An approach to initial constraints in general relativity

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Abstract

We present a 2+1 decomposition of the vacuum initial conditions. For a constant mean curvature one of the momentum constraints decouples and it can be solved by quadrature. The remaining momentum constraints are written in the form of the tangential Cauchy-Riemann equation. In several cases its solutions can be written in terms of integrals of known functions. We show how to obtain initial data with a marginally outer trapped surface. We also present a class of data with a non-constant mean curvature for which the full system of constraints reduces to two real equations in two variables.

Keywords: initial constraints, conformal method, marginally trapped surfaces

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1 Introduction

Vacuum initial data in general relativity consist of a Riemannian metric \( g = g_{ij}dx^idx^j \) and a symmetric tensor \( K = K_{ij}dx^idx^j \) given on a 3-dimensional manifold \( S \). These data have to satisfy the constraint equations

\[
\nabla_j \left( K^j_i - H \delta^j_i \right) = 0 \tag{1}
\]

\[
R + H^2 - K_{ij}K^{ij} = 0, \tag{2}
\]

where \( \nabla_i \) are covariant derivatives corresponding to \( g \), \( R \) is the Ricci scalar of \( g \) and \( H = K^i_i \). Tensors \( g \) and \( K \) are interpreted, respectively, as the induced metric and the external curvature of \( S \) embedded in a 4-dimensional spacetime developing from these data in accordance with the Einstein equations.

The conformal approach to the constraints of Lichnerowicz, Choquet-Bruhat and York (see [1-3] for a review) is based on the transformation

\[
g'_{ij} = \psi^4 g_{ij}, \quad K'_{ij} = \psi^{-2} K_{ij} + \frac{1}{3} H' \psi^4 g_{ij}, \tag{3}
\]
where $H = 0$. The momentum constraint (1) for the primed data yields equations for $g_{ij}$, $K_{ij}$, $H'$ and $\psi$ with no derivatives of $\psi$, whereas the Hamiltonian constraint (2) is equivalent to the Lichnerowicz equation

$$\triangle \psi = \frac{1}{8}R\psi - \frac{1}{8}K_{ij}K^{ij}\psi - \frac{1}{12}H'^2\psi$$

(4) where $\triangle$ is the covariant Laplace operator for metric $g$.

For constant mean curvature (CMC) data there is $H' = \text{const}$. Then the momentum constraint reduces to

$$\nabla_j K^j_i = 0, \quad H = 0.$$  

(5)

In this case one can first find a solution $(g, K)$ of (5) and then consider equation (4) for $\psi$. Equations (5) were solved analytically or reduced to a simpler system only for conformally flat initial metrics [4–7] or symmetric data [8–12]. Exact solutions of (4) are known for the simplest class of data with $K = 0$ and $g_{ij} \sim \delta_{ij}$. In other cases, the best what one can do is to prove the existence of solutions [13–16] or to find them numerically (see [17, 18] for a review).

The main aim of this paper is to simplify (5) for generic CMC data. We show that these equations decouple in coordinates adjusted to an arbitrary foliation of the initial manifold. One of them can be formally solved giving a component $W$ of $K$ as an indefinite integral of free data. Then the remaining two equations can be written in a simple way using the Cauchy-Riemann structure related to the initial metric. Under additional assumptions solutions of this equation can be also represented as integrals of known functions.

If $H' \neq \text{const}$ then the momentum constraint cannot be considered separately from the Hamiltonian constraint. Analytic solutions are not known and a proof of existence of solutions is much more difficult [3, 19]. We shortly present an idea of a non-conformal approach, in which the Hamiltonian constraint is treated as an algebraic equation for the mean curvature $H$. For a class of data the full system of constraints is reduced to 2 real equations in 2 variables even if tensor $K_{ij}$ depends on all coordinates.

In the last section we consider data admitting a marginally outer trapped surface (MOTS) which can be considered as an attribute of a black hole. Known constructions of such data are based on the puncture method of Brill and Lindquist [4], the conformal-imaging method of Misner [5] or the boundary condition method proposed by Thornburg [20]. In the spirit of Misner’s approach we define a class of maximal non-conformally flat data with a reflection symmetry which assures existence of MOTS. These data generalize the Kerr metric data but they don’t have to be axially symmetric.
The 2+1 decomposition of initial data

We would like to find coordinates in which the momentum constraint (5) decouples and can be partly integrated. Let us foliate the initial manifold $S$ into surfaces given by constant levels of a function $\varphi$. In coordinates $x^i = x^a, \varphi$, where $i = 1, 2, 3$ and $a = 1, 2$, arbitrary initial metric can be written in the form

$$g = g_{ab} dx^a dx^b + \alpha^2 (d\varphi + \beta_a dx^a)^2. \quad (6)$$

Coordinate transformations allow to impose up to 3 conditions on components of this metric. Most of them, including conditions satisfied in the Gauss coordinates, do not simplify equation (5). The method of trial and error shows that probably the best coordinates in this respect are $x^a, \varphi$ such that $2 \times 2$ metric $g_{ab}$ is conformally flat

$$g_{ab} = \rho^2 \delta_{ab}. \quad (7)$$

Throughout the paper we will assume that coordinates $x^a, \varphi$ satisfy (7). The foliation of $S$ into surfaces $\varphi = const$ can be still arbitrary. If we change $\varphi$ (and the foliation) we can further reduce a number of functions in $g$ to three.

Let us choose the following basis of 1-forms and the dual vector basis

$$\theta^a = dx^a, \quad \theta^3 = d\varphi + \beta_a dx^a \quad (8)$$
$$e_a = \partial_a - \beta \partial_{\varphi}, \quad e_3 = \partial_{\varphi}. \quad (9)$$

Given (7) it is convenient to define a complex coordinate $\xi$, a complex operator $\partial$ and functions $\beta, U, V, W$ as follows

$$\xi = x^1 + ix^2, \quad \beta = \frac{1}{2}(\beta_1 - i\beta_2), \quad \partial = \partial_\xi - \beta \partial_\varphi \quad (10)$$
$$U = \frac{1}{2} \alpha(K_{11} - K_{22}) - i\alpha K_{12}, \quad V = \alpha(K_{13} - iK_{23}), \quad W = K^a_a. \quad (11)$$

In coordinates $\xi, \bar{\xi}, \varphi$ the initial metric reads

$$g = \rho^2 d\xi d\bar{\xi} + \alpha^2 (d\varphi + \beta d\xi + \bar{\beta} d\bar{\xi})^2. \quad (12)$$

Note that complex functions $U$ and $V$ depend only on the traceless part of $K$.

**Proposition 2.1.** Let metric

$$g = \rho^2 \delta_{ab} dx^a dx^b + \alpha^2 (d\varphi + \beta_a dx^a)^2 \quad (13)$$
and functions $K_{a3}$ be given. Then the momentum constraint with $H = 0$ decouples into the following system of equations for $W$ and $U$

\[(\rho^3 W)_{,\varphi} = E\]  \hspace{1cm} (14)

\[U_{,\xi} - (\beta U)_{,\varphi} = F,\]  \hspace{1cm} (15)

where $E$ is given by (20) and $F$ is given up to $W$ by (21).

Proof. Let $\tilde{K}_a^b$ be the traceless part of $K_a^b$

\[K_a^b = \tilde{K}_a^b + \frac{1}{2} W \delta_a^b.\]  \hspace{1cm} (16)

In gauge (7) and basis (9) the momentum constraint (5) with $i = 3$ yields (14) with

\[E = \rho \alpha^{-1} e_a (\alpha \rho^2 K^a_3) - 2 \rho^3 \beta_{a,\varphi} K^a_3.\]  \hspace{1cm} (17)

For $i = a$ one obtains

\[\rho^2 (\alpha \tilde{K}_a^b)_{,b} - (\alpha \rho^2 \tilde{K}_a^b \beta_b)_{,\varphi} = F_a,\]  \hspace{1cm} (18)

where $\|_b$ denotes the covariant derivative with respect to metric (7) and

\[F_a = -\frac{1}{2} \rho^2 \alpha^{-2} e_a (\alpha^3 W) + \frac{3}{2} \rho^2 \alpha W \beta_{a,\varphi} + \rho^2 \alpha (\eta^{cd} e_c \beta_d) \eta_{ab} K^b_3 - (\alpha^{-1} \rho^2 K_{3a})_{,\varphi}.\]  \hspace{1cm} (19)

By means of (10) and (11) formula (17) can be written as

\[E = 2 \rho \alpha^{-1} Re(\bar{\partial}V - 2 \bar{\beta}_{,\varphi} V).\]  \hspace{1cm} (20)

A complex combination of equation (18) with $a = 1$ and $a = 2$ leads to (15) with $F = \frac{1}{2} (F_1 - iF_2)$ of the form

\[F = -\frac{1}{2} \rho^2 \alpha^{-2} \partial (\alpha^3 W) + \frac{3}{2} \rho^2 \alpha W \beta_{,\varphi} - 2i Im(\bar{\beta}) V - \frac{1}{2} (\alpha^{-2} \rho^2 V)_{,\varphi}.\]  \hspace{1cm} (21)

Equation (14) determines $W$ up to a real function $f$ of $\xi$ and $\bar{\xi}$

\[W = \rho^{-3} (\int_{\varphi_0}^{\varphi} Ed\varphi' + f(\xi, \bar{\xi})).\]  \hspace{1cm} (22)

Substituting (22) into (21) yields $F$ up to $f$. Then (15) becomes an equation for $U$. If it is solved all components of $K$ can be derived.
In view of (22) the momentum constraint reduces to equation (15) for $U$. Substituting $F = \hat{F}$ into (15) leads to the equation

$$\bar{\partial} \hat{U} = \hat{F}$$

(23)

for a function $\hat{U}$ such that $U = \hat{U}_\varphi$. Operator $\partial$ defines the Cauchy-Riemann (CR) structure (see [21] and references therein) on the initial manifold, not unique since locally there are many systems of coordinates with property (7). The example of Hans Lewy [22] shows that equation (23) can be unsolvable even for $C^\infty$ functions $\beta$ and $\hat{F}$. CR structures are known to appear in general relativity, especially in the context of algebraically special solutions of the Einstein equations (see e.g. [23]).

There are several cases in which solutions of (23) can be represented in an integral form. In all of them the CR structure is realizable. This means that, in addition to $\xi$, there exists a solution $\chi$ of equation $\bar{\partial} \chi = 0$ such that $\chi_\varphi \neq 0$. Equivalently, $\beta$ can be written in the form

$$\beta = \frac{\bar{\chi}}{\bar{\chi}_\varphi} \chi.$$

(24)

Given $\chi$ the initial manifold can be considered as a 3-dimensional real surface in space $C^2$ of pairs $(\xi, \chi)$.

**Proposition 2.2.** If $\beta = 0$ then $U$ is given by

$$U = \frac{1}{2\pi i} \int_\Omega \frac{F(\xi', \bar{\xi}', \varphi)}{\xi'^{-1} - \xi} \, d\xi' \wedge d\bar{\xi'} + h(\xi, \varphi),$$

(25)

where the integral is taken over a bounded open neighborhood $\Omega$ of $\xi$ in $C$ and $h$ is a function holomorphic in $\xi$.

**Proof.** For $\beta = 0$ equation (15) reads

$$U_{\xi} = F.$$

(26)

Integrating (26) with the fundamental solution $(\pi\bar{\xi})^{-1}$ for the Cauchy-Riemann operator $\partial_{\bar{\xi}}$ leads to a version of Cauchy’s integral formula (Theorem 1.2.1 in [24]) which can be converted into (25). The domain of integration can be extended to a whole support of $F$ provided that the r. h. s. of (25) still makes sense.

\[\square\]
Proposition 2.3. Let functions \( \rho, \alpha, \chi \) and \( V \) be analytic with respect to coordinates \( \phi \). Then

\[
U = \chi (\phi) \left( \frac{1}{2\pi i} \int_{\Omega} \frac{F(\xi', \check{\xi}', \phi(\xi', \check{\xi}, \chi))}{\xi' - \xi} \varphi_{,\chi}(\xi', \check{\xi}', \chi) d\xi' \wedge d\check{\xi}' + h(\xi, \chi) \right),
\]

where \( \chi \) depends on unprimed coordinates and \( \phi(\xi', \check{\xi}', \cdot) \) is the inverse function to \( \chi(\xi', \check{\xi}', \cdot) \).

Proof. Under assumptions of this proposition functions \( W, F \) and \( \hat{F} \) will be also analytic in \( \phi \). Thus, we can complexify \( \phi \) and pass to coordinates \( \xi, \check{\xi} \) and \( \chi \). In these coordinates equation (23) reads

\[
\hat{U}, \check{\xi} = \hat{F}(\xi, \check{\xi}, \phi(\xi, \check{\xi}, \chi)).
\]

Now, we can follow (26) and (25) in order to represent \( \hat{U} \) as

\[
\hat{U} = \frac{1}{2\pi i} \int_{\Omega} \frac{\hat{F}(\xi', \check{\xi}', \phi(\xi', \check{\xi}, \chi))}{\xi' - \xi} d\xi' \wedge d\check{\xi}' + \hat{h}(\xi, \chi).
\]

If we come back to coordinates \( \xi, \check{\xi}, \phi \) and differentiate (29) over \( \phi \) we obtain (27) with \( h = \hat{h}_{,\chi} \).

Formula (27) becomes much simpler if \( \beta_{,\phi} = 0 \). Then we can choose

\[
\chi = \phi + \chi^0(\xi, \check{\xi}),
\]

where \( \chi^0_{,\xi} = \beta \) (this condition can be always locally solved with respect to \( \chi^0 \)). Expression (27) takes the form

\[
U = \frac{1}{2\pi i} \int_{\Omega} \frac{F(\xi', \check{\xi}', \phi + \chi^0(\xi, \check{\xi}) - \chi^0(\xi', \check{\xi}') \phi(\xi', \check{\xi}, \chi)}{\xi' - \xi} d\xi' \wedge d\check{\xi}' + h(\xi, \phi + \chi^0(\xi, \check{\xi})).
\]

If all initial data are independent of \( \phi \) formula (31) reduces to

\[
U = \frac{1}{2\pi i} \int_{\Omega} \frac{F(\xi', \check{\xi}')}{\xi' - \xi} d\xi' \wedge d\check{\xi}' + h(\xi).
\]

In this case equation (14) is replaced by condition \( E = 0 \). It implies

\[
K^a_b = \alpha^{-1} \eta^{ab} \omega_{,b},
\]
where \( \omega \) is a function of coordinates \( x^a \) (a potential equivalent to \( \omega \) was also introduced in [9–11]). Now \( F \) is determined by metric \( g \) and functions \( W \) and \( \omega \).

If \( \chi \) and \( \hat{F} \) are analytic in all coordinates then one can solve (23) by integrating (28) over \( \xi \). Then, instead of (27) one obtains

\[
U = \chi \varphi \left( \int_{\xi_0}^{\xi} F(\xi, \xi', \varphi(\xi, \xi') \varphi (\xi, \xi') d\xi' + h(\xi, \chi) \right). \tag{34}
\]

This formula becomes particularly simple if data is independent of \( \varphi \)

\[
U = \int_{\xi_0}^{\xi} F(\xi, \xi') d\xi' + h(\xi). \tag{35}
\]

Results of this section have been obtained thanks to a simple structure of the momentum constraint in the conformal approach with \( H' = \text{const.} \). In order to complete the construction of initial data one should solve the Lichnerowicz equation (4) for the conformal factor \( \psi \). Existence theorems for this equation [15, 16] assume positivity of the Yamabe type invariant, but there is no practical device how to satisfy this condition. For some classes of data one can get it from the Sobolev inequalities admitted by the initial manifold [12]. The simplest way to assure existence and uniqueness of \( \psi \) is to assume that solution \((g, K)\) of the momentum constraint is asymptotically flat, the initial manifold is complete and \( R \geq 0 \). For instance, one can take the same initial manifold and metric \( g \) as in a known maximal \((H = 0)\) solution of the full set of initial conditions. It follows from the Hamiltonian constraint that inequality \( R \geq 0 \) must be everywhere satisfied. If we replace \( K \) by another solution of the momentum constraint we can be sure that a new solution \( \psi \) of the Lichnerowicz equation exists but there is still problem to find it numerically.

We are not able to extend above results to the case \( H' \neq \text{const.} \). Then equations (14) and (15) should be replaced by

\[
(\rho^3 W)_{,\varphi} = E - \frac{2}{3} \psi^6 \rho^3 H'_{,\varphi} \tag{36}
\]

\[
U_{,\xi} - (\beta U)_{,\varphi} = F + \frac{2}{3} \psi^6 \rho^2 \alpha \bar{\partial} H' \tag{37}
\]

( note that \( \beta, U, V \) and \( W \) are preserved by transformation (3) and \( \rho \rightarrow \psi^2 \rho, \alpha \rightarrow \psi^2 \alpha \)). Equations (36) and (37) have to be considered simultaneously with (4) except the case \( H'_{,\varphi} = 0 \) for which \( W \) is given by (22). Existence of solutions of this system is much more difficult to prove [3].
At the end of this section let us present an idea of a non-conformal approach to the constraints. For $H \neq 0$ the momentum constraint (1) yields the following generalization of equations (14) and (15)

\begin{align}
(\rho^3W)_{,\varphi} &= E + 2\rho^2 \rho_{,\varphi}H \\
U_{\xi} - (\beta U)_{,\varphi} &= F + \frac{2}{3} \rho^2 \alpha \bar{\partial} H .
\end{align}

(38) (39)

The Hamiltonian constraint (2) can be written as

\[ WH = 2W^2 + 2\alpha^{-2} \rho^{-4} U\bar{U} + 2\alpha^{-4} \rho^{-2} V\bar{V} - R . \]

(40)

If $W \neq 0$ then one can determine $H$ from (40) in terms of $g$, $U$, $V$ and $W$ and substitute the result into equations (38) and (39). In this way one obtains nonlinear equations for $W$ and $U$ whereas $g$ and $V$ can be arbitrarily chosen. These equations take a particularly simple form if $\rho_{,\varphi} = 0$. Then (38) yields formula (22) for $W$ and equation (39) leads to an equation for $U$ of the form

\[ U_{\xi} - (\beta U)_{,\varphi} - \rho^2 (fU\bar{U})_{,\xi} + \rho^2 (\bar{f}U\bar{U})_{,\varphi} = h , \]

(41)

where $f$ and $h$ are known functions (equation (41) can be replaced by 2 compatible third order equations for $|U|$). If $\beta = 0$ equation (41) reduces to

\[ U_{\xi} - \rho^2 (fU\bar{U})_{,\xi} = h , \]

(42)

where coordinate $\varphi$ appears as a parameter. Equation (42) looks much simpler than the full set of initial constraints with $\rho_{,\varphi} = \beta = 0$ in other formulations. A usefulness of (42) for numerical calculations remains an open question.

3 Horizons

In the theory of black holes it is important to construct initial data with one or more 2-dimensional surfaces which can represent horizons of black holes. This can be done within the conformal method by imposing an appropriate condition on the conformal factor $\psi$ on an internal boundary of the initial manifold [15]. This boundary becomes MOTS upon the conformal transformation (3). Existence of $\psi$ depends on the positivity of the Yamabe type invariant what is even more difficult to prove than in the case of an unbounded initial manifold (see [12] for partial results). Another problem is that a continuation of initial data through this
internal boundary is not assured. This problem does not appear in the inversion symmetry approach of Misner \cite{5}, however in this case a class of data is much more restricted. In this section we propose a construction of data which follows the latter method.

We would like to generalize the Kerr metric data at $t = \text{const}$, where $t$ is the Boyer-Lindquist time coordinate. The initial metric induced by the Kerr solution reads

$$g = \rho^2 \Delta^{-1} dr^2 + \rho^2 d\theta^2 + \rho^{-2} \Sigma^2 \sin^2 \theta d\varphi^2,$$

where

$$\rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 - 2Mr + a^2, \quad \Sigma^2 = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta.$$

Metric (43) can be written in the form (13) with $x^a = \tilde{r}, \theta$, where $\tilde{r}$ is related to $r$ by

$$r = M + \sqrt{M^2 - a^2 \cosh \tilde{r}}, \quad \tilde{r} \in [-\infty, \infty].$$

Domains where $\tilde{r} > 0$ or $\tilde{r} < 0$ are asymptotically flat. They are connected by the external Kerr horizon (the Einstein-Rosen bridge) located at $\tilde{r} = 0$. Only non-vanishing components of $K$ are given by (33) with $\omega = 4aM \rho^{-2} [2(r^2 + a^2) + (r^2 - a^2) \sin^2 \theta] \cos \theta$.

The Kerr initial data are analytic in coordinates $x^i = \tilde{r}, \theta, \varphi$. Tensors $g$ and $K$ are invariant under the reflection

$$x^1 \to -x^1$$

which corresponds to the inversion invariance of Misner. This and vanishing of $K^{11}$ at $x^1 = 0$ imply that surface $x^1 = 0$ is MOTS.

In order to generalize the Kerr data let us assume that $H = 0$, metric $g$ has the form (12) and

$$g \to g \quad K \to \epsilon K, \quad \epsilon = \pm 1$$

under reflection (47). Transformations (48)-(49) correspond to the following behaviour of components of $g$ and $K$

$$\alpha \to \alpha, \quad \rho \to \rho, \quad \beta \to -\tilde{\beta},$$

$$V \to -\epsilon \tilde{V}, \quad W \to \epsilon W, \quad U \to \epsilon \tilde{U}.$$
Proposition 3.1. (i) Let \( V \rightarrow -\varepsilon \bar{V}, \varepsilon = \pm 1 \), and metric (12) be invariant under reflection \( x^1 \rightarrow -x^1 \). Then constraints (14)-(15) and integral formulas (22), (25), (27), (34), (37), (35) are compatible with transformation (49) provided that
\[
\chi \rightarrow \bar{\chi}, \quad f \rightarrow \varepsilon f, \quad h \rightarrow \varepsilon \bar{h} \quad \text{if} \quad x^1 \rightarrow -x^1.
\] (52)

(ii) Let \( g, K \) satisfy (14)-(15), (48)-(49) and condition (only if \( \varepsilon = 1 \))
\[
\text{Re} U = -\frac{1}{2} \alpha \rho^2 W \quad \text{at} \quad x^1 = 0.
\] (53)

If equation (4) admits a unique conformal factor \( \psi \) then surface \( x^1 = 0 \) is MOTS with respect to ultimate initial data given by (3) with \( H' = 0 \).

Proof. A compatibility of the assumed transformations with (49) and equations (14)-(15) can be easily confirmed by means of (51) and (51). The same refers to integral formulas for \( W \) and \( U \) in section 2 if transformation (52) is taken into account.

The reflection invariance of \( g \) implies that the exterior curvature of surface \( x^1 = 0 \) embedded in the initial manifold vanishes. For \( \varepsilon = -1 \) function \( K^{11} \) is antisymmetric with respect to \( x^1 \), hence automatically
\[
K^{ij} n_i n_j = 0 \quad \text{at} \quad x^1 = 0,
\] (54)
where \( n_i \) is the normal vector of the surface. For \( \varepsilon = 1 \) equality (54) is assured by (53). The Lichnerowicz equation is invariant under transformation (49). If its solution \( \psi \) is unique it must be also invariant. Then properties (49) and (54) are preserved by the conformal transformation (3). Hence, the exterior curvature of surface \( x^1 = 0 \) with respect to metric \( g' \) also vanishes. This together with equation (54) and \( H' = 0 \) imply zeroing of expansions of null rays emitted outward or inward from \( x^1 = 0 \). Hence, this surface is MOTS with respect to data \((g, K)\) and \((g', K')\).

It follows from this proposition that one can relatively easily construct initial data with MOTS assuming symmetry (49) with \( \varepsilon = -1 \). For instance, one can adopt the Kerr initial metric (43) and assume that only non-vanishing components of a new tensor \( K \) are given by (33) with \( \omega \) which is an odd function of \( x^1 \). Since \( R \geq 0 \) the Lichnerowicz equation admits a unique solution which is reflection symmetric. Given \( \psi \) transformation (3) yields ultimate data which satisfies all
initial constraints. Note that we can choose $\omega$ which tends to (46) if $x^1 = \tilde{r} \to \infty$. Then the data approaches the Kerr data with the angular momentum $a$ if $x^1 \to \infty$ and with the angular momentum $-a$ if $x^1 \to -\infty$.

For $\epsilon = 1$ condition (53) imposes an inconvenient constraint on integral representations of $W$ and $U$ from section 2. Function $f$ in (22) cannot solve this constraint because $f$ is independent of $\varphi$. Function $h$ in integral formulas for $U$ can be used only for analytic fields. A simple way to satisfy (53) is to assume that $W = \Re U = \beta = 0$. Then equation (14) yields $V = 2i\omega,_{\xi}$ and (15) is equivalent to the following conjugate equations for $\Im U$:

$$\Im U(\Im U,_{\xi},_{\xi}) = -\left(\rho^2 \alpha^{-2} \omega,_{\xi},_{\varphi}\right), \quad (\Im U),_{\xi} = -(\rho^2 \alpha^{-2} \omega,_{\xi}),_{\varphi}. \quad (55)$$

The integrability condition for this system reads

$$(\rho^2 \alpha^{-2} \omega,_{1},_{2}) + (\rho^2 \alpha^{-2} \omega,_{2},_{1}) = f,_{2}, \quad (56)$$

where derivatives are taken with respect to real coordinates $x^a$ and $f$ is a $\varphi$-independent function. One can consider (56) as an equation for $\omega$, in which $\varphi$ is a parameter, or one can formally solve (56) to obtain

$$\rho^2 \alpha^{-2} \omega,_{1} - f = \gamma,_{1}, \quad \rho^2 \alpha^{-2} \omega,_{2} = -\gamma,_{2}, \quad (57)$$

where $\gamma$ is a new function. Let us treat (57) as equations for $\omega$ and $\rho^2 \alpha^{-2}$ whereas $\gamma$ and $f$ are given. Then $\omega$ has to satisfy the first order linear equation

$$\gamma,_{2} \omega,_{1} + (\gamma,_{1} + f) \omega,_{2} = 0. \quad (58)$$

If it is solved function $\rho^2 \alpha^{-2}$ is given by

$$\frac{\rho^2}{\alpha^2} = \frac{(\gamma,_{1} + f) \omega,_{1} - \gamma,_{2} \omega,_{2}}{\omega,_{1} \omega,_{1} + \omega,_{2} \omega,_{2}} \quad (59)$$

provided that the nominator and the denominator of the r. h. s. of (59) are both positive. Solving (58) with respect to $\omega$ is equivalent to finding integral lines of the vector field $v = \gamma,_{2} \partial_1 + (\gamma,_{1} + f) \partial_2$ in $\mathbb{R}^2$. The latter problem can be reduced to an ordinary differential equation with an arbitrary initial condition.

### 4 Summary

We have been considering initial constraints for the vacuum Einstein equations in the framework of the conformal approach. We assumed that for a foliation of an
initial manifold into surfaces $\varphi = const$ there exist coordinates in which metric takes the form (12). One can relate with these coordinates the Cauchy-Riemann operator $\partial$. The momentum constraint with $H = 0$ splits into equations (14) and (15) (proposition 2.1). Equation (14) can be directly integrated with respect to $W$ giving formula (22). In the case of fields analytic in coordinate $\varphi$ or data with $\beta = 0$ in metric (12) solutions of equation (15) can be also written as integrals of known functions (propositions 2.2 and 2.3). In order to complete the construction of initial data one has still to solve the Lichnerowicz equation (4) for the conformal factor $\psi$. Its existence and uniqueness can be easily proved in some cases, e.g. if the Ricci scalar of $g$ is nonnegative.

For arbitrary $H$ the momentum constraint and the Hamiltonian constraint are substantially coupled. Then one has to consider equation (4) together with (36) and (37). In the last part of section 2 we briefly discuss a nonstandard approach in which the full set of constraints is reduced to one complex equation (41) if $\rho, \varphi = 0$.

In section 3 we propose a construction of maximal data with a reflection symmetry which, together with (53), implies existence of a horizon in the form of a marginally outer trapped surface (proposition 3.1). As an example we present data obtained by a modification of the Kerr initial data.

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