QED Corrections for Polarized Elastic $\mu e$ Scattering$^{a,b}$

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ABSTRACT

We present a new study of polarized elastic muon-electron scattering. The Born cross-section is calculated for arbitrary polarization of muon and electron. The complete photonic $O(\alpha)$ radiative corrections are determined for the case of longitudinally polarized muons and electrons. All calculations are done by two methods: semianalytic, which allows an implementation of the experimental cuts used for the analysis of $\mu e$ scattering data from the beam polarimeter of the SMC experiment at CERN and completely analytic, which is used for cross checks. The FORTRAN code $\mu e\alpha$ realizes formulae of both approaches. We prove that certain experimental cuts lead to negligible radiative corrections in the muon beam polarization experiment.

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1 Introduction

Polarized elastic $\mu e$ scattering is being measured by the SMC collaboration at CERN as a monitor of muon beam polarization [1]. Since the measurement pretends to be very precise, the photonic corrections have to be taken into account.

The differential cross-section for this process in lowest order may be cast into the simple form [2]

$$\frac{d\sigma^{\text{BORN}}}{dy} = \frac{2\pi\alpha^2}{m_eE_\mu} \left[ \frac{(Y - y)}{y^2Y} (1 - yP_eP_\mu) + \frac{1}{2} (1 - P_eP_\mu) \right], \quad (1.1)$$

where the following notation is used:

$m_\mu, m_e$– muon and electron masses,

$P_\mu, P_e$– longitudinal polarizations of muon beam and electron target,

$y = y_\mu = 1 - \frac{E'_\mu}{E_\mu}$ – the measured energy loss of the muon,

$Y = \left(1 + \frac{m_\mu}{2E_\mu}\right)^{-1}$ – its kinematical maximum,

$E_\mu, E'_\mu, E'_e$– muon (initial, final), electron final energies in the laboratory frame.

The polarization dependence of $d\sigma$ is used to calculate the measured electron spin-flip asymmetry $A_{\mu e}^{\text{exp}}$

$$A_{\mu e}^{\text{exp}} = \frac{\frac{dy}{dy} \frac{d\sigma(\uparrow\downarrow)}{dy} - \frac{dy}{dy} \frac{d\sigma(\uparrow\uparrow)}{dy}}{\frac{dy}{dy} \frac{d\sigma(\uparrow\downarrow)}{dy} + \frac{dy}{dy} \frac{d\sigma(\uparrow\uparrow)}{dy}}. \quad (1.2)$$

The asymmetry is measured as a function of the variable $y_\mu$.

Previous calculations [3]-[6], presented results in terms of the variable $y_e = E'_e/E_\mu$, and, only Ref. [6] took into account the polarizations.

If elastic $\mu e$ scattering is treated in the Born approximation then

$$y_\mu = y_e = y, \quad (1.3)$$

and eq. (1.1) may be written in terms of either $y_\mu$ or $y_e$.

The situation changes drastically if one calculates QED corrections. Due to the emission of non-observed photons the identity (1.3) does not hold anymore, and one has to specify the variable to be used for the calculation of radiative corrections. Their numerical values may be very different in $y_\mu$ and $y_e$.

Since the measurement and the analysis were performed in terms of $y_\mu$ [1], the calculation of QED corrections must be done, of course, in terms of the same variable. This is why a new calculation was neccessary.

Our new calculation is the theoretical basis for the Fortran program $\mu e la$, [7]. It is a complete, order $O(\alpha^3)$, calculation. It takes into account longitudinal polarizations of both $\mu$ and $e$, finite muon mass effects (the electron mass is neglected wherever possible). In the semianalytic approach it is possible to apply all experimental cuts which were used in the analysis of the experimental data:

- a recoil electron energy cut, $E'_e \geq E'^{\text{RC}}_e$ ($E'^{\text{RC}}_e = 35$ GeV);
- an energy balance cut, $|E - E'_\mu - E'_e| \geq E^{\text{BC}}_e$ ($E^{\text{BC}}_e = 40$ GeV);
angular cuts on both $\mu$ and $e$, $\theta_\mu$ and $\theta_e$ in the laboratory system: $|\theta_\mu^{\text{meas}} - \theta_\mu^{\text{BORN}}| \leq \theta_{\text{min}}$, $|\theta_e^{\text{meas}} - \theta_e^{\text{BORN}}| \leq \theta_{\text{min}}$ ($\theta_{\text{min}} = 1$ mrad). In the above, $\theta_\mu^{\text{meas}}$ and $\theta_e^{\text{meas}}$ are the measured angles, while $\theta_\mu^{\text{BORN}}$ and $\theta_e^{\text{BORN}}$ are angular values calculated using BORN kinematics.

Muonic RC

Electronic RC

$\mu e$ - interference

Vacuum Polarization

Figure 1: Feynman diagrams for the elastic $\mu e$ scattering in order $O(\alpha^3)$. 

It order $\mathcal{O}(\alpha^3)$, the 14 Feynman graphs, shown in Fig. 1, contribute to the cross-section. The latter may be subdivided into $12=2 \times 6$ separately gauge invariant contributions:

$$\frac{d\sigma^{\text{QED}}}{dy_\mu} = \sum_{l=1}^{2} \sum_{k=1}^{6} \frac{d\sigma^l_k}{dy_\mu}, \quad (1.4)$$

where the indices $k$ and $l$ have the following meaning

- $l = 1$ — unpolarized contribution, $l = \text{unpol}$;
- $l = 2$ — polarized contribution (the terms proportional to $P_e P_\mu$ in (1.1)), $l = \text{pol}$.

$k = 1$ — Born cross-section, $k = b$;
- $k = 2$ — Radiative corrections (RC) for the muonic current: vertex + bremsstrahlung, $k = \mu\mu$;
- $k = 3$ — contribution of the anomalous magnetic moment of the muon, $k = \text{amm}$;
- $k = 4$ — RC for the electronic current: vertex + bremsstrahlung, $k = ee$;
- $k = 5$ — $\mu e$ interference: two-photon exchange +
  muon-electron bremsstrahlung interference, $k = \mu e$;
- $k = 6$ — Vacuum polarization correction, running $\alpha$, $k = \text{vp}$.

The resulting QED corrected cross-section is given by the sum

$$\frac{d\sigma^{\text{QED}}}{dy_\mu} = \sum_k \left( \frac{d\sigma^{\text{unpol}}_k}{dy_\mu} + P_e P_\mu \frac{d\sigma^\text{pol}_k}{dy_\mu} \right). \quad (1.5)$$

The cross-sections with $k = \mu\mu, ee, \mu e$ have similar generic structure

$$\frac{d\sigma_k}{dy_\mu} = \frac{\alpha}{\pi} \frac{\delta_{\text{VR}}}{\delta_k} \frac{d\sigma^\text{BORN}}{dy_\mu} + \frac{d\sigma^\text{BREM}}{dy_\mu}, \quad (1.6)$$

with a factorized part $\delta_{\text{VR}}^k$ originating from infrared divergent virtual (V) and real soft photon (R) contributions.

The main conclusion of this study is illustrated in Figs. 2 and 3, which present radiative corrections to the asymmetry as a function of the variable $y_\mu$ for two cases: without any cuts and with experimental cuts described above.

The asymmetry $A_{\mu e}^a$, and the radiative correction to it, $\delta_{y_\mu}^A$ are defined as follows:

$$A_{\mu e}^a = \frac{d\sigma^a(\uparrow\downarrow)}{dy_\mu} - \frac{d\sigma^a(\uparrow\uparrow)}{dy_\mu} + \frac{d\sigma^a(\uparrow\downarrow)}{dy_\mu} + \frac{d\sigma^a(\uparrow\uparrow)}{dy_\mu}, \quad a = \text{BORN, QED}, \quad \delta_{y_\mu}^A = \frac{A_{\mu e}^{\text{QED}}}{A_{\mu e}^{\text{BORN}}} - 1. \quad (1.7)$$

As is seen from the figures, the corrections without cuts are very large and reach up to -20%. When the four above mentioned cuts are taken into account they reduce $\delta$ to values below 1%. Actually, for a wide range of $y_\mu$ they are even well below 1%.

The main conclusion of our investigation is that one may safely neglect radiative corrections in the determination of the muon beam polarization with the SMC set-up.

---

1 In the following we will always present the formulae in the form (1.5), i.e. summed over $l$. 

Figure 2: QED corrections to the polarization asymmetry without experimental cuts.

Figure 3: QED corrections to the polarization asymmetry with experimental cuts: $E^{RC}_{RC} = 35$ GeV, $E^{BC}_{BC} = 40$, $\theta_{e,\min} = \theta_{\mu,\min} = 1$ mrad.
2 The Born cross-section

2.1 Kinematics and phase space

We consider the elastic scattering process

$$\mu(k_1) + e(p_1) \rightarrow \mu(k_2) + e(p_2), \quad (2.8)$$

in a fixed target experiment, i.e. with the initial state electron at rest, \( k_1 = (0, im_e) \).

Since the typical incident muon energy, \( E_\mu \), in present-day fixed target experiments is \( \mathcal{O}(10^2 - 10^3 \text{ GeV}) \), the maximal c.m.s. energy,

$$s = -(k_1 + p_1)^2 = m_\mu^2 + m_e^2 + 2m_e E_\mu, \quad (2.9)$$

is very small, \( \sqrt{s} \leq 1 \text{ GeV} \). Therefore, we may completely neglect Z-boson exchange.

In fact, for the energy used by the SMC collaboration, \( E_\mu = 190 \text{ GeV} \), the invariant \( s \) is only 20 times bigger than the muon mass squared. Therefore, we can not neglect effects of the finite muon mass. Of course, the electron mass may be completely neglected.

While calculating the Born cross-section, we will perform all derivations exactly even in \( m_e \), since the resulting expressions are very compact even if \( m_e \) is kept, but at the end of calculations we will neglect the electron mass \( m_e \). The Born process is characterized by one kinematical variable, besides \( s \). We will use the dimensionless variable \( y \):

$$y = \frac{p_1(k_1 - k_2)}{p_1 k_1} = \frac{k_0^1 - k_0^2}{k_0^1} \equiv 1 - \frac{E_\mu'}{E_\mu}, \quad (2.10)$$

We will introduce also the transferred momentum squared

$$Q^2 = (k_1 - k_2)^2 = -t. \quad (2.11)$$

It is easy to derive the identity:

$$Q^2 = Sy, \quad (2.12)$$

where

$$S = s - m_e^2 - m_\mu^2. \quad (2.13)$$

For the Born cross-section, we have

$$d\sigma^{\text{Born}} = \frac{1}{2\sqrt{\lambda_s}} |M^{\text{Born}}|^2 d\Gamma_2, \quad (2.14)$$

with

$$\lambda_s = S^2 - 4m_e^2 m_\mu^2, \quad (2.15)$$

$$d\Gamma_2 = (2\pi)^4 \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \delta(k_1 + p_1 - k_2 - p_2). \quad (2.16)$$

In terms of \( y \) the differential phase space reads

$$d\Gamma_2 = \frac{1}{16\pi^2} \frac{dQ^2}{\sqrt{\lambda_s}} d\varphi = \frac{S}{16\pi^2 \sqrt{\lambda_s}} dy d\varphi. \quad (2.17)$$

\(^2\) We refer to this approximation as to the “Ultra-Relativistic Approximation (URA) in \( m_e \)”.

7
2.2 Spin degrees of freedom

Since we are going to deal with the scattering of polarized particles, there will be additional essential variables, besides $s$ and $y$, which are supposed to describe the spin degrees of freedom of the problem. Their description uses the language of spin density matrix (for details we refer e.g. to Appendix C of [8]).

For non-polarized particles, we use projection operators in trace calculations, i.e. summing and averaging over spin indices looks as

$$\sum_s u^s(p)\bar{u}^s(p) = \frac{1}{2}\Lambda(p), \quad (2.18)$$

with

$$\Lambda(p) = -i\hat{p} + m. \quad (2.19)$$

For polarized particles, we use the spin density matrix instead

$$\sum_s u^s(p)\bar{u}^s(p) = \frac{1}{2}(1 + i\gamma_5\hat{\xi})\Lambda(p), \quad (2.20)$$

where $\xi$ is the polarization four-vector.

In the particle rest frame, $\vec{p} = 0$, it is:

$$\xi = (P\vec{n}, 0), \quad (2.22)$$

where $\vec{n}$ is a unit vector in the direction of spin quantization, and $P$ is the polarization, defining the degree of spin orientation along the direction $\vec{n}$. For instance, $P = 1$ means that the probability of a particle to have its spin projection along the direction $\vec{n}$ is equal to 1 (right handed longitudinal polarization, if vector $\vec{n}$ is chosen along particle momentum $\vec{p}$). From (2.22) in the particle rest frame, we have

$$\xi p = 0, \quad \xi^2 = P^2. \quad (2.23)$$

Due to Lorenz invariance, the properties (2.23) are fulfilled in any Lorenz frame.

The initial electron with the four-momentum $p_1$ is at rest in the laboratory frame. Using then the direction of incoming muon as the direction of spin quantization, i.e. $\vec{n} = \vec{k}_1/|\vec{k}_1|$, we get the four-vector of the electron polarization from (2.22)

$$\xi_e = P_e \left( \frac{\vec{k}_1}{|\vec{k}_1|}, 0 \right). \quad (2.24)$$

The four-vector $\xi_\mu$ may be obtained from the expression similar to (2.24) in the muon rest frame

$$\xi_\mu = P_\mu \left( \frac{\vec{k}_1}{|\vec{k}_1|}, 0 \right). \quad (2.25)$$

---

3 A naive use of longitudinal polarizations from the early beginning of calculations, i.e. use of the spin density matrices in the form

$$\sum_s u^s(p)\bar{u}^s(p) = \frac{1}{2}(1 + \lambda\gamma_5)\Lambda(p), \quad (2.21)$$

does not properly reproduce the finite muon mass terms in the $P_eP_\mu$ part of the cross-section.
by Lorenz boost to the electron rest frame along the beam axis:

\[ \xi_\mu = P_\mu \frac{k_1^0}{m_\mu} \left( \frac{\vec{k}_1}{|\vec{k}_1|}, \frac{|\vec{k}_1^0|}{k_1^0} \right). \]  

(2.26)

We will consider the Born cross-section with arbitrary orientations of electron and muon spins. We choose the laboratory frame with z-axis oriented along the incoming muon 3-momentum \( \vec{k}_1 \) and with the plane \((x,z)\) coinciding with the reaction plane. Another plane is spanned by the vectors \( \vec{k}_1 \) and the projection \((\vec{\xi}_e)_x \) of vector \( \vec{\xi}_e \) to a plane perpendicular to \( z \)-axis. In this frame, the relevant 4-vectors are written as follows:

\[ k_1 = \left( 0, 0, |\vec{k}_1|, k_1^0 \right), \]
\[ k_2 = \left( |\vec{k}_2| \sin \theta_\mu, 0, |\vec{k}_2| \cos \theta_\mu, k_2^0 \right), \]
\[ p_1 = \left( 0, 0, 0, m_e \right), \]
\[ p_2 = \left( |\vec{p}_2| \sin \theta_e, 0, |\vec{p}_2| \cos \theta_e, p_2^0 \right). \]  

(2.27)

For the spin vector \( \xi_e \) arbitrarily oriented in 3D-space, we have in the chosen laboratory frame instead of (2.24) the following generalization

\[ \xi_e = P_e \left( \sin \vartheta_e \cos \varphi_e, \sin \vartheta_e \sin \varphi_e, \cos \vartheta_e, 0 \right). \]  

(2.28)

We may identify the angle \( \varphi_e \) in (2.27) with \( \varphi \) of the phase space parametrization in (2.17) and therefore \( \varphi \) becomes an essential degree of freedom in the presence of a transverse polarization.

For arbitrarily oriented \( \xi_\mu \), we have instead of (2.25) in the corresponding rest frame

\[ \xi_\mu = P_\mu \left( \sin \vartheta_\mu \cos \varphi_\mu, \sin \vartheta_\mu \sin \varphi_\mu, \cos \vartheta_\mu, 0 \right). \]  

(2.29)

Now we boost \( \xi_\mu \) from the muon rest frame to the laboratory frame

\[ \xi_\mu = P_\mu \left( \sin \vartheta_\mu \cos \varphi_\mu, \sin \vartheta_\mu \sin \varphi_\mu, \frac{k_1^0}{m_\mu} \cos \vartheta_\mu, \frac{|\vec{k}_1|}{m_\mu} \cos \vartheta_\mu \right). \]  

(2.30)

Using the explicit representations (2.27), (2.28) and (2.30), we can easily derive all scalar products involving polarization vectors.

The doubly differential (in \( y \) and \( \varphi \)) cross-section exact in both masses reads:

\[ \frac{d\sigma^{\text{Born}}}{dyd\varphi} = \frac{2\alpha^2 S}{\lambda_S} \left\{ \frac{1}{y^2} - \frac{s}{yS} + \frac{1}{2} + P_e P_\mu \left[ \left( -\frac{1}{y} - \frac{1}{\sqrt{Y}} \right) \cos \vartheta_e \cos \vartheta_\mu \right. \right. \]
\[ + \left. \left. \frac{m_\mu |\vec{p}_2^\mu|}{\sqrt{\lambda_S} y} \left( 1 + \frac{2m_\mu^2}{S} \right) \sin \vartheta_\mu \cos \vartheta_e \sin \vartheta_\mu \cos \varphi_\mu \right] \frac{2m_e |\vec{k}_2^\mu|}{\sqrt{\lambda_S} y \sqrt{S}} \sin \vartheta_\mu \cos \vartheta_e \sin \vartheta_\mu \cos \varphi_\mu \]
\[ - \frac{m_e}{y^2 S} \sin \vartheta_e \sin \vartheta_\mu \left( \frac{|\vec{k}_2^\mu| |\vec{p}_2^\mu|}{S} \sin \vartheta_e \sin \vartheta_\mu \cos \varphi_e \cos \varphi_\mu \right. \right. \]
\[ \left. \left. + y \cos \delta_{(\varphi_e-\varphi_\mu)} \right\} \right\}. \]  

(2.31)

For \( \varphi \) two choices are possible: 1) \( \varphi = \varphi_e \), then \( \varphi_\mu = \varphi_e - \delta_{(\varphi_e-\varphi_\mu)} \) or 2) \( \varphi = \varphi_\mu \), then \( \varphi_e = \varphi_\mu + \delta_{(\varphi_e-\varphi_\mu)}. \)
The expressions for $\sin \theta_e$ and $\sin \theta_\mu$ exact in $m_e$ are

$$\sin \theta_e = \frac{2m_e \sqrt{S \hat{y}}}{\sqrt{\lambda_0}}, \quad (2.32)$$

$$\sin \theta_\mu = \frac{2m_e \sqrt{S \hat{y}}}{\sqrt{\lambda_l}}, \quad (2.33)$$

where

$$\lambda_l = S^2(1 - y)^2 - 4m_e^2m_\mu^2,$$

$$\lambda_0 = S^2y^2 + 4m_e^2Sy,$$  

$$\hat{y} = y \left(1 - \frac{y}{Y}\right) \quad (2.34)$$

and

$$Y = \frac{\lambda_s}{ss} \approx \left(1 + \frac{m_\mu^2}{s}\right)^{-1} \quad (2.35)$$

is the kinematical maximum of $y$-variation.

The substitution of these variables into (2.31) exhibits an interesting property of the general Born cross-section which becomes

$$\frac{d\sigma^{\text{Born}}}{dyd\varphi} = \frac{2\alpha^2 S}{\lambda_s} \left\{ \frac{1}{y^2} - \frac{s}{yS} + \frac{1}{2} + P_eP_\mu \left[ \left(1 - \frac{y}{Y} + \frac{1}{2}\right) \cos \varphi_e \cos \varphi_\mu \right. \right. $$

$$+ \frac{m_\mu \sqrt{S \hat{y}}}{y\sqrt{\lambda_s}} \left(1 + \frac{2m_e^2}{S}\right) \cos \varphi_e \sin \varphi_\mu \cos \varphi_\mu$$

$$- \frac{m_e \sqrt{S \hat{y}}}{y\sqrt{\lambda_s}} \left(1 + \frac{2m_\mu^2}{S}\right) \cos \varphi_\mu \sin \varphi_e \cos \varphi_e$$

$$- 2 \frac{m_e m_\mu}{yS} \sin \varphi_e \sin \varphi_\mu \left(1 - \frac{y}{Y}\right) \cos \varphi_e \cos \varphi_\mu + \cos \delta_{(\varphi_e-\varphi_\mu)} \right\} \right\}. \quad (2.37)$$

From the last presentation it is clearly seen that while terms related to the transverse electron polarization are small since they are suppressed by the electron mass (third and fourth lines), the term induced by the transverse muon beam polarization (second line) is not small, since it appears to be proportional to the muon mass.

By trivial algebra one may show that the expression (2.31) reduces to a very short form in two particular cases. In case of transverse electron and longitudinal muon polarizations in the URA in the electron mass we obtain

$$\frac{d\sigma^{\text{Born}}}{dyd\varphi} = \frac{2\alpha^2 S}{S} \left[ \frac{1}{y^2} - \frac{1}{yY} + \frac{1}{2} + P_eP_\mu \cos \varphi \sin \theta_\mu \frac{1 - y}{y} \left(\frac{1}{2} - \frac{1}{Y}\right) \right]. \quad (2.38)$$

The corresponding expression for the case of longitudinal polarization of both particles, but exact in both masses reads:

$$\frac{d\sigma^{\text{Born}}}{dy} = \frac{4\pi\alpha^2 S}{\lambda_s} \left[ \frac{1}{y^2} - \frac{s}{yS} + \frac{1}{2} + P_eP_\mu \left( -\frac{1}{y} + \frac{1}{Y} - \frac{1}{2} \right) \right]. \quad (2.39)$$
Having in mind that \( r_e = \alpha/m_e \) and \( S = 2m_eE_\mu \), we immediately identify (2.39) with the corresponding expressions from Section 2.2 of ref. \[2\] if one neglects terms of \( \mathcal{O}(m_e^2) \) here. In the URA in \( m_e \), equation (2.39) may be rewritten in a form, which is explicitly positive definite:

\[
\frac{d\sigma_{\text{BORN}}}{dy} = \frac{4\pi\alpha^2}{S} \left[ \frac{(Y - y)}{y^2Y} \left( 1 - yP_\mu P_\mu \right) + \frac{1}{2} \left( 1 - P_\mu P_\mu \right) \right].
\] (2.40)

3 Complete \( \mathcal{O}(\alpha) \) Radiative Corrections

3.1 Kinematics of \( \mu e \rightarrow \mu e\gamma \)

The reaction

\[
\mu(k_1) + e(p_1) \rightarrow \mu(k_2) + e(p_2)
\] (3.1)

is accompanied by the bremsstrahlung of non-observed photon(s)

\[
\mu(k_1) + e(p_1) \rightarrow \mu(k_2) + e(p_2) + (n)\gamma(p).
\] (3.2)

First of all we will study the kinematics of one-photon bremsstrahlung. We want \( y_\mu \) to be the last integration variable out of a set of four variables (besides \( S \)). There is some freedom in doing this.

We will use the definitions:

\[
Q_\mu^2 = (k_1 - k_2)^2, \quad y_\mu = \frac{p_1(k_1 - k_2)}{p_1k_1},
\] (3.3)

and

\[
Q_e^2 = (p_2 - p_1)^2, \quad y_e = \frac{p_1(p_2 - p_1)}{p_1k_1}.
\] (3.4)

The other invariants are:

\[
\begin{align*}
z_1 &= -2pk_1, & z_2 &= -2pk_2, \\
V_1 &= -2pp_1, & V_2 &= -2pp_2.
\end{align*}
\] (3.5)

Of course, only four of the invariants are independent.

Using 4-momentum conservation, it is easy to derive the following relations among them:

\[
\begin{align*}
z_1 + V_1 &= z_2 + V_2, \\
V_1 &= Sy_\mu - Q_\mu^2, \\
V_2 &= Sy_\mu - Q_\mu^2, \\
Q_e^2 &= Sy_e.
\end{align*}
\] (3.6)

We will use the following set of independent variables

\[
S, \quad Q_\mu^2, \quad y_\mu, \quad Q_e^2, \quad z_{2(1)}.
\] (3.7)

The last of eqs. (3.7) deserves a comment. Our choice of the 4-momentum \( p_1 \) in definitions (3.3) and (3.4) introduces the asymmetry between \( y_\mu \) and \( y_e \). This is reason why \( Q_\mu^2 \) and \( y_\mu \) may be chosen as independent variables, while there exists a relation between \( Q_e^2 \) and \( y_e \).
For the moduli of particle momenta and the energies, in the electron rest system \( \vec{p}_1 = 0 \), we obtain

\[
|\vec{p}| = \frac{\sqrt{\lambda_p}}{2m_e}, \quad p^0 = \frac{S(y_\mu - y_e)}{2m_e},
\]

\[
|\vec{k}_1| = \frac{\sqrt{\lambda_s}}{2m_e}, \quad k^0_1 = \frac{S}{2m_e},
\]

\[
|\vec{k}_2| = \frac{\sqrt{\lambda_l}}{2m_e}, \quad k^0_2 = \frac{S_1}{2m_e},
\]

\[
|\vec{Q}_{\mu}| = \frac{\sqrt{\lambda_\mu}}{2m_e}, \quad Q^0_\mu = \frac{Sy_\mu}{2m_e},
\]

\[
|\vec{p}_2| = |\vec{Q}_e| = \frac{\sqrt{\lambda_e}}{2m_e}, \quad p^0_2 = m_e + \frac{Sy_e}{2m_e},
\]

\[
\vec{k}_1 \cdot \vec{k}_2 = \frac{SS_1}{4m_e^2} - m^2_\mu,
\]

\[
\vec{k}_1 \cdot \vec{p}_2 = \frac{S(Q_e^2 + 2m_e^2)}{4m_e^2} - \frac{S - Q^2_e - z_2}{2},
\]

\[ (3.8) \]

with

\[ S_1 = S(1 - y_\mu). \]  

(3.9)

The corresponding equations for the Born kinematics (\( |\vec{p}| = 0 \)) can be easily derived from setting

\[ V_1 = V_2 = z_1 = z_2 = 0. \]  

(3.10)

In this limit

\[ Q_e^2 = Q_\mu^2 = Q^2 \quad \text{and} \quad y_e - y_\mu = y. \]  

(3.11)

Many relations and useful notations can now be taken from [9]. As usual, we introduce the relevant kinematical \( \lambda \)-functions:

\[
\lambda_p \equiv \lambda[-(p_1 + p)^2, -p_1^2, -p^2] = S^2(y_\mu - y_e)^2,
\]

\[
\lambda_s \equiv \lambda[-(p_1 + k_1)^2, -p_1^2, -k_1^2] = S^2 - 4m^2_\mu m_e^2,
\]

\[
\lambda_l \equiv \lambda[-(p_1 + k_2)^2, -p_1^2, -k_2^2] = S_1^2 - 4m^2_\mu m_e^2,
\]

\[
\lambda_\mu \equiv \lambda[-(p_1 + Q_\mu)^2, -p_1^2, -Q_\mu^2] = S^2y_\mu^2 + 4m^2_\mu Q^2_\mu,
\]

\[
\lambda_e \equiv \lambda[-(p_1 + p_2)^2, -p_1^2, -p_2^2] = S^2y_e^2 + 4m^2_\mu Q^2_e,
\]

\[ (3.12) \]

where

\[
\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz. \]  

(3.13)

### 3.2 Kinematic boundaries

The boundary conditions may be taken from [9]. The first one, (B.3) of [9], remains unchanged

\[
m_e^2Q^4_\mu + S^2y_\mu Q^2_\mu + m^2_\mu S^2y_\mu^2 - \lambda_\mu Q^2_\mu = 0,
\]

\[ (3.14) \]
while the second condition, (B.4) of [9], changes, since, contrary to [9], \( y_e \) is not an independent variable here. This is due to the fact that we are dealing now with elastic scattering rather than with the deep inelastic scattering in [9]. Using the last of eqs. (3.6), eq. (B.4) of [9] takes the form:

\[
S^2 y_e^2 Q_e^2 + Q_e^4 Q_-^2 - m_e^2 (Q_-^2 - Q_e^2)^2 - S y_e Q_+^2 (Q_-^2 + Q_e^2) = 0.
\] (3.15)

The physical region \( \mathcal{E}_\mu = (Q_-^2, y_e) \) is given by two inequalities (see [9], subsection B.2.1), which are derived from (3.14)

\[
0 \leq Q_-^2 \leq \frac{\lambda_S}{S + m_\mu + m_e} \equiv Q_-^2.
\] (3.16)

where

\[
y_\mu^{\text{min}}(Q_-^2) = \frac{Q_-^2}{S} \leq y_\mu \leq y_\mu^{\text{max}}(Q_-^2),
\] (3.17)

\[
y_\mu^{\text{max}}(Q_-^2) = \frac{1}{2 m_\mu^2} \left[ \frac{1}{S} \sqrt{\lambda_S \lambda_m - Q_-^2} \right],
\] (3.18)

and

\[
\lambda_m = Q_-^2 (Q_-^2 + 4 m_\mu^2).
\] (3.19)

The solution of eq.(3.15) is

\[
(Q_-^2)_{\text{max, min}} = \frac{S y_\mu (S y_\mu - Q_-^2) + 2 m_e Q_-^2 \pm (S y_\mu - Q_-^2) \sqrt{\lambda_m}}{2 (S y_\mu - Q_-^2 + m_e^2)}.
\] (3.20)

3.3 Another set of independent variables

Besides of (3.7) we use

\[
S, \quad y_\mu, \quad V_2, \quad V_1, \quad z_{2(1)},
\] (3.21)

To write down the limits in these invariants, we reorder first the physical region \( \mathcal{E}_\mu = (Q_-^2, y_e) \rightarrow (y_\mu, Q_-^2) \). Trivial manipulations with (3.10)–(3.18) lead to

\[
0 \leq y_\mu \leq y_\mu^{\text{max}},
\] (3.22)

\[
(Q_-^2)_{\text{min}} \leq Q_-^2 \leq \min\{ (Q_-^2)^{\text{max}}, S y_\mu \}.
\] (3.23)

Here

\[
(Q_-^2)_{\text{max, min}} = \frac{\lambda_S - S^2 y_\mu \pm \sqrt{\lambda_S \lambda_l}}{2 m_e}.
\] (3.24)

Solving the equation

\[
(Q_-^2)^{\text{max}} = S y_\mu,
\] (3.25)

we find a maximal value \( y_\mu^{\text{max}} \)

\[
y_\mu^{\text{max}} = \frac{\lambda_S}{S (S + m_\mu^2 + m_e^2)},
\] (3.26)
where the two upper limit branches of (3.23) meet each other.

From (3.23) and the definitions of $V_i$, we easily derive the limits of $V_2$ as function of $y_\mu$

$$0 \leq V_2 \leq \frac{Sy_\mu(S + 2m_e^2) - \lambda_s + \sqrt{\lambda_s \lambda_t}}{2m_e^2}. \tag{3.27}$$

The second solution

$$V_{2\min} = \frac{Sy_\mu(S + 2m_e^2) - \lambda_s - \sqrt{\lambda_s \lambda_t}}{2m_e^2}, \tag{3.28}$$

is unphysical. It is negative in the physical region (3.22) of $y_\mu$. Finally, from (3.20) we derive the limits of $V_1$ as functions of $y_\mu$ and $V_2$

$$V_{1\min} \leq V_1 \leq V_{1\max}, \tag{3.29}$$

where

$$V_{1\max,\min} = V_2 \frac{Sy_\mu + 2m_e^2 \pm \sqrt{\lambda_\mu}}{2(V_2 + m_e^2)}. \tag{3.30}$$

Examining (3.27), one may see that the invariant $V_2$ is positive only in the interval

$$0 \leq y_\mu \leq y_\mu^{\max}. \tag{3.31}$$

To complete the study of kinematics of the reaction (3.2), we have to give the limits of variation of the variable $z_{1(2)}$. We may take all the relevant formulae from [1] and simply list them for completeness:

$$z_{1(2)}^{\max,\min}(y_\mu, Q_\mu^2, y_e) = \frac{B_{1(2)} \pm \sqrt{D_z}}{A_{1(2)}} = \frac{C_{1(2)}}{B_{1(2)} \mp \sqrt{D_z}}, \tag{3.32}$$

and the Gram determinant

$$R_z = -A_1 z_1^2 + 2B_1 z_1 - C_1 \equiv -A_2 z_2^2 + 2B_2 z_2 - C_2, \tag{3.33}$$

$$A_2 = \lambda_\mu \equiv A_1, \tag{3.34}$$

$$B_2 = \left\{2m_e^2Q_\mu^2(Q_e^2 - Q_\mu^2) + S^2(1 - y_\mu)(y_\mu Q_e^2 - y_e Q_\mu^2) + S^2(1)Q_\mu^2(y_\mu - y_e)\right\}$$

$$\equiv -B_1 \{(1) \leftrightarrow -(1 - y_\mu)\}; \tag{3.35}$$

$$C_2 = \left\{S(1 - y_\mu)Q_e^2 -SQ_\mu^2[(1) - y_e]\right\}^2 + 4m_e^2[S^2(y_\mu - y_e)(y_\mu Q_e^2 - y_e Q_\mu^2) - m_e^2(Q_e^2 - Q_\mu^2)^2]$$

$$\equiv C_1 \{(1) \leftrightarrow -(1 - y_\mu)\}, \tag{3.36}$$

$$D_z = B_{1(2)}^2 - A_{1(2)}C_{1(2)}. \tag{3.37}$$

Here we now understand that $Q_e^2 \equiv Sy_e$. This makes no simplifications, therefore we did not substitute it in order not to destroy a nice symmetry of these equations.

The inequalities (3.31), (3.27), (3.30) and (3.32) are the kinematical limits in the sequence: $y_\mu, V_2, V_1, z_2$. 

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3.4 Phase spaces

In order to have a convincing proof that all the boundaries are correctly derived, we always write a supporting FORTRAN program to numerically check that the phase space volume in any sequence of variables is exactly the same. We calculated as many phase space integrals analytically as is possible.

For example, we realized the cross check of our sequence (3.21). Three integrations may be easily performed analytically, yielding:

$$\Gamma = \frac{\pi^2 S}{4\sqrt{\lambda_s}} \int_{\mu_0}^{\mu_{\max}} dy_\mu \left( V_2^{\max} - m_e^2 \ln \frac{V_2^{\max} + m_e^2}{m_e^2} \right)^3. \quad (3.38)$$

The numerical check returned for this integral exactly the same value as for the basic sequence, (C.1) of [9].

We will always integrate over $z_2$ first. Then, only three variables $V_1, V_2, y_\mu$ remain. The physical regions $\mathcal{E} = (y_\mu, V_2)$ and $\mathcal{I} = (V_2, V_1)$, as derived from (3.27) and (3.30), are shown in Figs. 4 and 5.

![Figure 4: $(V_2, y_\mu)$-plot](image1)

![Figure 5: $(V_1, V_2)$-plot](image2)

3.5 Bremsstrahlung cross section

For the normalized bremsstrahlung cross-section, we have

$$d\sigma^\text{BREM} = \frac{1}{2\sqrt{\lambda_s}} |M^\text{BREM}|^2 d\Gamma_3, \quad (3.39)$$

with the $2 \rightarrow 3$ phase space

$$d\Gamma_3 = (2\pi)^4 \frac{d^3 \vec{k}_2}{(2\pi)^3 2\vec{k}_2^0} \frac{d^3 \vec{p}_2}{(2\pi)^3 2\vec{p}_2^0} \frac{d^3 \vec{p}}{(2\pi)^3 2\vec{p}^0} \delta(k_1 + p_1 - k_2 - p_2 - p). \quad (3.40)$$
In terms of the variables (3.21) the differential phase space looks as follows

\[ d\Gamma_3 = \frac{S}{2^7\pi^4\sqrt{\lambda_s}} dy_{\mu} dV_2 dV_1 dz_1 \sqrt{R_z}. \]  

(3.41)

We developed two branches to present the final result:

- calculation within the Numerical Approach, with the possibility to impose arbitrary experimental cuts;

- calculation within an Analytic Approach with limited possibility to apply cuts.

We note that the integrals over \( z_1 \) were calculated analytically with arbitrary limits \( z_1^{\text{min}}, z_1^{\text{max}} \) (see, Appendix (A)), while integrals over \( V_{2(1)} \) were computed purely numerically. This is the essence of our numerical approach, the \textit{muenum} branch of \( \mu\text{ela} \).

We also performed the complete \( O(\alpha) \) RC calculations within an Analytic Approach, the \textit{mueana} branch of \( \mu\text{ela} \). In this description, we will present only a sketch of the calculations.

Within the analytic approach we have integrated the bremsstrahlung contribution over the variables \( z_1 \) and \( V_1 \) over the full photonic phase space without imposing experimental cuts. The aim of this is twofold: first, we performed an independent calculation of the bremsstrahlung cross section and its integration which was used to cross check the formulae of the numerical approach and their coding in the FORTRAN program \( \mu\text{ela} \). Second, these analytic formulae are rather elegant, although lengthy, in the Single Ultra Relativistic Approximation (SIURA) in the electron mass. Due to analytic integration, they run at the computer incredibly fast, several orders of magnitude faster than the numerical integration within the numerical approach. We note that this relative simplicity may be reached only if we integrate over the full phase space of photons. From formulae in the SIURA it is easy to derive the formulae in the DOubly Ultra Relativistic Approximation, DOURA. It is important to stress, that in the latter case satisfactory precision of the calculations is not ensured.

The derivation of analytic results is quite straightforward. We consequently integrate in the sequence \( dV_2 dV_1 dz_1 \). In reality, it is convenient to calculate and substitute the twofold integrals: \( dV_1 dz_1 \). These are presented in the Table in Appendix (D.1). After substituting the \( dV_1 dz_1 \) integrals, we arrive at another Table of integrals over \( V_2 \) which are listed in Appendix (D.2). They are much more complicated than the first, two fold integrals. The latter integrals were calculated in limits \([0, \hat{V}_2^{\text{max}}]\) with an arbitrary upper limit. This allows us to impose \( V_2 \)-dependent experimental cuts even within the analytic approach. An example is the muon angular cut.

A comment on DOURA is in due here. Since here we neglect \( m_\mu^2 \), a problem might arise from the fact that \( Q_\mu^{2\text{min}} \) is proportional to the muon mass. Apparently, \( m_\mu^2 \) can’t be neglected in \( Q_\mu^{2\text{min}} \), since it enters in the denominator of the photon propagator. However, we assume that a cut is imposed on \( V_2 \), which prevents \( Q_\mu^{2\text{min}} \) from reaching its kinematical limit. In other words we assume that

\[ Q_\mu^2 \geq (Q_\mu^2)^\text{cut} \equiv (Q_\mu^2)^c \gg m_\mu^2. \]

(3.42)

3.5.1 Final expression for the normalized bremsstrahlung cross section

Defining the \( z\)-integrated contributions,

\[ \left| \left| M_{\text{BREM}}^\text{BREM} \right| \right|_z^2 = \int_{z_1^{\text{min}}}^{z_1^{\text{max}}} \frac{dz_1}{\pi \sqrt{R_z}} \left| M_{\text{BREM}} \right|^2 \]

(3.43)
and neglecting \( m_e \) wherever possible, we finally obtain
\[
\frac{d\sigma_{\text{BREM}}}{dy_{\mu}} = \frac{\alpha^4}{S} \int_0^{\hat{V}_2^{\text{max}}} dV_2 \int_{V_1^{\text{min}}}^{V_1^{\text{max}}} dV_1 \left[ |M_{\text{BREM}}|^2 \right]_z. \tag{3.44}
\]

In (3.44), the limits \( \hat{V}_2^{\text{max}} \), \( z_1^{\text{min, max}} \) are functions of experimental cuts. In this way, the cuts are implemented within our numerical approach.

There are three gauge-invariant contributions to the bremsstrahlung differential cross section: \textit{muonic, electronic} and \textit{\( \mu e \) interference}, each of them is represented as the sum of an InfraRed divergent contribution \textquote{IR} and a finite (Regular) part \textquote{R}:
\[
\left[ \left| M_{\text{BREM}}^\mu\mu \right|^2 \right]_z = \frac{Q_e^2}{\mu} \left( B_{\mu\mu}^\text{IR} + S_{\mu\mu}^\text{R} \right), \tag{3.45}
\]
\[
\left[ \left| M_{\text{BREM}}^{ee} \right|^2 \right]_z = \frac{Q_e^2}{\mu} \left( B_{ee}^\text{IR} + S_{ee}^\text{R} \right), \tag{3.46}
\]
\[
\left[ \left| M_{\text{BREM}}^{\mu e} \right|^2 \right]_z = Q_e \left( B_{\mu e}^\text{IR} + S_{\mu e}^\text{R} \right), \tag{3.47}
\]

where \( B \) is the Born factor
\[
B = 4 \frac{Y - y}{y^2 Y} (1 - y P_e P_\mu) + 2 \left( (1 - P_\mu P_e) \equiv \frac{1}{16\pi^2 \alpha^2} |M_{\text{BORN}}|^2 \right). \tag{3.48}
\]

The \( F_i^{\text{IR}} \) are infrared factors:
\[
F_{\mu\mu}^{\text{IR}} = \left( S_{\mu\mu} + 2m_\mu^2 \right) \left[ \frac{1}{z_1 z_2} \right]_z - m_\mu^2 \left[ \frac{1}{z_1} \right]_z - m_\mu^2 \left[ \frac{1}{z_2} \right]_z, \tag{3.49}
\]
\[
F_{ee}^{\text{IR}} = \left( S_{ee} V_1 V_2 - m_e^2 \right) \left[ \frac{1}{z_1 z_2} \right]_z, \tag{3.50}
\]
\[
F_{\mu e}^{\text{IR}} = - S \left[ \frac{1}{z_1} \right]_z + S \left[ \frac{1}{z_2} \right]_z + \frac{S_1}{V_1} \left[ \frac{1}{z_1} \right]_z + \frac{S_1}{V_2} \left[ \frac{1}{z_2} \right]_z. \tag{3.51}
\]

In (3.43)-(3.51) and in the cumbersome functions \( S_i^{\text{IR}} \) given in Appendix (A), the \( z \)-integration \( \left[ \right]_z \) with arbitrary limits is assumed to be done.

Tables of \( z \)-integrals with cuts and equations for limits \( \hat{V}_2^{\text{max}} \), \( z_1^{\text{min, max}} \) in terms of experimental cuts are presented below in Appendix B: \( (B.93), (B.85), (B.104)-(B.108) \).

### 3.5.2 Treatment of the infrared divergent part

The three terms with \( F^{\text{IR}} \) in (3.43)-(3.47) cannot be simply integrated in (3.44) because of the infrared divergency at \( V_2 = 0 \). It is treated by dimensional regularization.

Substituting all terms with \( F^{\text{IR}} \) into (3.44), we define the \textbf{IR Part} of the bremsstrahlung cross-section.
\[
d\sigma^{\text{IR}} = \frac{2^7 \pi^2 \alpha^3}{\sqrt{\Lambda_S}} B \left( Q_\mu^2 F_{\mu\mu}^{\text{IR}} + Q_e Q_\mu F_{\mu e}^{\text{IR}} + Q_e^2 F_{ee}^{\text{IR}} \right) d\Gamma_3 \tag{3.52}
\]

\[
d\sigma^{\text{IR, soft}} + d\sigma^{\text{IR, hard}}.
\]

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We will treat \( d\sigma^{\text{IR}} \) in the R-frame which is defined in Appendix C by the condition
\[
\mathbf{p}_2 + \mathbf{p} = 0.
\] (3.53)

In this frame we find
\[
V_2 = -2p_2p_2 = -(p_2 + p)^2 - m_e^2 = (p_2^0 + p^0)^2 - m_e^2.
\] (3.54)

For sufficiently small \( \varepsilon \) we have at the point of separation of soft and hard photons, \( p^0 = \varepsilon \),
\[
V_2 = \bar{V}_2 = 2m_e\varepsilon,
\] (3.55)
which can be chosen to be much smaller then any typical invariant of the process.

From a study of the R-system it may be understood that there is an unique correspondence between angular integrations in the R-system and invariant integrations, i.e. we may use invariant limits of \( z \) in (3.43) and of \( V_1 \) in (3.44) instead of angular limits in the R-frame in order to compute the hard part of \( d\sigma^{\text{IR}} \)
\[
d\sigma^{\text{IR,hard}} = \frac{2\gamma^3\alpha^3}{\sqrt{\lambda_s}} B \left( q_{\mu \mu}^{\text{IR}}F_{\mu \mu}^{\text{IR}} + q_{\mu e}^{\text{IR}}F_{\mu e}^{\text{IR}} + q_{e e}^{\text{IR}}F_{e e}^{\text{IR}} \right) d\Gamma_3(p^0 - \varepsilon). \] (3.56)

This is very convenient, since hard photons are in general affected by experimental cuts and the cuts are implemented in our approach by means of limits of a numerical integration over \( z_1 \) and \( V_1 \). Finally, we get from (3.52)
\[
\frac{d\sigma^{\text{IR,hard}}}{dy_\mu} = \frac{\alpha^3}{S} B \int_{V_2}^{V_2^{\text{max}}} dV_2 \int_{V_1^{\text{min}}}^{V_1^{\text{max}}} dV_1 \left( q_{\mu \mu}^{\text{IR}}F_{\mu \mu}^{\text{IR}} z + q_{\mu e}^{\text{IR}}F_{\mu e}^{\text{IR}} z + q_{e e}^{\text{IR}}F_{e e}^{\text{IR}} z \right)
\]
\[
= \frac{d\sigma^{\text{BORN}}}{dy_\mu} \frac{\alpha}{\pi} \delta^{\text{IR,hard}}
\] (3.57)

with
\[
\delta^{\text{IR,hard}} = \int_{V_2}^{V_2^{\text{max}}} dV_2 \int_{V_1^{\text{min}}}^{V_1^{\text{max}}} dV_1 \left( q_{\mu \mu}^{\text{IR}}F_{\mu \mu}^{\text{IR}} z + q_{\mu e}^{\text{IR}}F_{\mu e}^{\text{IR}} z + q_{e e}^{\text{IR}}F_{e e}^{\text{IR}} z \right). \] (3.58)

Again, in (3.57)-(3.58) the \( z \)-integration (3.43) with experimental cuts is assumed to have been done.

### 3.5.3 A short form of the completely differential bremsstrahlung cross-section

The IR finite \( z \)-integrated contributions (A.71-A.73) are rather cumbersome indeed. For the unpolarized squared matrix elements of (3.45-3.47) with not yet separated infrared parts (3.49-3.51) a compact representation is known in the literature [10]. We present it here, in notations of our paper, even in a more compact form:

\[
\begin{align*}
\left[ M_{\text{BREM}}^{\text{IR,hard}} \right]_{\mu \mu}^{2 \text{unpol}} &= q_{\mu \mu}^{\text{IR}} \left[ \frac{F_{\mu \mu}^{\text{R}}}{Q_{\mu \mu}^2 z_1 z_2} + \frac{2}{Q_{\mu \mu}^4} \left( m_\mu^2 \Delta_{\mu \mu} + m_e^2 \Delta_{\mu e} \right) \right], \\
\left[ M_{\text{BREM}}^{\text{IR,hard}} \right]_{\mu e}^{2 \text{unpol}} &= q_{\mu e}^{\text{IR}} \left[ \frac{F_{\mu e}^{\text{R}}}{Q_{\mu e}^2 V_1 V_2} + \frac{2}{Q_{\mu e}^4} \left( m_\mu^2 \Delta_{\mu e} + m_e^2 \Delta_{\mu e} \right) \right], \\
\left[ M_{\text{BREM}}^{\text{IR,hard}} \right]_{ee}^{2 \text{unpol}} &= q_{\mu e}^{\text{IR}} \left[ -\frac{F_{\mu e}^{\text{R}}}{Q_{\mu e}^2 Q_{\mu}^2} \left( \frac{S}{V_1 z_1} + \frac{S'}{V_2 z_2} + \frac{U'}{V_1 z_2} + \frac{U'}{z_1 V_2} \right) - \frac{2m_e^2 \Delta_{\mu e}}{Q_{\mu e}^2 Q_{\mu}^2} \right]. 
\end{align*}
\] (3.59)
where

\[ \mathcal{F}^R = S^2 + S'^2 + U^2 + U'^2 \]
\[ \Delta_{\mu\mu}^e = -2S' \left( \frac{1}{z_1} - \frac{1}{z_2} \right) - 2m_\mu^2 Q_e^2 \left( \frac{1}{z_1} - \frac{1}{z_2} \right)^2, \]
\[ \Delta_{ee}^e = -S' + U'^2 z_2 \frac{z_1}{z_2} + 4m_\mu^2 Q_e^2 \left( \frac{1}{V_1^2} + \frac{1}{V_2^2} \right), \]
\[ \Delta_{\mu e}^e = -\left( Q_e^2 + Q_\mu^2 \right) \left( \frac{S}{V_1^2 z_1} + \frac{S'}{V_2^2 z_2} + \frac{U}{V_1^2 z_2} + \frac{U'}{z_1 V_2} \right), \]
\[ + 2S' - U \frac{z_1}{z_1} + 2U' - S \frac{z_2}{z_2}, \]
\[ \Delta_{\mu e}^\mu = -\left( S + U \right)^2 + \left( S' + U' \right)^2, \]
\[ \Delta_{\mu \mu}^e = \frac{(S + U)^2 + (S' + U')^2}{V_1 V_2}. \]

(3.60)

and

\[ S' = S - z_1 - V_1, \]
\[ U = -S + Q_e^2 + V_1, \]
\[ U' = -S + Q_e^2 + z_2. \]

(3.61)

The representation (3.60)-(3.61) is really very elegant, and it may be used in a Monte Carlo code. It can't be used, however, within the numerical approach of this paper for the following reasons:

1. It does not take into account polarizations (main reason).
2. It includes the IRD part which has to be separated out.
3. Our bremsstrahlung cross-section, as we emphasized already, is integrated once over the invariant \( z_{1,2} \) within arbitrary limits (by making use of a table of indefinite integrals). To do so, we need the canonical representation

\[ S_i^R = \sum_{i,j} f_{ij}(z_j) K_{ij}(S, y_{\mu}, V_2, V_1). \quad i = ee, e\mu, \mu\mu, \quad j = 1, 2, \quad l = 1 - 24, \]

(3.62)

see Appendix (D). We gain at least one order of magnitude in CPU-time, since instead of a three-fold numerical integration over \( V_2, V_1 \) and \( z_{1,2} \), we need only two-fold: \( (V_2, V_1) \). This gain of CPU-time makes our code very fast and user friendly. In order to arrive at this level, however, one has to substitute \( S', U, U' \), and the compact expressions, actually, become much more cumbersome (see Appendix A).

4. Finally, the canonical representation (3.62) seems to be the only road towards an analytically integrated differential cross-section, see the discussion at the beginning of this section.
3.6 The muon anomalous magnetic moment contribution

The contribution of the anomalous magnetic moment of the muon to the cross-section can be expressed as (here we follow decomposition of ref. [11]):

\[
\frac{d\sigma_{\text{amm}}}{dy} = \frac{\alpha^3}{S'} F_{\mu}^{\text{amm}} \left( -\frac{4m_{\mu}^2}{Sy'^2} \right) \left( \frac{1}{y} - \frac{1}{Y} \right) (2 - yP_eP_\mu),
\]

(3.63)

where

\[
F_{\mu}^{\text{amm}} = \frac{1}{\beta} \log \beta + \frac{1}{\beta - 1},
\]

(3.64)

and

\[
\beta = \sqrt{1 + \frac{4m_{\mu}^2}{Q^2}}.
\]

(3.65)

3.7 The running electromagnetic coupling

The correction due to the running QED coupling (vacuum polarization) can be implemented as

\[
\frac{d\sigma_{\text{vp}}}{dy} = \left[ \left( \frac{\alpha(Q^2)}{\alpha} \right)^2 - 1 \right] \frac{d\sigma_{\text{BORN}}}{dy} = \delta_{\text{vp}} \frac{d\sigma_{\text{BORN}}}{dy},
\]

(3.66)

where

\[
\alpha(Q^2) = \frac{\alpha}{1 - \Delta\alpha(Q^2)}.
\]

(3.67)

The correction \(\Delta\alpha\) contains two parts:

\[
\Delta\alpha(Q^2) = \Delta\alpha_l + \Delta\alpha_{udcsb},
\]

(3.68)

the first contribution is due to the charged leptons, the second one is due to the light quarks \((u, d, c, s, b)\). All details about the calculation of \(\delta\alpha_l\) are described in [12]. For the calculation of the \(\Delta\alpha_{udcsb}\) we use the parametrization of ref. [13].

3.8 The net \(\mathcal{O}(\alpha^3)\) cross-section

Now we have collected all the ingredients to construct the net QED cross-section up to order \(\mathcal{O}(\alpha^3)\). It has the form

\[
\frac{d\sigma_{\text{QED}}^{\mu\mu}}{dy_{\mu}} = \frac{d\sigma_{\text{BORN}}^{\mu\mu}}{dy_{\mu}} (1 + \delta_{\text{vp}}) + \frac{d\sigma_{\text{amm}}}{dy_{\mu}}
\]

\[
+ \sum_{k=\mu\mu,ee,\mu e} \left( \frac{\alpha}{\pi} \delta_{\text{VR}}^{\text{VR}_k} \frac{d\sigma_{\text{BORN}}^{\mu\mu}}{dy_{\mu}} + \frac{d\sigma_{\text{BREM}}^{\mu\mu}}{dy_{\mu}} \right) ,
\]

(3.69)

where the corrections \(\delta_{\text{VR}}^{\text{VR}_k}\) are given by (E.186), (E.196) and (E.215) and the last sum in (3.68) reads

\[
\sum_k \frac{d\sigma_{\text{BREM}}^{\mu\mu}}{dy_{\mu}} = \alpha^3 \frac{3}{S'} \int_{\hat{V}_2}^{\hat{V}_{\mu\mu}^{\max}} dV_2 \int_{V_{\mu\mu}^{\min}}^{V_1} dV_1 \left( Q_{\mu\mu}^2 S^{\mu\mu}_{\mu\mu} + Q_{ee}^2 S^{\mu\mu}_{ee} + Q_{e\mu} S^{\mu\mu}_{e\mu} \right),
\]

(3.70)

with \(S^{\mu\mu}_k\) given by (A.71), (A.72) and (A.73). The limits of integration in (3.70) are discussed in the Appendix B.
4 Acknowledgments

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A Finite contributions to the bremsstrahlung cross-section

Here we list three finite contributions, $S^R_{\mu}$, to the hard part of $d\sigma$ which enter the formulae (3.45-3.47):

$$S^R_{\mu} = P_e P_\mu \left\{ \frac{1}{z_1^2} \right\} 2 R_\mu \left[ -\frac{2 m^2 S}{Q_e} \left( V_1 y_\mu - 2 V_2 y_\mu + \frac{V_2^2}{S} \right) + \frac{S}{Q_e^2} \left( V_1 \left( 2 y_\mu - 3 + \frac{2}{y_\mu} \right) + V_2 \right) - V_1 \right]$$

$$+ \left\{ \frac{1}{z_2^2} \right\} 4 R_\mu S V_1$$

$$+ \left\{ \frac{1}{z_1 z_2} \right\} 2 m^2 \left( -S V_1 y_\mu (1 + 2 R_\mu)^2 + SV_2 y_\mu (3 + 8 R_\mu + 4 R_\mu^2) - V_2 y_\mu (3 + 4 R_\mu) + \frac{V_3^2}{S} \right)$$

$$+ \left\{ \frac{1}{z_1} \right\} \left[ \frac{4 m^2}{Q_e^2} \left( V_1 y_\mu (1 + R_\mu + 2 R_\mu^2) - V_2 y_\mu (2 + R_\mu) + \frac{V_2^2}{S} \right) - S y_\mu V_1 \left( 1 - \frac{2(2 + y_\mu)}{y_\mu} R_\mu \right) - V_2 \left( -3 y_\mu + 2 - 2 R_\mu \right) \right]$$

$$- V_1 (1 + 2 R_\mu) + V_2 (1 - 4 R_\mu)$$

$$+ \left\{ \frac{1}{z_1} \right\} \left[ \frac{4 m^2}{Q_e^2} \left( V_1 y_\mu (1 + R_\mu + 2 R_\mu^2) - V_2 y_\mu (2 + R_\mu) + \frac{V_2^2}{S} \right) - S y_\mu V_1 \left( 1 + \frac{2(2 + y_\mu)}{y_\mu} R_\mu \right) - V_2 \right]$$

$$- \frac{1}{Q_e^2} \left[ \frac{2 V_1}{y_\mu} \left( 1 - \left( 1 - \frac{2}{y_\mu} \right) R_\mu \right) + V_2 \right]$$

$$+ 2 \left( 1 - \frac{1}{y_\mu} + \frac{1}{y_\mu} \left( 1 - \frac{2}{y_\mu} \right) \right)$$
\[ S_{ee}^R = P_e P_{\mu} \left\{ \begin{array}{c}
\frac{4m_e^2}{SV^2 Q^2_{\mu}} \\
\frac{2m_e^2}{z_1} \left[ \frac{2m_e^2}{SV^2 Q^2_{\mu}} \left( 1 - 2S \frac{y_{\mu}(1 - y_{\mu} R_{\mu})}{Q^2_{\mu}} \right) - \frac{2}{V_2 Q^2_{\mu}} + \frac{2}{V_1 V_2} \right] \\
\frac{2m_e^2}{V_1^2 V_2 Q^2_{\mu}} \left[ 1 - \frac{2}{y_{\mu}} + 2 \left( \frac{S y_{\mu}}{Q^2_{\mu}} + 1 \right) R_{\mu} \right] - \frac{V_1}{Q^2_{\mu} V_2} \left( 1 - 2R_{\mu} \right) + \frac{2}{Q^2_{\mu} y_{\mu}} + 2 \left( \frac{S y_{\mu} V_2}{V_1 Q^2_{\mu}} - \frac{V_2}{V_1 V_2} \right) \left( 1 - \frac{1}{y_{\mu}} - 2R_{\mu} \right) \right] \\
\frac{2m_e^2}{z_1} \left( \frac{2m_e^2}{V_1 V_2} + 1 \right) \left( \frac{1}{Q^2_{\mu}} + \frac{1}{V_2 Q^2_{\mu}} \right) \\
\frac{2m_e^2}{z_1} \left( \frac{2m_e^2}{V_1^2 Q^2_{\mu}} - \frac{1}{V_1 V_2} \right) \left( \frac{2S}{Q^2_{\mu} - 1} + 1 \right) + \frac{1}{V_2 Q^2_{\mu}} \right\} \right\} 
\]
\[+ \frac{V_2}{Q_2^\mu y_\mu} \left( -1 + \frac{8SR_\mu}{Q_2^\mu} \right) + \frac{V_2}{V_1} \left[ \frac{S}{Q_2^\mu} \left( -3y_\mu + 9 - \frac{10}{y_\mu} + 4 \frac{2y_{\mu_1}R_\mu}{y_\mu} \right) - \frac{y_{\mu_1}}{y_\mu} \right] + \frac{V_1}{y_\mu} \left( \frac{y_{\mu_1}}{Q_2^\mu} + \frac{1}{V_2} \right) + \frac{2V_2}{Q_2^\mu y_\mu} (y_{\mu_1}^2 + (1 - 3y_\mu) R_\mu) + 2 \left( 1 - \frac{2}{y_\mu} + \frac{2}{y_\mu^2} - \frac{1}{y_\mu} \left( \frac{3}{4} - \frac{4}{y_\mu} \right) R_\mu \right) \] (A.73)

The indefinite integrals, in terms of which the quantities \( S_k \), are presented here, is the first series of integrals over \( z_1 \) or \( z_2 \), which is given in this paper. It corresponds to the first, innermost and the only semianalytical integration, within our numerical approach. In the R-frame, it corresponds to the integration over the angle \( \varphi_R \) of the photon, see eq. (C.127).

The three first integrals of this series are presented in Appendix D.1 of [9], eqns.(D.6)-(D.8). Here we recall the definition of integration, and present two additional integrals.

\[
\left[ A \right]_z = \frac{1}{\pi} \int_{z_{\text{min}}(1)}^{z_{\text{max}}(1)} \frac{dz_{1(2)}}{\sqrt{R_z}} A, \\
4) \left[ z_{1(2)} \right]_z = -\frac{1}{\pi \lambda_q} \sqrt{R_z(z_{1(2)})} \left[ 1 \right]_z, \\
5) \left[ z_{2(1)}^2 \right]_z = -\frac{1}{2\pi \lambda_q} \left( z_{1(2)} + 3 \frac{B_{1(2)}}{\lambda_q} \right) \sqrt{R_z(z_{1(2)})} \left[ 1 \right]_z + \frac{1}{2\lambda_q^2} \left( 3B_{1(2)}^2 - \lambda_q C_{1(2)} \right) \left[ 1 \right]_z.
\]

The functions \( A, B, C \) are defined in (3.34)-(3.36).

In the table, we introduced the abbreviation

\[
| \equiv \frac{z_{\text{max}}}{z_{\text{min}}}. \tag{A.74}
\]

The limits of the integration \( z_{\text{min,max}} \) may be arbitrary. The particular realization of these limits, depending on realistic experimental cuts, which we implement within our numerical approach, has been derived in Appendix B.3.

### B Numerical approach. Implementation of experimental cuts

Within our numerical approach, the cuts are applied via integration limits in (3.70). There are cuts in variables \( V_2, V_1 \) and \( z_1 \).

#### B.1 \( V_2 \)-dependent cut

- Angular cut of muon

Here we consider the cut on the angle of the scattered muon. This angle may be easily calculated from the Born and bremsstrahlung kinematics. In both cases

\[-2k_1k_2 = 2 \left( k_1^{0}k_2^{0} - |\vec{k}_1||\vec{k}_2| \cos \theta_\mu \right). \tag{B.75}\]
For the case of Born kinematics one can easily derive

$$\cos \theta_{\mu}^{\text{Born}} = \frac{S^2(1 - y_{\mu}) - 2m_e^2(Sy_{\mu} + 2m_{\mu}^2)}{\sqrt{\lambda_S} \sqrt{\lambda_l}}. \quad (B.76)$$

From eq. (B.76), keeping only terms $O(m_e^2)$, we get

$$\sin^2 \theta_{\mu}^{\text{Born}} = 1 - \cos^2 \theta_{\mu}^{\text{Born}} = \frac{4m_e^2V_{2}^{\text{max}}}{S^2(1 - y_{\mu})}, \quad (B.77)$$

so, one may derive the $\theta_{\mu}^{\text{Born}}$. Here

$$V_{2}^{\text{max}} = Sy_{\mu} - \frac{m_{\mu}^2 y_{\mu}^2}{1 - y_{\mu}} = \frac{S y_{\mu}}{1 - y_{\mu}} \left(1 - \frac{y_{\mu}}{\bar{Y}}\right), \quad (B.78)$$

is the upper limit of $V_2$ neglecting $m_e$. The exact expression was given in (3.27).

Using (B.8), one may receive from (B.73) the following expression for $\cos \theta_{\mu}^{\text{Brems}}$:

$$\cos \theta_{\mu}^{\text{Brems}} = \frac{S^2(1 - y_{\mu}) - 2m_e^2 \left(Sy_{\mu} - V_2 + 2m_{\mu}^2\right)}{\sqrt{\lambda_S} \sqrt{\lambda_l}}. \quad (B.79)$$

Keeping again only terms linear in $m_e^2$, we may derive the value of sine of the muon scattering angle in the bremsstrahlung process:

$$\sin^2 \theta_{\mu}^{\text{Brems}} = \frac{4m_e^2}{S^2(1 - y_{\mu})} (V_2^{\text{max}} - V_2). \quad (B.80)$$

It is obvious that:

$$\theta_{\mu}^{\text{Brems}} \leq \theta_{\mu}^{\text{Born}}. \quad (B.81)$$

Therefore, we may introduce the upper limit of the difference

$$\theta_{\mu}^{\text{Born}} - \theta_{\mu}^{\text{Brems}} \leq \bar{\theta}, \quad (B.82)$$

or

$$\left(\theta_{\mu}^{\text{Brems}}\right)^2 \geq \left(\theta_{\mu}^{\text{Born}}\right)^2 - 2\theta_{\mu}^{\text{Born}} \bar{\theta} + \bar{\theta}^2. \quad (B.83)$$

By substitution of expressions (B.77) and (B.80), we receive

$$V_2 \leq V_2^c = V_2^{\text{max}} \frac{\bar{\theta}(2\theta_{\mu}^{\text{Born}} - \bar{\theta})}{(\theta_{\mu}^{\text{Born}})^2}. \quad (B.84)$$

- Kinematical limitation

The kinematical limit on $V_2^{\text{max}}$ was written in (3.27).

So, the absolute max of $V_2$ should be a minimum of the two possible maximal values of $V_2$

$$V_2^{\text{max}} = \min \left[V_2^{\text{max}}, V_2^c\right]. \quad (B.85)$$
B.2 \( V_1 \)-dependent cuts

- Energy recoil cut \( E^{RC} \).

The electron energy in the final state is limited from below by: 
\[
E'_{el} = p_0^0 \geq E^{RC},
\]
where \( E^{RC} \) is some cut on recoil electron energy. In numerical calculation we use the SMC value \( E^{RC} = 35 \text{ GeV} \). So,
\[
2m_e p_0^0 \geq 2m_e E^{RC}. \tag{B.86}
\]

From the other hand:
\[
-2p_1 p_2 = 2m_e p_2^0 = Q_e^2 + 2m_e^2. \tag{B.87}
\]

This corresponds to
\[
S y_\mu - V_1 \geq 2m_e (E^{RC} - m_e), \tag{B.88}
\]
and we receive one of the possible cuts on \( V_1 \):
\[
V_1 \leq V_1^{E_{RC}} = S y_\mu - 2m_e (E^{RC} - m_e). \tag{B.89}
\]

- Energy balance cut \( E^{BC} \).

Photon energy \( p_0^0 \) also could be implicitly limited by an experimental condition, emerging from checking of overall energy balance:
\[
E_\mu + m_e - E'_\mu - E'_e = p_0 \leq E^{BC}. \tag{B.90}
\]

The SMC value is \( E^{BC} = 40 \text{ GeV} \). It is another energy cut related to the invariant \( V_1 \):
\[
2m_e p_0^0 \leq 2m_e E^{BC}, \tag{B.91}
\]
\[
V_1 \equiv -2p_1 p = 2m_e p_0^0 \leq 2m_e E^{BC} \equiv V_1^{E_{BC}}. \tag{B.92}
\]

- Kinematical limitation

Still we must take into account the kinematical limitation on \( V_1^{\text{max}} \). The boundaries of the allowed phase-space region are defined by (3.30).

These three cuts allow us to make the right choice for an absolute maximum of \( V_1 \)
\[
\dot{V}_1^{\text{max}} = \min [V_1^{\text{max}}, V_1^{E_{RC}}, V_1^{E_{BC}}]. \tag{B.93}
\]

B.3 \( z_1 \)-dependent cut

- Cut on the electron angle

The electron angle can be determined from
\[
-2k_1 p_2 = 2k_1^0 p_2^0 - 2|k_1||p_2| \cos \theta_e \tag{B.94}
\]

It differs for the Born and bremsstrahlung kinematics. We consider first the radiative case, where it may be rewritten as:
\[
\cos \theta_e^{\text{BREM}} = \frac{S [(Sy_\mu - V_1) + 2m_e^2] - [S(1 - y_\mu) + V_2 - z_1] 2m_e^2}{\sqrt{\lambda_S} \sqrt{\lambda_e}}
\]
\[
= \frac{(S + 2m_e^2)(Sy_\mu - V_1) + 2m_e^2 z_2}{\sqrt{\lambda_S} (Sy_\mu - V_1)^2 + 4m_e^2 (Sy_\mu - V_1)}. \tag{B.95}
\]
and

\[ \sin^2 \theta_e^{BREM} = \frac{4m_e^2}{\lambda_s \lambda_e} \left\{ Q_e^2 \left[ \lambda_s - Q_e^2(S + m_e^2 + m_\mu^2) - z_2(S + 2m_e^2) \right] - m_e^2 z_2 \right\}. \] (B.96)

In the URA in \( m_e^2 \), it becomes

\[ \sin^2 \theta_e^{BREM} \approx \frac{4m_e^2}{S^2Q_e^2} \left[ S^2 - Q_e^2 m_\mu^2 - S(S y_\mu - V_1 + z_2) \right]. \] (B.97)

We may extract the correct boundaries for the invariant \( z_1 \), from three quantities: \( (\theta_e^{BORN})^2 \), \( (\theta_e^{BREM})^2 \) and the difference \( \bar{\theta} \).

From (B.97) we get:

\[ (\theta_e^{BREM})^2 = \frac{4m_e^2}{S} \left[ \frac{S(1 - y_\mu) + V_2 - z_1}{S y_\mu - V_1} - \frac{m_\mu^2}{S} \right]. \] (B.98)

In the Born approximation \( V_1 = V_2 = z_1 = 0 \) it reduces to

\[ (\theta_e^{BORN})^2 = \frac{4m_e^2}{S} \left[ \frac{1 - y_\mu}{y_\mu} - \frac{m_\mu^2}{S} \right]. \] (B.99)

Due to the fact that the differences between \( |\theta_e^{BORN} - \theta_e^{BREM}| \) has to be smaller than some value, which is determined by experiment, we have:

\[ |\theta_e^{BORN} - \theta_e^{BREM}| \leq \bar{\theta}. \] (B.100)

One gets two cases

\[ \theta_e^{BORN} \geq \theta_e^{BREM}. \] (B.101)

and

\[ \theta_e^{BREM} \geq \theta_e^{BORN}. \] (B.102)

With (B.98), (B.99) and (B.101) we arrive at

\[ (z_1^{\text{max}})^2 = V_2 + V_1 \frac{(1 - y_\mu)}{y_\mu} + \frac{SQ_e^2}{4m_e^2} \bar{\theta} \left( 2\theta_e^{BORN} - \bar{\theta} \right), \] (B.103)

while from (B.102) we receive

\[ (z_1^{\text{min}})^2 = V_2 + V_1 \frac{(1 - y_\mu)}{y_\mu} - \frac{SQ_e^2}{4m_e^2} \bar{\theta} \left( \bar{\theta} + 2\theta_e^{BORN} \right). \] (B.104)

- Kinematical limitation

Also the kinematic boundaries \( z_1^{\text{max}} \) and \( z_1^{\text{min}} \) defined in (3.32), should be taken into account.
The $\theta_c$ cut conditions is active, if at least one of
\[
(z_1^c)^{\text{min}}, (z_1^c)^{\text{max}} \in [z_{1\text{min}}, z_{1\text{max}}],
\]
and
\[
(z_1^c)^{\text{min}} \leq (z_1^c)^{\text{max}}. \tag{B.106}
\]
Then the absolute maximum of $z_1$ should be a minimum of the two possible maxima of $z_1$
\[
z_1^{\text{max}} = \min [(z_1^{\text{max}}, (z_1^c)^{\text{max}})], \tag{B.107}
\]
and the absolute minimum of $z_1$ should be a maximum of the two possible minima of $z_1$
\[
z_1^{\text{min}} = \max [(z_1^{\text{min}}, (z_1^c)^{\text{min}})]. \tag{B.108}
\]

C R-frame kinematics

In two applications in this paper, we will need another frame than the laboratoy frame. When calculating the soft photon contribution and a table of two-fold integrals within the completely analytic approach, we will also make use of the so called R-frame, which is defined by
\[
\vec{p}_2 + \vec{p} = 0, \tag{C.109}
\]
or
\[
\vec{Q} = \vec{p}_1 + \vec{k}_1 - \vec{k}_2 = 0. \tag{C.110}
\]

First, we introduce coordinates of 4-vectors $p_1, k_1, k_2, p$, shown in the figure,
\[
\begin{align*}
p_1 &= (0, 0, |\vec{p}_1|, p_1^0) \\
k_i &= (0, |\vec{k}_i| \sin \theta_i, |\vec{k}_i| \cos \theta_i, k_i^0) \\
p &= p_0^0 \left( \sin \theta_R \sin \varphi_R, \sin \theta_R \cos \varphi_R, \cos \theta_R, 1 \right) \\
p_2 &= \left( -p_0^0 \sin \theta_R \sin \varphi_R, -p_0^0 \sin \theta_R \cos \varphi_R, -p_0^0 \cos \theta_R, p_2^0 \right). \tag{C.111}
\end{align*}
\]

As may be seen from figure 6, we have chosen the $z$-axis of the R-frame along the vector $\vec{p}_1$ and matched the R-frame $(z, y)$-plane with the plane spanned by vector $\vec{p}_1$ and one of the vectors $\vec{k}_1$ or $\vec{k}_2$. This trick is possible since we will use the R-frame only in analytical calculations of tables of bremsstrahlung hard and soft integrals, in which we will always ‘decouple’ $z_1$ and $z_2$ using an invariant partial fraction decomposition while calculating hard integrals and Feynman parametrization for the soft. Finally, the photonic vector $\vec{p}$ is directed arbitrarily and its angles are unlimited
\[
0 \leq \varphi_R \leq 2\pi, \\
0 \leq \theta_R \leq \pi. \tag{C.112}
\]

Actually we are using two R-frames, different for $z_1$ and $z_2$, containing integrands with different pairs of $\theta_R, \varphi_R$, both varying within the full solid angle $4\pi$.

The vector coordinates \((C.111)\) contain, apart from angular variables, the energies $p_1^0, k_{1,2}^0$ and vector moduli $|\vec{p}_1|, |\vec{k}_{1,2}|$. They can be derived using an invariant language. First we consider
\[
\tau = -(p_2 + p)^2 = (p_2^0 + p^0)^2. \tag{C.113}
\]
Therefore,

\[ p_2^0 + p^0 = \sqrt{\tau}. \]  

To derive an energetic coordinate of a 4-momentum, we consider the 4-scalar product of this 4-momentum with the four vector \(-2(p_2 + p)\) and write it twice: through invariants, and in the R-frame, using (C.114). We give the simplest example:

\[
-2(p_2 + p).p = V_2 \quad \text{through invariants,}
\]

\[
= 2(p_2^0 + p_0^0)p_0^0 = 2\sqrt{\tau}p^0 \quad \text{in the R-frame.}
\]  

Let us introduce the following \(\lambda\)-functions:

\[
\lambda_{k_1} \equiv \lambda \left[ -(p_2 + p - k_1)^2, \tau, -k_1^2 \right] = [S(1 - y_\mu) + V_2]^2 - 4m_\mu^2\tau,
\]

\[
\lambda_{k_2} \equiv \lambda \left[ -(p_2 + p - k_2)^2, \tau, -k_2^2 \right] = (S - V_2)^2 - 4m_\mu^2\tau,
\]

\[
\lambda_{p_1} \equiv \lambda \left[ -(p_2 + p - p_1)^2, \tau, -p_1^2 \right] = (Sy_\mu + 2m_e^2)^2 - 4m_e^2\tau = \lambda_\mu.
\]

In this way, the following table had been derived:

\[
\begin{align*}
|\vec{k}_1| &= \frac{\sqrt{\lambda_{k_1}}}{2\sqrt{\tau}}, & k_1^0 &= \frac{S(1 - y_\mu) + V_2}{2\sqrt{\tau}} = \frac{S_{k_1}}{2\sqrt{\tau}}, \\
|\vec{k}_2| &= \frac{\sqrt{\lambda_{k_2}}}{2\sqrt{\tau}}, & k_2^0 &= \frac{S - V_2}{2\sqrt{\tau}} = \frac{S_{k_2}}{2\sqrt{\tau}}, \\
|\vec{p}_1| &= \frac{\sqrt{\lambda_{p_1}}}{2\sqrt{\tau}}, & p_1^0 &= \frac{Sy_\mu + 2m_e^2}{2\sqrt{\tau}} = \frac{S_{p_1}}{2\sqrt{\tau}}, \\
|\vec{p}_2| &= \frac{V_2}{2\sqrt{\tau}}, & p_2^0 &= \frac{V_2 + 2m_e^2}{2\sqrt{\tau}}, \\
|\vec{p}| &= |\vec{p}_2|, & p^0 &= \frac{V_2}{2\sqrt{\tau}}.
\end{align*}
\]  

Figure 6: The R-frame.
Now we write invariant $S$ in terms of R-frame variables

$$S = -2p_1k_1 = 2(p^0_1k^0_1 - |\vec{p}_1||\vec{k}_1|\cos\theta_1). \quad (C.118)$$

By using (C.117), we get

$$S = \frac{S_{p_1}S_{k_1} - \sqrt{\lambda\mu}\lambda_{k_1}\cos\theta_1}{2\tau}, \quad (C.119)$$

from which we derive an expression for $\cos\theta_1$ in the R-frame in terms of invariants:

$$\cos\theta_1 = \frac{S_{p_1}S_{k_1} - 2S\tau}{\sqrt{\lambda\mu}\lambda_{k_1}}. \quad (C.120)$$

Then we write the invariant $S_1$

$$S_1 = -2p_1k_2 = 2(p^0_1k^0_2 - |\vec{p}_1||\vec{k}_2|\cos\theta_2). \quad (C.121)$$

By using (C.117), we get

$$S_1 = \frac{S_{p_1}S_{k_2} - \sqrt{\lambda\mu}\lambda_{k_2}\cos\theta_2}{2\tau}, \quad (C.122)$$

from where we derive an expression for $\cos\theta_2$ in R-frame:

$$\cos\theta_2 = \frac{S_{p_1}S_{k_2} - 2S_1\tau}{\sqrt{\lambda\mu}\lambda_{k_2}}. \quad (C.123)$$

Now we can write down invariants $z_1$ and $z_2$ in their two own R-frames correspondingly

$$z_1 = -2k_1p = 2p^0 \left(k^0_1 - |\vec{k}_1|\cos\theta_1\cos\theta_R - |\vec{k}_1|\sin\theta_1\sin\theta_R\cos\varphi_R \right), \quad (C.124)$$

$$z_2 = -2k_2p = 2p^0 \left(k^0_2 - |\vec{k}_2|\cos\theta_2\cos\theta_R - |\vec{k}_2|\sin\theta_2\sin\theta_R\cos\varphi_R \right). \quad (C.125)$$

The invariant $V_1$

$$V_1 = -2p_1p = 2p^0 \left(p^0_1 - |\vec{p}_1|\cos\theta_R \right) \quad (C.126)$$

looks formally the same in both R-frames.

Finally, relations (C.126), (C.125) allow to perform $\int dV_1 dz$ in term of $\int d\cos\theta_R d\varphi_R$

$$\int_{V_{1\min}}^{V_{1\max}} dV_1 \int_{z_{1\min}}^{z_{1\max}} \frac{dz_1}{\pi \sqrt{R_z}} \equiv \frac{V_2}{\tau} \int_{-1}^{+1} d\cos\theta_R \int_{0}^{2\pi} d\varphi_R. \quad (C.127)$$

This property was widely used when we discussed the treatment of the infrared divergent part.

**D Analytic approach. Tables of integrals**

**D.1 First and second analytic integrations. The R-integrals**

$$\left[ A \right]_R = \int_{V_{1\min}}^{V_{1\max}} dV_1 \int_{z_{1\min}}^{z_{1\max}} \frac{dz_1}{\pi \sqrt{R_z}} A \quad (D.128)$$
1) \[ \left[ 1 \right]_R = \frac{V_2}{\tau} \sqrt{\lambda_\mu} \quad \text{2) } \left[ V_1 \right]_R = \frac{1}{2} S_{p1} V_2^2 \sqrt{\lambda_\mu} \]

3) \[ \left[ \frac{1}{V_2} \right]_R = L_{p1} \quad \text{4) } \left[ \frac{m_e^2}{V_1^2} \right]_R = \frac{1}{V_2} \sqrt{\lambda_\mu} \]

5) \[ \left[ \frac{1}{Q_e^2} \right]_R = L_{Q_e} \quad \text{6) } \left[ \frac{m_e^2}{Q_e^4} \right]_R = \frac{V_2}{Q_\mu^4} \sqrt{\lambda_\mu} \]

7) \[ \left[ \frac{1}{z_1} \right]_R = L_{k1} \quad \text{8) } \left[ \frac{1}{z_2} \right]_R = L_{k2} \]

9) \[ \left[ \frac{m_\mu^2}{z_1^2} \right]_R = \frac{1}{V_2} \quad \text{10) } \left[ \frac{m_\mu^2}{z_2^2} \right]_R = \frac{1}{V_2} \]

11) \[ \left[ \frac{1}{z_1 z_2} \right]_R = \frac{2}{V_2} L_m \]

12) \[ \left[ \frac{m_\mu^2 V_1}{z_1^2} \right]_R = \frac{1}{\lambda_{k1}} \left[ SS_{k1} - 2m_\mu^2 S_{p1} + m_\mu^2 (S_{p1} S_{k1} - 2S\tau) \right] \left[ \frac{1}{z_1} \right]_R \]

13) \[ \left[ \frac{V_1}{z_1} \right]_R = \frac{S_{p1} S_{k1} - 2S\tau}{\lambda_{k1} \lambda_\mu} \left[ \frac{1}{z_1} \right]_R + \left( SS_{k1} - 2m_\mu^2 S_{p1} \right) \frac{V_2}{\lambda_{k1}} \left[ \frac{1}{z_1} \right]_R \]

14) \[ \left[ \frac{V_1}{z_2} \right]_R = \frac{S_{p1} S_{k2} - 2S_1 \tau}{\lambda_{k2} \lambda_\mu} \left[ \frac{1}{z_2} \right]_R + \left[ S_1 S_{k2} - 2m_\mu^2 S_{p1} \right] \frac{V_2}{\lambda_{k2}} \left[ \frac{1}{z_2} \right]_R \]

15) \[ \left[ \frac{1}{V_1 z_1} \right]_R = \frac{1}{V_2} L_S \quad \text{16) } \left[ \frac{1}{V_1 z_2} \right]_R = \frac{1}{V_2} L_{S_1} \]

17) \[ \left[ \frac{1}{Q_e z_1} \right]_R = \frac{1}{Q_\mu} L_{S_1} \quad \text{18) } \left[ \frac{1}{Q_e z_2} \right]_R = \frac{1}{Q_\mu} L_S \]
\[ 19) \quad \left[ \frac{1}{Q_{e z_1}^1} \right]_R = \frac{V_2 N_{k_1}}{m_e^6 Q_6^6 \lambda_t} + \frac{1}{Q_\mu^2 \lambda_t} \left[ S_1 S_{k_1} - 2m_e^2 \left( \frac{S y_\mu V_2}{Q_\mu^2} + 2m_e^2 \right) \right] \left[ \frac{1}{Q_{e z_1}^2} \right]_R \]

\[ 20) \quad \left[ \frac{1}{Q_{e z_2}^1} \right]_R = \frac{V_2 N_{k_2}}{m_e^6 Q_6^6 \lambda_S} + \frac{1}{Q_\mu^2 \lambda_S} \left[ S S_{k_2} - 2m_e^2 \left( \frac{S y_\mu V_2}{Q_\mu^2} + 2m_e^2 \right) \right] \left[ \frac{1}{Q_{e z_2}^2} \right]_R \]

\[ 21) \quad \left[ \frac{1}{Q_{e z_1}^2} \right]_R = \frac{N_{k_1}}{Q_\mu^2 \lambda_t} \left[ \frac{1}{Q_{e z_1}^z} \right]_R + \frac{1}{Q_\mu^2 \lambda_t} \left( \frac{S_1 V_2 + \lambda_t}{m_e^2 V_2} - \frac{2S y_\mu}{Q_\mu^2} \right) \]

\[ 22) \quad \left[ \frac{1}{Q_{e z_2}^2} \right]_R = \frac{N_{k_2}}{Q_\mu^2 \lambda_S} \left[ \frac{1}{Q_{e z_2}^z} \right]_R + \frac{1}{Q_\mu^2 \lambda_S} \left( \frac{-S V_2 + \lambda_S}{m_e^2 V_2} - \frac{2S y_\mu}{Q_\mu^2} \right) \]

\[ 23) \quad \left[ \frac{1}{Q_{e z_1}^1} \right]_R = \frac{1}{Q_\mu^2 \lambda_t} \left[ \frac{S V_2}{Q_\mu^2} - S_1 + 2m_e^2 \right. \]

\[ \left. + \frac{3N_{k_1}}{Q_\mu^2 \lambda_t} \left( \lambda_t + S_1 V_2 - \frac{2m_e^2 S y_\mu V_2}{Q_\mu^2} \right) \right] \left[ \frac{1}{Q_{e z_1}^2} \right]_R \]

\[ + \frac{1}{m_e^2 Q_\mu^4} \left( \frac{2S_1 + V_2 - 4m_e^2}{\lambda_t} + \frac{1}{V_2} \right) + \frac{V_2 \lambda_t}{m_e^2 Q_\mu^8 \lambda_t} \]

\[ + \frac{12V_2}{Q_\mu^4 \lambda_t} \left( m_\mu^2 \lambda_\mu + m_\mu^2 Q_\mu^4 - S S_1 Q_\mu^2 \right) \]

\[ 24) \quad \left[ \frac{1}{Q_{e z_2}^1} \right]_R = \frac{1}{Q_\mu^2 \lambda_S} \left[ \frac{S_1 V_2}{Q_\mu^2} - S - 2m_e^2 \right. \]

\[ \left. + \frac{3N_{k_2}}{Q_\mu^2 \lambda_S} \left( \lambda_S - S V_2 - \frac{2m_e^2 S y_\mu V_2}{Q_\mu^2} \right) \right] \left[ \frac{1}{Q_{e z_2}^2} \right]_R \]

\[ + \frac{1}{m_e^2 Q_\mu^4} \left( \frac{-2S + V_2 - 4m_e^2}{\lambda_S} + \frac{1}{V_2} \right) + \frac{V_2 \lambda_t}{m_e^2 Q_\mu^8 \lambda_S} \]

\[ + \frac{12V_2}{Q_\mu^4 \lambda_S} \left( m_\mu^2 \lambda_\mu + m_\mu^2 Q_\mu^4 - S S_1 Q_\mu^2 \right) \]

with

\[ S_{k_1} = S_1 + V_2, \quad S_{k_2} = S - V_2, \quad S_{p_1} = S y_\mu + 2m_e^2, \]

\[ L_{k_1} = \frac{1}{\sqrt{\lambda_{k_1}}} \ln \frac{(S_{k_1} + \sqrt{\lambda_{k_1}})^2}{m_\mu^2 \tau}, \quad L_{k_2} = \frac{1}{\sqrt{\lambda_{k_2}}} \ln \frac{(S_{k_2} + \sqrt{\lambda_{k_2}})^2}{4m_\mu^2 \tau}, \]

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\[
L_{\mu 1} = \ln \left( \frac{S y_\mu + \sqrt{\lambda_\mu} + 2 m_\mu^2}{4 m_\mu^2 \tau} \right), \\
L_{Q_e} = \ln \left( \frac{V_2 (S y_\mu + \sqrt{\lambda_\mu}) + 2 m_e^2 Q_\mu^2}{V_2 (S y_\mu - \sqrt{\lambda_\mu}) + 2 m_e^2 Q_\mu^2} \right), \\
L_m = \frac{1}{\sqrt{\lambda_m}} \ln \frac{\sqrt{\lambda_m} + Q_\mu^2}{\sqrt{\lambda_m} - Q_\mu^2}, \\
L_s = \frac{1}{\sqrt{\lambda_s}} \ln \frac{(S + \sqrt{\lambda_s})^2}{4 m_\mu^2 m_\mu^2}, \\
L_{S_1} = \frac{1}{\sqrt{\lambda_{S_1}}} \ln \frac{(S_1 + \sqrt{\lambda_{S_1}})^2}{4 m_\mu^2 m_\mu^2}, \\
N_{k_1} = SS_1 y_\mu - 2 m_\mu^2 Q_\mu^2, \\
S_1 = S(1 - y_\mu), \\
\lambda_{k_1} = S_{k_1} - 4 m_\mu^2 \tau, \\
\lambda_m = Q_\mu^4 + 4 m_\mu^2 Q_\mu^2.
\]

In (D.129) we defined the objects \(L_{k_1}, L_{k_2}, L_{\mu 1}, L_{Q_e}, L_m, L_s, L_{S_1}\), which will be used as integrands in the foregoing table. On top of them, we will also use two differences

\[
D_{k_1} = L_{k_1} - L_s, \quad D_{k_2} = L_{k_2} - L_{S_1}.
\]

### D.2 The third analytic integration. The \(V_2\)-integrals

\[
\begin{bmatrix}
A \\
V
\end{bmatrix} = \int_0^{V_{\text{max}}} dV_2 A
\]

1) \(\begin{bmatrix}
1 \\
V
\end{bmatrix} = S y_\mu - Q_\mu^{2\text{max}}\)

2) \(\begin{bmatrix}
1/Q_\mu^2 \\
V
\end{bmatrix} = - \ln \frac{Q_\mu^{2\text{max}}}{S y_\mu}\)

3) \(\begin{bmatrix}
1/Q_\mu^4 \\
V
\end{bmatrix} = \frac{1}{S} \left( \frac{S}{Q_\mu^{4\text{max}}} - \frac{1}{y_\mu} \right)\)

4) \(\begin{bmatrix}
1/Q_\mu^6 \\
V
\end{bmatrix} = \frac{1}{2 S^2} \left( \frac{S^2}{Q_\mu^{4\text{max}}} - \frac{1}{y_\mu} \right)\)

5) \(\begin{bmatrix}
1/Q_\mu^8 \\
V
\end{bmatrix} = \frac{1}{3 S^3} \left( \frac{S^3}{Q_\mu^{4\text{max}}} - \frac{1}{y_\mu} \right)\)

6) \(\begin{bmatrix}
L_m \\
V
\end{bmatrix} = \frac{1}{2} \left( \ln \frac{t_{\text{max}}^2 + 1}{t_{\text{max}}^2 - 1} - \ln \frac{t_{\text{min}}^2 + 1}{t_{\text{min}}^2 - 1} \right)\)

7) \(\begin{bmatrix}
L_m/Q_\mu^2 \\
V
\end{bmatrix} = \frac{1}{2 m_\mu^2} \left( t_{\text{min}}^2 \ln \frac{t_{\text{min}}^2 + 1}{t_{\text{min}}^2 - 1} - t_{\text{max}}^2 \ln \frac{t_{\text{max}}^2 + 1}{t_{\text{max}}^2 - 1} - \ln \frac{Q_\mu^{2\text{max}}}{S y_\mu} \right)\)
8) \[
\left[ \frac{L_{m\mu}}{Q_{\mu\mu}^a} \right]_V = -\frac{1}{12m_{\mu}} \left( \frac{2r_{\mu}}{y_{\mu}} - 1 \right) t_{\text{max}} \ln \frac{t_{\text{max}} + 1}{t_{\text{max}} - 1}
- \left( \frac{2Sr_{\mu}}{Q_{\mu}^{2\text{max}} - 1} \right) t_{\text{min}} \ln \frac{t_{\text{min}} + 1}{t_{\text{min}} - 1} - \ln \frac{Q_{\mu}^{2\text{max}}}{S y_{\mu}} - \frac{2S r_{\mu}}{Q_{\mu}^{2\text{max}}} + \frac{2r_{\mu}}{y_{\mu}}
\]

9.1) \[
\left[ \frac{1}{\lambda_1} \right]_V = \frac{1}{\sqrt{D_{k1}}} \left( \arctan \frac{B_{k1}}{\sqrt{D_{k1}}} - \arctan \frac{B_{k1} - V_{2\text{max}}}{\sqrt{D_{k1}}} \right)
\]

9.2) \[
\left[ \frac{1}{\lambda_2} \right]_V = \frac{1}{2\sqrt{D_{k2}}} \left( \ln \frac{B_{k2} - \sqrt{D_{k2}}}{B_{k2} + \sqrt{D_{k2}}} - \ln \frac{B_{k2} - V_{2\text{max}} - \sqrt{D_{k2}}}{B_{k2} - V_{2\text{max}} + \sqrt{D_{k2}}} \right)
\]

10) \[
\left[ \frac{1}{\lambda_2} \right]_V = \frac{1}{2\sqrt{D_{k2}}} \left( \ln \frac{B_{k2} - \sqrt{D_{k2}}}{B_{k2} + \sqrt{D_{k2}}} - \ln \frac{B_{k2} - V_{2\text{max}} - \sqrt{D_{k2}}}{B_{k2} - V_{2\text{max}} + \sqrt{D_{k2}}} \right)
\]

11) \[
\left[ \frac{V_2}{\lambda_1} \right]_V = \frac{1}{2} \ln \left[ \frac{(S_1 + V_{2\text{max}}^2 - 4m_{\mu}^2 V_{2\text{max}})}{S_1^2} \right] + B_{k1} \left[ \frac{1}{\lambda_1} \right]_V
\]

12) \[
\left[ \frac{V_2}{\lambda_2} \right]_V = \frac{1}{2} \ln \left[ \frac{(S - V_{2\text{max}}^2 - 4m_{\mu}^2 V_{2\text{max}})}{S^2} \right] + B_{k2} \left[ \frac{1}{\lambda_2} \right]_V
\]

13) \[
\left[ L_{k1} \right]_V = -\ln \frac{t_{1\text{max}} + 1}{t_{1\text{max}} - 1} \ln \frac{t_{1\text{max}} - 1}{2r_{\mu_1}} - Li_2 \left[ \frac{2(1 + r_{\mu_1})}{t_{1\text{max}} + 1} \right] + Li_2 \left( \frac{2r_{\mu_1}}{t_{1\text{max}} - 1} \right)
\]

14) \[
\left[ L_{k2} \right]_V = \ln \frac{t_{2\text{max}} + 1}{t_{2\text{max}} - 1} \ln \frac{t_{2\text{max}} - 1}{2r_{\mu}} + Li_2 \left[ \frac{2(1 + r_{\mu})}{t_{2\text{max}} + 1} \right] - Li_2 \left( \frac{2r_{\mu}}{t_{2\text{max}} - 1} \right)
\]

15) \[
\left[ \frac{1}{\lambda_1} L_{k1} \right]_V = -\frac{2}{S^2} \left[ \frac{t_{1\text{max}}^2}{\lambda_{k1}^2} \ln \left| \frac{2r_{\mu_1}(t_{1\text{max}} + 1)}{(t_{1\text{max}} - 1)(t_{1\text{max}} - t_{1\text{min}})} \right| \right.
+ \frac{1}{2(t_{1\text{min}} + 1)} \ln \left| \frac{(t_{1\text{max}} - t_{1\text{min}})^2(t_{1\text{max}} + 1)}{(t_{1\text{max}} + 1)\lambda_{k1}^2} \right|
+ \frac{1}{(t_{1\text{min}} - 1)} \ln \left| \frac{(t_{1\text{max}} - t_{1\text{min}})(t_{1\text{max}} - 1)}{\lambda_{k1}^2} \right|
- \frac{1}{t_{1\text{max}} - t_{1\text{min}}} \ln \left( \frac{t_{1\text{max}}^2 - t_{1\text{min}}^2}{t_{1\text{max}}^2 - t_{1\text{min}}} \right) + \frac{1}{t_{1\text{max}} - t_{1\text{min}}} \ln \left( \frac{t_{1\text{max}}^2 - t_{1\text{min}}^2}{t_{1\text{max}}^2 - t_{1\text{min}}} \right)
\]
16) \[
\frac{1}{\lambda_2} L_{k2}^{L} = \frac{2}{S^2} \left[ - \frac{t^0_2}{\lambda_2^{max}} \ln \left| \frac{2r_\mu(t_2^{max} + 1)}{(t_2^{max} - 1)(t_2^{max} - t^0_2)} \right| 
+ \frac{1}{2(t^0_2 + 1)} \ln \left| \frac{(t_2^{max} - t^0_2)^2(t_2^{max} - 1)}{(t_2^{max} + 1)\lambda_2^{max}} \right| 
+ \frac{1}{(t^0_2)^2 - 1} \ln \left| \frac{(t_2^{max} - t^0_2)(t_2^{max} - 1)}{\lambda_2^{max}} \right| 
- \frac{1}{t_{21} - t_{22}} \ln \left| \frac{t_2^{max} - t_{21}}{t_2^{max} - t_{22}} \right| \right] \]

17) \[
\frac{V_2}{\lambda_1} L_{k1}^{V} = \frac{2}{S_1} \left[ - \frac{1}{\lambda_1^{max}} \ln \left| \frac{2r_\mu(t_1^{max} + 1)}{(t_1^{max} - 1)(t_1^{max} - t^0_1)} \right| 
+ \frac{1}{(t^0_1)^2 - 1} \ln \left| \frac{(t_1^{max} - t^0_1)(t_1^{max} - 1)}{\lambda_1^{max}} \right| 
+ \frac{1}{2(t^0_1 + 1)} \ln \left| \frac{(t_1^{max})^2 - 1}{\lambda_1^{max}} \right| 
+ \frac{1}{t_{11} - t_{12}} \ln \left( \frac{t_1^{max} - t_{11}}{t_1^{max} - t_{12}} \right) + \frac{1}{t_{11} - t_{12}} \ln \left( \frac{t^0_1 - t_{11}}{t^0_1 - t_{12}} \right) \right] \]

18) \[
\frac{V_2}{\lambda_2} L_{k2}^{V} = \frac{2}{S} \left[ - \frac{1}{\lambda_2^{max}} \ln \left| \frac{2r_\mu(t_2^{max} + 1)}{(t_2^{max} - 1)(t_2^{max} - t^0_2)} \right| 
+ \frac{1}{(t^0_2)^2 - 1} \ln \left| \frac{(t_2^{max} - t^0_2)(t_2^{max} - 1)}{\lambda_2^{max}} \right| 
+ \frac{1}{2(t^0_2 + 1)} \ln \left| \frac{(t_2^{max})^2 - 1}{\lambda_2^{max}} \right| 
+ \frac{1}{t_{21} - t_{22}} \ln \left| \frac{t_2^{max} - t_{21}}{t_2^{max} - t_{22}} \right| \right] \]

19) \[
\frac{1}{Q_\mu^2} L_{k1}^{L} = - \frac{1}{d_1} \left[ F_{L_k}(t_{11}, t_{12}, t^0_1, t_1^{max}) - F_{L_k}(t_{11}, t_{12}, t^0_1, t_1^{max}) \right] \]

20) \[
\frac{1}{Q_\mu^2} L_{k2}^{L} = \frac{1}{d_2} \left[ F_{L_k}(t_{21}, t_{22}, t^0_2, t_2^{max}) - F_{L_k}(t_{21}, t_{22}, t^0_2, t_2^{max}) \right] \]

21) \[
\frac{1}{\tau^2} \left[ V \right] = \frac{1}{m_e^2} \]

22) \[
\frac{1}{\tau} \left[ V \right] = L_+ \]

23) \[
L_{p1}^{p1} \left[ V \right] = (L_+ + 1) \left[ \frac{1}{V} \right] \]

24) \[
\frac{1}{Q_\mu^2} L_{p1}^{p1} \left[ V \right] = L_- \left[ \frac{1}{Q_\mu^2} \right] + \text{Li}_2 \left( \frac{V_2^{max}}{S_{y_\mu}} \right) \]
25) \[
\frac{1}{Q_\mu^2} \left[ \frac{1}{Q_\mu^2} \right]_{V} = L - \left[ \frac{1}{Q_\mu^2} \right]_{V} + \frac{1}{S y_\mu} \left[ \frac{1}{Q_\mu^2} \right]_{V}
\]

26) \[
\left[ L_{Q_e} \right]_{V} = (L_e + 1) \left[ 1 \right]_{V} + 2 (V_{2\text{max}} - S y_\mu) \left[ \frac{1}{Q_\mu^2} \right]_{V}
\]

27) \[
\frac{1}{V_2} \left[ L_{Q_e} \right]_{V} = \frac{1}{2} L_+^2 + F_1 + 2 \text{Li}_2 \left( \frac{V_{2\text{max}}}{S y_\mu} \right)
\]

28) \[
\frac{1}{V_2} \left[ \frac{1}{V_2} \right]_{V} = \ln \frac{V_{2\text{max}}}{V_{2\text{min}}}
\]

29) \[
\frac{1}{V_2} \left[ L_{p_1} \right]_{V} = 2 \ln \frac{S y_\mu}{m_e} \left[ \frac{1}{V_2} \right]_{V} - \frac{1}{2} L_+^2 - F_1
\]

30) \[
\frac{1}{V_2} \left[ D_{k_1} \right]_{V} = \left[ \frac{1}{2} \ln \frac{S_1^2}{m_e^2 m_\mu^2} \ln \frac{S_1^2 m_e^2}{(V_{2\text{max}})^2 m_\mu^2} - F_1 - \frac{1}{2} \ln^2 \frac{S_1 (t_1^{\text{max}} - t_1^0)}{2(S + m_\mu^2)} \right]
\]

31) \[
\frac{1}{V_2} \left[ D_{k_2} \right]_{V} = \left[ \frac{1}{2} \ln \frac{S^2}{m_e^2 m_\mu^2} \ln \frac{S_1^2 m_e^2}{(V_{2\text{max}})^2 m_\mu^2} - \frac{1}{2} \ln^2 \frac{S (t_2^{\text{max}} - t_2^0)}{2(S + m_\mu^2)} - F_1 \right]
\]

32) \[
\frac{1}{V_2} \left[ \lambda_m \right]_{V} = \left[ \frac{1}{2} \ln \frac{\sqrt{\lambda_0^m}}{\sqrt{\lambda_0^m} - S y_\mu} \left[ \ln \frac{(t_0^{\text{max}} - t_0^a)(\lambda_0^m)^{3/2}}{(V_{2\text{max}})^2 m_\mu^2} + \ln \frac{V_{2\text{max}}}{V_{2\text{min}}} \right] \right]
\]

with

\[
Q_\mu^{2\text{min}} = \frac{m_e^2 y_\mu^2}{1 - y_\mu},
\]

\[
V_{2\text{max}} = S y_\mu - \frac{m_e^2 y_\mu^2}{1 - y_\mu}, \quad V_{2\text{min}} = 2 m_e \bar{\omega},
\]

\[
r_\mu = \frac{m_\mu^2}{S}, \quad r_{\mu_1} = \frac{m_\mu^2}{S_1},
\]

\[
Q_\mu^{2\text{max}} = S y_\mu - V_{2\text{max}}^\text{max},
\]

\[
\epsilon^{\text{max}} = \sqrt{1 + \frac{4 m_\mu^2}{S y_\mu}}, \quad \epsilon^{\text{min}} = \sqrt{1 + \frac{4 m_\mu^2}{Q_\mu^{2\text{max}}}},
\]

\[
D_{k_1} = 4 m_\mu^2 [S(1 - y_\mu) - m_\mu^2], \quad D_{k_2} = 4 m_\mu^2 (S + m_\mu^2),
\]

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and the additional notation

\begin{align*}
B_{k_1} &= -S_1 + 2m_{\mu}^2, & B_{k_2} &= S + 2m_{\mu}^2, \\
L_+ &= \ln \frac{S y_\mu}{m_e^2} + \ln \frac{V_{2\max}^2}{S y_\mu}, & L_- &= \ln \frac{S y_\mu}{m_e^2} - \ln \frac{V_{2\max}^2}{S y_\mu}, \quad (D.131)
\end{align*}

\[
\begin{align*}
t_1^0 &= 1 + 2r_{\mu_1}, \\
t_{11} &= 1 + 2 \left[ r_{\mu} + \sqrt{x_{\mu 1}(1 + x_{\mu 1})} \right], & t_{12} &= 1 + 2 \left[ r_{\mu_1} - \sqrt{x_{\mu 1}(1 + x_{\mu 1})} \right], \\
\lambda_{k1}^\max &= (t_1^{\max} - 1)^2 - 4r_{\mu_1} t_1^{\max}, \\
t_2^0 &= 1 + 2r_{\mu}, \\
t_{21} &= 1 + 2r_{\mu} + 2\sqrt{r_{\mu}(1 + r_{\mu})}, & t_{22} &= 1 + 2r_{\mu} - 2\sqrt{r_{\mu}(1 + r_{\mu})}, \\
\lambda_{k2}^\max &= (t_2^{\max} - 1)^2 - 4r_{\mu} t_2^{\max}, \\
t_{11} &= -S_1 + \sqrt{S_{d1}} \frac{1}{S y_\mu}, & t_{12} &= -S_1 - \sqrt{S_{d1}} \frac{1}{S y_\mu}, \\
t_{1}^{\max} &= \frac{\sqrt{(S_1 + V_{2\max}^2)^2 - 4m_{\mu}^2 V_{2\max}^2} - S_1}{V_{2\max}}, \\
t_{2}^{\max} &= \frac{S - \sqrt{(S - V_{2\max}^2)^2 - 4m_{\mu}^2 V_{2\max}^2}}{V_{2\max}}, \\
t_{1}^{\min} &= \frac{2S_1 + V_{2\min}^2 - 4m_{\mu}^2}{\sqrt{(S_1 + V_{2\min}^2)^2 - 4m_{\mu}^2 V_{2\min}^2} + S_1}, \\
t_{2}^{\min} &= \frac{2S - V_{2\min}^2 + 4m_{\mu}^2}{S + \sqrt{(S - V_{2\min}^2)^2 - 4m_{\mu}^2 V_{2\min}^2}}, \\
\lambda_{t}^0 &= S^2 y_\mu^2 + 4m_{\mu}^2 S y_\mu, \\
t_a^0 &= \frac{S y_\mu + 2m_{\mu}^2}{\sqrt{\lambda_{t}^0}}, \\
t_{a}^{\min} &= \frac{2(S y_\mu + 2m_{\mu}^2) - V_{2\min}}{\sqrt{\lambda_{t}^0 + \sqrt{\lambda_{t}^0 - 2V_{2\min}^2(S y_\mu + 2m_{\mu}^2) + (V_{2\min})^2}}}, \\
t_{a}^{\max} &= \frac{\sqrt{\lambda_{t}^0 - \sqrt{\lambda_{t}^0 - 2V_{2\max}^2(S y_\mu + 2m_{\mu}^2) + (V_{2\max})^2}}}{V_{2\max}}, \\
F_{L_k} &= \ln \frac{(t_0 - 1)(t_{11} + 1)}{(t_{11} - 1)(t_{11} - t_0)} \ln (t_a - t_{11}) - \ln \frac{(t_0 - 1)(t_{12} + 1)}{(t_{12} - 1)(t_{12} - t_0)} \ln (t_a - t_{12}) \\
&\quad - \text{Li}_2 \left( \frac{t_a - t_{11}}{-1 - t_{11}} \right) + \text{Li}_2 \left( \frac{t_a - t_{12}}{-1 - t_{12}} \right) + \text{Li}_2 \left( \frac{t_a - t_{11}}{1 - t_{11}} \right) \\
&\quad - \text{Li}_2 \left( \frac{t_a - t_{12}}{1 - t_{12}} \right) + \text{Li}_2 \left( \frac{t_a - t_{11}}{t_0 - t_{11}} \right) - \text{Li}_2 \left( \frac{t_a - t_{12}}{t_0 - t_{12}} \right). \quad (D.132)
\end{align*}
E Soft contribution to \(d\sigma^{\text{IR}}\)

In (3.52) the expression for \(d\sigma^{\text{IR,soft}}\) reads:

\[
d\sigma^{\text{IR,soft}} = \frac{27\pi^3\alpha^3}{\lambda_S} B \left( Q^2_{\mu} F^\text{IR}_{\mu\mu} + Q_{\mu} Q_{e} F^\text{IR}_{\mu e} + Q_e^2 F^\text{IR}_{ee} \right) \theta(\varepsilon - p^0) d\Gamma_3.
\]

(E.133)

We substitute now the phase space \(d\Gamma_3\), (3.40), which in the soft photon limit can be factorized

\[
d\Gamma_3 = (2\pi)^4 \frac{d^3k_2}{(2\pi)^3 2k_2^0} \frac{d^3p_2}{(2\pi)^3 2p_2^0} \frac{d^3p}{(2\pi)^3 2p^0} \delta(k_1 + p_1 - k_2 - p_2) = d\Gamma_2 \frac{d^3p}{(2\pi)^3 2p^0}.
\]

(E.134)

Using definition (2.14) and relation (3.48), we straightforwardly derive from (E.133)

\[
d\sigma^{\text{IR,soft}} \Rightarrow d\sigma^{\text{BORN}} \frac{\alpha}{\pi} \delta^{\text{IR,soft}},
\]

(E.135)

where (see also [9]):

\[
\delta^{\text{IR,soft}} = \frac{2}{\pi} \int \frac{d^3p}{2p^0} F^\text{IR} \theta(\varepsilon - p^0).
\]

(E.136)

Then, the function \(F^\text{IR}\) takes the unique form

\[
F^\text{IR} = Q^2_{\mu} \left( \frac{k_1}{2k_1p} - \frac{k_2}{2k_2p} \right)^2 + 2Q_{\mu} Q_e \left( \frac{k_1}{2k_1p} - \frac{k_2}{2k_2p} \right) \left( \frac{p_1}{2p_1p} - \frac{p_2}{2p_2p} \right)
\]

\[
+ Q_e^2 \left( \frac{p_1}{2p_1p} - \frac{p_2}{2p_2p} \right)^2
\]

\[
= Q^2_{\mu} F^\text{IR}_{\mu\mu} + Q_{\mu} Q_{e} F^\text{IR}_{\mu e} + Q_e^2 F^\text{IR}_{ee}.
\]

(E.137)

The symbol \(\Rightarrow\) in (E.135) means that instead of \(z\)-integrated IR-factors (3.49)-(3.51), we use them in a completely differential form (E.137); they depend on two photonic angles in the R-frame, \(\vartheta_R, \varphi_R\), and on the photonic energy, \(p^0\).

Since the energy of emitted soft photons is limited within a narrow interval

\[
0 \leq p^0 \leq \varepsilon
\]

(E.139)

it is always possible to choose \(\varepsilon\) small enough to ensure the phase space of soft photons be a sphere non-limited by experimental cuts. In other words, the phase space of soft photons is isotropic which allows to choose the \(z\)-axis along a convenient direction, differently for every term in (E.137), while performing angular integrations in (E.136). Therefore, the invariant variables \(V_1, V_2, z_1\) and \(z_2\) might be expressed in terms of only the polar angle \(\cos \vartheta_R \equiv \xi\) and made independent of the azimuthal angle \(\varphi_R\)

\[
V_2 = -2p_2p = 2p_2^0 = 2m_e p^0, \quad (E.140)
\]

\[
V_1 = -2p_1p = 2p^0(p_1^0 - |p_1^0|\xi), \quad (E.141)
\]

\[
z_1 = -2k_1p = 2p^0(k_1^0 - |k_1^0|\xi), \quad (E.142)
\]

\[
z_2 = -2k_2p = 2p^0(k_2^0 - |k_2^0|\xi). \quad (E.143)
\]
In the R-frame

\[ \vec{p}_2 + \vec{p} = 0 \]  \hspace{1cm} (E.144)

and in soft limit \( p \to 0 \) R-frame degenerates to the rest frame of \( \vec{p}_2 \)

\[ p^0_2 = m_e. \]  \hspace{1cm} (E.145)

This is why \( V_2 \) in (E.140) is angular independent.

We will use the dimensional regularization for the infrared divergences and rewrite the photonic phase space as follows (we took also \( 2p^0 \) out of every invariant variable in (E.140)-(E.143))

\[
\frac{2}{\pi} \int \frac{d^3\vec{p}}{(2p^0)^3} = \frac{4\pi}{(2\sqrt{\pi})^n \Gamma(n/2 - 1)} \int_0^{\xi} \frac{(p^0)^{n-5} dp^0}{\mu^{n-4}} \int_0^{2\pi} d\varphi \int_0^{\pi} (\sin \vartheta)^{n-3} d\vartheta. \]  \hspace{1cm} (E.146)

Here \( \gamma \) is the Euler constant and \( \mu \) is an arbitrary parameter with a dimension of mass.

Now integration in (E.136) over \( p^0 \) is performed straightforwardly:

\[
\delta_{IR,soft}^{\mu e} = \frac{1}{2} \int_{-1}^{+1} d\xi \left[ P_{IR} + \frac{\ln \xi}{\mu} + \frac{1}{2} \ln(1 - \xi^2) \right] F_{IR}. \]  \hspace{1cm} (E.147)

In (E.147)

\[ F_{IR} = (2p^0)^2 F_{IR} \]  \hspace{1cm} (E.148)

and the typical pole term

\[ P_{IR} = \frac{1}{n - 4} + \frac{1}{2} \gamma + \ln \frac{1}{2\sqrt{\pi}} \]  \hspace{1cm} (E.149)

represents the infrared divergences at \( n = 4 \).

From (E.140)-(E.143) entering (E.138) and from (E.147) we see that only those integrals over \( \xi \) may occur, which are presented in Appendix D.2 of [9]). Writing \( F_{IR} \) in a form similar to (E.138), we will calculate the three contributions \( \delta_{\mu \nu,soft}^{IR}, \delta_{\mu e,soft}^{IR}, \delta_{ee,soft}^{IR} \) separately. Before this, however, we present a collection of formulae in the R-frame in the soft photon limit.

**E.1 R-frame kinematics for the calculation of the soft photon contribution**

The R-frame is defined by

\[ \vec{p}_2 + \vec{p} = 0 \]  \hspace{1cm} (E.150)

or

\[ \vec{Q} = \vec{p}_1 + \vec{k}_1 - \vec{k}_2 = 0. \]  \hspace{1cm} (E.151)
Since the R-frame is isotropic, there is no need to fix its z-axis along a given direction, say along $\vec{p}_1$:

$$p_1 = \left(0, 0, |\vec{p}_1|, p_0^1\right). \quad (E.152)$$

It might be equally chosen along $\vec{k}_1$,

$$k_1 = \left(0, 0, |\vec{k}_1|, k_0^1\right), \quad (E.153)$$

or along $\vec{k}_2$, then

$$k_2 = \left(0, 0, |\vec{k}_2|, k_0^2\right), \quad (E.154)$$

or along any linear combination of any vectors, say $\vec{k}_\alpha = \alpha \vec{k}_1 + (1 - \alpha) \vec{k}_2$, then

$$k_\alpha = \left(0, 0, |\alpha \vec{k}_1 + (1 - \alpha) \vec{k}_2|, \alpha k_0^1 + (1 - \alpha) k_0^2\right). \quad (E.155)$$

So, we have indeed many R-frames, which differ one from another by a spatial rotation and when we write an arbitrarily oriented photonic 4-momentum as

$$p = p^0(\sin \vartheta_R \sin \varphi_R, \sin \vartheta_R \cos \varphi_R, \cos \vartheta_R, 1), \quad (E.156)$$

one should understand that in every R-frame one has its own angles $\vartheta_R, \varphi_R$ which vary within the same limits – covering the full solid angle. In this way, we arrive at equations (E.140)-(E.143) for invariants $V_2, V_1, z_1, z_2$ with formally one parameter $\xi$.

In the expression (E.147), which has to be integrated over $\xi$ with (E.148) and (E.137), enter the energies $p_1^0, k_{1,2}^0$ and moduli $|\vec{p}_1|, |\vec{k}_{1,2}|$ (see (E.140)-(E.143)).

All the energies and momenta moduli depend only on three invariants: $S, y$ and $V_2$. We make no distinction between $y$ and $y_\mu$ in the soft photon kinematics. In the soft photon problem, we neglect the small invariant $V_2$ as compared to the others. In this limit, the table (C.117) reduces to

$$|\vec{k}_1| = \frac{\sqrt{\lambda_l}}{2m_e}, \quad k_1^0 = \frac{S_1}{2m_e},$$

$$|\vec{k}_2| = \frac{\sqrt{\lambda_s}}{2m_e}, \quad k_2^0 = \frac{S}{2m_e},$$

$$|\vec{p}_1| = \frac{\sqrt{\lambda_w}}{2m_e}, \quad p_1^0 = \frac{Sy + 2m_e^2}{2m_e}, \quad (E.157)$$

$$|\vec{p}_2| = \frac{V_2}{2\sqrt{\tau}}, \quad p_2^0 = m_e,$$

$$|\vec{p}| = |\vec{p}_2|, \quad p^0 = \frac{V_2}{2m_e}.$$
E.2 Muonic current

In muonic current we are dealing with

\[ F_{\mu\mu}^{\text{IR}} = \left( \frac{k_1}{2k_1 p} \right)^2 + \left( \frac{k_2}{2k_2 p} \right)^2 - \frac{2k_1 k_2}{(2k_1 p)(2k_2 p)}, \]  \hspace{1cm} (E.158)

see (E.137). By applying the Feynman parameterization for the last term, we have

\[ F_{\mu\mu}^{\text{IR}} = -\frac{m_e^2}{(2k_1 p)^2} - \frac{m_e^2}{(2k_2 p)^2} - \int_0^1 d\alpha \frac{2k_1 k_2}{(2k_2 p)^2}, \]  \hspace{1cm} (E.159)

where a new 4-vector was introduced as discussed in (E.155):

\[ k_\alpha = \alpha k_1 + (1-\alpha) k_2. \]  \hspace{1cm} (E.160)

Therefore, using all soft machinery described above we can write \( \delta_{\mu\mu}^{\text{IR,soft}} \) as follows

\[ \delta_{\mu\mu}^{\text{IR,soft}} = Q_\mu^2 \left\{ \left( P^{\text{IR}} + \ln \frac{\varepsilon}{\mu} \right) \frac{1}{2} \int_0^1 d\alpha \int_{-1}^1 d\xi F_{\mu\mu}^{\text{IR}} + \frac{1}{4} \int_0^1 d\alpha \int_{-1}^1 d\xi \ln(1-\xi^2) F_{\mu\mu}^{\text{IR}} \right\}, \]  \hspace{1cm} (E.161)

see (E.147), with

\[ F_{\mu\mu}^{\text{IR}} = -\frac{m_e^2}{k_1^0 (1-\beta_1 \xi)^2} - \frac{m_e^2}{k_2^0 (1-\beta_2 \xi)^2} + \frac{S y + 2m_e^2}{k_\alpha^0} \frac{1}{(1-\beta_\alpha \xi)^2}. \]  \hspace{1cm} (E.162)

On passing, we used

\[ -2k_\alpha p = 2p^0 (k_\alpha^0 - |k_\alpha| \xi) = 2p^0 k_\alpha^0 (1-\beta_\alpha \xi), \]  \hspace{1cm} (E.163)

\[ z_1 = -2k_1 p = 2p^0 (k_1^0 - |k_1| \xi) = 2p^0 k_1^0 (1-\beta_1 \xi), \]  \hspace{1cm} (E.164)

\[ z_2 = -2k_2 p = 2p^0 (k_2^0 - |k_2| \xi) = 2p^0 k_2^0 (1-\beta_2 \xi), \]  \hspace{1cm} (E.165)

with three velocities

\[ \beta_1 = \frac{|\vec{k}_1|}{k_1^0} = \frac{\sqrt{\lambda_1}}{S_1}, \]  \hspace{1cm} (E.166)

\[ \beta_2 = \frac{|\vec{k}_2|}{k_2^0} = \frac{\sqrt{\lambda_s}}{S}, \]  \hspace{1cm} (E.167)

\[ \beta_\alpha = \frac{|\vec{k}_\alpha|}{k_\alpha^0}. \]  \hspace{1cm} (E.168)

We also have from (E.160) the following relations:

\[ k_\alpha^0 = \frac{S \alpha + S_1 (1-\alpha)}{2m_e}, \]  \hspace{1cm} (E.169)

\[ -k_\alpha^2 = m_e^2 + \alpha (1-\alpha) S y. \]  \hspace{1cm} (E.169)
Therefore

\[ \beta_\alpha = \frac{\sqrt{[S\alpha + S_1(1-\alpha)]^2 - 4m_e^2 \left[m_{\mu}^2 + \alpha(1-\alpha)S_y\right]}}{S\alpha + S_1(1-\alpha)}. \]  

(E.170)

Using the table of integrals from Appendix D.2 of [9] we obtain

\[ \delta_{\mu\mu}^{\text{IR,soft}} = Q_{\mu}^2 \int_0^1 d\alpha \left\{ \left(P^{\text{IR}} + \ln \frac{2\varepsilon}{\mu}\right) \left[ -\frac{m_{\mu}^2}{k_1^{\alpha^2}} \frac{1}{1-\beta_1^2} - \frac{m_{\mu}^2}{k_2^{\alpha^2}} \frac{1}{1-\beta_2^2} + \frac{S_y + 2m_{\mu}^2}{k_0^{\alpha^2}} \frac{1}{1-\beta_\alpha^2} \right] 
+ \frac{1}{2\beta_1} \frac{m_{\mu}^2}{k_1^{\alpha^2}} \frac{1}{1-\beta_1} \ln \frac{1 + \beta_1}{1 - \beta_1} + \frac{1}{2\beta_2} \frac{m_{\mu}^2}{k_2^{\alpha^2}} \frac{1}{1-\beta_2} \ln \frac{1 + \beta_2}{1 - \beta_2} 
- \frac{1}{2\beta_\alpha} \frac{S_y + 2m_{\mu}^2}{k_0^{\alpha^2}} \frac{1}{1-\beta_\alpha} \ln \frac{1 + \beta_\alpha}{1 - \beta_\alpha} \right\}. \]  

(E.171)

We have from (E.166)-(E.168)

\[ (k_1^0)^2(1-\beta_1^2) = -k_1^2 = m_{\mu}^2, \]  

(E.172)

\[ (k_2^0)^2(1-\beta_2^2) = -k_2^2 = m_{\mu}^2, \]  

(E.173)

\[ (k_\alpha^0)^2(1-\beta_\alpha^2) = -k_\alpha^2. \]  

(E.174)

And the expression (E.161) becomes

\[ \delta_{\mu\mu}^{\text{IR,soft}} = Q_{\mu}^2 \left\{ \left(P^{\text{IR}} + \ln \frac{2\varepsilon}{\mu}\right) \left[ -2 + \left(S_y + 2m_{\mu}^2\right) \int_0^1 \frac{d\alpha}{m_{\mu}^2 + \alpha(1-\alpha)S_y} \right] 
+ \frac{1}{2\beta_1} \ln \frac{1 + \beta_1}{1 - \beta_1} + \frac{1}{2\beta_2} \ln \frac{1 + \beta_2}{1 - \beta_2} 
+ \frac{S_y + 2m_{\mu}^2}{2} \int_0^1 \frac{d\alpha}{\beta_\alpha[m_{\mu}^2 + \alpha(1-\alpha)S_y]} \ln \frac{1 - \beta_\alpha}{1 + \beta_\alpha} \right\}. \]  

(E.175)

Now we use the URA in the electron mass

\[ \frac{1}{2\beta_1} \ln \frac{1 + \beta_1}{1 - \beta_1} \approx = \ln \frac{S_1}{m_em_{\mu}}, \]  

(E.176)

\[ \frac{1}{2\beta_2} \ln \frac{1 + \beta_2}{1 - \beta_2} \approx = \ln \frac{S}{m_em_{\mu}}. \]  

(E.177)

The first integral we calculated precisely in [9]

\[ \left(S_y + 2m_{\mu}^2\right) \int_0^1 \frac{d\alpha}{[m_{\mu}^2 + \alpha(1-\alpha)S_y]} = \frac{1 + \beta^2}{\beta} L_\beta \]  

(E.178)

with

\[ \beta = \sqrt{1 + \frac{4m_{\mu}^2}{S_y}}, \]

\[ L_\beta = \ln \frac{\beta + 1}{\beta - 1}. \]  

(E.179)
With the second integral the situation is more complicated. Defining it as in (9),

\[ S_\Phi \equiv \frac{1}{2} \left( Sy + 2m_\mu^2 \right) \int_0^1 \frac{d\alpha}{\beta_e[m_\mu^2 + \alpha(1-\alpha)Sy]} \ln \frac{1 - \beta_\alpha}{1 + \beta_\alpha}, \tag{E.180} \]

we cannot take over the analogous result from (9) since there that integral was calculated in the URA in the leptonic mass \( m \). Here we need result exact in \( m_\mu \) (see also (4)). Making use of URA in \( m_e \), with the aid of (E.170) we get

\[ S_\Phi \approx \left( Sy + 2m_\mu^2 \right) \int_0^1 \frac{d\alpha}{m_\mu^2 + \alpha(1-\alpha)Sy} \ln \frac{m_\mu^2 + \alpha(1-\alpha)Sy}{[S\alpha + S_1(1-\alpha)]^2}. \tag{E.181} \]

The expression for \( S_\Phi \) simplifies drastically and can be calculated straightforwardly

\[ S_\Phi = \frac{Sy + 2m_\mu^2}{\sqrt{\lambda_0 m}} \left( \ln \frac{m_\mu^2(Sy + 4m_\mu^2)}{S^2(1-y)(1-y\alpha_1)(1-y\alpha_2)} \ln \frac{\alpha_2}{(-\alpha_1)} \right. \]
\[- \frac{1}{2} \ln^2 \left( \frac{-\alpha_2}{(-\alpha_1)(1-y)} \right) + \ln(1-y) \ln \left( \frac{1-y}{1-y\alpha_1} \right) - Li_2(1) \]
\[ + Li_2 \left( \frac{-\alpha_1}{(1-y)\alpha_2} \right) - Li_2 \left( \frac{-\alpha_1}{\alpha_2} \right)^2 \right) \]. \tag{E.182} \]

with

\[ \alpha_{1,2} = \frac{1 + \beta}{2}. \tag{E.183} \]

Collecting all terms together, we finally have

\[ \delta_{\mu\mu}^{IR,soft} = Q_\mu^2 \left\{ 2 \left( P^{IR} + \ln \frac{2\varepsilon}{\mu} \right) \left( \frac{1 + \beta^2}{2\beta} L_\beta - 1 \right) + \ln \frac{S^2(1-y)}{m_e^2 m_\mu^2} + S_\Phi \right\}. \tag{E.184} \]

From the virtual photon correction to the muon vertex we have the contribution

\[ \delta_{\mu\mu}^{vert} = Q_\mu^2 \left\{ -2 \left( P^{IR} + \ln \frac{m_\mu}{\mu} \right) \left( \frac{1 + \beta^2}{2\beta} L_\beta - 1 \right) + \frac{3}{2} \beta L_\beta - 2 \right. \]
\[- \frac{1 + \beta^2}{2\beta} \left[ L_\beta \ln \frac{4\beta^2}{\beta^2 - 1} + Li_2 \left( \frac{1 + \beta}{1 - \beta} \right) - Li_2 \left( \frac{1 - \beta}{1 + \beta} \right) \right] \}. \tag{E.185} \]

The infrared divergence and the scale parameter \( \mu \) cancel exactly in the sum of these two contributions. The sum reads

\[ \delta_{\mu\mu}^{VR} = \delta_{\mu\mu}^{IR,soft} + \delta_{\mu\mu}^{vert} \]
\[ = Q_\mu^2 \left\{ \left[ \frac{1 + \beta^2}{2\beta} L_\beta - 1 \right] \left( 2 \ln \frac{V_{min}^2}{S} - \ln(1-y) \right) + \frac{3}{2} \beta L_\beta - 2 \right. \]
\[ + \frac{1 + \beta^2}{2\beta} \left[ L_\beta \ln \frac{1-y}{(1-y\alpha_1)(1-y\alpha_2)} - \frac{1}{2} \ln^2(1-y) \right. \]
\[ + \ln(1-y) \ln \left( \frac{1-y}{1-y\alpha_1} \right) + Li_2 \left[ \frac{(-\alpha_1)}{(1-y)\alpha_2} \right] \]
\[ + Li_2 \left[ \frac{(1-y)(-\alpha_1)}{\alpha_2} \right] - 2Li_2 \left[ \frac{(-\alpha_1)}{\alpha_2} \right] \}. \tag{E.186} \]

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E.3 Electronic current

The correction $\delta^{\text{IR,soft}}_{ee}$ is very easy to calculate.

$$F_{ee}^{\text{IR}} = \left( \frac{p_1}{2p_1p} \right)^2 + \left( \frac{p_2}{2p_2p} \right)^2 - \frac{2p_1p_2}{(2p_1p)(2p_2p)}. \quad \text{(E.187)}$$

Since $2p_2p$ is independent of $\xi = \cos \vartheta_R$, the $\alpha$-parameterization is not needed here and $\delta^{\text{IR,soft}}_{ee}$ reads

$$\delta^{\text{IR,soft}}_{ee} = Q_e^2 \left[ \left( P^{\text{IR}} + \ln \frac{\varepsilon}{\mu} \right) \frac{1}{2} \int_{-1}^{+1} d\xi F_{ee}^{\text{IR}} + \frac{1}{4} \int_{-1}^{+1} d\xi \ln(1 - \xi^2) F_{ee}^{\text{IR}} \right]. \quad \text{(E.188)}$$

Introducing

$$\beta = \frac{|\vec{p}_1|}{p_1^0} = \sqrt{\frac{S^2 y^2 + 4 m_e^2 S y}{S y + 2 m_e^2}}, \quad \text{(E.189)}$$

we receive

$$F_{ee}^{\text{IR}} = -\frac{m_e^2}{p_1^0 (1 - \beta \xi)^2} - 1 + \frac{2m_e p_1^0}{m_e p_1^0 (1 - \beta \xi)} = -1 + \frac{2}{1 - \beta \xi} - \frac{m_e^2}{p_1^0 (1 - \beta \xi)^2}. \quad \text{(E.190)}$$

Using the table of integrals from Appendix D.2 of [4] we derive

$$\delta^{\text{IR,soft}}_{ee} = Q_e^2 \left[ \left( P^{\text{IR}} + \ln \frac{2\varepsilon}{\mu} \right) \left[ -1 + \frac{1}{\beta} \ln \frac{1 + \beta}{1 - \beta} - \frac{m_e^2}{p_1^0 (1 - \beta^2)} \right] + \frac{1}{2\beta} \left[ \text{Li}_2 \left( \frac{2\beta}{\beta - 1} \right) - \text{Li}_2 \left( \frac{2\beta}{\beta + 1} \right) \right] + \frac{m_e^2}{p_1^0} \frac{1}{2\beta (1 - \beta^2)} \ln \frac{1 + \beta}{1 - \beta} \right]. \quad \text{(E.191)}$$

In the URA in $m_e^2$ we obtain

$$\frac{1}{\beta} \ln \frac{1 + \beta}{1 - \beta} \approx 2 \ln \frac{S y}{m_e^2} \quad \text{(E.192)}$$

and

$$\text{Li}_2 \left( \frac{2\beta}{\beta - 1} \right) - \text{Li}_2 \left( \frac{2\beta}{\beta + 1} \right) \approx -2 \text{Li}_2(1) - 2 \ln^2 \frac{S y}{m_e^2}. \quad \text{(E.193)}$$

So, we can write

$$\delta^{\text{IR,soft}}_{ee} = 2Q_e^2 \left[ \left( P^{\text{IR}} + \ln \frac{2\varepsilon}{\mu} \right) \left( -1 + \ln \frac{S y}{m_e^2} \right) + 1 + \ln \frac{S y}{m_e^2} - \ln^2 \frac{S y}{m_e^2} - \text{Li}_2(1) \right]. \quad \text{(E.194)}$$

The corresponding virtual photon correction to the electron vertex has the following form:

$$\delta^{\text{vert}}_{ee} = Q_e^2 \left[ 2 \left( P^{\text{IR}} + \ln \frac{m_e}{\mu} \right) \left( 1 - \ln \frac{S y}{m_e^2} \right) - 2 + \frac{3}{2} \ln \frac{S y}{m_e^2} - \frac{1}{2} \ln^2 \frac{S y}{m_e^2} + \text{Li}_2(1) \right]. \quad \text{(E.195)}$$

The complete answer is:

$$\delta_{ee}^{\text{VR}} = \delta^{\text{IR,soft}}_{ee} + \delta^{\text{vert}}_{ee} = Q_e^2 \left[ \left( \ln \frac{S y}{m_e^2} - 1 \right) \left( \ln \frac{S y}{m_e^2} + 2 \ln \frac{V_{\text{min}}}{m_e^2} \right) - 1 + \frac{3}{2} \ln \frac{S y}{m_e^2} - \frac{1}{2} \ln^2 \frac{S y}{m_e^2} \right]. \quad \text{(E.196)}$$
E.4 $\mu e$ interference

In the $\mu e$ interference the expression for $F^{\text{IR}}_{\mu e}$ reads

$$F^{\text{IR}}_{\mu e} = \frac{2k_1p_1}{(2k_1p)(2p_1p)} - \frac{2k_1p_2}{(2k_1p)(2p_2p)} - \frac{2k_2p_1}{(2k_2p)(2p_1p)} + \frac{2k_2p_2}{(2k_2p)(2p_2p)}. \tag{E.197}$$

We introduce the $\alpha$ parameterization with the aid of two new 4-vectors

$$k_{1\alpha} = k_1 + p_1(1 - \alpha), \quad (E.198)$$
$$k_{2\alpha} = k_2 + p_2(1 - \alpha), \quad (E.199)$$

resulting in

$$F^{\text{IR}}_{\mu e} = \frac{S_1}{2k_1p(1 - \beta_1 \xi)} - \frac{S}{2k_2p(1 - \beta_2 \xi)} + \frac{1}{0} d\alpha \left[ \frac{S_1}{2(k_{2\alpha}p)^2} - \frac{S}{2(k_{1\alpha}p)^2} \right]. \tag{E.200}$$

and $\delta^{\text{IR,soft}}_{\mu e}$ becomes

$$\delta^{\text{IR,soft}}_{\mu e} = Q_\mu Q_e \left[ (F^{\text{IR}} + \ln \frac{\varepsilon}{\mu}) \right] \frac{1}{2} \int_{0}^{1} d\alpha \int_{-1}^{1} d\xi F^{\text{IR}}_{\mu e} + \frac{1}{4} \int_{0}^{1} d\alpha \int_{-1}^{1} d\xi \ln(1 - \xi^2) F^{\text{IR}}_{\mu e} \tag{E.201}$$

with

$$F^{\text{IR}}_{\mu e} = \frac{2}{1 - \beta_1 \xi} - \frac{2}{1 - \beta_2 \xi} + \frac{S_1}{(k_{2\alpha}p)^2(1 - \beta_2 \xi)^2} - \frac{S}{(k_{1\alpha}p)^2(1 - \beta_1 \xi)^2}. \tag{E.202}$$

Here

$$-k_{1\alpha}^2 = m_\mu^2 \alpha^2 + m_e^2(1 - \alpha)^2 - 2k_1p_1 \alpha(1 - \alpha)$$
$$k_{1\alpha}^0 = \frac{S_1 \alpha + (Sy + 2m_e^2)(1 - \alpha)}{2m_e}. \tag{E.203}$$

$$-k_{2\alpha}^2 = m_\mu^2 \alpha^2 + m_e^2(1 - \alpha)^2 - 2k_2p_1 \alpha(1 - \alpha)$$
$$k_{2\alpha}^0 = \frac{S \alpha + (Sy + 2m_e^2)(1 - \alpha)}{2m_e}. \tag{E.204}$$

$$\beta_{1\alpha} = \frac{|k_{1\alpha}|}{k_{1\alpha}^0}, \quad (E.205)$$
$$= \sqrt{\left[ S \alpha + (Sy + 2m_e^2)(1 - \alpha) \right]^2 - 4m_e^2 \left[ m_\mu^2 \alpha^2 + m_e^2(1 - \alpha)^2 + S \alpha(1 - \alpha) \right]},$$

$$\beta_{2\alpha} = \frac{|k_{1\alpha}|}{k_{1\alpha}^0}, \quad (E.206)$$
$$= \sqrt{\left[ S \alpha + (Sy + 2m_e^2)(1 - \alpha) \right]^2 - 4m_e^2 \left[ m_\mu^2 \alpha^2 + m_e^2(1 - \alpha)^2 + S \alpha(1 - \alpha) \right]},$$
Using the table of integrals from Appendix D.2 of [4] we derive

\[
\delta^{\text{IR,soft}}_{\mu e} = \mathcal{Q}_e \mathcal{Q}_\mu \left\{ \left( P^{\text{IR}} + \ln \frac{2 \varepsilon}{\mu} \right) \left[ \frac{1}{2 \beta_1} \right] - \frac{1}{2 \beta_2} \frac{1 + \beta_1}{1 - \beta_1} - \frac{1 + \beta_2}{1 - \beta_2} - \mathcal{S} \int \frac{d\alpha}{-k_{1\alpha}^2} + \mathcal{S}_1 \int \frac{d\alpha}{-k_{2\alpha}^2} \right] - \frac{2}{\beta_2} \left[ \text{Li}_2 \left( \frac{2 \beta_2}{\beta_2 - 1} \right) - \text{Li}_2 \left( \frac{2 \beta_2}{\beta_2 + 1} \right) \right] + \frac{2}{\beta_1} \left[ \text{Li}_2 \left( \frac{2 \beta_1}{\beta_1 - 1} \right) - \text{Li}_2 \left( \frac{2 \beta_1}{\beta_1 + 1} \right) \right] - \frac{S}{2} \int \frac{d\alpha}{-k_{1\alpha}^2} \ln \frac{1 - \beta_1}{1 + \beta_1} + \mathcal{S}_1 \int \frac{d\alpha}{-k_{2\alpha}^2} \ln \frac{1 - \beta_2}{1 + \beta_2} \right\}.
\]  

(E.207)

In the URA in \( m_e^2 \)

\[
S \int_0^1 \frac{d\alpha}{-k_{1\alpha}^2} \approx \ln \frac{S^2}{m_e^2 m^2},
\]

(E.208)

\[
S_1 \int_0^1 \frac{d\alpha}{-k_{2\alpha}^2} \approx \ln \frac{S_1^2}{m_e^2 m^2},
\]

(E.209)

\[
\frac{S}{2} \int \frac{d\alpha}{-k_{1\alpha}^2} \ln \frac{1 - \beta_1}{1 + \beta_1} \approx S_\Phi(S, S_1),
\]

(E.210)

\[
\frac{S}{2} \int \frac{d\alpha}{-k_{2\alpha}^2} \ln \frac{1 - \beta_2}{1 + \beta_2} \approx S_\Phi(S_1, S).
\]

(E.211)

Where we introduced the generic function \( S_\Phi(I, \hat{I}) \)

\[
S_\Phi(I, \hat{I}) = \frac{I}{2} \int_0^1 \frac{d\alpha}{m_\mu^2 \alpha^2 + m_e^2 (1 - \alpha)^2 + I\alpha(1 - \alpha)} \ln \frac{m_e^2 [m_e^2 (1 - \alpha)^2 + I\alpha(1 - \alpha)]}{[\hat{I} \alpha + (S_\gamma + 2m_e^2)(1 - \alpha)]^2}
\]

\[
= \frac{1}{2} \ln \frac{m_e^2 I}{m_\mu^2 I} \ln \frac{\hat{I}}{m_\mu^2 I} - \frac{1}{4} \ln^2 \frac{m_e^2}{\hat{I}} - \frac{1}{4} \ln^2 \frac{m_e^2}{I} - \text{Li}_2 \left( 1 - \frac{S_\gamma m_e^2}{\hat{I}} \right)
\]

\[
+ \ln \frac{\hat{I}}{S_\gamma} \ln \frac{m_e^2}{\hat{I}} - \frac{1}{2} \ln^2 \frac{\hat{I}}{S_\gamma}.
\]

(E.212)

The expression (E.207) reduces in the URA in \( m_e \) to

\[
\delta^{\text{IR,soft}}_{\mu e} = \left( P^{\text{IR}} + \ln \frac{2 \varepsilon}{\mu} \right) 4 \ln(1 - y) + 2 \ln(1 - y) \left( \ln \frac{m_e^2}{S} - \ln y \right).
\]

(E.213)

In the next Appendix on the box contribution we derive

\[
\delta^{\text{box}}_{\mu e} = \mathcal{Q}_e \mathcal{Q}_\mu \left\{ \left( -P^{\text{IR}} - \frac{1}{2} \ln \frac{Q^2}{\mu^2} \right) 4 \ln(1 - y) \right\} + B^{\text{fin}}_{\mu e},
\]

(E.214)

and the following short final expression is obtained

\[
\delta_{\mu e} = \delta^{\text{IR,soft}}_{\mu e} + \delta^{\text{box}}_{\mu e} = \mathcal{Q}_e \mathcal{Q}_\mu \left\{ 4 \ln(1 - y) \ln \frac{V_{\min}}{S_y} + B^{\text{fin}}_{\mu e} \right\}.
\]

(E.215)

The finite contribution from box diagram, \( B^{\text{fin}}_{\mu e} \), is presented in Appendix F.
F Two photon exchange contribution

The two-photon exchange contribution is described by the two box diagrams: \textit{direct box} and \textit{crossed box}. For the sake of symmetry, it is convenient to duplicate the number of diagrams and to deal with four diagrams shown in the figure below:

\[
\begin{align*}
\begin{array}{c}
\alpha & \; k_1 + p & \beta & \; k_2 \\
p & \; q + p \\
p_1 & \; \alpha' & \; p_1 - p & \; \beta' & \; p_2
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
\beta & \; k_2 - p & \alpha & \; k_2 \\
q + p & \; p \\
p_1 & \; \beta' & \; p_2 + p & \; \alpha' & \; p_2
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
\alpha & \; k_1 + p & \beta & \; k_2 \\
p & \; q + p \\
p_1 & \; \beta' & \; p_2 + p & \; \alpha' & \; p_2
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
\beta & \; k_2 - p & \alpha & \; k_2 \\
q + p & \; p \\
p_1 & \; \alpha' & \; p_1 - p & \; \beta' & \; p_2
\end{array}
\end{align*}
\]

This is why the two-photon exchange matrix element is given by an expression, which contains an extra factor \(1/2\):

\[
M_{\text{box}}^{\text{box}} = \frac{1}{2} \int \frac{d^4p}{p^2(p + q)^2} \times \bar{u}(k_2) \left[ \frac{\gamma_\beta(2k_1\alpha + \hat{p}\gamma_\alpha)}{p^2 + 2pk_1} + \frac{(2k_2\alpha - \gamma_\alpha\hat{p})\gamma_\beta}{p^2 - 2pk_2} \right] u(k_1) \\
\times \bar{u}(p_2) \left[ \frac{(2p_2\alpha + \gamma_\alpha\hat{p})\gamma_\beta}{p^2 + 2pp_2} + \frac{\gamma_\beta(2p_1\alpha - \hat{p}\gamma_\alpha)}{p^2 - 2pp_1} \right] u(p_1).
\]  

(F.216)

For later use we introduce the short hand notation for propagators

\[
\begin{align*}
\Pi(k_1) &= p^2 + 2pk_1, \\
\Pi(k_2) &= p^2 - 2pk_2, \\
\Pi(p_1) &= p^2 - 2pp_1, \\
\Pi(p_2) &= p^2 + 2pp_2.
\end{align*}
\]  

(F.217)

Now we separate the infrared divergent (IRD) part of the two-photon exchange contribution. There are two IRD-poles: 1) one is located at \(p \rightarrow 0\) and 2) another one – at \((p + q) \rightarrow 0\).
At $p \to 0$, one has
\[
\frac{1}{2} \frac{1}{p^2 q^2} \left[ \frac{4k_1 p_2}{\prod(k_1) \prod(p_2)} + \frac{4k_1 p_1}{\prod(k_1) \prod(p_1)} + \frac{4k_2 p_2}{\prod(k_2) \prod(p_2)} + \frac{4k_2 p_1}{\prod(k_2) \prod(p_1)} \right] \gamma_\beta \otimes \gamma_\beta. \tag{F.218}
\]
While at $p \to q$, one has
\[
\frac{1}{2} \frac{1}{q^2(p+q)^2} \left[ \frac{4k_1 p_2}{\prod(k_1) \prod(p_2)} + \frac{4k_1 p_1}{\prod(k_1) \prod(p_1)} + \frac{4k_2 p_2}{\prod(k_2) \prod(p_2)} + \frac{4k_2 p_1}{\prod(k_2) \prod(p_1)} \right] \gamma_\beta \otimes \gamma_\beta. \tag{F.219}
\]
For the latter term, we perform the substitution $p = p' - q$ and observe that at this substitution the propagators transform as follows
\[
\prod(k_1) \to \prod(-k_2), \\
\prod(k_2) \to \prod(-k_1), \\
\prod(p_1) \to \prod(-p_2), \\
\prod(p_2) \to \prod(-p_1). \tag{F.220}
\]
With one more substitution $p' \to -p$, and using equalities $k_2.p_2 = k_1.p_1$, $k_2.p_1 = k_1.p_2$, we see that the second pole gives exactly the same contribution as the first one. Therefore, the full IRD-part of the two-photon exchange reads:
\[
\frac{1}{2} \frac{1}{p^2 q^2} \left[ \frac{4k_1 p_2}{\prod(k_1) \prod(p_2)} + \frac{4k_1 p_1}{\prod(k_1) \prod(p_1)} + \frac{4k_2 p_2}{\prod(k_2) \prod(p_2)} + \frac{4k_2 p_1}{\prod(k_2) \prod(p_1)} \right] \gamma_\beta \otimes \gamma_\beta. \tag{F.221}
\]
Now we add and subtract the IRD-part of the two-photon exchange and when adding it we use (F.221), while when subtracting it we use the sum of (F.218) and (F.219). Using the identity
\[
\frac{1}{p^2(p+q)^2} - \frac{1}{p^2 q^2} - \frac{1}{q^2(p+q)^2} \equiv \frac{-2p.(p+q)}{q^2 p^2 (p+q)^2}, \tag{F.222}
\]
we arrive at the final expression for the two-photon exchange contribution before integration over $d^4 p$.
\[
M_{\text{BOX}} \sim \frac{4}{q^2 p^2} \left[ \frac{-S}{\prod(k_1) \prod(p_1)} + \frac{-S_1}{\prod(k_1) \prod(p_2)} \right] \gamma_\beta \otimes \gamma_\beta \\
b - \frac{4p(p+q)}{p^2(p+q)^2 q^2} \left[ \frac{-S}{\prod(k_1) \prod(p_1)} + \frac{-S_1}{\prod(k_1) \prod(p_2)} \right] \gamma_\beta \otimes \gamma_\beta \\
+ \frac{1}{p^2(p+q)^2} \left[ \frac{k_{1\alpha}}{\prod(k_1)} + \frac{k_{2\alpha}}{\prod(k_2)} \right] \gamma_\beta \otimes \left[ \frac{\gamma_\alpha \gamma_\beta}{\prod(p_2)} - \frac{\gamma_\beta \gamma_\alpha}{\prod(p_1)} \right] \\
+ \frac{1}{p^2(p+q)^2} \left[ \frac{\gamma_\beta \gamma_\alpha}{\prod(k_1)} - \frac{\gamma_\alpha \gamma_\beta}{\prod(k_2)} \right] \otimes \gamma_\beta \left[ \frac{p_{2\alpha}}{\prod(p_2)} + \frac{p_{1\alpha}}{\prod(p_1)} \right] \\
+ \frac{1}{2p^2(p+q)^2} \left[ \frac{\gamma_\beta \gamma_\alpha}{\prod(k_1)} - \frac{\gamma_\alpha \gamma_\beta}{\prod(k_2)} \right] \otimes \gamma_\beta \left[ \frac{\gamma_\alpha \gamma_\beta}{\prod(p_2)} - \frac{\gamma_\beta \gamma_\alpha}{\prod(p_1)} \right]. \tag{F.223}
\]
The first raw of this formula represents the IRD-contribution, the second raw stands for the IR-free scalar, the next two represent the IR-free vector, and finally the last one is an IR-free tensor.

For the IR-divergent part we introduce the well-known presentation

$$J_{IR}^3(k_1, p_1) = \frac{16\pi^2}{i} \int \frac{d^4p}{(2\pi)^4 p^2 \Pi(k_1) \Pi(p_1)}$$

$$= P_{IR}(\mu) \int_0^1 \frac{dy}{(-K_y^2)} + \frac{1}{2} \int_1^1 \frac{dy}{(-K_y^2)} \ln \left(\frac{-K_y^2}{q^2}\right) + O(n-4). \quad (F.224)$$

where

$$-K_y^2 = m^2 y^2 + m^2 e(1-y)^2 - Sy(1-y). \quad (F.225)$$

We will use the short hand notations

$$P_S^{IR} \equiv P^{IR}(S) = -P_{IR}(\mu) \int_0^1 \frac{dy}{(-K_y^2)},$$

$$K_S \equiv K(S) = \int \frac{dy}{(-K_y^2)} \ln \left(\frac{-K_y^2}{q^2}\right). \quad (F.226)$$

Then all needed IRD-integrals are

$$J_{IR}^3(k_1, p_1) = J_{IR}^3(k_2, p_2) = -P_S^{IR} + \frac{K_S}{2}, \quad (F.227)$$

$$J_{IR}^3(k_1, p_2) = J_{IR}^3(k_2, p_1) = P_S^{IR} + \frac{K_S}{2}. \quad (F.228)$$

with

$$P^{IR}(S_1) = -P^{IR}(-S_1),$$

$$K_{S_1} = K(-S_1). \quad (F.229)$$

For the IR-finite scalars

$$J_{IR}^4(k_i p_j) = \frac{16\pi^2}{i} \int \frac{2p.(p + q)d^4p}{(2\pi)^4 p^2 (p + q)^2 \Pi(k_i) \Pi(p_j)}, \quad (F.230)$$

we derived

$$J_4(k_1, p_1) = J_4(k_2, p_2) = K_S, \quad (F.231)$$

$$J_4(k_1, p_2) = J_4(k_2, p_1) = K_{S_1}. \quad (F.232)$$

### F.1 Vectorial integrals

We begin with the reduction of vector integrals. First we define the coefficients of the vector reduction

$$V^{ij}_\mu = \frac{16\pi^2}{i} \int \frac{d^4p p_\mu}{(2\pi)^4 p^2 (p + q)^2 \Pi(k_i) \Pi(p_j)}$$

$$= (V_q)^{ij} q_\mu + (V_k)^{ij} (k_i)_\mu + (V_p)^{ij} (p_j)_\mu, \quad (F.233)$$
Define the coefficients of tensor reduction

\[ T^{ij}_{\mu\nu} = \frac{16\pi^2}{i} \int \frac{d^4 p}{(2\pi)^4 p^2 (p + q)^2 \Pi(k_i) \Pi(p_j)} \]

\[ = T_0^{\delta_{\mu\nu}} + T^{ij}_{q\mu} q_{\mu} + T^{ij}_{k\mu} (k_i)_{\mu} + T^{ij}_{q\nu} (p_i)_{\mu} (p_j)_{\nu} + T^{ij}_{p\mu} (q_{\mu} (k_i)_{\nu} + q_{\mu} (k_i)_{\mu}) + T^{ij}_{p\nu} (q_{\mu} (p_i)_{\nu} + q_{\mu} (p_j)_{\mu}) + T^{ij}_{q\nu} [(k_i)_{\mu} (p_j)_{\nu} + (k_i)_{\nu} (p_j)_{\mu}] \]

The system of equations for the coefficients \( T^{ij} \) was derived in the following way:

The system of linear equations for these coefficients in terms of scalar integrals, derived by contraction of (F.233) with \( q_\mu, (k_i)_\mu, (p_j)_\mu \), reads

\[ 2q^2 V^{ij}_q + 2S_{k_i} q \cdot k_i V^{ij}_k + 2S_{p_j} q \cdot p_j V^{ij}_p - J_4(k_i, p_j) + 2J^R_3(k_i, p_j) = 0 \]
\[ 2S_{k_i} q \cdot k_i V^{ij}_k + 2S_{k_i} 2k_i^2 V^{ij}_k + 2S_{p_j} k_i \cdot p_j V^{ij}_p - J_3(p_j) + 2J^R_3(k_i, p_j) = 0, \]
\[ 2S_{p_j} q \cdot p_j V^{ij}_p + 2S_{k_i} S_{p_j} k_i \cdot p_j V^{ij}_p + 2S_{p_j} 2p_j^2 V^{ij}_p - J_3(k_i) + 2J^R_3(k_i, p_j) = 0. \] (F.234)

Here

\[ S_{k_1} = -1, \quad S_{k_2} = +1, \]
\[ S_{p_1} = +1, \quad S_{p_2} = -1. \] (F.235)

Two additional types of scalar integrals

\[ J_3 (k_i) = \frac{16\pi^2}{i} \int \frac{d^4 p}{(2\pi)^4 p^2 (p + q)^2 \Pi(k_i)} \] (F.236)
and

\[ J_3 (p_j) = \frac{16\pi^2}{i} \int \frac{d^4 p}{(2\pi)^4 p^2 (p + q)^2 \Pi(p_j)} \] (F.237)
are introduced. They satisfy the equalities

\[ J_3 (k_1) = J_3 (k_2) \] (F.238)
and

\[ J_3 (p_1) = J_3 (p_2), \] (F.239)

and may be described by one generic formula:

\[ J_3 (q_i) = \int_0^1 \frac{dy}{-q^2 y - q_i^2 (1 - y) + (q - q_i)^2 y (1 - y)} \ln \left( \frac{(q - q_i)^2 y (1 - y) - q_i^2 (1 - y)}{q^2 y} \right). \] (F.240)

with \( q_i = (k_1, -k_2, p_2, -p_1) \).

The solution of the system (F.234) is presented in subsection F.4 of this Appendix.

### F.2 Tensorial integrals

Define the coefficients of tensor reduction

\[ T^{ij}_{\mu\nu} = \frac{16\pi^2}{i} \int \frac{d^4 p}{(2\pi)^4 p^2 (p + q)^2 \Pi(k_i) \Pi(p_j)} \]

\[ = T_0^{\delta_{\mu\nu}} + T^{ij}_{q\mu} q_{\mu} + T^{ij}_{k\mu} (k_i)_{\mu} + T^{ij}_{q\nu} (p_i)_{\mu} (p_j)_{\nu} + T^{ij}_{p\mu} (q_{\mu} (k_i)_{\nu} + q_{\mu} (k_i)_{\mu}) + T^{ij}_{p\nu} (q_{\mu} (p_i)_{\nu} + q_{\mu} (p_j)_{\mu}) + T^{ij}_{q\nu} [(k_i)_{\mu} (p_j)_{\nu} + (k_i)_{\nu} (p_j)_{\mu}] \] (F.241)
• the first equation was received by multiplying (F.241) with $\delta_{\mu\nu}$

$$J^{\text{IR}}_3(k_i p_j) = 4T_{00}^{ij} + q^2 T_{qq}^{ij} + (k_i)^2 T_{kk}^{ij} + (p_j)^2 T_{pp}^{ij} + q^2 T_{qp}^{ij} + I_{ij} T_{kp}^{ij}, \quad (F.242)$$

• the next three equations were received by multiplying (F.241) with $q_\mu$

$$\int \frac{d^4p}{(2\pi)^4} \frac{[2pq + p^2 - q^2 - p^2] \cdot p_\mu}{\Pi(k_i) \Pi(p_j)} = \int \frac{d^4p}{(2\pi)^4} \frac{p^\mu}{\Pi(k_i) \Pi(p_j)} - \int \frac{d^4p}{(2\pi)^4} \frac{p^\mu}{(p + q)^2 \Pi(k_i) \Pi(p_j)} - q^2 \int \frac{d^4p}{(2\pi)^4} \frac{p^\mu}{\Pi(k_i) \Pi(p_j)} = 2q_\mu \delta_0^{ij} + 2q^2 q_i T_{qq}^{ij} + q^2 T_{kk}^{ij} (k_i)_\mu + q^2 T_{pp}^{ij} (p_j)_\mu + T_{pq}^{ij} [q_\mu q^2 + 2q^2 (p_j)_\mu] + q^2 T_{kp}^{ij} [(k_i)_\mu + (p_j)_\mu], \quad (F.243)$$

• another six equations were received by multiplying (F.241) with $(k_i)_\nu$ and $(p_j)_\nu$.

The latter nine equations are

$$q^2 T_{kk}^{ij} + 2q^2 T_{qp}^{ij} + q^2 T_{kp}^{ij} + q^2 V_k (k_i, p_j) = 0$$
$$2k_i^2 T_{kk}^{ij} + q^2 T_{kk}^{ij} + 2S_{k_i} S_{p_j} k_i \cdot p_j T_{kk}^{ij} + V_{ik} (k_i, p_j) + q^2 V_q (k_i, p_j) + J_3^{IR} (k_i, p_j) = 0$$
$$2S_{k_i} S_{p_j} k_i \cdot p_j T_{kk}^{ij} + q^2 T_{kk}^{ij} + 2p_j^2 T_{kp}^{ij} - V_{2k} (k_i, p_j) + V_{1k} (k_i, p_j) = 0$$
$$q^2 T_{pp}^{ij} + 2q^2 T_{qp}^{ij} + q^2 T_{kp}^{ij} + q^2 V_p (k_i, p_j) = 0$$
$$2S_{k_i} S_{p_j} k_i \cdot p_j T_{pp}^{ij} + q^2 T_{pp}^{ij} + 2k_i^2 T_{kp}^{ij} - V_{3p} (k_i, p_j) + V_{1p} (k_i, p_j) = 0$$
$$2p_j \cdot p_j T_{pp}^{ij} + q^2 T_{pp}^{ij} + 2S_{k_i} S_{p_j} k_i \cdot p_j T_{pp}^{ij} + V_{1p} (k_i, p_j) + J_3^{IR} (k_i, p_j) + q^2 V_q (k_i, p_j) = 0$$
$$2q^2 T_{qq}^{ij} + q^2 T_{kk}^{ij} + q^2 T_{qp}^{ij} - V_{1k} (k_i, p_j) - V_{1p} (k_i, p_j) + 2q^2 V_q (k_i, p_j) = 0$$
$$q^2 T_{pp}^{ij} + 2k_i^2 T_{pp}^{ij} + 2S_{k_i} S_{p_j} k_i \cdot p_j T_{pp}^{ij} - V_{3q} (k_i, p_j) - V_{1q} (k_i, p_j) - J_3^{IR} (k_i, p_j) = 0$$
$$q^2 T_{qq}^{ij} + 2S_{k_i} S_{p_j} k_i \cdot p_j T_{qq}^{ij} + 2p_j^2 T_{qq}^{ij} - V_{2q} (k_i, p_j) - V_{1q} (k_i, p_j) - J_3^{IR} (k_i, p_j) = 0 \quad (F.244)$$

Together with the tenth equation (F.243), they can be solved and the answer may be expressed in terms $V^{ij}$ and additional vectors arising from various pinches of the box diagrams.

### F.3 The box contribution to the cross-section

Substituting all these solutions to the box amplitude and computing its interference with Born amplitude, we derive the contributions to the unpolarized and polarized cross-section parths from the two-photon exchange diagrams to the differential cross-section of elastic $\mu e$ scattering

$$\frac{d\sigma}{dy} = \frac{d\sigma}{dy}^{\text{BOX}}_{\text{SIURA}} \left[ -4P^{IR}(\mu) \ln(1 - y) \frac{d\sigma}{dy}^{\text{BORN}} + B_{\text{unp}} + P_\mu B_{\text{pol}} \right], \quad (F.245)$$
where unpolarized and polarized non-factorized contributions $B_{\text{unp}}$ and $B_{\text{pol}}$ are:

$$B_{\text{unp}} = \left( 1 - \frac{2}{y} + \frac{2R_\mu}{y} \right) (SK_S + S_1 K_{S_1}) + 4 \left[ -1 + \frac{2}{y} - 2R_\mu \left( 1 + \frac{1}{y_{\mu_1}} \right) \right]$$

$$+ 2 \left( \frac{y_{\mu_1}}{y} - 2R_\mu \right) l_S + 2 \left( \frac{1}{y} - \frac{R_\mu}{y_{\mu_1}} \right) l_{S_1}$$

$$+ 4R_\mu \left[ -1 - \frac{1}{y_{\mu_1}} + \frac{2(2 - y)}{y(y + 4R_\mu)} \right] \ln \frac{Q^2}{m_\mu^2} + 2 \left( -1 + \frac{2}{y} \right) S(K_S - yJ_P)$$

$$+ 2 \left[ -2 + y - 4R_\mu \left( \frac{y_{\mu_1}}{y} + \frac{1 + 2R_\mu}{y + 4R_\mu} \right) \right] S J_K,$$

$$B_{\text{pol}} = \left( -1 + \frac{2}{y} - 2R_\mu \right) (SK_S + S_1 K_{S_1})$$

$$+ 4 \left[ -1 + 2R_\mu \left( -1 + \frac{2}{y} + \frac{1 - R_\mu}{y_{\mu_1}} - R_\mu \right) \right]$$

$$+ 2 \left[ \frac{y_{\mu_1}}{y} (1 + 2R_\mu) - 2R_\mu^2 \right] l_S + 2 \left[ \frac{2R_\mu (1 - R_\mu)}{y_{\mu_1}} - \frac{1 - 2R_\mu}{y} \right] l_{S_1}$$

$$+ 4R_\mu \left( \frac{2}{y} + \frac{1 - R_\mu}{y_{\mu_1}} - \frac{2(1 + 2R_\mu)}{y + 4R_\mu} - R_\mu \right) \ln \frac{Q^2}{m_\mu^2}$$

$$+ 4R_\mu y \left( 1 + 2R_\mu \right) S J_K,$$  \hspace{1cm} (F.246)

with

$$R_\mu = \frac{m_\mu^2}{S},$$

$$Y = \frac{1}{1 + R_\mu}.$$  \hspace{1cm} (F.247)

There are two useful combinations:

$$S_1 K_{S_1} + SK_S = - \ln \frac{Q^4}{SS_1} \ln \frac{S_1}{S} - 2 \text{Li}_2 \left( 1 - \frac{m_\mu^2}{S_1} \right) + 2 \text{Li}_2 \left( 1 + \frac{m_\mu^2}{S} \right) + \frac{\pi^2}{3},$$  \hspace{1cm} (F.248)

$$S(K_S - yJ_P) = - \ln^2 \frac{Q^2}{S} + \frac{1}{2} \ln^2 \frac{Q^2}{m_\mu^2} + 2 \text{Li}_2 \left( 1 + \frac{m_\mu^2}{S} \right) + \frac{\pi^2}{3}.$$  \hspace{1cm} (F.249)

Here

$$J_K = J(m_\mu^2) = \frac{1}{\sqrt{\chi_m^0}} \left[ \ln \frac{m_\mu^2}{Q^2} \ln y_{\mu_2} - 2 \text{Li}_2 (1 - y_{\mu_2}) - \frac{1}{2} \ln^2 y_{\mu_2} + \frac{\pi^2}{3} \right],$$  \hspace{1cm} (F.250)

$$J_P = J(m_e^2).$$  \hspace{1cm} (F.251)

and

$$y_{\mu_1} = \frac{Q^2 + 2m_\mu^2 + \sqrt{\chi_m^0}}{2m_\mu^2},$$  \hspace{1cm} (F.252)

$$y_{\mu_2} = \frac{1}{y_{\mu_1}}.$$  \hspace{1cm} (F.253)
The two-photon exchange contribution simplifies drastically in the DOURA. We list the answer because of its elegance.

\[
\frac{d\sigma_{\text{BOX}}}{dy} = \frac{\alpha^3}{S} Q_{\mu} Q_e \left\{ - P^{IR}(\mu) \left[ \frac{2}{y^2} + (1 + P_e P_\mu) \left( 1 - \frac{2}{y} \right) \right] 8 \ln(1 - y) \\
+ (1 - P_e P_\mu) \left[ \left( 1 - \frac{2}{y} \right) \left( \ln^2(1 - y) - 2 \ln(1 - y) \ln y + \pi^2 \right) \right. \\
+ \frac{2}{y} \ln(1 - y) - 2 \ln y \right] \\
+ 2 \left( 1 - \frac{2}{y} \right) \ln^2(1 - y) - 4 \ln(1 - y) \left\} \right. \\
\] (F.254)

\[ \delta_{y_\mu} = \frac{d\sigma_{\text{QED}}}{dy_\mu} - \frac{d\sigma_{\text{BORN}}}{dy_\mu} - 1, \text{ } (\%) \] (G.255)

G Description of www-Figures

In March 1996, when the FORTRAN code \( \mu \text{ela} \) was completed, a lot of figures for the radiative corrections to the polarized cross-sections and the polarization asymmetry were produced and put to a home-page of the theory group of DESY-Zeuthen, http://www.ifh.de/\textasciitilde bardin.

In this Appendix, we give a short guide to these figures.

All the cross-sections, radiative corrections and asymmetries are shown as function of \( y_\mu \). The units are: cross-sections in microbarns, radiative corrections in percent, asymmetries in absolute, dimensionless units varying from 0 to 1.

Some legends:

- parallel, antiparallel means mutual longitudinal polarization orientations;
- \( P_e \) and \( P_\mu \) are modules of polarization degrees, always written in figures;
- **All corrections** mean that all 12 contributions (1.4) are taken into account;
- **AN** means Analytic calculations;
- when **AN** is not written, the numerical calculations are meant.

The \( \delta_{y_\mu} \) is defined by

\[ \delta_{y_\mu} = \frac{d\sigma_{\text{QED}}}{dy_\mu} - \frac{d\sigma_{\text{BORN}}}{dy_\mu} - 1, \text{ } (\%) \] (G.255)

SIURA-series of 43 figures contain following plots:

- 1-3 Born cross-section and Born asymmetry;
- 4-5 Results of completely integrated analytic calculations without cuts;
- 6-7 Results of completely numerical calculations without cuts;
• 8-17 Illustrate effects of some cuts separately, muon angular cut being treated both analytically and numerically;

• 18-19 are our main result: All corrections, All cuts (four indeed);

• 20-27 illustrate four cases: \( \mu + e \) corrections, \( \mu \) corrections, \( e \) corrections and \( \mu e \) interference corrections, correspondingly, without cuts;

• 28-35 the same, but with all cuts;

• 36-43 is a series of figures with ‘realistic’ \( P_e \) and \( P_\mu \). It is quite senseless for the asymmetry, where \( P_e \) and \( P_\mu \) cancel. We only ‘gain’ an instability due to cancellation of nearly equal numbers.

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