Solutions to the Conjectures of Pólya-Szegő and Eshelby

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March 30, 2022

Abstract

Eshelby showed that if an inclusion is of elliptic or ellipsoidal shape then for any uniform elastic loading the field inside the inclusion is uniform. He then conjectured that the converse is true, i.e. that if the field inside an inclusion is uniform for all uniform loadings, then the inclusion is of elliptic or ellipsoidal shape. We call this the weak Eshelby conjecture. In this paper we prove this conjecture in three dimensions. In two dimensions, a stronger conjecture, which we call the strong Eshelby conjecture, has been proved: If the field inside an inclusion is uniform for a single uniform loading, then the inclusion is of elliptic shape. We give an alternative proof of Eshelby’s conjecture in two dimensions using a hodographic transformation. As a consequence of the weak Eshelby’s conjecture, we prove in two and three dimensions a conjecture of Pólya and Szegő on the isoperimetric inequalities for the polarization tensors. The Pólya-Szegő conjecture asserts that the inclusion whose electrical polarization tensor has the minimal trace takes the shape of a disk or a ball.

Keywords: Polarization tensor, Isoperimetric inequality, Pólya and Szegő conjecture, Eshelby’ conjecture, Layer potential

1 Introduction

It is well known that amongst all inclusions occupying a given volume the sphere is the unique inclusion with maximum surface area. This raises the question as to whether the sphere is uniquely optimal with respect to other properties, such as electrical properties. It was conjectured by Pólya and Szegő [48] that the sphere would be the unique inclusion minimizing the trace of the electrical polarization tensor when the inclusion and matrix have isotropic electrical properties. Here we prove this conjecture.

A closely related conjecture is the Eshelby conjecture. It is connected to two problems. In the transformation problem a region (the “inclusion”) in a homogeneous medium undergoes a temperature change or phase change which in the absence of the confining surrounding medium (the “matrix”) would lead to a uniform strain \( \varepsilon_0 \). In the elastic polarization problem, the inclusion has different moduli to that of the matrix, and a uniform stress is applied at infinity. If one of these problems has been solved and the field in the

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inclusion is uniform then (by subtraction or addition of a constant field) one immediately has a solution to the other problem. Eshelby [14, 15], following earlier work in special cases by Mindlin and Cooper [38] and Robinson [50], showed that the stress in the inclusion was uniform for ellipsoids and furthermore stated (without proof) that “among closed surfaces the ellipsoid alone has this convenient property”. Here, for an isotropic matrix, we prove Eshelby’s conjecture (more precisely what we call the “weak Eshelby conjecture”) that the inclusion is necessarily ellipsoidal if the field in the inclusion is uniform for all transformation strains in the transformation problem, or equivalently if the field in the inclusion is uniform for all uniform loadings. In the transformation problem the strain $\varepsilon(x)$ inside the inclusion depends linearly on $\varepsilon_0$ so that we may write $\varepsilon(x) = S(x)\varepsilon_0$ where $S(x)$ is the fourth order Eshelby tensor field. The weak Eshelby conjecture states that if $S(x)$ is constant inside the inclusion then the inclusion is an ellipsoid. It follows from certain “trace properties” of the second derivative of the Green’s function associated with the problem (see, for example, equation (6.30) in [36]) that the isotropic part of $S(x)$ is always uniform and independent of the shape of the inclusion [61]. Consequently for any inclusion with a sufficiently high degree of symmetry the value of $S(x)$ at its center and the average of $S(x)$ over the inclusion equal the value of $S$ in a sphere (or circle in two-dimensions) [14, 27, 17].

For planar elasticity Sendeckyj [56], and for antiplane elasticity Ru and Schiavone [52], proved a stronger conjecture (what we call the “strong Eshelby conjecture”) that the inclusion is necessarily elliptical (ellipsoidal) if the field in the inclusion is uniform for a single transformation strain in the transformation problem, or equivalently for a single uniform loading in the elastic polarization problem.

Eshelby’s conjecture drew increased attention when it was claimed (see [40, 41] and references therein) that the field was uniform inside star-shaped polygonal inclusions in contradiction to the proof of Sendeckyj. Rodin [51] proved directly that the field cannot be uniform inside polygons or polyhedra, and exact expressions for these non-uniform fields were later obtained [44, 27, 45]. Markenscoff showed that the field cannot be uniform if any portion of the boundary was planar [33] and that the only small perturbations of any ellipsoid boundary that preserve field uniformity in the interior are those which perturb the ellipsoid into another ellipsoid [34]. Lubarda and Markenscoff [30] showed that the field cannot be uniform for inclusions bounded by polynomial surfaces of higher than second degree, nor for inclusions bounded by segments of two or more different surfaces, and argued that non-convex inclusions are also excluded.

We remark, in passing, that not only is the field in an ellipsoid uniform for uniform loadings but it is also polynomial for polynomial loadings. This was proved for an isotropic matrix by Eshelby [15] and for an anisotropic matrix independently by Willis in an unpublished essay [60], and by Asaro and Barnett [7].

Let us now put these conjectures in a precise mathematical framework. Consider in $\mathbb{R}^d$, $d = 2, 3$ an inclusion $\Omega$, which is a bounded Lipschitz domain being inserted into a homogeneous medium of conductivity 1 in which there existed a uniform electric field $E = -a$. We assume that the conductivity of $\Omega$ is $k \neq 1$. The insertion of the inclusion perturbs the uniform electric field and the perturbed electric field is given by $E = -\nabla u$ where the potential $u$ is the solution to

\[
\begin{cases}
    \nabla \cdot (1 + (k - 1)\chi(\Omega))\nabla u = 0 & \text{in } \mathbb{R}^d, \\
    u(x) - a \cdot x = O(|x|^{1-d}) & \text{as } |x| \to \infty,
\end{cases}
\]

(1.1)

where $a$ is a constant vector in $\mathbb{R}^d$ indicating the direction of the uniform field and $\chi(\Omega)$
denotes the indicator function of $\Omega$. The solution $u$ to (1.1) has a multipole asymptotic expansion at infinity, with the leading term being the dipolar one:

$$u(x) = a \cdot x + \frac{1}{\omega_d} \frac{(a, Mx)}{|x|^d} + O(|x|^{-d}), \quad \text{as } |x| \to \infty.$$  (1.2)

Here $\omega_d$ is the area of the $d-1$ dimensional unit sphere and $M$ is a constant $d \times d$ matrix independent of $a$ and $x$. The matrix $M = M(\Omega) := (M_{ij})$ is called the polarization tensor associated with the inclusion $\Omega$. See [5, 37].

In their book [48] Pólya and Szegő conjectured that the inclusion whose polarization tensor (PT) has the minimal trace take the shape of a disk or a ball. The purpose of this paper is to prove this conjecture in two and three dimensions. In fact, we prove a theorem much stronger than the Pólya and Szegő conjecture.

In connection with the Pólya and Szegő conjecture various kinds of isoperimetric inequalities for the PT have been obtained. See, for example, [46, 47, 55]. The optimal isoperimetric inequalities for the PT have been obtained by Lipton [29], and later by Capdeboscq-Vogelius [9] based on the variational argument in [28]. The bounds are called the Hashin-Shtrikman bounds after names of the scientists who first found the optimal bounds on the effective conductivity of isotropic two-phase composites [21], since as pointed out in the caption of Figure 2 of [35], the PT bounds for isotropic $M$ can be obtained as the low volume fraction limit of their bounds; more generally for non-isotropic $M$ the PT bounds can be obtained as the low volume fraction limit of the bounds of Lurie and Cherkaev [31, 32] and Murat and Tartar [42]. The PT bounds are given as follows: Let $|\Omega|$ denote the volume of $\Omega$. Then

$$\text{Tr}(M) \leq |\Omega|(k - 1)(d - 1 + \frac{1}{k}),$$  (1.3)

and

$$|\Omega|\text{Tr}(M^{-1}) \leq \frac{d - 1 + k}{k - 1},$$  (1.4)

where Tr denotes the trace.

In this paper we prove the following theorem.
Theorem 1.1 Let \( \Omega \) be a simply connected bounded Lipschitz domain in \( \mathbb{R}^d \), \( d = 2, 3 \). If the polarization tensor \( M(\Omega) \) of \( \Omega \) satisfies the equality in (1.4), then \( \Omega \) must be an ellipse or an ellipsoid.

Observe that if a PT \( M \) has a minimal trace, then \( M \) attains the equality in (1.5) and

\[
M = \frac{d(k - 1)}{k + d - 1} I
\]

where \( I \) is the \( d \times d \) identity matrix assuming that the volume \( |\Omega| = 1 \). In fact, it can be seen clearly from Figure 1 which is taken from [3]. In that figure, the horizontal and vertical axis represent the eigenvalues of the PT in two dimensions, and hence the constant trace lines are those with slope \(-1\). Thus the minimal trace occurs at the unique tangent point of the lower hyperbola and a line with slope \(-1\). This point is an eigenvalue pair of the PT associated with the disk. The same argument works for three dimensional case as well. Therefore, as an immediate consequence of Theorem 1.1 we obtain the following corollary.

Corollary 1.2 (The Pólya and Szegö conjecture) Let \( \Omega \) be a simply connected bounded Lipschitz domain in \( \mathbb{R}^d \), \( d = 2, 3 \). If

\[
\text{Tr} M(\Omega) = \min_D \text{Tr} M(D),
\]

where \( M(D) \) is the polarization tensor for the domain \( D \) and the minimum is taken over all the domain with Lipschitz boundary (simply connected or not) with the same volume as \( \Omega \), then \( \Omega \) is a disk or a ball.

The concept of the polarization tensor appears in various contexts such as the theory of composites (see [37] and references therein) and the study of potential flow [48]. Another important usage of the concept is for the inverse boundary value problem to detect diametrically small inclusions by means of boundary measurements. In fact, one can approximately detect, by boundary measurements, the location and the polarization tensor of the inclusion. Since the polarization tensor carries important geometric information, such as the volume of the inclusion, we are able to recover that information from boundary measurements. It was Friedman and Vogelius [18] who first used the polarization tensor for the detection of small inclusions. We refer to [5] and references therein for recent developments of this theory. It is worthwhile mentioning that the method works for detection of multiple closely spaced inclusions [6].

The main step in proving Theorem 1.1 is the following theorem from [24].

Theorem 1.3 Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^d \), \( d = 2, 3 \). If the polarization tensor \( M(\Omega) \) of \( \Omega \) satisfies the equality in (1.4), then for any vector \( a \in \mathbb{R}^d \) the solution \( u \) to (1.1) is linear in \( \Omega \).

Thanks to Theorem 1.3 Theorem 1.1 is now an immediate consequence of the following theorem.

Theorem 1.4 Let \( \Omega \) be a simply connected bounded Lipschitz domain in \( \mathbb{R}^d \), \( d = 2, 3 \). The solution \( u \) to (1.4) is linear in \( \Omega \) for any vector \( a \) if and only if \( \Omega \) is an ellipse or ellipsoid.
It should be noted that only the three dimensional case in Theorem 1.4 is new. In two
dimensions Ru and Schiavone [52] proved a stronger theorem using conformal mappings: If
the gradient of the solution to (1.1) is constant for a single non-zero direction \( a \), then \( \Omega \) is an
ellipse. This is the anti-plane elasticity case of the strong Eshelby conjecture for elasticity,
which we now explain in the context of two and three dimensional elasticity. Consider an
elastic inclusion \( \Omega \), whose Lamé parameters are \( \lambda, \mu \), embedded in a medium in \( \mathbb{R}^d \) with
Lamé parameters \( \tilde{\lambda}, \tilde{\mu} \). In [14], Eshelby showed that if \( \Omega \) is an ellipse or an ellipsoid, then
for any given uniform loading the elastic field inside \( \Omega \) is uniform, and in [15] he conjectured
that ellipses and ellipsoids are the only domains with this property, which is called Eshelby’s
uniformity property.

In order to explain Eshelby’s conjecture more precisely, let \( C = (C_{ijkl}) \) be the elasticity

tensor of the inclusion-matrix composite, namely,

\[
C_{ijkl} := \left( \lambda \chi(\mathbb{R}^d \setminus \overline{\Omega}) + \tilde{\lambda} \chi(\Omega) \right) \delta_{ij} \delta_{kl} + \left( \mu \chi(\mathbb{R}^d \setminus \overline{\Omega}) + \tilde{\mu} \chi(\Omega) \right) (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}).
\]

(1.7)

It is always assumed that

\[
\mu > 0, \quad d\lambda + 2\mu > 0, \quad \tilde{\mu} > 0 \quad \text{and} \quad d\tilde{\lambda} + 2\tilde{\mu} > 0,
\]

(1.8)

and for technical reasons we also assume that

\[
(\lambda - \tilde{\lambda})(\mu - \tilde{\mu}) \geq 0,
\]

(1.9)

which means \( \lambda - \tilde{\lambda} \) and \( \mu - \tilde{\mu} \) have the same signs. For given constants \( d \times d \) matrix \( A \),

consider the following problem for the Lamé system of the linear elasticity:

\[
\begin{aligned}
\nabla \cdot \left( C(\nabla u + \nabla u^T) \right) &= 0 \quad \text{in } \mathbb{R}^d, \\
\n\nabla u(x) - Ax &= O(|x|^{1-d}) \quad \text{as } |x| \to \infty.
\end{aligned}
\]

(1.10)

If \( u \) is the solution to (1.10), then \( \nabla u \) represents the field perturbed due to the presence
of the inclusion \( \Omega \) under the uniform loading given by \( \nabla (Ax) \). The conductivity model (1.1) in
two dimensions can be regarded as the anti-plane elasticity model of (1.10). What we call
the strong Eshelby conjecture asserts that if the solution \( u \) to (1.10) for a single nonzero \( A \) is
linear inside \( \Omega \), then \( \Omega \) is an ellipse or an ellipsoid. What we call the weak Eshelby conjecture
states that if the solution \( u \) to (1.10) is linear inside \( \Omega \) for all \( A \), then \( \Omega \) is an ellipse or an
ellipsoid. Theorem 1.4 can be regarded as a solution to the weak Eshelby conjecture for the
conductivity model. We prove the weak Eshelby conjecture for elasticity. We only state the
theorem in three dimensions:

**Theorem 1.5 (Weak Eshelby’s conjecture in 3D)** Let \( \Omega \) be a simply connected bounded
Lipschitz domain in \( \mathbb{R}^3 \). The solution \( u \) to (1.10) is linear in \( \Omega \) for all \( A \) if and only if \( \Omega \) is
an ellipsoid.

The strong Eshelby conjecture in two dimensions was proved by Sendeckyj for elasticity
[56]. In this paper we give a proof of the Eshelby conjecture in two dimensions which is
completely different from that in [56]. The novelty of our proof is the use of the hodographic
transformation. The same approach enables us to construct multiple inclusions satisfying
Eshelby’s uniformity property.
Theorem 1.6 (Strong Eshelby conjecture in 2D) Suppose that \( d = 2 \). Let \( \Omega \) be a simply connected bounded domain with the Lipschitz boundary. If the solution \( u \) to (1.10) is linear inside \( \Omega \) for a single nonzero \( A \), then \( \Omega \) must be an ellipse.

We also give an alternative proof of Ru and Schiavone’s theorem for the conductivity model:

Theorem 1.7 Suppose that \( d = 2 \). Let \( \Omega \) be a simply connected bounded Lipschitz domain. If the solution \( u \) to (1.1) for a single vector \( a \neq 0 \) is linear in \( \Omega \), then \( \Omega \) must be an ellipse.

Our proof uses a hodographic transformation. These have been widely used to solve free boundary problems in various problems in mechanics and fluid dynamics, to name one, the Saffman-Taylor fingering problem [54, 53, 8]. It is also appropriate to mention the Vigdergauz microstructure. Vigdergauz considered a periodic array of inclusions occupying a given volume fraction and found the inclusion shape with minimal overall elastic energy [58, 59] under certain loadings. He used the fact that the shape would be optimal for one of these loadings if the field inside the inclusion was uniform and hydrostatic. See [10] for a somewhat simpler treatment. Grabovsky and Kohn proved in the latter paper that the low volume fraction limit of the Vigdergauz microstructure is an ellipse. Thus one can expect that some variant of Vigdergauz’s complex analytic method might lead us to the proof of the strong Eshelby conjecture in two-dimensions. We regard the hodographic transformation as such a variant (see also section 23.9 of [37]). Incidentally, we remark that for a dilute periodic array of holes under shear loadings the ellipse is not the optimal energy minimizing shape [10].

It should be emphasized that the conjectures of Pólya-Szegő and Eshelby are true only for simply connected domains. In a forthcoming paper [23], we construct a family of structures with two inclusions in which fields are uniform, and their PT satisfies the lower equality in (1.4).

This paper is organized as follows. In section 2, we review basic facts about single layer potentials for harmonic equations and for linear elasticity. In section 3, we prove Theorem 1.1. We then prove Theorem 1.5 in section 3, and Theorem 1.6 and 1.7 in section 4.

2 Single layer potentials

We review some basic facts about single layer potentials for the harmonic equation and for isotropic elasticity. We will consider them only in three dimensions. For details of the materials presented here, we refer readers to [5].

The single layer potential for the harmonic equation on a bounded Lipschitz domain \( \Omega \) in \( \mathbb{R}^3 \) is defined to be

\[
S_{\Omega}[\phi](x) := \frac{1}{4\pi} \int_{\partial \Omega} \frac{\phi(y)}{|x-y|} d\sigma(y), \quad x \in \mathbb{R}^3,
\]

where \( \phi \) is a square integrable function on \( \partial \Omega \) and \( d\sigma(y) \) is the surface measure. Thus \( S_{\Omega}[\phi] \) is a function on \( \mathbb{R}^3 \) and \( S_{\Omega}(\phi)(x) \) denotes its value at \( x \). The following boundary behavior of the normal derivative of the single layer potential is well-known:

\[
\frac{\partial}{\partial n} S_{\Omega}[\phi] \bigg|_\pm (x) = \left( \pm \frac{1}{2} I + \kappa_{\Omega} \right) [\phi](x) \quad \text{a.e. } x \in \partial \Omega,
\]

(2.2)
where \( n = (n_1, n_2, n_3) \) is the outward unit normal to \( \Omega \), \( \frac{\partial}{\partial n} \) denotes the normal derivative, and \( K^*_\Omega \) is defined by
\[
K^*_\Omega(\phi)(x) = \frac{1}{4\pi} \text{p.v.} \int_{\partial \Omega} \frac{\langle x - y, n(x) \rangle}{|x - y|^3} \phi(y) \, d\sigma(y). \tag{2.3}
\]
Here the subscripts + and − denote the limits from the outside and inside \( \Omega \), respectively, and p.v. denotes the Cauchy principal value. See [16] for a proof of (2.2) when \( \partial \Omega \) is smooth and [57] when \( \partial \Omega \) is Lipschitz. It is known [26] (see also [5, Section 2.47]) that the solution \( u \) to (1.1) is given by
\[
u(x) = a \cdot x + S_\Omega(\phi)(x), \quad x \in \mathbb{R}^3, \tag{2.4}
\]
where
\[
\phi = \left( -\frac{k + 1}{2(k - 1)} I - K^*_\Omega \right)^{-1} [a \cdot n] \quad \text{on} \quad \partial \Omega. \tag{2.5}
\]
Furthermore, we have
\[
\phi = (k - 1) \frac{\partial u}{\partial n} \bigg|_{-}. \tag{2.6}
\]
The invertibility of the operator \( \frac{k + 1}{2(k - 1)} I - K^*_\Omega \) on \( L^2(\partial \Omega) \) is established in [12].

It is worthwhile to note that in view of the jump relation (2.2), the usage of the single layer potential is natural since (1.1) when \( d = 3 \) is equivalent to the following problem:
\[
\begin{aligned}
\Delta u &= 0 \quad \text{in} \quad \Omega \cup (\mathbb{R}^3 \setminus \Omega), \\
\left. u \right|_+ &= \left. u \right|_- \quad \text{on} \quad \partial \Omega, \\
\left. \frac{\partial u}{\partial n} \right|_+ &= k \left. \frac{\partial u}{\partial n} \right|_- \quad \text{on} \quad \partial \Omega, \\
\left. u \right|_+ - a \cdot x &= O(|x|^{-2}) \quad \text{as} \quad |x| \to \infty,
\end{aligned} \tag{2.7}
\]

We now review a similar representation formula for isotropic elasticity. The elastostatic system corresponding to the Lamé constants \( \lambda, \mu \) is defined by
\[
\mathcal{L}_{\lambda, \mu} u := \mu \Delta u + (\lambda + \mu) \nabla (\nabla \cdot u). \tag{2.8}
\]
The corresponding conormal derivative \( \partial u / \partial \nu \) on \( \partial \Omega \) is defined to be
\[
\frac{\partial u}{\partial \nu} := \lambda (\nabla \cdot u) n + \mu (\nabla u + \nabla u^T) n \quad \text{on} \quad \partial \Omega, \tag{2.9}
\]
where the superscript \( T \) denotes the transpose of a matrix. The Kelvin matrix \( \Gamma = (\Gamma_{ij})^{d}_{i,j=1} \) of the fundamental solution to the Lamé system \( \mathcal{L}_{\lambda, \mu} \) in three dimensions is given by
\[
\Gamma_{ij}(x) := -\frac{\alpha_1 \delta_{ij}}{4\pi |x|} - \frac{\alpha_2 x_i x_j}{4\pi |x|^{3}}, \quad x \neq 0, \tag{2.10}
\]
where
\[
\alpha_1 = \frac{1}{2} \left( \frac{1}{\mu} + \frac{1}{2\mu + \lambda} \right) \quad \text{and} \quad \alpha_2 = \frac{1}{2} \left( \frac{1}{\mu} - \frac{1}{2\mu + \lambda} \right). \tag{2.11}
\]
The single layer potentials of the density function \( \varphi \) on \( \partial \Omega \) associated with the Lamé parameters \((\lambda, \mu)\) are defined by

\[
\mathcal{S}_\Omega[\varphi](x) := \int_{\partial \Omega} \Gamma(x - y)\varphi(y) \, d\sigma(y), \quad x \in \mathbb{R}^3.
\] (2.12)

The single layer potential enjoys the following jump relation:

\[
\frac{\partial}{\partial \nu} \mathcal{S}_\Omega[\varphi] \bigg|_+ - \frac{\partial}{\partial \nu} \mathcal{S}_\Omega[\varphi] \bigg|_- = \varphi \quad \text{on} \ \partial \Omega,
\] (2.13)

where \( \partial / \partial \nu \) denotes the conormal derivative defined in (2.9).

Let \( \Psi \) be the vector space of all linear solutions of the equation \( \mathcal{L}_{\lambda, \mu} \mathbf{u} = 0 \) and \( \partial \mathbf{u} / \partial \nu = 0 \) on \( \partial \Omega \), or alternatively,

\[
\Psi := \left\{ \psi : \partial_i \psi_j + \partial_j \psi_i = 0, \quad 1 \leq i, j \leq d \right\}.
\] (2.14)

Here the \( \psi_i \) for \( i = 1, \ldots, d \), denote the components of \( \psi \). Define

\[
L^2_\Psi(\partial \Omega) := \left\{ \mathbf{f} \in L^2(\partial \Omega) : \int_{\partial \Omega} \mathbf{f} \cdot \psi \, d\sigma = 0 \text{ for all } \psi \in \Psi \right\}
\] (2.15)

which is a subspace of codimension 6 in \( L^2(\partial \Omega) \). Note that (1.10) is equivalent to the following problem:

\[
\begin{align*}
\mathcal{L}_{\lambda, \mu} \mathbf{u} &= 0 \quad \text{in} \ \mathbb{R}^3 \setminus \overline{\Omega} , \\
\mathcal{L}_{\lambda, \mu} \mathbf{u} &= 0 \quad \text{in} \ \Omega , \\
\mathbf{u} &= \mathbf{u} \quad \text{on} \ \partial \Omega , \\
\frac{\partial \mathbf{u}}{\partial \nu} &= \frac{\partial \mathbf{u}}{\partial \nu} \quad \text{on} \ \partial \Omega , \\
\mathbf{u} - A \mathbf{x} &= O(|\mathbf{x}|^{-2}) \quad \text{as} \ |\mathbf{x}| \to \infty,
\end{align*}
\] (2.16)

where \( \mathcal{L}_{\lambda, \mu} \) and \( \tilde{\nu} \) are the Lamé operator and the conormal derivative with respect to the Lamé constants \( (\lambda, \mu) \) of the inclusion. We denote by \( \mathcal{S}_\Omega \) and \( \tilde{\mathcal{S}}_\Omega \) the single layer potentials on \( \partial \Omega \) corresponding to the Lamé constants \( (\lambda, \mu) \) and \( (\tilde{\lambda}, \tilde{\mu}) \) of the matrix and inclusion, respectively. We then have the following representation formula for the solution \( \mathbf{u} \) to (1.10) or equivalently (2.10): There exists a unique pair \( (\varphi, \psi) \in L^2(\partial \Omega) \times L^2_\Psi(\partial \Omega) \) such that the solution \( \mathbf{u} \) of (2.16) is represented by

\[
\mathbf{u}(x) = \begin{cases} 
\frac{\tilde{\mathcal{S}}_\Omega[\varphi]}{A} \mathbf{x} + \mathcal{S}_\Omega[\psi](x), & x \in \mathbb{R}^3 \setminus \overline{\Omega}, \\
\mathcal{S}_\Omega[\varphi](x), & x \in \Omega,
\end{cases}
\] (2.17)

where the pair \( (\varphi, \psi) \) is the unique solution in \( L^2(\partial \Omega) \times L^2_\Psi(\partial \Omega) \) of

\[
\begin{align*}
\mathcal{S}_\Omega[\varphi] \bigg|_- - \tilde{\mathcal{S}}_\Omega[\psi] \bigg|_+ &= (A \mathbf{x}) |_{\partial \Omega} \quad \text{on} \ \partial \Omega , \\
\frac{\partial \tilde{\mathcal{S}}_\Omega[\varphi]}{\partial \tilde{\nu}} \bigg|_- - \frac{\partial \mathcal{S}_\Omega[\psi]}{\partial \nu} \bigg|_+ &= \frac{\partial (A \mathbf{x})}{\partial \nu} \bigg|_{\partial \Omega} \quad \text{on} \ \partial \Omega.
\end{align*}
\] (2.18)

The unique solvability of the integral equation (2.18) was proved in [13] under the condition (1.19), which is why we assumed this condition.
3 Proof of the Pólya-Szegö conjecture

We now prove Theorem 1.4. Theorem 1.1 follows as an immediate consequence. We consider only the three dimensional case because the same proof works for the two dimensional case. We begin with the following lemma.

Lemma 3.1 Suppose that $d = 3$ and that the solution $u$ to (1.1) is linear in $\Omega$ for any applied field $a \in \mathbb{R}^3$. Define a linear transformation $\Lambda$ on $\mathbb{R}^3$ by $\Lambda(a) = \nabla u_a|_{\Omega}$ where $u_a$ is the solution to (1.1). Then $\Lambda$ is one-to-one and onto.

Proof. Since the equation in (1.1) is linear, that $\Lambda$ is linear is obvious. Thus it suffices to show that $\Lambda$ is one-to-one. Suppose that $\Lambda(a) = 0$ for some $a \in \mathbb{R}^3$. Then $\nabla u_a = 0$ in $\Omega$.

Define $v(x) = u_a(x) - a \cdot x$ for $x \in \mathbb{R}^3$. Then $v$ is the solution to

\[ \begin{align*}
\nabla \cdot (1 + (k-1)\chi(\Omega)) \nabla v &= 0 \quad \text{in } \mathbb{R}^d, \\
v(x) &= O(|x|^{1-d}) \quad \text{as } |x| \to \infty.
\end{align*} \]

(3.1)

By the uniqueness of the solution to (3.1), $v \equiv 0$ and hence $a = 0$. This completes the proof. $\square$

Lemma 3.1 can be interpreted as follows: For any vector $b \in \mathbb{R}^3$ there exists $a$ such that the solution $u$ to (1.1) satisfies

\[ u(x) = b \cdot x + c, \quad x \in \Omega \]

for some constant $c$. It then follows from (3.1) and (3.2) that

\[ (b - a) \cdot x + c = (k-1)\mathcal{S}_\Omega[b \cdot n](x), \quad x \in \Omega. \]

(3.3)

In other words, $\mathcal{S}_\Omega[b \cdot n]$ is linear in $\Omega$ for any $b \in \mathbb{R}^3$. In particular, we have

\[ \mathcal{S}_\Omega[n_j](x) = \text{linear in } \Omega, \quad j = 1, 2, 3. \]

(3.4)

By reversing the arguments one can see that (3.4) is equivalent to the solution $u$ to (1.1) being linear in $\Omega$ for any vector $a$.

Now although we have only defined the action of the functional $\mathcal{S}_\Omega$ on scalar functions, the obvious generalization of (2.21) defines its action on vector valued functions. In particular we have

\[ \mathcal{S}_\Omega[n](x) = -\nabla \int_{\Omega} \frac{1}{4\pi|x-y|} dy, \quad x \in \Omega, \]

(3.5)

which can be seen using the divergence theorem. Thus we get from (3.3)

\[ \int_{\Omega} \frac{1}{4\pi|x-y|} dy = \text{a quadratic polynomial}, \quad x \in \Omega. \]

(3.6)

Since the property (3.6) is independent of the conductivity ratio $k \neq 1$, we have an interesting consequence.

Corollary 3.2 For $j = 1, 2, 3$ and $0 < k \neq 1 < \infty$, let $u_j^{(k)}$ be the solution to (1.1) with $a = e_j$ and conductivity ratio $k$ where $\{e_1, e_2, e_3\}$ is the standard basis for $\mathbb{R}^3$. If $\nabla u_j^{(k)}$ is constant in $\Omega$ for $j = 1, 2, 3$ and for some $k$, then $\nabla u_j^{(k)}$ is constant in $\Omega$ for all $k$. 

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Suppose that (3.6) holds, i.e., there is a symmetric matrix $A$, a constant vector $b$, and a constant $C$ such that
\[
\frac{1}{4\pi} \int_{\Omega} \frac{1}{|x-y|} \, dy = x \cdot Ax + b \cdot x + C, \quad x \in \Omega.
\] (3.7)

This identity has an interesting physical interpretation. If we think of $\Omega$ as a body of constant density in free space then the left hand side is (to within a proportionality factor) just the Newtonian gravitational potential at $x$, and the identity (3.7) then says that the gradient of this potential, which is the gravitational field, depends linearly on $x$ within the body. It has been shown by Dive [11] and Nikliborc [43] that ellipsoids are the only bodies which have this property. Before explaining their proof, let us establish some elementary results.

After a unitary transformation if necessary, we may assume $A$ is diagonal and
\[
\frac{1}{4\pi} \int_{\Omega} \frac{1}{|x-y|} \, dy = \frac{1}{2} \sum_{j=1}^{3} a_j x_j^2 + b \cdot x + C, \quad x \in \Omega,
\] (3.8)

for some constants $a_j$ and $C$, and a constant vector $b$. We claim that each $a_j$ is positive. In fact, it follows from (3.5) that
\[
S_{\Omega} [n_j] (x) = -a_j x_j - b_j, \quad x \in \Omega, \quad j = 1, 2, 3.
\] (3.9)

Therefore, by (2.2), we have
\[
\left( -\frac{1}{2} I + \mathcal{K}_{\Omega}^* \right) [n_j] = -a_j n_j \quad \text{on} \ \partial \Omega,
\] (3.10)

and hence
\[
\left( \frac{k+1}{2(k-1)} I - \mathcal{K}_{\Omega}^* \right)^{-1} [n_j] = \frac{k-1}{1 + (k-1)a_j} n_j \quad \text{on} \ \partial \Omega.
\] (3.11)

Since $\left( -\frac{1}{2} I + \mathcal{K}_{\Omega}^* \right)$ is invertible on $L^2_0(\partial \Omega) := \{ f \in L^2(\partial \Omega) : \int_{\partial \Omega} f = 0 \}$ as was proved in [57], one can see from (3.10) that $a_j \neq 0$. Moreover, the polarization tensor $M = (M_{ij})$ associated with $\Omega$ is given by
\[
M_{ij} = \int_{\partial \Omega} y_j \left( \frac{k+1}{2(k-1)} I - \mathcal{K}_{\Omega}^* \right)^{-1} [n_i](y) \, d\sigma(y)
\] (3.12)
as was proved in [4]. Therefore, we have
\[
M_{ij} = \delta_{ij} |\Omega| \frac{k-1}{1 + (k-1)a_j}, \quad i, j = 1, 2, 3.
\] (3.13)

Since the polarization $M$ is positive definite if $k > 1$ and negative definite if $k < 1$ regardless of $k$ (see [5]), we have $a_j > 0$, $j = 1, 2, 3$.

We can now make a complete square out of (3.8) and make a translation if necessary to conclude that
\[
\frac{1}{4\pi} \int_{\Omega} \frac{1}{|x-y|} \, dy = \frac{1}{2} \sum_{j=1}^{3} a_j x_j^2 + C, \quad x \in \Omega,
\] (3.14)

with $a_j > 0$.

The following theorem was proved by Dive [11] and Nikliborc [43] for a $C^1$ domain. The same theorem for Lipschitz domains can be proved by a slight variation of their arguments.
Theorem 3.3 Let \( \Omega \) be a bounded domain with a Lipschitz boundary. The relation (3.14) holds if and only if \( \Omega \) is an ellipsoid of the form
\[
\frac{x_1^2}{c_1^2} + \frac{x_2^2}{c_2^2} + \frac{x_3^2}{c_3^2} \leq 1.
\]
(3.15)

Proof. We briefly sketch the proof. Note that if \( \Omega \) is an ellipsoid of the form (3.15), then (3.14) holds with
\[
a_j = \frac{c_1 c_2 c_3}{2} \int_0^\infty \frac{ds}{(c_j^2 + s)\sqrt{(c_1^2 + s)(c_2^2 + s)(c_3^2 + s)}},
\]
(3.16)
for \( j = 1, 2, 3 \). To prove the converse, Suppose (3.14) holds with \( a_j > 0, j = 1, 2, 3 \). Then there is a unique triple \( c_1, c_2, c_3 \) satisfying the relation (3.16). Existence of such a triple was proved in [11, 43]. Let \( E \) be the ellipsoid given by (3.15). Then, defining for any region \( \Upsilon \)
\[
N_T(x) := \frac{1}{4\pi} \int_T \frac{1}{|x - y|} \, dy,
\]
(3.17)
we have
\[
N_E(x) = \frac{1}{2} \sum_{j=1}^3 a_j x_j^2 + C_1, \quad x \in E,
\]
(3.18)
for some constant \( C_1 \). For \( t > 0 \), let \( E_t := \{tx|x \in E\} \). Then by simple scaling one can see that
\[
N_{E_t}(x) = \frac{1}{2} \sum_{j=1}^3 a_j x_j^2 + C_t, \quad x \in E_t,
\]
(3.19)
for some constant \( C_t \) depending only on \( t \). Let \( t_0 \) be the smallest number such that \( \Omega \subset E_t \) for all \( t \geq t_0 \). Then there is a point \( Q \) which is contained in \( \partial E_{t_0} \cap \partial \Omega \), and
\[
N_{E_{t_0} \setminus \Omega}(x) = N_{E_{t_0}}(x) - N_{\Omega}(x) = \text{constant}, \quad x \in \Omega.
\]
(3.20)

Let \( n(Q) \) be the unit outward normal to \( E_{t_0} \) at \( Q \). Since \( E_{t_0} \setminus \Omega \) lies in one side of the tangent plane to \( E_{t_0} \) at \( Q \), we have
\[
\nabla N_{E_{t_0} \setminus \Omega}(Q) \cdot n(Q) = -\frac{1}{4\pi} \int_{E_{t_0} \setminus \Omega} \frac{\langle Q - y, n(Q) \rangle}{|Q - y|^3} \, dy < 0,
\]
provided that \( E_{t_0} \setminus \Omega \) is not empty.

If \( \partial \Omega \) is \( C^1 \), then \( n(Q) \) is also normal to \( \Omega \) at \( Q \) and the normal line goes through \( \Omega \). Here and in what follows, “goes through \( \Omega \)” means that there is \( s_0 > 0 \) such that the line segment \( \{Q - sn(Q)|0 < s < s_0\} \subset \Omega \). But by (3.20) we get \( \nabla N_{E_{t_0} \setminus \Omega}(Q) \cdot n(Q) = 0 \), and hence we can conclude that \( E_{t_0} \setminus \Omega = 0 \) by (3.21). This is the argument in [11, 43].

If \( \partial \Omega \) is only Lipschitz, then we can argue as follows. By (3.21), we have \( \nabla N_{E_{t_0} \setminus \Omega}(Q) \neq 0 \) provided that \( E_{t_0} \setminus \Omega \) is not empty. Thus for any unit vector \( v_0 \) and for any open neighborhood \( V \) of \( v_0 \) in \( S^2 \), the unit sphere, there is \( v \in V \) such that
\[
\nabla N_{E_{t_0} \setminus \Omega}(Q) \cdot v \neq 0.
\]
(3.22)
Choose \( v_0 \) so that the line in the direction \( v_0 \) passing through \( Q \) goes through \( \Omega \). Since \( \partial \Omega \) is Lipschitz, there is a neighborhood \( V \) in \( S^2 \) of \( v_0 \) such that any line in the direction of the vector in \( V \) passing through \( Q \) goes through \( \Omega \). Then for some \( v \in V \) \( (3.22) \) holds. But by \( (3.20) \) we get contradiction and hence \( E_{v_0} \setminus \Omega = \emptyset \). This completes the proof of Theorem 3.3 and hence Theorem 1.4.

4 Proof of the weak Eshelby conjecture

In this section we prove Theorem 1.5, the weak Eshelby conjecture for elasticity. We only will prove the three dimensional case because again the same proof works for the two dimensional case.

In fact there is a close link between the weak Eshelby conjecture for elasticity and the weak Eshelby conjecture for conductivity (which have just proved). In composite microstructures of two isotropic phases it is known that if for some periodic microgeometry the elastic field is uniform and hydrostatic (i.e. proportional to the identity) in phase one then that microstructure necessarily attains the “bulk modulus type trace bound”, equation (6.35) in [36], or the opposite inequality, depending on the moduli of the phases. Then as a consequence of an argument of Grabovsky [20] (see also section 25.6 of [37]) a solution to the conductivity problem (for any direction of the applied field) can be generated from that elasticity field, and the electric field in phase one is also necessarily uniform. This argument strongly suggests that if Eshelby’s uniformity property holds for all applied loadings, then there will be one loading for which the field in the inclusion is hydrostatic, and from which one can generate solutions to the conductivity problem. As a consequence the electric field in the inclusion will be uniform for all applied uniform electric fields, and hence the inclusion shape must be ellipsoidal.

Let us see this directly. Suppose that \( d = 3 \) and that the solution \( u \) to (1.10) is linear in \( \Omega \) for any \( 3 \times 3 \) matrix \( A \). Then in the representation formula (2.17), \( \tilde{S}_{\Omega} \phi \) is linear in \( \Omega \), say

\[
\tilde{S}_{\Omega}[\phi](x) = Bx + b, \quad x \in \Omega, \tag{4.1}
\]

for some \( 3 \times 3 \) matrix \( B \) and vector \( b \). It then follows from the integral equation (2.18) that

\[
\begin{align*}
\left[ \tilde{S}_{\Omega}[\psi] \right]_+ &= Bx - Ax + b & \text{on } \partial \Omega, \\
\left[ \frac{\partial}{\partial \nu} \tilde{S}_{\Omega}[\psi] \right]_+ &= \frac{\partial(Bx)}{\partial \nu} - \frac{\partial(Ax)}{\partial \nu} & \text{on } \partial \Omega. \tag{4.2}
\end{align*}
\]

Since \( L_{\lambda,\mu} \tilde{S}_{\Omega}[\psi] = L_{\lambda,\mu}(Bx - Ax + b) = 0 \) in \( \Omega \), the first relation in (4.2) implies that

\[
\tilde{S}_{\Omega}[\psi](x) = Bx - Ax + b, \quad x \in \Omega. \tag{4.3}
\]

It then follows from the jump relation (2.13)

\[
\psi = \left[ \frac{\partial}{\partial \nu} \tilde{S}_{\Omega}[\psi] \right]_+ - \left[ \frac{\partial}{\partial \nu} \tilde{S}_{\Omega}[\psi] \right]_- = \frac{\partial(Bx)}{\partial \nu} - \frac{\partial(Bx)}{\partial \nu} \quad \text{on } \partial \Omega. \tag{4.4}
\]

Substituting (4.3) into the first identity in (4.2), we arrive at

\[
\tilde{S}_{\Omega} \left[ \frac{\partial(Bx)}{\partial \nu} - \frac{\partial(Bx)}{\partial \nu} \right] (x) = \text{linear}, \quad x \in \Omega. \tag{4.5}
\]
The following theorem can be proved in the exactly same manner as Lemma 4.1.

**Lemma 4.1** Suppose that \( d = 3 \) and that the solution \( u \) to (1.10) is linear in \( \Omega \) for any \( 3 \times 3 \) matrix \( A \). Define a linear transformation \( \Lambda \) on \( V \), the vector space of all \( 3 \times 3 \) matrices, by \( \Lambda(A) = \nabla u_A |_{\Omega} \) where \( u_A \) is the solution to (1.10). Then \( \Lambda \) is one-to-one and onto.

Since Lemma 4.1 means that for any \( 3 \times 3 \) matrix \( B \), there is \( A \) such that the solution \( u \) to (1.10) satisfies \( \nabla u = B \) in \( \Omega \), we can conclude that (4.8) holds for all \( 3 \times 3 \) matrix \( B \).

Let us take a hydrostatic field \( B \), i.e. with \( B_{pq} = \delta_{pq} \) the identity matrix, in (4.5). Then we have \( Bx = x \). We claim that

\[
S_{ij} \left( \frac{\partial u}{\partial \nu} \right)(x) = -\frac{2\mu + 3\lambda}{2\mu + \lambda} S_{ij}[n_j](x), \quad x \in \Omega, \quad j = 1, 2, 3. \tag{4.6}
\]

where the subscript \( j \) indicates the \( j \)-th component. Similarly, we have

\[
S_{ij} \left( \frac{\partial u}{\partial \nu} - \frac{\partial u}{\partial x} \right)(x) = \frac{2\mu + 3\lambda - (2\overline{\mu} + 3\overline{\lambda})}{2\mu + \lambda} S_{ij}[n_j](x), \quad x \in \Omega, \quad j = 1, 2, 3. \tag{4.7}
\]

We will give proofs of (4.6) and (4.7) at the end of this section.

Because of (4.6) and (4.7), we get

\[
S_{ij} \left( \frac{\partial u}{\partial \nu} - \frac{\partial u}{\partial x} \right)(x) = \frac{(2\mu + 3\lambda) - (2\overline{\mu} + 3\overline{\lambda})}{2\mu + \lambda} S_{ij}[n_j](x), \quad x \in \Omega. \tag{4.8}
\]

Notice that the constant on the right-hand side of (4.8) is not zero unless \( 2\mu + 3\lambda - (2\overline{\mu} + 3\overline{\lambda}) \) which is excluded by our assumption (1.9). Since the left-hand side of (4.8) is linear in \( \Omega \), we deduce that \( S_{ij}[n_j] \) is linear in \( \Omega \) for \( j = 1, 2, 3 \). We now conclude from the result in the previous section that \( \Omega \) is an ellipsoid, and the proof is complete.

Let us now prove (4.8). Since

\[
\frac{\partial u}{\partial \nu} = \Lambda \nabla \cdot (x)n + \mu(\nabla(x) + \nabla(x)^T)n = (2\mu + 3\lambda)n, \tag{4.9}
\]

where \( n = (n_1, n_2, n_3) \) is the unit outward normal, we get, for \( j = 1, 2, 3 \),

\[
S_{ij} \left( \frac{\partial u}{\partial \nu} \right)(x) = (2\mu + 3\lambda) \left( \int_{\partial \Omega} \Gamma(x - y)n(y) \right) = (2\mu + 3\lambda) \sum_{\ell=1}^{3} \int_{\partial \Omega} \Gamma_{j\ell}(x-y)n_{\ell}(y). \tag{4.10}
\]

Using (2.10) we have theorem,

\[
\sum_{\ell=1}^{3} \int_{\partial \Omega} \Gamma_{j\ell}(x-y)n_{\ell}(y)d\sigma(y) = -\frac{\alpha_1}{4\pi} \int_{\partial \Omega} n_j(y)d\sigma(y) - \frac{\alpha_2}{4\pi} \int_{\partial \Omega} (x_j - y_j) \frac{\left| x - y, n(y) \right|}{\left| x - y \right|^3} d\sigma(y). \tag{4.11}
\]

By Green’s theorem, we have

\[
\int_{\partial \Omega} (x_j - y_j) \left[ \frac{\partial}{\partial n_y} \frac{1}{\left| x - y \right|} \right] d\sigma(y) = -\int_{\partial \Omega} (\frac{\partial}{\partial n_y} (x_j - y_j)) \frac{1}{\left| x - y \right|} d\sigma(y)
\]

\[
= \int_{\Omega} (x_j - y_j) \left[ \Delta_y \frac{1}{\left| x - y \right|} \right] dy - \int_{\Omega} \left| \Delta_y (x_j - y_j) \right| \frac{1}{\left| x - y \right|} dy = 0, \tag{4.12}
\]

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and hence
\[ \int_{\partial \Omega} (x_j - y_j) \frac{(x - y, n(y))}{|x - y|^3} d\sigma(y) = - \int_{\partial \Omega} \frac{n_j(y)}{|x - y|} d\sigma(y). \] (4.13)

It then follows from (4.11) and the fact that \( \alpha_1 - \alpha_2 = \frac{1}{2\mu + \lambda} \) that
\[ \sum_{i=1}^{3} \int_{\partial \Omega} \Gamma_{ji}(x - y)n_i(y) d\sigma(y) = - \frac{\alpha_1 + \alpha_2}{4\pi} \int_{\partial \Omega} \frac{n_j(y)}{|x - y|} d\sigma(y) = - \frac{1}{(2\mu + \lambda)} S_\Omega[n_j](x). \] (4.14)

By substituting this back in (4.10) we get (4.6).

To prove (4.7), it suffices to note that
\[ \frac{\partial x}{\partial \nu} = \tilde{\lambda} \nabla \cdot (x)n + \tilde{\mu}(\nabla(x) + \nabla(x)^T)n = (2\tilde{\mu} + 3\tilde{\lambda})n. \] (4.15)

The proof is now complete.

5 Alternative Proofs of the strong Eshelby conjecture in 2D

In this section we give alternative proofs of Theorem 1.6 and 1.7. We first prove a lemma on univalence of the analytic functions which will be used to prove Theorem 1.6 and 1.7.

The proof of the following lemma relies on the level curve argument which was used in various contexts. We particularly mention the work of Alessandrini and Nesi [2] in which they showed the univalence of \( \sigma \)-harmonic mappings in the context of periodic composite materials. The two phase inclusion-matrix problem in the free space can be viewed as a low volume limit of the periodic case.

**Lemma 5.1** Let \( \Omega \) be a simply connected domain with Lipschitz boundary in \( \mathbb{C} \) and let \( f \) be a analytic function in \( \mathbb{C} \setminus \overline{\Omega} \) such that there are constants \( \alpha \neq 0 \) and \( \beta \) such that \( f(z) - (\alpha z + \beta) \to 0 \) as \( |z| \to \infty \). If \( f(z) = ix_2 \) for \( z = x_1 + ix_2 \in \partial \Omega \), then \( f \) is univalent in \( \mathbb{C} \setminus \overline{\Omega} \).

**Proof.** Let \( u \) and \( v \) be the real and imaginary parts of \( f \), respectively. Observe that \( f \) maps \( \mathbb{C} \setminus \overline{\Omega} \) onto \( \mathbb{C} \setminus [ic_1, ic_2] \) where \( c_1 = \min_{z \in \partial \Omega} v(z) \) and \( c_2 = \max_{z \in \partial \Omega} v(z) \), and \( \mathbb{C} \) is the Riemann sphere. Since \( \mathbb{C} \setminus \overline{\Omega} \) and \( \mathbb{C} \setminus [ic_1, ic_2] \) are simply connected, it suffices to show that \( f'(z) \neq 0 \) for \( z \in \mathbb{C} \setminus \overline{\Omega} \) to prove univalence of \( f \). It is obvious that \( f'(\infty) \neq 0 \). We will show that \( \nabla u(z) \neq 0 \) for any \( z \in \mathbb{C} \setminus \overline{\Omega} \).

Suppose that \( \nabla u(z_0) = 0 \) for some \( z_0 \in \mathbb{C} \setminus \overline{\Omega} \). Then there is an integer \( m \geq 2 \) such that \( u \) takes the form
\[ u(z) - u(z_0) = \sum_{n=-m}^{\infty} |z - z_0|^n (a_n \cos n\theta + b_n \sin n\theta), \quad z \text{ near } z_0, \] (5.1)

where \( z - z_0 = |z - z_0|e^{i\theta} \) and \( a_m^2 + b_m^2 \neq 0 \). Therefore, there are \( 2m \) branches of level curves \( C(z_0) := \{ z \mid u(z) = u(z_0) \} \) coming out of the point \( z_0 \). Since \( u(z) = ax_1 + bx_2 + c + O(|z|^{-1}) \) as \( |z| \to \infty \) for some real constants \( a, b, \) and \( c \), at most two of these \( 2m \) branches are...
unbounded and extend to the infinity. For the other branches of \( C(z_0) \) which are bounded, one of the following two occurs: (i) a branch intersects \( \partial \Omega \), (ii) a branch meets another branch and makes a loop.

Suppose that (ii) occurs. Then the loop may encircle \( \overline{\Omega} \) or be contained in \( \mathbb{C} \setminus \overline{\Omega} \). If it is contained in \( \mathbb{C} \setminus \overline{\Omega} \), then by the maximum principle, \( u \) is constant inside the loop and we have contradiction. If the loop encircles \( \overline{\Omega} \), then \( u \) is constant (zero) on \( \partial \Omega \) and on the loop (the constants may be different). Let \( \Theta \) be the annular region enclosed by the loop and \( \partial \Omega \). Then by the Cauchy-Riemann equations, we have \( \frac{\partial u}{\partial n} = 0 \) on \( \partial \Theta \), and hence \( v \) is constant in \( \Theta \), which leads us to a contradiction.

Now suppose that (ii) does not occur for any bounded branch. Let \( l_j, j = 1, \ldots, k \) \((k \geq 2)\) be bounded branches of \( C(z_0) \). Let \( z_j \) be the point where \( l_j \) intersects \( \partial \Omega \). Then there are two branches, say \( l_1, l_2 \), such that \( l_1, l_2 \), and the arc on \( \partial \Omega \) connecting \( z_1 \) and \( z_2 \) make the boundary of a connected region \( \Omega \). Since \( u = u(z_1) = u(z_2) = 0 \) on \( l_1 \) and \( l_2 \), it follows from the maximum principle that \( u = 0 \) in \( \Omega \), which contradicts to the assumption that \( \alpha \neq 0 \). This completes the proof. \( \blacksquare \)

We now provide the alternative proofs of Theorem 1.6 and 1.7.

**Proof of Theorem 1.6.** Let \( u \) be the solution to (1.1) for some \( a \neq 0 \), and suppose that \( u \) is linear in \( \Omega \). Put \( u_\Omega := u \mid \Omega \) and \( u_\mathbb{R} := u \mid \mathbb{R}^2 \setminus \overline{\Omega} \). It is proved in [22] that there are functions \( U_\Omega \) and \( U_\mathbb{R} \) analytic in \( \Omega \) and \( \mathbb{R}^2 \setminus \overline{\Omega} \), respectively, such that \( \text{Re} \ U_\Omega = u_\Omega \) and \( \text{Re} \ U_\mathbb{R} = u_\mathbb{R} \), and

\[
\frac{k+1}{2} \ U_\Omega - \frac{k-1}{2} \ U_\mathbb{R} = U_\mathbb{R} + ic \ 	ext{ on } \partial \Omega,
\]

for some real constant \( c \). Since \( u_\mathbb{R} \) is linear in \( \Omega \), there are complex numbers \( \delta \) and \( \gamma \) such that

\[
U_\mathbb{R}(z) = \delta z + \gamma.
\]

Define \( f_\Omega \) by

\[
f_\Omega(z) := U_\mathbb{R}(z) - \frac{k+1}{2} (\delta z + \gamma) + ic, \quad z \in \mathbb{C} \setminus \overline{\Omega}.
\]

Then by (5.2) we have

\[
f_\Omega(z) = - \frac{k-1}{2} (\delta z + \gamma), \quad z \in \partial \Omega.
\]

We now define \( \psi_\Omega \) by

\[
\psi_\Omega(z) = \frac{1}{\delta(k-1)} \left[ f_\Omega(z) + \frac{k-1}{2} \gamma \right] + \frac{1}{2} \delta, \quad z \in \mathbb{C} \setminus \overline{\Omega}.
\]

Then one can see from (5.3) that

\[
\psi_\Omega(z) = ix_2, \quad \text{for } z = x_1 + ix_2 \in \partial \Omega.
\]

Since \( u_\mathbb{R}(x) - a \cdot x = O(|x|^{-1}) \) as \( |x| \to \infty \), \( \psi_\Omega \) takes the form

\[
\psi_\Omega(z) = \alpha z + \beta + \varphi_\Omega(z),
\]

for some complex numbers \( \alpha \) and \( \beta \), and an analytic function \( \varphi_\Omega \) in \( \mathbb{C} \setminus \overline{\Omega} \) such that \( \varphi_\Omega(z) = O(|z|^{-1}) \) as \( |z| \to \infty \).
We claim that $\alpha \neq 0$. In fact, if $\alpha = 0$, then $h := \text{Re} \psi_\Omega$ is the solution to
\[
\begin{cases}
\Delta h = 0 & \text{in } \mathbb{R}^2 \setminus \overline{\Omega}, \\
h = 0 & \text{on } \partial \Omega, \\
h(x) - c = O(|x|^{-1}) & \text{as } |x| \to \infty,
\end{cases}
\tag{5.9}
\]
for some real constant $c$. But the maximum principle (with the point at infinity being regarded as a point on the Riemann sphere) implies $c = 0$ and hence $h = 0$ in $\mathbb{R}^2 \setminus \overline{\Omega}$. Thus we conclude that $\alpha \neq 0$.

It then follows from Lemma 5.1 that $\psi_\Omega$ is a univalent mapping from $\mathbb{C}^* \setminus \overline{\Omega}$ onto $\mathbb{C}^* \setminus [ic_1, ic_2]$ where $c_1 = \min_{z \in \partial \Omega} \text{Im} \psi_\Omega(z)$ and $c_2 = \max_{z \in \partial \Omega} \text{Im} \psi_\Omega(z)$. We may assume that $c_1 = -1$ and $c_2 = 1$ by scaling if necessary.

If $D$ is the unit disk, we can construct such a mapping $\psi_D$ explicitly. In fact, it is the Koebe function
\[\psi_D(z) = \frac{1}{2}(z - \frac{1}{z}), \quad |z| > 1.\tag{5.10}\]
Then $\psi_D$ is a univalent mapping from $\mathbb{C}^* \setminus \overline{D}$ onto $\mathbb{C}^* \setminus [-i, i]$.

Let
\[F(z) := \psi_\Omega^{-1} \circ \psi_D(z), \quad |z| > 1,\tag{5.11}\]
in which $\psi_\Omega^{-1}$ is the hodographic transformation. Then $F$ is a univalent mapping from $|z| > 1$ onto $\mathbb{C} \setminus \overline{D}$ and $F(\infty) = \infty$. Moreover, $F(z)$ behaves as a linear analytic function at infinity. Since $\mathbb{C}^* \setminus \overline{D}$ is simply connected, it follows from Caratheodory’s theorem [19, P.18] that $F$ extends to $\mathbb{C}^* \setminus D$ as a continuous function and $F$ is a homeomorphism from $\partial D$ onto $\partial \Omega$.

Observe that if $|z| = 1$, then $F(z) = \psi_\Omega^{-1}(ix_2)$, and hence $\text{Im} F(z) = x_2$. In other words, $\text{Im} F$ is a solution to
\[
\begin{cases}
\Delta u = 0 & \text{in } \mathbb{R}^2 \setminus \overline{D}, \\
u(x) = x_2 & \text{on } \partial D, \\
u(x) - (b_1x_1 + b_2x_2 + c) = O(|x|^{-1}) & \text{as } |x| \to \infty,
\end{cases}
\tag{5.12}
\]
where $b_1$, $b_2$, and $c$ are real constants.

We claim that $c = 0$ and
\[\text{Im} F(z) = b_1x_1 + b_2x_2 + \frac{-b_1x_1 + (1 - b_2)x_2}{|z|^2}, \quad |z| > 1.\tag{5.13}\]
In fact, if we define $u$ by
\[u(z) = \text{Im} F(z) - \left[b_1x_1 + b_2x_2 + \frac{-b_1x_1 + (1 - b_2)x_2}{|z|^2}\right], \quad |z| > 1,\tag{5.14}\]
then $u$ is a solution to (5.12) with $\Omega$ replaced by $D$, and hence $c = 0$ and $u \equiv 0$.

We now get from (5.13) that
\[F(z) = \gamma z + C + \frac{\beta}{z}, \quad |z| > 1,\tag{5.15}\]
for some constants $\gamma \neq 0$ and $\beta$. One can easily see that the image of the unit disk under $F$ is an ellipse. This completes the proof of Theorem 1.7. \qed
Proof of Theorem 1.6 Let \( \mathbf{u} \) be the solution to (1.10) for some constant \( a_{ij} \), not all zero, and assume that \( \mathbf{u} \) is linear in \( \Omega \). We first invoke the following complex representation of the solution to (1.10) from [39] (see also [5, Theorem 6.20]): Let \( \mathbf{u} = (u, v) \) be the solution of (1.10) for \( d = 2 \) and let \( \mathbf{u}_c := \mathbf{u}|_{\mathbb{C} \setminus \overline{\Omega}} \) and \( \mathbf{u}_i := \mathbf{u}|_{\Omega} \). Then there are unique functions \( \varphi_c \) and \( \psi_c \) analytic in \( \mathbb{C} \setminus \overline{\Omega} \) and \( \varphi_i \) and \( \psi_i \) analytic in \( \Omega \) such that

\[
2\mu(u_c + iv_c)(z) = \kappa \varphi_c(z) - z\varphi'_c(z) - \psi_c(z), \quad z \in \mathbb{C} \setminus \overline{\Omega};
\]
\[
2\mu(u_i + iv_i)(z) = \kappa \varphi_i(z) - z\varphi'_i(z) - \psi_i(z), \quad z \in \Omega,
\]

where

\[
\kappa = \frac{\lambda + 3\mu}{\lambda + \mu}, \quad \kappa = \frac{\overline{\lambda} + 3\overline{\mu}}{\lambda + \mu}.
\]

Moreover, the following holds on \( \partial \Omega \):

\[
\frac{1}{2\mu} \left( \kappa \varphi_c(z) - z\varphi'_c(z) - \psi_c(z) \right) = \frac{1}{2\mu} \left( \kappa \varphi_i(z) - z\varphi'_i(z) - \psi_i(z) \right),
\]
\[
\varphi_c(z) + z\varphi'_c(z) + \psi_c(z) = \varphi_i(z) + z\varphi'_i(z) + \psi_i(z) + c,
\]

where \( c \) is a constant. Equation (5.19) is the continuity of the displacement and (5.20) is the continuity of the traction.

It follows from (5.19) and (5.20) that

\[
(\kappa + 1)\varphi_c(z) = \left( \frac{\mu \kappa}{\mu} + 1 \right) \varphi_i(z) + \left( 1 - \frac{\mu}{\mu} \right) \left( z\varphi'_i(z) + \psi_i(z) \right) + c, \quad z \in \partial \Omega.
\]

Since \( \mathbf{u} \) is linear in \( \Omega \), \( \varphi_i \) and \( \psi_i \) are linear analytic functions in \( \Omega \) by the uniqueness of \( \varphi_i \) and \( \psi_i \). It then follows from (5.21) that there are complex numbers \( \alpha, \beta, C \) such that

\[
\varphi_c(z) = \alpha z - \beta \overline{z} + C, \quad z \in \partial \Omega.
\]

Suppose that \( \beta \neq 0 \). Define \( f_\Omega \) by

\[
f_\Omega(z) = \frac{1}{2\beta} \left[ \varphi_c(z) - \alpha z - C \right] + \frac{1}{2} \overline{z}, \quad z \in \mathbb{C} \setminus \overline{\Omega}.
\]

Then \( f_\Omega(z) = ix \) for \( z = x_1 + ix_2 \in \partial \Omega \). Following the same argument as in the proof of Theorem 1.7, we conclude that \( \Omega \) is an ellipse.

If \( \beta = 0 \), then \( \varphi_c \) can be extended to the whole of \( \mathbb{C} \) as an entire function. In fact, if we define \( \varphi_c(z) = \alpha z + C \) in \( \Omega \), then \( \varphi_c \) is analytic in \( \mathbb{C} \setminus \partial \Omega \) and continuous on \( \partial \Omega \), and hence is an entire function (It can be proved using Morera’s theorem [1] that if a function defined in an open set \( U \) is analytic in \( U \) minus a Lipschitz curve and is continuous in \( U \), then that function is analytic in \( U \)). But, since \( \mathbf{u}(x) - \sum_{ij} a_{ij} x_i e_j = O(|x|^{-1}) \) as \( |x| \to \infty \), \( \varphi_c \) takes the form \( \varphi_c(z) = \gamma z + f_c(z) \) for some constant \( \gamma \) and a analytic function \( f_c \) in \( \mathbb{C} \setminus \overline{\Omega} \) such that \( f_c(z) = O(|z|^{-1}) \) as \( |z| \to \infty \). So \( f_c \) extends as a bounded entire function and hence \( f_c \) is constant and the constant is 0. Thus

\[
\varphi_c(z) = \gamma z \quad \text{for} \quad z \in \mathbb{C} \setminus \overline{\Omega}.
\]
It then follows from (5.19) and (5.20) that
\[
\frac{\kappa - \bar{\gamma}}{2\mu} z - \frac{1}{2\mu} \psi_e(z) = \frac{1}{2\mu} \left( \bar{\kappa} \varphi_{{\tilde{}}}(z) - z \varphi'_{{\tilde{}}}(z) - \psi'(z) \right),
\]
(5.25)
\[
(1 + \bar{\gamma}) z + \psi_e(z) = \varphi_i(z) + z \varphi'_i(z) + \psi_i(z) + c,
\]
(5.26)
on \partial \Omega. Therefore there are complex constant \(\delta, \eta, C\) such that
\[
\psi_e(z) = \delta z - \eta z + C.
\]
(5.27)
If \(\eta = 0\), then by the same reasoning as to derive (5.24), one can see \(\psi'(z)\) is constant. Therefore, \(u_e + iv_e\) is linear, and hence \(u(x) - \sum_{ij} a_{ij} x_i e_j = 0\) for all \(x \in \mathbb{C} \setminus \Omega\). This is possible only when \(\Omega\) is an empty set. Thus, \(\eta \neq 0\). Now let
\[
g_{\Omega}(z) = \frac{1}{2\eta} [\psi_e(z) - \delta z - C] + \frac{1}{2} z, \quad z \in \mathbb{C} \setminus \Omega.
\]
(5.28)
Then \(g_{\Omega}(z) = ix_2\) for \(z = x_1 + ix_2 \in \partial \Omega\). Following the same argument as in the proof of Theorem 1.7, we conclude that \(\Omega\) is an ellipse. This completes the proof. \(\square\)

We finally mention that the strong Eshelby conjecture for three dimensions (except in the special case where the field in the inclusion is hydrostatic) has not been proven, not even for the conductivity case.

**Acknowledgement.** We would like to thank Victor Isakov for informing us of the existence of the papers [11] and [43], Dave Barnett and Peter Schiavone for drawing our attention to the papers [56] and [52], Yves Capdeboscq for stimulating discussion on the Polya-Szego conjecture, and Hyundae Lee for pointing out an error in earlier draft of this paper. H.K.is grateful for partial support by the grant KOSEF R01-2006-000-10002-0 and G.W.M is grateful for support from the National Science Foundation through grant DMS-0411035.

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