The urban economy as a scale-free network

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We present empirical evidence that land values are scale-free and introduce a network model that reproduces the observations. The network approach to urban modelling is based on the assumption that the market dynamics that generates land values can be represented as a growing scale-free network. Our results suggest that the network properties of trade between specialized activities causes land values, and likely also other observables such as population, to be power law distributed. In addition to being an attractive avenue for further analytical inquiry, the network representation is also applicable to empirical data and is thereby attractive for predictive modelling.

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I. INTRODUCTION

The Zipf rank-size law for city sizes is one of the most widely known power laws in science[1]. It is also but one out of many similar power laws from systems in biology, economy and society. We continue this research by presenting empirical evidence that land values are scale-free. The data we use is based on a database delivered by Sweden Statistics that covers estimations of the market value of all land in Sweden (2.9 million data points).

Although power laws are common they are not easily reconstructed from realistic underlaying dynamics. Their ubiquity suggests that they could be caused by some general systemic property common to a range of systems. Recent research suggests that many empirically observed power laws may be due to fundamental properties of these systems viewed as networks of interacting nodes[2, 3, 4]. We investigate the mechanisms causing land values to follow these statistics and present a network model that reproduces the empirical results. The model is based on basic definitions of city formation in urban economics theory[4].

According to urban economics theory, the formation of modern cities is primarily caused by the advantages of trade between specialized producers. The exchange of goods and services between localized and largely immobile activities in trade economies makes a network representation natural: the nodes are units of land and the edges represent the exchange of goods and services between them.

It has been shown that the node degrees of a certain class of growing networks are power law distributed[2, 3]. This class of networks is important because their growth mechanisms can be mapped to the microscopic dynamics of several real-world systems. We demonstrate how trade in an urban system can be represented as a scale free network and that, as a consequence, land values can be expected to follow the same distribution. We also verify that the model retains these properties when spatial constraints are taken into account. We do this by using a spatial network model to reproduce the empirically observed distribution of land values.

The network approach solves a fundamental issue in the problem of modelling urban systems by representing the system at the level of the underlaying market structure. This allows us to produce prices in units of currency rather than undefined and subjective fitness measures. It thereby opens up doors for several extensions of the scope of the model and provides a natural interface for integrating it with other models.

When we refer to an urban system we do not necessarily refer to individual cities but rather to systems of specialized trading activities. To clarify further, our use of the term *activity* refers to trade gains in units of *currency per unit area and unit time*. Activities can be resolved to any resolution down to individual transactions.
II. THE URBAN ECONOMY AS A NETWORK

In section II A we define a non-spatial model of urban economic growth and in section II B we extend it to a spatial model where growth is mapped to a 2D surface. In section II C we motivate the model ontology and the basic assumptions on which we have based the model. In particular we discuss the connection between node degree and land value and how the urban system meets the criteria for being a scale-free-free network.

A. Formulation of the non-spatial model

A geographic area on which an urban economic system can grow is represented by an enumerated set of nodes, \{1, 2, \ldots, N\} corresponding to non-overlapping land areas. Trade of goods and service between activities in the nodes is represented by undirected edges. Since activities within the same node can trade with each other, an edge can connect a node with itself. The amount of activity of a site \(x_i\) is defined as the degree of the corresponding node. See figure 1.

The network is initialized by connecting \(n_0\) nodes so that each has a degree of \(x_0\). The \(N - n_0\) undeveloped nodes have no trade interactions and thus no activity.

At each time step we update the network as follows:

1. With probability \(q_1\) we add \(m\) edges between sites that already are developed: the first edge end-point is selected uniformly among developed nodes. The probability of a node \(i\) to be selected is

\[
\Pi_i^p = \delta_i^{(D)} \frac{1}{n_i^{(D)}},
\]

where \(n_i^{(D)}\) is the number of developed nodes at time \(t\) and \(\delta_i^{(D)} = 1\) if node \(i\) is developed and \(\delta_i^{(D)} = 0\) otherwise.

The second end-point is selected preferentially among developed nodes. Preferential selection corresponds to the uniform selection of an activity in the system and the subsequent location of its node. It was defined by Barabási et al [2] as

\[
\Pi_i^p = \frac{x_i}{\sum_j x_j},
\]

where \(\Pi_i^p\) is the probability of node \(i\) to be attached to a new edge and \(x_i\) is the degree of node \(i\).

2. With probability \(q_2\) we add edges between \(m\) pairs of nodes that are both selected preferentially according to Eq. (2).

3. With probability \(q_3\) we add \(m\) units of initial activity on land that is previously undeveloped: the first end-point is selected without bias similarly to Eq. 1 but among undeveloped nodes. However, since nodes have no properties beside their degree in the non-spatial model, any undeveloped node can be added. The second end-point is selected preferentially according to Eq. (2).

We will refer to the growth classes as type-1, type-2 and type-3 growth and to their relative rates as \(q_1\), \(q_2\) and \(q_3\) with \(q_1 + q_2 + q_3 = 1\) throughout the paper.

A continuum formulation

Instead of assuming any particular set of activities and interactions we can use a continuum formulation where we consider each edge end-point to be an average of a large set of urban activities. It follows that an average activity must be assumed to interact equally much with all other average activities. Rather than counting explicit interactions to determine the activity level, we study the evolution of the expected node degrees. The time evolution of activity on a developed site \(i\) follows the equation

\[
x_i(t+1) = x_i(t) + q_1 \frac{m}{n_i^{(D)}} + m(q_1 + 2q_2 + q_3) \frac{x_i(t)}{\sum_j x_j(t)},
\]

which is solved by the continuous-time method introduced by Barabási et al [3]. After sufficiently long time the degree distribution approaches the form

\[
P[x_i = x] \sim (x + A)^{-\gamma},
\]

with

\[
A = \frac{2mq_1}{q_3(1 + q_2)}.
\]
This is analogous to Eq. (2) but in the spatial model, transportation costs will bias the choice of trade partners. $D_{ij}$ is a matrix representing the interaction strength, which is assumed to decay with the transportation costs. For the results in this paper we have used $D_{ij} = (1 + cd(i,j))^{-\alpha}$, where $d(i,j)$ is the Euclidean distance between sites $i$ and $j$. The non-negative parameters $c$ and $\alpha$ controls the impact of spatial. Another function that can apply is $D_{ij} = \max(0, 1 - d(i,j))$ if the transportation characteristics of the activity is known, which can be the case in a model where activity types are modelled separately rather than as an average. Exponential $D_{ij} = e^{-d(i,j)}$ can also be interesting to the extent that shielding is important, i.e. an activity tends to trade exclusively with the nearest supplier.

3. Primary increase

When trade takes place there is a mutual benefit in efficiency that is often used for further increasing the activity in the city. This feedback process makes it possible for the amount of activity in the urban system to increase considerably faster than the population. It is useful to think of Eq. (7) as a black box system to which a driving force, primary increase, is applied. It should also be noted that the primary effects used for the model we present here are by no means neither exhaustive nor final: most earlier models of urban growth could be introduced as primary effects in our framework.

4. Primary effects in type-1, type-2 and type-3 growth

Network evolution in the spatial model is similar to the non-spatial model. Secondary increases are always preferential following Eq. (7) and primary increases now reflect the spatial distribution:

Type-1 growth: the primary uniform increase is identical to the non-spatial case following Eq. (1).

Type-2 growth: the primary preferential increase is identical to the non-spatial case following Eq. (2).

Type-3 growth is split into two related processes where one corresponds to growth in the perimeter of clusters and the other corresponds to growth in connection to infrastructure in the rural areas between clusters. Such infrastructure is not explicitly represented in our model and instead we use a parameter $\epsilon$ to tune the amount of ambient infrastructure and thus the rate with which seemingly isolated clusters will appear.

Type-3a growth: with a probability of $q_3(1 - \epsilon)$ we set the activity of a perimeter node to $m$. Perimeter nodes are nodes that are not developed but borders to at least one developed cell. The site of the new node is selected randomly and with uniform probability among

and

$$\gamma = 1 + \frac{2}{1 + q_2}. \tag{6}$$

According to Eq. (4) the node degree distribution will be power law distributed for the non-spatial model.

B. Formulation of the spatial model

The non-spatial model does not include any information about inhomogeneous relations in the network. An important departure from such a simple model that can be readily observed in real systems is of course the fact that different pairs of sites have different distances between them. Because of transportation cost optimization this has the potential to affect the edge distribution of the network model.

We will now extend the basic model to incorporate spatial interactions. This allows us to verify that the power law distribution of activities that is predicted by the non-spatial model is retained when space is introduced. Furthermore, it allows us to better map model output to empirical measurements. Scale-free spatial network models that have been studied recently are not directly applicable to urban economics since they require an a priori distribution of node locations.

1. Network evolution

In the continuum non-spatial model we were able to treat network evolution as a purely local phenomenon since interactions in the network are homogenous, see Eq. (3). In the spatial model we have to again explicitly separate the dynamics of the edge end-points. This is because the spatial context of the first end-point of a new edge modifies the probabilities with which other sites will become the second end-point (transportation costs).

The same types of growth that are used in the non-spatial model are also used in the spatial model. But, as stated, the two end-points here need to be dealt with separately. The first site (primary increase) is selected either uniformly or preferentially depending on growth type. This will be discussed shortly. The second end-points of the trade relation edges (secondary increase) are always preferential.

2. Secondary increase

The spatial model tells us how a given primary increase $p_j$ in activity at site $j$ causes secondary increases $s_i$ in activity at all other sites $i$ where

$$s_i = p_j \frac{D_{ij} x_i}{\sum_k D_{kj} x_k}. \tag{7}$$

This allows us to verify that the power law distribution of activities that is predicted by the non-spatial model, transportation costs will bias the choice of trade partners. $D_{ij}$ is a matrix representing the interaction strength, which is assumed to decay with the transportation costs. For the results in this paper we have used $D_{ij} = (1 + cd(i,j))^{-\alpha}$, where $d(i,j)$ is the Euclidean distance between sites $i$ and $j$. The non-negative parameters $c$ and $\alpha$ controls the impact of spatial. Another function that can apply is $D_{ij} = \max(0, 1 - d(i,j))$ if the transportation characteristics of the activity is known, which can be the case in a model where activity types are modelled separately rather than as an average. Exponential $D_{ij} = e^{-d(i,j)}$ can also be interesting to the extent that shielding is important, i.e. an activity tends to trade exclusively with the nearest supplier.
the perimeter sites on the grid

\[ \pi_i^a = \delta_i^{(P)} \frac{1}{n_i^{(P)}}, \tag{8} \]

where \( \pi_i^a \) is the probability with which node \( i \) is selected to undergo type-3a growth, \( \delta_i^{(P)} = 1 \) if the node \( i \) is on the perimeter, \( \delta_i^{(P)} = 0 \) if the node is not on the perimeter and \( n_i^{(P)} \) is the number of perimeter nodes at time \( t \).

Type-3b growth: With a probability of \( q \delta_i \) we set the activity of an external node to \( m \). An external node is a node that is undeveloped and that has no developed neighbors. The site of the new node is selected randomly and with uniform probability among external sites on the grid

\[ \pi_i^b = \delta_i^{(E)} \frac{1}{n_i^{(E)}}, \tag{9} \]

where \( \pi_i^b \) is the probability with which node \( i \) is selected to undergo type-3b growth, \( \delta_i^{(E)} = 1 \) if the node \( i \) is external, \( \delta_i^{(E)} = 0 \) if the node is not external and \( n_i^{(E)} \) is the number of external nodes at time \( t \).

5. A continuum formulation of the spatial model

A continuum formulation for the evolution of developed nodes in the spatial model can, as in section II A 1, be constructed by studying the time evolution of expected node degrees.

\[ x_i(t+1) = x_i(t) + E[p_i(t)] + E[s_i(t)], \tag{10} \]

where \( E[s_i] \) is the expected secondary increase which can be calculated by a weighted summation over the expected primary increments,

\[ E[s_i] = \sum_j E[p_j] \frac{D_{ij}x_i}{\sum_k D_{kj}x_k}. \tag{11} \]

For the special case of our simple model for the primary effects we have

\[ E[p_j] = \delta_j^{(D)} p_j^{(1)} + \delta_j^{(P)} p_j^{(2)} + \delta_j^{(3a)} p_j^{(3a)} + \delta_j^{(3b)} p_j^{(3b)}, \tag{12} \]

with

\[ p_j^{(1)} = q_1 \frac{m}{n_i^{(D)}}, \tag{13} \]

\[ p_j^{(2)} = q_2 \frac{x_j}{\sum_k D_{kj}x_k}, \tag{14} \]

\[ p_j^{(3a)} = q_3(1-\epsilon) \frac{m}{n_i^{(P)}}, \tag{15} \]

\[ p_j^{(3b)} = q_3 \epsilon \frac{m}{n_i^{(E)}}, \tag{16} \]

The total growth for a developed node can in this case be separated into a uniform and a preferential part as

\[ x_i(t+1) = x_i(t) + \zeta_i(t) + \eta_i(t)x_i(t), \tag{17} \]

with

\[ \zeta_i = q_1 \frac{m}{n_i^{(D)}}, \tag{18} \]

and

\[ \eta_i = \frac{q_2 m}{\sum_k x_k} + \sum_j E[p_j] \frac{x_j}{\sum_k D_{kj}x_k}. \tag{19} \]

Noting that \( \sum_i \eta_i x_i = m(q_1 + 2q_2 + q_3) \), equation (19) can be rewritten as

\[ x_i(t+1) = x_i(t) + \zeta_i(t) + m(q_1 + 2q_2 + q_3) \eta_i(t)x_i(t) \sum_j \eta_j(t)x_j(t), \tag{20} \]

which, in a comparison with equation (8), reveals that the only difference between the spatial and the non-spatial model is the site and time-dependent parameter \( \eta_i \). This is similar to the concept of node fitness, as presented in [11, 12], which can affect the node degree distribution. However, our simulation results indicate that \( \eta_i \) falls within a sufficiently narrow interval for the power law to be essentially preserved (Fig. 2). This is also supported by calculations of \( \eta_i \) for both simulated and empirical data.

C. The network model in an urban economics context

1. The connection between node degree and land value

An approximate linear relationship between node degree in the model and land value in the real system is crucial for the interpretation of our results. The motivation follows from i) market pricing of goods and services and ii) the connection between trade benefits and land value.

i) Market pricing of commodities provides an adaptive measure that allows us to compare the activities that generate them. Hence, on average, an edge contributes identically to the value of both nodes to which it connects. This contribution is exactly our definition of activity, which implies that the degree of a node is proportional to its benefits due to trade.

ii) This connection consists of two proportionality. For a node \( i \) we have

\[ v_i \propto r_i \propto x_i, \tag{21} \]

where \( v_i \) is the value of the corresponding land area, \( r_i \) is the bid-rent [13, 14], and \( x_i \) is the total trade benefits as outlined above. Capitalizing periodic rent income from
the site \( i \) gives land value \( v_i = \frac{r}{s} \) where \( i \) is the inter-

erest rate\footnote{5}. The second proportionality is a weak form

of the leftover principle from urban economics, which

states that, in a competitive land market, rent equals

the amount of money leftover after all expenses (except

rent) are paid. This amount of money equals the sum

of all trade benefits at the site. For our results it is suffi-

cient that, on average, a certain proportion of each new

unit of trade benefit goes to the landowner.

Together, i) and ii) suggest an approximately linear

relationship between node degrees and land values.

2. Types of growth

Urban activity can increase in essentially two ways: ei-

ther a new activity is related to, or it is unrelated to, an

existing activity at the site. In the former case (pre-

ferential growth) this could be a new employee hired as

a response to increased demand, in the latter case (uni-

form) it could be the establishment of a new firm. Pre-

ferential growth corresponds to a per-unit activity rate.

Uniform growth corresponds to establishment among lots

on a competitive land market where, for the average land

use, we can not expect any lot to be more profitable than

any other.

3. Reasons for treating perimeter growth separately from

internal growth

The jagged perimeter of urban areas exposes large ar-

eas of undeveloped land to urban infrastructure, thus

making it attractive for urban land use. Because perimeter

land is in ample supply and currently has a low rev-

nue, even land uses with a very low trade gain can be

competitive. Many low-activity land uses in the outskirts

of the urban area can likely just barely out-bid agricul-

ture and would not be viable in competition with other

urban land uses. Among high-activity land uses some

have very specific demands on land improvements and

can therefore not benefit from buying existing buildings

inside the urban area. This creates a special case for perim-

eter land. Note that just like for type-1 growth, com-

petition prevents prediction of where the next growth

event will take place among the perimeter nodes, and no

preferential growth is possible since there is no previous

activity that can expand.

4. Node fitness and growth

If we only regard trading activities the only difference

between two sites with identical activity is the value of

their spatial context. Therefore, in the spatial model,

secondary growth is not homogenous, see Eq. \( 17 \). Node

fitness (Eq. \( 17 \)) can be viewed as a local interest rate

that predicts the growth rate of activity investments made in

that site. The result of this is non-trivial growth predic-

tions since the expected amount of new local development

does not become a simple fraction of current develop-

ment.

Most notably, the model predicts the emergence of ur-

ban sub-centers. This is realized by examining the ex-

pected secondary effects in the spatial model (Eq. \( 11 \)).

Apart from being proportional to the amount of present

activity \( E[s_i] \) is also subject to site competition and will

increase for nodes that have high activity in relation to

their own neighborhoods. For each possible primary ef-

fect in \( j \), the node development \( x_j \) is weighted with the

fraction between trade intensity between site \( j \) and \( i \) and

the sum of the all other trading options for the primary

effect under consideration. Thus, nearby high-intensity

nodes will not necessarily benefit a small neighbor.

III. RESULTS

A. Land values are power law distributed

From empirical data, land values in Sweden are demon-

strated to follow a power-law distribution for the higher

range of land values (see figure \( 2 \)). The sharp transition

that can be observed around 60kSEK suggests that two

truly different mechanisms generate the prices below and

above this point. This is in agreement with the observa-

tion that the pricing mechanism we suggest would apply

only to trading activity.

The data we have used is based on the land value

component of market value estimations of about 2.9 mil-

lion units of real estate in Sweden. The data was origi-
nally compiled by the Swedish National Land Survey and

coded by Sweden Statistics to geographical coordinates.

The data points we use are aggregated land values into

100m \( \times \) 100m squares.

Our results are supported by a recent study by Kaizoji

that shows scale-free behavior of land prices in Japan,

with an exponent \( \gamma \) ranging from 2.53 to 2.76 \footnote{76[15]}.

B. Power law distributed prices are predicted by

the non-spatial model

We have developed a simple network model of the

urban economy based on the Barabási-Albert model

by mapping fundamental assumptions from urban eco-

nomics to the ontology of the network model (see section

\footnote{11A}). In our derivation, we have determined the ex-

ponent of the node degree distribution, and thus, per our

definition, also the predicted land values, to follow equa-

tion \footnote{5}. To the extent that our interpretation of the

underlying dynamics is correct this demonstrates why

urban land values can be expected to follow a power law

and how the exponent may depend on parameters.
FIG. 2: Double logarithmic histogram of simulation results and empirical data. Simulation results are are denoted with squares in the figure and they are a mean of the results of three runs of the spatial model. The exponent of the model output has been tuned to match the empirical data. As indicated in Eq. (6) this is done by setting the relative proportions of the growth types, in this case $q_1 = 0.1, q_2 = 0.6, \epsilon = 0.01, c = 0.2, \alpha = 1, m = 100kSEK$ and $t=170000$. The exponent is roughly 2.1 which is close to the value of $\gamma = 2.25$ that is predicted by Eq. (6) for these parameter values in the non-spatial model. A slightly different value for the spatial model must be expected because of spatial bias in the growth dynamics. The sharp transition that occurs around a land price of $60kSEK (\approx 5kUSD)$ per $100m \times 100m$ marks the difference in dynamics between trade based urban activities and rural activities whose values are not described by the presented model.

C. The spatial model retains power law statistics

As discussed in section II B 5, the impact of spatial constraints closely resembles that of node fitness. This could potentially result in the distribution for the spatial network model becoming a sum of many power laws with different exponents.

In figure 2 we plot results from simulations showing that node degrees in the spatial model follow a power law distribution. The model parameters have been tuned (see sec. II B 5) to reproduce the distribution of the observed land prices.

IV. CONCLUSION

We present empirical evidence that land prices follow a power law distribution for urban land uses, and present a generic model, based on the underlaying trade network, that reproduces this behavior. The model is a version of the Barabási-Albert scale free network that is also extended to incorporate spatial constraints.

The applicability of the network paradigm to urban growth suggests that scale invariance in urban systems is caused by: i) growth and ii) preferentiality in how new trade connections are formed between areas of land. Growth in this context refers to the continual development of new land. Preferentiality is a consequence of point-to-point interactions between activities occupying the land areas. Note that many other observables, such as population and urban land use intensity, might be highly correlated with land value.

The spatial model that we present can have more general applicability beyond urban economics. Other spatially growing networks are communication networks, transportation networks, electricity and utility networks. These can be expected to follow a similar type of growth since they are intimately connected to urban activity.

The network architecture is generic and allows for addition of any amount of detail. Also, being based on trade relations, the model produces output in units of currency. Because of this, such network models can provide a bridge between a microscopic dynamics that can be found empirically and emergent economic properties.

Further possible directions for research on urban economic networks include interpretation of other theoretical network results in terms of urban dynamics, finding empirical parameters for scenario predictions and model validation.

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[1] G.K. Zipf. *Human Behavior and the Principle of Least Effort*. Addison-Wesley, 1949.

[2] Albert-László Barabási and Réka Albert. Emergence of scaling in random networks. *Science*, 286:509–512, 1999.

[3] A.-L. Barabási, H. Jeong, R. Ravasz, Z. Néda, T. Vicsek, and A. Schubert. Statistical mechanics of complex networks. *Review of Modern Physics*, 74:47–97, 2002.

[4] S.N. Dorogovtsev and J.F.F. Mendes. Evolution of networks. *Adv. Phys.* 51:1079–1187, 2002.

[5] Arthur O’Sullivan. *Urban Economics*. McGraw Hill Higher Education, 2002.

[6] S.N. Dorogovtsev, J.F.F. Mendes, and A.N. Samukhin. Structure of growing networks: Exact solution of the barabasi-albert’s model. *Phys. Rev. Lett.*, 85, 2000.

[7] Réka Albert, Albert-László Barabási, and Hawoong Jeong. Mean-field theory for scale-free random networks. *Physica A*, 272:173–187, 1999.

[8] Marc Barthélemy. Crossover from scale-free to spatial networks. *arXiv:cond-mat/0212086*, 2002.

[9] R. Xulvi-Brunet and I. M. Sokolov. Evolving networks with disadvantaged long-range connections. *Physical Review E*, 66(026118), 2002.

[10] S. S. Manna and Parongama Sen. Modulated scale-free network in the euclidean space. *arXiv:cond-mat/0203216*, 2002.

[11] G. Bianconi and A.-L. Barabási. Competition and multiscaling in evolving networks. *Europhysics Letters*, 54 (4):436–442, 2001.

[12] G. Ergün and G. J. Rodgers. Growing random networks with fitness. *arXiv:cond-mat/0103423*, 2001.

[13] W. Alonso. Papers and proceedings of the regional science association. *Readings in Urban Economics*, 6:149–157, 1960.

[14] W. Alonso. *Location and Land Use*. Harvard University Press, 1964.

[15] Taisei Kaizoji. Scaling behavior in land markets. *arXiv:cond-mat/0302470*, 2003.