Euler equation of the optimal trajectory for the fastest magnetization reversal of nano-magnetic structures

X. R. Wang(a), P. Yan, J. Lu and C. He

Physics Department, The Hong Kong University of Science and Technology
Clear Water Bay, Hong Kong SAR, China

received 5 May 2008; accepted 10 September 2008
published online 14 October 2008

PACS 75.60.Jk – Magnetization reversal mechanisms
PACS 75.75.+a – Magnetic properties of nanostructures
PACS 85.70.Ay – Magnetic device characterization, design, and modeling

Abstract – The Euler equation of the optimal reversal trajectory for the fastest magnetization reversal is obtained for an arbitrary nano-magnetic structure. The Euler equation is useful in designing a magnetic field pulse and/or a polarized electric current pulse in magnetization reversal for two reasons. 1) It is straightforward to obtain the solution of the Euler equation, at least numerically, for a given magnetic nano-structure characterized by its magnetic anisotropy energy. 2) After obtaining the optimal reversal trajectory for a given magnetic nano-structure, finding a proper field/current pulse is an algebraic problem instead of the original non-linear differential equation.

The advent of miniaturization and fabrication of magnetic particles with single magnetic domains [1], called Stoner particles, makes the Stoner-Wohlfarth (SW) problem [2] very relevant to nano-technologies and nano-sciences. One current topic in nano-magnetism is the controlled manipulation of magnetization dynamics of Stoner particles. A magnetization state can be manipulated by a magnetic field [3,4], or by a spin-polarized electric current [5–11]. The magnetization dynamics of a Stoner particle under the influence of a magnetic field and/or a spin-polarized current is described by the so-called Landau-Lifshitz-Gilbert (LLG) equation that does not have analytical solutions. Important issues are to lower critical fields/currents required to reverse a magnetization [10] and to design a field/current pulse such that the magnetization can be switched from one state to another as quickly as possible [4,11]. Thus it is interesting to know the theoretical limits of the critical switching field/current, and the optimal field/current pulse for the fastest reversal.

The answers to the above questions were only known recently for the simplest Stoner particles of uniaxial magnetic anisotropy [4,11]. Unfortunately, those ideas and approaches are not applicable to a non-uniaxial Stoner particle. In fact, the method does not even work on an uniaxial Stoner particle when both a spin-polarized electric current and a static magnetic field non-collinear to its easy axis are presented. The optimal trajectory for the fastest reversal of an arbitrary Stoner particle under combined influences of a magnetic field and an electric current are the themes of this work. In this paper, the Euler equation of the optimal reversal trajectory is derived. This equation is very useful in designing field/current pulses for the fastest magnetization reversal. The generality of the present theory is demonstrated on a biaxial Stoner particle.

Consider a Stoner particle of a magnetization $\vec{M} = M \vec{m}$ under an external magnetic field $\vec{H}$ as well as a polarized electric current $I$, where $M$ is the saturated magnetization of the particle and $\vec{m}$ is the unit vector of $\vec{M}$. Theoretical studies [5–7] show that the STT $\Gamma$ is proportional to the current with following form:

$$\Gamma \equiv \left[ \frac{d(\vec{M}V)}{dt} \right]_{STT} = \frac{\gamma I}{\mu_0} \phi(P, \vec{m} \cdot \hat{s}) \vec{m} \times (\vec{m} \times \hat{s}),$$

where $\hat{s}$ is the current polarization. $V$, $h$, $\mu_0 = 4\pi \times 10^{-7}$ N/A$^2$, and $e$ denote the volume of the Stoner particle, the Planck constant, the vacuum magnetic permeability, and the electron charge, respectively. $\gamma = 2.21 \times 10^5$ (rd/s)/(A/m) is the gyromagnetic ratio. The exact microscopic formulation of STT is
still a debating subject \[6,7\]. Many experimental investigations \[8\] so far are consistent with the result of Słonczewski \[5\], \( g = 4P^{3/2}/[(1+P)^3(\vec{m} \cdot \vec{s})-16P^{3/2}] \), which will be assumed in this study.

The dynamics of the magnetization \( \vec{m} \) is governed by the generalized LLG equation \[11\]

\[
(1 + \alpha^2) \frac{d\vec{m}}{dt} = -\vec{m} \times \vec{h}_1 - \vec{m} \times (\vec{m} \times \vec{h}_2),
\]

where \( \vec{h}_1 = \vec{h}_i + ca_1 \vec{s} \), and \( \vec{h}_2 = a_0 \vec{h}_i - a_1 \vec{s} \). \( \alpha \) is the phenomenological damping constant and \( a_0 = \hbar g / (\mu_0 e M^2) \) is a dimensionless parameter of Słonczewski STT \[5\]. \( \vec{h}_i = \vec{H}_i/M = \vec{h} + \vec{h}_i \) is the total effective magnetic field that includes the applied magnetic field \( \vec{h} = \vec{H}/M \) and the internal field \( \vec{h}_i \) due to the magnetic anisotropic energy density \( w(\vec{m}) \), \( \vec{h}_i = -\nabla_m w(\vec{m})/\mu_0 \). \( t \) in eq. (1) is in the units of \((\gamma M)^{-1}\). Both magnetization and magnetic field are in the units of \( M \). \( \vec{h}_1 \) and \( \vec{h}_2 \) are in general non-collinear.

In terms of the polar angle \( \theta \) and the azimuthal angle \( \phi \) of \( \vec{m} \) in the spherical coordinates, eq. (1) becomes

\[
(1 + \alpha^2) \dot{\theta} = a_1 (\alpha s_\theta - s_\phi) - \alpha \left( \frac{\partial w}{\partial \theta} - h_\theta \right)
+ h_\phi - \frac{1}{\sin \theta} \frac{\partial w}{\partial \phi} \equiv F_1,

(1 + \alpha^2) \sin \theta \dot{\phi} = -a_1 (\alpha s_\theta + s_\phi) - \alpha \left( h_\phi + \frac{1}{\sin \theta} \frac{\partial w}{\partial \phi} \right) \equiv F_2,
\]

where \( \dot{f} \) means derivative of \( f \) with respect to time \( t \). Here \( s_r, s_\theta, s_\phi \) are the \( r, \theta, \phi \) components of \( \vec{s} \), satisfying

\[
G_1 \equiv s_\phi^2 + s_\theta^2 + s_r^2 - 1 = 0.
\]

Since \( h_r \) does not appear in the dynamics equation, it should be zero, \( h_r = 0 \), for an optimal minimal field pulse. However, \( s_r \) affects the magnetization dynamics through \( a_1 \) which is a function of \( P \) and \( s_r \). In general, \( F_i (i = 1,2) \) are functions of \( \theta, \phi, h_\theta, h_\phi, s_\theta, s_\phi, \) and \( s_r \). The reversal problem is as follows: before applying a field/current, magnetization \( \vec{m} \) has two stable states, \( \vec{m}_0 \) “north pole” (\( \theta = 0 \)) and \( -\vec{m}_0 \) “south pole” (\( \theta = \pi \)) along its easy axis (\( z \)-axis). Initially, the particle is in state \( \vec{m}_0 \), and the goal is to use a field pulse and a spin-polarized electric current pulse to switch the magnetization to \( -\vec{m}_0 \) quickly. Both the field direction and the spin polarization direction may vary with time.

Given a field/current pulse, there is a unique evolution path for the magnetization which is the solution of eq. (2). If the pulse is proper, the evolution path \( \phi(\theta) \) may pass through \( -\vec{m}_0 \). When this happens, the path is called a reversal trajectory of the magnetization, and the corresponding field/current pulses are called reversal pulses. Knowing a reversal trajectory \( \phi(\theta) \), passing through \( \theta = 0, \pi \), there are infinite number of reversal pulses satisfying, according to eq. (2),

\[
\phi' \sin \theta F_1 - F_2 = G_2 = 0,
\]

where \( \phi' \equiv d\phi/d\theta \). All \( h_i \) and \( s_i (i = r, \theta, \phi) \) satisfying eq. (4) form reversal pulses. Thus, there is a functional relationship between reversal pulses and reversal trajectories. What is more, given a reversal trajectory, magnetization “velocities” \( \theta \) and \( \phi \) are linear in \( \vec{h} \) and \( I \). Thus, the optimization problem (shortest reversal time) is meaningful only when one imposes additional conditions on \( h \) and \( I \). One sensible way is to fix \( I \) and \( h \), i.e.,

\[
G_3 \equiv h_\phi^2 + h_\theta^2 - h^2 = 0.
\]

The reversal time \( T \) is given by

\[
T = \int_0^\pi \frac{d\theta}{\sqrt{\alpha^2 + \gamma M}} = \int_0^\pi \frac{(1 + \alpha^2) d\theta}{F_1}.
\]

The optimization problem here is to find the optimal reversal trajectory \( \phi(\theta) \) and field/current pulses \( h_i \) and \( s_i \) (\( i = r, \theta, \phi \)) such that \( T \) is minimum under the constraints of (3), (4) and (5). Using the Lagrange multipliers method, the optimal reversal trajectory and optimal reversal field/current pulses are given by

\[
\frac{d}{d\theta} \left( \frac{\partial F}{\partial \phi'} \right) = \frac{\partial F}{\partial \phi'}, \quad (i = \theta, \phi),
\frac{d}{d\theta} \left( \frac{\partial F}{\partial s_i} \right) = \frac{\partial F}{\partial s_i}, \quad (i = r, \theta, \phi),
\]

\[
F = \frac{1 + \alpha^2}{F_1} + \lambda_1 G_1 + \lambda_2 G_2 + \lambda_3 G_3,
\]

\[
G_1 = 0, \quad G_2 = 0, \quad G_3 = 0,
\]

where \( ' \equiv d/d\theta \), thus, \( ' \) means derivative of \( f \) with respect to \( \theta \). \( \lambda_{1,2,3} \) are the Lagrange multipliers that can be determined self-consistently from eq. (7).

Equation (7) is the central result of this work. For a given \( I, P, h, \) and particle (\( w(\vec{m}) \)), the solution of the equations gives the optimal reversal trajectory \( \phi(\theta) \) and optimal reversal field/current pulses \( h_i \) and \( s_i (i = r, \theta, \phi) \). From the reversal field/current pulses, it is straightforward to find the theoretical limit of the minimum switching field/current. Although it is possible to simultaneously optimize the field and the current, one may be more interested in varying either the field or the current from the controlled manipulation viewpoint. Two cases are particularly interesting: A) fix \( I \) and \( \vec{s} \), and vary \( h_\theta \) and \( h_\phi \); B) fix the magnetic field \( \vec{h} \) and the magnitude of current \( I \), and vary the current polarization.

In order to demonstrate that eq. (7) is capable of finding the optimal reversal trajectory for an arbitrary Stoner
particle, let us consider case A), magnetic-field–induced magnetization reversal and $\alpha_1 = 0$. After some tedious but straightforward calculations, the Euler equations become

$$f_1 = \frac{\partial^2 L}{\partial \varphi^2} \phi' + \frac{\partial^2 L}{\partial \theta \partial \varphi} \frac{\partial \theta}{\partial \varphi} \phi' + \frac{\partial^2 L}{\partial \theta^2} \frac{\partial \theta}{\partial \varphi} = 0,$$

$$L = \frac{\sin \theta}{\sin \theta} \frac{\partial w}{\partial \theta} - \frac{1}{\sin \theta} \partial w = 0,$$

where $f_{1,2}$ are functions of $\theta$ and $\varphi$ determined by the magnetic anisotropy energy $w(\bar{m})$. Reference [4] corresponds to $w = -km^2/2$, and it can be shown that eq. (8) is

$$\phi' = \frac{2}{\alpha(\alpha + 1)(\sin \theta)}.$$

Its solution is

$$\phi = \phi_0 + \frac{1}{\alpha} \sqrt{\frac{h_c}{h_c - h_c}} \arctan \left[ \sqrt{\frac{2h_c}{h_c - h_c}} \right],$$

where $\phi_0$ is an arbitrary constant due to the $\phi$-symmetry of the system. This is exactly the same optimal trajectory equation obtained in ref. [4] with the theoretical limit of the switching field being $h_c = \max \{ \alpha k \cos \theta \sin \theta / \sqrt{1 + \alpha^2} \} = \alpha k / (2 \sqrt{1 + \alpha^2})$. All other results in ref. [4] can be re-derived from this optimal reversal trajectory. Similarly, one can also reproduce all the results in ref. [11] by applying eq. (7) to a special example of case B with system specificities of $h = 0$, $I = \text{const.}$, $P = \text{const.}$, and $w = w(cos \theta)$.

For a biaxial Stoner particle of $w(\theta, \varphi) = -(k_1/2) \cos^2 \theta + (k_2/2) \sin^2 \theta \cos^2 \varphi$, eq. (8) may not be easy to solve analytically, but numerical solutions are easy to find. The solutions for $\alpha = 0.1$, $k_1 = 1$, and $h = 1 (\gg h_c)$, which is about 0.05 for $k_2 = 0$, are presented in fig. 1 for $k_2 = 0, 1, 5, 10$, respectively. Figure 1a shows the optimal trajectories. It is interesting to note that the optimal trajectories for $k_2 = 0$ have rotational symmetry around the $z$-axis (an infinite number of solutions). The one shown in fig. 1a is just a particular solution with $\phi(t = 0) = 0$. For $k_2 \neq 0$, the rotational symmetry is broken, but there are still four equivalent trajectories due to the biaxial symmetry. The ones whose initial $\phi$'s are in the range of $[0, \pi/2]$ are used in the figure. Figures 1b and c are the time evolution of $\theta$ and $\phi$ along the trajectories. It is surprising to note that the switching time is shorter and shorter as $k_2$ increases, and the potential landscape is less smooth. Figures 1d and e are the corresponding optimal reversal pulses.

It should be pointed out that the Euler equation is more useful than the LLG equation in designing an optimal field/current pulse although the former is derived from the later. This is because the LLG equation is specified only when the field/current pulse is given. Thus, it will be extremely difficult, if not impossible, to find one reversal pulse from the LLG equation since most pulses cannot reverse a magnetization, not mentioning to pick out the best one from an infinite number of possibilities. Also, the present theory can deal with combined effects of a field and a current for an arbitrary Stoner particle.

In conclusion, a unified theory for the optimal magnetization reversal trajectories and reversal field and/or current pulses of an arbitrary Stoner particle is presented. The Euler equation of the optimal reversal trajectory along which the magnetization reversal is the fastest is derived. Early results on the critical switching field/current for a uniaxial Stoner particle are just two special cases of this unified theory. The optimal magnetization reversal of a biaxial Stoner particle is solved.

***

This work is supported by Hong Kong RGC/CERG grants (#603508 and #603007).

REFERENCES

[1] Pan M. H., Liu H., Wang J. Z., Jia J. F., Xue Q. K., Li J. L., Qin S., Mirdaidov U. M., Wang X. R., Market J. T., Zhang Z. Y. and Shihi C. K., Nano Lett., 5 (2005) 87.

[2] Hilkebrand B. and Ounadjela K. (Editors), Spin Dynamics in Confined Magnetic Structures I & II (Springer-Verlag, Berlin) 2001.
[3] He L. and Doyle W. D., *IEEE. Trans. Magn.*, 30 (1994) 4086; Acremann Y., Back C. H., Buess M., Portmann O., Vaterlaus A., Pescia D. and Melchior H., *Science*, 290 (2000) 492; Sun Z. Z. and Wang X. R., *Phys. Rev. B*, 71 (2005) 174430; 73 (2006) 092416; 74 (2006) 132401.

[4] Sun Z. Z. and Wang X. R., *Phys. Rev. Lett.*, 97 (2006) 077205.

[5] Slonczewski J., *J. Magn. & Magn. Mater.*, 159 (1996) L1; Berger L., *Phys. Rev. B*, 54 (1996) 9353.

[6] Sun J. Z., *Phys. Rev. B*, 62 (2000) 570.

[7] Brataas A., Nazarov Y. V. and Bauer G. E. W., *Phys. Rev. Lett.*, 84 (2000) 2481; Waintal X., Myers E. B., Brouwer P. W. and Ralph D. C., *Phys. Rev. B*, 62 (2000) 12317; Stiles M. D. and Zangwill A., *Phys. Rev. B*, 66 (2002) 014407.

[8] Tsoi M., Jansen A. G. M., Bass J., Chiang W.-C., Seck M., Tsoi V. and Wyder P., *Phys. Rev. Lett.*, 80 (1998) 4281; Myers E. B., Ralph D. C., Katine J. A., Louie R. N. and Buhrman R. A., *Science*, 285 (1999) 867; Katine J. A., Albert F. J., Buhrman R. A., Myers E. B. and Ralph D. C., *Phys. Rev. Lett.*, 84 (2000) 3149.

[9] Wetzels W., Bauer G. E. W. and Jouravlev O. N., *Phys. Rev. Lett.*, 96 (2006) 127203.

[10] Sun J., *J. Magn. & Magn. Mater.*, 202 (1999) 157; Nature, 425 (2003) 359; Lee K. J., Redon O. and Dieny B., *Appl. Phys. Lett.*, 86 (2005) 022505; Manschot J., Brataas A. and Bauer G. E. W., *Appl. Phys. Lett.*, 85 (2004) 3250; Kent A. D., Zyilmaz B. and del Barco E., *Appl. Phys. Lett.*, 84 (2004) 3897; Moriyama T., Cao R., Xiao J. Q., Lu J., Wang X. R., Wen Q. and Zhang H. W., *Appl. Phys. Lett.*, 90 (2007) 152503.

[11] Wang X. R. and Sun Z. Z., *Phys. Rev. Lett.*, 98 (2007) 077201.