\section{Introduction}

In contrast to the vast amount of literature regarding static properties (e.g., universal scaling laws) of phase transitions – both thermal and at zero temperature – we are just starting to understand their dynamical features, especially the behaviour during a time-dependent sweep (quench) through the critical point. This topic has attracted increasing interest in recent years, see, e.g., \cite{1,2,3}. and, again, some universal properties became evident \cite{4}. For example, during a symmetry-breaking (second-order) dynamical phase transition, the diverging response time inevitably entails nonequilibrium processes and so the initial quantum (and thermal) fluctuations are amplified strongly, ultimately determining the final order parameter distribution. If the final phase permits topological defects (e.g., vortices in superfluids), they will generally be created in such a quench via (the quantum version of) the Kibble-Zurek mechanism. The latter occurs in many diverse physical settings, for instance, in nonequilibrium phase transitions during the early universe \cite{5,6} or in condensed matter systems \cite{7}.

Unfortunately, due to the inherent computational complexity of such scenarios, explicit calculations are difficult in general, and thus often rather uncontrolled approximations and approximations (e.g., Gaussianity \cite{8,9}) have been invoked. For example, the correlation function after the transition has been used to infer the number of created quasiparticle excitations (see, e.g., \cite{10}). The quasiparticle number is, then, supposed to directly yield an estimate for the topological defect densities generated by the quench. For special cases such as the (exactly solvable) one-dimensional quantum Ising model (where the only excitations are topological defects, i.e., kinks \cite{11,12,13}), such an approach might give the correct answer – but in general, this will not be the case, as we will argue below.

In the following, we consider a rather general $O(N)$-symmetry breaking quantum quench and study the creation of topological defects (hedgehogs in the case considered) via calculating their winding number. In order to base our derivation on a well-defined expansion, we consider the large-$N$ limit. Apart from the large-$N$ limit, no further approximations will be needed, i.e., our results will be quite universal. Moreover, similar to analogous large-$N$ approaches in condensed matter and field theory (assuming that there is no critical value of $N$ where the system changes drastically), we expect our results to apply qualitatively also to finite $N$ \cite{10} which are accessible to experimental tests. Bose-Einstein condensates, in particular, permit the time-resolved observation of the defect formation mechanism due to the comparatively long quas-equilibration time scales of these dilute ultracold quantum gases \cite{14,15}.

\section{Effective Action}

As a first step, we construct a general effective action for an $O(N)$-model in terms of the $N$-component field $\phi = (\phi_1, \ldots, \phi_N)$, which determines the order parameter. To this end, we start from the equation of motion with an arbitrary function $f$

$$\ddot{\phi} = f(\phi, \dot{\phi}, \nabla^2 \phi, \nabla^2 \dot{\phi}, \nabla^4 \phi, \nabla^4 \dot{\phi}, \ldots) \quad (1)$$

In order to avoid run-away solutions and to facilitate a proper quantum description, we have assumed the absence of time derivatives of third or higher order. The initial state (before the transition) obeys the $O(N)$ symmetry: $\langle \phi_a \rangle = 0$ and $\langle \phi_a(x) \phi_b(x') \rangle \propto \delta_{ab}$, etc. As stated, in all of our calculations, we employ the large-$N$ limit assuming $N \gg 1$. In this case, $O(N)$ invariant combinations such as $\hat{\phi}^2 = \hat{\phi}_1^2 + \cdots + \hat{\phi}_N^2$ are sums of many independent quantities on an equal footing \cite{16}. Considering commutators of such combinations, we obtain the well-known fact that their leading contribution (in the large-$N$ limit) behaves as a c-number whereas the (classical and quantum) fluctuations scale with $\sqrt{N}$ (cf., the law of large numbers). Therefore, we may approximate

$$\hat{\phi}^2 = (\phi^2) + O(\sqrt{N}), \quad \langle \hat{\phi}^2 \rangle = O(N) \quad (2)$$

arriving at a semi-classical (mean-field) expansion valid in the large-$N$ limit. As a result, we may approximate the nonlinear terms in the equation of motion \cite{11}, for
example by $\tilde{\phi}^3 \approx (\tilde{\phi}^2)\tilde{\phi}$, arriving at a linearized description. This leads us to the most general linear and local $O(N)$ invariant effective action containing up to first time derivatives of the fields $\phi = (\phi_1, \ldots, \phi_N)$

$$\mathcal{L} = \frac{1}{2} \left( \tilde{\phi} \cdot F(-\nabla^2) \tilde{\phi} - \phi \cdot G(-\nabla^2)\phi \right),$$

(3)

with arbitrary Fourier space functions $F(k^2)$ and $G(k^2)$.

III. PHASE TRANSITION

From Eq. (3), we derive a Klein-Gordon type dispersion relation $[\text{to } O(k^2)]$ for the linearized fluctuations,

$$\omega^2(k) = \frac{G(k^2)}{F(k^2)} = m^2 c^4 + c^2 k^2 + O(k^4).$$

(4)

Initially, all modes are stable, $\omega^2(k) \geq 0$, since we linearize around the initial $[O(N)-symmetric] \text{ state. After the } O(N)\text{-symmetry breaking transition, however, the state } \langle \tilde{\phi} \rangle = 0 \text{ no longer stable and the system “wants” to roll down to a state with } \langle \tilde{\phi} \rangle \neq 0. \text{ Typically (for second-order transitions, i.e., without metastability), this implies that some of the modes become unstable, } \omega^2(k) < 0, \text{ cf. Fig. 1. Since Eq. (3) is already a result of the large-$N$ limit, we assume that } \omega^2(k) \text{ is independent of } N \geq 1 (\text{otherwise the group and phase velocities would either diverge or vanish in the limit } N \rightarrow \infty). \text{ Furthermore, modes with sufficiently large } k \text{ should be stable } \omega^2(k \uparrow \infty) > 0, \text{ so that the unstable interval in which } \omega^2(k) < 0 \text{ is assumed to be finite.}

So far, our results were independent of the number $D$ of spatial dimensions due to isotropy. In the following, we set $N = D$ in order to facilitate the creation of topological defects in the form of hedgehogs (see below). So strictly speaking, we consider the simultaneous limit $N \rightarrow \infty$ and $D \rightarrow \infty$ and assume that these limits commute. Since all relevant quantities such as $\langle \tilde{\phi}_a(r, t)\tilde{\phi}_b(r', t) \rangle$ depend on $|r - r'|$ only (isotropy), the large-$D$ limit basically just affects the integration measure $d^D k$, which strongly supports this assumption. With a Fourier expansion of Eq. (4), we obtain the two-point function after the quench

$$\langle \tilde{\phi}_a(r, t)\tilde{\phi}_b(r', t) \rangle = \frac{\delta_{ab}}{(2\pi)^{N/2}} \int dk k^{N-1} \frac{J_\nu(kL)}{(kL)^\nu} \left[C_k^+ e^{\pm 2i\omega_k t} + D_k \right],$$

(5)

with $L = |r - r'|$. The Bessel functions $J_\nu$ with index $\nu = N/2 - 1$ arise from the integration over all $k$-directions and the factors $C_k^\pm$ and $D_k$ depend on the initial state (for example the temperature) as well as quench dynamics, and are roughly independent of $N$. As expected, we obtain an exponential growth of the unstable modes, which have $\omega^2 < 0$, after the phase transition -- which then seeds the creation of topological defects. Of course, due to the growing modes, the linearization in Eq. (3) will fail eventually -- but for $N \uparrow \infty$, the time $t$ until which the linearization and thus Eq. (4) applies does also grow. Therefore, we may distinguish basically three phases following the quench: First, we get a period of exponential growth of the modes where the linearized description in Eqs. (3) till (5) applies. Then, nonlinear effects set in and lead to a saturation of this growth and possibly an oscillation around the new energy minimum. Finally, the topological defects created by the quench start to “feel” the attraction between hedgehogs and anti-hedgehogs leading to their approaching each other and eventual annihilation. This general picture has been qualitatively confirmed by numerical simulations [23] for $N = 2$. The topological defects are seeded in the first phase (exponential growth) and slowly disappear in the final phase. Therefore, we expect to obtain a good estimate for the maximum number of created defects at long length scales and intermediate times from our linearized analysis.

In addition to the exponentially growing (in $t$) modes at finite $k$, the integral (4) does also yield a huge contribution from large $k$, because the phase space factor $k^{N-1}$ rapidly rises with $k$ for large $N$. This gives rise to a strong $\text{UV singularity of the two-point function } \propto |r - r'|^{-O(N)}$, see the discussion below of regularizing this $\text{UV divergence.}

IV. TOPOLOGICAL DEFECTS

After the symmetry-breaking transition, the ground state is degenerate and can be specified by a nonvanishing expectation value $\langle \tilde{\phi} \rangle \neq 0$, which singles out a preferred direction given by the unit vector $n = \langle \tilde{\phi} \rangle / |\langle \tilde{\phi} \rangle|$. Thus the original $O(N)$ symmetry is broken down to $O(N-1)$, i.e., rotations around the $n$-axis, and the ground-state manifold corresponds to the surface $S_{N-1}$ of a sphere in

FIG. 1. Two generic examples for the evolution of the dispersion relation $[\text{during a symmetry-breaking phase transition, see also [10]}. \text{ Initially (dotted line), all } k \text{-values are stable, } \omega^2(k) > 0. \text{ At the critical point (dashed line), the dispersion relation touches the } k \text{-axis, and after the transition (solid line), modes in a finite } k \text{-interval become unstable, } \omega^2(k) < 0. \text{ The left panel corresponds to a case where } \omega^2(k = 0) = m^2 \text{ in Eq. (4) remains positive while } c^2 \text{ changes sign (see, e.g., [21]); whereas, in the right panel, } \omega^2(k = 0) = m^2 \text{ becomes negative. In both cases, however, there is a dominant wave vector } k_\ast \text{ (for large } N), \text{ as indicated by the vertical arrows.}
$N$ dimensions $O(N)/O(N-1) \simeq S_{N-1}$. Remembering the homotopy group $\pi_{N-1}(S_{N-1}) = \mathbb{Z}$, we see that topological point defects in the form of hedgehogs exist in $N$ spatial dimensions $[22, 23]$. These defects correspond to nontrivial mappings from the ground-state manifold $S_{N-1}$ onto the surface $S_{N-1}$ of a sphere in real space, characterized by a winding number $\mathcal{N} \in \mathbb{Z}$, which reads

$$\mathcal{N} = \frac{\varepsilon_{abc} \varepsilon^{\alpha\beta\gamma \cdots}}{\Gamma(N)} \langle \hat{S}_a n^a (\partial_\beta n^b)(\partial_\gamma n^c) \rangle \cdots , \quad (6)$$

where $\|S_{N-1}\| = 2\pi^{N/2}/\Gamma(N/2)$ is the surface area of the unit sphere in $N$ dimensions. Starting with the $O(N)$-symmetric state as the initial state, we cannot simply insert $n = \langle \hat{\phi} \rangle/\|\langle \hat{\phi} \rangle\|$ since $\langle \hat{\phi} \rangle$ vanishes. Therefore, we use a quantum operator $\hat{n}$ instead, which must be defined appropriately, and allows for a derivation of the probability distribution of the quantum winding number $\mathcal{N}$ in a given volume from the above general expression. In particular, the expectation value of the winding number is of course zero, $\langle \mathcal{N} \rangle = 0$, but its variance $\langle \mathcal{N}^2 \rangle$ is in general not. Setting directly $\hat{n} \propto \hat{\phi}$, we see that the variance $\langle \mathcal{N}^2 \rangle$ is plagued with UV divergences similar to other quantities containing products of quantum fields at the same space-time point. This is due to the fact that quantum fluctuations of $\hat{\phi}$ at arbitrary $k$-scales [cf. the strong UV singularity mentioned after Eq. (5)] would in general contribute to $\langle \mathcal{N}^2 \rangle$. In fact, even the $O(N)$-symmetric initial ground state can be viewed as a “quantum soup” of virtual hedgehog-anti-hedgehog pairs which are constantly popping in and out of existence. Here, we are not interested in those virtual short-lived defects, but in long-lived hedgehogs, which are created by the quantum quench. Therefore, we have to insert a time-averaged unit vector defined via

$$\hat{n}(r) = \frac{1}{2\pi} \int dt g(t) \hat{\phi}(t, r) , \quad (7)$$

with a smooth smearing function $g(t)$ and the normalization $Z = \langle [\int dt g(t) \hat{\phi}(t, r)]^2 \rangle^{1/2} \propto O(\sqrt{N})$, where we have used the large-$N$ (mean-field) expansion. This time-average now suppresses all (rapidly) oscillating modes with $\omega_k^2 > 0$ and only leaves the growing modes $\omega_k^2 < 0$. After this UV-regularization, the integral in the two-point function $[24]$ will be dominated by only a few modes in the vicinity of a certain wavenumber $k_*$. In view of the phasespace factor $k^{N-1}$ in $[24]$, the dominant contribution for large $N$ $[20]$ will arise from the largest $k$ value for which $\omega_k^2 < 0$, i.e., close to the zero of $\omega_k^2$, cf. Fig. [22].

Thus, we can evaluate $[24]$ in saddle point approximation and obtain the correlator for the time-averaged direction vector in $[27]$

$$\langle \hat{n}_a(r) \hat{n}_b(r') \rangle = 2^{\nu} \Gamma(\nu + 1) J_\nu(k_* L)/(k_* L)^\nu \delta_{ab} = f(L) \delta_{ab} \cdots , (8)$$

where higher-order terms $\propto N^{-3/2}$ stemming from the normalization $Z$ in $[27]$ have been omitted. These higher-order terms vanish for $N \to \infty$ so that $[27]$ becomes indeed exact and, in view of the asymptotic behavior of the Bessel functions $[27]$, a Gaussian correlator

$$f(L) = \frac{1}{N} \exp \left\{ -\frac{k_*^2 L^2}{2N} \right\} \cdots \quad (9)$$

follows. Thus, the typical linear domain size (correlation length) is given by $L_{\text{corr}} = O(\sqrt{N}/k_*)$. At extremely large distances $L = O(N/k_*)$ (where the first nontrivial zero of the Bessel function $J_\nu$ is located), there are oscillatory deviations, but in this regime, the correlator is already exponentially small. We emphasize that the Gaussian form of $f(L)$ stems from the large $N$ limit of the exact expression in $[27]$, and is not assumed $a \text{ priori}$. Furthermore, as may already be observed in Eq. (3), the emergence of a dominant wavevector $k_*$ implies the cancellation of all time-dependence, i.e., the time-dependence of the growing part of $[27]$ approximately separates such that the time-averaged unit vector $[27]$ becomes independent of $g(t)$ and thus stationary (in the regime under consideration). So the emergence of a dominant scale in the correlator, $k_*$, a posteriori justifies the introduction of the UV regulator $g(t)$, which only affects the rapidly oscillating modes at larger $k$ but not the observables we are interested in.

V. SCALING LAWS

Now we are in a position to derive the dependence of $\langle \mathcal{N}^2 \rangle$ on $N$ and the enclosed volume. Inserting Eq. (7) into the winding number variance $\langle \mathcal{N}^2 \rangle$ from Eq. (6), we obtain the expectation value of the product of $2N$ fields $\hat{\phi}_a$, which factorizes into $N$ two-point functions $[28]$. Since these functions are completely regular, we may apply Gauss’ law to the two surface integrals occurring in $\langle \mathcal{N}^2 \rangle$, and get after some algebra $[28]$

$$\langle \mathcal{N}^2 \rangle = \frac{NN!}{\|S_{N-1}\|^2} \int d^N r d^N r' \frac{1}{L^{N-1}} \frac{\partial}{\partial L} \left( \frac{\partial f}{\partial L} \right)^N . \quad (10)$$

For a sphere of radius $R$, $V = \{ r : r^2 < R^2 \}$, we can evaluate this expression and finally obtain a single integral of the form

$$\langle \mathcal{N}^2 \rangle = \frac{N!}{\pi} \int_0^{\pi/2} \sin \theta \left( -\cos \theta \frac{\partial f}{\partial L} (2R \sin \theta) \right)^N . \quad (11)$$

Note that these general expressions $[19]$ and $[11]$ for $\langle \mathcal{N}^2 \rangle$ are neither restricted to the Bessel functions $[28]$, nor to large $N$, but might be employed for any correlator of the form $\langle \hat{n}_a(r) \hat{n}_b(r) \rangle = \delta_{ab} f(L)$ in any dimension $N \geq 2$. Let us discuss the scaling of $\langle \mathcal{N}^2 \rangle$ with respect to $R$ and $N \gg 1$, using the Gaussian correlator $[28]$. For radii far
above the correlation length \( R \gg L_{\text{corr}} = \sqrt{N/k_s} \), the winding number variance \( \langle \hat{N}^2 \rangle \) behaves as

\[
\langle \hat{N}^2 \rangle = \left( e^{-3/2} \frac{k_s R}{\sqrt{N}} \left[ 1 + \mathcal{O}(1/\sqrt{N}) \right] \right)^{N-1}.
\]  

(12)

We observe that \( \langle \hat{N}^2 \rangle \) scales with the area \( R^{N-1} \) of the hyper-surface enclosing the defects. Apart from the prefactor \( e^{-3/2}/\sqrt{N} \), this area scaling is quite universal as it holds for spherical volumes in any dimension \( N \geq 2 \) – provided we assume short-range correlations – and can already be inferred from Eq. (13): If we calculate \( \langle \hat{N}^2 \rangle \) using (4), we obtain two hyper-surface integrals. Due to isotropy, the first one yields \( R^{N-1} \) while the second integral averages over the distance \( |r - r'| \) between the two points on the surface. Assuming short-range correlations only, this second integral becomes independent of \( R \) (for large \( R \)) and gives \( h(N)k_s^{N-1} \) with some function \( h(N) \).

Note, however, that the assumption of short-range correlations is crucial and nontrivial in this argument: For vortices in two dimensions, for example, we obtained logarithmic corrections to the ‘area’ scaling, \( \langle \hat{N}^2 \rangle \propto R \ln R \) [4], since the correlator fell off quite slowly at large \( L \).

If the radius \( R \) shrinks and approaches the correlation length \( R \sim L_{\text{corr}} = O(\sqrt{N/k_s}) \), the winding number variance decreases rapidly (for \( N \gg 1 \)). A sphere with \( R = O(\sqrt{N/k_s}) \) then would contain around one defect (or anti-defect) on average, \( \langle \hat{N}^2 \rangle = O(1) \), which determines the total defect density. For even smaller radii, far below the correlation length \( R \ll L_{\text{corr}} = O(\sqrt{N/k_s}) \), the above formulae would yield a scaling \( \langle \hat{N}^2 \rangle \sim R^{2N} \), i.e., an exponential suppression (for large \( N \)). However, the precise functional form should not be trusted upon in this regime since we have neglected \( O(1/\sqrt{N}) \)-corrections in our derivation, which is problematic if the final result is exponentially small. From a more physical point of view, the mean-field approximation breaks down near the core of a defect (where \( n \) becomes ill-defined), which renders Eq. (11) questionable for too small volumina. For small \( R \), one would expect a volume-type scaling \( \langle \hat{N}^2 \rangle \sim R^N \), i.e., the typical behaviour for uncorrelated defects, which should be the case if there is one hedgehog at most. The exponential suppression \( \langle \hat{N}^2 \rangle \sim \exp(-O(N)) \) for large \( N \) and small \( R \ll L_{\text{corr}} \) should still be correct, as this just reflects the diminishing probability of reversing field orientation in all directions when increasing \( N \).

The area scaling \( \langle \hat{N}^2 \rangle \) of the net defect number \( \langle \hat{N}^2 \rangle \) can be interpreted as the occurrence of a confined phase of bound defect-antidefect pairs. Only pairs where one of the partners is contained within while the other is outside the integration volume would yield net winding number, whereas those pairs entirely inside or outside do not contribute. Hence a scaling with surface area instead of volume is natural for short-ranged correlations. On the other hand, there could, in principle, also exist a de-confined phase of quasi-free hedgehogs similar to the quark-gluon plasma of quantum chromodynamics. Such a de-confined phase might occur if defect density and temperature are sufficiently high and any bound pairs are broken up again by thermal quasi-particles. In that case, defects and antidefects would be randomly distributed and a volume scaling of the winding number variance \( \langle \hat{N}^2 \rangle \) follows. Since we did not make any assumption in our derivation apart from the large \( N \) limit (where the results become exact), we can clearly distinguish between the two phases (confined or de-confined).

VI. STATISTICS

In a similar manner, we can calculate the higher moments of the winding number. Again, by exploiting the fact that we have short-range correlations, the large-\( N \) limit of the next nontrivial moment can be inferred from pure combinatorics in the analysis of the four integrals occurring in

\[
\langle \hat{N}^4 \rangle = 3(\langle \hat{N}^2 \rangle)^2 + O(R^{N-1}) = O(R^{2N-2}).
\]  

(13)

Analogously, the leading terms of \( \langle \hat{N}^4 \rangle \) are given by \( (2n-1)!!(\langle \hat{N}^2 \rangle)^n \) with \( (2n-1)!! = (2n-1)(2n-3) \ldots 1 \cdot 3 \). For large \( R \), the winding number \( \hat{N} \in \mathbb{Z} \) can be approximated by a continuous variable \( \hat{N} \in \mathbb{R} \) and thus its full statistics is given by the inverse Mellin transform of \( (2n-1)!! = 2^n\Gamma(n+1/2)/\sqrt{\pi} \), which yields the Gaussian probability distribution \( p(n) \propto \exp\{-\gamma^2(n^2)\} \), with \( 1/\gamma^2 = 2(\langle \hat{N}^2 \rangle) \). We note that, like in Eq. (12), the Gaussianity is not assumed but derived from first principles in a given limit – for small \( R \) (small \( \hat{N} \)), for example, there will be deviations from a Gaussian distribution.

VII. CONCLUSIONS

Based on a very general \( O(N) \)-invariant effective action, we presented an analytical derivation of the winding number counting the defects created by a symmetry-breaking quantum quench in the large-\( N \) limit. Consistent with previous calculations [25], our result (12) is nonperturbative, i.e., it does not admit a Taylor expansion in \( 1/N \). As another result, we find that the typical distance between defects scales with the correlation length \( O(\sqrt{N/k_s}) \). By contrast, the typical distance between quasiparticle excitations (e.g., Goldstone modes) does not increase with \( N \). This can be understood by recalling that the total energy of the system (which scales with \( N \)) in a given volume has to be distributed among all the quasiparticle excitations, whose typical energy is determined by the dispersion relation \( \omega^2(k) \) and thus independent of \( N \). Therefore, we conclude that the quasiparticle spectrum alone does not yield any direct information about the generation of topological defects in general. This situation is quite different in the one-dimensional quantum Ising model, where topological defects (kinks) are the only quasiparticle excitations [11, 12], which frequently led to the assumption in the literature that this
is generic. We demonstrated here that identifying quasiparticle excitation and defect numbers created by a quantum quench can be quite misleading.

The crucial difference between quasiparticles (whose number can be derived via a perturbative expansion in 1/N) and topological defects (which are nonperturbative) can be illustrated by the following intuitive picture: Considering a discrete regular lattice with a unit direction vector \( \mathbf{n}_i \) at each lattice site \( i \), a quasiparticle excitation occurs if \( \mathbf{n}_i \neq \mathbf{n}_j \) for two neighbours \( i, j \). A topological defect at the site \( i \), one the other hand, means that the unit vectors \( \mathbf{n}_j \) of all neighbouring sites either point away or towards the site \( i \). For large \( N \), this is obviously a much stronger condition.

It is also worth noting that the derived area scaling \( \langle R^2 \rangle \propto N^{-1} \) is inconsistent with the random defect gas model (where defects and anti-defects are distributed randomly in the sample volume, corresponding to a deconfined phase) since this model would predict a volume law, i.e., \( R^N \)-scaling. We remark in this connection that the area scaling \( \langle S \rangle \propto N^{-1} \) (corresponding to a confined phase) we obtain can be interpreted by a random \( \mathbf{n} \)-field model on the hyper-surface with the correlator Eq. (5), representing a generalization of the random phase walk model for \( N = 2 \) (cf. the result of \( \langle S \rangle \) in which reasonable agreement with the experiment reported in [18] was obtained).

Finally, we would like to stress that our result is quite universal, i.e., it is valid for very general dispersion relations of the \( O(N) \) model (cf. Fig. 1) and just relies on the large-\( N \) limit without any further approximations. Moreover, as indicated below Eq. (12), we expect that the general picture does still apply qualitatively for smaller, and thus experimentally accessible values of \( N \), for example \( N = 3 \). In particular, this should be true for fast quenches, where we have a well-defined period of exponential growth of the unstable linear modes, while non-linear effects (saturation of this growth, oscillations, and finally defect annihilation, see \[22\]) occur much later. In this case, one may find (instead of \( 1/N \)) another small parameter (e.g., the diluteness of the gas) in order to motivate the underlying effective action in analogy to Eq. (3).

For \( N \gg 1 \), universality will be partially lost and the dependence on the dispersion relation, for example, will be stronger. For instance, it might then be necessary to introduce a time-dependent critical \( k_\omega = k_\omega(t) \), which is not close to the zero of \( \omega^2(k) \), but near the actual minimum of \( \omega^2(k) \).

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[25] A. Rajantie and A. Tranberg, JHEP \textbf{11} (2006) 020, demonstrate that no signals of topological defects can be
found in the two-point (irreducible) correlator at next-to-leading order in $1/N$.

[26] It might be interesting to study possible relations between the area scaling obtained here and the area scaling of the ground-state entanglement entropy of a many-body system with short-range correlations. See, e.g., J. Eisert, M. Cramer, and M. B. Plenio, arXiv:0808.3773 [quant-ph], to appear in Rev. Mod. Phys. (2009).

[27] Handbook of Mathematical Functions edited by M. Abramowitz and I. Stegun (Dover, New York, 1970).

[28] More details of the calculation will be presented elsewhere, M. Uhlmann, R. Schützhold, and U. R. Fischer (in preparation).

[29] N. Horiguchi, T. Oka, and H. Aoki, J. Phys.: Conf. Ser. 150 (2009) 032007.

[30] For smaller $N$, however, the importance of the phase-space factor $k^{N-1}$ subsides and $k_*$ will be shifted towards the modes with the fastest growth in time. Note that then a different expansion parameter other than $1/N$ might be necessary in order to derive the quadratic action [3].