GLOBAL STABILITY AND UNIFORM PERSISTENCE FOR AN INFECTION LOAD-STRUCTURED SI MODEL WITH EXPONENTIAL GROWTH VELOCITY

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ABSTRACT. In this article is performed a global stability analysis of an infection load-structured epidemic model using tools of dynamical systems theory. An explicit Duhamel formulation of the semiflow allows us to prove the existence of a compact attractor for the trajectories of the system. Then, according to the sharp threshold $R_0$, the basic reproduction number of the disease, we make explicit the basins of attractions of the equilibria of the system and prove their global stability with respect to these basins, the attractiveness property being obtained using infinite dimensional Lyapunov functions.

1. Introduction. Since the first ODE epidemic model was introduced by Kermack and McKendrick [13, 14, 15], so-called SIR model (Susceptible-Infected-Recovered), similar models are frequently used to predict the temporal evolution of a disease and to prevent the apparition of epidemics. A common way to address such an issue lies in the determination through the model of a fundamental epidemiological threshold, called basic reproductive number and denoted $R_0$ (see [1, 5] for an introduction), that ensures ($R_0 < 1$) or not ($R_0 > 1$) the stability of a specific equilibrium point of the system, the "disease-free equilibrium". In the last decades, some epidemic models incorporating, adding to time, another continuous variable (the age of infection, the infection-load, the immunity level, the time remaining before disease detection... see [5, 18, 19, 20, 24, 25, 27, 29, 30] and references therein) have been proposed to discriminate the individuals of the infective population. These models, called structured population models, are described by systems of transport PDEs and consequently require, due to their infinite dimension, mathematical tools from spectral theory and dynamical systems theory to achieve some stability properties of their trajectories.

This article is devoted to the global analysis of an infection load-structured epidemiological model with an exponential growth of the disease, according to the only threshold $R_0$. This analysis aims to extend some weaker results exposed in [27],

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where the authors proved the local stability property of the equilibria as well the uniform persistence of the disease but under stricter conditions than \( R_0 > 1 \).

In all that follows, let us denote by \( i \) the infection load in the infective class, supposing that it grows exponentially fast according to the following evolution equation,

\[
\frac{di}{dt} = \nu i,
\]

where \( t \geq 0 \) denotes the time and the positive constant \( \nu \) denotes the growth velocity of the infection. Denoting by \( i_0 \) the minimal infection load, (1) then induces the following expression of the incubation time, i.e the time for the infectious process to reach the value \( i \) from the initial infection \( i_0 \):

\[
\theta : [i_0, +\infty) \to \mathbb{R}_+ \quad \forall i \mapsto \frac{1}{\nu} \ln \left( \frac{i}{i_0} \right)
\]

Taking into account a "\( \beta SI \)" mass action law as in Kermack-McKendrick models, the infection process is then described by a nonlinear system coupling an ODE with a PDE of transport type, given for \( t \geq 0 \) and \( i \in J = (i_0, +\infty) \subset \mathbb{R}_+ \) by

\[
\begin{align*}
\frac{dS(t)}{dt} &= \gamma - \mu_0 S(t) - S(t) \int_{i_0}^{+\infty} \beta(i) I(t, i) di, \\
\frac{\partial I(t, i)}{\partial t} &= -\frac{\partial(vi I)(t, i)}{\partial i} - \mu(i) I(t, i) + \Phi(i) S(t) \int_{i_0}^{+\infty} \beta(i) I(t, i) di, \\
I(t, i_0) &= 0, \\
S(0) &= S_0 \in \mathbb{R}_+, \quad I(0, \cdot) = I_0 \in L^1_+(J)
\end{align*}
\]

The biological justification of model (3) and its applicability are stated in [27], the meaning of the parameters are raised in Table 1, with the following assumptions:

(i) \( i_0, \mu_0, \nu \) and \( \gamma \) are positive constants;
(ii) function \( \Phi \in C^\infty_+(J) \) is such that \( \int_J \Phi(i) di = 1 \);
(iii) function \( \mu \in L^\infty_+(J) \) is such that \( \mu(i) \geq \mu_0 \) for almost every (f.a.e) \( i \in J \);
(iv) function \( \beta \in L^\infty_+(J) \) is a non-null function such that \( \beta(i) > 0 \) f.a.e \( i \in [\hat{i}, \bar{i}] \) with

\[
\hat{i} = \sup\{i \geq i_0, \left| \text{supp}(\beta) \cap (i_0, i) \right| = 0\}, \\
\bar{i} = \sup\{i \geq i_0, \left| \text{supp}(\beta) \cap (i, +\infty) \right| > 0\},
\]

where \( \left| \cdot \right| \) denotes the Lebesgue measure in \( \mathbb{R} \).

| Parameter definition | symbol |
|----------------------|--------|
| recruitment flux     | \( \gamma \) |
| minimal infection load| \( i_0 \) |
| basic mortality rate  | \( \mu_0 \) |
| disease mortality rate| \( \mu \) |
| horizontal transmission rate | \( \beta \) |
| infection load velocity | \( \nu \) |
| infection load distribution at contamination | \( \Phi \) |
Remark 1. Since $\beta$ is a non null function we necessarily have $i_0 \leq i < \bar{i} \leq +\infty$. Assumption (iv) then allows to consider the cases where there is no disease transmission for infection loads smaller, resp. bigger than the threshold $\underline{i}$, resp. $\bar{i}$ (see Figure 1). Furthermore, this assumption implies in the continuous case that the contagion process is ensured on certain range of infection load.

**Figure 1.** Three examples of shapes of function $\beta$ – left: $i_0 < \underline{i}$ and $\bar{i} = +\infty$; right: $i_0 < \underline{i}$ and $\bar{i} < +\infty$; down: $i_0 = \underline{i}$ and $\bar{i} < +\infty$

Global stability properties of SI epidemiological models described by ODEs have been extensively studied by means of Lyapunov function. We can find in [17, 16] an introduction of the concrete Lyapunov functions to study the global stability of the endemic equilibrium states for the SIR, SIRS and SIS models, a survey in [8] and more recently in [4] the suitable Lyapunov functions for the case of multi-strains SIS, SIR and MSIR models. In these latter articles, it appears that every suitable Lyapunov functional is defined using the following key function:

$$g : \mathbb{R}_+^* \rightarrow \mathbb{R}$$

$$x \mapsto x - 1 - \ln x \quad (4)$$

The first global stability results for structured epidemiological models have been initiated in 2010 in [20], where the authors prove the global stability of the equilibria of an age of infection structured model. The global analysis is performed into two steps: they first prove a result of uniform persistence of the semiflow, that induces the existence of a global compact attractor. The local stability of this latter is obtained as a direct consequence of the related result of Hale and Waltman [11]. In a second step they prove, using a Lyapunov function, that the attractor is reduced to the desired equilibria. Similarly to the finite dimensional case, this Lyapunov function is built using function $g$ defined in (4), but also incorporating integral operators related to the variable of age of infection. The authors recently extended this first result in [19] to the case of a two-group infection age structured model.
There are two main difficulties when dealing with such an approach in infinite dimensional system. The first one lies in the stability property of the compact attractor. Indeed, contrary to finite dimensional ODE systems, when using Lyapunov functions in infinite dimensional systems the stability property is not ensured even if the attractiveness property is (see Lasalle invariance principle [22]). Another difficulty holds in the second step described above, in checking the well-posedness of the Lyapunov function, since this latter may be undefined outside the compact attractor. One can check that a few other global stability results linked to age-structured epidemic models were recently proved by the use of Lyapunov functions ([6, 12, 21, 23, 31]), in which the authors prove the Lyapunov property, i.e the decrease of the function as in Proposition 4 of Section 4.2 in the present article, but where the authors don’t investigate the well-posedness of the Lyapunov function neither the stability property of the equilibria.

The local stability of equilibria of Problem (3) has been proved in [27] through the study of spectral properties of the linearized problem. Furthermore, some numerical simulations that were performed made us think that the stability was global. This article is consequently devoted to the proof of these global stability properties by using dynamical system theory. To achieve that goal, we prove the existence of a global attractor for the semiflow of (3). Unlike the initial work made by Magal, McCluskey and Webb in [20], in our case, we don’t get the existence of the global attractor as a consequence of the uniform persistence of the disease (as reminded above), but we take advantages of an explicit formulation of the semiflow that enables us to exhibit the compactness of the orbits. Then by extending the definition of “age of infection”-Lyapunov function of [20] to the infection-load structure case, we perform the global analysis of the equilibria of the problem with respect to a sharp threshold, the basic reproduction number \( R_0 \). Finally, the disease persistence result stated under the condition \( R_0 > 1 \) is obtained as a consequence of this global analysis.

This article is structured as follows: in Section 2 are stated some notations and preliminary results. Section 3 then deals with the existence of a compact attractor for the dynamical system and the determination of basins of attractions. Section 4 is dedicated to the global analysis. We first define the suitable Lyapunov functions and prove their well-posedness, and then conclude about the global stability using a Lasalle invariance theorem valid for infinite dimensional systems. Finally, Section 5 explains why our results generalize the ODE case obtained in [17], by carrying out the mathematical reformulation as an ODE model.

2. Preliminary results. In this section are introduced some notations and results of dynamical system theory as well on the PDE problem (6), that follow the sketch of [35], and necessary to perform the global analysis of the equilibria.

Let \( X = \mathbb{R} \times L^1(J) \) with product norm and denote by \( X_+ \) the positive cone \( X_+ = \mathbb{R}_+ \times L^1_+(J) \). Considering the differential operator \( A : D(A) \subset X \to X \) defined by

\[
D(A) = \{(x, \varphi) \in X, (i \varphi) \in W^{1,1}(J) \text{ and } \varphi(i_0) = 0\},
\]

\[
A = \begin{pmatrix}
-\mu_0 & 0 \\
0 & L
\end{pmatrix}, \quad L \varphi = -\frac{d}{dt}(\nu i \varphi) - \mu \varphi,
\]
and the function \( f : X \to X \) is given by
\[
    f(u, v) = \left( \begin{array}{c} 
    \gamma - u \int_j^t \beta v \\
    \Phi u \int_j^t \beta v 
    \end{array} \right) \tag{5}
\]
then Problem (3) rewrites as
\[
    \begin{align*}
    \frac{d}{dt} \begin{pmatrix} S(t) \\ I(t) \end{pmatrix} &= A \begin{pmatrix} S(t) \\ I(t) \end{pmatrix} + f(S(t), I(t)), \\
    S(0) &= S_0 \in \mathbb{R}_+, \\
    I(0, \cdot) &= I_0 \in L^1_+(J). 
    \end{align*} \tag{6}
\]

From standard results on semigroup theory we have the following results (see [26] and Theorem 3.16 of [27] for more details):

**Theorem 2.1.**

1. For every \( x = (S_0, I_0) \in X_+ \), Problem (3) has a unique mild solution \((S(t), I(t)) \in C(\mathbb{R}_+, X_+)\) that induces a continuous semiflow via
\[
    (t, x) \mapsto \phi_t(x) = (S(t), I(t)).
\]

2. Recalling the definition of \( \theta \) in (2), the semiflow \( \phi_t = (\phi_t^S, \phi_t^I) \) rewrites using Duhamel formulation as follows: \( \phi_t(x) = (0, \phi_t(x, x)) + (\phi_t^S(x), \phi_2(t, x)) \) with \( \phi_t^S(x) > 0 \) for every \( t > 0 \) and every \( x \in X_+ \) and

\[
    \begin{align*}
    \phi_1(t, x, i) &= \left( \int_0^t (ie^{-\mu t})e^{-\int_0^\xi (\mu(ie^{-\nu s})+\nu)ds}d\xi \right) \chi_{\{\theta(i) \geq t\}}, \\
    \phi_2(t, x, i) &= \left( \int_0^t (\int_0^\xi \beta(l)\phi_{\xi-l}^S(x; l)dl) \right) \Phi(ie^{-\nu s})e^{-\int_0^\xi (\mu(ie^{-\nu s})+\nu)ds}d\xi \chi_{\{\theta(i) < t\}}. 
    \end{align*} \tag{7} \tag{8}
\]

As a consequence of the last theorem, we get the following result:

**Corollary 1.** The infected stage satisfies the following asymptotic property,
\[
    \forall t \geq 0, \lim_{i \to +\infty} iI(t, i) = 0. \tag{9}
\]

**Proof.** Indeed it is clear that if \((S_0, I_0) \in D(A)\), then from Theorem 2.1 one gets \( iI(t, i) \in W^{1,1}(J) \) for every \( t \geq 0 \). Then from the equality
\[
iI(t, i) = \int_{0i}^{2i} \partial_u(uI(t, u)) \, du,
\]
one deduces that for every \( t \geq 0 \) there exists \( l(t) \geq 0 \) such that \( \lim_{i \to +\infty} iI(t, i) = l(t) \). Finally, since \( iI(t, i) \in W^{1,1}(J) \subset L^1(J) \), then necessarily \( l(t) = 0 \) for every \( t \geq 0 \). Now, if we take \((S_0, I_0) \in X_+\), then equation (7) the density of \( D(A) \) in \( X_+ \) yield (9). \( \square \)

Consider in all that follows the basic reproductive number \( R_0 \) given by
\[
    R_0 = \frac{\gamma}{\mu_0} \int_{i_0}^{+\infty} \beta(i)\pi(i) \, di \tag{10}
\]
where function \( \pi \in L^1(i_0, +\infty) \) is defined by
\[
    \pi(i) = \frac{1}{\nu_0} \int_{i_0}^i \Phi(s)e^{-\int_s^i \frac{\nu(s)}{\mu(s)} \, ds}, \tag{11}
\]
and remind the following theorem about the local stability of the equilibria that is proved in [27] :

**Theorem 2.2.**

- If \( \mathcal{R}_0 \leq 1 \) then Problem (3) has a unique equilibrium, \( E_0 = (S_F, 0) = \left( \frac{\gamma}{\mu_0}, 0 \right) \in X_+ \) (disease free equilibrium) which is locally asymptotically stable ;
- If \( \mathcal{R}_0 > 1 \) then Problem (3) has two equilibria in \( X_+ \), \( E_0 \) which is unstable and a locally asymptotically stable endemic equilibrium \( E^* \) given by \( E^* = (S^* , I^*) = \left( \frac{S_F}{\mathcal{R}_0}, \frac{\gamma (R_0 - 1)}{\mathcal{R}_0} \right) \).

### 3. Compact attractor and basins of attraction.

In all that follows, let us denote \( O_x = \{ \phi_t(x), t \geq 0 \} \) the orbit starting from \( x \in X_+ \) and \( \omega(x) = \bigcap_{\tau \geq 0} \{ \phi_t(x), t \geq \tau \} \) the \( \omega \)-limit set of \( x \).

The following lemma and proposition aim to prove the existence of a compact attractor for the trajectories of Problem (6).

**Lemma 3.1.** For every \( x \in X_+ \), the orbit \( O_x \subset X_+ \) is such that \( O_x \) is compact.

**Proof.** It follows from equation (7) that the following properties hold about the semiflow \( \phi_t \) :

(i) for every \( r > 0 \) and every \( x \in B_r \cap X_+ \), \( \| (0, \phi_1(t, x)) \| \leq re^{-(\mu_0 + \nu)t} \);

(ii) \( (\phi^1_t, \phi^2_t) \) maps the bounded sets of \( X_+ \) into sets with compact closure in \( X \).

The relative compactness of the orbit \( O_X \) is then a direct consequence of (i), (ii) and Proposition 3.13 in [33]. \( \square \)

A consequence of Lemma 3.1 is that the orbits are precompact leading to the existence of a compact attractor in the following sense (see Lemma 3.1. in [10] or Theorem 4.1. in [32]):

**Proposition 1.** For every \( x \in X_+ \),

(a) \( \omega(x) \) is nonempty, compact and connected;

(b) \( \omega(x) \) is invariant under \( \phi_t \), i.e. \( \phi_t(\omega(x)) = \omega(x) \);

(c) \( \omega(x) \) is an attractor, i.e. \( \lim_{t \to +\infty} d(\phi_t(x), \omega(x)) = 0 \).

The following result deals with the basins of attractions of the equilibria of the system.

**Proposition 2.** Consider the set \( S_0 = \{(S, I) \in X_+, \int_0^T I(i)di > 0\} \) and \( \partial S_0 = X_+ \setminus S_0 \).

1. The set \( \partial S_0 \) is positively invariant: \( \phi_t(\partial S_0) \subset \partial S_0 \) for \( t \geq 0 \);

2. The equilibrium \( E_0 \) is globally exponentially stable for \( \phi_t \) restricted to \( \partial S_0 \);

3. There exists \( \tau \geq 0 \) such that the following inequality holds for every \( x \in S_0 \) and every \( t \geq \tau \):

\[ \int_{\tau}^{t} \phi^1_t(x; i)di > 0. \]

Consequently, The set \( S_0 \) is asymptotically positively invariant: for every \( x \in S_0, \phi_t(x) \in S_0 \) for \( t \geq \tau \).
4. If $R_0 > 1$ then $\{ x \in S_0 \Rightarrow \omega(x) \subseteq S_0 \}$.

Proof. Let $x = (S_0, I_0) \in X_+$ and let us remind that the component in $I$ of the semiflow rewrites as $\phi_I^t(x) = \phi_1(t, x) + \phi_2(t, x)$ where $\phi_1, \phi_2$ are defined in (7).

(1) Suppose that $x = (S_0, I_0) \in \partial S_0$. Then standard majorations and the change of variables $u = ie^{-\mu t}$ imply that

$$\int_{t_0}^{\tau} \phi_1(t, x, i)di \leq \int_{t_0}^{\tau} I_0(i)di = 0. \quad (12)$$

Since $\beta$ is in $L^\infty$ then assumption (iv) on $\beta$ and (12) give

$$\int_{t_0}^{+\infty} \beta(i)\phi_1(t, x, i)di = \int_{t_0}^{\tau} \beta(i)\phi_1(t, x, i)di \leq \|\beta\|_\infty \int_{t_0}^{\tau} \phi_1(t, x, i)di = 0. \quad (13)$$

From here, using standard majorations and assumptions (ii) on $\Phi$ and (iii) on $\mu$, it is easy to check that the function $F(t) = \int_{t_0}^{+\infty} \beta(i)\phi_1^t(x; i)di$ satisfies

$$F(t) = \int_{t_0}^{+\infty} \beta(i)\phi_2(t, x, i)di \leq \|\beta\|_\infty \int_{t_0}^{t} F(t-s)\phi_2^s(x)ds.$$ 

Then a Gronwall argument states that $F(t) = 0$ for every $t \geq 0$. From here, one deduces the two following consequences:

1. equation (7) implies that $\phi_2 = 0$;
2. assumption (iv) on $\beta$ implies that $\int_{t_0}^{\tau} \phi_1^t(x; i)di = 0$.

From the two last points and (12) one then gets $\int_{t_0}^{\tau} \phi_1^t(x; i)di = 0$, and consequently $\partial S_0$ is positively invariant for $t \geq 0$.

(2) As proved above, for $x = (S_0, I_0) \in \partial S_0$ we have $F(t) = \int_{t_0}^{+\infty} \beta(i)\phi_1^t(x; i)di = 0$ for $t \geq 0$ and so Problem (3) and equation (7) yield that

$$\phi(t) = \left( e^{-\mu t}S_0 + \frac{\gamma}{\mu_0} (1 - e^{-\mu t}), \phi_1(t, x, i) \right)$$

and so

$$\|\phi(t) - E_0\| \leq e^{-\mu t} \left( \left| S_0 - \frac{\gamma}{\mu_0} \right| + \|I_0\|_1 \right), \quad (14)$$

which proves (2).

(3) Let us consider $x = (S_0, I_0) \in S_0$ and denote $\tau = \theta(\tilde{t})$.

Let us first remark the following equivalence that will be used several times in the sequel:

$$\int_{t_0}^{\tau} \phi_1^t(x; i)di = 0 \iff \int_{t_0}^{+\infty} \beta(i)\phi_1^t(x; i)di = 0, \quad \forall t \geq 0. \quad (15)$$

Indeed, if $\int_{t_0}^{\tau} \phi_1^t(x; i)di = 0$ then the positivity of the semiflow implies that $\phi_1^t(x; i) = 0$ f.a.e $i \in (\tilde{t}, \tilde{t})$ and then from the defnition of $\tilde{t}$ and $\tilde{t}$ we get $\beta(i)\phi_1^t(x; i) = 0$ f.a.e $i \in J$. The reciprocity lies on the same kind of argument, by using that $\beta(i) > 0$ f.a.e $i \in (\tilde{t}, \tilde{t})$ (see assumption (iv) on $\beta$).

Suppose now by contradiction that $\int_{t_0}^{\tau} \phi_1^t(x; i)di = 0$. Since $\tau = \theta(\tilde{t})$, equation (7) shows that $\phi_1(\tau, x, i) = 0$ for $i \leq \tilde{t}$, so using the change of variables $\sigma = \tau - s$
in the second equality of (7) one gets on \( \{ (\sigma, i) \in \mathbb{R}_+ \times [\tilde{\imath}, \tilde{\tau}], \tau - \theta(i) \leq \sigma \leq \tau \} \),
\[
\int_{t_0}^{+\infty} \beta(l) \phi'_\sigma(x;l) dl = 0. \tag{15}
\]
Moreover, since \( \tau = \theta(\tilde{\tau}) \), one deduces that this latter equation is satisfied for every \( \sigma \in [0, \tau] \) and equations (14) and (7) then imply that
\[
\int_{\tilde{\imath}}^{\tilde{\tau}} \phi_1(\sigma, x, i) di = 0, \quad \forall \sigma \in [0, \tau].
\]
From here, using again (7) and the change of variables \( u = ie^{-\nu_\sigma} \) one deduces that
\[
\int_{\Omega(\sigma)} \pi_0(u) H(\sigma, ue^{\nu_\sigma}) du = 0, \quad \forall \sigma \in [0, \tau],
\]
where \( H(\sigma, i) = e^{-\int_{0}^{\sigma} \mu(ie^{-\nu_\sigma}, i_0) di} \), \( \alpha(\sigma) = \max(ie^{-\nu_\sigma}, 0) \) and \( \pi(\sigma) = \tilde{\tau} e^{-\nu_\sigma} \). Since every \( \zeta \in [0, \tilde{\tau}] \) satisfies \( \zeta \in [\alpha(i), \pi(\sigma)] \) for \( \sigma \in \left[ \max(0, -\theta(\tilde{\imath})) - \theta(\zeta), \tau - \theta(\zeta) \right] \subset [0, \tau] \), this implies that \( \cup_{\sigma \in (0, \tau]} \{ \alpha(\sigma), \pi(\sigma) \} \) is a cover of \( [0, \tilde{\tau}] \). Since \( H \) is a positive function, equation (16) then implies that \( \int_{t_0}^{\tilde{\tau}} I_0(i) di = 0 \), that contradicts that \( x \in S_0 \), and so necessarily \( \int_{\tilde{\imath}}^{\tilde{\tau}} \phi'_\sigma(x;i) di > 0 \).

Let us now prove that the inequality can be extended on \( [\tau, +\infty[ \). Let us consider the non-empty set \( \Gamma = \{ t \geq \tau, \int_{\tilde{\imath}}^{t} \phi'_\sigma(x;i) di > 0, \forall \sigma \in [\tau, t] \} \). Clearly the continuity of \( t \mapsto \int_{\tilde{\imath}}^{t} \phi'_\sigma(x;i) di \) implies that \( \Gamma = [\tau, t_\infty) \) with \( t_\infty < +\infty \). Suppose by contradiction that \( t_\infty < +\infty \). So since on the one hand we have \( \theta(i) \leq \tau < t_\infty \) for \( i \in [\tilde{\imath}, \tilde{\tau}] \) and on the other hand \( t_\infty - s \leq \tau \) for \( s \in (0, t_\infty - \tau] \), one then deduces from equation (7) that \( \phi'_\sigma(x;i) = \phi_2(t_{\infty}, x, i) > 0 \) and consequently \( t_\infty \in \Gamma \). But then since \( \int_{\tilde{\imath}}^{\tilde{\tau}} \phi'_\sigma(x;i) di > 0 \) a continuity argument would then imply that the latter inequality can be extended on an open neighborhood of \( t_\infty \), contradicting the definition of \( t_\infty \).

Finally, \( \phi_\gamma(x) \in S_0 \) for every \( t \geq \tau \) is a direct consequence .

(4) Let us first prove that the stable manifold of \( E_0 \) is reduced to \( \partial S_0 \) when \( R_0 > 1 \). Focusing on the definition of \( R_0 \) in equation (10), then for \( R_0 > 1 \), one can consider \( \epsilon > 0 \) small enough such that
\[
\left( \frac{\gamma}{\mu_0} - \epsilon \right) \int_{t_0}^{+\infty} \beta(i) \pi(i) \, di > 1. \tag{17}
\]
Let us denote \( S_{0,\epsilon} = \{ x \in S_0, \| x - E_0 \| \leq \epsilon \} \). We aim to prove that for each \( x \in S_{0,\epsilon} \), there exists \( \tilde{t}(x) \) such that
\[
\| \phi_{\tilde{t}}(x) - E_0 \| > \epsilon. \tag{18}
\]
Suppose by contradiction that there exists \( x = (S_0, I_0) \in S_0 \) such that
\[
\| \phi_{\tilde{t}}(x) - E_0 \| \leq \epsilon, \quad \forall \tilde{t} \geq 0. \tag{19}
\]
Denoting for convenience \( \phi_\epsilon(x) = (S(t), I(t)) \), a consequence of the latter equation and the definition of \( E_0 \) is that
\[
S(t) \geq \frac{\gamma}{\mu_0} - \epsilon, \quad \forall t \geq 0,
\]
and consequently $I$ satisfies
\[
\begin{cases}
\partial_t I(t,i) + \partial_i (\nu I(t,i)) \geq -\mu(i)I(t,i) + \left(\frac{\gamma}{\mu_0} - \epsilon\right)\Phi(i) \int_{t_0}^{+\infty} \beta(i)I(t,i)di, \\
I(t,i_0) = 0, \\
I(0,\cdot) = I_0.
\end{cases}
\]

We then have $I(t,i) \geq \hat{I}(t,i)$ where $\hat{I}$ is solution of the last problem with an equality instead of the inequality in the pde part. Indeed, using the change of variables along the characteristics of the transport equation, i.e $(t,i) \mapsto (t, t - \theta(i))$, the partial differential inequality can be integrated leading to
\[
I(t,i) \geq I_0(i e^{-\nu t}) e^{-\int_0^t (\mu(e^{-\nu s} + \nu))ds}
\]
and the same holds for $\hat{I}$ but with an equality instead of the inequality. Multiplication by $\beta$, an integration in $i$ and Fubini's theorem then give the existence of a nonnegative quantities $g_1$ and $g_2$ such that
\[
T(\beta \hat{I})(t) \geq g_1(t) + \left(\frac{\gamma}{\mu_0} - \epsilon\right) \int_0^t g_2(s,t) T(\beta \hat{I})(s)ds
\]
and consequently to assumption (ii) on $\Phi$ and the positivity of parameters $\mu, \nu$ one gets for every $t \geq 0$ and every $s \leq t$,
\[
|g_2(s,t)| \leq \|\beta\|_{\infty}.
\]

Suppose now by contradiction that there exists $t_1 > 0$ such that $T(\beta I)(t_1) - T(\beta \hat{I})(t_1) < 0$. Then considering the non empty set $A = \{t \in [0,t_1], T(\beta I)(t) - T(\beta \hat{I})(t) \geq 0\}$ it is clear that $\bar{t} = \sup A$ exists, $\bar{t} \in A$ and $\bar{t} < t_1$. Then $t \in [\bar{t},t_1] \Rightarrow t \notin A$ and consequently to (21) and (22), for every $t \in [\bar{t},t_1]$ one gets
\[
0 < T(\beta \hat{I})(t) - T(\beta I)(t) \leq \left(\frac{\gamma}{\mu_0} - \epsilon\right) \int_0^t g_2(s,t)(T(\beta \hat{I}) - T(\beta I))(s)ds
\]
and a standard Gronwall argument would then imply that $T(\beta \hat{I})(t) - T(\beta I)(t) = 0$ on $[\bar{t},t_1]$ and so a contradiction. So, we necessarily have $T(\beta I)(t) \geq T(\beta \hat{I})(t)$ and the equation (20) then implies that $I(t,i) \geq \hat{I}(t,i)$.

Furthermore, one can check that $\hat{I}$ satisfies the linear problem $\frac{d}{dt} \hat{I}(t) = (L + B_\epsilon)\hat{I}(t)$ where, as stated in Section 2 (see (5) and above), $L$ is the differential operator and $B_\epsilon$, related to function $f$, is given by
\[
B_\epsilon \varphi = \left(\frac{\gamma}{\mu_0} - \epsilon\right) \Phi \int_{\bar{t}}^{2\bar{t}} \beta \varphi
\]
Now, it follows from (17) that the simple dominant eigenvalue of $L + B_\varepsilon$, solution of the characteristic equation
\[
\left( \frac{\gamma}{\mu_0} - \varepsilon \right) \int_{i_0}^{+\infty} \beta(i) \frac{1}{\nu_i} \int_{i_0}^{i} \Phi(s)e^{-\int_{s}^{i} \mu(l) + \lambda_0 dl} ds \, di = 1,
\]
is such that $\lambda_0 > 0$ (see proofs of Theorems 3.15-3.16 in [27] for details and in particular the derivation of the characteristic equation). A consequence of the asynchronous exponential growth property for the linear problem (see [3, 2, 7, 9, 34] for theoretical aspects as well as examples related to linear structured models) is that

$$
\Pi_{\lambda_0} \hat{I}(t) = e^{\lambda_0 t} \Pi_{\lambda_0} I_0, \quad t \geq 0,
$$

where $\Pi_{\lambda_0}$ is the projector associated to $\lambda_0$ on the generalized eigenspace of $L + B_\varepsilon$. One then gets

$$
\lim_{t \to +\infty} \|\Pi_{\lambda_0} \hat{I}(t, \cdot)\|_1 = +\infty,
$$

implying a contradiction with (19) since $I(t, i) \geq \hat{I}(t, i)$ and so (18) is then proved. Therefore, the stable manifold of $E_0$ does not intersect $S_0$ when $R_0 > 1$ and

$$
\left\{ x \in X_+, \lim_{t \to +\infty} \phi_t(x) = E_0 \right\} \cap S_0 = \emptyset.
$$

Let us now consider $x \in S_0$ and suppose, by contradiction, that there exists $y \in \omega(x) \cap \partial S_0$. The invariance of $\omega(x)$ then gives $\omega(y) = \omega(x)$ and so

$$
d(\omega(x), E_0) \leq d(\omega(y), \phi_t(y)) + d(\phi_t(y), E_0), \quad \forall t \geq 0.
$$

A consequence of Proposition 1 and (13) is that $d(\omega(x), E_0) = 0$ and so $\{E_0\} \subset \omega(x)$, which contradicts the fact that the stable manifold of $E_0$ does not intersect $S_0$. □

4. Global analysis. In this section, we aim to prove that the equilibria mentioned in Theorem 2.2 satisfy a global stability property, by the mean of Lyapunov functions.

In all that follows, let us define $\sigma \in L^\infty(J)$ the positive function

$$
\sigma(i) = \int_{i}^{+\infty} \beta(s) \frac{1}{\nu_s} e^{-\int_{s}^{i} \mu(l) + \lambda_0 dl} ds.
$$

Note that $\sigma$ satisfies $\|\sigma\|_\infty \leq \frac{\|\beta\|_\infty}{\mu_0}$, is solution of the differential equation

$$
\sigma'(i) = \frac{\mu(i)}{\nu_i} \sigma(i) - \frac{\beta(i)}{\nu_i}, \quad (23)
$$

and consequently to Fubini's lemma satisfies

$$
\int_{i_0}^{+\infty} \sigma(i) \Phi(i) di = \frac{R_0}{S_F}. \quad (24)
$$

To perform the global asymptotic analysis of the disease-free equilibrium $E_0$, we will use the function $L_0(x) = V_0(x) + W_0(x)$ formally defined for $x = (S, I) \in X$ by

$$
V_0(x) = S_F g \left( \frac{S}{S_F} \right)
$$

$$
W_0(x) = \int_{i_0}^{+\infty} \sigma_0(i) I(i) di
$$

with $\sigma_0 = S_F \sigma$. 

To perform the global asymptotic analysis of the endemic equilibrium \( E_* \), we will use the function \( L_*(x) = V_*(x) + W_*(x) \) formally defined for \( x = (S,I) \in X \) by
\[
V_*(x) = S_* g \left( \frac{S}{S_*} \right) \\
W_*(x) = \int_{t_0}^{+\infty} \sigma_*(i) I_*(i) g \left( \frac{I(i)}{I_*(i)} \right) di
\]
with \( \sigma_* = S_* \sigma \).

We recall, along with the definition of \( g \) in (4), that \( g \) is nonnegative and \( g(s) = 0 \) iif \( s = 1 \). The integrability of the \( g \)-part in \( \int^* \) is not always ensured, due to the presence of the "ln" in its definition (see equation (4)) and implies to determine domains for which \( \frac{I(i)}{I_*(i)} \) is faraway from zero. The following subsection is consequently devoted to a well-posedness study, that is a necessary prerequisite for the Lyapunov property of \( L_0 \) and \( L_* \).

4.1. Well-posedness of the functions \( L_0 \) and \( L_* \). The goal of Proposition 3 is to state an estimation that is uniform on the compact attractors \( \omega(x) \) for \( x \in \mathcal{S}_0 \), that will ensure that the restriction to \( \omega(x) \) of function \( L_* \) is well defined.

**Lemma 4.1.** Suppose that \( \mathcal{R}_0 > 1 \). Then for every \( x \in \mathcal{S}_0 \), there exists \( \delta(x) > 0 \) such that
\[
\int_{t_0}^{+\infty} \beta y i(y; t) dt \geq \delta(x), \quad \forall (t, y) \in \mathbb{R}_+ \times \omega(x).
\]

**Proof.** Let \( x \in \mathcal{S}_0 \). Since \( \mathcal{R}_0 > 1 \) then Proposition 2 implies that \( \omega(x) \subset \mathcal{S}_0 \) and so
\[
\int_{t_0}^{+\infty} \beta(i) y i(y; t) dt > 0, \quad \forall t \geq \tau, \forall y \in \omega(x). \tag{25}
\]
Let \( y = (y^S, y^I) \in \omega(x) \). The invariance of \( \omega(x) \) under the semiflow implies that for a fixed \( \tau \geq \tau \), there exists \( z \in \omega(x) \) such that \( y = \phi_t(z) \) and then (25) implies that
\[
\int_{t_0}^{+\infty} \beta(i) y i(y; t) dt = \int_{t_0}^{+\infty} \beta(i) y i(z; t) dt > 0.
\]
From here, a continuity argument and the compactness of \( \omega(x) \) then lead to the existence of \( \delta(x) > 0 \) such that \( \int_{t_0}^{+\infty} \beta(i) y i(y; t) dt \geq \delta(x) \) for every \( y \in \omega(x) \). Finally, the proposition is a direct consequence of the fact that \( \phi_t(\omega(x)) \subset \omega(x) \) for every \( t \geq 0 \).

**Proposition 3.** Suppose that \( \mathcal{R}_0 > 1 \). Then for every \( x \in \mathcal{S}_0 \) there exists \( c(x) > 0 \) such that for every \( (t, y) \in \mathbb{R}_+ \times \omega(x) \),
\[
0 \leq L_* g \left( \frac{\phi^t_i(y)}{I_*} \right) \leq c(x) \phi^t_i(y).
\]

**Proof.** Let \( x \in \mathcal{S}_0 \). Since the total population \( t \mapsto \phi^t_i(x) + \int_{t_0}^{+\infty} \phi^t_i(x; i) di \) is bounded on \( \mathbb{R}_+ \) (see Lemma 3.3 in [27]), then one can consider \( \psi(x) > 0 \) such that \( \int_{t_0}^{+\infty} \beta(i) \phi^t_i(x; i) dt \leq \psi(x) \). The integration of the differential equation in (6) then gives
\[
\phi^S_t(x) \geq e^{-(\mu_0 + \psi(x))t} S_0 + \frac{\gamma}{\mu_0 + \psi(x)} \left( 1 - e^{-(\mu_0 + \psi(x))t} \right)
\]
and so
\[
\liminf_{t \to +\infty} \phi^S_t(x) \geq \frac{\gamma}{\mu_0 + \psi(x)}. \tag{26}
\]
Moreover, the classical inequalities
\[ 1 - \frac{1}{r} \leq \ln r \leq r - 1, \quad \forall r > 0, \]
and the definition of function \( g \) yield that for every \( y \in \omega(x) \),
\[ 0 \leq I_*(i)g \left( \frac{\phi_t^i(y; i)}{I_*(i)} \right) \leq \phi_t^i(y; i) \left( \frac{I_*(i)}{\phi_t^i(y; i)} - 1 \right)^2, \quad (t, i) \in \mathbb{R}_+ \times J. \]

Consequently, to achieve the proof of the proposition, we have to check the existence of \( c(x) > 0 \) such that for every \( y \in \omega(x) \) and \( (t, i) \in \mathbb{R}_+ \times J \),
\[ \left( \frac{I_*(i)}{\phi_t^i(y; i)} - 1 \right)^2 \leq c(x). \quad (27) \]

Let us first suppose that \( t \in \mathbb{R}_+ \) and \( i \in J \) are such that \( t > \theta(i) \).

Then from (7), (26) and Lemma 4.1 one deduces that for every \( y \in \omega(x) \),
\[ \phi_t^i(y; i) \geq \frac{\delta \gamma}{\mu_0 + \psi(x)} \int_0^{\theta(i)} \Phi(i e^{-\nu \xi}) e^{- \int_0^s (\mu + \nu \xi) d\xi} ds. \]

Using the changes of variables \( l = i e^{-\nu \xi} \) and \( \lambda = i e^{-\nu s} \), the latter equation leads the following lower bound
\[ \phi_t^i(y; i) \geq \frac{\delta \gamma}{\mu_0 + \psi(x)} \mu \int_0^{i} \Phi(\lambda) e^{- \int_0^s \frac{\pi(\lambda)}{i}} d\lambda = \frac{\delta \gamma}{\mu_0 + \psi(x)} \pi(i). \]

Finally, the latter inequality and the expression of \( I_* \) in Theorem 2.2 imply that
\[ \phi_t^i(y; i) \geq \frac{\delta (R_0 - 1)}{(\mu_0 + \psi(x)) R_0}, \quad (28) \]
and consequently the existence of \( c(x) > 0 \) such that (27) is satisfied for every \( y \in \omega(x) \) and \( t > \theta(i) \).

Suppose now that \( t \in \mathbb{R}_+ \) and \( i \in J \) are such that \( t \leq \theta(i) \).

Since \( \omega(x) \) is invariant under the semiflow \( \phi_t \), it is a classical result (see for instance [28, p26]) that for every \( y \in \omega(x) \) there exists a full orbit \( \xi \mapsto u_y(\xi) \) defined for \( \xi \in \mathbb{R} \) passing through \( y \), i.e satisfying
\[
\begin{cases}
  u_y(\xi) \in \omega(x), & \forall \xi \in \mathbb{R}, \\
  u_y(0) = y, \\
  \phi_{\xi}(u_y(s)) = u_y(\xi + s), & \forall (\xi, s) \in \mathbb{R}_+ \times \mathbb{R}
\end{cases}
\]

From here, let us consider \( s \in \mathbb{R} \) such that \( t + s > \theta(i) \). Then since \( u_y(-s) \in \omega(x) \) one deduces from (28) that
\[ \frac{I_*(i)}{\phi_t^i(y; i)} = \frac{I_*(i)}{\phi_t^{i+s}(u_y(-s))} \leq \frac{\mu_0 + \psi(x)) R_0}{\delta (R_0 - 1)} \]
and consequently (27) is also satisfied for every \( y \in \omega(x) \) and \( t \leq \theta(i) \) which ends the proof.

The following result is a consequence of the latter proposition:

**Corollary 2.**

1. Function \( (t, x) \mapsto L_0(\phi_t(x)) \) is well-defined on \( \mathbb{R}_+ \times \mathbb{R}_+ \);
2. For every \( x \in S_0 \), function \( (t, y) \mapsto L_*(\phi_t(y)) \) is well-defined on \( \mathbb{R}_+ \times \omega(x) \) whenever \( R_0 > 1 \).
Proof. Note that, as a consequence of Theorem 2.2, the condition $\mathcal{R}_0 > 1$ is necessary to define $L_i$. Furthermore, the positivity of $\delta_i^0$ on $X_+$ (recalled in the proof of Lemma 3.1) implies that $V_0$ and $V_*$ are well-defined. Finally, the integrability on $J$ of $i \mapsto \sigma_i^0(x; i)$ for $(t, x) \in \mathbb{R}_+ \times X_+$ implies that $L_0$ is well-defined on $\mathbb{R}_+ \times X_+$ and $L_*$ is well-defined on $\mathbb{R}_+ \times \omega(x)$, $x \in S_0$, as a consequence of Proposition 3.

4.2. Global stability and persistence results. We recall the definition of a Lyapunov function for the semiflow $\phi_t$ in the case of infinite dimensional systems, as defined in [22]:

**Definition 4.2.** Let $D \subset X$. A function $L : X \to \mathbb{R}$ is called a Lyapunov function on $D$ if the following hold:

1. $L$ is continuous on $D$;
2. $L$ decreases along orbits starting in $D$, i.e. $t \mapsto L(\phi_t(x))$ is a nonincreasing function of $t \geq 0$ for every $x \in D$.

The following proposition shows that functions $L_0$ and $L_*$ are Lyapunov.

**Proposition 4.**

1. Suppose that $\mathcal{R}_0 \leq 1$. Then $L_0$ is a Lyapunov function on $X_+$.
2. Suppose that $\mathcal{R}_0 > 1$. Then $L_*$ is a Lyapunov function on $\omega(x)$ for every $x \in S_0$.

Proof. In all the proof, for better reading and understanding, we deliberately omit, in the functions $L_0(\phi_t(x))$ and $L_*(\phi_t(x))$, the dependence on $x$ for the semiflow, this latter being written as $(S(t), I(t))$.

1. Function $L_0$ is well-defined and clearly continuous on $X_+$. A differentation of $V_0$ w.r.t $t$ along (3) gives, using the equality $\gamma = \mu_0 S_F$,

$$
\frac{\partial}{\partial t}[V_0(\phi_t(x))] = \left( \gamma - \mu_0 S(t) - S(t) \int_{i_0}^{+\infty} \beta(i) I(t, i) di \right) \left( 1 - \frac{S_F}{S(t)} \right)
$$

$$
= -\frac{\mu_0}{S(t)} (S(t) - S_F)^2 - S(t) \int_{i_0}^{+\infty} \beta(i) I(t, i) di + S_F \int_{i_0}^{+\infty} \beta(i) I(t, i) di \quad (29)
$$

We also have

$$
\frac{\partial}{\partial t}[W_0(\phi_t(x))] = -\int_{i_0}^{+\infty} \sigma_0(i) \frac{\partial(\nu I)(t, i)}{\partial t} di - \int_{i_0}^{+\infty} \sigma_0(i) \mu(i) I(t, i) di
$$

$$
+ S(t) \left( \int_{i_0}^{+\infty} \beta(i) I(t, i) di \right) \left( \int_{i_0}^{+\infty} \sigma_0(i) \Phi(i) di \right)
$$

An integration by parts give, using the boundary condition in (3), the asymptotic property (9) and equation (23), implies that

$$
- \int_{i_0}^{+\infty} \sigma_0(i) \frac{\partial(\nu I)(t, i)}{\partial t} di = \int_{i_0}^{+\infty} \nu i \sigma_0(i) I(t, i) di
$$

$$
= \int_{i_0}^{+\infty} \sigma_0(i) \mu(i) I(t, i) di - S_F \int_{i_0}^{+\infty} \beta(i) I(t, i) di
$$

So from equation (24) one deduces that

$$
\frac{\partial}{\partial t}[W_0(\phi_t(x))] = \mathcal{R}_0 S(t) \int_{i_0}^{+\infty} \beta(i) I(t, i) di - S_F \int_{i_0}^{+\infty} \beta(i) I(t, i) di. \quad (30)
$$
From (29) and (30) one then gets
\[
\frac{\partial}{\partial t}[L_0(\phi_t(x))] = -\frac{\mu_0}{S(t)}(S(t) - S_F)^2 + (R_0 - 1)S(t)\int_{t_0}^{+\infty} \beta(i)I(t,i)di \quad (31)
\]
so that \(L_0\) is a decreasing function along the orbits starting in \(X_+\) whenever \(R_0 \leq 1\), and consequently \(L_0\) is a Lyapunov function on \(X_+\).

2. Suppose now that \(R_0 > 1\) and let \(x \in S_0\). Then \(L_*\) is well-defined on \(\omega(x)\) from corollary 2, and is clearly continuous.

Note that equation (24) then implies that \(\int_{t_0}^{+\infty} \sigma_*(i)\Phi(i)di = 1\). In all that follows, let us denote, for convenience, \(T\) the integral operator defined on \(L^1(i_0, +\infty)\) by \(T : f \mapsto \int_{t_0}^{+\infty} f(i)di\).

Similarly to the case \(R_0 \leq 1\), a differentiation of \(V_*\) gives, using the equality \(\gamma = \mu_0S_* + S_*T(\beta I_*)\),
\[
\frac{\partial}{\partial t}[V_*(\phi_t(x))] = -\frac{\mu_0}{S(t)}(S(t) - S_*)^2 + S_*T(\beta I_*) - \frac{S_*^2}{S(t)}T(\beta I_*) + S_*T(\beta I) - S(t)T(\beta I)
\]
\[
= -\frac{\mu_0}{S(t)}(S(t) - S_*)^2 + S_*\int_{t_0}^{+\infty} \beta(i)I_*(i) \left(1 - \frac{I(t,i)S(t)}{I_*(i)}\right)\frac{-S_*}{S(t)}\frac{I(t,i)}{I_*(i)} di
\]
Furthermore, a differentiation of \(W_*\) along trajectories of (3) gives
\[
\frac{\partial}{\partial t}[W_*(\phi_t(x))] = -\int_{t_0}^{+\infty} \sigma_*(i) \left(1 - \frac{I_*(i)}{I(t,i)}\right) \left(\partial_t(\nu i I(t,i)) + \mu(i)I_*(i)\Phi(i)ST(\beta I)\right) di
\]
\[
= -\int_{t_0}^{+\infty} \sigma_*(i) \left(1 - \frac{I_*(i)}{I(t,i)}\right) \left(\partial_t(\nu i I(t,i)) + \left(\mu(i) - \Phi(i)\frac{S_*T(\beta I_*)}{I_*(i)}\right)I(t,i)\right) di
\]
\[
= -\int_{t_0}^{+\infty} \sigma_*(i) \left(1 - \frac{I_*(i)}{I(t,i)}\right) \left(\Phi(i)\frac{S_*T(\beta I_*)I(t,i)}{I_*(i)} - \Phi(i)ST(\beta I)\right) di \quad (33)
\]
\[
:= [1] + [2]
\]
Now, noting that \(\nu I_0L_0(i_0) = 0 = \sigma_*(+\infty)\), using (34) and an integration by parts yields that the term \([1]\) in equation (33) satisfies
\[
[1] = \int_{t_0}^{+\infty} g \left(\frac{I(t,i)}{I_*(i)}\right) \left(\sigma_*(i)(\nu i I_*(i)) + \sigma_*(i)(\nu I_*'(i))\right) di
\]
Using the differential equation satisfied by \(\sigma_*\) that is derived from (23) and the equality \((\nu i I_*)' = -\mu I_* + \Phi S_*T(\beta I_*)\) one then gets
\[
[1] = \int_{t_0}^{+\infty} (-S_*\beta(i)L_* + \sigma_*(i)S_*T(\beta I_*))g \left(\frac{I(t,i)}{I_*(i)}\right) di
\]
constant. So a differentiation w.r.t. time implies that
\[ \partial_t \text{considering a fixed} \]
prove that
1. We use the Lasalle invariance principle (see Corollary 2.3 in [22]) to
Proof. 2. When
\[ y \]
and since
\[ g \]
then for every
\[ x \in \mathcal{S}_0 \].
Note that the previous proposition shows that \( L_\ast \) is a Lyapunov function only on the attractor \( \omega(x) \), that has no reason to be defined outside. However, it is sufficient to prove the global attractivity of the endemic equilibrium \( E_\ast \) with respect to the basin of attraction \( \mathcal{S}_0 \), as stated in the following theorem.

**Theorem 4.3.** 1. When \( R_0 \leq 1 \) the disease-free equilibrium \( E_0 \) is globally stable in \( X_+ \).
2. When \( R_0 > 1 \) the disease-free equilibrium \( E_0 \) is unstable and the endemic equilibrium \( E_\ast \) is globally stable in \( \mathcal{S}_0 \).

**Proof.** 1. We use the Lasalle invariance principle (see Corollary 2.3 in [22]) to prove that \( E_0 \) is globally attractive in \( X_+ \): consequently to Proposition 4, when considering a fixed \( x \in X_+ \) then for every \( y \in \omega(x) \) the function \( t \mapsto L_0(\phi_t(y)) \) is constant. So a differentiation w.r.t. time implies that \( \partial_t [L_0(\phi_t(y))] = 0 \). But then when \( R_0 < 1 \) equation (31) necessarily implies that \( \phi_t^1(y) = 0 \) and \( \phi_t^S(y) = S_F \), so that \( y = E_0 \) and consequently \( \omega(x) = \{ E_0 \} \). When \( R_0 = 1 \), then equation

\[
= -S_i \int_{t_0}^{\infty} \beta(i)I_i(t) \left[ g \left( \frac{I(t,i)}{L(i)} \right) - \int_{t_0}^{\infty} \sigma(s)g \left( \frac{I(t,s)}{I_s(s)} \right) \right]
\]

Then substituting the latter equality in (33) and using the fact that \( \int_{t_0}^{\infty} \sigma(i) \Phi(i)di = 1 \) gives

\[
\frac{\partial}{\partial t} [W_*(\phi_t(x))] = S_i \int_{t_0}^{\infty} \beta(i)I_i(t) \left[ -g \left( \frac{I(t,i)}{L(i)} \right) + \int_{t_0}^{\infty} \sigma(s)g \left( \frac{I(t,s)}{I_s(s)} \right) \right]
\]

Summing up the latter equality with (32), using again \( \int_{t_0}^{\infty} \sigma(i) \Phi(i)di = 1 \) and the definition of \( g \) yield

\[
\frac{\partial}{\partial t} [L_*(\phi_t(x))] = - \frac{I_0}{S(t)} (S(t) - S_\ast)^2
\]

Now, adding and substracting \( \ln \left( \frac{S_\ast}{S(t)} \right) \) and using the equality

\[
\ln \left( \frac{S(t)}{S_\ast(t)} \right) + \ln \left( \frac{I(t,i)}{I_\ast(i)} \right) + \int_{t_0}^{\infty} \sigma(s) \Phi(s) \ln \left( \frac{I(s)}{I(t,s)} \right) ds
\]

finally leads to

\[
\frac{\partial}{\partial t} [L_*(\phi_t(x))] = S_i \int_{t_0}^{\infty} \beta(i)I_i(t) \left[ -g \left( \frac{S_\ast}{S(t)} \right) - \int_{t_0}^{\infty} \sigma(s) \Phi(s) g \left( \frac{I(s)I(t,i)S(t)}{I_\ast(i)I(t,s)S_\ast} \right) ds \right]
\]

and since \( g \) is a non-negative function, \( L_\ast \) is a Lyapunov function on \( \omega(x) \) for every
\[ x \in \mathcal{S}_0. \]
2.2. case 1 the Lasalle invariance principle implies that $\frac{\partial S}{\partial t} = 0$ so necessarily $\phi^t_0(y) = 0$. Finally, we proved that $\omega(x) = \{E_0\}$ when $R_0 \leq 1$ for every $x \in X_+$. The stability result stated in Theorem 2.2 allows to conclude that $E_0$ is globally asymptotically stable.

2. We now prove the global stability of the $E_*$ with respect to basin of attraction $S_0$. Let $x \in S_0$ and consider $y \in \omega(x)$. Since $L_*$ is Lyapunov on $\omega(x)$, similarly to case 1 the Lasalle invariance principle implies that $\frac{\partial}{\partial t}[L_*(\phi^t_i(y))] = 0$. Consequently to (35), one clearly gets $y = \{E_*\}$ so $\omega(x) = \{E_*\}$ for every $x \in S_0$. The global stability of $E_*$ in the basin $S_0$ when $R_0 > 1$ is then a consequence of Theorem 2.2.

Finally, the uniform persistence of the disease, according to the condition $R_0 > 1$, is a direct consequence of Theorem 4.3, as stated in the following corollary:

**Corollary 3.** Suppose that $R_0 > 1$. Then the disease is uniformly persistent with respect to $S_0$, i.e. there exists $\epsilon > 0$ such that for every $(S_0, I_0) \in S_0$, $\liminf_{t \to +\infty} \int_{i_0}^{+\infty} I(t, i) di \geq \epsilon$.

5. **A final remark.** The goal of this short section is to show that the results we proved in this article generalize the ODE case, by recovering the result addressed by Korobeinikov and Wake in [17].

In the specific case where parameters are positive constant, $\mu(i) \equiv \mu_1, \beta(i) \equiv \beta_0$, then Problem (3) rewrites as the classical SI ODE model,

\[
\begin{cases}
\frac{dS(t)}{dt} = \gamma - \mu_0 S(t) - \beta_0 S(t)Y(t), \\
\frac{dY(t)}{dt} = \beta_0 S(t)Y(t) - \mu_1 Y(t), \\
S(0) = S_0, \quad Y(0) = \int_J I_0(i) di,
\end{cases}
\]

where $Y(t) = \int_J I(t, i) di$. Furthermore, integrations with respect to $i \in J$ show that the basic reproduction number is given by

\[
R_0 = \frac{\gamma \beta_0}{\mu_0 \mu_1}
\]

and the equilibria are $E_0 = \left(\frac{\gamma}{\mu_0}, 0\right)$ and $E_* = \left(\frac{\mu_1}{\beta_0}, \frac{\mu_0}{\beta_0} (R_0 - 1)\right)$.

Note that in this particular situation, $i = \text{sup}\{i \geq i_0, |\text{supp}(\beta) \cap (i, +\infty)| > 0\} = +\infty$ so that $S_0 = \{((S, I) \in X_+, \int_J I(i) di > 0\}$ and the basin of attraction of $E_*$ is consequently $R_+ \times R_+^*$. We then conclude from Theorem 4.3 that the following result holds, according to Theorem 1 in [17]: $R_0 \leq 1 \Rightarrow E_0$ is globally stable in $(R_+)^2$ and $R_0 > 1 \Rightarrow E_*$ is globally stable in $R_+ \times R_+^*$.

**REFERENCES**

[1] R. M. Anderson and R. M. May, *Infectious Diseases of Humans: Dynamics and Control*, Oxford University Press, 1991.

[2] O. Arino, A. Bertuzzi, A. Gandolfi, E. Sánchez and C. Sinisgalli, The asynchronous exponential growth property in a model for the kinetic heterogeneity of tumour cell populations, *Journal of Mathematical Analysis and Applications*, 302 (2005), 521–542.

[3] O. Arino, E. Sánchez and G. F. Webb, Necessary and sufficient conditions for asynchronous exponential growth in age structured cell populations with quiescence, *Journal of Mathematical Analysis and Applications*, 215 (1997), 499–513.
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[4] D. Bichara, A. Iggidr and G. Sallet, Global analysis of multi-strains SIS, SIR and MSIR epidemic models, J. Appl. Math. Comput., 44 (2014), 273–292.
[5] O. Diekmann and J. A. P. Heesterbeek, Mathematical Epidemiology of Infectious Diseases, Wiley Series in Mathematical and Computational Biology, John Wiley & Sons, 2000.
[6] X. Duan, S. Yuan, Z. Qiu and J. Ma, Global stability of an SVEIR epidemic model with ages of vaccination and latency, Comput. Math. Appl., 68 (2014), 288–308.
[7] Janet Dyson, Rosanna Villella-Bressan and G. F. Webb, Asynchronous exponential growth in an age structured population of proliferating and quiescent cells, Mathematical Biosciences, 177–178 (2002), 73–83.
[8] A. Fall, A. Iggidr, G. Sallet and J. J. Tewa, Epidemiological models and Lyapunov functions, Math. Model. Nat. Phenom., 2 (2007), 55–73.
[9] Jozsef Z. Farkas, Note on asynchronous exponential growth for structured population models, Nonlinear Analysis: Theory, Methods & Applications, 67 (2007), 618–622.
[10] J. K. Hale, Asymptotic Behavior of Dissipative Systems, W. O. Kermack and A. G. McKendrick, Contributions to the mathematical theory of epidemics, Proc. R. Soc. Lond. Ser. A, 115 (1927), 700–721.
[11] A. Korobeinikov and G. C. Wake, Lyapunov functions and global stability for SIR, SIRS, and SVEIR epidemic models, Appl. Math. Lett., 15 (2002), 955–960.
[12] B. Laroche and A. Perasso, Identifiability analysis of an epidemic model with immunity loss rate depending on the vaccine-age, Appl. Anal., 1058–1095.
[13] J. Math. Anal. Appl., 374 (2011), 154–165.
[14] A. Perasso and U. Razafison, Infection load structured si model with exponential velocity and external source of contamination, In Proceedings of the World Congress on Engineering (WCE), volume 1, pages 263–267, 2013.
[15] A. Perasso and U. Razafison, Asymptotic behavior and numerical simulations for an infection load-structured epidemiological models; application to the transmission of prion pathologies, SIAM J. Appl. Math., 74 (2014), 1571–1597.
[16] H. L. Smith, Monotone Dynamical Systems, volume 41 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 1995, An introduction to the theory of competitive and cooperative systems.
[17] H. L. Smith and H. R. Thieme, Dynamical Systems and Population Persistence, Graduate Studies in Mathematics 118, American Mathematical Society, 2011.
[18] H. R. Thieme and C. Castillo-Chavez, How may infection-age-dependent infectivity affect the dynamics of hiv/aids? SIAM J. Appl. Math., 53 (1993), 1447–1479.
[31] C. Vargas-De-León, E. Lourdes and A. Korobeinikov, Age-dependency in host-vector models: the global analysis, Appl. Math. Comput., 243 (2014), 969–981.
[32] J. A. Walker, Dynamical Systems and Evolution Equations, volume 20 of Mathematical Concepts and Methods in Science and Engineering, Plenum Press, New York-London, 1980, Theory and applications.
[33] G. F. Webb, Theory of Nonlinear Age-Dependent Population Dynamics, Marcel Dekker, New York, 1985.
[34] G. F. Webb, An operator-theoretic formulation of asynchronous exponential growth, Trans. Amer. Math. Soc., 303 (1987), 751–763.
[35] G. F. Webb, Population models structured by age, size, and spatial position, In Structured Population Models in Biology and Epidemiology, Lecture Notes in Math. 1936, pages 1–49. Springer, Berlin, 2008.

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