Perturbative Approach at Finite Temperature and the $\phi^4$ model

G. Germán

Instituto de Física, Laboratorio de Cuernavaca, Universidad Nacional Autónoma de México, Apartado Postal 48-3, 62251 Cuernavaca, Morelos, México

Abstract

We suggest that the $\phi^4$ model is only a polynomial approximation to a more fundamental theory. As a consequence the high temperature regime might not be correctly described by this model. If this turns out to be true then several results concerning e.g., critical temperatures, symmetry restoration at high temperature and high temperature expansions should be reconsidered. We illustrate our conjecture by using the Nambu-Goto string model. We compare a two-loop calculation of the free energy or quark-antiquark static potential at finite temperature with a previous exact calculation in the large-d limit and show how the perturbative expansion fails to reproduce important features in the neighborhood of the critical temperature. It becomes clear why this happens in the Nambu-Goto model and we suggest that perhaps something similar occurs with the $\phi^4$ model.
The purpose of this letter is to discuss the validity of the $\phi^4$ model in the high temperature regime where symmetry restoration might occur. The $\phi^4$ model is of great interest due to its wide range of applications, however, there are several problems related with the model which might hint to the possibility that it is only a polynomial approximation to a more fundamental theory. Although we have no proof of this we would like to illustrate our conjecture by using the Nambu-Goto string model as an example and discuss some issues related with the validity of the perturbative expansion in the vicinity of the critical temperature. For this we compare a two-loop calculation for a finite length Nambu-Goto string at arbitrary temperature with a previous exact calculation of the free energy in the limit of large-$d$ [1], $d$ being the number of dimensions of the embedding space, where the string evolves.

The Nambu-Goto model is defined by the following action in Euclidean space

$$A = M^2 \int d^2\xi \sqrt{g},$$

(0.1)

where $M^2$ is the string tension and $g$ is the determinant of the metric which is given in terms of the string coordinates by

$$g_{ij} = \partial_i x^\mu(\vec{\xi}) \partial_j x^\nu(\vec{\xi}) h_{\mu\nu}, \quad i = 0, 1,$$

(0.2)

being $h_{\mu\nu}$ the metric of the embedding Euclidean space where the string evolves. For a $d$-dimensional space we have that $\mu, \nu=0, 1, \ldots, d-1$ and $i, j = 0, 1$ for a string coordinate $x^\mu = x^\mu(\xi^0, \xi^1) = x^\mu(t, r)$. We work in a Monge parametrization or ”physical gauge”

$$x^\mu(\vec{\xi}) = (t, r, u^a),$$

(0.3)

where $u^a = u^a(t, r), a = 2, \ldots, d-1$ are the $(d-2)$ transverse oscillating modes of the string. The metric determinant can be written as [2]

$$g = detg_{ij} = 1 + \vec{u}_i^2 + \frac{1}{2} \vec{u}_i^4 - \frac{1}{2}(\vec{u}_i \cdot \vec{u}_j)^2.$$

(0.4)

For small field fluctuations and for the two-loop free energy we are interested in, the action given by Eq. (1) can be written as
\[ A = M^2 \int dt dr (1 + \frac{1}{2} \vec{u}^2 + \frac{1}{8} \vec{u}^4 - \frac{1}{4} (\vec{u}_i \cdot \vec{u}_j)^2). \] (0.5)

We now want to evaluate the functional integral

\[ Z = \int D\nu^a e^{-A}, \] (0.6)

for the quadratic part of the action. The interacting quartic terms are evaluated in the usual way. The resulting expression for the effective action is

\[ A_{eff} = \int drdt [M^2 + \frac{d - 2}{2} Tr ln(-\partial^2)] + < A_{int} >, \] (0.7)

where

\[ < A_{int} > = \frac{(d-2)^2}{8} M^2 \int drdt [(1 - \frac{2}{d-2}) < u_i u_i >^2 - 2 < u_i u_j >^2]. \] (0.8)

We now impose thermodynamic boundary conditions for a string with fixed ends at finite temperature. The Green functions are

\[ < u^a(t, r) u^b(t', r') > = \frac{2T \delta^{ab}}{M^2 R} \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \frac{e^{i\omega_m (t-t')}}{k_n^2 + \omega_m^2} \text{sink}_n r \text{sink}_m r', \] (0.9)

where \( T \) is the temperature and \( R \) the extrinsic length of the string. The momenta and frequencies are given by

\[ k_n = \frac{n\pi}{R}, \quad \omega_m = 2m\pi T \quad n = 1, 2, ... \quad m = 0,^+, 1,^+, 2, ... \] (0.10)

the trace is given by

\[ Tr ln(-\partial^2) = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} ln(k_n^2 + \omega_m^2). \] (0.11)

A similar expression for the trace has been obtained before (see Eqn. (2.17) and (2.20) of Ref. [1]). Also Eq. (8) above can be calculated along the lines of Ref. [2] but here the system is finite in both \( R \) and \( T \) directions. The resulting expression for the free energy or static potential at finite temperature is

\[ V(R, T) = V_0 + V_1 + V_2, \] (0.12)
where

\[ V_0 = M^2 R \]  
\[ V_1 = -\frac{(d-2)\pi}{24} \frac{1}{R} + (d-2)T \sum_{n=1}^{\infty} \ln(1 - e^{-\frac{n\pi}{RT}}) \]  
\[ V_2 = -\left[ \frac{(d-2)\pi}{24} \right]^2 \frac{144}{2M^2 R^3} \left[ \left( \sum_{n=1}^{\infty} n \coth \frac{n\pi}{2RT} \right)^2 + \frac{1}{d-2} \sum_{n=1}^{\infty} \left( n \coth \frac{n\pi}{2RT} \right)^2 \right] - \frac{(d-4)RT}{\pi} \sum_{n=1}^{\infty} n \coth \frac{n\pi}{2RT}. \]

An equivalent expression for \( V \) has been calculated before in terms of Dedekind functions by Dietz and Filk [3]. In that work analytic renormalization procedures for functional integrals was the subject of interest. Here we would like to compare with previous results in an exact large-\( d \) evaluation of the effective potential at finite temperature. The full potential, Eq. (12), is now shown in Fig. 1 for several values of the temperature in \( d = 4 \) dimensions and with the zero temperature string tension \( M^2 = M^2(T = 0) \) normalized to unity. In the Nambu-Goto model, the string tension at finite temperature for an infinitely long string was calculated some time ago by Pisarski and Alvarez [4] with the result that there is a critical temperature (the so-called deconfinement temperature) for which the string tension becomes vanishing, thus signaling a transition to a deconfined phase. The value of this temperature is

\[ T_{\text{dec}} = \sqrt{\frac{3}{(d-2)\pi}} |_{d=4} \sim 0.69. \] (0.14)

This result was obtained from and exact large-\( d \) calculation for an infinitely long string at arbitrary temperature. In an analogous calculation but for a finite length string we showed [1] that for \( T = T_{\text{dec}} \) the potential becomes flat for large \( R \) thus losing the "confinement" property or linear behavior of the string (see inset of Fig. 2a of Ref. [1]). This, as we argued before, is a perfectly consistent result with the identification of \( T_{\text{dec}} \) as a deconfinement temperature. In our two-loop calculation, however, this result does not show up. We investigated numerically the curve of Fig. 1 with \( T = 0.69 \) never becoming flat for large \( R \). A simple expansion of Eq. (12) for large \( R \) and the use of \( Z \)-function regularization to
evaluate the sums $\sum_{n=1}^{\infty} n = \zeta(0) = -\frac{1}{2}$ and $\sum_{n=1}^{\infty} \ln n = -\zeta'(0) = \frac{1}{2} \ln 2\pi$ shows that in fact the potential behaves like

$$V(R, T) \sim M^2 R - \frac{(d - 2)\pi}{6} RT^2 + \frac{(d - 2)}{2} T \ln 2RT,$$

(0.15)

thus never showing the constant potential obtained in the exact calculation. For temperatures bigger than the the critical value Fig. 2b of Ref. [1] shows that the potential exists although for a string up to certain length. In [1] this was understood as follows: for large $R$

$$V(R \to \infty, T) \sim R M^2(R \to \infty, T) = R^2(T) = R \sqrt{1 - \frac{T^2}{T_{dec}^2}}.$$

(0.16)

For $T = T_{dec}, M^2(T) = 0$ and in the competition of limits with $R \to \infty V(R \to \infty, T \to T_{dec})$ the potential results in a constant with an approximated value of 6.2. For $T \geq T_{dec}$ the potential starts "seeing" the linear behavior in $R$ with a slope becoming increasingly close to $\sqrt{1 - \frac{T^2}{T_{dec}^2}}$ thus appearing a negative radicand which "stops" the potential for a given $R$. In general the expression for the exact potential at finite temperature should be a very complicated function of $R$ and $T$ but for large $R$ a simpler expression of the form (16) should emerge. Thus for $T > T_{dec}$ we can no longer have strings of arbitrary length. In our two-loop result we see in Fig. 1 (curves with $T = 0.75$ and 0.8) that this interesting feature is not present as we can also see from the large-$R$ expansion of Eq. (15). In Fig. 2 we plot the ratio of the two-loop correction over the one-loop contribution for the same values of the temperature shown in Fig. 1. We see that for certain values of $R$ this ratio becomes bigger than one thus invalidating the loop calculation. We also see that the values of $R$ for which the potential stops in Fig. 2b of Ref. [1] lie in the region for which $V_2/V_1 > 1$, this ratio becoming again well behaved for larger values of $R$ which, however, is an artefact of the perturbative expansion; in the exact treatment of the problem this region does not exist, the string having probably decayed. Note that something similar happens in the $\phi^4$ model close to the critical temperature $T_c$ where $m_\phi(T) \to 0$. Here the problem is usually dealt with by including and infinite set of daisy and superdaisy diagrams. It is usually said that for $T > T_c$ higher-order corrections are again small and the results reliable. However, it has
been argued recently that the $\phi^4$ model has no particle interpretation for $m_\phi^2 > 0$, for large temperatures the $\phi$-particle probably disappears [5,6]. In the Nambu-Goto model we have seen that for $T > T_{dec}$ the ratio $V_2/V_1$ is also well behaved to the right of the peak, however because we are able to compare with the exact result we do not trust this region; as we said before it is only an artefact. Thus the perturbative result is jumping the singularity of the exact potential not showing any of the interesting behavior in the neighborhood of the critical temperature. It is easy to see why this happens: instead of beginning with Eq. (1) above we could have started with a model given by Eq. (5) without asking about its origin. Clearly everything would follow exactly the same and we would get Figs. 1 and 2 with the peculiar behavior noted before and we could agree that perturbation theory for $T > T_{dec}$ is playing a trick on us. Using the model given by Eq. (5) we could believe that, because $V_2/V_1 > 1$ only in a small region of $R$ values, everywhere else perturbation theory is fine even for $T > T_{dec}$. As it is, however, we can easily trace back the jumping of the singularity by the perturbative approach to the fact that from the very beginning we are expanding $\sqrt{g}$ in Eq. (1) to get Eq. (5), i.e., in a way Fig. 2 is telling us that we are working with an approximated model given by Eq. (5) instead of the full one Eq. (1). Having the model Eq. (1) and the exact treatment of the problem [1] we can see that in fact, for $T > T_{dec}$, all the region to the right of the peak $V_2/V_1 > 1$ where the ratio $V_2/V_1$ becomes again well behaved (see Fig. 2) is no longer reliable. Actually this phase is not even described by the Nambu-Goto model.

Now let us suppose that we are working with the $\phi^4$ model and that we find something similar to Fig. 2 but now with $V_2(\phi, T)/V_1(\phi, T)$ versus $\phi$ for various values of $T$. Our suggestion is that a similar behavior to the one described in this letter might be hinting towards a more fundamental origin of our field theory, something playing the role of our Eq. (1) above, and that the "failure" of the perturbative approach is in fact telling us that we are working with an approximated expanded model of a more fundamental theory. As a consequence the high temperature regime might not be correctly described by the $\phi^4$ model. If this turns out to be correct (and it should be investigated) then several results concerning
e.g., critical temperatures, symmetry restoration at high temperature and high temperature expansions should be reconsidered.
Figure Captions

Fig. 1
The two-loop effective potential for the Nambu-Goto string model is shown for various values of the temperature. If we compare with Fig. 2b of Ref. [1] we see that the perturbative calculation does not show the interesting behavior in the neighborhood of the critical temperature $T_{dec}$. The perturbative expansion “jumps” the singularity present in the exact result at $T = T_{dec}$.

Fig. 2
The two to one loop corrections ratio is here shown as a function of the string length for various values of the temperature. The perturbative expansion is no longer reliable when the temperature is in the vicinity of the critical value. The value of $R$ for which the potential ceases to exist in the exact result (Fig. 2b of Ref. [1]) lies in the region where $V_2/V_1 > 1$ above. In the perturbative approach the potential does not notice that it should not exist for some $R$ when $T > T_{dec}$, where the string has probably decayed.
References

1) A. Antillón and G. Germán, Phys. Rev. D47, 4567(1993).

2) G. Germán and H. Kleinert, Phys. Rev. D40, 1108(1989).

3) K. Dietz and T. Filk, Phys. Rev. D27, 2944(1983).

4) R.D. Pisarski and O. Alvarez, Phys. Rev. D26, 3735(1982).

5) H.A. Al-Kuwari, Phys. Lett. 375B, 217(1996).

6) B.A. Campbell, J. Ellis and K.A. Olive, Phys. Lett 235B, 325 (1990).