Exceptional points in coupled dissipative dynamical systems

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We study the transient behavior in coupled dissipative dynamical systems based on the linear analysis around the steady state. We find that the transient time is minimized at a specific set of parameter systems and show that at this parameter set, two eigenvalues and two eigenvectors of Jacobian matrix coalesce at the same time, this degenerate point is called the exceptional point. For the case of coupled limit cycle oscillators, we investigate the transient behavior into the amplitude death state, and clarify that the exceptional point is associated with a critical point of frequency locking, as well as the transition of the envelope oscillation.

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I. INTRODUCTION

In the eigenvalue problem of a non-Hermitian matrix, an exceptional point (EP) is a square-root branch point on a two-dimensional parameter space, at which not only eigenvalues but also the associated eigenvectors coalesce. The peculiar feature related to the EP is the exchange of eigenvalues and eigenvectors after a parameter variation encircling the EP once, of which topological structure is same as that of Möbius strip. The EPs and relating interesting phenomena have mainly been studied in open quantum systems described by non-Hermitian Hamiltonians such as atomic spectra in fields, microwave cavity experiments, chaotic optical microcavities, PT-symmetric quantum systems, and so on. Besides the open quantum systems, the EPs are also observed in coupled driven damped oscillators realized by electric circuits, which are purely classical systems.

The amplitude death (AD) is the complete suppression of oscillations of the entire system when the nonlinear dynamical systems are coupled. The AD has been observed in many coupled dynamical systems and the AD is achieved by various types of coupling interaction, i.e., the diffusive coupling in mismatched oscillators, delayed coupling, conjugate coupling, dynamical coupling, nonlinear coupling, etc. The AD has also been studied in networks of coupled oscillators and variety topologies such as a ring, small world, and scale free networks. Recently, the suppressions of oscillations are strictly classified into amplitude death and oscillation death, where the asymptotic steady state is homogeneous and inhomogeneous, respectively.

In this paper, we study the transient behaviors of coupled dissipative dynamical systems based on the linear analysis around the steady state. We find that the systems show the largest damping rate at an EP, which comes from the intrinsic feature of a square-root branch point. For the case of coupled limit cycle oscillators, the transient behavior into the amplitude death state is studied. We demonstrate that the EP is associated with a critical point of frequency locking, as well as the transition of the envelope oscillation.

This paper is organized as follows. In Sec. II, we show the occurrence of EP in coupled damped oscillators and discuss the damping behavior around the EP in a pedagogical way. In Sec. III, we present the transient behavior into the AD in coupled limit cycle oscillators, and it is explained based on the existence of an EP. Finally, we summarize our results in Sec. IV.

II. EXCEPTIONAL POINT IN COUPLED DAMPED OSCILLATORS

We consider the coupled damped oscillators,

\[ \ddot{x}_1 + \gamma_1 \dot{x}_1 + \omega_1^2 x_1 = -k x_2, \]
\[ \ddot{x}_2 + \gamma_2 \dot{x}_2 + \omega_2^2 x_2 = -k x_1, \]

where \( \gamma_i \) and \( \omega_i \) (\( i = 1, 2 \)) are damping ratio and undamped angular frequency of the \( i \)-th oscillator, and \( k \) is the coupling constant. Figure 1 shows the time series of \( x_1 \) and \( x_2 \) of Eq. \( i \) in the logarithmic scale when \( \omega_1 = \omega_2 = 1.0 \) and \( \gamma_0 = 0 \). First, we consider uncoupled case, \( k = 0 \). As we set \( \gamma_1 = 0 \) and \( \gamma_2 = 0.1 \), the time series of \( x_1 \) exhibits a stationary oscillation without damping, while an exponential damping appears in the time series of \( x_2 \), as shown in Fig. 1(a). Next, we consider a finite coupling strength of \( k = 0.1 \). In Fig. 1(b) with \( \gamma_1 = 0 \) and \( \gamma_2 = 0.1 \), both time series of \( x_1 \) and \( x_2 \) exhibit decays with envelope oscillations. Their decay rates, given by the slope of time series of \( x_1 \) and \( x_2 \) in the logarithmic plot, are equal. As \( \gamma_2 \) increases from 0.1, the period of the envelope oscillation and the decay rate increase. At \( \gamma_2 \sim 0.2 \), the envelope oscillation disappears and the decay rate reaches a maximum (see

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FIG. 1: (color online) Time series of $x_1$ (Black) and $x_2$ (Red) when $\omega_1 = \omega_2 = 1.0$ and $\gamma_1 = 0.0$. (a) No coupling case of $k = 0.0$ with $\gamma_2 = 0.1$. Coupling cases of $k = 0.1$ with (b) $\gamma_2 = 0.1$, (c) $\gamma_2 = 0.2$, and (d) $\gamma_2 = 0.3$. Insets show linearly scaled time series.

When $\gamma_2$ increases further, the decay rate decreases again. For example, the time series of the case with $\gamma_2 = 0.3$ is shown in Fig. 1(d). Although the amplitude of two oscillators are different, as shown in the inset, their decay rates are equal. In our work, we concentrate on the case that each uncoupled oscillators have zero or weak damping ratio so that their dampings are underdamped.

In order to understand the variation of decay rate with $\gamma_2$ and its maximum at $\gamma_2 \sim 0.2$, we analyze the eigenvalues of a stability matrix around the origin. Eq. (1) can be rewritten as

$$\begin{align*}
\dot{x}_1 &= y_1, \\
\dot{y}_1 &= -\gamma_1 y_1 - \omega_1^2 x_1 - k x_2, \\
\dot{x}_2 &= y_2, \\
\dot{y}_2 &= -\gamma_2 y_2 - \omega_2^2 x_2 - k x_1. 
\end{align*}$$

The stability matrix $M$ is then given by

$$M = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-\omega_1^2 & -\gamma_1 & -k & 0 \\
0 & 0 & 1 & 0 \\
-k & 0 & -\omega_2^2 & -\gamma_2 
\end{pmatrix}. \quad (3)$$

The eigenvalues $\lambda_i$ of $M$ are complex numbers, because the matrix $M$ is non-Hermitian. Since the time evolution of an eigenvector $\vec{e}_i$ is given as $\vec{e}_i(t) = \vec{e}_i e^{\lambda_i t}$, the real and imaginary parts of the eigenvalues correspond to the decay rates and the angular frequency of the corresponding time series, respectively.

The complex eigenvalues with positive imaginary parts are shown as a function of $\gamma_2$ in Fig. 2(a) and (b). When $\gamma_2 < 0.2$, real parts of two eigenvalues are very close but their imaginary parts are quite different, this means that the dynamics of eigenvectors would show almost same decay rate and different angular frequencies. In this range, the time series of $x_1$ and $x_2$ would show a constant overall slope given by the close real parts, but they would have an oscillatory envelope whose frequency is determined by the difference of the imaginary parts of eigenvalues. This behavior has been shown in Fig. 1(b). As $\gamma_2$ approaches to 0.2, the real parts of two eigenvalues decrease and the imaginary parts become closer with each other, which corresponds to the time series with a faster decay and a longer period of envelope oscillation, respectively.

As $\gamma_2$ goes further beyond 0.2, two real parts start to split but the difference of two imaginary parts become small. The splitting of two real parts indicates that the time series can be characterized by a combination of fast and slow decays. The fast decay might be seen only in

FIG. 2: (color online). (a) Real and (b) imaginary parts of two eigenvalues of which imaginary parts are positive as a function of $\gamma_2$ when $\omega_2 = 1.0$ (black) and $\omega_2 = 1.005$ (red) with $\gamma_1 = 0.0$ and $\omega_1 = 1.0$. (c) Real and (d) imaginary parts of the eigenvalues near EP as functions of $\omega_2$ and $\gamma_2$ when $\omega_1 = 1.0$ and $\gamma_1 = 0.0$. The black circle, red dotted line, and blue dotted line represent the EP, real value crossing line, and imaginary value crossing line, respectively.
the short time behavior and the slow decay, corresponding to the larger real part, dominates the long time behavior of the time series. Thus, although two imaginary parts are still different, there is no envelope oscillation due to the fast suppression of one eigen-component with the lower real part (see Fig. 1(d)). Note that the larger real part, governing long-time behavior, has a minimum value around at \( \gamma_2 \sim 0.2 \), which explains the maximum decay rate observed in Fig. 1(c).

Note that two complex eigenvalues are very close at \( \gamma_2 \sim 0.2 \) as shown by the black lines in Fig. 2(a) and (b). We can expect that there should be a degenerate point, called exceptional point (EP) \[1, 2\], where two complex eigenvalues coalesce, in the system parameter space. By adjusting \( \omega_2 \) a bit as \( \omega_2 = 1.005 \), we find an EP at \((\omega_2, \gamma_2) \sim (1.005, 0.2)\), which is shown by the red lines in Fig. 2(a) and (b). It is well known that two eigenvectors also coalesce at the EP and mathematically the EP is the square-root branch point. The EP can be characterized by a peculiar eigenvalue surfaces in a parameter plane. In Fig. 2(c) and (d), the surfaces of the two eigenvalues are plotted in \((\omega_2, \gamma_2)\) plane. Topology of the surface explains the exchange of two eigenvalues for a parameter variation encircling the EP \[3\]. It is emphasized that the larger real part becomes a local minimum at the EP, indicating the local maximum decay rate in the parameter plane.

III. EXCEPTIONAL POINT AND AMPLITUDE DEATH IN COUPLED LIMIT CYCLE OSCILLATORS

In this section, we study the role of the EP when the amplitude death (AD) occurs in coupled limit cycle oscillators. Let us start with the following system of two Stuart-Landau limit-cycle oscillators with diffusive coupling:

\[
\dot{z}_1 = (R_1 + i\omega_1 - |z_1|^2)z_1 + k(z_2 - z_1),
\]

\[
\dot{z}_2 = (R_2 + i\omega_2 - |z_2|^2)z_2 + k(z_1 - z_2),
\]

where \(z_j\) are complex variables, \(\omega_j\) are the intrinsic angular frequencies of uncoupled \(j\)-th limit cycle oscillators, and \(k\) is the coupling strength. Without coupling \((k = 0)\), two limit cycle oscillators are attracted to the limit cycle with radii \(\sqrt{R_j}\) for \(R_j > 0\) and the origin for \(R_j < 0\). Stuart-Landau limit-cycle oscillator is renowned as a paradigmatic model for studying the AD in coupled nonlinear oscillators because it is a prototypical system exhibiting a Hopf bifurcation that can reveal universal features of many practical systems. For instance, a variety of spatio-temporal periodic patterns can be created in two-dimensional lattice of delay-coupled Stuart-Landau oscillators \[31\].

![Image](325x588 to 554x740)

**FIG. 3:** (color online). Maximal values of real parts of eigenvalues with (a) \(R_2 = 1.0\), (b) \(R_2 = 0.0\), and (c) \(R_2 = -1.0\) when \(R_1 = 1.0\). The colored and white region represent negative and positive values, respectively. The blue dotted line represents the EP. (d) Real and (e) imaginary parts of two eigenvalues of which imaginary parts are positive as a function of \(k\) when \(\Delta \omega = 4.0\) and \(R_1 = 1.0\). Black, red, and green curves represent the cases of \(R_2 = 1.0, 0.0\), and \(-1.0\), respectively.

A. The amplitude death in coupled limit cycle oscillators

It has been well known that the AD occurs in coupled limit cycle oscillators at proper \(k\) if the \(\Delta \omega = \omega_2 - \omega_1\) is sufficiently large when \(R_1 = R_2 = 1.0\) \[10, 18\]. In order to obtain the AD region in the parameter space \((\Delta \omega, k)\), we calculate the Jacobian matrix \(J\) at the origin, which is given by

\[
J = \begin{pmatrix}
R_1 - k & -\omega_1 & k & 0 \\
\omega_1 & R_1 - k & 0 & k \\
k & 0 & R_2 - k & -\omega_2 \\
0 & k & \omega_2 & R_2 - k
\end{pmatrix}.
\]

The eigenvalues \(\lambda\) of \(J\) are complex numbers because the Jacobian matrix \(J\) is a non-Hermitian matrix. That is, the real and imaginary parts are the decay (or growing) rates and the angular frequency of the orbit near the origin, respectively.

The occurrence of AD is determined by the stability of the origin, which is related to the maximal value of the real parts of complex eigenvalues. If the maximal value is negative, the origin is stable fixed point and therefore the system exhibits the AD. The colored region in Fig. 3(a)-(c) where the maximal value is negative represent the AD regions when \(R_2 = 1.0, 0.0\), and \(-1.0\), respectively, with \(R_1 = 1.0\). As \(R_2\) decreases from 1.0 to \(-1.0\), the AD region becomes larger. Figure 3(d) clearly shows the transition between positive and negative values of maximal real parts as a function of \(k\) when \(\Delta \omega\) is fixed.
B. The exceptional point in coupled limit cycle oscillators

Similarly as the case of coupled damped oscillators, there also exists an EP in the coupled limit cycle oscillators. The EP occurs at \( k = 2.0 \) when \( \Delta \omega = 4.0 \) and \( R_1 = R_2 = 1.0 \), which is the double root position in Fig. 3(d) and (e). Considering \( R_1 = R_2 = R \), the eigenvalues of Eq. (5) are given by

\[
\lambda = \pm \sqrt{\frac{(\Delta \omega)^2}{2} + k^2 - \Delta},
\]

where \( \Delta = \frac{1}{2R}(\sqrt{\Delta \omega^2 - 4k^2 + 2\omega_1} + \omega_1(\sqrt{\Delta \omega^2 - 4k^2 + \omega_1}) \) and \( \Box = \frac{1}{2R}(\sqrt{\Delta \omega^2 - 4k^2 - 2\omega_1} + \omega_1(\sqrt{\Delta \omega^2 - 4k^2 - \omega_1}) \) are its minimum at the condition of EP, \( k = \Delta \omega/2 \). Because the decaying rate to the AD state can be considered as a maximal value of Re(\( \lambda \)) and the maximal value of Re(\( \lambda \)) has its minimum at the condition of EP, \( k = \Delta \omega/2 \). In order to confirm the role of the EP expected in the previous subsection, we obtain the time series of \( z_1 \) and \( z_2 \) as \( k \) increases. Figure 5 shows the time series of real parts of \( z_1 \) and \( z_2 \) with different \( k \) when \( \Delta \omega = 4.0 \) and \( R_1 = R_2 = 1.0 \). The coupling is turned on at \( t = 10.0 \) and the pair of Im(\( \lambda \)) equal each other, respectively. At \( k = 1.1 \), the AD occurs with transient behavior of envelope oscillation but there is no frequency locking on the transient behavior. At \( k = 2.0 \), the AD occurs without envelope oscillatory transient behavior and the decay is fastest because this is the condition of the EP. The 1:1 frequency locking on the transient behavior is also shown. At \( k = 2.4 \), the AD occurs without envelope oscillation and there is frequency locking on the transient behavior. The decay
FIG. 6: (color online). $-(1/t_{AD})$ as functions of $\Delta \omega$ and $k$ with (a) $R_2 = 1.0$, (b) 0.0, and (c) $-1.0$ when $R_1 = 1.0$. The colored and white region represent the AD and non-AD regions, respectively. The blue dotted line represents the EP. (d) $t_{AD}$ as a function of $k$ when $R_2 = 1.0$ (black), 0.0 (red), and $-1.0$ (green) when $\Delta \omega = 4.0$.

is slower than that in the case of $k = 2.0$. At $k = 3.0$, the AD does not occur but there is frequency locking. In the AD region ($1.0 < k < 2.5$), the EP is the transition point between decaying with and without envelope oscillations. Also, in this region, the EP is the transition point for frequency locking. The imaginary parts of oscillations. Also, in this region, the EP is the transition point between decaying with and without envelope oscillations. The imaginary parts of oscillations. Also, in this region, the EP is the transition point between decaying with and without envelope oscillations. The imaginary parts of oscillations. Also, in this region, the EP is the transition point between decaying with and without envelope oscillations. The imaginary parts of oscillations. Also, in this region, the EP is the transition point between decaying with and without envelope oscillations.

The important role of EP in AD is that the condition of EP guarantees the fastest attracting time to the origin, i.e., the AD state. We investigate the attracting time to the AD state, denoted by $t_{AD}$. Here, $t_{AD}$ is calculated by followings: If, at time $t$, the radii of two oscillators firstly become smaller than $c_{AD}$, i.e., the criterion for the AD state, and continue to be smaller than $c_{AD}$ for 200 seconds, then $t_{AD}$ equals to $t - t_{on}$ where $t_{on}$ is the time when the coupling is turned on. We set $c_{AD} = 0.001$ and $t_{on} = 10.0$. Figure 6 (a)-(c) show $-(1/t_{AD})$, with various $R_2$ when $R_1 = 1.0$, on the parameter space ($\Delta \omega$, $k$). Figure 6 (d) shows $-(1/t_{AD})$ as a function of $k$ when $R_1 = 1.0$ and $\Delta \omega = 4.0$ and the local minimum appears more clear when the parameters of system are closer to the EP. Contrary to the expectation from the maximal real parts of eigenvalues in Fig. 3, there are many wrinkled patterns when $R_2 = 1.0$. The wrinkled patterns gradually disappear as $R_2$ decreases and then there is no patterns when $R_2 = -1.0$. The different $t_{on}$ which means the different initial conditions makes the different wrinkled patterns. The wrinkled patterns when $k < \Delta \omega/2$ are caused by the oscillatory transient behavior. However, the reason of the wrinkled patterns when $k > \Delta \omega/2$ is that the transition from fast decay to slow decay occurs when the amplitudes of the oscillators are smaller than our critical value $c_{AD}$ and therefore the patterns disappear if $c_{AD}$ is sufficiently small.

Figure 7 shows the maximal values of real parts of eigenvalues and $-(1/t_{AD})$ with the parameter space ($\Delta R$, $k$) when $\Delta \omega = 4.0$. As shown in Fig. 7 (a) the EP exists at ($\Delta R$, $k$)=(0.0, 2.0), where the maximal values of real parts of eigenvalues are local minimum as shown in Fig. 7 (a). $t_{AD}$ are also local minimum at the EP. In Fig. 7 (b), the wrinkled patterns exist at $\Delta R = 0.0$ but they disappear as $\Delta R$ deviates from 0.0. In principle, for the long time behavior, the oscillation behavior such as underdamped case exists only on the line, $\Delta R = 0$ and $k < \Delta \omega/2$, because the real parts of two eigenvalues are same on the line in the parameter space ($\Delta R$, $k$). Different real parts of two eigenvalues mean the system has two different decay rates and therefore only one frequency is dominant for a long time behavior. It is noted that the EP is not local minimal point on the parameter space ($\Delta \omega$, $k$) because the EP forms the lines as shown in Fig. 7 (a) and Fig. 7 (a). That is, the maximal values of real parts of eigenvalues decrease as the $\Delta \omega$ increases on the EP line, $k = \Delta \omega/2$.

IV. SUMMARY

We have studied the exceptional point in dynamical systems and investigated the role of the exceptional point in the transient behaviors of amplitude death in coupled limit cycle oscillators. The exceptional point is associated with a critical point of frequency locking as well as the transition of the envelope oscillation, which also gives the fastest decay to the amplitude death in coupled limit cycle oscillators. In addition, for other examples (two Van der Pol oscillators interacting through mean-field diff-
fusive coupling, and coupled system of the Rössler and a linear oscillator), we have obtained the largest decay rates and transition behaviors at exceptional point (not shown here). As a result, the transient behaviors related to the exceptional point appear commonly for the coupled dissipative dynamical systems, independent of the specific properties of systems. We expect the exceptional point is important to the study on the various disciplines such as the nonequilibrium statistical mechanics and transient chaos because the exceptional point is not related to the stationary states but the transient behaviors.

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[1] T. Kato, Perturbation Theory of Linear Operators (Springer, Berlin, 1996).
[2] W.D. Heiss, J. Phys. A: Math. Theor. 45, 444016 (2012) and reference therein.
[3] W.D. Heiss, Eur. Phys. J. D 7, 1 (1999).
[4] O. Latinne, N.J. Kylstra, M. Dörr, J. Purvis, M. Teraodunseath, C.J. Joachain, P.G. Burke, and C.J. Noble, Phys. Rev. Lett. 74, 46 (1995).
[5] H. Cartarius, J. Main, and G. Wunner, Phys. Rev. Lett. 99, 173003 (2007).
[6] C. Dembowski, H.-D. Gräf, H.L. Harney, A. Heine, W.D. Heiss, H. Rehfeld, and A. Richter, Phys. Rev. Lett. 86, 787 (2001).
[7] C. Dembowski, B. Dietz, H.-D. Gräf, H.L. Harney, A. Heine, W.D. Heiss, and A. Richter, Phys. Rev. Lett. 90, 034101 (2003).
[8] S.-B. Lee, J. Yang, S. Moon, S.-Y. Lee, J.-B. Shim, S.W. Kim, J.-H. Lee, and K. An, Phys. Rev. Lett. 103, 134101 (2009).
[9] C.M. Bender and S. Boettcher, Phys. Rev. Lett. 80, 5243 (1998).
[10] S. Klaiman, U. Günther, and N. Moiseyev, Phys. Rev. Lett. 101, 080402 (2008).
[11] C.E. Rueter, K.G. Makris, R. El-Ganainy, D.N. Christodoulides, M. Segev, and D. Kip, Nat. Phys. 6, 192 (2010).
[12] W.D. Heiss, J. Phys. A: Math. Gen. 37, 2455 (2004).
[13] T. Stehmann, W.D. Heiss, and F.G. Scholtz, J. Phys. A: Math. Gen. 37, 7813 (2004).
[14] G. Saxena, A. Prasad, and R. Ramaswamy, Phys. Rep. 521, 205 (2012), and reference therein.
[15] K.B. Eli, J. Phys. Chem. 88, 3616 (1984).
[16] R.E. Mirollo and S. Strogatz, J. Stat. Phys. 60, 245 (1990).
[17] G.B. Ermentrout, Physica D 41, 219 (1990).
[18] D.G. Aronson, G.B. Ermentrout, and N. Kopell, Physica D 41, 403 (1990).
[19] D.V. Ramana Reddy, A. Sen, and G.L. Johnston, Phys. Rev. Lett. 80, 5109 (1998).
[20] D.V. Ramana Reddy, A. Sen, and G.L. Johnston, Physica D 129, 15 (1999).
[21] D.V. Ramana Reddy, A. Sen, and G.L. Johnston, Phys. Rev. Lett. 85, 3381 (2000).
[22] D.V. Ramana Reddy, A. Sen, and G.L. Johnston, Physica D 144, 335 (2000).
[23] W. Zou, D.V. Senthilkumar, Y. Tang, Y. Wu, J. Lu, and J. Kurths, Phys. Rev. E 88, 032916 (2013).
[24] R. Karnatak, N. Punetha, A. Prasad, and R. Ramaswamy, Phys. Rev. E 82, 046219 (2010).
[25] K. Konishi, Phys. Rev. E 68, 067202 (2003).
[26] A. Prasad, Y.C. Lai, A. Gavrielides, and V. Kovanis, Phys. Lett. A 318, 71 (2003).
[27] A. Prasad, M. Dhamala, B.M. Adhikari, and R. Ramaswamy, Phys. Rev. E 81, 027201 (2010).
[28] F.M. Atay, Physica D 41, 403 (1990).
[29] R. Dodla, A. Sen, and G.L. Johnston, Phys. Rev. E 69, 056217 (2004).
[30] K. Konishi, Phys. Rev. E 70, 066201 (2004).
[31] Z. Hou and H. Xin, Phys. Rev. E 68, 055103 (2003).
[32] W. Liu, X. Wang, S. Guan, and C.H. Lai, New J. Phys. 11, 093016 (2009).
[33] A. Koseska, E. Volkov, and J. Kurths, Phys. Rep. 531, 173 (2013).
[34] M. Kantner, E. Schöll, and S. Yanchuk, Sci. Rep. 5, 8522 (2015).
[35] R. Zwanzig, Nonequilibrium statistical mechanics (Oxford University Press, 2001).
[36] Y.-C. Lai and T. Tél, Transient Chaos: Complex Dynamics on Finite-Time Scales (Springer, New York, 2013).
[37] A.E. Motter, M. Gruiz, G. Károlyi, and T. Tél, Phys. Rev. Lett. 111, 194101 (2013).