A UNIQUE PRIME DECOMPOSITION RESULT FOR WREATH PRODUCT FACTORS

J. OWEN SIZEMORE AND ADAM WINCHESTER

Abstract. We use malleable deformations combined with spectral gap rigidity theory, in the framework of Popa’s deformation/rigidity theory to prove unique tensor product decomposition results for II$_1$ factors arising as tensor product of wreath product factors. We also obtain a similar result regarding measure equivalence decomposition of direct products of such groups.

Contents

Introduction 1
1. Preliminaries 2
2. Intertwining Techniques for Wreath Products 4
3. Proof of Main Theorems 6
References 7

Introduction

A major goal of the study of II$_1$ factors is the classification of these algebras based on the “input data” that goes into their construction. A significant landmark was the result, due to Connes [Co76], that all amenable II$_1$ factors are isomorphic. However, in the non-amenable realm there is a much greater variety, and a striking classification theory has developed.

One thrust of this research is to determine if some algebra which, a priori, is constructed in one manner, can be obtained in some other manner. For example, if we have a II$_1$ factor that we know to be a free product of two II$_1$ factors, is it also possible to be the tensor product of two (possibly different) II$_1$ factors?

In this vein we study whether certain factors can be written as a tensor product in two distinct ways. Such results go back to the study of prime factors, (ie. a factor which cannot be written as the tensor product of two other II$_1$ factors.) The first result was obtained by Popa in, [Po83], where he showed that the group von Neumann algebra of an uncountable free group is prime.

Later, in [Ge98], Ge proves that all group factors coming from finitely generated free groups are prime. Using C$^*$ techniques this was greatly generalized by Ozawa, [Oz03], to show that all i.c.c. Gromov hyperbolic groups give rise to prime factors. Also, using his deformation/rigidity theory, Popa showed in [Po06a] that all II$_1$ factors arising from the Bernoulli actions of nonamenable groups are prime. Further, Peterson used his derivation approach to deformation/rigidity ([Pe06]) to prove that

Date: October 16, 2018.
any $\mathrm{II}_1$ factor coming from a countable group with positive first $l^2$-betti number is also prime. Finally we should also note that using Popa’s deformation/rigidity theory, Chifan and Houdayer, [CH08], gave many more examples of prime $\mathrm{II}_1$-factors coming from amalgamated free products.

A natural question about prime factors is whether a tensor product of a finite number of such factors $P_1, P_2, \ldots, P_n$, has a “unique prime factor decomposition”, i.e., if $P_1 \overline{\otimes} \cdots \overline{\otimes} P_n = Q_1 \overline{\otimes} \cdots \overline{\otimes} Q_m$, for some other prime factors $Q_j$, forces $n = m$ and $P_i$ unitary conjugate to $Q_i$, modulo some permutation of indices and modulo some “rescaling” by appropriate amplifications of the prime factors involved. A first such result was obtained by Ozawa and Popa in [OP03], where a combination of $C^*$ techniques from [Oz03] and intertwining techniques from [Po03] is used to show that any $\mathrm{II}_1$ factor arising from a tensor product of hyperbolic group factors has such a unique tensor product decomposition.

In this paper we prove an analogous unique prime factor decomposition result for tensor products of wreath product $\mathrm{II}_1$ factors. More precisely, we prove the following result:

**Theorem 0.1.** Let $A_1, \ldots, A_n$ be non-trivial amenable groups; $H_1, \ldots, H_n$ be non-amenable groups; and $Q_1, \ldots, Q_k$ be diffuse von Neumann algebras such that

$$M = L(A_1 \wr H_1) \overline{\otimes} \cdots \overline{\otimes} L(A_n \wr H_n) = Q_1 \overline{\otimes} \cdots \overline{\otimes} Q_k$$

If $k \geq n$, then $n = k$, and after permutation of indices we have that $L(A_i \wr H_i) \simeq Q_i^{t_i}$ for some positive numbers $t_1, t_2, \ldots, t_n$ whose product is 1.

Also we have a natural generalization of this theorem to unique measure-equivalence decomposition results of finite products of wreath product groups. Such results were achieved for products of groups of the class $\mathcal{C}_{\text{reg}}$ by Monod and Shalom (Theorem 1.16 in [MS06]), for products of bi-exact groups by Sako (Theorem 4 in [Sa09]), and for products of groups in $Q\mathcal{H}_{\text{reg}}$ by Chifan and Sinclair (Corollary C in [CS10]).

**Corollary 0.2.** Let $A_1, \ldots, A_n$ be non-trivial amenable groups; $H_1, \ldots, H_n$ be non-amenable groups; and $K_1, \ldots, K_m$ be groups such that

$$A_1 \wr H_1 \times \cdots \times A_n \wr H_n \simeq_{\text{ME}} K_1 \times \cdots \times K_m$$

If $m \geq n$, then $n = m$, and after permutation of indices we have that $A_i \wr H_i \simeq_{\text{ME}} K_i$.

We prove these results by using deformation/rigidity theory. More precisely, we use the malleable deformation for wreath product group factors in [CPS11], combined with Popa’s spectral gap rigidity and intertwining by bimodules techniques.

**Acknowledgements:** We would like to thank Sorin Popa for suggesting this problem and his encouragement throughout.

**1. Preliminaries**

**Intertwining by Bimodules:** Let us recall Popa’s intertwining by bimodules technique. This is a crucial tool for locating subalgebras of $\mathrm{II}_1$-factors, and is summed up in the following theorem:
Theorem 1.1 (Popa, [Po03]). Let $P, Q \subset M$ be finite von Neumann algebras. Then the following are equivalent:

1. There exists nonzero projections $p \in P, q \in Q$, a nonzero partial isometry $v \in M$, and a *-homomorphism $\varphi : pPp \rightarrow qQq$ such that $vx = \varphi(x)v, \forall x \in pPp$.
2. There is a sub-$P - Q$-bimodule $\mathcal{H} \subset L^2(M)$ that is finitely generated as a right $Q$-module.
3. There is no sequence $u_n \in \mathcal{U}(P)$ such that $\lim_{n \to \infty} \|E_Q(xu_ny)\|_2 \rightarrow 0, \forall x, y \in M$.

If any of the above conditions hold we say that a corner of $P$ embeds in $Q$ inside $M$, denoted $P \preccurlyeq_M Q$.

Following [OP07] we have the following definition:

Definition 1.2. Let $P, Q \subset M$ be finite von Neumann algebras. We say that $P$ is amenable over $Q$ inside $M$, which we denote $P \preccurlyeq_M Q$, if there is a $P$-central state, $\varphi$, on $(M, e_Q)$ such that $\varphi|_M = \tau$, where $\tau$ is the trace on $M$.

Let us note that by Theorem 2.1 in [OP07] $P \preccurlyeq_M Q$ is equivalent to $L^2(P) \ltimes \bigoplus L^2((M, e_Q))$ as $P$-bimodules. Further, if $P \preccurlyeq_M Q$ then $L^2(M)$ contains a sub $P$-$Q$-module, $\mathcal{H}$, that is finitely generated as a right $Q$ module. Therefore, the projection onto this module will commute with the right action of $Q$ and will have finite trace. Therefore, then it will be a vector in $L^2((M, e_N))$. Further, it will also commute with $P$, so if we look at $L^2((M, e_N))$ as a $P$-bimodule, it will contain a central vector. Since strong containment implies weak containment we get the following observation.

Proposition 1.3. Let $P, Q \subset M$ be von Neumann algebras. If $P \preccurlyeq_M Q$ then $P \preccurlyeq_M Q$.

Deformation of Wreath Products: Let $A$ and $H$ be discrete groups. Then following standard notation we let $A \wr H = A^H \rtimes H$ denote the standard wreath product. Throughout this paper we will consider trace preserving actions of $A \wr H$ on a finite von Neumann algebra $N$, and we consider the resulting crossed product algebra $M = N \rtimes A \wr H$.

Let $\hat{A} = A \star \mathbb{Z}$. If we let $u \in L(\hat{A})$ denote the Haar unitary that generates $L(\mathbb{Z})$ then for every $t \in \mathbb{R}$, we define $u^t = \exp(ith) \in LZ$. This allows us to define $\theta_t \in \text{Aut}(L(\hat{A}))$ by $\theta_t(x) = u^tx(u^t)^t$. By applying this automorphism in each coordinate we can get an automorphism of $L(\hat{A}))$. Since the action of $H$ is by permuting the coordinates, it commutes with $\theta_t$ and so we can extend it to $L(\hat{A})$. If we now declare that the Haar unitaries in each coordinate do not act on the algebra $N$, then we can extend to an automorphism, which we still denote by $\theta_t$ of $\hat{M} = N \rtimes \hat{A} \wr H$.

It is easy to see that $\lim_{t \to 0} \|u^t - 1\|_2 = 0$ and hence we have $\lim_{t \to 0} \|\theta_t(x) - x\|_2 = 0$ for all $x \in \hat{M}$. Therefore, the path $(\theta_t)_{t \in \mathbb{R}}$ is a deformation by automorphisms of $\hat{M}$.

Next we show that $\theta_t$ admits a “symmetry”, i.e. there exists an automorphism $\beta$ of $\hat{M}$ satisfying the following relations:

$$\beta^2 = 1, \quad \beta|_M = id|_M, \quad \beta\theta_t\beta = \theta_{-t}, \quad \text{for all } t \in \mathbb{R}.$$
To see this, first define \( \beta_{\iota, A} = i \bar{d}_{L,A} \) and then for every \( h \in H \) we let \((u)_h \) be the element in \( L\hat{A}^H \) whose \( h^{th} \)-entry is \( u \) and 1 otherwise. On elements of this form we define \( \beta((u)_h) = (u^*)_h \), and since \( \beta \) commutes with the actions of \( H \) on \( A^H \), it extends to an automorphism of \( L(A \wr H) \) by acting identically on \( L(H) \). Finally, the automorphism \( \beta \) extends to an automorphism of \( \hat{M} \), still denoted by \( \beta \), which acts trivially on \( A \).

Let us note that, with this choice of \( \beta, \theta \) is an \( s \)-malleable deformation of \( \hat{M} \) in the sense of \( \text{Po03} \). In fact, this is the same deformation that the first author used in \( \text{CPS11} \), and is inspired by similar free malleable deformations in \( \text{Po01} \) \( \text{IPP05} \) \( \text{Io06} \), so we refer to this previous work for additional discussion.

2. Intertwining Techniques for Wreath Products

In this section we prove the necessary intertwining results for II\(_1\) factors arising from wreath product groups that we will need in order to prove our desired uniqueness of of tensor product decomposition.

The following proposition is a relative version of Lemma 4.2 in \( \text{CPS11} \), and will follow a similar proof.

Proposition 2.1. Let \( N \) be a finite von Neumann algebra. Let \( A, H \) be groups with \( A \) non-trivial amenable and \( H \) non-amenable. Let \( Q \subset N \rtimes A \wr H = M \) be an inclusion of von Neumann algebras. Assume \( Q \) is not amenable over \( N \) inside \( M \) then \( Q' \cap M^\omega \subset M^\omega \).

Proof. As mentioned above this proof follows closely the proof of Lemma 4.2 in \( \text{CPS11} \) as well as Lemma 5.1 in \( \text{Po06a} \) and other similar results in the literature.

We will prove the contrapositive so let us assume that \( Q' \cap M^\omega \not\subset M^\omega \). Then proceeding as in Lemma 5.1 in \( \text{Po06a} \) we see that

\[
L^2(Q) \prec L^2(\hat{M}) \otimes L^2(M)
\]

as \( Q \)-bimodules. Now we decompose \( L^2(\hat{M}) \otimes L^2(M) \) as an \( M \)-bimodule.

One can see that the above \( M \)-bimodule can be written as a direct sum of \( M \)-bimodules \( M\eta_2 M^\omega \), where the cyclic vectors \( \eta_2 \) correspond to an enumeration of all elements of \( \hat{A}^H \) whose non-trivial coordinates start and end with non-zero powers of \( u \).

Next, for every \( s \) we denote by \( \eta_s \) the element of \( A^H \) that remains from \( \eta_2 \) after deleting all nontrivial powers of \( u \). Also for every \( s \) let \( \Delta_s \subset H \) be the support of \( \eta_s \) and observe that if \( \text{Stab}_H(\eta_s) \) denotes the stabilizing group of \( \eta_s \) inside \( H \) then we have \( \text{Stab}_H(\eta_s)(H \setminus \Delta_s) \subset H \setminus \Delta_s \).

Hence we can consider the von Neumann algebra \( K_s = N \rtimes (A H \setminus \Delta, \text{Stab}_H(\eta_s)) \) and using similar computations as in Lemma 5.1 of \( \text{Po06a} \), one can easily check that the map \( x\eta_s y \rightarrow x\eta_s e_{K_s} y \) implements an \( M \)-bimodule isomorphism between \( M\eta_2 M^\omega \) and \( L^2((M, e_{K_s})) \).

Therefore, as \( M \)-bimodules, we have the following isomorphism

\[
L^2(\hat{M}) \otimes L^2(M) = \bigoplus L^2((M, e_{K_s})�).
\]

Thus we can get the following weak containment of \( Q \)-bimodules

\[
L^2(Q) \prec \bigoplus L^2((M, e_{K_s})).
\]
Notice that, since $\Delta_s$ is finite, and the action of $H$ on itself is free, then $\text{Stab}_H(\tilde{\eta}_s)$ is finite for all $s$. Also, since $A$ is an amenable group we have that $K_s \triangleleft N$ for all $s$. Thus for all $s$ we have the following weak containment of $K_s$-bimodules

$$L^2(K_s) \prec \bigoplus L^2((K_s, e_N)) \simeq \bigoplus L^2(K_s) \otimes_N L^2(K_s)$$

Now if we induce to $M$-bimodules and restrict to $Q$-bimodules and use continuity of weak containment under induction and restriction we get the following inclusions of $Q$-bimodules:

$$L^2(Q) \prec \bigoplus L^2(\langle M, e_K \rangle)$$

$$\simeq \bigoplus L^2(M) \otimes_K L^2(K_s) \otimes_K L^2(M)$$

$$\prec \bigoplus L^2(M) \otimes_K L^2(K_s) \otimes_N L^2(K_s) \otimes_K L^2(M)$$

$$\simeq \bigoplus L^2(M) \otimes_N L^2(M)$$

$$\simeq \bigoplus L^2(\langle M, e_N \rangle)$$

Thus $Q \triangleleft_M N \triangleleft M$.

We finish this section with a final theorem which allows us to locate regular subfactors with large commutant.

**Theorem 2.2.** Let $N$ be a finite von Neumann algebra. Let $A$ and $H$ be groups with $A$ non-trivial amenable and $H$ non-amenable. Let $Q \subset N \rtimes A \wr H = M$ be a subalgebra that is not amenable over $N$. Let $P = Q' \cap M$. If $P$ is a regular subfactor of $M$ then $P \triangleleft_M N$.

**Proof.** Applying Proposition 2.3 and following the proof of Theorem 4.1 in [CPS11] we see that the deformation $\theta_t$ converges uniformly on the unit ball of $P$, and thus by Theorem 3.1 in [CPS11] we have that $P \triangleleft_M N \rtimes A^H$ or $P \triangleleft_M N \rtimes H$.

Following the same argument as Theorem 4.1 [CPS11] if we assume that $P \triangleleft_M N \rtimes A^H$ and $P \not\triangleleft_M N$ then we get $Q \triangleleft_M N \rtimes A \wr H_0$ for some finite subgroup $H_0 \subset H$. Since $A$ is amenable and $H_0$ is finite then $N \rtimes A \wr H_0 \triangleleft_M N$. So since $Q \triangleleft_M N \rtimes A \wr H_0$ then by Proposition 1.2 we have $Q \triangleleft_M N \rtimes A \wr H_0$. Then by part 3 of Proposition 2.4 in [OP97] we have that $Q \triangleleft_M N$ contradicting our assumption.

Thus $P \triangleleft_M N \rtimes H$. Therefore, by Theorem 1.1 there exists nonzero projections $p, q \in N \rtimes H$, a nonzero partial isometry $v \in M$, and a *-homomorphism $\varphi : pPp \to q(N \rtimes H)q$ such that $vx = \varphi(x)v, \forall x \in pPp$. Furthermore we have that $v^*v = p$ and $vv^* = q \in \varphi(pPp)' \cap qMq$. Also, by Lemma 3.5 in [Po03] we know that $pPp$ is a regular subalgebra of $pMq$.

Then for all $u \in N_{pMq}(pPp)$ let us calculate:

$$\varphi(x)vuv^* = vuv^*$$

$$= vuv^*(u^* xu)v^*$$

$$= vu^*v(u^* xu)v^*$$

$$= vuv^*\varphi(u^* xu)v^*$$

$$= vv^*\varphi(u^* xu)$$
Now assume that $P \nsubseteq_N M$, then by part (2) of Lemma 2.4 in [CPS11] we have that $uv^* \in N \rtimes H$. Since $pP$ is regular in $pMp$ we would then get that $M \prec_M N \rtimes H$. However, this is impossible since the fact that $A$ is nontrivial implies that $[M : N \rtimes H] = \infty$.

\[ \Box \]

3. Proof of Main Theorems

In this section we prove our main theorem. Our main technical tool is the following, which is proposition 2.7 in [PV11]. Before we state the result let us recall that two von Neumann subalgebras $M_1, M_2 \subset M$ of a finite von Neumann algebra $M$ are said to form a commuting square if $E_{M_1}E_{M_2} = E_{M_2}E_{M_1}$.

**Theorem 3.1** (Popa-Vaes, [PV11]). Let $(M, \tau)$ be a tracial von Neumann algebra with von Neumann subalgebras $M_1, M_2 \subset M$. Assume that $M_1$ and $M_2$ form a commuting square and that $M_1$ is regular in $M$. If a von Neumann subalgebra $Q \subset pMp$ is amenable relative to both $M_1$ and $M_2$, then $Q$ is amenable relative to $M_1 \cap M_2$. Notice that this theorem allows us to eliminate the case where $Q$ is amenable over $M_1$. More specifically we have the following observation.

**Proposition 3.2.** Let $G_1$ and $G_2$ be groups. Let $A$ be a finite amenable von Neumann algebra with an action of $G_1 \times G_2$, and let $Q \subset A \rtimes G_1 \times G_2$ be a nonamenable subalgebra. Then there exists an $i$ such that $Q$ is not amenable over $A \rtimes G_i$.

**Proof.** If we let $A \rtimes G_i = M_i$ then it is easy to see that $M_1, M_2 \subset M$ form a commuting square. So if $Q$ is amenable over both $M_i$ we would have that it would be amenable over the intersection, which is $A$. Since $A$ is amenable this would imply that $Q$ is amenable. \[ \Box \]

Finally combining the above results we can prove our main theorem (Theorem 0.1).

**Proof.** First let us mention that for the case $n = 1$, this is equivalent to the primeness of $\Pi_1$-factors arising from Bernoulli shifts, which was proven in [Po06a].

Now notice that we can write $M$ as $M = N_1 \rtimes \sigma A_1 \rtimes H_1$, where $N_1 = L(A_1 \rtimes H_1) \otimes \ldots \otimes L(A_{i-1} \rtimes H_{i-1}) \otimes L(A_{i+1} \rtimes H_{i+1}) \otimes \ldots \otimes L(A_n \rtimes H_n)$ and $\sigma$ is the trivial action.

Let us define $\hat{Q}_i = (Q_i)’ \cap M = Q_1 \otimes \ldots \otimes Q_{i-1} \otimes Q_{i+1} \otimes \ldots \otimes Q_k$. Since $H_i \rtimes \Gamma_i$ does not have property Gamma for all $i$ this implies, in particular, that $Q_1$ is nonamenable. By proposition 3.2 where we let $A = C$, we know that there is an $i$ such that $Q_1$ is not amenable over $N_i$.

Since $\hat{Q}_1$ is a regular subalgebra of $M$, then by Theorem 2.2 we get that $\hat{Q}_1 \prec_M N$.

We complete the argument by following Proposition 12 and the induction argument of the proof of Theorem 1 in [OP03]. \[ \Box \]

Before we prove our final theorem let us recall the following definition:
Definition 3.3. We say that two group $\Gamma$ and $\Lambda$ are measure equivalent, $\Gamma \simeq_{ME} \Lambda$ if there is a diffuse abelian von Neumann algebra, $A$, and free ergodic trace preserving actions, $\sigma, \rho$ of $\Gamma$ and $\Lambda$, respectively, such that $A \rtimes \sigma \Gamma \simeq (A \rtimes \rho \Lambda)^t$, and the isomorphism takes $A$ onto $A^t$.

With this definition we can now prove our final result (Corollary [1,2]).

Proof. Let $A_1 \rtimes H_1, \ldots, A_n \rtimes H_n$ be as above, and let $K_1, \ldots, K_m$ be groups. Since $A_i \rtimes H_i \cong A \rtimes H \equiv A$ we have $A \equiv A$, and free ergodic trace preserving actions, $\sigma, \rho$ of $\Gamma$ and $\Lambda$, respectively, such that $A \rtimes \sigma \Gamma \simeq (A \rtimes \rho \Lambda)^t$, and the isomorphism takes $A$ onto $A^t$.

Let $N_i = A \rtimes H \equiv A$ be as above, and let $K_1, \ldots, K_m$ be groups. Since $A \equiv A$, and free ergodic trace preserving actions, $\sigma, \rho$ of $\Gamma$ and $\Lambda$, respectively, such that $A \rtimes \sigma \Gamma \simeq (A \rtimes \rho \Lambda)^t$, and the isomorphism takes $A$ onto $A^t$.

Now we know that there are actions on $L^\infty(X)$ such that $M = L^\infty(X) \rtimes A_1 \rtimes H_1 \times \cdots \times A_n \rtimes H_n \cong (L^\infty(X) \rtimes K_1 \times \cdots \times K_m)^t$. We may assume that $t = 1$.

Let $N_i = A \rtimes H \equiv A$ be as above, and let $K_1, \ldots, K_m$ be groups. Since $A \equiv A$, and free ergodic trace preserving actions, $\sigma, \rho$ of $\Gamma$ and $\Lambda$, respectively, such that $A \rtimes \sigma \Gamma \simeq (A \rtimes \rho \Lambda)^t$, and the isomorphism takes $A$ onto $A^t$.

Thus by Lemma 2.2 in [CPS11] we have that $A \rtimes K_2 \times \cdots \times K_m \times N_i \rtimes H_i$. Now since $A \equiv A$, and free ergodic trace preserving actions, $\sigma, \rho$ of $\Gamma$ and $\Lambda$, respectively, such that $A \rtimes \sigma \Gamma \simeq (A \rtimes \rho \Lambda)^t$, and the isomorphism takes $A$ onto $A^t$.

Notice that now we can follow exactly as in the proof of Corollary C in [CS10] to get our desired result.

□

References

[CH08] I. Chifan and C. Houdayer, Bass-Serre rigidity results in von Neumann algebras. Duke Math. J. 153 (2010), 23-54.

[CPS11] I. Chifan and S. Popa and J. O. Sizemore, Some OE and W*-rigidity results for actions by wreath product groups. (2011), Preprint. [arXiv:1110.2151v1].

[CS10] I. Chifan and T. Sinclair, On the structural theory of II$_1$ factors of negatively curved groups (2010), Preprint. [arXiv:1103.4299v2].

[Co76] A. Connes, Classification of injective factors. Cases II$_1$, II$_\infty$, III$_\lambda$, $\lambda \neq 1$. Ann. of Math. (2) 104 (1976), 73-115.

[Ge98] L. Ge, Applications of free entropy to finite von Neumann algebras II. Ann. of Math 147 (1998), 143-157.

[Io06] A. Ioana, Rigidity results for wreath product II$_1$ factors. J. Funct. Anal. 252 (2007), 763–791.

[IPP05] A. Ioana, J. Peterson and S. Popa, Amalgamated free products of weakly rigid factors and calculation of their symmetry groups. Acta Math. 200 (2008), 85–153.

[MS06] N. Monod and Y. Shalom, Orbit equivalence rigidity and bounded cohomology. Ann. of Math. (2) 164 (2006), 825878.

[Oz03] N. Ozawa, Solid von Neumann algebras. Acta Math. 192 (2004), 111-117.

[OP03] N. Ozawa and S. Popa, Some prime factorization results for type II$_1$ factors. Invent. Math. 156 (2004) 223–234.

[OP07] N. Ozawa and S. Popa, On a class of II$_1$ factors with at most one Cartan subalgebra. Ann. Math. 172 (2010) 713749.

[Pe06] J. Petersson, $L^2$-rigidity in von Neumann algebras. Invent. Math. 175 (2006), 417–433.

[Po83] S. Popa, Orthogonal pairs of $*$-subalgebras in finite von Neumann algebras. J. Operator Theory 9 (1983), 253–268.
[Po01] S. Popa, Some rigidity results for non-commutative Bernoulli shifts. *J. Funct. Anal.* **230** (2006), 273–328.

[Po03] S. Popa, Strong rigidity of II$_1$ factors arising from malleable actions of $w$-rigid groups, Part I. *Invent. Math.* **165** (2006), 369–408.

[Po06a] S. Popa, On the superrigidity of malleable actions with spectral gap. *J. Amer. Math. Soc.* **21** (2008), 981–1000.

[Po09] S. Popa, On the classification of inductive limits of II$_1$-factors with spectral gap. *Trans. AMS*, To appear.

[PV11] S. Popa and S. Vaes, Unique Cartan decomposition for II$_1$ factors arising from arbitrary actions of free groups (2011), Preprint, arXiv:1111.6951v1

[Sa09] H. Sako, Measure equivalence rigidity and bi-exactness of groups. *J. Funct. Anal.* **257** (2009), 3167–3202.

J. Owen Sizemore, UCLA, Math Sciences Building, Los Angeles, CA 90095-1555
E-mail address: sizemore@math.ucla.edu

Adam Winchester, UCLA, Math Sciences Building, Los Angeles, CA 90095-1555
E-mail address: lagwadam@math.ucla.edu