On the quantum origin of potentials

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The dynamics of a quantum particle is governed by its wavefunction, which in turn is determined by the classical potential to which it is subjected. However the wavefunction itself induces a quantum potential, the particle ‘sees’ the sum of the classical and quantum potentials, and there is no way to separate the two. Therefore in principle, part or whole of an observed potential may be attributable to a quantum potential. We examine this possibility and discuss implications.

The standard paradigm of quantum mechanics, non-relativistic or relativistic, consists of solving a suitable wave equation (such as Schrödinger, Klein-Gordon or Dirac) with an appropriate potential, for normalizable wavefunctions. However, there is a lesser known result that the wavefunction of a quantum particle itself generates a ‘quantum potential’ and the dynamics of the particle is governed by the sum of the classical and quantum potentials and not by them individually. Therefore, part or all of one can be replaced by the other, and in principle it is possible that particles are intrinsically in a flat spacetime without any potentials and what is normally perceived as a classical potential, for e.g. a harmonic oscillator, Coulomb or the gravitational potential, is in reality, a quantum potential generated by the wavefunction of the particle.

To explore this possibility, for simplicity, let us start with the Schrödinger equation (generalization to relativistic wave equations should be straightforward)

$$\begin{align*}
-\frac{\hbar^2}{2m} \nabla^2 \Psi + V \Psi &= i \frac{\partial \Psi}{\partial t}, \\
\end{align*}$$

and imaginary parts, yields two equations

$$\begin{align*}
\hat{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} &= 0, \\
m \frac{d \vec{v}}{dt} &= -\hat{\nabla} (V + V_Q) = -\hat{\nabla} V_{tot},
\end{align*}$$

where $\rho = |\Psi|^2$, $\vec{J} = \frac{\hbar}{2mi} \left[ \Psi^* \hat{\nabla} \Psi - \Psi \hat{\nabla} \Psi^* \right]$ are respectively the probability density and the probability current density, $\vec{v} := (h/m) \nabla S$ is the ‘velocity field’,

$$V_Q = -\frac{\hbar^2}{2m} \nabla^2 R,$$

is the ‘quantum potential’, and

$$V_{tot} \equiv V + V_Q.$$

Eq. (4) is simply the standard conservation equation for $\rho$ and $\vec{J}$, Eq. (4), on the other hand, shows that the quantum dynamics of a particle of mass $m$ is still governed by the classical Newton’s equation, albeit with the classical potential $V$ augmented by the wavefunction dependent ‘quantum potential’. For instance, while a stream of free electrons, governed just by a vanishing classical potential ($V = 0$), follows straight line trajectories, that governed by $V_{tot} = V + V_Q$, for an appropriate choice of $\Psi(\vec{x}, t)$ reproduces the familiar interference patterns. Thus, we see that a quantum particle subjected to $V_{tot}$ has no way of figuring out what part of it is ‘classical’ and what part ‘quantum’, and a part (or all) of the classical potential may in fact be the quantum potential generated by a suitable wavefunction and vice-versa. Going one step further, one can even say that observed potentials and wavefunctions are interchangeable and it may be a matter of choice as to which one is considered more fundamental.

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Before proceeding further, we present an even lesser known result which follows from Eqs. (1) and (5), namely, that for a stationary state, of energy \( E \) and described by the wavefunction \( \Psi(\vec{x}, t) = \psi(\vec{x}) e^{-iEt/\hbar} \), one finds

\[
V_Q = -V + E. \tag{7}
\]

The result is remarkable for two reasons. First, it shows that the \( \hbar \)'s cancel out from Eq. (5) and the induced quantum potential is simply equal and opposite to the starting classical potential \( V \) (up to an unimportant constant).

Accordingly, the quantum force is also equal and opposite to the starting classical force. Second, since as mentioned above, \( V \) and \( V_Q \) are interchangeable, it follows that if we had started with \( V_Q \) instead of \( V \) in the Schrödinger equation, it would have induced an equal and opposite \( V \) via quantum mechanics! In other words, what is to be treated as classical and what as quantum may just be a matter of choice, and the presence of any one would cause the other to emerge naturally, and this is true regardless of the exact form of the starting potential. For example, if one starts with an attractive gravitational potential, one would end up with a repulsive quantum potential. Alternatively, the observed attractive gravitational potential is equally likely to be, in reality, the quantum potential induced by a repulsive classical potential [3–7]!

Extending the idea further, we now ask a natural question: what if a particle experiences a potential, but the starting classical potential is zero, i.e. the particle is in free space, albeit with a wavefunction whose amplitude satisfies Eq. (5)? Evidently, the following form of Eq. (5)

\[
\nabla^2 \mathcal{R} + \frac{2mV_Q}{\hbar^2} \mathcal{R} = 0 \tag{8}
\]

makes it clear that the particle experiences a potential \( V_Q \). However, there is no way to tell whether this is a background potential, or that resulting from a specific form of the real part of the wave-amplitude (to be considered as time-independent and denoted by \( \mathcal{R}(\vec{x}) \)). Note that this does not completely specify the full wavefunction \( \Psi(\vec{x}, t) \), as its phase \( S(\vec{x}, t) \) remains undetermined. Nevertheless, the validity of the standard probability interpretation of quantum mechanics requires \( \Psi(\vec{x}, t) \), and hence \( \mathcal{R}(\vec{x}) \), to be normalizable in the first place. Therefore, one can take the route of finding normalizable solutions of Eq. (5) for a host of potentials. We however adopt a simpler approach in which \( V_Q \) in Eq. (8) is replaced by \( V_Q + E_0 \), where for simplicity, we take \( E_0 \) as the ground state energy of the system under consideration. So, the Eq. (8) becomes

\[
\nabla^2 \mathcal{R} + \frac{2m}{\hbar^2} \left( E_0 - \frac{m\omega^2 x^2}{2} \right) \mathcal{R} = 0. \tag{9}
\]

This is nothing but the time-independent Schrödinger equation corresponding to the ground state, but for a potential \(-V_Q\). However, the interpretation is quite different here. Let us assume that there exist normalizable solutions of this equation, for some reasonable potentials. Now, the requirement of \( \mathcal{R} \) to be real is guaranteed at least for the ground state [8]. We can therefore stipulate the potential \(-V_Q - E_0\) (or approximately, \(-V_Q\), for \(|V_Q| \gg |E_0|\)) to be that felt effectively by a particle, provided the amplitude of the particle’s wavefunction, naturally arising or otherwise, is a solution of Eq. (9).

As an example, consider an inverted linear harmonic oscillator and postulate that the ‘observed’ potential \( V = -m\omega^2 x^2/2 \) is in reality the quantum potential experienced by a free particle, described by an wave-function whose amplitude satisfies Eq. (9) with \( V_Q = V \), that now reads

\[
\nabla^2 \mathcal{R} + \frac{2m}{\hbar^2} \left( E_0 - \frac{m\omega^2 x^2}{2} \right) \mathcal{R} = 0. \tag{10}
\]

This is just the stationary state Schrödinger equation for a linear harmonic oscillator potential. The solution is of course the unique, normalizable ground state wavefunction given by

\[
\mathcal{R} = \left( \frac{m\omega}{\pi \hbar} \right)^{1/4} e^{-m\omega x^2/2\hbar}. \tag{11}
\]

Similarly, substituting for a repulsive Coulomb potential \( V_Q = e^2/r \), and replacing it with \( (V_Q + E_0) \) in Eq. (11), where \( E_0 = -me^4/2\hbar^2 \) is the ground state energy of the Hydrogen atom, one finds the unique normalizable solution

\[
\mathcal{R} = \frac{a_0^{-3/2}}{\sqrt{\pi}} e^{-r/a_0}, \tag{12}
\]

with \( a_0 = \hbar^2/(me^2) \). One may carry on this way by considering any other potential, and deriving the corresponding \( \mathcal{R} \) via Eq. (11).
To test the robustness of the method, we consider a couple of more examples and obtain the wavefunction required to produce a given quantum potential, using Eq. (8) directly. First, the ‘step potential’ (in one-dimension), defined by $V_Q = 0$ for $x < 0$, and $V_Q = V_0$ for $x \geq 0$, where $V_0$ is a constant. The corresponding wavefunction is given by $R = 1$ for $x < 0$, and $R = \cos(kx)$ for $x \geq 0$, where $k = \sqrt{2mV_0/\hbar^2}$. It can be easily verified that Eq. (5) gives the required step potential.

The last example we consider is a linear potential, also in one dimensions, given by $V_Q = \kappa x$, where $\kappa$ is a constant. In this case, substituting in Eq. (5) and solving for $R$, we get $R = Ai(-k_1^{1/3} x)$ or $R = Bi(-k_1^{1/3} x)$, where $Ai$ and $Bi$ are the Airy functions and $k_1 = 2m\kappa/\hbar^2$.

The above examples emphasize the suitability of the method, although we note that the number of exactly solvable examples from Eq. (5) is limited, and one would have to resort to numerical techniques for more complicated potentials. The plots of the various quantum potentials and the corresponding wavefunctions that give rise to them are shown in the Figures 1-4.

To summarize, we have shown that practically any pre-
sumptive classical potential can in fact be derived as the quantum potential of a suitable wavefunction. These could be familiar potentials such as the Coulomb, inverted harmonic oscillator or the gravitational potential, or exotic ones such as those which can effectively explain the flat galaxy rotation curves. Therefore it begs the questions as to whether the ones we observe as potentials and interactions are mere manifestations of appropriate wavefunctions. The question is particularly important in the context of the gravitational field, since the corresponding field theory is known to be non-renormalizable. Therefore, a more fundamental picture, with the observed gravitational fields as quantum potentials of particle wavefunctions would offer a fresh insight into the problem. It should even predict corrections to the $1/r$ potential, at very small or large length scales, beyond the regimes in which the Newtonian gravity has been verified experimentally. Of course, there are many questions to be answered. Some of these should be straightforward, for example, the relativistic generalization of our work starting from a relativistic wave equation (part of it has been done in ref. [3]). The others, for instance, the determination of the phase $\mathcal{S}$ and how the appropriate wavefunctions can be specified, require further studies.

To conclude, we list a few important observations and possible directions for future works.

1. **Excited states:** we remind that Eq. (9) is an approximation of Eq. (8), where the ground state energy $E_0$ was introduced in an attempt to guarantee a normalizable $\mathcal{R}$, albeit at the cost of ‘raising’ the quantum potential by a small amount. Similarly, one may replace $E_0 \rightarrow E_n$, an excited state energy, and $\mathcal{R} \rightarrow \mathcal{R}_n$, the corresponding excited state. This demonstrates the non-uniqueness of the wavefunction in general for a given quantum potential.

2. **Non-stationary states:** as Eq. (5) suggests, a time-dependent quantum potential $V_Q$ would require a time-dependent $\mathcal{R}$ as its source. Furthermore, one sees from Eq. (4) that such a $\mathcal{R}$ would entail a non-trivial phase $\mathcal{S}$.

3. **Other formulations:** It may be noted that a ‘modified potential’, distinct from, but related to the quantum potential, was defined and its implications explored in detail in [9]. A re-examination of our results in light of the aforementioned potential may shed further light on emergent potentials. Similarly, the possible origin of the Schrödinger equation purely from symmetry considerations was examined in [10] [11]. That study may also complement the current work, as it provides a justification for starting with the Schrödinger equation in the first place. It may also be useful in extending our results to other quantum wave equations.

4. **Future directions:** in addition to studying the above issues, namely excited states, non-stationary states and other formalisms, we will carefully investigate a few other things. This includes, the (special) relativistic generalization of our results, computing corrections to the gravitational potential at various length scales, studying the stability of the wavefunction, generalization to higher dimensions and last but not the least, examine whether the idea can be generalized to Yang-Mills potential. We hope to report on these elsewhere.

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