Amoebas of maximal area.

Grigory Mikhalkin *
Department of Mathematics
University of Utah
Salt Lake City, UT 84112, USA
mikhalkin@math.utah.edu

Hans Rullgård
Department of Mathematics
Stockholm University
S-10691 Stockholm, Sweden
hansr@matematik.su.se

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Abstract

To any algebraic curve \( A \) in \( (\mathbb{C}^*)^2 \) one may associate a closed infinite region \( \mathcal{A} \) in \( \mathbb{R}^2 \) called the amoeba of \( A \). The amoebas of different curves of the same degree come in different shapes and sizes. All amoebas in \( (\mathbb{R}^*)^2 \) have finite area and, furthermore, there is an upper bound on the area in terms of the degree of the curve.

The subject of this paper is the curves in \( (\mathbb{C}^*)^2 \) whose amoebas are of the maximal area. We show that up to multiplication by a constant in \( (\mathbb{C}^*)^2 \) such curves are defined over \( \mathbb{R} \) and, furthermore, that their real loci are isotopic to so-called Harnack curves.

1 Introduction.

Let \( f : \mathbb{C}^2 \to \mathbb{C} \) be a polynomial, \( f(z_1, z_2) = \sum_{j,k} a_{jk} z_1^j z_2^k \). Its zero set in \( (\mathbb{C}^*)^2 \) is a curve \( A = f^{-1}(0) \cap (\mathbb{C}^*)^2 \) (where \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \)). Let \( \Delta \subset \mathbb{R}^2 \) be the Newton polygon of \( f \), i.e. the convex hull of \( \{(j,k) \mid a_{jk} \neq 0\} \). Gelfand, Kapranov and Zelevinski introduced one more object associated to \( f \).

Definition 1 (Gelfand, Kapranov, Zelevinski [3]). The amoeba \( \mathcal{A} \subset \mathbb{R}^2 \) of \( f \) is \( \text{Log}(\mathcal{A}) \), where \( \text{Log} : (\mathbb{C}^*)^2 \to \mathbb{R}^2, (z_1, z_2) \mapsto (\log |z_1|, \log |z_2|) \).

It was remarked in [3] that every component of \( \mathbb{R}^2 \setminus \mathcal{A} \) is open and convex in \( \mathbb{R}^2 \). In particular, \( \mathcal{A} \) is closed and its (Lebesgue) area is well-defined.

Note that \( \mathcal{A} \) is never bounded in \( \mathbb{R}^2 \), since \( f^{-1}(0) \) must intersect the coordinate axes in \( \mathbb{C}^2 \). However it was shown by Passare and Rullgård [8] that the area of \( \mathcal{A} \) is always finite. Furthermore, it is bounded in terms of \( \Delta \).

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1In this paper we restrict our attention to functions of two variables. Amoebas are defined for functions of any number of variables.
Theorem (Passare, Rullgård [8]).

\[ \text{Area}(\mathcal{A}) \leq \pi^2 \text{Area}(\Delta). \]  
(1)

The main result of this paper is the extremal property of this inequality.

We say that a curve \( A \) is defined over \( \mathbb{R} \) if it is invariant under the complex conjugation \( \text{conj} : (\mathbb{C}^*)^2 \to (\mathbb{C}^*)^2 \), \( (z_1, z_2) \mapsto (\overline{z}_1, \overline{z}_2) \). In this case we may consider the real part of the curve \( \mathbb{R}A = A \cap (\mathbb{R}^*)^2 \) which is a real algebraic curve. We say that a curve \( A \) is real up to multiplication by a constant if there exist constants \( b_1, b_2 \in \mathbb{C}^* \) such that \( (b_1, b_2) \times A \subset (\mathbb{C}^*)^2 \) is defined over \( \mathbb{R} \). The condition that \( A \) is real up to multiplication by a constant is equivalent to the condition that there exist \( a, b_1, b_2 \in \mathbb{C}^* \) such that the polynomial \( af(\frac{z_1}{b_1}, \frac{z_2}{b_2}) \) has real coefficients. In this case we may also consider the real part \( \mathbb{R}A = \{(x_1, x_2) \in (\mathbb{R}^*)^2 \mid af(\frac{x_1}{b_1}, \frac{x_2}{b_2}) = 0\} \). We say that a map is at most 2-1 if the inverse image of any point in the target consists of at most 2 points. The main result of this paper is the following theorem.

**Theorem 1.** Suppose that \( \text{Area}(\Delta) > 0 \). Then the following conditions are equivalent.

1. \( \text{Area}(\mathcal{A}) = \pi^2 \text{Area}(\Delta) \).
2. The map \( \text{Log} \mid_A : A \to \mathbb{R}^2 \) is at most 2-1 and \( A \) is real up to multiplication by a constant.
3. The curve \( A \) is real up to multiplication by a constant and its real part \( \mathbb{R}A \) is a (possibly singular) Harnack curve (see Definitions 2 and 3) for the Newton polygon \( \Delta \).

Furthermore, these conditions imply that the non-singular locus of \( \mathbb{R}A \) coincides with \( A \cap \text{Log}^{-1}(\partial \mathcal{A}) \).

**Corollary 1.** The inequality (1) is sharp for any Newton polygon \( \Delta \).

The corollary follows from Theorem 1 and the Harnack-Itenberg-Viro Theorem (see section 2) on existence of Harnack curves.

**Remark 1.** A curve that is real up to multiplication by a constant may have more than one real part (other real parts may come as a result of multiplication by different constants). For instance, if \( f \) is a real polynomial which contains only even powers of \( z_2 \) then the pullback of \( f \) under \( (z_1, z_2) \mapsto (z_1, iz_2) \) is a real polynomial with a different real part.

The theorem implies that a Harnack curve is real up to multiplication by a constant in a unique way. Indeed, the choice of the real part is determined by the identity \( \mathbb{R}A = A \cap \text{Log}^{-1}(\partial \mathcal{A}) \).
2 Harnack curves in \((\mathbb{R}^*)^2\).

Let us fix a convex polygon \(\Delta \subset \mathbb{R}^2\) whose vertices have integer coordinates. Consider all possible real polynomials \(f\) whose Newton polygon is \(\Delta\). The same polynomial \(f\) may be viewed both as a function \((\mathbb{C}^*)^2 \to \mathbb{C}\) and as a function \((\mathbb{R}^*)^2 \to \mathbb{R}\).

Let \(\mathbb{R}A\) be the zero set of \(f\) in \((\mathbb{R}^*)^2\). Equivalently, \(\mathbb{R}A\) is a real part of the zero set \(A\) of \(f\) in \((\mathbb{C}^*)^2\). For a generic choice of coefficients of \(f\) the curve \(\mathbb{R}A\) is smooth. However the topology of \(((\mathbb{R}^*)^2, \mathbb{R}A)\) is different for different choices of coefficients of \(f\). In particular, the number of components of \(\mathbb{R}A\) may be different. Also the mutual position of the components may be different.

We may compactify the above setup. Recall (see e.g. [8]) that the polygon \(\Delta\) determines a toric surface \(\mathbb{CT}_\Delta \supset (\mathbb{C}^*)^2\). We denote the real part of \(\mathbb{CT}_\Delta\) with \(\mathbb{RT}_\Delta \supset (\mathbb{R}^*)^2\). The surface \(\mathbb{CT}_\Delta\) is a compactification of \((\mathbb{C}^*)^2\). Furthermore, the complement \(\mathbb{CT}_\Delta \setminus (\mathbb{C}^*)^2\) is a union of \(n\) (non-disjoint) lines, where \(n\) is the number of sides of \(\Delta\). Similarly, \(\mathbb{RT}_\Delta \setminus (\mathbb{R}^*)^2\) is a union of \(n\) real lines \(l_1, \ldots, l_n\). These lines are called the axes of \(\mathbb{RT}_\Delta\). We assume that the indexing of \(l_k\) is consistent with the natural cyclic order on the sides of \(\Delta\).

The closure \(\bar{A}\) of \(A \subset (\mathbb{C}^*)^2 \subset \mathbb{CT}_\Delta\) in \(\mathbb{CT}_\Delta\) is a compact curve whose real part is \(\mathbb{R}\bar{A} \supset \mathbb{RA}\). The topology of the triad \(((\mathbb{RT}_\Delta; \mathbb{R}\bar{A}, l_1, \ldots, l_n)\) carries all topological information on arrangement of \(\mathbb{R}A\) in \((\mathbb{R}^*)^2\).

The upper bound on the number of components of \(\mathbb{R}\bar{A} \subset \mathbb{RT}_\Delta\) is provided by Harnack’s inequality [8]. This number is never greater than one plus the genus of \(A\). Recall that by [8] the genus of \(A\) is equal to the number of lattice points in the interior of \(\Delta\). We denote this number with \(g\).

To deduce the upper bound on the number of components of \(\mathbb{R}A \subset (\mathbb{R}^*)^2\) we recall that \(\mathbb{RA} = \mathbb{R}\bar{A} \setminus (l_1 \cup \cdots \cup l_n)\), where \(l_k\) corresponds to a side \(\delta_k\) of \(\Delta\). Let \(d_k\) be the integer length of \(\delta_k\), i.e. the number of lattice points inside \(\delta_k\) plus one. Note that this length is an \(SL(2,\mathbb{Z})\)-invariant. The curve \(\mathbb{R}\bar{A}\) and the axis \(l_k\) intersect in no more that \(d_k\) points, since \(d_k\) is the intersection number of their complexifications. Therefore, \(\mathbb{R}A\) has no more than \(g + \sum_{k=1}^{n} d_k\) components.

**Definition 2 (Harnack curves, cf. [8].** A non-singular curve \(\mathbb{R}A \subset (\mathbb{R}^*)^2\) with the Newton polygon \(\Delta\) is called a Harnack curve if all the following conditions hold.

- The number of components of \(\mathbb{R}\bar{A}\) is equal to \(g + 1\) (where \(g\) is the number of lattice points in the interior of \(\Delta\)).
- All components of \(\mathbb{R}\bar{A}\) but one do not intersect \(l_1 \cup \cdots \cup l_n\).
- A component \(C\) of \(\mathbb{R}\bar{A}\) can be divided into \(n\) consecutive (with respect to the cyclic order on \(C\)) arcs \(\alpha_1, \ldots, \alpha_n\) so that for each \(k\) the intersections \(\alpha_k \cap l_k\) consists of \(d_k\) points, while \(\alpha_k \cap l_j = \emptyset\), \(j \neq k\).

Note that the first two conditions imply that the number of components of a Harnack curve \(\mathbb{R}A\) is equal to \(g + \sum_{k=1}^{n} d_k\).
Theorem (Mikhalkin [7]). For each Newton polygon $\Delta$ the topological type of the triad $(\mathbb{R} T_{\Delta}; \mathbb{R} A, l_1 \cup \cdots \cup l_n)$ is unique if $\mathbb{R} A$ is a Harnack curve.

Note that the above theorem implies that the topological type of the pair $((\mathbb{R}^*)^2, \mathbb{R} A)$ is also unique for each $\Delta$.

Theorem (Harnack, Itenberg, Viro, [4], [5], [7]). Harnack curves exist for any Newton polygon $\Delta$.

Harnack [4] proved this theorem for plane projective curves of arbitrary degree $d$. In our language this corresponds to the case when $\Delta$ is a triangle whose vertices are $(0,0)$, $(d,0)$, $(0,d)$. Harnack’s example was generalized to arbitrary Newton polyhedra $\Delta$ with the help of Viro’s patchworking described in [5], see Corollary A4 in [7]. The Harnack curves are a special case of the so-called T-curves, see [5].

We refer to [5] and [7] for illustrations of Harnack curves.

Recall that a point $p \in \mathbb{R} A \subset (\mathbb{R}^*)^2$ is called an ordinary real isolated double point of $\mathbb{R} A$ (or an $A^+_i$-point, see [3]) if there exist local coordinates $x_1, x_2$ at $p \subset (\mathbb{R}^*)^2$ such that $A$ is locally defined by equation $x_1^2 + x_2^2 = 0$.

Definition 3 (Singular Harnack curves). A singular curve $\mathbb{R} A \subset (\mathbb{R}^*)^2$ with the Newton polygon $\Delta$ is called a singular Harnack curve if

- the only singular points of $\mathbb{R} A$ are $A^+_i$-points (ordinary real isolated double points);
- the result of replacing of the singular points of $\mathbb{R} A$ with small ovals (which corresponds to replacing with the locus $x_1^2 + x_2^2 = \epsilon, \epsilon > 0$ in the local coordinates) gives a Harnack curve for $\Delta$.

In other words, a singular Harnack curve is the result of contraction to points of some ovals of a non-singular Harnack curve.

3 Monge-Ampère measure on $A$.

In the next section we prove the equivalence of conditions 1 and 2 in the main theorem. The proof is an extension of the proof of the inequality (1) given in [8]. We recapture in this section the main points in this proof. The idea is to construct a measure on the amoeba $A$, whose total mass is related to $\Delta$ and which can be computed explicitly in terms of the hypersurface $A$. This measure will be obtained as the real Monge-Ampère measure of a certain convex function associated to $f$.

We indicate briefly the definition of the real Monge-Ampère operator. Details may be found in [8]. Suppose $u$ is a smooth convex function defined in $\mathbb{R}^n$. Then $\text{grad } u$ defines a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$. The Monge-Ampère measure $M u$ of $u$ is defined by $M u(E) = \lambda(\text{grad } u(E))$ for any Borel set $E$, where $\lambda$ denotes Lebesgue measure on $\mathbb{R}^n$. That this is actually a measure requires a proof, since $\text{grad } u$ is in general not 1-to-1. If $u$ is convex but not necessarily smooth,
grad $u$ can still be defined as a multifunction, and the Monge-Ampère measure of $u$ is defined as in the smooth case. For smooth functions the Monge-Ampère measure is given by the determinant of the Hessian matrix,

$$\mu = |\text{Hess}(u)|\lambda,$$

where $\lambda$ is the Lebesgue measure.

Suppose now that $f$ is a given polynomial in two variables and define

$$N_f(x) = \frac{1}{(2\pi)^2} \int_{\text{Log}^{-1}(x)} \log |f(z)| \frac{dz_1 dz_2}{z_1^2 z_2^2}.$$

This is a real-valued function defined in $\mathbb{R}^2$, which is convex because $\log |f(z)|$ is plurisubharmonic. Define $\mu$ to be the Monge-Ampère measure of $N_f$.

**Lemma 1.** The measure $\mu$ has its support in $A$ and its total mass is equal to the area of $\Delta$.

**Proof.** It is not difficult to show that $N_f$ is affine linear in each connected component of $\mathbb{R}^2 \setminus A$ and that the gradient image $\text{grad} N_f(\mathbb{R}^2)$ is equal to $\Delta$ minus some of its boundary points. This readily implies the statement. For details we refer to [8].

Let $F$ denote the set of critical values of the mapping $\text{Log} : A \to \mathbb{R}^2$. Pick a point $x_0 \in A \setminus F$ and functions $\phi_j, \psi_j$ defined in a neighborhood $V$ of $x_0$, where $j$ ranges from $1$ to $n$ and $n$ is the cardinality of $\text{Log}^{-1}(x_0) \cap A$, such that $A \cap \text{Log}^{-1}(V) = \bigcup_{j=1}^n \{(\exp(x_1 + i\phi_j(x)), \exp(x_2 + i\psi_j(x))): x = (x_1, x_2) \in V\}$. The main step in the proof of the inequality is the following computation.

**Lemma 2.** With notations as above we have

$$\text{Hess}(N_f) = \frac{1}{2\pi} \sum_{j=1}^n \pm \begin{pmatrix} \partial \psi_j / \partial x_1 & \partial \psi_j / \partial x_2 \\ -\partial \phi_j / \partial x_1 & -\partial \phi_j / \partial x_2 \end{pmatrix}.$$  \hspace{1cm} (2)

The signs depend on the signs of the intersection numbers between $\text{Log}^{-1}(x_0)$ and $A$. Each term in the sum is a symmetric, positive definite matrix with determinant equal to $1$.

For the proof we refer to [8]. We remark that the fact that the matrices are symmetric with determinant equal to $1$ follows immediately when we know that $A$ is a complex analytic curve. The two last lemmas immediately imply the following corollary.

**Corollary 2.** If $\lambda$ denotes Lebesgue measure in $\mathbb{R}^2$, then $\mu \geq (\lambda / \pi^2)|A$. Hence the area of $A$ is not greater than $\pi^2$ times the area of $\Delta$. 
Proof. It is not difficult to show that for $2 \times 2$ symmetric, positive definite matrices $M_1, M_2$ the inequality
\[
\sqrt{\det(M_1 + M_2)} \geq \sqrt{\det M_1} + \sqrt{\det M_2}
\] (3)
holds, with equality precisely if $M_1$ and $M_2$ are real multiples of each other.
Applying this to the sum (2) and using the fact that it contains at least two terms for all $x_0 \in A \setminus F$, the first statement follows. Combining this with Lemma 1 yields the second part.

Remark 2. The inequality used in the previous proof follows as a special case of an inequality for positive definite matrices of arbitrary size, analogous to the Alexandrov-Fenchel inequality for mixed volumes. The general inequality can be found in [1].

4 Proof of Theorem 1: conditions 1 and 2 are equivalent.

We are now ready to prove the equivalence of conditions 1 and 2. Note that by Corollary 2, $\text{Area}(A) = \pi^2 \text{Area}(\Delta)$ if and only if $\mu = (\lambda/\pi^2)|_A$.

4.1 Implication $1 \implies 2$.
Suppose that $\mu = (\lambda/\pi^2)|_A$. We first show that $f$ is irreducible.

Lemma 3. If $\mu = (\lambda/\pi^2)|_A$, then $f$ is irreducible.

Proof. Let $K, L$ be compact convex subsets of $\mathbb{R}^2$. From the monotonicity properties of mixed volumes it follows that $\text{Area}(K + L) \geq \text{Area}(K) + \text{Area}(L)$ with strict inequality holding unless one of $K, L$ is a point or $K$ and $L$ are two parallel segments. Assume now that we have a non-trivial factorization $f = gh$ and let $\Delta_g, \Delta_h$ denote the Newton polytopes and $A_g, A_h$ the amoebas of $g$ and $h$ respectively. From Lemma 1 it follows that $\text{Area}(A) = \pi^2 \text{Area}(\Delta)$. On the other hand, since $A = A_g \cup A_h$ and $\Delta = \Delta_g + \Delta_h$, it follows from Corollary 2 that
\[
\text{Area}(A) \leq \text{Area}(A_g) + \text{Area}(A_h) \leq \pi^2(\text{Area}(\Delta_g) + \text{Area}(\Delta_h)) < \pi^2 \text{Area}(\Delta).
\]
This is a contradiction. □

From (3) it follows that for equality to hold in Corollary 2 it is necessary that $\log^{-1}(x)$ intersects $A$ in at most two points for all $x \notin F$. Hence the sum (2) contains two terms with opposite signs. For equality to hold in (3) applied to the sum (2) it is necessary that $\text{grad } \phi_1 = -\text{grad } \phi_2$ and $\text{grad } \psi_1 = -\text{grad } \psi_2$.
After a multiplication of each coordinate by a constant we may assume that $\phi_1 = -\phi_2, \psi_1 = -\psi_2$ in a neighborhood of a given point in $A \setminus F$. (The existence
of such points is guaranteed by the assumption that $\text{Area}(\Delta)$ and hence $\text{Area}(A)$ is positive.) But then $f(z)$ and $\overline{f(z)}$ have a common factor, and hence coincide up to a multiplicative constant since they are irreducible. Multiplying $f$ by a suitable constant, we obtain a polynomial with real coefficients.

To complete the proof we must show that $\log^{-1}(x_0)$ intersects $A$ in at most two points for all $x_0 \in F$. Note that $\log^{-1}(x_0) \cap A$ cannot contain more than 2 isolated points. Indeed, a small neighborhood in $A$ of an isolated point in $\log^{-1}(x_0) \cap A$ is mapped by $\log$ either onto a neighborhood of $x_0$, or in a 2-to-1 fashion onto a half-disk with $x_0$ on its boundary. In any case, the presence of more than 2 isolated points would imply that $\log^{-1}(x) \cap A$ contains more than two points for some $x \notin F$, which is a contradiction.

If $\log^{-1}(x_0) \cap f^{-1}(0)$ contains a curve $\gamma$ we consider two different cases. If $\gamma$ is of the form $\log^{-1}(x_0) \cap \{z_1^2 z_2^k = c\}$ for some $(j,k) \in \mathbb{Z}^2$ and $c \in \mathbb{C}$, then $f$ contains the factor $z_1^j z_2^k - c$, which is impossible by Lemma 3. Otherwise, $t \gamma := \{(t_1 z_1, t_2 z_2) ; (z_1, z_2) \in \gamma\}$ intersects $\gamma$ for all $t$ in an open set in the real torus $T^2$. By Theorem 5 in [8] (cf. the proof of Lemma 4) this implies that $\mu$ has a point mass at $x_0$, contradicting the assumptions. Hence we have shown that $\log : A \to \mathbb{R}^2$ is at most 2-to-1.

### 4.2 Implication 2 $\implies$ 1.

Conversely, assume that $\log : A \to \mathbb{R}^2$ is at most 2-to-1 and that $f$ has real coefficients. Since $A$ and $\mu$ are invariant under the changes of variables permitted in the theorem, this is no loss of generality. Then the sum (3) has two terms. Since $A$ is invariant under complex conjugation of the variables, it follows that $\phi_1 = -\phi_2, \psi_1 = -\psi_2$, hence the two terms are actually equal. This shows immediately that $\mu = (\lambda/\pi^2)|_A$ outside $F$. By the following Lemma neither $\mu$ nor $\lambda$ has any mass on $F$, so this equality holds everywhere.

**Lemma 4.** If $\log^{-1}(x) \cap A$ is a finite set for all $x$, then $\mu$ has no mass on $F$.

**Proof.** In Theorem 5 in [8] it is shown that $\mu(E)$ is proportional to the average number of solutions in $\log^{-1}(E)$ to the system of equations

$$f(z_1, z_2) = f(t_1 z_1, t_2 z_2) = 0$$

as $(t_1, t_2)$ ranges over the real torus $T^2 = \{t \in \mathbb{C}^2 ; |t_1| = |t_2| = 1\}$. Note that the set of critical values of the mapping $A \to \mathbb{R}^2 : (z_1, z_2) \mapsto (|z_1|^2, |z_2|^2)$ is a semialgebraic set. Thus it is contained in a real-algebraic curve $\tilde{F}$.

Consider the product space $\mathbb{C}^2 \times T^2$ with the two projections $\pi_1$ and $\pi_2$ onto $\mathbb{R}^2$ and $T^2$ defined by $\pi_1(z,t) = (|z_1|^2, |z_2|^2)$ and $\pi_2(z,t) = t$. Let $C = \pi_1^{-1}(\tilde{F}) \cap \{f(z_1, z_2) = f(t_1 z_1, t_2 z_2) = 0\} \subset \mathbb{C}^2 \times T^2$. Since the map $\pi_1 : C \to \tilde{F}$ has discrete fibers, it follows that $C$ is a real curve. Hence $\pi_2(C)$ is a null set in $T^2$. Since the equation (4) has no solutions in $\log^{-1}(F)$ for $t$ outside $\pi_2(C)$, it follows that $\mu(F) = 0$ as required. \qed
5 Proof of Theorem 1: conditions 2 and 3 are equivalent.

5.1 Implication 2 \(\Rightarrow\) 3.

By our assumption \(A\) is real up to multiplication by a constant. Thus multiplying by a suitable constant we may assume that \(A\) is already defined over \(\mathbb{R}\). In this case we may define the real part \(\mathbb{R}A\) as the fixed point set of the involution of complex conjugation \(\text{conj} : (z_1, z_2) \mapsto (\bar{z}_1, \bar{z}_2)\) restricted to \(A\).

Let \(\nu : \hat{A} \to A\) be the normalization of the curve \(A\). The involution \(\text{conj}|_{\hat{A}}\) can be lifted to an involution \(\text{conj}|_{\hat{A}}\) on the Riemann surface \(\hat{A}\). Let \(\mathbb{R}A\) be the real part of \(\hat{A}\). Note that \(\nu(\mathbb{R}\hat{A}) \subset \mathbb{R}A\), but real isolated (singular) points of \(\mathbb{R}A\) are not contained in \(\nu(\mathbb{R}\hat{A})\).

Since \(\log|_{\hat{A}}\) is at most 2-1 we can view the map \(\log \circ \nu : \hat{A} \to A\) as a branched double covering. Let \(F \subset A\) be the branch locus of this covering, i.e. the set of points whose inverse image under \(\log|_{\hat{A}}\) consists of one point.

**Lemma 5.** The involution \(\text{conj}|_{\hat{A}}\) is the deck transformation of the branched double covering \(\log \circ \nu\).

**Proof.** The Lemma follows from the fact that \(\log\) maps conjugate points to the same point, \(\log \circ \text{conj} = \log\).

**Corollary 3.** \(A = \hat{A}/\text{conj}|_{\hat{A}}, \text{while } F = \log(\nu(\mathbb{R}\hat{A})) = \partial A.\)

**Proof.** The curve \(\hat{A}\) is non-singular and therefore \(\hat{A}/\text{conj}|_{\hat{A}}\) is a smooth surface with the boundary \(\partial \hat{A}\).

Thus \(\partial A\) consists of the images of components of \(\mathbb{R}\hat{A}\). These components are of two types, closed components, called ovals, and non-compact components. Accordingly, each oval of \(\mathbb{R}A\) which does not contain singular points corresponds to a hole in \(A\).

Consider first the case when \(A\) is a non-singular curve, so that \(\hat{A} = A\). Let \(l\) be the number of ovals of \(\mathbb{R}A\). Then \(\chi(A) = 1 - l\), where \(\chi\) stands for the homology Euler characteristic, i.e. the alternated sum of Betti numbers (we specify that since \(A\) is not compact). On the other hand, by additivity of Euler characteristic for compact spaces, \(\chi(\hat{A}) = 2\chi(A) = 2 - 2l\) (recall that \(\hat{A}\) is a compactification of \(A\) in a suitable toric surface, see Section 2). But \(\chi(\hat{A}) = 2 - 2g\) and, therefore, \(l = g\).

To ensure that \(\mathbb{R}A\) has the right number of non-compact components we recall that \(A\) intersect the complexification of \(l_k\) in \(d_k\) points. Each such intersection corresponds to a "tentacle" of \(A\) which goes to infinity (see 3). Therefore \(\mathbb{R}^2 \setminus A\) has \(\sum_{k=1}^{n} d_k\) non-compact components and each of them must be bounded by a non-compact component of \(\mathbb{R}A\).
To finish the proof in the case when $A$ is non-singular we need to show that these $g + \sum_{k=1}^{n} d_k$ components of $RA$ are arranged in $(\mathbb{R}^*)^2$ in the Harnack way. This follows from Lemma 11 of [7]. Compactifying with $l_1 \cup \cdots \cup l_n$ we obtain that $(RT_A; RA, l_1 \cup \cdots \cup l_n)$ is a Harnack arrangement.

Now we consider a general case where $A$ might have singular points.

**Lemma 6.** $A$ has no singularities other than real isolated double points.

**Proof.** We claim that the singular points of $A$ may only arise as the intersection points of two non-singular branches of $\tilde{A}$. Consider the map $\tilde{A} \to A \to A$.

Over $A \setminus F$ each of the two branches of $\tilde{A}$ must be non-singular. Indeed, it maps 1-1 to $A \setminus F$ and, therefore, the link of each point of this branch is an unknot.

By a similar reason branches of $\tilde{A}$ cannot have singular points over $F$. Indeed, the links of such points are unknots since neighborhoods of those points map 2-1 to small half-disks from $\tilde{A}/\text{conj}\tilde{A}$.

By Lemma 1 of [7] the image of each branch of $\tilde{A}$ under Log has a convex complement. Therefore the images of branches of $\mathbb{R}\tilde{A}$ cannot intersect (that would produce points of $A$ with at least 4 inverse images under $\text{Log}|_A$).

Thus the only singularities of $A$ are intersection points $p$ of a pair of conjugate non-singular imaginary branches. If these branches are not transverse then they have a real tangent line $\tau$. The points of $\tau$ close to $p$ will be covered at least twice by each of the two branches of $\tilde{A}$ which leads to a contradiction. We conclude that the only singularities of $A$ are $A^+_1$-singularities.

Now we may replace each $A^+_1$-point with a small oval that corresponds to its local perturbation and proceed similar to the case of non-singular curves.

**5.2 Implication 3 $\implies$ 2.**

This implication is contained in the proof of the main theorem in [7]. Indeed, a Harnack curve is in cyclically maximal position (see Theorem 3 of [7]). By Lemmas 5 and 8 of [7] we know that $F = \text{Log}(RA) = \partial A$ and by Lemma 9 $\text{Log}|_{RA}$ is an embedding. Therefore the only singularities of $\text{Log}|_A$ are folds and $\text{Log}|_A$ is at most 2-1.

**References**

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