The running coupling from lattice gauge theory

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Abstract
I discuss some calculations of the running coupling in SU(N) gauge theories from lattice simulations, centering on the work of the UKQCD collaboration. This talk is introductory in nature; full details have been published elsewhere.

1. Introduction
At the moment, lattice simulations are the most popular way of extracting truly non-perturbative results from quantum field theories. Their commonest uses are, quite naturally, calculations with some direct experimental relevance, such as the spectroscopy of hadrons and the calculation of matrix elements.

However, we are quite at liberty to examine more basic aspects of the discrete theory, in order to assure ourselves that the results are as we expect. (It would be more interesting if they were not, but in QCD this is increasingly unlikely.) In this talk I shall describe a (successful) attempt to look at the behaviour of the running coupling of the field theory. This allows us to make direct contact with the standard formalism of perturbation theory and the renormalisation group.

Phenomenologically, the most interesting field theory is quantum chromodynamics (QCD), with its strong interaction and significant scale-dependence over the region where the physics is most interesting. The proper lattice formalism is that of SU(3) gauge theory with dynamical quarks, incorporating Feynman diagrams with internal quark loops. This is quite simply too difficult for us with our present-day technology and computer resources. Even generating an equilibrated lattice (one on which samples of the fields are guaranteed to be representative of the true vacuum with the right coupling) is very costly and can take many months even on the largest machines.

However, simulations in the quenched approximation, the theory without dynamical quarks but retaining fully non-perturbative gluonic contributions, are proving more tractable and we are seeing results which in many cases agree with experiment at the level of 10% or so. One should of course be careful; at some stage the effect of quenching is likely to dominate our errors and in some calculations to change the nature of the physics completely. We are not yet at the stage where we can determine the limits of the quenched approximation and in general the effect is not predicted. Preliminary indications are that where light quarks are unimportant the dominant effect is simply a uniform rescaling of the results.

This need not worry us if we are engaged in an exercise in field theory. In fact, in that case we can make one further simplification and use the gauge group SU(2) instead of SU(3), resulting in roughly...
an order of magnitude reduction in the computing effort required for a similar standard of results. We shall also see that the results for the two groups are qualitatively very similar.

This talk deals for the most part in the pure gauge theory, which does not involve fermions at any stage. This is not a further approximation beyond quenching: it simply means we are looking at the gluonic sector. In fact, the pure gauge theory is a true field theory (to the best of our knowledge) while the quenched fermionic theory is not since it involves an unnatural treatment of quarks on a background gauge field with which they do not interact properly.

The technology for producing pure-gauge results from lattice simulations is now quite well advanced. Using improved operators we can extract quite accurate numbers for, among other things, the interquark potential which will be our probe here.

The method outlined in the major part of the talk first appeared in [1]. Full details of the calculations by the UK QCD Grand Challenge (UKQCD) using this method are given in ref. [2] (for SU(2)) and ref. [3].

2. The running coupling and asymptotic freedom

In QCD, or any quantum field theory, the physically-meaningful coupling — the one which relates directly to the strength of the interaction — is the renormalised value. This is commonly referred to as the running coupling, as the process of renormalisation makes the coupling apparently change its value in low-order perturbation theory. One way of looking at this new quantity is that it parametrises our ignorance about what is happening at very high energies when QCD is no longer valid, replacing it with a cut-off and an effective theory.

In four dimensions, where the coupling is dimensionless, we see the rather surprising phenomenon known as dimensional transmutation whereby the mechanics of renormalisation introduces a fixed scale, namely the lambda parameter, $\Lambda_{\text{QCD}}$. The “interesting” physics of QCD, by which I mean the battleground of different causes and effects, happens in processes involving momenta $q \sim \Lambda_{\text{QCD}}$. The second important consequence of dimensional transmutation is that the renormalised coupling is itself scale-dependent: $g_{\text{phys}} = g_{\text{phys}}(q)$. As we shall see later on, the connection between $g_{\text{phys}}$ (I shall drop the “phys” suffix) and the bare coupling $g_0$ which appears in the original Lagrangian can be a bit obscure. Actually, this is true not just on the lattice: it is, after all, why we need renormalisation in the first place.

The SU($N$) theories have the feature that as the momentum scale is increased, the physical coupling decreases (asymptotic freedom). This means that in the limit of large momentum the theory becomes perturbative. That is how we shall connect the non-perturbative lattice results with analytic calculations in perturbation theory.

In SU($N$) to two loops the running coupling is given by

$$\alpha(q) \equiv \frac{4\pi}{g^2} = \frac{4\pi}{4\pi (b_0 \log(q/\Lambda))^2 + (b_1/b_0) \log \log(q/\Lambda)^2}$$

where $b_0$ and $b_1$ depend only on $N$. This is for the pure gauge theory with no quarks; adding a few species of quarks does not change the expression qualitatively, though asymptotic freedom is weakened and eventually (with 17 quarks in SU(3)) the sign of $b_0$ and the character of the theory change.

Instead of a momentum scale, one can express the results in terms of an appropriate length scale $R \sim q^{-1}$. This is more appropriate to a static configuration like the one we use for extracting potentials. We need to change our renormalisation scheme to do this; the scheme appropriate to potentials uses the separation $R$ between the quark and antiquark, as described in the next paragraph, as the scale parameter. It turns out that this is close to $\overline{\text{MS}}$ in the sense that the $\Lambda$-parameters are related by a small factor; this is not true of the usual scheme for lattice regularisation.
In the perturbative limit, the force between a static (infinitely massive) quark and antiquark of opposite colour (so the pair is colourless) a distance $R$ apart is Coulombic:

$$-F_{\text{Coulomb}} = \frac{dV}{dR} = C_f \frac{\alpha(R)}{R^2}$$

($C_f$ is a combinatoric factor containing $N$; it derives from the quadratic Casimir operator). Hence if we form a dimensionless quantity

$$\alpha_{\text{eff}} \equiv -\frac{F_{\text{calc}}R^2}{C_f}$$

a lattice calculation of $F_{\text{calc}}$ at small $R$ is enough to tell us the running coupling. We use the force instead of the potential as it removes an uninteresting constant. On the lattice, the force is extracted from the potential by a finite difference; this actually introduces negligible extra errors. The significant thing about the lattice calculation is that, although we are trying to show agreement with the perturbative analysis, the calculation is at every step fully non-perturbative.

3. Lattices and scales

Our lattice formulation is the standard one of Wilson. The gauge fields live on the links of a hypercubic lattice in Euclidean space-time, that is the lines joining nearest-neighbour points of the simple cubic structure like a stack of wire-framed cubes. Each link has an associated matrix which is an element of the gauge group. Line integrals between points are replaced by simply matrix multiplication of the appropriate links; taking the trace of such a product which corresponds to a closed loop produces a gauge-invariant object, the Wilson loop. These loops are essentially the only gauge-invariant quantity in the pure theory; the quarks, if present, would live at the sites of the lattice and act as sources and sinks of colour.

The action for the theory is a sum over all plaquettes: the plaquette is the smallest possible Wilson loop consisting of four links about an elementary square of the lattice. This preserves exact gauge invariance and in the continuum limit it goes over smoothly to the usual continuum action.

The scale is fixed by the lattice spacing $a$, the length of one link. This is determined by the bare coupling $g_0$ which we choose for our interaction (in fact, we usually work with $\beta = 2N/g_0^2$, which appears as the multiplier of the sum of plaquettes in the action). The relationship between $a$ at different values of the coupling is determined by the renormalisation group beta-function (two completely unrelated uses of the symbol beta, unfortunately); because of asymptotic freedom, a small $g_0$ corresponds to a small lattice spacing. Decreasing $g_0$ therefore makes the lattice less and less important and takes us nearer the real world where $a\Lambda$ is small, or in other words the scale $\Lambda$ is very much less than the momentum space cut-off.

One can think of a rectangular Wilson loop (as in figure 1) which extends $R$ in a spatial direction and $T$ in the time direction in the following way. A source and a sink of colour in the fundamental representation of the gauge group, both infinitely massive, are created instantaneously a distance $R$ apart. They propagate in this fashion for time $T$ and then are instantaneously annihilated. We refer to the colour sources as a (static) quark-antiquark pair even though the calculation does not involve fermions at any stage. We need some experimental input to determine $a$ (or — which is equivalent in theory if not always in practice — $\Lambda_{QCD}$) in physical units.
4. Measuring the quark-antiquark potential

To extract the potential between the two, we simply need the expectation value of the loops. This is related to the Euclidean propagator \( C(T) \sim \sum_i c_i \exp(-V_i T) \) for the various energy levels \( V_i \) of the quark-antiquark system. At large \( T \) the main contribution is from the ground state and the ratio \( C(T)/C(T-1) \) tells us the potential \( V \equiv V_0 \). Unfortunately the statistical errors increase with \( T \), also the decay of the correlators is faster than typically found in (for example) calculations of light hadron masses, so choosing a suitable value of \( T \) is something of an art form and introduces significant systematic errors.

We should note two other major sources of systematic error in our calculations, again deriving from limitations on computing resources. First of all the lattice size \( L \) is finite as we can only fit \( L/a \) sites on a side of our lattice. If this is too small the fields will feel the effects of the boundaries of the lattice, being squashed into the box. In our case we have chosen sizes (from previous experience) such that this is not expected to be a problem.

Secondly, \( a \) is finite, generating cut-off effects. This is significant for us, since we are attempting to probe the region at small \( R \) where perturbation theory is expected to become valid: our results are for quark separations of only a few lattice spacings. We have used a one-parameter fit to smooth out the bumpiness caused by finite \( a \); the smoothness of the result, together with the agreement between different lattice spacings as described below and shown in figure 2, assures us that this has worked.

Our Monte Carlo simulation generates different samples of the QCD vacuum for us to measure by **local updating**: changing one link at a time until the set of all links is sufficiently different from the previous sample. The process involves subjecting each link to two different procedures. The first is a “heatbath”, closely analogous to the same concept in statistical mechanics, in which the link is made aware of the coupling (corresponding to the temperature) of the links around it. This includes the element of randomness that drives our stochastic process, like the random kinetic motion in a real
Figure 2: The scaling properties of the potential in SU(2). Diamonds are $\beta = 2.85$ data, triangles $\beta = 2.7$ and squares $\beta = 2.4$. The $R$ axis (only) is adjusted for the latter two.

heatbath.

The second procedure is “over-relaxation”. This uses the fact that the SU(2) group manifold is a sphere and that we can flip the gauge element for any link about the (scaled) gauge element representing the combined effect of the other plaquettes in which the chosen link appears without changing the action. Hence we can do this as often as we like without affecting the statistical-mechanical properties of the simulation.

In both procedures, each link of the lattice is updated in turn throughout the whole lattice (one sweep). The SU(3) simulation is similar: for each link we perform a heatbath in each of the three possible SU(2) subgroups in turn, and similarly for over-relaxation. Typically we perform four over-relaxation sweeps for every heatbath sweep; some other groups perform more, but our heatbath code is well optimised so that the extra computer time over the over-relaxation code (which is simpler) is fairly low. This appears to be the most efficient way of generating distinct configurations at present.

The sets of data we have used are in each case separated by several hundred sweeps (largely for logistical reasons) and statistical correlations between different sets are found to be completely negligible.

Having extracted values for the potential we should like to decide whether results on lattices at different inverse couplings $\beta_1, \beta_2$ give equivalent results — in other words, whether the ratios of measured quantities (say, masses) $m_X(\beta_1)/m_X(\beta_2)$ are the same for all possible measurements $X$. If this happens, the results are said to scale and we know that any dependence on $a$ in the expansion of the ratio has disappeared. (However, we cannot necessarily make the stronger statement that the individual quantities $m_X$ have an expansion which behaves according to the RG beta function in low order. This requirement — asymptotic scaling — is mentioned later.)

One way of showing this from the potential data is by forming “$\alpha_{\text{eff}}$” as in eq. (3). We do not require at this point that this $\alpha_{\text{eff}}$ should be the true running coupling; we are simply using it as a convenient
dimensionless physical quantity. If scaling holds we should be able to plot this against the separation $R$ and achieve a single curve simply by rescaling the $R$ axis as appropriate. Figure 2 shows this for three different couplings in SU(2): $\beta = 4/g_0^2 = 2.4 \ [4], \ 2.7 \ [1]$and $2.85 \ [2]$. The scaling factor in $R$ shows that the lattice spacing is some four times smaller at $2.85$ than at $2.4$. This is a strong indication that we have control over all finite lattice spacing effects — any scaling violation in the potential measurement should show up clearly in this plot.

5. The running coupling

We now proceed to the running coupling itself (following the method of ref. [1]). The potential shown in figure 2 was over a wide range of scales which includes a linearly-rising potential $V \sim KR$ that dominates at large $R$ (we confirm this by suitable fits to the data; the value of $K$ is well-determined). We concentrate on our smallest lattice spacing, with $\beta = 2.85$ and 48 lattice sites in each spatial direction, for small separations. As the lattice spacing is unphysical, we set the scale instead by using our measured value of $K$ on the same lattice. This result is shown in figure 3 (the points with error bars). We show the corresponding analysis for SU(3) in figure 4; in this case $\beta = 6.5$ and the lattice has 36 sites in each spatial direction.

We should like to compare this with analytic results. We do this by choosing a value of $a\Lambda$ that fits our results. The lines show the running coupling from perturbation theory to two loops at the largest and smallest values of $a\Lambda$ which seem consistent (this was done by eye). When this is done our lattice effective coupling seems to agree well with the perturbative expressions (note that to the right of the diagram non-perturbative effects are beginning to enter). In other words, we are seeing real perturbative field theory from non-perturbative calculations on the lattice.

We can use our chosen value of $a\Lambda$ to extract $\Lambda$ provided we can express $a$ in physical units. To do this we turn back to $K$. In SU(3), we can equate this with the “string tension” in the Regge picture of hadrons as a quark-antiquark pair connected by a tube of relativistic glue. (There is no more formal justification for this; however, there is no good reason to suppose this is a bad way to set the scale either.) In this picture $\sqrt{K} = 440$ MeV. From this, we extract the $\Lambda$ parameter in the renormalisation scheme appropriate to the potential picture. This is a small multiplicative factor away from the more familiar $\Lambda_{\overline{MS}}$; we deduce that $\Lambda_{\overline{MS}} = 256 \pm 20$ MeV. However, remember this is in SU(3) without dynamical quarks. We have not attempted to correct for this; we do not believe that in our case the systematic errors are sufficiently under control.

We can also use $K$ to give more meaning to the horizontal scale. With the same value 440 MeV, a 5 GeV momentum scale corresponds to $R\sqrt{K} \sim 0.9$ on the lower scale of figure 4, so $\alpha \sim 0.16$. This is very rough because apart from all the other errors I have not bothered to do a proper conversion from the $R$ scheme to $\overline{MS}$, though the difference is small; the physics is in the running itself, not necessarily the actual values we extract.

6. Other calculations

The same calculations have been performed in SU(3) by Bali and Schilling [6]; their results are very similar to ours, although they have more data. Among their lattices are some with smaller physical extent than ours; this does not change the running coupling behaviour, so it seems that the small distance physics is not strongly affected by finite size effects. This was not a priori obvious, though it is not too surprising a result if one believes in the separation of physics at different scales.

Other calculations (ref. [7]; see also reviews in ref. [8]) for $\alpha$ have been performed using the charmonium potential (any such colourless, heavy quark-antiquark system will do as the results are nearly mass-independent). Here too the potential is expected to be in the perturbative region. In this calcula-
Figure 3: The running coupling in pure SU(2) lattice gauge theory. All results are at $\beta = 4/g_0^2 = 2.85$. Error bars are both statistical and systematic; the latter are dotted. The upper scale on the horizontal axis shows the separation in lattice spacings; the lower scale relates it to the string tension. The upper and lower lines are two-loop perturbative predictions with $a\Lambda = 0.044$ and 0.038 respectively.

Figure 4: The running coupling in pure SU(3) lattice gauge theory. The plot is similar to figure 3. Here diamonds are data at $\beta = 6.5$ [3] and triangles at 6.2 [5]; the upper scale on the horizontal axis is only appropriate to the former. The upper and lower lines in this case are for $a\Lambda = 0.070$ and 0.060 respectively.
tion the scale does not appear in an explicit non-perturbative fashion (as in our calculations) and one has to make sure the scale is chosen appropriately, as emphasised by Stan Brodsky at this workshop, and that the problems with perturbation theory mentioned in the next section are correctly handled. These authors have chosen to correct for the effect of quenching (amongst other systematic effects), that is to predict the effect of four light quarks using the renormalisation group; this increases the value of $\alpha$ at 5 GeV from $0.140 \pm 0.004$ to $0.170 \pm 0.010$. Of course it can then be run in the same way to any interesting scale.

7. Lattices and perturbation theory

What might appear more surprising is that we have such good agreement with perturbation theory at all. The naïve way of comparing with perturbative results is to insert our coupling $g_0$ into perturbative expressions (such as the beta function) and see what comes out; in the asymptotic scaling limit we would find agreement. In practice this is not seen. For example, the beta function produces a ratio of lattice spacings between $a(\beta = 2.7)/a(\beta = 2.85)$ which is very different from our value (by almost 20%).

The problem arises from the use of the bare coupling. All our previous analysis of the running coupling was done without any perturbative input; $\beta$ was used only to label our different lattices and so $g_0$ never appeared in the calculations. There is clearly a problem with perturbation theory on the lattice which does not appear if one restricts oneself to physical quantities as we have done.

This problem has been elucidated during the last couple of years by Lepage and Mackenzie [9]. The lattice regularisation is somewhat unnatural for a field theory; large constants from tadpole diagrams are introduced into perturbative series which become only painfully convergent. Provided a physical coupling is used there is no problem. Lepage and Mackenzie give methods for improving perturbative calculations along these lines, using some non-perturbative input such as the average plaquette as a renormalisation factor.

A further result of this is that attempts to extrapolate to zero lattice spacing (i.e. zero bare coupling) by some groups [6,10] has required more sophistication. One method of proceeding is to define an improved coupling with real physics in it. The “$\beta_E$” effective coupling scheme, descended originally from a suggestion of Parisi [11,10], is one way of doing this; in fact, it is very much in the spirit of the Lepage-Mackenzie programme. Any physically quantity (in this case the action, which is easily measured on the lattice — in fact it very nearly emerges as a by-product of the way links are updated) may be expressed as a power series in the coupling.

$$\langle S_{\text{plaq}} \rangle = \frac{c_1}{\beta} + \frac{c_2}{\beta^2} + \frac{c_3}{\beta^3} + \cdots$$ (4)

This series may have a poor convergence. However, one can truncate at some low order and invert it, expressing the coupling as some function of the action, and use this truncated expression to define an effective coupling:

$$\beta_E^{(1)} = \frac{c_1}{S}$$ (5)

(this is the first order $\beta_E$ scheme; one can truncate at higher order). By missing out the higher terms, one hopes the new coupling is more appropriate to use in low order, having resummed any non-perturbative contributions. This does seem to help the link to asymptotic behaviour.

8. The approach of Lüscher et al.

Lüscher, Sommer, Weisz and Wolff implemented a recursive strategy which involves measuring a suitable observable on lattices of different sizes, allowing them to calculate the running coupling with no a priori assumptions. They have results for both SU(2) [12], and SU(3) [13].
Increasing physical size

Figure 5: The strategy of Lüscher, Sommer, Weisz and Wolff.

The idea (see figure 5) is that one performs simulations on two lattices at a fixed coupling, so that one can change the size of the lattice (used to set the scale) by an exact factor simply by adding more points in each direction. By inspired guesswork one can then find a lattice at a different (higher) coupling which is roughly the same physical size; any discrepancy can be handled with only small errors by the renormalisation group. Hence the size can be increased stepwise. The smallest lattice used is well in the perturbative region: a small box has small length scales which means it is perturbative by virtue of asymptotic freedom. Note that the choice of the measured observable is important: the response to a chromoelectric field forced onto the lattice by the boundary conditions is used.

This technique avoids our problem of having all the scales on the one lattice, so that having a finite cut-off is less of a problem. The obvious point against is that there are more technical details to understand and bring under control (for example, a lattice renormalisation of the chosen observable used to extract the running coupling).

The measured running couplings agree much better with naïve perturbation theory than the UKQCD results in both SU(2) and SU(3). The reasons are not understood; it may simply be (as Lüscher et al. observe [13]) that their chosen observable has particularly good asymptotic scaling properties.

9. Summary
As a summary of the results in SU(2) (as these are computationally easier to extract; those in SU(3) look very similar), here is a table of various estimates for the ratio of lattice spacings between two inverse couplings \( \beta = 2.7 \) and 2.85. This deliberately uses the bare coupling, except where explicitly improved, to show the discrepancies; as explained above this is not generally the smartest thing to do. They range from the UKQCD results, through our results improved by \( \beta_E \), the results of ref. [12] (which can be read off from a graph in that paper using the authors’ own fit), to the naïve perturbative result at two loops. Apart from the UKQCD result (which includes an estimate of systematic errors), the errors are negligible.
Quantity
UKQCD: Full fit to the potential, including systematic errors
Perturbation theory with first-order \( \beta_E \) improvement
Ref. [12]
Ordinary perturbation theory at two loops

\[ a(2.7)/a(2.85) \]
1.60(6)
1.53
1.48
1.46

10. Conclusions

Attempts to see the running coupling in lattice gauge theory and to relate it to perturbation theory have been successful; the necessity of dealing with the problems of perturbation theory on the lattice is now clear. By setting a physical scale \( \Lambda_{QCD} \) can be extracted. As time goes by we hope to have more control over the systematic errors involved. It is already claimed by analysts of the heavy quark data [8] that the lattice is a better place to extract the running coupling than experiment; lattice theorists will need to substantiate this by continuing to refine their methods.

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