ELIMINATE OBSTRUCTIONS: CURVES ON A 3-FOLD

SEN YANG

Abstract. By using higher K-theory, we reinterpret and generalize an idea on eliminating obstructions of deformation of cycles, which is known to Mark Green and Phillip Griffiths [3] and TingFai Ng [5].

As an application, we show how to eliminate obstructions of deformation of curves on a 3-fold in an explicit way. This answers affirmatively an open question by TingFai Ng [5].

Contents

1. Introduction: Ng’s question 1  
2. Reformulate Ng’s question 4  
3. Answer Ng’s question 6  
4. Acknowledgments 14  
References 14

1. Introduction: Ng’s question

Let $X$ be a smooth projective variety over a field $k$ of characteristic 0. For $Y \subset X$ a subscheme of codimension $p$, $Y$ can be considered as an element of the Hilbert scheme $\text{Hilb}^p(X)$ and the Zariski tangent space $T_Y \text{Hilb}(X)$ can be identified with $H^0(N_{Y/X})$. It is well-known that $\text{Hilb}^p(X)$ may be nonreduced at $Y$. In other words, let $Y'$ be a first order lifting of $Y$ in $X[\varepsilon]/(\varepsilon^2)$, the lifting of $Y'$ to $X[\varepsilon]/(\varepsilon^3)$ may be obstructed.

However, Green-Griffiths predicts that we can eliminate obstructions in their program [3], by considering $Y$ as a cycle. That is, instead of considering $Y$ as an element of $\text{Hilb}^p(X)$, considering $Y$ as an element of the cycles class group $Z^p(X)$ can eliminate obstructions. For $p = 1$, Green-Griffiths’ idea was realized by TingFai Ng in his Ph.D thesis [5]:

\textbf{Theorem 1.1.} [5] The divisor class group $Z^1(X)$ is smooth.

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In down-to-earth terms, Ng’s theorem says the following. For \( Y \in Z^1(X) \), let \( Y' \) be a first order lifting of \( Y \) in \( X[\varepsilon]/(\varepsilon^2) \), then we can lift \( Y' \) (as a cycle) to \( X[\varepsilon]/(\varepsilon^3) \) and then lift it to higher order successively. Ng’s method is to use Bloch’s semi-regularity map and we sketch it briefly as follows.

For \( Y \) a locally complete intersection, Bloch constructs semi-regularity map in [2],
\[
\pi : H^1(Y, N_{Y/X}) \to H^{p+1}(X, \Omega^{p-1}_{X/k}).
\]
In particular, for \( p = 1 \), this map \( \pi : H^1(Y, N_{Y/X}) \to H^2(O_X) \) agrees with the boundary map in the long exact sequence
\[
\cdots \to H^1(O_X(Y)) \to H^1(Y, N_{Y/X}) \to H^2(O_X) \to \cdots,
\]
associated to the short exact sequence:
\[
0 \to O_X \to O_X(Y) \to N_{Y/X} \to 0.
\]
Let \( W \) be an ample divisor such that \( H^1(O_X(Y + W)) = 0 \). Since the subscheme \( Y \cup W \) is still a locally complete intersection, we have the semi-regularity map \( \pi : H^1(Y \cup W, N_{Y\cup W/X}) \to H^2(O_X) \), which agrees with the boundary map in the long exact sequence
\[
\cdots \to H^1(O_X(Y + W)) \to H^1(Y \cup W, N_{Y\cup W/X}) \to H^2(O_X) \to \cdots.
\]

Let \( (Y\cup W)' \) denote the first order lifting of \( Y \cup W \). Since \( H^1(O_X(Y + W)) = 0 \), the kernel of \( \pi \) is 0. In other words, \( Y \cup W \) is semi-regularity in \( X \), so according to Theorem 7.3 of [2], the Hilbert scheme \( \text{Hilb}(X) \) is smooth at the point corresponding to \( Y \cup W \). By infinitesimal lifting property, we can lift \( (Y \cup W)' \) to a second order lifting in \( X[\varepsilon]/(\varepsilon^3) \).

As a cycle, \( Y \) can be formally written as
\[
Y = (Y + W) - W
\]
and then we may write \( Y' \), a first order lifting of \( Y \), as
\[
Y' = (Y + W)' - W',
\]
where \( W' \) is a first order lifting of \( W \) in \( X[\varepsilon]/(\varepsilon^2) \). For our purpose, in order to avoid bringing new obstructions, we fix \( W \), that is, we take \( W' = W \subset X[\varepsilon]/(\varepsilon^2) \) and \( W' \) can be lifted to \( W'' = W \subset X[\varepsilon]/(\varepsilon^3) \).

Idea: \( W \) helps to eliminate obstructions, without introducing new obstructions.
For related discussions and intuitive pictures, see Green-Griffiths [3](page 188-189). In Section 1.5 of [5], TingFai Ng asks whether we can extend the above method beyond divisor case, e.g., curves on a 3-fold $X$.

Suppose $C' \subset X[\varepsilon]/(\varepsilon^2)$ is a first order lifting of a curve $C$ on a 3-fold $X$. Such a lifting $C'$ corresponds to a vector field in $H^0(C, N)$. While the lifting of $C'$ to $X[\varepsilon]/(\varepsilon^3)$ may be obstructed (as a subscheme), one can ask whether a lift of $C'$ as a cycle always exists:

**Question 1.2.** [5] Given a smooth closed curve $C$ in a 3-fold $X$ and a normal vector field $v$, we wish to know whether it is always possible to find a nodal curve $\tilde{C}$ in $X$, of which $C$ is a component, (i.e. $\tilde{C} = C \cup D$ for some residue curve $D$) and a normal vector field $\tilde{v}$ on $\tilde{C}$ such that

1. $\tilde{v}|_C = v$,
2. the first order deformation given by $(\tilde{C}, \tilde{v})$ extends to second order, and
3. the first order deformation given by $(D, v')$ extends to second order.

**Cycle-theoretically,** we have $(C, v) = (\tilde{C}, \tilde{v})-(D, v')$. So we are asking whether $(C, v)$ as a first order deformation of cycles always extends to second order.

The key to answer Ng’s question is to interpret it, especially the word **Cycle-theoretically,** in an appropriate way. For this purpose, we reformulate Ng’s question in the framework of [8] and answer it affirmatively.

**Remark 1.3.** A similar question on eliminating obstructions for $X$ of any dimension and $Y$ of any codimension has been asked by Mark Green and Phillip Griffiths [3], which has been reformulated and has been answered affirmatively in [8].

We remark that Ng’s question above and its reformulation in Question 2.4 below, is different from Green – Griffiths’ question on obstruction issues reformulated in [8]. That’s mainly because we don’t know whether the map $\mu$ in Definition 2.2 is surjective or not. The author learned this subtlety from Spencer Bloch.

**Notations and conventions.**

1. K-theory used in this note will be Thomason-Trobaugh non-connective K-theory, if not stated otherwise. For any abelian group $M$, $M_{\mathbb{Q}}$ denotes the image of $M$ in $M \otimes_{\mathbb{Z}} \mathbb{Q}$. $(a, b)^T$ denotes the transpose of $(a, b)$.

1We will remove this hypothesis later in Question 2.4
(2). $X_j$ denotes the $j$-th trivial infinitesimal deformation of $X$, i.e., $X_j = X \times_k \text{Spec}(k[\varepsilon]/\varepsilon^{j+1})$. In particular, $X_0 = X$, $X_1 = X[\varepsilon]/(\varepsilon^2)$, and $X_2 = X[\varepsilon]/(\varepsilon^3)$.

To fix notations, $D_{\text{perf}}^{(i)}(X_j)$ denotes the derived category obtained from the exact category of perfect complex on $X_j$ and $L_{(i)}(X_j)$ is defined to be

$$L_{(i)}(X_j) := \{ E \in D_{\text{perf}}^{(i)}(X_j) \mid \text{codim}_{\text{Krull}}(\text{supp}(E)) \geq -i \}.$$

Let $(L_{(i)}(X_j)/L_{(i-1)}(X_j))^\#$ denote the idempotent completion of the Verdier quotient $L_{(i)}(X_j)/L_{(i-1)}(X_j)$.

**Theorem 1.4.** If For each $i \in \mathbb{Z}$, localization induces an equivalence

$$(L_{(i)}(X_j)/L_{(i-1)}(X_j))^\# \simeq \bigsqcup_{x \in X_j^{(-i)}} D_{x_j}^{\text{perf}}(X_j)$$

between the idempotent completion of the quotient $L_{(i)}(X_j)/L_{(i-1)}(X_j)$ and the coproduct over $x \in X_j^{(-i)}$ of the derived category of perfect complexes of $O_{X_j,x_j}$-modules with homology supported on the closed point $x \in \text{Spec}(O_{X_j,x_j})$. Consequently, localization induces an isomorphism

$$K_0((L_{(i)}(X_j)/L_{(i-1)}(X_j))^\#) \simeq \bigoplus_{x \in X_j^{(-i)}} K_0(O_{X_j,x_j} \text{ on } x_j).$$

2. **Reformulate Ng’s question**

Let $X$ be a nonsingular projective 3-fold over a field $k$ of characteristic 0 and let $Y \subset X$ be a curve (not necessarily locally complete intersection or smooth) with generic point $y$. For a point $x \in Y \subset X$ with local uniformizer $f, g, h$, the local ring $O_{X,x}$ is a regular local ring of dimension 3 and the maximal ideal $m_{X,x}$ is generated by a regular sequence $f, g, h$. To fix notations, we further assume $Y$ is generically defined by $f$ and $g$. So the local ring $O_{X,y} = (O_{X,x})(f,g)$ is a regular local ring of dimension 2 and the maximal ideal $m_{X,y}$ is generated by the regular sequence $f, g$. Since $h \notin (f, g)$, $h^{-1}$ exists in $O_{X,y}$.

Let $\varepsilon$ be a nilpotent satisfying $\varepsilon^2 = 0$ in this section. Let $Y' \subset X[\varepsilon]$ be a first order infinitesimal deformation of $Y$, that is, $Y'$ is flat over $\text{Spec}(k[\varepsilon])$ and $Y' \otimes_{k[\varepsilon]} k \cong Y$. While $Y'$ is not necessary to be a trivial deformation, it is generically trivial. Let $\mathcal{I}_{Y'}$ be the ideal sheaf of $Y'$, the localization at the generic point $(\mathcal{I}_{Y'})_y = (f + \varepsilon f_1, g + \varepsilon g_1)$, where $f_1, g_1 \in O_{X,y}$. See [9] for related discussions if necessary.
We keep the notations in [9] and use \( F_\bullet(f + \varepsilon f_1, g + \varepsilon g_1) \) to denote the Koszul complex associated to the regular sequence \( f + \varepsilon f_1, g + \varepsilon g_1 \), which is a resolution of \( O_{X,y}[\varepsilon]/(f + \varepsilon f_1, g + \varepsilon g_1) \):

\[
0 \rightarrow O_{X,y}[\varepsilon] \xrightarrow{(g+\varepsilon g_1,-f-\varepsilon f_1)^T} O_{X,y}[\varepsilon]^{\oplus 2} \xrightarrow{(f+\varepsilon f_1,g+\varepsilon g_1)} O_{X,y}[\varepsilon].
\]

Recall that Milnor K-group with support is rationally defined as certain eigenspaces of K-groups in [7],

\[
K_0^M(O_{X,y} \text{ on } y) := K_0(2)(O_{X,y} \text{ on } y)_\mathbb{Q},
\]

\[
K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) := K_0(2)(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_\mathbb{Q}.
\]

**Lemma 2.1.**

\[
K_0^M(O_{X,y} \text{ on } y) = K_0(O_{X,y} \text{ on } y)_\mathbb{Q},
\]

\[
K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) = K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_\mathbb{Q}.
\]

**Proof.** According to Riemann-Roch without denominator [6],

\[
K_0(2)(O_{X,y} \text{ on } y)_\mathbb{Q} \cong K_0(0)(k(y))_\mathbb{Q},
\]

where \( k(y) \) is the residue field. This forces \( K_0(j)(O_{X,y} \text{ on } y)_\mathbb{Q} = 0 \), except for \( j = 2 \). So we have

\[
K_0^M(O_{X,y} \text{ on } y) = K_0(O_{X,y} \text{ on } y)_\mathbb{Q}.
\]

Since \( K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_\mathbb{Q} = K_0(O_{X,y} \text{ on } y)_\mathbb{Q} \oplus H^2_y(\Omega^1_{X/\mathbb{Q}}) \), one can check that \( K_0(j)(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_\mathbb{Q} = 0 \), except for \( j = 2 \). That is,

\[
K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) = K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_\mathbb{Q}.
\]

\( \square \)

**Definition 2.2.** [9] We define a map \( \mu : H^0(Y,N_{Y/X}) \rightarrow K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_\mathbb{Q} \) as follows:

\[
\mu : H^0(Y,N_{Y/X}) \rightarrow K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_\mathbb{Q},
\]

\[
Y' \rightarrow F_\bullet(f + \varepsilon f_1, g + \varepsilon g_1).
\]

Now, we recall Milnor K-theoretic cycles:

**Definition 2.3.** [7, 8] Let \( X \) be a nonsingular projective 3-fold over a field \( k \) of characteristic 0, for \( i = 0,1,2 \), the 2nd Milnor K-theoretic cycles on \( X_i \) are defined as follows:

\[
Z_2^M(D_{\text{Perf}}(X_i)) = \text{Ker}(d^2_{1,X_i}),
\]

where \( d^2_{1,X_i} \) are the differentials in Theorem 3.1 and Theorem 3.3 below.
Now, we are ready to rewrite Ng’s question as follows:

**Question 2.4.** Let $X$ be a nonsingular projective 3-fold over a field $k$ of characteristic 0. Given a curve $Y$ in $X$ and a first order infinitesimal deformation $Y'$ of $Y$, let $\mu(Y') \in K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])$ denote the image of $Y'$ under the map $\mu$ in Definition 2.2. Is it always possible to find an element $\gamma \in Z_2^M(D_{\text{Perf}}(X))$ such that $\gamma = \mu(Y) + \mu(Z)$ for some curve $Z \subset X$ and a first order deformation $\gamma'$ of $\gamma$, in the sense of 8 (Definition 2.10), such that

1. $\gamma' = \mu(Y') + \mu(Z') \in Z_2^M(D_{\text{Perf}}(X_1))$, with $Z'$ a first order infinitesimal deformation of $Z$,
2. the first order deformation $\gamma'$ extends to second order $\gamma'' \in Z_2^M(D_{\text{Perf}}(X_2))$, and
3. the first order deformation given by $\mu(Z')$ extends to second order.

**Cycle-theoretically**, we have $\mu(Y') = \gamma' - \mu(Z')$. So we are asking whether $\mu(Y')$ as a first order deformation of cycles always extends to second order.

**Definition 2.5.** The ideal $(h, g) \subset O_{X,x}$ defines another curve $Z$ on $X$:

$$Z := \text{Spec}(O_{X,x}/(h, g)).$$

We fix the notation $Z$ and use $z$ to denote the generic point of $Z$. Then $O_{X,z} = (O_{X,x})_{(h,g)}$. Since $f \notin (h, g)$, $f^{-1}$ exists in $O_{X,z}$.

### 3. Answer Ng’s Question

**Theorem 3.1.** For $X$ a nonsingular projective 3-fold over a field $k$ of characteristic 0 and for $q = 2$ and $j = 1$ in Theorem 3.14 of 7, we have the following commutative diagram (For $w_i = x_i, y_i, z_i$ in the diagram below with $i = (1, 2)$ and 1, we have $K^M_*(O_{X_0,w_1} \text{ on } w_1) \cong K_*(O_{X_0,w_0} \text{ on } w_0)\mathbb{Q}$. This may be explained (in a similar way) by Lemma 2.11. We omit the subscript $\mathbb{Q}$ in the diagram below):

---

1. $\mu(Y)$ and $\mu(Z)$ are defined in Definition 2.2. E.g., to define $\mu(Y)$, we take $f_1 = g_1 = 0$.
2. By convention, $w_0 = w$ and $X_0 = X$. 

ELIMINATE OBSTRUCTIONS: CURVES ON A 3-FOLD

$$0 \xrightarrow{} \Omega^1_{k(X)/Q} \xleftarrow{\text{Chern}} K^2_M(k(X)[\varepsilon]) \xrightarrow{\varepsilon=0} K^2_M(k(X))$$

$$\bigoplus_{z \in X(1)} H^1_z(\Omega^1_{X/Q}) \xleftarrow{\text{Chern}} \bigoplus_{z[\varepsilon] \in X'[z(1)]} K^1(O_{X,z}[\varepsilon] \text{ on } z[\varepsilon]) \xrightarrow{\varepsilon=0} \bigoplus_{z \in X(1)} K^1(O_{X,z} \text{ on } z)$$

$$\bigoplus_{y \in X(2)} H^2_y(\Omega^1_{X/Q}) \xleftarrow{\text{Chern}} \bigoplus_{y[\varepsilon] \in X'[y(2)]} K^0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \xrightarrow{\varepsilon=0} \bigoplus_{y \in X(2)} K^0(O_{X,y} \text{ on } y)$$

$$\bigoplus_{x \in X(3)} H^3_x(\Omega^1_{X/Q}) \xleftarrow{\text{Chern}} \bigoplus_{x[\varepsilon] \in X'[x(3)]} K^{-1}(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]) \xrightarrow{\varepsilon=0} \bigoplus_{x \in X(2)} K^{-1}(O_{X,x} \text{ on } x)$$

$$0 \xrightarrow{} 0 \xrightarrow{} 0 \xrightarrow{} 0.$$
Since $h^{-1}$ exists in $(O_{X,x})_{(f,g)}$ and $f^{-1}$ exists in $(O_{X,x})_{(g,h)}$, the above two complexes are quasi-isomorphic to the following complexes respectively, still called $L_1^\bullet$ and $L_2^\bullet$ by abuse of notations,

$$L_1^\bullet: 0 \to (O_{X,x})_{(f,g)}[\varepsilon] \xrightarrow{(g, -(f+\varepsilon)h)} (O_{X,x})_{(f,g)}[\varepsilon]\oplus (O_{X,x})_{(f,g)}[\varepsilon],$$

and

$$L_2^\bullet: 0 \to (O_{X,x})_{(h,g)}[\varepsilon] \xrightarrow{(g, -(h+\varepsilon)f)} (O_{X,x})_{(h,g)}[\varepsilon]\oplus (O_{X,x})_{(h,g)}[\varepsilon].$$

Noting $O_{X,y} = (O_{X,x})_{(f,g)}$, we have $L_1^\bullet \in K_0(O_{X,y}[[\varepsilon]]$ on $y[[\varepsilon]]$). Similarly, $L_2^\bullet \in K_0(O_{X,z}[[\varepsilon]]$ on $z[[\varepsilon]]$. Mimicking the algorithm by Green-Griffiths [3], page 131), the image of $L_1^\bullet$ under the Chern map

$$\text{Chern} : \bigoplus_{y[[\varepsilon]] \in X[[\varepsilon]]} K_0(O_{X,y}[[\varepsilon]] \to \bigoplus_{y \in X} H^2_y(\Omega^1_{X/y}/Q),$$

may be described as follows. The following diagram, associated to $L_1^\bullet$,

$$\begin{align*}
(O_{X,x})_{(f,g)} \xrightarrow{(g, -(f+\varepsilon)h)} (O_{X,x})_{(f,g)} \oplus (O_{X,x})_{(f,g)} &\to (O_{X,x})_{(f,g)}/(f, g) \twoheadrightarrow 0 \\
(O_{X,x})_{(f,g)} \xrightarrow{\varepsilon h} \Omega^1_{(O_{X,x})_{(f,g)}}/Q,
\end{align*}$$

gives an element $\alpha$ in $\text{Ext}^2_{(O_{X,x})_{(f,g)}}((O_{X,x})_{(f,g)}/(f, g), \Omega^1_{(O_{X,x})_{(f,g)}}/Q)$. Noting that

$$H^2_y(\Omega^1_{(O_{X,x})_{(f,g)}}/Q) = \lim_{n \to \infty} \text{Ext}^2_{(O_{X,x})_{(f,g)}}((O_{X,x})_{(f,g)}/(f, g)^n, \Omega^1_{(O_{X,x})_{(f,g)}}/Q),$$

the image $[\alpha]$ of $\alpha$ under the limit is in $H^2_y(\Omega^1_{(O_{X,x})_{(f,g)}}/Q)$ and it is the image of $L_1^\bullet$ under the Chern map.

Similarly, the following diagram, associated to $L_2^\bullet$,

$$\begin{align*}
(O_{X,x})_{(h,g)} \xrightarrow{(g, -(h+\varepsilon)f)} (O_{X,x})_{(h,g)} \oplus (O_{X,x})_{(h,g)} &\to (O_{X,x})_{(h,g)}/(h, g) \twoheadrightarrow 0 \\
(O_{X,x})_{(h,g)} \xrightarrow{\varepsilon f} \Omega^1_{(O_{X,x})_{(h,g)}}/Q,
\end{align*}$$

gives an element $\beta$ in $\text{Ext}^2_{(O_{X,x})_{(h,g)}}((O_{X,x})_{(h,g)}/(h, g), \Omega^1_{(O_{X,x})_{(h,g)}}/Q)$. Noting that

$$H^2_z(\Omega^1_{(O_{X,x})_{(h,g)}}/Q) = \lim_{n \to \infty} \text{Ext}^2_{(O_{X,x})_{(h,g)}}((O_{X,x})_{(h,g)}/(h, g)^n, \Omega^1_{(O_{X,x})_{(h,g)}}/Q),$$
the image \([\beta]\) of \(\beta\) under the limit is in \(H^2_x(\Omega^1_{(O_{X,x}(f,g)/Q)})\) and it is the image of \(L^*_2\) under the Chern map.

Mimicking Green-Griffiths [3], page 103, \(\partial^2_{1,-2}\) maps \(\alpha\) in \(H^3_x(\Omega^1_{X/Q})\) to:

\[
\begin{aligned}
&O_{X,x} \xrightarrow{M_1} O_{X,x}^\oplus_3 \xrightarrow{M_2} O_{X,x}^\oplus_3 \xrightarrow{M_3} O_{X,x} \xrightarrow{\partial^2_{1,-2}} O_{X,x}/(f,g,h) \xrightarrow{0} \\
&O_{X,x} \xrightarrow{w_1dg} \Omega^1_{O_{X,x}/Q},
\end{aligned}
\]

where \(M_1, M_2\) and \(M_3\) are matrices associated to the Koszul resolution of \(O_{X,x}/(f,g,h)\):

\[
M_1 = \begin{pmatrix} f & 0 & -h \\ -g & -h & 0 \\ h & 0 & f \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & -h & -g \\ -h & 0 & f \\ g & f & 0 \end{pmatrix}, \quad M_3 = (f,g,h).
\]

Similarly, \(\partial^2_{1,-2}\) maps \(\beta\) in \(H^3_x(\Omega^1_{X/Q})\) to:

\[
\begin{aligned}
&O_{X,x} \xrightarrow{N_1} O_{X,x}^\oplus_3 \xrightarrow{N_2} O_{X,x}^\oplus_3 \xrightarrow{N_3} O_{X,x} \xrightarrow{\partial^2_{1,-2}} O_{X,x}/(h,g,f) \xrightarrow{0} \\
&O_{X,x} \xrightarrow{w_1dg} \Omega^1_{O_{X,x}/Q},
\end{aligned}
\]

where \(N_1, N_2\) and \(N_3\) are matrices associated to the Koszul resolution of \(O_{X,x}/(h,g,f)\):

\[
N_1 = \begin{pmatrix} h & 0 & -f \\ -g & 0 & h \\ f & h & 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & -f & -g \\ -f & 0 & h \\ g & h & 0 \end{pmatrix}, \quad N_3 = (h,g,f).
\]

Mimicking the argument by Green-Griffiths [3], page 103, and noting the commutative diagram below:

\[
\begin{aligned}
O_{X,x} & \xrightarrow{M_1} O_{X,x}^\oplus_3 & \xrightarrow{M_2} O_{X,x}^\oplus_3 & \xrightarrow{M_3} O_{X,x} & \xrightarrow{\partial^2_{1,-2}} O_{X,x}/(f,g,h) & \xrightarrow{0} \\
& \xrightarrow{-1} & \xrightarrow{w_1} & \xrightarrow{w_2} & \xrightarrow{1} & \xrightarrow{\equiv} \\
O_{X,x} & \xrightarrow{N_1} O_{X,x}^\oplus_3 & \xrightarrow{N_2} O_{X,x}^\oplus_3 & \xrightarrow{N_3} O_{X,x} & \xrightarrow{\partial^2_{1,-2}} O_{X,x}/(h,g,f) & \xrightarrow{0},
\end{aligned}
\]

where \(W_1\) and \(W_2\) stand for the following matrices:

\[
W_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},
\]

one can see that \(\partial^2_{1,-2}(\alpha)\) and \(\partial^2_{1,-2}(\beta)\) are negative of each other in \(\text{Ext}^3_{O_{X,x}}(O_{X,x}/(f,g,h), \Omega^1_{O_{X,x}/Q})\). Hence, \(\partial^2_{1,-2}(\alpha + \beta)\) is 0 in \(H^3_x(\Omega^1_{X/Q})\). Therefore, \(\partial^2_{1,-2}(L^*_2) = 0\) because of the commutative diagram:
Theorem 3.3. For $X$ a nonsingular projective 3-fold over a field $k$ of characteristic 0 and for $j = 2$ and $q = 2$ in Theorem 3.14 in [7], we have the following commutative diagram (For $w_i = x_i, y_i, z_i$ below with $i = 1$ and 2, $K_*^M(O_{X_i}, w_i)$ on $w_i) \cong K_*^M(O_{X}, w_i)$. This may be explained (in a similar way) by Lemma 2.1. We omit the subscript $\mathcal{Q}$ in the diagram below), the left arrows are induced by Chern character from $K$-theory to negative cyclic homology:

For $X$ a nonsingular projective 3-fold over a field $k$ of characteristic 0 and for $j = 2$ and $q = 2$ in Theorem 3.14 in [7], we have the following commutative diagram (For $w_i = x_i, y_i, z_i$ below with $i = 1$ and 2, $K_*^M(O_{X_i}, w_i)$ on $w_i) \cong K_*^M(O_{X}, w_i)$. This may be explained (in a similar way) by Lemma 2.1. We omit the subscript $\mathcal{Q}$ in the diagram below), the left arrows are induced by Chern character from $K$-theory to negative cyclic homology:

Now, we consider $O_{X,x}[\varepsilon]/(fh + \varepsilon w_1 + \varepsilon^2 w_2, g)$, where $w_1, w_2$ are arbitrary elements of $O_{X,x}$ and $\varepsilon^3 = 0$. The Koszul resolution of $O_{X,x}[\varepsilon]/(fh + \varepsilon w_1 + \varepsilon^2 w_2, g)$,

$L'^* : 0 \to O_{X,x}[\varepsilon] \xrightarrow{(g, -(fh + \varepsilon w_1 + \varepsilon^2 w_2))^T} O_{X,x}[\varepsilon] \xrightarrow{(fh + \varepsilon w_1 + \varepsilon^2 w_2, g)} O_{X,x}[\varepsilon]$, 

defines an element of $K_0(\mathcal{L}_2(X[\varepsilon]/\mathcal{L}_3(X[\varepsilon])/\varepsilon))) \cong \bigoplus_{y\varepsilon \in X[\varepsilon][2]} K_0(O_{X,y}[\varepsilon]/y[\varepsilon])$, with $\varepsilon^3 = 0$. 

\[ \bigoplus_{y \in X^{(2)}} H^2_{\text{Chern}}(\Omega^{(2)}_{X/Q}) \xrightarrow{\partial^2_{1-2}} \bigoplus_{x \in X^{(3)}} H^2_{\text{Chern}}(\Omega^{(3)}_{X/Q}) \cong K_0(O_{X,y}[\varepsilon]/y[\varepsilon]) \xrightarrow{\partial^2_{1-2}} K_0(O_{X,x}[\varepsilon]/x[\varepsilon]). \]
Under the isomorphism in Theorem 1.4 for $j = 2$ and $i = -2$, 
\[ K_0((L_{(-2)}(X[\varepsilon])/L_{(-3)}(X[\varepsilon]))^\#) \simeq \bigoplus_{y[\varepsilon] \in X[\varepsilon]^{(2)}} K_0(\mathcal{O}_{X,y}[\varepsilon] \text{ on } y[\varepsilon]), \text{ with } \varepsilon^3 = 0, \]

$L''^\bullet$ decomposes into the direct sum of
\[
L''_1^\bullet : 0 \to (\mathcal{O}_{X,x}(f,g)[\varepsilon]) \xrightarrow{\left( \frac{g}{h} \right)^{\left( \frac{h}{f} + \varepsilon^2 \frac{w_2}{h} \right)}} (\mathcal{O}_{X,x}(f,g)[\varepsilon]^{\oplus 2}) \xrightarrow{\left( \frac{f + \varepsilon \frac{w_1}{h} + \frac{w_2}{h}}{g} \right)} (\mathcal{O}_{X,x}(f,g)[\varepsilon]),
\]
and
\[
L''_2^\bullet : 0 \to (\mathcal{O}_{X,x}(h,g)[\varepsilon]) \xrightarrow{\left( \frac{g}{h} \right)^{\left( \frac{h}{f} + \varepsilon^2 \frac{w_2}{h} \right)}} (\mathcal{O}_{X,x}(h,g)[\varepsilon]^{\oplus 2}) \xrightarrow{\left( \frac{f + \varepsilon \frac{w_1}{h} + \frac{w_2}{h}}{g} \right)} (\mathcal{O}_{X,x}(h,g)[\varepsilon]).
\]

Since $h^{-1}$ exists in $(\mathcal{O}_{X,x}(f,g))$, $L''_1^\bullet$ is quasi-isomorphic to the following complex, still called $L''_1^\bullet$ by abuse of notations,
\[
L''_1^\bullet : 0 \to (\mathcal{O}_{X,x}(f,g)[\varepsilon]) \xrightarrow{\left( \frac{g}{h} \right)^{\left( \frac{h}{f} + \varepsilon^2 \frac{w_2}{h} \right)}} (\mathcal{O}_{X,x}(f,g)[\varepsilon]^{\oplus 2}) \xrightarrow{\left( \frac{f + \varepsilon \frac{w_1}{h} + \frac{w_2}{h}}{g} \right)} (\mathcal{O}_{X,x}(f,g)[\varepsilon]).
\]

Noting $\mathcal{O}_{X,y} = (\mathcal{O}_{X,x}(f,g))$, we have $L''_1^\bullet \in K_0(\mathcal{O}_{X,y}[\varepsilon] \text{ on } y[\varepsilon])$. The image of $L''_1^\bullet$ under the Chern map
\[
\text{Chern} : \bigoplus_{y_2 \in X^{(2)}} K_0(\mathcal{O}_{X,y_2} \text{ on } y_2) \to \bigoplus_{y \in X^{(2)}} H^2_y((\Omega^1_{X/Q})^{\oplus 2})
\]
may be described similarly as the Chern map in Theorem 3.2.

The following diagram, associated to $L''_1^\bullet$,
\[
\begin{align*}
(\mathcal{O}_{X,x}(f,g)) & \xrightarrow{(g,f)^T} (\mathcal{O}_{X,x}(f,g)^{\oplus 2}) \xrightarrow{(f,g)} (\mathcal{O}_{X,x}(f,g)) \to (\mathcal{O}_{X,x}(f,g)/(f,g)) \to 0, \\
(\mathcal{O}_{X,x}(f,g)) & \xrightarrow{-w_1 dq + w_2 dq} (\Omega^1_{(\mathcal{O}_{X,x}(f,g)/Q)^{\oplus 2}},
\end{align*}
\]
gives an element $\alpha$ in $\text{Ext}^2((\mathcal{O}_{X,x}(f,g)/(f,g)), (\Omega^1_{(\mathcal{O}_{X,x}(f,g)/Q)^{\oplus 2}})$. Noting that
\[
H^2_y((\Omega^1_{(\mathcal{O}_{X,x}(f,g)/Q)^{\oplus 2}}) = \lim_{n \to \infty} \text{Ext}^2((\mathcal{O}_{X,x}(f,g)/(f,g))^n, (\Omega^1_{(\mathcal{O}_{X,x}(f,g)/Q)^{\oplus 2}}),
\]
the image $[\alpha]$ of $\alpha$ under the limit is in $H^2_y((\Omega^1_{(\mathcal{O}_{X,x}(f,g)/Q)^{\oplus 2}})$ and it is the image of $L''_1^\bullet$ in $H^2_y((\Omega^1_{(\mathcal{O}_{X,x}(f,g)/Q)^{\oplus 2}})$. We have the similar description of $L''_2^\bullet$. By repeating the argument in Theorem 3.2, we can show

**Theorem 3.4.** $L'' \in Z^M_2(D\text{Perf}(X_2))$, i.e., $L'' \in \text{Ker}(d^M_{2,1,X_2})$. 
Now, by taking \( w_1 = f_1 h \), the Koszul complex \( L^\bullet \) in Theorem 3.2 is of the form

\[
L^\bullet : 0 \to O_{X,x}[\varepsilon] \xrightarrow{(g, -(f h + \varepsilon f_1 h)^T)} O_{X,x}[\varepsilon]^\oplus 2 \xrightarrow{(f h + \varepsilon f_1 h, g)} O_{X,x}[\varepsilon].
\]

Under the isomorphism in Theorem 1.4, the complex \( L^\bullet \) decomposes into the direct sum of \( L_1^\bullet \) and \( L_2^\bullet \),

\[
L^\bullet = L_1^\bullet + L_2^\bullet,
\]

where \( L_1^\bullet \) and \( L_2^\bullet \) are of the forms

\[
L_1^\bullet : 0 \to (O_{X,x}(f,g)[\varepsilon] \xrightarrow{(g, -(f + \varepsilon f_1)^T)} (O_{X,x}(f,g)[\varepsilon]^\oplus 2 \xrightarrow{(f + \varepsilon f_1, g)} (O_{X,x}(f,g)[\varepsilon],
\]

and

\[
L_2^\bullet : 0 \to (O_{X,x}(h,g)[\varepsilon] \xrightarrow{(h, -(f h + \varepsilon f_1 h)^T)} (O_{X,x}(h,g)[\varepsilon]^\oplus 2 \xrightarrow{(h + \varepsilon f_1 h, g)} (O_{X,x}(h,g)[\varepsilon].
\]

Now, by taking \( w_1 = f_1 h \) and \( w_2 = f_2 h \), where \( f_2 \) is an arbitrary element of \( O_{X,x} \), the Koszul complex \( L''^\bullet \) in Theorem 3.4 is of the form

\[
L''^\bullet : 0 \to O_{X,x}[\varepsilon] \xrightarrow{(g, -(f h + \varepsilon f_1 h)^T)} O_{X,x}[\varepsilon]^\oplus 2 \xrightarrow{(f h + \varepsilon f_1 h, g)} O_{X,x}[\varepsilon].
\]

Under the isomorphism in Theorem 1.4, the complex \( L''^\bullet \) decomposes into the direct sum of \( L_1''^\bullet \) and \( L_2''^\bullet \),

\[
L''^\bullet = L_1''^\bullet + L_2''^\bullet,
\]

where \( L_1''^\bullet \) and \( L_2''^\bullet \) are of the forms

\[
L_1''^\bullet : 0 \to (O_{X,x}(f,g)[\varepsilon] \xrightarrow{(g, -(f + \varepsilon f_1 + f_2 h)^T)} (O_{X,x}(f,g)[\varepsilon]^\oplus 2 \xrightarrow{(f + \varepsilon f_1 + f_2 h, g)} (O_{X,x}(f,g)[\varepsilon),
\]

and

\[
L_2''^\bullet : 0 \to (O_{X,x}(h,g)[\varepsilon] \xrightarrow{(h, -(f h + \varepsilon f_1 h + f_2 h)^T)} (O_{X,x}(h,g)[\varepsilon]^\oplus 2 \xrightarrow{(h + \varepsilon f_1 h + f_2 h, g)} (O_{X,x}(h,g)[\varepsilon].
\]

**Lemma 3.5.** \( L_2^\bullet \in Z_2^M(D\text{Perf}(X_1)) \) and \( L_2''^\bullet \in Z_2^M(D\text{Perf}(X_2)) \)

**Proof.** The proof can be done by mimicking the argument in Theorem 3.2. We sketch it for readers’ convenience.
The following diagram, associated to $L_2^*$,
\begin{equation}
\begin{aligned}
\left\{\begin{array}{c}
(\Omega_{X,x}^1)_{(h,g)} \xrightarrow{(g-h)^T} (\Omega_{X,x}^2)_{(h,g)} \xrightarrow{(h,g)} (\Omega_{X,x})_{(h,g)} \xrightarrow{h} (\Omega_{X,x})_{(h,g)}/\langle h, g \rangle \xrightarrow{0}
\end{array}\right.
\end{aligned}
\end{equation}
gives an element $\beta$ in $\text{Ext}^2_{(\Omega_{X,x})_{(h,g)}}((\Omega_{X,x})_{(h,g)}/\langle h, g \rangle, \Omega_{X,x}^1_{(h,g)}/\langle h, g \rangle)$. Noting $h^\perp/\langle h, g \rangle = h_{(h,g)}$, $\beta \equiv 0 \in \text{Ext}^2_{(\Omega_{X,x})_{(h,g)}}((\Omega_{X,x})_{(h,g)}/\langle h, g \rangle, \Omega_{X,x}^1_{(h,g)}/\langle h, g \rangle)$. Hence the image $[\beta]$ of $\beta$ in $H^2_{\perp}(\Omega_{X,x}^1_{(h,g)}/\langle h, g \rangle)$ is 0. In other words, the image of $L_1^*$ is 0 under the Chern map. It is trivial that $\partial^3_1([\beta]) = 0$ in $H^2_{\perp}(\Omega_{X,x}^1_{/\langle h, g \rangle})$.

Therefore, $d_{1,X_1}^2(L_1^*) = 0$ because of the commutative diagram:
\begin{equation}
\begin{array}{ccc}
\bigoplus_{y \in X^{(2)}} \Omega_{X,y}^2 & \xleftarrow{\text{Chern}} & \bigoplus_{y \in X^{(2)}} K_0(\Omega_{X,y}^1) \\
\partial^2_1 & \downarrow & \partial^3_1 \downarrow \\
\bigoplus_{x \in X^{(3)}} \Omega_{X,x}^3 & \xleftarrow{\text{Chern}} & \bigoplus_{x \in X^{(3)}} K_1(\Omega_{X,x}^2) \\
\end{array}
\end{equation}

Similar argument works for $L_2^*$. \hfill \square

Let $L^*$ denote the following complex, which is the Koszul resolution of $O_{X,x}/(fh, g)$
\begin{equation}
L^* : 0 \rightarrow O_{X,x} \xrightarrow{(g-f)^T} O_{X,x}^2 \xrightarrow{(f-h)} O_{X,x}.
\end{equation}
Under the isomorphism in Theorem 1.4, $L^*$ decomposes into the direct sum of $L_1^*$ and $L_2^*$, which are (quasi-isomorphic to) of the forms
\begin{equation}
L_1^* : 0 \rightarrow (O_{X,x})_{(f,g)} \xrightarrow{(g-f)^T} (O_{X,x})^2_{(f,g)} \xrightarrow{(f-h)} (O_{X,x})_{(f,g)}
\end{equation}
and
\begin{equation}
L_2^* : 0 \rightarrow (O_{X,x})_{(g,h)} \xrightarrow{(g-h)^T} (O_{X,x})^2_{(g,h)} \xrightarrow{(h-g)} (O_{X,x})_{(g,h)}.
\end{equation}

**Theorem 3.6.** The answer to TingFai Ng’s Question 24 is positive:
We can take $\gamma = L^*(3.15) \in Z^3_{\perp}(D_{\text{Perf}}(X))$, it satisfies that $L^* = L_1^* + L_2^*$, where $L_1^* = \mu(Y)$ and $L_2^* = \mu(Z)$. The complex $L^*(3.6)$ is a first order deformation of $\gamma$, satisfying:
1. $L^* = L_1^* + L_2^*(3.7)$, where in fact $L_1^* = \mu(Y')$ and $L_2^* = \mu(Z')$, with $Z'$ a first order infinitesimal deformation of $Z$, which is generically
given by \((h + \varepsilon \frac{f_1 h}{f}, g) \subset O_{X,z}[\varepsilon] = (O_{X,z})(h,g)[\varepsilon]\), with \(\varepsilon^2 = 0\). (Please note \((h + \varepsilon \frac{f_1 h}{f}, g) \not\subset O_{X,z}[\varepsilon]\), since \(f_1 h / f \not\in O_{X,z}\)).

2. the first order deformation given by \(L'^{\bullet}(3.6)\) extends to second order \(L'^{\bullet}(3.10) \in Z^2_M(DPerf(X_2))\), and

3. the first order deformation given by \(\mu(Z')\) extends to second order \(L''^{\bullet}(3.13) \in Z^2_M(DPerf(X_2))\).

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Yau Mathematical Sciences Center, Tsinghua University, Beijing, China
E-mail address: syang@math.tsinghua.edu.cn; senyangmath@gmail.com