The properties of bordered matrix of symmetric block design *

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Abstract

Let \( X = \{p_1, p_2, \cdots, p_v\} \) be a \( v \)-set (a set of \( v \) elements), called points, and let \( B = \{B_1, B_2, \cdots, B_v\} \) be a finite collection of subsets of \( X \), called blocks. The pair \((X, B)\) is called a symmetric \((v, k, \lambda)\) design if the following conditions hold:

(i) Each \( B_i \) is a \( k \)-subset of \( X \).
(ii) Each \( B_i \cap B_j \) is a \( \lambda \)-subset of \( X \) for \( 1 \leq i \neq j \leq v \).
(iii) The integers \( v, k, \lambda \) satisfy \( 0 < \lambda < k < v - 1 \).

Problem 0.1 One of the major unsolved problems in combinatorics is the determination of the precise range of values of \( v, k, \) and \( \lambda \) for which a symmetric \((v, k, \lambda)\) design exists.

The symmetric \((n^2 + n + 1, n + 1, 1)\) block design is a projective plane of order \( n \). Projective planes of order \( n \) exist for all prime powers \( n \) but for no other \( n \) is a construction known. Thus Conjecture 0.2 is the famous long-standing conjecture of finite projective planes.

Conjecture 0.2 (O. Veblen and W.H. Bussey, 1906; R.H. Bruck and H.J. Ryser, 1949) If a finite projective plane of order \( n \) exists, then \( n \) is a power of some prime \( p \).

It was proved in 1989 by a computer search that there does not exist any projective plane of order 10 by Lam, C.W.H., Thiel, L. and Swiercz, S. Whether there exists any projective plane of order 12 is still open.

The author introduces the bordered matrix of a \((v, k, \lambda)\) symmetric design, which preserves some row inner product property, and gives some new necessary conditions for the existence of the symmetric \((v, k, \lambda)\) design. Thus Theorem 0.3 generalizes Schutzenberger’s theorem and the Bruck-Ryser-Chowla theorem on the existence of symmetric block designs. This bordered matrix has been a breakthrough idea since 1950.

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Theorem 0.3 Let \( C \) be a \( w \times w+d \) nonsquare rational matrix without any column of \( k \cdot 1^t_{w+d} \), where \( 1^t_{w+d} \) is the \( w+d \)-dimensional all 1 column vector and \( k \) is a rational number, \( \alpha, \beta \) be positive integers such that matrix \( \alpha I_w + \beta J_w \) is the positive definite matrix with plus \( d \) congruent factorization

\[
CC^t = \alpha I_w + \beta J_w.
\]

Then the following cases hold.

**Case 1** If \( w \equiv 0 \pmod{4} \), \( d = 1 \), then \( \beta \) is a perfect square.

**Case 2** If \( w \equiv 0 \pmod{4} \), \( d = 2 \), then \( \beta \) is a sum of two squares.

**Case 3** If \( w \equiv 2 \pmod{4} \), \( d = 1 \) and \( \alpha = a^2 + b^2 \), where \( a, b \) are integers, then \( \beta \) is a perfect square.

**Case 4** If \( w \equiv 2 \pmod{4} \), \( d = 2 \) and \( \alpha = a^2 + b^2 \), where \( a, b \) are integers, then \( \beta \) is a sum of two squares.

**Case 5** If \( w \equiv 1 \pmod{4} \), \( d = 1 \) and \( \alpha = a^2 + b^2 \), where \( a, b \) are integers, then \( \alpha^* = \beta^* \), where \( m^* \) denotes the square-free part of the integer \( m \).

**Case 6** If \( w \equiv 1 \pmod{4} \), \( d = 2 \) and \( \alpha = a^2 + b^2 \), where \( a, b \) are integers, then the equation

\[
\alpha z^2 = -x^2 + \beta y^2
\]

must have a solution in integers, \( x, y, z \), not all zero.

**Case 7** If \( w \equiv 3 \pmod{4} \), \( d = 1 \) \( \alpha^* = \beta^* \), where \( m^* \) denotes the square-free part of the integer \( m \).

**Case 8** If \( w \equiv 3 \pmod{4} \), \( d = 2 \) \( \alpha z^2 = -x^2 + \beta y^2 \)

must have a solution in integers, \( x, y, z \), not all zero.

As an application of the above theorem the following theorems are obtained.

**Theorem 0.4** Conjecture 0.2 holds if finite projective plane of order \( n \leq 33 \).

**Theorem 0.5** If \( (v, k, \lambda) \) are parameters \( (49, 16, 5), (154, 18, 2) \) and \( (115, 19, 3) \), then each symmetric \( (v, k, \lambda) \) design does not exist.

For Problem 0.1 or Conjecture 0.2 Lam’s algorithm is an exponential time algorithm. But the proof of main Theorem 0.3 is just the Ryser-Chowla elimination procedure. Thus author’s algorithm is a polynomial time algorithm. It fully reflects his algorithm high efficiency.

**Keywords** symmetric design; bordered matrix; finite projective plane; polynomial time algorithm; exponential time algorithm; the Ryser-Chowla elimination procedure.

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1 Introduction

An incidence structure consists simply of a set $X$ of points and a set $B$ of blocks, with a relation of incidence between points and blocks. Symmetric block designs have an enormous literature and discussions of their basic properties are readily available in [4, 14, 16].

Definition 1.1 Let $v, k$ and $\lambda$ be integers. Let $X = \{p_1, p_2, \cdots, p_v\}$ be a $v$-set (a set of $v$ elements), called points, and let $B = \{B_1, B_2, \cdots, B_v\}$ be a finite collection of subsets of $X$, called blocks. The pair $(X, B)$ is called a symmetric $(v, k, \lambda)$ design if the following conditions hold:

(i) Each $B_i$ is a $k$-subset of $X$.
(ii) Each $B_i \cap B_j$ is a $\lambda$-subset of $X$ for $1 \leq i \neq j \leq v$.
(iii) The integers $v, k, \lambda$ satisfy $0 < \lambda < k < v - 1$.

The set $\{v, k, \lambda\}$ is called the set of parameters of the symmetric $(X, B)$ design. We also use the notation $\mathcal{D} = (X, B)$.

Definition 1.2 Define a $v \times v$ 0-1 matrix

$$A = (a_{ij})_{1 \leq i \leq v, 1 \leq j \leq v},$$

whose rows are indexed by the points $p_1, p_2, \cdots, p_v$ and columns are indexed by the blocks $B_1, B_2, \cdots, B_v$, by

$$a_{ij} = \begin{cases} 1, & \text{if } p_i \in B_j, \\ 0, & \text{otherwise.} \end{cases}$$

Then $A$ is called the incidence matrix of the symmetric $(v, k, \lambda)$ design. We set $n = k - \lambda$ and call $n$ the order of the symmetric $(v, k, \lambda)$ design.

$A^t$ denotes the transpose of $A$. $J_v$ and $I_v$ are the $v \times v$ all 1's matrix and the identity matrix, respectively.

Let $A$ be a $v \times v$ 0-1 matrix. Then $A$ is the incidence matrix of a symmetric $(v, k, \lambda)$ design if and only if

$$AA^t = \lambda J_v + (k - \lambda)I_v.$$

In this paper we introduce the bordered matrix $C$ of the $(v, k, \lambda)$ symmetric design and prove some new necessary conditions for the existence of the symmetric design.

Then the bordered matrix $C$ of the $(v, k, \lambda)$ symmetric design is obtained from $A$ by adding many rows rational vectors and many columns rational vectors such that

$$CC^t = (\lambda + l)J_{v+s} + (k - \lambda)I_{v+s}$$
for some positive integers \( l, s \), where \( C \) is a nonsquare rational matrix and is of full row rank.

Let \( \alpha, \beta \) be positive integers. A matrix \( \alpha I_w + \beta J_w \) is called the positive definite matrix with plus \( d \) congruent factorization property if there exists a nonsquare rational \( w \) by \( w + d \) matrix \( C \) such that

\[
\alpha I_w + \beta J_w = C C^t,
\]

where \( d \) is the difference between the number of columns and the number of rows of \( C \).

In this paper we consider more the positive definite matrix \( \alpha I_w + \beta J_w \) with the above plus \( d \) congruent factorization property.

What happens for the positive definite matrix \( \alpha I_w + \beta J_w \) with plus \( d \) congruent factorization property if \( \alpha, \beta \) are two positive integers?

If \( d = 1 \) or \( 2 \) then we obtain the following main theorem 1. As an application of the main theorem 1 it is easy to determine that there does not exist finite projective plane of order \( n \) if \( n \) is each of the first open values 10, 12, 15, 18, 20, 24, 26 and 28, for which the Bruck-Ryser-Chowla Theorem can not be used. For large \( n \) the new method is also valid. Also some symmetric designs are excluded by the new method.

Structure of the paper: some elementary definitions and results are summarized and the main theorems are stated in §2 below. Some theorems from number theory are needed for our work in combinatorial analysis in §3. The proof of main theorem 1 will be given in §4. An application of main theorem 1 will be given in §5 and §6. Some concluding remarks will be given in §7.

2 Background and statement of the main results

Symmetric block designs have an enormous literature and discussions of their basic properties are readily available in [4, 14, 16].

**Notation 2.1**

\( \mathbb{Z} \) denotes the set of integers.

\( \mathbb{Q} \) denotes the field of rational numbers.

\( m^* \) denotes the square-free part of the integer \( m \).

Let \( A \) be a matrix. \( A^t \) denotes the transpose of \( A \).

\( J_v \) and \( I_v \) are the \( v \times v \) all 1’s matrix and the identity matrix, respectively.

\( 1_v \) is the \( v \)-dimensional all 1 row vector.

**Proposition 2.2** Let \( A \) be a \( v \times v \) 0-1 matrix. Then \( A \) is the incidence matrix of a symmetric \( (v, k, \lambda) \) design if and only if

\[
AA^t = A^t A = \lambda J_v + (k - \lambda)I_v.
\]
Proposition 2.3 In a symmetric \((v,k,\lambda)\) design, the integers \(v,k,\) and \(\lambda\) of the design must satisfy the following relations

\[\begin{align*}
(1) & \quad \lambda(v-1) = k(k-1), \\
(2) & \quad k^2 - v\lambda = k - \lambda, \text{ and} \\
(3) & \quad (v-k)\lambda = (k-1)(k-\lambda).
\end{align*}\]

Theorem 2.4 (Schützenberger) Suppose there exists a symmetric \((v,k,\lambda)\) design with an incidence matrix \(A\). If \(v\) is even, then \(k-\lambda\) must be a perfect square.

The Bruck-Ryser-Chowla Theorem gives a necessary condition for the existence of a symmetric design.

Theorem 2.5 (Bruck-Ryser-Chowla) Suppose there exists a symmetric \((v,k,\lambda)\) design. If \(v\) is odd, then the equation

\[x^2 = (k-\lambda)y^2 + (-1)^{(v-1)/2}\lambda z^2\]

must have a solution in integers, \(x,y,z\), not all zero.

Remark 2.6 If \(v\) is odd, then the equation

\[x^2 = (k-\lambda)y^2 + (-1)^{(v-1)/2}\lambda z^2\]

must have a solution in integers, \(x,y,z\), not all zero. We also say that it has a nontrivial integral solution.

Problem 2.7 One of the major unsolved problems in combinatorics is the determination of the precise range of values of \(v,k,\) and \(\lambda\) for which a symmetric \((v,k,\lambda)\) design exists.

Theorem 2.8 (Bruck-Ryser-Chowla) Let \(C\) be a square rational \(w \times w\) matrix, \(\alpha, \beta\) be positive integers such that

\[CC^t = \alpha I_w + \beta J_w.\]

If \(w\) is odd, then the equation

\[z^2 = \alpha x^2 + (-1)^{(w-1)/2}\beta y^2\]

must have a solution in integers, \(x,y,z\), not all zero.

Definition 2.9 Let \(n\) be a positive integer. A finite projective plane of order \(n\) is a symmetric \((n^2 + n, n + 1, 1)\) design. A block in a finite projective plane is called a line.
The theorem of Desargues is universally valid in a projective plane if and only if the plane can be constructed from a three-dimensional vector space over a field. These planes are called Desarguesian planes, named after Girard Desargues. The projective planes that can not be constructed in this manner are called non-Desarguesian planes, and the Moulton plane is an example of one. The $PG(2, K)$ notation is reserved for the Desarguesian planes, where $K$ is some field.

**Theorem 2.10** (O.Veblen and W.H.Bussey, 1906, see[22]) Let $q$ be a prime power. Then $PG(2, \mathbb{F}_q)$ is a finite projective plane of order of $q$.

From Theorem 2.5 we deduce

**Theorem 2.11** (Bruck-Ryser) Let $n$ be a positive integer and $n \equiv 1, 2 \ (mod\ 4)$ and let the squarefree part of $n$ contain at least one prime factor $p \equiv 3 \ (mod\ 4)$. Then there does not exist a finite projective plane of order $n$.

As an application, consider projective planes. Here $\lambda = 1$ and $v = n^2 + n + 1$ is odd. If $n \equiv 0$ or $3 \ (mod\ 4)$, the Bruck-Ryser-Chowla equation always has the solution $(0, 1, 1)$ and thus the theorem excludes no values of $n$. However, if $n \equiv 1$ or $2 \ (mod\ 4)$, the equation becomes $nx^2 = y^2 + z^2$, which has a nontrivial integral solution if and only if $n$ is the sum of two squares of integers. Projective planes of order 6, 14, 21, 22, 30 or 33 therefore cannot exist.

**Conjecture 2.12** (O.Veblen and W.H.Bussey, 1906; R.H.Bruck and H.J. Ryser, 1949)
If a finite projective plane of order $n$ exists, then $n$ is a power of some prime $p$.

Despite much research no one has uncovered any further necessary conditions for the existence of a symmetric $(v, k, \lambda)$ design apart from the equation $(v - 1)\lambda = k(k - 1)$, Schutzenberger’s Theorem and the Bruck-Ryser-Chowla Theorem. For no $(v, k, \lambda)$ satisfying these requirements has it been shown that a symmetric $(v, k, \lambda)$ design does not exist.

It is possible that these conditions are sufficient. As a matter of fact, this is true the seventeen admissible $(v, k, \lambda)$ with $v \leq 48$[14], Notes to Chapter 2); the first open case as of early 1982 is $(49, 16, 5)$.

Projective planes of order $n$ exist for all prime powers $n$ (aside from $PG(2, n)$ a host of other constructions are known ) but for no other $n$ is a construction known. The first open values are $n = 10, 12, 15, 18, 20, 24, 26$ and 28. It was proved by a computer search that there does not exist any projective plane of order 10, cf. Lam, C.W.H., Thiel, L. and Swiercz, S. [13]. Whether there exists any projective plane of order 12 is still open.
Now we introduce the bordered matrix of the $(v, k, \lambda)$ symmetric design in Definition 2.14 and prove some new necessary conditions for the existence of the symmetric design.

**Remark 2.13** The condition (1) in Proposition 2.2 for an incidence matrix $A$ of the symmetric $(v, k, \lambda)$ design is equivalent to the following two conditions

(i) the inner product of any two distinct rows of $A$ is equal to $\lambda$;

(ii) and the inner product of any rows with themselves of $A$ is equal to $k$.

**Definition 2.14** Let $A$ be an incidence matrix of the symmetric $(v, k, \lambda)$ design. Then the bordered matrix $C$ of $A$ for some positive integers $l$ and $d$ is obtained from $A$ by adding many rows rational vectors and many columns rational vectors such that

(i) the inner product of any two distinct rows of $C$ is equal to $\lambda + l$;

(ii) and the inner product of any rows with themselves of $C$ is equal to $k + l$;

where $C$ is a $w$ by $w + d$ nonsquare rational matrix and is of full row rank.

**Remark 2.15** It is difficult to construct a square bordered matrix of $A$. The author does this by computer computation in Maple. But it is easy to construct a nonsquare bordered matrix of $A$. If it exists then it is not unique for some positive integers $l$ and $d$. The author also does this by computer computation in Maple.

Theorem 2.8 gives a necessary condition for the existence of positive definite matrix $\alpha I_w + \beta J_w$, which is congruent to identity matrix over rational field for the positive integers $\alpha, \beta$.

**Definition 2.16** Let $\alpha, \beta$ be positive integers. A matrix $\alpha I_w + \beta J_w$ is called the positive definite matrix with plus $d$ congruent factorization property if there exists a nonsquare rational $w$ by $w + d$ matrix $C$ such that

$$\alpha I_w + \beta J_w = C C^t. \quad (3)$$

**Remark 2.17** The matrix equation (3) implies $C$ is always of rank $w$, i.e., of full row rank if $\alpha, \beta$ are two positive integers and $C$ is a $w$ by $w + d$ matrix over rational field $\mathbb{Q}$.

In this paper we consider more the positive definite matrix $\alpha I_w + \beta J_w$ with plus $d$ congruent factorization property.

**Problem 2.18** What happens for the positive definite matrix $\alpha I_w + \beta J_w$ with plus $d$ congruent factorization property if $\alpha, \beta$ are two positive integers?
The matrix equation (3) is of fundamental importance. But it is difficult to deal with this matrix equation in its full generality. Nevertheless, if $d = 1$ or 2 then we obtain the following main theorem. Thus Main Theorem 1 generalizes Schützenberger’s theorem and the Bruck-Ryser-Chowla theorem on the existence of symmetric block designs.

We are now prepared to state our main conclusions.

**Main Theorem 1** Let $C$ be a $w$ by $w + d$ nonsquare rational matrix without any column of $k \cdot 1_{w+d}^t$, where $1_{w+d}^t$ is the $w + d$-dimensional all 1 column vector and $k$ is a rational number, $\alpha, \beta$ be positive integers such that matrix $\alpha I_w + \beta J_w$ is the positive definite matrix with plus $d$ congruent factorization

$$C C^t = \alpha I_w + \beta J_w.$$ 

Then the following cases hold.

**Case 1** If $w \equiv 0 \pmod{4}$, $d = 1$, then $\beta$ is a perfect square.

**Case 2** If $w \equiv 0 \pmod{4}$, $d = 2$, then $\beta$ is a sum of two squares.

**Case 3** If $w \equiv 2 \pmod{4}$, $d = 1$ and $\alpha = a^2 + b^2$, where $a, b$ are integers, then $\beta$ is a perfect square.

**Case 4** If $w \equiv 2 \pmod{4}$, $d = 2$ and $\alpha = a^2 + b^2$, where $a, b$ are integers, then $\beta$ is a sum of two squares.

**Case 5** If $w \equiv 1 \pmod{4}$, $d = 1$ and $\alpha = a^2 + b^2$, where $a, b$ are integers, then $\alpha^* = \beta^*$, where $m^*$ denotes the square-free part of the integer $m$.

**Case 6** If $w \equiv 1 \pmod{4}$, $d = 2$ and $\alpha = a^2 + b^2$, where $a, b$ are integers, then the equation

$$\alpha z^2 = -x^2 + \beta y^2$$

must have a solution in integers, $x, y, z$, not all zero.

**Case 7** If $w \equiv 3 \pmod{4}$, $d = 1$, then $\alpha^* = \beta^*$, where $m^*$ denotes the square-free part of the integer $m$.

**Case 8** If $w \equiv 3 \pmod{4}$, $d = 2$, then the equation

$$\alpha z^2 = -x^2 + \beta y^2$$

must have a solution in integers, $x, y, z$, not all zero.

As an application of the main theorems it is easy to determine that there does not exist finite projective plane of order $n$ if $n$ is each of the first open values 10, 12, 15, 18, 20, 24, 26 and 28, for which the Bruck-Ryser-Chowla Theorem can not be used. For large $n$ the new method is also valid. Also some symmetric designs are excluded by the new method.

**Main Theorem 2** Conjecture 2.12 holds if finite projective plane of order $n \leq 33$. 

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Main Theorem 3 If \((v, k, \lambda)\) are parameters \((49, 16, 5), (154, 18, 2)\) and \((115, 19, 3)\), then the symmetric \((v, k, \lambda)\) designs do not exist.

3 Some theorems from number theory

In this section we shall state some theorems from number theory that are needed for our work in combinatorial analysis. No proofs will be given, but references will be given to books where the proofs may be found.

Lemma 3.1 (Lagrange, Sum of Four Squares Theorem, [20]) Every positive integer is the sum of four integral squares.

Lemma 3.2 (Sum of Two Squares Theorem, [21], Theorem 27.1) Let \(m\) be a positive integer. Factor \(m = p_1 p_2 \cdots p_r M^2\) with distinct prime factors \(p_1, p_2, \ldots, p_r\). Then \(m\) can be written as a sum of two integral squares exactly when each \(p_i\) is either 2 or is congruent to 1 modulo 4.

Lemma 3.3 (Sum of Three Squares Theorem, [20]) Positive integer \(n\) is the sum of three integral squares if \(n \equiv 1, 2, 3, 5, \text{ or } 6 \pmod{8}\).

We shall also use the following elementary identities which can be verified by direct multiplication.

Lemma 3.4 (Two Squares Identity, [21], Chapter 26) \((b_1^2 + b_2^2)(x_1^2 + x_2^2) = y_1^2 + y_2^2\), where
\[
y_1 = b_1 x_1 - b_2 x_2, \\
y_2 = b_2 x_1 + b_1 x_2.
\]

Lemma 3.5 (Four Squares Identity, [14], §2.1) \((b_1^2 + b_2^2 + b_3^2 + b_4^2)(x_1^2 + x_2^2 + x_3^2 + x_4^2) = y_1^2 + y_2^2 + y_3^2 + y_4^2\), where
\[
y_1 = b_1 x_1 - b_2 x_2 - b_3 x_3 - b_4 x_4, \\
y_2 = b_2 x_1 + b_1 x_2 - b_4 x_3 + b_3 x_4, \\
y_3 = b_3 x_1 + b_4 x_2 + b_1 x_3 - b_2 x_4, \\
y_4 = b_4 x_1 - b_3 x_2 + b_2 x_3 + b_1 x_4.
\]
Let $m$ and $b$ be nonzero two integers and $(b, m) = 1$. The integers $b$ are divided into two classes called quadratic residues and quadratic nonresidues according as $x^2 \equiv b \pmod{m}$ does or does not have a solution $x \pmod{m}$.

Let $p$ be an odd prime. The integers $b$ with $p \nmid b$ are divided into two classes called quadratic residues and quadratic nonresidues according as $x^2 \equiv b \pmod{p}$ does or does not have a solution $x \pmod{p}$. This property is expressed in term of the Legendre symbol $(\frac{b}{p})$ by the rules

$$(\frac{b}{p}) = +1 \text{ if } b \text{ a quadratic residue modulo } p,$$

$$(\frac{b}{p}) = -1 \text{ if } b \text{ a quadratic nonresidue modulo } p.$$

**Lemma 3.6** (Legendre, [14], §2.1) Let $a, b, c$ be all positive, coprime to each other and square-free integers. The equation

$$ax^2 + by^2 = cz^2 \tag{4}$$

has solutions in integers $x, y, z$ not all zero if and only if $bc, ac$ and $-ab$ are quadratic residues mod(a), mod(b) and mod(c) respectively.

**Lemma 3.7** (Legendre, [14], §2.1) Consider the equation

$$Ax^2 + By^2 + Cz^2 = 0 \tag{5}$$

and assume initially that $A, B, \text{ and } C$ are square-free integers, pairwise relatively prime. Necessary conditions for the existence of a nontrivial integral solution are that, for all odd primes $p$,

1. If $p \mid A$, then the Legendre symbol $(\frac{-BC}{p}) = 1$,
2. If $p \mid B$, then the Legendre symbol $(\frac{-AC}{p}) = 1$,
3. If $p \mid C$, then the Legendre symbol $(\frac{-AB}{p}) = 1$,
and, of course,
4. $A, B, \text{ and } C$ do not all have the same sign.

It is a classical theorem, due to Legendre that these simple necessary conditions are sufficient.

**Remark 3.8** ([14], §2.1) If $A, B, \text{ and } C$ do not satisfy our assumptions above we may slightly modify the equation (5). Henceforth, let $m^*$ denote the square-free part of the integer $m$. Then (5) has a nontrivial integral solution if and only if

$$A^*x^2 + B^*y^2 + C^*z^2 = 0 \tag{6}$$

has a nontrivial integral solution.
Remark 3.9 ([L4], §2.1) If $p$ divides all three coefficients; we may divide it out and if $p$ divides only $A$ and $B$, then (5) has a nontrivial integral solution if only if

$$\frac{A}{p}x^2 + \frac{B}{p}y^2 + (pC)z^2 = 0$$

does. Hence (6) can always be transformed into an equation to which Legendre’s result applies.

4 Proof of main Theorem 1

We are going to complete the proof of main theorem 1 in this section and to give some examples of the existence of bordered matrix of symmetric $(v, k, \lambda)$ designs. We will see that the proof of main Theorem 1 is just like the Ryser-Chowla elimination procedure in [3]. Also it is just like the Gaussian elimination procedure for solving the homogeneous linear equations.

Lemma 4.1 (Case 1 of Main Theorem 1) Let $C$ be a $w$ by $w + 1$ nonsquare rational matrix without any column of $k \cdot 1_{w+1}$, where $1_{w+1}$ is the $w + 1$-dimensional all $1$ column vector and $k$ is a rational number, $\alpha, \beta$ be two positive integers. Suppose the matrix $\alpha I_w + \beta J_w$ is the positive definite matrix with plus $1$ congruent factorization property such that

$$CC^t = \alpha I_w + \beta J_w.$$ (7)

If $w \equiv 0 \pmod{4}$, then $\beta$ is a perfect square.

Proof By the assumption we have the identity

$$CC^t = \alpha I_w + \beta J_w$$

for the rational matrix $C$. The idea of the proof is to interpret this as an identity in quadratic forms over the rational field.

Suppose that $w \equiv 0 \pmod{4}$. If $x$ is the row vector $(x_1, x_2, \cdots, x_w)$, then the identity for $CC^t$ gives

$$xC C^t x^t = \alpha(x_1^2 + x_2^2 + \cdots + x_w^2) + \beta(x_1 + x_2 + \cdots + x_w)^2.$$ (8)

Putting $f = xC$, $f = (f_1, f_2, \cdots, f_w, f_{w+1})$, $z = x_1 + x_2 + \cdots + x_w$, we have $ff^t = xC C^t x^t$ and

$$f_1^2 + f_2^2 + \cdots + f_w^2 + f_{w+1}^2 = \alpha(x_1^2 + x_2^2 + \cdots + x_w^2) + \beta z^2.$$ (8)
\begin{align*}
f_1 &= c_1 x_1 + c_2 x_2 \cdots + c_w x_w, \\
f_2 &= c_1 x_1 + c_2 x_2 \cdots + c_w x_w, \\
\vdots & \quad \\
f_w &= c_1 x_1 + c_2 x_2 \cdots + c_w x_w, \\
f_{w+1} &= c_1 x_1 + c_2 x_2 \cdots + c_w x_w; \\
z &= x_1 + x_2 + \cdots + x_w.
\end{align*}

Thus the cone (8) of variables \( f_1, f_2, \cdots, f_w, f_{w+1}, x_1, x_2, \cdots, x_w, z \) has some nontrivial rational points. We will get a nontrivial rational point for \( y^2 = \beta z^2 \) such that \( y \neq 0 \) by the Ryser-Chowla elimination procedure for the above homogeneous equations.

Now define a linear mapping \( \sigma \) from \( \mathbb{Q}^w \) to \( \mathbb{Q}^{w+1} \)

\[ \sigma : x \mapsto x C. \]

The image space \( \sigma(\mathbb{Q}^w) \) is a vector subspace of \( \mathbb{Q}^{w+1} \). Let \( \gamma_1, \gamma_2, \cdots, \gamma_w \) be the row vectors of \( C \). Thus the row space \( R(C) \) is subspace of \( \mathbb{Q}^{w+1} \) spanned by \( \gamma_1, \gamma_2, \cdots, \gamma_w \). So \( \sigma(\mathbb{Q}^w) = R(C) \). By Remark 2.17, since \( C \) is of full row rank, \( \text{dim}_{\mathbb{Q}}(R(C)) = w \). So \( \sigma \) is an one-one linear mapping from \( \mathbb{Q}^w \) to \( R(C) \).

The equation (8) is an identity in \( x_1, x_2, \cdots, x_w \). Each of the \( f \)'s is a rational combination of the \( x \)'s, since \( f = x C \). By Remark 2.17, since \( C \) is of full row rank, each of the \( x \)'s is a rational combination of the \( f \)'s for any \( f \in R(C) \). Thus the equation (8) is an identity in the variables \( f_1, f_2, \cdots, f_w, f_{w+1} \) for any \( f \in R(C) \).

We express the integer \( \alpha \) as the sum of four squares by Lemma 3.1, and bracket the terms \( x_1^2 + \cdots + x_w^2 \) in fours. Each product of sums of four squares is itself a sum of four squares, i.e. Lemma 3.5, and so (8) yields

\[
\begin{align*}
f_1^2 + f_2^2 + \cdots + f_w^2 + f_{w+1}^2 \\
= y_1^2 + y_2^2 + \cdots + y_w^2 + \beta z^2,
\end{align*}
\]

where \( z = x_1 + x_2 + \cdots + x_w \), and the \( y \)'s are related to the \( x \)'s by an invertible linear transformation with rational coefficients. Since the \( y \)'s are rational linear combinations of the \( x \)'s, it follows that the \( y \)'s (and \( z \)) are rational linear combinations of the \( f \)'s for any \( f \in R(C) \). Thus the equation (9) is an identity in the variables \( f_1, f_2, \cdots, f_w, f_{w+1} \) for any \( f \in R(C) \).

Suppose that \( y_i = b_{i1} f_1 + \cdots + b_{iw} f_w + b_{i,w+1} f_{w+1}, 1 \leq i \leq w \). We can define \( f_1 \) as a rational linear combination of \( f_2, \cdots, f_{w+1} \), in such a way that \( y_1^2 = f_1^2 \): if \( b_{11} \neq 1 \) we set \( f_1 = \frac{1}{1-b_{11}} (b_{12} f_2 + \cdots + b_{1,w+1} f_{w+1}) \), while if \( b_{11} = 1 \) we set \( f_1 = \frac{1}{1-b_{11}} (b_{12} f_2 + \cdots + b_{1,w+1} f_{w+1}) \).
\( \cdots + b_{w+1}f_{w+1} \). Now we know that \( y_2 \) is a rational linear combination of the \( f' \)'s, and, using the relevant expression for \( f_1 \) found above, we can express \( y_2 \) as a rational linear combination of \( f_2, \cdots, f_{w+1} \). As before, we fix \( f_2 \) as a rational combination of \( f_3, \cdots, f_{w+1} \) in such a way that \( y_2^2 = f_2^2 \). Continuing thus, we eventually obtain \( y_1, \cdots, y_w \) and \( f_1, \cdots, f_w \) as rational multiples of \( f_{w+1} \), satisfying \( f_i^2 = y_i^2 \) (\( 1 \leq i \leq w \)).

We reduce the equations step by step in this way until a truncated triangle of equations is obtained, say

\[
\begin{align*}
  f_1 &= d_{12} f_2 + \cdots + d_{1w+1} f_{w+1}, \\
  f_2 &= d_{23} f_3 + \cdots + d_{2w+1} f_{w+1}, \\
  &\vdots \\
  f_w &= d_{w \ w+1} f_{w+1}; \\
  f_i^2 &= y_i^2, (1 \leq i \leq w);
\end{align*}
\]

where \( d_{ij} \in \mathbb{Q} \).

For any \( x \in \mathbb{Q}^w \) and \( x \neq 0 \), by Remark 2.17, since \( C \) is of full row rank, \( f = x C \) implies \( f \neq 0 \). Let the last one \( f_{w+1} \neq 0 \) with suitable renumberings, if necessary. Choose any non-zero rational value for \( f_{w+1} \). All the \( y \)'s, the remaining \( f' \)'s, and \( z \), are determined as above, and substituting these values in (9) we obtain

\[
f_{w+1}^2 = \beta z^2. \tag{10}
\]

Multiplying by a suitable constant we have that \( \beta \) is a perfect square. So the theorem is proved. \( \square \)

**Example 4.2** The projective plane of order 5 is the symmetric \((31,6,1)\) design. Let \( A \) be its incidence matrix, which is a 31 by 31 matrix. Choose its bordered matrix \( C \) is a 32 by 33 matrix as the following matrix.

\[
C = \begin{bmatrix}
A_{1\ 1} & A_{1\ 2} \\
A_{2\ 1} & A_{2\ 2}
\end{bmatrix},
\]

\( A_{1\ 1} = A \).

\( A_{1\ 2} \) is a 31 by 2 matrix and

\[
A_{1\ 2} = (2 \cdot 1_{31}^t, 2 \cdot 1_{31}^t).
\]

\( A_{2\ 1} \) is a 1 by 31 matrix and

\[
A_{2\ 1} = \left[ \frac{1}{12} \cdot 1_{31} \right].
\]
$A_{2 \times 2}$ is a 1 by 2 matrix and

$$A_{2 \times 2} = \begin{bmatrix} 7 & 11 \\ 12 & 3 \end{bmatrix}.$$ 

It is easy to check that $C$ has the property of row inner products, i.e.,

(i) the inner product of any two distinct rows of $C$ is equal to 9;

(ii) and the inner product of any rows with themselves of $C$ is equal to 14.

It follows that

$$CC^t = 5I_{32} + 9J_{32}$$

and $C$ is exactly the bordered matrix of the symmetric $(31,6,1)$ design. You wish to verify this by hand or electronic computation. We have that 9 is a perfect square just as the assertion of the above theorem.

**Lemma 4.3 (Case 2 of Main Theorem 1)** Let $C$ be a $w$ by $w + 2$ nonsquare rational matrix without any column of $k \cdot 1_{w+2}$, where $1_{w+2}$ is the $w + 2$-dimensional all 1 column vector and $k$ is a rational number, $\alpha, \beta$ be positive integers. Suppose the matrix $\alpha I_w + \beta J_w$ is the positive definite matrix with plus 2 congruent factorization property such that

$$CC^t = \alpha I_w + \beta J_w.$$  \hfill (11)

If $w \equiv 0 \pmod{4}$, then $\beta$ is a sum of two squares.

**Proof** By the assumption we have the identity

$$CC^t = \alpha I_w + \beta J_w$$

for the rational matrix $C$. The idea of the proof is to interpret this as an identity in quadratic forms over the rational field.

Suppose that $w$ is an even integer with $w \equiv 0 \pmod{4}$. If $x$ is the row vector $(x_1, x_2, \cdots, w)$, then the identity for $CC^t$ gives

$$xC C^t x^t = \alpha(x_1^2 + x_2^2 + \cdots + x_w^2) + \beta(x_1 + x_2 + \cdots + x_w)^2.$$

Putting $f = x C$, $f = (f_1, f_2, \cdots, f_w, f_{w+1}, f_{w+2})$, $z = x_1 + x_2 + \cdots + x_w$, we have $f f^t = xCC^t x^t$ and

$$f_1^2 + f_2^2 + \cdots + f_w^2 + f_{w+1}^2 + f_{w+2}^2 = \alpha(x_1^2 + x_2^2 + \cdots + x_w^2) + \beta z^2, \quad (12)$$

$$f_1 = c_1 x_1 + c_2 x_2 + \cdots + c_{w+1} x_{w+1},$$

$$f_2 = c_1 x_1 + c_2 x_2 + \cdots + c_{w+2} x_{w+2},$$

$$\cdots \cdots \cdots$$
Thus the cone (12) of variables \(f_1, f_2, \ldots, f_w, f_{w+1}, f_{w+2}, x_1, x_2, \ldots, x_w, z\) has some non-trivial rational points. We will get a nontrivial rational point for \(f_{w+1}^2 + f_{w+2}^2 = \beta z^2\) such that \(f_{w+2}^2 \neq 0\) by the Ryser-Chowla elimination procedure for the above homogeneous equations.

Now define a linear mapping \(\sigma\) from \(\mathbb{Q}^w\) to \(\mathbb{Q}^{w+2}\)

\[
\sigma : x \mapsto x \, C.
\]

The image space \(\sigma(\mathbb{Q}^w)\) is a vector subspace of \(\mathbb{Q}^{w+2}\). Let \(\gamma_1, \gamma_2, \ldots, \gamma_w\) be the row vectors of \(C\). Thus the row space \(R(C)\) is subspace of \(\mathbb{Q}^{w+2}\) spanned by \(\gamma_1, \gamma_2, \ldots, \gamma_w\). So \(\sigma(\mathbb{Q}^w) = R(C)\). By Remark 2.17, since \(C\) is of full row rank, \(\dim_\mathbb{Q}(R(C)) = w\). So \(\sigma\) is an one-one linear mapping from \(\mathbb{Q}^w\) to \(R(C)\).

The equation (12) is an identity in \(x_1, x_2, \ldots, x_w\). Each of the \(f\)'s is a rational combination of the \(x\)'s, since \(f = xC\). By Remark 2.17, since \(C\) is of full row rank, each of the \(x\)'s is a rational combination of the \(f\)'s. Thus the equation (12) is an identity in the variables \(f_1, f_2, \ldots, f_w, f_{w+1}, f_{w+2}\) for any \(f \in R(C)\).

We express the integer \(\alpha\) as the sum of four squares by Lemma 3.1, and bracket the terms \(x_1^2 + \cdots + x_w^2\) in fours. Each product of sums of four squares is itself a sum of four squares, i.e. Lemma 3.5, and so (12) yields

\[
f_1^2 + f_2^2 + \cdots + f_w^2 + f_{w+1}^2 + f_{w+2}^2 = y_1^2 + y_2^2 + \cdots + y_w^2 + \beta z^2, \tag{13}
\]

where \(z = x_1 + x_2 + \cdots + x_w\), and the \(y\)'s are related to the \(x\)'s by an invertible linear transformation with rational coefficients. Since the \(y\)'s are rational linear combinations of the \(x\)'s, it follows that the \(y\)'s (and \(z\)) are rational linear combinations of the \(f\)'s. Thus the equation (13) is an identity in the variables \(f_1, f_2, \ldots, f_w, f_{w+1}, f_{w+2}\) for any \(f \in R(C)\).

Suppose that \(y_i = b_{1i} f_1 + \cdots + b_{wi} f_w + b_{i,w+1} f_{w+1} + b_{i,w+2} f_{w+2} + b_{i,w+3} f_{w+3}, 1 \leq i \leq w\). We can define \(f_1\) as a rational linear combination of \(f_2, \ldots, f_{w+1}, f_{w+2}\), in such a way that \(y_1^2 = f_1^2\): if \(b_{11} \neq 1\) we set \(f_1 = \frac{1}{b_{11}} (b_{12} f_2 + \cdots + b_{1,w+1} f_{w+1} + b_{1,w+2} f_{w+2})\), while if \(b_{11} = 1\) we set \(f_1 = \frac{1}{1-b_{11}} (b_{12} f_2 + \cdots + b_{1,w+1} f_{w+1} + b_{1,w+2} f_{w+2})\). Now we know that \(y_2\) is a rational linear combination of the \(f\)'s, and, using the relevant expression for \(f_1\)
found above, we can express \( y_2 \) as a rational linear combination of \( f_2, \cdots, f_{w+1}, f_{w+2} \).

As before, we fix \( f_2 \) as a rational combination of \( f_3, \cdots, f_{w+1}, f_{w+2} \) in such a way that \( y_2^2 = f_2^2 \). Continuing thus, we eventually obtain \( y_1, \cdots, y_w \) and \( f_1, \cdots, f_w \) as rational combinations of \( f_{w+1}, f_{w+2} \), satisfying \( f_i^2 = y_i^2 \) (1 \( \leq \) i \( \leq \) w).

We reduce the equations step by step in this way until a truncated triangle of equations is obtained, say

\[
\begin{align*}
f_1 &= d_{12}f_2 + \cdots + d_{1w+1}f_{w+1} + d_{1w+2}f_{w+2}, \\
f_2 &= d_{23}f_3 + \cdots + d_{2w+1}f_{w+1} + d_{2w+2}f_{w+2}, \\
\vdots & \quad \vdots \\
f_w &= d_{w,w+1}f_{w+1} + d_{w,w+2}f_{w+2}; \\
f_i^2 &= y_i^2, \quad (1 \leq i \leq w);
\end{align*}
\]

where \( d_{ij} \in \mathbb{Q} \).

For any \( x \in \mathbb{Q}^w \) and \( x \neq 0 \), by Remark 2.17, since \( C \) is of full row rank, \( f = xC \) implies \( f \neq 0 \). Let the last one \( f_{w+2} \neq 0 \) with suitable renumberings, if necessary. Choose any non-zero rational value for \( f_{w+2} \). All the \( y_i \)'s, the remaining \( f'_i \)'s, and \( z \), are determined as above, and substituting these values in (13) we obtain

\[
f_{w+1}^2 + f_{w+2}^2 = \beta z^2. \tag{14}
\]

Multiplying by a suitable constant we have that \( \beta \) is a sum of two squares. So the theorem is proved. \( \square \)

**Example 4.4** The projective plane of order 5 is the symmetric \((31, 6, 1)\) design. Let \( A \) be its incidence matrix, which is a 31 by 31 matrix. Choose its bordered matrix \( C \) is a 32 by 34 matrix as the following matrix.

\[
C = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},
\]

\( A_{11} = A \).

\( A_{12} \) is a 31 by 3 matrix and

\[
A_{12} = (1 \ast 1_{31}^t, 0 \ast 1_{31}^t, 0 \ast 1_{31}^t).
\]

\( A_{21} \) is a 1 by 31 matrix and

\[
A_{21} = \left[ \frac{1}{3} \cdot 1_{31} \right].
\]
A_2 = \begin{bmatrix} 0 & \frac{4}{3} \\ \frac{4}{3} & \frac{3}{3} \end{bmatrix}.

It is easy to check that C has the property of row inner products, i.e.,
(i) the inner product of any two distinct rows of C is equal to 2;
(ii) and the inner product of any rows with themselves of C is equal to 7.

It follows that
\[ CC^t = 5I_{32} + 2J_{32} \]
and C is exactly the bordered matrix of the symmetric (31, 6, 1) design. We have that 2 is a sum of two squares just as the assertion of the above theorem.

**Lemma 4.5 (Case 3 of Main Theorem 1)** Let C be a w by w + 1 nonsquare rational matrix without any column of k · 1^t_{w+1}, where 1^t_{w+1} is the w + 1-dimensional all 1 column vector and k is a rational number, α, β be positive integers and \( \alpha = a^2 + b^2 \), where a, b are integers. Suppose the matrix \( \alpha I_w + \beta J_w \) is the positive definite matrix with plus 1 congruent factorization property such that
\[ CC^t = \alpha I_w + \beta J_w. \] (15)
If \( w \equiv 2 \pmod{4} \), then \( \beta \) is a perfect square.

**Proof** By the assumption we have the identity
\[ CC^t = \alpha I_w + \beta J_w \]
for the rational matrix C. The idea of the proof is to interpret this as an identity in quadratic forms over the rational field.

Suppose that \( w \equiv 2 \pmod{4} \). If x is the row vector \( (x_1, x_2, \ldots, x_w) \), then the identity for \( CC^t \) gives
\[ xCC^t x^t = \alpha(x_1^2 + x_2^2 + \cdots + x_w^2) + \beta(x_1 + x_2 + \cdots + x_w)^2. \]
Putting \( f = xC, \ f = (f_1, f_2, \ldots, f_w, f_{w+1}), \ z = x_1 + x_2 + \cdots + x_w \), we have \( ff^t = xCC^t x^t \) and
\[ f_1^2 + f_2^2 + \cdots + f_w^2 + f_{w+1}^2 = \alpha(x_1^2 + x_2^2 + \cdots + x_w^2) + \beta z^2; \] (16)
\[ f_1 = c_1 x_1 + c_2 x_2 + \cdots + c_w x_w, \]
\[ f_2 = c_1 x_1 + c_2 x_2 + \cdots + c_w x_w, \]
\[ \ldots \ldots \ldots \ldots \]
The equation (16) is an identity in $x_1, x_2, \ldots, x_w$. Each of the $f$’s is a rational combination of the $x$’s, since $f = xC$. By Remark 2.17, since $C$ is of full row rank, $\dim_{\mathbb{Q}}(R(C)) = w$. So $\sigma$ is an one-one linear mapping from $\mathbb{Q}^w$ to $R(C)$.

The equation (16) is an identity in $x_1, x_2, \ldots, x_w$. Each of the $f$’s is a rational combination of the $x$’s, since $f = xC$. By Remark 2.17, since $C$ is of full row rank, each of the $x$’s is a rational combination of the $f$’s. Thus the equation (16) is an identity in the variables $f_1, f_2, \ldots, f_w, f_{w+1}$ for any $f \in R(C)$.

We express the integer $\alpha$ as the sum of two squares by the assumption, and bracket the terms $x_1^2 + \cdots + x_w^2$ in twos. Each product of sums of two squares is itself a sum of two squares, i. e. Lemma 3.4, and so (16) yields

$$f_1^2 + f_2^2 + \cdots + f_w^2 + f_{w+1}^2 = y_1^2 + y_2^2 + \cdots + y_w^2 + \beta z^2,$$

where $z = x_1 + x_2 + \cdots + x_w$, and the $y$’s are related to the $x$’s by an invertible linear transformation with rational coefficients. Since the $y$’s are rational linear combinations of the $x$’s, it follows that the $y$’s (and $z$) are rational linear combinations of the $f$’s. Thus the equation (17) is an identity in the variables $f_1, f_2, \ldots, f_w, f_{w+1}$ for any $f \in R(C)$.

Suppose that $y_i = b_{i1}f_1 + \cdots + b_{iw}f_w + b_{i,w+1}f_{w+1}, 1 \leq i \leq w$. We can define $f_1$ as a rational linear combination of $f_2, \ldots, f_{w+1}$, in such a way that $y_1^2 = f_1^2$: if $b_{11} \neq 1$ we set $f_1 = \frac{1}{1 - b_{11}}(b_{12}f_2 + \cdots + b_{1,w+1}f_{w+1})$, while if $b_{11} = 1$ we set $f_1 = \frac{1}{1 - b_{11}}(b_{11}f_1 + \cdots + b_{1,w+1}f_{w+1})$. Now we know that $y_2$ is a rational linear combination of the $f$’s, and, using the relevant expression for $f_1$ found above, we can express $y_2$ as a rational linear combination of $f_2, \ldots, f_{w+1}$. As before, we fix $f_2$ as a rational combination of $f_3, \ldots, f_{w+1}$ in such a way that $y_2^2 = f_2^2$. Continuing thus, we eventually obtain $y_1, \ldots, y_w$ and $f_1, \ldots, f_w$ as rational multiples of $f_{w+1}$, satisfying $f_i^2 = y_i^2 (1 \leq i \leq w)$. 

Thus the cone (16) of variables $f_1, f_2, \ldots, f_w, f_{w+1}, x_1, x_2, \ldots, x_w, z$ has some nontrivial rational points. We will get a nontrivial rational point for $y^2 = \beta z^2$ such that $y \neq 0$ by the Ryser-Chowla elimination procedure for the above homogeneous equations.

Now define a linear mapping $\sigma$ from $\mathbb{Q}^w$ to $\mathbb{Q}^{w+1}$

$$\sigma : x \mapsto x C.$$ 

The image space $\sigma(\mathbb{Q}^w)$ is a vector subspace of $\mathbb{Q}^{w+1}$. Let $\gamma_1, \gamma_2, \ldots, \gamma_w$ be the row vectors of $C$. Thus the row space $R(C)$ is subspace of $\mathbb{Q}^{w+1}$ spanned by $\gamma_1, \gamma_2, \ldots, \gamma_w$. So $\sigma(\mathbb{Q}^w) = R(C)$. By Remark 2.17, since $C$ is of full row rank, $\dim_{\mathbb{Q}}(R(C)) = w$. So $\sigma$ is an one-one linear mapping from $\mathbb{Q}^w$ to $R(C)$.
We reduce the equations step by step in this way until a truncated triangle of equations is obtained, say

\[ f_1 = d_{12}f_2 + \cdots + d_{1w+1}f_{w+1}, \]

\[ f_2 = d_{23}f_3 + \cdots + d_{2w+1}f_{w+1}, \]

\[ \cdots \cdots \cdots \]

\[ f_w = d_{ww+1}f_{w+1}; \]

\[ f_i^2 = y_i^2, \quad (1 \leq i \leq w); \]

where \( d_{ij} \in \mathbb{Q} \).

For any \( x \in \mathbb{Q}^w \) and \( x \neq 0 \), by Remark 2.17, since \( C \) is of full row rank, \( f = xC \) implies \( f \neq 0 \). Let the last one \( f_{w+1} \neq 0 \) with suitable renumberings, if necessary. Choose any non-zero rational value for \( f_{w+1} \). All the \( y' \)s, the remaining \( f' \)s, and \( z \), are determined as above, and substituting these values in (17) we obtain

\[ f_{w+1}^2 = \beta z^2. \]  

(18)

Multiplying by a suitable constant we have that \( \beta \) is a perfect squarer. So the theorem is proved. \( \square \)

**Example 4.6** There is a symmetric \((45, 12, 3)\) design (see [14]). Let \( A \) be its incidence matrix, which is a 45 by 45 matrix. Choose its bordered matrix \( C \) is a 46 by 47 matrix as the following matrix.

\[
C = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix},
\]

\( A_{11} = A \).

\( A_{12} \) is a 45 by 2 matrix and

\( A_{12} = (1 \ast 1_{45}^t, 0 \ast 1_{45}^t) \).

\( A_{21} \) is a 1 by 45 matrix and

\( A_{21} = \left[ \begin{array}{c} \frac{2}{9} \cdot 1_{45} \\ \end{array} \right] \).

\( A_{22} \) is a 1 by 2 matrix and

\( A_{22} = \left[ \begin{array}{c} \frac{4}{3} \\ 3 \\end{array} \right] \).

It is easy to check that \( C \) has the property of row inner products, i.e.,

(i) the inner product of any two distinct rows of \( C \) is equal to 4 ;
(ii) and the inner product of any rows with themselves of \( C \) is equal to 13. It follows that

\[
CC^t = 9I_{46} + 4J_{46}
\]

and \( C \) is exactly the bordered matrix of the symmetric \((45,12,3)\) design. We have that 4 is a square just as the assertion of the above theorem.

**Lemma 4.7 (Case 4 of Main Theorem 1)** Let \( C \) be a \( w \times w \) nonsquare rational matrix without any column of \( k \cdot 1_{w+2} \), where \( 1_{w+2} \) is the \( w+2 \)-dimensional all 1 column vector and \( k \) is a rational number, \( \alpha, \beta \) be positive integers and \( \alpha = a^2 + b^2 \), where \( a, b \) are integers. Suppose the matrix \( \alpha I_w + \beta J_w \) is the positive definite matrix with plus 2 congruent factorization property such that

\[
CC^t = \alpha I_w + \beta J_w. \tag{19}
\]

If \( w \equiv 2 \pmod{4} \), then \( \beta \) is a sum of two squares.

**Proof** By the assumption we have the identity

\[
CC^t = \alpha I_w + \beta J_w
\]

for the rational matrix \( C \). The idea of the proof is to interpret this as an identity in quadratic forms over the rational field.

Suppose that \( w \equiv 2 \pmod{4} \). If \( x \) is the row vector \((x_1, x_2, \ldots, x_w)\), then the identity for \( CC^t \) gives

\[
xCC^t x^t = \alpha (x_1^2 + x_2^2 + \cdots + x_w^2) + \beta (x_1 + x_2 + \cdots + x_w)^2.
\]

Putting \( f = xC, f = (f_1, f_2, \ldots, f_w, f_{w+1}, f_{w+2}) \), \( z = x_1 + x_2 + \cdots + x_w \), we have \( ff^t = xCC^t x^t \) and

\[
f_1^2 + f_2^2 + \cdots + f_w^2 + f_{w+1}^2 + f_{w+2}^2 = \alpha (x_1^2 + x_2^2 + \cdots + x_w^2) + \beta z^2, \tag{20}
\]

\[
f_1 = c_1 x_1 + c_2 x_2 + \cdots + c_w x_w,
\]

\[
f_2 = c_1 x_1 + c_2 x_2 + \cdots + c_w x_w,
\]

\[
\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots
\]
Thus the cone (20) of variables \(f_1, f_2, \cdots, f_w, f_{w+1}, f_{w+2}, x_1, x_2, \cdots, x_w, z\) has some non-trivial rational points. We will get a nontrivial rational point for \(f_{w+1}^2 + f_{w+2}^2 = \beta z^2\) such that \(f_{w+2}^2 \neq 0\) by the Ryser-Chowla elimination procedure for the above homogeneous equations.

Now define a linear mapping \(\sigma\) from \(Q^w\) to \(Q^{w+2}\)

\[
\sigma : x \mapsto x \; C.
\]

The image space \(\sigma(Q^w)\) is a vector subspace of \(Q^{w+2}\). Let \(\gamma_1, \gamma_2, \cdots, \gamma_w\) be the row vectors of \(C\). Thus the row space \(R(C)\) is subspace of \(Q^{w+2}\) spanned by \(\gamma_1, \gamma_2, \cdots, \gamma_w\). So \(\sigma(Q^w) = R(C)\). By Remark 2.17, since \(C\) is of full row rank, \(\dim_Q(R(C)) = w\). So \(\sigma\) is an one-one linear mapping from \(Q^w\) to \(R(C)\).

The equation (20) is an identity in \(x_1, x_2, \cdots, x_w\). Each of the \(f\)'s is a rational combination of the \(x\)'s, since \(f = x \; C\). By Remark 2.17, since \(C\) is of full row rank, each of the \(x\)'s is a rational combination of the \(f\)'s. Thus the equation (20) is an identity in the variables \(f_1, f_2, \cdots, f_w, f_{w+1}, f_{w+2}\) for any \(f \in R(C)\).

We express the integer \(\alpha\) as the sum of two squares by the assumption, and bracket the terms \(x_1^2 + \cdots + x_w^2\) in twos. Each product of sums of two squares is itself a sum of two squares, and so (20) yields

\[
f_1^2 + f_2^2 + \cdots + f_w^2 + f_{w+1}^2 + f_{w+2}^2 = y_1^2 + y_2^2 + \cdots + y_w^2 + \beta z^2, \tag{21}\]

where \(z = x_1 + x_2 + \cdots + x_w\), and the \(y\)'s are related to the \(x\)'s by an invertible linear transformation with rational coefficients. Since the \(y\)'s are rational linear combinations of the \(x\)'s, it follows that the \(y\)'s (and \(z\)) are rational linear combinations of the \(f\)'s. Thus the equation (21) is an identity in the variables \(f_1, f_2, \cdots, f_w, f_{w+1}, f_{w+2}\) for any \(f \in R(C)\).

Suppose that \(y_i = b_i f_1 + \cdots + b_i w f_w + b_{i w+1} f_{w+1} + b_{i w+2} f_{w+2}, 1 \leq i \leq w\). We can define \(f_1\) as a rational linear combination of \(f_2, \cdots, f_{w+1}, f_{w+2}\), in such a way that \(y_1^2 = f_1^2\): if \(b_{11} \neq 1\) we set \(f_1 = \frac{1}{b_{11}} (b_{12} f_2 + \cdots + b_{1 w+1} f_{w+1} + b_{1 w+2} f_{w+2})\), while if \(b_{11} = 1\) we set \(f_1 = \frac{1}{1-b_{11}} (b_{12} f_2 + \cdots + b_{1 w+1} f_{w+1} + b_{1 w+2} f_{w+2})\). Now we know that \(y_2\) is a rational linear combination of the \(f\)'s, and, using the relevant expression for \(f_1\) found above, we can express \(y_2\) as a rational linear combination of \(f_2, \cdots, f_{w+1}, f_{w+2}\). As before, we fix \(f_2\) as a rational combination of \(f_3, \cdots, f_{w+1}, f_{w+2}\) in such a way that \(y_2^2 = f_2^2\). Continuing thus, we eventually obtain \(y_1, \cdots, y_w\) and \(f_1, \cdots, f_w\) as rational combinations of \(f_{w+1}, f_{w+2}\), satisfying \(f_i^2 = y_i^2 (1 \leq i \leq w)\).
We reduce the equations step by step in this way until a truncated triangle of equations is obtained, say

\[ f_1 = d_{12} f_2 + \cdots + d_{1w+1} f_{w+1} + d_{1w+2} f_{w+2}, \]
\[ f_2 = d_{23} f_3 + \cdots + d_{2w+1} f_{w+1} + d_{2w+2} f_{w+2}, \]
\[ \vdots \]
\[ f_w = d_{w, w+1} f_{w+1} + d_{w, w+2} f_{w+2}; \]
\[ f_i^2 = y_i^2, \quad (1 \leq i \leq w); \]

where \(d_{ij} \in \mathbb{Q}\).

For any \(x \in \mathbb{Q}^w\) and \(x \neq 0\), by Remark 2.17, since \(C\) is of full row rank, \(f = x C\) implies \(f \neq 0\). Let the last one \(f_{w+2} \neq 0\) with suitable renumberings, if necessary. Choose any non-zero rational value for \(f_{w+2}\). All the \(y_i's\), the remaining \(f_i's\), and \(z\), are determined as above, and substituting these values in (21) we obtain

\[ f_{w+1}^2 + f_{w+2}^2 = \beta z^2. \tag{22} \]

Multiplying by a suitable constant we have that \(\beta\) is a sum of two squares. So the theorem is proved. \(\square\)

**Example 4.8** There is a symmetric \((45, 12, 3)\) design (see [14]). Let \(A\) be its incidence matrix, which is a 45 by 45 matrix. Choose its bordered matrix \(C\) is a 46 by 48 matrix as the following matrix.

\[
C = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix},
A_{11} = A.
\]

\(A_{12}\) is a 45 by 3 matrix and

\[ A_{12} = (1 \ast 1_{45}^t, 1 \ast 1_{45}^t, 0 \ast 1_{45}^t). \]

\(A_{21}\) is a 1 by 45 matrix and

\[ A_{21} = \begin{bmatrix} 0 \cdot 1_{45} \end{bmatrix}. \]

\(A_{22}\) is a 1 by 3 matrix and

\[ A_{22} = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}. \]

It is easy to check that \(C\) has the property of row inner products, i.e.,

\(i\) the inner product of any two distinct rows of \(C\) is equal to 5 ;

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(ii) and the inner product of any rows with themselves of \( C \) is equal to 14. It follows that

\[
C C^t = 9I_{46} + 5J_{46}
\]

and \( C \) is exactly the bordered matrix of the symmetric \((45, 12, 3)\) design. We have that 5 is a sum of two squares just as the assertion of the above theorem.

Lemma 4.9 (Case 5 of Main Theorem 1) Let \( C \) be a \( w \) by \( w + 1 \) nonsquare rational matrix without any column of \( k \cdot 1_{w+1}^t \), where \( 1_{w+1}^t \) is the \( w + 1 \)-dimensional all 1 column vector and \( k \) is a rational number, \( \alpha, \beta \) be positive integers and \( \alpha = a^2 + b^2 \), where \( a, b \) are integers. Suppose the matrix \( \alpha I_w + \beta J_w \) is the positive definite matrix with plus 1 congruent factorization property such that

\[
C C^t = \alpha I_w + \beta J_w.
\]

(23)

If \( w \equiv 1 \pmod{4} \), then \( \alpha^* = \beta^* \).

Proof By the assumption we have the identity

\[
C C^t = \alpha I_w + \beta J_w
\]

for the rational matrix \( C \). The idea of the proof is to interpret this as an identity in quadratic forms over the rational field.

Suppose that \( w \equiv 1 \pmod{4} \). If \( x \) is the row vector \((x_1, x_2, \cdots, x_w)\), then the identity for \( C C^t \) gives

\[
xC C^t x^t = \alpha(x_1^2 + x_2^2 + \cdots + x_w^2) + \beta(x_1 + x_2 + \cdots + x_w)^2.
\]

Putting \( f = x C, f = (f_1, f_2, \cdots, f_w, f_{w+1}), z = x_1 + x_2 + \cdots + x_w \), we have \( ff^t = x C C^t x^t \) and

\[
f_1^2 + f_2^2 + \cdots + f_w^2 + f_{w+1}^2 = \alpha(x_1^2 + x_2^2 + \cdots + x_w^2) + \beta z^2; \quad (24)
\]

\[
f_1 = c_{11}x_1 + c_{21}x_2 + \cdots + c_{w1}x_w,
\]

\[
f_2 = c_{12}x_1 + c_{22}x_2 + \cdots + c_{w2}x_w,
\]

\[\vdots\]

\[
f_w = c_{1w}x_1 + c_{2w}x_2 + \cdots + c_{ww}x_w,
\]

\[
f_{w+1} = c_{1w+1}x_1 + c_{2w+1}x_2 + \cdots + c_{ww+1}x_w;
\]

\[
z = x_1 + x_2 + \cdots + x_w.
\]
Thus the cone (24) of variables $f_1, f_2, \cdots, f_w, f_{w+1}, x_1, x_2, \cdots, x_w, z$ has some nontrivial rational points. We will get a nontrivial rational point for $\alpha y^2 = \beta z^2$ such that $y \neq 0$ by the Ryser-Chowla elimination procedure for the above homogeneous equations.

Now define a linear mapping $\sigma$ from $\mathbb{Q}^w$ to $\mathbb{Q}^{w+1}$

$$\sigma : x \mapsto x C.$$ 

The image space $\sigma(\mathbb{Q}^w)$ is a vector subspace of $\mathbb{Q}^{w+1}$. Let $\gamma_1, \gamma_2, \cdots, \gamma_w$ be the row vectors of $C$. Thus the row space $R(C)$ is subspace of $\mathbb{Q}^{w+1}$ spanned by $\gamma_1, \gamma_2, \cdots, \gamma_w$. So $\sigma(\mathbb{Q}^w) = R(C)$. By Remark 2.17, since $C$ is of full row rank, $\dim_{\mathbb{Q}}(R(C)) = w$. So $\sigma$ is an one-one linear mapping from $\mathbb{Q}^w$ to $R(C)$.

The equation (24) is an identity in $x_1, x_2, \cdots, x_w$. Each of the $f$'s is a rational combination of the $x$'s, since $f = x C$. By Remark 2.17, since $C$ is of full row rank, each of the $x$'s is a rational combination of the $f$'s. Thus the equation (24) is an identity in the variables $f_1, f_2, \cdots, f_w, f_{w+1}$ for any $f \in R(C)$.

We express the integer $\alpha$ as the sum of two squares by the assumption, and bracket the terms $f_1^2 + \cdots + f_w^2 + f_{w+1}^2$ in twos. Each product of sums of two squares is itself a sum of two squares, and so (24) yields

$$\alpha(y_1^2 + y_2^2 + \cdots + y_w^2 + y_{w+1}^2) = \alpha(x_1^2 + x_2^2 + \cdots + x_w^2) + \beta z^2, \quad (25)$$

where $z = x_1 + x_2 + \cdots + x_w$, and the $y$'s are related to the $f$'s by an invertible linear transformation with rational coefficients. Thus $\det(P) \neq 0, y = f P$.

Now define a linear mapping $\tau$ from $\mathbb{Q}^{w+1}$ to $\mathbb{Q}^{w+1}$

$$\tau : f \mapsto f P.$$ 

The image space $\tau \sigma(\mathbb{Q}^w) = V$ is a vector subspace of $\mathbb{Q}^{w+1}$ and $\dim_{\mathbb{Q}} V = w$.

Since the $x$'s are rational linear combinations of the $f$'s, it follows that the $x$'s (and $z$) are rational linear combinations of the $y$'s. Thus the equation (25) is an identity in the variables $y_1, y_2, \cdots, y_w, y_{w+1}$ for any $y \in V$.

Suppose that $x_i = b_{11} y_1 + \cdots + b_{iw} y_w + b_{i,w+1} y_{w+1}, 1 \leq i \leq w$. We can define $y_1$ as a rational linear combination of $y_2, \cdots, y_{w+1}, y_{w+1}, y_{w+1}$, in such a way that $x_i^2 = y_i^2$: if $b_{11} \neq 1$ we set $y_1 = \frac{1}{1-b_{11}}(b_{12} y_2 + \cdots + b_{i+1} y_{w+1})$, while if $b_{11} = 1$ we set $y_1 = \frac{1}{-1-b_{11}}(b_{12} y_2 + \cdots + b_{i+1} y_{w+1})$. Now we know that $x_2$ is a rational linear combination of the $y$'s, and, using the relevant expression for $y_1$ found above, we can express $x_2$ as a rational linear combination of $y_2, \cdots, y_{w+1}$. As before, we fix $x_2$ as a rational combination of $y_3, \cdots, y_{w+1}$ in such a way that $x_2^2 = y_2^2$. Continuing thus, we eventually obtain $x_1, \cdots, x_w$ and $y_1, \cdots, y_w$ as rational multiples of $y_{w+1}$, satisfying $x_i^2 = y_i^2 (1 \leq i \leq w)$. 

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We reduce the equations step by step in this way until a truncated triangle of equations is obtained, say

\[ y_1 = d_{12}y_2 + \cdots + d_{1w+1}y_{w+1}, \]

\[ y_2 = d_{23}y_3 + \cdots + d_{2w+1}y_{w+1}, \]

\[ \cdots \cdots \cdots \]

\[ y_w = d_{w\,w+1}y_{w+1}; \]

\[ x_i^2 = y_i^2, \quad (1 \leq i \leq w); \]

where \(d_{ij} \in \mathbb{Q}.\)

For any \(x \in \mathbb{Q}^w\) and \(x \neq 0,\) by Remark 2.17, since \(C\) is of full row rank, \(f = xC,\) and \(\text{det}(P) \neq 0,\) \(y = fP,\) it implies \(y \neq 0.\) Let the last one \(y_{w+1} \neq 0\) with suitable renumberings, if necessary. Choose any non-zero rational value for \(y_{w+1}.\) All the \(x'\)s, the remaining \(y'\)s, and \(z,\) are determined as above, and substituting these values in (25) we obtain

\[ \alpha y_{w+1}^2 = \beta z^2. \quad (26) \]

Multiplying by a suitable constant we have that \(\alpha^* = \beta^*.\) So the theorem is proved. □

**Example 4.10** There is a symmetric \((36, 15, 6)\) design (see [14]). Let \(A\) be its incidence matrix, which is a 36 by 36 matrix. Choose its bordered matrix \(C\) is a 37 by 38 matrix as the following matrix.

\[ C = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \]

\[ A_{11} = A. \]

\(A_{12}\) is a 36 by 2 matrix and

\[ A_{12} = (3 \ast 1_{36}^t, 1 \ast 1_{36}^t). \]

\(A_{21}\) is a 1 by 36 matrix and

\[ A_{21} = \begin{bmatrix} \frac{7}{9} \cdot 1_{36} \end{bmatrix}. \]

\(A_{22}\) is a 1 by 2 matrix and

\[ A_{22} = \begin{bmatrix} 14/15 & 23/15 \end{bmatrix}. \]

It is easy to check that \(C\) has the property of row inner products, i.e.,

(i) the inner product of any two distinct rows of \(C\) is equal to 16 ;
(ii) and the inner product of any rows with themselves of $C$ is equal to 25. It follows that

$$CC^t = 9I_{37} + 16J_{37}$$

and $C$ is exactly the bordered matrix of the symmetric $(36, 15, 6)$ design. We have that $9^* = 16^*$ just as the assertion of the above theorem.

**Lemma 4.11 (Case 6 of Main Theorem 1)** Let $C$ be a $w$ by $w + 2$ nonsquare rational matrix without any column of $k\cdot 1_{w+2}$, where $1_{w+2}$ is the $w+2$-dimensional all 1 column vector and $k$ is a rational number, $\alpha, \beta$ be positive integers and $\alpha = a^2 + b^2$, where $a, b$ are integers. Suppose matrix $\alpha I_w + \beta J_w$ is the positive definite matrix with plus 2 congruent factorization property such that

$$CC^t = \alpha I_w + \beta J_w.$$  \hspace{1cm} (27)

If $w \equiv 1 \pmod{4}$, then the equation

$$\alpha z^2 = -x^2 + \beta y^2$$

must have a solution in integers, $x, y, z$, not all zero.

**Proof** By the assumption we have the identity

$$CC^t = \alpha I_w + \beta J_w$$

for the rational matrix $C$. The idea of the proof is to interpret this as an identity in quadratic forms over the rational field.

Suppose that $w \equiv 1 \pmod{4}$. If $x$ is the row vector $(x_1, x_2, \ldots, x_w)$, then the identity for $CC^t$ gives

$$xCC^tx^t = \alpha(x_1^2 + x_2^2 + \cdots + x_w^2) + \beta(x_1 + x_2 + \cdots + x_w)^2.$$ 

Putting $f = xC$, $f = (f_1, f_2, \ldots, f_w, f_{w+1}, f_{w+2})$, $z = x_1 + x_2 + \cdots + x_w$, we have $f^t xCC^tx^t$ and

$$f_1^2 + f_2^2 + \cdots + f_w^2 + f_{w+1}^2 + f_{w+2}^2 = \alpha(x_1^2 + x_2^2 + \cdots + x_w^2) + \beta z^2;$$ \hspace{1cm} (28)

$$f_1 = c_1 x_1 + c_2 x_2 + \cdots + c_w x_w,$$

$$f_2 = c_1 x_1 + c_2 x_2 + \cdots + c_w x_w,$$

$$\cdots\cdots$$

$$f_w = c_1 x_1 + c_2 x_2 + \cdots + c_w x_w,$$
Thus the cone (28) of variables \( f_1, f_2, \cdots, f_w, f_{w+1}, f_{w+2}, x_1, x_2, \cdots, x_w, z \) has some nontrivial rational points. We will get a nontrivial rational point for \( \alpha y^2 + x^2 = \beta z^2 \) such that \( x \neq 0 \) by the Ryser-Chowla elimination procedure for the above homogeneous equations.

Now define a linear mapping \( \sigma \) from \( \mathbf{Q}^w \) to \( \mathbf{Q}^{w+2} \)

\[
\sigma : x \mapsto x C.
\]

The image space \( \sigma(\mathbf{Q}^w) \) is a vector subspace of \( \mathbf{Q}^{w+2} \). Let \( \gamma_1, \gamma_2, \cdots, \gamma_w \) be the row vectors of \( C \). Thus the row space \( R(C) \) is subspace of \( \mathbf{Q}^{w+2} \) spanned by \( \gamma_1, \gamma_2, \cdots, \gamma_w \).

So \( \sigma(\mathbf{Q}^w) = R(C) \). By Remark 2.17, since \( C \) is of full row rank, \( \dim_{\mathbf{Q}}(R(C)) = w \). So \( \sigma \) is an one-one linear mapping from \( \mathbf{Q}^w \) to \( R(C) \).

The equation (28) is an identity in \( x_1, x_2, \cdots, x_w \). Each of the \( f \)'s is a rational combination of the \( x \)'s, since \( f = x C \). By Remark 2.17, since \( C \) is of full row rank, each of the \( x \)'s is a rational combination of the \( f \)'s. Thus the equation (28) is an identity in the variables \( f_1, f_2, \cdots, f_w, f_{w+1}, f_{w+2} \) for any \( f \in R(C) \).

We express the integer \( \alpha \) as the sum of two squares by the assumption, and bracket the terms \( f_1^2 + \cdots + f_w^2 + f_{w+1}^2 \) in twos. Each product of sums of two squares is itself a sum of two squares, and so (28) yields

\[
\alpha(y_1^2 + y_2^2 + \cdots + y_w^2 + y_{w+1}^2) + y_{w+2}^2
= \alpha(x_1^2 + x_2^2 + \cdots + x_w^2) + \beta z^2,
\]

where \( z = x_1 + x_2 + \cdots + x_w, f_{w+2} = y_{w+2}, \) and the \( y \)'s are related to the \( f \)'s by an invertible linear transformation with rational coefficients. Thus \( det(P) \neq 0, y = f P \).

Now define a linear mapping \( \tau \) from \( \mathbf{Q}^{w+2} \) to \( \mathbf{Q}^{w+2} \)

\[
\tau : f \mapsto f P.
\]

The image space \( \tau \sigma(\mathbf{Q}^w) = V \) is a vector subspace of \( \mathbf{Q}^{w+2} \) and \( \dim_{\mathbf{Q}}V = w \).

Since the \( x \)'s are rational linear combinations of the \( f \)'s, it follows that the \( x \)'s (and \( z \)) are rational linear combinations of the \( y \)'s. Thus the equation (29) is an identity in the variables \( y_1, y_2, \cdots, y_w, y_{w+1}, y_{w+2} \) for any \( y \in V \).

Suppose that \( x_i = b_{i1} y_1 + \cdots + b_{iw} y_w + b_{i(w+1)} y_{w+1} + b_{i(w+2)} y_{w+2}, \) \( 1 \leq i \leq w \). We can define \( y_1 \) as a rational linear combination of \( y_2, \cdots, y_{w+1}, y_{w+2} \), in such a way that
\( x_1^2 = y_1^2 \): if \( b_{11} \neq 1 \) we set \( y_1 = \frac{1}{1-b_{11}}(b_{12}y_2 + \cdots + b_{1w+1}y_{w+1} + b_{1w+2}y_{w+2}) \), while if \( b_{11} = 1 \) we set \( y_1 = \frac{1}{1-b_{11}}(b_{12}y_2 + \cdots + b_{1w+1}y_{w+1} + b_{1w+2}y_{w+2}) \). Now we know that \( x_2 \) is a rational linear combination of the \( y_i \)'s, and, using the relevant expression for \( y_1 \) found above, we can express \( x_2 \) as a rational linear combination of \( y_2, \ldots, y_{w+1}, y_{w+2} \).

As before, we fix \( x_2 \) as a rational combination of \( y_3, \ldots, y_{w+1}, y_{w+2} \) in such a way that \( x_2^2 = y_2^2 \). Continuing thus, we eventually obtain \( x_1, \ldots, x_w \) and \( y_1, \ldots, y_w \) as rational linear combinations of \( y_{w+1}, y_{w+2}, \) satisfying \( x_i^2 = y_i^2 \) (\( 1 \leq i \leq w \)).

We reduce the equations step by step in this way until a truncated triangle of equations is obtained, say

\[
\begin{align*}
y_1 &= d_{12}y_2 + \cdots + d_{1w+1}y_{w+1} + d_{1w+2}y_{w+2}, \\
y_2 &= d_{23}y_3 + \cdots + d_{2w+1}y_{w+1} + d_{2w+2}y_{w+2}, \\
&\quad \quad \quad \quad \quad \vdots \\
y_w &= d_{w-1w+1}y_{w+1} + d_{w}y_{w+2}, \\
x_i^2 &= y_i^2, \quad (1 \leq i \leq w);
\end{align*}
\]

where \( d_{ij} \in \mathbb{Q} \).

For any \( x \in \mathbb{Q}^w \) and \( x \neq 0 \), by Remark 2.17, since \( C \) is of full row rank, \( f = xC \), and \( \det(P) \neq 0 \), \( y = fP \), it implies \( y \neq 0 \). Let the last one \( y_{w+2} \neq 0 \) with suitable renumberings, if necessary. Choose any non-zero rational value for \( y_{w+2} \). All the \( x' \)'s, the remaining \( y' \)'s, and \( z \), are determined as above, and substituting these values in (29) we obtain

\[
\alpha y_{w+1}^2 + y_{w+2}^2 = \beta z^2.
\] (30)

Multiplying by a suitable constant we have that

\[
\alpha z^2 = -x^2 + \beta y^2.
\]

So the theorem is proved. \( \square \)

**Example 4.12** There is a symmetric (36, 15, 6) design (see[14]). Let \( A \) be its incidence matrix, which is a 36 by 36 matrix. Choose its bordered matrix \( C \) is a 37 by 39 matrix as the following matrix.

\[
C = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix},
\]

\( A_{11} = A \).

\( A_{12} \) is a 36 by 3 matrix and

\[
A_{12} = (2 * 1_{136}^t, 0 * 1_{36}^t, 0 * 1_{36}^t).
\]

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\( A_{21} \) is a 1 by 36 matrix and
\[
A_{21} = \begin{bmatrix}
\frac{8}{15} \\
1
\end{bmatrix}.
\]
\( A_{22} \) is a 1 by 3 matrix and
\[
A_{22} = \begin{bmatrix}
1 & 1 & \frac{13}{5}
\end{bmatrix}.
\]
It is easy to check that \( C \) has the property of row inner products, i.e.,
(i) the inner product of any two distinct rows of \( C \) is equal to 10;
(ii) and the inner product of any rows with themselves of \( C \) is equal to 19.

It follows that
\[
CC^t = 9I_{37} + 10J_{37}
\]
and \( C \) is exactly the bordered matrix of the symmetric \((36,15,6)\) design. We have that the equation
\[
9z^2 = -x^2 + 10y^2
\]
must have a solution in integers, \( x,y,z \), not all zero just as the assertion of the above theorem.

**Lemma 4.13** (Case 7 of Main Theorem 1) Let \( C \) be a \( w \) by \( w+1 \) nonsquare rational matrix without any column of \( k \cdot 1^t_{w+1} \), where \( 1^t_{w+1} \) is the \( w+1 \)-dimensional all 1 column vector and \( k \) is a rational number, \( \alpha, \beta \) be positive integers. Suppose the matrix \( \alpha I_w + \beta J_w \) is the positive definite matrix with plus 1 congruent factorization property such that
\[
CC^t = \alpha I_w + \beta J_w. \tag{31}
\]
If \( w \equiv 3 \pmod{4} \), then \( \alpha^* = \beta^* \).

**Proof**  By the assumption we have the identity
\[
CC^t = \alpha I_w + \beta J_w
\]
for the rational matrix \( C \). The idea of the proof is to interpret this as an identity in quadratic forms over the rational field.

Suppose that \( w \equiv 3 \pmod{4} \). If \( x \) is the row vector \( (x_1,x_2,\ldots,x_w) \), then the identity for \( CC^t \) gives
\[
xCC^t x^t = \alpha(x_1^2 + x_2^2 + \cdots + x_w^2) + \beta(x_1 + x_2 + \cdots + x_w)^2.
\]
Putting \( f = xC, f = (f_1,f_2,\ldots,f_w,f_{w+1}), z = x_1 + x_2 + \cdots + x_w \), we have \( ff^t = xCC^t x^t \) and
\[
f_1^2 + f_2^2 + \cdots + f_w^2 + f_{w+1}^2 = \alpha(x_1^2 + x_2^2 + \cdots + x_w^2) + \beta z^2; \tag{32}
\]
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\[f_1 = c_1x_1 + c_2x_2 + \cdots + c_w x_w,\]
\[f_2 = c_1x_1 + c_2x_2 + \cdots + c_w x_w,\]
\[\cdots\]
\[f_w = c_1x_1 + c_2x_2 + \cdots + c_w x_w,\]
\[f_{w+1} = c_1x_1 + c_2x_2 + \cdots + c_{w+1} x_{w+1};\]
\[z = x_1 + x_2 + \cdots + x_w.\]

Thus the cone \((32)\) of variables \(f_1, f_2, \cdots, f_w, f_{w+1}, x_1, x_2, \cdots, x_w, z\) has some nontrivial rational points. We will get a nontrivial rational point for \(\alpha y^2 = \beta z^2\) such that \(y \neq 0\) by the Ryser-Chowla elimination procedure for the above homogeneous equations.

Now define a linear mapping \(\sigma\) from \(Q^w\) to \(Q^{w+1}\)
\[\sigma : x \mapsto x C.\]

The image space \(\sigma(Q^w)\) is a vector subspace of \(Q^{w+1}\). Let \(\gamma_1, \gamma_2, \cdots, \gamma_w\) be the row vectors of \(C\). Thus the row space \(R(C)\) is subspace of \(Q^{w+1}\) spanned by \(\gamma_1, \gamma_2, \cdots, \gamma_w\). So \(\sigma(Q^w) = R(C)\). By Remark 2.17, since \(C\) is of full row rank, \(dim_Q(R(C)) = w\). So \(\sigma\) is an one-one linear mapping from \(Q^w\) to \(R(C)\).

The equation \((32)\) is an identity in \(x_1, x_2, \cdots, x_w\). Each of the \(f\)'s is a rational combination of the \(x\)'s, since \(f = x C\). By Remark 2.17, since \(C\) is of full row rank, each of the \(x\)'s is a rational combination of the \(f\)'s. Thus the equation \((32)\) is an identity in the variables \(f_1, f_2, \cdots, f_w, f_{w+1}\) for any \(f \in R(C)\).

We express the integer \(\alpha\) as the sum of four squares by Lemma 3.1, and bracket the terms \(f_1^2 + \cdots + f_w^2 + f_{w+1}^2\) in fours. Each product of sums of four squares is itself a sum of four squares, and so \((32)\) yields
\[\alpha(y_1^2 + y_2^2 + \cdots + y_w^2 + y_{w+1}^2)\]
\[= \alpha(x_1^2 + x_2^2 + \cdots + x_w^2) + \beta z^2,\]
\((33)\)
where \(z = x_1 + x_2 + \cdots + x_w\), and the \(y\)'s are related to the \(f\)'s by an invertible linear transformation with rational coefficients. Thus \(det(P) \neq 0, y = f P\).

Now define a linear mapping \(\tau\) from \(Q^{w+1}\) to \(Q^{w+1}\)
\[\tau : f \mapsto f P.\]

The image space \(\tau \sigma(Q^w) = V\) is a vector subspace of \(Q^{w+1}\) and \(dim_Q V = w\).

Since the \(x\)'s are rational linear combinations of the \(f\)'s, it follows that the \(x\)'s (and \(z\)) are rational linear combinations of the \(y\)'s. Thus the equation \((33)\) is an identity in the variables \(y_1, y_2, \cdots, y_w, y_{w+1}\) for any \(y \in V\).
Suppose that $x_i = b_{i1}y_1 + \cdots + b_{iw}y_w + b_{i+1w}y_{w+1}$, $1 \leq i \leq w$. We can define $y_1$ as a rational linear combination of $y_2, \ldots, y_{w+1}$, in such a way that $x_1^2 = y_1^2$: if $b_{11} \neq 1$ we set $y_1 = \frac{1}{1-b_{11}}(b_{12}y_2 + \cdots + b_{1w+1}y_{w+1})$, while if $b_{11} = 1$ we set $y_1 = \frac{1}{1-b_{11}}(b_{12}y_2 + \cdots + b_{1w+1}y_{w+1})$. Now we know that $x_2$ is a rational linear combination of the $y_i$'s, and, using the relevant expression for $y_1$ found above, we can express $x_2$ as a rational linear combination of $y_1, \ldots, y_{w+1}$. As before, we fix $x_2$ as a rational combination of $y_3, \ldots, y_{w+1}$ in such a way that $x_2^2 = y_2^2$. Continuing thus, we eventually obtain $x_1, \ldots, x_w$ and $y_1, \ldots, y_w$ as rational multiples of $y_{w+1}$, satisfying $x_i^2 = y_i^2$ ($1 \leq i \leq w$).

We reduce the equations step by step in this way until a truncated triangle of equations is obtained, say

\[
y_1 = d_{12}y_2 + \cdots + d_{1w+1}y_{w+1},
\]
\[
y_2 = d_{23}y_3 + \cdots + d_{2w+1}y_{w+1},
\]
\[\vdots\]
\[
y_w = d_{w+1w}y_{w+1};
\]
\[
x_i^2 = y_i^2, \quad (1 \leq i \leq w);
\]

where $d_{ij} \in \mathbb{Q}$.

For any $x \in \mathbb{Q}^w$ and $x \neq 0$, by Remark 2.17, since $C$ is of full row rank, $f = xC$, and $\det(P) \neq 0$, $y = fP$, it implies $y \neq 0$. Let the last one $y_{w+1} \neq 0$ with suitable renumberings, if necessary. Choose any non-zero rational value for $y_{w+1}$. All the $x$'s, the remaining $y$'s, and $z$, are determined as above, and substituting these values in (33) we obtain

\[
\alpha y_{w+1}^2 = \beta z^2.
\] (34)

Multiplying by a suitable constant we have that $\alpha^* = \beta^*$. So the theorem is proved. □

**Example 4.14** The projective plane of order 7 is the symmetric $(57, 8, 1)$ design. Let $A$ be its incidence matrix, which is a 57 by 57 matrix. Choose its bordered matrix $C$ is a 59 by 60 matrix as the following matrix.

\[
C = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},
\]

$A_{11} = A$.

$A_{12}$ is a 57 by 3 matrix and

\[
A_{12} = (2 \ast 1_{57}^t, 1 \ast 1_{57}^t, 1 \ast 1_{57}^t).
\]
$A_{2,1}$ is a 2 by 57 matrix and

$$A_{2,1} = \begin{bmatrix} 0 \cdot 1_{57} \\ 0 \cdot 1_{57} \end{bmatrix}.$$  

$A_{2,2}$ is a 2 by 3 matrix and

$$A_{2,2} = \begin{bmatrix} 1 & 3 & 2 \\ \frac{11}{5} & -\frac{2}{5} & 3 \end{bmatrix}.$$  

It is easy to check that $C$ has the property of row inner products, i.e.,

(i) the inner product of any two distinct rows of $C$ is equal to 7;

(ii) and the inner product of any rows with themselves of $C$ is equal to 14.

It follows that

$$CC^t = 7I_{59} + 7J_{59}$$

and $C$ is exactly the bordered matrix of the symmetric $(57,8,1)$ design. We have that $\alpha = 7, \beta = 7$ and $\alpha^* = \beta^*$ just as the assertion of the above theorem.

Lemma 4.15 (Case 8 of Main Theorem 1) Let $C$ be a $w$ by $w + 2$ nonsquare rational matrix without any column of $k \cdot 1_{w+2}$, where $1_{w+2}$ is the $w + 2$-dimensional all 1 column vector and $k$ is a rational number, $\alpha, \beta$ be two positive integers. Suppose the matrix $\alpha I_w + \beta J_w$ is the positive definite matrix with plus 2 congruent factorization property such that

$$CC^t = \alpha I_w + \beta J_w.$$  \hspace{1cm} (35)

If $w \equiv 3 \pmod{4}$, then the equation

$$\alpha z^2 = -x^2 + \beta y^2$$

must have a solution in integers, $x, y, z$, not all zero.

Proof  By the assumption we have the identity

$$CC^t = \alpha I_w + \beta J_w$$

for the rational matrix $C$. The idea of the proof is to interpret this as an identity in quadratic forms over the rational field.

Suppose that $w \equiv 3 \pmod{4}$. If $x$ is the row vector $(x_1, x_2, \cdots, x_w)$, then the identity for $CC^t$ gives

$$xC C^t x^t = \alpha(x_1^2 + x_2^2 + \cdots + x_w^2) + \beta(x_1 + x_2 + \cdots + x_w)^2.$$
Putting \( f = xC, f = (f_1, f_2, \cdots, f_w, f_{w+1}, f_{w+2}), z = x_1 + x_2 + \cdots + x_w \), we have \( f f^t = xC C^t x^t \) and

\[
f_1^2 + f_2^2 + \cdots + f_w^2 + f_{w+1}^2 + f_{w+2}^2 = \alpha(x_1^2 + x_2^2 + \cdots + x_w^2) + \beta z^2; \quad (36)
\]

\[
f_1 = c_{11}x_1 + c_{21}x_2 + \cdots + c_{w1}x_w,
\]

\[
f_2 = c_{12}x_1 + c_{22}x_2 + \cdots + c_{w2}x_w,
\]

\[\cdots\cdots\cdots\]

\[
f_w = c_{1w}x_1 + c_{2w}x_2 + \cdots + c_{ww}x_w,
\]

\[
f_{w+1} = c_{1w+1}x_1 + c_{2w+1}x_2 + \cdots + c_{ww+1}x_w,
\]

\[
f_{w+2} = c_{1w+2}x_1 + c_{2w+2}x_2 + \cdots + c_{ww+2}x_w;
\]

\[
z = x_1 + x_2 + \cdots + x_w.
\]

Thus the cone (36) of variables \( f_1, f_2, \cdots, f_w, f_{w+1}, f_{w+2}, x_1, x_2, \cdots, x_w, z \) has some nontrivial rational points. We will get a nontrivial rational point for \( \alpha y^2 + x^2 = \beta z^2 \) such that \( x \neq 0 \) by the Ryser-Chowla elimination procedure for the above homogeneous equations.

Now define a linear mapping \( \sigma \) from \( \mathbb{Q}^w \) to \( \mathbb{Q}^{w+2} \)

\[
\sigma : x \mapsto x C.
\]

The image space \( \sigma(\mathbb{Q}^w) \) is a vector subspace of \( \mathbb{Q}^{w+2} \). Let \( \gamma_1, \gamma_2, \cdots, \gamma_w \) be the row vectors of \( C \). Thus the row space \( R(C) \) is subspace of \( \mathbb{Q}^{w+2} \) spanned by \( \gamma_1, \gamma_2, \cdots, \gamma_w \). So \( \sigma(\mathbb{Q}^w) = R(C) \). By Remark 2.17, since \( C \) is of full row rank, \( dim_{\mathbb{Q}}(R(C)) = w \). So \( \sigma \) is an one-one linear mapping from \( \mathbb{Q}^w \) to \( R(C) \).

The equation (36) is an identity in \( x_1, x_2, \cdots, x_w \). Each of the \( f \)'s is a rational combination of the \( x \)'s, since \( f = xC \). By Remark 2.17, since \( C \) is of full row rank, each of the \( x \)'s is a rational combination of the \( f \)'s. Thus the equation (36) is an identity in the variables \( f_1, f_2, \cdots, f_w, f_{w+1}, f_{w+2} \) for any \( f \in R(C) \).

We express the integer \( \alpha \) as the sum of four squares by Lemma 3.1, and bracket the terms \( f_1^2 + \cdots + f_w^2 + f_{w+1}^2 \) in fours. Each product of sums of four squares is itself a sum of four squares, and so (36) yields

\[
\alpha(y_1^2 + y_2^2 + \cdots + y_w^2 + y_{w+1}^2) + y_{w+2}^2 = \alpha(x_1^2 + x_2^2 + \cdots + x_w^2) + \beta z^2;
\]

\[
\alpha(y_1^2 + y_2^2 + \cdots + y_w^2 + y_{w+1}^2) + y_{w+2}^2 = \alpha(x_1^2 + x_2^2 + \cdots + x_w^2) + \beta z^2, \quad (37)
\]

where \( z = x_1 + x_2 + \cdots + x_w, f_{w+2} = y_{w+2} \), and the \( y \)'s are related to the \( f \)'s by an invertible linear transformation with rational coefficients. Thus \( det(P) \neq 0, y = f P \).
Now define a linear mapping $\tau$ from $\mathbb{Q}^{w+2}$ to $\mathbb{Q}^{w+2}$

$$
\tau : f \mapsto f \cdot P.
$$

The image space $\tau \sigma(\mathbb{Q}^w) = V$ is a vector subspace of $\mathbb{Q}^{w+2}$ and $\dim_{\mathbb{Q}} V = w$.

Since the $x'$s are rational linear combinations of the $f'$s, it follows that the $x'$s (and $z$) are rational linear combinations of the $y'$s. Thus the equation (37) is an identity in the variables $y_1, y_2, \ldots, y_w, y_{w+1}, y_{w+2}$ for any $y \in V$.

Suppose that $x_i = b_{i1}y_1 + \cdots + b_{iw}y_w + b_{i w+1}y_{w+1} + b_{i w+2}y_{w+2}$, $1 \leq i \leq w$. We can define $y_1$ as a rational linear combination of $y_2, \ldots, y_w, y_{w+1}, y_{w+2}$, in such a way that $x_1^2 = y_1^2$: if $b_{11} \neq 1$ we set $y_1 = \frac{1}{1-b_{11}}(b_{12}y_2 + \cdots + b_{1 w+1}y_{w+1} + b_{1 w+2}y_{w+2})$, while if $b_{11} = 1$ we set $y_1 = \frac{1}{1-b_{11}}(b_{12}y_2 + \cdots + b_{1 w+1}y_{w+1} + b_{1 w+2}y_{w+2})$. Now we know that $x_2$ is a rational linear combination of the $y'$s, and, using the relevant expression for $y_1$ found above, we can express $x_2$ as a rational linear combination of $y_2, \ldots, y_w, y_{w+1}, y_{w+2}$. As before, we fix $x_2$ as a rational combination of $y_3, \ldots, y_{w+1}, y_{w+2}$ in such a way that $x_2^2 = y_2^2$. Continuing thus, we eventually obtain $x_1, \ldots, x_w$ and $y_1, \ldots, y_w$ as rational linear combinations of $y_{w+1}, y_{w+2}$, satisfying $x_i^2 = y_i^2$ ($1 \leq i \leq w$).

We reduce the equations step by step in this way until a truncated triangle of equations is obtained, say

\[
\begin{align*}
    y_1 &= d_{12}y_2 + \cdots + d_{1 w+1}y_{w+1} + d_{1 w+2}y_{w+2}, \\
    y_2 &= d_{23}y_3 + \cdots + d_{2 w+1}y_{w+1} + d_{2 w+2}y_{w+2}, \\
    \vdots & \vdots \\
    y_w &= d_{w w+1}y_{w+1} + d_{w w+2}y_{w+2}; \\
    x_i^2 &= y_i^2, \ (1 \leq i \leq w);
\end{align*}
\]

where $d_{i j} \in \mathbb{Q}$.

For any $x \in \mathbb{Q}^w$ and $x \neq 0$, by Remark 2.17, since $C$ is of full row rank, $f = x \cdot C$, and $\det(P) \neq 0$, $y = f \cdot P$, it implies $y \neq 0$. Let the last one $y_{w+2} \neq 0$ with suitable renumberings, if necessary. Choose any non-zero rational value for $y_{w+2}$. All the $x'$s, the remaining $y'$s, and $z$, are determined as above, and substituting these values in (37) we obtain

$$
\alpha y_{w+1}^2 + y_{w+2}^2 = \beta z^2.
$$

(38)

Multiplying by a suitable constant we have that

$$
\alpha z^2 = -x^2 + \beta y^2.
$$

So the theorem is proved. \qed
Example 4.16 The projective plane of order 7 is the symmetric \((57,8,1)\) design. Let \(A\) be its incidence matrix, which is a \(57\) by \(57\) matrix. Choose its bordered matrix \(C\) is a \(59\) by \(61\) matrix as the following matrix.

\[
C = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix},
\]

\(A_{11} = A\).

\(A_{12}\) is a \(57\) by \(4\) matrix and

\[
A_{12} = (1 \cdot 1_{57}^t, 0 \cdot 1_{57}^t, 0 \cdot 1_{57}^t, 0 \cdot 1_{57}^t).
\]

\(A_{21}\) is a \(2\) by \(57\) matrix and

\[
A_{21} = \begin{bmatrix}
0 \cdot 1_{57} \\
0 \cdot 1_{57}
\end{bmatrix}.
\]

\(A_{22}\) is a \(2\) by \(4\) matrix and

\[
A_{22} = \begin{bmatrix}
2 & 2 & 1 & 0 \\
2 & 0 & -2 & 1
\end{bmatrix}.
\]

It is easy to check that \(C\) has the property of the property of row inner products, i.e.,

(i) the inner product of any two distinct rows of \(C\) is equal to \(2\);

(ii) and the inner product of any rows with themselves of \(C\) is equal to \(9\).

It follows that

\[
C C^t = 7I_{59} + 2J_{59}
\]

and \(C\) is exactly the bordered matrix of the symmetric \((57,8,1)\) design. We have that \(\alpha = 7, \beta = 2\) and the equation

\[
7z^2 = -x^2 + 2y^2
\]

must have a solution in integers, \(x, y, z\), not all zero just as the assertion of the above theorem.

Proof of Main Theorem 1 By the above Lemmas we finish the proof of Main Theorem 1. \(\square\)

Suppose there exists a \((v,k,\lambda)\) symmetric design with an incidence matrix \(A\). It is difficult to construct a square bordered matrix of \(A\). The author does this by computer computation in Maple. But it is easy to construct a nonsquare bordered matrix of \(A\). The author also does this by computer computation in Maple just as the following remarks.
Remark 4.17 Suppose there exists a \((v, k, \lambda)\) symmetric design with an incidence matrix \(A\). Further suppose there exists a positive integer \(l\) such that \(l = a^2 + b^2\), where \(a, b\) are two integers. By Lemma 3.7 and computation in Maple we can choose \(l\) and construct the bordered matrix \(v+1 \times v+2\) \(C\) of the incidence matrix \(A\) as the following matrix

\[
C = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix},
\]

\(A_{12}\) is a \(v \times 2\) matrix and

\[
A_{12} = (a \cdot 1_v^t, b \cdot 1_v^t).
\]

\(A_{21}\) is a \(1 \times v\) matrix and

\[
A_{21} = \begin{bmatrix}
x_1 \cdot 1_v
\end{bmatrix},
\]

\(A_{22} = (x_2, x_3)\) is some \(1 \times 2\) matrix, where \(x_1, x_2, x_3\) are some rational numbers.

Remark 4.18 Suppose there exists a \((v, k, \lambda)\) symmetric design with an incidence matrix \(A\). Further suppose there exists a positive integer \(l\) such that \(l = a^2 + b^2\), where \(a, b\) are two integers. By Lemma 3.7 and computation in Maple we can choose \(l\) and construct the bordered matrix \(v+1 \times v+3\) \(C\) of the incidence matrix \(A\) as the following matrix

\[
C = \begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23}
\end{bmatrix},
\]

\(A_{12}\) is a \(v \times 3\) matrix and

\[
A_{12} = (a \cdot 1_v^t, b \cdot 1_v^t, 0 \cdot 1_v^t).
\]

\(A_{21}\) is a \(1 \times v\) matrix and

\[
A_{21} = \begin{bmatrix}
x_1 \cdot 1_v
\end{bmatrix},
\]

\(A_{22} = (x_2, x_3, x_4)\) is some \(1 \times 3\) matrix, where \(x_1, x_2, x_3, x_4\) are some rational numbers.

Remark 4.19 Suppose there exists a \((v, k, \lambda)\) symmetric design with an incidence matrix \(A\). Further suppose there exists a positive integer \(l\) such that \(l = a^2 + b^2\), where \(a, b\) are two integers. By Lemma 3.7 and computation in Maple we can choose \(l\) and construct the bordered matrix \(C\) of the incidence matrix \(A\) as the following matrix

\[
C = \begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23}
\end{bmatrix},
\]
\[
\det(C \ C^t) \neq 0,
\]
\[A_{11} = A.\]

\(A_{12}\) is a \(v\) by 2 matrix and
\[A_{12} = (a \cdot 1_v^t, b \cdot 1_v^t).\]

\(A_{13}\) is a \(v\) by \(s\) zero matrix, where \(s\) is 1 or 2. \(A_{21}\) is a 2 by \(v\) matrix and
\[A_{21} = \begin{bmatrix} c \cdot 1_v \\ d \cdot 1_v \end{bmatrix},\]
where \(c, d\) are some two rational numbers. \(A_{22}\) is some 2 by 2 matrix. \(A_{23}\) is a 2 by \(s\) matrix, where \(s\) is 1 or 2.

It is easy to construct the above bordered matrix by the computer using Lemma 3.6, Lemma 3.7, Remark 3.8 and Remark 3.9 if it does exist just as In §5 and §6.

Remark 4.20 Let \(A\) be the incidence matrix of a symmetric \((v, k, \lambda)\) design. Then the bordered matrix of \(A\) may not exist. If it exists then it is not unique for positive integers \(s, l\).

5 Proof of Main Theorem 2

Theorem 5.1 Projective planes of order 6, 14, 21, 22, 30 and 33 do not exist.

Proof In these cases we can use Theorem 2.11 (the Bruck-Ryser Theorem). □

In order to use main theorem 1 to show that symmetric designs with certain parameters cannot exist, we must show that the corresponding bordered matrix exist and the corresponding equation has no integral solution.

It is easy to construct the above bordered matrix by the computer using Lemma 3.6, Lemma 3.7, Remark 3.8 and Remark 3.9 in Maple if it does exist. It should be remarked that one does not need to trust the computer blindly. Although the proofs are discovered by the computer, it produces a proof certificate that can easily be checked by hand, if so desired.

Theorem 5.2 There does not exist finite projective plane of order 10.

Proof In this case we can not use the Bruck-Ryser Theorem but can use case 1 of Main Theorem 1. Suppose that a symmetric \((111, 11, 1)\) design exists. Let \(A\) be its
incidence matrix, which is a 111 by 111 matrix. Choose its bordered matrix $C$ is a 112 by 113 matrix as the following matrix.

$$C = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

$A_{11} = A$.

$A_{12}$ is a 111 by 2 matrix and

$$A_{12} = (10^t \cdot 1_{111}^t, 0 \cdot 1_{111}^t).$$

$A_{21}$ is a 1 by 111 matrix and

$$A_{21} = \begin{bmatrix} -\frac{2129}{11221} \cdot 1_{111} \end{bmatrix}.$$

$A_{22}$ is a 1 by 2 matrix and

$$A_{22} = \begin{bmatrix} \frac{115674}{11221} & 6 \end{bmatrix}.$$

It is easy to check that $C$ has the property of row inner products, i.e.,

(i)  the inner product of any two distinct rows of $C$ is equal to 101 ;  
(ii)  and the inner product of any rows with themselves of $C$ is equal to 111.  

It follows that the property of

$$C \cdot C^t = 10I_{112} + 101J_{112}.$$

Thus $C$ is exactly the bordered matrix of the symmetric $(111, 11, 1)$ design if $A$ exists. But by case 1 of Main Theorem 1 we have that 101 is a perfect square, which is a contradiction. So there does not exist finite projective plane of order 10. \qed

**Theorem 5.3** There does not exist finite projective plane of order 12.

**Proof** In this case we can not use the Bruck-Ryser Theorem but can use case 8 of Main Theorem 1. Suppose that a symmetric $(157, 13, 1)$ design exists. Let $A$ be its incidence matrix, which is a 157 by 157 matrix. Choose its bordered matrix $C$ is a 159 by 161 matrix as the following matrix.

$$C = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix},$$

$A_{11} = A$.

$A_{12}$ is a 157 by 2 matrix and

$$A_{12} = (2 \cdot 1_{157}^t, 0 \cdot 1_{157}^t).$$
$A_{1 \, 3}$ is a $157$ by $2$ matrix and

$$A_{1 \, 3} = (0 \ast 1_{157}^t, 0 \ast 1_{157}^t).$$

$A_{2 \, 1}$ is a $2$ by $157$ matrix and

$$A_{2 \, 1} = \begin{bmatrix} \frac{285}{2191} & 1_{157} \\ -\frac{7}{1} & 1_{157} \end{bmatrix}.$$

$A_{2 \, 2}$ is a $2$ by $2$ matrix and

$$A_{2 \, 2} = \begin{bmatrix} \frac{3625}{2191} & 11 \\ \frac{24}{7} & 10 \end{bmatrix}.$$

$A_{2 \, 3}$ is a $2$ by $2$ matrix and

$$A_{2 \, 3} = \begin{bmatrix} \frac{669}{313} & \frac{669}{313} \\ 0 & 0 \end{bmatrix}.$$

It is easy to check that $C$ has the property of row inner products, i.e.,

(i) the inner product of any two distinct rows of $C$ is equal to $5$;

(ii) the inner product of any rows with themselves of $C$ is equal to $17$.

It follows that the property of

$$C \, C^t = 12I_{159} + 5J_{159}.$$ 

Thus $C$ is exactly the bordered matrix of the symmetric $(157, 13, 1)$ design if $A$ exists. But by case 8 of main Theorem 1 the equation

$$12z^2 = -x^2 + 5y^2$$

must have a solution in integers, $x, y, z$, not all zero. It implies that, by Lemma 3.6, Lemma 3.7, Remark 3.8 and Remark 3.9, the Legendre symbol $(\frac{5}{p}) = 1$, which is a contradiction. So there does not exist finite projective plane of order $12$.

\[\square\]

**Theorem 5.4** There does not exist finite projective plane of order $15$.

**Proof** In this case we can not use the Bruck-Ryser Theorem but can use case 8 of Main Theorem 1. Suppose that a symmetric $(241, 16, 1)$ design exists. Let $A$ be its incidence matrix, which is a $241$ by $241$ matrix. Choose its bordered matrix $C$ is a $243$ by $245$ matrix as the following matrix.

$$C = \begin{bmatrix} A_{1 \, 1} & A_{1 \, 2} & A_{1 \, 3} \\ A_{2 \, 1} & A_{2 \, 2} & A_{2 \, 3} \end{bmatrix},$$
$A_{11} = A.$  

$A_{12}$ is a 241 by 2 matrix and

$A_{12} = (7 \cdot 1_{241}^t, 0 \ast 1_{241}^t).$

$A_{13}$ is a 241 by 2 matrix and

$A_{13} = (0 \ast 1_{241}^t, 0 \ast 1_{241}^t).$

$A_{21}$ is a 2 by 241 matrix and

$A_{21} = \begin{bmatrix} 1432 & 1_{241} \\ -23 & 1_{241} \end{bmatrix}.$

$A_{22}$ is a 2 by 2 matrix and

$A_{22} = \begin{bmatrix} 353234 & -1 \\ 2741 & 381 \end{bmatrix}.$

$A_{23}$ is a 2 by 2 matrix and

$A_{23} = \begin{bmatrix} 120 & 486 \\ 131 & 0 \end{bmatrix}.$

It is easy to check that $C$ has the property of row inner products, i.e.,

(i) the inner product of any two distinct rows of $C$ is equal to 50 ;

(ii) and the inner product of any rows with themselves of $C$ is equal to 65.

It follows that the property of

$C \cdot C^t = 15I_{243} + 50J_{243}.$

Thus $C$ is exactly the bordered matrix of the symmetric $(241, 16, 1)$ design if $A$ exists. But by case 8 of Main Theorem 1 the equation

$15z^2 = -x^2 + 50y^2$

must have a solution in integers, $x, y, z$, not all zero. It implies that, by Lemma 3.6, Lemma 3.7, Remark 3.8 and Remark 3.9, the Legendre symbol $(\frac{2}{5}) = 1$, which is a contradiction. So there does not exist finite projective plane of order 15.

**Theorem 5.5** There does not exist finite projective plane of order 18.

**Proof** In this case we can not use the Bruck-Ryser Theorem but can use case 1 of Main Theorem 1. Suppose that a symmetric $(343, 19, 1)$ design exists. Let $A$ be its
incidence matrix, which is a 343 by 343 matrix. Choose its bordered matrix $C$ is a 344 by 345 matrix as the following matrix.

$$C = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

$A_{11} = A$.

$A_{12}$ is a 343 by 2 matrix and

$$A_{12} = (6 \cdot 1^t_{343}, 0 \ast 1^t_{343})$$

$A_{21}$ is a 1 by 343 matrix and

$$A_{21} = \begin{bmatrix} -\frac{23}{355} \cdot 1_{343} \end{bmatrix}.$$

$A_{22}$ is a 1 by 2 matrix and

$$A_{22} = \begin{bmatrix} 2262 \cdot 18 \\ 5 \end{bmatrix}.$$

It is easy to check that $C$ has the property of row inner products, i.e.,

(i) the inner product of any two distinct rows of $C$ is equal to 37;

(ii) and the inner product of any rows with themselves of $C$ is equal to 55.

It follows that the property of

$$C \ C^t = 18I_{344} + 37J_{344}.$$

Thus $C$ is exactly the bordered matrix of the symmetric $(343, 19, 1)$ design if $A$ exists. But by case 1 of Main Theorem 1 we have 37 is a perfect square, which is a contradiction. So there does not exist finite projective plane of order 18. □

**Theorem 5.6** There does not exist finite projective plane of order 20.

**Proof** In this case we can not use the Bruck-Ryser Theorem but can use case 8 of Main Theorem 1. Suppose that a symmetric $(421, 21, 1)$ design exists. Let $A$ be its incidence matrix, which is a 421 by 421 matrix. Choose its bordered matrix $C$ is a 423 by 425 matrix as the following matrix.

$$C = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix},$$

$A_{11} = A$.

$A_{12}$ is a 421 by 2 matrix and

$$A_{12} = (8 \cdot 1^t_{421}, 0 \ast 1^t_{421}).$$
$A_{1\ 3}$ is a 421 by 2 matrix and
\[ A_{1\ 3} = (0 \ast 1_{421}^t, 0 \ast 1_{421}^t). \]

$A_{2\ 1}$ is a 2 by 421 matrix and
\[ A_{2\ 1} = \begin{bmatrix} -\frac{549047}{30808725} & \cdot 1_{421} \\ \frac{257419}{2231} & \cdot 1_{421} \end{bmatrix}. \]

$A_{2\ 2}$ is a 2 by 2 matrix and
\[ A_{2\ 2} = \begin{bmatrix} 83919088 & -\frac{2}{25} \\ \frac{8977928}{237419} & 210 \end{bmatrix}. \]

$A_{2\ 3}$ is a 2 by 2 matrix and
\[ A_{2\ 3} = \begin{bmatrix} 3808 & 23614 \\ 5625 & 5625 \\ 0 & 0 \end{bmatrix}. \]

It is easy to check that $C$ has the property of row inner products, i.e.,
(i) the inner product of any two distinct rows of $C$ is equal to 65;
(ii) and the inner product of any rows with themselves of $C$ is equal to 85.

It follows that the property of
\[ C\ C^t = 20I_{423} + 65J_{423}. \]

Thus $C$ is exactly the bordered matrix of the symmetric $(421, 21, 1)$ design if $A$ exists. But by case 8 of Main Theorem 1 the equation
\[ 20z^2 = -x^2 + 65y^2 \]

must have a solution in integers, $x, y, z,$ not all zero. It implies that, by Lemma 3.6, Lemma 3.7, Remark 3.8 and Remark 3.9, the Legendre symbol $(\frac{13}{7}) = 1$, which is a contradiction. So there does not exist finite projective plane of order 20.

**Theorem 5.7** There does not exist finite projective plane of order 24.

**Proof** In this case we can not use the Bruck-Ryser Theorem but can use case 8 of Main Theorem 1. Suppose that a symmetric $(601, 25, 1)$ design exists. Let $A$ be its incidence matrix, which is a 601 by 601 matrix. Choose its bordered matrix $C$ is a 603 by 605 matrix as the following matrix.
\[ C = \begin{bmatrix} A_{1\ 1} & A_{1\ 2} \\ A_{2\ 1} & A_{2\ 2} \end{bmatrix}, \]
\[ A_{11} = A. \]

\( A_{12} \) is a 601 by 4 matrix and
\[ A_{12} = (1 \cdot 1_{601}^T, 0 \ast 1_{601}^T, 0 \ast 1_{601}^T, 0 \ast 1_{601}^T). \]

\( A_{21} \) is a 2 by 601 matrix and
\[ A_{21} = \begin{bmatrix} \frac{3}{25} \cdot 1_{601} \\ \frac{13}{185} \cdot 1_{601} \end{bmatrix}. \]

\( A_{22} \) is a 2 by 4 matrix and
\[ A_{22} = \begin{bmatrix} -1 & 46 & 18 & 0 \\ 9 & -\frac{284}{37} & \frac{168}{37} & 0 \end{bmatrix}. \]

It is easy to check that \( C \) has the property of row inner products, i.e.,

(i) the inner product of any two distinct rows of \( C \) is equal to 2;

(ii) the inner product of any rows with themselves of \( C \) is equal to 26.

It follows that the property of
\[ C \cdot C^t = 24I_{603} + 2J_{603}. \]

Thus \( C \) is exactly the bordered matrix of the symmetric \((601, 24, 1)\) design if \( A \) exists. But by case 8 of Main Theorem 1 the equation
\[ 24z^2 = -x^2 + 2y^2 \]

must have a solution in integers, \( x, y, z \), not all zero. It implies that, by Lemma 3.6, Lemma 3.7, Remark 3.8 and Remark 3.9, the Legendre symbol \((\frac{2}{3}) = 1\), which is a contradiction. So there does not exist finite projective plane of order 24. \( \blacksquare \)

**Theorem 5.8** There does not exist finite projective plane of order 26.

**Proof** In this case we can not use the Bruck-Ryser Theorem but can use case 1 of Main Theorem 1. Suppose that a symmetric \((703, 27, 1)\) design exists. Let \( A \) be its incidence matrix, which is a 703 by 703 matrix. Choose its bordered matrix \( C \) is a 704 by 705 matrix as the following matrix.
\[ C = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \]
\[ A_{11} = A. \]
$A_{1 \ 2}$ is a 703 by 2 matrix and

$$A_{1 \ 2} = (3 \cdot 1_{703}^t, 0 \ast 1_{703}^t).$$

$A_{2 \ 1}$ is a 1 by 703 matrix and

$$A_{2 \ 1} = \left[ -\frac{4}{17} \cdot 1_{703} \right].$$

$A_{2 \ 2}$ is a 1 by 2 matrix and

$$A_{2 \ 2} = \left[ \frac{526}{17}, \frac{100}{21} \right].$$

It is easy to check that $C$ has the property of row inner products, i.e.,

(i) the inner product of any two distinct rows of $C$ is equal to 10;

(ii) and the inner product of any rows with themselves of $C$ is equal to 36.

It follows that the property of

$$C \ C^t = 26I_{704} + 10J_{704},$$

and $C$ is exactly the bordered matrix of the symmetric $(703, 27, 1)$ design if $A$ exists. But by case 1 of Main Theorem 1 we have that 10 is a perfect square, which is a contradiction. So there does not exist finite projective plane of order 26. \hfill \square

**Theorem 5.9** There does not exist finite projective plane of order 28.

**Proof** In this case we can not use the Bruck-Ryser Theorem but can use case 8 of Main Theorem 1. Suppose that a symmetric $(813, 29, 1)$ design exists. Let $A$ be its incidence matrix, which is a 813 by 813 matrix. Choose its bordered matrix $C$ is a 815 by 817 matrix as the following matrix.

$$C = \left[ \begin{array}{cc} A_{1 \ 1} & A_{1 \ 2} \\ A_{2 \ 1} & A_{2 \ 2} \end{array} \right],$$

$$A_{1 \ 1} = A.$$

$A_{1 \ 2}$ is a 813 by 4 matrix and

$$A_{1 \ 2} = (1 \cdot 1_{813}^t, 2 \ast 1_{813}^t, 0 \ast 1_{813}^t, 0 \ast 1_{813}^t).$$

$A_{2 \ 1}$ is a 2 by 813 matrix and

$$A_{2 \ 1} = \left[ \frac{1}{7} \cdot 1_{813}, \frac{291}{2990} \cdot 1_{813} \right].$$
$A_{2,2}$ is a 2 by 4 matrix and

$$A_{2,2} = \begin{bmatrix} -23 & 18 & 0 & 0 \\ 5991 & 3 & 336 & 287 \\ 2590 & 14 & 336 & 287 \end{bmatrix}.$$ 

It is easy to check that $C$ has the property of row inner products, i.e.,

(i) the inner product of any two distinct rows of $C$ is equal to 6;

(ii) and the inner product of any rows with themselves of $C$ is equal to 34.

It follows that the property of

$$C \cdot C^t = 28I_{815} + 6J_{815}.$$ 

So $C$ is exactly the bordered matrix of the symmetric $(813,29,1)$ design if $A$ exists. But by case 8 of Main Theorem 1 the equation

$$28z^2 = -x^2 + 6y^2$$

must have a solution in integers, $x,y,z$, not all zero. It implies that, by Lemma 3.6, Lemma 3.7, Remark 3.8 and Remark 3.9, the Legendre symbol $\left(\frac{6}{7}\right) = 1$, which is a contradiction. So there does not exist finite projective plane of order 28. $\square$

**Proof of Main Theorem 2**  By the above theorems we finish the proof of Main Theorem 2. $\square$  

### 6  Proof of Main Theorem 3

In order to use the main theorems to show that symmetric designs with certain parameters cannot exist, we must show that the corresponding bordered matrix exist and the corresponding equation has no integral solution.

**Theorem 6.1** There does not exist symmetric $(49,16,5)$ design.

**Proof** In this case we can not use the Bruck-Ryser-Chowla Theorem but can use case 8 of Main Theorem 1. Suppose that a symmetric $(49,16,5)$ design exists. Let $A$ be its incidence matrix, which is a 49 by 49 matrix. Choose its bordered matrix $C$ is a 51 by 53 matrix as the following matrix.

$$C = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix},$$

$$A_{1,1} = A.$$
$A_{1\ 2}$ is a 49 by 4 matrix and

$$A_{1\ 2} = (1 \ast 1_{49}^t, 0 \ast 1_{49}^t, 0 \ast 1_{49}^t, 0 \ast 1_{49}^t).$$

$A_{2\ 1}$ is a 2 by 49 matrix and

$$A_{2\ 1} = \begin{bmatrix}
\frac{1}{3} \cdot 1_{49} \\
\frac{144}{425} \cdot 1_{49}
\end{bmatrix}.$$

$A_{2\ 2}$ is a 2 by 4 matrix and

$$A_{2\ 2} = \begin{bmatrix}
\frac{2}{3} & \frac{10}{3} & 0 & 0 \\
\frac{86}{425} & -\frac{2}{125} & \frac{422}{425} & \frac{6787}{2125}
\end{bmatrix}.$$

It is easy to check that $C$ has the property of row inner products, i.e.,

(i) the inner product of any two distinct rows of $C$ is equal to 6 ;

(ii) and the inner product of any rows with themselves of $C$ is equal to 17.

It follows that

$$C \ C^t = 11 I_{51} + 6 J_{51},$$

and $C$ is exactly the bordered matrix of the symmetric $(49, 16, 5)$ design if $A$ exists. But by case 8 of main Theorem 1 the equation

$$11z^2 = -x^2 + 6y^2$$

must have a solution in integers, $x, y, z$, not all zero. It implies that, by Lemma 3.6, Lemma 3.7, Remark 3.8 and Remark 3.9, the Legendre symbol $(6 \mid 11^t) = 1$, which is a contradiction. So there does not exist symmetric $(49, 16, 5)$ design. \hfill \Box

Theorem 6.2 There does not exist symmetric $(154, 18, 2)$ design.

Proof In this case we can not use the Bruck-Ryser Theorem but can use case 8 of main Theorem 1. Suppose that a symmetric $(154, 18, 2)$ design exists. Let $A$ be its incidence matrix, which is a 154 by 154 matrix. Choose its bordered matrix $C$ is a 155 by 157 matrix as the following matrix.

$$C = \begin{bmatrix}
A_{1\ 1} & A_{1\ 2} \\
A_{2\ 1} & A_{2\ 2}
\end{bmatrix},$$

$$A_{1\ 1} = A.$$

$A_{1\ 2}$ is a 154 by 3 matrix and

$$A_{1\ 2} = (1 \ast 1_{154}^t, 0 \ast 1_{154}^t, 0 \ast 1_{154}^t).$$
$A_{2 \, 1}$ is a 1 by 154 matrix and
\[
A_{2 \, 1} = \left[ \frac{3}{20} \cdot 1_{154} \right].
\]

$A_{2 \, 2}$ is a 1 by 3 matrix and
\[
A_{2 \, 2} = \left[ \frac{3}{10} \, 47 \, 63 \, 20 \right].
\]

It is easy to check that $C$ has the property of row inner products, i.e.,
(i) the inner product of any two distinct rows of $C$ is equal to 3;
(ii) and the inner product of any rows with themselves of $C$ is equal to 19.

It follows that
\[
C \, C^t = 16I_{155} + 3J_{155},
\]
and $C$ is exactly the bordered matrix of the symmetric $(154, 18, 2)$ design if $A$ exists.

But by case 8 of Main Theorem 1 the equation
\[
16z^2 = -x^2 + 3y^2
\]
must have a solution in integers, $x, y, z$, not all zero. It implies that, by Lemma 3.6, Lemma 3.7, Remark 3.8 and Remark 3.9, the Legendre symbol $(\frac{-1}{3}) = 1$, which is a contradiction. So there does not exist symmetric $(154, 18, 2)$ design. \hfill \Box

**Theorem 6.3** There does not exist symmetric $(115, 19, 3)$ design.

**Proof** In this case we can not use the Bruck-Ryser Theorem but can use case 1 of Main Theorem 1. Suppose that a symmetric $(115, 19, 3)$ design exists. Let $A$ be its incidence matrix, which is a 115 by 115 matrix. Choose its bordered matrix $C$ is a 116 by 117 matrix as the following matrix.

\[
C = \left[ \begin{array}{cc} A_{1 \, 1} & A_{1 \, 2} \\ A_{2 \, 1} & A_{2 \, 2} \end{array} \right],
\]

$A_{1 \, 1} = A$.

$A_{1 \, 2}$ is a 115 by 2 matrix and
\[
A_{1 \, 2} = (3 \ast 1_{115}^t, 0 \ast 1_{115}^t).
\]

$A_{2 \, 1}$ is a 1 by 115 matrix and
\[
A_{2 \, 1} = \left[ \frac{3}{7} \cdot 1_{115} \right].
\]

$A_{2 \, 2}$ is a 1 by 2 matrix and
\[
A_{2 \, 2} = \left[ \frac{8}{7} \, 16 \, 7 \right].
\]
It is easy to check that $C$ has the property of row inner products, i.e.,

(i) the inner product of any two distinct rows of $C$ is equal to 12;

(ii) and the inner product of any rows with themselves of $C$ is equal to 28.

It follows that

$$C \ C^t = 16I_{116} + 12J_{116},$$

and $C$ is exactly the bordered matrix of the symmetric $(115,19,3)$ design if $A$ exists. But by case 1 of Main Theorem 1 we have that 12 is a perfect square, which is a contradiction. So there does not exist any symmetric $(115,19,3)$ design.  

\(\boxdot\)

**Proof of Main Theorem 3** By the above theorems we finish the proof of Main Theorem 3.  

\(\boxdot\)

7 Concluding remarks

We conclude the discussion on block designs by mentioning the very short proof of the Bruck-Ryser-Chowla theorem on the existence of symmetric block designs, which is motivated at least in part by the matrix equation of set intersections\cite{18}. Let $A$ be the incidence matrix of the symmetric $(v,k,\lambda)$ design. Ryser dealt only with the case of symmetric $(v,k,\lambda)$ designs with $v$ odd. The criterion for $v$ even is elementary. He formed the following bordered matrix of order $v + 1$\cite{18}

$$A^* = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

(39)

where $A_{11} = A$, $A_{12}$ is a column vector $1_v^t$, $A_{21}$ is a row vector $1_v$ and $A_{22} = \frac{k}{\lambda}$. He also defined the following diagonal matrices $D$ and $E$ of order $v + 1$

$$D = \text{diag}(l, \cdots, 1, -\lambda), \ E = \text{diag}(k - \lambda, \cdots, k - \lambda, -\frac{k - \lambda}{\lambda}).$$

Then it follows that the matrices $D, E, \text{ and } A^*$ are interrelated by the equation

$$A^*DA^* = E.$$

Thus the existence of the symmetric $(v,k,\lambda)$ design implies that the diagonal matrices $D$ and $E$ of order $v + 1$ are congruent to one another over the field of rational numbers. The remainder of the argument proceeds along standard lines and utilizes the Witt cancellation law. He just gave a new proof and did not obtain new necessary conditions on the existence of symmetric $(v,k,\lambda)$ designs.
In this paper we consider the bordered matrix $C$ of the symmetric $(v, k, \lambda)$ design with preserving some row inner product property for some positive integer $l$, which is different from the above one, such that

$$C = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

$$CC^t = (k - \lambda)I_w + (\lambda + l)J_w. \quad (40)$$

where $A_{11} = A$, $A_{12}$, $A_{21}$ and $A_{22}$ are submatrices over $\mathbb{Q}$.

The matrix equation (40) is of fundamental importance. But it is difficult to deal with this matrix equation in its full generality. In this paper $C$ maybe nonsquare matrix. The equation (40) implies positive definite matrix $(k - \lambda)I_w + (\lambda + l)J_w$ of order $w$ is quasi-congruent to the identity matrix of order $w + d$ with plus $d$ over the field of rational numbers. The equation (40) certainly contains much more information than (39). The difficulty lies in utilizing this information in an effective manner. So the bordered matrix of $C$ of the symmetric $(v, k, \lambda)$ design, which preserves some row inner product property for some positive integer $l$, is just considered more property of $(0, 1)$-matrix. Let $d$ be the difference between the number of columns and the number of rows of $C$ in (40). If $d > 2$, then we do not obtain the Diophantine equations of Legendre type. Thus in this paper we just consider that $d$ is 1 or 2. This has been the key breakthrough since 1950.

It was proved by a computer search that there does not exist any projective plane of order 10 by Lam, C.W.H., Thiel, L. and Swiercz, S. This is not the first time that a computer has played an important role in proving a theorem. A notable earlier example is the four-color theorem. It is easy to construct the above bordered matrix by the computer using Lemma 3.6, Lemma 3.7, Remark 3.8 and Remark 3.9 in Maple in Theorem 5.1. It should be remarked that one does not need to trust the computer blindly. Although the proofs are discovered by the computer, it produces a proof certificate that can easily be checked by hand, if so desired. So we obtain a proof in the traditional mathematical sense for nonexistence of finite projective plane of order 10 and some other cases.

For Problem 2.7 or Conjecture 2.12 Lam’s algorithm[13] is an exponential time algorithm. But the proof of main Theorem 1 is just the Ryser-Chowla elimination procedure in [3]. Thus our algorithm is a polynomial time algorithm. It fully reflects our algorithm high efficiency.
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