On the tensor-to-scalar ratio in large single-field inflation models

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We show that generically the tensor-to-scalar ratio in large single-field inflation scenarios is bounded to be larger than $O(10^{-3})$ for the spectral index in the range favored by observations.

INTRODUCTION

In scenarios of inflation, a necessary step to explain various features of the present universe, the number of $e$-foldings ($N_e$) is conventionally taken to be around 50-60 depending on the cosmological history after inflation. The lower bound of $N_e$ can be pushed down as long as the reheating temperature after inflation is high enough to provide the relevant physics afterwards (e.g., baryogenesis, production of dark matter, a successful Big-Bang nucleosynthesis). However, the upper bound is saturated at about 60, since the upper bound of the tensor-to-scalar ratio ($r_T < 0.1$ [1]) coming from the non-observations of the tensor mode perturbations, puts an upper bound on the energy scale of inflation i.e., $V^{1/4}_e \lesssim 2 \times 10^{16}$ GeV.

In conventional Einstein gravity, the tensor-to-scalar ratio is simply proportional to the slow-roll parameter associated with the slope of the inflaton potential. For small field inflation in which the excursion scale of the inflaton can be lowered down far below Planck scale, $r_T$ can be extremely small. However, in large-field inflation scenarios, this can not be the case since the $e$-foldings are likely to turn out too large unless the potential has an unphysical sudden change. Hence, $r_T$ is likely to be lower-bounded in such scenarios.

In this work, we show that the $e$-foldings required to match observations set a lower bound of $r_T$ at around $O(10^{-3})$ in most of realistic models of large single-field inflation. Our result is consistent with a recent work, Ref. [2], although the approach is different.

INFLATION AND THE NUMBER OF $e$-FOLDINGS FOR A GIVEN SCALE

In slow-roll inflation scenario with a potential $V$ of a single scalar field (playing the role of inflaton), the slow-roll parameters are defined as

$$\epsilon \equiv \frac{1}{2} \left| \frac{M_P V'}{V} \right|^2, \quad \eta \equiv \frac{M_P^2 V''}{V}, \quad \xi^2 \equiv \frac{M_P^4 V'''}{V^2}$$

(1)

where $M_P = 2.4 \times 10^{18}$ GeV is the reduced Planck mass, and $'$ represents derivative with respect to the inflaton field. The power spectrum of scalar and tensor mode perturbations are given by

$$P_R = \left( \frac{H}{2\pi} \right)^2 \frac{1}{2\epsilon M_P^2}, \quad P_T = \left( \frac{H}{2\pi} \right)^2 \frac{8}{M_P^2}$$

(2)

respectively, giving a tensor-to-scalar ratio

$$r_T \equiv \frac{P_T}{P_R} = 16\epsilon$$

(3)

The spectral indices of these modes are

$$n_s = 1 - 6\epsilon + 2\eta, \quad n_T = -2\epsilon$$

(4)

Note that $n_T = -r_T/8$, a consistency relation of inflation in Einstein gravity. For inflation to end smoothly, $\epsilon$ should depend on the scale; although $\eta$ can be constant. The spectral running, $\alpha \equiv dn_s/d\ln k$, is given by

$$\alpha = 16\epsilon \eta - 24\epsilon^2 - 2\xi^2$$

(5)

The number of $e$-foldings generated until the end of inflation, since a given cosmological scale (denoted with a subscript ‘$\ell$’) exits the horizon during inflation, is given by

$$N_{e,\ell}^\text{th} \equiv \int_*^\ell H dt \approx -\frac{1}{M_P^2} \int_*^\ell d\phi \frac{V}{V'}$$

(6)

with $\phi$ being the inflaton field. For a given model of inflation (i.e., a specific form of potential $V$), the observed $n_s$ fixes $\phi_*$ and hence $N_{e,\ell}^\text{th}$ is also fixed. Meanwhile, in order for inflation to solve the smoothness problem of Big-Bang cosmology, any two points within our Hubble patch had to be in casual contact at some stage before the end of inflation. That requires a minimum amount of $e$-foldings, that we will denote $N_{e,\ell}^\text{obs}$. In order for an inflation scenario to work, $N_{e,\ell}^\text{th} = N_{e,\ell}^\text{obs}$ is required, and this constrains the possible models of inflation. If there were no more entropy production after the reheating following inflation, the $e$-foldings associated with the Planck pivot scale ($k_* = 0.05\text{Mpc}^{-1}$) are given by [3]

$$N_{e,\ell}^\text{obs} = \frac{1}{3} \ln \left[ \frac{2\sqrt{6\pi}^3 s_0}{3} \frac{(V_*)^3}{k_*^3} \left( \frac{T_d}{V_*^4} \right) \right]$$

$$\simeq 57.87 - \frac{1}{3} \ln \left[ \frac{(V_*)^3}{V_*^4} \left( \frac{10^{16} \text{GeV}}{V_*^4} \right) \frac{V_*^4}{T_d} \right]$$

(7)
where $s_0$ is the present entropy density

$$s_0 = \frac{2\pi^2}{45} g_{*s0} T_0^3$$  \hspace{1cm} (8)$$

with $g_{*s0} = 3.9091$ being the number of relativistic degrees of freedom and $T_0 = 2.726K$ the CMB temperature at the present universe. The subscript ‘$c$’ in the right-hand side denotes the value at the end of inflation, and $T_d$ is the decay temperature of the inflaton. The current upper-bound on the tensor-to-scalar ratio $r_T < 0.1$ \cite{1} implies

$$V_+^{1/4} \lesssim 2 \times 10^{16} \text{GeV}$$  \hspace{1cm} (9)$$

and the reheating temperature is bounded as

$$1 \text{MeV} \lesssim T_d \lesssim V_+^{1/4}$$  \hspace{1cm} (10)$$

Then, setting $V_+ = V_c$, one finds

$$N^{\text{obs}}_{e,\ast} \lesssim 58.56$$  \hspace{1cm} (11)$$

The lower bound of $N^{\text{obs}}_{e,\ast}$ depends on the post-inflation cosmology as well as $V_+$ and $T_d$. In particular, there can be an $\mathcal{O}(10)$ additional contribution to the $e$-foldings of inflation from, for example, thermal inflation \cite{4,5}. In this case, the required $e$-foldings from primordial inflation are reduced by that amount. In principle, there can be multiple stages of thermal inflation, providing a few tens of $e$-foldings. However, thermal inflation with more than two stages is non-trivial to realize (or not so realistic), and too much extra $e$-foldings may cause some inconsistency with observations of small scale structure (see for example \cite{6}). In realistic models, we expect extra $e$-foldings of around 20 or less \cite{7,8,9,10,11}. Hence, in this letter we take $N_{e,\ast} = 15$ as the plausible maximal extra $e$-foldings with $T_d = V_+^{1/4}$ for simplicity in the case where thermal inflation is considered.

The number of $e$-foldings in Eq. (6) can be separated into contributions from slow-roll and non-slow-roll (or fast-roll) regions, denoted respectively as $\Delta N_{e,\text{sr}}$ and $\Delta N_{e,\text{fr}}$, i.e.,

$$N^{\text{th}}_{e,\ast} = \Delta N_{e,\text{sr}} + \Delta N_{e,\text{fr}}$$  \hspace{1cm} (12)$$

where

$$N_{e,\text{sr}} = \int_x^\infty \frac{d\phi}{M_P \sqrt{2\epsilon}}, \quad N_{e,\text{fr}} = \int_x^{e_x} \frac{d\phi}{M_P \sqrt{2\epsilon}}$$  \hspace{1cm} (13)$$

The value of $e_x$ for which the slow-roll approximation breaks down depends on the potential and is rather ambiguous. However, we can take $e_x \equiv e(e_x) \sim \mathcal{O}(0.1)$, and in this case

$$N_{e,\text{fr}} < \frac{\Delta \phi/M_P}{e_x} \sim 2.24 \left(\frac{0.1}{e_x}\right)^{1/2} \frac{\Delta \phi}{M_P}$$  \hspace{1cm} (14)$$

where $\Delta \phi \equiv |\phi_c - \phi_x|$ with $\phi_c$ being the value of the inflaton at the end of inflation. In large-field models, generically, $\Delta \phi \gtrsim M_P \sim 1$, being the precise value model dependent. Hence, as a simple conservative constraint, we require

$$\Delta N_{e,\ast} < N^{\text{th}}_{e,\ast} - \Delta N_{e}$$  \hspace{1cm} (15)$$

with $\Delta N_{e} = 0, 15$ depending on the existence of thermal inflation.

**CLASSES OF POTENTIALS**

There are numerous different shapes of potentials for slow-roll inflation. However, in the region of field values where the slow-roll approximation is still valid, inflaton potentials generally fall into one of the following forms,

- Chaotic monomial: $V_{ch} = V_0 x^p$
- Inverse-Hilltop: $V_{ht} = V_0 (1 - 1/x^p + \ldots)$
- Starobinsky-like: $V_{st} = V_0 (1 - e^{-x} + \ldots)$
- Hilltop: $V_{ht} = V_0 (1 - x^p + \ldots)$

where $x = \phi/\mu$ with $\mu$ being a scale characterizing the end of inflation, and $p$ is assumed to be positive definite. Note that, even if the potentials listed above are approximated froms, it is clear that they all are significantly changed as $x \to 1$. Hence, unless the potentials have non-trivial complications around the end of inflation, we expect that inflation ends at $x \sim 1$. In the following subsections, we show the approximate forms of slow-roll parameters and $N^{\text{th}}_{e,\ast}$ or $N_{e,\text{sr}}$ (depending on its relevance) which are expected to be valid as long as $\epsilon$ and $\eta$ are much smaller than unity in the region where $V \approx V_0$. These approximate expressions are useful to get an idea of the parametric dependences of the relevant quantities. The discrepancies one would get when considering a complete potential are minor and do not change our argument.

**Chaotic monomial**

In this case, the slow-roll parameters take the (exact) form

$$\epsilon = \frac{p^2}{2x^2} \left(\frac{M_P}{\mu}\right)^2, \quad \eta = \frac{p(p-1)}{x^2} \left(\frac{M_P}{\mu}\right)^2$$  \hspace{1cm} (16)$$

giving $\eta = 2(p-1)\epsilon/p$, and from Eq. (4)

$$\epsilon_\ast = \left(\frac{p}{p+2}\right) \frac{1 - n_s}{2}$$  \hspace{1cm} (17)$$
which does not depend on $\mu$. The $e$-foldings are

$$N_{e,s} = \frac{p}{4} \left( \frac{1}{\epsilon_s} - 1 \right) \approx \frac{p + 2}{2(1 - n_s)}$$  \hspace{1cm} (18)

It is clear that, as $p$ becomes larger than 1, $\eta$ becomes larger than zero, and similarly $\epsilon$. Also, only $p \lesssim 4$ is allowed, otherwise too much $e$-foldings are expected. As can be seen from Eq. (17), $\epsilon$ can be lowered down by taking a small $p$. However, in large field scenarios in which $x_\epsilon > 1$ with $\mu \geq M_p$, we are constrained to have $p \gtrsim O(10^{-2})$. Hence, $r_T$ is lower-bounded at $O(10^{-3})$.

**Inverse-Hilltop potential**

For $V \approx V_0$, the slow-roll parameters can be approximated as

$$\epsilon \approx \frac{p^2}{2x^{2(p+1)}} \left( \frac{M_p}{\mu} \right)^2, \quad \eta \approx \frac{-p(p+1)}{x^{p+2}} \left( \frac{M_p}{\mu} \right)^2$$  \hspace{1cm} (19)

From Eq. (19), we find

$$\eta = -\frac{2(p+1)}{p} \left[ \frac{2}{p^2(M_p/\mu)^2} \right]^{\gamma^{-1}} e^\epsilon$$  \hspace{1cm} (20)

where

$$\frac{1}{2} < q \equiv \frac{p+2}{2(p+1)} < 1$$  \hspace{1cm} (21)

Using Eq. (4) with Eq. (20), one can find $\epsilon_s$, at least numerically, as a function of $p$ and $n_s$ for a given $\mu$. Note that $\eta/\epsilon = -2x(p+1)/p$ in this potential, which means that for a given $\eta$, as $p$ goes away from a value around one (or $\mu$ is lowered down), $\epsilon$ decreases. The $e$-foldings for slow-roll regime are

$$N_{e,\text{sr}} \approx \frac{x_p^{p+2} - x_P^{p+2}}{p(p+2)} \left( \frac{\mu}{M_p} \right)^2 \approx \frac{1}{\eta_s} - \frac{1}{\eta_s} \left( \frac{p+1}{p+2} \right)$$  \hspace{1cm} (22)

where $\eta_\text{x}$ is found from Eq. (20) with $\epsilon_s = 0.1$.

We can explore the limiting cases of $p$ leading to small $\epsilon_s$. When $p \to 0$, one finds

$$\epsilon_s \approx \frac{p}{4} (1 - n_s), \quad N_{e,\text{sr}} \approx \frac{1}{1 - n_s}$$  \hspace{1cm} (23)

Hence, taking a small $p$, one can lower down $\epsilon_s$, but $N_{e,\text{sr}} \gtrsim 50$ requires $n_s \gtrsim 0.98$ which is out of the 2-$\sigma$ region. On the other hand, if $p \to \infty$,

$$\epsilon_s^{1/2} \approx -\frac{\gamma}{\sqrt{2}} \left( 1 - \frac{6(1 - n_s)}{\gamma^2} \right), \quad N_{e,\text{sr}} \approx \frac{2}{1 - n_s}$$  \hspace{1cm} (24)

with $\gamma \equiv \sqrt{2p(M_p/\mu)}$. It may be natural and tempting to set $\mu$ to be the Planck scale. However, in principle $\mu$ can be much larger than $M_p$ although it may need some fine-tuning or non-trivial (or unnatural) realizations.

**Starobinsky-like**

In this case, one finds

$$\epsilon \approx \frac{e^{2x}}{2} \left( \frac{M_p}{\mu} \right)^2, \quad \eta \approx -e^{-x} \left( \frac{M_p}{\mu} \right)^2$$  \hspace{1cm} (25)

Note that, as $\mu$ is increased, $\epsilon_s$ matching observations increases as well. So, we take $\mu = M_P$ to see the smallest allowed $\epsilon_s$. From Eq. (25), one finds

$$\eta = -\sqrt{2\epsilon}^{1/2}$$  \hspace{1cm} (26)

leading to

$$\epsilon_s^{1/2} = -\frac{\sqrt{2}}{6} \left( 1 - \sqrt{1 + 3(1 - n_s)} \right)$$  \hspace{1cm} (27)

which is the same as $\epsilon_s^{1/2}$ in Eq. (24) with $\mu/M_P = p$.

The number of $e$-foldings for slow-roll regime are

$$N_{e,\text{sr}} \approx \frac{1}{\eta_s} - \frac{1}{\eta_s}$$  \hspace{1cm} (28)

where $\eta_\text{x}$ is obtained from Eq. (26) with $\epsilon_\text{x} = 0.1$.

**Hilltop potential**

In this case, for $0 < p < 1$ inflaton should be located at a particular region in $0 < x < 1$ as the initial condition, and this is non-trivial to realize. Also, as $p$ increases, the $\epsilon_s$ needed to match observation becomes smaller. So, we consider only $p \geq 2$ in order to avoid irrelevant complications. Slow-roll parameters are given as

$$\epsilon \approx \frac{p^2 x^{2(p-1)}}{2} \left( \frac{M_p}{\mu} \right)^2, \quad \eta \approx -p(p-1)x^{p-2} \left( \frac{M_p}{\mu} \right)^2$$  \hspace{1cm} (29)

Note that for $p = 2$, $\eta$ is a constant depending on $\mu$ exclusively, and

$$\frac{\mu}{M_p} = \frac{2}{\sqrt{1 - n_s - 6\epsilon_s}} \geq \frac{2}{\sqrt{1 - n_s}} \sim O(10)$$  \hspace{1cm} (30)

For $p \neq 1, 2$, $\eta$ can be found in terms of $\epsilon$ from Eqs. (20) and (21) with $p \to p$. The number of $e$-foldings is

$$N_{e,\text{sr}} \approx \left\{ \begin{array}{ll} \frac{1}{2} \eta \ln \left( \frac{\epsilon_s}{\eta_s} \right) \left( \frac{1}{\eta_s} - \frac{1}{\eta_s} \right) \left( \frac{p-1}{p} \right) : p = 2 \\ \frac{1}{2} \eta \ln \left( \frac{\epsilon_s}{\eta_s} \right) \left( \frac{1}{\eta_s} - \frac{1}{\eta_s} \right) \left( \frac{p-1}{p} \right) : p \neq 1, 2 \end{array} \right.$$  \hspace{1cm} (31)

Similarly to the case of $V_{\text{start}}$, as $\mu$ decreases, $\epsilon_s$ decreases, and we consider $\mu/M_P \geq 1$ here too. Note that, if $\mu/M_P = p$, as $p \to \infty$, $\epsilon_s$ collapses to the one in Eq. (27).
FIG. 1: The lower bound of $r_T$ (gray line(s)) as a function of $n_s$. Green regions are 1, 2, 3-$\sigma$ bands of $n_s$ from Planck observations (Planck TT+lowP) [12]. Red lines are obtained by imposing the constraint on the $e$-foldings, Eq. (15). Red dashed lines are for inverse-Hilltop potential with $p = 0.1, 4$ from right to left. Red dot-dashed line is for Hilltop potential with $p = 4$. Red solid line is for $p \to \infty$ in both inverse-Hilltop and Hilltop potentials. For each value of $p$, the right-side of the red line is excluded because of too much $e$-foldings. Green line(s) is the bound obtained from the completions of potentials, Eqs. (32), (34) and (33). Blue dot is the prediction of Starobinsky-like potential with $\mu = M_P$. Upper: $\mu = M_P$. Lower: $\mu = pM_P$ for $p \geq 1$, but $\mu = M_P$ for $p < 1$.

**THE LOWERBOUND OF $r_T$: NUMERICAL RESULTS**

In order to find a lower bound of $r_T$, we used the expressions obtained in the previous section and performed a numerical analysis. Also, for comparison, we used the following completions of potentials:

\[
V_{\text{ht}} = V_0 \left(1 - \frac{1}{2x^p}\right)^2 \\
V_{\text{st}} = V_0 \left(1 - e^{-x}/2\right)^2 \\
V_{\text{ht}} = V_0 \left(1 - x^p/2\right)^2
\]

The result is shown in Fig. 1. We found that, if $\mu \sim M_P$ (upper panels), when $p \gg 4$ in inverse-Hilltop and Hilltop potentials, it is possible to lower-down $r_T$ by many orders of magnitude relative to the current upper bound. However, for $p \leq 4$ which is likely to be the case, in the region of interest the field value is far away from the end point of inflation, and either $r_T \gtrsim \mathcal{O}(10^{-3})$ or $n_s$ is out of the 3-$\sigma$ bound of observations (in Hilltop). As shown in the lower panels of Fig. 1, if we take $\mu$ larger than $M_P$, the case of $p \leq 4$ in Hilltop potential can be within the preferred region of observations, but $r_T$ is pushed up to $\mathcal{O}(10^{-3})$. So, we can conclude that, including chaotic monomials and Starobinsky-like potentials, in realistic models (probably) with $p \leq 4$, $r_T$ is lower-bounded at about $10^{-3}$ or $n_s$ is out-of the 3-$\sigma$ allowed band. Fig. 1 also shows that, if the decay temperature of the inflaton is low or there is an extra contribution to $e$-foldings, the lower-bound of $r_T$ is pushed up.

**CONCLUSIONS**

In this paper, we examined the lower bound of the tensor-to-scalar ratio in large single-field scenarios of inflation. For inflaton field values associated to the relevant cosmological scales, the inflaton potential may be approximated to either chaotic monomial, inverse-Hilltop, Hilltop, or Starobinsky-like potentials in which the leading field-dependent term is $(\phi/\mu)^{2-4p}$ with $0 < p \leq 4$ or $e^{-\phi/\mu}$. We showed that, if the dimensionful scale $\mu$ characterizing the end of inflation is Planck scale or larger, which is the case of large single-field scenarios, the tensor-to-scalar ratio is lower-bounded at $\mathcal{O}(10^{-3})$ for the range of the spectral index favored by observations. Therefore, even if it will not be done in the near future, most large single-field inflation models will be probed as experiments reach a $r_T$ at the level of $10^{-3}$.

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[1] P. A. R. Ade et al. [Planck Collaboration], arXiv:1502.01589 [astro-ph.CO].
[2] Q. G. Huang, Phys. Rev. D 91, no. 12, 123532 (2015) [arXiv:1503.04513 [astro-ph.CO]].
[3] E. W. Kolb and M. S. Turner Addison-Wesley Pub. Co. (1994)
[4] D. H. Lyth and E. D. Stewart, Phys. Rev. Lett. 75, 201 (1995) [hep-ph/9502417].
[5] D. H. Lyth and E. D. Stewart, Phys. Rev. D 53, 1784 (1996) [hep-ph/9510204].
[6] S. E. Hong, H. J. Lee, Y. J. Lee, E. D. Stewart and H. Zoe, JCAP 1506, no. 06, 002 (2015)
[7] D. h. Jeong, K. Kadota, W. I. Park and E. D. Stewart, JHEP 0411, 046 (2004) [hep-ph/0406136].
[8] S. Kim, W. I. Park and E. D. Stewart, JHEP 0901, 015 (2009) [arXiv:0807.3607 [hep-ph]].
[9] K. Choi, K. S. Jeong, W. I. Park and C. S. Shin, JCAP 0911, 018 (2009) [arXiv:0908.2154 [hep-ph]].
[10] W. I. Park, JHEP 1007, 085 (2010) [arXiv:1004.2326 [hep-ph]].
[11] K. Choi, W. I. Park and C. S. Shin, JCAP 1303, 011 (2013) [arXiv:1211.3755 [hep-ph]].
[12] P. A. R. Ade et al. [Planck Collaboration], arXiv:1502.02114 [astro-ph.CO].