Data-driven control of switched linear systems with probabilistic stability guarantees

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Abstract—This paper tackles state feedback control of switched linear systems under arbitrary switching. We propose a data-driven control framework that allows to compute a stabilizing state feedback using only a finite set of observations of trajectories with quadratic and sum of squares (SOS) Lyapunov functions. We do not require any knowledge on the dynamics or the switching signal, and as a consequence, we aim at solving uniform stabilization problems in which the feedback is stabilizing for all possible switching sequences. In order to generalize the solution obtained from trajectories to the actual system, probabilistic guarantees on the obtained quadratic or SOS Lyapunov function are derived in the spirit of scenario optimization. For the quadratic Lyapunov technique, the generalization relies on a geometric analysis argument, while, for the SOS Lyapunov technique, we follow a sensitivity analysis argument. In order to deal with high-dimensional systems, we also develop parallelized schemes for both techniques. We show that, with some modifications, the data-driven quadratic Lyapunov technique can be extended to LQR control design. Finally, the proposed data-driven control framework is demonstrated on several numerical examples.

Index Terms—Switched linear systems, stabilization, data-driven control, scenario optimization

I. INTRODUCTION

Switched systems are typical hybrid dynamical systems which consist of a number of dynamics modes and a switching rule selecting the current mode. The jump from one mode to another often causes complicated hybrid behaviors resulting in significant challenges in stability analysis and control design, see, e.g., [1]–[3]. This paper focuses on stabilization and control of switched linear systems.

Many control techniques have been proposed for switched systems depending on the assumptions on the switching rule. In the case where the switching signal is the only control input, a standard technique for achieving stabilization is to impose constraints on the switching sequence, e.g., dwell time [4]–[6] and average dwell time [7], [8]. More advanced techniques use state-dependent switching rules to achieve stabilization, see, e.g., [5], [9], [10] and the references therein. In [11], [12], necessary and sufficient conditions for stabilizability are also provided. For switched systems with affine control inputs, the stabilization problem becomes even more complicated due to extra freedom. In [13], [14], the (time) varying nature of the dynamics is considered as uncertainty and uniform state feedback stabilization laws are proposed for all possible switching sequences. When both the affine control input and the switching signal are accessible, exponential stabilization can be achieved for instance by using piecewise quadratic Lyapunov functions which are essentially solutions to switched LQR problems [15]. In the presence of state and input constraints, optimal control of switched linear systems is also addressed under the framework of model predictive control [16], [17]. However, these stabilization methods all require a model of the underlying switched system.

While there exist hybrid system identification techniques [18], identification of state-space models of switching systems is in general cumbersome and computationally demanding. More specifically, identifying a switched linear system is NP-hard [19] and most of currently available techniques as mentioned in [18] rely on heuristics and lack of formal guarantees. In recent years, data-driven analysis and control under the framework of black-box systems has received a lot of attention, see, e.g., [20]–[23]. For instance, probabilistic stability guarantees are provided in [21], [24], [25] for black-box switched linear systems, based on the observation of a finite set of trajectories. Let us also mention that, although data-driven techniques for controlling linear systems already exist (see, e.g., [26]), they are not able to tackle switched systems.

In this paper, we address feedback control design for switched linear systems without knowing the model of the system or the switching signal. As the switching is considered as a source of uncertainty, we need to design a uniform state feedback controller allowing to stabilize the system in the worst case, similar to [13], [14]. To do this, we compute a state feedback controller and a common Lyapunov function for all the switching modes of the closed-loop system using a finite set of trajectories. We use both quadratic and sum of squares (SOS) Lyapunov functions, which lead to constrained nonlinear optimization problems with a large number of Lyapunov inequalities. For numerical tractability, we then design algorithms to solve these problems by making use of the underlying structure. For example, with quadratic Lyapunov
functions, the biconvex Lyapunov inequalities allow to use alternating minimization where each iteration solves convex problems.

One major issue of this data-based feedback control design is that it is usually only valid for the regions where the data is sampled but may not stabilize the actual system in the whole space. In order to formally describe the properties of the controller, we derive probabilistic stability guarantees in the spirit of scenario optimization [27]–[29]. In this context, one trajectory can be considered as a scenario and the stabilization problem formulated based on a set of trajectories is a sampled problem. As our problem is non-convex, the convex chance-constrained theorems in [27] are not applicable. While chance-constrained theorems for nonlinear optimization problems also exist in [28], [29], their probabilistic bounds rely on the knowledge of the essential set (which is basically the set of irremovable constraints). Identifying this set can be very expensive for general nonlinear problems, in particular for nonlinear semidefinite problems. Hence, the techniques in [28], [29] are not suitable for our case which involves a large number of polynomial constraints and linear/bilinear matrix inequalities. Instead, probabilistic stability guarantees in this paper are derived relying on the notions of set covering and packing (see, e.g., [30, Chapter 27]) and geometric/sensitivity analysis of the underlying problem. Similar probabilistic guarantees are also developed in [21], [24], [25] for autonomous systems. However, the probabilistic guarantees in [21], [24], [25] all require the optimality of the obtained solution, while our techniques work with any feasible solution of the underlying optimization problem. This allows to parallelize our algorithms to substantially speed up the computations. Finally, we show that the proposed data-driven Lyapunov framework can be extended to switched LQR problems.

A preliminary version of this paper appears as a conference paper in [31] which only considers quadratic stabilization, and does not contain complete proofs. In this paper, we provide complete detailed proofs of all the results in [31]. In particular, the present paper contains the first published proof of our main Theorem 1. Furthermore, we present an extension to SOS stabilization which calls for a new technique for deriving probabilistic stability guarantees. Besides the stabilization problem, we also consider a switched LQR problem. In addition, to circumvent computational issues, we also present a parallelized scheme for both quadratic and SOS stabilization.

The rest of the paper is organized as follows. This section ends with the notation, followed by Section II on the review of preliminary results on stability of switched linear systems and the formulation of the state feedback stabilization problem. Section III presents the proposed data-driven quadratic Lyapunov technique, including an alternating minimization algorithm, probabilistic stability analysis, and a parallelized scheme. In Section IV, the SOS Lyapunov technique is considered with a similar alternating minimization algorithm. In Section V, we extend the data-driven Lyapunov framework to switched LQR design. Numerical results are provided in Section VI.

**Notation.** The set of non-negative integers is denoted by \( \mathbb{Z}^+ \). For a square matrix \( Q \), \( Q \succ (\succeq) 0 \) means that \( Q \) is symmetric and positive definite (semi-definite). \( \mathbb{S} \) and \( \mathbb{B} \) are the unit sphere and the unit (closed) ball in \( \mathbb{R}^n \) respectively. \( \mu (-) \) denotes the uniform spherical measure on \( \mathbb{S} \) with \( \mu (\mathbb{S}) = 1 \). For any square matrix \( P \), \( \text{tr}(P) \) denotes the trace of \( P \). For any symmetric matrix \( P \), we denote by \( \lambda_{\max} (P) \) and \( \lambda_{\min} (P) \) the largest and smallest eigenvalues of \( P \) respectively. For any matrix \( P \succ 0 \), let \( \kappa (P) := \lambda_{\max} (P)/\lambda_{\min} (P) \) be the condition number. For any \( p \geq 1 \), the \( \ell_p \) norm of a vector \( x \in \mathbb{R}^n \) is \( \|x\|_p (\|x\| \text{ is the } \ell_2 \text{ norm by default}) \) with \( \|x\|^2_Q = x^T Q x \) for any \( Q \succeq 0 \). Finally, given \( x \in \mathbb{S} \) and \( \theta \in [0, \pi/2] \), let \( \text{Cap}(x, \theta) := \{v \in \mathbb{S} : |x^T v| \geq \cos (\theta)\} \) be the symmetric spherical cap with direction \( x \) and angle \( \theta \).

**II. PRELIMINARIES AND PROBLEM STATEMENT**

We consider the following switched linear system

\[
x(t + 1) = A_{\sigma(t)} x(t) + Bu(t), \quad t \in \mathbb{Z}^+,
\]

where \( x(t) \in \mathbb{R}^n \) is the state vector, \( u(t) \in \mathbb{R}^m \) is the input and \( \sigma : \mathbb{Z}^+ \to \mathcal{M} := \{1, 2, \ldots, M\} \) is a time-dependent switching signal that indicates the current active mode of the system among \( M \) possible modes in \( \mathcal{A} := \{A_1, A_2, \ldots, A_M\} \). In this paper, we consider the case in which the switching signal is changing arbitrarily and cannot be observed, i.e., the information on the switching signal is not available. Note that the input matrix \( B \) is constant. Our goal is to find a static linear state feedback stabilizing the system under arbitrary switching, that is, a feedback matrix \( K \in \mathbb{R}^{m \times n} \) such that the closed-loop system below is stable for all switching signals

\[
x(t + 1) = (A_{\sigma(t)} + BK) x(t), \quad t \in \mathbb{Z}^+.
\]

For notational convenience, let \( \mathcal{A}_K := \{A_1 + BK, A_2 + BK, \ldots, A_M + BK\} \) for a given \( K \in \mathbb{R}^{m \times n} \). The stability of System (2) under arbitrary switching can be described by the joint spectral radius (JSR) [32] of the matrix set \( \mathcal{A}_K \) defined by

\[
\rho(\mathcal{A}_K) := \lim_{k \to \infty} \max_{\sigma(k) \in \mathcal{M}^k} \|\mathcal{A}_{\sigma(k)}(K)\|^{1/k}
\]

where \( \sigma(k) := \{\sigma(0), \sigma(1), \ldots, \sigma(k - 1)\} \) and \( \mathcal{A}_{\sigma(k)}(K) = (A_{\sigma(k - 1)} + BK) \cdots (A_{\sigma(1)} + BK)(A_{\sigma(0)} + BK) \). System (2) is asymptotically stable when \( \rho(\mathcal{A}_K) < 1 \). Hence, the state feedback stabilization problem for System (1) amounts to finding a \( K \in \mathbb{R}^{m \times n} \) such that \( \rho(\mathcal{A}_K) < 1 \). However, the computation of the JSR of a set of matrices is known to be a very challenging problem in general, let alone its optimization in the context of control design. For this reason, we use tractable upper bounds on the JSR, providing sufficient conditions for stability or stabilization, see [32]. The following proposition provides a sufficient condition based on a common quadratic Lyapunov function which can be computed via semidefinite programming (SDP) [33].

**Proposition 1 ([32, Prop. 2.8]):** Consider the closed-loop matrices \( \mathcal{A}_K \) for some state feedback \( K \in \mathbb{R}^{m \times n} \). If there exist \( \gamma \geq 0 \) and \( P \succ 0 \) such that \( A^T PA \preceq \gamma^2 P, \forall A \in \mathcal{A}_K \), then \( \rho(\mathcal{A}_K) \leq \gamma \).
From this proposition, we formulate the following nonlinear semidefinite optimization problem for stabilization of switched linear systems:

\[
\gamma^* := \min_{\gamma \geq 0, P, K} \gamma \quad \text{s.t.} \quad (A + BK)^T P (A + BK) \preceq \gamma^2 P, \quad \forall A \in \mathcal{A} \quad \text{(4a)}
\]

\[
P \succ 0. \quad \text{(4b)}
\]

Using the Schur complement formula [34, Theorem 1.12] with \( \gamma \) known, Problem (5) can be efficiently solved via semidefinite optimization problem for stabilization of switched linear systems when the matrices \( A \) are unknown.

By homogeneity, one can restrict the constraints in (6b) to the set of points in the unit sphere \( \mathbb{S} \) instead of the whole space \( \mathbb{R}^n \). As we will show later, the formulation in (6b) allows us to develop a model-free and data-based control design. For that, we make the following assumption about the observations available.

**Assumption 1:** The state \( x(t) \) can be fully observed for all \( t \in \mathbb{Z}^+ \), the input matrix \( B \) is time-invariant and known, and the number of modes (or an upper bound) is available.

The assumption that \( B \) is time-invariant is not restrictive in many applications, for instance, when the switching only occurs in some parameters of the dynamics. Such an assumption is often made in the literature, see, e.g., [13], [36].

### III. State feedback stabilization

This section presents our model-free quadratic Lyapunov technique for stabilizing black-box switched linear systems. We first formulate a sample-based stabilization problem, which consists of a set of biconvex constraints. Then, to solve this problem, we present an alternating minimization algorithm that generates feasible iterates. With the concepts of covering/packing numbers, probabilistic guarantees on the obtained solution are then provided using geometric analysis. Finally, we also show that the algorithm can be parallelized to speed up the computation.

#### A. Sampled stabilization problem

For the model-free design, we sample a finite number of pairs of the initial state and switching mode. More precisely, we randomly and uniformly generate \( N \) initial states on \( \mathbb{S} \) and \( N \) modes in \( \mathcal{M} \), which are denoted by \( \omega_N := \{(x_i, \sigma_i) \in \mathbb{S} \times \mathcal{M} : i = 1, 2, \ldots, N\} \). From this random sampling, we observe the trajectories of the open-loop system of System (1) with \( u = 0 \) and obtain the observed data set \( \{(x_i, A_{\sigma_i}, x_i) : i = 1, 2, \ldots, N\} \), where \( A_{\sigma_i}, x_i \) is the successor of the initial state \( x_i \) with respect to mode \( \sigma_i \). Note that although \( A_{\sigma_i}, x_i \) depends on \( \sigma_i, \sigma_i \) is not directly observed.

For the given sample set \( \omega_N \), we define the following *sampled* problem:

\[
\min_{\gamma \geq 0, P, K} \gamma \quad \text{s.t.} \quad (A_{\sigma} + BKx)^T P (A_{\sigma} + BKx) \preceq \gamma^2 x^T P x, \quad \forall (x, \sigma) \in \omega_N \quad \text{(7b)}
\]

\[
P \succ 0. \quad \text{(7c)}
\]

From the homogeneity property of (7b), the inequality \( P \succ 0 \) can be replaced by \( P \succeq I \) due to the fact that the feasibility of \( P \) implies the feasibility of \( P/\lambda_{\min}(P) \). When the size of \( \omega_N \) is small, the sampled problem (7) can be solved using polynomial optimization toolboxes [37]–[39]. As \( P \) is invertible, the Schur complement formula [34, Theorem 1.12] can be applied to (7b) using the reformulation below

\[
\min_{\gamma \geq 0, P, K} \gamma \quad \text{s.t.} \quad (A_{\sigma} + BKx)^T P (A_{\sigma} x + BKx) \preceq \gamma^2 x^T P x, \quad \forall (x, \sigma) \in \omega_N. \quad \text{(8)}
\]

We can then convert the constraints in (7b) into a set of bilinear matrix inequalities (BMI) and reformulate Problem (7) as the following BMI problem

\[
\min_{\gamma \geq 0, P \succeq I, K} \gamma \quad \text{s.t.} \quad \gamma^2 x^T P x \leq (A_{\sigma} x + BKx)^T P (A_{\sigma} x + BKx) \preceq (A_{\sigma} x + BKx)^T P P^{-1} (A_{\sigma} x + BKx), \quad \forall (x, \sigma) \in \omega_N. \quad \text{(9b)}
\]

This reformulation allows us to solve Problem (7) using BMI solvers [35], [40], [41].

#### B. An alternating algorithm

As the size of \( \omega_N \) increases, it becomes numerically intractable to find a (local) optimum of (7) or (9) using the aforementioned polynomial or BMI solvers. Since we do not seek to have optimality, we can use a less costly approach described below. We propose an alternating minimization algorithm between \( P \) and \( K \) for its numerical tractability and simple implementation. As we will show later, this alternating algorithm also enables us to parallelize the computation. Given
of the unit ball $\mathbb{B}$ enclosed by $\text{Cap}(x, \theta)$ and the hyperplanes $|x^Tv| = \cos(\theta)$, expressed as $\{z \in \mathbb{B} : |x^Tz| \geq \cos(\theta)\}$. We recall again from [42] that $\delta_v(\theta)$ can be given by

$$\delta_v(\theta) = \mathcal{I}(\sin^2(\theta); n + 1, 1, 2), \quad \forall \theta \in [0, \pi/2].$$

Similarly, the function $\delta_v$ is also strictly increasing with $\theta$ and its inverse is denoted by $\delta_v^{-1}$.

![Fig. 1. Measure $\mu$ of the symmetric spherical cap $\text{Cap}(x, \theta)$ in $\mathbb{R}^n$ for different values of $n$.](image)

C. Probabilistic stability guarantees

We now derive formal stability guarantees on the solution obtained from Algorithm 1. Let us first introduce a few definitions and notation. For any $\theta \in [0, \pi/2]$ and any $x \in \mathbb{S}$, we denote $\delta(\theta)$ denote the relative area of the symmetric spherical cap $\text{Cap}(x, \theta)$, given by $\{v \in \mathbb{S} : |x^Tv| \geq \cos(\theta)\}$. From [42], it holds that

$$\delta(\theta) = \mathcal{I}(\sin^2(\theta); n - 1, 1, 2)$$

where $\mathcal{I}(x; a, b)$ is the regularized incomplete beta function defined as

$$\mathcal{I}(x; a, b) := \int_0^x \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt,$$  \hspace{1cm} (13)

The function $\delta$ is strictly increasing with $\theta$, and thus we can define its inverse, denoted by $\delta^{-1}$, see also Figure 1 for an illustration. Let $\delta_v(\theta)$ denote the relative volume of the portion of $\mathbb{B}$ enclosed by $\text{Cap}(x, \theta)$ and the hyperplanes $|x^Tv| = \cos(\theta)$, expressed as $\{z \in \mathbb{B} : |x^Tz| \geq \cos(\theta)\}$.

![Fig. 1. Measure $\mu$ of the symmetric spherical cap $\text{Cap}(x, \theta)$ in $\mathbb{R}^n$ for different values of $n$.](image)

Let us then extend the definition of $\epsilon$-covering to the joint set $\mathbb{S} \times \mathcal{M}$ as follows.

**Definition 3:** Given $\epsilon \in (0, 1)$, a set $\mathbb{S} \times \mathcal{M}$ is called an $\epsilon$-covering of $\mathbb{S} \times \mathcal{M}$ if, for any $(x, \sigma) \in \mathbb{S} \times \mathcal{M}$, there exists $(z, \sigma) \in \mathbb{S} \times \mathcal{M}$, with the solution of (11) being the

![Fig. 1. Measure $\mu$ of the symmetric spherical cap $\text{Cap}(x, \theta)$ in $\mathbb{R}^n$ for different values of $n$.](image)
The following lemma shows probabilistic properties of the sample set $\omega_N$, which will be needed below in order to achieve formal guarantees on the controller.

**Lemma 2:** Given $N \in \mathbb{Z}^+$, let $\omega_N = \{(x_i, \sigma_i)\}_{i=1}^N$ be independent and identically distributed (i.i.d) with respect to the uniform distribution $P$ over $S \times M$. Then, given any $\epsilon \in (0,1)$, with probability no smaller than $1 - B(\epsilon; N)$, $\omega_N$ is an $\epsilon$-covering of $S \times M$, where

$$B(\epsilon; N) := \frac{M \left(1 - \delta \left(\frac{1}{2} \delta^{-1}(\epsilon)\right)\right)^N}{\delta \left(\frac{1}{2} \delta^{-1}(\epsilon)\right)}.$$  

(16)

**Proof:** Consider a maximal $\epsilon$-packing $Z$ of $S$ with $\epsilon' = \delta \left(\frac{1}{2} \delta^{-1}(\epsilon)\right)$ and let $\theta = \delta^{-1}(\epsilon') = \frac{1}{2} \delta^{-1}(\epsilon)$. From the proof of Lemma 1, $\{\cap(z, \theta)\}_{z \in Z}$ covers $S$. Suppose $\omega_N$ is sampled randomly according to the uniform distribution, then the probability that each set in $\{\cap(z, \theta)\}_{z \in Z}$ contains $M$ points with $M$ different modes is no smaller than $1 - N_\sigma(\epsilon')M(1 - \frac{1}{M})^N \geq 1 - B(\epsilon; N)$. When this happens, for any $(x, \sigma) \in S \times M$, there exists a pair $(z, \sigma) \in \omega_N$ such that $|x^T z| \geq \cos(\theta)$. This completes the proof.

We are now able to present a key result of this section, which provides a stability certificate from the solution of the sampled problem (7) assuming sufficient covering by the sample set.

**Theorem 1:** Given a sample set $\omega_N \subset S \times M$, consider Problem (7). Let $(\gamma, P, K)$ be a feasible solution to Problem (7). Suppose that $\omega_N$ is an $\epsilon$-covering of $S \times M$ for some $\epsilon \in (0, 1)$. Then,

$$\rho(A_K) \leq \frac{\gamma}{\max \{\varphi_P(\epsilon), \psi_P(\epsilon)\}}$$  

(17)

where $\rho(A_K)$ is defined in (3) and

$$\varphi_P(\epsilon) := 1 - \kappa(P)(1 - \cos(\delta^{-1}(\epsilon)))$$

$$\psi_P(\epsilon) := \cos \left(\delta^{-1}(1 - \sqrt{\frac{\det(P)}{\lambda_{\max}(P)^n} \cos(\delta^{-1}(\epsilon))^n}}\right)$$  

(18)

(19)

with $\delta(\cdot)$ and $\delta_n(\cdot)$ being given in (12) and (14) respectively.

**Proof:** We drop the subscript $N$ in $\omega_N$ in the proof for convenience. Let $P = L^TL$ be the Cholesky decomposition of $P$, and let

$$\tilde{\omega} := \left\{ \left(\frac{Lz}{\|Lz\|}, \sigma\right) : (z, \sigma) \in \omega \right\} \subset S \times M.$$  

(20)

We first show that $\rho(A_K) \leq \frac{\gamma}{\varphi_P(\epsilon)}$. The proof is divided into two steps.

**Step 1:** We show that if $\omega$ is an $\epsilon$-covering of $S \times M$, then $\tilde{\omega}$ is an $\epsilon$-covering of $S \times M$ for some $\epsilon > 0$ defined below, that is, for any $(\tilde{x}, \tilde{\sigma}) \in \tilde{\omega}$, we want to show that there exists $(\tilde{x}, \sigma) \in \omega$ such that $|\tilde{x}^TL\tilde{x}| \geq \cos(\tilde{\theta})$ where $\tilde{\theta} = \delta^{-1}(\tilde{\epsilon})$. Note that any $\tilde{x} \in \tilde{\omega}$ can be uniquely expressed as $\tilde{x} = Lx/\|Lx\|$ for some $x \in S$. Let $\tilde{x} = Lx/\|Lx\| \in S$. Since $\omega$ is an $\epsilon$-covering of $S \times M$, from the definition, there exists $(x, \sigma) \in \omega$ such that $|x^Tz| \geq \cos(\theta)$ when $\theta = \delta^{-1}(\epsilon)$. Hence, it implies that $|x^Tz| \geq \sqrt{2 - 2\cos(\theta)}$ or $|x^TLx| \leq \sqrt{2 - 2\cos(\theta)}$. Now, let us look at the value $|\tilde{x}^TL\tilde{x}|$ where

$$\tilde{z} = Lz/\|Lz\| \in \tilde{\omega}.$$  

Without loss of generality, we consider the case that $|x - z| \leq \sqrt{2 - 2\cos(\theta)}$. Hence,

$$\frac{|Lz^TLz|}{\|Lz\|^2} = \frac{|x^TPz|}{\|Lz\|^2} - \frac{|(x-z)^TP(x-z)|}{2\|Lz\|^2} \geq \frac{|x^TPz + z^TPz - |(x-z)^TP(x-z)|}{2\|Lz\|^2} \geq 1 - \frac{|(x-z)^TP(x-z)|}{2\|Pz\|^2}.$$  

(21)

$$\geq 1 - \frac{|x^TPz|}{2\lambda_{\min}(P)} \geq 1 - \frac{|x^TPz|}{\kappa(P)(1 - \cos(\theta))} = \varphi_P(\epsilon).$$  

(22)

(23)

where (21) follows from the fact that $x^TPz + z^TPz \geq 2\|Pz\|^2$ and (22) is a direct consequence of the inequality $\lambda_{\min}(P)I \preceq P \preceq \lambda_{\max}(P)I$. Hence, $\tilde{\omega}_N$ is a $\epsilon$-covering of $S \times M$ with $\tilde{\epsilon} = \delta(\tilde{\theta})$ and $\cos(\tilde{\theta}) = 1 - \kappa(P)(1 - \cos(\theta))$. From Definition 3, we know that $\tilde{\omega}_N^\sigma$ is an $\epsilon$-covering of $S$ for all $\sigma \in M$. This implies that $\cos(\tilde{\theta})B \subseteq \text{conv}(\pm\tilde{\omega}_N)$ for all $\sigma \in M$, where $\pm\tilde{\omega}_N$ denotes the union of $\tilde{\omega}_N$ and $-\tilde{\omega}_N$, i.e., $\tilde{\omega}_N \cup -\tilde{\omega}_N$. Hence,

$$\cos(\tilde{\theta})B \subseteq \bigcup_{\sigma \in M} \text{conv}(\pm\tilde{\omega}_N).$$  

(25)

Let $\tilde{A}_\sigma := LA_\sigma L^{-1} + LBKL^{-1}$ for all $\sigma \in M$. It holds that for all $\sigma \in M$ and $z \in \pm\tilde{\omega}_N^\sigma, |\tilde{A}_\sigma z| \leq \gamma \|z\|$. This, together with (25), implies that, $\forall \sigma \in M$,

$$\cos(\tilde{\theta})A_\sigma B \subseteq \frac{1}{\gamma}A_\sigma \text{conv}(\pm\tilde{\omega}_N) \subseteq \frac{1}{\gamma} \text{conv}(\pm A_\sigma \tilde{\omega}_N) \subseteq B.$$  

As a consequence, we obtain that $\forall \sigma \in M$,

$$\langle A_\sigma + BK \rangle^TP(A_\sigma + BK) \leq \left(\frac{\gamma}{\cos(\tilde{\theta})}\right)^2 P.$$  

(26)

Finally, by combining (26) with Proposition 1, we get that $\frac{\cos(\tilde{\theta})}{\varphi_P(\epsilon)}$ is an upper bound on $\rho(A_K)$.

We then prove that $\rho(A_K) \leq \frac{\gamma}{\psi_P(\epsilon)}$. Similarly, let $\omega^\sigma := \{x : (x, \sigma) \in \omega\}, \forall \sigma \in M$. We consider an arbitrary $\sigma \in M$. Since $\omega^\sigma$ is an $\epsilon$-covering of $S$, $\cos(\delta^{-1}(\epsilon))B \subseteq \text{conv}(\pm\omega^\sigma)$. Hence,

$$\lambda(\text{conv}(\pm\omega^\sigma)) \geq \cos(\delta^{-1}(\epsilon))\lambda(B).$$  

(27)

where $\lambda(\cdot)$ denotes the Lebesgue measure. Note that $\omega^\sigma$ as defined in (24) can be expressed as $\omega^\sigma = \{\frac{Lz}{\|Lz\|} : z \in \omega^\sigma\}$, which leads to the following relation

$$\text{conv}(\pm\omega^\sigma) \supseteq \frac{L}{\sqrt{\lambda_{\max}(P)}} \text{conv}(\pm\omega^\sigma).$$  

(28)
Combining (27) and (28) yields
\[
\frac{\lambda(\text{conv}(\pm \tilde{\omega}^\sigma))}{\lambda(B)} \geq \sqrt{\frac{\det(P)}{\lambda_{\text{max}}(P)n}} \cos(\delta^{-1}(\epsilon))^n \quad (29)
\]
which implies that
\[
\frac{\lambda(B \setminus \text{conv}(\pm \tilde{\omega}^\sigma))}{\lambda(B)} \leq 1 - \sqrt{\frac{\det(P)}{\lambda_{\text{max}}(P)n}} \cos(\delta^{-1}(\epsilon))^n \quad (30)
\]
We claim that the distance from \(\partial \text{conv}(\pm \tilde{\omega}^\sigma))\) to the origin is bounded from below by
\[
\psi_p(\epsilon) = \cos(\delta^{-1}(1 - \sqrt{\frac{\det(P)}{\lambda_{\text{max}}(P)n}} \cos(\delta^{-1}(\epsilon))^n))
\]
which is valid by contradiction. Suppose there exists a point \(x \in \partial \text{conv}(\pm \tilde{\omega}^\sigma))\) such that \(|x| < \psi_p(\epsilon)\). Then, there exists a hyperplane \(h^\top x = 1\) such that \(\frac{1}{\partial \text{conv}(\pm \tilde{\omega}^\sigma))} < \psi_p(\epsilon)\) and \(h^\top x \leq 1\) for any \(x \in \text{conv}(\pm \tilde{\omega}^\sigma))\). By symmetry, it also holds that \(-h^\top x \leq 1\) for any \(x \in \text{conv}(\pm \tilde{\omega}^\sigma))\). Let \(\hat{B}\) denote the set \(\{x \in B : h^\top x \geq 1\} \cup \{x \in B : -h^\top x \geq 1\}\). By construction, \(B \subseteq B \setminus \text{conv}(\pm \tilde{\omega}^\sigma))\), which means that \(\lambda(\hat{B}) \leq (1 - \sqrt{\frac{\det(P)}{\lambda_{\text{max}}(P)n}} \cos(\delta^{-1}(\epsilon))^n)\lambda(B))\) from (30).

We recall from (42) that the volume of \(\hat{B}\) is \(T(1 - \frac{1}{\lambda(\tilde{\omega}^\sigma)}; \frac{n+1}{2}, \frac{1}{2})\lambda(B))\), which, from the condition that \(\frac{1}{\partial \text{conv}(\pm \tilde{\omega}^\sigma))} < \psi_p(\epsilon)\), implies that \(\lambda(\hat{B}) \geq (1 - \psi_p(\epsilon)^2; \frac{n+1}{2}, \frac{1}{2})\lambda(B) = (1 - \sqrt{\frac{\det(P)}{\lambda_{\text{max}}(P)n}} \cos(\delta^{-1}(\epsilon))^n)\lambda(B)\), where the equality is by the definition of \(\psi_p(\epsilon)\) in (19). This leads to a contradiction. Therefore, we conclude that \(\psi_p(\epsilon)B \subseteq \text{conv}(\pm \tilde{\omega}^\sigma))\). As \(\sigma\) is chosen arbitrarily, this implies that
\[
\frac{\psi_p(\epsilon)}{\gamma} \leq \frac{1}{\gamma} \tilde{\omega}^\sigma \subseteq \text{conv}(\pm \tilde{\omega}^\sigma)) \subseteq B.
\]
Thus, \(\forall \sigma \in M\),
\[
(A_\sigma + BK)^T P(A_\sigma + BK) \leq \left(\frac{\gamma}{\psi_p(\epsilon)}\right)^2 P \quad (31)
\]
Putting (26) and (31) together, we arrive at (17).

The result in Theorem 1 allows to establish a probabilistic stability certificate from the solution of the sampled problem (7).

**Corollary 1:** Given \(N \in \mathbb{Z}^+\), let \(\omega_N = \{(x_i, \sigma_i)\}_{i=1}^N\) be i.i.d with respect to the uniform distribution \(P\) over \(\mathbb{S} \times M\). Let \(\gamma(\omega_N)\), \(P(\omega_N)\), and \(K(\omega_N)\) be obtained from Algorithm 1. Then, for any \(\epsilon \in (0, 1)\), with probability no smaller than \(1 - B(\epsilon; N)\),
\[
\rho(A_{K(\omega_N)}) \leq \frac{\gamma(\omega_N)}{\max(\psi_p(\omega_N)(\epsilon), \psi_p(\omega_N)(\epsilon))} \quad (32)
\]
deriving \(\rho(A_{K(\omega_N)})\) is defined in (3) with \(K = K(\omega_N)\) and \(B(\epsilon; N), \varphi_p(\omega_N)(\epsilon)\) and \(\psi_p(\omega_N)(\epsilon)\) are given in (16), (18) and (19) respectively.

**Proof:** From Lemma 2, with probability no smaller than \(1 - B(\epsilon; N)\), \(\omega_N\) is an \(\epsilon\)-covering of \(\mathbb{S} \times M\). Combining this with Theorem 1, we obtain the result above, since Algorithm 1 always generates feasible iterations.

**Remark 2:** The results above bear some similarities with the probabilistic stability guarantees in [21], [24], [25] which are concerned with autonomous systems, the major difference is that the bound in this paper is applicable to any feasible solution while [21], [24], [25] rely on the optimality of the solution. Let us also highlight that the bound in Theorem 1 can be considered as an improvement to the one in [31] in the sense that the additional term \(\varphi_p(\epsilon)\) prevents the bound from going unbounded when \(\kappa(P)(1 - \cos(\delta^{-1}(\epsilon))) \geq 1\), which happens when \(\epsilon\) is not sufficiently small. From a practical point of view, \(\epsilon\) has to be small for the bound in (17) to be meaningful. In such cases, \(\varphi_p(\epsilon)\) is often larger than \(\psi_p(\epsilon)\). Hence, the bound that is really used in practice is \(\gamma(\omega_N)\) in (17).

**Remark 3:** In some practical situations, what we receive is a set of input-state data, i.e., \(\{(x_i, u_i, x_i^+) : i = 1, 2, \ldots, N\}\) where \(x_i^+ = A_{\sigma_i} x_i + Bu_i\) and \(u_i\) is the \(i\)th input. As \(B\) is known, we can convert this data set into \(\{(x_i^+, x_i^+ - Bu_i) : i = 1, 2, \ldots, N\}\). We can then apply our approach on this converted data set. Furthermore, the states may not lie on the unit sphere. While the solution of the sampled problem in (7) does not change from a theoretical point of view, we can use the scaled data \(\{(x_i/\|x_i\|, (x_i^+ - Bu_i)/\|x_i\|) : i = 1, 2, \ldots, N\}\) to improve numerical stability. If the samples follow an isotropic Gaussian distribution centered at zero (with the covariance matrix being a scalar variance multiplied by the identity matrix) and are generated independently, the scaled points are uniformly distributed on the unit sphere and hence our probabilistic guarantees in Corollary 1 are still valid.

**D. Parallel computation**

To achieve high confidence in Theorem 3, a large number of samples are typically needed. As a result, the problems (11) and (10) at each iteration of Algorithm 1 quickly become demanding due to the increasing number of constraints. To circumvent this issue, inspired by the stochastic gradient descent and its variants [44], we propose a parallelized scheme for Algorithm 1 by dividing these constraints into small batches. More precisely, given \(L \in \mathbb{Z}^+\), we build \(L\) disjoint subsets of \(\omega_N\), denoted by \(\omega_N^l\), \(l = 1, \ldots, L\), with \(\bigcup_{l=1}^L \omega_N^l = \omega_N\). The choice of \(L\) and \(\{\omega_N^l\}_{l=1}^L\) depends on the number of computing resources available and their computation power. With this partition, we then solve the alternating minimization problems as defined in (11) and (10) individually for each batch, as shown in Algorithm 2. The solution of each subproblem provides a candidate descent direction. We then choose the solution that provides the lowest convergence rate via a line search heuristic (33)-(34), which guarantees feasibility of the iterate for all the constraints. The line search step also guarantees that \(\{\gamma_k\} \) does not increase along iterations. Note that (33) and (34) are both scalar optimization problems that can be easily solved. The probabilistic guarantee in Corollary 1 is still applicable as it only requires a feasible solution, though Algorithm 2 may produce a more conservative solution \(\gamma(\omega_N)\) compared with Algorithm 1.

**IV. SOS LYAPUNOV FRAMEWORK**

Quadratic stabilization of switched systems can be very restrictive in some cases. To reduce conservatism, we can use sum of squares (SOS) techniques, which have already been
used in [45], [46] to improve the bound on the JSR. In the framework of data-driven stability optimization, the application of SOS optimization has already proven useful for the case of autonomous systems in [24]. Here, we want to show that SOS techniques are also applicable for the stabilization problem.

Let us first recall some definitions in SOS optimization. We refer to [45] for the details. Given \( x \in \mathbb{R}^n \) and \( d \in \mathbb{Z}^+ \), let \( x[d] \in \mathbb{R}^\left(\frac{n(n+d-1)}{2}\right) \) denote the \( d \)-lift of \( x \) which consists of all possible monomials of degree \( d \), indexed by all the possible variables and \( N \) polynomial constraints, which is much more computationally demanding than Problem (7) depending on the degree \( d \). For a small data set, this problem can be solved by polynomial toolboxes [37]–[39] or general nonlinear solvers, such as IPOPT [47] and NLopt [48]. For a large data set, to reduce the complexity, we again make use of the structure in (38b) to develop an alternating minimization algorithm between \( P \) and \( K \) as Algorithm I for the quadratic case. For a given \( K \), we define:

\[
\mathcal{P}_d(\omega_N; K) \coloneqq \min_{\gamma \geq 0, P \succeq 1, \mathbb{R}^n} \gamma \quad \text{s.t.} \quad (A_\sigma x + B K x)[d] \succeq \gamma x[d] \lor P x[d], \quad \forall \sigma \in \omega_N
\]  

(38b)

From the definition of \( d \)-lift of vectors above, this is a polynomial optimization problem with \( mn + \binom{n+d-1}{d} \) variables and \( N \) polynomial constraints, which is much more computationally demanding than Problem (7) depending on the degree \( d \). For a small data set, this problem can be solved by polynomial toolboxes [37]–[39] or general nonlinear solvers, such as IPOPT [47] and NLopt [48]. For a large data set, to reduce the complexity, we again make use of the structure in (38b) to develop an alternating minimization algorithm between \( P \) and \( K \) as Algorithm I for the quadratic case. For a given \( K \), we define:

\[
\mathcal{P}_d(\omega_N; K) \coloneqq \min_{\gamma \geq 0, P \succeq 1, \mathbb{R}^n} \gamma \quad \text{s.t.} \quad (A_\sigma x + B K x)[d] \succeq \gamma x[d] \lor P x[d], \quad \forall \sigma \in \omega_N
\]  

(38b)

Indeed, one may observe that Problem (39) can be solved efficiently by using SDP solvers (like Mosek [49]) and bisection on \( \gamma \). While Problem (40) is still a polynomial optimization problem, there are only \( mn \) decision variables, which makes it easier to handle than Problem (38). To solve the problems in (39) and (40), we typically need an initial solution, in particular for the polynomial problem (40). In this alternating minimization algorithm, we set the initial solution to be the solution from the previous iteration, which leads to a convergent sequence of \( \gamma \). The details of this procedure is given in Algorithm 3.
Algorithm 3 Alternating minimization for SOS stabilization

Input: \((x_i, A \in \mathbb{R}^{m \times n})\) \(i=1, \ldots, N\), \(d\), and some tolerance \(\epsilon_{ tol} > 0\)
Output: \(\gamma_{ sos}(\omega_N), P_{ sos}(\omega_N), A_{ sos}(\omega_N)\)

1: Initialization: \(k \leftarrow 0, K_k \leftarrow 0, \gamma_k \leftarrow \max_{i \in [N]} \|A_{i}x_{i}\|\)
2: Obtain \(P_{k+1}\) by solving \(\mathcal{P}_d(\omega_N; K_k)\) via bisection on \(\gamma\) starting from \(\max_{i \in [N]} \|A_{i}x_{i}\|\)
3: Obtain \(\gamma_{k+1}\) and \(K_{k+1}\) by solving \(\mathcal{P}_d(\omega_N; K_{k+1})\) initialized at \(K = K_k\)
4: if \(\|\gamma_{k+1} - \gamma_k\| < \epsilon_{tol}\) then
5: \(\gamma_{ sos}(\omega_N) \leftarrow \gamma_{k+1}, P_{ sos}(\omega_N) \leftarrow P_{k+1}, A_{ sos}(\omega_N) \leftarrow K_{k+1}\)
6: Terminate;
7: else
8: Let \(k \leftarrow k + 1\) and go to Step 2.
9: end if

B. Stability guarantees via sensitivity analysis

With the aforementioned alternating minimization algorithm for SOS stabilization, we get a feasible solution, denoted by \((\gamma_{ sos}(\omega_N), P_{ sos}(\omega_N), A_{ sos}(\omega_N))\). Similar to Section III-C, we now derive stability guarantees for this solution. However, due to the polynomial lifting in (38), we lose convexity in the Lyapunov function, which prevents us from applying the geometric results in Section III-C to the SOS framework. In particular, the reasoning about (26) and (31) in the proof of Theorem 1 does not hold. Hence, we use an alternative way to derive probabilistic guarantees on the solution obtained from Algorithm 3.

Given any \(\epsilon \in (0, 1)\) and \(d \in \mathbb{Z}^+\) \((d \geq 1)\), let us define

\[
\phi(\epsilon, d) := \sum_{k=1}^{d} \binom{d}{k} (2 - 2 \cos (\delta^{-1}(\epsilon)))^{\frac{k}{2}}
\]

Similar to Theorem 1, we derive a stability certificate for SOS stabilization as stated in the following theorem.

**Theorem 2:** Given a sample set \(\omega_N \subset \mathbb{S} \times \mathbb{M}\) and an integer \(d \geq 1\), consider Problem (38). Let \((\gamma, P, K)\) be a feasible solution to Problem (38). Suppose \(\omega_N\) is an \(\epsilon\)-covering of \(\mathbb{S} \times \mathbb{M}\) for some \(\epsilon > 0\). Then,

\[
\rho(A_K) \leq \sqrt{\gamma} + \gamma d + \rho(A_K) d \sqrt{\kappa(P)} \phi(\epsilon, d)
\]

where \(\rho(A_K)\) is defined in (3), \(\phi(\epsilon, d)\) is given in (41), and

\[
\rho(A_K) := \max_{A \in A_K} \|A\|.
\]

**Proof:** We drop the subscript \(N\) in \(\omega_N\) in the proof for convenience. Since \(\omega\) is an \(\epsilon\)-covering of \(\mathbb{S} \times \mathbb{M}\), from Definition 3, for any \((x, \sigma) \in \mathbb{S} \times \mathbb{M}\), there exists \((z, \sigma) \in \omega\) such that \(\|z - x\| \geq \cos (\delta^{-1}(\epsilon))\), which implies that \(\|x - z\| \leq \sqrt{2 - 2 \cos (\delta^{-1}(\epsilon))}\) or \(\|z + x\| \leq \sqrt{2 - 2 \cos (\delta^{-1}(\epsilon))}\). Due to the fact that \((-z, \sigma)\) also satisfies (38b), we consider the case that \(\|x - z\| \leq \sqrt{2 - 2 \cos (\delta^{-1}(\epsilon))}\) without loss of generality.

From [45], it can be shown that

\[
\|x^{[d]} - z^{[d]}\| = \|x^{[d]} - z^{[d]}\| \
\leq \sum_{k=1}^{d} \binom{d}{k} \|x - z\|^k z^{d-k}
\]

where \(x^{[d]}\) denotes the \(d\)-fold Kronecker product. With this and some manipulations, we get that

\[
\|A_{\sigma}x + BKx\| \leq \|A_{\sigma}z + BKz\| + \|A_{\sigma}z + BKz\| (\|x^{[d]} - z^{[d]}\|)
\]

From Proposition 2, we then obtain (42).

**Remark 4:** When \(d = 1\), it becomes exactly the quadratic case, which means that Theorem 2 provides an alternative bound for the quadratic case. However, by numerical simulation, for reasonable values of \(P\) and \(\epsilon\), the bound in (17) is better than the one in (42) with \(d = 1\). In addition, Theorem 2 requires the information of \(\bar{\rho}(A_K)\), which is yet to be estimated.

Following the same arguments as in Section III-C, we can then establish probabilistic guarantees for this data-driven SOS framework. However, the bound in (42) relies on \(\bar{\rho}(A_K)\) which is not available. To handle this issue, we use the data set to estimate \(\bar{\rho}(A_K)\). Given a sample set \(\omega_N \subset \mathbb{S} \times \mathbb{M}\) and any \(K \in \mathbb{R}^{m \times n}\), we define the following problem:

\[
\eta^*(\omega_N, K) := \min_{\eta \geq 0} \eta
\]

s.t. \(\|A_{\sigma}x + BKx\| \leq \eta, \forall (x, \sigma) \in \omega_N\).

An upper bound on \(\bar{\rho}(A_K)\) is then given in the following proposition.

**Proposition 3:** Given a sample set \(\omega_N \subset \mathbb{S} \times \mathbb{M}\) and any \(K \in \mathbb{R}^{m \times n}\), let \(\eta^*(\omega, K)\) be defined as in (44). Suppose \(\omega_N\) is an \(\epsilon\)-covering of \(\mathbb{S} \times \mathbb{M}\) for some \(\epsilon > 0\). Then,

\[
\bar{\rho}(A_K) \leq \frac{\eta^*(\omega_N, K)}{\cos (\delta^{-1}(\epsilon))}
\]

where \(\bar{\rho}(A_K)\) is given in (43).

**Proof:** This result is a special case of Theorem 1 where \(P = I\), which implies that \(\varphi_P(\epsilon) = \cos (\delta^{-1}(\epsilon))\).

Based on the results above, we now present the main result of this section, which is a probabilistic stability certificate from the solution of the sampled problem (38).

**Theorem 3:** Given \(N \in \mathbb{Z}^+\), let \(\omega_N\) be i.i.d with respect to the uniform distribution \(\mathbb{P}\) over \(\mathbb{S} \times \mathbb{M}\). Let \((\gamma_{ sos}(\omega_N), P_{ sos}(\omega_N), A_{ sos}(\omega_N))\) be obtained from Algorithm
3 and \( \eta^*(\omega_N, K_{sos}(\omega_N)) \) be defined as in (44). For any \( \epsilon \in (0, 1) \), with probability no smaller than \( 1 - B(\epsilon; N) \),
\[
\rho(A_{K_{sos}(\omega_N)}) \leq \gamma_{sos}(\omega_N) \sqrt{1 + \left( 1 + \frac{\eta(\omega_N)}{\gamma_{sos}(\omega_N)} \right)} \sqrt{\kappa(P_{sos}(\omega_N))} \phi(\epsilon, d)
\]
where \( \phi(\epsilon, d) \) is given in (41), \( B(\epsilon; N) \) is defined as in (16), and
\[
\eta(\omega_N) := \eta^*(\omega_N, K_{sos}(\omega_N)) \left( \frac{\cos(\delta^{-1}(\epsilon))}{\delta^{-1}(\epsilon)} \right)
\]
where \( \delta(\cdot) \) denotes the measure of a symmetric spherical cap as shown in (12).

**Proof:** From Lemma 2, \( \omega_N \) is an \( \epsilon \)-covering of \( S \times M \) with probability no smaller than \( 1 - B(\epsilon; N) \). Combining Theorem 2 and Proposition 3, we obtain the result above. ■

### C. Parallelized scheme for SOS stabilization

Similar to Algorithm 2, we can also use a parallelized scheme for SOS stabilization. In fact, it is even more beneficial to use a parallelized scheme as Algorithm 3 involves a large number of polynomial constraints of order \( 2d \) at each iteration. Suppose we build \( L \) disjoint subsets \( \{\omega_N^i\}_{i=1}^L \) from \( \omega_N \) for a given \( L \in \mathbb{Z}^+ \). We solve the optimization problems in (39) and (40) with these subsets individually using SDP solvers (like Mosek [49]) and nonlinear solvers (like IPOPT [47] and NLopt [48]) respectively. At each iteration, the solver is initialized with the solution from the last iteration, which, together with the line search heuristics as shown in (48) and (49), guarantees a convergent sequence \( \{\gamma_k\}_{k \in \mathbb{Z}^+} \). The parallelized scheme for SOS stabilization is given in Algorithm 4.

### V. Data-driven switched LQR

In this section, we show that the proposed data-driven framework can be extended to infinite-horizon LQR problems of arbitrary switched linear systems. We consider the following infinite-horizon quadratic cost
\[
J_\infty(x, u, \sigma) = \sum_{\ell=0}^{\infty} L(x(\ell), u(\ell))
\]
where \( x, u \) and \( \sigma \) denote the state, control and switching sequences respectively, and \( L(x, u) = x^T Q x + u^T R u \) is the stage cost with \( Q > 0 \) and \( R > 0 \). With these definitions, the infinite-horizon LQR problem of System (1) can be cast as
\[
J^*(x) := \inf \sup_{u, \sigma \in M} J_\infty(x, u, \sigma)
\]
with \( x(0) = x \). Without any information on the switching signal, we only consider static feedback in the infinite-horizon LQR problem. Our goal is to find a quadratic upper bound \( x^T P x \) of \( J^*(x) \) with a static feedback \( u = K x \), i.e., we want to find a pair \((K, P)\) such that \( J_\infty(x, u, \sigma) \leq \|x(0)\|^2_P \) for all \( \sigma \in M^\infty \) with \( u(\ell) = K x(\ell) \) for all \( \ell \in \mathbb{Z}^+ \). When \((K, P)\) satisfies
\[
(A + BK)^T (P + BK) \preceq P - Q - K^T R K
\]
for all \( \sigma \in A \), following standard manipulations (see, e.g., [50], [51]), it can be shown that, for any switching sequence and any \( k \in \mathbb{Z}^+ \),
\[
\|x(0)\|^2_P - \sum_{\ell=0}^{k} L(x(\ell), u(\ell)) \geq \|x(k+1)\|^2_P
\]
with \( u(\ell) = K x(\ell) \), which implies that \( J^*(x) \leq \|x\|^2_P \) for all \( x \in \mathbb{R}^n \). For linear systems (when \( A \) is a singleton), \((K, P)\) can be obtained by solving the Algebraic Riccati Equation, see [52, Chapter 2] for details. For multiple dynamics matrices, using the Schur complement formula with \( S = P^{-1} \) and \( Y = K S \), a model-based solution of (52) can be obtained by solving

---

**Algorithm 4 Parallel alternating minimization for SOS stabilization**

**Input:** \( \{(x_i, A_i, x_i)\}_{i=1}^N \), and some tolerance \( \epsilon_{tot} > 0 \)

**Output:** \( \gamma_{sos}(\omega_N), P_{sos}(\omega_N), K_{sos}(\omega_N) \)

1: **Initialization:** Create a partition \( \{\omega_N^i\}_{i=1}^N \) on \( \omega_N \): Let \( k \leftarrow 0 \), \( K_k \leftarrow 0 \) and \( P_k \leftarrow I \).

2: **Minimization on** \( P \) ........................................

3: for \( \ell = 1, 2, \ldots, L \) do

4: Solve \( \bar{P}_d(\omega_N^i; K_k) \) and let the solution be denoted by \( \bar{P}(\omega_N^i; P_k) \);

5: Compute
\[
\gamma_k^i \leftarrow \min_{\lambda \in [0,1]} \max_{(x, \sigma) \in \omega_N} \|\bar{A}_x x + B K_k x \|_d \|P_k^2(\lambda)\| \leq \gamma_{sos}(\omega_N), P_{sos}(\omega_N), K_{sos}(\omega_N) \rightarrow K_{sos}(\omega_N), \text{ and terminate};
\]
the following LMI problem, see [53, Theorem 1],
\[
\begin{aligned}
\min_{S,Y} \quad & -\log \det(S) \\
\text{s.t.} \quad & \begin{pmatrix}
SP + YB
& S
\end{pmatrix} \succeq 0,
\end{aligned}
\tag{54a}
\]
\[
\begin{pmatrix}
AS + BY
& S
\end{pmatrix} \begin{pmatrix} S \\ 0 \end{pmatrix} = 0,
\]
\[
\begin{pmatrix}
0
& Q^{-1}
\end{pmatrix} \begin{pmatrix} Y \\ 0 \end{pmatrix} = 0,
\] 
\[
\forall A \in \mathcal{A}.
\tag{54b}
\]

### A. Sampled LQR problem

In the model-free case, given a sample set \( \omega_N \subset \mathbb{S} \times \mathcal{M} \), a sample-based relaxation of (52) is given as follows
\[
\begin{align}
(A_x + BKx)^\top P(A_x + BKx) & \leq x^\top (P - Q - K^\top RK)x, \forall \omega \in \omega_N.
\end{align}
\tag{55}
\]

However, a solution to (55) may not be valid for (52) due to the randomness in the sampling. To deal with this issue, we introduce a scaling parameter \( \xi \in (0, 1) \) and formulate the following problem:
\[
\begin{align}
\min_{P, K} \quad & \text{tr}(P) \\
\text{s.t.} \quad & (A_x + BKx)^\top P(A_x + BKx) \\
& \leq \xi^2 x^\top (P - Q - K^\top RK)x, \forall \omega \in \omega_N,
\end{align}
\tag{56a}
\]
\[
P \succeq Q + K^\top RK
\tag{56b}
\]

where the constraint (56c) is imposed as (56b) does not guarantee that \( P - Q - K^\top RK \) is positive semidefinite. When the data set is sufficiently rich, the problem (56) leads to the following robustness property.

**Theorem 4:** Given a sample set \( \omega_N \subset \mathbb{S} \times \mathcal{M} \), and \( \xi \in (0, 1) \), let \( (P, K) \) be a feasible solution to Problem (56) and \( Z = P - Q - K^\top RK \). Suppose \( \omega_N \) is an \( \varepsilon \)-covering of \( \mathbb{S} \times \mathcal{M} \). Then it follows that (56) leads to the following robustness property.

For the sake of simplicity, we denote the Cholesky decompositions of \( P - Q - K^\top RK \) and \( L^\top L \) as \( L = \operatorname{chol}(P - Q - K^\top RK) \). From the same reasoning as in the proof of Theorem 1, it can be shown that \( \omega \) is an \( \varepsilon \)-covering of \( \mathbb{S} \times \mathcal{M} \) with \( \varepsilon = \delta(\bar{\theta}) \) and 
\[
\cos(\theta) = \varphi_Z(\varepsilon) = 1 - \kappa(Z)(1 - \cos(\bar{\theta})) \text{ where } \theta = \delta^{-1}(\varepsilon).
\]

Again, as in (24), we define
\[
\mathcal{S} := \left\{ \tilde{z} \in \mathbb{S} : (\bar{z}, \bar{\sigma}) \in \omega \right\} \subset \mathbb{S} \times \mathcal{M}.
\tag{58}
\]

From the same reasoning as in the proof of Theorem 1, it can be shown that \( \mathcal{S} \) is an \( \varepsilon \)-covering of \( \mathbb{S} \times \mathcal{M} \) with \( \varepsilon = \delta(\bar{\theta}) \) and
\[
\cos(\theta) = \varphi_Z(\varepsilon) = 1 - \kappa(Z)(1 - \cos(\bar{\theta})) \text{ where } \theta = \delta^{-1}(\varepsilon).
\]

Again, as in (24), we define
\[
\mathcal{S} := \left\{ z : (x, \sigma) \in \omega \right\}, \forall \sigma \in \mathcal{M}.
\tag{59}
\]

Following (25), we have \( \varphi_Z(\varepsilon) \mathcal{S} \subseteq \bigcup_{\sigma \in \mathcal{M}} \mathcal{S}(x, \sigma) \subseteq \varphi_Z(\varepsilon) \mathcal{S} \). With Cholesky decomposition of \( P = L^\top L \), it also holds that \( \|A_{\sigma x}z\| \leq \|z\| \) for all \( \sigma \in \mathcal{M} \) and \( z \in \mathcal{S}^{\sigma} \), where
\[
\varphi_Z^*(\varepsilon, \alpha) := 1 - \alpha(1 - \cos(\delta^{-1}(\varepsilon))).
\tag{63}
\]

With this definition, it can be verified that, when \( \kappa(P - Q - K^\top RK) \leq \bar{\kappa} \) and \( \xi \leq \xi^*(\varepsilon, \bar{\kappa}) \), the constraint (62c) is satisfied.

We now present the overall procedure. At the initialization, we find the value of \( \xi \) that is closest to 1 such that (56b) is
feasible with $K = 0$, denoted by $\xi_0$. Note that, given the a-priori upper bound $\bar{\kappa}$, the constraint $\kappa(P - Q - K^\top R K) \leq \bar{\kappa}$ can be rewritten as $\lambda_{\text{max}}(P - Q - K^\top R K) \leq \bar{\kappa} \nu$ and $\lambda_{\text{min}}(P - Q - K^\top R K) \geq \nu$ for some $\nu > 0$, which are equivalent to $\nu I \leq P - Q - K^\top R K \leq \bar{\kappa} \nu I$. We then proceed by optimizing $(K, \xi)$ and $P$ in an alternating way. In the optimization of $(K, \xi)$, the goal is to decrease $\xi$ to a certain value below $1$ but not to minimize $\xi$ as much as possible. For this reason, we penalize the distance of the current $\xi$ to the previous value at each iteration in order to generate a smooth sequence $\{\xi_k\}$. Even when $P$ is fixed, the constraints (56b) and (56c) are still nonlinear. Again, we use the Schur complement formula to convert these constraints into LMIs as follows:

\[
\begin{pmatrix}
  x^\top P x - x^\top Q x & (A_x x + BK x)^\top P x + x^\top K^\top \\
  (A_x x + BK x) & \xi^2 P \\
  K x & 0 \\
  K^\top R^{-1} & 0
\end{pmatrix} \geq 0, \quad \forall (x, \sigma) \in \omega_N,
\]

\[
\begin{pmatrix}
  P - Q & K^\top \\
  K & R^{-1}
\end{pmatrix} \geq 0.
\]

The optimization problem involving $(K, \xi)$ is formulated in (67). Replacing $\xi^2$ with a new variable $\eta$ yields a convex problem. To ensure the constraint $\kappa(P - Q - K^\top R K) \leq \bar{\kappa}$ at each iteration, the optimization of $(K, \xi)$ is followed by a backtracking step. After $K$ and $\xi$ are updated, we optimize over $P$ subject to the constraints (56b), (56c) and (66c). The details are given in Algorithm 5. This algorithm can also be parallelized following the same idea as in Algorithm 2 and Algorithm 4.

Based on Theorem 4, a probabilistic guarantee is presented below for the proposed data-driven LQR.

**Corollary 2:** Consider Problem (52) with $Q, R > 0$. Given $N \in \mathbb{Z}^+$, let $\omega_N = \{(x_i, \sigma_i)\}_{i=1}^N$ be i.i.d with respect to the uniform distribution $P$ over $\mathbb{S} \times M$. For any $\epsilon \in (0, 1)$, we define $\xi^*(\epsilon)$ as in (63). Let $(\xi(\omega_N), P(\omega_N), K(\omega_N))$ be a solution obtained from Algorithm 5 and $Z(\omega_N) := P(\omega_N) - Q - K(\omega_N)^\top R K(\omega_N)$. Then, with probability no smaller than $1 - B(\epsilon; N)$, the following statement holds: if $\xi(\omega_N) \leq \xi^*(\epsilon, \nu (Z(\omega_N)))$, $(P(\omega_N), K(\omega_N))$ is a feasible solution to (52).

**Proof:** From Lemma 2, with probability no smaller than $1 - B(\epsilon; N)$, $\omega_N$ is an $\epsilon$-covering of $\mathbb{S} \times M$. Then, according to Theorem 4, if $\xi(\omega_N) \leq \xi^*(\epsilon, \nu (Z(\omega_N)))$, $(P(\omega_N), K(\omega_N))$ is a feasible solution.

**Remark 5:** From (67), the sequence $\{\xi_k\}$ obtained from Algorithm 5 is monotonically non-increasing. With the constraint in (66d), the sequence $\{\xi_k\}$ is bounded from below and thus is convergent.

**VI. Numerical Experiments**

In this section, we demonstrate the proposed data-driven control framework on several numerical examples. The codes are written in Julia and available at [https://github.com/zhemingwang/DataDrivenSwitchControl](https://github.com/zhemingwang/DataDrivenSwitchControl).

### A. Quadratic stabilization for multiple examples

We consider quadratic stabilization for several systems. We first consider an example that can be quadratically stabilized.

Consider the following switched linear system with $n = 2$, $m = 1$ and $M = 3$:

\[
A_1 = \begin{pmatrix} 0.7 & 0.16 \\ 1.1 & -1.1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.4 & -0.84 \\ 0.83 & 0.35 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0.37 & 0.96 \\ 0.34 & -1.2 \end{pmatrix},
\]

\[
B = \begin{pmatrix} -0.9 \\ -1.2 \end{pmatrix}.
\]

To show that this is not a trivial example, we compute $\rho(A) = 1.544$, the ISR of the open-loop system, using the ISR toolbox [54]. We also compute $\gamma^* = 0.8756$ as defined in (4) by solving (5) with bisection on $\gamma$ to check feasibility of quadratic stabilization for this example.

First, let $N = 2000$ and set the confidence level to $B(\epsilon; N) = 0.01$. The corresponding value of $\epsilon$ can be computed via bisection using (16). With this setting, the upper
bound in (17) is valid with probability larger than 99%. For convenience, let

$$\gamma(\omega_N) := \frac{\gamma(\omega_N)}{1 - \kappa(P(\omega_N))(1 - \cos(\delta^{-1}(\epsilon)))}.$$  

We then sample $\omega_N$ according to the uniform distribution on $\mathbb{S} \times \mathcal{M}$ and apply Algorithm 1 with the tolerance being $\epsilon_{tol} = 10^{-3}$. The obtained solution is:

$$\gamma(\omega_N) = 0.8836, \quad K(\omega_N) = \begin{pmatrix} 0.6626 & -0.4359 \end{pmatrix},$$

$$P(\omega_N) = \begin{pmatrix} 1.1302 & 0.5480 \\ 0.5480 & 3.3064 \end{pmatrix}. \quad \text{ (17)}$$

The bound obtained from (17) is $\gamma(\omega_N) = 0.8873$. To empirically verify the solution, we compute $\rho(A_{K(\omega_N)}) = 0.8767$, the JSR of the closed-loop system with $K(\omega_N)$, using the JSR toolbox [54].

We then apply the proposed approach to higher dimensional examples. Again, the confidence level is set to 0.01, i.e., $B(\epsilon; N) = 0.01$. We choose different values of $N$ and compute the corresponding $\epsilon$ that satisfies $B(\epsilon; N) = 0.01$. The dynamics matrices $\mathcal{A}$ and the input matrix $B$ are generated randomly in a way that each entry is chosen from the uniform distribution over $[-1, 1]$. Note that these random examples may not be stabilizable by a static linear feedback. Hence, in the simulation, we only compute the upper bound $\gamma(\omega_N)$ and compare it to the true solution $\gamma^*$ defined in Problem (6). Finally, we divide the data set into a number of subsets of size 1000 and use the parallelized scheme in Algorithm 2. The results are shown in Figure 2. As expected, the sample size needed to reach the true white-box solution increases as the system dimension and the number of modes increase.

where SOS stabilization outperforms quadratic stabilization. Consider the following switched linear system with 2 modes, which is generated randomly with a procedure similar as above:

$$A_1 = \begin{pmatrix} -1.6856 & -0.1665 \\ 0.7785 & -1.6321 \end{pmatrix}, \quad B = \begin{pmatrix} 0.1975 \\ 0.8640 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -0.2915 & -3.2824 \\ 3.9761 & -0.0274 \end{pmatrix}. \quad \text{ (17)}$$

For different values of $N$, let $\epsilon$ be chosen such that $B(\epsilon; N) = 0.01$. We then compute the upper bounds on the JSR according to the results in Section III-C and Section IV-B. For SOS stabilization, we consider the case of $d = 2$. To speed up the computation, we use parallelized schemes in Algorithm 2 and Algorithm 4 where the size of the subsets is set to be 1000. The results are given in Figure 3, which shows that the JSR upper bound from SOS stabilization becomes tighter as $N$ increases. When $N = 25000$, the solutions for the two algorithms are: $K(\omega_N) = [-2.0257, 0.5407]$ and $K_{\text{sos}}(\omega_N) = [-2.2517, 0.9063]$. In addition to the probabilistic bounds in Figure 3, we also compute the actual JSR of the closed-loop system using the JSR toolbox [54]: $\rho(A_{K(\omega_N)}) = 2.6002$ and $\rho(A_{K_{\text{sos}}(\omega_N)}) = 2.1188$. As a reference, we also solve the white-box quadratic stabilization problem as given in (5) and obtain the white-box solution $K^* = [-0.2831 - 0.2965]$. With this, we again compute the actual JSR [54]: $\rho(A_{K^*}) = 2.5582$, as indicated by the dashed line in Figure 3. From these computations, we can see that the data-driven solution from SOS stabilization even outperforms the solution from the white-box quadratic stabilization.

![Fig. 2. Convergence of the sample-based solution to the true solution for systems of different dimensions and modes.](image)

**B. Quadratic stabilization versus SOS stabilization**

For stability analysis of switched linear systems, SOS Lyapunov functions often provide tighter bounds than quadratic Lyapunov functions, see [45], [46] for a few examples. Recently, we have also found out that it is beneficial to use SOS Lyapunov functions in data-driven stability analysis in [24] with numerical examples. Here, we show that this is also the case in the stabilization problem. We give such an example

![Fig. 3. Comparison of quadratic stabilization and SOS stabilization: the dashed line is the JSR of the white-box quadratic stabilization solution.](image)

**C. Building temperature regulation with LQR**

Buildings can be viewed as complex control systems with both continuous and discrete dynamics. For instance, events like opening/closing doors and windows instantly affect the dynamic evolution of the zone temperature. To capture these hybrid behaviors, hybrid RC networks are often used in the modeling of building systems [55], [56]. Consider a building
with three zones as shown in Figure 4. Its thermal RC model is described as below, \(i = 1, 2, 3,\)
\[
c_i T_i = \sum_{j \neq i} \frac{T_j - T_i}{R_{ji}} + \frac{T_0 - T_i}{R_i^0} + m_i c_p (T_s - T_i) + q_i
\]
where \(i\) is the index of the zone, \(T_i\) is the temperature of zone \(i\), \(T_0\) is the temperature of outside air, \(c_i\) is the thermal capacitance of the air in zone \(i\), \(R_{ij}\) denotes the thermal resistances between zone \(i\) and zone \(j\), \(R_i^0\) denotes the thermal resistance between zone \(i\) and the outside environment, \(c_p\) is the specific heat capacity of air, \(T_s\) is the temperature of the supply air delivered to zone \(i\), \(m_i\) is the flow rate into zone \(i\) and \(q_i\) is the thermal disturbance from internal loads like occupants and lighting. The temperatures of the supply air and the outside environment are known, i.e., \(T_s\) and \(T_o\) are available. The thermal disturbance is estimated as:
\(q_1 = 0.11\text{kJ/s}, q_2 = 0.11\text{kJ/s}, q_3 = 0.12\text{kJ/s}.\) Other system parameters are given in Table I. The control objective is to steer the temperature of each zone to \(T_{\text{target}} = 24^\circ\text{C}.\)

![System schematic of the building](image)

**Fig. 4.** System schematic of the building

| Symbol     | Value          | Units          |
|------------|----------------|----------------|
| \(c_i, \forall i\) | 1.375 \times 10^3 | kJ/K          |
| \(c_p\)    | 1.012          | kJ/(kg \cdot \text{K}) |
| \(R_{12} = R_{21}\) | 1.5           | K/kW          |
| \(R_{13} = R_{31}\) | 3             | K/kW          |
| \(R_{23} = R_{32}\) | 2.7           | K/kW          |
| \(T_s\)    | 16             | °C            |
| \(T_o\)    | 32             | °C            |

**TABLE I**

**SYSTEM PARAMETERS**

By letting \(Q_i^{AC} = m_i c_p (T_s - T_i)\), an equivalent linearized model is obtained:
\[
c_i T_i = \sum_{j \neq i} \frac{T_j - T_i}{R_{ji}} + \frac{T_0 - T_i}{R_i^0} + Q_i^{AC} + q_i, i = 1, 2, 3.
\]
The steady input for the given targeted temperature is \(Q_i^{AC} = -\frac{T_o - T_{\text{target}}}{R_i^0} - q_i\) for all \(i\). Let \(x_i = T_i - T_{\text{target}}\) and \(u_i = Q_i^{AC} - Q_i^{AC}\) for all \(i\). We get
\[
\dot{x}_i = \sum_{j \neq i} \frac{x_j - x_i}{c_i R_{ji}} - \frac{x_i}{c_i R_i^0} + \frac{1}{c_i} u_i, i = 1, 2, 3.
\]

As the doors are frequently and unpredictably open and closed, this is a typical switching system in which the thermal resistances \(R_{13}\) (or \(R_{31}\)) and \(R_{23}\) (or \(R_{32}\)) are changing arbitrarily. For each door, we consider two modes: "open" and "closed". Hence, for the overall system, there are 4 modes in total, see Table II. The values of these thermal resistances are assumed to be unknown. In the simulation, we only use them to generate synthetic data.

|            | "open" | "closed" |
|------------|--------|----------|
| \(R_{13} = R_{21}\) (K/kW) | 0.8    | 1.2      |
| \(R_{23} = R_{32}\) (K/kW) | 0.8    | 1.2      |

**TABLE II**

**THERMAL RESISTANCES OF DIFFERENT MODES.**

We discretize the continuous-time system with the sampling time \(\tau = 3\) minutes. Here, we use the Euler forward method for its simple implementation. See [57] for an extensive study on different discretization methods for buildings. The discretized system is given below:
\[
x_i(t + 1) - x_i(t) = \sum_{j \neq i} \frac{x_j(t) - x_i(t)}{c_i R_{ji}} - \frac{x_i(t)}{c_i R_i^0} + \frac{u_i(t)}{c_i}, i = 1, 2, 3.
\]

We now design the LQR for the switching system above in a data-driven fashion. Let the LQR parameters be \(Q = I\) and \(R = 0.02I\). For different values of \(N\), we generate the data set \(\omega_N\). The confidence level is set to be \(\bar{B}(\epsilon; N) = 0.01\) in all the cases and the corresponding \(\epsilon\) is computed via bisection. We then use Algorithm 5 with \(\bar{K} = 100\) to obtain a feasible solution to the sampled LQR problem in (56). Let

\[
\xi(\omega_N) := \xi^{\star}(\epsilon, \kappa; (P(\omega_N) - Q - K(\omega_N)^\top RK(\omega_N)))
\]

From the discussions in Section V, the value \(\xi(\omega_N)\) can be considered as an indicator for feasibility (provided that \(\omega_N\) is an \(\epsilon\)-covering of \(S \times M\): \((P(\omega_N), K(\omega_N))\) is a feasible solution when \(\xi(\omega_N) \leq 1\). We show the values of \(\xi(\omega_N)\) as the size of the data set \(\omega_N\) increases in Figure 5. From this curve, we can see that \(\xi(\omega_N)\) becomes less than 1 when \(N \geq 8000\). We also show the LQR solution when \(N = 12000\) below

\[
K(\omega_N) = \begin{pmatrix}
-3.3773 & -0.5579 & -0.6681 \\
-0.5580 & -3.3763 & -0.6660 \\
-0.6683 & -0.6686 & -3.2397
\end{pmatrix}
\]

\[
P(\omega_N) = \begin{pmatrix}
1.4041 & 0.1138 & 0.1333 \\
0.1138 & 1.4041 & 0.1333 \\
0.1333 & 0.1333 & 1.3787
\end{pmatrix}
\]

As a way to validate the sample-based solution, we compute the white-box LQR solution by solving the LMI inequalities.
in (54) and the solution is given below

$$K^* = \begin{pmatrix} -3.1736 & -0.4840 & -0.5938 \\ -0.4840 & -3.1736 & -0.5938 \\ -0.5882 & -0.5882 & -3.0320 \end{pmatrix},$$

$$P^* = \begin{pmatrix} 1.3844 & 0.1085 & 0.1270 \\ 0.1085 & 1.3844 & 0.1270 \\ 0.1270 & 0.1270 & 1.3602 \end{pmatrix}. $$

The relative differences between the two solutions are given by $\|K(\omega_N) - K^*\|/\|K^*\| \times 100\% = 8.44\%$ and $\|P(\omega_N) - P^*\|/\|P^*\| \times 100\% = 1.93\%$, which suggests that the sample-based solution is a quite good approximation. Finally, we show the evolution of the temperatures of the three zones with the controller $u = K(\omega_N)x$ in Figure 6.

With SOS Lyapunov functions, we end up with a nonlinear optimization problem with a set of polynomial constraints. We then propose alternating minimization algorithms to solve these problems by making use of the underlying structure. In both cases, we develop parallelized schemes that allow to handle high-dimensional systems. Using the notions of covering/packing numbers, we also provide probabilistic stability guarantees via geometric analysis for the quadratic Lyapunov technique and sensitivity analysis for the SOS Lyapunov technique. Finally, we show that the proposed data-driven framework can be extended to LQR design of switched linear systems.

Fig. 5. Feasibility measure of the LQR solution

Fig. 6. Temperatures of the three zones with the LQR controller.

**VII. Conclusions**

We have presented a data-driven control framework for stabilization of black-box switched linear systems in which the dynamics matrices and the switching signal are unknown using quadratic and SOS Lyapunov functions. With quadratic Lyapunov functions, the stabilization problem is formulated as a biconvex problem using a finite number of trajectories. With SOS Lyapunov functions, we end up with a nonlinear optimization problem with a set of polynomial constraints. We then propose alternating minimization algorithms to solve these problems by making use of the underlying structure. In both cases, we develop parallelized schemes that allow to handle high-dimensional systems. Using the notions of covering/packing numbers, we also provide probabilistic stability guarantees via geometric analysis for the quadratic Lyapunov technique and sensitivity analysis for the SOS Lyapunov technique. Finally, we show that the proposed data-driven framework can be extended to LQR design of switched linear systems.

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