A Lie-algebra model for a noncommutative space time geometry

B.-D. Dörfel
Institut für Physik, Humboldt-Universität zu Berlin
Invalidenstraße 110, D-10115 Berlin, Germany

May 16, 2002

Abstract

We propose a Lie-algebra model for noncommutative coordinate and momentum space. Based on a rigid commutation relation for the commutators of space time operators the model is quite constrained if one tries to keep Lorentz invariance as much as possible. We discuss the question of invariants esp. the definition of a mass.

Keyword(s): noncommutative geometry, Lie algebra

1 Introduction

After the motivation for a noncommutative geometry of space time has been demonstrated in numerous papers (s. e. g. [1-3]) there exists now a well established form of quantum field theory on noncommutative spaces [4-10] which allows to calculate Feynman diagrams and related values. Those theories show a characteristic mixing of infrared and ultraviolet divergencies [10,11], but the study of planar diagrams exhibits the fact that they cannot be handled without the usual ultraviolet cutoff in momentum space necessary for the treatment of ordinary renormalizable quantum field theories. On the other hand it was one of the motivations of noncommutative geometry to generate such a cutoff within the theory what has now definitely failed for the class of models mentioned above [11]. Therefore it seems still necessary to look for other possibilities to employ noncommutative space time geometry.

Generally speaking, such models are mainly contained in three classes depending on the structure of the commutator for space time coordinates.
a) The canonical structure:
\[ [X^\lambda, X^\mu] = i\Theta^{\lambda\mu} \]  
(1.1)

b) The Lie-algebra structure:
\[ [X^\lambda, X^\mu] = i\epsilon^{\lambda\mu}_\nu X^\nu \]  
(1.2)

c) The quantum space structure:
\[ [X^\lambda, X^\mu] = id^{\lambda\mu}_{\nu\sigma} X^\nu X^\sigma \]  
(1.3)

The last class has been intensively treated in papers like [12-15], as a general feature space time becomes discrete avoiding most renormalization problems for possible quantum field theories. The case a) is usually connected with string theory while case b) has been worked out in papers like [16-19].

Our approach shall be viewed as a combination of a) and b) because our main ingredient is the postulate of a Lie algebra including all operators of physical interest. As a consequence we demand the \( \Theta \) to belong to that algebra as central operators. We understand that this postulate is in disagreement with nearly all other models existing but their drawbacks seem to provide enough arguments for testing new conceptions. The current paper is therefore mainly devoted to illustrating the consequences of our approach.

Because standard quantum field theory on noncommutative spaces of type a) is also not Lorentz covariant, it is worthwhile to re-examine the role of Lorentz invariance in noncommutative field theory. After we have taken as a basis the existence of a Lie algebra we found it straightforward to demand the Lorentz group to be the second factor in the semi-direct product of Lie groups (s. next sect.). This condition seems neither too weak nor too strong and its consequences are worth of being worked out.

The paper is organized as follows. In sect.2 we define our model and determine the possible structure of \( \Theta \). In sect.3 we present an explicit representation of our noncommutative Lie algebra. Sect.4 is devoted to the inclusion of momentum operators and sect.5 deals with the problem of physical invariants. The last section 6 contains our conclusions.

2 The model

We consider a flat four-dimensional space time with operators \( X^0, X^1, X^2, X^3 \) obeying the commutation relations
\[ [X^\lambda, X^\mu] = i\Theta^{\lambda\mu}, \quad \Theta^{\lambda\mu} = -\Theta^{\mu\lambda} \]  
(2.1)

where the \( \Theta^{\lambda\mu} \) are considered to be ”true” (real) numbers. Herewith we mean that they are considered to be proportional to the unit operator \( E \) of a Lie algebra. Hence this operator
has to commute with all other operators of the algebra considered. It is this condition which restricts the model to a great extent, what is viewed here as an advantage taking into account the huge variety of models possible.

Therefore, besides the ordinary condition

\[ [\Theta^{\mu\nu}, X^\lambda] = 0 \]  

we demand for the commutator

\[ [\Theta^{\mu\nu}, M^{\rho\sigma}] = 0, \]  

where the \( M^{\rho\sigma} \) (with \( M^{\rho\sigma} = -M^{\sigma\rho} \)) are the generators of the Lorentz or \( SO(3,1) \) group obeying the standard commutation relations among each other. It might seem more natural instead of eq. (2.3) to postulate the commutator to be given by a tensor representation of \( SO(3,1) \). Imposing further Lie algebra conditions those models are consistent with commuting momentum operators and allow the definition of mass and spin in the usual manner. They are realized in standard noncommutative field theory (s. e.g. [20,21], remind also some recently raised criticism baised on different schemes of renormalization of a non-local field theory in [26]). Nevertheless this paper deals with models based on eq. (2.3) and is devoted to studying the consequences of those conditions.

The set of physical operators which is postulated to form a Lie algebra \( L \) contains 11 operators, besides the six Lorentz generators and the four space time operators we also must keep the unit operator \( E \). So our algebra \( L \) has a non-vanishing centre and is therefore nilpotent and hence solvable. Due to a standard theorem in Lie algebra theory [22] \( L \) can be decomposed into a semi-direct sum

\[ L = I_L \oplus S \]  

where \( I_L \) is the largest solvable ideal and \( S \) is semi-simple. The part of \( S \) is obviously played here by the algebra \( so(3,1) \) and therefore \( I_L \) is given by the set of operators \( \{ E, X^\mu \} \) being nilpotent and hence solvable.

To fulfill the property of an ideal the commutators of \( M \) and \( X \) must belong to \( I_L \) :

\[ [M^{\mu\nu}, X^i] = i f_j^{\mu\nu} X^j \]  

We have used Greek and Latin indices to manifest their different character in further discussion. (The rather unphysical inclusion of \( E \) on the r.h.s. of eq. (2.5) has been disregarded.) We understand the Latin indices as operating in a 4x4 matrix space while the Greek indices label the six different matrices \( F^A, A = 1...6 \). Then the semi-direct sum (2.4) forces those matrices (to be precise with a factor \((-1)\)) to form a representation of \( so(3,1) \)

\[ [F^A, F^B] = -C^{AC}_{\phantom{AC}B} F^C \]
where $C^{AB}_C$ are the structure constants of the algebra $so(3,1)$. The last equation can be understood as the Jacobi condition for two $M$ and one $X$ operators. The second condition derived from the semi-direct sum is equivalent to the Jacobi identity for two $X$ and one $M$ operators. Taking into account eq. (2.5) it can be written as a true matrix condition

$$F\Theta = -\Theta F^T$$

(2.7)

For clearness we add that the indices of $\Theta$ should to be understood as Latin ones.

Now we look for solutions of eqs. (2.6) and (2.7) with unknown matrices $F$ and $\Theta$. Exploiting two times the symplectic algebra $sp(2)$ we have found the solution

$$\Theta^{mn} = \begin{pmatrix} 0 & 0 & A & B \\ 0 & 0 & B & -A \\ -A & -B & 0 & 0 \\ -B & A & 0 & 0 \end{pmatrix}$$

(2.8)

or explicitly

$$\Theta^{02} = -\Theta^{13} = A, \quad \Theta^{03} = \Theta^{12} = B$$

$$\Theta^{01} = \Theta^{23} = 0$$

(2.9)

$A$ and $B$ are arbitrary real constants not fixed within our model. The rank of $\Theta$ is four as long as they do not vanish both. The electric and magnetic components of $\Theta$ are of equal value. The Lie algebra $F$ is composed of all matrices of the form

$$F = \begin{pmatrix} J & -C & -D + K & -E - G \\ C & J & E + G & -D + K \\ D + K & -E + G & -J & C \\ E - G & D + K & -C & -J \end{pmatrix}$$

(2.10)

where $C, D, E, J, G, K$ are arbitrary numbers.

The use of $sp(2)$ is due to the fact, that there is no non-trivial solution, if $so(3)$ is used instead of $so(3,1)$. The same happens if one tries to substitute the last condition of eq. (2.9) by introducing a further constant. The choice of the six matrices $F^A$ is not unique, we have taken the $M$-operators to be antisymmetric and the $N$-operators to be symmetric. There is still an obvious freedom for a three-dimensional rotation. Now we can give the explicit view of the $F$-matrices describing the transformation of space time operators in our model. For shortness we write $M$ and $N$ for the generators of the $F$-algebra but one has to keep in mind that the "true" representation of $so(3,1)$ is given by $-M$ and $-N$ in our notation.

$$M^1 = 1/2 \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad M^2 = 1/2 \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

(2.4)
\[
M^3 = 1/2 \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad N^1 = 1/2 \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\] (2.11)

\[
N^2 = 1/2 \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad N^3 = 1/2 \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}
\]

Those equations are to be read (s. eq.(2.5)) like

\[
[M^1, X^0] = -i/2 X^1, \quad [M^1, X^2] = i/2 X^3,
\]

\[
[M^1, X^1] = i/2 X^0, \quad [M^1, X^3] = -i/2 X^2
\]

and analogously for \(M^2, M^3, N^1, N^2, N^3\). The first question is of course what kind of representation of \(so(3,1)\) is given by \(F\). In ordinary commutative theory the space time operators are transformed by the usual vector representation of \(so(3,1)\) that is in the \(D(i,j)\) notation by the representation \(D(1/2,1/2)\). Our representation is reducible which can be seen quickly by calculating the Lorentz invariant \(\vec{M}\vec{N}\) which is here proportional to the matrix

\[
\begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

excluding irreducibility.

Applying the non-singular transformation matrix

\[
T = \frac{1}{2} \begin{pmatrix} 1 & -i & 0 & 0 \\ 0 & 0 & 1 & -i \\ 1 & i & 0 & 0 \\ 0 & 0 & 1 & i \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ i & 0 & -i & 0 \\ 0 & 1 & 0 & 1 \\ 0 & i & 0 & -i \end{pmatrix}
\] (2.13)

the matrices (2.11) can be brought to the form \(M' = TMT^{-1}\)

\[
M'^1 = \frac{i}{2} \begin{pmatrix} -\sigma_z & 0 & 0 \\ 0 & \sigma_z & 0 \\ 0 & 0 & -\sigma_z \end{pmatrix}, \quad M'^2 = \frac{i}{2} \begin{pmatrix} -\sigma_y & 0 & 0 \\ 0 & -\sigma_y & 0 \\ 0 & 0 & \sigma_y \end{pmatrix}, \quad M'^3 = \frac{i}{2} \begin{pmatrix} -\sigma_x & 0 & 0 \\ 0 & \sigma_x & 0 \\ 0 & 0 & -\sigma_x \end{pmatrix}
\] (2.14)

\[
N'^1 = -\frac{1}{2} \begin{pmatrix} \sigma_z & 0 & 0 \\ 0 & \sigma_z & 0 \\ 0 & 0 & -\sigma_z \end{pmatrix}, \quad N'^2 = -\frac{1}{2} \begin{pmatrix} \sigma_y & 0 & 0 \\ 0 & -\sigma_y & 0 \\ 0 & 0 & \sigma_y \end{pmatrix}, \quad N'^3 = -\frac{1}{2} \begin{pmatrix} \sigma_x & 0 & 0 \\ 0 & \sigma_x & 0 \\ 0 & 0 & -\sigma_x \end{pmatrix}
\]

which clearly shows the decomposition into \(D^{(1/2,0)} \oplus D^{(0,1/2)}\), the reducible representation used in Dirac’s equation. Nevertheless we mention that in our case both subspaces obtain
the same orientation only after an additional rotation around the $y$-axis in generator space by the angle of $\pi$. At the end of this section we comment about the relation of our model to other ones.

At first, exploiting another representation as usual for the transformation of space time operators means to depart from viewing noncommutativity as a deformation of commutative theory, i.e. for $\Theta \to 0$ we do not obtain the standard theory because there is no smooth way from an irreducible representation to a reducible one (in general to any other non-equivalent). We deeply believe that if live is noncommutative, there is no analyticity to be expected in $\Theta$. This is quite analogous to perturbation theory in ordinary quantum mechanics. We expect the picture to resemble some kind of phase transition when one passes from commutativity to noncommutativity.

The second difference consists in the fact, that $\Theta$ cannot be changed by Lorentz transformations (s. eq. (2.3)) and therefore plays the role of a constant external field, which of course breaks Lorentz invariance. From eq. (2.8) it follows, that the $x$-dimension is treated differently from $y$ and $z$, which are handled on the same footing. In a further study the latter ones could be compactified. This leads to the idea to consider our model as a toy model for higher compactified dimensions. Here the important question arises whether noncommutativity might be connected with those extra dimensions only.

3 Explicit structure of our noncommutative Lie algebra

For any finite-dimensional Lie algebra there exists a true matrix representation. It is interesting to look at it for our five-dimensional nilpotent algebra $I_L$. It can be easily seen that the five 4x4 matrices

$$X^0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -B & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad X^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ A & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & B & 0 \end{pmatrix}$$

$$X^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad X^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

(3.1)
\[ E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \]

fulfill the commutation relations

\[ [X^0, X^2] = A \cdot E, \quad [X^0, X^3] = B \cdot E, \quad [X^1, X^2] = B \cdot E, \quad (3.2) \]

\[ [X^1, X^3] = -A \cdot E, \quad [X^i, E] = 0 \]

One can get rid of the constants \( A \) and \( B \) after introducing so-called renormalized light-cone coordinates \( \tilde{X}^0 \) and \( \tilde{X}^1 \) by

\[ \tilde{X}^0 = -\frac{B X^0 + A X^1}{A^2 + B^2}, \quad \tilde{X}^1 = \frac{A X^0 + B X^1}{A^2 + B^2} \quad (3.3) \]

resulting in the easier commutation relations

\[ [\tilde{X}^0, X^3] = -1, \quad [\tilde{X}^1, X^2] = 1 \quad (3.4) \]

All other commutators vanish. In the matrix representation \( \tilde{X}^0 \) and \( \tilde{X}^1 \) take the form

\[ \tilde{X}^0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{X}^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (3.5) \]

As a next step we shall integrate the algebra \( I_L \) and construct the Lie group (to be exact the universal covering Lie group) of the Lie algebra \( I_L \). We call this five parameter group \( \tilde{X}^4 \) which is the noncommutative generalization of the four-dimensional space time coordinate group. One way of doing that is the following. Let \( g \in \tilde{X}^4 \) be a function of the parameters \( y_0, \tilde{y} \) and a phase \( \phi \) and

\[ g(y_0, \tilde{y}, \phi) = \exp(i X^0 y_0 + i X^1 \tilde{y}_1 + i X^2 y_2 + i X^3 y_3 + i E \phi) \quad (3.6) \]

Using Baker-Campbell-Hausdorff’s formula for operators whose commutators commute with the original ones the composition law and the inverse element can be defined in the way:

\[ g(y, \phi)g(z, \sigma) = g(y + z, \phi + \sigma + \frac{1}{2}i A [y_0 z_0 - y_2 \tilde{z}_2 - y_1 \tilde{z}_3 + y_3 \tilde{z}_1] + \frac{1}{2}i B [y_0 \tilde{z}_3 - y_3 z_0 + y_1 z_2 - y_2 z_1]) \]

\[ g^{-1} = g(-y, -\phi) \quad (3.7) \]
Now one may ask how to construct finite dimensional irreducible representations of the Lie group $\tilde{X}^4 \times SO(3,1)$. A standard theorem [22] tells us that we have to look for all characters $\chi$ of $\tilde{X}^4$ fulfilling the property

$$\chi(g) = \chi(g_s^{-1}gg_s) \quad (3.8)$$

where

$$g_s \in SO(3,1)$$

Eq. (3.8) is fulfilled only by characters depending on $\phi$ solely. But the composition law eq. (3.7) does not allow such characters to exist. Hence all finite dimensional irreducible representations of our semi-direct product are (like in commutative case) equivalent to the finite dimensional irreducible representations of $SO(3,1)$. We shall return to this important point after the inclusion of momenta in the next section.

4 The inclusion of momentum operators

Now it is straightforward to include momentum operators in our noncommutative Lie algebra. To do that in a consistent way one has to care for all possible Jacobi conditions to be fulfilled by the commutators invented. At first we suppose the commutators of $M$ and $P$ operators to be given by a representation of $so(3,1)$, that is

$$[M^\mu_\nu, P^i] = i\theta_{ij}^{\mu\nu} P^j \quad (4.1)$$

where the matrices $H^A$ have to obey the condition of eq. (2.6) equivalent to $(MMP)$ Jacobi identity. Next we wish to preserve the canonical quantization commutation relations

$$[X^i, P^j] = -ig^{ij} \quad (4.2)$$

where $g^{ij}$ is the Minkowski metric tensor. Now the $(MPX)$ Jacobi condition leads to the matrix constraint

$$H = -gF^T g \quad (4.3)$$

which determines $H$ fully.

It follows in our model that momentum operator transformation is given by an equivalent (but not identical) to coordinate operator transformation representation of $so(3,1)$. From eq. (2.7) we derive

$$FA = AH \quad (4.4)$$

with $A = \Theta g$. (Remember that $A$ has a non-vanishing determinant.) We remind the reader that in commutative case for representation $D^{(1/2,1/2)}$ $H$ is simply identical to $F$. 
It is now consistent with our approach to assume
\[ [P^\mu, P^\lambda] = i\tilde{\Theta}^{\mu\nu} \] (4.5)
and
\[ [\tilde{\Theta}^{\mu\nu}, M^{\rho\sigma}] = 0 \] (4.6)
The concrete form of \( \tilde{\Theta} \) is up to now not specified, but the \((PPM)\) Jacobi condition yields
\[ H\tilde{\Theta} = -\tilde{\Theta}H^T \] (4.7)
Introducing \( \tilde{\Theta}' \) by \( \tilde{\Theta}' = g\tilde{\Theta}g \) eq. (4.7) is equivalent to
\[ F^T\tilde{\Theta}' = -\tilde{\Theta}'F \] (4.8)
which can be easily solved for given \( F \) from eq. (2.10). The result is
\[
\tilde{\Theta} = \begin{pmatrix}
0 & 0 & C & D \\
0 & 0 & -D & C \\
-C & D & 0 & 0 \\
-D & -C & 0 & 0
\end{pmatrix}
\] (4.9)
where \( C \) and \( D \) are arbitrary (real) constants. The \((XXP)\), \((PPX)\) and \((PPP)\) Jacobi conditions are fulfilled if we put
\[ [\Theta^{\mu\nu}, P^i] = [\tilde{\Theta}^{\mu\nu}, X^i] = [\tilde{\Theta}^{\mu\nu}, P^i] = 0 \] (4.10)
We stress the fact that our model is consistent even if \( C \) and \( D \) vanish both. In that case noncommutativity is connected with coordinate space only and not with momentum space. This point reminds strongly what happens in quantum field theory of noncommutative spaces even though our model is different from theirs [23,24]. The consequences of non-vanishing \( C \) and \( D \) will be analyzed in the next section.

5 Invariants and the mass problem

In this section we study the noncommutative generalization of the Poincare group \( \tilde{T}^4 \times SO(3, 1) \) where \( \tilde{T}^4 \) is the generalization of the commutative four dimensional translation group \( T^4 \). Its Lie algebra is generated by the operators \( P^0, P^1, P^2, P^3 \) and \( E \). That algebra can be integrated in the same way as in section 3.

Hence the Lie algebra of our generalized Poincare group is nilpotent and therefore solvable. The usual construction of invariants for the Lie group of our algebra does not apply because the Cartan-Weyl tensor is singular. We expect the existence of two independent central operators for our Lie algebra but in the literature [22,25] there was
even no theorem making statements about the number of invariants in case of solvable Lie algebras. Therefore we shall construct below the bilinear central operator by hand.

We consider the operator
\[
\tilde{I} = i_{mn} P^m P^n
\] (5.1)
with numbers \(i_{mn}\) forming the matrix \(I\). From eq. (4.5) we calculate
\[
[\tilde{I}, P^l] = P^m (i_{mn} \Theta^{nl} + i_{mn} \Theta^{nl})
\] (5.2)
In the same way after eq. (4.1) we find
\[
[\tilde{I}, M^\mu\lambda] = -P^m (h^T_{\mu\lambda n} i_{nl} + i_{mn} h^T_{\mu\lambda n}) P^l
\] (5.3)
which leads to the matrix condition
\[
H^T I = -IH
\] (5.4)
This condition has to be fulfilled by all six matrices \(H\). To solve this condition we apply a non-singular transformation matrix \(\tilde{T} = gT^{-1} g\) in the way
\[
H' = \tilde{T} H \tilde{T}^{-1}
\] (5.5)
and therefore
\[
I' = \tilde{T}^{-1} T I \tilde{T}^{-1}
\] (5.6)
We shall obtain \(H'\) which are the six matrices of eqs. (2.14) with several signs reversed. Nevertheless it is evident that any \(I'\) fulfilling eq. (5.4) is given by
\[
I' = \begin{pmatrix} \sigma_y & 0 \\ 0 & \lambda \sigma_y \end{pmatrix}
\] (5.7)
which after retransformation leads to
\[
I = \lambda_1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix}
\] (5.8)
with all \(\lambda\) being arbitrary constants. (This \(I\) is antisymmetric and therefore obeys eq. (5.2).) But now we have
\[
\tilde{I} = 2i(\lambda_1 C + i \lambda_2 D)
\] (5.9)
That means \(\tilde{I} \sim E\), which may have been expected from the very beginning. We have done the calculation in an explicit way to show that here is no bilinear central operator besides \(E\). This is one of the intrinsic problems of constructing central operators for a nilpotent Lie algebra which by definition contains a non-trivial centre. The only way to interprete the above result in a physical manner is, that the mass operator is a constant and therefore all particles have the same mass. This consequence, which at the first moment seems to rule out the model, must be viewed in context with the arguments of
sect. 2 about a necessary phase transition. Then it is really very unlikely that masses and other particle parameters resemble each other on different sides of the transition point. (For vanishing $C$ and $D$ the masses also vanish.) It is not surprising that the square of the Pauli-Lubanski vector $\epsilon_{l m n k} M^{m n} P^k$ is not central in our Lie algebra. By direct calculation we have established that the commutator with $P^a$ does not vanish (It is a non-zero combination of $\tilde{\Theta}$, $M$ and $P$.) while the one with $M^{rs}$ does.

To construct infinite dimensional irreducible representations of our generalized Poincare group one has to adjust the standard procedure (for Abelian $T^4$) via orbits to the non-commutative case which seems to be an interesting task for the future.

6 Conclusions

We have presented a Lie-algebra model for noncommutative coordinate and momentum space which contains four unrelated parameters not to be determined within the model. After redefinition the number of parameters can be reduced to two, namely $\sqrt{A^2 + B^2}$ and $\sqrt{C^2 + D^2}$, the former setting the scale for $\Theta$ and the latter for the mass.

While Lorentz covariance is broken by $\Theta$, $\Theta$ itself is considered as a Lorentz scalar.

The representation under which coordinate and momentum operators are transformed can be no longer $D(1/2,1/2)$, the usual vector representation. It turns out that the easiest possibility is the reducible spinor representation $D(1/2,0) \oplus D(0,1/2)$, well known from Dirac’s equation. That forces us to adopt the occurence of a phase transition between noncommutative and commutative world instead of the usual conception of a smooth deformation. This point has to be worked out in further research. The phase transition also has to generate the mass spectrum because in our approach the only suitable mass operator is a constant. The main open question is the construction of a second invariant which is thought to replace ordinary spin. This problem is under consideration now.

7 Acknowledgements

This work has been supported by DFG.

The author thanks J. Wess, D. Lüst, H. Dorn and M. Karowski for helpful discussions and J. Lukierski for critical advice.

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