1. Introduction

Gravitational theories in which the Einstein–Hilbert (EH) term is supplemented with curvature-squared terms generically propagate negative energy (ghost) modes in addition to the desired spin-2 graviton modes. An exception is the ‘\((R + R^2)\)’ theory, where \(R\) is the curvature scalar, because this can be shown to be equivalent to a scalar field coupled to gravity, with a potential that provides a mass for the associated spin-zero particle in the Minkowski vacuum [1]; we shall refer to this as ‘scalar massive gravity’ (SMG). Recently, three of us showed that there...
is another exception in three spacetime dimensions [2]: ghosts are avoided if (i) the EH term appears with the ‘wrong-sign’ and (ii) the curvature-squared scalar is

\[ K = R_{\mu\nu} R^{\mu\nu} - \frac{3}{8} R^2, \]

where \( R_{\mu\nu} \) is the Ricci tensor. In its Minkowski vacuum, this ‘new massive gravity’ (NMG) model propagates, unitarily, two massive modes of helicities \( \pm 2 \). Unitarity has since been confirmed in [3, 4]. A more general model obtained by adding a Lorentz Chern–Simons (LCS) term was also considered in [2], and this model propagates the two spin-2 modes with different masses; by taking one of the two masses to infinity one gets the ‘topologically massive gravity’ (TMG) of [5].

This ‘general massive gravity’ (GMG) model has an obvious extension to include a cosmological constant, and this ‘cosmological’ GMG was investigated briefly in [2], allowing for either sign of the EH term as in studies of cosmological TMG [6–9]. In a subsequent work [10] the issue of unitarity and stability in de Sitter (dS) and anti-de Sitter (adS) vacua was considered in detail for the ‘cosmological’ NMG model. Other investigations of this model include [11]. Of particular interest are adS vacua since these could be associated with potentially novel 2D conformal field theories (CFTs) on the adS boundary. A major result of [10] was the finding that the central charge of this boundary CFT field theory is negative whenever the bulk theory is unitary, and vice-versa, with the exception of one case in which the bulk gravitons are absent and the central charge vanishes. Essentially the same difficulty arises in cosmological TMG; in this context, the ‘chiral gravity’ program initiated in [6] may yield a resolution but this remains unclear. One motivation for the study of 3D supergravity models with curvature-squared terms is that supersymmetric adS vacua may be ‘better behaved’ than generic adS vacua.

The \( N=1 \) supergravity extension of the NMG model was already considered briefly in [2], as was the more general model with generic curvature-squared-term of the form \((aK + bR^2)\). The off-shell supermultiplet containing the metric (actually dreibein) and gravitino field also contains an ‘auxiliary’ field \( S \) [12] which really is auxiliary when \( b = 0 \), in the sense that its equation of motion is algebraic. As noted in [2], the fully nonlinear supergravity theory must contain either an \( S^4 \) or an \( S^2R \) term, or both, and the (non-zero) constant value of \( S \) in any adS vacuum depends on the coefficients of these terms. Thus, we expect a cubic equation for \( S \) with \( R \)-dependent coefficients. It is not difficult to see that \( S = 0 \) is necessarily a solution in the absence of a cosmological term in the action, and this is sufficient to deduce that the Minkowski vacuum of the GMG model is supersymmetric. In fact, the Minkowski vacuum with \( S = 0 \) is supersymmetric quite generally, so linearization about this vacuum of any of the ‘\((aK + bR^2)\)’ models must yield a theory in which all modes (particles in the quantum theory) form supermultiplets. Any massive particles must have a definite helicity, and it was shown in [2] (by adapting earlier results of [13]) that super-GMG propagates one supermultiplet of helicities \( \left( 2, \frac{3}{2} \right) \) and another of helicities \( \left( -2, -\frac{3}{2} \right) \), generically with different masses.

In this paper we construct, in detail, the off-shell supersymmetric \( N=1 \) 3D supergravity model with both generic curvature-squared terms and cosmological constant. To be specific, we construct the 3D supergravity theory with action of the form

\[ I = \frac{1}{\kappa^2} \int d^3 x \left\{ e \left[ ML_C + \sigma L_{\text{EH}} + \frac{1}{m^2} L_K + \frac{1}{8\tilde{m}^2} L_{R^2} \right] + \frac{1}{\mu} L_{\text{top}} \right\}, \]

where \( e \) is the volume scalar density, \((M, m, \tilde{m}, \mu)\) are mass parameters, \( \sigma \) is a dimensionless parameter and \( \kappa \) is the 3D gravitational coupling required to ensure that \( I \) has dimensions of an action. The individual Lagrangians in this action are separately supersymmetric and they
take the form
\begin{align}
L_C &= S + \text{fermions}, \\
L_{(EH)} &= R - 2S^2 + \text{fermions}, \\
L_K &= K - \frac{1}{2}S^2 R - \frac{1}{2}S^4 + \text{fermions}, \\
L_{R^2} &= -16 \left( (\partial S)^2 - \frac{9}{4} (S^2 + \frac{1}{3} R)^2 \right) + \text{fermions}, \\
L_{\text{top}} &= -\frac{1}{2} \epsilon^{\mu\nu\rho} \left( R_{\mu \nu}^{ab}(\omega) \omega_{\rho ab} + \frac{2}{7} \omega_{\mu}^{a b} \omega_{\nu}^{b \rho c} \omega_{\rho \mu}^{c a} \right) + \text{fermions},
\end{align}

(1.3)

where the ‘fermions’ provide the $N=1$ supersymmetric completion, and $\omega$ is the usual spin connection.

Note that $S$ does not appear in $L_{\text{top}}$, which is the supersymmetric extension of the Lorentz Chern–Simons (LCS) term [14–16]; this is because of the superconformal invariance of this term. Note also that $S$ is indeed auxiliary as long as the $L_{R^2}$ term is absent, which is achieved by taking the limit $\tilde{m}^2 \to \infty$. As summarized above, this limit yields a unitary theory of massive gravitons in the Minkowski vacuum with $S = 0$ (which is a solution when $M = 0$) but the presence of an $S^2 R$ term suggests that the ‘effective’ balance of the two possible curvature-squared terms in a given (a)dS vacuum could depend on the (constant) value of $S$ in this vacuum, and this possibility motivates consideration of the generic theory. We do not know, a priori, which (if any) combination of curvature-squared terms will allow a unitary theory in a non-Minkowski vacuum.

An important aspect of our construction is the full dependence of the bosonic action on the ‘auxiliary’ field $S$, as given above. This is crucial both for a classification of the possible maximally symmetric vacua, and for a determination of whether a given adS vacuum is supersymmetric (since one needs to know the value of $S$ in it). In the absence of the curvature-squared terms, i.e., in the limit that $m^2 \to \infty$ and $\tilde{m}^2 \to \infty$, the $S$ field may be trivially eliminated and the resulting action then has a cosmological constant proportional to $\Lambda_1$. Otherwise, the relation between $\Lambda$ and $M^2$ is more complicated. In fact, the possible maximally symmetric vacua correspond to points on two curves in the $(\Lambda, M^2)$ plane. All supersymmetric vacua lie on one of these two curves (actually a half-line) which (remarkably) is the same for all $\tilde{m}$; in other words, the presence or absence of the $L_{R^2}$ term in the action has no effect on supersymmetric vacua, although we expect that it will affect the fluctuations and hence the analysis of unitarity/stability. The endpoint of the ‘supersymmetric curve’ in the $(\Lambda, M^2)$ plane is the origin, corresponding to the supersymmetric Minkowski vacuum with $S = 0$. Apart from this one special vacuum, there is no obvious way to compare results with those found in [10] for the purely bosonic theory (without the $S$ field). This unusual feature is due to the nonlinearity of the $S$ equation of motion. It means that the unitarity/stability analysis of [10] must be undertaken anew, but this is encouraging because there is therefore the possibility of an improved outcome in regard to boundary CFT central charges.

The supersymmetric Minkowski and adS vacua are special cases of supersymmetric solutions of the field equations. Because we have an off-shell supersymmetry (in the sense that the supersymmetry transformations close without the need to invoke field equations) one can separate the question of whether a given field configuration is supersymmetric from the question of whether it solves any particular set of field equations. Here we present a complete analysis of the possible supersymmetric configurations, generalizing an analysis presented in [17] for what amounts to the special case in which the scalar field $S$ is constant and non-zero. In this special case, we recover the generic supersymmetric pp-wave configurations that generalize the adS vacua. Particular subcases are known to solve the field equations of conformal 3D gravity [18], TMG [19, 17] and NMG [20]; here we find the supersymmetric pp-wave solutions of the generic 3D supergravity theory of the type under consideration.

We
will comment on some features that can be read off from these solutions as, for instance, critical values of the adS length $\ell$ at which the pp-wave solution becomes locally diffeomorphic to adS, indicating a generalized notion of ‘chiral gravity’.

In the last part of this paper we present a classification of all supergravity theories of $(a K + b R^2)$ type that are unitary in a Minkowski vacuum, together with a detailed analysis of their fermionic sectors. We begin with a ‘canonical’ analysis along the lines of [4]; this throws up three classes of unitary theories, which are the supersymmetrizations of the following three classes of bosonic models.

- General massive gravity (GMG). As summarized above, this propagates two massive gravitons of helicities $\pm 2$, generically with different masses. This includes TMG and NMG as special cases.
- Scalar massive gravity (SMG). This is the parity-preserving theory with $a = 0$ and ‘right-sign’ EH term, equivalent to a scalar field coupled to gravity. To the best of our knowledge, its supersymmetrization has not been considered previously.
- ‘New topologically massive gravity’ or NTMG. This is a model in which the EH term is omitted. It involves the ‘new’ scalar $K$ but otherwise turns out to propagate a single helicity 2 mode, like TMG, hence the name. The massless limit yields the ‘pure-K’ theory considered in [4].

This analysis does not yield the helicity content of the massive modes, so we then reconsider each of these three classes of unitary theories using covariant methods. In particular, we present a new proof that the GMG model propagates, unitarily, two spin-2 modes, and we verify the supermultiplet content of its supersymmetric extension. The novel feature of super-SMG is a third-order equation for a vector spinor field that propagates, unitarily, two spin-1/2 modes. The NTMG theory is new, even as a purely bosonic theory, so we consider this in more detail; in particular, we show that the linearized theory propagates a single massive mode of helicity 2, just like TMG, and this becomes a supermultiplet of helicities $(2, \frac{1}{2})$ or $(-2, -\frac{1}{2})$, in the supersymmetric case.

Finally, we consider the linearized $\mathcal{N} = 2$ super-GMG model. This is an obvious first step in an investigation of $\mathcal{N} > 1$ 3D massive supergravities. It is also of interest in that it unifies the new spin-2 models with well-known spin-1 models.

2. $\mathcal{N} = 1$ massive supergravity

In this section we are going to determine the full nonlinear $\mathcal{N} = 1$ supersymmetric off-shell invariants corresponding to the action (1.2). First, we give the off-shell $\mathcal{N} = 1$ supergravity multiplet together with the known invariants corresponding to the Einstein–Hilbert action with cosmological constant and the LCS term. Next, we determine the curvature-square invariants.

Our conventions are as follows. The metric signature is ‘mostly plus’. All fermions are two-component Majorana spinors. We may choose the Dirac matrices $\gamma^a$ ($a = 0, 1, 2$), which satisfy the anticommutation relation $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$, to be real $2 \times 2$ matrices, in which case the Majorana spinors are also real. The Ricci tensor is $R_{\mu\nu} \equiv R_{\mu\nu}^{\rho\sigma} = \partial_{\mu} \Gamma_{\rho\sigma}^{\nu} + \cdots$.

2.1. Off-shell $\mathcal{N} = 1$ supergravity multiplet

The $\mathcal{N} = 1$ supergravity multiplet in 3D consists of the dreibein $e_{\mu}^a$ and the gravitino $\psi_\mu$, neither of which propagates any modes in ‘pure’ supergravity but both will start propagating once higher derivative terms are added. Off-shell closure requires a real scalar auxiliary field $S$. The supersymmetry transformation rules are
\[\delta e^a_\mu = \frac{1}{2} \bar{\epsilon} \gamma^a \psi_\mu, \quad (2.1)\]
\[\delta \psi_\mu = D_\mu (\hat{\omega}) \epsilon + \frac{1}{2} S \gamma_\mu \epsilon, \quad (2.2)\]
\[\delta S = \frac{1}{2} \bar{\epsilon} \gamma^{\alpha \nu} \psi_{\alpha \nu} - \frac{1}{2} \bar{\epsilon} \gamma^\mu \psi_\mu S, \quad (2.3)\]

where
\[D_\mu (\omega) \epsilon = \partial_\mu \epsilon + \frac{1}{4} \omega_{ab} \epsilon \gamma^a \gamma^b \epsilon, \quad \psi_{\mu \nu} (\omega) = \frac{1}{2} (D_\mu (\omega) \psi_\nu - D_\nu (\omega) \psi_\mu). \quad (2.4)\]

The spin connection \(\hat{\omega}\) is the spin-connection with torsion determined by the super-torsion constraint
\[D_\mu (\hat{\omega}) e^a_\nu - D_\nu (\hat{\omega}) e^a_\mu = \frac{1}{2} \bar{\psi}_\mu \gamma^a \psi_\nu. \quad (2.5)\]

Its solution reads
\[\hat{\omega}_{\mu ab} (\epsilon, \psi) = \frac{1}{2} (R_{\mu ab} - R_{ab \mu} + R_{b \mu a}), \quad (2.6)\]
\[R_{\mu \nu} = 2 \hat{\omega}_{\mu \nu} - \frac{1}{2} \bar{\psi}_\mu \gamma^a \psi_\nu. \quad (2.7)\]

In the following we denote by \(D_\mu\) the covariant derivative with respect to the standard spin connection \(\omega = \omega (\epsilon)\) with vanishing torsion. Whenever another connection is used, this will be explicitly indicated.

The Lagrangians corresponding to cosmological constant, Einstein–Hilbert and Lorentz Chern–Simons term have been constructed long ago, and they are given by [12, 14–16]
\[L_C = S + \frac{1}{4} \bar{\psi}_\mu \gamma^{\alpha \nu} \psi_\nu, \quad (2.8)\]
\[L_{(EH)} = R - \bar{\psi}_\mu \gamma^{\alpha \nu} D_\nu (\hat{\omega}) \psi_\mu - 2 S^2, \quad (2.9)\]
\[e^{-1} L_{\text{top}} = e^{-1} \epsilon^{\mu \nu \rho \sigma} \left( R_{\mu \nu}^{ab} (\hat{\omega}) \hat{\omega}_{\rho \sigma} + \frac{2}{3} \hat{\omega}_{\rho \sigma} e^{c} \hat{\omega}_{\mu}^{\ a} \right) + 2 \bar{\psi}_\mu \gamma^{\nu} \gamma^{\alpha} \gamma^\rho R_\nu, \quad (2.10)\]

where we defined the dual of the gravitino curvature,
\[\mathcal{R}_\mu = e^{-1} \epsilon^{\mu \nu \rho \sigma} D_\nu (\hat{\omega}) \psi_\rho. \quad (2.11)\]

### 2.2. Yang–Mills multiplets and the Riemann invariant

Here we are going to determine the \(N = 1\) supersymmetric extension of the square of the Riemann tensor. This can be done very efficiently by introducing a torsionful spin connection which allows the problem to be reduced to one of coupling Yang–Mills multiplets to supergravity.

The off-shell Yang–Mills multiplet consists of a vector field and a Majorana spinor, both transforming in the adjoint representation of some gauge group. We denote these fields by \(A_\mu^I\) and \(\chi^I\), where \(I\) is a Lie algebra index. The supersymmetry transformations are
\[\delta A_\mu^I = -\bar{\epsilon} \gamma_\mu \chi^I, \quad \delta \chi^I = \frac{1}{3} \bar{\epsilon} \gamma^{\mu \nu} \tilde{F}_{\mu \nu}^I \epsilon, \quad (2.12)\]

with the super-covariant field strength
\[\tilde{F}_{\mu \nu}^I = \partial_\mu A_\nu^I - \partial_\nu A_\mu^I + f_{IL}^J A_\mu^J A_\nu^L + 2 \bar{\psi}_{[\mu} \gamma_{\nu]} \chi^I. \quad (2.13)\]

\[\]
The locally supersymmetric $F^2$ invariant reads
\[ \mathcal{L}_{\text{SYM}} = -\frac{1}{4} e F^{\mu
u I} F_{\mu
u}^I - 2 e \bar{\chi'}^{\mu} \gamma^\mu (D_{\mu} \chi')^I + \frac{1}{2} e F_{\mu\nu}^I \bar{\psi}_{\rho} \gamma^{\mu
u} \gamma^\rho \chi'_{I} + e S \bar{\chi'}^{\mu} \chi'_{I}. \] (2.14)

Let us note that here the covariant derivative acting on $\chi'$ is the ordinary covariant derivative with respect to the torsionless spin connection. Introducing the super-covariant spin connection $\hat{\omega}$ would change the coefficient of the second four-fermi term. However, the quartic fermion couplings cannot be fully absorbed into the spin connection and so we keep the standard covariant derivative.

The next step is to realize that the spin connection can be redefined such that it transforms as a Yang–Mills gauge potential. For this we use the auxiliary field $S$ to define a torsionful connection as follows:
\[ \Omega_{\mu}^{ab} = \hat{\omega}_{\mu}^{ab} \pm S e_{\mu}^{ab}. \] (2.15)

The supersymmetry transformations on $\psi_{\mu}$ and $S$ can in turn be rewritten as
\[ \delta \psi_{\mu} = D_{\mu}(\Omega^{-}) \epsilon, \quad \delta S = \frac{1}{2} \epsilon \gamma^\mu \psi_{\mu}(\Omega^{-}). \] (2.16)

Here we have introduced the gravitino curvature with respect to $\Omega^{-}$, i.e. explicitly
\[ \psi_{\mu\nu}(\Omega^{-}) = \frac{1}{2} (D_{\mu}(\Omega^{-}) \psi_{\nu} - D_{\mu}(\Omega^{-}) \psi_{\nu}). \] (2.17)

While the original spin connection $\omega(e, \psi)$ transforms under supersymmetry as
\[ \delta \hat{\omega}_{\mu}^{ab} = -\frac{1}{2} \bar{\epsilon}(\gamma_{\mu} \psi_{\nu}^{ab}(\Omega^{-}) - 2 \gamma_{\mu} \psi_{\nu}^{ab}(\Omega^{-})) + \frac{1}{2} S \bar{\psi}_{\mu} Y_{ab} \epsilon, \] (2.18)
the supersymmetry rule for $\Omega^{*}$ simplifies to
\[ \delta \Omega_{\mu}^{ab} = -\bar{\epsilon} \gamma_{\mu} \psi_{\nu}^{ab}(\Omega^{-}). \] (2.19)

We observe that there is no mixing left between Lorentz and world indices. Consequently, the supersymmetry rule coincides with the one for the gauge potential (2.12) if we treat the Lorentz indices as Yang–Mills indices and if we identify $\psi^{ab}(\Omega^{-})$ with the fermionic partner. To prove that $(\Omega_{\mu}^{ab}, \psi^{ab}(\Omega^{-}))$ transforms as a Yang–Mills vector multiplet it remains to check the supersymmetry variation of $\psi^{ab}(\Omega^{-})$. We first observe that
\[ \delta \psi_{ab}(\Omega^{-}) = \frac{1}{2} e_{\mu}^{ab} e_{b} e_{a} \bar{R}_{\mu\nu}^{abcd}(\Omega^{-}) \gamma^{cd} \epsilon. \] (2.20)

This is almost of the required form, except that the connection is $\Omega^{-}$ instead of $\Omega^{*}$ and that the index pairs are in the ‘wrong’ order. However, due to the torsionful connection the standard Bianchi identity no longer holds but rather we have
\[ \bar{R}_{ab,cd}(\Omega^{*}) = \bar{R}_{cd,ab}(\Omega^{-}), \] (2.21)
where we have introduced the super-covariant form of the Riemann tensor
\[ \bar{R}_{\mu\nu}^{ab}(\Omega^{*}) = R_{\mu\nu}^{ab}(\Omega^{*}) + 2 \bar{\psi}_{\mu \gamma}^{ab}(\Omega^{-}) \gamma^{\gamma \nu} \epsilon. \] (2.22)

The generalized Bianchi identity (2.21) can be easily derived by writing out the explicit $S$ dependence
\[ R_{\mu\nu}^{ab}(\Omega^{*}) = R_{\mu\nu}^{ab}(\omega) \pm \frac{1}{2} S \bar{\psi}_{\mu \gamma}^{ab} \psi_{\nu} \pm 2 \partial_{\mu} S e_{\nu}^{ab} + 2S^{2} e_{\mu}^{ab} e_{\nu}^{ab} \] (2.23)
In total this implies that $(\Omega_{\mu}^{ab}, \psi^{ab}(\Omega^{-}))$ transforms precisely as a Yang–Mills vector multiplet.

Finally, we can give the supersymmetric extension of the square of the Riemann tensor simply by specializing (2.14) to the multiplet $(\Omega_{\mu}^{ab}, \psi^{ab}(\Omega^{*}))$,
\[ \mathcal{L} = -\frac{1}{4} e R^{\mu\nu \rho \sigma}(\Omega^{*}) R_{\mu\nu\rho\sigma}(\Omega^{*}) - 2 e \bar{\psi}_{ab}(\Omega^{-}) \gamma^{\mu} D_{\nu} \psi^{ab}(\Omega^{-}) \]
\[ + \frac{1}{2} e R^{\mu\nu \rho \sigma}(\Omega^{*}) \bar{\psi}_{\rho} \gamma^{\mu \nu} \gamma^{\rho} \psi^{ab}(\Omega^{-}) + e S \bar{\psi}_{ab}(\Omega^{-}) \psi^{ab}(\Omega^{-}) \]
\[ - \frac{1}{2} e \bar{\psi}_{ab}(\Omega^{-}) \gamma^{\mu} D_{\nu} \psi^{ab}(\Omega^{-}) + \frac{1}{2} e \bar{\psi}_{ab}(\Omega^{-}) \psi^{ab}(\Omega^{-}) \psi^{\gamma \mu \nu} \psi_{\nu}. \] (2.24)
Here we stress again that unless stated differently the covariant derivative is with respect to $\omega(e)$. Since the Riemann tensor is equivalent to the Ricci tensor in 3D this result amounts to supersymmetrizing $R^\mu_\nu R^\mu_\nu$. Using

$$\varepsilon^{\mu\nu\rho\sigma}R_{\rho\sigma\delta\epsilon} = 4G^{\mu\alpha}, \quad R_{\mu\nu\alpha\beta} = \varepsilon_{\mu\nu\rho\sigma}G^{\rho\sigma},$$

(2.25)

where $G_{\mu\nu}$ is the Einstein tensor, one finds for the bosonic action

$$e^{-1}L = -(R^\mu_\nu R^\mu_\nu - \frac{1}{4} R^2) + 2\partial^\mu S\partial^\mu S - S^2 R - 3S^4.$$  

(2.26)

### 2.3. Scalar multiplets and the Ricci scalar invariant

After having determined the supersymmetric extension of the square of the Riemann tensor, and hence of the Ricci tensor, the only independent invariant left in 3D is the supersymmetrization of the square of the Ricci scalar $R$. This can be reduced to the problem of coupling an off-shell scalar multiplet to supergravity, in a similar way that we reduced the earlier problem to one of coupling a Yang–Mills multiplet to supergravity.

An off-shell $\mathcal{N} = 1$ scalar multiplet in 3D consists of a real scalar $\phi$, a Majorana fermion $\lambda$ and a real auxiliary scalar $f$. Its Lagrangian, after coupling to supergravity, reads

$$L = -eg_{\mu\nu}\partial^\mu\phi\partial^\nu\phi - \frac{1}{4}e\bar{\lambda}\gamma^\mu D^\mu\lambda + \frac{1}{16}ef^2 + \frac{1}{8}eS\bar{\lambda}\lambda + \frac{1}{32}e\bar{\lambda}\lambda\bar{\psi}_\mu\psi^\mu,$$

(2.27)

The supersymmetry rules are

$$\delta\phi = \frac{1}{2}\bar{\epsilon}\lambda,$$

(2.28)

$$\delta\lambda = \hat{D}\phi - \frac{1}{2}f\epsilon,$$

(2.29)

$$\delta f = -\bar{\epsilon}\hat{D}\lambda + \frac{1}{8}S\bar{\epsilon}\lambda,$$

(2.30)

where the super-covariant derivatives are given by

$$\hat{D}_\mu\phi = \partial_\mu\phi - \frac{1}{2}\bar{\psi}_{\mu}\lambda,$$

(2.31)

$$\hat{D}_\mu\lambda = D_\mu(\hat{\omega})\lambda - \hat{D}\phi\psi_\mu + \frac{1}{4}f\psi_\mu.$$  

(2.32)

We will now show that

$$(\phi, \lambda, f) \equiv (S, \gamma^{\mu\nu}\psi^\mu(\Omega^-), \hat{R}(\Omega^\pm)),$$

(2.33)

where

$$\hat{R}(\Omega^\pm) = R(\hat{\omega}) + 6S^2 + 2\bar{\psi}_\mu\gamma_\nu\psi^{\mu\nu}(\Omega^-) + \frac{1}{2}S\bar{\psi}_\mu\gamma^{\mu\nu}\psi^\nu,$$

(2.34)

transforms under local supersymmetry precisely as required by (2.28), (2.29) and (2.30). First, we infer from (2.16) that $S$ transforms as the scalar component. Moreover, it is easily checked that

$$\delta(\gamma^{\mu\nu}\psi_{\mu\nu}(\Omega^-)) = \hat{D}Se - \frac{1}{2}\hat{R}(\Omega^\pm)\epsilon,$$

(2.35)

i.e. the gamma trace of $\psi_{\mu\nu}(\Omega^-)$ transforms as the spinor component. It takes a little bit more work to check the supersymmetry variation of $\hat{R}(\Omega^\pm)$. Using

$$\gamma_\mu D_\mu\psi^{\mu\nu}(\Omega^-) = \frac{1}{2}\hat{D}(\gamma^{\mu\nu}\psi_{\mu\nu}(\Omega^-)) - \frac{1}{2}\varepsilon^{\mu\nu}\mu_\nu(\Omega^-)$$

(2.36)

and

$$\varepsilon^{\mu\nu}\mu_\nu(\Omega^-) = \frac{1}{2}G_{\mu\nu}\gamma_\mu\psi_\nu - \frac{1}{2}\varepsilon^{\mu\nu}\mu_\nu(S\psi_\nu)\gamma_\mu,$$

(2.37)
one may verify that
\[ \delta \hat{R}(\Omega^\pm) = -\hat{\epsilon} \hat{Y}^{\mu\nu} \psi_{\mu\nu}(\Omega^-) + \frac{1}{2} \hat{S} \hat{\gamma}^{\mu\nu} \psi_{\mu\nu}(\Omega^-). \]  
(2.38)
as required. Thus, we can use the supersymmetry of (2.27) to construct directly the \( R^2 \) invariant
\[ \mathcal{L}_{R^2} = \frac{1}{16} e^{R^2} - e \partial^\mu S \partial_\mu S + \frac{3}{2} e S^2 R + \frac{3}{4} e S^4. \]  
(2.40)
In total we have determined the complete supersymmetrization of the bosonic actions given by (2.26) and (2.40) from which the form (1.3) given in the introduction readily follows.

### 3. Supersymmetric configurations

Before proceeding to consider solutions of the field equations, we shall first determine which bosonic field configurations are supersymmetric. By definition, these are configurations that admit a Killing spinor, defined as a non-zero solution for \( \kappa \) to the equation
\[ (D_\mu + \frac{1}{2} \gamma_\mu S) \kappa = 0, \]  
(3.1)
which is obtained by setting to zero the supersymmetry variation of the gravitino field, specializing to bosonic field configurations and replacing the anticommuting spinor \( \epsilon \) by the commuting spinor field \( \kappa \). The \( S \) term may be viewed as a torsion part of the spin connection. The integrability condition of the Killing spinor equation is
\[ (G^{\mu\nu} - g^{\mu\nu} S^2 - e^{-1} \epsilon^{\mu\nu\rho\sigma} \partial_{\rho} S) \gamma_\nu \kappa = 0. \]  
(3.2)
It follows from this equation that the only maximally supersymmetric field configurations are Minkowski space, with \( S = 0 \), and anti-de Sitter space, with \( G_{\mu\nu} = S^2 g_{\mu\nu} \) for constant non-zero \( S \), so the main interest in what follows will be in other configurations that preserve \( 1/2 \) supersymmetry.

To begin with, we may easily deduce some other relations from (3.2). By contracting with \( \gamma_\mu \) one finds that
\[ \gamma^\mu \kappa \partial_\mu S = \frac{1}{2} (S^2 + \frac{1}{6} R) \kappa, \]  
(3.3)
which in turn implies that
\[ (\partial S)^2 = \frac{1}{2} (S^2 + \frac{1}{6} R)^2. \]  
(3.4)
Remarkably, this is equivalent to the vanishing of the bosonic part of \( L_{R^2} \). This does not mean that the \( L_{R^2} \) term is irrelevant to the field equations because its variation could still be non-zero. In the case of maximally symmetric supersymmetric vacua, for which \( S \) is constant, even the variation of \( L_{R^2} \) is zero, so the possibilities for such vacua are unaffected by the presence of the \( L_{R^2} \) term. Moreover, all contributions of the curvature squared terms to the field equations, including those of \( K \), vanish when evaluated for maximally symmetric supersymmetric configurations, as we will show in section 4.1. However, these contributions could affect other non-supersymmetric vacua, and supersymmetric non-vacuum solutions. Also the second variation, of relevance to perturbative unitarity and stability, is generically non-vanishing.
3.1. The null Killing vector field

To make further progress, we observe that the existence of a Killing spinor implies the existence of a null vector field:

\[ V^\mu = \bar{\kappa} \gamma^\mu \kappa, \quad V^2 = 0. \]  

(3.5)

Note that since \( \bar{\kappa} \kappa \equiv 0 \), a direct consequence of (3.3) is the relation

\[ V^\mu \partial_\mu S = 0. \]  

(3.6)

In other words, \( S \) is constant on orbits of \( V \). Similarly, an immediate consequence of (3.2) is the relation

\[ (G^{\mu \nu} - g^{\mu \nu} S^2 - e^{-1} \varepsilon^{\mu \nu \rho} \partial_\rho S)V_\nu = 0, \]  

(3.7)

which implies, in particular, that \( G^{\mu \nu} V_\mu V_\nu = 0 \).

The vector field \( V \) is covariantly constant with respect to the connection with torsion defined by the Killing spinor equation. Explicitly, this condition reads

\[ e D^\mu V^\nu = S \varepsilon^{\mu \nu \rho} V_\rho. \]  

(3.8)

This implies that \( D(\mu V_\nu) = 0 \) and hence that \( V \) is a Killing vector field (KVF). It also implies that

\[ \varepsilon^{\mu \nu \rho} \partial_\nu V_\rho = -2eSV^\mu. \]  

(3.9)

3.2. Adapted coordinates

The full implications of (3.7) and (3.9) can be analysed by choosing coordinates that are adapted to the null KVF. The general 3-metric with null Killing vector \( V = \partial_v \) takes the form

\[ g = h_{ij} dx^i dx^j + 2A_i dx^i dv \quad (i, j = 1, 2), \]  

(3.10)

where the (not necessarily invertible) symmetric 2-tensor field \( h_{ij} \) and the 1-form \( A_i dx^i \) are independent of \( v \). We may choose new coordinates \( x' = (u, x) \) such that

\[ dx = \sqrt{h_{22}} dx^2 + F dx^1, \quad A_i dx^i = f(x, u) du, \]  

(3.11)

for some positive function \( f \), and function \( F \) such that \( \partial_2 F = \partial_1 \sqrt{h_{22}} \). We may then shift \( v \) by a function of \( x \) and \( u \) so as to remove the \( du dx \) term in the metric. We thus arrive at a metric of the form

\[ g = dx^2 + 2f(x, u) du dv + h(x, u) du^2, \]  

(3.12)

where \( f(x, u) \) is everywhere positive. For this metric we have

\[ \sqrt{|g|} = f, \quad V_u = f, \quad V_v = V_t = 0. \]  

(3.13)

We are now in a position to analyse the full content of (3.9). The \( u \)-component is an identity. The \( x \)-component tells us that \( \partial_v f = 0 \), which we already know. The \( v \) component involves a choice of sign for \( \varepsilon^{xuv} \), which amounts to a choice of one of the two irreducible representations of the Clifford algebra spanned by the 3D Dirac matrices and their products. For the choice

\[ \varepsilon^{xuv} = 1, \]  

(3.14)

we find that \( S = -\partial_v \log \sqrt{f} \).

\[ \text{This relation was previously derived in [17] under the assumption of constant } S. \]

\[ \text{The sign of } S \text{ differs for the other choice, such that the restrictions on the metric implied by supersymmetry become independent of the choice of Dirac matrices.} \]
A computation of the Ricci tensor yields the result
\[ R_{\mu\nu} \, dx^\mu \, dx^\nu = -2(S^2 - \partial_x S) \, dx^2 - 2 f(2S^2 - \partial_x S) \, du \, dv + 2 \partial_u S \, dx \, du \]
\[ - (5 \partial_x h + 2 \partial_u S^2 + \frac{1}{2} \partial_u^2 h) \, du^2, \]
where we have used (3.15). This gives the Ricci scalar
\[ R = -6S^2 + 4\partial_u S, \]
in agreement with (3.4). We then find that
\[ (G - S^2 g)^{\mu\nu} \partial_\mu \partial_\nu = \frac{2}{f} \partial_u S \, f \, \partial_x \partial_v - \frac{1}{f^2} \left( S \partial_x h + \frac{1}{2} \partial_u^2 h \right) \partial_v \partial_v \partial_u. \]
(3.18)
We can now use this in the integrability condition (3.7). The \( u \) and \( v \) components are identities. The \( x \)-component implies that
\[ \partial_v S = 0, \]
in agreement with (3.6). We then find that
\[ \left( G - S^2 g \right)_{\mu\nu} \partial_\mu \partial_\nu = \frac{2}{f} \partial_u S \, f \, \partial_x \partial_v - \frac{1}{f^2} \left( S \partial_x h + \frac{1}{2} \partial_u^2 h \right) \partial_v \partial_v \partial_u. \]
(3.18)

3.3. Constant \( S \)

Let us now spell out the condition (3.19) for the case that \( S \) is constant. If we set \( S = \pm 1/\ell \), for finite constant \( \ell \), then \( f(u, x) = A(u) \exp(\mp 2x/\ell) \) for some function \( A(u) \), which we may set to unity without loss of generality; we then have the metric
\[ ds^2 = dx^2 + 2 f(u, x) \, du \, dv + h(u, x) \, du^2, \]
(3.19)
where the functions \( f \) and \( h \) are arbitrary, except that \( f \) is everywhere positive, and the sign of \( S \) depends on the choice of Dirac matrices.

This has the general form of a pp-wave metric; the special case of \( h \equiv 0 \) yields a metric that is locally isometric to \( \text{adS} \), for either choice of sign. Each choice yields a chart that extends to a horizon (at \( x \to \pm \infty \)) that separates the two charts. Taken together, the two charts cover the whole of \( \text{adS} \) except for the horizon, although the sign of \( S \) changes across the horizon. Thus, it is really \( S^2 \) that is constant in the \( \text{adS} \) vacuum, rather than \( S \). In the limiting case that \( \ell \to \infty \) (i.e. \( S \to 0 \)) we find the metric
\[ ds^2 = dx^2 + 2 \, du \, dv + h(u, x) \, du^2, \]
(3.20)
which is the pp-wave in a Minkowski background.

Here we shall find the Killing spinor admitted by the general (\( \text{adS} \)) pp-wave configuration. Starting from the metric (3.20) with lower sign in the exponent for concreteness, setting \( \ell = 1 \) for notational simplicity, and changing coordinates as \( \exp = r \, dv \), \( e^- = r \, du \), \( e^2 = \frac{1}{r} \, dr \).

(3.23)
It follows that the only non-vanishing components of the spin connection 1-form are
\[ \omega^{+2} = r \, dv + r \partial_v \left( \frac{1}{2r} \right) \, du, \quad \omega^{+2} = r \, du. \]
(3.24)

The ± labels denote flat indices. To be specific, given a vector \( v_i \) in the tangent space, we define the light-cone indices in a local Lorentz frame as \( v_{\pm} = \frac{1}{\sqrt{2}}(v_0 + v_1) \).
The Killing spinor equation \((d + \frac{1}{2}\omega^{ab}\gamma_{ab} - \frac{1}{2}e^a\gamma_a)\kappa = 0\) takes the form
\[
d\kappa + \frac{1}{2}(\omega^{12}\gamma_{12} + \omega^{-2}\gamma_{-2})\kappa - \frac{1}{2}((e^3\gamma_4 + e^{-}\gamma_- + e^2\gamma_2)\kappa = 0. \tag{3.25}
\]
A convenient choice of \(\gamma\) matrices is
\[
\gamma_0 = i\sigma_2, \quad \gamma_1 = \sigma_1, \quad \gamma_2 = \sigma_3. \tag{3.26}
\]
Writing the spinor parameter as \(\kappa = \begin{pmatrix} \psi \\ \chi \end{pmatrix}\),
\[
d\psi = \sqrt{2r}\chi\, d\nu + \frac{1}{\sqrt{2}}\chi(1 + r\partial_r)\left(\frac{h}{2r}\right)\, du + \frac{1}{2r}\psi\, dr, \tag{3.28}
\]
\[
d\chi = -\frac{1}{2r}\chi\, dr. \tag{3.29}
\]
The solution to these equations is given by
\[
\psi = \psi_0\sqrt{r}, \quad \chi = 0, \tag{3.30}
\]
where \(\psi_0\) is an arbitrary constant. This means that half of supersymmetry is broken, in the sense that we have a Killing spinor \(\kappa_0\) given by
\[
\kappa_0 = \sqrt{r}\eta_-, \tag{3.31}
\]
where \(\eta_-\) is a single Majorana–Weyl spinor in 1 + 1 dimensions satisfying \(\gamma_2\eta_- = -\eta_-\).
Note also that since \(\chi = 0\), the term containing the function \(h(u, r)\) in (3.28) drops out and consequently the Killing spinor (3.31) exists for a generic pp-wave solution, not depending on the detailed form of \(h(r, u)\).

If we specialize to the adS_3 metric, which amounts to setting \(h = 0\), the solution is given by
\[
\psi = \psi_0\sqrt{r} + \sqrt{2}\psi_0\sqrt{r}\chi_0, \quad \chi = \frac{\chi_0}{\sqrt{r}}, \tag{3.32}
\]
where \(\psi_0\) and \(\chi_0\) are arbitrary constants. As expected, this means a symmetry enhancement, since the Killing spinor now takes the form [17]
\[
\kappa_0 = r^{-1/2}\eta_- + r^{1/2}(\eta_+ + v\gamma_4\eta_-), \tag{3.33}
\]
where \(\eta_{\pm}\) are constant spinors satisfying \(\gamma_2\eta_{\pm} = \pm\eta_{\pm}\), and \(\kappa_0\) now decomposes into two independent Majorana–Weyl spinors from the (1 + 1)-dimensional point of view.

4. Field equations and solutions

From (1.2) and (1.3) we see that the bosonic action of the generic 3D supergravity theory of interest is
\[
I = \frac{1}{\kappa^2}\int d^3x \left\{ e \left[ MS + \sigma(R - 2S^2) + \frac{1}{m^2}\left(K - \frac{1}{2}S^2R - \frac{3}{2}S^4\right)\right] - \frac{2}{m^2}e \left( \partial S \right)^2 - \frac{9}{4} \left( S^2 + \frac{1}{6}R \right)^2 + \frac{1}{\mu}L_{\text{LCS}} \right\}. \tag{4.1}
\]

7 Note that this result is considerably simpler in form than that found in [17] due to our different choice of basis 1-forms.
Note that the $S$ field is auxiliary in the limit that $\tilde{m}^2 \to \infty$, but with an equation that is not linear, in contrast to the usual auxiliary fields of supergravity theories. Note also that there is an $S^2 R$ term, which means that elimination of $S$ could alter the ‘effective’ curvature squared term in a vacuum with non-zero $S$. In this general model with both $L_K$ and $L_{R^2}$ terms, there is a further very special case: that for which

$$\tilde{m}^2 = 3m^2.$$  

(4.2)

This can be viewed as the limit in which $\hat{m}^2 \to \infty$ where the mass parameter $\hat{m}$ is defined by

$$\frac{1}{\hat{m}^2} = \frac{1}{m^2} - \frac{3}{\tilde{m}^2}.$$  

(4.3)

The $S^2 R$ and $S^4$ terms cancel in the $\hat{m}^2 \to \infty$ limit, and the curvature-squared terms become proportional to the square of the trace-free tensor $R_{\mu\nu} - \frac{1}{3} g_{\mu\nu} R$.

In this section we will give the equations of motion, find some solutions and the amount of supersymmetry they preserve. From (4.1) we find that the metric equation of motion is

$$0 = \left( -\frac{1}{2} MS + \sigma S^2 \right) g_{\mu\nu} + \sigma G_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} + \frac{1}{2m^2} K_{\mu\nu} + \frac{1}{2\tilde{m}^2} L_{\mu\nu}$$

$$- \frac{2}{\tilde{m}^2} \left[ \partial_\mu S \partial_\nu S - \frac{1}{2} g_{\mu\nu} (\partial S)^2 \right]$$

$$- \frac{1}{2\tilde{m}^2} \left[ G_{\mu\nu} S^2 - \frac{3}{2} g_{\mu\nu} S^4 - (D_\mu D_\nu - g_{\mu\nu} D^2) S^2 \right],$$  

(4.4)

where

$$\sqrt{|g|} C_{\mu\nu} = \epsilon_{\mu}^{\tau\rho} D_\tau S_{\rho\nu}, \quad S_{\mu\nu} = R_{\mu\nu} - \frac{1}{3} g_{\mu\nu} R,$$

(4.5)

$$K_{\mu\nu} = 2D^2 R_{\mu\nu} - \frac{1}{2} D_\mu D_\nu R - \frac{1}{2} g_{\mu\nu} D^2 R - \frac{13}{8} g_{\mu\nu} R^2$$

$$+ \frac{9}{2} RR_{\mu\nu} - 8R_{\mu}^\rho R_{\nu\rho} + 3g_{\mu\nu}(R^\rho_\sigma R_{\rho\sigma}),$$

(4.6)

$$L_{\mu\nu} = -\frac{1}{2} D_\mu D_\nu R + \frac{1}{2} g_{\mu\nu} D^2 R - \frac{1}{8} g_{\mu\nu} R^2 + \frac{1}{2} RR_{\mu\nu}.$$  

(4.7)

The tensor $C_{\mu\nu}$ is the Cotton tensor, which is a derivative of the (3D) Schouten tensor $S_{\mu\nu}$; this term arises from variation of the LCS term in the action. The tensor $K_{\mu\nu}$ is the tensor given in [2]; it arises from variation of the $K$ term in the action. The tensor $L_{\mu\nu}$ arises from variation of the $R^2$ term in the action. The trace of the metric equation can be written as

$$(M - 4\sigma S) S + 2\sigma \left( S^2 + \frac{1}{6} R \right) - \frac{1}{3m^2} \left( K + \frac{1}{2} S^2 R + \frac{9}{2} S^4 \right)$$

$$+ \frac{9}{2m^2} \left( S^2 + \frac{1}{6} R \right) \left( S^2 - \frac{1}{18} R \right) = \frac{2}{3m^2} D^2 S^2 + \frac{1}{5m^2} [2(\partial S)^2 + D^2 R].$$  

(4.8)

The $S$ equation of motion is

$$(M - 4\sigma S) - \frac{6}{m^2} S \left( S^2 + \frac{1}{6} R \right) = -\frac{4}{m^2} D^2 S.$$  

(4.9)

4.1. Maximally (super)symmetric vacua

We will now consider in detail the possibilities for maximally symmetric, but not necessarily supersymmetric, vacua, for which $S$ is constant and

$$G_{\mu\nu} = -\Lambda g_{\mu\nu}.$$  

(4.10)
for cosmological constant \( \Lambda \), which has dimensions of mass squared. For such solutions the condition (3.3) for supersymmetry reduces to

\[
\Lambda + S^2 = 0. 
\]

(4.11)

This was derived as a necessary condition for supersymmetry but it is also sufficient within the class of maximally symmetric vacua. Naturally, it implies that \( \Lambda \leq 0 \) so that only Minkowski and adS vacua can be supersymmetric.

The \( S \) equation of motion for maximally symmetric solutions, with constant \( S \), reduces to

\[
(M - 4\sigma S) - \frac{6}{\hat{m}^2} S(S^2 + \Lambda) = 0.
\]

(4.12)

Only the trace of the metric equation is needed, and this is

\[
S(M - 4\sigma S) + (\Lambda + S^2) \left[ 2\sigma + \frac{1}{2\hat{m}^2}(\Lambda - 3S^2) \right] = 0.
\]

(4.13)

Note that both these equations simplify dramatically in the limit that \( \hat{m}^2 \to \infty \). In this special case, there is a unique vacuum for given \( M \), with \( S = M/(4\sigma) \) and \( \Lambda = -M^2/16 \). This vacuum is Minkowski for \( M = 0 \) and adS for \( M \neq 0 \), and supersymmetric in either case. In the Minkowski vacuum the linearized theory is non-unitary.

In a next step, let us assume that \( \hat{m}^2 \) is finite, which amounts to finding solutions for the generic curvature-squared theory. We observe that equations (4.12) and (4.13) imply that

\[
(M - 4\sigma S)(9S^2 + 4\sigma \hat{m}^2 + \Lambda) = 0.
\]

(4.14)

This leads to two branches of vacua. One comes from setting \( M = 4\sigma S \). In this case \( \Lambda + S^2 = 0 \), so we have a supersymmetric vacuum when \( \Lambda < 0 \). In a plot of \( \Lambda \) against \( M^2 \), the vacua of this branch lie on a half-line, in the \( \Lambda < 0 \) sector, that starts at the origin.

The other branch of vacua arises from solutions of \( 9S^2 = -(\Lambda + 4\sigma \hat{m}^2) \). Substituting for \( S \) in (4.12) we learn that

\[
(\Lambda + 4\sigma \hat{m}^2) \left( \Lambda + \frac{1}{4}\sigma \hat{m}^2 \right)^2 + \left( \frac{9\hat{m}^2 M}{16} \right)^2 = 0,
\]

(4.15)

which is a cubic equation for \( \Lambda \). Let us consider in turn the two possible signs for \( \sigma \).

\begin{itemize}
  \item \( \sigma < 0 \): there is no solution for \( \Lambda \) unless

    \[
    \Lambda < 4\hat{m}^2.
    \]

    (4.16)

If we plot \( \Lambda \) against \( M^2/16 \), we see that the cubic curve that gives the vacua on this branch just touches the \( M = 0 \) axis at \( \Lambda = \frac{1}{4}\hat{m}^2 \). This means that \( M = 0 \) allows two dS vacua (in addition to the supersymmetric Minkowski vacuum of the other branch); one has \( \Lambda = \frac{1}{4}\hat{m}^2 \) and the other has \( \Lambda = 4\hat{m}^2 \). These are connected in the sense that they lie on the same curve in the \( M^2 \geq 0 \) region of the \( (\Lambda, M^2/16) \) plane. This cubic curve also cuts the \( \Lambda = 0 \) axis, so there is a non-supersymmetric Minkowski vacuum (with non-zero \( S \)) in addition to the supersymmetric Minkowski vacuum on the other branch. The cubic curve intersects the line \( \Lambda + M^2/16 = 0 \) in two points given by \( M^2 = 8\hat{m}^2 \) and \( M^2 = 8(3\sqrt{3} - 5)\hat{m}^2 \), so these two adS vacua are supersymmetric.

\begin{itemize}
  \item \( \sigma > 0 \): for \( M = 0 \) there is a non-supersymmetric adS vacuum with \( \Lambda = -\frac{1}{4}\hat{m}^2 \). This is ‘isolated’ because the part of the cubic curve with \( M^2 < 0 \) is unphysical. All other solutions on this branch are such that

    \[
    \Lambda \leq -4\hat{m}^2.
    \]

    (4.17)

The limiting \( \Lambda = -4\hat{m}^2 \) adS vacuum occurs for \( M^2 = 0 \). All these adS vacua are non-supersymmetric, with one exception, corresponding to the point in the \( (\Lambda, M^2/16) \) plane at which the cubic equation cuts the line of supersymmetric adS vacua.
These possibilities are displayed in figures 1 and 2.

Let us finally note that there is a neat geometrical interpretation for the existence of supersymmetric adS solutions with $\Lambda = -\frac{M^2}{16}$. Remarkably, this is precisely the value one gets for pure Einstein–Hilbert plus cosmological constant. In other words, the higher derivative contributions to the field equations drop out for maximally supersymmetric solutions. This can be directly understood by noting that the connection $\Omega^+$ gives rise to the so-called parallelizing torsion for maximally supersymmetric configurations. To be precise, from (2.23) we infer that the curvature with respect to the torsionful connection vanishes when evaluated for supersymmetric adS solutions:

$$R_{\mu\nu}^{ab}(\Omega^+)_{|S^a = \Lambda} = 0.$$  \hfill (4.18)

Since the supersymmetric curvature-square actions have been computed as the squares of $R(\Omega^+)$, it follows directly from (4.18) that their contribution to the field equations obtained by varying this action vanishes for supersymmetric configurations. Explicitly, we have for the $K$ invariant the factorization

$$L_K = (G_{\mu\nu} - S^2 g_{\mu\nu}) [ (G_{\mu\nu} - S^2 g_{\mu\nu}) + \frac{1}{4} (R + 6S^2) g_{\mu\nu} ].$$  \hfill (4.19)

Let us stress that since the first factor vanishes only for maximally symmetric geometries, the variation of $L_K$ will not vanish for general supersymmetric configurations. This is in contrast to $L_{K F}$ whose variation vanishes for all supersymmetric solutions with constant $S$ by virtue of (3.4).
Figure 2. Maximally symmetric vacua for $\sigma = 1$ and $\hat{m}^2 = 1$, the straight line representing the supersymmetric vacua; this is the same straight line as in figure 1, despite appearances, because of the different scales for the $\Lambda$ axes. There is also an isolated adS vacuum at $\Lambda = -\frac{1}{4}$, $M = 0$, which is not indicated.

4.2. pp-wave solutions

We now aim to find a supersymmetric pp-wave metric (3.20) that solves the metric equation (4.4) and $S$-equation (4.9). It is straightforward to verify that the $vv$ and $vx$ components of the metric equation are automatically satisfied. Thus, we have to consider the $uu$, $uv$, $ux$ and $xx$ components of the metric equation, and the $S$-equation. The latter, upon the use of (3.15) and (3.17), takes the form

$$\frac{1}{\hat{m}^2} \partial_r^2 S - \left(\frac{1}{m^2} - \frac{1}{\hat{m}^2}\right) S \partial_r S - \frac{1}{4} M = 0.$$  \hspace{1cm} (4.20)

Setting $S = -1/\ell$, we see from (4.20) that $\ell = -4\sigma/M$. The non-vanishing components of the Ricci tensor $R_{ab}$ and Cotton tensor $C_{ab}$ are

$$R_{++} = R_{22} = -2,$$

$$R_{--} = -\frac{1}{2r^2} \left( r^2 \partial_r^2 r - r \partial_r \right) h,$$

$$C_{--} = (r \partial_r + 1) R_{--},$$

where we set, from now, on $\ell = 1$. Turning to the metric equations, using these results and (3.24), we find that they are all trivially satisfied except the $uu$ component which takes the form

$$\left[ \frac{1}{m^2} r^2 \partial_r^2 + \left( \frac{3}{m^2} + \frac{1}{\mu} \right) r \partial_r + \left( \sigma + \frac{1}{\mu} \right) \right] R_{uu} = 0.$$  \hspace{1cm} (4.24)
To solve this equation, we substitute \( n = r^n \). The resulting characteristic polynomial is

\[
(n - 2) \left( \frac{1}{m^2} n(n - 2) + \frac{1}{\mu} (n - 1) + \sigma \right) = 0.
\]

(4.25)

Thus we find the solutions

\[
h_{\pm}(u, r) = r^n \pm f_1(u) + r^2 f_2(u) + f_3(u),
\]

(4.26)

where \( f_{1,2,3} \) are arbitrary functions of \( u \) only and

\[
n_{\pm} = 1 - \frac{m^2}{2\mu} \pm \sqrt{1 + \frac{m^4}{4\mu^2} - \sigma m^2}.
\]

(4.27)

The functions \( f_2 \) and \( f_3 \) can be removed by local coordinate transformations (see, for example, [17]). Therefore, we shall take \( n_\pm \neq 0, 2 \) and write the general solution as

\[
h(u, r) = h_+(u) r^{n_+} + h_-(u) r^{n_-},
\]

(4.28)

where \( h_\pm(u) \) are arbitrary functions of \( u \), and the exponents \( n_\pm \) are as given in (4.27).

Next, we observe that in the bosonic NMG, the characteristic equation obtained in [20] has an additional factor of \( 1/(2m^2) \) in the parenthesis multiplying \( n(n - 2) \) in (4.25). In the massive supergravity model we are considering, however, there is an additional contribution coming from the term proportional to \( G_{\mu\nu} S^2 \) in (4.4). As a consequence, we obtain the characteristic equation (4.25), and the roots (4.27) differ from those in [20] in that the first term under the square root is 1 instead of \( \frac{1}{2} \). This difference has interesting consequences, as we shall see below.

To begin with, let us consider the roots of (4.27) and examine the parameter values for which degeneracies arise. In such cases, as is well known, additional logarithmic solutions appear. The doubly degenerate roots \( n_+ = n_- \) arise for

\[
m^2_{\pm} = 2\mu^2 \left( \sigma \pm \sqrt{\sigma^2 - \frac{1}{\mu^2}} \right),
\]

(4.29)

where, again, we have suppressed the adS radius, which can easily be re-introduced by dimensional analysis, for notational simplicity. In this case, the following additional solutions arise

\[
h(r, u) = r^{k_{\pm}} [h_1(u) \log r + h_2(u)],
\]

(4.30)

where \( k_{\pm} = 1 - (m^2/2\mu) \) takes the form

\[
k_{\pm} = 1 - \mu \sigma \mp \sqrt{\mu^2 \sigma^2 - 1},
\]

(4.31)

and \( h_1(u), h_2(u) \) are arbitrary functions of \( u \).

Considering the root \( n = 0 \) of (4.25), it becomes triply degenerate for \( \mu \sigma = +1 \), and the root \( n = 2 \) becomes triply degenerate for \( \mu \sigma = -1 \), since

\[
k_{\pm} = \begin{cases} 0 & \text{if } \mu \sigma = +1, \\ 2 & \text{if } \mu \sigma = -1. \end{cases}
\]

(4.32)

This means that the solutions becomes adS3, and that the following additional solutions arise:

\[
\mu \sigma = +1 : \quad h(r, u) = \log r [h_1(u) \log r + h_2(u)],
\]

(4.33)

\[
\mu \sigma = -1 : \quad h(r, u) = r^2 \log r [h_1(u) \log r + h_2(u)].
\]

(4.34)

This is remarkable because \( \mu \sigma = \pm 1 \) are precisely the critical points which arise in the chiral gravity limit of TMG [6] in which, apart from the logarithmic modes that do not obey
the standard Brown–Henneaux boundary conditions [8], the usual graviton mode ceases to propagate in the bulk. In the case of ordinary bosonic NMG, on the other hand, it can be shown that critical points arise for those values at which the central charges of bosonic NMG vanish [20]. We shall comment further on various aspects of our critical points $\mu \sigma = \pm 1$ in the conclusions.

5. Linearization about a supersymmetric Minkowski vacuum

We now wish to investigate the propagating degrees of freedom and their multiplet structure around a supersymmetric Minkowski vacuum, which requires a linearization about this background. Linearized 3D $N = 1$ supergravity theories are constructed from the symmetric tensor $h_{\mu \nu}$, the anticommuting vector spinor $\psi_\mu$ (which is a Majorana spinor) and the ‘auxiliary’ scalar $S$ (which may actually propagate, depending on the details of the action). We insist on gauge invariance with respect to the following linear gauge transformations

$$h_{\mu \nu} \rightarrow h_{\mu \nu} + \partial(\mu v_{\nu}), \quad \psi_\mu \rightarrow \psi_\mu + \partial_\nu \zeta,$$

where $v$ is an arbitrary vector field and $\zeta$ an arbitrary Majorana spinor field. It is convenient to define

$$h_\mu = \eta^{\nu \rho} \partial_\rho h_{\mu \nu}, \quad h = \eta^{\mu \nu} h_{\mu \nu},$$

and to introduce the following gauge invariant ‘field strengths’

$$R_{\mu \nu}^{(\text{lin})} = -\frac{1}{2} \square h_{\mu \nu} - 2 \partial_\mu h_{\nu \rho} + \partial_\nu h_{\mu \rho}, \quad R_{\mu}^{(\text{lin})} = \eta^{\nu \rho} \partial_\nu \psi_\rho.$$

The first of these is the linearized Ricci tensor and the second is the Rarita–Schwinger field strength. The linearized Einstein tensor is

$$G_{\mu \nu}^{(\text{lin})} = R_{\mu \nu}^{(\text{lin})} - \frac{1}{2} \eta_{\mu \nu} R^{(\text{lin})}, \quad R^{(\text{lin})} = \eta^{\mu \nu} R_{\mu \nu}^{(\text{lin})}.$$

Also useful is the linearized Cotton tensor

$$C_{\mu \nu}^{(\text{lin})} = \varepsilon_{\mu \tau \rho} \partial_\tau C_{\rho \nu}^{(\text{lin})}, \quad S_{\mu \nu}^{(\text{lin})} = R_{\mu \nu}^{(\text{lin})} - \frac{1}{2} \eta_{\mu \nu} R^{(\text{lin})},$$

and its fermionic counterpart, the ‘Cottino tensor’

$$C_\mu^{(\text{lin})} = \gamma^\nu \partial_\nu R_\mu^{(\text{lin})} + \varepsilon^{\mu \nu \rho} \partial_\nu R_\rho.$$

Note the identities

$$\gamma_\mu C_\mu^{(\text{lin})} = 0, \quad \partial_\mu C^{\mu \nu} = 0.$$

The linearized off-shell supersymmetry transformations may now be written as

$$\delta_\epsilon h_{\mu \nu} = \bar{\epsilon} \gamma_{\mu \nu} \psi_\nu, \quad \delta_\epsilon S = \frac{1}{4} \bar{\epsilon} \gamma_{\rho} R_\rho^{(\text{lin})}, \quad \delta_\epsilon \psi_\mu = \left[ -\frac{1}{4} \varepsilon^{\rho \mu \nu} \gamma_\sigma \partial_\rho h_{\mu \nu} + \frac{1}{2} S \gamma_\mu \right] \epsilon.$$

The following four quadratic Lagrangians yield actions that are both gauge invariant and supersymmetric, up to surface terms:

$$L_{(\tilde{E})}^{(2)} = -\frac{1}{2} h_{\mu \nu} G_{\mu \nu}^{(\text{lin})} - 2 S^2 - \bar{\psi}_\mu R_\mu^{(\text{lin})},$$

$$L_{\text{top}}^{(2)} = \frac{1}{2} h_{\mu \nu} C_{\mu \nu}^{(\text{lin})} + \frac{1}{2} \bar{\psi}_\mu C_\mu^{(\text{lin})},$$

$$L_5^{(2)} = -\frac{1}{2} \varepsilon^{\mu \nu \rho} \partial_\mu h_{\rho \nu} - \frac{1}{2} \bar{\psi}_\mu (\gamma^\rho \partial_\nu) C_\rho^{(\text{lin})},$$

$$L_{R^2}^{(2)} = R_{\mu \nu}^{(\text{lin})} + 16 S \square S = 4 (R_{\mu \nu}^{(\text{lin})} \cdot \gamma) (\gamma \cdot \partial) (\gamma \cdot R_{\mu \nu}^{(\text{lin})}).$$

One can show that

$$R_{\mu \nu}^{(\text{lin})} R_{\mu \nu}^{(\text{lin})} = \frac{1}{8} R_{\mu \nu}^{(\text{lin})} + \text{total derivative},$$

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so the Lagrangian $L^{(3)}_K$ is indeed the quadratic approximation to the supersymmetrization of the Lagrangian $L_K$. Similarly, the Lagrangian $L^{(3)}_{\text{top}}$ is the quadratic approximation to the LCS term since its variation yields the linearized Cotton tensor.

In the following we shall consider the general linear combination of these four Lagrangians, which are parametrized by a dimensionless constant $\sigma$ and three mass parameters $(\mu, m, \tilde{m})$:

$$L^{(2)} = \sigma L^{(2)}_{\text{EH}} + \frac{1}{\mu} L^{(2)}_{\text{top}} + \frac{1}{m^2} L^{(2)}_K + \frac{1}{8\tilde{m}^2} L^{(2)}_R.$$  \hspace{1cm} (5.11)

On setting $\sigma = 1$ and taking all mass parameters to infinity, one gets the linearization of the standard $\mathcal{N} = 1$ 3D supergravity, which has no propagating modes. Allowing finite $\mu$ leads to a unitary theory if $\sigma < 0$ and one may then choose $\sigma = -1$ without loss of generality; this is the linearization of topologically massive supergravity, which propagates modes of helicities $(2, 3/2, -1)$, the sign depending on the sign of $\mu$. Of principal interest here will be the models for which either $m^2$ or $\tilde{m}^2$ is finite; as we shall see, unitarity requires that we take either $m^2$ or $\tilde{m}^2$ to infinity, but this is merely a necessary condition for unitarity, not a sufficient one. Our aim here is to determine all possible unitary theories within the class of models considered.

### 5.1. Canonical decomposition

There are three gauge-invariant components of the metric, which we may write, following Deser [4] but in terms of slightly different variables $(N, \xi, \varphi)$ as

$$h_{ij} = -\epsilon^{ik} \epsilon^{jl} \partial_k \partial_l \nabla^2 \varphi, \quad h_{0i} = -\epsilon^{ij} \frac{1}{\sqrt{-g}} \partial_j \xi, \quad h_{00} = \frac{1}{\sqrt{-g}} (N + \Box \varphi).$$  \hspace{1cm} (5.12)

Observe that this decomposition implies the gauge choice

$$\partial_i h_{ij} \equiv 0, \quad \partial_i h_{0i} \equiv 0.$$  \hspace{1cm} (5.13)

We may make a similar decomposition of the anticommuting vector spinor $\psi_\mu$ in terms of anticommuting spinors $(\eta, \chi)$ by writing

$$\psi_i = \gamma_i \chi, \quad \psi_0 = \gamma_0 \left( \frac{1}{\sqrt{-g}} \partial_i \eta + \chi \right).$$  \hspace{1cm} (5.14)

This implies the gauge choice

$$\gamma^i \psi_i = 2 \gamma^i \frac{\partial_i}{\sqrt{-g}} \bar{\psi}_j,$$  \hspace{1cm} (5.15)

which is non-standard but simplifies the subsequent analysis.

In terms of the variables $(N, \xi, \varphi)$, the components of the linearized Einstein tensor are

$$G^{\text{lin}}_{00} = \frac{1}{2} \nabla^2 \varphi, \quad G^{\text{lin}}_{0i} = \frac{1}{2} \left( \partial_0 \bar{\psi} + \epsilon^{ij} \partial_j \xi \right),$$

$$G^{\text{lin}}_{ij} = -\frac{1}{2} (\delta_{ij} \Box \varphi - \partial_i \partial_j \varphi) - \frac{1}{2} \left( \delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2} \right) N + \frac{1}{2} \left( \epsilon^{ik} \frac{\partial_k \partial_j}{\nabla^2} + \epsilon^{jk} \frac{\partial_k \partial_i}{\nabla^2} \right) \bar{\xi},$$  \hspace{1cm} (5.16)

and hence

$$R^{\text{lin}} = N + 2 \Box \varphi.$$  \hspace{1cm} (5.17)

The components of the linearized Cotton tensor are

$$C^{\text{lin}}_{00} = \frac{1}{2} \nabla^2 \bar{\xi}, \quad C^{\text{lin}}_{0i} = \frac{1}{2} \partial_0 \bar{\xi} - \frac{1}{4} \epsilon^{ij} \partial_j N,$$

$$C^{\text{lin}}_{ij} = \frac{1}{2} (\delta_{ij} \Box + \partial_i \partial_j) \bar{\xi} + \frac{\partial_i \partial_j}{\nabla^2} \bar{\xi} - \frac{1}{4} \left( \epsilon^{ik} \frac{\partial_k \partial_j}{\nabla^2} + \epsilon^{jk} \frac{\partial_k \partial_i}{\nabla^2} \right) N.$$  \hspace{1cm} (5.18)
In terms of the anticommuting spinor variables \((\eta, \chi)\), the components of the Rarita–Schwinger field strength are

\[ R^0_{(\text{lin})} = \gamma^0 \chi, \quad R^i_{(\text{lin})} = \gamma^i \left( \varepsilon^{ij} \partial_j \left( \chi + \frac{\gamma^k \partial_k}{\sqrt{\gamma^2}} \eta \right) + \gamma^j \chi \right), \tag{5.19} \]

and hence

\[ \gamma^0 R^0_{(\text{lin})} = \gamma^0 \eta + 2 \gamma^\mu \partial_\mu \chi. \tag{5.20} \]

The components of the fermionic counterpart of the Cotton tensor are

\[ C^0 = \gamma^0 \chi, \quad C^i = \varepsilon^{ij} \partial_j \left( \gamma^0 \eta + \frac{\gamma^k \partial_k}{\sqrt{\gamma^2}} \eta \right) - \gamma^0 \partial_i \left[ \gamma^k \partial_k \eta \right]. \tag{5.21} \]

Using these results, one finds that

\[
\begin{align*}
L^{(2)}_{\text{EH}} &= -\frac{1}{2}(\varphi N + \varphi \Box \varphi - \xi^2) - 2S^2 + 2\bar{\xi}(\gamma^\mu \partial_\mu)\chi + 2\bar{\chi} \eta, \\
L^{(2)}_{\text{top}} &= \frac{1}{2} N^2 - \frac{1}{2} \bar{\eta} \eta, \\
L^{(2)}_K &= \frac{1}{2} N^2 + \frac{1}{2} \xi^2 \Box - \frac{1}{2} \bar{\eta} (\gamma^\mu \partial_\mu) \eta, \\
L^{(2)}_{K'} &= (N + 2\Box \varphi)^2 + 16S \Box S - 4\bar{\eta} (\gamma^\mu \partial_\mu) \eta - 16\bar{\eta} \Box \chi - 16\bar{\chi} (\gamma^\mu \partial_\mu) \chi.
\end{align*}
\]

Note that both \(L^{(2)}_{\text{top}}\) and \(L^{(2)}_K\) are independent of both \(\varphi\) and \(\chi\). For \(L^{(2)}_{\text{top}}\), this is a consequence of its superconformal invariance. For \(L^{(2)}_K\), it is a consequence of an ‘accidental’ linearized superconformal invariance that is not a feature of the full action. The combination of these Lagrangians corresponding to (5.11) can be written as

\[ L^{(2)} = L^{(2)}_{\text{(bos)}} + L^{(2)}_{\text{(term)}}, \tag{5.23} \]

where

\[
L^{(2)}_{\text{(bos)}} = -\frac{1}{2} \frac{\sigma^2}{\mu^2} (\varphi N + \varphi \Box \varphi - \xi^2) + \frac{1}{8m^2} N^2 + \frac{1}{2m^2} \xi \Box \xi \\
+ \frac{1}{8m^2} (N + 2\Box \varphi)^2 - \frac{2}{m^2} S (\Box - \sigma \Box S) \tag{5.24}
\]

and

\[
L^{(2)}_{\text{(term)}} = 2\frac{\sigma}{\mu} [\bar{\xi} \gamma^\mu \partial_\mu \chi + \bar{\chi} \eta] - \frac{1}{2\mu} \bar{\eta} \eta - \frac{1}{2m^2} \bar{\eta} \gamma^\mu \partial_\mu \eta \\
+ \frac{1}{m^2} [-4\bar{\eta} (\gamma^\mu \partial_\mu) \eta - 16\bar{\eta} \Box \chi - 16\bar{\chi} (\gamma^\mu \partial_\mu) \chi]. \tag{5.25}
\]

A notable feature of the above Lagrangians is that they can be interpreted as Lorentz invariant Lagrangians in their own right, despite the initial time–space split that was used to arrive at them. In this context, we would interpret the bosonic fields as Lorentz scalars and the fermionic fields as Lorentz spinors. However, the stress tensor of this scalar–spinor theory is not the same as that of the ‘original’ theory, and hence the integral for angular momentum is quite different to that of the original theory, so one cannot read off the spins of the propagated modes in the original theory in any obvious way. However, the formalism is well-suited to the task of determining all possible unitary theories. Once we have these theories, other methods must be used to determine the helicity content (in the case of massive modes, because helicity is not defined for massless particles in 3D).
5.1.1. Check of supersymmetry. To determine the supersymmetry transformations of the variables \((N, \xi, \phi)\) and \((\eta, \chi)\), we must consider the combined transformations

\[
\delta_h^{\mu\nu} = \delta_\epsilon h^{\mu\nu} + \partial(\mu \nu (comp)) , \quad \delta \psi_\mu = \delta_\epsilon \psi_\mu + \partial_\mu \zeta^{(comp)},
\]

where the \(\delta_\epsilon\) variations are those of (5.8) and the parameters of the (compensating) gauge transformations must be chosen such that the combined transformations preserve the gauge choices (5.13) and (5.15). This requirement implies that

\[
v(0) = \bar{\epsilon} \gamma^\mu \partial_\mu \eta, \quad v_i = -\frac{1}{2} \bar{\epsilon} \nabla^2 \partial_i \chi,
\]

and that

\[
\zeta^{(comp)} = -\frac{1}{4} \left[ \phi + \gamma^i \partial_i \bar{\epsilon} \left( \gamma^0 \phi - \xi \right) \right] \epsilon.
\]

One then finds that

\[
\delta N = -\bar{\epsilon} \gamma^\mu \partial_\mu \eta, \quad \delta \xi = -\frac{1}{2} \bar{\epsilon} \eta, \quad \delta \phi = -\bar{\epsilon} \chi, \quad \delta S = \frac{1}{2} \bar{\epsilon} \gamma^\mu \partial_\mu \chi + \frac{1}{4} \bar{\epsilon} \eta,
\]

and that

\[
\delta \chi = -\frac{1}{4} \gamma^\mu \epsilon \partial_\mu \phi + \frac{1}{2} \bar{\epsilon} \epsilon + \frac{1}{4} \bar{\epsilon} \epsilon, \quad \delta \eta = -\frac{1}{2} N \epsilon - \frac{1}{2} \gamma^\mu \epsilon \partial_\mu \xi.
\]

One may verify that all four Lagrangians (5.22) are invariant under these transformations.

5.2. Unitarity

We now use the above results to find all unitary theories within the class of the theories parametrized by \((\sigma, \mu, m, \tilde{m})\). We shall do this separately for the bosonic part and the fermionic bilinear part.

5.2.1. Bosonic part. The \(N\) field is auxiliary in (5.24) and can be eliminated to yield the equivalent Lagrangian

\[
L^{bos} = \frac{1}{2} (m^2 + \tilde{m}^2) (\Box \phi)^2 + \frac{1}{2m^2} \xi \Box \xi - \frac{m^2}{(m^2 + \tilde{m}^2) \mu} \xi \Box \phi - \frac{\sigma}{2} \left( \frac{\tilde{m}^2 - m^2}{m^2 + \tilde{m}^2} \right) \phi \Box \phi
\]

\[
- \frac{m^4 \tilde{m}^2}{2(m^2 + \tilde{m}^2)} \left[ \sigma^2 \phi^2 - 2 \sigma \mu \phi \xi + \left( \frac{1}{\mu^2} - \frac{(m^2 + \tilde{m}^2) \sigma}{m^2 m^2} \right) \xi^2 \right]
\]

\[
+ \frac{2}{\tilde{m}^2} S (\Box - \sigma \tilde{m}^2) S.
\]

There are ghosts unless the \((\Box \phi)^2\) term is absent, which requires that \(m^2 + \tilde{m}^2 \to \infty\). We may take \(\tilde{m}^2 \to \infty\) keeping \(m^2\) fixed or vice versa. We shall consider these two possibilities in turn.

- \(\tilde{m}^2 \to \infty\): in this case it is convenient to set

\[
\xi = m \zeta,
\]

after which the Lagrangian becomes

\[
L^{bos} = \frac{1}{2} \left[ -\sigma \phi \Box \phi + \zeta \Box \zeta \right] - \frac{1}{2} m^2 \left[ \sigma^2 \phi^2 + 2 \sigma \frac{m}{\mu} \phi \zeta + \left( \frac{m^2 - \sigma \mu^2}{\mu^2} \right) \zeta^2 \right].
\]

\(^8\) There are special cases for which \(N\) occurs only linearly, in which case it is a Lagrange multiplier for a constraint, but the solution of the constraint turns out to yield models that can also be obtained as limits of the generic ones obtained by integrating out \(N\).
This result generalizes that of [4] to allow for $\sigma \neq -1$ and $|\mu| \neq \infty$. We see that $\sigma \leq 0$ is necessary for unitarity.

Consider first the $\sigma < 0$ case; we may then choose $\sigma = -1$ without loss of generality. In terms of the row 2-vector $\Phi^T = (\varphi, \zeta)$, the Lagrangian takes the form

$$L_{\text{bos}} = \frac{1}{2} \Phi^T \Box \Phi - \frac{1}{2} \Phi^T M^2 \Phi,$$

where $M^2$ is a mass matrix with eigenvalues $m_{\pm}$ such that

$$m_{+} m_{-} = m^2, \quad |m_{+} - m_{-}| = \frac{m^2}{|\mu|}.$$  \hfill (5.34)

We thus find agreement with [2], although it is not obvious from this analysis that both modes have spin 2.

When $\sigma = 0$ we get the Lagrangian

$$L_{\text{bos}} = \frac{1}{2} \left[ \zeta \Box \zeta - \left( \frac{m^2}{\mu} \right)^2 \zeta^2 \right] - \frac{1}{2} m^2 \varphi^2.$$  \hfill (5.35)

The variable $\varphi$ is now auxiliary so we have a single mode with mass $m^2/\mu$; it will be shown that this mode has spin 2, so the model is, at least at the linearized level, a ‘new topologically massive gravity’ (NTMG).

• $m^2 \to \infty$: in this case we have

$$L_{\text{bos}} = -\frac{1}{\mu} \xi \Box \varphi + \frac{\sigma}{2} \varphi \Box \varphi - \frac{1}{2} \tilde{m}^2 \left[ \sigma^2 \varphi^2 - 2 \frac{\sigma}{\mu} \varphi \xi + \left( \frac{1}{\mu^2} - \frac{\sigma}{\tilde{m}^2} \right) \xi^2 \right]$$

$$+ \frac{2}{\tilde{m}^2} S(\Box - \sigma \tilde{m}^2) S.$$ \hfill (5.36)

Given that $\sigma \neq 0$, we may simplify the Lagrangian by using the new variables $(\varphi', \zeta')$ defined by

$$\varphi = \varphi' + \frac{\zeta'}{\sigma}, \quad \xi = -\mu \zeta'.$$ \hfill (5.37)

One then finds that

$$L_{\text{bos}} = \sigma \frac{\varphi'(\Box - \sigma \tilde{m}^2) \varphi'}{2\sigma} - \frac{1}{2\sigma} \zeta'(\Box - \sigma \tilde{m}^2) \zeta' + \frac{2}{\tilde{m}^2} S(\Box - \sigma \tilde{m}^2) S.$$ \hfill (5.38)

We see that either $\varphi'$ or $\zeta'$ is a ghost mode, but we can still get a unitary theory by taking the ghost mass to infinity. Returning to (5.36) and taking $\mu^2 \to \infty$ we get the Lagrangian

$$L_{\text{bos}} = \sigma \frac{\varphi(\Box - \sigma \tilde{m}^2) \varphi}{2\sigma} + \frac{2}{\tilde{m}^2} S(\Box - \sigma \tilde{m}^2) S + \frac{\sigma}{2\tilde{m}^2} \xi^2.$$ \hfill (5.39)

The variable $\xi$ is now auxiliary and may be trivially eliminated, resulting in a theory that is unitary and tachyon-free for $\sigma > 0$; we may choose $\sigma = 1$ without loss of generality. This unitary ‘scalar massive gravity’ (SMG) theory propagates two scalar modes of mass $\tilde{m}$; one mode comes from the metric and the other comes from the ‘auxiliary’ scalar $S$.

If $\sigma = 0$ then (5.36) becomes

$$L_{\text{bos}} = -\frac{1}{\mu} \xi \Box \varphi - \frac{\tilde{m}^2}{2\mu^2} \xi^2 + \frac{2}{\tilde{m}^2} S \Box S.$$ \hfill (5.40)

We see that $\xi$ is auxiliary again, but its elimination now yields the non-unitary Lagrangian

$$L_{\text{bos}} = \frac{1}{2\tilde{m}^2} (\Box \varphi)^2 + \frac{2}{\tilde{m}^2} S \Box S.$$ \hfill (5.41)
To summarize, there are essentially just three ways to get a unitary Lagrangian when either \(m^2\) or \(\tilde{m}^2\) is finite. These are

1. \(\tilde{m}^2 \rightarrow \infty\) and \(\sigma = -1\). This yields GMG.
2. \(\tilde{m}^2 \rightarrow \infty\) and \(\sigma = 0\). This yields ‘new topologically massive gravity’ (NTMG), but this model may have problems at the interacting level. The massless version is the ‘pure-K’ model considered by Deser [4].
3. \(m^2 \rightarrow \infty\) and \(\mu^2 \rightarrow \infty\), and \(\sigma = 1\). This is the bosonic sector of SMG; it is equivalent to 3D gravity coupled to a scalar field with a particular potential that linearizes to give a particle of mass \(\tilde{m}\), plus an ‘auxiliary’ scalar describing another particle of mass \(\tilde{m}\).

5.2.2. Fermionic part. It is convenient to rewrite the \(1/\tilde{m}^2\) contribution to (5.25) so that

\[
L^{(2)}_{\text{ferm}} = -\frac{1}{2\tilde{m}^2} \bar{\eta} \gamma^\mu \partial_\mu \eta - \frac{1}{2\tilde{m}^2} \bar{\beta} \gamma^\mu \partial_\mu \beta + 2\sigma \bar{\chi} \gamma^\mu \partial_\mu \chi - \frac{1}{\tilde{m}^2} \bar{\xi} \gamma^\mu \partial_\mu \xi
+ 2\sigma \bar{\chi} \eta - \frac{1}{2\mu} \bar{\eta} \eta - \frac{1}{2\tilde{m}^2} \bar{\xi} (\eta - \beta).
\] (5.42)

This involves two new spinor variables \((\beta, \lambda)\) but \(\lambda\) is a Lagrange multiplier that imposes the constraint \(\beta = \eta + 2\gamma^\mu \partial_\mu \chi\), whereupon the Lagrangian reduces to the previous one of (5.25).

The kinetic terms for \((\chi, \lambda)\) can be brought to diagonal form in new variables but the result is that there is a ghost unless either (i) \(\tilde{m}^2 \rightarrow \infty\) or (ii) \(m^2 \rightarrow \infty\) and \(\mu^2 \rightarrow \infty\). We shall consider in turn these two possibilities.

- \(\tilde{m}^2 \rightarrow \infty\): the fermionic Lagrangian simplifies to

\[
L^{\text{ferm}} = -\frac{1}{2m^2} \bar{\eta} \gamma^\mu \partial_\mu \eta + 2\sigma \bar{\eta} \gamma^\mu \partial_\mu \chi + 2\sigma \bar{\chi} \eta - \frac{1}{2\mu} \bar{\eta} \eta.
\] (5.43)

Unitarity requires \(\sigma < 0\) and we may choose \(\sigma = -1\) without loss of generality. By setting

\[
\eta = m\eta', \quad \chi = \frac{1}{2} \chi',
\] (5.44)

and introducing a row 2-vector \(\Xi^T = (\eta', \chi')\), we can put the Lagrangian in the form

\[
L^{\text{ferm}} = -\frac{1}{2} \Xi (\gamma^\mu \partial_\mu - M) \Xi,
\] (5.45)

where \(M\) is a diagonalizable mass matrix such that

\[
\det M^2 = m^4, \quad \text{tr} M^2 = \frac{m^2(m^2 + 2\mu^2)}{\mu^2}.
\] (5.46)

This implies that \(M^2\) has eigenvalues \(m^2\), the squared masses of GMG. Supersymmetry implies that the two propagated modes have spin 3/2, but this fact is not obvious from this approach.

When \(\sigma = 0\) the Lagrangian (5.43) simplifies to

\[
L^{\text{ferm}} = -\frac{1}{2} \bar{\eta} \left( \gamma^\mu \partial_\mu + \frac{m^2}{\mu} \right) \eta'.
\] (5.47)

This is the fermionic part of NTMG. As expected, it propagates a single mode of mass \(m^2/\mu\). Supersymmetry implies that this mode has spin 3/2.
\( m^2 \to \infty \) and \( \mu^2 \to \infty \): taking the limit \( m^2 \to \infty \) does not immediately remove the ghost modes from (5.25) but it removes the kinetic term for \( \eta \). If we also remove the mass term by taking \( |\mu| \to \infty \) then \( \eta \) becomes a Lagrange multiplier for the constraint
\[
\lambda = 4 \bar{m}^2 \sigma \chi.
\] (5.48)

Using this we arrive at the Lagrangian
\[
L_{\text{ferm}} = -\frac{1}{2 \bar{m}^2} \bar{\beta} \gamma^\mu \partial_\mu \beta - \frac{1}{m^2} \bar{\chi} \gamma^\mu \partial_\mu \chi + 2 \sigma \bar{\beta} \chi.
\] (5.49)

We now see that unitarity also requires \( \sigma \geq 0 \). When \( \sigma > 0 \) we may choose \( \sigma = 1 \) without loss of generality. By setting
\[
\beta = \bar{m} \beta', \quad \chi = \frac{1}{2} \chi',
\] (5.50)

and again introducing a row 2-vector \( \Xi = (\eta', \chi') \), we can again put the Lagrangian in the form (5.45) but now with a mass matrix \( M \) such that \( M^2 \) has both eigenvalues equal to \( \bar{m}^2 \). This is to be expected because in the supersymmetrization of SMG the ‘auxiliary’ scalar \( S \) propagates with mass \( \bar{m} \), so we need two spin-1/2 modes of this mass.

When \( \sigma = 0 \), we get the very simple Lagrangian
\[
L_{\text{ferm}} = -\frac{1}{2} \bar{\beta} \gamma^\mu \partial_\mu \beta',
\] (5.51)

which propagates a single massless mode. This is the superpartner to the ‘Deser’ mode of the ‘pure-K’ theory.

To summarize, the fermionic Lagrangian provides exactly the modes implied by supersymmetry given our earlier bosonic results.

6. The three unitary theories

Our investigations so far can be summarized by saying that among the generic ‘higher-derivative’ supergravity theories there are three classes of unitary theories.

- **GMSG or ‘general massive supergravity’**: this is obtained by setting \( \sigma = -1 \) and \( \bar{m}^2 = \infty \), so that
\[
I_{\text{GMSG}} = \frac{1}{\kappa^2} \int d^4x \left\{ e \left[ -L_{(EH)} + \frac{1}{m^2} L_K \right] + \frac{1}{\mu} \mathcal{L}_{\text{LCS}} \right\} + \text{fermions}.
\] (6.1)

This includes the supersymmetric extensions of both ‘new massive gravity’ (NMG) and ‘topologically massive gravity’ (TMG), obtained as the limiting cases in which \( \mu^2 \to \infty \) or \( m^2 \to \infty \), respectively.

- **NTMSG or ‘new topologically massive supergravity’**: this is obtained by setting \( \sigma = 0 \) and \( \bar{m}^2 = \infty \), and so
\[
I_{\text{NTMSG}} = \frac{1}{\kappa^2} \int d^4x \left\{ \frac{1}{m^2} e L_K + \frac{1}{\mu} \mathcal{L}_{\text{LCS}} \right\} + \text{fermions}.
\] (6.2)

The bosonic action might be considered as a limit of GMG in which \( \sigma \to 0 \) but there are various reasons for considering it separately. In contrast to NMG and TMG, one cannot get to the theory with \( \sigma = 0 \) just by taking limits of particle masses. Also, there is an ‘accidental’ superconformal invariance of the linearized theory when \( \sigma = 0 \), and this means that the quadratic approximation leads to a linearized Minkowski space field theory with a ‘missing’ field equation. Interpretation of the linearized results is therefore not straightforward. Nevertheless, we will show here that this linearized theory has many features in common with TMG; hence the name we choose for it. In particular, it propagates a single spin-2 mode, and its fermionic counterpart propagates a single spin-3/2 mode.
SMSG or ‘scalar massive supergravity’. This is obtained by setting $\sigma = 1$ and both $\mu = \infty$ and $m^2 = \infty$, so that
\[ I_{\text{SMSG}} = \frac{1}{k^2} \int d^3x \ e \left[ L_{(\text{EH})} + \frac{1}{8\hbar^2} L_{R^2} \right] + \text{fermions}. \] (6.3)

In the context of the purely bosonic theory, and ignoring the supergravity ‘auxiliary’ field $S$, this is known to be equivalent to a scalar field coupled to gravity with a potential that gives the scalar field a mass $\tilde{m}$ in the linearized limit (see [22] for a review). This model has never been supersymmetrized, to our knowledge.

We shall now consider in turn these three classes of unitary supergravity theories and determine the helicities of the different fields.

### 6.1. General massive supergravity

The quadratic approximation to the Lagrangian of the ‘general massive supergravity’ model is
\[ L^{(2)}_{\text{GMSG}} = L^{(2)}_{\text{(bos)}} + L^{(2)}_{\text{(ferm)}}, \] (6.4)
where
\[ L^{(2)}_{\text{(bos)}} = \frac{1}{2} h_{\mu\nu} G^{\text{lin}}_{\mu\nu} + 2S^2 + \frac{1}{2\mu} h_{\mu\nu} C^{\text{lin}}_{\mu\nu} - \frac{1}{2m^2} e^{\mu\rho\sigma} h_{\mu\nu} \partial_\sigma C^{\text{lin}}_{\rho\nu}, \] \[ L^{(2)}_{\text{(ferm)}} = \bar{\psi}_\mu R^{\text{lin}}_{\mu} + \frac{1}{2\mu} \bar{\psi}_\mu C^\mu - \frac{1}{2m^2} \bar{\psi}_\mu (\gamma^\nu \partial_\nu) C^\mu. \] (6.5)
The field $S$ is genuinely auxiliary and may be trivially eliminated. It was observed in [2] that the metric perturbation field equation can be written as
\[ \mathcal{O}(-m_-) \mathcal{O}(m_+) h^\mu_{\nu} = 0, \quad \eta^{\mu\nu} h_{\mu\nu} = 0, \quad \partial^\mu h_{\mu\nu} = 0, \] (6.6)
where the masses $m_{\pm}$ are given by
\[ m^2 = m_+ m_-, \quad \mu = \frac{m_+ m_-}{(m_+ - m_-)}, \] (6.7)
and $\mathcal{O}$ is the following operator, defined for arbitrary mass $m$:
\[ [\mathcal{O}(m)]^\nu_\mu = \delta^\nu_\mu + \frac{1}{m} \varepsilon^\nu_{\mu\sigma} \partial_\sigma. \] (6.8)
Because of the linearized Bianchi identity $\partial^\mu G^{\text{lin}}_{\mu\nu} = 0$, equations (6.6) propagate two spin-2 modes, with masses $m_+$ for helicity $+2$ and mass $m_-$ for helicity $-2$. Here we shall present a novel proof of this fact.

Consider first the special case with $m_+ = m_-; \in$ this case we need to prove that equations (6.6) are equivalent to the 3D version of the standard Fierz–Pauli (FP) equation [21]. Actually, Fierz and Pauli presented their results in terms of one dynamical equation and two subsidiary conditions. For a 3D symmetric tensor field $\tilde{h}$, these equations are
\[ \left( \Box - m^2 \right) \tilde{h}_{\mu\nu} = 0, \quad \eta^{\mu\nu} \tilde{h}_{\mu\nu} = 0, \quad \partial^\mu \tilde{h}_{\mu\nu} = 0. \] (6.9)
We may solve the differential subsidiary condition by writing
\[ \tilde{h}_{\mu\nu} = G^{\text{lin}}_{\mu\nu}(h), \] (6.10)
where $G^{\text{lin}}$ is the linearized Einstein tensor for a new symmetric tensor field $h$. The remaining subsidiary constraint and the dynamical equation are, when expressed as equations for $h$,
precisely those of (6.6) in the special case that \( m_+ = m_- \). This proves the equivalence of linearized NMG to the 3D FP theory. To obtain the analogous result for GMG, one must start from the parity-violating modification of the 3D FP equation found by replacing the wave equation for \( \tilde{h} \) with the equation

\[
O(m_-) \partial_x O(m_+) \frac{1}{\mu^2} \mu^2 R_{\mu \nu} = 0. \tag{6.11}
\]

Given this result for the bosonic Lagrangian, supersymmetry implies that the two modes of masses \( m_\pm \) propagated by the fermionic Lagrangian must have either spin \( \frac{3}{2} \) or spin \( \frac{5}{2} \). We shall now show that these modes have spin \( \frac{3}{2} \). The \( \psi_{\mu} \) field equation is

\[
\frac{1}{2\mu} \left( \gamma^\tau \partial_\tau \right) C_{\mu \nu}^{(\text{lin})} - \frac{1}{m^2} \epsilon^{\mu \nu \rho \tau} \partial_\rho C_{\tau \nu}^{(\text{lin})} = 0. \tag{6.12}
\]

Observe that this equation implies that

\[
\gamma \cdot R_{\mu \nu} = 0. \tag{6.13}
\]

To go further it is convenient to consider first the limiting case in which \( m^2 \to \infty \): in this case we have the equation

\[
(\gamma \cdot \partial) R_{\mu \nu}^{(\text{lin})} = -2\mu R_{\nu}^{(\text{lin})} - \epsilon^{\mu \nu \rho \tau} \partial_\rho R_{\tau \nu}^{(\text{lin})} \quad (m^2 = \infty), \tag{6.14}
\]

which can be written as

\[
[\hat{O}(\mu)] R_{\nu}^{(\text{lin})} = 0, \tag{6.15}
\]

where

\[
\hat{O}(\mu)_{\nu}^{\mu} = \delta_{\nu}^{\mu} + \frac{1}{2\mu} \left( \delta_{\nu}^{\mu} (\gamma \cdot \partial) - \epsilon_{\nu}^{\tau \mu} \partial_\tau \right). \tag{6.16}
\]

We know from studies of super-TMG that this equation must propagate a single spin \( \frac{3}{2} \) mode of mass \( \mu \) [14, 23]. Next, we observe that the generic field equation (6.12) can be written in the form

\[
[\hat{O}(\mu)] R_{\nu}^{(\text{lin})} = 0. \tag{6.17}
\]

There is a precise parallel with our analysis of the spin-2 equation of GMG, as expected from supersymmetry. The helicity +2 propagated with mass \( m_+ \) is accompanied by a helicity +\( \frac{3}{2} \) mode of the same mass, and the same for the negative helicity states but with mass \( m_- \).

6.2. New topologically massive supergravity

The quadratic approximation to the Lagrangian of the ‘new topologically massive supergravity’ model is

\[
L_{\text{NTMSG}}^{(2)} = L_{\text{bos}}^{(2)} + L_{\text{ferm}}^{(2)}, \tag{6.18}
\]

where

\[
L_{\text{bos}}^{(2)} = \frac{1}{2\mu} h_{\mu \nu} C_{\nu \tau}^{(\text{lin})} - \frac{1}{2m^2} \epsilon^{\mu \tau \rho \nu} h_{\rho \tau} \partial_\tau C_{\nu \tau}^{(\text{lin})}, \tag{6.19}
\]

\[
L_{\text{ferm}}^{(2)} = \frac{1}{2\mu} \bar{\psi}_\mu C_{\nu \tau}^{(\text{lin})} - \frac{1}{2m^2} \bar{\psi}_\mu (\gamma \cdot \partial) C_{\nu \tau}^{(\text{lin})}. \tag{6.19}
\]

As we have seen, this model propagates one bosonic mode and one fermionic mode, both of mass

\[
\hat{\mu} = m^2 / \mu. \tag{6.20}
\]

We now show that these modes have spin 2 and spin \( \frac{3}{2} \), respectively.
The linearized field equation for $h$ can be written as
\[ [\mathcal{O}(\tilde{\mu})]_{\mu}^{\nu} C_{\mu\nu}^{\text{lin}} = 0, \quad [\mathcal{O}(\tilde{\mu})]_{\nu}^{\nu} = \delta_{\mu}^{\nu} + \frac{1}{\tilde{\mu}} \epsilon_{\mu}^{\nu} \partial_{\nu}. \] (6.21)

The tensor operator $\mathcal{O}(\tilde{\mu})$ is the ‘square-root’ of the ‘Proca’ operator [24]. Despite appearances, the tensor $\mathcal{O}(\mathcal{M})C_{\mu\nu}^{\text{lin}}$ is symmetric by virtue of the tracelessness of $C_{\mu\nu}^{\text{lin}}$ and the ‘Bianchi’ identity
\[ \partial_{\mu} C_{\mu\nu}^{\text{lin}} = 0. \] (6.22)

As a consequence of this identity, we have the further identity
\[ -\tilde{\mu}^{2} [\mathcal{O}(\tilde{\mu}) C_{\mu\nu}^{\text{lin}}]_{\mu\nu} = (\Box - \tilde{\mu}^{2}) C_{\mu\nu}^{\text{lin}}, \] (6.23)

from which it follows that the field equation $\mathcal{O}(\tilde{\mu}) C_{\mu\nu}^{\text{lin}} = 0$ implies that
\[ \Box C_{\mu\nu}^{\text{lin}} = 0. \] (6.24)

The combination of this equation with (6.22) is equivalent to the FP equation for the symmetric tensor $C_{\mu\nu}^{\text{lin}}$. This is not the independent field, of course, but this does not matter because the equation $C_{\mu\nu}^{\text{lin}} = 0$ implies that $h$ is pure gauge. One may expand on this argument along the lines presented for NMG in [2], but here we present an alternative argument that extends the one used above for GMG. Starting with the FP equations in the form (6.9) for the symmetric tensor field $\tilde{\mu}_{\mu\nu}$, we may solve both of the subsidiary conditions by writing
\[ \tilde{h}_{\mu\nu} = C_{\mu\nu}^{\text{lin}}(h), \] (6.25)

where $C_{\mu\nu}^{\text{lin}}(h)$ is the Cotton tensor for a new symmetric tensor field $h$. The remaining dynamical equation for $\tilde{h}$ is, when expressed as an equation for $h$, precisely (6.24).

We now turn to the linearized equation for the vector spinor field:
\[ (\gamma^{\nu} \partial_{\nu} - \tilde{\mu}) C_{\mu}^{\nu} = 0. \] (6.26)

This propagates spin $3/2$ because the spin-1/2 components are absent as a consequence of the identities $\partial \cdot C \equiv 0$ and $\gamma \cdot C \equiv 0$.

6.3. Scalar massive supergravity

The quadratic approximation to the Lagrangian of the ‘scalar massive supergravity’ model is
\[ L_{\text{SMSSG}}^{(2)} = L_{\text{(bos)}}^{(2)} + L_{\text{(ferm)}}^{(2)}, \] (6.27)

where
\[ L_{\text{(bos)}}^{(2)} = -\frac{1}{2} h^{\mu\nu} G_{\mu\nu}^{(\text{lin})} - 2 S^{2} + \frac{1}{\tilde{m}} \left[ \frac{1}{8} R_{\text{lin}}^{2} + 2 S \Box S \right], \] (6.28)

\[ L_{\text{(ferm)}}^{(2)} = -\bar{\psi}_{\mu} R_{\mu}^{(\text{lin})} - \frac{1}{2 \tilde{m}} \bar{\gamma} (\bar{R}_{\text{lin}} \cdot \gamma)(\gamma \cdot \partial)(\gamma \cdot R_{\text{lin}}). \]

In this case the field $S$ is not actually auxiliary; it propagates a spin-zero mode of mass $\tilde{m}$. It is known that the one mode of mass $\tilde{m}$ propagated by the metric part of the bosonic Lagrangian also has spin zero, so supersymmetry implies that the fermionic part must propagate two spin-1/2 modes of mass $\tilde{m}$. To verify this, we rewrite the ‘fermionic’ Lagrangian as
\[ L_{\text{(ferm)}}^{(2)} = -\bar{\psi}_{\mu} R_{\mu}^{(\text{lin})} - \frac{1}{2} \bar{\rho} \gamma \cdot \partial \rho + \bar{\lambda} (\tilde{m} \rho - \gamma \cdot R_{\text{lin}}). \] (6.29)

where the new spinor field $\lambda$ is a Lagrange multiplier field that constrains the other new spinor field $\rho$ to equal $\gamma \cdot R_{\text{lin}}/\tilde{m}$. The general solution of the $\psi_{\mu}$ field equation is
\[ \psi_{\mu} = \frac{i}{2} \gamma_{\mu} \lambda + \partial_{\mu} \epsilon. \] (6.30)
Thus, $\psi$ is determined in terms of $\lambda$ up to an irrelevant gauge transformation. Using this result, the $\lambda$ equation becomes

$$\gamma^\tau \partial_\tau \lambda = \tilde{m} \rho,$$

while the $\rho$ field equation is

$$\gamma^\tau \partial_\tau \rho = \tilde{m} \lambda.$$

It follows that

$$(\gamma^\tau \partial_\tau \pm \tilde{m})(\lambda \pm \rho) = 0.$$  

which implies two spin-1/2 modes of mass $\tilde{m}$. 

7. $\mathcal{N} > 1$ massive supergravities

Our results for $\mathcal{N} = 1$ 3D supergravities can be extended to $\mathcal{N} = 2$. The linearized limit of the general parity-preserving curvature-squared model was considered in [13] and those results were adapted in [2] to deduce some features of the $\mathcal{N} = 2$ extension of the new massive gravity model. Here we present more details and give the extension to GMG, i.e. we allow for parity-violating terms.

Any $\mathcal{N} = 2$ model can be viewed in $\mathcal{N} = 1$ terms. In the context of the GMG models, this involves a decomposition of the $\mathcal{N} = 2$ graviton multiplet into an $\mathcal{N} = 1$ graviton multiplet and another $\mathcal{N} = 1$ multiplet that propagates helicities $\pm(\frac{3}{2}, 1)$. We begin by presenting this new multiplet.

7.1. The spin $(3/2, 1)$ multiplet

Consider the following infinitesimal supersymmetry transformations connecting a ‘second’ gravitino field $\psi'_\mu$ to a vector field $A_\mu$ and a ‘second’ scalar auxiliary field $S'$:

$$\delta \psi'_\mu = \frac{1}{4} \gamma^\tau \gamma_\mu \epsilon A_\tau + \frac{1}{2} \gamma_\mu \epsilon S', \quad \delta A_\mu = \frac{1}{2} \epsilon^{\mu\nu\rho} A_\nu F_{\rho\nu},$$
$$\delta S' = \frac{1}{4} \epsilon \cdot (\gamma^\tau \partial_\tau) R_{(\text{lin})}' - \frac{1}{2} F_{\mu\nu} F^{\mu\nu}.$$

Putting this together we get the following Lagrangian:

$$L' = L'_\text{(bos)} + L'_\text{(ferm)},$$

where

$$L'_\text{(bos)} = -\frac{1}{4m^2} F^{\mu\nu} F_{\mu\nu} - \frac{1}{4}\epsilon^{\mu\nu\rho} A_\mu F_{\nu\rho} - \frac{1}{2} A^\mu A_\mu,$$
$$L'_\text{(ferm)} = \bar{\psi}'_\mu (\gamma^\tau \partial_\tau) C_{(\text{lin})}'^\mu + \frac{1}{2\mu} \bar{\psi}'_\mu R'_{\mu\nu},$$
$$L' = \bar{\psi}'_\mu (\gamma^\tau \partial_\tau) C_{(\text{lin})}'^\mu + \frac{1}{2\mu} \bar{\psi}'_\mu R'_{\mu\nu}.$$
This Lagrangian propagates one helicity \( \left( \frac{3}{2}, 1 \right) \) supermultiplet with mass \( m_+ \) and one helicity \( \left( -\frac{1}{2}, -1 \right) \) supermultiplet with mass \( m_- \). In the special case that \( m_- \to \infty \) for fixed \( m_+ \), which corresponds to the \( m^2 \to \infty \) limit, we have a supersymmetrization of the ‘odd-dimensional self-dual’ (or ‘Proca square-root’) model of [24].

7.2. Linearized \( \mathcal{N} = 2 \) massive supergravity

The fields of the off-shell linearized \( \mathcal{N} = 2 \) supergravity are the metric perturbation \( h_{\mu\nu} \), two gravitini \( \psi^a_\mu (a = 1, 2) \), a vector \( A_\mu \) and an auxiliary scalar field \( S^{ab} \) that is symmetric and traceless in its two indices, which we can interpret as indices of the \( SO(2) \) automorphism group of the \( \mathcal{N} = 2 \) supersymmetry algebra. The \( \mathcal{N} = 2 \) infinitesimal supersymmetry transformations of these fields, with anticommuting Majorana spinor parameters \( \epsilon^a \), are

\[
\begin{align*}
\delta h_{\mu\nu} &= \epsilon^a Y_{(\mu}(\psi_{\nu)}^a), \\
\delta \psi^a_\mu &= -\frac{1}{2} \gamma^{\rho\sigma} \partial_\rho h_{\mu\sigma} \epsilon^a - \frac{1}{4} \epsilon^{ab} \gamma^\tau \gamma^\mu \gamma^\nu \epsilon^b A_\tau + \frac{1}{2} \gamma^\mu \epsilon^b S^{ab}, \\
\delta A_\mu &= \frac{1}{2} \epsilon^{ab} \epsilon^a \gamma_\nu \gamma_\mu R^{\nu b}, \\
\delta S^{ab} &= \frac{1}{2} \epsilon^a \gamma^\nu \cdot R^{\nu b} - \frac{1}{2} \epsilon^{ab} \epsilon^\nu \gamma^\tau \cdot R^{\tau c}.
\end{align*}
\]  

(7.8)

The following three Lagrangians are invariant under these transformations

\[
\begin{align*}
L_{N=2}^{\text{grav}} &= \frac{1}{2} h_{\mu\nu} G_{\mu\nu}^{\text{(lin)}} + \frac{1}{2} \epsilon^a \gamma_\mu C_a^{\text{(lin)}}, \\
L_{N=2}^{\text{form}} &= \frac{1}{2} \psi^a_\mu \cdot R_a^{\text{(lin)}}, \\
L_{N=2}^{\text{aux}} &= -\frac{1}{2} A^a A_\mu A^{\mu a},
\end{align*}
\]

(7.9)

Putting these results together we get the following Lagrangian for the \( \mathcal{N} = 2 \) supersymmetric extension of linearized GMG:

\[
L_{\text{GMG}}^{N=2} = L_{\text{grav}}^{N=2} + L_{\text{form}}^{N=2} + L_{\text{aux}}^{N=2},
\]

(7.10)

where

\[
\begin{align*}
L_{\text{grav}}^{N=2} &= \frac{1}{2} h_{\mu\nu} G_{\mu\nu}^{\text{(lin)}} + \frac{1}{2} \epsilon^a \gamma_\mu C_a^{\text{(lin)}}, \\
L_{\text{form}}^{N=2} &= \frac{1}{4} A^a A_\mu A^{\mu a}, \\
L_{\text{aux}}^{N=2} &= \frac{1}{2} \psi^a_\mu \cdot R_a^{\text{(lin)}},
\end{align*}
\]

(7.11)

These formulas show that \( \mathcal{N} = 2 \) supersymmetry concisely combines the different mechanisms in 3D of assigning mass to modes of spin 1, \( \frac{3}{2} \) and 2.

8. Conclusions and outlook

Motivated by recent work on massive gravity theories in three dimensions, we have constructed the full off-shell supersymmetric \( \mathcal{N} = 1 \) 3D supergravity theory with cosmological and Lorentz–Chern–Simons terms, and general curvature-squared terms. The general model of this type is parametrized by four mass parameters \( (M, \mu, m, \tilde{m}) \) and a dimensionless coefficient \( \sigma \) of the Einstein–Hilbert term that is unity for standard 3D general relativity. We have found that the maximally symmetric vacua, with cosmological constant \( \Lambda \), are characterized by two curves in the \((\Lambda, M^2)\) plane, and all vacua on one of them are supersymmetric. This family...
of supersymmetric vacua includes the Minkowski vacuum as a limiting case. Apart from this Minkowski vacuum, the overall picture is remarkably different from that found in [10] for the non-supersymmetric ‘new massive gravity’ (NMG) model. This is due to the new ‘auxiliary’ field in the supergravity theory; although it really is auxiliary in the NMG case, its equation of motion is cubic with coefficients that depend on the scalar curvature $R$. Because of this, it is unclear whether any of the conclusions of [10] concerning unitarity in adS vacua, and the central charges of the boundary CFTs, will still apply in the supergravity case. Thus, one obvious direction for further research is a unitarity/stability analysis for adS vacua.

In the context of a possible adS/CFT relation, a crucial role is played by the central charges of the asymptotic Virasoro algebra. While in this paper we did not attempt to compute these charges from first principles (as could be done, e.g., by following the original Brown–Henneaux argument [25]) a natural conjecture emerges from an application of a formula of [26], and of [27], who have demonstrated its applicability for generic (parity-preserving) higher curvature Lagrangians $L_3$ with adS$_3$ vacuum. This formula is

$$c = \frac{\ell}{2G_3} \frac{\delta L_3}{\delta R_{\mu\nu}},$$

(8.1)

where $G_3$ is Newton’s constant determined by $\kappa^2 = 16\pi G_3$. It is not clear to us whether this formula is still applicable in our case, in which there are also terms that couple curvature-squared terms to the extra scalar $S$. Nevertheless, if we assume that it is applicable, at least for the supersymmetric adS vacua with $S^2 = -\Lambda$, then we deduce that

$$c_L = \frac{3\ell}{2G_3} \left( \sigma + \frac{1}{\mu\ell} \right), \quad c_R = \frac{3\ell}{2G_3} \left( \sigma - \frac{1}{\mu\ell} \right),$$

(8.2)

where we have also included the known contribution of the parity-violating Lorentz–Chern–Simons term [28]. We note, in particular, that the values of the central charges coincide with those of pure TMG; in other words, the extra contributions due to the curvature-squared terms to the extra scalar $S$. Nevertheless, if we assume that it is applicable, at least for the supersymmetric adS vacua with $S^2 = -\Lambda$, then we deduce that

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(8.2)
have called it ‘new topologically massive gravity’ (NTMG). However, it is currently unclear whether this linearized theory is still consistent when interactions are included because the interactions break an accidental gauge invariance of the linearized theory.

We have also constructed the linearized $\mathcal{N} = 2$ massive supergravity, which propagates both a multiplet of helicities $(2, \frac{1}{2}, 1)$ and a multiplet of helicities $(-2, -\frac{1}{2}, -1)$, in general with different masses $m_{\pm}$. This model unifies the GMG model of [2] with the general spin-1 theory, i.e. the 3D Proca theory with a CS term. In particular the spin-1 sector of the $\mathcal{N} = 2$ super TMG is the self-dual spin-1 model of [24] whereas the spin-1 sector of the $\mathcal{N} = 2$ super NTMG is the topologically massive spin-1 theory of [5]. For parity-preserving models the representation theory of the super-Poincaré group is essentially the same for massive 3D particles as it is for massless 4D particles, so we expect that there is an $\mathcal{N} = 8$ massive supergravity theory and that $\mathcal{N} = 8$ is maximal. For parity-violating models the maximal value of $\mathcal{N}$ must be less than this.

An obvious next step is the construction of the full $\mathcal{N} = 2$ massive supergravity model. Given that the options for maximally symmetric vacua for $\mathcal{N} = 1$ are so different from those for $\mathcal{N} = 0$, one might think that they would again be different for $\mathcal{N} = 2$. However, a cosmological term in an $\mathcal{N} = 2$ theory could involve at most one scalar and would therefore break the $SO(2)$ symmetry. It therefore seems likely that vacua for $\mathcal{N} = 2$, and by extension for $\mathcal{N} > 2$, are determined by the truncation to $\mathcal{N} = 1$. Thus, we expect the results obtained here to survive the extension to higher $\mathcal{N}$.

One important motivation for our work that we have not yet mentioned is the possibility that some massive supergravity might be ultra-violet finite. The situation for NMG, to take the simplest case, is unclear to us. On the one hand it has been argued in [30] that NMG is super-renormalizable (as one might expect from the known renormalizability of the 4D ‘$R+\kappa$’ theory [31]). On the other hand, it was argued in [5] that NMG is not even renormalizable, but even if this is true it is still likely that super-NMG will be better behaved than NMG.

Finally we would like to mention that in the context of massive 3D Poincaré supersymmetry, an unconventional multiplet shortening may arise due to the possibility of non-central charges in the superalgebra [32]. It would be interesting to see whether such a mechanism can be realized for massive supergravity models of the type considered in this paper.

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Note added. Shortly after the first version of this paper appeared on the archives, a paper of Dalmazi and Mendonca appeared [33], in which the model that we have here called ‘new topologically massive gravity’ was discussed. (See also [34].)

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