AT LEAST HALF OF ALL GRAPHS SATISFY $\chi \leq \frac{1}{4}\omega + \frac{3}{4}\Delta + 1$

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ABSTRACT. We prove that for any graph $G$ at least one of $G$ or $\bar{G}$ satisfies $\chi \leq \frac{1}{4}\omega + \frac{3}{4}\Delta + 1$. In particular, self-complementary graphs satisfy this bound.

1. Introduction

In [5] Reed conjectured that

(1) $\chi \leq \left\lceil \frac{\omega + \Delta + 1}{2} \right\rceil$.

In the same paper he proved that there exists $\epsilon > 0$ such that

$\chi \leq \epsilon \omega + (1 - \epsilon)\Delta + 1$,

holds for every graph. The $\epsilon$ used in the proof is quite small (less than $10^{-8}$).

We prove the following.

**Main Result.** Let $G$ be a graph. Then at least one of $G$ or $\bar{G}$ satisfies

$\chi \leq \frac{1}{4}\omega + \frac{3}{4}\Delta + 1$.

To prove this we combine a result from [4] on graphs containing a doubly critical edge with the following lemma.

**Key Lemma.** Every graph satisfies $\chi \leq \iota + \omega + \Delta + n + 2$.

Here $\iota$ is the maximum number of singleton color classes appearing in an optimal coloring of the graph (formally defined below).

2. Stinginess

In [4] it was shown that a doubly critical edge is enough to imply an upper bound on the chromatic number that is slightly weaker than Reed’s conjectured upper bound.

**Lemma 2.1.** If $G$ is a graph containing a doubly critical edge, then

$\chi(G) \leq \frac{1}{3}\omega(G) + \frac{2}{3}(\Delta(G) + 1)$.

The following two lemmas were proved in [1] using matching theory results.

**Lemma 2.2.** If $G$ is a graph with $\chi(G) > \left\lceil \frac{|G|}{2} \right\rceil$, then

$\chi(G) \leq \frac{\omega(G) + \Delta(G) + 1}{2}$. 

Lemma 2.3. If \( G \) is a graph with \( \alpha(G) \leq 2 \), then
\[
\chi(G) \leq \left\lceil \frac{\omega(G) + \Delta(G) + 1}{2} \right\rceil.
\]

Lemma 2.4. Let \( G \) be a graph for which every optimal coloring has all color classes of order at most 2. Then
\[
\chi(G) \leq \left\lceil \frac{\omega(G) + \Delta(G) + 1}{2} \right\rceil.
\]

Proof. If \( \alpha(G) \leq 2 \), the result follows by Lemma 2.3. Hence we may assume that we have an independent set \( I \subseteq G \) with \( |I| \geq 3 \). Put \( H = G \setminus I \). Since \( G \) has no optimal coloring containing a color class of order \( \geq 3 \), we have
\[
\chi(H) = \chi(G) = \frac{|G|}{2} = \frac{|H| + 3}{2} > \left\lceil \frac{|H|}{2} \right\rceil.
\]
Hence, by Lemma 2.2, we have
\[
\chi(G) = \chi(H) \leq \frac{\omega(H) + \Delta(H) + 1}{2} \leq \frac{\omega(G) + \Delta(G) + 1}{2}.
\]
The lemma follows.

Definition 1. The stinginess of a graph \( G \) (denoted \( \iota(G) \)) is the maximum number of singleton color classes appearing in an optimal coloring of \( G \). An optimal coloring of \( G \) is called stingy just in case it has the maximum number of singleton color classes.

Lemma 2.5. Let \( G \) be a graph and \( H \) an induced subgraph of \( G \). If \( \chi(G) = \chi(G \setminus H) + \chi(H) \), then \( \iota(G) \geq \iota(G \setminus H) + \iota(H) \).

Proof. Assume that \( \chi(G) = \chi(G \setminus H) + \chi(H) \). Then patching together any optimal coloring of \( G \setminus H \) with any optimal coloring of \( H \) yields an optimal coloring of \( G \). The lemma follows.

Lemma 2.6. Let \( G \) be a graph. Then \( \chi(G) \leq \frac{\iota(G) + |G|}{2} \).

Proof. Let \( C = \{I_1, \ldots, I_m, \{s_1\}, \ldots, \{s_{\iota(G)}\}\} \) be a stingy coloring of \( G \). Since \( |I_j| \geq 2 \) for \( 1 \leq j \leq m \), we have \( \chi(G) \leq \iota(G) + \frac{|G| - \iota(G)}{2} = \frac{|G| + \iota(G)}{2} \).

3. Respectfully Greedy Partial Colorings

Definition 2. Let \( G \) be a graph. A partial coloring \( C \) of \( G \) is called \( r \)-greedy just in case every color class has order at least \( r \).

Definition 3. Let \( G \) be a graph. A partial coloring of \( C \) of \( G \) is called respectful just in case \( \chi(G \setminus \cup C) = \chi(G) - |C| \).

Lemma 3.1. Let \( G \) be a graph and \( C \) a respectful 3-greedy partial coloring of \( G \) with \( |G \setminus \cup C| \) minimal. Then
\[
\chi(G) \leq \frac{\omega(G) + \Delta(G) + 1}{2} + \frac{|C| + 1}{2}.
\]
Proof. Put \( H = G \setminus \cup C \). By the minimality of \( |H| \), every optimal coloring of \( H \) has all color classes of order at most 2. Thus, by Lemma 2.4, we have

\[
\chi(H) \leq \frac{\omega(H) + \Delta(H) + 1}{2} + \frac{1}{2}.
\]

Using the minimality of \( |H| \) again, we see that every vertex of \( H \) is adjacent to at least one vertex in each element of \( C \). Hence \( \Delta(H) \leq \Delta(G) - |C| \). Putting it all together, we have

\[
\chi(G) = \chi(H) + |C|
\leq \frac{\omega(H) + \Delta(H) + 1}{2} + \frac{1}{2} + |C|
\leq \frac{\omega(H) + \Delta(G) - |C| + 1}{2} + \frac{1}{2} + |C|
\leq \frac{\omega(G) + \Delta(G) - |C| + 1}{2} + \frac{1}{2} + |C|
= \frac{\omega(G) + \Delta(G) + 1}{2} + \frac{|C| + 1}{2}.
\]

□

**Key Lemma.** Every graph satisfies \( \chi \leq \frac{\iota + \omega + \Delta + n + 2}{4} \).

Proof. Let \( C \) be a respectful 3-greedy partial coloring of a graph \( G \) with \( |G \setminus \cup C| \) minimal. Since \( \chi(G \setminus \cup C) = \chi(G) - |C| \) we have \( \iota(G \setminus \cup C) \leq \iota(G) \) (by Lemma 2.5). Applying Lemma 2.6 yields

\[
\chi(G) = \chi(G \setminus \cup C) + |C|
\leq \frac{\iota(G) + |G| - |\cup C|}{2} + |C|
\leq \frac{\iota(G) + |G| - |C|}{2}.
\]

Adding this inequality with the inequality in Lemma 3.1 gives

\[
2\chi(G) \leq \frac{\iota(G) + \omega(G) + \Delta(G) + |G| + 2}{2}.
\]

The lemma follows.

□

4. The Main Results

**Theorem 4.1.** Let \( G \) be a graph. Then at least one of the following holds,

1. \( \chi(G) \leq \frac{1}{3}\omega(G) + \frac{2}{3}(\Delta(G) + 1) \),
2. \( \chi(G) \leq \frac{\omega(G) + |G| + \Delta(G) + 3}{4} \).

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Proof. Assume that (1) does not hold. Then, by Lemma 2.1, we have \( \iota(G) < 2 \). Applying the Key Lemma gives

\[
\chi(G) \leq \frac{1 + \omega(G) + \Delta(G) + |G| + 2}{4}.
\]

The theorem follows. \( \square \)

**Corollary 4.2.** Let \( G \) be a graph satisfying \( \Delta \geq \frac{n}{2} \). Then \( G \) also satisfies

\[
\chi \leq \frac{1}{4} \omega + \frac{3}{4} (\Delta + 1).
\]

**Proof.** By Theorem 4.1, \( G \) satisfies

\[
\chi \leq \max \left\{ \frac{1}{3} \omega + \frac{2}{3} (\Delta + 1), \frac{\omega + n + \Delta + 3}{4} \right\}
\]

\[
\leq \max \left\{ \frac{1}{3} \omega + \frac{2}{3} (\Delta + 1), \frac{\omega + n + \Delta + 3}{4} \right\}
\]

\[
\leq \max \left\{ \frac{1}{3} \omega + \frac{2}{3} (\Delta + 1), \frac{\omega + 3 \Delta + 3}{4} \right\}
\]

\[
= \frac{1}{4} \omega + \frac{3}{4} (\Delta + 1).
\]

\( \square \)

We would like to find an upper bound on the chromatic number that must hold for a graph or its complement. The previous corollary is not quite good enough for this purpose since it doesn’t handle \( \frac{n-1}{2} \)-regular graphs. Instead, we use the following.

**Corollary 4.3.** Let \( G \) be a graph satisfying \( \Delta \geq \frac{n-1}{2} \). Then \( G \) also satisfies

\[
\chi \leq \frac{1}{4} \omega + \frac{3}{4} \Delta + 1.
\]

**Proof.** By Theorem 4.1, \( G \) satisfies

\[
\chi \leq \max \left\{ \frac{1}{3} \omega + \frac{2}{3} (\Delta + 1), \frac{\omega + n + \Delta + 3}{4} \right\}
\]

\[
\leq \max \left\{ \frac{1}{3} \omega + \frac{2}{3} (\Delta + 1), \frac{\omega + n + \Delta + 3}{4} \right\}
\]

\[
\leq \max \left\{ \frac{1}{3} \omega + \frac{2}{3} (\Delta + 1), \frac{\omega + 3 \Delta + 3}{4} \right\}
\]

\[
= \frac{1}{4} \omega + \frac{3}{4} \Delta + 1.
\]

\( \square \)

Since every graph satisfies \( \Delta + \bar{\Delta} \geq \Delta + n - 1 - \Delta = n - 1 \), combining the pigeonhole principle with Corollary 4.3 proves the following.

**Main Result.** Let \( G \) be a graph. Then at least one of \( G \) or \( \bar{G} \) satisfies

\[
\chi \leq \frac{1}{4} \omega + \frac{3}{4} \Delta + 1.
\]

5. **Some Related Results**

In [3] the following was proven.

**Lemma 5.1.** If \( G \) is a graph with \( \iota(G) > \frac{\omega(G)}{2} \), then

\[
\chi(G) \leq \frac{\omega(G) + \Delta(G) + 1}{2}.
\]
Theorem 5.2. Let $G$ be a graph. Then at least one of the following holds,

1. $\chi(G) \leq \frac{\omega(G) + \Delta(G) + 1}{2}$,
2. $\chi(G) \leq \frac{3}{8} \omega(G) + \frac{|G| + \Delta(G) + 2}{4}$.

Proof. Assume that (1) does not hold. Then, by Lemma 5.1, we have $\iota(G) \leq \frac{\omega(G)}{2}$. Applying the Key Lemma gives

$$\chi(G) \leq \frac{\omega(G)}{2} + \omega(G) + \Delta(G) + |G| + 2.$$ 

The theorem follows. \qed

References

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