Detailing the model and objective function for the task of optimizing the power distribution process

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Abstract. In this article, for the spectral expansion of periodic functions by means of a truncated segment of the Fourier series, the authors demonstrated options for their representation in the form of sets of coefficients and showed the relationship between the latter. On the basis of mathematical calculations, the authors have prepared an appropriate mechanism for organizing calculations, for which a number of linear matrix operations have been formed. The application of a new computing mechanism based on a discrete expansion of continuous periodic functions is demonstrated in detailing the problem of combinatorial optimization of the power distribution process, as well as the corresponding load model and target function.

1. Introduction
The world practice of spectral analysis and signal processing has developed quite a long time ago. It mainly relies on the use of discrete Fourier transform operators. Representation of periodic signals in the form of a set of Fourier coefficients is widely used in calculations in various subject areas [1, 2]. However, the need to develop new mechanisms for using the Fourier apparatus still remains [3, 4]. There is a need for solving the practical problem of optimizing the process of distributing electricity between single-phase consumers at the level of a transformer substation, and for detailing the corresponding load model and target function, in particular. It is important to note that the solution of the problem must be performed using binary optimization algorithms, operating in the search for solutions by variables in the form of discrete binary vectors.

We propose a corresponding solution for consideration. Section 2 describes the mechanisms of the spectral expansion of continuous periodic functions and shows the corresponding variants of linear operations. Paragraph 3 presents the formulation of the problem of optimizing the power distribution process at the level of criteria and the developed data presentation. Section 4 shows the load and objective function models for the optimization problem, detailed using the mechanisms of discrete transformation of periodic functions.

2. Discrete expansion of continuous periodic functions
Let \( a = x(t) \) be a continuous periodic function with the main period \( T \), that is \( a(t + T) = x(t) \). A discrete expansion of a function \( x(t) \) is its representation in the form:

\[
a_n(t) = c_1 e_1(t) + c_2 e_2(t) + \cdots + c_n e_n(t) + \cdots = \sum_{k=1}^{\infty} c_k e_k(t)
\]

In the case \( \|a - a_n\| \to 0 \), where the norm is defined by the equation
\[ \|a - a_n\| = \int_0^T (a(t) - a_n(t))^2 \, dt, \]

usually, on the right-hand side of equality (1) we are limited to a finite number \( n \) of terms of the series. The resulting truncated sum of a series of the form

\[ \bar{a}(t) = \sum_{k=1}^n c_k e_k(t) \]  

is taken as an approximation of the function \( a(t) \), and the set of coefficients \( \bar{c} = (c_1, c_2, \ldots, c_n) \) is called its spectrum over the set of functions \( \{e_k(t)\}_{k=1}^n \).

The most common spectral decomposition of periodic functions is their representation in the form of a truncated segment of the Fourier series \([5, 6]\) of the form:

\[ \bar{a}(t) = a_0 + \sum_{k=1}^n a_k \sin k\omega t + \sum_{k=1}^n a_{n+k} \cos k\omega t \]  

where \( \omega = 2\pi f; \ f = \frac{1}{T} \); 

\[ a_0 = \frac{1}{T} \int_0^T x(t) \, dt ; \]

\[ a_k = \frac{2}{T} \int_0^T x(t) \sin k\omega t \, dt ; \]

\[ a_{n+k} = \frac{2}{T} \int_0^T x(t) \cos k\omega t \, dt ; \]

\[ k = \overline{1,n}. \]

wherein \( e_1 = 1; \ e_k = \sin k\omega t; e_{n+k} = \cos k\omega t. \)

Spectral decomposition of the form (2) can be specified differently. Let us introduce a set of functions \( \{e_k(t)\}_{k=1}^m \) of the form:

\[ e_k(t) = \begin{cases} 
h - t, & t \in [0, h]; \\
h, & t \in [h, 2h]; \\
(k-1)h - t, & t \in [(k-1)h, kh]; \\
(k-1)h, & t \in [kh, (k+1)h]; \\
T + t - h, & t \in [T - h, T]; \\
h, & t \in [T - h, T], 
\end{cases} \]

where \( h = \frac{T}{m-1} \).

Figure 1. Function graph \( e_k(t) \)

The functions \( e_k(t) \) on the segment \([0,T]\) are defined by equation

\[ e_k(t) = \begin{cases} 
h - t, & t \in [0, h]; \\
h, & t \in [h, 2h]; \\
(k-1)h - t, & t \in [(k-1)h, kh]; \\
(k-1)h, & t \in [kh, (k+1)h]; \\
T + t - h, & t \in [T - h, T]; \\
h, & t \in [T - h, T], 
\end{cases} \]  

where \( h = \frac{T}{m-1} \).
For the system of functions (5), the segment (2) of the series can be represented as
\[ \tilde{a}(t) = \sum_{k=1}^{m} b_k e_k(t), \]  
(6)
where 
\[ b_k = e_k(t_k); \]
\[ t_k = (k - 1)h; \]
\[ k = \overline{1, m}. \]

The operation of transition from a continuous function \( x(t) \) to its scalar representation in the form of a segment of the Fourier series (3), which is specified by a set of values of the Fourier coefficients \( \tilde{a} = (a_0, a_1, \ldots, a_n, a_{n+1}, \ldots, a_{2n}) \) in accordance with equation (4), will be denoted by \( S_f \). In this case
\[ S_f x(t) = \tilde{a} \]  
(7)
Similarly, the operation of transition from the function \( x(t) \) to its expansion (6), which is formally specified by the set \( \tilde{b} = (b_1, b_2, \ldots, b_m) \), we will denote as \( S_d \). In this case
\[ S_d x(t) = \tilde{b} \]  
(8)
Spectral decomposition of representations (7), (8) is performed by linear operations.

Indeed, if \( y(t) \) is a periodic function with Fourier coefficients (4), (6) respectively equal to \( \tilde{c} = (c_0, c_1, \ldots, c_n, c_{n+1}, \ldots, c_{2n}) \), \( \tilde{d} = (d_1, d_2, \ldots, d_m) \), then
\[ S_f (x(t) + y(t)) = \tilde{a} + \tilde{c}; \quad S_f (ax(t)) = a\tilde{a}; \]
\[ S_d (x(t) + y(t)) = \tilde{b} + \tilde{d}; \quad S_d (ax(t)) = a\tilde{b}; \]
where \( a \) is an arbitrary number.

For a function of the form \( \tilde{a}(t) \) of the form (3) the following also holds:
\[ \frac{1}{2} \int_0^T \tilde{x}^2(t) dt = a_0^2 + \frac{1}{2} \sum_{k=1}^{n} (a_k^2 + a_{n+k}^2) \]  
(9)
It must be noted that the sets of coefficients \( \tilde{a}, \tilde{b} \), corresponding to transformations (7), (8) are related to each other. Indeed, using the trapezoidal formula \( x_k = x((k - 1)h) \) in finding integrals (4), we obtain:
\[ a_0 = \frac{h}{2} (x_1 + 2x_2 + \cdots + 2x_{m-1} + x_m) = \frac{h}{2} (b_1 + 2b_2 + \cdots + 2b_{m-1} + b_m); \]
\[ a_k = 2h (x_2 \sin k\omega h + x_3 \sin 2k\omega h + \cdots + x_{m-1} \sin 2k\omega(T - h)) = \]
\[ = 2h (b_2 \sin k\omega h + b_3 \sin 2k\omega h + \cdots + b_{m-1} \sin 2k\omega(T - h)); \]
\[ a_{h+k} = \frac{h}{2} (x_1 + 2x_2 \cos k\omega h + \cdots + 2x_{m-1} \cos 2k\omega(T - h) + x_m) = \]
\[ = \frac{h}{2} (b_1 + 2b_2 \cos k\omega h + \cdots + 2b_{m-1} \cos 2k\omega(T - h) + b_m). \]

Similarly, the components of the vector \( \tilde{b} \) are expressed through the components of the vector \( \tilde{a} \). This follows formally from the congruence
\[ \tilde{x}_i = a_0 + \sum_{k=1}^{n} a_k \sin k\omega t_i + \sum_{k=1}^{n} a_{n+k} \cos k\omega t_i \]  
(11)
where 
\[ t_i = (i - 1)h; \]
\[ i = \overline{1, m}. \]
From formulas (10), (11) we obtain the equalities
\[ S_f x(t) = A_f \tilde{b} = A_f S_d x(t); \]
\[ S_d x(t) = A_d \ddot{a} = A_d S_f x(t); \]

where matrices \( A_f, A_d \) are determined according to formulas (10), (11), respectively.

3. Formulation of the optimization problem for electricity distribution process

3.1. Optimization criteria

The determining criteria for solving the problem of optimizing the process of electricity distribution [7, 8] are:

1) equalization of the effective total for each of the three phases of the currents \( I_A, I_B, I_C \) of all the loads connected to them;

2) ensuring a decrease in the coefficients of nonlinear distortion of currents in the phases (Total Harmonic Distortion, THD [9, 10]), equal to

\[ K_{THD} = \frac{\sqrt{l_1^2 + l_2^2 + \ldots + l_s^2}}{l_E}; \quad E = A; B; C, \tag{12} \]

where \( l_1, l_2, l_3, l_E \) are first, second, third and \( s \)-th, accordingly, the harmonics of the effective total current in the \( E \)-th phase.

3.2. Data representation

Let there be \( p \) consumers in total, the instantaneous and effective values of the load currents of which are respectively equal

\[ i_q(t), l_q, q = \overline{1, p}. \tag{13} \]

To identify the connected consumers to each of the three phases \( A, B, C \) we will use a binary vector \( \overline{Z} \) of dimension \( 3p \) of the form

\[ \overline{Z} = (z_1, z_2, \ldots z_p, z_{p+1}, \ldots z_{2p}, z_{2p+1}, \ldots z_{3p}). \tag{14} \]

Vector \( \overline{Z} \) has three segments of length \( p \), corresponding to consumers and connection phases. For the \( j \)-th consumer, one of the values \( z_j, z_{p+j}, z_{2p+j} \) is equal to one in the segment that corresponds to the connection phase. The rest of the values are zero. With this formalization, the phase currents are

\[ i_A(t) = \sum_{j=1}^{p} z_j i_j(t); \]
\[ i_B(t) = \sum_{j=p+1}^{2p} z_j i_j(t); \]
\[ i_C(t) = \sum_{j=2p+1}^{3p} z_j i_j(t). \tag{15} \]

Factoring in the linearity of the operation \( S_f \) we obtain from (15)

\[ S_f i_A(t) = \sum_{j=1}^{p} z_j S_f i_j(t); \]
\[ S_f i_B(t) = \sum_{j=p+1}^{2p} z_j S_f i_j(t); \]
\[ S_f i_C(t) = \sum_{j=2p+1}^{3p} z_j S_f i_j(t). \tag{16} \]

It follows that the Fourier spectra of the phase currents are equal to the sum of the spectra connected to each of the phases of the load currents. This allows the objective functions to be expressed through the specified spectra of the harmonics of phase currents and variables \( z_j, j = \overline{1, 3p} \), which must satisfy additional conditions:

\[ z_j + z_{p+j} + z_{2p+j} = 1, j = \overline{1, p}. \tag{17} \]
Limitations (17) express the fact that each of the loads can be connected to only one phase (line). Obviously:

\[ z_{2p+j} = 1 - (z_j + z_{p+j}), \; j = \overline{1, p}, \]  

which indicates the sufficiency of storing data on \( Z \) in the form of a vector of length \( 2p \); in the course of software data processing, it is necessary to store only \( z_j, z_{p+j} \), \( j = \overline{1, p} \), and the values \( z_{2p+j}, j = \overline{1, p} \) can be uniquely determined based on formula (16). It is also worth noting that the described data presentation is very convenient for organizing calculations based on binary optimization algorithms.

4. Proposed solution

4.1. Detailing the load model

For each of the three phases A, B and C, the load current is

\[ i_E(t), \; E = A; B; C \]  

in accordance with (3) and (7) is uniquely determined by the set of values of the Fourier coefficients \( \tilde{a} = (a_0, a_1, ..., a_n, a_{n+1}, ..., a_{2n}) \).

Let’s consider \( a_0, a_1, ..., a_n, a_{n+1}, ..., a_{2n} \) as random variables. In this case \( i_E(t) \) is a random process, and \( \tilde{a} = (a_0, a_1, ..., a_n, a_{n+1}, ..., a_{2n}) \) is a random vector.

To build a specific load model (19), factoring in (15), it becomes necessary to preliminary estimate the statistical parameters of the distribution \( a_0, a_1, ..., a_n, a_{n+1}, ..., a_{2n} \). The solution to the estimation problem assumes the presence of \( \gamma \) corresponding statistical observations, for which, using (3), a statistical sample of the form

\[
\tilde{a}_1 \leftrightarrow \{a_{0,1}, a_{1,1}, ..., a_{n,1}, a_{n+1,1}, ..., a_{2n,1}\} \\
\vdots \\
\tilde{a}_n \leftrightarrow \{a_{0,n}, a_{1,n}, ..., a_{n,n}, a_{n+1,n}, ..., a_{2n,n}\},
\]

The statistical estimates obtained on the basis of (20) will make it possible to determine the parameters of the distribution functions:

\[ \eta_k = F_k^\xi(\zeta_k), \; \zeta_k \in [0; 1] \]

for each of the \( 2n \) values. Naturally, the larger the value of statistical measurements \( \gamma \), the smaller the statistical error of parameter estimates (21) are.

Thus, it is possible to generate random vectors:

\[ \tilde{A} \leftrightarrow \{a_0, a_1, ..., a_n, a_{n+1}, ..., a_{2n}\}, \]

defining characteristics of load currents (13) for a given number \( p \) of individual consumers when constructing, according to the rules (14), a model of load current consumption (19), corresponding to a statistical sample (20).

4.2. Detailing the objective function

In the task of optimizing the electricity distribution process, the target function is determined based on 3 criteria [11]:

\[ F(Z, \tilde{A}) = c_1 \cdot f_1 + c_2 \cdot f_2 + c_3 \cdot f_3 \rightarrow \min, \]  

where \( c_1, c_2, c_3 \) are criteria weights \( (c_1 + c_2 + c_3 \leq 1) \);

\( f_1 \) is a criterion reflecting the level of asymmetry / uneven distribution of the effective values of currents in the phases (current Unbalance Factor, IUF [12, 13]);

\( f_2 \) is a criterion reflecting the average estimate of the nonlinear distortion coefficients of currents in the phases (12);
\( f_3 \) is a criterion reflecting the ratio of the number of required switchings of single-phase electricity consumers between phases to their total number.

However, in practice, for the optimization algorithm to work, it is sufficient to use only two criteria out of three: \( f_1 \) и \( f_2 \). At the same time, it is logical to take the second criterion into restrictions and solve the problem in the following setting:

\[
f_1(Z, \bar{A}) \rightarrow \min
\]

\[
f_{2,E}(Z, \bar{A}) \leq K_{THD_{\text{max}}}, E = A, B, C,
\]

where \( K_{THD_{\text{max}}} \) is the maximum permissible value of the harmonic distortion factor of currents in the phases, specified as a constant.

Using the operation on (7), the phase currents (16) obtained for each of the three phases \( A, B, \) and \( C \) are determined by means of the corresponding sets of Fourier coefficients:

\[
S_f i_E(t) = \overline{a_E} = (a_0, a_1, \ldots, a_n, a_{n+1}, \ldots, a_{2n}), E = A; B; C. \tag{24}
\]

Then the \( f_1 \) values will be calculated based on the formula:

\[
f_1 = \frac{I_A^2 + I_B^2 + I_C^2}{(I_A + I_B + I_C)^2},
\]

where, based on the formula for calculating the effective current value from the amplitude spectrum of its signal \([14, 15]\) and by virtue of (3) and (24)

\[
I_E = \sqrt{a_0^2 + \sum_{k=0}^{2n} \frac{a_k^2}{2}}, \quad E = A; B; C,
\]

Calculations of \( f_2 \) values will be performed according to formula (12), for which, due to (3) and (24)

\[
l_{k_E}^2 = a_k^2 + a_{n+k_E}^2.
\]

5. Results

Organization of calculations based on binary optimization algorithms requires discrete representation of variables in the appropriate format. The possibility of transition from continuous functions to their scalar representation is given by the Fourier transform. The paper exhibits variants of linear operations of such a transition as applied to a specific optimization problem from the field of electric power industry. It was found that with the described approach the Fourier spectra of the phase currents are equal to the sum of the spectra connected to each of the phases of the load currents. In turn, we were able to express the objective functions for the optimization problem through the spectra of the harmonics of the phase currents and the vector of binary variables that identify the connected consumers to each of the three phases. It has been shown that for the organization of computations it is necessary and sufficient to use a binary vector, the length of which is only twice the number of consumers.

6. Discussion

The described mechanism for organizing calculations is illustrated with examples, detailed models and objective functions in relation to the problem of optimizing the electricity distribution process. This mechanism corresponds to the rules for performing numerical calculations based on the discrete Fourier transform \([16, 17]\) and expands the scope of their application. It is quite possible that this approach will be successfully applied to problems of another class, in which the processes occurring are also initially described by continuous periodic functions \([18, 19, 20]\). However, in each specific case, the expressions for the objective functions are likely to be different. The authors of the work provide an opportunity for other researchers to test the described mechanisms based on the Fourier transform for detailing their own models and target functions when solving optimization problems from other subject areas.
7. Conclusions
This work provides variants of discrete expansion of continuous periodic functions and their representation in the form of sets of Fourier coefficients. There is an obvious relationship between these concepts, which was proven by the calculations. A corresponding mechanism for organizing calculations has been prepared, for which a number of linear matrix operations has been formed.

The use of the presented computing mechanism based on the discrete expansion of continuous periodic functions made it possible to detail the problem of optimizing the process of power distribution between single-phase consumers at the level of a transformer substation in order to establish its effective solution through binary optimization algorithms. For this task, at the level of the linear matrix operations prepared by the authors, the corresponding load model and the goal function were also detailed.

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