Tailored holograms for superimposed vortex states

K Eickhoff, D Köhnke, L Feld, L Englert, T Bayer and M Wollenhaupt

Carl von Ossietzky Universität Oldenburg, Institut für Physik, Carl-von-Ossietzky-Straße 9-11, D-26129 Oldenburg, Germany

* Author to whom any correspondence should be addressed.

E-mail: matthias.wollenhaupt@uni-oldenburg.de

Keywords: spatial light modulation, vortex superposition state, OAM states, tailored computer generated holograms, optical information transfer

Abstract

We present the generation, optimization and full control of superimposed optical vortices (SOVs) using tailored computer generated holograms by utilizing a 2D liquid crystal spatial light modulator. To perform full radial and azimuthal control over the targeted SOVs we apply spatial amplitude modulation via window functions as well as radial and azimuthal phases, encoded in the diffraction mask. In particular we discuss the influence of spatial linear and quadratic radial phases, which is supplemented by an analytical description. The developed formalism further permits the direct shaping in $k$-space which is highlighted by the radial and azimuthal confinement of SOV states. Our technique enables full real-time control over the spatial structure, the symmetry and azimuthal orientation of the generated SOVs in a common path geometry, which is useful in the context of optical information transfer. We also study the topological properties, i.e. the orbital current $\vec{S}_O$ to determine the topological charge $\ell$ of the generated SOVs.

1. Introduction

Orbital angular momentum (OAM) beams have opened up a new degree of freedom in diverse fields of physics, ranging from fundamental optics [1, 2] via stimulated emission depletion (STED) microscopy [3, 4] to twisted electron beams [5] and coherent control of matter waves [6, 7]. Recently, optical OAM states have emerged as promising candidates for classical [8–10] and quantum communication [11, 12], allowing for optical links with high information transfer rates [13, 14] and quantum key distribution (QKD) [15–17]. In the last years the improved multiplexing and demultiplexing of OAM beams in combination with advanced machine-learning techniques [18, 19] offered new possibilities for free-space [20–22] and underwater [23, 24] optical links. These versatile applications emphasize the importance of OAM states as an ambitious photonic technology for high speed encrypted communication.

The generation of OAM states using spiral phase plates [25, 26], solid diffractive structures [26], 2D liquid crystal spatial light modulators (LC-SLMs) [27, 28] or customized on-chip twisted light emitters [29] has been established for years and found numerous of applications [30–34]. For real-time control of the created OAM states an LC-SLM, enabling the application of computer generated holograms (CGHs), is a powerful tool which has also been applied to fs-laser pulses [27, 28, 35, 36]. Especially in the context of optical communication the azimuthal orientation as well as the symmetry of the generated superimposed optical vortex (SOV) states has been utilized to carry information [37, 38], underscoring the importance of full SOV control for an efficient and robust information transfer.

In this paper we present a holographic diffraction technique for the generation and full control of SOV states utilizing a 2D LC-SLM. The applied CGHs, initially proposed for the generation of pure OAM beams [39] (denoted as pure CGHs in the following) and refined in [5, 27, 28, 40–43], are specifically tailored for SOVs and optimized to improve the target state’s quality and intensity. Our approach provides additional control over the SOVs via azimuthal- and radial phases. The refined CGHs can also be binarized and therefore used for the generation of superimposed OAM electron states in a transmission electron microscope (TEM) [5, 43–46]. In particular we demonstrate spatial light shaping by encoding radial and azimuthal phases using amplitude modulation. While the radial structures of the target states are cleaned...
(i.e. Bessel-type oscillations are suppressed) by a suitable real valued window function [40–42], the phase manipulation enables azimuthal and radial control of the target states [47]. In addition we present a CGH-matching method, allowing for SOV shaping directly in k-space. The combination of CGH-matching and phase modulation enables full radial and rotational control of SOVs, creating a manifold of tailored SOV states at different radial positions with full access to each azimuthal degree of freedom. Hence, our approach supports the generation and full manipulation of SOVs in a common-path geometry on a millisecond time-scale, only limited by the frame rate and the resolution of the LC-SLM. To demonstrate the structural enhancement of the target state and its individual rotational and radial control, we exemplify our technique on different states with $c_{20}$-, $c_7$-, $c_6$- and $c_4$-rotational symmetry.

The paper is structured as follows. We start in section 2 by presenting our experimental methods. In section 2.1 we introduce the tailored mask function and describe its diffraction pattern. We show that rotational, radial and symmetry control of SOVs is achieved by tailored CGHs utilizing azimuthal and radial phases in combination with suitable window functions. Subsequently, in section 2.2 we investigate the orbital current $S_0$ and the topological charge (TC) $\ell$ to characterize the angular momenta of the introduced SOVs. In section 2.3 we show our experimental setup. The experimental results along with simulations are presented in section 3 which is subdivided into three parts. First, we compare the SOV generation with pure and tailored CGHs for different window functions and point out the significance to clean the target SOV structure and increase its intensity. In the second part we describe the control of SOVs by using combined azimuthal and radial phases to provide full radial and rotational control of multi-vortex states. Finally, in the third part we study the orbital current and the TC of specific SOVs to characterize their topological properties and their manipulation.

2. Methods

2.1. Computer generated holograms

Optical beams which carry OAM exhibit a spiral phase front resulting in a phase singularity at the center, leading to a non-vanishing TC. Their electric field $\vec{E}$ is generally described by [48]

$$\vec{E}(r, \phi, z) = \mathcal{E}(r, \phi)e^{ikz} \vec{e}_x = |\mathcal{E}(r, \phi)|e^{i\Phi(r, \phi)} e^{ikz} \vec{e}_x,$$

with the amplitude $\mathcal{E}(r, \phi) \in \mathbb{C}$ of the corresponding linear polarized component (in x-direction) and the laser electric field’s partial OAM state

$$\psi(r, \phi) = e^{i\Phi(r, \phi)}.$$

In the simplest case the phase function is $r$-independent and linear in $\phi$, i.e. $\Phi(r, \phi) = \ell \phi$ where $\ell \in \mathbb{Z}$ is the respective TC [48], associated with a torus shaped intensity profile. In general a superposition of $J$ partial states yields superimposed vortices with a specific symmetry. Such an SOV state is described by

$$\Psi_{\{m_1, \ldots, m_J\}}(r, \phi) \propto \sum_{j=1}^{J} \psi_{m_j}(r, \phi) = \sum_{j=1}^{J} e^{i\Phi(r, \phi)} e^{i\phi_j(r)} ,$$

with the phases $\Phi_{m_j}(r, \phi) = m_j \phi + \gamma_j + \varphi_j(r)$ and the respective TCs $m_j \in \mathbb{Z}$ for the $j$th partial state. We further introduce the constant phases $\gamma_j \in \mathbb{R}$ for an additional rotational control and $r$-dependent phases $\varphi_j(r) \in \mathbb{R}$ for radial manipulation of the resulting SOV states.

To generate these SOVs we use a diffraction mask technique by employing a 2D LC-SLM, as depicted in figure 1. The amplitude and phase of the target state and an additional plane wave, separating the modulated beam from the zeroth diffraction order, are imprinted in a CGH [49]. Whereas in ultrafast spectral pulse-shaping, amplitude and phase modulation of a laser pulse [50] is achieved using double-layer LC-SLMs [51], this diffraction mask technique proposed by [39] and refined by [3, 27, 40–43] enables both amplitude and phase shaping with single-layer LC-SLMs. The diffraction masks introduced for pure OAM states $\Psi_{\ell} = e^{i\phi}$ in [5], lead to undesired contributions in the diffraction pattern when they are applied for the generation of SOVs. These contributions result from inter-grating interferences of the respective partial states (see for details appendix C). Hence, this approach is not optimal for superposition states composed of several partial states $\psi_{m_j}$. For this reason we introduce CGHs tailored for the generation of SOVs similar to [40–42]. In section 3.1 we compare them to the pure ones on the example of a $\Psi_{\{1, \ldots, 4\}}$ superposition state in order to demonstrate the enhancement of the target structures and the throughput. The interferences between the partial states in the CGH are circumvented by a superposition of the respective masks (cf equations (C.1) and (C.2)). We show that these CGHs further exhibit an enlarged active (transmissive)
mask area, resulting in an improved intensity of the generated target states. The SOV-tailored CGHs are described by

$$M(r, \phi) = \sum_{j=1}^{J} w_j(r) |\psi_{m_j}(r, \phi) + e^{i\gamma_j u(x, y; \kappa_j)}|^2,$$

(4)

with arbitrary radial dependent window functions \(w_j(r) \in \mathbb{R}_+.\) In addition to the constant phases \(\gamma_j\) of the partial states we introduce the phases \(\kappa_j \in \mathbb{R}\) for the individual rotational control of the plane wave orientation \(u(x, y; \kappa_j)\) with wave number \(k_0\) (see appendix A for details). Because \(k_0\) is equal for all partial states the generated SOVs are located on the same circle in \(k\)-space. The generalized description in equation (4) allows for more versatile CGHs \(M(r, \phi)\), resulting in a higher degree of control of diffraction patterns \(D(k, \xi)\). It especially enables amplitude modulation via \(w_j(r)\) and phase control via \(\gamma_j, \kappa_j\) and \(\gamma_j\) imprinted in the CGH. The far-field diffraction pattern \(D\) is described by the power spectral density of the 2D Fourier transform \(\mathcal{F}\) of the CGH [52]

$$D(\vec{k}) = |\mathcal{F}\{M\}|^2,$$

(5)

where the SOV state \(\Psi_{(m_1, \ldots, m_J)}\) is projected into the first diffraction orders \(D^{(\pm 1)}\) [39]. Since \(M \in \mathbb{R}\), Friedel’s law [53] holds and \(D\) obeys the point-symmetry

$$D(\vec{k}) = D(-\vec{k})$$

(6)

with respect to the zeroth diffraction order \(D^{(0)}\). See figure 1 for a schematic representation of the setup and figure 2 for experimental data. In \(k\)-space polar coordinates \((k, \xi)\), \(D^{(\pm 1)}\) is given by [43, 54, 55] (for details see appendix A)

$$D^{(\pm 1)}(k, \xi) \propto \sum_{j=1}^{J} |\hat{I}_{j,m_j}(k^\pm)|^2 + 2 \sum_{j>\ell} \hat{I}_{\text{eff}}(k^\pm) \cos \left( \Delta_{j,\ell}^{(m_j)} \left[ \xi_j^\pm \pm \frac{\pi}{2} \right] + \Delta_{j,\ell}^{(\gamma_j)} + \Delta_{j,\ell}^{(\kappa_j)}(k^\pm) \right),$$

(7)

with \(\hat{I}_{\text{eff}}(k^\pm) = |\hat{I}_{j,m_j}(k^\pm)| \hat{I}_{\ell,m_j}(k^\pm)|\) as an effective amplitude structure, where

$$\hat{I}_{j,m_j}(k) = \int_{0}^{\infty} w_j(r) e^{i\phi_j(r)} J_m(kr) r \, dr = \mathcal{H}_{m_j} \{ w_j \} \otimes \mathcal{H}_0 \{ e^{i\phi_j(r)} \}. $$

(8)

The latter expression denotes the Hankel-type convolution of the respective Hankel-transforms \(\mathcal{H}\) of \(w_j\) and \(e^{i\phi_j(r)}\) [54, 55]. Moreover we used the short hand notation \(\Delta_{j,\ell}^{(m_j)} = m_\ell - m_j\) and analogously for \(\Delta_{j,\ell}^{(\gamma_j)}\) and \(\Delta_{j,\ell}^{(\kappa_j)}(k^\pm)\) with \(\bar{\phi}_j(k^\pm) = \arg \left[ \hat{I}_{j,m_j}(k^\pm) \right]\) as well as shifted frequency coordinates \(k^\pm\) and \(\xi_j^\pm\) (see appendix A for details). The first sum in equation (7) represents a torus-shaped background ensuring a positive signal whereas the second sum describes the target state. For simplicity we assumed \(\kappa_j = \kappa_j\) for all partial states \(\psi_j\) interfering with the reference state \(\psi_j\), which creates an SOV at the coordinates \((k^\pm, \xi_j^\pm)\). The azimuthal rotations due to constant and radial dependent phases are represented by the quantities \(\Delta_{j,\ell}^{(\gamma_j)}\) and \(\Delta_{j,\ell}^{(\kappa_j)}(k^\pm)\). For each partial state combination equation (7) reveals an \(\Delta_{j,\ell}^{(\gamma_j)}\)-fold rotational symmetry of the target state, whereas the chosen window functions \(w_j(r)\) and radial phases \(\phi_j(r)\) affect the radial part via their Hankel-transform.
To specify the topological character of the generated SOVs we examine the orbital current

\[ J_{\mathcal{O}}(r, \phi) = \frac{c}{8\pi k} \mathcal{J}(\vec{E}^\times(r, \phi) \cdot (\nabla) \vec{E}(r, \phi)) \]

with the invariant Berry notation \( \vec{E}^\times \cdot (\nabla) \vec{E} \) [57]. In contrast to the spin flow density, determined by the polarization state, this orbital current does not vanish for linearly polarized light [48]. We further introduce the TC \( \ell \) which describes the accumulated phase along a contour \( \mathcal{C} \) enclosing the phase singularity and is given by [56]

\[ \ell = \frac{1}{2\pi} \oint_{\mathcal{C}} \arg (\Psi_{(m_1, \ldots, m_j)}(r, \phi)) \, d\vec{r}. \]

Since the SOV states from equation (3) are general we exemplify the quantities \( J_{\mathcal{O}} \) and \( \ell \) on superpositions composed of two states without radial phases. In this case we obtain a local TC [7, 43, 58, 59]

\[ \ell_k = \frac{1}{2} (\text{sgn} \left[ A_1^2 - A_2^2 \right] (m_1 - m_2) + (m_1 + m_2)) \]

at the radius \( k \), where \( A_j \) represent the amplitudes of the partial waves \( \psi_j \) in the target at this radius \( k \).

Especially, for SOVs consisting of only two states with equal amplitudes, the local TC takes the (fractional) value \( \ell_k = \frac{m_1 + m_2}{2} \). In this context we distinguish between ‘balanced’ SOVs with a vanishing local TC (\( m_1 = -m_2 \)) and ‘unbalanced’ SOVs composed of topologically different (\( |m_1| \neq |m_2| \)) partial states, leading to \( \ell_k \neq 0 \).

2.3. Experimental setup

The experimental setup, depicted in figure 1, is based on spatial amplitude modulation of a polarization adjusted diode laser (\( \lambda = 532 \text{ nm}, P = 1.85 \text{ mW} \)) with a collimated beam diameter of 20 mm using a 2D LC-SLM with an active area of \( (36.9 \times 27.6) \text{ mm}^2 \). The twisted nematic 2D LC-SLM (Holoeye SLM LC2012, \( 1024 \times 768 \)) px\(^2\)-XGA, pixel size: 36 \( \mu \text{m}, 8 \) bit) in combination with the polarizers \( P1 \) and \( P2 \) enables a programmable spatial amplitude modulation using a customized CGH. The adaptation of the CGHs with the frame rate of 60 Hz permits real-time spatial control of the generated SOVs. The optical setup is completed by the two lenses \( L1 \) (focal length \( f = 300 \text{ mm} \)) and \( L2 \) (\( f = -25 \text{ mm} \)) to image the modulated phase front into \( k \)-space by focusing the laser beam and projecting the diffraction pattern \( D \) on a screen.

The generated SOVs are captured by a charge-coupled device (CCD) camera [IDS, \( 3684 \times 4912 \)] px\(^2\), exposure time: 212.8 ms, pixel rate: 96 MHz, pixel size: 1.25 \( \mu \text{m} \). The measured data is slightly smoothed using a 2D Gaussian filter with an FWHM of 5 px.
3. Results and discussion

In the following we present SOVs with different symmetries resulting from pure and tailored CGHs. Analogous to established pulse-shaper setups in ultrafast laser science [51, 60, 61], we decompose the SOV manipulation into amplitude and phase modulation, which are applied via the window functions \( w_j(r) \) and the phases \( \varphi_j(r) \), \( \gamma_j \) and \( \kappa_j \), respectively. Below we use the notation \( \gamma_{j_1,\ldots,j_n} \) for \( \gamma_{j_1} = \cdots = \gamma_{j_n} \) where \( j_1, \ldots, j_n \) are the respective SOV state indices. This notation is also used for \( \kappa \) and \( \varphi \).

3.1. Amplitude modulation of SOVs

In this section we discuss the generation and control of SOV beams with tailored CGHs, compare them with pure CGHs and investigate the influence of different types of window functions (see insets figure 2(a)). In the following sections 3.1.1 and 3.1.2 we use for a quantitative analysis a mask radius \( R = 10 \text{ mm} \) and a wave number of \( k_0 = 3.6 \text{ mm}^{-1} \).

3.1.1. Intensity enhancement of SOVs

First, we compare tailored and pure [5, 43] CGHs, using a \( j \)-independent circular window function

\[
\tag{12}
w_j(r) = \text{circ}_k(r) \equiv \Theta(r - R),
\]

with the radial Heaviside step function \( \Theta \). As discussed in section 2.1 the tailored CGHs are described by a superposition of all partial diffraction masks, avoiding the generation of inter-grating interferences and causing an increased intensity of the target state. The results for a \( \Psi_{\{−3,4\}} \) state are depicted in figure 2(a), showing the measured (top) and simulated (bottom) target state in \( D^{(1)} \) for a pure (left frames) and an SOV-tailored (middle frames) CGH. The insets display the applied CGHs (top insets) and the simulated SOV phase patterns (bottom insets). The increased intensity is made visible by normalizing the data to the maximum of the tailored SOV. For a detailed investigation of the intensity improvement and its experimental exemplification we choose SOVs with up to six different partial states (\( \Psi_{\{−3,4,2,−5,6,−7\}} \)). The experimental results for the intensity study are depicted in the insets of figure 2(b). We observe a significant intensity difference between the generated SOVs in favor of the tailored CGHs. Furthermore, for pure CGHs the structures in \( D^{(1)} \) smear out for \( J \geq 3 \) and therefore deviate from the desired target state defined by equation (3). For a quantitative analysis we introduce the intensity ratio

\[
\tag{13}
Q = \frac{D^{(1)}_{\text{tailored}}}{D^{(1)}_{\text{pure}}},
\]

where \( D^{(1)} \) is the integrated intensity of the target state extracted from the first diffraction order for tailored and pure CGHs, respectively. Note that for the measured data the extracted target state does not contain the surrounding Bessel-type oscillations, since a distinction between those oscillations and other disturbing structures is hardly accessible. The graph in figure 2(b) depicts the intensity ratio for the measured six \( \Psi_{\{−3,4,2,−5,6,−7\}} \) SOV states are depicted in figure 2(a), where we used \( R_0 = 0.5 \text{ R} = 5 \text{ mm} \) and \( \sigma = 2.4 \text{ mm} \) (cf upper right inset in

3.1.2. Structural optimization of SOVs

The radial Bessel-type oscillations around \( D^{(1)} \) (green arrows in the left and middle frame of figure 2(a)), originating from the sharp cutoff by the circular window function (cf equation (12)), are reduced by apodization [62] using a softened CGH [40–42]. To minimize these radial structures around the desired SOV state the inner region is faded out, while the emerging edges are softened. This apodization is implemented with a shifted radial Gaussian window given by

\[
\tag{14}
w_j(r) = e^{-\frac{\sigma - r_0}{2\sigma^2}},
\]

with a width parameter \( \sigma \) and a radial shift \( r_0 \). The measured and simulated \( \Psi_{\{−3,4\}} \) SOV states are depicted in the right frame of figure 2(a) where we used \( r_0 = 0.5 \text{ R} = 5 \text{ mm} \) and \( \sigma = 2.4 \text{ mm} \) (cf upper right inset in
Figure 3. Diffraction patterns of a $\Psi^{\{1,\ldots,10\}}$ state with and without CGH-matching. The generated $\Psi^{\{1,\ldots,10\}}$ state using a Gaussian window function $w_j(r)$ according to equation (14) exhibits a radial smearing due to the different radii of its partial states (left). In contrast, the CGH-matching method enables the radial localization of the target state corresponding to the aimed $k$-distribution $w_j^{\text{target}}(k)$ in equation (15) (right). The radial and azimuthal confinement of the SOV is emphasized in the bottom insets where the left frame shows the $k$-sections along the target states and the right frame the $\xi^+$-sections around the SOV origin. The measured data is normalized on the respective maximum $I_{\text{max}}$.

While the intensity of the target state is slightly decreased due to the reduced active mask area, the radial Bessel-type oscillations are efficiently suppressed. For a quantitative discussion of the measured data we introduce the target state quality $\Gamma$ as the intensity ratio between the extracted target state $\bar D^{(1)}$ and the Bessel-type oscillations with other disturbing structures. For a pure CGH with a circular window function those oscillations in combination with inter-grating interferences result in an target state quality of $\Gamma_{\text{circ\ pure}} = 0.85 \pm 0.15$. Whereas for a tailored CGH the inter-grating interferences vanish, resulting in $\Gamma_{\text{circ\ tailored}} = 1.3 \pm 0.15$. The application of a Gaussian instead of a circular window function, using the parameters mentioned above, introduces the apodization which leads to $\Gamma_{\text{gau}} = 2.0 \pm 0.3$ and therefore improves the target state quality and the corresponding phase structure [40, 41, 63]. The additionally observed change in the SOV diameter, corresponds to the Gaussian parameters which enables the radial control of the SOV in addition to its TC. However, since $w_j(r)$ affects the diffraction pattern solely via $H_{m_j}\{w_j(r)\}$, the azimuthal SOV symmetry is invariant under changes in $w_j(r)$. Although mechanically manufactured diffraction masks, e.g. in a TEM, are binary [5, 44, 46], discontinuous circular edges can also be circumvented by choosing a Gaussian window function before applying a binarization. This preserves the modified grating structure and simplifies the fabrication process, resulting in a more stable mask. In the following sections we restrict to a qualitative discussion and therefore slightly adapt the CGH-parameters for the respective scenario.

3.1.3. SOV shaping in $k$-space

So far the SOV shaping is implemented in position space [40, 41, 63] which leads to a spreading in $k$-space due to different orders of Hankel-transformation for the various involved partial states (cf equation (8)). However, our approach also permits the direct structural optimization of the SOVs in $k$-space by using the $|m_j|$th-order Hankel-transform (involutive map) of the targeted $k$-distribution $w_j^{\text{target}}(k)$ as a window function

$$w_j(r) = H_{m_j}\{w_j^{\text{target}}(k)\},$$

in position space. Note that the absolute value of $m_j$ is chosen to ensure the same rotational alignment as in the unmatched case. This CGH-matching enables pre-defined radial SOV-structuring independent of the involved TCs. Note that phases in $k$-space can be introduced in the same way by replacing each $w_j$ by $\hat w_j$ (see appendix A for details), similar to [43]. In figure 3 we exemplify this procedure on a $\Psi^{\{1,\ldots,10\}}$ state.
(with equidistantly spaced TCs \( \Delta m_j = 1, \ldots, 9 \)) analogously to the principle of mode-locking in ultrafast laser science [64]. We compare the CGH-matched mask (shifted Gaussian window in \( k \)-space) with an unmatched spatial Gaussian windowed one. The SOV state exhibits in both cases a localized narrow azimuthal distribution with \( \epsilon_1 \)-rotational symmetry due to the interference of the partial states in azimuthal direction. For a pure Gaussian window the distribution smears out in radial direction due to a spreading in \( k \)-space, which results from Hankel-transformations of different orders (cf figure 3). However, in the matched case with full overlap of all partial states in \( k \)-space the radial confinement is increased. The different radii in \( k \)-space, leading to the interference of adjacent partial states, prevent further narrowing in azimuthal direction. Hence, in the unmatched case an increase of the number of participating OAM states does not yield a stronger localization. CGH-matching is an example of radial mask tailoring to optimize azimuthal interference for maximum localization, as shown in figure 3. The mode-locking example shows that a superposition of an arbitrary number of OAM states in \( k \)-space, enables a high degree of localization.

In section 3.3 we point out the meaning of the CGH-matching also for the topological structure of unbalanced SOV states. This matching-procedure is crucial for the radial control of SOVs composed of partial states with \( |\Delta m_j| \gg 1 \). In the following we discuss SOVs with \( |\Delta m_j| \leq 1 \) therefore CGH-matching is not necessary, so we use spatial Gaussian window functions according to equation (14).

### 3.2. Phase modulation of SOVs

So far, we have discussed the generation of SOVs and the manipulation of their symmetries and structures. In the next step we focus on the rotational and radial control of the generated SOV beams by phase modulation.

#### 3.2.1. Azimuthal phase control of SOVs

In this section we demonstrate the rotational control of SOVs where the results are presented in figure 4. To this end we stack multiple CGHs in radial direction, where an additional twist of each plane wave component via \( \kappa_j \) results in an overall rotation of the respective diffraction pattern about the angle \( \kappa_j \) around the center \( D^{(1)} \). The rotational-alignment of the individual SOV state is maintained such that only the origin of the SOV is rotated but not the SOV itself. In contrast constant target state phases \( \gamma_j \) lead to a rotation about

\[
\beta_{f,j} = \frac{\gamma_f - \gamma_j}{m_f - m_j} = \frac{\Delta \gamma_j}{\Delta m_j},
\]

(16)

around the center of the respective SOV (cf rotation formula in \([6]\)), similar to \([65]\). Hence, the independent quantities \( \kappa_j \) and \( \beta_{f,j} \) are used to control the azimuthal orientation of the target states, either of the whole diffraction pattern or only of a specific SOV state. These two types of SOV rotations were also studied in \([66]\) in a TEM in the context of Lamor- and Gouy-rotations.

The stacked CGH is tailored for the generation of the target states \( \Psi_{\{-10,10\}}, \Psi_{\{-3,4\}} \) and \( \Psi_{\{1,2\}} \), which are partially stacked in radial direction and twisted against each other about \( \frac{\pi}{2} \). The states \( \Psi_{\{-10,10\}} \) and \( \Psi_{\{1,2\}} \) are radially stacked using their intrinsic radial distribution, leading to the \( \Psi_{\{1,2,-10,10\}} \) SOV. The simulated diffraction pattern together with the magnified SOV states is shown in figure 4(a) and compared

![Figure 4](image-url)

Figure 4. Fanned out multi-vortex generation by applying CGH-stacking. The tailored CGH contains the holograms for the generation of the SOV states \( \Psi_{\{-10,10\}}, \Psi_{\{-3,4\}} \) and \( \Psi_{\{1,2\}} \), which are partially stacked in radial direction and twisted against each other about \( \frac{\pi}{2} \). The simulated and measured diffraction patterns (a) and (b) exhibit the imprinted states from the CGH, fanned out on a circle. The insets show the zoomed in generated SOVs from \( D^{(1)} \). Further applied azimuthal phases \( \gamma_j \) enable individual rotational control of the SOV (according to equation (16)), as exemplified for \( \Psi_{\{1,2\}} \).
to the measurement in (b). To demonstrate the azimuthal phase control we highlight the rotation of a $\Psi_{1,2}^{(3,4)}$ state about an angle $\kappa_{1,2} = \frac{\pi}{2}$, while the individual rotation about $\beta_{3,8} = -\frac{\pi}{2}$ of a $\Psi_{1,2}^{(1,2)}$ SOV, by applying $\gamma_j = \frac{\pi}{2}$ to the partial state $\psi_j$, is highlighted in green. In contrast to the CGH-stacking with linear radial phases, presented in the following section 3.2.2, the stacking approach leads to a conservation of the SOV’s phase structure, also in radial direction as depicted in the inset of figure 4(a).

3.2.2. Radial phase control of SOVs

In this section we present the shaping of the generated SOVs by encoding radial phases in the CGH. The modification of the real-valued radial window function $w_j(r)$ leaves the azimuthal symmetry unaltered and introduces no additional phases (cf insets figure 2). In contrast, radial phases do not only manipulate the SOV structure via the order of the corresponding Hankel-transform $|\tilde{Z}_j|$, they also introduce additional azimuthal phases $\phi_j(k^\pm)$, causing a $k$-dependent rotation of the modulated target state (cf section 2.1). Similar to the approach in ultrafast optics [51, 60, 67], we approximate the in general arbitrary radial phase function by a Taylor polynomial introduced in the appendix B, a linear phase introduces a radial shift in $c_2$-rotational symmetry by applying $\varphi_j^{(1)}$ to either both or only one of its partial states $\psi_{-3}$ and $\psi_4$. As detailed in the appendix B, a linear phase introduces a radial shift in $k$-space of the respective partial state to larger radii. Especially, when $\varphi_j^{(1)}$ is the diameter of the whole SOV is enlarged. Since this radial expansion of the SOVs is reminiscent of a linear axicon in optics [72], we denote this kind of modulation as an axicon-type phase modulation.

In figure 5(a) we present the measurements and simulations, for three different linear phase functions $\varphi_j^{(1)}(r)$. The resulting SOV radius is enlarged and the structure is rotated in counter-clockwise-direction for increasing linear phases ($\varphi_j^{(1)} > 0$). The azimuthal rotation indicated in figures 5(a) and (b) originates from the additional phases $\Delta \phi_j^{(2)}$ in equation (7) which also results in a spiral-shaped phase structure of the target SOV state (cf insets figure 5(a)). In figure 5(b) we present the experimental results and simulations, for three different values of $\phi_j^{(1)}(\geq 0)$ applied to the partial state $\psi_{-3}$. We observe a torsion within the SOV structure where the tilt of the SOV lobes increases for larger values of $\phi_j^{(1)}$. This effect originates from the superposition of two differently twisted and radially shifted OAM beams. The corresponding vortices of matter waves has recently been introduced as photoelectron vortices [73] and have been studied experimentally in [6, 74, 75]. A direct analogy with these photoelectron vortices is achievable by using the CGH-matching technique, applying a linear radial phase and appropriate window functions $w_j^{\text{target}}$ directly in $k$-space.

A combination of (linear) radial- and azimuthal phase manipulation enables the generation of radially stacked and fully rotational controllable SOVs. We exemplify this technique on a $\Psi_{1,2,3,4}^{(1,2)}$ multi-vortex-structure composed of an inner $\Psi_{1,2}^{(1,2)}$ (crescent-shaped) and outer $\Psi_{3,4}^{(3,4)}$ SOV state. To this end we utilize the axicon-type phases $\varphi_j^{(1)}(r)$ for radial enlargement of the outer SOV and combine it with azimuthal phases $\gamma_j$ allowing for full rotational control of all states. The results are presented in figure 5(c), where both SOVs are individually rotated about their respective center in opposite directions (green and blue arrows).

Quadric radial phases. In this section we examine the effect of quadratic radial phases on a $\Psi_{1,2,3,4}^{(1,2)}$ SOV state, with $\varphi_j^{(2)} = \varphi_j^{(2)}$ applied to all of its partial states. We show in appendix B that the Hankel-transform of a quadratic radial phase converges asymptotically to a Fresnel-type representation of the $\delta$-distribution which is known from (electron-)optics, emulating a non-aberrated focusing symmetric lens [62, 76]. For this reason we denote the quadratic radial phase as a lens-type modulation. Analogous to the dispersive Fourier transform [59, 69, 77] for $\varphi_j^{(2)} \to \infty$, the states in the diffraction orders $D^{(\pm)}$ in $k$-space...
Figure 5. CGHs and corresponding diffraction patterns for linear radial (axicon-type) phase manipulation of SOVs. (a) Linear radial phases applied to both partial states of a $\Psi_{(-3,4)}$ SOV induce an enlargement of the SOV diameter which increases with $\phi(1)$, and introduce a $k$-dependence in the SOV phase via $\Delta \hat{\phi}_{1,2}$. (b) Same as (a) but the radial phases are only applied to one partial state leading to a spiral shape in the intensity and phase of the SOV, where the tilt increases with $\phi(1)$. (c) The states $\Psi_{(-3,4)}$ and $\Psi_{(1,2)}$ are stacked by applying a linear radial phase to the outer SOV state. The rotational control is achieved by azimuthal phase manipulation with $\gamma_{j}$ and demonstrated for several values of $\gamma_{1,2}$ and $\gamma_{3,4}$, where the SOVs are rotated in opposite directions. In all figures the color scales are adjusted in terms of $I_{0}$ which is defined as the maximum value within the respective image set.

reproduce the initial windowed wave function in position-space [68], i.e.

$$\lim_{\varphi(2) \to \infty} D^{(\pm 1)}(\vec{k}) = \left( \sum_{j=1}^{l} w_{j}(\vec{k}) |\psi_{j}(\vec{k})| \right)^{2}.$$  \hfill (18)

The results are depicted in figure 6(a) showing the SOV state $\Psi_{(1,2,-3,4)}$ (left frame), which evolves with increasing $\varphi(2)$ into an image of the windowed partial states (magenta contour in the bottom frame) according to equation (18), exhibiting a Fresnel-type phase structure (right inset). During this process the diameter of the SOV is magnified continuously.

Lenses-type phases are also used to introduce an astigmatism into the CGH by applying the quadratic phase in one Cartesian direction $\varphi_{\text{ast}}(x) = \varphi(2)x^{2}$. Astigmatism has been utilized to retrieve the TC $\ell$ from pure OAM states by counting the minima in the diffraction structure [70, 71]. We exemplify the astigmatism technique on a pure $\Psi_{(-3)}$ state, depicted in figure 6(b). Counting the minima and taking into account the rotational orientation (representing the sign of $\ell$) reveals the TC of $\ell = -3$. It should be noted that this approach for the experimental determination of $\ell$ is generally applied to pure OAM states. Nevertheless, we examine the applicability of the astigmatism technique to either balanced ($\Psi_{(-3,3)}$) or unbalanced ($\Psi_{(-2,4)}$) SOVs and show that a distinction between them is feasible. In general, structured SOVs do not support astigmatism-induced minima, which number corresponds to its TC, since additional minima occur due to the $\Delta m_{j}$-fold rotational symmetry. However, the structural minima of the SOV are superimposed with the ones originating due to its topology. An astigmatic phase applied to a balanced SOV ($\ell = 0$) creates structural caused minima occurring in an axial-symmetric pattern. In contrast, an unbalanced SOV state ($\ell \neq 0$) exhibits not only minima due to the structured state itself, but also destructive contributions originating from the self-interference of both partial OAM states. Hence, the diffraction pattern is not axial- but point-symmetric and a distinction from the balanced case is possible. The experimental results together with simulations are shown in figure 6(c) for a $\Psi_{(-3,3)}$ (bottom frame) and $\Psi_{(1,2,-4)}$ (top frame) state. Therefore this technique permits the decision whether an SOV carries OAM or not (cf section 2.2).

### 3.3. Topological structure of tailored SOV states

In order to investigate the topological structure of the generated SOVs we calculate their orbital current $\vec{S}_{O}$ [78] and determine the corresponding local TC $\ell_{k}$. Following the discussion in section 2.2 we focus on balanced and unbalanced SOVs which are composed of two OAM states. The topological properties are
Figure 6. CGHs and corresponding diffraction patterns for quadratic radial (lens-type) phase manipulation of SOVs. (a) Quadratic radial phases applied to all partial states of a $\Psi_{(1,2,-3,4)}$ SOV. The $D^{(1)}$ contribution reproduces its own windowed partial states (magenta contour) due to the dispersive Fourier transform [68, 69] for large values of $\varphi^{(2)}$. (b) Astigmatism applied to a pure $\Psi_{(-3)}$ state via quadratic phases $\varphi^{(2)}(x)$ in one Cartesian direction. For sufficiently large values of $\varphi^{(2)}$, we observe a contraction and self-interference of the state, resulting in countable minima and representing its TC $\ell = -3$ [70, 71]. (c) Same as (b) but for a $\Psi_{(-3,3)}$ and a $\Psi_{(-2,4)}$ state. Although a direct retrieval of the TCs is not feasible, the astigmatism allows to distinguish between balanced ($\Psi_{(-3,3)}$) and unbalanced ($\Psi_{(-2,4)}$) states. In all figures the color scales are adjusted in terms of $I_0$ which is defined as the maximum value within the respective image set.

Given by the superposition of the partial state’s topological characteristics, where the radial and azimuthal components of $\vec{S}_O$ are not only determined by the involved TCs $m_j$ but also by the radial phases $\varphi_j(r)$ and window functions $w_j(r)$. Above, we showed that the diameter of an OAM state with a TC of $m_j$ is determined by the $m_j$th order Hankel-transform (cf section 2.1). Consequentially, balanced SOVs exhibit a vanishing local TC since their partial states show the same radial behavior and interfere destructively in a topological sense. Unbalanced SOVs, however, show a radial topological structure where inner and outer TCs are determined by the respective partial state. At their intersection the SOV forms a fractional local TC according to equation (11). Our results of the simulations are depicted in figure 7. The top frame in (a) shows the unbalanced $\Psi_{(-3,4)}$ SOV state together with its orbital current $\vec{S}_O$ (blue arrows). While the inner region (green) exhibits a TC of $\ell_k = -3$, the outer region (orange) has a TC of $\ell_k = 4$, corresponding to the respective partial state’s current. In the intermediate regime ‘turbulences’ between both states occur (highlighted in magenta). In the top frame of (b) we present the balanced $\Psi_{(-3,3)}$ state which does not exhibit an orbital current. Hence, the latter state has a vanishing TC.

To avoid the azimuthal orbital current annihilation even for balanced SOV states and the bicircular topological structure of unbalanced ones we utilize the radial phases and window functions introduced in the previous sections. The radial shift between the two partial states within an unbalanced SOV is compensated by the CGH-matching method introduced in section 3.1.2. This procedure enables a pre-defined overlap between $\psi_{-3}$ and $\psi_4$, avoiding the bicircular topological structure, which is depicted for $\Psi_{(-3,4)}$ in the bottom frame of figure 7(a). The effective local TC of the generated SOV takes the fractional value $\ell_k = 0.5$. While the CGH-matching does not affect the orbital current of the partial states, it changes the target state’s radial dependence. We showed that a radial shift is also implemented by applying a linear radial phase to the respective partial state. This procedure is exemplified on the balanced $\Psi_{(-3,3)}$ SOV, where the $\psi_{-3}$ state is shifted in positive radial direction. The resulting radial separation uncovers the partial states $\psi_3$ and $\psi_{-3}$, leading to a non-vanishing inner (outer) local TC of $\ell_k = 3$ ($\ell_k = -3$) (cf bottom frame of figure 7(b)). It should be noted that in the case of photoelectron vortices created with MPI [6, 7, 74, 75] the linear radial phase does not introduce a shift of the partial states in $k$-space, since the radial part is determined by a Fourier- instead of a Hankel-transform. For this reason the SOV states created in [6, 7, 59, 74, 75] are essentially free of ‘turbulences’ in the probability current.
4. Conclusion

In this paper we presented the generation and full control of SOV states with tailored CGHs using a 2D liquid crystal spatial light modulator (LC-SLM). Based on the holographic diffraction mask approach [5, 27, 39] we imprinted the amplitude and phase of the target state in a CGH which spatially modulates a polarization adjusted diode laser beam. We specifically tailored our CGHs for the generation of SOVs with higher intensities and apodized target state structures, compared to the SOVs generated by pure CGHs. The radial and rotational controllability of the generated SOVs was enabled by adding radial- and azimuthal phases in the CGH design. Furthermore we examine the topological structure of the manipulated SOVs, highlighting the role of phase and amplitude modulation for the respective TC and the orbital current.

We demonstrated our SOV-tailored CGH technique on states with $c_{20-}$, $c_{7-}$, $c_{6-}$ and $c_{1-}$ rotational symmetry and compared the SOVs generated by our tailored to those generated by pure CGHs in terms of structural and intensity improvements. Subsequently we demonstrate the radial- and azimuthal confinement of SOVs, analogously to mode-locking, using tailored k-space window functions (CGH-matching method). Moreover we exemplified the rotational control of both the entire SOV pattern and the individual SOV states by imprinting azimuthal phases into the CGH and showed that this technique permits the differential controllability of the SOV orientation. In addition, we manipulated the SOV’s radial intensity distribution by applying linear (axicon-type) radial dependent phases. This mechanism was utilized to radially stack and rotationally control target states as well as generate vortex structures similar to [6, 73–75]. Using quadratic (lens-type) radial phases the diffraction pattern in k-space evolves in its own windowed position-space representation [68] due to the dispersive Fourier transform [69, 77]. As a further application we determine the TC via astigmatic phases, as also done by [70, 71], and extended this to superimposed states where our technique facilitates to decide if an SOV carries a TC at all. Finally, we study the orbital current $\vec{S}_O$ and the (fractional) local TC $\ell_k$ of specific SOVs under the influence of CGH-matching and linear radial phases.

The presented technique goes beyond the generation of SOV states enabling the control of their radial and topological structure as well as of their alignment in a common-path geometry on a millisecond time scale. These features and the additional degrees of freedom of the SOVs, e.g. the rotation-angle [79, 80],
slope of tilted vortex lobes [75] and the topological SOV structure, could be advantageous in optical information transfer with such states. Here, a possible extension of the demonstrated phase control of SOVs is proposed in ultrafast laser science [51, 60, 61, 67] by using other polynomial or sinusoidal radial phases and combinations of those. An application of the astigmatism phase mask could be the pre-compensation and measurement of astigmatism which is introduced by the experimental setup or other optical elements, utilizing a well defined SOV state as a reference. Furthermore, by appropriate binarization many of the presented CGHs and control mechanisms are also applicable in a TEM [5, 44–46, 56, 66].

Funding

This research was funded by the Deutsche Forschungsgemeinschaft via the priority program SPP1840 QUTIF.

Availability of data and materials

The datasets used during the current study are available from the corresponding author on reasonable request.

Acknowledgments

Stimulating discussions with C Rathje and S Schäfer on the principles of two-dimensional electron beam diffraction during the cooperation on ‘Orbital angular momentum superposition states in transmission electron microscopy and bichromatic multiphoton ionization’ [43] are gratefully acknowledged.

Appendix A. Diffraction pattern of an SOV-tailored CGH

In this section we discuss the diffraction theory for the generation of arbitrary superimposed OAM states

\[ \Psi_{\{m_1,\ldots,m_I\}} \propto \sum_{j=1}^{J} e^{i(m_j\phi + \gamma_j + \phi_j(r))}, \]  

(A.1)

composed of \( J \) partial states, using the polar coordinates \( r \) and \( \phi \). The respective partial state with TC \( m_j \) contains, beside a constant phase \( \gamma_j \), an additional and radial-dependent phase \( \phi_j(r) \). A corresponding generalized tailored CGH is given by

\[ \mathcal{M}(r, \phi) = \sum_{j=1}^{J} w_j(r) |e^{i(m_j\phi + \gamma_j + \phi_j(r))} + e^{ik_0u(x,y;\kappa_j)}|^2, \]  

(A.2)

with arbitrary window functions \( w_j(r) \in \mathbb{R}_+ \) introducing an amplitude modulation and a generalized plane wave phase

\[ u(x,y;\kappa_j) = -x \sin(\kappa_j) + y \cos(\kappa_j). \]  

(A.3)

The phase of the cosine and sine functions \( \kappa_j \) represents a rotation of the respective plane wave’s direction, whereas \( k_0 \) is the corresponding wave number. Since the diffraction of the CGH is described by a Fourier transformation in the far-field regime [52], we examine the CGH’s 2D Fourier-transform in polar coordinates, given by [54, 55]

\[ \mathcal{F}\{f\}(k, \xi) = \int_{0}^{\infty} \int_{-\pi}^{\pi} f(r) e^{-ikr \cos(\xi - \phi)} r \, dr \, d\phi, \]  

(A.4)

with \( k = |\vec{k}| \) and the angle \( \xi = \arctan(k_y/k_x) \). Whereby \( k_x \) and \( k_y \) are the Cartesian frequency coordinates. For non-radially symmetric functions \( f(\vec{r}) \) it is advantageous to use the Fourier decomposition

\[ f(\vec{r}) = f(r, \phi) = \sum_{q=-\infty}^{\infty} f_q(r) e^{i\phi q}, \]  

(A.5)
especially the CGH can be written as \( \mathcal{M} = \mathcal{M}_0 + \mathcal{M}_1 + \mathcal{M}_2 \) with

\[
\mathcal{M}_0 = 2 \sum_{j=1}^{J} w_j(r), \tag{A.6}
\]

\[
\mathcal{M}_1 = \sum_{j=1}^{J} \tilde{w}_j^*(r) e^{-i(m_j\phi + \gamma_j)} e^{i\beta(r,x_j;k_j)}, \tag{A.7}
\]

\[
\mathcal{M}_2 = \mathcal{M}_1^*, \tag{A.8}
\]

using \( \tilde{w}_j(r) = w_j(r) e^{i\phi_j(r)} \). With the 2D Fourier transform of equation (A.5) \cite{54}

\[
\tilde{f}(\vec{k}) = \sum_{q=-\infty}^{\infty} \sum_{a=-\infty}^{\infty} 2\pi i q \int_{0}^{\infty} f_q(r) J_q(kr) r \, dr, \tag{A.9}
\]

we directly obtain the Fourier transformed CGH components

\[
\tilde{\mathcal{M}}_0 = 4\pi \sum_{j=1}^{J} I_{ja}(k), \tag{A.10}
\]

\[
\tilde{\mathcal{M}}_1 = 2\pi \sum_{j=1}^{J} i^{-m_j} e^{i(m_j\xi^+ + \gamma_j)} \mathcal{I}_{ja}(k^+), \tag{A.11}
\]

\[
\tilde{\mathcal{M}}_2 = 2\pi \sum_{j=1}^{J} i^{-m_j} e^{i(m_j\xi^- + \gamma_j)} \mathcal{I}_{ja}(k^-). \tag{A.12}
\]

Here, we used a short hand notation for the Hankel-transformation of \( \tilde{w}_j \)

\[
\mathcal{I}_{ja}(k) = \int_{0}^{\infty} \tilde{w}_j(r) J_a(kr) r \, dr, \tag{A.13}
\]

and \( w_j(r) \)

\[
I_{ja}(k) = \int_{0}^{\infty} w_j(r) J_a(kr) r \, dr, \tag{A.14}
\]

with the \( n \)th order Bessel-function of the first kind \( J_n(kr) \) and its identity \( J_{-n}(kr) = (-1)^n J_n(kr) \). We further introduced shifted frequency coordinates

\[
k^\pm = \sqrt{(k_x \pm k_0 \sin(\kappa_j))^2 + (k_y \mp k_0 \cos(\kappa_j))^2},
\]

\[
\xi^+_j = \arctan \left( \frac{k_y \mp k_0 \cos(\kappa_j)}{k_x \pm k_0 \sin(\kappa_j)} \right). \tag{A.15}
\]

The coordinate shift originates from the 2D Fourier shift theorem

\[
\mathcal{F}\{e^{ik_x x} e^{ik_0 y} f\}(\vec{k}) = \tilde{f}(k_x \mp k_0, k_y \pm k_0). \tag{A.16}
\]

For an approximation of the diffraction pattern

\[
\mathcal{D} = |\mathcal{F}\{\mathcal{M}\}|^2 = \sum_{a=0}^{2} |\tilde{\mathcal{M}}_a|^2 + \sum_{a=0}^{2} \tilde{\mathcal{M}}_a \tilde{\mathcal{M}}_a^* \approx \sum_{a=0}^{2} |\tilde{\mathcal{M}}_a|^2, \tag{A.17}
\]

we neglect the mixing terms \( \tilde{\mathcal{M}}_a \tilde{\mathcal{M}}_a^* \) which represent the overlap between different CGH components in the spectral domain. This approximation is valid for a sufficiently large separation of the target states.
\[ |\tilde{\mathcal{M}}_{1,2}|^2 \]

from the zeroth diffraction order \(|\tilde{\mathcal{M}}_0|^2\), i.e., for large values of \(k_0\). Hence, the main contributions are given by

\[
|\tilde{\mathcal{M}}_0|^2 = 16\pi^2 \sum_{j'\neq j} \mathcal{I}_{j,0}(k) \mathcal{I}_{j',0}(k) = \frac{\pi}{k_0} \sum_{j'\neq j} I_{j',0}(k) I_{j,0}(k)
\]

\[
|\tilde{\mathcal{M}}_{1,2}|^2 = 4\pi^2 \sum_{j=1}^4 \tilde{I}_{j,0}(k^\pm)^2 + 8\pi^2 \sum_{j' > j} \tilde{I}_{eff}(k^\pm) \cos \left( \frac{\Delta_{j,j'}^\beta + \Delta_{j,j'}^\gamma + \Delta_{j,j'}^\delta}{\pi} \right)
\]

with \(\tilde{I}_{eff}(k^\pm) = |\tilde{I}_{j,0}(k^\pm)|\tilde{I}_{j,0}(k^\pm)(k)\) as an effective \(k\)-distribution and the phases \(\tilde{\psi}(k^\pm) = \arg \left( \tilde{I}_{j,0}(k^\pm) \right)\).

The first index of \(|\tilde{\mathcal{M}}_{1,2}|^2\) in equation \((A.19)\) corresponds to the plus and the second to the minus sign. In the following we use the notation \(D^{(0)}(k,\xi) = |\tilde{\mathcal{M}}_0|^2\) for the zeroth and \(D^{(\pm)}(k^\pm,\xi^\pm) = |\tilde{\mathcal{M}}_{1,2}|^2\) for the plus/minus first diffraction order. The desired superimposed OAM states with \(\Delta_{j,j'}^\beta\)-fold rotational symmetry appear in \(D^{(\pm)}(k^\pm,\xi^\pm)\), shifted about \(\pm k_0\) in direction of the generalized plane wave. In equation \((A.19)\) we assumed that \(\kappa_j = \kappa_{j'}\) to ensure the rotational overlap of different SOVs. Furthermore, we used the notations \(\Delta_{j,j'}^\beta = m_j - m_{j'}, \Delta_{j,j'}^\gamma = \gamma_j - \gamma_{j'}\) and \(\Delta_{j,j'}^\delta(k^\pm) = \hat{\psi}(k^\pm) - \hat{\psi}(k^\pm)\). While \(\Delta_{j,j'}^\beta\) determines the rotational symmetry of the respective partial state combination, the phase terms \(\Delta_{j,j'}^\gamma\) and \(\Delta_{j,j'}^\delta(k^\pm)\) lead to an azimuthal rotation of this very same state. Whereas \(\Delta_{j,j'}^\gamma\) originates from constant phases and rotates the respective target structure about an angle \(\beta_{j,j'}\) (cf. equation \((16)\)), the term \(\Delta_{j,j'}^\delta(k^\pm)\) results from radial dependent phases and yields an analogous but \(k\)-dependent rotation.

To check whether the neglect of the mixing terms for sufficiently large values of \(k_0\) in equation \((A.17)\) is justified, we compare the far field diffraction pattern of the tailored CGH obtained by numerical 2D Fourier transform described by equation \((5)\) to the analytical result equation \((A.17)\). We provide a specific example similar to the mode-locking scenario discussed in section \(3.1.2\) to facilitate the reproducibility of our results. To this end, we consider the superposition of four partial states \(\psi_n\) with TCs \(n = 1, \ldots, 4\) and the respective radially phase modulated window functions \(w_n(r) = r^n e^{-r^2} e^{i\phi_n r^2}\). Our formalism provides access to the respective distribution in \(k\)-space which is given by

\[
\hat{I}_{n,n}(k) = \frac{k^n e^{i\frac{r^2}{2}}}{\left(2 - 2i\psi_n^{(2)}\right)^{n+1}}.
\]

---

**Figure A1.** Comparison of analytically and numerically calculated SOVs. To emphasize the agreement between the analytical calculation of the diffraction pattern using equations \((A.18)\) and \((A.19)\) and the numerics, we compare the results for a phase modulated \(\Psi_{[1,2,3,4]}\) state, using a window function according to equation \((A.20)\) the phases \(\psi_1^{(2)} = 0.95, \psi_2^{(2)} = 1.00, \psi_3^{(2)} = 1.50\) and \(\gamma_1 = 0.95\pi, \gamma_2 = 1.10\pi, \gamma_3 = -0.20\pi, \gamma_4 = 1.00\pi\) as well as \(\kappa_{[1,2,3,4]} = \kappa\). The resulting SOV states show an excellent agreement, underscoring the validity of the approximation in equation \((A.17)\).
The parameters used to generate the diffraction pattern in figure A1 are \( \varphi_1^{(2)} = 0.95, \varphi_2^{(2)} = 1.00, \varphi_3^{(2)} = 1.50, \varphi_4^{(2)} = 1.90 \) for the quadratic radial phase modulation and \( \gamma_1 = 0.95 \pi, \gamma_2 = 1.10 \pi, \gamma_3 = -0.20 \pi, \gamma_4 = 1.00 \pi \) for the azimuthal phases of the individual contributions. The plane wave has a value of \( k_0 = 14 \) and an angle of \( \kappa_{[1,2,3,4]} = \pi/4 \). The excellent agreement of the numerical and analytical diffraction pattern presented in figure A1 confirms the validity of our approximations and highlights the accuracy of both approaches.

Appendix B. Radial phase control

Despite the fact that specific examples allow for an analytic solution (cf equation (A.20)), the intertwining of the window functions \( w_j \) and the radial phases \( \varphi_j(r) \) in equation (A.13) does not allow for a general and intuitive radial phase discussion. This can be overcome by using the convolution theorem for the 2D Fourier transform \([54]\) leading to a separation of window and radial phase effects. For this reason we rewrite equation (A.13) with the short hand notation \( p_j(r) = e^{j\varphi_j(r)} \), such that

\[
\hat{I}_{j,m}(k) = \int_0^\infty w_j(r)p_j(r)J_m(kr) \, dr
= \int_0^\infty \int_0^\infty \tilde{W}_j(v)S^m_{0}(v,k,u)p_j(u)v \, dv \, du
\equiv (\tilde{W}_j \otimes \tilde{P}_j)(k)
\]  

Introducing the Hankel-transforms

\[
w_j(r) = \int_0^\infty \tilde{W}_j(v)J_m(ur)v \, dv \quad (B.2)
\]

\[
p_j(r) = \int_0^\infty \tilde{P}_j(u)J_0(ur)u \, du \quad (B.3)
\]

and the so called ‘shift-operator’

\[
S^m_{0}(v,k,u) = \int_0^\infty J_0(\varphi_0)J_m(kr)r \, dr.
\]  

Note that the Hankel-transform is an involutive map. Since the integral in equation (B.1) represents a Hankel-type convolution of \( \tilde{W}_j \) and \( \tilde{P}_j \), the influence of radial phases on the distribution in \( k \)-space is determined by its Hankel-transform \( \tilde{P}_j \). For a linear radial phase \( \varphi_j(r) = j\varphi_j^{(1)}r \), we obtain from equation (B.3) \([81]\)

\[
\tilde{P}_{j}^{lin}(k) = \frac{\delta(k - \varphi_j^{(1)})}{k} + 2 \sum_{m=1}^{\infty} \int_0^\infty J_m(\varphi_j^{(1)}r)J_0(kr)r \, dr.
\]

This corresponds mainly to a radial expansion via a ring \( \delta \)-function and is reminiscient of an axicon. For this reason we also denote a linear phase as a so called axicon-type phase. Nevertheless, the residual terms lead to additional amplitude and phase contributions which are not discussed in the present paper. For a quadratic radial phase \( \varphi_j(r) = \varphi_j^{(2)}r^2 \) we find

\[
\tilde{P}_{j}^{quad}(k) = -\frac{1}{2\varphi_j^{(2)}} e^{-\frac{k^2}{4\varphi_j^{(2)}}},
\]  

similar to a softened \( \delta \)-sequence in Fresnel-representation around \( k = 0 \). Since quadratic phases in \( k \)-space are known in the context of non-aberrated focusing circular symmetric lenses, we denote quadratic radial phases as lens-type phases.

Appendix C. Comparison of pure and tailored CGHs for SOV generation

In the following section we compare pure and tailored CGHs, especially with regard to the target state’s intensity. For the sake of clarity and for ease of notation we neglect the amplitude modulation via a window function \( w_j(r) \) and restrict ourselves to a constant circular window (cf equation (12)) in the following. We
also assume $\gamma_j = \varphi_j(r) = 0$. Again, we focus on the generation of SOV states taking the form of equation (A.1) from appendix A. While a tailored mask $\mathcal{M}$ can be written as

$$\mathcal{M}(r, \phi) = \text{circ}_R(r) \sum_{j=1}^{J} |\psi_j(r, \phi)|^2 + e^{i\Phi_j}|^2,$$

(C.1)

a pure mask $M$ is given by

$$M(r, \phi) = \text{circ}_R(r) \sum_{j=1}^{J} |\psi_j(r, \phi)|^2.$$ (C.2)

By expanding both equations and normalize the expressions to the respective maximum at the origin $\vec{r} = (0, 0)$ we find

$$\mathcal{M}(r, \phi) = \text{circ}_R(r) \frac{2J}{4} + 2R\{e^{-i\Phi_j} \Psi_j(r, \phi)\},$$ (C.3)

$$M(r, \phi) = \text{circ}_R(r) \frac{(J+1) + 2R\{e^{-i\Phi_j} \Psi_j(r, \phi)\}}{(J+1)^2} + \text{circ}_R(r) \frac{B(r, \phi)}{(J+1)^2},$$ (C.4)

where the term

$$B(r, \phi) = \sum_{j \neq j'} \psi_j(r, \phi)\psi^*_j(r, \phi)$$ (C.5)

represents the inter-grating beating terms of all involved target states. Following the procedure and the notation from appendix A the corresponding diffraction patterns can be written as

$$D_{\mathcal{M}}(k, \xi) = \frac{D^{(0)}}{4} + \frac{D^{(1)} + D^{(-1)}}{16J^2},$$ (C.6)

$$D_{M}(k, \xi) = \frac{D^{(0)}}{4} + \frac{D^{(1)} + D^{(-1)} + D^{(\text{inter})}}{(J+1)^2},$$ (C.7)

where we further introduced $D^{(\text{inter})} \equiv |\vec{B}|^2$. In the diffraction pattern these contributions $D^{(\text{inter})}$ lead to pronounced inter-grating interferences which are discernible in figures 2(a) and C1(a). For this reason and due to energy conservation, the intensity for a pure CGH has to be distributed on more terms resulting in a lower intensity of the target state. In figure C1 we show the analytically calculated diffraction patterns for a $\Psi_{(-3,4]}$ SOV state using pure (a) and SOV-tailored (b) CGHs. To quantify the intensity enhancement of the tailored CGH, we use the intensity ratio $Q(J)$ between the respective generated SOVs in $D^{(1)}$ using pure and tailored CGHs (cf equation (13)). Using equations (C.6) and (C.7) and the approximation in equation (A.17) we obtain an intensity ratio of

$$Q_{\text{theo}}(J) = \frac{D^{(1)}_{\text{tailored}}}{D^{(1)}_{\text{pure}}} = \frac{(J+1)^4}{16J^2},$$ (C.8)

with the corresponding integrated target state signals

$$D^{(1)} = \int D^{(1)}(k, \xi) d^2k.$$ (C.9)

The analytical expression equation (C.8) is depicted in figure C1(c) for $J \in [1, 20]$ (green line) and compared to simulated (blue squares) and measured (red circles) data. Hence, the more states contribute to the SOV, the larger is the intensity optimization of the tailored CGHs and therefore its significance for vortex superposition. This intensity enhancement of the target states can be physically described by the enlarged active (transmissive) mask area compared to a pure CGH (cf insets in figure C1(c)). While the analytical estimate is in good accordance with the simulation within the region of $J \leq 14$, the simulated data shows a stagnating behavior for $Q(J)$ when the number of partial states is further increased, which can be lead back to the finite CGH area. For this reason, both the pure and the tailored CGHs have an intensity of almost zero in $D^{(1)}$ leading to a value of $Q = 1$ for a sufficiently large number of partial states. To compare the results with experimental data, we capture six different SOVs with pure and tailored CGHs, respectively. We choose an individual signal-threshold to isolate the target SOV state and therefore ensure the integration of the correct contributions in the diffraction pattern (cf equation (C.9)). The errors are calculated by using the respective threshold range in which the target state remains unaffected. The
comparison with measured data is limited by the resolution of the LC-SLM, whereby binarization effects occur in the CGH for a large number ($J \gtrsim 5$) of partial states, which leads to a distorted representation of the desired target state. The uncertainties appear through the extraction of the target state during the data processing and the distinction from mixing terms and other disturbing structures.

A generalization of the CGH design from equation (C.2) similar to equation (4) is difficult to achieve since individual window functions do not take into account the plane wave contribution. This circumstance can be overcome by choosing an overall window function in front of the absolute square, leading to an integral equation for the distribution and phase in $k$-space. Nevertheless, for binarized diffraction masks this effect can be minimized with an appropriate choice of the binarization threshold and target structure width [43, 44].

References

[1] Molina-Terriza G, Torres J P and Torner L 2007 Twisted photons Nat. Phys. 3 305
[2] Shen Y, Wang X, Xie Z, Min C, Fu X, Liu Q, Gong M and Yuan X 2019 Optical vortices 30 years on: OAM manipulation from topological charge to multiple singularities Light Sci. Appl. 8 1–29
[3] Hell S W and Wichmann J 1994 Breaking the diffraction resolution limit by stimulated emission: stimulated-emission-depletion fluorescence microscopy Opt. Lett. 19 780–2
[4] Zou J W 2007 Far-field optical nanoscopy Science 316 1153–8
[5] Verbeeck J, Tian H and Schattschneider P 2010 Production and application of electron vortex beams Light Sci. Appl. 8 1–29
[6] Kerbstadt S, Eickhoff K, Bayer T and Wollenhaupt M 2019 Odd electron wave packets from cycloidal ultrashort laser fields Nature 467 301–4
[7] Kerbstadt S, Eickhoff K, Bayer T and Wollenhaupt M 2019 Control of free electron wave packets by polarization-tailored ultrashort bichromatic laser fields Adv. Phys. X 4 1672583
[8] Wang J 2016 Advances in communications using optical vortices Photon. Res. 4 B14–28
[9] Torres J P 2012 Multiplexing twisted light Nat. Photon. 6 420
[10] Willner A E et al 2015 Optical communications using orbital angular momentum beams Adv. Opt. Photon. 7 66–106
[11] Ndagano B, Nape I, Cox M A, Rosales-Guzman C and Forbes A 2017 Creation and detection of vector vortex modes for classical and quantum communication J. Lightwave Technol. 36 292–301
[12] Krenn M, Handsteiner J, Fink M, Fickler R and Zeilinger A 2015 Twisted photon entanglement through turbulent air across Vienna Proc. Natl Acad. Sci. USA 112 14197–201
[13] Ren Y et al 2016 Experimental characterization of a 400 gbit/s orbital angular momentum multiplexed free-space optical link over 120 m Opt. Lett. 41 622–5
[14] Wang J et al 2012 Terabit free-space data transmission employing orbital angular momentum multiplexing Nat. Photon. 6 488
[15] Willner A E et al 2015 Optical communications using orbital angular momentum beams Adv. Opt. Photon. 7 66–106
[16] Willner A E et al 2015 High-dimensional quantum cryptography with twisted light New J. Phys. 17 033033
[17] Villoresi P 2014 Free-space quantum key distribution by rotation-invariant twisted photons Phys. Rev. Lett. 113 060503
[18] Doster T and Watnik A T 2017 High-dimensional intracity quantum cryptography with structured photons Optica 4 1006–10
[19] Doster T and Watnik A T 2017 Machine learning approach to OAM beam demultiplexing via convolutional neural networks Appl. Opt. 56 3586–96

Figure C1. Analytically calculated diffraction patterns and intensity study for pure and SOV-tailored CGHs. (a) Calculated diffraction pattern for a $\Psi_{-3,4}$ state and respective integrated signals for a pure CGH, exhibiting a pronounced $c_{14}$-fold signal $D^{(inter)}$. (b) Same as (a) but for an SOV-tailored CGH, which avoids the $c_{14}$-fold contribution and increases the target states integrated intensities by about 27% compared to the pure ones. (c) Intensity ratio $Q(J)$ for several numbers of states $J$ according to equation (C.8) (green line) together with the simulation (blue squares) and the measurement (red circles). The insets show the respective pure and tailored CGHs (bottom) and the measured SOVs (top).
[19] Liu Z, Yan S, Liu H and Chen X 2019 Superhigh-resolution recognition of optical vortex modes assisted by a deep-learning method Phys. Rev. Lett. 123 183902
[20] Gong L, Zhao Q, Zhang H, Hu Y-Y, Huang K, Yang J-M and Li Y-M 2019 Optical orbital-angular-momentum-multiplexed data transmission under high scattering Light Sci. Appl. 8 1–11
[21] Huang H et al 2014 100 bit/s free-space data link enabled by three-dimensional multiplexing of orbital angular momentum, polarization, and wavelength Opt. Lett. 39 197–200
[22] Trichili A, Park K-H, Zghal M, Ooi B S and Alouini M-S 2019 Communicating using spatial mode multiplexing: potentials, challenges, and perspectives IEEE Commun. Surv. Tutorials 21 3175–203
[23] Baghdady J, Miller K, Morgan K, Byrd M, Oser S, Ragusa R, Li W, Cochenour B M and Johnson E G 2016 Multi-gigabit/s underwater communication link using orbital angular momentum multiplexing Opt. Express 24 9794–805
[24] Ren Y et al 2016 Orbital angular momentum-based space division multiplexing for high-capacity underwater optical communications Sci. Rep. 6 33306
[25] Li H, Phillips D B, Wang X, Ho Y-L D, Chen L, Zhou X, Zhu J, Yu S and Cai X 2015 Orbital angular momentum vertical-cavity surface-emitting lasers Optica 2 547–52
[26] Wang X, Nie Z, Liang Y, Wang J, Li T and Jia B 2018 Recent advances on optical vortex generation Nanophotonics 7 1533–56
[27] Stoyanov L, Topuzoski S, Stefanov I, Janicijevic L and Dreischuh A 2015 Far field diffraction of an optical vortex beam by a fork-shaped grating Opt. Commun. 350 301–8
[28] Carpentier A V, Michinel H, Salgueiro J R and Olivieri D 2008 Making optical vortices with computer-generated holograms Am. J. Phys. 76 916–21
[29] Xie Z et al 2018 Ultra-broadband on-chip twisted light emitter for optical communications Light Sci. Appl. 7 18001
[30] Franke-Arnold S, Allen L and Padgett M 2008 Advances in optical angular momentum Laser Photon. Rev. 2 299–313
[31] Simpson N B, Allen L and Padgett M J 1996 Optical tweezers and optical spanners with Laguerre–Gaussian modes J. Mod. Opt. 43 2485–91
[32] Liesener J, Reichter M, Hais t T and Tiziani H J 2000 Multi-functional optical tweezers using computer-generated holograms Opt. Commun. 185 77–82
[33] Li M, Yan S, Liang Y, Zhang P and Yao B 2018 Spinning of particles in optical double-vortex beams J. Opt. 20 025401
[34] Ashkin A and Dziedzic J 1987 Optical trapping and manipulation of viruses and bacteria Science 235 1517–20
[35] Bezuhanov K, Dreischuh A, Paulus G G, Schaitel M G and W alther H 2004 Vortices in femtosecond laser fields Opt. Lett. 29 1942–4
[36] Zürch M, Kern C, Hansinger P, Dreischu a and Spielmann C 2012 Strong-field physics with singular light beams Nat. Phys. 8 743–6
[37] Krenn M, Fickler R, Fink M, Handsteiner J, Malik M, Scheidl T, Ursin R and Zeilinger A 2014 Communication with spatially modulated light through turbulent air across Vienna New J. Phys. 16 113028
[38] Krenn M, Handsteiner J, Fink M, Fickler R, Ursin R and Zeilinger A 2016 Communication with spatially modulated light through turbulent air across Vienna New J. Phys. 16 113028
[39] Bazhenov V Y, Vlasetsov M V and Soksin M S 1990 Laser beams with screw dislocations in their wavefronts JETP Lett. 52 429–31
[40] Trichili A, Rosales-Guzmán C, Dudley A, Ndagano B, Salem A B, Zghal M and Forbes A 2016 Optical communication beyond orbital angular momentum Sci. Rep. 6 27674
[41] Rosales-Guzmán C, Buhbne N and Forbes A 2017 Simultaneous generation of multiple vector beams on a single sdm Opt. Express 25 25697–706
[42] Rosales-Gu zmán C, Ndagano B and Forbes A 2018 A review of complex vector light fields and their applications J. Opt. 20 123001
[43] Eickhoff K, Rathie C, Köhne D, Kerbstad t S, Englert L, Bayer T, Schäfer S and Wollenhaupt M 2020 Orbital angular momentum superposition states in transmission electron microscopy and bichromatic multiphoton ionization New J. Phys. 22 103045
[44] Harvey T R, Pierce J S, Agrawal A K, Ercius P, Linck M and McMorr an B J 2014 Efficient diffractive phase optics for electrons New J. Phys. 16 093039
[45] McMorr an B J, Harvey T R and Lavery M J P 2017 Efficient sorting of free electron orbital angular momentum New J. Phys. 19 023053
[46] Johnson C W, Pierce J S, Moraski R C, Turner A E, Greenberg A T, Parker W S and McMorr an B J 2020 Exact design of complex amplitude holograms for producing arbitrary scalar fields Opt. Express 28 17334–46
[47] Zhao H, Quan B, Wang X, Gu C, Li J and Zhang Y 2017 Demonstration of orbital angular momentum multiplexing and demultiplexing based on a metasurface in the terahertz band ACS Photon. Sci. 3 1726–32
[48] Torres J P and Torner L 2011 Twisted Photons: Applications of Light with Orbital Angular Momentum (New York: Wiley)
[49] Gabor D 1972 Holography, 1948–1971 Science 177 299–313
[50] Träger F and Träger F 2012 Short and Ultrashort Laser Pulses vol 2 (Berlin: Springer) pp 1047–94
[51] Kerbstad t S, Englert L, Bayer T and Wollenhaupt M 2017 Ultrashort polarization-tailored bichromatic fields J. Mod. Opt. 64 1010
[52] Saleh B E A and Teich M C 1991 Fundamentals of Photonics (New York: Wiley)
[53] Juchtmans R, Guzzinati G and Verbeeck J 2016 Extension of Friedel’s law to vortex-beam diffraction Phys. Rev. A 94 033858
[54] Baddour N 2011 Two-dimensional Fourier Transforms in Polar Coordinates vol 165 (Amsterdam: Elsevier) pp 1–45
[55] Baddour N 2009 Operational and convolution properties of two-dimensional Fourier transforms in polar coordinates J. Opt. Soc. Am. A 26 1767–77
[56] Bliokh K Y et al 2017 Theory and applications of free-electron vortex states Phys. Rep. 690 1–70
[57] Berry M V 2009 Optical currents J. Opt. A: Pure Appl. Opt. 11 094001
[58] Kerbstad t S, Eickhoff K, Bayer T and Wollenhaupt M 2020 Bichromatic Control of Free Electron Wave Packets (Berlin: Springer) pp 43–76
[59] Bayer T, Philipp C, Eickhoff K and Wollenhaupt M 2020 Atomic photoionization dynamics in ultrashort cycloidal laser fields Phys. Rev. A 102 013104
[60] Weiner A M 2000 Femtosecond pulse shaping using spatial light modulators Rev. Sci. Instrum. 71 1929–60
[61] Mommaryant A, Weber S and Chatel B 2010 A newcomer’s guide to ultrashort pulse shaping and characterization J. Phys. B: At. Mol. Opt. Phys. 43 103001–34
[62] Goodman J W 1996 Introduction to Fourier Optics vol 2 (New York: McGraw-Hill)
[63] Bolduc E, Bent N, Santamato E, Karimi E and Boyd R W 2013 Exact solution to simultaneous intensity and phase encryption with a single phase-only hologram Opt. Lett. 38 3546–9
[64] Rulliere C 2005 Femtosecond Laser Pulses. Principles and Experiments (Berlin: Springer)
[65] Maji S, Jacob P and Brundavanam M M 2019 Geometric phase and intensity-controlled extrinsic orbital angular momentum of off-axis vortex beams Phys. Rev. Appl. 12 054053
[66] Guzzinati G, Schattschneider P, Bliokh K Y, Nori F and Verbeeck J 2013 Observation of the Larmor and Gouy rotations with electron vortex beams Phys. Rev. Lett. 110 093601
[67] Köhler J, Wollenhaupt M, Bayer T, Sarpe C and Baumert T 2011 Zeptosecond precision pulse shaping Opt. Express 19 11638–53
[68] Winter M, Wollenhaupt M and Baumert T 2006 Coherent matter waves for ultrafast laser pulse characterization Opt. Commun. 264 285–92
[69] Goda K and Jalali B 2013 Dispersive Fourier transformation for fast continuous single-shot measurements Nat. Photon. 7 102
[70] Shutova M, Zhdanova A A and Sokolov A V 2017 Detection of mixed OAM states via vortex breakup Phys. Lett. A 381 408–12
[71] Kotlyar V V, Kovalev A A and Porfirov A P 2017 Astigmatic transforms of an optical vortex for measurement of its topological charge Appl. Opt. 56 4095–104
[72] Friberg A T 1996 Stationary-phase analysis of generalized axicons J. Opt. Soc. Am. A 13 743–50
[73] Ngoko Djiokap J M, Hu S X, Madsen L B, Manakov N L, Meremianin A V and Starace A F 2015 Electron vortices in photoionization by circularly polarized attosecond pulses Phys. Rev. Lett. 115 113004
[74] Pengel D, Kerbstadt S, Johannmeyer D, Englert L, Bayer T and Wollenhaupt M 2017 Electron vortices in femtosecond multiphoton ionization Phys. Rev. Lett. 118 053003
[75] Pengel D, Kerbstadt S, Englert L, Bayer T and Wollenhaupt M 2017 Control of three-dimensional electron vortices from femtosecond multiphoton ionization Phys. Rev. A 96 043426
[76] Zuo J M and Spence J C H 2017 Advanced Transmission Electron Microscopy (Berlin: Springer)
[77] Solli D R, Chou J and Jalali B 2008 Amplified wavelength-time transformation for real-time spectroscopy Nat. Photon. 2 48–51
[78] Maji S and Brundavanam M M 2018 Topological transformation of fractional optical vortex beams using computer generated holograms J. Opt. 20 045607
[79] Pfeiffer A N, Cirelli C, Smolarski M and Keller U 2013 Recent attoclock measurements of strong field ionization Chem. Phys. 414 84–91
[80] Eickhoff K, Kerbstadt S, Bayer T and Wollenhaupt M 2020 Dynamic quantum state holography Phys. Rev. A 101 013430
[81] Cuyt A A M, Petersen V, Verdronk B, Waandel H and Jones W B 2008 Handbook of Continued Fractions for Special Functions (Berlin: Springer)