CALCULATING THE DISSONANCE OF A CHORD
ACCORDING TO HELMHOLTZ THEORY

Giorgio Dillon
Dipartimento di Fisica, Università di Genova
INFN, Sezione di Genova
dillon@ge.infn.it

ABSTRACT
Following the ideas of von Helmholtz and Plomp-Levelt, an algorithm for calculating the dissonance of complex sounds, free from logical inconsistencies and useful for comparing different chords, is proposed. The method is tested by comparing different tunings of the same major triad. Some interesting conclusions from this calculation may be drawn.

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1. INTRODUCTION
According to von Helmholtz [1] the dissonance between two simultaneously played complex tones is related to the roughness or rapid beating between their adjacent partials. Because a musical sound is harmonic (i.e., the frequencies of its partials are multiple of an audible fundamental frequency) it follows that the number of beating partials between two simultaneous musical sounds is minimized when the fundamental frequencies are related by a ratio of small integers. This explains, on physical grounds, the longstanding puzzle of why some musical intervals (those with simple ratios for their frequencies) are sensed as pleasant while others are judged dissonant.

After the precise definition of the critical band provided by the work of Fletcher [2], Plomp and Levelt [3] reviewed and clarified the above explanation of consonance relating it to the critical bandwidth (CB). They established experimentally a standard curve which represents the “tonal consonance” between two pure (sinusoidal) tones, on an arbitrary scale, as a function of the ratio between the frequency difference and the CB. The maximal tonal dissonance was found at intervals of about 25% of the CB at the mean frequency of the two tones. In Fig. 1 the function \( g(x) \) reproduces the Plomp-Levelt standard curve turned upside down to give dissonance instead of consonance. Accordingly the dissonance of a dyad of pure tones of frequencies \( f_1 \) and \( f_2 \) lying in the same CB \( b(f) \) centered at the mean frequency, is given by

\[
d(1, 2) = g(x)
\]

where \( x = |f_1 - f_2|/b(f) \).

Given the dissonance of a single dyad, the problem arises how to calculate the total dissonance of a complex sound or of a chord of musical tones. The simplest assumption is the naive addition of the dissonances of each dyad present in the complex sound [3]. Anyhow, in summing the contributions from the beating of all nearby partials an appropriate weighting factor related to their intensities \( I \), or better to their loudness \( l \), is required:

\[
d_{\text{tot}} = \sum_{i<j} f(l_i, l_j) d(i, j)
\]

Different criteria have been adopted in the past. For instance, in [4, 5] each pure tone is supposed to contribute in proportion to its amplitude. In [6] the weighting factor is taken to be proportional to the loudness of the weaker component.
However we note that (2) cannot be correct in general and that the above criteria hide a logical inconsistency. This may be seen as follows: Consider the dissonance of a dyad of pure tones as given by (1). As a matter of fact one of them (or both) may be considered as the result of two superimposed pure tones with the same frequency and halved intensities. Then we could use (2) to calculate the total dissonance and should recover the same result as (1). This is not so. In fact, to avoid such a paradox, the weighting factor in (2) should be proportional to the intensities (i.e. to the power per unit area) of both components, rather than to the amplitudes.

In [7] the concept of dissonance intensity was introduced as a physical counterpart of the psychological dissonance sensation and a power law is suggested to hold between the two according to the general psychophysics law of sensation postulated by Stevens [8]. This is just the same kind of law that relates, for example, the intensity (or the sound pressure level) to the loudness, that will be exploited in Sec.2. In this hypothesis Eq. (2) does not hold since the summation concerns the dissonance intensities and not the dissonances as perceived by our hearing apparatus. Once the power law is established, the rule for calculating the total dissonance ensues. However, in spite of the accurate experiments and analysis that include the effect of the noise [7, 9], the resulting exponents lead to a framework that is plagued with analogous inconsistencies as (2). Indeed one may check the method calculating the effect of two identical superimposed dyads (resulting in a total of 4 partials and 6 dyads) whose dissonance output should be the same as an equivalent single dyad of pure tones with doubled intensities. Once again the discrepancy is sizable.

Following these premises we propose here a practical and mathematically consistent method for calculating the dissonance of a complex sound. Our recipe is intended to provide a relative measure for the dissonance useful to compare different chords. We shall test our method by comparing different tunings for the central major triad corresponding to historical temperaments.

2. WHICH EXPONENT?

We begin this section giving a parameterization of the Plomp-Levelt curve of Fig.1. Though many parameterizations already exist [5, 6], for our purpose the following simple polynomial fit is most suitable:

$$g(x) = Nx(1.2 - x)^4$$

that yields, with $N = 4.906$, $g(x) = 1$ at $x = 0.25$. The fit (3) assumes that $g(x)$ vanishes identically at $x \geq 1.2$.

To avoid mathematical inconsistencies the $f(l_i, l_j)$ must necessarily be symmetric in the two indices and vanish when the loudness of one component vanishes. Then the simplest possible form is

$$f(l_i, l_j) \propto (l_i \cdot l_j)^\alpha$$

with $\alpha > 0$.

The loudness are approximately related to the intensities by [10]

$$l_{i,j} \propto \left( \frac{I_{i,j}}{I^*} \right)^{1/3}$$

where $I^*$ is a reference intensity (depending on the frequency).

Now when two or more tones are sounded simultaneously, the total loudness depends on whether they lie or not within the same CB. If they do, the intensities are to be summed and the total loudness is

$$l_{\text{tot}} \propto \left( \frac{1}{T^*} \sum I_i \right)^{1/3}$$

In the opposite case the loudness should add [10]. However, since our parameterization for $g(x)$ vanishes identically outside the CB, we are not to worry about

Figure 1: The function that quantifies the dissonance experienced when two pure tones of frequencies $f_1$ and $f_2$ are sounded simultaneously, as derived by Plomp-Levelt (compare with Fig. 10 of [3]). The variable $x$ is defined as $x = |f_1 - f_2|/b(\bar{f})$ where $b(\bar{f})$ is the CB at the mean frequency.
tones differing in frequency more than the relevant CB.

Then we are led to write

\[ d_{tot} = \frac{1}{l^*} \left( \sum_{i,j} (d_i \cdot l_j)^3 d(i,j)^{3/\alpha} \right)^{\alpha/3} \] (7)

where \( l^* \) is an arbitrary reference loudness. Equation (7), because of (3), is free from any drawbacks of the kind outlined in Sec.1 and consistent with the hypothesis put forward in [7].

The problem is now how to choose \( \alpha \). Note that for \( \alpha = 3 \) one recovers the simple sum (2) for the elementary dissonances \( d(i,j) \) with the weighting factors proportional to the intensities, as already observed. We tested different values for \( \alpha \) and concluded that the most sensible one is \( \alpha = \frac{1}{2} \). The choice \( \alpha = 1/2 \) is also appealing because it amounts to set

\[ f(l_i, l_j) \propto \sqrt{l_i l_j} \] (8)

that corresponds to a mean loudness for the dyad \((i,j)\).

3. COMPARING DIFFERENT TEMPERAMENTS FOR THE MAJOR TRIAD

The major triad is considered as the most consonant chord build up from the simultaneous sounding of three tones and is most frequently encountered in tonal music. The fundamental frequencies of the three tones are in the ratios 4:5:6, which correspond to the musical intervals of a major third (5/4), a minor third (6/5), and a perfect fifth (3/2). Actually these ratios refer to the just intonation for these intervals, since, in practical usage, they must be slightly adjusted (tempered). Though nowadays the so-called equal temperament that stems from the division of the octave in 12 equal parts has been definitively adopted in western music, different temperaments have been employed in the past according to the requirements and the taste of the music at the time. In this section we test [7] (with \( \alpha = 1/2 \)) by comparing different tunings for the central major triad \((C_4 - E_4 - G_4)\). The notation \( C_4 \) singles out the note \( C \) of the 4\(^{th}\) octave that corresponds to the \( C \) at the middle of a piano keyboard.

The spectra of the three tones are supposed to be the same as that of a sawtooth wave characteristic of a bowed string [11] (the amplitude of the \( n \)th harmonic being \( \propto 1/n \)). We take the same loudness \( l^* \) for the fundamentals and limit the summation to 6 harmonics. So the expression for this calculation takes the form

\[ d_{tot} = \left( \sum_{i<j} \sum_{m,n=1}^{6} \frac{1}{n^m m^n} d(n_i, n_j)^6 \right)^{1/6} \] (9)

where \( n_i \) stands for the \( n \)th harmonic of tone \( i \), \( i, j = 1, 2, 3 \); \( 1 \) standing for \( C_4 \), \( 2 \) for \( E_4 \) and \( 3 \) for \( G_4 \) and \( m, n = 1, \ldots, 6 \). The values of the CB are taken from [12]. At this stage we neglected the frequency dependence of \( l^* \) (see [5]), that may equivalently be taken into account by some appropriate correction of the amplitudes in (9). The results are displayed in Table 1. The central \( C \) is tuned at 260Hz and the following temperaments are compared: Just Intonation (J.I.), Meantone (M.T.), Werkmeister III (W. III), 12-Tone Equal Temperament (Eq.), Pythagorean (Pyt.). In the 2nd (3rd) column the contributions to the dissonance from the fundamentals (from the higher harmonics) is displayed separately. Each tone consists of 6 harmonics with amplitudes \( \propto 1/n \).

| Tuning          | \( C_4 - E_4 - G_4 \) | fund. | h.h. | \( d_{tot} \) |
|-----------------|------------------------|-------|------|---------------|
| J.I.            | 260 – 325 – 390        | 328   | 415  | 430           |
| M.T.            | 260 – 325 – 388.8      | 340   | 412  | 431           |
| W. III          | 260 – 326 – 388.8      | 346   | 417  | 437           |
| Eq.             | 260 – 327.6 – 389.6    | 352   | 424  | 444           |
| Pyt.            | 260 – 329 – 390        | 367   | 434  | 457           |

Table 1: The dissonance of the central major triad in different tunings: Just Intonation (J.I.), Meantone (M.T.), Werkmeister III (W. III), 12-Tone Equal Temperament (Eq.), Pythagorean (Pyt.).
the 4th harmonic of tone 2. Then the unpleasant effect of the Pythagorean third may be attributed to the beating of such harmonics not more in tune. Our calculation does not support this hypothesis. Instead it suggests that the Pythagorean major third is not as nice as that of just intonation mainly because the 3rd harmonic of tone 2 dangerously approaches the 4th harmonic of tone 1 (rather than a missed coincidence of the 4th and the 5th harmonics). Of course things may drastically change with different spectra [7, 6].

4. CONCLUSIONS

In the present paper we aimed at giving a consistent algorithm for calculating and comparing the dissonances of composite sounds. Following the ideas of Plomp-Levelt [3] we proposed a recipe of how to “sum” dissonances. Since physically one should sum the intensities, to avoid logical inconsistencies dissonances must be composed according to (7). We chose \( \alpha = 1/2 \). Actually the results are quite sensitive to this exponent. For instance for \( \alpha = 1 \) the harmonics but the fundamentals would give a negligible contribution to the total dissonance (in contrast with von Helmholtz’s hypothesis). Conversely too much weight would be attributed to higher partials for a lesser \( \alpha \). So \( \alpha = 1/2 \) does seem a well gauged value. We think that this method may be useful for comparing chords from sounds with known spectra and for further analyses.

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