QHI Theory, I:
3-Manifolds Scissors Congruence Classes and Quantum Hyperbolic Invariants

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Abstract
For any triple \((W, L, \rho)\), where \(W\) is a closed connected and oriented 3-manifold, \(L\) is a link in \(W\) and \(\rho\) is a flat principal \(B\)-bundle over \(W\) (\(B\) is the Borel subgroup of \(SL(2, \mathbb{C})\)), one constructs a \(D\)-scissors congruence class \(c_D(W, L, \rho)\) which belongs to a (pre)-Bloch group \(P(D)\). The class \(c_D(W, L, \rho)\) may be represented by \(D\)-triangulations \(T = (T, H, D)\) of \((W, L, \rho)\). For any \(T\) and any odd integer \(N > 1\), one defines a “quantization” \(T_N\) of \(T\) based on the representation theory of the quantum Borel subalgebra \(W_N\) of \(U_q(sl(2, \mathbb{C}))\) specialized at the root of unity \(\omega_N = \exp(2\pi i/N)\). Then one defines an invariant state sum \(K_N(W, L, \rho) := K(T_N)\) called a quantum hyperbolic invariant (QHI) of \((W, L, \rho)\). One introduces the class of hyperbolic-like triples. They carry also a classical scissors congruence class \(c_I(W, L, \rho)\), that belongs to the classical (pre)-Bloch group \(P(I)\) and may be represented by explicit idealizations \(T_I\) of some \(D\)-triangulations \(T\) of a special type. One shows that \(c_I(W, L, \rho)\) lies in the kernel of a generalized Dehn homomorphism defined on \(P(I)\), and that it induces an element of \(H^3_{\mathbb{Z}}(PSL(2, \mathbb{C}); \mathbb{Z})\) (discrete homology). One proves that \(\lim_{N \to \infty} (2\pi i/N^2) \log[K_N(W, L, \rho) \rho] = G(W, L, \rho)\) essentially depends of the geometry of the ideal triangulations representing \(c_I(W, L, \rho)\), and one motivates the strong reformulation of the Volume Conjecture, which would identify \(G(W, L, \rho)\) with the evaluation \(R(c_I(W, L, \rho))\) of a certain refinement of the classical Rogers dilogarithm on the \(I\)-scissors class.

Keywords: quantum dilogarithm, hyperbolic 3-manifolds, scissors congruences, state sum invariants, volume conjecture.

1 Introduction

In a series of papers \([27]-[31]\), Kashaev proposed an infinite family \(\{K_N\}, N > 1\) being any odd positive integer, of conjectural complex valued topological invariants for pairs \((W, L)\), where \(L\) is a link in a closed connected and oriented 3-manifold \(W\). These invariants should be computed as a state sum \(K_N(T)\) supported by some kind of decorated triangulation \(T\) of \((W, L)\). The main ingredients of the state sum were
the quantum-dilogarithm 6j-symbols at the \(N\)-th-root of unity \(\omega = \exp(2\pi i/N)\), suitably associated to the decorated tetrahedra of \(\mathcal{T}\). On the “quantum” side, the state sum is reminiscent of the Turaev-Viro one \([1]\). On the “classical” side, it relates to the computation of the volume of non-compact finite volume hyperbolic 3-manifolds by the sum of the volumes of the ideal tetrahedra of any of their ideal triangulations. The decorated triangulations \(\mathcal{T}\) were not formally defined; anyway, it was clear that they should fulfill certain non trivial global constraints. Hence even their existence was not evident a priori. Beside this neglected existence problem, a main question left unsettled was the invariance of \(K_N(\mathcal{T})\) when \(\mathcal{T}\) varies. On the other hand, Kashaev proved the invariance of \(K_N(\mathcal{T})\) under certain “moves” on \(\mathcal{T}\), which we call decorated transits.

In \([28, 29]\) Kashaev also derived solutions of the Yang-Baxter equation from the pentagon identity satisfied by the quantum-dilogarithm 6j-symbols. Then, by means of this \(R\)-matrix and the usual planar link diagrams, he constructed a family \(Q_N\) of invariants for links in \(S^3\). It is commonly accepted that \(Q_N^3\) is an instance of \(K_N\), when \(W = S^3\) (for more details on this point, see \([1]\)). More recently, Murakami-Murakami \([40]\) have shown that \(Q_N\) actually equals a specific coloured Jones invariant \(J_N\), getting, by the way, another proof that it is a well-defined invariant for links in \(S^3\).

The so called Volume Conjecture \([30, 39, 40, 52]\) predicts that when \(L\) is a hyperbolic knot, one can recover the volume of \(S^3\setminus L\) from the asymptotic behaviour of \(J_N\) when \(N \to \infty\). If confirmed, it would establish a deep interaction between the 3-dimensional topological quantum field theory and the theory of geometric (hyperbolic) 3-manifolds.

The reformulation of Kashaev’s invariants for links in \(S^3\) within the main stream of Jones polynomials was an important achievement, but it also had the negative consequence of putting aside the original purely 3-dimensional and more geometric set-up for links in an arbitrary \(W\), willingly forgetting the complicated and somewhat mysterious decorations. In our opinion, this set-up deserved to be understood and developed, also in the perspective of finding the “right” conceptual framework for a reasonable general version of the Volume Conjecture. The present paper, which is the first of a series, establishes some fundamental facts of this program.

Notations. Let \(W\) be a closed, connected and oriented 3-manifold; \(L\) is a link in \(W\), considered up to ambient isotopy, \(U(L)\) is a tubular neighbourhood of \(L\) in \(W\), and \(M = W \setminus \text{Int}(U(L))\).

We denote by \(\rho\) a flat principal \(B\)-bundle on \(W\), considered up to isomorphisms of flat bundles, where \(B = B(2, \mathbb{C})\) is the Borel subgroup of upper triangular matrices of \(SL(2, \mathbb{C})\). Equivalently \(\rho\) can be interpreted as an element of \(\chi_B(W) = \text{Hom}(\pi_1(W), B)/B\), where \(B\) acts by inner automorphisms. Flat \(B\)-bundle isomorphisms over \(W\) that define homeomorphisms of pairs \((W, L)\) and preserve the orientation of \(W\) induce an equivalence relation on triples \((W, L, \rho)\).

We shall consider \((W, L, \rho)\) up to this relation.

Description of the paper. We first give a well-understood geometric definition of the decorated triangulations \(\mathcal{T} = (T, H, \mathcal{D})\), which we call \(\mathcal{D}\)-triangulations. Then we prove that every triple \((W, L, \rho)\) has \(\mathcal{D}\)-triangulations, and even full \(\mathcal{D}\)-triangulations. The \(\mathcal{D}\)-triangulations are distinguished triangulations of \((W, L)\), i.e. \(L\) is realized as a subcomplex of the triangulation \(T\) of \(W\), such that it contains all the vertices; “full” implies that any edge of \(T\) has distinct end-points. The \(\mathcal{D}\)-triangulations have a decoration \(\mathcal{D} = (b, z, c)\) that consists of three components: a branching \(b\) (which is a particular system of orientations on the edges of \(T\)), a \(B\)-valued 1-cocycle \(z\) on \(T\) representing the bundle \(\rho\), and an integral charge \(c\). The
Theorem 1.1 The element $c_\rho$ is an element of $P$ in the particular case when $P$ induce further algebraically independent relations in $P$. Hence, in order to prove Th. 1.1, we also have to show that the other moves do not induce any relations in $P$.

The topological invariants $K$ are defined on hyperbolic ideal tetrahedra. Every $P$-triangulation $\mathcal{T}$ of $(W, L, \rho)$ induces an element $c_\rho(\mathcal{T}) \in P$. We prove

**Theorem 1.2** The value of $K(\mathcal{T}_N)$ does not depend on the $P$-triangulation $\mathcal{T}$. Hence $K_N(\mathcal{T}) = K(\mathcal{T}_N)$ is a well-defined complex valued invariant of $(W, L, \rho)$, called a $P$-scissors congruence class.

The proofs of Th. 1.1 and 1.2 are similar, but there are some important differences. In order to connect different $P$-triangulations of a given triple via $P$-transits, one applies several kinds of local moves. On the other hand, we define the (pre)-Bloch group $P(D)$ by using only five-terms relations related to the specific $2 \to 3$ move. Hence, in order to prove Th. 1.1, we have to show that the other moves do not induce further algebraically independent relations in $P(D)$. Moreover, all the moves must be “positive”. Indeed, in general “negative” moves do not allow branching transits. To prove Th. 1.2, we are forced to use only full $P$-triangulations, and full $P$-transits (that is transits which preserve the fullness condition). This is rather demanding. However full $P$-triangulations carry special branchings associated to total orderings on the set of vertices of $T$, which transit without any problem, and this is a technical advantage.

In $\mathfrak{B}$, we state some properties of the QHI related to a change of the orientation of $W$, and to the important (still open) problem of understanding the QHI as functions of the $q$-argument. We show that in several cases one can extend the definition of the $P$-scissors class and of the QHI when the bundle $\rho$ is defined on $M = W \setminus U(L)$ but not necessarily on the whole of $W$. For this we use a so-called $q$-Dehn surgery along $L$, which is reminescent of Thurston’s hyperbolic Dehn surgery. It allows to modify $(W, L)$ in such a way that the bundle is defined on the whole of the surgered manifold, so that the results of the present paper apply. We refer to $\mathfrak{B}$ for a more extensive approach to this problem.

Since $K_N(\mathcal{T}) = K(\mathcal{T}_N)$ for any full representative $\mathcal{T}$ of $c_\rho(\mathcal{T})$, roughly speaking $K_N(\mathcal{T})$ may be considered as a function $K_N(c_\rho(\mathcal{T}))$ of the $P$-scissors congruence class (see $\mathfrak{B}$). A main problem in QHI theory is to determine the
nature of the asymptotic behaviour of $K_N(W, L, \rho)$ when $N \to \infty$, which should depend on the $D$-scissors class. One expects that this behaviour becomes more geometrically transparent when the triples $(W, L, \rho)$ carry some kind of hyperbolic structure. In this perspective, we define in $\S 3$ a version $\mathcal{P}(I)$ of Neumann’s classical extended (pre)-Bloch group $\hat{\mathcal{P}}(\mathbb{C})$ built on hyperbolic ideal tetrahedra $\mathbb{I}_2$. 12, 14, and we point out a remarkable specialization $\mathcal{P}(\mathbb{I}_D)$ of $\mathcal{P}(D)$ which maps onto $\mathcal{P}(I)$ via an explicit homomorphism called the idealization. Then, in $\S 3$ we introduce the so-called hyperbolic-like triples, whose $D$-scissors congruence class $c_D(W, L, \rho)$ belongs to $\mathcal{P}(\mathbb{I}_D)$. Using the idealization map, to any hyperbolic-like triple one can also associate a $I$-scissors congruence class $c_I(W, L, \rho) \in \mathcal{P}(I)$. It is represented by any $I$-triangulation $T_I$ obtained via the idealization of a $D$-triangulation $\mathcal{T}_D$ of a special kind. It turns out, in particular, that such a special $\mathcal{T}$ is decorated by a $Par(B)$-valued cocycle, where $Par(B)$ is the parabolic abelian subgroup of $B$. Apparently, $Par(B)$ plays a distinguished role in our approach to the volume conjecture for hyperbolic-like triples. A natural problem is to understand the relationship between hyperbolic-like and usual hyperbolic structures. In $\S 8$ we obtain a contribution in this direction by showing that $c_I(W, L, \rho)$ belongs to an enriched version $B(I)$ of the classical Bloch group $B(\mathbb{C})$; we define $B(I)$ as the kernel of a suitable refinement of the classical Dehn homomorphism. (For details on the “classical” notions, see e.g. 22, 42 and the references therein). Moreover, we show

**Theorem 1.3** For any hyperbolic-like triple $(W, L, \rho)$, the $I$-scissors congruence class $c_I(W, L, \rho)$ defines a cohomology class in $H^2_3(PSL(2, \mathbb{C}); \mathbb{Z})$ ($\delta$ for discrete homology).

In $\S 8$, we interpret the Volume Conjecture using the preceding results. Given a hyperbolic-like triangle $(W, L, \rho)$, let $\mathcal{T}$ and $T_I$ be as above. The state sum expression $K(\mathcal{T}_N)$ of $K_N(W, L, \rho)$ and the explicit idealization $T_I$ allow us to prove a “qualitative” part of the volume conjecture for hyperbolic-like triples:

$$\lim_{N \to \infty} \frac{1}{2\pi} \log[K_N(W, L, \rho)] = G(T_I).$$

This formally means that the limit is a function $G$ (a priori very complicated) which essentially depends on the geometry of the ideal tetrahedra of $T_I$; as $\mathcal{T}_N$ is arbitrary, the limit may be roughly considered as a function $G(c_I(W, L, \rho))$ of the $I$-scissors congruence class. At a qualitative level, this is just what any generalization of the volume conjecture would predict. It rests to properly “identify” the function $G$.

For this, consider a non-compact and finite volume hyperbolic 3-manifold $N$. Recall that the Chern-Simons invariant is an invariant of compact Riemannian manifolds with values in $\mathbb{R}/2\pi^2 \mathbb{Z}$ 18, 17, 42, 10, 53, which was extended to non-compact finite volume hyperbolic 3-manifolds with values in $\mathbb{R}/\pi^2 \mathbb{Z}$ by Meyerhoff 43. Using hyperbolic ideal triangulations of $N$ equipped with suitable integral charges, one can define a refined scissors congruence class $\beta(N) \in \mathcal{P}(I)$, and $\beta(N)$ is formally defined in exactly the same way as $c_I(W, L, \rho)$ 42, 13, 14. Moreover, we have

$$R(\beta(N)) = i(Vol(N) + iCS(N)), \quad (1)$$

where $Vol$ is the volume of $N$, $CS$ is the Chern-Simons invariant and

$$R : \mathcal{P}(I) \to \mathbb{C}/(\pi^2 \mathbb{Z})$$

is a natural lift on $\mathcal{P}(I)$ of the classical Rogers dilogarithm. (Neumann’s $\hat{\mathcal{P}}(\mathbb{C})$ in 42, 13 is somehow different from $\mathcal{P}(I)$; nevertheless, it can be shown that $R$ is
also well-defined on $\mathcal{P}(T)$, see \[9\]. The map $R$ can be viewed as a refinement of the Bloch regulator, defined on $B(\mathbb{C})$ (see e.g. \[23\] and the references therein), or as a refinement of the volume of the conjugacy class of the discrete and faithful representation of $\pi_1(N)$ into $PSL(2, \mathbb{C})$, associated with the canonical hyperbolic structure on $N$ (see \[49\] for this point of view). When $N$ is compact, formula (1) was proved mod($\pi_2\mathbb{Q}$) by Dupont \[20\], as a consequence of the evaluation on the fundamental class $[N]$ of the Cheeger-Chern-Simons class for flat $SL(2, \mathbb{C})$-bundles associated with the second Chern polynomial.

It is known that the leading term of the asymptotic expansion of the quantum-dilogarithm $6j$-symbols is related to the Rogers dilogarithm. Some details on this point are given in \[9\]. This and the preceding discussion support the following reformulation of the volume conjecture for hyperbolic-like triples, and (1) give us some hints about the eventual geometric meaning of the limit $G(T_{I}(W,L,\rho))$:

**Conjecture 1.4** We have $G(c_{T}(W,L,\rho)) = R(c_{T_{I}}(W,L,\rho))$.

We stress that, for hyperbolic-like triples, the transition from $T$ to $T_{I}$ (hence from $c_{T}(W,L,\rho)$ to $c_{T_{I}}(W,L,\rho)$, from $K_N(W,L,\rho)$ to $G(T_{I})$), is explicit and geometric and does not involve any “optimistic” \[30, 39, 52\] computation. On the other hand the actual identification of $G(T_{I})$ with $R(c_{T}(W,L,\rho))$ still sets serious analytic problems (see \[4\]). Since $(J_{N})^{N}$ is an instance of $K_N$ (see above), this conjecture, in particular for the factor $1/N^2$, is formally coherent with the current Volume Conjecture for hyperbolic knots in $S^3$ and the coloured Jones polynomial $J_{N}$. A natural complement to Conj. \[1, 4\] is to properly understand the relationship between hyperbolic-like and usual hyperbolic structures. All these problems trace some lines of development of our program.

One could try to extend the above constructions to other groups beside the Borel group $B$; for instance it should be rather natural to consider the whole group $SL(2, \mathbb{C})$. The theory of cyclic representations and the computation of the related $6j$-symbols are certainly complicated, but we believe that the theory of the quantum coadjoint action of De Concini, Kac and Procesi \[19\] provides a framework to achieve it. On the other hand, the above discussion on the hyperbolic-like triples suggests that $B$-bundles ($Par(B)$-bundles indeed) already cover most of the relevant geometric features of the QHI, for what concerns the asymptotic behaviour.

The Appendix provides a detailed account, from both the algebraic and geometric points of view, of the definitions, the properties and the explicit formulas concerning the $6j$-symbols needed for the construction of the state sum $K_N(T)$. We refer to \[1\] for the proofs of all the statements contained in this Appendix.

Finally we point out that another leading idea on the background of our work is to look at it as part of an “exact solution” of the Euclidean analytic continuation of $(2 + 1)$ quantum gravity with negative cosmological constant, that was outlined in \[54\]. This is as a gauge theory with gauge group $SO(3, 1)$ and an action of Chern-Simons type. Hyperbolic 3-manifolds are the empty “classical solutions”. The volume conjecture Conj. \[1, 4\] perfectly agrees with the expected “classical limits” of the partition functions of this theory (see pag. 77 of \[54\]). The Turaev-Viro state sum invariants are similarly intended with respect to the positive cosmological constant \[13\]. We will not indulge on more circumstantial speculations on this point; however, it is for us, at least, a very meaningful heuristic support.

# 2 Distinguished and quasi-regular triangulations

Using the Hauptvermutung, depending on the context, we will freely assume that 3-dimensional manifolds are endowed with a (necessarily unique) PL or smooth
structure. In particular, we shall refer to the topological space underlying a 3-dimensional simplicial complex as its *polyhedron*. We first recall the definition and some properties of standard spines of 3-manifolds. For the foundation of this theory, including the existence, the reconstruction of manifolds from spines and the complete “calculus” of spine-moves, one refers to [14], [37], [47]. As other references one can look at [6], [9], and one finds also a clear treatment of this material in [51] (note that sometimes the terminologies do not agree).

Consider a tetrahedron $\Delta$ with its usual triangulation with 4 vertices, and let $C$ be the interior of the 2-skeleton of the dual cell-decomposition. A *simple polyhedron* $P$ is a 2-dimensional compact polyhedron such that each point of $P$ has a neighbourhood which can be embedded into an open subset of $C$. A simple polyhedron $P$ has a natural stratification given by singularity; depending on the dimensions, we call the components of this stratification *vertices*, *edges* and *regions* of $P$. A simple polyhedron is *standard* (in [51] one uses the term *cellular*) if it contains at least one vertex and all the regions of $P$ are open 2-cells.

Every compact 3-manifold $Y$ (which for simplicity we assume connected) with non-empty boundary $\partial Y$ has *standard spines* [14], that is standard polyhedra $P$ embedded in $\text{Int} Y$ such that $Y$ collapses onto $P$ (i.e. $Y$ is a regular neighbourhood of $P$). Moreover, $Y$ can be reconstructed from any standard spine. Standard spines of oriented 3-manifolds are characterized among standard polyhedra by the property of carrying an *orientation*, that is a suitable “screw-orientation” along the edges [9]. Also, such an oriented 3-manifold $Y$ can be reconstructed (up to orientation preserving homeomorphisms) from any of its oriented standard spines. From now on we assume that $Y$ is oriented, and we shall only consider oriented standard spines of it.

A *singular triangulation* of a polyhedron $Q$ is a triangulation in a weak sense, namely self-adjacencies and multiple adjacencies between 3-simplices are allowed. For any $Y$ as above, let us denote by $Q(Y)$ the polyhedron obtained by collapsing each component of $\partial Y$ to a point. An *ideal triangulation* of $Y$ is a singular triangulation $T$ of $Q(Y)$ such that the vertices of $T$ are precisely the points of $Q(Y)$ which correspond to the components of $\partial Y$.

For any ideal triangulation $T$ of $Y$, the 2-skeleton of the dual cell-decomposition of $Q(Y)$ is a standard spine $P(T)$ of $Y$. This procedure can be reversed, so that we can associate to each (oriented) standard spine $P$ of $Y$ an ideal triangulation $T(P)$ of $Y$ such that $P(T(P)) = P$. Thus standard spines and ideal triangulations are dual equivalent viewpoints which we will freely intermingle. Note that, by removing small neighbourhoods of the vertices of $Q(Y)$, any ideal triangulation leads to a cell-decomposition of $Y$ by truncated tetrahedra which induces a singular triangulation of the boundary of $Y$.

Consider now a closed oriented 3-manifold $W$. For any $r_0 \geq 1$ let $W_{r_0} = W \setminus r_0 D^3$, that is the manifold with $r_0$ spherical boundary components obtained by removing $r_0$ disjoint open balls from $W$. By definition $Q(W_{r_0}) = W$ and any ideal triangulation of $W_{r_0}$ is a singular triangulation of $W$; moreover, it is easily seen that all singular triangulations of $W$ are obtained in this way. We shall adopt the following terminology.

**Notations.** A singular triangulation of $W$ is simply called a *triangulation*. Ordinary triangulations (where neither self-adjacencies nor multi-adjacencies are allowed) are said to be *regular*.

The main advantage in using singular triangulations (resp. standard spines) instead of only ordinary triangulations consists of the fact that there exists a *finite* set of
moves which are sufficient in order to connect (by means of finite sequences of these moves) singular triangulations (resp. standard spines) of the same manifold. Let us recall some elementary moves on the triangulations (resp. simple spines) of a polyhedron $Q(Y)$ that we shall use throughout the paper; see Fig. 1 - Fig. 2.

**The 2 → 3 move.** Replace the triangulation $T$ of a portion of $Q(Y)$ made by the union of 2 tetrahedra with a common 2-face $f$ by the triangulation made by 3 tetrahedra with a new common edge which connect the two vertices opposite to $f$. This move corresponds dually to “blowing up” some edge $e$ of $P(T)$, or equivalently sliding some region of $P(T)$ along $e$ until it bumps into another one.

**The bubble move.** Replace a face of a triangulation $T$ of $Q(Y)$ by the union of two tetrahedra glued along three faces. This move corresponds dually to the gluing of a closed 2-disk $D$ via its boundary $\partial D$ on the standard spine $P(T)$, with exactly two transverse intersection points of $\partial D$ along some edge of $P(T)$. The new triangulation thus obtained is dual to a spine of $Y \setminus D^3$, where $D^3$ is an open ball in the interior of $Y$.

**The 0 → 2 move.** Replace two adjacent faces of a triangulation $T$ of $Q(Y)$ by the union of two tetrahedra glued along two faces, so that the other faces match the two former ones. The dual of this move is the same as for the 2 → 3 move, except that the sliding of the region is done without intersecting any edge of $P(T)$ before it bumps into another one.

Standard spines of the same compact oriented 3-manifold $Y$ with boundary and with at least two vertices (which, of course, is a painless requirement) may always be connected by the (dual) move $2 \to 3$ and its inverse. In order to handle trian-
gulations of a closed oriented 3-manifold we also need a move which allows us to vary the number of vertices. The shortest way is to use the bubble move. Although the $2 \to 3$ move and the bubble move generate a “complete calculus” of standard spines, it is useful to introduce the $0 \to 2$ move. It is also known as lune move and is somewhat similar to the second Reidemeister move on link diagrams.

We say that a move which increases the number of tetrahedra is positive, and its inverse negative. Note that the inverse of the lune move is not always admissible because one could lose the standardness property when using it. We shall see in §3.1 that in some situations it may be useful to handle only positive moves. In this sense we recall the following technical result due to Makovetskii [36]:

**Proposition 2.1** Let $P$ and $P'$ be standard spines of $Y$. There exists a spine $P''$ of $Y$ such that $P''$ can be obtained from both $P$ and $P'$ via a finite sequence of positive $0 \to 2$ and $2 \to 3$ moves.

We introduce now the notion of distinguished triangulation for a pair $(W, L)$, where, as we have stipulated, $W$ is a closed, connected and oriented 3-manifold and $L$ is a link in $W$.

**Definition 2.2** A distinguished triangulation $(T, H)$ of the pair $(W, L)$ is a triangulation $T$ of $W$ such that $L$ is realized as a Hamiltonian subcomplex, that is $H$ is union of edges and contains all the vertices of $T$. Note that at each vertex of $T$ there are exactly two “germs” of edges of $H$ (the two germs could belong to the same edge of $H$).

Here is a description of distinguished triangulations in terms of spines:

**Definition 2.3** Let $Y$ be as before. Let $S$ be any finite family of $r$ disjoint simple closed curves on $\partial Y$. We say that $Q$ is a quasi-standard spine of $Y$ relative to $S$ if:

(i) $Q$ is a simple polyhedron with boundary $\partial Q$ consisting of $r$ circles. These circles bound (unilaterally) $r$ annular regions of $Q$. The other regions are cells.
(ii) \((Q, \partial Q)\) is properly embedded in \((Y, \partial Y)\) and transversely intersects \(\partial Y\) at \(S\).

(iii) \(Q\) is is a spine of \(Y\).

Lemma 2.4 Quasi-standard spines of \(Y\) relative to \(S\) do exist.

Proof. Let \(\tilde{P}\) be any standard spine of \(Y\). Consider a normal retraction \(r : Y \to \tilde{P}\). Recall that \(Y\) is the mapping cylinder of \(r\). For each region \(R\) of \(\tilde{P}\), \(r^{-1}(R) = R \times I\); for each edge \(e\), \(r^{-1}(e) = e \times \{\text{"tripode"}\}\); for each vertex \(v\), \(r^{-1}(v) = \{\text{a "quadrilope"}\}\). We can assume that \(S\) is in general position with respect to \(r\), so that the mapping cylinder of the restriction of \(r\) to \(S\) is a simple spine of \(Y\) relative to \(S\). Possibly after doing some \(0 \to 2\) moves we obtain a quasi-standard \(Q\). □

Definition 2.5 Let \(M = W \setminus U(L)\), where \(U(L)\) is an open tubular neighbourhood of \(L\) in \(W\), and \(S\) be formed by the union of \(t_i \geq 1\) parallel copies on \(\partial M\) of the meridian \(m_i\) of the component \(L_i\) of \(L\), \(i = 1, \ldots, n\). A spine of \(M\) adapted to \(L\) of type \(t = (t_1, \ldots, t_n)\) is a quasi-standard spine of \(M\) relative to such an \(S\).

Let \(Q\) be a spine of \(M\) adapted to \(L\). Filling each boundary component of \(Q\) by a 2-disk we get a standard spine \(P = \tilde{P}(Q)\) of \(W' = W \setminus rD^3\), \(r = \sum_i t_i\). Since the dual triangulation \(T(P)\) of \(W\) contains \(L\) as a Hamiltonian subcomplex, it is a distinguished triangulation of \((W, L)\). Conversely, starting from any distinguished \((T, H)\) and removing an open disk in the dual region to each edge of \(H\), we pass from \(P = \tilde{P}(T)\) to a spine \(Q = \tilde{P}(P)\) of \(M\) adapted to \(L\). So adapted spines and distinguished triangulations are equivalent objects. One deduces from Lemma 2.4 that distinguished triangulations of \((W, L)\) exist.

Next we consider moves on distinguished triangulations. Let \((T, H)\) be a distinguished triangulation. Any positive \(0 \to 2\) or \(2 \to 3\) move \(T \to T'\) naturally specializes to a move \((T, H) \to (T', H')\): in fact \(H' = H\) is still Hamiltonian. We consider also the inverse negative moves \((T', H') \to (T, H)\). In the bubble move case, we assume that an edge \(e\) of \(H\) lies in the boundary of the involved face \(f\); \(e\) lies in the boundary of a unique 2-face \(f'\) of \(T'\) containing the new vertex. Then we get the Hamiltonian \(H'\) just by replacing \(e\) by the other two edges of \(f'\). Sometimes we will refer to these moves as “distinguished” moves.

Let \((T, H)\) and \((T', H')\) be distinguished triangulations of \((W, L)\) such that the associated quasi-standard spines \(Q, Q'\) of \(M\) adapted to \(L\) are relative to \(S\) and \(S'\) and are of the same type \(t\). Up to isotopy, we can assume that \(S = S'\) and that the “germs” of \(Q\) and \(Q'\) at \(S\) coincide. Using Th. 3.4.B of [51], one obtains the following relative version of Lemma 2.4 for adapted spines:

Lemma 2.6 Let \(P\) and \(P'\) be quasi-standard spines of \(M\) adapted to \(L\) relative to \(S\) of type \(t = (t_1, \ldots, t_n)\). There exists a spine \(P''\) of \(M\) adapted to \(L\), such that \(P''\) can be obtained from both \(P\) and \(P'\) via a finite sequence of positive \(0 \to 2\) and \(2 \to 3\) moves, and at each step one still has spines of \(M\) adapted to \(L\) and of type \(t\).

Possibly using distinguished bubble moves, we deduce from lemma 2.6 and the correspondence between adapted spines and distinguished triangulations that:

Corollary 2.7 For any \(r \geq n\) there exist distinguished triangulations of \((W, L)\) with \(r\) vertices. Given any two distinguished triangulations \((T, H)\) and \((T', H')\) of \((W, L)\) there exists a distinguished triangulation \((T'', H'')\) which may be obtained from both \((T, H)\) and \((T', H')\) via a finite sequence of positive bubble, \(0 \to 2\) and \(2 \to 3\) moves, and at each step we still have distinguished triangulations.
In §5 we shall need a more restricted type of distinguished triangulations, generalizing at the same time the regular triangulations:

**Definitions 2.8** A quasi-regular triangulation $T$ of a closed 3-manifold $W$ is a (singular) triangulation where all edges have distinct vertices. A move $T \rightarrow T'$ on a quasi-regular triangulation $T$ is quasi-regular if $T'$ is quasi-regular.

**Proposition 2.9** For any pair $(W,L)$ there exist quasi-regular distinguished triangulations.

*Proof.* Let $(T,H)$ be a distinguished triangulation of $(W,L)$. It is not quasi-regular if some edge $e$ of $T$ is a loop, i.e. if the ends of $e$ are identified. In the cellulation $D(T)$ of $W$ dual to $T$, this means that the spine $P = P(T)$ contains some region $R = R(e)$ which has the same 3-cell $C$ on both sides: the boundary of $C$ is a sphere $S$ immersed at $R$. Let us say that $R$ is bad. We construct a quasi-regular and distinguished $(T',H')$ by doing some bubble moves on $(T,H)$ (thus adding new 3-cells to $D(T)$), and then sliding their “capping” discs off the bad regions, so that one desingularizes all the boundary 2-spheres.

Let us formalize this argument. Any (dual) bubble move $P \rightarrow P'$ is done by gluing a closed 2-disc $D^2$ along its boundary $\partial D^2$, with two transverse intersection points of $\partial D^2$ with an edge $e$ of $P$ (see the second move in Fig. 3). Denote by $A$ and $B$, $A \cup B = \partial D^2$, the two arcs thus defined. The bubble move is distinguished if at least one of $A$ or $B$ lies on a region $R_H$ of $P$ dual to an edge of $H$. The two new regions of $P'$ dual to edges of $H$ are $D^2$ and the region enclosed by $D^2$ and adjacent to $R_H$. We call $D^2$ the capping disc of the bubble move. Note that a bubble move does not increase the number of bad regions, and that any $2 \leftrightarrow 3$ move on $D^2$ also has this property as long as $D^2$ stays embedded.

Let now $R \in S$ be a bad region (top right of Fig. 3). Using distinguished bubble moves we may always assume that each connected component of $H$ has at least two vertices. Since $(T,H)$ is distinguished, there are exactly two regions $R_H$ and $R'_H$ in the cellular decomposition of $S$ which are dual to edges of $H$. As above, do a bubble move that involves $R_H$ (for instance), and slide its capping disc $D^2$ isotopically via $2 \leftrightarrow 3$ moves along the 1-skeleton of $S$, until it reaches a vertex of $R$. This is obviously always possible, and at each step we still have (dual) distinguished triangulations with no more bad regions. Next expand $D^2$ over $R$ by doing further $2 \rightarrow 3$ moves along the edges of $\partial R$. If $R$ is embedded in $S$, we can choose such a sequence of moves so that $D^2$ is embedded at each step and finally covers $R$ completely (bottom right of Fig. 3). Both $R$ and $D^2$ are in the boundary of the 3-cell introduced by the bubble move, which followed $D^2$ during the sequence. Thus we eventually finish with a spine dual to a distinguished triangulation and having one less bad region than $P$.

If $R$ is immersed on its boundary (for example if it looks like an annulus with one edge that joins the boundary circles), note that the complementary regions of $R$ in $S$ form a non-empty set, since $R_H$ is distinct from $R$. Then $R$ is contained inside a disc embedded in $S$, and we may still find a sequence of $2 \leftrightarrow 3$ moves (possibly arranged so that they give $0 \leftrightarrow 2$ moves) ending with a spine dual to a distinguished triangulation and having one less bad region than $P$. Iterating this procedure, we get the conclusion. \qed

**Proposition 2.10** Any two quasi-regular distinguished triangulations $(T,H)$ and $(T',H')$ of $(W,L)$ may be obtained from each other by a finite sequence of $2 \rightarrow 3$ moves, bubble moves and their inverses, so that at each step we have quasi-regular distinguished triangulations.
Proof. We use the same terminology as in Prop. 2.9. Let \( s : (T, H) \rightarrow \cdots \rightarrow (T^\prime, H^\prime) \) be a sequence of moves as in Cor. 2.7. We may assume, up to further subdivisions of \( s \), that there are no \( 0 \rightarrow 2 \) moves. We divide the proof in two steps. We first prove that there exists a sequence \( s^\prime : P = P(T) \rightarrow \cdots \rightarrow P^\prime = P(T^\prime) \) with only quasi-regular moves and such that the spine \( P^\prime \) is obtained from \( P^\prime = P(T^\prime) \) by gluing some 2-discs \( \{ D^2_i \} \) along their boundaries. Then we show that we may get \( P^\prime \) from \( P^\prime \) just by using distinguished bubble moves and quasi-regular moves. By combining both sequences we will get the conclusion.

Bubble moves are always quasi-regular. Consider the first non quasi-regular move \( m \) in \( s \). It produces a bad region \( R \); see the top of Fig. 3, where we indicate \( R \) by dashed lines and we underline the sliding arc \( a \). Alternatively, a step before \( m \) we may do a distinguished bubble move and slide its capping disc \( D^2 \) as in Prop. 2.9, until it covers \( a \). Next, make the arc \( a \) sliding as in \( s \); see the bottom of Fig. 3. These moves are quasi-regular and give distinguished dual triangulations: starting with the moves of \( s \) and turning \( m \) into this sequence, we define the first part of \( s^\prime \).

We wish to complete it with the following moves of \( s \), applying the same procedure each time a non quasi-regular move would be done. But suppose that one of these moves would have affected \( a \), and let \( b \) be the sliding arc responsible for it. Then in \( s^\prime \) we just have first to slide \( b \) “under” \( D^2 \), pushing it up. The arc \( b \) is then in the same position w.r.t. \( a \) than it has in \( s \). (The new region produced by this move would appear in the bottom right picture of Fig. 3 inside the closed 3-cell). With this rule there are no obstruction to complete the desired sequence \( s^\prime \). The images in \( T^\prime \) of all the capping discs form the set \( \{ D^2_i \} \).

Let us now turn to the second claim. In the dual cellulation \( D(T^\prime) \) of \( W \) consider the boundary spheres \( S_j \) obtained by removing the discs \( D^2_i \). Fix one of them \( S \) and, reversing this procedure, let \( D^2 \in \{ D^2_i \} \) (considered with its gluings) be the first disc glued to it. Do a distinguished bubble move on \( S \). We may slide its capping disc isotopically via \( 2 \rightarrow 3 \) moves along the 1-skeleton of \( S \), so that it finally reaches the position of \( D^2 \) in \( P^\prime \). We may repeat this argument inductively on the \( D^2_i \)’s. Since all these moves are quasi-regular, this proves our claim. \( \square \)

3 Decorations

In this section we consider triples \( (W, L, \rho) \) and define the decorations \( D = (b, z, c) \) of distinguished triangulations \((T, H)\) of \((W, L)\). We shall say that \( T = (T, H, D) \) is a
$D$-triangulation of $(W,L,\rho)$. We shall also describe the moves on $D$-triangulations, which shall be called $D$-transits.

### 3.1 Branchings

Let $P$ be a standard spine of a compact oriented 3-manifold $Y$ with boundary, and consider the dual ideal triangulation $T = T(P)$. By an abstract tetrahedron $\Delta$ of $T$, we mean the simplicial complex formed by the standard triangulation of $\Delta$ with four vertices, without considering the self-gluings that may eventually happen in $T$.

A branching $b$ of $T$ is a system of orientations of the edges of $T$ such that each abstract tetrahedron of $T$ has one source and one sink on its 1-skeleton. This is equivalent to saying that, for any 2-face $f$ of $T$, the edge-orientations do not induce an orientation of the boundary of $f$. In dual terms, a branching is a system of orientations of the regions of $P$ such that for each edge of $P$ we have the same induced orientation only twice. In particular, note that each edge of $P$ has an induced orientation.

Branchings, mostly in terms of spines, have been widely studied and applied in $[6, 7, 8]$. One can see that a branching of $P$ gives it the extra structure of an embedded and oriented (hence normally oriented) branched surface in $\text{Int}(Y)$. Moreover a branched $P$ carries a suitable positively transverse combing of $Y$ (ie. a vector field without zeros up to homotopy). We recall here part of the branching’s combinatorial content. A branching $b$ allows us to define an orientation on any cell of $T$, not only on the edges. Indeed, consider any abstract tetrahedron $\Delta$. For each vertex of $\Delta$ consider the number of incoming $b$-oriented edges in the 1-skeleton. This gives us an ordering $b_\Delta : \{0, 1, 2, 3\} \to V(\Delta)$ of the vertices which reproduces the branching on $\Delta$ by stipulating that the edge $[v_i, v_j]$ is positively oriented iff $j > i$. We can take $v_3 = b_\Delta(3)$ as base vertex; the ordered triple of edges incoming into $v_3$, gives an orientation of $\Delta$. This orientation may or may not agree with the orientation of $Y$. In the first case we say that $\Delta$ is of index 1, and it is of index $-1$ otherwise. The ordering $b_\Delta$ induces also orderings of the edges; we shall precise in §3 one which is convenient for us.

The 2-faces can be named by their opposite vertices. We orient them by working as above on the boundary of each abstract 2-face $f$. There is an ordering $b_f : \{0, 1, 2\} \to V(f)$, a base vertex $v_0 = b_f(0)$, and finally an orientation of $f$ which induces on $\partial f$ the prevailing orientation among the three (oriented) edges. This 2-face orientation can be described in another equivalent way. Let us consider the 1-cochain $s_b$ such that $s_b(e) = 1$ for each $b$-oriented edge. Then there is a unique way to orient any 2-face $f$ such that the coboundary $\delta s_b(f) = 1$.

**Branching’s existence and transit.** In general a given ideal triangulation $T$ of $Y$ could admit no branching. Given any system of edge-orientations $g$ on $T$ and any move $T \to T'$, a transit $(T,g) \to (T',g')$ is given by any system $g'$ of edge-orientations on $T'$ which agrees with $g$ on the “common” edges. We use the same terminology for moves on standard spines. One has

**Proposition 3.1** $[1]$ Th. 3.4.9 *For any $(T,g)$ there exists a finite sequence of positive $2 \to 3$ transits such that the final $(T',g')$ is actually branched.*

This applies in particular to any distinguished triangulation $(T,H)$ of $(W,L)$ and ensures the existence of branched distinguished triangulations. On the other hand, any quasi-regular triangulation of $(W,L)$ admits branchings of a special type, given by fixing any total ordering of its vertices and by stipulating that the edge $[v_i, v_j]$ is positively oriented iff $j > i$. 

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Definition 3.2 Given a distinguished triangulation \((T, H)\) of \((W, L)\), the first component of a decoration \(D = (b, z, c)\) is a branching \(b\) of \(T\).

Let us come to the notion of branching transit. We have already defined the edge-orientation transit. We have a branching transit if both the involved systems of orientations are branchings. Any quasi-regular move which preserves the number of vertices also preserves the total orderings on the set of vertices, hence it obviously induces total-ordering-branching transits. If it increases the number of vertices, one can extend to the new vertex, in several different ways, the old total orderings of the set of vertices. Any of these ways induces again a total-ordering-branching transit. If \((T, b)\) is any branched triangulation (that is \(T\) is not necessarily quasi-regular nor \(b\) is total-ordering) and \(T \rightarrow T'\) is either a positive \(2 \rightarrow 3\), \(0 \rightarrow 2\) or bubble move, then it can be completed, sometimes in different ways, to a branched transit \((T, b) \rightarrow (T', b')\). Any of these ways is a possible transit. On the contrary, it is easily seen that a negative \(3 \rightarrow 2\) or \(2 \rightarrow 0\) move may not be “branchable” at all. This shows that it is important, when dealing with triangulations which are not quasi-regular, to use only positive moves.

Figure 4: 2 \(\rightarrow\) 3 sliding moves.

Figure 5: 2 \(\rightarrow\) 3 bumping move.

Figure 6: branched lune-moves.

In Fig. 4 - Fig. 6 we show the whole set of 2 \(\rightarrow\) 3 and 0 \(\rightarrow\) 2 (dual) branched transits on standard spines, up to evident symmetries (in those figures, it is intended that the vertical vector field is the combing transverse to the initial spine). Note that the middle sliding move in Fig. 4 corresponds dually to the branched triangulation
move shown in Fig. 8. Following [6] one could distinguish two quite different kinds of branched transits: the sliding moves, which actually preserve the positively transverse combing mentioned at the beginning of this section, and the bumping moves, which eventually change it. We shall not exploit this difference in the present paper.

3.2 Cocycles

Let \((T, H, b)\) be a branched distinguished triangulation of \((W, L)\). Recall that \(B = B(2, \mathbb{C})\) denotes the subgroup of upper triangular matrices of \(SL(2, \mathbb{C})\). Consider the set \(Z^1(T; B)\) of \(B\)-valued (cellular) 1-cocycles on \((T, b)\). This means that the values of \(z\) are specified with the \(b\)-orientation. We often write \(z(e) = (t(e), x(e))\) for

\[
z(e) = \begin{pmatrix} t(e) & x(e) \\ 0 & t(e)^{-1} \end{pmatrix}.
\]

We denote by \([z]\) the equivalence class of \(z \in Z^1(T; B)\) up to (cellular) coboundaries. A 0-cochain \(\lambda\) is a \(B\)-valued function defined on the set of vertices \(V(T)\) of \(T\). Then the 1-cocycles \(z\) and \(z'\) are equivalent if they differ by the coboundary of some 0-cochain \(\lambda\): for any \(b\)-oriented edge \(e\) with ordered end points \(v_0, v_1\), one has

\[
z'(e) = \lambda(v_0)^{-1}z(e)\lambda(v_1).
\]

We denote this quotient set by \(H^1(T; B)\). It is well-known that it can be identified with the set of isomorphism classes of flat principal \(B\)-bundles on \(W\).

There are two distinguished abelian subgroups of \(B\). They are:

1) the Cartan subgroup \(C = C(B)\) of diagonal matrices; it is isomorphic to the multiplicative group \(\mathbb{C}^*\) via the map which sends \(A = (a_{ij}) \in C\) to \(a_{11}\);

2) the parabolic subgroup \(Par(B)\) of matrices with double eigenvalue 1; it is isomorphic to the additive group \(\mathbb{C}\) via the map which sends \(A = (a_{ij}) \in Par(B)\) to \(x = a_{12}\).

Denote by \(G\) any such abelian subgroup. There is a natural map \(H^1(T; G) \to H^1(T; B)\), induced by the inclusion, and \(H^1(T; G)\) is endowed with the natural abelian group structure. Note that \(H^1(T; Par(B)) = H^1(T; \mathbb{C})\) is isomorphic to the ordinary (singular or de Rham) first cohomology group of \(W\).

From now on we consider a triple \((W, L, \rho)\), where \(\rho\) is a flat principal \(B\)-bundle on \(W\).

Definitions 3.3 1) Given a branched distinguished triangulation \((T, H, b)\) of \((W, L)\) and \(\rho \in H^1(W; B)\), the second component of a decoration \(D = (b, z, c)\) is a \(B\)-valued 1-cocycle on \(T\) representing \(\rho\). We say that \(z\) is full if for every edge \(e\) of \(T\) with \(z(e) = (t(e), x(e))\) one has \(x(e) \neq 0\).

2) We say that a triangulation of \(W\) is fullable if it carries a full 1-cocycle representing the trivial flat \(B\)-bundle over \(W\). We say that \((T, H, (b, z))\) is full if \(T\) is fullable and \(z\) is full.

Remark that a triangulation \(T\) is fullable iff it is quasi-regular. Moreover, if \(T\) is fullable then for any flat \(B\)-bundle \(\rho\) on \(W\) it carries a full 1-cocycle representing \(\rho\). Indeed, a complex valued injective function \(u\) defined on the vertices of \(T\) may be viewed as a \(Par(B)\)-valued 0-cochain, and its coboundary \(z = du\) is a full cocycle representing \(0 \in H^1(T; \mathbb{C})\) (whence the trivial flat \(B\)-bundle in \(H^1(T; B)\)). Such \(du\)'s form a dense subset of the coboundaries on \((T, g)\). Hence, using them, we can generically perturb any 1-cocycle \(z\) on \((T, g)\) so that the resulting \(z' = u^{-1}\cdot z\cdot u\) is full.
**Transit of (full) cocycles.** Let \( T \) be a triangulation of \( W \) with any edge-orientation system \( g \). Let \( z \) be a 1-cocycle on \( (T, g) \). For any transit of orientations \((T, g) \rightarrow (T', g'), (T, g, z) \rightarrow (T', g', z')\) is an associated *transit of cocycle* if \( z \) and \( z' \) agree on the common edges of \( T \) and \( T' \) and the isomorphism \( H^1(T; B) \cong H^1(T'; B) \) maps \([z]\) to \([z']\). Abusing of the notations, we shall write \([z] = [z']\). Assume now that \( z \) is full. We say that a cocycle transit \((T, g, z) \rightarrow (T', g', z')\) is full if and only if also the final \( z' \) is full.

For bubble moves \((T, g) \rightarrow (T', g')\) there is an infinite set of possible transits \((T, g, z) \rightarrow (T', g', z')\). Moreover, given one such \((T, g, z) \rightarrow (T', g', z')\) with \( z \) a full cocycle, we can always turn \( z' \) into an equivalent full \( z'' \) via the coboundary of some 0-cochain with support consisting of the new vertex \( v \) of \( T' \). Hence for bubble moves there always exists an infinite set of full cocycle transits.

In the \( 2 \rightarrow 3 \) or \( 0 \rightarrow 2 \) cases and their inverses, there is a unique \( z' \) on \((T', g')\) which agrees with \( z \) on the common edges. This uniquely defines a transit \((T, g, z) \rightarrow (T', g', z')\) with \([z] = [z']\). Assume now that \( z \) is full: the trouble is that, in general, \( z' \) is no longer full.

The following lemma shows that quasi-regular triangulations have *generically* a good behaviour with respect to the existence and the transit of full cocycles.

**Lemma 3.4** Let \((T, g)\) be a quasi-regular triangulation of \( W \) endowed with any edge-orientation system \( g \). Let \((T, g) = (T_1, g_1) \rightarrow \cdots \rightarrow (T_s, g_s) = (T', g')\) be any finite sequence of quasi-regular 2 \( \leftrightarrow \) 3 edge-orientation transits. Then for each \( T_i \) there exists an open dense set \( U_i \) of full cocycles (in the topology induced by the Zariski topology of \( B^1(T_i) \)) such that the transit \( T_i \rightarrow T_{i+1} \) maps \( U_i \) into \( U_{i+1} \). Moreover each class \( \alpha \in H^1(T_i; B) \) can be represented by cocycles in \( U_i \).

**Proof.** Each cocycle transit \((T_i, z_i) \rightarrow (T_{i+1}, z_{i+1})\) can be regarded as an algebraic bijective map from \( Z^1(T_i; B) \) to \( Z^1(T_{i+1}; B) \). Since all edges of \( T_{i+1} \) have distinct vertices, there are no trivial cocycle relations on \( T_{i+1} \). Hence the set of full cocycles for which the full elementary transit fails is contained in a proper algebraic subvariety of \( Z^1(T_i; B) \). Working by induction on \( s \) we get the conclusion. \( \square \)

### 3.3 Integral charges

The definition of integral charges emerges from Neumann’s work on Cheeger-Chern-Simons classes of hyperbolic manifolds and scissors congruences classes of hyperbolic polytopes \([11, 13]\). This relationship shall be developed in \([4, 7, 10]\) and \([8]\) so we only give here the definition of integral charge and of charge transit.

Let \((T, H)\) be a distinguished triangulation of \((W, L)\). Denote by \( E_\Delta(T) \) the set of all edges of all abstract tetrahedra of \( T \), and give it an auxiliary ordering. Denote by \( E(T) \) the set of all edges of \( T \); there is a natural identification map \( \epsilon : E_\Delta(T) \rightarrow E(T) \). Let \( s \) be a simple closed curve in \( W \) in general position with respect to the \( T \). We say that \( s \) has no back-tracking with respect to \( T \) if it never departs a tetrahedron of \( T \) across the same 2-face by which it entered. Thus each time \( s \) passes through a tetrahedron, it selects the edge between the entering and departing faces.

**Definition 3.5** An integral charge on \((T, H)\) is a map \( \epsilon : E_\Delta(T) \rightarrow \mathbb{Z} \) which satisfies the following properties:

1. For each 2-face \( f \) of any abstract \( \Delta \) with edges \( e_1, e_2, e_3 \) we have \( \sum_{i} c(e_i) = 1 \),

2. for each \( e \in E(T) \setminus E(H) \) we have \( \sum_{e' \in \epsilon^{-1}(e)} c(e') = 2 \),

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for each $e \in E(H)$ we have $\sum_{e' \in \epsilon^{-1}(e)} c(e') = 0$.

(2) Let $s$ be any curve which has no back-tracking with respect to $T$. Each time $s$ enters a tetrahedron of $T$ the map $c$ associates an integer to the selected edge. Let $c(s)$ be the sum of these integers. Then, for each $s$ we have $c(s) \equiv 0 \pmod{2}$.

We call charge of an edge $e \in T$ the value $c(e)$.

**Definition 3.6** Given a distinguished triangulation $(T, H)$ of $(W, L)$, the third component of a decoration $D = (b, z, c)$ is an integral charge $c$ of $(T, H)$.

**Remark 3.7** The meaning of Def. 3.5 (2) is that any map $c : E_\Delta(T) \to \mathbb{Z}$ satisfying Def. 3.5 (1) induces an element $[c] \in H^1(W; \mathbb{Z}/2\mathbb{Z})$ and one prescribes that $[c] = 0$. We give here another description of $[c]$ in terms of spines. Take the dual spine $P = P(T)$.$^9$ “Blow-up” each vertex $v$ of $P$ as in Fig. 7; this replaces $v$ by a tetrahedron. Denote by $P'$ the standard spine thus obtained. This blowing-up corresponds to a so-called $1 \to 4$ move on the tetrahedron $\Delta(v)$ dual to $v$: subdivide this tetrahedron into the union of 4 tetrahedra obtained by taking the cone over the 1-skeleton of $\Delta(v)$, from an interior point which is a new vertex. Such a move is a composition of a bubble move and a $2 \to 3$ move. Then each of the new edges of $P'$ coming from the vertex $v$ is opposite to an edge of $\Delta(v)$. Give it the same charge, and associate to the other edges of $P'$ the value 0. This defines a 1-cochain on $P'$. The relations in Def. 3.5 (1) implies that this is actually a mod(2) cocycle that represents $[c]$, via the natural isomorphism between $H^1(P'; \mathbb{Z}/2\mathbb{Z})$ and $H^1(W; \mathbb{Z}/2\mathbb{Z})$.

![Figure 7: a vertex $v$ of $P$ viewed from the top, and a 1 → 4 move on it; the two dotted arcs represent the transverse discs.](image)

**Charge's existence.** The existence of integral charges is obtained by adapting, almost verbatim, Neumann’s proof of the existence of combinatorial flattenings of ideal triangulations of compact 3-manifolds whose boundary is a union of tori (Th. 2.4.(i) and Lemma 6.1 of [11]). In Neumann’s situation there is no link but the manifold has a non empty boundary; only the first condition of Def. 3.5 (1) is present, and there is a further condition in Def. 3.5 (2) about the behaviour of the charges on the boundary. In our situation, as $W$ is a closed manifold, this further condition is essentially empty, and the second condition in Def. 3.5 (1) together with the realization of $H$ as a Hamiltonian subcomplex of $T$ replaces the role of the non-empty boundary in the combinatorial algebraic considerations that lead to the existence of charges. All the details of this adaptation are contained in [4, Prop. 2.2.5]. Note that the charge existence holds for distinguished triangulations which are not necessarily branchable. Anyway, we have finally proved that $D$-triangulations $T = (T, H, D = (b, z, c))$ do exist.

**Charge transit.** Next we describe the structure of integer lattice of the set of integral charges on $(T, H)$. This is again an adaptation to the present situation.
of Neumann’s results. The first condition in Def. 3.5 (1) implies that for each $\Delta$ opposite edges have the same charge. Then there are only two degrees of freedom in choosing the charges of the edges of $\Delta$. Let us assume that $T$ is branched and use the branching $b$ of $T$ to order these edges. The first three edges belong to the face $f$ opposite to the vertex $v_3$ of $\Delta$, with the ordering defined by the oriented path starting from the base vertex of $f$. The last three edges are the opposite ones, ordered correspondingly. Hence, given a branching $b$ on $T$ there is a preferred ordered pair of charges $(c_1^\Delta, c_2^\Delta)$ for each abstract $\Delta$, consisting of the values of $c$ on the two first edges of $\Delta$.

Set $w_1^\Delta := c_1^\Delta$ and $w_2^\Delta = -c_2^\Delta$. Let $r_0$ and $r_1$ be respectively the number of vertices and edges of $T$. An easy computation with the Euler characteristic shows that there are exactly $r_1 - r_0$ tetrahedra in $T$ (see the proof of Prop. 6.2). If we order the tetrahedra of $T$ in a sequence $\{\Delta^i\}_{i=1}^{r_1-r_0}$, one can write down an integral charge on $(T, b)$ as a vector $c = c(w) \in \mathbb{Z}^{2(r_1-r_0)}$ with

$$c = (w_1^{\Delta^1}, \ldots, w_1^{\Delta^{r_1-r_0}}, w_2^{\Delta^1}, \ldots, w_2^{\Delta^{r_1-r_0}})^t.$$  

**Proposition 3.8** [Cor. 2.2.7] There exist determined $w(e) \in \mathbb{Z}^{2(r_1-r_0)}$, $e \in E_\Delta(T)$, such that given any integral charge $c$ all the other integral charges $c'$ are of the form

$$c' = c + \sum_{e} \lambda_e w(e)$$

where for any $e \in E_\Delta(T)$ we have $\lambda_e \in \mathbb{Z}$.

The vectors $w(e)$ have the following form. For any abstract $\Delta^i$ glued along a specific $e$, define $r_1^{\Delta^i}(e)$ (resp. $r_2^{\Delta^i}(e)$) as the number of occurrences of $w_1^{\Delta^i}$ (resp. $w_2^{\Delta^i}$) in $e^{-1}(e) \cap \Delta^i$. Then $w(e) = (r_1^{\Delta^1}, \ldots, r_1^{\Delta^{r_1-r_0}}, -r_2^{\Delta^1}, \ldots, -r_2^{\Delta^{r_1-r_0}})^t \in \mathbb{Z}^{2(r_1-r_0)}$.

**Example 3.9** Consider the situation described on the right of Fig. 8. Denote by $\Delta^i$ the tetrahedron which does not contain the $j$th vertex. We have

$$r_1^{\Delta^0}(e) = -1 \quad r_1^{\Delta^2}(e) = 0 \quad r_1^{\Delta^4}(e) = -1$$

$$r_2^{\Delta^0}(e) = 1 \quad r_2^{\Delta^2}(e) = -1 \quad r_2^{\Delta^4}(e) = 1,$$

where $e$ is the central edge. Then $w(e) = (1, -1, 1, 0, 1)^t$.

**Charge transit.** Let $(T_1, H_1) \to (T_2, H_2)$ be a $2 \to 3$ move. Let $c_1$ be an integral charge on $(T_1, H_1)$ and $e$ be the edge that appears. Consider the two abstract tetrahedra $\Delta', \Delta''$ of $T_1$ involved in the move. They determine a subset $E(T_1)$ of $E_\Delta(T_1)$. Denote by $\tilde{c}_1$ the restriction of $c_1$ to $E(T_1)$. Let $c_2$ be an integral charge on $(T_2, H_2)$, and consider the three abstract tetrahedra of $T_2$ involved in the move. Define $\tilde{E}(T_2)$ and $\tilde{c}_2$ as above. Denote by $\tilde{E}(T_1)$ the complement of $E(T_1)$ in $E_\Delta(T_1)$ and by $\tilde{c}_1$ the restriction of $c_1$ on $\tilde{E}(T_1)$. Do similarly for $\tilde{E}(T_2)$ and $\tilde{c}_2$. Clearly $\tilde{E}(T_1)$ and $\tilde{E}(T_2)$ can be naturally identified.

**Proposition 3.10** i) We have a charge transit $(T_1, H_1, c_1) \to (T_2, H_2, c_2)$ if:

1. $\tilde{c}_1$ and $\tilde{c}_2$ agree on $\tilde{E}(T_1) = \tilde{E}(T_2)$,
2. for each common edge $e_0 \in e_{T_1}(\tilde{E}(T_1)) \cap e_{T_2}(\tilde{E}(T_2))$ we have

$$\sum_{e' \in e_{T_1}^{-1}(e_0)} \tilde{c}_1(e') = \sum_{e'' \in e_{T_2}^{-1}(e_0)} \tilde{c}_2(e'').$$

The same two conditions allow one to define charge-transits also for $0 \to 2$ and bubble moves.
ii) Fix the integral charge $c_2$ on $(T_2, H_2)$ and put

$$C(e, c_2, T) = \{c'_2 = c_2 + \lambda w(e), \lambda \in \mathbb{Z}\},$$

where $e$ is the edge that appears and $w(e)$ is as in Example 3.3. The integral charges $c'_2$ obtained by varying the charge transit exactly span $C(e, c_2, T)$.

**Proof.** i) We have to show that (1)-(2) actually define integral charges. We claim that $\Delta''$.

Consider first $2 \to 3$ moves. By (1) we can restrict our attention to Star($e$, $T_2$). Consider Fig. 8. Denote by $c'$ the integral charge on $\Delta'$ and by $c'_{jk}$ the value of $c'$ on the edge with vertices $v_j$ and $v_k$. By (2) we have:

$$c'_{02} + c'_{24} + c'_{40} = (c^3_{02} - c^3_{02}) + (c^3_{24} - c^3_{24}) + (c^3_{40} - c^3_{40}) = c^{40}_{13} + c^{0}_{13} + c^{2}_{13} - (c^3_{02} + c^3_{24} + c^3_{40}),$$

where in the second equality we use the fact that opposite edges of a tetrahedron have the first condition in 3.5 (1). Next consider $0 \to 2$ moves. Any non-branched $0 \to 2$ move $(T_1, H_1) \to (T_2, H_2)$ is a composition of $2 \to 3$ and $3 \to 2$ moves [47]. In particular, the negative moves in this composition do not involve the edges of $E(T_1) \cap E(T_2)$. Since integral charges do not depend on branchings, our previous conclusion for $2 \to 3$ moves (which obviously still holds for $3 \to 2$ moves) holds for $0 \to 2$ charge transits. For such a transit $(T_1, H_1, c_1) \to (T_2, H_2, c_2)$, denote by $\Delta'$ and $\Delta''$ the new tetrahedra. It is easy to verify (see Lemma 3.4) that it is explicitly defined by $s_1 := c_2(e^{-1}(e) \cap \Delta') + c_2(e^{-1}(e) \cap \Delta'') = 0$ for each $e \in E(T_1) \cap E(T_2)$, $s_2 := c_2(e^{-1}(e) \cap \Delta') + c_2(e^{-1}(e) \cap \Delta'') = 2$ on the new interior edge $e_c$, and $s_3 := c_2(e^{-1}(e') \cap \Delta') + c_2(e^{-1}(e'') \cap \Delta'') = 2$ on the edges $e'$ and $e''$ opposite to $e_c$ in $\Delta'$ and $\Delta''$ respectively. (Of course, we also have the first condition in 3.3 (1)).

Figure 8: $2 \to 3$ charge transits are generated by Neumann’s vectors $w(e)$. 


Finally consider bubble moves. Remark that a bubble move \((T_1, H_1) \rightarrow (T_2, H_2)\) is \textit{abstractly} obtained from the final configuration of a \(0 \rightarrow 2\) move by gluing two more faces. Namely, denote by \(\Delta'\) and \(\Delta''\) the tetrahedra produced by the bubble move: we may view the face of \(\Delta' \cap \Delta''\) containing the two edges of \(H_2\) as obtained by gluing two faces in the final configuration of the \(0 \rightarrow 2\) move. Define the charge transit for a bubble move via the very same formulas as for a \(0 \rightarrow 2\) move. This makes sense, for the sum of the charges is equal to \(s_1 = 0\) along each of the two new edges of \(H_2\), to \(s_2 = 2\) along the other interior edge of \(\Delta' \cap \Delta''\), and to \(s_3 = 2\) along the edge of \(H_1\). This proves our claim, since nothing else is altered.

\[ -\alpha_1 + \alpha_2 - \alpha_3 = (\alpha_4 + \alpha_6 - 1) + (2 - \alpha_4 - \alpha_2) + (\alpha_5 + \alpha_7 - 1) = \alpha_6 + \alpha_7, \]

where we use the first two relations of Def. 3.5 (1) for \(c_2\).

\[
\begin{align*}
\alpha_1 & = 0, \\
\alpha_2 & = \alpha_4 + \alpha_6 - 1, \\
\alpha_3 & = 2 - \alpha_4 - \alpha_2, \\
\alpha_4 & = \alpha_5 + \alpha_7 - 1, \\
\alpha_5 & = \alpha_6 + \alpha_7.
\end{align*}
\]

Figure 9: proof of 3.5 (2) for \(c_2\).

Let us show that \(c_2\) also verifies Def. 3.5 (2). As above, it is enough to consider a \(2 \rightarrow 3\) move \((T_1, H_1) \rightarrow (T_2, H_2)\). Denote by \(e\) the edge that appears. We have to show that for any simple closed curve \(s\) without back-tracking with respect to \(T_1\) and \(T_2\) we have \(c_2(s) \equiv 0 \mod(2)\). Fig. 8 shows an instance of \(s\) in a section of the three tetrahedra of \(T_2\) glued along \(e\). In this picture the charges \(\alpha_i\) are attached to the dihedral angles of the tetrahedra. We have

\[ -\alpha_1 + \alpha_2 - \alpha_3 = (\alpha_4 + \alpha_6 - 1) + (2 - \alpha_4 - \alpha_2) + (\alpha_5 + \alpha_7 - 1) = \alpha_6 + \alpha_7, \]

where we use the first two relations of Def. 3.5 (1) for \(c_2\). Then \(c_2(s) = c_1(s) \equiv 0\).

ii) Consider the situation of Fig. 8, and use the notations of Example 3.9. The symbols \(E, D, F, A, C, B\) denote the charges on the top edges of \(\Delta_0, \Delta_2\) and \(\Delta_4\) respectively. The space of solutions of the system of equations (1)-(2), which defines \(c_2\) from \(c_1\), is one-dimensional. Hence there is a single degree of freedom in choosing these charges. Fix a particular choice for them (whence for \(c_2\)). If \(c_2'\) is defined by decreasing \(B\) by 1, we have

\[
\begin{align*}
\alpha_1 & = 0, \\
\alpha_2 & = \alpha_4 + \alpha_6 - 1, \\
\alpha_3 & = 2 - \alpha_4 - \alpha_2, \\
\alpha_4 & = \alpha_5 + \alpha_7 - 1, \\
\alpha_5 & = \alpha_6 + \alpha_7.
\end{align*}
\]

This shows that the integral charges on \(T_2\) obtained by varying the charge transit may only differ by a \(\mathbb{Z}\)-multiple of \(w(e)\).

\[
\begin{align*}
\alpha_1 & = 0, \\
\alpha_2 & = \alpha_4 + \alpha_6 - 1, \\
\alpha_3 & = 2 - \alpha_4 - \alpha_2, \\
\alpha_4 & = \alpha_5 + \alpha_7 - 1, \\
\alpha_5 & = \alpha_6 + \alpha_7.
\end{align*}
\]

Summarizing the last two sections we have proved that every triple \((W, L, \rho)\) admits \(\mathcal{D}\)-triangulations \((T, H, \mathcal{D} = (b, z, c))\). We have also proved that it admits \textit{full} \(\mathcal{D}\)-triangulations. Moreover, we have carefully described the (full) \(\mathcal{D}\)-transits.
Example 3.11 The tunnel construction. Here is nice example of fullable \( \mathcal{D} \)-triangulations of \((S^3, L \cup m)\) derived from link diagrams, where \(m\) is a meridian of an arbitrary component of the link \(L\). Remove two open 3-balls \(B^3\) from \(S^3\) away from the link \(L\); we get a manifold homeomorphic to \(S^2 \times [-1,1]\) with the 2-sphere \(\Sigma = S^2 \times \{0\}\) as simple spine. Consider, as usual, a generic projection \(\pi(L)\) of \(L \subset S^2 \times [-1,1]\) onto \(\Sigma\), considered up to isotopy, and encode it by a link diagram on \(\Sigma\) with support \(\pi(L)\). Dig tunnels on \(\Sigma\) around \(\pi(L)\), by respecting the under/over crossings, as in Fig. 10. Glue 2-discs inside the tunnels, between each of the tunnel junctions, such that their boundaries span a meridian of \(L\). Denote by \(P\) the standard spine thus obtained, and by \(T\) the dual triangulation of \(S^3\). The edges of \(T\) duals to the glued 2-discs make \(L\) (up to isotopy).

Using Seifert’s algorithm one may easily settle out two 2-regions \(R_{h_1}, R_{h_2}\) of \(P\), embedded in \(\Sigma\), that are dual to edges \(h_1\) and \(h_2\) of \(T\) which make (up to isotopy) the meridian \(m\) of \(L\). Let \(H\) be the union of these edges with the ones making \(L\). Then \((T,H)\) is a distinguished triangulation of \((S^3, L \cup m)\). It is clearly quasi-regular but not regular - look for instance at the gluings near the tunnel junctions. Note that a very particular branching on \(T\) can be obtained from a fixed orientation of \(\Sigma\), that is from an ordering on the boundary components of \(S^2 \times [-1,1]\), together with an orientation of \(L\). The glued discs are positively oriented in accordance with the orientations of \(L\) and \(S^3\). The other regions of \(P\) are positively oriented w.r.t. the transverse flow traversing \(S^2 \times [-1,1]\) from the first component towards the second one.

This construction is the core of the geometric part of 4.

4 Scissors congruences

4.1 The \(\mathcal{D}\)-pre-Bloch group

For the sake of clarity and for future reference we need to recall in an abstract setup the decorations of tetrahedra occurring in the \(\mathcal{D}\)-triangulations \(\mathcal{T} = (T, H, \mathcal{D} = (b, z, c))\) of \((W, L, \rho)\).

Let us fix a base embedded regular tetrahedron \(\Delta\) in the Euclidean 3-space \(\mathbb{R}^3\). One identifies \(\Delta\) with \(\psi(\Delta)\) whenever \(\psi\) is any orientation preserving cellular self-homeomorphism of \(\Delta\) which induces the identity map on the set of vertices of \(\Delta\). For any branching \(b\) we denote by \(E\) the set of oriented edges of \((\Delta, b)\). In \((\Delta, b)\) we select the ordered triple of oriented edges

\[ (e_0 = [v_0, v_1], \ e_1 = [v_1, v_2], \ e_2 = [v_0, v_2] = -[v_2, v_0]) \]
which are contained in the face opposite to the vertex $v_3$. For every $e \in E$ we denote by $e'$ the opposite edge.

![Figure 11: $(\Delta, b^+)$ and $(\Delta, b^-)$.](image)

Consider the two branchings $b^+$ and $b^-$ of $\Delta$ shown in Fig. 11. Up to orientation preserving Euclidean isometries of $\mathbb{R}^3$, any branching of $\Delta$ is of this form: accordingly, we say that $b$ is equivalent to $b^+$ resp. $b^-$, and write $b \sim b^+$ resp. $b \sim b^-$. When considered as an ordered triple of vectors at $v_3$, the triple $(e'_0, e'_1, e'_2)$ is for $b \sim b^+$ (resp. $b \sim b^-$) a positive (resp. negative) basis of $\mathbb{R}^3$ w.r.t. its standard orientation, given by the left-handed screw-rule. Thus it determines an orientation of $\Delta$, which we indicate by a sign $\ast = \pm$: $\ast(\Delta, b)$ if $b \sim b^+$.

**Definition 4.1** A $D$-decoration of $\ast(\Delta, b)$ is given by a $B$-valued 1-cocycle $z$ on $(\Delta, b)$ and an integral charge $c$, i.e. a map $c : E \to \mathbb{Z}$ such that $c(e) = c(e')$, $c(e) = c(-e)$, and $c(e_0) + c(e_1) + c(e_2) = 1$.

We denote by $D^*$ the set of $D$-tetrahedra $\ast(\Delta, b, z, c)$ and put $D = D^+ \cup D^-$. Let $S_4$ be the group of permutations on four elements, and $\varepsilon(s)$ be the signature of $s \in S_4$. Changing the branching of $\Delta$ induces a natural action

$$p_D : S_4 \times D \to D$$

$$p_D(s, \ast(\Delta, b, z, c)) = \ast \varepsilon(s) (\Delta, s(b), s(z), s(c)),$$

(2)

where $s(b)$, $s(z)$ and $s(c)$ are defined respectively from $b$, $z$ and $c$ just by permuting the ordered vertices of $(\Delta, b)$ in accordance with $s$. This means that forgetting the branching each edge $e \in E$ keeps the same charge, and that $z(e) = s(z)(e)$ iff $e$ has the same orientation w.r.t. both $b$ and $s(b)$ and $z(e) = s(z)(e)^{-1}$ iff the two orientations are opposite one to each other. But note for instance that the edge $e_1$ for $b^+$ coincides with $e_2$ for $b^-$ and so on.

**Definition 4.2** A singular $D$-tetrahedron is a continuous surjective map

$$\phi : \ast(\Delta, b, z, c) \to \phi(\Delta),$$

where $\phi(\Delta)$ is a compact subset of some oriented 3-manifold $W$ and $\phi$ satisfies the following properties:

1) The restriction of $\phi$ to every open $j$-cell, $j = 0, 1, 2, 3$, of the natural cell-decomposition of $\Delta$ is a homeomorphism.

2) The restriction of $\phi$ on $\text{Int}(\Delta)$ preserves the orientation if and only if the branching $b$ of $\mathcal{R}$ is equivalent to $b^+$: $b \sim b^+$. 

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3) \( \phi(\text{Int}(\Delta)) \cap \phi(\partial \Delta) = \emptyset \).

4) For any open 2-faces \( F_1 \) and \( F_2 \) of \( \Delta \), either \( \phi(F_1) \cap \phi(F_2) = \emptyset \), or \( \phi(F_1) = \phi(F_2) \), and at most two different open 2-faces can share the same image.

The set \( \phi(\Delta) \) is a quotient space of \( \Delta \) obtained by pairwise identifications of some faces of \( \Delta \). One identifies \( \phi \) with \( \phi \psi \) whenever \( \psi \) is any orientation preserving cellular self-homeomorphism of \((\Delta, b_\Delta)\) which induces the identity map on the set of vertices of \( \Delta \) (note that \( \psi \) preserves the orientation of each edge of \( \Delta \)). So we will often confuse such a class of singular tetrahedra with any of its representative. From now on we will consider only singular tetrahedra and we will abuse of the notations by not specifying the homeomorphism \( \phi \), which should be clear from the context.

Any \( \mathcal{D} \)-triangulation \( \mathcal{T} = (T, H, \mathcal{D} = (b, z, c)) \) of a triple \((W, L, \rho)\) may be seen as a collection of \( W \)-valued singular decorated tetrahedra such that the \( \phi(\Delta)'s \) form a singular triangulation of \( W \). The compatibility of the decorations of the \( \phi(\Delta)'s \) is a supplementary strong global constraint.

Let us denote by \( \mathbb{Z}[\mathcal{D}] \) the free \( \mathbb{Z} \)-module generated by \( \mathcal{D} \). We stipulate that the signs of the \( \mathcal{D} \)-tetrahedra (the generators) are compatible with the algebraic sum: if \( b_\Delta \sim b'_\Delta \), then

\[-(\sim(\Delta, b_\Delta, c_\Delta)) = (\Delta, b_\Delta, c_\Delta)\]

and so on. We now define the pre-Bloch group \( \mathcal{P}(\mathcal{D}) \), which is a quotient of \( \mathbb{Z}[\mathcal{D}] \) by (the linear extension of) the action \( p_\mathcal{D} \) and by a set of 5-terms relations. These relations are the algebraic analogues of all the instances of \( 2 \to 3 \) \( \mathcal{D} \)-transits.

In Fig. 12 - Fig. 13 one can see all the details for one instance of \( 2 \to 3 \) \( \mathcal{D} \)-transit, component by component: branching \( b \), integral charge \( c \) and cocycle \( z \). For the sake of simplicity we have used \( \text{Par}(B) \)-valued cocycles in Fig. 13.

**Figure 12**: An instance of charge transit.

**Definition 4.3** The \( \mathcal{D} \)-pre-Bloch group \( \mathcal{P}(\mathcal{D}) \) is the quotient of \( \mathbb{Z}[\mathcal{D}] \) by the linear extension of the action \( p_\mathcal{D} \) of \( S_4 \) and by the ideal generated by the 5-terms relations

\[ S(T_0) = S(T_1) \]  \hspace{1cm} (3)
Figure 13: An instance of cocycle transit.

where \( T_0 \rightarrow T_1 \) is any instance of 2 \( \rightarrow \) 3 \( \mathcal{D} \)-transit between decorated singular triangulations and \( S(T_i) \) denotes the formal sum of decorated tetrahedra of \( T_i \).

The relation (3) corresponding to the 2 \( \rightarrow \) 3 \( \mathcal{D} \)-transit of Fig. 12 - Fig. 13 is:

\[
\left( \Delta_1, b_1, (b-c, a-b+c, a), (\alpha(1) + \beta(2), \alpha(2) + \alpha(3)) \right) + \\
\left( \Delta_2, b_2, (d, b-c-d, b-c), (\alpha(1) + \alpha(3), \beta(2) + \beta(3)) \right) = \\
\left( \Delta_3, b_3, (d, a-d, a), (\alpha(1), \theta(1)) \right) + \left( \Delta_5, b_5, (d, b-c-d, b-c), (\alpha(3), \beta(3)) \right) \\
+ \left( \Delta_4, b_4, (b-c-d, a-b+c, a-d), (\beta(2), \alpha(2)) \right),
\]

where \( \Delta_1 \) and \( \Delta_2 \) (from top to bottom) are the two tetrahedra in \( T_0 \), and \( \Delta_3, \Delta_4 \) and \( \Delta_5 \) (from left to right and then behind) are the tetrahedra in \( T_1 \). Here we denote the cocycle \( z_i \) (resp. the charge \( c_i \)) of \( \Delta_i \) by its three (resp. two) first values.

One can repeat all the constructions of this section by replacing the group \( B \) with the parabolic subgroup \( \text{Par}(B) \). One defines in this way the (pre)-Bloch group \( \mathcal{P}(\mathcal{D}_P) \), and there is a natural homomorphism

\[
J : \mathcal{P}(\mathcal{D}_P) \rightarrow \mathcal{P}(\mathcal{D})
\]

induced by the inclusion of \( \text{Par}(B) \) into \( B \). This will play an important role in §7 and §8.

4.2 The \( \mathcal{D} \)-class

As before, let \((W, L, \rho)\) be a triple formed by a closed, connected and oriented 3-manifold \( W \), a non-empty link \( L \) in \( W \) and a flat principal \( B \)-bundle \( \rho \) on \( W \). To any \( \mathcal{D} \)-triangulation \( T = (T, H, \mathcal{D} = (b, z, c)) \) of \((W, L, \rho)\) one can associate an element \( \epsilon_P(T) \in \mathbb{Z}[^\mathcal{D}] \). It is defined as the formal sum of the singular decorated tetrahedra of \( T \) endowed with the decorations \( \mathcal{D}_\Delta = (b_\Delta, z_\Delta, c_\Delta) \) induced by \( \mathcal{D} \):
\[ c_D(T) = \sum_{\Delta \in T} \ast (\Delta, D_\Delta), \]

where \( \ast = + \) if \( b_\Delta \sim b_\Delta^+ \) and \( \ast = - \) if \( b_\Delta \sim b_\Delta^- \). In the rest of this section we prove that the class \( c_D(T) \in P(D) \) does not depend on the choice of \( T \). We begin by a technical lemma which shows that no moves on \( D \)-triangulations may introduce relations in \( \mathbb{Z}[D] \) that are independent from (3) - see Cor. 4.5.

Fix a \( D \)-decoration \((\Delta, b, z, c)\) on the base tetrahedron \( \Delta \) and an arbitrary pair \((u, u')\) of opposite edges of \( \Delta \). Let \((\Delta!, b!)\) be the branched tetrahedron obtained by deforming \( u \) until it passes through \( u' \). For any oriented edge \( e \) of \((\Delta, b)\) denote by \( e! \) the image of \( e \) in \( \Delta! \). Put \( z!(e!) = z(e) \). Let \( \alpha \) and \( \beta \) be respectively the charges of two consecutively oriented edges \( v \) and \( w \) of \( \Delta \), with \( v \) and \( w \) distinct from \( u \) and \( u' \), and set \( c!(v!) = -\alpha \) and \( c!(w!) = -\beta \). (This determines uniquely \( c! \) on \( \Delta! \)). We say that \((\Delta!, b!, z!, c!)\) is the mirror image of \((\Delta, b, z, c)\) w.r.t. \( u' \). Recall that we do not distinguish \( \Delta \) and \( \Delta! \) as bare tetrahedra. Then we will drop out the symbol "!" for \( \Delta! \).

Lemma 4.4 The following relation holds in \( P(D) \):

\[ (\Delta, b, z, c) = (\Delta, b!, z!, c!). \]

Proof. We are going to prove a particular instance of (3). All the other instances may be obtained in exactly the same way, for there is no restriction on the specific branching we choose in the arguments below. These arguments are done using a pictorial encoding with singular decorated tetrahedra, but this is no loss of generality since the corresponding algebraic relations in \( P(D) \) may be thought as between abstract elements.

Consider the sequence of 2 \( \rightarrow \) 3 \( D \)-transits in Fig. 14, where the first \( D \)-transit is the one of Fig. 12 - Fig. 13. We shall specify below the decorations in the second 2 \( \rightarrow \) 3 \( D \)-transit. Denote by \( D_i = (b_i, z_i, c_i) \) the decoration of \( \Delta_i \). We call \((\Delta_5, D_5)\) and \((\Delta_8, D_8)\) the two decorated tetrahedra glued along two faces in the final configuration (see Fig. 15): \((\Delta_8, D_8)\) is glued to \((\Delta_6, D_6)\) and \((\Delta_7, D_7)\) along \( f_1 \) and \( f_2 \).

Figure 14: how to produce two-terms relations.

The branchings and the cocycles of \((\Delta_6, D_6)\) and \((\Delta_7, D_7)\) are respectively the same as for \((\Delta_1, D_1)\) and \((\Delta_2, D_2)\). We choose the second 2 \( \rightarrow \) 3 \( D \)-transit in Fig. 14 so that the same holds for the integral charges. This is always possible due to Prop. 3.10 i) (2): the sum of charges in any angular sector of an edge stays equal during a charge transit. Hence we may identify \((\Delta_6, D_6)\) with \((\Delta_1, D_1)\) and \((\Delta_7, D_7)\) with
(Δ₂, D₂), when considered as singular decorated tetrahedra. As a consequence of the five-terms relations in P(D), we deduce that the composition of 2 → 3 D-transits in Fig. 14 translates in P(D) into the equality

\[(Δ₁, D₁) + (Δ₂, D₂) = (Δ₅, D₅) - (Δ₈, D₈) + (Δ₆, D₆) + (Δ₇, D₇)\]

\[= (Δ₅, D₅) - (Δ₈, D₈) + (Δ₁, D₁) + (Δ₂, D₂),\]

where we notice that D₈ gives a negative orientation to Δ₈. This yields

\[(Δ₅, D₅) - (Δ₈, D₈) = 0.\]  

(6)

The mirror image of (Δ₅, D₅) w.r.t. the interior edge in the left of Fig. 15 is (Δ₈, D₈). Namely, it is obvious that b₇! = b₈ and z₅! = z₆. Using Prop. 3.10 i) (2) two times we get

\[c₂(e₀) = c₅(e₀) + c₃(e₀) = c₅(e₀) + c₈(e₀) + c₆(e₀)\]
\[= c₅(e₁) + c₄(e₀) = c₅(e₁) + c₈(e₁) + c₆(e₁),\]

where cₙ(eᵢ) is the charge of the j-th edge of Δᵢ w.r.t. bᵢ. We have chosen the second 2 → 3 D-transit in Fig. 14 so that c₆(e₀) = c₂(e₀) and c₆(e₁) = c₂(e₁). Then c₅(e₀) = −c₈(e₀) and c₅(e₁) = −c₈(e₁). This shows that c₅! = c₈, concluding the proof of the lemma.

**Corollary 4.5** The relations in Z[D] corresponding to the 0 → 2 and distinguished bubble D-transits are consequences of the relations corresponding to 2 → 3 D-transits.

**Proof.** For any instance of 0 → 2 and distinguished bubble D-transit we assume that the faces in the initial configurations are endowed with branchings and cocycles. A 0 → 2 D-transit leads to singular tetrahedra (Δ₁, D₁) and (Δ₂, D₂) with mirror decorations, as defined before Lemma 4.4. This is clear for the branchings and the cocycles. For the integral charges, this follows, as in Lemma 4.4, from Prop. 3.10 i) (2): the sum of charges in any angular sector of an edge stays equal during a charge transit. This gives c₁(e₀) = −c₂(e₀) and c₁(e₁) = −c₂(e₁), where cᵢ(eᵢ) is the charge of the j-th edge of Δᵢ w.r.t. bᵢ, and we suppose that (Δ₂, D₂) is the mirror image of (Δ₁, D₁) w.r.t. (e₂, e₂'). With Lemma 4.4 we get the conclusion for 0 → 2 D-transits.

If we cut open the final configuration of a distinguished bubble D-transit along the interior face enclosed by the edges of H, we obtain the final configuration of a 0 → 2 D-transit. This is coherent with the definition of integral charges, which sum up
Theorem 4.6 The element $c_D(T) \in \mathcal{P}(D)$ does not depend on the choice of $T$.

For any $\mathcal{D}$-triangulation $\mathcal{T}$ of $(W, L, \rho)$ we call $c_D(W, L, \rho) = c_D(T)$ the $D$-scissors congruence class, or $D$-class, of $(W, L, \rho)$. This terminology is justified in §8. Note that we do not require $\mathcal{T}$ to be a full $\mathcal{D}$-triangulation. Since $(T, H)$ defines $(W, L)$ up to (PL) orientation preserving homeomorphisms, the $D$-class $c_D(W, L, \rho)$ only depends on $(W, L, \rho)$ up to homeomorphisms of pairs $\theta : (W, L) \to (W', L')$ which map $\rho$ to $\rho'$ and preserve the orientations of $W$ and $W'$.

**Proof.** Fix a model of $W$ and a flat $B$-bundle $\rho$ on $W$, with $L \subset W$ considered up to ambient isotopy.

Consider two $\mathcal{D}$-triangulations $\mathcal{T}$ and $\mathcal{T}'$ of $(W, L, \rho)$. Up to some distinguished bubble moves we can assume that $T$ and $T'$ have the same vertices and that the corresponding spines of $W$ adapted to $L$ have the same type (Def. 2.3); and coincide along $L$. Let us apply (dually) Prop. 2.6 to $(T, H)$ and $(T', H')$; we find $(T'', H'')$; as Prop. 2.6 uses only positive moves, the branchings transit. There are (non necessarily full) cocycle transits, and Prop. 3.10 i) implies that the charges also transit. Summing up, we have two decorated transits $\mathcal{T} \to \mathcal{T}_1$, $\mathcal{T}' \to \mathcal{T}_2$, where $\mathcal{T}_1 = (T'', H'', \mathcal{D}_1)$ and $\mathcal{T}_2 = (T'', H'', \mathcal{D}_2)$ and possibly $\mathcal{D}_1 \neq \mathcal{D}_2$. From Cor. 4.5 we know that $c_D(\mathcal{T}) = c_D(\mathcal{T}_1)$ and $c_D(\mathcal{T}') = c_D(\mathcal{T}_2)$. If $c_D(\mathcal{T}_1) = c_D(\mathcal{T}_2)$ the theorem follows. We now prove that given any $\mathcal{T} = (T, H, \mathcal{D} = (b, z, c))$ the $D$-class $c_D(\mathcal{T})$ does not depend on the specific decoration $\mathcal{D}$.

**Branching invariance.** A different choice of the branching $b$ induces a corresponding action (2) of $S_b$ on the decorated singular tetrahedra of $\mathcal{T}$. The definition of $c_D(\mathcal{T})$ as a sum of signed decorated tetrahedra implies that its class in $\mathcal{P}(\mathcal{D})$ is invariant under this action.

**Charge invariance.** We will localize the problem. Fix any edge $e$ of $T \setminus H$. Consider the set of integral charges which differ from $c$ only on $\text{Star}(e, T)$. It is of the form (we use the notations of Prop. 3.8)

$$C(e, c, T) = \{ c' = c + \lambda w(e), \; \lambda \in \mathbb{Z} \}.$$ 

Thanks to Prop. 3.8, it is enough to prove that $c_D(T, H, (b, z, c)) = c_D(T, H, (b, z, c'))$ when $c'$ varies in $C(e, c, T)$. This result is an evident consequence of the following facts:

1. Let $(\mathcal{T}, c) \to (\mathcal{T}'', c'')$ be any $2 \to 3$ charge transit such that $e$ is a common edge of $T$ and $T''$. Then the result holds for $C(e, c, T)$ if and only if it holds for $C(e, c'', T'')$.

2. There exists a sequence of $2 \to 3$ transits $\mathcal{T} \to \cdots \to \mathcal{T}''$ such that $e$ persists at each step, and $\text{Star}(e, T'')$ is like the final configuration of a $2 \to 3$ move with $e$ playing the role of the central common edge of the 3 tetrahedra.

3. If $\text{Star}(e, T)$ is like $\text{Star}(e, T'')$ in (2), then the result holds for $C(e, c, T)$.

Property (1) is a consequence of the following facts: $C(e, c, T)$ transits to $C(e', c', T'')$ due to Prop. 3.8 and Prop. 3.10 and $c_D(\mathcal{T})$ is not altered by $\mathcal{D}$-transits.
To prove (2) it is perhaps easier to think, for a while, in dual terms. Consider the dual region $R = R(e)$ in $P = P(T)$. The final configuration of $e$ in $T''$ corresponds dually to $R$ being an embedded triangle. More generally, there is a natural notion of geometric multiplicity $m(R,a)$ of $R$ at each edge $a$ of $P$, and $m(R,a) \in \{0, 1, 2, 3\}$. We say that $R$ is embedded in $P$ if for each $a$, $m(R,a) \in \{0, 1\}$. If $R$ has a loop in its boundary, a suitable $2 \rightarrow 3$ move at a proper edge of $P(T)$ having a common vertex with the loop puts proper edges in place of the loop. Each time $R$ has a proper (i.e. with two distinct vertices) edge $a$ with $m(R,a) \in \{2, 3\}$, the (non-branched) $2 \rightarrow 3$ move at $a$ put new edges $a'$ with $m(R,a') \leq 2$ in place of $a$. In the situation where this is an equality, remark that if we first blow up an edge $b$ adjacent to $a$ and such that $m(R,b) = 2$, and then apply the $2 \rightarrow 3$ move along $a$, we get $m(R,a') = 1$ (look at Fig. 16). By induction, up to $2 \rightarrow 3$ moves, we can assume that $R$ is an embedded polygon. To obtain the final configuration of $e$ in $T''$ let us come back to the dual situation. We possibly have more than 3 tetrahedra around $e$. It is not hard to reduce the number to 3, via some further $2 \rightarrow 3$ moves.

![Figure 16: evolution of the geometric multiplicity of $R$ when blowing-up $a$.](image)

Concerning property (3), we would like to do first a $3 \rightarrow 2$ $D$-transit on $e$ and then a $2 \rightarrow 3$ $D$-transit, varying the charge transit $(T,c) \rightarrow (T',c')$. By Prop. 3.10 ii) we know that the charges $c''$ exactly describe $C(e,c',T')$. Since $c_D(T)$ is not altered by $D$-transits, this would conclude. But there is a little subtlety: in general, the branching $b$ does not transit during a $3 \rightarrow 2$ move. Anyway, we can modify the branching $b$ on the 3 tetrahedra around $e$ in such a way that the $3 \rightarrow 2$ move becomes branchable. So we have on $T$ the original branching $b$ and another system of edge-orientations $g$. Applying Prop. 3.11 to $(T,g)$, we find $T'$ with two branchings: $b'$ by the transit of $b$, and the branching $b''$ obtained from $g$. Note that $e$ persists, since Prop. 3.1 uses only positive moves, and $\text{Star}(e,T') = \text{Star}(e,T)$. Moreover we have a charge transit $(T,c) \rightarrow (T',c')$ with $c$ and $c'$ which agree on $\text{Star}(e,T')$. So, using the branching invariance of the $D$-class, we may assume that the $3 \rightarrow 2$ move is branchable. Since $e \in T \setminus H$, this move is also possible at the level of integral charges and the charge invariance is thus proved.

Suppose now that $e \in H$. The analogue of (1) for distinguished bubble moves is true for the same reasons. Then, applying a distinguished bubble move on a face of $T$ containing $e$ we are brought back to the previous situation.

**Cocycle-invariance.** Let $T$ and $T'$ be two $D$-triangulations of $(W,L)$ which only differ by cocycles $z, z'$ representing the same cohomology class. We have to show that $c_D(T) = c_D(T')$. The two cocycles differ by a coboundary $\delta \lambda$, and it is enough to consider the elementary case when the 0-cochain $\lambda$ is supported by one vertex $v_0$ of $T$. Again we have “localized” the problem. The invariance of $c_D(T)$ for distinguished bubble $D$-transits gives the result in the special situation when $v_0$ is the new vertex after the move. We will reduce the general case to this special one by means of $D$-transits.
It is enough to show that we can modify Star($v_0, T$) to reach the star-configuration of the special situation. But Star($v_0, T$) is the cone over Link($v_0, T$) (which is homeomorphic to $S^2$). So Star($v_0, T$) is determined by the triangulation of Link($v_0, T$).

Recall the definition of $1 \to 4$ moves from Remark 3.7. One sees that performing $2 \to 3$ and $1 \to 4$ moves around $v_0$ in such a way that $v_0$ persists, their traces on Link($v_0, T$) are $2 \to 2$ moves (2-dimensional analogues of the $2 \to 3$ moves) or $1 \to 3$ moves (2-dimensional analogues of the $1 \to 4$ moves). It is well known that these moves are sufficient to connect any two triangulations of a given surface. We only have to take into account the technical complication due to the fact that, in our situation, the $1 \to 4$ moves must be moves of distinguished triangulations. For this, we need to involve some edges of $H$ in the $1 \to 4$ moves, which is always possible.

Returning to the beginning of the proof, we deduce that $c_D(T_1) = c_D(T_2)$. This concludes.

If $(W, L, \rho)$ is a triple such that $\rho \in H^1(W; \text{Par}(B)) \cong H^1(W; \mathbb{C})$, the same arguments allow to define a $D_p$-scissors congruence class $c_{D_p}(W, L, \rho)$, such that $c_D(W, L, \rho) = J(c_{D_p}(W, L, \rho))$, where $J$ is the homomorphism defined in [4].

## 5 Quantum hyperbolic invariants

Let $(W, L, \rho)$ be as usual, and fix any full $D$-triangulation $T = (T, H, (b, z, c))$ of $(W, L, \rho)$. Let $N > 1$ be an odd integer. Fix a determination of the $N$-th root which holds for all the matrix entries $t(e)$ and $x(e)$ of $z(e)$, for all the edges $e$ of $T$. The reduction mod($N$) $T_N$ of $T$ consists in changing the decoration of each edge $e$ of $T$ as follows:

- $(a(e) = t(e)^{1/N}, y(e) = x(e)^{1/N})$ instead of $z(e) = (t(e), x(e))$;
- $c_N(e) = c(e)/2 \text{ mod}(N)$ instead of $c(e)$ (it makes sense because $N$ is odd).

Let us interpret this new decoration. Details and explicit formulae concerning the quantum data are given in the Appendix; for the proofs we refer to [3, Ch. 3].

Each $(a(e), y(e))$ describes an irreducible $N$-dimensional cyclic representation $r_N(e)$ of a quantum Borel subalgebra $W_N$ of $U_q(sl(2, \mathbb{C}))$, specialized at the root of unit $\omega_N = \exp(2\pi i/N)$. By “cyclic representation” we mean that the generators of $W_N$ act as automorphisms.

Call $F(T)$ the set of 2-faces of $T$. A $N$-state of $T$ is a function $\alpha : F(T) \to \{0, 1, \ldots, N - 1\}$ (in fact one often identifies $\{0, 1, \ldots, N - 1\}$ and $\mathbb{Z}/N\mathbb{Z}$). The state $\alpha$ can be considered as a family of functions $\alpha_i : F(\Delta_i) \to \{0, 1, \ldots, N - 1\}$ which are compatible with the face pairings.

Consider on each branched tetrahedron $(\Delta_i, b_i)$ of $T$ the ordered triple of oriented edges $(e_0 = [v_0, v_1], e_1 = [v_1, v_2], e_2 = [v_2, v_0])$ which are the opposite edges to the vertex $v_3$. The cocycle property of $\alpha$ and the fullness assumption (this is crucial at this point, due to the algebraic structure of the cyclic representations of $W_N$) imply that $r_N(e_0) \otimes r_N(e_1)$ coincides up to isomorphism with the direct sum of $N$ copies of $r_N(e_2)$. This set of data allows us to associate to every $(\Delta_i, b_i, r_{N,i}, \alpha_i)$ a 6j-symbol $R(\ast(\Delta_i, b_i, r_{N,i}, \alpha_i)) \in \mathbb{C}$, that is a matrix element of a suitable “intertwiner” operator. The reduced charge $c_N$ is used to slightly modify this operator in order to get its (partial) invariance up to branching changes. In this way one gets the (partially) symmetrized $c$-6j-symbols $\Psi(\ast(\Delta_i, (D_N), \alpha_i)) = \Psi(\ast(\Delta_i, b_i, r_{N,i}, c_{N,i}, \alpha_i)) \in \mathbb{C}$.

We are now ready to define the state sum, which is a weighted operator trace. Set
\[
\Psi(T_N) = \sum_{\alpha} \prod_{i} \Psi(\Delta_i, (D_N)_i, \alpha_i) \\
H(T_N) = \Psi(T_N) \cdot N^{-r_0} \prod_{e \in E(T) \setminus E(H)} x(e)^{(1-N)/N},
\]
where \(r_0\) is the number of vertices of \(T\).

**Proposition 5.1** Let \(T = (T, H, (b, z, c))\) be a full \(D\)-triangulation of \((W, L, \rho)\) and \(T \to T'\) be a full \(D\)-transit. Fix a determination of the \(N\)-th root that holds for all entries of both \(z\) and \(z'\). Up to \(N\)-th roots of unity we have \(H(T_N) = H(T'_N)\).

**Proof.** This follows immediately from the behaviour of the c-6j-symbols w.r.t. \(D\)-transits. See Prop. 5.3, 9.10 and 9.11. Beware that these statements are given for specific branchings, and that for other branchings they would involve \(N\)-th roots of unity, due to Prop. 9.11. \(\Box\)

**Theorem 5.2** Let \(T = (T, H, D = (b, z, c))\) be a full \(D\)-triangulation of \((W, L, \rho)\). Up to \(N\)-th roots of unity \(H(T_N)\) neither depends on the choice of \(T\) nor on the determination of the \(N\)-th root. Hence \(K_N(W, L, \rho) := K(T_N) := H(T_N)^N\) is a well-defined invariant of \((W, L, \rho)\).

As for \(D\)-classes, the state sum invariants \(K_N(W, L, \rho)\) only depend on \((W, L, \rho)\) up to orientation preserving homeomorphisms of pairs \(\theta : (W, L) \to (W', L')\) which map \(\rho\) to \(\rho'\).

**Proof.** The state sum \(H(T_N)\) does not depend on the choice of the determination of the \(N\)-th root because the functions \(h\) and \(\omega\) in the c-6j-symbols, defined in Prop. 5.6, are homogeneous of degree 0 (see (16) and (19)). Lemma 5.7 also shows that if we change the branching one multiplies \(H(T_N)\) by a \(N\)-th root of unity. As we are free to choose the branching, we stipulate that from now on we use only total-ordering branchings on our quasi-regular triangulations.

By Prop. 5.3 we know that \(H(T_N)\) is not altered by full \(D\)-transits. Also, Lemma 3.4 implies that given any quasi-regular sequence \((T, H, b, z) \to \ldots \to (T', H', b', z')\) with \(z\) and \(z'\) full cocycles, one may generically change \(z\) and \(z'\) in order to guarantee full transits. This will cause no trouble because, by continuity, it is enough to prove the present statement for full cocycles arbitrarily close to the one of \(T\). Then the rest of the proof is almost the same as for Th. 4.6. We only have to verify that, without altering the arguments, one can turn the sequences of moves used in Th. 4.6 for proving the charge and the cocycle invariance into distinguished quasi-regular ones.

For this, do distinguished bubble moves and slide their capping discs as in Prop. 2.10. For the charge invariance this is always possible because these moves may not increase the geometric multiplicity of the the edges of the region \(R\) under consideration. Also, remark that we do not need property (3), since negative \(3 \to 2\) moves are branchable for total-ordering branchings. For the cocycle invariance, recall that we only have to control the modifications of \(\text{Link}(v_0, T)\). Since \(T\) is quasi-regular \(\text{Link}(v_0, T)\) is a 2-dimensional quasi-regular triangulation. Proving that we may find distinguished quasi-regular sequences of moves in \(\text{Star}(v_0, T)\) that make it like in a distinguished bubble move is equivalent to proving the same result for \(\text{Link}(v_0, T)\).

Then we need a 2-dimensional analogue of Prop. 2.10. It is done as follows. Recall that the traces on \(\text{Link}(v_0, T)\) of \(2 \to 3\) and \(1 \to 4\) moves in \(\text{Star}(v_0, T)\) for which \(v_0\) persists are \(2 \to 2\) moves (2-dimensional analogues of the \(2 \to 3\) moves) or \(1 \to 3\) moves (2-dimensional analogues of the \(1 \to 4\) moves). Consider an arbitrary
sequence of moves in Link($v_0, T$), induced from a distinguished one in Star($v_0, T$), and that makes it like in a distinguished bubble move. View it a sequence $s: \ldots \rightarrow P \xrightarrow{m_0} P_1 \xrightarrow{m_1} P_2 \xrightarrow{m_2} \ldots$

between the (1-dimensional) dual spines. A non quasi-regular move on a spine is the flip of an edge that makes it the frontier of a same region. Let $m_0$ be the first non quasi-regular move in $s$. A step before $m_0$ let us first apply the “relative” $r_P(m_1)$ of $m_1$ on $P$, where by “relative” we mean the move along the same edge; we get $Q$. Then apply $r_Q(m_0)$; see the bottom sequence of Fig. 17. Note that $r_Q(m_0)$ is necessarily quasi-regular, for otherwise $m_0$ would not be the first non quasi-regular move in $s$. We claim that $r_P(m_1)$ is also quasi-regular. Indeed, in $P$ we necessarily have one of the two situations of Fig. 18, where the dotted arcs represent boundary edges. In the first situation, $r' = r''$ is impossible. In the second one, if $r' = r''$ then $r' = r$ and $m_0$ is not the first non quasi-regular move in $s$, thus giving a contradiction. Hence the sequence $r_Q(m_0) \circ r_P(m_1)$ is necessarily quasi-regular. Moreover we have (see Fig. 17):

$$P' = r_P(m_0) \circ m_1 \circ m_0 (P) = r_Q(m_0) \circ r_P(m_1) (P).$$

Figure 17: the 2-dimensional analogue of Prop. 2.10.

Figure 18: the proof that $r_P(m_1)$ is quasi-regular.

This implies that we can modify $s$ locally so as to obtain

$$s': \ldots \rightarrow P \xrightarrow{r_P(m_1)} Q \xrightarrow{r_Q(m_0)} P' \xrightarrow{r_P'(m_0)} P_2 \xrightarrow{m_2} \ldots,$$

where the first possible non quasi-regular move is $r_P'(m_0)$. The length of $s'$ after $r_P'(m_0)$ is less than the length of $s$ after $m_0$. Then, working by induction on the
length, replacing each non quasi-regular move as above and noting that $1 \rightarrow 3$
moves are always quasi-regular, we get a quasi-regular $s'$. It induces a distinguished and
quasi-regular sequence of moves in $\text{Star}(v_0, T)$. Thus we have proved the $2$-
dimensional analogue of Prop. 2.11 which concludes the proof of the theorem.
\[\Box\]

6 Complements on QHI

Duality. There is a natural involution on the argument $W$ of the triple $(W, L, \rho)$, obtained just by changing its orientation. One has another involution on the bundle argument, by passing to the bundle with complex conjugate representative cocycles. The QHI duality property relates these involutions. Let $\mathcal{T} = (T, H, \mathcal{D})$ and $z$ be as in (31). Let us denote by $z^*$ the complex conjugate full cocycle, $\rho^* = \overline{[z]}$, $\mathcal{D}^* = (b, z^*, c)$ and $\hat{W}$ the manifold $W$ with the opposite orientation.

Proposition 6.1 \((K_N(W, L, \rho))^* = K_N(\hat{W}, L, \rho^*)\). In particular, if $\rho$ is a flat $B(2, \mathbb{R})$-bundle, then

\[K_N(W, L, \rho))^* = K_N(\hat{W}, L, \rho) .\]

Proof. Given a matrix $A$ let $^TA$ denote its transpose. Changing the orientation of $W$ turns a c-6j-symbol $\Psi(*(\Delta_i, (\mathcal{D}_i)_i, \alpha_i))$ into $^T\Psi(\overline{*}(\Delta_i, (\mathcal{D}_i)_i, \alpha_i))$. But Prop. 3.2 shows that

\[^T\Psi(\overline{*}(\Delta_i, (\mathcal{D}_i)_i, \alpha_i)) = (\Psi(*(\Delta_i, (\mathcal{D}_i)_i, -\alpha_i)))^* .\]

Since the state sums $K(\mathcal{T}_N)$ do not depend on $\alpha$, this yields the conclusion. \[\Box\]

QHI projective invariance. A main problem concerning the QHI is to understand $K_N(W, L, \cdot)$ as a function of the bundle argument $\rho$. Here is a very partial contribution in this direction. Let $\mathcal{T}$ be a full $\mathcal{D}$-triangulation of $(W, L, \rho)$, and write $z(e) = (t(e), x(e))$ as in (32). For any $\lambda \neq 0$ we can turn $z$ into $z_{\lambda}$, where for any $e \in E(T)$ we put $t_{\lambda}(e) = t(e)$ and $x_{\lambda}(e) = \lambda x(e)$. In this way we define a full $\mathcal{D}$-triangulation $\mathcal{T}_\lambda$ for some $(W, L, \rho_\lambda)$.

Proposition 6.2 For each $\lambda \neq 0$ we have $K_N(W, L, \rho_\lambda) = K_N(W, L, \rho)$.

Proof. The functions $h$ and $\omega$ in the c-6j-symbols are homogeneous of degree 0 (see (14), (15)). Then for each tetrahedron $\Delta_i$ of $T$, $\Psi(*(\Delta_i, (\mathcal{D}_{\lambda,N})_i, \alpha_i))$ adds a factor $\lambda^{-1}$ in the state sum $K((\mathcal{T}_\lambda)_N)$. Denote by $r_i$ the number of $i$-simplices of $T$; remark that $r_0$ is also the number of edges of $H$. The Euler characteristic of $W$ is 0:

\[\chi(W) = r_0 - r_1 + r_2 - r_3 = 0 .\]

Each tetrahedron $\Delta_i$ has four faces and each face belongs to exactly two tetrahedra. Hence $r_2 = 2r_3$, and $r_3 = r_1 - r_0$. Since there are only $r_1 - r_0$ edge contributions coming from $T \setminus H$ in the formula of $K((\mathcal{T}_\lambda)_N)$, each one being equal to $\lambda^{-2r_0}$, this yields the conclusion. \[\Box\]

QHI as invariants of the $\mathcal{D}$-scissors congruence class. As the value of $K_N(W, L, \rho) = K(\mathcal{T}_N)$ does not depend on the choice of any full representative $\mathcal{T}$ of $\mathfrak{c}_D(W, L, \rho)$, one would like to consider $K_N(W, L, \rho)$ as a function of the $\mathcal{D}$-scissors congruence class. This is not completely correct because the face pairings between the $\mathcal{D}$-tetrahedra of $\mathcal{T}$ are not encoded in the representatives $\mathfrak{c}_D(\mathcal{T})$ of
\[ \mathcal{C}(W, L, \rho) \], which is just a formal linear combination of \( \mathcal{D} \)-tetrahedra. But the “states” as well as the non-\( \Psi(T_N) \) factors in the right-hand side of (\[ \mathcal{D} \]) depend on the face pairings. This is a technical point which can be overcome by looking at \( K(T_N) \) as a well-defined function of an “augmented” \( \mathcal{D} \)-class \( \mathcal{C}(W, L, \rho) \). This class belongs to an “augmented” (pre)-Bloch-like group \( \mathcal{P}(\mathcal{D}) \) which dominates \( \mathcal{P}(\mathcal{D}) \) via a natural “forgetting map” \( f \) such that \( \mathcal{C}(W, L, \rho) = f(\mathcal{C}(W, L, \rho)) \). All the details of this construction are given in [3], where some results of the present paper have been announced, and it is not so important to reproduce them here. Then, roughly speaking, we may consider \( K_N(W, L, \rho) \) as a function of the \( \mathcal{D} \)-class.

\[ \rho \text{-Dehn surgery.} \] Let us consider more general triples \((W, L, \rho)\), where \( \rho \) is a flat \( B \)-bundle defined on \( W \setminus L \), and not necessarily on the whole of \( W \). In other words, \( \rho \) may have a non trivial holonomy along the meridians of \( L \). Here we show that using an elementary procedure which is reminiscent of Thurston’s hyperbolic Dehn surgery (see e.g. [3], Ch. E)), several of these more general triples can be transformed into usual ones to which the results of the present paper apply. We face more extensively the complete extension of the QHI theory to bundles with non trivial holonomy in [3].

To simplify the notations, let us assume that \( L \) is a knot; everything works similarly for an arbitrary link. As usual, let \( M \) be the complement in \( W \) of an open tubular neighbourhood \( U(L) \) of \( L \). Denote by \( Z = \partial M \) the boundary torus. Fix on orientation of \( L \) and a basis \((m, l)\) of \( \pi_1(Z) \cong H_1(Z; \mathbb{Z}) \). We orient the meridian \( m \) of \( L \) positively w.r.t. the orientation of \( L \) and the orientation of \( W \). The longitude \( l \) is oriented in such a way that \((m, l)\) gives the boundary orientation of \( Z \).

Abusing of notations, we denote by \( \rho \) any representative of its conjugacy class in \( \text{Hom}(\pi_1(W), B)/B \). Up to conjugation, we may assume that the elements \( \alpha = \rho(m) \) and \( \beta = \rho(l) \) of \( B \) either belong to the parabolic subgroup \( \text{Par}(B) \cong (\mathbb{C},+) \) or to the Cartan subgroup \( C(B) \cong \mathbb{C}^* \). In any case, for every \((s, r) \in \mathbb{Z}^2\) we have

\[ \rho(sm + rl) = \alpha^r \beta^s . \]

We are looking for \((s, r) \in \mathbb{Z}^2\) with \( \gcd(s, r) = 1 \) and \( \alpha^r \beta^s = 1 \). If \( \alpha = (1, x) \) and \( \beta = (1, y) \) belong to \( \text{Par}(B) \), the equation \( \alpha^r \beta^s = 1 \) reads

\[ sx + ry = 0 , \]

and a solution \((s, r) \) exists iff \( x/y \) belongs to \( \mathbb{Q} \). Similarly, if \( \alpha = (t, 0) \) and \( \beta = (z, 0) \) belong to \( C(B) \), the equation \( \alpha^r \beta^s = 1 \) reads

\[ t^s z^r = 1 , \]

and a solution \((s, r) \) exists iff \( \log(t)/\log(z) \in \mathbb{Q} \).

When such a pair \((s, r) \) exists, it is called a \( \rho \)-surgery coefficient. Let us denote by \( W' = W_{(s, r)} \) the closed manifold obtained from \( M \) by the Dehn filling of \( Z \) with coefficient \((s, r) \). The bundle \( \rho \) extends as \( \rho' = \rho_{(s, r)} \) on the whole of \( W' \). If \( L' \) denotes the core of the filling, then \((W', L', \rho') \) is a triple canonically associated to \((W, L, \rho) \). Indeed, since \( \text{Aut}(H_1(Z; \mathbb{Z})) = \text{SL}(2, \mathbb{Z}) \), the existence of \( \rho \)-surgery coefficients is an intrinsic property of \( \rho \), and the construction of \((W', L', \rho') \) does not depend on the choice of the basis \((m, l)\) but only on the isotopy class \([c]\) of the oriented curve \( c = sm + rl \). Hence we may apply the constructions of the paper to \((W', L', \rho') \). In particular, one may define the \( \rho \)-surgery invariants

\[ \mathcal{C}_D(M, [c], \rho) := \mathcal{C}_D(W', L', \rho') , \quad K_N(M, [c], \rho) := K_N(W', L', \rho') \]

of \((M, [c], \rho) \).
Fix a basis \((m, l)\) as above. It is well-known that the kernel of the map
\[
i_* : H_1(Z; \mathbb{Q}) \to H_1(M, \mathbb{Q})
\]
is a Lagrangian subspace \(\mathcal{L}\) of \(H_1(Z; \mathbb{Q})\) w.r.t. the intersection form. Up to a change of base, we can assume that \(\mathcal{L}\) is generated by the homology class of \(pm + ql\), where \(p, q \in \mathbb{Z}\) and \(\text{gcd}(p, q) = 1\). Identifying \(H^1(M; \text{Par}(B))\) with \(H^1(M; \mathbb{C})\), it follows that \((p, q)\) is a \(\rho\)-surgery coefficient for \(\rho \in H^1(M; \text{Par}(B))\), and that for any \(\rho \in H^1(M; \text{Par}(B))\) there exist \(\rho\)-surgery invariants. Note that \(W = W(p, q)\) when \(q = 0\). Also, using the map \(\exp : H^1(M; \mathbb{C}) \to H^1(M; \mathbb{C}^*) \cong H^1(M; C(B))\) induced by the exponential \(\exp : \mathbb{C} \to \mathbb{C}^*\), we see that for any \(\rho \in H^1(M; C(B))\) there exist \(\rho\)-surgery invariants.

We shall see in \(\S 8\) that \(\text{Par}(B)\)-bundles play a very significant role in QHI theory. This shows that the above generalization of the QHI to bundles on \(M\) coming from the abelian simplicial cohomology is meaningful. In particular, the case of bundles that are trivial on the boundary of \(M\) belong to this specialization, and for them any Dehn filling is \(\rho\)-admissible.

Finally, note that one can specialize the choice of the link. For example, we may take \(L\) as the trivial knot embedded in an open ball of \(W\). We obtain QHI invariants of closed oriented manifolds, possibly endowed with non-trivial bundles \(\rho\).

\section{The \(I\)-pre-Bloch group and the idealization}

In this section we define a group \(\mathcal{P}(I)\) which is a version of Neumann’s extended pre-Bloch group \(\mathcal{D}(I)\). We show that a remarkable specialization of the \(\mathcal{D}\)-pre-Bloch group \(\mathcal{P}(\mathcal{D})\) defined in \(\S 4.1\) maps onto \(\mathcal{P}(I)\). We call the resulting homomorphism the \textit{idealization}. It is the key ingredient in the formulation of the Volume Conjecture given in \(\S 7\).

We use the notations of \(\S 4.1\). For every complex number \(w\), let \(\text{Im}(w)\) denotes its imaginary part and set
\[
\Pi^+ = \{w \in \mathbb{C} ; \text{Im}(w) > 0\} , \quad \Pi^- = \{w \in \mathbb{C} ; \text{Im}(w) < 0\} , \quad \Pi^0 = \mathbb{R} \setminus \{0, 1\} .
\]

\begin{definition}
An \(I\)-decoration (or ideal decoration) of \(*(\Delta, b)\) is given by an integral charge \(c\) as in Def. \(4.1\) and a map \(w : \mathcal{E} \to \Pi^+ \cup \Pi^0\) such that:

1) For every \(e \in \mathcal{E}\), \(w(e) = w(e')\).

2) Set \(w_1 = w(e_1), i = 0, 1, 2\). Then
\[
w_0w_1w_2 = -1 \quad \text{and} \quad w_0w_1 - w_1 = -1 .
\]

Let \(I^*\) be the set of \(I\)-tetrahedra \(*(\Delta, b, w, c)\) and put \(I = I^+ \cup I^-\). We denote \(Z[I]\) for the free \(\mathbb{Z}\)-module generated by \(I\). There is a natural action \(p_D\) of \(\mathcal{S}_4\) on \(I\) which acts as \(p_D\) in \(\mathcal{E}\) on \(b\), \(*\) and \(c\); moreover \(s(w)(e) = w(e)^{c(s)}\).

Clearly, \(w_1 = (1 - w_0)^{-1}, w_2 = (1 - w_1)^{-1}\) and \(w_0 = (1 - w_2)^{-1}\), so that it is enough to specify \(w = w_0\) in order to completely determine the map \(w\) of an ideal decoration (whence the abuse of notation). Sometimes we shall write also \((w, w', w'')\) instead of \((w_0, w_1, w_2)\). In fact, \((w_0, w_1, w_2)\) may be considered as a \textit{modular triple} of an ideal tetrahedron of the hyperbolic space \(\mathbb{H}^3\), with ordered vertices \(v_0, v_1, v_2, v_3\) on the boundary \(\partial \mathbb{H}^3 = \mathbb{C}P^1\) and with
\[
w = \frac{(v_2 - v_1)(v_3 - v_0)}{(v_2 - v_0)(v_3 - v_1)} .
\]
Remark 7.2 Our orientation convention in Def. 7.1 states that when the imaginary part of $w$ is not zero and the branching $b$ of $\Delta$ is equivalent to $b^+,$ then $\text{Im}(w) > 0$ and the ideal tetrahedron $(\Delta, b, w, c)$ is positively oriented and with positive volume. The same convention holds for $b \sim b^-$ with the opposite sign. Geometrically degenerated flat tetrahedra corresponding to real modular triples are allowed, but also in this case the branching specifies the orientation.

$\mathcal{I}$-transits. One define $\mathcal{I}$-transits in the same way as $\mathcal{D}$-transits. The transits of branchings and integral charges are the same for both kinds of decorations. We call ideal transit the transit of modular triples. In Fig. 19 one can see an instance of $2 \to 3$ ideal transit. Only some members of the modular triples are indicated; See Fig. 12 for the charge transit.

\[ T_0 \quad \Rightarrow \quad T_1 \]

$T_0$ $T_1$

**Figure 19:** An instance of ideal transit.

The initial and the final configurations of an $\mathcal{I}$-transit describe the decomposition of a branched hyperbolic ideal polyhedron $Q$ in two ways, by means of $2$ or $3$ branched hyperbolic ideal tetrahedra respectively. The branching is used in order to associate in a coherent way one term of a modular triple to each edge. It is well-known that the modular triple determines and is determined by the dihedral angles at the edges of the corresponding ideal tetrahedron. Then, in terms of dihedral angles, an ideal transit can be formally expressed by the equations Prop. 3.10 i) (1)-(2), providing that the charges are interpreted as dihedral angles (hence real positive numbers). In particular, the dihedral angles satisfy the first two relations in Def. 3.5 if one replaces $1$ and $2$ respectively by $\pi$ and $2\pi$ in these relations. The second relation is in agreement with the fact that the composition of the pairings of faces in $Q$ is an hyperbolic isometry, well-defined along the central common edge.

**Definition 7.3** The $\mathcal{I}$-pre-Bloch group $\mathcal{P}(\mathcal{I})$ is the quotient of $\mathbb{Z}[\mathcal{I}]$ by the linear extension of the action $p_{\mathcal{I}}$ of $S_4$ and by the ideal generated by the $5$-terms relations

\[ S(T_0) = S(T_1) , \] (8)

where $T_0 \to T_1$ is any instance of $2 \to 3$ $\mathcal{I}$-transit and $S(T_i)$ denotes the formal sum of decorated tetrahedra occurring in $T_i.$
Remarks 7.4 1) Each ideal tetrahedron of \( \mathbb{Z}[I] \) has a natural base vertex (for example we can stipulate that it is the vertex \( v_3 \)). We can stipulate that such a base ideal vertex is the point \( \infty \) in the half-space model of \( \mathbb{H}^3 \). The group of isometries of \( \mathbb{H}^3 \) which fix the point \( \infty \) is the Borel group \( B \) occurring in \( P(D) \).

2) Neumann’s extended pre-Bloch group \( \hat{P}(\mathbb{C}) [32, 33] \) is obtained by identifying two \( I \)-tetrahedra whenever they have the same modular triples, and then by taking the quotient of \( \mathbb{Z}[I] \) by the ideal generated by the relations \( [8] \) and further simple relations \( \text{(rel)} \) in order to eliminate a “mod 2” ambiguity. The latter relations only depend on the integral charge component, and are merely used for showing that the Bloch-Wigner map mentioned in Remark 8.9.1) lifts as an isomorphism. We think that defining \( \hat{P}(\mathbb{C}) \) as \( P(I)/(\text{rel}) \) would be more convenient, because this group is orientation-sensitive, as a quotient by the full action \( p_I \) (and not only by the action of \( S_4 \) preserving the modular triples).

Next we show the relation between ideal and \( D \)-tetrahedra, and we construct the idealization homomorphism.

Definition 7.5 A pseudo-ideal decoration of \( *(\Delta, b, c) \) is given by a map \( a : E \to \mathbb{C} \) such that:

1) For every \( e \in E, a(e) = a(e') \).

2) Set \( a_i = a(e_i), i = 0, 1, 2. \) Then

\[
    a_0a_1a_2 = -1 .
\]

We denote by \( PI^* \) the set of decorated tetrahedra \( *(\Delta, b, a, c) \) and set \( PI = PI^+ \cup PI^- \). Clearly \( I^* \subsetneq PI^* \).

Put \( p_0 = x(e_0)x(e_0'), p_1 = x(e_1)x(e_1') \) and \( p_2 = -x(e_2)x(e_2') \), where as usual we write \( z = (t, x) \). A straightforward computation using the cocycle property of \( z \) implies

\[
    p_0 + p_1 + p_2 = x(e_0)x(e_0') + x(e_1)x(e_1') - x(e_2)x(e_2') = 0 .
\]

So one can define a map

\[
    f_+ : D^+ \to PI^+ ,
\]

\[
    f_+(\Delta, b, z, c) = (\Delta, b, a_+(z), c) ,
\]

with \( a_+(z)(e_j) = \exp(p_j + c_j \pi i), j = 0, 1, 2. \) Similarly, define

\[
    f_- : D^- \to PI^- ,
\]

\[
    f_- (-(\Delta, b, z, c)) = -(\Delta, b, a_-(z), c) ,
\]

with \( a_-(z)(e_j) = \exp(p_j - c_j \pi i), j = 0, 1, 2. \) Finally set

\[
    f = (f_+, f_-) : D \to PI \ .
\]

Definition 7.6 We say that \( *(\Delta, b, z, c) \in D^+ \) is pre-ideal if \( f_+ \( *(\Delta, b, z, c) \)) \in I^* \).

Remark 7.7 Due to the relation \( a_0a_1 - a_1 = -1 \) in Definition 7.1, the cocycle \( z \) of a pre-ideal \( D \)-tetrahedron is necessary full, i.e. for each \( e \in E \) the upper diagonal entry \( x(e) \) of \( z(e) \) is non-zero.

We denote by \( ID = ID^+ \cup ID^- \) the set of pre-ideal \( D \)-tetrahedra.
Lemma 7.8 We have \( f_*(\mathcal{D}^*) = \mathcal{P}\mathcal{I}^* \), whence \( f_*(\mathcal{I}\mathcal{D}^*) = \mathcal{I}^* \).

Proof. Given a \( \text{Par}(B) \)-valued 1-cocycle \( z \) on \((\Delta, b)\), let us denote it by the set \( \{x(e)\} \) of the upper diagonal entries of \( \{z(e)\} \). We are going to prove that \( f_* \) covers \( \mathcal{P}\mathcal{I}^* \) even using only full \( \text{Par}(B) \)-valued 1-cocycles. Let us show this for \( b^+ \); the proof is the same for any other branching.

Fix \((\Delta, b^+, a, c) \in \mathcal{P}\mathcal{I}^+ \). We are looking for a full cocycle \( \{x(j)\} = \{x(e_j)\} \) such that the following relations are satisfied (see Fig. 20):

\[
\exp(x(0)x(2) + c_0 \pi i) = a_0 \\
\exp(x(1)(x(0) + x(1) + x(2)) + c_1 \pi i) = a_1 \\
\exp(-(x(0) + x(1))(x(1) + x(2)) + c_2 \pi i) = a_2 .
\]

Figure 20: A \( \mathbb{C} \)-1-cocycle \( \{x(j)\} \) on \((\Delta, b^+)\).

If we pass to the natural logarithm, we get relations of the form

\[
x(0)x(2) = s_0 \\
x(1)(x(0) + x(1) + x(2)) = s_1 \\
-(x(0) + x(1))(x(1) + x(2)) = s_2
\]

where we can assume that every \( s_j \neq 0 \) and \( s_0 + s_1 + s_2 = 0 \). So it is enough to set \( x(2) = t \neq 0, x(0) = s_0/t \) and to solve the equation \( x(1)^2 + (t + s_0/t)x(1) = s_1 \) to conclude (as \( s_j \neq 0 \), the solution actually is a full cocycle). \( \square \)

Consider the free \( \mathbb{Z} \)-module \( \mathbb{Z}[\mathcal{I}\mathcal{D}] \) generated by \( \mathcal{I}\mathcal{D} \). Extending by linearity the map \( f = (f_+, f_-) \) defined in (9), we get by Lemma 7.8 a surjective map

\[F : \mathbb{Z}[\mathcal{I}\mathcal{D}] \to \mathbb{Z}[\mathcal{I}] .\]

In general, the defining relations (3) of \( \mathcal{P}(\mathcal{D}) \) do not specialize on \( \mathbb{Z}[\mathcal{I}\mathcal{D}] \). For this we have to consider only the submodule \( \mathbb{Z}[\mathcal{I}\mathcal{D}_P] \) of \( \mathbb{Z}[\mathcal{I}\mathcal{D}] \) generated by the tetrahedra endowed with \( \text{Par}(B) \)-valued (full) 1-cocycles. The proof of Lemma 7.8 shows that we still have \( F(\mathbb{Z}[\mathcal{I}\mathcal{D}_P]) = \mathbb{Z}[\mathcal{I}] \).

Proposition 7.9 1) A decorated tetrahedron \( X = (\Delta, b, z, c) \in \mathcal{D} \) is pre-ideal iff \( p_\mathcal{D}(s, X) \) is pre-ideal for any \( s \in S_4 \). Furthermore, in such a case we have

\[p_\mathcal{I}(s, F(X)) = F(p_\mathcal{D}(s, X)) .\]

2) For any \( 2 \to 3 \) \( \mathcal{D} \)-transit \( T_0 \to T_1 \) the \( \mathcal{D} \)-tetrahedra occurring in \( T_0 \) belong to \( \mathcal{I}\mathcal{D}_P \) if and only if those occurring in \( T_1 \) belong to \( \mathcal{I}\mathcal{D}_P \).

3) If \( T_0 \to T_1 \) is a \( 2 \to 3 \) \( \mathcal{I}\mathcal{D}_P \)-transit, then \( F(T_0) \to F(T_1) \) is a \( 2 \to 3 \) \( \mathcal{I} \)-transit.
Corollary 7.10 The relations which define $\mathcal{P}(\mathcal{D})$ properly specialize on $\mathbb{Z}[\mathcal{I}D_p]$, producing the group $\mathcal{P}(\mathcal{I}D_p)$. The map $F$ induces a well-defined surjective homomorphism
\[ \bar{F} : \mathcal{P}(\mathcal{I}D_p) \to \mathcal{P}(\mathcal{I}). \]

Proof of Proposition 7.9. We prove 1) in the case where $b = b^+$ and $s$ is the permutation between the vertices $v_0$ and $v_1$ of $(\Delta, b^+)$. The proof of the other cases is the same. Then we have $X = (\Delta, b^+, x, c)$ and $\text{pp}(s, X) = (\Delta, b^-, s(x), s(c))$. Let $(w, w', w'')$ be the modular triple associated to $(\Delta, b^+, x, c)$, which we assume to be pre-ideal. We have to show that the pseudo-modular triple $(a_0, a_1, a_2)$ associated to $(\Delta, b^-, s(x), s(c))$ via the map $f$ in (9) is actually a modular triple, and that $a_0 = 1/w$. For the sake of simplicity (and because it suffices for the rest of the proposition), we shall work only with $\text{Par}(B)$-valued cocycles $z$, but the proof works also for arbitrary $B$-valued cocycles. As before, we identify $z$ with the set of the upper diagonal terms of $\{z(e)\}$. Set $x = x(e_0), y = x(e_1), t = x(e_0)$ w.r.t. $b^+$ (see Fig. 21). Just by applying the definitions one has
\[ w = \exp(xt + c_0\pi i), \quad a_0 = \exp(-xt - c_0\pi i). \]

Hence $wa_0 = 1$. Now we have to verify that $a_2 = 1 - \frac{1}{a_0} = 1 - w$. In fact
\[ a_2 = \exp(-y(x + y + t) - c_1\pi i) \]
\[ 1 - w = (w')^{-1} = \exp(-y(x + y + t) - c_1\pi i). \]

A similar computation holds for $a_1$. This proves 1).

![Figure 21: Action of $(v_0v_1) \in S_4$ on $(\Delta, b^+, z, c)$.](image)

Several purely algebraic computations give 2) and 3) simultaneously. As above, for any $\text{Par}(B)$-valued cocycle $z$ we identify $z$ with the set of the upper diagonal terms of $\{z(e)\}$. We refer to Fig. 12, 13 and 14. Set
\[ x = \exp(d(x + (\alpha(1) + \alpha(3))\pi i)) \]
\[ y = \exp((b - a)(b - c) + (\alpha(1) + \beta(2))\pi i). \]

Assume that the two decorated tetrahedra of $T_0$ are pre-ideal. Then we have
\[ (1 - x)^{-1} = \exp(b(b - c - d) + (\beta(2) + \beta(3))\pi i) \]
\[ (1 - y)^{-1} = \exp(b(a - b + c) + (\alpha(2) + \alpha(3))\pi i). \]
One verifies directly that
\[
\frac{x(y-1)}{y} = \exp\left( (a(1) + \alpha(3) - \alpha(2) - \alpha(3) - \alpha(1) - \beta(2) + 1) \pi i \right)
\]
\[
= \exp\left( (a-d) + (1 - \alpha(2) - \beta(2)) \pi i \right)
\]
\[
= \exp\left( (a-d) + \theta(2) \pi i \right),
\]
and
\[
\frac{y(x-1)}{x} = \exp\left( (a-d)(c-b) + (1 - \alpha(3) - \beta(3)) \pi i \right)
\]
\[
= \exp\left( (a-d)(c-b) + \theta(3) \pi i \right),
\]
where the first relation in Def. 3.5 1) is used in the last equalities. Both results are in agreement with (9). The second relation in Def. 3.5 1) is hidden, as can be seen for instance from the alternative computation
\[
\frac{x(y-1)}{y} = \exp\left( (b-a)(b-c) - b(b-c-d) - dc + (\alpha(1) + \beta(2) - \beta(2) - \alpha(1) - \beta(3) + 1) \pi i \right)
\]
\[
= \exp\left( (a-d)(c-b) + (1 - \alpha(3) - \beta(3)) \pi i \right)
\]
\[
= \exp\left( (a-d)(c-b) + \theta(3) \pi i \right),
\]
We get similarly
\[
\frac{1}{(x-1)(y-1)} = \exp\left( (b(a-d) + \theta(3) \pi i \right).
\]
Thus we have checked some of the components of the pseudo-modular triples associated to the tetrahedra of $T_1$. One can continue in the same way. Alternatively, all these components are solutions of the polynomial system of compatibility equations which determines uniquely the modular triples in $T_1$ from those of $T_0$. For instance, consider in Fig. 19 the tetrahedra $\Delta$ and $\Delta'$ with modular components $x(y-1)/y$ and $y(x-1)/x$ respectively. Let $e$ be the bottom edge of the "common" face of $\Delta$ and $\Delta'$. Denote by $x_\Delta(e)$ and $x_{\Delta'}(e)$ the moduli associated to $e$. One has
\[
x_\Delta(e) = \exp\left( (b-a)(b-c-d) + \beta(2) \pi i \right)
\]
\[
x_{\Delta'}(e) = \exp\left( (a-b-c-d) + \beta(3) \pi i \right),
\]
whence the compatibility equation
\[
x_\Delta(e) \cdot x_{\Delta'}(e) = \exp\left( (b-b-c-d) + (\beta(2) + \beta(3)) \pi i \right) = \frac{1}{1-x}.
\]
The whole set of compatibility equations for all the edges finally determine all
moduli. For instance we have
\[
x_{\Delta}(e) = \frac{y}{y(1 - x) + x} = \frac{1}{1 - \frac{x(y - 1)}{y}}
\]
\[
x_{\Delta'}(e) = \frac{y(x - 1) - x}{y(x - 1)} = 1 - \frac{x}{y(x - 1)},
\]
where the second equalities follow from a direct computation using Fig. 3 - Fig. 4 and 5. We have performed some of the verifications which show that the \(\mathcal{ID}_p\)-transit and the \(I\)-transit fit well together. The other verifications are similar. □

Note that there are natural homomorphisms \(J_\rho : \mathcal{P}(\mathcal{ID}_p) \to \mathcal{P}(\mathcal{D}_p)\) and \(J_D : \mathcal{P}(\mathcal{ID}_p) \to \mathcal{P}(\mathcal{D})\) such that \(J_D = J \circ J_\rho\), where \(J\) is defined in (4).

## 8 Hyperbolic-like triples

In this section we introduce the hyperbolic-like triples \((W, L, \rho)\), for which we can define \(I\)-classes in \(\mathcal{P}(I)\). Then, we show that the \(D\)-class (resp. \(I\)-class) of a hyperbolic-like triple belongs to the kernel of a suitably generalized Dehn homomorphism. This kernel is a refinement of the usual Bloch group. Finally, we prove that the \(I\)-class of a hyperbolic-like triple naturally defines a cohomology class in \(H^1_c(PSL(2, \mathbb{C}); \mathbb{Z})\) (discrete homology).

**Definition 8.1** Let \((W, L, \rho)\) be a triple such that \(\rho \in H^1(W; \text{Par}(B))\). One says that \((W, L, \rho)\) is hyperbolic-like if there exists a \(\mathcal{ID}_p\)-triangulation \(\mathcal{T} = (T, H, D)\) for \((W, L, \rho)\). This means that \(\mathcal{T}\) is a full \(D\)-triangulation, the cocycle \(z\) of \(D\) belongs to \(Z^1(T; \text{Par}(B))\), and all the \(D\)-tetrahedra of \(\mathcal{T}\) actually belong to \(\mathcal{ID}_p\).

Assume that \((W, L, \rho)\) is hyperbolic-like. Every \(\mathcal{ID}_p\)-triangulation \(\mathcal{T}\) defines an element \(\epsilon_{\mathcal{ID}_p}(\mathcal{T}) \in \mathcal{P}(\mathcal{ID}_p)\) and an element \(\epsilon_{\mathcal{D}}(\mathcal{T}) = F(\epsilon_{\mathcal{ID}_p}(\mathcal{T}))\), where \(\mathcal{T}\) is the \(I\)-triangulation obtained via the idealization of every tetrahedron of \(\mathcal{T}\). Clearly
\[
\epsilon_{\mathcal{D}}(\epsilon_{\mathcal{ID}_p}(\mathcal{T})) = \epsilon_D(\mathcal{T}) \in \mathcal{P}(\mathcal{D}).
\]
In fact, using \(0 \to 2\) and bubble \(\mathcal{ID}_p\)-transits (which induce on \(\mathbb{Z}[\mathcal{ID}_p]\) relations which are consequences of the \(2 \to 3\) \(\mathcal{ID}_p\)-transit relations), and \(0 \to 2\) and bubble \(I\)-transits, one can adapt the proof of Th. 4.3 so that we get

**Proposition 8.2** Let \((W, L, \rho)\) be a hyperbolic-like triple. The class \(\epsilon_{\mathcal{ID}_p}(\mathcal{T})\) does not depend on the \(\mathcal{ID}_p\)-triangulation \(\mathcal{T}\) of \((W, L, \rho)\). Hence \(\epsilon_{\mathcal{ID}_p}(W, L, \rho) := \epsilon_{\mathcal{ID}_p}(\mathcal{T}) \in \mathcal{P}(\mathcal{ID}_p)\) is an invariant of \((W, L, \rho)\), called the \(\mathcal{ID}_p\)-scissors congruence class. Moreover, \(\epsilon_{\mathcal{I}}(W, L, \rho) := F(\epsilon_{\mathcal{ID}_p}(W, L, \rho))\) is also an invariant of \((W, L, \rho)\), called the \(I\)-scissors congruence class.

**Generalized Dehn homomorphisms.** Our approach is very close to [42, 43]: the homomorphism \(\delta_T\) below is essentially equivalent to \(\delta_{\mathcal{C}}\) and \(\nu\) in [23] and [43] respectively. Let us introduce the notation \(c'\) for combinatorial flattenings, following Neumann’s terminology. They are defined by: for any \((\Delta, b, z, c)\), set \(c'_0 = c_0\), \(c'_1 = c_1\) and \(c'_2 = c_2 - 1\). The interest of combinatorial flattenings is that they allow to reparametrize any charge transit in a convenient way. Consider the homomorphisms
\[
\delta_{\mathcal{ID}_p} : \quad \mathbb{Z}[\mathcal{ID}_p] \quad \to \quad \mathbb{C} \wedge \mathbb{C}
\]
\[
* (\Delta, b, z, c) \quad \mapsto \quad * (x(e_0)x(e'_0) + *i\pi c'_0) \wedge (x(e_1)x(e'_1) + *i\pi c'_1),
\]

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\[ \delta_I : \mathbb{Z}[\mathcal{I}] \rightarrow \mathbb{C} \wedge \mathbb{Z} (\mathbb{C}/i\pi \mathbb{Z}) \]
\[ \ast(\Delta, b, w, c) \mapsto \ast \left( \log(w_0) + \ast i\pi c'_0 \right) \wedge (\log(w_1) + \ast i\pi c'_1), \]
where \( \log \) is any determination of the logarithm, and the notations for \( z, w, x \) and \( e_i \) are the usual ones. We call \( \delta_{D_p} \) (resp. \( \delta_I \)) the \( D_p \)-Dehn map (resp. \( I \)-Dehn map). Below we write \( \wedge \) for \( \wedge \mathbb{Z} \).

**Proposition 8.3** The Dehn maps \( \delta_{D_p} \) and \( \delta_I \) induce homomorphisms

\[
\begin{align*}
\delta_{D_p} : \mathcal{P}(D_p) & \rightarrow \mathbb{C} \wedge \mathbb{C} \\
\delta_I : \mathcal{P}(I) & \rightarrow \mathbb{C} \wedge (\mathbb{C}/i\pi \mathbb{Z}).
\end{align*}
\]

Set \( \delta_{ID_p} = \delta_{D_p} \circ J_p \). We also have the factorization \( \delta_{ID_p} = \delta_I \circ \bar{F} \).

The proof shows that the five-terms functional relations for \( \delta_{D_p} \) and \( \delta_I \) characterize the \( 2 \rightarrow 3 \) \( D_p \)-transit relations and the \( 2 \rightarrow 3 \) \( I \)-transit relations respectively. Since we work with \( e \) (see (2)). They all follow from the relations for the permutations \( (01), (12) \) and \( (23) \) as a lift of \( \delta_I \) that removes this ambiguity. Similar computations show that \( \delta_{D_p} \) do not extend to the whole of \( \mathcal{P}(D) \).

**Proof.** Consider the symmetry relations in \( \mathcal{P}(D_p) \), that come from the action \( p_D \) (see (2)). They all follow from the relations for the permutations \( (01), (12) \) and \( (23) \) of the vertices \((v_0, v_1), (v_1, v_2) \) and \((v_2, v_3) \) respectively. Recall that \( p_0 = x(e_0)x(e'_0), \)
\[ p_1 = x(e_1)x(e'_1) \text{ and } p_2 = -x(e_2)x(e'_2). \]
We have:

\[
\begin{align*}
\delta_{D_p}(p_D((01), \ast(\Delta, D))) & = \delta_{D_p}(\ast(\Delta, (01)D)) = \ast(p_0 - \ast i\pi c'_0) \wedge (p_2 - \ast i\pi c'_2) \\
& = \ast(p_0 + \ast i\pi c'_0) \wedge (p_1 + \ast i\pi c'_1) = \delta_{D_p}(\ast(\Delta, D)) \\
\delta_{D_p}(p_D((12), \ast(\Delta, D))) & = \ast(-p_2 - \ast i\pi c'_2) \wedge (-p_1 - \ast i\pi c'_1) \\
& = \ast(p_0 + p_1 + \ast i\pi c'_0 + c'_1) = \delta_{D_p}(\ast(\Delta, D)) .
\end{align*}
\]

The computation for the permutation \( (23) \) is the same as for \( (01) \). Then \( \delta_{D_p} \) is well-defined on \( \mathbb{Z}[D_p]/p_D \). Next, consider the five-term functional relation for \( \delta_{D_p} \) corresponding to the cocycle transit of Fig. 22, which is convenient for this computation (note that the branchings give a negative orientation to some tetrahedra in this figure). There are three kinds of summands: they are of the form \( x(e_0)x(e'_0) \wedge x(e_1)x(e'_1), \)
\[ i\pi c'_0 \wedge x(e_1)x(e'_1) \text{ or } x(e_0)x(e'_0) \wedge i\pi c'_1, \]
and \( i\pi c'_0 \wedge i\pi c'_1 \).

Since we work with \( \wedge = \wedge \mathbb{Z} \), the l.h.s. and the r.h.s. they read

\[
\begin{align*}
-\ast ac \wedge b(a + b + c) + \ast dc \wedge b(b + c + d),
\end{align*}
\]
and in the r.h.s. they read

\[
(a - d)(b + c) \wedge d(a + b + c) - b(a - d) \wedge d(a + b) - c(a - d) \wedge (b + d)(a + b + c) .
\]
Now we have

\[
-\ast c(a - d) \wedge (b + d)(a + b + c) = -\ast ac \wedge b(a + b + c) - \ast ac \wedge d(a + b + c) + \ast cd \wedge b(b + c) + \ast cd \wedge ab + \ast cd \wedge d(a + b + c) .
\]
Thus, the equality of both sides is equivalent to
\[(a - d)(b + c) \land d(a + b + c) - b(a - d) \land d(a + b) - ac \land d(a + b + c) + \]
\[cd \land ab + cd \land (a + c)d = 0 . \]
An easy computation shows that this holds true. Finally, consider the summands of the form \(i\pi c_2 \land x(e_1)x(e'_1)\) or \(x(e_2)x(e'_2) \land i\pi c'_1\). In the l.h.s. they read
\[i\pi(\alpha_1 + \alpha_3) \land b(b + c + d) + dc \land i\pi(\beta_2 + \beta_3) + \]
\[ac \land i\pi(\alpha_2 + \alpha_3) + i\pi((\beta_1 + \beta_3) \land b(a + b + c) . \]
In the r.h.s. they read
\[i\pi(\beta_3 - 2) \land d(a + b + c) + (a - d)(b + c) \land i\pi\alpha_3 - (-i\pi)\theta_1 \land d(a + b) \]
\[-b(a - d) \land (-i\pi)\alpha_1 - (-i\pi)\theta_2 \land (b + d)(a + b + c) - c(a - d) \land (-i\pi)\alpha_2 . \]
Note that in these equalities, the only combinatorial flattening that is different from the corresponding charge is \(\theta_3 = 2\), in place of \(\theta_3\) (compare with Fig. 22 - the branching does not matter). A straightforward computation using the whole set of relations between the charges shows that the difference between these summands in both sides vanishes identically. Since \(\delta_{D_p}\) is well-defined on \(\mathbb{Z}[D_p]/p\mathbb{D}\), the five-term functional relation for \(\delta_{D_p}\), corresponding to any other branching is also true. Since the cocycle was arbitrary and \(\delta_{D_p}\) does not depend on the charge, we thus have proved the first claim for \(\delta_{D_p}\).

The proof that \(\delta_I : \mathbb{P}(\mathbb{I}) \to \mathbb{C} \land (\mathbb{C}/i\pi\mathbb{Z})\) is well-defined follows from the very same arguments, using computations similar to those above. The only difference is that the cocycle property of \(z\) is replaced by the modularity property of \(w\). All the instances of \(2 \to 3 \mathbb{I}\)-transit relations still hold true, but the symmetry relations only hold mod\((i\pi)\) (when fixing a determination of the logarithm in \(\delta_I\)). For instance, \(\delta_I(p_I((01), *(\Delta, b, w, c)))\) is equal to
\[
* \left(-\log(w_0) - *i\pi c_0' \land (-\log(w_2) - *i\pi c_3') \right)
\]
\[= *(\log(w_0) + *i\pi c_0') \land (\log(w_0) + \log(w_1) + i\pi - *i\pi(-c'_0 - c'_1) \right) \]
\[= *(\log(w_0) + *i\pi c_0') \land (\log(w_1) + *i\pi c_1') \land (\log(w_0) + *i\pi c_0') \land i\pi . \]
The factorization \(\delta_{ID_p} = \delta_I \circ F\) is an immediate consequence of (f). This concludes the proof. \(\square\)

**Definitions 8.4** The \(D_p\)-Bloch group, the \(ID_p\)-Bloch group and the \(I\)-Bloch group are defined by \(B(D_p) = \text{Ker}(\delta_{D_p}), B(ID_p) = \text{Ker}(\delta_{ID_p}), \) and \(B(I) = \text{Ker}(\delta_I)\) respectively.

In order to clarify our results, let use precise some relationships between \(\delta_{D_p}, \delta_I\) and the “classical” Dehn homomorphisms for hyperbolic scissors congruences. Consider the group \(\mathbb{P}(\mathbb{C})\) defined from \(\mathbb{P}(\mathbb{I})\) just by forgetting the charge components. It is isomorphic to the group of orientation-preserving isometry classes of convex ideal polyhedra in \(\mathbb{H}^3\) with triangulated faces, up to scissors congruences [44 App]; if \(P\) is a polyhedra obtained by gluing \(P_1\) and \(P_2\) along faces with compatible triangulations, then \([P] = [P_1] + [P_2]\).

Denote by \(\mathbb{P}(\mathbb{H}^3)\) and \(\mathbb{P}(\mathbb{H}^3)\) the scissors congruence groups of isometry classes of non-flat geodesic polyhedra in \(\mathbb{H}^3\) and \(\mathbb{H}^3\) respectively, and by \(\mathbb{P}(\partial \mathbb{H}^3)\) the scissors congruence group of non-flat hyperbolic ideal polyhedra. For future reference, let us quote the following results of Dupont and Sah:
The Dehn invariant for hyperbolic tetrahedra is the homomorphism

$$\delta : \mathcal{P}(\mathbb{H}^3) \to \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\pi \mathbb{Z},$$

where the sum is over all the edges $e$ of $P$, and $l(e)$ and $\theta(e)$ are the length and dihedral angles (in radians) at $e$. Due to (R2), $\delta$ extends uniquely to

$$\delta : \mathcal{P}(\mathbb{H}^3) \to \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\pi \mathbb{Z}.$$

This may be viewed geometrically as follows. If $P \in \mathcal{P}(\mathbb{H}^3)$ has a vertex $v$ at infinity, we delete a horoball around $v$; for any edge $e$ of $P$ ending at $v$ the length $l(e)$ is defined only up to the horosphere. Since the sum of angles at the edges ending at $v$ is a multiple of $\pi$, this undeterminacy vanishes in $\delta$. So $\delta$ is also defined for ideal polyhedra, and (R3) implies that it is actually determined by the values on such polyhedra.

The Bloch-Wigner complex Dehn invariant [22, Th. 4.10] is the homomorphism

$$\delta_{\mathbb{C}} : \mathcal{P}(\mathbb{C}) \to \mathbb{C}^* \wedge \mathbb{C}^* \quad \ast (\Delta, b, w) \mapsto \ast w_0 \wedge w_1^{-1}.$$

Neumann proved in [42, §3.2] that $\delta$ is twice the “imaginary part” of $\delta_{\mathbb{C}}$, using the decomposition

$$\mathbb{C}^* \wedge \mathbb{C}^* \cong \mathbb{R} \wedge_{\mathbb{Z}} \mathbb{R} \oplus \mathbb{R}/2\pi \mathbb{Z} \wedge_{\mathbb{Z}} \mathbb{R}/2\pi \mathbb{Z} \oplus \overbrace{\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\pi \mathbb{Z}}^{\text{Imaginary subspace}}.$$
induced by the isomorphism \( C^* \rightarrow \mathbb{R} \oplus \mathbb{R}/2\pi\mathbb{Z} \) defined by \( z \mapsto \log |z| + \arg(z) \).

Let \( \zeta : C \times C \rightarrow C^* \times C^* \) be the linear extension of the map given by \( \zeta(x \wedge y) = \exp(x) \wedge \exp(-y) \). Denote by \( \pi : \mathcal{P}(\mathcal{L}) \rightarrow \mathcal{P}(\mathcal{C}) \) the natural projection which forgets the integral charges. For any \( \alpha = \sum_i \ast_i (\Delta_i, b_i, z_i, c_i) \in \mathbb{Z}[\mathcal{ID}_p] \), consider the map \( \gamma : \mathbb{Z}[\mathcal{ID}_p] \rightarrow C^* \times C^* \) defined by

\[
\gamma(\alpha) = \sum_i \ast_i \zeta (\delta_{\mathcal{ID}_p}(\Delta_i, b_i, z_i, c_i)) \in C^* \times C^*.
\]

Then, one has the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{F}(D_p) & \xrightarrow{\delta_{\mathcal{F}_p}} & \mathcal{C} \times \mathcal{C} \\
\downarrow J_p & & \downarrow \delta_{\mathcal{F}_p} \\
\mathcal{F}(\mathcal{ID}_p) & \xrightarrow{\gamma} & \mathcal{C} \times \mathcal{C}^*/\mathbb{Z}/2 \\
\end{array}
\]

where \( \mathbb{Z}/2 \) acts on \( C^* \times C^* \) by multiplication with \( \pm 1 \). Note that the commutativity of the bottom triangle implies that \( \gamma \), when taking values in \( C^* \times C^*/\mathbb{Z}/2 \), is well-defined on \( \mathcal{F}(\mathcal{ID}_p) \). Due to the symmetry relations in \( \mathcal{F}(\mathcal{ID}_p) \), it is difficult to compare \( \gamma \) and \( \delta_{\mathcal{ID}_p} \).

Let \( M \) be an oriented hyperbolic 3-manifold. If \( M \) is compact, denote by \( T \) a geodesic triangulation of \( M \), and let \( \mathfrak{c}(T) \) be the (signed) formal sum of one lift in \( \mathbb{H}^3 \) for each (oriented) tetrahedron of \( T \). One can consider the scissors congruence class \( \mathfrak{c}(M) = \mathfrak{c}(T) \in \mathcal{P}(\mathbb{H}^3) \). If \( M \) is non-compact, let \( T \) be a geodesic ideal triangulation, where possibly some tetrahedra are geometrically flat (such triangulations exist by \( \text{(23)} \)). Use (R1) and (R3) to represent flat tetrahedra in \( \mathcal{P}(\mathbb{H}^3) \cong \mathcal{P}(\mathbb{H}^3) \), and set \( \mathfrak{c}(M) = \mathfrak{c}(T) \in \mathcal{P}(\mathbb{H}^3) \). In both cases we have \( \delta(\mathfrak{c}(M)) = 0 \), since the contributions coming from each edge \( e \) of \( T \) sum up to \( l(e) \otimes 2\pi \). More generally we have:

**Proposition 8.5** For any triple \((W, L, \rho)\) with \( \rho \) a \( \text{Par}(B) \)-bundle, we have

\[
\delta_{\mathcal{F}_p}(\mathfrak{c}_3(W, L, \rho)) = 0.
\]

In particular, the \( D \)-class (resp. the \( I \)-class) of a hyperbolic-like triple \((W, L, \rho)\) lies in the \( \mathcal{ID}_p \)-Bloch group: \( \mathfrak{c}_3(W, L, \rho) \in \mathcal{B}(\mathcal{ID}_p) \) (resp. \( \mathfrak{c}_1(W, L, \rho) \in \mathcal{B}(I) \)).

**Proof.** Given a \( D_p \)-triangulation \( T = (T, H, b, z, c) \) of \((W, L, \rho)\), let \( e \) be any edge of \( T \). As in the proof of Th. \( \text{(14)} \), one can find a sequence of \( 2 \rightarrow 3 \) \( D_p \)-transits which put \( \text{Star}(e, T) \) in the configuration of 3 tetrahedra glued along \( e \) (e.g. as in the right of Fig. \( \text{(23)} \)). By Prop. \( \text{(8.3)} \), we know that \( \delta_{\mathcal{ID}_p} \) satisfies the \( 2 \rightarrow 3 \) \( D_p \)-transit relations. Hence the above sequence do not introduce contributions for the new edges, and the contributions in \( \delta_{\mathcal{ID}_p} \) coming from \( e \) sum up to \( 0 \). This proves that \( \delta_{\mathcal{ID}_p}(\mathfrak{c}_3(W, L, \rho)) = 0 \). From the factorization \( \delta_{\mathcal{ID}_p} = \delta_I \circ \tilde{F} \), we know that \( \tilde{F}(\mathcal{B}(\mathcal{ID}_p)) \subset \mathcal{B}(I) \). Hence we deduce that \( \delta_I(\mathfrak{c}_1(W, L, \rho)) = 0 \in C \wedge (C/2\pi\mathbb{Z}) \). \( \square \)

The \( D_p \)-Dehn invariant may be viewed as a 1-dimensional measure of the geometric rigidity of polyhedra: when there does not exist local (abelian) geometric degrees of freedom on a polyhedron \( Q \), so that one may apply \( 2 \rightarrow 3 \) \( D_p \)-transits on it, then \( \delta_{\mathcal{ID}_p}(Q) \) vanishes.

**Homology of \( PSL(2, \mathbb{C}) \) and hyperbolic-like triples.** We now prove that one can associate a well-defined cohomology class \( \alpha(W, L, \rho) \in H^3_3(PSL(2, \mathbb{C}); \mathbb{Z}) \) to a
hyperbolic-like triple. We shall need the following lemma. As usual, for any \( \mathcal{TD}_p \)-triangulation \( \mathcal{T} = (T,H,(b,z,e)) \) of a triple \( (W,L,\rho) \), if \( (\Delta_j, b_j, z_j, c_j) \in \mathcal{T} \), we put \( (\Delta_j, b_j, w_j, c_j) = F(\Delta_j, b_j, z_j, c_j) \).

**Lemma 8.6** Let \( \mathcal{T} \) be a \( \mathcal{TD}_p \)-triangulation of a hyperbolic-like triple \( (W,L,\rho) \). For any edge \( e \) of \( T \), denote by \( \Delta_1, \ldots, \Delta_n \) the tetrahedra glued along \( e \). We have

\[
\prod_{i=1}^n w_i(e) = 1 ,
\]

**Proof.** Using (1) we have

\[
\prod_{i=1}^n w_i(e) = \exp \left( z(e) \left( \sum_{i=1}^n \epsilon_i(e'_i) \right) + i\pi \left( \sum_{i=1}^n \epsilon'_i c_i(e) \right) \right) ,
\]

where \( e'_i \) is the edge of \( \Delta_i \) opposite to \( e \), \( c_i(e) \) is the charge of \( e \) for \( \Delta_i \), and \( \epsilon_i(e'_i) \) and \( \epsilon'_i \) are obtained as follows. The orientation defined by the modulus \( w_{\Delta_i}(e) \in \Pi \) is coherent with the orientation of \( (\Delta_i, b_{\Delta_i}) \) (see Remark 7.2). If \( e = e_2 \) (resp. \( e = e'_2 \)) w.r.t. \( b_{\Delta_i} \), this orientation is opposite to the one defined by the direction of \( e' \) w.r.t. the vertex \( v_0 \) (resp. \( v_1 \)); see Fig. 11. By (1) we deduce in that case that \( \epsilon_i(e'_i) \) is \( 1 \) if \( i = 1, \ldots, n \). On the contrary if \( e = e_0, e_1, e_0' \) or \( c_1 \), the orientation defined by \( w_{\Delta_i}(e) \) is the same as the one defined by the direction of \( e' \) w.r.t. the vertices \( v_0, v_1, v_2 \) or \( v_0 \) respectively.

We know that for any \( e \in E(T) \) we have

\[
\sum_{i=1}^n c_i(e) \equiv 0 \pmod{2} .
\]

Hence \( \sum_{i=1}^n \epsilon'_i c_i(e) \equiv 0 \pmod{2} \). Since the simplicial chain \( \sum_{i=1}^n e'_i \) is a \( \mathbb{Z} \)-boundary, the cocycle property of \( z \) gives

\[
\sum_{i=1}^n z(e'_i) = 0 .
\]

These two relations together give (11). \( \square \)

**Remark 8.7** The statement of Lemma 8.6 is also true, more generally, for \( \mathcal{D}_p \)-triangulations of triples \( (W,L,\rho) \) with \( \rho \) a \( \text{Par}(B) \)-bundle, replacing the moduli \( w_i \) with the corresponding pseudo-moduli (see Def. 7.3).

Let us recall the normalized standard chain complex for the discrete homology of a group \( G \) [11, §1.5]. Consider \( G \) as an infinite dimensional simplex, where the vertices are the elements of \( G \) and every finite subset of \( G \) is a simplex. Denote by \( F_n \) the free \( \mathbb{Z} \)-module generated by the \( (n+1) \)-tuples \( \langle g_0, \ldots, g_n \rangle \) of elements of \( G \), with the \( G \)-action given by \( g \cdot \langle g_0, \ldots, g_n \rangle = \langle gg_0, \ldots, gg_n \rangle \), and set \( F_{-1} = \mathbb{Z} \). Let \( 1 \) be the identity of \( G \). The map \( \partial_n : F_n \rightarrow F_{n-1} \) defined by

\[
\partial_n(g_0, \ldots, g_n) = \sum_{i=0}^n (-1)^i(g_0, \ldots, \hat{g}_i, \ldots, g_n)
\]

is a boundary operator for the augmented chain complex

\[
\cdots \rightarrow F_3 \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial} \mathbb{Z} \rightarrow 0 .
\]
where the augmentation $\epsilon$ is given by $\epsilon(g_0) = 1$. The $\mathbb{Z}$-homomorphism $h : F_n \to F_{n+1}$ for the underlying complex $F'_n$ of $\mathbb{Z}$-modules, and defined by

$$h(g_0, \ldots, g_n) = (1, g_0, \ldots, g_n) \quad \text{if } n \geq 0$$

$$h(1) = (1) \quad \text{if } n \geq -1$$

is a contracting homotopy for $F'_n$ (i.e. $h$ verifies $\partial_{n+1} h + h \partial_n = \text{id}_{F'_n}$). This shows that $F_i$ is acyclic; it is called the the standard (free) resolution of $\mathbb{Z}$ over $\mathbb{Z}$. Consider now the “degenerate” subcomplex $D_*$ of $F_*$ generated by the elements $(g_0, \ldots, g_n)$ such that $g_i = g_j$ for some $i \neq j$. The contracting homotopy $h$ carries $D_i$ into itself. Then $h$ induces a contracting homotopy of the complex $C_* = F_*/D_*$, which is the normalized standard (free) resolution of $\mathbb{Z}$ over $\mathbb{Z}$. Hence the chain complex $C_G$

\[ \cdots \xrightarrow{\partial_3} C_2 \otimes_{\mathbb{Z}} G \xrightarrow{\partial_2} C_1 \otimes_{\mathbb{Z}} G \xrightarrow{\partial_1} C_0 \otimes_{\mathbb{Z}} G \xrightarrow{\partial_0} 0 \]

computes the (discrete) homology $H^\delta_3(G; \mathbb{Z})$ of $G$, where $\mathbb{Z}$ is endowed with the trivial $\mathbb{Z}G$-module structure. Remark that $C_n \otimes_{\mathbb{Z}} G$ is a free $\mathbb{Z}$-module generated by the “homogeneous” simplices $(g_0 : \ldots : g_n)$, where the $g_i$ are distinct elements of $G$ and $(g_0 : \ldots : g_n) = (g'_0 : \ldots : g'_n)$ iff there exists $g \in G$ with $gg_i = g'_i$ for $i = 0, \ldots, n$.

For $G = \text{PSL}(2, \mathbb{C})$, there is a natural geometric interpretation of $3$-chains. Namely, any $\alpha \in H^\delta_3(\text{PSL}(2, \mathbb{C}); \mathbb{Z})$ may be represented by a sum

$$\sum_i \epsilon_i (g^i_0 : \ldots : g^i_3),$$

where $\epsilon_i = \pm 1$ and $g^i_j \in \text{PSL}(2, \mathbb{C})$. There is a natural bijective map $\lambda$ between $C_3 \otimes_{\mathbb{Z}} G$ and the free $\mathbb{Z}$-module generated by orientation-preserving isometry classes of hyperbolic ideal tetrahedra. It is defined as follows. Let $[\Delta(z_0, z_1, z_2, z_3)]$ denote the orientation-preserving isometry class of the hyperbolic ideal tetrahedron with vertices $z_0, \ldots, z_3 \in \mathbb{C}P^1 = \partial \mathbb{H}^3$. Then we set

$$\lambda(g_0 : \ldots : g_3) = [\Delta(g_0 z, g_1 z, g_2 z, g_3 z)],$$

where $z \in \mathbb{C}P^1$ is any point such that the $g_i z$ are pairwise distinct. Conversely, given $[\Delta(g_0 z, g_1 z, g_2 z, g_3 z)]$, consider its representative $\Delta(0, 1, \infty, z)$. Choose any $g_0 \in \text{PSL}(2, \mathbb{C})$, and let $g_1, g_2, g_3 \in \text{PSL}(2, \mathbb{C})$ be such that

$$g_1g_0^{-1}(0) = 1, \quad g_2g_0^{-1}(0) = \infty, \quad g_3g_0^{-1}(0) = z.$$

Since $\text{PSL}(2, \mathbb{C})$ acts 3-transitively on $\mathbb{C}P^1 = \partial \mathbb{H}^3$, the 4-tuple $(g_0, \ldots, g_3)$ is uniquely determined from $\Delta(0, 1, \infty, z)$. Hence

$$\lambda^{-1}[\Delta(0, 1, \infty, z)] = (g_0 : \ldots : g_3)$$

is well-defined. Recall that $\pi : \mathbb{Z}[\mathbb{Z}] \to \mathbb{Z}[\mathbb{C}]$ is the natural projection which forgets the integral charges. For every $(\Delta, b, w)$, one can define a hyperbolic ideal tetrahedron in the half-space model of $\mathbb{H}^3$, with base vertices the points $v_0 = 1$, $v_1 = w(e_0)$, $v_2 = 0$ and $v_3 = \infty$. The orientation-preserving isometry class of $(\Delta, b, w)$ is completely determined by the cross-ratio $v_1 = w_i(e_0) \in \mathbb{C} \setminus \{0, 1\}$. Thus, any element $x \in \mathbb{Z}[\mathbb{C}]$ may be seen as an element of the free $\mathbb{Z}$-module generated by orientation-preserving isometry classes of hyperbolic ideal tetrahedra, and we can consider $\lambda^{-1}(x)$ as a 3-chain on $\text{PSL}(2, \mathbb{C})$. 

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The relations (10) imply that one can glue in a coherent way along a geodesic line $l$ the representatives $\Delta_1, \ldots, \Delta_n$ of isometry classes of tetrahedra in $\pi(e(T))$ that come from $\text{Star}(e, T)$, for some edge $e$ of $T$. Hence the “2-faces” in the boundary of $\lambda^{-1} \circ \pi(e(T))$ may be paired and eventually cancel out: for any $(g_0^i : g_1^i : g_2^i) \in \partial_l(\lambda^{-1} \circ \pi(e(T)))$ with $\epsilon_i = \pm 1$, there exists $(g_0^i : g_1^i : g_2^i) = (g_0^j : g_1^j : g_2^j)$ with $\epsilon_j = \mp 1$ and $j \neq i$, for otherwise $l$ would belong to $\pm \partial(\Delta_1 \cup \cdots \cup \Delta_n)$. So, $\lambda^{-1} \circ \pi(e(T))$ is a 3-cycle. The symmetry relations in $\mathbb{Z}C$ are consequences of the five-term relations [22, §5], which themselves are the images by $\pi$ of the 2→3 $\mathcal{I}$-transit relations. Namely, they imply
\[
(\Delta, w_0) = (\Delta, w_1) = (\Delta, w_2) = -(\Delta, w_0^{-1}) = -(\Delta, w_1^{-1}) = -(\Delta, w_2^{-1}),
\]
where to simplify the notations we omit the branchings, and $w_1 = 1/(1 - w_0)$ and $w_2 = 1/(1 - w_1) = 1 - 1/w_0$. Moreover, the image via $\lambda^{-1} \circ \pi$ of all the instances of $2 \rightarrow 3 \mathcal{I}$-transit relations in $\mathbb{Z}C$ are $\partial_l$-boundary relations in the complex $C_G$, for $C = \text{PSL}(2, \mathbb{C})$. Hence, varying $T$ we obtain 3-cycles $\lambda^{-1} \circ \pi(e(T))$ which are equal up to boundaries. This proves the theorem.

\[\text{Remark 8.9} \] Using [11, 14], one can prove that $\lambda$ induces a surjective homomorphism $H_3^0(\text{PSL}(2, \mathbb{C}); \mathbb{Z}) \rightarrow \mathbb{B}(\mathcal{I})$. In that construction, one can simplify Neumann’s arguments, replacing Lemma 5.1 of [43] by our proof of charge invariance of the $\mathcal{D}$-class (whence of the $\mathcal{I}$-class), see Th 4.6 and Th. 8.2. However, we did not use this result above.

\section{On the volume conjecture}

Let $(W, L, \rho)$ be as usual, and fix an arbitrary $\mathcal{D}$-triangulation $T$ of $(W, L, \rho)$. In this section, we show by elementary means that the leading term $G(W, L, \rho)$ of the asymptotic expansion, when $N \to \infty$, of $K_N(W, L, \rho)$ only depends on the pseudoideal tetrahedra of $f(T)$, where $f$ is defined in [4]. Then we discuss the analytic problems underlying the determination of $G(W, L, \rho)$, and the relationships between $G(W, L, \rho)$ and classical dilogarithm functions. This leads us to the reformulation of the volume conjecture for hyperbolic-triples, stated and discussed in the introduction.

Recall from Th. 5.2 that $K_N(W, L, \rho) := K(T_N) := H(T_N)^N$, where
\[
H(T_N) = \Psi(T_N) \ N^{-r_0} \prod_{e \in \mathcal{E}(T) \setminus \mathcal{E}(H)} x(e)^{1-N}/N
\]
\[
\Psi(T_N) = \sum_{\alpha} \prod_i \Psi(\ast(\Delta_i, (D_N)_i, \alpha_i))
\]
and $r_0$ is the number of vertices of $T$. The definition of $\Psi(\ast(\Delta_i, (D_N)_i, \alpha_i))$ is given in [20], see also [19]. Prop. 9.5 and Prop. 9.6. Extract from $\Psi(\ast(\Delta_i, (D_N)_i, \alpha_i))$ each scalar of the form $[x]$ and $y^p_{\text{ou}}$ $(N = 2p + 1)$. Beware that $y^p_{\text{ou}}$ refers to $x^{1/N}$ for some edge $e$ of $T$, see [20]. This defines $\Psi'(\ast(\Delta_i, (D_N)_i, \alpha_i))$ and
\[
\Psi'(T_N) = \sum_{\alpha} \prod_i \Psi'(\ast(\Delta_i, (D_N)_i, \alpha_i)).
\]
We immediately see that
\[ \lim_{N \to \infty} (2i\pi/N^2) \log(K_N(W,L,\rho)) = \lim_{N \to \infty} (2i\pi/N) \log(\Psi'(T_N)) \]
for any determination of the logarithm. The explicit formula of \( \Psi'(+(\Delta_i, (D_N)_i, \alpha_i)) \) is
\[
h_{rN, i(e_0), rN, i(e_1), rN, i(e_0')} = 2 \omega_{c_i(e_1)(\alpha_i+\alpha_{i_0})-c_i(e_0)c_i(e_1)/2} \times \omega(\gamma_1, \gamma_2, \gamma_3) \]
\[
\omega(y_1(e_i)y_2(e_i'), y_1(e_0)y_2(e_0'), y_1(e_2)y_2(e_2')|\alpha_i - c_1(e_0), \alpha_{i_0}) \quad \delta(\alpha_i - c_1(e_0), \alpha_{i_0} - \alpha_i),
\]
where
\[
h'_{rN, i(e_0), rN, i(e_1), rN, i(e_0')} = g(1) g \left( \frac{y_1(e_2)y_2(e_2')}{y_1(e_1)y_2(e_1')} \right),
\]
and \( g(x) := \prod_{j=1}^{N-1} (1 - x\omega_j)^{1/N} \). Remark that the idealization map \( f \) in (11) gives
\[
y_1(e_j)y_2(e_j') = \log a_i(e_j) - i\pi c_i(e_j),
\]
where \( a_i(e_j) \) is the pseudo-modulus of \( e_j \) for \( \Delta_i \) (see Def. 7.3). One can do similar observations for \( \Psi'(-(\Delta_i, (D_N)_i, \alpha_i)) \). Hence, \( \Psi'(T_N) \) is actually a function \( \Psi'((f(T_N)) \) of the pseudo-ideal tetrahedra \( *(\Delta_i, b_i, a_i, c_i, \alpha_i) \), endowed with the states \( \alpha_i \). For a \( i\mathbb{D}p \)-triangulation \( T \) of a hyperbolic-like triple \( (W,L,\rho), f(T) = T_\mathbb{Z} \) is an ideal triangulation. Thus we get
\[
\lim_{N \to \infty} (2i\pi/N^2) \log(K_N(W,L,\rho)) = G(T_\mathbb{Z}) \quad (12)
\]
for some function \( G \). As explained in [26], \( K_N(W,L,\rho) \) is a function of an augmented \( \mathcal{D} \)-scissors congruence class (see [26] for details). The expression (12) implies that, asymptotically, this “augmentation” only concerns the states, i.e. \( G(T_\mathbb{Z}) \) is a function of a class \( \mathcal{C}_D(W,L,\rho) \), which is an enriched version of \( \mathcal{C}_D(W,L,\rho) \) whose definition only involves the states as further arguments.

Recall that the Rogers dilogarithm is the function defined for \( x \in (0,1) \) by
\[
L(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} + \frac{1}{2} \log(x) \log(1-x) - \frac{\pi^2}{6}
\]
\[
= - \int_0^x \frac{\log(1-t)}{t} dt + \frac{1}{2} \log(x) \log(1-x) - \frac{\pi^2}{6}.
\]
We add the summand \(-\pi^2/6\) in order to improve the symmetry of the functional relations of \( L \), by setting \( L(1) = 0 \). Using the integral on the right-hand side, which is called the Euler dilogarithm \( \text{Li}_2(z) \), \( L \) can be continued as a complex analytic function on \( \mathbb{C} \setminus \{(-\infty, 0) \cup (1; \infty)\} \). It is well-known [35, p. 6-7] that we have the five-term functional relation (the “Rogers identity”)
\[
L(x) + L(y) - L(xy) - L \left( \frac{x(1-y)}{1-xy} \right) - L \left( \frac{y(1-x)}{1-xy} \right) = 0 \quad (13)
\]
for \( x, y \in (0,1) \). Up to normalization (that we have fixed by \( L(1) = 0 \), the Rogers identity determines \( L \) as a function of class \( C^3((0,1)) \) (see e.g. [20, App] for a proof). The Rogers identity implies
\[
L(x) = -L(x^{-1}), \quad L(x) = -L(1-x) \quad (14)
\]
Consider the map \( R : \mathbb{Z}[\mathcal{I}] \rightarrow \mathbb{C}/(\pi^2 \mathbb{Z}) \) given by

\[
R(*(\Delta, b, w, c)) = *L(w_0) + \frac{i\pi}{2}(-c'_0 \log(w_1) + c'_1 \log(w_0))
\]

for any determination of the logarithm, where \( c'_i \) is defined in \([8]\). The following result is essentially due to Neumann (one has to further verify that the symmetry relations hold \( \mod(\pi^2/2) \) to obtain this statement, and this uses \([14]\)):

*Proposition 9.1* \([13]\) Prop. 2.5 \( R \) induces a homomorphism

\[
R : \mathcal{P}(\mathcal{I}) \rightarrow \mathbb{C}/(\pi^2 \mathbb{Z}) .
\]

There is a generalization of the Euler dilogarithm \( \text{Li}_2(z) \), called the *non-compact quantum dilogarithm* \( S_\gamma(p) \) \([23]\). For real \( \gamma \), it is a meromorphic function of \( p \) whose leading term of the asymptotic expansion for \( \gamma \to 0 \) is equal to \( \exp(\text{Li}_2(-e^{ip})/2i\gamma) \).

The properties of \( S_\gamma(p) \) are well-known: they all follow from standard complex analysis techniques in one variable \([10, 32, 33]\). One of the most interesting features of \( S_\gamma(p) \) is that it satisfies an analogue of \([11]\), i.e. a \( \gamma \)-deformation of the Rogers identity that recovers it in the limit \( \gamma \to 0 \).

Let \((W, L, \rho)\) be a hyperbolic-like triple, and denote by \( \mathcal{T} \) a \( \mathcal{I} \)-triangulation of \((W, L, \rho)\). Using \( S_\gamma, \) one can write down explicitly the leading term of the asymptotic expansion for \( N \to \infty \) of each of the \( c-6j \)-symbols in \( K(W, L, \rho), \) *when considered independently from the others.* It is of the form

\[
\exp \left( \frac{N}{2i\pi} \left( \text{Li}_2(e^{i(u+g)}) + f(u, v) \right) \right),
\]

where: \( u = -i(\log(y_1(e_1)y_1(e'_1)) - \log(y_1(e_2)y_1(e'_2))), \ v = -i(\log(y_1(e_0)y_1(e'_0)) - \log(y_1(e_2)y_1(e'_2))) \) for the corresponding ideal tetrahedron \(*(\Delta, b, w, c)\) of \( \mathcal{T} \) (see \([11]\)): \( g \) is a constant that depends on \( N \), the charge and the states; \( f(u, v) \) is a degree two polynomial that also depends on \( N \), the states and the charges. Despite the fact that we do not know how to hold the asymptotic behaviour of the whole state sum, and not only of its local ingredients, the \( c-6j \)-symbols, this and our preceding results lead us to formulate the following conjecture for hyperbolic-like triples \((W, L, \rho)\):

*Conjecture 9.2* We have \( G(c_\mathcal{I}(W, L, \rho)) = R(c_\mathcal{I}(W, L, \rho)) \).

One purely analytic way to prove it would be to find an integral representation for \( \Psi'(\mathcal{T}) \), in order to apply a pluri-dimensional steepest descent method - if such a method would exist. Then one should suitably deform the cycle of integration, so that the leading term of the asymptotic expansion of \( \Psi'(\mathcal{T}) \) is determined by some stationary points of the *phase* \( \Phi \) of the integrand. (Remark that this is related, for hyperbolic-like triples, to showing that \( G(\mathcal{T}) \) does only depend on \( c_\mathcal{I}(W, L, \rho) \), and not on its augmentation.) One expects that \( \Phi \) may be reduced to expressions involving only the Rogers dilogarithm, when evaluated on stationary points. Such phenomena have been formally verified on typical situations by several people \([29, 26, 39, 24]\). Finally, one should identify, among the whole set of stationary points of \( \Phi \), those which have dominant contributions to the integral. This, as well as the deformation of the cycle of integration, should be related to the choice of the \( B \)-bundle \( \rho \), and even to the choice of the particular cocycle that represents it. In our opinion, all of this seems to be very difficult.
Appendix: quantum data

In this Appendix we give a detailed account, from both the algebraic and geometric points of view, of the definition and the properties of the 6j-symbols (resp. c-6j-symbols) needed for the construction of the QHI. All the explicit formulae are originally due to Kashaev [29]. We refer to [1, Ch. 3] for the proofs.

Recall that \( \omega = \exp(2\pi i/N) \) for an odd positive integer \( N > 1 \), where \( N = 2p+1 \), \( p \in \mathbb{N} \). Fix the determination \( \omega^{1/2} = \omega^{p+1} \) for its square root. We shall henceforth denote \( 1/2 := p + 1 \mod N \). All other notations for manifolds, triangulations and spines are as in the rest of the paper.

Cyclic representations of \( \mathcal{W}_N \). The Weyl algebra \( \mathcal{W}_N \) is the unital algebra over \( \mathbb{C} \) generated by elements \( E, E^{-1}, D \) satisfying the commutation relation \( ED = \omega DE \).

It is well-known that \( \mathcal{W}_N \) can be endowed with a structure of Hopf algebra isomorphic to the simply-connected (non-restricted) integral form of a Borel subalgebra of \( U_q(sl(2, \mathbb{C})) \) [37, §9], specialized in \( \omega \), with the following co-multiplication, co-unit and antipode maps:

\[
\Delta(E) = E \otimes E , \quad \Delta(D) = E \otimes D + D \otimes 1 , \\
\epsilon(E) = 1 , \quad \epsilon(D) = 0 , \quad S(E) = E^{-1} , \quad S(D) = -E^{-1}D .
\]

A \( N \)-dimensional irreducible representation \( \rho : \mathcal{W}_N \to \text{End}(V_\rho) \) is called cyclic if \( \rho(E), \rho(D) \in \text{GL}(V_\rho) \). Denote by \( C \) this set of representations; we write \( \rho \sim \mu \) when \( \rho \) is isomorphic to \( \mu \). A sequence \( \rho_1, \ldots, \rho_n \) of irreducible cyclic representations of \( \mathcal{W}_N \) is regular if \( \rho_i \otimes \ldots \otimes \rho_{i+j}, 1 \leq i \leq n, 1 \leq j \leq n-i \) is cyclic.

Let \( \delta_{ij} \) be Kronecker’s symbol, and denote by \( X \) and \( Z \) the \( N \times N \) matrices with components \( X_{ij} = \delta_{i,j+1} \) and \( Z_{ij} = \omega^i \delta_{i,j} \) in the standard basis of \( \mathbb{C}_N \). Define a standard representation \( \rho \in C \) by:

\[
\rho(E) = a_\rho^2 Z , \quad \rho(D) = a_\rho y_\rho X ,
\]

where \( a_\rho, y_\rho \in \mathbb{C}^* \). The complex conjugate representation \( \rho^* \) and the inverse representation \( \bar{\rho} \) are the standard representations with

\[
a_{\bar{\rho}} = 1/a_\rho , \quad y_{\bar{\rho}} = -y_\rho ,
\]

\[
a_{\rho^*} = (a_\rho)^* , \quad y_{\rho^*} = (y_\rho)^* .
\]

Proposition 9.3

i) Two standard representations \( \rho, \mu \in C \) are isomorphic iff

\[
a_{\rho}^{2N} = a_{\mu}^{2N} , \quad a_{\rho}^N y_{\rho}^N = a_{\mu}^N y_{\mu}^N .
\]

ii) Any \( \rho \in C \) is isomorphic to a standard representation.

iii) Fix a determination of the \( N \)-th root. If \( (\rho, \mu) \in C^2 \) is regular, then \( \rho \otimes \mu : \mathcal{W}_N \to \text{End}(V_\rho) \otimes \text{End}(V_\mu) \) splits as a direct sum of \( N \) representations isomorphic to the standard representation \( \rho \otimes \mu \in C \) defined by

\[
a_{\rho \mu} = a_\rho a_\mu , \quad y_{\rho \mu} = \left( a_\rho^N y_{\mu}^N + y_{\rho}^N a_\mu^{-N} \right)^{1/N} .
\]

Recall that \( B \) is the Borel subgroup of \( SL(2, \mathbb{C}) \) of upper triangular matrices. This proposition allows to define a faithful map \( \Phi : C/\sim \to B \), where \( \text{Im}(\Phi) \) is the set of non diagonal elements of \( B \) and

\[
\Phi([\rho]) = \begin{pmatrix} a_{\rho}^N & y_{\rho}^N \\ 0 & a_{\rho}^{-N} \end{pmatrix} .
\]
Property iii) is equivalent to $\Psi(\rho\mu) = \Psi(\rho) \cdot \Psi(\mu)$ for any regular pair $(\rho, \mu)$.

**Clebsch-Gordan operators.** The *multiplicity module* of representations $\rho, \mu \in C$ is the set

$$M_{\rho,\mu} = \text{End}_{W_N} (V_\rho, V_\mu) = \{ U : V_\rho \to V_\mu \mid U \rho(a) = \mu(a)U, \forall a \in W_N \} ,$$

which is formed by the *intertwiners* of $\rho$ and $\mu$. Prop. 9.3 ii)-iii) imply that for any regular pair $(\rho, \mu)$, $\text{dim}_C(M_{\rho,\rho\otimes\mu})$ is equal to $N$ if $\nu$ is isomorphic to $\rho\mu$, and zero otherwise. Similarly for $M_{\nu,\nu\otimes\mu,\nu}$ (which are embeddings) are called *Clebsch-Gordan operators* (CGO), and the elements of $M_{\rho\otimes\nu,\nu}$ (which are projectors) are called *dual Clebsch-Gordan operators*.

Let us give an explicit basis of the CGO for a regular pair of standard representations; for the dual operators, see Prop. 9.3. Denote by $[x, y, z]$ the homogeneous coordinates of $\mathbb{CP}^2$. Consider the curve $\Gamma \subset \mathbb{CP}^2$ which is the zero set of the equation $x^N + y^N = z^N$, and define for any positive integer $n$ a rational function $\omega$ (not to be confused with the root of unity $\omega$ !) by:

$$\omega(x, y, z|n) = \prod_{j=1}^{n} \frac{y^j}{z - x\omega^j}, \ [x, y, z] \in \Gamma \setminus \{(1, 0, \omega^j), j = 1, \ldots, n\}. \quad (16)$$

Set $\omega(x, y, z|m, n) = \omega(x, y, z|m-n)\omega^{n^2/2}$. Define also a periodic Kronecker symbol by $\delta(n) = 1$ if $n \equiv 0 \mod N$, and $\delta(n) = 0$ otherwise. Note that the function $\omega$ is periodic in its integer argument, with period $N$. (See [3] for a summary of some properties of the function $\omega$).

**Proposition 9.4.** Let $(\rho, \mu)$ be a regular pair of standard representations. For any non-zero complex number $h_{\rho,\mu}$, the set $\{K_\alpha(\rho, \mu), \alpha = 0, \ldots, N-1\}$ of linear operators with components

$$K_\alpha(\rho, \mu)_{i,j}^k = h_{\rho,\mu} \omega^{\alpha j} \omega(a_\rho y_\mu, y_\mu|\alpha) \delta(i+j+k)$$

is a basis of $M_{\rho\mu,\rho\otimes\mu}$.

The factor $h_{\rho,\mu}$ is of course inessential here, but it will be justified below.

**6j-symbols.** For any regular triple $(\rho, \mu, \nu)$ set

$$M_{\rho,\mu,\nu} = \text{End}_{W_N} (V_\rho \otimes (V_\mu \otimes V_\nu))
\quad M_{(\rho,\mu),\nu} = \text{End}_{W_N} ((V_\rho \otimes V_\mu) \otimes V_\nu) .$$

One has

$$M_{\rho,\mu,\nu} \cong M_{\rho\mu,\rho\otimes\nu} \otimes M_{\mu\nu,\mu\otimes\nu}$$

$$M_{(\rho,\mu),\nu} \cong M_{\rho\mu,\rho\otimes\nu} \otimes M_{\rho\nu,\nu\otimes\nu} .$$

The isomorphism of $W_N$-modules $(V_\rho \otimes (V_\mu \otimes V_\nu)) \cong ((V_\rho \otimes V_\mu) \otimes V_\nu)$ induces an isomorphism

$$R(\rho, \mu, \nu) : M_{(\rho,\mu),\nu} \cong M_{(\rho,\mu),\nu} \quad (17)$$

which we call a 6j-symbol (strictly speaking, only matrix components usually have this name). The 6j-symbols satisfy a 3-cocycloid relation, the *pentagon equation*,

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where \( R \) is the mutativity of the diagram in the basis of CGO of Prop. 9.4. The isomorphism (17) translates into the commutativity of the diagram

\[
\begin{align*}
\rho\mu\nu & \quad \rightarrow \quad \rho \otimes \mu \otimes \nu \\
\end{align*}
\]

where \((\rho, \mu, \nu)\) is a regular triple of standard representations and the arrows denote embeddings of representations. The families of maps \( \{(id \otimes K_\delta(\mu, \nu)) \circ K_\gamma(\rho, \mu)\}_{\delta, \gamma=0} \) and \( \{(K_\alpha(\rho, \mu) \otimes id) \circ K_\beta(\rho, \mu)\}_{\alpha, \beta} \) form two linear basis of the space of embeddings of \( \rho\mu\nu \) into \( \rho \otimes \mu \otimes \nu \). With respect to these basis, the 6\(j\)-symbol \( R(\rho, \mu, \nu) \) reads

\[
K_\alpha(\rho, \mu) K_\beta(\rho\mu, \nu) = \sum_{\delta, \gamma=0}^{N-1} R(\rho, \mu, \nu)_{\gamma, \delta}^{\alpha, \beta} K_\delta(\mu, \nu) K_\gamma(\rho, \mu\nu) .
\]

Consider the functions

\[
\forall x \in \mathbb{C}^\ast, \quad g(x) := \prod_{j=1}^{N-1} (1 - x \omega^j)^{1/N}, \quad h(x) := x^{-p} \frac{g(x)}{g(1)} .
\]

where \( p \) is defined by \( N = 2p + 1 \), and the function \( g \) is understood as the analytic continuation of the formal power series over \( x \) into the whole complex plane with cuts from the points \( x = \exp(2i\pi k\epsilon/N) \), \( k = 0, \ldots, N-1 \), \( \epsilon \in \mathbb{R} \), to infinity. (We have \( |g(1)| = N^{N/2} \)). Fix the scalar \( h_{\rho, \mu} \) in Prop. 9.4 as \( h_{\rho, \mu} = h\left(\frac{y_{\rho\mu}}{y_{\mu}y_{\mu}}\right) \), and set

\[
h_{\rho, \mu, \nu} = h\left(\frac{y_{\rho\mu}y_{\mu\nu}}{y_{\rho\mu\nu}}\right), \quad [x] = N^{-1} \left(\frac{1 - x^N}{1 - x}\right).
\]

Hereafter we will implicitly assume in the definition of \( g \) that the cuts are away from the points where \( g \) is explicitly evaluated (these are the parameters of a finite number of fixed standard representations of \( \mathcal{W}_N \)). Note that \( h_{\rho, \mu} \) is such that with Prop. 9.5 one has:

\[
K_{\alpha}(\rho, \mu)_{i, j}^{k} = R(\rho, \mu)_{i, j}^{k, \alpha} .
\]

In fact, one can prove that the 6\(j\)-symbols and the CGO are representations of the canonical element of the Heisenberg double of \( \mathcal{W}_N \) (which is a twisted quantum dilogarithm), acting on \( M_{\rho\mu\nu, \rho \otimes \mu \otimes \nu} \) [\( \mathfrak{I}, \S 3.2-3.3 \)].

**Proposition 9.5** In the basis of CGO of Prop. 9.4, the 6\(j\)-symbols of regular sequences of cyclic representations of \( \mathcal{W}_N \) read

\[
R(\rho, \mu, \nu)_{\gamma, \delta}^{\alpha, \beta} = h_{\rho, \mu, \nu} \omega^{\alpha\delta} \omega(y_{\rho\mu\nu}, y_{\rho\mu\nu}, y_{\rho\mu\nu}) \delta(\gamma + \delta - \beta) ,
\]

and their inverses are given by

\[
\check{R}(\rho, \mu, \nu)_{\gamma, \delta}^{\alpha, \beta} = \frac{h_{\rho, \mu, \nu}}{h_{\rho, \mu, \nu}} \omega^{-\alpha\delta} \frac{\delta(\gamma + \delta - \beta)}{\omega(y_{\rho\mu\nu}, y_{\rho\mu\nu}, y_{\rho\mu\nu}) \omega(\gamma, \alpha)} .
\]
Symmetries. We are forced to *symmetrize* the 6j-symbols, that is, as in (20) below, to make them equivariant in some way w.r.t. the branching (see the discussion about the QHI phase factor in [3]). For this we have to extend their definition. This is done as follows. Define \( N \times N \) matrices \( A = \{ A_{m,n} \} \) and \( A^{-1} = \{ A^{m,n} \} \), resp. \( B = \{ B_{m,n} \} \) and \( B^{-1} = \{ B^{m,n} \} \), which are inverse one to each other, by \((\zeta \in \mathbb{C})\)

\[
\begin{align*}
A_{m,n} &= \zeta^{-1} \omega^{m^2/2} \delta(m + n), \\
A^{m,n} &= \zeta \omega^{-m^2/2} \delta(m + n), \\
B_{m,n} &= N^{-1/2} \omega^{mn}, \\
B^{m,n} &= N^{-1/2} \omega^{-mn}.
\end{align*}
\]

Proposition 9.6 Given \( a, c \in \mathbb{Z}/N\mathbb{Z} \) and a regular sequence \((\rho, \mu, \nu)\) of standard representations, consider the c-6j-symbols defined by

\[
\begin{align*}
R(\rho, \mu, \nu|a, c)_{\gamma, \delta}^{\alpha, \gamma'} &\equiv (y_{\rho \rho} y_{\mu \nu})^p \omega^{c(\gamma - a - \alpha)/2} R(\rho, \mu, \nu)_{\alpha, \beta}^{\gamma - \alpha, \delta}, \\
\tilde{R}(\rho, \mu, \nu|a, c)_{\alpha, \gamma}^{\alpha, \beta} &\equiv (y_{\rho \rho} y_{\mu \nu})^p \omega^{c(\alpha - a + \nu)/2} \tilde{R}(\rho, \mu, \nu)_{\gamma + \alpha, \delta}^{\alpha, \beta + \delta},
\end{align*}
\]

where \( N = 2p + 1 \). We have the following relations:

\[
\begin{align*}
\sum_{\alpha', \gamma'} R(\rho, \mu, \nu|a, c)_{\alpha', \gamma'}^{\gamma, \delta} A_{\gamma, \gamma'} A^{\alpha, \alpha'} &\equiv \omega^{a/4} \tilde{R}(\rho, \rho \mu, \nu|a, b)_{\gamma, \delta}^{\alpha, \alpha'}, \\
\sum_{\alpha', \delta'} R(\rho, \mu, \nu|a, c)_{\alpha', \delta'}^{\gamma, \delta'} A_{\delta, \delta'} B^{\alpha, \alpha'} &\equiv \omega^{-c/4} \tilde{R}(\rho \mu, \rho \nu|b, c)_{\beta, \delta}^{\alpha, \gamma}, \\
\sum_{\beta', \delta'} R(\rho, \mu, \nu|a, c)_{\alpha', \beta'}^{\gamma, \delta'} B_{\delta, \delta'} B^{\beta, \beta'} &\equiv \omega^{a/4} \tilde{R}(\rho \nu, \rho \nu|a, b)_{\gamma, \delta}^{\alpha, \alpha'},
\end{align*}
\]

where \( b = 1/2 - a - c \in \mathbb{Z}/N\mathbb{Z} \) and \( \zeta \) is some \( N \)-th root of unity.

Let us interpret these relations. Let \((W, L, \rho)\) be as usual: \( W \) is a closed oriented 3-manifold, \( L \) is a link in \( W \) and \( \rho \) is flat principal \( B \)-bundle over \( W \). Choose a \( D \)-triangulation \( T = (T, H, D = (b, z, c)) \) of \((W, L, \rho)\). Fix a common determination of the \( N \)-th root for all the matrix entries of \( \{ z(e) \} \). As in [5] denote by \( \alpha : \mathcal{F}(T) \to \mathbb{Z}/N\mathbb{Z} \) a \( N \)-state of \( T \) and by \( \alpha_1 \) the restriction of \( \alpha \) to \( \Delta_1 \subset T \); we write \( \alpha_{ij} = \alpha_1(f_j) \) for the face \( f_j \) opposite to the \( j \)-th vertex w.r.t. \( b_i \). Finally, let \((\mathcal{D}_N) = (b_N, i_N, i_N, \ldots)\), where \( r_{N,i} \) is the standard representation defined (using Prop. 7.8) on each edge \( e \in T \) by \((a(e), g(e)) = (t(e)^{1/N}, x(e)^{1/N})\). Set \( \Psi(\ast(\Delta_1, (\mathcal{D}_N), \alpha_i)) \) equal to

\[
\begin{align*}
\{ R(r_{N,i}(e_0), r_{N,i}(e_1), r_{N,i}(e_0'))_{c(e_0), c(e_1)}^{\alpha_{12}, \alpha_{0}} &\text{ if } * = +1, \\
\tilde{R}(r_{N,i}(e_0), r_{N,i}(e_1), r_{N,i}(e_0'))_{c(e_0), c(e_1)}^{\alpha_{13}, \alpha_{01}} &\text{ if } * = -1.
\}
\end{align*}
\]

(20)

The matrices \( A \) and \( B \) satisfy

\[
\begin{align*}
B^4 &= \text{id}, & B^2 &= \zeta' (BA)^3,
\end{align*}
\]

where \(|\zeta'| = 1\). Hence they define a projective \( N \)-dimensional representation \( \theta \) of \( SL(2, \mathbb{Z}) \), which admits a presentation of the form \((a, b| b^4 = 1, b^2 = (ba)^3)\). One can rewrite Prop. 9.5 as follows. Put

\[
\begin{align*}
M_+(\rho, \mu, \nu) &= \text{End} \left( M_{(\rho, \mu, \nu)}, M_{(\rho, \mu), \nu} \right), \\
M_-(\rho, \mu, \nu) &= \text{End} \left( M_{(\rho, \mu, \nu)}, M_{(\rho, \mu), \nu} \right).
\end{align*}
\]

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One has $\Psi((\Delta_i,(D_N)_i,\alpha_i)) \in M_4(r_{N,i}(e_0),r_{N,i}(e_1),r_{N,i}(e_0))$. Denote by $\Upsilon$ the set of transpositions of the latter space, and consider the maps

$$\Upsilon \times EM_{\mu,\mu \otimes \nu} \times EM_{\rho,\rho \otimes \nu} \times EM_{\rho,\rho \otimes \mu} \times EM_{\mu,\mu \otimes \mu} \times M_4(\rho,\mu,\nu) \downarrow \text{End}$$

$$\chi^+(t,f_0,f_1,f_2,f_3,\Psi) = t \circ ((f_3 \otimes f_1) \circ \Psi \circ (f_2 \otimes f_0))$$

$$\Upsilon \times EM_{\mu,\mu \otimes \nu} \times EM_{\rho,\rho \otimes \nu} \times EM_{\rho,\rho \otimes \mu} \times EM_{\mu,\mu \otimes \mu} \times M_4(\rho,\mu,\nu) \downarrow \text{End}$$

$$\chi^-(t,f_0,f_1,f_2,f_3,\Psi) = t \circ ((f_2 \otimes f_0) \circ \Psi \circ (f_3 \otimes f_1)),$$

where $EV$ stands for $End_C(V)$ and $End$ for morphisms of complex vector spaces. Denote by $U(1)_N$ the group of $N$-th roots of unity. Prop. 9.6 says that there exist maps $\pi : S_4 \to \Upsilon$ and $\Theta : S_4 \times D \to SL(2,\mathbb{Z})^4$ such that the following diagram is commutative:

$$S_4 \times D^* \xrightarrow{p_D} D$$

$$\Upsilon \times SL(2,\mathbb{Z})^4 \times End \xrightarrow{\chi^o(id \times \theta^4 \times id)} \text{End}/U(1)_N$$

where $p_D$ is defined in (3). A permutation of the vertices of $\Delta_i$ induces a permutation of the states $\alpha_j$; this defines the map $\pi$. Here is a recipe for defining $\Theta$ [27]. Each $f \in F(\Delta_i)$ inherits an orientation from $b_i$: set $\epsilon(f) = 1$ if this orientation is the one induced by $\Delta_i$ as a boundary, and $\epsilon(f) = -1$ otherwise. Given two vertices $v_j, v_{j+1} \in f$, set $\lambda(f) = 1$ if the vertex of $f$ distinct from $v_j$ and $v_{j+1}$ is greater than $j+1$, and $\lambda(f) = -1$ if it is less than $j$. Finally, for the permutation $\sigma = (v_j, v_{j+1})$ put

$$\sigma_{ab}(f) = \frac{(1 + a \lambda(f))(1 + b \epsilon(f))}{4},$$

where $a, b = \pm$. For instance, if $b_i \sim b_i^+$ and we consider the permutation $\sigma = (v_0, v_1)$, then $\sigma_{ab}(f_2) \neq 0$ if $a = b = +$ (where $f_2$ is opposite to $v_2$). Denote by $\Theta_j$ the $j$-th component of $\Theta$. Then, for any $\sigma = (v_i, v_{i+1})$ and $f_j \in F(\Delta_i)$, we set

$$\Theta_j(\sigma, (\Delta_i,D_i)) = \sigma_{++}(f_j) a + \sigma_{+-}(f_j) a^{-1} + \sigma_{-+}(f_j) b + \sigma_{--}(f_j) b^{-1} \in SL(2,\mathbb{Z}).$$

Since the elementary transpositions $(i,i+1)$ generate the whole group $S_4$, this completely determines the map $\Theta$. Moreover, the matrix $\theta \circ \Theta_j(\sigma, (\Delta_i,D_i))$ acts on $\Psi((\Delta_i, (D_N)_i,\alpha_i))$ by composition through the tensor factor corresponding to the face $f_j$.

**Branching invariance of the state sum.** As in [3] consider the state sum

$$\Psi(T_N) = \sum_\alpha \prod_i \Psi((\Delta_i,(D_N)_i,\alpha_i)).$$

Choose a maximal tree $\Gamma$ in the 1-skeleton of the cell decomposition of $W$ dual to $T$. Denote by $\Gamma(T)$ the polyhedron obtained by gluing the abstract tetrahedra
\( \Delta_i \in T \) along the faces dual to the edges of \( \Gamma \). Let \( \alpha_{\Gamma} : \mathcal{F}(\Gamma(T) \setminus \partial \Gamma(T)) \to \mathbb{Z}/N\mathbb{Z} \) be a \textit{N-state} of \( \Gamma(T) \). There are an even number of faces in \( \partial \Gamma(T) \), and they are naturally paired via the identifications in \( T \). Each pair consists of a “top” face and a “bottom” face (fix a choice arbitrarily). Consider the operator

\[
\Gamma(T_N) = \sum_{\Delta_i} \prod_i \Psi(\ast(\Delta_i, (\mathcal{P}_N)_i, \alpha_i)),
\]

which is viewed as a morphism from the tensor product of the \textit{N}-dimensional complex vector spaces attached to the “top” faces, and with values in the same tensor product for the “bottom” faces. We have \( \Gamma(T_N) \in \text{End}(\mathbb{C}^{n_{\Gamma}}) \), where \( n_{\Gamma} \) is the number of univalent vertices of \( \Gamma \). Identifying bottom and top faces we clearly get

\[
\Psi(T_N) = \text{tr}(\Gamma(T_N)),
\]

where \( \text{tr} \) is the trace on \( \text{End}(\mathbb{C}^{n_{\Gamma}}) \).

**Lemma 9.7** Up to multiplication by \( N \)-th roots of unity, \( \Psi(T_N) \) does not depend on the branching \( b \) in \( T \).

**Proof.** Any change of branching translates on each \( \Delta_i \) as a composition of transpositions of vertices. By Prop. 10.3 such transpositions induce an action of the matrices \( A^\pm 1 \) and \( B^\pm 1 \) on \( \Psi(\ast(\Delta_i, (\mathcal{P}_N)_i, \alpha_i)) \); moreover, there is a power of \( \omega^{1/4} \) appearing in factor. Now remark that for any \( f \in \mathcal{F}(\Delta_i) \), changing \( \varepsilon(f) \) turns \( \sigma_+(f) \) into \( \sigma_-(f) \), and this implies the action of a dual matrix on the vector space attached to \( f \). Since the tetrahedron glued to \( \Delta_i \) along \( f \) gives it the opposite orientation, we deduce that a change of branching may only alter \( \Psi(T_N) \) by a \( N \)-th root of unity. (Equivalently, this means that a change of branching turns \( \Gamma(T_N) \), up to \( N \)-th roots of unity, into a conjugate operator, and thus \( \Psi(T_N) = \text{tr}(\Gamma(T_N)) \) does not depend on \( b \) up to \( N \)-th roots of unity). \( \square \)

Directly from the definition of the c-6j-symbols one may prove :

**Proposition 9.8** Let \( (\rho, \mu, \nu) \) be a regular sequence of standard representations. We have the following identity :

\[
\hat{R}(\rho^*, \mu^*, \nu^*| a, c)^{\alpha, \beta}_{\gamma, \delta} = (R(\rho, \mu, \nu| a, c)^{-\gamma, -\delta}_{\alpha, -\beta})^*,
\]

where * denotes the complex conjugation.

**Functional properties of the c-6j-symbols.** Let us use the notations of Fig. 8. Denote by \( c^i \) the integral charge on \( \Delta^i \) and by \( c_{jk}^i \) the value of \( c^i \) on the edge with vertices \( v_j \) and \( v_k \). First we give the relation between the operators associated via (20) to the left (LHS) and the right (RHS) hand side of Fig. 8, where we suppose that the \( 2 \to 3 \) \( D \)-transit is full. The resulting equation is called the \textit{extended pentagon relation} (EP relation below).

There are exactly four degrees of freedom in choosing the charges of the LHS (e.g. \( c_{03}^1, c_{01}^3, c_{23}^3 \) and \( c_{21}^3 \)), and there is one more independent one for the RHS (e.g. \( c_{03}^3 \)). Let us consider the following set of independent charges:

\[
i = c_{01}^4, \quad j = c_{01}^2, \quad k = c_{12}^0, \quad l = c_{23}^1, \quad m = c_{12}^3.
\]

One can easily show that

\[
i + m = c_{13}^2, \quad l - i = c_{23}^0, \quad j + k = c_{02}^1, \quad i + j = c_{01}^3, \quad m - k = c_{12}^4.
\]

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Proposition 9.9 Let \((\rho, \mu, \nu, \upsilon)\) be a regular sequence of standard representations. The following EP relation holds:

\[
R_{12}^1(\rho, \mu, \nu | i, m - k) R_{13}^2(\rho, \mu \upsilon, \nu | j, l + m) R_{01}^3(\mu, \nu, \upsilon | k, l - i) = y_{\rho \mu}^{2p} R_{24}^1(\mu \nu, \upsilon | j + k, l) R_{12}^3(\rho, \mu, \nu \upsilon | i + j, m),
\]

where \(R^i\) is associated to \(\Delta^i\) via (20), and we set \(R^1_{12} = R^i \otimes \text{id}\) and so on.

One can read a relation corresponding to a \(0 \rightarrow 2\) \(D\)-transit for \(\Delta^4\) by comparing the identities obtained by applying first a \(2 \rightarrow 3\) \(D\)-transit on \(\Delta^0\) and \(\Delta^2\), and then a \(3 \rightarrow 2\) transit on \(\Delta^0, \Delta^2\) and \(\Delta^4\) (this is possible, since the final configuration is branchable; this argument is somehow similar to the one used in the proof of Lemma 4.4).

Proposition 9.10 Let \((\rho, \mu, \nu)\) be a regular sequence of standard representations. The following orthogonality relation holds:

\[
R(\rho, \mu, \nu | a, c) \tilde{R}(\rho, \mu, \nu | -a, -c) = (y_{\rho \mu} y_{\mu \nu})^{2p} \text{id} \otimes \text{id}.
\]

A branched relation corresponding to a distinguished bubble move is obtained by taking the partial trace over one of the tensor factors in the orthogonality relation:

Proposition 9.11 Let \((\rho, \mu, \nu)\) be a regular sequence of standard representations. The following bubble relation holds:

\[
\text{tr}_i \left( R(\rho, \mu, \nu | a, c) \tilde{R}(\rho, \mu, \nu | -a, -c) \right) = N (y_{\rho \mu} y_{\mu \nu})^{2p} \text{id},
\]

where \(i\) is equal to 1 or 2.

Using Prop. 9.9 one can easily derive from Prop. 9.9, 9.10 and 9.11 the whole set of EP, orthogonality and bubble relations, for any branching. Note, however, that they involve \(N\)-th roots of unity as phase factors.

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