A Simple Approach to Increase the Maximum Allowable Transmission Interval*

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**Abstract:** When designing Networked Control Systems (NCS), the maximum allowable transmission interval (MATI) is an important quantity, as it provides the admissible time between two transmission instants. An efficient procedure to compute a bound on the MATI such that stability can be guaranteed for general nonlinear NCS is the emulation of a continuous-time controller. In this paper, we present a simple but efficient modification to the well-established emulation-based approach from Carnevale et al. (2007) to derive a bound on the MATI. Whilst only minor technical changes are required, the proposed modification can lead to significant improvements for the MATI bound as compared to Carnevale et al. (2007). We revisit two numerical examples from literature and demonstrate that the improvement may amount to more than 100%.

**Keywords:** Networked Nonlinear Systems, Control of Nonlinear Systems, Control of Networks

1. INTRODUCTION

Networked Control Systems (NCS) are dynamical systems, where hard-wired links in the feedback loop are replaced by a shared communication medium. A networked architecture for control systems has several advantages, as, e.g., lower cost of installation and maintenance and a greater flexibility. In turn, network-induced effects like a limited packet rate and varying transmission intervals need to be considered when designing NCS (Hespanha et al. (2007)).

An important research topic in the field of NCS is therefore to characterize conditions on the communication channel and on the transmission protocol of the network such that stability guarantees can be obtained for the NCS despite network effects. When considering time-varying sampling intervals, stability guarantees can typically be given as long as the time between two transmissions stays below a maximum allowable transmission interval (MATI). Being able to guarantee stability for a preferably large MATI allows to save network resources and is therefore important in the design of NCS. Several approaches, based on a wide variety of methods have been developed to derive such bounds on the MATI, cf. Hetel et al. (2017) for an overview. An efficient procedure is to emulate a continuous-time controller, i.e., to design first the controller assuming perfect communication and to derive then in a second step a bound on the MATI, taking into account the communication channel and the transmission protocol (Walsh et al. (2002)).

For general nonlinear NCS, an emulation-based approach for computing a bound on the MATI, such that stability can be guaranteed, has been proposed in Nesic and Teel (2004) and improved in Carnevale et al. (2007); Nesic et al. (2009) using hybrid Lyapunov functions. The wide variety of work that is based on the findings from Carnevale et al. (2007); Nesic et al. (2009) illustrates the importance of the approach, see Heemels et al. (2010); Postoyan et al. (2014); Dolk et al. (2017) for some examples. There have also been various attempts to improve the results from Carnevale et al. (2007); Nesic et al. (2009). Most approaches are restricted to special system classes or require additional assumptions for the communication network, see, e.g., Bauer et al. (2012); Hertneck et al. (2020); Heijmans et al. (2020). Recently, a notable improvement for the MATI bound of up to 15 % for the setup from Carnevale et al. (2007) could be achieved in Heijmans et al. (2017, 2018), using more general hybrid Lyapunov function approaches.

In this paper, we propose a simple modification of the main result from Carnevale et al. (2007) that leads to a significantly improved bound on the MATI for which stability is guaranteed. In particular, we relax (Carnevale et al., 2007, Assumption 1) by considering a more general function $H(x, e)$ instead of restricting ourselves to $H(x)$. Using the proposed modification, the MATI bound can be increased significantly, for some typical benchmark examples from literature even by more than 100%. Hence, the proposed modification is very beneficial to determine a large bound on the MATI. In the proofs of Carnevale et al. (2007), only minor modifications are required due to the more general choice of $H(x, e)$. This fact is also of great importance, as it makes it easy to apply the modification to many results that build on the approach.
from Carnevale et al. (2007), as, e.g., Nesic et al. (2009); Heemels et al. (2010); Postoyan et al. (2014); Dolk et al. (2017), leading there also to significant improvements.

We illustrate the proposed modification with two numerical benchmark examples from literature, and demonstrate for this examples that it leads to an improvement in the range of 66-108% in comparison to the approach from Carnevale et al. (2007); Nesic et al. (2009). Additionally, we demonstrate for linear systems how our approach can be used systematically to improve the MATI in comparison to Carnevale et al. (2007); Nesic et al. (2009).

The remainder of this paper is structured as follows. First, we recap the setup and problem formulation from Carnevale et al. (2007) in Section 2. Then, we present our approach to compute the MATI, give stability guarantees and illustrate the approach with the nonlinear example from Nesic et al. (2009) in Section 3. In Section 4, we refine our main result for linear NCS using an LMI, and revisit the example from Heijmans et al. (2017). The paper is concluded in Section 5.

**Notation and Definitions**

The positive real numbers are denoted by $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0} := \mathbb{R}_{>0} \cup \{0\}$. The positive natural numbers are denoted by $\mathbb{N}$, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. A continuous function $\alpha : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is a class $\mathcal{K}$ function if it is strictly increasing and $\alpha(0) = 0$. It is a class $\mathcal{K}_{\infty}$ function if it is of class $\mathcal{K}$ and it is unbounded. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ is a class $\mathcal{KL}$ function if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each $t \geq 0$ and $\beta(s, \cdot)$ is nonincreasing and satisfies $\lim_{t \to \infty} \beta(s, t) = 0$ for each $s \geq 0$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ is a class $\mathcal{KL}_L$ function if for each $r \geq 0$, $\beta(\cdot, r, \cdot)$ and $\beta(\cdot, \cdot, r)$ belong to class $\mathcal{KL}$. We denote by $\| \cdot \|$ the Euclidean norm.

As in Carnevale et al. (2007); Nesic et al. (2009), we will use the following definitions, that are originally taken from Goebel and Teel (2006), to characterize a hybrid model of the considered NCS.

**Definition 1.** Carnevale et al. (2007) A compact hybrid time domain is a set $D \subset \mathbb{R}_{\geq 0} \times \mathbb{N}_0$ given by:

$$D = \bigcup_{j=0}^{J-1} \{[t_j, t_{j+1}], j\}$$

where $J \in \mathbb{N}_0$ and $0 = t_0 \leq t_1 \leq \cdots \leq t_J$. A hybrid time domain is a set $D \subset \mathbb{R}_{\geq 0} \times \mathbb{N}_0$ such that, for each $(T, J) \in D$ where $D \cap \{0, T\} \times \{0, \ldots, J\}$ is a compact hybrid time domain.

**Definition 2.** Carnevale et al. (2007) A hybrid trajectory is a pair $(\xi, \xi)$ consisting of the hybrid time domain domain $\xi$ and a function $\xi$ defined on domain $\xi$ that is continuously differentiable in $t$ on $(\xi) \cap (\mathbb{R}_{\geq 0} \times \{j\})$ for each $j \in \mathbb{N}_0$.

**Definition 3.** Carnevale et al. (2007) For the hybrid system $\mathcal{H}$ given by the open state space $O \subset \mathbb{R}^n$ and the data $(F, G, C, D)$ where $F : O \to \mathbb{R}^n$ is continuous, $G : O \to O$ is locally bounded and $C$ and $D$ are subsets of $O$, a hybrid trajectory $\xi : \mathcal{H} \to \mathcal{H}$ is a solution to $\mathcal{H}$ if

(1) for all $j \in \mathbb{N}_0$ and for almost all $t \in I_j := \{t \in (\mathbb{R}_{\geq 0} \times \{j\}) : \xi(t, j) \in C \land \xi(t, j) = F(\xi(t, j))\}$.

For further details on these definitions, see Goebel and Teel (2006). Omitting time arguments, the hybrid system model is described by

$$\mathcal{H} = \begin{cases} \xi = F(\xi) & \xi \in C \\ \xi^+ = G(\xi) & \xi \in D, \end{cases}$$

(1)

where $\xi^+$ denotes $\xi(t, j+1)$. Note that typically $C \cap D = \emptyset$ and therefore, the hybrid model we consider may have nonunique solutions.

2. SETUP

We consider the same NCS model as in Carnevale et al. (2007). The dynamics of plant and controller are given by

$$\begin{align*}
\dot{x}_P &= f_P(x_P, \hat{u}) \\
y &= g_P(x_P)
\end{align*}$$

(2)

and

$$\begin{align*}
\dot{x}_C &= f_C(x_C, \bar{y}) \\
u &= g_C(x_C)
\end{align*}$$

(3)

where $x_P \in \mathbb{R}^n$ and $x_C \in \mathbb{R}^m$ denote the plant and controller states, $y \in \mathbb{R}^m$ is the plant output, $u \in \mathbb{R}^m$ is the controller output and $\bar{y} \in \mathbb{R}^m$. We denote by $\| \cdot \|$ the Euclidean norm.

The sensors and actuators of the plant are spread over $l$ nodes with index $i \in \{1, 2, \ldots, l\}$. Let the sequence of transmission times be given by $(t_j)$, $j \in \mathbb{N}_0$, satisfying $0 < \epsilon_j \leq \epsilon_{j+1} \leq \tau_{\text{max}}$ for all $j \in \mathbb{N}_0$ and some fixed $\epsilon_{\text{min}} < \epsilon_{\text{max}}$. Here, $\tau_{\text{max}}$ is the maximum allowable transmission interval (MATI) and $\epsilon$ is an arbitrarily small bound that excludes Zeno behavior. At each transmission time $t_j$, a scheduling protocol grants network access to one of the nodes. Depending on which node was granted network access, $\bar{y}$ and $\hat{u}$ are updated at transmission times as

$$\begin{align*}
\bar{y}(t_j^+) &= y(t_j) + h_y(i, \epsilon(t_j)) \\
\hat{u}(t_j^+) &= u(t_j) + h_u(i, \epsilon(t_j)),
\end{align*}$$

where $i$ is the index of the respective node. The function $h := [h_y, h_u]^T$ with $h : \mathbb{N}_0 \times \mathbb{R}^{n+nu} \to \mathbb{R}^{n+nu}$ thus models the scheduling protocol Walsh et al. (2002); Nesic and Teel (2004). For many protocols as, e.g., try-once-discard (TOD) and Round Robin (RR), the part of the state error that is measured by the node with network access is reset to 0.

Combining $x := [x_P, x_C]^T$ and $\epsilon := [\epsilon_y, \epsilon_u]^T$, we can write the overall NCS in the (even more general) form

$$\begin{align*}
\dot{x} &= f(x, \epsilon) \quad \forall t \in [t_{j-1}, t_j] \\
\dot{\epsilon} &= g(x, \epsilon) \quad \forall t \in [t_{j-1}, t_j]
\end{align*}$$

(4)

and

$$\begin{align*}
\epsilon(t_j^+) &= h(j, \epsilon(t_j)),
\end{align*}$$

(5)

(6)

for all $j \in \mathbb{N}_0$, where $x \in \mathbb{R}^{n+nu}$ and $\epsilon \in \mathbb{R}^{n+nu}$ with $n_x = n_P + n_C$ and $n_{\epsilon} = n_y + n_u$. If $\bar{y}$ and $\hat{u}$ are kept constant between
transmission times, i.e., for the zero-order-hold (ZOH) case with
\[
\begin{align*}
\dot{y} &= 0 \quad t \in [t_{j-1}, t_j] \\
\dot{u} &= 0 \quad t \in [t_{j-1}, t_j]
\end{align*}
\]
for all \( j \), we obtain
\[
\begin{align*}
f(x, e) &= \begin{bmatrix} f_P(x_P, g_C(x_C) + e_u) \\ f_C(x_C, g_P(x_P) + e_y) \end{bmatrix} \\
g(x, e) &= \begin{bmatrix} \frac{\partial f_P}{\partial x} f_P(x_P, g_C(x_C) + e_u) \\ \frac{\partial f_C}{\partial x} f_C(x_C, g_P(x_P) + e_y) \end{bmatrix}
\end{align*}
\]
The problem that we consider is the same as in Carnevale et al. (2007).

**Problem 1.** Carnevale et al. (2007) Suppose that the controller (3) was designed for the plant (2) so that the closed-loop system (2), (3) without network is globally asymptotically stable. Determine the value of \( \tau_{\text{MATI}} \) so that for any \( \epsilon \in (0, \tau_{\text{MATI}}) \) and all \( \tau_{\text{max}} \in [\epsilon, \tau_{\text{MATI}}] \), we have that the NCS described by (4), (5), (6) is stable in the last transmission and the counter \( \kappa \in \mathbb{N}_0 \) for the number of transmissions. The resulting system is of the form
\[
\begin{align*}
\dot{x} &= f(x, e) \\
\dot{e} &= g(x, e) \\
\tau &= 1 \\
\kappa &= 0
\end{align*}
\]
Details on (8) can be found in Carnevale et al. (2007), where the same model is used. We shall also use Standing Assumption 1 from Carnevale et al. (2007).

**Standing Assumption 1.** Carnevale et al. (2007) \( f \) and \( g \) are continuous and \( h \) is locally bounded.

Note that \( \tau \) and \( \kappa \) are artificially introduced states. Like in Carnevale et al. (2007), we aim therefore at guaranteeing uniform global asymptotic stability for the set \( \{ x, e, \tau, \kappa : x = 0, e = 0 \} \) as defined next.

**Definition 4.** (Carnevale et al., 2007, Definition 4) For the hybrid system (8), the set \( \{ x, e, \tau, \kappa : x = 0, e = 0 \} \) is uniformly globally asymptotically stable (UGAS) if there exists \( \beta \in \mathcal{KL} \) such that, for each initial condition \( \tau(0), \kappa(0), x(0), e(0) \in \mathbb{R}^n, e(0, 0) \in \mathbb{R}^n \), and each corresponding solution
\[
\| x(t, j) \| \leq \beta \| x(0, 0) \| e(t, j)
\]
for all \((t, j)\) in the solutions domain. The set is uniformly globally exponentially stable (UGES), if \( \beta \) can be taken to have the form \( \beta(s, t, k) = Ms \exp(\lambda(t + k)) \) for some \( M > 0 \) and \( \lambda > 0 \).

### 3.1 Improving the MATI

We shall use the following relaxed version of (Carnevale et al., 2007, Assumption 1) in order to derive guarantees for UGAS or UGES.

**Assumption 1.** There exists a function \( W : \mathbb{N}_0 \times \mathbb{R}^n \to \mathbb{R}_+ \) that is locally Lipschitz in its second argument, a locally Lipschitz, positive definite, radially unbounded function \( V : \mathbb{R}_+ \to \mathbb{R}_+ \), a continuous function \( H : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+ \), real numbers \( \lambda \in (0, 1), L > 0, \gamma > 0 \), \( \omega_W, \omega_W \in \mathbb{K}_\infty \) and a continuous, positive definite function \( g \) such that, \( \forall \epsilon \in \mathbb{N}_0 \) and \( e \in \mathbb{R}^n \)
\[
\begin{align*}
\omega_W(\| e \|) &\leq W(\kappa, e) \leq \omega_W(\| e \|) \\
W(\kappa + 1, h(\kappa, e)) &\leq W(\kappa, e)
\end{align*}
\]
and for all \( \kappa \in \mathbb{N}_0, x \in \mathbb{R}^n \) and almost all \( e \in \mathbb{R}^n \)
\[
\left( \frac{\partial W(\kappa, e)}{\partial e} g(x, e) \right) \leq LW(\kappa, e) + H(x, e)
\]
moreover, for all \( e \in \mathbb{R}^n \), all \( \kappa \in \mathbb{N}_0 \) and almost all \( x \in \mathbb{R}^n \),
\[
\langle \nabla V(x), f(x, e) \rangle \leq -g(\| e \|) - g(W(\kappa, e)) - H^2(x, e) + \gamma^2 W^2(e, \kappa, e).
\]

Based on Assumption 1, we will show that uniform UGAS can be guaranteed if
\[
\tau_{\text{MATI}} \leq \begin{cases} 
\frac{1}{\gamma} \arctan\left( \frac{r(1-\lambda)}{2\sqrt{\gamma} + \gamma + 1} \right) & \gamma > L \\
\frac{1}{\gamma} \arctan\left( \frac{r(1-\lambda)}{2\sqrt{\gamma} + \gamma + 1} \right) & \gamma = L \\
\frac{1}{\gamma} \arctan\left( \frac{r(1-\lambda)}{2\sqrt{\gamma} + \gamma + 1} \right) & \gamma < L,
\end{cases}
\]
where
\[
r := \sqrt{\left( \frac{2}{L} \right)^2 - 1}.
\]
Remark 2. Note that (14) and (15) are identical to (2) and (3) from Carnevale et al. (2007). However, the more general function $H(x,e)$ in Assumption 1 can lead to significantly smaller values for $\gamma$ and $L$ in Assumption 1 in comparison to the minimum possible values for (Carnevale et al., 2007, Assumption 1). Since the value of the bound on $\tau_{\text{MATI}}$ in (14) gets larger when $\gamma$ and $L$ get smaller (cf. (Nesic et al., 2009, Remark 6)), the more general choice of $H(x,e)$ leads then to a significant increase of the bound $\tau_{\text{MATI}}$. We will demonstrate this later for two benchmark examples.

Remark 3. The setup of Wang et al. (2020) admits to use a more general function $H(x,e)$. However, this additional degree of freedom is not exploited therein.

We can now state the equivalent to (Carnevale et al., 2007, Theorem 1) for our setup.

Theorem 1. Under Assumption 1, if $\tau_{\text{MATI}}$ in (8) satisfies the bound (14) and $0 < \varepsilon \leq \tau_{\text{MATI}}$ then, for the system (8), the set \{$x,e,\tau,\kappa : x = 0, e = 0\}$ is UGAS. If, in addition, there exist strictly positive real numbers $\varpi_W, \varpi_W, a_1, a_2$ and $a_3$, such that $\varpi_W |e| \leq W(e), c \leq \varpi_W |e|, a_1 |x|^2 \leq V(x) \leq a_2 \|x\|^2$, and $g(s) \leq a_3 s^2$, then this set is UGES.

Proof. See Appendix A. \qed

Note that the wording of Theorem 1 is identical to the wording of (Carnevale et al., 2007, Theorem 1). A difference to (Carnevale et al., 2007, Theorem 1) with significant impact on the bound $\tau_{\text{MATI}}$ is however the choice of $H$ in Assumption 1.

Remark 4. The fact that only minor changes are required in the proof of Theorem 1 in comparison to (Carnevale et al., 2007, Theorem 1) implies that the same modification can also be used for many results that build on Carnevale et al. (2007), as e.g. Heemels et al. (2010); Postoyan et al. (2014); Dolk et al. (2017).

Remark 5. To obtain a sufficient condition for uniform local asymptotic stability based on Assumption 1, Theorem 2 from Carnevale et al. (2007) and its proof can easily be modified using Assumption 1 instead of (Carnevale et al., 2007, Assumption 1).

Remark 6. For sampled-data systems, i.e., for systems for which $e(t_{j+1}) = 0$, which implies that (11) holds even with $\lambda = 0$, the stability results from Theorem 1 apply for any $\tau_{\text{MATI}}$ that satisfies

$$\tau_{\text{MATI}} < \begin{cases} \frac{L}{\gamma} \arctan(r) & \gamma > L \\ 0 & \gamma = L \\ \frac{L}{\gamma} \text{arctanh}(r) & \gamma < L \end{cases}$$

with $r$ according to (15). The modifications, that are required for this case in our setup and in the proof of Theorem 1, are similar as those in Nesic et al. (2009).

We are now ready to demonstrate for the example from Nesic et al. (2009) that the proposed modification for $H$ can lead to a significantly larger bound on $\tau_{\text{MATI}}$ in comparison to the approach from Carnevale et al. (2007); Nesic et al. (2009).

3.2 Example 1

We employ the example system from Karafyllis and Kravaris (2009), that was also used in Nesic et al. (2009), for which the plant is given by $\dot{x}_p = dx^3_p - x^3_p + u$ for some unknown parameter $d$ with $|d| \leq 1$. The controller is given by $u = -2x_p$, where $\hat{x}_p$ is the most recently received value of $x_p$. Note that, even though our plant model (2), (3) is not stated in such a generality, this setup can easily be modeled as an NCS of the form (4)-(6) as it has been demonstrated in Nesic et al. (2009).

For $e := \hat{x}_p - x_p$ and $x := x_p$, we obtain $f(x,e) = -2x + dx^2 - x^3 - 2e$ and $g(x,e) = -f(x,e)$. We chose $W(e) := |e|$, which satisfies for all $e \neq 0$ and any fixed $k \in [0,2]$

$$\langle \frac{\partial W(e)}{\partial e} , g(x,e) \rangle = \text{sign}(e)g(e,x) \leq |2x + dx^2 - x^3 - 2e| \leq L_k W(e) + H_k(x,e)$$

where $H_k(x,e) = |2x + dx^2 - x^3 - ke|$ and $L_k = 2 - k$. Here $k$ is a tunable parameter that can be varied to obtain a large MATI bound. As a result, (10) and (12) hold globally. Since the system has only one node, (11) even holds with $\lambda = 0$ and therefore we employ the MATI bound for sampled-data systems from (16). It hence remains, depending on the tunable parameter $k$, to find a preferably small value $\gamma_k$, such that (13) is satisfied. We chose $g(s) = \delta s^2$ for some fixed $\delta > 0$. Note that $\delta^2$ determines the worst-case convergence speed for the closed-loop system. We can rewrite (13) for polynomial $V(x)$ as

$$-\langle \nabla V(x), f(x,e) \rangle - \delta x^2 - \delta^2 e^2 - 2x + dx^2 - x^3 - ke)^2 + \gamma_k e^2 = p_1(x,e)\delta^2 + p_2(x,e)d + p_3(x,e) \geq 0,$$  

where $p_1(x,e), p_2(x,e)$ and $p_3(x,e)$ are polynomials in $x$ and $e$. Therefore (17) holds for polynomial $V(x)$, if its left hand side is sum of squares (SOS). We can moreover conclude that the left-hand side of (17) is SOS for any $d$ with $|d| \leq 1$, if it is SOS for all the combinations from $(d,d^2) \in \{(1,0),(1,1),(-1,0),(-1,1)\}$. Note that this procedure is inspired by the second example from Omran et al. (2016).

Since we consider a sampled-data system, we will now compare our approach with the approach from Nesic et al. (2009), which is the sampled-data version of Carnevale et al. (2007). As in Nesic et al. (2009), we consider $\delta = 0.1$. Using Theorem 1, UGAS can be guaranteed for any $\tau_{\text{MATI}} < 0.7099$, which results from $\gamma = 1.544$ and $L = 0.738$, for which (17) and thus also (13) can be verified using SOSTOOLS and the above described procedure with $V(x) = 0.3578 x^4 + 1.431 x^2$. In Nesic et al. (2009), the bound on the MATI was given for the considered example as $\tau_{\text{MATI}} \leq 0.3688$. To determine $\gamma$ and $L$, an approach that includes some conservative estimates was used in Nesic et al. (2009). These estimates can be circumvented by the above described procedure based on SOSTOOLS. Therefore the best value for the bound on $\tau_{\text{MATI}}$ for the approach from Nesic et al. (2009), that we could find is $\tau_{\text{MATI}} < 0.4762s$, which is attained for $\gamma = 2.151$ and $L = 2$ with $V(x) = 0.5x^4 + 2x^2$. Thus, our approach
leads to an improvement of more than 66 % in comparison to the best value that we could find for the approach from Nesic et al. (2009). A second example for NCS with multiple nodes and an even higher improvement of the MATI will be given in the next section.

4. THE LINEAR CASE

For general nonlinear systems, it is a challenging task to find the smallest possible values for $\gamma$ and $L$ for which Assumption 1 holds. For the special case of linear systems, of the form

$$f(x, e) = Ax + Ec, \quad g(x, e) = Cx + Fe. \quad (18)$$

we can however state a systematic procedure for finding $\gamma$ and $L$ based on an LMI. The procedure is inspired by Heijmans et al. (2017).

4.1 A systematic procedure to compute $\gamma$ and $L$

We shall reformulate (10) and (13) in terms of an LMI, that allows an efficient search for small values of $\gamma$ and $L$ that satisfy Assumption 1. The following derivations are inspired by Heijmans et al. (2017). We assume that $\left\| \frac{\partial W(x, e)}{\partial e} \right\| \leq M_w$ and that $\bar{\alpha}_w(s) = \bar{\alpha}_w(s)$ for constants $M_w > 0$, $\bar{\alpha}_w > 0$. This is a reasonable assumption for many scheduling protocols, cf. (Heemels et al., 2010, Section V). For RR and TOD, it holds with $M_w = \sqrt{T}$, $M_w = \sqrt{T}$. The following derivations lead to

$$L_k = M_w \tilde{\alpha}_w^{-1} \left\| (1 - k)F \right\| \quad (19)$$

and $H_k(x, e) = M_w \left\| C x + kFe \right\|$ in (12). We consider $\delta(s) = \delta_n^2 s^2$ for some chosen $\delta > 0$. Note that $\delta^2$ determines again the worst-case convergence speed for the closed-loop system. We assume for simplicity 2 that $\gamma_k^2 \geq \delta^2$, implying $(\gamma_k^2 - \delta^2) W^2(c) \geq \left( \gamma_k^2 - \delta^2 \right) \bar{\alpha}_w M_w \left\| e \right\|^2$ and thus we obtain for the right-hand side of (13)

$$-\phi(s) = \delta_n^2 s^2 (\gamma_k^2 - \delta^2) \bar{\alpha}_w M_w \left\| e \right\|^2$$

and that

$$J(k) :=\begin{bmatrix} -\delta^2 I_{n_w} - M_w^2 C^T C & -kM_w^2 F^T F \\ -kM_w^2 F^T C & \bar{\alpha}_w (\gamma_k^2 - \delta^2) I_{n_e} - M_w^2 k^2 F^T F \end{bmatrix}.$$

Moreover, for $V(x) = x^T P x$ with $P \in \mathbb{R}^{n_x \times n_x}$, we obtain for the left-hand side of (13) $\left\langle \nabla V(x), Ax + Ec \right\rangle = x^T (A^T P + PA) x + 2x^T P e$. Therefore, (13) holds for the linear setup (18) if $\Lambda_{11} = 0$, $\Lambda_{12} = \Lambda_{21}$, and $\Lambda_{22}$ has the form

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{21} \\ \Lambda_{12} & \Lambda_{22} \end{bmatrix} \leq 0 \quad (20)$$

where $\Lambda_{11} = A^T P + PA + \delta^2 I_{n_w} + M_w^2 C^T C$, $\Lambda_{21} = kM_w^2 C^T F + P E$, $\Lambda_{12} = \Lambda_{21}^T$ and $\Lambda_{22} = -\bar{\alpha}_w (\gamma_k^2 - \delta^2) I_{n_e} + M_w^2 k^2 F^T F$. This leads to the following corollary of Theorem 1.

**Corollary 1.** Assume (10) and (11) hold for some $W, \bar{\alpha}_w$ and $\bar{\alpha}_w$ with $\left\| \frac{\partial W(x, e)}{\partial e} \right\| \leq M_w$, $\bar{\alpha}_w(s) = \bar{\alpha}_w(s)$ and $\bar{\alpha}_w(s) = \bar{\alpha}_w(s)$ for constants $M_w > 0$, $\bar{\alpha}_w > 0$ and $\bar{\alpha}_w > 0$. Suppose $f$ and $g$ satisfy (18) and let (20) hold for some $\gamma_k > 0$, $P > 0$, $\delta > 0$ and $k \in [0, 1]$. If $\tau_{MATI}$ in (8) satisfies for $L = L_k$ according to (19) and $\gamma = \gamma_k$ the bound (14) and $0 < \epsilon < \epsilon_{MATI}$ then, for the system (8), the set $\{x, e, \tau, k\} : x = 0, e = 0$ is UGES.

**Proof.** Follows with the preceding derivations from Theorem 1.

\[ \square \]

For any fixed parameter $k$, we can find a suitable value for $\gamma_k$ by minimizing it subject to the LMI constraint (20). $P$ can be used as an additional decision variable. To find preferably good values for $\gamma_k$ and $L_k$, we can therefore test different values $k \in [0, 1]$ and compute $L_k$ and $\gamma_k$ for each of these values. To select the values of $k$ for testing, for example a uniform grid can be used. For each pair $(\gamma_k, L_k)$, a bound on $\tau_{MATI}$ can then be computed with (14), and we can therefore use the pair $(\gamma_k, L_k)$ that leads to the largest bound.

**Remark 7.** For $k = 0$, (20) reduces to the LMI that was considered in (Heijmans et al., 2017, Theorem 3), which can be used to determine the MATI bound for the approach from Carnevale et al. (2007). Computing $\gamma_k$ and $L_k$ for any fixed parameter $k$ requires hence solving an LMI of the same dimension as for the approach from Carnevale et al. (2007). Thus the computational effort for our proposed approach scales, in comparison to the approach from Carnevale et al. (2007), linearly with the number of grid points that are considered.

4.2 Example 2

To illustrate our approach for linear systems, we consider the numerical example from Heijmans et al. (2017), for which plant and controller are given by $\dot{x}_p = A_P x_p + B_P u$ and $u = -Kz$ with $A_P = \begin{bmatrix} 1 & -4 \\ -2 & 3 \end{bmatrix}$, $B_P = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, and $K = -0.2$, i.e., $\hat{x}_p$ contains the most recently received values for the plant states. Each plant state is measured by a separate node. For $e = \hat{x}_p - x_p$, we obtain therefore $A = A_P - B_P K$, $E = -B_P K, C = -A, F = -E$ and $l = 2$. Moreover, when we consider the TOD protocol, we can use $\lambda = \sqrt{1/\tau}$ and $W(\kappa, e) = \|e\|$ with $\tilde{\alpha}_w = M_w = 1$ (cf. Heemels et al. (2010)).

A comparison of our approach with the approach from Carnevale et al. (2007) is given in Table 1. In the first column of Table 1, the bound on $\tau_{MATI}$ for the approach from Carnevale et al. (2007) is given (i.e., the value for $k = 0$, cf. also Heijmans et al. (2017)). In the second column of Table 1, we state values for the bound on $\tau_{MATI}$ according to Corollary 1. To find these bounds on the MATI, we have gridded the interval $k \in [0, 1]$ uniformly with a step size of 0.001 and minimized $\gamma_k$ subject to the constraint (20) for each value of $k$ from this grid, using YALMIP. Sturm (2004) and Sedumi 1.3. Sturm (1999). Then, we have computed the bound on $\tau_{MATI}$ from (14) for each resulting combination of $\gamma_k$ and $L_k$. The value for the bound on $\tau_{MATI}$ in Table 1 is for each $\delta$ the largest bound.
that we found using this procedure. In the last column of Table 1, the improvement for the bound on \( T_{\text{MATI}} \) that can be achieved using our approach, is given. It can be seen that an improvement of over 100% is possible in some cases.

5. CONCLUSION

We have proposed a simple, yet efficient modification for the approach to compute the MATI from Carnevale et al. (2007); Nesic et al. (2009). The key feature is a slightly more general version of the main assumption from Carnevale et al. (2007). Due to this modification, the approach that we have presented in this paper can result in significantly larger bounds on the MATI, as it was demonstrated with two examples from the literature. An improvement of more than 100% is possible in some cases. This modification can also easily be applied to the setups of many works that are based on the results of Carnevale et al. (2007); Nesic et al. (2009), providing there similar improvements. Moreover, combining our results with different approaches to improve the MATI bound as, e.g., those from Hertneck et al. (2020); Heijmans et al. (2020, 2017) can lead to an even further improvement of the MATI bound.

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Appendix A. PROOF OF THEOREM 1

**Proof.** This proof is essentially the same as the proof of (Carnevale et al., 2007, Theorem 1), and requires only minor modifications. Nevertheless, we state it here for the sake of completeness. We consider the solution \( \phi : [0, T_{\text{MATI}}] \to \mathbb{R} \) to

\[
\dot{\phi} = -L_\phi - \gamma (\phi^2 + 1), \quad \phi(0) = \lambda^{-1}.
\]

(\ref{eq:phi})

From Carnevale et al. (2007), we know that \( \phi(\tau) \in [\lambda, \lambda^{-1}] \) for all \( \tau \in [0, T_{\text{MATI}}] \). We denote \( \xi := [x^\top, e^\top, \tau, \kappa] \) and \( F(\xi) := [f(x, e)^\top, g(x, e)^\top, 1, 0]^\top \), and use subsequently the function \( U(\xi) := V(x) + \gamma \phi(\tau) W^2(\kappa, e) \). Note that
We observe from (A.2) that

\[ U(x^+) = V(x^+) + \gamma \phi(\tau^+) W^2(\kappa^+, e^+) \]
\[ = V(x) + \gamma \phi(0) W^2(\kappa + 1, h(\kappa, e)) \]
\[ \leq V(x) + \gamma \lambda W^2(\kappa, e) \leq U(\xi). \quad (A.2) \]

We also have\(^3\) using (12), (13) and (A.1), for all \((\tau, \kappa)\) and almost all \((x, e)\) that

\[ \langle \nabla U(\xi), F(\xi) \rangle \leq -\varrho(||x||) - \varrho(W(\kappa, e)) - H^2(x, e) \]
\[ + \gamma^2 W^2(\kappa, e) \]
\[ + 2\gamma \phi(\tau) W(\kappa, e) (LW(\kappa, e) + H(x, e)) \]
\[ - \gamma W^2(\kappa, e) (2L\phi(\tau) + \gamma (\phi^2(\tau) + 1)) \]
\[ \leq -\varrho(||x||) - \varrho(W(\kappa, e)) - H^2(x, e) \]
\[ + 2\gamma \phi(\tau) W(\kappa, e) H(x, e) \]
\[ - \gamma^2 W^2(\kappa, e) \phi^2(\tau) \]
\[ \leq -\varrho(||x||) - \varrho(W(\kappa, e)). \quad (A.3) \]

Note that (A.3) is the only part of this proof where \(H\) and therefore the modified part of Assumption 1 come into play and thus the only part of this proof that is different to the proof of (Carnevale et al., 2007, Theorem 1). The function \(\varrho\) is positive definite, and \(V\) is positive definite and radially unbounded. Therefore, since \(\phi(\tau) \in [\lambda, \lambda^{-1}]\) for \(\tau \in [\epsilon, \tau_{\text{MATI}}]\), there exists a continuous, positive definite function \(\tilde{\varrho}\) such that \(\langle \nabla U(\xi), F(\xi) \rangle \leq -\tilde{\varrho}(U(\xi)).\) This implies, using standard arguments for continuous-time systems (see e.g. (Khalil, 2002, p. 146)), that there is \(\beta \in KL\) such that

\[ U(\xi(t, j)) \leq \beta(U(\xi(t_1, j)), t - t_j), \forall(t_j, j) \subseteq (t, j) \in \text{dom } \xi, \quad (A.4) \]

where \((t_j, j) \subseteq (t, j)\) means \(t_j \leq t,\) with \(\beta\) satisfying also

\[ \beta(s, t_1 + t_2) = \beta(\beta(s, t_1), t_2), \]

\[ \forall(s, t_1, t_2) \in R_{\geq 0} \times R_{\geq 0} \times R_{\geq 0}. \quad (A.5) \]

We observe from (A.2) that \(U(\xi(t_{j+1}, j + 1)) \leq U(\xi(t_{j+1}, j))\) for all \(j\) such that \((t, j) \in \text{dom } \xi\) for some \(t \geq 0.\) Using in addition (A.4) and (A.5), we obtain \(U(\xi(t, j)) \leq \beta(U(\xi(0, 0)), t), \forall(t, j) \in \text{dom } \xi,\) and thus, as \(t_{j+1} - t_j \geq \epsilon,\)

\[ U(\xi(t_{j+1}, j)) \leq \beta(U(\xi(0, 0)), 0.5t + 0.5\epsilon j), \forall(t_j, j) \in \text{dom}(\xi). \]

Note, since \(V\) is positive definite, using additionally the bounds on \(W(e)\) from (10) and the fact that \(\phi(\tau) \in [\lambda, \lambda^{-1}]\) for \(\tau \in [0, \tau_{\text{MATI}}],\) global uniform asymptotic stability of the set \(\{x, e, \tau, \kappa : x = 0, e = 0\}\) follows.

To prove uniform exponential stability, the additional assumptions for the theorem can be used to show that \(\tilde{\varrho}\) can be chosen linear, which allows us to choose \(\beta(s, t) = Ms\exp -\lambda t\) in (A.4) for some \(\lambda > 0\) and some \(M > 0.\) This guarantees together with the additional quadratic bounds on \(V(x)\) and \(W(e),\) uniform exponential stability. \(\square\)

\(^3\) As in Carnevale et al. (2007), we use here \(\langle \nabla U(\xi), F(\xi) \rangle\) by a slight abuse of notation, even though \(W\) is not differentiable with respect to \(\kappa.\) This is justified since the corresponding component of \(F(\xi)\) is zero.