Multifractal Measures in Iterative Maps

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We investigate chaotic and multi-fractal properties of a two parameter map of the unit interval onto itself – the Kim-Kong map. These results are compared with similar properties in well known one parameter maps of the unit interval onto itself.

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Over the past few decades, following the discovery of chaotic behaviour in deterministic nonlinear dynamical systems, there has been considerable interest in the study of nonlinear differential equations and nonlinear difference equations with applications ranging across physics, chemistry, biology and economics. In many of these studies a knowledge of the properties of simple one-dimensional maps such as the logistic map, the dissipative standard map, the Lorentz map, the Hénon map, the tent map, the quadratic map, etc., has provided fundamental insights\[1−5\]. This has been particularly the case in understanding deterministic chaos, understanding various transitions to chaos as a control parameter is varied, and identifying universality classes for the onset of chaos similar to universality in the theory of critical phenomena.

Most detailed studies of one-dimensional maps have been confined to maps with a single control parameter and many of the chaotic and universal features of these maps are now well understood. For example, Feigenbaum\[5\] identified universal values for convergence ratios in the period-doubling route to chaos in certain one-dimensional maps. Universality has also been identified in the quasiperiodic route to chaos and in intermittency\[3\]. Two of the most useful quantitative measurements of dynamical properties of single parameter one-dimensional maps have been furnished by Lyapunov exponents and fractal dimensions. In this paper we present the results of our measurements of these and related quantities, multifractal spectrum, for a one-dimensional map with two control parameters - the Kim-Kong map\[6\]. The motivation for this work is twofold; firstly explore multifractal analysis for discrete dynamical systems and secondly to investigate the possibly competing effects of two control parameters in discrete dynamical systems.

The majority of the applications of multifractal analysis\[7−9\] to date have been to physical systems. For example in recent applications, Kim and Kong used the box-counting method to compute generalized dimensions and scaling exponents for mountain heights\[10\] and sea-bottom depths\[11\] in Korea.
The Kim-Kong map is a two parameter map of the unit interval onto itself defined by

\[ x(n + 1) = f(x(n), \gamma, \beta) \]
\[ = \gamma \exp\left[-\beta (\log x(n))^2\right]\left[1 - \exp\left[-\beta (\log x(n))^2\right]\right] \]

with \( x(n) \in (0, 1) \), where \( \gamma \) and \( \beta \) are the two control parameters. To begin with we note that the Kim-Kong map is a dissipative unimodal map with a locally quadratic maximum. The maximum value is \( f(x) = \frac{\gamma}{4} \) which occurs at \( x = \exp\left(-\sqrt{\frac{\ln 2}{\beta}}\right) \). A series expansion about the maximum yields

\[ f(x) \approx \frac{\gamma}{4} - \gamma \beta \exp\left(-2 \sqrt{\frac{\ln 2}{\beta}}\right) (x - \exp\left(-2 \sqrt{\frac{\ln 2}{\beta}}\right))^2, \]

and the Schwarzian derivative is negative for all \( x \in (0, 1) \). If \( \frac{\gamma}{4} > \exp\left(-\sqrt{\frac{\ln 2}{\beta}}\right) \) then the map has two fixed points \( x_1 \) and \( x_2 \) on the interval \((0, 1)\) with one on either side of the maximum, i.e., \( 0 < x_1 < \exp\left(-\sqrt{\frac{\ln 2}{\beta}}\right) < x_2 < 1 \). The map also has a fixed point at \( x = 0 \) and while this is outside the domain \((0, 1)\) it is an attracting fixed point for \( x \in (0, x_1) \). Numerical underflow can be avoided by selecting parameter values \( \gamma \) and \( \beta \) together with the initial point \( x(0) \) so that the two conditions \( x_1 < f\left(\frac{\gamma}{4}\right) \) and \( x_1 < x(0) < \frac{\gamma}{4} \) are satisfied. Future iterates will be confined to \( x \in (x_1, \frac{\gamma}{4}) \).

The function \( f(x) \) is displayed in Fig. 1 for \( \gamma = 3.78 \) and several values of \( \beta = 0.2, 0.6, 1.0, 1.5, 2.0 \). From this figure it can be seen that the main effect of increasing \( \beta \) at a fixed value of \( \gamma \) is to shift the position of the maximum to higher values of \( x \). In contrast to the logistic map, the Kim-Kong map is not symmetric about the maximum. In Fig. 2 we have displayed an empirical unimodal one-dimensional map that was obtained from Belousov-Zhabotinskii reaction experiments\[12–14\]. By comparing Figs. 1 and 2 it can be seen that the functional form of the Kim-Kong map for \( 0 < \beta < 0.2 \) and \( \gamma = 3.78 \) is similar to the experimentally obtained map.

The Kim-Kong map undergoes a period doubling cascade to chaos at fixed \( \beta \) as \( \gamma \) is increased. An example of the period doubling cascade is shown in in Fig. 2 of [6] for \( \beta = 0.2 \).
and $0 < \gamma < 3.78$. Denoting $\gamma_k$ as the parameter at which the period $2^{k-1}$ cycle becomes unstable we compute the sequence

$$\delta_k = \frac{\gamma_{k+1} - \gamma_k}{\gamma_{k+2} - \gamma_{k+1}}$$

from which we deduce that $\delta = \lim_{k \to \infty} \delta_k \approx 4.66920$ in agreement with Feigenbaum’s universal scaling result[5] for dissipative uni-modal maps. The other scaling exponent, $\alpha \approx 2.50290$ for successive branching widths is also recovered in the Kim-Kong map.

We now consider a multifractal analysis[7–9] of chaotic orbits of the Kim-Kong map. First we review the definition of generalized dimensions in the multifractal formalism. Suppose that we divide the unit interval into $M(\epsilon)$ cells of size $\epsilon = \frac{1}{M(\epsilon)}$. For a given chaotic orbit of length $N$ let $p_i = \frac{n_i}{N}$ denote the fractional number of points in the $i$th cell, $[(i-1)\epsilon, i\epsilon]$. The generalized dimensions are now defined by

$$D_q = \lim_{\epsilon \to 0} \frac{\log \sum_{i=1}^{M(\epsilon)} p_i^q}{(q-1) \log \epsilon}. \quad (3)$$

and the multifractal measures $f_q$ and $\alpha_q$ are related to $D_q$ via the Legendre transform:

$$f_q = q \frac{d}{dq} [(q-1)D_q] - (q-1)D_q \quad (4)$$

and

$$\alpha_q = \frac{d}{dq} [(q-1)D_q]. \quad (5)$$

We have made use of Eqs.(3)-(5) to compute the multifractal measures for the Kim-Kong map and the fractal dimension, $D_0$, which is numerically compared with that of other maps.

In our numerical computations we have restricted ourselves to three cases for the two control parameters in the chaotic regime; (a) $\gamma = 3.0$ and $\beta = 0.12$, (b) $\gamma = 3.4$ and $\beta = 0.12$, and (c) $\gamma = 3.5$ and $\beta = 0.12$. The multifractal measures that we have computed are based on $2 \times 10^4$ iterations of the map in Eq.(1). Figs. 4 and 5 are, respectively, the plots of $D_q$ versus $q$ and the spectrum $f_q$ versus $\alpha_q$ for chaotic orbits in the Kim-Kong map. The fractal dimension $D_0$ or $f_0$ (i.e., the maximum value of $D_q$ or $f_q$) of the Kim-Kong map is numerically compared with that of other maps, in Table 1.
In conclusion, we have computed chaotic and multifractal properties of the Kim-Kong map - a two-parameter map of the unit interval onto itself. Although this map was investigated as a mathematical example in this work the map could prove useful as a model for the transport of quantum excitations, directed polymer problems, and the electron localization in quantum mechanics\cite{18-20}. In future work on the Kim-Kong map we plan to investigate chaotic orbits of this map as examples of deterministic diffusion including the possibility of biased diffusion and anomalous diffusion. The chaotic orbits of the Kim-Kong maps will also be compared with randomness in a reaction-diffusion system with multi-species reactants\cite{21-22}.

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FIGURE CAPTIONS

Fig. 1. A plot of $x(n+1)$ verse $x(n)$ graph in the Kim-Kong map\[6\], $x(n+1) = \gamma \exp[-\beta(\log x(n))^2][1-\exp[-\beta(\log x(n))^2]]$, for $\gamma = 3.78$ and several values of $\beta = 0.2, 0.6, 1.0, 1.5, 2.0$.

Fig. 2. Plots of two types of maps for chaotic states with impure malonic acid in the Belousov-Zhabotinskii reaction[12], where the circle and triangle data are the different U sequence of periodic states, respectively.

Fig. 3. Plots of $D_q$ and $q$ for $\gamma = 3.0$ and $\beta = 0.12$ (thick solid line), $\gamma = 3.4$ and $\beta = 0.12$ (dashed line), and $\gamma = 3.5$ and $\beta = 0.12$ (thick dashed line).

Fig. 4. Plots of $f_q$ versus $\alpha_q$ obtained from the chaotic sequences of the Kim-Kong map for three cases of the parameters; (a) $\gamma = 3.0$ and $\beta = 0.12$ (thick solid line), (b) $\gamma = 3.4$ and $\beta = 0.12$ (dashed line), and (c) $\gamma = 3.5$ and $\beta = 0.12$ (thick dashed line).

TABLE CAPTIONS

Table 1. Summary of values of the fractal dimensions of chaotic orbits in the Kim-Kong map and other maps.
| Maps                        | Control Parameter        | Fractal Dimension |
|-----------------------------|--------------------------|-------------------|
| Kim-Kong Map                | (a) $\gamma = 3.0, \beta = 0.12$ | 0.8395            |
|                             | (b) $\gamma = 3.4, \beta = 0.12$ | 0.9790            |
|                             | (c) $\gamma = 3.5, \beta = 0.12$ | 0.9951            |
| Hénon map[15]               | $a = 1.4, b = 0.3$       | 1.26              |
| Kaplan-Yorke Map[15]        | $\alpha = 0.2$          | 1.431             |
| Logistic Map[16]            | $b = 3.5699456...$       | 0.538             |
| Lorentz Equation[15, 17]    |                          | $2.06 \pm 0.01$   |
| Zaslavskii Map[16]          |                          | 1.39              |
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