Existence and uniqueness of solutions for coupled system of fractional differential equation by means of topological degree method

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Abstract

In this article, we study the existence and uniqueness of solutions for system of fractional hybrid differential equations of order \( n-1 < \nu \leq n \), without compactness of operator for the given toppled system

\[
\begin{align*}
D^\nu (x(t) - \Theta(t, x(t))) &= \Phi(t, y(t), I^\nu y(t)), \\
D^\nu (y(t) - \Theta(t, y(t))) &= \Phi(t, x(t), I^\nu x(t)), \quad a.e \ t \in \vartheta, \quad n-1 \leq \nu < n, \\
x(0) = x'(0) = x''(0) = \cdots = x^{n-2}(0) = 0, \quad x(1) = \psi(x(\eta)), \\
y(0) = y'(0) = y''(0) = \cdots = y^{\eta-2}(0) = 0, \quad y(1) = \phi(y(\zeta)), 0 < \eta, \zeta < 1
\end{align*}
\]

where \( \psi, \phi : [0, 1] \to \mathbb{R} \) linear and \( D^\nu \) is the R-L fractional derivative of order \( \nu \), \( \vartheta = [0, 1] \), and the functions \( \Theta : \vartheta \times \mathbb{R} \to [0, 1], \Theta(0, 0) = 0, \) and \( \Phi : \vartheta \times \mathbb{R} \times \mathbb{R} \to [0, 1] \) are continuous and satisfy certain conditions. We established sufficient conditions for the existence and uniqueness of solutions using fixed point theorem on topological degree method. We provide an example to justify the obtained results.

Keywords: Coupled systems, Boundary value problems, Fractional differential equations, Existence results, Hybrid fixed point theorems.

Mathematics subject classification: 26A33, 34A08, 35B40.

1 Introduction and Preliminaries

The area of nonlinear fractional differential equations and boundary value problems is still in its initial stages. This area is need to be explored by many aspects. Wang et al. [16], used topological degree method and studied the existence and uniqueness of solutions for a class of nonlocal Cauchy problems considering Caputo’s fractional derivative. Chen et al. [3], studied sufficient conditions for existence results for two point boundary value problem by exploring this problem. Again, Wang et al. [15], studied the two point boundary value problem for fractional differential equations with different boundary conditions. Dhage and Lakshmikantham [4], studied the existence and uniqueness theorems of the solution to the ordinary first-order hybrid differential equation with perturbation of first type. Dhage and Jadhav [6], studied the existence and uniqueness theorems of the solution of the ordinary first-order hybrid differential
equation with perturbation of second type
In [9], author discuss the existence of solutions to hybrid fractional differential equations in both types using the Caputo fractional derivative instead of the classical one in both.

Recently, existence solutions to boundary value problems for coupled systems of fractional order differential equations have also attracted many mathematician, for more detail see [2, 10, 11, 12, 13].

In [14], author used topological degree theory to obtain necessary and sufficient condition for the solution to the following problem

\[
\begin{align*}
D^\nu x(t) &= \Theta(t,x(t)) \text{ a.e } t \in [0,T], \\
x(0) + \Phi(x) &= x_0 \in \mathbb{R}.
\end{align*}
\]

(1.2)

Where \(0 < \nu \leq 1, x_0 \in \mathbb{R}, \Phi \in (C([0,1]), \mathbb{R}) \to \mathbb{R} \) be given function and \(\Theta: [0,1] \times \mathbb{R} \to \mathbb{R}\) is continuous function. Some important results can be found in the literature which use degree theory method for the existence of solutions to boundary value problems [1, 5].

Motivated with the above works, our purpose in this paper is to prove the existence of solution to the following system of fractional hybrid differential equations of order for \(n - 1 < \nu \leq n\)

\[
\begin{align*}
D^\nu (x(t) - \Theta(t,x(t))) &= \Phi(t,y(t),t^\nu y(t)), \\
D^\nu (y(t) - \Theta(t,y(t))) &= \Phi(t,x(t),t^\nu x(t)), \text{ a.e } t \in \emptyset, \\
x(0) &= x(0) = x'(0) = \cdots = x^{n-2}(0) = 0, x(1) = \psi(x(\xi)), \\
y(0) &= y(0) = y'(0) = \cdots = y^{n-2}(0) = 0, y(1) = \phi(y(\zeta)),
\end{align*}
\]

(1.3)

where \(0 < \eta, \zeta < 1\), and \(\psi\) and \(\phi\) are linear mapping. Next, we present some important definitions, propositions and theorem from [5]. Let \(\mathcal{A}\) be a Banach space and \(\mathcal{B} \subset \mathcal{P}(\mathcal{A})\) denotes the family of all bounded subsets of \(\mathcal{A}\).

**Definition 1.1.** The Kuratowski measure of noncompactness \(\kappa: \mathcal{A} \to \mathbb{R}_+\) is defined as

\[
\kappa(A) = \inf\{d > 0 \mid A \cap B(d, A) \neq \emptyset\}.
\]

where \(A \in \mathcal{A}\) admits a finite cover by sets of diameter \(d\).

**Proposition 1.1.** The Kuratowski measure \(\kappa\) satisfy the following properties:

(i) \(\mathcal{B}\) is relatively compact iff \(\kappa(\mathcal{A}) = 0\).

(ii) \(\kappa\) is a seminorm, i.e., \(\kappa(\lambda \mathcal{A}) = |\lambda| \kappa(A), \lambda \in \mathbb{R}\) and \(\kappa(\mathcal{A} + \mathcal{A}) \leq \kappa(\mathcal{A}) + \kappa(\mathcal{A})\).

(iii) \(A_1 \subset A_2\) implies \(\kappa(A_1) \leq \kappa(A_2)\), \(\kappa(A_1 \cup A_2) = \max\{\kappa(A_1), \kappa(A_2)\}\).

(iv) \(\kappa(\text{conv}A) = \kappa(A)\).

(v) \(\kappa(A) = \kappa(A)\).

**Definition 1.2.** Let the function \(\mathcal{A}_{11}: \Psi \to \mathcal{A}\) be a continuous bounded map, where \(\Psi \subset \mathcal{A}\). Then \(\mathcal{A}_{11}\) is \(\kappa\)-Lipschitz if there exists \(k \geq 0\) such that

\[
\kappa(\mathcal{A}_{11}(Y)) \leq k \kappa(Y), \forall Y \subset \Psi \text{ bounded.}
\]

Further, \(\mathcal{A}_{11}\) will be \(\kappa\)-contraction if \(k < 1\).

**Definition 1.3.** The function \(\mathcal{A}_{11}\) is \(\kappa\)-condensing if

\[
\kappa(\mathcal{A}_{11}(Y)) < \kappa(Y), \forall Y \subset \Psi \text{ bounded with } \kappa(Y) > 0.
\]

In other words, \(\kappa(\mathcal{A}_{11}(Y)) \geq \kappa(Y)\) implies \(\kappa(Y) = 0\).

Here, we denote the class of all strict \(\kappa\)-contractions \(\mathcal{A}_{11}: \Psi \to \mathcal{A}\) by \(\Theta(\mathcal{C}_k(\Psi))\) and the class of all \(\kappa\)-condensing maps \(\mathcal{A}_{11}: \Psi \to \mathcal{A}\) by \(\mathcal{C}_k(\Psi)\).

**Remark 1.1.** \(\Theta(\mathcal{C}_k(\Psi)) \subset \mathcal{C}_k(\Psi)\) and every \(\mathcal{A}_{11} \in \mathcal{C}_k(\Psi)\) is \(\kappa\)-Lipschitz with constant \(k = 1\).

Moreover, we recall that \(\mathcal{A}_{11}: \Psi \to \mathcal{A}\) is Lipschitz if there exists \(k > 0\) such that

\[
\|\mathcal{A}_{11}(x) - \mathcal{A}_{11}(y)\| \leq k|x - y|, \quad \forall x, y \in \Psi
\]

and that if \(k < 1\), then \(\mathcal{A}_{11}\) is a strict contraction.
Proposition 1.2. Assume the maps \( \varphi_{11}, \varphi_{22} : \Psi \rightarrow \Lambda \) are \( \kappa \)-Lipschitz with constants \( k \) and \( k' \) respectively, then \( \varphi_{11} + \varphi_{22} : \Psi \rightarrow \Lambda \) are \( \kappa \)-Lipschitz with constants \( k + k' \).

Proposition 1.3. If \( \varphi_{11} : \Psi \rightarrow \Lambda \) is compact, then \( \varphi_{11} \) is \( \kappa \)-Lipschitz with constant \( k = 0 \).

Proposition 1.4. If \( \varphi_{11} : \Psi \rightarrow \Lambda \) is Lipschitz with constant \( k \), then \( \varphi_{11} \) is \( \kappa \)-Lipschitz with the same constant \( k \).

Here, we present basic properties of topological degree method. For more details, we refer the readers to Isaia [7]. Let

\[
F = \{(I - \varphi_{11}, \Psi, y) : \Psi \subset \Lambda \text{ open and bounded, } \varphi_{11} \in C_{c}(\Psi), y \in \Lambda \setminus (I - \varphi_{11})(\partial \Psi)\},
\]

be the family of the admissible triplets.

**Theorem 1.1.** There exists one degree function \( D : F \rightarrow \mathbb{Z} \) which satisfies the following properties:

(D1) **Normalisation:** \( D(I, \Psi, y) = 1 \) for every \( y \in \Psi \).

(D2) **Additivity on domain:** For every disjoint, open sets \( \Psi_1, \Psi_2 \subset \Psi \) and every \( y \) not belonging to \( (I - \varphi_{11})(\Psi_1 \cup \Psi_2) \), we have

\[
D(I - \varphi_{11}, \Psi, y) = D(I - \varphi_{11}, \Psi_1, y) + D(I - \varphi_{11}, \Psi_2, y).
\]

(D3) **Invariance under homotopy:** \( D(I - H((t, \cdot), \Psi, y(t))) \) is independent of \( t \in [0, 1] \) for every continuous, bounded map \( H : [0, 1] \times \Psi \rightarrow \Lambda \) which satisfies

\[
k(H([0, 1] \times \Psi)) < k(\Psi), \quad \forall \, \Psi \subset \bar{\Psi} \text{ with } k(\Psi) > 0
\]

and every continuous function \( y : [0, 1] \rightarrow \Psi \) which satisfies

\[
y(t) \neq x - H(t, x), \quad \forall \, t \in [0, 1], \quad \forall \, x \in \partial \Psi.
\]

(D4) **Existence:** \( D(I - \varphi_{11}, \Psi, y) \neq 0 \) implies

\[
y \in (I - \varphi_{11})(\Psi).
\]

(D5) **Exclusion:** \( D(I - \varphi_{11}, \Psi, y) = D(I - \varphi_{11}, \Psi_1, y) \) for every open set \( \Psi_1 \subset \Psi \) and every \( y \) not belongs to \( (I - \varphi_{11})(\Psi \setminus \Psi_1) \).

**Theorem 1.2.** Let \( \varphi_{11} : \Lambda \rightarrow \Lambda \) be \( \kappa \)-condensing and

\[
\Theta_{11} = \{x \in \Lambda : \exists \, \lambda \in [0, 1] \text{ such that } x = \lambda \varphi_{11} x\}.
\]

If \( \Theta_{11} \) is a bounded set in \( \Lambda \), so there exists \( r > 0 \) such that \( \Theta_{11} \subset \Upsilon_r(0) \), then

\[
D(I - \lambda \varphi_{11}, \Upsilon_r(0), 0) = 1, \quad \forall \, \lambda \in [0, 1].
\]

Consequently, \( \varphi_{11} \) has at least one fixed point and the set of the fixed points of \( \varphi_{11} \) lies in \( \Upsilon_r(0) \).

**Definition 1.4.** The form of Riemann-Liouville fractional integral operator of order \( \nu > 0 \) of function \( \Theta \in L^1(\mathbb{R}^+) \) is defined as

\[
I^\nu \Theta(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t - s)^{\nu - 1} \Theta(s)ds.
\]

**Definition 1.5.** Let \( \nu \) be a positive real number, such that \( m - 1 \leq \nu < m, m \in \mathbb{N} \) and \( \Theta^{(m)}(x) \) exists a function of class \( C \). Then Caputo’s fractional derivative of \( \Theta \) is defined as

\[
D^\nu \Theta(t) = \frac{1}{\Gamma(m - \nu)} \int_0^t (t - s)^{m - \nu - 1} \Theta^{(m)}(s)ds.
\]

**Definition 1.6.** The Riemann-Liouville fractional derivative of order \( \nu > 0 \) of continuous function \( \Theta : \mathbb{R}^+ \rightarrow \mathbb{R}, \Theta \in L^1(\mathbb{R}^+) \) is defined as

\[
D^\nu \Theta(t) = \frac{1}{\Gamma(n - \nu)} \left( \frac{d}{dt} \right)^m \int_0^t (t - s)^{m - \nu - 1} \Theta(s)ds.
\]

**Lemma 1.1.** Let \( 0 < \nu < 1 \) and \( \Theta \in L^1((0, 1)) \). Then

\[
D^\nu I^\nu \Theta(t) = \Theta(t)
\]

hold.

\[
I^\nu D^\nu \Theta(t) = \Theta(t) - \frac{[D^{\nu - 1} \Theta](t)]_{t=0}{\Gamma(\nu)}^{\nu - 1}.
\]

hold almost everywhere on \( \Theta \).
Lemma 1.2. [8] The fractional differential equation of order $q > 0$

$$^{c}D^{q}y(t) = 0, \ n-1 < q \leq n,$$

has a unique solution of the form $y(t) = \beta_{0} + \beta_{1}t + \beta_{2}t^{2} + \ldots + \beta_{n-1}t^{n-1}$, where $\beta_{i} \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$.

Lemma 1.3. [8] The following result holds for fractional differential equations

$$^{Ic}D^{q}y(t) = y(t) + \beta_{0} + \beta_{1}t + \beta_{2}t^{2} + \ldots + \beta_{n-1}t^{n-1},$$

for arbitrary $\beta_{k} \in \mathbb{R}, \ i = 0, 1, 2, \ldots, n-1$.

Lemma 1.4. Assume that hypothesis (C0) holds. Then, for any $y \in C(\bar{\theta}, \mathbb{R})$ and $0 < \nu < 1$, the function $\Theta \in C(\bar{\theta}, \mathbb{R})$ with $\Theta(0,0) = 0$ and $\frac{\partial \Theta(t,x(t))}{\partial t}|_{t=0} = 0$, $i = 1, 2, \ldots, n-2$. Then the unique solution of the toppled system of FHDE

$$D^{\nu}(x(t) - \Theta(t,x(t))) = \Phi(t,x(t),I^{\nu}y(t)),
D^{\nu}(y(t) - \Theta(t,y(t))) = \Phi(t,x(t),I^{\nu}x(t)), \ a.e \ t \in \bar{\theta}, \ n-1 \leq \nu < n,$$

$$x(0) = x'(0) = \ldots = x^{(\nu-2)}(0) = 0, \ x(1) = \psi(x(\eta)),
\ y(0) = y'(0) = \ldots = y^{(\nu-2)}(0) = 0, \ y(1) = \phi(y(\zeta)),
$$

where $0 < \eta < 1$ and $\psi$ and $\phi$ are linear functions.

Proof. Let $x$ be a solution of the Cauchy problem (1.7). We apply the Riemann-Liouville fractional integral $I^{\nu}$ on both sides of (1.7).

$$I^{\nu}[^{c}D^{\nu}u(t)] = u(t) + \beta_{0} + \beta_{1}t + \beta_{2}t^{2} + \ldots + \beta_{n-1}t^{n-1},$$

where $n = [\nu] + 1$ and $\beta_{k} \in \mathbb{R}^{+}$ and $\nu$ is integral operator of fractional order.

$$I^{\nu}[^{c}D^{\nu}[x(t) - \Theta(t,x(t))]] = I^{\nu}[\Phi(t,x(t),I^{\nu}y(t))],
\ x(t) - \Theta(t,x(t)) = I^{\nu}[\Phi(t,x(t),I^{\nu}y(t))] + \beta_{0} + \beta_{1}t + \beta_{2}t^{2} + \ldots + \beta_{n-1}t^{n-1},
\ x(t) = \Theta(t,x(t)) + I^{\nu}[\Phi(t,x(t),I^{\nu}y(t))] + \beta_{0} + \beta_{1}t + \beta_{2}t^{2} + \ldots + \beta_{n-1}t^{n-1},
\ x'(t) = \Theta_{1}(t,x(t)) + I^{\nu-1}[\Phi(t,y(t),I^{\nu}y(t))] + \beta_{1} + 2\beta_{2}t + \ldots + (n-1)\beta_{n-1}t^{n-2},
\ x''(t) = \Theta_{2}(t,x(t)) + I^{\nu-2}[\Phi(t,y(t),I^{\nu}y(t))] + 2\beta_{2} + 3\beta_{3}t + \ldots + (n-1)(n-2)\beta_{n-1}t^{n-3},
\ x^{(n-2)}(t) = \Theta_{n-2}(t,x(t)) + I^{\nu-(n-2)}[\Phi(t,y(t),I^{\nu}y(t))] + (n-2)(n-3)\beta_{n-2} + \ldots + (n-1)(n-2)\beta_{n-1},$$

If we assume $\Theta(0,0) = 0$, from 2nd equation of above system (1.9), $c_{0} = 0$. If assume $\frac{\partial \Theta(t,x(t))}{\partial t}|_{t=0} = 0$, then $c_{1} = 0$ using 3rd equation of above system (1.9). Similarly in this way in last if assume $\frac{\partial^{n-2} \Theta(t,x(t))}{\partial t^{n-2}}|_{t=0} = 0$, then $c_{n-2} = 0$. Therefore, using values of $c_{0}, c_{1}, \ldots, c_{n-2}$, we have

$$x(t) = \Theta(t,x(t)) + I^{\nu}[\Phi(t,y(t),I^{\nu}y(t))] + \beta_{n-1}t^{n-1}.$$
for arbitrary $u, v \in C(\varnothing, R)$. (A2) There exists a constant $C_{\psi, m_{\psi}, q_{1}}, q_{1} \in [0, 1], C_{\psi, m_{\psi}} > 0$ such that

$$|\psi(u)| \leq C_{\psi}||u||^{q_{1}}_{C} + m_{\psi}, |\psi(u)| \leq C_{\psi}||u||^{q_{1}}_{C} + m_{\psi}, \text{ for } u \in C(\varnothing, R).$$

(A3) For arbitrary $u, v \in C(\varnothing, R)$, there exists a constant $K_{\Theta}$ such that

$$|\Theta(t, u) - \Theta(t, v)| \leq K_{\Theta}|u - v|.$$
(A5) There exists a constant $L_0 > 0$ such that
$$|\Phi(t,x_2,y_2) - \Phi(t,x_1,y_1)| \leq L_0(|x_2 - x_1| + |y_2 - y_1|).$$

To show that equation (0.1) has at least one solution $u \in C(\bar{\theta}, \mathbb{R})$ based on assumptions (A1) – (A5), we define the following operators;

Define operators $A_1, A_2 : \Lambda \to \Lambda$ and $B_1, B_2 : \Lambda \to \Lambda$ $\mathcal{O}_{11}, \mathcal{O}_{22}, T : \Lambda \times Y \to \Lambda \times Y$ as

$$
\begin{align*}
\mathcal{O}_{11}(x(t),y(t)) &= (A_1 x(t), A_2 y(t)) \\
\mathcal{O}_{22}(x(t),y(t)) &= (B_1 x(t), B_2 y(t)) \\
T(x(t),y(t)) &= \mathcal{O}_{11}(x(t),y(t)) + \mathcal{O}_{22}(x(t),y(t)),
\end{align*}
$$

where

$$
\begin{align*}
A_1 x(t) &= \Theta(t,x(t)) + r^{n-1} \left[ \frac{\psi \left( \Theta(x_1) \right) - \Theta(1,x(1))}{1 - \psi(\eta^{n-1})} \right] \\
B_1 x(t) &= I^\nu \Phi(t,x(t),I^\nu x(t)) + r^{n-1} \left[ \frac{\psi \left( I^\nu \Phi(\eta,x(\eta)),I^\nu x(\eta) \right) - I^\nu \Phi(1,x(1)),I^\nu x(1))}{1 - \phi(\xi^{n-1})} \right].
\end{align*}
$$

\begin{align*}
A_2 y(t) &= \Theta(t,y(t)) + r^{n-1} \left[ \frac{\phi \left( \Theta(y_1) \right) - \Theta(1,y(1))}{1 - \phi(\zeta^{n-1})} \right] \\
B_2 y(t) &= I^\nu \Phi(t,y(t),I^\nu y(t)) + r^{n-1} \left[ \frac{\phi \left( I^\nu \Phi(\eta,y(\eta)),I^\nu y(\eta) \right) - I^\nu \Phi(1,y(1)),I^\nu y(1))}{1 - \psi(\eta^{n-1})} \right].
\end{align*}

Lemma 1.5. The operator $\mathcal{O}_{11} : \Lambda \times Y \to \Lambda \times Y$ is Lipschitz under assumption $\|x(\eta) - x'(\eta)\| \leq \|x(t) - x'(t)\|$. Moreover, the operator $\mathcal{O}_{11}$ satisfies the following growth condition:

$$
\|\mathcal{O}_{11}(x,y)\| \leq C\|x,y\|^{\nu} + M, \quad (1.17)
$$

under assumption

$$
\frac{\psi \left( \Theta(x_1) \right) - \Theta(1,x(1))}{1 - \psi(\eta^{n-1})} \leq L_1, \quad \frac{\psi \left( \Theta(y_1) \right) - \Theta(1,y(1))}{1 - \phi(\zeta^{n-1})} \leq L_2.
$$
Proof. For \((x,y),(x',y')\) \(\in A \times Y\), Using (A1) and (A3) and let \(\Delta = 1 - \psi(\eta^{n-1})\), we have

\[
|A_1x(t) - A_1x'(t)| = \left| \Theta(t,x(t)) + t^{n-1} \left[ \frac{\psi(\Theta(\eta,x(\eta))) - \Theta(1,x(1))}{1 - \psi(\eta^{n-1})} \right] - \Theta(t,x'(t)), \right| \\
= t^{n-1} \left[ \frac{\psi(\Theta(\eta,x(\eta))) - \Theta(1,x(1))}{1 - \psi(\eta^{n-1})} \right] - \Theta(t,x(t)) \\
\leq |\Theta(t,x(t)) - \Theta(t,x'(t))| + \frac{\psi(\Theta(\eta,x(\eta))) - \Theta(1,x(1)) - \psi(\Theta(\eta,x'(\eta))) + \Theta(1,x'(1))}{\Delta} \\
= |\Theta(t,x(t)) - \Theta(t,x'(t))| + \frac{\psi(\Theta(\eta,x(\eta))) - \Theta(1,x(1)) - \psi(\Theta(\eta,x'(\eta))) + \Theta(1,x'(1))}{\Delta} \\
\leq |\Theta(t,x(t)) - \Theta(t,x'(t))| + \frac{k_{y'}(\Theta(\eta,x(\eta)) - \Theta(\eta,x'(\eta)))}{\Delta} + k_{o}|x'(1) - x(1)| \\
\leq k_{x}[|x(t) - x'(t)|] + \frac{|k_{o}K_{y'}||x(\eta) - x'(\eta)|| + |k_{o}||\psi(x(\eta)) - \psi(x'(\eta))|||}{\Delta} \\
\leq k_{x}[|x(t) - x'(t)|] + \frac{|k_{o}K_{y'}||x(\eta) - x'(\eta)|| + |k_{o}K_{y'}||\psi(x(\eta)) - \psi(x'(\eta))|||}{\Delta} \\
\leq k_{x}[|x(t) - x'(t)|] + |K'| |x(t) - x'(t)|| + \frac{|K'||(x(t))(x'(t))||}{\Delta} \\
\leq k_{x}[|x(t) - x'(t)|] \\
\]

where \(K' = \frac{k_{o}K_{y'}}{\Delta}\) and 
\(k = \max\{k_{o}, K'\}\)

Similarly, we can show that \(A_2\) is Lipschitz under same assumption replacing \(\psi\) by \(\phi\). Thus proved that \(\sigma_{11}\) is \(\nu\)-Lipschitz with constant \(K\). Now to prove the growth condition. Consider

\[
\|\sigma_{11}(x,y)\| \leq \|(A_1x(t),A_2y(t))\| \\
\]

Now,

\[
|A_1x(t)| = \left| \Theta(t,x(t)) + t^{n-1} \left[ \frac{\psi(\Theta(\eta,x(\eta))) - \Theta(1,x(1))}{1 - \psi(\eta^{n-1})} \right] \right| \\
\leq |\Theta(t,x(t))| + t^{n-1} \left[ \frac{\psi(\Theta(\eta,x(\eta))) - \Theta(1,x(1))}{1 - \psi(\eta^{n-1})} \right] \\
\leq |\Theta(t,x(t))| + t^{n-1} \left[ \frac{\psi(\Theta(\eta,x(\eta))) - \Theta(1,x(1))}{1 - \psi(\eta^{n-1})} \right] \\
\leq C_{1o}^y|y|^q + m_{o} + \frac{\psi(\Theta(\eta,x(\eta))) - \Theta(1,x(1))}{1 - \psi(\eta^{n-1})} |(1.19)\]

Similarly we can show that

\[
|A_2y(t)| \leq C_{1o}^y|y|^q + m_{o} + \frac{\phi(\Theta(\eta,y(\zeta))) - \Theta(1,y(1))}{1 - \psi(\eta^{n-1})} |(1.20)\]

Consider \(C = C_{1o}^y\) and \(M = \max\{m_{o},L_1,L_2\}\). From 1.19 and 1.20 we get the required growth condition. □
Lemma 1.6. Under assumption $A_3$ the operator $\mathcal{D}_2 : \Lambda \times Y \to \Lambda \times Y$ is continuous and satisfies the following growth condition for every $(x,y) \in \Lambda \times Y$

$$\|\mathcal{D}_2(x,y)\| \leq C_V \| (x,y) \|^{p_0} + \| (I^\nu x, I^\nu y) \|^{p_2} + M, \quad q_1, q_2 \in (0, 1)$$

(1.21)

under assumption

$$\frac{\psi \left( I^\nu \Phi(x(s), y(s)) \right) - I^\nu \Phi(x(1), y(1))}{1 - \phi(\zeta^{n-1})} \leq N_1$$

$$\frac{\phi \left( I^\nu \Phi(x(s), y(s)) \right) - I^\nu \Phi(x(1), y(1))}{1 - \psi(\eta^{n-1})} \leq N_2.$$

Proof. Let $\{z_n = (x_n, y_n)\}$ be a sequence of a bounded set $B_k = \{ (x, y) \in \Lambda \times Y \}$ such that $z_n \to z = (x, y)$ as $n \to \infty$ in $B_k$. We need to show that $\|\mathcal{D}_2 z_n - \mathcal{D}_2 z\| \to 0$.

$$|B_1 x_n(t) - B_1 x(t)| = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} \Phi(x_n(x(s)), I^\nu x_n(s)))ds$$

$$+ \int_0^t (t-s)^{\nu-1} \left[ -I^\nu \Phi(1, x_n(1), I^\nu x_n(1)) + \psi \left( I^\nu \Phi(x(s), y(s)) \right) \right] ds$$

$$- \int_0^t (t-s)^{\nu-1} \Phi(x(t), I^\nu x(t)))ds$$

$$\leq \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} \Phi(x_n(x(s)), I^\nu x_n(s)))ds - \int_0^t (t-s)^{\nu-1} \Phi(x(t), I^\nu x(t)))ds$$

$$+ \int_0^t (t-s)^{\nu-1} \left[ \psi \left( I^\nu \Phi(x(s), y(s)) \right) - \psi \left( I^\nu \Phi(x(s), y(s)) \right) \right] ds$$

From the continuity of $\Phi$, it follows that

$$\Phi(x_n(x(s), I^\nu x_n(s))) \to \Phi(x(x(s), I^\nu x(s))$$

as $n \to \infty$ for $t \in \Theta$, using $A_4$, we obtain

$$| (t-s)^{\nu-1} \Phi(x_n(x(s), I^\nu x_n(s))) - \Phi(x(x(s), I^\nu x(s))) \| \leq (t-s)^{\nu-1} 2(C\|k\|^{q_1} + C\|k\|^{q_2} + m_{\Phi}).$$

which implies the integrability for $s, t \in \Theta$ and by using the Lebesgue dominated convergence theorem, we obtain

$$\int_0^t (t-s)^{\nu-1} \left( \Phi(x_n(x(s), I^\nu x_n(s))) - \Phi(x(x(s), I^\nu x(s))) \right) ds \to 0.$$

Which implies that

$$|B_1 x_n(t) - B_1 x(t)| \to 0 \text{ as } n \to \infty.$$
Similarly, under the same assumption replacing $\psi$ by $\phi$ we can show that

$$|B_{2Y_1}(t) - B_{2Y}(t)| \to 0.$$  

Which implies that

$$\|\mathcal{O}_{22}\zeta_n - \mathcal{O}_{22}\zeta\| \to 0.$$  

Now, to prove the growth condition.

$$|B_1(x(t))| = \left| I^x\Phi(t, x(t), I^x x(t)) + t^{n-1} \left[ \frac{\psi(t)\Phi(t, x(t), I^x x(t))}{1 - \phi(t)} - 1 \right] \right|$$

$$\leq \frac{1}{\Gamma(v)} \left| \int_0^s (s - t)^{v-1} (\Phi(s, x(s), I^x x(s))) ds \right| + \left| \frac{\psi(t)\Phi(t, x(t), I^x x(t))}{1 - \phi(t)} - 1 \right|$$

$$\leq \frac{1}{\Gamma(v + 1)} \left| \Phi(s, x(s), I^x x(s)) \right| + \left| \frac{\psi(t)\Phi(t, x(t), I^x x(t))}{1 - \phi(t)} - 1 \right|$$

$$|B_1(x(t))| \leq \frac{1}{\Gamma(v + 1)} \left[ (C^1 |x|^{q_1} + |C^2| |x|^2 |x|^{q_1} + m_\phi) + N_1 \right] \quad (1.22)$$

Similarly

$$|B_2(x(t))| \leq \frac{1}{\Gamma(v + 1)} \left[ (C^1 |y|^{q_1} + |C^2| |y|^2 |y|^{q_1} + m_\phi) + N_2 \right] \quad (1.23)$$

From equations 1.22 and 1.23, we have

$$\|\mathcal{O}_{22}(x, y)\| \leq C_v \left[ (|x|^{q_1} + ||y||^{q_2}) + M \right]$$

where $C_v = \max\{\frac{1}{\Gamma(v + 1)} C^1, \frac{1}{\Gamma(v + 1)} C^2\}$ and $M = \max\{N_1, N_2, \frac{1}{\Gamma(v + 1)} m_\phi\}$.

**Lemma 1.7.** The operator $\mathcal{O}_{22} : \Lambda \times Y \to \Lambda \times Y$ is compact

Consider a bounded set $\Lambda \subset D_1 \subset \Lambda \times Y$. We need to prove that $\mathcal{O}_{22}(\Lambda)$ is relatively compact in $\Lambda \times Y$. For any $z_0 = (x_0, y_0) \in \Lambda \subset D_1$, the growth condition 1.21 implies that

$$\|\mathcal{O}_{22}(x_0, y_0)\| \leq C_v (k^{q_1} + k^{q_2}) + M_\theta, \quad q_1 \in (0, 1).$$
That is \( \mathfrak{G}_{22}(A) \) is uniformly bounded. Now, we have to show that \( \mathfrak{G}_{22} \) is equi-continuous. For this, let \( 0 \leq t_1 < t_2 \leq 1 \), then

\[
\|B_1 x(t_1) - B_1 x(t_2)\| = \left| \frac{1}{\Gamma(\nu)} \int_0^{t_1} (t_1 - s)^{\nu-1} \Phi(x(s), t_1) \int_0^{t_2} (t_2 - s)^{\nu-1} \Phi(x(s), t_2) ds \right|
\]

\[
+ \int_0^{t_2} \left[ \frac{-t_2^\nu \Phi(1, x(1), t_2^\nu x(1)) + \psi \left( I^\nu \Phi(\eta, x(\eta), t_1^\nu x(\eta)) \right)}{1 - \phi(\zeta^\nu - 1)} \right] ds
\]

\[
- \int_0^{t_2} \left[ \frac{-t_2^\nu \Phi(1, x(1), t_2^\nu x(1)) + \psi \left( I^\nu \Phi(\eta, x(\eta), t_1^\nu x(\eta)) \right)}{1 - \phi(\zeta^\nu - 1)} \right] ds
\]

\[
\leq \left| \frac{1}{\Gamma(\nu)} \int_0^{t_1} (t_1 - s)^{\nu-1} \Phi(x(s), t_1) \int_0^{t_2} (t_2 - s)^{\nu-1} \Phi(x(s), t_2) ds \right|
\]

\[
+ \int_0^{t_2} \left[ \frac{-t_2^\nu \Phi(1, x(1), t_2^\nu x(1)) + \psi \left( I^\nu \Phi(\eta, x(\eta), t_1^\nu x(\eta)) \right)}{1 - \phi(\zeta^\nu - 1)} \right] ds
\]

\[
- \int_0^{t_2} \left[ \frac{-t_2^\nu \Phi(1, x(1), t_2^\nu x(1)) + \psi \left( I^\nu \Phi(\eta, x(\eta), t_1^\nu x(\eta)) \right)}{1 - \phi(\zeta^\nu - 1)} \right] ds
\]

\[
\leq \left| \frac{1}{\Gamma(\nu)} \int_0^{t_1} (t_1 - s)^{\nu-1} \Phi(x(s), t_1) \int_0^{t_2} (t_2 - s)^{\nu-1} \Phi(x(s), t_2) ds \right|
\]

\[
+ \int_0^{t_2} \left[ \frac{-t_2^\nu \Phi(1, x(1), t_2^\nu x(1)) + \psi \left( I^\nu \Phi(\eta, x(\eta), t_1^\nu x(\eta)) \right)}{1 - \phi(\zeta^\nu - 1)} \right] ds
\]

\[
- \int_0^{t_2} \left[ \frac{-t_2^\nu \Phi(1, x(1), t_2^\nu x(1)) + \psi \left( I^\nu \Phi(\eta, x(\eta), t_1^\nu x(\eta)) \right)}{1 - \phi(\zeta^\nu - 1)} \right] ds
\]

Which implies

\[
\|B_1 x(t_1) - B_1 x(t_2)\| \rightarrow 0 \text{ as } t_2 \rightarrow t_1
\]

Similarly it is easy to show

\[
\|B_2 y(t_1) - B_2 y(t_2)\| \rightarrow 0 \text{ as } t_2 \rightarrow t_1
\]
Therefore, from
\[
\|\sigma_{22}(x_n,y_n) - \sigma_{22}(x,y)\| = \| (B_1 x(t), B_2 y(t)) - (B_1 x_n(t), B_2 y_n(t)) \|
\]
we have
\[
\|\sigma_{22}(x_n,y_n) - \sigma_{22}(x,y)\| \to 0
\]
Hence, \(\sigma_{22}\) is equicontinuous. Therefore, \(\sigma_{22}(\cdot)\) is relatively compact in \(C(\bar{\theta}, \mathbb{R})\) by Arzela-Ascoli theorem. Furthermore, by proposition (1.3), \(\sigma_{22}\) is \(\nu\)-Lipschitz with constant zero.

From now onward, we will prove our main results.

**Theorem 1.3.** Assuming that (A1)-(A5) holds, then equation 0.1 has at least one solution \(u \in C(\bar{\theta}, \mathbb{R})\). Moreover, the set of solutions for 0.1 is bounded in \(C(\bar{\theta}, \mathbb{R})\).

**Proof.** From Lemma 1.5 \(\sigma_{11}\) is \(\kappa\) Lipschitz with constant \(K\) and by Lemma 1.7 \(\sigma_{22}\) is \(\kappa\) Lipschitz with constant 0. \(T\) is \(\kappa\) Lipschitz with constant \(K\). Consider the following set
\[
S_0 = \{ (u,v) \in \Lambda \times Y : \exists \lambda \in [0,1] \text{ such that } u = \lambda \mathcal{T}(u,v) \}.
\]
We need to prove that \(S_0\) is bounded in \(C(\bar{\theta}, \mathbb{R})\). For this, consider
\[
\| (u,v) \| = \| \lambda \mathcal{T}(u,v) \| = \lambda \| T(u,v) \| \leq \lambda (\| \sigma_{11}(u,v) \| + \| \sigma_{22}(u,v) \|)
\]
Which implies that
\[
\| u \|_\nu \leq \lambda \left( \| \sigma_{11}(u,v) \| + \| \sigma_{22}(u,v) \| \right)
\]
Similarly
\[
\| v \| \leq \lambda \left( \| \sigma_{22}(u,v) \| + \| \sigma_{22}(v,v) \| \right)
\]
Using (1.24) and (1.25), together with \(q_1 < 1\) show that \(S_0\) is bounded \(\Lambda \times Y\). Therefore, by Theorem 1.2 \(T\) has at least one fixed point and the set of fixed points is bounded in \(C(\bar{\theta}, \mathbb{R})\).

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