Certification of minimal approximant bases
Pascal Giorgi, Vincent Neiger

To cite this version:
Pascal Giorgi, Vincent Neiger. Certification of minimal approximant bases. ISSAC 2018 - 43rd International Symposium on Symbolic and Algebraic Computation, Jul 2018, New York, United States. pp.167-174, 10.1145/3208976.3208991. hal-01701861v2

HAL Id: hal-01701861
https://unilim.hal.science/hal-01701861v2
Submitted on 17 May 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Certification of Minimal Approximant Bases
Pascal Giorgi
LIRMM, Université de Montpellier, CNRS
Montpellier, France
pascal.giorgi@lirmm.fr

Vincent Neiger
Univ. Limoges, CNRS, XLIM, UMR 7252
F-87000 Limoges, France
vincent.neiger@unilim.fr

ABSTRACT
For a given computational problem, a certificate is a piece of data that one (the prover) attaches to the output with the aim of allowing efficient verification (by the verifier) that this output is correct. Here, we consider the minimal approximant basis problem, for which the fastest known algorithms output a polynomial matrix of dimensions \(m \times m\) and average degree \(D/m\) using \(O(m^e D/m)\) field operations. We propose a certificate which, for typical instances of the problem, is computed by the prover using \(O(m^{e+1} D)\) additional field operations and allows verification of the approximant basis by a Monte Carlo algorithm with cost bound \(O(m^{e+2} + D)\).

Besides theoretical interest, our motivation also comes from the fact that approximant bases arise in most of the fastest known algorithms for linear algebra over the univariate polynomials; thus, this work may help in designing certificates for other polynomial matrix computations. Furthermore, cryptographic challenges such as breaking records for discrete logarithm computations or for integer factorization rely in particular on computing minimal approximant bases for large instances: certificates can then be used to provide reliable computation on outsourced and error-prone clusters.

KEYWORDS
Certification; minimal approximant basis; order basis; polynomial matrix; truncated product.

1 INTRODUCTION
Context. For a given tuple \(d = (d_1, \ldots, d_n) \in \mathbb{Z}_{\geq 0}^n\) called order, we consider an \(m \times n\) matrix \(F\) of formal power series with the column \(j\) truncated at order \(d_j\). Formally, let \(F \in \mathbb{K}[X][m \times n]\) be a matrix over the univariate polynomials over a field \(\mathbb{K}\), such that the column \(j\) of \(F\) has degree less than \(d_j\). Then, we consider the classical notion of minimal approximant bases for \(F\) [1, 27]. An approximant is a polynomial row vector \(p \in \mathbb{K}[X]^{1 \times n}\) such that

\[
pF = 0 \mod X^d, \quad \text{where} \quad X^d = \text{diag}(X^{d_1}, \ldots, X^{d_n}); \quad (1)
\]

here \(pF = 0 \mod X^d\) means that \(pF = qX^d\) for some \(q \in \mathbb{K}[X]^{1 \times n}\).

The set of all approximants forms a (free) \(\mathbb{K}[X]\)-module of rank \(m\),

\[
\mathcal{A}_d(F) = \left\{ p \in \mathbb{K}[X]^{1 \times n} \mid pF = 0 \mod X^d \right\}.
\]

A basis of this module is called an approximant basis (or sometimes an order basis or a \(\sigma\)-basis); it is a nonsingular matrix in \(\mathbb{K}[X]^{m \times m}\) whose rows are approximants in \(\mathcal{A}_d(F)\) and generate \(\mathcal{A}_d(F)\).

The design of fast algorithms for computing approximant bases has been studied throughout the last three decades [1, 14, 15, 26–28]. Furthermore, these algorithms compute minimal bases, with respect to some degree measure specified by a shift \(s \in Z^n\). The best known cost bound is \(O(m^e D)\) operations in \(\mathbb{K}\) [15] where \(D\) is the sum \(D = |d| = d_1 + \cdots + d_n\). Throughout the paper, our complexity estimates will fit the algebraic RAM model counting only operations in \(\mathbb{K}\), and we will use \(O(n^e)\) to refer to the complexity of the multiplication of two \(m \times m\) matrices, where \(\omega < 2.373\) [4, 21].

Here, we are interested in the following question:

\textbf{How to efficiently certify that some approximant basis algorithm indeed returns an \(s\)-minimal basis of \(\mathcal{A}_d(F)\)?}

Since all known fast approximant basis algorithms are deterministic, it might seem that a posteriori certification is pointless. In fact, it is an essential tool in the context of unreliable computations that arise when one delegates the processing to outsourced servers or to some large infrastructure that may be error-prone. In such a situation, and maybe before concluding a commercial contract to which this computing power is attached, one wants to ensure that he will be able to guarantee the correctness of the result of these computations. Of course, to be worthwhile, the verification procedure must be significantly faster than the original computation.

Resorting to such computing power is indeed necessary in the case of large instances of approximant bases, which are a key tool within challenging computations that try to tackle the hardness of some cryptographic protocols, for instance those based on the discrete logarithm problem (e.g. El Gamal) or integer factorization (e.g. RSA). The computation of a discrete logarithm over a 768-bit prime field, presented in [20], required to compute an approximant basis that served as input for a larger computation which took a total time of 355 core years on a 4096-cores cluster. The approximant basis itself took 1 core year. In this context, it is of great interest to be able to guarantee the correctness of the approximant basis before launching the most time-consuming step.

Linear algebra operations are good candidates for designing fast verification algorithms since they often have a cost related to matrix multiplication while their input only uses quadratic space. The first example one may think of is linear system solving. Indeed, given a solution vector \(x \in \mathbb{K}^n\) to a system \(Ax = b\) defined by \(A \in \mathbb{K}^{n \times n}\) and \(b \in \mathbb{K}^n\), one can directly verify the correctness by checking the equations at a cost of \(O(n^2)\) operations in \(\mathbb{K}\). Comparatively, solving the system with the fastest known algorithm costs \(O(n^3)\).

Another famous result, due to Freivalds [11], gives a method to verify a matrix product. Given matrices \(A, B, C \in \mathbb{K}^{n \times n}\), the idea is to check \(uC = (uA)B\) for a random row vector \(u \in \{0, 1\}^{1 \times n}\).
rather than \( C = AB \). This verification algorithm costs \( O(n^3) \) and is false-biased one-sided Monte-Carlo (it is always correct when it answers "false"); the probability of error can be made arbitrarily small by picking several random vectors.

In some cases, one may require an additional piece of data to be produced together with the output in order to prove the correctness of the result. For example, Farkas' lemma [10] certifies the infeasibility of a linear program thanks to an extra vector. Although the verification is deterministic in this example, the design of certificates that are verified by probabilistic algorithms opened a line of work for faster certification methods in linear algebra [7, 8, 17, 18].

In this context, one of the main challenges is to design optimal certificates, that is, ones which are verifiable in linear time. Furthermore, the time and space needed for the certificate must remain negligible. In this work, we seek such an optimal certificate for the problem of computing shifted minimal approximant bases.

Here, an instance is given by the input \((d, F, s)\) which is of size \( O(mD) \): each column of \( F \) contains at most \( md_i \) elements of \( \mathbb{K} \), and the order sums to \( d_1 + \cdots + d_n = |d| = D \). We neglect the size of the shift \( s \), since one may always assume that it is nonnegative and such that \( \max(s) < mD \) (see [15, App. A]). Thus, ideally one would like to have a certificate which can be verified in time \( O(mD) \).

In this paper, we provide a non-interactive certification protocol which uses the input \((d, F, s)\), the output \( P \), and a certificate which is a constant matrix \( C \in \mathbb{K}^{m \times n} \). We design a Monte-Carlo verification algorithm with cost bound \( O(mD + m^{\omega + 1}(m + n)) \); this is optimal as soon as \( D \) is large compared to \( m \) and \( n \) (e.g. when \( D > m^2 + mn \)), which is most often the case of interest. We also show that the certificate \( C \) can be computed in \( O(m^{\omega+1}D) \) operations in \( \mathbb{K} \), which is faster than known approximant basis algorithms.

### Degrees and size of approximant bases.

For \( P \in \mathbb{K}[X]^{m \times m} \), we denote the row degree of \( P \) as \( \text{rdeg}(P) = (r_1, \ldots, r_m) \) where \( r_i = \text{deg}(P_{i, \cdot}) \) is the degree of the row \( i \) of \( P \) for \( 1 \leq i \leq m \). The column degree \( \text{cdeg}(P) \) is defined similarly. More generally, we will consider row degrees shifted by some additive column weights: for a shift \( s = (s_1, \ldots, s_m) \in \mathbb{Z}^m \), the \( s \)-row degree of \( P \) is \( \text{rdeg}_s(P) = (r_1 + s_1, \ldots, r_m + s_m) \).

We use \( |\cdot| \) to denote the sum of integer tuples: for example \( |\text{rdeg}_s(P)| \) is the \( s \)-row degree of \( P \) (note that this sum might contain negative terms). The comparison of integer tuples is entrywise: \( \text{cdeg}(F) < d \) means that the column \( j \) of \( F \) has degree less than \( d_j \), for \( 1 \leq j \leq n \). When adding a constant to a tuple, say for example \( s - 1 \), this stands for the tuple \( (s_1 - 1, \ldots, s_m - 1) \).

In existing approximant basis algorithms, the output bases may take different forms: essentially, they can be s-minimal (also called \( s \)-reduced [27]), \( s \)-weak Popov [23], or \( s \)-Popov [3]. For formal definitions and for motivating the use of shifts, we direct the reader to these references and to those above about approximant basis algorithms; here the precise form of the basis will not play an important role. What is however at the core of the efficiency of our algorithms is the impact of these forms on the degrees in the basis.

In what follows, by size of a matrix we mean the number of field elements used for its dense representation. We define the quantity

\[
\text{Size}(P) = m^2 + \sum_{1 \leq i, j \leq m} \max(0, \text{deg}(p_{ij}))
\]

for a matrix \( P = [p_{ij}] \in \mathbb{K}[X]^{m \times m} \). In the next paragraph, we discuss degree bounds on \( P \) when it is the output of any of the approximant basis algorithms mentioned above; note these bounds all imply that \( P \) has size in \( O(mD) \).

There is no general degree bound for approximant bases: any unimodular matrix is a basis of \( \mathcal{A}(0) = \mathbb{K}[X]^{1 \times m} \). Still, a basis \( P \) of \( \mathcal{A}(F) \) always satisfies \( \text{deg} \left( \text{det}(P) \right) \leq D \). Now, for an \( s \)-minimal \( P \), we have \( |\text{cdeg}(P)| \in O(D) \) as soon as \( |s - \min(s)| \in O(D) \) [27, Thm. 4.1], and it was shown in [28] that \( P \) has size in \( O(mD) \) if \( |\max(s) - s| \in O(D) \). Yet, without such assumptions on the shift, there are \( s \)-minimal bases whose size is in \( \Omega(m^2D) \) [15, App. B], ruling out the feasibility of finding them in time \( O(m^{\omega+1}D) \). In this case, the fastest known algorithms return the more constrained \( s \)-Popov basis \( P \), for which \( |\text{cdeg}(P)| \leq D \) holds independently of \( s \).

### Problem and contribution.

Certifying that a matrix \( P \) is an \( s \)-minimal approximant basis for a given instance \((d, F, s)\) boils down to the following three properties of \( P \):

1. **Minimal**: \( P \) is in \( s \)-reduced form. By definition, this amounts to testing the invertibility of the so-called \( s \)-leading matrix of \( P \) (see Step 1 of Algorithm 1 for the construction of this matrix), which can be done using \( O(m^\omega) \) operations in \( \mathbb{K} \).
2. **Approximant**: the rows of \( P \) are approximants. That is, we should check that \( P F = 0 \mod X^d \). The difficulty is to avoid computing the full truncated product \( P F \mod X^d \), since this costs \( O(m^{\omega+1}D) \). In Section 3, we give a probabilistic algorithm which verifies more generally \( P F = G \mod X^d \) using \( O(\text{Size}(P) + mD) \) operations, without requiring a certificate.
3. **Basis**: the rows of \( P \) generate the approximant module. For this, we prove that it suffices to verify first that \( \text{det}(P) \) is of the form \( cX^\delta \) for some \( c \in \mathbb{K} \setminus \{0\} \) and where \( \delta = |\text{rdeg}(P)| \), and second that some constant \( m \times (m + n) \) matrix has full rank; this matrix involves \( P(0) \) and the coefficient \( C \) of degree 0 of \( PFX^{-d} \). In Section 2, we show that \( C \) can serve as a certificate, and that a probabilistic algorithm can assess its correctness at a suitable cost.

Our (non-interactive) certification protocol is as follows. Given \((d, F, s)\), the Prover computes a matrix \( P \), supposedly an \( s \)-minimal basis of \( \mathcal{A}(F) \), along with a constant matrix \( C \in \mathbb{K}^{m \times n} \), supposedly the coefficient of degree 0 of the product \( PFX^{-d} \). Then, the Prover communicates these results to the Verifier who must solve Problem 1 within a cost asymptotically better than \( O(m^{\omega+1}D) \).

---

1This is not implied by (1) and (2): for \( d = \max(d) \), then \( X^d \text{Im} \) is \( s \)-reduced and \( X^d \text{ImF} \equiv 0 \mod X^d \) holds; yet, \( X^d \text{Im} \) is not a basis of \( \mathcal{A}(P) \) for most \((d, F)\).

---

**Problem 1: Approximant Basis Certification**

**Input:**
- order \( d \in \mathbb{Z}^n \)
- matrix \( P \in \mathbb{K}[X]^{m \times n} \) with \( \text{cdeg}(F) < d \)
- shift \( s \in \mathbb{Z}^m \)
- matrix \( P \in \mathbb{K}[X]^{m \times m} \)
- certificate matrix \( C \in \mathbb{K}^{m \times n} \)

**Output:**
- True if \( P \) is an \( s \)-minimal basis of \( \mathcal{A}(F) \) and \( C \) is the coefficient of degree 0 of \( PFX^{-d} \), otherwise False.
The main result in this paper is an efficient solution to Problem 1.

**Theorem 1.1.** There is a Monte-Carlo algorithm which solves Problem 1 using $O(md + m^{o(\log m)}(m + n))$ operations in $\mathbb{K}$, assuming $\text{Size}(P) \in O(mD)$. It chooses $m + 2$ elements uniformly and independently at random from a finite subset $S \subset \mathbb{K}$. If $S$ has cardinality at least $2(D + 1)$, then the probability that a true answer is incorrect is less than $1/2$, while a false answer is always correct.

A detailed cost bound showing the constant factors is described in Proposition 2.5. If $\text{Size}(P) \in O(mD)$, then the size bound above is therefore optimal (up to constant factors) as soon as $m^{o(\log m)}(m + n) \in O(D)$.

If $\mathbb{K}$ is a small finite field, there may be no subset $S \subset \mathbb{K}$ of cardinality $\# S \geq 2(D + 1)$. Then, our approach still works by performing the probabilistic part of the computation over a sufficiently large extension of $\mathbb{K}$. Note that an extension of degree about $1 + \lceil \log_2(D) \rceil$ would be suitable; this would increase our complexity estimates by $O(d)$.

Here, we present our certification algorithm. Its properties, given in Proposition 2.5, are summarized in the following result.

**Theorem 2.1.** Let $d \in \mathbb{N}_{>0}$, let $F \in \mathbb{K}[X]^{m \times n}$ with $d \in D$ and $m \in O(D)$, and let $P \in \mathbb{K}[X]^{m \times m}$. If $|\text{deg}(P)| \in O(D)$ or $|\text{deg}(P)| \in O(d)$, then there is a deterministic algorithm which computes the coefficient of degree $0$ of $\text{PFX}^{-d}$ using $O(m^{d-1}D)$ operations in $\mathbb{K}$ if $m \geq n$ and $O(m^{d-1}D)$ operations in $\mathbb{K}$ if $m < n$.

Note that the assumption $m \in O(D)$ in this theorem is commonly made in approximant basis algorithms, since when $D \leq m$ most entries of a minimal approximant basis have degree in $O(1)$ and the algorithms then rely on methods from dense $\mathbb{K}$-linear algebra.

## 2 CERTIFYING APPROXIMANT BASES

Here, we present our certification algorithm. Its properties, given in Proposition 2.5, prove Theorem 1.1. One of its core components is the verification of truncated polynomial matrix products; the details of this are in Section 3 and are taken for granted here.

First, we show the basic properties behind the correctness of this algorithm, which are summarized in the following result.

**Theorem 2.2.** Let $d \in \mathbb{N}_{>0}$, let $F \in \mathbb{K}[X]^{m \times n}$, and let $s \in \mathbb{Z}^m$. A matrix $P \in \mathbb{K}[X]^{m \times m}$ is an $s$-minimal basis of $\mathcal{A}_d(F)$ if and only if the following properties are all satisfied:

(i) $P$ is s-reduced;

(ii) $\text{det}(P)$ is a nonzero monomial in $\mathbb{K}[X]$;

(iii) the rows of $P$ are in $\mathcal{A}_d(F)$, that is, $PF = 0$ mod $X^d$;

(iv) $[P(0) \quad C] \in \mathbb{K}^{m \times (m+n)}$ has full rank, where $C$ is the coefficient of degree $0$ of $\text{PFX}^{-d}$.

We remark that having both $PF = 0$ mod $X^d$ and the constant coefficient of $\text{PFX}^{-d}$ is equivalent to the single truncated identity $PF = CX^d$ mod $X^d$, where $t = (d_1 + 1, \ldots, d_t + 1)$.

As mentioned above, the details of the certification of the latter identity is deferred to Section 3, where we present more generally the certification for truncated products of the form $PF = G$ mod $X^t$.

Concerning Item (ii), the fact that the determinant of any basis of $\mathcal{A}_d(F)$ must divide $X^D$, where $D = |d|$, is well-known; we refer to [2, Sec. 2] for a more general result.

The combination of Items (i) and (iii) describes the set of matrices $P \in \mathbb{K}[X]^{m \times m}$ which are s-reduced and whose rows are in $\mathcal{A}_d(F)$.
thus $U(0)$ has rank $< m$. Since $[P \; Q] = U[A \; AF^{-d}]$, it directly follows that $[P(0) \; C]$ has rank $< m$. □

**Proof of Theorem 2.1.** If $P$ is an $s$-minimal basis of $\mathcal{A}_d(F)$, then by definition Items (i) and (iii) are satisfied. Since the rows of $X_{\text{max}}^d 1_m$ are in $\mathcal{A}_d(F)$ and $P$ is a basis, the matrix $X_{\text{max}}^d 1_m$ is a left multiple of $P$ and therefore the determinant of $P$ divides $X_{\text{max}}^d 1_m$, it is a nonzero monomial. Then, according to Lemma 2.4, $[P(0) \; C]$ has full rank. Conversely, if Items (ii) to (iv) are satisfied, then Lemma 2.4 states that $P$ is a basis of $\mathcal{A}_d(F)$; thus, if furthermore Item (i) is satisfied then $P$ is an $s$-minimal basis of $\mathcal{A}_d(F)$. □

**Algorithm 1: CertifApproxBasis**

**Input:**
- $d = (d_1, \ldots, d_n) \in \mathbb{Z}^n_{\geq 0}$
- matrix $F \in \mathbb{K}[X]^{m \times n}$ with cdeg($F$) $< d$
- shift $s = (s_1, \ldots, s_m) \in \mathbb{Z}^m$
- matrix $P \in \mathbb{K}[X]^{m \times m}$
- certificate matrix $C \in \mathbb{K}^{m \times n}$

**Output:** True if $P$ is an $s$-minimal basis of $\mathcal{A}_d(F)$ and $C$ is the constant term of $PF$'s $-d$-reduced otherwise True or False.

1. /* $P$ not in $s$-reduced form */
   $L \leftarrow$ the matrix in $\mathbb{K}^{m \times n}$ whose entry $i,j$ is the coefficient of degree $\text{cdeg}(P_{i,j}) - s_j$ of the entry $i,j$ of $P$
   If $L$ is not invertible then return False
2. /* rank($P(0)$) not full rank */
   $r \leftarrow \text{rank}(P(0) C)$ < $m$ then return False
3. /* det($P$) not a nonzero monomial */
   $S \leftarrow$ a finite subset of $\mathbb{K}$
   $\Delta \leftarrow \text{deg}(P) - |S|
   \alpha \leftarrow \text{chosen uniformly at random from } S$
   If det($P(\alpha)$) $\neq \text{det}(P(1))\alpha^\Delta$ then return False
4. /* certify truncated product */
   $PF \stackrel{\text{mod }}{=} CX$ $*$
   $t \leftarrow (d_1 + 1, \ldots, d_n + 1)$
   Return VerifTruncMatProd($t, P, F, CX$)

In order to provide a sharp estimate of the cost of Algorithm 1, we recall the best known bound with constant factors of the LQUP factorization of an $n \times n$ matrix over $\mathbb{K}$, which we use for computing ranks and determinants. Assuming $m \leq n$, we have:

$$C(m, n) = \left\lceil \frac{n!}{m!} \frac{1}{2^{d+1} - 2 - 2^{d+1}} \right\rceil MM(m)$$

operations in $\mathbb{K}$ [6, Lem. 5.1], where $MM(m)$ is the cost for the multiplication of $m \times m$ matrices over $\mathbb{K}$.

**Proposition 2.5.** Algorithm 1 uses at most $3\text{Size}(P) + 2m(D + \text{max}(d)) + 3C(m, m) + C(m, m + n)$ $+ (4m + 1)n + 4\log_2(Dd_1 \cdots d_n)$ operations in $\mathbb{K}$, where $D = |d|$. It is a false-biased Monte Carlo algorithm. If $P$ is not an $s$-minimal basis of $\mathcal{A}_d(F)$, then the probability that it outputs True is less than $\frac{D+1}{8S}$, where $S$ is the finite subset of $\mathbb{K}$ from which random field elements are drawn.

**Proof.** By definition, $P$ is $s$-reduced if and only if its $s$-leading matrix $L$ computed at Step 1 is invertible. Thus, Step 1 correctly tests the property in (i) of Theorem 2.1. It uses at most $C(m, m)$ operations in $\mathbb{K}$. Furthermore, Step 2 correctly tests the first part of Item (iv) of Theorem 2.1 and uses at most $C(m, m + n)$ operations.

Step 3 performs a false-biased Monte Carlo verification of Item (ii) of Theorem 2.1. Indeed, since $P$ is $s$-reduced (otherwise the algorithm would have exited at Step 1), we know from [16, Sec. 6.3.2] that deg(det($P$)) $= \Delta = \text{deg}_{\mathcal{A}_d}(P) - |S|$. Thus, det($P$) is a nonzero monomial if and only if det($P$) $= \text{det}(P(1))\alpha^\Delta$. Step 3 tests the latter equality by evaluation at a random point $\alpha$. The algorithm only returns False if det($P(\alpha)$) $\neq \text{det}(P(1))\alpha^\Delta$, in which case det($P$) is indeed not a nonzero monomial. Furthermore, if we have det($P$) $\neq \text{det}(P(1))\alpha^\Delta$, then the probability that the algorithm fails to detect this, meaning that det($P(\alpha)$) $= \text{det}(P(1))\alpha^\Delta$, is at most $\frac{\Delta}{S}$. Since $\Delta \leq D$ according to [27, Thm. 4.1], this is also at most $\frac{D+1}{8S} < \frac{D+1}{8S}$.

The evaluations $P(\alpha)$ and $P(1)$ are computed using respectively at most $2(\text{Size}(P) - m^2)$ operations and at most $\text{Size}(P) - m^2$ additions. Then, computing the two determinants det($P(\alpha)$) and det($P(1)$) uses at most $2C(m, m) + 2m$ operations. Finally, computing det($P(1))\alpha^\Delta$ uses at most $2\log_2(\Delta) + 1 \leq 2\log_2(D) + 1$ operations.

Summing the cost bounds for the first three steps gives:

$$3(\text{Size}(P) - m^2) + 3C(m, m) + C(m, m + n) + 2m + 2\log_2(D) + 1 \leq 3\text{Size}(P) + 3C(m, m) + C(m, m + n) + 2\log_2(D).$$

(2)

Step 4 tests the identity $PF \stackrel{\text{mod }}{=} CX$, which corresponds to both Item (ii) of Theorem 2.1 and the second part of Item (iv). Proposition 3.2 ensures that:

- If the call to VerifTruncMatProd returns False, we have $PF \not\equiv CX$ mod $X^t$, and Algorithm 1 correctly returns False.

- If $PF \not\equiv CX$ mod $X^t$ holds, the probability that Algorithm 1 fails to detect this (that is, the call at Step 4 returns True) is less than $\frac{\text{max}(d+1)}{4S}$. A cost bound for Step 4 is given in Proposition 3.2, with a minor improvement for the present case given in Remark 3.3. Summing it with the bound in Eq. (2) gives a cost bound for Algorithm 1, which is bounded from above by that in the proposition.

Thanks to Theorem 2.1, the above considerations show that when the algorithm returns False, then $P$ is indeed not an $s$-minimal basis of $\mathcal{A}_d(F)$. On the other hand, if $P$ is not an $s$-minimal basis of $\mathcal{A}_d(F)$, the algorithm returns True if and only if one of the probabilistic verifications in Steps 3 and 4 take the wrong decision. According to the probabilities given above, this may happen with probability less than $\frac{\text{max}(d+1)}{8S}$. □

3 VERIFYING A TRUNCATED PRODUCT

In this section, we focus on the verification of truncated products of polynomial matrices, and we give the corresponding algorithm VerifTruncMatProd used in Algorithm 1.

Given a truncation order $t$ and polynomial matrices $P, F, G$, our goal is to verify that $PF \equiv G$ mod $X^t$ holds with good probability. Without loss of generality, we assume that the columns of $F$ and $G$ are already truncated with respect to the order $t$, that is, cdeg($F$) $< t$ and cdeg($G$) $< t$. Similarly, we assume that $P$ is truncated with respect to $\delta = \text{max}(t)$, that is, $\text{deg}(P) < \delta$. 

with regards to truncation. A similar issue was tackled in [13] which computes the truncated product \(uA\). Thus, altogether the probability that
\[
\text{uPF rem } X^i(\alpha) = \alpha^{t-1} \sum_{k=0}^{i-1} c_{i-1-k} f_k,
\]
where we define, for \(0 \leq k < \delta\),
\[
c_k = (\text{uPF rem } X^{k+1})(\alpha^{-1}) = \sum_{i=0}^{k} p_{k-i} \alpha^{-i} \in \mathbb{K}^{1\times m}.
\]
These identities give an algorithm to compute the truncated product evaluation \((\text{uPF rem } X^i)(\alpha)\), which we sketch as follows:

- apply Horner’s method to the reversal of \(uA\) rem \(X^i\) at the point \(\alpha^{-1}\), storing the intermediate results which are exactly the \(t\) vectors \(c_0, \ldots, c_{\delta-1}\);
- compute the scalar products \(\lambda_k = c_{i-1-k} f_k\) for \(0 \leq k < t\);
- compute \(\alpha^{-t-1}\) and then \(\alpha^{-t-1} \sum_{0 \leq k < \delta} \lambda_k\).

The last step gives the desired evaluation according to Eq. (3). In our case, this will be applied to each column \(f = F_{s,j}\) for \(1 \leq j \leq n\). We will perform the first item only once to obtain the \(\delta\) vectors \(c_0, \ldots, c_{\delta-1}\), since they do not depend on \(f\).

**Proposition 3.2. Algorithm 2 uses at most**

\[2\text{Size}(P) + (6m + 1)\lceil t \rceil + 2n \log_2(\delta) \in O(\text{Size}(P) + m\lceil t \rceil + n \log_2(\delta))\]

operations in \(\mathbb{K}\), where \(\delta \leq \lceil t \rceil\) is the largest of the truncation orders. It is a false-biased Monte Carlo algorithm. If \(PF \neq G\) mod \(X^i\), the probability that it outputs True is less than \(\frac{5}{9\delta}\), where \(S\) is the finite subset of \(\mathbb{K}\) from which random field elements are drawn.

**Proof.** The discussion above shows that this algorithm correctly computes \(\{e_j\}_{1 \leq j \leq n} = uA(\alpha)\) and \(\{e_j'\}_{1 \leq j \leq n} = (\text{uPF rem } X^i)(\alpha)\). If it returns False, then there is at least one \(j\) for which \(e_j' \neq e_j\), thus we must have \(uA\) \# \(uG\) and therefore \(PF \neq G\) mod \(X^i\). Besides, the algorithm correctly returns True if \(PF = G\) mod \(X^i\).

The analysis of the probability of failure for the algorithm returns True while \(PF \neq G\) mod \(X^i\) is a direct consequence of Lemma 3.1.

Step 2 uses at most \(2\text{Size}(P) + (2m - 1)\lceil t \rceil\) operations in \(\mathbb{K}\). The Horner evaluations at Steps 3 and 4 require at most \(2\lceil \frac{t}{n} \rceil - n\) and at most \(1 + 2m(\delta - 1)\) operations, respectively. Now, we consider the \(j\)-th iteration of the loop at Step 5. The scalar products \(\lambda_k = \sum_{0 \leq k < \delta} f_k\) are computed using at most \((2m - 1) t_j\) operations; the sum and multiplication by \(\alpha^{t-1}\) giving \(e_j'\) use at most \(t_j + 2\log_2(t_j - 1)\) operations. Summing over \(1 \leq j \leq n\), this gives a total of at most
2m|t| + 2 \log_2((t_1 - 1) \cdots (t_n - 1)) operations for Step 5. Finally, Step 6 uses at most n comparisons of two field elements. Summing these bounds for each step yields the cost bound

2\text{Size}(P) + (4m + 1)|t| + 2m(\delta - 1) - n + 2 \log_2((t_1 - 1) \cdots (t_n - 1)),

which is at most the quantity in the proposition. □

In the certification of approximant bases, we want to verify a truncated matrix product in the specific case where each entry in the column of G is simply zero or a monomial of degree \(t_j - 1\). Then, a slightly better cost bound can be given, as follows.

**Remark 3.3.** Assume that \(t = (t_1, \ldots, t_n) \in \mathbb{Z}^n_{>0}\), matrix \(P \in \mathbb{K}[X]_{m \times m}\) such that \(\text{deg}(P) < \delta = \text{max}(t)\), matrix \(F = [f_{ij}] \in \mathbb{K}[X]_{m \times m}^{n \times n}\) with \(\text{deg}(F) < t\), and matrix \(G \in \mathbb{K}[X]_{m \times n}\) with \(\text{deg}(G) < t\).

Output: True if \(PF = G \mod X^\delta\), otherwise True or False.

1. /* Main objects for verification */
   \(S \leftarrow \) a finite subset of \(\mathbb{K}\)
   \(\alpha \leftarrow \) element of \(\mathbb{K}\) chosen uniformly at random from \(S\)
   \(u \leftarrow \) vector in \(\mathbb{K}^{\times m}\) with entries chosen uniformly and independently at random from \(S\)

2. /* Freivalds: row dimension becomes 1 */
   \(p \leftarrow uP\) // in \(\mathbb{K}[X]^{1 \times m}\), degree < \(\delta\)
   \(g \leftarrow uG\) // in \(\mathbb{K}[X]^{1 \times n}\), \(\text{deg}(g) < t\)

3. /* Evaluation of right-hand side: \(uPF \mod X^\delta\) */
   write \(g = [g_1, \ldots, g_n] \) with \(g_j \in \mathbb{K}[X]\) of degree < \(t\)
   for \(j\) from 1 to \(n\):
   \(e_j \leftarrow g_j(\alpha)\)

4. /* Evaluation of left-hand side: \(uPF \mod X^\delta\) */
   write \(p = \sum_{0 \leq k < \delta} p_k X^k\) with \(p_k \in \mathbb{K}^{1 \times m}\)
   \(c_0 \leftarrow p_0\)
   for \(k\) from 1 to \(\delta - 1\):
   \(c_k \leftarrow c_k \cdot g_k(\alpha)\)

5. /* Freivalds: column dimension becomes 1 */
   write \(F_{\alpha} = \sum_{0 \leq k < \delta} f_{i,k} X^k\)
   \((\lambda_k)_{0 \leq k < \delta} \leftarrow c_k - 1 \cdot \lambda_k\)
   \(e_j' \leftarrow \alpha^{-1} \sum_{0 \leq k < t_j} \lambda_k\)
   for \(j\) from 1 to \(n\):
   if \(e_j' = 0\) then return False
   else return True

4 COMPUTING THE CERTIFICATE

4.1 Context

In this section, we show how to efficiently compute the certificate \(C \in \mathbb{K}^{m \times n}\), which is the term of degree \(0\) of the product \(PF - d\), whose entries are Laurent polynomials (they are in \(\mathbb{K}[X]\) if and only if the rows of \(P\) are approximants). Equivalently, the column \(C_{\alpha,j}\) is the term of degree \(d_j\) of the column \(j\) of \(PF\), where \(d = (d_1, \ldots, d_n)\).

We recall the notation \(D = d_1 + \cdots + d_n\). Note that, without loss of generality, we may truncate \(P\) so that \(\text{deg}(P) \leq \text{max}(d)\).

For example, suppose that the dimensions and the order are balanced: \(m = n = (D/m, \ldots, D/m)\). Then, \(C \in \mathbb{K}^{m \times n}\) is the coefficient of degree \(D/m\) of the product \(PF\), where \(P\) and \(F\) are \(m \times m\) matrices over \(\mathbb{K}\). Thus, \(C\) can be computed using \(D/m\) multiplications of \(m \times m\) matrices over \(\mathbb{K}\), at a total cost \(O(m^{\omega-1}D)\).

Going back to the general case, the main obstacle to obtain similar efficiency is that both the degrees in \(P\) and the order \(d\) (hence the degrees in \(F\)) may be unbalanced. Still, we have \(\text{deg}(F) < \text{deg}(P)\) with sum \(|d| = D\) and, as stated in the introduction, we may assume that either \(|\text{deg}(P)| \in O(D)\) or \(|\text{deg}(P)| \leq D\) holds. In this context, both \(F\) and \(P\) are represented by \(O(mD)\) field elements.

We will generalize the method above for the balanced case to this general situation with unbalanced degrees, achieving the same cost \(O(m^{\omega-1}D)\). As a result, computing the certificate \(C\) has negligible cost compared to the fastest known approximant basis algorithms. Indeed, the latter are in \(O(m^{\omega-1}D)\), involving logarithmic factors in \(D\) coming both from polynomial arithmetic and from divide and conquer approaches. We refer the reader to [28, Thm. 5.3] and [15, Thm. 1.4] for more details on these logarithmic factors.

We first remark that \(C\) can be computed by naive linear algebra using \(O(m^2D)\) operations. Indeed, writing \(\text{rdeg}(P) = (r_1, \ldots, r_m)\), we have the following explicit formula for each entry in \(C\): 

\[
C_{i,j} = \sum_{k=1}^{\min(r_i, d_j)} P_{i,k} \cdot P_{i,j,k} \cdot X^{j-k},
\]

where \(P_{i,k}\) is the coefficient of degree \(k\) of the row \(i\) of \(P\) and similar notation is used for \(F\). Then, since \(\min(r_i, d_j) \leq d_j\), the column \(C_{\alpha,j}\) is computed via \(md_j\) scalar products of length \(m\), using \(O(m^2d_j)\) operations. Summing this for \(1 \leq j \leq n\) yields \(O(m^2D)\).

This approach considers each column of \(F\) separately, allowing us to truncate at precision \(d_j + 1\) for the column \(j\) and thus to rule out the issue of the unbalancedness of the degrees in \(P\). However, this also prevents us from incorporating fast matrix multiplication. In our efficient method, we avoid considering columns or rows separately, while still managing to handle the unbalancedness of the degrees in both \(P\) and \(F\). Our approach bears similarities with algorithms for polynomial matrix multiplication with unbalanced degrees (see for example [31, Sec. 3.6]).

4.2 Sparsity and degree structure

Below, we first detail our method assuming \(|\text{rdeg}(P)| \in O(D)\); until further notice, \(\gamma \geq 1\) is a real number such that \(|\text{rdeg}(P)| \leq \gamma D\).

To simplify the exposition, we start by replacing the tuple \(d\) by the uniform bound \(d = \text{max}(d)\). To achieve this, we consider the matrix \(H = FX^{d-d}\), where \(d\) and \(d\) stands for \((d-d_1, \ldots, d-d_n)\); then, \(C\) is the coefficient of degree \(d\) in \(PH\).
Since $\text{cdeg}(F) < d$, we have $\text{deg}(H) < d$. The fact that $F$ has column degree less than $d$ translates into the fact that $H$ has column valuation at least $d - d$ (and degree less than $d$); like $F$, this matrix $H$ is represented by $mD$ field elements. Recalling the assumption $\text{deg}(P) \leq d$, we can write $P = \sum_{k=d}^{\infty} P_k X^k$ and $H = \sum_{k=d}^{\infty} H_k X^k$, where $P_k \in \mathbb{K}^{m \times mD}$ and $H_k \in \mathbb{K}^{m \times m}$ for all $k$ (note that $H_d = 0$).

Then, our goal is to compute the matrix

$$C = \sum_{k=1}^{d} P_k H_{d-k}.$$  

The essential remark to design an efficient algorithm is that each matrix $P_k$ has only few nonzero rows when $k$ becomes large, and each matrix $H_{d-k}$ has only few nonzero columns when $k$ becomes large. To state this formally, we define two sets of indices, for the rows of degree at least $k$ in $P$ and for the orders at least $k$ in $d$:

$$\mathcal{R}_k = \{i \in [1, \ldots, m] \mid \text{rdeg}(P_{i,*}) \geq k\},$$

$$\mathcal{D}_k = \{j \in [1, \ldots, n] \mid d_j \geq k\}.$$  

The latter corresponds to the set of indices of columns of $F$ which are allowed to have degree $\geq k - 1$ or, equivalently, to the set of indices of columns of $H$ which are allowed to have valuation $\leq d - k$.

**Lemma 4.1.** For a given $k \in [1, \ldots, d]$: if $i \notin \mathcal{R}_k$, then the row $i$ of $P_k$ is zero; if $j \notin \mathcal{D}_k$, then the column $j$ of $H_{d-k}$ is zero. In particular, $P_k$ has at most $|\mathcal{R}_k| \leq \gamma d/k$ nonzero rows and $H_{d-k}$ has at most $|\mathcal{D}_k| \leq D/k$ nonzero columns.

**Proof.** The row $i$ of $P_k$ is the coefficient of degree $k$ of the row $i$ of $P$. If it is nonzero, we must have $i \in \mathcal{R}_k$. Similarly, the column $j$ of $H_{d-k}$ is the coefficient of degree $d-k$ of the column $j$ of $H = FX^d$. If it is nonzero, we must have $d-k \geq d-j$, hence $j \in \mathcal{D}_k$.

The upper bounds on the cardinalities of $\mathcal{R}_k$ and $\mathcal{D}_k$ follow by construction of these sets: we have $k \cdot |\mathcal{D}_k| \leq |d| = D$, and also $k \cdot |\mathcal{R}_k| \leq |\text{rdeg}(P)|$ with $|\text{rdeg}(P)| \leq \gamma D$ by assumption. \hfill $\square$

### 4.3 Algorithm and cost bound

Following Lemma 4.1, in the computation of $C$ based on Eq. (6) we may restrict our view of $P_k$ to its submatrix with rows in $\mathcal{R}_k$, and our view of $H_k$ to its submatrix with columns in $\mathcal{D}_k$. For example, if $k > \gamma D/m$ and $d > D/n$, the matrices in the product $P_k H_k$ have dimensions at most $\lceil \gamma D/k \rceil \times m$ and $m \times \lceil D/k \rceil$. These remarks on the structure and sparsity of $P_k$ and $H_k$ lead us to Algorithm 3.

**Proposition 4.2.** Algorithm 3 is correct. Assuming that $m \in O(D)$ and $|\text{rdeg}(P)| \in O(D)$, where $D = |d|$, it uses $O(m^{-\omega+1} D)$ operations in $\mathbb{K}$ if $n \leq m$ and $O(m^{-\omega+1} D \log(n/m))$ operations in $\mathbb{K}$ if $n > m$.

**Proof.** For the correctness, note that for all $j$ the coefficient of degree $d_j - k$ of $F_{k,j}$ is the coefficient of degree $d-k$ of $H_{d-k}$. Thus, using notation from Section 4.2, the matrix $B$ at the iteration $k$ of the loop is exactly the submatrix of $H_{d-k}$ of its columns in $\mathcal{D}_k$. Therefore, the loop in Algorithm 3 simply applies Eq. (6), discarding from $P_k$ and $F_{k,j}$ rows and columns which are known to be zero.

Now, we estimate the cost of updating $C$ at each iteration of the loop. Precisely, the main task is to compute $AB$, where the matrices $A$ and $B$ have dimensions $\# \mathcal{R} \times m$ and $m \times t$. Then, adding this product to the submatrix $C_{\mathcal{R},\mathcal{D}}$ only costs $\# \mathcal{R} \cdot t$ additions in $\mathbb{K}$.

Consider $\gamma = [\text{rdeg}(P)/D] \geq 1$ (indeed, if $\text{rdeg}(P) = 0$, then $P$ is constant and $C = 0$). By Lemma 4.1, at the iteration $k$ we have $\# \mathcal{R} \leq \min(m, \gamma D/k)$ and $t = \# \mathcal{D} \leq \min(n, D/k)$. We separate the cases $n \leq m$ and $n > m$, and we use the bound $\lceil \gamma D/m \rceil \in O(D/m)$, which comes from our assumptions $m \in O(D)$ and $\gamma \in O(1)$.

First, suppose $n \leq m$. At the iterations $k < \lceil \gamma D/m \rceil$ the matrices $A$ and $B$ both have dimensions at most $m \times m$, hence their product can be computed in $O(m^{\omega+1})$ operations. These iterations have a total cost of $O(m^{\omega+1} \lceil \gamma D/m \rceil) \subseteq O(m^{\omega+1} D)$. At the iterations $k \geq \lceil \gamma D/m \rceil$, $A$ and $B$ have dimensions at most $(\gamma D/k) \times m$ and $m \times (D/k)$, with $D/k \leq \gamma D/k \leq m$; computing their product costs $O((D/k)^{\omega+1} m) \subseteq O(m D^{\omega+1} k^{1-\omega})$. Thus, the total cost for these iterations is in

$$O(m D^{\omega+1} \sum_{k=\gamma D/m}^{\max(d)} k^{1-\omega}) \subseteq O(m D^{\omega+1} (\gamma D/m)^{1-\omega} \sum_{k=\gamma D/m}^{\infty} n^{2-\omega} \leq 1).$$

For the first inclusion, we apply Lemma 4.3 with $\mu = [\gamma D/m], \nu = \max(d)$, and $\theta = 1 - \omega$. For the second, the sum is finite since $2^{2-\omega} < 1$. Hence Algorithm 3 costs $O(m^{\omega+1} D)$ in the case $n \leq m$.

Now, suppose $n > m$. At the iterations $k < [D/n]$, $A$ and $B$ have dimensions at most $m \times m$ and $m \times n$, hence their product can be computed in $O(m^{\omega+1} n)$. The total cost is in $O(m^{\omega+1} D)$ since there are $[D/n] - 1 < D/n$ iterations (with $n \leq D$ by definition). For the iterations $k \geq [D/n]$, we repeat the analysis done above for the same values of $k$: these iterations cost $O(m^{\omega+1} D)$ here as well.

Finally, for the iterations $[D/n] \leq k < \lceil \gamma D/m \rceil$, $A$ and $B$ have dimensions at most $m \times m$ and $m \times (D/k)$, with $D/k \leq n$. Thus the product $AB$ can be computed in $O(m^{\omega+1} + m^{\omega+1} D/k)$ operations. Summing the term $m^{\omega+1}$ over these $O(D/m)$ iterations yields the cost $O(m^{\omega+1} D)$. Summing the other term gives the cost $O(m^{\omega+1} D \log(n/m))$, by the last claim of Lemma 4.3, we have

$$\sum_{k=[D/n]}^{\lceil \gamma D/m \rceil} k^{1-\omega} \leq 1 + \log_2 \left( \frac{[\gamma D/m][\gamma D/m]}{[D/n]} \right) \leq 1 + \log_2(\gamma n/m).$$

Adding the costs of the three considered sets of iterations, we obtain the announced cost for Algorithm 3 in the case $n > m$ as well. \hfill $\square$
Lemma 4.3. Given integers $0 < \mu < \nu$ and a real number $\theta \leq 0$,
\[
\sum_{k=\mu}^{\nu} k^\theta \leq \nu^{\theta+1} \sum_{i=0}^{\ell-1} \nu^{i(\theta+1)},
\]
holds, where $\ell = \lceil \log_2(\nu/\mu) \rceil + 1$. In particular, $\sum_{k=\mu}^{\nu} k^{-1} \leq \ell$.

Proof. Note that $\ell$ is chosen such that $2^\ell \mu - 1 \geq \nu$. Then, the upper bound is obtained by splitting the sum as follows:
\[
\sum_{k=\mu}^{\nu} k^\theta \leq \sum_{k=\mu}^{2\mu} k^\theta \leq \sum_{k=\mu}^{2\mu} \sum_{i=0}^{\ell-1} \nu^{i(\theta+1)},
\]
where the second inequality comes from the fact that $x \mapsto x^\theta$ is decreasing on the positive real numbers.

Finally, we describe minor changes in Algorithm 3 to deal with the case of small average column degree $\text{cdeg}(P) \in O(D)$; precisely, we replace the assumption $|\text{cdeg}(P)| \leq yD$ by $|\text{cdeg}(P)| \leq yD$. Then, instead of the set $R_k$ used above, we rather define
\[
C_k = \{ j \in \{ 1, \ldots, m \} \mid \text{cdeg}(P_{x,j}) \geq k \}.
\]
Then we have the following lemma, analogous to Lemma 4.1.

Lemma 4.4. For $k \in \{ 1, \ldots, m \}$ and $j \notin C_k$, the column $j$ of $P_k$ is zero. In particular, $P_k$ has at most $\#C_k \leq yD$ nonzero columns.

Thus, we can modify Algorithm 3 to take into account the column degree of $P$ instead of its row degree. This essentially amounts to redefining the matrices $A$ and $B$ in the loop as follows:

- $A \in \mathbb{R}^{m \times \#C_k}$ is the coefficient of degree $k$ of $P_k$.
- $B \in \mathbb{R}^{\#C_k \times n}$ is such that for all $i \in C_k$ and $1 \leq j \leq t$, $B_{i,j}$ is the coefficient of degree $d_j - k$ of $P_{x,j}$.

These modifications have obviously no impact on the correctness. Furthermore, it is easily verified that the same cost bound holds since we obtain a similar matrix multiplication cost at each iteration.

5 Perspectives

As noted in the introduction, our certificate is almost optimal since we can verify it at a cost $O(mD + m^{\omega-1}(m + n))$ while the input size is $mD$. One should notice that the extra term $O(m^{\omega-1}(m + n))$ corresponds to certifying problems of linear algebra over $\mathbb{R}$, namely the rank and the determinant. These could actually be dealt with in $O(m(m + n))$ operations using interactive certificates built upon the results in [7, 9, 18], thus yielding an optimal certificate. Still, for practical applications, our simpler certification should already be significantly faster than the approximant basis computation, since the constants involved in the cost are small as we have observed in our estimates above. We plan to confirm this for the approximant bases implementations in the LinBox library.

Finally, our verification protocol needs $(m + 2) \log_2(\#S)$ random bits, yielding a probability of failure less than $\frac{D_2+1}{D_2}$. The majority of these bits is required by Algorithm 2 when choosing $m$ random elements for the vector $u$. As proposed in [19], it may be worthwhile to pick a single random value $\zeta$ and to use $u = [1 \; \zeta \; \cdots \; \zeta^{m-1}]$. In the case where $\max(d) < D/2$, this choice would not affect the probability of failure while decreasing the number of random bits to $3 \log_2(\#S)$. In particular, at the price of the same number of bits as we currently use in our algorithm, we could run our verification $(m + 2)/3$ times and decrease the probability of failure to $(D_2+1)^{-2}$. 

References

[1] B. Beckermann and G. Labahn. 1994. A Uniform Approach for the Fast Computation of Matrix-Type Padé Approximants. SIAM J. Matrix Anal. Appl. 15, 3 (1994), 804–823.
[2] B. Beckermann and G. Labahn. 1997. Recursiveness in matrix rational interpolation problems. J. Comput. Appl. Math. 77, 1–2 (1997), 5–34.
[3] B. Beckermann, G. Labahn, and G. Villard. 1999. Shifted Normal Forms of Polynomial Matrices. In ISSAC’99 ACM, 189–196.
[4] D. Coopermuth and S. Winograd. 1990. Matrix multiplication via arithmetic progressions. J. Symbolic Comput. 9, 3 (1990), 251–280.
[5] R. A. DeMillo and R. J. Lipton. 1978. A Probabilistic Remark on Algebraic Program Testing. Inform. Process. Lett. 7, 4 (1978), 193–195.
[6] J.-G. Dumas, P. Giorgi, and C. Pernet. 2008. Dense Linear Algebra over Word-Size Prime Fields: The FFLAS and FFPACK Packages. ACM Trans. Math. Softw. 35, 3, Article 19 (2008), 42 pages.
[7] J.-G. Dumas and E. Kaltofen. 2014. Essentially Optimal Interactive Certificates in Linear Algebra. In ISSAC’14 ACM, 146–153.
[8] J.-G. Dumas, E. Kaltofen, E. Thomé, and G. Villard. 2016. Linear Time Interactive Certificates for the Minimal Polynomial and the Determinant of a Sparse Matrix. In ISSAC’16 ACM, 199–206.
[9] J.-G. Dumas, D. Lucas, and C. Pernet. 2017. Certificates for Triangular Equivalence and Rank Profiles. In ISSAC’17 ACM, 135–140.
[10] Julius Farkas. 1902. Theorie der einfachen Ungleichungen. J. Reine Angew. Math. 124 (1902), 1–27. http://eudml.org/doc/149129
[11] B. P. Frevelds. 1979. Fast probabilistic algorithms. In Mathematical Foundations of Computer Science, Vol. 74. Springer Berlin Heidelberg, 57–69.
[12] J. von zur Gathen and J. Gerhard. 2013. Modern Computer Algebra (third edition). Cambridge University Press.
[13] P. Giorgi. 2017. Certification of Polynomial Middle Product. (2017). Available at https://hal-lirmm.ccsd.cnrs.fr/lirmm-01538452 (accessed in May 2018).
[14] P. Giorgi, C.-P. Jeannerod, and G. Villard. 2003. On the complexity of polynomial matrix computations. In ISSAC’03 ACM, 135–142.
[15] C.-P. Jeannerod, V. Neiger, E. Schost, and G. Villard. 2016. Fast computation of minimal interolation bases in Popov form for arbitrary shifts. In ISSAC’16 ACM, 295–302.
[16] T. Kaivala. 1980. Linear Systems: Prentice-Hall.
[17] E. Kaltofen, B. Li, Z. Yang, and E. Zhi. 2012. Exact certification in global polynomial optimization via sums-of-squares of rational functions with rational coefficients. J. Symbolic Comput. 47, 1 (2012), 1–15.
[18] E. Kaltofen, M. Nebeving, and D. Saunders. 2011. Quadratic-time Certificates in Linear Algebra. In ISSAC’11 ACM, 171–176.
[19] T. Kimbrel and R. K. Sinha. 1993. A probabilistic algorithm for verifying matrix products using $O(m^2)$ time and $\log_2(m+n)+O(1)$ random bits. Inform. Process. Lett. 52, 2 (1995), 107–110.
[20] T. Kleinjung, C. Diem, A. K. Lenstra, C. Priplata, and C. Stahlke. 2017. Computation of a 768-Bit Prime Field Discrete Logarithm. In Eurocrypt 2017. Springer International Publishing, 185–201.
[21] F. Le Gall. 2014. Powers of Tensors and Fast Matrix Multiplication. In ISSAC’14 ACM, 296–303.
[22] R. Motwani and P. Raghavan. 1995. Randomized Algorithms. Cambridge University Press, New York, NY, USA.
[23] T. Mulders and A. Storjohann. 2003. On lattice reduction for polynomial matrices. J. Symbolic Comput. 35 (2003), 377–401. Issue 4.
[24] V. Neiger. 2016. Bases of relations in one or several variables: fast algorithms and applications. Ph.D. Dissertation. École Normale Supérieure de Lyon.
[25] J. T. Schwartz. 1980. Fast Probabilistic Algorithms for Verification of Polynomial Identities. J. ACM 27, 4 (1980), 761–777.
[26] A. Storjohann. 2006. Notes on computing minimal approximant bases. In Challenges in Symbolic Computation Software (Dugasith Seminar Proceedings).
[27] M. Van Barel and A. Bultheel. 1992. A general module theoretic framework for vector $M$-Padé and matrix rational interpolation. Numer. Algorithms 3 (1992), 451–462.
[28] W. Zhou and G. Labahn. 2012. Efficient Algorithms for Order Basis Computation. J. Symbolic Comput. 47, 7 (2012), 793–819.
[29] W. Zhou and G. Labahn. 2013. Computing Column Bases of Polynomial Matrices. In ISSAC’13 ACM, 379–386.
[30] W. Zhou and G. Labahn. 2014. Unimodular Completion of Polynomial Matrices. In ISSAC’14 ACM, 413–420.
[31] W. Zhou, G. Labahn, and A. Storjohann. 2012. Computing Minimal Nullspace Bases. In ISSAC’12 ACM, 366–373.
[32] R. Zippel. 1979. Probabilistic algorithms for sparse polynomials. In EUROSAM’79 (LNCS), Vol. 72. Springer, 216–226.