the neighbouring spacetime through which it proceeds such as implanting its cylindrical geometry upon it and giving rise to some trapped surface [38, 39, 40, 41]. We follow the development of this cylindrical GW, once its probabilistic chances were greatly increased, and find its properties in the transverse-traceless (TT) gauge and also calculate the geometry of the generated trapped surface [38, 39, 40, 41].

In Section II we introduce the principal expressions [11] related to the Einstein-Rosen cylindrical GW [42] as, especially, represented in [11, 36]. In Section III we discuss the cylindrical GW in the commutation number representation [9, 10] so that it may be thought of as composed of a very large number of gravitational quanta which, like the quantum ones [9, 10], are created and destroyed by the corresponding gravitational creation and destruction operators. In Section IV we relate the cylindrical lapse and shift functions [7, 14, 15] to the mentioned gravitational creation and destruction operators. Note that the lapse and shift functions participate (as emphasized in [14] (see Sections 21.8 and 21.9 there)) in the determination of the geometry of spacetime hypersurfaces just as the mentioned gravitational
creation and destruction operators determine this geometry through creating or (and) destroying the quantum components of the generating GW. We note that the cylindrical lapse and shift functions were related in [11] to the Einstein-Rosen parameters (see Eqs (29)-(30) in [11]). The detailed calculation relating the cylindrical lapse and shift functions to the creation and destruction operators is shown in Appendix A. In Section V we show that, beginning from a gravitational cylindrical geometry in some hypersurface, the probability to find the same geometry upon this hypersurface tends to unity in the limit of the space Zeno effect [31]. As mentioned, we rather discuss the space Zeno effect and not the (intrinsic) time analogue of it because in the cylindrical geometry one discusses [11] the extrinsic time variable which is canonically conjugate to momentum (Sec VII in [11]) just as is any spatial variable. We note that it has been shown [43], using the examples of the quantum bubble and open-oyster processes [37, 44], that the mentioned static and dynamic quantum Zeno effects are also valid in quantum field theory [37, 44]. We also note in this respect that the quantum Zeno effect were discussed [45] in the framework of gravitomagnetism [46]. The detailed calculations of the appropriate probability is shown in Appendix B. In Section VI we graphically corroborate our theoretical results so that one may see how the probability approach unity in the Zeno limit. In Section VII we discuss the cylindrical GW in the transverse-traceless (TT) gauge which is characterized by a very simplified formalism [14] in which, for example, the number of independent components of the related GW is minimal [14]. In Section VIII we discuss, using the method in [38], the related embedded trapped surface [14, 38, 39, 40, 41] resulting from the passing cylindrical GW. In Section IX we summarize our discussion in a Concluding Remarks Section.

2 The Einstein-Rosen Cylindrical gravitational wave

A spacetime is considered to be cylindrically symmetric [11] if and only if one can show that there exists a coordinate system \((t, r, \phi, z)\), \(-\infty < t < +\infty, \infty > r \geq 0, 2\pi > \phi \geq 0\),
\(-\infty < z < +\infty\) in which the line element becomes

\[
ds^2 = -(N^2 - e^{(\psi - \gamma)} N_1^2) dt^2 + 2 N_1 dt dr + e^{(\gamma - \psi)} dr^2 + R^2 e^{-\gamma} d\phi^2 + e^\gamma dz^2,
\]

where \(R \geq 0\) and \(\gamma, \psi, N, N_1\) are functions of \(t\) and \(r\). The former dependence of the nonzero metric tensor components \(g_{11}, g_{22}, g_{33}, g_{00}, g_{01}\) upon the functions \(\gamma, \psi, R, N, N_1\) is, especially, designed \([11]\) to suit the ADM \([7]\) canonical formulation of general relativity. Thus, the \(N\) and \(N_1\) are, respectively, the known ADM lapse and radial shift functions \([11, 14]\).

The coordinates \(\phi\) and \(z\) are, essentially, fixed except for a possible trivial transformation of \(\bar{\phi} = \pm \phi + \phi_0\) and \(\bar{z} = az + z_0\) whereas \(t\) and \(r\) may be subject, without changing the form of the line element from \((1)\), to the more general transformation

\[
\bar{t} = \ell(t, r), \quad \bar{r} = \ell(t, r)
\]

One may show \([11]\), using Killing vectors in the \((t, r, \phi, z)\) system, that the functions \(R\) and \(\gamma\) are scalars. Thus, since the metric tensor coefficients depends, as mentioned, only upon \(t\) and \(r\) one may write \([11]\) the \((t, r)\) part of the line element \((1)\) in the following conformally flat form

\[
ds^2 = e^{(\bar{\gamma} - \psi)} (-d\bar{t}^2 + d\bar{r}^2) + R^2 e^{-\psi} d\phi^2 + e^\psi dz^2,
\]

where the \(R, \psi, \phi\) and \(z\) are not barred due to their mentioned essential invariancy. Now, as emphasized in \([11]\), if one writes the Einstein field equations for the line element \((3)\) one may realize that \(R\) must be a harmonic function which satisfies \(\frac{\partial^2 R}{\partial \bar{r}^2} - \frac{\partial^2 R}{\partial \bar{t}^2} = 0\). Thus, one may assume \([11]\) \(R\) to be a new radial coordinate and \(T\) the time coordinate corresponding to it. That is, as emphasized in \([11]\), the Einstein-Rosen coordinates can be uniquely and rigorously defined by invariant prescriptions so that the line element \((3)\) may be written as

\[
ds^2 = e^{(\Gamma - \psi)} (-dT^2 + dR^2) + R^2 e^{-\psi} d\phi^2 + e^\psi dz^2
\]
In such case the Einstein vacuum equations are considerably simplified and reduce to the following three equations

\[
\frac{\partial^2 \psi}{\partial T^2} - \frac{\partial^2 \psi}{\partial R^2} - R^{-1} \frac{\partial \psi}{\partial R} = 0
\]  

(5)

\[
\frac{\partial \Gamma}{\partial R} = \frac{1}{2} R \left( \left( \frac{\partial \psi}{\partial T} \right)^2 + \left( \frac{\partial \psi}{\partial R} \right)^2 \right)
\]  

(6)

\[
\frac{\partial \Gamma}{\partial T} = R \frac{\partial \psi}{\partial T} \frac{\partial \psi}{\partial R}
\]  

(7)

As emphasized in [11], Eq (5) has exactly the same form as the wave equation of the cylindrically symmetric massless scalar field \( \psi \) advancing in a Minkowskian spacetime whereas Eqs (6) and (7) are, respectively, the energy density and the radial momentum density of this field in cylindrical coordinates. The solution of Eq (5) is obtained by using the separation of variables method [36] so that the resulting wave function for a particular wave number \( k \) is

\[
\psi_k(R, T) = J_0(kR) \left( A(k)e^{(ikT)} + A^*(k)e^{-(ikT)} \right),
\]  

(8)

where \( J_0(kR) \) is the bessel function of order zero [47] and \( A(k), A^*(k) \) are the amplitude and its complex conjugate of the solution to the time part of Eq (5). Note that here we assume, as generally done in the relevant literature, that \( c = \hbar = 1 \) so that \( w = k = \rho \) where \( w, k, \rho \) are respectively the frequency, wave number and momentum of some mode. Since \( k \) is a continuous parameter one may obtain the general solution to Eq (5) by integrating over all the modes \( k \). Thus, the relevant general wave function is

\[
\psi(R, T) = \int_0^\infty dk J_0(kR) \left( A(k)e^{(ikT)} + A^*(k)e^{-(ikT)} \right)
\]  

(9)

The canonical conjugate momentum \( \pi_\psi(T, R) \) may be obtained [36] by using the Hamilton
\( \frac{\partial \psi}{\partial t} = \{ \psi, H \} \), \hspace{1cm} (10)

where \( \psi \) is given by Eq (9), the curly brackets at the right denote the Poisson brackets and the Hamilton function \( H \) is

\[
H = \int_0^\infty dr \left( \tilde{N} \tilde{H} + \tilde{N}_1 \tilde{H}_1 \right) \tag{11}
\]

The quantities \( \tilde{H} \) and \( \tilde{H}_1 \) are respectively the rescaled superHamiltonian and supermomentum which where shown in [11] (see Eqs (93)-(97) and (106)-(108) in [11]) to be

\[
\tilde{H} = R_r \Pi_T + T_r \Pi_R + \frac{1}{2} R^{-1} \frac{\partial^2}{\partial \psi^2} + \frac{1}{2} R \psi^2_r \tag{12}
\]

\[
\tilde{H}_1 = T_r \Pi_T + R_r \Pi_R + \psi_r \pi_{\psi},
\]

where the suffixed apostroph denote differentiation with respect to \( r \) and \( \Pi_T, \Pi_R \) are the respective momenta canonically conjugate to \( T \) and \( R \). The quantities \( \tilde{N} \) and \( \tilde{N}_1 \) respectively denote the rescaled lapse and shift function \( N, N_1 \) (see Eq (96) in [11]). Thus, \( \pi_{\psi}(T, R) \) were shown [36] to have the form

\[
\pi_{\psi}(T, R) = iRR_r \int_0^\infty dk \kappa_0(kR) \left( A(k)e^{(ikT)} - A^*(k)e^{-(ikT)} \right) - RT_r \int_0^\infty dk J_1(kR) \left(A(k)e^{(ikT)} + A^*(k)e^{-(ikT)} \right), \tag{13}
\]

where \( J_1(kR) \) is the first order Bessel function [17] which may be obtained by differentiating \( J_0(kR) \) with respect to \( R \) as \( J_0(kR)_R = -kJ_1(kR) \). The initial data for \( \psi(T, R) \) and \( \pi_{\psi}(T, R) \) are calculated for \( T = 0 \) and \( R = r \) and are, respectively, denoted by \( Q(r) \) and \( P(r) \) as follows

\[
Q(r) = \psi_0(r) = \psi(T, R)|_{T=0, R=r} = \int_0^\infty dk J_0(kr) \left(A(k) + A^*(k) \right) \tag{14}
\]
\[ P(r) = \pi_\psi(r) = \pi_\psi(T, R)|_{T=0, R=r} = ir \int_0^\infty dk k J_0(kr)(A(k) - A^*(k)) \] (15)

Solving the last two equations for \( A(k) \) and \( A^*(k) \) one obtains

\[ A(k) = \frac{1}{2} \int_0^\infty dr J_0(kr)(krQ(r) - iP(r)) \] (16)

\[ A^*(k) = \frac{1}{2} \int_0^\infty dr J_0(kr)(krQ(r) + iP(r)) \] (17)

One may show, using Eqs (16)-(17), that the variables \( A(k) \), \( A^*(k) \) satisfy the following Poisson brackets

\[ \{A(k), A^*(k')\} = \int_0^\infty dr \left[ \frac{\delta(A(k)) \delta(A^*(k'))}{\delta(Q(r)) \delta(P(r))} - \frac{\delta(A(k)) \delta(A^*(k))}{\delta(P(r)) \delta(Q(r))} \right] = \frac{i\delta(k - k')}{2} \] (18)

where use was made of the relation \[ \int_0^\infty dr J_n(kr)J_n(k'r) = \frac{1}{k'}\delta(k - k'), \quad n = 0, 1, 2, \ldots \]

In a similar manner, using Eqs (14)-(15), it is possible to show that the variables \( Q(r) \) and \( P(r) \) satisfy the following Poisson brackets

\[ \{Q(r), P(r')\} = \int_0^\infty dk \left[ \frac{\delta(Q(r)) \delta(P(r'))}{\delta(A(k)) \delta(A^*(k))} - \frac{\delta(Q(r)) \delta(P(r'))}{\delta(A^*(k)) \delta(A(k))} \right] = -2i\delta(r - r') \]

\[ \{Q(r), Q(r')\} = \{P(r), P(r')\} = 0, \] (19)

where use was made of the relation \[ \int_0^\infty dk k J_n(kr)J_n(k'r) = \frac{\delta(r - r')}{r'}, \quad n = 0, 1, 2, \ldots \]

Using Eqs (6), (9) and the unnumbered relation written just after Eq (18) one may obtain
the following expression for the energy $\Gamma$ \cite{36}

$$\Gamma = 2 \int_{0}^{\infty} dk kA(k)A^*(k) \quad (20)$$

As shown in \cite{36} the former expression (20) is for $R$ at infinity and it shows that the energy remains finite for this case. We are interested in other expression for the energy in terms of $Q(r)$ and $P(r)$ which is derived from the expression (20) by using Eqs (16)-(17) and the Poisson brackets (19)

$$\Gamma - k = \int_{0}^{\infty} dr \left( \frac{P^2(r)}{2r} + \frac{rk^2Q^2(r)}{2} \right), \quad (21)$$

where the $k$ term is obtained by using the relation (19) and the unnumbered relation just after it. The term under the integral sign involving $P^2(r)$ and $Q^2(r)$ is obtained by using the following orthogonality expression which is valid for any $n = 0, 1, 2, \ldots$

$$\int_{0}^{\infty} dr' r' \int_{0}^{\infty} dk k J_n(kr)J_n(kr')f(r') = f(r) \quad (22)$$

Note that the energy at the right hand side of Eq (21) corresponds to the energy of the harmonic oscillator \cite{9} $H_{\text{harmonic oscil}} = \frac{p^2}{2m} + \frac{1}{2}mw^2r^2$ not only in its form but also in its role in the following commutation relations (23)-(24) and in Eqs (25)-(32).

3 The occupation number representation for cylindrical field

In this section we apply the occupation number formalism \cite{9,10} for the cylindrical GW. As known \cite{9,10}, the passage from the classical domain to the quantum one entails regarding the classical variables as operators \cite{9,10} and the change, for any function $f$, of $f^* \rightarrow f^+$ where $f^+$ is the hermitian adjoint of $f$ \cite{9,10}. Also, the known Poisson brackets $\{f, f^*\}$ are replaced by the quantum commutation ones $[f, f^+]$ obtained through $\{f, f^*\} \rightarrow \frac{[f, f^+]}{\hbar}$.
where, as mentioned, we assign the unity value to $\hbar$. In the following we use Eqs (18) for calculating the commutation relations between $Q(r)$, $P(r)$ from Eq (14)-(15) and the energy $\int_0^\infty dr \left( \frac{P^2(r)}{2r} + \frac{rk^2Q^2(r)}{2} \right)$ from Eq (21). Thus, the commutation relation between the observable $Q(r)$ from Eq (14) and the former energy $(\Gamma - k)$ from Eq (21) is

$$[Q(r), (\Gamma - k)] = \left[ \int_0^\infty dk J_0(kr) \left( A(k) + A^+(k) \right), 2 \int_0^\infty dk'k' A(k')A^+(k') - k \right] =$$

$$= \int_0^\infty \int_0^\infty dkd' J_0(kr)k' \left( A(k')\delta(k - k') - A^+(k')\delta(k' - k) \right) =$$

$$= \int_0^\infty dk'k' J_0(k'r) \left( A(k') - A^+(k') \right) = \frac{P(r)}{ir} \tag{23}$$

Likewise, using again Eqs (18) and the expression (21) for the relevant energy $(\Gamma - k)$, we calculate the commutation relation between $P(r)$ from Eq (15) and this energy

$$[P(r), (\Gamma - k)] = \left[ ir \int_0^\infty dkk J_0(kr) \left( A(k) - A^+(k) \right), 2 \int_0^\infty dk'k' A(k')A^+(k') - k \right] =$$

$$= -\frac{r}{i} \int_0^\infty dkk^2 J_0(kr) \left( A(k) + A^+(k) \right) = -\frac{rk^2Q(r)}{i} \tag{24}$$

Using the Dirac’s ket and bra notation [9, 10] for the matrix representation of the observables $Q(r)$, $P(r)$, $(\Gamma - k)$ and taking into account that $|j><j|$ is unit operator one may obtain from the two ends of the first commutation relation (23)

$$<k|Q|j> <j|(\Gamma - k)|l> - <k|(\Gamma - k)|j> <j|Q|l> =$$

$$= (E_l - E_k) <k|Q|l> = \frac{<k|P|l>}{ir}, \tag{25}$$

where $E_l$ and $E_k$ are the eigenvalues of the energy operator $(\Gamma - k)$ which, respectively, correspond to the kets $|l>$ and $|k>$. In a similar manner one may obtain from the two ends
of the second commutation relation \([24]\)

\[
<k|P|j><j|(\Gamma - k)|l> - <k|(\Gamma - k)|j><j|P|l> = (E_l - E_k) <k|P|l> = i k^2 r <k|Q|l>
\]  

(26)

Solving the last equation for \(<k|Q|l>\) and substituting in \((25)\) one obtains for \((E_l - E_k)\)

\[
(E_l - E_k) = \pm k
\]  

(27)

Now, multiplying Eq \((25)\) by \(-irk\) and adding to \((26)\) one obtains

\[
(E_l - E_k - k) <k|P|l> + irk (E_l - E_k) <k|Q|l> = 0
\]  

(28)

From the last equation one may realize that \(<k|(P + i rkQ)|l>\) is different from zero only when \(E_k = E_l - k\). That is, operating with the operator \((P + i rkQ)\) on the ket \(|l>\) results in some multiple of the ket \(|k>\) with an energy lower by \(k\) than that of the ket \(|l>\). If, on the other hand, Eq \((25)\) is multiplied by \(irk\) and then added to \((26)\) one comes with the result that operating with the hermitian adjoint operator \((P - i rkQ)\) on the ket \(|l>\) results in some multiple of the ket \(|k>\) with an energy higher by \(k\) than that of the ket \(|l>\). Note that for the harmonic oscillator these two operators change roles so that \((P + i rkQ)\) is the energy raising operator whereas \((P - i rkQ)\) is the energy lowering one (see Section 25 in \([9]\)). Now, since the energy must be positive one can not apply indefinitely the lowering operator \((P + i rkQ)\) on any ket unless there exists a lowest energy eigenstate \(|0>\) so that \((P + i rkQ)|0> = 0\) (compare with the analogous discussion in \([9]\) regarding the harmonic oscillator). The lowest energy eigenvalue corresponding to the lowest eigenstate may be found by first operating
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with $(P - i r k Q)$ on $(P + i r k Q)|0>$ and using the quantum version of the relations (19)

$$(P - i r k Q)(P + i r k Q)|0> = \left( P^2 + r^2 k^2 Q^2 - i r k (Q P - P Q) \right)|0> =$$

$$= 2r \left( \frac{P^2}{2r} + \frac{r k^2 Q^2}{2} - k \delta(r - r') \right)|0> = 0$$

We, now, divide the last expression by $2r$ and integrate the obtained result with respect to $r$

$$\left\{ \int_0^\infty dr \left( \frac{P^2}{2r} + \frac{r k^2 Q^2(r)}{2} - k \delta(r - r') \right) \right\}|0> = \left\{ \int_0^\infty dr \left( \frac{P^2}{2r} + \frac{r k^2 Q^2(r)}{2} \right) - (30) - k \right\}|0> = 0$$

We, thus, see that the eigenvalue which corresponds to the lowest energy eigenstate $|0>$ is $k$.

Note that in the harmonic oscillator case one obtains a value of $\frac{k}{2}$ for the lowest eigenvalue (see Section 25 in [9]) because the value obtained there for the commutation relation $[Q, P]$ is half the value obtained here. As mentioned, application of the raising operator $(P - i r k Q)$ on an arbitrary ket results in raising its energy by $k$ and repeated application of it generates a sequence of eigenstates that may appropriately be denoted by $|n>$. The energy eigenvalues of this sequence may be expressed by

$$E_n = (n + 1)k, \quad n = 0, 1, 2, 3, \ldots$$

(31)

Now, if we, respectively, multiply the lowering and raising operators $(P + i r k Q), (P - i r k Q)$ by $-i\frac{J_0(kr)}{r}$ and $i\frac{J_0(kr)}{r}$ and integrating the resulting expressions with respect to $r$ from $r = 0$ to $r = \infty$ one, actually, obtains the variables $A(k)$ and $A^+(k)$ as seen from Eqs (16)-(17).

Thus, one may call the variable $A(k)$, in analogy with the harmonic oscillator case [9, 10], the energy lowering or destruction operator and the variable $A^+(k)$ may, correspondingly, be called the energy raising or creation operator. This may also be seen from Eqs (20)-(21).
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which may be read

\[ 2 \int_0^\infty dk k A(k) A^+(k) = \int_0^\infty dr \left( \frac{P^2(r)}{2r} + \frac{rk^2Q^2(r)}{2} \right) + k, \]

and also to read, when the places of \( A(k) \) and \( A^+(k) \) are commuted,

\[ 2 \int_0^\infty dk k A^+(k) A(k) = \int_0^\infty dr \left( \frac{P^2(r)}{2r} + \frac{rk^2Q^2(r)}{2} \right) - k \]

The last equation may be written as

\[ \left( 2 \int_0^\infty dk A^+(k) A(k) + 1 \right) k = \int_0^\infty dr \left( \frac{P^2(r)}{2r} + \frac{rk^2Q^2(r)}{2} \right) \]  \hspace{1cm} (32)

Comparing Eqs (31) and (32) one may realize, as done in the corresponding harmonic oscillator case \([9]\), that to the eigenvalues \( n \) of Eq (31) there corresponds the operator \( N \)

\[ N = \int_0^\infty dk N_k = \int_0^\infty dk (2A^+(k) A(k)), \]  \hspace{1cm} (33)

so that \( N_k = 2A^+(k) A(k) \). Now, as realized from all the former equations, the variable \( k \) was considered to be continuous but one may refer to it as a discrete parameter (see, for example, the discussion in P. 502 at \([9]\) and Eqs (54.1) and (55.11) there) in which case one obtains the same former expressions and results except for replacing all the integrals over \( k \) by corresponding summations. In such case Eq (33) may be written as \( N = \sum_k N_k = \sum_k 2A^+_k A_k \) where \( N_k = 2A^+_k A_k \). Thus, it may be shown, as for the corresponding harmonic oscillator case, that considering the representation in which each \( N_k \) is diagonalized, the states of the quantized field may be represented, as done in \([9]\), by the kets \( |n_1, n_2, n_3, \ldots, n_k, \ldots \rangle \) in which each \( n_k \) is a positive integer or zero and it is an eigenvalue of \( N_k \). Also, since in this representation only the diagonal matrix elements of \( A^+(k) A(k) \) are nonzero one may write
such an element as

\begin{equation}
<n_k|A^+(k)A(k)|n_k> = <n_k|A^+(k)|n'_k><n'_k|A(k)|n_k> = |\lambda_n|\bar{n} = n,
\end{equation}

(34)

where a summation over a complete set \( n'_k \) is meant and \( \lambda_n \) is equal to \( n^{\frac{1}{2}} \). Thus, from the last equation one may obtain the relations

\begin{align}
A(k)|n_1, n_2, n_3, \ldots, n_k, \ldots> &= n^\frac{1}{2}_k|n_1, n_2, n_3, \ldots, n_k - 1, \ldots>
\end{align}

(35)

\begin{align}
A^+(k)|n_1, n_2, n_3, \ldots, n_k, \ldots> &= (n_k + 1)^{\frac{1}{2}}|n_1, n_2, n_3, \ldots, n_k + 1, \ldots>
\end{align}

(36)

The last two equations shows that \( A(k) \) and \( A^+(k) \) are, respectively, the destruction and creation operators for the state \( k \) of the field. We, thus, have shown that the quantized cylindrical GW may be discussed in the commutation number representation in which it may be thought of as composed of a large number of particle-like components inhabiting a certain finite spacetime region. This special representation enables us, as mentioned, to discuss the cylindrical GW in space Zeno terminology as we do in the following Sections V-VI.

4 The lapse and shift function corresponding to the cylindrical gravitational wave

For finding the appropriate lapse and shift functions \( N^a \) which correspond to the cylindrical GW’s in the canonical formalism we begin from the Hamilton equation (10) in its quantum version \( \dot{\psi} = \frac{i}{\hbar}[\psi, H] \) where \( \psi \) is given by Eq (9) and the square brackets at the right denote the quantum commutation relations. The expression for the energy \( H \) which involves the operators \( A(k) \) and \( A^+(k) \) is given by Eqs (20)-(21) so substituting from Eqs (9) and (20)-(21)
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in \( \dot{\psi} = \frac{i}{\hbar} [\psi, H] \) one obtains

\[
\frac{\partial}{\partial t} \left( \int_0^\infty dk J_0(kR) \left( A(k) e^{(ikT)} + A^+(k) e^{-(ikT)} \right) \right) = \frac{1}{i} \left[ \int_0^\infty dk J_0(kR) \left( A(k) e^{(ikT)} + A^+(k) e^{-(ikT)} \right), 2 \int_0^\infty dk k A(k) A^+(k) - k \right]
\]

(37)

In the following we introduce the lapse and shift functions (see, especially, Eqs (38)-(39)) using the method described in [36]. Thus, the time differentiation at the left hand side of the last equation may be performed by using the embedding [36] \( X^\alpha = (T(r), R(r), \phi, z) \) which maps the hypersurface \( \Sigma \) at \( t = \text{constant} \) into the flat spacetime as \( X^\alpha : \Sigma \rightarrow \mathbb{R}^4 \). The embeddings \( X^\alpha = (T(r), R(r), \phi, z) \) are cylindrically symmetric slices in spacetime where, due to this symmetry, \( T(r) \) and \( R(r) \) do not depend on \( \phi \) and \( z \). Thus, the time differentiation \( \frac{\partial \psi(r)}{\partial t} \) may be written as

\[
\frac{\partial \psi(r)}{\partial t} = \frac{\partial \psi(r)}{\partial X^\alpha(r)} \frac{dX^\alpha(r)}{dt} = N^\alpha \frac{\partial \psi(r)}{\partial X^\alpha(r)},
\]

(38)

where \( dX^\alpha(r)/dt = N^\alpha \). Thus, using the last equation, the relation [47] \( \frac{dJ_0(r)}{r} = -J_1(r) \) and the fact that \( \psi \) does not depend on \( \phi \) and \( z \) one may calculate the commutation relation from Eq (37) as

\[
\frac{\partial \psi(r)}{\partial t} = N^0 \frac{\partial \psi(r)}{\partial T(r)} + N^1 \frac{\partial \psi(r)}{\partial R(r)} = iN^0 \int_0^\infty dk k J_0(kR) (A(k) e^{ikT}) - A^+(k) e^{-(ikT)}) - N^1 \int_0^\infty dk k J_1(kR) (A(k) e^{ikT} + A^+(k) e^{-(ikT)}) =
\]

(39)

\[
= 2 \int_0^\infty dk \int_0^\infty dk' J_0(kr) k' \left( e^{(ikT)} A^+(k') \frac{\delta(k - k')}{2} - e^{-(ikT)} A^+(k') \frac{\delta(k' - k)}{2} \right)
\]

Since the last equation involves real and imaginary expressions we have to decompose it into two equations one of which relates the real expressions among themselves and the second
the imaginary ones. Thus, the equation involving the real expressions is

\[- N^0 \int_0^\infty dk k J_0(kR) \sin(kT) \left(A(k) + A^+(k)\right) - N^1 \int_0^\infty dk k J_1(kR) \cos(kT) \left(A(k) + A^+(k)\right) = \int_0^\infty dk k J_0(kR) \cos(kT) \left(A(k) - A^+(k)\right)\]

(40)

And that involving the imaginary ones is

\[N^0 \int_0^\infty dk k J_0(kR) \cos(kT) \left(A(k) - A^+(k)\right) - N^1 \int_0^\infty dk k J_1(kR) \sin(kT) \left(A(k) - A^+(k)\right) = \int_0^\infty dk k J_0(kR) \sin(kT) \left(A(k) + A^+(k)\right)\]

(41)

We integrate both sides of the last two equations (40)-(41) over \( r \) from \( r = 0 \) to \( r = \infty \) and also for avoiding the intricacy of the resulting expressions and facilitating the following calculation we label these expressions as

\[C_1 = \int_0^\infty dr \int_0^\infty dk k J_0(kr) \sin(kT) \left(A(k) + A^+(k)\right)\]

\[C_2 = \int_0^\infty dr \int_0^\infty dk k J_1(kr) \cos(kT) \left(A(k) + A^+(k)\right)\]

(42)

\[C_3 = \int_0^\infty dr \int_0^\infty dk k J_0(kr) \cos(kT) \left(A(k) - A^+(k)\right)\]

\[C_4 = \int_0^\infty dr \int_0^\infty dk k J_1(kr) \sin(kT) \left(A(k) - A^+(k)\right)\]

Thus, Eqs (40)-(41) integrated over \( r \), may be compactly written as

\[- C_1 N^0 - C_2 N^1 = C_3, \quad C_3 N^0 - C_4 N^1 = C_1\]

(43)

Solving the last two equations for \( N^0 \) and \( N^1 \) one obtains

\[N^0 = \frac{(C_1 C_2 - C_4 C_3)}{(C_1 C_4 + C_2 C_3)}, \quad N^1 = -\frac{(C_1 C_1 + C_3 C_3)}{(C_1 C_4 + C_2 C_3)}\]

(44)
The expressions \((C_1C_4 + C_2C_3), (C_1C_2 - C_4C_3), (C_1C_1 + C_3C_3)\) are calculated in Appendix A and are, respectively, given by Eqs \((A_2)-(A_4)\) there. Thus, substituting these results of Appendix A in Eqs \((44)\) we obtain the following results for \(N^0\) and \(N^1\).

\[
N^0 = \frac{\int_0^\infty dk k \sin(2Tk) \left( 2A(k)A^+(k) - \frac{1}{2} \right)}{\int_0^\infty dk k \left( A(k)A(k) - A^+(k)A^+(k) - \frac{1}{2} \right)} \tag{45}
\]

\[
N^1 = 2 \int_0^\infty dk k^2 \left\{ \cos(2Tk) \left( A(k)A(k) + A^+(k)A^+(k) \right) - \left( 2A(k)A^+(k) - \frac{1}{2} \right) \right\} \frac{r \int_0^\infty dk k \left( A(k)A(k) - A^+(k)A^+(k) - \frac{1}{2} \right)}{r \int_0^\infty dk k \left( A(k)A(k) - A^+(k)A^+(k) - \frac{1}{2} \right)} \tag{46}
\]

The last Eqs \((45)-(46)\) express the lapse and shift functions, which are an inherent part of the 4-dimensional metric tensor \([14]\), in terms of the gravitational cylindrical creation and destruction operators \(A(k), A^+(k)\) which determine the constituents of the related GW and, therefore, the geometry it imposes upon the neighbouring space-time.

5 Space Zeno effect for the cylindrical GW

For demonstrating the Zeno effect in the cylindrical GW we use the Dirac quantization of the cylindrical GW in the half parametrized formalism as represented in \([11]\). We may use the ADM quantization \([11]\) but we prefer to discuss the mentioned Dirac one which is more general \([11]\) than that of ADM. This quantization in the half parametrized formalism is equivalent, as shown in \([11]\) (see Section XII there), to the the full parametrized one and so one may discuss the former formalism without losing anything. In this formalism the function \(r(R)\) is equated to \(R\) so that the two remaining canonical coordinates \(T(r(R)), \psi(r(R))\) may be written as \(T(r(R)) = T(R)\) and \(\psi(r(R)) = \psi(R)\). These variables, as well as their canonically conjugate momenta \(\Pi_T(R)\) and \(\pi_\psi(R)\), are replaced in the quantum
theory [9] by operators so that one may write, for example, the conjugate momenta as the variational derivatives

\[
\Pi_T(R) = -i \frac{\delta}{\delta T(R)}, \quad \pi_\psi(R) = -i \frac{\delta}{\delta \psi(R)} \tag{47}
\]

In the representation in which the former canonical coordinates \(T(R)\) and \(\psi(R)\) are diagonal the state functional \(\Psi\) depends on these two functions and its behaviour changes according to the following Schrödinger-type equation

\[
\frac{i}{\delta T(R)} \frac{\delta \Psi(T(R), \psi(R))}{\delta T(R)} = \mathcal{H}(T_{,R}(R), \psi(R), \pi_\psi(R)) \Psi(T(R), \psi(R)), \tag{48}
\]

where in the half parametrized formalism \(\mathcal{H}\) is given by [11]

\[
\mathcal{H} = \frac{1}{2} \left(1 - T_{,R}^2(R)\right)^{-1} \left( -i R^{-\frac{1}{2}} \frac{\delta}{\delta \psi(R)} - R^{\frac{3}{2}} T_{,R}(R) \psi_{,R}(R) \right)^2 + \frac{1}{2} R \psi_{,R}^2(R) =
\]

\[
= \frac{1}{2(1 - T_{,R}^2(R))} \left( R^{-1} \pi_\psi^2(R) - 2 T_{,R}(R) \pi_\psi(R) \psi_{,R}(R) + \frac{1}{2} R \psi_{,R}^2(R) \right) \tag{49}
\]

The last result was obtained by using the second equation of (47) for the operator \(\pi_\psi(R)\) and noting that the commutation relation between \(\pi_\psi(R)\) and \(\psi_{,R}(R)\) is zero at the same point, i.e., \([\psi_{,R}(R), \pi_\psi(R')] = i \frac{\delta \psi_{,R}(R)}{\delta \psi(R')} = i \frac{\delta \psi(R)}{\delta \psi(R')} = i \frac{d \delta(R-R')}{dR} = 0\) because of the antisymmetry of the \(\delta\) function by which one have \(\frac{d \delta(0)}{dR} = 0\) (see the beginning of Section XI in [11] and the first two unnumbered equations there). Note that the variational derivative at the left of (48) is with respect to the extrinsic time \(T(R)\) and \(T_{,R}(R), \psi_{,R}(R)\) denote derivatives of \(T(R)\) and \(\psi(R)\) with respect to \(R\). One may also note that although Eq (48) is a functional differential equation which is a whole set of equations, one for each value of \(R\), it may be discussed for the purpose of this section, which is the demonstration of the space Zeno effect for the cylindrical GW (as emphasized after the following Eq (51)), as if it was a partial differential equation. This is, especially, true when assigning to \(R\) some specific value. In
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such case the solution to the Schroedinger-type equation (48) is

\[ \Psi(T(R), \psi(R)) = \exp\left(-i\mathcal{H}(T_R(R), \psi(R))T(R)\right)\Psi(0, \psi(R)), \]  

(50)

where \( \Psi(0, \psi(R)) \) is the state functional on the hypersurface \( T(R) = 0 \). Indeed, differentiating \( \Psi(T(R), \psi(R)) \) from the last equation with respect to \( T(R) \) and taking into account that \( \mathcal{H}(T_R(R), \psi(R)) \) does not depend explicitly on \( T(R) \) one obtains the Schroedinger-type equation (48). As emphasized in [11], the set of equations for all values of \( R \) which are implied by the functional differential Eq (48) are not mutually independent and may be reduced, as shown in [11], to a single differential equation as follows.

\[ i\frac{\partial}{\partial t}(\bar{\Psi}(T(R, t), \psi(R))) = \bar{H}_t(\psi(R), \pi_\psi(R))\bar{\Psi}(T(R, t), \psi(R)), \]

where

\[ \bar{H}_t(\psi(R), \pi_\psi(R)) = \int_0^\infty dR \mathcal{H}(T_R(R, t), \psi(R), \pi_\psi(R))\frac{\partial(T(R, t))}{\partial t} \]

The \( t \) is a labelling parameter from some closed range, for example, \([0, 1]\) and \( \mathcal{H}(T_R(R, t), \psi(R), \pi_\psi(R)) \) is the same as that of Eq (48)-(50). The solution to the former single differential equation is [11]

\[ \bar{\Psi}(T(R), \psi(R)) = P \exp(-i \int_0^1 dt \bar{H}_t)\bar{\Psi}_0(\psi(R)), \]

where \( P \) is a time-ordering operator. As mentioned, we prefer, for demonstrating the space Zeno effect for the cylindrical GW, to directly discuss the Hamiltonian density \( \mathcal{H}(T(R), \psi(R), \pi_\psi(R)) \) and Eq (48) with the understanding that all the results obtained may be applied also to the single differential equation. This is realized when noting from the former equations that \( \bar{\Psi}(T(R), \psi(R)) \) is in effect a very large number, over all values of \( R \) and \( t \) (in the range \( 0 \leq t \leq 1 \)), of multiplying products of the kind of \( \Psi(T(R), \psi(R)) \) from Eq (50). Thus, since it has been proved in Appendix B and the following equations that for
the function $\Psi(T(R), \psi(R))$ the relevant Zeno probability $P_{r(1)}(|\Psi(0, \psi(R)) >, \rho)$ tends to unity so it may be seen that for the function $\Psi(T(R), \psi(R))$ the relevant Zeno probability $P_{r(1)}(|\Psi(0, \psi(R)) >, \rho)$ becomes a very large number of multiplying products each tending to the unity value and, therefore, this probability tends as well to this value. This, of course, takes into account that $\Psi(T(R), \psi(R))$ shares the same characteristics with $\Psi(T(R), \psi(R))$, especially, the normalization property (see the discussions before Eq (B14) and after Eq (B12) in Appendix B).

We show here that under the influence of space Zeno effect the state $\Psi(T(R), \psi(R))$ remain stable in space. For that purpose we have calculated in Appendix B, using the shift operator $e^{iP\rho}$, the probability $P_{r(1)}(|\Psi(0, \psi(R)) >, \rho)$ for the state function $\Psi(0, \psi(R))$ from Eq (50) (we also use the Dirac’s bra notation $\langle 9, 10 | \Psi(0, \psi(R)) >$) to remain at the same state after shifting it by the small amount $\rho$. This probability, after the single shift, is given in (B15) in Appendix B as

$$
P_{r(1)}(|\Psi(0, \psi(R)) >, \rho) = \left| \langle \Psi(0, \psi(R)) | e^{iP\rho} | \Psi(0, \psi(R)) \rangle \right|^2 = 1 + 4\eta^2 + 6\eta^4 + \rho^2\eta^4 \left( \frac{B\pi_\psi(R)}{D} \right)^2 + \eta^2 \left( \frac{\rho^2}{D^2} \left( 2A\pi_\psi(R) + B \right)^2 + \sin \left( \frac{\rho}{D} B\pi_\psi(R) \right) \right) \cdot \left\{ 2\eta^3 \left( \frac{\rho}{D} B\pi_\psi(R) \right) \left( 2A\pi_\psi(R) + B \right) - 2\eta^2 \frac{\rho B\pi_\psi(R)}{D} (1 + 2\eta^2) \right\} -$$

$$- \cos \left( \frac{\rho}{D} B\pi_\psi(R) \right) \left( 2\eta^3 \frac{\rho^2 B}{D^2\pi_\psi(R)} \left( 2A\pi_\psi(R) + B \right) + 2\eta^2 (1 + 2\eta^2) \right) +$$

$$+ 2\eta^3 \frac{\rho}{D\pi_\psi^2(R)} \left( 2A\pi_\psi(R) + B \right) \sin \left( \frac{\rho}{D} A\pi_\psi(R) \right) - 2\eta^2 (1 + 2\eta^2) \right\} +$$

$$+ 2\eta^3 \left( A\pi_\psi^2(R) - 2\eta^4 \frac{\rho B\pi_\psi(R)}{D} \sin \left( \frac{\rho}{D} \left( A\pi_\psi^2(R) - B\pi_\psi(R) \right) \right) \right) +$$

$$+ 2\eta^4 \cos \left( \frac{\rho}{D} \left( A\pi_\psi^2(R) - B\pi_\psi(R) \right) \right)$$

where $A, B, C, D$ are given by Eqs (B6) in Appendix B. The relevant state which we want to keep fixed represents the cylindrical GW. We, now, wish to generalize this single shift in one region of space to an arbitrary number of identical shiftings simultaneously performed in
a large number of independent regions of space. For that we construct an operator, as done in [31], which regulate the simultaneous identical shiftings in separate regions of spacetime in terms of the known quantum projection operator $|\phi\rangle\langle\phi|$ and its translate in shifts of amount $\rho$ which is denoted here for the single shift as $P_{tr}^{(1)} = e^{-iP\rho}pe^{iP\rho} = e^{-iP\rho} <\phi|e^{iP\rho}|\phi>$. Taking the absolute square of the last expression one obtains

$$|P_{tr}^{(1)}|^2 = |e^{-iP\rho}pe^{iP\rho}|^2 = |<\phi|e^{iP\rho}|\phi>|^2$$

which, assuming for $\phi >$ the state functional $|\Psi(0,\psi(R))>$, amount to the probability $Pr^{(1)}(|\Psi(0,\psi(R))>,\rho)$ from Eq (51). That is, $Pr^{(1)}(|\Psi(0,\psi(R))>,\rho) = |P_{tr}^{(1)}|^2$. For the double identical shifts of the state functional $|\Psi(0,\psi(R))>$, each of amount $\rho$, simultaneously performed in two separate regions of space-time one may write for $P_{tr}^{(2)}$ and its absolute square $|P_{tr}^{(2)}|^2$

$$P_{tr}^{(2)} = e^{-i2P\rho}pe^{i2P\rho}e^{-iP\rho}pe^{iP\rho} = e^{-i2P\rho}pe^{iP\rho}pe^{iP\rho}$$

$$|P_{tr}^{(2)}|^2 = |e^{-i2P\rho}pe^{iP\rho}|^2 = |e^{-i2P\rho}(<\Psi(0,\psi(R))|e^{iP\rho}|\Psi(0,\psi(R))>)|^2 =$$

$$= |(<\Psi(0,\psi(R))|e^{iP\rho}|\Psi(0,\psi(R))>)|^2,$$

which is identical to the probability to remain at the same state $|\Psi(0,\psi(R))>$ after these two simultaneous shiftings performed in two separate regions of space-time. That is, $|P_{tr}^{(2)}|^2 = Pr^{(2)}(|\Psi(0,\psi(R))>,\rho) = Pr^{(1)}(|\Psi(0,\psi(R))>,\rho)^2$. The generalization to $n$ arbitrary shiftings of $|\Psi(0,\psi(R))>$, each of amount $\rho$, in $n$ separate subregions of space, all included
in the finite larger region denoted by \( X \), is

\[
\mathcal{P}^{(n)}_{tr} = e^{-inP\rho}e^{inP\rho}e^{-i(n-1)P\rho}e^{i(n-1)P\rho} \cdots e^{-iP\rho}e^{iP\rho} = e^{-inP\rho}\Psi(0,\psi(R)) > \left( <\Psi(0,\psi(R))|e^{iP\rho}|\Psi(0,\psi(R)) > \right)^{n-1} (54)
\]

\[
\cdot <\Psi(0,\psi(R))|e^{iP\rho} = e^{-inP\rho}\left( <\Psi(0,\psi(R))|e^{iP\rho}|\Psi(0,\psi(R)) > \right)^n
\]

The absolute square of the last expression is equal to the probability to remain at the same state after \( n \) simultaneous shiftings, each of amount \( \rho \), in \( n \) separate independent regions of spacetime. That is, raising the probability \( Pr^{(1)}(|\Psi(0,\psi(R)) >, \rho) \) from Eq (51) to the \( n \)-th power one obtains

\[
|\mathcal{P}_{tr}^{(n)}|^2 = \left| e^{-inP\rho}\Psi(0,\psi(R)) > \left( <\Psi(0,\psi(R))|e^{iP\rho}|\Psi(0,\psi(R)) > \right)^{n-1} <\Psi(0,\psi(R))|e^{iP\rho} > \right|^2 = \left| \left( <\Psi(0,\psi(R))|e^{iP\rho}|\Psi(0,\psi(R)) > \right)^n \right| = \left( Pr^{(1)}(|\Psi(0,\psi(R)) >, \rho) \right)^n = \left\{ 1 + 4n^2 + 6n^4 + \rho^2 \eta^2 \left( \frac{B_{\pi\psi}(R)}{D} \right)^2 + \eta^2 \frac{\rho B_{\pi\psi}(R)}{D^2} \left( 2A_{\pi\psi}(R) + B \right) \right. \\
+ \sin \left( \frac{\rho}{D} B_{\pi\psi}(R) \right) \left\{ 2n^3 \frac{\rho}{D^2\pi\psi(R)} \left( 2A_{\pi\psi}(R) + B \right) - 2n^2 \frac{\rho B_{\pi\psi}(R)}{D} \left( 1 + 2n^2 \right) \right. \}
\]

\[
- \cos \left( \frac{\rho}{D} B_{\pi\psi}(R) \right) \left\{ 2n^3 \frac{\rho^2 B}{D^2\pi\psi(R)} \left( 2A_{\pi\psi}(R) + B \right) + 2n^2 \left( 1 + 2n^2 \right) \} + \\
+ 2n^3 \frac{\rho}{D^2\pi\psi(R)} \left( 2A_{\pi\psi}(R) + B \right) \sin \left( \frac{\rho}{D} A_{\pi\psi}(R) \right) - 2n^2 \left( 1 + 2n^2 \right) \cos \left( \frac{\rho}{D} A_{\pi\psi}(R) \right) - \\
- 2n^4 \frac{\rho B_{\pi\psi}(R)}{D} \sin \left( \frac{\rho}{D} \left( A_{\pi\psi}(R) - B_{\pi\psi}(R) \right) \right) + 2n^4 \cos \left( \frac{\rho}{D} \left( A_{\pi\psi}(R) - B_{\pi\psi}(R) \right) \right) \right\} \frac{X}{\rho},
\]

where the \( n \), which refers to the \( n \) equal shiftings each of amount \( \rho \), were written as \( \frac{X}{\rho} \) in which \( X \) is the finite region of space which includes all the \( n \) smaller subregions. We, now,
approach the Zeno limit of $\lim_{\rho \to 0}$ and note that in this limit one have

\[
\lim_{\rho \to 0} \left( \cos\left(\frac{\rho}{D} A\pi^2_\psi(R)\right) \right) = \lim_{\rho \to 0} \left( \cos\left(\frac{\rho}{D} B\pi_\psi(R)\right) \right) = 1
\]

\[
\lim_{\rho \to 0} \left( \sin\left(\frac{\rho}{D} A\pi^2_\psi(R)\right) \right) = \lim_{\rho \to 0} \left( \sin\left(\frac{\rho}{D} B\pi_\psi(R)\right) \right) = 0
\]

Thus, using the last limits, one obtains for the overall probability $(Pr^{(1)}(\Psi(0, \psi(R)), \rho))^n$ from Eq (55) in the limit $\rho \to 0$

\[
\lim_{\rho \to 0} \left( Pr^{(1)}(\Psi(0, \psi(R)), \rho) \right)^n = 1,
\]

where we use the mentioned equality $n = \frac{X}{\rho}$. Note that the last result is obtained for any value of the eigenvalue $\eta$. Thus, it is shown that in the limit of space Zeno effect the cylindrical GW, which is interpreted in the framework of quantum mechanics in probability terms, have a unity probability for being found in some spacetime region and, therefore, if it is strong enough for implanting its geometry upon this neighbourhood.

6 Graphical representation of the gravitational space

Zeno effect

We, now, wish to corroborate our former results through numerical and graphical representation. For that we should note that the idea of the space Zeno effect, which is to remain with the same cylindrical geometry across all space, necessitates a constancy of the function $T(R)$ across all the $n$ subregions of space. Thus, one may assume $\frac{dT(R)}{dR} \approx 0$ which entails $\frac{d^2T(R)}{dR^2} = \left(\frac{dT(R)}{dR}\right)^2 = 0$. In such case the functions $A, B, C, E$ from Eqs (66) assume a very
simple form. That is

\[ A_{(T, R=0)} = T(R), \quad B_{(T, R=0)} = 0 \]

\[ C_{(T, R=0)} = -T(R) \left( \frac{R^3}{2} + R \rho(R + \frac{\rho}{2}) \right) \left( \rho \left( \frac{d^2 \psi(R)}{dR^2} \right)^2 + 2 \psi(R) \frac{d^2 \psi(R)}{dR^2} \right) - \frac{R}{2} (R + \rho) T(R) \psi_R^2(R) \]

\[ D_{(T, R=0)} = 2R(R + \rho) \]

Substituting from the last \( A_{(T, R=0)}, B_{(T, R=0)} \) and \( D_{(T, R=0)} \) in Eq (55) for the probability one obtains

\[ \left( P^{(1)}_{(T, R=0)}(\Psi(0, \psi(R)), \rho) \right) \frac{\pi}{\rho} = \left( \left| <\Psi(0, \psi(R)) | e^{iP \rho} | \Psi(0, \psi(R)) > \right|_{(T, R=0)} \right) \frac{\pi}{\rho} \]

\[ = \left( 1 + 4\eta^2 + 6\eta^4 + 4\eta^2 \frac{\rho^2}{D^2} \pi^2(R) - 2\eta^2 (1 + 2\eta^2) - 2\eta^2 (1 + \eta^2) \cos \left( \frac{\rho}{D} T(R) \pi^2(R) \right) + 4\eta^3 \frac{\rho}{D} \pi \frac{\rho}{D} T(R) \pi^2(R) \right) \frac{\pi}{\rho} \]

(58)

For \( \psi(R) \) we use the quantum version of Eq (8) for a specific value of \( k \) and for \( \pi \psi(R) \) we take into account that we discuss here \( T, R = 0 \) so one may write the quantum version of \( \pi \psi(R) \) from Eq (13) as

\[ \pi \psi(R) = iRk J_0(kR) \left( A(k) e^{i(kT)} - A^+(k) e^{-i(kT)} \right) R, R, \]

(59)

where \( R, R = 1 \) because \( R = r \) here. In the equality \( n = \frac{X}{\rho} \), which relates \( \rho \) to the number of subregions \( n \) included in the larger region \( X \), we assume \( X = 20R \). Note the difference between the amplitudes \( A(k), A^+(k) \) and \( A \) given by Eq (57) and by the first of Eqs (B6). We also assume, for avoiding complex expressions and easing the graphical introduction of our results, that the amplitude \( A(k) \) is equal to its complex conjugate \( A^+(k) \) so that multiplying the last equation (59) by its conjugate and taking the square root of the resulting product
one obtains
\[ \pi_\psi(R) = RkA(k)J_0(kR) \left( 2 \left( 1 - \cos(2kT) \right) \right)^{\frac{1}{2}} \] (60)

Substituting the last expression for \( \pi_\psi(R) \) in Eq (58) for the probability and graphing the resulting expression for certain values of \( T, k, A \) and \( R \) one may visually see how the probability approach unity as \( \rho \) approach zero. Note that although a probability should be in the range of \((0, 1)\) we obtain here some graphs which exceed unity because of the calculations, such as that for \( \pi_\psi(R) \) in Eq (60), used to assign values to the parameters required for these graphs. One may also realize, for all the Figures and Subfigures shown here, that the drawn graphs and surfaces are at the neighbourhood of unity for the assumed ranges of \( \rho \) and all approach unity as \( \rho \to 0 \). We note that although \( A(k) A^+(k) \) depend upon \( k \) we refer to them, for the present numerical discussion, as constants and assume \( A = A^+ \).

Also, since, as mentioned, \( A \) and \( A^+ \) are thought in the quantum regime to be creation and destruction operators and since these, respectively, raise and decrease the relevant energy by no more than unity the corresponding values assumed for \( A, A^+ \) in the numerical discussion here should be in the order of magnitude of unity.

In Figure 1 we see two Subfigures which show the probability from Eq (58) as function of \( \rho \) in the range \( 1 \geq \rho \geq 0 \) and for \( k = 1, R = 100, T = 1 \) and \( a = 1 \). The left Subfigure 1, \( a \) is drawn for \( \eta = 1 \) and the right one for \( \eta = 9 \). One may realize that for small values of the eigenvalue \( \eta \), such as in Subfigure 1, \( a \), the probability is linear whereas for relatively larger values of \( \eta \), such as in Subfigure 1, \( b \), it deviates from linearity.

In Figure 2 we show three Subfigures which show the probability from Eq (58) as function of \( \rho \) in the range \( 0.2 \geq \rho \geq 0 \), for \( k = \frac{1}{2}, T = 6, \eta = 1, A = A^+ = 1 \) and for certain different ranges of \( R \). Thus, in Subfigure (1, \( a \)) we see this probability for \( 0.2 \geq \rho \geq 0, 20 \geq R \geq 10 \) and one may see that as \( \rho \) becomes smaller the probability approach unity for all values of \( R \). Also, one may realize that the larger becomes \( R \), even for the larger values of the drawn range of \( \rho \), the probability tends to the unity value. Similar situations are also shown in Subfigures (2, \( b \)) and (2, \( c \)) which are, respectively, drawn for intermediate \( 120 \geq R \geq 100 \)
and larger values $1020 \geq R \geq 1000$ of $R$ where it is clearly shown that the surfaces drawn for these $R$’s have, essentially, a unity value for the whole range of $0.2 \geq \rho \geq 0$ and become proper unity as $\rho \to 0$.

In Subfigure (3, a) of Figure 3 we show a three-dimensional surface of the probability from Eq (58) as function of $\rho$ in the range $0.5 \geq \rho \geq 0$, of the eigenvalue $\eta$ in the range $2 \geq \rho \geq 0$ and for $k = \frac{1}{2}$, $T = 6$, $R = 150$ and $A = A^+ = 1$. As realized from Eqs (56) even the general probability from Eq (55) tends to unity as $\rho \to 0$ for any value of $\eta$. We have, nevertheless, show how the probability from Eq (58) approach unity in this limit. As seen in panel (3, a) for a value of $\rho$ which is very close to zero and for certain values of $\eta$ the probability discontinuously decreases to zero and immediately as $\rho$ becomes zero it increases, for the same values of $\eta$, to unity. Subfigure (3, b) shows again this behaviour of the probability for the comparatively extended ranges of $1 \geq \rho \geq 0$, $9 \geq \eta \geq 0$ and for $R = 100$ whereas $k$, $T$ and $A = A^+$ remain as for Panel (3, a). One may again see how for a $\rho$ just before zero the probability discontinuously fall towards zero and then at $\rho = 0$ this probability discontinuously increases to unity. In Subfigure (3, c) we see the probability from Eq (58) as function of $\rho$ and $T$ in the respective ranges of $1 \geq \rho \geq 0$ and $100 \geq T \geq 0$ and for $k = \frac{1}{2}$, $R = 100$, $\eta = 1$ and $A = A^+ = 1$.

7 The cylindrical GW in the TT gauge

We discuss here a linearized version of general relativity in which the cylindrical GW is considered as a small perturbation in the otherwise flat Minkowskian metric. That is, the appropriate metric tensor may be written as

$$g_{\mu \nu} = \epsilon_{\mu \nu} + h_{\mu \nu}, \quad |h| << 1,$$

where the GW is identified with the small perturbation $h$. Thus, the exponents in the cylindrical metrics of Eq (4) may be expanded in a Taylor series in which we retain only the

$$g_{\mu \nu} = \epsilon_{\mu \nu} + h_{\mu \nu} + \frac{1}{2} h_{\mu \nu} h^{\rho \sigma} \epsilon_{\rho \sigma} + \cdots,$$

and so on.
Figure 1: The left panel a shows the probability from Eq (58) as function of ρ in the range $1 \leq \rho \leq 0$ and for $k = 1$, $R = 100$, $\eta = 1$, $T = 1$ and $A = A^+ = 1$. The right panel b shows this probability for the same values of the former parameters except for $\eta = 9$. One may see how the graph changes its linear form, which is valid for small $\eta$, to the nonlinear one as $\eta$ relatively increases. See also the discussion after Eq (60).

first two terms so that taking into account the flat Minkowskian background (in cylindrical coordinates [14]) ($\varepsilon_{\mu\nu} = \text{diag}(-1, 1, R^2, 1)$) one may write the metrics as

$$ds^2 = (1 + 1 + \Gamma - \psi)(dT^2 + dR^2) + (2 + 2\ln(R^2) - \psi)d\phi^2 + (1 + 1 + \psi)dz^2,$$

where

$$ds^2_{\phi\phi} = \varepsilon_{\phi\phi} + h_{\phi\phi} = \left( R^2 + R^2e^{-\psi} \right)d\phi^2 = \left( e^{\ln(R^2)} + e^{(\ln(R^2)-\psi)} \right)d\phi^2 =$$

$$= \left( 1 + \ln(R^2) + 1 + \ln(R^2) - \psi \right)d\phi^2 = \left( 2 + 2\ln(R^2) - \psi \right)d\phi^2$$

From Eqs (61)-(62) one may determine the components $h_{\mu\nu}$ which characterize the cylindrical GW.

$$h_{00} = -(1 + \Gamma - \psi), \quad h_{11} = (1 + \Gamma - \psi), \quad h_{22} = (1 + \ln(R^2) - \psi), \quad h_{33} = 1 + \psi,$$  

(63)
Figure 2: The three panels of this figure show three-dimensional surfaces of the probability from Eq (58) as function of $\rho$ in the range $0.2 \geq \rho \geq 0$, for $k = \frac{1}{2}$, $T = 6$, $\eta = 1$, $A(k) = A^\dagger = 1$ and for 3 different ranges of $R$. As seen, for all the three panels the relevant probability tends to unity as $\rho \to 0$. In Panel a the range of $R$ is $20 \geq R \geq 10$ and those of Panels b and c are respectively $120 \geq R \geq 100$ and $1020 \geq R \geq 1000$. Note that the surfaces shown in these panels and also in Subfigure 3, c have, essentially, a unity value for all their ranges.
Figure 3: The panel a shows a three-dimensional surface of the probability from Eq (58) as function of $\rho$ in the range $0.5 \geq \rho \geq 0$, $\eta$ in the range $2 \geq \rho \geq 0$ and for $k = \frac{1}{2}$, $T = 6$, $R = 150$ and $A = A^+ = 1$. Note that for a value of $\rho$ which is very close to zero and for certain values of $\eta$ the probability is practically zero and discontinuously increases to unity as $\rho$ becomes zero. Panel b shows again this probability but now for the relatively extended ranges of $1 \geq \rho \geq 0$, $9 \geq \eta \geq 0$ and for $R = 100$ whereas $K$, $T$ and $A$, $A^+$ remain as for Panel a. One may again see the discontinuous jumps in the probability just before and at $\rho = 0$. Panel c shows a three-dimensional surface of the probability from Eq (58) as function of $\rho$ in the range $0.5 \geq \rho \geq 0$ and $T$ in the range $100 \geq \rho \geq 1$. One may discern the cyclicity and monotony by which the probability changes with $T$ in the neighbourhood of unity.
where $\psi$ and $\Gamma$ are respectively given by Eqs (9) and (20)-(21) and the suffixes $(0, 1, 2, 3)$ are $(t, r, \phi, z)$. Now, since we discuss here pure GW’s in which the matter terms in Einstein’s field equations are zero we may use [14] the transverse traceless (TT) gauge [2, 14] which is characterized with a minimum number of components [14] for the metric tensor $h_{\mu\nu}$ so that any component of it, except the spatial ones, vanishes, i.e; $h_{\mu0}^{TT} = 0$. The TT gauge is also characterized with the following properties [14]: $h_{kj,j}^{TT} = 0$, so that these components are divergence-free and are also trace-free, e.g., $h_{kk}^{TT} = 0$. Thus, since, as mentioned, the GW is identified with $h_{jk}^{TT}$ it, naturally, shares the same properties. In the linearized version of general relativity the components of the Riemann curvature tensor are found to be [14]

$$R_{\alpha\mu\beta\nu} = \frac{1}{2} \left( h_{\alpha\nu,\mu\beta} + h_{\mu\beta,\nu\alpha} - h_{\mu\nu,\alpha\beta} - h_{\alpha\beta,\mu\nu} \right)$$  \hspace{1cm} (64)$$

Also, it has been shown (see Eq (35.10) in [14]) that in the TT gauge the time-space components of the Riemann curvature tensor have an especially simple form (see Eq (35.10) in [14])

$$R_{j0k0} = R_{0j0k} = -R_{j0k0} = -R_{0j0k} = -\frac{1}{2} h_{jk,00}^{TT}$$  \hspace{1cm} (65)$$

Thus, substituting in (64) $\alpha = j$, $\beta = k$, and $\mu = \nu = 0$ one obtains

$$R_{j0k0} = \frac{1}{2} \left( h_{j0,0k} + h_{0k,0j} - h_{00,jk} - h_{jk,00} \right)$$  \hspace{1cm} (66)$$

From the last two equations (65)-(66) and the vanishing of any nondiagonal element of the metric tensor $h$ (see Eq (63)) one may realize that if $h_{00} = 0$ then $h_{jk} = h_{jk}^{TT}$ which means that in the linearized version of general relativity the cylindrical GW can be expressed only in the TT gauge. But since, as seen from Eq (63), $h_{00} \neq 0$ one may use Eqs (65)-(66) to find an explicit expression for $h_{jk}^{TT}$

$$h_{jk}^{TT} = c_1 + c_2 t + h_{jk} + \int dt \int dt' h_{00,jk},$$  \hspace{1cm} (67)$$
where, as realized from Eq (63), \( j = k \) and \( c_1, c_2 \) are constants of integration. Thus, substituting in Eq (67) for \( h_{jk} \) and \( h_{00} \) from Eq (63), taking into account that these \( h \)'s depend only upon \( t \) and \( r \) (through \( \psi \), the \( \Gamma \) does not depend upon them as seen from Eqs (20)-(21)), using Eqs (49) and (20) for \( \psi \) and \( \Gamma \), performing the double integration over \( t \) and using the Bessel’s derivatives \[ 47 \]
\[
\frac{dJ_0(x)}{dx} = -J_1(x), \quad \frac{dJ_1(x)}{dx} = \frac{1}{2}(J_0(x) - J_2(x))
\]
one may obtain for the components of \( h_{TT}^{jk} \)

\[
\begin{align*}
  h_{rr}^{TT} &= \Re \left\{ (c_1 + c_2 t + h_{rr} + \int dt \int dt' h_{00,rr}) \right\} \\
  &= \Re \left\{ c_1 + c_2 t + \Gamma - \psi + \int dt \int dt' (\psi - \Gamma - 1)_{rr} \right\} \\
  &= \Re \left\{ c_1 + c_2 t + \Gamma - \frac{1}{2} \int_0^\infty dk \left( J_0(kr) + J_2(kr) \right) \cdot \left( A(k) e^{ikT} + A^+(k) e^{-ikT} \right) \right\} \\
  &= c_1 + c_2 t - \frac{1}{2} \int_0^\infty dk \left( J_0(kr) + J_2(kr) \right) \cdot \left( A(k) + A^+(k) \right) (\cos(kT) - 4kA(k)A^+(k)) \tag{68}
\end{align*}
\]

\[
\begin{align*}
  h_{\phi\phi}^{TT} &= \Re \left\{ c_1 + c_2 t + h_{\phi\phi} \right\} = \Re \left\{ c_1 + c_2 t + \ln(R^2) - \psi \right\} \\
  &= c_1 + c_2 t + \ln(R^2) - \left( \int_0^\infty J_0(kr) \cos(kT)(A(k) + A^+(k)) \right) \tag{69}
\end{align*}
\]

\[
\begin{align*}
  h_{zz}^{TT} &= \Re \left\{ c_1 + c_2 t + h_{zz} \right\} = c_1 + c_2 t + \int_0^\infty J_0(kr) \cos(kT)(A(k) + A^+(k)) \tag{70}
\end{align*}
\]

where the unity terms of Eqs (63) are included in the constants \( c_1 \) and \( \Re \) is the real part of the relevant expressions. These components of the cylindrical GW are, naturally, polarized along their respective directions just as the electromagnetic waves are polarized along their spatial axes. We should, however, take into account that these GW’s are tensors and therefore their polarization directions have, likewise, tensorial \[ 49 \] characters. The corresponding unit polarization tensors were introduced in \[ 14 \], with respect to plane GW propagating along a
The unit polarization tensors in the \( (\hat{n}, \hat{n}_1, \hat{n}_2) \) system, as

\[
e_{+n_1n_2} = e_{n_1} \otimes e_{n_1} - e_{n_2} \otimes e_{n_2} = -\left( e_{n_2} \otimes e_{n_2} - e_{n_1} \otimes e_{n_1} \right) = -e_{+n_2n_2} \tag{71}
\]
\[
e_{\times n_1n_2} = e_{n_1} \otimes e_{n_2} + e_{n_2} \otimes e_{n_1} = \left( e_{n_2} \otimes e_{n_1} + e_{n_1} \otimes e_{n_2} \right) = e_{\times n_2n_1},
\]

The unit polarization tensors in the \( (\hat{x}, \hat{y}, \hat{z}) \) system may be obtained from Eqs (71) by substituting, \( \hat{n} = \hat{z}, \hat{n}_1 = \hat{x}, \hat{n}_2 = \hat{y} \)

\[
e_{+xx} = e_x \otimes e_x - e_y \otimes e_y = -\left( e_y \otimes e_y - e_x \otimes e_x \right) = -e_{+yy} \tag{72}
\]
\[
e_{\times xy} = e_x \otimes e_y + e_y \otimes e_x = \left( e_y \otimes e_x + e_x \otimes e_y \right) = e_{\times xy},
\]

The corresponding unit polarization tensors in the cylindrical system \((\hat{r}, \hat{\phi}, \hat{z})\) for a GW which advances in the \( \hat{z} \) direction are detaily derived in [40, 41] from Eqs (72) by using the transformation equations for \( e_x \) and \( e_y \)

\[
e_x = \cos(\phi)e_\rho - \sin(\phi)e_\phi, \quad e_y = \sin(\phi)e_\rho + \cos(\phi)e_\phi, \quad e_z = e_z \tag{73}
\]

That is, using the last Eqs (73) and the trigonometric identities \((\cos^2(\phi) - \sin^2(\phi)) = \cos(2\phi), 2\cos(\phi)\sin(\phi) = \sin(2\phi)\), one may obtain, as done in [40, 41], the corresponding unit polarization tensors in the cylindrical system \((\hat{r}, \hat{\phi}, \hat{z})\), denoted here as \(e^{(z)}_{+rr} = -e^{(z)}_{+\phi\phi}\) and \(e^{(z)}_{\times r\phi} = e^{(z)}_{\times \phi r}\),

\[
e^{(z)}_{+rr} = \cos(2\phi)\left( e_r \otimes e_r - e_\phi \otimes e_\phi \right) - \sin(2\phi)\left( e_r \otimes e_\phi + e_\phi \otimes e_r \right) = -e^{(z)}_{+\phi\phi} \tag{74}
\]
\[
e^{(z)}_{\times r\phi} = \sin(2\phi)\left( e_r \otimes e_r - e_\phi \otimes e_\phi \right) + \cos(2\phi)\left( e_r \otimes e_\phi + e_\phi \otimes e_r \right) = e^{(z)}_{\times \phi r}
\]

Note that since the two systems \((\hat{r}, \hat{\phi}, \hat{z})\) and \((\hat{x}, \hat{y}, \hat{z})\) are orthogonal the former unit polarization tensors \(e^{(z)}_{+rr}, e^{(z)}_{+\phi\phi}, e^{(z)}_{\times r\phi}, e^{(z)}_{\times \phi r}\) for a GW advancing in the \( \hat{z} \) direction of the \((\hat{r}, \hat{\phi}, \hat{z})\)
system are derived here, as mentioned, from the corresponding unit polarization tensors $e_{x,\phi}, e_{y,\phi}, e_{y,\phi}, e_{x,\phi}$ of a GW advancing in the $z$ direction of the $(\hat{x}, \hat{y}, \hat{z})$ system.

We discuss here a cylindrical GW advancing in a general direction which may be decomposed along the $(\hat{z}, \hat{r}, \hat{\phi})$ axes so that each of these components may contribute its polarizing part to its respective perpendicular plane. Thus, the contribution to the unit polarization tensors in the $(\hat{r}, \hat{\phi})$ plane resulting from the component along the $z$ axis are given by Eqs (74). The contribution to the unit polarization tensors in the $(\hat{r}, \hat{z})$ plane resulting from the component along the $\phi$ axis may be calculated in a similar manner, using Eqs (73), as

$$e_{+r\phi}^{(r)} = \cos^2(\phi)(e_\phi \otimes e_\phi) + \sin^2(\phi)(e_{\phi} \otimes e_{\phi}) + \frac{1}{2}\sin(2\phi)\left(e_r \otimes e_\phi + e_\phi \otimes e_r\right) -$$

$$- (e_z \otimes e_z) = -e_{+zz}^{(r)}$$

$$e_{\times r\phi}^{(r)} = \cos(\phi) \left(e_r \otimes e_z + e_z \otimes e_r\right) - \sin(\phi) \left(e_\phi \otimes e_z + e_z \otimes e_\phi\right) = e_{\times r\phi}^{(r)}$$

And the corresponding contribution to the unit polarization tensors in the $(\hat{\phi}, \hat{z})$ plane resulting from the component along the $\hat{r}$ axis is

$$e_{+r\phi}^{(r)} = \cos^2(\phi)(e_\phi \otimes e_\phi) + \sin^2(\phi)(e_{\phi} \otimes e_{\phi}) + \frac{1}{2}\sin(2\phi)\left(e_r \otimes e_\phi + e_\phi \otimes e_r\right) -$$

$$- (e_z \otimes e_z) = -e_{+zz}^{(r)}$$

$$e_{\times r\phi}^{(r)} = \cos(\phi) \left(e_{\phi} \otimes e_z + e_z \otimes e_{\phi}\right) + \sin(\phi) \left(e_r \otimes e_z + e_z \otimes e_r\right) = e_{\times r\phi}^{(r)}$$

Thus, the total unit polarization tensor in the $\hat{r}\hat{r}$ direction may be found from Eqs (74)-(75) as

$$e_{+rr}^{(total)} = e_{+rr}^{(z)} + e_{+rr}^{(\phi)} = \left(\cos(2\phi) + \cos^2(\phi)\right)(e_r \otimes e_r) + \left(\sin^2(\phi) - \cos(2\phi)\right)\cdot$$

$$\cdot \left(e_\phi \otimes e_\phi\right) - \frac{3}{2}\sin(2\phi)\left(e_r \otimes e_\phi + e_\phi \otimes e_r\right) - (e_z \otimes e_z)$$

And the total unit polarization tensor in the $\hat{\phi}\hat{\phi}$ direction may, analogously, be found from
Eqs (74) and (76) as

\[
e^{(total)}_{\phi+\phi} = e^{(x)}_{\phi+\phi} + e^{(r)}_{\phi+\phi} = \left( \sin^2(\phi) - \cos(2\phi) \right) (e_{\hat{r}} \otimes e_{\hat{r}}) + \left( \cos^2(\phi) + \cos(2\phi) \right) \cdot (e_{\hat{\phi}} \otimes e_{\hat{\phi}}) + \frac{3}{2} \sin(2\phi) \left( e_{\hat{\rho}} \otimes e_{\hat{\rho}} + e_{\hat{\phi}} \otimes e_{\hat{r}} \right) - (e_{\hat{z}} \otimes e_{\hat{z}}),
\]

Likewise, the total unit polarization tensor in the \( \hat{z}\hat{z} \) direction may be found from Eqs (73)- (76) as

\[
e^{(total)}_{\hat{z}\hat{z}} = e^{(r)}_{\hat{z}\hat{z}} + e^{(\phi)}_{\hat{z}\hat{z}} = 2(e_{\hat{z}} \otimes e_{\hat{z}}) - (e_{\hat{r}} \otimes e_{\hat{r}} + e_{\hat{\phi}} \otimes e_{\hat{\phi}}).
\]

As for the polarization tensors in the directions \( \hat{j}\hat{k} \) where \( \hat{j} \neq \hat{k} \) these should certainly be generated by the mixed GW’s \( h_{jk\neq k}^{TT} \) but as seen from Eqs (63) and (68)-(70) no such cylindrical GW’s exist in either the TT gauge or in the linearized version introduced by Eqs (61)-(64). Thus, the general cylindrical GW, denoted here \( h_{(+,\times)jk}^{TT} \), may be simplified as

\[
h_{(+,\times)jk}^{TT} = h_{+,jk}^{TT} + h_{\times,jk}^{TT} = A_{+,jk} e^{(total)}_{+,jk} h_{jk\neq k}^{TT} + A_{\times,jk} e^{(total)}_{\times,jk} h_{jk\neq k}^{TT} = A_{+,jk} e^{(total)}_{+,jk} h_{jk\neq k}^{TT},
\]

where \( A_{+,jk} \) and \( A_{\times,jk} \) are constants [14, 40, 41] related, respectively, to \( h_{jk\neq k}^{TT} \) and \( h_{jk\neq k}^{TT} \) and \( e^{(total)}_{+,jk} \) are given by Eqs (77)-(79). Substituting in Eq (80) from Eqs (68)-(70) for \( h_{jk\neq k}^{TT} \) and from Eqs (77)-(79) for \( e^{(total)}_{+,jk} \) one obtains

\[
h_{+,rr}^{TT} = A_{+,rr} e^{(total)}_{+,rr} h_{rr}^{TT} = A_{+,rr} \left\{ \left( \cos(2\phi) + \cos^2(\phi) \right) (e_{\hat{\phi}} \otimes e_{\hat{\phi}}) + \left( \sin^2(\phi) - \cos(2\phi) \right) (e_{\hat{\rho}} \otimes e_{\hat{\rho}}) - \frac{3}{2} \sin(2\phi) \left( e_{\hat{\rho}} \otimes e_{\hat{\rho}} + e_{\hat{\phi}} \otimes e_{\hat{r}} \right) - (e_{\hat{z}} \otimes e_{\hat{z}}) \right\}.
\]

\[
\cdot \left\{ c_1 + c_2 t - \frac{1}{2} \int_0^\infty dk \left( \left( J_0(kr) + J_2(kr) \right) \cdot \cos(kT) \left( A(k) + A^+(k) \right) - 4kA(k)A^+(k) \right) \right\}
\]

\[
- 4kA(k)A^+(k) \right) \right\}
\]

\[
- 4kA(k)A^+(k) \right) \right\}
\]
The corresponding cylindrical metric in the TT gauge may, now, be written as

\[
    (ds_{TT}^2)_{(\hat{r}, \hat{\phi}, \hat{z})} = h_{+\phi}^{TT} dr^2 + h_{+\phi}^{TT} d\phi^2 + h_{+z}^{TT} dz^2, \tag{84}
\]

where \( h_{+r}^{TT}, h_{+\phi}^{TT}, \) and \( h_{+z}^{TT} \) are, respectively, given by Eqs (81)-(83).

## 8 The embedded trapped surface generated by the cylindrical gravitational wave

We first note in the context of trapped surfaces that they are, generally, generated by very strong gravitational waves [14, 38] and not by the weak ones discussed in the former sections. But, as emphasized in [40, 41], if these weak GW’s persist somehow in some spacetime region for a long time then these GW’s are added and aggregated [14] upon each other in such a way that they may influence a spacetime region as if a very strong GW passes this region in a short time. Thus, we discuss here, as in [40, 41], GW which dwell in some spacetime region enough time to generate trapped surface.

We, now, calculate and find the embedded trapped surface generated by the cylindrical GW. We note that since it is difficult to embed the whole trapped surface [38] one resorts to
the simpler task of embedding the equatorial plane due to its rotational symmetry. We use for that the method in [38] and begin by requiring the metric on the equator to be equal to that of a surface of rotation $z(x, y)$ in Euclidean space.

$$x = F(r) \cos(\phi), \quad y = F(r) \sin(\phi), \quad z = G(r)$$

Thus, taking into account that on the equator $d^2z = 0$ we obtain for the metrics

$$ds^2 = dx^2 + dy^2 + dz^2 = \left( F_\rho(r)^2 + G_{,\rho}(r)^2 \right) dr^2 + F^2(r) d\phi^2 =$$

$$= h_{rr}^{TT} dr^2 + h_{\phi\phi}^{TT} d\phi = A_{rr}^{\text{total}} h_{rr}^{TT} dr^2 + A_{\phi\phi}^{(\text{total})} h_{\phi\phi}^{TT} d\phi =$$

$$= A_{rr} \left\{ \left( \cos(2\phi) + \cos^2(\phi) \right) (e_\rho \otimes e_\rho) + \left( \sin^2(\phi) - \cos(2\phi) \right) \cdot \right.$$

$$\left. \cdot (e_\phi \otimes e_\phi) - \frac{3}{2} \sin(2\phi) \left( e_\tau \otimes e_\tau + e_\phi \otimes e_\tau \right) - (e_2 \otimes e_2) \right\}. \quad (86)$$

$$\cdot \left\{ c_1 + c_2 t - \frac{1}{2} \left[ \int_0^\infty dk \left( J_0(kr) + J_2(kr) \right) \cdot \cos(kT) (A(k) + A^+(k)) -$$

$$- 4kA(k)A^+(k) \right) \right\} \right\} d^2r + A_{\phi\phi} \left\{ \left( \sin^2(\phi) - \cos(2\phi) \right) (e_\tau \otimes e_\tau) +$$

$$+ \left( \cos^2(\phi) + \cos(2\phi) \right) (e_\phi \otimes e_\phi) + \frac{3}{2} \sin(2\phi) \left( e_\tau \otimes e_\tau + e_\phi \otimes e_\tau \right) -$$

$$- (e_2 \otimes e_2) \right\} \left\{ c_1 + c_2 t + \ln(R^2) - \left( \int_0^\infty dk J_0(kr) \cos(kT) (A(k) + A^+(k)) \right) \right\} d^2\phi$$

From the last equation one obtains for the quantities $F(r), F_\rho(r)$ and $G(r)$

$$F(r) = A_{\phi\phi} \left\{ \left( \sin^2(\phi) - \cos(2\phi) \right) (e_\tau \otimes e_\tau) + \left( \cos^2(\phi) + \cos(2\phi) \right) \right.$$

$$\cdot (e_\phi \otimes e_\phi) + \frac{3}{2} \sin(2\phi) \left( e_\tau \otimes e_\tau + e_\phi \otimes e_\tau \right) - (e_2 \otimes e_2) \right\} \right.$$

$$\cdot \left\{ c_1 + c_2 t + \ln(R^2) - \left( \int_0^\infty dk J_0(kr) \cos(kT) (A(k) + A^+(k)) \right) \right\} \right.$$
CONCLUDING REMARKS

\[ F_{,r}(r) = A_{+,\phi}\left(\frac{2}{R} + \left(\int_0^\infty dk k J_1(kr) \cos(kT) \left(A(k) + A^+(k)\right)\right)\right) \cdot \left\{ \left(\sin^2(\phi) - \cos(2\phi)\right)(e_\phi \otimes e_r) + \left(\cos^2(\phi) + \cos(2\phi)\right) \cdot \left( e_\phi \otimes e_\phi + e_\phi \otimes e_\phi \right) - (e_z \otimes e_z) \right\}, \]

where the Bessel’s derivative \( \frac{dJ_0(r)}{dr} = -J_1(r) \) is used.

\[ G(r) = \int dr \left\{ A_{+,rr}\left\{ \left(\cos(2\phi) + \cos^2(\phi)\right)(e_\rho \otimes e_r) + \left(\sin^2(\phi) - \cos(2\phi)\right) \cdot \left( e_\phi \otimes e_\phi + e_\phi \otimes e_r \right) - (e_z \otimes e_z) \right\} \right\} \cdot \left\{ c_1 + c_2t - \frac{1}{2} \left[ \int_0^\infty dk \left( J_0(kr) + J_2(kr) \right) \cdot \cos(kT) \left(A(k) + A^+(k)\right) \right] \right\} - F_{,r}^2(r) \right\}^{\frac{1}{2}}, \]

where \( F_{,r}(r) \) is given by Eq (88). The expressions \( F(r) \), \( F_{,r}(r) \) and \( G(r) \) determine the geometry of the cylindrical trapped surface.

9 Concluding Remarks

We have discussed quantum aspects of the cylindrical GW and, especially, the theoretical possibility of increasing its quantum probability. We use for that the spatial version [31] of the Zeno effect [28, 29, 30] which is affected by performing the same experiment in a large number of nonoverlapping separate regions of space all included in a finite total region so that in the limit in which these subregions become infinitesimal, keeping the total region fixed, the quantum state becomes constant in space. The last spatial method of the Zeno effect is more appropriate for the cylindrical GW since it is related [11] to the extrinsic time variable which is canonically conjugate to momentum [11] just as the spatial coordinate is canonically conjugate to it. Thus, we have shown that, beginning with some cylindrical GW in some certain subregion of space-time included in a larger one and if the measurement of
this field is similarly done in other neighbouring space-time subregions then at the limit in which these subregions become infinitesimally small, keeping the including larger one fixed, one obtains the Zeno result in which the cylindrical GW is fixed in all these subregions. For that we have first detaily shown that the cylindrical GW may be appropriately discussed in the commutation number representation [9, 10]. In this representation the cylindrical GW is shown to be composed from a large ensemble of particle-like components each one inhabits some space-time point so that it may be discussed in terms of space Zeno effect which also requires a large number of similar components to dwell in similar points all confined in some finite region of space-time.

As known [14], in the canonical formulation of general relativity the lapse and shift functions, which are fixed (by observer) upon some space-time hypersurface, determine [14] the later evolution and geometry of space-time. This is reminiscent of the similar role of the creation and destruction operators in the commutation number representation which also determine space-time geometry through controlling the number of gravitational constituents of the GW which impose its geometry upon space-time. In Section IV and Appendix A we have expressed these lapse and shift functions in terms of the creation and destruction operators. Moreover, we were not contented in only finding the conditions through which the probability of the GW increases but also follow the "realized" (with unity-value probability) GW in its passage through space-time. We have, thus, discussed its properties in the (TT) gauge and have calculated the trapped surface generated by it.

It must be noted that although the time version of the quantum Zeno effect was experimentally validated [32, 33] no such corroboration exists, for now, for its spatial version [31] and less for the gravitational application of it that we use here. This is because the GW itself were not experimentally, up to now, detected. One hope that some future technology will detect not only GW but also a possible Zeno effect for them. That is, one may hope that as this effect proves itself so efficiently in the quantum regime it may, likewise, also be efficient in fixing and classicalizing GW and by that fixing its imposed geometry upon the
surrounding spacetime.

A Calculation of the expressions \((C_1C_4 + C_2C_3), (C_1C_2 - C_4C_3), (C_1C_1 + C_3C_3)\) from Eq \((44)\)

We, now, calculate and simplify the expressions \((C_1C_4 + C_2C_3), (C_1C_2 - C_4C_3), (C_1C_1 + C_3C_3)\). We begin with \((C_1C_4 + C_2C_3)\) and use the commutation relations \((18)\) for \(A(k)\), the trigonometric identities: 
\[
\sin(a)\sin(b) = \frac{1}{2} \left( \cos(a - b) - \cos(a + b) \right), \quad \cos(a)\cos(b) = \frac{1}{2} \left( \cos(a - b) + \cos(a + b) \right)
\]
and the expression (see integral 11.4.42 in P. 487 in [47])

\[
\int_0^\infty J_\mu(ar)J_{\mu-1}(br)dr = \begin{cases} 
\frac{b^{(\mu-1)}}{a^\mu} & \text{for } (0 < b < a) \text{ and } \Re(\mu) > 0 \\
\frac{1}{2b} & \text{for } (0 < b = a) \text{ and } \Re(\mu) > 0 \\
0 & \text{for } (b > a > 0) \text{ and } \Re(\mu) > 0
\end{cases} \quad (A_1)
\]
Thus, the expression \((C_1C_4 + C_2C_3)\) is

\[
C_1C_4 + C_2C_3 = \int_0^\infty dr \int_0^\infty dk \int_0^\infty dk' kk' J_0(kr)J_1(k'r) \left\{ \sin(kT) \sin(k'T) \right. \\
+ \left. \left( A(k)A(k') - A(k)A^+(k') + A^+(k)A(k') - A^+(k)A^+(k') \right) + \cos(kT) \cos(k'T) \right. \\
+ \left. \left( A(k')A(k) - A(k')A^+(k) + A^+(k')A(k) - A^+(k')A^+(k) \right) \right\} = \\
= \int_0^\infty dr \int_0^\infty dk \int_0^\infty dk' kk' J_0(kr)J_1(k'r) \left\{ \cos\left(\frac{T(k - k')}{2}\right) \left(2A(k)A(k') - \frac{\delta(k - k')}{2} \right) + \frac{1}{2} \cos(T(k + k')) \left( A^+(k')A(k) + A(k)A^+(k') - A^+(k)A(k') - A(k')A^+(k) \right) \right\} = \\
= \frac{1}{2} \int_0^\infty dkk \left( A(k)A(k) - A^+(k)A^+(k) - \frac{1}{2} \right)
\]

Note that in the last result we have equated \(k\) to \(k'\) and use the middle relation from Eqs (A1) otherwise if we assume \(k \neq k'\) we would get either an imaginary result for \((C_1C_4 + C_2C_3)\) if we use the first relation of (A1) or \((C_1C_4 + C_2C_3) = 0\) if we use the third relation of it. These outcomes would make \(N^0\) and \(N^1\) from Eqs (44) either imaginary because of the first of (A1) or undefined because of the third of it.

In a similar manner, using the commutation relations (18) for \(A(k)\), the trigonometric identities \(\sin(a) \cos(b) = \frac{1}{2}(\sin(a - b) + \sin(a + b))\), \(\sin(a - b) = -\sin(b - a)\) and the integral
As realized from the last result we have assumed $k = k'$ otherwise, as noted after Eq (A2), we would obtain either an imaginary or zero value for $(C_1C_4 + C_2C_3)$. These would make $N^0$ and $N^1$, as noted after Eq (A2), either imaginary or undefined.

The remaining expression $C_1C_1 + C_3C_3$ is calculated by using the former product trigonometric identities and the orthogonality relation (see the unnumbered expression after Eq (19)).
\[ \int_0^\infty dk k J_n(kr) J_n(k'r) = \frac{\delta(r-r')}{r'} \]

\[ C_1 C_1 + C_3 C_3 = \int_0^\infty dr \int_0^\infty dk \int_0^\infty dk' kk' J_0(kr) J_0(k'r) \left\{ \sin(kT) \sin(k'T) \right\} \]

\[ \cdot \left( A(k)A(k') + A(k)A^+(k') + A^+(k)A(k') + A^+(k)A^+(k') \right) + \cos(kT) \cos(k'T) \]

\[ \cdot \left( A(k')A(k) - A(k')A^+(k) - A^+(k')A(k) + A^+(k')A^+(k) \right) \right\} = \delta(r-r') \]

\[ \int_0^\infty dr \int_0^\infty dk \int_0^\infty dk' kk' J_0(kr) J_0(k'r) \left\{ \cos(\frac{T(k-k')}{2}) \left( 2A(k)A^+(k') - \frac{\delta(k-k')}{2} - \frac{\delta(k'-k)}{2} \right) - \frac{1}{2} \cos(\frac{T(k+k')}{2}) \right\} \]

\[ = \int_0^\infty dr \int_0^\infty \frac{dr-r}{r} kk \left\{ \frac{1}{2} \left( 4A(k)A(k) - 1 \right) - \cos(2Tk) \left( A(k)A(k) + A^+(k)A^+(k) \right) \right\} \]

\[ B \quad \text{Derivation of the probability } Pr^{(1)}(|\Psi(0, \psi(R))>, \rho) \text{ from Eq (51)} \]

We use the shift operator \[ e^{i\rho P} \] where \( P \) denote the momentum and \( \rho \) is, as mentioned, a very small amount by which the state functional shifts \[ e^{i\rho P} \]. Note, as mentioned, that the extrinsic time variable related to the cylindrical GW is, like any spatial variable, canonically conjugate \[ e^{i\rho P} \] to momentum. Thus, one may write for this probability \[ Pr^{(1)}(|\Psi(0, \psi(R))>, \rho) \]
\[ P_r^{(1)}(|\Psi(0, \psi(R)) >, \rho) = |\Psi(T(R), \psi(R))| e^{\frac{i\omega p}{h}} |\Psi(T(R), \psi(R))|^2 = |\Psi(0, \psi(R))| \cdot \exp \left\{ i\left[ \frac{T(R)}{2(1 - T^2_R(R))} \left( R^{-1}\pi^2_\psi(R) - 2T_R(R)\pi_\psi(R)\psi_R(R) + \frac{1}{2}R^2\pi^2_\psi(R) \right) \right] \right\} \cdot \exp \left\{ \frac{T(R)}{2R(1 - T^2_R(R))} \right\} \exp \left\{ -i \left[ \frac{T(R)}{2(R + \rho)} \left( \frac{1}{1 - T^2_R(R + \rho)} \left( \pi^2_\psi(R + \rho) - 2(R + \rho)T(R + \rho) \pi_\psi(R + \rho) \right) \right) \right\} \left[ \frac{T(R)}{2(R + \rho)} \left( \frac{1}{1 - T^2_R(R + \rho)} \left( \pi^2_\psi(R + \rho) - 2(R + \rho)T(R + \rho) \pi_\psi(R + \rho) \right) \right) \right] \right\} |\Psi(0, \psi(R + \rho))|^2 \]

Taking common denominator for the exponential expressions one obtains from the last result

\[ P_r^{(1)}(|\Psi(0, \psi(R)) >, \rho) = \left| \Psi(0, \psi(R)) \right|^2 \exp \left\{ \frac{1}{2R(1 - T^2_R(R)) \left( 1 - T^2_R(R + \rho) \right)} \right\} \cdot \left\{ T(R) \left( R + \rho \right) \left( 1 - T^2_R(R + \rho) \right) \pi^2_\psi(R) - 2R(R + \rho) \left( 1 - T^2_R(R + \rho) \right) \pi_\psi(R) \right\} - \pi_\psi(R \pi_\psi(R) \psi_R(\pi_\psi(R) \psi_R) + \frac{1}{2}(R + \rho)^2 \pi^2_\psi(R + \rho) \right] \right\} \left( \pi_\psi(R + \rho) \pi_\psi(R + \rho) \psi_R(R + \rho) + \frac{1}{2}(R + \rho)^2 \pi^2_\psi(R + \rho) \right) \right\} \right\} |\Psi(0, \psi(R + \rho))|^2 \]

In order to be able to calculate the last expression we exploit the fact that the shift \( \rho \) is very small so one may expand \( \psi(R + \rho), T(R + \rho), \psi_R^2(R + \rho), T^2_R(R + \rho) \) in a Taylor series and keeping the first two terms as follows
$$\psi(R + \rho) = \psi(R) + \rho \frac{d\psi(R)}{dR}$$

$$T(R + \rho) = T(R) + \rho \frac{dT(R)}{dR}$$

$$(B3)$$

$$\psi^2_R(R + \rho) = (\psi_R(R) + \rho \frac{d\psi(R)}{dR})^2 = (\psi_R(R))^2 +$$

$$+ \rho^2 \left( \frac{d^2\psi(R)}{d^2R} \right)^2 + 2 \rho \psi(R) \frac{d^2\psi(R)}{dR^2}$$

$$T^2_R(R + \rho) = (T_R(R) + \rho \frac{dT(R)}{dR})^2 = (T_R(R))^2 +$$

$$+ \rho^2 \left( \frac{d^2T(R)}{d^2R} \right)^2 + 2 \rho T(R) \frac{d^2T(R)}{dR^2}$$

Note that $\pi_\psi(R + \rho)$ and its square $\pi^2_\psi(R + \rho)$, given by the second of Eqs (47), are variational derivatives operators that operate upon an arbitrary function of the shifted state $\psi(R + \rho)$, i.e., $f(\psi(R + \rho))$ so it is obvious that operating with either $\pi_\psi(R + \rho)$ or $\pi^2_\psi(R + \rho)$ upon $f(\psi(R + \rho))$ is identical with respectively operating with either $\pi_\psi(R)$ or $\pi^2_\psi(R)$ upon it. That is, using the second of Eq (47), one have

$$\pi_\psi(R + \rho) f(\psi(R + \rho)) = -i \frac{\delta(f(\psi(R + \rho)))}{\delta(\psi(R + \rho))} =$$

$$= -i \frac{\delta(f(\psi(R + \rho)))}{\delta(\psi(R))} = \pi_\psi(R) f(\psi(R + \rho))$$

$$(B4)$$

$$\pi^2_\psi(R + \rho) f(\psi(R + \rho)) = -\frac{\delta^2(f(\psi(R + \rho)))}{\delta^2(\psi(R + \rho))} =$$

$$= -\frac{\delta^2(f(\psi(R + \rho)))}{\delta^2(\psi(R))} = \pi^2_\psi(R) f(\psi(R + \rho))$$

Using Eqs (B3)-(B4) one may write the probability from Eq (B2) as

$$Pr^{(1)}(|\Psi(0, \psi(R)|, \rho) = \frac{\langle \Psi(0, \psi(R)) \rangle \exp \left\{ \frac{i}{\rho} \left( \frac{A\pi^2_\psi(R) + B\pi_\psi(R) + C}{D} \right) \right\}}{|\Psi(0, \psi(R + \rho)|}^2, \quad (B5)$$
where \( A, B, C \) and \( D \) are given by

\[
A = (1 - T_{R^2}(R)) (T(R) - \frac{dT(R)}{dR}) - T(R)(R + \rho) \left( \rho \frac{d^2T(R)}{dR^2} \right)^2 + \\
+ 2T_{R^2}(R) \frac{d^2T(R)}{dR^2}
\]

\[
B = 2R(R + \rho) \left\{ T_{R^2}(R) \psi_{R^2}(R) T(R) \left( \rho \frac{d^2T(R)}{dR^2} \right)^2 + 2T_{R^2}(R) \frac{d^2T(R)}{dR^2} \right\} + \\
+ \frac{1 - T_{R^2}(R)}{2} \left\{ \frac{d^2T(R)}{dR^2} \right\} \left[ \psi_{R^2}(R) + \frac{d^2\psi(R)}{dR^2} \right] + \\
+ T_{R^2}(R) \frac{d^2\psi(R)}{dR^2} + \rho T_{R^2}(R) \frac{d^2\psi(R)}{dR^2} \right\}
\]

\[
C = -\frac{R^2}{2} (R + \rho) \psi_{R^2}(R) T(R) \left( \rho \frac{d^2T(R)}{dR^2} \right)^2 + 2T_{R^2}(R) \frac{d^2T(R)}{dR^2} - \\
- \frac{R}{2} (R + \rho)^2 \left( 1 - T_{R^2}(R) \right) T(R) + \rho \frac{dT(R)}{dR} \left( \rho \frac{d^2\psi(R)}{dR^2} \right)^2 + \\
+ 2\psi_{R^2}(R) \frac{d^2\psi(R)}{dR^2} - \frac{R}{2} (R + \rho) \psi_{R^2}(R) T(R) + \\
+ (R + \rho) \frac{dT(R)}{dR}
\]

\[
D = 2R(R + \rho) \left( 1 - T_{R^2}(R) \right) \left( 1 - T_{R^2}(R + \rho) \right)
\]

In order to continue we must overcome the commutation problem resulting from the presence of \( \pi_{\psi}(R), \pi_{\psi}^2(R), \langle \Psi(0, \psi(R)) \rangle \) and \( |\Psi(0, \psi(R + \rho))\rangle \) in Eq \((B_5)\) (see the second of Eqs \((47)\)). For this purpose one may begin from expanding the exponential functions of the probability expression Eq \((B_5)\) in Taylor series and then use the commutation relations between the resulting integral powers of \( \pi_{\psi} \) and \( |\Psi(0, \psi(R))\rangle \) which may be derived by using the second of Eqs \((47)\). Thus, one may, for example, see that the following respective commutation relations between \( |\Psi(0, \psi(R))\rangle \) and \( \pi_{\psi}(R), \pi_{\psi}^2(R), \pi_{\psi}^3(R) \) hold
\[ \begin{align*}
[|\Psi(0, \psi(R))>, \pi_\psi(R)] &= i\frac{d(|\Psi(0, \psi(R))>)}{d(\psi(R))} \\
[|\Psi(0, \psi(R))>, \pi_\psi^2(R)] &= 2i\frac{d(|\Psi(0, \psi(R))>)}{d(\psi(R))}\pi_\psi + \frac{d^2(|\Psi(0, \psi(R))>)}{d^2(\psi(R))} \pi_\psi(R) \\
[|\Psi(0, \psi(R))>, \pi_\psi^3(R)] &= 3i\frac{d(|\Psi(0, \psi(R))>)}{d(\psi(R))}\pi_\psi^2 + \frac{d^2(|\Psi(0, \psi(R))>)}{d^2(\psi(R))}\pi_\psi(R)
\end{align*} \]

Note that although we refer to \(|\Psi(0, \psi(R))>\) as a state we should remember that it is actually the solution (50) of the Schroedinger-type equation (48) and is a function of the operators \(T(R), \psi(R)\) so one may calculate the commutation between it and the operator \(\pi_\psi(R)\) (see similar relations at P. 177-178 in [9]). Thus, using mathematical induction and the last relations one may realize that for any integral value \(n\) the following commutation relation is obtained
\[ \begin{align*}
[|\Psi(0, \psi(R))>, \pi_\psi^n(R)] &= ni\frac{d(|\Psi(0, \psi(R))>)}{d(\psi(R))}\pi_\psi^{(n-1)}(R) + \\
&+ \frac{d^2(|\Psi(0, \psi(R))>)}{d^2(\psi(R))}\pi_\psi^{(n-2)}(R)
\end{align*} \]

Note that the commutation relations in Eqs \((B_7)-(B_8)\) hold for any general function of \(\psi(R)\) and not only for \(|\Psi(0, \psi(R))>\). Using the last equations one may find the appropriate expressions for operating with the operators \(\exp(i\rho \frac{A\pi_\psi^2(R)}{E})\) and \(\exp(i\rho \frac{B\pi_\psi(R)}{E})\) upon \(|\Psi(0, \psi(R))>\)
as

\[
\exp(i\rho A E \pi^2_\psi(R))|\Psi(0, \psi(R))\rangle = \left( 1 + i\rho A E \pi^2_\psi(R) - \rho^2 A^2 E^2 \pi^4_\psi(R) - \right.
\]

\[
- i\rho^3 \frac{A^3}{3! E^3} \pi^3_\psi(R) + \ldots \bigg) |\Psi(0, \psi(R))\rangle = |\Psi(0, \psi(R))\rangle + i\rho A E \pi^2_\psi(R) - \rho^2 A^2 E^2 \pi^4_\psi(R) -
\]

\[
- 2i \frac{d(|\Psi(0, \psi(R))\rangle)}{d(\psi(R))} \pi_\psi(R) - \frac{d^2(|\Psi(0, \psi(R))\rangle)}{d^2(\psi(R))} \pi^2_\psi(R) - \rho^2 A^2 E^2 \pi^4_\psi(R) -
\]

\[
\cdot \pi^4_\psi(R) - 4i \frac{d(|\Psi(0, \psi(R))\rangle)}{d(\psi(R))} \pi^3_\psi(R) - \frac{d^2(|\Psi(0, \psi(R))\rangle)}{d^2(\psi(R))} \pi^2_\psi(R) + \ldots = \tag{B_9}
\]

\[
= |\Psi(0, \psi(R))\rangle \exp(i\rho A E \pi^2_\psi(R)) - \frac{1}{\pi^2_\psi(R)} \frac{\delta^2(|\Psi(0, \psi(R))\rangle)}{\delta^2(\psi(R))} \exp(i\rho A E \pi^2_\psi(R)) +
\]

\[
+ \frac{1}{\pi^2_\psi(R)} \frac{\delta^2(|\Psi(0, \psi(R))\rangle)}{\delta^2(\psi(R))} + 2\rho A E \pi^2_\psi(R) \frac{\delta(|\Psi(0, \psi(R))\rangle)}{\delta(\psi(R))} \exp(i\rho A E \pi^2_\psi(R))
\]

\[
\exp(i\rho B E \pi_\psi(R))|\Psi(0, \psi(R))\rangle = \left( 1 + i\rho B E \pi_\psi(R) - \rho^2 B^2 E^2 \pi^2_\psi(R) - \right.
\]

\[
- i\rho^3 \frac{B^3}{3! E^3} \pi^3_\psi(R) + \ldots \bigg) |\Psi(0, \psi(R))\rangle = |\Psi(0, \psi(R))\rangle + i\rho B E \pi_\psi(R) - \rho^2 B^2 E^2 \pi^2_\psi(R) -
\]

\[
- i \frac{d(|\Psi(0, \psi(R))\rangle)}{d(\psi(R))} - \rho^2 B^2 E^2 \left( |\Psi(0, \psi(R))\rangle \pi^2_\psi(R) - 2i \frac{d(|\Psi(0, \psi(R))\rangle)}{d(\psi(R))} \pi_\psi(R) -
\]

\[
- \frac{d^2(|\Psi(0, \psi(R))\rangle)}{d^2(\psi(R))} \right) + \ldots = |\Psi(0, \psi(R))\rangle \exp \left( i\rho B E \pi_\psi(R) \right) - \tag{B_10}
\]

\[
- \frac{1}{\pi^2_\psi(R)} \frac{\delta^2(|\Psi(0, \psi(R))\rangle)}{\delta^2(\psi(R))} \frac{\exp(i\rho B E \pi_\psi(R)) + \frac{1}{\pi^2_\psi(R)} \frac{\delta^2(|\Psi(0, \psi(R))\rangle)}{\delta^2(\psi(R))}} +
\]

\[
+ i\rho B E \pi_\psi \frac{\delta^2(|\Psi(0, \psi(R))\rangle)}{\delta^2(\psi(R))} + B \frac{\delta(|\Psi(0, \psi(R))\rangle)}{\delta(\psi(R))} \exp(i\rho B E \pi_\psi(R))
\]

Using Eqs \(B_9\)-\(B_{10}\) one may obtain for the combined operation
\[ \exp \left( i \rho \frac{A}{E} \pi^2(R) \right) \exp \left( i \rho \frac{B}{E} \pi \psi(R) \right) \text{ upon } |\Psi(0, \psi(R)) > \]

\[ \exp \left( i \rho \frac{A}{E} \pi^2(R) \right) \exp \left( i \rho \frac{B}{E} \pi \psi(R) \right) |\Psi(0, \psi(R)) > = \exp \left( i \rho \frac{A \pi^2(R)}{E} \right) \left\{ |\Psi(0, \psi(R)) > \cdot \exp \left( i \rho \frac{B}{E} \pi \psi(R) \right) - \frac{1}{\pi^2(R)} \frac{\delta^2(|\Psi(0, \psi(R)) > | \delta^2(\psi(R))} + \frac{B}{E} \frac{\delta(|\Psi(0, \psi(R)) > | \delta(\psi(R))} \right\} \]

\[ \cdot \exp \left( i \rho \frac{B}{E} \pi \psi(R) \right) \]

\[ + \frac{1}{\pi^2(R)} \frac{\delta^2(|\Psi(0, \psi(R)) > | \delta^2(\psi(R))} + \frac{B}{E} \frac{\delta(|\Psi(0, \psi(R)) > | \delta(\psi(R))} \right\} \]

\[ \cdot \exp \left( i \rho \frac{B}{E} \pi \psi(R) \right) \}

\[ = |\Psi(0, \psi(R)) > \exp \left( i \rho \frac{A}{E} \pi^2(R) \right) \exp \left( i \rho \frac{B}{E} \pi \psi(R) \right) + \frac{1}{\pi^2(R)} \frac{\delta^2(|\Psi(0, \psi(R)) > | \delta^2(\psi(R))} \right\} \exp \left( i \rho \frac{B}{E} \pi \psi(R) \right) + \left( \frac{2 \rho \pi \psi(R) + B}{E} \right) \frac{\delta(|\Psi(0, \psi(R)) > | \delta(\psi(R))} \right\} \]

\[ \cdot \exp \left( i \rho \frac{A}{E} \pi^2(R) \right) \exp \left( i \rho \frac{B}{E} \pi \psi(R) \right) + i \rho \frac{B}{E} \frac{\delta^2(|\Psi(0, \psi(R)) > | \delta^2(\psi(R))} \right\} \exp \left( i \rho \frac{A}{E} \pi^2(R) \right) \]

Substituting from the last equations (\(B_9\))-(\(B_{11}\)) into Eq (\(B_5\)) for the probability one obtains
Note that since, generally, different powers of the same operator commute with each other we have equated the exponent of the sum of $\pi_\psi^2(R)$ and $\pi_\psi(R)$ to the product of the exponents of the corresponding operators. Now, in order to simplify the following calculation we denote the two complex factors in Eq \((B_{12})\) which respectively multiply $\delta(\Psi(0, \psi(R + \rho)) >, \delta(\Psi(0, \psi(R + \rho)))$ by $F(\rho)$ and $G(\rho)$ and use the second of Eqs \((17)\) to write these variational derivatives as $\frac{\delta(\Psi(0, \psi(R + \rho)) >)}{\delta(\psi(R))} = i\pi_\psi(R)\Psi(0, \psi(R + \rho)) >$ and $\frac{\delta^2(\Psi(0, \psi(R +\rho)) >)}{\delta^2(\psi(R))} = -\pi_\psi^2(R)\Psi(0, \psi(R + \rho)) >$. Thus, one may write the two expressions in Eq \((B_{12})\) related to these variational
derivatives as

\[ \frac{1}{\pi_{\psi}^2(R)} \frac{\delta^2(\Psi(0, \psi(R + \rho)))}{\delta^2(\psi(R))} \left[ \exp \left( i \rho \frac{B}{D} \pi_{\psi}(R) \right) + \exp \left( i \rho \frac{A}{D} \pi_{\psi}^2(R) \right) \left( 1 + i \rho \frac{B}{D} \pi_{\psi}(R) \right) - 2 \exp \left( i \rho \frac{A}{D} \pi_{\psi}^2(R) \right) \exp \left( i \rho \frac{B}{D} \pi_{\psi}(R) \right) \right] = -\pi_{\psi}^2(R) |\Psi(0, \psi(R + \rho)) > F(\rho) \]  

\[ \rho \frac{2A\pi_{\psi}(R) + B}{D} \frac{\delta(\Psi(0, \psi(R + \rho)))}{\delta(\psi(R))} \exp \left( i \rho \frac{A}{D} \pi_{\psi}^2(R) \right) \exp \left( i \rho \frac{B}{D} \pi_{\psi}(R) \right) = i \pi_{\psi}(R) |\Psi(0, \psi(R + \rho)) > G(\rho) \]

Note that we discuss here a half-parametrized formalism and a representation in which the canonical coordinates \( T(R) \) and \( \psi(R) \) are diagonal (see discussion after Eq (17)) so the state functional depends only upon these two coordinates i.e., \( \Psi(T(R), \psi(R)) \). This representation enables one to introduces the Dirac constraint as the Schroedinger-type form of Eq (48) and also to write the corresponding eigenvalue constraint \( \pi_{\psi}(R) |\Psi(0, \psi(R + \rho)) > = \eta |\Psi(0, \psi(R + \rho)) > \) where \( \eta \) is an eigenvalue related to the operator \( \pi_{\psi}(R) \). Note that a similar momentum eigenvalue equation is discussed in [11] (see Eq (64) in [11]). Thus, using the last eigenvalue relation, the normalization [11] of the state functionals \( \Psi(0, \psi(R)) \) and the complex character of the expressions \( F(\rho) \) and \( G(\rho) \) from Eq (B3) which enables one to represent them in terms of their real and imaginary parts one may write the former probability from Eq (B12) as

\[ Pr^{(1)}(|\Psi(0, \psi(R)) > , \rho) = \left| \exp(i \rho \frac{C}{D} \right) < \Psi(0, \psi(R)) |\Psi(0, \psi(R + \rho)) > \cdot \left( \exp \left( i \rho \frac{A}{D} \pi_{\psi}^2(R) + B \pi_{\psi}(R) \right) \right) + i \pi_{\psi}(R) |\Psi(0, \psi(R + \rho)) > G(\rho) - \pi_{\psi}^2(R) |\Psi(0, \psi(R + \rho)) > F(\rho) \right|^2 = \left| < \Psi(0, \psi(R)) |\Psi(0, \psi(R + \rho)) > \exp(i \rho \frac{C}{D} \right|^2 \cdot \left[ \exp \left( i \rho \frac{A}{D} \pi_{\psi}^2(R) + B \pi_{\psi}(R) \right) \right] + i \eta \left( \Re G(\rho) + i \Im G(\rho) \right) \right|^2 \cdot \eta^2 \left( \Re F(\rho) + i \Im F(\rho) \right) \]

The real and imaginary parts of \( G(\rho) \) and \( F(\rho) \) denoted, respectively, by \( \Re G(\rho) \), \( \Re F(\rho) \), \( \Im G(\rho) \) and \( \Im F(\rho) \), are given by Eq (B3). Thus, substituting from Eq (B3) into Eq (B14) and
using the following trigonometric product identities; \( \sin(x) \sin(y) = \frac{1}{2} \left( \cos(x - y) - \cos(x + y) \right) \), \( \cos(x) \cos(y) = \frac{1}{2} \left( \cos(x - y) + \cos(x + y) \right) \), \( \sin(x) \cos(y) = \frac{1}{2} \left( \sin(x - y) + \sin(x + y) \right) \)

and \( \sin^2(x) + \cos^2(y) = 1 \) one obtains for the probability \( P_{r1}(|\Psi(0, \psi(R)) >, \rho) \)

\[
P_{r1}(|\Psi(0, \psi(R)) >, \rho) = 1 + 4\eta^2 + 6\eta^4 + \rho^2 \eta^4 \left( \frac{B\pi(R)}{D} \right)^2 + \eta^2\rho^2 \left( \frac{2A\pi(R)}{D} + B \right)^2 + \sin \left( \frac{\rho}{D} B\pi(R) \right) \left\{ 2\eta^3 \frac{\rho}{D\pi^2(R)} \left( 2A\pi(R) + B \right) - 2\eta^2 \rho \frac{B\pi(R)}{D} \right\} (1 + 2\eta^2) - \cos \left( \frac{\rho}{D} B\pi(R) \right) \left\{ 2\eta^3 \frac{\rho^2 B}{D^2\pi^2(R)} \left( 2A\pi(R) + B \right) + 2\eta^2 (1 + 2\eta^2) \right\} + \left( B_{15} \right) + 2\eta^3 \frac{\rho D}{D\pi^2(R)} \left( 2A\pi(R) + B \right) \sin \left( \frac{\rho}{D} A\pi^2(R) \right) - 2\eta^2 (1 + 2\eta^2) \cos \left( \frac{\rho}{D} A\pi^2(R) \right) - 2\eta^4 \rho \frac{B\pi(R)}{D} \sin \left( \frac{\rho}{D} \left( A\pi^2(R) - B\pi(R) \right) \right) + 2\eta^4 \cos \left( \frac{\rho}{D} \left( A\pi^2(R) - B\pi(R) \right) \right)
\]

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[1] The problem of quantization is found in the literature to refer to gravitational field and also to gravitational wave. Since we use here some results from we also adopt the terminology there of gravitational wave.

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