A Combinatorial Approach to Rauzy-type Dynamics II: the Labelling Method and a Second Proof of the KZB Classification Theorem

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Abstract. Rauzy-type dynamics are group actions on a collection of combinatorial objects. The first and best known example (the Rauzy dynamics) concerns an action on permutations, associated to interval exchange transformations (IET) for the Poincaré map on compact orientable translation surfaces. The equivalence classes on the objects induced by the group action have been classified by Kontsevich and Zorich, and by Boissy through methods involving both combinatorics, algebraic geometry, topology and dynamical systems. Our precedent paper [DS17] as well as the one of Fickenscher [Fic16] proposed an ad hoc combinatorial proof of this classification.

However, unlike those two previous combinatorial proofs, we develop in this paper a general method, called the labelling method, which allows one to classify Rauzy-type dynamics in a much more systematic way. We apply the method to the Rauzy dynamics and obtain a third combinatorial proof of the classification. The method is versatile and will be used to classify three other Rauzy-type dynamics in follow-up articles.

Another feature of this paper is to introduce an algorithmic method to work with the sign invariant of the Rauzy dynamics. With this method we can prove most of the identities appearing in the literature so far ([KZ03], [Del13], [Boi13], [DS17], ...) in an automatic way.
# Contents

1 Permutational diagram monoids and groups  

2 The labelling method  
   2.1 Definition and application  
   2.2 Difficulties and implementation  

3 Definition of the Rauzy dynamics  
   3.1 The Rauzy dynamics  
   3.2 Definition of the invariants  
   3.3 Exceptional classes  
   3.4 The classification theorems  

4 Proof overview  

5 Amenability of $\mathcal{S}_n$ to the labelling method  
   5.1 Boosted dynamics  
   5.2 Preliminaries: Definition of a consistent labelling  
   5.3 The amenability  

6 Cycle invariant and edge addition  

7 Shift-irreducible standard family  

8 The sign invariant  
   8.1 Arf functions for permutations  
   8.2 Automatic proofs of Arf identities  
   8.3 Arf relation for the induction  

9 A $I_2X$-permutation for every $(\lambda, r, s)$  

10 The induction  
   10.1 Every non exceptional class has a shift-irreducible family  
   10.2 Proof of theorem  
   10.3 Existence: for every valid invariant $(\lambda, r, s)$ there exists a permutation with invariant $(\lambda, r, s)$.  
   10.4 Proposition First step of the labelling method  
   10.5 Completeness: Every pair of permutations $(\sigma, \sigma')$ with invariant $(\lambda, r, s)$ are connected  
   10.6 Proposition Every non exceptional class contains a $I_2X$-permutation  
   10.7 The 2-point monodromy theorem  

A Exceptional classes
1 Permutational diagram monoids and groups

The labelling method that we introduce in section 2 is a method to classify Rauzy-type dynamics. In this section we define what we mean by Rauzy-type dynamics.

Let $X_I$ be a set of labelled combinatorial objects, with elements labelled from the set $I$. We shall identify it with the set $[n] = \{1, 2, \ldots, n\}$ (and use the shortcut $X_n \equiv X_{[n]}$). The symmetric group $\mathfrak{S}_n$ acts naturally on $X_n$, by producing the object with permuted labels.

Vertex-labeled graphs (or digraphs, or hypergraphs) are a typical example. Set partitions are a special case of hypergraph (all vertices have degree 1). Matchings are a special case of partitions, in which all blocks have size 2. Permutations $\sigma$ are a special case of matchings, in which each block $\{i, j\}$ has $i \in \{1, 2, \ldots, n\}$ and $j = \sigma(i) + n \in \{n + 1, n + 2, \ldots, 2n\}$.

We will consider dynamics over spaces of this type, generated by operators of a special form that we now introduce:

Definition 1 (Monoid and group operators). We say that $A$ is a monoid operator on set $X_n$, if, for a map $\alpha : X_n \to \mathfrak{S}_n$, it consists of the map on $X_n$ defined by

$$A(x) = \alpha_x x,$$

where the action $\alpha_x x$ is in the sense of the symmetric-group action over $X_n$. We say that $A$ is a group operator if, furthermore, $\alpha_{A(x)} = \alpha_x$.

Said informally, the function $\alpha$ “poses a question” to the structure $x$. The possible answers are different permutations, by which we act on $x$. Note that in all our applications the set $Y_n = \{\alpha_x | x \in X_n\}$ of all possible permutations has a much smaller cardinality than $X_n$ and $\mathfrak{S}_n$, i.e. very few ‘answers’ are possible. In the Rauzy case, $|X_n| = |\mathfrak{S}_n| = n!$ while $|Y_n| = n$. The asymptotic behaviour is similar ($|X_n|$ is at least exponential in $n$, while $|Y_n|$ is linear) in all of our applications. Clearly we have:

Proposition 2. Group operators are invertible.

Proof. For a given value of $n$, let $A$ be a group operator on the set $X_n$. The property $\alpha_{A(x)} = \alpha_x$ implies that, for all $k \in \mathbb{N}$, $A^k(x) = (\alpha_x)^k x$. Thus, for all $x$ there exists an integer $d_A(x) \in \mathbb{N}^+$ such that $A^{d_A(x)}(x) = x$. More precisely, $d_A(x)$ is a (divisor) of the l.c.m. of the cycle-lengths of $\alpha_x$. Call $d_A = \text{lcm}_{x \in X_n} d_A(x)$ (i.e., more shortly, the l.c.m. over $y \in Y_n$ of the cycle-lengths of $\alpha_y$). Then $d_A$ is a finite integer, and we can pose $A^{-1} = A^{d_A - 1}$. The reasonings above show that $A$ is a bijection on $X_n$, and $A^{-1}$ is its inverse. \hfill $\Box$

Definition 3 (monoid and group dynamics). We call a monoid dynamics the datum of a family of spaces $\{X_n\}_{n \in \mathbb{N}}$ as above, and a finite collection $A = \{A_i\}$ of monoid operators. We call a group dynamics the analogous structure, in which all $A_i$’s are group operators.

1The use of $I$ instead of just $[n]$ is a useful notation when considering substructures: if $x \in X_n$, and $x' \subseteq x$ w.r.t. some notion of inclusion, it may be convenient to say that $x' \in X_I$ for $I$ a suitable subset of $[n]$, instead that the canonical one.
For a monoid dynamics on the datum $S_n = (X_n, A)$, we say that $x, x' \in X_n$ are strongly connected, $x \sim x'$, if there exist words $w, w' \in A^*$ such that $wx = x'$ and $w'x' = x$.

For a group dynamics on the datum $S_n = (X_n, A)$, we say that $x, x' \in X_n$ are connected, $x \sim x'$, if there exists a word $w \in A^*$ such that $wx = x'$.

Here the action $wx$ is in the sense of monoid action. Being connected is clearly an equivalence relation, and coincides with the relation of being graph-connected on the Cayley Graph associated to the dynamics, i.e. the digraph with vertices in $X_n$, and edges $x \leftrightarrow_i x'$ if $A_i^{\pm 1} x = x'$. An analogous statement holds for strong-connectivity, and the associated Cayley Digraph.

We call such dynamics Rauzy-type dynamics since the original Rauzy dynamics that inspired this definition is also of this type.

**Definition 4** (classes of configurations). Given a dynamics as above, and $x \in X_n$, we define $C(x) \subseteq X_n$, the class of $x$, as the set of configurations connected to $x$, $C(x) = \{ x' : x \sim x' \}$.

We will call those classes the Rauzy classes of the dynamics.

**Definition 5** (Invariant of classes). Given a dynamics as above, we say that $f : X_n \to G_n$ is an invariant of the dynamics if $f(x) = f(x')$ for every pair $(x, x')$ of connected combinatorial structures.

Thus given a dynamics $(X_n, A)$ our goal is to find a family of invariant $(f_i)$ such that two combinatorial structures are in the same Rauzy class if and only if they have the same invariants.

### 2 The labelling method

The labelling method propose a sequence of steps to classify a dynamics $(X_n, (A_k))$ for which we already know the set of invariants. That is to say, we wish to prove that the invariants we identified completely characterize the Rauzy classes.
Figure 2: $S'$ is a boosted sequence of $S$ for $(x, c)$, because red $\circ S'$ gives the same result as $S \circ $red. Note that $S'$ may be not unique: in the diagram we show a second boosted sequence of $S$ for $(x, c)$, namely $S''$. It is not necessarily the case that $(x', c') := S'(x, c)$ coincides with $(x'', c'') := S''(x, c)$, but merely that their reductions coincide (they must be both $y'$).

2.1 Definition and application

We place ourself in the framework of section §1. Let $X_n$ be a combinatorial set and $(A_k)$ be the set of operators of the dynamics. Before introducing the labelling method, let us lay down a number of necessary definitions.

**Definition 6 ((k, r)-coloring and reduction).** Let $x \in X_n$ be a combinatorial structure, and $k, r$ integers with $k + r = n$. A $(k, r)$-coloring $c$ of $x$ is a coloring of the $n$ vertices of $x$ into a black set of $k$ vertices and a gray set of $r$ vertices, such that, calling $y$ the restriction of $x$ to the black vertices, $y$ is also a combinatorial structure (thus $y \in X_k$). We call $y$ the reduction of $x$.

For example, for a matching $m \in M_n = X_{2n}$, in principle we shall color in black and gray the $2n$ points. However, in order for $y$ to be a matching, in a valid $(k, r)$-coloring we need that the endpoints of an edge of the matching are either both black or both gray (in particular $r$ and $k$ are even).

Our first important notion is that of a boosted sequence.

**Definition 7 (Boosted sequence and boosted dynamics).** Let $x \in X_n$ and let $y$ be the reduction of a $(k, r)$-coloring $c$ of $x$. Let $S$ be a sequence of operators in $X_k$ such that $S(y) = y'$.

If a sequence $S'$ in $X_n$ is such that $y'$ is the reduction of $(x', c') = S'(x, c)$, we say that $S'$ is a boosted sequence of $S$ for $(x, c)$. See figure 2.

A dynamics $((X_n)_n, (A_k)_k)$ has a boosted dynamics if for every $n$, $x \in X_n$, $(k, r)$-coloring $c$ of $x$ and sequence $S$ as above, there exists a boosted sequence $S'$.

Thus in the following diagram: red $\downarrow \quad y \quad S \quad y'$, a boosted sequence $S'$ is a lift of $S$ red $\downarrow \quad y \quad S \quad y'$. such that the square becomes commutative: red $\downarrow \quad y \quad S \quad y'$.
Figure 3: A representation of a combinatorial structure $x \in X_n$ with a labelling $\Pi_b$. The intervals are represented by the bottoms arcs.

Note that, for a given $(n, x, c, S)$ as above, the boosted sequence $S'$ is not necessarily unique. This is rather obvious when the dynamics is a group dynamics, as in this case, even without boosting, the set of sequences $S : x_1 \to x_2$ is a coset (w.r.t. sequence concatenation) within the group of all possible sequences, i.e., if $Sx_1 = x_2$, every sequence $S' : x_2 \to x_2$ is such that $(S')^k Sx_1 = x_2$ for all $k \in b\mathbb{Z}$. Nonetheless, with abuse of notation, in our definition of the boosted dynamics we will often refer to ‘the’ boosted sequence $S'$, by this referring to the most natural boosted sequence, w.r.t. some notion changing from case to case and depending on the invariants.

As we said in the introduction, we represent combinatorial structures in $X_n$ as $n$ vertices on the real line (indexed from left to right) with the combinatorial structure (be it matching, set partition, graph or hypergraph) placed above in the upper half plane. This is done in order to allow the dynamics operator to act graphically from below (they will be represented as permutations, contained within a horizontal strip), as is customarily the case for diagram algebras (like the partition algebra, the braid group, and so on).

This graphical representation is also useful in order to produce a certain construction on configurations, which we now describe. Place two new vertices, with label 0 and $n + 1$, to the left and right of the existing vertices 1, ..., $n$, respectively. Then, consider the intervals between the pairs of adjacent vertices $(i, i+1)$, for $i = 0, \ldots, n$. This modification of the structure $x \in X_n$ will be called a combinatorial structure with intervals.

If we have a $(k, r)$ coloring of $x$, in the construction of the combinatorial structure with intervals we will set the two extra points, 0 and $n + 1$, as black.

Let $\Sigma = \{b_0, \ldots, b_n\}$ be an alphabet of distinct symbols. We define a labelling of $x \in X_n$ to be a bijection

$$\Pi_b : \{0, \ldots, n\} \to \Sigma$$

i.e., a labelling of the intervals between the vertex of $x$ in the construction above, with the symbols from $\Sigma$. See figure 3.

In the following treatment, once $\Pi_b$ is given, it will be convenient to designate intervals either by their positions $\beta \in \{0, \ldots, n\}$ or by their labels $b \in \Sigma$, depending from the situation. As a convention, and in order to avoid confusion, we will use greek and latin letters as above in the two cases.

We need a definition for a consistent action of the dynamics on the labelling. Recall that in section 3 we have defined an operator $A$ of the dynamics as the action
of an application $\alpha : X_n \to \mathcal{S}_n$, so that $Ax = \alpha_x x$ is the symmetric-group action of $\alpha_x$ on $x$.

**Definition 8** (Labelling and dynamics). For $x \in X_n$, let $(x, \Pi_b)$ be a combinatorial structure with labelling. The operator $\hat{A}$ is the labelled extension of $A$ if it acts on $(x, \Pi_b)$ as follows:

$$\hat{A}(x, \Pi_b) = (\alpha_x(x), \Pi_b \circ \alpha'_y)$$

where, in analogy with the Definition $1 \alpha : X_n \to \mathcal{S}_n \alpha' : X_n \to \mathcal{S}_{n+1}$.

Thus, in this case, not only $A$ chooses a permutation $\alpha_x$, depending on $x$, by which it acts on the $n$ vertices of $x$, but it also chooses a permutation $\alpha'_y$, again depending only on $x$, by which it acts on the set of $n+1$ labels, in both cases in the sense of the symmetric-group action.

When there will be no confusion, we will often write $\hat{A}$ for $A$.

Such a definition is not unique, the choice will be made so as to verify the following property:

**Definition 9** (Labelling, vertices, and boosted dynamics). Let $x \in X_n$, $c$ a $(k,r)$-coloring of $x$, $y \in X_k$ the corresponding reduction, and $\Pi_b$ a labelling of $y$.

We say that a gray vertex $\ell$ of $x$ is within an interval $\beta$ of $y$, and write $\ell \in b$, if the black vertices of $x$, corresponding to the vertices $\beta$ and $\beta + 1$ of $y$, are the first black vertices to the left and to the right of $\ell$ (note that $\beta$ and $\beta + 1$ may be 0 and $k + 1$, i.e. the external vertices added in the interval construction). We say that the labelling is compatible with the boosted dynamics if the following statement holds:

Let $x$, $c$, $y$ and $\Pi_b$ as above, $S$ a sequence in $X_k$, and $S'$ a boosted sequence of $S$ for $(x,c)$. If $v$ is a gray vertex of $x$ within the interval $b \in \Sigma$ of $(y, \Pi_b)$, then the image of $v$ in $(x', c') = S'(x, c)$ must be within the image of the interval $b$ in $(y', \Pi'_b) = S(y, \Pi_b)$. In other words $v$ is within the interval with position $\Pi_b^{-1}(b)$ in $(y', \Pi'_b)$.

In other words, if the labelling is compatible with the boosted dynamics, we can keep track of the positions of the gray vertices by understanding how the labelling of a reduced combinatorial structure $y \in X_k$ evolves in the extended labelled dynamics.

**Definition 10.** Given a combinatorial structure $x \in X_n$ with a labelling $\Pi_b$, let $L(x, \Pi_b)$ be defined as

$$L(x, \Pi_b) = \{\Pi'_b \mid \exists S \text{ with } S(x, \Pi_b) = (x, \Pi'_b)\},$$

that is, the set of all the labellings that are reachable from $(x, \Pi_b)$ by a loop $S$ of the dynamics.

We say that the $r$-point monodromy of $x$ is known if the following problem is solved: for every ordered $r$-uple of intervals $(b_1, \ldots, b_r)$ in the structure with intervals $(x, \Pi_b)$, and every $r$-uple $(\beta_1, \ldots, \beta_r)$ of distinct integers in $\{0, \ldots, n\}$, we know whether there exists a $\Pi'_b \in L(x, \Pi_b)$, and thus an $S$ such that $S(x, \Pi_b) = (x, \Pi'_b)$, with the property that $S(b_a) = \beta_a$ for all $a = 1, \ldots, r$.

The definition above is somewhat redundant. In fact it is clear that, for $\pi \in \mathcal{S}_{n+1}$,

$$L(x, \Pi_b \circ \pi) = L(x, \Pi_b) \pi$$

(3)
(and in particular, by choosing $\pi = \Pi_b^{-1}$, it suffices to describe the set $L$ for the canonical left-to-right labelling). Also, if we have a group dynamics, the dependence from $x$ is only through the class of $x$, more precisely, if $S_0$ is a sequence in the labelled dynamics from $(x, \Pi_b)$ to $(x', \Pi'_b)$, then

$$L(x, \Pi_b) = L(x', \Pi'_b)$$

(4)

because, for each $\Pi''_b \in L(x', \Pi'_b)$ realised with the sequence $S$, then $\Pi''_b \in L(x, \Pi_b)$ is realised with the sequence $S_0^{-1}SS_0$, and similarly with $x \leftrightarrow x'$ and $S_0 \leftrightarrow S_0^{-1}$.

The interest in these properties is highlighted in the crucial definition below, of the ‘labelling method’ as a whole:

**Definition 11.** A dynamics $((X_n), (A_k))$ is amenable to the labelling method if:

1. It has a boosted dynamics.
2. There is a labelling compatible with the boosted dynamics.
3. For a suitable value of $r > 0$, the $r$-point monodromy of any class is known.

When proving a classification theorem for a group dynamics, we shall usually proceed in two steps: first, guessing the ‘right’ set $\text{Inv}_n$ of invariants for the dynamics on $X_n$, and then, proving that the set is right. In order to do the latter, i.e. to prove that there exists exactly one class per invariant, we need to show two things:

**existence** For every $\text{inv} \in \text{Inv}_n$ there exists a combinatorial structure $x \in X_n$ with invariant $\text{inv}$.

**completeness** The invariant discriminates the classes, i.e. for every pair $x_1, x_2 \in X_n$ with the same invariant $\text{inv} \in \text{Inv}_n$ there exists a sequence $S$ such that $S(x_1) = x_2$.

The first item of the list is often achieved relatively easily, by constructing a candidate $x \in X_n$ either directly or by induction. The difficulty of the construction depends mainly on how tractable and explicit the invariant set is.

The second part, which is sensibly more complicated, is where the labelling method comes forth.

The method is aimed at constructing an inductive step, so we assume that the invariant is complete up to sizes $n' < n$, and that the $r$-point monodromy problem (with an appropriate value of $r$ specified below) is solved for all configurations up to sizes $n' < n$, and we analyse how we can prove that the invariant is complete at size $n$. At a later moment, we shall solve the $r$-point monodromy problem (with the same value of $r$) at size $n$, in light of the $r$-point monodromy problem at smaller size, and the completeness at size up to $n$. The proof strategy will change from dynamics to dynamics but we will indicate some elements of proof that remain mostly identical in the next section.

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2 In case of a monoid dynamics the goal needs to be modified accordingly, and there is no general recipe. If the Cayley digraphs of the classes has a strongly connected component, we can restrict the study to this portion of the classes, with small difference w.r.t. the group case. Another possible sensible approach consists in modifying the monoid dynamics into a group dynamics, by adding new or modifying existant operators, so that the weak-connectedness in the original Cayley digraph of the classes turns into ordinary connectedness in the new Cayley graph.
So, we have two arbitrary $x_1, x_2 \in X_n$, combinatorial structure with the same invariant $\text{inv} \in \text{Inv}_n$, and we want to show that they are connected by the dynamics. Our first goal is \textbf{find $x'_1$ and $x'_2$ in $X_n$}, connected to $x_1$ and $x_2$ respectively, with the following property: there exists two $(k, r)$-colorings $c_1$ and $c_2$ of $x'_1$ and $x'_2$ respectively, such that $y_1 := \text{red}(x'_1, c_1)$ and $y_2 := \text{red}(x'_2, c_2)$ have the same invariant $\text{inv}' \in \text{Inv}_k$. Which, by induction hypothesis, implies that they are in the same class. Call $S_1$ and $S_2$ the sequences such that $x'_1 = S_1 x_1$ and $x'_2 = S_2 x_2$.

We call $x'_1$ and $x'_2$ normal forms as their structure will often be rather specific in order to prove the required property. Thus we can rename our first goal as 'finding normal forms for every pair $(x_1, x_2)$' or 'normalizing $x_1$ and $x_2$'.

Then, use induction to \textbf{find a sequence $S$ such that $S(y_1) = y_2$}.

Since the dynamics is amenable to the labelling method, there exists a boosted dynamics and a labelling compatible with it. \textbf{Let us choose such a labelling.}

Let $\Pi_y$ be a labelling of $y_1$. Say that the $r$ gray vertices of $(x'_1, c_1)$ are within the intervals with labels $b^1, \ldots, b^r$ of $(y_1, \Pi_y)$, in this order. In addition, the $r$ gray vertices of $(x'_2, c_2)$ are within the intervals with position, say, $\beta^1, \ldots, \beta^r$ of $y_2$, in this order. \textbf{We will be concerned with these two $r$-uples.}

By the boosted dynamics, there exists a sequence $S'$ which is \textbf{a boosted sequence of $S$ for $(x'_1, c_1)$}. Let $(x_3, c_3) = S'(x'_1, c_1)$. If it were $(x_3, c_3) = (x'_2, c_2)$, we would be done, but, this needs not to be the case. So, we know that $(x_3, c_3)$ and $(x'_2, c_2)$ are not equal in general, nonetheless the reduction of both of them is $y_2$.

By compatibility of the labelling with the boosted dynamics, \textbf{we know the location of the $r$ gray vertices of $(x_3, c_3)$}. Namely, by virtue of our notation (in which the alphabet $\Pi_y$ changes alongside the dynamics), we have that the $r$ gray vertices are within the intervals with labels $b^1, \ldots, b^r$ of $(y_2, \Pi'_y) = S(y_1, \Pi_y)$. Thus $x_3$ and $x'_2$ possibly differ by the position of the $r$ vertices which are gray in the colouring, that are within the intervals with position $(\Pi'_y)^{-1}(b^a)$ and $\beta^a$ of $y_2$, for $x_3$ and $x'_2$, respectively, for $1 \leq a \leq r$.

Since the $r$-point monodromy problem is solved for $y_2$, \textbf{we know whether there exists a sequence $S_*$ that sends the label $b^a$ to the positions $\beta^a$}. More precisely, a sequence $S_* \in L(y_2, \Pi'_y)$ such that the labeling $\Pi''_y = S_*(\Pi'_y)$ has $(\Pi''_y)^{-1}(b^a) = \beta^a$ for all $1 \leq a \leq r$.

\textbf{If $S_*$ exists}, let $S'_* be the boosted sequence of $S_*$ for $(x_3, c_3)$. Then, again by compatibility of the labelling with the boosted dynamics, we have $S'_*(x_3, c_3) = (x'_2, c_2)$.

Thus we have shown that $x_1$ and $x_2$ are connected, namely

$$x_2 = (S_2)^{-1} S'_* S' S_1 x_1$$ (5)

See figure 4.

Conversely, if $S^*$ does not exist, the labelling method has failed.

\subsection{2.2 Difficulties and implementation}

Let us discuss the possible difficulties one can encounter when trying to employ the labelling method. We can identify two main tasks: proving that the dynamics is

\footnote{We shall adopt $r$ to be the smallest value such that $X_k$ is a valid combinatorial set, and in most of our applications it is just $r = 1$ or $r = 2$.}
Figure 4: Outline of the proof of connectivity between $x_1$ and $x_2$, using the labelling method. The sequence $S$ sends $y_1$ to $y_2$, however the intervals containing the gray vertices of $x'_1$ may not be at their correct place, in order to match with those of $x'_2$. The sequence $S_*$ corrects for this. Thus the boosted sequence $S'_*S'$ sends $x'_1$ to $x'_2$, and $x_1$ and $x_2$ are connected.

amenable to the method, and then applying it to the ‘proof that $x_1 \sim x_2$’ outlined above. More precisely, those two tasks are divided into fives items:

- Proving the existence of a boosted dynamics.
- Finding a labelling compatible with it.
- Proving the $r$-point monodromy for some $r$.
- Initiating the labelling method by finding normal forms.
- And finally carrying out the rest of the labelling method.

Each of which has its own difficulties.

In the first part:

- Proving the existence of a boosted dynamics is often easy. The dynamics we study (at least as far as our investigations have led us) are ‘regular’, in the sense that the operators in $X_{n+1}$ and in $X_n$ do not differ much from each other. Thus, given an operator $S = A_i$ on $X_{n-r}$ acting on $y$, the corresponding operator $A_i$ on $X_n$ will act on $(x, c)$ in much the same way and we only need to correct the sequence a ‘little’ to obtain a valid boosted sequence.

For example in three dynamics that we have studied the boosted sequences of an operator $A_i$ has been respectively $A_i, A_j^2A_i$ for some $j$ and $A_jA_iA_j^{-1}$ for some $A_j$ (with $A_j$ possibly the identity).

This heuristic may not necessarily work in the case of a monoid dynamics, since, for a given operator $A_i$ on $X_{n-r}$, it may be not the case that $A_i$ is well-defined on $X_n$. Thus when working with monoid dynamics the verification that the boosted dynamics exists is a crucial step that can very well fail (which in turn dashes any hope of applying the method).
• Proving that there is a labelling compatible with the boosted dynamics is normally straightforward. The objective is to extend the actions of the operators \((A_i)_i\) on the labelling so that the property that the labelling on \(X_{n-r}\) keeps track of the gray vertices on \(X_n\) is verified. Since, at this point, we have already defined the boosted dynamics we know how the gray vertices moves when applying the boosted sequence on \(X_n\) corresponding to an operator on \(X_{n-r}\). Thus we just define the action of \(A_i\) on the labelling so that the labels of the intervals containing the gray vertices are moved along the gray vertices. Moreover, given the regularity of the operators, \(\alpha'(\cdot)\) is rather similar to \(\alpha(\cdot)\).

• Solving the \(r\)-point monodromy problem is the most difficult part of the amenability conditions. We need to investigate the structure of the set \(L(x, \Pi_b)\) which depends heavily on the invariant set.

Let us recall that our goal is to find a set of loops \(L_g(x)\) on \(x \in X_n\) such that the group generated by \(L_g(x)\) is either all of \(L(x, \Pi_b)\) (in which case we have all the information we can ever obtain from the monodromy group) or a subgroup containing at least enough information to answer the \(r\)-point monodromy problem for some \(r\).

Let us note that we can choose (at least in the case of a group dynamics) a specific \(x_0\) since all the \(L(x, \Pi_b)\) are isomorphic\(^4\). Thus we can choose a combinatorial structure \(x_0\) for which loops that generate the monodromy are easy to obtain (either by directly finding such sequences or by choosing a combinatorial structure that works well with an induction).

Thus the proof is performed by induction at size \(n\), after the induction step for the classification theorem has been established at size \(n\). I.e. we suppose both statements true at size \(n' < n\) then we apply the labelling method to prove the classification theorem at size \(n\), and finally we work out the monodromy problem at size \(n\), in this very order for all of our applications.

Indeed, in this way we will only need to construct a combinatorial structure \(x_0\) with the required structure for every valid invariant \(inv\) and then the classification theorem at size \(n\) will implies that for any starting combinatorial structure \(x\) with invariant \(inv\) there is a path to \(x_0\).

Finally let us conclude with a caveat that will be of importance for the fifth task. Recall that the \(r\)-point monodromy problem is stated as follows: Given \((x, \Pi_b)\), labels \(b_1, \ldots, b_r \in \Sigma\) and positions \(\beta_1, \ldots, \beta_r\) is there a loop that sends, for every \(1 \leq k \leq r\), the label \(b_{ik}\) to the position \(\beta_k\) ?

However the answer to this question can be somewhat implicit, that is, the answer is yes if and only if \(P((x, \Pi_b), b_1, \ldots, b_r, \beta_1, \ldots, \beta_r)\) is true for a given property \(P\) depending on the structure of the invariant.

For the application of the labelling method there are two steps which can necessitate an important amount of non-automatised work.

• At the very beginning, we shall certify that we can reach configurations \(x'_1\) and \(x'_2\) which are of some normal forms.

\(^4\)In case of a monoid dynamics the situation can be a bit more difficult but as long as the dynamics is regular enough, a similar argument should be useable.
They are two approaches to this problem. The first one, used in [Rau79] for the standardisation procedure (see also lemma 3.2 in [DS17]), [DS17] for the T-structure and [Fic16] for the piece-wise order reversing permutation of a given type, consists in directly finding a sequence connecting $x$ to a certain $x'$ in normal form. We refer to the three previously cited article to see implementations of this method.

The second approach is a very nice trick to make this kind of problem easier. Let $T_A$ be the set of normal forms and let $T_B \subseteq T_A$. We will want to choose $T_B$ such that if $y \in T_B$ and $(x,c)$ has reduction $y$ then $x \in T_A$ or a similar property.

We want to prove that for every $x$ there exists $x' \in T_A$ such that $x \sim x'$. We proceed in two steps:

First we construct for every invariant $inv \in Inv_n$ a combinatorial structure $x_0 \in T_B$. This construction is often done by induction and can replace the existence step of the labelling method (indeed we are exactly constructing a candidate $x \in X_n$ of a very special form for every $inv \in Inv_n$). Note that since combinatorial structure in $T_B$ have a very constrained structure, the proof of the existence of an $x_0 \in T_B$ for every invariant can be long. In this paper a whole section (Sections 9) has to be dedicated to it.

Then the proof is performed by induction at size $n$, before the induction step for the classification theorem has been established at size $n$.

More precisely, since the classification theorem is proven at size $n' < n$, we know that every class (at this size) contains a $x_0 \in T_B$. We start from our $x$ at size $n$, we choose a $(n-r, r)$-coloring $c$. Then the reduction $y$ is connected to a $y_0 \in T_B$ by a sequence $S$. Consider $x' = S(x,c)$ either $x' \in T_A$ due to our choice of $T_B$ (that is the best case scenario) or at least it should be close to a $x'' \in T_A$ with only the gray vertices not at the correct positions. Then using the $r$-point monodromy (as we did in the labelling method) one should be able to prove that $x' \sim x''$.

Clearly this is only a heuristic since we need to construct the correct $T_B$ for this to work. If $T_A$ is a very large class then we may just have $T_B = T_A$ i.e. a local modification (by adding $r$ vertices) of a combinatorial structure $x$ in $T_A$ remains in $T_A$.

We will employ this strategy for the labelling proof of the KZB classification in this article. In this case $T_A$ is the set of shift-irreducible standard families and $T_B$ is the set of $I_2X$ permutations.

Exceptional classes also cause problems, as they often break the “one class per invariant” landscape (which is modified into “one non-exceptional class per invariant”). As a result, even with the normals forms at hand, we may have that $y_1$ and $y_2$ share the same invariant but are not in the same class, because one of the two is in an exceptional class. When we had to face this issue, we could solve it by choosing $x_1'$ and $x_2'$ appropriately so that $y_1$ and $y_2$ are easily checked to not be in an exceptional class (thus constraining even more the normal forms). However, in the cases we have studied so far, certifying this has required the complete characterisation of the structure of the exceptional classes, a knowledge which may remain elusive in more complicated dynamics.
• The step concerning the sequence $S_*$ also needs a certain amount of care. Indeed, as we have anticipated, the solution to the $r$-point monodromy problem involve a property $P$ that could be difficult to exploit due to its potentially implicit form.

At this point of the proof we have the following: $(x_3, c_3)$ and $(x_2', c_2)$ with the same invariant, their reduction $y_2$ with a labelling $\Pi'_b$ and $r$ labels $b_1, \ldots, b_r$ and $r$ positions $\beta_1, \ldots, \beta_r$ such the gray vertices of $(x_3, c_3)$ are within the interval with labels $(b_i)$ and the gray vertices of $(x_2', c_2)$ are within the interval with positions $(\beta_i)$. The question is whether this information is enough to deduce that $P((y, \Pi'_b), b_1, \ldots, b_r, \beta_1, \ldots, \beta_r)$ is true.

In this article $P$ is rather explicit so we have enough information and the step is straightforward. However let us quickly describe an example for which it is non-trivial. The involution dynamics (studied in [DS18]) has a unique invariant which is a graph. Moreover the labelling of the intervals of a combinatorial structure $x$ correspond bijectively to the labelling of the edges of the graph. Then the property associated to the 1-point monodromy is the following: $P((x, \Pi_b), b, \beta)$ is true if and only if the edges corresponding to $b$ and $\beta$ are isomorphic. Thus to prove that $P$ is true we must construct an automorphism $g$ of the graph $G$ of $(y_2, \Pi'_b)$ that sends the edge $b$ to the edge $\beta$. However the only information we dispose of is that the graphs $G_1$ and $G_2$ associated to $(x_3, c_3)$ and $(x_2', c_2)$ are isomorphic by some $f$ and that they are obtained from $G$ by a local modification involving the edges $b$ and $\beta$ respectively. It is the case that without further information on the structure of $G$ and $G_1$, $f$ is not enough to recover an automorphism $g$ of $G$ sending $b$ to $\beta$. In that case our solution is to consider a subset of classed for which the graph invariants have enough structure to recover $g$ from $f$. Then we prove that the classification for this subset of classes yields a classification for all the classes.

For the rest, and assuming that the sequence $S_*$ with the appropriate properties always exists (otherwise, as we said, we would be in trouble), all the other steps are elementary, and follow a non-ambiguous roadmap, with no need of case-to-case inventions.

There is one last thing to point out. In the labelling proof we implicitly assumed that no gray vertices were adjacent since otherwise there would be two or more gray vertices within the same interval and thus the number of intervals of interest would be strictly less than $r$ the number of gray vertices. This assumption does not cost much: it suffices to have normal forms for which the $(k, r)$-coloring does not produce adjacent gray vertices. In our application $r$ is small so such property is easy to satisfy.

Nevertheless, we could consider the case where more than one gray vertices are within the same interval. The same reasoning as above occurs with the following difference: within an interval the order of the vertices might get permuted by the dynamics thus the data of interest are the labels $b_1, \ldots, b_{r_1}$ of the $r_1 < r$ intervals containing the gray vertices plus a permutation $\pi_1, \ldots, \pi_{r_1}$ indicating the order of the vertices within the same interval.

The permutations $\pi_i$ depend on the dynamics: for the Rauzy dynamics the order remains fixed so we have $\pi_i = id$. For the involution dynamics [DS18] we have
\(\pi_i = id\) or \(\pi_i = \omega\) the order reversing permutation. We will make use of this slight generalisation for the involution dynamics so we refer to this paper for an example.

We have spend some time detailling how to handle the different problems that can arise in the labelling method, now we summarize this discussion by presenting the organisation of such proof.

We organise the proof in two steps: the preparation of the induction and its execution.

- Preparation of the induction:
  - Part 1: Definition of the invariants and study of it. The study answers the question what happens to the invariants when adding/removing vertices to a combinatorial structure.
  - Part 2: Definition of a boosted dynamics and definition of a compatible labelling and statement of the \(r\)-point monodromy.
  - Part 3: Construction of a \(x \in T_b\) for every invariant \(inv\).

- Execution of the induction:
  - Part 4: Proof that every \((x_1, x_2)\) are connected to a \((x'_1, x'_2)\) in normal forms.
  - Part 5: Proof of the classification theorem by the labelling method
  - Part 6: Proof of the \(r\)-point monodromy.

A final remark. The labelling method can only be applied if the set of invariants is known and such set changes completely from case to case. However computing the set \(L(x, \Pi_b)\) is of use for find the invariants as the structure of the monodromy group gives indirect information on the underlying invariant.

For example in this article, the cycle invariant can really be found thanks to the monodromy. Indeed, by studying the monodromy one would notice that the labels are partitionned into \(k\) subset of a given length \(\lambda_1, \ldots, \lambda_k\) and that the labels of a given subset can shift in a cyclic way (i.e. one loop can shift all the labels of a given subset by 1 or 2 in a cyclic way and leave the others in place). This information is quite enough to reconstruct the full structure of the cycle invariant.

3 Definition of the Rauzy dynamics

We will now apply the labelling method defined in the previous section to give an original proof of the classification of the Rauzy dynamics. Other proofs have been achieved in [Boi12] (Boissy uses the classification proof for the extended Rauzy dynamics appearing in [KZ03]) using geometric methods and in [Fic16] and [DS17] using combinatorial methods.

The extended Rauzy classes (classified in [KZ03]) are of interest in the translation surface field as they are in one-to-one correspondance with the connected components of the strata of the moduli space of abelian differentials (see [Vee82]). As for the non-extended Rauzy classes, they are in one-to-one correspondance with the connected components of the strata of the moduli space of abelian differentials with a marked zero as shown in [Boi12].
3.1 The Rauzy dynamics

Let $\mathfrak{S}_n$ denote the set of permutations of size $n$, and $\mathfrak{M}_n$ the set of matchings over $[2n]$, thus with $n$ arcs. Let us call $\omega$ the permutation $\omega(i) = n + 1 - i$.

A permutation $\sigma \in \mathfrak{S}_n$ can be seen as a special case of a matching over $[2n]$, in which the first $n$ elements are paired to the last $n$ ones, i.e. the matching $m_{\sigma} \in \mathfrak{M}_n$ associated to $\sigma$ is $m_{\sigma} = \{(i, \sigma(i) + n) \mid i \in [n]\}$.

We say that $\sigma \in \mathfrak{S}_n$ is irreducible if $\omega \sigma$ doesn’t leave stable any interval $\{1, \ldots, k\}$, for $1 \leq k < n$, i.e. if $\{\sigma(1), \ldots, \sigma(k)\} \neq \{n - k + 1, \ldots, n\}$ for any $k = 1, \ldots, n - 1$. We also say that $m \in \mathfrak{M}_n$ is irreducible if it does not match an interval $\{1, \ldots, k\}$ to an interval $\{2n - k + 1, \ldots, 2n\}$. Let us call $\mathfrak{S}_n^{irr}$ and $\mathfrak{M}_n^{irr}$ the corresponding sets of irreducible configurations.

We represent matchings over $[2n]$ as arcs in the upper half plane, connecting pairwise $2n$ points on the real line (see figure 5, top left). Permutations, being a special case of matching, can also be represented in this way (see figure 5, top right), however, in order to save space and improve readability, we rather represent them as arcs in a horizontal strip, connecting $n$ points at the bottom boundary to $n$ points on the top boundary (as in Figure 5, bottom left). Both sets of points are indicised from left to right. We use the name of diagram representation for such representations.

We will also often represent configurations as grids filled with one bullet per row and per column (and call this matrix representation of a permutation). We choose here to conform to the customary notation in the field of Permutation Patterns, by adopting the algebraically weird notation, of putting a bullet at the Cartesian coordinate $(i,j)$ if $\sigma(i) = j$, so that the identity is a grid filled with bullets on the anti-diagonal, instead that on the diagonal. An example is given in figure 5, bottom

\[ m = ((16)(24)(37)(58)) \in \mathfrak{M}_8 \quad \sigma = [41583627] \in \mathfrak{S}_8 \subseteq \mathfrak{M}_{16} \]

matching diagram representation

diagram representation

\[ \sigma = [41583627] \in \mathfrak{S}_8 \]

matrix representation

Figure 5: Diagram representations of matchings and permutations, and matrix representation of permutations.
Let us define a special set of permutations (in cycle notation)

\[ \gamma_{L,n}(i) = ( i - 1 \ i - 2 \ \cdots \ 1 )( i ) ( i + 1 ) \cdots ( n ) ; \]  
\[ \gamma_{R,n}(i) = ( 1 ) ( 2 ) \cdots ( i ) ( i + 1 + 2 \ \cdots \ n ) ; \]  

i.e., in a picture in which the action is diagrammatic, and acting on structures \( x \in X_n \) from below,

\[ \gamma_{L,n}(i) : \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \gamma_{R,n}(i) : \]

Of course, \( \omega \gamma_{L,n}(i) \omega = \gamma_{R,n}(n + 1 - i) \).

The Rauzy dynamics \( S_n \) that we study in this article is defined as a special case of the \( M_n \) dynamics:

**\( M_n \) :** The space of configuration is \( \mathfrak{M}_n^{irr} \), irreducible \( n \)-arc matchings. There are two generators, \( L \) and \( R \), with \( \alpha_{L}(m) = \gamma_{L,2n}(m(1)) \) and \( \alpha_{R}(m) = \gamma_{R,2n}(m(2n)) \) (\( \alpha \) is as in Definition 1). See Figure 6, top.

**\( S_n \) :** The space of configuration is \( \mathfrak{S}_n^{irr} \), irreducible permutations of size \( n \). Again, there are two generators, \( L \) and \( R \). If permutations are seen as matchings such that indices in \( \{1, \ldots, n\} \) are paired to indices in \( \{n + 1, \ldots, 2n\} \), the dynamics coincide with the one given above. See Figure 6, bottom.

The motivation for restricting to irreducible permutations and matchings shall be clear at this point: a non-irreducible permutation is a grid with a non-trivial block-decomposition. The operators \( L \) and \( R \) only act on the first block (say, of
size $k$), so that the study of the dynamics trivially reduces to the study of the $S_k$ dynamics on these blocks (see figure 8).

This simple observation, however, comes with a disclaimer: To apply the labelling method we must guarantee that the outcome of our manipulations on irreducible configurations is still irreducible during the induction steps in the classification theorem. We explain this problem in the overview section. (The solution will be to choose normal forms that take this problem into account).

3.2 Definition of the invariants

The main purpose of this article, is to characterise the classes appearing in the so-called Rauzy dynamics $S_n$ using the labelling method. This gives a third combinatorial proof of this result, the two preceding ones can be found in [DS17] and [Fic16].

In this section we recall the definition of the invariants, the proof of their invariance can be found in [DS17].

3.2.1 Cycle invariant

Let $\sigma$ be a permutation, identified with its diagram. An edge of $\sigma$ is a pair $(i^-,j^+)$, for $j = \sigma(i)$, where $-$ and $+$ denote positioning at the bottom and top boundary of the diagram. Perform the following manipulations on the diagram: (1) replace each edge with a pair of crossing edges; more precisely, replace each edge endpoint, say $i^-$, by a black and a white endpoint, $i^-_b$ and $i^-_w$ (the black on the left), then introduce the edges $(i^-_b,j^+_w)$ and $(i^-_w,j^+_b)$. (2) connect by an arc the points $i^+_w$ and $(i+1)^+_b$, for $i = 1, \ldots, n-1$, both on the bottom and the top of the diagram; (3) connect by an arc the top-right and bottom-left endpoints, $n^-_w$ and $1^-_b$. Call this arc the “$-1$ mark”.

The resulting structure is composed of a number of closed cycles, and one open path connecting the top-left and bottom-right endpoints, that we call the rank path. If it is a cycle that goes through the $-1$ mark (and not the rank path), we call it the principal cycle. Define the length of an (open or closed) path as the number of top (or bottom) arcs (connecting a white endpoint to a black endpoint) in the path. These numbers are always positive integers (for $n > 1$ and irreducible permutations). The
length \( r \) of the rank path will be called the \textit{rank} of \( \sigma \), and \( \lambda = \{ \lambda_i \} \), the collection of lengths of the cycles, will be called the \textit{cycle structure} of \( \sigma \). Define \( \ell(\sigma) \) as the number of cycles in \( \sigma \) (this does not include the rank). See Figure 9 for an example.

Note that this quantity does not coincide with the ordinary path-length of the corresponding paths. The path-length of a cycle of length \( k \) is \( 2k \), unless it goes through the \(-1\) mark, in which case it is \( 2k + 1 \). Analogously, if the rank is \( r \), the path-length of the rank path is \( 2r + 1 \), unless it goes through the \(-1\) mark, in which case it is \( 2r + 2 \). (This somewhat justifies the name of “\(-1\) mark” for the corresponding arc in the construction of the cycle invariant.)

In the interpretation within the geometry of translation surfaces, the cycle invariant is exactly the collection of conical singularities in the surface (we have a singularity of \( 2k\pi \) on the surface, for every cycle of length \( \lambda_i = k \) in the cycle invariant, and the rank corresponds to the ‘marked singularity’ of \cite{Boi12}, see article \cite{DS17}).

It is easily seen that

\[
r + \sum_i \lambda_i = n - 1,
\]

this formula is called the \textit{dimension formula}. Moreover, in the list \( \{ r, \lambda_1, \ldots, \lambda_\ell \} \), there is an even number of even entries. This is part of theorem \ref{thm:13} stated next section and proven during the induction step.

We have

\textbf{Proposition 12.} The pair \((\lambda, r)\) is invariant in the \( S \) dynamics.

For a proof, see \cite{DS17} section 3.1.

We have also shown in \cite{DS17} appendix B that cycles of length 1 have an especially simple behaviour and can thus be omitted from the classification theorem. Thus \textbf{all the classes we consider in this article have a cycle invariant} \( \lambda \) with no parts of length 1.
3.2.2 Sign invariant

For \( \sigma \) a permutation, let \([n]\) be identified to the set of edges (e.g., by labeling the edges w.r.t. the bottom endpoints, left to right). For \( I \subseteq [n] \) a set of edges, define \( \chi(I) \) as the number of pairs \( \{i', i''\} \subseteq I \) of non-crossing edges. Call

\[
\mathcal{A}(\sigma) := \sum_{I \subseteq [n]} (-1)^{|I|} + \chi(I)
\]  

(8)

the Arf invariant of \( \sigma \) (see Figure 10 for an example). Call \( s(\sigma) = \text{Sign}(\mathcal{A}(\sigma)) \in \{-1, 0, +1\} \) the sign of \( \sigma \).

Both the quantity \( \mathcal{A}(\cdot) \) and \( s(\cdot) \) are invariant for the dynamics \( \mathcal{S} \). The proof can be found in [DS17] section 4.2. However, in order to illustrate how to automatically compute the arf identities, the proof will be given once more in equation (16).

There exists an important relationship between the arf invariant and the cycle invariant that we describe below. The proof of this theorem will be done during the induction (see the proof overview section for more details).

**Theorem 13.** Let \( \sigma \) be a permutation with cycle invariant \((\lambda, r)\) and let \( \ell \) be the number of cycles (not including the rank) of \( \sigma \) i.e. \( \ell = |\lambda| \).

- The list \( \lambda \cup \{r\} \) has an even number of even parts.
- \( \mathcal{A}(\sigma) = \begin{cases} 
\pm 2^{n-\ell} & \text{if there are no even parts in the list } \lambda \cup \{r\} \\
0 & \text{otherwise.} \end{cases} \)

as a consequence we have:

**Proposition 14.** The sign of \( \sigma \) can be written as \( s(\sigma) = 2^{-n+\ell} \mathcal{A}(\sigma) \), where \( \ell \) is the number of cycles of \( \sigma \).

3.3 Exceptional classes

the invariants described above allow to characterise all classes for the dynamics on irreducible configurations, with two exceptions. These two exceptional classes are called Id\(_n\) and Id\(_n'\).
Table 1: Cycle, rank and sign invariants of the exceptional classes. The sign $s \in \{-1, 0, +1\}$ is shortened into $\{-, 0, +\}$.

|        | $n$ even | $n$ odd |
|--------|----------|---------|
| $(\lambda, r)$ of $\text{Id}_n$ | $(\emptyset, n-1)$ | $((n-1)/2, (n-1)/2)$ |
| $(\lambda, r)$ of $\text{Id}'_n$ | $((n-2)/2, (n-2)/2)$ | $(\{n-2\},1)$ |

Id$_n$ is called the ‘hyperelliptic class’, because the Riemann surface associated to Id$_n$ is hyperelliptic. Similarly Id’$_n$ is often referred to as the ‘hyperelliptic class’ with a marked point.

Those two classes were studied in details in Appendix C of [DS17]. In this article, and as outlined in the labelling method section, we will need to certify that the reduced permutations $y_1$ and $y_2$ we obtain from $(x'_1, c_1)$ and $(x'_2, c_2)$ do not fall into exceptional classes. This will not be very difficult but we will need a short lemma that we included into appendix A. Moreover, to make sense of this lemma we reproduce in this appendix a few results of the appendix C.1 of the paper [DS17].

The cycle and sign invariants of these classes depend from their size mod 4, and are described in Table 1.

3.4 The classification theorems

For the case of the $\mathcal{S}$ dynamics, we have a classification involving the cycle structure $\lambda(\sigma)$, the rank $r(\sigma)$ and sign $s(\sigma)$ described in Section 3.2.

**Theorem 15.** Besides the exceptional classes Id and Id’, which have cycle and sign invariants described in Table 1, the number of classes with cycle invariant $(\lambda, r)$ (no $\lambda_i = 1$) depends on the number of even elements in the list $\{\lambda_i\} \cup \{r\}$, and is, for $n \geq 9$,

- **zero**, if there is an odd number of even elements;
- **one**, if there is a positive even number of even elements; the class then has sign 0.
- **two**, if there are no even elements at all. The two classes then have non-zero opposite sign invariant.

For $n \leq 8$ the number of classes with given cycle invariant may be smaller than the one given above, and the list in Table 2 gives a complete account.

As a consequence, two permutations $\sigma$ and $\sigma'$, not of Id or Id’ type, are in the same class iff they have the same cycle and sign invariant.

The original theorem from Kontsevich and Zorich [KZ03] classifies the extended Rauzy dynamics $\mathcal{S}^{\text{ex}}$, however we have shown in section 6.5 of [DS17] that this theorem is a simple corollary of Theorem 15.
Table 2: List of invariants $(\lambda, r, s)$ for $n \leq 8$, for which the corresponding class in the $S_n$ dynamics exists. We shorten $s$ to $\{-,+\}$ if valued $\{-1, +1\}$, and omit it if valued 0.

4 Proof overview

In this section we present a proof of the classification of the Rauzy classes of the dynamics $S_n$ by applying the labelling method.

A proof of classification via the labelling method, always proceeds in two parts. First we prove the amenability of the dynamics $S_n$ in section 5 (although, as usually occurs in this family of proofs, the proof of the $r$-point monodromy must be deferred to the main induction). Then we carry out the main induction in section 10, in which we apply the labelling method to prove the classification theorem and the $r$-point monodromy theorem.

In an instance where the labelling method can be applied ‘smoothly’ (as is the case with the involution dynamics in [DS18], though it does have some interesting specific difficulties), the main induction is constituted of three statements: the existence and completeness part of the classification theorem, and then the $r$-point monodromy of the set $L(x, \Pi_b)$. Moreover, nothing prevents from presenting the main induction immediately after that the amenability has been established, i.e. no complicated normal forms have to be defined.

The dynamics $S_n$ is not such a smooth instance. Therefore the main induction, that we now detail, will include a certain number of unavoidable technical statements, and several sections (i.e. Sections 6 to 9) are needed in preparation of it. In particular, we will construct our normal forms using the trick introduced in the section 2.2 of the labelling method. The set $T_A$ will be the set of shift-irreducible permutations and the set $T_B \subseteq T_A$ that of the $I_2 X$-permutations.

Main induction: We demonstrate by induction on $n$ the seven following statements:

1. **Proposition 16.** Every class contains a shift-irreducible standard family (cf definition 42).

2. **Theorem 13** on the relationship between the arf invariant and the cycle invariant.

3. **Proposition 17** (Existence part of the classification theorem). For every valid invariant $(\lambda, r, s)$ (i.e., every invariant in the list of Thm. 15) there exists a permutation with invariant $(\lambda, r, s)$.  

21
4. **Proposition 18** (First step of the labelling method: the normal forms). Let \( \sigma_1, \sigma_2 \in S_n \) be two irreducible permutation with invariant \( (\lambda, r, s) \). There exist \( \sigma'_1 \) and \( \sigma'_2 \), connected to \( \sigma_1 \) and \( \sigma_2 \) respectively, with the following property:

let \( c_1 \) be the \((2n - 2, 2)\)-coloring of \( \sigma'_1 \) where the edge \( e'_1 = (\sigma'_1 - 1(1), 1) \) is grayed, and call \( \tau_1 \) the reduction of \( (\sigma'_1, c_1) \); define the analogous quantities for \( \sigma'_2 \) (i.e., the coloring \( c_2 \), the edge \( e'_2 = (\sigma'_2 - 1(1), 1) \), and the configuration \( \tau_2 \)); then \( \tau_1 \) and \( \tau_2 \) are irreducible, have the same invariant \( (\lambda', r', s') \), and none of them is in an exceptional class.

5. **Proposition 19** (Completeness part of the classification theorem). Every pair of permutations \( \sigma \) and \( \sigma' \) with the same invariant \( (\lambda, r, s) \) are connected.

6. **Proposition 20**. Every non exceptional class contains a \( I_2X \)-permutation (Defined above corollary 45).

7. **Theorem 33** on the 2-point monodromy problem for this dynamics.

The three major steps of the present proof are to establish Propositions 17 and 19 and Theorem 33.

As we commented in the Section 2.2, the most difficult step in the organisation of the abstract labelling method, when specialised to the \( S_n \) dynamics, is finding the normal forms. In this case, this step takes the form of Proposition 18 within the main induction using both Propositions 16 and 20 as support.

This proposition is difficult to establish because it requires that \( \tau_1 \) and \( \tau_2 \) satisfy simultaneously four unrelated properties: being irreducible, having the same cycle invariant \( (\lambda', r', s') \), having the same sign invariant \( s' \), and not being in an exceptional class. Let us outline the steps of the proof of this proposition:

**Cycle invariant:** In Section 7 (Proposition 40), we observe that if \( \sigma'_1 \) and \( \sigma'_2 \) are standard and are both of type \( X(r, i) \) or of type \( H(r_1, r_2) \), then their reductions have the same cycle invariant \( (\lambda', r') \).

**Irreducibility:** We introduce a special subset of standard families that we call the shift-irreducible standard families (see Definition 42). By definition, if \( \sigma \) is in a shift-irreducible standard family, then the reduction \( \tau \) of \( (\sigma, c) \) is (almost always and in our case always) irreducible, where \( c \) is the \((2n - 2, 2)\)-coloring of \( \sigma \) where edge \( e = (\sigma^{-1}(1), 1) \) is grayed.

**Sign invariant:** In Section 8.3 we prove Proposition 55. Applied to our case, it shows that if \( \sigma'_1 \) and \( \sigma'_2 \) have invariant \( (\lambda, r, s) \) with \( s \neq 0 \) and are shift-irreducible standard permutation of type \( X(r, i) \) then \( \tau_1 \) and \( \tau_2 \) also have sign invariant \( s \). There are two other cases concerning the permutations of type \( H \), for which the underlying mechanism is too technical to be described concisely. Let us only note here that they make use of Theorem 13 and Proposition 57 (also established in Section 8.3), respectively.

**Exceptional classes:** Finally we must certify that neither \( \tau_1 \) nor \( \tau_2 \) are in an exceptional class. This is achieved by showing that, in a standard family \( (\sigma_i)_{1 \leq i \leq n-1} \),
at most one of the reductions of the permutation \((\sigma_i, c_i)\) where \(c_i\) is the \((2n - 2, 2)\)-coloring in which the edge \((\sigma_i^{-1}(1), 1)\) is gray, is in a exceptional class (a lemma established in the appendix discussing the structure of exceptional classes). As a result, we can always avoid this case.

Thus, by choosing \(\sigma'_1\) and \(\sigma'_2\) in shift-irreducible standard families, we can take care simultaneously of those four properties and prove proposition \([18]\).

However, contrarily to the existence of standard families, proving that every irreducible permutation is connected to a shift-irreducible standard family by directly finding a sequence is too hard. Thus we make use of the trick introduced in section \([2.2]\) and we find a special subset \(T_B \subseteq T_A\) of standard permutations, named \(I_2X\) permutations, for which the associated standard family is shift-irreducible (see corollary \([45]\)). We dedicate the whole Section \([9]\) to the construction of one such permutation for every \((\lambda, r, s)\), which, on top of this, are not in an exceptional class (this result is the content of Theorems \([61]\) and \([66]\)).

Then, as we outlined in section \([2.2]\), in the main induction we use the stronger induction hypothesis of Proposition \([20]\) (statement 6) in order to prove Proposition \([16]\) (statement 1). Finally Proposition \([20]\) (statement 6) is demonstrated by applying the completeness part of the classification theorem (Proposition \([19]\) statement 5) to the existence of a \(I_2X\) permutation for every \((\lambda, r, s)\) (proven in Section \([9]\) Theorems \([61]\) and \([66]\)).

Thus we can organise this article as follows

- **Section 5**: We define the boosted dynamics, a labelling compatible with it and state the 2-point monodromy theorem.

- **Sections 6-7**: We study the cycle invariant when adding or removing edges on permutations and define both the shift-irreducible family and the \(I_2X\)-permutations.

- **Section 8**: We study the arf invariant when adding or removing edges on permutations and some other local modification.

- **Section 9**: Construct a \(I_2X\) permutation for every possible \((\lambda, r, s)\).

- **Section 10**: Proceed with the induction, proving the 7 statements presented in this overview.

We note that the structure is very close to the organisation presented at the very end of the labelling section: Part 1 corresponds to sections 6-7-8, part 2 to section 5, part 3 to section 9 and parts 4,5,6 to section 10.

This organisation will reappear in all proofs using the labelling method with some minor differences on a case-by-case basis.

## 5 Amenity of \(S_n\) to the labelling method

In this section we introduce the notions necessary to show that \(S_n\) is amenable to the labelling method. Thus our task is to define a boosted dynamics, determine a labelling \(\Pi_b\) compatible with it as well as to state the 2-point monodromy theorem.
for set $L(\sigma, \Pi_b)$ for any permutation $\sigma$.

Let us overview the content of the section:

We begin the section by introducing the boosted dynamics.

Then the second subsection introduces the definition of a special set of labellings, that we call consistent labellings, and proves a number of properties. In short, consistent labellings are labellings on an alphabet that respects the structure of the cycle invariant.

The third subsection extends the dynamics to the labelled case, proves its compatibility with the boosted dynamics (cf. Theorem 32), and introduces the theorem pertinent to the 2-point monodromy problem (cf. Corollary 34).

5.1 Boosted dynamics

Call pivots of $\sigma$ the two edges $(1, \sigma(1))$ and $(\sigma^{-1}(n), n)$. For a pair $(\sigma, c)$, we say that $\sigma$ is proper if no gray edge of $\sigma$ is a pivot. In this case, the dynamics on $\tau$ (the reduction of $\sigma$) extends to the boosted dynamics on $\sigma$, as follows:

for every operator $H$ (i.e., $H \in \{L, R\}$), we define $\alpha_H(\sigma, c)$ as the smallest positive integer such that $H^{\alpha_H(\sigma, c)}(\sigma)$ is proper, and, for a sequence $S = H_k \cdots H_2 H_1$ acting on $\tau$, the sequence $B(S)$, the boosted sequence of $S$, acting on $\sigma$ is $B(S) = H_k^{\alpha_k} \cdots H_2^{\alpha_2} H_1^{\alpha_1}$ for the appropriate set of $\alpha_j$’s.

A simple verification (by induction on the number of gray edges) shows that $B(S)$ makes the following square commute.

Thus $B(S)$ is a well-defined boosted sequence.

5.2 Preliminaries: Definition of a consistent labelling

In the labelling method we labelled the intervals of a combinatorial structure with intervals. In this particular case, the intervals corresponds to the bottom and top arcs introduced in the construction of the cycle invariant. Thus in the following we will rather label the arcs (top and bottom), and instead of saying ‘the gray vertex is within the interval with label $b$’ we will say ‘the gray edge is within the arcs with labels $t$ and $b$ (for the top arc and bottom arc respectively)’.

Also note the change of terminology from graying vertices to graying edges. Indeed, recall definition 6: when choosing a coloring $c$ for a permutation we must guarantee that the reduction is still a permutation and thus we will always gray edges (or equivalently the two vertices connected by the edge) rather than just vertices.)

Let $\sigma$ be a permutation of size $n$. The procedure to construct the cycle invariant $(\lambda, r)$ (as described in Section 3.2.1) involves the introduction of $n - 1$ top and bottom arcs connecting adjacent top and bottom vertices. As our proof requires it, we also add one top arc, to the left of the other top arcs (i.e to the left of the edge $(\sigma^{-1}(1), 1)$) and one bottom arc, to the right of the bottom arcs (i.e to the right of
the edge \((n, \sigma(n))\). Those two arcs are clearly added at the two endpoints of the rank path (in red in the figure below).

We number the top and bottom arcs from left to right, and refer to them by their position: the bottom arc \(\beta \in \{1, \ldots, n\}\) is the \(\beta\)-th bottom arc, counting from the left.

![Diagram](image)

By convention, the variables used to name the positions of the top (bottom) arcs will be \(\alpha\) (respectively \(\beta\)), in order not to make confusion with other parts of the diagram (for which we will use \(i, j, \ldots\) or \(x, y, \ldots\)).

**Definition 21.** We say that two (bottom) arcs \(\beta, \beta'\) are consecutive (in a cycle) if they are inside the same cycle (or the rank path), and they are consecutive in the cyclic order induced by the cycle (or the total order induced by the path). This occurs in one of the three graphical patterns:

![Consecutive Arcs Diagram](image)

In formulas:

\[
\beta' = \sigma^{-1}(\sigma(\beta + 1) + 1) \quad \text{if} \quad \sigma(\beta + 1) < n \quad \text{and} \quad \beta' = \sigma^{-1}(\sigma(1) + 1) \quad \text{if} \quad \sigma(\beta + 1) = n
\]

We define consecutive arcs for top arcs similarly.

**Remark 22.** As we have seen above, when representing graphically the consecutive arcs, we need three figures depending on the different cases (edges crossing or not, and edges ending at the north-east corner of the diagram). However, these cases are treated in a very similar way, and, in the graphical explanation of our following properties, we shall mostly draw consecutive arcs by representing the case of non-crossing and non-corner edges, i.e. the left-most of the drawings above. It is intended that the underlying reasonings remain valid for the other cases.

Next we define suitable alphabets used to label the top and bottom arcs of a permutation.

**Notation 1.** For all \(j\), let \(\Sigma_{t,j} = \{b_{0,i,j}, \ldots, b_{i-1,i,j}\}\) and \(\Sigma'_{t,j} = \{t_{0,i,j}, \ldots, t_{i-1,i,j}\}\) be a pair of alphabets which label the bottom arcs and the top arcs respectively of a cycle of length \(i\), and let \(\Sigma_r = \{b_{r}^{0}, \ldots, b_{r}^{rk}\}\) and \(\Sigma'_r = \{t_{r}^{0}, \ldots, t_{r}^{rk}\}\) be the pair of alphabets used to label the bottom arcs and the top arcs respectively of the rank path.

Note that the labels of the rank range from 0 to \(r\), instead of form 0 to \(r - 1\), since we added a left-most top arc and a right-most bottom arc, which are now part of the rank, according to the construction of the arcs of a permutation outlined above.

Finally, we can introduce our notion of consistent labelling.
Definition 23 (Consistent labelling). Let \( \sigma \) be a permutation with invariant \( (\lambda = \{\lambda_1^{m_1}, \ldots, \lambda_k^{m_k}\}, r) \) and define a consistent labelling to be a pair \((\Pi_b, \Pi_t)\) of bijections:

\[
\Pi_b : \{1, \ldots, n\} \to \Sigma_b = \Sigma_r \cup \left[ \bigcup_{i=1}^{k} \left( \bigcup_{j=1}^{m_i} \Sigma_{\lambda_i,j} \right) \right]
\]

\[
\Pi_t : \{1, \ldots, n\} \to \Sigma_t = \Sigma'_r \cup \left[ \bigcup_{i=1}^{k} \left( \bigcup_{j=1}^{m_i} \Sigma'_{\lambda_i,j} \right) \right]
\]

such that

1. Two arcs within the same cycle have labels within the same alphabet. Thus if \( S_b = \{(b_k)_{1 \leq k \leq \lambda_i}\} \) and \( S_t = \{(t_k)_{1 \leq k \leq \lambda_i}\} \) are the sets of bottom (respectively top) arcs of a cycle of length \( \lambda_i \), then \( \Pi_b(S_b) = \Sigma_{\lambda_i,j} \) and \( \Pi_t(S_t) = \Sigma'_{\lambda_i,j} \) for some \( 1 \leq j \leq m_i \).

2. Two consecutive arcs of a cycle of length \( \lambda_i \) have labels with consecutive indices: if \( \beta \) and \( \beta' \) are consecutive, then \( \Pi_b(\beta) = b_k, \lambda_i, j \) for some \( k \leq \lambda_i \) and \( j \leq m_i \) and \( \Pi_t(\beta') = b_{k+1}, \lambda_i, j \), where \( k + 1 \) is intended modulo \( \lambda_i \). Likewise for top arcs.

3. The bottom right arc \( \beta \) and the top left arc \( \alpha \) of an edge \( i \) are labeled by the same indices:

\[
\text{if } \beta = i, \alpha = \sigma(i), \text{ then } \begin{cases} 
\Pi_t(\alpha) = t_{k,\lambda_i,j} & \iff \Pi_b(\beta) = b_{k,\lambda_i,j} \\
\Pi_t(\alpha) = t_{rk} & \iff \Pi_b(\beta) = b_{rk} 
\end{cases}
\]

4. Let \((\alpha_i)_{0 \leq i \leq r} \) and \((\beta_i)_{0 \leq i \leq r} \) be the top (respectively bottom) arcs of the rank ordered along the path (thus \( \alpha_0 = 1, \beta_0 = \sigma^{-1}(1), \alpha_1 = \sigma(\beta_0 + 1) + 1 \ldots \alpha_r = \sigma(n), \beta_r = n \)). Then

\[
\forall \ 0 \leq i \leq r, \ \Pi_t(\alpha_i) = t_{rk} \text{ and } \Pi_b(\beta_i) = b_{rk}
\]
Figure [Π] provides two examples of consistent labellings.

**Lemma 24.** Let $\sigma$ be a permutation and $\Pi_b : \{1, \ldots, n\} \to \Sigma_b$ a labelling of bottom arcs verifying property 1, 2 and 4 of Definition 23. Then there exists a unique $\Pi_t : \{1, \ldots, n\} \to \Sigma_t$ such that $(\Pi_b, \Pi_t)$ is a consistent labelling.

**Proof.** Let $(\Pi_b, \Pi_t)$ a be consistent labelling. Then by property 3 we must have $\Pi_t(\alpha) = \begin{cases} t_{i, \lambda_i, j} & \text{if } \Pi_b(\sigma^{-1}(\alpha)) = b_{i, \lambda_i, j} \\ \ell_{i}^{r_k} & \text{if } \Pi_b(\sigma^{-1}(\alpha)) = \ell_{i}^{r_k} \end{cases}$. This uniquely defines $\Pi_t$. \hfill $\square$

This lemma implies that the data $(\sigma, \Pi_b)$ or $(\sigma, \Pi_t)$ are sufficient to reconstruct $(\sigma, (\Pi_b, \Pi_t))$. Thus, occasionally, we will consider just $(\sigma, \Pi_b)$ rather than $(\sigma, (\Pi_b, \Pi_t))$.

**Lemma 25.** Let $(\sigma, (\Pi_b, \Pi_t))$ be a permutation with a labelling. We have that $(\Pi_b, \Pi_t)$ is a consistent labelling (i.e. it verifies properties 1 to 4) if property 2 is true for bottom arcs, property 3 is true and $\Pi_t(1) = t_0^{r_k}$.

**Proof.** Clearly property 1 is implied by property 2 and 3, and property 4 is implied by property 2 and 3 applied to the rank, plus the fact that $\Pi_t(1) = t_0^{r_k}$, which, by property 2, fixes the order of the other labels of the rank. \hfill $\square$

It is clear from the definition that two consistent labellings of a permutation can only differ by the following operations: a cyclic shift of the labels within a cycle (due to property 2), and the permutation of the ‘$j$’ labels of two cycles of same size (due to property 1). In particular, the labels of the rank, and their order, coincide in all consistent labellings of a given permutation (by property 4).

We state this property more formally in the following definition and proposition.

**Definition 26.** Let $\sigma$ be a permutation with invariant $(\lambda = \{\lambda_1^{m_1}, \ldots, \lambda_k^{m_k}\}, r)$, and define operators on consistent labelling $\Pi_b : \{1, \ldots, n\} \to \Sigma_b$.

- The shift operator $\text{Sh}_{\lambda_i,j}^m$, that shifts by $m$ the labels of the $j$th cycle of length $\lambda_i$:

$$\forall m \geq 1, \forall 1 \leq i \leq k, \forall 1 \leq j \leq m_i \quad \text{Sh}_{\lambda_i,j}^m(\Pi_b)(\beta) = \begin{cases} b_{\ell + m \lambda_i, j} & \text{if } \exists \ell / \Pi_b(\beta) = b_{\ell \lambda_i, j} \\ \Pi_b(\beta) & \text{otherwise}. \end{cases}$$

Figure 11: Two consistent labellings $(\Pi_b, \Pi_t)$ and $(\Pi_b', \Pi_t')$ of a permutation $\sigma$ with cycle invariant $(\{2, 2, 2\}, 2)$. Following definition 26 we have $(\Pi_b', \Pi_t') = \text{Sh}_{2,1}^1((\Pi_b, \Pi_t))$.
The exchange operator $\text{Ex}_{\lambda_i,j_1,j_2}$, that exchanges the labels of the $j_1$th and $j_2$th cycles of length $\lambda_i$:

$$\forall 1 \leq i \leq k, \forall 1 \leq j_1, j_2 \leq m_i$$

$$\text{Ex}_{\lambda_i,j_1,j_2}(\Pi_b)(\beta) = \begin{cases} 
  b_{\ell,\lambda_i,j_2} & \text{if } \exists \ell / \Pi_b(\beta) = b_{\ell,\lambda_i,j_1} \\
  b_{\ell,\lambda_i,j_1} & \text{if } \exists \ell / \Pi_b(\beta) = b_{\ell,\lambda_i,j_2} \\
  \Pi_b(\beta) & \text{otherwise}
\end{cases}$$

**Proposition 27** (Set of consistent labelings). Let $\sigma$ be a permutation with invariant $(\lambda = \{\lambda_1^{m_1}, \ldots, \lambda_k^{m_k}\}, r)$. Two consistent labelings are obtained from one another by a sequence of shift and exchange operators.

**Lemma 28.** Let $\sigma$ be a permutation with invariant $(\lambda = \{\lambda_1^{m_1}, \ldots, \lambda_k^{m_k}\}, r)$ and let $\Pi$ be the set of consistent labelings. Then

$$|\Pi| = \prod_{i=1}^{k} (m_i!\lambda_i^{m_i}).$$

### 5.3 The amenability

Let us define two families of permutations (in cycle notation):

$$\gamma_{t,n}(i) = (1)(2) \cdots (i)(i+1 \ n \ n-1 \ \cdots \ i+2)$$

$$\gamma_{b,n}(i) = (2 \ 3 \ \cdots \ i-1 \ 1)(i)(i+1) \cdots (n)$$

Note that, with this notation, $L(\sigma) = \gamma_{t,n}^{-1}(\sigma(1)) \circ \sigma$ and $R(\sigma) = \sigma \circ \gamma_{b,n}(\sigma^{-1}(n))$.

The dynamics $S_n$ can be naturally extended in order to act also on the labelling.

**Definition 29** (Action of the dynamics $S_n$ on the labelling). Let $(\sigma, (\Pi_b, \Pi_t))$ be a permutation with a consistent labelling. Then $L(\sigma, (\Pi_b, \Pi_t)) = (L(\sigma), (\Pi_b, \Pi_t \circ \gamma_{t,n}(\sigma(1)))$ and $R(\sigma, (\Pi_b, \Pi_t)) = (R(\sigma), (\Pi_b \circ \gamma_{b,n}(\sigma^{-1}(n)), \Pi_t))$.

Refer to Figure 12 for an illustration.

There are two good reasons for this to be ‘the correct way’ of extending the dynamics to the labelled case. First, it is compatible with the structure of the cycle invariant, as the image by the dynamics of a consistent labelling is another consistent labelling (this is the content of Theorem 30). Moreover, as is crucially needed by our methods, it is compatible with the boosted dynamics (see Theorem 32).

**Theorem 30.** Let $(\sigma, (\Pi_b, \Pi_t))$ be a permutation with a consistent labelling. Then $L(\sigma, (\Pi_b, \Pi_t))$ and $R(\sigma, (\Pi_b, \Pi_t))$ are permutations with a consistent labelling.

**Proof.** By symmetry, we can consider just the action of the operator $L$. Let us show that $(L(\sigma), (\Pi'_b = \Pi_b, \Pi'_t = \Pi_t \circ \gamma_{t,n}(\sigma(1))))$ is a permutation with a consistent labelling. First note that

$$L(\sigma)(i) = \begin{cases} 
  \sigma(i) & \text{if } \sigma(i) \leq \sigma(1), \\
  \sigma(i) + 1 & \text{if } \sigma(1) + 1 \leq \sigma(i) \leq n - 1 \\
  \sigma(1) + 1 & \text{if } \sigma(i) = n,
\end{cases}$$
By Lemma 25, we need to check that $\Pi'_0$ verifies property 2, $(\Pi'_b, \Pi'_t)$ verifies property 3, and $\Pi'_t(1) = t_0^k$.

Let $\alpha > 1$ be a top arc. There are three possibilities: $\alpha \leq \sigma(1)$, $\alpha = \sigma(1) + 1$ and $\alpha \geq \sigma(1) + 2$.

- If $\alpha \leq \sigma(1)$ then $\Pi'_t(\alpha) = \Pi_t(\alpha)$ by definition of $\Pi'_t$ (cf. Figure 12 right). Let $\beta = L(\sigma)^{-1}(\alpha - 1)$ and $\beta' = L(\sigma)^{-1}(\alpha)$ be the two consecutive bottom arcs associated to $\alpha$ in $L(\sigma)$. We also have $\beta = \sigma^{-1}(\alpha - 1)$ and $\beta' = \sigma^{-1}(\alpha)$ by definition of $L(\sigma)$, since $\alpha \leq \sigma(1)$. Thus $\Pi'_0(\beta) = b_{i, \lambda} \cdot (\text{or } b^k_i)$ and $\Pi'_0(\beta') = b_{i+1} \cdot \lambda_{\sigma(1)} \cdot j \cdot (\text{or } b^k_{i+1})$, by property 2 for $(\sigma, \Pi_b)$, since $\Pi'_0 = \Pi_b$ and $\beta$ and $\beta'$ are consecutive in $\sigma$.

Likewise, $\Pi'_t(\alpha) = t_{i+1} \cdot \lambda_{\sigma(1)} \cdot j \cdot (\text{or } t_k_{i+1})$ and $\Pi'_t(\beta') = b_{i+1} \cdot \lambda_{\sigma(1)} \cdot j \cdot (\text{or } b^k_{i+1})$ by property 3 for $(\sigma, \Pi_t)$, since $\Pi'_t = \Pi_t$ and $\Pi'_t(\alpha) = \Pi_t(\alpha)$.

Thus in this case $\Pi'_0 = \Pi_b$ verifies property 2 and $(\Pi'_b, \Pi'_t)$ verifies property 3 (see also figure 13).

Note that the same reasoning applies for $\alpha = 1$, from which we deduce that $\Pi'_t(1) = t_0^k$ and $\Pi'_0(\sigma(1)^{-1}(1)) = b_0^k$.

- If $\alpha = \sigma(1) + 1$, let $\alpha' = n$. Then $\Pi'_t(\alpha) = \Pi_t(\alpha')$ by definition of $\Pi'_t$ (cf. Figure 12). Let $\beta' = L(\sigma)^{-1}(n)$ and $\beta' = L(\sigma)^{-1}(\alpha)$ be the two consecutive bottom arcs associated to $\alpha$ in $L(\sigma)$ (this is the special case of consecutive arcs, cf. Figure 12).
the third figure in Definition \[21\]. We have $\beta = \sigma^{-1}(n-1)$ and $\beta' = \sigma^{-1}(n)$ by definition of $L(\sigma)$, thus $\beta$ and $\beta'$ are also the two consecutive bottom arcs associated to $\alpha$ in $\sigma$.

Thus $\Pi'_b(\beta) = b_{i,\lambda_r,j}$ (or $b_{ik}$) and $\Pi'_b(\beta') = b_{i+1 \ mod \ \lambda_r,\lambda_r,j}$ (or $b_{ik}$), by property 2 for $(\sigma, \Pi_b)$, since $\Pi'_b = \Pi_b$ and $\beta$ and $\beta'$ are consecutive in $\sigma$.

Likewise, $\Pi'_t(\alpha) = t_{i+1 \ mod \ \lambda_r,\lambda_r,j}$ (or $t_{ik}$) and $\Pi'_t(\beta') = b_{i+1 \lambda_r,\lambda_r,j}$ (or $b_{ik}$), by property 3 for $(\sigma, (\Pi_b, \Pi_t))$, since $\Pi'_b = \Pi_b$ and $\Pi'_t(\alpha = \sigma(1)+1) = \Pi_t(\alpha' = n)$.

Thus, also in this case $\Pi'_b = \Pi_b$ verifies property 2 and $(\Pi'_b, \Pi'_t)$ verifies property 3 (see also Figure 14).

Figure 14: The proof that $\Pi'_b$ verifies property 2 and $(\Pi'_b, \Pi'_t)$ verifies property 3 for the case $\alpha = \sigma(1) + 1$.

- If $\alpha \geq \sigma(1) + 2$, let $\alpha' = \alpha - 1$. Then $\Pi'_t(\alpha) = \Pi'_t(\alpha')$ by definition of $\Pi'_t$ (cf. Figure 12 right). Let $\beta = L(\sigma)^{-1}(\sigma - 1)$ and $\beta' = L(\sigma)^{-1}(\alpha)$ be the two consecutive bottom arcs associated to $\alpha$. We have $\beta = \sigma^{-1}(\alpha')$ and $\beta' = \sigma^{-1}(\alpha' - 1)$ by definition of $L(\sigma)$, thus $\beta$ and $\beta'$ are also the two consecutive bottom arcs associated to $\alpha$ in $\sigma$.

Thus $\Pi'_b(\beta) = b_{i,\lambda_r,j}$ (or $b_{ik}$) and $\Pi'_b(\beta') = b_{i+1 \ mod \ \lambda_r,\lambda_r,j}$ (or $b_{ik}$), by property 2 for $(\sigma, \Pi_b)$, since $\Pi'_b = \Pi_b$ and $\beta$ and $\beta'$ are consecutive in $\sigma$.

Likewise $\Pi'_t(\alpha) = t_{i+1 \ mod \ \lambda_r,\lambda_r,j}$ (or $t_{ik}$) and $\Pi'_t(\beta') = b_{i+1 \lambda_r,\lambda_r,j}$ (or $b_{ik}$) by property 3 for $(\sigma, (\Pi_b, \Pi_t))$, since $\Pi'_b = \Pi_b$ and $\Pi'_t(\alpha) = \Pi_t(\alpha')$.

\[\square\]

**Corollary 31.** Let $(\sigma, \Pi_b)$ be a permutation with a consistent labelling, and let $S$ be a ‘loop’ (i.e. a sequence in $\{L, R\}^*$ such that $S(\sigma) = \sigma$). Then $S(\Pi_b)$ is a consistent labelling on $\sigma$, and is obtained from $\Pi_b$ by a sequence of shift and exchange operators.

**Proof.** This is immediate from Theorem 30 and Proposition 27 \[\square\]

Let us now prove the compatibility of the labelling with the boosted dynamics.

**Theorem 32** (The labelling is compatible with the boosted dynamics). Let $\sigma \in S_n$ be a permutation, and let $\tau \in S_k$ be the reduction of a $(2k, 2r)$-coloring $c$ of $\sigma$. Let $(\Pi_b, \Pi_t)$ be a consistent labelling of $\tau$, and let $S$ be a sequence of operators for the dynamics $S_k$, acting on $(\tau, (\Pi_b, \Pi_t))$ as described in Definition 27. Let $e$ be a gray
edge of $\sigma$ with endpoints within the arcs with labels $t = \Pi_t(\alpha)$ and $b = \Pi_b(\beta)$ of $\tau$, for some $\alpha$ and $\beta$.

Then, in $(\sigma', c') = B(S)(\sigma, c)$ the gray edge $e$ has its endpoints within the arcs with labels $t$ and $b$ of $(\tau', (\Pi_b', \Pi_t')) = S(\tau, (\Pi_b, \Pi_t))$. See Figure 16.

Proof. By induction on the length of the sequence, and by symmetry, we can consider just the case $S = L$.

Let us consider $\sigma \in \mathcal{S}_n$, and a $(2k, 2r)$-coloring of $\sigma$, $\tau \in \mathcal{S}_k$ the corresponding reduction and $(\Pi_b, \Pi_t)$ a consistent labelling of $\tau$, as in the statement of the theorem.

Let $e \in (t, b)$ be a gray edge of $(\sigma, c)$, which is inserted within the top arc with label $t$ and bottom arc with label $b$ of $(\tau, (\pi_b, \Pi_t))$.

We must show that the gray edge $e$ of $(\sigma', c') = B(L)(\sigma, c)$ is still within the arcs with labels $t$ and $b$ of $(\tau', (\Pi_b', \Pi_t')) = L(\tau, (\Pi_b, \Pi_t))$.

First, let $k_2$ be the size of the block of gray edges immediately to the left of the right pivot (that is, the edge $(\sigma^{-1}(n), n)$). By the mechanisms of the boosted dynamics, in $(\sigma, c)$ we have $B(L) = L^{k_2}$ (see Figure 16). We shall consider three cases, depending on the position of $\alpha$, the arc labelled by $t$:

- If $\tau(1) < \alpha = \Pi_t^{-1}(t) < k_2$, then for $(\tau', (\Pi_b', \Pi_t')) = L(\tau, (\Pi_b, \Pi_t))$ we have $\Pi_t'^{-1}(t) = \alpha + 1$. Since $e = (i, \sigma(i))$ is within the arcs $t$ and $b$ of $(\tau, (\Pi_b, \Pi_t))$, we have $\sigma(1) < \sigma(i) < n - k_2$, thus $L^{k_2}(e) = (i, \sigma(i) + k_2)$ in $(\sigma', c') = L^{k_2}(\sigma, c)$. Therefore in $(\sigma', c')$ the edge $e$ is inserted within the arcs with labels $t$ and $b$ of $(\tau', (\Pi_b', \Pi_t'))$. See Figure 16 with the edge $e_1$ inserted between $t_1$ and $b_1$.

- If $\alpha = \Pi_t^{-1}(t) < \tau(1)$, we are in a situation which is almost identical to the previous one, and we omit to discuss it.

- If $\alpha = \Pi_t^{-1}(t) = n$, then for $(\tau', (\Pi_b', \Pi_t')) = L(\tau, (\Pi_b, \Pi_t))$ we have $\Pi_t'^{-1}(t) = \tau(1) + 1$. Since $e = (i, \sigma(i))$ is inserted within $t$ and $b$ in $(\tau, (\Pi_b, \Pi_t))$, we have $n - k_2 \leq \sigma(i) < n$, thus $L^{k_2}(e) = (i, j)$ with $j$ such that $s(1) < j \leq \sigma(1) + k_2$ in $(\sigma', c') = L^{k_2}(\sigma, c)$. Thus in $(\sigma', c')$ the gray edge $e$ is inserted within the arcs with labels $t$ and $b$ of $\tau'$. See Figure 16 with the edge $e_2$ inserted within $t_2$ and $b_2$. 

$\square$
The last main task of this section is to introduce the 2-point monodromy theorem. As announced in the overview section, Sec. [4] this theorem is the seventh (and last) statement in the organisation of the main induction, in Section [10].

**Theorem 33.** Let \((\sigma, \Pi_b)\) be a permutation with invariant \((\lambda = \{\lambda_1^{m_1}, \ldots, \lambda_k^{m_k}\}, r)\), equipped of a consistent labelling.

*Cycle 1-shift:* Suppose \(\lambda\) does not contain any even cycle, or it has 2 or more even cycles. Let \(i\) be a cycle of length \(\lambda_i\) in \(\lambda\). Then there exists a loop \(S\) in the dynamics such that \(\Pi'_b = S(\Pi_b)\) is a consistent labeling, and verifies \(\Pi'_b(\beta) = b_{\ell+1} \mod \lambda_i, \lambda_i, i \iff \Pi_b(\beta) = b_{\ell, \lambda_i, i}\). In other words, there exists a loop in the dynamics that shifts the labels of the cycle \(i\) by one (and thus, by taking powers, by any integer \(m\)). The positions of the other labels are, in principle, unknown, nonetheless they are constrained by the fact that \(\Pi'_b\) is also a consistent labelling.

*Cycle 2-shift:* Suppose now that \(\lambda\) has exactly one even cycle, of length \(\lambda_i\). Then there exists a loop \(S\) such that \(\Pi'_b = S(\Pi_b)\) verifies \(\Pi'_b(\beta) = b_{\ell+2} \mod \lambda_i, \lambda_i, 1 \iff \Pi_b(\beta) = b_{\ell, \lambda_i, 1}\). In other words, there exists a loop that shifts the labels of the unique even cycle by two (and thus by any even integer \(2m\)). This is consistent with the fact that, if and only if \(\lambda_i\) is odd, then iterated shifts by 2 ultimately produce a shift by 1.

*Cycle jump:* For any two cycles \(j_1, j_2\) of the same length \(\lambda_i\), there exists a loop \(S\) such that \(\Pi'_b = S(\Pi_b)\) verifies \(\Pi'_b(\beta) = b_{\ell, \lambda_i, j_2} \iff \Pi_b(\beta) = b_{\ell, \lambda_i, j_1}\). In other words, there exists a loop that sends the labels of the cycle \(j_2\) on the positions of the labels of the cycle \(j_1\), preserving their ordering.

It might not be clear a priori why such theorem implies the 2-point monodromy. Indeed, in our proof we will choose the label of the top arc to be \(t_0^k\), thus, since it is fixed by Theorem [30], we only need to consider the label of the bottom arc. In this case we have the immediate corollary:
Corollary 34 (2-point monodromy theorem). Let \((\sigma, \Pi_b)\) be a permutation with invariant \((\lambda = \{\lambda_1^m, \ldots, \lambda_k^m\}, r)\) and with a consistent labelling. Let \((t, b) \in \Sigma_t \times \Sigma_b\) such that \(t = t_0^k\).

- If \(b = b_{t,\lambda, j_1}^k\) for some \(i\), there exists a loop \(S\) such that \(\Pi_b' = S(\Pi_b)\) verifies \(\Pi_b'(\beta) = b\) if and only if \(\Pi_b(\beta) = b\).

- Let us now distinguish two cases:
  1. \(\lambda\) has no even cycles or has at least two of them. Then:
     - If \(b = b_{\ell, \lambda, j_1}\) for some triple \((\ell, \lambda, j_1)\), there exists a loop \(S\) such that \(\Pi_b' = S(\Pi_b)\) verifies \(\Pi_b'(\beta) = b\) if and only if \(\Pi_b(\beta) = b_{\ell', \lambda, j_2}\) for some \((\ell', j_2)\).
  2. \(\lambda\) has exactly one even cycle, the cycle \(i\) of length \(\lambda_i\). Then
     - If \(b = b_{\ell, \lambda, j_1}\) for some \((\ell, \lambda, j_1)\) with \(j \neq i\), there exists a loop \(S\) such that \(\Pi_b' = S(\Pi_b)\) verifies \(\Pi_b'(\beta) = b\) if and only if \(\Pi_b(\beta) = b_{\ell', \lambda, j_2}\) for some \((\ell', j_2)\).
     - If \(b = b_{\ell, \lambda, 1}\) for some \((\ell, \lambda, 1)\), there exists a loop \(S\) such that \(\Pi_b' = S(\Pi_b)\) verifies \(\Pi_b'(\beta) = b\) if \(\Pi_b(\beta) = b_{\ell', \lambda_1, 1}\) with \(\ell' \equiv \ell \pmod{2}\).

In both cases, \(\Pi_t(0) = \Pi_t'(0) = t_0^k\) since the labels of the rank are fixed in a consistent labelling.

Let us rephrase the content of this corollary. If we choose the label of the top arc to be \(t_0^k\), then if the bottom label is in the rank it is fixed, otherwise, if it is in a cycle, we can find a loop \(S\) such that the label can move to any bottom label of a cycle of the same length, with one exception: if the bottom label is in the unique cycle of even length. In this case, we can only find a loop \(S\) such that the label can move to the bottom labels of the cycle which has the same parity of the index.

Proof. By Theorem 30, if \(b = b_{t,\lambda, j_1}^k\) then, since \(\Pi_b'\) is a consistent labelling and the labels of the rank are fixed, \(\Pi_b'(\beta) = b\) if and only if \(\Pi_b(\beta) = b\).

If \(b = b_{\ell, \lambda, j_1}\), by Theorem 30 if \(\Pi_b' = S(\Pi_b)\) for some loop \(S\), then it is a consistent labelling, thus we must have \(\Pi_b'(\beta) = b\) if and only if \(\Pi_b'(\beta) = b_{\ell', \lambda, j_2}\) for some \(\ell'\) and \(j_2\), this by Corollary 31.

The rest of the statement is a straightforward application of Theorem 33.

We can actually completely characterise the set \(L(\sigma, (\Pi_b, \Pi_t))\). The following theorem answers this problem:

Theorem 35 (Characterisation of \(L(\sigma, (\Pi_b, \Pi_t))\)). Let \(\sigma\) be a permutation with invariant \((\lambda = \{\lambda_1^m, \ldots, \lambda_k^m\}, r, s)\), and let \((\Pi_b, \Pi_t)\) be a consistent labelling. Then \(L(\sigma, (\Pi_b, \Pi_t))\) consists of the labellings generated by:

- Any exchange operators \(Ex_{\lambda_i, j_1, j_2}\).
- Any 1-shift operators for cycles of odd length \(\lambda_i\) \(Sh_{\lambda_i, j}^1\).
- Any 2-shift operators for cycles of even length \(\lambda_i\) \(Sh_{\lambda_i, j}^2\).

33
• Any 1-shift operators on a pair of cycles of even lengths $\lambda_{i_1}, \lambda_{i_2}$ $Sh_{\lambda_{i_1}, j_1}^1 \circ Sh_{\lambda_{i_2}, j_2}^1$.

This theorem was the content of [Boi13] (however we cannot use it in this article as the proof require the classification theorem). We can derive an alternative proof using the techniques of section 10.7 (see remark 77) but we will not do so in this paper.

6 Cycle invariant and edge addition

This preliminary section study the change of the cycle invariant when inserting a few consecutives edges in a permutation. The results of this section will be used in sections 8.3, 9, 10.

First we reintroduce a notation from [DS17] (it was also defined as $m|a$ and $m_l \cdot m_r$ in [Del13]).

**Definition 36.** A permutation $\sigma$ is of type $H$ if the rank path goes through the $-1$ mark, and of type $X$ otherwise. In the case of a type-$X$ permutation, we call principal cycle the cycle going through the $-1$ mark.

See also figure 17.

**Notation 2.** let $\sigma$ be a permutation and let $\alpha$ be a top arc and $\beta$ be a bottom arc, we define $\sigma|_{1, \alpha, \beta}$ to be the permutation obtained from $\sigma$ by inserting $i \in \mathbb{N}$ consecutive and parallel edges within $\alpha$ and $\beta$. (see figure 18 for an example with $i = 1$).

For the special case of $i = 2$ we call the two added edges a double-edge.

Figure 17: Left: a schematic representation of a permutation of type $H(r_1, r_2)$. Right: a representation of a permutation of type $X(r, i)$. These configurations have rank $r_1 + r_2 - 1$ and $r$, respectively.

Figure 18: The insertion of one edge within the arcs $\alpha$ and $\beta$. 
**Proposition 37** (One edge insertion). Let $\sigma$ be a permutation with cycle invariant $(\lambda, r)$.

Let $\sigma|_{1,\alpha,\beta}$ be the permutation resulting from the insertion of an edge within two arcs of two different cycles of respective length $\ell$ and $\ell'$. Then the cycle invariant of $\sigma|_{1,\alpha,\beta}$ is $(\lambda \setminus \{\ell, \ell\} \cup \{\ell + \ell' + 1\}, r)$.

Let $\sigma|_{1,\alpha,\beta}$ be the permutation resulting from the insertion of an edge within an arc of the rank path and an arc of a cycle (principal cycle or not) of length $\ell$. Then the cycle invariant of $\sigma|_{1,\alpha,\beta}$ is $(\lambda \setminus \{\ell\}, r + \ell + 1)$.

**Proof.** See figure 19.

![Figure 19](image)

**Figure 19:** The first line represents the case: Top arc : any cycle. Bottom arc: any cycle. The second line represents the case: Top arc : rank path. Bottom arc: principal cycle.

Some cases are not represented in the figure. The missing cases are:

- Top arc: principal cycle. Bottom arc: any cycle.
- Top arc: any cycle. Bottom arc: principal cycle.
- Top arc: rank path. Bottom arc: any cycle.
- Top arc: principal cycle. Bottom arc: rank path.
- Top arc: any cycle. Bottom arc: rank path.

Their proof is nearly identical to the ones represented in the figure and are thus omitted.

**Proposition 38** (double-edge insertion). Let $\sigma$ be a permutation with cycle invariant $(\lambda, r)$.

Inserting a double-edge within two arcs of the same cycle (or rank path) increases the length of the cycle (or rank) by two.

Likewise inserting a double-edge within two arcs of different cycles (or rank path) increases the length of each by 1.

**Proof.** See figure 20.
7 Shift-irreducible standard family

In section 3.3 of [DS17] we introduced (or reintroduced since it was already well-known in the literature) the notions of standard permutation and standard family. We reproduce them here without proof for ease of reference.

This section then concerns itself with the more specific notion of shift-irreducible standard family. Briefly a shift-irreducible standard family is a standard family \((\sigma_i)\) for which all but two permutations are irreducible after removing the edge \((\sigma_i)^{-1}(1), 1)\). They are extremely useful for the main induction as they resolve the irreducibility issue of proposition 18 in the proof overview section 4. They are also our set \(T_A\).

The permutation \(\sigma\) is standard if \(\sigma(1) = 1\). It is well known that every irreducible class contains a standard permutation and we can define the standard family of a standard permutation \(\sigma\) as the collection of \(n-1\) permutations \(\{\sigma(i)\}_{0 \leq i \leq n-2}\).

Those notions allow one to prove the important following proposition (the original proposition 39 can be found page 43 of [DS17].)

**Proposition 39** (Properties of the standard family). Let \(\sigma\) be a standard permutation with cycle invariant \((\lambda, r)\), and \(S = \{\sigma(i)\}_i = \{L^i(\sigma)\}_i\) be its standard family. The latter has the following properties:

1. Every \(\tau \in S\) has \(\tau(1) = 1\);
2. The \(n-1\) elements of \(S\) are all distinct;
3. There is a unique \(\tau \in S\) such that \(\tau(n) = n\);
4. Let \(m_i\) be the multiplicity of the integer \(i\) in \(\lambda\) (i.e. the number of cycles of length \(i\)), and \(r\) the rank. There are \(i m_i\) permutations of \(S\) which are of type...
\[ X(r, i), \text{ and } 1 \text{ permutation of type } H(r - j + 1, j), \text{ for each } 1 \leq j \leq r \]  

Let us introduce a convenient notation.

**Notation 3.** Let \( \sigma \) be a permutation, we define \( d(\sigma) \) to be the permutation obtained from \( \sigma \) by discarding the edge \((\sigma^{-1}(1), 1)\), thus if \( \sigma = \sigma(1), \ldots, \sigma(n) \) then \( d(\sigma) = \sigma(1) - 1, \ldots, \sigma(\sigma^{-1}(1)), \ldots, \sigma(n) - 1 \).

Thus \( d(\sigma) \) is the reduction of \((\sigma, c)\) where \( c \) is the \((2n - 2, 2)\)-coloring where the edge \((\sigma^{-1}(1), 1)\) is gray.

Finally we describe the cycle invariant \((\lambda', r')\) of \( d(\sigma^i) \) in function of the cycle invariant \((\lambda, r)\) of \( \sigma^i \).

**Proposition 40.** Let \((\sigma^i)\) be a standard family with cycle invariant \((\lambda, r)\) then \( \forall i \):

- If \( \sigma^i \) has type \( X(r, j) \), \( d(\sigma^i) \) has type \( H(j, r) \) and cycle invariant \((\lambda \setminus \{j\}, r + j - 1)\).
- If \( \sigma^i \) has type \( H(r_1, r_2) \), \( d(\sigma^i) \) has type \( X(r_2 - 1, r_1 - 1) \) and cycle invariant \((\lambda \cup \{r_1 - 1\}, r_2 - 1)\).

**Proof.** The first case of this proposition was proven in lemma 5.16 in [DS17]. The second case is proven likewise: This is the reverse implication of case \( \odot \) in table \( 3 \) in [DS17] (reproduced here), specialised to \( s = 0 \).

This proposition is very useful for the induction since it implies the following lemma.

**Corollary 41.** Let \( \sigma_1 \) and \( \sigma_2 \) be two standard permutations. If \( \sigma_1 \) and \( \sigma_2 \) have invariant \((\lambda, r, s)\) and same type \( X(r, i) \) or \( H(r_1, r_2) \) then the reducted permutations \( \tau_1 = d(\sigma_1) \) and \( \tau_2 = d(\sigma_2) \) have same cycle invariant.

This property (i.e same cycle invariant for \( \tau_1 \) and \( \tau_2 \)) was the first of the three properties we where looking for in our proposition [18] of the proof overview.

For the purpose of guarantying the irreducibility of \( \tau_1 \) and \( \tau_2 \), we now define a more precise notion than just the standard family, we call it a *shift-irreducible standard family*.

**Definition 42.** Let \( \sigma \) be a standard permutation and \( S = (\sigma^i = L^i(\sigma))_{0 \leq i \leq n - 2} \) be its standard family. We say that \( S \) is shift-irreducible if

\[
\forall i \in \{0, \ldots, n - 2\} \setminus \{n - \sigma(2), n - \sigma(n) + 1\} \text{ } d(\sigma^i) \text{ is irreducible.}
\]

In other words, a standard family \( S \) is shift-irreducible if every \( d(\sigma^i) \) that can be irreducible is indeed irreducible. \( d(\sigma^{n-\sigma(2)}) \) and \( d(\sigma^{n-\sigma(n)+1}) \) are both always reducible since \( \sigma^{n-\sigma(2)}(2) = n \) thus \( d(\sigma^{n-\sigma(2)})(1) = n - 1 \) and \( \sigma^{n-\sigma(n)+1}(n) = 2 \) thus \( d(\sigma^{n-\sigma(n)+1})(n - 1) = 1 \). Figure [21] provides an example of a shift-irreducible family.

---

Note that, as \( \sum_i m_i + r = n - 1 \) by the dimension formula [16], this list exhausts all the permutations of the family.
|     | Once per |     |     |     |
|-----|----------|-----|-----|-----|
| 1   | \(1 \leq s \leq r_1\) | \(2s - 1\) | \(2r_1 - 2s + 1\) | \(2r_2\) |
| 2   | \(\ell\) times per each cycle of \(\lambda\) of length \(\ell\) | \(2r_1\) | \(2r_2\) | \(2\ell\) |
| 3   | \(0 \leq s \leq r_2\) | \(2r_1\) | \(2r_2 - 2s\) | \(2s\) |
| 4   | \(1 \leq s \leq r\) | \(2s - 1\) | \(2i + 1\) | \(2r - 2s\) |
| 5   | \(\ell\) times per each cycle of \(\lambda\) of length \(\ell\) | \(2i + 1\) | \(2r + 1\) | \(2\ell\) |
| 6   | \(0 \leq s \leq i\) | \(2i - 2s + 1\) | \(2r + 1\) | \(2s\) |

|     | \(H(r', r'_2)\) | \(2r_1 = r_1 - s + 1\) | \(r'_2 = r_2\) |
|-----|-----------------|-----------------|-----------------|
| 2   | \(H(r'_1, r'_2)\) | \(r'_1 = r_1 + \ell + 1\) | \(r'_2 = r_2\) |
| 3   | \(X(r', i')\) | \(r' = r_2 - s\) | \(i' = r_1 + s\) |
| 4   | \(X(r', i')\) | \(r' = r - s\) | \(i' = i\) |
| 5   | \(X(r', i')\) | \(r' = r + \ell + 1\) | \(i' = i\) |
| 6   | \(H(r'_1, r'_2)\) | \(r'_1 = i - s + 1\) | \(r'_2 = r + s + 1\) |

Table 3: Modification to the cycle invariant between \(\sigma\) and \(\sigma|_{1,0,i}\), for every possible \(i\). In green, the newly-added edge. In red, parts which get added to the rank path. In blue, parts which are singled out to form a new cycle.
Figure 21: An example of a shift-irreducible family $S = (\sigma^i)_{0 \leq i \leq n-2}$ with its two unavoidably reducible permutations $d(\sigma^{n-\sigma(2)})$ and $d(\sigma^{n-\sigma(n)+1})$. For a proof of the shift-irreducibility, the characterisation proposition 44 is helpful.

Proposition 43. Let $\sigma$ be a standard permutation with cycle invariant $(\lambda, r)$, and $S = (\sigma^i = L^i(\sigma))_{0 \leq i \leq n-2}$ its standard family. Then $S$ is shift irreducible if and only if

- For every $i \in \lambda$ the image by $d$ of the $i \cdot m_i$ permutations of $S$ of type $X(r, i)$ are irreducible (where $m_i$ is the multiplicity of $i$ in $\lambda$).
- For all $1 < j < r$ the image by $d$ of the permutation of $S$ of type $H(r - j + 1, j)$ is irreducible.

Proof. By proposition 39, the permutations of $S$ are exactly the $im_i$ permutations of type $X(r, i)$ for every $i \in \lambda$ and the permutations of type $H(r - j + 1, j)$ for all $1 \leq j \leq r$. Clearly the permutations of type $H(1, r)$ and $H(r, 1)$ are respectively $\sigma^{n-\sigma(2)}$ and $\sigma^{n-\sigma(n)+1}$ (see figure 22). Thus if the image by $d$ of every permutation besides those two are irreducible the family is shift-irreducible and reciprocally if the family is shift-irreducible the image by $d$ of every permutation besides those two are irreducible.

Figure 22: The two permutations $\sigma^{n-\sigma(2)}$ and $\sigma^{n-\sigma(n)+1}$ are of type $H(1, r)$ and $H(r, 1)$ respectively.

As mentioned above the two permutations $\sigma^{n-\sigma(2)}$ and $\sigma^{n-\sigma(n)+1}$ - whose images by $d$ are reducible - have type $H(1, r)$ and $H(r, 1)$. Thus $d(\sigma^{n-\sigma(2)})$ has cycle invariant $(\lambda \cup \{0\}, r - 1)$ and $d(\sigma^{n-\sigma(n)+1})$ has cycle invariant $(\lambda \cup \{r - 1\}, 0)$ by proposition 40. Neither are cycle invariants that we allow for in the classification theorem (we never consider cycles or rank of length 0 since it always implies reducibility).

In other term, every single one of the permutations $d(\sigma^i)$ that can be used for the induction are irreducible if $(\sigma^i)_i$ is an shift-irreducible standard family.

It is interesting that the fruitful notion of shift-irreducible family has a very simple characterisation in term of just the permutation $\sigma$ of the family with $\sigma(1) = 1$ and $\sigma(2) = 2$. 

39
**Proposition 44** (Characterisation of shift-irreducible families). *Let $\sigma$ be a standard permutation with $\sigma(2) = 2$ and $S = (\sigma^i = L^i(\sigma))_{0 \leq i \leq n-2}$ its standard family. Then $S$ is shift-irreducible if and only if $\sigma$ does not have the following form:*

\[
\begin{array}{c}
\emptyset \\
\emptyset \\
\emptyset \\
\emptyset \\
\end{array}
\]

\[
\begin{array}{c}
\emptyset \\
\emptyset \\
\emptyset \\
\end{array}
\]

\[
\begin{array}{c}
\emptyset \\
\emptyset \\
\end{array}
\]

\[
\begin{array}{c}
\emptyset \\
\emptyset \\
\end{array}
\]

**Proof.** We show that $S$ is not shift-irreducible if and only if $\sigma$ has the form described in the proposition. If $S$ is not shift-irreducible then it means that there exists a $i \in \{0, \ldots, n-2\} \setminus \{n - \sigma(2), n - \sigma(n) + 1\}$ such that $d(\sigma^i)$ is reducible. Thus $d(\sigma^i)$ has the following form

\[
\begin{array}{c}
\emptyset \\
\emptyset \\
\emptyset \\
\end{array}
\]

\[
\begin{array}{c}
\emptyset \\
\emptyset \\
\end{array}
\]

The blocks with $\neq \emptyset$ must not be empty otherwise we would have either $\sigma^i = \sigma^{n-\sigma(2)}$ or $\sigma^{n-\sigma(n)+1}$. $\sigma^i$ has then the form

\[
\begin{array}{c}
\emptyset \\
\emptyset \\
\end{array}
\]

\[
\begin{array}{c}
\emptyset \\
\end{array}
\]

and since by definition $\sigma^i = L^i(\sigma)$, $\sigma$ must have the form

\[
\begin{array}{c}
\emptyset \\
\emptyset \\
\emptyset \\
\end{array}
\]

as predicted. The reverse implication is obtained by reversing the steps of the proof. □

In light of this proposition we will often say *shift-irreducible permutation* to denote the permutation $\sigma$ of the shift-irreducible standard family with $\sigma(1) = 1$ and $\sigma(2) = 2$. Moreover to prove the existence of a shift-irreducible standard family we will just construct a shift-irreducible permutation.

A particular type of shift-irreducible permutation is the $I_2 X$-permutation:

**Corollary 45.** *Let $\sigma$ be a standard permutation with $\sigma(2) = 2$ If $\sigma$ have the following form:*

\[
\begin{array}{c}
\emptyset \\
\emptyset \\
\emptyset \\
\emptyset \\
\end{array}
\]

\[
\begin{array}{c}
\emptyset \\
\emptyset \\
\end{array}
\]

\[
\begin{array}{c}
\emptyset \\
\emptyset \\
\end{array}
\]

\[
\begin{array}{c}
\emptyset \\
\end{array}
\]

\[
\begin{array}{c}
\emptyset \\
\emptyset \\
\end{array}
\]

\[
\begin{array}{c}
\emptyset \\
\emptyset \\
\end{array}
\]

\[
\begin{array}{c}
\emptyset \\
\emptyset \\
\end{array}
\]
Then \( \sigma \) is a shift-irreducible permutation. We call \( \sigma \) a \( I_2X \)-permutation.

**Proof.** It is clear that a \( I_2X \)-permutation verifies the condition of the characterisation proposition 44.

\[ \square \]

# 8 The sign invariant

## 8.1 Arf functions for permutations

For \( \sigma \) a permutation in \( S_n \), let

\[
\chi(\sigma) = \# \{ 1 \leq i < j \leq n \mid \sigma(i) < \sigma(j) \}
\]

i.e. \( \chi(\sigma) \) is the number of pairs of non-crossing edges in the diagram representation of \( \sigma \).

Let \( E = E(\sigma) \) be the subset of \( n \) edges in \( K_{n,n} \) described by \( \sigma \). For any \( I \subseteq E \) of cardinality \( k \), the permutation \( \sigma|_I \in \mathcal{G}_k \) is defined in the obvious way, as the one associated to the subgraph of \( K_{n,n} \) with edge-set \( I \), with singletons dropped out, and the inherited total ordering of the two vertex-sets.

Define the two functions

\[
A(\sigma) := \sum_{I \subseteq E(\sigma)} (-1)^{\chi(\sigma|_I)} ; \quad \overline{A}(\sigma) := \sum_{I \subseteq E(\sigma)} (-1)^{|I| + \chi(\sigma|_I)} .
\]

When \( \sigma \) is understood, we will just write \( \chi_I \) for \( \chi(\sigma|_I) \). The quantity \( A \) is accessory in the forthcoming analysis, while the crucial fact for our purpose is that the quantity \( \overline{A} \) is invariant in the \( S_n \) dynamics.

In the following section, we define a technique to demonstrate identities of the arf invariant involving different configurations.

## 8.2 Automatic proofs of Arf identities

We will not try to evaluate Arf functions of large configurations starting from scratch. We will rather compare the Arf functions of two (or more) configurations, which differ by a finite number of edges, and establish linear relations among their Arf functions. The method we develop here, gives an algorithm to find and check Arf identities.

In order to have the appropriate terminology for expressing this strategy, let us define the following: Given a permutation \( \sigma \) define \( \sigma_{k,\ell} \) to be a permutation with \( k \) marks on its bottom line and \( \ell \) marks on its top line. The marks are all at distinct positions and do not touch the corners of the permutation. These marks break the bottom (respectively top) line into \( k + 1 \) open interval \( P_{-1}, \ldots, P_{-k+1} \) (respectively \( \ell + 1 \) open interval \( P_{+1}, \ldots, P_{+\ell+1} \)).
For example if \( k = 1, \ell = 3 \):

\[
\sigma_{k,\ell} = \begin{array}{cccc}
P_{+,1} & P_{+,2} & P_{+,3} & P_{+,4} \\
\circ & \circ & \circ & \circ \\
P_{-,1} & P_{-,2} & & 
\end{array}
\]

Let \( \sigma_{k,\ell,E'} \) be the permutation obtained by adding a set of edges \( E' \) on the marks of permutation \( \sigma_{k,\ell} \) with the following convention: an edge \( e \in E' \) is a pair \((i,x,j,y)\). The edge connects the \( i \)th bottom mark and the \( j \)th top mark, and it is ordered as the \( x \)th edge within the bottom mark and the \( y \)th edge within the top mark. Note that if \( i = 0 \) of \( i = k + 1 \) (likewise of \( j \)) this implies that the edge is connected to a corner of the permutation.

For example if \( k = 1, \ell = 3 \) and \( E' = \{(0.1, 2.2), (1.1, 3.1), (1.2, 1.1), (1.3, 2.1), (2, 1.2)\} \):

\[
\sigma_{k,\ell,E'} = \begin{array}{cccc}
P_{+,1} & P_{+,2} & P_{+,3} & P_{+,4} \\
\circ & \circ & \circ & \circ \\
P_{-,1} & P_{-,2} & & 
\end{array}
\]

We will define an algorithm that allows one to check if, for all \( \sigma \), we have \( \sum_{i=1}^{n} K_i A(\sigma_{k,\ell,E'}) = 0 \) or \( \sum_{i=1}^{n} K_i A(\sigma_{k,\ell,E'}) = 0 \) for some \( k, \ell, (E')_i, (K_i)_i, n \).

**Definition 46.** Let \( \sigma_{k,\ell,E'}, P_{-,1}, \ldots, P_{-,k+1} \) and \( P_{+,1}, \ldots, P_{+,\ell+1} \) be as defined above. Then define the \( m \times (k+1)(\ell+1) \) matrix valued in \( GF_2 \)

\[
Q_{e,ij} := \begin{cases} 
1 & \text{edge } e \in E' \text{ does not cross the segment connecting } P_{-,i} \text{ to } P_{+,j}; \\
0 & \text{otherwise.}
\end{cases}
\]

(11)

For \( v \in GF_2^{(k+1)(\ell+1)} \), let \( |v| \) be the number of entries equal to 1. Similarly, identify \( v \) with the corresponding subset of \( [(k+1)(\ell+1)] \). Given such a construction, introduce the following functions on \((GF_2)^{(k+1)(\ell+1)}\)

\[
A_{k,\ell,E'}(v) := \sum_{u \in (GF_2)^{E'}} (-1)^{x_u+(u,Qv)}; \quad \overline{A}_{k,\ell,E'}(v) := \sum_{u \in (GF_2)^{E'}} (-1)^{|u|+x_u+(u,Qv)}.
\]

(12)

The construction is illustrated in Figure 23.

Let us comment on the reasons for introducing such a definition. The quantities \( A_{k,\ell,E'}(v) \) (respectively \( \overline{A}_{k,\ell,E'}(v) \)) do not depend on \( \sigma \) and allows to sum together many contributions to the function \( A(\sigma_{k,\ell,E'}) \). Our goal is to have \( E' \) of fixed size, while \( E \) (the edge set of \( \sigma \)) is arbitrary and of unbounded size, so that the verification of our properties, as it is confined to the matrix \( Q \), involves a finite data structure. Thus the algorithm will be exponential in \( |E'| \) which will not be a problem for small sizes.

Indeed, let us split in the natural way the sum over subsets \( I \) of \( E \cup E' \) the edge set of \( \sigma_{k,\ell,E'} \) namely

\[
\sum_{I_0 \subseteq E \cup E'} f(I_0) = \sum_{I \subseteq E} \sum_{I' \subseteq E'} f(I \cup I')
\]

42
Figure 23: The permutation $\sigma_{1,4,\{(0.1,0.1),(0.2,3,1),(0.3,1,1),(1,1.4,1),(1.2,2,1)\}}$. We cannot show the full matrix $Q$ for such a big example, but we can give one row, for the edge which has the label $e$ in the drawing. The row $Q_e$ reads $(Q_e)_{11,12,\ldots,15,21,\ldots,25} = (1,1,0,0,0,0,0,1,1,1)$.

For $I$ and $J$ two disjoint sets of edges, call $\chi_{I,J}$ the number of pairs $(i,j) \in I \times J$ which do not cross. Then clearly

$$\chi_{I\cup J} = \chi_I + \chi_J + \chi_{I,J}$$

Now let $u(I') \in \{0,1\}^{E'}$ be the vector with entries $u_e = 1$ if $e \in I'$ and 0 otherwise. Let $m(I) = \{m_{ij}(I)\}$ be the $(k+1) \times (\ell+1)$ matrix describing the number of edges connecting the intervals $P_{-i}$ to $P_{+j}$ in $\sigma$, and let $v(I) = \{v_{ij}(I)\}$, $v_{ij} \in \{0,1\}$ be the parities of the $m_{ij}$'s. Call $I_{ij}$ the restriction of $I$ to edges connecting $P_{-i}$ and $P_{+j}$. Clearly,

$$\chi_{I',i} = \sum_{ij} \chi_{I',I_{ij}} = \sum_{e,ij} u_e Q_{e,ij} m_{ij} = (u(I'),Qm(I)),$$

which has the same parity as the analogous expression with $v$'s instead of $m$'s. I.e. we have

$$(-1)^{\chi_{I',i}} = (-1)^{(u(I'),Qv(I))},$$

Now, while the $m$'s are in $\mathbb{N}$, the vector $v$ is in a linear space of finite cardinality, which is crucial for allowing a finite analysis of our expressions.

As a consequence,

$$A(\sigma_{k,\ell,E'}) = \sum_{I \subseteq E} (-1)^{|I|} A_{k,\ell,E'}(v(I));$$  \hspace{1cm} (13)

$$\overline{A}(\sigma_{k,\ell,E'}) = \sum_{I \subseteq E} (-1)^{|I|+1 \chi} A_{k,\ell,E'}(v(I)).$$  \hspace{1cm} (14)

Thus we have the following proposition:

**Theorem 47.** Let $k, \ell \in \mathbb{N}$ and let $(E_i)_{1 \leq i \leq n}$ be a family of edge set. Then the two following statements are equivalent:

1. For all $v \in GF(k+1)(\ell+1)$, we have $\sum_{i=1}^{n} K_i A_{k,\ell,E'}(v) = 0$.

2. for all $\sigma$, we have $\sum_{i=1}^{n} K_i A(\sigma_{k,\ell,E'}) = 0$.

The same statement holds for $\overline{A}$.

43
Proof. Statement 1 implies 2 due to equation (13). Let us show the converse: If $\sigma$ has no edge then $\sum_{i=1}^{n} K_i A(\sigma_{k,\ell,E'}) = 0$ is equivalent to $\sum_{i=1}^{n} K_i A_{k,\ell,E'}(v) = 0$ with $v$ being the zero vector of $(GF_2)^{(k+1)(\ell+1)}$.

Then we choose the family of permutations $(\sigma^{a,b})_{a,b}$ with exactly one edge connecting $P_{-a}$ to $P_{+b}$. Then if we define $v_{a,b}$ to be the vector $(GF_2)^{(k+1)(\ell+1)}$ with exactly one 1 at position $ab$ we have

$$\sum_{i=1}^{n} K_i A(\sigma^{a,b}_{k,\ell,E'}) = 0$$

$$\iff \sum_{i=1}^{n} K_i (A_{k,\ell,E'}(0) + A_{k,\ell,E'}(v_{a,b})) = 0$$

$$\iff \sum_{i=1}^{n} K_i A_{k,\ell,E'}(v_{a,b}) = 0$$

in the last line we have used that $\sum_{i=1}^{n} K_i A_{k,\ell,E'}(0) = 0$.

Thus inductively we show that $\sum_{i=1}^{n} K_i A_{k,\ell,E'}(v) = 0$ for any $v \in (GF_2)^{(k+1)(\ell+1)}$ with at most $p \leq n$ ones. □

This theorem is very important since it reduces the problem of calculating Arf identities for permutations of any size to a check of an Arf identity for $2^{(k+1)(\ell+1)}$ values. Thus in exponential time in $k\ell$ we can calculate $A_{k,\ell,E'}(v)$ for every $v \in GF_2^{(k+1)(\ell+1)}$ and decide if a given Arf identity is correct.

We can even do better:

Proposition 48. Let $k, \ell \in \mathbb{N}$ and let $(E')_{1 \leq i \leq n}$ be a family of edge set. We can decide (in exponential time in $k\ell$) if there exists $x_1, \ldots, x_n$ such that $\sum_{i=1}^{n} x_i A(\sigma_{k,\ell,E'}) = 0$.

Proof. For every $v$ we have an equation $\sum_{i=1}^{n} x_i A_{k,\ell,E'}(v) = 0$ with the $n$ unknown variables. So there are $2^{(k+1)(\ell+1)}$ equations. We can find the subspace of solution in time exponential in $k\ell$. □

The previous proposition can be used when we suspect a relation between a few configurations without knowing the coefficients. The algorithm demands little more than the previous one for the verification so it remains usable for small $|E'|$.

Finally we can actually enumerate all the possible Arf identities:

Proposition 49. There is an algorithm that enumerate all the arf identities with at most $n$ terms and on an edge set $E'$ of size at most $h$.

This algorithm is really not practiceable. However it can be used in the following case: we have two terms and we want to find an arf identity relating them to one another but the previous algorithm failed (i.e there are no identity containing only those two terms). Then we use this algorithm to find a third term (or a fourth etc...) for which an identity exists.

We can even propose a generalisation of this framework: let us choose two permutations $\pi_-$ and $\pi_+$ of size $k+1$ and $\ell+1$ respectively then $\sigma_{k,\ell,E'} \pi_-, \pi_+$ is obtained from $\sigma_{k,\ell,E'}$ by permuting the $P_{-,i}$ with $\pi_-$ and $P_{+,i}$ with $\pi_+$. 44
For example if $k = 1, \ell = 3$ and $E' = \{(0.1, 2.2), (1.1, 3.1), (1.2, 1.1), (1.3, 2.1), (2, 1.2)\}$:

\[
\sigma_{k, \ell, E'} = \begin{pmatrix}
P_{+, 1} & P_{+, 2} & P_{+, 3} & P_{+, 4} \\
P_{-, 1} & P_{-, 2} & & \\
& & & \\
& & & 
\end{pmatrix} \quad \sigma_{k, \ell, E', \pi_- , \pi_+} = \begin{pmatrix}
P_{+, \pi_+ 1} & P_{+, \pi_+ 2} & P_{+, \pi_+ 3} & P_{+, \pi_+ 4} \\
P_{-, \pi_- 1} & P_{-, \pi_- 2} & & \\
& & & \\
& & & 
\end{pmatrix}.
\]

For an example with $\pi_- = id_2$ and $\pi_+ = (2, 1)$ (the reversing permutation $\omega$ at size 2) we have:

\[
\sigma_{1, 1, E'} = \begin{pmatrix}
P_{+, 1} & P_{+, 2} \\
P_{-, 1} & P_{-, 2} \\
& & & \\
& & & 
\end{pmatrix} \quad \sigma_{1, 1, E', \pi_- , \pi_+} = \begin{pmatrix}
P_{+, 1} & P_{+, 2} \\
P_{-, 1} & P_{-, 2} \\
& & & \\
& & & 
\end{pmatrix}.
\]

It is easily checked that the previous theorems continue to hold for this generalisation once we introduce for $v \in GF_2^{(k+1)(\ell+1)}$ the following function on $(GF_2)^{(k+1)(\ell+1)}$ (similar definition for $A_{k, \ell, E', \pi_- , \pi_+}$)

\[
A_{k, \ell, E', \pi_- , \pi_+}(v) := \sum_{u \in (GF_2)^E} (-1)^{\chi_u + P_{\pi_- v P_{\pi_+}}}; \quad (15)
\]

Where in the expression $(P_{\pi_- v P_{\pi_+}})$, $v$ is identified to the matrix of size $(k+1) \times (\ell+1)$ and $P_{\pi_-}$ and $P_{\pi_+}$ are the permutation matrices associated to $\pi_-$ and $\pi_+$.

The framework of automatic proofs of Arf identities we have developed is rather general. Most of the identities found in the literature (see [KZ03], [Boi13], [DS17], [Del13], [Gut17]) can be obtained in this setting.

Let us now apply the algorithm to find Arf identities.

It is convenient to introduce the notation $\vec{A}(\sigma) = \begin{pmatrix} \pi(\sigma) \\ A(\sigma) \end{pmatrix}$. We have

**Proposition 50.**

\[
\vec{A}(\tau) = \vec{A}(\sigma) = \begin{pmatrix}
\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}
\end{pmatrix} \\
\begin{pmatrix}
\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}
\end{pmatrix} \\
\begin{pmatrix}
\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}
\end{pmatrix} \\
\begin{pmatrix}
\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}
\end{pmatrix} \\
\begin{pmatrix}
\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}
\end{pmatrix}
\]

Clearly equation [16] prove the invariance of the arf invariant for the dynamics since $\sigma = L(\tau)$ and the case $R$ is deduced by symmetry.

**8.3 Arf relation for the induction**

This section lists the arf identities required for the inductive proof using the labelling method.
This section contains four statements of note: propositions 51, 55 and 57 will be used in a technical way during the induction and proposition 53 will be applied in section 9 to construct pairs of permutations with same cycle invariant but opposite sign invariant.

We have a first proposition, involving the evaluation of $\mathcal{A}$ on three distinct configurations

**Proposition 51.**

\[
\mathcal{A}\left(\sigma = \begin{array}{l}
e f  \\
g\end{array}\right) + \mathcal{A}\left(\tau = \begin{array}{l}2f \\
ge'\end{array}\right) = 2\mathcal{A}\left(\rho = \begin{array}{l}f \\
g\end{array}\right)
\] (19)

\[
\mathcal{A}\left(\sigma = \begin{array}{l}
e f  \\
g\end{array}\right) + \mathcal{A}\left(\tau = \begin{array}{l}2f \\
ge'\end{array}\right) = 2\mathcal{A}\left(\rho = \begin{array}{l}f \\
g\end{array}\right)
\] (20)

**Proof.** Clearly we are in the framework developped in the previous section so the proof can be checked by proposition 47.

Those identities are also found in [Del13] lemmas 4.9 and 4.10 and [Boi12] proposition 4.2. The correspondance is not exact due to the difference of language but the proof of those proposition/lemmas involve the use of those identities (once translated in their language).

Let $\sigma = \begin{array}{l}2f \\
ge\end{array}$, we define $\sigma_{i,j} = \begin{array}{l}2f \\
i\end{array}$

With this notation, the first equation of proposition 51 can be rewritten

$$\mathcal{A}(\sigma_{1,0}) + \mathcal{A}(\sigma_{0,1}) = 2\mathcal{A}(\sigma_{0,0}).$$

**Lemma 52.**

\[
\mathcal{A}(\sigma_{2,0}) - \mathcal{A}(\sigma_{0,2}) = 2 (\mathcal{A}(\sigma_{1,0}) - \mathcal{A}(\sigma_{0,1}))
\] (21)

\[
\mathcal{A}(\sigma_{3,0}) + \mathcal{A}(\sigma_{0,3}) = 3\mathcal{A}(\sigma_{2,0}) + 3\mathcal{A}(\sigma_{0,2}) - 4\mathcal{A}(\sigma_{0,0})
\] (22)

Once again we are in the framework so the proposition can be verified by proposition 47.

The main use of those two lemmas is to compare the Arf invariant of two configurations having the same cycle invariant and differing by just a few consecutives and parallel edges. The exact statement is the content of the next proposition.

**Proposition 53 (Opposite sign).** Let $\sigma$ be a permutation with exactly two even cycles (one being possibly the rank). Let $\alpha$ be a top arc of the first cycle and $\beta$ and $\beta'$ be two consecutive bottom arcs of the second cycle.

Define $\sigma|_{i,\alpha,\beta}$ and $\sigma|_{i,\alpha,\beta'}$ following notation 2 (as a reminder they are the two permutations obtained by adding $i$ parallel and consecutive edges within $\alpha, \beta$ and
within $\alpha, \beta'$ respectively). See figure 24 for the case $i = 1$ and $\sigma$ has two even cycles (none being the rank path).

Then for $i \leq 3$, $\sigma|_{i,\alpha,\beta}$ and $\sigma|_{i,\alpha,\beta'}$ have invariant $(\lambda', r', s)$ and $(\lambda', r', -s)$ respectively.

To be more precise:

1. If $\sigma$ has cycle invariant $(\lambda \cup \{2\ell, 2\ell'\}, r)$ we have

|    | $i = 1$                                      | $i = 2$                                      | $i = 3$                                      |
|----|---------------------------------------------|---------------------------------------------|---------------------------------------------|
| $\sigma$ | $(\lambda, r, s)$ of $\sigma|_{1,\alpha,\beta}$ | $(\lambda \cup \{2\ell + 2\ell' + 1\}, r, s)$ | $(\lambda \cup \{2\ell + 2\ell' + 3\}, r, s)$ |
| $\sigma$ | $(\lambda, r, s)$ of $\sigma|_{1,\alpha,\beta'}$ | $(\lambda \cup \{2\ell + 1, 2\ell' + 1\}, r, s)$ | $(\lambda \cup \{2\ell + 2\ell' + 3\}, r, s)$ |

2. If $\sigma$ has cycle invariant $(\lambda \cup \{2\ell\}, 2r)$ we have

|    | $i = 1$                                      | $i = 2$                                      | $i = 3$                                      |
|----|---------------------------------------------|---------------------------------------------|---------------------------------------------|
| $\sigma$ | $(\lambda, r, s)$ of $\sigma|_{1,\alpha,\beta}$ | $(\lambda, 2r + 2\ell + 1, s)$              | $(\lambda, 2r + 2\ell + 3, s)$              |
| $\sigma$ | $(\lambda, r, s)$ of $\sigma|_{1,\alpha,\beta'}$ | $(\lambda \cup \{2\ell + 1\}, 2r + 1, s)$ | $(\lambda, 2r + 2\ell + 3, -s)$ |

Figure 24: From left to right: the permutations $\sigma$, $\sigma|_{1,\alpha,\beta}$ and $\sigma|_{1,\alpha,\beta'}$. The proposition 53 says that $\sigma|_{1,\alpha,\beta}$ and $\sigma|_{1,\alpha,\beta'}$ have same cycle invariant and opposite sign invariant.

Proof. Let us start with the cycle invariant. For the case $i = 1$ and $i = 2$ this is a strict application of propositions 37 and 38 respectively.

For $i = 3$ note that $\sigma|_{3,\alpha,\beta}$ is obtained from $\sigma|_{1,\alpha,\beta}$ by adding a double-edge. In the first case $\sigma|_{1,\alpha,\beta}$ has cycle invariant $(\lambda \cup \{2\ell + 2\ell' + 1\}, r)$ and the double-edge is inserted on two arcs of the cycle of length $2\ell + 2\ell' + 1$ thus by proposition 38 the cycle invariant of $\sigma|_{3,\alpha,\beta}$ is $(\lambda \cup \{2\ell + 2\ell' + 3\}, r)$.

In the second case $\sigma|_{1,\alpha,\beta}$ has cycle invariant $(\lambda, 2r + 2\ell + 1)$ and the double-edge is inserted on two arcs of the rank path of length $2r + 2\ell + 1$ thus by proposition 38 the cycle invariant of $\sigma|_{3,\alpha,\beta}$ is $(\lambda, 2r + 2\ell + 3)$.

The same reasoning applies to $\sigma|_{3,\alpha,\beta'}$.

In the following the reasoning applies to both cases ($\sigma$ has cycle invariant $(\lambda \cup \{2\ell, 2\ell'\}, r)$ and $\sigma$ has cycle invariant $(\lambda \cup \{2\ell\}, 2r)$ so we will no longer differentiate.

By hypothesis on $\sigma$, $\lambda$ has no even part thus it can be verified on the two tables that the cycle invariant of $\sigma|_{1,\alpha,\beta}$ and $\sigma|_{1,\alpha,\beta'}$ has no even cycle for $i \leq 3$.

Consequently, $\overline{A}(\sigma|_{1,\alpha,\beta}) = \pm 2^{n_i + \ell_i}$ and $\overline{A}(\sigma|_{1,\alpha,\beta'}) = \pm 2^{n_i + \ell_i}$ where $n_i$ is the size of $\sigma|_{1,\alpha,\beta}$ and $\ell_i$ is the number of cycle (not including the rank) of $\sigma|_{1,\alpha,\beta}$ by theorem 13.
Let \( n \) be the size of \( \sigma \) and \( \ell_0 \) be the number of cycles (not including the rank) of \( \sigma \). Then by inspecting the tables we know that

\[
\overline{A}(\sigma|_{1,\alpha,\beta}) = \pm 2^{n+\ell_0} = \pm \overline{A}(\sigma|_{1,\alpha,\beta'})
\]

(23)

and that

\[
\overline{A}(\sigma|_{2,\alpha,\beta}) = \pm 2^{n+\ell_0+1} = \pm \overline{A}(\sigma|_{2,\alpha,\beta'})
\]

(24)

by theorem 13 since \( \sigma \) has even cycles. Moreover by applying proposition 51 to \( \sigma|_{1,\alpha,\beta}, \sigma|_{1,\alpha,\beta'} \) and \( \sigma \) we have that

\[
\overline{A}(\sigma|_{1,\alpha,\beta}) + \overline{A}(\sigma|_{1,\alpha,\beta'}) = 2\overline{A}(\sigma)
\]

(25)

thus

\[
\overline{A}(\sigma|_{1,\alpha,\beta}) = -\overline{A}(\sigma|_{1,\alpha,\beta'})
\]

(26)

Now by applying lemma 52 (equation 21) to \( \sigma|_{2,\alpha,\beta}, \sigma|_{2,\alpha,\beta'} \) and \( \sigma|_{1,\alpha,\beta}, \sigma|_{1,\alpha,\beta'} \) we have

\[
\overline{A}(\sigma|_{2,\alpha,\beta}) - \overline{A}(\sigma|_{2,\alpha,\beta'}) = 2(\overline{A}(\sigma|_{1,\alpha,\beta}) - \overline{A}(\sigma|_{1,\alpha,\beta'}))
\]

(27)

by eq 26

Thus by equations 23 and 24 we must have

\[
\overline{A}(\sigma|_{2,\alpha,\beta}) = -\overline{A}(\sigma|_{2,\alpha,\beta'})
\]

(27)

Finally by applying lemma 52 (equation 22) to \( \sigma|_{3,\alpha,\beta}, \sigma|_{3,\alpha,\beta'} \) and \( \sigma|_{2,\alpha,\beta}, \sigma|_{2,\alpha,\beta'} \) and \( \sigma \) we have

\[
\overline{A}(\sigma|_{3,\alpha,\beta}) + \overline{A}(\sigma|_{3,\alpha,\beta'}) = 3(\overline{A}(\sigma|_{2,\alpha,\beta}) + \overline{A}(\sigma|_{2,\alpha,\beta'})) - 4\overline{A}(\sigma)
\]

(28)

thus

\[
\overline{A}(\sigma|_{3,\alpha,\beta}) = -\overline{A}(\sigma|_{3,\alpha,\beta'}).
\]

(28)

The key point of the proposition is really that for \( i \leq 3 \), \( \sigma|_{i,\alpha,\beta} \) and \( \sigma|_{i,\alpha,\beta'} \) have same cycle invariant and opposite sign invariant. We will use this proposition to construct \( I_2X \)-permutations with same cycle invariant and opposite sign in the next section.

**Remark 54.** The proof of proposition 53 makes use of theorem 13. This theorem will only be proved during the induction, thus every time we use proposition 53 we must check that we have already proved theorem 13 or indicate that the newly proven proposition is also dependent on theorem 13.

48
Proposition 55. Let $\tau, (\Pi_b, \Pi_t)$ be a permutation with a consistent labelling and invariant $(\lambda, r, s)$, then

$$A(\tau_{|1, t_{rk}^k, b_{rk}^k}) = 0$$

(29)

More generally for $0 \leq i \leq r$,

$$A(\tau_{|1, t_{rk}^k, b_{rk}^r - i}) = \begin{cases} 0, & \text{if } i \equiv 0 \mod 2 \\ 2A(\tau), & \text{Otherwise.} \end{cases}$$

(30)

Note that the first equation corresponds to the case $i = 0$ of the second equation.

Proof. The first equation is a straightforward application of proposition 47. The second equation is derived from the first by induction on $i$. As noted above the base case ($i = 0$) of the induction is equation 29. For the inductive step, we suppose equation 30 true for $i$.

First case: if $i + 1$ is even. Then $A(\tau_{|1, t_{rk}^k, b_{rk}^r - (i + 1)}) = 2A(\tau)$ by induction, applying proposition 51, we have that

$$A(\tau_{|1, t_{rk}^k, b_{rk}^r - (i + 1)}) + A(\tau_{|1, t_{rk}^k, b_{rk}^r - i}) = 2A(\tau)$$

since the arcs $\beta = \Pi_b^{-1}(b_{rk}^r - (i + 1))$ and $\beta' = \Pi_b^{-1}(b_{rk}^r - i)$ are consecutive. Thus $A(\tau_{|1, t_{rk}^k, b_{rk}^r - (i + 1)}) = 0$.

Second case: if $i + 1$ is odd is proved similarly.

Part of this identity can be found in the second half of proposition 5.11 of [Del13].

Now we have a relation that use the generalisation of the framework where the $(P_{\pm, i})$ can be permuted.

Lemma 56.

$$A(\tau_1) + A(\tau_2) = A(\rho)$$

(31)

The letters $A$ and $B$ above the configurations denote that fact that, contrarily to what was the case up to this point, the gray part of the diagrams are permuted.

Proof. We are in the extended framework: $\tau_1 = \sigma_{1,1,E_1}$ and $\tau_2 = \sigma_{1,1,E_2,\pi_-\pi_+}$ with $E_1 = \{(1.1,1.1),(1.2,2.1)\}$, $E_2 = \{(1.1,0.1),(1.1,1.1)\}$, $\pi_- = id_2$ and $\pi_+ = (2,1)$. □

We can now deduce a corollary, which is more relevant in the following, and that involves three distinct configurations (of course could prove it directly with the theorem 47 but since this was the first time we used the extended framework we choose to start with a simple case.)

Proposition 57.

$$A(\sigma) + A(\tau) + A(\rho) = 0$$

(32)

Proof. Combine equation (17) (or rather the mirror of it), and equation (31) applied to $\rho$. □
9 A \( I_2X \)-permutation for every \((\lambda, r, s)\)

In this section, we will construct a non exceptional \( I_2X \)-permutation for every invariant \((\lambda, r, s)\).

The tools we will use from the preceding sections are:

- The double-edge insertion proposition \( \text{38} \) to obtain the correct cycle invariant \((\lambda, r)\).
- The opposite sign proposition \( \text{53} \) to obtain the correct sign invariant \( s \).

Let us give an overview of the section before detailing the actual construction of an \( I_2X \)-permutation for every a given \((\lambda, r, s)\).

Proposition \( \text{58} \) and \( \text{64} \) allows to add a (or a pair of) cycle of any given length into a permutation without breaking the property that is it \( I_2X \).

Propositions \( \text{60} \) and \( \text{65} \) construct \( I_2X \)-permutations for some specific values of \((\lambda', r')\), more precisely for every \( r' \) and for some \( \lambda' \) of small cardinalities. We call those permutations base permutations.

Finally Propositions \( \text{53} \) and \( \text{59} \) allows us to produce an \( I_2X \)-permutation with invariant \((\lambda, r, -s)\) from a \( I_2X \)-permutation with invariant \((\lambda, r, s)\).

The construction of a \( I_2X \)-permutation \( \sigma \) for a given \((\lambda, r, s)\), proceeds in three steps. Of course, at each step of the construction we guarantee that the resulting permutations are still \( I_2X \).

1. We choose a base permutation \( \sigma^0 \) with invariant \((\lambda', r)\) such that \( \lambda' \subseteq \lambda \).
2. We add cycles on \( \sigma^0 \) until the new permutation \( \sigma^1 \) has invariant \((\lambda, r)\).
3. If \( \sigma^1 \) has no even cycle then the sign of \( \sigma^1 \) is \( \pm 1 \) (refer to theorem \( \text{13} \) for a reminder of the relationship between the sign invariant \( s \) and the cycle invariant \((\lambda, r)\)). Since we have no control on whether it is \( +1 \) or \( -1 \), we need to construct another permutation \( \sigma^2 \) with opposite sign so as to insure that either \( \sigma^1 \) or \( \sigma^2 \) has invariant \((\lambda, r, s)\).
   This is done by using proposition \( \text{59} \) in most cases and by proposition \( \text{53} \) for a few remaining cases. The construction is the subject of theorem \( \text{61} \).
   If \( \sigma^1 \) has even cycles then the sign of \( \sigma^1 \) is necessarily 0 and we are done. The construction is the subject of theorem \( \text{66} \).
4. Finally we will verify (in a remark after the theorem) that the constructed permutations are not exceptional.

Let us define \( C_p \) for any \( p \in \mathbb{N} \) the cross permutation

\[
C_p = \begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\]

50
Proposition 58 (Adding cycles 1). Let \( \sigma \) be a permutation with cycle invariant \((\lambda, r)\), we call \( \sigma_i(C_p) \) the permutation obtained by replacing the \( i \)th edge of \( \sigma \) by a cross permutation \( C_p \) (see figure 25). The cycle invariant \((\lambda', r)\) of \( \sigma_i(C_p) \) depends on \( p \) in the following way:

- if \( p = 4k \) then \( \lambda' = \lambda \cup \{p + 1\} \).
- if \( p = 4k + 1 \) then \( \lambda' = \lambda \cup \{2k + 1, 2k + 1\} \).
- if \( p = 4k + 2 \) then \( \lambda' = \lambda \cup \{p + 1\} \).
- if \( p = 4k + 3 \) then \( \lambda' = \lambda \cup \{2k + 2, 2k + 2\} \).

Proof. By induction on \( p \). The base cases are for \( p = 0 \) and \( p = 1 \).

For those cases, the permutation \( \sigma \) with invariant \((\lambda, r)\) becomes respectively \( \sigma_i(C_0) \) with invariant \((\lambda \cup \{1\}, r)\) and \( \sigma_i(C_1) \) with invariant \((\lambda \cup \{1,1\}, r)\), as displayed in figure 26.

Then the statement follows by induction from the insertion of a double-edge (the resulting change of the cycle invariant are described in proposition 38).

Note that if \( p = 4k + 3 \) the two cycles of the \( C_p \) structure attached on \( \sigma' = \sigma_i(C_p) \) are even. Let \( \alpha \) be the first top arc of the \( C_p \) structure of \( \sigma' \) and \( \beta \) and \( \beta' \) its first and third bottom arcs (refer to figure 27 left). Then by proposition 53 for \( j = 1, 2, 3 \) the sign invariant of \( \sigma'|_{j,\alpha,\beta} \) and \( \sigma'|_{j,\alpha,\beta'} \) are opposite while their cycle invariants remain equal (figure 27 middle and right).

For clarity, let us call \( C_{p,j} \) for any \( p, j \) the permutation:

![Figure 26](image_url)
Then we have $\sigma'|_{j,\alpha,\beta} = \sigma_i(C_{p+j})$ and $\sigma'|_{j,\alpha,\beta'} = \sigma_i(C_{p,j})$ and our discussion implies the following statement:

**Proposition 59** (Two opposite signs). Let $\sigma$ be a permutation and let $p = 4k + j$ with $0 \leq j < 3$ and $k > 0$. Then $\sigma_i(C_p)$ and $\sigma_i(C_{p-(j+1),j+1})$ have invariant $(\lambda, r, s)$ and $(\lambda', r, -s)$ respectively for some $(\lambda, r, s)$.

This statement will be crucial for our construction in theorem 61. Indeed, as outlined in the beginning of the section, we will construct for every $(\lambda, r, s)$ a permutation with invariant $(\lambda, r, s)$ by adding cycles on a base permutation with invariant $(\lambda' \subseteq \lambda, r)$. This construction will be performed through the means of proposition 58.

Thus in the final step of this procedure, we construct $\sigma_1 = \sigma_i(C_p)$ with invariant $(\lambda, r)$. In order to insure that our constructed permutation has invariant $s$, we also consider $\sigma_1' = \sigma_i(C_{p-(j+1),j+1})$ for a correct $j$. Then by proposition 59 either $\sigma_1$ or $\sigma_1'$ will have invariant $(\lambda, r, s)$.

The last ingredient of our proof of theorem 61 are the base permutations.

The following proposition provides the base permutations for the case where $(\lambda, r, s)$ has no even cycle (first line of figure 28). It will also allow us to apply proposition 53 to obtain two permutations with opposite sign (second line of figure 28) for the few cases not covered by proposition 59.

Let us define the permutations $X_{p,p',p''}$, $X_{p,p',p''}$ for any $p, p', p''$ to be

Then we have:

**Proposition 60** (Base permutations, no even cycle). For every $k \geq 1$.

The permutations $X_{1,2k}$, $X_{2,1,2k}$ and $X_{2,2k}$ (described in the first line of figure 28) have cycle invariant $(\lambda = \{2k+1\}, r = 1)$, $(\lambda = \{2k+1\}, r = 3)$ and $(\lambda = \emptyset, r = 2k+3)$ respectively.

The permutations $X_{1,4k+3}$, $X_{2,1,4k+3}$ and $X_{2,2,4k+3}$ (described in the second line of figure 28) have cycle invariant $(\lambda = \{2k+2, 2k+2\}, r = 1)$, $(\lambda = \{2k+2, 2k+2\}, r = 3)$ and $(\lambda = \{2k+2\}, r = 2k+4)$ respectively.
Proof. By induction on $k$. The base cases for the first family are respectively:

$\lambda = \{2k+1\}, r = 1$

$\lambda = \{2k+1\}, r = 3$

$\lambda = \emptyset, r = 2k+3$

Then the statement follows by induction from the insertion a double-edge within $\alpha$ and $\beta$ (the resulting change of the cycle invariant are described in proposition 38).

The base cases for the second family are respectively:

$\lambda = \{2k+2, 2k+2\}, r = 1$

$\lambda = \{2k+2, 2k+2\}, r = 3$

$\lambda = \{2k+2\}, r = 2k+4$

Figure 28: Two families of base permutations with their respective cycle invariant.

$\lambda = \{2k+2, 2k+2\}, r = 1$

$\lambda = \{2k+2, 2k+2\}, r = 3$

$\lambda = \{2k+2\}, r = 2k+4$
Figure 29: The permutation resulting from adding a $C_p$ on the last edge of $\sigma^0$ or one of its successor. It is clearly also $I_2X$.

Then the statement follows by induction from the insertion of two double-edges within $\alpha$ and $\beta$ (the resulting change of the cycle invariant are described in proposition [38]).

We can finally state and prove the first theorem of this section.

**Theorem 61.** Let $(\lambda, r, s)$ be an invariant with no even cycle, then there exists an $I_2X$-permutation with invariant $(\lambda, r, s)$.

**Proof.** We can always consider that the size of the permutation is at least 10, for smaller size the result can be obtained by automatic search.

Let $(\lambda, r, s)$ be an invariant with no even cycles. Following the proof sketch of the introduction, we first construct a base permutation $\sigma^0$ with invariant $(\lambda' \subseteq \lambda, r)$ then add cycles to obtain an $I_2X$-permutation $\sigma_1$ with invariant $(\lambda, r)$ and finally we use either proposition 59 or proposition 53 to obtain two $I_2X$-permutations with opposite sign.

- If the rank $r = 1$, then the base permutation $\sigma^0$ with invariant $(\lambda' = \{2\ell + 1\}, 1)$ for any $\ell \geq 1$ is exactly $X_{2\ell}$ according to proposition 60 (first line, first case of figure 28).
- If the rank $r = 3$, then the base permutation $\sigma^0$ with invariant $(\lambda' = \{2\ell + 1\}, 3)$ for any $\ell \geq 1$ is exactly $X_{2,1,2\ell}$ according to proposition 60 (first line, second case of figure 28).
- If the rank $r > 5$, then the base permutation $\sigma^0$ with invariant $(\lambda' = \emptyset, r)$ is exactly $X_{2,r-3}$ according to proposition 60 (first line, third case of figure 28 indeed $r$ odd implies $r - 3$ is even).

Next we add (by the means of proposition 58) cycles one by one and then by pair on the last edge of the current permutation $\sigma^i$ so as to make sure that the last $C_p$ attached is not a $C_2$. This is always possible unless $\lambda = \lambda'$ (in which case $\sigma_1 = \sigma^0$) or $\lambda' = \lambda \cup \{3\}$. In this case we only have to add a cycle a length 3 to the permutation and the procedure begins and ends with the attachment of a $C_2$: $\sigma_1 = \sigma_n^0(C_2)$.

The procedure must finish with the addition of a $C_p$ with $p > 2$ in order to make the application of proposition [59] possible.

Note that since we always attach $C_p$ on the last edge $(|\sigma^i|, \sigma^i(|\sigma^i|))$ of the current permutation $\sigma^i$, the successive permutations from $\sigma^0$ to $\sigma_1$ are all $I_2X$. See figure 28.

Let $2\ell_1 + 1 < \ldots < 2\ell_k + 1$ be the length and $(m_i)_{1 \leq i \leq k}$ the multiplicity of the cycles to be added on $\sigma^0$ (i.e. the cycles of $\lambda \setminus \lambda'$). The procedure is divided into two steps:
Figure 30: The two constructed \( I_2X \)-permutations \( \sigma_1 = \sigma_{|\sigma|}(C_p) \) and \( \sigma'_1 = \sigma_{|\sigma|}(C_{p-2},2) \) with invariant \((\lambda, r, \pm s)\). In this case \( p = 4k + 1 \).

- First we look at the parity of \( m_i \) from \( i = 1 \) to \( i = k \) and if \( m_i \) is odd we attach a \( C_{2\ell_i} \) on the last edge of the current permutation (which adds a cycle of length \( 2\ell_i + 1 \) to the cycle invariant).

More precisely, let \( i_1, \ldots, i_m \) be the indices such that the \((m_{i_j})_j \) are odd, we define:

\[
\sigma^j = \sigma^{j-1}_{|\sigma^j-1|}(C_{2\ell_i}) \quad \text{for} \quad 1 \leq j \leq m.
\]

Let \((\lambda^j, r)\) be the cycle invariant of \( \sigma^j \), by proposition \( \text{58} \) we have \( \lambda^j = \lambda^{j-1} \cup \{2\ell_i + 1\} \). Thus the multiplicities \( (2m_{i_j}^j)_{1 \leq i \leq k} \) of the cycles of length \( (2\ell_i + 1)_{1 \leq i \leq k} \) to be added on \( \sigma^m \) are all even.

- In the second step, we attach a \( C_{4k+1} \) on the last edge of the current permutation (which add two cycles of length \( 2\ell_i + 1 \) to the cycle invariant) consecutively \( m_i/2 \) time for \( i = 1 \) to \( i = k \).

More specifically, we define \( \sigma^{j,i} \) and its cycle invariant \((\lambda^{j,i}, r)\) by

\[
\begin{align*}
\sigma^{j,i} &= \sigma^{j-1,i}_{|\sigma^{j-1,i}|}(C_{4k+1}) \quad \text{and} \quad \lambda^{j,i} = \lambda^{j-1,i} \cup \{2\ell_i + 1, 2\ell_i + 1\} \quad \text{for} \quad 1 \leq j \leq m_i/2 \quad \text{and} \quad 1 \leq i \leq k \\
\sigma^{0,i} &= \sigma^{m_i/2,i-1} \quad \text{and} \quad \lambda^{0,i} = \lambda^{m_i/2,i-1} \quad \text{for} \quad 1 < i \leq k, \\
\sigma^{0,1} &= \sigma^m \quad \text{and} \quad \lambda^{0,1} = \lambda^m.
\end{align*}
\]

Once again, the values of the cycle invariants are justified by proposition \( \text{58} \).

Let us call the permutation obtained by the procedure \( \sigma_1 \).

By construction \( \sigma_1 \) has cycle invariant \((\lambda, r)\) and the last added \( C_p \) by the procedure is \( C_2 \) if and only if there was just one cycle of length \( 3 \) to be added.

Let us now deal with the sign invariant.

We can divide the problem in two cases: The last \( C_p \) attached has \( p > 2 \) or not (the latter case corresponds to attaching exactly one \( C_2 \) or not attaching anything, thus remaining with the base permutation). Moreover, in accordance with the procedure, \( p = 4k + j, k > 0 \) and \( 0 \geq j < 3 \). Thus \( p \) and \( k \) verify the condition

* If the last \( C_p \) attached has \( p > 2 \). We have \( \sigma_1 = \sigma_{|\sigma|}(C_p) \) for some \( \sigma \) (more specifically \( \sigma = \sigma^{m_p/2-1,k} \) the second to last permutation of the procedure) and \( \sigma_1 \) has the form shown in figure \( \text{29} \).

Moreover, in accordance with the procedure, \( p = 4k + j, k > 0 \) and \( 0 \geq j < 3 \). Thus \( p \) verify the condition of proposition \( \text{59} \) and \( \sigma'_1 = \sigma_{|\sigma|}(C_{p-(j+1),j+1}) \) has invariant cycle \((\lambda, r)\) and sign invariant opposite to \( \sigma_1 \), thus either one of them has invariant \((\lambda, r, s)\). By construction \( \sigma'_1 \) is also \( I_2X \). (See figure \( \text{30} \) for a representation of \( \sigma_1 \) and \( \sigma'_1 \) for \( j = 2 \).)
* If no $C_p$ or just one $C_2$ are attached. We will only consider the case no $C_p$ are added, if a $C_2$ is added on the last edge, the reasoning is strictly identical since adding a $C_p$ does not change the already existing cycles and arcs of the permutation (it only adds cycles and consecutive arcs). Futhermore among the three cases $X_{1,2\ell}$, $X_{2,1,2\ell}$ and $X_{2, r-3}$ we will only handle the first one, the other two being similar.

Let $\sigma^0 = \sigma_1 = X_{1,2\ell}$ for some $\ell \geq 2$ (this is always the case since otherwise $X_{1,2\ell}$ has size 5)

- If $2\ell = 4k$ then removing the first parallel edge of the $2\ell$ consecutives and parallel edges of $\sigma_1 = X_{1,2\ell}$ we get $\tau = X_{1,4(k-1)+1}$ by proposition 53 and proposition 60 (second line first case of figure 28) the permutations $\tau|_{1,\alpha,\beta} = \sigma^1$ and $\tau|_{1,\alpha,\beta'}$ have same cycle invariant and opposite sign.

- If $2\ell = 4k+2$ then removing the first three parallel edge of the $2\ell$ consecutives and parallel edges of $\sigma_1 = X_{1,2\ell}$ we get $\tau = X_{1,4(k-1)+1}$ by proposition 53 and proposition 60 (second line first case of fig 28) the permutations $\tau|_{3,\alpha,\beta} = \sigma^1$ and $\tau|_{3,\alpha,\beta'}$ have same cycle invariant and opposite sign.

Schematically the two cases (for $X_{1,2\ell}$, $X_{2,1,2\ell}$ and $X_{2, r-3}$ with or without a $C_2$ attached on the last edge) are:

\[
\begin{align*}
\tau|_{1,\alpha,\beta} &= \sigma^1 \\
\tau|_{1,\alpha,\beta'} &= \sigma^2
\end{align*}
\]

and

\[
\begin{align*}
\tau|_{3,\alpha,\beta} &= \sigma^1 \\
\tau|_{3,\alpha,\beta'} &= \sigma^2
\end{align*}
\]

Finally the permutations obtained are clearly $I_2 X$. \[\square\]

Remark 62. Theorem 61 makes use of proposition 53 directly and indirectly in its use of proposition 59. Therefore theorem 61 is also dependent on theorem 13 and we must check that we have proved theorem 13 before using theorem 61.

Remark 63. It is unfortunate that in the case $r = 1$ and $\lambda = \{2\ell + 1\}$ one of the two permutations with invariant $(\lambda, r, \pm s)$ produced by theorem 61 is exactly $X_{1,2\ell} = id_{2\ell+3}$ i.e. a permutation from an exceptional class. We rectify this by constructing a third permutation with the same invariant as $X_{1,2\ell}$.

- if $2\ell = 4k$

\[
\begin{align*}
(\{2\ell + 1\}, 1, s) \\
(\{2\ell + 1\}, 1, -s) \\
(\{2\ell + 1\}, s)
\end{align*}
\]
The first two permutations are the ones with invariant \( (\{2\ell + 1\}, 1, \pm s) \) constructed by theorem 61. The invariant of the third one are justified by applying proposition 53 on \( \alpha, \beta' \) and \( \beta'' \) since \( \alpha \) is part of a even cycle of length \( 2k \) and \( \beta'' \) and \( \beta' \) are consecutive arc of another even cycle of length \( 2k \) by proposition 60 (second line third case of figure 28).

\[ \text{if } 2\ell = 4k + 2: \]

\[ \begin{array}{ccc}
\sigma & \sigma_{i} & \sigma' = \sigma_{i}(C_{2p}), \ (\lambda \cup \{2p + 1\}, r) \\
\sigma_{i}(C_{2p}) \cup C_{2p'} & (\lambda \cup \{2p + 2, 2p' + 2\}, r) & \sigma''_{j}(C_{2p'}), \ (\lambda \cup \{2p + 1, 2p' + 1\}, r) \\
\end{array} \]

No other permutation produced by theorem 61 are exceptional since by appendix C in [DS17] (see also appendix A in this article), there are only two exceptional permutations starting with \( \sigma(1) = 1 \) and \( \sigma(2) = 2 \) those are \( \text{id}_{n} \) and \( \text{id}'_{n} \). We just solved the case of \( \text{id}'_{n} \) and \( \text{id}_{n} \) is not \( I_{2}X \).

We now consider the case \( (\lambda, r) \) contains even cycles. For that purpose, proposition 58 is not sufficient since it does not allow one to add pairs of even cycles of differing lengths. Recall that by theorem 13 (and more generally the classification theorem) that even cycles must be in even number in a permutation, thus they can only be added in at least pairs.

The following proposition complements proposition 58 and makes it possible to add pair of even cycles of differing lengths.

**Proposition 64 (Adding cycles 2).** Let \( \sigma \) be a permutation with cycle invariant \( (\lambda, r) \) and let \( p > p' \), we call \( \sigma_{i}(C_{2p} \cup C_{2p'}) \) the permutation obtained by : (see also figure 27)

1. replacing the \( i \)th edge of \( \sigma \) by the cross permutation \( C_{2p} \)
2. replacing the first parallel edge of \( C_{2p} \) by the cross permutation \( C_{2p'} \)
3. and finally inserting a double-edge within \( \alpha \) the leftmost top arc of \( C_{2p'} \) and \( \beta \) the leftmost bottom arc of \( C_{2p} \).

The cycle invariant \( (\lambda', r) \) of \( \sigma_{i}(C_{2p} \cup C_{2p'}) \) verifies \( \lambda' = \lambda \cup \{2(p + 1), 2(p' + 1)\} \):

\[ \sigma, \ (\lambda, r) \downarrow \sigma_{i}(C_{2p} \cup C_{2p'}), \ (\lambda \cup \{2p + 2, 2p' + 2\}, r) \]

\[ \sigma' = \sigma_{i}(C_{2p}), \ (\lambda \cup \{2p + 1\}, r) \xrightarrow{2} s'' = \sigma'_{j}(C_{2p'}), \ (\lambda \cup \{2p + 1, 2p' + 1\}, r) \]
The modifications of the cycle invariants are justified in step 1 and 2 by the proposition \ref{prop:prop1} and in step 3 by the double-edge insertion proposition \ref{prop:prop2} since \(\sigma_i(C_{2p} \cup C_{2p'}) = \sigma''\) and \(\alpha\) and \(\beta\) are two arcs of two different cycles of length \(2p + 1\) and \(2p' + 1\) respectively.

The second theorem handles the case \(\sigma\) has even cycles. Fortunately since the sign is 0 in this case there are no need two produce two permutations with opposite sign. However there are more base permutations to consider so the proof is not much shorter.

**Proposition 65** (Base permutations 2). For every \(k, k' \geq 0\),

The permutations \(X_{2k+1, 2k'}\) and, \(X_{2k, 2k'+1}\) (described in the first line of figure \ref{fig:fig1}) have cycle invariant \((\lambda = \{k+2k'+1\}, r = k+1)\) and \((\lambda = \{k'+1\}, r = 2k+k'+1)\) respectively.

The permutations \(X_{2k, 3, 2k}, X_{3, 2k, 2k}\) and \(X_{2, 2, 3}\) (described in the second line of figure \ref{fig:fig1}) have cycle invariant \((\lambda = \{2k+2\}, r = 2k+2)\), \((\lambda = \{2, 2k+3\}, r = 2)\) and \((\lambda = \{2, 2, 2\}, r = 2)\) respectively.

---

**Figure 31:** The scheme to add two different even cycles to a permutation.

**Figure 32:** Two families of \(I_2X\)-permutations with their respective cycle invariant.
Proof. By induction on \( k, k' \). In order to highlight the structure of the cycle invariant we start the base cases \( k, k' \geq 1 \). The base cases for the first line are respectively:

And they have cycle invariant \((\lambda, r) : (\{4\}, 2), (\{2\}, 4)\) as shown just below:

Then the statement follows by induction from the insertion a double-edge within \( \alpha \) and \( \beta \) or within \( \alpha' \) and \( \beta' \) (the resulting change of the cycle invariant are described in proposition \[38\]).

The base cases for the second family are respectively (for the second we make \( k \) start at 0 since the structure of the cycle invariant is just as explicit here):

And they have cycle invariant \((\lambda, r) : (\{4\}, 4), (\{2, 3\}, 2), (\{2, 2, 2\}, 2)\) as shown below:

Then the statement follows by induction from the insertion of a pair of double-edges within \( \alpha \) and \( \beta \) and within \( \alpha' \) and \( \beta' \) for the first permutation. For the second permutation the double-edge must be inserted within \( \alpha \) and \( \beta \) (the resulting change of the cycle invariant are described in proposition \[38\]). \( \square \)

Theorem 66. Let \((\lambda, r, 0)\) be an invariant with even cycle, then there exists a shift-irreducible family with invariant \((\lambda, r, 0)\).

Proof. We can always consider that the size of the permutation is at least 10, for smaller size the result can be obtained by automatic search.

Let \((\lambda, r, s)\) be an invariant with even cycles. Following the proof sketch of the introduction, we first construct a base permutation \(\sigma^0\) with invariant \((\lambda' \subseteq \lambda, r)\) then add cycles to obtain a \(I_2X\)-permutation \(\sigma_1\) with invariant \((\lambda, r, 0)\).

- If the rank is odd and there is at least one odd cycle \(2\ell + 1\), then the base permutation is either \(X_{1,2\ell}, X_{2,1,2\ell}\) or \(X_{2,r-3}\) as in theorem \[61\].
• If the rank is odd and there are no odd cycles:
  
  – If the rank \( r = 1 \), then there are two base cases: If \( \lambda \) has 2 even cycles of the same length \( 2\ell \), the base permutation \( \sigma^0 \) with invariant \( (\lambda' = \{2\ell, 2\ell\}, 1) \) for any \( \ell \geq 1 \) is \( X_{1,4(\ell-1)+3} \) according to lemma 60 (second line, first case).

If \( \lambda \) does not have two even cycles of the same length, the base permutation \( \sigma^0 \) with invariant \( (\lambda' = \{2\ell, 2\ell' + 2\ell\}, 1) \) is obtained in two steps. First we take, as above, \( \sigma^0 = X_{1,4(\ell-1)+3} \), it has invariant \( (\lambda' = \{2\ell, 2\ell\}, 1) \). Then we choose the two arcs \( \alpha \) and \( \beta \) as below:

\[
X_{1,4(\ell-1)+3} = \begin{array}{c}
\alpha \\
\ldots \\
\beta \\
\ldots \\
\alpha \\
\beta \\
\ldots
\end{array}
\]

\( \alpha \) and \( \beta \) are in the same cycle of length \( 2\ell \), thus by the double-edge insertion proposition \( 38 \) \( \sigma^0 |_{2\ell', \alpha, \beta} \) the permutation resulting from the insertion of \( \ell \) double-edges within \( \alpha \) and \( \beta \) has invariant \( (\lambda' = \{2\ell, 2\ell' + 2\ell\}, 1) \).

– If the rank \( r = 3 \), then there are two base permutations:

If \( \lambda \) has 2 even cycle of the same length \( 2\ell \), the base permutation \( \sigma^0 \) with invariant \( (\lambda' = \{2\ell, 2\ell\}, 3) \) for any \( \ell \geq 1 \) is \( X_{2,1,4(\ell-1)+3} \) according to lemma 60 (second line, second case).

If \( \lambda \) does not have two even cycles of the same length, the base permutation \( \sigma^0 \) with invariant \( (\lambda' = \{2\ell, 2\ell' + 2\ell\}, 1) \) is obtained in two steps. First we take, as above, \( \sigma^0 = X_{2,1,4(\ell-1)+3} \), it has invariant \( (\lambda' = \{2\ell, 2\ell\}, 3) \). Then we choose the two arcs \( \alpha \) and \( \beta \) as below:

\[
X_{2,1,4(\ell-1)+3} = \begin{array}{c}
\alpha \\
\ldots \\
\beta \\
\ldots \\
\alpha \\
\beta \\
\ldots
\end{array}
\]

\( \alpha \) and \( \beta \) are in the same cycle of length \( 2\ell \), thus by the double-edge insertion proposition \( 38 \) \( \sigma^0 |_{2\ell', \alpha, \beta} \) the permutation resulting from the insertion of \( \ell \) double-edges within \( \alpha \) and \( \beta \) has invariant \( (\lambda' = \{2\ell, 2\ell' + 2\ell\}, 3) \).

– If the rank \( r > 5 \), then the base permutation \( \sigma^0 \) with invariant \( (\lambda' = \emptyset, r) \) is exactly \( X_{2,r-3} \) according to proposition 60 (first line, third case of figure 28 since \( r \) is odd implies \( r - 3 \) is even).

• If the rank \( r \) is even and the longest even cycle of \( \lambda \) has length \( 2\ell + r \), \( \ell \geq 1 \) then \( \sigma^0 \) is \( X_{2r-1,2\ell} \) and has invariant \( (\lambda = \{2\ell + r\}, r) \) by proposition 65 (first line, first case).

• If the rank \( r \) is even and the longest even cycle of \( \lambda \) has length \( 2\ell \), \( r > 2\ell \geq 1 \) then \( \sigma^0 \) is \( X_{r-2\ell,4\ell-1} \) and has invariant \( (\lambda = \{2\ell\}, r) \) by proposition 65 (first line, second case).
• If the rank \( r \) is even and the longest even cycle of \( \lambda \) has also length \( r \):
  
  - If \( r > 2 \) then \( \sigma^0 \) is \( X_{r-2,3,r-2} \) and has invariant \((\lambda = \{r\}, r)\) by proposition 65 (second line, first case).
  
  - If \( r = 2 \) and there is an odd cycle of length \( 2\ell' + 1 \), \( \ell' \geq 1 \) in \( \lambda \) then\( \sigma^0 \) is \( X_{3,2\ell'-1,3} \) and has invariant \((\lambda = \{2, 2\ell' + 1\}, 2)\) by proposition 65 (second line, second case).
  
  - If \( r = 2 \) and there are no odd cycle (thus every cycle has length two) in \( \lambda \) then \( \sigma^0 \) is \( X_{2,1,3} \) and has invariant \((\lambda = \{2, 2, 2\}, 2)\) by proposition 65 (second line, third case).

\( \square \)

10 The induction

Let us list a few lemma before beginning the induction.

**Lemma 67.** Let \( \sigma \) be a permutation and let \( c \) be a \((2k, 2r)\)-coloring such that either the edge \( e_1 = (\sigma^{-1}(1), 1) \) or \( e_2 = (n, \sigma(n)) \) are grayed. Let \( \tau \) be the corresponding reduction and let \( S \) be a sequence. Define \( \tau' = S(\tau) \).

Then \((\sigma', c') = B(S)(\sigma, c)\) has the following property: the gray edge \( e_1 \) is \((\sigma'^{-1}(1), 1)\) or the gray edge \( e_2 \) is \((n, \sigma'(n))\) in \((\sigma', c')\).

**Proof.** Let us do the case for \( e_1 \), the case for \( e_2 \) is identical.

Let \((\Pi_b, \Pi_t)\) be a consistent labelling for \( \tau \) then the gray edge \( e_1 \) of \( \sigma, c \) is inserted within the arcs with label \( r_0^{rk} \) and \( b \in \Sigma_b \) (since \( e_1 = (\sigma'^{-1}(1), 1) \)). By theorem 30 the image of a consistent labelling is a consistent labelling thus \( \Pi'_t = S(\Pi_t) \) verifies \( \Pi'^{-1}_t(r_0^{rk}) = 1 \) by definition. Since the labelling is compatible with the boosted dynamics (cf proposition 32) the top endpoint of the gray edge \( e_1 \) of \((\sigma', c')\) is still within \( r_0^{rk} \) and thus inserted within the top arc with position 1 in \((\tau', \Pi'_b)\). \( \square \)

In other words, the leftmost top endpoint of the edge \((\sigma^{-1}(1), 1)\) and the rightmost bottom endpoint of the edge \((n, \sigma(n))\) are fixed by the dynamics. It had already been proved many times but this proof is a good illustration of how we will employ the labelling and the boosted dynamics.

**Lemma 68 (\( d(\sigma) \) for \( \sigma \) of type \( X \)).** Let \( \sigma \) be a standard permutation with invariant \((\lambda, r, s)\) of type \( X(r, i) \) and let \( \tau = d(\sigma) \). Then \( \tau \) has cycle invariant \((\lambda \setminus \{i\}, r + i - 1)\) and type \( H(i, r) \). Moreover for any consistent labelling \((\Pi_b, \Pi_t)\) of \( \tau \) we have \( \sigma_1 = R(\sigma) = \tau|_{1, t_0^{rk} b_{i-1}^{rk}} \).

**Proof.** Let \( \sigma \) and \( \tau \) be as in the lemma. Then \( \tau \) has indeed cycle invariant \((\lambda \setminus \{i\}, r + i - 1)\) and type \( H(i, r) \), by proposition 40.

Let \((\Pi_b, \Pi_t)\) be a consistent labelling for \( \tau \) and define \( \beta = \tau^{-1}(n) - 1 \) the bottom arc to the left of the edge \((\tau^{-1}(n), n)\). Then clearly \( \sigma_1 = R(\sigma) \) is \( \tau_{1,1,\beta} \). Moreover \( \Pi_b(\beta) = b_{i-1}^{rk} \) since \( \tau \) has type \( H(i, r) \) and therefore the top part of the rank (connecting the top left corner to the top right corner) has length \( i \).

Thus we also have \( \sigma_1 = \tau_{1, t_0^{rk} b_{i-1}^{rk}} \) for \( \tau, (\Pi_b, \Pi_t) \). See figure 33. \( \square \)
Figure 33: The case $\sigma$ has type $X(r, i)$ of lemma 68. We have $\sigma_1 = \tau_1, \alpha_1 = 1, \beta = \tau_1, t_{rk}, b_{i-1}, b_{i-2}, \ldots$

**Lemma 69** ($d(\sigma)$ for $\sigma$ of type $H$). Let $\sigma$ be a standard permutation with invariant $(\lambda, r, s)$ of type $H(i, r_2)$ and let $\tau = d(\sigma)$. Then $\tau$ has cycle invariant $(\lambda \cup \{i - 1\}, r_2 - 1)$ and type $X(r_2 - 1, i - 1)$. Moreover there exists a consistent labelling $(\Pi_b, \Pi_t)$ of $\tau$ such that we have $\sigma_1 = R(\sigma) = \tau|_{1, t_{rk}, b_{i-2, i-1}}$.

**Proof.** Let $\sigma$ and $\tau$ be as in the lemma. Then $\tau$ has indeed cycle invariant $(\lambda \cup \{i - 1\}, r_2 - 1)$ and type $H(i - 1, r_2 - 1)$, by proposition 40.

Let $(\Pi_b, \Pi_t)$ be a consistent labelling for $\tau$ such that the label of the first top arc of the principal cycle (i.e the arc with position $\alpha = \sigma(1) + 1$) is $t_{rk}, i - 1, 1$. Such a labelling exists since the principal cycle has length $i - 1$.

Then consider $\beta' = \Pi^{-1}(b_{2,3,1})$ and $\beta = \Pi^{-1}(b_{3,3,1})$ the last two consecutives bottom arcs (in this order) of the principal cycle. Thus $\beta = \tau^{-1}(n - 1) - 1$ the bottom arc to the left of the edge $(\tau^{-1}(n - 1), n - 1)$ ($\tau$ has size $n - 1$). See figure 34 bottom.

Clearly $\sigma_1 = R(\sigma)$ is $\tau_1, i, \beta = \tau_1, t_{rk}, b_{i-2, i-1}$. See figure 34 top: right and middle.

Let us proceed with the induction. The statements 1 to 7 are true at small size $< 10$.

**Inductive case:** By induction we suppose that statements 1 to 7 are true at size up to $n - 1$, let us prove that they are true at size $n$:

10.1 **Statement 1:** Every non exceptional class has a shift-irreducible family

Let $C$ be a non exceptional class with invariant $(\lambda, r, s)$ and $\sigma \in C$ a standard permutation with $\sigma(2) = 2$. Let $c$ be the $(n - 2, 2)$-coloring in which the edge $e = (n, \sigma(n))$ of $\sigma$ is grayed and $\tau$ the corresponding reduction. Clearly $\tau$ is irreducible since $\tau(1) = 1$ (and $\tau(2) = 2$).

We distinguish two cases: $\tau$ is not in an exceptional class or it is.
Figure 34: The case $\sigma$ has type $H(i, r_2)$ of lemma 69. We have $\sigma_1 = \tau_1, \alpha_1 = 1, \beta = \tau_1, t_2, b_{i-1, i-1, 1}$.

- **Suppose that $\tau$ is not in an exceptional class.** By induction (proposition 20 statement 6) there exists a $I_2X$ permutation $\tau'$ in the class of $\tau$. Thus there exists $S$ such that $S(\tau) = \tau'$.

  Let $B(S)$ be the boosted sequence of $S$, we have $B(S)(\sigma, c) = (\sigma', c')$ with $c'$ having the following property: the gray edge $e$ is $(n, \sigma'(n))$ by lemma 67.

  Thus the class $C$ contains a permutation $\sigma'$ such that removing the last edge $(n, \sigma'(n))$ gives a $I_2X$-permutation $\tau'$. By the characterisation proposition for shift-irreducible permutations (proposition 44) a simple case study on the value of $\sigma'(n)$ shows that the standard family of $\sigma'$ is shift-irreducible.

**Remark 70.** Let us stop here for a moment. We introduced a trick to finding normals forms in section 2.2 by defining two sets $T_A$ and $T_B$ and in the proof overview we specified that $T_A$ were the shift-irreducible standard families and $T_B$ the $I_2X$-permutations. We have finally used the trick here: by the assumption that $I_2X$-permutation exists at size $n' < n$ we prove that shift-irreducible standard families exists at size $n$. One can verify that if we had the weaker induction hypothesis that shift-irreducible standard families exists at size $n' < n$ then the case study on the value of $\sigma'(n)$ would show that the standard family of $\sigma'$ is not always shift-irreducible.

  Of course to complete the trick we still need to prove that every class contains a $I_2X$-permutation at size $n$, but as will be shown in section 10.6, this is direct consequence of the construction of an $I_2X$-permutation for every valid invariant $(\lambda, r, s)$ done in section 7 and the classification theorem at size $n$.

If $\tau$ is in an exceptional class then it is either in $Id'_n$ or $Id_n$.

- **Suppose $\tau \in Id'_{n-1}$.** By proposition 78 since $\tau(1) = 1$ and $\tau(2) = 2$, we must have $\tau = id_{n-1}'$. But then, the standard family of $\sigma$ is shift-irreducible by the characterisation proposition 44 since $\sigma$ has the following form:
Suppose \( \tau \in \text{Id}_n \). By proposition 78 since \( \tau(1) = 1 \) and \( \tau(2) = 2 \), we must have \( \tau = \text{id}_{n-1} \), thus \( \sigma \) has the form:

\[
\sigma = \begin{array}{ccccc}
& \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast
\end{array} \quad i > 0, j > 0
\]

We must have \( i > 0 \) since otherwise \( \sigma = \text{id}'_{n-1} \) and \( j > 0 \) since otherwise \( \sigma = \text{id}_n \), in both cases \( \sigma \) would be in an exceptional class which is false by hypothesis.

Now the permutation \( \sigma' = R^{-1}L^iR^{-j}L^j(\sigma) \) has the form:

\[
\sigma' = \begin{array}{ccccc}
& \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast
\end{array} \quad i, j > 0.
\]

and the standard family of \( \sigma' \) is shift-irreducible by the characterisation proposition 44.

10.2 Statement 2: Proof of theorem 13

* Let \( C \) be a class with invariant \((\lambda, r)\), we show that the list \( \lambda \cup \{r\} \) has an even number of even parts.

Let \( \sigma \) be a standard permutation with \( \sigma(2) = 2 \), then \( \tau = d(\sigma) \) is irreducible for it is standard and has invariant \((\lambda', r')\).

By induction hypothesis the list \( \lambda' \cup \{r'\} \) has an even number of even parts. Moreover, by inspection of table 3 it is easy to see that the parity of the number of even parts is preserved when adding an edge \((i, 1)\) to a permutation for every \( i \). Thus since \( d(\sigma) = \tau \), \( \sigma \) is obtained by adding the edge \((1, 1)\) to \( \tau \) and the list \( \lambda \cup \{r\} \) must therefore contain an even number of even parts.

10.2.0.1 * Let \( \sigma \in C \) be a permutation with cycle invariant \((\lambda, r)\) we must prove that:

\[
\mathcal{A}(\sigma) = \begin{cases} 
\pm 2^{\frac{n+\ell}{2}} & \text{If the number of even parts of the list } \lambda \cup \{r\} \text{ is 0.} \\
0 & \text{Otherwise.}
\end{cases}
\]

Where \( \ell \) is the number of parts in \( \lambda \).

We distinguish two cases:

* Suppose \( \lambda = \emptyset \). We must prove that \( \mathcal{A}(C) = \pm 2^{\frac{n}{2}} \) since there are no even cycles and \( \ell = 0 \). **Proof idea:** We apply proposition 51 to two permutations \( \sigma \) and \( \sigma' \) of \( C \) and conclude by induction.
More formally, the proof proceeds from the two following lemma:

**Lemma.** Let $C$ be as above, There exists $\tau$ with $\overline{A}(\tau) = \pm 2^{\frac{n}{2}}$ and two consecutive bottom arcs $\beta, \beta'$ such that $\tau|_{1, 1, \beta}$ and $\tau|_{1, 1, \beta'}$ are in $C$.

and

**Lemma.** Let $\tau$ be a permutation and let $\beta, \beta'$ be two consecutive bottom arcs such that $\tau|_{1, 1, \beta}$ and $\tau|_{1, 1, \beta'}$ are in the same class $C$ then $\overline{A}(C) = \overline{A}(\tau)$.

Clearly the two lemmas put together prove the statement.

**Proof of the first lemma.** Let $St$ be a shift-irreducible family of $C$ and let $\sigma \in St$ the unique permutation with type $H(4, r - 4 + 1)$ (it exists by proposition 39). Let $\tau = d(\sigma)$, $(\Pi_b, \Pi_t)$ and $\sigma_1 = R(\sigma) = \tau|_{1, b_{1, 3, 1}^*, b_{2, 3, 1}}$ be as in lemma 69. Then $\tau$ has cycle invariant $\{3\}, r - 4$, type $X(r - 4, 3)$ and is irreducible since the family is shift-irreducible (proposition 43). Moreover by induction it has $\overline{A}(d(\sigma)) = \pm 2^{\frac{n-1}{4}}$. Let us say wlog $\overline{A}(d(\sigma)) = 2^\tau$.

Consider $\beta' = \Pi^{-1}(b_{1, 3, 1})$ and $\beta = \Pi^{-1}(b_{2, 3, 1})$ the last two consecutive bottom arcs of the principal cycle. Thus $\beta = \tau^{-1}(n - 1) - 1$ is the bottom arc to the left of edge $(\tau^{-1}(n - 1), n - 1)$. See figure 35 bottom: middle.

By induction we apply the corollary 34 of the monodromy theorem 33 which is true by induction hypothesis. Thus there exists a loop $S$ on $\tau$ such that $\Pi_b' = S(\Pi_b)$ verifies $\Pi_b'(\beta') = b_{2, 3, 1}$ since the cycle is odd (has length 3).

Let $c$ be $(n - 2, 2)$-coloring of $\sigma_1$ in which the edge $e = (\sigma_1^{-1}(1), 1)$ is grayed (by choice of $c$ the reduction is $\tau$). Thus $e \in (r_{d}^{k}, b_{2, 3, 1})$ in $\tau, (\Pi_b, \Pi_t)$. See figure 35 top: right and middle.
By lemma [67] and since the labelling is compatible with the boosted dynamics (theorem [32]) \((s_1', c') = B(S) (σ_1, c)\) verifies \(σ_1' = τ_{1,1,β'}\). Indeed the gray edge \(e ∈ (τ_{0,b}^k, b_{2,3,1})\) in \(S(τ, (Π_b, Π_t)) = τ(Π_b^t, Π_t^r)\) i.e. the edge \(e\) is inserted within the top arc with position 1 and bottom arc with position \(β' = Π_b^{-1}(b_{2,3,1})\). See figure 35 top and bottom right.

The permutations constructed \(σ_1', σ_1\) and \(τ\) prove the lemma. □

Proof of the second lemma. Let us apply proposition [51] to the permutations \(τ|_{1,1,β}, τ|_{1,1,β'}\) and \(τ\).

\[\overline{A}(τ|_{1,1,β}) + \overline{A}(τ|_{1,1,β'}) = 2\overline{A}(τ)\]

We know that

\[\overline{A}(τ|_{1,1,β}) = \overline{A}(τ|_{1,1,β'})\]

since they are in the same class.

Thus

\[\overline{A}(C) = \overline{A}(τ|_{1,1,β}) = \overline{A}(τ)\]

□

Let us now consider the second case.

• **Suppose \(λ ≠ ∅\).**  **Proof idea:** Since \(λ ≠ ∅\) there must be a shift-irreducible standard permutation \(σ\) of \(C\) of type \(X(r, i)\). Then the result follows from an application of proposition [57] and lemma [68]

Let \(σ\) be a standard shift-irreducible permutation of type \(X(r, i)\), and let \(τ = d(σ)\) and \(σ_1 = R(σ) = τ|_{1,τ_{0,b}^k,τ_{r,b}^k}\) be as in lemma [68]. Then \(τ\) is irreducible since \(σ\) is shift-irreducible (proposition [43]) with invariant \((λ \setminus \{i\}, r' = r + i - 1)\) and type \(H(i, r)\).

Let us choose \(j = r' - i + 1 = r\) then \(r' - j = i - 1\). According to lemma [55] we have

\[\overline{A}(σ_1 = τ|_{1,τ_{0,b}^k,τ_{r,b}^k}) = \begin{cases} 0, & \text{if } j ≡ 0 \mod 2 \\ 2\overline{A}(τ) & \text{Otherwise.} \end{cases} \quad (33)\]

Clearly if \(j = r\) is even \(\overline{A}(σ_1) = 0\). However since \(r\) is even and the list \(λ \cup \{r\}\) contains even parts.

If \(j = r\) is odd and λ does contain some even cycles then the list \(λ \setminus \{i\} \cup \{r'\}\) also contains even parts and by induction \(\overline{A}(τ) = 0\), thus \(\overline{A}(R(σ)) = 2\overline{A}(τ) = 0\).

If \(j = r\) is odd and λ does not contain even cycles then the list \(λ \setminus \{i\} \cup \{r'\}\) does not contain even parts either, thus by induction \(\overline{A}(τ) = ±2^{n-1+\ell′} (\text{where } \ell′ \text{ is the number of parts of } λ \setminus \{i\})\)

\[\overline{A}(σ_1) = 2 \cdot ±2^{n-1+\ell′} = ±2^{n+\ell+1} = ±2^{n+\ell′}\]

where \(\ell′ = \ell + 1\) is the number of parts of \(λ\).

Note that we have proven the following lemma (that will be used in the section just below)

**Lemma 71.** Let \(σ\) be a shift-irreducible permutation with invariant \((λ ≠ ∅, r, s)\) of type \(X(r, i)\), let \(τ = d(σ)\) and suppose the list \(λ \cup \{r\}\) contains no even cycle. Then \(τ\) is irreducible and has invariant \((λ \setminus \{i\}, r' = r + i - 1, s)\).
10.3 Statement 3, Existence: For every valid invariant \((\lambda, r, s)\) there exists a permutation with invariant \((\lambda, r, s)\).

The statement is a direct consequence of theorems \([61, 66]\) where we constructed a \(I_2\)-permutation for every valid \((\lambda, r, s)\).

Note that we can indeed apply theorem \([61]\) since we have just proved theorem \([13]\) up to \(n\) (cf warning from dependency remarks \([62, 54]\)).

10.4 Statement 4, Proposition \([18]\): First step of the labelling method

We already outlined the proof in the proof overview section.

Let \(\sigma_a\) and \(\sigma_b\) with invariant \((\lambda, r, s)\). We shall find \(\sigma'_a\) and \(\sigma'_b\) two standard shift-irreducible permutations of the same type \((X\text{ or } H)\) connected to \(\sigma_a\) and \(\sigma_b\) respectively, such that \(\tau_1 = d(\sigma'_a)\) and \(\tau_2 = d(\sigma'_b)\) have the same invariant \((\lambda', r', s')\).

Then the proposition \([18]\) is proven by taking \(\sigma'_1 = R(\sigma'_a)\) and \(\sigma'_2 = R(\sigma'_b)\) as in lemma \([68, 69]\). Indeed if \(\sigma'_a\) has type \(X(r_i)\) by lemma \([68]\) \(\sigma'_1 = R(\sigma'_a) = \tau_1|_{\{e_{r_i}^0, e_{r_i}^1\}}\). Let \(c_1\) be the \((n - 2, 2)\)-coloring of \(\sigma'_1\) where the edge \(e = (\sigma'_1^{-1}(1), 1)\) is grayed. Then the reduction of \((\sigma'_1, c_1)\) is \(\tau_1\). The same reasoning apply to \(\sigma'_b\) which gives a \((\sigma'_2, c_2)\) with reduction \(\tau_2\).

The case \(H(i, r_2)\) is similar with the use of lemma \([69]\) instead.

Remark 72. There is a reason for our normal forms not to be the standard permutation \((\sigma'_a, \sigma'_b)\) but the permutations \(\sigma'_1 = R(\sigma'_a)\) and \(\sigma'_1 = R(\sigma'_b)\) instead; despite the fact that the reductions are \(\tau_1\) and \(\tau_2\) in both cases. If we chose \((\sigma'_a, \sigma'_b)\) the gray edge would be the edge \((1, 1)\) which is a pivot, thus \((\sigma'_a, c_a)\) and \((\sigma'_b, c_b)\) are not proper (cf section \([5.1]\) on the boosted dynamics) while \((\sigma'_1, c_1)\) and \((\sigma'_2, c_2)\) are.

Case 1: Suppose \(\lambda \neq \varnothing\) and \(\lambda \cup \{r\}\) has no even parts. Let \(i \in \lambda\), let \(\sigma'_a, \sigma'_b\) be two standard shift-irreducible permutation of type \(X(r, i)\) in the class of \(\sigma_a\) and \(\sigma_b\) respectively (they exist by proposition \([39]\)).

By lemma \([71]\) \(\tau_1 = d(\sigma'_a)\) and \(\tau_2 = d(\sigma'_b)\) are irreducible and have the same invariant. \(\square\)

Case 2: Suppose \(\lambda \neq \varnothing\) and \(\lambda\) has at least two even cycles (equivalently \(\lambda \cup \{r\}\) has at least four even parts). Let \(i \in \lambda\), let \(\sigma'_a, \sigma'_b\) be two standard shift-irreducible permutations of type \(X(r, i)\) in the class of \(\sigma_a\) and \(\sigma_b\) respectively (they exist by proposition \([39]\)).

By lemma \([68]\) \(\tau_1\) and \(\tau_2\) have cycle invariant \((\lambda \setminus \{i\}, r + i - 1)\), thus the list \(\lambda \setminus \{i\} \cup \{r + i - 1\}\) must still contains even cycles and by theorem \([13]\) \(\tau_1\) and \(\tau_2\) have sign 0. Thus they have the same invariant. Moreover they are irreducible since \(\sigma'_a\) and \(\sigma'_b\) are shift-irreducible (proposition \([16]\)). \(\square\)

Case 3: Suppose \(r \geq 4\). Let \(\sigma'_a, \sigma'_b\) be two standard shift-irreducible permutations of type \(H(3, r - 2)\) in the class of \(\sigma_a\) and \(\sigma_b\) respectively (they exist by proposition \([39]\) since \(r \geq 4\)).

By lemma \([69]\) \(\tau_1\) and \(\tau_2\) have cycle invariant \((\lambda \cup \{2\}, r - 3)\). Thus the list \(\lambda \cup \{2\}\cup \{r - 3\}\) contains even cycles and by theorem \([13]\) \(\tau_1\) and \(\tau_2\) have sign 0.
\[ \sigma = L^{n-j+1} \quad \text{and} \quad \sigma_{\text{next}} = \]

\[ d(\sigma) = L^{n-j+1} \quad \text{and} \quad d(\sigma_{\text{next}}) = \]

\[ \sigma_{\text{next}} = L^{n-j+1}(\sigma) \quad \text{is of type } X(r, i). \]

\[ \text{Moreover } A(d(\sigma_{\text{next}})) = -A(d(\sigma)). \]

\[ \text{Proof. Clearly } \sigma_{\text{next}} \text{ is of type } X(r, i) \text{ (see figure 36).} \]

\[ \text{For the second part of the statement we apply proposition 57 to } d(\sigma_{\text{next}}), d(\sigma) \text{ and } \sigma. \text{ we have } A(d(\sigma_{\text{next}})) + A(d(\sigma)) = A(\sigma). \text{ Thus } A(d(\sigma_{\text{next}})) = -A(d(\sigma)) \text{ since } A(\sigma) = 0. \]

\[ \text{Let } \sigma'_a, \sigma'_b \text{ be two standard shift-irreducible permutation of type } X(r, 2k) \text{ in the class of } \sigma_a \text{ and } \sigma_b \text{ respectively (they exist by proposition 39). Applying lemma 73 to } \sigma'_a \text{ and } \sigma'_b \text{ we obtain respectively } \sigma'_{a,\text{next}} \text{ and } \sigma'_{b,\text{next}} \text{ of type } X(r, 2k) \text{ such that } d(\sigma'_a) \text{ and } d(\sigma'_{a,\text{next}}) \text{ as well as } d(\sigma'_b) \text{ and } d(\sigma'_{b,\text{next}}) \text{ have opposite sign.} \]

\[ \text{Thus choosing } \tau_1 = d(\sigma'_a) \text{ and } \tau_2 = d(\sigma'_b) \text{ or } \tau_2 = d(\sigma'_{b,\text{next}}), \tau_1 \text{ and } \tau_2 \text{ have the same cycle invariant by lemma 68 and the same sign invariant (by choice of } \tau_2). \text{ They are also irreducible since } \sigma'_a \text{ and } \sigma'_b \text{ or } \sigma'_{b,\text{next}} \text{ are shift-irreducible (proposition 16).} \]

The four cases overlap somewhat, however they do cover all possibilities. Indeed ’no cycles’ is covered by case 3, ’no even parts and some cycles’ by case 1, ’at least four even parts’ by case 2, ’exactly two even parts’ by case 3 (for \( r \geq 4 \)) and case 4 (for \( r = 2 \)). Recall that there are always an even number of even parts in \( \lambda \cup \{r\} \) by theorem 13 so every single possible invariant is handled.

**Case 4:** Suppose \( \lambda = 2k \) and \( r = 2 \). We will make use of proposition 57. We first prove the following lemma:

**Lemma 73.** Let \( \sigma \) be a standard permutation with invariant \( (\lambda, r, 0) \) of type \( X(r, i) \). Let \( e = (\sigma^{-1}(n), n) \) and \( e' = (\sigma^{-1}(n) - 1, \sigma(\sigma^{-1}(n) - 1)) \) and define \( j = \sigma(\sigma^{-1}(n) - 1) \) then \( \sigma_{\text{next}} = L^{n-j+1}(\sigma) \) is of type \( X(r, i) \).

Moreover \( A(d(\sigma_{\text{next}})) = -A(d(\sigma)). \)

**Proof.** Clearly \( \sigma_{\text{next}} \) is of type \( X(r, i) \) (see figure 36).

For the second part of the statement we apply proposition 57 to \( d(\sigma_{\text{next}}), d(\sigma) \) and \( \sigma \). we have \( A(d(\sigma_{\text{next}})) + A(d(\sigma)) = A(\sigma). \) Thus \( A(d(\sigma_{\text{next}})) = -A(d(\sigma)) \) since \( A(\sigma) = 0. \)

**Case 4:** Suppose \( \lambda = 2k \) and \( r = 2 \). We will make use of proposition 57. We first prove the following lemma:

**Lemma 73.** Let \( \sigma \) be a standard permutation with invariant \( (\lambda, r, 0) \) of type \( X(r, i) \). Let \( e = (\sigma^{-1}(n), n) \) and \( e' = (\sigma^{-1}(n) - 1, \sigma(\sigma^{-1}(n) - 1)) \) and define \( j = \sigma(\sigma^{-1}(n) - 1) \) then \( \sigma_{\text{next}} = L^{n-j+1}(\sigma) \) is of type \( X(r, i) \).

Moreover \( A(d(\sigma_{\text{next}})) = -A(d(\sigma)). \)

**Proof.** Clearly \( \sigma_{\text{next}} \) is of type \( X(r, i) \) (see figure 36).

For the second part of the statement we apply proposition 57 to \( d(\sigma_{\text{next}}), d(\sigma) \) and \( \sigma \). we have \( A(d(\sigma_{\text{next}})) + A(d(\sigma)) = A(\sigma). \) Thus \( A(d(\sigma_{\text{next}})) = -A(d(\sigma)) \) since \( A(\sigma) = 0. \)

Let \( \sigma'_a, \sigma'_b \) be two standard shift-irreducible permutation of type \( X(r, 2k) \) in the class of \( \sigma_a \) and \( \sigma_b \) respectively (they exist by proposition 39). Applying lemma 73 to \( \sigma'_a \) and \( \sigma'_b \) we obtain respectively \( \sigma'_{a,\text{next}} \) and \( \sigma'_{b,\text{next}} \) of type \( X(r, 2k) \) such that \( d(\sigma'_a) \) and \( d(\sigma'_{a,\text{next}}) \) as well as \( d(\sigma'_b) \) and \( d(\sigma'_{b,\text{next}}) \) have opposite sign.

Thus choosing \( \tau_1 = d(\sigma'_a) \) and \( \tau_2 = d(\sigma'_b) \) or \( \tau_2 = d(\sigma'_{b,\text{next}}), \tau_1 \) and \( \tau_2 \) have the same cycle invariant by lemma 68 and the same sign invariant (by choice of \( \tau_2 \)). They are also irreducible since \( \sigma'_a \) and \( \sigma'_b \) or \( \sigma'_{b,\text{next}} \) are shift-irreducible (proposition 16).

The four cases overlap somewhat, however they do cover all possibilities. Indeed 'no cycles' is covered by case 3, 'no even parts and some cycles' by case 1, 'at least four even parts' by case 2, 'exactly two even parts' by case 3 (for \( r \geq 4 \)) and case 4 (for \( r = 2 \)). Recall that there are always an even number of even parts in \( \lambda \cup \{r\} \) by theorem 13 so every single possible invariant is handled.

**We must now justify that the \( \tau_i \) obtained are not in an exceptional class.**

First note that in case 1, 2 and 4 the rank of \( \tau_i \) is strictly more than one and in case 3
\(\tau\) has a cycle of length 2 thus \(\tau \not\in \text{Id}_{n-1}'\) since the cycle invariants are not compatible. Likewise in case 3, \(\tau \not\in \text{Id}_{n-1}'\) since the cycle invariants are not compatible.

It thus remains to show that \(\tau \not\in \text{Id}_{n-1}'\) in the cases 1, 2, 4 all with a rank < 4. Consider the following lemma (proven in the appendix A).

**Lemma 74.** Let \(C\) be a non-exceptional class and let \(St\) be a standard family. Then at most one \(\sigma \in St\) has \(d(\sigma) \in \text{Id}_{n-1}'\).

Since the rank is small, there must be either many cycles or a cycle of large length (since \(n\) is at least 10 in the induction). If we are unlucky and the permutation \(\sigma'_a\) or \(\sigma'_b\) of type \(X(r, i)\) has \(d(\sigma'_a) \in \text{Id}_{n-1}'\) or \(d(\sigma'_b) \in \text{Id}_{n-1}'\) we know that it is the only such one in the standard family and we can choose another of type \(X(r, i)\) (since there are \(im\) permutation of type \(X(r, i)\) by proposition 39).

### 10.5 Statement 5, Completeness: Every pair of permutations \((\sigma, \sigma')\) with invariant \((\lambda, r, s)\) are connected.

The completeness statement is demonstrated by the labelling method. (Refer to section 2)

- Let \(\sigma_a\) and \(\sigma_b\) be two irreducible permutations with invariant \((\lambda, r, s)\). By proposition 18 there exists \((\sigma'_1, c_1)\) and \((\sigma'_2, c_2)\) connected to \(\sigma_a\) and \(\sigma_b\) respectively with the following property:

  \(c_1\) and \(c_2\) are the \((2n - 2, 2)\)-coloring of \(\sigma'_1\) and \(\sigma'_2\) where the edge \(e_1 = (\sigma'_1^{-1}(1), 1)\) and \(e_2 = (\sigma'_2^{-1}(1), 1)\) are grayed respectively and \(\tau_1\) and \(\tau_2\) the reductions of \((\sigma'_1, c_1)\) and \((\sigma'_2, c_2)\) are irreducible, have the same invariant \((\lambda', r', s')\) and are not in exceptional classes.

  Moreover by the proof of proposition 18 \(\sigma'_a = R^{-1}(\sigma'_1)\) and \(\sigma'_b = R^{-1}(\sigma'_2)\) are standard and have the same type \(X(r, i)\) or \(H(i, r_2)\).

  If they have type \(X(r, i)\) then by lemma 68 there exists a consistent labelling \((\Pi_b, \Pi_i)\) of \(\tau_1\) and \((\Pi'_b, \Pi'_i)\) of \(\tau_2\) such that \(\sigma'_1 = \tau_1|_{1, t_0^k, b_{i-1}}\) and \(\sigma'_2 = \tau_2|_{1, t_0^k, b_{i-1}}\).

  Define \(t = t_0^k\) and \(b = b_{i-1}\), then \(e_1\) is within \((t, b)\) in \((\tau_1, (\Pi_b, \Pi_i))\) and \(e_2\) is within \((t, b)\) in \((\tau_2, (\Pi'_b, \Pi'_i))\).

  If they have type \(H(i, r_2)\) then by lemma 69 there exists a consistent labelling \((\Pi_b, \Pi_i)\) of \(\tau_1\) and \((\Pi'_b, \Pi'_i)\) of \(\tau_2\) such that \(\sigma'_1 = \tau_1|_{1, t_0^k, b_{i-1}, 1}^{-1}\) and \(\sigma'_2 = \tau_2|_{1, t_0^k, b_{i-1}, 1}\).

  Define \(t = t_0^k\) and \(b = b_{i-1}\), then \(e_1\) is within \((t, b)\) in \((\tau_1, (\Pi_b, \Pi_i))\) and \(e_2\) is within \((t, b)\) in \((\tau_2, (\Pi'_b, \Pi'_i))\). Moreover \(\lambda' = \lambda \cup i - 1\) (lemma 69 again).

- Since \(\tau_1\) and \(\tau_2\) are irreducible and have the same invariant they are in the same class by the classification theorem (which is true by induction hypothesis). Therefore there exists \(S\) such that \(\tau_2 = S(\tau_1)\).

- Since there exists a boosted dynamics and the labelling is compatible with the boosted dynamics (theorem 32) there exists \(B(S)\) such that the reduction of \((\sigma'_3, c_3)\) = \(B(S)(\sigma'_1, c_1)\) is \(\tau_2\) and the gray edge \(e_1\) of \((\sigma'_3, c_3)\) is inserted within the arcs with labels \(t\) and \(b\) of \(\tau_2\), \((\Pi'_b, \Pi'_i) = S(\tau_1, (\Pi_b, \Pi_i))\). Moreover by theorem 30 \((\Pi'_b, \Pi'_i)\) is a consistent labelling.
Remark 75. The next point is the last difficulty we emphasized in section 2.2: we need to prove that $P(\tau_2, (\Pi''_b, \Pi'_t), t, b, \alpha, \beta)$ is true. Where $P(\tau_2, (\Pi''_b, \Pi'_t), t, b, \alpha, \beta)$ is true if and only if there exists a loop $S$ such that $\tau_2, (\Pi''_b, \Pi'_t) = S(\tau_2, (\Pi_b^s, \Pi'_t))$. Let us apply proposition 51 to $\sigma$ cycle. Let us apply proposition 51 to $\sigma$ cycle.

Contrariwise to the involution dynamics (see [DTS]) the propriety $P$ can be verified. Essentially the proof is based on the fact that the labellings are consistent: if one endpoint of a gray edge is within an arc of a cycle of length $\lambda_i$ in $x, (\Pi_b, \Pi_t)$ then this endpoint will still be within an arc of a cycle of length $\lambda_i$ in $S(x, (\Pi_b, \Pi_t))$ for any $S$. This remark combined with the 2-point monodromy theorem (corollary 34) is enough to deduce that the sequence $S_*$ always exists.

• First case: $b = b_{j-1}$. Both $(\Pi''_b, \Pi'_t)$ and $(\Pi_b^s, \Pi'_t)$ are consistent labellings of $\tau_2$ and the labels of the rank are fixed by definition, thus we have $\beta = \Pi''_b(b_{j-1}) = \Pi'_{b-1}(b_{j-1})$ (and $\alpha = 1 = \Pi''_t(t_{0}) = \Pi'_{t-1}(t_{0})$). Therefore the gray edge $e_1$ of $(\sigma'_3, c_3)$ and $e_2$ of $(\sigma'_2, c_2)$ are both inserted within the arcs with position $\alpha = 1$ and $\beta$. Thus $(\sigma'_2, c_2) = (\sigma'_3, c_3)$.

Second case: $b = b_{i-2, i-1}$. Since $(\Pi_b^s, \Pi'_t)$ is a consistent labelling of $\tau_2$, the arc with position $\beta = \Pi''_{b-1}(b_{i-2, i-1})$ is an arc of a cycle of length $i - 1$ (More precisely it is the last bottom arc of the principal cycle by lemma 69). Since $(\Pi''_b, \Pi'_t)$ is also a consistent labelling of $\tau_2$, $\Pi''_b(\beta) = b_{c,i-1,d}$ for some $0 \leq c < i - 1$ and $1 \leq d \leq m_{i-1}$ where $m_{i-1}$ is the multiplicity of $i - 1$ in $\lambda'$.

If $\lambda'$ has no even cycle or at least two even cycles, then by corollary 34 of theorem 33 (which is true by induction hypothesis) there exists a loop $S_*$ of $\tau_2$ such that $\Pi''_b = S_*(\Pi'_t)$ verifies $\Pi''_b(\beta) = b_{i-2, i-1}$. Let $B(S_*)$ be the boosted sequence of $S_*$ then the gray edge $e_1$ of $(s'_3, c_3) = B(S_*)(s'_3, c_3)$ is inserted within the arcs with position $\alpha = 1$ and $\beta$ therefore $(s'_2, c_2)$. Let us show that $\Pi''_b(\beta) = b_{0,i-1}$. Refer also to figure 37.

We constructed $\sigma'_2 = t_{2}\Pi''_b b_{i-2, i-1}$ and $(\Pi''_b, \Pi'_t)$ by lemma 69 therefore, as said above, $\beta$ is the last arc of the principal cycle of $\tau_2$ (to the left of the edge $(\tau^{-1}_{2}(n - 1), n - 1)$) and $\lambda = \lambda' \setminus \{i - 1\}$, thus the list $\lambda \cup \{r\}$ has no even parts and $s \neq 0$.

If $\Pi''_b(\beta) = b_{0,i-1}$, let $\beta' = \Pi''_{b-1}(b_{i-2, i-1})$ then $\beta'$ and $\beta$ are consecutive (in this order) since $\Pi''_{b}$ is a consistent labelling and $i - 2$ and $0$ are consecutive (modulo $i - 1$) indices. Moreover $(\sigma'_4, c_4) = B(S_*)(\sigma'_4, c_3)$ verifies $\sigma'_4 = t_{2}\Pi''_b(\beta')$ since the gray edge $e_1$ of $(\sigma'_4, c_4)$ is inserted within the arcs with labels $t_{0}^{k_0}$ and $b_{i-2, i-1}$ of $\tau_2, (\Pi''_b, \Pi'_t)$.

However, we have $\mathcal{A}(\sigma'_4) = \mathcal{A}(\sigma'_2) = \pm 2^{\frac{2+4}{2}}$ since both have invariant $(\lambda'_r, s)$ and $\mathcal{A}(\sigma'_2) = \pm 2^{\frac{2+4}{2}}$ by theorem 13. Moreover $\mathcal{A}(\tau_2) = 0$ since $\lambda'$ has an even cycle. Let us apply proposition 51 to $\sigma'_4, \sigma'_2$ and $\tau'$.

$$\mathcal{A}(\sigma'_4) + \mathcal{A}(\sigma'_2) = 2 \cdot \pm 2^{\frac{2+4}{2}} = 0$$
Figure 37: $(σ'_4, c_4)$ are both permutations with invariant $(λ, r, s ≠ 0)$ and $τ_2$ has sign invariant 0. However proposition 51 applied to $σ'_4, σ'_2$ and $τ_2$ implies that $σ'_4$ should have invariant $–s$. This is contradictory and thus the case $Π''_b(β) = b_0, i-1, 1$ cannot happen.

this is contradictory thus $Π''_b(β) = b_0, i-1, 1$ cannot happen.

The classification theorem (theorem 15) is a consequence of statements 3 and 5.

10.6 Statement 6, Proposition 20. Every non exceptional class contains a $I_2 X$-permutation.

The statement is a direct consequence of theorems 61 and 61 where we constructed a $I_2 X$ permutation for every valid $(λ, r, s)$ and of the classification theorem 15 which says that there is only one non-exceptional class per invariant.

10.7 Statement 7: the 2-point monodromy theorem 33

Before tackling the monodromy theorem, let us consider the following problem: we have two standard permutations $σ$ and $σ'$ of type $X(r, i)$ and a labelling $(Π_b, Π_t)$ of $σ$. The labels of the principal cycle of $σ$ are $t_{c, i, j}, t_{c+i-1, i, j}$ in that order.

We wish to find a sequence $S'$ such that $S'((Π_b, Π_t))$ is a labelling of $σ'$ and the labels of the principal cycle of $σ'$ are also $t_{c, i, j}, t_{c+i-1, i, j}$ in that order.

The following proposition tells us that it is possible if $τ = d(σ)$ and $τ' = d(σ')$ are in the same class.

Proposition 76. Let $σ$ and $σ'$ be two standard permutations with invariant $(λ, r, s)$ and type $X(r, i)$. Let $(Π_b, Π_t)$ be a consistent labelling of $σ$, and let $(α_1, \ldots, α_i)$ and $(α'_1, \ldots, α'_i)$ be the arcs (in that order) of the principal cycle of $σ$ and $σ'$ respectively. Then

$$Π_t(α_1) = t_{c, i, j}, \ldots, Π_t(α_i) = t_{c+i-1, i, j} \mod i, i, j$$

for some $c$ and $j$.

Let $τ = d(σ)$ and $τ' = d(σ')$ and suppose there exists $S$ such that $τ' = S(τ)$.

Finally let $σ', (Π'_b, Π'_t) = R^{-1}B(S)R(σ, (Π_b, Π_t))$. We have

$$Π'_t(α'_1) = t_{c, i, j}, \ldots, Π'_t(α'_i) = t_{c+i-1, i, j} \mod i, i, j.$$
Figure 38: $\sigma$ and $\sigma'$ are standard of type $X(r,i)$. The labels of the principal cycle of $\sigma$ are send to the arcs of the principal cycle of $\sigma'$ by the sequence $R^{-1}B(S)R$. Indeed they are attached to the ith first labels of the rank of $\tau$ and the labels of the rank are fixed. Thus they are also attached to ith first the labels of the rank of $\tau'$ which are the arcs of the principal cycle of $\sigma'$.

In the figure, we choose $\Pi_t(\alpha) = t_{0,i,j}$ instead of $t_{c,i,j}$ for some $c$ for space-saving purpose.

**Proof.** Let $\tau, (\Pi_b', \Pi_t')$, $\sigma_1 = R(\sigma) = \tau|_{t_{0,k}^k b_{r_{i,-1}}^k}$ and $\sigma', (\Pi_b', \Pi_t')$, $\sigma_1 = R(\sigma') = \tau'|_{t_{0,k}^k b_{r_{i,-1}}^k}$ be as in lemma 68.

Let $c$ and $c'$ be the $(n - 2, 2)$-coloring of $\sigma_1$ and $\sigma_1'$ such that the corresponding reductions are $\tau$ and $\tau'$ (i.e. $e \in (t_{0,k}^k, b_{r_{i,-1}}^k)$ in $\tau, (\Pi_b', \Pi_t')$ likewise for $e'$).

Let $B(S)$ be the boosted sequence of $S$ then $B(S)(\sigma_1, c) = (\sigma_1', c')$ since both $e$ and $e'$ are inserted within $t_{0,k}^k, b_{r_{i,-1}}^k$ in $\tau', (\Pi_b', \Pi_t') = S(\tau, (\Pi_b', \Pi_t'))$ and $\tau', (\Pi_b', \Pi_t')$ respectively, and the labels of the rank are fixed. (This part of the proof follows that of statement 5).

For the following, refer to figure 38.

In $\sigma, \Pi_t(\alpha_1) = t_{c,i,j}, \ldots, \Pi_t(\alpha_i) = t_{c+i-1} \mod i,i,j$ however in $\tau, \Pi_t(\alpha_1) = t_{0,k}^k, \ldots, \Pi_t(\alpha_i) = t_{i-1}^k$. Thus the labels of the principal cycle of $\sigma$ are attached to the ith first labels of the rank of $\tau$ and they will move along with them in the boosted sequence.

In $\tau', (\Pi_b', \Pi_t')$ we have $\Pi_t(\alpha_1') = t_{0,k}^k, \ldots, \Pi_t(\alpha_i') = t_{i-1}^k$ since the labels of the rank are fixed, thus for $\sigma', (\Pi_b', \Pi_t') = R^{-1}B(S)R(\sigma, (\Pi_b', \Pi_t'))$, we must also have $\Pi_t(\alpha_1') = t_{c,i,j}, \ldots, \Pi_t(\alpha_i') = t_{c+i-1} \mod i,i,j$.

We now prove the monodromy theorem 33. Let $C$ be a class with invariant $(\lambda, r, s)$, if we establish the theorem for a given permutation with a consistent la-
Then we conclude by applying proposition 76 to the following properties:

- Let us demonstrate by the statement on the cycle 1-shift. In this case \( \lambda \) has no even cycles or at least 2 even cycles. Let \( i \in \lambda \), let \( \sigma \) a standard shift-irreducible permutation of type \( X(r, i) \) and let \( (\Pi_b, \Pi_t) \) be a chosen consistent labelling of \( \sigma \). Let \( \alpha_1, \ldots, \alpha_3 \) be the top arcs of the principal cycle, we have \( \Pi_t(\alpha_1) = t_{0, i, j} \) and \( \Pi_t(\alpha_2) = t_{1, i, j} \), \( \Pi_t(\alpha_3) = t_{i-1, i, j} \).

We construct a loop \( S \) of such that \( (\Pi_b', \Pi_t') = S(\Pi_b, \Pi_t) \) verifies \( \Pi_t'(\alpha_1) = t_{1, i, j} \) and \( \Pi_t'(\alpha_2) = t_{2, i, j} \) \( \Pi_t'(\alpha_3) = t_{0, i, j} \).

Let \( k = \alpha_2 - 2 \) then \( \sigma' \), \( (\Pi_b', \Pi_t') = L^{-k}(\sigma, (\Pi_b, \Pi_t)) \) is a standard shift-irreducible permutation of type \( X(r, i) \). Let \( \alpha'_1, \ldots, \alpha'_r \) be the arc of the principal cycle, we have \( \Pi_t'(\alpha'_1) = t_{1, i, j}, \Pi_t'(\alpha'_2) = t_{2, i, j} \), \( \Pi_t'(\alpha'_3) = t_{0, i, j} \). See figure 39 left and middle.

By the proof of proposition 13 case 1 and 2, \( \tau = d(\sigma) \) and \( \tau' = d(\sigma') \) are irreducible and have the same invariant, thus there is \( S \) such that \( S(\tau') = \tau \) by the classification theorem. Therefore by proposition 76 let us define \( \sigma, (\Pi_b', \Pi_t') = R^{-1}(S)R(\sigma', (\Pi_b', \Pi_t')) \), we have \( \Pi_t'(\alpha_1) = t_{1, i, j} \), \( \Pi_t'(\alpha_2) = t_{2, i, j} \), \( \Pi_t'(\alpha_i) = t_{0, i, j} \) as expected. See figure 39 right.

Figure 39: The case of the cycle 1-shift. We apply proposition 76 on \( \sigma' \) and \( \sigma \).
• Let us demonstrate the statement on the cycle jump. In this case λ has at least two cycles of length i Let i ∈ λ, let σ a standard shift-irreducible permutation of type \(X(r, i)\) and let (\(Π_b, Π_t\)) be a chosen consistent labelling of σ. Let \(α_1, \ldots, α_i\) be the top arcs of the principal cycle, we have \(Π_t(α_1) = t_{0, i,j}\) and \(Π_t(α_2) = t_{1, i,j} \ldots α_i = t_{i-1, i,j}\). Let \(α'\) be an arc of another cycle of length i, \(Π_t(α') = t_{c,i,j}\) for some \(c, j'\).

We construct a loop \(S\) of \(σ\) such that \((Π_b, Π_t') = S(Π_b, Π_t)\) verifies \(Π_t'(α_1) = t_{c,i,j'}\) and \(Π_t'(α_2) = t_{c+1, i+1, i,j} \ldots, Π_t'(α_i) = t_{c+i-1, mod i, j}\).

Let \(k = α' - 2\) then \(σ', (Π_b', Π_t') = L^{-k}(σ, (Π_b, Π_t))\) is a standard shift-irreducible permutation of type \(X(r, i)\). Let \(α'_1, \ldots, α'_i\) be the arc of the principal cycle, we have \(Π_t'(α'_1) = t_{c,i,j'}, Π_t'(α'_2) = t_{c+1, i+1, i,j'} \ldots, Π_t'(α'_i) = t_{c+i-1, mod i, j'}\). See figure 40 left and middle.

By the proof of proposition \(18\) case 1 and 2, \(τ = d(σ)\) and \(τ' = d(σ')\) are irreducible and have the same invariant, thus there is \(S\) such that \(S(τ') = τ\) by the classification theorem. Therefore, by proposition \(76\) let us define \(σ, (Π_b', Π_t') = R^{-1}B(σ, (Π_b, Π_t), Π_t')\), we have \(Π_t'(α_1) = t_{c,i,j'}, Π_t'(α_2) = t_{c+1, i+1, i,j'} \ldots, Π_t'(α_i) = t_{c+i-1, mod i, j'}\) as expected. See figure 40 right.

• Let us demonstrate the statement on the cycle 2-shift. In this case λ has at exactly one cycle of even length, the cycle of length i. Let σ a standard shift-irreducible permutation of type \(X(r, i)\) and let \((Π_b, Π_t)\) be a chosen consistent labelling of σ. Let \(α_1, \ldots, α_i\) be the top arcs of the principal cycle, we have \(Π_t'(α_1) = t_{0, i,j}\) and \(Π_t'(α_2) = t_{1, i,j} \ldots, Π_t'(α_i) = t_{i-1, i,j}\).

We construct a loop \(S\) of \(σ\) such that \((Π_b', Π_t') = S(Π_b, Π_t)\) verifies \(Π_t(α_1) = t_{2,i,j}\) and \(Π_t(α_2) = t_{3,i,j} \ldots α_i = t_{1, mod i, j}\).

Let \(k = α_3 - 2\) then \(σ', (Π_b', Π_t') = L^{-k}(σ, (Π_b, Π_t))\) is a standard shift-irreducible permutation of type \(X(r, i)\). Let \(α'_1, \ldots, α'_i\) be the arc of the principal cycle, we
have \( \Pi_t'(\alpha'_1) = t_{2,i,j}, \Pi_t'(\alpha'_2) = t_{3,i,j}, \ldots, \Pi_t'(\alpha'_i) = t_{1,i,j} \). See figure 41 left and middle.

We must establish that \( \tau = d(\sigma) \) and \( \tau' = d(\sigma') \) are irreducible and have the same invariant. For the irreducibility and the cycle invariant, this follows from lemma \ref{lem:irreducibility} and the fact that \( \sigma \) and \( \sigma' \) are shift irreducible.

The sign invariant is slightly more complicated, we employ lemma \ref{lem:sign-invariant} as we already did in case 4 of the proof of proposition \ref{prop:sign-invariant}. In a few words, define \( \sigma'' = \sigma_{\text{next}} \) as in the lemma, then we have \( \sigma''_{\text{next}} = \sigma \) thus \( A(\sigma) = -A(\sigma'' = A(\sigma') \).

Thus there is \( S \) such that \( S(\tau') = \tau \) by the classification theorem. Then by proposition \ref{prop:classification} let \( \sigma, (\Pi'_b, \Pi'_t) = R^{-1}B(S)R(\sigma', (\Pi'_b, \Pi'_t)) \), we have \( \Pi'_t(\alpha_1) = t_{3,i,j}, \Pi'_t(\alpha_2) = t_{4,i,j}, \ldots, \Pi'_t(\alpha_i) = t_{1,i,j} \) as expected. See figure 41 right.

This complete the proof of theorem \ref{thm:irreducibility}.

**Remark 77.** Once the classification theorem is proven we can prove theorem \ref{thm:classification}. Indeed If instead of using the induction hypothesis of theorem \ref{thm:classification} we use the stronger induction hypothesis of that of theorem \ref{thm:classification} then the proof above (with some added technicalities) transforms into a proof of theorem \ref{thm:classification}.

### A Exceptional classes

In this appendix, when using a matrix representation of configurations, it is useful to adopt the following notation: The symbol \( \epsilon \) denotes the \( 0 \times 0 \) empty matrix. The symbol \( \Box \) denotes a square block in a matrix (of any size \( \geq 0 \)), filled with an identity matrix. A diagram, containing these special symbols and the ordinary bullets used through the rest of the appendix, describes the set of all configurations that could be obtained by specifying the sizes of the identity blocks. In such a syntax, we can write equations of the like

\[
\text{id} := \Box = \epsilon \cup \Box = \epsilon \cup \Box; \quad \text{id}' := \Box .
\]

The sets \( \text{id} \) and \( \text{id}' \) contain one element per size, \( \text{id}_n \) and \( \text{id}'_n \), for \( n \geq 0 \) and \( n \geq 3 \) respectively.

The two exceptional classes \( \text{Id}_n \) and \( \text{Id}'_n \) contain the configurations \( \text{id}_n \) and \( \text{id}'_n \), respectively.

We have the following proposition:

**Proposition 78.** The permutation \( \sigma = \text{id}_n \) (respectively \( \sigma = \text{id}'_n \)) is the only permutation of \( \text{Id}_n \) (respectively \( \text{Id}'_n \)) with \( \sigma(1) = 1 \) and \( \sigma(2) = 2 \).

The structure of the classes \( \text{Id}_n \) is summarised by the following relation:

\[
\text{Id} := \bigcup_n \text{Id}_n = \left( \bigcup_{k \geq 1} (X^k_{\text{RL}} \cup X^k_{\text{LR}} \cup X^k_{\text{LL}} \cup X^k_{\text{RR}}) \right) \cup \text{id}
\]

where the configurations \( X^k \) are defined as in figure 42 (discard colours for the moment).
We can now prove the lemma 74 that we introduced in section 10.4.

**Proof of Lemma 74.** This is equivalent to say that there are no pairs of permutations \( \sigma_1, \sigma_2 \in \text{Id}_n \) which allow for a block decomposition

\[
\sigma_1 = \begin{array}{c}
A \\
B
\end{array} \quad \quad \sigma_2 = \begin{array}{c}
B \\
A
\end{array}
\]  

If the block \( A \) has \( \ell \) rows, we say that \( \sigma_2 \) is the result of shifting \( \sigma_1 \) by \( \ell \).

Clearly, at the light of the structure of configurations that we have presented (refer in particular to Figure 42), this pattern is incompatible with \( \sigma_1 \) or \( \sigma_2 \) being \( \text{id}_n \) (as a non-trivial shift produces a configuration which is not even irreducible), so we have excluded the cases in which, still with reference to the figure, we have only one violet block, and the number of black points is at least 3, for \( X_{LL}^{(k)} \) and \( X_{RR}^{(k)} \), and at least 4, for \( X_{LR}^{(k)} \) and \( X_{RL}^{(k)} \). Note that the black points are the positions in the grid which are south-west or north-east extremal (i.e., positions \((i', j') \in \sigma \) such that there is no \((i', j') \in \sigma \) with \(i' < i \) and \( j' < j \), or the analogous statement with \(i' > i \) and \(j' > j \)). Let us call number of records, \( \rho(\sigma) \), this parameter. Thus we have that configurations in \( X_{LL}^{(k)} \) and \( X_{RR}^{(k)} \) have \( \rho = 2k + 1 \), and configurations in \( X_{LR}^{(k)} \) and \( X_{RL}^{(k)} \) have \( \rho = 2k + 2 \).

Now, if we perform a shift within one block of consecutive ascents, it is easily seen, by investigation of the sub-configuration at the right of the entry of the new configuration in the bottom-most row, or the one at the left of the entry of the new configuration in the top-most row, that the resulting structure is incompatible.
with the structure of \Id. The same reasoning apply if we perform the shift at the beginning/end of a non-empty diagonal block, which is not the one at the bottom-right/top-left. On the other side, if we perform a shift in any other configuration, we have a new configuration in which $\rho$ has strictly decreased. As $\sigma_2$ is a non-trivial shift of $\sigma_1$, and $\sigma_1$ is a non-trivial shift of $\sigma_2$, we can thus conclude. □

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