PAPER

Quantum control for high-fidelity multi-qubit gates

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Abstract
Quantum control for error correction is critical for the practical use of quantum computers. We address quantum optimal control for single-shot multi-qubit gates by framing it as a feasibility problem for the Hamiltonian model that is then solved with standard global optimization software. Our approach yields faster high-fidelity (>99.99%) single-shot three-qubit-gate control than obtained previously, and it has also enabled us to solve the quantum-control problem for a fast high-fidelity four-qubit gate.

1. Introduction
Quantum computing could convert certain intractable computational problems, such as number factorization, into efficiently solvable problems by replacing binary representations and Boolean logic with quantum-bit (qubit) strings and quantum gates [1]. A procedure in a quantum algorithm is expressed as a quantum circuit comprising a universal primitive instruction set of quantum gates, such as the Hadamard (H), π/8 (T), and two-qubit entangling controlled-not gates, or, alternatively, the H gate along with the three-qubit Toffoli gate, which effects controlled–controlled–not operation [1]. Scalable quantum computing is achieved if its components, such as preparation of qubits, quantum logic gates, and measurements, achieve a minimum performance threshold, typically expressed in terms of average gate fidelity but rigorously required to be the diamond-norm-based error rate for the fault-tolerance theorem [2, 3]. If the performance threshold is met, then the quantum information is encoded into a quantum codespace, processed in that codespace by suitably encoded quantum gates, and decoded at the output.

Although the Solovay–Kitaev theorem guarantees efficient decomposition of quantum gates into a primitive instruction set [4], concatenating faulty gates unfortunately compounds error. Fortunately, however, these errors can be ameliorated by fault-tolerant methods [2]. Even if high-quality gates are created, the multi-qubit operations to map quantum information into and out of the quantum codespace could fail due to accumulated errors. Our aim is to devise versions of single-shot multi-qubit gates [5–8], where single-shot refers to a controlled, uninterrupted, continuous-time evolution to realize the quantum gate. We solve for such gates based on a Hamiltonian description of the physical system such that a suitable time-dependent driving term delivers a high-fidelity approximation to the desired quantum gate using error avoidance to achieve the requisite threshold for efficient quantum error correction. Fast single-shot multi-qubit gates are vital to achieving high fidelity because they are able to operate faster than the decoherence time. The alternative to decomposing a multi-qubit gate into a sequence of one- and two-qubit gates is necessarily much slower and thus is affected significantly by decoherence.

To this end, some of us have designed an algorithm based on differential evolution that delivers a fast high-fidelity Toffoli gate [9] and other fast high-fidelity three-qubit gates [10] in contrast to alternative approaches that decompose a three-qubit gate into single- and two-qubit gates. We use average fidelity gate because the figure of merit for the quality of the gate as this figure of merit is widely used and amenable to experimental
testing. Whereas this approach is shown to deliver fast high-fidelity three-qubit gates, extending to four-qubit gates was infeasible for our available computational resources, and adoption of this approach by other researchers was impeded by the need to learn and code our specialized algorithm that we dubbed subspace-selective self-adaptive differential evolution (SuSSADE)\cite{10,11}.

Here we report a new method that is superior in that it reproduces previous results, yields faster, higher-fidelity three-qubit gates, and solves fast, high-fidelity four-qubit gates. Our method is also superior in that we use standard optimization software rather than specialized in-house code, thus making our method easily accessible to other researchers. Solving the quantum-control procedure in this way takes patience because many time-consuming runs are required, but our demonstration of better three-qubit gates and the first solution for a single-shot four-qubit gate (rather than decomposing the four-qubit gate into concatenated fewer-qubit gates \cite{12-14}) via quantum control proves that this method works and is viable with modest computational resources.

We have formulated the quantum-control problems as feasibility problems that define the solution only in terms of criteria (i.e., design constraints) that the multi-qubit gate must possess. These criteria are typically expressed mathematically as sets of equations or inequalities. The feasibility problem is then solved using standard global optimization software. Most quantum-control problems are solved using greedy algorithms \cite{13}, which either rely on the optimization problem being convex or on local solutions being good enough, but our aim for high-fidelity performance renders the response surface highly complex, thus making global optimization algorithms attractive \cite{15}. Thus, our approach offers two advantages: the first is using global optimization to obtain better pulse sequences for gate optimization, and the second is avoiding the decomposition into fewer-qubit gates that would slow operation and thus enhance the deleterious effect of decoherence.

As an application of our quantum-controlled multi-qubit gate approach, we focus on a superconducting-circuit realization because remarkable progress has been achieved with superconducting artificial atoms, realized as transmons or similar, such as the demonstration of nine coherently coupled superconducting transmons \cite{16}. A high-fidelity two-qubit entangling gate is achieved via exploiting energy levels beyond the two-level qubit space and employing the method of avoided level crossings \cite{17}. The essence of the avoided-level-crossing-level gate is that the frequency of each qubit is tuned such that energy levels approach each other but remain non-degenerate through the whole evolution. Avoided-level-crossing evolution mixes the energy population and dynamical phases such that the final evolution of the system leads to the desired quantum gates.

Our strategy for designing quantum control to achieve multi-qubit gates is based on avoided level crossings but for a higher dimension \cite{9,10}. We first pose quantum control for avoided level crossings as a feasibility problem. Then we describe how we use the Global Optimization Toolbox in MATLAB\textsuperscript{\textregistered} to solve the feasibility problem and extract the external pulses. We characterize the accuracy of quantum gates by the intrinsic fidelity, from which the error rate can be inferred. A quantum gate design is defined as feasible if its intrinsic fidelity satisfies a sufficiently high threshold, which here is taken to be 99.99%.

The outline of our paper is as follows. The superconducting-circuit model for realizing a Toffoli gate is described in section 2 by giving a mathematical description of the mechanics involved. Avoided-level-crossing-based quantum gates are described in detail in section 3. This description is followed by an explanation of the quantum-control procedure and the computational methods involved in finding high-fidelity procedures in section 4. Results are presented in section 5. Conclusions and directions for future research are given in section 6.

2. Model

We consider $n$ coupled superconducting artificial atoms \cite{18}, with parameters appropriate for capacitively coupled transmon systems \cite{19,20}. Each transmon has $j$ energy states, and transmon locations are denoted by $k \in [n] = \{1, 2, \ldots, n\}$. Anharmonicities of the second and third energy levels are parametrized by $\eta$ and $\eta'$, where we assume

$$\eta = 200 \text{ MHz}, \quad \eta' \approx 3\eta,$$

which is appropriate under the cubic approximation of the potential well \cite{17}. Capacitive coupling between the transmons yields an $XY$ interaction between adjacent transmons (in the rotating frame) with a coupling strength of $g = 30$ MHz.

In the ideal gate, a pulse is sent into the transmon system to correct for errors. The shifted frequency for the pulse applied to the gate is the set of shifted frequencies

$$\varepsilon(t) = \{\varepsilon_k(t)\}_{k=1}^{n}, \quad t \in [0, \Theta],$$

with each shifted frequency bounded by

$$-2.5 \text{ GHz} \leq \varepsilon_k(t) \leq 2.5 \text{ GHz}$$

for each $k = 1, 2, \ldots, n$. The pulse duration is $\Theta$ seconds, and the shifted frequencies are applied with a delay of $\tau$ seconds. The problem of finding the optimal pulse sequence is then formulated as a feasibility problem, where the solution is the set of shifted frequencies that satisfies the design constraints.

The feasibility problem is solved using the Global Optimization Toolbox in MATLAB\textsuperscript{\textregistered}, which employs an adaptive differential evolution algorithm (SuSSADE)\cite{10,11} to find the optimal pulse sequence. The solution is obtained by minimizing an objective function that represents the error rate of the quantum gate. The error rate is calculated as the difference between the desired and actual final states of the transmon system, and the objective function is designed to minimize this difference.

The final design of the quantum gate is presented in section 5, along with the results of the feasibility problem solution. The conclusions of the research and directions for future work are presented in section 6.
in our numerical simulations. The Hamiltonian for \( n \) capacitively coupled transmons is represented as the \( j^n \)-dimensional block-diagonal matrix [17]

\[
\frac{H(t)}{\hbar} := \hat{H}(t) = \sum_{k=1}^{n} P^{\otimes n}_{k}(\text{diag}(0, \varepsilon_k(t), 2\varepsilon_k(t) - \eta, 3\varepsilon_k(t) - \eta'))
\]

\[
+ \frac{g}{2} \sum_{k=1}^{n-1} P^{\otimes (n-1)}(X_k \otimes X_{k+1} + Y_k \otimes Y_{k+1}),
\]

where \( I \) is the identity matrix and each block corresponds to a fixed number of excitations and acts on the Hilbert space \( \mathcal{H}_i^{\otimes n} \). The promotion operator \( P^{\otimes n}(\mathcal{B}) \) is defined as the sum of all possible Kronecker products of \( \mathcal{B} \) with \( (n - 1) \) copies of \( A \).

The coupling operators

\[
X_k = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & \sqrt{2} & 0 \\
0 & \sqrt{2} & 0 & \sqrt{3} \\
0 & 0 & \sqrt{3} & 0
\end{pmatrix}, \quad Y_k = \begin{pmatrix}
0 & -1 & 0 & 0 \\
0 & 1 & -\sqrt{2} & 0 \\
0 & \sqrt{2} & 0 & -\sqrt{3} \\
0 & 0 & \sqrt{3} & 0
\end{pmatrix},
\]

are the generalized Pauli operators [17]. Under the assumption of uniform coupling, these operators are in fact independent of \( k \).

The block-diagonal property of the Hamiltonian permits equation (3) to be reduced to a subspace in which at most \( n \) excitations are present

\[
\hat{H}_p(t) = C_n \hat{H}(t) C_n^\dagger,
\]

for \( C_n \) denoting the operator that truncates the Hamiltonian (3). Hamiltonian (5) evolution proceeds over the gate time \( \Theta \) such that the resultant unitary operator is

\[
U(\Theta) = T \exp \left\{ \int_0^\Theta dt \frac{H_p(t)}{i\hbar} \right\} = T \exp \left\{ -2\pi i \int_0^\Theta dt \hat{H}_p(t) \right\},
\]

with \( T \) the time-ordering operator [21].

Transmon states are defined over a computational subspace \( U(\Theta) \) of at most three excitations, where the resulting projection

\[
U_{\text{cs}}(\Theta) = \mathcal{P} U(\Theta) \mathcal{P}^\dagger,
\]

yields a computational subspace for unitary operator (6) with (matrix) dimension \( 2^n \). The projected unitary operator (7) is used to determine the intrinsic fidelity by comparing it to the target gate \( U_{\text{target}} \) of the system. However, for superconducting artificial atoms, local \( z \) rotations can be performed quickly and accurately; therefore, we construct a local \( z \)-equivalence class for an arbitrary element (gate) in SU(4) and then specialize to the ideal gate.

Subspaces \( U(\Theta) \) and \( U'(\Theta) \) of SU(4) are equivalent if

\[
U'(\Theta) = U U(\Theta) U^\dagger,
\]

with \( 2^n \)-dimensional diagonal matrix

\[
U_{lr}(\beta_1, \beta_2, \ldots, \beta_k): = \bigotimes_{r=1}^{k} R_z(\beta_r), \quad R_z(\beta) = \text{diag}(1, e^{-i\beta})
\]

for \( \beta_r \) the angle between diagonal entries of unitary evolution matrix (6). For example, the diagonals of \( U_{lr} \) for the three-qubit case are represented explicitly as

\[
U_{lr} = \text{diag}(1, e^{-i\beta_1}, e^{-i\beta_2}, e^{-i(\beta_1+\beta_2)}, e^{-i\beta_3}, e^{-i(\beta_1+\beta_3)}, e^{-i(\beta_2+\beta_3)}, e^{-i(\beta_1+\beta_2+\beta_3)}).
\]

This matrix (10) is employed to perform phase compensation in our numerical simulation.

By comparing the gates, the intrinsic fidelity is given by the trace formula

\[
\mathcal{F}(\Theta, n; \varepsilon(t)) = \frac{1}{2^n} |\text{tr}(U_{\text{target}}^\dagger U_{\text{cs}}(\Theta))| \in [0, 1]
\]

and is a performance figure of merit for the quantum-gate operation for given shifted frequencies that does not include decoherence. Our aim is to find a set of shifted frequencies (2) such that

\[
\mathcal{F}(\Theta, n; \varepsilon(t)) \geq 99.99\%
\]

for multi-qubit gates of size \( n = 3, 4 \). The set (2) of shifted frequencies is defined to be the solution to a feasibility problem.
3. Avoided-level-crossing-based quantum gates

We now explain the avoided-level-crossing-based controlled-Z (CZ) gate for the physical model of two capacitively coupled frequency-tunable transmons, with $\eta$ and $g (g \ll \eta)$ being the anharmonicity and coupling strength, respectively. We clarify why finding a theoretical solution for multi-qubit gates is challenging and motivates our quantum-control approach.

Whether in the sudden or the adiabatic regime, the idea of engineering a pulse for the avoided-level-crossing-based gate in a superconducting architecture remains the same. To design a control pulse for the two-qubit CZ gate, we tune the transmon frequencies such that the $|1\rangle$ state mixes with the other states in the second excitation subspace while the states in zero- and single-excitation subspaces remain detuned from each other. However, for practical implementations, both the sudden and adiabatic regimes are unsuitable for obtaining the threshold fidelity required for the fault tolerance (at least 99.99% for topological error correction [22]), and one needs to resort to advanced quantum-control techniques to engineer the feasible pulse; this is the primary motivation for our work.

One idea for designing three-qubit gates is to couple three transmons via a superconducting cavity, usually referred to as the circuit-quantum-electrodynamics (cQED) architecture [23] and tune the transmon frequencies in the dispersive regime such that the time-evolution operator gives rise to the target three-qubit gate at the end of the operation. Whereas such an approach has already been used to demonstrate a Toffoli gate [7], we do not consider cQED hardware here because the architecture only contains a few transmons inside a superconducting cavity and is thus not evidently scalable.

Instead, we consider a one-dimensional chain of $n$ transmons with nearest-neighbor coupling. One might seek to avoid the advanced quantum control approach and resort to a theoretical approach for designing multi-qubit gates [17], but this approach could fail for designing multi-qubit gates. In our approach, we initially detune the transmon frequencies from each other, making all eigenstates of the system non-degenerate. In order to obtain a high-fidelity three-qubit gate, we then vary the frequencies of all the qubits simultaneously such that the $|111\rangle$ state mixes strongly with other states in the third excitation subspace, while all other computational-basis states are detuned from each other.

Figure 1 shows the energy spectrum of the three-transmon Hamiltonian (truncated up to the third excitation subspace) as a function of the frequency of the second transmon while other frequencies are kept fixed at 4.8 GHz and 6.8 GHz. It is clear from figure 1 that devising an analytic quantum-control technique for a three-transmon system where all the transmons are driven simultaneously is a difficult task because of the existence of so many avoided crossings in the energy spectra.

We can follow the same argument to explain an avoided-level-crossing-based approach for designing a four-qubit gate. For this case, we initially detune the frequency of four individual transmons to certain finite values and then drive all of them simultaneously such that the $|1111\rangle$ state mixes strongly with the other states in the fourth excitation subspace, with all other computational-basis states detuned from each other.

From what we have described for a three-qubit gate, the avoided-level-crossing-based four-qubit-gate design problem is evidently much harder. For the case of the three-qubit gate, the truncated Hamiltonian includes the first 20 levels of the energy spectrum, whereas this number is 66 for a four-qubit gate. The increased number of energy levels increases the dimension of the optimization problem for the design of the four-qubit gate, and the complexity in solving optimization problems scales exponentially with dimension. For example, the increased number of excitations in each subspace also makes it significantly more difficult relative to two- or three-qubit gates to find an optimal region to establish a working avoided-level-crossing-based procedure to design four-qubit gates. This is why we employ a quantum-control scheme to devise a policy for designing such multi-qubit gates [9].

4. Quantum control

The following subsections discuss how our quantum-control procedure is realized. Approaches in the literature have formulated the problem as an optimization problem. The procedure here is based on finding a solution to a feasibility problem for an $n$-qubit circuit where shifted pulse frequencies are chosen such that the only requirement is that the intrinsic fidelity meets or exceeds a given threshold. We describe how the feasibility problem is stated and solved using the GlobalSearch solver in in MATLAB® for the three-qubit and four-qubit cases.

4.1. Feasibility problem

We formally state the feasibility problem for the design of a high-fidelity $n$-qubit gate as follows. Given a circuit with $n$ qubits and a gate time $\Theta$, 
approaches based on local to solving quantum control problems each feasibility problem duration Newton \[\text{Inequality}\] shifted frequencies to obtain a feasible solution. transmons more available degrees of freedom and hence a greater chance of optimization has advantages dramatically with practice. This, combined with the fact that the degree of non-convexity of the response surface increases problem, it may not make sense to optimize anything, and in fact doing so may hinder or even prevent finding a solution is not biased by an objective function when trying to greedy methods. However, multi-start methods and genetic algorithms were ultimately ineffective in solutions to the squares of the constraint violations. The idea is that in the face of multiple (properly scaled) constraints, the solution is not biased by an objective function when trying to find a solution to a feasibility problem. For a given problem, it may not make sense to optimize anything, and in fact doing so may hinder or even prevent finding a

\[\text{find } e(t), \quad t \in [0, \Theta],\]  
subject to \[-2.5 \text{ GHz} \leq e_k(t) \leq 2.5 \text{ GHz}, \quad k = 1, 2, \ldots, n,\]  
\[\mathcal{F}(\Theta, n; e(t)) \geq 99.99\%.\]  

The response surface defined by the average gate fidelity (11) is highly non-convex. Despite this issue, approaches based on local (greedy) algorithms such as GRAPE and CRaB are mentioned as preferred approaches to solving quantum control problems [24]. Local algorithms such as Nelder–Mead [25], Krotov [26], and quasi-Newton [27] fail [15]. Accordingly, we turn to global optimization strategies to search the domain (13b) of the shifted frequencies to obtain a feasible solution.

The basic globalization strategy of employing a multiple-restart (or multi-start) of a (local) quasi-Newton optimization has advantages [28] with respect to quality of solution and required computational resources over greedy methods. However, multi-start methods and genetic algorithms were ultimately ineffective in finding solutions to (13) [15]. The most effective approach [15] is based on differential evolution [29] and led to an effective approach dubbed SuSSADE [9].

Discrepancies between algorithmic performance is frequent in scientific computing; robustness and speed often must be traded off for each other, where robustness in this context refers to the reliability of obtaining a consistent solution. On the one hand, purely global optimization algorithms such as SuSSADE are slow and steady. In the long run, they are expected to give the best results as defined by the final value of the objective function. On the other hand, local-optimization strategies such as multi-start can be relatively fast and give good answers. The final values of the objective function, however, are typically inferior to solutions provided by purely global optimization algorithms applied to non-convex optimization problems, given sufficient time and computational resources.

The frequency components (2) in (13) are represented as piecewise-constant (step) functions with time step duration \(\Delta t = 1\) ns. The gate time \(\Theta\) is assumed to be an integer number of \(\Delta t\) for a given simulation. Thus, each feasibility problem (13) has associated with it \(n\Theta/\Delta t\) degrees of freedom from (13b) that are used to satisfy Inequality (13c). The number of degrees of freedom is the same as the dimension of the control space. It is a straightforward observation that decreasing the time step duration \(\Delta t\) or increasing the gate time \(\Theta\) results in more available degrees of freedom and hence a greater chance of finding a solution to (13) for a given number of transmons \(n\). However, there are limits on how small \(\Delta t\) can be as well as how large \(\Theta\) can be for viability in practice. This, combined with the fact that the degree of non-convexity of the response surface increases dramatically with \(n\), makes (13) a challenging problem to solve.

In principle, feasibility problems do not have a formal objective function, in that sense distinguishing them from optimization problems. In practice, however, feasibility problems are typically solved as optimization problems. Often, an objective function is created as a penalty function of constraint violations, e.g., the sum of the squares of the constraint violations. The idea is that in the face of multiple (properly scaled) constraints, the solution is not biased by an objective function when trying to find a solution to a feasibility problem. For a given problem, it may not make sense to optimize anything, and in fact doing so may hinder or even prevent finding a

Figure 1. Energy spectrum of three nearest-neighbor-coupled transmons, where energy levels that have at most 3 excitations are shown. The first and third transmon frequencies are fixed at 4.8 GHz and 6.8 GHz, respectively. The frequency of the second transmon varies from 4.5 GHz to 7.5 GHz.
feasible solution. In the special case of (13), with only one constraint (13c), efficiency of the search can in practice be enhanced through minimizing an appropriate objective function such as infidelity

\[1 - \mathcal{F}(\Theta, n; \varepsilon(t))\]

or the angular deviation in Hilbert space from the target gate

\[\arccos(\mathcal{F}(\Theta, n; \varepsilon(t))).\]  \hspace{1cm} (14)

We find angular deviation (14) to yield the fastest convergence.

4.2. Algorithm

Here, the three-qubit and four-qubit cases are optimized using the GlobalSearch solver from the Global Optimization Toolbox in MATLAB®. The optimization is subject to a nonlinear constraint arising from the feasibility condition (13c). This solver is based on the scatter search/nonlinear programming solver as implemented in [30]. Scatter search can be thought of as a sophisticated multi-start algorithm for solving global optimization problems; i.e., the restart points are chosen adaptively as the information about the response surface becomes available. In contrast, classical multi-start algorithms, such as that described in [28] or the MultiStart algorithm in MATLAB®, simply start local optimization algorithms from uniformly distributed or random points in the design space.

Although in principle a sensible global optimization procedure should eventually converge to a global optimum, the amount of time and computational resources required to realize such convergence may not be realistic. Besides using GlobalSearch, we experimented with a number of global optimization solvers: SuSSADE, the DIRECT algorithm [31], particle swarm optimization [32] (and variants thereof), and MultiStart. We found GlobalSearch to be the most effective by a rather wide margin: it was the only algorithm that was ultimately able to find previously undiscovered feasible solutions. Optimization instances for three-, four-, or five-qubit circuits with \(\Theta \leq 100\) ns typically require only about 500 MB of RAM, meaning that from this perspective they can be run using modest hardware, such as a laptop computer. Serial computation times required to find solutions vary considerably, from a few minutes for the solution of the three-qubit problem with \(\Theta = 26\) ns to a few weeks for the four-qubit problem with \(\Theta = 70\) ns to months for the three-qubit problem with \(\Theta = 24\) ns. Parallel processing would speed up these times, the extent to which depends on the number of processors available. In such situations, high-performance computing hardware, e.g., in the form of clusters, would be more appropriate than desktop computing.

Beyond the goal of solving the feasibility problem defined by (13c), we are also interested in devising gates with short gate times \(\Theta\). Note that, although we seek small values of \(\Theta\), our approach is not optimization per se but rather employs a constraint on fidelity for fixed reasonable gate time to compute a solution. Posing the problem as finding the shortest gate time subject to the constraint of sufficiently high fidelity is equivalent to a single-objective optimization problem. In our work, the gate time is reduced directly by attempting to solve the feasibility problem for smaller and smaller \(\Theta\) values until solutions could no longer be found. A form of parameter continuation was employed such that feasible solutions for longer gate times were used as starting points in the solver for shorter gate times in the global search algorithm. This enabled more robust and more efficient convergence to a solution.

5. Results

Using the quantum-control process described in section 4, pulses are generated for the design of a single-shot high-fidelity three-qubit Toffoli gate that lead to a feasible solution of (13) over the minimal duration time of 23 ns. This is an improvement of 11.5% over the gate time of 26 ns reported in [9]. Figure 2 shows the resulting piecewise-constant pulses as a function of \(t\).

The quantum-control process is also able to produce a solution to the feasibility problem (13) for a four-qubit CCCZ gate. Figure 3 shows the pulses that lead to a feasible solution of (13) for piecewise-constant pulses as a function of \(t\). The resultant four-qubit gate operates in 70 ns. This is the first design to satisfy (13) with \(n = 4\). Suitable designed single-shot four-qubit gates could be valuable for four-qubit encoding to correct one erasure error [33] and sets the stage for progressing to higher-order single-shot multiqubit gates such as a five-qubit gate that could encode in a single shot for the five-qubit code [34].

6. Conclusions

We have formulated the problem of designing multi-qubit gates as a feasibility problem. This allows for a formally proper problem formulation because at its core the only defining feature of a solution is that the gate...
has a sufficiently high intrinsic fidelity \( \mathcal{F}(\Theta, n; \varepsilon(t)) \geq 99.99\% \). Because feasibility problems are often solved as optimization problems and there is only one such defining feature, convergence to a solution to (13) can be enhanced through the use of an appropriate objective function with a non-zero gradient at the solution. The control pulses for the transmon-shifted frequencies are discretized using piecewise-constant representations in time steps of 1 ns. The degrees of freedom of the feasibility problem are the parameters that represent the piecewise-constant values. Determining small gate times \( \Theta \) is done directly by solving a feasibility problem for fixed \( \Theta \) then reducing it until a feasible solution could no longer be found. Extensive tests were performed for the three-qubit system, so we have a high degree of confidence in the minimality of the gating time attained. The testing with the four-qubit system has thus far been more limited, and the problem is much more challenging; hence we make no strong claims as to the minimality of \( \Theta \) in this case.

We have employed this approach to design single-shot high-fidelity quantum gates, including the Toffoli and CCCZ gates. The operation time of these three- and four-qubit gates is comparable with the time of entangling two-qubit CZ [18] gates under the same experimental constraints. We have designed the multi-qubit gates for a simple architecture of linearly capacitively coupled superconducting atoms that can be a module of

![Figure 2](image-url)  
**Figure 2.** Piecewise-constant pulse representation of the frequencies \( \varepsilon_k \) for the superconducting atoms versus the evolution time \( t \). The size of time step is \( \Delta t = 1 \) ns. The generated pulses enable the design of a high-fidelity \( \mathcal{F}(\Theta, n; \varepsilon(t)) \geq 99.99\% \) Toffoli gate that operates in 23 ns. The solid dots show the control parameters used through the feasibility process to tune the shape of the pulses. The data used to create this figure are available.

![Figure 3](image-url)  
**Figure 3.** Piecewise-constant pulse representation of the frequencies \( \varepsilon_k \) for the superconducting atoms versus the evolution time \( t \). The size of time step is \( \Delta t = 1 \) ns. The generated pulses enable the design of a high-fidelity \( \mathcal{F}(\Theta, n; \varepsilon(t)) \geq 99.99\% \) CCCZ gate that operates in 70 ns. The solid dots show the control parameters used through the feasibility process to tune the shape of the pulses. The data used to create this figure are available.
any 1D or 2D architecture. Our approach can be used in any quantum-control approach when the underlying problem can be formulated as a feasibility problem. Here our approach assumes that Hamiltonian evolution provides a sound description of the quantum dynamics. Noise and parameter variability may be important in practice, however, in which case quantum master equations or other descriptions could be used if the noise is fully understood [9, 10]; otherwise statistical or black-box techniques such as reinforcement learning could be adopted [11].

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