Research Article

Martingale Decomposition and Backward Stochastic Dynamic Equations on Time Scales

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The paper aims to establish the related backward stochastic dynamic equations on time scales, BS∇Es for short, concerning to ∇-integral on time scales. We present the martingale decomposition theorem on time scales and prove the existence and uniqueness theorem of solutions to BS∇Es. This work can be considered as a unification and a generalization of similar results in backward stochastic difference equations and backward stochastic differential equations.

1. Introduction

In 1988, Hilger [1] introduced the calculus of measure chains to unify continuous and discrete analysis. Since then, this topic, mainly the deterministic analysis, has attracted much attention [2–4]. For the stochastic calculus on time scales, a lot of work [5–8] focused on ∇-integration on the semi-open intervals of the form [t_i, t_{i+1}). Meanwhile, by the requirement of the predictable integrand (for the martingale property of stochastic integral), it is easy to consider semi-open intervals (t_i, t_{i+1}). Du and Dieu [9, 10] established the stochastic calculus on time scales for the ∇ case. Some basic problems such as stochastic integration, Doob-Meyer decomposition theorem, Ito’s formula, and stochastic differential equations on time scales have been studied carefully. Zhu [11] studied stochastic optimal control problems on time scales. The corresponding adjoint equations were backward stochastic dynamic equations on time scales. However, there are few studies focused on BS∇Es, which stimulates us to discover more in the field.

The theory of backward stochastic differential equations on continuous-time (BSDEs) is a mature field. The general nonlinear BSDEs were first studied in the Brownian framework by Pardoux and Peng [12]:

\[
\begin{align*}
Y_t &= Y_T + \int_t^T g(u,Y_u,Z_u)\,du - \int_t^T Z_u\,dW_u, \quad t \in [0,T]_R; \\
Y_T &= \xi.
\end{align*}
\]  

(1)

A solution of equation (1), associated with the terminal value ξ and generator \( g(\omega,t,y,z) \), is a couple of adapted stochastic processes \((Y_t,Z_t)_{t \in [0,T]_R}\) which satisfy equation (1). It was followed by a long series of contributions; see, for example, [13] for a survey on BSDEs with jumps and applications to finance.

The formal studies of discrete counterpart BSDEs focus on the order of convergence as a numerical scheme, rarely the discrete scheme itself. By switching from the continuous-time Brownian motion to discrete-time, we lose the predictable representation property (PRP). It is well known that we need to include in the formulation of the BSDEs on discrete-time additional orthogonal martingales terms [14]:

\[
\begin{align*}
Y_{t+1} - Y_t &= -g(t,Y_t,Z_t) + Z_t(W_{t+1} - W_t) + N_t - N_t, \quad t \in \mathbb{N}; \\
Y_T &= \xi, \quad T \in \mathbb{N}.
\end{align*}
\]  

(2)

By the Galtchouk–Kunita-Watanabe theorem, the solution of (2) is a triple tuple \((Y_T,Z_t,N_t)_{t \in \mathbb{N}}\) where \((Z_t)_{t \in \mathbb{N}}\) is
predictable and $(N_t)_{t \in \mathbb{N}}$ is an orthogonal martingale to the integrals w.r.t. the driven process $(W_t)_{t \in \mathbb{N}}$. Bielecki [15] first studied the existence and uniqueness of the solutions of discrete BSDEs (2) by the Galtchouk–Kunita–Watanabe decomposition. For the discrete BSDEs, based on the driving process, there are mainly two formulations (see [16–22]). One is driving by a finite state process taking values from the basis vectors as in [16–18] and the other is driving by a martingale with independent increments as in [15, 19, 20]. We are more interested in the second case. The theory for the discrete-time counterpart of BSDEs is still a developing field.

We point out that finding solutions to the BSDEs (1) and (2) is equivalent to finding the martingale representation or martingale decomposition property of the random variable $\xi$. Before introducing BS V Es, a very natural and fundamental question in the time scales framework is as follows:

What is the form of the Martingale Representation Theorem on time scales?

We obtain that every square-integral martingale $M \in \mathbb{M}^2_\mathbb{T}$ with $M_0 = 0$, can be written as follows:

$$M_t = I_t(X) + N_t, \quad t \in \mathbb{T},$$

where $I_t(X)$ denotes the stochastic integral w.r.t. Brownian motion on time scales and $(N_t)_{t \in \mathbb{T}}$ is a square-integrable martingale with $N_0 = 0$ satisfying a suitable orthogonality condition that we will make more precise later. As an important consequence, we will see that decomposition (3) allows us to construct a solution to special equations driven by the Brownian motion on time scales, of the following form:

$$Y_t = \xi + \int_t^T g_0(u)\nabla u - \int_t^T Z_u \nabla W_u - (N_T - N_t), \quad t \in \mathbb{T},$$

where $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ denotes the terminal condition, $(Z_t)_{t \in \mathbb{T}}$ is predictable, and $(N_t)_{t \in \mathbb{T}}$ is a square-integrable martingale with $N_0 = 0$, orthogonal to $(W_t)_{t \in \mathbb{T}}$ in a weak suitable sense.

This paper aims to establish the existence and uniqueness of solutions to general BS V Es as follows:

$$\begin{align*}
Y_t &= Y_T + \int_t^T g(u, Y_u, Z_u)\nabla u - \int_t^T Z_u \nabla W_u - (N_T - N_t), \\
Y_T &= \xi.
\end{align*}$$

A triplet of processes $(Y_t, Z_t, N_t)_{t \in \mathbb{T}}$ will satisfy the equations (5), where $(Z_t)_{t \in \mathbb{T}}$ is predictable and $(N_t)_{t \in \mathbb{T}}$ is orthogonal to the driving processes $(W_t)_{t \in \mathbb{T}}$. The BS V Es driven by the Brownian motion on time scales are similar to traditional BSDEs driven by general $\sigma$-martingales beyond the Brownian setting [14, 23, 24].

The paper is organized as follows. In Section 2, we introduce basic notations of analysis on time scales. In Section 3, we first give some stochastic notations and results on time scales and then prove the martingale decomposition theorem on time scales. Section 4 is devoted to obtain the existence and uniqueness of solutions of BS V Es which is our main result. Also, finally, in Section 5, we apply BS V Es to financial hedging problems.

2. Preliminaries

A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. The distance between the points $t, s \in \mathbb{T}$ is defined as the normal distance on $\mathbb{R}$: $|t - s|$. In this paper, we always suppose $\mathbb{T}$ is bounded with $0 = \inf \mathbb{T}, T = \sup \mathbb{T} > 0$. The forward jump operator $\sigma$ and backward jump operator $\rho$ are, respectively, defined by the following:

$$\sigma(t) = \inf \{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup \{s \in \mathbb{T} : s < t\}.$$

We say that $t$ is right-scattered (left-scattered, right-dense, left-dense), if $\sigma(t) > t$, $\rho(t) < t$, $\sigma(t) = t$, $\rho(t) = t$ holds. $\mu(t) = \sigma(t) - t$ is called fineness, $\nu(t) = t - \rho(t)$ is called backward fineness. Consider the following:

$$I = \{t : t \text{ is left-} \text{scattered}\}.$$

The set of left-scattered points of a time scale is at most countable.

For $a, b \in \mathbb{T}$ with $a \leq b$, define the closed interval in $\mathbb{T}$ by $[a, b]_\mathbb{T} = \{t \in \mathbb{T} : a \leq t \leq b\}$.

A function $f$ is called right-dense continuous (rd-continuous) on $\mathbb{T}$ if it is continuous at every right-dense point and the left-sided limit exists at every left-dense point. Denote $\lim_{t \to (t^+)} f(s)$ by $f(t^+)$ and $\lim_{t \to (t^-)} f(s)$ by $f(t^-)$, respectively, if limits exist. It is to be noted that on the right-scattered points, $f(t^+)$ may not exist for continuous functions on time scales. If $t$ is left-scattered, then $f(t^-) = f(\sigma(t))$, right-scattered, then $f(t^-) = f(\rho(t))$.

Let $A$ be an increasing right-continuous function of finite variation defined on $\mathbb{T}$. We denote $\mu^0_A$ as the Lebesgue $\mu$-measure associated with $A$. For any $\mu^0_A$-measurable function $f : \mathbb{T} \to \mathbb{R}$, we write $\int_0^t f(t^+) dA_t$ for the integral of $f$ w.r.t. $\mu^0_A$. It is seen that the function $t \to \int_0^t f(t^+) dA_t$ is c a d la $g$ for details, please refer to [9].

For any continuous function on time scales $p : \mathbb{T} \to \mathbb{R}$, with $1 + p(t) \mu(t) \neq 0 \quad \forall t \in \mathbb{T}^k$, the $\Delta$-exponential function $e_p(t, t_0)$, defined by (12), Definition 1.38, is the solution $y(\cdot)$ of the initial value problem:

$$y^\Delta(t) = p(t) y(t), \quad \forall t > t_0, t \in \mathbb{T}, \quad y(t_0) = 1,$$
and \( e_p(t,t_0) = (1 + u(t)p(t))e_p(t,t_0) \), \( e_p(t,t_0) = (1 + v(t)p(t−))e_p(t−,t_0) \). Denote \( e_{op}(t,s) = 1/(e_p(t,s)) \).

2.1. Stochastic Calculus on Time Scales. Let \( \mathbb{R}^k \) be the \( k \)-
dimensional Euclidean space, equipped with the standard inner product \((\cdot, \cdot)\), and the Euclidean norm \(|\cdot|\), \( \mathbb{R}^{\mathbb{N}^k} \) be the collection of all \( k \times d \) real matrices, and \( z = (z_{ij})_{k \times d} \) be the matrix with \( z_{ij} = (z_{ij1}, \ldots, z_{ijd}) \) and \( |z| = \sqrt{\text{tr}(z^T z^T)} \), where \( z^T \) represents the transpose of \( z \). \( E[X][\mathcal{F}_t] \) or \( E^F \) is denoted as the conditional expectation w.r.t. the filtration \( \mathcal{F}_t \). Assume that we are working on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) with the filtration \( \{\mathcal{F}_t\}_{t \in \mathbb{T}} \) satisfying the usual conditions.

\[
E\left( (M_t - M_s)^2 \mid \mathcal{F}_s \right) = E\left( \sum_{t} (M_t - M_{t-})^2 \mid \mathcal{F}_t \right)_{t \rightarrow \infty} E[M_t - M_s] \mathcal{F}_s. \tag{11}
\]

It means \( M^2 - [M] \) is an \( F \)-martingale, which implies that \([M] - \langle M \rangle\) is also an \( F \)-martingale.

The one-dimensional Brownian motion is given in [25]. In view of the fact that the multidimensional Brownian motion can be constructed through the classical product space [26], we give a result about multidimensional Brownian motion on time scales.

\[
\langle M^1, M^j \rangle = t \delta_{ij}; 1 \leq i, j \leq d,
\]

\[
\left[ M^i, M^j \right] = \delta_{ij} \lambda ([t_0, t] \cap \mathbb{T}) + \sum_{t_0 \leq a = b < t} (W_{bn} - W_{an})^2, \forall t \in \mathbb{T}, \tag{13}
\]

where \( \lambda \) is the classical Lebesgue measure and \( \cup_{n=1}^{\infty} (a_n, b_n) = [t_0, \infty) \cap \mathbb{T} \) is the expression for the open subset \([t_0, \infty) \cap \mathbb{T} \) of \( \mathbb{R} \) as the countable of disjoint open intervals [27]. Furthermore, the vector of martingales \( M = (M^1, \ldots, M^d) \) is independent of \( \mathcal{F}_0 \).

Remark 1. The Lévy martingale characterization of Brownian motion fails on time scales, that’s a continuous martingale with \( \langle X^i, X^j \rangle = t \delta_{ij} \), cannot be a Brownian motion.

The stochastic integral on time scales in this paper is based on [9], in which the authors established the stochastic \( \mathcal{V} \)-integral w.r.t. square integral martingales and extended to special semimartingales [10]. Consider the integral w.r.t. Brownian motion \( W \) on time scales. Let \( \mathcal{L}_2^2 ((0,t]; W) \) be the space of all real-valued, predictable processes \( \phi = \{\phi_t\}_{t \in \mathbb{T}} \) satisfying the following:

\[
\|\phi\|^2_{\mathcal{L}_2^2} = E \int_0^t |\phi_t| \sqrt{\mathcal{V}(W)_t} = E \int_0^t |\phi_t| \sqrt{\mathcal{V}s_s} < \infty. \tag{14}
\]

Based on [9], Definition 3.6, define the integral as follows:

Denoted by \( \mathcal{M}^2_\mathbb{T} \), the space of square integrable martingales with \( E\{[M]_t\} < \infty \) and consider \( M \in \mathcal{M}^2_\mathbb{T} \). Since \( M^2 \) is a submartingale, following the Doob–Meyer decomposition theorem on time scales [9], there exists uniquely a natural increasing process \( \langle M \rangle = \langle \langle M \rangle \rangle_{\mathbb{T}} \), such that \( M^2 - \langle M \rangle \) is an \( F \)-martingale. The natural increasing process \( \langle M \rangle \) is called characteristic of the martingale \( M \).

Define the quadratic co-variation \( [M, N]_t \) of two processes similar to \( [\cdot, \cdot] \) [9], Definition 3.13. If \( N = M \) we write \([M, M]_t \), for \([M, M]_t \), and call it the quadratic variation of \( M \). For partitions \( \{s = t_0^0 < t_1^0 < \cdots < t_i^0 \cdots < t_n^0 = t \} \) of \([s,t]_\mathbb{T} \) with \( \max_i (p(t_i) - t_{i-1}) \leq 2^{-n}, \)

\[
\text{Lemma 1. Let } \{W_i = (W^1_i, W^2_i, \ldots, W^d_i), \mathcal{F}_i; t \in \mathbb{T} \} \text{ be a } d \text{-dimensional Brownian motion. The processes are as follows:}
\]

\[
M^i_t = W^i_t - W^i_0, \mathcal{F}_i; \quad t \in \mathbb{T}, 1 \leq i \leq d, \tag{12}
\]

are continuous, square-integrable martingales, with the following:

\[
I_t (X) = \int_0^t X_s \sqrt{\mathcal{V}s_s}, \quad X \in \mathcal{L}_2^2 ((0,t]; W). \tag{15}
\]

The space \( \mathcal{L}_2^2 ((0,t]; W) \) is actually the \( L^2 \) space under the measure given by \( \sqrt{\mathcal{V}(W)} \times d\mathbb{P} \). Now, let us define the multidimensional Stochastic Integral:

Definition 1. Let \( \{W_i, t \in \mathbb{T} \} \) be a \( d \)-dimensional Brownian motion on time scales, \( \{X_i, t \in \mathbb{T} \} \) is \( k \times d \) matrix process, \( \forall 1 \leq i \leq k, 1 \leq j \leq d, X^{ij} \in \mathcal{L}_2^2 ((0,t]; W^j) \). Define the following:

\[
I_t (X) = \int_0^t X_s \sqrt{\mathcal{V}s_s} = \int_0^t \begin{pmatrix} X_{11} & \cdots & X_{1d} \\ \vdots & \ddots & \vdots \\ X_{n1} & \cdots & X_{nd} \end{pmatrix} \begin{pmatrix} \sqrt{\mathcal{V}s_s}^1 \\ \vdots \\ \sqrt{\mathcal{V}s_s}^d \end{pmatrix}, \tag{16}
\]

to be the multidimensional stochastic integral for \( X \), also \( I_t (X) \) for short. The i-th component of \( I_t (X) \) is as follows:

\[
I^i_t (X) = \sum_{j=1}^d \int_0^t X^i_j \sqrt{\mathcal{V}s_s}, 1 \leq i \leq k, t \in \mathbb{T}. \tag{17}
\]
Clearly it can be seen that $I_t(X)$ belongs to $\mathcal{M}^2_{T,k}$, the $\mathbb{R}^k$-valued square-integral martingale on time scales. For more properties about the stochastic integral, readers could see [9, 10]. Now, we list the Itô’s formula on time scales [10].

**Theorem 1.** Let $f \in C^1(\mathbb{T} \times \mathbb{R}^d)$ and $X = (X_1, \ldots, X_d)$ be a $d$-dimensional semimartingales defined by the following:

$$
X_t^i = X_0^i + \int_0^t \phi^i_s W_s + \int_0^t \phi^i_s V_s^i,
$$

where $\int_0^t \phi^i_s V_s < \infty$, a.s. and $\phi^i \in L^2((0, t]; \mathbb{W})$. For all $t \in \mathbb{T}$, then $f(t, X(t))$ is a semimartingale and the following formula holds:

$$
f(t, X_t) = f(t_0, X_0) + \int_{(0,t]} f^\tau(s, X(s-))V_s + \sum_{i=1}^d \frac{\partial f}{\partial X_i}(s, X(s-))V X_i(s) + \frac{1}{2} \sum_{i,j} \frac{\partial^2 f}{\partial X_i \partial X_j}(s, X(s-))V X_i X_j(s)
$$

$$
+ \sum_{i} (f(s, X_i) - f(s, X_{i-})) - \sum_{i} \frac{\partial f}{\partial X_i}(s, X_{i-})V^* X_i(s)
$$

$$
= f(t_0, X_0) + \int_{(0,t]} f^\tau(s, X(s-))V_s + \sum_{i=1}^d \frac{\partial f}{\partial X_i}(s, X(s-))\phi^i_t V_s^i + \frac{1}{2} \sum_{i,j} \frac{\partial^2 f}{\partial X_i \partial X_j}(s, X(s-))\phi^i_t \phi^j_t V X_i X_j(s)
$$

$$
+ \sum_{i} (f(s, X_i) - f(s, X_{i-})) - \sum_{i} \frac{\partial f}{\partial X_i}(s, X_{i-})\phi^i_t V^* W_s^i
$$

$$
- \frac{1}{2} \sum_{i,j} \frac{\partial^2 f}{\partial X_i \partial X_j}(s, X_{i-})\phi^i_t \phi^j_t (V^* W_s^i)(V^* W_s^j),
$$

where $V^* X_i^i = X_i^i - X_{i-}^i = \phi^i_t V(t) + \phi^i_t (W_s^i - W_{s-}^i)$ for all $s \in (0, t_\mathbb{T}].$

2.2. Summary of Notations. For readers’ convenience, we collect some spaces and notations used in the paper. For any integer $k$,

$L^2_j(\mathcal{F}_T)$, short for $L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^k)$: the space of $\mathbb{R}^k$-valued random vectors $X$ that are $\mathcal{F}_t$-measurable and satisfy $E[|X|^2] < \infty$,

$L^2_{k,d}(\mathcal{F}_T)$, short for $L^2((0, t]; \mathbb{R}^k)$: the set of $(\mathcal{F}_t)_{t\in \mathbb{T}}$-predictable, $\mathbb{R}^k$-valued integrands processes satisfying $E\int_0^t |X_t|V_t < \infty$,

$\mathcal{M}^2_{T,k}$: the continuous $\mathbb{R}^k$-valued square-integral martingales on time scales with $M_0 = 0$, $\mathbb{P}$-a.s.,

$\mathcal{M}^2_{T,k}$: the subset of $\mathcal{M}^2_{T,k}$, such that for each $M \in \mathcal{M}^2_{T,k}$, there exists $X \in \mathcal{L}^2_{T,k,d}$ and $M_t = I_t(X)$,

$\mathcal{M}^2_{T,k}$: short for $\mathcal{M}^2_{T,k}$; the set of $\mathbb{R}^k$ valued, adapted and continuous processes $(\phi_t)_{t\in [0, \mathbb{T}]}$ such that $E[\sup_{t\in [0, \mathbb{T}]} |\phi_t|^2] < \infty$.

$\mathcal{M}^2_{T,k}$: subset of $\mathcal{M}^2_{T,k}$, the set of all the $k$-dimensional martingales, such that each martingale is orthogonal to that in $\mathcal{M}^2_{T,k}$.

3. Martingale Decomposition Theorem on Time Scales

In this section, we come back to the martingale decomposition problem on time scales. The fundamental tool on time scales analysis is the countable dense subset. The countable dense subset will play the same role as the dyadic rational numbers that played in the classical analysis from discrete to continuous time. For any $\delta > 0$, consider a partition of $[0, \mathbb{T}]$ inductively by letting $t_0 = 0$ and for $i = 1, 2, \ldots$, set as follows:
\[ t_i = \begin{cases} \sup B_{t_i} & \text{if } B_t \neq \emptyset, \\ \sigma(t_{i-1}) & \text{if } B_t = \emptyset, \end{cases} \]

The partition is given in [3, 9]. On time scales, the size of the interval \((t_{i-1}, t_i)\) will not converge to zero \((|t_i - t_{i-1}| \to 0)\) if \(\sigma(t_{i-1}) = t_i\). A more specific result was given by David Grow [25]. Now, we provide the optional sampling theorem on time scales to show the basic analysis method on general time scales.

**Lemma 2** (Optional Sampling Theorem on Time Scales). If \(X\) is a right-continuous martingale (submartingale) on bounded time scales \(T\) with a last element \(X_T\) and \(S_1, S_2\) are two bounded stopping times with \(S_1 \leq S_2\) on \(T\), then
\[
E[X_{S_1} | \mathcal{F}_{S_2}] = X_{S_1} \quad \text{P-a.s..} \tag{21}
\]

**Proof.** Let \([0, T]_T\) be a time scale, and let \(\Pi_n = \{t_0, t_1, \ldots, t_n\} \subseteq [0, T]_T\) be a partition of \(T\), where \(0 = t_0 < t_1 < \cdots < t_n = T\). Consider the following sequence of random times:
\[
S^n_i(w; P_n) = \rho(t_{i+1}) \quad \text{if } t_i < S_1(w) < t_{i+1}, \tag{22}
\]
and the similarly defined sequences \(\{S^n_i\}\). These are stopping times. For every fixed integer \(n \geq 1\), both \(S^n_1\) and \(S^n_2\) take on a countable number of values and we also have \(S^n_1 \leq S^n_2\). Therefore, by the discrete optional sampling theorem, we have \(\int_{\mathfrak{A}_n} X^\omega dP \leq \int_{\mathfrak{A}_n} X^{\omega'} dP\) for every \(A \in \mathcal{F}_n\). \(S_1 \leq S^n_1\) implies \(\mathcal{F}_{S^n_1} \subseteq \mathcal{F}_{S_1}\), the preceding inequality also holds for every \(A \in \mathcal{F}_{S^n_1}\).

The discrete martingale results show that the sequence of random variables \(\{X_{S^n_1}\}\) is uniformly integrable, and the same is of course true for \(\{X_{S^n_2}\}\). \(X_{S_1} = \lim_{n \to \infty} X_{S^n_1}(\omega)\) and \(X_{S_2} = \lim_{n \to \infty} X_{S^n_2}(\omega)\) hold for a.e. \(\omega \in \Omega\). It follows from uniform integrability that \(X_{S_1}, X_{S_2}\) are integrable and that
\[
\int_{\Omega} X_{S_1} dP \leq \int_{\Omega} X_{S_2} dP \quad \text{holds for every } A \in \mathcal{F}_{S_1}. \quad \Box
\]

Via the identity \(M_t = E[M_T | \mathcal{F}_T]\), each \(M_t \in \mathcal{M}_T^2\) can be identified with its terminal value \(M_T \in L^2(\Omega, \mathbb{F}_T, \mathbb{P})\) (in general the terminal variables can be extended to \(M_\infty\), if exists). \(\mathcal{M}_T^2\) becomes a Hilbert space isomorphic to \(L^2(\Omega, \mathcal{F}_T, \mathbb{P})\), if endowed with the following inner product:
\[
(M, N)_{\mathcal{M}_T^2} = E[M_T N_T], \quad \|M\|_{\mathcal{M}_T^2} = \|M_T\|_{L^2}. \tag{23}
\]

Indeed, if \(M^n\) is a Cauchy sequence for \(\| \cdot \|_{\mathcal{M}_T^2}\), then the sequence \(M^n\) is Cauchy in \(L^2(\Omega, \mathcal{F}_T, \mathbb{P})\) and so goes to a limit \(M_T\) in this space; then if \(M\) is the martingale with terminal variable \(M_T\), it belongs to \(\mathcal{M}_T^2\) and \(\|M^n - M\| \to 0\). The set of all continuous elements of \(\mathcal{M}_T^2\) denoted by \(\mathcal{M}_T^{2c}\), is a closed subspace of the Hilbert space \(\mathcal{M}_T^2\). Now, define a measure \(\mu\) on \((T \times \Omega, \mathcal{B}(T) \otimes \mathcal{F})\) by the following:
\[
\mu_B(A) = E\left[\int_A 1_A(s, \omega)\mathcal{W}(\mathcal{W})_s\right]. \tag{24}
\]

Similar to Lemma 2.2 chapter 3 in [26], we have the following:

**Lemma 3.** The \(\mathcal{L}^2_{T \times \omega}\) space is a closed space with the norm \(\| \cdot \|_{\mathcal{L}^2_{T \times \omega}}\).

**Proof.** We define a Hilbert space \(\mathcal{H}_T = L^2(T \times \Omega, \mathcal{B}(T) \otimes \mathcal{F}_T, \mu_T).\) Obviously, \(\mathcal{L}^2_{T \times \omega}\) is a subspace of \(\mathcal{H}_T\). Now, we prove that it is closed. Suppose that \(\{X^{m}\}_{m=1}^\infty\) is a convergent sequence in \(\mathcal{L}^2_{T \times \omega}\) with limit \(X \in \mathcal{H}_\omega\). Thus, the sequence has a convergent subsequence which converges almost surely under \(\mu\), also denoted by \(\{X^{m}\}_{m=1}^\infty\). Therefore \(\mu_T(t, \omega; \lim_{n \to \infty} X^n(\omega) \neq X(\omega)) \cdot \mathcal{H}_T\). the sequence has a convergent subsequence which converges almost surely under \(\mu\), also denoted by \(\{X^{m}\}_{m=1}^\infty\). Therefore \(\mu_T(t, \omega; \lim_{n \to \infty} X^n(\omega) \neq X(\omega)) \cdot \mathcal{H}_T\). Therefore, \(X\) is predictable and belongs to \(\mathcal{L}^2_{T \times \omega}\). The proof is complete. \(\Box\)

Now, define an inner product on \(\mathcal{L}^2_{T \times \omega}\) by \((X, Y) = E\left[\int_0^T (X_t, Y_t) \mathcal{W}\right].\) Recall the inner product on \(L^2(\mathcal{F}_T), (I_1, I_2) = E[I_1 I_2].\) Consider the mapping \(X \mapsto I_t(X)\) from \(\mathcal{L}^2_{T \times \omega}\) to \(L^2(\mathcal{F}_T).\) From the definition of the stochastic integral on time scales, the mapping is injective. This mapping preserves following inner products:
\[
(X, Y)_t = E\left[\int_0^T (X_t, Y_t) \mathcal{W}\right] = E[I_t(X) I_t(Y)] = (I_t(X), I_t(Y)). \tag{25}
\]

Denote \(\mathcal{B}_k(\mathcal{F}_T) = \{I_T(X) : X \in \mathcal{L}^2_{T \times \omega}\}\). Since any convergent sequence in \(\mathcal{B}_k(\mathcal{F}_T)\) is also Cauchy, its pre-image sequence in \(\mathcal{L}^2_{T \times \omega}\) must have a limit in \(\mathcal{L}^2_{T \times \omega}\). It follows that \(\mathcal{B}_k(\mathcal{F}_T)\) is closed in \(L^2(\mathcal{F}_T).\) Let us denote by \(\mathcal{M}_{T \times \omega}^{2c}\) the subset of \(\mathcal{M}_{T \times \omega}^{2c}\) which consists of stochastic integrals \(I_t(X) = \int_0^t X_t \mathcal{W}_t; \quad 0 \leq t \leq T, \quad t \in T,\) of processes \(X \in \mathcal{L}^2_{T \times \omega}\):
\[
\mathcal{M}_{T \times \omega}^{2c} = \{I(X); X \in \mathcal{L}^2_{T \times \omega}\} \subseteq \mathcal{M}_{T \times \omega}^{2c} \subseteq \mathcal{M}_{T \times \omega}. \tag{26}
\]

The following result is the "fundamental decomposition theorem" for the martingales w.r.t Brownian motion on time scales.

**Theorem 2.** For every \(M \in \mathcal{M}_{T \times \omega}^{2c}\) with \(M_0 = 0, \mathbb{P}\) - a.s., we have the following decomposition:
\[
M_t = I_t(X) + N_t, \quad \forall t \in T, \tag{27}
\]
where \(X \in \mathcal{L}^2_{T \times \omega}; I_t(X) \in \mathcal{M}_{T \times \omega}^{2c}, N_t \in \mathcal{M}_{T \times \omega}^{2c}\) with \(N_0 = 0\) and \(N_t\) is orthogonal to every element of \(\mathcal{M}_{T \times \omega}^{2c}\).
Proof. We have to show the existence of a process $Y \in \mathcal{D}_T^{2,k,d}$ such that $M_t = I_t(Y) + N_t$, where $N \in \mathcal{M}_T^{2,k}$ has the property

$$\langle I(X), N \rangle_T = 0, \quad \forall X \in \mathcal{D}_T^{2,k,d}. \tag{28}$$

Such a decomposition is unique (up to indistinguishability); indeed, if we have $M = I(Y') + N' = I(Y'') + N''$ with $Y', Y'' \in \mathcal{D}_T^{2,k,d}$ and both $N'$ and $N''$ satisfy the property, then

$$Z \equiv N'' - N' = I(Y' - Y''). \tag{29}$$

is in $\mathcal{M}_T^{2,k}$ with $Z_0 = 0$ and $\langle Z \rangle = \langle Z(0), Y'(0) - Y''(0) \rangle = 0$. Then $P[Z_t = 0, \forall t \in T] = 1$ from Lemma 2. The decomposition is unique up to indistinguishability.

Since $\mathcal{R}_T(\mathcal{F}_T)$ is a closed subspace of $L_2^1(\mathcal{F}_T)$, we can denote its orthogonal complement by $\mathcal{R}_T^\perp(\mathcal{F}_T)$. The random variable $M_T$ is in $L_2^1(\mathcal{F}_T)$, so it admits the following decomposition:

$$M_T = I_T(Y) + N_T, \tag{30}$$

where $Y \in \mathcal{D}_T^{2,k,d}$ and $N_T \in L_2^1(\mathcal{F}_T)$ satisfies $E[N_T I_T(X)] = 0; \forall X \in \mathcal{D}_T^{2,k,d}$. We construct a martingale through $N_T$ by $N_T = E(N_T | \mathcal{F}_T)$. Obviously $N \in \mathcal{M}_T^{2,k}$. Taking conditional expectation under $\mathcal{F}_T$ on $M_T$, we obtain the following:

$$M_t = I_t(Y) + N_t, \quad t \in T. \tag{31}$$

It remains to show that $N$ is orthogonal to every square-integrable martingale of the form $\{I(X); X \in \mathcal{D}_T^{2,k,d}\}$, or equivalently, that $\{N_t I_t(X); \mathcal{F}_T\}_{t \in T}$ is a martingale. It is to be noted that each martingale has a right continuous modification. So, now, we suppose that $N$ is right continuous.

According to Lemma 2, we only need to prove the following:

$$E[N_t I_S(X)] = E[N_t I_S(X)] = 0. \tag{32}$$

holds for every stopping time $S$ of the filtration $[\mathcal{F}_T]_{t \leq T}$, with $S < T$ (since $I_0(X) = 0$). The integral has $I_S(X) = I_T(\bar{X})$, where $\bar{X}(\omega) = X_t(\omega)1_{t \leq S(\omega)}$ is a process in $\mathcal{D}_T^{2,k,d}$. Therefore, by Lemma 2,

$$E[N_t I_S(X)] = E[E(N_T | \mathcal{F}_S) I_S(X)] = E[N_T I_T(\bar{X})] = 0. \tag{33}$$

The proof is complete. □

Remark 2.

(1) Note $f(t) \neq f(t-)$ at left-scattered points for continuous function on time scales, $N \in \mathcal{M}_T^{2,k}$ is continuous, but $\nabla N \neq \theta$ at left-scattered points.

(2) Let $T = \mathbb{R}, k = 1$, (Karatzas [26], Proposition 4.14 one-dimension decomposition), $\mathcal{M}_\mathbb{R}^{2,k}$ and $\mathcal{M}_\mathbb{R}^{2, \perp}$ actually coincide, the component $N_t$ in the decomposition is actually $\nabla N_t = 0$. The predictable process space is isomorphic to the adapted process space [28].

(3) For the Brownian motion on general time scales, even on the augmentation filtration of the filtration generated by $W_t$, $\mathcal{M}_T^{2,k}$ and $\mathcal{M}_T^{2, \perp}$ do not coincide, see the following example.

(4) Let $T = \mathbb{N}$ (Follmer, Hans, [29], Theorem 10.18), it is the discrete time version of the Kunita–Watanabe decomposition w.r.t a sequence of normal distribution random variables.

(5) The orthogonality condition given in Theorem 2 is the weak orthogonality condition in continuous-time, but we call it strong orthogonality condition on time scales. That is, for scattered points $t$: $E[(X_{a(t)} - X_t)(Y_{a(t)} - Y_t)|\mathcal{F}_T] = 0$.

Example 1. For a martingale $W_T^2 - t$ on time scales $T = [0,1] \cup \{3,4,5\}$, we have the following:

$$W_T^2 - t = (W_t - W_s)^2 - (t-s) + 2W_s(W_t - W_s) + W_t^2 - s. \tag{34}$$

For $t = 3, M_t = I_3 + N_3$,

$$I_3 = \int_0^1 W_t^2 \nabla W_s + W_t^2 \nabla W_s + 2W_1 (W_3 - W_1) = W_3^2 - 1 + 2W_1 (W_3 - W_1),$$

$$N_3 = (W_3 - W_s)^2 - (3-1),$$

with $E[(I_3 - I_1)(N_3 - N_1)|\mathcal{F}_T] = E[(2W_1 (W_3 - W_1))(W_3 - W_1)^2 - 2)|\mathcal{F}_T] = 0$.

4. BS $\nabla$ Es Driven by Brownian Motion on Time Scales

In this section, we denote $(\Omega, \mathcal{F}, F, P)$ to be a probability space equipped with a complete filtration $F = (\mathcal{F}_t)_{t \in T}$ generated by a d-dimensional Brownian motion $W_t$ on time scales, and augmented by all the $P$-null sets in $\mathcal{F}_T$.

For simplicity, we consider the following general BS $\nabla$ Es on time scales:

$$Y_t = Y_T + \int_t^T g(u,Y_u,Z_u)\nabla u - \int_t^T Z_u \nabla W_u - (N_T - N_t), \quad Y_T = \xi. \tag{35}$$

We call $g$ the driver of the BS $\nabla$ Es and the pair $(\xi, g)$ the data of the BS $\nabla$ Es.

A solution to BS $\nabla$ E is a triple of processes $(Y, Z, N)$ satisfying (35), such that $Y$ is a $\mathbb{R}^d$-valued, continuous and adapted process, $Z$ is a $\mathbb{R}^{d \times d}$-valued and predictable process and $N$ is a martingale orthogonal to $W$. For terminal condition $\xi$ and generator $g$, we make the following assumptions:

Assumption 1. We assume that for any inter $k$,
(H1) $g$ is defined as $\Omega \times \mathcal{T} \times \mathbb{R}^k \times \mathbb{R}^{kd} \rightarrow \mathbb{R}^k$, such that $\forall (y, z) \in \mathbb{R}^k \times \mathbb{R}^{kd}$, $g(\cdot, y, z)$ is $\mathcal{F}_t$ progressively measurable,

(H2) $\int_0^T |g(\cdot, 0, 0)|^2 \text{d}t \in L^2(\mathcal{F}_T)$, \hfill (36)

(H3) $g$ satisfies Lipschitz condition w.r.t. $(y, z)$: there exists a constant $L > 0$, $\forall y, y' \in \mathbb{R}^k$, $z, z' \in \mathbb{R}^{kd}$, s.t.

\[ |g(t, y, z) - g(t, y', z')| \leq L(|y - y'| + |z - z'|), \] \hfill (37)

(H4) $\xi \in L^2(\mathcal{F}_T)$.

**Theorem 3.** (Main Result) If Assumption 1 is satisfied, BSDE with parameters $\xi$ and $g_0$ exists a unique triple solution $(Y, Z, N) \in \mathcal{F}^2 \times L_{\mathcal{F}T}^{1, k,d} \times \mathcal{M}^{2, k,d}$. Moreover,

\[ |Y_t|^2 + \mathbb{E}^\mathcal{F} \int_0^T \left\{ \frac{1}{2} Y_{s-}^2 + |Z_s|^2 \right\} e^\beta(s, t) \text{d}s + \int_0^T e^\beta(s, t) \mathbb{E}^\mathcal{F} \text{d}s \leq \mathbb{E}^\mathcal{F} \int_0^T |g_0(s)|^2 e^\beta(s, t) \mathbb{E}^\mathcal{F} \text{d}s, \] \hfill (38)

where $\beta$ is a positive constant with $1 + \mu(t, \beta) > 0$ for all $t \in \mathcal{T}_T$.

**Proof.** Existence: We denote $M_t = \mathbb{E}^\mathcal{F}_t[\xi + \int_0^T g_0(s) \mathbb{E}^\mathcal{F} \text{d}s]$. Apparently, $M_t$ is a square integrable martingale. Following the martingale decomposition theorem on time scales, there exists a unique predictable process $Z_t \in L_{\mathcal{F}T}^{1, k,d}$ and $N \in \mathcal{M}^{2, k,d}$, s.t. $M_t = M_0 + \int_0^T Z_s \mathbb{V}W_s + N_t$, $M_T = M_0 + \int_0^T Z_s \mathbb{V}W_s + N_T$. Then $M_t = M_T - \int_t^T Z_s \mathbb{V}W_s - (N_T - N_t)$. As also $M_T = \xi + \int_0^T g_0(s) \mathbb{E}^\mathcal{F} \text{d}s$, denote $Y_t = M_t - \int_t^T g_0(s) \mathbb{E}^\mathcal{F} \text{d}s$ and the differential form: $\mathbb{V}Y_t = -g_0(t) \mathbb{V}t + Z_t \mathbb{V}W_t + \mathbb{V}N_t$. We have the following:

\begin{align}
|Y_t|^2 |e^\beta(T, t) &= |Y_t|^2 + \int_t^T \beta e^\beta(s, t) Y_{s-} \mathbb{V}Y_s + 2 \int_t^T e^\beta(s, t) Y_{s-} \mathbb{V}Y_s + \int_t^T e^\beta(s, t) \mathbb{V}[Y]_s + S \\
&= |Y_t|^2 + \int_t^T \beta e^\beta(s, t) Y_{s-} \mathbb{V}Y_s - 2 \int_t^T e^\beta(s, t) Y_{s-} g_0(s) \mathbb{V}s + 2 \int_t^T e^\beta(s, t) Y_{s-} \mathbb{V}Z_s + 2 \int_t^T e^\beta(s, t) \mathbb{V}N_s + S \\
&+ 2 \int_t^T e^\beta(s, t) Y_{s-} \mathbb{V}N_s + \int_t^T e^\beta(s, t) Z_s^2 \mathbb{V}[W]_s + \int_t^T e^\beta(s, t) \mathbb{V}[N]_s + S,
\end{align}

where

\[ S = \sum_{s \in (T, T]} e^\beta(s, t) \left\{ |Y_s|^2 - |Y_{s-}|^2 - 2 |Y_{s-}| (|Y_s| - |Y_{s-}|) - (|Y_s| - |Y_{s-}|)^2 \right\} = 0. \] \hfill (40)

Thus, we have
\[ |Y_t|^2 + \int_t^T \beta \phi(s, t) Y_s^2 \, ds + \int_t^T \phi(s, t) Z_s^2 \, d\mathbb{V}s + \int_t^T \phi(s, t) \mathbb{V}[N_s] \]
\[ = \|\xi\|^2 \phi(T, t) + 2 \int_t^T \phi(s, t) Y_s g_0(s) \, ds - 2 \int_t^T \phi(s, t) Y_s Z_s \, d\mathbb{V}s - 2 \int_t^T \phi(s, t) Y_s \, d\mathbb{V}N. \]  

(41)

Since \( g_0(s) \) is \( \mathcal{F}_s \)-predictable, the above integrals with respect to \( \mathbb{V} \) and \( \mathbb{N} \) all belong to \( \mathcal{M}^{2, k}_{T,k} \). By taking the expectation with respect to \( \mathcal{F}_t \), we obtain the following:

\[ |Y_t|^2 + E\mathcal{F}_t \left[ \begin{align*} &|\delta Y_s|^2 + |\delta Z_s|^2 \right] \phi(s, t) \, ds + E\mathcal{F}_t \int_t^T \phi(s, t) \mathbb{V}[N_s] \]
\[ \leq E\mathcal{F}_t |\xi|^2 \phi(T, t) + E\mathcal{F}_t \left[ \begin{align*} &\frac{\beta}{2} |\delta Y_s|^2 + \frac{2}{\beta} |g_0(s)|^2 \right] \phi(s, t) \, ds. \]  

(42)

So, we obtain the inequality.

Now, we suppose there are two solutions \((Y^1, Z^1, N^1)\) and \((Y^2, Z^2, N^2)\). Define \( \delta Y_t = Y^1_t - Y^2_t, \) \( \delta Z_t = Z^1_t - Z^2_t, \)
\( \delta N_t = N^1_t - N^2_t \), then,

\[ \begin{align*}
\delta Y_t &= - \int_t^T \delta Z_s \mathbb{V}W_s - \delta N_T + \delta N_t, \\
\delta Y_T &= 0.
\end{align*} \]  

(43)

Then, \( (\delta Y, \delta Z, \delta N) \) satisfy BSDE(35) with \( \xi = 0 \) and \( g = 0 \). By the inequality,

\[ |\delta Y_0|^2 + E \int_0^T \left[ \frac{\beta}{2} |\delta Y_s|^2 + |\delta Z_s|^2 \right] \phi(s, 0) \, ds \]
\[ + \int_0^T \phi(s, 0) \mathbb{V}[\delta N] \leq 0. \]  

(44)

By the continuity of \( \delta Y_s, \delta Z_s = 0, \delta N_s = 0 = 0, \) \( \) \( \) \( \mathcal{P} \)-a.s. The uniqueness is proved. \( \square \)

Introduce the new norms on spaces:

\[ \mathcal{S}^2_{T,k}: \|Y\|_{\mathcal{S},T,k} = \left( E \int_0^T |Y_s|^2 \phi(s, 0) \, ds \right)^{1/2}, \]
\[ \mathcal{L}^2_{T,k}: \|Z\|_{\mathcal{L},T,k} = \left( E \int_0^T |Z_s|^2 \phi(s, 0) \, ds \right)^{1/2}, \]
\[ \mathcal{M}^2_{T,k}: \|N\|_{\mathcal{M},T,k} = \left( E \int_0^T \phi(s, 0) \mathbb{V}[N_s] \right)^{1/2}. \]  

(45)

for any positive integer \( k \). Apparently, for each \( \beta > 0 \), \( \| \cdot \|_{\mathcal{S},T,k} \)

is equivalent to \( \| \cdot \|_{\mathcal{S},T} \) which is the original norm on the corresponding space. Now, we start to prove Theorem 3:

\[ \begin{align*}
Y_t &= \xi + \int_t^T g(s, Y_s, Z_s) \, ds - \int_t^T Z_s \mathbb{V}W_s - N_T + N_t, \quad \forall t \in [0, T].
\end{align*} \]  

(46)

Hence, we can define an operator as follows:

\[ (Y, Z, N) = I([y, z, n]): \mathcal{S}^2_{T,k} \times \mathcal{L}^2_{T,k} \times \mathcal{M}^2_{T,k} \rightarrow \mathcal{S}^2_{T,k} \times \mathcal{L}^2_{T,k} \times \mathcal{M}^2_{T,k} \]  

(47)

We can prove that \( I \) forms a contraction mapping on the Banach space \( \mathcal{S}^2_{T,k} \times \mathcal{L}^2_{T,k} \times \mathcal{M}^2_{T,k} \). Take any \((y^1(\cdot), z^1(\cdot), n^1(\cdot)), (y^2(\cdot), z^2(\cdot), n^2(\cdot)) \in \mathcal{S}^2_{T,k} \times \mathcal{L}^2_{T,k} \times \mathcal{M}^2_{T,k}, \) we denote the following:

\[ \begin{align*}
(Y^1, Z^1, N^1) &= I([y^1, z^1, n^1]), \\
(Y^2, Z^2, N^2) &= I([y^2, z^2, n^2]).
\end{align*} \]  

(48)

and \( \delta y = y^1 - y^2, \) \( \delta z = z^1 - z^2, \) \( \delta n = n^1 - n^2. \) By equation (38), we obtain the following:
\[
E[\delta Y_0]^2 + E\left\{ \int_0^T \left[ \frac{1}{2} |\delta Y_s|^2 + |\delta Z_s|^2 \right] e_\beta(s,0) \, ds + \int_0^T e_\beta(s,0) \delta Y_s, \delta N_s \right\} \\
\leq \frac{2}{\beta} E \int_0^T e_\beta(s,0) \left| g(s, y_s^1, z_s^1) - g(s, y_s^2, z_s^2) \right|^2 \, ds \\
\leq \frac{2}{\beta} E \int_0^T e_\beta(s,0) |L| |\delta y_s|^2 + |L| |\delta z_s|^2 \, ds \\
\leq \frac{2}{\beta} E \int_0^T e_\beta(s,0) \left[ 2L^2 |\delta y_s|^2 + 2L^2 |\delta z_s|^2 \right] \, ds.
\]

(49)

We obtain as follows:

\[
E\left\{ \int_0^T \left[ \frac{1}{2} |\delta Y_s|^2 + |\delta Z_s|^2 \right] e_\beta(s,0) \, ds + \int_0^T e_\beta(s,0) \delta Y_s, \delta N_s \right\} \\
\leq \frac{4L^2}{\beta} E \int_0^T e_\beta(s,0) \left| \delta y_s + |\delta z_s|^2 \right| \, ds.
\]

(50)

Let \( \beta = 8(1 + L^2) \),

\[
|\delta Y|_{\mathcal{T}, \mathcal{F}, T}^2 + |\delta Z|_{\mathcal{T}, \mathcal{F}, T}^2 + |\delta N|_{\mathcal{T}, \mathcal{F}, T}^2 \leq \frac{1}{2} \left( |\delta y|_{\mathcal{T}, \mathcal{F}, T}^2 + |\delta z|_{\mathcal{T}, \mathcal{F}, T}^2 \right).
\]

(51)

We can obtain that the operator \( I \) is contractive. Hence, in this case, there exists a unique fixed point \( (Y, Z, N) \in \mathcal{D}^2 \times \mathcal{D}^2 \times \mathcal{N}^2 \) for the map \( I \). The proof is complete.

□

4.1. Linear BSVEs. As a particular Lipschitz case, we give a linear case result. First, we cite the exponential martingale on time scales [9].

Example 2. Let \( M \) be a bounded martingale satisfying \( \nabla M_s = M_s - M_{s-} \neq -1 \) for any \( s \in (0, \mathcal{T}] \), \( s \in \mathbb{T} \). Set

\[ \tilde{M}_t = M_t - \sum_{s \in (0,t]} \left( M_s - M_{s-} \right), \]

\[ \mathcal{E}_t = \left\{ \prod_{s \in (0,t]} \left( 1 + \nabla M_s \right) \right\} \exp \left\{ \tilde{M}_t - \frac{1}{2} \tilde{M}_t \right\}. \]

(52)

Then, \( \mathcal{E}_t \) satisfies the Doelen’s exponential equation

\[ \mathcal{E}_t = 1 + \int_{(0,t]} \mathcal{E}_s \nabla M_s. \]

(53)

Lemma 5. Let \( a, b, c \) be predictable bounded processes, let \( \mathcal{E} \) be the exponential martingale of the martingale \( M_t = \int_0^t b_s \nabla W_s \), define \( y_t = e_a(t, 0), \Gamma_t = y_t \mathcal{E} \). Suppose that \( \mathcal{E} \) is a positive uniformly integrable martingale;

\[ \mathbb{E}[(\sup_{t \in [0,T]} y_t) \mathcal{E}^2] < \infty. \]

If the linear backward equation

\[ -\nabla Y_t = (a + b_t Z_t + c_t) \nabla Y_t - Z_t \nabla W_t - \nabla N_t, Y_T = \xi, \]

has a solution \( (Y, Z, N) \), then \( Y \) is given by

\[ Y_t = E \left[ \xi_T + \int_t^T \Gamma_s \nabla s | \mathcal{F}_t \right]. \]

(55)

Proof. By Girsanov’s theorem on time scales [30], \( \tilde{W}_t = W_t - \int_0^t b_s \nabla s \) is a \( \mathcal{Q} \)-martingale, the probability measure \( \mathcal{Q} \) which has density \( \mathcal{E} \) with respect to \( \mathcal{P} \) on \( \mathcal{F}_T \). Under the measure \( \mathcal{Q} \), the equation becomes

\[ -\nabla Y_t = (a + b_t Z_t + c_t) \nabla Y_t - Z_t \nabla W_t - \nabla N_t, Y_T = \xi, \]

Applying Itô’s formula to the process \( yY \), we obtain the following:

\[ y_t Y_t = y_0 Y_0 + \int_0^t y_s Z_s \nabla W_s + \int_0^t y_s \nabla N_s - \int_0^t c_s y_s \nabla [\tilde{W}_s] + S, \]

(56)

where \( S = \sum_{s \in (0,t]} (y_s Y_s - y_s Y_{s-} - (y_s (Y_s - Y_{s-}))) = 0 \). Since

\[ \int_0^t y_s Z_s \nabla W_s \text{ and } \int_0^t y_s \nabla N_s \]

are martingales,

\[ y_t Y_t = E\left[ \xi_T + \int_t^T c_s y_s \nabla s | \mathcal{F}_t \right], \]

(57)

and consequently,

\[ \Gamma_t Y_t = E\left[ \xi_T + \int_t^T c_s \nabla s | \mathcal{F}_t \right]. \]

(58)

This concludes the proof.

□

4.2. Comparison Theorem. The linear case allows us to obtain a comparison theorem on time scales.

Theorem 4. Let \( (Y_1, Z_1, N_1), (Y_2, Z_2, N_2) \) be the corresponding solutions to the data \( (g^1(t, Y_1^1, Z_1^1), \xi_1^1), (g^2(t, Y_2^1, Z_2^1), \xi_2^1) \) and suppose \( g^1, g^2 \) satisfy conditions \((H2)\), \((H3)\). Assume that \( \xi_1 \geq \xi_2 \) and satisfy \((H4)\),

\[ g^1(t, Y_1^2, Z_1^2) \geq g^2(t, Y_2^2, Z_2^2) \text{ a.s.} - \mathcal{P}. \]

Then, for any \( t \), we have the following:

\[ Y_1^t \geq Y_2^t \text{ a.s.} - \mathcal{P}. \]

(59)

Proof. We define \( \delta Y_t = Y_1^t - Y_2^t, \delta Z_t = Z_1^t - Z_2^t, \delta N_t = N_1^t - N_2^t, \).

Then, \( \delta Y \) solves the following linear BSVE

\[ -\nabla Y_t = (a + b_t Z_t + c_t) \nabla Y_t - Z_t \nabla W_t - \nabla N_t, Y_T = \xi, \]

has a solution \( (Y, Z, N) \), then \( Y \) is given by

\[ Y_t = E \left[ \xi_T + \int_t^T \Gamma_s \nabla s | \mathcal{F}_t \right]. \]
\[ -\nabla Y_t = (a_t \delta Y_t + b_t \delta Z_t + c_t) \nabla t - \delta Z_t \nabla W_t - \nabla \delta N_t, \]
\[ \delta Y_t = \xi_1 - \xi_2, \]
where
\[ a_t = g_1(t, Y^1_t, Z^1_t) - g_1(t, Y^1_t, Z^1_t), \]
\[ b_t = g_2(t, Y^2_t, Z^2_t) - g_2(t, Y^2_t, Z^2_t), \]
\[ c_t = g_3(t, Y^3_t, Z^3_t) - g_3(t, Y^3_t, Z^3_t). \]

According to the linear BS \( \diamond \) Es, if the coefficients \( a_t, b_t, c_t \) satisfy the suitable conditions, we have the following:
\[ \Gamma_t \delta Y_t = E \left[ \Gamma_0 (\xi_1 - \xi_2) + \int_0^T \delta c_t \Gamma_t \nabla s_t | \mathcal{F}_t \right]. \]

Assume now that \( \xi_1 \geq \xi_2 \), and for any \( t, c_t \geq 0 \) a.s. - \( \mathcal{P} \).
Then, for any \( t, Y^1_t \geq Y^2_t \) a.s. - \( \mathcal{P} \).

**Example 3.** Consider \( T = [0,1] \cup [2,3] \), the BS \( \diamond \) E:
\[ \begin{cases} 
Y_t = Y_t + \int_0^T g(u, Y_u, Z_u) \nabla u - \int_0^T Z_u \nabla W_u - (N_t - N_0), \\
Y_T = \xi. 
\end{cases} \]

The solution \( (Y_t, Z_t, N_t) \) on (2.3) equals to the classical solution on continuous-time with terminal condition \( Y_3 = \xi \) and \( N_3 - N_0 = 0 \). For right-scattered point \( t = 1 \),
\[ Y_1 = Y_2 + \int_1^2 g(u, Y_u, Z_u) \nabla u - \int_1^2 Z_u \nabla W_u - (N_2 - N_1) = Y_2 + g(2, Y_2, Z_2) - Z_2 (W_2 - W_1) - (N_2 - N_1). \]

In the conditional expectation form:
\[ Y_1 = E[Y_2 + g(2, Y_2, Z_2) | \mathcal{F}_1]. \]

Recall (Example 1), in a particular case, when
\[ Y_2 + g(2, Y_2, Z_2) = W^2 - 2, \]
\[ W^2 - 2 = W^2_1 - 1 + 2W_1 (W_2 - W_1) + (W^2_2 - W^2_1) - 1. \]
\[ Y_1 = W^2_1 - 1, \quad Z_2 = 2W_1, \quad N_T - N_0 = (W^2_2 - W^2_1) - 1. \]

### 5. Applications to the Financial Market

Consider the financial market on general time scales, such as \( \{0,1\} \cup [2,3] \), the market is no longer complete with any time gap. Let us consider a riskless bond \( P^0_t \) solution to
\[ \nabla P^0_t = r_t P^0_t \nabla t, \]
with \( P^0(0) = 1 \) and a risky asset \( S_t \)
\[ \begin{cases} 
\nabla S_t = b_t S_t \nabla t + \sigma_t S_t \nabla W_t, \\
S_0 = s, 
\end{cases} \]
where \( W \) is a standard Brownian motion on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). The solution to equation (71) is given in [10]. We will denote by \( V_t = \{ V_t; t \in [0,T] \} \) the wealth stochastic process representing the total value of the investor’s portfolio at time \( t \), given an initial wealth \( V_0 > 0 \). In particular, the investor, at a given time \( t \), holds \( \pi_t \) share of the risky stock. The trading strategy \( (V_t, \pi) \) is called self-financing if
\[ V_t = V_0 + \int_0^t (V_t - \pi_t) \frac{\nabla P^0}{P^0} + \pi_t \nabla S_t S_t^{-1}, \]
or equivalently,
\[ V_t = V_0 + \int_0^t ([r_t V_t + (b_t - r_t) \pi_t] \nabla t + \sigma_t \pi_t \nabla W_t), \]
where \( \pi_t \) is a predictable process.

In incomplete markets, Föllmer [31] introduced the broader concept of the mean-self-financing strategy. The cost process \( C \) is defined by the difference
\[ C_t = V_t - \int_0^t ([r_t V_t + (b_t - r_t) \pi_t] \nabla t + \sigma_t \pi_t \nabla W_t), \]
the hedging strategy \( (\hat{V}_t, \hat{\pi}) \) against contingent claims \( H \in L^2(\mathcal{F}_T) \) is called mean-self-financing if the corresponding cost process \( C \) is a martingale. That is,
\[ \begin{cases} 
\hat{V}_t = V_T - \int_0^T ([r_t \hat{V}_t + (b_t - r_t) \pi(t)] \nabla t + \sigma_t \pi(t) \nabla W_t) - (C_T - C_0), \\
\hat{V}_T = H, 
\end{cases} \]
where \( C_t \) is a martingale orthogonal to \( \int_0^t \sigma_t \nabla W_t \). The process \( C_t \) is the tracking error. In particular, at the terminal time, the tracking error measures the spread between the contingent claim and the portfolio value, and \( C_t \) corresponds to the cost process introduced by Föllmer and Schweizer [32]. Notice that the tracking error of the self-financing hedging strategy equals to zero.

When hedging the contingent claim with terminal payoff given by \( H = h(S_T) \), we assume the risk-free rate \( r_t \), the trend \( b_t \), and the volatility \( \sigma_t \) to be constant. We have the following BS \( \diamond \) E:
\[ \begin{cases} 
\hat{V}_t = V_T - \int_0^t ([r_t \hat{V}_t + (b_t - r_t) \pi(t)] \nabla t + \sigma_t \pi(t) \nabla W_t) - (C_T - C_0), \\
\hat{V}_T = h(S_T), 
\end{cases} \]
that is, a linear BS \( \diamond \) E. Applying the linear BS\ E(54), we have that the value \( V_0 \) of the portfolio at initial time is given by (57) under some measure \( Q \):
\[ Y_0 = E_Q[\epsilon_{0,T} h(S_T)]. \]

\( Q \) is the minimal martingale measure introduced by Föllmer and Schweizer [32], which coincides with variance-optimal signed martingale measure [33].
Remark 4.
(1) On continuous time, the market driven by the Brownian motion is complete. The option price is the well-known.
\[
Y_0 = E_Q[\exp(-rT)h(S_T)],
\]
where the $Q$ turn to be the unique equivalent probability measure. Let $\theta = (b - r/\sigma), (dQ/dP)|_{\mathcal{F}_t} = \exp(-\int_0^t \theta dW_t - (1/2) \int_0^t \theta^2 dt).
(2) On discrete-time, the market driven by the Brownian motion is incomplete. In incomplete market, the equivalent probability measure contains more than one measure according to the so-called Second Fundamental Theorem of Asset Pricing. Upon imposing additional assumptions on the martingale measure, one can distinguish a unique measure and hence, a unique price. The discrete case can be seen in [29].

6. Conclusions
In this paper, we provided martingale decomposition on time scales. This allows us to prove the existence and uniqueness of solution for the backward stochastic dynamic equations on time scales.

Data Availability
The data used to support the findings of this study are included within the article.

Disclosure
This work was preprinted on arxiv named “Martingale Decomposition and BSDE on Time Scales” [34].

Conflicts of Interest
The authors declare that there are no conflicts of interest regarding the publication of this paper.

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