DENSITY THEOREMS FOR EXCEPTIONAL EIGENVALUES
FOR CONGRUENCE SUBGROUPS

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Abstract. Using the Kuznetsov formula, we prove several density theorems for exceptional Hecke and Laplacian eigenvalues of Maass cusp forms of weight 0 or 1 for the congruence subgroups \( \Gamma_0(q), \Gamma_1(q), \) and \( \Gamma(q) \). These improve and extend upon results of Sarnak and Huxley, who prove similar but slightly weaker results via the Selberg trace formula.

1. Introduction

Let \( \kappa \in \{0, 1\} \), let \( \Gamma \) be a congruence subgroup of \( \text{SL}_2(\mathbb{Z}) \), and let \( \chi \) be a congruence character of \( \Gamma \) satisfying \( \chi(-I) = (-1)^\kappa \). Let \( -I \) be a member of \( \Gamma \). Denote by \( A_\kappa(\Gamma, \chi) \) the space of Maass cusp forms of weight \( \kappa \), level \( \Gamma \), and nebentypus \( \chi \), namely the space of smooth functions \( f: \mathbb{H} \to \mathbb{C} \) satisfying

- \( f(\gamma z) = \chi(\gamma)j_\gamma(z)^\kappa f(z) \) for all \( \gamma \in \Gamma \) and \( z \in \mathbb{H} \), where for \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \),
  \[ j_\gamma(z) = \frac{cz + d}{|cz + d|}, \]

- \( f \) is an eigenfunction of the weight \( \kappa \) Laplacian
  \[ \Delta_\kappa = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + i\kappa y \frac{\partial}{\partial x}, \]

- \( f \) is of moderate growth,

and the constant term in the Fourier expansion of \( f \) at every singular cusp \( \alpha \) with respect to \( \chi \) of \( \Gamma \setminus \mathbb{H} \) is zero.

When \( \chi \) is the trivial character, we merely write this as \( A_\kappa(\Gamma) \).

We may choose a basis \( B_\kappa(\Gamma, \chi) \) of the complex vector space \( A_\kappa(\Gamma, \chi) \) consisting of Hecke eigenforms. For \( f \in B_\kappa(\Gamma, \chi) \), we let \( \lambda_f = 1/4 + t_f^2 \) denote the eigenvalue of the weight \( \kappa \) Laplacian, where either \( t_f \in [0, \infty) \) or \( it_f \in (0, 1/2) \). Similarly, we let \( \lambda_f(p) \) denote the eigenvalue of the Hecke operator \( T_p \) at a prime \( p \), so that \( |\lambda_f(p)| \leq p^{1/2} + p^{-1/2} \). The generalised Ramanujan conjecture states that \( t_f \) is real and that \( |\lambda_f(p)| \leq 2 \) for every prime \( p \); exceptions to this conjecture are called exceptional eigenvalues, and exceptional Laplacian eigenvalues cannot occur if \( \kappa = 1 \). The best current bounds towards the generalised Ramanujan conjecture are due to Kim and Sarnak; they show that

\[ \lambda_f \geq \frac{1}{4} - \left( \frac{7}{64} \right)^2, \quad |\lambda_f(p)| \leq p^{7/64} + p^{-7/64}. \]

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In this paper, we use the Kuznetsov formula to prove density results for exceptional eigenvalues for the congruence subgroups

$$\Gamma_0(q) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{q} \right\},$$

$$\Gamma_1(q) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : a, d \equiv 1 \pmod{q}, \ c \equiv 0 \pmod{q} \right\},$$

$$\Gamma(q) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : a, d \equiv 1 \pmod{q}, \ b, c \equiv 0 \pmod{q} \right\},$$

with $\chi$ equal to the trivial character.

Recall that

$$\operatorname{vol}(\Gamma \backslash \mathbb{H}) = \frac{\pi}{3} \left[ \text{SL}_2(\mathbb{Z}) : \Gamma \right] = \begin{cases} 
\frac{\pi}{3} q \prod_{p|q} \left( 1 + \frac{1}{p} \right) & \text{if } \Gamma = \Gamma_0(q), \\
\frac{\pi}{3} q^2 \prod_{p|q} \left( 1 - \frac{1}{p^2} \right) & \text{if } \Gamma = \Gamma_1(q), \\
\frac{\pi}{3} q^3 \prod_{p|q} \left( 1 - \frac{1}{p^2} \right) & \text{if } \Gamma = \Gamma(q). 
\end{cases}$$

We define

$$C_\Gamma := \begin{cases} 
4 & \text{if } \Gamma = \Gamma_0(q), \\
3 & \text{if } \Gamma = \Gamma_1(q), \\
8 & \text{if } \Gamma = \Gamma(q). 
\end{cases}$$

**Theorem 1.1.** For any finite collection of primes $\mathcal{P}$ not dividing $q$, any $\alpha_p > 2$ and $0 \leq \mu_p \leq 1$ for all $p \in \mathcal{P}$ with $\sum_{p \in \mathcal{P}} \mu_p = 1$, we have that

$$\# \left\{ f \in \mathcal{B}_c(\Gamma) : t_f \in [0, T], \ |\lambda_f(p)| \geq \alpha_p \text{ for all } p \in \mathcal{P} \right\} \ll \varepsilon \operatorname{vol}(\Gamma \backslash \mathbb{H})^{1-C_{\Gamma}} \left( \sum_{p \in \mathcal{P}} \mu_p \frac{\log \alpha_p/2}{\log p} \right) + \varepsilon (T^2)^{1-4 \left( \sum_{p \in \mathcal{P}} \mu_p \frac{\log \alpha_p/2}{\log p} \right)} + \varepsilon.$$

For $\Gamma = \text{SL}_2(\mathbb{Z})$ and $\mathcal{P}$ consisting of a single prime $p$, this theorem is a result of Blomer, Buttcane, and Raulf [BBR14, Proposition 1], improving on a slightly weaker result of Sarnak [Sar87, Theorem 1.1], who uses the Selberg trace formula in place of the Kuznetsov formula and obtains instead

$$\# \left\{ f \in \mathcal{B}_c(\text{SL}_2(\mathbb{Z})) : t_f \in [0, T], \ |\lambda_f(p)| \geq \alpha \right\} \ll (T^2)^{1 - \frac{\log \alpha}{\log p}}.$$

**Theorem 1.2.** For any finite (possibly empty) collection of primes $\mathcal{P}$ not dividing $q$, any $\alpha_0 > 0$, $\alpha_p > 2$, and $0 \leq \mu_0, \mu_p \leq 1$ for all $p \in \mathcal{P}$ with $\mu_0 + \sum_{p \in \mathcal{P}} \mu_p = 1$, we have that

$$\# \left\{ f \in \mathcal{B}_0(\Gamma) : t_f \in (\alpha_0, 1/2), \ |\lambda_f(p)| \geq \alpha_p \text{ for all } p \in \mathcal{P} \right\} \ll \varepsilon \operatorname{vol}(\Gamma \backslash \mathbb{H})^{1-C_{\Gamma}} \left( \mu_0 \alpha_0 + \sum_{p \in \mathcal{P}} \mu_p \frac{\log \alpha_p/2}{\log p} \right) + \varepsilon.$$

When $\mathcal{P}$ is empty, this improves on a result of Huxley [Hux86], who uses the Selberg trace formula in place of the Kuznetsov formula and obtains instead this result with $C_{\Gamma} = 2$ for each of the three congruence subgroups. For $\Gamma = \Gamma_0(q)$ and $\mathcal{P}$ empty, this is a result of Iwaniec [Iwa02, Theorem 11.7].
The background on automorphic forms and notation in this section largely follows [DF102]; see [DF102, Section 4] for more details. Let $\kappa \in \{0, 1\}$, and let $\chi$ be a Dirichlet character modulo $q$ of conductor $q$, where $q \chi$ divides $q$, satisfying $\chi(-1) = (-1)^{\kappa}$; this defines a congruence character of $\Gamma_0(q)$ via $\chi(\gamma) := \chi(d)$ for $\gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(q)$. We denote by $L^2(\Gamma_0(q) \backslash \mathbb{H}, \kappa, \chi)$ the $L^2$-completion of the space of all smooth functions $f: \mathbb{H} \to \mathbb{C}$ that are of moderate growth and satisfy $f(\gamma z) = \chi(\gamma) \vartheta(z)^s f(z)$. This space has the spectral decomposition

$$L^2(\Gamma_0(q) \backslash \mathbb{H}, \kappa, \chi) = \mathcal{A}_\kappa(q, \chi) \oplus \mathcal{E}_\kappa(q, \chi)$$

with respect to the weight $\kappa$ Laplacian, where $\mathcal{A}_\kappa(q, \chi) := \mathcal{A}_\kappa(\Gamma_0(q), \chi)$ is the space of Maass cusp forms of weight $\kappa$, level $q$, and nebentypus $\chi$, and $\mathcal{E}_\kappa(q, \chi)$ denotes the continuous spectrum spanned by incomplete Eisenstein series parametrised by the cusps $\mathfrak{a}$ of $\Gamma_0(q) \backslash \mathbb{H}$ that are singular with respect to $\chi$.

We denote by $\mathcal{B}_\kappa(q, \chi)$ any basis of Maass cusp forms $f \in \mathcal{A}_\kappa(q, \chi)$ normalised to have $L^2$-norm 1:

$$\|f\|^2_{L^2(\Gamma_0(q) \backslash \mathbb{H})} := \int_{\Gamma_0(q) \backslash \mathbb{H}} |f(z)|^2 \, d\mu(z) = 1,$$

where $d\mu(z) = \frac{dx \, dy}{y^2}$ is the $\text{SL}_2(\mathbb{R})$-invariant measure on $\mathbb{H}$. Later we will use the Atkin–Lehner decomposition of $\mathcal{A}_\kappa(q, \chi)$ in order to specify that $\mathcal{B}_\kappa(q, \chi)$ can be chosen to consist of Hecke eigenforms. The Fourier expansion of $f \in \mathcal{A}_\kappa(q, \chi)$ is

$$f(z) = \sum_{\substack{n \in \mathbb{Z} \setminus \{0\}}} \rho_f(n) W_{\varphi(n)\pi, \alpha_j}(4\pi|n|y)e(nx),$$

where $W_{\alpha, \beta}$ is the Whittaker function. The continuous spectrum $\mathcal{E}_\kappa(q, \chi)$ is the space of functions $g \in L^2(\Gamma_0(q) \backslash \mathbb{H}, \kappa, \chi)$ that are orthogonal to every cusp form $f \in \mathcal{A}_\kappa(q, \chi)$ and have the spectral expansion

$$(1) \quad g(z) = \frac{1}{\text{vol}(\Gamma_0(q) \backslash \mathbb{H})} \langle g, \delta_{\chi \chi_0} \rangle + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \langle g, \mathcal{E}_\mathfrak{a} \left( \frac{1}{2} + it, \chi \right) \rangle \, d\mu(t, \chi)$$

for all $z \in \mathbb{H}$, where $\delta_{\chi \chi_0}$ is 1 if $\chi$ is the principal character and 0 otherwise, the summation is over the cusps $\mathfrak{a}$ of $\Gamma_0(q) \backslash \mathbb{H}$ that are singular with respect to $\chi$, and $\mathcal{E}_\mathfrak{a}(z, s, \chi) := \sum_{\gamma \in \Gamma_\mathfrak{a} \backslash \Gamma_0(q)} \chi(\gamma) j_{\sigma_\mathfrak{a}^{-1}, \gamma}(z)^{-s} \Im(\sigma_\mathfrak{a}^{-1} \gamma z)^s$ is an Eisenstein series with Fourier expansion

$$\delta_\mathfrak{a} y^{1/2+it} + \varphi_\mathfrak{a} \left( \frac{1}{2} + it, \chi \right) y^{1/2-it} + \sum_{\substack{n = -\infty \atop n \neq 0}}^{\infty} \rho_\mathfrak{a}(n, t, \chi) W_{\varphi(n)\pi, \alpha_j}(4\pi|n|y)e(nx)$$

for $s = 1/2 + it$ with $t \in \mathbb{R}$.

The pre-Kuznetsov formula is the following.
**Theorem 2.1** ([DFI02, Proposition 5.2]). For \( m, n \geq 1 \) and \( r \in \mathbb{R} \),

\[
\sum_{f \in \mathcal{B}_n(q, \chi)} \frac{4\pi \sqrt{mn\pi f(m)\rho_f(n)}}{\cosh \pi(r - t_f)\cosh \pi(r + t_f)} + \sum_{a} \int_{-\infty}^{\infty} \frac{\sqrt{mn\pi a(m, t, \chi)\rho_a(n, t, \chi)}}{\cosh \pi(r - t)\cosh \pi(r + t)} \, dt
\]

\[= \left| \Gamma \left( 1 - \frac{\kappa}{2} - i r \right) \right|^2 \left( \delta_{m,n} + \sum_{c=1}^{\infty} \frac{S_h(m, n; c)}{c} I_{\kappa} \left( \frac{4\pi \sqrt{mn}}{c}, r \right) \right),\]

where

\[S_h(m, n; c) := \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^n} \chi(d)e \left( \frac{md + nd}{c} \right),\]

\[I_{\kappa}(t, r) := -2t \int_{-1}^{1} (-i\zeta)^{\kappa-1} K_{2\kappa}(\zeta t) \, d\zeta,\]

with the latter integral being over the semicircle \(|z| = 1, \Re(z) > 0\).

By the reflection formula for the gamma function, we have that for \( r \in \mathbb{R} \),

\[\left| \Gamma \left( 1 - \frac{\kappa}{2} - i r \right) \right|^2 = \begin{cases} \frac{\pi r}{\sinh \pi r} & \text{if } \kappa = 0, \\ \frac{\pi}{\cosh \pi r} & \text{if } \kappa = 1. \end{cases}\]

Given a sufficiently well-behaved function \( h \), we may multiply both sides of the pre-Kuznetsov formula for \( \kappa = 0 \) by

\[\frac{1}{2} \left( h \left( r + \frac{i}{2} \right) + h \left( r - \frac{i}{2} \right) \right) \cosh \pi r\]

and then integrate both sides from \(-\infty\) to \(\infty\) with respect to \( r \). This yields the following Kuznetsov formula.

**Theorem 2.2** ([BHM07, Section 2.1.4], [IK04, Theorem 16.3], [KL13, Equation (7.32)]). Let \( \delta > 0 \), and let \( h \) be a function that is even, holomorphic in the vertical strip \(|\Im(t)| \leq 1/2 + \delta\), and satisfies \( h(t) \ll (|t| + 1)^{-2-\delta} \). Then

\[
\sum_{f \in \mathcal{B}_n(q, \chi)} 4\pi \sqrt{mn\pi f(m)\rho_f(n)} \frac{h(t_f)}{\cosh \pi t_f}
\]

\[+ \sum_{a} \int_{-\infty}^{\infty} \frac{\sqrt{mn\pi a(m, t, \chi)\rho_a(n, t, \chi)} h(t)}{\cosh \pi t} \, dt
\]

\[= \delta_{m,n} g_0 + \sum_{c=1}^{\infty} \frac{S_h(m, n; c)}{c} g_0 \left( \frac{4\pi \sqrt{mn}}{c} \right),\]

where

\[g_0 := \frac{1}{\pi} \int_{-\infty}^{\infty} rh(r) \tanh \pi r \, dr,
\]

\[g_0(x) := 2i \int_{-\infty}^{\infty} J_{2\kappa}(x) \cosh \pi r \, dr.
\]

The left-hand side of the Kuznetsov formula is called the spectral side; the first term is the contribution from the discrete spectrum, while the second term is the contribution from the continuous spectrum. The right-hand side of the Kuznetsov formula is called the geometric side; the first term is the delta term and the second term is the Kloosterman term.
3. Decomposition of Spaces of Modular Forms

3.1. Eisenstein Series and Hecke Operators. The continuous spectrum $E_n(q, \chi)$ consists of functions $g \in L^2(T_0(q) \setminus \mathbb{H}, \kappa, \chi)$ having the spectral expansion (1). We may instead choose a different spanning set of Eisenstein series in this spectral expansion; in place of the set of Eisenstein series $E_n(z, s, \chi)$ with $\chi$ a singular cusp, we may instead choose a spanning set of Eisenstein series of the form $E(z, s, f)$ with Fourier expansion

$$c_1 f(t) g_{1/2+it} + c_2 f(t) g_{1/2-it} + \sum_{n=\infty}^{\infty} \rho_f(n, t, \chi) W_{\text{sgn}(n)} \frac{1}{\pi} \sqrt{n} e(\pi n y)e(nx)$$

for $s = 1/2 + it$ with $t \in \mathbb{R}$, where $B(\chi_1, \chi_2) \ni f$ with $\chi_1 \chi_2 = \chi$ is some finite set depending on $\chi_1, \chi_2$ corresponding to an orthonormal basis in the space of the induced representation constructed out of the pair $(\chi_1, \chi_2)$; see [BHM07, Section 2.1.1] or [KL13, Chapter 5]. For our purposes, we need not be more specific about $B(\chi_1, \chi_2)$, other than noting that for each $f \in B(\chi_1, \chi_2)$, the Eisenstein series $E(z, 1/2 + it, f)$ is an eigenfunction of the Hecke operators $T_n$ for $(n, q) = 1$, where for $g: \mathbb{H} \to \mathbb{C}$ a periodic function of period one,

$$(T_n g)(z) := \frac{1}{\sqrt{n}} \sum_{a \equiv n} \chi(a) \sum_{b \equiv n} g \left( \frac{az + b}{d} \right).$$

The Hecke eigenvalues of $E(z, 1/2 + it, f)$ are given by

$$\lambda_f(n, t) = \sum_{a \equiv n} \chi(a) a^{it} \chi_2(b) b^{-it}.$$ 

So for $f \in B(\chi_1, \chi_2)$,

(2) $\lambda_f(m, t) \lambda_f(n, t) = \sum_{d | (m, n)} \chi(d) \lambda_f \left( \frac{mn}{d^2}, t \right),$

(3) $\overline{\lambda_f}(n, t) = \overline{\chi(n) \lambda_f(n, t)},$

(4) $\rho_f(1, t) \lambda_f(n, t) = \sqrt{n} \rho_f(n, t)$

whenever $m, n \geq 1$ with $(mn, q) = 1$ and $s = 1/2 + it$.

Lemma 3.1. For any prime $p \nmid q$ and positive integer $\ell$, we have that

(5) $|\lambda_f(p, t)|^{2\ell} = \sum_{j=0}^{\ell} \alpha_{2j, 2\ell} \chi(p)^j \lambda_f \left( p^{2j}, t \right)$

for any $f \in B(\chi_1, \chi_2)$ and $s = 1/2 + it$, where

(6) $\alpha_{2j, 2\ell} = \frac{2j + 1}{\ell + j + 1} \binom{2\ell}{\ell + j} = \begin{cases} \binom{2\ell}{\ell - j} - \binom{2\ell}{\ell - j - 1} & \text{if } 0 \leq j \leq \ell - 1, \\ \binom{2\ell}{1} & \text{if } j = \ell, \end{cases}$

so that each $\alpha_{2j, 2\ell}$ is positive and satisfies

(7) $\sum_{j=0}^{\ell} \alpha_{2j, 2\ell} = \binom{2\ell}{\ell} \leq 2^{2\ell}.$

Proof. That (7) follows from (6) is clear. For (5), we have that

$\overline{\chi(p)^j} \lambda_f \left( p^{2j}, t \right) = U_j \left( \frac{\overline{\chi(p)^j} \lambda_f \left( p^{2j}, t \right)}{2} \right)$,
where $U_j$ is the $j$-th Chebyshev polynomial of the second kind, because $U_j$ satisfies $U_0(x/2) = 1$, $U_1(x/2) = x$, and the recurrence relation

$$U_{j+1} \left( \frac{x}{2} \right) = xU_j \left( \frac{x}{2} \right) - U_{j-1} \left( \frac{x}{2} \right)$$

for all $j \geq 1$, and $\chi(p)^{j/2} \lambda_f(p^j, t)$ satisfies the same recurrence relation from (2). Since

$$\frac{2}{\pi} \int_{-1}^{1} U_j(x) U_k(x) \sqrt{1 - x^2} \, dx = \delta_{j,k},$$

we have that

$$x^{2\ell} = \sum_{j=0}^{2\ell} \alpha_{j, 2\ell} U_j \left( \frac{x}{2} \right),$$

where

$$\alpha_{j, 2\ell} = \frac{2^{2\ell+1}}{\pi} \int_{-1}^{1} x^{2\ell} U_j(x) \sqrt{1 - x^2} \, dx.$$  

This vanishes if $j$ is odd as $U_j(-x) = (-1)^j U_j(x)$, while for $j$ even we have the identity (6) from [GR07, 7.311.2]. Combined with (3), this proves (5).

\[ \square \]

3.2. Atkin–Lehner Decomposition for $\Gamma_0(q)$. Similarly, we may choose a basis of $A_\kappa(q, \chi)$ consisting of Hecke eigenforms. Let $B_\kappa(q, \chi)$ denote the set of newforms of weight $\kappa$, level $q$, and nebentypus $\chi$, and let $A_\kappa^*(q, \chi)$ denote the subspace spanned by these newforms. Recall that a newform $f \in B_\kappa^*(q, \chi)$ is an eigenfunction of the weight $\kappa$ Laplacian $\Delta_\kappa$ with eigenvalue $1/4 + t_f^2$ and of every Hecke operator $T_n$, $n \geq 1$, with eigenvalue $\lambda_f(n)$, as well as the operator $Q_{1/2+it_f, \kappa}$ as defined in [DF102, Section 4], with eigenvalue $\epsilon_f \in \{-1, 1\}$; we say that $f$ is even if $\epsilon_f = 1$ and $f$ is odd if $\epsilon_f = -1$.

The Atkin–Lehner decomposition states that

$$A_\kappa(q, \chi) = \bigoplus_{q' \equiv 0 \pmod{q_0}} \bigoplus_{d|q} \iota_{d, q', \kappa} A_\kappa^*(q', \chi),$$

where $\iota_{d, q', \kappa} : A_\kappa(q', \chi) \to A_\kappa(q, \chi)$ is the map $\iota_{d, q', \kappa} f(z) = f(dz)$. In particular, this gives a natural (as yet unnormalised) basis $B_\kappa(q, \chi)$ of $A_\kappa(q, \chi)$ of the form

$$B_\kappa(q, \chi) := \bigcup_{q' \equiv 0 \pmod{q_0}} \bigcup_{d|q} \iota_{d, q', \kappa} B_\kappa^*(q', \chi).$$

Each $f \in B_\kappa(q, \chi)$ is an eigenfunction of the Hecke operators $T_n$ for which $(n, q) = 1$ with eigenvalue $\lambda_f(n)$, from which one can show that

$$\lambda_f(m) \lambda_f(n) = \sum_{d|(m, n)} \chi(d) \lambda_f \left( \frac{mn}{d^2} \right),$$

$$\overline{\lambda_f(n)} = \overline{\chi(n)} \lambda_f(n),$$

$$\rho_f(1) \lambda_f(n) = \sqrt{n} \rho_f(n)$$

whenever $m, n \geq 1$ and $(mn, q) = 1$. Using (9) and (10), we have the following.

Lemma 3.2. For any prime $p \nmid q$ and positive integer $\ell$, we have that

$$\lambda_f(p) = \sum_{j=0}^{\ell} \alpha_{j, 2\ell} \left( \frac{p}{2j} \right)$$

for any $f \in B_\kappa(q, \chi)$, where once again $\alpha_{j, 2\ell}$ is given by (6).
The map \( \iota_{d,q',q} \) commutes with the weight \( k \) Laplacian \( \Delta_k \), and the Hecke operators \( T_n \) whenever \( n \geq 1 \) and \( (n, q) = 1 \). It follows that if \( g = \iota_{d,q',q}f \) for some \( f \in B_k^\ast(q', \chi) \), then \( t_g = t_f \) and \( \lambda_g(n) = \lambda_f(n) \) whenever \( n \geq 1 \) and \( (n, q) = 1 \). Note, however, that \( \rho_g(1) = 0 \) unless \( d = 1 \), in which case \( \rho_g(1) = \rho_f(1) \) and
\[
\frac{\|g\|_{L^2(\Gamma_0(q) \backslash \mathbb{H})}}{\text{vol}(\Gamma_0(q) \backslash \mathbb{H})} = \frac{\|f\|_{L^2(\Gamma_0(q') \backslash \mathbb{H})}}{\text{vol}(\Gamma_0(q') \backslash \mathbb{H})}.
\]

3.3. Explicit Kuznetsov Formula. We may use the explicit basis (8) together with (10) and (11) to rewrite the discrete part of the Kuznetsov formula. Similarly, the continuous part can be rewritten in terms of the Eisenstein spanning set \( B(\chi_1, \chi_2) \) with \( \chi_1 \chi_2 = \chi \) together with (3) and (4). This yields the following explicit versions of the pre-Kuznetsov and Kuznetsov formulas.

**Proposition 3.3.** When \( m, n \geq 1 \) with \( (mn, q) = 1 \), the pre-Kuznetsov formula has the form
\[
\sum_{q' \equiv 0 (\text{mod } q_0)} \sum_{g \in B_k^\ast(q', \chi)} 4\pi |\rho_f(1)|^2 \frac{\overline{\chi}(m)\lambda_f(m)\lambda_f(n)}{\cosh \pi(r-t_f)\cosh \pi(r+t_f)}
\]
\[
+ \sum_{\chi \mid q} \sum_{\chi \mid \chi_2} \int_{-\infty}^{\infty} |\rho_f(1, t)|^2 \frac{\overline{\chi}(m)\lambda_f(m, t)\lambda_f(n, t)}{\cosh \pi(r-t)\cosh \pi(r+t)} \, dt
\]
\[
= \frac{\Gamma(1 - \frac{r}{2} + ir)}{\pi^2} \left( \delta_{mn} + \sum_{c \equiv 0 (\text{mod } q)} S_\chi(m, n; c) \frac{\pi}{c} I_\kappa \left( \frac{4\pi\sqrt{mn}}{c}, r \right) \right)
\]
for \( \kappa \in \{0, 1\} \), while the Kuznetsov formula for \( \kappa = 0 \) has the form
\[
\sum_{q' \equiv 0 (\text{mod } q_0)} \sum_{g \in B_k^\ast(q', \chi)} 4\pi |\rho_f(1)|^2 \frac{\overline{\chi}(m)\lambda_f(m)\lambda_f(n)h(t_f)}{\cosh \pi t_f}
\]
\[
+ \sum_{\chi \mid q} \sum_{\chi \mid \chi_2} \int_{-\infty}^{\infty} |\rho_f(1, t)|^2 \frac{\overline{\chi}(m)\lambda_f(m, t)\lambda_f(n, t)h(t)}{\cosh \pi t} \, dt
\]
\[
= \delta_{mn}g_0 + \sum_{c \equiv 0 (\text{mod } q)} S_\chi(m, n; c) \frac{\pi}{c} g_0 \left( \frac{4\pi\sqrt{mn}}{c}, r \right),
\]
where each \( f \in B_k^\ast(q', \chi) \) is normalised such that
\[
\|f\|_{L^2(\Gamma_0(q') \backslash \mathbb{H})} = \frac{\text{vol}(\Gamma_0(q') \backslash \mathbb{H})}{\text{vol}(\Gamma_0(q) \backslash \mathbb{H})}.
\]

3.4. Atkin–Lehner Decomposition for \( \Gamma_1(q) \). We also recall the decomposition
\[
\mathcal{A}_\kappa(\Gamma_1(q)) = \bigoplus_{\chi \equiv \chi_0 (\text{mod } q)} \mathcal{A}_\kappa(q, \chi),
\]
which follows from the fact that \( \Gamma_1(q) \) is a normal subgroup of \( \Gamma_0(q) \) with quotient group isomorphic to \( (\mathbb{Z}/q\mathbb{Z})^\kappa \), noting that \( \mathcal{A}_\kappa(q, \chi) = \{0\} \) if \( \chi(-1) \neq (-1)^\kappa \). From this, we obtain the natural basis of \( \mathcal{A}_\kappa(\Gamma_1(q)) \) given by
\[
\mathcal{B}_\kappa(\Gamma_1(q)) := \bigcup_{\chi \equiv \chi_0 (\text{mod } q)} \bigcup_{q' \equiv 0 (\text{mod } q_0)} \bigcup_{d | q} \iota_{d,q',q} B_k^\ast(q', \chi).
\]
3.5. Atkin–Lehner Decomposition for $\Gamma(q)$. A similar decomposition also holds for $A_\kappa(\Gamma(q))$. In this case, the fact that

$$\Gamma_0\left(q^2\right) \cap \Gamma_1(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : a, d \equiv 1 \pmod{q}, \ c \equiv 0 \pmod{q^2} \right\}$$

implies that

$$A_\kappa(\Gamma(q)) = \iota_{q^{-1}} A_\kappa(\Gamma_0\left(q^2\right) \cap \Gamma_1(q)),$$

where $\iota_{q^{-1}} : A_\kappa(\Gamma_0\left(q^2\right) \cap \Gamma_1(q)) \to A_\kappa(\Gamma(q))$ is the map $\iota_{q^{-1}}f(z) = f(q^{-1}z)$. As $\Gamma_0\left(q^2\right) \cap \Gamma_1(q)$ is a normal subgroup of $\Gamma_0\left(q^2\right)$ with quotient group isomorphic to $(\mathbb{Z}/q\mathbb{Z})^\times$, we obtain the decomposition

$$A_\kappa(\Gamma(q)) = \bigoplus_{\chi \mod{q}} \iota_{q^{-1}} A_\kappa\left(q^2, \chi\right),$$
	hereby allowing us to choose an explicit basis of $A_\kappa(\Gamma(q))$ of the form

$$B_\kappa(\Gamma(q)) := \bigcup_{\chi \mod{q}} \bigcup_{q' | q^2} \bigcup_{q'' \equiv 0 \pmod{q_0}} \iota_{q^{-1}} B_\kappa^q\left(q', \chi\right).$$

4. Bounds for Fourier Coefficients of Newforms

In the Kuznetsov formula (14), the Fourier coefficients $|\rho_f(1)|^2$ appear naturally. To remove these weights, we obtain lower bounds for $|\rho_f(1)|$ if $f$ is a newform. Such bounds are well-known, appearing in some generality in [DFI02, Equation (7.16)]; nevertheless, we take this opportunity to correct some of the minor numerical errors in this proof, as well as greatly streamline the proof via the recent work of Li [Li09] on obtaining upper bounds for $L$-functions at the edge of the critical strip.

For $f \in A_\kappa(q, \chi)$, we define

$$\nu_f := \Gamma\left(\frac{1 + \kappa}{2} + it_f\right) \Gamma\left(\frac{1 + \kappa}{2} - it_f\right) |\rho_f(1)|^2.$$

Note that

$$\Gamma\left(\frac{1 + \kappa}{2} + it\right) \Gamma\left(\frac{1 + \kappa}{2} - it\right) = \begin{cases} \frac{\pi}{\cos \pi t} & \text{if } \kappa = 0, \\ \frac{\pi t}{\sinh \pi t} & \text{if } \kappa = 1. \end{cases}$$

**Lemma 4.1.** Suppose that $f \in B_\kappa^q(q, \chi)$. Then

$$\frac{\|f\|^2_{L^2(\Gamma_0(q) \backslash \mathbb{H})}}{\text{vol}(\Gamma_0(q) \backslash \mathbb{H})} = \nu_f \sum_{s=1}^{\infty} \frac{|\lambda_f(n)|^2}{n^s}.$$

The work of Li [Li09, Corollary 1] shows that

$$\sum_{s=1}^{\infty} \frac{|\lambda_f(n)|^2}{n^s} \ll \exp\left(C \left(\log q \left(\frac{1}{4} + t_f^2\right)\right)^{1/4} \left(\log \log q \left(\frac{1}{4} + t_f^2\right)\right)^{1/2}\right)$$

for some absolute constant $C > 0$.

**Corollary 4.2.** Suppose that $f \in B_\kappa^q(q, \chi)$. Then

$$\nu_f \gg \varepsilon \frac{\|f\|^2_{L^2(\Gamma_0(q) \backslash \mathbb{H})}}{\text{vol}(\Gamma_0(q) \backslash \mathbb{H})} \left(q \left(\frac{1}{4} + t_f^2\right)\right)^{-\varepsilon}.$$


Proof of Lemma 4.1. We let
\[ E(z, s) := \sum_{\gamma \in \Gamma \setminus \Gamma_0(q)} \Im(\gamma z)^s \]
and define
\[ F(s) := \langle E(\cdot, s)f,f \rangle = \int_{\Gamma_0(q) \backslash \mathbb{H}} E(z, s)|f(z)|^2 \, d\mu(z). \]

Unfolding the integral and using Parseval’s identity,
\[
F(s) = \int_0^\infty y^{s-1} \sum_{n=-\infty}^{\infty} |\rho_f(n)|^2 W_{\text{sgn}(n)\nu}(4\pi|n|y)^2 \frac{dy}{y}.
\]

From (11) and the fact from [DFI02, Equation (4.70)] that
\[ \rho_f(-n) = \epsilon_f \frac{\Gamma\left(\frac{1+\kappa}{2} + it_f\right)}{\Gamma\left(\frac{1+\kappa}{2}ight)} \rho_f(n) \]
for \( n \geq 1 \), where \( \epsilon_f \in \{-1, 1\} \), we find that \( F(s) \) is equal to
\[
\frac{|\rho_f(1)|^2}{(4\pi)^{s-1}} \sum_{n=1}^{\infty} \frac{|\lambda_f(n)|^2}{n^s} \int_0^\infty y^{s-1} \left( W_{\frac{\lambda_f}{4\pi\nu}}(y)^2 + \frac{\Gamma\left(\frac{1+\kappa}{2} + it_f\right)}{\Gamma\left(\frac{1+\kappa}{2}ight)} \right)^2 \frac{dy}{y}.
\]

We have by [GR07, 7.611.4] that for \( \kappa \in \mathbb{C} \) and \( -1/2 < \Re(it) < 1/2 \),
\[
\int_0^\infty W_{\frac{\lambda_f}{4\pi\nu}}(y)^2 \frac{dy}{y} = \frac{\pi}{\sin 2\pi it} \frac{\psi\left(\frac{1+\kappa}{2} + it\right) - \psi\left(\frac{1+\kappa}{2} - it\right)}{\Gamma\left(\frac{1+\kappa}{2} + it\right) \Gamma\left(\frac{1+\kappa}{2} - it\right)}
\]
where \( \psi \) is the digamma function; note that a slightly erroneous version of this appears in [DFI02, Equation (19.6)]. By the gamma and digamma reflection formulæ, we find that
\[
\int_0^\infty \left( W_{\frac{\lambda_f}{4\pi\nu}}(y)^2 + \frac{\Gamma\left(\frac{1+\kappa}{2} + it_f\right)}{\Gamma\left(\frac{1+\kappa}{2}ight)} \right)^2 \frac{dy}{y} = \Gamma\left(\frac{1+\kappa}{2} + it_f\right) \Gamma\left(\frac{1+\kappa}{2} - it_f\right)
\]
assuming that \( t_f \in [0, \infty) \) if \( \kappa = 1 \) and \( t_f \in [0, \infty) \) or \( it_f \in (0, 1/2) \) if \( \kappa = 0 \). The result then follows by taking the residue at \( s = 1 \) of \( F(s) \) and using the fact that the residue of \( E(z, s) \) at \( s = 1 \) is \( 1/\text{vol}(\Gamma_0(q) \backslash \mathbb{H}) \).

5. Bounds for Sums of Kloosterman Sums

We denote by
\[ S(m, n; c) := \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^\times} e\left(\frac{md + nd}{c}\right) \]
the usual Kloosterman sum with trivial character, for which the Weil bound holds:
\[ |S(m, n; c)| \leq \tau(c) \sqrt{(m, n, c)c}. \]
We also require bounds for Kloosterman sums with nontrivial character. For \( c \equiv 0 \pmod{q} \), \( m, n \geq 1 \), and \( (a, q) = 1 \), we have that
\[
\sum_{\chi \left( \frac{m,n}{c} \right)} \chi(1) S_\chi(m,n;c) \\
\cdot \left( \sum_{\chi \left( \frac{m,n}{c} \right)} \chi(-1) \chi(-d) = \frac{1}{2} \sum_{d \in \left( \frac{\mathbb{Z}}{c\mathbb{Z}} \right)^\times} \sum_{\chi \left( \frac{m,n}{c} \right)} \chi(d) \chi(-d) e \left( \frac{md + nd}{c} \right) \right).
\]
We break this up into two sums. In the second sum, we can replace \( d \) with \(-d\) and \( \chi \) with \( \overline{\chi} \) to see that
\[
\sum_{\chi \left( \frac{m,n}{c} \right)} \chi(1) S_\chi(m,n;c) = \begin{cases} \varphi(q) \Re \left( S_{a(q)}(m,n;c) \right) & \text{if } \kappa = 0, \\
\iota \varphi(q) \Im \left( S_{a(q)}(m,n;c) \right) & \text{if } \kappa = 1,
\end{cases}
\]
where we set
\[
S_{a(q)}(m,n;c) := \sum_{d \in \left( \frac{\mathbb{Z}}{c\mathbb{Z}} \right)^\times} e \left( \frac{md + nd}{c} \right).
\]
If \( c = c_1c_2 \) with \( (c_1, c_2) = 1 \) and \( c_1c_2 \equiv 0 \pmod{q} \), then we let \( d = c_2z_1d_1 + c_1z_2d_2 \), where \( d_1 \in \left( \frac{\mathbb{Z}}{c_1\mathbb{Z}} \right)^\times \), \( d_2 \in \left( \frac{\mathbb{Z}}{c_2\mathbb{Z}} \right)^\times \), and \( c_2d_2 \equiv 1 \pmod{c_1} \), \( c_1d_1 \equiv 1 \pmod{c_2} \). By the Chinese remainder theorem,
\[
S_{a(q)}(m,n;c) = S_{a(q,c_1)}(m/n, m/n; c_1) S_{a(q,c_2)}(m/n, m/n; c_2).
\]
To bound \( S_{a(q)}(m,n;c) \), it therefore suffices to find bounds for \( S_{a(p^\beta)}(m,n;p^\beta) \) for any prime \( p \) and any \( \beta \geq \alpha \geq 1 \). The trivial bound is merely \( p^{3-\alpha} \); somewhat surprisingly, this is sufficient for our needs. Indeed, we cannot do better than this when \( \beta = \alpha \), and in our applications, this will be the dominant contribution.

**Lemma 5.1.** When \( (m,n) = 1 \), we have that
\[
\sum_{c \leq 4 \sqrt{mn}} \frac{|S(m,n;c)|}{c^{3/2}} \ll \frac{\tau(q)(\log(mn + 1))^2}{q},
\]
\[
\sum_{c \leq 4 \sqrt{mn}} \frac{|S_{a(q)}(m,n;c)|}{c^{3/2}} \ll \frac{(\log(mn + 1))^2}{q^{3/2}} \prod_{p \parallel q} \frac{1}{1 - p^{-1/2}},
\]
and
\[
\sum_{c \leq 4 \sqrt{mn}} \frac{|S_{a(q)}(m,n;c)|}{c^{3/2}} \ll \frac{(\log(mn + 1))^2}{q^2} \prod_{p \parallel q} \frac{1}{1 - p^{-1/2}}.
\]

**Proof.** Via the Weil bound, the left-hand side of (19) is bounded by
\[
\frac{\tau(q)\sqrt{|m,n,q|}}{q} \sum_{c \leq 4 \sqrt{mn}} \frac{\tau(c)\sqrt{|m,n,c|}}{c},
\]
which, if \( (m,n) = 1 \), is readily seen to be bounded by a constant multiple of
\[
\frac{\tau(q)(\log(mn + 1))^2}{q}.
\]
For (20), we write \( q = p_1^{\alpha_1} \cdots p_t^{\alpha_t} \), so that the left-hand side of (20) is

\[
\sum_{\beta_1 = \alpha_1}^{\infty} \cdots \sum_{\beta_t = \alpha_t}^{\infty} \frac{1}{p_1^{\beta_1} \cdots p_t^{\beta_t}} \sum_{c \leq 4\pi \sqrt{mn p_1^{\beta_1} \cdots p_t^{\beta_t}}} 1 \sum_{c \leq 4\pi \sqrt{\tau(m, n; c)}} 1 \sum_{c \leq 4\pi \sqrt{mn \pi_{\beta_1}^{\beta_1} \cdots \pi_{\beta_t}^{\beta_t}}} S_a(q) \left( \frac{m\pi_{\beta_1}^{\beta_1} \cdots \pi_{\beta_t}^{\beta_t}}{c} \right).
\]

Using the Weil bound for the first Kloosterman sum and the trivial bound for the second, we find that this is bounded by

\[
\sum_{\beta_1 = \alpha_1}^{\infty} \cdots \sum_{\beta_t = \alpha_t}^{\infty} \frac{1}{p_1^{\beta_1} \cdots p_t^{\beta_t}} \sum_{c \leq 4\pi \sqrt{mn}} \sum_{(c, q) = 1} \frac{\tau(c) \sqrt{(m, n; c)}}{c}.
\]

If \((m, n) = 1\), the inner sum is bounded by a constant multiple of \((\log(mm+1))^2\), and so the sum is bounded by a constant multiple of

\[
(\log(mm+1))^2 \sum_{\beta_1 = \alpha_1}^{\infty} \cdots \sum_{\beta_t = \alpha_t}^{\infty} \frac{1}{p_1^{\beta_1} \cdots p_t^{\beta_t}} \sum_{c \leq 4\pi \sqrt{mn \pi_{\beta_1}^{\beta_1} \cdots \pi_{\beta_t}^{\beta_t}}} \tau(c) \sqrt{(m, n; c)}.
\]

which yields (20) upon evaluating these geometric series. (21) follows similarly. \( \square \)

**Lemma 5.2.** When \((m, n) = 1\), we have that

\[
(22) \quad \sum_{c \leq 4\pi \sqrt{mn/c}} \frac{|S(m, n; c)|}{c^2} \left( 1 + \log \frac{c}{4\pi \sqrt{mn}} \right) \ll \frac{(\log(mm+1))^2}{(mn)^{1/4} q}.
\]

\[
(23) \quad \sum_{c \leq 4\pi \sqrt{mn/c}} \frac{|S_a(m, n; c)|}{c^2} \left( 1 + \log \frac{c}{4\pi \sqrt{mn}} \right) \ll \frac{(\log(mm+1))^2}{(mn)^{1/4} q^2} \prod_{p \mid q} \frac{1}{1 - p^{-1/2}}.
\]

\[
(24) \quad \sum_{c \leq 4\pi \sqrt{mn/c}} \frac{|S_a(m, n; c)|}{c^2} \left( 1 + \log \frac{c}{4\pi \sqrt{mn}} \right) \ll \frac{(\log(mm+1))^2}{(mn)^{1/4} q^2} \prod_{p \mid q} \frac{1}{1 - p^{-1/2}}.
\]

**Proof.** Again by the Weil bound, the left-hand side of (22) is bounded by

\[
\frac{\tau(q) \sqrt{(m, n; q)}}{q^{3/2}} \sum_{c \leq 4\pi \sqrt{mn/c}} \frac{\tau(c) \sqrt{(m, n; c)}}{c^{3/2}} \left( 1 + \log \frac{cq}{4\pi \sqrt{mn}} \right),
\]

which in turn is bounded by a constant multiple of

\[
\frac{\tau(q)(\log(mm+1))^2}{q}.
\]

With \( q = p_1^{\alpha_1} \cdots p_t^{\alpha_t} \), the left-hand side of (23) is bounded by

\[
\sum_{\beta_1 = \alpha_1}^{\infty} \cdots \sum_{\beta_t = \alpha_t}^{\infty} \frac{1}{p_1^{\beta_1} \cdots p_t^{\beta_t}} \sum_{c \leq 4\pi \sqrt{mn p_1^{\beta_1} \cdots p_t^{\beta_t}}} \sum_{(c, q) = 1} \tau(c) \sqrt{(m, n; c) \log c}.
\]

If \((m, n) = 1\), then the inner sum is bounded by a constant multiple of

\[
\frac{\sqrt{p_1^{\beta_1} \cdots p_t^{\beta_t}} \log(c) \log(mm+1)^2}{(mn)^{1/4}}.
\]
It follows that the sum is bounded by a constant multiple of
\[
\frac{(\log(mn+1))^2}{(mn)^{1/4}} \sum_{\beta_1 = \alpha_1}^\infty \cdots \sum_{\beta_k = \alpha_k}^\infty p_1^{\beta_1 - \alpha_1} \cdots p_k^{\beta_k - \alpha_k},
\]
which gives (23). The proof of (24) is analogous. \(\square\)

**Lemma 5.3.** When \((m, n) = 1\) and for all \(1/2 < \sigma \leq 1\),
\[
\left(25\right) \sum_{\substack{c=1 \atop c \equiv 0 \mod q}}^\infty \frac{|S(m,n;c)|}{e^{c+\sigma}} \leq \frac{4\pi(q)}{(2\sigma - 1)^2 q^{1/2+2\sigma}},
\]
\[
\left(26\right) \sum_{\substack{c=1 \atop c \equiv 0 \mod q}}^\infty \frac{|S(q)(m,n;c)|}{e^{c+\sigma}} \leq \frac{4}{(2\sigma - 1)^2 q^{1+\sigma}} \prod_{p|q} \frac{1}{1 - p^{-\sigma}},
\]
\[
\left(27\right) \sum_{\substack{c=1 \atop c \equiv 0 \mod q^2}}^\infty \frac{|S(q)(m,n;c)|}{e^{c+\sigma}} \leq \frac{4}{(2\sigma - 1)^2 q^{1+2\sigma}} \prod_{p|q} \frac{1}{1 - p^{-\sigma}}.
\]

**Proof.** The inequality (25) follows from the proof of [IK04, Equation (16.50)]. For (26), once again writing \(q = p_1^{\alpha_1} \cdots p_k^{\alpha_k}\) and bounding the Kloosterman sums,
\[
\sum_{\substack{c=1 \atop c \equiv 0 \mod q}}^\infty \frac{|S(q)(m,n;c)|}{e^{c+\sigma}} \leq \sum_{\substack{c=1 \atop (c,q)=1}}^\infty \frac{\tau(c) \sqrt{(m,n,c)}}{e^{c+2\sigma}} \prod_{\beta_1 = \alpha_1}^\infty \cdots \prod_{\beta_k = \alpha_k}^\infty \left(\frac{p_1^{\beta_1 - \alpha_1} \cdots p_k^{\beta_k - \alpha_k}}{e^{c+\sigma}}\right) \cdot \frac{1}{q^{1+\sigma}} \prod_{p|q} \frac{1}{1 - p^{-\sigma}}
\]
\[
= \sum_{\substack{c=1 \atop (c,q)=1}}^\infty \frac{\tau(c) \sqrt{(m,n,c)}}{e^{c+2\sigma}} \prod_{p|q} \frac{1}{1 - p^{-\sigma}}
\]
\[
\leq \zeta \left(\sigma + 1\right)^2 \frac{1}{q^{1+\sigma}} \prod_{p|q} \frac{1}{1 - p^{-\sigma}}
\]
\[
\leq \frac{4}{(2\sigma - 1)^2 q^{1+2\sigma}} \prod_{p|q} \frac{1}{1 - p^{-\sigma}}.
\]
This proves (26). The inequality (27) follows by a similar argument. \(\square\)

6. **Sarnak’s Density Theorem for Exceptional Hecke Eigenvalues**

**Lemma 6.1.** For \(T \geq 1\), let
\[
h_{\kappa,T}(t) := \frac{\pi^2}{\Gamma\left(\frac{1+\kappa}{2} + it\right) \Gamma\left(\frac{1+\kappa}{2} - it\right)} \int_0^T r \left|G\left(1 - \frac{k}{2} + ir\right)\right|^{-2} \frac{\cosh \pi(r-t) \cosh \pi(r+t)}{\cosh \pi(r-t) \cosh \pi(r+t)} dr
\]
\[
= \begin{cases} 
\cosh \pi t \int_0^T \frac{\sinh \pi r}{\cosh \pi(r-t) \cosh \pi(r+t)} dr & \text{if } \kappa = 0, \\
\sinh \pi t \int_0^T \frac{r \cosh \pi r}{\cosh \pi(r-t) \cosh \pi(r+t)} dr & \text{if } \kappa = 1.
\end{cases}
\]

Then \(h_{\kappa,T}(t)\) is positive for all \(t \in \mathbb{R}\) and additionally, should \(\kappa\) be equal to 0, for it \(\in (-1/2, 1/2)\). Furthermore, \(h_{\kappa,T}(t) \gg 1\) for \(t \in [0,T]\).

**Proof.** Using the fact that
\[
\cosh \pi(r-t) \cosh \pi(r+t) = \cosh^2 \pi t + \sinh^2 \pi r = \sinh^2 \pi t + \cosh^2 \pi r,
\]
it is clear that \(h_{\kappa,T}(t)\) is positive for all \(t \in \mathbb{R}\) and additionally, should \(\kappa\) be equal to 0, if \(it \in (-1/2, 1/2)\).
For $\kappa = 0$, we have that
\[
h_{0,T}(t) = \frac{\cosh \pi t}{\pi} \int_1^{\cosh \pi T} \frac{1}{x^2 + \sinh^2 \pi t} \, dx
= \frac{\coth \pi t}{\pi} \int_0^{\sinh \pi T} \frac{\arctan \left( \frac{\sinh \pi t (\cosh \pi T - 1)}{\sinh^2 \pi t + \cosh \pi T} \right)}{\sinh \pi t (\cosh \pi T - 1)} \, dx,
\]
where the second line follows from the arctangent subtraction formula. The first expression shows that $h_{0,T}(t) \gg 1$ when $t$ is small, while when $t$ is large, the argument of arctan is essentially
\[
e^{\pi(T+t)} - e^{\pi t} - e^{2\pi t} + e^{\pi(T+t/2)},
\]
and this is bounded from below provided that $t \leq T$, so that again $h_{0,T}(t) \gg 1$.

For $\kappa = 1$, we can similarly show via integration by parts that
\[
h_{1,T}(t) = \frac{\sinh \pi t}{\pi^2 t} \int_0^{\sinh \pi T} \frac{\arcsinh x}{x^2 + \cosh^2 \pi t} \, dx
= \frac{\tanh \pi t}{\pi^2 t} \int_0^{\sinh \pi T} \frac{\arctan \left( \frac{\sinh \pi T - \sinh \frac{\pi t}{2}}{\cosh ^2 \pi t + \sinh \pi T \sinh \frac{\pi t}{2}} \right)}{\sqrt{x^2 + 1}} \, dx.
\]
The first expression shows that $h_{1,T}(t) \gg 1$ when $t$ is small, while when $t$ is large, we break up the second expression into two integrals: one from $0$ to $\sinh \frac{\pi t}{2}$ and one from $\sinh \frac{\pi t}{2}$ to $\sinh \pi T$. Trivially bounding the numerator in each integral, we find that
\[
h_{1,T}(t) \geq \frac{\tanh \pi t}{2\pi} \left( \arctan \frac{\sinh \pi T}{\cosh \pi t} - \arctan \frac{\sinh \frac{\pi t}{2}}{\cosh \pi t} \right)
= \frac{\tanh \pi t}{2\pi} \arctan \frac{\cosh \pi t \left( \sinh \pi T - \sinh \frac{\pi t}{2} \right)}{\cosh^2 \pi t + \sinh \pi T \sinh \frac{\pi t}{2}}.
\]
The argument of arctan is essentially
\[
e^{\pi(T+t)} - e^{\pi t/2} + e^{2\pi t} - e^{\pi(T+t/2)},
\]
and this is bounded from below provided that $t \leq T$, while $\tanh \pi t$ is bounded from below provided that $t$ is larger than some fixed constant. It follows again that $h_{1,T}(t) \gg 1$.\qed

**Lemma 6.2.** For $\kappa \in \{0,1\}$ and $T > 0$, we have the bound
\[
\int_0^T r I_\kappa(a,r) \, dr \ll \begin{cases} \sqrt{a} & \text{if } a \geq 1, \\ a \left(1 + \log \frac{1}{a}\right) & \text{if } 0 < a < 1 \end{cases}
\]
uniformly in $T$.

**Proof.** From [Kuz81, Equation (5.13)], we have that
\[
\int_0^T r I_0(a,r) \, dr = a \int_0^\infty \frac{\tanh \xi}{\xi} (1 - \cos 2T\xi) \sin(a \cosh \xi) \, d\xi.
\]
Similarly, using the fact that $K_{2r}(\zeta) = \int_0^\infty e^{-\zeta \cosh \xi} \cos 2r\xi \, d\xi$ for $r \in \mathbb{R}$ and $\Re(\zeta) > 0$ from [GR07, 8.432.1], we have that
\[
\int_0^T r I_1(a,r) \, dr = -2a \int_0^\infty \int_0^T r \cos 2r\xi \, dr \int_{-i}^i e^{-\zeta a \cosh \xi} \, d\zeta \, d\xi.
\]
Evaluating each of the inner integrals and then integrating by parts, we find that
\[
\int_0^T r I_1(a, r) \, dr = i a \int_0^\infty \frac{\tanh \xi}{\xi} (1 - \cos 2T \xi) \cos(a \cosh \xi) \, d\xi
- i \int_0^\infty \frac{\tanh \xi}{\xi} (1 - \cos 2T \xi) \frac{\sin(a \cosh \xi)}{\cosh \xi} \, d\xi.
\]
From here, one can show via stationary phase on subintervals of \((0, \infty)\) that \(\int_0^T r I_0(a, r) \, dr\) and the first term in the above expression for \(\int_0^T r I_1(a, r) \, dr\) both are bounded by a constant multiple of
\[
\sqrt{a} \quad \text{if } a \geq 1,
\]
\[
a \left(1 + \log \frac{1}{a}\right) \quad \text{if } 0 < a < 1;
\]
see [Kuz81, Equation (5.14)]. The second term in the expression for \(\int_0^T r I_1(a, r) \, dr\) is uniformly bounded for \(a \geq 1\), so we need only consider when \(0 < a < 1\). In this case, the fact that \(|\sin x| \leq \min\{1, |x|\}\) for \(x \in \mathbb{R}\) implies that this is bounded by
\[
2a \int_0^{\log \frac{1}{a}} \frac{\tanh \xi}{\xi} \, d\xi + 2 \int_{\log \frac{1}{a}}^\infty \frac{\tanh \xi}{\cosh \xi} \, d\xi \ll a \left(1 + \log \frac{1}{a}\right).
\]

**Proof of Theorem 1.1 for \(\Gamma_1(q)\).** By Rankin’s trick,
\[
\# \{f \in B_\kappa(\Gamma_1(q)) : t_f \in [0, T], |\lambda_f(p)| \geq \alpha_p \text{ for all } p \in \mathcal{P}\} \leq \prod_{p \in \mathcal{P}} \alpha_p^{-2\ell_p} \sum_{t_f \in [0, T]} \prod_{p \in \mathcal{P}} |\lambda_f(p)|^{2\ell_p}
\]
for any nonnegative integers \(\ell_p\) to be chosen. Using the explicit basis (15) of \(\mathcal{A}_\kappa(\Gamma_1(q))\) together with the lower bound (17) for \(\nu_f\),
\[
\sum_{t_f \in [0, T]} \prod_{f \in B_\kappa(\Gamma_1(q))} |\lambda_f(p)|^{2\ell_p} = \sum_{\chi \pmod{q} \chi(1) = (-1)^n} \left(\sum_{q' \mid q, q' \equiv 0 \pmod{q}} \sum_{t_f \in [0, T]} \prod_{p \in \mathcal{P}} |\lambda_f(p)|^{2\ell_p}\right) \ll q^{1 + \varepsilon} T^{\varepsilon} \sum_{\chi \pmod{q} \chi(1) = (-1)^n} \left(\sum_{q' \mid q, q' \equiv 0 \pmod{q}} \sum_{t_f \in [0, T]} \nu_f \prod_{p \in \mathcal{P}} |\lambda_f(p)|^{2\ell_p}\right).
\]

We take \(m = 1\) and \(n = \prod_{p \in \mathcal{P}} p^{2j_p}\) in the pre-Kuznetsov formula (13), multiply both sides by \(\prod_{p \in \mathcal{P}} \alpha_{2j_p, 2\ell_p} \chi(p)^{j_p}\), and sum over all \(0 \leq j_p \leq \ell_p\), over all \(p \in \mathcal{P}\), and over all Dirichlet characters \(\chi \pmod{q}\) satisfying \(\chi(-1) = (-1)^n\). We then multiply both sides by \(\pi^2 r |\Gamma \left(1 - \frac{s}{2} + ir\right)|^{-2}\) and integrate both sides with respect to \(r\) from 0 to \(T\).

On the spectral side, (2), (5), and Lemma 6.1 allow us to use positivity to discard the contribution from the continuous spectrum, while we may discard the contribution of the discrete spectrum with \(t \notin [0, T]\) via (9), (12), and Lemma 6.1, so that the spectral side is bounded from below by a constant multiple of
\[
\sum_{\chi \pmod{q} \chi(1) = (-1)^n} \left(\sum_{q' \mid q, q' \equiv 0 \pmod{q}} \sum_{t_f \in [0, T]} \nu_f \prod_{p \in \mathcal{P}} |\lambda_f(p)|^{2\ell_p}\right).
\]

On the geometric side, we only pick up the delta term when \(j_p = 0\) for all \(p \in \mathcal{P}\), in which case the term is bounded by a constant multiple of \(q T^2 \prod_{p \in \mathcal{P}} \alpha_{0, 2\ell_p}\). For
We then take
\[
\frac{\varphi(q)}{\pi} \sum_{p \in \mathcal{P}} \prod_{j_p = 0}^{\ell_p} \alpha_{2j_p, 2\ell_p} \sum_{c=1}^{\infty} \sum_{c \equiv 0 \pmod{q}} \Re \left( S_{\prod_{p \in \mathcal{P}} p^{p_p}(q)} \left( 1, \prod_{p \in \mathcal{P}} p^{2j_p}; c \right) \right) c
\]
and \((\ast)\) to bound the summation over \(c\), so that the Kloosterman term is bounded by a constant multiple of
\[
\prod_{p \in \mathcal{P}} \frac{1}{1 - p^{j_p - 1/2}} \sum_{p \in \mathcal{P}} \prod_{j_p = 0}^{\ell_p} \alpha_{2j_p, 2\ell_p} p^{j_p/2} \left( \log \left( \prod_{p \in \mathcal{P}} p^{j_p/2} + 1 \right) \right)^2.
\]
We bound the summation over \(j_p\) and over \(p \in \mathcal{P}\) via (7), thereby obtaining
\[
\# \{ f \in B_n(\Gamma(q)) : t_f \in [0, T], \; |\lambda_f(p)| \geq \alpha_p \text{ for all } p \in \mathcal{P} \}
\]
\[
\ll_{\varepsilon} q^{1/2 + \varepsilon} T^2 \prod_{p \in \mathcal{P}} \left( \frac{\alpha_p}{2} \right)^{-2\ell_p} \left( q^{3/2} T^2 + \prod_{p \in \mathcal{P}} p^{\ell_p/2} \left( \log \prod_{p \in \mathcal{P}} p^{\ell_p/2} \right)^2 \prod_{p \in \mathcal{P}} \frac{1}{1 - p^{j_p - 1/2}} \right).
\]
It remains to take
\[
\ell_p = \left\lfloor \frac{\mu_p \log \left( \text{vol} (\Gamma(1) \backslash \mathbb{H})^{3/2} T^4 \right)}{\log p} \right\rfloor.
\]

**Proof of Theorem 1.1 for \(\Gamma(q)\).** We use (16), (21), and (24) in place of (15), (20), and (23), thereby finding that
\[
\# \{ f \in B_n(\Gamma(q)) : t_f \in [0, T], \; |\lambda_f(p)| \geq \alpha_p \text{ for all } p \in \mathcal{P} \}
\]
\[
\leq \prod_{p \in \mathcal{P}} \alpha_p^{-2\ell_p} \sum_{f \in B_n(\Gamma(q))} \prod_{t_f \in [0, T]} |\lambda_f(p)|^{2\ell_p},
\]
with
\[
\sum_{f \in B_n(\Gamma(q))} \prod_{t_f \in [0, T]} |\lambda_f(p)|^{2\ell_p} = \sum_{\chi \pmod{q}} \sum_{\chi(-1) = (-1)^k} \sum_{q' \equiv 0 \pmod{q_\chi}} \chi(q') \prod_{t_f \in [0, T]} |\lambda_f(p)|^{2\ell_p}
\]
\[
\ll_{\varepsilon} q^{1 + \varepsilon} T^2 \prod_{p \in \mathcal{P}} q^{2\ell_p} \left( q^{3/2} T^2 + \prod_{p \in \mathcal{P}} p^{\ell_p/2} \left( \log \prod_{p \in \mathcal{P}} p^{\ell_p/2} \right)^2 \prod_{p' \in \mathcal{P}} \frac{1}{1 - p'^{j_p - 1/2}} \right).
\]
We then take
\[
\ell_p = \left\lfloor \frac{\mu_p \log \left( \text{vol} (\Gamma(1) \backslash \mathbb{H})^{4/3} T^4 \right)}{\log p} \right\rfloor.
\]
1.1

The result follows upon taking

with

\[
\sum_{f \in B_n(\Gamma_0(q)) \cap [0,T]} \prod_{p \in P} |\lambda_f(p)|^{2\ell_p} = \sum_{q \mid q \in P} \sum_{f \in B_n(\Gamma_0(q'))} \tau\left(\frac{q}{q'}\right) \prod_{p \in P} |\lambda_f(p)|^{2\ell_p}
\]

\[
\ll \epsilon q^\epsilon T^\epsilon \prod_{p \in P} q^{2\ell_p} \left(q^2 + \tau(q) \prod_{p \in P} p^{\ell_p/2} \left(\log \prod_{p \in P} p^{\ell_p/2}\right)^2\right).
\]

The result follows upon taking

\[
\ell_p = \left\lfloor \frac{2\mu_p \log \left(\text{vol}(\Gamma_0(q) \setminus \mathbb{H}) T^2\right)}{\log p} \right\rfloor.
\]

7. HUXLEY’S DENSITY THEOREM FOR EXCEPTIONAL LAPLACIAN EIGENVALUES

Proof of Theorem 1.2 for \(\Gamma_1(q)\). We again use Rankin’s trick with nonnegative integers \(\ell_p\) and a positive real number \(X \geq 1\) to be chosen:

\[
\# \{f \in B_0(\Gamma_1(q)) : it_f \in (\alpha_0, 1/2), |\lambda_f(p)| \geq \alpha_p \text{ for all } p \in P\} \leq X^{-2\alpha_0} \prod_{p \in P} \alpha_p^{-2\ell_p} \sum_{f \in B_0(\Gamma_1(q)) \cap [0,1/2]} X^{2it_f} \prod_{p \in P} |\lambda_f(p)|^{2\ell_p}.
\]

Again using (15) and (17),

\[
\sum_{f \in B_0(\Gamma_1(q)) \cap [0,1/2]} X^{2it_f} \prod_{p \in P} |\lambda_f(p)|^{2\ell_p} = \sum_{\chi \chi(x \equiv 1) \chi^{-1} \equiv 1} \sum_{q' \mid q} \left(\frac{q}{q'}\right) \prod_{p \in P} |\lambda_f(p)|^{2\ell_p}
\]

\[
\ll \epsilon q^{1+\epsilon} \sum_{\chi \chi(x \equiv 1) \chi^{-1} \equiv 1} \sum_{q' \mid q} \nu_f X^{2it_f} \prod_{p \in P} |\lambda_f(p)|^{2\ell_p}.
\]

We take \(m = 1, n = \prod_{p \in P} p^{2\ell_p} \), and

\[
h(t) = h_X(t) = \left(\frac{X^it + X^{-it}}{t^2 + 1}\right)^2.
\]

in the Kuznetsov formula (14), multiply both sides by \(\prod_{p \in P} \alpha_{2p,2\ell_p} x(p)\), and sum over all \(0 \leq \ell_p \leq \ell_p\), over all \(p \in P\), and over all even Dirichlet characters modulo \(q\). On the spectral side, we discard all but the discrete spectrum for which
it \in (0, 1/2) via positivity, so that the spectral side is bounded from below by a constant multiple of
\[
\sum_{\chi \pmod{q}} \sum_{\substack{q \mid q' \mid \chi \pmod{q} \mid q' \equiv 0 \pmod{q} \mid \nu \in \mathcal{B}_0(q', \chi) \mid \nu \in (0, 1/2)}} \nu f X^{2t_f} \prod_{p \in \mathcal{P}} |\lambda_f(p)|^{2t_p}.
\]

We only pick up the delta term on the geometric side when \( j_p = 0 \) for all \( p \in \mathcal{P} \), in which case the term is bounded by a constant multiple of \( q \prod_{p \in \mathcal{P}} 2^{2t_p} \). We write the Kloosterman term in the form
\[
\frac{\varphi(q)}{2\pi i} \sum_{\nu \in \mathcal{B}_0(q', \chi)} \prod_{c=1}^{t_p} \alpha_{2j_p, 2t_p} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\Re \left( S_{\prod_{p \in \mathcal{P}} p^{j_p} (q)} \left( 1, \prod_{p \in \mathcal{P}} p^{2j_p} ; c \right) \right)}{c} \prod_{p \in \mathcal{P}} \frac{p^{j_p} \sigma}{\log \nu} ds.
\]
for any \( 1/2 < \sigma \leq 1 \). We have, via [GR07, 8.411.4], the bound
\[
J_s(x) \ll \frac{x^\sigma}{\Gamma (s + \frac{1}{2})} \ll e^{\pi |s|/2} \left( \frac{x}{|s|} \right)^\sigma,
\]
and so the integral in the Kloosterman term is bounded by a constant multiple of
\[
\prod_{p \in \mathcal{P}} p^{j_p} \sum_{c=1}^{t_p} \frac{\left| S_{\prod_{p \in \mathcal{P}} p^{j_p} (q)} \left( 1, \prod_{p \in \mathcal{P}} p^{2j_p} ; c \right) \right|^2}{c^{1+\sigma}} \int_{\sigma/2-i\infty}^{\sigma/2+i\infty} \left| \alpha_{2j_p, 2t_p} \right|^2 \prod_{p \in \mathcal{P}} \frac{1}{p^{j_p}} \prod_{p' \mid q} \frac{1}{p'^{j_p}} ds.
\]

We take
\[
\sigma = \frac{1}{2} + \frac{1}{\log \left( X \prod_{p \in \mathcal{P}} p^{j_p} \right)},
\]
so that the integral is bounded by a constant multiple of \( \sqrt{X} \), and use (26) to bound the summation over \( c \) and (7) to bound the summation over \( j_p \) and \( p \in \mathcal{P} \) in order to find that
\[
\# \left\{ f \in \mathcal{B}_0 (\Gamma_1 (q)) : \nu f \in (\alpha_0, 1/2), \ |\lambda_f(p)| \geq \alpha_p \right\}
\ll \varepsilon q^{1/2+\varepsilon} X^{-2\alpha_0} \prod_{p \in \mathcal{P}} \left( \frac{\alpha_p}{2} \right)^{2t_p}
\times \left( q^{3/2} + \sqrt{X} \prod_{p \in \mathcal{P}} p^{j_p/2} \left( \log \left( X \prod_{p \in \mathcal{P}} p^{j_p} \right) \right) \prod_{p' \mid q} \frac{1}{1 - p'^{-1/2}} \right).
\]
The result follows upon taking
\[
X = \text{vol} (\Gamma_1 (q) \backslash \mathbb{H})^{|\gamma|_{\mathbb{H}}}, \quad t_p = \left\lfloor \frac{3\mu_p \log \text{vol} (\Gamma_1 (q) \backslash \mathbb{H})}{2 \log p} \right\rfloor.
\]
Proof of Theorem 1.2 for $\Gamma(q)$. By using (16) and (27) in place of (15) and (26), we obtain

$$
\# \{ f \in B_0(\Gamma(q)) : \text{it}_f \in (\alpha_0, 1/2), |\lambda_f(p)| \geq \alpha_p \text{ for all } p \in \mathcal{P} \} 
\ll q^{1+\epsilon} X^{-2\alpha_0} \prod_{p \in \mathcal{P}} \left( \frac{\alpha_p}{2} \right)^{-2t_p} \times \left( q^2 + \sqrt{X} \prod_{p \in \mathcal{P}} p^{t_p/2} \left( \log \left( X \prod_{p \in \mathcal{P}} p^{t_p} \right) \right)^2 \prod_{p \mid q} \frac{1}{1-p^{t_p/2}} \right),
$$

and it remains to take

$$
X = \text{vol}(\Gamma(q) \backslash \mathbb{H})^{\beta_0/3}, \quad t_p = \left\lfloor \frac{4\mu_p \log \text{vol}(\Gamma(q) \backslash \mathbb{H})}{3 \log p} \right\rfloor.
$$

\qed

Proof of Theorem 1.2 for $\Gamma_0(q)$. We use (25) in place of (26) to find that

$$
\# \{ f \in B_0(\Gamma_0(q)) : \text{it}_f \in (\alpha_0, 1/2), |\lambda_f(p)| \geq \alpha_p \text{ for all } p \in \mathcal{P} \} 
\ll q^\epsilon X^{-2\alpha_0} \prod_{p \in \mathcal{P}} \left( \frac{\alpha_p}{2} \right)^{-2t_p} \left( q + \tau(q) \sqrt{X} \prod_{p \in \mathcal{P}} p^{t_p/2} \left( \log \left( X \prod_{p \in \mathcal{P}} p^{t_p} \right) \right)^2 \right),
$$

and so we take

$$
X = \text{vol}(\Gamma_0(q) \backslash \mathbb{H})^{\beta_0}, \quad t_p = \left\lfloor \frac{2\mu_p \log \text{vol}(\Gamma_0(q) \backslash \mathbb{H})}{3 \log p} \right\rfloor.
$$

\qed

8. Enlarging $C_{\Gamma_i(q)}$ via twisting

Let $f \in B_\epsilon(q, \chi)$ and let $\psi$ be a primitive Dirichlet character modulo $q_\psi$, where $q_\psi \mid q$. By [AL78, Proposition 3.1], the twisted cusp form

$$(f \otimes \psi)(z) := \frac{1}{\tau(\psi)} \sum_{d \mod q_\psi} \overline{\psi}(d) f \left( \frac{z+d}{q_\psi} \right)$$

is a nontrivial element of $A_\epsilon(q^2, \chi \psi^2)$, where $\tau(\psi)$ denotes the Gauss sum of $\psi$. Note that twisting by a Dirichlet character preserves the Laplacian eigenvalue $\lambda_f = 1/4 + t_f^2$ and the absolute value $|\lambda_f(n)|$ of the Hecke eigenvalues of $f$ for all $(n, q) = 1$. Moreover, if $f_1 \in B_\epsilon(q, \chi_1)$, $f_2 \in B_\epsilon(q, \chi_2)$ are such that there exist primitive Dirichlet characters $\psi_1$ modulo $q_{\psi_1}$ and $\psi_2$ modulo $q_{\psi_2}$ with $q_{\psi_1}, q_{\psi_2} \mid q$ such that

$$f_1 \otimes \psi_1 = f_2 \otimes \psi_2,$$

then $f_2 = f_1 \otimes \psi_1 \psi_2$.

Lemma 8.1. If $q$ is squarefree, $\psi$ is a primitive Dirichlet modulo $q_\psi$, where $q_\psi \mid q$, and $f \in B_\epsilon(q, \chi)$, then $f_1 \otimes \psi \in A_\epsilon(\Gamma_1(q))$ if and only if $\psi$ divides $\chi$, in the sense that $\psi \chi$ has conductor dividing $q_\psi$.

Proof. This follows via the methods of [Hum15]. For $p \mid q$, let $\pi_p$ be the local component of the automorphic representation $\pi$ associated to $f$, so that the central character $\omega_p$ of $\pi_p$ is the local component of the Hecke character $\omega$ that is the idelic lift of $\chi$. As $q$ is squarefree, $\pi_p$ is either a principal series representation or a special representation.

In the former case, $\pi_p = \omega_{p,1} \boxplus \omega_{p,2}$ with central character $\omega_p = \omega_{p,1} \omega_{p,2}$, where $\omega_{p,1}, \omega_{p,2}$ are characters of $\mathbb{Q}_p^*$ with conductor exponents $c(\omega_{p,1}), c(\omega_{p,2}) \in \{0, 1\}$ such that the conductor exponent $c(\pi_p)$ of $\pi_p$ is $c(\omega_{p,1}) + c(\omega_{p,2}) \in \{0, 1\}$. The twist $\pi_p \otimes \omega'_{p}$ of $\pi_p$ by a character $\omega'_{p}$ of $\mathbb{Q}_p^*$ of conductor exponent $c(\omega'_{p}) \in \{0, 1\}$
\{0, 1\} is \(\omega_p, \omega_p' \oplus \omega_p, 2\omega_p' \) with corresponding conductor exponent \(c(\pi_p \otimes \omega_p') = c(\omega_p, 1\omega_p') + c(\omega_p, 2\omega_p, p)\). For this to be at most 1, either \(\omega_p'\) is unramified, or one of \(c(\omega_p, 1\omega_p'), c(\omega_p, 2\omega_p, p)\) must be equal to 0, so that \(\omega_p'\) is equal to \(\omega_p, 1\) or \(\omega_p, 2\) up to multiplication by an unramified character.

In the latter case, \(\pi_p = \omega_p, 1\) with central character \(\omega_p = \omega_p^2\) such that \(c(\omega_p, 1) = 0\), so that \(c(\pi_p) = 1\). The twist of \(\pi_p\) by \(\omega_p'\) is \(\omega_p, 1\omega_p'\) St, corresponding conductor exponent \(c(\pi_p \otimes \omega_p') = \max\{1, 2c(\omega_p, 1\omega_p')\}\). For this to be at most 1, \(\omega_p'\) must be unramified.

It follows that the Hecke character \(\omega'\) is the id\'elic lift of \(\psi\), then the conductor of \(\pi \otimes \omega'\) divides \(q\) if and only if the conductor of \(\omega'\omega\) divides the conductor of \(\omega\).

From this, we see that there are at most \(\tau(q)\) nontrivial members of \(B_n(q, \chi)\) that can be twisted by a Dirichlet character modulo \(q\) to give the same member of \(A_n (\Gamma_1(q))\). So by considering the twist of each \(f \in B_n (\Gamma_1(q))\) by each primitive Dirichlet character of conductor dividing \(q\), together with the explicit bases (15) and (16), we thereby obtain the following.

**Lemma 8.2.** Let \(q\) be squarefree, let \(P\) be a finite collection of primes not dividing \(q\), let \(A_0\) be a measurable subset of \([0, \infty) \cup i(0, 1/2)\), and let \(A_p\) be a measurable subset of \([0, \infty)\) for each \(p \in P\). Then

\[
\# \{ f \in B_n (\Gamma_1(q)) : tf \in A_0, |\lambda_f(p)| \leq A_p \text{ for all } p \in P \} \leq \frac{\tau(q)}{\varphi(q)} \# \{ f \in B_n (\Gamma_1(q)) : tf \in A_0, |\lambda_f(p)| \leq A_p \text{ for all } p \in P \}.
\]

Combining this with the fact that \(\text{vol}(\Gamma_1(q) \backslash \mathbb{H}) = q \text{ vol}(\Gamma_1(q) \backslash \mathbb{H})\), we can improve Theorems 1.1 and 1.2 when \(\Gamma = \Gamma_1(q)\) with \(q\) squarefree.

**Corollary 8.3.** When \(q\) is squarefree, Theorems 1.1 and 1.2 hold for \(\Gamma = \Gamma_1(q)\) with \(C_{\Gamma_1(q)} = 4\).

It is likely that a more careful analysis could obtain this same result even when \(q\) is not squarefree via the methods in [Hum15].

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