Problems of the Strategy of Regions

V.A. Smirnov

Nuclear Physics Institute of Moscow State University
Moscow 119899, Russia

Abstract

Problems that arise in the application of general prescriptions of the so-called strategy of regions for asymptotic expansions of Feynman integrals in various limits of momenta and masses are discussed with the help of characteristic examples of two-loop diagrams. The strategy is also reformulated in the language of alpha parameters.

\[^{1}\text{E-mail: smirnov@theory.npi.msu.su}\]
1 Introduction

When analyzing the leading asymptotic behaviour in various limits a standard strategy of regions is often used: instead of the integration over the whole space of loop momenta only the integration over some specific regions is considered. This strategy turns out to be possible because the information about the leading behaviour is somehow encoded in this integration. A classical example of this procedure is the analysis and summation of the leading logarithms in the Sudakov limit [1]. It was argued (and demonstrated for the threshold expansion) in [2] that it is reasonable to use the strategy of regions in the most general form and apply it to the whole asymptotic expansion, i.e. for any powers and logarithms in an arbitrary limit. In this generalized form, the strategy reduces to the following prescriptions:

(i) Consider various regions of the loop momenta and expand, in every region, the integrand in Taylor series with respect to the parameters that are considered small in the given region;

(ii) Integrate the integrand expanded, in every region in its own way, over the whole integration domain of the loop momenta;

(iii) Put to zero any scaleless integral.

In the case of off-shell and off-threshold limits, this strategy leads to the well-known explicit prescriptions [3, 4] (see a brief review [5]) based on the strategy of subgraphs. Some subtle points in the application of these general prescriptions of strategy of regions to on-shell limits were discussed in ref. [6]. The purpose of this paper is to further discuss the status of the strategy of regions using characteristic examples of two-loop diagrams. We present two examples of vertex diagrams in Section 2. In Section 3, we reformulate the strategy of regions in the language of alpha parameters. In conclusion, we discuss the status of the strategy of regions and present various recipes for its application.

2 Two examples

Our first example is the diagram of Fig. 1 with the masses $m_1 = m_2 = m_3 = m_4 = m$, $m_5 = m_6 = 0$, the external momenta $p_1^2 = p_2^2 = m^2$ and $(p_1 - p_2)^2 = -Q^2$ in the limit $m/Q \to 0$.

The Feynman integral can be written as

$$F_1(Q, m, \epsilon) = \int \int \frac{d^d k d^d l}{(l^2 - 2p_1 l)(l^2 - 2p_2 l)(k^2 - 2p_1 k)(k^2 - 2p_2 k)k^2(k - l)^2}. \quad (1)$$

We use dimensional regularization [7] with $d = 4 - 2\epsilon$. When presenting our results we shall omit $i\pi^{d/2}$ per loop and, when writing down separate contributions through expansion in $\epsilon$, we shall also omit $\exp(-\gamma_E \epsilon)$ per loop ($\gamma_E$ is the Euler constant).
Figure 1: (a) Two-loop planar vertex diagram.

Let us choose, for convenience, the external momenta as follows:

\[ p_1 = \tilde{p}_1 + \frac{m^2}{Q^2} \tilde{p}_2, \quad p_2 = \tilde{p}_2 + \frac{m^2}{Q^2} \tilde{p}_1, \quad \tilde{p}_{1,2} = (Q/2, \mp Q/2, 0, 0) \quad (2) \]

so that \( p_i^2 = m^2, \quad \tilde{p}_i^2 = 0, \quad 2\tilde{p}_1\tilde{p}_2 = 2\tilde{p}_1p_2 = Q^2. \) In the given limit, the following regions happen to be typical [8]:

- **hard (h):** \( k \sim Q, \)
- **1-collinear (1c):** \( k_+ \sim Q, \quad k_- \sim m^2/Q, \quad k \sim m, \)
- **2-collinear (2c):** \( k_- \sim Q, \quad k_+ \sim m^2/Q, \quad k \sim m, \)
- **soft (s):** \( k \sim m, \)
- **ultrasoft (us):** \( k \sim m^2/Q. \)

Here \( k_{\pm} = k_0 \pm k_1, \quad \underline{k} = (k_2, k_3). \) We mean by \( k \sim Q, \) etc. that any component of \( k_\mu \) is of order \( Q. \)

In the leading order, \( 1/Q^4, \) we obtain the following five contributions generated by the regions from this list: (h-h), (1c-h), (2c-h), (1c-1c), (2c-2c). (We indicate regions for the loop momenta \( k \) and \( l \) in [1] respectively in the first and the second place.) In contrast to the diagram with another distribution of the masses and assignments of the external momenta, \( m_1 = \ldots = m_4 = 0, \quad m_5 = m_6 = m, \quad p_1^2 = p_2^2 = 0, \) where the last four contributions are not regulated dimensionally so that it is necessary to introduce an auxiliary analytic regularization [9], every term can be now considered separately, and we symbolically have (2c-h)=(1c-h), (2c-2c)=(1c-1c).

The (h-h) region generates terms obtained by Taylor expanding the integrand in the expansion parameter, \( m. \) In the leading order, this is the value of the massless planar diagram at \( p_i^2 = p_2^2 = 0 \) evaluated in [10]:

\[ C_{(h-h)}^{(1)} = \frac{C_{(h-h)}^{(1)}}{(Q^2)^{2+2\epsilon}}, \]

2
\[ c_{(h-h)}^{(1)} = \frac{1}{4\epsilon^4} + \frac{5\pi^2}{24\epsilon^2} + \frac{29\zeta(3)}{6\epsilon} + \frac{3\pi^4}{32} + O(\epsilon). \] (8)

Then we have

\[ C_{(1c-1c)}^{(1)} = \int \int \frac{d^dkd^dl}{(-2\tilde{p}_1l)(l^2 - 2p_2l)(-2\tilde{p}_1k)(k^2 - 2p_2k)k^2(l - l)^2}, \]
\[ C_{(1c-h)}^{(1)} = \int \int \frac{d^dkd^dl}{(l^2 - 2\tilde{p}_1l)(l^2 - 2\tilde{p}_2l)(-2\tilde{p}_1k)(k^2 - 2p_2k)} \times \frac{1}{k^2(l^2 - (2\tilde{p}_1k)(2\tilde{p}_2k)/Q^2)}. \] (9)

Using the technique of alpha parameters and Mellin-Barnes representation we obtain the following results:

\[ C_{(1c-1c)}^{(1)} = \frac{c_{(1c-1c)}^{(1)}}{(Q^2)^2(m^2)^2}, \]
\[ c_{(1c-1c)}^{(1)} = -\frac{\pi^2}{24\epsilon^2} + \frac{5\zeta(3)}{4\epsilon} + \frac{\pi^4}{48} + O(\epsilon), \] (10)
\[ C_{(1c-h)}^{(1)} = \frac{c_{(1c-h)}^{(1)}}{(Q^2)^{2+\epsilon}}(m^2)^{\epsilon}, \]
\[ c_{(1c-h)}^{(1)} = -\frac{1}{6\epsilon^4} - \frac{\pi^2}{6\epsilon^2} - \frac{41\zeta(3)}{9\epsilon} - \frac{37\pi^4}{180} + O(\epsilon). \] (11)

If we combine all these five contributions we shall obviously obtain a wrong result because the poles of the fourth and the third order will not cancel. It turns out that the following non-standard region has to be included into the list:

**1-ultracollinear (1uc):** \( k_+ \sim m^2/Q, \ k_- \sim m^4/Q^3, \ k \sim m^3/Q^2. \) (12)

The missing contributions are (2uc-1c) and (1uc-2c) which are equal to each other and easily evaluated by alpha parameters for general \( \epsilon. \) In the leading order, we have

\[ C_{(2uc-1c)}^{(1)} = \int \int \frac{d^dkd^dl}{(-2\tilde{p}_1l)(l^2 - 2p_2l)(-2\tilde{p}_1k)(k^2 - 2p_2k)k^2(l^2 - (2\tilde{p}_1l)(2\tilde{p}_2k)/Q^2)} \]
\[ = \frac{c_{(2uc-1c)}^{(1)}}{(Q^2)^{2-2\epsilon}}(m^2)^{4\epsilon}, \]
\[ c_{(2uc-1c)}^{(1)} = -\Gamma(\epsilon)\Gamma(2\epsilon)\Gamma(3\epsilon)\Gamma(-4\epsilon) \]
\[ = \frac{1}{24\epsilon^4} + \frac{5\pi^2}{48\epsilon^2} + \frac{7\zeta(3)}{18\epsilon} + \frac{493\pi^4}{2880} + O(\epsilon). \] (13)

Collecting all the seven contributions together we observe that the poles of the third and the fourth order in \( \epsilon \) cancel, and we come to the following result

\[ (Q^2)^{2+2\epsilon}F_1(Q, m, \epsilon) \sim Q \to \infty \]
for the external momenta. Let us choose the external momenta as integral $F_{MN}$ and Mellin-Barnes representation we come to the following result:

$$
C_{(h-h)}^{(1)} + 2C_{(1c-h)}^{(1)} \left( \frac{Q^2}{m^2} \right)^\epsilon + 2C_{(1c-1c)}^{(1)} \left( \frac{Q^2}{m^2} \right)^{2\epsilon} + 2C_{(2uc-1c)}^{(1)} \left( \frac{Q^2}{m^2} \right)^{4\epsilon}
$$

$$
= \ln^2 \frac{m^2}{Q^2} \frac{1}{2\epsilon^2} - \left( \frac{5}{6} \ln^3 \frac{m^2}{Q^2} + \frac{\pi^2}{3} \ln \frac{m^2}{Q^2} + \frac{\zeta(3)}{15} \right) \frac{1}{\epsilon}
$$

$$
+ \frac{7}{8} \ln^4 \frac{m^2}{Q^2} + \frac{4\pi^2}{3} \ln^2 \frac{m^2}{Q^2} + \frac{\zeta(3)}{\epsilon} \ln \frac{m^2}{Q^2} + \frac{\pi^4}{15} + O(\epsilon). \quad (14)
$$

with the proper coefficient at the double pole which can be evaluated starting from the full diagram.

Note that in the case of the non-planar diagram in the considered limit, with $p_1^2 = p_2^2 = m^2$, there are exactly the same problems with dimensional regularization as in the limit with $p_1^2 = p_2^2 = 0$.

The second example is the same diagram Fig. 1 with the masses $m_1 = m_3 = M$, $m_2 = m_4 = m$, $m_5 = m_6 = 0$, the external momenta $p_1^2 = M^2$, $p_2^2 = m^2$ and $Q^2 = (p_1 - p_2)^2 = 0$ in the limit $m/M \to 0$. The corresponding Feynman integral $F_2(M, m, \epsilon)$ has literally the same form as (1), with the other assignments for the external momenta. Let us choose the external momenta as $p_1 = (M, 0), p_2 = Mn_1 + \frac{m^2}{M}n_2$, with $n_{1,2} = (1/2, \mp 1/2, 0, 0)$. We have $2n_{1,2}k = k_\pm$ for any $d$-vector $k$.

No 'unusual' regions are relevant here. In the leading order, there are four non-zero contributions, corresponding to (h-h), (2c-h),(2c-2c) and (us-2c) regions. (Here the same characterization of the regions (3)--(7) in terms of the components $k_\pm$ and $k$, with the substitution $Q \to M$, is implied.) The (h-h) contribution is obtained by expanding the integrand in Taylor series in $m^2$. Using again the technique of alpha parameters and Mellin-Barnes representation we come to the following result:

$$
C_{(h-h)}^{(2)} = \frac{C_{(h-h)}^{(2)}}{(M^2)^{2+2\epsilon}},
$$

$$
c_{(h-h)}^{(2)} = \frac{1}{12\epsilon^4} + \frac{\pi^2}{12\epsilon^2} + \frac{91\zeta(3)}{36 \epsilon} + \frac{179\pi^4}{1440} + O(\epsilon). \quad (15)
$$

The contribution of the (2c-2c) region is obtained due to the following prescriptions:

(a) Expand the propagators $1/(k^2 - 2p_1k)$ and $1/(l^2 - 2p_1l)$ in Taylor series respectively in $k^2$ and $l^2$.

(b) Expand each resulting term, which is a function of three kinematical invariants, $p_1^2 = M^2, p_2^2 = m^2$ and $2p_1p_2 = m^2 + M^2$, in a Taylor series at the point $p_1^2 = 0$ and $2p_1p_2 = M^2$ (do not touch $p_2^2 = m^2$).

It might seem that we expand in the large mass $M$ but this is just an illusion. The (2c-h) contribution is of an intermediate character. In the leading order, we have

$$
C_{(2c-2c)}^{(2)} = \int \int \frac{d^d k d^d l}{(-2P_2l)(l^2 - 2p_2l)(-2P_2k)(k^2 - 2p_2k)k^2(k - l)^2},
$$
\[ C_{(2c-h)}^{(2)} = \int \int \frac{d^4 k d^4 l}{(l^2 - 2p_1 l)(l^2 - 2p_2 l)(l^2 - 2P_2 k)(k^2 - 2p_2 k)k^2(l^2 - (2n_1 l)^2/l^2)}, \quad (16) \]

where \( P_1 = M n_1, \ P_2 = M n_2 \). These contributions are evaluated by the same techniques as before, with the following results:

\[ C_{(2c-2c)}^{(2)} = \frac{c_{(2c-2c)}^{(2)}}{(M^2)^2(m^2)^{2\epsilon}}, \]

\[ c_{(2c-2c)}^{(2)} = c_{(2c-2c)}^{(1)} = c_{(1c-1c)}^{(1)}, \quad (17) \]

\[ C_{(2c-h)}^{(2)} = \frac{c_{(2c-h)}^{(2)}}{(M^2)^{2+\epsilon}(m^2)^{\epsilon}}, \]

\[ c_{(2c-h)}^{(2)} = \frac{1}{8\epsilon^4} - \frac{7\pi^2}{48\epsilon^2} - \frac{31\zeta(3)}{6\epsilon} - \frac{871\pi^4}{2880} + O(\epsilon). \quad (18) \]

The (us-2c) contribution

\[ C_{(us-2c)}^{(2)} = \int \int \frac{d^4 k d^4 l}{(l^2 - 2P_2 l)(l^2 - 2p_2 l)(-2p_1 k)(-2P_1 k)k^2(l^2 - (2n_1 k)^2/l^2)m^2)} \quad (19) \]

is evaluated in gamma functions, with a result which is closely related to the (2uc-1c) and (1uc-2c) contributions in Example 1:

\[ C_{(us-2c)}^{(2)} = \frac{c_{(us-2c)}^{(2)}}{(Q^2)^2(m^2)^{3\epsilon}}, \]

\[ c_{(us-2c)}^{(2)} = c_{(2uc-1c)}^{(1)} = c_{(1uc-2c)}^{(1)}. \quad (20) \]

Collecting all the four contributions together we observe that the poles of the third and the fourth order in \( \epsilon \) cancel, and we come to the following result

\[ (M^2)^{2+2\epsilon} F_2(M, m, \epsilon) \sim \infty \]

\[ \epsilon_{(h-h)}^{(2)} + \epsilon_{(2c-h)}^{(2)} \left( \frac{M^2}{m^2} \right)^{\epsilon} + \epsilon_{(2c-2c)}^{(2)} \left( \frac{M^2}{m^2} \right)^{2\epsilon} + \epsilon_{(us-2c)}^{(2)} \left( \frac{M^2}{m^2} \right)^{3\epsilon} \]

\[ = \ln^2 \frac{m^2}{M^2} \frac{1}{8\epsilon^2} - \left( \frac{1}{6} \ln^3 \frac{m^2}{M^2} + \frac{\pi^2}{12} \ln \frac{m^2}{M^2} + \zeta(3) \right) \frac{1}{\epsilon} \]

\[ + \frac{13}{96} \ln^4 \frac{m^2}{M^2} + \frac{5\pi^2}{16} \ln^2 \frac{m^2}{M^2} + \frac{3\zeta(3)}{2} \ln \frac{m^2}{M^2} + \frac{\pi^4}{72} + O(\epsilon). \quad (21) \]

with the proper coefficient at the double pole which can be evaluated starting from the full diagram.
3 The language of the alpha representation

There are also contributions of the (h-2c) region (in both examples) and (h-1c) region (in the first example) which can be evaluated in gamma functions for general $\epsilon$ and start from the next-to-leading order, $m^2$. Consider now another choice of the loop momenta where the momentum $l$ is chosen as the momentum of the 2nd line. Then we can recognize a non-zero contribution from the (h-s) region ($k$ is hard and this new momentum $l$ is soft). However this is nothing but a double counting because these contributions are identical: this can be seen by analyzing this shift of the variables $l \to l - p_2$. This example shows that one has to be careful when testing different choices of the loop momenta. Of course, there is no sense of choosing different assignments of the loop momenta for the hard regions. But for other types of regions certain choices of the loop momenta are crucial — see, e.g., two-loop examples in the case of the threshold expansion in [2], in particular, in the case of the ultrasoft regions. Here is an another example where a different assignment of the loop momenta is important: consider Fig. 1 with $m_1 = \ldots = m_4 = 0$, $m_5 = m_6 = m$, $p_1^2 = p_2^2 = 0$ and $m^2/(p_1 - p_2)^2 \to 0$. Then it is necessary to take into account the (h-s) region with hard $k$ and the soft momentum of the 6th line [9]. (It also contributes from the next-to-leading order.)

There is a possibility to completely avoid such double counting by turning to the alpha representation and using a similar strategy of regions in the language of alpha parameters. This representation for a $h$-loop scalar diagram $\Gamma$ with powers of propagators $a_l$ has the form

$$F_{\Gamma}(q_1, \ldots, q_n) = (-1)^{a_D} \frac{\int_0^\infty d\alpha_1 \ldots \int_0^\infty d\alpha_L \prod_l \alpha_l^{a_l-1} D^{-d/2} e^{iA/D-i\sum m_l^2 \alpha_l}}{\prod_l \Gamma(a_l)} ,$$

(22)

where

$$D = \sum_T \prod_{l \in T} \alpha_l ,$$

(23)

$$A = \sum_T \left( \prod_{l \in T} \alpha_l \left( \sum_i q_i \right)^2 \right) .$$

(24)

In (23), the sum runs over trees of the given graph, i.e. maximal connected subgraphs without loops, and, in (24), over 2-trees, i.e. subgraphs that do not involve loops and consist of two connectivity components. The sum of momenta present in (24) goes over the external momenta that flow into one of the connectivity components of the 2-tree $T$. The products of the alpha parameters involved are taken over the lines that do not belong to the given tree $T$. In the non-scalar case, there appear additional factors in the integrand of the alpha representation — see, e.g., [11].

6
The strategy of regions in the alpha representation is formulated in the same way as in the integrals in the loop momenta. One has to consider alpha parameters to be of different order measured in terms of given masses and kinematical invariants. Let us illustrate the new language using the two above examples. The function (23) is the same in both cases

\[ D = (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4 + \alpha_5) + \alpha_6(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5). \]  

(25)

The functions (24) are

\[
A_1 = [(\alpha_1 + \alpha_3)(\alpha_2 + \alpha_4)\alpha_6 + (\alpha_1 + \alpha_2)\alpha_3\alpha_4 + \alpha_1\alpha_2(\alpha_3 + \alpha_4 + \alpha_5)]Q^2 \\
+ [(\alpha_1 + \alpha_2)((\alpha_3 + \alpha_4)(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) \\
+ (\alpha_1 + \alpha_2)\alpha_5 + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)^2\alpha_6)m^2, \\
A_2 = [M^2(\alpha_1 + \alpha_3) + m^2(\alpha_2 + \alpha_4)][(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\alpha_6 + (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4)] \\
+ [M^2\alpha_1 + m^2\alpha_2](\alpha_1 + \alpha_2)\alpha_5. \]  

(26)

The contribution of the momentum space (h-h) region is obtained when considering all the \( \alpha_i \) as being of the same order. (Note that the functions in the alpha representation are homogeneous so that it suffices to take into account only the relative order of the alpha parameters.) Thus the form \( D \) is not expanded and the part of the exponent proportional to \( m^2 \) is expanded in Taylor series in \( m \).

In Example 1, we reproduce the contributions of the previously considered momentum space regions as follows:

\[
(1c-h) \rightarrow \{\alpha_{4,5} \sim m^0, \alpha_{1,2,3,6} \sim m^2\}; \\
(1c-1c) \rightarrow \{\alpha_{2,4,5,6} \sim m^0, \alpha_{1,3} \sim m^2\}; \\
(h-1c) \rightarrow \{\alpha_2 \sim m^0, \alpha_{1,3,4,5,6} \sim m^2\}; \\
(2uc-1c) \rightarrow \{\alpha_5 \sim m^0, \alpha_3 \sim m^2, \alpha_{2,4,6} \sim m^4, \alpha_1 \sim m^6\}. \]  

(27)

The alpha parameters have dimension \( m^{-2} \) but to simplify relations we put \( Q \) and \( M \) to one and express all the magnitudes in powers of \( m \). The rest of the regions is obtained by permutations.

In Example 2, we have

\[
(2c-h) \rightarrow \{\alpha_{4,5} \sim m^0, \alpha_{1,2,3,6} \sim m^2\}; \\
(2c-2c) \rightarrow \{\alpha_{2,4,5,6} \sim m^0, \alpha_{1,3} \sim m^2\}; \\
(h-1c) \rightarrow \{\alpha_2 \sim m^0, \alpha_{1,3,4,5,6} \sim m^2\}; \\
(us-2c) \rightarrow \{\alpha_5 \sim m^0, \alpha_{2,3,4,6} \sim m^2, \alpha_1 \sim m^4\}. \]  

(28)

Note that here we do not have risk to perform double counting of the (h-2c) and (h-s) (with another choice of the second loop momentum) regions which was observed above. Moreover, the description of the regions in the alpha parametric language is
manifestly Lorentz-invariant. However, a visible drawback of this language is rather non-trivial description of the (us-2c) and especially (2uc-1c) regions. To find them one would need to consider an enormous quantity of various possibilities.

Let us turn to one more example: the on-shell double box diagram shown in Fig. 2. We have $p_i^2 = 0$, $i = 1, 2, 3, 4$. It was evaluated analytically in ref. $[12]$. An explicit analytical algorithm for on-shell double boxes with arbitrary numerators and integer powers of propagators was presented in ref. $[13]$. In $[13]$ these results were also checked against first terms of the expansion in $t/s$, where $s = (p_1 + p_2)^2$ and $t = (p_1 + p_3)^2$ are Mandelstam variables, with the help of the strategy of regions. The external momenta were chosen as

$$p_{1,2} = (\mp Q/2, Q/2, 0, 0), \quad r \equiv p_1 + p_3 = (-t/Q, 0, \sqrt{-t + t^2/Q^2}, 0),$$  

where $s = -Q^2$. The non-zero contributions to the asymptotic expansion in the limit $t/s \to 0$ are (h-h), (1c-1c) and (2c-2c). \(\text{(The description of the collinear regions is given by eqs. (4) and (5), with the replacement } m^2 \to -t.\) The (1c-1c) and (2c-2c) contributions are not individually regularized by dimensional regularization; a natural way to deal with regularized quantities is to introduce and auxiliary analytic regularization which is eventually switched off in the sum.

When analyzing the limit $s/t \to 0$, it is reasonable to rotate the diagram and perform the substitutions $t \leftrightarrow s$, $p_1 \to p_2$, $p_2 \to p_4$, etc. The non-zero contributions to the asymptotic expansion in the limit $s/t \to 0$ are (h-h), (1c-1c), (2c-2c) and (2c-1c). Here the introduction of an auxiliary analytic regularization is also reasonable. However the corresponding poles in the analytic regularization parameter happen to be here up to the second order, as in an example of a non-planar vertex diagram in ref. $[4]$.

Let us now describe the contributions to the asymptotic expansion in the limits $t/s \to 0$ and $s/t \to 0$ in the language of alpha parameters. The functions (23) and (24) for the double box are

$$D_3 = (\alpha_1 + \alpha_2 + \alpha_7)(\alpha_3 + \alpha_4 + \alpha_5) + \alpha_6(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_7),$$

$$A_3 = [\alpha_1\alpha_2(\alpha_3 + \alpha_4 + \alpha_5) + \alpha_3\alpha_4(\alpha_1 + \alpha_2 + \alpha_7) + \alpha_6(\alpha_1 + \alpha_3)(\alpha_2 + \alpha_4)]s + \alpha_5\alpha_6\alpha_7t.$$  

The (h-h) contributions are reproduced by considering all the alpha parameters of the same order. Let us again turn to dimensionless alpha parameters by putting

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{double_box_diagram.png}
\caption{Double box diagram.}
\end{figure}
\( s = -1 \) \( (t = -1) \) in the first (second) limit. In the limit \( t/s \to 0 \), we reproduce the contributions of the previously considered collinear momentum space regions as follows:

\[
(1c-1c) \to \{ \alpha_{2,4,5,6,7} \sim t^0, \alpha_{1,3} \sim -t \}; \\
(2c-2c) \to \{ \alpha_{1,3,5,6,7} \sim t^0, \alpha_{2,4} \sim -t \},
\]

and, in the limit \( t/s \to 0 \), we have

\[
(1c-1c) \to \{ \alpha_{1,2,3,4,6,7} \sim s^0, \alpha_{5} \sim -s \}; \\
(2c-2c) \to \{ \alpha_{1,2,3,4,5,6} \sim s^0, \alpha_{7} \sim -s \}; \\
(2c-1c) \to \{ \alpha_{1,2,3,4,5,7} \sim s^0, \alpha_{6} \sim -s \}.
\]

We see that, in this example, the description of the collinear contributions in the language of alpha parameters is certainly simpler than in the momentum space.

4 Discussion

Let us realize that the very word ‘region’ is understood in the ‘physical’ sense. In fact, it indicates relations between components of the loop momenta expressed in terms of the given masses and kinematical invariants. This is clearly not the mathematical sense where the region is determined by inequalities. We even do not bother about the decomposition of unity, i.e. that our initial integral in the whole space of the loop momenta is decomposed into a sum of integrals over all the possible regions which, presumably, have zero measure in the intersection of any pair, with their union being the whole integration space.

But the most non-trivial step in the strategy of regions is, probably, the last one when all the scaleless integrals are put to zero. Note that this step generally does not refer to the use of dimensional regularization. For example, some integrals generated by potential contributions within threshold expansion \([2]\) were not at all regularized. Still the rule to put such scaleless integrals to zero was experimentally checked through examples.

Up to now there are no counterexamples that would show that the strategy of regions generally does not work. Still mathematical proofs in the general case are also absent. An indirect confirmation of the strategy is the fact that such proof is indeed available \([4]\) in the case of off-shell and off-threshold limits. (These are limits typical for Euclidean space. For this class of regions, it suffices to consider any loop momentum to be either hard or soft.) The decisive point of this proof is the analysis of convergence of the Feynman integrals. To be more precise, this is the resolution of singularities of integrands which is usually performed in the alpha representation. To resolve singularities of \((22)\) the whole integration domain is decomposed into sectors. These are either \(\alpha_1 \leq \ldots \leq \alpha_L\) plus sectors obtained by permutations, or more advanced sectors associated with one-particle-irreducible subgraphs (and their infrared...
analogs of the given graph. After a suitable change of variables in each sector, the integrand is factorized and the analysis of convergence reduces to power counting in one-dimensional integrals.

To carry out the analysis of convergence for a given on-shell or threshold limit we first need to invent appropriate sectors and sector variables which would provide the factorization of the corresponding integrand. At least the sectors mentioned above are here insufficient — this can be seen in one-loop examples. It should be also stressed that, for each on-shell or threshold limit, this problem of resolution of singularities should be solved separately, with sectors and sector variables that are specific for it.

Suppose that we have to expand a Feynman diagram in some limit typical for Minkowski space. The first step is to find all relevant regions that generate non-zero contributions. (For the off-shell limits, there is no problem here: we use the well-known graph-theoretical language for writing down prescription for expanding the given Feynman diagram and just list all relevant subgraphs that are taken from a certain family. This task can be even done by computer. In some partial cases, the prescriptions for on-shell limits can be also formulated in a graph-theoretical language.

The problem to successfully go through this step is a matter of experience and (both physical and mathematical) intuition. The physical flavour of the problem is a correspondence of the class of the regions to certain operators and subsequent translating the prescriptions for the asymptotic expansion into the operator language.

When testing various regions it is necessary to be aware of possible double counting. The combination of this search of the relevant regions both in momentum space and alpha integrals looks rather reasonable. But how can we decide that we have found all the contributions to the asymptotic expansion? Unfortunately, there is no definite answer to this question. At least we can check our results by comparing them with one-loop examples where explicit analytical results can be obtained. Sometimes comparisons with analytical two-loop results are also available — see, e.g., examples in [14]. Another important partial check is the cancellation of poles in $\epsilon$, up to a certain order, and the analysis of the coefficient at the highest pole which can be evaluated by an independent method.

Thus, if the situation with the poles is unsatisfactory there are at least two options:
(a) to decide that the strategy of regions breaks down in the given limit;
(b) to look for missing regions.

Since the first option has been never realized, it looks reasonable to stay always optimistic and continue the search of regions. The regions can be rather non-trivial indeed: the ultracollinear region which can be symbolically described as $1uc = (m^2/Q^2) \times 1c$ is a strange example. By the way, any 1-ultra-...-ultracollinear region $1uc = (m^2/Q^2)^h \times 1c$ is also a reality for an arbitrary $h$. Consider the $h$-loop ladder diagram in the limit of Example 1. Then the contributions of $(2u^{h-1}c-1u^{h-2}c-...-1c)$ and $(1u^{h-1}c-2u^{h-2}c-...-2c)$ regions are non-zero (here $h$ is supposed to be even, for definiteness).

But imagine now that our terms of expansion satisfy the check of poles. The
success is not yet guaranteed because we cannot exclude the existence of a region that enters with simple poles in $\epsilon$ or even without poles which is insensitive to this check. Then one could use numerical checks with numerical evaluation of the initial diagram (and, of course, stay optimistic).

After this advice, let us stress that there is at least an example of the on-shell double box diagrams when it turns out easier to evaluate them analytically, rather than expand them up to a desired order [12, 13]. So, the last advice is to try to evaluate the diagrams analytically, without any expansion. Still in this case, we can use the strategy of regions for crucial checks — see, e.g., [13].

Acknowledgments. I am grateful to M. Beneke and A. Hoang for useful discussions. The work was supported by the Volkswagen Foundation, contract No. 1/73611, and by the Russian Foundation for Basic Research, project 98–02–16981.

References

[1] V.V. Sudakov, Zh. Eksp. Teor. Fiz. 30 (1956) 87.

[2] M. Beneke and V.A. Smirnov, Nucl. Phys. B522 (1998) 321.

[3] S.G. Gorishny, preprints JINR E2–86–176, E2–86–177 (Dubna 1986); Nucl. Phys. B319 (1989) 633; K.G. Chetyrkin, Teor. Mat. Fiz. 75 (1988) 26; 76 (1988) 207; K.G. Chetyrkin, preprint MPI-PAE/PTh 13/91 (Munich, 1991).

[4] V.A. Smirnov, Commun. Math. Phys. 134 (1990) 109; V.A. Smirnov, Renormalization and asymptotic expansions (Birkhäuser, Basel, 1991).

[5] V.A. Smirnov, Mod. Phys. Lett. A 10 (1995) 1485.

[6] V.A. Smirnov and E.R. Rakhmetov, Teor. Mat. Fiz. 120 (1999) 64.

[7] G. ‘t Hooft and M. Veltman, Nucl. Phys. B44 (1972) 189; C.G. Bollini and J.J. Giambiagi, Nuovo Cim. 12B (1972) 20.

[8] J.C. Collins, in Perturbative QCD, ed. A.H. Mueller, 1989, p. 573.

[9] V.A. Smirnov, Phys. Lett. B404 (1997) 101; Proceedings of 5th International Conference on Physics Beyond the Standard Model (Balholm, Norway, 29 April – 4 May 1997), p. 354. AIP, 1997. (hep-ph/9708423).

[10] R.J. Gonsalves, Phys. Rev. D28 (1983) 1542.

[11] P. Breitenlohner and D. Maison, Commun. Math. Phys. 52 (1977) 39.

[12] V.A. Smirnov, hep-ph/9905323, to appear in Phys. Lett. B.
[13] V.A. Smirnov and O.L. Veretin, hep-ph/9907385.

[14] K.G. Chetyrkin and V.A. Smirnov, Phys. Lett. 144B (1984) 419.

[15] V.A. Smirnov, Phys. Lett. B394 (1997) 205; A. Czarnecki and V.A. Smirnov, Phys. Lett. B394 (1997) 211.