Lagrangian for the Frenkel electron

Alexei A. Deriglazov

Depto. de Matemática, ICE, Universidade Federal de Juiz de Fora, MG, Brasil
and
Laboratory of Mathematical Physics, Tomsk Polytechnic University, 634050 Tomsk, Lenin Ave. 30, Russian Federation

We found Lagrangian action which describes spinning particle on the base of non-Grassmann vector and involves only one auxiliary variable. It provides the right number of physical degrees of freedom and yields generalization of the Frenkel and BMT equations to the case of an arbitrary electromagnetic field. We briefly discuss the differences in the behavior of a particle satisfying the exact and approximate equations.

I. INTRODUCTION

Relativistic description of rotational degrees of freedom of a body starting from the proper Lagrangian has a long history, see [1–4] and references therein. Since the spin operators in quantum theory satisfy the angular-momentum algebra, closely related problem consist in establishing of variational formulation which should lead to classical equations of spinning electron [5–7]. One possibility here is to assume the Frenkel spin-tensor [6, 7], leading to a theory with the number and algebraic structure of constraints different from those of free theory. So the quantization procedure, this leads to quantum models essentially different from the Dirac electron. In the recent works [14–17] we partially solved this task by presenting a number of equivalent Lagrangians with the right physical covariants.

Though a number of vector models [5, 9, 11–13] yield Frenkel and BMT equations, they also contain extra degrees of freedom. At the classical level one can simply ignore them. However, they should be taken into account during quantization procedure, this leads to quantum models essentially different from the Dirac electron. In the recent works [14–17] we partially solved this task by presenting a number of equivalent Lagrangians with the right physical degrees of freedom and yields generalization of the Frenkel and BMT equations to the case of an arbitrary electromagnetic field. We briefly discuss the differences in the behavior of a particle satisfying the exact and approximate equations.

The last term in (1) represents velocity-independent constraint which is well known from classical mechanics. So, we might follow the classical-mechanics prescription to exclude the auxiliary variable $g_4$ from the formulation. But this would lead to lose of manifest covariance of the formalism.
II. LAGRANGIAN AND HAMILTONIAN FORMULATIONS

Consider spinning particle with mass \( m \), electric charge \( e \) and magnetic moment \( \mu \) interacting with an arbitrary electromagnetic field

\[
F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = (F_{0i} = E_i, \quad F_{ij} = \epsilon_{ijk} B_k), \quad E_i = -\frac{1}{c} \partial_t A_i + \partial_i A_0, \quad B_i = \frac{1}{2} \epsilon_{ijk} F_{jk}. \tag{3}
\]

We study the following Poincare and reparametrization invariant Lagrangian action on configuration space with coordinates \( x^\mu(\tau) \), \( \omega^\mu(\tau) \) and \( g(\tau) \):

\[
S = \int d\tau \frac{1}{4g} \left[ \dot{x} N \dot{x} + g D\omega N D\omega - \sqrt{|\dot{x} N \dot{x} + g D\omega N D\omega|^2 - 4g(\dot{x} N D\omega)^2} \right] - \frac{g}{2} m^2 c^2 + \frac{\alpha}{\sqrt{2} \omega^2} + \frac{e}{c} A \dot{x}. \tag{4}
\]

This depends on one free parameter \( \alpha \) which determines spin of the particle. We take \( \alpha = \frac{3\hbar^2}{4} \), this corresponds to spin one-half particle, see below. The only auxiliary variable is \( g \), this provides the mass-shell condition \( (17) \). It has been denoted

\[
\tilde{N}^{\mu \nu} \equiv \eta^{\mu \nu} - \frac{\omega^{\mu} \omega^{\nu}}{\omega^2}, \quad \text{then} \quad N^{\mu \nu} \omega_\nu = 0. \tag{5}
\]

Together with \( \tilde{N}^{\mu \nu} \equiv \frac{\omega^{\mu} \omega^{\nu}}{\omega^2} \), this forms a pair of projectors: \( N + \tilde{N} = 1, \quad N^2 = N, \quad \tilde{N}^2 = \tilde{N}, \quad N \tilde{N} = 0 \). The square-root appeared in the Lagrangian seem to be typical structure [2] for the models which imply the Frenkel-type constraint \( J^{\mu \nu} F_{\nu} = 0 \).

To introduce coupling of the position variable \( x \) with electromagnetic field we have added the minimal interaction \( \frac{e}{c} A \mu \dot{x}^\mu \). As for spin, it couples with \( A^\mu \) through the term

\[
D \omega^\mu \equiv \omega^\mu - g \frac{e \mu}{c} F^{\nu \mu} \omega_\nu. \tag{6}
\]

This is the only term we have found compatible with symmetries and constraints which should be presented in the theory. In particular, under reparametrizations the variable \( g \) transforms as \( g = \frac{\partial^\tau}{\partial \tau'} g' \). This implies homogeneous transformation law of \( D \omega \), \( D \omega = \frac{\partial^\tau}{\partial \tau'} D' \omega' \), and, at the end, reparametrization invariance of the Lagrangian. In turn, this provides the expected mass-shell condition \( \tilde{P}^2 - \frac{\tilde{g}}{\tilde{N}} (FJ) + m^2 c^2 = 0 \), see below.

Switching off the spin variables \( \omega^\mu \) from Eq. (4), we arrive at \( L = \frac{1}{2g} \dot{x}^2 - \frac{g}{2} m^2 c^2 + \frac{\tilde{g}}{c} A \dot{x} \). Integrating over the auxiliary variable \( g \) we obtain \( L = -mc \sqrt{-\dot{x}^2 + \frac{\tilde{g}}{c} A \dot{x}} \). This is equivalent to the standard Lagrangian of spinless particle in terms of physical variables \( \tilde{x}(t) \), \( L = -mc \sqrt{\dot{x}^2 - \frac{\tilde{g}}{c} A \dot{x}} + e A_0 + \frac{\tilde{g}}{c} A \dot{x} \), if we restrict ourselves to the class of increasing parametrizations of the world-line. This implies positive \( g(\tau) \). So we study [4] under the assumptions \( \frac{d\tau}{d\tau'} > 0, g(\tau) > 0 \). In the presence of spin, our Lagrangian is a complicated function of \( g \) even in the case of free theory.

Let us construct Hamiltonian formulation of the model. Conjugate momenta for \( x^\mu \), \( \omega^\mu \) and \( g \) are denoted as \( p^\mu \), \( \pi^\mu \) and \( \pi_g \). Besides, we use the condensed notation \( \sqrt{|\dot{x} N \dot{x} + g D\omega N D\omega|^2 - 4g(\dot{x} N D\omega)^2} \equiv \sqrt{\tilde{P}^2} \) and \( \tilde{P}^\mu \equiv p^\mu - \frac{\tilde{g}}{c} A^\mu \).

Contrary to \( \pi_g = \frac{\partial L}{\partial \dot{g}} = 0 \), the momentum \( \pi_g \) represents the primary constraint, \( \pi_g = 0 \). Expressions for the remaining momenta, \( p^\mu = \frac{\partial L}{\partial \dot{x}^\mu} \) and \( \pi^\mu = \frac{\partial L}{\partial \dot{\omega}^\mu} \), can be written in the form

\[
\tilde{P}^\mu = \frac{1}{2g} (N \dot{x}^\mu - K^\mu), \quad K^\mu \equiv \frac{[\dot{x} N \dot{x} + g D\omega N D\omega] (N \dot{x})^\mu - 2g(\dot{x} N D\omega)(N D\omega)^\mu}{\sqrt{\tilde{P}^2}}, \tag{7}
\]

\[
\pi^\mu = \frac{1}{2} (N D\omega^\mu - R^\mu), \quad R^\mu \equiv \frac{[\dot{x} N \dot{x} + g D\omega N D\omega] (N D\omega)^\mu - 2(\dot{x} N D\omega)(N \dot{x})^\mu}{\sqrt{\tilde{P}^2}}. \tag{8}
\]

The functions \( K^\mu \) and \( R^\mu \) obey the following remarkable identities

\[
K^2 = \dot{x} N \dot{x}, \quad R^2 = D\omega N D\omega, \quad K R = -\dot{x} N D\omega, \quad \dot{x} R + D\omega K = 0, \quad \dot{x} K + g D\omega R = \sqrt{-}. \tag{9}
\]
Due to Eq. (5), contractions of the momenta with $\omega^{\mu}$ vanish, that is we have the primary constraints $\omega^\pi = 0$ and $\mathcal{P}\omega = 0$. One more primary constraint, $\mathcal{P}T_i = 0$, is implied by (9).

Hence we deal with a theory with four primary constraints. Hamiltonian has the form

$$H = p\dot{x} + \pi\dot{\omega} - L + \lambda_i T_i, \quad (10)$$

where $\lambda_i$ are the Lagrangian multipliers for the primary constraints $T_i$. To construct manifest form of the Hamiltonian, we note the equalities $\mathcal{P}^2 = \frac{1}{2}\mathcal{P}[\dot{\mathcal{P}} - \dot{x}K]$ and $\pi^2 = \frac{1}{2}[D\omega ND\omega - D\omega R]$. Then, using (9) we obtain

$$\frac{g}{2}\mathcal{P}^2 + \frac{1}{2}\pi^2 = L_0, \quad (11)$$

where $L_0$ is the first line in Eq. (4). Further, using Eqs. (9) we have

$$p\dot{x} + \pi\dot{\omega} \equiv \mathcal{P}\dot{x} + \frac{e}{c}\mathcal{P} x + \pi D\omega + g\frac{\mu}{c}(\pi F\omega) = 2L_0 + \frac{e}{c}\mathcal{P} x - g\frac{\mu}{4c}(FJ), \quad (12)$$

where the Frenkel-type spin-tensor appeared

$$J^{\mu\nu} = 2(\omega^{\mu\pi}_\nu - \omega^{\nu\pi}_\mu). \quad (13)$$

Using (12) and (11) in (10) we arrive at the Hamiltonian

$$H = \frac{g}{2}\left(\mathcal{P}^2 - \frac{\mu}{2c}(FJ) + m^2 c^2\right) + \frac{1}{2}\left(\pi^2 - \frac{\alpha}{\omega^2}\right) + \lambda_5(\omega\pi) + \lambda_6(\mathcal{P}\omega) + \lambda_7(\mathcal{P}\pi) + \lambda_9\pi g. \quad (14)$$

The fundamental Poisson brackets $\{x^{\mu}, p^{\nu}\} = \eta^{\mu\nu}$ and $\{\omega^{\mu}, \pi^{\nu}\} = \eta^{\mu\nu}$ imply

$$\{x^{\mu}, \mathcal{P}^{\nu}\} = \eta^{\mu\nu}, \quad \{\mathcal{P}^{\mu}, \pi^{\nu}\} = \frac{e}{c}F^{\mu\nu}, \quad (15)$$

$$\{J^{\mu\nu}, J^{\alpha\beta}\} = 2(\eta^{\mu\alpha}J^{\nu\beta} - \eta^{\mu\beta}J^{\nu\alpha} - \eta^{\nu\alpha}J^{\mu\beta} + \eta^{\nu\beta}J^{\mu\alpha}). \quad (16)$$

According to Eq. (16) the spin-tensor is generator of Lorentz algebra $SO(1,3)$. As $\omega\pi$, $\omega^2$ and $\pi^2$ are Lorentz-invariants, they have vanishing Poisson brackets with $J^{\mu\nu}$. To reveal the higher-stage constraints we write the equations $\dot{T}_i = \{T_i, H\} = 0$. The Dirac procedure stops on third stage with the following equations

$$\pi g = 0 \quad \Rightarrow \quad \mathcal{P}^2 - \frac{\mu}{2c}(FJ) + m^2 c^2 = 0 \quad \Rightarrow \quad \lambda_6 C + \lambda_7 D = 0, \quad (17)$$

$$(\omega\pi) = 0 \quad \Rightarrow \quad \pi^2 - \frac{\alpha}{\omega^2} = 0, \quad (18)$$

$$(\mathcal{P}\omega) = 0 \quad \Rightarrow \quad \lambda_7 = \frac{gC}{M^2 c^2}, \quad (19)$$

$$(\mathcal{P}\pi) = 0 \quad \Rightarrow \quad \lambda_6 = -\frac{gD}{M^2 c^2}. \quad (20)$$

We have denoted

$$M^2 = m^2 + \frac{e(2\mu + 1)}{4c^2}(FJ), \quad C = -\frac{e(\mu - 1)}{c}(\omega F\mathcal{P}) + \frac{e\mu}{4c}(\omega\vartheta)(FJ), \quad D = -\frac{e(\mu - 1)}{c}(\pi F\mathcal{P}) + \frac{e\mu}{4c}(\pi\vartheta)(FJ). \quad (21)$$

The last equation from (17) turns out to be a consequence of (19) and (21) and can be omitted. Hence the Dirac procedure revealed two secondary constraints written in Eqs. (17) and (18), and fixed the Lagrangian multipliers $\lambda_6$ and $\lambda_7$. The multipliers $\lambda_9$ and $\lambda_5$ and the auxiliary variable $g$ have not been determined. $H$ vanishes on the complete constraint surface, as it should be in a reparametrization-invariant theory.

We summarized the algebra of Poisson brackets between constraints in the Table I. The constraints $\pi g$, $T_1$, $T_3$ and $T_5$ form the first-class subset, while $T_6$ and $T_7$ represent a pair of second class. The presence of two primary first-class constraints $\pi g$ and $T_5$ is in correspondence with the fact that two lagrangian multipliers remain undetermined within the Dirac procedure. This also indicates on two local symmetries which must be presented in the theory. One of them is the standard reparametrization invariance. Another is the spin-plane symmetry discussed in the next section.

Hamiltonian (14) determines evolution of the basic variables according the following equations

$$\dot{x}^{\mu} = g(T^\mu_{\nu}\mathcal{P}^\nu + Y^{\mu}), \quad \dot{\mathcal{P}}^{\mu} = \frac{e}{c}(F\dot{x})^{\mu} + \frac{g\mu}{4c}\vartheta^{\mu}(FJ), \quad (22)$$
for any solution to the equations of motion. Besides, the constrain ts
fixed value of spin
\(\mu\) of anomalous magnetic moment
have two-parametric ambiguity due to
\(\pi\lambda\) with a gradient of field is proportional to
moment, 

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symmetry of the theory.

The equations (22) and (25) follow from this
Dirac bracket for the second-class pair
T

The variables

 form a closed system. The remaining ambiguity due to \(g\) presented in these equations reflects the reparametrization symmetry of the theory.

The term \(\frac{a}{2m^2}\) in the Lagrangian (4) provides the constraint \(\omega^2 J^2 = \alpha = \frac{3\hbar^2}{4}\). Together with \(\omega \pi = 0\), this implies fixed value of spin

for any solution to the equations of motion. Besides, the constraints \(\omega \pi = 0\) imply the Pirani condition [18–20]

The variables \(x, \pi\) and \(J\) have vanishing Poisson brackets with the second and third terms in (14). Hence these terms do not contribute into equations (22) and (25), and can be omitted from Hamiltonian. Further, we can construct the Dirac bracket for the second-class pair \(T_5\) and \(T_7\), after that they also can be omitted from (14). Then the relativistic Hamiltonian acquires an expected form

The equations (22) and (25) follow from this \(H\) with use of Dirac bracket, \(\dot{z} = \{z, H\}_{DB}\).

The first equation from (22) together with \(T_2\)-constraint can be used to exclude the variables \(\pi\) and \(g\). For \(g\) we obtain \(\sqrt{-g_{\mu\nu} \dot{\pi}^\mu \dot{\pi}^\nu}\), where the effective metric \(g_{\mu\nu}\) is given by (34). So, the presence of \(g\) in Eq. (6) implies highly non-linear interaction of spinning particle with electromagnetic field. Excluding \(\pi\) and \(g\), we obtain closed system of Lagrangian equations for the set \(x, J\)

| \(T_1 = p^2 - \frac{\mu^2}{c^2}(FJ) + m^2 c^2\) | \(T_3 = \pi^2 - \frac{\mu^2}{c^2}\) | \(T_5 = 0\) | \(T_6 = 0\) | \(T_7 = 0\) |
|---|---|---|---|---|
| 0 | 0 | 0 | -2C | -2D |
| 0 | 0 | -2| 0 | -2T_7 \approx 0 |
| 0 | 2T_3 \approx 0 | 0 | -T_6 \approx 0 | T_7 \approx 0 |
| 2C | 2T_7 \approx 0 | T_6 \approx 0 | 0 | T_1 - M^2 c^2 \approx -M^2 c^2 |
| 0 | 0 | -\frac{2\alpha}{(\omega_7)^2} T_6 \approx 0 | -T_7 \approx 0 | -T_1 + M^2 c^2 \approx M^2 c^2 |

**TABLE I**: Algebra of constraints.

\[ \dot{\omega} = \frac{e\mu}{c}(F\omega)^\mu + g \frac{C}{M^2 c^2} \pi^\mu + \pi^\mu + \lambda_5 \omega^\mu, \quad \dot{\pi}^\mu = \frac{e\mu}{c}(F\pi)^\mu + g \frac{D}{M^2 c^2} \pi^\mu - \frac{\alpha}{(\omega_7)^2} \omega^\mu - \lambda_5 \pi^\mu, \] (23)

We have denoted

\[ T_{\mu\nu} = \eta_{\mu\nu} - (\mu - 1)a(JF)^{\mu\nu}, \quad Y^\mu = \frac{\mu a}{4} J^{\mu\nu} \partial_\nu(JF), \quad a = \frac{e}{2M^2 c^3} = \frac{-2e}{4m^2 c^3 - e(2\mu + 1)(JF)}. \] (24)

The interaction leads to modification of the Lorentz-force equation. Only for the “classical” value of magnetic moment, \(\mu = 1\), and constant electromagnetic field the constraints (19) and (20) would be the same as in the free theory, \(\lambda_6 = \lambda_7 = 0\). Then \(T_{\mu\nu} = \eta_{\mu\nu}, Y^\mu = 0\), and four-velocity becomes proportional to \(\pi\), see (22). Contribution of anomalous magnetic moment \(\mu \neq 1\) to the difference between \(\dot{x}\) and \(\pi\) is proportional to \(\frac{1}{c^2} \sim \frac{\hbar}{c^2}\), while the term with a gradient of field is proportional to \(\frac{\omega^2}{c^2} \sim \frac{\hbar^2}{c^2}\).

All the basic variables have ambiguous evolution. \(x^\mu\) and \(\pi^\mu\) have one-parametric ambiguity due to \(g\) while \(\omega\) and \(\pi\) have two-parametric ambiguity due to \(g\) and \(\lambda_5\). The quantities \(x^\mu, \pi^\mu\) and the spin-tensor \(J^{\mu\nu}\) turn out to be invariant under spin-plane symmetry. So they can be observable quantities. Equations (22) together with

\[ J^{\mu\nu} = g \frac{e\mu}{c}(FJ)^{\mu\nu} + 2\pi^{[\mu\nu]}, \] (25)

form a closed system. The remaining ambiguity due to \(g\) presented in these equations reflects the reparametrization symmetry of the theory.

The term \(\frac{a}{2m^2}\) in the Lagrangian (4) provides the constraint \(\omega^2 \pi^2 = \alpha = \frac{3\hbar^2}{4}\). Together with \(\omega \pi = 0\), this implies fixed value of spin

\[ J^{\mu\nu} J_{\mu\nu} = 8(\omega^2 \pi^2 - (\omega \pi)^2) = 6\hbar^2, \] (26)

for any solution to the equations of motion. Besides, the constraints \(\omega \pi = 0\) imply the Pirani condition [18–20]

\[ J^{\mu\nu} \pi^\nu = 0. \] (27)

The first equation from (22) together with \(T_2\)-constraint can be used to exclude the variables \(\pi\) and \(g\). For \(g\) we obtain \(\sqrt{-g_{\mu\nu} \dot{\pi}^\mu \dot{\pi}^\nu}\), where the effective metric \(g_{\mu\nu}\) is given by (34). So, the presence of \(g\) in Eq. (6) implies highly non-linear interaction of spinning particle with electromagnetic field. Excluding \(\pi\) and \(g\), we obtain closed system of Lagrangian equations for the set \(x, J\).
\[ D_g \left[ m_r (\bar{T} D_g x)^\mu \right] = \frac{e}{c^2} (F D_g x)^\mu + \frac{\mu e}{4 m_r c^3} \partial^\mu (J F) + D_g \left( \frac{b}{a c} Y^\mu \right), \]  

\[ j^{\mu \nu} = \frac{e \mu \sqrt{-x g T}}{m_r c^2} (F J)^{[\mu \nu]} - \frac{2 b (\mu - 1) m_r c}{\sqrt{-x g T}} \epsilon^{[\mu \nu]} (J F \dot{x})^\nu + \frac{2 b}{a} \dot{x}^{[\mu} Y^\nu]. \]  

Besides, all solutions satisfy the Lagrangian analog of Pirani condition

\[ J^{\mu \nu} (\dot{T} \dot{x})_\nu - \frac{b}{a m_r c} \sqrt{-x g T} Y^\nu = 0, \]  

as well as to the value-of-spin condition \( J^{\mu \nu} J_{\mu \nu} = 6 \hbar^2 \). We have denoted by

\[ \dot{T}^{\mu \nu} = \eta^{\mu \nu} + (\mu - 1) b (J F)^{\mu \nu}, \quad b = \frac{2 a}{2 + (\mu - 1) a (J F)} = \frac{-2 e}{4 m^2 c^3 - 3 e \mu (J F)}, \]  

the inverse matrix for \( T \), Eq. (24). Interaction of spin with the external field yields the radiation mass \( m_r \)

\[ m_r^2 = m^2 - \frac{\mu e}{2 c^3} (J F) - \frac{(Y g Y)}{c^2}, \]  

as well as the effective metric

\[ g_{\mu \nu} = (\ddot{T} T)_{\mu \nu} = [\eta + b (\mu - 1) (J F + F J) + b^2 (\mu - 1)^2 F J F]_{\mu \nu}, \quad D_g = \frac{1}{\sqrt{-x g T}} \frac{d}{d \tau}. \]  

The equations (29)-(31) coincide with those obtained in [16] from the Lagrangian with four auxiliary variables. In the approximation \( O^3 (J, F, \partial F) \) and when \( \mu = 1 \) they coincide with Frenkel equations.

Let us specify the equation for spin precession to the case of uniform and stationary field, supposing also \( \mu = 1 \) and taking physical time as the parameter, \( \tau = t \). Then (31) reduces to the Frenkel condition, \( J^{\mu \nu} \dot{x}_\nu = 0 \), while (30) reads \( j^{\mu \nu} = \frac{e \mu \sqrt{-x g T}}{m_r c^2} (F J)^{[\mu \nu]} \). We decompose spin-tensor on electric dipole moment \( \vec{D} \) and Frenkel spin-vector \( \vec{S} \) as follows:

\[ J^{\mu \nu} = (J^{\cdot 0} = D^i, \quad J^{ij} = 2 \epsilon_{ijk} S_k). \]  

Then \( \vec{D} = -\frac{2}{e} \vec{S} \times \vec{v} \), while precession of \( \vec{S} \) is given by

\[ \frac{d \vec{S}}{dt} = \frac{e \sqrt{c^2 - \vec{v}^2}}{m_r c^3} \left[ -\vec{E} \times (\vec{v} \times \vec{S}) + e \vec{S} \times \vec{B} \right]. \]  

### III. SPIN SURFACE AND ASSOCIATED SPIN FIBER BUNDLE \( T^4 \).

While spin-sector of our model consists of the basic variables \( \omega^\mu \) and \( \pi^\mu \), quantum mechanics obtained in terms of spin-tensor \( J^{\mu \nu} \). The passage from \( \omega \) and \( \pi \) to \( J \) is not a change of variables, and acquires a natural interpretation in the geometric construction described below. Generalization of this construction on the case of \( SO(k, n) \) Lie-Poisson manifold can be found in [21].

In the previous section we have obtained the following constraints in spin-sector:

\[ P_\omega = 0, \quad P_{\pi} = 0, \]  

\[ \omega \pi = 0, \quad \pi^2 - \frac{\alpha}{\omega^2} = 0, \]  

It should be noticed that the Lagrangian (2) implies \( \omega \pi = 0 \), \( \pi^2 - a_3 = 0 \) and \( \omega^2 - a_4 = 0 \) instead of (38). So, the Lagrangian (1) does not appear from (2) by removing the auxiliary variables \( g_3 \), \( g_4 \) and \( g_7 \).
To see the meaning of Lorentz-invariant constraints (37) and (38), we consider this surface in Lorentz frame which implies \( \mathcal{P}^\mu = (P^0, 0) \). Then Eqs. (37) mean \( \omega^0 = \pi^0 = 0 \). Taking this into account, the constraints (38) determines the following surface in \( \mathbb{R}^6(\vec{\omega}, \vec{\pi}) \)

\[
T^4 = \{ \vec{\omega}\vec{\pi} = 0, \quad \vec{\pi}^2 - \frac{\alpha}{\vec{\omega}^2} = 0 \},
\]

(39)

that is \( \vec{\omega} \) and \( \vec{\pi} \) represent a pair of orthogonal vectors with ends attached to the hyperbola \( y = \frac{x^2}{c^2} \). The constraints (37) imply \( J^{\mu\nu}P_\nu = 0 \). In the rest frame this gives \( J^{0i} = 0 \), that is the spin-tensor has only three components which we identify with non-relativistic spin-vector, \( J_{ij} = 2\epsilon_{ijk}S_k \). Due to the constraints (39) the spin-vector belong to two-dimensional sphere of radius \( \sqrt{\alpha} \)

\[
J_{ij}J_{ij} = 8\alpha, \quad \text{or} \quad \vec{S}^2 = \alpha, \quad \text{so we assume} \quad \alpha = \frac{3\hbar^2}{4}.
\]

(40)

We call this the spin surface. The chosen value of parameter corresponds to spin one-half particle.

Hence, to describe spin in the rest frame, we have six-dimensional space of basic variables \( \mathbb{R}^6(\vec{\omega}, \vec{\pi}) \), the spin-tensor space \( \mathbb{R}^3(J_{ij} \sim \vec{S}) \) and the map

\[
f : \mathbb{R}^6 \to \mathbb{R}^3, \quad f : (\vec{\omega}, \vec{\pi}) \to \vec{S} = \vec{\omega} \times \vec{\pi}, \quad \text{rank} \frac{\partial (S_i)}{\partial (\omega_j, \pi_k)} = 3.
\]

(41)

According to previous section, all trajectories \( \vec{\omega}(\tau), \vec{\pi}(\tau) \) lie in the manifold (39) of \( \mathbb{R}^6 \). \( f \) maps the manifold \( T^4 \) onto spin surface, \( f(T^4) = \vec{S}^2 \).

Denote \( \mathbb{F}_2^S \subset T^4 \) preimage of a point \( \vec{S} \in \mathbb{S}^2 \), \( \mathbb{F}_2^S = f^{-1}(\vec{S}) \). Let \( (\vec{\omega}, \vec{\pi}) \in \mathbb{F}_2^S \). Then the two-dimensional manifold \( \mathbb{F}_2^S \) consist of the pairs \( (k\vec{\omega}, \frac{1}{k}\vec{\pi}) \), \( k \in \mathbb{R}^+ \), as well as those obtained by rotation of \( (k\vec{\omega}, \frac{1}{k}\vec{\pi}) \) in the plane of vectors \( \vec{\omega} \) and \( \vec{\pi} \). So elements of \( \mathbb{F}_2^S \) are related by two-parametric transformations

\[
\vec{\omega}' = \vec{\omega}k \cos \beta + \vec{\pi}k|\vec{\omega}| \sin \beta, \quad \vec{\pi}' = -\vec{\omega}|\vec{\pi}| \sin \beta + \vec{\pi} \frac{1}{k} \cos \beta.
\]

(42)

In the result, the manifold \( T^4 \) acquires natural structure of fiber bundle \( \mathbb{T}^4 = (∑^2, \mathbb{F}^2, f) \) with base \( S^2 \), standard fiber \( \mathbb{F}^2 \), projection map \( f \) and structure group given by transformations (42). As local coordinates of \( T^4 \) adjusted with the structure of fiber bundle we can take \( k, \beta \), and two coordinates of the vector \( \vec{S} \). By construction, the structure-group transformations leave inert points of base, \( \delta S_i = 0 \).

The Lorentz-invariant equations (37), (38) together with the map \( J^{\mu\nu} = 2\omega^{[\mu}\pi^{\nu]} \) represent this construction in an arbitrary Lorentz frame. In the dynamical realization given in previous section, structure group acts independently at each instant of time and turn into the local symmetry. \( k \)-transformations provide reparametrization invariance of the action (11). The spin-plane rotations \( \beta \) are associated with the first-class constraints \( T_3 \) and \( T_5 \) and selects \( J \) as the physical (observable) variable.

IV. DISCUSSION

We obtained a generalization (29) and (30) of Frenkel and BMT equations to the case of an arbitrary electromagnetic field. They follow from the Lagrangian (4) which also yields the constraints (17), (26) and (31), providing the right number of physical degrees of freedom. Some relevant comments are in order.

The relativistic equation (30) automatically incorporates the Thomas precession (4). Indeed, let in instantaneous rest frame of the particle we have \( F^{\mu\nu} = (\vec{E}' = \text{const}, \vec{B}' = 0) \). Then Eq. (30) tell us that spin does not experience a torque in the rest frame, \( \frac{d\vec{S}}{dt} = 0 \). Consider a frame where the particle has velocity \( \vec{v} \). In this frame the field is \( F^{\mu\nu} = (\vec{E}, \vec{B} = \frac{1}{c^2}\vec{v} \times \vec{E}) \), where \( \vec{E} \) is determined by Lorentz boost of \( \vec{E}' \) (24). An observer in the laboratory frame detects the Thomas precession (30). Expressing \( \vec{B} \) through \( \vec{E} \), the equation (30) can be written as follows: \( \frac{d\vec{S}}{dt} = \frac{e}{mc^2}\frac{\vec{v} \times \vec{E}}{m^2c^2} \).

Classical analog of the Pauli Hamiltonian (22) contains the term \( \frac{1}{2}\vec{S} \cdot \vec{E} \times \vec{v} + c\vec{S} \cdot \vec{B} \), while the relativistic theory (25) implies \( \frac{i}{\hbar}F^{\mu\nu}J_{\mu\nu} = \vec{S} \cdot \vec{E} \times \vec{v} + c\vec{S} \cdot \vec{B} \). Both Hamiltonians are written in a laboratory system. The difference is the famous one-half factor. Our analysis clearly shows the origin of this discrepancy on the classical level: we deal with two different sets of variables. Our variables obey noncommutative Dirac brackets while in the Pauli theory...
the brackets supposed to be canonical. To compare the Hamiltonians, we need manifest form of (time-dependent) canonical transformation among the two formulations. Probably, the projection operator method for diagonalization of Dirac brackets \[26 \] could be used to this aim.

Even for uniform fields, behavior of our spinning particle with anomalous magnetic moment \((\mu \neq 1)\) differs from that of Frenkel and BMT. This is due to two structural modifications implied by the Lagrangian which provides the necessary constraints\(^2\). First, velocity is not proportional to the canonical momentum, see Eq. \((22)\). Second, in interacting theory we necessarily have the Pirani condition \(J^{\mu \nu} P_{\nu} = 0\) on the place of Frenkel condition \(J^{\mu \nu} \dot{x}_{\nu} = 0\). In the Lagrangian formulation this leads to the equation \[\left[\frac{\ddot{x}}{\sqrt{1 - \ddot{x}^2}}\right] = f\], which has the structure different from that of Frenkel and BMT, \[\left[\frac{\ddot{x}}{\sqrt{1 - \ddot{x}^2}}\right] = f\]. This results in extra contribution to the standard expression for the Lorentz force, \(\ddot{x} \sim F \ddot{x} + O(J)\). So the complete theory implies an extra spin-orbit interaction as compared with the approximate Frenkel and BMT equations. For instance, BMT electron in a constant magnetic field moves around a circle on the plane orthogonal to the field. For our particle, the circular motion is perturbed by slow oscillations along the magnetic field \[16\].

As we have seen, in interacting theory the Frenkel condition on spin-tensor necessarily turns into the Pirani condition. Frenkel condition implies \(\ddot{D} = 0\) in the rest frame, that is zero electric dipole moment. In contrast, the Pirani condition \[31\] predicts small non-vanishing electric dipole moment \(\ddot{D} \sim \dddot{S} \times (\dddot{S} \times \dddot{E})\).

As it should be in a Lorentz-invariant theory, the speed of light \(c\) represents the invariant scale in our model: if one observer concludes that a particle has the speed \(c\), all other inertial observers will make the same conclusion. At the same time, when \(\mu \neq 1\) our equations of motion necessarily involve the factor \(\sqrt{-\ddot{x}^2}\) instead of the standard relativistic-contraction factor \(\sqrt{-\ddot{x}^2}\). Computing the acceleration implied by \((29)-(31)\), we obtain \(\dddot{a} \sim \sqrt{-\ddot{x}^2} \dddot{f}\) with \(\dddot{f}\) being non-singular function as \(\ddot{x} \dddot{g} \rightarrow 0\). So the factor determines critical speed which the spinning particle can not overcome during its evolution in external field. The critical speed is determined as a solution to \(\ddot{x} \dddot{g} \dddot{x} = 0\). This surface is slightly different from the sphere \(c^2 - \dddot{x}^2 = 0\). So, the critical speed of our particle can be different from the speed of light. Similar conclusion has been made by Hanson and Regge with respect to their relativistic spherical top \[2\].

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