Complete moment convergence of moving average processes for m-WOD sequence

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Abstract
In this paper, the complete moment convergence for the partial sum of moving average processes \( \{X_n = \sum_{i=\infty}^{\infty} a_i Y_i\}, n \geq 1 \) is established under some mild conditions, where \( \{Y_i, -\infty < i < \infty\} \) is a sequence of m-widely orthant dependent (m-WOD, for short) random variables which is stochastically dominated by a random variable \( Y \), and \( \{a_i, -\infty < i < \infty\} \) is an absolutely summable sequence of real numbers. These conclusions promote and improve the corresponding results from m-extended negatively dependent (m-END, for short) sequences to m-WOD sequences.

Keywords: Moving average processes; m-WOD; Complete moment convergence

1 Introduction and main results
Let \( \{Y_i, -\infty < i < \infty\} \) be a sequence of random variables and \( \{a_i, -\infty < i < \infty\} \) be an absolutely summable sequence of real numbers, and for \( n \geq 1 \) set \( X_n = \sum_{i=\infty}^{\infty} a_i Y_i\). The limit properties of the moving average process \( \{X_n, n \geq 1\} \) have been extensively investigated by many authors. For example, Burton and Dehling [1] obtained a large deviation principle, Ibragimov [2] established the central limit theorem, Račkauskas and Suquet [3] proved the functional central limit theorems for self-normalized partial sums of linear processes, and An [4], Chen et al. [5], Kim and Ko [6], Li et al. [7], Li and Zhang [8], Wang and Hu [9], Yang and Hu [10], Zhang [11], Zhou [12], Zhou and Lin [13], Zhang [14], Zhang and Ding [15], Song and Zhu [16, 17] got the complete (moment) convergence of moving average process based on a sequence of different dependent (or mixing) random variables, respectively. But few results for moving average process based on m-WOD random variables are known. Firstly, we introduce some definitions.

Definition 1.1 A sequence \( \{Y_i, -\infty < i < \infty\} \) of random variables is said to be stochastically dominated by a random variable \( Y \) if there exists a constant \( C \) such that

\[
P(\{|Y_i| > x\}) \leq CP(\{|Y| > x\}), \quad x \geq 0, -\infty < i < \infty.
\]

Definition 1.2 A real-valued function \( f(x) \), positive and measurable on \([a, \infty), a > 0\), is said to be slowly varying at infinity if, for each \( \lambda > 0 \), \( \lim_{x \to \infty} \frac{f(\lambda x)}{f(x)} = 1 \).
The concept of widely orthant dependence structure was introduced by Wang et al. [18] as follows.

**Definition 1.3** For the random variables \( \{X_n, n \geq 1\} \), if there exists a finite positive sequence \( \{g_U(n), n \geq 1\} \) satisfying, for each \( n \geq 1 \) and for all \( x_i \in R, 1 \leq i \leq n \),

\[
P(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n) \leq g_U(n) \prod_{i=1}^{n} P(X_i > x_i), \quad (1.1)
\]

then we say that the random variables \( \{X_n, n \geq 1\} \) are widely upper orthant dependent (WUOD, for short); if there exists a finite positive sequence \( \{g_L(n), n \geq 1\} \) satisfying, for each \( n \geq 1 \) and for all \( x_i \in R, 1 \leq i \leq n \),

\[
P(X_1 < x_1, X_2 < x_2, \ldots, X_n < x_n) \leq g_L(n) \prod_{i=1}^{n} P(X_i < x_i), \quad (1.2)
\]

then we say that the random variables \( \{X_n, n \geq 1\} \) are widely lower orthant dependent (WLLOD, for short); if they are both WUOD and WLOD, then we say that the random variables \( \{X_n, n \geq 1\} \) are widely orthant dependent (WOD, for short), and \( g_U(n), g_L(n), n \geq 1 \), are called dominated coefficients.

Inspired by WOD and m-NA, Fan et al. [19] introduced the following notion.

**Definition 1.4** Let \( m \geq 1 \) be a fixed integer. A sequence of random variables \( \{X_n, n \geq 1\} \) is said to be m-WOD if, for any \( n \geq 2 \) and \( i_1, i_2, \ldots, i_n \) such that \( |i_k - i_j| \geq m \) for all \( 1 \leq k \neq j \leq n \), we have that \( X_{i_1}, X_{i_2}, \ldots, X_{i_n} \) are WOD.

By (1.1) and (1.2), we can see that \( g_U(n) \geq 1 \) and \( g_L(n) \geq 1 \). Recall that when \( g_U(n) = g_L(n) = M \) for some positive constant \( M \) and any \( n \geq 1 \), then the random variables \( \{X_n, n \geq 1\} \) are called extended negatively dependent (END, for short). The definition of END was introduced by Liu [20]. If both (1.1) and (1.2) hold for \( g_U(n) = g_L(n) = 1 \) for any \( n \geq 1 \), then the random variables \( \{X_n, n \geq 1\} \) are called negatively orthant dependent (NOD, for short), which was introduced by Ebrahimi and Ghosh [21]. It is well known that negatively associated (NA, for short) random variables are NOD. Hu [22] pointed out that negatively superadditive dependent (NSD, for short) random variables are NOD. Hence, the class of m-WOD random variables includes independent sequence, m-NA sequence, NSD sequence, m-NOD sequence, and m-END sequence as special cases. Studying the probability limit theory and its applications for m-WOD random variables is of great interest. But there are few results on the complete moment convergence of moving average process based on an m-WOD sequence. Therefore, in this paper, we establish some results on the complete moment convergence for partial sums for moving average process.

Throughout the sequel, \( C \) represents a positive constant although its value may change from one appearance to the next, \( I(A) \) denotes the indicator function of the set \( A \), \( [x] \) denotes the integer part of \( x \), \( X^+ = \max(X, 0) \), \( X^- = \max([-X, 0]) \).

### 2 Preliminary lemmas

In this section, we give some lemmas which will be useful to prove our main results.
Lemma 2.1 (Fang et al. [19]) Let \( \{X_n, n \geq 1\} \) be a sequence of \( m\)-WOD random variables with dominating coefficients \( g(n) = \max\{g_1(n), g_2(n)\} \). If \( \{f_n, n \geq 1\} \) are all nondecreasing (or nonincreasing), then \( \{f_n(X_n), n \geq 1\} \) are still \( m\)-WOD with dominating coefficients \( g(n), n \geq 1 \).

Lemma 2.2 (Fang et al. [19]) For a positive real number \( q \geq 2 \), if \( \{X_n, n \geq 1\} \) is a sequence of mean zero \( m\)-WOD random variables with dominating coefficients \( g(n) = \max\{g_1(n), g_2(n)\} \). If \( E|X_i|^q < \infty \) for every \( i \geq 1 \), then for all \( n \geq 1 \) there exist positive constants \( C_1(m, q) \) and \( C_2(m, q) \) depending on \( q \) and \( m \) such that

\[
E\left( \left| \sum_{i=1}^{n} X_i \right|^q \right) \leq C_1(m, q) \sum_{i=1}^{n} E|X_i|^q + C_2(m, q) g(n) \left( \sum_{i=1}^{n} EX_i^2 \right)^{\frac{q}{2}}.
\]

Lemma 2.3 (Zhou [12]) If \( l \) is slowly varying at infinity, then

1. \( \sum_{n=m}^{\infty} n^{l(n)} \leq Cn^{q+1}l(m) \) for \( s > -1 \) and positive integer \( m \),
2. \( \sum_{n=m}^{\infty} n^{l(n)} \leq Cn^{q+1}l(m) \) for \( s < -1 \) and positive integer \( m \).

Lemma 2.4 (Wang et al. [23]) Let \( \{X_n, n \geq 1\} \) be a sequence of random variables which is stochastically dominated by a random variable \( X \). Then, for any \( a > 0 \) and \( b > 0 \),

\[
E|X|^a I\{|X| \leq b\} \leq C|E|X|^a I\{|X| \leq b\} + b^aP\{|X| > b\},
\]

\[
E|X|^a I\{|X| > b\} \leq CE|X|^a I\{|X| > b\}.
\]

3 Main results and proofs

Theorem 3.1 Let \( l \) be a function slowly varying at infinity, \( p \geq 1 \), \( \alpha > 1/2 \), \( ap > 1 \). Assume that \( \{a_i, -\infty < i < \infty\} \) is an absolutely summable sequence of real numbers. Suppose that \( \{X_n = \sum_{i=1}^{\infty} a_i Y_{i+n}, n \geq 1\} \) is a moving average process generated by a sequence \( \{Y_i, -\infty < i < \infty\} \) of \( m\)-WOD random variables with dominating coefficients \( g(n) = O(n^\delta) \) for some \( \delta \geq 0 \) which is stochastically dominated by a random variable \( Y \). If \( E|Y| = 0 \) for \( 1/2 < \alpha \leq 1 \), \( E|Y|^p I\{|Y|^{1/p} < \infty \} \) for \( p > 1 \), and \( E|Y|^{1+\lambda} < \infty \) for \( p = 1 \) and some \( \lambda > 0 \), then for any \( \varepsilon > 0 \)

\[
\sum_{n=1}^{\infty} n^{ap-2-\alpha} l(n) E\left\{ \left| \sum_{j=1}^{n} X_j - \varepsilon n^\alpha \right|^q \right\} < \infty.
\]  

(3.1)

Proof Let \( f(n) = n^{ap-2-\alpha} l(n) \) and \( Y_{ij}^{(1)} = -xI\{Y_j < -x\} + Y_j I\{|Y_j| \leq x\} + xI\{Y_j > x\} \) and \( Y_{ij}^{(2)} = Y_j - Y_{ij}^{(1)} \) be the monotone truncations of \( \{Y_j, -\infty < j < \infty\} \) for \( x > 0 \). Then, by Lemma 2.1, it is easy to know that \( \{Y_{ij}^{(1)} - Y_{ij}^{(2)} - \infty < j < \infty\} \) and \( \{Y_{ij}^{(2)} - \infty < j < \infty\} \) are two sequences of \( m\)-WOD random variables. Note that \( \sum_{k=1}^{n} X_k = \sum_{i=\infty}^{\infty} a_i \sum_{j=1}^{i+n} Y_j \) and \( \sum_{i=\infty}^{\infty} |a_i| < \infty \), then by Lemma 2.4 we have, for \( x > n^\alpha \), if \( \alpha > 1 \)

\[
x^{-1} E\left\{ \sum_{i=\infty}^{\infty} a_i \sum_{j=1}^{i+n} Y_{ij}^{(1)} \right\} \leq x^{-1} \sum_{i=\infty}^{\infty} |a_i| \sum_{j=1}^{i+n} \left[ E|Y_j|^q I\{|Y_j| \leq x\} + xP\{|Y_j| > x\} \right]
\]
\[
\leq C x^{-1} n \left[ E |Y| \mathbb{1} \{|Y| \leq x\} + xP(|Y| > x) \right] \leq C n^{1-\alpha} \to 0, \quad \text{as } n \to \infty.
\]

If \(1/2 < \alpha \leq 1\), note that \(\alpha p > 1\), this means \(p > 1\). By \(E|Y|^p I(|Y|^{1/p}) < \infty\) and \(I\) is slowly varying at infinity, it is easy to conclude that, for any \(0 < \varepsilon < p - 1/\alpha\), we have \(E|Y|^{p-\varepsilon} < \infty\).

Then, noting \(EY_i = 0\), by Lemma 2.4 we can obtain

\[
x^{-1} \left| E \sum_{i=\infty}^{n} a_i \sum_{j=1}^{i+n} Y_{ij}^{(1)} \right| = x^{-1} \left| E \sum_{i=\infty}^{n} a_i \sum_{j=1}^{i+n} Y_{ij}^{(2)} \right| \\
\leq C x^{-1} n \sum_{i=\infty}^{n} |a_i| \sum_{j=1}^{i+n} E|Y_j| \mathbb{1} \{|Y_j| > x\} \\
\leq C x^{1/\alpha - 1} n E|Y| \mathbb{1} \{|Y| > x\} \\
\leq E|Y|^{p-\varepsilon} I \mathbb{1} \{|Y| > x\} \to 0, \quad \text{as } x \to \infty.
\]

Therefore, by the above discussion, for \(x > n^{\varepsilon}\) large enough, we know

\[
x^{-1} \left| E \sum_{i=\infty}^{n} a_i \sum_{j=1}^{i+n} Y_{ij}^{(1)} \right| < \varepsilon/4.
\]

Then

\[
\sum_{n=1}^{\infty} f(n) E \left\{ \left\| X_n \right\| - \varepsilon n^{\alpha} \right\}^+ \\
\leq \sum_{n=1}^{\infty} f(n) \int_{\varepsilon n^{\alpha}}^{\infty} P \left\{ \left\| X_n \right\| \geq x \right\} dx \\
\leq C \sum_{n=1}^{\infty} f(n) \int_{\varepsilon n^{\alpha}}^{\infty} P \left\{ \left\| X_n \right\| \geq \varepsilon x \right\} dx \\
\leq C \sum_{n=1}^{\infty} f(n) \int_{\varepsilon n^{\alpha}}^{\infty} P \left\{ \sum_{i=\infty}^{n} a_i \sum_{j=i+1}^{i+n} Y_{ij}^{(2)} \geq \varepsilon x/2 \right\} dx \\
+ C \sum_{n=1}^{\infty} f(n) \int_{\varepsilon n^{\alpha}}^{\infty} P \left\{ \sum_{i=\infty}^{n} a_i \sum_{j=i+1}^{i+n} (Y_{ij}^{(1)} - EY_{ij}^{(1)}) \geq \varepsilon x/4 \right\} dx \\
=: I_1 + I_2. \tag{3.2}
\]

Firstly we prove \(I_1 < \infty\). Noting \(|Y_{ij}^{(2)}| < |Y_j| \mathbb{1} \{|Y_j| > x\}\), then by Markov’s inequality and Lemma 2.4, we have

\[
I_1 \leq C \sum_{n=1}^{\infty} f(n) \int_{\varepsilon n^{\alpha}}^{\infty} x^{-1} E \left\| \sum_{i=\infty}^{n} a_i \sum_{j=i+1}^{i+n} Y_{ij}^{(2)} \right\| dx \\
\leq C \sum_{n=1}^{\infty} f(n) \int_{\varepsilon n^{\alpha}}^{\infty} x^{-1} \sum_{i=\infty}^{n} \left| a_i \sum_{j=i+1}^{i+n} E Y_{ij}^{(2)} \right| dx
\]
\[
\leq C \sum_{n=1}^{\infty} n f(n) \int_{n^{\alpha}}^{\infty} x^r E|Y| I\{ |Y| > x \} \, dx
\]

\[
= C \sum_{n=1}^{\infty} n f(n) \sum_{m=n}^{\infty} m^{(m+1)^\alpha} x^r E|Y| I\{ |Y| > x \} \, dx
\]

\[
= C \sum_{n=1}^{\infty} \int_{n^\alpha}^{\infty} n^\alpha x^r E|Y| I\{ |Y| > n^\alpha \} \, dx
\]

\[
= C \sum_{n=1}^{\infty} m^{1-\alpha} I(m) E|Y| I\{ |Y| > m^\alpha \} \sum_{n=1}^{m^{1-\alpha} I(m)} n^{\alpha-1} l(n).
\]

If \( p > 1 \), then \( \alpha p - 1 - \alpha > -1 \), by Lemma 2.3, we can get

\[
I_1 \leq C \sum_{m=1}^{\infty} m^{\alpha p-1-\alpha} I(m) E|Y| I\{ |Y| > m^\alpha \}
\]

\[
= C \sum_{m=1}^{\infty} m^{\alpha p-1-\alpha} I(m) \sum_{k=m}^{\infty} E|Y| I\{ k^\alpha < |Y| \leq (k+1)^\alpha \}
\]

\[
= C \sum_{k=1}^{\infty} E|Y| I\{ k^\alpha < |Y| \leq (k+1)^\alpha \} \sum_{m=1}^{k} m^{\alpha p-1-\alpha} I(m)
\]

\[
\leq C \sum_{k=1}^{\infty} m^{\alpha p-1-\alpha} I(k) E|Y| I\{ k^\alpha < |Y| \leq (k+1)^\alpha \}
\]

\[
\leq CE|Y|^{\alpha p} I(|Y|^{1/\alpha}) < \infty.
\]

If \( p = 1 \), \( E|Y|^{1+\lambda} < \infty \) implies \( E|Y|^{1+\lambda'} I(|Y|^{1/\alpha}) < \infty \) for any \( 0 < \lambda' < \lambda \), then by Lemma 2.3 we get

\[
I_1 \leq C \sum_{m=1}^{\infty} m^{-1} E|Y| I\{ |Y| > m^\alpha \} \sum_{n=1}^{m} n^{-1} l(n)
\]

\[
\leq C \sum_{m=1}^{\infty} m^{-1} E|Y| I\{ |Y| > m^\alpha \} \sum_{n=1}^{m} n^{-1} \alpha \lambda' l(n)
\]

\[
\leq C \sum_{m=1}^{\infty} m^{\lambda' - 1} I(m) E|Y| I\{ |Y| > m^\alpha \}
\]

\[
\leq CE|Y|^{1+\lambda'} I(|Y|^{1/\alpha}) < \infty.
\]

So, we conclude

\[
I_1 < \infty. \quad (3.3)
\]

Next we show \( I_2 < \infty \). By Markov’s inequality, Hölder’s inequality, and Lemma 2.2, we can obtain

\[
I_2 \leq C \sum_{n=1}^{\infty} f(n) \int_{n^\alpha}^{\infty} x^r E \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=1}^{i+n} (Y_{xj} - EY_{xj}) \right| \, dx
\]
\[
\begin{align*}
&\leq C \sum_{n=1}^{\infty} f(n) \int_{n^p}^{\infty} x^{-r} E \left[ \sum_{i=0}^{\infty} \left( |a_i| \right)^{r} \left( \left| \sum_{j=i+1}^{n} (Y_{i,j}^{(1)} - EY_{i,j}^{(1)}) \right| \right)^{r} \right] dx \\
&\leq C \sum_{n=1}^{\infty} f(n) \int_{n^p}^{\infty} x^{-r} \left( \sum_{i=0}^{\infty} |a_i| \left( \sum_{j=i+1}^{n} \left| \sum_{j=i+1}^{n} (Y_{i,j}^{(1)} - EY_{i,j}^{(1)}) \right| \right)^{r} \right) dx \\
&\leq C \sum_{n=1}^{\infty} f(n) \int_{n^p}^{\infty} x^{-r} \left( \sum_{i=0}^{\infty} |a_i| \left( \sum_{j=i+1}^{n} \left| \sum_{j=i+1}^{n} (Y_{i,j}^{(1)} - EY_{i,j}^{(1)}) \right| \right)^{r} \right) dx \\
&+ C \sum_{n=1}^{\infty} f(n) g(n) \int_{n^p}^{\infty} x^{-r} \left( \sum_{i=0}^{\infty} |a_i| \left( \sum_{j=i+1}^{n} \left| \sum_{j=i+1}^{n} (Y_{i,j}^{(1)} - EY_{i,j}^{(1)}) \right| ^{2} \right) ^{r/2} \right) dx \\
&= I_{21} + I_{22},
\end{align*}
\]

where \( r \geq 2 \) will be given later.

For \( I_{21} \), if \( p > 1 \), taking \( r > \max(2, p) \), then by \( C_{r} \) inequality, Lemma 2.3, and Lemma 2.4, we know

\[
I_{21} \leq C \sum_{n=1}^{\infty} f(n) \int_{n^p}^{\infty} x^{-r} \sum_{i=0}^{\infty} |a_i| \sum_{j=i+1}^{n} \left[ E|Y_i'|I\{|Y_i| \leq x\} + x^r P(|Y_i| > x) \right] dx \\
\leq C \sum_{n=1}^{\infty} nf(n) \int_{n^p}^{\infty} x^{-r} \left[ E|Y_i'|I\{|Y| \leq x\} + x^r P(|Y| > x) \right] dx \\
\leq C \sum_{n=1}^{\infty} nf(n) \sum_{m=0}^{n} \int_{n^p}^{\infty} \left[ x^{-r} E|Y_i'|I\{|Y| \leq x\} + P(|Y| > x) \right] dx \\
\leq C \sum_{n=1}^{\infty} nf(n) \sum_{m=0}^{n} \left[ m^{(1-r)-1} E|Y_i'|I\{|Y| \leq (m+1)^{a}\} + m^{r-1} P(|Y| > m^{a}) \right] \\
= C \sum_{m=1}^{\infty} \left[ m^{(1-r)-1} E|Y_i'|I\{|Y| \leq (m+1)^{a}\} + m^{r-1} P(|Y| > m^{a}) \right] \sum_{n=1}^{\infty} nf(n) \\
\leq C \sum_{m=1}^{\infty} m^{(p-r)-1} l(m) \sum_{k=1}^{m} E|Y_i'|I\{k^a < |Y| \leq (k+1)^{a}\} \\
+ C \sum_{m=1}^{\infty} m^{p-1} l(m) \sum_{k=0}^{m} E|Y_i'|I\{k^a < |Y| \leq (k+1)^{a}\} \\
= C \sum_{k=1}^{\infty} E|Y_i'|I\{k^a < |Y| \leq (k+1)^{a}\} \sum_{m=k}^{\infty} m^{(p-r)-1} l(m) \\
+ C \sum_{k=1}^{\infty} E|Y_i'|I\{k^a < |Y| \leq (k+1)^{a}\} \sum_{m=1}^{k} m^{p-1} l(m) \\
\leq C \sum_{k=1}^{\infty} k^{(p-r)} l(k) E|Y_i'|I\{|Y| > (k+1)^{a}\} \\
+ C \sum_{k=1}^{\infty} k^{p} l(k) E|Y_i'|I\{|Y| > (k+1)^{a}\} \\
\leq CE|Y^p| l(|Y|^{1/\alpha}) < \infty.
\]
For $I_{21}$, if $p = 1$, taking $r > \max\{1 + \lambda', 2\}$, where $0 < \lambda' < \lambda$, then by the same argument as above we know

$$I_{21} \leq C \sum_{m=1}^{\infty} \left[ m^{\alpha(1-r)-1} E|Y|^2 I\left(|Y| \leq (m+1)^\alpha \right) + m^{\alpha-1} P\left(|Y| > m^\alpha \right) \right] \sum_{n=1}^{m} n f(n)$$

$$\leq C \sum_{m=1}^{\infty} \left[ m^{\alpha(1-r)-1} E|Y|^2 I\left(|Y| \leq (m+1)^\alpha \right) + m^{\alpha-1} P\left(|Y| > m^\alpha \right) \right] \sum_{n=1}^{m} n^{-1+\alpha'} I(n)$$

$$\leq C \sum_{m=1}^{\infty} m^{\alpha(1-r+\lambda')-1} l(m) E|Y|^2 I\left(|Y| \leq (m+1)^\alpha \right)$$

$$+ m^{\alpha(1+\lambda')-1} l(m) E I\left(|Y| > m^\alpha \right)$$

$$\leq CE |Y|^{1+\lambda'} I(|Y|^{1/\alpha}) < \infty. \quad (3.6)$$

For $I_{22}$, if $1 < p < 2$, noting that $g(n) = O(n^\beta)$, taking $r > 2$ such that $\alpha p + r/2 - \alpha p r/2 - 1 + \delta = (\alpha p - 1)(1 - r/2) + \delta < 0$, then by $C_r$ inequality, Lemma 2.3, and Lemma 2.4, we obtain

$$I_{22} \leq C \sum_{n=1}^{\infty} n^{r/2} f(n) g(n) \int_{x_0}^{\infty} x^{-\tau} \left[ E|Y|^2 I\left(|Y| \leq x \right) \right]^{r/2} + x^{r/2} \left[ E|Y|^2 I\left(|Y| > x \right) \right] dx$$

$$\leq C \sum_{n=1}^{\infty} n^{r/2} f(n) g(n) \sum_{m=n}^{\infty} \left[ m^{\alpha(1-r)-1} \left( E|Y|^2 I\left(|Y| \leq (m+1)^\alpha \right) \right)^{r/2} + m^{\alpha-1} P^{r/2} \left(|Y| > m^\alpha \right) \right]$$

$$\leq C \sum_{m=1}^{\infty} \left[ m^{\alpha(1-r)-1} \left( E|Y|^2 I\left(|Y| \leq (m+1)^\alpha \right) \right)^{r/2} + m^{\alpha-1} P^{r/2} \left(|Y| > m^\alpha \right) \right] \sum_{n=1}^{m} n^{r/2} f(n) g(n)$$

$$\leq C \sum_{m=1}^{\infty} m^{\alpha(p-r)+r/2-\delta-2} l(m) \left( E|Y|^p I\left(|Y|^{1-p} \right) \right)^{r/2} \left( E|Y|^2 I\left(|Y| \leq (m+1)^\alpha \right) \right)^{r/2}$$

$$+ C \sum_{m=1}^{\infty} m^{\alpha(p-r)+r/2-\delta-2} l(m) \left( E|Y|^p I\left(|Y|^p > m^\alpha \right) \right)^{r/2}$$

$$\leq C \sum_{m=1}^{\infty} m^{\alpha(p-r)+r/2-\delta-2} l(m) \left( E|Y|^p \right)^{r/2} < \infty. \quad (3.7)$$

For $I_{22}$, if $p \geq 2$, noting that $g(n) = O(n^\beta)$, taking $r > (\alpha p - 1)/(\alpha - 1/2) \geq p$ such that $\alpha (p - r) + r/2 + \delta - 1 < 0$, then by $C_r$ inequality, Lemma 2.3, and Lemma 2.4, similar to the proof of (3.7), one gets

$$I_{22} \leq C \sum_{m=1}^{\infty} \left[ m^{\alpha(1-r)-1} \left( E|Y|^2 I\left(|Y| \leq (m+1)^\alpha \right) \right)^{r/2} \right.$$
Let \( l \) be a function slowly varying at infinity.

**Theorem 3.2** Let \( l \) be a function slowly varying at infinity. Assume that 
\[
\sum_{n=1}^{\infty} n^p f(n) g(n) + m^p \frac{1}{p(1/p)} \left( |Y| > m^p \right) \sum_{n=1}^{m} n^{1/p} f(n) g(n)
\]
\[
\leq C \sum_{m=1}^{\infty} m^p \sum_{m+1}^{2m+2} l(m) \left( |Y|^{2p} I \left( |Y| \leq (m+1)^p \right) \right)^{1/r}
\]
\[
+ C \sum_{m=1}^{\infty} m^p \sum_{m+1}^{2m+2} l(m) \left( |Y|^{2p} I \left( |Y| > m^p \right) \right)^{1/r}
\]
\leq C \sum_{m=1}^{\infty} m^{p+r/2} l(m) \left( |Y|^{2p} \right)^{1/r} < \infty.

(3.8)

Thus, (3.1) can be deduced immediately by combining (3.2)–(3.8).

The next theorem will discuss the case \( ap = 1 \).

**Theorem 3.2** Let \( l \) be a function slowly varying at infinity, \( 1 \leq p < 2 \). Assume that 
\[
\sum_{i=-\infty}^{\infty} |a_i| < \infty,
\]
where \( \theta \) belongs to \((0,1)\) if \( p = 1 \) and \( \theta = 1 \) if \( 1 < p < 2 \). Suppose that 
\( \{X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}, n \geq 1 \} \) is a moving average process generated by a sequence \( \{Y_i, -\infty < i < \infty\} \) of \( m \)-WOD random variables with dominating coefficients \( g(n) = O(n^\alpha) \) for some \( 0 \leq \delta < (2-p)/p \) which is stochastically dominated by a random variable \( Y \). If \( EY_i = 0 \) and \( E|Y|^{p(1+\delta)} I(|Y| < \infty) \), then for any \( \epsilon > 0 \)
\[
\sum_{n=1}^{\infty} n^{-1/p} l(n) E \left\{ \left| \sum_{j=1}^{k} X_j \right| - \epsilon n^{1/p} \right\}^+ < \infty.

(3.9)

**Proof** Let \( h(n) = n^{-1/p} l(n) \). Similar to the proof of (3.2), we obtain
\[
\sum_{n=1}^{\infty} h(n) E \left\{ \left| \sum_{j=1}^{n} X_j \right| - \epsilon n^{1/p} \right\}^+
\]
\[
\leq C \sum_{n=1}^{\infty} h(n) \int_{n^{1/p}}^{\infty} P \left\{ \left| \sum_{i=-\infty}^{\infty} a_i \sum_{i+j=n}^{i+n} Y_{ij}^{(2)} \right| \geq \epsilon x/2 \right\} dx
\]
\[
+ C \sum_{n=1}^{\infty} h(n) \int_{n^{1/p}}^{\infty} P \left\{ \left| \sum_{i=-\infty}^{\infty} a_i \sum_{i+j=n}^{i+n} \left( Y_{ij}^{(1)} - EY_{ij}^{(1)} \right) \right| \geq \epsilon x/4 \right\} dx
\]
\[
=: J_1 + J_2.

(3.10)

For \( J_1 \), by Markov’s inequality, \( C_r \) inequality, Lemma 2.3, and Lemma 2.4, one gets
\[
J_1 \leq C \sum_{n=1}^{\infty} h(n) \int_{n^{1/p}}^{\infty} x^{-\theta} E \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=-\infty}^{j=n} Y_{ij}^{(2)} \right| \left| x^{-\theta} \right| \left| x^{-\theta} \right| dx
\]
\[
\leq C \sum_{n=1}^{\infty} h(n) \int_{n^{1/p}}^{\infty} x^{-\theta} E|Y|^{\theta} I \left( |Y| > x \right) \left| x^{-\theta} \right| \left| x^{-\theta} \right| dx
\]
\[
= C \sum_{n=1}^{\infty} h(n) \int_{n^{1/p}}^{(n+1)^{1/p}} x^{-\theta} E|Y|^{\theta} I \left( |Y| > x \right) \left| x^{-\theta} \right| \left| x^{-\theta} \right| dx
\]
For $J_2$, as the same argument of $J_2$, noting that $g(n) = O(n^\delta)$ for some $0 \leq \delta < (2 - p)/p$, taking $r = 2$, by Lemma 2.2, Lemma 2.3, and Lemma 2.4, we conclude

$$J_2 \leq C \sum_{n=1}^{\infty} [m^{-1/p}E|Y|^2I\{|Y| \leq (m+1)^{1/p}\} + m^{1/p-1}P(|Y| > m^{1/p})] \sum_{n=1}^{m} nh(n)(1 + g(n))$$
\[
\leq C \sum_{k=1}^{\infty} E|Y|^2 I\{k^{1/p} < |Y| \leq (k + 1)^{1/p}\} \sum_{m=k}^{\infty} m^{-2/p+\delta} l(m)
+ C \sum_{k=1}^{\infty} E I\{k^{1/p} < |Y| \leq (k + 1)^{1/p}\} \sum_{m=1}^{k} m^{\delta} l(m)
\leq C \sum_{k=1}^{\infty} k^{-2/p+\delta+1} l(k) E|Y|^2 I\{k^{1/p} < |Y| \leq (k + 1)^{1/p}\}
+ C \sum_{k=1}^{\infty} k^{\delta+1} l(k) E I\{k^{1/p} < |Y| \leq (k + 1)^{1/p}\}
\leq C \sum_{k=1}^{\infty} l(k) E|Y|^{p(1+\delta)} I\{k^{1/p} < |Y| \leq (k + 1)^{1/p}\}
\leq C E|Y|^{p(1+\delta)} l(|Y|^p) < \infty. \tag{3.12}
\]

Hence, by combining (3.10)–(3.12), (3.9) holds. \[\square\]

For the complete convergence, we have the following corollary from the above theorems immediately.

**Corollary 3.3** Under the assumptions of Theorem 3.1, for any \( \varepsilon > 0 \), we have

\[
\sum_{n=1}^{\infty} n^{\alpha p - 2} l(n) P \left( \left| \sum_{j=1}^{n} X_j \right| > \varepsilon n^p \right) < \infty. \tag{3.13}
\]

Under the assumptions of Theorem 3.2, for any \( \varepsilon > 0 \), we have

\[
\sum_{n=1}^{\infty} n^{-1} l(n) P \left( \left| \sum_{j=1}^{n} X_j \right| > \varepsilon n^{1/p} \right) < \infty. \tag{3.14}
\]

**Remark 3.4** Since m-WOD random variables include independent, m-NA, NSD, WOD, m-NOD, and m-END random variables, so our results also hold for independent, m-NA, NSD, WOD, m-NOD, and m-END random variables, and therefore Theorem 3.1 and Theorem 3.2 improve upon the known results.

**Remark 3.5** Obviously, the assumption that \( \{Y_i, -\infty < i < \infty\} \) is stochastically dominated by a random variable \( Y \) is weaker than the assumption of identical distribution of the random variables \( \{Y_i, -\infty < i < \infty\} \), therefore the results of Theorem 3.1 and Theorem 3.2 also hold for identically distributed random variables.

**Remark 3.6** Let \( a_0 = 1, a_i = 0, i \neq 0 \), then \( S_n = \sum_{k=1}^{n} X_k = \sum_{k=1}^{n} Y_k \). Hence the results of Theorem 3.1 and Theorem 3.2 also hold when \( \{X_k, k \geq 1\} \) is a sequence of m-WOD random variables which is stochastically dominated by a random variable \( Y \).

**Remark 3.7** The results obtained by this paper and Fang et al. [19] are different. In our paper, we mainly discuss the complete moment convergence of moving average processes for an m-WOD sequence, Fang et al. [19] proved the asymptotic approximations of ratio moments based on the m-WOD sequence.
4 Conclusions

In this paper, using the moment inequality for m-WOD sequences and truncation method, the complete moment convergence for the partial sum of moving average processes \( \{X_n = \sum_{i=-\infty}^{\infty} a_i Y_{n+i}, n \geq 1 \} \) is established, where \( \{Y_{n+i}, -\infty < i < \infty \} \) is a sequence of m-WOD random variables which is stochastically dominated by a random variable \( Y \), and \( \{a_i, -\infty < i < \infty \} \) is an absolutely summable sequence of real numbers. These conclusions obtained extend and improve the corresponding results from m-END sequences to m-WOD sequences.

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Availability of data and materials

Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

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