MOTIVIC CLASSES OF CLASSIFYING STACKS OF SOME SEMI-DIRECT PRODUCTS

IVAN MARTINO AND FEDERICO SCAVIA

Abstract. Let $k$ be a field, let $G$ be a finite group, and let $T$ be a split $k$-torus on which $G$ acts multiplicatively. For every $m \geq 1$ denote by $T[m]$ the $m$-torsion subgroup of $T$. Under a suitable assumption on $m$, we show that the motivic class of $B(T[m] \rtimes G)$ in $K_0(\text{Stacks}_k)$ equals that of $BG$. As a consequence, we prove that the motivic class of $BW$ is trivial for a large class of complex reflection groups $W$.

1. Introduction

Let $k$ be a field, and let $G$ be a finite group scheme over $k$. This paper contributes to the computation of the motivic class $\{BG\}$ of the classifying stack of $G$ in the Grothendieck ring of algebraic stacks $K_0(\text{Stacks}_k)$. More specifically, we are interested in the triviality of $\{BG\}$, following up on work of Ekedahl [8, 7], the computations of the first author [13], and the results of [1, 6, 22, 23].

Before we go any further, we provide the context where our results are relevant and several motivations to tackle this motivic computations.

The Noether Problem. Let $V$ be a faithful $G$-representation, finite-dimensional over $k$. The Noether Problem for $G$ and $V$ is the question of whether the quotient variety $V/G$ is $k$-rational, that is, birational to some affine space over $k$. In 1917, Noether [16] studied this question in the case when $G$ is constant (i.e. a finite group). In this case, the problem amounts to determine whether the field of invariants $k(V)^G$ is rational (i.e. purely transcendental) over $k$. It is not hard to produce many examples for which the answer is affirmative. However, no negative examples were found for more than fifty years.

The first example of a group for which the Noether Problem has negative answer is due to Swan [21], who proved that over $k = \mathbb{Q}$ the field of invariants of the regular representation of $\mathbb{Z}/47\mathbb{Z}$ is not rational.

Over an algebraically closed field $k$ of characteristic zero, the first negative example is due to Saltman [18]; his methods were subsequently refined by Bogomolov [2]. The examples of Saltman and Bogomolov are certain $p$-groups of order $p^9$ and of order $p^6$, respectively. More recently Hoshi, Kang and Kunyavskiǐ [9] found examples of order $p^5$, where $p$ is odd.

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The strategy of Saltman is as follows. If $K/k$ is a field extension, one can consider the unramified Brauer group $\text{Br}_{nr}(K) := H^2_{nr}(K, \mathbb{Q}/\mathbb{Z}(1))$. It is an abelian group, and it is trivial if $K/k$ is rational. To find a negative example to the Noether Problem, Saltman exhibited $G$ and $V$ for which he was able to show that $\text{Br}_{nr}(k(V)^G) \neq 0$. Later, Bogomolov [2] described $\text{Br}_{nr}(k(V)^G)$ purely in terms of the group cohomology of $G$, as a subgroup of the Schur multiplier $H^3(G, \mathbb{Z})$. For this reason the group $\text{Br}_{nr}(k(V)^G)$ is also known as the Bogomolov multiplier of $G$, and is sometimes denoted by $B_0(G)$.

Triviality of the motivic class of $BG$. In 2009, Ekedahl [8] considered the motivic class $\{BG\}$ in the Grothendieck ring of stacks $K_0(\text{Stacks}_k)$. Furthermore, when $k$ has characteristic zero, he used the refined Euler characteristic introduced in [8] to construct geometric invariants of $G$, one for each integer $n \geq 1$. For $n = 2$, his construction recovers $B_0(G)$. These invariants are called the Ekedahl invariants of $G$ in [13, 14], to which we refer for an overview of this topic.

We say that $BG$ has trivial motivic class if $\{BG\} = 1$ in $K_0(\text{Stacks}_k)$. Triviality of the class $\{BG\}$ entails triviality of the Ekedahl invariants and, in particular, of the Bogomolov multiplier; see [8, Theorem 5.1]. This implies that in the aforementioned negative examples to the Noether Problem over an algebraically closed field of characteristic zero, $\{BG\}$ is not trivial in $K_0(\text{Stacks}_k)$. The connection between the triviality of $\{BG\}$ and the Noether Problem is intriguing, but so far remains largely unexplained.

A further point of interest in the triviality of $\{BG\}$ comes from recent work of Totaro [23], which suggests that it might be related to five other interesting properties of finite groups: stable rationality of $BG$, triviality for the birational motive of the quotient varieties $V/G$, the Chow Künneth property of $BG$, the Chow Künneth property of $BG$ and the mixed Tate property of $BG$. As explained by Totaro, it is entirely possible that all these properties are equivalent, when $k$ is algebraically closed of characteristic zero. For example, if the Bogomolov multiplier $B_0(G)$ is non-trivial, then all the above properties fail for $G$.

We note that over non-algebraically closed fields the above properties are in general not equivalent. For example, if $k$ is a field of characteristic zero admitting a biquadratic field extension, there exist non-constant finite group schemes $G$ over $k$ such that $\{BG\} \neq 1$ and $BG$ is stably rational; see [19].

The known instances of triviality for $\{BG\}$ mainly come from work of Ekedahl. They are:

- the group schemes $\mu_n$ of $n$-th roots of unity, for every $n \geq 1$ (see [8, Proposition 3.2]);
- the symmetric groups $S_n$, $n \geq 1$ (see [8, Theorem 4.3]);
- all finite subgroups of the group of affine transformations of $\mathbb{A}^1_k$, assuming $k$ algebraically closed (see [8, p. 8, Example ii) on page 8]).

Subsequently, when $k$ is algebraically closed of characteristic zero, the first author proved triviality of $\{BG\}$ in $K_0(\text{Stacks}_k)$ for

- the finite subgroups of $\text{GL}_3$, [13, Theorem 2.4].

The proofs of all of these results run along similar lines: one considers a faithful representation $V$ of $G$, stratifies $V$ according to the stabilizer, computes $\{U/G\}$ as a polynomial in $L := \{A^1_k\}$, where $U \subseteq V$ is the open subset where $G$ acts freely, and inductively computes the other strata.
Semi-direct products. We now come to the new results of this paper. We devise a new method for establishing the triviality of the classes of the classifying stacks of certain finite groups, based on integral representation theory.

Let $G$ be a finite constant group, and let $M$ be a $G$-lattice, that is, $M$ is a finitely generated free abelian group on which $G$ acts additively. Consider a short exact sequence

$$0 \to N \to P \to M \to 0$$

where $P$ is a permutation $G$-lattice and $N$ is a coflasque $G$-lattice. Recall that a $G$-lattice is permutation if it admits a basis which is stable under the $G$-action, and that it is coflasque if $H^1(H, N) = 0$ for every subgroup $H$ of $G$. We refer the reader to [4, Proposition 1.3] for the construction of a sequence (1.1).

We denote by $e(M)$ the period of the class of (1.1) in $\text{Ext}^1_G(M, N)$. The invariant $e(M)$ has a very interesting geometric interpretation: if $L/F$ is a finite Galois extension with Galois group $G$, and $U$ is an $F$-torus split by $L$ and whose character $G$-lattice is isomorphic to $M$, by a theorem of Merkurjev [15] the number $e(M)$ equals the period of a generic $U$-torsor $X \to \text{Spec} K$, as an element of the group $H^1(K, U)$; see [15] for the precise definitions. In particular, $e(M)$ does not depend on the choice of (1.1). It is clear that $e(M)$ is a divisor of $|G|$.

**Theorem 1.1.** Let $k$ be a field, let $G$ be a finite group, let $T \cong \mathbb{G}_m^n$ be a split $k$-torus on which $G$ acts multiplicatively, and let $\hat{T}$ be the character lattice of $T$, viewed as a $G$-lattice via the induced action. Let $m \equiv \pm 1 \pmod{e(\hat{T})}$ be a non-negative integer, and denote by $T[m]$ the $m$-torsion subgroup of $T$.

Then we have $\{B(T[m] \rtimes G)\} = \{BG\}$ in $K_0(\text{Stacks}_k)$.

In particular, if $\{BG\} = 1$ in $K_0(\text{Stacks}_k)$, then $\{B(T[m] \rtimes G)\} = 1$. We note that Theorem [11] makes no assumptions on the base field $k$: the group scheme $T[m]$ is allowed to be non-constant and even non-reduced.

As we have already mentioned, all previous results on the triviality of $\{BG\}$ have been proved by choosing a faithful representation $V$ of $G$ and by computing the class of $[V/G]$ using a suitable stratification. Theorem [11] is of a different nature: its proof is arithmetic, and makes use of multiplicative invariant theory, the theory of (non-split) algebraic tori, and Galois cohomology.

Finite reflection groups. As an application of Theorem [11] we prove the triviality of a large number of motivic classes of classifying stacks of finite complex reflection groups.

Recall that a finite constant group $G$ is called a complex reflection group if there exists a faithful complex representation $V$ of $G$ such that $G$ is generated by pseudoreflections (i.e. elements which fix some complex hyperplane of $V$ pointwise); a useful reference on the subject is [17]. If $G$ is a complex reflection group, and $V$ is a reflection representation for $V$, it follows from the Chevalley-Shephard-Todd Theorem that the Noether Problem for $G$ and $V$ has affirmative answer, and so $BG$ is stably rational. It is then natural to wonder if some of the other properties listed by Totaro are also true. In particular, do we have $\{BG\} = 1$? If $G = S_n$, then the answer is affirmative by [8, Theorem 4.3], however the proof does not seem to generalize to other reflection groups.
For every field $k$ and all integers $m, p, n \geq 1$ such that $p$ divides $m$, consider the $k$-group scheme

$$G(m, p, n) := \{(\zeta_1, \ldots, \zeta_n, \sigma) \in \mu_m^n \rtimes S_n : \prod_{i=1}^n \zeta_i^{m/p} = 1\}.$$ 

If $k = \mathbb{C}$, the $G(m, p, n)$ form the infinite family of irreducible finite complex reflection groups.

**Corollary 1.2.** Let $k$ be a field. Assume that $p = 1$, or that $p = m \equiv \pm 1 \pmod{n}$. Then we have $\{BG(m, p, n)\} = 1$ in $K_0(\text{Stacks}_k)$.

The proof of Corollary 1.2 in the case $p = 1$ is particularly simple, and it implies that $\{BG\} = 1$ for every finite reflection group $G$ of type $B_n$; this case may also be deduced from the $\text{Symm}$ formalism of $[3]$.

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**2. Motivic classes of classifying stacks**

We begin by recalling that the Grothendieck ring of algebraic varieties $K_0(\text{Var}_k)$ is the group generated by the isomorphism classes $\{X\}$ of $k$-schemes of finite type $X$, subject to the relation $\{X\} = \{Y\} + \{X \setminus Y\}$ for every closed embedding $Y \hookrightarrow X$. We define a product on $K_0(\text{Var}_k)$ by setting $\{X\} \cdot \{Y\} := \{X \times Y\}$ and extending by bilinearity. This makes $K_0(\text{Var}_k)$ into a commutative ring with identity $1 = \{\text{Spec } k\}$. We denote by $\mathbb{L}$ the class of the affine line $\mathbb{A}_k^1$ in $K_0(\text{Var}_k)$.

In $[7]$, Ekedahl constructed a Grothendieck ring of algebraic stacks as follows.

**Definition 2.1.** Let $\mathcal{S}$ be an algebraic stack of finite type over $k$. The Grothendieck ring of algebraic stacks over $\mathcal{S}$, denoted $K_0(\text{Stacks}_\mathcal{S})$, is the quotient of the free abelian group generated by equivalence classes $\{X\}$ of algebraic stacks $X$ finitely presented over $\mathcal{S}$ and with affine stabilizers by the following relations:

1. $\{X\} = \{Z\} + \{X \setminus Z\}$ for every closed embedding $Z \hookrightarrow X$;
2. $\{E\} = \{X \times_\mathcal{S} E\}$ for every vector bundle $E$ of constant rank $n$ over $X$.

The product on $K_0(\text{Stacks}_\mathcal{S})$ is defined by $\{X\}\{Y\} := \{X \times_\mathcal{S} Y\}$. The class $\{\mathbb{A}_\mathcal{S}^1\}$ in $K_0(\text{Stacks}_\mathcal{S})$ is denoted by $L$, or by $L_\mathcal{S}$ if reference to $\mathcal{S}$ is necessary.

We will be especially interested in the cases, when $\mathcal{S} = \text{Spec } k$ or $\mathcal{S} = BG$ for some finite group scheme $G$ over $k$. There is a natural ring homomorphism $K_0(\text{Var}_k) \rightarrow K_0(\text{Stacks}_k)$, which by $[7]$ Theorem 4.1 induces an isomorphism:

$$K_0(\text{Stacks}_k) \cong K_0(\text{Var}_k)[L^{-1}, \{(L^n - 1)^{-1}, n \geq 1\}].$$

If $f : \mathcal{S}' \rightarrow \mathcal{S}$ is a morphism of stacks, we get a natural ring homomorphism $f^* : K_0(\text{Stacks}_\mathcal{S}') \rightarrow K_0(\text{Stacks}_\mathcal{S})$ by pulling back stacks along $f$. In particular, $f^*(L_\mathcal{S}) = L_{\mathcal{S}'}$. If $f$ is finitely presented, we also have a group homomorphism $f_* : K_0(\text{Stacks}_\mathcal{S}') \rightarrow K_0(\text{Stacks}_\mathcal{S})$, which sends the class of a stack $X \rightarrow \mathcal{S}'$ to the
class of $\mathcal{X} \to S'$ \xrightarrow{f} S$. Note that $f_*$ is not a ring homomorphism, however there is a projection formula

$$f_*(f^*C \cdot C') = C \cdot f_*C', \quad C \in K_0(\text{Stacks}_S), C' \in K_0(\text{Stacks}_{S'})$$

To prove it, one may reduce to the case when $C = \{\mathcal{X}\}$ and $C' = \{\mathcal{X}'\}$, where $\mathcal{X}$ is a stack over $S$ and $\mathcal{X}'$ is a stack over $S'$, in which case the claim follows from the $S$-isomorphism $\mathcal{X} \times S' \times S. \mathcal{X}' \cong \mathcal{X} \times S' \mathcal{X}'$.

Let $S$ be an algebraic stack and let $G$ be a linear algebraic group scheme over $S$, that is, $G$ is flat over $S$ and is a closed subgroup of $\text{GL}_{n,S}$ for some $n \geq 1$. We denote by $B_S G$ the classifying stack of $G$ over $S$. If $T$ is a scheme over $S$, then by definition $B_S G(T)$ is the groupoid whose objects are $G_T$-torsors $P \to T$, and whose arrows are isomorphisms of $G_T$-torsors over $T$. If $S = S$ is a scheme, this is the usual quotient stack $[S/G]$, where $G$ acts trivially on $S$. If $S = \text{Spec} k$, then we relax our notation and simply write $B G$ instead of $B_{\text{Spec} k} G$.

The structure morphism $B_S G \to S$ sends a $G_T$-torsor $P \to T$ to its base $T$. The natural morphism $S \to B_S G$, sending a scheme $T \to S$ to the split $G_T$-torsor, is a $G$-torsor.

The next lemma is an immediate generalization to an arbitrary base stack $S$ of results that are well-known when $S = \text{Spec} k$.

**Lemma 2.2.** Let $S$ be an algebraic stack of finite type over $k$. Then:

(a) $\{\text{GL}_{n,S}\} = \prod_{i=0}^{n-1} (L^n - L^i)$ in $K_0(\text{Stacks}_S)$;

(b) if $\mathcal{X} \to \mathcal{Y}$ is a $\text{GL}_{n,S}$-torsor of algebraic stacks over $S$, then $\mathcal{X} = \{\text{GL}_{n,S}\}\{\mathcal{Y}\}$ in $K_0(\text{Stacks}_S)$;

(c) we have $\{B_S \text{GL}_{n,S}\}\{\text{GL}_{n,S}\} = 1$ in $K_0(\text{Stacks}_S)$. In particular $\{\text{GL}_{n,S}\}$ is invertible in $K_0(\text{Stacks}_S)$.

**Proof.**

(a) By [1 Proposition 1.1(i)], the relation $\{\text{GL}_{n,k}\} = \prod_{i=0}^{n-1} (L^n - L^i)$ holds in $K_0(\text{Stacks}_k)$. The desired formula follows by pulling back along the structure morphism $S \to \text{Spec} k$ of $S$.

(b) Assume first that $\{\mathcal{X}\} = \{\text{GL}_{n,Y}\}\{\mathcal{Y}\}$ in $K_0(\text{Stacks}_Y)$, and denote by $f : \mathcal{Y} \to S$ the structure morphism. Then, using the projection formula, we deduce

$$f_*(\{\mathcal{X}\}) = f_*(f^*\{\text{GL}_{n,Y}\}\{\mathcal{Y}\}) = f_*(f^*\{\text{GL}_{n,S}\}\{\mathcal{Y}\}) = \{\text{GL}_{n,S}\} f_*\{\mathcal{Y}\},$$

showing that $\{\mathcal{X}\} = \{\text{GL}_{n,S}\}\{\mathcal{Y}\}$ in $K_0(\text{Stacks}_S)$. Therefore, we may assume that $\mathcal{Y} = S$, and the claim becomes $\{\mathcal{X}\} = \{\text{GL}_{n,S}\}$ in $K_0(\text{Stacks}_S)$. In this case, even though in [1 Proposition 2.2] and [7 Proposition 1.1(ii)] it is claimed that $\{\mathcal{X}\} = \{\text{GL}_{n,S}\}$ in $K_0(\text{Stacks}_k)$, both proofs show that the equality holds in $K_0(\text{Stacks}_S)$ as well.

(c) The natural map $S \to B_S \text{GL}_{n,S}$ is a $\text{GL}_{n,S}$-torsor, hence the conclusion follows from (b). \qed

The next lemma will be essential to derive the results of Section 3. It will allow us to reduce questions about classes in $K_0(\text{Stacks}_{BG})$ of representable stacks over $BG$ to questions about schemes.
Lemma 2.3. Let $k$ be a field, and let $G$ be a linear algebraic group over $k$. Then there exist a $k$-variety $X$ and a morphism $f : X \to BG$ such that the pullback map $f^* : K_0(\text{Stacks}_{BG}) \to K_0(\text{Stacks}_X)$ is injective.

Proof. Choose an embedding of $G \hookrightarrow \text{GL}_n$ for some $n \geq 1$, and set $X := \text{GL}_n / G$. The variety $X$ is a homogeneous space under $\text{GL}_n$ and has a $k$-point with stabilizer isomorphic to $G$, hence $[X / \text{GL}_n] \cong BG$. The canonical map $f : X \to [X / \text{GL}_n] = BG$ is a $\text{GL}_n$-torsor. For a stack $\varphi : \mathcal{X} \to BG$, let $f^* \mathcal{X} := \mathcal{X} \times_{\varphi, BG, f} X$, so that $f^* \{ \mathcal{X} \} = \{ f^* \mathcal{X} \}$ in $K_0(\text{Stacks}_X)$. The first projection $f^* \mathcal{X} \to \mathcal{X}$ is a $\text{GL}_n$-torsor, hence by Lemma 2.2(b)

(2.1) $f_* f^* \{ \mathcal{X} \} = f_* \{ f^* \mathcal{X} \} = \{ \mathcal{X} \} \{ \text{GL}_n \}$ in $K_0(\text{Stacks}_{BG})$.

Let $C \in K_0(\text{Stacks}_{BG})$. We may write $C = \sum_i \{ \mathcal{X}_i \} - \sum_j \{ \mathcal{Y}_j \}$, for some algebraic stacks $\mathcal{X}_i$ and $\mathcal{Y}_j$ over $BG$. Using (2.1) on each term, we get

$$f_* f^* C = \sum \{ \mathcal{X}_i \} \{ \text{GL}_n \} - \sum \{ \mathcal{Y}_j \} \{ \text{GL}_n \} = \sum \{ (\mathcal{X}_i) - \{ \mathcal{Y}_j \} \} \{ \text{GL}_n \} = C \{ \text{GL}_n \}$$

in $K_0(\text{Stacks}_{BG})$. In other words, $f_* f^* : K_0(\text{Stacks}_{BG}) \to K_0(\text{Stacks}_{BG})$ is the map of multiplication by $\{ \text{GL}_n \}$. By Lemma 2.3(c) $\{ \text{GL}_n \}$ is invertible, hence $f^*$ is injective. $\square$

Let $S$ be a scheme, and let $G$ be a linear algebraic group scheme over $S$. We say that $G$ is special if $H^1_{\text{fppf}}(T, G) = H^2_{\text{zar}}(T, G)$ for every $S$-scheme $T$, that is, if $G$-torsors are Zariski-locally trivial over any $S$-scheme. The following result has been proved in [1] in the case, when $S = \text{Spec } k$ for some field $k$.

Lemma 2.4. Let $S$ be a scheme and $G$ be a special $S$-group.

(a) If $\pi : X \to Y$ is a $G$-torsor of $S$-schemes, then $\{ X \} = \{ G \} \{ Y \}$ in $K_0(\text{Stacks}_S)$.

(b) If $\mathcal{X} \to \mathcal{Y}$ is a $G$-torsor of $S$-stacks, then $\{ \mathcal{X} \} = \{ G \} \{ \mathcal{Y} \}$ in $K_0(\text{Stacks}_S)$.

(c) We have $\{ B_S G \} \{ G \} = 1$ in $K_0(\text{Stacks}_S)$.

Proof. (a) The proof of [1] Proposition 2.1] immediately generalizes; we include it for completeness. Since $G$ is special, there exists a non-empty open subscheme $X_1$ of $X$ such that $Y_1 := \pi^{-1}(X_1)$ is isomorphic to $X_1 \times_S G$. Iterating this procedure, we eventually obtain a stratification $X = \coprod_i Y_i$, such that $Y_i := \pi^{-1}(X_i) \cong X_i \times_S G$. By the scissor relation, we have $\{ X \} = \sum \{ X_i \}$ and $\{ Y \} = \sum \{ Y_i \}$. Since $\{ Y_i \} = \{ X_i \} \{ G \}$ for every $i$, we conclude that

$$\{ Y \} = \sum \{ Y_i \} = \sum \{ X_i \} \{ G \} = \left( \sum \{ X_i \} \right) \{ G \} = \{ X \} \{ G \}.$$  

(b) [1] Proposition 2.2], the proof of [1] Proposition 2.3] adapts without difficulties. Assume that $Z$ is a closed substack of $\mathcal{Y}$, with open complement $\mathcal{U}$, and that the claim holds for $\mathcal{X}_Z \to Z$ and $\mathcal{X}_U \to \mathcal{U}$. Then, by the scissor relation

$$\{ \mathcal{X} \} = \{ \mathcal{X}_U \} + \{ \mathcal{X}_Z \} = \{ \mathcal{U} \} \{ G \} + \{ Z \} \{ G \} = \{ \mathcal{Y} \} \{ G \}.$$  

By noetherian induction, it is enough to show the claim for a non-empty open substack of $\mathcal{X}$. By [1] Proposition 3.5.9], $\mathcal{Y}$ is stratified by stacks of the form
\[ [U/ \text{GL}_{n,S}], \text{ where } U \text{ is a scheme over } S. \] Hence we may assume that \( \mathcal{Y} = [Y/ \text{GL}_{n,S}] \), where \( n \geq 1 \) and \( Y \) is a scheme. We have a cartesian diagram

\[
\begin{array}{ccc}
\mathcal{X}_Y & \xrightarrow{\varphi} & \mathcal{X} \\
\downarrow & & \downarrow \\
\mathcal{Y} & \xrightarrow{f} & \mathcal{Y}'
\end{array}
\]

where the horizontal maps are \( \text{GL}_{n,S} \)-torsors. It follows from Lemma 2.2 that
\( \{Y\} = \{\text{GL}_{n,S}\}\{Y\} \) and \( \{\mathcal{X}_Y\} = \{\text{GL}_{n,S}\}\{\mathcal{X}\} \). Since \( Y \) is a scheme, by (a) we have
\( \{\mathcal{X}_Y\} = \{\text{GL}_{n,S}\}\{Y\} \). Combining these relations, we arrive to
\( \{\text{GL}_{n,S}\}\{\mathcal{X} - \mathcal{G}\}\{Y\} = 0. \)

By Lemma 2.2(c) \( \{\text{GL}_{n,S}\} \) is invertible, hence \( \{\mathcal{X} - \mathcal{G}\}\{Y\} = 0. \)

If \( T \) is a scheme and \( T \to B_SG \) is a morphism, corresponding to a \( G \)-torsor \( P \to T \), by (a) we have \( \{P\} = \{G\}\{T\} \). Applying (b) to \( S \to B_SG \) and \( C = \{G\} \), we get \( 1 = \{S\} = \{B_SG\}\{G\} \) in \( K_0(\text{Stacks}_S) \).

3. Groups of multiplicative type over stacks

Let \( G \) be a finite group and \( M \) be a \( G \)-module of rank \( n \). We define \( T_M := [\text{Spec } k[M]/G] \) and \( B_M := B(\text{Spec } k[M] \times G) \).

The representable morphism \( T_M \to BG \) exhibits \( T_M \) as a group object over \( BG \).

If \( f : X \to BG \) is a morphism, corresponding to a \( G \)-torsor \( \pi : Y \to X \), then
\( T_M := X \times f, BG, \pi T_M \) is a group of multiplicative type split by \( Y \). It is obtained by twisting \( \text{Spec } k[M] \times X \) by \( \pi \) (using the \( G_X \)-action on \( \text{Spec } k[M] \times X \)). We will be mostly interested in the case, when \( M \) is torsion-free. In that case \( T_M \) is a relative torus over \( X \).

If \( \pi \) is a geometric point of \( X \), then \( \pi \) corresponds to a surjection \( \pi_1(X, \pi) \to G \), and \( T_M \) corresponds to the representation of \( \pi_1(X, \pi) \to G \to \text{GL}(M) \).

The projection \( B_M \to BG \) admits a section, making \( B_M \) into a neutral gerbe over \( BG \). Its fibers are the twists of \( \text{Spec } k[M] \) using the \( G \)-action, as we now explain.

More generally, let us consider the semidirect product \( N \rtimes H \) of two linear algebraic \( k \)-groups \( N \) and \( H \). If \( f : X \to BH \) is a morphism, corresponding to an \( H \)-torsor \( \pi : Y \to X \), we have a cartesian diagram

\[
\begin{array}{ccc}
[X/\pi N] & \xrightarrow{\varphi} & B(N \rtimes H) \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & BH.
\end{array}
\]

Here \( \pi N_X \) is the twist of \( N_X \) by the torsor \( \pi : Y \to X \), using the \( H_X \)-action. It is a group scheme over \( X \). The morphism \( \varphi \) is constructed in the following way: twist the inclusion \( N_X \to (N \rtimes H)_X \) by the \( (N \rtimes H)_X \)-torsor \( Y \times^H (N \rtimes H) \to X \) to obtain a morphism \( \pi N_X \to \pi (N \rtimes H)_X \). Since \( \pi (N \rtimes H)_X \) is an inner form of \( (N \rtimes H)_X \), we have \( [X/\pi (N \rtimes H)_X] \cong [X/(N \rtimes H)_X] \). We define \( \varphi \) as the composition

\[
[X/\pi N] \to [X/\pi (N \rtimes H)_X] \cong [X/(N \rtimes H)_X] \cong X \times B(N \rtimes H) \xrightarrow{pr_2} B(N \rtimes H).
\]
Specialization of diagram \((3.1)\) to our situation yields a cartesian diagram
\[
\begin{array}{ccc}
B_X T_M & \longrightarrow & B_M \\
\downarrow & & \downarrow \\
X & \longrightarrow & BG
\end{array}
\]
(3.2)
for every morphism \(f : X \to BG\).

Recall that, by definition, an invertible \(G\)-lattice \(M\) is a direct summand of a permutation lattice \([3][12]\).

**Lemma 3.1.** Let \(M\) be an invertible \(G\)-lattice (for example, a permutation \(G\)-lattice). Then:

(a) for every morphism \(f : X \to BG\), the \(X\)-torus \(f^* T_M\) is special;
(b) \(\{T_M\}\{B_M\} = 1\) in \(K_0(\text{Stacks}_{BG})\);

**Proof.** Let \(P\) be a permutation \(G\)-lattice such that \(M\) is a direct summand of \(P\).

Let \(Y \to X\) be the \(G\)-torsor corresponding to \(f\). Then \(f^* T_P = R_{Y/X}(\mathbb{G}_m,Y), f^* B_P = B_X(R_{Y/X}(\mathbb{G}_m,Y))\) and \(f^* T_M\) is a direct factor of \(f^* T_P\).

\([a]\) By Shapiro’s lemma for étale cohomology \([10]\) Lemma 29.6], the \(X\)-torus \(R_{Y/X}(\mathbb{G}_m,Y)\) is special. Since \(f^* T_M\) is a direct factor of \(f^* T_P\), the cohomology of \(f^* T_M\) is a direct summand of the cohomology of \(f^* T_P\), hence \(f^* T_M\) is also special.

For the rest of the proof we fix a \(k\)-variety \(X\) and a morphism \(f : X \to BG\) such that \(f^* : K_0(\text{Stacks}_{BG}) \to K_0(\text{Stacks}_X)\) is injective, by Lemma \([2][3]\)

\([b]\) By Lemma \([2][4]\) we obtain that
\[
f^*[\{T_M\}\{B_M\}] = \{f^* T_M\}\{f^* B_M\} = \{R_{Y/X}(\mathbb{G}_m,Y)\}\{B(R_{Y/X}(\mathbb{G}_m,Y))\} = 1
\]
in \(K_0(\text{Stacks}_X)\). Since \(f^*\) is injective, we conclude that \(\{T_M\}\{B_M\} = 1\). \(\square\)

**Proposition 3.2.** Let
\[
0 \to M' \to M \to M'' \to 0
\]
be a short exact sequence of \(G\)-modules. If either \(M\) or \(M''\) is an invertible \(G\)-lattice, then
\[
\{T_{M'}\} = \{T_M\}\{B_{M''}\}
\]
in \(K_0(\text{Stacks}_{BG})\).

**Proof.** Lemma \([2][3]\) gives a \(k\)-variety \(X\) and a morphism \(f : X \to BG\) such that \(f^* : K_0(\text{Stacks}_{BG}) \to K_0(\text{Stacks}_X)\) is injective. Fix a geometric point \(\pi\) of \(X\).

The morphism \(f\) corresponds to a \(G\)-torsor \(Y \to X\), and as discussed this in turn corresponds to a surjection \(\pi_1(X,\pi) \to G\). The sequence \(0 \to M' \to M \to M'' \to 0\) is then a sequence of integral \(\pi_1(X,\pi)\)-representations, hence we obtain a short exact sequence of algebraic tori over \(X\):
\[
1 \to T_{M''} \to T_M \to T_{M'} \to 1.
\]
Since \(f^*\) is injective, it is enough to show that \(f^*\{T_{M'}\} = f^*[\{T_M\}\{B_{M''}\}]\), that is
\[
(3.3) \quad \{T_{M'}\} = \{T_M\}\{B_X T_{M''}\} \text{ in } K_0(\text{Stacks}_X).
\]
If \(M''\) is invertible, by Lemma \([3.1][b]\) \(T_{M''}\) is special. Since \(T_M \to T_{M'}\) is a \(T_{M''}\)-torsor, \((3.3)\) follows from Lemma \([3.1]\).

If \(M\) is invertible, then \((3.3)\) follows from \([1]\) Proposition 2.9]; we repeat the argument here. The map \(T_{M''} \to \{T_{M'/T_M}\}\) is a \(T_{M'}\)-torsor. Since \(T_M\) acts transitively on \(T_{M'}\) and \(T_{M'}\) admits an \(X\)-point with stabilizer \(T_{M''}\) (for example, the identity
section), we have \([T_{M'}/T_M] \cong B_X T_{M''}\). By Lemma 3.1(b), \(T_M\) is special. It follows that \(\{T_{M'}\} = \{T_M\}/(T_{M'}/T_M) = \{T_M\}/\{B_X T_{M''}\} \).

\[\square\]

4. PROOF OF THEOREM 1.1

In this section, we prove the main results of this paper. The proof of Theorem 1.1 was inspired by the second author’s previous work [20], where it was shown that if \(T\) is a non-split \(k\)-torus, the stable rationality of \(BT[n]\) has an answer that is periodic in \(n\), with period dividing the period of the generic \(T\)-torsor; see [15] for the definition. The period of the generic \(T\)-torsor equals the number \(e(M)\) that was defined in the Introduction.

**Proof of Theorem 1.1.** Let \(M\) be the character \(G\)-lattice of \(T\). Consider the following diagram with exact rows and columns

\[
\begin{array}{ccccccc}
0 & & & & & & 0 \\
& & & & & & \\
0 & \rightarrow & N & \rightarrow & P & \rightarrow & M & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & N_m & \rightarrow & P & \rightarrow & M/mM & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & M & \rightarrow & 0 & & \\
& & \downarrow & & \downarrow & & \\
& & 0 & & & & \\
\end{array}
\]

(4.1)

where the first row is (1.1). Denote by \(\alpha\) the class of (1.1) in \(\text{Ext}^1_G(M, N)\). By [20, Proposition 3.1] the class of

\[
0 \rightarrow N \rightarrow N_m \rightarrow M \rightarrow 0
\]

in \(\text{Ext}^1_G(M, N)\) is \(ma\). The order of \(\alpha\) in \(\text{Ext}^1_G(M, N)\) divides \(e(M)\) (it actually equals \(e(M)\), by [15, Theorem 3.1]). It follows that \(N_m \cong N_{m'}\) when \(e(M) \mid m - m'\).

Recall that if \(\gamma \in \text{Ext}^1_G(M, N)\) is the class of

\[
0 \rightarrow N \rightarrow Q \xrightarrow{\eta} M \rightarrow 0,
\]

then \(-\gamma\) is represented by

\[
0 \rightarrow N \rightarrow Q \xrightarrow{-\eta} M \rightarrow 0.
\]

It follows that \(N_m \cong N_{m'}\) when \(e(M) \mid m + m'\) as well.

Assume that \(m \equiv \pm 1\) (mod \(e(M)\)). Then \(N_m \cong N_1 \cong P\) and the second row of (4.1) becomes

\[
0 \rightarrow P \rightarrow P \rightarrow M/mM \rightarrow 0.
\]

Since \(P\) is a permutation \(G\)-lattice, by Proposition 3.2 we deduce that \(\{B(T[m] \rtimes G)\} = 1\) in \(K_0(\text{Stacks}_{BG})\). This implies that \(\{B(T[m] \rtimes G)\} = \{BG\}\) in \(K_0(\text{Stacks}_{k})\), as desired. \(\square\)
We now prove our main result concerning reflection groups $G(m, p, n)$. As we anticipated in the introduction, the case $p = 1$ contains all groups of type $B_n$ and is particularly easy.

**Proof of Corollary 1.2** Denote by $U_n$ the $S_n$-lattice $\mathbb{Z}[S_n/S_{n-1}]$, and let $e_1, \ldots, e_n$ be the standard basis of $U_n$.

Assume first that $p = 1$. We have $BG(m, 1, n) = B_{U_n/mU_n}$. Applying Proposition 3.2 to the short exact sequence

$$0 \to U_n \xrightarrow{x_m} U_n \to U_n/mU_n \to 0,$$

we obtain $\{BG(m, 1, n)\} = 1$ in $K_0(\text{Stacks}_{BS_n})$. Since $\{BS_n\} = 1$ in $K_0(\text{Stacks}_k)$ by [8] Theorem 4.3, we conclude that $\{BG(m, 1, n)\} = 1$ in $K_0(\text{Stacks}_k)$.

Assume now that $p = m$. Let $A_{n-1}$ be the $S_n$-module defined by the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\epsilon} U_n \to A_{n-1} \to 0,$$

where $\epsilon(1) = \sum_{i=1}^{n} e_i$. Let $S_n$ act on $G^m_n$ by permutation of the coordinates, and consider the subtorus of $G^m_n$ given by

$$T_n := \{(x_1, \ldots, x_n) \in G^m_n : \prod_{i=1}^{n} x_i = 1\}.$$

The character $S_n$-lattice of $T_n$ is $A_{n-1}$, and $G(m, m, n) = T_n[m] \rtimes S_n$. In view of Theorem 1.1 to complete the proof it is enough to show that $\epsilon(A_{n-1}) = n$. Since $\{4.2\}$ is a coflasque resolution of $A_{n-1}$, it is enough to show that the class of $\{4.2\}$ in $\text{Ext}^1_{S_n}(A_{n-1}, \mathbb{Z})$ has order $n$. This is equivalent to showing that the class of the dual of $\{4.2\}$ has order $n$ in $\text{Ext}^1_{S_n}(\mathbb{Z}, A'_{n-1})$, where $A'_{n-1}$ is the $S_n$-lattice dual to $A_{n-1}$. This is well-known; see e.g. [15] Example 4.1. It follows from Theorem 1.1 that $\{BG(m, m, n)\} = \{BS_n\}$ in $K_0(\text{Stacks}_k)$ when $m \equiv \pm 1 \mod n$. We have $\{BS_n\} = 1$ by [8] Theorem 4.3, and the conclusion follows.

We conclude by stating the following conjecture.

**Conjecture 4.1.** Let $W$ be a complex reflection group. Then $\{BW\} = 1$ in $K_0(\text{Stacks}_C)$.

This statement was claimed as the main theorem of the first online version of the preprint [5] of Emanuele Delucchi and the first author, but its proof was later shown to be defective.

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Department of Mathematics, Royal Institute of Technology, Stockholm, Sweden.

E-mail address: imartino@kth.se

Department of Mathematics, University of British Columbia, Vancouver, BC, Canada.

E-mail address: scavia@math.ubc.ca