Almost Vanishing Polynomials for Sets of Limited Precision Points.

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Abstract

From the numerical point of view, given a set \( X \subset \mathbb{R}^n \) of \( s \) points whose coordinates are known with only limited precision, each set \( \tilde{X} \) of \( s \) points whose elements differ from those of \( X \) of a quantity less than the data uncertainty can be considered equivalent to \( X \). We present an algorithm that, given \( X \) and a tolerance \( \varepsilon \) on the data error, computes a set \( \mathcal{G} \) of polynomials such that each element of \( \mathcal{G} \) "almost vanishing" at \( X \) and at all its equivalent sets \( \tilde{X} \). Even if \( \mathcal{G} \) is not, in the general case, a basis of the vanishing ideal \( \mathcal{I}(X) \), we show that, differently from the basis of \( \mathcal{I}(X) \) that can be greatly influenced by the data uncertainty, \( \mathcal{G} \) can determine a geometrical configuration simultaneously characterizing the set \( X \) and all its equivalent sets \( \tilde{X} \).

Keywords: Vanishing ideal, border and Gröbner bases, limited precision data.

1 Introduction

Let \( P = \mathbb{R}[x_1, \ldots, x_n] \) be the polynomial ring in \( n \) indeterminates over the reals and let \( X = \{p_1, \ldots, p_s\} \) be a finite set of points of \( \mathbb{R}^n \).

It is well known [9, 10] that the vanishing ideal \( \mathcal{I}(X) \subset P \) of all polynomials which vanish at \( X \) can be described by a Gröbner basis [3], if a term ordering is chosen, or by a border basis, if an appropriate basis of the quotient space \( P/\mathcal{I}(X) \) is given.

However, it is also well known that small perturbations of the points of \( X \) can cause structural changes in the bases of \( \mathcal{I}(X) \) [10, 14] as illustrated in the following example.

Example 1.1 Let \( \sigma \) be the DegLex term ordering with \( x > y \). Given the set of points \( X = \{(1,1), (3,2), (5,1,3)\} \), the \( \sigma \)-Gröbner basis \( GB \) of the vanishing ideal \( \mathcal{I}(X) \) is given by:

\[
GB = \left\{ \begin{array}{l}
y^2 - 20x + 37y - 18 \\
x y - 43x + 81y - 39 \\
x^2 - 90.1x + 172.2y - 83.1
\end{array} \right.
\]

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The set $GB$ is also the border basis of $I(\mathcal{X})$, founded on the set $\mathcal{O} = \{1, y, x\}$ whose residue classes span $P/I(\mathcal{X})$.

A slightly perturbation of the point $(5,1,3)$ leads to a new set of points $\mathcal{X} = \{(1,1), (3,2), (5,3)\}$. The $\sigma$-Gröbner basis $\widetilde{GB}$ of the vanishing ideal $I(\mathcal{X})$ is completely different from $GB$:

$$\widetilde{GB} = \begin{cases} x - 2y + 1, \\ y^3 - 6y^2 + 11y - 6, \end{cases}$$

Further, all the border bases of $I(\mathcal{X})$ also present a structural discontinuity, since the residue classes of the set $\mathcal{O}$ do not span the space $P/I(\mathcal{X})$.

In the previous example the structural changes happen since the input points $\mathcal{X}$ are almost aligned, while the slightly perturbed points $\mathcal{X}$ are exactly aligned. This example also illustrates that, if we deal with a set $\mathcal{X}$ of points known with limited precision, the exact bases of $I(\mathcal{X})$ could not highlight some pleasant geometrical properties almost satisfied by the points $\mathcal{X}$.

In this paper we present an algorithm that computes, given a set of points known with limited precision, a set of polynomials allowing to recognize if such points almost lie on a particularly simple geometrical configuration.

Given a set $\mathcal{X}$ of points whose coordinates are known with limited precision, each $p$ of $\mathcal{X}$ represents a “cloud” of points: every point $\tilde{p}$ which differs from $p$ by less than the data uncertainty can be considered computationally equivalent to $p$. Analogously, an input set obtained from $\mathcal{X}$ replacing some $p$ by its perturbation $\tilde{p}$ can be considered an admissible perturbation computationally equivalent to $\mathcal{X}$. It is then clear that the knowledge of $\mathcal{X}$ with limited precision, combined with the structural discontinuity of a basis, points out that a significant characterization of $I(\mathcal{X})$ can be a very tricky problem. In fact the structure of a basis can drastically change choosing different admissible input sets and moreover a blindly choice of a basis can hidden significant geometrical properties of $\mathcal{X}$. For this reason exact methods applied to limited precision data can produce meaningless results.

The problem of the characterization of the vanishing ideal of a set of perturbed points has been studied by several authors from different points of view. In [11], Sauer describes a method, suitable for numerical computations, which computes a small degree algebraic variety containing the input points. In [7], Heldt et al. present an algorithm, based on the singular value decomposition of matrices, that computes, without using explicitly the estimation of the data error, a set of polynomials which assume particularly small values at the input points.

In [2], Abbott et al. present an algorithm that computes, explicitly using the tolerance on the data error, a monomial set $\mathcal{O}$ which, in most cases, is a basis of $P/I(\mathcal{X})$ and of $P/I(\mathcal{X})$ for all the admissible perturbations $\mathcal{X}$ so that the $\mathcal{O}$-border basis of all the vanishing ideals $I(\mathcal{X})$ can be obtained.
Given a set $X$ of limited precision points and a tolerance $\varepsilon$ on the data error, we focus our attention on the possibility of simultaneously characterizing, with a set of polynomials, the set $X$ together with all its admissible perturbations. To this aim, we present an algorithm that computes an order ideal $O$ and a polynomial set $G$, whose supports are defined by $O$, having the following properties.

1. The elements of $G$ are almost vanishing, w.r.t. the norm of their coefficient vectors, at $X$ and at each admissible perturbation $\tilde{X}$.
2. For each admissible perturbation $\tilde{X}$, the set $\{r(\tilde{X})|r \in O\}$ consists of independent vectors, up to the first order error analysis.
3. For each leading term $t$ of $g \in G$ there could be an admissible perturbation $\tilde{X}_g$ such that $t(\tilde{X}_g)$ depends on $\{r(\tilde{X}_g)|r \in O\}$.

Condition 3 implies that for each $g \in G$ there could exist a polynomial $\hat{g}$ with the same support of $g$ and similar coefficients which vanishes at $\tilde{X}_g$. If it is the case, the algorithm determines a geometrical structure, given by $g$, almost satisfied by all the admissible perturbations of $X$ and similar to a geometrical structure, given by $\hat{g}$, exactly satisfied by the admissible perturbation $\tilde{X}_g$.

As illustrated in the numerical examples in Section 6, it can happen that there exists a single admissible perturbation $\tilde{X}$ satisfying the previous property for all the polynomials $g \in G$. In this case it is very natural to consider $X$ as a possible exact input set, that is the input in absence of data error. Moreover, even if in general $G$ is not a basis of $I(X)$, it is analogously natural to consider $G$ as a common characterization of all the admissible perturbations of $X$.

Finally, once again as we will show in Section 6, it can happen that $\tilde{X}$ turns out to be the exact zero set of the polynomials of $G$ so that, in this case, $G$ is a Gröbner basis of $I(\tilde{X})$.

There is an evident open problem regarding the algorithm and its results: the existence and possibly the determination of the admissible perturbation $\tilde{X}$. We will show, once again in Section 6, that there are cases when $\tilde{X}$ does not exist. Then an open problem is to find conditions for the existence of $\tilde{X}$ and, in case of existence, to determine it explicitly. The numerical tests suggest that in case of non existence, this can be due to two possible causes: the algorithm does not recognize a possible element of $O$ or it detects some geometrical configurations, close to the points of $X$, which are incompatible with each other. The study of such open problem will be the subject of our future work.

The paper improves and formalizes a 2005 Preprint of the author [5] and it is organized as follows. Section 2 shows some basic concepts. Section 3 contains the description of the algorithm. Section 4 and Section 5 describe the numerical properties of $O$ and $G$, respectively. Finally, Section 6 presents some examples illustrating the behaviour of our method.
2 Preliminaries

In order to formalize the idea of perturbed points, we recall the definitions of empirical point and of admissible perturbation [14, 2].

**Definition 2.1** Let \( p = (c_1, \ldots, c_n) \) be a point of \( \mathbb{R}^n \) and let \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \), with each \( \varepsilon_i \in \mathbb{R}^+ \), be the vector of the componentwise tolerances. An empirical point \( p^\varepsilon \) is the pair \( (p, \varepsilon) \), where we call \( p \) the specified value and \( \varepsilon \) the tolerance. A point \( \tilde{p} = (\tilde{c}_1, \ldots, \tilde{c}_n) \in \mathbb{R}^n \) is called an admissible perturbation of \( p \) if \( \tilde{c}_i = c_i + e_i, |e_i| < \varepsilon_i, i = 1, \ldots, n. \)

Given a finite set \( \mathcal{X}^\varepsilon \) of empirical points all sharing the same tolerance \( \varepsilon \), we can formalize the concept of a set \( \tilde{\mathcal{X}} \) “equivalent” to \( \mathcal{X} \) w.r.t. the data accuracy.

**Definition 2.2** Let \( \mathcal{X}^\varepsilon = \{p_1^\varepsilon, \ldots, p_s^\varepsilon\} \) be a set of empirical points with uniform tolerance \( \varepsilon \) and with \( \mathcal{X} \subset \mathbb{R}^n \). A set of points \( \tilde{\mathcal{X}} = \{\tilde{p}_1, \ldots, \tilde{p}_s\} \subset \mathbb{R}^n \) is called an admissible perturbation of \( \mathcal{X} \) if each \( \tilde{p}_i \) is an admissible perturbation of \( p_i \).

Finally, we recall (see [9, 10]) some basic concepts related to the polynomial ring \( P = \mathbb{R}[x_1, \ldots, x_n] \).

**Definition 2.3** Let \( \mathcal{X} = \{p_1, \ldots, p_k\} \) be a non-empty finite set of points of \( \mathbb{R}^n \) and let \( G = \{g_1, \ldots, g_k\} \) be a non-empty finite set of polynomials.

- The \( \mathbb{R} \)-linear map \( \text{eval}_\mathcal{X} : P \to \mathbb{R}^s \) defined by \( \text{eval}_\mathcal{X}(f) = (f(p_1), \ldots, f(p_s)) \) is called the evaluation map associated to \( \mathcal{X} \). For brevity, we write \( f(\mathcal{X}) \) to mean \( \text{eval}_\mathcal{X}(f) \).

- The evaluation matrix (or vector if \( k = 1 \)) of \( G \) associated to \( \mathcal{X} \), written \( M_G(\mathcal{X}) \) (or \( g_1(\mathcal{X}) \)), is defined as having entry \( (i, j) \) equal to \( g_j(p_i) \).

**Definition 2.4** Let \( \mathbb{T}^n \) be the monoid of power products of \( P \) and let \( \mathcal{O} \) be a non-empty subset of \( \mathbb{T}^n \).

- The set \( \mathcal{O} \) is called an order ideal if \( \mathcal{O} = \overline{\mathcal{O}} \), where \( \overline{\mathcal{O}} \) is the set of all power products in \( \mathbb{T}^n \) which divide some power product of \( \mathcal{O} \).

- Given an order ideal \( \mathcal{O} \), the corner set of \( \mathcal{O} \) is the set \( \mathcal{C}[\mathcal{O}] = \{t \in \mathbb{T}^n : t \notin \mathcal{O}, x_i | t \Rightarrow t/x_i \in \mathcal{O}, i = 1 \ldots n\} \)

Later on we suppose the reader familiar with the concepts of Gröbner basis and border basis of a vanishing ideal. Regarding these arguments, the reader is referred to the literature (see, among the others, [3, 9, 10]).
3 The Numerical Algorithm

Before processing a set $X$ of limited precision points, it is possible to mitigate some negative effects of the data uncertainty, replacing with a single representative point the elements of $X$ which differ each other by less than the data accuracy, since they can be regarded as different perturbations of the same empirical value. Later on we suppose w.l.o.g. that the set $X$ does not present such “redundancy”. If it is not the case, it is possible to preprocess the input data to obtain well-separated points, using for instance the algorithms described in [1] and included in CoCoALib [4]. Nevertheless the preprocessing of the input data is not sufficient to eliminate the instabilities of the exact bases of the vanishing ideal $I(X)$, as illustrated in Example 1.1, where the points $X$ are well separated.

We base the construction of our algorithm on the Buchberger-Möller one [2] which computes, given a set $X$ of points and a term ordering $\sigma$, the $\sigma$-Gröbner basis $GB$ of $I(X)$ as follows. At each step, if $O = \{t_1, \ldots, t_k\}$ is the order ideal computed at the previous steps, a power product $t > \sigma t_i$ is chosen. If the vector $t(X)$ is linearly independent of the vectors $\{t_1(X), \ldots, t_k(X)\}$, $t$ is added to $O$. Otherwise, the polynomial $g = t - \sum_{i=1}^k c_i t_i$ is put into $GB$. Nevertheless, since the test of linear dependence is crucially affected by even very small variations of the input data, when we deal with points known with limited precision, small perturbations of the input data may lead to different choices in the Buchberger-Möller algorithm.

In order to solve this drawback, we present an algorithm which checks the linear dependence in a robust way w.r.t. the data uncertainty. Since every admissible perturbation $\tilde{X}$ is computationally equivalent to $X$, the vector $t(\tilde{X})$ can be considered numerically dependent on $\{t_1(\tilde{X}), \ldots, t_k(\tilde{X})\}$ if there exists an $\tilde{X}$ such that $t(\tilde{X})$ exactly depends on the vectors $\{t_1(\tilde{X}), \ldots, t_k(\tilde{X})\}$. Formally we have the following definition.

**Definition 3.1** Given a set $O = \{t_1, \ldots, t_k\}$ and a power product $t$, the vector $t(X)$ **numerically depends** on $\{t_1(X), \ldots, t_k(X)\}$ if there exists an $\tilde{X}$ such that $t(\tilde{X})$ is a null vector.

3.1 Sensitivity of the least squares problem

In order to detect the numerical linear dependency of a set of evaluation vectors, we need some results concerning the sensitivity of the least squares problem

$$M_O(X)\alpha = t(X)$$

(1)

First of all we recall some results, based on the componentwise perturbation analysis (see [8]), about the sensitivity of a generic least squares problem.

Given an $h \times k$ matrix $A$, we denote by $A^+$ its pseudoinverse, that is $A^+ = (A^tA)^{-1}A^t$, and by $|A|$ the matrix consisting of the absolute values of the elements of $A$; given an $h \times k$ matrix $B$, we assume that $|A| < |B|$ means
that the relation holds componentwise. Moreover, given a real value $\eta \ll 1$ we denote by $O(\eta^m)$, $m \in \mathbb{N}$, an $h \times k$ matrix $W(\eta) = (w_{i,j}(\eta))$ (or a real function if $h = k = 1$) such that, for each $(i, j)$, $|w_{i,j}(\eta)|/\eta^m$ is bounded near 0.

**Theorem 3.2** Let $A$ and $A + \Delta A$ both be $p \times q$, $p > q$, full rank matrices and let $b$ and $b + \Delta b$ be two vectors of $\mathbb{R}^p$ such that $|\Delta A| \leq \eta E$ and $|\Delta b| \leq \eta f$, $\eta \in \mathbb{R}^+$, $\eta \ll 1$. Consider the least squares problems

$$Ax = b \quad \text{with} \quad \rho = b - Ax \quad \text{and} \quad (A + \Delta A)(x + \Delta x) = b + \Delta b \quad \text{with} \quad \rho + \Delta \rho = b + \Delta b - (A + \Delta A)(x + \Delta x)$$

We have that

$$\Delta x = A^+(\Delta b - \Delta Ax) + (A^tA)^{-1}(\Delta A)^t \rho + O(\eta^2)$$

$$\Delta \rho = (I - AA^+)(\Delta b - \Delta Ax) - (A^+)^t(\Delta A)^t \rho + O(\eta^2)$$

where $I$ is the $p \times p$ identity matrix.

**Proof.** It is possible (see [6]) to express the least squares problem in the form

$$\begin{bmatrix} I & A \\ A^t & 0 \end{bmatrix} \begin{bmatrix} \rho \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

and so

$$\begin{bmatrix} I & (A + \Delta A) \\ (A + \Delta A)^t & 0 \end{bmatrix} \begin{bmatrix} \rho + \Delta \rho \\ x + \Delta x \end{bmatrix} = \begin{bmatrix} b + \Delta b \\ 0 \end{bmatrix}$$

Taking the difference of the previous equations, we have

$$\begin{bmatrix} I & A \\ A^t & 0 \end{bmatrix} \begin{bmatrix} \Delta \rho \\ \Delta x \end{bmatrix} = \begin{bmatrix} \Delta b - \Delta A(x + \Delta x) \\ -(\Delta A)^t(\rho + \Delta \rho) \end{bmatrix}$$

Since

$$\begin{bmatrix} I - AA^+ & (A^+)^t \\ A^+ & -(A^tA)^{-1} \end{bmatrix}$$

is the inverse matrix of

$$\begin{bmatrix} I & A \\ A^t & 0 \end{bmatrix}$$

we obtain

$$\Delta \rho = (I - AA^+)(\Delta b - \Delta A(x + \Delta x)) - (A^+)^t(\Delta A)^t(\rho + \Delta \rho)$$

$$\Delta x = A^+(\Delta b - \Delta A(x + \Delta x)) + (A^tA)^{-1}(\Delta A)^t(\rho + \Delta \rho)$$

(2)

Supposing $|\Delta A| \leq \eta E$ and $|\Delta b| \leq \eta f$, the absolute values of $\Delta x$ and $\Delta \rho$ satisfy

$$|\Delta \rho| \leq \eta \left( |I - AA^+| (f + E|x + \Delta x|) + |A^+|^tE^t|\rho + \Delta \rho| \right)$$

$$|\Delta x| \leq \eta \left( |A^+| (f + E|x + \Delta x|) + |(A^tA)^{-1}|E^t|\rho + \Delta \rho| \right)$$

so we have that

$$\Delta A \Delta x = O(\eta^2) \quad \text{and} \quad (\Delta A)^t \Delta \rho = O(\eta^2)$$

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and the conclusion follows. 

Since we are interested in the behaviour of the least squares problem (1), we present an estimation of the sensitivity of the matrix \( M_\mathcal{O}(X) \) and of the vector \( t(X) \) to slight perturbations of the set \( X \).

Given the power product \( t = x_1^{\beta_1} \ldots x_n^{\beta_n} \) and the monomial set \( \mathcal{O} \), we denote by \( \varepsilon_M = \max\{\varepsilon_i, i = 1 \ldots n\} \), by \( \deg(x_k, t) = \beta_k \) the degree of \( x_k \) into \( t \), by \( \partial_t t = \deg(x_k, t)x_1^{\beta_1} \ldots x_k^{\beta_k-1} \ldots x_n^{\beta_n} \) and by \( \partial_t \mathcal{O} = \{\partial_t t : t \in \mathcal{O}\} \).

The following result concerns the sensitivity of the evaluation vector \( t(X) \).

**Lemma 3.3** Let \( t \) be a power product of \( \mathbb{T}^n \). Given a set \( X^\varepsilon = \{p_1^\varepsilon, \ldots, p_s^\varepsilon\} \) of empirical points and an admissible perturbation \( \overline{X} = \{\overline{p}_1, \ldots, \overline{p}_s\} \) of \( X \), we have that the vector \( \Delta t = t(\overline{X}) - t(X) \) satisfies

\[
\Delta t = \sum_{k=1}^{n} E_k \partial_t t(X) + O(\varepsilon_M^2)
\]

where \( E_k = \text{Diag}(e_{1,k}, \ldots, e_{s,k}) \) is a diagonal matrix and \( e_{i,k}, |e_{i,k}| < \varepsilon_k \), is a perturbation the \( k \)-th coordinate of \( p_i \).

**Proof.** First of all we consider a point \( p = (c_1, \ldots, c_n) \in \mathbb{R}^n \) and an admissible perturbation \( \overline{p} = (\overline{c}_1, \ldots, \overline{c}_n) \) of \( p \) w.r.t the tolerance \( \varepsilon \). Given \( t = x_1^{\beta_1} \ldots x_n^{\beta_n} \), we have that

\[
t(\overline{p}) - t(p) = (c_1 + e_1)^{\beta_1} \ldots (c_n + e_n)^{\beta_n} - c_1^{\beta_1} \ldots c_n^{\beta_n} = \sum_{k=1}^{n} e_k \beta_k c_k^{\beta_k-1} \prod_{h=1, h \neq k}^{n} c_h^{\beta_h} + O(\varepsilon_M^2) = \sum_{k=1}^{n} e_k \partial_t t(p) + O(\varepsilon_M^2)
\]

then we obtain

\[
t(\overline{p}_i) - t(p_i) = \sum_{k=1}^{n} e_{i,k} \partial_t t(p_i) + O(\varepsilon_M^2)
\]

and so, since \( t(\overline{p}_i) - t(p_i) \) is the \( i \)-th coordinate of \( t(\overline{X}) - t(X) \), we conclude that

\[
t(\overline{X}) - t(X) = \sum_{k=1}^{n} E_k \partial_t t(X) + O(\varepsilon_M^2)
\]

The following result concerns the sensitivity of the evaluation matrix \( M_\mathcal{O}(X) \).

**Lemma 3.4** Let \( \mathcal{O} \) be an order ideal. Given a set \( X^\varepsilon = \{p_1^\varepsilon, \ldots, p_s^\varepsilon\} \) of empirical points and an admissible perturbation \( \overline{X} = \{\overline{p}_1, \ldots, \overline{p}_s\} \) of \( X \), we have that the matrix \( \Delta M = M_\mathcal{O}(\overline{X}) - M_\mathcal{O}(X) \) satisfies

\[
\Delta M = \sum_{k=1}^{n} E_k M \partial_t \mathcal{O}(X) + O(\varepsilon_M^2)
\]

where \( E_k = \text{Diag}(e_{1,k}, \ldots, e_{s,k}) \) is the diagonal matrix of Lemma 3.3.
Proof. Since the $j$-th column of $M_\mathcal{O}(\mathcal{X})$ is given by $t_j(\mathcal{X})$, $t_j \in \mathcal{O}$, Lemma 3.3 implies that the $j$-th column of $M_\mathcal{O}(\mathcal{X}) - M_\mathcal{O}(\tilde{\mathcal{X}})$ is

$$t_j(\mathcal{X}) - t_j(\tilde{\mathcal{X}}) = \sum_{k=1}^n E_k \partial_k t_j(\mathcal{X}) + O(\varepsilon_M^2)$$

The conclusion follows since $\partial_k t_j(\mathcal{X})$ is the $j$-th column of the evaluation matrix of the set $\partial_k \mathcal{O}$. \hfill ∎

The next theorem, based on Theorem 3.2, Lemma 3.3 and Lemma 3.4, presents a componentwise estimation of the sensitivity of the problem (1) to the data perturbations. Further, it shows a componentwise upper bound of the absolute value of the residual, when there exists an admissible perturbation $\tilde{\mathcal{X}}$ such that the perturbed least squares problem $M_\mathcal{O}(\tilde{\mathcal{X}})\hat{\alpha} = t(\tilde{\mathcal{X}})$ has a zero residual.

**Theorem 3.5** Let $\mathcal{X}^\varepsilon$ be a set of $s$ empirical points and let $\tilde{\mathcal{X}}$ be an admissible perturbation of $\mathcal{X}$. Let $\mathcal{O}$ be an order ideal such that $M_\mathcal{O}(\mathcal{X})$ and $M_\mathcal{O}(\tilde{\mathcal{X}})$ are full rank matrices. Given the least squares problems

$$M_\mathcal{O}(\mathcal{X})\alpha = t(\mathcal{X}) \quad \text{with residual} \quad \rho(\mathcal{X}) = t(\mathcal{X}) - M_\mathcal{O}(\mathcal{X})\alpha$$

and

$$M_\mathcal{O}(\tilde{\mathcal{X}})\tilde{\alpha} = t(\tilde{\mathcal{X}}) \quad \text{with residual} \quad \rho(\tilde{\mathcal{X}}) = t(\tilde{\mathcal{X}}) - M_\mathcal{O}(\tilde{\mathcal{X}})\tilde{\alpha}$$

then the vectors $\Delta \alpha = \tilde{\alpha} - \alpha$ and $\Delta \rho = \rho(\tilde{\mathcal{X}}) - \rho(\mathcal{X})$ satisfy

$$\Delta \rho = (I - M_\mathcal{O}(\mathcal{X})M_\mathcal{O}(\mathcal{X})^t) \sum_{k=1}^n E_k \partial_k t(\mathcal{X}) - M_\mathcal{O}(\mathcal{X})\alpha$$

$$- (M_\mathcal{O}(\mathcal{X}))^t \left( \sum_{k=1}^n M_\mathcal{O}(\mathcal{X})^t E_k \right) \rho(\mathcal{X}) + O(\varepsilon_M^2)$$

$$\Delta \alpha = M_\mathcal{O}(\mathcal{X}) \sum_{k=1}^n E_k \partial_k t(X) - M_\mathcal{O}(\mathcal{X})\alpha$$

$$+ (M_\mathcal{O}(\mathcal{X})M_\mathcal{O}(\mathcal{X}))^{-1} \left( \sum_{k=1}^n M_\mathcal{O}(\mathcal{X})E_k \right) \rho(\mathcal{X}) + O(\varepsilon_M^2)$$

Moreover, if there exists an admissible perturbation $\tilde{\mathcal{X}}$ of $\mathcal{X}$ such that the residual $\rho(\tilde{\mathcal{X}})$ of the least squares problem $M_\mathcal{O}(\tilde{\mathcal{X}})\tilde{\alpha} = t(\tilde{\mathcal{X}})$ is a zero vector, then the residual $\rho(\mathcal{X})$ satisfies

$$|\rho(\mathcal{X})| \leq |I - M_\mathcal{O}(\mathcal{X})M_\mathcal{O}(\mathcal{X})^t| \sum_{k=1}^n \varepsilon_k |\partial_k t(\mathcal{X}) - M_{\mathcal{O},\mathcal{O}}(\mathcal{X})\alpha| + O(\varepsilon_M^2)$$

Proof. Since $|\Delta M| < \varepsilon_M E$, $|\Delta t| < \varepsilon_M f$ and $M_\mathcal{O}(\mathcal{X})$ and $M_\mathcal{O}(\tilde{\mathcal{X}})$ have full rank, from Theorem 3.2 we obtain

$$\Delta \rho = (I - M_\mathcal{O}(\mathcal{X})M_\mathcal{O}(\mathcal{X})^t)(\Delta t - \Delta M\alpha) - (M_\mathcal{O}(\mathcal{X}))^t(\Delta M)^t \rho(\mathcal{X}) + O(\varepsilon_M^2)$$

$$\Delta \alpha = M_\mathcal{O}(\mathcal{X})(\Delta t - \Delta M\alpha) + (M_\mathcal{O}(\mathcal{X})M_\mathcal{O}(\mathcal{X}))^{-1}(\Delta M)^t \rho(\mathcal{X}) + O(\varepsilon_M^2)$$
and so, from Lemma 3.3 and Lemma 3.4, Equations (3) and (4) follow.

Moreover, if \( \rho(\tilde{X}) \) is a zero vector from formula (2) we have

\[
\rho(X) = -\Delta \rho = (M_O(X)M_O^+(X) - I) \sum_{k=1}^{n} E_k (\partial_k t(X) - M_{\partial_k}O(X)\alpha) + O(\varepsilon_M^2)
\]

and if we consider the componentwise absolute value of \( \rho(X) \) we obtain

\[
|\rho(X)| \leq |I - M_O(X)M_O^+(X)| \sum_{k=1}^{n} |E_k| |\partial_k t(X) - M_{\partial_k}O(X)\alpha| + O(\varepsilon_M^2)
\]

\[
\leq |I - M_O(X)M_O^+(X)| \sum_{k=1}^{n} \varepsilon_k |\partial_k t(X) - M_{\partial_k}O(X)\alpha| + O(\varepsilon_M^2) \quad \diamond
\]

### 3.2 The NBM Algorithm

Theorem 3.5 shows a sufficient condition for the numerical independency of \( t(X) \) of the columns of \( M_O(X) \). In fact if the residual \( \rho(X) \) of the least squares problem \( M_O(X)\alpha = t(X) \) satisfies

\[
|\rho(X)| > |I - M_O(X)M_O^+(X)| \sum_{k=1}^{n} \varepsilon_k |\partial_k t(X) + M_{\partial_k}O(X)\alpha| + O(\varepsilon_M^2)
\]

then there are no admissible perturbations \( \tilde{X} \) of \( X \) such that the residual of the least squares problem \( M_O(\tilde{X})\alpha = t(\tilde{X}) \) is a null vector. So from Definition 3.4 it follows that \( t(X) \) is numerically independent of \( \{r(X) : r \in O\} \). In particular, this implies that if \( M_O(\tilde{X}) \) is a full rank matrix then \( [M_O(\tilde{X})t(\tilde{X})] \) is a full rank matrix too, for each admissible perturbation \( \tilde{X} \). By exploiting this idea, we develop the Numerical Buchberger Möller algorithm, whose main check is based on condition (4). In particular, since we assume the tolerance on the data error is relatively small, we neglect the errors of order \( O(\varepsilon_M^2) \) focusing our attention on a first order error analysis of the problem.

**The Numerical Buchberger Möller (NBM) Algorithm.**

**Input.** A set \( X^s \) of \( s \) empirical points and a term ordering \( \sigma \).

**Output.** An order ideal \( O \) and a polynomial set \( \mathcal{G} \).

At the first step \( O = \{1\} \) and \( \mathcal{G} \) is an empty set. A generic step can be described as follows. Let \( O = \{t_1, \ldots, t_k\} \) be the order ideal computed at the previous steps and let \( t \) be the current power product, \( t >_{\sigma} t_1, \ldots, t_k \), chosen according to the strategy of the Buchberger-Möller algorithm.

1. Solve the least square problem \( M_O(X)\alpha = t(X) \) and compute the residual \( \rho(X) = t(X) - M_O(X)\alpha \).

2. If \( \rho(X) \) satisfies

\[
|\rho(X)| > |I - M_O(X)M_O^+(X)| \sum_{k=1}^{n} \varepsilon_k |\partial_k t(X) - M_{\partial_k}O(X)\alpha|
\]

(6)
3. Otherwise, put the polynomial $g = t - \sum_{i=1}^{k} \alpha_i t_i$ into $\mathcal{G}$. ♦

The NBM algorithm stops after finitely many steps and computes an order ideal $\mathcal{O}$, since the strategy to choose the power products to analyze is the same as in the Buchberger-Möller algorithm.

Note that the term ordering $\sigma$ is only a computational tool for obtaining a set $\mathcal{O}$ closed under taking divisors. In fact in the general case $\mathcal{O}$ is different from $\mathcal{O}_\sigma$, the quotient basis determined by the $\sigma$-Gröbner basis of $\mathcal{I}(\mathbb{X})$. Moreover it can happen that, for each possible term ordering $\tau$, $\mathcal{O}$ does not coincide to any $\mathcal{O}_\tau$ corresponding to the $\tau$-Gröbner basis of $\mathcal{I}(\mathbb{X})$ (see Example 6.4). For this reason any different strategy for building an order ideal can be used in the NBM algorithm instead to fix a term ordering.

4 Properties of the order ideal $\mathcal{O}$

A first important property of the order ideal $\mathcal{O}$ computed by the NBM algorithm is its invariance w.r.t. the scaling and the translation of the points $\mathbb{X}$, as shown in the following theorem.

**Theorem 4.1** Let $\mathbb{X}^\varepsilon$ be a set of empirical points with
\[
\mathbb{X} = \{p_1, \ldots, p_s\} \quad p_i = (c_{i,1}, \ldots, c_{i,n}) \quad \text{and} \quad \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)
\]

Let $\mathbb{X}^\delta_S$ be the set of scaled empirical points such that $\mathbb{X}^\delta_S = \{\overline{p}_1, \ldots, \overline{p}_s\}$ and
\[
\overline{p}_i = (d_1c_{i,1}, \ldots, d_nc_{i,n}) \quad \text{with} \quad (d_1, \ldots, d_n) \in \mathbb{R}^n \quad \text{and} \quad \delta = (|d_1|\varepsilon_1, \ldots, |d_n|\varepsilon_n)
\]

Let $\mathbb{X}^\tau_T$ be the set of translated empirical points such that $\mathbb{X}^\tau_T = \{\hat{p}_1, \ldots, \hat{p}_s\}$ and
\[
\hat{p}_i = (c_{i,1} + v_1, \ldots, c_{i,n} + v_n) \quad \text{with} \quad (v_1, \ldots, v_n) \in \mathbb{R}^n \quad \text{and} \quad \tau = \varepsilon
\]

Then the NBM algorithm computes the same order ideal $\mathcal{O}$ for all the input sets $\mathbb{X}^\varepsilon$, $\mathbb{X}^\delta_S$ and $\mathbb{X}^\tau_T$.

**Proof.** We prove that the NBM algorithm computes the same order ideals at each step independently of the input sets $\mathbb{X}^\varepsilon$, $\mathbb{X}^\delta_S$ or $\mathbb{X}^\tau_T$.

At the first step it is true, since $\mathcal{O} = \{1\}$. Let us suppose that, at the current step, with all the three input sets the same order ideal $\mathcal{O} = \{t_1, \ldots, t_k\}$ has been computed and that the term $t$ has to be processed.

Let us consider the set $\mathbb{X}_S$ of the scaled points.

Given a term $r = x_1^{\beta_1} \ldots x_n^{\beta_n}$, denoting by $r(d) = d_1^{\beta_1} \ldots d_n^{\beta_n}$, we have
\[
r(\overline{p}_i) = r(d)r(p_i) \quad \text{and} \quad \partial_k r(\overline{p}_i) = \frac{r(d)}{d_k} \partial_k r(p_i)
\]
so that, denoting by $D_\mathcal{O}$ the diagonal matrix $Diag(r(d) : r \in \mathcal{O})$, we have

$$t(\mathcal{X}_S) = t(d)t(\mathcal{X}) \quad \text{and} \quad M_\mathcal{O}(\mathcal{X}_S) = M_\mathcal{O}(\mathcal{X})D_\mathcal{O}$$

$$\partial_t t(\mathcal{X}_S) = \frac{t(d)}{d_k}\partial_t t(\mathcal{X}) \quad \text{and} \quad M_{\partial_t}\mathcal{O}_S(\mathcal{X}_S) = \frac{1}{d_k}M_{\partial_t}\mathcal{O}(\mathcal{X})D_\mathcal{O}$$

The least squares problems $M_\mathcal{O}(\mathcal{X})\alpha = t(\mathcal{X})$ and $M_\mathcal{O}(\mathcal{X}_S)\alpha_S = t(\mathcal{X}_S)$ solved with input sets $\mathcal{X}\varepsilon$ and $\mathcal{X}_S\delta$ are such that

$$M_\mathcal{O}(\mathcal{X})D_\mathcal{O}\alpha = t(d)t(\mathcal{X}) \Rightarrow D_\mathcal{O}\alpha = t(d)M_\mathcal{O}(\mathcal{X})t(\mathcal{X}) \Rightarrow \alpha = t(d)D_\mathcal{O}^{-1}\alpha$$

$$\rho(\mathcal{X}_S) = t(\mathcal{X}_S) - M_\mathcal{O}(\mathcal{X}_S)\alpha_S = t(d)t(\mathcal{X}) - t(d)M_\mathcal{O}(\mathcal{X})D_\mathcal{O}^{-1}\alpha = t(d)\rho(\mathcal{X})$$

If we consider the upper bound (6) of Step 2 computed for the scaled empirical points $\mathcal{X}_S\delta$, straightforward computations show that

$$I - M_\mathcal{O}(\mathcal{X}_S)M_\mathcal{O}^\top(\mathcal{X}_S) = I - M_\mathcal{O}(\mathcal{X})M_\mathcal{O}^\top(\mathcal{X})$$

$$\partial_t t(\mathcal{X}_S) - M_{\partial_t}\mathcal{O}_S(\mathcal{X}_S)\alpha_S = \frac{t(d)}{d_k} \left[ \partial_t t(\mathcal{X}) - M_{\partial_t}\mathcal{O}(\mathcal{X})\alpha \right]$$

It follows that $t$ satisfies condition (6) with input set $\mathcal{X}_S\delta$ if and only if

$$|t(d)||\rho(\mathcal{X})| > |t(d)||I - M_\mathcal{O}(\mathcal{X})M_\mathcal{O}^\top(\mathcal{X})| \sum_{k=1}^n \frac{\delta_k}{|d_k|} \left| \partial_t t(\mathcal{X}) - M_{\partial_t}\mathcal{O}(\mathcal{X})\alpha \right|$$

that is if and only if $t$ satisfies condition (6) with input set $\mathcal{X}\varepsilon$ since $\delta_k = |d_k|\varepsilon_k$. We conclude that the NBM algorithm puts $t$ into $\mathcal{O}$ processing the input $\mathcal{X}\varepsilon$ if and only if $t$ is added to $\mathcal{O}$ using the input $\mathcal{X}_S\delta$.

Let us consider the set $\mathcal{X}_T$ of the translated points.

Given a term $r = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$, there exist (see [13]) a set $R = \{ r_j : r_j \rho \}$ of power products and a set $\{ \gamma_j : \gamma_j = \gamma_j(v_1, \ldots, v_n) \}$ of coefficients such that for each $p = (c_1, \ldots, c_n)$ and $\hat{p} = (c_1 + v_1, \ldots, c_n + v_n)$

$$r(\hat{p}) = r(p) + \sum_{r_j \in R} \gamma_j r_j(p)$$

Furthermore, let $F(v_1, \ldots, v_n) : \mathbb{R}^n \to \mathbb{R}$ be a function such that

$$F(v_1, \ldots, v_n)(x_1, \ldots, x_n) = (x_1 + v_1)^{\beta_1} \cdots (x_n + v_n)^{\beta_n} - x_1^{\beta_1} \cdots x_n^{\beta_n} - \sum_{r_j \in R} \gamma_j r_j(x_1, \ldots, x_n)$$

Since $F(v_1, \ldots, v_n)(p) = 0$ for each point $p \in \mathbb{R}^n$ we obtain

$$0 = \frac{\partial F(v_1, \ldots, v_n)}{\partial x_k}(p) = \frac{\partial r}{\partial x_k}(\hat{p}) - \frac{\partial r}{\partial x_k}(p) - \sum_{r_j \in R} \gamma_j \frac{\partial r_j}{\partial x_k}(p)$$

that is, using our notation,

$$\partial_t r(\hat{p}) = \partial_t r(p) + \sum_{r_j \in R} \gamma_j \partial_t r_j(p)$$
Now, let us consider at the current step the set $\mathcal{O}$ and the power product $t$. By construction $\mathcal{O}$ is factor closed, so that for each $r \in \mathcal{O} \cup C[\mathcal{O}]$ the set $R$ is a subset of $\mathcal{O}$. Since $t \in C[\mathcal{O}]$, we have

$$t(\hat{p}_i) = t(p_i) + \sum_{t_j \in \mathcal{O}} \lambda_j t_j(p_i) \quad \text{and} \quad \partial_k t(\hat{p}_i) = \partial_k t(p_i) + \sum_{t_j \in \mathcal{O}} \lambda_j \partial_k t_j(p_i)$$

so that, denoting by $\lambda$ the vector which consists of the values $\lambda_j$,

$$t(\mathcal{X}_T) = t(\mathcal{X}) + M_O(\mathcal{X}) \lambda \quad \text{and} \quad \partial_k t(\mathcal{X}_T) = \partial_k t(\mathcal{X}) + M_{\partial_k \mathcal{O}}(\mathcal{X}) \lambda$$

Analogously, analyzing each column of the matrices $M_O(\mathcal{X}_T)$ and $M_{\partial_k \mathcal{O}}(\mathcal{X}_T)$, there exists a square matrix $\Lambda$ such that

$$M_O(\mathcal{X}_T) = M_O(\mathcal{X}) + M_O(\mathcal{X}) \Lambda \quad \text{and} \quad M_{\partial_k \mathcal{O}}(\mathcal{X}_T) = M_{\partial_k \mathcal{O}}(\mathcal{X}) + M_{\partial_k \mathcal{O}}(\mathcal{X}) \Lambda \quad (7)$$

The least squares problems $M_O(\mathcal{X}) \alpha = t(\mathcal{X})$ and $M_O(\mathcal{X}_T) \alpha_T = t(\mathcal{X}_T)$ solved with the input sets $\mathcal{X}^\varepsilon$ and $\mathcal{X}_T^\varepsilon$ are such that

$$M_O(\mathcal{X})(I + \Lambda) \alpha_T = t(\mathcal{X}) + M_O(\mathcal{X}) \lambda \Rightarrow (I + \Lambda) \alpha_T = \alpha + \lambda$$

$$\rho(\mathcal{X}_T) = t(\mathcal{X}_T) - M_O(\mathcal{X}_T) \alpha_T = t(\mathcal{X}) + M_O(\mathcal{X}) \lambda - M_O(\mathcal{X})(\alpha + \lambda) = \rho(\mathcal{X})$$

Since the residual of least squares problem is invariant w.r.t. the translation and $M_O(\mathcal{X})$ has full rank then $M_O(\mathcal{X}_T)$ is a full rank matrix too. It follows from (7) that $M_O(\mathcal{X})(I + \Lambda) = M_O(\mathcal{X}_T)$, and so $I + \Lambda$ is a non singular matrix.

If we consider the upper bound (6) of Step 2, computed for the translated empirical points $\mathcal{X}_T^\varepsilon$ straightforward calculations lead to

$$I - M_O(\mathcal{X}_T) M_O^+(\mathcal{X}_T) = I - M_O(\mathcal{X}) M_O^+(\mathcal{X})$$

Furthermore since

$$\partial_k t(\mathcal{X}_T) - M_{\partial_k \mathcal{O}}(\mathcal{X}_T) \alpha_T = \partial_k t(\mathcal{X}) + M_{\partial_k \mathcal{O}}(\mathcal{X}) \lambda - M_{\partial_k \mathcal{O}}(\mathcal{X})(I + \Lambda) \alpha_T = \partial_k t(\mathcal{X}) + M_{\partial_k \mathcal{O}}(\mathcal{X}) \lambda - M_{\partial_k \mathcal{O}}(\mathcal{X})(\alpha + \lambda) = \partial_k t(\mathcal{X}) - M_{\partial_k \mathcal{O}}(\mathcal{X}) \alpha$$

it follows that $t$ satisfies condition (6) with input $\mathcal{X}_T^\varepsilon$ if and only if $t$ satisfies condition (6) with input $\mathcal{X}^\varepsilon$. We conclude that the NBM algorithm puts $t$ into $\mathcal{O}$ processing $\mathcal{X}^\varepsilon$ if and only if $t$ is added to $\mathcal{O}$ using $\mathcal{X}_T^\varepsilon$. \hfill \top

In order to analyze the stability properties of the order ideal $\mathcal{O}$ we recall some basic concepts (see [2]).

**Definition 4.2** An order ideal $\mathcal{O}$ is stable w.r.t. $\mathcal{X}^\varepsilon$ if the evaluation matrix $M_O(\mathcal{X})$ has full rank for each admissible perturbation $\mathcal{X}$ of $\mathcal{X}^\varepsilon$.

Heuristically speaking an order ideal $\mathcal{O}$ can be considered stable w.r.t. the data uncertainty if the linear independency of the evaluation vectors of its elements is not affected by slight perturbations of $\mathcal{X}$. 

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It is well known that each order ideal is stable providing the values of ε are sufficiently small. Nevertheless in our problem the tolerance ε is given a priori and then not all the order ideals turn out to be stable.

By the very nature of the NBM algorithm, no formal results about the stability of $O$ can be stated. In fact, when the numerical independence of $\{r(X) : r \in O\}$ is tested using condition (5), Theorem 3.5 ensures that $O$ is stable. Unfortunately, for implementative reasons, the NBM algorithm checks the numerical independence of $\{r(X) : r \in O\}$ using the first order approximation (6). So the stability of $O$ is not guaranteed.

Nevertheless, in most cases, the numerical tests show that the upper bound (6) is satisfied with a wide margin, widely greater than $O(\varepsilon^2 M)$: then, although we cannot have the complete certainty, there is an high probability that the order ideal $O$ is stable.

We recall that (see [2]) it is possible to compute a stable order ideal by using the SOI algorithm. Its elevated computational cost, widely greater than the computational cost of the NBM algorithm, makes the SOI algorithm not particularly suitable for all who are not mainly interested in stability.

However, the possibility of comparing the results of the SOI and the NBM algorithms points out a comforting behaviour of the NBM algorithm. In fact in several numerical tests the order ideals computed by the algorithms coincide. This result supports the fact that the NBM algorithm, although without certainty, often gives stable order ideals.

5 Properties of the polynomial set $G$

First of all, we formalize the idea of almost vanishing polynomials introducing the following definition. Theorem 4.1 allows to restrict our attention to set of points whose coordinate belong to $[-1, 1]$.

**Definition 5.1** Given a set $X^e$ of empirical points whose coordinates belong to $[-1, 1]$ a polynomial $g$, with coefficient vector $c$, is almost vanishing at $X$ if

$$\frac{\|g(X)\|_2}{\|c\|_2} < O(\varepsilon_M)$$

Obviously in the general case $G$ is not a basis of $\mathcal{I}(X)$, since $G$ can contain polynomials that do not exactly vanish at $X$. However the following theorem shows that $G$ exhibits interesting properties w.r.t. the data uncertainty.

**Theorem 5.2** Let $X^e$ be a set of $s$ empirical points and let $\tilde{X}$ be an admissible perturbation of $X$. The polynomial set $G$ satisfies the following properties.

**P1** If $g$ is a polynomial of $G$ of degree $\deg(g)$ and coefficient vector $c$, then

$$\frac{\|g(X)\|_2}{\|c\|_2} < s \deg(g) \sum_{k=1}^{n} \varepsilon_k \quad \text{and} \quad \frac{\|g(\tilde{X})\|_2}{\|c\|_2} < 2s \deg(g) \sum_{k=1}^{n} \varepsilon_k + O(\varepsilon_M^2)$$

Therefore, $g$ is almost vanishing at $X$ and at $\tilde{X}$. 

\textbf{P2} If the zero set of $\mathcal{G}$ is an admissible perturbation $\hat{X}$ such that $M_{\mathcal{O}}(\hat{X})$ has full rank, then $\mathcal{G}$ is the $\sigma$-Gröbner basis of $I(\hat{X})$.

\textbf{P3} If $\#\mathcal{O} = s$, each polynomial $g \in \mathcal{G}$ corresponds to a unique polynomial $g_b$ of the $\mathcal{O}$-border basis of $I(X)$ such that the support of $g$ is a subset of the support of $g_b$. Furthermore, if $c$ and $c_b$ are respectively the coefficient vectors of $g$ and $g_b$ then

$$\frac{\|c_b - [c, 0\ldots, 0]\|_2}{\|c\|_2} \leq \deg(g)\|M_{\mathcal{O}}(X)\|_2\|M_{\mathcal{O}}^{-1}(X)\|_2\sum_{k=1}^{n} \varepsilon_k$$

\textbf{Proof.}

\textbf{P1} Let us consider Step 3 of the NBM algorithm where the polynomial $g$ is computed. Let $t$ be the monomial analyzed at such step and let $\mathcal{O}_t$ be the order ideal obtained at the previous ones. Since the polynomial $g$ is added to $\mathcal{G}$ if the residual $\rho(X)$ of the least squares problem $M_{\mathcal{O}_t}(X)\alpha = t(X)$ does not satisfy condition (6) and since $g(X) = \rho(X)$, we have

$$\|g(X)\|_2 < \|I - M_{\mathcal{O}_t}(X)M_{\mathcal{O}_t}^+(X)\|_2 \sum_{k=1}^{n} \varepsilon_k \|\partial_k t(X) - M_{\partial_k \mathcal{O}_t}(X)\alpha\|_2$$

First of all we prove that $\|I - M_{\mathcal{O}_t}(X)M_{\mathcal{O}_t}^+(X)\|_2 = 1$. In fact let $A$ be a $p \times q$ full rank matrix, $p > q$ and let $A = USV^t$ be its singular values decomposition. It is well known (see [8]) that $U$ and $V$ are square orthonormal matrices and $\Sigma$ is a block matrix of the form $\Sigma^t = [\Sigma_1, 0]$, where $\Sigma_1$ is the square diagonal matrix of the singular values, and so $\|I - AA^+\|_2 = \|I - \Sigma(\Sigma^t\Sigma)^{-1}\Sigma\|_2 = 1$.

Later on we denote by $\hat{M}_k$ the matrix $[\partial_k t(X) \ M_{\partial_k \mathcal{O}_t}(X)]$, which consists of the vectors $\partial_k r(X)$ with $r \in \mathcal{O}_t \cup \{t\}$. Obviously, if $\deg(x_k, r) = 0$ the corresponding column of $\hat{M}_k$ is a null vector. Moreover, for each $q, r \in \mathcal{O}_t \cup \{t\}$ such that $q \neq r$, $\deg(x_k, r) \neq 0$ and $\deg(x_k, q) \neq 0$, we have $\partial_k r/\deg(x_k, r) \neq \partial_k q/\deg(x_k, q)$ and, since $\mathcal{O}_t$ is factor closed and $t \in \mathcal{C}[\mathcal{O}]$, $\partial_k r/\partial(x_k, r) \in \mathcal{O}_t$.

It follows that each column $\partial_k r(X)$ of $\hat{M}_k$ is a null vector or it corresponds to a unique column of $M_{\mathcal{O}_t}(X)$ multiplied by $\deg(x_k, r)$. Since $\|\hat{M}_k\|_2$ is equal to the norm of its submatrix consisting of the non zero columns and since $\deg(x_k, r) \leq \deg(g)$ we have that

$$\|\hat{M}_k\|_2 \leq \deg(g)\|M_{\mathcal{O}_t}(X)\|_2$$

Finally, since $c = [1, -\alpha]^t$ is the coefficient vector of $g$, we have

$$\|\partial_k t(X) - M_{\partial_k \mathcal{O}_t}(X)\alpha\|_2 = \|\hat{M}_k c\|_2 \leq \deg(g)\|M_{\mathcal{O}_t}(X)\|_2\|c\|_2$$

(8)
so that
\[ \| g(X) \|_2 \leq \deg(g) \| M_{O_t}(X) \|_2 \| c \|_2 \sum_{k=1}^{n} \varepsilon_k \quad (9) \]

Since the coordinates of the points belong to \([-1, 1]\), we have \( \| M_{O_t}(X) \|_2 \leq \| M_{O_t}(X) \|_F \leq s \), where \( \| \cdot \|_F \) is the Frobenius matrix norm. Then the first upper bound of P1 follows immediately.

Further, in order to show the result about \( g(\bar{X}) \), note that
\[ g(\bar{X}) = g(X) + \Delta t - M_\alpha \]

So from Lemma 3.3 and Lemma 3.4 we have
\[ g(\bar{X}) = g(X) + \sum_{k=1}^{n} E_k (\partial_k t(X) - M_{\partial_k O_t}(X) \alpha) + O(\varepsilon_M^2) \]

and, computing the norm of \( g(\bar{X}) \), we obtain
\[ \| g(\bar{X}) \|_2 \leq \| g(X) \|_2 + \deg(g) \| M_{O_t}(X) \|_2 \| c \|_2 \sum_{k=1}^{n} \varepsilon_k + O(\varepsilon_M^2) \]
\[ \leq 2s \deg(g) \| c \|_2 \sum_{k=1}^{n} \varepsilon_k + O(\varepsilon_M^2) \]

that is the second upper bound of P1.

**P2** If the zero set of \( G \) is an admissible perturbation \( \tilde{X} \), since the residuals associated to the elements of \( G \) vanish at \( \tilde{X} \) and \( M_{O_t}(\tilde{X}) \) has full rank, the NBM algorithm computes the polynomial set \( G \) with input set \( \tilde{X} \) and tolerance \( \varepsilon = (0, \ldots, 0) \). Then Property P2 follows immediately because the NBM algorithm with a zero tolerance coincides with the exact Buchberger-Möller one.

**P3** Since \( \#O = s \) then \( O \) is the quotient basis of \( P/I(X) \) and so there exists the \( O \)-border basis of \( I(X) \) (see [10]). By construction, each polynomial \( g \in G \) with leading term \( t \) and support contained in \( \{t\} \cup O_t \) corresponds to a polynomial \( g_b \) of the \( O \)-border basis of \( I(X) \) whose support is contained in \( \{t\} \cup O \). If we order the elements of \( O_t \) and \( O \) in an increasing way w.r.t. \( \sigma \), then the columns of \( M_{O_t}(X) \) coincide with the first \( \#O_t \) columns of \( M_{O}(X) \) and the coefficient vectors \( c = [1, -\alpha] \) of \( g \) and \( c_g = [1, -\beta] \) of \( g_b \) obey \( \| c_g - [c, 0, \ldots, 0] \| = \| \beta - [\alpha, 0, \ldots, 0] \| \). Moreover they are such that
\[ M_{O_t}(X) \begin{bmatrix} \alpha \\ 0 \end{bmatrix} = t(X) + \rho(X) \]
\[ M_{O_t}(X) \beta = t(X) \]

Then we obtain
\[ \begin{bmatrix} \alpha \\ 0 \end{bmatrix} - \beta = M_{O_t}^{-1}(X) \rho(X) \]

From \( \rho(X) = g(X) \) and the upper bound (9), the thesis follows.
Let us point out two pleasant properties of the polynomial set \( \mathcal{G} \) easily following by P1 and P3.

Property P1 immediately implies that \( \mathcal{G} \) can contain almost vanishing polynomials even if the coordinates of \( X \) do not belong to \([-1, 1]\). In fact it is sufficient that the coordinates of \( X \) are “not too elevated” (see the examples of Section 6).

In the case when the condition number \( \|M^{-1}(X)\|_2\|MO(X)\|_2 \) (see [6]) of the matrix \( MO(X) \) is “not too elevated”, Property P3 implies that \( g \) is “close” to a polynomial \( g_b \) vanishing at \( X \). Then \( X \) can be considered a pseudozero set of \( \mathcal{G} \), in the sense given by Stetter (see [12, 13]).

The formal results proved above allow us to justify in detail the heuristic properties of \( \mathcal{G} \) described in the Introduction and in particular the reasons why \( \mathcal{G} \) characterizes the input points \( X \).

First of all note that Property P1 implies that each element \( g \) of \( \mathcal{G} \) assumes small values, and then it is almost vanishing, at \( X \) and at each admissible perturbation (of course w.r.t. the norm of its coefficient vector).

Moreover, we recall that, by construction, each element \( g \) of \( \mathcal{G} \) with leading term \( t \) corresponds to a least squares problem \( MO_t(X)\alpha = t(X) \), \( \mathcal{O}_t \subset \mathcal{O} \), whose residual \( \rho \) does not satisfy condition (6). Note that, since Theorem 3.5 involves only sufficient conditions on the residual, the fact that \( \rho \) does not satisfy condition (6) gives essentially the same information of the fact that \( \rho \) does not satisfy condition (5).

Given \( g \) of \( \mathcal{G} \), let us suppose that there exists an admissible perturbation \( \hat{X}_g \) such that \( \rho(\hat{X}_g) \) is a null vector. This is a possible case because condition (6) is not satisfied. Moreover, let us suppose that the order ideal \( \mathcal{O} \) is stable, so that the matrix \( MO_t(\hat{X}_g) \) has full rank. Then there exists a polynomial \( \hat{g} \), given by the solution of \( MO_t(\hat{X}_g)\hat{\alpha} = t(\hat{X}_g) \), having the following properties:

- \( \hat{g} \) exactly vanishes at \( \hat{X}_g \);
- \( \hat{g} \) has the same support of \( g \);
- \( \hat{g} \) and \( g \) have “similar” coefficients, if the condition number \( \|MO(X)\|\|M^1(X)\| \) is “not too elevated”. In fact from relations (6) and (8) we have

\[
\frac{\| [1, -\hat{\alpha}] - [1, -\alpha] \|_2}{\| [1, -\alpha] \|_2} \leq \frac{\sum_{k=1}^n \varepsilon_k \| \partial_{\hat{\alpha}} t(X) - M_{\hat{\alpha}}(\hat{X}_g) \|_2}{\| [1, -\alpha] \|_2} \leq \frac{\| M^1(\hat{X}_g) \|\| MO(X) \| \text{deg}(g)}{\| [1, -\alpha] \|_2} \sum_{k=1}^n \varepsilon_k
\]

In this sense \( g \) can selects a geometrical configuration \( \hat{X}_g \) of points, close to \( X \), that can be considered an “approximate” representation of the input points independent of the data errors.

Furthermore, as we will show in the examples of Section 6, it can happen that the whole set of polynomials \( g \) of \( \mathcal{G} \) selects a unique geometrical configuration \( \hat{X} \).
Therefore the polynomials \( \hat{g} \) constitute a Gröbner basis of \( I(\hat{X}) \). We can then conclude that \( \hat{g} \) can be viewed as an approximation of a Gröbner basis of an ideal of points \( \hat{X} \) close to \( X \) and the set \( \hat{X} \) can be considered as a possible “exact” configuration corresponding to the absence of data uncertainty.

We point out that, once again as shown in Section 6, it can happen that each \( \hat{g} \) coincides with \( g \) and then \( \hat{G} \) itself is a Gröbner basis for \( I(\hat{X}) \).

Let us conclude this section with a short recall of the open problems already presented in the Introduction. They are essentially related to the existence and possibly the determination of \( \hat{X} \). The numerical examples show that there are cases when \( \hat{X} \) does not exist. This seems to be due to two possible causes. One is because the NBM algorithm could not recognize a possible element of \( O \) so that a polynomial \( g \) which never vanishes at any admissible perturbation is added to \( \hat{G} \). The second reason is when the points of \( X \) are close to different incompatible geometrical configurations. However, in our numerical examples, in this case the NBM algorithm explicitly detects these incompatible geometrical configurations.

6 Numerical examples

The following numerical tests are performed using a prototype version of the NBM algorithm. An improved version of it will be included soon in CoCoALib [4]. In all the examples the term ordering DegLex, \( y < x \) is used; in addition, the coordinates of the points and the coefficients of the polynomials are displayed with a finite number of digits, but all computations are performed in exact arithmetic using CoCoA 4.7 [4].

In Example 6.1 the NBM algorithm computes an exact Gröbner basis of a vanishing ideal of an admissible perturbation.

**Example 6.1** Given the same data of Example 1.1, that is the set of points \( X = \{(1, 1), (3, 2), (5.1, 3)\} \), if the tolerance is \( \varepsilon = (0.15, 0) \), the NBM algorithm computes the quotient basis \( O = \{1, y, y^2\} \) and the polynomial set \( \hat{G} \):

\[
\hat{G} = \begin{cases} 
  g_1 & = x - 2.05y + 1.06 \\
  g_2 & = y^3 - 6y^2 + 11y - 6 
\end{cases}
\]

The polynomial \( g_2 \) vanishes at \( \hat{X} \) while \( g_1 \), with coefficient vector \( c_1 \), is almost vanishing at \( X \) since \( \|g_1(X)\|/\|c_1\| = 0.0162 \).

Since \( \hat{G} \) is the \( \sigma \)-Gröbner basis of \( I(\hat{X}) \) which corresponds to the admissible perturbation \( \hat{X} = \{(0.98\overline{3}, 1), (3.0\overline{5}, 2), (5.08\overline{3}, 3)\} \) consisting of aligned points, we conclude that the points \( X \) are misaligned because of data inaccuracy.

In Example 6.2, a set \( X \) of 20 points close to a circumference is processed and the NBM algorithm detects this geometrical configuration.
Example 6.2 Let $X$ be a set of points obtained varying the coordinates of a set of 20 points lying on the circumference $C$ of equation $x^2 + y^2 - 1 = 0$, with componentwise perturbations less than $10^{-4}$.

The $\sigma$-Gröbner basis of $\mathcal{I}(X)$ does not detect that the points $X$ are close to a circumference. On the contrary, the NBM algorithm, processing the set $X$ with tolerance $\varepsilon = (0.0001, 0.0001)$, computes the stable quotient basis

$$
\mathcal{O} = \{1, y, x, y^2, xy, y^3, xy^2, y^4, x^2y, y^5, xy^3, y^6, xy^4, y^7, x^3y, y^8, xy^5, y^9, xy^6, y^{10}\}
$$

and the set $\mathcal{G}$ of polynomials

$$
\mathcal{G} = \begin{cases} 
    g_1 &= x^2 + 0.99999y^2 - 1.00002 \\
    g_2 &= xy^2 - 2.00006xy^7 + 1.31256xy^5 - 0.31251xy^3 + 0.01953xy \\
    g_3 &= y^{11} - 3.00006y^9 + 3.3126y^7 - 1.6251y^5 + 0.3320y^3 - 0.0195y 
\end{cases}
$$

The set $\mathcal{G}$ is not a Gröbner basis, but $g_2$ and $g_3$ vanish at $X$ and $g_1$, with coefficient vector $c_1$, is almost vanishing at $X$ since $\|g_1(X)\|/\|c_1\| \approx 10^{-4}$.

Moreover, since the coefficient vector of $g_1$ is close to those of the circumference $C$ we conclude that the elements of $X$ are “almost lying” on $C$.

In Example 6.3 the NBM algorithm processes the same set of points with two different tolerances. In the first case it detects two incompatible geometrical configurations close to $X$. In the second case, choosing a smaller tolerance, the NBM algorithm computes a set $\mathcal{G}$ very similar to a Gröbner basis of $\mathcal{I}(X_1)$, where $X_1$ is an admissible perturbation of $X$.

Example 6.3 Given the set $X$ of points

$$
X = \{(1, 6), (2, 3), (2.449, 2.449), (3, 2), (6, 1)\}
$$

we consider two different tolerances.

Firstly if $\varepsilon = (0.018, 0.018)$ the NBM algorithm computes the stable quotient basis $\mathcal{O} = \{1, y, x, y^2, y^3\}$ and the set $\mathcal{G}$ of polynomial which is not a basis of a vanishing ideal since its zero set is empty:

$$
\mathcal{G} = \begin{cases} 
    g_1 &= xy + 0.00008y^2 - 0.00064x - 0.00125y - 5.99501 \\
    g_2 &= x^2 + 0.99199y^2 - 11.94095x - 11.88550y + 46.54436 \\
    g_3 &= y^4 - 14.477y^3 + 76.7241y^2 - 14.8620x - 188.4194y + 214.3446 
\end{cases}
$$

In this case $g_1$ and $g_2$ highlight that the points of $X$ almost lie on the hyperbola $xy - 6$ and on the circumference $x^2 + y^2 - 12x - 12y + 47$. In fact we have that both sets of points

$$
X_1 = \{(1, 6), (2, 3), (\sqrt{5}, \sqrt{5}, \sqrt{6}) \}, (3, 2), (6, 1)\} \quad \text{and}
$$

$$
X_2 = \{(1, 6), (2, 3), (6 - 2.5\sqrt{2}, 6 - 2.5\sqrt{2}, 6) \), (3, 2), (6, 1)\}
$$

are admissible perturbations of $X$. Nevertheless the configurations corresponding to $X_1$ and $X_2$ are incompatible, since $\#X = 5$ while the intersection between an hyperbola and a circumference consists of at most 4 points.
If we choose a smaller tolerance, the configuration of points near to the circumference is not detected by the algorithm. In fact, if \( \varepsilon = (0.001, 0.001) \) we obtain the stable quotient basis \( \mathcal{O} = \{1, y, x, y^2, x^2\} \) and the set \( \mathcal{G} \) of polynomial, with an empty zero set:

\[
\mathcal{G} = \begin{cases}
g_1 = xy + 0.00008y^2 - 0.00064x - 0.00125y - 5.9950 \\
g_2 = y^3 - 2.3444x^2 - 14.3444y^2 + 34.1336x + 75.1336y - 182.1901 \\
g_3 = x^3 - 14.3444x^2 - 2.3444y^2 + 34.1336x + 34.1336y - 182.1901 
\end{cases}
\]

In this case the \( \sigma \)-Gröbner basis \( GB_1 \) of \( \mathcal{I}(X_1) \)

\[
GB_1 = \begin{cases}
xy - 6 \\
y^3 - 2.4494x^2 - 14.4494y^2 + 35.3938x + 76.3938y - 187.1260 \\
x^3 - 14.4494x^2 - 2.4494y^2 + 76.3938x + 35.3938y - 187.1260 
\end{cases}
\]

consists of polynomials “similar” to the elements of \( \mathcal{G} \). Since \( X_1 \) is an admissible perturbation also w.r.t. the tolerance \((0.01, 0.01)\) then \( \mathcal{G} \) highlight that the points \( \mathcal{X} \) almost lie on a hyperbola.

Example 6.4 shows that the term ordering \( \sigma \) is only a computational tool for building a factor closed set. In fact, given the set \( \mathcal{X} \), the NBM algorithm computes the order ideal \( \mathcal{O} \) which cannot be obtained by the exact Buchberger-Möller algorithm working on \( \mathcal{X} \) with any term ordering.

**Example 6.4** Let \( \mathcal{X} = \{(1.1, 1.1), (0.9, -1.1), (-0.9, 0.9), (-1.1, -0.9)\} \) be the input points and let \( \varepsilon = (0.12, 0.12) \) be the tolerance. Since the vector space \( P/\mathcal{I}(\mathcal{X}) \) has dimension 4, the possible quotient bases are

\[
\mathcal{O}_1 = \{1, x, x^2, x^3\} \quad \mathcal{O}_2 = \{1, y, y^2, y^3\} \\
\mathcal{O}_3 = \{1, y, x, x^2\} \quad \mathcal{O}_4 = \{1, y, x, y^2\} \quad \mathcal{O}_5 = \{1, y, x, xy\}
\]

Each quotient basis \( \mathcal{O}_j, j = 1 \ldots 4 \), is associated to the \( \sigma_j \)-Gröbner basis of \( \mathcal{I}(\mathcal{X}) \), where \( \sigma_1 = \sigma_2 = Lex \) with \( y > x \) or \( x > y \) respectively, and \( \sigma_3 = \sigma_4 = DegLex \) with \( y > x \) or \( x > y \) respectively. Nevertheless these sets are not stable quotient bases, since each evaluation matrix \( M_{\mathcal{O}_j}(\mathcal{X}) \), \( j = 1 \ldots 4 \), is singular for the admissible perturbation \( \mathcal{X} = \{(1, 1), (1, -1), (-1, 1), (-1, -1)\} \).

Vice versa \( \mathcal{O}_5 \), computed by the NBM algorithm, cannot be obtained using the exact Buchberger-Möller algorithm w.r.t. any term ordering \( \tau \). In fact the vector \( t(\mathcal{X}) \), with \( t = x^2 \) or \( t = y^2 \) is independent of \( \{r(\mathcal{X}) : r \in \{1, x, y\}\} \) so that \( \mathcal{O}_3 \), if \( x^2 \prec_\tau xy \), or \( \mathcal{O}_4 \), if \( y^2 \prec_\tau xy \), is built. It follows that the set \( \mathcal{G} \) computed by the NBM algorithm

\[
\mathcal{G} = \begin{cases}
y^2 - 0.19998x + 0.01980y - 1.01 \\
x^2 - 0.20199xy + 0.00201x + 0.01999y - 0.98980,
\end{cases}
\]

is not the \( \tau \)-Gröbner basis of \( \mathcal{I}(\mathcal{X}) \), for any term ordering \( \tau \). Nevertheless, \( \mathcal{G} \) is the \( \sigma \)-Gröbner basis of \( \mathcal{I}(\mathcal{X}) \), where the zero set \( \mathcal{X} \) of \( \mathcal{G} \) is the admissible perturbation:

\[
\mathcal{X} = \{(1.099, 1.099), (0.899, -1.100), (-0.899, 0.901), (-1.099, -0.898)\}
\]
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References

[1] J. Abbott, C. Fassino, M. Torrente. Thinning Out Redundant Empirical Data. Mathematics in Computer Science, 1(2):375–392, 2007.

[2] J. Abbott, C. Fassino, M. Torrente. Stable Border Bases for Ideals of Points. Journal of Symbolic Computation, 2008. To appear.

[3] B. Buchberger, H.M. Möller. The construction of multivariate polynomials with preassigned zeros. In EUROCAM '82: Proc. of the European Computer Algebra Conf., Lect. Notes Comput. Sci. 144: 24–31, 1982.

[4] CoCoA Team, CoCoA: a system for doing computations in Commutative Algebra, Available at http://cocoa.dima.unige.it/

[5] C. Fassino. An approximation of the Gröbner basis of ideals of perturbed points, part I. Preprint n. 535, Dipartimento di Matematica, Università di Genova, 2005 Available at http://arxiv.org

[6] G.H. Golub, C. Van Loan. Matrix computations - Second edition. The Johns Hopkins University Press 1991.

[7] D. Heldt, M. Kreuzer, S. Pokutta, H. Poulisse, Approximate computation of zero-dimensional polynomial ideals, 2006 Preprint available at http://staff.fim.uni-passau.de/algebraic-oil/en/index.html

[8] N.J. Higham. A survey of componentwise perturbation theory. Manchester Institute for Mathematical Sciences, The University of Manhchester, 1994 Available at http://www.manchester.ac.uk/mims/eprints

[9] M. Kreuzer, L. Robbiano. Computational commutative algebra I. Springer, 2000.

[10] M. Kreuzer, L. Robbiano. Computational commutative algebra II. Springer, 2005.

[11] T. Sauer, Approximate varieties, approximate ideals and dimension reduction. Numer. Algorithms, 45(1–4): 295–313, 2007.
[12] J.H. Stetter. Polynomials with coefficients of limited accuracy. In Computer Algebra in Scientific Computing, Springer: 409–430, 1999.

[13] J.H. Stetter. The nearest polynomial with a given zero, and similar problems SIGSAM Bull., 33(4): 2–4, 1999.

[14] H.J. Stetter. Numerical polynomial algebra. Siam 2004.

[15] M. Torrente. Application of algebra to oil industry. Ph.D. Thesis, Scuola Normale Superiore, Pisa, (in preparation).