Reliable Detection of Causal Influences in Dynamical Systems

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Knowledge about existence, strength, and dominant direction of causal influences is of paramount importance for understanding complex systems. With limited amounts of realistic data, however, current methods for investigating causal links among different observables from dynamical systems suffer from ambiguous results. Missing is a statistically well defined approach that avoids false positive detections while being sensitive for weak interactions. Ideally, it should be able to infer directed causal influences also when synchronizations occur. The proposed method exploits local inflations of manifolds to obtain estimates of upper bounds on the information loss among state reconstructions from two observables. It comes with a test for the absence of causal influences. Simulated data demonstrate that it is robust to intrinsic noise, copes with synchronizations, and tolerates also measurement noise.

I. INTRODUCTION

Simple cause-effect notions of causality are misleading when interactions are reciprocal. This is particularly clear for deterministic dynamical systems where Takens’ Theorem [1-3] shows that different observables individually contain all information about the entire system’s state. Here, state reconstructions from different observables are generically equivalent which reflects the fact that the systems are non-separable and behave as wholes. Still, it is of considerable interest to gain knowledge about the strength of influences among selected components of a complex dynamical system.

Several methods proposed for this purpose rely on phase space reconstructions [4-6]. In essence, they are based on the following simple consideration: Let a system X uni-directionally influence another system Y. Obviously, then Y receives information about X. Consequently, states of Y will contain information about the state of X, while states of X by assumption cannot provide full ‘knowledge’ about Y. Thereby states Y can be expected to predict observables of X better than vice versa. This heuristics was put on solid mathematical grounds in [5] leading to a method that allowed to capture also mutual interactions. The measure of Topological Causality was introduced for inference of the intensity of directed effective influences among observables. It relies on local expansions of the mappings between state reconstructions. While mathematically transparent, Topological Causality was hitherto estimated from fitting the corresponding manifold inflations not only capture possible dimensional conflicts among state reconstructions [6], but also allow for a statistical criterion to control for false positive detections of causal influences.

After introducing the method it is tested on a time discrete model system demonstrating the relation to coupling constants in its dependency on intrinsic noise, limited measurements, and measurement noise. Then we demonstrate that the method reliably detects the absence of causal links and copes with synchronization when applied to time series of realistic length from time continuous systems. A first application to heart rate and breathing rate data is found to deliver surprisingly unambiguous results.

II. MANIFOLD INFLATION AS A PROXY FOR INFORMATION LOSS

We consider dynamical systems composed of two subsystems X and Y, governed by

\[ \dot{x} = f(x, w_{xy} \mu_x(y)) \]
\[ \dot{y} = g(y, w_{yx} \mu_y(x)) \]

where \( \mu_i(i) \) denote fixed scalar functions and \( w_{ij} \) coupling constants. It was shown by Takens [1] that a topologically equivalent portrait of the attractor can be reconstructed from lagged coordinates.

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For this purpose a $m$-dimensional vector of delayed copies of a single observable are merged, giving $r^x(t) = [x(t), \ldots, x(t + (m - 1)\tau)]$. The dimension $m$ is sufficient if $m > 2D$ where $D$ is the dimension of the attractor. Formally the time delay $\tau > 0$ is arbitrary, in practice, however, there are methods for finding suitable values of $m$ and $\tau$ [10][11]. Note that to be a valid reconstruction of the overall attractor, there must be incoming connections $w_{xi}$ linking the subsystem $X$ to the whole system. In this case there also exists a unique mapping between reconstructions from different observables e.g. from $r^x$ to $r^y$ [12].

Here we use a combination of local and global properties of the relations among nearest neighbours to reference points in both reconstructions for estimating how much the causally affected system 'knows' about the state of the system influencing it. Each point is identified by its time index and its location in the respective reconstruction. For each reference point $t$ we determine the $k$ nearest neighbours in both reconstructed spaces $t^x_l$ and $t^y_l$, $l = 1, \ldots, k$ and the (euclidean) distances from the reference point to these neighbours both, in their origin space $L_x(t, t^x_l)$ and the distance to the putative neighbours based on proximity in the respective other space $L_x(t, t^y_l)$.

A set of random neighbours is generated by Monte-Carlo-Simulation staggering time indices in one space by $\pm 2m\tau$. In doing so temporal correlations within an observable are preserved and relations between different observables are destroyed, e.g. a reference point in $X$ is then associated with a set of random points $t^y_l$ and the distances to these points are $L_x(t, t^y_l)$. Also we introduce $d^i_l(k)$ as mean logarithmic size of the neighbourhood for all $L_i(t, t^i_l)$:

$$d^i_l(k) = \left(\log(\max([L_i(t, t^i_l)]_{l=1..k}))\right)_{t \in E}$$

$i, j = x, y, x^*, y^*$

The random neighbourhoods are here averaged over both, the Monte-Carlo ensembles $E$ and the reference points, whereas for the non-shifted neighbours the average is only over the reference points.

As an example consider two coupled logistic maps

$$x(t + 1) = x(t)[R_x(1 - x(t)) + w_{xy}y(t)] + \eta_x(t)$$
$$y(t + 1) = y(t)[R_y(1 - y(t)) + w_{yx}x(t)] + \eta_y(t)$$

with reflecting boundaries, $R_x$ being system parameters and subjected to additive Gaussian noise $\eta(t) \in \mathcal{N}(0, \sigma)$. The simplest case is the noise free unilaterally coupled system ($w_{xy} = 0$ and $w_{yx} = 0.3$). Here, the time-delay reconstruction of subsystem $X$ is one-dimensional and we can visualize the whole system in three dimensions by showing $x_t$ over the $(y_t, y_{t+1})$-plane. (FIG. 1). Since $w_{yx} > 0$ information about $X$ is contained in $Y$ and the images of neighbours in $Y$ are also localized in $X$. In contrast, $X$ doesn’t contain full information about $Y$ and thus the image of these neighbours is spread over the $y_t$-$y_{t+1}$-plane. With increasing coupling $w_{yx}$ neighbours become more localized, in the limiting case of perfect information preservation the neighbours are identical and $d^x_l(k) = d^y_l(k)$. In the other limiting case no information about $X$ is included in $Y$ and $d^y_l(k)$ will on average be identical with the random neighbourhood $d^y_l(k)$.

These relations between neighborhood sizes are visible in plots of the mean logarithmic neighbourhood sizes $d^i_l(k)$ as functions of $k = \psi(k) = -\log(N)$ FIG. 2, where $k$ is the number of neighbours and $N$ the amount of data and $\psi$ is the psi-gamma-function. This particular choice of the abscissa allows for an unbiased estimate of the fractal dimensions of the subsystems by taking the slope of $\psi(k)$ versus $d^i_x(k)$ (resp. $d^i_y(k)$) [13]. In the present example the independent observable $X$ has a smaller dimension than the influenced observable. While dimensional conflicts provide a sufficient criterion for the direction of causal influence in unilaterally coupled deterministic systems [6], they are useless for mutually coupled systems.

In the general case a different criterion for determining the dominant direction of causal influence is needed. With $d^x_l(k)$ providing a lower bound and $d^x_l(k)$ an upper bound for the size of the $k$-the neighbourhood, the size $d^x_l(k)$ can be used to define a measure for how much information is preserved within in the neighbourhood in $y$. We use the ratio of the distance between $d^x_l(k)$ and $d^y_l(k)$ and the chance-level $d^y_l(k)$:

$$I_{x\rightarrow y}(k) = \frac{d^x_l(k) - d^y_l(k)}{d^x_l(k) - d^y_l(k)}$$

![FIG. 1. $x_t$ over $y_t$ and $y_{t+1}$ for the noise free unilaterally coupled logistic maps with $w_{xy} = 0, w_{yx} = 0.3, R_x = R_y = 3.82$, together with the projection of the manifold on the $y_{t+1}$-$y_t$-plane and the $x_t$-axis. 10^6 data points are shown in grey, the 10 nearest neighbours of a reference point are shown in grey in both, the manifold and the projections. The 10 neighbours searched in Y (X) are shown in red (blue).](image)
do not overlap.

k for significance that at least 15 out of
results we apply the fivefold SE and furthermore require
are needed to detect significant causal influences. If the
coupling for different amounts of data (Fig. 4 (a)
significant causal influence is detectable. We varied the
error of the mean logarithmic neighbourhood sizes
For determining significance we use the simple standard
error is shown for each
dline, \(d_y^\kappa(k)\) shown as dotted line and the respective
level

\(\kappa\) shown as solid line, \(d_y^\alpha(k)\) shown as solid line, \(d_x^\alpha(k)\)
phys and \(d_x^\kappa(k)\) (dashed line). Furthermore the fivefold standard
error is shown for each \(d_x^\kappa(k)\) as a grey shade.

Note that \(0 \leq I_{x \rightarrow y} \leq 1\) and therefore \(I_{x \rightarrow y} \simeq 0\) means
that \(x\) does not influence \(y\) at all. If \(I_{x \rightarrow y} \simeq 1\) \(y\)'s 
knows everything' about \(x\), which suggests that \(x\) has a strong
influence on \(y\). Furthermore we introduce a measure for
the asymmetry of causal influences \(\alpha\):

\[
\alpha = \frac{I_{y \rightarrow x}(k) - I_{x \rightarrow y}(k)}{I_{y \rightarrow x}(k) + I_{x \rightarrow y}(k)}
\]

For determining significance we use the simple standard
error of the mean logarithmic neighbourhood sizes \(\sigma_d^\kappa\).
For significance the standard errors \(\sigma_{d_y^\kappa}\) and \(\sigma_{d_x^\kappa}\) must
not overlap, e.g. as shown in FIG. 2 (a)). In practice
the errors overlap for large \(k\). Therefore, in all following
results we apply the fivefold SE and furthermore require
for significance that at least 15 out of \(k = 1..20\) values
do not overlap.

III. RESULTS

Firstly, we investigate sensitivity and specificity of the
method. For this purpose two logistic maps are coupled
bilaterally and the causal influence is determined for
different coupling weights as shown in Fig. 3. The causal
influence \(I_{y \rightarrow x}\) is a monotonic function of the weight \(w_{y \rightarrow x}\),
while the other direction \(I_{x \rightarrow y}\) is largely unaffected by
this weight even over three different magnitudes of
coupling strength. However, for couplings smaller \(10^{-3}\) no
significant causal influence is detectable. We varied the
coupling for different amounts of data (Fig. 4 (a), (d))
which for this system shows that at least \(10^3\) data points
are needed to detect significant causal influences. If the
coupling is small the required amount of data increases.
Note that, regardless of the amount of data, no false pos-

itives are detected. To demonstrate noise robustness we
injected intrinsic and external additive Gaussian noise in
the logistic maps Fig. 3 (e) - (f). While noise lowers
the causal influence in both cases, it still correctly de-
dpends on the coupling and even for strong noise no false
positives are induced.

Next we demonstrate the correct identification of the
direction of causal influence for time continuous systems
using two coupled Lorenz-systems. Here, we additionally
introduced external and internal noise to demonstrate
noise resistance also in this case (Fig. 5). The Lorenz
systems used were coupled by their \(x\)-components and
are given by:

\[
\begin{align*}
\dot{x}_i &= -\mu_i(x_i - y_i) + w_{ij}x_j + \sigma \eta^x_i \\
\dot{y}_i &= \rho_i x_i - y_i - x_i z_i + \sigma \eta^y_i \\
\dot{z}_i &= -\theta_i z_i + x_i y_i + \sigma \eta^z_i \\
\text{with} & \quad < \eta^i \eta^{i'} > = \delta_{ij} \delta(t - t')
\end{align*}
\]

and the parameters \(\mu_i = 28, \rho_i = 8/3\) and \(\theta_i = 10\). For
both, the noise free and noise polluted system, the correct
direction of causal influence is estimated correctly by \(\alpha\).
However, in the presence of noise the causal influence
is weakened and the asymmetry less pronounced, which
was also observed for the logistic maps.

Finally, we determine the causal influence for strong
couplings, where the systems tend to synchronize. To
quantify synchronization we estimate the information di-
mension from \(d_x^\kappa(k)\). For unilateral coupling this is a sen-
tive measure, since for complete synchronization the
dimension of the driven system will drop to the one of
the driving system.

The first example, a Roessler-Lorenz-System, uses the
same model analysed in [3]. As a second example we con-
sider two coupled FitzHugh-Nagumo Neurons governed
with the parameters $c_1 \approx 0.088$, $c_2 \approx 0.119$, $I_{\text{ext}} = 1$, $a_i = 0.7$ and $b_i = 0.8$. For both example systems synchronization occurs at a unilateral coupling strength $w_{\text{Rössler}} \approx 2.5$ ($w_{\text{FHN}} \approx 0.35$). Still, we are also here able to detect the correct direction of causal influence, even with degrees of synchronization where the dimensional drop dramatically (Fig. 5), which is in sharp contrast to other approaches (Krakovská Fig. 4 [9]). Interestingly, we found that intrinsic noise improves the detectability of influence asymmetries (asteriks in Fig. 6).

FIG. 4. Bilaterally coupled logistic maps $R_x = R_y = 3.92$, $w_{x \rightarrow y} = 0.05$ and varying $w_{y \rightarrow x}$ between 0 and 0.1. All time series were embedded with $\tau = 1$ and $m = 4$ and $10^4$ ensembles were generated to estimate chance-level. (a) & (d) Varying amount of data between $10^3$ and $10^4$, in (a) the causal influence $I_{y \rightarrow x}$ associated with the coupling $w_{yx}$ and (d) the reverse direction $I_{x \rightarrow y}$. For noise polluted time-series $10^4$ data points were used. (b) & (e) Additive internal noise is injected into the system. The $x$-axis shows the ratio of the standard deviations of noise and the unpolluted system varying between 0 and roughly 8% noise. (b) shows the causal influence $I_{y \rightarrow x}$ associated with the coupling $w_{yx}$ and (e) the reverse direction $I_{x \rightarrow y}$. (c) & (f) Additive external noise is added to the observed time-series. The $x$-axis shows the ratio of the standard deviations of noise and the unpolluted system varying between 0 and roughly 8% noise. (c) shows the causal influence $I_{y \rightarrow x}$ associated with the coupling $w_{yx}$ and f the reverse direction $I_{x \rightarrow y}$. For better visualisation contour lines mark lines of equal causal influence.

FIG. 5. Asymmetry index between obtained from $N = 10^4$ data points of two Lorenz oscillators with slightly different frequencies and coupled by their $y$-components ($\theta_1/j = 10, \rho_1 = 28.5, \rho_2 = 27.5, \theta_1/j = 8/3$). The time series were embedded with $m = 9$ and $\tau = 10$. The noise free asymmetry is shown in black. Coloured circles represent internal noise with $\sigma = 2$ (green) and additive external noise with $\sigma = 2$ (red).

FIG. 6. (a,c) Causal Influence $I_{x \rightarrow y}$ (blue) and $I_{y \rightarrow x}$ (red) for unilateral coupling $X \rightarrow Y$. (b,d) The asymmetry for the noise free (solid lines) and for the system perturbed by Gaussian white noise with $\sigma = 1$. Grey Lines show the Information Dimension estimated from the time series of the two subsystems. (a,b) Results for the unilaterally coupled Roessler $\rightarrow$ Lorenz-System. $10^3$ reference points where chosen from a total of $10^4$ data points that were embedded with $\tau = 2$ and $m = 7$. For chance-level-estimation $10^3$ ensembles were used. (c,d) As (a,b), however, for the unilaterally coupled pair of Fitzhugh-Nagumo oscillators embedded with $\tau = 9$ and $m = 4$.

IV. DISCUSSION:

When a system’s component (say X) unilaterally influences another component (say Y) then Y receives information about X. Typically, X will not have as much information about Y. This simple heuristics is opposite to standard approaches based on prediction including
Granger Causality \[14\] and Transfer Entropy \[15\]. Here, we consider the relative amounts of information about the systems' states in the directions opposite to the influences. The method estimates metric inflations of neighbourhoods and their projections by the putatively homeomorphic mappings among manifolds of reconstructed system states as a proxy for information loss. In the past a related idea was used for determining the quality of mappings \[15\] \[16\], where the ratio of the inflations within the respective spaces was expected to be close to one for homeomorphy while for topology violations it systematically deviates from one. In other words, while the core of method presented here is information theoretic its particular sensitivity for metric topology violations could have been expected from previous work. A recent work \[5\] similarly used local properties of the mappings among state reconstructions to establish a measure of causal influence (termed Topological Causality). In stark contrast to Topological Causality, however, the current method exploits expansions within and not among reconstructions. Basing the method on the relation of distances within the same space solves a range of problems of TC and related methods including Convergent Cross Mapping \[4\]. In particular, the method presented here is much less sensitive to synchronizations where to our knowledge previous methods often fail to deliver correct results \[7\] \[8\].

We are confident that applications also to real data will provide more reliable results than previous methods. First tests using heart rate versus breathing rate data from an apnoe patient revealed a substantial asymmetry of influence from heart rate to breathing rate (not shown), a statistically unambiguous result that is much more pronounced than when determined e.g. with Transfer Entropy (Fig. 4 in \[15\]).

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