A new method for judging the stability and boundedness of continuous-time fractional positive systems

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Abstract. By decomposing the state matrix into a symmetric part and an antisymmetric part, this paper investigates the stability of continuous-time fractional positive systems, and a sufficient and necessary condition for continuous-time fractional positive systems to be bounded in a given interval is obtained. The results show that when the system corresponding to its symmetric part is asymptotically stable, continuous-time fractional positive systems is also asymptotically stable.

1. Introduction

Fractional-order positive systems refer to fractional systems with non-negative states, because these states represent actual physical quantities, and these physical quantities are not negative numbers, such as concentration, water level and height in [1]. the fractional-order derivative was put forward by Riemann and Liouville in the 19th century in [2]. In recent years, the research on fractional-order positive systems has received more attention from engineers in [3-5]. This paper investigates the stability of continuous-time fractional-order positive systems, and gives a sufficient and necessary condition for continuous-time fractional positive systems to be bounded in a given interval, by decomposing the state matrix into a symmetric part and an antisymmetric part. The results show that when the system corresponding to its symmetric part is asymptotically stable, the continuous-time fractional positive systems is also asymptotically stable. This method is simple and feasible, and it is also feasible for systems with higher dimensionality. Numerical examples show that this method is effective and feasible.

2. Preliminaries

Consider the continuous-time fractional-order linear system:

\[
D^\alpha x(t) = Ax(t), \quad A \in M_n, \quad 0 < \alpha \leq 1, \quad x(0) = x_0 > 0, \tag{1}
\]
where $x(t) \in \mathbb{R}^n$ is the state, $\alpha$ is the order of the fractional derivative and $D_0^\alpha x(t)$ represents the Riemann-Liouville fractional derivative of $x(t)$ is defined by:

$$D_0^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left( \int_0^t \frac{x(\tau)}{(t-\tau)^\alpha} d\tau \right),$$

$$0 < \alpha \leq 1.$$  

The Gamma function is defined by:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad z \in \mathbb{R}.$$  

The solution of system (1) is given by: $x(t) = E_\alpha (A t^\alpha) x_0$, where $E_\alpha (\cdot)$ represents the Mittag-Leffler function, which is defined as

$$E_\alpha (A t^\alpha) = \sum_{k=0}^{\infty} \frac{(At^\alpha)^k}{\Gamma(k\alpha + 1)}.$$  

**Definition 1.1:** [5] Given any nonnegative initial condition $x_0 \in \mathbb{R}_+$, if the corresponding trajectory is never negative, the system (1) is called positive, that is $x(t) \in \mathbb{R}_+$ for all $t \geq 0$.

**Lemma 1.2:** [5] If there exists a vector $\lambda \in \mathbb{R}^n$, $\lambda > 0$ such that $A \lambda < 0$, then the system (1) is asymptotically stable.

### 3. Main results

By decomposing the state matrix of the positive fractional-order linear continuous system (1) into a symmetric part and an anti-symmetric part, we considered its stability, and the sufficient and necessary conditions for it to be bounded in a given interval are obtained.

Consider the matrix $A = [a_{ij}] \in M_n$, which is not symmetrical in general. As we all know, a matrix can be decomposed into a symmetric part $A_s$ and an antisymmetric part $A_a$, that is $A = A_s + A_a$, where

$$A_s = \frac{A + A^T}{2}, \quad A_a = \frac{A - A^T}{2}.$$  

**Theorem 3.1:** If there exists a vector $\lambda > 0$ such that $A_s \lambda < 0$, the system (1) is asymptotically stable.

Proof: suppose that $A \lambda \geq 0$ when $A_s \lambda < 0$. if $A \lambda \geq 0$, then $(2A_s - A^T) \lambda \geq 0$, that is $2A_s \lambda - A^T \lambda \geq 0$.

Since $A \lambda \geq 0$, we can get $A^T \lambda < 0$, that is $A \lambda < 0$, which is contradicted with the known, so the hypothesis does not hold, so $A \lambda < 0$.

**Lemma 3.2:** [5] for any matrix $A \in \mathbb{R}^{n,n}$, we have $E_\alpha (A t^\alpha) I = D^{-\alpha} (E_\alpha (A t^\alpha) A)$.

**Theorem 3.3:** Consider continuous-time fractional system (1), for a given $\overline{x} > 0$, we obtained $\forall t \geq 0$, $0 \leq x(t) \leq \overline{x}$ for any initial condition $x_0$ satisfying $0 \leq x_0 \leq \overline{x}$ if and only if $A_s \overline{x} \leq 0$.

Proof: **Necessity.** Supposed that $0 \leq x(t) \leq \overline{x}$, the solution of system (1) with the initial condition $\overline{x}$ is given by $x(t) = E_\alpha (A t^\alpha) \overline{x}$, $x(t) = E_\alpha (A t^\alpha) \overline{x} \leq \overline{x}$, that is $(E_\alpha (A t^\alpha) - I) \overline{x} \leq 0$. Since it has been proved in reference [5] that when $A$ is a Metzler matrix, $E_\alpha (A t^\alpha)$ is non-negative, so
By applying the Lemma 3.2, we can get $D^\alpha(E_\alpha(At^\alpha)A)x \leq 0$. Therefore $A^T x \leq 0$, $A^T x \leq 0$.

**Sufficiency.** Assume that $A^T x \leq 0$. when $A$ is a Metzler matrix, $E_\alpha(At^\alpha)$ is non-negative, The solution of the system (1) is $x(t) = E_\alpha(At^\alpha)x_0 \leq E_\alpha(At^\alpha)x$. By applying the Lemma 3.2, we can get

$$x(t) \leq E_\alpha(At^\alpha)x = (I + D^\alpha(E_\alpha(At^\alpha)A))x$$

$$= x + D^\alpha(E_\alpha(At^\alpha)A)x.$$

By applying the Theorem 3.1, we have $A^T x \leq 0$. Since $D^\alpha(E_\alpha(At^\alpha)) \geq 0$, so we can get $x(t) \leq x$. For $x(t) = E_\alpha(At^\alpha)x_0$, when $A$ is a Metzler matrix, $E_\alpha(At^\alpha)$ is non-negative, and $x_0 \geq 0$, it is easy to get $x(t) \geq 0$. Therefore, we can get $0 \leq x(t) \leq x$.

**4. Numerical examples**

**Example 1:** Consider the system (1) determined by the following matrix

$$A = \begin{bmatrix} -5 & 2 & 1 \\ 4 & -7 & 0 \\ 1 & 4 & -9 \end{bmatrix}$$

Choosing the vector $\lambda = [0.7, 0.5, 0.4]^T$ for the matrix, we obtained

$$A' = \frac{A + A^T}{2} = \begin{bmatrix} -5 & 3 & 1 \\ 3 & -7 & 2 \\ 1 & 2 & -9 \end{bmatrix}$$

$$A_\alpha = \begin{bmatrix} -5 & 3 & 1 \\ 3 & -7 & 2 \\ 1 & 2 & -9 \end{bmatrix} \begin{bmatrix} 0.7 \\ 0.5 \\ 0.6 \end{bmatrix} = \begin{bmatrix} -1.4 \\ -0.2 \\ -4.6 \end{bmatrix},$$

which satisfying the theorem 3.1

$$A_\lambda = \begin{bmatrix} -5 & 2 & 1 \\ 4 & -7 & 0 \\ 1 & 6 & -9 \end{bmatrix} \begin{bmatrix} 0.7 \\ 0.5 \\ 0.6 \end{bmatrix} = \begin{bmatrix} -2.1 \\ -0.7 \\ -0.4 \end{bmatrix}.$$

**Example 2:** Consider the system (1) determined by the following matrix

$$A = \begin{bmatrix} -0.4 & 0.3 \\ 0.06 & -0.5 \end{bmatrix}.$$

Choosing the vector

$$\bar{x} = \begin{bmatrix} 20 \\ 15 \end{bmatrix}, \quad \alpha = 0.5,$$
which satisfying the conditions of Theorem 3.3

\[
A_s = \frac{A + A^T}{2} = \begin{bmatrix}
-0.4 & 0.18 \\
0.18 & -0.52
\end{bmatrix},
\]

where the time step is 0.1s, the total time is 35s, and k=1000 in the Mittag-Leffler exponential function.

Choosing non-negative initial conditions are \(x_{01} = [2 \ 3]^T\), \(x_{02} = [3 \ 4]^T\), \(x_{03} = [5 \ 3]^T\), respectively. The state trajectory figure of the system (1) is shown below.

![State trajectories](image)

**Fig.1 State trajectories of the system (1) in example 2 from three different initial values**

5. Summary

This paper focuses on the stability and boundedness of continuous-time fractional-order positive system. First, the necessary and sufficient conditions for fractional-order positive systems to maintain positiveness are given. Then, based on the theory of continuous-time fractional positive system, a sufficient condition for judging the stability of continuous-time fractional positive system and a necessary and sufficient condition for making positive fractional-order linear continuous system bounded are obtained, and related proofs are completed. Numerical examples are given to show that this method is effective and feasible.

References

[1] Virnik, E. (2008) Stability analysis of positive descriptor systems. Linear Algebra Appl, 429, 2640–2659.
[2] Busłowicz, B. (2011) Stability of state space models of linear continuous-time fractional order systems. Acta Mechanica et Automatica, Vol. 5, no. 2, pp 15–22.
[3] Kaczorek, T. (2011) Necessary and sufficient stability conditions of fractional positive continuous-time linear systems. in Acta Mechanica et Automatica, Vol. 5, no. 2, pp 52–54.
[4] Kaczorek, T. (2008) Fractional positive continuous-time linear systems and their reachability. Int. J. Appl. Math. Comput. Sci., 18(2), 223–228.
[5] Benzaouia, A. Hmamed, A. Mesquine, F. Benhayoun M. and Tadeo, F. (2014) Stabilization of Continuous-time Fractional Positive Systems by using Lyapunov function. IEEE Trans. Automat. Contr., vol. 59, no. 8, pp. 2203–2208.