MULTIPLICITY RESULTS FOR NONHOMOGENEOUS ELLIPTIC EQUATIONS WITH SINGULAR NONLINEARITIES

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ABSTRACT. This paper is concerned with the study of multiple positive solutions to the following elliptic problem involving a nonhomogeneous operator with nonstandard growth of $p$-$q$ type and singular nonlinearities

\[
\begin{aligned}
- \mathcal{L}_{p,q}u &= \lambda \frac{f(u)}{u^\gamma}, & u > 0 & \text{in } \Omega, \\
& & u = 0 & \text{on } \partial \Omega,
\end{aligned}
\]

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with $C^2$ boundary, $N \geq 1$, $\lambda > 0$ is a real parameter,

\[\mathcal{L}_{p,q}u := \text{div}(|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u),\]

$1 < p < q < \infty$, $\gamma \in (0, 1)$, and $f$ is a continuous nondecreasing map satisfying suitable conditions. By constructing two distinctive pairs of strict sub and super solution, and using fixed point theorems by Amann, we prove existence of three positive solutions in the positive cone of $C_0^\infty(\Omega)$ and in a certain range of $\lambda$.

Key words: singular nonlinearities, nonstandard growth, $p$-$q$ Laplacian, multiplicity results, infinite positone problems, fixed point theorems.

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1. INTRODUCTION

1.1. State of the art. The study of problems involving nonhomogeneous operators of $\mathcal{L}_{p,q}$ type emerges from the work of Zhikov [42, 43] where the models of strongly anisotropic materials were considered in the context of homogenization. In particular, Zhikov considered
the following model of functional in relationship to the Lavrentiev phenomenon

\[ I(u) := \int_{\Omega} |\nabla u|^p + r(x)|\nabla u|^q \, dx, \quad 0 \leq r(x) \leq L, \quad 1 < p < q \]

where the modulating coefficient \( r(x) \) dictates the geometry of the composite made up by two materials with hardening exponent \( p \) and \( q \) respectively.

According to Marcellini’s terminology \[36, 37\], the functional \( I \) falls into the following category of so-called functionals with nonstandard growth of \( p-q \) type

\[ u \mapsto - \int_{\Omega} f(x, \nabla u) \, dx \]

where the energy density function \( f \) satisfies

\[ |t|^p \leq f(x, t) \leq 1 + |t|^q, \quad 1 \leq p \leq q. \]

The pioneering contributions of Marcellini \[36, 37\] and a series of remarkable papers by Mingione et al. \[11, 12\] have brought to light the subject of studying such functionals with nonstandard growth, and have subsequently motivated many works involving the equations and systems with a nonhomogeneous operator of \( L_{p,q} \) type.

In this paper, we study the existence and multiplicity results of the following problem involving the nonhomogeneous operator \( L_{p,q} \) with nonstandard growth of \( p-q \) type and singular nonlinearities

\[
(P_\lambda) \begin{cases} 
- L_{p,q} u = \lambda \frac{f(u)}{u^\gamma}, & u > 0 \quad \text{in } \Omega, \\
 u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with \( \partial \Omega \in C^2 \), \( N \geq 1 \), \( \lambda > 0 \), \( \gamma \in (0, 1) \),

\[ L_{p,q}(u) := \text{div}(|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u), \quad 1 < p < q < \infty. \]

The differential operator \( L_{p,q} \) is also known as \((p,q)\)-Laplacian operator and appears in a variety of physical models such as reaction diffusion systems, elasticity theory, quantum physics and transonic flows. The singular problem \((P_\lambda)\) is also known as infinite positone problem (see \[30\]). For a more comprehensive description of applications, we refer to the work \[5, 6, 10\], a survey article \[35\] and its references.

The study of elliptic equations with singular nonlinearities started mainly with the seminal work of Crandall, Rabinowitz and Tarter \[13\]. Later on, much attention has been paid to the subject, leading to an abundant literature investigating a large spectrum of issues (see \[13, 16, 17\]) and surveys \[19, 27\]. We cite here some related work with no intent to furnish an exhaustive list. For \( p = 2 \), multiplicity results for the equation involving singular nonlinearities have been dealt in \[25, 29\] for critical nonlinearity and in \[14\] for exponential nonlinearity. In \[15\], authors proved existence of three solutions for the problem \((P_\lambda)\), under the assumption of existence of two different pairs of sub and super solutions. For the quasilinear case i.e. \( p \neq 2 \), Giacomoni et al. \[22\] have studied the problem \((P_\lambda)\) when \( f(u) \equiv \text{Cst} \) and perturbed with subcritical and critical nonlinearities. Using variational methods, they proved the existence of multiple solutions in \( C^{1,\alpha}(\Omega) \) when \( \gamma \in (0, 1) \), and figured out the boundary behavior of weak solutions by constructing suitable sub and super solutions. In \[30\], Ko et al. studied the problem \((P_\lambda)\) and for a certain range of \( \lambda \), they
proved the existence of two solutions via constructing two pairs of sub and super solutions. For more contrasting results with different summability conditions on \( f \), we refer to the work \([7, 9, 20, 31]\) and their references within. For a further detailed review of elliptic equations involving singular nonlinearities we refer to the monograph \([18]\) and the overview article \([28]\).

Turning to the equations involving the nonhomogeneous operators and singular nonlinearities, in particular, the operator \( L_{p,q} \) have been recently investigated in \([20, 31]\) and \([38, 40]\). In \([38, 31]\) the authors studied the following singular problem

\[
-L_{p,q} u = \frac{g(x)}{u^\gamma}, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ in } \partial \Omega \tag{1.1}
\]

perturbed with subcritical and critical growth nonlinearities respectively. Precisely, for \( g(x) = \lambda \) and \( \gamma \in (0,1) \), Papageorgiou et al. \([38]\) proved bifurcation type theorem via variational methods, and Kumar et al. \([31]\) proved the existence of at least two solutions by splitting the Nehari manifold set. Very recently, in \([20]\), Giacomoni et al. proved Sobolev and Hölder regularity results for the minimal weak solution \( u \) of the problem \((1.1)\). The study of three solutions for the singular problem involving nonhomogeneous operator with unbalanced growth of \( p-q \) type and singular nonlinearities was completely open till now, even in the case of \( p = q \). In this regard our work brings new results. To achieve the goal, we use the fixed point theorem by Amann \([1]\) by constructing two different pairs of strict sub and super solutions. For more results concerning the singular problem for nonlocal and nonlinear operators, we refer to \([2, 3]\) and their reference within.

The main difficulty here is because of the singular nature of the reaction term near the boundary which, in turn, prevents the operator associated to \((P_\lambda)\) to be monotone. To handle this, we transform our problem to an equivalent problem by absorbing the singularity into the operator. Precisely, we formulate a new problem \((\hat{P}_\lambda)\) (see Page \([11]\)) and prove the operator \( \hat{T} \) (see Page \([11]\)) associated to it, is completely continuous, increasing and has strict invariant property. Similar type of ideas are used by Dhanya et al. \([15]\) in the local case \( i.e. \ p = q = 2 \), and Giacomoni et al. \([21]\) in the nonlocal case. They have showed the operator \( \hat{T} \) is strongly increasing with the help of strong comparison principle in both local and nonlocal cases, respectively. But here due to the nonlinearity and nonhomogeneity of the operator \( L_{p,q} \), their approach cannot be applied to the problem \((\hat{P}_\lambda)\) (most substantially \([15, Theorem 3.6]\) and \([21, Theorem 4.7]\)).

To overcome these issues, we explicitly provide the construction of two distinctive pairs of strict sub and super solutions \((u_0, v_0), (v_0, u_0)\), which by its own nature and a comparison principle in \([35, Proposition 6]\) implies the strict invariant property of the map \( \hat{T} \) in suitable retracts of \( C^0_\delta(\Omega) \) (see page \([14]\)). Unlike the case of \( p \)-Laplacian, the non-homogeneous nature of the principal operator \( L_{p,q} \) does not allow us to use the scaling of eigenfunctions of \((-\Delta)_p\) for the construction of sub and super solution. This necessitates to look for different choices of scaling function in the construction process. The non-trivial task of finding scaling function is carried out by analyzing the qualitative properties of the weak solution of the singular problem involving a doubly parametrized nonhomogeneous operator \( L_{p,q}^{\alpha,\beta} \) (see problem \((P_{\alpha,\beta})\)). By considering the nonhomogeneous property of principal operator \( L_{p,q} \) and new types of scaling of the weak solution of the problem \((P_{\alpha,\beta})\), with appropriate choices of the parameters \( \alpha \) and \( \beta \) in \( L_{p,q}^{\alpha,\beta} \), the existence of first pair of the sub and super
solution is proved. The existence of second pair of sub and super solution is derived by studying a related non-singular problem in a ball $B(0, R)$ of radius $R$ inscribed in $\Omega$ and by borrowing some ODE techniques from \[30\] to suitably extending it to $\Omega$. Lastly, the main result of this work is accomplished by using the fixed point theorem by Amann \[1\].

1.2. Functional spaces and description of main results. We start by introducing some functional spaces and notations which are used throughout the text. Assume $\Omega$ is a bounded domain in $\mathbb{R}^N$ with $C^2$ boundary, $N \geq 1$ and denote

$$\delta(x) := \inf_{y \in \partial \Omega} |x - y|$$ and $\Omega_\nu := \{x \in \Omega : \text{dist}(x, \partial \Omega) < \nu\}$ for some $\nu > 0$.

For a given positive function $\rho \in C_0(\overline{\Omega})$,

$$C_\rho(\overline{\Omega}) := \{u \in C_0(\overline{\Omega}) : \text{there exists a } c \geq 0 \text{ such that } |u(x)| \leq c \rho(x) \text{ for all } x \in \Omega\}$$

equipped with the norm

$$\|u\|_{C_\rho(\overline{\Omega})} := \left\| \frac{u}{\rho} \right\|_{L^\infty(\Omega)}$$

is a Banach space. Furthermore, $C_\rho(\overline{\Omega})$ is an ordered Banach space (OBS) whose associated positive cone $C_\rho(\overline{\Omega})^+ := \{u \in C_\rho(\overline{\Omega}) : u(x) \geq 0 \text{ for all } x \in \Omega\}$ has nonempty interior and normal (see \[1\] Theorem 1.5). We also define a open convex subset of the positive cone $C_{\rho(\overline{\Omega})}^+$ as

$$C_{\rho(\overline{\Omega})}^+ := \left\{ u \in C_{\rho(\overline{\Omega})}^+ : \inf_{x \in \Omega} \frac{u(x)}{\rho(x)} > 0 \right\}.$$

For $1 < p < q < \infty$ and $\vartheta > 0$, set

$$\mathcal{F}(\vartheta) := \frac{q\vartheta}{2C(N,q)} \min \left\{ 1, \left( \frac{q\vartheta}{2C(N,q)} \right)^{\frac{p-q}{p-1}} \right\} \text{ and } C(N,q) := \left( \frac{(N + q - 1)N + q - 1}{N^N} \right)^{\frac{1}{q-1}}.$$

We impose the following assumptions on the function $f$:

(f$_0$) $f \in C^1([0, \infty))$ such that $f(0) > 0$.

(f$_1$) $f$ is nondecreasing in $\mathbb{R}^+$.

(f$_2$) $\lim_{t \to \infty} \frac{f(t)}{t^p} = 0$.

(f$_3$) There exists $\theta_1, \theta_2 > 0$ such that $0 < \theta_1 < \min\{\theta_2, \mathcal{F}(\theta_2)\}$ and $\frac{f(t)}{t^p}$ is nondecreasing in $(\theta_1, \theta_2)$.

We define a function $\hat{f}$ on $[0, \infty)$ as

$$\hat{f}(t) = \begin{cases} \lambda \left( \frac{f(t) - f(0)}{t^p} \right) & \text{if } t \neq 0, \\ 0 & \text{if } t = 0, \end{cases}$$

and assume

(f$_4$) There exists a constant $\hat{k}$ such that $\hat{f}(t) + \hat{k}t$ is increasing in $[0, \infty)$.

**Example:** The function $f$ defined by $f(t) = \exp\left( \frac{kt}{t + t} \right)$ for any $t \geq 0$ with $k \gg 1$ satisfy assumptions (f$_0$)-(f$_4$).

The notion of weak solution for the problem $(P_\lambda)$ is understood in the following sense:

**Definition 1.1.** A function $u \in W^{1,q}_0(\Omega)$ is said to be a weak sub solution (or super solution) of the problem $(P_\lambda)$ if
(1) for every $K \in \Omega$ there exists a constant $c_K > 0$ such that $\inf_K u(x) \geq c_K$ and $f(u) \in L^q_{\text{loc}}(\Omega)$.

(2) for every $\xi \in W := \bigcup_{\Omega^\prime \in \Omega} W^{1,q}_0(\Omega^\prime)$, $\xi \geq 0$ following holds

$$
\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \xi + |\nabla u|^{q-2} \nabla u \cdot \nabla \xi \, dx \leq (\text{or } \geq) \lambda \int_{\Omega} \frac{f(u)}{u^\gamma} \xi \, dx.
$$

(1.2)

A function $u \in W^{1,q}_0(\Omega)$ which is both weak sub solution and weak super solution of $(P_\lambda)$ is called a weak solution of $(P_\lambda)$.

**Remark 1.1.** If $u(x) \geq c \delta(x)$ for some $c > 0$ and $f(u) \in L^q(\Omega)$ then by using Hardy’s inequality, the set of test functions in [1,2] can be extended from $W$ to $W^{1,q}_0(\Omega)$.

For the singular problem $(P_\lambda)$, first we prove the following existence result under the weaker assumption $(f_2)'$:

$$(f_2)' \lim_{t \to 0} \frac{f(t)}{t^{1+\gamma}} = 0.$$

**Theorem 1.1.** Let $f$ satisfies $(f_0)$-$\text{(f}_1)$, $(f_2)'$ and $(f_4)$. Then, for every $\lambda > 0$, the problem $(P_\lambda)$ has a minimal weak solution in $W^{1,q}_0(\Omega)$. Furthermore, $u \in C^{1,1}(\overline{\Omega}) \cap C(\overline{\Omega})^+$ for some $l \in (0,1)$.

Secondly, we prove the following multiplicity result for the singular problem $(P_\lambda)$.

**Theorem 1.2.** Let $f$ satisfies $(f_0)$-$\text{(f}_4)$. Then there exists constants $0 < \lambda_* < \lambda^*$ such that for every $\lambda \in [\lambda_*, \lambda^*]$, the problem $(P_\lambda)$ has at least three solutions $u_i \in W^{1,q}_0(\Omega)$. Additionally, $u_i \in C^{1,1}(\overline{\Omega}) \cap C(\overline{\Omega})^+$ for some $l \in (0,1)$ and $i = 1, 2, 3$.

Turning to the layout of the paper, in section 2 we explicitly construct two pairs of sub and supersolution for the singular problem $(P_\lambda)$. In section 3.1 we prove our existence (Theorem 1.1) and multiplicity results (Theorem 1.2).

2. Construction of Strict Sub and Super Solutions Pairs

To construct the pairs of strict sub and super solution, first we investigate the following problem involving a doubly parametrized nonhomogeneous operator $-\mathcal{L}_{p,q}^{\alpha,\beta}$ and singular nonlinearities:

$$
(P_{\alpha,\beta}) \begin{cases} 
-\mathcal{L}_{p,q}^{\alpha,\beta} u = \frac{\lambda}{u^\gamma}, & u > 0 \text{ in } \Omega, \\
0 & u = 0 \text{ on } \partial\Omega.
\end{cases}
$$

where $\mathcal{L}_{p,q}^{\alpha,\beta} u := \text{div}(\alpha|\nabla u|^{p-2}\nabla u + \beta|\nabla u|^{q-2}\nabla u)$ for $\alpha, \beta > 0$ and $1 < p < q < \infty$.

We start by stating the following existence and regularity result for the problem $(P_{\alpha,\beta})$ whose detailed proof is presented in Appendix.

**Lemma 2.1.** For every $\alpha, \beta > 0$ and $\gamma \in (0,1)$, there exists a unique minimal weak solution $u_{\alpha,\beta} \in W^{1,q}_0(\Omega)$ of the problem $(P_{\alpha,\beta})$. Furthermore, $u_{\alpha,\beta} \in C(\overline{\Omega})^+ \cap C^{1,1}(\overline{\Omega})$ for some $l \in (0,1)$.

**Theorem 2.1.** Let $f$ satisfies $(f_0)$-$\text{(f}_1)$ and $(f_2)'$. Then, for any $\lambda > 0$, there exists a pair of strict weak sub solution and weak super solution $(u_0, u^0)$ of the problem $(P_\lambda)$. In addition, $u_0, u^0 \in C^{1,1}(\overline{\Omega}) \cap C(\overline{\Omega})^+$ for some $l \in (0,1)$.
Proof. First, we construct our subsolution \( u_0 \). For \( \eta > 0 \), let \( w_\eta \in C_0^1(\Omega) \cap C_\delta(\Omega)^+ \) be the unique weak solution of the problem

\[
\begin{cases}
-\mathcal{L}_{p,q} w_\eta = \eta, \quad w_\eta > 0 & \text{in } \Omega, \\
w_\eta = 0 & \text{on } \partial \Omega.
\end{cases}
\]

The existence of the unique solution \( w_\eta \) for \( \eta \in (0, 1) \) can be proved via Proposition 3.1 such that \( w_\eta \in C_\delta(\Omega)^+ \) and \( w_\eta \to 0 \) in \( C_0^1(\Omega) \) as \( \eta \to 0^+ \). Since (f0) holds, we can choose \( \eta = \eta(\lambda) \) small enough such that \( \eta \) and \( w_\eta \) satisfies

\[
\eta \leq \frac{\lambda f(w_\eta)}{2} \quad \text{in } \Omega.
\]

Hence, by defining \( u_0 := w_\eta \) as first strict weak sub solution, we obtain, \( u_0 \) satisfies

\[-\mathcal{L}_{p,q} u_0 \leq \frac{\lambda f(u_0)}{2} \leq \frac{\lambda f(u_0)}{u_0} - \chi^\# \quad \text{weakly in } \Omega\]

where \( 0 < \chi^\# \leq \frac{\lambda}{2} \min_{x \in \Omega} f(w_\eta)(x) w_\eta^{-\gamma}(x) \).

For the construction of supersolution \( u^0 \), let \( u_\alpha := u_{\alpha,1} \) be the solution of the problem

\[-\mathcal{L}_{p,q}^\alpha u_\alpha = \frac{2}{u_\alpha^{q}}, u_\alpha > 0 \text{ in } \Omega \text{ and } u_\alpha = 0 \text{ on } \partial \Omega
\]

for some \( \alpha > 0 \). From Remark 4.1, we know that \( \|u_\alpha\|_{L^\infty(\Omega)} = C(\alpha) \) and \( \lim_{\alpha \to 0} C(\alpha) < \infty \), and \( f \) satisfy (f2)', so we choose \( \alpha = \alpha_s^{q-\gamma} \) for \( \alpha_s > 0 \) large enough such that

\[
\frac{f(\alpha_s \|u_\alpha\|_{L^\infty(\Omega)})}{(\alpha_s \|u_\alpha\|_{L^\infty(\Omega)})^{q+\gamma-1}} \leq \frac{1}{\lambda \|u_\alpha\|_{L^\infty(\Omega)}}.
\]

Define \( u^0 := \alpha_s u_\alpha \). Then,

\[-\mathcal{L}_{p,q} u^0 = -\text{div}(\|\nabla u^0\|^{p-2} \nabla u^0 + |\nabla u^0|^{q-2} \nabla u^0) = \alpha_s^{q-1}(-\text{div}(\|\nabla u_\alpha\|^{p-2} \nabla u_\alpha)) + \alpha_s^{q-1}(-\text{div}(\|\nabla u_\alpha\|^{q-2} \nabla u_\alpha))
\]

\[
= \alpha_s^{q-1}(-\text{div}(\|\nabla u_\alpha\|^{p-2} \nabla u_\alpha)) + \alpha_s^{q-1}(-\text{div}(\|\nabla u_\alpha\|^{q-2} \nabla u_\alpha))
\]}

\[
\geq \alpha_s^{q-1}(-\mathcal{L}_{p,q}^\alpha u_\alpha) \geq \frac{2\alpha_s^{q-1}}{u_\alpha^q} = \frac{2\alpha_s^{q-1}}{(u^0)^\gamma}.
\]

Since (f1)-(f2)' holds and \( \alpha_s \) satisfies (2.2), therefore we get

\[
\mathcal{L}_{p,q} u^0 \geq \frac{\alpha_s^{q-1+\gamma}}{(u^0)^\gamma} + \chi^\# \geq \frac{\lambda f^*(\alpha_s \|u_\alpha\|_{L^\infty(\Omega)})}{(u^0)^\gamma} + \chi^\# \geq \frac{\lambda f(u^0)}{(u^0)^\gamma} + \chi^\# \quad \text{in } \Omega.
\]

where \( 0 < \chi^\# \leq \alpha_s^{q-1} \min_{x \in \Omega} u_\alpha^{-\gamma}(x) \).

Denote \( R \) be the radius of the ball \( B(0, R) \) inscribed in \( \Omega \) with \( R \leq 1 + \frac{N}{q-1} \) and let \( \vartheta^* \in (0, \vartheta_1] \) such that \( \bar{f}(\vartheta^*) = \min_{0 < t \leq \vartheta^*} \frac{f(t)}{2t} \) and define \( h \in C([0, \infty)) \) given by

\[
h(t) = \begin{cases}
\bar{f}(\vartheta^*) & \text{if } t \leq \vartheta^*, \\
\frac{f(t)}{2t} & \text{if } t \geq \vartheta_1,
\end{cases}
\]

so that the function \( h \) is nondecreasing on \([0, \vartheta_1]\) and \( h(t) \leq \frac{f(t)}{2t} \) for \( t \geq 0 \). Set \( \epsilon := \frac{NR}{N+q-1} \).
Theorem 2.2. There exists \( \lambda_*, \lambda^* > 0 \) and a function \( \phi \in C^1(B(0, R)) \) such that for every \( \lambda \in [\lambda_*, \lambda^*] \) the function \( \phi \) satisfies

\[
\begin{align*}
- \mathcal{L}_{p,q}(x) &\leq \lambda h(\phi), \quad \phi > 0 \quad \text{in } B(0, R), \\
\phi &\equiv 0 \quad \text{on } \partial B(0, R),
\end{align*}
\]

and \( \| \phi \|_{L^\infty(B(0,R))} \in [\vartheta_1, \vartheta_2] \). (2.3)

Proof. For \( \chi, \kappa > 1 \), define \( \Upsilon : [0, R] \to [0, 1] \) by

\[
\Upsilon(r) = \begin{cases} 
1 & \text{if } r \in [0, \epsilon], \\
1 - \left(1 - \left(\frac{R-r}{R-\epsilon}\right)^\chi\right) & \text{if } r \in (\epsilon, R].
\end{cases}
\]

Let \( \vartheta \in (\vartheta_1, \min\{\vartheta_2, \mathcal{F}(\vartheta_2)\}) \) (see (f3)) and define \( v(r) = \vartheta \Upsilon(r) \) such that \( |v'(r)| \leq \vartheta \frac{\chi}{R-\epsilon} \).

Now we prove that there exists a radially symmetric solution \( \Phi \in C^1(B(0, R)) \) of the following problem:

\[
(P_{sym}) \begin{cases} 
- \mathcal{L}_{p,q}(x) = \lambda h(v(|x|)), \quad \phi > 0 & \text{in } B(0, R), \\
\phi &\equiv 0 \quad \text{on } \partial B(0, R).
\end{cases}
\]

From elementary calculations, it is easy to see that \( \phi(|x|) \) is radially symmetric solution of \( (P_{sym}) \) iff \( \Phi(1) = \phi(|x|) \) for \( |x| = r \) is the solution of the following equivalent problem:

\[
(P'_{sym}) \begin{cases} 
- \mathcal{L}_{p,q}(x)(\Phi'(r))' = \lambda r^{N-1} h(v(r)), \quad \Phi > 0 & \text{in } r \in [0, R), \\
\Phi'(0) = 0, \quad \Phi(R) = 0.
\end{cases}
\]

where \( \mathbb{L}_{p,q}(t) = |t|^{p-2} t + |t|^{q-2} t \). By integrating from 0 to \( r < R \), we get

\[
- \lambda r^{N-1} \mathbb{L}_{p,q}(\Phi'(r)) = \lambda \int_0^r t^{N-1} h(v(t)) \, dt.
\]

Using intermediate value theorem, we get the nonhomogeneous function \( \mathbb{L}_{p,q} \) is bijective, monotone and continuous, therefore \( \mathbb{L}_{p,q}^{-1} \) is well defined and continuous. Hence, we have

\[
- \Phi'(r) = \mathbb{L}_{p,q}^{-1} \left( \frac{\lambda}{r^{N-1}} \int_0^r t^{N-1} h(v(t)) \, dt \right). \quad (2.4)
\]

Now, by again integrating, we get

\[
\Phi(r) = \Phi(R) + \int_r^R \mathbb{L}_{p,q}^{-1} \left( \frac{\lambda}{s^{N-1}} \int_0^s t^{N-1} h(v(t)) \, dt \right) \, ds. \quad (2.5)
\]

By taking \( \Phi(R) = 0 \) and using the fact that

\[
\Phi'(0) = \lim_{r \to 0} \Phi'(r) = - \lim_{r \to 0} \mathbb{L}_{p,q}^{-1} \left( \frac{\lambda}{r^{N-1}} h(\vartheta) \int_0^r t^{N-1} \, dt \right) = - \lim_{r \to 0} \mathbb{L}_{p,q}^{-1} \left( \frac{\lambda h(\vartheta)}{N} r \right) = 0
\]

implies \( \Phi \) defined in (2.5) is the solution of the problem \( (P'_{sym}) \) and \( \phi(x) = \Phi(|x|) \) is the solution of \( (P_{sym}) \). Now, we claim that there exists \( \lambda_*, \lambda^* > 0 \) such that for all \( \lambda \in [\lambda_*, \lambda^*] \) the following holds

\[
\phi(r) \geq v(r) \quad \text{for all } r \in [0, R] \quad \text{and } \| \phi \|_{L^\infty(\Omega)} \leq \vartheta_2 \quad (2.6)
\]
and which further implies that $\phi$ is the subsolution of the nonsingular problem (2.3) since $h$ is nondecreasing function in $[0, \vartheta_2]$. We observe that $\phi(R) = v(R) = 0$, so in order to prove our claim (2.6) it is enough to prove that

$$\phi'(r) \leq v'(r) \text{ for every } r \in [0, R].$$

For $r \in [0, \epsilon]$, $v'(r) = 0$ and $\phi'(r) \leq 0$, so the claim holds and for $r \in [\epsilon, R]$ we have

$$-\phi'(r) = L^{-1}_{p,q} \left( \frac{\lambda}{r^{N-1}} \int_0^r t^{N-1} h(v(t)) \, dt \right) \geq L^{-1}_{p,q} \left( \frac{\lambda}{R^{N-1}} \int_0^r t^{N-1} h(v(t)) \, dt \right)$$

$$= L^{-1}_{p,q} \left( \frac{\lambda h(\vartheta)}{R^{N-1}} \int_0^r t^{N-1} \, dt \right) = L^{-1}_{p,q} \left( \frac{\lambda h(\vartheta)}{R^{N-1}} \frac{\epsilon^N}{N} \right).$$

From the definition of the function $v$ and by the choice of $R$ and $\epsilon$, we get

$$|v'(r)| \leq \frac{\vartheta \chi \kappa}{R - \epsilon} \text{ and } \frac{2(\chi \kappa)^{q-1}}{(R - \epsilon)^{q-1} \epsilon^N} \geq \frac{1}{\epsilon} L^{-1}_{p,q} \left( \frac{\kappa \chi}{R - \epsilon} \right).$$

Hence, if we choose $\chi, \kappa$ close to $1$ and $\lambda$ such that

$$\lambda \geq \lambda_* := \max \{ \vartheta \chi \kappa, q^{-1} \} \frac{2R^{N-1} N}{(R - \epsilon)^{q-1} \epsilon^N h(\vartheta)},$$

we get $-\phi'(r) \geq -v'(r)$ and therefore the claim holds. Now, in order to prove the $L^\infty$ bound of the function $\phi$ in (2.4), we integrate the equation (2.4) for any $s \in [0, R]$. Since $\vartheta \in (\vartheta_1, \vartheta_2)$ and the function $h$ in increasing in $[0, \vartheta_2]$, therefore we have

$$\phi(s) = \int_s^R L^{-1}_{p,q} \left( \frac{\lambda}{r^{N-1}} \int_0^r t^{N-1} h(v(t)) \, dt \right) \, dr \leq \int_s^R L^{-1}_{p,q} \left( \frac{\lambda h(\vartheta)}{r^{N-1}} \int_0^r t^{N-1} \, dt \right) \, dr$$

$$= \int_s^R L^{-1}_{p,q} \left( \frac{\lambda h(\vartheta)}{N} \right) \, dr.$$
Proof. First, we construct the supersolution \( v^0 \) such that \( \|v^0\|_{L^\infty(\Omega)} \leq \vartheta_1 \) (see (f3)). Let \( u_\beta := u_{1,\beta} \) is the solution of

\[
-\mathcal{L}_{p,q}^{1,\beta} u_\beta = \frac{2}{u_\beta}, u_\beta > 0 \text{ in } \Omega \text{ and } u_\beta = 0 \text{ on } \partial \Omega.
\]

From Remark 1.1, we know that \( \|u_\beta\|_{L^\infty(\Omega)} \leq C(\beta) \) and \( \lim_{\beta \to 0} C(\beta) < \infty \), and since (f2) holds, so we choose \( \beta = \beta^* := m_\lambda^{1-p} \) small enough such that

\[
m_\lambda \|u_{\beta^*}\|_{L^\infty(\Omega)} \leq \vartheta_1 \quad \text{and} \quad m_\lambda^{p-1+\gamma} \geq \lambda f(m_\lambda C(\beta^*)).
\]

Define

\[
v^0(x) = m_\lambda u_{\beta^*} \quad \text{where } m_\lambda \text{ satisfies (2.7)}.
\]

Then by using (f1)-(f2), we obtain

\[
-\mathcal{L}_{p,q} v^0 = - \text{div}(|\nabla v^0|^{p-2} \nabla v^0 + |\nabla v^0|^{q-2} \nabla v^0)
\]

\[
= m_\lambda^{p-1} (- \text{div}(|\nabla u_{\beta^*}|^{p-2} \nabla u_{\beta^*})) + m_\lambda^{p-1} (- \text{div}(|\nabla u_{\beta^*}|^{q-2} \nabla u_{\beta^*}))
\]

\[
\geq m_\lambda^{p-1} (-\mathcal{L}_{p,q}^{1,\beta^*} u_{\beta^*}) \geq m_\lambda^{p-1} \frac{2}{u_{\beta^*}} \geq \frac{f(m_\lambda \|u_{\beta^*}\|_{L^\infty(\Omega)})}{(v^0)^\gamma} + m_\lambda^{p-1} \geq \frac{f(v^0)}{(v^0)^\gamma} + \varepsilon^*
\]

where \( 0 < \varepsilon^* \leq m_\lambda^{p-1} \min_{x \in \Omega} u_{\beta^*}^{-\gamma} \).

Now, to construct the second subsolution \( v_0 \), we consider the following nonsingular problem in \( \Omega \):

\[
(P_{NS}) \left\{ \begin{array}{ll}
-\mathcal{L}_{p,q} u = \lambda h(u), & u > 0 \quad \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{array} \right.
\]

Let \( R \) be the radius of the ball \( B(0, R) \) inscribed in \( \Omega \) and the function \( \phi \) satisfies (2.3). Herewith, extend the function \( \phi \) in \( \mathbb{R}^N \setminus B(0, R) \) and define

\[
\zeta(x) = \begin{cases} 
\phi(x) & \text{if } x \in B(0, R), \\
0 & \text{otherwise}.
\end{cases}
\]

such that \( \zeta \in W_0^{1,q}(\Omega) \cap C^1(\Omega) \) and \( \zeta \) satisfies \( -\mathcal{L}_{p,q} \zeta \leq \lambda h(\zeta) \) in \( \Omega \) and \( \zeta = 0 \) on \( \partial \Omega \). In order to obtain the strictly positive solution of the nonsingular problem \((P_{NS})\), we iterate the subsolution \( \zeta \) in the following way. Let \( \psi \) be a weak solution of the following problem

\[
(P_{p,q}) \left\{ \begin{array}{ll}
-\mathcal{L}_{p,q} \psi + \Theta_\lambda \mathbb{I}_{p,q}(\psi) = g(\zeta), & \psi > 0 \quad \text{in } \Omega, \\
\psi = 0 & \text{on } \partial \Omega
\end{array} \right.
\]

where \( g(t) = \lambda h(t) + \Theta_\lambda \mathbb{I}_{p,q}(t) \) where \( \Theta_\lambda \) is chosen in such a way that \( g \) is an increasing function for all \( t \geq 0 \). The existence of weak solution of the problem \((P_{p,q})\) can be proved by finding the minimizer of the energy functional \( J_\Theta \) defined on \( W_0^{1,q}(\Omega) \) as

\[
J_\Theta(\psi) = \int_{\Omega} \left( \frac{\nabla \psi|^p}{p} + \frac{|\nabla \psi|^q}{q} \right) dx + \Theta_\lambda \int_{\Omega} \left( \frac{|\psi|^p}{p} + \frac{|\psi|^q}{q} \right) dx - \int_{\Omega} g(\zeta) \psi dx.
\]

Since \( g(z) \in L^\infty(\Omega) \), and \( W_0^{1,q}(\Omega) \hookrightarrow W_0^{1,p}(\Omega) \cap L^q(\Omega) \), \( J_\Theta \) is continuous and coercive on \( W_0^{1,q}(\Omega) \). Therefore, there exists a global minimizer \( \psi \in W_0^{1,q}(\Omega) \) and since \( J_\Theta \in C^1 \), \( \psi \) is
the weak solution of the problem \((P_{p,q})\) in the sense that
\[
\int_{\Omega} |\nabla \psi|^{p-2} \nabla \psi \cdot \nabla \xi + |\nabla \psi|^{q-2} \nabla \psi \cdot \nabla \xi \, dx + \Theta_{\lambda} \int_{\Omega} |\psi|^{p-2} u\xi + |\psi|^{q-2} u\xi \, dx
= \int_{\Omega} g(z)\xi \, dx \quad \text{for} \quad \xi \in W^{1,q}_{0}(\Omega).
\]
Since \(g(\zeta) \geq 0\) and \(J(\psi^+) \leq J(\psi), \, \psi \geq 0\). By using elliptic regularity theory, we get \(\psi \in L^\infty(\Omega)\) (see [32, Page 286]), \(\psi \in C^{1,k}(\Omega)\) (see [31, Page 22, Theorem 1.7]) and \(\psi \in C^0(\Omega^+\) (see [39, Page 111, 120]). Now, by using the monotonicity of the operator \(\mathcal{L} + \Theta_{\lambda}^{\lambda}\mathcal{L}\) and monotonicity of the function \(h\) we get
\[
\zeta \leq \psi \quad \text{and} \quad -\mathcal{L}_{p,q}\psi + \Theta_{\lambda}^{\lambda}\mathcal{L}_{p,q}(\psi) = g(\zeta) \leq g(\psi) = \lambda h(\psi) + \Theta_{\lambda}^{\lambda}\mathcal{L}_{p,q}(\psi).
\]
Since, \(h(t) \leq \frac{f(t)}{2t^2}\) for all \(t \geq 0\), we obtain \(v_0 := \psi\) is the subsolution of the singular problem of \((P)\) for all \(\lambda \in [\lambda_*, \lambda^*]\) and satisfies
\[
-\mathcal{L}_{p,q}v_0 \leq \lambda \frac{f(v_0)}{v_0^2} - \varepsilon_* \quad \text{in} \quad \Omega
\]
where \(0 < \varepsilon_* \leq \frac{1}{2} \min_{x \in \Omega} f(\psi)(x)\psi^{-\gamma}(x)\).

3. Proof of main results

3.1. Existence and multiplicity results. Before proving our main result, we recall some definitions and results from [1].

Definition 3.1. A nonempty subset \(E\) of a metric space \(X\) is called a retract of \(X\) if there exists a continuous map \(r : X \to E\) such that \(r|_E = \text{id}|_E\).

Definition 3.2. Let \(X\) be a nonempty subset of some Banach space and \(f\) be a map from \(X\) into a second Banach space. Then \(f\) is called compact if it is continuous and if \(f(X)\) is relatively compact. The map \(f\) is called completely continuous if \(f\) is continuous and maps bounded subsets of \(X\) into compact sets.

Remark 3.1. Every nonempty closed convex subset \(E\) of a Banach space \(X\) is a retract of \(X\) and every compact map is completely continuous, and the two notions coincide if \(E\) is bounded.

Theorem 3.1 (Lemma 4.1, [1]). Let \(X\) be a retract of some Banach space and let \(f : X \to X\) be a compact map. Suppose that \(X_1\) and \(X_2\) are disjoint retracts of \(X\), and let \(U_k, k = 1, 2,\) be open subsets of \(X\) such that \(U_k \subset X_k,\) \(k = 1, 2\). Moreover, suppose that \(f(X_k) \subset X_k\) and that \(f\) has no fixed points on \(X_k \setminus U_k,\) \(k = 1, 2\). Then \(f\) has at least three distinct fixed points \(x_0, x_1, x_2\) with \(x_k \in X_k,\) \(k = 1, 2,\) and \(x_0 \in X \setminus (X_1 \cup X_2)\).

Corollary 3.1 (Corollary 6.2, [1]). Let \(X\) be an ordered Banach space and let \([y_1, y_2]\) be an ordered interval in \(X\). Let \(f : [y_1, y_2] \to X\) be an increasing compact map such that \(f(y_1) \geq y_1\) and \(f(y_2) \leq y_2\). Then \(f\) has a minimal fixed point \(x_*\) and a maximal fixed point \(x^+\).

Now, we prove our existence and multiplicity results for the problem \((P)\) via a critical point theorem in [1]. For this, we begin by introducing an equivalent formulation of our original problem \((P)\). Since \(f \in C^1([0, \infty))\), then \(\hat{f}\) can be treated as continuous function on
Using the same arguments in the proof of [31, Lemma 3.2], we obtain
\( \hat{f}(t) = \lambda f'(t_0)t^{1-\gamma} \) for some \( t_0 \in (0,t) \). Then the fact \( \lim_{t \to 0} |f'(t)| < \infty \) and \( \gamma \in (0,1) \) implies \( \hat{f}(0) = 0 \). Herewith, we formulate our equivalent problem as

\[
(\hat{P}_\lambda) \begin{cases} 
-\mathcal{L}_{p,q}u - \frac{\lambda f(0)}{u^\gamma} = \hat{f}(u), & u > 0 \quad \text{in } \Omega, \\
\quad u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

We begin by introducing the notion of weak solution of the problem \((\hat{P}_\lambda)\) as:

**Definition 3.3.** A function \( u \in W^{1,p}_0(\Omega) \) is said to be a weak solution of the problem \((\hat{P}_\lambda)\) if

1. for every \( K \subseteq \Omega \) there exists a constant \( c_K > 0 \) such that \( \inf_K u(x) \geq c_K \) and \( \hat{f}(u) \in L^1_{loc}(\Omega) \).
2. for every \( \xi \in \mathfrak{W} \), \( u \) satisfies

\[
\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla \xi + |\nabla u|^{p-2}\nabla u \cdot \nabla \xi \ dx - \lambda \int_\Omega \frac{f(0)}{u^\gamma} \xi \ dx = \int_\Omega \hat{f}(u) \xi \ dx.
\]

We extend the function \( f \) and \( \hat{f} \) in a continuous manner such that \( f(t) = f(0) \) and \( \hat{f}(t) = \hat{f}(0) \) for all \( t \leq 0 \). To use critical point theorem in [1], we introduce a map \( \hat{T} : C_0(\Omega) \to C_0(\Omega) \) as \( \hat{T}(u) = w \) if and only if \( w \) is the weak solution of

\[
(\hat{P}_{\lambda,u}) \begin{cases} 
-\mathcal{L}_{p,q}w - \frac{\lambda f(0)}{w^\gamma} = \hat{f}(u), & w > 0 \quad \text{in } \Omega, \\
\quad w = 0 & \text{on } \partial \Omega.
\end{cases}
\]  \( \tag{3.1} \)

Without loss of generality, we can assume \( \hat{k} = 0 \) in \((f_4)\) (i.e. \( \hat{f} \) is increasing on \( \mathbb{R}^+ \)). If not, then instead of \((P_{\lambda,u})\), we can study

\[
(\hat{P}_{\lambda,u}) \begin{cases} 
-\mathcal{L}_{p,q}w - \frac{\lambda f(0)}{w^\gamma} + \hat{k}w = \hat{f}(u), & w > 0 \quad \text{in } \Omega, \\
\quad w = 0 & \text{on } \partial \Omega.
\end{cases}
\]

and establish the same results for \((\hat{P}_{\lambda,u})\) by defining the map \( \hat{T}(u) = w \) iff \( w \) is a weak solution of \((\hat{P}_{\lambda,u})\).

**Proposition 3.1.** A function \( u \in W^{1,p}_0(\Omega) \cap C_0(\Omega) \cap C_0(\Omega) \) is weak solution of \((P_{\lambda})\) iff \( u \) is a fixed point of the map \( \hat{T} \).

**Proof.** Let \( u \in W^{1,p}_0(\Omega) \cap C_0(\Omega) \cap C_0(\Omega) \) is weak solution of \((P_{\lambda})\), then it implies \( u \) is the fixed point of the map \( \hat{T} \). Conversely, let \( u \) is the fixed point of the map \( \hat{T} \) i.e. \( \hat{T}(u) = u \) for some \( u \in C_0(\Omega) \), then \( u \) is the weak solution of \((P_{\lambda})\) but it remains to prove that \( u \in C_0(\Omega) \cap C_0(\Omega) \). Since \( u > 0 \) and \( f \) satisfies \((f_0)-(f_2)\) then we have

\[
-\mathcal{L}_{p,q}u = \lambda \frac{f(u)}{u^\gamma} \leq \lambda \left( \frac{f(u)}{u^\gamma} \chi_{\{|u| \leq K\}} + C(K)u^{p-1} \right) \leq C(\lambda, f, K) \left( \frac{1}{u^\gamma} + u^{p-1} \right) \quad \text{weakly in } \Omega.
\]

Using the same arguments in the proof of [31, Lemma 3.2], we obtain \( u \in L^\infty(\Omega) \). Now, let \( v_1, v_2 \in W^{1,p}_0(\Omega) \cap C_0(\Omega) \) be the weak solution of

\[
-\mathcal{L}_{p,q}v_1 = \frac{M_1}{v_1^\gamma} \quad \text{and} \quad -\mathcal{L}_{p,q}v_2 = \frac{M_2}{v_2^\gamma} \tag{3.2}
\]
for $M_1 \leq \lambda f(0)$ and $M_2 \geq \lambda f(||u||_{L^\infty(\Omega)})$. The existence of weak solutions $v_1, v_2$ can be proved via [20, Theorem 1.4]. Since $f(0) \leq f(u) \leq f(||u||_{L^\infty(\Omega)})$, then by using weak comparison principle (see [20, Theorem 1.5]) we get $C_1 \delta(x) \leq v_1(x) \leq u(x) \leq v_2(x) \leq C_2 \delta(x)$ for all $x \in \Omega$. Finally, [20, Theorem 1.7] gives $u \in C^{1,l}(\overline{\Omega})$ for some $l \in (0, 1)$.

**Proposition 3.2.** The map $\hat{T} : C_0(\overline{\Omega}) \rightarrow C_\delta(\overline{\Omega})^+$ is well defined, completely continuous and increasing.

**Proof.** To show the map $\hat{T}$ is well defined, we have to claim that for every $u \in C_0(\overline{\Omega})$, the problem $(\hat{P}_{\lambda,u})$ has a unique solution in $C_\delta(\overline{\Omega})^+$. Let $u \in C_0(\overline{\Omega})$ and $\hat{v} = \hat{f}(w)$. Then $\hat{v} \in C_0(\overline{\Omega})$ and $v \geq 0$ in $\Omega$. Let us consider the following energy functional defined on $W^{1,0}_0(\Omega)$

$$E(w) = \int_{\Omega} \left( \frac{|
abla w|^p}{p} + \frac{|
abla w|^q}{q} \right) \, dx - \lambda f(0) \int_{\Omega} (w^+)^{1-\gamma} \, dx - \int_{\Omega} \hat{v} w \, dx.$$ 

It is easy to see that the functional $E$ is coercive and weakly lower semi-continuous on $W^{1,0}_0(\Omega)$. Therefore, there exists a global minimizer $w \in W^{1,0}_0(\Omega)$ and owing to $E(0) = 0 > E(\hat{w}u)$ for every $\epsilon$ small enough and $E(w) \geq E(\hat{w}u)$, we get $w \neq 0$ and $w \geq 0$.

Now, we will prove that $w$ is the weak solution of the problem $(\hat{P}_{\lambda,u})$. Let $u_\eta \in W^{1,0}_0(\Omega) \cap C_\delta(\overline{\Omega})^+$ be the weak solution of [20, (1.4)]. Due to fact that $u_\eta \in C_\delta(\overline{\Omega})^+$, $u_\eta \rightarrow 0$ in $C_0^1(\overline{\Omega})$ as $\eta \rightarrow 0^+$ and Hardy’s inequality, we obtain that functional $E$ is Gâteaux differentiable at $u_\eta$ and hence for $\eta$ small enough

$$E'(u_\eta) = -\mathcal{L}_{p,q}u_\eta - \lambda \frac{f(0)}{u_\eta} = \eta - \lambda \frac{f(0)}{u_\eta} < 0.$$ 

Define $g : (0, 1] \rightarrow \mathbb{R}$ as $g(t) = E(w + tv)$ where $v = (u_\eta - w)^+$. Since $w + tv \geq tu_\eta$ for $t \in (0, 1]$ and $g$ is strictly convex, we get $g$ is differentiable on $(0, 1]$ and $g'$ is nonnegative and nondecreasing. Therefore, if $|\text{supp}(w)| \neq 0$

$$0 \leq g'(1) - g'(t) \leq g'(1) = E(u_\eta) < 0$$

is a contradiction. Thus, $c_\delta(x) \leq w(x)$ in $\Omega$ and $E$ is differentiable at $w$. This further implies that $w$ is the weak solution of $(\hat{P}_{\lambda,u})$. Now, to prove the upper boundary behavior and $C_0^1(\overline{\Omega})$ regularity of weak solution $w$, we construct a supersolution of the problem $(\hat{P}_{\alpha,\beta})$. Define $\overline{w} = Mu_\alpha$ where $u_\alpha := u_{\alpha,1} \in W^{1,q}_0(\Omega) \cap C_\delta(\overline{\Omega})^+$ is the unique solution of the problem $(P_{\alpha,\beta})$ with $\alpha = M^{p-q}$, $\beta = 1$ and $M$ is chosen large enough such that

$$M^{q-1} \left( \frac{1}{u_\alpha} - \lambda \frac{f(0)}{(Mu_\alpha)^\gamma} \right) \geq \hat{v}.$$ 

Then,

$$-\mathcal{L}_{p,q} \overline{w} - \lambda \frac{f(0)}{\overline{w}} = -M^{q-1} \left( \text{div}(\alpha |\nabla u_\alpha|^{p-2}\nabla u_\alpha) + \text{div}(|\nabla u_\alpha|^{q-2}\nabla u_\alpha) \right) - \lambda \frac{f(0)}{\overline{w}} = -M^{q-1} \mathcal{L}_{p,q} u_\alpha - \lambda \frac{f(0)}{(Mu_\alpha)^\gamma} = M^{q-1} \left( \frac{1}{u_\alpha} - \lambda \frac{f(0)}{(Mu_\alpha)^\gamma} \right) > \hat{v}.$$ 

Now, by using the comparison principle (with minor changes in the proof of [20, Theorem 1.5]), we get $w(x) \leq C \delta(x)$ in $\Omega$. Finally [20, Theorem 1.7] implies $w \in C^{1,l}(\overline{\Omega})$ and $\hat{T}$ maps bounded subsets of $L^\infty(\Omega)$ to bounded subsets of $C^{1,l}(\overline{\Omega})$ for some $l \in (0, 1)$. 


To prove the continuity of the operator $\hat{T}$, let $u_n$ be a bounded sequence in $C_0(\overline{\Omega})$ such that $u_n \to u$ in $C_0(\overline{\Omega})$ as $n \to \infty$. Since $f$ is a continuous function and $\hat{f}(0) = 0$, $\hat{f}(u_n) \to \hat{f}(u)$ in $C_0(\overline{\Omega})$ as $n \to \infty$. Denote $w_n = T(u_n)$ and $w := T(u)$. As we know that $w_n, w \in W^{1,q}_0(\Omega) \cap C^{1,l}(\overline{\Omega}) \cap C^3(\overline{\Omega})^+$ for some $l \in (0,1)$, then by taking $(w_n - w)$ as a test function in the weak formulation of problem $(\hat{P}_\lambda,u)$ and Lemma 4.4, we get: for $q \geq 2$

\[
C \int_\Omega |\nabla (w_n - w)|^q \, dx \leq \int_\Omega \left( |\nabla w_n|^{|p-2|} \nabla w_n + |\nabla w|^{|q-2|} \nabla w \right) \nabla (w_n - w) \, dx
\]

\[
- \lambda \hat{f}(0) \int_\Omega \left( \frac{1}{w_n^{q-1}} - \frac{1}{w^{q-1}} \right) (w_n - w) \, dx
\]

\[
= \int_\Omega \left( \hat{f}(u_n) - \hat{f}(u) \right) (w_n - w) \, dx
\]

\[
\leq \|f(u_n) - f(u)\|_{L^q(\Omega)} \|w_n - w\|_{L^q(\Omega)}
\]

\[
\leq C\|\hat{f}(u_n) - \hat{f}(u)\|_{L^q(\Omega)} \|w_n - w\|_{W^{1,q}_0(\Omega)}.
\]

Since, $w_n$ is uniformly bounded in $C^{1,l}(\overline{\Omega})$ for some $l \in (0,1)$ and by using interpolation inequality \cite{33} Corollary 1.3]) for any $\theta \in (0,1)$, we get

\[
\|w_n - w\|_{C^{1,l}(\overline{\Omega})} \leq C_1\|w_n - w\|_{C^{1+l}(\overline{\Omega})}^{1-\theta}\|w_n - w\|_{W^{1,q}_0(\Omega)}^\theta
\]

\[
\leq C_2\|\hat{f}(u_n) - \hat{f}(u)\|_{L^q(\Omega)}^{\theta}\|w_n - w\|_{W^{1,q}_0(\Omega)}^{\theta} \to 0 \text{ as } n \to \infty
\]

Since, $||\nabla w_n| + |\nabla w||_{L^q(\Omega)} \leq C_0$, $C_0$ is independent of $n$, then by again using interpolation inequality and Lemma 4.4 for $q \leq 2$, we get

\[
C_0^{-1} \int_\Omega |\nabla (w_n - w)|^2 \, dx \leq C\|\hat{f}(u_n) - \hat{f}(u)\|_{L^q(\Omega)} \|w_n - w\|_{W^{1,2}_0(\Omega)}
\]

and $\|w_n - w\|_{C^{1,l}(\overline{\Omega})} \to 0$ as $n \to \infty$. Since $C^{1,l}(\overline{\Omega}) \subseteq C^{1,l}(\overline{\Omega})$ we have $\hat{T} : C_0(\overline{\Omega}) \to C^1(\overline{\Omega})$ is completely continuous. Let $u_1, u_2 \in C^1(\overline{\Omega})^+$ such that $u_1 \leq u_2$ in $\Omega$ then there exists $s \in (0,\|u_2\|_{L^q(\Omega)})$ such that $\hat{f}(t) = \lambda f(s)^{1-\gamma}$ and $\hat{f}$ is increasing in $[0,\|u_2\|_{L^q(\Omega)}]$ and $\hat{f}(u_1) \leq \hat{f}(u_2)$. Then, by weak comparison principle we get $\hat{T}(u_1) \leq \hat{T}(u_2)$ i.e. the map $\hat{T}$ is increasing.

\textbf{Proof of Theorem 1.1} First, we prove the existence of a fixed point of map $\hat{T}$ defined in \cite{33} i.e the existence of weak solution of the equivalent problem $(\hat{P}_\lambda)$.

Let $X = [u_0, v^0]$ where $u_0, v^0$ are first pair of strict sub and super solution constructed in Theorem 2a. Using the same arguments as in Proposition 3.2 we get that the map $\hat{T} : X \to X$ is increasing, completely continuous and self invariant. By applying Corollary 3.1 we get the existence of a minimal fixed point $u_\lambda \in C^1(\overline{\Omega})^+ \cap C^{1,l}(\overline{\Omega})$ of map $\hat{T}$ in $X$ for some $l \in (0,1)$. Finally, Proposition 3.1 implies that the fixed point $u_\lambda$ is a weak solution of the problem $(\hat{P}_\lambda)$.

\textbf{Proof of Theorem 1.2} Define

$X = [u_0, u^0], \quad X_1 = [u_0, v^0], \quad X_2 = [v_0, u^0]$. 


The sets $X_i$ for each $i = 1, 2$ and $X$ form retracts of $\mathcal{C}_0(\Omega)$, since they are nonempty, closed and convex subsets of Banach space $\mathcal{C}_0(\Omega)$. The functions $u_0, u^0$ and $\mathring{T}(u_0), \mathring{T}(u^0)$ satisfies the following inequalities

$$-\mathcal{L}_{p,q}u_0 - \lambda \frac{f(0)}{u_0^\gamma} \leq \dot{f}(u_0) - a^* < \frac{f(0)}{(T(u^0))^{\gamma}}$$

$$\leq -\mathcal{L}_{p,q}\mathring{T}(u^0) - \lambda \frac{f(0)}{(T(u^0))^{\gamma}} = \dot{f}(u^0)$$

$$< \dot{f}(u^0) + a^* \leq -\mathcal{L}_{p,q}u^0 - \lambda \frac{f(0)}{(u^0)^{\gamma}}, \text{ in } \Omega$$

for $0 < \alpha_* \leq \min\{\varepsilon_*, \chi^*\}$ and $a^* \leq \min\{\varepsilon^*, \chi^*\}$. Since $u_0, u^0$ are ordered sub and super solutions of $(P_\lambda)$, respectively and $\dot{f}$ in increasing on $[0, \infty)$, then by using strong comparison principle (see [38, Proposition 6]) we get

$$\mathring{T}(u_0) - u_0, \mathring{T}(u^0) \in \mathcal{C}_0(\Omega)^+ \text{ and } \mathring{T}(X) \subset X.$$ 

Using the same arguments as above, Proposition 3.2, Corollary 3.1 we get

$$\hat{T}(v_0) - v_0, \mathring{T}(u^0) \in \mathcal{C}_0(\Omega)^+, \hat{T}(X_i) \subset X_i \text{ for } i = 1, 2$$

and there exists a minimal fixed point $\hat{T}(u_1) = u_1 \in X_1$ such that $u_1 \in (u_0, v^0)$ and a maximal fixed point $\hat{T}(u_2) = u_2 \in X_2$ such that $u_2 \in (v_0, u^0)$. Since the map $\hat{T}$ is increasing, then there exist positive constant $c_i > 0$ for $i = 1, 2$ and $\Theta \in \mathcal{C}_0(\Omega)^+$ such that

$$u_0 + c_1 \Theta \leq \hat{T}(u_0) \leq \hat{T}(u_1) = u_1, \quad v^0 - u_1 = v^0 - \hat{T}(u_1) \geq v^0 - \hat{T}(v^0) \geq c_1 \Theta$$

and

$$u^0 + c_2 \Theta \geq \hat{T}(u^0) \geq \hat{T}(u_2) = u_2, \quad u_2 - v_0 = \hat{T}(u_2) - v_0 \geq \hat{T}(v^0) - v_0 \geq c_2 \Theta.$$ 

For fixed points $u_i$, we define the open ball $B_i$ in $X$ by

$$B_i := X \cap \{ \phi \in \mathcal{C}_0(\Omega)^+ : \| u_i - \phi \|_{\mathcal{C}_0(\Omega)} \leq c_i \}.$$ 

Then for every $i = 1, 2$, we have $u_i + B_i \subset X_i$ i.e., $X_i$ have nonempty interior. So, we construct open balls around each fixed points of the map $\hat{T}$ in $X_i$ and by taking union of all such balls $B_i$ say $B_i$, such that $X_i\setminus B_i$ contains no fixed map of $\hat{T}$. Finally, by using Theorem 3.3 we get the existence of third point $u_3$ in $X \setminus (X_1 \cup X_2)$.

4. Appendix

In this section, we recall suitable inequalities due to Simon [11] and prove some preliminary results:

**Lemma 4.1.** For any $u, v \in \mathbb{R}^N$, we have

$$\langle |\nabla u|^{q-2}u - |\nabla v|^{q-2}v, u - v \rangle \geq c(q)|\nabla u - \nabla v|^q \quad \text{if } q \geq 2$$

$$\langle |\nabla u|^{q-2}u - |\nabla v|^{q-2}v, u - v \rangle \geq c(q)\frac{|\nabla u - \nabla v|^2}{(|\nabla u| + |\nabla v|)^{2-q}} \quad \text{if } 1 < q < 2.$$
Lemma 4.2. Let $1 < q < 2$. Then there exists a constant $C > 0$ such that for any $u_1, u_2 \in W^{1,q}_0(\Omega)$:

$$
\int_{\Omega} \left( |\nabla u_2|^q - |\nabla u_1|^q \right) \nabla (u_2 - u_1) \, dx \\
\geq C \left( \int_{\Omega} |\nabla (u_2 - u_1)|^q \, dx \right) \left( \frac{1}{\left\| (|\nabla u_2| + |\nabla u_1|^\Theta) \right\|_{L^2(\Omega)}} \right)^{\frac{q}{2}}
$$

(4.1)

where $\Theta = \frac{a(2-a)}{2}$.

Proof. First, using Hölder inequality, we obtain:

$$
\int_{\Omega} |\nabla (u_2 - u_1)|^q \, dx \left( \frac{1}{\left\| (|\nabla u_2| + |\nabla u_1|^\Theta) \right\|_{L^2(\Omega)}} \right)^{\frac{q}{2}} \leq C \left\| (|\nabla u_2 - u_1|^q) \right\|_{L^\frac{2}{q}(\Omega)} (4.2)
$$

On the other hand, from Lemma 4.1:

$$
\int_{\Omega} \left( |\nabla u_2|^q - |\nabla u_1|^q \right) \nabla (u_2 - u_1) \, dx \geq C(q) \int_{\Omega} \frac{|\nabla u_2 - u_1|^2}{|\nabla u_2| + |\nabla u_1|^{q-2}} \, dx (4.3)
$$

Now, by combining (4.2) and (4.3), we obtain our claim. \qed

Before proving the existence and regularity result for the doubly parameterized problem $(P_{\alpha,\beta})$, we recall the following lemma from [22] for the construction of the barrier function for the problem $(P_{\alpha,\beta})$:

Lemma 4.3 (Lemma A.7, [22]). There exists a $C^1$ function $\Xi_r : [0, R_r] \to [0, \infty]$ satisfying the initial value problem

$$
\begin{cases}
- \frac{d}{dr}(\Xi_r'(r)) \cdot \Xi_r^\alpha = \frac{1}{\Xi_r^\beta(r)} \cdot w_n > 0 & \text{for } r \in (0, R_r), \\
\Xi_r(0) = 0, \quad \Xi_r'(0) = \tau > 0.
\end{cases}
$$

(4.4)

where $R_r = \sup\{s \in (0, \infty) : \Xi_r(t) > 0 \text{ for all } t \in (0, s)\}$. The function $\Xi_r$ is represented in the form

$$
\Xi_r(r) = \tau^{\frac{\alpha}{\beta-\alpha}} \Xi_1(\tau^{\frac{\alpha}{\beta}} r) \text{ for } 0 \leq r \leq R_r := \tau^{\frac{\alpha}{\beta}} R_1 > \text{diam}(\Omega),
$$

where the function $\Xi_1$ and $R_1$ are given by [22] (A.14) and [22] (A.15) respectively. Furthermore, $\Xi_r$ is strictly increasing in $[0, R_r]$ and $\Xi_r'$ is strictly decreasing in $[0, R_r]$ and

$$
- \frac{d}{dr}(\Xi_r'(r)) \cdot \Xi_r^\alpha(r) \geq 0 \text{ for } r \in (0, R_r). (4.5)
$$

Proof of Lemma 4.1

The existence of solution $u_{\alpha,\beta} \in W^{1,q}_0(\Omega) \cap C_{\text{loc}}(\Omega) \cap C^{1,1}(\Omega)$ can be proved by adopting same arguments from [20] Lemma 2.6 and Proposition 2.7 and [31] Lemma 3.2. For the sake of completeness, we provide a brief sketch of the proof. To prove the existence of a solution, we define the energy functional $\mathcal{J} : W^{1,q}_0(\Omega) \to \mathbb{R}$ as

$$
\mathcal{J}(u) := \frac{\alpha}{p} \int_{\Omega} |\nabla u|^p \, dx + \frac{\beta}{q} \int_{\Omega} |\nabla u|^q \, dx - \frac{\lambda}{1-\gamma} \int_{\Omega} |u|^{1-\gamma} \, dx.
$$

Using Sobolev’s embedding and lower semi-continuity of the norms, we infer that $\mathcal{J}$ is coercive and weakly lower semi-continuous. Therefore there exists a global minimizer $u \in \mathbb{R}$.
Now, by combining (4.4) and (4.5) together with above estimates, we get

\[ W \]

hence \( u \) is Gâteaux differentiable at \( u \) (for more details see [20, Lemma 2.6]), hence \( u \) is weak solution of \((P_\alpha)\). By following the same lines of the proof of [26, Theorem 2] and [31, Lemma 3.2], we get \( \|u_{\alpha,\beta}\|_{L^\infty(\Omega)} \leq C(\alpha,\beta) \).

To prove the upper boundary behavior, we define the barrier function

\[ w(x) := M_\lambda \Xi_\tau(\delta(x)) \quad \text{for } x \in \Omega \]

where the choice of \( M_\lambda := M(\lambda) \geq 1 \) will be determined later. Since \( \partial \Omega \in C^2 \), from [24, Lemma 14.6] there exists a constant \( \nu > 0 \) such that \( \delta \in C^2(\Omega_\nu) \) and by using the fact that \( |\nabla \delta| = 1 \), we get for \( 0 \leq \xi \in C^\infty_c(\Omega_\nu) \)

\[
\int_{\Omega_\nu} -L^{\alpha,\beta}_{p,q} w \xi \, dx = \beta M_\lambda^{-1} \int_{\Omega_\nu} (\Xi'_\tau(\delta))^p \xi \, dx + \alpha M_\lambda^{-1} \int_{\Omega_\nu} (\Xi'_\tau(\delta))^q \xi \, dx
\]

\[
= -\beta M_\lambda^{-1} \int_{\Omega_\nu} \nabla (\Xi'_\tau(\delta))^p \nabla \xi \, dx - \alpha M_\lambda^{-1} \int_{\Omega_\nu} \nabla (\Xi'_\tau(\delta))^q \nabla \xi \, dx
\]

\[
\geq -\beta M_\lambda^{-1} \int_{\Omega_\nu} \left( (\Xi'_\tau(\delta))^p \nabla^2 \xi - \|\Delta \delta\|_{L^\infty(\Omega_\nu)} (\Xi'_\tau(\delta))^q \right) \xi \, dx
\]

\[
- \alpha M_\lambda^{-1} \int_{\Omega_\nu} \left( (\Xi'_\tau(\delta))^q \nabla^2 \xi - \|\Delta \delta\|_{L^\infty(\Omega_\nu)} (\Xi'_\tau(\delta))^p \right) \xi \, dx.
\]

Now, by combining (4.4) and (4.5) together with above estimates, we get

\[
-L^{\alpha,\beta}_{p,q} w \geq \beta M_\lambda^{-1} \left( (1-q)(\Xi'_\tau(\delta))^q \Xi''(\delta) - \|\Delta \delta\|_{L^\infty(\Omega_\nu)} (\Xi'_\tau(\delta))^q \right)
\]

\[
+ \alpha M_\lambda^{-1} \left( (1-p)(\Xi'_\tau(\delta))^p \Xi''(\delta) - \|\Delta \delta\|_{L^\infty(\Omega_\nu)} (\Xi'_\tau(\delta))^p \right)
\]

\[
\geq M_\lambda^{-1} \left[ \beta(1-q)(\Xi'_\tau(\delta))^q \Xi''(\delta) - \|\Delta \delta\|_{L^\infty(\Omega_\nu)} \left( (\Xi'_\tau(\delta))^q + \alpha(\Xi'_\tau(\delta))^p \right) \right]
\]

\[
\geq M_\lambda^{-1} \left[ \frac{\beta}{\Xi''(\tau)} - \|\Delta \delta\|_{L^\infty(\Omega_\nu)} \left( \beta(\Xi'_\tau(\delta))^q + \alpha(\Xi'_\tau(\delta))^p \right) \right]
\]

weakly in \( \Omega_\nu \). Since \( \Xi_\tau(0) = 0 \), \( \Xi'_\tau(0) = \tau \), the concavity of the function \( \Xi_\tau \) implies

\[
\Xi_\tau(\delta) \leq \tau \delta \quad \text{and} \quad \Xi'_\tau(\delta) \leq \tau.
\]

Now by choosing \( \nu \) small enough such that \( \frac{1}{\Xi''(\tau)} \geq \frac{\lambda}{\beta} \left( \|\Delta \delta\|_{L^\infty(\Omega_\nu)} \tau^{p-1}(\beta \tau^{q-p} + \alpha) \right) \) in \( \Omega_\nu \), we get

\[
-L^{\alpha,\beta}_{p,q} w \geq M_\lambda^{-1} \frac{\beta}{\Xi''(\tau)} - \|\Delta \delta\|_{L^\infty(\Omega_\nu)} \left( \beta(\Xi'_\tau(\delta))^q + \alpha(\Xi'_\tau(\delta))^p \right) \quad \text{weakly in } \Omega_\nu.
\]

Now by choosing \( M_\lambda \) large enough such that \( M_\lambda^{-1} \beta \geq \lambda \) and using the fact that \( u_{\alpha,\beta} \in L^\infty_{loc}(\Omega) \), weak comparison principle gives \( u_{\alpha,\beta} \leq C\delta \) in \( \Omega \) for some \( C > 0 \). Finally, [20, Theorem 1.7] implies \( u_{\alpha,\beta} \in C^{1,l}(\Omega) \) for some \( l \in (0,1) \).

**Remark 4.1.** Let \( u_{\alpha,\beta} \) be weak solution solution of the problem \((P_{\alpha,\beta})\) such that

\[
\|u_{\alpha,\beta}\|_{L^\infty(\Omega)} \leq C_1(\alpha,\beta) \quad \text{and} \quad \|u_{\alpha,\beta}\|_{C^{1,l}(\overline{\Omega})} \leq C_2(\alpha,\beta).
\]
Then, by looking discreetly the proof of [26, Theorem 2], [31, Lemma 3.2] and using [20, Remark 1.10], we get
\[ \lim_{\alpha \to 0} C_1(\alpha, \beta) \leq C(\beta) \text{ and } \lim_{\alpha \to 0} C_2(\alpha, \beta) < C(\beta). \]

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