Quantum fields in the static de Sitter universe

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Abstract

We construct explicit mode expansions of various tree-level propagators in the Rindler-De Sitter universe, also known as the static (or compact) patch of the de Sitter spacetime. We construct in particular the Wightman functions for thermal states having a generic temperature $T$. We give a fresh simple proof that the only thermal Wightman propagator that respects the de Sitter isometry is the restriction to the Rindler de Sitter wedge of the Bunch-Davies states and has temperature with $T = (2\pi R)^{-1}$ in the units of de Sitter curvature. Propagators with $T \neq (2\pi R)^{-1}$ are only time translation invariant and have extra singularities on the boundary of the Static Patch. We also construct the expansions for the so-called alpha-vacua in the Static Patch and discuss the flat limit. Loop corrections are discussed in a companion paper.
1 Introduction

One of the most peculiar properties of the de Sitter spacetime is that, notwithstanding its maximal symmetry, it does not possess a globally defined time–like Killing vector field. This fact is at the very root of the various difficulties arising in de Sitter Quantum Field Theory (QFT).

The maximal symmetry allows for the existence of several interesting coordinate systems that may be used to parametrize various regions of the de Sitter manifold, which may or may not contain a complete Cauchy surface and have different physical implications.

The coordinate system which is of interest for us here was introduced as early as 1917 by Willem de Sitter himself in the course of the famous debate on the relativity of inertia [1]. Einstein criticised de Sitter’s first paper [2] by objecting that the model found by his friend and competitor was not static. To answer that criticism and better compare his model with the cylindrical static universe of Einstein, de Sitter wrote his line element in new coordinates [3]. In the new coordinate system the components of the metric tensor are time-independent but the temporal component is not constant and vanishes on the “equator” of the spherical space. Einstein wrongly argued that this unacceptable singular behavior was pointing towards the presence of matter on the equator.

From a group theoretical viewpoint the new time coordinate introduced by de Sitter is nothing but the parameter of the one-dimensional subgroup of the de Sitter symmetry group stabilizing the equator. By applying that subgroup to points of any spherical spatial section containing the equator one obtains coordinates for the two opposite static patches which may also be called the Rindler - de Sitter wedges (see Fig. 1; in two dimensions the spatial sections are ellipses; the equator degenerates in the two points where all the ellipses meet).

![Figure 1: The static patch and it opposite seen as right and left Rindler-de Sitter wedges.](image)

A celebrated result by Gibbons and Hawking [4, 5, 6, 7, 8] says that the restriction to the Rindler - de Sitter wedge of the Bunch-Davies state [9, 10, 11, 12, 13] is a thermal equilibrium state w.r.t. the time coordinate of the static patch, the temperature is $1/2\pi R$. This result was actually predated by another important result by Figari, Hoegh-Krohn and Nappi [14] who studied interacting quantum fields in the wedge in two dimensions by applying constructive methods on the Euclidean sphere.

A century after the Einstein-de Sitter debate the role of coordinates in General Relativity and Cosmology is much better understood; on the other hand, when quantizing fields in curved spacetimes (and, even worse, fields and gravity together) things are yet not so clear, because the field dynamics may depend on the choice of coordinates. Indeed, one method to calculate loop corrections to the field correlation functions in curved spacetimes is the Schwinger–Keldysh diagrammatic technique [15, 16]. One has to choose an initial Cauchy surface (the spacetime is...
supposed to be globally hyperbolic) and an initial state at \( t = t_0 \). The presence of \( t_0 \) is highly important for tracing of the destiny of the initial state [17, 18, 19].

In the de Sitter case one mainly considers a flat non-compact initial Cauchy surface of the exponentially expanding (the so-called Poincaré) coordinate system [20, 21, 22], or a compact spherical surface of the global coordinate system [21, 22, 23]. Furthermore, to build the correlation functions using the Schwinger–Keldysh technique one usually considers a Bunch–Davies initial condition or a finite density perturbation of it (see e.g. [17] and references therein). In both cases the free Hamiltonian is time–dependent due to the explicit time–dependence of the components of the metric tensor.

Only if the initial state has very peculiar properties (that have to be investigated case by case) the loop corrections do not depend on the initial time \( t_0 \) [17]. In general, even in the situations where there is time translation invariance at the field algebraic level, the time translation symmetry is broken in the loops and sooner or later they become comparable to the tree–level contributions. The reason for this is that in non–stationary situations the semi–classical approximation does not work and loop corrections grow with time.

The Rindler-de Sitter wedge is itself a globally hyperbolic spacetime but a Cauchy surface for the wedge is incomplete w.r.t. the whole de Sitter manifold, being only ”one half” of a bona fide Cauchy surface, see Figs. (1) and (2). Quantization in the static coordinates has however distinct features in comparison with any other coordinate system on the de Sitter manifold because the Hamiltonian operator is time independent.

In this paper we construct all the time invariant states building their correlation functions (at tree–level). In particular we construct all the thermal mixed states and show that only at \( T = 1/2\pi R \) the complete de Sitter invariance is recovered. Note that also the zero temperature vacuum (pure) state is not de Sitter invariant. For the sake of completeness we also construct the so-called alpha-vacua by expanding them into the modes of the static patch. All the above states but the de Sitter invariant ones have extra singularities at the horizon, giving retrospectively some support to Einstein’s suspicions about the equator.

We restrict our attention to the two–dimensional case just to simplify the equations. Most of our results can be straightforwardly extended to the general dimensional case. Loop corrections to tree level propagators will be investigated in a companion paper.

2 Geometry

The two-dimensional de Sitter space can be most easily visualized as the one-sheeted hyperboloid embedded in a three dimensional ambient Minkowski space:

\[
dS^2 = \{ X \in \mathbb{R}^3, \ X^\alpha X_\alpha = X_0^2 - X_1^2 - X_2^2 = -R^2 \} \tag{2.1}
\]

(capital \( X^\alpha \) denote the coordinates of a given Lorentzian frame of the ambient spacetime; we set the radius \( R \) of the de Sitter space equal to one). A suitable coordinate system for the static patch is:

\[
X \left( \frac{t}{R}, \frac{x}{R} \right) = \begin{cases} 
X^0 = R \sinh \frac{t}{R} \cosh \frac{x}{R} \\
X^1 = R \tanh \frac{t}{R} = u \\
X^2 = R \cosh \frac{t}{R} \cosh \frac{x}{R} 
\end{cases}, \quad t \in (-\infty, \infty), \ x \in (-\infty, \infty). \tag{2.2}
\]

In the following we will set \( R = 1 \) and \( \tanh x = u \).

The above coordinates cover only the region \( \{|X^1| < 1\} \cap \{X^2 > |X^0|\} \) of the real de Sitter manifold, the shaded region in Fig. 2 or else the right wedge in Fig. 1. The metric

\[
ds^2 = \frac{dt^2 - dx^2}{\cosh x^2} \tag{2.3}
\]

is time independent and conformal to the flat metric. The static patch is bordered by a bifurcate Killing horizon

\[
x \to \pm \infty, \quad t = \pm x
\]
Figure 2: Penrose diagram of the de Sitter manifold with Cauchy surfaces of different patches. The static patch is bordered by a bifurcate Killing horizon.

where the metric degenerates. The corresponding Killing vector is not time-like when extended outside the static patch. The de Sitter invariant scalar product is given by

$$\zeta = \zeta_{12} = X_1^a X_2^a = - \frac{\cosh(t_1 - t_2) + \sinh x_1 \sinh x_2}{\cosh x_1 \cosh x_2}. \quad (2.4)$$

The geodesic distance $L$ and $\zeta$ are related as follows: $\zeta = -\cosh(L)$ for time-like geodesics, $\zeta = \cos(L)$ for space-like ones; $\zeta = -1$ for light-like separations or coincident points.

### 3 Canonical Quantization

In this section we outline the canonical quantization of the Klein-Gordon field in the static chart coordinates:

$$\left( \partial_t^2 - \partial_x^2 + \frac{m^2}{\cosh^2 x} \right) \phi(t, x) = 0,$$

$$\left[ \phi(t, x_1), \phi(t, x_2) \right] = 0, \quad \left[ \phi(t, x_1), \dot{\phi}(t, x_2) \right] = i\delta(x_1 - x_2). \quad (3.2)$$

The static chart is in itself a globally hyperbolic manifold, though geodesically incomplete. We may apply standard methods of canonical quantization and look for a complete set of modes by separating the variables. Of course the so constructed set of modes will be incomplete when considered w.r.t. the whole de Sitter manifold [24, 25].

Let us consider factorized modes which have positive frequencies w.r.t. the time coordinate $t$:

$$\varphi(t, x) = e^{-i\omega t} \psi_\omega(u), \quad u = \tanh x. \quad (3.3)$$

$\psi_\omega(u)$ are eigenfunctions of the continuous spectrum of the well-known quantum mechanical scattering problem:

$$\left[ - \partial_x^2 + \frac{m^2}{\cosh^2 x} \right] \psi_\omega(u) = \omega^2 \psi_\omega(u), \quad u = \tanh x, \quad m^2 = \frac{1}{4} + \nu^2. \quad (3.4)$$

For any given $\omega \geq 0$ the Ferrers functions $P_{\nu}^{i\omega}(\pm u)$ – also known as Legendre functions on the cut [26] – are two independent solutions of the above equation. The double degeneracy of the
energy level $\omega$ points towards the introduction of two pairs of creation and annihilation operators for each level:

$$[a_{\omega_1}, a_{\omega_2}^\dagger] = \delta(\omega_1 - \omega_2), \quad [b_{\omega_1}, b_{\omega_2}^\dagger] = \delta(\omega_1 - \omega_2), \quad [a_{\omega_1}, b_{\omega_2}] = [a_{\omega_1}, b_{\omega_2}^\dagger] = 0.$$ (3.5)

The mode expansion of the field operator $\phi(t, x)$ can then be written as follows:

$$\phi(t, x) = \int_0^\infty \frac{d\omega}{2\pi} \left[ e^{-i\omega t} \left( \psi_\omega(u) a_\omega + \psi^*_\omega(-u) b_\omega \right) + e^{i\omega t} \left( \psi^*_\omega(u) a_\omega^\dagger + \psi^*_\omega(-u) b_\omega^\dagger \right) \right]$$ (3.6)

where

$$\psi_\omega(u) \equiv \sqrt{\sinh(\pi \omega)} \Gamma\left(\frac{1}{2} + i\omega - i\nu\right) \Gamma\left(\frac{1}{2} - i\omega - i\nu\right) P_{-\frac{1}{2} + i\nu}^\omega(u).$$ (3.7)

The normalization has been chosen according with the completeness relation (A.1) shown in Appendix A. At large positive $x$ the wave

$$\psi_\omega(tanh x) \sim e^{i\omega x} \quad x \to \infty$$ (3.8)

is purely right moving (at large negative $x \to -\infty$ the wave $\psi_\omega(-tanh x) \sim e^{-i\omega x}$ is purely left moving).

By normal ordering w.r.t. the vacuum of the $a_\omega$ and $b_\omega$ operators we get the free Hamiltonian in the standard form

$$H := \int_{-\infty}^{+\infty} dx \sqrt{g} : T_0^a := \int_0^{+\infty} d\omega \, \omega \left( a_\omega^\dagger a_\omega + b_\omega^\dagger b_\omega \right).$$ (3.9)

Note that the energy operator is by zero (rather than $m$ as for a massive field in flat space). This is because the "mass" term in the action vanishes near the horizon (recall that $\sqrt{g} = (\cosh x)^{-2}$).

### 3.1 Thermal two-point functions

The quantum mechanical average over a thermal state of inverse temperature $\beta$ is given by

$$\langle \mathcal{O} \rangle_\beta = \frac{\text{Tr} \, \rho \, \mathcal{O}}{\text{Tr} \, \rho}, \quad \rho \equiv e^{-\beta H}. \quad (3.10)$$

Although the previous expression is ill-defined in quantum field theory, it still allows to compute the thermal two-point function at inverse temperature $\beta$ by assuming the Bose-Einstein distribution of the energy levels

$$\langle a_\omega^\dagger a_{\omega'} \rangle_\beta = \langle b_\omega^\dagger b_{\omega'} \rangle_\beta = (e^{\beta \omega} - 1)^{-1} \delta(\omega - \omega'). \quad (3.11)$$

Eqs. (3.6) and (3.9) give the following expression:

$$W_\beta(t_1 - t_2, x_1, x_2) = \langle \phi(t_1, x_1) \phi(t_2, x_2) \rangle_\beta = \int_0^\infty \frac{d\omega}{4\pi^2} \left[ e^{-i\omega(t_1 - t_2)} \left( \psi_\omega(u_1) \psi^*_\omega(u_2) + \psi^*_\omega(-u_1) \psi^*_\omega(-u_2) \right) \right]$$

where

$$\tilde{P}_\nu(\omega, u_1, u_2) = \frac{e^{\pi \omega}(P^{\omega}_{-\frac{1}{2} + i\nu}(u_1) P^{\omega}_{-\frac{1}{2} - i\nu}(u_2) + P^{\omega}_{-\frac{1}{2} + i\nu}(-u_1) P^{\omega}_{-\frac{1}{2} - i\nu}(-u_2))}{8 \cosh \pi(\nu - \omega) \cosh \pi(\nu + \omega)}.$$. (3.13)
The states defined by the above two-point functions are mixed. The only pure state is obtained in the limit \( \beta \to \infty \).

In section 5 we will prove that for \( \beta = 2\pi \) the above two-point function is de Sitter invariant and coincides with the restriction to the static patch of the Bunch-Davies two-point function:

\[
W_{2\pi}(t_1 - t_2, x_1, x_2) = W_{BD}(\zeta) = \frac{1}{4 \cosh \pi \nu} P_{\frac{\nu}{2} + i \nu}(\zeta),
\]

where \( \zeta \) is the de Sitter invariant variable defined in (2.4). On the other hand for arbitrary \( \beta \) the two–point function \( W_{\beta}(t_1 - t_2, x_1, x_2) \) and its permuted function do not respect the de Sitter isometry because their periodicity thermal property in imaginary time \( t \to t + i \beta \) is incompatible with the geometry of the global de Sitter manifold, the only exception being \( \beta = 2\pi \).

4 Mode expansion of the holomorphic plane waves

Let us now move to the complex two-dimensional de Sitter spacetime:

\[
dS^2_c = \{ Z \in \mathbb{C}^3, \ Z^2_0 - Z^2_1 - Z^2_2 = -1 \}. \tag{4.1}
\]

We may use the same coordinate chart as in Eq. (2.2):

\[
Z(t, x) = \begin{cases} 
Z^0 = \sinh t \ \text{sech} \ x \\
Z^1 = \tanh x \\
Z^2 = \cosh t \ \text{sech} \ x
\end{cases}.
\]

but now \( t \) and \( x \) are complex. In particular

1. For \( 0 < \text{Im} \ t < \pi \) and \( x \in \mathbb{R} \) the point \( Z(t, x) \) belongs to the forward tube

\[
T_+ = \{ Z = X + iY \in dS^2_c, \ Y^2 > 0, \ Y^0 > 0 \}. \tag{4.3}
\]

2. For \( -\pi < \text{Im} \ t < 0 \) and \( x \in \mathbb{R} \) the point \( Z(t, x) \) belongs to the backward tube

\[
T_- = \{ Z = X + iY \in dS^2_c, \ Y^2 > 0, \ Y^0 < 0 \}. \tag{4.4}
\]

There exists a remarkable set of solutions of the de Sitter Klein Gordon equation which may be interpreted as de Sitter plane waves [6, 7, 27]. Their definition makes no appeal to any particular coordinate system and may be given just in terms of the ambient spacetime coordinates: given a forward pointing lightlike real vector \( \xi \) in the ambient spacetime\(^1\) and a complex number \( \lambda \in \mathbb{C} \) let us construct the homogeneous function

\[
Z \in dS^2_c : \ Z \mapsto (\xi \cdot Z)^\lambda. \tag{4.5}
\]

For any given \( \xi \) and \( \lambda \) the above functions are holomorphic in the tuboids \( T_\pm \) [6, 7] and satisfy the massive (complex) de Sitter Klein-Gordon equation:

\[
(\Box - \lambda(\lambda - 1))(\xi \cdot Z)^\lambda = 0 \tag{4.6}
\]

(we may write \( \lambda = -\frac{1}{2} + iv\zeta \) in the following we will take for simplicity \( \nu \in \mathbb{R} \)). The boundary values

\[
(\xi \cdot X)^\lambda_\pm = \lim_{Z \in T_\pm, \ Z \to X} (\xi \cdot Z)^\lambda \tag{4.7}
\]

are homogeneous distributions of degree \( \lambda \) in the ambient spacetime and their restrictions to the real manifold \( dS_2 \) are solutions of the real de Sitter Klein-Gordon equation. All these objects are entire functions of \( \lambda \).

\(^1\) \( \xi \) is a real vector belonging to the forward lightcone \( C^+ = \{ \xi \in \mathbb{R}^3, \ (\xi^0)^2 - (\xi^1)^2 - (\xi^2)^2 = 0, \ \xi^0 > 0 \} \).
Let us now expand the above plane wave into modes of the static patch. The first thing to be done is to choose a basis manifold of the forward light-cone; the convenient choice is the hyperbolic basis $\Gamma = \Gamma_l \cup \Gamma_r$ “parallel” to the coordinate system (2.2) of the static chart:

$$\xi_l(w) = \begin{cases} \xi_0^l = \cosh w \\ \xi_1^l = -1 \\ \xi_2^l = \sinh w \end{cases}, \quad \xi_r(w) = \begin{cases} \xi_0^r = \cosh w \\ \xi_1^r = +1 \\ \xi_2^r = -\sinh w \end{cases}. \quad (4.8)$$

With all the above specifications, we get

$$\xi_l \cdot Z = \tanh x + \sech x \sinh(t - w), \quad \xi_r \cdot Z = -\tanh x + \sech x \sinh(t + w). \quad (4.9)$$

Let us take $Z(t + i\epsilon, x)$ with $t$ real. Since $Z(t + i\epsilon, x) \in T^+$ the wave $(\xi_l \cdot Z)^\lambda$ is a regular function of $t$ decreasing at infinity; its Fourier transform is given by

$$\int_{-\infty}^{\infty} e^{-it\omega} (\xi_l(w) \cdot Z(t + i\epsilon, x))^{\frac{\lambda}{2} + i\nu} dt = \frac{2e^{-i\omega\nu}}{\Gamma(\frac{\lambda}{2} - i\nu + i\omega)} e^{\frac{i}{2}\pi\omega} \left( e^{-i\omega Q_{-\frac{\lambda}{2} + i\nu}(u + i\epsilon)} \right); \quad (4.10)$$

where $Q$ is the associated Legendre function of the second kind$^2$ [26] defined on the complex plane cut on the real axis from $-\infty$ to $1$. Inversion gives

$$\left(\xi_l(w) \cdot Z(t + i\epsilon, x)\right)^{\frac{\lambda}{2} - i\nu} = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{i\omega(t - w)} \frac{\Gamma(\frac{\lambda}{2} - i\nu + i\omega)}{\Gamma(\frac{\lambda}{2} + i\nu)} e^{\frac{i}{2}\pi\omega} \left( e^{-i\omega Q_{-\frac{\lambda}{2} + i\nu}(u + i\epsilon)} \right) d\omega. \quad (4.12)$$

For $Z \in T^-$ an analogous computation gives

$$\left(\xi_l(w) \cdot Z(t - i\epsilon, x)\right)^{\frac{\lambda}{2} + i\nu} = \frac{1}{\pi} \int e^{-i\omega(t - w)} \frac{\Gamma(\frac{\lambda}{2} + i\nu - i\omega)}{\Gamma(\frac{\lambda}{2} + i\nu)} e^{\frac{i}{2}\pi\omega} \left( e^{-i\omega Q_{\frac{\lambda}{2} - i\nu}(u - i\epsilon)} \right) d\omega. \quad (4.13)$$

Similarly

$$\left(\xi_r(w) \cdot Z(t \pm i\epsilon, x)\right)^{-\frac{\lambda}{2} + i\nu} = \pm e^{-i\nu\pi} \int e^{i\omega(t \pm w)} \frac{\Gamma(\frac{\lambda}{2} \pm i\nu \pm i\omega)}{\Gamma(\frac{\lambda}{2} + i\nu)} e^{\frac{i}{2}\pi\omega} \left( e^{-i\omega Q_{\frac{\lambda}{2} \mp i\nu}(u \mp i\epsilon)} \right) d\omega. \quad (4.14)$$

### 5 Mode expansion of the maximally analytic two-point function

The maximally analytic (Bunch-Davies) two-point function admits the following global manifestly de Sitter invariant integral representation, valid for $Z_1 \in T^-$ and $Z_2 \in T^+$ [6, 7]:

$$W_{BD}(Z_1, Z_2) = \frac{e^{\pi\nu}}{8\pi \cosh \pi\nu} \int_{\Sigma} (\xi \cdot Z_1)^{-\frac{\lambda}{2} - i\nu}(\xi \cdot Z_2)^{-\frac{\lambda}{2} + i\nu} d\sigma(\xi). \quad (5.1)$$

Here $\Sigma$ is any basis manifold of the forward lightcone $C_+$ and $d\sigma$ the corresponding induced measure [6]. In the symbol $W_{BD}$ referring to the Bunch-Davies Wightman function we left the mass parameter $m = \sqrt{\frac{\lambda}{2} + \nu^2}$ implicit.

By using the static coordinates (2.2), the basis $\Gamma = \Gamma_l \cup \Gamma_r$ for the lightcone (with $d\sigma_{\Gamma} = dw$) and by inserting Eqs. (4.12 - 4.14) in Eq. (5.1) we get that the boundary value on the reals in

$^2$Note that the above Legendre functions are related by complex conjugation as follows:

$$\left( Q_{-\frac{\lambda}{2} + i\nu}(u + i\epsilon) \right)^* = e^{2i\nu\pi} Q_{\frac{\lambda}{2} + i\nu}(u - i\epsilon). \quad (4.11)$$
the static chart of the above global holomorphic two-point function can be represented as follows:

\[ W_{BD}(X_1, X_2) = \int_{-\infty}^{\infty} e^{-i\omega(t_1 - t_2)} \overline{W}_{BD}(\omega, u_1, u_2) d\omega \]  \hspace{1cm} (5.2)

\[ \overline{W}_{BD}(\omega, u_1, u_2) = \frac{e^{\pi \omega}}{4\pi^2 \cosh \pi(\nu - \omega)} \left[ e^{i\pi \nu Q_\omega^{\nu} - \frac{\omega}{4} + i\nu} (u_1 - i\epsilon) Q_{\omega}^{-\frac{i\omega}{2} - i\nu} (u_2 + i\epsilon) + e^{-i\pi \nu Q_\omega^{\nu} - \frac{\omega}{4} + i\nu} (u_1 + i\epsilon) Q_{\omega}^{-\frac{i\omega}{2} - i\nu} (u_2 - i\epsilon) \right] \]  \hspace{1cm} (5.3)

By using the identity [26]

\[ Q_{-\frac{1}{4} + i\nu}(u \pm i\epsilon) = \frac{\pi}{2 \cosh(\pi(\nu + \omega))} \left( e^{i\pi \nu P_\omega^{\nu} - \frac{\omega}{4} + i\nu} (u) + P_{\omega}^{\nu} - \frac{\omega}{4} + i\nu} (-u) \right) \]  \hspace{1cm} (5.4)

a straightforward calculation shows that

\[ \overline{W}_{BD}(\omega, u_1, u_2) = \tilde{P}_\nu(\omega, u_1, u_2). \]  \hspace{1cm} (5.5)

When \( \beta = 2\pi \) Eqs. (3.12) and (5.3) do coincide proving the claimed identification.

The permuted two-point function is in turn represented as follows:

\[ W_{BD}(X_2, X_1) = \int_{-\infty}^{\infty} e^{-i\omega(t_1 - t_2)} \tilde{P}_\nu(-\omega, u_1, u_2) d\omega = \int_{-\infty}^{\infty} e^{-i\omega(t_1 - t_2)} e^{-2\pi \omega} \tilde{P}_\nu(\omega, u_1, u_2) d\omega. \]  \hspace{1cm} (5.6)

In the above chain of identities we changed the integration variable \( \omega \to -\omega \) and - in the second step - used the symmetry of the two-point function \( \nu \to -\nu \). As a by product - by the Riemann-Lebesgue theorem - we get also the following crucial identity (which may also be checked directly):

\[ \tilde{P}_\nu(-\omega, u_2, u_1) = e^{-2\pi \omega} \tilde{P}_\nu(\omega, u_1, u_2). \]  \hspace{1cm} (5.7)

Eq. (5.6) encodes the Kubo-Martin-Schwinger property of the restriction of the maximal analytic two-point function (5.3) to the static patch: a geodetic observer in the static patch "perceives" a thermal bath of particles at inverse temperature \( 2\pi R \).

6 More about the vacuum of the static geodetic observer

By using Eqs. (5.2) and (5.6) we obtain the following new integral representation of the covariant commutator in the static chart:

\[ C_\nu(X_1, X_2) = W_{BD}(X_1, X_2) - W_{BD}(X_2, X_1) = \int_{-\infty}^{\infty} e^{-i\omega(t_1 - t_2)} \tilde{C}_\nu(\omega, u_1, u_2) d\omega \]  \hspace{1cm} (6.1)

where

\[ \tilde{C}_\nu(\omega, u_1, u_2) = (1 - e^{-2\pi \omega}) \tilde{P}_\nu(\omega, u_1, u_2) = -\tilde{C}_\nu(-\omega, u_2, x_1) \]  \hspace{1cm} (6.2)

Let us take the zero temperature limit \( \beta \to \infty \) in Eq. (3.12); only positive energies survive:

\[ W_\infty(X_1, X_2) = \int_0^{\infty} e^{-i\omega(t_1 - t_2)} (1 - e^{-2\pi \omega}) \tilde{P}_\nu(\omega, u_1, u_2) d\omega \]

\[ = \int_{-\infty}^{\infty} e^{-i\omega(t_1 - t_2)} \theta(\omega) \tilde{C}_\nu(\omega, u_1, u_2) d\omega. \]  \hspace{1cm} (6.3)

\( \theta(\omega) \) is Heaviside’s step function. The above equation points towards the following natural family of Rindler-de Sitter positive frequency modes (\( \omega \geq 0 \)):

\[ \varphi_{\omega,1}(t, u) = e^{\frac{i}{2} \pi \nu} e^{\frac{i}{2} \pi \omega} \sqrt{\sinh \pi \nu / 2\pi^2} \Gamma \left( \frac{1}{2} - i\nu + i\omega \right) e^{-i\omega t} Q_\omega^{\nu} - \frac{i\omega}{2} + i\nu (u - i\epsilon) \]
\[ \varphi_{\omega,2}(t, u) = e^{-\frac{1}{2}\pi \nu}e^{\pi \omega} \sqrt{\frac{\sinh \pi \omega}{2\pi^3}} \Gamma \left( \frac{1}{2} - i\nu + i\omega \right) e^{-i\omega t}Q^{\frac{1}{2}+i\nu}(u + i\epsilon), \]
\[ \varphi_{\omega,1}(t, u) = e^{\frac{1}{2}\pi \nu}e^{-\pi \omega} \sqrt{\frac{\sinh \pi \omega}{2\pi^3}} \Gamma \left( \frac{1}{2} + i\nu - i\omega \right) e^{i\omega t}Q^{\frac{1}{2}-i\nu}(u + i\epsilon), \]

\[ \varphi_{\omega,2}(t, u) = e^{-\frac{1}{2}\pi \nu}e^{-\pi \omega} \sqrt{\frac{\sinh \pi \omega}{2\pi^3}} \Gamma \left( \frac{1}{2} + i\nu - i\omega \right) e^{i\omega t}Q^{\frac{1}{2}-i\nu}(u + i\epsilon). \] (6.4)

equivalent to the one used in Sect. 3. Using the above modes we may represent the field operator in the static patch in the usual way

\[ \phi(t, x) = \int_0^\infty \left( \varphi_{\omega,1}(t, u)a_1(\omega) + \varphi_{\omega,2}(t, u)a_2(\omega) + \varphi_{\omega,1}^\ast(t, u)a_1^\dagger(\omega) + \varphi_{\omega,2}^\ast(t, u)a_2^\dagger(\omega) \right) d\omega. \] (6.5)

The state \( W_\infty(X_1, X_2) \) is characterized by the conditions \( a_1(\omega)\Psi_0 = a_2(\omega)\Psi_0 = 0 \); it is a pure state

\[ W_\infty(X_1, X_2) = \sum_i \int_0^\infty \varphi_{\omega,i}(t_1, u_1)\varphi_{\omega,i}^\ast(t_2, u_2) d\omega. \] (6.6)

Positive-definiteness is also clear from (6.6). The state defined by \( W_\infty \) may be interpreted as the vacuum state for the geodesic observer in the Rindler-de Sitter wedge and is the close analogous of the Fulling vacuum of Rindler QFT [28, 29].

Finally, the covariant commutator (6.1), which is of course independent from the chosen state, can be written as follows:

\[ C_{\nu}(X_1, X_2) = W_\infty(X_1, X_2) - W_\infty(X_2, X_1) = \sum_i \int_0^\infty \left[ \varphi_{\omega,i}(t_1, u_1)\varphi_{\omega,i}^\ast(t_2, u_2) - \varphi_{\omega,i}(t_2, u_2)\varphi_{\omega,i}^\ast(t_1, u_1) \right] d\omega. \] (6.7)

7. Other time-translation invariant states.

More generally, we may introduce the two-point functions

\[ W_F(X_1, X_2) = \int_{-\infty}^\infty e^{-i\omega(t_1-t_2)}F(\omega)\tilde{C}_\nu(\omega, u_1, u_2) d\omega \] (7.1)

\[ W_F(X_2, X_1) = \int_{-\infty}^\infty e^{-i\omega(t_1-t_2)}F(-\omega)\tilde{C}_\nu(-\omega, u_2, u_1) d\omega \] (7.2)

where \( F(\omega) \) is a real function or a distribution such that the product \( F(\omega)\tilde{C}(\omega, u_1, u_2) \) is well defined. Eq. (6.2) implies that it must be

\[ F(\omega) + F(-\omega) = 1. \] (7.3)

In particular

1. The vacuum (6.3) of the static geodetic observer correspond to

\[ F(\omega) = \theta(\omega). \] (7.4)

2. The Bunch-Davies maximally analytic state (5.3) correspond to

\[ F(\omega) = \frac{1}{1 - e^{-2\pi \omega}}. \] (7.5)

3. An antisymmetric function \( \beta(-\omega) = -\beta(\omega) \) defines a time invariant state

\[ F(\omega) = \frac{1}{1 - e^{-\beta(\omega)}}. \] (7.6)
4. The thermal equilibrium state \((3.12)\) at inverse temperature \(\beta\) corresponds to \(\beta(\omega) = \beta \omega\).

All the above two-point functions have the following general structure

\[
W(X_1, X_2) = \sum_{i=1,2} \int_0^\infty \cosh^2(\gamma(\omega)) \varphi_{\omega,i}(t_1, u_1) \varphi_{\omega,i}^*(t_2, u_2) d\omega + \sum_{i=1,2} \int_0^\infty \sinh^2(\gamma(\omega)) \varphi_{\omega,i}^*(t_1, u_1) \varphi_{\omega,i}(t_2, u_2) d\omega \tag{7.7}
\]

with in particular \(\cosh(\gamma(\omega)) = (1 - e^{-\beta \omega})^{-\frac{1}{2}}\) for the thermal state

\[
W_\beta(X_1, X_2) = \sum_{i=1,2} \int_0^\infty \varphi_{\omega,i}(t_1, u_1) \varphi_{\omega,i}^*(t_2, u_2) d\omega + \sum_{i=1,2} \int_0^\infty \varphi_{\omega,i}^*(t_1, u_1) \varphi_{\omega,i}(t_2, u_2) e^{\beta \omega} d\omega. \tag{7.8}
\]

The latter formula in turn allows to write \(W_\beta(t, x_1, x_2)\) as a Matsubara sum over imaginary frequencies as follows:

\[
W_\beta(X_1, X_2) = \sum_{n=0}^\infty W_\infty(t_1 - i n \beta, x_1, t_2, x_2) + \sum_{n=1}^\infty W_\infty(t_2, x_2, t_1 + i n \beta, x_1). \tag{7.9}
\]

The above representations clearly shows that all such states (but the vacuum \(\gamma = 0\)) are mixed states. In particular the maximally analytic Bunch-Davies two-point function is written

\[
W_{BD}(X_1, X_2) = \sum_{i=1,2} \int_0^\infty \varphi_{\omega,i}(t_1, u_1) \varphi_{\omega,i}^*(t_2, u_2) \frac{1}{1 - e^{-\beta \omega}} d\omega + \sum_{i=1,2} \int_0^\infty \varphi_{\omega,i}^*(t_1, u_1) \varphi_{\omega,i}(t_2, u_2) \frac{e^{\beta \omega} - 1}{e^{2 \pi \omega} - 1} d\omega. \tag{7.10}
\]

These are simple examples of what has been called a "Generalized Bogoliubov transformation" \([24, 25]\), a construction that directly provides mixed states by suitably extending the canonical quantization formalism.

8 Alpha–states in the static chart

The set of states \((7)\) does not contain every time translation invariant state. There is still the freedom to add to the two-point function a symmetric part that, as such, does not contribute to the commutator. The so-called \(\alpha\)-vacua \([12, 30, 32, 33, 34]\) belong to this second class of states. Let us briefly sketch their construction in the static patch coordinates.

The two-point Wightman functions of the \(\alpha\)-vacua may be written in terms of the Bunch-Davies two-point function as follows \([33, 34]\):

\[
W^{(\alpha)}(X_1, X_2) = \cosh^2 \alpha W_{BD}(X_1, X_2) + \frac{1}{2} \sinh 2 \alpha \left[ W_{BD}(X_1, -X_2) + W_{BD}(-X_1, X_2) \right] \tag{8.1}
\]

We are left with the task of expanding \(W_{BD}(X_1, -X_2)\) in the modes \((6.4)\). To do it, let us introduce the parity automorphism of the static patch:

\[
X(t, x) \rightarrow \tilde{X}(t, x) = X(t, -x) \tag{8.2}
\]

The curve \(s \rightarrow \tilde{X}(t + is, x)\) for \(0 < s < \pi\) is entirely contained in \(T_+\) and ends at

\[
\tilde{X}(t + i \pi, x) = -X(t, x) \tag{8.3}
\]

in the left Rindler–de Sitter wedge (see Fig. \((1)\)). Similarly, the curve \(s \rightarrow \tilde{X}(t + is, x)\) for \(0 > s > -\pi\) is entirely contained in \(T_-\) and ends again at \(\tilde{X}(t - i \pi, x) = -X(t, x)\) but from the opposite tube.
Given any two points $X_1$ and $X_2$ in the right Rindler–de Sitter wedge we may use again the maximally analytic global two-point function (5.1) and get

$$W_{BD}(X_1, -X_2) = W_{BD}(X_1, X_2) = W_{BD}(-X_1, X_2) = W_{BD}(X_1, -X_1, X_2)$$

$$= -i \frac{4}{4\pi^2} \int_{-\infty}^{\infty} e^{-i\omega(t_1-t_2)} Q^{\omega}_{\frac{1}{2}+iv}(u_1 - i\epsilon)Q^{-\omega}_{-\frac{1}{2}-iv}(u_2 - i\epsilon) d\omega$$

$$+ i \frac{4}{4\pi^2} \int_{-\infty}^{\infty} e^{-i\omega(t_1-t_2)} Q^{\omega}_{\frac{1}{2}+iv}(u_1 + i\epsilon)Q^{-\omega}_{-\frac{1}{2}-iv}(u_2 + i\epsilon) d\omega \quad (8.4)$$

$$= -i \int_{0}^{\infty} \varphi_{\omega,1}(t_1, x_1) \varphi^*_{\omega,2}(t_2, x_2) d\omega$$

$$+ i \int_{0}^{\infty} \varphi^*_{\omega,1}(t_1, u_1)\varphi_{\omega,2}(t_2, u_2) - \varphi_{\omega,2}(t_1, x_1) \varphi^*_{\omega,1}(t_2, x_2) \frac{2\sinh \pi \omega}{2 \sinh \pi \omega} d\omega. \quad (8.5)$$

In the second step we used Eq. (7.10) and the following relations:

$$\varphi_{\omega,1}(t \pm i\pi, -u) = ie^{\pm i\omega} \varphi_{\omega,2}(t, u), \quad \varphi_{\omega,2}(t \pm i\pi, -u) = -ie^{\pm i\omega} \varphi_{\omega,1}(t, u),$$

$$\varphi^*_{\omega,1}(t \pm i\pi, -u) = -ie^{\mp i\omega} \varphi^*_{\omega,2}(t, u), \quad \varphi^*_{\omega,2}(t \pm i\pi, -u) = ie^{\mp i\omega} \varphi^*_{\omega,1}(t, u). \quad (8.6)$$

Integration is (8.5) the over positive energies only. Putting everything together we get:

$$W^{(\alpha)}(X_1, X_2) = \sum_i \int_{0}^{\infty} \cosh^2(\gamma(\omega)) \varphi_{\omega,i}(t_1, u_1) \varphi^*_{\omega,i}(t_2, u_2) \frac{2\sinh \pi \omega}{2 \sinh \pi \omega} d\omega$$

$$+ \sum_i \int_{0}^{\infty} \sinh^2(\gamma(\omega)) \varphi^*_{\omega,i}(t_1, u_1) \varphi_{\omega,i}(t_2, u_2) \frac{2\sinh \pi \omega}{2 \sinh \pi \omega} d\omega$$

$$- i \sinh 2\alpha \int_{0}^{\infty} \varphi_{\omega,1}(t_1, x_1) \varphi^*_{\omega,2}(t_2, x_2) - \varphi_{\omega,2}(t_1, x_1) \varphi^*_{\omega,1}(t_2, x_2) \frac{2\sinh \pi \omega}{2 \sinh \pi \omega} d\omega$$

$$+ i \sinh 2\alpha \int_{0}^{\infty} \varphi^*_{\omega,1}(t_1, u_1)\varphi_{\omega,2}(t_2, u_2) - \varphi_{\omega,2}(t_1, u_1)\varphi^*_{\omega,1}(t_2, u_2) \frac{2\sinh \pi \omega}{2 \sinh \pi \omega} d\omega. \quad (8.7)$$

where

$$\cosh(\gamma(\omega)) = \sqrt{\frac{e^{i\omega} \cosh^2 \alpha + e^{-i\omega} \sinh^2 \alpha}{2 \sinh(\pi \omega)}}. \quad (8.8)$$

As expected, the $\alpha$-vacua are translation invariant w.r.t. the time variable of the Rindler–de Sitter wedge; here the generalized Bogoliubov transformation of the positive energy modes is more general than the one exhibited in Eq. (7.7). The extra terms which do not contribute to the commutator are altogether symmetric in the exchange of $X_1$ and $X_2$.

9 More about thermal propagators

In this section we examine some properties of the thermal correlation functions and discuss various limiting behaviors. This study is to better characterize them and also to lay the ground for the study of the IR loop contributions which will be the matter of a companion paper.

9.1 Wightman propagators for large time–like separation

Let us consider the limit $t = t_1 - t_2 \to \infty$, $x_1 = x_2 = 0$ (the general case $x_1 \neq x_2$ being essentially the same). The integrand in Eq. (3.12) has poles at

$$\omega = \pm \nu - \frac{i}{2} + in, \quad n \in \mathbb{Z}, \quad \omega = \frac{2\pi ik}{\beta}, \quad k \in \mathbb{Z}, \quad k \neq 0. \quad (9.1)$$
In Eq. (3.12) there is no pole at \( \omega = 0 \); still, \( \omega = 0 \) has a role to play in calculating the spacelike asymptotics.

In the limit \( t \to \infty \) the leading contributions come from the poles which are closer to the real axis:

\[
W_\beta(t, x_1 = x_2 = 0) \approx \begin{cases} 
  e^{-\frac{\pi}{2} (C_+ e^{i\nu t} + C_- e^{-i\nu t})} & \text{for } \beta < 4\pi, \\
  C_\beta e^{-t \frac{\pi}{\beta}} & \text{for } \beta > 4\pi,
\end{cases}
\]

(9.2)

where

\[
C_+ = C_- = \frac{1 - e^{-2\pi (\nu + \frac{i}{2})}}{1 - e^{-\beta (\nu + \frac{i}{2})}} e^{-\pi \nu} \Gamma \left( \frac{i}{2} - i \nu \right) \Gamma \left( \frac{1}{2} - i \nu \right),
\]

(9.3)

\[
C_\beta = \frac{\sin \left( \frac{2\pi^2}{\beta} \right)}{4\pi^2 \beta} |\Gamma \left( \frac{1}{4} - \frac{i \nu}{\beta} \right) \Gamma \left( \frac{1}{4} - \frac{i \nu}{2} + \frac{\pi}{\beta} \right) |^2.
\]

(9.4)

The asymptotic behavior of the propagator changes at \( \beta = 4\pi \). In the limit \( \beta \to \infty \) the constant \( C_\beta \) tends to zero and the Wightman function asymptotics is given again the upper line in (9.2).

The first set of poles in (9.1) is also actually related to the transmission and reflection coefficients of the quantum mechanical scattering problem (3.4). For instance a straightforward computation based on Eqs. (9.6) and (9.7) gives the transmission coefficient

\[
T = \frac{\sinh^2 (\pi \omega)}{\cosh[\pi (\nu - \omega)] \cosh[\pi (\nu + \omega)]}.
\]

(9.5)

The second set of poles in (9.1) depends also on the inverse temperature \( \beta \). As \( \beta \) increases the poles move towards the real axis. When \( \beta > 4\pi \) poles of the second set dominate and the large \( t \) behavior of the propagator changes accordingly.

### 9.2 Large spacelike separation

We now evaluate the asymptotic behaviour of the correlators when one of the spacelike coordinates goes to infinity in two distinct ways.

By use of the asymptotic behaviour of the Ferrers function at \( x \to \infty \) we get

\[
P^{\omega}_{\frac{1}{2} + i\nu} (\tanh x) \approx x^{-\infty} e^{i\omega x} \Gamma(1 - i \omega),
\]

(9.6)

\[
P^{\omega}_{\frac{1}{2} + i\nu} (-\tanh x) \approx x^{-\infty} \left[ \Gamma \left( \frac{1}{2} + i \nu - i \omega \right) e^{-i\omega x} \frac{\cosh(\nu \pi) \Gamma(\nu \pi) e^{i \omega x}}{\pi} \right].
\]

(9.7)

The singularities at \( \omega = 0 \) in the latter equation cancel each other, but, in the limit under consideration the two terms contribute separately. By substituting the above expressions into (3.12) and making the shift \( \omega \to \omega + i \epsilon \) we see that the dominant contribution comes from the lower half plane. We get that in this limit the Wightman propagator still depends on the temperature:

\[
\lim_{x_2 \to \infty} W_\beta(t_1 - t_2, x_1, x_2) = \frac{2\pi}{\beta} \frac{1}{4 \cosh \nu \pi} P_{\frac{1}{2} + i\nu} (-\tanh x_1) = \frac{2\pi}{\beta} W_{BD} (-\tanh x_1).
\]

(9.8)

Alternatively, we may consider the formal Taylor expansion of Eq. (3.12):

\[
W_\beta(t, x_1, x_2) = \frac{2\pi}{\beta} W_{BD}(\zeta) + \left( \pi - \frac{2\pi^2}{\beta} \right) i \frac{\partial}{\partial \zeta} W_{BD}(\zeta)
- \left( \frac{\pi \beta}{6} + \frac{4\pi^3}{3\beta} - \frac{\pi^2}{3} \right) \frac{\partial^2}{\partial \zeta^2} W_{BD}(\zeta) - \left( \frac{\pi^2 \beta}{6} - \frac{2\pi^4}{3\beta} + \frac{2\pi^3}{3} \right) i \frac{\partial^3}{\partial \zeta^3} W_{BD}(\zeta) + \ldots
\]

(9.9)

For \( \beta = 2\pi \) all the de Sitter breaking terms (i.e. every term but the first) at the RHS cancel, as expected. Also, when \( t = t_1 - t_2 \) is held constant and either \( x_1 \) or \( x_2 \) tend to plus or minus infinity, only the first terms at the RHS survives, with \( \zeta = - \tanh x \).
3.12 When the temperature is an integer multiple of the Hawking-Gibbons temperature, i.e. when \( |\omega| \approx |\beta| \), the limit \( |\omega| \to +\infty \) to one. The approximation works for \( |\omega| \) much larger than \( m \) and \( R \). The dependence on the temperature is lost in this high energy limit: only the Hadamard term survives.

3.14 Extra-Singularities at the horizon

When the temperature is an integer multiple of the Hawking-Gibbons temperature, i.e. when \( \beta = 2\pi/N \), we may use Eq. (3.12) to derive another representation of the two-point function as a finite sum of Legendre functions (as opposed to the infinite Matsubara-type series (7.9)); this is obtained by translating the Birrel-Davies maximal analytic two-point function in the imaginary time variable within the analyticity strip \(-2\pi < \Im t < 0\) (see also [30, 31]):

\[
W_{BD}(t, x_1, x_2) = \int_{-\infty}^{\infty} e^{-i\omega(t_1-t_2)} \frac{1-e^{-2\pi\omega}}{1-e^{-\pi\omega}} \tilde{P}_{\omega,\nu}(u_1, u_2) d\omega =
\]

\[
= \frac{1}{4 \cosh \pi \nu} P_{\nu+\omega} \left( \zeta \left( t_1 - t_2 - i\epsilon, x_1, x_2 \right) \right) + \frac{1}{4 \cosh \pi \nu} \sum_{n=1}^{N-1} P_{\nu+\omega} \left( \zeta \left( t_1 - t_2 - i\frac{2\pi n}{N}, x_1, x_2 \right) \right). \quad (9.11)
\]

The first term on the RHS is exactly the BD de Sitter invariant Wightman function; this is singular at \( \zeta = -1 \). The extra terms become singular when the two points approach either the left or the right horizon:

\[
X_1 = X(\lambda + c_1, \lambda), \quad X_2 = X(\lambda + c_2, \lambda + \Delta \lambda) \quad (9.12)
\]

In the limit \( \lambda \to \pm \infty \) the above events belong to the horizons. Then:

\[
\zeta \left( c_1 - c_2 - i\frac{2\pi n}{N}, \lambda, \Delta \lambda \right) = -\cosh \left( c_1 - c_2 - i\frac{2\pi n}{N} \right) \cosh(\lambda + \Delta \lambda) \to -1.
\]

For generic \( \beta \), the limit \( \lambda \to \infty \) may be obtained by performing manipulations similar to those which led to (9.8):

\[
W_{BD}(\lambda \to \infty) \approx -\frac{1}{2} \int d\omega \frac{1}{(e^{2\pi(\omega+i0)} - 1) \sinh \pi (\omega + i0)} e^{-2i\omega \lambda},
\]

Due to presence of the double pole at \( \omega = -i0 \) the answer is as follows:

\[
W_{BD}(\lambda \to \infty) \approx \frac{2\pi \lambda}{\beta} \approx \frac{2\pi}{\beta} W_{BD}(\lambda \to \infty),
\]

14
Note that taking in Eq. (9.11) the horizon limit also gives $W_{BD}^*(\lambda \to \infty) \approx NW_{BD}(\lambda \to \infty)$.

A remarkable fact is the following: for light-like separations inside the static patch the dominant contribution to the propagator comes from large $\omega$’s; on the contrary, at the horizon small $\omega$’s provide the leading contribution. This is because the horizon is the boundary of the patch; the main contribution comes from the infrared rather than ultraviolet frequencies.

### 9.5 Flat space limit

Here we consider the flat space limit, i.e. we let the de Sitter radius go to infinity ($R \to \infty$). Let us start by discussing the flat limit of the modes (4.5) and of the BD two-point function, following the treatment given in [6]. To this aim it is better to use another orbital basis of the forward lightcone $C^+$:

$$
\xi_+(k) = \begin{cases} 
\xi_0 = \sqrt{k^2 + m^2}/m \\
\xi_1 = k/m \\
\xi_2 = -1 
\end{cases} \quad \xi_-(k) = \begin{cases} 
\xi_0 = \sqrt{k^2 + m^2}/m \\
\xi_1 = -k/m \\
\xi_2 = +1 
\end{cases} \quad (9.13)
$$

$$
\lim_{R \to \infty} \left( \frac{\xi_+(k) \cdot X(t_1 - i\nu R, x_1/R)}{R} \right)^{-\frac{1}{2} - imR} = e^{-it\sqrt{k^2 + m^2} + i\nu x} \quad (9.14)
$$

$$
\lim_{R \to \infty} \left( \frac{\xi_-(k) \cdot X(t_2 + i\nu R, x_2/R)}{R} \right)^{-\frac{1}{2} - imR} = 0 \quad (9.15)
$$

and so on (recall that $\nu = \sqrt{m^2 R^2 - \frac{1}{4}}$).

It follows that when $R \to \infty$ [6]

$$
W_{BD}(X(t_1 - i\nu R/R, x_1/R), X(t_2 + i\nu R/R, x_2/R)) \to \frac{1}{4\pi} \int_{-\infty}^{\infty} e^{-it\sqrt{k^2 + m^2}(t_2 - t_1 - i\nu x)} e^{-\frac{1}{2} - \frac{1}{2} imR} \frac{dk}{\sqrt{k^2 + m^2}} \quad (9.16)
$$

which is the standard Fourier representation of the (positive energy) Wightman function in Minkowski space.

To find the flat limit of the Wightman function $W_{BD}$ for arbitrary $\beta$ in the same way we may start by rewriting the integral representation (3.12) in the following way:

$$
W_{\beta}(t, x_1, x_2) = \int_0^{\infty} d\omega \left[ e^{-i\omega t} P^R_{\nu}(\omega, x_1, x_2) \frac{1 - e^{-2\pi R\omega}}{1 - e^{-\beta\omega}} + e^{i\omega t} \tilde{P}^R_{\nu}(\omega, x_1, x_2) \frac{1 - e^{-2\pi R\omega}}{e^{\beta\omega} - 1} \right] \quad (9.17)
$$

the superscript $R$ indicates explicitly the restored dependence of (3.12) on the radius $R$. For $\beta = 2\pi R$ the limit $R \to \infty$ (formally) gives

$$
\lim_{R \to \infty} W_{BD}(t, x_1, x_2) = \int_0^{\infty} d\omega e^{-i\omega t} \tilde{P}^\infty_{\nu}(\omega, x_1, x_2). \quad (9.18)
$$

Taking into account Eq. (9.16) it follows that

$$
\lim_{R \to \infty} W_{\beta}(t, x_1, x_2) = \int_0^{\infty} d\omega \left[ e^{-i\omega t} P^\infty_{\nu}(\omega, x_1, x_2) \frac{1}{1 - e^{-\beta\omega}} + e^{i\omega t} \tilde{P}^\infty_{\nu}(\omega, x_1, x_2) \frac{1}{e^{\beta\omega} - 1} \right] =
$$

$$
= \int_{-\infty}^{\infty} \frac{dk}{4\pi \sqrt{k^2 + m^2}} \left[ e^{-i\sqrt{k^2 + m^2} t + i\nu x} \frac{1}{1 - e^{-\beta\sqrt{k^2 + m^2}}} + e^{i\sqrt{k^2 + m^2} t - i\nu x} \frac{1}{e^{\beta\sqrt{k^2 + m^2}} - 1} \right] \quad (9.19)
$$

which is precisely the flat space thermal propagator with temperature $1/\beta$. In the the Bunch-Davies the temperature scales together with $R$ and this maintains invariance at every stage, while in generic case $\beta$ does not scale with $R$. On the other hand scaling $\beta = \beta' R$ with constant $\beta'$ provides in the in the vacuum positive energy Wightman function. This is true also for $\beta = \infty$. 

15
10 Conclusions and outlook

Cauchy surfaces in the Rindler-de Sitter wedge universe are not Cauchy’s for the geodetically complete global de Sitter universe. Giving initial data on such surfaces completely determines the classical dynamics of fields in the Rindler-de Sitter universe. By applying the formalism of canonical quantization and Bogoliubov transformations we may construct all the pure Fock states representing quantum Klein Gordon fields in the wedge. Generalized Bogoliubov transformations however allow for the construction of a much wider set of states which are generally speaking mixed. In this paper we have explicitly constructed all the above states by separating the variables in the static chart (2.2); the construction was exhibited for the two–dimensional de Sitter space not to burden the presentation with unnecessary complications. In particular, we gave integral representations of all the KMS states including the Bunch Davies state at temperature \( T = 1/2\pi R \); all of them are directly seen to be mixed states, the only pure state in that family being obtained in the zero temperature limit. We also provided explicit formulae for the alpha vacua which include also non diagonal terms.

The thermal propagators have unusual pathological singularities on the horizons (vaguely remembering Einstein’s suspicions about the presence of matter on the horizons [1]). We mention also that, while these propagators obey the fluctuation-dissipation theorem, the de Sitter invariant Bunch-Davies state, restricted to the wedge, does not possess at least one of the properties of Minkowskian thermal states [36] because de Sitter invariance forbids Debye screening. So there is room for further study.

The important question for cosmology is: what about the initial state of our Universe? The difference between the static patch, the Poincaré patch and the global de Sitter universe [17], [18] will appear in the infrared loops which are sensitive to the initial (and to the boundary conditions).

In flat space–time (at least in a box) an initial arbitrary state (within a reasonable class) will thermalize sooner or latter. The temperature depends on the initial conditions and may be arbitrary. What about thermalization in de Sitter space? Is there thermalization to a state with an arbitrary temperature? How does the answer to these questions depends on the choice of patch (type of initial Cauchy surface)?

Boltzmann’s equation allows to sum up leading secularly growing corrections from all loops [17]. What is the analog of flat space Boltzmann’s equation in the static de Sitter space? We will address some of the above questions in a forthcoming companion paper.

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A Appendix

A.1 Completeness relation of Associated Legendre Functions on the cut

Here we provide an explicit (formal) calculation of the Canonical Commutation Relations (3.2) which, by introducing \( \cos \theta = \tanh x = u \), we rewrite as follows:
\[
\sin \theta_1 \sin \theta_2 \delta(\cos \theta_1 - \cos \theta_2) = \int_{-\infty}^{\infty} \frac{\omega \, d\omega}{4\pi \sinh(\pi \nu)} \Gamma\left(\frac{1}{2} + i \nu - i \omega\right) \Gamma\left(\frac{1}{2} - i \nu + i \omega\right) \times \\
\times \left[ P_{-\frac{1}{2} + i \nu}^{i \omega + i \nu}(\cos \theta_1) \left( P_{-\frac{1}{2} + i \nu}^{i \omega - i \nu}(\cos \theta_2) \right)^* + P_{-\frac{1}{2} + i \nu}^{i \omega - i \nu}(-\cos \theta_1) \left( P_{-\frac{1}{2} + i \nu}^{i \omega - i \nu}(-\cos \theta_2) \right)^* \right]. \tag{A.1}
\]

Using the holomorphic plane waves introduced in Sec. (4) we get the following integral representation for \( P_{-\frac{1}{2} + i \nu}^{i \omega}(\cos \theta) \) (see Eq. (4.10) and the following ones):

\[
P_{-\frac{1}{2} + i \nu}^{i \omega}(\cos \theta) = \frac{i \Gamma\left(\frac{1}{2} + i \nu\right)}{2 \pi \Gamma\left(\frac{1}{2} + i \nu - i \omega\right)} \int_{-\infty}^{\infty} dt e^{-i \omega t} \Delta f(t, \theta) \tag{A.2}
\]

where we set

\[
f_\pm(t, \theta) = (\xi(t) \cdot Z(t \pm i \epsilon, \theta))^\pm_{-i \nu} = [\cos \theta + \sinh(t \pm i \epsilon)]^{-\frac{1}{2} - i \nu}, \tag{A.3}
\]

\[
\Delta f(t, \theta) = (f_+(t, \theta) - f_-(t, \theta)). \tag{A.4}
\]

\( P_{-\frac{1}{2} + i \nu}^{i \omega}(\cos \theta) \) is therefore the Fourier transform of the discontinuity of the holomorphic plane waves on the real de Sitter manifold. Let us insert (A.2) in Eq. (A.1); let us consider for instance the first term on the rhs of Eq. (A.1). By performing the integration over \( \omega \) we get

\[
(A.1) = - \frac{i}{16 \pi \sinh^2 \pi \nu} \int_{-\infty}^{\infty} dt [\partial_1 \Delta f(t, \theta_1)] \Delta f(t, \theta_1)^* - \Delta f(t, \theta_1) \partial_1 \Delta f(t, \theta_2)^* + \\
- \frac{i}{16 \pi \sinh^2 \pi \nu} \int_{-\infty}^{\infty} dt [\partial_2 \Delta f(t, \pi - \theta_1)] \Delta f(t, \pi - \theta_1)^* - \Delta f(t, \pi - \theta_1) \partial_2 \Delta f(t, \pi - \theta_2)^* = \\
= - \frac{i}{16 \pi \sinh^2 \pi \nu} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} dt [(\partial_1 f_k(t, \theta_1)) f_k(t, \theta_1)^* - f_k(t, \theta_1) \partial_1 f_k(t, \theta_2)^*] + \\
- \frac{i}{16 \pi \sinh^2 \pi \nu} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} dt [(\partial_2 f_k(t, \pi - \theta_1)) f_k(t, \pi - \theta_1)^* - f_k(t, \pi - \theta_1) \partial_2 f_k(t, \theta_2)^*]. \tag{A.5}
\]

In the second step we used the analyticity properties of the plane waves; this simplification is valid in the two-dimensional spacetime and in any even dimensional spacetime as well. By introducing the Mellin representation of the plane wave:

\[
f_\pm(t, \theta) = \frac{e^{\mp \frac{i}{2} \pi \left(\frac{1}{2} + i \nu\right)}}{\Gamma\left(\frac{1}{2} + i \nu\right)} \int_{0}^{\infty} du \, u^{-\frac{1}{2} + i \nu} e^{\pm i u (\cos \theta + \sin \theta \sinh(t \pm i \epsilon))}, \quad 0 < \theta < \pi, \tag{A.6}
\]

a few easy integrations show the validity of Eq. (A.1) and the completeness of the modes.
References

[1] M. Janssen, *The Einstein-de Sitter debate and its aftermath*, HSci/Phys - Lorentz.leidenuniv.nl (2016).

[2] W. de Sitter, *Koninklijke Akademie van Wetenschappen te Amsterdam. Proceedings* 19 (1916-17): 1217-1225.

[3] W. de Sitter, *Koninklijke Akademie van Wetenschappen te Amsterdam. Proceedings* 20 (1917-18): 229243.

[4] G. W. Gibbons and S. W. Hawking, *Phys. Rev.* D15, 2738 (1977).

[5] G. Sewell *Ann. Phys.* 141, 202 (1982)

[6] J. Bros and U. Moschella, *Rev. Math. Phys.* 8, 327 (1996)

[7] J. Bros, U. Moschella and J. P. Gazeau, *Phys. Rev. Lett.* 73, 1746 (1994).

[8] H. Narnhofer, I. Peter and W. E. Thirring, Int. J. Mod. Phys. B10, 1507-1520 (1996).

[9] W. E. Thirring, Acta Phys. Aust. Suppl. IV, 269 (1967)

[10] O. Nachtmann, Österr. Akad. Wiss. Math.-Naturw. Kl. Abt. II 176, 363379 (1968)

[11] N. A. Chernikov and E. A. Tagirov, *Ann. Inst. H. Poincaré Phys. Theor.* A9, 109 (1968).

[12] C. Schomblond, and P. Spindel, *Annales de l’I.H.P. Physique thoriqve* 25, 67-78 (1976)

[13] T. S. Bunch and P. C. W. Davies, *Proc. Roy. Soc. Lond.* A360, 117 (1978).

[14] R. Figari, R. Hoegh-Krohn and C.R. Nappi, Comm. Math. Phys. 44, 265-278 (1975).

[15] L. D. Landau and E. M. Lifshitz, Theoretical Physics Vol. 10. Pergamon Press, Oxford (1975).

[16] A. Kamenev, *Many-body theory of non-equilibrium systems* Cambridge University Press, Cambridge (2011).

[17] E. T. Akhmedov, Int. J. Mod. Phys. D 23, 1430001 (2014)

[18] E. T. Akhmedov, U. Moschella and F. K. Popov, Phys. Rev. D 99 (2019) no.8, 086009

[19] E. T. Akhmedov, U. Moschella, K. E. Pavlenko and F. K. Popov, Phys. Rev. D 96, no. 2, 025002 (2017)

[20] G. Lemaitre, *Journal of Mathematics and Physics*, 4, 188 (1925).

[21] E. Schrodinger, *Expanding Universes*, Cambridge University Press, Cambridge (1956).

[22] U. Moschella, Prog. Math. Phys. 47, 120-133 (2006).

[23] K. Lanczos, Welt. Phys. Z. 24, 539 (1922)

[24] U. Moschella and R. Schaeffer, JCAP 02, 033 (2009)

[25] U. Moschella and R. Schaeffer, AIP Conf. Proc. 1132, no.1, 303-332 (2009)

[26] *Higher Transcendental Functions [Volumes I-III]* Bateman, Harry (1953) Higher Transcendental Functions [Volumes I-III]. Vol.I-III. McGraw-Hill Book Company, New York.

[27] I.M.Gel’fand, M. I. Graev and N. Ya. Vilenkin, *Generalized Functions Vol 5: Integral geometry and representation theory*, Academic Press, New York (1964)
[28] S. A. Fulling, Phys. Rev. D7, 28502862 (1973).

[29] S. A. Fulling, J. Phys. A10, 917951 (1977).

[30] E. T. Akhmedov, K. V. Bazarov, D. V. Diakonov, U. Moschella, F. K. Popov and C. Schubert, Phys. Rev. D 100 (2019) no.10, 105011 doi:10.1103/PhysRevD.100.105011 [arXiv:1905.09344 [hep-th]].

[31] M. Bertola, V. Gorini and M. Zeni, ”hep-th/9508004 (1995)

[32] E. Mottola, Phys. Rev. D 31, 754 (1985). doi:10.1103/PhysRevD.31.754

[33] B. Allen, Phys. Rev. D 32, 3136 (1985). doi:10.1103/PhysRevD.32.3136

[34] H. Epstein and U. Moschella, Commun. Math. Phys. 336, no.1, 381-430 (2015)

[35] Sebastian Bielski (2013) Orthogonality relations for the associated Legendre functions of imaginary order, Integral Transforms and Special Functions, 24:4, 331-337, DOI: 10.1080/10652469.2012.690097

[36] F. K. Popov, JHEP 06, 033 (2018) doi:10.1007/JHEP06(2018)033 [arXiv:1711.11010 [hep-th]].