ON THE NONLOCAL CURVATURES OF SURFACES WITH OR WITHOUT BOUNDARY

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Abstract. For surfaces without boundary, nonlocal notions of directional and mean curvatures have been recently given. Here, we develop alternative notions, special cases of which apply to surfaces with boundary. Our main tool is a new fractional or nonlocal area functional for compact surfaces.

1. Introduction. In the standard mathematical modelling of thin elastic structures such as plates and shells a central role is played by the local-curvature fields over their mid surfaces. On learning about the notion of nonlocal mean curvature for surfaces without boundaries, we wondered whether consideration of nonlocal curvatures would allow for capturing certain phenomenologies that are beyond the reach of standard models, so much so when the thin structures under study have a peculiar constitution, such as, say, plates and shells made of polymeric gels. This was our original motivation to try and develop the nonlocal notions of directional and mean curvature we propose in this paper, which are different from those found in the literature and, at variance with them, apply also to surfaces with boundary embedded in $\mathbb{R}^n$. To set the stage, we begin by recalling some facts.

Within the framework of the theory of functions with bounded variation on $\mathbb{R}^n$, the perimeter of a bounded set $E$ with nice boundary $\partial E$ equals the $(n-1)$-dimensional Hausdorff measure of $\partial E$:

$$\text{Per}(E) = \mathcal{H}^{n-1}(\partial E) \equiv \text{Area}(\partial E)$$

(1)

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A relation akin to (1) between generalized perimeter and area functionals plays a central role in our paper, because it is from the stationarity of the generalized area functionals we consider that we derive the nonlocal notion of mean curvature we propose.

In recent years, Caffarelli and coworkers [6, 7, 8, 9, 11] have motivated the study of s-perimeter functionals \(0 < s < 1/2\), a family of functionals over subsets of \(\mathbb{R}^n\), whose stationarity condition suggests a definition of nonlocal mean curvature for the closed surface that bounds a candidate minimizer. The regularity of minimizers, called \(s\)-minimal surfaces, has been investigated by Valdinoci and collaborators [10, 15, 18, 27]. Among other things, it is known that \(s\)-minimal surfaces are smooth off of a singular set of dimension at most \(n-8\) for \(s\) sufficiently close to \(1/2\). While this is in agreement with a well-known result for classical minimal surfaces, \(s\)-minimal surfaces may have features different from their classical counterparts, in that they may stick to the boundary instead of being transversal to it [15, 16]. The motion of surfaces by nonlocal mean curvature has been investigated using level set methods [12, 13, 14, 22]. For an interesting application, the nonlocal notion of perimeter has been used to modify the Gauss free-energy functional used in capillarity theory [23]. A physical motivation for studying this topic is provided by the fact that surfaces with vanishing nonlocal mean curvature arise as limit interfaces of phase-coexistence models with long-tail interactions [26].

The functional delivering the \(s\)-perimeter of a measurable set \(E\) admits the following alternative representations: for \(0 < s < 1/2\), for \(\alpha_{n-1}\) the volume of the unit ball in \(\mathbb{R}^{n-1}\), and for \(CE\) the complement of \(E\) in \(\mathbb{R}^n\),

\[
s\text{-Per}(E) = \frac{1}{\alpha_{n-1}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\chi_{CE}(x)\chi_{E}(y)}{|x-y|^{n+2s}} \, dx \, dy,
= \int_{E} \int_{CE} \kappa(x,y) \, dx \, dy, \quad \kappa(x,y) = \frac{1}{\alpha_{n-1}} \frac{1}{|x-y|^{n+2s}}.
\]

As \(s \to 1/2^-\), \(s\)-Per tends, in a sense to be specified later, to the classical perimeter functional studied in [21]. The first representation in (2) makes it explicit how \(s\text{-Per}(E)\) is related to the \(H^s(\mathbb{R}^n)\)-norm of the characteristic function \(\chi_{E}\) of \(E\). The second allows \(s\text{-Per}(E)\) to be interpreted as the evaluation of a distance interaction between a bounded set \(E\) and its complement \(CE\) in \(\mathbb{R}^n\), in terms of an integral norm that assigns maximum weight to short-distance pairs \((x, y)\) while keeping track of long two-point correlations.

The value at \(E\) of \(s\text{-Per}\) is not finite if \(E\) is unbounded. In that case, a bounded set \(\Omega\) is fixed and the \(s\)-perimeter of \(E\) relative to \(\Omega\) is defined in terms of the interaction functional

\[
I(A, B) := \int_{A} \int_{B} \kappa(x,y) \, dx \, dy, \quad A \cap B = \emptyset,
\]

in the following way:

\[
s\text{-Per}(E, \Omega) := I(E \cap \Omega, CE \cap \Omega) + I(E \cap \Omega, CE \cap \Omega) + I(E \cap C\Omega, CE \cap \Omega); \quad (4)
\]

this definition coincides, up to the multiplicative constant \((\alpha_{n-1})^{-1}\), with that given in [8]. We recap those properties of \(s\text{-Per}\) functionals that are relevant to our present developments in Section 2.
In this paper, $\mathcal{S}$ denotes a $(n-1)$-dimensional surface embedded in $\mathbb{R}^n$, with or without boundary; in the latter instance, we regard $\mathcal{S}$ as the complete boundary $\partial E$ of a bounded open set $E$ in $\mathbb{R}^n$. Our first goal is to develop a notion of $s$-area for whatever $\mathcal{S}$. Clearly, when $\mathcal{S} \equiv \partial E$, it would be expedient to have a representation of $s$-Per alternative to (2), according to which the evaluation of the $s$-perimeter of $E$ depended only on $\partial E$. In Section 3 we motivate and discuss the following choice:

$$s\text{-Per}(E) = \frac{1}{2} \int_{\mathcal{X}(\partial E)} \kappa(x, y) \, dxdy,$$

where $\mathcal{X}(\partial E)$ consist of all pairs $(x, y)$ such that the straight-line segment $[x, y]$ both has an odd number of cross intersections with $\partial E$ and is not tangent to $\partial E$. However, to define the $s$-area of a compact smooth surface $\mathcal{S}$ using this formula with $\partial E$ replaced by $\mathcal{S}$ is not viable because the integral on the right in general diverges when $\mathcal{S}$ is not the boundary of a set. Thus, similar to what is done for the $s$-perimeter of an unbounded set, we define the $s$-area of $\mathcal{S}$ relative to a chosen bounded set $\Omega$. Once we have a definition for $s$-area functionals over compact surfaces, we are able to close Section 3 by showing that the $s$-area converges, in an appropriate sense, to the classical notion of area, as $s$ approaches $1/2$ from below. Next, in Section 4, we calculate the first variation of the $s$-area functional. The emerging definition of nonlocal mean curvature, which is meaningful for any surface, compact or otherwise, is laid down and discussed in Section 5, where we also adapt to our context the notion of nonlocal directional curvature [1, 25].

Here is a quick introduction to the new notion of nonlocal mean curvature we propose. Let $\mathcal{S}$ be an oriented smooth surface. The nonlocal mean curvature at $z \in \mathcal{S}$ is

$$H_s(z) := \frac{1}{\omega_{n-2}} \text{PV} \int_{\mathbb{R}^n} \chi_{\mathcal{S}}(z, y) |z - y|^{-n+2s} \, dy,$$

where, for $x \in \mathbb{R}^n$,

$$\chi_{\mathcal{S}}(z, x) := \begin{cases} +1 & \text{if } x \in \mathcal{A}_i(z, 1), \\
-1 & \text{if } x \in \mathcal{A}_e(z, 1). \end{cases}$$

The sets $\mathcal{A}_i(z, 1)$ and $\mathcal{A}_e(z, 1)$ are defined by means of the set $\mathcal{X}(\mathcal{S})$, and can be respectively interpreted as the ‘interior’ and the ‘exterior’ of the surface $\mathcal{S}$ relative to the point $z$; while a precise definition is to be found in Section 4, Fig. 1 offers a representation of these sets in a particular case. By the use of a formula of Cabré et al. [5], the mean curvature of $\mathcal{S}$ can be given the following alternative expression:

$$H_s(z) = \frac{1}{s \omega_{n-2}} \text{PV} \int_{\mathcal{S}} |z - y|^{-(n+2s)}(z - y) \cdot \mathbf{n}_{\mathcal{A}_i}(y) \, dy,$$

where $\mathbf{n}_{\mathcal{A}_i}$ is the outward unit normal to $\mathcal{A}_i(z, 1)$.

2. $s$-perimeter and nonlocal curvatures of surfaces without boundary.

Let $B_R$ denote the ball of radius $R$ centered at the origin of $\mathbb{R}^n$; throughout the paper we take $n \geq 2$. Caffarelli and Valdinoci [11] proved that, if for some $R > 0$ the set $\partial E \cap B_R$ is $C^{1, \beta}$ for some $\beta \in (0, 1)$, then

$$\lim_{s \to 1/2^-} (1 - 2s) s\text{-Per}(E, B_r) = \text{Per}(E, B_r)$$

(5)

for almost every $r \in (0, R)$ (for another result along this line, see [4]). The regularity assumption on the boundary of $E$ made in this statement is natural for minimizers of the $s$-perimeter functional. A set $E \subset \mathbb{R}^n$ that minimizes $s$-Per$(E, \Omega)$ among all
the measurable sets $\bar{E} \subset \mathbb{R}^n$ such that $E \setminus \Omega = \bar{E} \setminus \Omega$ is called $s$-minimal. It is proved in [8] that, if $E$ is $s$-minimal, then $\partial E \cap \Omega$ is of class $C^{1,\beta}$ for some $\beta \in (0,1)$, up to a set of Hausdorff codimension in $\mathbb{R}^n$ at least equal to 2. It is also proved in [8] that, if $E$ is an $s$-minimal set in $\Omega$ and $\partial E$ is smooth enough, then $E$ satisfies the Euler-Lagrange equation of the $s$-perimeter functional:

$$H_s = 0 \quad \text{on} \quad \partial E.$$  

Here, the nonlocal mean curvature of $E$ at $z \in \partial E$ is defined to be

$$H_s(z) := \frac{1}{\omega_{n-2}} \text{PV} \int_{\mathbb{R}^n} \tilde{\chi}_E(y) |z - y|^{-(n+2s)} dy,$$

where $\text{PV}$ stands for the principal value of the integral, $\omega_{n-2}$ is the Hausdorff measure of the $(n-2)$-dimensional unit sphere, and

$$\tilde{\chi}_E(y) := \begin{cases} +1 & \text{if } y \in E, \\ -1 & \text{if } y \in CE. \end{cases}$$

This definition of nonlocal mean curvature coincides with that given in [1]; the definition given in [5] is the same, to within a multiplicative constant.

Following [1], we now define the nonlocal directional curvature. Let $y \in \partial E$, let $e$ be a unit vector tangent to $\partial E$ at $z$, and let

$$\pi(z,e) := \{ y \in \mathbb{R}^n \mid y = z + \rho e + h n(z), \ \rho > 0, \ h \in \mathbb{R} \}$$

be the half-plane through $z$ defined by the unit vector $e$ and the normal $n(z)$ (Figure 2); moreover, let a superscript prime denote the points of the straight line through $z$ in the direction of $e$ when they are obtained by projection in the direction of $n(z)$ of points of $E \cap \pi(z,e)$, so that $y' = z + \rho e$. The nonlocal directional curvature of $E$ at $z$ in the direction $e$ is

$$K_{s,e}(z) := \text{PV} \int_{\pi(z,e)} |y' - z|^{n-2} \tilde{\chi}_E(y) |z - y|^{-(n+2s)} dy, \quad s \in (0,1/2).$$

In the present instance,

$$\text{PV} \int_{\mathbb{R}^n} \tilde{\chi}_E(y) |z - y|^{-(n+2s)} dy = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(z)} \tilde{\chi}_E(y) |z - y|^{-(n+2s)} dy.$$
It is proved in [1] that the nonlocal directional and mean curvatures tend to their local counterparts pointwise in the limit when $s \to 1/2^-$; precisely,
\[
\lim_{s \to 1/2^-} (1 - 2s)K_{s,e} = K_e, \quad \lim_{s \to 1/2^-} (1 - 2s)H_s = H.
\]  
(9)

3. The nonlocal area functional. To motivate the definition of an $s$-area functional related to the $s$-perimeter functional, recall the definition of the $s$-perimeter for a bounded set $E$. To evaluate the integrals in (2), one has to identify pairs of points $x, y \in \mathbb{R}^n$ such that one point is in $E$ and the other in $CE$. Now, we would like to write the $s$-perimeter functional as an integral over a region depending only on $\partial E$:
\[
s\text{-Per}(E) = \frac{1}{2} \int_{\mathcal{X}(\partial E)} \kappa(x, y) \, dx \, dy, \quad \mathcal{X}(\partial E) \subset \mathbb{R}^n \times \mathbb{R}^n.
\]  
(10)

In preparation for choosing such a region, let us consider Figure 3. We see that point $x_1$ is internal to $E$, point $y_1$ external, and the straight-line segment $[x_1, y_1]$, defined by
\[
[x_1, y_1] := \{(1 - \lambda)x_1 + \lambda y_1 \mid \lambda \in [0, 1]\},
\]  
has an odd number of points in common with $\partial E$. We also see that segment $[x_2, y_2]$, which connects two points external to $E$, has an even number of points in common
with \( \partial E \). However, the set of pairs with one point in \( E \) and the other in \( CE \) cannot be characterized by only looking at the parity of the number of common points the connecting straight-line segment has with \( \partial E \). In fact, not all segments with one end in and the other end out of \( E \) have an odd number of points in common with \( \partial E \); see e.g. the segments \([x_3,y_3]\) and \([x_4,y_4]\) having, respectively, two and infinitely many common points with \( \partial E \).

We let \( \mathcal{X}(\partial E) \) consist of all pairs \((x,y)\) such that the straight-line segment \([x,y]\) both has an odd number of cross intersections with \( \partial E \) and is not tangent to \( \partial E \). While it is true that \( \mathcal{X}(\partial E) \subset (E \times CE) \cup (CE \times E) \), the previous discussion shows that the reverse inclusion does not hold. However, as stated in Proposition 3.2 below, the set \( \mathcal{X}(\partial E) \) differs from \((E \times CE) \cup (CE \times E)\) by a set of \( H^{2n} \)-measure zero. This result validates formula (10), the main tool to put together our definition of a nonlocal area functional for a compact surface, with or without boundary.

Lemma 3.1. Let \( S \) be a compact \((n-1)\)-dimensional \( C^1 \) manifold in \( \mathbb{R}^n \) and let \( U_n \) denote the unit sphere in \( \mathbb{R}^n \). Consider the function
\[
\Phi : S \times U_n \times \mathbb{R}^+ \times \mathbb{R}^- \rightarrow \mathbb{R}^n \times \mathbb{R}^n
\]
defined by
\[
\Phi(z,u,\xi,\eta) := (z + \xi u, z + \eta u) \quad \text{for all } (z,u,\xi,\eta) \in S \times U_n \times \mathbb{R}^+ \times \mathbb{R}^-,
\]
where \( \mathbb{R}^+ \) and \( \mathbb{R}^- \) denote the sets \((0, +\infty)\) and \((-\infty, 0)\). If \( A \) is a subset of \( S \times U_n \times \mathbb{R}^+ \times \mathbb{R}^- \) and \( f : \Phi(A) \rightarrow \mathbb{R} \) is any positive integrable function, then
\[
\iint_{\Phi(A)} f(x,y) dxdy \leq \int \int \int_A |f(z + \xi u, z + \eta u)| |\xi - \eta|^{n-1} |u \cdot n(z)| dz dud\xi d\eta,
\]
where \( n(z) \) is a normal to the surface \( S \) at the point \( z \). Moreover, if the restriction of the function \( \Phi \) to \( A \) is injective, then (12) holds with an equality sign.

Proof. See Figure 4 for a depiction of how \( \Phi \) associates \((z,u,\xi,\eta)\) with points \( x \) and \( y \) in \( \mathbb{R}^n \).

![Figure 4](image_url)

**Figure 4.** How the mapping \( \Phi \) in (11) associates \((z,u,\xi,\eta)\) with the pair of points \( x \) and \( y \) in \( \mathbb{R}^n \).

It suffices to prove the Lemma for a set \( A = S_A \times U_A \times A_\xi \times A_\eta \) where \( S_A \subset S \), \( U_A \subset U_n \), \( A_\xi \subset \mathbb{R}^+ \), and \( A_\eta \subset \mathbb{R}^- \). We may further suppose that the sets \( S_A \) and \( U_A \) can be covered by just one chart (the general case may be reduced to this by means of a partition of unity). This is tantamount to asserting that there are two sets \( P_A, U_A \subset \mathbb{R}^{n-1} \) and two \( C^1 \) bijective mappings
\[
P_A \ni p \mapsto \varphi(p) = z \in S_A
\]
and

\[ U_A \ni u \mapsto \psi(u) = u \in U_A. \]

For later use, we recall that the integral of a function \( g \) over \( S_A \) is defined by

\[
\int_{S_A} g(z) \, dz = \int_{P_A} g(\varphi(p)) J_\varphi \, dp,
\]

(13)

where \( J_\varphi = \sqrt{\det \nabla \varphi^T \nabla \varphi} \) is the Jacobian of \( \varphi \); the integral over \( U_A \) is defined similarly.

Set \( A = P_A \times U_A \times \{\xi, \eta\} \) and define \( \tilde{\Phi} : A \to \Phi(A) \) by

\[
\tilde{\Phi}(p, u, \xi, \eta) = \Phi(\varphi(p), \psi(u), \xi, \eta) = (\varphi(p) + \xi \psi(u), \varphi(p) + \eta \psi(u)),
\]

and \( \tilde{f} : A \to \mathbb{R} \) by

\[
\tilde{f}(p, u, \xi, \eta) := f(\varphi(p) + \xi \psi(u), \varphi(p) + \eta \psi(u)).
\]

By the Area Formula (see Theorem 3.9 of Evans and Gariepy [19]), it follows that

\[
\iint_{\Phi(A)} \left[ \sum_{(p, u, \xi, \eta) \in \Phi^{-1}(x, y)} \tilde{f}(p, u, \xi, \eta) \right] \, dx \, dy = \iiint_A \tilde{f}(p, u, \xi, \eta) \left| \det \nabla \Phi \right| dp \, du \, d\xi \, d\eta.
\]

Let \( \tilde{\Phi}^{-1}(x, y) \) be the pre-image through \( \tilde{\Phi}^{-1} \) of the point \((x, y)\); since by definition \( \tilde{f} = f \circ \tilde{\Phi} \), for any \((p, u, \xi, \eta) \in \tilde{\Phi}^{-1}(x, y)\) we have that \( \tilde{f}(p, u, \xi, \eta) = f(x, y) \) and hence

\[
\iint_{\Phi(A)} f(x, y) \, dx \, dy \leq \iint_{\Phi(A)} \left[ \sum_{(p, u, \xi, \eta) \in \Phi^{-1}(x, y)} \tilde{f}(p, u, \xi, \eta) \right] \, dx \, dy.
\]

Notice that, if the function \( \Phi \) restricted to \( A \) is injective, then the above equation holds with an equality sign. Thus

\[
\iint_{\Phi(A)} f(x, y) \, dx \, dy \leq \iint_{\Phi(A)} \left[ \sum_{(p, u, \xi, \eta) \in \Phi^{-1}(x, y)} \tilde{f}(p, u, \xi, \eta) \right] \, dx \, dy.
\]

from which, taking into account (13), the Lemma follows provided that

\[
| \det \nabla \tilde{\Phi} | = |\xi - \eta|^{n-1} |u \cdot n| J_\varphi J_\psi.
\]

(14)

To prove this identity, we note that the gradient of \( \tilde{\Phi} \) is:

\[
\nabla \tilde{\Phi} = \begin{pmatrix} n-1 & n-1 & 1 & 1 \\ \nabla \varphi & \xi \nabla \psi & \psi & \left( \begin{array}{ccc} 0 & & \end{array} \right) n \end{pmatrix} \cdot \begin{pmatrix} \nabla \varphi \\ \xi \nabla \psi \\ \eta \nabla \psi \\ \psi \end{pmatrix}.
\]
Thus,

\[ |\det \nabla \tilde{\Phi}| = \left| \det \begin{pmatrix} \nabla \varphi & \xi \nabla \psi & \psi & 0 \\ \nabla \varphi & \eta \nabla \psi & 0 & \psi \end{pmatrix} \right| \]

\[ = \left| \det \begin{pmatrix} \nabla \varphi & \xi \nabla \psi & \psi & 0 \\ 0 & (\eta - \xi) \nabla \psi & -\psi & \psi \end{pmatrix} \right| \]

\[ = \left| \det \begin{pmatrix} \nabla \varphi & \xi \nabla \psi & \psi & 0 \\ 0 & (\eta - \xi) \nabla \psi & 0 & \psi \end{pmatrix} \right| \]

\[ = \left| \det \begin{pmatrix} \nabla \varphi & \psi & \xi \nabla \psi & 0 \\ 0 & 0 & (\eta - \xi) \nabla \psi & \psi \end{pmatrix} \right|. \]

For upper-block triangular matrices, the following identity holds:

\[ \det \begin{pmatrix} A \\ B \\ 0 \\ C \end{pmatrix} = \det(A) \det(C). \]

Consequently,

\[ |\det \nabla \tilde{\Phi}| = |\det(\nabla \varphi|\psi) \det((\eta - \xi) \nabla \psi|\psi)| \]

\[ = |\eta - \xi|^{n-1} \det(\nabla \varphi|\psi) \det(\nabla \psi|\psi)|; \quad (15) \]

here \((D|d)\) denotes the \(n \times n\) matrix whose last column is \(d \in \mathbb{R}^n\). Let \(\text{cof}\nabla \varphi\) denote the vector whose \(i\)-th component is equal to \((-1)^{n+i}\) times the determinant of the \((n - 1) \times (n - 1)\) matrix obtained by deleting the \(i\)-th row from \(\nabla \varphi\). We observe that, for every \(i = 1, \ldots, n - 1\), we have that

\[ 0 = \det(\nabla \varphi \frac{\partial \varphi}{\partial p_i}) = \text{cof}\nabla \varphi \cdot \frac{\partial \varphi}{\partial p_i}, \]

which implies that \(\text{cof}\nabla \varphi\) is orthogonal to the surface \(S\). By the Cauchy–Binet theorem, it follows that \(|\text{cof}\nabla \varphi| = J_\varphi\); hence,

\[ n = \frac{1}{J_\varphi}\text{cof}\nabla \varphi \]

is a unit vector orthogonal to the surface \(S\). Similarly we can show that the unit normal \(\psi\) to the surface \(\mathcal{U}_n\) is given by

\[ \psi = \frac{1}{J_\psi}\text{cof}\nabla \psi. \]

Then

\[ \det(\nabla \varphi(p)|\psi(u)) = \text{cof}\nabla \varphi \cdot \psi(u) = J_\varphi n \cdot u, \]

\[ \det(\nabla \psi(u)|\psi(u)) = J_\psi \psi(u) \cdot \psi(u) = J_\psi, \]
and hence from (15) we deduce (14).

Notice that Lemma 3.1 holds also for non-orientable surfaces, in which case a discontinuous normal field may have to be used. The change-of-variables formula (12) is used to prove the next proposition.

**Proposition 3.2.** Let \( S \) be a compact \((n-1)\)-dimensional \( C^1 \) manifold in \( \mathbb{R}^n \). The set consisting of all pairs of points \( (x, y) \in \mathbb{R}^n \times \mathbb{R}^n \) that satisfy at least one of the following conditions:

1. either \( x \in S \) or \( y \in S \);
2. the straight-line segment \([x, y]\) has an infinite number of common points with \( S \);
3. the straight-line segment \([x, y]\) is tangent to \( S \);

has \( \mathcal{H}^{2n} \)-measure zero.

**Proof.** Those pairs of points that satisfy Condition 1 are contained in the set \( S \times \mathbb{R}^n \cup \mathbb{R}^n \times S \), which has \( \mathcal{H}^{2n} \)-measure zero because \( \mathcal{H}^{n-1}(S) < \infty \).

Consider now a pair of points \( x, y \in \mathbb{R}^n \) such that Condition 2 holds. Since both the straight-line segment \([x, y]\) and \( S \) are compact, the set of intersection points of \([x, y]\) with \( S \) has a cluster point \( z \) that is also a point of intersection. If this cluster point is \( x \) or \( y \), we are back in the case of Condition 1. If \( z \) is an interior point of \([x, y]\), then either that straight-line segment is tangent to \( S \) at that point, and hence we are in the case of Condition 3, or it is not. This latter situation cannot occur, because such points are isolated, in the sense that there is a neighborhood of \( z \) in which \([x, y]\) intersects \( S \) only once. This follows from the fact that since \( S \) is a \( C^1 \) surface, there is a neighborhood of \( z \) such that \( S \) can be approximated by the tangent space \( T_z(S) \). Thus, if a line intersects \( S \) at \( z \) and is not tangent to \( S \) at \( z \), then there is a neighborhood of \( z \) such that the line only intersects \( S \) once in that neighborhood. This contradicts the fact that \( z \) is a cluster point of intersections.

To prove the proposition it remains for us to show that the set \( X_{\text{tan}} \) of all pairs of points satisfying Condition 3, has \( \mathcal{H}^{2n} \)-measure zero. Let \( S_R \) denote the set of all points in \( \mathbb{R}^n \) within a distance \( R > 0 \) from \( S \). Using the function (11), it follows that

\[
X_{\text{tan}} \cap (S_R \times S_R) \subset \bigcup_{z \in S} \Phi\{\{z\} \times (T_z(S) \cap U_n) \times [0, d_R] \times [-d_R, 0]\},
\]

where \( d_R = \text{diameter}(S) + R \) and \( T_z(S) \) denotes the tangent space of \( S \) at \( z \). Using the change of variables in (12), we have

\[
\mathcal{H}^{2n}(X_{\text{tan}}) = \lim_{R \to \infty} \mathcal{H}^{2n}(X_{\text{tan}} \cap (S_R \times S_R))
= \lim_{R \to \infty} \int_{X_{\text{tan}} \cap (S_R \times S_R)} dxdy
\leq \lim_{R \to \infty} \int_S \int_{T_z(S) \cap U_n} \int_{-d_R}^{0} \int_{-d_R}^{d_R} d\eta d\xi |u \cdot n(z)| dz.
\]

The last integral is zero because \( u \in T_z(S) \).
own end points. Proposition 3.2 and the discussion at the beginning of this section guarantee that $\mathcal{X}(\partial E)$ and $(E \times CE) \times (CE \times E)$ differ by a set of $\mathcal{H}^{2n}$-measure zero, and thus (10) holds if $E$ has a $C^1$-boundary.

Notice that the right-hand side of (10) depends on $E$ only through its boundary. If we defined the $s$-area of a compact smooth surface $S$ using the same formula with $\partial E$ replaced by $S$, then the integral would diverge whenever $S$ is not the boundary of a bounded set. Thus, similar to what is done for the $s$-perimeter of an unbounded set, we define the $s$-area of $S$ relative to an open and bounded set $\Omega$ by

$$s\text{-Area}(S, \Omega) := \frac{1}{2} \int_{\mathcal{X}(S)} \kappa(x, y) \max\{\chi_\Omega(x), \chi_\Omega(y)\} \, dx \, dy.$$  \hspace{1cm} (16)

To see that the $s$-area relative to $\Omega$ is finite, first notice that, since $\kappa(x, y) = \kappa(y, x)$,

$$s\text{-Area}(S, \Omega) = \frac{1}{2} \int_{\mathcal{X}(S)} \kappa(x, y) \chi_\Omega(x, y) \, dx \, dy + \int_{\mathcal{X}(S)} \kappa(x, y) \chi_{\Omega \times \Omega}(x, y) \, dx \, dy$$

$$= \frac{1}{2} \int_\Omega \int_{\mathcal{X}(S, y) \cap \Omega} \kappa(x, y) \, dx \, dy + \int_\Omega \int_{\mathcal{X}(S, y) \cap S_R} \kappa(x, y) \, dx \, dy,$$  \hspace{1cm} (17)

where: $\mathcal{X}(S, y) = \{x \in \mathbb{R}^n \mid (x, y) \in \mathcal{X}(S)\}$. Hence,

$$s\text{-Area}(S, \Omega) \leq \int_\Omega \int_{\mathcal{X}(S, y) \cap \Omega} \kappa(x, y) \, dx \, dy$$

$$+ \int_\Omega \int_{\mathcal{X}(S, y) \cap S_R} \kappa(x, y) \, dx \, dy + \int_\Omega \int_{\mathcal{X}(S, y) \setminus S_R} \kappa(x, y) \, dx \, dy, \hspace{1cm} (18)$$

where $S_R$ is the set of all points within a distance $R > 0$ from $S$. Now, the integrals on the right in (18) turn out to be finite. Indeed, as to the first, choose $R > 0$ and large enough so that $\Omega \subset S_R$ and utilize the change of variables in Lemma 3.1 to find that

$$\int_\Omega \int_{\mathcal{X}(S, y) \cap S_R} \frac{1}{|x - y|^{n+2s}} \, dx \, dy \leq \int_\Omega \int_{\mathcal{X}(S, y) \cap S_R} \frac{|u \cdot n(z)|}{|\xi - \eta|^{1+2s}} \, d\xi \, d\eta \, dz$$

$$= \int_\Omega \int_{\mathcal{X}(S, y) \cap S_R} \frac{2R^{1-2s} - (2R)^{1-2s}}{2s(1-2s)} |u \cdot n(z)| \, dz \, du < \infty;$$

as to the second, use spherical coordinates centered at $y$ to obtain

$$\int_\Omega \int_{\mathcal{X}(S, y) \setminus S_R} \frac{\omega_{n-1}}{|x - y|^{n+2s}} \, dx \, dy \leq \int_\Omega \int_{R} \frac{1}{r^{1+2s}} \, dr \, dy < \infty.$$

Unsurprisingly, the $s$-area of a surface $S$ relative to $\Omega$ satisfies a relation similar to (5). Assume that $\Omega$ is chosen so that $S \subset \Omega$.

**Theorem 3.3.** If $S$ is a compact $(n-1)$-dimensional $C^1$ manifold and $\Omega \subset \mathbb{R}^n$ is an open and bounded set, then

$$\lim_{s \to 1/2^-} (1 - 2s) \ s\text{-Area}(S, \Omega) = \text{Area}(S).$$  \hspace{1cm} (19)

**Proof.** To begin with, set $\varepsilon = \sqrt{1 - 2s}$, so that as $s$ goes to $1/2$ from the left, $\varepsilon$ goes to zero from the right. Put

$$\mathcal{X}_\varepsilon(S) = \{(x, y) \in \mathcal{X}(S) \mid |x - y| < \varepsilon\}.$$
Notice that
\[
\lim_{s \to 1/2^-} \int_{\mathcal{X}(\mathcal{S}) \setminus \mathcal{X}_e(\mathcal{S})} \frac{1 - 2s}{|x - y|^{n+2s}} \max\{\chi_\Omega(x), \chi_\Omega(y)\} \, dx \, dy \\
\leq \lim_{s \to 1/2^-} 2 \int_{\Omega} \int_{\{|x - y| \geq \varepsilon\}} \frac{1 - 2s}{|x - y|^{n+2s}} \, dx \, dy \\
= \lim_{s \to 1/2^-} 2\omega_{n-1} \int_{\Omega} \int_{\varepsilon} \frac{1 - 2s}{r^{n+2s}} \, dr \, dy \\
= 0.
\]

Since \(\max\{\chi_\Omega(x), \chi_\Omega(y)\} = 1\) for \((x, y) \in \mathcal{X}_e(\mathcal{S})\) and small \(\varepsilon\), it follows that
\[
\lim_{s \to 1/2^-} (1 - 2s) s\text{-Area}(\mathcal{S}, \Omega) = \lim_{s \to 1/2^-} \frac{1}{2\omega_{n-1}} \int_{\mathcal{X}_e(\mathcal{S})} \frac{1 - 2s}{|x - y|^{n+2s}} \, dx \, dy, \tag{20}
\]
where, as previously defined, \(\alpha_{n-1}\) is the volume of the unit ball in \(\mathbb{R}^{n-1}\). Even for \(s\) close to \(1/2\), and hence \(\varepsilon\) close to 0, it is possible for the straight-line segment connecting a pair of points \((x, y) \in \mathcal{X}_e(\mathcal{S})\) to cross \(\mathcal{S}\) more than once; hence, such a pair \((x, y)\) is not naturally associated with a unique point on the surface \(\mathcal{S}\). However, for each pair of points \((x, y) \in \mathcal{X}_e(\mathcal{S})\) we can arbitrarily choose a point \(z \in \mathcal{S}\) that lies on the straight-line segment joining \(x\) and \(y\). Denote this point by \(c(x, y)\). One can think of \(c\) as a function from \(\mathcal{X}_e(\mathcal{S})\) to \(\mathcal{S}\) that singles out a particular crossing for the pair \((x, y) \in \mathcal{X}_e(\mathcal{S})\). There are many such functions, here we choose one. For each \(z \in \mathcal{S}\) and for each \(u \in \mathcal{U}_n\), set
\[
C_\varepsilon(z, u) := \{((\xi, \eta) \in \mathbb{R}^+ \times \mathbb{R}^- \mid (z + \xi u, z + \eta u) \in \mathcal{X}_e(\mathcal{S})
\text{ and } c(z + \xi u, z + \eta u) = z\}. \tag{21}
\]
Notice that the function \(\Phi\) defined in Lemma 3.1 is injective on the set
\[
\bigcup_{(z, u) \in \mathcal{S} \times \mathcal{U}_n} \{z\} \times \{u\} \times C_\varepsilon(z, u).
\]
To see this, consider two quadruplets \((z_1, u_1, \xi_1, \eta_1)\) and \((z_2, u_2, \xi_2, \eta_2)\) in this set that
\[
\Phi(z_1, u_1, \xi_1, \eta_1) = \Phi(z_2, u_2, \xi_2, \eta_2) = (x, y) \in \mathcal{X}_e(\mathcal{S}).
\]
We know that the straight-line segment \([x, y]\) crosses \(\mathcal{S}\) an odd number of times. Since both quadruplets get mapped to \((x, y)\) and \((\xi_1, \eta_1)\) and \((\xi_2, \eta_2)\) belong to \(C_\varepsilon(z, u)\), we must have that
\[
c(z_1 + \xi_1 u_1, z_1 + \eta_1 u_1) = z = c(z_2 + \xi_2 u_2, z_2 + \eta_2 u_2).
\]
It then follows from (21) that \(z = z_1 = z_2\). Moreover, since \(\xi_1\) and \(\xi_2\) are positive, the equality chain
\[
x = z + \xi_1 u_1 = z + \xi_2 u_2
\]
holds only for \(\xi_1 = \xi_2\) and \(u_1 = u_2\); similarly, it follows that \(\eta_1 = \eta_2\). Thus, \(\Phi\) is injective.
Now, using the change of variables (12) with equality sign, we have that
\[
\frac{1}{2} \int_{X_{\varepsilon}(S)} \frac{1-2s}{|x-y|^{n+2s}} dxdy = \frac{1}{2} \int_S \int_{C_{\varepsilon}(z,u)} \frac{1-2s}{|\xi-\eta|^{1+2s}} |u \cdot n(z)| d\xi d\eta dz
\]
\[
= \frac{1}{2} \int_S \int_{\{u \in U_n \mid u \cdot n(z) > 0\}} \int_{C_{\varepsilon}(z,u)} \frac{1-2s}{|\xi-\eta|^{1+2s}} u \cdot n(z) d\xi d\eta dz
\]
\[
- \frac{1}{2} \int_S \int_{\{u \in U_n \mid u \cdot n(z) < 0\}} \int_{C_{\varepsilon}(z,u)} \frac{1-2s}{|\xi-\eta|^{1+2s}} u \cdot n(z) d\xi d\eta dz.
\]
(22)

For each \( z \in S \) and \( u \in U_n \) such that \( u \cdot n(z) > 0 \) there exists an \( \varepsilon_0 > 0 \) such that the segment \( \{ z + su \mid |s| \leq \varepsilon_0 \} \) intersects the surface \( S \) only at \( z \). Hence, for \( \varepsilon \leq \varepsilon_0 \) we have that
\[
C_{\varepsilon}(z, u) = \{ (\xi, \eta) \in \mathbb{R}^+ \times \mathbb{R}^- \mid \xi - \eta \leq \varepsilon \},
\]
so that
\[
\lim_{s \to 1/2^-} \int_{C_{\varepsilon}(z,u)} \frac{1-2s}{|\xi-\eta|^{1+2s}} d\xi d\eta = \lim_{s \to 1/2^-} \int_{-\varepsilon}^{\varepsilon+\eta} \frac{1-2s}{|\xi-\eta|^{1+2s}} d\xi d\eta
\]
\[
= \lim_{s \to 1/2^-} \frac{\varepsilon^{1-2s}}{2s} = 1.
\]

The same result can be reached for \( z \in S \) and \( u \in U_n \) such that \( u \cdot n(z) < 0 \). These facts together with the dominated convergence theorem allows for the calculation of the limit of (22). Namely, if \( B^{n-1} \) is the unit ball in \( \mathbb{R}^{n-1} \), then
\[
\lim_{s \to 1/2^-} \frac{1}{2\alpha_{n-1}} \int_{X_{\varepsilon}(S)} \frac{1-2s}{|x-y|^{n+2s}} dxdy
\]
\[
= \frac{1}{2\alpha_{n-1}} \int_S \int_{\{u \in U_n \mid u \cdot n(z) > 0\}} u \cdot n(z) du dz
\]
\[
- \frac{1}{2\alpha_{n-1}} \int_S \int_{\{u \in U_n \mid u \cdot n(z) < 0\}} u \cdot n(z) du dz
\]
\[
= \frac{1}{\alpha_{n-1}} \int_S \int_{\{u \in U_n \mid u \cdot n(z) > 0\}} u \cdot n(z) du dz
\]
\[
= \frac{1}{\alpha_{n-1}} \int_S \int_{B^{n-1}} \int_0^{\pi/2} \sin(\theta) d\theta dAdz
\]
\[
= \text{Area}(S).
\]

Putting this together with (20) shows that (19) holds.

\[\square\]

4. The first variation of the \( s \)-area functional. Motivated by the connection between the local mean curvature of a surface and the first variation of the area functional, we calculate the first variation of the \( s \)-area functional.

From now on in this section we restrict attention to compact \((n - 1)\)-dimensional \(C^1\) manifold in \( \mathbb{R}^n\), which we choose to be orientable. Let \( S \) be such a surface with \( n \) its chosen normal field, let \( \Omega \) be an open bounded set that contains \( S \), and let \( \phi : S \to \mathbb{R} \) be a normal variation of \( S \), that is, a continuously differentiable function that is zero on \( \partial S \). For \( \varepsilon > 0 \), define
\[
S_{\varepsilon} := \{ z + \varepsilon \phi(z) n(z) \mid z \in S \};
\]
note that $\partial S_\varepsilon = \partial S$ and that $S_\varepsilon$ is a compact $(n-1)$-dimensional manifold for small $\varepsilon$. We wish to find a characterization of those $S$ that satisfy
\[
\lim_{\varepsilon \to 0^+} \frac{s\text{-Area}(S_\varepsilon, \Omega) - s\text{-Area}(S, \Omega)}{\varepsilon} = 0 \quad (23)
\]
for all normal variations $\phi$.

**Figure 5.** A depiction of $S$, $S_\varepsilon$, and $V_\varepsilon$.

Let $V_\varepsilon$ denote the region inclosed by the surfaces $S_\varepsilon$ and $S$, so that
\[
V_\varepsilon := \{ z + \zeta n \mid z \in S, \phi(z) \neq 0, 0 < \zeta/\phi(z) < \varepsilon \}
\]
(see Figure 5), and let
\[
W_\varepsilon := (V_\varepsilon \times CV_\varepsilon) \cup (CV_\varepsilon \times V_\varepsilon).
\]
In view of Proposition 3.2, for $\mathcal{H}^{2n}$-almost every pair $(x, y) \in W_\varepsilon$, the number of points the straight-line segment connecting $x$ and $y$ has in common with $S$ (not counting its end points) differs by an odd number from the number of points it has in common with $S_\varepsilon$. Indeed, for $\mathcal{H}^{2n}$-almost every pair $(x, y) \in W_\varepsilon$, the segment $[x, y]$ intersects $\partial V_\varepsilon$ an odd number of times; let us denote by $\#_\partial V_\varepsilon$ this odd number. Let $\#_S$ and $\#_{S_\varepsilon}$ denote the number of times that $[x, y]$ intersect $S$ and $S_\varepsilon$ respectively. Since $\partial V_\varepsilon = S \cup S_\varepsilon$ we have that $\#_\partial V_\varepsilon = \#_S + \#_{S_\varepsilon}$. But the parity of $\#_S - \#_{S_\varepsilon}$ coincides with the parity of $\#_S + \#_{S_\varepsilon}$ and hence it is odd just like $\#_\partial V_\varepsilon$.

Thus, up to a set of $\mathcal{H}^{2n}$-measure zero,
\[
\mathcal{X}(S_\varepsilon) = (C\mathcal{X}(S) \cap W_\varepsilon) \cup (\mathcal{X}(S) \setminus W_\varepsilon).
\]
With
\[
D_\eta := \{(x, y) \mid |x - y| > \eta\},
\]
for $\eta > 0$, it follows that, for
\[
f(x, y) := |x - y|^{-n-2}\chi_{\max\{\chi_{\Omega}(x), \chi_{\Omega}(y)\}\chi_{D_\eta}}(x, y) = f(y, x),
\]
we have that
\[
\left( \int_{\mathcal{X}(S_\varepsilon)} - \int_{\mathcal{X}(S)} \right) f(x, y) dxdy
\]
\[
= \left( \int_{C\mathcal{X}(S) \cap \mathcal{W}_\varepsilon} + \int_{\mathcal{X}(S) \setminus \mathcal{W}_\varepsilon} - \int_{\mathcal{X}(S) \cap \mathcal{W}_\varepsilon} - \int_{\mathcal{X}(S) \setminus \mathcal{W}_\varepsilon} \right) f(x, y) dxdy
\]
\[
= \left( \int_{C\mathcal{X}(S) \cap \mathcal{W}_\varepsilon} - \int_{\mathcal{X}(S) \setminus \mathcal{W}_\varepsilon} \right) f(x, y) dxdy.
\]
Hence, \((23)\) is equivalent to the condition
\[
\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left( \int_{C_X(S) \cap W_\varepsilon} f(x,y) \, dx \, dy - \int_{X(S) \cap W_\varepsilon} f(x,y) \, dx \, dy \right) = 0. 
\tag{25}
\]

Moreover, as \(f(x,y) = f(y,x)\),
\[
\int_{C_X(S) \cap W_\varepsilon} f(x,y) \, dx \, dy = 2 \int_{V_\varepsilon} \int \{ y \in CV_\varepsilon \mid (x,y) \in C_X(S) \} f(x,y) \, dy \, dx;
\]
similarly,
\[
\int_{X(S) \cap W_\varepsilon} f(x,y) \, dx \, dy = 2 \int_{V_\varepsilon} \int \{ y \in CV_\varepsilon \mid (x,y) \in X(S) \} f(x,y) \, dy \, dx.
\]

Now, for all \(x \in V_\varepsilon\) define
\[
f_{C_X}(x) := \int \{ y \in CV_\varepsilon \mid (x,y) \in C_X(S) \} f(x,y) \, dy, \quad \quad f_X(x) := \int \{ y \in CV_\varepsilon \mid (x,y) \in X(S) \} f(x,y) \, dy,
\]
and, for \(z \in S\), define
\[
A_e(z, \phi) := \{ y \in \mathbb{R}^n \mid ((z,y) \in \mathcal{X}(S) \text{ and } \phi(z)(z-y) \cdot n(z) > 0) \
\quad \quad \text{or } ((z,y) \in C_X(S) \text{ and } \phi(z)(z-y) \cdot n(z) < 0) \},
\]
\[
A_i(z, \phi) := \{ y \in \mathbb{R}^n \mid ((z,y) \in \mathcal{X}(S) \text{ and } \phi(z)(z-y) \cdot n(z) > 0) \
\quad \quad \text{or } ((z,y) \in X(S) \text{ and } \phi(z)(z-y) \cdot n(z) < 0) \};
\]
(see Figures 6 and 1). When \(\phi(z) \neq 0\), the sets \(A_e(z, \phi)\) and \(A_i(z, \phi)\) complement each other up to a set of \(\mathcal{H}^n\)-measure zero. Moreover, using Proposition 3.62 in [3], it can be shown that \(A_e(z, \phi)\) and \(A_i(z, \phi)\) locally have finite perimeter and so have an exterior unit normal on their reduced boundary. In particular, if \(S\) is the boundary of a set \(E\), \(n\) is the exterior normal, and \(\phi > 0\), then, up to a set of \(\mathcal{H}^n\)-measure zero, \(A_e(z, \phi)\) consists of the points outside of \(E\) and \(A_i(z, \phi)\) consists of the points in \(E\).
We state a useful generalization of a result by Weyl [28] (third to last formula on page 464), which can be obtained by use of the area formula: given an integrable function $g$ defined on $Y_v$, we have

$$\int_{Y_v} g(x) \, dx = \int_S \int_{-\varepsilon \phi^-(z)}^{+\varepsilon \phi^+(z)} g(z + \xi \mathbf{n}(z)) \det \tau_z(S)(I - \xi \mathbf{L}(z)) \, d\xi \, dz \quad (26)$$

where $\phi^+$ and $\phi^-$ are the positive and negative parts of $\phi$, $\det \tau_z(S)$ is the determinant function for linear mappings from $T_z(S)$, the tangent space of $S$ at $z$, into itself, and $\mathbf{L}$ is the curvature tensor for $S$, which is defined as $-\nabla_S \mathbf{n}$, the surface gradient of the normal vector field.

With the use of (26), the limit in (25) can be computed:

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{\mathcal{C}\mathcal{X}(S) \cap \Omega_v} f(x, y) \, dxdy = \lim_{\varepsilon \to 0^+} \frac{2}{\varepsilon} \int_{Y_v} f_{\varepsilon}^{CX}(x) \, dx$$

$$= 2 \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_S \int_{-\varepsilon \phi^-(z)}^{+\varepsilon \phi^+(z)} f_{\varepsilon}^{CX}(z + \xi \mathbf{n}(z)) \det \tau_z(S)(I - \xi \mathbf{L}(z)) \, d\xi \, dz$$

$$= 2 \lim_{\varepsilon \to 0^+} \int_S \int_{-\phi^-(z)}^{+\phi^+(z)} f_{\varepsilon}^{CX}(z + \xi \mathbf{n}(z)) \det \tau_z(S)(I - \xi \mathbf{L}(z)) \, d\xi \, dz.$$

For every $z \in S$ and $\xi \in \mathbb{R}$ such that $\phi(z)\xi > 0$, let

$$E_\varepsilon := \{y \in \mathcal{C}\mathcal{V}_v \mid (z + \varepsilon \mathbf{n}(z), y) \in \mathcal{C}\mathcal{X}(S)\}.$$

Then, $\chi_{E_\varepsilon} \rightarrow \chi_{A_{\varepsilon}(z, \phi)}$ in $L^1(\mathbb{R}^n)$ and hence, for almost every $z$,

$$\lim_{\varepsilon \to 0} f_{\varepsilon}^{CX}(z + \varepsilon \mathbf{n}(z)) = \int_{A_{\varepsilon}(z, \phi)} f(z, y) \, dy.$$

Since

$$|f(x, y)| \leq \frac{1}{\eta^{n+2}} \max\{\chi_\Omega(x), \chi_\Omega(y)\},$$

by the dominated convergence theorem, we conclude that

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{\mathcal{C}\mathcal{X}(S) \cap \Omega_v} f(x, y) \, dxdy = 2 \int_S \int_{-\phi^-(z) \cap \varepsilon}^{+\phi^+(z) \cap \varepsilon} f(z, y) \, dy \, d\zeta \, dz$$

$$= 2 \int_S (\phi^+(z) + \phi^-(z)) \int_{A_{\varepsilon}(z, \phi)} f(z, y) \, dy \, dz$$

$$= 2 \int_S |\phi(z)| \int_{A_{\varepsilon}(z, \phi)} f(z, y) \, dy \, dz.$$

Similarly,

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{\mathcal{C}\mathcal{X}(S) \cap \Omega_v} f(x, y) \, dxdy = 2 \int_S |\phi(z)| \int_{A_{\varepsilon}(z, \phi)} f(z, y) \, dy \, dz.$$

Thus, (25) is equivalent to

$$\int_S |\phi(z)| \int_{A_{\varepsilon}(z, \phi)} f(z, y) \, dy \, dz = \int_S |\phi(z)| \int_{A_{\varepsilon}(z, \phi)} f(z, y) \, dy \, dz,$$

a condition which holds whatever $\phi$ if and only if, for all $z \in S$,

$$\int_{A_{\varepsilon}(z, \phi)} f(z, y) \, dy = \int_{A_{\varepsilon}(z, \phi)} f(z, y) \, dy,$$
where $\mathcal{A}_i(z, 1) := \mathcal{A}_i(z, \phi(\cdot) = 1)$ and $\mathcal{A}_e(z, 1) := \mathcal{A}_e(z, \phi(\cdot) = 1)$. Equation (28) is found by fixing a $z \in \mathcal{S}$ and considering variations induced by a positive-valued $\phi$, whose support is contained in a small neighborhood of $z$. Considering variations associated with negative-valued $\phi$ would lead to the same condition, because

$$\mathcal{A}_e(z, -1) = \mathcal{A}_i(z, 1) \quad \text{and} \quad \mathcal{A}_i(z, -1) = \mathcal{A}_e(z, 1).$$

Recalling the definition of $f$ and that $z \in \mathcal{S} \subset \Omega$, (28) writes as

$$\int_{A_i(z, 1) \setminus B_n(z)} |z - y|^{-n-2s} dy = \int_{A_i(z, 1) \setminus B_n(z)} |z - y|^{-n-2s} dy,$$

and letting $\eta$ go to zero we find the following result.

**Theorem 4.1.** Let $\mathcal{S}$ be an orientable compact $(n - 1)$-dimensional $C^1$ manifold in $\mathbb{R}^n$ and let $\Omega$ be an open bounded set that contains $\mathcal{S}$. A necessary and sufficient condition for the vanishing of the first variation with respect to surfaces with the same boundary of the s-area of $\mathcal{S}$ relative to $\Omega$ is

$$PV \int_{A_i(z, 1)} |z - y|^{-n-2s} dy = PV \int_{A_i(z, 1)} |z - y|^{-n-2s} dy,$$

for each $z \in \mathcal{S}$.

## 5. Nonlocal curvatures of surfaces.

Motivated by condition (29) for the vanishing of the first variation of the s-area relative to $\Omega$, we define as follows the nonlocal mean curvature of an orientable $C^1$ surface $\mathcal{S}$, which need not be compact, at its point $z$:

$$H_s(z) := \frac{1}{\omega_{n-2}} PV \int_{\mathbb{R}^n} \widehat{\mathcal{S}}(z, y) |z - y|^{-n-2s} dy,$$

where, for $x \in \mathbb{R}^n$,

$$\widehat{\mathcal{S}}(z, x) := \begin{cases} +1 & \text{if } x \in \mathcal{A}_i(z, 1), \\ -1 & \text{if } x \in \mathcal{A}_e(z, 1). \end{cases}$$

Notice that $H_s(z)$ does not depend on the choice of $\Omega$. When the surface $\mathcal{S}$ is the boundary of an open set, formulas (30)-(31) are consistent with formulas (6)-(7) holding for surfaces without boundary.

Cabré et al. [5] noticed that

$$|z - y|^{-(n+2s)} = \frac{1}{2s} \text{div}_y \left[ |z - y|^{-(n+2s)} (z - y) \right],$$

which, together with the divergence theorem, allows the nonlocal-mean-curvature formula (6) for a surface without boundary to be written as

$$H_s(z) = \frac{1}{s \omega_{n-2}} PV \int_{\partial \mathcal{E}} |z - y|^{-(n+2s)} (z - y) \cdot n(y) dy.$$

We now show that an analogous result also holds for formula (30).

**Proposition 5.1.** Let $\mathcal{S}$ be an oriented $C^1$ surface. For $z \in \mathcal{S}$, let $n_{\mathcal{A}_i}$ be the outward unit normal to $\mathcal{A}_i(z, 1)$. The nonlocal mean curvature of $\mathcal{S}$ at $z$ satisfies

$$H_s(z) = \frac{1}{s \omega_{n-2}} PV \int_{\mathcal{S}} |z - y|^{-(n+2s)} (z - y) \cdot n_{\mathcal{A}_i}(y) dy.$$
Proof. We start by noticing that $\mathcal{S}$ is contained in the boundary $\partial \mathcal{A}_i(z,1)$ of $\mathcal{A}_i(z,1)$, and that if
\[
y \in \partial \mathcal{A}_i(z,1) \setminus (\mathcal{S} \cup \partial \mathcal{S}) \implies (z - y) \cdot n_{\mathcal{A}_i}(y) = 0. \tag{34}
\]
Indeed, assume that there is a point $y \in \partial \mathcal{A}_i(z,1) \setminus (\mathcal{S} \cup \partial \mathcal{S})$ for which $(z - y) \cdot n_{\mathcal{A}_i}(y) \neq 0$. Then, there exists an $\varepsilon > 0$ such that $B(y, \varepsilon)$, the ball of radius $\varepsilon$ centered at $y$, does not intersect the surface $\mathcal{S}$. Also, from $(z - y) \cdot n_{\mathcal{A}_i}(y) \neq 0$ we deduce that there are two points $y^i, y^c \in B(y, \varepsilon)$ that lie on the straight line passing through the points $z$ and $y$ and such that $y^i \in \mathcal{A}_i(z,1)$ and $y^c \notin \mathcal{A}_i(z,1)$. Since $B(y, \varepsilon) \cap \mathcal{S} = \emptyset$ either both $(z, y^i)$ and $(z, y^c)$ belong to $\mathcal{X}(\mathcal{S})$ or they both belong to $\mathcal{N}(\mathcal{S})$. But since
\[
(z - y^i) \cdot n(z) = (z - y^c) \cdot n(z),
\]
because $y^i$ and $y^c$ lie on the line passing through the points $z$ and $y$, we deduce that both points $y^i$ and $y^c$ belong either to $\mathcal{A}_i(z,1)$ or to $\mathcal{A}_i(z,1)$. This contradicts the fact that $y^i \in \mathcal{A}_i(z,1)$ and $y^c \notin \mathcal{A}_i(z,1)$; this contradiction proves (34).

By (32) and the divergence theorem,
\[
H_s(z) = \frac{1}{2s \omega_{n-2}} PV \int_{\mathbb{R}^n} \hat{\chi}_s(z,y) \text{div}_y [(z - y)|z - y|^{-(n+2s)}] \, dy
= \frac{1}{s \omega_{n-2}} PV \int_{\partial \mathcal{A}_i(z,1)} |z - y|^{-(n+2s)} (z - y) \cdot n_{\mathcal{A}_i}(y) \, dy
= \frac{1}{s \omega_{n-2}} PV \int_{\mathcal{S}} |z - y|^{-(n+2s)} (z - y) \cdot n_{\mathcal{A}_i}(y) \, dy,
\]
where the last equality follows from (34).

Remark 1. For a surface without boundary, the outward normal to $\mathcal{A}_i(z,1)$ coincides with the normal to the surface. For a surface $\mathcal{S}$ with boundary, at points $z \in \mathcal{S}$ such that all the half-lines starting at $z$ intersect $\mathcal{S}$ in at most one point (not counting $z$) the normal $n_{\mathcal{A}_i}$ coincides with the normal $n$ to $\mathcal{S}$. Indeed, for these particular kinds of surfaces, (33) can be written as
\[
H_s(z) = \frac{1}{s \omega_{n-2}} PV \int_{\mathcal{S}} |z - y|^{-(n+2s)} (z - y) \cdot n(y) \, dy.
\]

Finally, similar to what is done in [1] for surfaces without boundary, we define the nonlocal directional curvature at $z$ in the direction of $e \in T_z(\mathcal{S})$ by
\[
K_{s,e}(z) := PV \int_{\pi(z,e)} |y' - z|^{n-2} \hat{\chi}_s(z,y) |z - y|^{-(n+2s)} dy, \tag{35}
\]
where the notation is consistent with that used in (8). When the surface $\mathcal{S}$ is the boundary of an open set, this formula for the nonlocal directional curvature is consistent with (8). The nonlocal mean and directional curvatures are related through the formula
\[
H_s(z) = \frac{1}{\omega_{n-2}} \int_{\{e \in T_z(\mathcal{S}) \mid |e| = 1\}} K_{s,e}(z) \, de,
\]
meaning that the nonlocal mean curvature is the average of the nonlocal directional curvatures. Unsurprisingly, just like the nonlocal curvatures for surfaces without boundary, the nonlocal quantities converge to their local counterparts in the appropriate limit.
Proposition 5.2. Let $S$ be an oriented $C^1$ surface with orientation $n$. For all $z \in \mathcal{S}$,
\[
\lim_{s \to 1/2^-} (1 - 2s)K_{s,e}(z) = K_e(z) \quad \text{and} \quad \lim_{s \to 1/2^-} (1 - 2s)H_s(z) = H(z). \quad (36)
\]

Proof. Only a proof of $(36)_1$ will be given, because the proof of $(36)_2$ is similar. Fix $\varepsilon > 0$ and notice that
\[
\left| \int_{\pi(z,\varepsilon) \cap B_s(z)} \frac{|y - z|^{n-2}}{|y - z|^{n+2s}} \chi_\mathcal{S}(z, y) \, dy \right| \leq \int_{\pi(z,\varepsilon) \cap B_s(z)} \frac{1 - 2s}{|y - z|^{2+2s}} \, dy 
= \int_{\pi(z,\varepsilon) \cap B_s(0)} \frac{1 - 2s}{|y|^{2+2s}} \, dy,
\]
the last integral goes to zero as $s$ goes to $1/2$. Thus,
\[
\lim_{s \to 1/2^-} (1 - 2s)K_{s,e}(z) = \lim_{s \to 1/2^-} \int_{\pi(z,\varepsilon) \cap B_s(z)} (1 - 2s) \frac{|y - z|^{n-2}}{|y - z|^{n+2s}} \chi_\mathcal{S}(z, y) \, dy,
\]
meaning that, for $s$ approaching $1/2$, $(1 - 2s)K_{s,e}(z)$ only depends on the surface $\mathcal{S}$ in a small neighborhood of $z$. Choose $\varepsilon$ small enough so that the part of $\mathcal{S}$ inside $B_s(z)$ is diffeomorphic to a disk. It is possible to find an open set $E$ with smooth boundary such that $\mathcal{S} \cap B_s(z) = \partial E \cap B_s(z)$. We denote by $K_{s,e}^\partial E(z)$ the nonlocal directional curvature of $\partial E$ at $z$, so that
\[
\lim_{s \to 1/2^-} (1 - 2s)K_{s,e}(z) = \lim_{s \to 1/2^-} (1 - 2s)K_{s,e}^\partial E(z) = K_{e}^\partial E(z),
\]
where (9) has been utilized. Since $K_{e}^\partial E(z) = K_e(z)$, we have, as desired, that
\[
\lim_{s \to 1/2^-} (1 - 2s)K_{s,e}(z) = K_e(z).
\]

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