PERPETUAL AMERICAN OPTIONS WITH ASSET-DEPENDENT DISCOUNTING

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ABSTRACT. In this paper we consider the following optimal stopping problem

\[ V_\omega^A(s) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_s [e^{-\int_0^\tau \omega(S_t)dt} g(S_\tau)], \]

where the process \( S_t \) is a jump-diffusion process, \( \mathcal{T} \) is a family of stopping times and \( g \) and \( \omega \) are fixed payoff function and discounting function, respectively. In a financial market context, if \( g(s) = (K - s)^+ \) or \( g(s) = (s - K)^+ \) and \( \mathbb{E} \) is the expectation taken with respect to a martingale measure, \( V_\omega^A(s) \) describes the price of a perpetual American option with a discount rate depending on the value of the asset process \( S_t \).

If \( \omega \) is a constant, the above problem produces the standard case of pricing perpetual American options. In the first part of this paper we find sufficient conditions for the convexity of the value function \( V_\omega^A(s) \).

This allows us to determine the stopping region as a certain interval and hence we are able to identify the form of \( V_\omega^A(s) \). We also prove a put-call symmetry for American options with asset-dependent discounting. In the case when \( S_t \) is a geometric Lévy process we give exact expressions using the so-called omega scale functions introduced in [49]. We prove that the analysed value function satisfies HJB and we give sufficient conditions for the smooth fit property as well. Finally, we analyse few cases in detail performing extensive numerical analysis.

KEYWORDS. American option • Lévy process • diffusion • Black-Scholes market • optimal stopping problem • convexity

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1. Introduction

In this paper the uncertainty associated with the stock price $S_t$ is described by a jump-diffusion process defined on a complete filtered risk-neutral probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, where $\mathcal{F}_t$ is a natural filtration of $S_t$ satisfying the usual conditions and $\mathbb{P}$ is a risk-neutral measure under which the discounted (with respect to the risk-free interest rate) asset price process $S_t$ is a local martingale. We point out that, as noted in [24], introducing jumps into the model, implies lost of completeness of the market which results in the lack of a uniqueness of equivalent martingale measure. The main goal is the analysis of the situation in this paper with respect to the risk-free interest rate) asset price process defined on a complete filtered risk-neutral probability space.

Later we focus on the case when $S_t$ is a geometric spectrally negative Lévy process. In this case, using the fluctuation theory of Lévy processes, we identify value function (1) in terms of the omega scale functions.

A financial model is considered, that is if $g(s) = (K - s)^+$ and $\mathbb{P}$ is a martingale measure, this function can be interpreted as the value function of the perpetual American option with functional discounting $\omega$ and payoff function $g$. It is a function of the perpetual American option’s value function with constant discount rate $r$.

The main objective of this paper is to find a closed expression of (1) and identify the optimal stopping rule $\tau^*$ for which supremum is attained. To do this we start from proving in Theorem 2 an inheritance of convexity property from the payoff function to the value function. This preserves the convexity by the solution of a certain obstacle problem.

Using this observation and classical optimal stopping theory presented e.g. in [50] one can identify the optimal stopping region as an interval $[\bar{t}, u^*]$, that is, $\tau^* = \inf\{t \geq 0 : S_t \in [\bar{t}, u^*]\}$. Hence, in general, one can obtain in this case a double continuation region.

Later we focus on the case when $S_t$ is a geometric spectrally negative Lévy process. In this case, using the fluctuation theory of Lévy processes, we identify value function (1) in terms of the omega scale functions introduced in [19].

For optimal stopping problem (1) we formalise the classical approach here as well. In particular, in Theorem 6 we prove that the value function $V^\omega_A(s)$ is the unique solution of a certain Hamiltonian-Jacobi-Bellman (HJB) system. Moreover, in the case of geometric Lévy process of the asset price $S_t$, we prove that the regularity of 1 for the half-lines $(-\infty, 1)$ and $(1, +\infty)$ gives the smooth fit property at the ends of the stopping region. In Theorem 8 we show the put-call symmetry as well.

These theoretical results allow us to find the price of the perpetual American option with asset-dependent discounting for some particular cases. We take for example a put option, that is $g(s) = (K - s)^+$, and a geometric Brownian motion for the asset price $S_t$. We model $S_t$ also by the geometric Lévy process with exponentially distributed downward jumps. We analyse various discounting functions $\omega$. In Section 3 we provide extensive numerical analysis.

The discount rate changing in time or a random discount rate are widely used in pricing derivatives in financial markets. They have proved to be valuable and flexible tools to identify the value of various options.
Usually, either the interest rate is independent from the asset price or this dependence is introduced via taking correlation between gaussian components of these two processes. Our aim is slightly different. We want to understand an extreme case when we have strong, functional dependence between the interest rate and the asset price. For example, for the American put option, if the asset price is in 'higher' region one can expect that the interest rate will be higher as well. The opposite effect one expect for 'smaller' range of asset’s prices.

One can look at optimisation problem (1) from a wider perspective though. The killing by potential \( \omega \) has been known widely in physics and other applied sciences. Then (1) can be seen as a certain functional describing gain or energy and the goal is to optimise it by choosing some random time horizon. We focus here on financial applications only.

Our research methodology is based on combining the theory of partial differential equations with the fluctuation theory of Lévy processes.

To prove the convexity we start from proving in Theorem 15 the convexity of

\[
V_E^{\omega}(s, t) := \mathbb{E}_{s,t} \left[ e^{-\int_{t}^{T} \omega(S_w)dw} g(S_T) \right]
\]

for fixed time horizon \( T \), where \( \mathbb{E}_{s,t} \) is the expectation \( \mathbb{E} \) with respect to \( \mathbb{P} \) when \( S_t = s \). In the proof we follow the idea given by Ekström and Tysk in [34]. Namely, the value function \( V_E^{\omega}(s, t) \) given in (2) can be presented as the unique viscosity solution of a certain Cauchy problem for some second-order operator related to the generator of the process \( S_t \). In fact, applying similar arguments like in [58 Proposition 5.3] and [34 Lemma 3.1], one can show that, under some additional assumptions, this solution can be treated as the classical one. Then we can formulate the sufficient locally convexity preserving conditions for the infinitesimal preservation of convexity at some point. This characterisation is given in terms of a differential inequality on the coefficients of the considered operator. It also allows to prove the convexity of \( V_E^{\omega}(s, t) \).

Then, in Theorem 2 and Lemma 19 we apply the dynamic programming principle (see [33]) in order to generalise the convexity property of \( V_E^{\omega}(s, t) \) to the value function \( V^{\omega}(s) \).

Later we focus on the American put option, hence when \( g(s) = (K - s)^+ \) for some strike price \( K > 0 \). Using the convexity property mentioned above we can conclude that the optimal stopping rule is defined as the first entrance of the process \( S_t \) to the interval \([l, u]\), that is,

\[
\tau_{l,u} := \inf \{ t \geq 0 : S_t \in [l, u] \}.
\]

In the next step, one has to identify

\[
v^{\omega}_{A_{\tau_{l,u}}}(s, l, u) := \mathbb{E}_s \left[ e^{-\int_{0}^{\tau_{l,u}} \omega(S_w)dw} (K - S_{\tau_{l,u}})^+ \right]
\]

and take maximum over levels \( l \) and \( u \) to identify the optimal stopping rule \( \tau^* \) and to find the price \( V^{\omega}_{A_{\tau^*}}(s) \) of the American put option when \( S_0 = s \). This is done for the geometric spectrally negative Lévy process \( S_t = e^{X_t} \) where \( X_t \) is a spectrally negative Lévy process starting at \( X_0 = \log S_0 = \log s \). We recall that spectrally negative Lévy processes do not have positive jumps. Hence, in particular, our analysis could be used for Black-Scholes market where \( X_t \) is a Brownian motion with a drift. To execute this plan we express \( v^{\omega}_{A_{\tau_{l,u}}}(s, l, u) \) in terms of the laws of the first passage times and then we use the fluctuation theory developed in [49]. In the whole analysis the use of the change of measure technique developed in [54] is crucial as well. Optimal levels \( l^* \) and \( u^* \) of the stopping region \([l^*, u^*]\) and the price \( V^{\omega}_{A_{\tau^*}}(s) \) of the American put option could be found by application of the appropriate HJB equation. We prove this HJB equation and the smooth fit condition relying on the classical approach of [47] and [60].

Finally, to find the price of the American call option we prove the put-call symmetry in our set-up. The proof is based on the exponential change of measure introduced in [54].
We analyse in detail the Black-Scholes model and the case when a logarithm of the asset price is a linear drift minus compound Poisson process with exponentially distributed jumps and various discounting functions \( \omega \). The first example shows behaviour of the American prices in a gaussian and continuous market while the latter is to model the market including downward shocks in the assets’ behaviour. In this paper we present some numerical analysis for these two cases. In particular, we show how two different approaches, namely solving HJB equation, finding function \( v^{\omega}_{A^+}(s,l,u) \) and maximising it over \( l \) and \( u \), can be executed in daily practice.

Our paper seems to be the first one analysing the optimal problem of the form (1) in this generality for jump-diffusion processes. For the classical diffusion process Lamberton in [48] proved that the value function in (1) is continuous and can be characterised as the unique solution of a variational inequality in the sense of distributions. Another crucial paper for our considerations is [11] which introduced discounting via a positive continuous additive functional of the process \( S_t \) and used the technique of Bensoussan and Lions [12] to characterise the value function. Note that \( t \to \int_0^t \omega(S_w)dw \) is indeed additive functional. Another interesting paper of Rodosthenous and Zhang [59] who studied the optimal stopping of an American call option in a random time-horizon under a geometric spectrally negative Lévy model. The random time-horizon is modeled by Omega default clock which is in their case the first time when the occupation time of the asset price below a fixed level \( y \) exceeds an independent exponential random variable with mean \( 1/\varrho \). This corresponds to the special case of our discounting with \( \omega(s) = r + \varrho 1\{s \leq y\} \), where \( r \) is a risk-free interest rate.

The convexity of the value function and convexity preserving property, which is a key ingredient of our analysis, have been studied quite extensively, see e.g. [13, 14, 20, 32, 36, 37, 40, 41] for diffusion models, and [35, 39] for one-dimensional jump-diffusion models.

We model dynamics of the asset price in a financial market by the jump-diffusion process. The reason to take into account more general class of stochastic processes of asset prices than in the seminal Black-Scholes market is the empirical observation that the log-prices of stocks have a heavier left tail than the normal distribution, on which the seminal Black-Scholes model is founded. The introduction of jumps in the financial market dates back to [51], who added a compound Poisson process to the standard Brownian motion to better describe dynamics of the logarithm of stocks. Since then, there have been many papers and books working in this set-up, see e.g. [24, 61] and references therein. In particular, [24, Table 1.1, p. 29] gives many other reasons to consider this type of market. Apart from the classical Black-Scholes market one can consider the normal inverse Gaussian model of [53], the hyperbolic model of [30], the variance gamma model of [50], the CGMY model of [18], and the tempered stable process analysed in [15, 43]. American options in the jump-diffusion markets have been studied in many papers as well; see e.g. [1, 2, 4, 9, 15, 19, 21, 42, 52].

Identifying the solution of the optimal stopping problem by solving the corresponding HJB equation (as it is done in this paper as well) has been widely used; see [44, 56] for details. In the context of American options with constant discounting both methods of variational inequalities and viscosity solutions of the boundary value problems in the spirit of Bensoussan and Lions [12] are also well-known; see e.g. [17, 57, 58]. To determine the unknown boundary of stopping region usually the smooth fit condition is used; see e.g. [46, 47] for the geometric Lévy process of asset prices. As Lamberton and Mikou [47] and Kyprianou and Surya [46] showed the continuous fit is always satisfied but not necessary the smooth fit property. Therefore we focus on identifying the sufficient conditions for the smooth fit in our model which are generalisations of the classical ones. What we want to underline here is that using our approach (proving convexity and maximising over ends \( l \) and \( u \) of the stopping interval \([l, u]\)) one can avoid these calculations. Apart of this, the interval form of the stopping region (hence producing double-sided continuation region) is much more
rare. It comes from the fact that when at time $t = 0$ the discount rate is negative then it is worth to wait since discounting might increase the value of payoff. This phenomenon has been already observed for fixed negative discounting (see [6, 7, 8, 26, 63]) or in the case of American capped options with positive interest rate (see [16, 27]).

In this paper we also prove that in this general setting of asset-dependent discounting, one can express the price of the call option in terms of the price of the put option. It is called the put-call symmetry (or put-call parity). Our finding supplements [31, 38] who extend to the Lévy market the findings by [17]. An analogous result for the negative discount rate case was obtained in [6, 7, 8, 26]. A comprehensive review of the put-call duality for American options is given in [28]. We also refer to [29, Section 7] and other references therein for a general survey on the American options in the jump-diffusion model.

The paper is organised as follows. In Section 2 we introduce basic notations and assumptions that we use throughout the paper and we give the main results of this paper. In Section 3, we perform the numerical analysis for the case of put option and Black-Scholes market and the market with prices being modeled by geometric Lévy process with downward exponential jumps. Section 4 contains proofs of all main theorems. We put into Appendix proofs of auxiliary lemmas. The last section includes our concluding remarks.

2. Main results

2.1. Jump-diffusion process. In this paper we assume a jump-diffusion financial market defined formally as follows. On the basic probability space we define a couple $(B_t, v)$ adapted to the filtration $\mathcal{F}_t$, where $B_t$ is a standard Brownian motion and $v = v(dt, dz)$ is an independent of $B_t$ homogeneous Poisson random measure on $\mathbb{R}_+^+ \times \mathbb{R}$ for $\mathbb{R}_+^+ = [0, +\infty)$. Then the stock price process $S_t$ solves the following stochastic differential equation

$$dS_t = \mu(S_{t-}, t)dt + \sigma(S_{t-}, t)dB_t + \int_{\mathbb{R}} \gamma(S_{t-}, t, z)\tilde{v}(dt, dz),$$

where

- $\tilde{v}(dt, dz) = (v - q)(dt, dz)$ is a compensated jump martingale random measure of $v$,
- $v$ is a homogenous Poisson random measure defined on $\mathbb{R}_+^+ \times \mathbb{R}$ with intensity measure $q(dt, dz) = dt m(dz)$.

If additionally, the jump-diﬀusion process has ﬁnite activity of jumps, i.e. when

$$\lambda := \int_{\mathbb{R}} m(dz) < \infty,$$

then $N_t = v([0, t] \times \mathbb{R})$ is a Poisson process and $m$ can be represented as

$$m(dz) = \lambda E(e^{Y_i} - 1 | dz),$$

where $\{Y_i\}_{i \in \mathbb{N}}$ are i.i.d. random variables independent of $N_t$ with distribution $\mu_Y$. Note that $B_t$ and $N_t$ are independent of each other as well. When additionally $\mu(s, s) = \mu s$, $\sigma(s, t) = \sigma s$ and $\gamma(s, t, z) = sz$, then the asset price process $S_t$ is the geometric Lévy process, that is,

$$S_t = e^{X_t},$$

where $X_t$ is a Lévy process starting at $x = \log s$ with a triple $(\zeta, \sigma, \Pi)$ for

$$\zeta := \mu - \frac{\sigma^2}{2}, \quad \Pi(dx) := \lambda \mu_Y(dx).$$

This observation follows straightforward from Itô’s rule.
2.2. Assumptions. Before we present the main results of this paper, we state now the assumptions on the model parameters used later. We denote $\mathbb{R}^+ := (0, +\infty)$.

Assumptions (A)

(A1) The drift parameter $\mu: \mathbb{R}^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ and the diffusion parameter $\sigma: \mathbb{R}^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ are continuous functions, while the jump size $\gamma: \mathbb{R}^+ \times \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ is measurable and for each fixed $z \in \mathbb{R}$, the function $(s, t) \to \gamma(s, t, z)$ is continuous.

(A2) There exists a constant $C > 0$ such that

$$\mu^2(s, t) + \sigma^2(s, t) + \gamma^2(s, t, z) \leq Cs^2$$

for all $(s, t, z) \in \mathbb{R}^+ \times \mathbb{R}_0^+ \times \mathbb{R}$.

(A3) There exists a constant $C > 0$ such that

$$|\mu(s_2, t) - \mu(s_1, t)| + |\sigma(s_2, t) - \sigma(s_1, t)| + |\gamma(s_2, t, z) - \gamma(s_1, t, z)| \leq C|s_2 - s_1|$$

for all $(s, t, z) \in \mathbb{R}^+ \times \mathbb{R}_0^+ \times \mathbb{R}$.

(A4) There exists a constant $C > -1$ such that

$$\gamma(s, t, z) > Cs$$

for all $(s, t, z) \in \mathbb{R}^+ \times \mathbb{R}_0^+ \times \mathbb{R}$.

(A5) $g(s) \in C_{\text{pol}}(\mathbb{R}^+)$, where $C_{\text{pol}}(\mathbb{R}^+)$ denotes the set of functions of at most polynomial growth. $\omega(s)$ is bounded from below.

Assumptions (A1), (A2), (A3) guarantee that there exists a unique solution to (5). Moreover, (A2) and (A4) imply that

$$\mathbb{P}(S_t \leq 0 \text{ for some } t \in \mathbb{R}_0^+) = 0$$

which is a natural assumption since the process $S_t$ describes the stock price dynamics and its value has to be positive. Additionally, assumptions (A5) and (A6) give that $V_{\omega}^E(s, t)$ is finite.

Remark 1. Note that assumptions (A1)–(A4) are all satisfied for the geometric Lévy process.

2.3. Convexity of the value function. Our first main result concerns the convexity of the value function $V_{\omega}^E(s)$.

Theorem 2. Let Assumptions (A) hold. Assume that the payoff function $g$ is convex, $\omega$ is concave, the stock price process $S_t$ follows (5), and the following inequalities are satisfied

$$(8) \quad \frac{\partial^2 \gamma(s, t, z)}{\partial s^2} \geq 0,$$

$$(9) \quad \left( \frac{\partial^2 \mu(s, t)}{\partial s^2} - 2 \frac{d\omega(s)}{ds} \right) \frac{\partial V_{\omega}^E(s, t)}{\partial s} \geq 0,$$

where $V_{\omega}^E(s, t)$ is defined in (2). Then the value function $V_{\omega}^E(s)$ is convex as a function of $s$.

The proof of the above theorem is given in Section [3].

Remark 3. We give now sufficient conditions in terms of model parameters for (9) to be satisfied. If $S_t$ is the geometric Lévy process (hence $\mu(s, t) = \mu s$, $\sigma(s, t) = \sigma s$ and $\gamma(s, t, z) = sz$) then (8) is satisfied. Let additionally $g(s) = (K - s)^+$. Then our optimal stopping problem is equivalent to pricing American put option with functional discounting. If $\omega$ is increasing function then the function $s \to V_{\omega}^E(s, t)$ is decreasing.
Hence in this case condition \( \psi \) is satisfied as well. Concluding, if \( \omega \) is concave and increasing, then the value function of American put option in geometric Lévy market is convex as a function of the initial asset price.

2.4. American put option and the optimal exercise time. Assume now the particular case of \( \psi \) with the payoff function

\[
g(s) = (K - s)^{+},
\]

that is, the value function \( \psi_{\text{Put}}(s) \) gives the price of American put option. The value function for this special choice of payoff function is denoted by

\[
V_{\text{Put}}(s) := \sup_{\tau \in \mathcal{T}} \mathbb{E}_s \left[ e^{-\int_0^\tau \omega(S_u)\,du} (K - S_\tau) \right].
\]

Note that above we used the fact that the option will not be realised when it equals to zero, hence the plus in the payoff function could be skipped.

From [56, Thm. 2.7, p. 40] it follows that the optimal stopping rule is of the form

\[
\tau^* = \inf\{ t \geq 0 : V_{\text{Put}}(S_t) = (K - S_t) \}.
\]

From Theorem 2 we know that \( V_{\text{Put}}(s) \) is convex. Moreover, from the definition of the value function it follows that \( V_{\text{Put}}(s) \geq (K - s) \). Having both these facts in mind together with linearity of the payoff function, it follows that \( V_{\text{Put}}(s) \) and \( g \) can cross each other in at most two points. This observation produces straightforward the following main result. We recall that in (3) and (4) we introduced the entrance time \( \tau_{l,u} = \inf\{ t \geq 0 : S_t \in [l, u] \} \) into interval \([l, u]\) and the corresponding value function

\[
v_{\text{Put}}(s, l, u) = \mathbb{E}_s \left[ e^{-\int_{\tau_{l,u}} \omega(S_u)\,du} (K - S_{\tau_{l,u}}) \right],
\]

respectively.

**Theorem 4.** The value function defined in (10) equals to

\[
V_{\text{Put}}(s) = v_{\text{Put}}(s, l^*, u^*),
\]

where

\[
v_{\text{Put}}(s, l^*, u^*) := \sup_{0 \leq l \leq u \leq K} v_{\text{Put}}(s, l, u).
\]

The optimal stopping rule is \( \tau_{l^*, u^*} \), where \( l^*, u^* \) realise the supremum above.

Theorem 4 indicates that the optimal stopping rule in our problem is the first time when the process \( S_t \) enters the interval \([l^*, u^*]\) for some \( l^* \leq u^* \). In the case when \( l^* = u^* \) the interval becomes a point which is possible as well. In some cases the above observation allows to identify the value function in a much more transparent way. Finally, note that if the discounting function \( \omega \) is positive, then it is never optimal to wait to exercise the option for small asset prices, that is, always \( l^* = 0 \) in this case and the stopping region is one-sided.

2.5. Spectrally negative geometric Lévy process. We can express the value function \( V_{\text{Put}}(s) \) explicitly for the spectrally negative geometric Lévy process defined in (3), that is when

\[
S_t = e^{X_t},
\]

where \( X_t \) is a spectrally negative Lévy process with \( X_0 = x = \log s \) and hence \( S_0 = s \). This means that \( X_t \) does not have positive jumps which corresponds to the inclusion of the support of Lévy measure \( m \) on the negative half-line. This is very common assumption which is justified by some financial crashes; see e.g. [2, 5, 19]. One can easily observe that the dual case of spectrally positive Lévy process \( X_t \) can be also handled in a similar way. We decided to skip this analysis and focus only on a more natural
For any measurable function \( \theta \)

To introduce these functions let us define first the Laplace exponent via

\[
\psi(\theta) := \frac{1}{t} \log \mathbb{E}[e^{\theta X_t} | X_0 = 0],
\]

which is finite at least for \( \theta \geq 0 \) due to downward jumps. This function is strictly convex, differentiable, equals to zero at zero and tends to infinity at infinity. Hence there exists its right inverse \( \Phi(q) \) for \( q \geq 0 \).

The key functions for the fluctuation theory are the scale functions; see [23]. The first scale function \( \mathcal{H}^{(1)}(x) \) equals to zero at zero and tends to infinity at infinity. Hence there exists its right inverse \( \mathcal{W}^{(1)}(x) \) for \( x \geq \Phi(q) \).

For any measurable function \( \xi \) we define the \( \xi \)-scale functions \( \{\mathcal{W}^{(\xi)}(x), x \in \mathbb{R}\} \), \( \{\mathcal{Z}^{(\xi)}(x), x \in \mathbb{R}\} \) and \( \{\mathcal{H}^{(\xi)}(x), x \in \mathbb{R}\} \) as the unique solutions to the following renewal-type equations

\[
\mathcal{W}^{(\xi)}(x) = W(x) + \int_0^x W(x-y)\xi(y)\mathcal{W}^{(\xi)}(y)dy,
\]

\[
\mathcal{Z}^{(\xi)}(x) = 1 + \int_0^x W(x-y)\xi(y)\mathcal{Z}^{(\xi)}(y)dy,
\]

\[
\mathcal{H}^{(\xi)}(x) = e^{\Phi(c)x} + \int_0^x W(c)(x-z)(\xi(z) - c)\mathcal{H}^{(\xi)}(z)dz,
\]

where \( W(x) = W^{(0)}(x) \) is a classical zero scale function and in equation (14) it is additionally assumed that \( \xi(x) = c \) for all \( x \leq 0 \) and some constant \( c \in \mathbb{R} \). We also need function \( \{\mathcal{W}^{(\xi)}(x,z), (x,z) \in \mathbb{R}^2\} \) solving the following equation

\[
\mathcal{W}^{(\xi)}(x,z) = W(x-z) + \int_z^x W(x-y)\xi(y)\mathcal{W}^{(\xi)}(y,z)dy.
\]

We introduce the following \( S_t \) counterparts of the scale functions (12), (13), (14) and (15)

\[
\mathcal{W}^{(\xi)}(s) := \mathcal{W}^{(\xi,\exp)}(\log s),
\]

\[
\mathcal{Z}^{(\xi)}(s) := \mathcal{Z}^{(\xi,\exp)}(\log s),
\]

\[
\mathcal{H}^{(\xi)}(s) := \mathcal{H}^{(\xi,\exp)}(\log s),
\]

\[
\mathcal{W}^{(\xi)}(s,z) := \mathcal{W}^{(\xi,\exp)}(\log s, z),
\]

where \( \xi \circ \exp(x) := \xi(e^x) \).

For \( \alpha \) for which the Laplace exponent is well-defined we can define a new probability measure \( \mathbb{P}^{(\alpha)} \) via

\[
\frac{d\mathbb{P}^{(\alpha)}}{d\mathbb{P}} |_{\mathcal{F}_t} = e^{\alpha X_t - \psi(\alpha) t}.
\]

By [51], under \( \mathbb{P}^{(\alpha)} \), the process \( X_t \) is again spectrally negative Lévy process with the new Laplace exponent

\[
\psi^{(\alpha)}(\theta) := \psi(\theta + \alpha) - \psi(\alpha).
\]

For this new probability measure \( \mathbb{P}^{(\alpha)} \) we can define \( \xi \)-scale functions which are denoted by adding subscript \( \alpha \) to the regular counterparts, hence we have \( \mathcal{W}_{\alpha}^{(\xi)}(s) \), \( \mathcal{Z}_{\alpha}^{(\xi)}(s) \), \( \mathcal{H}_{\alpha}^{(\xi)}(s) \) and \( \mathcal{W}_{\alpha}^{(\xi)}(s,z) \).

Let

\[
\omega_u(s) := \omega(su) \quad \text{and} \quad \omega_{\alpha}^u(s) := \omega_u(s) - \psi(\alpha).
\]
The main result is given in terms of the resolvent density at \( z \) of \( X_t \) starting at \( \log s - \log u \) killed by the potential \( \omega_u \) and on exiting from positive half-line given by

\[
(22) \quad r(s, u, z) := \frac{\mathcal{J}(\omega_u)(\log s - \log u) c_{\omega_u}(z)}{\mathcal{J}(\omega_u)(\log s - \log u) - \mathcal{J}(\omega_u)(\log s - \log u, z)},
\]

where

\[
c_{\omega_u}(z) := \lim_{y \to \infty} \frac{\mathcal{J}(\omega_u)(\log y, z)}{\mathcal{J}(\omega_u)(\log y)}.
\]

**Theorem 5.** Assume that the stock price process \( S_t \) is described by \((6)\) with \( X_t \) being the spectrally negative Lévy process. Let \( \omega \) be a measurable function such that

\[
\omega(s) = c \text{ for all } s \in (0, 1] \text{ and some constant } c \in \mathbb{R}.
\]

Then

\[
v_A^{\omega_u}(s, l, u) = \frac{\mathcal{J}(\omega)(s)}{\mathcal{J}(\omega)(l)} (K - l) \mathbf{1}_{\{s < l\}} + (K - s) \mathbf{1}_{\{s \in [l, u]\}}
\]

\[
+ \left\{ \int_0^\infty \int_0^\infty \frac{\mathcal{J}(\omega_u)((\frac{u}{1+y}) \land l)}{\mathcal{J}(\omega_u)(l)} (K - e^{\log l \cdot (\log u - y)^{\Pi}}) r(s, u, z) \Pi(-z - dy) dz
\]

\[
+ (K - u) \left( \lim_{\alpha \to \infty} \left( \mathcal{J}_\alpha^{\omega_u}(s) - c_{\omega_u}^{\omega_u} \right) \right) \mathbf{1}_{\{s > u\}} \right\}
\]

where

\[
c_{\omega_u}^{\omega_u}(z) = \lim_{z \to \infty} \mathcal{J}_\alpha^{\omega_u}(z)
\]

and \( r(s, u, z) \) is given in \((22)\).

The proof of the above theorem is given in Section 4.

2.6. **HJB, smooth and continuous fit properties.** The classical approach via HJB system is possible in our set-up as well. More precisely, as before in \((6)\) we have

\[
S_t = e^{X_t}
\]

for the Lévy process \( X_t \) with the triple \((\zeta, \sigma, \Pi)\). We start from the observation that using \((6)\) Thm. 31.5, Chap. 6] and Itô’s formula one can conclude that the process \( S_t \) is a Markov process with an infinitesimal generator

\[
A f(s) = A^C f(s) + A^J f(s),
\]

where \( A^C \) is the linear second-order differential operator of the form

\[
A^C f(s) = \frac{\sigma^2 s^2}{2} f''(s) + \left( \zeta + \frac{\sigma^2}{2} \right) s f'(s)
\]

and \( A^J \) is the integral operator given by

\[
A^J f(s) = \int_{(-\infty, 0)} \left( f(se^z) - f(s) - s|z| f'(s) \mathbf{1}_{\{|z| \leq 1\}} \right) \Pi(dz).
\]

The domain \( D(A) \) of this generator consists of the functions belonging to \( C^2(\mathbb{R}^+) \) if \( \sigma > 0 \) and \( C^1(\mathbb{R}^+) \) if \( \sigma = 0 \). In this paper we prove that \( V_A^{\omega_u}(s) \) satisfies the following HJB equation with appropriate smooth fit conditions.
Theorem 6. Assume that $V_A^\omega(s) \in D(A)$ and that $g(s) \in C^1(\mathbb{R}^+)$. Then $V_A^\omega(s)$ solves uniquely the following equations

\begin{equation}
\begin{cases}
A V_A^\omega(s) - \omega(s) V_A^\omega(s) = 0, & s \notin [l, u], \\
V_A^\omega(s) = g(s), & s \in [l, u].
\end{cases}
\end{equation}

Moreover, if 1 is regular for $(-\infty, 1)$ and for the process $S_t$ then there is a smooth fit at the right end of the stopping region

$$(V_A^\omega)^'(u) = g'(u).$$

Similarly, if 1 is regular for $(1, +\infty)$ and for the process $S_t$ then there is a smooth fit at the left end of the stopping region

$$(V_A^\omega)^'(l) = g'(l).$$

Remark 7. Let us consider the American put option. Then from Theorems 4 and 5, we can conclude that smoothness of the value function $V_A^{\omega^u}(s)$ corresponds to the smoothness of the scale functions for $\omega$, $\omega_u$ and $\omega^u$. From the definition of these scale functions given in (12), (13) and (14) it follows that the smoothness of the latter functions is equivalent to the smoothness of the first scale function observed under measures $P$ and $P^{(\omega)}$. By [23, Lem. 8.4] the smoothness of the first scale function does not change under the exponential change of measure (20). Thus from [23] Lem. 2.4, Thms 3.10 and 3.11 it follows that

- if $\sigma > 0$ then $V_A^{\omega^u}(s) \in C^2(\mathbb{R}^+)$;
- if $\sigma = 0$ and the jump measure $\Pi$ is absolutely continuous or $\int_0^1 |x|\Pi(dx) = +\infty$, then $V_A^{\omega^u}(s) \in C^1(\mathbb{R}^+)$.\n
Moreover, by [2] Prop. 7, 1 is regular for both $(-\infty, 1)$ and $(1, +\infty)$ if $\sigma > 0$. Hence HJB system (24) with the smooth fit property could be used without any additional assumptions as long as $\sigma > 0$. If one has single continuation region $[u^*, +\infty)$ and $\sigma = 0$ then by [2] Prop. 7 to get the smooth fit condition at $u^*$ it is sufficient to assume that the drift $\zeta$ of the process $X_t$ is strictly negative.

2.7. Put-call symmetry. The put-call parity allows to calculate the American call option price having the put one. We formulate this relation again for $S_t$ being a general geometric Lévy process defined in (5), that is, $S_t = e^{X_t}$, for $X_t$ being a general Lévy process having triple $(\zeta, \sigma, \Pi)$

for $\zeta$ and $\Pi$ defined in (7) and starting position $X_0 = \log S_0 = s$. Apart from function

$v_A^{\omega^u}(s, K, \zeta, \sigma, \Pi, l, u) := E_s[e^{-\int_{l}^{u} \omega(S_w)dw} (K - S_{\Pi, t})^+ ]$

defined in (4) we denote

$v_A^{\omega^c}(s, K, \zeta, \sigma, \Pi, l, u) := E_s[e^{-\int_{l}^{u} \omega(S_w)dw} (S_{\Pi, t} - K)^+]$.

Theorem 8. Assume that $\psi(1) = \log E e^{X_1} = \log ES_1$ is finite. Let $l \leq u \leq K$. Then we have

$v_A^{\omega^c}(s, K, \zeta, \sigma, \Pi, l, u) = v_A^{\omega^c}(K, s, -\zeta, \sigma, \Pi, \frac{\zeta}{u}, l, K)$,

where

\begin{equation}
\hat{\Pi}(dx) := e^{-x} \Pi(-dx),
\end{equation}

$\vartheta^{(1)}(\cdot) := \omega \left( \frac{1}{s} \cdot sK \right) - \psi(1).$
Remark 9. Note that price of the American call option is expressed in terms of the American put option calculated for the Lévy process $\hat{X}_t$ being dual to $X_t$ process observed under the measure $\mathbb{P}^{(1)}$. In particular, the jumps of the process $\hat{X}_t$ have opposite direction to the jumps of the process $X_t$ for which the call option is priced.

2.8. Black-Scholes model. We can give more detailed analysis in the case of classical Black–Scholes model in which the stock price process $S_t$ satisfies the following stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

with a constant $\mu \in \mathbb{R}$ and $\sigma > 0$. That is,

$$S_t = se^{(\mu - \frac{\sigma^2}{2})t + \sigma B_t}$$

is a geometric Brownian motion.

Theorem 10. Let (A6) hold and assume that the stock price process $S_t$ follows (26). Then the function $v_{A,Put}^{\omega}(s, l, u)$ defined in (4) is given by

$$v_{A,Put}^{\omega}(s, l, u) = \frac{h(s)}{h(l)}(K - l)1_{\{s < l\}} + (K - s)1_{\{s \in [l, u]\}} + \frac{h(s)}{h(u)}(K - u)1_{\{s > u\}},$$

where $h(s)$ is a solution to

$$\frac{\sigma^2 s^2}{2}h''(s) + \mu sh'(s) - \omega(s)h(s) = 0$$

which satisfies

$$\begin{cases} h(s) = g(s), & s \in [l^*, u^*], \\
\lim_{s \to \infty} h(s) = \text{const.} \end{cases}$$

Remark 11. The optimal boundaries $l^*$ and $u^*$ can be found from the smooth fit property given in Theorem 6.

2.9. Exponential crashes market. We can construct more explicit equation for the value function for the case of classical Black–Scholes model with additional downward exponential jumps, that is, as in (6), $S_t = e^{X_t}$ for

$$X_t = \log s + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t - \sum_{i=1}^{N_t} Y_i,$$

where $N_t$ is the Poisson process with intensity $\lambda > 0$ independent of Brownian motion $B_t$ and $\{Y_i\}_{i \in \mathbb{N}}$ are i.i.d. random variables independent of $B_t$ and $N_t$ having exponential distribution with mean $1/\varphi > 0$. For this model the price of American put option is easier to determine.

Theorem 12. Assume that $\omega$ is nonnegative, concave and increasing. For geometric Lévy model with $X_t$ given in (29) we have $l^* = 0$. Furthermore, (i)

$$(30) \quad V_{A,Put}^{\omega}(s) = \sup_{u > 0} \left\{ \left( K - \frac{u\varphi}{\varphi + 1} \right) \left( 1 - e^{\gamma(\omega_u)} \left( \frac{s}{u} \frac{\gamma(\omega_u)}{\omega_u} \right) \right) \right\}$$
if \( \sigma = 0 \), where \( W^{(\omega_u)} \) and \( \mathcal{Z}^{(\omega_u)} \) are given in \([16]\) and \([17]\), respectively and

\[
(31) \quad c_{\mathcal{Z}^{(\omega_u)}} = \lim_{z \to \infty} \frac{\mathcal{Z}^{(\omega_u)}(z)}{W^{(\omega_u)}(z)}.
\]

(ii)

\[
(32) \quad V_{X_{\phi}}^{\omega_u}(s) = \sup_{u > 0} \left\{ \left( K - \frac{u}{\varphi + 1} \right) \left( \mathcal{Z}^{(\omega_u)} \left( \frac{s}{u} \right) - c_{\mathcal{Z}^{(\omega_u)}} \right) \right. \\
+ \left. \left( K - u \right) \left( \lim_{\alpha \to \infty} \left( 2^{(\omega_u)} \left( \frac{s}{u} \right) - c_{2^{(\omega_u)}} \mathcal{W}^{(\omega_u)} \left( \frac{s}{u} \right) \right) \right) \right\}
\]

if \( \sigma > 0 \). In this case the optimal boundary \( u^* \) is determined by the smooth fit condition

\[
(V_{X_{\phi}}^{\omega_u})'(u^*) = -1.
\]

Thus to identify the price \( V_{X_{\phi}}^{\omega_u}(s) \) of American put option we have to identify the scale functions \( W^{(\xi)}(s) \) and \( \mathcal{Z}^{(\xi)}(s) \) for \( \xi \) equals to \( \omega_u \) or \( \omega_u^\alpha \) under measure \( \mathbb{P} \) and measure \( \mathbb{P}^{(\alpha)} \) defined in \([20]\). Note that from \([21]\) (see also \([54]\), Prop. 5.6) with the Laplace exponent

\[
(33) \quad \psi(\theta) = \left( \mu - \frac{\sigma^2}{2} \right) \theta + \frac{\sigma^2}{2} \theta^2 - \frac{\lambda \theta}{\varphi + \theta}
\]

of the process \( X_t \), under \( \mathbb{P}^{(\alpha)} \), the Lévy process \( X_t \) given in \([29]\) is of the same form with \( \mu \) and \( \sigma \) unchanged and with new intensity of Poisson process \( \lambda^{(\alpha)} := \lambda \varphi/(\varphi - \alpha) \) and new parameter of exponential distribution of \( Y_t \) given by \( \varphi^{(\alpha)} := \varphi - \alpha \). To find the scale functions \( W^{(\xi)}(s) \) and \( \mathcal{Z}^{(\xi)}(s) \) it enough then to identify them under original measure \( \mathbb{P} \). To do so, we recall that in \([16]\) and \([17]\) we introduced them via regular omega scale functions, that is \( W^{(\xi)}(s) = W^{(\text{exp})}(x) \) and \( \mathcal{Z}^{(\xi)}(s) = \mathcal{Z}^{(\text{exp})}(x) \) for \( x = \log s \). It suffices to find omega scale functions \( W^{(\xi)}(x) \) and \( \mathcal{Z}^{(\xi)}(x) \) for given generic function \( \xi \). We recall that both omega scale functions are given as the solutions of renewal equations \([12]\) and \([13]\) formulated in terms of the classical scale function \( W(x) \). From the definition of the first scale function given in \([11]\) with \( q = 0 \) and \([33]\) we derive

\[
W(x) = \sum_{i=1}^{3} T_i e^{\gamma_i x},
\]

where \( \gamma_i \) solves

\[
\psi(\gamma_i) = 0
\]

and

\[
T_i := \frac{1}{\varphi(\gamma_i)}.
\]

If \( \sigma = 0 \) then \( T_3 := 0, \gamma_1 := 0, \gamma_2 := \frac{\lambda - \varphi \mu}{\mu}, T_1 := -\frac{\varphi}{\lambda - \varphi \mu} \) and \( T_2 := \frac{\lambda}{\mu \lambda - \varphi \mu} \). Observe that \( T_1 + T_2 = \frac{1}{\mu} \).

Next theorem gives the ordinary differential equations whose solutions are the omega scale functions. We use this result later in the numerical analysis.

**Theorem 13.** We assume that the function \( \xi \) is continuously differentiable. For geometric Lévy model \([6]\) with \( X_t \) given in \([29]\) we have

(i) If \( \sigma = 0 \) then the function \( W^{(\xi)}(x) \) solves the following ordinary differential equations

\[
(34) \quad W^{(\xi)^{''}}(x) = ((T_1 + T_2) \xi(x) + \gamma_2) W^{(\xi)'}(x) + ((T_1 + T_2) \xi(x) - \gamma_2 T_1 \xi(x)) W^{(\xi)}(x)
\]
Moreover, the function $Z^{(\xi)}(x)$ solves the same equation (34) with the following boundary conditions

$$Z^{(\xi)}(0) = 1$$

and

$$Z^{(\xi)'}(0) = (Y_1 + Y_2)\xi(0)Z^{(\xi)}(0).$$

(ii) If $\sigma > 0$ then the function $W^{(\xi)}(x)$ solves the following ordinary differential equations

$$W^{(\xi)'''}(x) = ((Y_1 + Y_2 + Y_3)\xi(x) + \gamma_2 + \gamma_3)W^{(\xi)''}(x)$$

$$+ (2(Y_1 + Y_2 + Y_3)\xi'(x) + Y_2(\gamma_2 - \gamma_3)\xi(x) - (Y_1 + Y_2 + Y_3)\gamma_2\xi(x) - \gamma_2\gamma_3 - \gamma_3 Y_1\xi(x)) W^{(\xi)'}(x)$$

$$+ ((Y_1 + Y_2 + Y_3)\xi''(x) + Y_2(\gamma_2 - \gamma_3)\xi'(x) - \gamma_2(Y_1 + Y_2 + Y_3)\xi'(x) + \gamma_2\gamma_3 Y_1\xi(x) - \gamma_3 Y_1\xi'(x)) W^{(\xi)}(x)$$

with

$$W^{(\xi)}(0) = Y_1 + Y_2 + Y_3,$$

$$W^{(\xi)'}(0) = Y_2\gamma_2 + Y_3\gamma_3 + (Y_1 + Y_2 + Y_3)\xi(0)W^{(\xi)}(0)$$

and

$$W^{(\xi)''}(0) = Y_2(\gamma_2 - \gamma_3) + (Y_1 + Y_2 + Y_3)(\xi'(0)W^{(\xi)}(0) + \xi(0)W^{(\xi)'}(0))$$

$$+ \gamma_2(\gamma_2 - \gamma_3)\xi(0)W^{(\xi)}(0) + \gamma_3 W^{(\xi)'}(0) - \gamma_3 Y_1\xi(0)W^{(\xi)}(0).$$

Moreover, the function $Z^{(\xi)}(x)$ solves the same equation (38) with the following boundary conditions

$$Z^{(\xi)}(0) = 1,$$

$$Z^{(\xi)'}(0) = (Y_1 + Y_2 + Y_3)\xi(0)Z^{(\xi)}(0)$$

and

$$Z^{(\xi)''}(0) = (Y_1 + Y_2 + Y_3)(\xi'(0)Z^{(\xi)}(0) + \xi(0)Z^{(\xi)'}(0))$$

$$+ Y_2(\gamma_2 - \gamma_3)\xi(0)Z^{(\xi)}(0) + \gamma_3 Z^{(\xi)'}(0) - \gamma_3 Y_1\xi(0)Z^{(\xi)}(0).$$

Remark 14. Note that assumption (23) is not required in Theorem 13 because we do not use the function $\mathcal{H}^{(w)}(s)$ in the expression for value function (30) and (32).

3. Numerical analysis

In this section we present the closed forms of value function (10) for the particular $\omega$ and for the Black-Scholes model and Black-Scholes model with downward exponential jumps. In the first scenario, we take into account only the case of negative $\omega$ which implies a double continuation region, while in the second example we focus on the positive $\omega$. 
3.1. Black-Scholes model revisited. Let
\[
\omega(s) = -\frac{C}{s + 1} - D,
\]
where \(C\) and \(D\) are some positive constants. Applying Theorem\textsuperscript{[10]} we obtain
\[
\psi_{A_{\text{put}}}(s, l, u) = \frac{h(s)}{h(l)} (K - l) \mathbb{I}_{\{s \in (0, l]\}} + (K - s) \mathbb{I}_{\{s \in [l, u]\}} + \frac{h(s)}{h(u)} (K - u) \mathbb{I}_{\{s \in (u, +\infty)\}},
\]
where \(h\) is a solution to
\[
\frac{\sigma^2 s^2}{2} h''(s) + \mu s h'(s) - \left(-\frac{C}{s + 1} - D\right) h(s) = 0
\]
which satisfies
\[
\begin{align*}
\lim_{s \to \infty} h(s) &= \text{const}, \\
\lim_{s \to 0} h(s) &= g(s), \quad s \in [l^*, u^*],
\end{align*}
\]
We first solve above equation and then we look for boundaries \(l^*\) and \(u^*\) such that \(\psi_{A_{\text{put}}}(s, l^*, u^*) = \sup_{l, u} \psi_{A_{\text{put}}}(s, l, u)\). The general solution to (39) is given by
\[
h(s) = K_2 s^{d_2} F_1(a_2, b_2; c_2; -s) + K_1 s^{d_1} F_1(a_1, b_1; c_1; -s),
\]
where \(L := \frac{1}{2} - \frac{\mu}{\sigma^2}\), \(M := \sqrt{L^2 - \frac{2D}{\sigma^2}}\), \(G := \sqrt{L^2 - \frac{2(C + D)}{\sigma^2}}\), \(a_2 := -M + G\), \(b_2 := M + G\), \(c_2 := 1 + 2G\), \(d_2 := G + L\), \(a_1 := -M - G\), \(b_1 := -M - G\), \(c_1 := 1 - 2G\). Using formula (41) and the boundary conditions given in (40) we can identify the form of value function (10). Since we consider the negative \(\omega\) we obtain a double continuation region. We take one of the summands from (41) for \(s \in (0, l^*)\) and the second one for \(s \in (u^*, +\infty)\). This choice is made in such a way that on the given interval we impose to have a greater function of these two. Hence we derive
\[
V_{A_{\text{put}}}^{\text{opt}}(s) = \begin{cases} K_2 s^{d_2} F_1(a_2, b_2; c_2; -s), & s \in (0, l^*), \\ K - s, & s \in [l^*, u^*], \\ K_1 s^{d_1} F_1(a_1, b_1; c_1; -s), & s \in (u^*, +\infty). \end{cases}
\]
Using the smooth and continuous fit properties we can find \(K_1\) and \(K_2\) and show that \(l^*\) and \(u^*\) solve the following equation
\[
1 + 2F_1(a_i, b_i, c_i; -s) K_i D_i + s^{d_i} P_i = 0,
\]
where
\[
K_i := (K - s) \frac{1}{2} s^{-d_i} F_1(a_i, b_i, c_i; -s), \\
D_i := d_i s^{d_i - 1}, \\
P_i := -a_i b_i c_i + 1 + b_i + 1 + c_i + 1 - s
\]
for \(i = 1, 2\). We numerically calculate the roots of (42) for \(i = 1, 2\). We assign the smaller result to \(l^*\), and the greater one to \(u^*\).
Let us assume that \(C = 0.1\%\), \(D = 1\%\), \(K = 20\), \(\mu = 5\%\) and \(\sigma = 20\%\). Above numerical procedure produces \(l^* \approx 7.23\) and \(u^* \approx 8.34\). The figure of the value function is depicted in Figure\textsuperscript{[11]}.
Figure 1. The value and payoff functions for the given set of parameters: $C = 0.1\%$, $D = 1\%$, $K = 20$, $\mu = 5\%$ and $\sigma = 20\%$.

3.2. Exponential crashes market revisited. We consider the stock price process $S_t$ given in (6) with $X_t$ defined in (29) for $\sigma = 0$, i.e.

$$X_t = x + \mu t - \sum_{i=1}^{N_t} Y_i$$

where $x = \log s$, $N_t$ is the Poisson process with intensity $\lambda > 0$ and $\{Y_i\}_{i \in \mathbb{N}}$ are i.i.d. exponential random variables with parameter $\varphi > 0$. Let

$$\omega(s) = Cs,$$

where $C$ is some positive constant. Note that this discounting function is nonegative. Hence from Theorem 12 it follows that $l^* = 0$. Let

$$\eta(x) := \omega(e^x) = \omega(s) \quad \text{and} \quad \eta_u(x) := \eta(x + \log u).$$

From Theorem 12 (see equation (30)) with $\sigma = 0$ and using (16) and (17) we can conclude that

$$V^{\omega}_{\mathcal{A}^*}(s) = \sup_{u > 0} V^{\omega}_{\mathcal{A}^*}(s, 0, u),$$

where

$$v^{\omega}_{\mathcal{A}^*}(s, 0, u) = \left( \frac{u \varphi}{\varphi + 1} \right) \left( Z(\eta_u)(x - \log u) - c_{Z(\eta_u)/W^{(\eta_u)}} W^{(\eta_u)}(x - \log u) \right),$$

and from (31)

$$c_{Z(\omega)/W(\omega)} = c_{Z(\eta_u)/W^{(\eta_u)}} := \lim_{z \to \infty} \frac{Z(\eta_u)(z)}{W^{(\eta_u)}(z)}.$$

From Theorem 13 it follows that $W^{(\eta)}(x)$ solves the following ordinary differential equation

$$W^{(\eta)''}(x) = (Ae^x + B)W^{(\eta)'}(x) + De^x W^{(\eta)}(x),$$
with $A := \frac{C}{\mu}$, $B := \frac{\lambda - \varphi \mu}{\mu}$, and $D := C^{\frac{1 + \varphi}{\mu}}$. The above equation is also satisfied by $Z^{(\eta)}(x)$.

From (35)–(37) we conclude that

$$
\begin{align*}
W^{(\eta)}(0) &= \frac{1}{\mu}, \\
W^{(\eta)'}(0) &= \frac{C + \lambda}{\mu^2},
\end{align*}
$$

and

$$
\begin{align*}
Z^{(\eta)}(0) &= 1, \\
Z^{(\eta)'}(0) &= \frac{C}{\mu}.
\end{align*}
$$

We solve (45) numerically which allows us to plot its solution. In this way we can produce figures of $W^{(\eta)}(x)$ and $Z^{(\eta)}(x)$. By shifting these scale functions by $\log u$ we can produce figures of $W^{(\eta_u)}(x - \log u)$ and $Z^{(\eta_u)}(x - \log u)$. Then $c_{Z^{(\eta_u)}}/W^{(\eta_u)}$ is calculated numerically as the ratio of the scale functions for large enough arguments (when the ratio stops changing). In this way we can derive value function (44). Finally, by the continuous fit condition we choose the optimal $u$ in such a way that at $s = u$ the value function is equal to the payoff function.

Let us assume that $C = 1$, $K = 20$, $\mu = 5\%$, $\sigma = 20\%$, $\lambda = 6$, $\varphi = 2$. Above numerical procedure produces $u^* \approx 11.5$. The figure of the value function is shown in Figure 2.

**Figure 2.** The value and payoff functions for the given set of parameters: $C = 1$, $K = 20$, $\mu = 5\%$, $\sigma = 20\%$, $\lambda = 6$, $\varphi = 2$.

### 4. Proofs

Before we prove main Theorem 2 we show the convexity of European option price $V_E^\omega(s, t)$ defined in (2) as a function of $s$. It is done in Theorems 15 and 18. In the proof we apply idea demonstrated in [34, Prop. 4.1]. Later, in the proof of Theorem 2 we use a variant of the maximum principle.

Let us introduce a set $E \subset \mathbb{R} \times [0, T]$. We use the following notations

- $C_\alpha(E)$ is the set of locally Hölder($\alpha$) functions with $\alpha \in (0, 1)$,
• \( C_{\text{pol}}(E) \) is the set of functions of at most polynomial growth in \( s \),
• \( C^{p,q}(E) \) is the set of functions for which all the derivatives \( \frac{\partial^k}{\partial s^k} f(s,t) \) with \(|k| + 2l \leq p\) and \(0 \leq l \leq q\) exist in the interior of \( E \) and have continuous extensions to \( E \),
• \( C_{\text{pol}}^{p,q}(E) \) and \( C_{\alpha}^{p,q}(E) \) are the sets of functions \( f \in C^{p,q}(E) \) for which all the derivatives \( \frac{\partial^k}{\partial s^k} \left( \frac{\partial f(s,t)}{\partial t} \right) \) with \(|k| + 2l \leq p\) and \(0 \leq l \leq q\) belong to \( C_{\text{pol}}(E) \) and \( C_\alpha(E) \), respectively.

We need the following conditions in the proofs.

**Assumptions (B)**
There exist constants \( C > 0 \) and \( \alpha \in (0,1) \) such that

\[ (B1) \quad \mu(s,t) \in C^{2,1}_\alpha(\mathbb{R}^+ \times [0, T]); \]
\[ (B2) \quad \sigma^2(s,t) \geq Cs^2 \text{ for all } (s,t) \in \mathbb{R}^+ \times [0, T]; \]
\[ (B3) \quad \sigma(s,t) \in C^{2,1}_\alpha(\mathbb{R}^+ \times [0, T]); \]
\[ (B4) \quad \gamma(s,t,z) \in C^{2,1}_\alpha(\mathbb{R}^+ \times [0, T]) \text{ with the Hölder continuity being uniform in } z; \]
\[ (B5) \quad \omega(s) \leq C \text{ for all } s \in \mathbb{R}^+; \]
\[ (B6) \quad \omega(s) \in C^2_\alpha(\mathbb{R}^+); \]
\[ (B7) \quad g(s) \text{ is Lipschitz continuous}; \]
\[ (B8) \quad g(s) \in C^4_\alpha(\mathbb{R}^+). \]

**Assumptions (C)**
There exist a constant \( C > 0 \) such that

\[ (C1) \quad |\frac{\partial \mu(s,t)}{\partial t}| \leq Cs, \quad |\frac{\partial^2 \mu(s,t)}{\partial s^2}| \leq \frac{C}{s} \text{ for all } (s,t) \in \mathbb{R}^+ \times [0, T]; \]
\[ (C2) \quad |\frac{\partial \sigma(s,t)}{\partial t}| \leq Cs, \quad |\frac{\partial^2 \sigma(s,t)}{\partial s^2}| \leq \frac{C}{s} \text{ for all } (s,t) \in \mathbb{R}^+ \times [0, T]; \]
\[ (C3) \quad |\frac{\partial \gamma(s,t,z)}{\partial s}| \leq Cs, \quad |\frac{\partial^2 \gamma(s,t,z)}{\partial s^2}| \leq \frac{C}{s} \text{ for all } (s,t,z) \in \mathbb{R}^+ \times [0, T] \times \mathbb{R}; \]
\[ (C4) \quad \left| \frac{d \omega(s)}{ds} \right| \leq \frac{Cs}{s^2}, \quad \left| \frac{d^2 \omega(s)}{ds^2} \right| \leq \frac{Cs}{s^3} \text{ for all } s \in \mathbb{R}^+; \]
\[ (C5) \quad g(s) \in C^3_{\text{pol}}(\mathbb{R}^+). \]

**Theorem 15.** Let all assumptions of Theorem 2 be satisfied. We assume additionally that conditions (B) and (C) hold true. Then \( V_E^n(s,t) \) is convex with respect to \( s \) at all times \( t \in [0,T] \).

**Proof.** The first part of the proof proceeds in a similar way as the proof of [34, Prop. 4.1]. Let

\[ \mathcal{L}V_E^n(s,t) = -\frac{\partial V_E^n(s,t)}{\partial t} - A_t^1 V_E^n(s,t) - A_t^2 V_E^n(s,t) + \omega(s)V_E^n(s,t), \]

where \( A_t \) is the linear second-order differential operator of the form

\[ A_t^1 V_E^n(s,t) = \beta(s,t) \frac{\partial^2 V_E^n(s,t)}{\partial s^2} + \mu(s,t) \frac{\partial V_E^n(s,t)}{\partial s} \]

with \( \beta(s,t) = \frac{\sigma^2(s,t)}{2} \) and \( A_t^2 \) is the integro-differential operator given by

\[ A_t^2 V_E^n(s,t) = \int_{\mathbb{R}} \left( V_E^n(s + \gamma(s,t,z), t) - V_E^n(s,t) - \gamma(s,t,z) \frac{\partial V_E^n(s,t)}{\partial s} \right) m(dz). \]

**Lemma 16.** Let Assumptions (A) and (B) hold and assume that the stock price process \( S_t \) follows (5). Then \( V_E^n(s,t) \in C^{4,1}_\alpha(\mathbb{R}^+ \times [0, T]) \cap C_{\text{pol}}(\mathbb{R}^+ \times [0, T]) \) and it is the solution to the Cauchy problem

\[ (46) \quad \begin{cases} \mathcal{L}V_E^n(s,t) = 0, & (s,t) \in \mathbb{R}^+ \times [0,T], \\ V_E^n(s,T) = g(s), & s \in \mathbb{R}^+. \end{cases} \]
Lemma 17. Let Assumptions (A), (B) and (C) hold and assume that the stock price process $S_t$ follows \[. Then there exist constants $n > 0$ and $K > 0$ such that the value function $V_E^\omega(s, t)$ satisfies

$$\left| \frac{\partial^2 V_E^\omega(s, t)}{\partial s^2} \right| \leq K(s^{-n} + s^n)$$

for all $(s, t) \in \mathbb{R}^+ \times [0, T]$.

Proofs of both above lemmas are given in Appendix.

We introduce the function $u^\omega : \mathbb{R}^+ \times [0, T] \rightarrow \mathbb{R}^+$ of the form

$$u^\omega(s, t) := V_E^\omega(s, T - t)$$

and we prove convexity of $u^\omega(s, t)$ with respect to $s$. Note that it is equivalent to the convexity of the value function $V_E^\omega(s, t)$ in $s$. Furthermore, based on Lemma 16, the function $u^\omega(s, t)$ solves the Cauchy problem of the form

$$\begin{cases}
\frac{\partial u^\omega(s, t)}{\partial t} = \hat{L}u^\omega(s, t), & (s, t) \in \mathbb{R}^+ \times (0, T], \\
u^\omega(s, 0) = g(s), & s \in \mathbb{R}^+,
\end{cases}$$

where

$$\hat{L}u^\omega(s, t) = \beta(s, t) \frac{\partial^2 u^\omega(s, t)}{\partial s^2} + \mu(s, t) \frac{\partial u^\omega(s, t)}{\partial s} - \omega(s) u^\omega(s, t)$$

$$+ \int_{\mathbb{R}} \left( u^\omega(s + \gamma(s, t, z), t) - u^\omega(s, t) - \gamma(s, t, z) \frac{\partial u^\omega(s, t)}{\partial s} \right) m(dz)$$

with $\beta(s, t) = \frac{s^2(s, t)}{2}$. Observe that by Lemma 17 there exist constants $n > 0$ and $K > 0$ such that

$$\left| \frac{\partial^2 u^\omega(s, t)}{\partial s^2} \right| \leq K(s^{-n} + s^n)$$

for all $(s, t) \in \mathbb{R}^+ \times [0, T]$.

Let us now define a convex function $\kappa : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ of the form

$$\kappa(s) := s^{n+3} + s^{-n+1}$$

with

$$\frac{d^2 \kappa(s)}{ds^2} = (n + 3)(n + 2)s^{n+1} + n(n - 1)s^{-n-1}$$

and

$$\frac{d^2 (\hat{L}\kappa(s))}{ds^2} = \frac{d^2 \beta(s, t)}{ds^2} + 2 \frac{d \beta(s, t)}{ds} \frac{d^3 \kappa(s)}{ds^3} + \beta(s, t) \frac{d^4 \kappa(s)}{ds^4}$$

$$+ \frac{d^2 \mu(s, t)}{ds^2} \frac{d \kappa(s)}{ds} + 2 \frac{d \mu(s, t)}{ds} \frac{d^3 \kappa(s)}{ds^3} + \mu(s, t) \frac{d^4 \kappa(s)}{ds^4}$$

$$- \frac{d^2 \omega(s)}{ds^2} \frac{\kappa(s)}{\omega(s)} - 2 \frac{d \omega(s)}{ds} \frac{d \kappa(s)}{ds} - \omega(s) \frac{d^2 \kappa(s)}{ds^2}$$

$$+ \int_{\mathbb{R}} \left( \frac{d^2 \kappa(s + \gamma(s, t, z))}{ds^2} \right) \left( 1 + \frac{\partial \gamma(s, t, z)}{\partial s} \right)^2$$

$$+ \frac{d \kappa(s + \gamma(s, t, z))}{ds} \frac{d^2 \gamma(s, t, z)}{ds^2} - \gamma(s, t, z) \frac{d^3 \kappa(s)}{ds^3}$$

$$- \left( 1 + 2 \frac{\partial \gamma(s, t, z)}{\partial s} \right) \frac{d^2 \kappa(s)}{ds^2} - \frac{d^2 \gamma(s, t, z)}{ds^2} \frac{\partial \kappa(s)}{\partial s} \right) m(dz).$$
The assumptions that we put on the coefficients μ, σ and function ω and their derivatives imply that each component of the above expression grows at most like $s^{n+1}$ for large $s$ and like $s^{-n-1}$ for small $s$. The same behaviour characterises $\frac{d^2 \kappa(s)}{ds^2}$.

In addition, we define the function $\vartheta : \mathbb{R}^+ \times [0, T] \rightarrow \mathbb{R}$ given by

$$\vartheta(s, t) := \left( \frac{\partial^2 \mu(s, t)}{\partial s^2} - 2 \frac{d\omega(s)}{ds} \right) \frac{d\kappa(s)}{ds}$$

which also behaves like $\frac{d^2 (L\kappa(s))}{ds^2}$ at $+\infty$ and $-\infty$.

Hence we claim that there exist a positive constant $C$ such that

\begin{equation}
C \frac{d^2 \kappa(s)}{ds^2} - \frac{d^2 (L\kappa(s))}{ds^2} > -\vartheta(s, t).
\end{equation}

In the second part of the proof, we define the auxiliary function

\begin{equation}
u^\varepsilon(s, t) := u^\omega(s, t) + \varepsilon e^{Ct} \kappa(s)
\end{equation}

for some $\varepsilon > 0$.

We carry out a proof by contradiction. Let us then assume that $\nu^\varepsilon(s, t)$ is not convex. For this purpose, we denote by $\Lambda$ the set of points for which $\nu^\varepsilon(s, t)$ is not convex, i.e.

$$\Lambda := \{(s, t) \in \mathbb{R}^+ \times [0, T] : \frac{\partial^2 \nu^\varepsilon(s, t)}{\partial s^2} < 0\}$$

and we assume that the set $\Lambda$ is not empty.

From Lemma [17] we know that $u^\omega(s, t)$ satisfies (47). Due to this fact and using (49) we claim that there exist a positive constant $R$ such that $\Lambda \subseteq [R^{-1}, R] \times [0, T]$. This is a direct consequence of such a choice of $u^\varepsilon(s, t)$ in (49) so that $\frac{d^2 \kappa(s)}{ds^2}$ grows faster than $\frac{\partial^2 u^\varepsilon(s, t)}{\partial s^2}$ for both large and small values of $s$.

Consequently, the set $\Lambda$ is a bounded set. Since the closure of a bounded set is also bounded, we conclude that the closure of $\Lambda$, i.e. $\text{cl}(\Lambda)$, is compact.

Due to the fact that a compact set always contains its infimum we can define

$$t_0 := \inf\{t \geq 0 : (s, t) \in \text{cl}(\Lambda) \text{ for some } s \in \mathbb{R}^+\}.$$ 

From the initial condition, i.e. $u^\omega(s, 0) = g(s)$ and convexity of $g$ we have

$$\frac{d^2 u^\varepsilon(s, 0)}{ds^2} = \frac{d^2 (g(s) + \varepsilon \kappa(s))}{ds^2} \geq \varepsilon \frac{d^2 \kappa(s)}{ds^2} > 0$$

for all $s \in \mathbb{R}^+$. Hence we can conclude that $t_0 > 0$.

Moreover, at the point when the infimum is attained, i.e. $(s_0, t_0)$ for some $s_0 \in \mathbb{R}^+$

$$\frac{\partial^2 u^\varepsilon(s_0, t_0)}{\partial s^2} = 0.$$

This is a consequence of the continuity of the function $\frac{\partial^2 u^\varepsilon(s, t)}{\partial s^2}$ in $s$. In addition, for $t \in [0, t_0)$ we have $\frac{\partial^2 u^\varepsilon(s_0, t)}{\partial s^2} > 0$ and thus, by applying the symmetry of second derivatives at $t = t_0$, we derive

\begin{equation}
\frac{\partial^2}{\partial s^2} \left( \frac{\partial u^\varepsilon(s_0, t)}{\partial t} \right) = \frac{\partial}{\partial t} \left( \frac{\partial^2 u^\varepsilon(s_0, t)}{\partial s^2} \right) \leq 0.
\end{equation}
Furthermore, at \((s_0, t_0)\) we also have
\[
\frac{\partial^2 \hat{V}^\omega\left(s_0, t_0\right)}{\partial s^2} = \frac{\partial^2 \beta(s_0, t_0)}{\partial s^2} \frac{\partial^2 u^\omega(s_0, t_0)}{\partial s^2} + 2 \frac{\partial \beta(s_0, t_0)}{\partial s} \frac{\partial^3 u^\omega(s_0, t_0)}{\partial s^3}
\]
\[
+ \beta(s_0, t_0) \frac{\partial^4 u^\omega(s_0, t_0)}{\partial s^4} + \frac{\partial^2 \mu(s_0, t_0)}{\partial s^2} \frac{\partial u^\omega(s_0, t_0)}{\partial s}
\]
\[
+ 2 \frac{\partial \mu(s_0, t_0)}{\partial s} \frac{\partial^2 u^\omega(s_0, t_0)}{\partial s^2} + \frac{\mu(s_0, t_0)}{\partial s} \frac{\partial^3 u^\omega(s_0, t_0)}{\partial s^3}
\]
\[
- \frac{d^2 \omega(s_0)}{ds^2} u^\omega(s_0, t_0) - 2 \frac{d \omega(s_0)}{ds} \frac{\partial u^\omega(s_0, t_0)}{\partial s} - \omega(s_0) \frac{\partial^2 u^\omega(s_0, t_0)}{\partial s^2}
\]
\[
+ \int_{\mathbb{R}} \left( \frac{\partial^2 u^\omega(s_0 + \gamma(s_0, t_0), z)}{\partial s^2} \left( 1 + \frac{\partial \gamma(s_0, t_0, z)}{\partial s} \right)^2
\]
\[
+ \frac{\partial u^\omega(s_0 + \gamma(s_0, t_0, z), t_0)}{\partial s} \frac{\partial^2 \gamma(s_0, t_0, z)}{\partial s^2} - \gamma(s_0, t_0, z) \frac{\partial^3 u^\omega(s_0, t_0)}{\partial s^3}
\]
\[
- \left( 1 + 2 \frac{\partial \gamma(s_0, t_0, z)}{\partial s} \right) \frac{\partial^2 u^\omega(s_0, t_0)}{\partial s^2} - \frac{\partial^2 \gamma(s_0, t_0, z)}{\partial s^2} \frac{\partial u^\omega(s_0, t_0)}{\partial s}\right)m(dz).
\]
Since \(\frac{\partial u^\omega(s_0, t_0)}{\partial s^2} = 0\) and \(\frac{\partial^3 u^\omega(s_0, t_0)}{\partial s^3} = 0\) has a local minimum at \(s = s_0\), we have \(\frac{\partial^2 u^\omega(s_0, t_0)}{\partial s^2} = 0\) and
\[
\frac{\partial u^\omega(s_0, t_0)}{\partial s} = 0.
\]
Thus
\[
\frac{\partial^2 \hat{V}^\omega\left(s_0, t_0\right)}{\partial s^2} \geq \frac{\partial^2 \mu(s_0, t_0)}{\partial s^2} \frac{\partial u^\omega(s_0, t_0)}{\partial s}
\]
\[
- \frac{d \omega(s_0)}{ds} \frac{\partial u^\omega(s_0, t_0)}{\partial s}
\]
\[
+ \left( \frac{\partial u^\omega(s_0 + \gamma(s_0, t_0, z), t_0)}{\partial s} \frac{\partial^2 \gamma(s_0, t_0, z)}{\partial s^2}
\]
\[
- \frac{\partial u^\omega(s_0, t_0)}{\partial s} \frac{\partial^2 \gamma(s_0, t_0, z)}{\partial s^2} \right)m(dz).
\]
Since \(u^\omega(s, t_0)\) is convex in \(s\) and \(\frac{\partial^3 u^\omega(s_0, t_0)}{\partial s^3} = 0\), applying \(\[\]\) we can conclude that the integral part of the above expression is nonnegative. Moreover, the concavity of \(\omega\) and \(\[\]\) imply that
\[
\frac{\partial^2 \hat{V}^\omega\left(s_0, t_0\right)}{\partial s^2} \geq \varepsilon e^{C_{t_0}} \left( \frac{\partial^2 \mu(s_0, t_0)}{\partial s^2} \frac{\partial^2 \gamma(s_0, t_0, z)}{\partial s^2} \right) \frac{d \gamma(s_0, t_0)}{ds} = \varepsilon e^{C_{t_0}} \vartheta(s_0, t_0).
\]
Combining \(\[\]\) with \(\[\]\) and \(\[\]\) at \((s_0, t_0)\) we derive that
\[
\frac{\partial^2}{\partial s^2} \left( \frac{\partial u^\omega(s_0, t_0)}{\partial t} - \hat{V}^\omega\left(s_0, t_0\right) \right) = \varepsilon e^{C_{t_0}} \frac{d^2}{ds^2} \left( C \gamma(s_0) - \hat{V}^\omega\left(s_0, t_0\right) \right)
\]
\[
> -\varepsilon e^{C_{t_0}} \vartheta(s_0, t_0) \geq \frac{\partial^2}{\partial s^2} \left( \frac{\partial u^\omega(s_0, t_0)}{\partial t} - \hat{V}^\omega\left(s_0, t_0\right) \right)
\]
which is a contradiction. This confirms that the set \(\Lambda\) is empty, and thus \(u^\omega(s, t)\) is a convex function. Finally, letting \(\varepsilon \to 0\) we conclude that \(u^\omega(s, t)\) is convex in \(s\) for all \(t \in [0, T]\). \(\square\)

Using the same arguments like in the proof of \(\[\]\) Thm. 4.1, we can resign from Assumptions (B) and (C) in Theorem 15 that is, the following theorem holds true.

**Theorem 18.** Let assumptions of Theorem 2 hold true. Then \(V^\omega_E(s, t)\) is convex with respect to \(s\) at all times \(t \in [0, T]\).
We are ready to give the proof of our first main result.

**Proof of Theorem 2.** As noted in [34, Sec. 7], under conditions (A1)–(A4) for each $p \geq 1$ there exists a constant $C$ such that the stock price process given in (5) satisfies

$$\mathbb{E}_s \left[ \sup_{0 \leq t \leq T} |S_t|^p \right] \leq C(1 + s^p).$$

Together with (A5) and (A6) it implies that the value function given by

$$V^\omega_{A,T}(s,t) := \sup_{\tau \in \mathcal{T}_T} \mathbb{E}_s \left[ e^{-\int_\tau^T \omega(S_u) du} g(S_{\tau}) \right]$$

is well-defined, where $\mathcal{T}_T$ is the family of $\mathcal{F}_t$-stopping times with values in $[t,T]$ for fixed maturity $T > 0$. Moreover, we denote

$$V^\omega_{A}(s) := V^\omega_{A,T}(s,0).$$

Let us define now a Bermudan option with the value function of the form

$$V^\omega_{B,\Xi}(s,t) := \sup_{\tau \in \mathcal{T}_\Xi} \mathbb{E}_s \left[ e^{-\int_\tau^T \omega(S_u) du} g(S_{\tau}) \right],$$

where $\mathcal{T}_\Xi$ is the set of stopping times with values in

$$\mathcal{B}_\Xi = \left\{ \frac{n}{2^\Xi}(T-t) + t : n = 0, 1, ..., 2^\Xi \right\},$$

where $\Xi$ is some positive integer number. To simplify the notation, we denote

$$V^\omega_{B,\Xi}(s) := V^\omega_{B,\Xi}(s,0).$$

In contrast to the American options, the Bermudan options are the options that can be exercised at one of the finitely many number of times.

Now we show that $V^\omega_{B,\Xi}(s,t)$ inherits the property of convexity from its European equivalent $V^\omega_{E}(s,t)$. Next, we generalise this result to the American case $V^\omega_{A}(s)$.

**Lemma 19.** Let assumptions of Theorem 2 hold true. Then $V^\omega_{B,\Xi}(s,t)$ is convex with respect to $s$ at all times $t \in [0,T]$.

Its proof is given in Appendix.

As the possible exercise times of the Bermudan option get denser, the value function $V^\omega_{B,\Xi}(s,t)$ converges to $V^\omega_{A,T}(s,t)$. To formalise this result, we proceed as follows. For a given stopping time $\tau^T_0$ that takes values in $[0,T]$, we define

$$\tau_{\Xi} := \inf \{ t \in \mathcal{B}_\Xi : t \geq \tau^T_0 \}.$$ 

Then $\tau_{\Xi} \in \mathcal{B}_\Xi$ is a stopping time and $\tau_{\Xi} \to \tau^T_0$ almost surely as $\Xi \to +\infty$. Moreover, by the dominated convergence theorem, we obtain

$$\mathbb{E}_s \left[ e^{-\int_{\tau_{\Xi}}^{\tau^T_0} \omega(S_u) du} g(S_{\tau_{\Xi}}) \right] - \mathbb{E}_s \left[ e^{-\int_{\tau^T_0}^{\tau^T_0} \omega(S_u) du} g(S_{\tau^T_0}) \right] \leq \mathbb{E}_s \left| e^{-\int_{\tau_{\Xi}}^{\tau^T_0} \omega(S_u) du} g(S_{\tau_{\Xi}}) - e^{-\int_{\tau^T_0}^{\tau^T_0} \omega(S_u) du} g(S_{\tau^T_0}) \right| \to 0$$

as $\Xi \to \infty$. Therefore, it follows that

$$\liminf_{\Xi \to \infty} V^\omega_{B,\Xi}(s) \geq V^\omega_{A,T}(s).$$

Since it is obvious that

$$V^\omega_{B,\Xi}(s) \leq V^\omega_{A,T}(s),$$
we finally derive

\[ V_{t_x}^\omega(s) \to V_{t_y}^\omega(s) \]

as \( \Xi \to \infty \). To receive our claim we take the maturity \( T \) tending to infinity. \hfill \Box

**Proof of Theorem 5** Before we proceed to the actual proof let us remind the main exit identities from 49. Let

\[ \sigma_a^+ := \inf\{ t > 0 : X_t \geq a \}, \quad \sigma_a^- := \inf\{ t > 0 : X_t \leq a \} \]

for some \( a \in \mathbb{R} \). Then for the function \( \eta \) defined in \( 43 \) we have

\( 52 \)

\[ \mathbb{E} \left[ e^{-\int_0^{\tau_a^+} \eta(X_s) \mathrm{d}s} \right| X_0 = x] = \frac{\mathcal{H}(\eta)(x)}{\mathcal{H}(\eta)(a)}, \]

\( 53 \)

\[ \mathbb{E} \left[ e^{-\int_0^{\tau_a^-} \eta(X_s) \mathrm{d}s} \right| X_0 = x] = \mathcal{Z}(\eta)(x) - c_{\mathcal{Z}(\eta)/\mathcal{W}(\eta)} \mathcal{W}(\eta)(x), \]

where \( c_{\mathcal{Z}(\eta)/\mathcal{W}(\eta)} = \lim_{x \to \infty} \frac{\mathcal{Z}(\eta)(x)}{\mathcal{W}(\eta)(x)} \) and we use condition \( \eta(x) = c \) for all \( x \leq 0 \) and some constant \( c \in \mathbb{R} \) in the first identity. Denoting

\[ \tau_a^+ := \inf\{ t > 0 : S_t \geq a \}, \quad \tau_a^- := \inf\{ t > 0 : S_t \leq a \} \]

and keeping in mind that \( S_t = e^{X_t} \), from \( 52 \) and \( 53 \) we can conclude that

\( 54 \)

\[ \mathbb{E}_s \left[ e^{-\int_0^{\tau_a^+} \omega(S_s) \mathrm{d}s} \tau_a^+ < \infty \right] = \frac{\mathcal{H}(\omega)(s)}{\mathcal{H}(\omega)(a)}; \]

\[ \mathbb{E}_s \left[ e^{-\int_0^{\tau_a^-} \omega(S_s) \mathrm{d}s} \tau_a^- < \infty \right] = \mathcal{Z}(\omega)(s) - c_{\mathcal{Z}(\omega)/\mathcal{W}(\omega)} \mathcal{W}(\omega)(s), \]

where \( \omega(s) = \omega(e^x) = \eta(x) \) and the functions \( \mathcal{Z}(\omega)(s), \mathcal{W}(\omega)(s), \mathcal{H}(\omega)(s) \) were defined in \( 16, 17 \) and \( 18 \).

We consider three possible cases of a position of the initial state \( S_0 = s \) of the process \( S_t \).

1. \( s < l \): As the process \( S_t \) is spectrally negative and starts below the interval \([l, u] \), it can enter this interval only in a continuous way and hence \( \tau_{l,u} = \tau_1^+ \) and \( S_{\tau_{l,u}} = l \). Thus from \( 54 \)

\[ v_{\mathcal{A}_{\mathcal{P}^0}}^\omega(s, l, u) = \mathbb{E}_s \left[ e^{-\int_0^{\tau_1^+} \omega(S_s) \mathrm{d}s}; S_{\tau_1^+} = l \right] (K - l) = \frac{\mathcal{H}(\omega)(s)}{\mathcal{H}(\omega)(l)} (K - l). \]

2. \( s \in [l, u] \): If the process \( S_t \) starts inside the interval \([l, u] \) which is an optimally stopping, we decide to exercise our option immediately, i.e. \( \tau_{l,u} = 0 \). Therefore, we have

\[ v_{\mathcal{A}_{\mathcal{P}^0}}^\omega(s, l, u) = K - s. \]

3. \( s > u \): There are three possible cases of entering the interval \([l, u] \) by the process \( S_t \) when it starts above \( u \): either \( S_t \) enters \([l, u] \) continuously going downward or it jumps from \((u, +\infty) \) to \((l, u) \) or \( S_t \) jumps from the interval \((u, +\infty) \) to the interval \((0, l) \) and then, later, enters \([l, u] \) continuously. We can distinguish these cases in the following way

\[ v_{\mathcal{A}_{\mathcal{P}^0}}^\omega(s, l, u) = \mathbb{E}_s \left[ e^{-\int_0^{\tau_u^{-}} \omega(S_s) \mathrm{d}s} (K - S_{\tau_u^-}) \right] + \mathbb{E}_s \left[ e^{-\int_0^{\tau_l^-} \omega(S_s) \mathrm{d}s} (K - S_{\tau_l^-}) \right]. \]
To analyse the first component in (55), note that
\[
\mathbb{E}_s \left[ e^{-\int_0^{\tau^+} \omega(S_u) du} (K - S_{\tau^+}) ; \tau^+ < \tau_1^- \right] = \mathbb{E}_s \left[ e^{-\int_0^{\tau^+} \omega(S_u) du} (K - S_{\tau^-}) ; S_{\tau^-} \in [l, u] \right] = \int_{(l, u)} (K - z) \mathbb{E}_s \left[ e^{-\int_0^{\tau^+} \omega(S_u) du} ; S_{\tau^-} \in dz \right] + (K - u) \mathbb{E}_s \left[ e^{-\int_0^{\tau^+} \omega(S_u) du} ; S_{\tau^-} = u \right].
\]

We express now above formulas in terms of formulas (56) and (57). From the compensation formula for Lévy processes given in [45, Thm. 4.4] we have
\[
\mathbb{E}_s \left[ e^{-\int_0^{\tau^+} \omega(S_u) du} (K - S_{\tau^+}) ; \tau^+ < \tau_1^- \right] = \int_{(l, u)} (K - e^x) \mathbb{E}_s \left[ e^{-\int_0^{\tau^+} \omega(S_u) du} (K - S_{\tau^-}) ; S_{\tau^-} \in dz \right] + (K - u) \mathbb{E}_s \left[ e^{-\int_0^{\tau^+} \omega(S_u) du} \eta(X_{\sigma^+}) ; X_{\sigma^+} = 0 \mid X_0 = x \right]
\]
\[
= \int_{(l, u)} (K - e^{u - y}) \mathbb{E}_s \left[ e^{-\int_0^{\tau^+} \omega(S_u) du} \eta(X_{\sigma^+}) ; -X_{\sigma^+} \in dy \mid X_0 = x - u \right] + (K - u) \mathbb{E}_s \left[ e^{-\int_0^{\tau^+} \omega(S_u) du} \eta(X_{\sigma^+}) ; X_{\sigma^+} = 0 \mid X_0 = x - u \right].
\]

From the compensation formula for Lévy processes given in [45, Thm. 4.4] we have
\[
\mathbb{E}_s \left[ e^{-\int_0^{\tau^+} \omega(S_u) du} (K - S_{\tau^+}) ; \tau^+ < \tau_1^- \right] = \int_0^\infty r^{(\eta)}(x - \log u, z) \Pi(-z - dy) dz,
\]
where \( r^{(\eta)}(x - \log u, z) \) is the resolvent density of \( X_t \) killed by potential \( \eta \) and on exiting from positive half-line which is, by [49, Thm. 2.2], given by
\[
r^{(\eta)}(x - \log u, z) = \mathcal{W}^{(\eta)}(x - \log u) \lim_{y \to \infty} \frac{\mathcal{W}^{(\eta)}(y, z)}{\mathcal{W}^{(\eta)}(y)} - \mathcal{W}^{(\eta)}(x - \log u, z).
\]

Note that \( r^{(\eta)}(\log s - \log u, z) = r(s, u, z) \) for \( r(s, u, z) \) given in (22). To find \( \mathbb{E}_s \left[ e^{-\int_0^{\tau^+} \eta_\alpha(X_u) du} X_{\sigma^+} = 0 \mid X_0 = x - u \right] \), we consider
\[
\mathbb{E}_s \left[ e^{-\int_0^{\tau^+} \eta_\alpha(X_u) du} \alpha^{X_{\sigma^+}} \sigma^- < \infty \mid X_0 = x - u \right]
\]
for some \( \alpha > 0 \). Note that using the change of measure given in (20) it is equal to
\[
\mathbb{E}_s^{(\alpha)} \left[ e^{-\int_0^\infty \eta_\alpha(X_u) du} \sigma^- < \infty \mid X_0 = x - u \right]
\]
where \( \mathbb{E}_s^{(\alpha)} \) is an expectation with respect to \( \mathbb{P}^{(\alpha)} \) and \( \eta_\alpha(x) := \eta(x) - \psi(\alpha) \). From (53) we know that
\[
\mathbb{E}_s^{(\alpha)} \left[ e^{-\int_0^{\tau^+} \eta_\alpha(X_u) du} \sigma^- < \infty \mid X_0 = x - u \right] = \mathcal{Z}_\alpha^{(\eta_\alpha)}(x - \log u) - \frac{\mathcal{Z}_\alpha^{(\eta_\alpha)}(x - \log u)}{\mathcal{W}_\alpha^{(\eta_\alpha)}(x - \log u)} \mathcal{W}_\alpha^{(\eta_\alpha)}(x - \log u).
\]
Moreover, observe that
\[
\mathbb{E} \left[ e^{-f_0^\sigma_{\omega} \eta_w(X_w)dw + \alpha X_{\sigma_w^-}}; \sigma_w^- < \infty \mid X_0 = x - \log u \right] = \mathbb{E} \left[ e^{-f_0^\sigma_{\omega} \eta_w(X_w)dw}; X_{\sigma_w^+} = 0 \mid X_0 = x - \log u \right]
\]
\[
+ \mathbb{E} \left[ e^{-f_0^\sigma_{\omega} \eta_w(X_w)dw + \alpha X_{\sigma_w^-}}; X_{\sigma_w^-} < 0 \mid X_0 = x - \log u \right].
\]

Taking the limit \( \alpha \to \infty \) and using (58) we derive
\[
\lim_{\alpha \to \infty} \mathbb{E}^{(\alpha)} \left[ e^{-f_0^\sigma_{\omega} \eta_w(X_w)dw}; \sigma_w^- < \infty \mid X_0 = x - \log u \right] = \mathbb{E} \left[ e^{-f_0^\sigma_{\omega} \eta_w(X_w)dw}; X_{\sigma_w^-} = 0 \mid X_0 = x - \log u \right]
\]
and therefore we have
\[
\mathbb{E} \left[ e^{-f_0^\sigma_{\omega} \eta_w(X_w)dw}; X_{\sigma_w^-} = 0 \mid X_0 = x - \log u \right] = \lim_{\alpha \to \infty} \left( \mathcal{Z}_\omega^{(\eta_w)}(x - \log u - c_{\omega}^{(\eta_w)} \mathcal{W}_\omega^{(\eta_w)})(x - \log u) \right).
\]

Furthermore, the second component of (55) equals to
\[
\mathbb{E}_\omega \left[ e^{-f_0^{\tau_{l,u}} \log(S_t)dw} (K - S_{\tau_{l,u}}); \tau_{l,u} = \tau_{l,u}^- \right] = \mathbb{E} \left[ e^{-f_0^{\tau_{l,u}} \eta_w(X_w)dw} (K - e^{X_{\tau_{l,u}}^-}); \sigma_{\log u}^- = \sigma_{\log u}^- \mid X_0 = x \right]
\]
\[
= \mathbb{E} \left[ e^{-f_0^{\tau_{l,u}} \eta_w(X_w)dw} (K - e^{X_{\tau_{l,u}}^-}); X_{\sigma_{\log u}^-} < \log l \mid X_0 = x \right]
\]
\[
= \mathbb{E} \left[ e^{-f_0^{\sigma_{\log u}} \eta_w(X_w)dw} \mathbb{E}_{X_{\log u}^-} \left[ e^{-f_0^{\tau_{l,u}} \eta_w(X_w)dw} (K - e^{X_{\tau_{l,u}}^-}) \mid X_{\sigma_{\log u}^-} < \log l \mid X_0 = x \right] \right]
\]
\[
= \int_{\log u - \log l}^{\infty} \mathbb{E} \left[ e^{-f_0^{\sigma_{\log u}} \eta_w(X_w)dw} \mathbb{E}_{X_{\log u}^-} \left[ e^{-f_0^{\tau_{l,u}} \eta_w(X_w)dw} (K - e^{X_{\tau_{l,u}}^-}) \mid X_0 = \log u - y \right] \right] \frac{\mathcal{H}(\eta_w)(\log u - y)}{\mathcal{H}(\eta_w)(\log l)} dy.
\]

Now we have to express all scale functions in terms of the \( S_t \) scale functions defined in (16)-(19) with \( x = \log s \) and using (43). Finally, using (55) together with (56), (57), (59) and (61) completes the proof.

\( \square \)

**Proof of Theorem 6** From the fact that \( V^\omega_A(s) \in D(\mathcal{A}) \) and using classical arguments it follows that \( V^\omega_A(s) \) solves uniquely equation (21); see [33, Thm. 2.4, p. 37], [3] and [62, Thm. 1] for details. More formally, our function as a convex function is continuous (in whole domain). Since our boundary is sufficiently regular we know that the Dirichlet/Poisson problem can be solved uniquely in \( D(\mathcal{A}) \). This solution can then be identified with the value function \( V^\omega_A(s) \) itself using the stochastic calculus or infinitesimal generator techniques in the continuation set; see [50, p. 131] for further details. We are left with the proof of the smoothness at the boundary of stopping set. We prove it at \( u \). The proof at lower end follows exactly in the same way. We choose to follow the idea given in [47] although one can also apply [3] or similar arguments as the ones given in [26].

Suppose then that 1 is for \( (-\infty, 1) \). Since \( V^\omega_A(s) \geq g(s) \) and \( V^\omega_A(u) = g(u) \), we have
\[
\frac{V^\omega_A(u + h) - V^\omega_A(u)}{h} \geq \frac{g(u + h) - g(u)}{h}.
\]
Hence
\[
\lim\inf_{h \downarrow 0} \frac{V^\omega_A(u + h) - V^\omega_A(u)}{h} \geq g'(u).
\]
To get the opposite inequality we introduce
\[ \tau_h = \inf \{ t \geq 0 : S_t \in [l, u] | S_0 = u + h \}. \]
By assumed regularity, \( \tau_h \to 0 \) a.s. as \( h \downarrow 0 \). Moreover, by Markov property
\[ V^\omega_{\hat{X}}(u) \geq E_{\log u} \left[ e^{-\int_0^{\tau_u} \omega(S_w) dw} (S_{\tau_u}) \right]. \]
Then by \([B5]\) and the space homogeneity of \( \log S_t \),
\[
\begin{align*}
\frac{V^\omega_{\hat{X}}(u + h) - V^\omega_{\hat{X}}(u)}{h} &\leq \frac{E_{u+h} \left[ e^{-\int_0^{\tau_u} \omega(S_w) dw} (S_{\tau_u}) \right] - E_u \left[ e^{-\int_0^{\tau_u} \omega(S_w) dw} (S_{\tau_u}) \right]}{h} \\
&\leq \frac{E_{u+h} \left[ e^{-\int_0^{\tau_u} \omega(S_w) dw} ((u+h)S_{\tau_u}) \right] - E_u \left[ e^{-\int_0^{\tau_u} \omega(S_w) dw} (uS_{\tau_u}) \right]}{h}
\end{align*}
\]
and
\[
\limsup_{h \downarrow 0} \frac{V^\omega_{\hat{X}}(u + h) - V^\omega_{\hat{X}}(u)}{h} \leq g'(u),
\]
where we use the fact that \( g \) is continuously differentiable at \( u \) in the last step. This completes the proof.

**Proof of Theorem 8** We recall that
\[
V^\omega_{\hat{X}}(s, K, \zeta, \sigma, \Pi, l, u) = E_s [e^{-\int_0^{\tau_l,u} \omega(S_w) dw} (S_{\tau_{l,u}} - K)^+] = E[e^{-\int_0^{\tau_{l,u}} \eta(X_w) dw} (e^{X_{\tau_{l,u}}} - K)^+ | X_0 = x].
\]
By our assumption for general Lévy process \( X_t \) we can define new measure \( \mathbb{P}^{(1)} \) via
\[
\frac{d\mathbb{P}^{(1)}}{d\mathbb{P}} |_{X_t} = e^{X_t - \psi(1)t},
\]
see also \([20]\) (considered there only for spectrally negative Lévy process). Let \( x = \log S_0 = \log s \). Then
\[
\begin{align*}
E \left[ e^{-\int_0^{\tau_l,u} \eta(X_w) dw} (e^{X_{\tau_l,u}} - K)^+ | X_0 = x \right] \\
= E^{(1)} \left[ e^{-\int_0^{\tau_l,u} \eta(K, X_s) dw} (e^{X_{\tau_l,u}} - K)^+ | X_0 = \log K \right],
\end{align*}
\]
where \( \hat{S}_t = e^{\hat{X}_t} \) and \( \hat{X}_t = -X_t \) is the dual process to \( X_t \) and from \([26, 38, 54]\) it follows that under \( \mathbb{P}^{(1)} \) it is again Lévy process with the triple \((-\zeta, \sigma, \hat{\Pi})\) for \( \hat{\Pi} \) defined in \([25]\). This completes the proof.

**Proof of Theorem 10** We prove that for the function \( h \) satisfying \([27]\) we have
\[
E_s \left[ \frac{h(S_{\tau_l,u})}{h(s)} e^{-\int_0^{\tau_l,u} \omega(S_w) dw} \right] = 1.
\]
Since process \( S_t \) is continuous in Black-Scholes model, \( S_{\tau_{l,u}} \) equals either to \( l \) or \( u \) depending on the initial state of \( S_t \). We can distinguish three possible scenarios.
(1) $s < l$: As the process $S_t$ is a continuous process and starts below the interval $[l, u]$, then $\tau_{l,u} = \tau_l^+$ and $S_{\tau_{l,u}} = l$. Thus, we get

$$v^\omega_{A\text{Put}}(s, l, u) = \mathbb{E}_s \left[ e^{-\int_0^{\tau_l^+} \omega(S_w) dw}; S_{\tau_l^+} = l \right] (K - l)$$

(63)

$$= \frac{h(s)}{h(l)} (K - l).$$

(2) $s \in [l, u]$: If the process $S_t$ starts inside the interval $[l, u]$ which is the optimal stopping region, we decide to exercise our option immediately, i.e. $\tau_{l,u} = 0$. Therefore, we have

$$v^\omega_{A\text{Put}}(s, l, u) = K - s.$$

(64)

(3) $s > u$: Similarly to the case when $s < l$, the process $S_t$ can enter $[l, u]$ only via $u$ and thus $\tau_{l,u} = \tau_u^-$ and $S_{\tau_{l,u}} = u$. Therefore,

$$v^\omega_{A\text{Put}}(s, l, u) = \mathbb{E}_s \left[ e^{-\int_0^{\tau_u^-} \omega(S_w) dw}; S_{\tau_u^-} = u \right] (K - u)$$

(65)

$$= \frac{h(s)}{h(u)} (K - u).$$

Identities (63), (64) and (65) give the first part of the assertion of the theorem. Note that boundary condition (28) follows straightforward from the definition of the value function of the American put option. We are left with the proof of (62). Consider strictly positive and bounded by some $C$ function $h \in C^2(\mathbb{R}^+) \subset D(A)$. Then by [54, Prop. 3.2] the process

$$E^h(t) := \frac{h(S_t)}{h(S_0)} e^{-\int_0^t \frac{Ah(S_w)}{h(S_w)} dw},$$

is a mean-one local martingale, where in the case of Black-Scholes model

$$Ah(s) = \mu sh'(s) + \frac{\sigma^2 s^2}{2} h''(s).$$

Observe that equation (27) is equivalent to

$$\omega(s) = \frac{Ah(s)}{h(s)}.$$

Let

$$\tau_{l,u}^M := \tau_{l,u} \land M$$

for some fixed $M > 0$. Applying the optional stopping theorem for bounded stopping time, we derive

$$\mathbb{E}_s \left[ \frac{h(S_{\tau_{l,u}^M})}{h(s)} e^{-\int_0^{\tau_{l,u}^M} \omega(S_w) dw} \right] = 1.$$
It easy to notice that $\mathbb{P}(\tau_{l,u}^M \leq \tau_{\text{last}}(K)) = 1$. Then, from the boundedness of $h$, lower boundedness of $\omega$ and Cauchy–Schwarz inequality we obtain
\[
I_1 \leq \frac{C}{h(s)} \mathbb{E} \left[ e^{\omega \tau_{\text{last}}(K)}; \tau_{l,u} > M \right] = \frac{C}{h(s)} \mathbb{E} \left[ e^{\omega \tau_{\text{last}}(K)} \mathbb{1}_{\tau_{l,u} > M} \right] \leq \frac{C}{h(s)} \sqrt{\mathbb{E} \left[ e^{-2\omega \tau_{\text{last}}(K)} \right]} \mathbb{P}(\tau_{l,u} > M),
\]
where $\omega := \min_{s \in \mathbb{R}} \omega(s)$. Note that $\sqrt{\mathbb{E} \left[ e^{-2\omega \tau_{\text{last}}(K)} \right]} < \infty$ by [10 Thm. 2] because $\mathbb{E} e^{-2\omega B_t} < \infty$ for any $t \geq 0$. Thus $\lim_{M \to \infty} I_1 = 0$. Moreover,
\[
0 < I_2 \leq \frac{C}{h(s)} \mathbb{E} \left[ e^{\omega \tau_{\text{last}}(K)}; \tau_{l,u} < M \right].
\]
Hence by (60) and the dominated convergence we get (62) as long as $h$ is positive and bounded. Finally, since $S_{\tau_{l,u}}$ equals either to $l$ or $u$, the boundedness assumption could be skipped. This completes the proof.

Proof of Theorem 12 From Theorem 2 and Remark 3 it follows that the optimal exercise time is the first entrance to the interval $[l^*, u^*]$ and by Theorem 4 the value function $V_{l^*, u^*}^\omega(s)$ equals to the maximum over $l$ and $u$ of $V_{l^*, u^*}^\omega(s, l, u)$ defined in (4). We recall the observation that if the discounting function $\omega$ is positive, then it is never optimal to wait to exercise option for small asset prices, that is, always $\sigma = 0$, by the lack of memory of exponential random variable, using similar analysis like in the proof of Theorem 3 we have
\[
v_{l^*, u^*}^\omega(s, 0, u) = \mathbb{E} \left( K - e^{\log u - Y} \right) + \mathbb{E} \left[ e^{-\int_0^\tau \omega(S_w) dw}; \tau_u < \infty \right]
\]
\[
= \mathbb{E} \left( K - e^{\log u - Y} \right) + \mathbb{E} \left[ e^{-\int_0^\tau \eta_u(X_w) dw}; \sigma_0^- < \infty \mid X_0 = x - \log u \right]
\]
\[
= \mathbb{E} \left( K - e^{\log u - Y} \right) + \left( \mathbb{P}(\omega \in \omega_u(x - \log u)) - e^{\mathbb{P}(\omega \in \omega_u(x - \log u))} \right)
\]
\[
= \mathbb{E} \left( K - e^{\log u - Y} \right) + \left( \mathbb{P}(\omega \in \omega_u(x - \log u)) - e^{\mathbb{P}(\omega \in \omega_u(x - \log u))} \right).
\]
Observing that
\[
\mathbb{E} \left( K - e^{\log u - Y} \right) = K - \frac{u \varphi}{\varphi + 1}
\]
completes the proof of part (i).

If $\sigma > 0$ then
\[
v_{l^*, u^*}^\omega(s, 0, u) = \mathbb{E} \left( K - e^{\log u - Y} \right) + \mathbb{E} \left[ e^{-\int_0^\tau \eta_u(X_w) dw}; \sigma_0^- < \infty, X_{\sigma_0^-} = 0 \mid X_0 = x - \log u \right]
\]
\[
+ (K - u) \mathbb{E} \left[ e^{-\int_0^\tau \eta_u(X_w) dw}; \sigma_0^- < \infty, X_{\sigma_0^-} = 0 \mid X_0 = x - \log u \right].
\]
The first increment can be analysed like in the case of $\sigma = 0$. The expression for the second component follows from (60).

Finally, the smooth fit condition follows straightforward from Theorem 6.\qed
Proof of Theorem 13. Assume first that $\sigma = 0$. Then

\begin{equation}
W(x) = \Upsilon_1 + \Upsilon_2 e^{\gamma_2 x}.
\end{equation}

To produce ordinary differential equation for $W^{(\xi)}(x)$ we start from equation (12). Putting (67) there gives

\begin{equation}
W^{(\xi)}(x) = \Upsilon_1 + \Upsilon_2 e^{\gamma_2 x} + \Upsilon_1 \int_0^x \xi(y) W^{(\xi)}(y) dy + \Upsilon_2 \int_0^x e^{\gamma_2 (x-y)} \xi(y) W^{(\xi)}(y) dy.
\end{equation}

Taking the derivative of both sides gives

\begin{equation}
W^{(\xi)'}(x) = \Upsilon_2 \gamma_2 e^{\gamma_2 x} + \Upsilon_1 \xi(x) W^{(\xi)}(x) + \Upsilon_2 \left( \xi(x) W^{(\xi)}(x) + \gamma_2 \int_0^x e^{\gamma_2 (x-y)} \xi(y) W^{(\xi)}(y) dy \right).
\end{equation}

From (68) we have

\begin{equation}
\int_0^x e^{\gamma_2 (x-y)} \xi(y) W^{(\xi)}(y) dy = \frac{1}{\Upsilon_2} \left( W^{(\xi)}(x) - \Upsilon_1 - \Upsilon_2 e^{\gamma_2 x} - \Upsilon_1 \int_0^x \xi(y) W^{(\xi)}(y) dy \right).
\end{equation}

From (67) we have

\begin{equation}
W^{(\xi)'}(x) = ((\Upsilon_1 + \Upsilon_2) \xi(x) + \gamma_2) W^{(\xi)}(x) - \gamma_2 \Upsilon_1 - \gamma_2 \Upsilon_1 \int_0^x \xi(y) W^{(\xi)}(y) dy.
\end{equation}

We take the derivative of both sides again to get equation (34).

From (12) and (67) we obtain first boundary condition (35) and from (69) we derive second boundary condition (36). Similar analysis can be done for the $Z^{(\xi)}(x)$ scale function producing equation (34) and its boundary conditions. This completes the proof of the case (i).

In the case when $\sigma > 0$ observe that

\begin{equation}
W(x) = \Upsilon_1 + \Upsilon_2 e^{\gamma_2 x} + \Upsilon_3 e^{\gamma_3 x},
\end{equation}

thus from (12) $W^{(\xi)}(x)$ satisfies the following equation

\begin{equation}
W^{(\xi)}(x) = \Upsilon_1 + \Upsilon_2 e^{\gamma_2 x} + \Upsilon_3 e^{\gamma_3 x} + \int_0^x (\Upsilon_1 + \Upsilon_2 e^{\gamma_2 (x-y)} + \Upsilon_3 e^{\gamma_3 (x-y)}) \xi(y) W^{(\xi)}(y) dy.
\end{equation}

We simplify it deriving

\begin{equation}
W^{(\xi)}(x) = \Upsilon_1 + \Upsilon_2 e^{\gamma_2 x} + \Upsilon_3 e^{\gamma_3 x} + \Upsilon_1 \int_0^x \xi(y) W^{(\xi)}(y) dy + \Upsilon_2 \int_0^x e^{\gamma_2 (x-y)} \xi(y) W^{(\xi)}(y) dy + \Upsilon_3 \int_0^x e^{\gamma_3 (x-y)} \xi(y) W^{(\xi)}(y) dy.
\end{equation}

In the next step we take derivative of both sides to get

\begin{equation}
W^{(\xi)'}(x) = \Upsilon_2 \gamma_2 e^{\gamma_2 x} + \Upsilon_3 \gamma_3 e^{\gamma_3 x} + \Upsilon_1 \xi(x) W^{(\xi)}(x) + \Upsilon_2 \left( \xi(x) W^{(\xi)}(x) + \gamma_2 \int_0^x e^{\gamma_2 (x-y)} \xi(y) W^{(\xi)}(y) dy \right) + \Upsilon_3 \left( \xi(x) W^{(\xi)}(x) + \gamma_3 \int_0^x e^{\gamma_3 (x-y)} \xi(y) W^{(\xi)}(y) dy \right).
\end{equation}

From (71) we have

\begin{equation}
\int_0^x e^{\gamma_3 (x-y)} \xi(y) W^{(\xi)}(y) dy = \frac{1}{\Upsilon_3} \left( W^{(\xi)}(x) - \Upsilon_1 - \Upsilon_2 e^{\gamma_2 x} - \Upsilon_3 e^{\gamma_3 x} \right) + \Upsilon_1 \int_0^x \xi(y) W^{(\xi)}(y) dy - \Upsilon_2 \int_0^x e^{\gamma_2 (x-y)} \xi(y) W^{(\xi)}(y) dy.
\end{equation}
We put it into (72) deriving
\[ W^{(\xi)}(x) = \Upsilon_2(\gamma_2 - \gamma_3)e^{\gamma_2 x} + (\Upsilon_1 + \Upsilon_2 + \Upsilon_3)\xi(x)W^{(\xi)}(x) \]
+ \( \Upsilon_2(\gamma_2 - \gamma_3) \int_0^x e^{\gamma_2(x-y)}\xi(y)W^{(\xi)}(y)dy + \gamma_3 W^{(\xi)}(x) - \gamma_3 \Upsilon_1 \]
- \( \gamma_3 \Upsilon_1 \int_0^x \xi(y)W^{(\xi)}(y)dy. \)
(73)

Taking again derivative of both sides produces
\[ W^{(\xi)''}(x) = \Upsilon_2(\gamma_2 - \gamma_3)\gamma_2 e^{\gamma_2 x} + (\Upsilon_1 + \Upsilon_2 + \Upsilon_3)(\xi'(x)W^{(\xi)}(x) + \xi(x)W^{(\xi)'}(x)) \]
+ \( \Upsilon_2(\gamma_2 - \gamma_3) \left( \xi(x)W^{(\xi)}(x) + \gamma_2 \int_0^x e^{\gamma_2(x-y)}\xi(y)W^{(\xi)}(y)dy \right) + \gamma_3 W^{(\xi)'}(x) \]
- \( \gamma_3 \Upsilon_1 \xi(x)W^{(\xi)}(x). \)
(74)

From (73) we have
\[ \int_0^x e^{\gamma_2(x-y)}\xi(y)W^{(\xi)}(y)dy = \frac{1}{\Upsilon_2(\gamma_2 - \gamma_3)} \left( W^{(\xi)'}(x) - \Upsilon_2(\gamma_2 - \gamma_3)e^{\gamma_2 x} \right) \]
- \( (\Upsilon_1 + \Upsilon_2 + \Upsilon_3)\xi(x)W^{(\xi)}(x) - \gamma_3 W^{(\xi)}(x) + \gamma_3 \Upsilon_1 \]
+ \( \gamma_3 \Upsilon_1 \int_0^x \xi(y)W^{(\xi)}(y)dy. \)

We put it into (74) to get
\[ W^{(\xi)''}(x) = (\Upsilon_1 + \Upsilon_2 + \Upsilon_3)(\xi'(x)W^{(\xi)}(x) + \xi(x)W^{(\xi)'}(x)) \]
+ \( \Upsilon_2(\gamma_2 - \gamma_3)\xi(x)W^{(\xi)}(x) \]
+ \( \gamma_2 \left( W^{(\xi)'}(x) - (\Upsilon_1 + \Upsilon_2 + \Upsilon_3)\xi(x)W^{(\xi)}(x) - \gamma_3 W^{(\xi)}(x) \right) \]
+ \( \gamma_3 \Upsilon_1 + \gamma_3 \Upsilon_1 \int_0^x \xi(y)W^{(\xi)}(y)dy \)
+ \( \gamma_3 W^{(\xi)'}(x) - \gamma_3 \Upsilon_1 \xi(x)W^{(\xi)}(x). \)

Taking again derivative and simplifying gives
\[ W^{(\xi)'''}(x) = ((\Upsilon_1 + \Upsilon_2 + \Upsilon_3)\xi(x) + \gamma_2 + \gamma_3)W^{(\xi)''}(x) \]
+ \( (2(\Upsilon_1 + \Upsilon_2 + \Upsilon_3)\xi'(x) + \Upsilon_2(\gamma_2 - \gamma_3)\xi(x) - (\Upsilon_1 + \Upsilon_2 + \Upsilon_3)\gamma_2 \xi(x) - \gamma_2 \gamma_3 - \gamma_3 \Upsilon_1 \xi(x))W^{(\xi)'}(x) \)
+ \( ((\Upsilon_1 + \Upsilon_2 + \Upsilon_3)\xi''(x) + \Upsilon_2(\gamma_2 - \gamma_3)\xi'(x) - \gamma_2(\Upsilon_1 + \Upsilon_2 + \Upsilon_3)\xi'(x) + \gamma_2 \gamma_3 \Upsilon_1 \xi(x) - \gamma_3 \Upsilon_1 \xi'(x))W^{(\xi)}(x) \)

which is the equation that we wanted to prove.

From (70) and (12) we have
\[ W^{(\xi)}(0) = \Upsilon_1 + \Upsilon_2 + \Upsilon_3. \]

From (73) it follows that
\[ W^{(\xi)'}(0) = \Upsilon_2 \gamma_2 + \Upsilon_3 \gamma_3 + (\Upsilon_1 + \Upsilon_2 + \Upsilon_3)^2 \xi(0). \]

Finally, from (74) we have
\[ W^{(\xi)''}(0) = \Upsilon_2 \gamma_2(\gamma_2 - \gamma_3) + (\Upsilon_1 + \Upsilon_2 + \Upsilon_3)(\xi'(0)W^{(\xi)}(0) + \xi(0)W^{(\xi)'}(0)) \]
+ \( \Upsilon_2(\gamma_2 - \gamma_3)\xi(0)W^{(\xi)}(0) + \gamma_3 W^{(\xi)'}(0) - \gamma_3 \Upsilon_1 \xi(0)W^{(\xi)}(0). \)
The analysis for $Z^{(g)}(x)$ can be done in the same way. This completes the proof. □

5. Appendix

**Proof of Lemma** 16 Firstly, we define the function $f : \mathbb{R}^+ \to \mathbb{R}$ of the form

$$
f(s) = \begin{cases} 
-\frac{1}{s}, & s \in (0, 1], \\
1, & s \in [2, +\infty)
\end{cases}
$$

such that $f(s) \in C^2(\mathbb{R}^+)$ and $f'(s) > 0$ for all $s \in \mathbb{R}^+$.

Taking $Y_t = f(S_t)$ and applying Itô’s lemma on (5), we obtain

$$
dY_t = \mu(Y_{t-}, t)dt + \sigma(Y_{t-}, t)dB_t + \int_{\mathbb{R}} \gamma(Y_{t-}, t, z)\tilde{v}(dt, dz),
$$

where

$$
\begin{align*}
\mu(y, t) &= \mu(f^{-1}(y), t)f'(f^{-1}(y)) + \frac{\sigma^2(f^{-1}(y), t)}{2} f''(f^{-1}(y)) \\
&\quad + \int_{\mathbb{R}} (\gamma(y, t, z) - f'(f^{-1}(y))\gamma(f^{-1}(y), t, z)) m(dz), \\
\sigma(y, t) &= f'(f^{-1}(y))\sigma(f^{-1}(y), t), \\
\gamma(y, t, z) &= f(f^{-1}(y) + \gamma(f^{-1}(y), t, z)) - y.
\end{align*}
$$

We define also the function

$$
\tilde{\omega}(y) := \omega(f^{-1}(y))
$$

and

$$
\tilde{g}(y) := g(f^{-1}(y)).
$$

We can now verify that the functions $\tilde{\mu}(y, t), \tilde{\sigma}(y, t), \tilde{\gamma}(y, t, z)$ and $\tilde{g}(y)$ satisfy conditions (2.2) – (2.5) from Section 2]. Let

$$
v(y, t) := V^{\gamma}(f^{-1}(y), t).
$$

From Theorem 3.1 it follows that $v(y, t)$ is a viscosity solution to

$$
\begin{align*}
\begin{cases}
\hat{L}v(y, t) = \hat{f}(y, t), & (y, t) \in \mathbb{R} \times [0, T), \\
v(y, T) = \tilde{g}(y), & y \in \mathbb{R},
\end{cases}
\end{align*}
$$

where

$$
\hat{L}v(y, t) = -\frac{\partial v(y, t)}{\partial t} - \frac{\sigma^2(y, t)}{2} \frac{\partial^2 v(y, t)}{\partial y^2} - \mu(y, t) \frac{\partial v(y, t)}{\partial y} + \tilde{\omega}(y)v(y, t)
$$

with

$$
\mu(y, t) = \tilde{\mu}(y, t) - \int_{\mathbb{R}} \tilde{\gamma}(y, t, z)m(dz)
$$

and

$$
\hat{f}(y, t) = -\int_{\mathbb{R}} (v(y + \gamma(y, t, z), t) - v(y, t)) m(dz).
$$

In addition, using Prop. 3.3 yields that $v(y, t) \in C(\mathbb{R} \times [0, T])$ and it satisfies

$$
|v(y_2, t_2) - v(y_1, t_1)| \leq C((1 + |y_2|)|t_2 - t_1|^{\frac{3}{2}} + |y_2 - y_1|)
$$

for some $C > 0$ and for all $t_1, t_2 \in [0, T]$ and $y_1, y_2 \in \mathbb{R}$. Based on Thm. 14] and assumptions put on $\gamma$ we can conclude that $\hat{f}(y, t) \in C_\alpha(\mathbb{R} \times [0, T]) \cap C_{pol}(\mathbb{R} \times [0, T])$. Then applying Thm. 14] give us the existence of a unique classical solution $w(y, t)$ to such that $w(y, t) \in C^{2, 1}(\mathbb{R} \times [0, T]) \cap C_{pol}(\mathbb{R} \times [0, T])$. In view
of the fact that \( w(y, t) \) is continuous, we can observe that \( f(y, t) \) is Lipschitz continuous in \( y \), uniformly in \( t \). Hence from [33] Lem. 3.1 we know that that \( w(y, t) \) is also Lipschitz continuous in \( y \), uniformly in \( t \). From the uniqueness result given in [33] Thm. 4.1 we can deduce that \( v(y, t) = w(y, t) \). Applying [39] Thm. A.18, we find that \( v(y, t) \in C_{\alpha}^{1,1}(\mathbb{R} \times [0, T]) \). Changing back to the original coordinates, it follows that \( V^\omega_E(s, t) \in C_{\alpha}^{1,1}(\mathbb{R}^+ \times [0, T]) \cap C_{pol}(\mathbb{R}^+ \times [0, T]) \) and it satisfies [46].

**Proof of Lemma 17** The proof follows in the same way as the proof of Lemma 16. However, this time we apply [39] Thm. A.20 which guarantees the existence of a unique classical solution \( w(y, t) \) of (79) satisfying \( w(y, t) \in C^{2,1}_{pol}(\mathbb{R} \times [0, T]) \). Hence, coming back to the original coordinates, we have that \( V^\omega_E(s, t) \in C^{2,1}_{pol}(\mathbb{R}^+ \times [0, T]) \). Therefore, there exist constants \( n > 0 \) and \( K > 0 \) such that

\[
\left| \frac{\partial^2 V^\omega_E(s, t)}{\partial s^2} \right| \leq K(s^{-n} + s^n)
\]

for all \( (s, t) \in \mathbb{R}^+ \times [0, T] \). This completes the proof.

**Proof of Lemma 19** By the dynamic programming principle formulated e.g. in [33], the value function \( V^\omega_B(s, t) \) satisfies

1. At time \( t = T \), the value function \( V^\omega_B(s, t) \) is equal to \( g(s) \).
2. Given the price \( V^\omega_B(s, t_n) \) at the time \( t_n = \frac{n}{T} T \), the price at time \( t_{n-1} = \frac{n-1}{T} T \) is \( V^\omega_B(s, t_{n-1}) = \max\{E_{s,t_{n-1}}[e^{-\int_{t_{n-1}}^{t_n} \omega(S_u)du} V^\omega_B(S_{t_{n}}, t_n)], g(s)\} \).

Thus, the price \( V^\omega_B(s, t_{n-1}) \) of a Bermudan option at \( t = t_{n-1} \) can be calculated inductively as the maximum of the payoff function \( g \) and the price of a European option with expiry \( t_n \) and payoff function \( V^\omega_B(s, t_n) \). From Theorem 15 we know that the value function of European option is convex in \( s \) provided the payoff function is convex, and since the maximum of two convex functions is again a convex function, we conclude that the Bermudan option price \( V^\omega_B(s, t) \) is convex in \( s \) for all \( t \in [0, T] \).

6. **Concluding remarks**

In this paper, we have identified the value function in the optimal stopping problem with functional discounting. We have performed a numerical analysis as well. It is tempting to analyse other discounting functions. For example \( \omega \) might be a random function or just simply a random variable dependent on the asset process \( S_t \). One can take other processes as a discount rate where the dependence is introduced not only via correlation between gaussian components but via common jump structure. This jump-dependence is crucial since crashes in the market affect large portion of business at the same time; see e.g. [22].

One can take Poisson version of American options where exercise is possible only at independent Poisson epochs as well. First attempt for classical perpetual American options has been already made in [55]. We believe that present analysis can be generalised to this set-up.

Obviously, it would be good to work out details for different payoff functions, hence for various options. One could think of barrier options, Russian, Israeli or Swing options. What is maybe even more interesting for the future analysis is taking into account Markov switching markets and using omega scale matrices introduced in [25]. We expect that in this setting the optimal exercise time is also the first entrance time to the interval which ends depend on the governing Markov chain.

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