EULER SUMS OF HYPERHARMONIC NUMBERS

Ayhan Dil
Department of Mathematics, Akdeniz University, 07058 Antalya Turkey
e-mail: adil@akdeniz.edu.tr
Khristo N. Boyadzhiev
Department of Mathematics and Statistics, Ohio Northern University Ada, Ohio 45810, USA,
e-mail: k-boyadzhiev@onu.edu

Abstract
The hyperharmonic numbers \( h_n^{(r)} \) are defined by means of the classical harmonic numbers. We show that the Euler-type sums with hyperharmonic numbers:

\[
\sigma(r, m) = \sum_{n=1}^{\infty} \frac{h_n^{(r)}}{n^m}
\]

can be expressed in terms of series of Hurwitz zeta function values. This is a generalization of a result of Mező and Dil. We also provide an explicit evaluation of \( \sigma(r, m) \) in a closed form in terms of zeta values and Stirling numbers of the first kind. Furthermore, we evaluate several other series involving hyperharmonic numbers.

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1 Introduction
In this paper we are interested in Euler-type sums with hyperharmonic numbers \( \sigma(r, m) \) (see definitions (1) and (2) below). Such series could be of interest in analytic number theory. We will show that these sums are related to the values of the Riemann zeta function. In [7] the authors considered the case \( r = 1 \). Here we extend this result to \( r > 1 \).

In the second section we express \( \sigma(r, m) \) as a special series involving zeta values. In the third section we evaluate \( \sigma(r, m) \) as a finite sum including Stirling numbers of the first kind, zeta values, and values of the digamma (psi) function.
In the last fourth section we use certain integral representations to evaluate several series with hyperharmonic numbers. For example,

\[ \sum_{n=1}^{\infty} \frac{h_n^{(r)}}{n(n+1)\ldots(n+r)} = \frac{\pi^2}{6r!} \]

and

\[ \sum_{n=1}^{\infty} h_n^{(r)} B(r+1, n+1) = 1. \]

### 1.1 Definitions and notation

The \( n \)-th harmonic number is defined by the \( n \)-th partial sum of the harmonic series

\[ H_n := \sum_{k=1}^{n} \frac{1}{k} \quad (n \in \mathbb{N} := \{1, 2, 3, \ldots\}), \]

where the empty sum \( H_0 \) is conventionally understood to be zero.

Starting with \( h_n^{(0)} = \frac{1}{n} \quad (n \in \mathbb{N}) \), the \( n \)-th hyperharmonic number \( h_n^{(r)} \) of order \( r \) is defined by (see \([4]\), see also \([7]\)):

\[ h_n^{(r)} := \sum_{k=1}^{n} h_k^{(r-1)} \quad (r \in \mathbb{N}). \]

It is easy to see that \( h_n^{(1)} := H_n \quad (n \in \mathbb{N}) \).

These numbers can be expressed in terms of binomial coefficients and ordinary harmonic numbers (see \([4, 7]\)):

\[ h_n^{(r)} = \binom{n + r - 1}{r - 1}(H_{n+r-1} - H_{r-1}). \]

The well-known generating functions of the harmonic and hyperharmonic numbers are given as

\[ \sum_{n=1}^{\infty} H_n x^n = -\frac{\ln (1-x)}{1-x} \]

and

\[ \sum_{n=1}^{\infty} h_n^{(r)} x^n = -\frac{\ln (1-x)}{(1-x)^{r+1}}. \]

Euler discovered the following formula (see, e.g., \([5, 10]\)):

\[ 2 \zeta_H (m) = (m + 2) \zeta (m + 1) - \sum_{n=1}^{m-2} \zeta (m - n) \zeta (n + 1) \quad (m \in \mathbb{N} \setminus \{1\}), \]

where \( \zeta_H (m) = \sum_{n=1}^{\infty} \frac{H_n^{(m)}}{n^m} \) and \( \zeta (s) \) is the Riemann zeta function, and, throughout this paper the empty sum understood to be nil.
For certain pairs of positive integers $r$ and $m$, several authors have evaluated the Euler sums

$$S(r, m) = \sum_{n=1}^{\infty} \frac{H_n^{(r)}}{n^m},$$

where $H_n^{(r)}$ denotes the generalized harmonic numbers of order $r$ defined by

$$H_n^{(r)} = 1 + \frac{1}{2^r} + \cdots + \frac{1}{n^r} = \sum_{k=1}^{n} \frac{1}{k^r}$$

(see [1, 5] and for an elementary procedure [3]). In the earlier works have been done with $S(r, m)$, in this paper, we want to give a closed form of the following sum:

$$\sigma(r, m) = \sum_{n=1}^{\infty} \frac{h_n^{(r)}}{n^m}.$$  \hspace{1cm} (6)

Using the relation (3), Mezö and Dil ([7], Corollary 3) found that the series $\sigma(r, m)$ converges for $m > r$, i.e.,

$$\sigma(r, m) = \sum_{n=1}^{\infty} \frac{h_n^{(r)}}{n^m} < \infty \quad (m > r).$$

A rearrangement transforms $\sigma(r, m)$ into the following sum ([7], Theorem 4):

$$\sigma(r, m) = \sum_{n=1}^{\infty} h_n^{(r-1)} \zeta(m, n) \quad (r \in \mathbb{N}; m \geq r + 1), \hspace{1cm} (7)$$

where $\zeta(s, a)$ is the Hurwitz (or generalized) zeta function defined by

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s} \quad (Re(s) > 1; a \notin \mathbb{Z}_0^-),$$

where $\mathbb{Z}_0^- := \{0, -1, -2, \ldots\}$.

Using (6) and (7), Mezö and Dil ([7], p. 364) obtained the following identity:

$$2 \sum_{k=1}^{\infty} \frac{\zeta(m, k)}{k} = (m + 2) \zeta(m + 1) - \sum_{n=1}^{m-2} \zeta(m - n) \zeta(n + 1) \quad (m \in \mathbb{N}\setminus\{1\}). \hspace{1cm} (8)$$

2 Euler Sums of Hyperharmonic Numbers

The following theorem provides a general version of the Equation (7).

\hspace{1cm}
Theorem 1 For $0 \leq k < r < m$ ($k, r, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$), we have

$$\sigma (r, m) = \sum_{n=1}^{\infty} \frac{h_n^{(r)}}{n^m} = \sum_{n=1}^{\infty} h_n^{(r-k-1)} \sum_{n \leq i_1 \leq i_2 \leq \cdots \leq i_k < \infty} \zeta (m, i_k),$$

where

$$\sum_{n \leq i_1 \leq i_2 \leq \cdots \leq i_k < \infty} \zeta (m, i_k) = \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \cdots \sum_{i_k=i_{k-1}}^{\infty} \zeta (m, i_k).$$

Proof. From (7) we have

$$\sigma (r, m) = \sum_{n=1}^{\infty} \frac{h_n^{(r)}}{n^m} = \sum_{n=1}^{\infty} \sum_{j=1}^{n} h_j^{(r-2)} \zeta (m, n).$$

A rearrangement of (11) gives

$$\sigma (r, m) = \sum_{n=1}^{\infty} h_n^{(r-2)} \sum_{i_1=n}^{\infty} \zeta (m, i_1).$$

Using a similar argument, after $k$-steps we obtain the desired result.

Remark 2 Let us consider two special cases of (9).

To express the Euler sums of hyperharmonic numbers $\sigma (r, m)$ in terms of the multiple sums of Hurwitz zeta function we set $k = r - 1$ to have

$$\sigma (r, m) = \sum_{n=1}^{\infty} \frac{h_n^{(r)}}{n^m} \sum_{n \leq i_1 \leq i_2 \leq \cdots \leq i_{r-1} < \infty} \zeta (m, i_{r-1}).$$

Also the case $k = r - 2$ gives

$$\sigma (r, m) = \sum_{n=1}^{\infty} H_n \sum_{n \leq i_1 \leq i_2 \leq \cdots \leq i_{r-2} < \infty} \zeta (m, i_{r-2}),$$

which is a representation of $\sigma (r, m)$ in terms of harmonic numbers and the multiple sums of Hurwitz zeta function.

3 A closed form of $\sigma (r, m)$

We shall present a closed form evaluation of the following sum:

$$\sigma (r, m) = \sum_{n=1}^{\infty} \frac{h_n^{(r)}}{n^m}, \text{ where } m \geq r + 1.$$
Lemma 3 (12) For every positive integer $m$ and every $r > 0$

$$
\mu(m, r) = \sum_{k=1}^{m-1} \frac{(-1)^{k-1}}{r^k} \zeta(m - k + 1) + \frac{(-1)^{m-1}}{r^m} (\Psi(r + 1) + \gamma),
$$

where $\Psi(s)$ is the Psi (or digamma) function defined by $\Psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}$ and $\gamma = -\Psi(1)$ is the Euler-Mascheroni constant.

Theorem 4 For $r, m \in \mathbb{N}$ with $r < m$, we have

$$
\sigma(r, m) = \frac{1}{(r-1)!} \sum_{k=1}^{r} \left\lfloor \frac{r}{k} \right\rfloor.
$$

Proof. Using the well-known expansion:

$$
x (x + 1) \ldots (x + n - 1) = \sum_{k=0}^{n} \binom{n}{k} x^k,
$$

we get

$$
\binom{n + r}{r} = \frac{1}{r!} \sum_{k=1}^{r+1} \left\lfloor \frac{r+1}{k} \right\rfloor n^{k-1}.
$$

Hence,

$$
h_n^{(r+1)} = \frac{1}{r!} \sum_{k=1}^{r+1} \left\lfloor \frac{r+1}{k} \right\rfloor n^{k-1} (H_{n+r} - H_r)
= \frac{1}{r!} \sum_{k=1}^{r+1} \left\lfloor \frac{r+1}{k} \right\rfloor n^{k-1} \left( H_n + \frac{1}{n+1} + \cdots + \frac{1}{n+r} - H_r \right).
$$

Now we can write

$$
\frac{h_n^{(r)}}{n^m} = \frac{1}{(r-1)!} \sum_{k=1}^{r} \left\lfloor \frac{r}{k} \right\rfloor \left( \frac{H_n}{n^m-k+1} - \frac{H_{r-1}}{n^{m-k+1}} + \sum_{j=1}^{r-1} \frac{1}{n^m-k+1(n+j)} \right).
$$

Therefore,

$$
\sum_{n=1}^{\infty} \frac{h_n^{(r)}}{n^m} = \frac{1}{(r-1)!} \sum_{k=1}^{r} \left\lfloor \frac{r}{k} \right\rfloor \left\{ \sum_{n=1}^{\infty} \frac{H_n}{n^{m-k+1}} - H_{r-1} \sum_{n=1}^{\infty} \frac{1}{n^{m-k+1}} + \sum_{j=1}^{r-1} \sum_{n=1}^{\infty} \frac{1}{n^{m-k+1}(n+j)} \right\}.
$$

5
In view of (15), we obtain (16).

Next we consider two particular cases $r = 2$ and $3$ of $\sigma (r, m)$.

Case $r = 2$.

By considering the case $r = 2$ in (16) we get

$$\sigma (2, m) = \zeta_H (m - 1) + \zeta_H (m) - \zeta (m - 1).$$

With the aid of Theorem 1, equation (17) can be written as

$$\sigma (2, m) = \sum_{n=1}^{\infty} H_n \zeta (m, n) = \zeta_H (m - 1) + \zeta_H (m) - \zeta (m - 1).$$

Setting $m = 3$ in (18) we obtain

$$\sigma (2, 3) = \sum_{n=1}^{\infty} H_n \zeta (3, n) = 2\zeta (3) + \frac{5}{4} \zeta (4) - \zeta (2)$$

and for $m = 4$ we have

$$\sigma (2, 4) = \sum_{n=1}^{\infty} H_n \zeta (4, n) = \frac{5}{4} \zeta (4) + 3\zeta (5) - \zeta (2) \zeta (3) - \zeta (3)$$

and so on.

Case $r = 3$.

By (9) and (16) we have

$$\sigma (3, m) = \sum_{n=1}^{\infty} h_n^{(2)} \zeta (m, n)$$

$$= \frac{1}{2} \zeta_H (m - 2) + \frac{3}{2} \zeta_H (m - 1) + \zeta_H (m) - \frac{5}{4} \zeta (m - 1) - \frac{3}{4} \zeta (m - 2)$$

Setting $m = 4$ in the above equation we obtain

$$\sigma (3, 4) = \sum_{n=1}^{\infty} h_n^{(2)} \zeta (4, n) = \frac{15}{6} \zeta (4) + \zeta_H (4) - \frac{1}{4} \zeta (3) - \frac{3}{4} \zeta (2).$$

4 Some series with hyperharmonic numbers

In this section we evaluate some specific series involving hyperharmonic numbers.

Proposition 5

$$\sum_{n=1}^{\infty} \frac{h_n^{(r)}}{(n + 1)(n + 2) \ldots (n + r + 1)} = \frac{1}{r!} \quad (r \in \mathbb{N}_0).$$
Proof. Using the formula (see [6])

\[ \frac{1}{r!} \int_0^1 t^n (1-t)^r \, dt = \frac{1}{(n+1)(n+2)\ldots(n+r+1)}, \]  

(23)

we can write

\[ \frac{1}{r!} \int_0^1 h_n^{(r)} t^n (1-t)^r \, dt = \frac{h_n^{(r)}}{(n+1)(n+2)\ldots(n+r+1)}. \]  

(24)

With the help of (23), we get

\[ \sum_{n=1}^{\infty} \frac{h_n^{(r)}}{n(n+1)(n+2)\ldots(n+r+1)} = -\frac{1}{r!} \int_0^1 \ln (1-t) \, dt. \]  

(25)

This equation completes the proof, since

\[ \int_0^1 \ln (1-t) \, dt = -1. \]  

(26)

Proposition 6

\[ \sum_{n=1}^{\infty} \frac{h_n^{(r)}}{n(n+1)\ldots(n+r)} = \frac{\pi^2}{6r!} \]  

(27)

Proof. In the same way as in the proof of Proposition 5 using (23), we can write

\[ \frac{1}{r!} \int_0^1 h_n^{(r)} t^n (1-t)^r \, dt \frac{dt}{t} = \frac{h_n^{(r)}}{n(n+1)\ldots(n+r)} \]  

(28)

from which it follows that

\[ \sum_{n=1}^{\infty} \frac{h_n^{(r)}}{n(n+1)\ldots(n+r)} = -\frac{1}{r!} \int_0^1 \frac{\ln (1-t)}{t} \, dt. \]  

(29)

Now, using the following known formula (see [6]):

\[ \int_0^1 \frac{\ln (1-t)}{t} \, dt = -\frac{\pi^2}{6}, \]  

(30)

we obtain (27). ■

Remark 7 Our results in Propositions 5 and 6 are involved in the Beta function \( B(x,y) \) defined by (see [9]):

\[ B(x,y) = \int_0^1 t^{x-1} (1-t)^{y-1} \, dt, \]  

(31)
where \( \text{Re}(x) > 0 \) and \( \text{Re}(y) > 0 \). In view of the following relation:

\[
B(r+1, n+1) = \frac{\Gamma(r+1)\Gamma(n+1)}{\Gamma(r+n+2)} = \frac{r!n!}{(r+n+1)!} = \frac{r!}{(n+1)(n+2)\ldots(n+r+1)},
\]

the Equations (22) and (27), respectively, can be written in the following forms:

\[
\sum_{n=1}^{\infty} h_n^{(r)} B(r+1, n+1) = 1 \quad \text{(32)}
\]

and

\[
\sum_{n=1}^{\infty} h_n^{(r)} B(r+1, n) = \frac{\pi^2}{6}.
\]

At the end of this section we give two specific series associated with harmonic numbers. Their proofs consist of routine manipulation with the generating function \[5\]. Therefore we omit proofs.

**Proposition 8** The following equations hold.

\[
\sum_{m=1}^{\infty} \frac{(-1)^{m+1}(2m+2n+3)}{(m+1)(m+2n+2)} H_m = 2 \ln 2 \sum_{k=0}^{n} \frac{1}{2k+1} - \sum_{j=1}^{2n+1} \frac{1}{j} \sum_{k=1}^{j} \frac{(-1)^{k-1}}{k},
\]

and

\[
\sum_{m=1}^{\infty} \frac{(-1)^{m+1}2n}{(m+1)(m+2n+1)} H_m = 2 \ln 2 \sum_{k=0}^{n-1} \frac{1}{2k+1} - \sum_{j=1}^{2n} \frac{1}{j} \sum_{k=1}^{j} \frac{(-1)^{k-1}}{k}.
\]

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