Coalescence Hidden Variable Fractal Interpolation Functions and its Smoothness Analysis

A.K.B.CHAND AND G.P.KAPOOR
Department of Mathematics,
Indian Institute of Technology Kanpur,
Kanpur 208016,India
Email: chand@iitk.ac.in ; gp@iitk.ac.in
Phone: 91-512-2597609
Fax : 91-512-2597500

ABSTRACT : We construct a coalescence hidden variable fractal interpolation function(CHFIF) through a non-diagonal iterated function system(IFS). Such a FIF may be self-affine or non-self-affine depending on the parameters of the defining non-diagonal IFS. The smoothness analysis of the CHFIF has been carried out by using the operator approximation technique. The deterministic construction of functions having order of modulus continuity \( O\left( \log \log t \right)^m \) (m a non-negative integer and \( 0 < \frac{1}{m} \)) is possible through our CHFIF. The bounds of fractal dimension of CHFIFs are obtained first in certain critical cases and then, using estimation of these bounds, the bounds of fractal dimension of any FIF are found.

KEYWORDS : FIF, IFS, Coalescence, Hidden Variable, Self-affine, Non-self-affine, Operator approximation, Smoothness analysis, Fractal dimension.

AMS Classification : 28A80,65D05,37C25,41A35,26A16,37L30.

1. INTRODUCTION

The fractal curves arise during several applications in various disciplines such as Natural Science [1–4], Engineering Applications [5], Economics [6] etc. To approximate these curves, Barnsley [7, 8] constructed a fractal interpolating function (FIF) arising from a suitable iterated function system (IFS). FIFs are generally self-affine in nature and the Hausdorff-Besicovitch dimensions of their graphs are non-integers. To approximate non-self-affine patterns, the hidden

\footnote{The present research is partially supported by CSIR Grant No: 9/92(160)/98-EMR-I, India.}
variable FIFs (HFIFs) are constructed in [8–10] by projection of vector valued FIF from generalized interpolation data. However, in practical applications of FIF, the interpolation data might be generated simultaneously from self-affine and non-self-affine functions. Thus, the question whether it is possible to construct an IFS that is capable of generating both of the self-affine or non-self-affine FIFs simultaneously needs to be settled. The hidden variable bivariate fractal interpolation surfaces are studied in [11] by introducing the concept of constrained free variables. In the present work a Coalescence Hidden Variable FIF (CHFIF) that is self-affine or non-self-affine depending on the parameters of defining IFS is constructed.

Since FIFs are continuous but generally nowhere differentiable functions, their analysis can not be done satisfactorily by restricting to classical analytic tools. For the applications of FIF theory, in general, an expansion of the FIF in terms of a suitable function system is usually considered. Barnsley and Harrington [12] used shifted composition to express affine FIFs and computed their fractal dimensions. However, this representation is somewhat difficult to use. Zhen [13] gave another series representation of self-affine FIF through a new function to study the Hölder property of FIF. Since, the function has too many points of discontinuity, it is slightly tedious to analyze it in applications. Zhen and Gang [14] expanded equidistant FIF on $[0;1]$ by using Haar-wavelet function system and obtained their global Hölder property, when the number of interpolation points is $N = 2^p + 1$, $p$ being a definite positive integer. Gang [15] employed the technique of operator approximation to characterize the Hölder continuity of self-affine FIFs on a general set of nodes on $[0;1]$. Bedford [16] obtained the Hölder exponent $h$ of a self-affine fractal function that has non-linear scaling, using code space of $n$ symbols associated with the IFS. He also showed the existence of a larger Hölder exponent $h$ defined at almost every point with respect to Lebesgue measure. The distribution of points where the FIF has strongest singularity is found by Maslyuk [17] that helps in calculating the parameters of an IFS with aid of wavelet-based techniques, such as modulus maxima lines tracing. The Hölder exponent needed in smoothness analysis of non-self-affine FIF is not yet studied due to interdependence of the components of vector valued FIF in the construction of HFIFs.

It is seen in the present paper that, contrary to the observation of Barnsley [8] that ‘the graph of HFIF is not self-similar or self-affine or self-anything’, CHFIF is indeed self-affine under certain conditions even though the class of CHFIFs is a subclass of the class of HFIFs. Our approximation of CHFIF is obtained through an operator found with integral averages on each subinterval of the FIF. Using this approximation, the Hölder exponent of the non-self-affine
functions arising from IFS is found for the first time. The bounds of Fractal dimension of the CHFIF in critical cases obtained in the present paper help to calculate the bounds of Fractal dimension of any FIF by converting the CHFIF to a self-affine FIF.

The organization of the paper is as follows: In Section 2, we construct a coalescence hidden variable FIF. For this purpose, an IFS is constructed in $\mathbb{R}^3$ with the introduction of constrained free variable. The projection of the attractor of our IFS on $\mathbb{R}^2$ is a CHFIF or a self-affine FIF depending upon choices of hidden variables. The Hölder continuity of CHFIFs (both self-affine and non-self-affine) is investigated in Section 3 by using the operator approximation technique. The bounds on fractal dimension of CHFIFs in critical cases are obtained in Section 4. The results found in the present work through Sections 2-4 are illustrated in Section 5 with the help of suitably chosen examples.

2. CONSTRUCTION OF CHFIF

2.1. Construction of IFS for CHFIF

Let the interpolation data be $f(x_i; y_i) \in \mathbb{R}^2 : i = 0; 1; 2; \ldots; N$ $g_i$ where $1 < x_0 < x_1 < \ldots < x_i 1$. For constructing an interpolation function $f_1 : [x_0; x_N]$ $\in \mathbb{R}$ such that $f_1(x_1) = y_1$ for all $i = 0; 1; 2; \ldots; N$; consider a generalized set of data $f(x_i; y_i; z_i) \in \mathbb{R}^3$ $i = 0; 1; 2; \ldots; N$ $g_i$ where $z_i$ $i = 0; 1; 2; \ldots; N$ are real parameters. The following notations are used throughout the sequel: $I = [x_0; x_N]$ $i = [x_{i-1}; x_i]$; $g_i = M \in \gamma_i$; $g_2 = M \in \gamma_i$; $h_1 = M \in z_i$; $h_2 = M \in z_i$ and $K = I \setminus D$; where $D = J_1 \cup J_2$; $J_1 \cup J_2$ are suitable compact sets in $\mathbb{R}$ such that $[g_1; g_2]$ $[h_1; h_2]$ $D$; Let $L_1 : I \setminus I_i$ be a contractive homeomorphism and $F_1 : K \setminus I$ be a continuous vector valued function such that

$$L_1(x_0) = x_1 \quad \text{and} \quad x_1 = \left. \frac{\partial L_1}{\partial x} \right|_{x_0} = 0$$

$$F_1(x_0; y_0; z_0) = (y_1; z_1); \quad F_1(x_N; y_N; z_N) = (y_i; z_i)$$

and

$$d(F_1(x; y; z); F_1(x; y; z)) = \sum_{i=1}^N d_e((y; z); (y; z))$$

for all $i = 1; 2; \ldots; N$ where, $c$ and $s$ are positive constants with $0 < s < 1$; $(x; y; z); (x; y; z); (x; y; z) \in K$, $d$ is the sup. metric on $K$ and $d_e$ is the Euclidean metric on $\mathbb{R}^2$. For defining the required CHFIF, the functions $L_1$ and $F_1$ are chosen to be of the form $L_1(x) = a_1 x + b_1$ and

$$F_1(x; y; z) = A_1(y; z)^T + \left( \varphi_1(x); \varphi_2(x) \right)^T$$

(2.3)
where, \( A_1 \) is an upper triangular matrix and \( p_i(x), q_i(x) \) are continuous functions having at least two unknowns. We choose \( j \) as free variable with \( j_1 < 1 \) and \( j \) as constrained free variable with respect to \( j \) such that \( j_1 + j \leq 1 \). The generalized IFS that is needed for construction of CHFIF corresponding to the data \( f(x; y; z) \) is now defined as

\[
f(x; y; z) = (L_1(x); F_1(x; y; z)); i = 1, 2, \ldots, N \quad (2.4)
\]

It is shown in the sequel that projection of the attractor of IFS (2.4) on \( R^2 \) is the desired CHFIF.

### 2.2. Existence and Uniqueness of CHFIF

It is known [9] that the IFS defined in (2.4) associated with the data \( f(x; y; z) \) is hyperbolic with respect to a metric \( d \) on \( R^3 \) equivalent to the Euclidean metric. In particular, there exists a unique nonempty compact set \( G \subset R^3 \) such that

\[
G = \bigcup_{i=1}^{N} \Gamma_i(G)
\]

The following proposition gives the existence of a unique vector valued function \( f \) that interpolates the generalized interpolation data and also establishes that the graph of \( f \) equals the attractor \( G \) of the generalized IFS:

**Proposition 2.1.** The attractor \( G \) (c.f. (2.5)) of the IFS defined in (2.4) is the graph of the continuous vector valued function \( f : I \to D \) such that \( f(x_i) = (y_i; z_i) \) for all \( i = 1, 2, \ldots, N \) i.e. \( G = \{ f(x; y; z) : x \in I, f(x) = (y(x); z(x)) \} \)

**Proof.** Consider the family of functions, \( F = ff : I \to D \) and \( f(x_0) = (y_0; z_0); f(x_N) = (y_N; z_N) \) where, \( \sup_k f(x) = \sup_k g(x) \) denotes the Euclidean norm on \( R^2 \). Then, \( F; f \) is a complete metric space. Now, for \( x \in I \), define the Read-Bajraktarević operator \( T \) on \( F; f \) as

\[
(T f)(x) = F_1(L_{1}^{-1}(x); y(L^{-1}_{1}(x)); z(L^{-1}_{1}(x)))
\]

For \( f \in F; f \) using (2.1), \( (T f)(x_0) = F_1(L_{1}^{-1}(x_0); y(L^{-1}_{1}(x_0)); z(L^{-1}_{1}(x_0))) = F_1(x_0; y_0; z_0) = (y_0; z_0) \): Similarly, \( (T f)(x_N) = (y_N; z_N) \). The function \( T f \) is clearly continuous on each of the subinterval \( (x_1; x_i) \) for \( i = 1, 2, \ldots, N \). Also, from (2.1), it follows that \( T f(x_1) = T f(x_1^{-}) \) for
each $i$. Consequently, $TF$ is continuous on $I$; Thus, $TF \in F$ : This proves that $T$ maps $F$ into itself.

Next, we prove that $T$ is a contraction map on $F$: For $f \in F$; define $y_f(x)$; $z_f(x)$ as the $y$-value and $z$-value of the vector valued function $TF$ at $x$. Let $f,g \in F$ and $x \in I_i$; Then,

$$\langle T f; T g \rangle = \sup_{x \in I_i} f(x) \cdot T g(x) \cdot kg$$

$$= \sup_{x \in I_i} f(y_f(L^{-1}_i(x))) \cdot y_g(L^{-1}_i(x)) + g(z_f(L^{-1}_i(x))) \cdot z_g(L^{-1}_i(x)) \cdot kg$$

$$s \sup_{x \in I_i} f(y_f(L^{-1}_i(x))) \cdot y_g(L^{-1}_i(x)) + g(z_f(L^{-1}_i(x))) \cdot z_g(L^{-1}_i(x)) \cdot kg$$

$$= \langle f; g \rangle$$

where, in view of the conditions on $i$; $j$; $k$ in Section 2.1, $s = \max_{i, j, k} f_j \cdot j \cdot g_j \cdot k < 1$. This shows that $T$ is a contraction mapping. By fixed point theorem, $T$ has a unique fixed point, i.e. there exists a unique vector valued function $f \in F$ such that for all $x \in I_i$; $\langle T f \rangle(x) = f(x)$:

Now, for all $i = 1; 2; \cdots; N$

$$f(x_i) = \langle T f \rangle(x_i) = F_i(x_i; y_i; z_i) = y_i(L^{-1}_i(x_i)) + z_i(L^{-1}_i(x_i)) = F_{i+1}(x_0; y_0; z_0) = (y_i; z_i);$$

which establishes that $f$ is the function interpolating the data $f(x_i; y_i; z_i)$ $ji = 0, 1; \cdots; N$.

It remains to show that the graph $G$ of the vector valued function $f$ is the attractor of the IFS defined in (2.4). To this end, observe that for all $x \in I_i$; $i = 1, 2, \cdots, N$; and $f \in F$;

$$\langle T f \rangle(L^{-1}_i(x)) = F_i(x; y; z) = (y_i + e_i x + c_i ; z_i + d_i + e_i x + f_i)$$

and

$$!_i(x; y; z) = (L^{-1}_i(x) ; F_i(x; y; z)) = (L^{-1}_i(x) ; T f(L^{-1}_i(x))) = (L^{-1}_i(x) ; f(L^{-1}_i(x)))$$

which implies that $G$ satisfies the invariance property, i.e. $G = \bigcup_{i=1}^{N} !_i(G)$. Since the nonempty compact set that satisfies the invariance property is unique, it follows that $G = \mathcal{G}$: This proves $G$ is the graph of the vector valued function $f$ such that $G \equiv f(x; y; z) \in \mathcal{C}$.

Let the vector valued function $f : I \to \mathbb{R}^2$ in Proposition 2.1 be written as $f(x) = (f_1(x); f_2(x))$. The required CHFIF is now defined as follows:
Definition 2.1. Let \( f(x; f_1(x)) : x \to \mathbb{R}^2 \) be the projection of the attractor \( G \) (c.f. (2.5)) on \( \mathbb{R}^2 \). Then, the function \( f_1(x) \) is called coalescence hidden variable FIF (CHFIF) for the given interpolation data \( f(x_i; y_i) \) where \( i = 0; 1; \ldots; N \).

Remark 2.1. 1. Although, the attractor \( G \) (c.f. (2.5)) of the IFS defined in (2.4) is a union of affine transformations of itself, the projection of the attractor is not always union of affine transformations of itself. Hence, CHFIFs are generally non-self-affine in nature.

2. By choosing \( y_i = z_i \) and \( i + j = 1 \); CHFIF \( f_1(x) \) obtained as the projection on \( \mathbb{R}^2 \) of the attractor of the IFS (2.4) coincides with a self-affine FIF \( f_2(x) \) for the same interpolation data. Hence, the CHFIF is self-affine in this case, in contrast to the observation of Barnsley [8] that the graph of a HFIF is not self-similar or self-affine or self-anything.

3. For a given set of interpolation data, if an extra dimension is added to construct the CHFIF, we have 1 free variable in the 3\(^{rd}\) co-ordinate whereas, in the 2\(^{nd}\) co-ordinate, we have 1 free variable and 1 constrained variable. In the case \( y_i = z_i \), the resulting scaling factor of CHFIF is \( i + j \). As \( j_i j < 1 \) and \( j_i j + j_i j < 1 \), taking \( j_i j < \) for sufficiently small \( \), the scaling factor of the CHFIF is found to lie between \( 2^* \) and \( 2^* \).

4. If extra \( n \) dimensions are added to interpolation data to get the CHFIF, \( (n + 2)^{th} \) co-ordinate has 1 free variable, \( (n + 1)^{th} \) co-ordinate has 1 free variable and may have at most 1 constrained free variable, \( n^{th} \) co-ordinate has 1 free variable and may have at most 2 constrained free variables, \( \ldots \). Continuing, the \( 2^{nd} \) co-ordinate has 1 free variable and may have at most \( n \) constrained free variables. So, in this extension, there are \( n \) free variables and at most \( 1 + 2 + 3 + \ldots + n \) free variables in the CHFIF. Due to the restrictions on free variables and constrained free variables, the scaling factor of the CHFIF lies between \( (n + 1)^* \) and \( (n + 1) \). Thus, one can expect a wider range of CHFIFs in higher dimension extensions.

3. SMOOTHNESS ANALYSIS OF CHFIF

In this section, the smoothness of CHFIFs are studied by using their operator approximations. The Hölder exponent of CHFIFs are calculated in the proof of our main Theorems 3.1-3.3. We take the interpolation data on X-axis as \( 0 = x_0 < x_1 < \ldots < x_N = 1 \); Let the function \( F_1 \) of the IFS (2.4) be of the form

\[
F_1(x; y; z) = (i y + i z + p_i(x); j z + q_i(x))
\]
where \( j \cdot j < 1 \); \( j \cdot j + j \cdot j < 1 \); \( p_{i} \cdot \| \cdot \| (0 < i < 1) \) and \( q_{i} \cdot \| \cdot \| (0 < i < 1) \): From (2.6) and (3.1), for \( \mathbf{x} \in \mathbb{I}_{0} \), the fixed point \( f \) of \( T \) satisfies

\[
T f (x) = E_{1} (L_{1}^{-1} (x); f_{1} (L_{1}^{-1} (x)); f_{2} (L_{1}^{-1} (x)))
\]

\[
f (x) = E_{1} (L_{1}^{-1} (x); f_{1} (L_{1}^{-1} (x)); f_{2} (L_{1}^{-1} (x)))
\]

\[
(f_{1} (x); f_{2} (x)) = (i f_{1} (L_{1}^{-1} (x)) + i f_{2} (L_{1}^{-1} (x)) + p_{1} (L_{1}^{-1} (x)); i f_{2} (L_{1}^{-1} (x)) + q_{1} (L_{1}^{-1} (x))
\]

Consequently, for all \( \mathbf{x} \in \mathbb{I}_{0} \);

\[
(f_{1} (L_{1} (x)); f_{2} (L_{1} (x))) = (i f_{1} (x) + i f_{2} (x) + p_{1} (x); i f_{2} (x) + q_{1} (x))
\]

Following Proposition 2.1, the CHIF in this case can be written as

\[
f_{1} (L_{1} (x)) = i f_{1} (x) + i f_{2} (x) + p_{1} (x)
\]

(3.2)

where, the self-affine fractal function \( f_{2} (x) \) is given by

\[
f_{2} (L_{1} (x)) = i f_{2} (x) + q_{1} (x)
\]

(3.3)

Let \( \mathbf{I}_{R_{1}} = [x_{R_{1}}; \mathbf{x}_{R_{1}}] = L_{R_{1}} (\mathbb{I}) \): Then, \( \mathbf{I}_{R_{1}} \cdot L_{R_{1}} (0) + \mathbb{J}_{R_{1}} \mathbb{J} ; \) where \( \mathbb{J}_{R_{1}} \cdot \mathbf{x}_{R_{1}} \) is the length of \( \mathbf{I}_{R_{1}} \); \( R_{1} \cdot N \): Similarly, \( \mathbf{I}_{R_{1}} \cdot L_{R_{2}} (0) + \mathbb{J}_{R_{2}} \mathbb{J} ; \) where \( \mathbb{J}_{R_{1}} \cdot \mathbf{I}_{R_{2}} = \mathbb{J}_{R_{1}} \cdot \mathbb{J} \cdot \mathbb{J} \cdot \mathbb{J} \cdot \mathbb{J} ; \) \( \mathbb{J}_{R_{2}} \) is the length of \( \mathbf{I}_{R_{2}} \); \( R_{2} \cdot N \): In general,

\[
\mathbf{I}_{R_{1} \cdot R_{2} \cdot \cdots \cdot R_{n}} = L_{R_{n}} (0) + \mathbb{J}_{R_{n}} \mathbb{J}_{R_{1} \cdot R_{2} \cdot \cdots \cdot R_{n-1}} = L_{R_{n}} L_{R_{n-1}} \cdots L_{R_{1}} (\mathbb{I}) = L_{R_{1} \cdot R_{2} \cdot \cdots \cdot R_{n}}
\]

(3.4)

where, \( \mathbb{J}_{R_{1} \cdot R_{2} \cdot \cdots \cdot R_{n}} = \mathbb{J}_{R_{1}} \mathbb{J} \mathbb{J}_{R_{1}} \mathbb{J}_{R_{2}} \cdots \mathbb{J}_{R_{n}} \) and \( R_{1} \cdot R_{2} ; \cdots ; R_{n} \cdot N \):

We need the following lemmas for our main results:

**Lemma 3.1.** Let \( f_{1} \) be defined as in (3.2) and \( f_{2} (x) = \frac{R}{I_{R_{1} \cdot R_{2} \cdot \cdots \cdot R_{n}}} f_{1} (x) \cdot dx \); Then,

\[
E_{R_{1} \cdot R_{2} \cdot \cdots \cdot R_{n}} = \frac{1}{k! \cdot (j+1)} \int_{I_{R_{1} \cdot R_{2} \cdot \cdots \cdot R_{n}}} \int_{I_{R_{1} \cdot R_{2} \cdot \cdots \cdot R_{n}}} \cdots \int_{I_{R_{1} \cdot R_{2} \cdot \cdots \cdot R_{n}}} f_{1} (x) \cdot dx
\]

(3.5)

where, \( I_{R_{0}} = \mathbf{I} \) and \( a_{R_{1} \cdot R_{2} \cdot \cdots \cdot R_{n}} = \frac{R}{I_{R_{1} \cdot R_{2} \cdot \cdots \cdot R_{n}}} f_{2} (x) \cdot dx \):
Lemma 3.2. \( f_1(x)dx \), a change of the variable \( x \) by \( x = L_{\infty} (0) + \mathcal{L}_{m} j \) gives

\[
\mathcal{B}_{1,2} = \int_{I_1} f_1 \left( L_{\infty} (0) + \mathcal{L}_{m} j \right) \mathcal{L}_{m} j \, dx = \mathcal{L}_{m} j \int_{I_1} f_1 \left( L_{\infty} (0) \right) \, dx
\]

The proof follows immediately by using Mean Value Theorem.

Since \( f_1(x) \) is continuous, the integral average \( \mathcal{B}_{1,2} = \mathcal{L}_{m} j \) can be taken as a good approximation of \( f_1(x) \) in the subinterval \( I_{1,2} \), when \( m \) is very large, leading to the following definition of the approximating operator \( Q_m \) on the interval \( I \):

Definition 3.1. Let

\[
Q_m (f_1;x) = \int_{I_1} f_1 \, dx
\]

where, \( I_{1,2} \) is defined by (3.4), \( b_{1,2} \) is defined by (3.5) and

\[
Q_m (\varphi) = \begin{cases} 
1 & \text{if } I_{1,2} = I \\
0 & \text{if } I_{1,2} \subset I \
\end{cases}
\]

Lemma 3.2. The operator \( Q_m (f_1;x) \), given by (3.6), converges to \( f_1(x) \) uniformly on \( I \) as \( m \to \infty \).

Proof. The proof follows immediately by using Mean Value Theorem.

The following notations are needed throughout in the sequel:

\[
= m \text{ axf}_{j} j : i = 1;2;\cdots;N \ g; \quad = m \text{ axf}_{j} j : i = 1;2;\cdots;N \ g; \quad = m \inf_{i} : \]
follows that of the above two intervals, we assume that \( i = 3 \) operator approximations \( 3.2 \), it is sufficient to find an upper bound on the difference between functional values of their \( i \) magnitude of \( i \).

Let Theorem 3.1.

Proof. In view of the above lemmas and notations, we now prove our smoothness results according to the magnitude of 

Theorem 3.1. Let \( f_1(x) \) be the CHFIF defined by (3.2) with \( \theta > 1 \). Then, (a) for \( \theta > 1 \) and \( \theta = 1, f_1 \) 2 Lip \( (b) for \) \( \theta = 1 \) or \( \theta > 1 \). \( (f_1,t) = (\theta \log \theta) \). for suitable values of 2 \( (0;1) \)

For 0 \( x < x \) \( 1 \), there exists a least \( m \), such that \( Q_{m-\theta} \) is the largest interval contained in \( (x;x) \). So, either \( x \) or \( x \) \( 2 I_{1} \) : Assume that \( x \) \( 2 I_{1} \) \( 1 \); \( x \) \( 2 I_{2} \) \( 1 \); \( \theta = 1 \) or \( \theta \) \( > 1 \). \( t \) \( \theta > 1 \) : \( N \) : Let \( n > m \). \( N \) and \( n > m \) : Taking further refinement of the above two intervals, we assume that \( x \) \( 2 I_{1} \) \( 1 \); \( x \) \( 2 I_{2} \) : It now follows that

\[
Q_{n}(f_1;x) = \frac{1}{\prod_{i=1}^{n} (\prod_{j=1}^{r} \prod_{k=1}^{m} a_{i}^{a_{x_1}}) \prod_{k=1}^{m} a_{x_1}} \int_{0}^{1} f_1(x) dx
\]

\[
= \frac{1}{\prod_{i=1}^{n} (\prod_{j=1}^{r} \prod_{k=1}^{m} a_{i}^{a_{x_1}}) \prod_{k=1}^{m} a_{x_1}} \int_{0}^{1} (f_1)^{r_1} \prod_{j=1}^{r} a_{x_1}^{a_{x_1}} \int_{0}^{1} (f_1)^{r_1} \prod_{j=1}^{r} a_{x_1}^{a_{x_1}}
\]

\[
= \prod_{i=1}^{n} (\prod_{j=1}^{r} \prod_{k=1}^{m} a_{i}^{a_{x_1}}) \prod_{k=1}^{m} a_{x_1} \int_{0}^{1} f_1(x) dx
\]

\[
= \prod_{i=1}^{n} (\prod_{j=1}^{r} \prod_{k=1}^{m} a_{i}^{a_{x_1}}) \prod_{k=1}^{m} a_{x_1} \int_{0}^{1} f_1(x) dx
\]
Similarly, the expression for \( Q_n(f_1;x) \) can be written as

\[
Q_n(f_1;x) = X^m \begin{pmatrix} Y \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} Z \end{pmatrix} \begin{pmatrix} P_k(\cdots)d \end{pmatrix} + \begin{pmatrix} a_{v_1};n \end{pmatrix} \begin{pmatrix} tr^2 ; \cdots ; tr_k \end{pmatrix} \begin{pmatrix} \sum_{j=1}^{k} a_{v_1};n \end{pmatrix} \begin{pmatrix} tr^2 ; \cdots ; tr_k \end{pmatrix} \begin{pmatrix} f_1(\cdots)d \end{pmatrix} + \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} Y \end{pmatrix} \begin{pmatrix} Z \end{pmatrix} \begin{pmatrix} P_k(\cdots)d \end{pmatrix}
\]

To estimate \( Q_n(f_1;x) \), observe that

\[
Q_n(f_1;x) = X^m \begin{pmatrix} Y \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} Z \end{pmatrix} \begin{pmatrix} P_k(\cdots)d \end{pmatrix} + \begin{pmatrix} a_{v_1};n \end{pmatrix} \begin{pmatrix} tr^2 ; \cdots ; tr_k \end{pmatrix} \begin{pmatrix} \sum_{j=1}^{k} a_{v_1};n \end{pmatrix} \begin{pmatrix} tr^2 ; \cdots ; tr_k \end{pmatrix} \begin{pmatrix} f_1(\cdots)d \end{pmatrix} + \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} Y \end{pmatrix} \begin{pmatrix} Z \end{pmatrix} \begin{pmatrix} P_k(\cdots)d \end{pmatrix}
\]

(3.7)

Since [15],

\[
a_{e_1 e_2} = \begin{pmatrix} f_2(\cdots)d \end{pmatrix}
\]

it follows that

\[
\begin{align*}
\frac{a_{u_1};n}{J_{u_1};n} & = \frac{1}{J_{u_1};n} \begin{pmatrix} f_2(\cdots)d \end{pmatrix} \\
& = \frac{1}{J_{u_1};n} \begin{pmatrix} \sum_{j=1}^{k} a_{v_1};n \end{pmatrix} \begin{pmatrix} f_2(\cdots)d \end{pmatrix} + \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} Y \end{pmatrix} \begin{pmatrix} Z \end{pmatrix} \begin{pmatrix} P_k(\cdots)d \end{pmatrix}
\end{align*}
\]
Similarly,
\[
\frac{a_{v_1 \cdots v_n} \cdot x_{j+1} \cdots x_{k_l}}{J(v_1 \cdots v_n \cdot x_2 \cdots x_{k_{l-1}})} = \frac{X^1}{Z} \cdot \frac{Y^1}{Z} \cdot \frac{1}{J(v_1 \cdots v_n \cdot x_2 \cdots x_{k_{l-1}})} \cdot \frac{\alpha_1(\cdot d}{J(v_1 \cdots v_n \cdot x_2 \cdots x_{k_{l-1}})}
\]
\[
+ (r_1) \frac{f_2(\cdot d}{J(v_1 \cdots v_n \cdot x_2 \cdots x_{k_{l-1}})}
\]
Consequently,
\[
\frac{j_{a_{v_1 \cdots v_n} \cdot x_{j+1} \cdots x_{k_l}}}{J(u_1 \cdots u_n \cdot x_{j+1} \cdots x_{k_l})} = \frac{X^1}{Z} \cdot \frac{Y^1}{Z} \cdot \frac{1}{J(u_1 \cdots u_n \cdot x_{j+1} \cdots x_{k_l})} \cdot \frac{\alpha_1(\cdot d}{J(u_1 \cdots u_n \cdot x_{j+1} \cdots x_{k_l})}
\]
\[
+ (r_1) \frac{f_2(\cdot d}{J(u_1 \cdots u_n \cdot x_{j+1} \cdots x_{k_l})}
\]
\[
\frac{1}{J(u_1 \cdots u_n \cdot x_{j+1} \cdots x_{k_l})} \cdot \frac{\alpha_1(\cdot d}{J(u_1 \cdots u_n \cdot x_{j+1} \cdots x_{k_l})}
\]

where, \( M_1 \) is Lipschitz bound and \( M_2 = 2k_2k_1 \): Using (3.8) in (3.7),
\[
\frac{\gamma_n}{\gamma_n} \cdot \frac{\gamma_n}{\gamma_n} \cdot \frac{X^m}{Y^n} \cdot \frac{Y^n}{Z} \cdot \frac{Z}{J(u_1 \cdots u_n \cdot x_{j+1} \cdots x_{k_l})} \cdot \frac{\alpha_1(\cdot d}{J(u_1 \cdots u_n \cdot x_{j+1} \cdots x_{k_l})}
\]
\[
+ (r_1) \frac{\alpha_1(\cdot d}{J(u_1 \cdots u_n \cdot x_{j+1} \cdots x_{k_l})}
\]
\[
\frac{X^m}{Y^n} \cdot \frac{Y^n}{Z} \cdot \frac{Z}{J(u_1 \cdots u_n \cdot x_{j+1} \cdots x_{k_l})} \cdot \frac{\alpha_1(\cdot d}{J(u_1 \cdots u_n \cdot x_{j+1} \cdots x_{k_l})}
\]
\[
+ (r_1) \frac{\alpha_1(\cdot d}{J(u_1 \cdots u_n \cdot x_{j+1} \cdots x_{k_l})}
\]
where, $M_3$ is Lipschitz bound and $M_4 = \frac{2k\ell_1k_1}{\ell}$.
From the above inequality it follows that

$$
\begin{align*}
\mathcal{Q}_n (f_1; x) & \quad Q_n (f_1; x) j \quad M_3 \quad \left( \begin{array}{c}
\begin{array}{c}
X^n \\
\ell
\end{array}
\end{array}\right) \times \left( \begin{array}{c}
\begin{array}{c}
\ell
\end{array}
\end{array}\right) + \frac{M_3}{\ell} \mathcal{J}_{x_j} \mathcal{Q} \times Q_k + M_4 \left( \begin{array}{c}
\begin{array}{c}
\ell
\end{array}
\end{array}\right) \\
\mathcal{Q}_n (f_1; x) & \quad Q_n (f_1; x) j \quad M_3 \quad \left( \begin{array}{c}
\begin{array}{c}
\ell
\end{array}
\end{array}\right) \times \left( \begin{array}{c}
\begin{array}{c}
\ell
\end{array}
\end{array}\right) + \frac{M_3}{\ell} \mathcal{J}_{x_j} \mathcal{Q} \times Q_k
\end{align*}
$$

Therefore,

$$
\begin{align*}
\mathcal{Q}_n (f_1; x) & \quad Q_n (f_1; x) j \quad M_5 \quad \left( \begin{array}{c}
\begin{array}{c}
X^n \\
\ell
\end{array}
\end{array}\right) \times \left( \begin{array}{c}
\begin{array}{c}
\ell
\end{array}
\end{array}\right) + \frac{M_3}{\ell} \mathcal{J}_{x_j} \mathcal{Q} \times Q_k + M_6 \left( \begin{array}{c}
\begin{array}{c}
\ell
\end{array}
\end{array}\right) \\
\mathcal{Q}_n (f_1; x) & \quad Q_n (f_1; x) j \quad M_5 \quad \left( \begin{array}{c}
\begin{array}{c}
X^n \\
\ell
\end{array}
\end{array}\right) \times \left( \begin{array}{c}
\begin{array}{c}
\ell
\end{array}
\end{array}\right) + \frac{M_3}{\ell} \mathcal{J}_{x_j} \mathcal{Q} \times Q_k
\end{align*}
$$

where, $M_5 = M \cdot \frac{\ell}{(\ell_n \in \mathbb{J})}$ and $M_6 = M \cdot \frac{\ell}{(\ell_n \in \mathbb{J})}$. The above inequality gives

$$
\begin{align*}
\mathcal{Q}_n (f_1; x) & \quad Q_n (f_1; x) j \quad M_5 \quad \left( \begin{array}{c}
\begin{array}{c}
X^n \\
\ell
\end{array}
\end{array}\right) \times \left( \begin{array}{c}
\begin{array}{c}
\ell
\end{array}
\end{array}\right) + \frac{M_3}{\ell} \mathcal{J}_{x_j} \mathcal{Q} \times Q_k + M_6 \left( \begin{array}{c}
\begin{array}{c}
\ell
\end{array}
\end{array}\right) \\
\mathcal{Q}_n (f_1; x) & \quad Q_n (f_1; x) j \quad M_5 \quad \left( \begin{array}{c}
\begin{array}{c}
X^n \\
\ell
\end{array}
\end{array}\right) \times \left( \begin{array}{c}
\begin{array}{c}
\ell
\end{array}
\end{array}\right) + \frac{M_3}{\ell} \mathcal{J}_{x_j} \mathcal{Q} \times Q_k
\end{align*}
$$

Since $\mathcal{Q}_n (f_1; x) < 1$ (3.9) further reduces to

$$
\begin{align*}
\mathcal{Q}_n (f_1; x) & \quad Q_n (f_1; x) j \quad M_5 \quad \left( \begin{array}{c}
\begin{array}{c}
X^n \\
\ell
\end{array}
\end{array}\right) \times \left( \begin{array}{c}
\begin{array}{c}
\ell
\end{array}
\end{array}\right) + \frac{M_3}{\ell} \mathcal{J}_{x_j} \mathcal{Q} \times Q_k + M_6 \left( \begin{array}{c}
\begin{array}{c}
\ell
\end{array}
\end{array}\right) \\
\mathcal{Q}_n (f_1; x) & \quad Q_n (f_1; x) j \quad M_5 \quad \left( \begin{array}{c}
\begin{array}{c}
X^n \\
\ell
\end{array}
\end{array}\right) \times \left( \begin{array}{c}
\begin{array}{c}
\ell
\end{array}
\end{array}\right) + \frac{M_3}{\ell} \mathcal{J}_{x_j} \mathcal{Q} \times Q_k
\end{align*}
$$

**Case (a).** $\theta \geq 1$ and $\theta \leq 1$: The desired Hölder exponents are found individually for each of the following subcases

I. $< 1$ and $< 1$ \hspace{1cm} \begin{align*}
\mathcal{Q}_n (f_1; x) & \quad Q_n (f_1; x) j \quad M_5 \quad \left( \begin{array}{c}
\begin{array}{c}
X^n \\
\ell
\end{array}
\end{array}\right) \times \left( \begin{array}{c}
\begin{array}{c}
\ell
\end{array}
\end{array}\right) + \frac{M_3}{\ell} \mathcal{J}_{x_j} \mathcal{Q} \times Q_k + M_6 \left( \begin{array}{c}
\begin{array}{c}
\ell
\end{array}
\end{array}\right) \\
\mathcal{Q}_n (f_1; x) & \quad Q_n (f_1; x) j \quad M_5 \quad \left( \begin{array}{c}
\begin{array}{c}
X^n \\
\ell
\end{array}
\end{array}\right) \times \left( \begin{array}{c}
\begin{array}{c}
\ell
\end{array}
\end{array}\right) + \frac{M_3}{\ell} \mathcal{J}_{x_j} \mathcal{Q} \times Q_k
\end{align*}

where, $M_5 = m \cdot \frac{\ell}{(\ell_n \in \mathbb{J})}$ and $M_6 = \frac{\ell}{(\ell_n \in \mathbb{J})}$. Thus, as $n \to 1$, the above inequality together with Lemma 3.2 gives $f_1 \in L^p$ with $\theta = 1$.

II. $> 1$ and $> 1$ \hspace{1cm} \begin{align*}
\mathcal{Q}_n (f_1; x) & \quad Q_n (f_1; x) j \quad M_5 \quad \left( \begin{array}{c}
\begin{array}{c}
X^n \\
\ell
\end{array}
\end{array}\right) \times \left( \begin{array}{c}
\begin{array}{c}
\ell
\end{array}
\end{array}\right) + \frac{M_3}{\ell} \mathcal{J}_{x_j} \mathcal{Q} \times Q_k + M_6 \left( \begin{array}{c}
\begin{array}{c}
\ell
\end{array}
\end{array}\right) \\
\mathcal{Q}_n (f_1; x) & \quad Q_n (f_1; x) j \quad M_5 \quad \left( \begin{array}{c}
\begin{array}{c}
X^n \\
\ell
\end{array}
\end{array}\right) \times \left( \begin{array}{c}
\begin{array}{c}
\ell
\end{array}
\end{array}\right) + \frac{M_3}{\ell} \mathcal{J}_{x_j} \mathcal{Q} \times Q_k
\end{align*}

(3.10)
Suppose $1 > 0$ such that $\lambda_x x_j m = \lambda_x x_j$. Then,

$$1 + \frac{m \log \log \lambda_x x_j}{\log \lambda_x x_j} x_j$$

(3.12)

Further, $\lambda_{f_{1:x}}\lambda_{1} x_j x_j < 1$ implies $\lambda_{m\in\mathcal{J}} \lambda_{f_{1:x}} x_j x_j \frac{1}{m \log \lambda_{f_{m\in\mathcal{J}}}} x_j x_j$.

Also, $\frac{\log \log \log \lambda_{f_{m\in\mathcal{J}}}}{\log \lambda_{f_{m\in\mathcal{J}}}}$ is such that $\frac{1}{\log \lambda_{f_{m\in\mathcal{J}}}} x_j x_j x_j$.

Therefore, by (3.12), $x_j x_j = \frac{1}{\log \lambda_{f_{m\in\mathcal{J}}}} x_j x_j$.

Similarly, if $2 > 0$ is such that $\lambda_x x_j m = \lambda_x x_j$, then $x_j x_j = \frac{1}{\log \lambda_{f_{m\in\mathcal{J}}}} x_j x_j$.

Let $3 = m \in \mathcal{J}$, $\frac{\log \log \log \lambda_{f_{m\in\mathcal{J}}}}{\log \lambda_{f_{m\in\mathcal{J}}}}$, $\mathcal{Q}_{n} (f_{1;x}) Q_{n} (f_{1;x}) j M_{5} \lambda_x x_j + \frac{M_{6}}{2} \lambda_x x_j$.

Hence, as $n > 1$, the last inequality together with Lemma 3.2 gives $f_{1} \log \lambda_{m \in \mathcal{J}}$ with $= 3$.

IV. $< 1$ and $1 > 1$ implies $\mathcal{Q}_{n} (f_{1;x}) Q_{n} (f_{1;x}) j M_{10} \lambda_x x_j x_j$.

where, $M_{10} = m \in \mathcal{J}$, $\frac{M_{6}}{2} \lambda_x x_j x_j$.

Thus, as $n > 1$, the above inequality together with Lemma 3.2 gives $f_{1} \log \lambda_{m \in \mathcal{J}}$ with $= 4$.

Case (b). $= 1$ or $= 1$: The desired Hölder exponents are found individually for each of the following subcases

I. $= 1$ and $1$ or $< 1$ and $= 1$:

$$\mathcal{Q}_{n} (f_{1;x}) Q_{n} (f_{1;x}) j \emptyset M_{5} \lambda_x x_j + \frac{M_{6}}{2} \lambda_x x_j \emptyset (m \in \mathcal{J})$$

$$= \emptyset M_{5} \lambda_x x_j + \frac{M_{6}}{2} \lambda_x x_j \emptyset \frac{\log \lambda_x x_j}{\log \lambda_{f_{m\in\mathcal{J}}}}$$

$$M_{11} \lambda_x x_j x_j x_j$$

where, $M_{11} = \frac{M_{6}}{2} \lambda_x x_j$. As $n > 1$, the last inequality together with Lemma 3.2 gives $(f_{1};t) = (\lambda x j \log \lambda j)$ with $= 1$. For $= 1$ and $< 1$, $\mathcal{Q}_{n} (f_{1;x}) Q_{n} (f_{1;x}) j M_{12} \lambda_x x_j x_j x_j$.

where $M_{12} = m \in \mathcal{J}$, $\frac{M_{6}}{2} \lambda_x x_j x_j x_j$.

Hence, as $n > 1$, the above inequality together with Lemma 3.2 gives $(f_{1};t) = (\lambda x j \log \lambda j) (\lambda x j \log \lambda j)$ with $= 1$. The estimate for $< 1$ and $= 1$ follows using analogous arguments.

II. $> 1$ and $= 1$:

$$\mathcal{Q}_{n} (f_{1;x}) Q_{n} (f_{1;x}) j M_{13} \lambda_x x_j x_j x_j$$

where $M_{13} = m \in \mathcal{J}$, $\frac{M_{6}}{2} \lambda_x x_j x_j x_j$.

Making $n > 1$, the above
inequality together with Lemma 3.2 gives \((f_1; t) = (j^3 (1 + \log j) + \log j) = 3\).

III. \(1 \leq j \leq 1 : \square_n (f_1; x) \quad \square_n (f_1; x) j \quad M_5 x j \quad x^j (\frac{m}{\log \frac{x}{n}}) x^j M_6 x j x^j = 1\)

\(M_{14} x j x^j (1 + \log j x j); \text{where } M_{14} = m \text{axf} \frac{M_5}{\log \frac{x}{n}} x^j (1 + \log j x j). \text{ So, as } n \leq 1, \text{ the above inequality together with Lemma 3.2 gives } \(f_1; t) = (j^3 (1 + \log j); \quad (j^3 \log j) = 4\).

Theorem 3.1 now follows from the above cases with suitable values of \(j Q\) as found in various subcases. 

Theorem 3.2. Let \(f_1(x)\) be the CHFIF defined by (3.2) with \(j = 1\). Then, (a) for \(j = 1\) and \(j = 1\), \((f_1; t) = (j^3 (1 + \log j) + \log j) = 1\) or \(j = 1, \quad (f_1; t) = (j^3 (1 + \log j); j) = 2\) for suitable values of \(2 (0; 1)\).

Proof. Since \(j = 1, (3.9)\) gives,

\[\square_n (f_1; x) \quad \square_n (f_1; x) j \quad M_5 x j \quad x^j (\frac{m}{\log \frac{x}{n}}) x^j M_6 x j x^j = 1\]

The rest of proof is similar to that of Theorem 3.1 with the respective values of \(j Q\) as in different cases of Theorem 3.1.

Finally, the smoothness results for the class of CHFIFs for \(j > 1\) are given by the following:

Theorem 3.3. Let \(f_1(x)\) be the CHFIF defined by (3.2) with \(j > 1\). Then, (a) for \(j = 1\) and \(j = 2\) \(L \) \(j \) \(b) f o r \(j = 1\) or \(j = 1\), \((f_1; t) = (j^3 (1 + \log j); j) = 2\) for suitable values of \(2 (0; 1)\).

Proof. Inequality (3.9) for \(j > 1\) gives

\[\square_n (f_1; x) \quad \square_n (f_1; x) j \quad M_5 x j \quad x^j (\frac{m}{\log \frac{x}{n}}) x^j M_6 x j x^j = 1\]

Let \(a > 0\) be such that \(x j m \quad x j^3. \text{ Then,}\)

\[4 + \frac{m}{\log \frac{x}{n}} x j \quad \frac{\log \frac{x}{n}}{\log \frac{x}{n}^2} \quad x^j\]

Since \(1\) in Theorem 3.1 satisfies \(1 \quad \frac{\log \frac{x}{n}}{\log \frac{x}{n}^2}\), we can choose \(a = 1\) so that (3.14) reduces to

\[\square_n (f_1; x) \quad \square_n (f_1; x) j \quad M_5 x j \quad x^j (\frac{m}{\log \frac{x}{n}}) x^j M_6 x j x^j = 1\]
The rest of the proof is similar to that of Theorem 3.1 by considering (3.15) in place of (3.10). As in Theorem 3.1, the value of \( p \) in different cases are given by Case (a): I. \( p = m \ln (1 ; 1) \), II. \( p = 6 \) where, \( 6 = \frac{\log \ln \mathcal{J}^n_{m n}}{\log \ln \mathcal{J}^n_{m n}} \); III. \( p = \gamma \) where, \( \gamma = \frac{\log \ln \mathcal{J}^n_{m n}}{\log \ln \mathcal{J}^n_{m n}} \); IV. \( p = 8 = m \ln (1 ; 6) \); and Case (b): I. \( p = 5, II. = \gamma, III. = 8 \).

**Remark 3.1.** 1. It follows from Theorems 3.1-3.3 that the smoothness of the CHFIF depends on the free variables \( \varepsilon, \iota \) and the Lipschitz exponents \( \iota, \iota \).

2. If \( p_1(\varepsilon, \iota) \) and \( q_1(\varepsilon, \iota) \) belong to the same function space, then \( \iota = \iota \) (A) For \( \varepsilon = \iota ; \iota > 1 \); \( \iota = \iota \) (B) For \( \varepsilon < 1 \); \( \iota = \iota ; \iota = \iota \) (C) For \( \varepsilon > 1 \); \( \iota = \iota \) where \( \iota = \iota \) (A) For \( \varepsilon = \iota ; \iota > 1 \); \( \iota = \iota \) (B) For \( \varepsilon < 1 \); \( \iota = \iota ; \iota = \iota \) (C) For \( \varepsilon > 1 \); \( \iota = \iota \).

3. Let \( \varepsilon = \iota \) and \( \varepsilon < 1 \). Then, \( \varepsilon < 1 \) (A) For \( \varepsilon = \iota ; \iota > 1 \); \( \varepsilon = \iota \) (B) For \( \varepsilon < 1 \); \( \varepsilon = \iota \) (C) For \( \varepsilon > 1 \); \( \varepsilon = \iota \).

4. Suppose \( \varepsilon = \iota \) and \( \varepsilon < 1 \). Then, \( \varepsilon = \iota \) (A) For \( \varepsilon = \iota ; \iota > 1 \); \( \varepsilon = \iota \) (B) For \( \varepsilon < 1 \); \( \varepsilon = \iota \) (C) For \( \varepsilon > 1 \); \( \varepsilon = \iota \).

5. Let \( \varepsilon = \iota \) and \( \varepsilon < 1 \). Then, \( \varepsilon = \iota \) (A) For \( \varepsilon = \iota ; \iota > 1 \); \( \varepsilon = \iota \) (B) For \( \varepsilon < 1 \); \( \varepsilon = \iota \) (C) For \( \varepsilon > 1 \); \( \varepsilon = \iota \).

6. If \( \varepsilon_1(\varepsilon, \iota) = \varepsilon_2(\varepsilon, \iota) \); then \( \varepsilon_1(\varepsilon, \iota) \) is also self-affine and in such case, \( \varepsilon_1 = \varepsilon_2, \iota_1 = \iota_2 \) and \( \varepsilon_1(\varepsilon, \iota) = \varepsilon_2(\varepsilon, \iota) \).

**Remark 3.2.** In this case gives the smoothness result as follows: (A) For \( \varepsilon = \iota ; \iota > 1 \); \( \varepsilon = \iota \) (B) For \( \varepsilon < 1 \); \( \varepsilon = \iota \) (C) For \( \varepsilon > 1 \); \( \varepsilon = \iota \).

**Remark 3.3.** In the case \( \varepsilon = \iota ; \iota > 1 \); \( \varepsilon = \iota \) which gives \( \varepsilon = \iota ; \iota > 1 \); \( \varepsilon = \iota \) (C) For \( \varepsilon > 1 \); \( \varepsilon = \iota \).

Since, the class of CHFIFs for \( \varepsilon < 1 \) is contained in the class of CHFIFs for \( \varepsilon = 1 \) and \( \varepsilon > 1 \), the smoothness results in [15] for self-affine function \( \varepsilon_2(\varepsilon) \)
Follows as special case of our smoothness results derived in the above Remarks 3-5.

4. FRACTAL DIMENSION AND CHFIF

The following definitions are needed in the sequel: The conditions \( r = 1, s = 1 \) or \( t = 1 \) are called critical conditions. The CHFIF \( f_1(x) \) with any one of these conditions is called critical CHFIF. Let \( N(\beta) \) be the smallest number of closed balls of radius \( > 0 \) needed to cover \( A \). Then, the Fractal dimension of \( A \) is defined by \( D_B(\beta) = \lim_{\beta \to 0} \frac{\log N(\beta)}{\log \beta} \), whenever the limit exists. Our following theorems give bounds of the fractal dimension for the critical CHFIFs.

**Theorem 4.1.** Let CHFIF \( f_1(x) \) be defined by (3.2). Then, for the critical condition \( r = 1 \),

\[
\frac{\log \beta}{\log (\mathcal{J}_{m \mathcal{A}})} \sum_{j=1}^{k-1} \frac{\log j \cdot j}{j} D_B(\text{graph}(f_1)) 1 \frac{\log N}{\log \mathcal{J}_{m \mathcal{A}} j} (4.1)
\]

and for the critical condition \( s = 1 \);

\[
\frac{\log \beta}{\log (\mathcal{J}_{m \mathcal{A}})} \sum_{j=1}^{k-1} \frac{\log j \cdot j}{j} D_B(\text{graph}(f_1)) 1 \frac{\log N}{\log \mathcal{J}_{m \mathcal{A}} j} (4.2)
\]

where, takes suitable values as in the subcases in Theorems 3.1, 3.2

**Proof.** Let \( r = 1 \) and \( s = 1 \). Since \( (f_1 \cdot t) = (f_1 \cdot \log \mathcal{J}) \); (c.f. Theorem 3.1), for all \( x \in \mathcal{A} \), \( x; x \in 2 \mathcal{I} \); there exist constants \( C_1; C_2 \) such that

\[
C_1 jk \cdot j^i \cdot f_1(x) \cdot f_1(x) \cdot j \cdot j^i \cdot \log jk \cdot j (4.3)
\]

Suppose, \( G_{r_1, r_2; \cdots; r_m} = f(\xi; f_1(\xi); f_2(\xi)) \cdot j \cdot x \cdot I_{r_1, r_2; \cdots; r_m} \) \( g \). Define, \( \mathcal{A} j = \sup_j k \cdot x j \cdot j k (x, y; z) \); \( x, y, z \) \( 2 \mathcal{A} g \); \( \mathcal{A} x = \sup \mathcal{J} \cdot j \cdot j \cdot j (x, y; z); (x, y, z) \) \( 2 \mathcal{A} g \); for any \( A \) \( \mathbb{R}^3 \): Since, \( \mathcal{F}_{r_1, r_2; \cdots; r_m} \cdot j = \mathcal{F}_{r_1, r_2; \cdots; r_m} \cdot j (4.4) \) reduces to

\[
C_1 j^i \mathcal{F}_{r_1, r_2; \cdots; r_m} \cdot j^i \mathcal{F}_{r_1, r_2; \cdots; r_m} \cdot j^i \mathcal{F}_{r_1, r_2; \cdots; r_m} \cdot j (4.4)
\]

Choose \( m \) large such that \( \mathcal{J}_{m \mathcal{A}} j < \frac{1}{2} \); \( > 0 \); Since, \( r_j = 1 \) implies \( j \cdot j \cdot j \cdot j \cdot j \cdot j \); \( j \cdot j \cdot j \cdot j \cdot j \) it follows by (4.4) that

\[
C_1 j \cdot j \cdot j \cdot j \cdot j \cdot j \mathcal{F}_{r_1, r_2; \cdots; r_m} \cdot j \cdot j \cdot j \cdot j \cdot j \cdot j = j \cdot j \cdot j \cdot j \cdot j \cdot j (4.5)
\]
Theorem 4.2. Let CHFIF $f_1(x)$ be defined in (3.2) with $\alpha = 1$. Then, for $\beta \leq 1$ or $\beta \geq 1$,  
\[
\log \frac{\beta}{\log j_{m ax} j} \leq \log \frac{n}{\log j_{m ax} j} \leq \log \frac{N}{\log j_{m ax} j}
\]
where, $\beta$ takes suitable values as in Theorem 3.2.

Proof. The proof is similar to the case $\beta < 1$ of Theorem 4.1. 

Theorems 4.1, 4.2 lead to the following bounds on fractal dimension of equally spaced critical CHFIFs.
Corollary 4.1. Let CHFIF \( f_1(x) \) be defined by (3.2). Then, for \( \gamma = 1 \) or \( \gamma = 1 \),

\[
1 + \frac{\log \prod_{k=1}^{n} j_k j}{\log N} = \frac{D_B (\text{graph}(f_1))}{2} < 1.
\]

Further, for \( \gamma = 1 \),

\[
1 + \frac{\log \prod_{k=1}^{n} j_k j}{\log N} = \frac{D_B (\text{graph}(f_1))}{2} < 1.
\]

where, \( \gamma \) takes suitable values as in Theorems 3.1-3.3.

Corollary 4.2. Let the equidistant CHFIF \( f_1(x) \) be defined by (3.2). Then, \( D_B (\text{graph}(f_1)) = 1 \) in the following cases:

1. \( \gamma = 1 \), either \( \gamma = 1 \) or \( \gamma = 3 \), and either \( \prod_{k=1}^{n} j_k j = 1 \) or \( \prod_{k=1}^{n} j_k j = 1 \).
2. \( \gamma = 4 \), either \( \gamma = 1 \) or \( \gamma = 1 \), and \( \prod_{k=1}^{n} j_k j = 1 \).
3. \( \gamma = 1 \), \( \gamma = 2 \), and \( \prod_{k=1}^{n} j_k j = 1 \).
4. \( \gamma > 1 \), either \( \gamma = 5 \) or \( \gamma = 8 \), and \( \prod_{k=1}^{n} j_k j = 1 \).
5. \( \gamma > 1 \), and \( \prod_{k=1}^{n} j_k j = 1 \).

Remark 4.1. 1. In the critical case \( \gamma = 1 \); the fractal dimension bounds of CHFIF \( f_1(x) \) found in (4.2) coincide with the fractal dimension bounds of FIF \( f_2(x) \) found in [15], if \( f_1(x) \) is also self-affine.
2. Choosing \( \gamma \) suitably, \( \gamma = 1 \) for any self-affine CHFIF \( f_1 = f_2 \). In the resulting critical cases, the bounds on fractal dimension of any self-affine FIF \( f_1 \) can be found by (4.1) with suitable choice of hidden variables even if \( \gamma \not= 1 \).
3. In Corollary 4.2, critical CHFIF \( f_1(x) \) is considered as fractal function, since \( f_1(x) \) satisfies \( \gamma \prod_{k=1}^{n} j_k j \): Consequently, fractal functions having \( D_B (\text{graph}(f_1)) = 1 \) can be constructed by using this corollary.

5. EXAMPLES

Consider the interpolation data \( f(0.2),(0.35,7),(0.75,4),(1,9) \). Here, for simplicity, we construct affine CHFIFs. Since, in this case \( \gamma = \gamma = 1 \); it follows that \( \gamma = \gamma \) : In Figs. 1-3, the
generalized set of data is chosen such that \( z_i = y_i \) and in Figs. 4-16, the generalized set of data chosen such that \( z_i \notin Y_i \). The values of \( \alpha \) and \( \beta \) chosen for the computer generation of affine CHFIFs for all these figures are given in Table 1. Fig. 1 gives the self-affine CHFIF \( f_1(x) \) for the given interpolation data whenever \( \alpha + \beta = \gamma \). Fig. 2 and Fig. 3 show respectively the effect on the CHFIF for suitable choices of \( \alpha \), \( \beta \) and \( \gamma \) when the effective scaling factor is close to \( 2^+ \) and 2.

**Table 1**: Free variables and constrained free variables in the construction of affine CHFIFs

| Figures | 1  | 2  | 3  | 1  | 2  | 3  | 1  | 2  | 3  |
|---------|----|----|----|----|----|----|----|----|----|
| 1       | 0.8| 0.7| 0.3| -0.3| -0.4| -0.2| 0.5| 0.3| 0.6|
| 2       | 0.99| 0.99| 0.99| 0.99| 0.99| 0.99| 0.005| 0.005| 0.005|
| 3       | -0.999| -0.999| -0.999| -0.99| -0.99| -0.005| -0.005| -0.005| -0.005|
| 4       | 0.2| 0.38| 0.2| 0.4| 0.35| 0.5| 0.3| 0.3| 0.24|
| 5       | 0.2| 0.4| 0.22| 0.4| 0.35| 0.5| 0.35| 0.3| 0.2|
| 6       | 0.4| 0.3| 0.5| 0.4| 0.35| 0.5| 0.3| 0.5| 0.4|
| 7       | 0.2| 0.38| 0.2| 0.4| 0.35| 0.5| 0.3| 0.5| 0.4|
| 8       | 0.4| 0.3| 0.5| 0.4| 0.35| 0.5| 0.3| 0.5| 0.4|
| 9       | 0.2| 0.4| 0.22| 0.4| 0.35| 0.5| 0.3| 0.3| 0.24|
| 10      | 0.2| 0.38| 0.2| 0.4| 0.35| 0.5| 0.35| 0.3| 0.2|
| 11      | 0.2| 0.4| 0.22| 0.4| 0.35| 0.5| 0.3| 0.5| 0.4|
| 12      | 0.4| 0.3| 0.5| 0.4| 0.35| 0.5| 0.35| 0.3| 0.2|
| 13      | 0.2| 0.38| 0.2| -0.6| -0.45| -0.4| 0.3| 0.3| 0.24|
| 14      | 0.4| 0.3| 0.5| -0.6| -0.45| -0.4| 0.3| 0.5| 0.4|
| 15      | 0.4| 0.3| 0.5| -0.6| -0.45| -0.4| 0.3| 0.3| 0.24|
| 16      | 0.4| 0.3| 0.5| -0.6| -0.45| -0.4| 0.3| 0.5| 0.4|

Figs. 4-12 with the fixed values of \( \alpha \) (as in Table 1) and \( z_i \) \((3,1,8,5\) respectively for interpolation data points\) illustrate the nature of non-self-affine CHFIF, depending upon various cases of smoothness analysis in Theorems 3.1-3.3 when \( \beta = \gamma \). As expected, CHFIFs in these figures have the same type of shape since the underlying function spaces are independent of \( \alpha \) and \( \beta \): Figs. 13-15 give the effect of change in the values of \( \alpha \) (as in Table 1) on the shape of non-self-affine CHFIF. Fig. 16 shows the effect of change in the value of \( z_i \) \((7,9,10,8\) respectively...
for interpolation data points) on the shape of non-self-affine CHFIF. On comparing Fig. 4 with Fig. 13, Fig. 6 with Fig. 14 and Fig. 11 with Fig. 15, it is found that although CHFIFs are in the same function spaces, these are very much different in shape due to changes in the values of $i$ (as given in Table 1). The underlying function spaces are the same because these spaces depend only on the values of $i; i; i$ and $i$; $i$. Further, comparing Fig. 14 with Fig. 16, it is observed that by keeping all the other values fixed and changing only the values of $z_i$ from 3, 1, 8, 5 respectively to 7, 9, 10, 8 in generalized interpolation data, the shape of the CHFIF changes arbitrarily.

6. CONCLUSION

A generalized IFS is constructed in the present paper for generating coalescence hidden variable FIF. The existence and uniqueness of the CHFIF is proved by choosing suitable values of the variables $i; i; i$ and $i$ and the parameter $z_i$. Our IFS gives CHFIFs that may be self-affine or non-self-affine depending on free variables, constraints free variable and the parameters $z_i$. When construction of the CHFIF is carried out by adding $n$ dimensions linearly in generalized interpolation data, $(n + 1)$ free variables and at most $(1 + 2 + \ldots + n)$ constrained free variables can be chosen. If all of the extra $n$ dimensions take the same values of $z_i$, the scaling factor of the CHFIF lies between $(n + 1)^+$ and $(n + 1)^{-}$. Besides using the generalized IFS for construction of CHFIFs in the present work, it can also be used in other scientific applications to capture the self-affine and non-self-affine nature simultaneously for the relevant curves.

It is seen that the smoothness of CHFIF $f_i(\kappa)$ depends on free variables $i$ and $i$ as well as on the smoothness of $p_i(\kappa)$ and $q_i(\kappa)$. Although, $z_i$ and $i$ are responsible for the shape of the CHFIF, these are found not to affect its smoothness. In general, the deterministic construction of functions having order of modulus of continuity $0 (1 \log (1/\varepsilon)^m)$ $(m$ a non-negative integer, and $0 < 1)$ is possible through the CHFIF. The fact that CHFIFs are different in shape although they are in the same function spaces may enable considering them in more general function spaces such as Besov and Triebel-Lizorkin spaces apart from Lipschitz spaces. These former spaces have additional indices that ‘fine-tune’ a function. Our bounds of fractal dimension of CHFIFs are found in different critical conditions. Finally, it is proved that by suitable choices of the hidden variables, the fractal dimension bounds for any self-affine FIF can be found using the bounds obtained with the critical condition $= = 1$. 
REFERENCES

1. B. B. Mandelbrot, Fractals: From Chance and Dimension, W.H.Fremman (1977).
2. P. M. Iannaccone and M. Khokha, Fractal Geometry in Biological Systems, CRC Press (1996).
3. V. V. Mourzenko, J.-F. Thovert, and P. M. Adler, Percolation and conductivity of self-affine fractures, Phys. Rev. E, 59(4), 4265-4284 (1999).
4. J. Santamaria, M. E. Gómez, J. L. Vicent, K M. Krishnan, and I. K. Schuller, Scaling of the Interface Roughness in Fe-Cr Superlattices: Self-Affine versus Non-Self-Affine, Phys. Rev. Lett., 19, 190601 (2002).
5. J. L. Vehél, E. Lutton and C. Tricot, Fractals in Engineering: From Theory to Industrial Applications, Springer Verlag (1997).
6. Y. Kumagai, Fractal Structure of Financial High Frequency Data, Fractals, 10(1), 13-18 (2002).
7. M. F. Barnsley, Fractal Functions and Interpolations, Constr. Approx. 2, 303-329 (1986).
8. M. F. Barnsley, Fractals Everywhere, Academic Press, Orlando, Florida, (1988).
9. M. F. Barnsley, J. Elton, D. P. Hardin and P. R. Massopust, Hidden variable fractal interpolation functions, SIAM J. Math. Anal., 20(5), 1218-1242 (1989).
10. P. R. Massopust, Fractal Functions, Fractal Surfaces and Wavelets, Academic Press (1994).
11. A. K. B. Chand and G. P. Kapoor, Hidden Variable Bivariate Fractal Interpolation Surfaces. Fractals, 11(3), 277-288 (2003).
12. M.F.Barnsley and A.N.Harrington, The Calculus of Fractal Interpolation Functions, J. Approx. Theory, 57, 14-34 (1989).
13. S. Zhen, Hölder property of Fractal functions, Approx. Theory Appl., 4, 73-88 (1993).
14. S. Zhen, and C. Gang. Haar expansion of a class of fractal functions and their logical derivatives, Approx. Theory Appl., 4, 73-88 (1993).
15. C. Gang. The Smoothness and Dimension of Fractal Interpolation Functions. Appl-Math. J. Chinese Univ. Ser. B, 11, 409-428 (1996).
16. T.Bedford, Hölder exponents and box dimension for self-affine fractal functions, Constr. Approx., 5, 33-48 (1989).
17. L. I. Levkovich-Maslyuk, Wavelet based determination of generating matrices for fractal interpolation functions, Regular and Chaotic Dynamics, 3(2), 20-29, 1998.
Fig. 1 Self-affine CHFIF $f_1(x)$.  
Fig. 2 CHFIF $f_1(x)$ with scaling factor $2^*$.  
Fig. 3 CHFIF $f_1(x)$ with scaling factor 2.  
Fig. 4 CHFIF with $= < 1; < 1$.  
Fig. 5 CHFIF with $= 1; = 1$.  
Fig. 6 CHFIF with $> 1; > 1$.  
Fig. 7 CHFIF with $= < 1; > 1$.  
Fig. 8 CHFIF with $> 1; < 1$.  
Fig. 9 CHFIF with \( i_1 = 1; < 1 \)

Fig. 10 CHFIF with \( i_1 = < 1; = 1 \)

Fig. 11 CHFIF with \( i_1 = 1; > 1 \)

Fig. 12 CHFIF with \( i_1 = > 1; = 1 \)

Fig. 13 CHFIF with \( i_1 = < 1; < 1 \) with a different set of \( i_1 \)

Fig. 14 CHFIF with \( i_1 = > 1; > 1 \) with a different set of \( i_1 \)

Fig. 15 CHFIF with \( i_1 = > 1; < 1 \) with a different set of \( i_1 \)

Fig. 16 CHFIF with \( i_1 = > 1; > 1 \) with a different set of \( z_i \)