STRONG FACTORIZATION AND THE BRAID ARRANGEMENT FAN

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Abstract. We establish strong factorization for pairs of smooth fans which are refined by the braid arrangement fan. This includes normal fans of generalized permutahedra. We show that the realization of the toric variety of the permutahedron as an iterated blow-up of projective space can be achieved by considering a sequence of polytopes known as hyper-permutahedra. To any poset we associate a cone which is the union of some Weyl chambers of type A. We give conditions for when a toric variety defined by such a cone is Gorenstein and for the existence of a crepant resolution.

1. Introduction

Here we use the combinatorics of the braid arrangement fan to study the toric varieties defined by fans which are refined by the braid arrangement fan. The braid arrangement fan is the fan whose maximal cones are the Weyl chambers of type A. Our main result is a proof of Oda’s strong factorization conjecture in the special case where we are given two complete smooth fans that are refined by the braid arrangement fan. All cones we encounter are naturally indexed by preposets. In addition to considering factorization of birational maps, we also give criteria for when toric varieties defined by these cones indexed by preposets are Gorenstein and when there is a crepant resolution. We now recall Oda’s strong factorization conjecture is its combinatorial form.

Conjecture 1.1 (Oda78). Given two smooth fans $\Sigma_1$ and $\Sigma_2$ with the same support, there exists a third fan $\Sigma_3$ which can be obtain from both $\Sigma_1$ and $\Sigma_2$ by sequences of smooth star subdivisions.

In Section 2 we give some background on toric varieties and convex geometry. The correspondence between normal toric varieties and fans is recalled in Section 2.1. The braid arrangement fan is defined in Section 2.2 and review some of its combinatorics which will be of use to us. Section 3 contains the proof of our main result which establishes Conjecture 1.1 in the case that $\Sigma_1$ and $\Sigma_2$ are refined by the braid arrangement fan. We study Gorenstein singularities and crepant resolutions in Section 4. In the remainder of this introduction section to briefly discuss how Conjecture 1.1 and our work here is related to the factorization of birational maps.

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Consider a birational map $\phi : X \to Y$ between smooth complete varieties which is an isomorphism on a common open set $U$. We want to find a factorization of $\phi$ as a sequence of birational maps

$$X \to X_1 \to X_2 \to \cdots \to X_\ell \to Y$$

so that each is a blow-up or blow-down at a smooth center disjoint from $U$. Furthermore, if $X$ and $Y$ are both projective we ask that each $X_i$ also be projective. A strong factorization consists of a sequence of blow-ups followed by a sequence of blow-downs. A weak factorization allows blow-ups and blow-downs in any order. When $X$ and $Y$ are smooth complete varieties over an algebraically closed field of characteristic zero a weak factorization exists [AKMW02, W03].

Equivariant versions of the factorization problems for smooth toric varieties were conjectured in [Oda78] and became known as “Oda’s strong factorization conjecture” and “Oda’s weak factorization conjecture.” As with many problems in toric geometry, Oda’s conjectures can be phrased in terms of convex geometry as we do in Conjecture 1.1. In these terms blow-ups become star subdivisions. Oda’s weak factorization conjecture has been solved by Włodarczyk [W97] and Morelli [Mor96]. Oda’s strong factorization conjecture is still open. Da Silva and Karu [DSK11] propose an algorithm to produce a strong factorization from a weak factorization, but this proposed algorithm is not guaranteed to terminate.

2. Convex geometry and toric varieties

In this section we recall the general dictionary between convex geometry and toric varieties. We also introduce the braid arrangement fan and generalized permutahedra which will be the particular cases of interest to us.

2.1. Toric varieties, fans, and blowing up. We discuss only what will be needed for our purposes, but for a more in depth treatment of toric varieties we refer the reader to book [CLS11]. Given any lattice $N$ with corresponding vector space $N_\mathbb{R} = \mathbb{R} \otimes \mathbb{Z} N$, any fan $\Sigma$ of strongly convex rational polyhedral cones defines a toric variety we denote by $X_\Sigma$. For a cone $\sigma$, which we always assume is strongly convex rational polyhedral cone, we let $\sigma(1)$ denote to the set of ray generators of $\sigma$. Give a set of lattice vectors $A \subseteq N$ we let

$$\text{cone}(A) = \{ \sum_{v \in A} \lambda_v v : \lambda_v \in \mathbb{R}, \lambda_v \geq 0 \} \subseteq N_\mathbb{R}$$

and thus $\sigma = \text{cone}(\sigma(1))$. The support of the fan $\Sigma$ is denoted $|\Sigma|$ and consists of all $v \in N_\mathbb{R}$ such that $v \in \sigma$ for some $\sigma \in \Sigma$. The fan is called complete if $|\Sigma| = N_\mathbb{R}$. A complete fan $\Sigma$ corresponds to a complete variety $X_\Sigma$. A cone is called simplicial if its ray generators can be extended to a basis of the vector space $N_\mathbb{R}$. If its ray generators can be extended to a basis of lattice $N$, then the cone is called smooth. A fan inherits the adjective
simplicial or smooth if each cone is the fan is simplicial or smooth. Smooth fans define smooth toric varieties while simplicial fans define orbifolds.

We also consider the dual lattice $M = \text{Hom}_\mathbb{Z}(N, \mathbb{Z})$ and corresponding vector space $M_\mathbb{R} = \mathbb{R} \otimes \mathbb{Z} M$. For any lattice polytope $P \subseteq M_\mathbb{R}$ its normal fan $\mathcal{N}(P)$ is a complete fan in $N_\mathbb{R}$. The toric variety $X_\Sigma$ is projective if and only if $\Sigma = \mathcal{N}(P)$ for a lattice polytope $P$. We may write $X_P$ in place of $X_{\mathcal{N}(P)}$.

We now define the operation on fans which will be our focus. Consider a smooth cone $\tau \in \Sigma$ with $\tau(1) = \{v_1, v_2, \ldots, v_k\}$ and let $v_0 = v_1 + v_2 + \cdots + v_k$. Assume further that all cones $\sigma \in \Sigma$ containing $\tau$ are also smooth. For each $\tau \subseteq \sigma$ the (smooth) star subdivision of the cone $\sigma$ relative to $\tau$ is the fan

$$\Sigma^*_\sigma(\tau) = \{\text{cone}(A) : A \subseteq \sigma(1) \cup \{v_0\}, \tau(1) \not\subseteq A\}.$$  

The star subdivision of the fan $\Sigma$ relative to $\tau$ is

$$\Sigma^*(\tau) = \{\sigma \in \Sigma : \tau \not\subset \sigma\} \cup \bigcup_{\tau \subseteq \sigma} \Sigma^*_\sigma(\tau).$$

Figure 1 shows two possible star subdivisions of the fan shown in the middle with ray generators $\{v_1, v_2, v_3, v_4\}$. The star subdivision on the left is induced by adding the ray generator $v_2 + v_3$, and the star subdivision on the right is induced by adding the ray generator $v_2 + v_3 + v_4$. If $\dim N = n$, then $X_\Sigma$ is an $n$-dimensional variety. Since $\tau$ is smooth we have $\dim \tau = k$. Under the standard orbit-cone correspondence, the cone $\tau$ corresponds to an $(n-k)$-dimensional smooth subvariety $V(\tau)$ which is the closure of a torus orbit inside $X_\Sigma$. The toric variety $X_{\Sigma^*(\tau)}$ is the blow up of $X_\Sigma$ along $V(\tau)$.

2.2. The braid arrangement and generalized permutahedra. Let $[n] = \{1, 2, \ldots, n\}$ and consider the lattice $\mathbb{Z}^n \subseteq \mathbb{R}^n$ with standard basis $\{e_i : i \in [n]\}$. For any $A \subseteq [n]$ we let $e_A = \sum_{i \in A} e_i$. We then define the lattice $N = \mathbb{Z}^n / \mathbb{Z} e_{[n]}$ and vector space $N_\mathbb{R} = \mathbb{R} \otimes \mathbb{Z} N$. For any $1 \leq i < j \leq n$ let $H_{ij}$ denote the hyperplane in $N_\mathbb{R}$ defined by $x_i = x_j$. The braid arrangement is the hyperplane arrangement consisting of the hyperplanes $\{H_{ij}\}_{1 \leq i < j \leq n}$. We let $\hat{\Sigma}(n)$ denote the braid arrangement fan in $N_\mathbb{R}$. Thus the maximal
cones in $\tilde{\Sigma}(n)$ are Weyl chambers of type $A_{n-1}$. We let $\Pi(n) \subseteq M_\mathbb{R}$ denote the permutahedron which is a lattice polytope with $\mathcal{N}(\Pi(n)) = \tilde{\Sigma}(n)$. Identifying both $N$ and $M$ with $\mathbb{Z}^n / \mathbb{Z}[e_n]$ (i.e. consider coordinates which are $n$-tuples of integers modulo addition by $e_n = (1, 1, \ldots, 1)$) the vertices of $\Pi(n)$ are all permutations of the vector $(1, 2, \ldots, n)$ and the rays in $\mathcal{N}(\Pi(n)) = \tilde{\Sigma}(n)$ are $e_A$ for each $A \subseteq [n]$ such that $A \not\in \{\emptyset, [n]\}$. The permutahedron $\Pi(3)$ and the fan $\tilde{\Sigma}(3)$ with rays label by minimal lattice points are shown in Figure 2.

A generalized permutahedron \cite{Pos09} is a polytope in $M_\mathbb{R}$ whose normal fan is refined by $\tilde{\Sigma}(n)$ \cite[Propositon 3.2]{PRW08}. A preposet is a binary relation which is reflexive and transitive which will be written either by ordered pairs $(i, j)$ or $i \preceq j$. Postnikov, Reiner, and Williams have given a correspondence between cones in fans which refined by $\tilde{\Sigma}(n)$ and preposts on $[n]$ \cite[Section 3]{PRW08}. When not otherwise stated we will assume that any preposet has $[n]$ as its underlying set. The correspondence works by observing that any cone in such a fan is determine by inequalities of the form $x_i \leq x_j$ where $(x_1, x_2, \ldots, x_n)$ are coordinates for $N = \mathbb{Z}^n / \mathbb{Z}[e_n]$. Hence $x_i \leq x_j$ exactly maps to $i \preceq j$. For cones which are not of maximal dimension there will be pairs $x_i \leq x_j$ and $x_j \leq x_i$ giving equality $x_i = x_j$. A poset is prepost which is also antisymmetric. A poset is called a tree poset.
if its Hasse diagram is a tree. Recall the Hasse diagram of the poset has a vertex for each element of the poset and an edge for edge covering relation.

Any preposet determines an equivalence relation on \([n]\) by declaring \(i \sim i\) for all \(i \in [n]\) and \(i \sim j\) for \(i \neq j\) whenever \(i \preceq j\) and \(j \preceq i\). For any preposet we define its Hasse diagram to be the transitive reduction of the graph (with loops and multiple edges removed) which has a vertex for each equivalence class and directed edge from the class of \(a\) to the class of \(b\) for each relation \(a \preceq b\). A preposet is called connected if its Hasse diagram is connected. We will assume throughout that all preposets we encounter are connected, this ensures that the cones they define are strongly convex [PRW08, Proposition 3.5 (7)]. When drawing a Hasse diagram we will omit the direction of the edges and draw so that each edge should be oriented upward. A tree preposet is defined to be a preposet whose Hasse diagram is a tree. Maximal cones will be labeled by posets. In general the dimension of a cone labeled by a preposet will be one less than the number of equivalence classes on \([n]\) the preposet determines.

A fan is smooth if and only if each maximal cone is labeled by a tree preposet [PRW08, Corollary 3.10]. In the smooth case, containment of cones can be found by contracting edges in Hasse diagrams. When we contract an edge in a Hasse diagram we merge the equivalence classes labeling to the two vertices of the edge. Hence, for a smooth fan each cone will be labeled by a tree preposet. Figure 3 shows an example of a smooth fan which is refined by \(\hat{\Sigma}(3)\) which some cones labeled by both tree preposets as well as their Hasse diagrams.

A linear order is a poset in which any two elements are comparable. A linear extension of a poset \(P\) is any linear order which is compatible with \(P\) in the sense that if \(i \preceq j\) in \(P\) then \(i\) must be below \(j\) in the linear order. Under our cone poset correspondence linear order index the Weyl chambers. The cone indexed by a poset \(P\) is the union of the Weyl chambers indexed by all its linear extensions [PRW08 Proposition 3.5 (10)].

An upset of a poset \(P\) is a subset \(A\) of \([n]\), the underlying set of \(P\), such that if \(a \in A\) and \(b \in [n]\), then \(b \in A\) whenever \(a \preceq b\). A downset is defined analogously. Upsets and downsets will be relevant to us. Note any element \(a \in [n]\) generates an upset (downset) consisting of all elements greater than \(a\) (less than \(a\)). The down-degree of \(a\) is the number of edges \(b \rightarrow a\) in the Hasse diagram while the up-degree is the number of edges \(a \rightarrow b\).

3. Smooth fans and strong factorization

Before proving our main theorem, we prove a lemma to describe the ray generators of a smooth cone labeled by a tree preposet. It will be useful to us to have the ray generators explicitly described in terms of the Hasse diagram of a tree preposet. The lemma could also be deduced from Lemma [L1] which we will prove later and applies to cones which are not necessarily smooth.
First let us define some notation. Given a tree preposet $P$ with Hasse diagram $D$ for each edge $a \rightarrow b$ in $D$ we will associate an certain element of the lattice $N$ we will now describe. If we remove the edge $a \rightarrow b$ from the Hasse diagram we obtain two connected components. Let $B$ denote the union of all the equivalence classes of corresponding to the vertices in the component with $b$. The to the edge $a \rightarrow b$ we associate the lattice vector $v_{a \rightarrow b}$ which is defined as $v_{a \rightarrow b} := e_B$.

**Lemma 3.1.** Let $\sigma$ be a smooth cone labeled by a tree preposet $P$ with Hasse diagram $H$. The ray generators of $\sigma$ are $\{v_{a \rightarrow b} : a \rightarrow b$ is an edge in $H\}$.

**Proof.** Let us consider a maximal dimensional cone $\sigma$. Cones of smaller dimension can be treated by applying the same argument projected onto a smaller dimension space. Each edge $a \rightarrow b$ of the Hasse diagram gives a facet of the cone $\sigma$ on the hyperplane $H_{ab}$. Since our cone is smooth it has $n-1$ facets and $n-1$ ray generators. Each ray generator is obtained by the intersection of $n-2$ facets. Choose any edge $a \rightarrow b$ of the Hasse diagram. We will now show that $v_{a \rightarrow b}$ is the ray generator obtained by taking the intersection of the facets corresponding to the other $n-2$ edges. Let $A$ denote the union of all vertices in the component of $a$ in the Hasse diagram with $a \rightarrow b$ removed. Similarly, let $B$ denote the union of the vertices in
the component with $b$ after removing $a \to b$. The intersection of the $n - 2$
hyperplanes is the line defined by $x_i = x_j$ for $i, j \in A$ and $x_k = x_\ell$
for $k, \ell \in B$. Considering again the edge $a \to b$ we see within the cone $\sigma$
that on this line $x_i \leq x_k$ for $i \in A$ and $k \in B$. Therefore it follows that
$v_a \to b = e_B$ is a ray generator for cone $\sigma$.

**Theorem 3.2.** Let $\Sigma$ be a complete smooth fan refined by $\hat{\Sigma}(n)$, then there
exists a sequence of fans

$$(\Sigma_0, \Sigma_1, \ldots, \Sigma_\ell)$$

such that $\Sigma_0 = \Sigma$, $\Sigma_\ell = \hat{\Sigma}(n)$, and $\Sigma_i$ is obtained from $\Sigma_{i-1}$ by a star
subdivision for each $1 \leq i \leq \ell$.

**Proof.** Let $\Sigma = \Sigma_i$ at some $i$ in the proposed sequence of fans. Note that
$\Sigma = \hat{\Sigma}(n)$ if and only if every maximal cone is labeled by a linear order. So,
assume some maximal cone is not labeled by a linear order. Since $\Sigma$ is
smooth this maximal cone of is labeled by a tree poset which is not a linear
order. Choose a cone $\sigma \in \Sigma$ labeled by a tree poset $P$ with Hasse diagram
$D$ which is not a linear order. This means in $D$ there is a vertex with either
down-degree or up-degree strictly greater than 1. Let us assume we have a
vertex $b$ with down-degree $k > 1$. We may assume we are in the case of
down-degree greater than 1 since we could consider $P^{op}$ and the change of
coordinates exchanging $x_i$ and $-x_i$ for $1 \leq i \leq n$.

Let $a_i \leq b$ for $1 \leq i \leq k$ be the covering relations in $P$ with $b$ as the
greater element. Now set $B$ to be the upset generated by $b$ and $A_i$ to be
the downset generated by $a_i$ for each $1 \leq i \leq k$. We can then contract
the Hasse diagram to a Hasse diagram of a tree preposet having vertex set
$\{A_1, A_2, \ldots, A_k, B\}$ and edge set $\{A_i \to B : v1 \leq i \leq k\}$. This tree preposet
indexes a face $\tau$ such that $\Sigma(\tau)$ is refined by $\hat{\Sigma}(n)$, then there
exists sequence of fans

$$\Sigma(\tau) = \hat{\Sigma}(n)$$

We now verify that $\Sigma^*(\tau)$ is still refined by $\hat{\Sigma}(n)$. To do this we let

$S = \{v_{i \to j} : i \to j \in D, i \to j \neq a_1 \to b\} \cup \{e_B\}$

and will show that $\sigma' = \text{cone}(S)$ is the cone indexed by the poset with Hasse
diagram $D'$ that has edges

$$\{i \to j \in D : j \neq b\} \cup \{a_1 \to b\} \cup \{a_i \to a_1 : 1 < i \leq k\}.$$ 

The labeling of the $\{a_i : 1 \leq i \leq k\}$ is arbitrary it suffices the consider $i = 1$
as we have in $\sigma'$. Furthermore it did not matter which cone $\sigma$ containing
the face $\tau$ we originally chose. A local picture of the Hasse diagrams $D$ and
$D'$ can be found in Figure 4.
Let \( v'_{i \rightarrow j} \) denote the ray generators of \( \sigma' \) corresponding to edges of \( D' \). We need to show

\[
\{ v'_{i \rightarrow j} : i \rightarrow j \in D' \} = S.
\]

First note if \( i \rightarrow j \in D \) and \( i \rightarrow j \in D' \), then \( v_{i \rightarrow j} = v'_{i \rightarrow j} \). Next we see that \( v_{a_1 \rightarrow b} = v'_{a_1 \rightarrow a_1} \) for \( 1 < i \leq k \). Finally we observe that \( e_B = v'_{a_1 \rightarrow b'} \).

It follows that \( \sigma' \) is indeed the cone indexed by the poset with Hasse diagram \( D' \). Thus, \( \Sigma^* (\tau) \) is refined by \( \hat{\Sigma} (n) \). Let \( \Sigma_{i+1} = \Sigma^* (\tau) \). The theorem is proven by iterating the process we have described. \( \square \)

Corollary 3.3. Conjecture 1.1 holds whenever \( \Sigma_1 \) and \( \Sigma_2 \) are two complete smooth fans refined by \( \hat{\Sigma} (n) \). Moreover, in this case the third fan \( \Sigma_3 \) can always be taken to be \( \hat{\Sigma} (n) \).

We conclude this section with an example of the blow-ups used in the proof of Theorem 3.2. Given any \( A \subseteq [n] \) we associated the simplex \( \Delta_A \) in \( \mathbb{R}^n \) which is the convex hull of \( \{ e_i : i \in A \} \). A hypergraph \( H \) on the vertex set \( [n] \) is collection of subsets of \( [n] \). Given a hypergraph \( H \) its hypergraphic polytope is

\[
P_H := \sum_{A \in H} \Delta_A
\]

where \( \sum \) denotes Minkowski addition. If \( H \) consists of \( m \) subsets of \( [n] \), then the hypergraphic polytope \( P_H \) lives in the affine subspace of \( \mathbb{R}^n \) defined by \( x_1 + x_2 + \cdots + x_n = m \). This affine subspace can be identified with \( M_\mathbb{R} \). Hypergraphic polytopes are generalized permahedra, and thus there normal fans are refined by \( \hat{\Sigma} (n) \). The normal fans of hypergraphic polytopes are described in [BBM]. The hyper-permutahedron \( \Pi (k, n) \) is the hypergraphic polytope \( P_H \) where \( H \) consists of all \( k \)-subsets of \( [n] \) [Agn17]. The polytope \( \Pi (2, n) \) is the permahedron and the polytope \( \Pi (n, n) \) is a simplex. So, the toric variety corresponding to \( \Pi (n, n) \) is projective space.
In the literature the toric variety $X_{\Pi(n)} = X_{\Sigma(n)}$ is often described as an iterated blow-up of projective space [Kap93, Theorem 4.3.13]. This can be realized by the sequence of toric varieties $X_{\Pi(n,n)}, X_{\Pi(n-1,n)}, \ldots, X_{\Pi(2,n)}$ each associated to a hyper-permutahedron. The preposets indexing the cones in $\mathcal{N}(\Pi(k,n))$ are described in [BBM, Proposition 3.9] (see also [Agn17, Theorem 4.10]). A poset indexing a maximal cone corresponds to an ordered set partition $(A, \{i_1\}, \{i_2\}, \ldots, \{i_{n-k+1}\})$ where $|A| = k - 1$ and edges of the Hasse diagram are $(a, i_1)$ for $a \in A$ along with $(i_j, i_{j+1})$ for $1 \leq j \leq n - k$. An example of the posets indexing maximal cones over the vertices of a facet of $\Pi(4,3)$ is included in Figure 5.

The only vertex with up-degree or down-degree greater than 1 is $i_1$. Thus $A$ is exactly $A$ in the proof of Theorem 3.2 and $b = i_1$ so $B = \{i_1, i_2, \ldots, i_{n-k+1}\}$. This preposet obtained by contraction represents a dimension $n-k$ torus invariant subvariety containing the points corresponding to ordered set partitions $(A, \{i_{\pi(1)}\}, \{i_{\pi(2)}\}, \ldots, \{i_{\pi(n-k+1)}\})$ for permutations $\pi$ of $[n-k]$. After performing star subdivisions at all faces of this type in $\mathcal{N}(\Pi(k,n))$, the maximal cones of the fan obtained will then correspond to ordered set partitions of the form $(A, \{i_1\}, \{i_2\}, \ldots, \{i_{n-k+2}\})$. This is exactly the fan $\mathcal{N}(\Pi(k-1,n))$.

4. Singular cones

In Section 3 we considered only smooth toric varieties. We now turn our attention to some properties of singular toric varieties defined by fans which are refined by the braid arrangement fan. Let us restrict our attention to affine varieties. Given a cone $\sigma$ we let $U_\sigma$ denote the affine toric variety defined by $\sigma$. A variety is $\mathbb{Q}$-Gorenstein if some multiple of its canonical
divisor in Cartier. A variety is \textit{Gorenstein} if it is Cohen-Macaulay and its canonical divisor is Cartier. If \(v_1, v_2, \ldots, v_k \in \mathbb{N}\) are the ray generators of \(\sigma\), then \(U_\sigma\) is \(\mathbb{Q}\)-Gorenstein if and only if the exists \(u \in M\) such that \(\langle u, v_i \rangle = r\) for all \(1 \leq i \leq k\) and some \(r \in \mathbb{Z}\). The \textit{index} of a \(\mathbb{Q}\)-Gorenstein toric variety in the minimal possible \(r > 0\) which can be taken. If the index is \(r = 1\), then the toric variety is \textit{Gorenstein}.

If \(\Sigma\) is a smooth fan refining a cone \(\sigma\) then \(X_\Sigma \rightarrow U_\sigma\) gives a resolution of singularities. Take a \(\mathbb{Q}\)-Gorenstein toric variety \(U_\sigma\) of index \(r\) and \(u \in M\) such that \(\langle u, v_i \rangle = r\) for all ray generators \(v_1, v_2, \ldots, v_k\) of \(\sigma\). If \(\Sigma\) is a smooth fan refining \(\sigma\) such that any ray generator \(v\) of \(\Sigma\) also has \(\langle u, v \rangle = r\), then this resolution is called \textit{crepant}. A crepant resolution does not change the canonical class.

We first develop some combinatorics which will allow us to describe lattice points inside and ray generators of cones defined by posets. To a poset \(P\) we let \(\bar{P}\) denote the cone corresponding to this poset. We also let \(\hat{P}\) denote the fan refining \(P\) which is the union of all Weyl chambers \(\sigma_L\) for each linear extension \(L\) of \(P\). We wish to describe when \(U_{\sigma_P}\) is Gorenstein and when \(U_{\bar{\sigma}_P}\) is a crepant resolution of \(U_{\sigma_P}\).

For a binary relation \(R\), let \(R^{\text{op}}\) denote the opposite binary relation where \((i, j) \in R^{\text{op}}\) if and only if \((j, i) \in R\). Generalizing contracting edges in the Hasse diagram of a tree poset, we define a \textit{contraction} of a poset \(P\) to the transitive closure of \(P \cup R^{\text{op}}\) for some \(R \subseteq P\). For example, taking the poset \(P = \{2 \leq 1, 2 \leq 3\}\) and \(R = \{2 \leq 1\}\) we find the transitive closure of \(P \cup R^{\text{op}}\) to be \(\{1 \leq 2, 2 \leq 1, 1 \leq 3, 2 \leq 3\}\). We can see in Figure \ref{fig:contraction} how this example corresponds to contracting an edge in a Hasse diagram of a tree poset. A key fact is that if \(P\) corresponds to a cone \(\sigma\), then \(P'\) corresponds to a face \(\tau \subseteq \sigma\) if and only if \(P'\) is a contraction of \(P\) \cite[Proposition 3.5 (b)]{PRW08}.

Given an upset \(A\) we define
\[
R_A := \{(i, j) \in P : i, j \in A \text{ or } i, j \notin A\}
\]
and let \(P_A\) be the preposet which is the transitive closure of \(P \cup R_A^{\text{op}}\). We also define the \textit{dimension} of \(A\) with respect to \(P\), denoted \(\dim A\), to be one less than the number of equivalence classes determined by the preposet \(P_A\). The dimension \(\dim A\) can be computed by
\[
\dim A = c(A) + c(\overline{A}) - 1
\]
where \(c(A)\) and \(c(\overline{A})\) are the number of connected components of the Hasse diagram of restricted to \(A\) and \(\overline{A}\) respectively. Here \(\overline{A}\) denotes the complement of \(A\). Figure \ref{fig:dimension} shows a Hasse diagram of a poset \(P\) on the left and of the contraction \(P_A\) for the upset \(A = \{1, 2\}\) on the right. In this example we find that \(\dim A = 1\). To any poset \(P\) we let \(\sigma_P\) denote the cone corresponding to this poset.

\textbf{Lemma 4.1.} If \(P\) is a poset, then the lattice point \(e_A\) is contained in \(\sigma_P\) if and only if \(A\) is an upset. Moreover, for an upset \(A\) the lattice point \(e_A\) is contained in the relative interior of a face \(\tau \subseteq \sigma_P\) with \(\dim \tau = \dim A\).
Proof. It follows immediately for the definitions that $e_A \in \sigma_P$ if and only if $A$ is an upset. Now assume that $A$ is an upset so that $e_A \in \sigma_P$. Then there is a face $\tau \subseteq \sigma_P$ with $\dim \tau = \dim A$ indexed by the contraction $P_A$. Furthermore, we have $e_A \in \tau$ by construction. It only remains to show that $e_A$ is in the relative interior of $\tau$. That is, we must show $e_A$ is not in any face properly contained in $\tau$. Such faces properly contains in $\tau$ will correspond to contractions of $P_A$. Take any $(i, j) \in P_A$. If $(j, i) \in P_A$ this relation will not matter in any further contractions. If $(j, i) \not\in P_A$, then it follows that $j \in A$ and $i \notin A$. For any contraction which is the transitive closure of $P_A \cup R^{op}$ with $(i, j) \in R$ we must have $x_j \leq x_i$ in the corresponding cone. However, since $j \in A$ and $i \notin A$ we see that $e_A$ does not satisfy this inequality. Therefore $e_A$ is in the relative interior of $\tau$ as desired. \qed

Figure 6. A Hasse diagram of a poset and a contraction.

Proposition 4.2. Let $P$ be a poset. The toric variety $U_{\sigma_P}$ is $\mathbb{Q}$-Gorenstein with index $r$ if and only if the poset $P$ has an $r$-Gorenstein labeling but has no $s$-Gorenstein labeling for any $0 < s < r$.

Proof. Given a poset $P$ let

$\{A_1, A_2, \ldots, A_k\} = \{A : A$ is an upset and $\dim A = 1\}$

so by Lemma 4.1 the ray generators of $\sigma_P$ are $e_{A_1}, e_{A_2}, \ldots, e_{A_k}$. We observe that $u \in M$ with $\langle u, e_{A_i} \rangle = r$ for $1 \leq i \leq k$ is equivalent to an $r$-Gorenstein labeling $\phi$ where $\phi(i) = \langle u, e_i \rangle$ for $1 \leq i \leq n$. \qed

Let $P$ be a poset for which $U_{\sigma_P}$ is $\mathbb{Q}$-Gorenstein with index $r$. A crepant labeling of $P$ is a function $\phi : [n] \to \mathbb{Z}$ such that

$\sum_{i \in [n]} \phi(i) = 0$

and

$\sum_{i \in A} \phi(i) = r$

for all upsets $A$ with $\dim A = 1$. We call a 1-Gorenstein labeling a Gorenstein labeling.

Given a poset $P$ on $[n]$ an $r$-Gorenstein labeling is a function $\phi : [n] \to \mathbb{Z}$ such that

$\sum_{i \in [n]} \phi(i) = 0$

and

$\sum_{i \in A} \phi(i) = r$

for all upsets $A$ with $\dim A = 1$. We call a 1-Gorenstein labeling a Gorenstein labeling.
and

$$\sum_{i \in A} \phi(i) = r$$

for all upsets $A$ with $\dim A > 1$. Any crepant labeling is by definition an $r$-Gorenstein labeling. The converse is not true, an $r$-Gorenstein labeling need not be a crepant labeling. For the poset given by the Hasse diagram in Figure 6 the labeling given by

$$\begin{align*}
\phi(1) &= \phi(3) = 1 \\
\phi(2) &= \phi(5) = \phi(6) = 0 \\
\phi(4) &= \phi(7) = -1
\end{align*}$$

is a Gorenstein labeling which is not a crepant labeling.

**Proposition 4.3.** Let $P$ be a poset for which $U_{\sigma_P}$ is $\mathbb{Q}$-Gorenstein. If $P$ has a crepant labeling, then $X_{\hat{\sigma}_P} \rightarrow U_{\sigma_P}$ is crepant.

**Proof.** By Lemma 4.1 we see that the rays added when refining $\sigma_P$ to $\hat{\sigma}_P$ will be exactly $e_A$ such that $A$ is an upset with $\dim A > 1$. It then follows by the definition of a crepant labeling that $X_{\hat{\sigma}_P} \rightarrow U_{\sigma_P}$ is crepant. \(\square\)

If a poset has a unique maximum element we will denote this element by $\hat{1}$. Thus if $\hat{1}$ exists we have $i \preceq \hat{1}$ for all elements $i$ of the poset. Similarly, if a poset has a unique minimum element it will be denoted $\hat{0}$. In the case $\hat{0}$ exists we have $0 \preceq i$ for each $i$ in the poset.

**Proposition 4.4.** If $P$ is a poset with both $\hat{1}$ and $\hat{0}$, then $U_{\sigma_P}$ is Gorenstein and $X_{\hat{\sigma}_P} \rightarrow U_{\sigma_P}$ is crepant.

**Proof.** By Propositions 4.2 and 4.3 it suffices to produce a Gorenstein labeling of $P$ which is also a crepant labeling. Let us set $\phi(\hat{1}) = 1$, $\phi(\hat{0}) = -1$ and $\phi(i) = 0$ for $i \in [n]$ with $i \notin \{0, \hat{1}\}$. We observe that $\phi$ as defined is a Gorenstein labeling which is also a crepant labeling. Indeed, for any upset

![Hasse diagram and cone](image_url)
A with \( \dim A > 0 \) we have \( \hat{1} \in A \) and \( \hat{0} \notin A \). Thus, for such an \( A \)
\[
\sum_{i \in A} \phi(i) = \phi(\hat{1}) = 1
\]
as needed. \( \square \)

In Figure 7 on the left we have the Hasse diagram of the smallest poset which has \( \hat{0} \) and \( \hat{1} \) but is not tree poset. On the right we have the cone indexed by the poset. The dashed line gives the crepant resolution of singularities from Proposition 4.4. In this case the cone is simply the union of the two Weyl chambers indexed by the two possible linear extensions.

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