Breakdown of Self-Organized Criticality

Maria de Sousa Vieira
Department of Biochemistry and Biophysics, University of California, San Francisco, CA 94143-0448.

We introduce two sandpile models which show the same behavior of real sandpiles, that is, an almost self-organized critical behavior for small systems and the dominance of large avalanches as the system size increases. The systems become fully self-organized critical as the system parameters are changing, showing that these systems can make a bridge between the well known theoretical and numerical results and what is observed in real experiments. A simple mechanism determines the boundary where self-organized criticality can or cannot be found, which is the existence of local chaos.

The concept of Self-Organized Criticality (SOC) was introduced by Bak, Tang and Wiesenfeld (BTW) in 1987 to denote a phenomenon in which out of equilibrium, multidimensional systems, drive themselves to a critical state characterized by a power-law distribution of event sizes \[\Delta x\]. Until then, the studies of fractal structures were related to equilibrium systems where this kind of distribution appears only at special parameter values where a phase transition takes place. In that pioneering work, the concept of SOC was illustrated by a model for sandpiles and since then an enormous amount of research on SOC, both theoretically and experimentally, has been done. Among other phenomena in which SOC has been connected with are earthquakes and evolution.

The existence of SOC in an experiment with a quasi-one-dimensional pile of rice was demonstrated by Frette et al. They found that the occurrence of SOC depends on the shape of the rice. Only with sufficient elongated grains avalanches with a power-law distribution occurred. For more symmetric grains a stretched exponential distribution was seen. Christensen et. al introduced a model for the elongated rice pile experiment in which the local critical slope varies randomly between 1 and 2. They found that their model, known as the Oslo rice pile model, reproduced well the experimental results on the quasi-one-dimensional rice pile. In a recent publication, we introduced a fully deterministic one-dimensional SOC system, which presents the same qualitative and quantitative behavior of the Oslo system. In other words, they belong to the same universality class.

When one goes to sandpiles with geometry of two-dimensions a different picture emerges. That is, the models predict the presence of power-law distributions and the experiments do not display them. The most well known sandpile experiments can be classified in two types: (a) local dropping of sand in the center of the pile and (b) uniform driving, more specifically the rotating drum experiments. In class (a) it was found that small systems present scaling almost consistent with SOC, but in large systems another regime with big avalanches belonging to a different distribution appears. In class (b) one sees small avalanches that seems to display a power-law distribution of limited size. These small avalanches are interwoven with big system-wide avalanches which belong to a different distribution. There are studies of avalanches in natural settings, which also show similar distribution as the ones observed in the sandpile experiments.

Although SOC has not been observed in sandpile experiments is well known that power law distributions does exist in Nature, one of the most well known cases being Gutenberg-Richter law for earthquakes in agreement with SOC. It is clear that there is a missing piece in this puzzle. That is, would there be simple models that would display the observations in real sandpiles, and still present SOC in other parameter regions? To our knowledge it still missing in the literature such models and it is the aim of this letter to introduce them. Our models makes a bridge between the SOC and non SOC behavior, and the boundary that characterizes the separation between the two regimes is the absence or presence of local chaos. However, we will see that local chaos is a necessary but not sufficient condition for SOC to be destroyed.

Our models are inspired by the model introduced in Ref. and is also governed by coupled lattice map (that is, systems characterized by discrete time and continuous values for the space variables). Here we increase the dimensionality and change the drive and the relaxation rules. Having in mind the sandpiles experiments we first introduce a model for the local dropping of sand. We assume a two-dimensional square lattice of linear size \(L\) and to each site \(i, j\) in the lattice there is associated to it a variable \(x_{i,j}\) with \(x \in [0, +\infty)\), which is to represent the local slope of the pile. The dynamics of the model is described by the following algorithm:

1. Start the system by assigning random initial values for the variables \(x_{i,j}\), so the they are below a chosen, fixed, threshold \(x_{th}\).

2. Choose a nearly central site of the lattice and update it slope according to \(x'_{i,j} = x_{i,j} - x_{th}\).

3. Check the slope in each element. If an element \(i, j\) has \(x_{i,j} \geq x_{th}\), update \(x_{i,j}\) according to \(x'_{i,j} = \phi(x_{i,j} - x_{th})\), where \(\phi\) is a given nonlinear function that has two parameters \(a\) and \(d\). Increase the slope in all its nearest neighboring element according to \(x'_{nn} = x_{nn} + \Delta x/4\), where \(\Delta x = x_{i,j} - x'_{i,j}\) and \(nn\) denotes nearest neighbors.
(4) If \( x'_{i,j} < x_{th} \) for all the elements, go to step (2) (the event, or avalanche, has finished). Otherwise, go to step (3) (the event is still evolving).

Without losing generality, we can take \( x_{th} = 1 \). In our simulation in step (2) we have chosen the site with \( i = j = L/2 \). The nonlinear function we use is

\[
\phi(x) = \begin{cases} 1 - d - ax, & \text{if } x < (1 - d)/a, \\ 0, & \text{otherwise.} \end{cases}
\]

The parameter \( d \) would be associated with the minimum drop in energy after an event involving one single element and \( a \) would be the parameter associated with the amount of friction between the grains. That is, the smaller the \( a \), the larger the friction and the smaller the change in the slope of the pile. We have tested several other functions and found that the quantitative and qualitative results we show here are robust. The important ingredient being the shape of \( \phi(x) \) in the vicinity of \( x = 0 \).

In contrast with the one-dimensional case [3], it is not required here that \( \phi(x) \) be periodic in order to find the presence SOC.

We have chosen to evolve the system using parallel dynamics with open boundary conditions. It is beyond the scope of the present letter to study the several possible variations of our models. Further results on these models will be presented in a future publication [12]. The distribution of time duration of the avalanches will also be presented in a future publication [12]. The distributions of our models. Further results on these models have been included in the statistics. In (a) we show \( P(s) \) for \( L = 64, d = 0.1 \) and vary \( a \) and in (b), we use \( a = 1.5 \) and vary \( d \), keeping \( L = 64 \). We notice the existence of two regimes. For small \( d \), power-law distributions, that is SOC, appears only when \( a \leq 1 \), whereas if \( d \) is large, we see SOC even with \( a > 1 \). As in real sandpiles [3], the almost SOC regime is characterized by a region with an apparent scaling for small events, and the big events belong to a different distribution. To illustrate this, we show in Fig. 1(c) simulations with \( a = 1.5 \) and \( d = 0.1 \) and varying \( L \). In small systems \( P(s) \) can be fit to a scaling form of the type

\[
P(s, L) = L^{-\beta} G(s/L^{\nu}),
\]

as shown in Fig. 2(d), where we have used \( \beta = 6 \) and \( \nu = 3 \). The function \( G \) is not well fit by a power-law, since one can clearly see in the figure that it is curved. We have found [12] that it is consistent with a stretched exponential, as in real sandpiles [13]. The observations of Himalayan avalanches [10] also display the kind of distribution shown in Fig. 3(c) for large systems.

The second model we introduce here is for the rotating drum experiment [8,9], which we call uniform drive, since the slope of the pile increases uniformly for all the grains. The algorithm is similar to the one described above, with the exception of step (2), which is now replaced by

(2) Find the element in the lattice that has the largest \( x \), denoted here by \( x_{\max} \). Then update all the lattice elements according to \( x'_{i,j} = x_{i,j} + x_{th} - x_{\max} \).

We show examples of \( P(s) \) for this model in Fig. 3. There, in (a) we fix \( d \) and vary \( a \), and in (b) we fix \( a \) and vary \( d \). In both cases we have used \( L = 64 \) and the events that involve all the elements of the system have been excluded. Distinctly from the model of local dropping, it seems here that there is a power-law distribution for any parameter value. However, the behavior is not exactly SOC. We have found that SOC is only seen if \( a < 1 \) or \( d \) is large enough, as in the case of the local dropping. When \( a > 1 \) and \( d \) is smaller than a given value, we see a SOC-like behavior only for small values of \( L \). As \( L \) grows, there is a transition to a different behavior, in which the larger the system, the smaller the power-law region, as Fig. 3(c) shows.

System-wide avalanches have been reported in the rotating drum experiment [8,9] that belong to a different distribution than the one of the small avalanches. This is exactly what we see in this model for \( a > 1 \) and small \( d \). In Fig. 3(a) we show all the events of the system including the ones that involve all the elements (for \( a = 3, d = 0.1 \) and varying \( L \)). We see that the scaling region does not get bigger as the system size increases, and a peak related to the events involving all the elements is seen. We have found that the intermediate size events (10 \( \leq s \leq 100 \)) can be fit by a scaling of the type \( P(s, L) = L^{-\beta} G(s/L^{\nu}) \) with \( \beta = 1 \) and \( \nu = 0 \), as shown in Fig. 3(b). As in the rotating drum experiment [8,9] the function \( G \) in this case is closer to a power-law than in the case of the experiment with local dropping of sand.

For given \( a \) and \( d \) the slope of the power-law distribution seems to be the same for both models, that is, local dropping of sand and uniform drive, but the slope varies with \( a \), as shown in Fig. 3 and 4. Consequently, the universality class of these models vary with the parameters. In Fig. 4(a) we show the scaling given by Eq. (3) with \( \beta = 3.55 \) and \( \nu = 2.70 \) for the local dropping (ld) and \( \beta = 3.55 \nu = 2.85 \) for the uniform drive (ud). In the limit \( a \to 0 \) or \( d \to 1 \) the slope of the power-law distribution is the same as the one in the BTW model [1] and the conservative OFC model [4], that is, \( P(s) \sim s^{-1.245} \).

We next investigate what would happen if the relaxation function is just a random number generator. In other words, instead of using \( \phi \) in step 3 of the above algorithm we now use \( x'_{i,j} = \rho \), where \( \rho \) is a random number uniformly distributed in the interval \([0, 1 - d]\). We have
found SOC for any \( d \in (0, 1] \) with the same exponents as the BTW model. This is displayed in Fig. 3(b) where we show the size distribution for \( d = 0.01 \) in the cases of local dropping (ld) and uniform drive (ud). Therefore nonlinearities in \( \phi \) and consequently non ergodicity are necessary for the SOC behavior to be destroyed in these models. In that figure, we also show the case in which \( x'_{i,j} = 0 \), which corresponds to \( d = 1 \). In this limit we recover the OFC model for the case of uniform drive.

The reason why \( a = 1 \) determines a special boundary, in which SOC may or may not be present, is due to the fact that it marks the boundary in which local chaos exists. By “local” here we mean that is is at the grain level, no matter what happens at the system level. Chaos is defined as the exponential divergence of trajectories that start with almost the same initial conditions. In fact, if we consider two copies of a given system, copy 1 and copy 2, with all the elements having the same \( x \) except that in copy 1 the element \( x'_{i,j} = x^* \), whereas in copy 2 \( x'_{i,j} = x^* + \Delta \). It is not difficult to find that after one iteration by \( \phi \), the separation of the two elements instead of \( \Delta \) will be \( a\Delta \). So, if \( a > 1 \) the separation increases (that is, we have local chaos) whereas if \( a < 1 \) the separation decreases. We have seen from the above results that, since SOC can exist even with \( a > 1 \), if \( d \) is large enough or \( L \) is smaller than a given value, this means that local chaos is a necessary but not sufficient condition for SOC to be broken. Consequently, the true boundary between SOC and non SOC depends on three parameters, that is, \( a \), \( d \) and \( L \). Further studies are necessary to understand this boundary more clearly.

In summary, we have introduced a model for sandpiles, or other systems that present avalanche like behavior, that reproduce very well the observation in real sandpiles. We have found that a simple mechanism, i.e local chaos, can explain the breakdown of SOC in those systems. Based on our results, we believe that with an appropriate choice of grains SOC will be seen even in real sandpiles. Grains with large friction would be the best candidates for this.

FIG. 1. (a) Frequency of events involving \( s \) updates for (a) variable \( a \) with \( d = 0.1 \), and (b) variable \( d \) and \( a = 1.5 \) with \( L = 64 \). In (c) we vary the system size and use \( a = 1.5 \) and \( d = 0.1 \), and in (d) we show the fitting using the scaling of Eq. 2 for small systems, with \( \beta = 6 \) and \( \nu = 3 \). In this figure we are using the model for local dropping of sand. We have used \( 4 \times 10^6 \) avalanches in all the simulations of this paper.

FIG. 2. (a) Frequency of events involving \( s \) updates for (a) variable \( a \) with \( d = 0.1 \), and (b) variable \( d \) and \( a = 1.5 \) with \( L = 64 \). In (c) we vary the system size and use \( a = 1.5 \) and \( d = 0.1 \). In this figure we are using the model for uniform drive (that is, the rotating drum experiment).

FIG. 3. (a) Frequency of events involving \( s \) updates for \( a = 3 \) and \( d = 0.1 \) with varying \( L \). The peaks in the distribution are the events involving all the elements of the system. In (b) we show the fitting using the scaling of Eq. 2 for the region of avalanches of intermediate sizes, using \( \beta = 1 \) and \( \nu = 0 \). In this figure we are using the model for uniform drive.

FIG. 4. (a) Frequency of events involving \( s \) updates for \( a = 0.3 \) and \( d = 0.1 \) with varying \( L \) using the scaling of Eq. 2. For the case of local dropping we have used \( \beta = 3.55 \) and \( \nu = 2.75 \). For uniform drive we have used \( \beta = 3.55 \) and \( \nu = 2.85 \). In (b) we show the cases in which after relaxation the variable \( x \) is a given random number uniformly distributed between 0 and 0.99 for local dropping (ld) and for uniform drive (ud). We also show the cases in which \( x \) is relaxed to 0 after an event, which is the same as having \( d = 1 \).
Fig. 1(a)

(a)
local dropping
\[ d=0.1, L=64 \]

\[ P(s,L) \]

Fig. 1(b)

(b)
local dropping
\[ a=1.5, L=64 \]

\[ P(s,L) \]
Fig. 1(c)

\[ P(s, L) \]

local dropping
\[ a = 1.5, d = 0.1 \]

Fig. 1(d)

\[ P(s, L)L^\beta \]

local dropping
\[ a = 1.5, d = 0.1 \]
Fig. 2(a)

uniform drive

\(d=0.01\), \(L=64\)

\(d=0.1\), \(L=64\)

\(d=0.5\)

\(d=0.9\)

Fig. 2(b)

uniform drive

\(a=1.5\), \(L=64\)
Fig. 2(c)

Graph showing the dependence of $P(s,L)$ on $s$ for different values of $L$. The graph indicates a uniform drive with $a=1.5$ and $d=0.1$. The data is presented on a logarithmic scale for both axes.
Fig. 3(a)

Fig. 3(b)

uniform drive

$a=3, d=0.1$
Fig. 4(a)

\[ P(s,L) \]

\[ \beta \]

\[ s/L^\nu \]

(a)

\[ a=0.3, \ d=0.1 \]

Fig. 4(b)

\[ P(s,L) \]

\[ L=64 \ (ld) \]

\[ L=64 \ (ud) \]

\[ L=32 \ (ld) \]

\[ L=32 \ (ud) \]

\[ L=128 \ (ld) \]

\[ L=128 \ (ud) \]

\[ d=0.01 \ (ld, \ random) \]

\[ d=0.01 \ (ud, \ random) \]

\[ d=1 \ (ld) \]

\[ d=1 \ (ud - OFC) \]