The Completeness and Incompleteness of Financial Markets in Economies Driven by Diffusion Processes

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Received 8 January 2019; Revised 13 March 2019; Accepted 20 March 2019; Published 7 April 2019

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We investigate sufficient conditions for the completeness and incompleteness of financial markets with multiple goods and heterogeneous agents in continuous-time economies driven by diffusion processes. It is demonstrated that, under certain conditions, two types of utility functions lead to the incompleteness of markets, while the invertibility of the Jacobian matrix of dividend rates' volatility in units of the numeraire good gives rise to the completeness of markets. In particular, it is shown that if a type of special utility functions is adopted, then the change of numeraire good leads to the conversion of completeness of markets.

1. Introduction

The classical method to set up the equilibria of financial markets with heterogenous agents in continuous-time economies is the innovative technique in Duffie [1]. The method is composed of three steps. The first step is to calculate the Arrow-Debreu equilibrium, the second is to give a definition of candidate stock prices by the pricing kernel or the consumption price process, and the last step is to confirm whether these stock prices make the markets complete or not. The most complicated step is the last one, because these candidate stock prices cannot be obtained in explicit form. Duffie [2] shows that the financial markets are complete if and only if the volatility matrix of stock prices is invertible almost everywhere. Anderson and Raimondo [3] demonstrate that if the volatility matrix of dividends at terminal time is nonsingular, then the candidate stock prices lead to completeness of the financial markets. The models in Anderson and Raimondo [3] require that the dividends of all stocks are paid at only terminal time. Hugonnier et al. [4], and Riedel and Herzberg [5], Kramkov and Predoiu [6] find weak conditions for the completeness of markets with only one consumption good.

Ehling and Larsen [7] pay attention to conditions for the incompleteness of markets and get a class of utility function for incomplete financial markets in which stock prices follow the process of a geometric Brownian motion. Ehling and Larsen [7] demonstrate that this class of utility function leads to the incompleteness of markets.

We extend parts of the results in Hugonnier et al. [4] and Ehling and Larsen [7]. Specifically, we present two types of utility functions that give rise to the incompleteness of financial markets. The first one is similar to that of Ehling and Larsen [7]. If the utility function satisfies a partial differential equation, we find that one of the stock prices is replicable. In other words, one of the stock prices is the linear combination of other stock prices. It brings about the incompleteness of financial markets. Solving the partial differential equation, we get the first type of utility functions that lead to the incompleteness of the markets. Using a similar method, if the utility function satisfies another differential equation and several conditions, we derive that certain risks in the economy will not appear in the stock prices, which also is conducive to the incompleteness of financial markets. A new sufficient condition for the incompleteness of the market is
obtained. Examples are given to explain the inherent reason for the incompleteness of the markets. We derive that the incompleteness of markets in the second case depends on the choice of numeraire good, while the first case does not.

Our results about the complete market are similar to Riedel and Herinzberg [5], in which meaningful results about the completeness of markets in Radner equilibria are presented. They show that if the asset markets are potentially complete, there exist Radner equilibria with dynamically complete markets in continuous-time economies with a single consumption good, while we concentrate on multigood economies.

The rest of the sections of this paper are organized as follows. In Section 2, we construct the model of economy and calculate the diffusion coefficient matrix of the commodity prices and dividends in unit of the numeraire good. In Section 3, we present two conditions that lead to the incompleteness of financial markets and compare these two conditions. In Section 4, based on the calculations in Section 2, we give a sufficient condition for complete markets. We conclude our work in Section 5.

2. The Model

In this section, we give a basic setup for our continuous-time economy in which the time span is \([0, T]\), where \(T\) is a finite number (our model originates from the macro-finance model in Duffie [8]). The risks in our model are represented by an \(N\)-dimensional Brownian motion \(Z = (Z_1, Z_2, \ldots, Z_N)\), which is constructed on a probability space \((\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})\).

There are total \(N + 1\) assets in our model which includes one risk-free bond and \(N\) stocks. The risk-free bond is risk-free in units of the numeraire good instead of money. All the stocks pay dividends continuously which are in units of corresponding good. Specifically, there are \(N\) stocks numbered as \(i = 1, 2, \ldots, N\), and they pay dividends in consumption good \(i\) at rate \(g_i(X(t))\), where \(g_i(\cdot)\) is a positive real analytic function and \(X(t)\) denotes the state variables, which is an \(N\)-dimensional vector with the following (we assume that all the equalities hold in the “everywhere (almost surely)” sense and all the stochastic processes are measurable):

\[
dX(t) = \mu_X(X(t), t) dt + \sigma_X(X(t), t)dZ(t),
\]

where \(\mu_X : \mathbb{R}^N \times [0, T] \rightarrow \mathbb{R}^N, \sigma_X : \mathbb{R}^N \times [0, T] \rightarrow \mathbb{R}^{N \times N},\) and \(X(0) \in \mathbb{R}^N\).

The following assumptions are necessary for our analysis.

Assumption 1 (see [7]). \(\text{rank}(\sigma_X(x, t)) = N\) for all \((x, t) \in \mathbb{R}^N \times [0, T]\).

Assumption 2 (see [7]). \(g_i(x) \in \mathbb{R}^d\).

From (1) and the above assumptions, we find that the volatility of dividend, which is denoted by \(\sigma_{g_i}(t)\), is equal to \((\partial g_i(X(t))/\partial x)\sigma_X(X(t), t)\), i.e., \(\sigma_{g_i}(t) = (\partial g_i(X(t))/\partial x)\sigma_X(X(t), t)\) (we write \(F(X(t), t)\) as \(F(t)\) sometimes for conciseness).

To make sure that the uncertainty of the state variables spreads over all the dividends, which makes the market potentially complete, the next assumption is necessary.

Assumption 3 (see [7]). \(\text{rank}((\partial g(X(t))/\partial x)\sigma_X(X(t), t)) = N\), where \(g(X(t)) = (g_1(t), g_2(t), \ldots, g_N(t))\).

Let \(p_i(t)\) be the price of good \(i\) at time \(t\), and let \(p(t) = 1\) be the price of the numeraire good. Thus, we have the vector of prices \(P = \{p_1, p_2, \ldots, p_N\}\). The process of \(p_i\) is given by

\[
dp_i(t) = p_i(t) \mu_{p_i}(t) dt + p_i(t) \sigma_{p_i}(t) dZ(t).
\]

From (2), we define the process of dividend rate in units of the numeraire good: \(\tilde{g}_i(t) = g_i(t)p(t)\).

The process price of stock \(i\) is given by

\[
dS_i(t) = S_i(t) \mu_{S_i}(S_i(t), t) dt - \tilde{g}_i(t) dt + S_i(t) \sigma_{S_i}(S_i(t), t) dZ(t).
\]

Note that the processes of stock prices \(S_i\) are different from those in Ehling and Larsen [7], in which the process of prices is subject to a geometric Brownian motion.

The risk-free bond pays out in the numeraire good, and its price process \(B(t)\) is subject to

\[
dB(t) = r(t) B(t) dt,
\]

where \(r\) is the risk-free rate and is determined endogenously in equilibrium.

Let \(c^k\) be the amount of good \(k\) consumed by agent \(k\), and let the vector of consumption of agent \(k\) at time \(t\) be \(C^k(t) = (c^k_1(t), c^k_2(t), \ldots, c^k_K(t))\), where \(k = 1, 2, \ldots, K\). We assume that the utility function of agent \(k\) is \(U_k\) with

\[
U_k(C^k(t)) = E \left[ \int_0^T e^{-\rho t} u_k(C^k(t)) dt \right],
\]

where \(\rho\) is a positive constant which means the discount rate of utility and \(u_k\) is a von Neumann-Morgenstern utility function.

Assumption 4 (see [7]). The utility function \(u_k : (0, \infty)^N \rightarrow \mathbb{R}\) is an increasing, strictly concave function in class \(C^3\) and satisfies the multidimensional Inada conditions.

Assume that \(W^k(t)\) is the total wealth process of agent \(k\) in units of the numeraire good, and \(W^k(t) = \alpha^k(t) B(t) + \pi^k(t) \top S(t)\), where \(\alpha^k(t)\) represents the amount of the risk-free asset and \(\pi^k(t) = (\pi^k_1(t), \pi^k_2(t), \ldots, \pi^k_K(t))\) denotes the amount of units of stocks held by agent \(k\) at time \(t\). It is well known that the object of every individual agent is to maximize his own utility function under the limitation of his budget constraint. In other words, the wealth process of agent \(k\) must satisfy the dynamic budget constraint:

\[
dW^k(t) = W^k(t) r(t) dt - P(t) \top C^k(t) dt + \pi^k(t) \top [I_{\alpha^k(t)} \mu_S(S(t), t) - r(t) 1_{\alpha^k(t)}] dt + \pi^k(t) \top I_{\pi^k(t)} \sigma_S(S(t), t) dZ(t),
\]
where $I_{N \times N}$ is an $N \times N$ matrix with $S_j(t)$ as element $(i, i)$ and zero elsewhere and $1_{N \times 1}$ is an $N$-dimensional vector of ones.

To study the completeness of the whole market, we begin with the social planner’s, or, equivalently, the government’s, problem, which is to give each agent a utility weight $a_k$, $k = 1, 2, \ldots, K$, and maximize the weighted total utility of all agents. The problem is defined as follows.

Definition 5 (see [7]). The social planner’s problem is

$$u(a, g) = \max_{\sum_{k=1}^{K} C^k = g} \sum_{k=1}^{K} a_k u_k(C^k),$$

(7)

where $a = \{a_1, a_2, \ldots, a_K\}$ is the $K$-dimensional unit simplex and means the Pareto weights.

We cite the definition of the Arrow-Debreu equilibrium, which not only solves the social planner’s problem, but also needs that all agents’ individual consumption allocations are affordable if they have fixed initial wealth.

Definition 6 (see [7]). An Arrow-Debreu equilibrium is a state price density $\xi$, commodity prices $P$, and consumption allocations $(C^k)_{k=1}^{K}$ such that the following statements hold:

(a) All agents maximize their expected utility function under their own static budget constraints:

$$\max \quad U_k(C^k)$$

s.t. $E \left[ \int_0^T \xi(t) \left( P(t) C^k(t) - \eta^k(t) g(t) \right) d\tau \right] \leq 0,$

$$\sum_{k=1}^{K} C^k(t) = g(t)$$

(8)

(9)

(b) The commodity market clears:

$$\sum_{k=1}^{K} \eta^k(t) = 1,$$

$$\sum_{k=1}^{K} \eta^k_B(t) = 0,$$

(10)

where $\eta^k_B(t)$ is the risk-free bond held by agent $k$.

If the vector of utility weight $a$ is given, then the Arrow-Debreu equilibrium agrees with finding the allocations of initial wealth $\eta$ which satisfy the maximization of (7) and the agents’ individual budget constraints.

We know that the social planner’s problem is well defined no matter whether the financial market is complete or not. When we discuss the sufficient conditions for the incompleteness of financial markets in Section 3, we commence with (7) and show that the candidate stock prices do not complete the market under certain conditions. In Section 4, we derive sufficient conditions for the invertibility of the volatility matrix of candidate stock prices. Under these conditions, the Arrow-Debreu equilibrium and complete market are achieved by trading stocks and the risk-free bond.

Assumption 7 (see [7]). There exists an Arrow-Debreu equilibrium in which the state price density is given by

$$\xi(t) = e^{-\rho t} \frac{\partial u(a, g(t))}{\partial \eta_l},$$

(12)

and the price vector $P(t) = \{p_1(t), p_2(t), \ldots, p_N(t)\}^\top$ is given by

$$P(t) = \frac{\nabla u(a, g(t))}{\partial \eta_l}.$$

(13)

The elements of the vector of utility weights $a$ are given by the solutions of

$$E \left[ \int_0^T \xi(a, \tau) \cdot \left( P(a, \tau)^\top C^k(a, \tau) - \eta^k g(a, \tau) \right) d\tau \right] = 0,$$

(14)

for $k = 1, 2, \ldots, K$.

Proof. The proof follows that of Proposition 1 in Ehling and Larsen [7]. Here we omit the proof.

In Proposition 8, we find that the vector of commodity prices is a function of $g(t)$, which is a stochastic process associated with $X_t$. Then we use the well-known Itô lemma to get the diffusion coefficient of the price vector $P$. We give the following lemma to illustrate this.

Lemma 9. The diffusion coefficient of the vector of commodity prices $\sigma_P(t)$ is given by

$$\sigma_P(t) = \Gamma_{P0}^{-1} \varepsilon(t) \sigma_g(t),$$

(15)
where $I_{P(t)}$ is an $N \times N$ matrix with $p_i(t)$ as element $(i,i)$ and zero elsewhere, $\sigma_g(t)$ is the diffusion matrix of dividend rate vector $g(t)$, and $\epsilon(t)$ is the Jacobian matrix of $P(g)$, i.e.,

$$\epsilon(t) = \begin{bmatrix}
\frac{\partial p_1}{\partial g_1} & \frac{\partial p_1}{\partial g_2} & \ldots & \frac{\partial p_1}{\partial g_N} \\
\frac{\partial p_2}{\partial g_1} & \frac{\partial p_2}{\partial g_2} & \ldots & \frac{\partial p_2}{\partial g_N} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial p_N}{\partial g_1} & \frac{\partial p_N}{\partial g_2} & \ldots & \frac{\partial p_N}{\partial g_N}
\end{bmatrix}.$$  

(16)

Proof. Note that $P(t) = \mathbb{V}(a, g(t))/\partial u(a, g(t))/\partial g_j)$ and $g(t)$ is a stochastic process. We apply Itô's lemma to complete the proof.

Using Itô's lemma again, we get the following proposition.

**Proposition 10.** The diffusion coefficient of the process of dividend rate in units of the numeraire good $\tilde{g}(t)$ is given by

$$\sigma_{\tilde{g}} = [I_g \epsilon(t) + I_p] \times \sigma_g,$$

where $I_g$ is an $N \times N$ matrix with $g_i(t)$ as element $(i,i)$ and zero elsewhere.

Furthermore, $\sigma_{\tilde{g}}$ is the product of the Jacobian matrix of $\tilde{g}(g)$ and the volatility matrix of dividends, $\sigma_g$, i.e.,

$$\sigma_{\tilde{g}} = \frac{\partial \tilde{g}}{\partial g} \times \sigma_g = \begin{bmatrix}
\frac{\partial \tilde{g}_1}{\partial g_1} & \frac{\partial \tilde{g}_1}{\partial g_2} & \ldots & \frac{\partial \tilde{g}_1}{\partial g_N} \\
\frac{\partial \tilde{g}_2}{\partial g_1} & \frac{\partial \tilde{g}_2}{\partial g_2} & \ldots & \frac{\partial \tilde{g}_2}{\partial g_N} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial \tilde{g}_N}{\partial g_1} & \frac{\partial \tilde{g}_N}{\partial g_2} & \ldots & \frac{\partial \tilde{g}_N}{\partial g_N}
\end{bmatrix} \times \sigma_g.$$  

(18)

Proof. The proof is the direct calculations by applying Itô's lemma to the consumption process in units of the numeraire. We find that the element $(i,j)$ of the matrix $\sigma_{\tilde{g}}$ is given by

$$\sigma_{\tilde{g}}(i,j) = \begin{cases}
p_i + g_i \frac{\partial p_i}{\partial g_i} = \frac{\partial \tilde{g}_i}{\partial g_i} \times \sigma_g & (i = j), \\
\frac{\partial p_i}{\partial g_j} = \frac{\partial \tilde{g}_j}{\partial g_j} \times \sigma_g & (i \neq j),
\end{cases}$$

(19)

which is the Jacobian matrix of $\tilde{g}(g)$.

Note that, in Assumption 3, $\sigma_{\tilde{g}}(g(X_i),t)$ is assumed to be invertible at all times. Thus, we deduce that $\tilde{g}(t)$ is invertible if and only if $I_{\tilde{g}(t)}\epsilon(t) + I_{p(t)}$ or the Jacobian matrix $\partial \tilde{g}/\partial g$ is invertible. In other words, even if the financial market is potentially complete, the market can be incomplete since $\partial \tilde{g}/\partial g$ is noninvertible.

In the Arrow-Debreu equilibrium, a reasonable form of stock prices is the expected discounted value of all the dividends that will be paid in the future:

$$S_j(t) = E_t \left[ \int_t^T \frac{\xi(\tau)}{\xi(t)} p_i g_i(X_\tau) d\tau \right], \quad i = 1, 2, \ldots, N.$$  

(20)

### 3. Incomplete Markets

In this section, we investigate sufficient conditions for the incompleteness of financial market that we mentioned in Section 2.

As is well known, the completeness of markets is closely linked with the volatilities of stock prices in a continuous-time model. Specifically, one can show that the volatility matrix of stock prices $\sigma_g$ is invertible if and only if the market is complete. Relatively, proving that the market is incomplete is equivalent to proving that the volatility matrix of stock prices $\sigma_g$ is noninvertible (see Duffie [2] for more details). This gives us a concrete way to verify the completeness of the financial markets.

Moreover, one can show that if an $N \times N$ matrix $A$ is noninvertible, or, identically, $|A| = 0$, then either of the following two conditions must be satisfied. Firstly, if all the vertical vectors of $A$ are nonzero, then the transversal vectors of $A$ are linearly dependent. Secondly, at least one of the vertical vectors is zero.

If we regard $\sigma_g$ as the matrix $A$, then we implicitly divide the incomplete markets into two cases corresponding to the two conditions above. The first one is that although stock prices carry on all the risks in the state variables $X_i$, at least one of the stocks is expressed as a linear combination of other stocks. This means that some stocks are replicable by using the portfolio that consists of other stocks, i.e.,

$$S_j(t) = \sum_{j=1,j\neq i}^{N} \beta_j S_j(t),$$

(21)

where $i$ is an arbitrary natural number in $\{1, N\}$ and $\beta_j \in \mathbb{R}$, for $j = 1, 2, \ldots, i-1, i+1, \ldots, N$, and they cannot all equal zero. In this case, $\sigma_g$ must be noninvertible. Hence the financial market is incomplete. The second condition implies the case in which the financial market loses at least one sort of risks represented by a Brownian motion during the process in which the risks transfer from the state variables $X(t)$ to the stock prices $S(t)$. This inevitably causes that at least one of the vertical vectors in $\sigma_g$ equals zero. Therefore, $\sigma_g$ is noninvertible and the incompleteness of financial market emerges.

In the first case, we emphasize that all the vertical vectors of $\sigma_g$ are nonzero. It is due to the fact that stock prices cannot be zero. Meanwhile, in the second case, we need that at least one of the vertical vectors is zero because we focus on the volatility of stock prices, which do not have to be nonzero.

The rest of this section is dedicated to debating these two cases, respectively. We start with the first one.

**Lemma 11.** If the social planner’s utility function $u(t)$ and the vector of dividends rate $g(t)$ satisfy the partial differential equation

$$\frac{\partial u(g(t))}{\partial g_j} g_j(t) = \sum_{j=1,j\neq i}^{N} \beta_j \frac{\partial u(g(t))}{\partial g_j} g_j(t),$$

(22)
then the price of stock $i$ is a linear combination of the prices of other stocks, i.e.,

$$S_i(t) = \sum_{j=1,j\neq i}^{N} \beta_j S_j(t).$$  \hfill (23)

Thus, the financial market is incomplete.

Proof. We assume that the solution of (22) exists. Multiplying both sides of (22) by $e^{-pt}$ and integrating from $t$ to $T$, we get

$$\int_t^T e^{-p\tau} \frac{\partial u(g(\tau))}{\partial g_i} g_i(\tau) \, d\tau = \sum_{j=1,j\neq i}^{N} \int_t^T e^{-p\tau} \left( \beta_j \frac{\partial u(g(\tau))}{\partial g_j} g_j(\tau) \right) \, d\tau,$$

and we take conditional expectation on both sides to obtain

$$E_t \left[ \int_t^T e^{-p\tau} \frac{\partial u(g(\tau))}{\partial g_i} g_i(\tau) \, d\tau \right] = \sum_{j=1,j\neq i}^{N} E_t \left[ \int_t^T e^{-p\tau} \left( \beta_j \frac{\partial u(g(\tau))}{\partial g_j} g_j(\tau) \right) \, d\tau \right].$$ \hfill (24)

Dividing the equation by $\xi(t)$ and writing it as the candidate stock price formula in (20), we get $S_i(t) = \sum_{j=1,j\neq i}^{N} \beta_j S_j(t)$. We find that (22) is a first-order partial differential equation. Solving this equation, we get Lemma 12.

Lemma 12. The general solution to (22) is

$$u(g) = \varphi(g_1^{\beta_1}, g_2^{\beta_2}, \ldots, g_i^{\beta_i}, \ldots, g_N^{\beta_N}),$$

where $\varphi(\cdot) : \mathbb{R}^{N-1} \rightarrow \mathbb{R}$ and it is an arbitrary real analytic function such that $u(\cdot)$ satisfies Assumption 4.

Proof. Solving partial differential equation (22), we complete the proof.

Now we present Conclusion 13.

Conclusion 13 (see [7]). If the social planner's utility function $u(g(t))$ satisfies (26), then the financial market is incomplete.

Proof. Using Lemmas 11 and 12, we complete the proof of Conclusion 13.

Conclusion 13 presents utility functions which make the financial markets incomplete. We can check the incompleteness of the financial market by verifying whether the social planner's utility function from Definition 5 (see (7)) satisfies the condition in Conclusion 13. Comparing our method with the techniques which need to calculate the volatility matrix of stock prices for all time and states and check the invertibility of the matrix, we find that our method is not only easier but also more convenient.

We give an example to illustrate this conclusion, which is different from that in Ehling and Larsen [7].

Example 14. Assume that there are three goods in the market, and the social planner's utility function is

$$u(g_1(t), g_2(t), g_3(t)) = \beta_1 g_1(t) g_2^{\beta_2}(t) g_3^{\beta_3}(t),$$

where $\beta_1, \beta_2,$ and $\beta_3$ are positive constants such that Assumption 4 holds. Let the first good be the numeraire. When the market achieves equilibrium, we get

$$p_1(t) = 1,$$

$$p_2(t) = \frac{\beta_2 g_1(t)}{\beta_1 g_2(t)},$$

$$p_3(t) = \frac{\beta_3 g_1(t)}{\beta_1 g_3(t)}.$$ \hfill (28)

Then we obtain that the dividends in units of the numeraire at time $t$ are given by

$$\bar{g}_1(t) = g_1(t),$$

$$\bar{g}_2(t) = \frac{\beta_2 g_1(t)}{\beta_1},$$

$$\bar{g}_3(t) = \frac{\beta_3 g_1(t)}{\beta_1}.$$ \hfill (29)

We see that dividend rates in units of good 1 are only related to $g_1(t)$. Specifically, they are all proportional to $g_1(t)$ for all times and states of the economy. Therefore, we conclude that stock 2 and stock 3 are colinear with stock 1 in the sense of candidate stock price. This gives rise to the incompleteness of financial market. Practically, if we let good 2 or 3 be the numeraire good, instead of good 1, then we find that the financial market is incomplete too.

To apply Conclusion 13, we change the social planner’s utility function into

$$u(g_1(t), g_2(t), g_3(t)) = \varphi \left( g_2(t) g_3^{\beta_3/2\beta_1}(t) \right)^{\beta_1} \left( g_2(t) g_1^{\beta_1/2\beta_3}(t) \right)^{\beta_3},$$

$$u(g_1(t), g_2(t), g_3(t)) = \varphi (g_2(t) g_1(t)^{\alpha_1}, g_3(t) g_1(t)^{\alpha_2}),$$ \hfill (31)

where $\varphi(x, y) = x y, \alpha_1 = \beta_1/2\beta_2, \alpha_2 = \beta_1/2\beta_3$. Therefore, Conclusion 13 is available and brief to demonstrate that the financial market is incomplete.

As we see in Example 14, one can check whether the utility function leads to the incomplete market or not by applying Conclusion 13. This method is also useful to verify that the financial model in Serrat [9] is incomplete, in which the Cobb-Douglas utility function is adopted to describe the market.
Conclusion 13 gives a sufficient, while not necessary, condition for the first situation that leads to the incompleteness of financial markets. We note that it is hard to find a both necessary and sufficient condition for the incompleteness market since \( g(t) \) in (22) is a vector of stochastic processes, and the equality holds in the “almost surely” sense. That is to say, if \( u(g(t)) \) satisfies (22), then the measure of \( (\partial u(g(t))/\partial g_j)g_j(t) \neq \sum_{j=1,j \neq i}^{N} \beta_j(\partial u(g(t))/\partial g_j)g_j(t) \) is zero. While (23) is a function of expectation, it still holds without the condition that the measure of \( (\partial u(g(t))/\partial g_j)g_j(t) \neq \sum_{j=1,j \neq i}^{N} \beta_j(\partial u(g(t))/\partial g_j)g_j(t) \) is zero. This means that the condition of (22) is stronger than that in (23), and thus the condition in Conclusion 13 is only sufficient while not necessary for the incompleteness of the market.

Now we study the second case. Similar to the first one, we derive a differential equation and prove that this equation leads to the incompleteness of markets.

**Lemma 15.** If the dividend rate vector \( g_j(t) \) has no connection with \( X_j(t) \) for all \( i \neq j \), i.e.,

\[
\frac{\partial g_j(t)}{\partial X_j(t)} = 0 \quad (32)
\]

for \( i = 1, 2, \ldots, j-1, j+1, \ldots, N \), the social planner’s utility function \( u(g(t)) \) and the dividend rate vector \( g(t) \) satisfy the differential equation

\[
g_j(t) \frac{\partial^2 u(g(t))}{\partial g_j^2} + \frac{\partial u(g(t))}{\partial g_j} = 0 \quad (33)
\]

and good \( j \) is not the numeraire good, then the financial market is incomplete.

**Proof.** From the first condition in Lemma 15, we know that the risks in \( X_j \) will not appear in stock prices except for \( S_j \). As long as we make sure that the risks in \( X_j \) do not appear in \( S_j \), the market is incomplete.

Note that \( (\partial u(t)/\partial g_j(t))g_j(t) \) is all the random part in the expectation in (20), and we have

\[
d \frac{\partial u(t)}{\partial g_j(t)}g_j(t) = g_j(t) \frac{\partial u(t)}{\partial g_j(t)} + \frac{\partial u(t)}{\partial g_j(t)}dg_j(t)
\]

\[
+ \frac{\partial u(t)}{\partial g_j(t)}dg_j(t).
\]

We only care about \( g_j(t) \) since it is the only term that may carry the risks of \( X_j \). Using Itô’s lemma we know that the coefficient of \( dg_j(t) \) in (34) is \( g_j(t)(\partial^2 u(g)/\partial g_j^2) + \partial u(g)/\partial g_j \). If \( g_j(t)(\partial^2 u(g)/\partial g_j^2) + \partial u(g)/\partial g_j = 0 \), then the financial market has no connection with \( X_j \), which makes the market incomplete.

Condition (32) is required to prevent the risk of \( X_j \) from spreading out to dividends except for \( g_j(t) \). This condition seems rigorous, but it is reasonable for some industries which have less connection with others. For example, it is reasonable that, in special high-tech companies, their outputs rely heavily on the innovations of scientific researchers who occasionally have outstanding ideals, while those innovations barely influence other companies because the companies try to protect their property right or apply a patent.

The next lemma gives the solution of (33).

**Lemma 16.** The general solution of differential equation (33) is

\[
u(g) = C \ln g_j + \varphi \left( g_1, g_2, \ldots, g_{j-1}, g_{j+1}, \ldots, g_N \right), \quad (35)
\]

where \( C \) is a constant and \( \varphi(\cdot) \) is an arbitrary real analytic function such that \( u(t) \) satisfies Assumption 4.

**Proof.** The proof is completed by solving the differential equation in Lemma 15.

**Theorem 17.** If the social planner’s utility function \( u(g(t)) \) satisfies (35), the state variable vector \( X(t) \) and the dividend rate vector \( g(t) \) satisfy (32) for \( i = 1, 2, \ldots, j-1, j+1, \ldots, N \) and the good \( j \) is not the numeraire good, then the financial market is incomplete.

**Proof.** The proof is completed by directly using Lemmas 15 and 16.

Inducing the condition in Lemma 15 and the special form of the utility function in Lemma 16, the financial market eventually gets rid of the risks in \( X_j \) when equilibrium is achieved. This leads one of the column vectors of \( \sigma_S \) to be zero, and thus \( |\sigma_S| = 0 \). Therefore, the financial market is incomplete.

We give an example to illustrate Theorem 17.

**Example 18.** Similar to Example 14, we assume that there are three goods in the market, and the social planner’s utility function is

\[
u(g_1, g_2, g_3) = \beta_1 \ln (g_1) + g_2^\beta_2 g_3^\beta_3,
\]

where \( \beta_1, \beta_2, \) and \( \beta_3 \) are constants such that Assumption 4 holds. The dividend rate is

\[
g_i(t) = c^{X_i(t)}, \quad i = 1, 2, 3.
\]

The state variables are given by

\[
X_1(t) = \mu_1 (X_1) dt + \sigma_1 (X_1) dZ_1 (t),
\]

\[
X_2(t) = \mu_2 (X_2, X_3) dt + \sigma_2 (X_2, X_3) dZ_2 (t),
\]

\[
X_3(t) = \mu_3 (X_2, X_3) dt + \sigma_3 (X_2, X_3) dZ_3 (t).
\]
Taking good 3 as the numeraire good, we get that the dividend rates in the units of good 3 are given by

\[ \tilde{g}_1(t) = \frac{\beta_1}{\beta_3} g_2(t), \]
\[ \tilde{g}_2(t) = \frac{\beta_2}{\beta_3} g_3(t), \]
\[ \tilde{g}_3(t) = g_3(t). \]  

(41)

We find that \( \tilde{g}_3(t) \) is absent in the above equations, while it is the unique factor that carries the risk in \( X_3(t) \). Hence, the diffusion matrix of dividend rates \( \sigma_3 \) in units of good 3 must be noninvertible. Then it can be shown that the volatility matrix of stock prices \( \sigma_3 \) is also noninvertible because of the loss of a risk, which certainly leads to the incompleteness of financial market.

Different from Example 14, we find that the change of numeraire good makes a real difference. If we take good 1 as the numeraire good instead of good 2 or good 3, then we get

\[ \tilde{g}_1(t) = g_1(t), \]
\[ \tilde{g}_2(t) = \frac{\beta_2}{\beta_1} g_1(t) g_2^\beta_1(t) g_3^\beta_1(t), \]
\[ \tilde{g}_3(t) = \frac{\beta_3}{\beta_1} g_1(t) g_2^\beta_1(t) g_3^\beta_1(t). \]  

(42)

We find that the dividend rates in units of the numeraire good carry all the risk in state variables, and they will be transported to the stock prices. Therefore, the financial market is complete.

One obvious difference between Examples 14 and 18 is whether the switch of numeraire good changes the completeness of markets or not. The reason for the difference is that when we change the numeraire good, we actually change the set of available assets as well, and the definition of available assets will impact the completeness of financial markets intensely.

To regularize the observation, we assume that the amount of available assets is fixed while we change the numeraire good. In order to preserve the assumption that the risk-free bond pays interests in the units of numeraire good, we convert the structure of assets to adapt this requirement. This suggests that the previous risk-free bond cannot be traded if we change the numeraire good. Under these assumptions, we define the numeraire good indifference.

\textbf{Definition 19 (see [7]).} The economy defined in Section 2 is numeraire good indifference if an arbitrary choice of the numeraire good has no effect on the rank of the volatility matrix of stock prices \( \sigma_3 \) or the completeness of the financial markets.

We notice that, in the first example, no matter what good we choose as the numeraire, the market is always incomplete. Meanwhile, in the second example, the rank of \( \sigma_3 \) drops if good 2 or good 3 is the numeraire good. Therefore, the economy in Example 14 shows the attribute of numeraire good indifference, and Example 18 does not.

The next proposition aims to illustrate this observation.

\textbf{Proposition 20.} If the utility function satisfies the condition in Conclusion 13, then the financial markets exhibit the attribute of numeraire good indifference. If the utility function satisfies the condition in Theorem 17, then the financial markets do not exhibit the attribute of numeraire good indifference.

\textit{Proof.} Note that the condition in Conclusion 13 is irrelevant to the numeraire good, while the condition in Theorem 17 involves the choice of numeraire good. The proof is completed.

Apart from Example 18, Theorem 17 is able to demonstrate the example in Ehling and Larsen [7], in which they assume that the social planner’s utility function \( u(g_1(t), g_2(t)) = \ln(g_1(t)) + g_2(t)^{1-\gamma}/(1-\gamma) \), and other conditions in Theorem 17 are also satisfied. Applying Theorem 17, we find that the financial market of this model is incomplete without difficult calculations.

To summarize, our conclusion and theorem can help those who want to build complete financial markets to avoid some incorrect settings and assumptions about utility functions and give suggestions to those who want to set up incomplete financial markets.

\section{4. Complete Markets}

In this section, we are going to investigate the sufficient conditions for completeness of financial markets. Before we present our results, several additional assumptions are imposed. They are similar to assumptions in Hugonnier et al. [4] and Ehling and Larsen [7].

\textbf{Assumption 21 (see [7]).} The unique solution to (1) admits a transition density \( \phi(t, x, \tau, y) \) that is smooth for \( t \neq \tau \).

\textbf{Assumption 22 (see [7]).} There are locally bounded function \( (K, L) \), a metric \( d \) that is locally equivalent to the Euclidean metric, and constants \( \varepsilon, \iota, \varphi > 0 \) such that \( \phi(t, x, \tau, y) \) is analytic with respect to \( t \neq \tau \) in the set

\[
\mathcal{P}_\varepsilon \equiv \left\{ (t, \tau) \in \mathbb{R}^2 : \Re(t) \geq 0, 0 \leq \Re(\tau) \leq T \, \text{ and } \, |\varrho| \leq c \Re(\tau - t) \right\},
\]  

and satisfies

\[
|\phi(t, x, \tau, y)| \leq K(x) L(y)|\tau - t|^{-\varepsilon} e^{d(x,y)^2/|\tau - t|} \equiv \overline{\phi}(t, x, \tau, y),
\]  

for all \( (t, \tau, x, y) \in \mathcal{P}_\varepsilon \times \mathbb{R}^2 \).

\textbf{Assumption 23 (see [7]).} The dividend rates \( g_i(x) \), for \( i = 1, 2, \ldots, N \), are real analytic functions.
Assumption 24 (see [7]). The utility function of agent \( k, u_k \), for \( k = 1, 2, \ldots, K \), is analytic and there are constants \( R \leq \rho \) and \( v > 1 \) such that

\[
\int_0^T \sum_{k=1}^K \left( e^{-R\tau \partial u_k(g(y)/K)} \frac{\partial g(y)}{\partial g} \right) d\tau < \infty.
\]  

Definition 25 (see [4]). Given the state price density \( \phi(t) = \psi(t)P(t) \), the consumption allocation \( C^k \) is feasible for agent \( k \) if there exists an investment strategy \( (\pi^k_1, \pi^k_2) \) that finances \( C^k \) at the initial cost of \( \eta(T)S(0) \) and ensures that the process \( \xi(t)W_t + \int_0^t \xi_s c_s d\tau \) is a martingale with \( W_T \geq 0 \).

Based on the definition of the feasible consumption allocation, we define the Radner equilibrium, which is an extension of the Arrow-Debreu equilibrium.

Definition 26 (see [10]). A Radner equilibrium is made up of the state price density \( \xi(t) \), the commodity prices \( P(t) \), a set of price processes \((B, S)\), consumption allocations \((C^k)_{k=1}^K\) and a set of investment strategies \((\pi^k_1, \pi^k_2)_{k=1}^K\) such that the following statements hold:

(a) \( C^k \) maximizes \( U_k \) over the feasible consumption allocations and is financed by \((\pi^k_1, \pi^k_2)\)
(b) All markets clear

As is shown in Dana and Jeanblanc [10], the Arrow-Debreu equilibrium can be implemented as a Radner equilibrium if the markets are dynamically complete. Riedel and Herzberg [5] use this method to prove that, in smooth Markovian continuous-time economies, if the candidate stock price function is real analytic and the volatility matrix of terminal dividends is invertible, there exist Radner equilibria with dynamic complete markets.

Under the above assumptions and definitions, we give the condition of the social planner’s utility function for complete markets.

Theorem 27. Let Assumptions 1–24 hold. If the social planner’s utility function \( u(g(t)) \) and the numeraire good \( l \) make the Jacobian matrix of \( \bar{g}(g(T, x)) \) invertible, i.e.,

\[
\frac{\partial \bar{g}}{\partial g} \neq 0
\]  

for at least one \( x \in \mathcal{X} \), then there exists a Radner equilibrium with dynamically complete financial market.

Proof. Applying Theorem B.2 in Anderson and Raimondo [3] and the analytic implicit function theorem in Krantz and Parks [11], we derive that the candidate price function \( S \) is also joint real analytic in \( (t, x) \in (0, T) \times \mathcal{X} \). Using Proposition 2 in Hugonnier et al. [4], we know that the candidate price function \( S \) is jointly real analytic in \( (t, x) \in (0, T) \times \mathcal{X} \). Following the work in Hugonnier et al. [4], it is shown that the volatility matrix of stock prices \( \sigma_S \) is

\[
\sigma_S = (T-t)\sigma_{\bar{g}(g(T, x))} + o(T-t)
\]  

which derives that \( \sigma_S \) is jointly real analytic and is proportional to the volatility of the dividends in units of numeraire good when time approaches \( T \). Then \( |\sigma_S| = |\partial \bar{g}/\partial g| \times |\sigma_g| \neq 0 \), while \( |\sigma_g| \) is not zero by our assumptions. Therefore, using Proposition 3 in [4], we conclude that a sufficient condition for the existence of the Radner equilibrium with dynamically complete financial market is \( |\partial \bar{g}/\partial g| \neq 0 \) for at least one \( x \in \mathcal{X} \).

Note that the sufficient condition for the completeness of financial markets is different from those in Hugonnier et al. [4], Elie and Larsen [7], and Riedel and Herzberg [5]. We import the Jacobian matrix \( \partial \bar{g}(g)/\partial g \) to test the completeness of markets. This matrix is not only simple to write but also easy to calculate. If \( \partial \bar{g}(g)/\partial g \neq 0 \) at one point, then we deduce that the market is complete. Even if the social planner’s utility function is unknown in closed form, it is also easy to verify the completeness of markets by applying Theorem 27. Using numerical methods, one can solve the social planner’s utility function and check whether the market is complete or not. However, if we compute the volatilities of stock prices for all possible realization \( (t, x) \in [0, T] \times \mathcal{X} \) instead of using Theorem 27, it may be too complex and even infeasible to get the result.

We know that the social planner’s utility function \( u(g) \) is hard or even impossible to derive while Theorem 27 provides the possibility of verifying the completeness of financial markets without knowing the concrete form of \( u(g) \). Since the data of dividends and dividends in units of numeraire good are easy to obtain and available to calculate, we can
approximately calculate $\frac{\partial \tilde{g}(g)}{\partial g}$ and apply Theorem 27 to verify the completeness of financial markets.

5. Conclusion

In previous sections, we discuss the attributes and sufficient conditions of incomplete and complete financial markets in a continuous-time economy with multiple agents and commodities. Based on the work in Anderson and Raimondo [3] and Hugonnier et al. [4], we give sufficient conditions for the completeness of markets. Through checking whether the economy satisfies this condition or not, one can verify the completeness of the markets without difficult calculations. A sufficient condition for the incompleteness of market is investigated. In this condition, the crucial step is to check the invertibility of the Jacobian matrix of the dividends rate in the units of numeraire good for just one realization of the state variables. The conditions in this paper for incomplete and complete markets should be helpful in the domain of asset pricing.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

Acknowledgments

The authors are very grateful to the reviewers for their helpful comments of the paper and confirm that this work is supported by the Fundamental Research Funds for the Central Universities [JBK120504].

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