THE COARSE GEOMETRIC $\ell^p$-NOVIKOV CONJECTURE FOR SUBSPACES OF NON-POSITIVELY CURVED MANIFOLDS

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Abstract. In this paper, we prove the coarse geometric $\ell^p$-Novikov conjecture for metric spaces with bounded geometry which admit a coarse embedding into a simply connected complete Riemannian manifold of nonpositive sectional curvature.

1. Introduction

The coarse geometric Novikov conjecture provides an algorithm of determining non-vanishing of the higher index for elliptic differential operators on noncompact Riemannian manifolds. It implies Gromov’s conjecture stating that a uniformly contractible Riemannian manifold with bounded geometry cannot have uniformly positive scalar curvature, and the zero-in-the-spectrum conjecture stating that the Laplacian operator acting on the space of all $L^2$-forms of a uniformly contractible Riemannian manifold has zero in its spectrum [9,13,17,22-30].

Recently, an $\ell^p$ analog of the coarse geometric Novikov conjecture for $1 < p < \infty$ was introduced in [8]. Although the $\ell^p$ analog of the coarse Novikov conjecture has no known geometric or topological applications when $p \neq 2$, this study contributes to the general interests of understanding of the $K$-theory of some operator algebras. For example, Zhang and Zhou proved that $K$-theory for $L^p$ Roe algebra of a finite asymptotic dimensional metric space does not depend on $p$ in [31]. More related references are included in [8].
In this paper, we prove the following result.

**THEOREM 1.** Let $\Gamma$ be a discrete metric space with bounded geometry. If $\Gamma$ admits a coarse embedding into a simply-connected complete Riemannian manifold of non-positive sectional curvature, then the coarse geometric $\ell^p$-Novikov conjecture holds for $\Gamma$, i.e., the index map

$$e_* : \lim_{d \to \infty} K_*(B^p_L(P_d(\Gamma))) \to K_*(B^p(\Gamma))$$

is injective.

Recall that for two metric spaces $X$ and $Y$, a map $f : X \to Y$ is said to be a **coarse embedding** [12] if there exist non-decreasing functions $\rho_1$ and $\rho_2$ from $\mathbb{R}_+ = [0, \infty)$ to $\mathbb{R}_+$ such that:

1. $\rho_1(d(x, y)) \leq d(f(x), f(y)) \leq \rho_2(d(x, y));$
2. $\lim_{r \to \infty} \rho_i(r) = \infty$ for $i = 1, 2.$

### 2. The coarse geometric Novikov conjecture

In this section, we shall recall the concepts of the $\ell^p$-Roe algebra [22], Yu’s $\ell^p$-localization algebras [27] and the coarse geometric $\ell^p$-Novikov conjecture.

Let $X$ be a metric space. $X$ is called **proper** if every closed ball is compact. When $X$ is discrete, $X$ has **bounded geometry** if for any $R > 0$, there exists $M_R > 0$ such that for any $x \in X$ the cardinality $|B(x; R)|$ is less than or equal to $M_R$. For $r > 0$, a $r$-net in $X$ is a discrete subset $Y \subset X$ such that for any $y_1, y_2 \in Y$, $d(y_1, y_2) \geq r$ and for any $x \in X$ there is a $y \in Y$ such that $d(x, y) < r$. A general metric space $X$ is called to have bounded geometry if $X$ has a $r$-net $Y$ for some $r > 0$ such that $Y$ has bounded geometry.
Throughout the paper, \( p > 1 \). And \( \mathcal{K}_p = \mathcal{K}(\ell^p) \), the set of all compact operators over \( \ell^p \).

**DEFINITION 2** ([22][3]). Let \( X \) be a proper metric space, and fix a countable dense subset \( Z \subseteq X \). Let \( T \) be a bounded operator on \( \ell^p(Z, \ell^p) \), and write \( T = (T(x, y))_{x, y \in Z} \) so that each \( T(x, y) \) is a bounded operator on \( \ell^p \). \( T \) is said to be locally compact if

- each \( T(x, y) \) is a compact operator on \( \ell^p \);
- for every bounded subset \( B \subseteq X \), the set
  \[
  \{(x, y) \in (B \times B) \cap (Z \times Z) : T(x, y) \neq 0\}
  \]
  is finite.

The propagation of \( T \) is defined to be

\[
propagation(T) = \inf\{S > 0 : T(x, y) = 0 \text{ for all } x, y \in Z \text{ with } d(x, y) > S\}.
\]

The algebraic \( \ell^p \) Roe algebra of \( X \), denoted by \( B^\text{alg}_{\ell^p}(X) \), is the subalgebra of \( \mathcal{L}(\ell^p(Z, \ell^p)) \) consisting of all finite propagation, locally compact operators. The \( \ell^p \) Roe algebra of \( X \), denoted by \( B^p(X) \), is the closure of \( B^\text{alg}_{\ell^p}(X) \) in \( \mathcal{L}(\ell^p(Z, \ell^p)) \).

\( B^p(X) \) does not depend on the choice of \( Z \). See [31] for a proof.

**DEFINITION 3** ([27]). The \( \ell^p \)-localization algebra \( B^p_L(X) \) is the norm-closure of the algebra of all bounded and uniformly norm-continuous functions \( g : [0, \infty) \to B^p(X) \) such that

\[
propagation(g(t)) \to 0 \quad \text{as} \quad t \to \infty.
\]

The evaluation homomorphism \( e \) from \( B^p_L(X) \) to \( B^p(X) \) is defined by \( e(g) = g(0) \) for \( g \in B^p_L(X) \).
**Definition 4.** Let $\Gamma$ be a discrete metric space and $d \geq 0$. The Rips complex of $\Gamma$ at scale $d$, denoted by $P_d(\Gamma)$, is the simplicial complex with vertex set $\Gamma$, and where a finite subset $E \subset \Gamma$ spans a simplex if and only if $d(g, h) \geq d$ for all $g, h \in E$.

Points in $P_d(\Gamma)$ can be written as formal linear combinations $\sum_{g \in \Gamma} t_g g$, where $t_g \in [0, 1]$ for each $g$ and $\sum_{g \in \Gamma} t_g = 1$. $P_d(\Gamma)$ is equipped with the $\ell^1$ metric, i.e.,

$$d\left(\sum_{g \in \Gamma} t_g g, \sum_{g \in \Gamma} s_g g\right) = \sum_{g \in \Gamma} |t_g - s_g|.$$

To define the assembly map, we recall that when $p = 2$, Yu in [27] proved that the local index map from $K$-homology to $K$-theory of localization algebra is an isomorphism for finite-dimensional simplicial complexes. Qiao and Roe in [21] later generalized this isomorphism to general locally compact metric spaces. Therefore for $p \in (1, \infty)$, considering the analogs of $\ell^p$-Roe algebra and $\ell^p$-localization algebra, we get the following assembly map which is equivalent to the original map when $p = 2$. The following conjecture is called the coarse geometric $\ell^p$-Novikov conjecture:

**Conjecture 5.** If $\Gamma$ is a discrete metric space with bounded geometry, then the index map

$$e_* : \lim_{d \to \infty} K_*(B^p_L(P_d(\Gamma))) \to \lim_{d \to \infty} K_*(B^p(P_d(\Gamma))) \cong K_*(B^p(\Gamma))$$

is injective.

3. An $\ell^p$ coarse Mayer-Vietoris principle

In this section, we present an $\ell^p$ coarse Mayer-Vietoris principle similar to the argument in [14].
DEFINITION 6 ([14]). Let $X$ be a proper metric space, and let $A$ and $B$ be closed subspace with $X = A \cup B$. We say that $(A, B)$ is a $w$-excisive couple, or that $X = A \cup B$ is a $w$-excisive decomposition, if for each $R > 0$ there is some $S > 0$ such that

$$\text{Pen}(A; R) \cap \text{Pen}(B; R) \subset \text{Pen}(A \cap B; S),$$

where $\text{Pen}(A; R)$ is the set of points in $X$ of distance at most $R$ from $A$.

DEFINITION 7 ([14]). Let $A$ be a closed subspace of a proper metric space $X$. Denote by $B^p(A; X)$ the operator-norm closure of the set of all locally compact, finite propagation operators $T$ whose support is contained in $\text{Pen}(A; R) \times \text{Pen}(A; R)$, for some $R > 0$ (depending on $T$).

One can see that $B^p(A; X)$ is a two-sided ideal of $B^p(X)$. For $s, t \in [0, \infty)$ with $s < t$, let $i_{s,t} : \text{Pen}(A; s) \to \text{Pen}(A; t)$ be the inclusion map. Then $i_{s,t}$ induces a map

$$i_{s,t}^* : \ell^p(\text{Pen}(A; t), \mathcal{K}_p) \to \ell^p(\text{Pen}(A; s), \mathcal{K}_p).$$

Then the induced map $i_{s,t,*} : B^p(\text{Pen}(A; s)) \to B^p(\text{Pen}(A; t))$ is defined by, for any $f \in \ell^p(\text{Pen}(A; t), \mathcal{K}_p)$ and $T \in B^p(\text{Pen}(A; s))$,

$$i_{s,t,*}(T)(f) = T(i_{s,t}^*(f)).$$

By definition, we have $i_{t,r,*} \circ i_{s,t,*} = i_{s,r,*}$ for any $s < t < r$. And $\|i_{s,t,*}(T)\|_t \leq \|T\|_s$ for $T \in B^p(\text{Pen}(A; s))$. Here we identify $A$ as $\text{Pen}(A; 0)$. Since $B^p(A; X) = \lim_{n \to \infty} B^p(\text{Pen}(A; n))$, we define that, for any $T \in B^p(A)$, $i_\ast : K_\ast(B^p(A)) \to K_\ast(B^p(A; X))$ by

$$i_\ast(T) = \lim_{n \to \infty} i_{0,n,*}(T).$$
**Lemma 8** ([14]). The induced map

\[ i_* : K_*(B^p(A)) \to K_*(B^p(A;X)) \]

is an isomorphism.

**Proof.** This is because that the inclusions \( A \subset Pen(A; n) \) and \( Pen(A; n) \subset Pen(A; n + 1) \) are coarsely equivalent, hence the induced maps on \( K \)-theory are all isomorphisms. \( \square \)

Let \( X = A \cup B \). Let \( I = B^p(A;X) \) and \( J = B^p(B;X) \). Define \( U : A \times A \to K_p \) such that

\[
U(x, y) = 0 \text{ if } x \neq y,
\]

\[
U(x, x) = \left( \begin{array}{cc}
I_{r(x)} & 0 \\
0 & 0
\end{array} \right),
\]

where \( I_{r(x)} \) is a rank \( r(x) \) identity matrix for some \( r(x) \in \mathbb{N} \). We define a partial order on all such \( U \) by the following: \( U_2 \leq U_1 \) if \( \text{rank}(U_2(x,x)) \leq \text{rank}(U_1(x,x)) \) for all \( x \in X \). Let \( \mathcal{U} \) be the set of all such operators \( U \) with this order.

**Proposition 9.** The collection \( \mathcal{U} \) is an approximate unit of \( I \).

**Proof.** Let \( T \in I \), for any \( \epsilon > 0 \), for any \( x, y \in X \), \( T(x,y) \) is either a zero operator or a compact operator over \( \ell^p \). \( X \times X \) is countable, so each pair \( (x,y) \in X \times X \) has a corresponding integer \( n \).

Let \( F(x,y) \) be a finite rank operator over \( \ell^p \) such that \( \|T(x,y) - F(x,y)\| < \frac{1}{2^n} \epsilon \) when \( T(x,y) \neq 0 \) and \( n \) is the corresponding integer of \( (x,y) \), and \( F(x,y) = 0 \) when \( T(x,y) = 0 \). Then \( F = (F(x,y)) \) is a locally finite rank operator of finite propagation with \( \|T - F\| < \epsilon \).

For each fixed \( x \), since \( F \) has finite propagation, there are only finitely many \( y \) such that \( F(x,y) \neq 0 \). Let \( U(x,x) = \left( \begin{array}{cc}
I_{r(x)} & 0 \\
0 & 0
\end{array} \right) \) be a finite-rank projection for some \( r(x) \in \mathbb{N} \) such that \( U(x,x)F(x,y) = F(x,y) \).
for all \( y \) with \( F(x, y) \neq 0 \). Then define a \( U_\lambda = \{ U(x, y) = 0 \text{ if } x \neq y \} \), and \( U(x, x) = \begin{pmatrix} I_{r(x)} & 0 \\ 0 & 0 \end{pmatrix} \) where \( U(x, x) \) is defined ahead.

Then

\[
\|U_\lambda T - T\| \leq \|U_\lambda T - U_\lambda F\| + \|U_\lambda F - F\| + \|F - T\|.
\]

Here \( \|F - T\| \leq \epsilon \), \( U_\lambda F - F = 0 \) and \( \|U_\lambda T - U_\lambda F\| \leq \|U_\lambda\| \|F - T\| < \epsilon \). So \( \|U_\lambda T - T\| \leq 2\epsilon \) and the proof is done. \( \Box \)

**PROPOSITION 10.**

1. \( B^p(A; X) + B^p(B; X) = B^p(X) \) for any decomposition \( A \) and \( B \) of \( X \);
2. \( B^p(A; X) \cap B^p(B; X) = B^p(A \cap B; X) \) if \( A \) and \( B \) are \( w \)-excisive.

Proof. Naturally, \( B^p(A; X) + B^p(B; X) \subset B^p(X) \). For any \( T \in B^p_{\text{alg}}(X) \), assume that \( \text{propagation}(T) = R \). Let \( \chi_A \) be the characteristic function of \( A \). Then \( T \cdot \chi_A \in B^p(A; X) \) and \( T \cdot (1 - \chi_A) \in B^p(B; X) \), and \( T = T\chi_A + T(1 - \chi_A) \in B^p(A; X) + B^p(B; X) \). Hence \( B^p(A; X) + B^p(B; X) = B^p(X) \).

For the second part, \( B^p(A \cap B; X) \subset B^p(A; X) \cap B^p(B; X) \) holds for any decomposition pair \( (A, B) \). By Proposition 9, one can easily see that \( B^p(A; X) \cap B^p(B; X) = B^p(A; X)B^p(B; X) \). For \( T_A \in B^p_{\text{alg}}(A; X) \) and \( T_B \in B^p_{\text{alg}}(B; X) \) with

\[
\text{Supp}(T_A) \subset \text{Pen}(A; R') \times \text{Pen}(A; R');
\]

\[
\text{Supp}(T_B) \subset \text{Pen}(B; R'') \times \text{Pen}(B; R'').
\]

\((A, B)\) is \( w \)-excisive, then there exists \( S > 0 \) such that

\[
\text{Supp}(T_A T_B) \subset \text{Pen}(A \cap B; S) \times \text{Pen}(A \cap B; S).
\]

Hence \( B^p(A; X)B^p(B; X) \subset B^p(A \cap B; X) \). \( \Box \)
Combining these lemmas, we have the following $\ell^p$ coarse Mayer-Vietoris principle.

**PROPOSITION 11.** Let $A$ and $B$ be a $w$-excisive decomposition of $X$, then the following sequence is exact:

\[
\cdots \to K_j(B^p(A \cap B)) \to K_j(B^p(A)) \oplus K_j(B^p(B)) \to K_j(B^p(X)) \to K_{j-1}(B^p(A \cap B)) \to \cdots
\]

4. **Twisted $\ell^p$-Roe algebras and twisted $\ell^p$-localization algebras**

In this section, we shall define the twisted $\ell^p$-Roe algebras and the twisted $\ell^p$-localization algebras for bounded geometry spaces which admit a coarse embedding into a simply connected complete Riemannian manifold of nonpositive sectional curvature. The construction of these twisted $\ell^p$-algebras is similar to those twisted algebras introduced in [29].

Let $M$ be a simply connected complete Riemannian manifold of nonpositive sectional curvature. In the following, we shall assume that the dimension of $M$ is even. If $\dim(M)$ is odd, we can replace $M$ by $M \times \mathbb{R}$. Indeed, the product manifold $M \times \mathbb{R}$ is also a simply connected complete Riemannian manifold with nonpositive sectional curvature. And if $f : \Gamma \to M$ is a coarse embedding, then the induced map $f' : \Gamma \to M \times \mathbb{R}$ defined by $f'(\gamma) = (f(\gamma), 0)$ is also a coarse embedding so that we can replace $f$ by $f'$. Thus, without loss of generality, we assume $\dim M = 2n$ for some integer $n > 0$.

Let $\mathcal{A} = C_0(M, \text{Cliff}_\mathbb{C}(TM))$ be the $C^*$-algebra of continuous functions $a$ on $M$ which have value $a(x) \in \text{Cliff}_\mathbb{C}(T_xM)$ at each point $x \in M$ and vanish at infinity, where $\text{Cliff}_\mathbb{C}(T_xM)$ is the complexified Clifford
algebra \cite{2,18} of the tangent space $T_xM$ at $x \in M$ with respect to the inner product on $T_xM$ given by the Riemannian structure of $M$. Then $\text{Cliff}_C(TM)$ is the Clifford bundle over $M$. Meanwhile, for any $x \in M$, $\text{Cliff}_C(T_xM)$ is also a Hilbert space, so that $\text{Cliff}_C(TM)$ is also a Hilbert space bundle. Let $\mathcal{H} = L^2(M, \text{Cliff}_C(TM))$, the set of all $L^2$ sections of $\text{Cliff}_C(TM)$, which is a Hilbert space. $\mathcal{A}$ acts on $\mathcal{H}$ by pointwise multiplication. For $a \in \mathcal{A}$ and $h \in \mathcal{H}$. Define $a_{\text{max}} = \max\{\|a(x)\| : x \in M\}$. Then $\|a \cdot h\| \leq a_{\text{max}}\|h\|$ and $\mathcal{A} \subset L(\mathcal{H})$. For $n \in \mathbb{N}$, define $\mathcal{H}_{n,p} = \mathcal{H} \oplus_p \cdots \oplus_p \mathcal{H}$, the $\ell^p$-direct sum of $n$ copies of $\mathcal{H}$. The $\ell^p$-norm of $\mathcal{H}_{n,p}$ is defined as

$$\|(f_1, \cdots, f_n)\|_p = \sqrt[p]{\sum_{i=1}^{n} \|f_i\|^p}, \quad \text{for } f_1, \cdots, f_n \in \mathcal{H}.$$ 

Let $M_n(\mathcal{A})$ be the set of $n \times n$ matrices with entries in $\mathcal{A}$. Then elements of $M_n(\mathcal{A})$ act on $\mathcal{H}_{n,p}$ by matrix multiplication. For $a = (a_{i,j})_{i,j \in \{1, \cdots, n\}} \in M_n(\mathcal{A})$ and $h_n \in \mathcal{H}_{n,p}$, $\|a \cdot h_n\| \leq \max_{i,j \in \{1, \cdots, n\}} \{(a_{i,j})_{\text{max}}\} \cdot \|h_n\|$. Hence $M_n(\mathcal{A}) \subset L(\mathcal{H}_{n,p})$. Let $r_{n,n+1} : \mathcal{H}_{n+1,p} \to \mathcal{H}_{n,p}$ be the projection map defined by $r_{n,n+1}(h_1, \cdots, h_n, h_{n+1}) = (h_1, \cdots, h_n)$, for $(h_1, \cdots, h_{n+1}) \in \mathcal{H}_{n,p}$. Then $r_{n,n+1}^*(M_n(\mathcal{A})) \subset M_{n+1}(\mathcal{A})$. This is equivalent to embed $M_n(\mathcal{A})$ into $M_{n+1}(\mathcal{A})$ by placing matrices at the top left corner and inserting 0 at the right column and the bottom line. And $\|r_{n,n+1}^*(M)\| \leq \|M\|$ for all $M \in M_n(\mathcal{A})$. Let $M_{\infty,p}(\mathcal{A})$ be the inductive limit of $\{M_n(\mathcal{A})\}_{n=1}^{\infty}$. Define $\mathcal{H}_{\infty,p}$ to be the $\ell^p$-direct sum of infinite copies of $\mathcal{H}$ with the $\ell^p$-norm

$$\|\{f_i\}_{i=1}^{\infty}\|_p = \sqrt[p]{\sum_{i=1}^{\infty} \|f_i\|^p}, \quad \text{for } \{f_i\}_{i=1}^{\infty} \in \mathcal{H}_{\infty,p}.$$
Then $\mathcal{H}_{\infty,p} \cong \ell^p(\mathbb{N}, \mathcal{H}) \cong \ell^p(\mathbb{N}) \otimes_p \mathcal{H}$ and all $M_n(\mathcal{A})$ can be considered as subalgebras of $\mathcal{L}(\mathcal{H}_{\infty,p})$. Denote by $\mathcal{K}_p \otimes_{alg} \mathcal{A}$ the algebraic tensor product of $\mathcal{K}_p$ and $\mathcal{A}$. Naturally $\mathcal{K}_p \otimes_{alg} \mathcal{A}$ acts on $\mathcal{H}_{\infty,p}$ and $\mathcal{K}_p \otimes_{alg} \mathcal{A} \subset \mathcal{L}(\mathcal{H}_{\infty,p})$. Let $\mathcal{K}_p \otimes_p \mathcal{A} = \mathcal{K}_p \otimes_{alg} \mathcal{A} \mathcal{L}(\mathcal{H}_{\infty,p})$. It follows that $\mathcal{K}_p \otimes_p \mathcal{A} \cong M_\infty(\mathcal{A})$.

Let $\Gamma$ be a discrete metric space with bounded geometry. Let $f : \Gamma \to M$ be a coarse embedding.

For each $d > 0$, we shall extend the map $f$ to the Rips complex $P_d(\Gamma)$ in the following way. Note that $f$ is a coarse map, i.e., there exists $R > 0$ such that for all $\gamma_1, \gamma_2 \in \Gamma$,

$$d(\gamma_1, \gamma_2) \leq d \implies d_M(f(\gamma_1), f(\gamma_2)) \leq R.$$

For any point $x = \sum_{\gamma \in \Gamma} c_\gamma \gamma \in P_d(\Gamma)$, where $c_\gamma \geq 0$ and $\sum_{\gamma \in \Gamma} c_\gamma = 1$, we choose a point $f_x \in M$ such that

$$d(f_x, f(\gamma)) \leq R$$

for all $\gamma \in \Gamma$ with $c_\gamma \neq 0$. The correspondence $x \mapsto f_x$ gives a coarse embedding $P_d(\Gamma) \to M$, also denoted by $f$.

Choose a countable dense subset $\Gamma_d$ of $P_d(\Gamma)$ for each $d > 0$ in such a way that $\Gamma_d \subset \Gamma_{d'}$ when $d < d'$.

Let $B^p_{alg}(P_d(\Gamma), \mathcal{A})$ be the set of all functions

$$T : \Gamma_d \times \Gamma_d \to \mathcal{K}_p \otimes_p \mathcal{A} \subset \mathcal{L}(\ell^p \otimes_p L^2(M, \text{Cliff}_{\mathbb{C}}(TM)))$$

such that

1. there exists $C > 0$ such that $\|T(x, y)\| \leq C$ for all $x, y \in \Gamma_d$;
2. there exists $R > 0$ such that $T(x, y) = 0$ if $d(x, y) > R$;
3. there exists $L > 0$ such that for every $z \in P_d(\Gamma)$, the number of elements in the following set

$$\{(x, y) \in \Gamma_d \times \Gamma_d : d(x, z) \leq 3R, d(y, z) \leq 3R, T(x, y) \neq 0\}$$

is at most $L$. 


is less than $L$.

(4) there exists $r > 0$ such that

$$\text{Supp}(T(x, y)) \subset B(f(x), r)$$

for all $x, y \in \Gamma_d$, where $B(f(x), r) = \{m \in M : d(m, f(x)) < r\}$

and, for all $x, y \in \Gamma_d$, the entry $T(x, y) \in \mathcal{K}_p \otimes_p A$ is a function on $M$ with $T(x, y)(m) \in \mathcal{K}_p \otimes_p \text{Cliff}_C(T_m M)$ for each $m \in M$

so that the support of $T(x, y)$ is defined by

$$\text{Supp}(T(x, y)) := \{m \in M : T(x, y)(m) \neq 0\}.$$ 

For $f \in \ell^p(\Gamma_d, \mathcal{H}_{\infty, p})$, we define

$$Tf(x) = \sum_{y \in \Gamma_d} T(x, y)f(y).$$

Then $T = (T(x, y)) \in \mathcal{L}(\ell^p(\Gamma_d, \mathcal{H}_{\infty, p})).$

**DEFINITION 12.** The twisted $\ell^p$-Roe algebra $B^p(P_d(\Gamma), A)$ is defined to be the operator norm closure of $B^p_{\text{alg}}(P_d(\Gamma), A)$ in $\mathcal{L}(\ell^p(\Gamma_d, \mathcal{H}_{\infty, p})).$

The above definition of the twisted $\ell^p$-Roe algebra is similar to that in [29].

Let $B^p_{L,\text{alg}}(P_d(\Gamma), A)$ be the set of all bounded, uniformly norm-continuous functions

$$g : \mathbb{R}_+ \to B^p_{\text{alg}}(P_d(\Gamma), A)$$

such that

(1) there exists a bounded function $R(t) : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t \to \infty} R(t) = 0$ such that $(g(t))(x, y) = 0$ whenever $d(x, y) > R(t)$;

(2) there exists $L > 0$ such that for every $z \in P_d(\Gamma)$, the number of elements in the following set

$$\{(x, y) \in \Gamma_d \times \Gamma_d : d(x, z) \leq 3R, d(y, z) \leq 3R, g(t)(x, y) \neq 0\}$$
is less than $L$ for every $t \in \mathbb{R}_+$.

(3) there exists $r > 0$ such that $\text{Supp}((g(t))(x, y)) \subset B(f(x), r)$ for all $t \in \mathbb{R}_+$, $x, y \in \Gamma_d$, where $f : P_d(\Gamma) \to M$ is the extension of the coarse embedding $f : \Gamma \to M$ and $B(f(x), r) = \{m \in M : d(m, f(x)) < r\}$.

**DEFINITION 13.** The twisted $\ell^p$-localization algebra $B^p_L(P_d(\Gamma), \mathcal{A})$ is defined to be the norm completion of $B^p_{L,\text{alg}}(P_d(\Gamma), \mathcal{A})$, where $B^p_{L,\text{alg}}(P_d(\Gamma), \mathcal{A})$ is endowed with the norm

$$\|g\|_\infty = \sup_{t \in \mathbb{R}_+} \|g(t)\|_{B^p(P_d(\Gamma), \mathcal{A})}.$$

The above definition of the twisted $\ell^p$-localization Roe algebra is similar to that in [29]. The evaluation homomorphism $e$ from $B^p_L(P_d(\Gamma), \mathcal{A})$ to $B^p(P_d(\Gamma), \mathcal{A})$ defined by $e(g) = g(0)$ induces a homomorphism at $K$-theory level:

$$e_* : \lim_{d \to \infty} K_*(B^p_L(P_d(\Gamma), \mathcal{A})) \to \lim_{d \to \infty} K_*(B^p(P_d(\Gamma), \mathcal{A})).$$

**THEOREM 14.** Let $\Gamma$ be a discrete metric space with bounded geometry which admits a coarse embedding $f : \Gamma \to M$ into a simply connected, complete Riemannian manifold $M$ of non-positive sectional curvature. Then the homomorphism

$$e_* : \lim_{d \to \infty} K_*(B^p_L(P_d(\Gamma), \mathcal{A})) \to \lim_{d \to \infty} K_*(B^p(P_d(\Gamma), \mathcal{A})).$$

is an isomorphism.

The proof of Theorem 14 is similar to the proof of Theorem 6.8 in [29]. To begin with, we need to discuss ideals of the twisted algebras associated to open subsets of the manifold $M$.

**DEFINITION 15.**
(1) The support of an element \( T \) in \( B_{alg}^p(P_d(\Gamma), \mathcal{A}) \) is defined to be
\[
\text{Supp}(T) = \left\{ (x, y, m) \in \Gamma_d \times \Gamma_d \times M : m \in \text{Supp}(T(x, y)) \right\} = \left\{ (x, y, m) \in \Gamma_d \times \Gamma_d \times M : (T(x, y))(m) \neq 0 \right\}.
\]

(2) The support of an element \( g \) in \( B_{L,alg}^p(P_d(\Gamma), \mathcal{A}) \) is defined to be
\[
\bigcup_{t \in \mathbb{R}^+} \text{Supp}(g(t)).
\]

Let \( O \subset M \) be an open subset of \( M \). Define \( B_{alg}^p(P_d(\Gamma), \mathcal{A})_O \) to be the subalgebra of \( B_{alg}^p(P_d(\Gamma), \mathcal{A}) \) consisting of all elements whose supports are contained in \( \Gamma_d \times \Gamma_d \times O \), i.e.,
\[
B_{alg}^p(P_d(\Gamma), \mathcal{A})_O = \{ T \in B_{alg}^p(P_d(\Gamma), \mathcal{A}) : \text{Supp}(T(x, y)) \subset O, \forall x, y \in \Gamma_d \}.
\]

Define \( B^p(P_d(\Gamma), \mathcal{A})_O \) to be the norm closure of \( B_{alg}^p(P_d(\Gamma), \mathcal{A})_O \). Similarly, let
\[
B_{L,alg}^p(P_d(\Gamma), \mathcal{A})_O = \left\{ g \in B_{L,alg}^p(P_d(\Gamma), \mathcal{A}) : \text{Supp}(g) \subset \Gamma_d \times \Gamma_d \times O \right\}
\]
and define \( B^p_L(P_d(\Gamma), \mathcal{A})_O \) to be the norm closure of \( B_{L,alg}^p(P_d(\Gamma), \mathcal{A})_O \) under the norm \( \|g\|_\infty = \sup_{t \in \mathbb{R}^+} \|g(t)\|_{B^p(P_d(\Gamma), \mathcal{A})} \).

Note that \( B^p(P_d(\Gamma), \mathcal{A})_O \) and \( B^p_L(P_d(\Gamma), \mathcal{A})_O \) are closed two-sided ideals of \( B^p(P_d(\Gamma), \mathcal{A}) \) and \( B^p_L(P_d(\Gamma), \mathcal{A}) \), respectively. We also have an evaluation homomorphism \( e : B^p_L(P_d(\Gamma), \mathcal{A})_O \to B^p(P_d(\Gamma), \mathcal{A})_O \) given by \( e(g) = g(0) \).

**Lemma 16.** For any two open subsets \( O_1, O_2 \) of \( M \), we have
\[
B^p(P_d(\Gamma), \mathcal{A})_{O_1} + B^p(P_d(\Gamma), \mathcal{A})_{O_2} = B^p(P_d(\Gamma), \mathcal{A})_{O_1 \cup O_2},
\]
\[
B^p(P_d(\Gamma), \mathcal{A})_{O_1} \cap B^p(P_d(\Gamma), \mathcal{A})_{O_2} = B^p(P_d(\Gamma), \mathcal{A})_{O_1 \cap O_2},
\]
\[
B^p_{L}(P_d(\Gamma), \mathcal{A})_{O_1} + B^p_{L}(P_d(\Gamma), \mathcal{A})_{O_2} = B^p_{L}(P_d(\Gamma), \mathcal{A})_{O_1 \cup O_2},
\]
\[
B^p_{L}(P_d(\Gamma), \mathcal{A})_{O_1} \cap B^p_{L}(P_d(\Gamma), \mathcal{A})_{O_2} = B^p_{L}(P_d(\Gamma), \mathcal{A})_{O_1 \cap O_2}.
\]
Consequently, we have the following commuting diagram connecting two Mayer-Vietorís sequences at $K$-Theory level:

\[
\begin{array}{cccccc}
AL_0 & \rightarrow & BL_0 & \rightarrow & CL_0 & \\
\downarrow & & \downarrow & & \downarrow & \\
CL_1 & \rightarrow & BL_1 & \rightarrow & AL_1 & \\
\downarrow & & \downarrow & & \downarrow & \\
A_0 & \rightarrow & B_0 & \rightarrow & C_0 & \\
\downarrow & & \downarrow & & \downarrow & \\
C_1 & \rightarrow & B_1 & \rightarrow & A_1 &
\end{array}
\]

where, for $* = 0, 1$,

\[
\begin{align*}
AL_* &= K_*(B^p_L(P_d(\Gamma), \mathcal{A})_{O_1 \cap O_2}), \\
CL_* &= K_*(B^p_L(P_d(\Gamma), \mathcal{A})_{O_1 \cup O_2}), \\
A_* &= K_*(B^p(P_d(\Gamma), \mathcal{A})_{O_1 \cap O_2}), \\
C_* &= K_*(B^p(P_d(\Gamma), \mathcal{A})_{O_1 \cup O_2}), \\
B_* &= K_*(B^p(P_d(\Gamma), \mathcal{A})_{O_1}) \bigoplus K_*(B^p(P_d(\Gamma), \mathcal{A})_{O_2}).
\end{align*}
\]

Proof. We shall prove the first equality. Other equalities can be proved similarly. Then the two Mayer-Vietorís exact sequences follow from Lemma 2.4 of [14].

To prove the first equality, it suffices to show that

\[
B^p_{alg}(P_d(\Gamma), \mathcal{A})_{O_1 \cup O_2} \subseteq B^p_{alg}(P_d(\Gamma), \mathcal{A})_{O_1} + B^p_{alg}(P_d(\Gamma), \mathcal{A})_{O_2}.
\]

Now suppose $T \in B^p_{alg}(P_d(\Gamma), \mathcal{A})_{O_1 \cup O_2}$. Take a continuous partition of unity $\{\varphi_1, \varphi_2\}$ on $O_1 \cup O_2$ subordinate to the open over $\{O_1, O_2\}$ of $O_1 \cup O_2$. Define two functions

\[
T_1, T_2 : \Gamma_d \times \Gamma_d \longrightarrow \mathcal{K}_p \otimes_p \mathcal{A}
\]

by

\[
T_1(x, y)(m) = \varphi_1(m)(T(x, y)(m)),
\]

\[
T_2(x, y)(m) = \varphi_2(m)(T(x, y)(m)).
\]
\[ T_2(x, y)(m) = \varphi_2(m) \left( T(x, y)(m) \right) \]

for \( x, y \in \Gamma_d \) and \( m \in M \).

Then \( T_1 \in B^p_{\text{alg}}(P_d(\Gamma), \mathcal{A})_{O_1}, T_2 \in B^p_{\text{alg}}(P_d(\Gamma), \mathcal{A})_{O_2} \), and

\[ T = T_1 + T_2 \in B^p_{\text{alg}}(P_d(\Gamma), \mathcal{A})_{O_1} + B^p_{\text{alg}}(P_d(\Gamma), \mathcal{A})_{O_2} \]

as desired. \( \square \)

It would be convenient to introduce the following notion associated with the coarse embedding \( f : \Gamma \to M \).

**DEFINITION 17.** Let \( r > 0 \). A family of open subsets \( \{ O_i \}_{i \in J} \) of \( M \) is said to be \((\Gamma, r)\)-separate if

1. \( O_i \cap O_j = \emptyset \) if \( i \neq j \);
2. there exists \( \gamma_i \in \Gamma \) such that \( O_i \subseteq B(f(\gamma_i), r) \subset M \) for each \( i \in J \).

**LEMMA 18.** If \( \{ O_i \}_{i \in J} \) is a family of \((\Gamma, r)\)-separate open subsets of \( M \), then

\[ e_* : \lim_{d \to \infty} K_*(B^p_E(P_d(\Gamma), \mathcal{A})_{\sqcup_{i \in J} O_i}) \to \lim_{d \to \infty} K_*(B^p(P_d(\Gamma), \mathcal{A})_{\sqcup_{i \in J} O_i}) \]

is an isomorphism, where \( \sqcup_{i \in J} O_i \) is the (disjoint) union of \( \{ O_i \}_{i \in J} \).

We will prove Lemma 18 in the next section. Granting Lemma 18 for the moment, we are able to prove Theorem 14.

**Proof of Theorem 14.** (29). For any \( r > 0 \), we define \( O_r \subset M \) by

\[ O_r = \bigcup_{\gamma \in \Gamma} B(f(\gamma), r), \]

where \( f : \Gamma \to M \) is the coarse embedding and \( B(f(\gamma), r) = \{ p \in M : d(p, f(\gamma)) < r \} \).
For any \( d > 0 \), if \( r < r' \) then \( B^p(P_d(\Gamma), \mathcal{A})_{O_r} \subseteq B^p(P_{d'}(\Gamma), \mathcal{A})_{O_{r'}} \) and \( B^p_L(P_d(\Gamma), \mathcal{A})_{O_r} \subseteq B^p_L(P_{d'}(\Gamma), \mathcal{A})_{O_{r'}} \). By definition, we have

\[
B^p(P_d(\Gamma), \mathcal{A}) = \lim_{r \to \infty} B^p(P_d(\Gamma), \mathcal{A})_{O_r},
\]

\[
B^p_L(P_d(\Gamma), \mathcal{A}) = \lim_{r \to \infty} B^p_L(P_d(\Gamma), \mathcal{A})_{O_r}.
\]

On the other hand, for any \( r > 0 \), if \( d < d' \) then \( \Gamma_d \subseteq \Gamma_{d'} \) in \( P_d(\Gamma) \subseteq P_{d'}(\Gamma) \) so that we have natural inclusions \( B^p(P_d(\Gamma), \mathcal{A})_{O_r} \subseteq B^p(P_{d'}(\Gamma), \mathcal{A})_{O_{r'}} \) and \( B^p_L(P_d(\Gamma), \mathcal{A})_{O_r} \subseteq B^p_L(P_{d'}(\Gamma), \mathcal{A})_{O_{r'}} \). These inclusions induce the following commuting diagram

\[
\begin{array}{ccc}
K_*(B^p_L(P_d(\Gamma), \mathcal{A})_{O_r}) & \xrightarrow{\epsilon_*} & K_*(B^p(P_d(\Gamma), \mathcal{A})_{O_r}) \\
\downarrow & & \downarrow \\
K_*(B^p_L(P_{d'}(\Gamma), \mathcal{A})_{O_r}) & \xrightarrow{\epsilon_*} & K_*(B^p(P_{d'}(\Gamma), \mathcal{A})_{O_r}) \\
\downarrow & & \downarrow \\
K_*(B^p_L(P_{d'}(\Gamma), \mathcal{A})_{O_{r'}}) & \xrightarrow{\epsilon_*} & K_*(B^p(P_{d'}(\Gamma), \mathcal{A})_{O_{r'}})
\end{array}
\]

which allows us to change the order of limits from \( \lim_{d \to \infty} \lim_{r \to \infty} \) to \( \lim_{d \to \infty} \lim_{r \to \infty} \) in the second piece of the following commuting diagram

\[
\begin{array}{ccc}
\lim_{d \to \infty} K_*(B^p_L(P_d(\Gamma), \mathcal{A})) & \xrightarrow{\epsilon_*} & \lim_{d \to \infty} K_*(B^p(P_d(\Gamma), \mathcal{A})) \\
\downarrow \cong \downarrow & & \downarrow \cong \\
\lim_{d \to \infty} \lim_{r \to \infty} K_*(B^p_L(P_d(\Gamma), \mathcal{A})_{O_r}) & \xrightarrow{\epsilon_*} & \lim_{d \to \infty} \lim_{r \to \infty} K_*(B^p(P_d(\Gamma), \mathcal{A})_{O_r}) \\
\downarrow \cong \downarrow & & \downarrow \cong \\
\lim_{r \to \infty} \lim_{d \to \infty} K_*(B^p_L(P_d(\Gamma), \mathcal{A})_{O_r}) & \xrightarrow{\epsilon_*} & \lim_{r \to \infty} \lim_{d \to \infty} K_*(B^p(P_d(\Gamma), \mathcal{A})_{O_r})
\end{array}
\]

So, to prove Theorem 14 it suffices to show that, for any \( r > 0 \),

\[
\epsilon_* : \lim_{d \to \infty} K_*(B^p_L(P_d(\Gamma), \mathcal{A})_{O_r}) \to \lim_{d \to \infty} K_*(B^p(P_d(\Gamma), \mathcal{A})_{O_r})
\]

is an isomorphism.
Let $r > 0$. Since $\Gamma$ has bounded geometry and $f : \Gamma \to M$ is a coarse embedding, there exist finitely many mutually disjoint subsets of $\Gamma$, say $\Gamma_k := \{\gamma_i : i \in J_k\}$ with some index set $J_k$ for $k = 1, 2, \ldots, k_0$, such that $\Gamma = \bigcup_{k=1}^{k_0} \Gamma_k$ and, for each $k$, $d(f(\gamma_i), f(\gamma_j)) > 2r$ for distinct elements $\gamma_i, \gamma_j$ in $\Gamma_k$.

For each $k = 1, 2, \ldots, k_0$, let

$$O_{r,k} = \bigcup_{i \in J_k} B(f(\gamma_i), r).$$

Then $O_r = \bigcup_{k=1}^{k_0} O_{r,k}$ and each $O_{r,k}$, or an intersection of several $O_{r,k}$, is the union of a family of $(\Gamma,r)$-separate (Definition 17) open subsets of $M$.

Now Theorem 14 follows from Lemma 18 together with a Mayer-Vietoris sequence argument by using Lemma 19. \hfill \Box

5. Strong Lipschitz homotopy invariance

In this section, we shall present Yu’s arguments about strong Lipschitz homotopy invariance for $K$-theory of the twisted localization algebras [29], and prove Lemma 18 of the previous section.

Let $f : \Gamma \to M$ be a coarse embedding of a bounded geometry discrete metric space $\Gamma$ into a simply connected complete Riemannian manifold $M$ of nonpositive sectional curvature, and let $r > 0$. Let \{\{O_i\}_{i \in J} be a family of $(\Gamma, r)$-separate open subsets of $M$, i.e.,

(1) $O_i \cap O_j = \emptyset$ if $i \neq j$;

(2) there exists $\gamma_i \in \Gamma$ such that $O_i \subseteq B(f(\gamma_i), r) \subset M$ for each $i \in J$.

For $d > 0$, let $X_i, i \in J$, be a family of closed subsets of $P_d(\Gamma)$ such that $\gamma_i \in X_i$ for every $i \in J$ and \{\{X_i\}_{i \in J} is uniformly bounded in the sense
that there exists $r_0 > 0$ such that $\text{diameter}(X_i) \leq r_0$ for each $i \in J$. In particular, we will consider the following three cases of $\{X_i\}_{i \in J}$:

1. $X_i = B_{P_d(\Gamma)}(\gamma_i, R) := \{x \in P_d(\Gamma) : d(x, \gamma_i) \leq R\}$, for some common $R > 0$ for all $i \in J$;
2. $X_i = \Delta_i$, a simplex in $P_d(\Gamma)$ with $\gamma_i \in \Delta_i$ for each $i \in J$;
3. $X_i = \{\gamma_i\}$ for each $i \in J$.

For each $i \in J$, let $A_{O_i}$ be the subalgebra of $A = C_0(M, \text{Cliff}_C(TM))$ generated by those functions whose supports are contained in $O_i$. We define

$$A(X_i : i \in J) = \prod_{i \in J} B^p(X_i) \otimes A_{O_i}$$

$$= \left\{ \bigoplus_{i \in J} T_i \bigg| T_i \in B^p(X_i) \otimes A_{O_i}, \sup_{i \in J} \|T_i\| < \infty \right\}$$

Similarly we define $A_L(X_i : i \in J)$ to be the subalgebra of

$$\left\{ \bigoplus_{i \in J} b_i \bigg| b_i \in B^p_L(X_i) \otimes A_{O_i}, \sup_{i \in J} \|b_i\| < \infty \right\}$$

generated by elements $\bigoplus_{i \in J} b_i$ such that

1. the function

$$\bigoplus_{i \in J} b_i : \mathbb{R}_+ \rightarrow \prod_{i \in J} B^p(X_i) \otimes A_{O_i}$$

is uniformly norm-continuous in $t \in \mathbb{R}_+$.

2. there exists a bounded function $c(t)$ on $\mathbb{R}_+$ with $\lim_{t \to \infty} c(t) = 0$ such that $(b_i(t))(x, y) = 0$ whenever $d(x, y) > c(t)$ for all $i \in J$, $x, y \in X_i$ and $t \in \mathbb{R}_+$.

For each natural number $s > 0$, let $\Delta_i(s)$ be the simplex with vertices $\{\gamma \in \Gamma : d(\gamma, \gamma_i) \leq s\}$ in $P_d(\Gamma)$ for $d > s$.

**Lemma 19.** Let $O = \sqcup_{i \in J} O_i$ be the (disjoint) union of a family of $(\Gamma, r)$-separate open subsets $\{O_i\}_{i \in J}$ of $M$ as above. Then
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(1) $B^p(P_d(\Gamma), A)_O \cong \lim_{R \to \infty} A(\{x \in P_d(\Gamma) : d(x, \gamma_i) \leq R\} : i \in J)$;

(2) $B^p_L(P_d(\Gamma), A)_O \cong \lim_{R \to \infty} A_L(\{x \in P_d(\Gamma) : d(x, \gamma_i) \leq R\} : i \in J)$;

(3) $\lim_{d \to \infty} B^p(P_d(\Gamma), A)_O \cong \lim_{s \to \infty} A(\Delta_i(s) : i \in J)$;

(4) $\lim_{d \to \infty} B^p_L(P_d(\Gamma), A)_O \cong \lim_{s \to \infty} A_L(\Delta_i(s) : i \in J)$.

Proof. (29) Let $A_O$ be the subalgebra of $A = C_0(M, \text{Cliff}_C(TM))$ generated by elements whose supports are contained in $O$. Let $H_O = L^2(O, \text{Cliff}_C(TM))$ and $H_{O,\infty,p}$ be the $\ell^p$-direct sum of infinite copies of $H_O$ with the $\ell^p$-norm

$$\|(f_1, \cdots, f_n)\|_p = \sqrt[p]{\sum_{i=1}^{n} \|f_i\|^p}, \quad \text{for } f_1, \cdots, f_n \in H_O.$$ 

$K_p \otimes_p A_O$ acts on $H_{O,\infty,p}$ and $B^p(P_d(\Gamma), A)_O$ acts on $\ell^p(\Gamma_d, H_{O,\infty,p})$. We have a decomposition

$$\ell^p(\Gamma_d, H_{O,\infty,p}) = \bigoplus_{i \in J} \ell^p(\Gamma_d, H_{O_i,\infty,p}).$$

Each $T \in B^p_{\text{alg}}(P_d(\Gamma), A)_O$ has a corresponding decomposition

$$T = \bigoplus_{i \in J} T_i$$

such that there exists $R > 0$ for which each $T_i$ is supported on

$$\{(x, y, p) : p \in O_i, x, y \in \Gamma_d, d(x, \gamma_i) \leq R, d(y, \gamma_i) \leq R\}.$$ 

On the other hand, the Banach algebra $B^p(\{x \in P_d(\Gamma) : d(x, \gamma_i) \leq R\}) \otimes_p A_{O_i}$ acts on

$$\ell^p(\{x \in \Gamma_d : d(x, \gamma_i) \leq R\}, H_{O_i,\infty,p}),$$

so that on $\ell^p(\Gamma_d, H_{O,\infty,p})$, for each $R > 0$, the algebra $A(\{x \in P_d(\Gamma) : d(x, \gamma_i) \leq R\} : i \in J)$ can be represented as a subalgebra of $B^p(P_d(\Gamma), A)_O$. 
In this way, the decomposition $T = \bigoplus_{i \in J} T_i$ induces a Banach algebra isomorphism

$$B^p(P_d(\Gamma), A) \cong \lim_{R \to \infty} A\{x \in P_d(\Gamma) : d(x, \gamma_i) \leq R \} : i \in J$$

as desired in (1). Then (2),(3),(4) follows easily from (1). \qed

Now we turn to recall the notion of strong Lipschitz homotopy [27, 28, 29].

Let $\{Y_i\}_{i \in J}$ and $\{X_i\}_{i \in J}$ be two families of uniformly bounded closed subspaces of $P_d(\Gamma)$ for some $d > 0$ with $\gamma_i \in X_i$, $\gamma_i \in Y_i$ for every $i \in J$. A map $g : \bigsqcup_{i \in J} X_i \to \bigsqcup_{i \in J} Y_i$ is said to be Lipschitz if

1. $g(X_i) \subseteq Y_i$ for each $i \in J$;
2. there exists a constant $c$, independent of $i \in J$, such that

$$d(g(x), g(y)) \leq c \, d(x, y)$$

for all $x, y \in X_i$, $i \in J$.

Let $g_1, g_2$ be two Lipschitz maps from $\bigsqcup_{i \in J} X_i$ to $\bigsqcup_{i \in J} Y_i$. We say $g_1$ is strongly Lipschitz homotopy equivalent to $g_2$ if there exists a continuous map

$$F : [0, 1] \times (\bigsqcup_{i \in J} X_i) \to \bigsqcup_{i \in J} Y_i$$

such that

1. $F(0, x) = g_1(x)$, $F(1, x) = g_2(x)$ for all $x \in \bigsqcup_{i \in J} X_i$;
2. there exists a constant $c$ for which $d(F(t, x), F(t, y)) \leq c \, d(x, y)$ for all $x, y \in X_i$, $t \in [0, 1]$, where $i$ is any element in $J$;
3. $F$ is equicontinuous in $t$, i.e., for any $\varepsilon > 0$ there exists $\delta > 0$ such that $d(F(t_1, x), F(t_2, x)) < \varepsilon$ for all $x \in \bigsqcup_{i \in J} X_i$ if $|t_1 - t_2| < \delta$.

We say $\{X_i\}_{i \in J}$ is strongly Lipschitz homotopy equivalent to $\{Y_i\}_{i \in J}$ if there exist Lipschitz maps $g_1 : \bigsqcup_{i \in J} X_i \to \bigsqcup_{i \in J} Y_i$ and $g_2 : \bigsqcup_{i \in J} Y_i \to$
Define $A_{L,0}(X_i : i \in J)$ to be the subalgebra of $A_L(X_i : i \in J)$ consisting of elements $\oplus_{i \in J} b_i(t)$ satisfying $b_i(0) = 0$ for all $i \in J$.

**Lemma 20** ([29]). If $\{X_i\}_{i \in J}$ is strongly Lipschitz homotopy equivalent to $\{Y_i\}_{i \in J}$ then $K_*(A_{L,0}(X_i : i \in J))$ is isomorphic to $K_*(A_{L,0}(Y_i : i \in J))$.

Let $e$ be the evaluation homomorphism from $A_L(X_i : i \in J)$ to $A(X_i : i \in J)$ given by $\oplus_{i \in J} g_i(t) \mapsto \oplus_{i \in J} g_i(0)$.

**Lemma 21** ([29]). Let $\{\gamma_i\}_{i \in J}$ be as above, i.e., $O_i \subseteq B(f(\gamma_i), r) \subset M$ for each $i$. If $\{\Delta_i\}_{i \in J}$ is a family of simplices in $P_d(\Gamma)$ for some $d > 0$ such that $\gamma_i \in \Delta_i$ for all $i \in J$, then

$$e_*(K_*(A_L(\Delta_i : i \in J)) \rightarrow K_*(A(\Delta_i : i \in J))$$

is an isomorphism.

**Proof.** ([29]). Note that $\{\Delta_i\}_{i \in J}$ is strongly Lipschitz homotopy equivalent to $\{\gamma_i\}_{i \in J}$. By an argument of Eilenberg swindle, we have $K_*(A_{L,0}(\{\gamma_i\} : i \in J)) = 0$. Consequently, Lemma 21 follows from Lemma 20 and the six term exact sequence of Banach algebra $K$-theory.

We are now ready to give a proof to Lemma 18 of the previous section.

**Proof of Lemma 18**. By Lemma 19 we have the following commuting diagram

$$\lim_{d \to \infty} B^p_L(P_d(\Gamma), A)_{\bigcup_{i \in J} O_i} \xrightarrow{e} \lim_{d \to \infty} B^p(P_d(\Gamma), A)_{\bigcup_{i \in J} O_i}
\xrightarrow{\cong} \lim_{s \to \infty} A_L(\Delta_i(s)_i : i \in J) \xrightarrow{e} \lim_{s \to \infty} A(\Delta_i(s)_i : i \in J)$$
which induces the following commuting diagram at $K$-theory level

\[
\begin{array}{ccc}
\lim_{d \to \infty} K_* \left( B^p_L(P_d(\Gamma), \mathcal{A}) \cup_{i \in J} \mathcal{O}_i \right) & \overset{e^*}{\longrightarrow} & \lim_{d \to \infty} K_* \left( B^p(P_d(\Gamma), \mathcal{A}) \cup_{i \in J} \mathcal{O}_i \right) \\
\cong & & \cong \\
\lim_{s \to \infty} K_* \left( A_L(\Delta_i(s) : i \in J) \right) & \overset{e^*}{\longrightarrow} & \lim_{s \to \infty} K_* \left( A(\Delta_i(s) : i \in J) \right)
\end{array}
\]

Now Lemma \[18\] follows from Lemma \[21\].

6. Almost flat Bott elements and Bott maps

In this section, we shall construct uniformly almost flat Bott generators for a simply connected complete Riemannian manifold with nonpositive sectional curvature, and define a Bott map from the $K$-theory of the Roe algebra to the $K$-theory of the twisted Roe algebra and another Bott map between the $K$-theory of corresponding localization algebras. We show that the Bott map from the $K$-theory of the $\ell^p$-localization algebra to the $K$-theory of the twisted $\ell^p$-localization algebra is an isomorphism (Theorem \[25\]).

Let $M$ be a simply connected complete Riemannian manifold with nonpositive sectional curvature. As remarked at the beginning of Section 4, without loss of generality, we assume in the following $\dim(M) = 2n$ for some integer $n > 0$.

Recall that $\mathcal{A} := C_0(M, \text{Cliff}_C(TM))$ and $\dim M = 2n$, the exponential map

\[\exp_x : T_x M \cong \mathbb{R}^{2n} \to M\]

at any point $x \in M$ induces an isomorphism

\[C_0(M, \text{Cliff}_C(TM)) \cong C_0(\mathbb{R}^{2n}) \otimes \mathcal{M}_2^n(\mathbb{C}).\]

Similarly, we define $\mathcal{B} := C_b(M, \text{Cliff}_C(TM))$ to be the Banach algebra of all bounded functions $a$ on $M$ with $a(x) \in \text{Cliff}(T_x M)$ at all $x \in M$.
Let $x \in M$. For any $z \in M$, let $\sigma : [0, 1] \to M$ be the unique geodesic such that
\[
\sigma(0) = x, \quad \sigma(1) = z.
\]
Let $v_x(z) := \frac{\sigma'(1)}{\|\sigma'(1)\|} \in T_zM$. For any $c > 0$, take a continuous function $\phi_{x,c} : M \to [0, 1]$ satisfying
\[
\phi_{x,c}(z) = \begin{cases} 
0, & \text{if } d(x, z) \leq \frac{c}{2} \\
1, & \text{if } d(x, z) \geq c.
\end{cases}
\]
For any $z \in M$, let
\[
f_{x,c}(z) := \phi_{x,c}(z) \cdot v_x(z) \in T_zM.
\]
Then $f_{x,c} \in B$. The following result describes certain “uniform almost flatness” of the functions $f_{x,c}$ ($x \in M$, $c > 0$).

**Lemma 22.** For any $R > 0$ and $\varepsilon > 0$, there exist a constant $c > 0$ and a family of continuous function $\{\phi_{x,c}\}_{x \in M}$ satisfying the above condition (1) such that, if $d(x, y) < R$, then
\[
\sup_{z \in M} \|f_{x,c}(z) - f_{y,c}(z)\|_{T_zM} < \varepsilon.
\]

**Proof.** Let $c = \frac{2R}{\varepsilon}$. For any $x \in M$, define $\phi_{x,c} : M \to [0, 1]$ by
\[
\phi_{x,c}(z) = \begin{cases} 
0, & \text{if } d(x, z) \leq \frac{R}{\varepsilon} \\
\frac{\varepsilon}{R} d(x, z) - 1, & \text{if } \frac{R}{\varepsilon} \leq d(x, z) \leq \frac{2R}{\varepsilon} \\
1, & \text{if } d(x, z) \geq \frac{2R}{\varepsilon}.
\end{cases}
\]
Let $x, y \in M$ such that $d(x, y) < R$. Then we have several cases for the position of $z \in M$ with respect to $x, y$.

Consider the case where $d(x, z) > c = \frac{2R}{\varepsilon}$ and $d(y, z) > c = \frac{2R}{\varepsilon}$. Since $\phi_{x,c}(z) = \phi_{y,c}(z) = 1$, we have
\[
f_{x,c}(z) - f_{y,c}(z) = v_x(z) - v_y(z).
\]
Without loss of generality, assume $d(x, z) \leq d(y, z)$. Then there exists a unique point $y'$ on the unique geodesic connecting $y$ and $z$ such
that \( d(y', z) = d(x, z) \). Then \( d(y', y) < R \) since \( d(x, y) < R \), so that \( d(x, y') < 2R \).

Let \( \exp_z^{-1} : M \to T_z M \) denote the inverse of the exponential map
\[
\exp_z : T_z M \to M
\]
at \( z \in M \). Then we have
\[
(\alpha) \| \exp_z^{-1}(x) \| = d(x, z) = d(y', z) = \| \exp_z^{-1}(y') \| > c = \frac{2R}{\varepsilon};
\]
\[
(\beta) \| \exp_z^{-1}(x) - \exp_z^{-1}(y') \| \leq d(x, y') < 2R, \text{ since } M \text{ has nonpositive sectional curvature};
\]
\[
(\gamma) v_x(z) = -\frac{\exp_z^{-1}(x)}{\| \exp_z^{-1}(x) \|} \text{ and } v_y(z) = -\frac{\exp_z^{-1}(y')}{\| \exp_z^{-1}(y') \|}.
\]
Hence, for any \( z \in M \), we have
\[
\| f_{x,c}(z) - f_{y,c}(z) \| = \| v_x(z) - v_y(z) \| < 2R/(2R/\varepsilon) = \varepsilon
\]
whenever \( d(x, y) < R \). Similarly, we can check the inequality in other cases where \( z \in M \) satisfies either \( d(x, z) \leq c \) or \( d(y, z) \leq c \).

Now let’s consider the short exact sequence
\[
0 \to A \to B \overset{\pi}{\to} B/A \to 0,
\]
where \( A = C_0(M, \text{Cliff}_\mathbb{C}(TM)) \) and \( B = C_b(M, \text{Cliff}_\mathbb{C}(TM)) \). For any \( f_{x,c} (x \in M, c > 0) \) constructed above, it is easy to see that \([f_{x,c}] := \pi(f_{x,c})\) is invertible in \( B/A \) with its inverse \([\tilde{f}_{x,c}]\). Thus \([f_{x,c}]\) defines an element in \( K_1(B/A) \). With the help of the index map
\[
\partial : K_1(B/A) \to K_0(A),
\]
we obtain an element \( \partial([f_{x,c}]) \) in
\[
K_0(A) = K_0 \left( C_0(M, \text{Cliff}_\mathbb{C}(TM)) \right) \cong K_0 \left( C_0(\mathbb{R}^{2n}) \otimes \mathcal{M}_{2n}(\mathbb{C}) \right) \cong \mathbb{Z}.
\]
It follows from the construction of \( f_{x,c} \) that, for every \( x \in M \) and \( c > 0 \), \( \partial([f_{x,c}]) \) is just the Bott generator of \( K_0(A) \).
The element \( \partial([f_{x,c}]) \) can be expressed explicitly as follows. Let

\[
W_{x,c} = \begin{pmatrix} 1 & f_{x,c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ f_{x,c} & 1 \end{pmatrix} \begin{pmatrix} 1 & f_{x,c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
\]

\[
b_{x,c} = W_{x,c} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W_{x,c}^{-1},
\]

\[
b_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Then both \( b_{x,c} \) and \( b_0 \) are idempotents in \( \mathcal{M}_2(A^+) \), where \( A^+ \) is the algebra jointing a unit to \( A \). It is easy to check that

\[
b_{x,c} - b_0 \in C_c(M, \text{Cliff}_C(TM)) \otimes \mathcal{M}_2(C),
\]

the algebra of \( 2 \times 2 \) matrices of compactly supported continuous functions, with

\[
\text{Supp}(b_{x,c} - b_0) \subset B_M(x, c) := \{ z \in M : d(x, z) \leq c \},
\]

where for a matrix \( a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \) of functions on \( M \) we define the support of \( a \) by

\[
\text{Supp}(a) = \bigcup_{i,j=1}^2 \text{Supp}(a_{i,j}).
\]

Now we have the explicit expression

\[
\partial([f_{x,c}]) = [b_{x,c}] - [b_0] \in K_0(A).
\]

**Lemma 23** (Uniform almost flatness of the Bott generators). The family of idempotents \( \{b_{x,c}\}_{x \in M, c > 0} \) in \( \mathcal{M}_2(A^+) = C_0(M, \text{Cliff}_C(TM))^+ \otimes \mathcal{M}_2(C) \) constructed above are uniformly almost flat in the following sense:

for any \( R > 0 \) and \( \varepsilon > 0 \), there exist \( c > 0 \) and a family of continuous functions \( \{\phi_{x,c} : M \to [0, 1]\}_{x \in M} \) such that, whenever \( d(x, y) < R \), we have

\[
\sup_{z \in M} \|b_{x,c}(z) - b_{y,c}(z)\|_{\text{Cliff}_C(TM) \otimes \mathcal{M}_2(C)} < \varepsilon,
\]
where \( b_{x,c} \) is defined via \( W_{x,c} \) and \( f_{x,c} = \phi_{x,c}v_x \) as above, and \( \text{Cliff}_C(T_z M) \) is the complexified Clifford algebra of the tangent space \( T_z M \).

**Proof.** Straightforward from Lemma 22.

It would be convenient to introduce the following notion:

**DEFINITION 24.** For \( R > 0, \varepsilon > 0, c > 0 \), a family of idempotents \( \{b_x\}_{x \in M} \) in \( \mathcal{M}_2(A^+) = C_0(M, \text{Cliff}_C(TM))^+ \otimes \mathcal{M}_2(\mathbb{C}) \) is said to be \((R, \varepsilon; c)\)-flat if

1. for any \( x, y \in M \) with \( d(x, y) < R \) we have
   \[
   \sup_{z \in M} \|b_x(z) - b_y(z)\|_{\text{Cliff}_C(T_z M) \otimes \mathcal{M}_2(\mathbb{C})} < \varepsilon.
   \]
2. \( b_x - b_0 \in C_c(M, \text{Cliff}_C(TM)) \otimes \mathcal{M}_2(\mathbb{C}) \) and
   \[
   \text{Supp}(b_x - b_0) \subset B_M(x, c) := \{z \in M : d(x, z) \leq c\}.
   \]

**Construction of the Bott map \( \beta_* \):**

Now we shall use the above almost flat Bott generators for
\[
K_0(A) = K_0\left(C_0(M, \text{Cliff}_C(TM))\right)
\]
to construct a “Bott map”
\[
\beta_* : K_*(B^p(P_d(\Gamma))) \to K_*(B^p(P_d(\Gamma), A)).
\]

To begin with, we give a representation of \( B^p(P_d(\Gamma)) \) on \( \ell^p(\Gamma_d, \ell^p) \), where \( \Gamma_d \) is the countable dense subset of \( P_d(\Gamma) \) and \( H_0 \) is the Hilbert space as in the definition of \( B^p(P_d(\Gamma), A) \).

Let \( B^p_{alg}(P_d(\Gamma)) \) be the algebra of functions
\[
Q : \Gamma_d \times \Gamma_d \to \mathcal{K}_p
\]
such that

1. there exists \( C > 0 \) such that \( \|Q(x, y)\| \leq C \) for all \( x, y \in \Gamma_d \);
2. there exists \( R > 0 \) such that \( Q(x, y) = 0 \) whenever \( d(x, y) > R \);
(3) there exists $L > 0$ such that for every $z \in P_d(\Gamma)$, the number of elements in the following set

$$\{ (x, y) \in \Gamma_d \times \Gamma_d : d(x, z) \leq 3R, d(y, z) \leq 3R, Q(x, y) \neq 0 \}$$

is less than $L$.

The product structure on $B_{alg}^p(P_d(\Gamma))$ is defined by

$$(Q_1 Q_2)(x, y) = \sum_{z \in P_d} Q_1(x, z) Q_2(z, y).$$

The algebra $B_{alg}^p(P_d(\Gamma))$ acts on $\ell^p(P_d(\Gamma))$. The operator norm completion of $B_{alg}^p(P_d(\Gamma))$ with respect to this action is isomorphic to $B^p(P_d(\Gamma))$ when $\Gamma$ has bounded geometry.

Note that $B^p(P_d(\Gamma))$ is stable in the sense that $B^p(P_d(\Gamma)) \cong B^p(P_d(\Gamma)) \otimes \mathcal{M}_k(\mathbb{C})$ for all natural number $k$. Any element in $K_0(B^p(P_d(\Gamma)))$ can be expressed as the difference of the $K_0$-classes of two idempotents in $B^p(P_d(\Gamma))$. To define the Bott map $\beta_* : K_0(B^p(P_d(\Gamma))) \to K_0(B^p(P_d(\Gamma), \mathcal{A}))$, we need to specify the value $\beta_*([P])$ in $K_0(B^p(P_d(\Gamma), \mathcal{A}))$ for any idempotent $P \in B^p(P_d(\Gamma))$.

Now let $P \in B^p(P_d(\Gamma)) \subseteq \mathcal{B}(\ell^p(\Gamma_d), \ell^p)$ be an idempotent. Denote $\|P\| = N$. For any $0 < \varepsilon_1 < 1/100$, take an element $Q \in B_{alg}^p(P_d(\Gamma))$ such that

$$\|P - Q\| < \frac{\varepsilon_1}{2N + 2}.$$ 

Then $\|Q\| < \|P - Q\| + \|P\| < N + 1$, hence

$$\|Q - Q^2\| \leq \|Q - P\| + \|P\| \|P - Q\| + \|P - Q\| \|Q\| \leq \varepsilon_1$$

and there is $R_{\varepsilon_1} > 0$ such that $Q(x, y) = 0$ whenever $d(x, y) > R_{\varepsilon_1}$. For any $\varepsilon_2 > 0$, take by Lemma 23 a family of $(R_{\varepsilon_1}, \varepsilon_2; c)$-flat idempotents $\{b_x\}_{x \in M}$ in $\mathcal{M}_2(\mathcal{A}^+)$ for some $c > 0$. Define

$$\tilde{Q}, \; \tilde{Q}_0 : \Gamma_d \times \Gamma_d \to K_{\mathbb{P}} \otimes_K \mathcal{A}^+ \otimes \mathcal{M}_2(\mathbb{C})$$
by
\[ \tilde{Q}(x, y) = Q(x, y) \otimes b_x \]
and
\[ \tilde{Q}_0(x, y) = Q(x, y) \otimes b_0, \]
respectively, for all \((x, y) \in \Gamma_d \times \Gamma_d\), where \(b_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\). Then
\[ \tilde{Q}, \tilde{Q}_0 \in B^p_{\text{alg}}(P_d(\Gamma), A^+ \otimes \mathcal{M}_2(\mathbb{C})) \cong B^p_{\text{alg}}(P_d(\Gamma), A^+) \otimes \mathcal{M}_2(\mathbb{C}) \]
and
\[ \tilde{Q} - \tilde{Q}_0 \in B^p_{\text{alg}}(P_d(\Gamma), A) \otimes \mathcal{M}_2(\mathbb{C}). \]

Since \(\Gamma\) has bounded geometry, by the almost flatness of the Bott generators (Lemma 23), we can choose \(\varepsilon_1\) and \(\varepsilon_2\) small enough to obtain \(\tilde{Q}, \tilde{Q}_0\) as constructed above such that \(\|\tilde{Q}^2 - \tilde{Q}\| < 1/5\) and \(\|\tilde{Q}_0^2 - \tilde{Q}_0\| < 1/5\).

It follows that the spectrum of either \(\tilde{Q}\) or \(\tilde{Q}_0\) is contained in disjoint neighborhoods \(S_0\) of 0 and \(S_1\) of 1 in the complex plane. Let \(f : S_0 \cup S_1 \to \mathbb{C}\) be a holomorphic function such that \(f(S_0) = \{0\}, f(S_1) = \{1\}\). Let \(\Theta = f(\tilde{Q})\) and \(\Theta_0 = f(\tilde{Q}_0)\). Then \(\Theta\) and \(\Theta_0\) are idempotents in \(B^p(P_d(\Gamma), A^+) \otimes \mathcal{M}_2(\mathbb{C})\) with
\[ \Theta - \Theta_0 \in B^p(P_d(\Gamma), A) \otimes \mathcal{M}_2(\mathbb{C}). \]

Note that \(B^p(P_d(\Gamma), A) \otimes \mathcal{M}_2(\mathbb{C})\) is a closed two-sided ideal of \(B^p(P_d(\Gamma), A^+) \otimes \mathcal{M}_2(\mathbb{C})\).

At this point we need to recall the difference construction in K-theory of Banach algebras introduced by Kasparov-Yu [17]. Let \(J\) be a closed two-sided ideal of a Banach algebra \(B\). Let \(p, q \in B^+\) be idempotents such that \(p - q \in J\). Then a difference element \(D(p, q) \in B^p(P_d(\Gamma), A^+) \otimes \mathcal{M}_2(\mathbb{C})\).
$K_0(J)$ associated to the pair $p, q$ is defined as follows. Let

$$Z(p, q) = \begin{pmatrix} q & 0 & 1-q & 0 \\ 1-q & 0 & 0 & q \\ 0 & 0 & q & 1-q \\ 0 & 1 & 0 & 0 \end{pmatrix} \in \mathcal{M}_4(B^+) .$$

We have

$$(Z(p, q))^{-1} = \begin{pmatrix} q & 1-q & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1-q & 0 & q & 0 \\ 0 & q & 1-q & 0 \end{pmatrix} \in \mathcal{M}_4(B^+) .$$

Define

$$D_0(p, q) = (Z(p, q))^{-1} \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & 1-q & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} Z(p, q) .$$

Let

$$p_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

Then $D_0(p, q) \in \mathcal{M}_4(J^+)$ and $D_0(p, q) = p_1$ modulo $\mathcal{M}_4(J)$. We define the difference element

$$D(p, q) := [D_0(p, q)] - [p_1]$$

in $K_0(J)$.

Finally, for any idempotent $P \in B^p(P_d(\Gamma))$ representing an element $[P]$ in $K_0(B^p(P_d(\Gamma)))$, we define

$$\beta_*([P]) = D(\Theta, \Theta_0) \in K_0(B^p(P_d(\Gamma), A)) .$$

The correspondence $[P] \mapsto \beta_*([P])$ extends to a homomorphism, the Bott map

$$\beta_* : K_0(B^p(P_d(\Gamma))) \to K_0(B^p(P_d(\Gamma), A)) .$$

By using suspension, we similarly define the Bott map

$$\beta_* : K_1(B^p(P_d(\Gamma))) \to K_1(B^p(P_d(\Gamma), A)) .$$
Construction of the Bott map \((\beta_L)_*\) :

Next we shall construct a Bott map for \(K\)-theory of \(\ell^p\)-localization algebras:

\[
(\beta_L)_*: K_*(B^p_L(P_d(\Gamma))) \to K_*(B^p_L(P_d(\Gamma), A)).
\]

Let \(B^p_{L,alg}(P_d(\Gamma))\) be the algebra of all bounded, uniformly continuous functions

\[
g : \mathbb{R}_+ \to B^p_{alg}(P_d(\Gamma)) \subset B(\ell^p(\Gamma_d, \ell^p))
\]

with the following properties:

(1) there exists a bounded function \(R : \mathbb{R}_+ \to \mathbb{R}_+\) with \(\lim_{t \to \infty} R(t) = 0\) such that \(g(t)(x, y) = 0\) whenever \(d(x, y) > R(t)\) for every \(t\);

(2) there exists \(L > 0\) such that for every \(z \in P_d(\Gamma)\), the number of elements in the following set

\[
\{(x, y) \in \Gamma_d \times \Gamma_d : d(x, z) \leq 3R, d(y, z) \leq 3R, g(t)(x, y) \neq 0\}
\]

is less than \(L\) for every \(t \in \mathbb{R}_+\).

The \(\ell^p\)-localization algebra \(B^p_L(P_d(\Gamma))\) is isomorphic to the norm completion of \(B^p_{L,alg}(P_d(\Gamma))\) under the norm

\[
\|g\|_\infty := \sup_{t \in \mathbb{R}_+} \|g(t)\|
\]

when \(\Gamma\) has bounded geometry. Note that \(B^p_L(P_d(\Gamma))\) is stable in the sense that \(B^p_L(P_d(\Gamma)) \cong B^p_L(P_d(\Gamma)) \otimes \mathcal{M}_k(\mathbb{C})\) for all natural number \(k\). Hence, any element in \(K_0(B^p_L(P_d(\Gamma)))\) can be expressed as the difference of the \(K_0\)-classes of two idempotents in \(B^p_L(P_d(\Gamma))\). To define the Bott map \((\beta_L)_*: K_0(B^p_L(P_d(\Gamma))) \to K_0(B^p_L(P_d(\Gamma), A))\), we need to specify the value \((\beta_L)_*([g])\) in \(K_0(B^p_L(P_d(\Gamma), A))\) for any idempotent \(g \in B^p_L(P_d(\Gamma))\) representing an element \([g] \in K_0(B^p_L(P_d(\Gamma)))\).
Now let \( g \in B^p_L(P_d(\Gamma)) \) be an idempotent with \( \|g\| = N \). For any \( 0 < \varepsilon_1 < 1/100 \), take an element \( h \in B^p_{L,alg}(P_d(\Gamma)) \) such that

\[
\|g - h\|_\infty < \frac{\varepsilon_1}{2N + 2}.
\]

Then \( \|h - h^2\|_\infty < \varepsilon_1 \) and there is a bounded function \( R_{\varepsilon_1}(t) > 0 \) with \( \lim_{t \to \infty} R_{\varepsilon_1}(t) = 0 \) such that \( h(t)(x, y) = 0 \) whenever \( d(x, y) > R_{\varepsilon_1}(t) \) for every \( t \). Let \( R_{\varepsilon_1} = \sup_{t \in \mathbb{R}_+} R(t) \). For any \( \varepsilon_2 > 0 \), take by Lemma 23 a family of \((R_{\varepsilon_1}, \varepsilon_2; c)\)-flat idempotents \( \{b_x\}_{x \in M} \) in \( M_2(\mathcal{A}^+) \) for some \( c > 0 \). Define

\[
\tilde{h}, \tilde{h}_0 : \mathbb{R}_+ \to B^p_{L,alg}(P_d(\Gamma), \mathcal{A}^+) \otimes M_2(\mathbb{C})
\]

by

\[
\left( \tilde{h}(t) \right)(x, y) = \left( h(t)(x, y) \right) \otimes b_x \in \mathcal{A}^+ \otimes K_p \otimes M_2(\mathbb{C}),
\]

\[
\left( \tilde{h}_0(t) \right)(x, y) = \left( h(t)(x, y) \right) \otimes \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \in \mathcal{A}^+ \otimes K_p \otimes M_2(\mathbb{C})
\]

for each \( t \in \mathbb{R}_+ \). Then we have

\[
\tilde{h}, \tilde{h}_0 \in B^p_{L,alg}(P_d(\Gamma), \mathcal{A}^+) \otimes M_2(\mathbb{C})
\]

and

\[
\tilde{h} - \tilde{h}_0 \in B^p_{L,alg}(P_d(\Gamma), \mathcal{A}) \otimes M_2(\mathbb{C}).
\]

Since \( \Gamma \) has bounded geometry, by the almost flatness of the Bott generators, we can choose \( \varepsilon_1 \) and \( \varepsilon_2 \) small enough to obtain \( \tilde{h}, \tilde{h}_0 \), as constructed above, such that \( \|\tilde{h}^2 - \tilde{h}\|_\infty < 1/5 \) and \( \|\tilde{h}_0^2 - \tilde{h}_0\| < 1/5 \). The spectrum of either \( \tilde{h} \) or \( \tilde{h}_0 \) is contained in disjoint neighborhoods \( S_0 \) of \( 0 \) and \( S_1 \) of \( 1 \) in the complex plane. Let \( f : S_0 \sqcup S_1 \to \mathbb{C} \) be the function such that \( f(S_0) = \{0\}, f(S_1) = \{1\} \). Let \( \eta = f(\tilde{h}) \) and \( \eta_0 = f(\tilde{h}_0) \). Then \( \eta \) and \( \eta_0 \) are idempotents in \( B^p_L(P_d(\Gamma), \mathcal{A}^+) \otimes M_2(\mathbb{C}) \) with

\[
\eta - \eta_0 \in B^p_L(P_d(\Gamma), \mathcal{A}) \otimes M_2(\mathbb{C}).
\]
Thanks to the difference construction, we define

$$(\beta_L)_*([g]) = D(\eta, \eta_0) \in K_0(B^p_L(P_d(\Gamma), A)).$$

This correspondence $[g] \mapsto (\beta_L)_*([g])$ extends to a homomorphism, the Bott map

$$(\beta_L)_* : K_0(B^p_L(P_d(\Gamma))) \to K_0(B^p_L(P_d(\Gamma), A)).$$

By suspension, we similarly define

$$(\beta_L)_* : K_1(B^p_L(P_d(\Gamma))) \to K_1(B^p_L(P_d(\Gamma), A)).$$

This completes the construction of the Bott map $(\beta_L)_*$.

It follows from the constructions of $\beta_*$ and $(\beta_L)_*$, we have the following commuting diagram

$$
\begin{array}{ccc}
K_*(B^p_L(P_d(\Gamma))) & \xrightarrow{(\beta_L)_*} & K_*(B^p_L(P_d(\Gamma), A)) \\
\downarrow{\epsilon_*} & & \downarrow{\epsilon_*} \\
K_*(B^p(P_d(\Gamma))) & \xrightarrow{\beta_*} & K_*(B^p(P_d(\Gamma), A))
\end{array}
$$

**THEOREM 25.** For any $d > 0$, the Bott map

$$(\beta_L)_* : K_*(B^p_L(P_d(\Gamma))) \to K_*(B^p_L(P_d(\Gamma), A))$$

is an isomorphism.

**Proof.** Note that $\Gamma$ has bounded geometry, and both the $\ell^p$-localization algebra and the twisted $\ell^p$-localization algebra have strong Lipschitz homotopy invariance at the $K$-theory level. By a Mayer-Vietoris sequence argument and induction on the dimension of the skeletons [27, 5], the general case can be reduced to the 0-dimensional case, i.e., if $D \subset P_d(\Gamma)$ is a $\delta$-separated subspace (meaning $d(x, y) \geq \delta$ if $x \neq y \in D$) for some $\delta > 0$, then

$$(\beta_L)_* : K_*(B^p_L(D)) \to K_*(B^p_L(D, A))$$
is an isomorphism. But this follows from the facts that
\[ K_\ast(B^p_L(D)) \cong \prod_{\gamma \in D} K_\ast(B^p_L(\{\gamma\})) , \]
\[ K_\ast(B^p_L(D, A)) \cong \prod_{\gamma \in D} K_\ast(B^p_L(\{\gamma\}, A)) \]
and that \((\beta_L)_\ast\) restricts to an isomorphism from \(K_\ast(B^p_L(\{\gamma\})) \cong K_\ast(K_p)\) to
\[ K_\ast(B^p_L(\{\gamma\}, A)) \cong K_\ast(K_p \otimes A) \]
at each \(\gamma \in D\) by the classic Bott periodicity.

\[ \square \]

7. Proof of the Main Theorem

Proof of Theorem 1. We have the commuting diagram
\[
\begin{array}{ccc}
\lim_{d \to \infty} K_\ast(B^p_L(P_d(\Gamma))) & \xrightarrow{(\beta_L)_\ast} & \lim_{d \to \infty} K_\ast(B^p_L(P_d(\Gamma), A)) \\
\downarrow e_\ast & & \downarrow e_\ast \\
\lim_{d \to \infty} K_\ast(B^p(P_d(\Gamma))) & \xrightarrow{\beta_\ast} & \lim_{d \to \infty} K_\ast(B^p(P_d(\Gamma), A)).
\end{array}
\]
Hence, \(\beta_\ast \circ e_\ast = e_\ast \circ (\beta_L)_\ast\). It follows from Theorem 14 and Theorem 25 that \(\beta_\ast \circ \text{ind}\) is an isomorphism. Consequently, the index map
\[ e_\ast : \lim_{d \to \infty} K_\ast(B^p_L(P_d(\Gamma))) \to \lim_{d \to \infty} K_\ast(B^p(P_d(\Gamma))) \cong K_\ast(B^p(\Gamma)) \]
is injective. \[ \square \]

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