The Radon-Nikodym problem for approximately proper equivalence relations

Jean Renault

November 23, 2002

Abstract

We study the Radon-Nikodym problem for approximately proper equivalence relations and more specifically the uniqueness of certain Gibbs states. One of our tools is a variant of the dimension group introduced in the study of AF algebras. As applications, we retrieve sufficient conditions for the uniqueness of traces on AF algebras and parts of the Perron-Frobenius-Ruelle theorem.¹

1 Introduction

The motivation of this work is a problem in the theory of C*-algebras, namely the study of the KMS states of some automorphism groups of the Cuntz algebras and their generalizations studied in [17]. While the crucial role of the Perron-Frobenius theorem in this problem has been noticed from the beginning (see for example [11, 13]), the application of Ruelle’s version of this theorem is more recent ([27, 15, 34]). The existing proofs of the Perron-Frobenius-Ruelle theorem, in particular [20], use heavily a sequence of expectations associated with the asymptotic algebra. The purpose of this work is to use the formalism of groupoids (cf. [33]), approximately proper equivalence relations and dimension groups (cf. [24]) to provide a convenient setting for these proofs.

Given a groupoid $G$ on a space $X$ and a cocycle $D \in Z^1(G, R_+^\times)$, the Radon-Nikodym problem mentioned in the title is the study of the probability measures on $X$ which are quasi-invariant with respect to $G$ (the definition is recalled in Section 2) and which admit this cocycle as Radon-Nikodym derivative. When $G = R$ is an approximately proper (abbreviated as AP) equivalence relation, i.e. an increasing union of proper equivalence relations $R_n$, the cocycle defines a sequence of expectations $E_n$ with range $C(X/R_n)$. The solutions of the Radon-Nikodym problem are exactly the measures which factor through $E_n$ for all $n$.

¹ 1991 Mathematics Subject Classification. Primary: 37D35 Secondary: 46L85.

Key words and phrases. Equivalence relations. C*-algebras. Cocycles. Radon-Nikodym derivative. Dimension groups. Perron-Frobenius theorem.
A classical example of this situation is provided by statistical mechanics on a lattice $\Lambda$. The sequence $(R_n)$ is defined by an increasing sequence of finite sets $\Lambda_n$ with union $\Lambda$. The cocycle is the energy cocycle, as in Section II.5 of [33]. In this setting, the solutions of the Radon-Nikodym problem are called Gibbs states (this definition has been introduced by D. Capocaccia in [5]).

Our main concern is the uniqueness of the solution of the Radon-Nikodym problem on an AP equivalence relation. We give sufficient conditions for uniqueness in two cases. First we consider quasi-product cocycles on AF equivalence relations. The data consist of a Bratteli diagram and a labeling of its edges. Then a convenient condition on this labeling (Corollary 4.3, (ii)) guarantees uniqueness. Our result covers (and was inspired by) A. Török’s work [35] (pointed to me by O. Bratteli) on uniqueness of traces on AF $C^*$-algebras. The second case is the classical setting of Ruelle’s Perron Frobenius theorem and gives the part of P. Walters’ Theorem 8 in [37] concerning the transpose of the Ruelle operator (Corollary 6.2 and Proposition 7.2). The eigenvalue problem for this operator amounts to a Radon-Nikodym problem on the semi-direct groupoid $G(X,T)$. Our method, outlined in Section 4.1 of [34], is to solve first the Radon-Nikodym problem on the asymptotic equivalence relation $R(X,T)$ which is approximately proper. If it has a unique solution, this is also a solution of the initial problem.

This work is organized as follows. The definition and some examples of AP equivalence relations are given in the first section. The second section deals with cocycles. We are only concerned here with cocycles with values in the multiplicative group $\mathbb{R}_+^*$ of strictly positive real numbers. Let $D$ be a cocycle defined on the AP equivalence relation $R = \cup R_n$ on the compact space $X$. Then its restriction to $R_n$ can be written as a coboundary. This provides a normalized potential $\rho_n$ and an expectation $E_n$ from $C(X)$ to $C(X/R_n)$, where $C(X)$ denotes the space of real-valued continuous functions on the compact space $X$. This is the sequence of expectations mentioned earlier. A necessary and sufficient condition for the unique ergodicity of $D$ is that for all $f \in C(X)$, the variation of $E_n(f)$ tends to 0. The theory of dimension groups (this means here inductive limits of $C(X_n)$’s, viewed as ordered vector spaces, under positive linear maps), as developed by K. Goodearl in [24], is well suited to our problem. Indeed, the solutions of the Radon-Nikodym problem are the states of the dimension group associated to the sequence $(E_{n,n-1})$ (where $E_n = E_{n,n-1} \circ E_{n-1}$). Therefore, we recall in an appendix some results of this theory. When the vector spaces $C(X_n)$ have finite dimension, there is an elementary sharp estimate of the rate at which Markovian (i.e. unital) operators contract the variation (Lemma A.5) (this is essentially the same estimate which is used by Török in [35]; it is so natural that it must have been noticed by other people). Rather surprisingly, this contraction rate admits an easy estimate in terms of non-unital positive linear maps (Lemma A.6). This gives an elementary proof of the Perron-Frobenius theorem for primitive matrices. This estimate is also used in the third section devoted to quasi-product cocycles on AF equivalence relations. This section is very close to the sections 3 and 4 of [14], where the emphasis is on measures rather than on cocycles. The main result is Corollary 4.3, which gives a sufficient condition for unique ergodicity as mentioned earlier. The section four introduces some
definitions such as π-cover, which are useful when dealing with local homeomorphism on arbitrary compact spaces. When $X$ is a compact space, we find more convenient to use the entourages associated to a finite open cover rather than those defined by a compatible metric. This section also contains some estimates of the variation. The section five studies the case of a stationary system defined by a single surjective local homeomorphism $T : X \to X$, where $X$ is compact. The main result, Theorem 6.1, is well-known: it gives the unique ergodicity of the cocycle $D$ defined by a potential $g \in C(X, \mathbb{R}_+^\infty)$ (or by a sequence of potentials $(g_n)$) under the usual assumptions on the dynamical system ($T$ is assumed to be expansive and $R(X, T)$ to be minimal) and on the potentials (Walter’s condition). The proof is more elementary than most in the sense that it does not use the Schauder-Tychonoff theorem. However, the key step relies on the Ascoli-Arzela theorem just as in [37]. It may be that the estimate of the contraction rate of the variation gives a more precise proof, with an estimate of the speed of convergence; but this is not done here. In the last section, the results are applied to the transfer operator.

Acknowledgments

A major part of this work was done at the Centre for Advanced Study in Oslo while I held a Visiting Fellowship under the program "Non-commutative Phenomena in Mathematics and Theoretical Physics". I heartily thank its staff for an excellent stay and the organizers, in particular M. Landstad, for their invitation. I am also grateful to E. Alfsen, O. Bratteli, R. Exel, A. Kunjian and C. Skau for fruitful discussions and suggestions.

2 AP equivalence relations

In this section, $X$ is a locally compact second countable Hausdorff space and $R$ is an equivalence relation on $X$. For the sake of simplicity, we only consider equivalence relations with countable equivalence classes. We also denote by $R \subset X \times X$ its graph.

Definition 2.1. The equivalence relation $R$ on $X$ will be called proper and étale if its quotient space is Hausdorff and its quotient map $X \to X/R$ is a local homeomorphism.

Endowed with the product topology of $X \times X$, $R$ is a locally compact groupoid which is étale. This simply means that the projection maps $r, s : R \to X$ are local homeomorphisms. Every open cover $\mathcal{U}$ of $X$ by open sections of $\pi$ gives a cover $\{(U \times V) \cap R, U, V \in \mathcal{U}\}$ of $R$ by open bisections (i.e. simultaneous sections of the projection maps $r, s$). The construction of [33] applies and gives the C*-algebra $C^*(R)$. Moreover this groupoid is proper in the sense of [1]. It is known, (see [32]) that $C^*(R)$ has continuous trace and that it is Rieffel-Morita equivalent to $C_0(X/R)$. Since our study is limited to étale
equivalence relations and groupoids, we shall often omit the word étale and say “proper” rather than “proper and étale”.

**Definition 2.2.** The equivalence relation $R$ on $X$ will be called *approximately proper*, abbreviated AP, if there exists a sequence

$$X_0 \xrightarrow{\pi_{1,0}} X_1 \rightarrow \ldots \rightarrow X_{n-1} \xrightarrow{\pi_{n,n-1}} X_n \rightarrow \ldots$$

where $X_0 = X$ and for each $n \geq 1$, $X_n$ is a Hausdorff space and $\pi_{n,n-1}$ is a surjective local homeomorphism such that

$$R = \{(x, y) \in X \times X : \exists n \in \mathbb{N} : \pi_n(x) = \pi_n(y)\}$$

where $\pi_n = \pi_{n,n-1} \circ \pi_{n-1,n-2} \ldots \circ \pi_{1,0}$.

In the context of the above definition, we let $R_n$ be the equivalence relation on $X$ defined by $\pi_n$.

**Proposition 2.1.** Let $R$ be the equivalence relation defined by the sequence $(\pi_n)$ as above.

(i) $(R_n)$ is an increasing sequence of subsets of $X \times X$ and $R = \bigcup R_n$.

(ii) For $m \leq n$, $R_m$ is a closed and open subset of $R_n$.

(iii) Endowed with the inductive limit topology, $R$ is an étale locally compact groupoid.

**Proof.** The assertion (i) is obvious. For the assertion (ii), we introduce the equivalence relation $S$ on $X_m$ defined by the map $\pi_{n,m}$. The diagonal $\Delta_{X_m}$ is closed and open in $S$. Therefore, $R_m = (\pi_m \times \pi_m)^{-1}(\Delta_{X_m})$ is open and closed in $R_n = (\pi_m \times \pi_m)^{-1}(S)$. For the assertion (iii), according to [33], it suffices to construct a cover of $R$ consisting of open locally compact bisections. For each $n \in \mathbb{N}$, let $U_n$ be a cover of $X$. Then, the family $\{(U \times V) \cap R_n\}$, where $U, V \in U_n$, is an open cover of $R_n$ and the union of these covers is the desired cover of $R$.

**Corollary 2.2.** Let $R = \bigcup R_n$ be an AP equivalence relation as above. Then, $C^*(R)$ is the inductive limit of the $(C^*(R_n))$’s. More precisely, $C^*(R_n)$ can be identified with a sub $C^*$-algebra; these subalgebras are increasing and their union is dense.

On the other hand, for all $n \in \mathbb{N}$, $C(X_n)$ can be identified with the subalgebra $(\pi_n^* C(X_n))$ of $C(X)$ and this sequence of subalgebras is decreasing. Its intersection is the fixed point subalgebra, as defined below.

**Definition 2.3.** Let $(X, R)$ be an étale equivalence relation. The **fixed point subalgebra** is the subalgebra of $C(X)$:

$$C(X)^R = \{f \in C(X) : (x, y) \in R \Rightarrow f(x) = f(y)\}.$$ 

Its spectrum will be denoted by $X^\infty$. One says that $(X, R)$ is **irreducible** if $C(X)^R$ consists only of constant functions.
Thus, when \((X, R)\) is an AP equivalence relation defined by a sequence \((X_n)\), the fixed point subalgebra \(C(X)_R\) is the intersection of the \(\pi_n^*(C(X_n))\)'s. The inclusion \(C(X)_R \subset \pi_n^* C(X_n)\) defines a continuous surjective map \(\pi_{\infty,n} : X_n \to X_\infty\). In particular, we write \(\pi_\infty = \pi_{\infty,0} : X \to X_\infty\). These maps identify \(X_\infty\) to the inductive limit of the sequence \((\pi_{n,n-1} : X_{n-1} \to X_n)\). In the irreducible case, this space is reduced to a point while the equivalence relation \(R\) is non-trivial.

**Example 2.1. AF equivalence relations.** (See [34] and [23].) Let \((V, E)\) be a Bratteli diagram. Recall that this means an oriented graph, where the vertices are stacked on levels \(n = 0, 1, 2, \ldots\) and the edges run from a vertex of level \(n-1\) to a vertex of level \(n\). We denote by \(V(n)\) the set of vertices of the level \(n\) and by \(E(n)\) the set of edges from level \(n-1\) to level \(n\). We assume that every vertex \(v\) emits finitely many, but at least one, edges and that every vertex on a level \(n \geq 1\) receives at least one edge. An infinite path is a sequence of connected edges \(x = x_1x_2 \ldots\), where \(x_1\) starts from level 0. The space \(X\) of infinite paths has a natural totally disconnected topology, with the cylinder sets \(Z(x_1x_2 \ldots x_n)\) as a basis. We define similarly the space \(X_n\) of infinite paths starting from level \(n\) and we have the obvious projection maps \(\pi_{n,n-1} : X_{n-1} \to X_n\). This defines an AP equivalence relation \(R\) on \(X\) called the tail equivalence relation of the Bratteli diagram. In the sequel, following [23], we shall use the notation \((X = X(V, E), R = R(V, E))\) to designate this AP equivalence relation; we shall call it the tail equivalence relation of the Bratteli diagram \((V, E)\). An explicit construction of matrix units in \(C^*(R)\) shows that it is an AF \(C^*\)-algebra admitting \((V, E)\) as Bratteli diagram. Conversely, it is shown in [23] (and in [34] when the space \(X\) is compact) that every AP equivalence relation on a space \(X\) which is locally compact and totally disconnected is the tail equivalence relation of a Bratteli diagram. Such an equivalence relation is called an AF equivalence relation. Properties of AF equivalence relations are studied in [23], where an equivalent definition is used.

**Example 2.2. Stationary equivalence relations.** Let \(X\) be a locally compact space and \(T : X \to X\) a local homeomorphism which is onto. We form the stationary sequence
\[
X \xrightarrow{T} X \xrightarrow{T} X \xrightarrow{T} \ldots
\]
The corresponding AP equivalence relation
\[
R = \{(x, y) \in X \times X : \exists n \in \mathbb{N} : T^n x = T^n y\}
\]
plays an essential role in the study of \(T\) when it is large enough. This example will be developed later.

The definition of an AP equivalence relation makes an implicit reference to a defining sequence
\[
X_0 \xrightarrow{\pi_{1,0}} X_1 \to \ldots \to X_{n-1} \xrightarrow{\pi_{n,n-1}} X_n \to \ldots
\]
Definition 2.4. We shall say that two sequences

\[ X_0 \to X_1 \to \ldots \to X_m \to \ldots \]
\[ Y_0 \to Y_1 \to \ldots \to Y_n \to \ldots \]

as in Definition 2.2 are equivalent if there are subsequences \((m_k)\) and \((n_k)\) and surjective local homeomorphisms \(X_{m_k} \to Y_{n_k}\) and \(Y_{n_k} \to X_{m_{k+1}}\) making commutative diagrams.

This is an equivalence relation. An other way to obtain the same definition is to define first the contraction of a sequence \((X_n)\): it is the new sequence \((\overline{X}_k)\) defined by a subsequence \((n_k)\); explicitly, \(\overline{X}_k = X_{n_k}\) and \(\overline{\pi}_{k,k-1} = \pi_{n_k,n_{k-1}}\). Then, we say that the original sequence is a dilation of the new sequence. Two sequences are equivalent iff they admit contractions which have a common dilation.

Proposition 2.3. Let \(X\) be a locally compact space.

(i) Equivalent defining sequences \((X_n)\) and \((Y_n)\) as above with \(X_0 = Y_0 = X\) define the same equivalence relation \(R\) on \(X\) and the same topology on \(R\).

(ii) Conversely, if \(X\) is compact, two sequences \((X_n)\) and \((Y_n)\) as above with \(X_0 = Y_0 = X\) which define the same equivalence relation \(R\) are equivalent.

Proof. Let us call \((R_n)\) [resp. \((S_n)\)] the sequence of equivalence relations on \(X\) defined by \((X_n)\) [resp. \((Y_n)\)]. If \((X_n)\) and \((Y_n)\) are equivalent and if we have subsequences \((m_k)\) and \((n_k)\) as in the definition, we have the inclusions \(R_{m_k} \subset S_{n_k} \subset R_{m_{k+1}}\) for all \(k\). We also know from Proposition 2.4 \((ii)\) that these inclusion maps are open. Therefore the sequences \((R_n)\) and \((S_n)\) have the same union \(R\) and the inductive limit topology is the same. Suppose now that \(X\) is compact and that the sequences \((R_n)\) and \((S_n)\) have the same union \(R\). Let us endow \(R\) with the inductive limit topology of the \((S_n)\)'s. Let us fix \(m\). Since \(R_m\) is closed in \(X \times X\), it is a compact subset of \(R\). Since it is covered by the union of the open sets \(S_n\)'s, it is contained in some \(S_n\). Similarly, any \(S_n\) is contained in some \(R_m\). Therefore one can construct subsequences \((m_k)\) and \((n_k)\) such that \(R_{m_k} \subset S_{n_k} \subset R_{m_{k+1}}\) for all \(k\). These inclusions give the desired maps \(X_{m_k} \to Y_{n_k}\) and \(Y_{n_k} \to X_{m_{k+1}}\).

3 Cycles

We shall only consider cocycles with values in the group \(A = \mathbb{R}\) or equivalently \(A = \mathbb{R}^*_+\). If \(R\) is an equivalence relation on \(X\), a cocycle with values in \(A\) is a map \(c : R \to A\) satisfying \(c(x,y) + c(y,z) = c(x,z)\) for all \((x,y),(y,z) \in R\). A cocycle \(c\) is a coboundary if there is a map \(b : X \to A\), called a potential, such that \(c(x,y) = b(x) - b(y)\). Two cocycles are cohomologous if their difference is a coboundary. In our setting, \(R\) is a topological groupoid \(A\) is a topological group and we assume that \(c\) and \(b\) as above are continuous. We denote by \(Z^1(R,A)\) the
set of continuous $A$-valued cocycles. We shall use the convention $c \in Z^1(R, A)$ and $D \in Z^1(R, R^+_1)$.

One defines similarly cocycles on arbitrary groupoids. Let $G$ be a locally compact groupoid with a continuous Haar system (cf. [33], Chapter 1). A measure $\mu$ on its unit space $X = G^{(0)}$ is said to be quasi-invariant if the measures $\mu \circ \lambda$ and $(\mu \circ \lambda)^{-1}$ on $G$ are equivalent. Then, the Radon-Nikodym derivative $D_\mu = \frac{d(\mu \circ \lambda)}{d(\mu \circ \lambda)^{-1}}$ satisfies the cocycle identity almost everywhere. When $G$ is an étale groupoid, for example a proper and étale equivalence relation or an AP equivalence relation as above, it carries the canonical continuous Haar system consisting of counting measures $\lambda^x$ on the fibers $G^x = r^{-1}(x)$ of the range map $r$. A measure $\mu$ on $X = G^{(0)}$ is quasi-invariant if and only if for all open bisections $S \subseteq G$, the measures $\sigma(S)_* (\mu|_{s(S)})$ and $\mu|_{r(S)}$ are equivalent, where $\sigma(S) : s(S) \to r(S)$, such that $\sigma(S)(x) = r(Sx)$, is the map induced by $S$. Moreover $d\sigma(S)_* \mu(y) = D_\mu^{-1}(y S) d\mu(y)$.

The Radon-Nikodym problem for $D \in Z^1(G, R^+_1)$, where $G$ is a locally compact groupoid with Haar system $\lambda$ and compact unit space $X$, is to determine the set $S_{G,D}(X)$ (or $S_{D}(X)$ if there is no ambiguity on $G$) of probability measures $\mu$ on $X$ which are quasi-invariant and admit $D$ as their Radon-Nikodym derivative. We denote by $S(X)$ the set of probability measures on $X$.

### 3.1 Cocycles on proper equivalence relations

Let us first look at cocycles on proper equivalence relations.

**Proposition 3.1.** Let $R$ be a proper equivalence relation on $X$ defined by $\pi : X \to \Omega$ and let $c \in Z^1(R, A)$, where $A = R$ or $R^+_1$. Then

(i) $c$ is a coboundary.

(ii) If $b$ and $b'$ are two potentials for $c$, they differ by a function of the form $h \circ \pi$, where $h : \Omega \to A$ is continuous.

(iii) If $X$ is compact and $D \in Z^1(R, R^+_1)$, there is a unique potential $\rho \in C(X, R^+_1)$ such that $\sum_\omega \rho(x) = 1$ for all $\omega \in \Omega$.

**Proof.** For (i) and (ii), we assume that $A = R$. To prove the first assertion, we choose a locally finite open cover $\{V_j\}$ of $\Omega$, continuous sections $\sigma_j : V_j \to X$ of $\pi$ and a partition of unity $\{h_j\}$ subordinate to $\{V_j\}$. We define $b_j : \pi^{-1}(V_j) \to R$ by $b_j(x) = c(x, \sigma_j(\pi(x)))$ and $b : X \to R$ by $b(x) = \sum_j h_j(\pi(x)) b_j(x)$. Then $c(x, y) = b(x) - b(y)$. For the second assertion, we notice that if $b$ and $b'$ satisfy $b'(x) - b'(y) = b(x) - b(y)$ for all $(x, y) \in R$, then $b' - b$ is constant on the equivalence classes, hence of the form $h \circ \pi$. For (iii), we pick an arbitrary potential $\rho' \in C(X, R^+_1)$ and define $Z(\omega) = \sum_\omega \rho'(x)$. Then $Z \in C(\Omega, R^+_1)$ and $\rho = \rho' / Z \circ \pi$ is the desired potential. Because of (ii), two potentials satisfying this normalization agree. \qed
Therefore, via the introduction of this normalized potential, we relate a cocycle $D$ with a family of probability measures along the fibers of the map $\pi : X \to \Omega$ or equivalently, a conditional expectation:

**Definition 3.1.** Given a proper equivalence relation $R$ on a compact space $X$ and $D \in Z^1(R, R_+^*)$, the potential $\rho = \rho_D \in C(X, R_+^*)$ satisfying the normalization $\sum_\omega \rho(x) = 1$ for all $\omega \in \Omega$ is called the normalized potential of $D$. It defines a Markovian (Definition A.1) operator $E = E_D : C(X) \to C(\Omega)$ according to $E(f)(\omega) = \sum_\omega \rho(x)f(x)$, called the expectation relative to $D$.

**Remark 3.1.**
- Our definition carries an abuse of language. It is not $E$ itself but $\pi^* \circ E$ which is a conditional expectation from $C(X)$ to $C(\Omega)$.
- I owe to R. Exel the following observation. The above expectation $E$ has finite index. Conversely, given a continuous surjection $\pi : X \to \Omega$, there exists a conditional expectation $E : C(X) \to C(\Omega)$ of finite index if and only if $\pi$ is a local homeomorphism. Then, there exists $D \in Z^1(R, R_+^*)$ such that $E = E_D$.

The Radon-Nikodym problem is easily solved for proper equivalence relations. The following proposition says that the solutions are the measures on $X$ admitting $E_D$ as conditional expectation.

**Proposition 3.2.** Let $R$ be a proper equivalence relation on a compact space $X$ and $D \in Z^1(R, R_+^*)$. Then

$$S_D(X) = \{\Lambda \circ E_D, \Lambda \in S(\Omega)\},$$

where $E_D$ is the expectation relative to $D$. More precisely, $E_D^* : C(\Omega)^* \to C(X)^*$ identifies $S_D(X)$ and $S(\Omega)$ as compact convex sets.

**Proof.** Let us first show that, for $\Lambda \in S(\Omega)$, $\mu = \Lambda \circ E \in S_D(X)$. We denote by $\alpha$ the system of counting measures on the fibers of $\pi : X \to \Omega$, by $\lambda_r$ the system of counting measures on the fibers of $r : R \to X$ (the first projection) and by $\lambda_s$ the system of counting measures on the fibers of $s : R \to X$ (the second projection). Then, we have for $f \in C_c(R)$,

$$\mu \circ \lambda_r(f) = \Lambda \circ \alpha(\rho \lambda_r(f)) = \Lambda \circ \alpha \circ \lambda_r((\rho \circ r)f),$$

$$\mu \circ \lambda_s(f) = \Lambda \circ \alpha(\rho \lambda_s(f)) = \Lambda \circ \alpha \circ \lambda_s((\rho \circ s)f).$$

The obvious equality $\alpha \circ \lambda_r = \alpha \circ \lambda_s$ gives the result. Conversely, let us suppose that $\mu \in S_D(X)$. Then, for every $f \in C(X \times X)$, we have

$$\int \sum_{\pi(y) = \pi(x)} f(x, y)d\mu(x) = \int \sum_{\pi(x) = \pi(y)} f(x, y)D(x, y)d\mu(y).$$

If we choose $f(x, y) = \rho(y)f(y)$, where $f \in C(X)$, we get

$$\int \sum_{\pi(y) = \pi(x)} \rho(y)f(y)d\mu(x) = \int f(y)d\mu(y),$$

which says that $\pi_* \mu \circ E = \mu$. \qed
**Definition 3.2.** Let $X \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_2} X_2$ be surjective local homeomorphisms. Let $R_1$ [resp.$R_2$] be the equivalence relation defined by $\pi_1$ [resp.$\pi_2 = \pi_2 \circ \pi_1$]. Let $D_2 \in Z^1(R_2, R^*_1)$ with potential $\rho_2 \in C(X,R^*_1)$ and $D_1 = D_{|R_1}$ with potential $\rho_1$. Then, there exists a unique $\rho_{2,1} \in C(X_1, R^*_1)$ such that $\rho_2 = \rho_1(\rho_{2,1} \circ \pi_1)$. Moreover, if $\rho_2$ and $\rho_1$ are normalized, then so is $\rho_{2,1}$, i.e. $\sum_{x_1 \in D_1} \rho_{2,1}(x_1) = 1$ for all $x_1 \in X_2$.

**Proof.** The uniqueness of $\rho_{2,1}$ results from the surjectivity of $\pi_{2,1}$. Since $\rho_2$ is also a potential for $D_1$, its existence results from Proposition 3.1 (ii). Let us assume that $\rho_2$ and $\rho_1$ are normalized. Then for $x_2 \in X_2$, we have

$$1 = \sum_{x_1 \in D_1} \rho_{2,1}(x_1) \sum_{x_2 \in D_2} \rho_2(x_2) \rho_1(x_1) \pi_1(x_1) = \sum_{x_1 \in D_1} \rho_{2,1}(x_1) \rho_1(x_1)$$

3.2 **Cocycles on AP equivalence relation**

Let us consider now the case of an AP equivalence relation $R$ on a compact space $X$ with a defining sequence $(X_n)$. We use the notation of Definition 2.2; in particular, we denote by $R_n$ the proper equivalence relation defined by $\pi_n : X \rightarrow X_n$. Given $D \in Z^1(R, R^*_n)$, we consider the sequence of its restrictions $D_n = D_{|R_n} \in Z^1(R_n, R^*_n)$. For each $n$, we pick a potential $\rho_n$ for $D_n$. From Lemma 3.3, we obtain $\rho_{n,n-1} \in C(X_{n-1}, R^*_n)$ such that $\rho_n = \rho_{n-1}(\rho_{n,n-1} \circ \pi_{n-1})$. Equivalently, $\rho_n(x)/\rho_{n-1}(x)$ depends only on $\pi_{n-1}(x)$. In particular, we can choose the sequence of normalized potentials; in that case $\rho_{n,n-1}$ is also normalized.

**Definition 3.2.** Let $R$ be an AP equivalence relation on a compact space $X$ defined by a sequence $(X_n)$.

(i) A **compatible sequence of potentials** is a sequence $(\rho_n \in C(X,R^*_n))$ such that $\rho_n(x)/\rho_{n-1}(x)$ depends only on $\pi_{n-1}(x)$. Equivalently, a compatible sequence of potentials is defined by an initial potential $\rho_0 \in C(X,R^*_1)$ and a sequence of local potentials $\rho_{n,n-1} \in C(X_{n-1}, R^*_n)$; then $\rho_n$ is given by

$$\rho_n = (\rho_0)(\rho_{1,0})(\rho_{2,1} \circ \pi_1) \cdots (\rho_{n,n-1} \circ \pi_{n-1}).$$

If for all $n$ and for all $x_n \in X_n$, $\sum_{x_n} \rho_n(x) = 1$, we speak of a sequence of normalized potentials.

(ii) Given $D \in Z^1(R, R^*_n)$, a sequence of potentials $\rho_n$ of $D_n = D_{|R_n}$ is called a sequence of **potentials** of $D$. The associated sequence of functions $(\rho_{n,n-1})$ is called a sequence of **local potentials** of $D$. In particular, we can consider the sequence of normalized potentials of $D$.  

Note that a compatible sequence of potentials \((\rho_n)\) defines a cocycle \(D \in Z^1(R, R^*_+\)) such that \(D(x, y) = \rho_n(x)/\rho_n(y)\) for \((x, y) \in R_n\) and that every cocycle \(D \in Z^1(R, R^*_+)\) is defined by a compatible sequence of potentials.

**Remark 3.2.** The sequence of normalized potentials of a cocycle \(D \in Z^1(R, R^*_+)\) is unique. However, it is sometimes advantageous to consider arbitrary sequences of potentials. For example, a change of initial potential amounts to replacing the cocycle by a cohomologous cocycle. Although we restrict the discussion below to the normalized potentials, the theory of dimension groups and their state space applies as well to the general case.

**Proposition 3.4.** The correspondence which associates to a cocycle \(D\) its sequence of normalized potentials \((\rho_n)\) [resp. its sequence of normalized local potentials \((\rho_{n,n-1})\)] is a bijection from \(Z^1(R, R^*_+)\) onto the set of compatible sequences of normalized potentials [resp. onto the set of sequences of normalized local potentials].

The normalized potentials \(\rho_n\) define conditional expectations \(E_n : C(X) \to C(X_n)\) and the normalized local potentials \((\rho_{n,n-1})\) define conditional expectations

\[
E_{n,n-1} : C(X_{n-1}) \to C(X_n).
\]

For \(m \leq n\), we set:

\[
E_{n,m} = E_{n,n-1}E_{n-1,n-2} \ldots E_{m+1,m}.
\]

Then, we have \(E_{n} = E_{n,0}\).

**Proposition 3.5.** Let \(R = \cup R_n\) be an AP equivalence relation on a compact space \(X\), \(D \in Z^1(R, R^*_+)\) and \(D_n = D|_{R_n} \in Z^1(R_n, R^*_+)\) as above. Then, the sequence of conditional expectations \((P_n = \pi^*_nE_n)\) is a (reversed) martingale, i.e. it satisfies \(P_nP_m = P_mP_n\) for \(m \leq n\).

**Proof.** This is an immediate consequence from the fact that \(E_m\pi_n^*\) is the identity map of \(C(X_m)\) and of the factorizations \(\pi_n = \pi_{n,m} \circ \pi_m\) and \(E_n = E_{n,m}E_m\). □

Thus we obtain a sequence of Markovian operators

\[
\underline{E} = (E_{n,n-1} : C(X_{n-1}) \to C(X_n))
\]

and we can apply the results of the Appendix. We denote by \(\mathcal{E} = \mathcal{E}(\underline{E})\) its dimension group and by \(S = S(\underline{E})\) its state space. Recall that it is a non-empty compact convex Choquet simplex. We denote by \(E_{\infty,n} : C(X_n) \to \mathcal{E}\) the canonical morphisms; they are surjective. In particular, we write \(E_{\infty} = E_{\infty,0} : C(X) \to \mathcal{E}\). We have a convenient description of the state space \(S(\underline{E})\). Indeed, given a reversed martingale \((P_n)\), Brown and Dooley define in [4] a G-measure as a probability measure \(\mu\) on \(X\) satisfying \(\mu = (P_n)^*\mu\) for all \(n\). On the other hand, by Definition A.2, \(S(\underline{E})\) is the set of sequences \((\mu_n)\), where \(\mu_n \in S(X_n)\) and \(\mu_{n-1} = \mu_n \circ E_{n,n-1}\). For such a sequence, \(\mu = \mu_0 = E_0^*\mu_0\) does not depend on \(n\) and is a G-measure. Conversely, one recovers the sequence \((\mu_n)\) from the G-measure \(\mu\) by setting \(\mu_n = (\pi_n)_*\mu\). Thus:
**Lemma 3.6.** The map \((\mu_n) \mapsto \mu_0\) identifies the state space \(S(E)\) to the set of \(G\)-measures.

Moreover, we have seen previously that the measures of the form \(E^*_n\mu_n\), where \(\mu_n \in S(X_n)\) are exactly the probability measures admitting \(D_n\) as Radon-Nikodym derivative. This gives the next proposition. In the context of equilibrium states in statistical mechanics, the equivalence of the definitions of a Gibbs measure as a \(G\)-measure or as a quasi-invariant measure is well known ([25], Theorem 5.2.4.(a)).

**Proposition 3.7.** Let \((X, R), D \in Z^1(R, R^*_+)\), \(D_n, E_n\) and \(E\) as above.

(i) The sequence \((S_{D_n}(X))\) is decreasing and its intersection is \(S_D(X)\).

(ii) The isomorphisms \(E^*_n : S(X_n) \to S_{D_n}(X)\) induce an isomorphism of compact convex sets from \(S = S(E)\) onto \(S_D(X)\). In other words, the quasi-invariant probability measures admitting \(D\) as Radon-Nikodym derivative are exactly the \(G\)-measures of the martingale \((P_n = \pi_n^*E_n)\).

**Proof.** (i) A measure \(\mu \in S(X)\) belongs to \(S_{D_n}(X)\) if and only if it is quasi-invariant with respect to \(R_n\) and admits \(D_n\) as its Radon-Nikodym derivative. Since \(R_{n-1} \subset R_n\) and \(D_{n-1} = D_n|_{R_{n-1}}\), a measure \(\mu \in S(X)\) which belongs to \(S_{D_n}(X)\) also belongs to \(S_{D_{n-1}}(X)\). Moreover \(\mu \in S(X)\) belongs to \(S_D(X)\) if and only if, for all \(n\), it is quasi-invariant with respect to \(R_n\) with Radon-Nikodym derivative \(D_n\). Therefore, the intersection of the \((S_{D_n}(X))\)'s is \(S_D(X)\).

(ii) Let \((\mu_n) \in S(E)\) and let \(\mu = E^*_n\mu_n\) the associated \(G\)-measure. Then \(\mu\) belongs to \(S_{D_n}(X)\) for all \(n\) and therefore belongs to \(S_D(X)\). Conversely, if \(\mu\) belongs to \(S_D(X)\), we define \(\mu_n = \pi_n^*\mu \in S(X_n)\) for all \(n\). Then the sequence \((\mu_n)\) belongs to \(S(E)\) and \(\mu = E^*_n\mu_n\). Therefore, the map which sends \((\mu_n) \in S(E)\) into \(\mu_0 \in S(X)\) is an isomorphism of \(S(E)\) onto \(S_D(X)\). \(\square\)

We reformulate Lemma A.2 of the Appendix in this setting.

**Corollary 3.8.** Let \((X, R), D \in Z^1(R, R^*_+)\), \(E_n\) be as above. For \(f \in C(X)\), the following conditions are equivalent:

(i) \(\|E_n(f)\|\) tends to 0 as \(n\) tends to \(\infty\),

(ii) \(\mu(f) = 0\) for all \(\mu \in S_D(X)\).

With the notation of the Appendix, we denote by \(\text{Aff}(S)\) the space of continuous functions on the compact convex space \(S\) and by \(\theta : E \to \text{Aff}(S_D(X))\) the evaluation map. Then \(\theta_0 = \theta \circ E_\infty : C(X) \to \text{Aff}(S_D(X))\) is the evaluation map \(\theta_0(f)(\mu) = \mu(f)\) for \(\mu \in S_D(X)\).

**Proposition 3.9.** Let \((X, R)\) be an AP equivalence relation and let \(D\) be a continuous cocycle in \(Z^1(R, R^*_+)\). Then the evaluation map \(\theta_0\) induces an isomorphism of ordered Banach spaces with order-unit from \(C(X)/\text{Ker}\theta_0\) onto \(\text{Aff}(S_D(X))\).
Proposition 3.11. Let \((X, R)\) be an AP equivalence relation and let \(D \in Z^1(R, R^*_+)\) be as above. The following assertions are equivalent:

(i) \(C(X) = C(X)^R + \text{Ker}\theta_0\).

(ii) The restriction of \(\theta_0\) to \(C(X)^R\) is an isomorphism from \(C(X)^R\) onto \(\text{Aff}(S_D(X))\).

(iii) The restriction map \((\pi_\infty)_*: S_D(X) \to S(X_\infty)\) is an isomorphism from \(S_D(X)\) onto \(S(X_\infty)\).

(iv) There is a conditional expectation \(E_\infty : C(X) \to C(X_\infty)\) such that
\[
S_D(X) = \{\Lambda \circ E_\infty : \Lambda \in S(X_\infty)\}.
\]

(v) For all \(f \in C(X)_+\), \((\pi_\infty^*E_n(f))\) converges uniformly.

(vi) For all \(f \in C(X)\), \((\pi_\infty^*E_n(f))\) is equicontinuous.

Proof. (i) \(\Leftrightarrow\) (ii) This results from Proposition 3.9 and Lemma 3.10.

(ii) \(\Leftrightarrow\) (iii) We can identify the state space of \(\text{Aff}(S_D(X))\) with \(S_D(X)\) and the state space of \(C(X)^R\) with \(S(X_\infty)\). The map from \(S_D(X)\) into \(S(X_\infty)\) induced by \(\theta_0 : C(X)^R \to \text{Aff}(S_D(X))\) is the restriction map \((\pi_\infty)_*\). Therefore, if (ii) holds, this restriction map is an isomorphism of convex compact sets. Conversely, let us assume that the restriction map \((\pi_\infty)_* : S_D(X) \to S(X_\infty)\) is an isomorphism of compact convex sets. It induces an isomorphism of affine spaces from \(\text{Aff}(S(X_\infty)) = C(X_\infty)\) onto \(\text{Aff}(S_D(X))\). But this map coincides with \(\theta_0\).
(i) ⇒ (iv) We let $P$ be the projection of $C(X)$ onto $C(X)^R$ along $\text{Ker}\theta_0$ and $E_\infty : C(X) \to C(X_\infty)$ be the composition of $P$ and the isomorphism $C(X)^R \to C(X_\infty)$. Then, for all $\mu \in SD(X)$ and $f \in C(X)$, we have the equalities

$$<\mu, \pi^*_\infty E_\infty(f)>=<\mu, Pf>=<\mu, f>.$$

and therefore $\mu = (\pi_\infty)_* \circ E_\infty$. Since the restriction map $(\pi_\infty)_*$ is a bijection from $SD(X)$ onto $S(X_\infty)$, we have the result (iv).

(iv) ⇒ (v) Let $f \in C(X)$. Then, using the characterization of $SD(X)$ given in (iv), we see that $f - \pi^*_\infty E_\infty(f)$ belongs to $\text{Ker}\theta_0$. According to Corollary 3.8, this implies that $\|\pi^*_n E_n(f) - \pi^*_\infty E_\infty(f)\| = \|E_n(f - \pi^*_\infty E_\infty(f))\|$ tends to 0.

(v) ⇒ (vi) is clear.

(vi) ⇒ (i) To ease the notation, we introduce $P_n = \pi^*_n E_n : C(X) \to C(X)$.

Let $f \in C(X)$. Since $(P_n(f))$ is equicontinuous and bounded, there is a subsequence $(P_{n_k}(f))$ converging to $g \in C(X)$. For a fixed $n$, $(P_n P_{n_k}(f))$ converges to $P_n(g) \in C(X)$ by continuity of $P_n$. But since $P_n P_{n_k}(f) = P_{n_k}(f)$ for $k$ sufficiently large, the sequence converges also to $g$, hence $g = P_n(g)$ and belongs to $\pi^*_n C(X_n)$. Thus $g$ belongs to $C(X)^R$. It remains to show that $f - g$ belongs to $\text{Ker}\theta_0$. According to Corollary 3.8, it suffices to check that $\|E_n(f - g)\|$ tends to 0. But this is clear, since for $n \geq n_k$, we have the inequality

$$\|E_n(f - g)\| \leq \|P_{n_k}(f) - g\|.$$  

Let us specialize the above proposition to the irreducible case, i.e. when $X_\infty$ is reduced to one point. It will give a necessary and sufficient condition for the uniqueness of the solution of the Radon-Nikodym problem.

The following corollary of Proposition 3.8, which is well known (see for example [4] or Proposition 1 of [20]), will be our basic tool to show unique ergodicity.

**Corollary 3.12.** Let $(X, R = \cup R_n)$ and $D \in Z^1(R, R^*_+)$ be as above. The following assertions are equivalent.

(i) $D$ is uniquely ergodic, i.e. $SD(X)$ contains one element.

(ii) $(X, R)$ is irreducible and $D$ is equicontinuous (i.e. for all $f \in C(X)$, $(\pi^*_n E_n(f))$ is equicontinuous).

(iii) For all $f \in C(X)$, $(\pi^*_n E_n(f))$ converges uniformly to a constant function.

(iv) For all $f \in C(X)$, the variation $\text{var}_+(E_n(f))$ of $E_n(f)$ (i.e. the difference between its maximum and its minimum) tends to 0.

**Proof.** We first observe that (i) or (ii) imply that $C(X)^R = \mathbb{R}1$. For (i), this is a consequence of Lemma 3.10. For (ii), this comes from the fact that $\pi^*_n E_n$ acts as the identity on $C(X)^R$. Then, one can see that these first three conditions are a reformulation of the conditions of Proposition 3.11 under the assumption that $R$ is irreducible. The equivalence of (i) and (iv) is a particular case of Corollary A.3.
4 Quasi-product cocycles on AF relations.

Let $(V, E)$ be a Bratteli diagram. We denote by $X = X(V, E)$ its infinite path space, by $X_n$ the space of infinite paths starting at level $n$ and by $R = R(V, E) = \cup R_n$ the tail equivalence relation on $X$. A function $\Phi : E \to \mathbb{R}^*_+$ defines a cocycle $D \in Z^1(R, \mathbb{R}^*_+)$ according to the formula

$$D(x, y) = \lim_{n \to \infty} \frac{\Phi(x_1)\Phi(x_2)\ldots\Phi(x_n)}{\Phi(y_1)\Phi(y_2)\ldots\Phi(y_n)}.$$

(Note that the sequence is stationary.)

**Definition 4.1.** Given a Bratteli diagram $(V, E)$, a cocycle $D$ on the tail equivalence relation $R(V, E)$ of its infinite path space $X(V, E)$ is called a quasi-product cocycle if it is of the above form.

**Remark 4.1.** It is shown in [34] that every continuous cocycle on an AP equivalence relation on a compact totally disconnected space is cohomologous to a quasi-product cocycle relative to some Bratteli diagram. However, this result gives little information about the Bratteli diagram, in particular about its growth. We shall characterize later the cocycles which are cohomologous to a quasi-product cocycle relative to some contraction of a given Bratteli diagram.

We assume from now on that the set of vertices $V(n)$ of each level $n$ is finite (and as before, that every vertex emits finitely many, but at least one, edges and that every vertex of a level $n \geq 1$ receives at least one edge). Then the space of infinite paths $X$ of the diagram is compact. Let $D$ be a quasi-product cocycle given by $\Phi : E \to \mathbb{R}^*_+$. Let us apply to $D$ the general approximation procedure. The normalized potential of $D_n = D|_{R_n}$ is the function $\rho_n \in C(X)$ given by

$$\rho_n(x) = \Phi(x_1)\Phi(x_2)\ldots\Phi(x_n)/Z_n(r(x_n)),$$

where we have introduced the normalization factor

$$Z_n(v) = \sum \Phi(y_1)\Phi(y_2)\ldots\Phi(y_n)$$

where $v \in V(n)$ and the sum is over all the finite paths $y_1y_2\ldots y_n$ of length $n$ ending at $v$. The sequence of normalized local potentials of $D$ is given by

$$\rho_{n, n-1}(y_nx_{n+1}\ldots) = Z_n \circ r(y_n)^{-1}\Phi(y_n)Z_{n-1} \circ s(y_n).$$

Note that $\rho_{n, n-1}$ depends only on $y_n$. Let us point out this elementary property in the following proposition.

**Proposition 4.1.** Let $(X, R)$ be the AF equivalence relation defined by the Bratteli diagram $(V, E)$ and let $D \in Z^1(R, \mathbb{R}^*_+)$. Then, the following conditions are equivalent:

(i) $D$ is a quasi-product cocycle relative to $(V, E)$. 


(ii) For all $n$, its normalized local potential $\rho_{n,n-1}$ depends only on the first variable $x_n$. 

(iii) For all $n$, its normalized potential $\rho_n$ depends only on the first $n$ variables: $x_1 x_2 \ldots x_n$. 

(iv) $D$ admits an initial potential $\rho_0 = 1$ and a sequence of local potentials $(\rho_{n,n-1})$ such that $\rho_{n,n-1}(x_n \ldots)$ depends only on $x_n \in E(n)$. 

Proof. The equivalence of (i) and (iv) is clear. It is also clear that (iv) implies (ii), and that (ii) implies (iii), because 

$$\rho_n(x) = \rho_{1,0}(x) \ldots \rho_{n,n-1} \circ \pi_n(x).$$ 

Finally, (iii) implies (iv) because $\rho_{n,n-1} \circ \pi_n(x) = \rho_n(x)/\rho_{n-1}(x)$. 

The conditional expectation $E_n : C(X) \rightarrow C(X_n)$ is given by 

$$E_n(f)(x_{n+1} \ldots) = \sum \rho_n(y_1 \ldots y_n, x_{n+1} \ldots f(y_1 \ldots y_n, x_{n+1} \ldots)$$ 

where the sum is over all the finite paths $y_1 y_2 \ldots y_n$ of length $n$ ending at $s(x_{n+1})$. The conditional expectation $E_{n,n-1} : C(X_{n-1}) \rightarrow C(X_n)$ is given by 

$$E_{n,n-1}(f)(x_{n+1} \ldots) = \sum \rho_{n,n-1}(y_n, x_{n+1} \ldots f(y_n, x_{n+1} \ldots)$$ 

where the sum is over all the $y_n \in E(n)$ ending at $s(x_{n+1})$. Thus we have as usual a sequence $E = (E_{n,n-1} : C(X_{n-1}) \rightarrow C(X_n))$ of Markovian operators. On the other hand, $\Phi$ defines a sequence of matrices $A = (A_n : C(V(n-1)) \rightarrow C(V(n)))$, where the coefficients of $A_n$ are $A_n(w, v) = \sum \Phi(e)$, the sum being over all the $e \in E(n)$ starting at $v \in V(n-1)$ and ending at $w \in V(n)$. The state space $S(A)$ of this sequence is defined in the Appendix. It is the state space of the inductive limit dimension group. By definition, it consists of sequences $(\rho_n)$ of positive numbers such that 

$$\rho_{n-1}(v) = \sum_{s(e)=v} \Phi(e) \rho_n \circ r(e) \quad (n = 1, 2, \ldots v \in V(n-1)) \quad (1)$$

$$1 = \sum_{V(0)} \rho_0(v). \quad (2)$$

Such a sequence $(\rho_n)$ defines a Markov measure $\mu$ such that 

$$\mu(Z(x_1 \ldots x_n)) = \rho_0(s(x_1))p_1(x_1) \ldots p_n(x_n)$$

where $p_n(e) = (\rho_{n-1} \circ s(e))^{-1} \Phi_n(e) \rho_n \circ r(e)$ for $n = 1, 2, \ldots, e \in E(n)$. 

Proposition 4.2. ([34], Proposition 3.3) Let $D$ be a quasi-product cocycle. The above construction defines an isomorphism from the state space $S(A)$ of the sequence of matrices $A = (A_n : C(V(n-1)) \rightarrow C(V(n)))$ to the space $S_D(X)$ of solutions of the $(RN)$ equation $D_\mu = D$. In other words, the quasi-invariant probability measures admitting $D$ as Radon-Nikodym derivative are exactly the above Markov measures.
Proof. We refer the reader to [34]. It easy to check that the Markov measure \( \mu \) constructed above belongs to \( S_D(X) \). It is shown in [34] that conversely every \( \mu \in S_D(X) \) is of that form. The maps so defined are continuous and preserve convex combinations. \( \square \)

Note that the maps \( J_n : C(V(n)) \to C(X_n) \) defined by \( J_n(f)(x_{n+1}\ldots) = Z_n \circ s(x_{n+1})^{-1} f \circ s(x_{n+1}) \) yield a morphism from \( A \) to \( E \), hence a morphism of their dimension groups \( J : \mathcal{E}(A) \to \mathcal{E}(E) \). This morphism induces the above isomorphism of their state spaces. However \( J \) itself is not necessarily an isomorphism. For example, let the Bratteli diagram \((V, E)\) be a tree (and \( \Phi : E \to R_n^* \) be identically one). Then \( \pi_{n,n-1} : X_{n-1} \to X_n \) is a bijection and \( E_{n,n-1} : C(X_{n-1}) \to C(X_n) \) is the transposed map \( \pi_{n,n-1}^* \). The inductive limit \( E \) can be identified with \( C(X) \). On the other hand the image of \( J \) consists of locally constant functions.

Taking into account this proposition, Corollary A.3 and Corollary A.7 become:

**Corollary 4.3.** Let \( R \) be the tail equivalence relation on the infinite path space \( X \) of a Bratteli diagram \((V, E)\) and let \( D \in Z^1(R, R_n^*) \) be a quasi-product cocycle defined by \( \Phi : E \to R_n^* \). Define the matrix \( A_n(w, v) = \sum \Phi(e) \), the sum being over all the \( e \in E(n) \) starting at \( v \in V(n-1) \) and ending at \( w \in V(n) \) as above.

(i) A necessary and sufficient condition for \( S_D(X) \) to have exactly one element is that, for any fixed \( m \) and \( v \in V(m) \), the variation of the function \( w \in V(n) \mapsto B_n\ldots B_{m+1}(v, w) \) goes to \( 0 \) when \( n \) goes to infinity, where \( u_n(w) = \sum_{v \in V(n)} A_n\ldots A_1(w, v) \) and \( B_n(w, v) = u_n(w)^{-1} A_n(w, v) u_{n-1}(v) \) for \( w \in V(n) \) and \( v \in V(n-1) \).

(ii) A sufficient condition for \( S_D(X) \) to have exactly one element is that the serie \( \sum \epsilon_n \) diverges, where \( \epsilon_n \) is the ratio of the smallest element of \( A_n \) over its largest element.

The most studied quasi-product cocycle is \( D \equiv 1 \). Then, the elements of \( S_D(X) \) are the invariant (with respect to the tail equivalence relation) probability measures on the path space \( X \) of the Bratteli diagram \((V, E)\) (they are called central measures in [36]). They correspond to the tracial states of the AF algebra of the Bratteli diagram. We choose \( \Phi \equiv 1 \). The matrices \( A_n \) are the adjacency matrices of the graph. They have coefficients in \( N \) and the inductive limit of the \( (A_n : C(V(n-1, Z)) \to C(V(n), Z))'s \) is the usual dimension group of the Bratteli diagram. Then Proposition 4.2 gives the well known correspondence between invariant probability measures and states of the dimension group. In that case, parts of Corollary 4.3 appear in the work [35] of A. Török. The necessary and sufficient condition \((i)\) also appears in [36]. Here are a few examples.

**Example 4.1.** An example of Fack and Maréchal. In their work [18] on the symmetries of UHF algebras, T. Fack and O. Maréchal study the dimension.
group defined by the sequence of matrices $A = (A_n)$, where

$$A_n = \begin{pmatrix} p_n & r_n \\ r_n & p_n \end{pmatrix}$$

and $(p_n)$ and $(r_n)$ are two sequences of integers such that $0 < r_n \leq p_n$ for all $n$. Let $\epsilon_n = r_n/p_n$. According to the above, a sufficient condition for $\mathcal{E}(A)$ having a unique state is that $\sum \epsilon_n = \infty$. Fack and Maréchal show by an explicit computation of the dimension group that this condition is necessary. This can also be deduced from Corollary 4.3 (i).

**Example 4.2.** Pascal’s triangle. The simple random walk on $\mathbb{Z}$ provides another example. We let $X = \prod_{1}^{\infty} \{0, 1\}$ be the space of increments. We introduce $X_n = \{0, 1, \ldots, n\} \times \prod_{n+1}^{\infty} \{0, 1\}$ and $\pi_n : X \to X_n$ defined by

$$\pi_n(x_1x_2\ldots) = (x_1 + \ldots + x_n, x_{n+1}x_{n+2}\ldots).$$

Its first coordinate is the position of the walker at time $n$ (assuming that his initial position is 0). We let $R$ be the AP equivalence relation on $X$ defined by the $\pi_n$’s. Note that $(X, R)$ admits the infinite Pascal triangle as Bratteli diagram and that $D \equiv 1$ is the quasi-product cocycle defined by the function $\phi \equiv 1$. It is known that the state space $S_1$ of invariant probability measures is isomorphic to the space of probability measures on $[0,1]$; see for example Section VII.4 of [19], [26] or the Appendix of [33], which contains an explicit computation of the dimension group of the infinite Pascal triangle. The following comment of the proof is inspired by Section 5 of [40]. First, it is immediate to check that for $t \in [0,1]$, the product measure

$$\mu_t = \prod_{1}^{\infty}((1-t)\delta_0 + t\delta_1)$$

is invariant and that $\mu_t(Z(n,k)) = C_n^k(t)(1-t)^{n-k}$, where $Z(n,k)$ is the set of paths having position $k$ at time $n$. An elementary estimate using the expansion of a polynomial of degree not greater than $n$ in the basis of Bernstein polynomials $\{t^k(1-t)^{n-k}, k = 0, 1, \ldots, n\}$ shows that for all $f \in C(X)$,

$$\lim_{n \to \infty} \sup_{(k,x) \in X_n} |E_n(f)(k,x) - \hat{f}(k/n)| = 0,$$

where $E_n(f)(k,x)$ is the average of the $f(a_1 \ldots a_n x)$ over all $a_1 \ldots a_n$ such that $a_1 + \ldots + a_n = k$ and $\hat{f}(t) = \mu_t(f)$. Using Corollary 3.8, one deduces that $\hat{f} = 0$ if and only if $\mu(f) = 0$ for all invariant probability measures $\mu$. Therefore, the measures $\mu_t, t \in [0,1]$ are exactly the extremal elements of $S_1$. Since the equivalence relation $R$ is irreducible (it has dense orbits), $C(X)^R = C1$ and the condition (ii) of Proposition 3.11 is not realized. One can also see that for $x \in X$, $\pi_n^*E_n(f)(x)$ converges to $\hat{f}(t)$ for all $f \in C(X)$ if and only if the path $x = x_1x_2\ldots$ has the property that $(x_1 + \ldots + x_n)/n$ tends to $t$. The strong law of large numbers (or also the martingale convergence theorem) says that, with respect to the measure $\mu_t$, this set is conull.
Example 4.3. Stationary cocycles. The Bratteli diagram \((V, E)\) is called stationary if for all \(n \geq 1\), \(V(n) = V(0)\) and \(E(n) = E(1)\). In that case, the one-sided shift \(T(x_1x_2\ldots) = x_2x_3\ldots\) acts on the infinite path space \(X\) of the diagram. Then, \(\Phi : E \to \mathbb{R}^*_+\) is called stationary if it does not depend on the level \(n\). The associated quasi-product cocycle is also called stationary. The sufficient condition \((ii)\) of Corollary 4.3 is always satisfied for a stationary quasi-product cocycle \(D\). Therefore \(S_D(X)\) is reduced to one element.

5 Beyond quasi-product cocycles.

As mentioned previously, every continuous cocycle \(D \in Z^1(R, \mathbb{R}^*_+)\) of an AF equivalence relation \(R\) on a compact space \(X\) is cohomologous to a quasi-product cocycle with respect to some Bratteli diagram. However, in general we do not have sufficient information on the Bratteli diagram in order to apply Corollary 4.3. When the cocycle \(D\) satisfies an appropriate condition of equicontinuity relative to a given Bratteli diagram, it is approximately a quasi-product cocycle with respect to the same Bratteli diagram (or a contraction of it); then one can use the technique of Corollary 4.3 to obtain unique ergodicity. Moreover, the idea of approximating a cocycle by a quasi-product cocycle is also fruitful for arbitrary AP equivalence relations. The definitions below are adapted from [34].

**Definition 5.1.** Let \(\pi : X \to Y\) be a surjective local homeomorphism. We say that an open subset \(V\) of \(Y\) is well-covered if \(\pi^{-1}(V)\) is the disjoint union of a family of open sets \(\{U_i, i \in I\}\) which all map homeomorphically onto \(V\). By definition, a \(\pi\)-cover will consist of a cover \(\mathcal{V}\) of \(Y\) by well-covered open subsets and for each \(V \in \mathcal{V}\) a partition of \(\pi^{-1}(V)\) by open subsets \(U\) of \(X\) which all map homeomorphically onto \(V\). We denote by \(\mathcal{U}\) the cover of \(X\) by these open sets \(U\).

When \(X\) is compact, \(\pi\) admits finite \(\pi\)-covers. Moreover, if an open cover \(\mathcal{W}\) of \(X\) is given, we can construct our \(\pi\)-cover such that \(\mathcal{U}\) strictly refines \(\mathcal{W}\) (notation: \(\mathcal{U} \prec \mathcal{W}\)), in the sense that for each \(U \in \mathcal{U}\), there exists \(W \in \mathcal{W}\) such that \(\overline{U} \subset W\).

In the sequel, we shall only consider finite \(\pi\)-covers. Recall from [2] that a finite open cover \(\mathcal{U}\) of a compact space \(X\) defines the entourage
\[
\Delta_{\mathcal{U}} = \cup_{\mathcal{U}} U \times U
\]
of the canonical uniform structure of \(X\).

Given \(\Delta \subset X \times X\) and \(\varphi \in C(X, R)\), we define the additive variation of \(\varphi\) over \(\Delta\) as
\[
var_+(\varphi, \Delta) = \sup_{\Delta} |\varphi(x) - \varphi(x')|.
\]
Similarly, given \(g \in C(X, \mathbb{R}^*_+)\), we define the multiplicative variation of \(g\) over \(\Delta\) as
\[
var_*(g, \Delta) = \sup_{\Delta} \left| \frac{g(x)}{g(x')} - 1 \right|.
\]
Lemma 5.1. Let \( \pi : X \to Y \) be a surjective local homeomorphism. Let \((V, U)\) be a \(\pi\)-cover with associated entourages \(\delta = \Delta_V, \Delta = \Delta_U\). Consider a potential \(g \in C(X, R^+_\pi)\) and its normalized potential \(\rho = (Z \circ \pi)^{-1} g\), where \(Z(y) = \sum_{\pi^{-1}(y)} g(x)\) for \(y \in Y\). If \(\var_* (g, \Delta) < \epsilon < 1\), then \(\var_* (\rho, \Delta) < 2\epsilon (1 - \epsilon)^{-1}\).

Proof. For all \((x, x') \in \Delta\), we have

\[
(1 - \epsilon)g(x') < \rho(x) < (1 + \epsilon)g(x').
\]

Let \((y, y') \in \delta\). We have by construction a bijection \(x \in \pi^{-1}(y) \mapsto x' \in \pi^{-1}(y')\) such that \((x, x') \in \Delta\). Summing above inequalities above \(\pi^{-1}(y)\), we obtain

\[
(1 - \epsilon)Z(y') < Z(y) < (1 + \epsilon)Z(y').
\]

Therefore, for all \((x, x') \in \Delta\), we have

\[
\frac{1 - \epsilon}{1 + \epsilon} \rho(x') < \rho(x) < \frac{1 + \epsilon}{1 - \epsilon} \rho(x')
\]

and the above inequality. \(\square\)

We keep the same notation as above. We introduce the positive linear map \(E : C(X) \to C(Y)\) such that

\[
E(f)(y) = \sum_{\pi^{-1}(y)} \rho(x) f(x).
\]

Lemma 5.2. (cf. [37], Lemma 1) Let \( \pi : X \to Y \) be a surjective local homeomorphism. Let \((V, U)\) be a \(\pi\)-cover with associated entourages \(\delta, \Delta\). Consider a normalized potential \(\rho \in C(X, R^+_\pi)\) and its expectation \(E : C(X) \to C(Y)\).

Then, for every \(f \in C(X)\),

\[
\var_+ (E(f), \delta) \leq \var_+ (f, \Delta) + \|f\| \var_* (\rho, \Delta).
\]

Proof. Let \((y, y') \in \delta\). By construction, we have a bijection \(x \in \pi^{-1}(y) \mapsto x' \in \pi^{-1}(y')\) such that \((x, x') \in \Delta\). Then,

\[
E(f)(y) = \sum_{\pi^{-1}(y)} \rho(x) f(x) = \sum_{\pi^{-1}(y)} \rho(x) f(x) - f(x') + [\rho(x) - \rho(x')] f(x') = \sum_{\pi^{-1}(y)} \rho(x) [f(x) - f(x')] + [\rho(x) - \rho(x')] f(x')
\]

and

\[
\sum_{\pi^{-1}(y)} \rho(x) [f(x) - f(x')] = \max_{\pi(x) = y} \|f\| \max_{\pi(x) = y} |\rho(x)| - 1 \leq \var_+ (f, \Delta) + \|f\| \var_* (\rho, \Delta).
\]

\(\square\)

Let us consider now an AP equivalence relation \(R\) on a compact space \(X\) defined by a sequence

\[
X_0 \xrightarrow{\pi_{1,0}} X_1 \to \ldots \to X_{n-1} \xrightarrow{\pi_{n,n-1}} X_n \to \ldots
\]

where \(X_0 = X\) and for each \(n \geq 1\), \(X_n\) is a Hausdorff space and \(\pi_{n,n-1}\) is a surjective local homeomorphism. We can construct inductively \(\pi_{n,n-1}\)-covers \((V^n, U^n)\) such that \(U^n \prec V^{n-1}\).
**Definition 5.2.** Such a sequence \((V^n, \mathcal{U}^n)\) of \(\pi_{n,n-1}\)-covers will be called a tower relative to the sequence \((X_n)\).

Given \(m < n\) and a sequence \(a = (U_{m+1}, U_{m+2}, \ldots, U_n)\), where \(U_k \in \mathcal{U}^k\), such that \(\overline{U}_{k+1} \subseteq \pi_{k,k-1}(U_k)\) for \(k = m + 1, \ldots, n - 1\), we define the following subset of \(X_m\):

\[
U_n = U_{m+1} \cap \pi_{m+1,m}^{-1}(U_{m+2}) \cap \ldots \cap \pi_{n-1,m}^{-1}(U_n).
\]

To avoid redundancies, we implicitly choose once for all, for \(n = 1, 2, \ldots\) and \(U \in \mathcal{U}_n\) a set \(V = V(U) \in \mathcal{V}_{n-1}\) such that \(\overline{U} \subseteq V\). We shall only consider sequences \(a = (U_{m+1}, U_{m+2}, \ldots, U_n)\) with \(U_k \in \mathcal{U}^k\) such that \(\pi_{k,k-1}(U_k) = V(U_{k+1})\) for \(m < k < n\).

**Lemma 5.3.** Let \((V^n, \mathcal{U}^n)\) be a tower. Given \(m < n\), we define \(\mathcal{U}^{m,n}\) as the family of open sets \(U_a\), where

\[
a = (U_{m+1}, \ldots, U_n), \quad U_k \in \mathcal{U}^k, \quad \pi_{k,k-1}(U_k) = V(U_{k+1}), \quad m < k < n.
\]

Then \((V^n, \mathcal{U}^{m,n})\) is a \(\pi_{n,m}\)-cover.

**Proof.** Let \(V \in \mathcal{V}^n\). Let us show that \(\pi_{n,m}^{-1}(V)\) is the disjoint union of the family of \(U_a\)'s, where \(a = (U_{m+1}, U_{m+2}, \ldots, U_n)\) where \(U_k \in \mathcal{U}^k\), \(\pi_{k,k-1}(U_k) = V(U_{k+1})\) for \(m < k < n\) and \(V = \pi_{n,n-1}(U_n)\) and that \(\pi_{n,m}\) maps each \(U_a\) homeomorphically onto \(V\). Let \(x_m \in \pi_{n,m}^{-1}(V)\). We define \(x_k = \pi_{k,m}(x_m)\) for \(m < k \leq n\). Since \(x_m \in V\), there is a unique \(U_n \in \mathcal{U}_n\) mapping homeomorphically onto \(V\) and containing \(x_{n-1}\). We define \(V_{n-1} = V(U_n) \in \mathcal{V}_{n-1}\) and proceed by induction to construct \(V_{n-1} \subseteq \cdots \subseteq U_{m+1} \subseteq U_m\). Then \(x_m \in U_a\) where \(a = (U_{m+1}, U_{m+2}, \ldots, U_n)\). If \(a = (U_{m+1}, U_{m+2}, \ldots, U_n)\) and \(a' = (U'_{m+1}, U'_{m+2}, \ldots, U'_n)\) are distinct, there exists a larger \(k \leq n\) such that \(U_k \neq U'_k\). Then, \(U_k\) and \(U'_k\) are disjoint and so are \(U_a\) and \(U_{a'}\). The restriction of \(\pi_{n,m}\) to \(U_a\) is a composition of homeomorphisms and therefore maps \(U_a\) homeomorphically onto \(V\). \(\square\)

Thus, if we contract the initial sequence of spaces by means of a subsequence \((n_k)\), our initial tower \((V^n, \mathcal{U}^n)\) provides a tower \((V^{n_k}, \mathcal{U}^{n_k})\) relative to the sequence \((X_{n_k})\): we set \(V^{n_k} = \mathcal{V}^{n_k}\) and \(\mathcal{U}^{n_k} = \mathcal{U}^{n_k,m_k}\).

In particular, we will denote \(U_n = \mathcal{U}^{0,n}\). Note that \((V^n, U_n)\) is a \(\pi_n\)-cover and that \(U_n \prec U_{n-1}\).

Given a tower \((V^n, \mathcal{U}^n)\) as above for the defining sequence \((X_n)\), we define the following entourages in \(X \times X\):

\[
\Delta_n = \Delta_{U_n} = \cup \{ U \times U, \quad U \in \mathcal{U}_n \}
\]

and in \(X_n \times X_n\):

\[
\delta_n = \Delta_{V^n} = \cup \{ V \times V, \quad V \in \mathcal{V}^n \}.
\]

Note that \(\Delta_n\) is a sequence of neighborhoods of the diagonal \(\Delta_X\) of \(X \times X\) such that \(\Delta_n \subseteq \Delta_n \subseteq \Delta_{n-1}\).
**Definition 5.3.** We say that the tower \((V^n, \mathcal{U}^n)\) for the defining sequence \((X_n)\) is a **generator** if \(\cap \Delta_n\) is reduced to the diagonal \(\Delta_X\) of \(X \times X\). In other words, a tower \((V^n, \mathcal{U}^n)\) is a generator iff \((\Delta_n)\) is a fundamental system of neighborhoods of \(\Delta_X\).

**Example 5.1.** AF equivalence relation defined by a Bratteli diagram. Let \((V, E)\) be a Bratteli diagram, let \(X\) be its infinite path space and \(X_n\) the space of paths starting at level \(n\). The Bratteli diagram defines a tower \((V^n, \mathcal{U}^n)\), where \(V^n\) is the space of paths starting at level \(n\) and \(\mathcal{U}^n\) is the partition of \(X_n\) by the cylinder sets

\[Z^n(v) = \{x_{n+1}x_{n+2} \ldots \in X_n : s(x_{n+1}) = v\},\]

where \(v \in V(n)\) and \(\mathcal{U}^n\) is the partition of \(X_{n-1}\) by the cylinder sets

\[Z^{n-1}(e) = \{x_nx_{n+1} \ldots \in X_{n-1} : x_n = e\},\]

where \(e \in E(n)\). Then \(\mathcal{U}_n\) is the partition of \(X\) by the cylinder sets \(Z(a)\), where \(a = a_1 \ldots a_n\) is a path from level 0 to level \(n\). We have, for \(x, y \in X\),

\[(x, y) \in \Delta_n \iff x_1 = y_1, \ldots, x_n = y_n\]

and for \(x, y \in X_n\),

\[(x, y) \in \delta_n \iff s(x) = s(y),\]

i.e. \(x\) and \(y\) start from the same vertex.

Let \(D \in Z^1(R, R_+^*)\) and let \((\rho_n)\) be the sequence of its normalized potentials. We have

\[\text{var}_*(\rho_n, \Delta_n) = \sup \left| \frac{\rho_n(ax)}{\rho_n(ay)} - 1 \right|\]

where the supremum is taken over all paths \(a\) from level 0 to level \(n\) and all infinite paths \(x, y\) starting at \(r(a)\). Note that if \(D\) is a quasi-product cocycle relative to \((V, E)\), then for all \(n\) and for all paths \(a\) from level 0 to level \(n\), we have \(\frac{\rho_n(ax)}{\rho_n(ay)} = 1\) and therefore \(\text{var}_*(\rho_n, \Delta_n) = 0\). The estimates of this section can be used to extend the results of Corollary 4.3 concerning quasi-product cocycles to cocycles for which there is a good control of \(\text{var}_*(\rho_n, \Delta_n)\).

Let us return to the general case of an AP equivalence relation \(\mathcal{R}\) with a given tower \((V^n, \mathcal{U}^n)\) and let us consider the cocycles \(D\) on \(\mathcal{R}\) which satisfy \(\lim \text{var}_*(\rho_n, \Delta_n) = 0\), where the \(\Delta_n = \Delta_{\mathcal{U}^n}\) and where \((\rho_n)\) is a sequence of potentials for \(D\). It results from Lemma 5.1 that if this condition is satisfied by a sequence of (unnormalized) potentials \((g_n)\), it is also satisfied by the sequence of normalized potentials \((\rho_n)\). Let us also observe that, when the tower \((V^n, \mathcal{U}^n)\) is a generator, this condition is invariant under cohomology. Indeed, if

\[D(x, y) = b(x)D(x, y)b(y)^{-1}\]

for some \(b \in C(X, R_+^*)\), it admits the sequence of potentials \((bp_n)\). The condition \(\lim \text{var}_*(\rho_n, \Delta_n) = 0\) and the uniform continuity of \(b\) imply that
lim $\text{var}_s(b\rho_n, \Delta_n) = 0$. Thus a cocycle $D$ on an AF equivalence relation given by a Bratteli diagram $(V, E)$ which is cohomologous to a quasi-product cocycle with respect to the diagram must satisfy $\lim \text{var}_s(\rho_n, \Delta_n) = 0$. One can prove a weak converse: if $\lim \inf \text{var}_s(\rho_n, \Delta_n) = 0$, then $D$ is cohomologous to a quasi-product cocycle relative to a contraction of $(V, E)$. The proof is similar to that of Theorem 3.1 of [34].

6 Stationary systems

We shall now apply the above estimates to stationary systems, in the following sense. We say that the sequence

$$X_0 \xrightarrow{\pi_1, 0} X_1 \rightarrow \ldots \rightarrow X_{n-1} \xrightarrow{\pi_n, n-1} X_n \rightarrow \ldots$$

is stationary if for all $n \in \mathbb{N}$, $X_n = X$ and for all $n \geq 1$, $\pi_{n-1} = T$. As usual, we assume that $X$ is compact and that $T$ is a local homeomorphism and a surjection. The AP equivalence relation defined by this sequence is

$$R = R(X, T) = \{(x, y) \in X \times X : \exists n \in \mathbb{N} : T^n x = T^n y\}.$$ 

Then $R$ is a subgroupoid of the semi-direct product groupoid (see for example [34]) of the dynamical system

$$G(X, T) = \{(x, m - n, y) \in X \times \mathbb{Z} \times X : m, n \in \mathbb{N} : T^m x = T^n y\}$$

We say that a cocycle $D \in Z^1(R, \mathbb{R}_+^\times)$ is stationary if it is the restriction of a cocycle in $Z^1(G(X, T), \mathbb{R}_+^\times)$. Because $G(X, T)$ is singly generated (see 4.1 of [34]), cocycles in $Z^1(G(X, T), \mathbb{R}_+^\times)$ are in a one-to-one correspondence with functions $g \in C(X, \mathbb{R}_+^\times)$. However, the cocycles defined by $g$ and $\lambda g$, where $\lambda \in \mathbb{R}_+^\times$, (or more generally $\lambda \in C(X, \mathbb{R}_+^\times)^R$) will have the same restriction to $R$. More explicitly, a cocycle $D \in Z^1(R, \mathbb{R}_+^\times)$ is stationary if it admits a sequence of potentials $(g_n)$ of the form:

$$g_n(x) = g(x)g(Tx) \ldots g(T^{n-1}x)$$

where $g$ is a given function in $C(X, \mathbb{R}_+^\times)$. Then the normalized potentials are given by

$$\rho_n = \frac{g_n}{Z_n \circ T^n}, \text{ where } Z_n(x) = \sum_{T^ny = x} g_n(y)$$

are the partition functions. The normalized local potentials are given by

$$\rho_{n-1}(x) = (Z_n(Tx))^{-1}g(x)Z_{n-1}(x).$$

We say that a tower $(\mathcal{V}^n, \mathcal{U}^n)$ for the stationary sequence $(X_n = X, \pi_{n-1} = T)$ is stationary if it does not depend on $n$. Thus, a stationary tower is given by a $T$-cover $(V, \mathcal{U})$ such that $\mathcal{U} \prec \mathcal{V}$. As before, we define $\mathcal{U}_n$ as the cover of $X$ by the open sets

$$U_a = U_1 \cap T^{-1}(U_2) \cap \ldots \cap T^{-(n-1)}(U_n),$$
where \( a = (U_1, \ldots, U_n) \), \( U_1, \ldots, U_n \in \mathcal{U} \), \( \overline{U}_{k+1} \subset T(U_k) \). We also define the entourages \( \Delta_n = \Delta_{U_n} \subset X \times X \) and \( \delta_1 = \Delta_{V_1} \subset X \times X \). Note that \( \delta_n \) does not depend on \( n \); we call it \( \delta \). Remember that the tower is called a generator if \( \cap \Delta_n \) is reduced to the diagonal \( \Delta_X \) of \( X \times X \). We then say that the corresponding \( T \)-cover \((V, U)\) is a generator. In the classical terminology of [38], the cover \( U \) itself is called a generator if \( \cap \Delta_n \) is reduced to the diagonal \( \Delta_X \).

**Definition 6.1.** Let \( T : X \to X \) be a continuous map on a compact space. One says that \( g \in C(X, R^*_+) \) satisfies Walters’ condition if for all \( \epsilon > 0 \), there exists an entourage \( \Delta \) of the uniform structure of \( X \) such that the sequence

\[
\Delta_n = \{(x, y) \in X \times X : \forall k = 1, \ldots, n-1, (T^k x, T^k y) \in \Delta\}
\]

decreases to the diagonal and for all \( n \), \( \text{var}_*(g_n, \Delta_n) \leq \epsilon \).

An equivalent definition is that, given a generator \((V, U)\), for all \( \epsilon \), there exists an integer \( N \) such that for all \( n \), \( \text{var}_*(g_n, \Delta_{U_n} + n) \leq \epsilon \).

We can make a similar definition for an arbitrary cocycle on an “almost stationary” AP equivalence relation \((X, R)\).

**Definition 6.2.** Let \( R \) be an AP equivalence relation on a compact space \( X \) with a defining sequence \((X_n, \pi_n)\) where \( X_n = X \) for all \( n \). We shall say that \( D \in Z^1(R, R^*_+) \) satisfies Walters’ condition if for all \( \epsilon > 0 \), there exists a generator \((V_n, U_n)\) with \( V_n = V_1 \) for all \( n \geq 1 \) and a sequence of potentials \((g_n)\) for \( D \) such that for all \( n \), \( \text{var}_*(g_n, \Delta_{U_n}) \leq \epsilon \).

**Theorem 6.1.** Let \( R \) be an AP equivalence relation on a compact space \( X \) with a defining sequence \((X_n, \pi_n)\) where \( X_n = X \) for all \( n \) and let \( D \in Z^1(R, R^*_+) \). Assume that:

(i) the AP equivalence relation \( R \) is minimal;

(ii) \( D \) satisfies Walters’ condition.

Then, \( D \) is uniquely ergodic.

**Proof.** The proof is essentially the same as in Theorem 6 of [37] (see also [41]). We introduce the normalized potentials \((\rho_n)\) and their expectations \((E_n)\). According to Lemma 5.1, they also satisfy Walters’ condition. Let \( \epsilon > 0 \) be given; there exists a generator \((V_n, U_n)\) such that for all \( n \geq 1 \), \( V_n = V_1 \) and \( \text{var}_*(\rho_n, \Delta_n) \leq \epsilon \). We apply Lemma 6.1 with \( \pi = \pi_n \) and \((V_1, U_n)\): given \( f \in C(X) \), we have

\[
\text{var}_+(E_n(f), \delta) \leq \text{var}_+(f, \Delta_n) + \|f\| \text{var}_*(\rho_n, \Delta_n),
\]

where \( \delta = \Delta_{V_1} \) is an entourage of the uniform structure of \( X \). This show the equicontinuity of the sequence \((E_n(f))\) in \( C(X) \). Since this sequence is also
bounded, there exists a sequence \((n_k)\) tending to infinity and \(f^* \in C(X)\) such that \(E_{n_k}(f)\) converges uniformly to \(f^*\). Then, for any \(m\), \(E_{m+n_k}(f)\) converges uniformly to \(E_m(f^*)\). Since the sequence \((E_n(f)_{max})\) is decreasing, it converges to \(f_{max}^* = E_m(f^*)_{max}\). Let \(S_m\) be the set of points where \(E_m(f^*)\) takes its maximum. Note that, because of the strict positivity of \(\rho_{m+k,m}\), \(\pi_{m+k,m}(S_{m+k}) \subset S_m\). Therefore \((\pi_{m}^{-1}(S_{m}))\) is a decreasing sequence of non-empty closed sets and has a non-empty closed intersection \(S\). Moreover \(S\) is invariant under \(R\). Because of (i), \(S = X\) and \(f^*\) is a constant function. This implies that \(\text{var}_+(E_n(f))\) tends to zero. Since the sequence \((\var(E_n(f))\) is decreasing, it also tends to zero. Then Corollary 3.12 gives the unique ergodicity. \(\square\)

Remark 6.1. Under the assumptions of the theorem, let \(\mu\) be the unique quasi-invariant measure on \(X\) admitting \(D\) as Radon-Nikodym derivative. Then, for \(f \in C(X)\), \((E_n(f))\) converges uniformly to \(\mu(f)1_X\). In particular, \((E_n(f)(x))\) converges to \(\mu(f)\) for all \(x \in X\).

In the case of a stationary cocycle, one retrieves the well-known uniqueness result:

**Corollary 6.2.** (Theorem 6,[37]) Let \(X\) be a compact space, let \(T : X \to X\) be a surjective local homeomorphism and let \(g \in C(X, R^*_+)\). Assume that:

(i) the AP equivalence relation \(R(X,T)\) is minimal;

(ii) there exists an integer \(L \geq 1\) and a \(T^L\)-cover \((\mathcal{V}, \mathcal{U})\) which is a generator in the above sense for the sequence defined by \(T^L\);

(iii) there exists an integer \(M \geq 1\) such that \(g_M\) satisfies Walters’ condition with respect to \(T^M\).

Then, the stationary cocycle \(D\) defined by \(g\) on \(R(X,T)\) is uniquely ergodic.

**Proof.** Let \(N = LM\). If \((\mathcal{V}, \mathcal{U})\) is a generator for \(T^L\), then \((\mathcal{V}, \mathcal{U}_M)\) is a generator for \(T^N\). If \(g_M\) satisfies Walters’ condition with respect to \(T^M\), \(g_N\) satisfies Walters’ condition with respect to \(T^N\). Moreover \(R(X,T^N) = R(X,T)\). Therefore, replacing \(T\) by \(T^N\), we may assume that \(L = M = 1\). Then, the AP equivalence relation \(R(X,T)\) defined by \((X,T)\) and the stationary cocycle \(D \in Z^1(R, R^*_+)\) satisfy the assumptions of the theorem. \(\square\)

**Example 6.1.** Subshifts of finite type. Let \(\Gamma = (\Gamma^{(0)}, \Gamma^{(1)})\) be a finite graph. We assume that each vertex receives and emits at least one edge. We let \(X\) be the space of one-sided infinite paths \(x = x_1x_2\ldots\) and \(T : X \to X\) be the one-sided shift \(T(x_1x_2\ldots) = x_2\ldots\). For \(v \in \Gamma^{(0)}\), we let \(Z(v)\) be the set of paths starting at \(v\) and for \(e \in \Gamma^{(1)}\), we let \(Z(e)\) be the set of paths having \(e\) as initial edge. We define \(\mathcal{V} = \{Z(v), v \in \Gamma^{(0)}\}\) and \(\mathcal{U} = \{Z(e), e \in \Gamma^{(1)}\}\). Then \((\mathcal{V}, \mathcal{U})\) is a generator for the sequence defined by \(T\). This is a particular case of Example 5.1 where \((\mathcal{V}, E)\) is the stationary Bratteli diagram defined by \(\Gamma\). It is known that \(R(X,T)\) is minimal if and only if the graph \(\Gamma\) is primitive (i.e.
there exists an integer $L \geq 1$ such that every pair of vertices $(v, w)$ can be joined by a path of length $L$. This is also equivalent to $T$ being topologically mixing. Walters’ condition for $g \in C(X, \mathbb{R}_+^*)$ reads here:

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : x_1 \ldots x_{n+N} = y_1 \ldots y_{n+N} \Rightarrow \left| \frac{g_n(x)}{g_n(y)} - 1 \right| \leq \epsilon.$$  

This is also a particular case of the next example.

**Example 6.2.** *Expansive maps.* Let $X$ be a compact metric space and let $T : X \to X$ be surjective and a local homeomorphism. One says that $T$ is (positively) expansive if there exists $\epsilon > 0$ such that for every pair of points of $X$, $x \neq y$, there is $n \in \mathbb{N}$ such that $d(T^nx, T^ny) \geq \epsilon$.

**Lemma 6.3.** Let $T : X \to X$ be a surjective and expansive local homeomorphism. Then, there exist an integer $L \in \mathbb{N}$ and a $T^L$-cover $(\mathcal{V}, \mathcal{U})$ which is a generator for $T^L$.

**Proof.** Replacing the metric by a topologically equivalent metric, we may assume that $T$ is locally expanding: there exists $\tau > 0$ and $\lambda < 1$ such that $d(x, y) < \tau \Rightarrow d(Tx, Ty) \leq \lambda d(x, y)$. Let $(\mathcal{V}, \mathcal{U})$ be a finite $T$-cover such that the elements of $\mathcal{U}$ have diameter strictly less than $\tau$. Let $c$ be an upper bound for the diameter of the elements of $\mathcal{V}$. For each $L \in \mathbb{N}$, $(\mathcal{V}, \mathcal{U}_L)$ is a $T^L$-cover. For $L$ sufficiently large, the diameter of each element of $\mathcal{U}_L$, which is majorized by $\lambda^L c$, will be strictly less than the Lebesgue number of the cover $\mathcal{V}$ and we will have $\mathcal{U}_L \prec \mathcal{V}$. Let $\Delta_n = \Delta_{\mathcal{U}_n}$. Since $d(\Delta_n) \leq \lambda^n c$ tends to 0, $(\mathcal{V}, \mathcal{U}_L)$ is a generator.

One says that $T : X \to X$ is **exact** if for any non-empty open set $U$, there is an integer $n > 0$ such that $T^n(U) = X$. This condition is equivalent to the minimality of $R(X, T)$ (for equivalent conditions, see [30]).

Let $g \in C(X, \mathbb{R}_+^*)$ and let $(g_n)$ be the corresponding sequence of potentials. We fix a compatible metric $d$ on $X$ and define

$$d_n(x, y) = \max\{d(Tx, Ty), \ldots, d(T^{n-1}x, T^{n-1}y)\}.$$  

The usual Walters condition for $g$ is that for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $n$, $\text{var}_n(g_n, \Delta_n) \leq \epsilon$, where

$$\Delta_n = \{(x, y) \in X \times X : d_n(x, y) < \delta\}.$$  

(This condition is often expressed in terms of $\varphi = \log g$.) Since the sets

$$\Delta = \{(x, y) \in X \times X : d(x, y) < \delta\}$$  

form a fundamental system of entourages of the uniform structure of $X$, this condition is equivalent to ours. Thus one retrieves the well-known result that if $T$ is expansive and exact and if for some integer $M \geq 1$, $g_M$ satisfies Walters’ condition with respect to $T^{3M}$, then the stationary cocycle $D$ defined by $g$ on $R(X, T)$ is uniquely ergodic.
The above theorem establishes the unique ergodicity of quasi-product cocycles on stationary Bratteli diagrams:

**Corollary 6.4.** Let \((V, E)\) be a stationary Bratteli diagram defined by a finite primitive graph. Let \(X = X(V, E)\) and \(R = R(V, E)\). Then every quasi-product cocycle \(D \in Z^1(X, R)\) is uniquely ergodic.

**Proof.** As we have seen, \(R(V, E)\) is minimal. The condition \(\lim_{n \to \infty} \text{var}_r(\rho_n, \Delta_n) = 0\), where we use the notation of Example 5.1, is trivially satisfied for a quasi-product cocycle and implies Walters’ condition for \(D\).

**Remark 6.2.** This result shows that the sufficient condition \((ii)\) of Corollary 4.3 is not necessary.

## 7 Application to the transfer operators

We show in this section how our above results on the unique ergodicity of cocycles on AP equivalence relations imply parts of the Perron-Frobenius-Ruelle theorem (as stated for example in [37]) namely the existence and the uniqueness of the Perron eigenvalue and eigenvector of the Ruelle operator

\[ L^*_g : C(X)^* \to C(X)^*. \]

The setting is a dynamical system \((X, T)\), where \(X\) is a compact space and \(T\) is a surjective local homeomorphism of \(X\) onto itself.

An arbitrary continuous function \(g \in C(X, R^*_+)\) defines both a cocycle \(D_g\) in \(Z^1(G, R^*_+)\), where the definition of \(G = G(X, T)\) has been recalled earlier, and a **Ruelle** (or **transfer**) operator \(L_g : C(X) \to C(X)\) according to

\[ L_g(f)(x) = \sum_{T y = x} g(y) f(y). \]

The relation between the Ruelle operator \(L_g\) and the cocycle \(D_g\) is given by the following elementary result.

**Proposition 7.1.** (Proposition 4.2 of [34]) Let \(\mu\) be a probability measure on \(X\). The following conditions are equivalent:

1. \(\mu\) is quasi-invariant with respect to \(G(X, T)\) and admits \(D_g\) as Radon-Nikodym derivative;
2. \(L_g^* \mu = \mu\).

The quasi-invariance with respect to \(G(X, T)\) is defined in the same way that the quasi-invariance with respect to \(R(X, T)\): it means that the measures \(r^* \mu\) and \(s^* \mu\) are equivalent, where \(r, s : G(X, T) \to X\) are the projections. Since \(R(X, T) \subset G(X, T)\), the quasi-invariance with respect to \(G(X, T)\) implies the quasi-invariance with respect to \(R(X, T)\) and the restriction of a Radon-Nikodym derivative relative to \(G(X, T)\) is a Radon-Nikodym derivative relative to \(R(X, T)\).
Proposition 7.2. Let \((X, T)\) be a dynamical system as above and let \(g \in C(X, \mathbb{R}_+^*).\) Let \(G(X, T), R(X, T), D_g\) and \(L_g\) be as above. Assume that the restriction \(D\) of \(D_g\) to \(R(X, T)\) is uniquely ergodic. Then the eigenvalue problem
\[ L_g^* \mu = \lambda \mu, \]
where \(\mu\) is a probability measure, admits one and only one solution \(\mu.\)

Proof. Let us first show the uniqueness of \(\mu.\) Suppose that the probability measure \(\mu\) is an eigenvector of the transpose of the Ruelle operator: \(L_g^* \mu = \lambda \mu.\) Then \(\lambda > 0\) and \(L_g^* - 1 \mu = \mu.\) Therefore, \(\mu\) is quasi-invariant with respect to \(G(X, T)\) with Radon-Nikodym derivative \(D_g^{-1}.\) Since \(D_g^{-1}\) and \(D_g\) have the same restriction \(D\) to \(R(X, T), \mu\) is solution of the Radon-Nikodym problem for \(D.\) This shows that \(\mu,\) if it exists, is unique.

Let us show that the unique solution \(\mu\) of the Radon-Nikodym problem for \(D\) is a solution of the eigenvalue problem. The cocycle \(D\) on the AP equivalence relation \(R(X, T)\) admits the sequence \((g_n)\) as (unnormalized) potentials. The corresponding local potentials are \(g_{n-1} = g\) for all \(n.\) In other words, according to Proposition A.2 and the Appendix, the solutions of the Radon-Nikodym problem for \(D\) are in one-to-one correspondence with the states of the inductive limit of the sequence
\[ C(X) \xrightarrow{L_g} C(X) \rightarrow \ldots \rightarrow C(X) \xrightarrow{L_g} C(X) \rightarrow \ldots \]
Explicitly, a state is given by a sequence of measures \((\mu_n)\) such that \(\mu_0\) is a probability measure and \(\mu_n = L_g^* \mu_{n+1}\) for all \(n \in \mathbb{N}.\) The correspondence is \((\mu_n) \mapsto \mu_0.\) It is then clear that, if \((\mu_n)\) is a state, so is \((\mu'_n = L_g^* \mu_n / \lambda, where \(\lambda = \mu_0(L_g(1)).\) By unique ergodicity of \(D,\) we obtain that the above measure \(\mu\) satisfies \(\mu = L_g^* \mu / \lambda, where \(\lambda = \mu_0(L_g(1)).\)\]

A State space of dimension groups

Since the work [10] of G. Elliott, dimension groups have been used as a convenient tool in the study of AF-algebras and topological Markov chains ([28]). Our focus will be the state space of the dimension group, which describes the traces of the AF-algebra. More generally, we shall use dimension groups in the study of Radon-Nikodym cocycles on AP relations. Most results of this section are not new. Those concerning dimension groups can be found in the monograph [24] by K. Goodearl, in the general setting of (partially) ordered abelian groups. Another basic reference on dimension groups is [9] by E. Effros. We also give a slight improvement of a result of A. Török [35] on the uniqueness of a trace on an AF-algebra. The purpose of this appendix is to present the results we need, in the setting the most appropriate to the Radon-Nikodym problem. We first introduce some notation.

When \(X\) is a compact space, \(C(X)\) designates the real vector space of real-valued continuous functions on \(X\) endowed with the uniform norm \(\|f\| =\)
sup \_{X} |f(x)|. We write \( f \geq 0 \) if \( f(x) \geq 0 \) for all \( x \in X \) and we denote by \( 1 = 1_{X} \) the constant function \( 1(x) = 1 \). This turns \((C(X), 1_{X})\) into an ordered real vector space with order-unit. We write \( f > 0 \) if \( f(x) > 0 \) for all \( x \in X \).

**Definition A.1.** A bounded linear operator \( A : C(X) \to C(Y) \), where \( X, Y \) are compact spaces is called

- **positive** if \( f \geq 0 \Rightarrow Af \geq 0 \);
- **strongly positive** if \( f > 0 \Rightarrow Af > 0 \);
- **Markovian** if it is positive and \( A1_{X} = 1_{Y} \).

When \( X = \{1, \ldots, q\} \) and \( Y = \{1, \ldots, p\} \), \( C(X) = \mathbb{R}^{q}, C(Y) = \mathbb{R}^{p} \) and \( A \) is given by a matrix \((a(i,j)) \in M_{p,q}(\mathbb{R})\). Positivity means \( a(i,j) \geq 0 \) for all \( i, j \) and strong positivity means positivity and non-zero rows; it is a weaker condition than the strict positivity of the \( a(i,j) \)'s.

Let \((X_{n}), n \in \mathbb{N}\) be a sequence of compact spaces and let \( A = (A_{n} : C(X_{n-1}) \to C(X_{n})) \), \( n = 1, 2, \ldots \) be a sequence of strongly positive operators. Our problem is to study the set of sequences \( \mu = (\mu_{n}) \), where \( \mu_{n} \) is a positive linear functional on \( C(X_{n}) \), i.e. a measure on \( X_{n} \), satisfying the recurrence relation \( \mu_{n-1} = A_{n}^{*} \mu_{n} \) for all integers \( n \geq 1 \) as well as the normalization condition \( \mu_{0}(1_{X_{0}}) = 1 \).

We introduce
\[
\mu_{n} = A_{n}A_{n-1} \ldots A_{1}1_{X_{0}}.
\]
We view \((\mathcal{E}_{n} = C(X_{n}), \mu_{n})\) as an ordered real vector space with order-unit. The sequence \( A \) defines an inductive system of ordered real vector spaces with order-unit and we can consider its inductive limit \( \mathcal{E} = \mathcal{E}(A) = \lim \mathcal{E}_{n} \). It is an ordered real vector spaces with order-unit \( u \). We denote by \( j_{n} : \mathcal{E}_{n} \to \mathcal{E} \) the canonical morphisms.

Transposition gives the projective system \((A_{n}^{*} : C(X_{n-1})^{*} \leftarrow C(X_{n})^{*})\), \( n = 1, 2, \ldots \).

We introduce the state space of \((\mathcal{E}_{n}, \mu_{n})\):
\[
S_{n} = \{ \tau \in C(X_{n})^{*} : \text{positive and } \tau(u_{n}) = 1 \}.
\]
It is a convex compact subset of \( C(X_{n})^{*} \) in the \( \ast \)-weak topology, in fact it is a Choquet simplex. We are interested in its projective limit:
\[
S = S(A) = \lim S_{n} = \{ (\tau_{n}) : \tau_{n} \in S_{n}; \tau_{n} = \tau_{n+1}A_{n+1}, n \in \mathbb{N} \}.
\]

It is known ([24], 6.14) that \( S \) is the state space of \((\mathcal{E}, u)\), i.e. the convex set of positive homomorphisms \( \tau : \mathcal{E} \to \mathbb{R} \) such that \( \tau(u) = 1 \). The value of \( \tau = (\tau_{n}) \in S \) on \( f = j_{n}(f_{n}) \in \mathcal{E} \) is \( \tau(f) = \tau_{n}(f_{n}) \). It is also known ([24], 10.21) that \( S \) is a (non-empty!) convex compact set, and a Choquet simplex.

**Definition A.2.** We call \( \mathcal{E} = \mathcal{E}(A) = \lim \mathcal{E}_{n} \) the dimension group and \( S = S(A) \) the state space of the sequence \( A = (A_{n}) \).
Remark A.1. When $\mathcal{A} = (A_n)$ is a sequence of matrices with coefficients in $\mathbb{N}$, the dimension group $\mathcal{E}(\mathcal{A}) = \lim\downarrow (A_n : \mathbb{C}(X_{n-1}) \to \mathbb{C}(X_n))$ is a coarser invariant than the usual dimension group $K_0(\mathcal{A}) = \lim\downarrow (A_n : \mathbb{C}(X_{n-1}, \mathbb{Z}) \to \mathbb{C}(X_n, \mathbb{Z}))$ introduced by Elliott. More precisely $\mathcal{E}(\mathcal{A})$ is the realization of $K_0(\mathcal{A})$. However, the state spaces are the same.

Let $\mathcal{A} = (A_n : \mathbb{C}(X_{n-1}) \to \mathbb{C}(X_n))$ and $\mathcal{B} = (B_n : \mathbb{C}(Y_{n-1}) \to \mathbb{C}(Y_n))$ be two sequences of strongly positive operators as above. A morphism from $\mathcal{B}$ to $\mathcal{A}$ is a sequence $\mathcal{D} = (D_n)$ of positive operators $D_n : \mathbb{C}(Y_n) \to \mathbb{C}(X_n)$ such that $D_0(1_{Y_0}) = 1_{X_0}$ and which makes the following diagram commutative:

$$
\begin{array}{cccc}
C(X_0) & \xrightarrow{A_1} & \ldots & \xrightarrow{A_n} & C(X_n) \\
\uparrow D_0 & & & \uparrow D_{n-1} & \uparrow D_n \\
C(Y_0) & \xrightarrow{B_1} & \ldots & \xrightarrow{B_n} & C(Y_n)
\end{array}
$$

This implies that the $D_n$'s are strongly positive. It induces a morphism $D : \mathcal{E}(\mathcal{B}) \to \mathcal{E}(\mathcal{A})$ and a morphism of their state spaces $D^* : S(\mathcal{A}) \to S(\mathcal{B})$ which sends $(\tau_n)$ into $(\tau_nD_n)$.

Definition A.3. A contraction of a sequence of strongly positive operators $A_n : \mathbb{C}(X_{n-1}) \to \mathbb{C}(X_n)$ is a sequence $(B_k = A_{n_k}A_{n_k-1} \ldots A_{n(k-1)+1})$, where $(n_k)$ is a strictly increasing sequence of integers.

Note that $(A_n)$ and a contraction $(B_k)$ give the same inductive limit $\mathcal{E}$ and the same state space. The following elementary observation reduces the problem to the Markovian case.

Proposition A.1. Let $(A_n)$ be a sequence of strongly positive operators. Then, there exists an isomorphic sequence consisting of Markovian operators.

Proof. Let $D_n$ be the operator of multiplication by $u_n$ and define

$$B_n = D_n^{-1}A_nD_{n-1}.$$ 

\[\square\]

In the study of the inductive limit and its state space, we may therefore assume that we have a sequence $\mathcal{B} = (B_n)$ of Markovian operators. That is what we do from now on.

The following result is well known in the theory of dimension groups (see [9], Corollary 4.2. and [24], Corollary 4.10), where its proof is based on an ordered group analogue of the Hahn-Banach theorem due to Goodearl and Handelman). We use here a compactness argument modelled after [39], Theorem 2.9. Albeit elementary, it is one of the most useful tools to compute the state space of a dimension group.

Lemma A.2. Let $\mathcal{B} = (B_n)$ be a sequence of Markovian operators, $\mathcal{E} = \mathcal{E}(\mathcal{B})$ and $S = S(\mathcal{B})$ as above. Let $f = j_m(f_m) \in \mathcal{E}$ and let $f_n = B_nB_{n-1} \ldots B_{m+1}f_m$ for $n \geq m + 1$. The following conditions are equivalent:
Let us show that \( (i) \) implies the condition \( \mu(f) = 0 \) for all \( \mu \in S \).

Proof. \( (i) \Rightarrow (ii) \)

Given \( \epsilon > 0 \), there exists \( n \) such that \( \|f_n\| < \epsilon \). Therefore,

\[
\|\mu(f)\| = |\mu(f_n)| \leq \|f_n\| < \epsilon
\]

and \( \mu(f) = 0 \).

\( (ii) \Rightarrow (i) \)

Suppose that \( (i) \) is not satisfied. Since the sequence \( \|f_n\| \), \( n = m, m+1, \ldots \), is decreasing, there exists \( \epsilon > 0 \) such that \( \|f_n\| \geq \epsilon \) for all \( n \geq m \). Therefore, for \( n \geq m \), the set \( K_n = \{ \mu_n \in S_n : |\mu_n(f_n)| \geq \epsilon \} \) is not empty. Consider the sequence of subsets of \( \prod_0^\infty S_n \):

\[
S^N = \{ (\mu_n) : \mu_{n-1} = \mu_n B_n \text{ for } n \leq N \text{ and } \mu_n \in K_n \text{ for } n \geq N \}
\]

where \( N \geq m \). It is a decreasing sequence of non-empty compact sets. Then, an element \( \mu = (\mu_n) \) of its intersection belongs to \( S \) and \( \mu(f) \neq 0 \) since \( |\mu(f)| = |\mu_m(f)| \geq \epsilon \).

Let us first apply this lemma to study when the inductive system \( A = (A_n) \) has a unique state, that is, when \( S(A) \) is reduced to one element.

Given \( f \in C(X) \), where \( X \) is a compact space, we define:

\[
f_{\min} = \min_x f(x), \quad f_{\max} = \max_x f(x), \quad \text{var}(f) = f_{\max} - f_{\min}.
\]

**Corollary A.3.** Let \( B = (B_n) \) be a sequence of Markovian operators. The following conditions are equivalent:

\( (i) \) For all \( m \in \mathbb{N} \) and all \( f \in C(X_m) \), \( \text{var}(B_n B_{n-1} \ldots B_{m+2} B_{m+1} f) \) tends to zero as \( n \) tends to infinity.

\( (ii) \) For all \( m \in \mathbb{N} \) and all \( f \in C(X_m) \), there exists \( \mu_m(f) \in \mathbb{R} \) such that \( \|B_n B_{n-1} \ldots B_{m+2} B_{m+1} f - \mu_m(f) 1_{X_n}\| \) tends to zero as \( n \) tends to infinity.

\( (iii) \) The state space \( S(B) \) defined above consists of a single element. If \( \nu = (\nu_n) \) is this element, then for all \( m \in \mathbb{N} \), \( \nu_m \) agrees with \( \mu_m \) defined in \( (ii) \).

Proof. Given \( f_m \in C(X_m) \), we define \( f_n = B_n B_{n-1} \ldots B_{m+1} f \) for \( n \geq m + 1 \). Let us show that \( (i) \Rightarrow (ii) \). More precisely, we show that the condition \( (i) \) with \( m = 0 \) implies the condition \( (ii) \) with \( m = 0 \). Let \( f \in C(X_0) \). The sequences \( (f_n^{\min}) \) and \( (f_n^{\max}) \) are adjacent. If moreover \( f_n^{\max} - f_n^{\min} \) tends to zero, they converge to the same limit \( \mu(f) \in \mathbb{R} \). The inequality

\[
\|f_n - \mu(f) 1_{X_n}\| \leq \text{var}(f_n)
\]

gives the conclusion. The reverse implication \( (ii) \Rightarrow (i) \) is clear.
Let us show that \((ii) \Rightarrow (iii)\). We know that \(S(B)\) is not empty. Let \((\nu_n) \in S(B)\). We fix \(m \in \mathbb{N}\). For all \(n \in \mathbb{N}, n \geq m + 1\), we can write \(\nu_m = \nu_n B_n B_{n-1} \ldots B_{m+1}\). Therefore for all \(f \in C(X_m)\),

\[
\nu_m(f) - \mu_m(f) = \nu_n(B_n \ldots B_{m+1} f - \mu_m(f)1_{X_n}),
\]

\[
|\mu_0(f) - \mu(f)| \leq \|B_n \ldots B_{m+1} f - \mu_m(f)1_{X_n}\|.
\]

One concludes that \(\nu_m(f) = \mu_m(f)\) and \(\nu_m = \mu_m\).

Let us show that \((iii) \Rightarrow (ii)\). Let \(\nu = (\nu_n)\) be the single element of \(S\). For \(f_m \in C(X_m)\), \(j_m(f_m - \nu_m(f_m)1_{X_m})\) belongs to \(S_{\perp}\). According to the lemma, this implies \((ii)\).

Thus the above corollary gives a necessary and sufficient condition for an inductive system \(A = (A_n)\) to have a unique state (other necessary and sufficient conditions can be found in [24], Chapter 4, in the general setting of partially ordered abelian groups). We shall give a more practical sufficient condition, along the lines of the work [35] of A. Török, who studied the uniqueness of traces on AF C*-algebras. We first replace the sequence \(A = (A_n)\) by the Markovian sequence \(B = (B_n)\), where \(B_n = D_n^{-1} A_n D_{n-1}\) and \(D_n\) is the operator of multiplication by \(u_n = A_n A_{n-1} \ldots A_1 1_{X_0}\) in order to apply \((i)\) of Corollary A.3. It will be used under the following form.

**Corollary A.4.** Let \(B = (B_n)\) be a sequence of Markovian operators. Suppose that there exists a sequence of positive numbers \((\epsilon_n)\) such that \(\sum \epsilon_n = \infty\) and \(\text{var}(B_n f) \leq (1 - \epsilon_n) \text{var}(f)\) for all \(n\) and all \(f \in C(X_{n-1})\). Then \(B\) has a unique state.

**Proof.** By induction, for all \(m\), all \(f \in C(X_m)\) and all \(n \geq m + 1\),

\[
\text{var}(B_n B_{n-1} \ldots B_{m+1} f) \leq \prod_{k=m+1}^n (1 - \epsilon_k) \text{var}(f).
\]

This tends to 0 when \(n\) tends to \(\infty\).

It remains to establish an estimate on the variation \(\text{var}(B_n f)\). We shall only consider here the case when the \(X_n\)'s are finite sets (i.e. the \(B_n\)'s are Markovian matrices). Let us start with the following lemma.

**Lemma A.5.** (cf. [35], Lemma 2.) Let \(I, J\) be finite sets and let \(B : C(I) \to C(J)\) be a Markovian operator defined by a Markovian matrix \(b : J \times I \to \mathbb{R}\). Let

\[
\epsilon = \min\{\sum_{i \in I_1} b(j, i) + \sum_{i \notin I_1} b(j', i) : j, j' \in J, I_1 \subset I\}.
\]

Then, for all \(f \in C(I)\), \(\text{var}(Bf) \leq (1 - \epsilon) \text{var}(f)\).
Proof. We first note that \( \text{var} : C(I) \to \mathbb{R}_+ \) is a semi-norm. It is invariant under translation by \( \mathbb{R}1_I \) and induces a norm on the quotient space \( C(I)/\mathbb{R}1_I \). If \( \text{var}(f) = 0, f = c1_I \), where \( c \in \mathbb{R}, Bf = c1_J \) and \( \text{var}(Bf) = 0 \). The case \( \text{var}(f) \neq 0 \) can be reduced to the case \( f_{\text{min}} = 0, f_{\text{max}} = 1 \) by considering \( g = (f - f_{\text{min}}1_I)/\text{var}(f) \). We will get our estimate if we show that \( \text{var}(Bf) \leq (1 - \epsilon) \) for all \( f \) in the convex set
\[
C = \{ f \in C(I) : \forall i \in I, 0 \leq f(i) \leq 1 \}.
\]
Since the map \( f \mapsto \text{var}(Bf) \) is convex, its maximum is attained on the set \( \partial C \) of extremal points of \( C \):
\[
\max_C \text{var}(Bf) = \max_{I_1 \subset I} \text{var}(B1_{I_1}).
\]
For \( I_1 \subset I \) and \( j, j' \in J \), we have
\[
B1_{I_1}(j) - B1_{I_1}(j') = \sum_{i \in I_1} b(j, i) - \sum_{i \in I_1} b(j', i) = 1 - \sum_{i \notin I_1} b(j, i) - \sum_{i \in I_1} b(j', i).
\]
Since \( \sum_{i \notin I_1} b(j, i) + \sum_{i \in I_1} b(j', i) \geq \epsilon \), \( \text{var}(B1_{I_1}) \leq 1 - \epsilon \). \( \square \)

Remark A.2. This lemma gives, in the case \( I = J \), an estimate of the spectral gap of the Markovian matrix \( B \) and a Perron-Frobenius theorem for such a matrix. As noted in [35], the matrix need not be primitive: a necessary and sufficient condition for \( \epsilon > 0 \) is that \( B \) does not have any pair of orthogonal rows.

We deduce from the above lemma the following result, valid for strongly positive matrices rather than Markovian matrices, which will be used to show the convergence of stationary systems.

Lemma A.6. Let \( J, I \) be finite sets and let \( A : C(I) \to C(J) \) be an operator defined by a strongly positive matrix \( a : J \times I \to \mathbb{R}_+^* \). Let \( u \in C(I) \) be a strictly positive vector and let \( B : C(I) \to C(J) \) be the Markovian operator defined by the matrix
\[
b(j, i) = (Au(j))^{-1}a(j, i)u(i).
\]
Then, for all \( f \in C(I) \),
\[
\text{var}(Bf) \leq (1 - \frac{a_{\text{min}}}{a_{\text{max}}}) \text{var}(f).
\]
Proof. It suffices to show that the constant \( \epsilon \) of the previous lemma satisfies \( \epsilon \geq a_{\text{min}}/a_{\text{max}} \). This is immediate, because of the obvious inequalities
\[
\sum_{i \in I_1} a(j, i)u(i) \geq a_{\text{min}} \sum_{i \in I_1} u(i) \quad \text{and} \quad Au(j) \leq a_{\text{max}} \sum_{i \in I} u(i),
\]
which are valid for all \( I_1 \subset I \) and \( j \in J \). \( \square \)
Thus one gets a convenient condition on the sequence \((A_n)\) ensuring unique state. A similar result is given in [35] but ours has the advantage to be expressed directly in terms of the \((A_n)\)'s rather than the \((B_n)\)'s.

**Corollary A.7.** Let \(A = (A_n : C(I_{n-1}) \to C(I_n))\) be a sequence of strongly positive operators, where the \(I_n\)'s are finite spaces. Let \(\epsilon_n\) be the ratio of the smallest matrix coefficient of \(A_n\) over its largest matrix coefficient. If \(\sum \epsilon_n = \infty\), then \(A\) has a unique state.

As an application, let us study the case of a stationary sequence \((A_n = A)\).

**Example A.1.** The Perron-Frobenius theorem for primitive matrices. Let \(I\) be a finite set and let \(a : I \times I \to \mathbb{R}_+\) be a primitive matrix; this means that there exists a positive integer \(L\) such that \(a^L\) is strictly positive. The matrix \(a\) defines a strongly positive linear operator \(A : C(I) \to C(I)\). Let us first show that the stationary sequence \(A = (A_n = A)\) has a unique state. Since contracting the stationary sequence \(A = (A_n = A)\) has a unique state. Since contracting does not change the state space, we consider instead the stationary sequence \(\hat{A} = (\hat{A}_n = A^L)\). Using above notation, \(\epsilon_n = \epsilon = \frac{a_{\min}}{a_{\max}}\) is constant and strictly positive; therefore the condition \(\sum \epsilon_n = \infty\) is satisfied. This shows that \(S(A)\) has a unique element \(\mu = (\mu_n)\). Recall that this is a sequence of (positive) measures on \(I\) satisfying \(\mu_{n-1} = A^\ast \mu_n\) and \(\mu_0(1) = 1\). Since the sequence \(\nu = (\nu_n)\) where \(\nu_n = \lambda \mu_{n+1}\) and \(\lambda = 1/\mu_1(1)\) is also a state, we must have \(\mu = \nu\). This shows that \(\mu_0\) is an eigenvector of \(A^\ast\) for the eigenvalue \(\lambda\) and that \(\mu_n = \lambda^{-n} \mu_0\). Conversely, if \(\nu_0\) is a probability measure on \(I\) which is an eigenvector of \(A^\ast\) for the eigenvalue \(\rho\), then \(\nu = (\rho^{-n} \nu_0)\) is a state of \(A\). By uniqueness of the state, \(\nu_0 = \mu_0\) and \(\rho = \lambda\). This shows the existence and the uniqueness of the Perron-Frobenius eigenvector and eigenvalue of the transpose \(A^\ast\). This also gives the same result for \(A\) since \(A\) is primitive if and only if \(A^\ast\) is so. Our proof of the Perron-Frobenius-Ruelle theorem will follow the same pattern: we first establish that the state space of the stationary system defined by the Ruelle operator \(L_\theta : C(X) \to C(X)\) has a single element and then deduce the existence and the uniqueness of the Perron-Frobenius eigenvector and eigenvalue of the transpose \(L_\theta^\ast\).

To conclude, let us quote some results on the affine representation of dimension groups. They can be found in [24] in a more general framework. Given an arbitrary sequence \(B = (B_n)\) of Markovian operators, we want to relate the inductive limit \(E = E(B)\) and the state space \(S = S(B)\). As usual in the theory of compact convex sets, we define \(Aff(S)\) as the space of continuous real-valued affine functions on \(S\). It is a closed subspace of \(C(S)\). We have the evaluation map \(\theta : f \in E \mapsto \hat{f} \in Aff(S)\) defined by \(\hat{f}(\mu) = \mu(f)\) for \(\mu \in S\). The elements of the kernel of \(\theta\) are called infinitesimals.

**Proposition A.8.** (see [24], Ch. 7) With the above notation,

(i) The evaluation map \(\theta\) has dense range in \(Aff(S)\); moreover the image of positive cone of \(E\) is dense in the positive cone of \(Aff(S)\).
(ii) $\text{Aff}(S)$ is isomorphic as an ordered Banach space with unit to the completion of $\mathcal{E}/\ker\theta$ with respect to the norm
\[ \|f\| = \inf\{\|f_n\| : f_n \in C(X_n), j_n(f_n) = f\}. \]

(iii) $(\text{Aff}(S), (\theta \circ j_n))$ is universal among the $(F, (\varphi_n))$ where $F$ is an ordered Banach space with an order unit 1 and $\varphi_n : C(X_n) \to F$ are positive unital linear maps compatible with the $B_n$'s which have the following property: if $f_k \in C(X_k)$ and $g_l \in C(X_l)$ are such that
\[ \|B_n \ldots B_{k+1}(f_k) - B_n \ldots B_{k+1}(g_l)\| \]
tends to 0 when $n$ tends to $\infty$, then $\varphi_k(f_k) = \varphi_l(g_l)$.

**Proof.** The statements of (i) are proved in [24], Theorem 7.9.

For (ii), let $f \in \mathcal{E}$ and $f_n \in C(X_n)$ such that $f = j_n(f_n)$. For $\mu = (\mu_n) \in S$, we have $f(\mu) = \mu_n(f_n)$, hence $\|\hat{f}\| \leq \|f\|$. The equality $\|f\| = \|\hat{f}\|$ results from [24], Proposition 7.12 (e), and the equality of the norm $\|\|_n$ defined there and our norm $\|f\|$.

For (iii), we first observe that $(\text{Aff}(S), (\theta_n = \theta \circ j_n))$ has the required properties. Let us just check the last property. Let $f_k \in C(X_k)$ and $g_l \in C(X_l)$ be as above. Let $\mu \in K$ and $n \geq k, l$. Then,
\[
|\theta_k(f_k)(\mu) - \theta_l(g_l)(\mu)| = |\mu(f_k \circ \pi_k - g_l \circ \pi_l)|
\]
\[
= |(\pi_n)^*\mu(B_n \ldots B_{k+1}(f_k) - B_n \ldots B_{l+1}(g_l))|
\]
\[
\leq \|B_n \ldots B_{k+1}(f_k) - B_n \ldots B_{l+1}(g_l)\|
\]

Since this last quantity goes to 0 when $n$ goes to $\infty$, we obtain $\theta_k(f_k) = \theta_l(g_l)$. Suppose now that $(F, (\varphi_n))$ satisfies also the required properties. We have to define $\varphi : \text{Aff}(S) \to F$ such that $\varphi_n = \varphi \circ \theta_n$. We show that $\varphi_0(f)$ depends only on $\theta(f)$. Indeed, if $\theta(f) = \theta(g)$, then $f - g \in K_\perp$. According to the lemma, $\|B_n \ldots B_1(f) - B_n \ldots B_1(g)\|$ goes to 0. By hypothesis, this implies that $\varphi_0(f) = \varphi_0(g)$. Therefore, there is a well-defined map $\varphi : \text{Aff}(S) \to F$ such that $\varphi \circ \theta(f) = \varphi_0(f)$. The property $\varphi_n = \varphi \circ \theta_n$ follows. \qed

Recall that a partially ordered abelian group $G$ is called archimedean if
\[ x, y \in G \quad nx \leq y \quad \forall n \in \mathbb{N} \Rightarrow x \leq 0. \]

Note that the ordered real vector spaces $C(X)$ and $\text{Aff}(S)$ are archimedean.

**Proposition A.9.** ([24], Theorem 7.7) With the above notation, the following properties are equivalent

(i) 0 is the only infinitesimal element of $\mathcal{E}$.

(ii) $\mathcal{E}$ is archimedean.

(iii) The evaluation map $\theta : \mathcal{E} \to \text{Aff}(S)$ is injective.
References

[1] C. Anantharaman-Delaroche and J. Renault: *Amenable groupoids*, Monographie de l’Enseignement Mathématique No 36, Genève, 2000.

[2] N. Bourbaki: *Topologie générale, chapitres 1 à 4*, Eléments de Mathématique, Diffusion C.C.L.S., Paris, 1971.

[3] T. Bousch: *La condition de Walters*, Janvier 2000

[4] G. Brown and A. H. Dooley: *Odometer actions on G-measures*, Ergodic Theory Dyn. System, 11 (1991), 279–307.

[5] D. Capocaccia: *A definition of Gibbs state for a compact set with $\mathbb{Z}^d$ action*, Commun. Math. Phys., 48 (1976), 85–88.

[6] J. Cuntz: *A class of C*-algebras and topological Markov chains II: reducible chains and the Ext-functor for C*-algebras*, Inventiones Math., 63 (1981), 25–40.

[7] J. Cuntz and W. Krieger: *A class of C*-algebras and topological Markov chains*, Inventiones Math., 56 (1980), 251–268.

[8] E. Effros: *Dimensions and C*-algebras*, CBMS Regional Conf. Series in Math., 46, Amer. Math. Soc., 1981.

[9] G. Elliott: *On the classification of inductive limits of sequences of semisimple finite-dimensional algebras*, J. Algebra, 38 (1976), 29–44.

[10] R. Exel: *KMS states for generalized gauge actions on Cuntz-Krieger algebras*, preprint, (2001).

[11] R. Exel and M. Laca: *Cuntz-Krieger algebras for infinite matrices*, J. reine angew. Math. (Crelle) 512 (1999), 119–172.

[12] R. Exel: *Partial Dynamical Systems and the KMS Condition*, preprint, (2000).
[18] T. Fack and O. Maréchal: Sur la classification des symétrie des $C^*$-algèbres UHF, Canadian Journal of Math.

[19] W. Feller: An introduction to probability theory and its applications, vol. II, John Wiley, N.Y.

[20] A. H. Fan: A proof of the Ruelle theorem, Reviews Math. Phys. 7, no. 8 (1995), 1241–1247.

[21] A. H. Fan and Y. P. Jiang: On Ruelle-Perron-Frobenius Operators. I. Ruelle theorem, Commun. Math. Phys. 223 (2001), 125–141.

[22] H.-O. Georgii: Gibbs measures and phase transitions, de Gruyter Studies in Mathematics, 1988.

[23] T. Giordano, I. Putnam and C. Skau: Affable equivalence relations and orbit structure of Cantor dynamical systems, preprint 2002.

[24] K. R. Goodearl: Partially ordered abelian groups with interpolation, Mathematical Surveys and Monographs, 20, Amer. Math. Soc., 1986.

[25] G. Keller: Equilibrium States in Ergodic Theory, London Mathematical Society Student Texts 42, Cambridge University Press (1998).

[26] S. Kerov: The boundary of Young lattice and random Young tableaux, DIMACS Ser. Discrete Math. Theoret. Comput. Sci. 24, Amer. Math. Soc., (1996), 133–158.

[27] D. Kerr and C. Pinzari: Noncommutative pressure and the variational principle in Cuntz-Krieger type $C^*$-algebras, J. Funct. Anal., 188 (2002), 156-215.

[28] W. Krieger: On dimension functions and topological Markov chains, Inventiones Math., 56 (1980), 239–250.

[29] A. Kumjian: On localizations and simple $C^*$-algebras, Pacific J. Math. 112 (1984), 141–192.

[30] A. Kumjian and J. Renault: KMS states on $C^*$-algebras associated to expansive maps, in preparation.

[31] D. Lind and B. Marcus: An introduction to symbolic dynamics and coding, Cambridge University Press, Cambridge, 1995.

[32] P. Muhly, D. Williams: Continuous trace groupoid $C^*$-algebras, Math. Scand. 66 (1990), 231–241.

[33] J. Renault: A groupoid approach to $C^*$-algebras, Lecture Notes in Mathematics, Vol. 793 Springer-Verlag Berlin, Heidelberg, New York (1980).
[34] J. Renault: *AF equivalence relations and their cocycles*, to appear in the Proceedings of the OAMP Conference (Constanta, 2001), the Theta Foundation.

[35] A. Török: *AF-algebras with unique trace*, Acta. Sci. Math. (Szeged) **55**, No 1-2 (1991), 129–139.

[36] A. Vershik and S. Kerov: *Asymptotic character theory of the symmetric group*, Funct. Analysis and its Applications **15**, (1981), 25–36.

[37] P. Walters: *Invariant measures and equilibrium states for some mappings which expand distances*, Trans. Amer. Math. Soc., **236**, (1978), 121–153.

[38] P. Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics, **79**, Springer-Verlag, New York-Berlin, 1982.

[39] P. Walters: *Convergence of the Ruelle operator for a function satisfying Bowen’s condition*, Trans. Amer. Math. Soc., **353**, No 1 (2000), 327–347.

[40] A. Wassermann: *Automorphic actions of compact groups on operator algebras*, Ph.D thesis, U. of Pennsylvania (1981).

[41] M. Zinsmeister: *Formalisme thermodynamique et systèmes dynamiques holomorphes*, Panoramas et Synthèses, **4**, Société Mathématique de France, 1996.