TWO-LOOP SELF-DUAL QED

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We present explicit closed-form expressions for the two-loop Euler-Heisenberg Lagrangians in a constant self-dual field, for both spinor and scalar QED. The simplicity of these representations allows us to examine in detail the asymptotic properties of these Lagrangians, and to construct their imaginary part using Borel dispersion relations. In particular, for this self-dual case we obtain the explicit form of the Lebedev-Ritus functions appearing in the Schwinger representation of the imaginary part at two loops. Using the connection between self-duality and helicity, we also obtain explicit formulas for the low energy limits of the ‘all + helicity’ $N$ photon amplitudes, in scalar and spinor QED at one and two loops.

1 Introduction

In gauge theory a very special role is played by self-dual fields, i.e. fields satisfying the condition

$$F_{\mu\nu} = \tilde{F}_{\mu\nu} \equiv \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$$

(1.1)

Fields of this type are prominent in QCD for a number of different reasons:

• Instantons are self-dual.

• Among all covariantly constant gluon backgrounds, only the self-dual quasi-abelian background is stable (at one-loop) under fluctuations [1].

• Large classes of integrable models can be obtained by dimensional reduction starting from self-dual Yang-Mills theories [2].

• Self-dual fields are helicity eigenstates, so that the effective action in such a field carries the information on the corresponding gluon amplitudes with all equal helicities [3]. Such ‘all +’ amplitudes generally exhibit a particularly simple structure, at the tree level [4] and beyond [5].
In the abelian case most of this motivation does not exist, with the exception of the last point mentioned, which holds in the abelian case as well. Correspondingly, little use has been made so far of self-dual fields in QED. In the present contribution, we will consider the case of constant self-dual fields in QED, and show that the self-duality condition leads to very remarkable simplifications for the effective action in such a background field [6].

2 One-loop Euler-Heisenberg Lagrangians

Before presenting these two-loop results, let us shortly recapitulate some facts about Euler-Heisenberg Lagrangians. At one-loop, the on-shell renormalized effective Lagrangians in a constant background field, for spinor and scalar QED, are given by the well-known formulas [7, 8]

\[
\mathcal{L}^{(1)}_{\text{spin}} = -\frac{1}{8\pi^2} \int_0^\infty \frac{dT}{T} e^{-m^2 T} \left[ \frac{e^2 ab}{\tanh(eaT) \tan(ebT)} - \frac{e^2}{3} (a^2 - b^2) \right]
\]

\[
\mathcal{L}^{(1)}_{\text{scal}} = \frac{1}{16\pi^2} \int_0^\infty \frac{dT}{T} e^{-m^2 T} \left[ \frac{e^2 ab}{\sinh(eaT) \sin(ebT)} + \frac{e^2}{6} (a^2 - b^2) \right]
\]

(2.2)

Here \(a, b\) are related to the two invariants of the Maxwell field by \(a^2 - b^2 = B^2 - E^2, \ ab = E \cdot B\).

These effective Lagrangians are real for a purely magnetic field, while in the presence of an electric field there is an imaginary (absorptive) part, indicating the process of electron–positron (resp. scalar–antiscalar) pair creation by the field. For example, in the case of a purely electric field, \(E\), the effective Lagrangians (2.2) have imaginary parts given by:

\[
\text{Im}\mathcal{L}^{(1)}_{\text{spin}}(E) = \frac{m^4}{8\pi^3} \beta^2 \sum_{k=1}^\infty \frac{1}{k^2} \exp \left[-\frac{\pi k}{\beta} \right]
\]

\[
\text{Im}\mathcal{L}^{(1)}_{\text{scal}}(E) = -\frac{m^4}{16\pi^3} \beta^2 \sum_{k=1}^\infty \frac{(-1)^k}{k^2} \exp \left[-\frac{\pi k}{\beta} \right]
\]

(2.3)

where \(\beta = eE/m^2\). These expressions are clearly non-perturbative in terms of the field and coupling. Their physical interpretation is that the coefficient of the \(k\)-th exponential can be directly identified with the rate for the coherent production of \(k\) pairs by the field [8].

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3 Two-loop Euler-Heisenberg Lagrangians

The two-loop corrections to the Euler-Heisenberg Lagrangians, taking into account an additional photon exchange in the loop, where first calculated by Ritus in the seventies [9] (see also [10]). More recently these Lagrangians have been recalculated [11] using the ‘string-inspired’ formalism [12, 13, 14, 15]. However, in all cases the results are not nearly as explicit as the one-loop formulas (2.2); they involve two-parameter integrals and a counterterm from one-loop mass renormalization. As far as the magnetic case is concerned, where the effective Lagrangian is real, the complicated form of these presentations is perhaps not too bothersome. For values of the magnetic field small compared to the ‘critical’ field strength $B_{c} \equiv \frac{m^{2}}{e}$, the Lagrangian can be computed using the weak field expansion, whose coefficients are easy to compute to fairly high orders [11, 16]:

$$\mathcal{L}^{(2)}_{\text{spin}}(B) = \frac{\alpha m^{4}}{(4\pi)^{3}} \frac{1}{81} \left[ 64 \left( \frac{B}{B_{c}} \right)^{4} - \frac{1219}{25} \left( \frac{B}{B_{c}} \right)^{6} + \frac{135308}{1225} \left( \frac{B}{B_{c}} \right)^{8} - \ldots \right]$$

(3.4)

For general values of the field strength numerical integration can be used.

Calculating the imaginary part of the corresponding Lagrangian for the electric field case is a more difficult matter. At the one-loop level, the imaginary parts (2.3) can be obtained from (2.2) by a simple application of the residue theorem. The analogous analysis for the two-loop parameter integrals is already highly nontrivial. Nevertheless, Ritus and Lebedev [9, 17] were able to obtain along these lines the following two-loop generalization of the Schwinger decompositions (2.3):

$$\text{Im}\mathcal{L}^{(1)}_{\text{spin}}(E) + \text{Im}\mathcal{L}^{(2)}_{\text{spin}}(E) = \frac{m^{4}}{8\pi^{3}} \beta^{2} \sum_{k=1}^{\infty} \left[ \frac{1}{k^{2}} + \alpha \pi K_{k}^{\text{spin}}(\beta) \right] \exp \left[ -\frac{\pi k}{\beta} \right]$$

$$\text{Im}\mathcal{L}^{(1)}_{\text{scal}}(E) + \text{Im}\mathcal{L}^{(2)}_{\text{scal}}(E) = -\frac{m^{4}}{16\pi^{3}} \beta^{2} \sum_{k=1}^{\infty} \left[ \frac{1}{k^{2}} + \alpha \pi K_{k}^{\text{scal}}(\beta) \right] \exp \left[ -\frac{\pi k}{\beta} \right]$$

(3.5)

where $\alpha = \frac{e^{2}}{4\pi}$ is the fine-structure constant. The coefficient functions $K_{k}^{\text{spin,scal}}(\beta)$ appearing here were not obtained explicitly by [17]. However, it was shown that they have small $\beta$ expansions of the following form:
$$K_{k}^{\text{spin,scal}}(\beta) = -\frac{c_k}{\sqrt{\beta}} + 1 + O(\sqrt{\beta})$$

$$c_1 = 0, \quad c_k = \frac{1}{2\sqrt{k}} \sum_{l=1}^{k-1} \frac{1}{\sqrt{l(k-l)}}, \quad k \geq 2 \quad (3.6)$$

Note that to the order given $K_{k}^{\text{spin}} = K_{k}^{\text{scal}}$. For $k \geq 2$, these expansions start with terms that are singular in the limit of vanishing field $\beta \to 0$, which seems to be at variance with the fact that these coefficients have a direct physical meaning. In [17] a physically intuitive solution was offered to this dilemma. Its basic assumption is that, if one would take into account contributions from higher loop orders to the prefactor of the $k$-th exponential, then one would find the two lowest order terms in the small – $\beta$ expansion of $K_{k}^{\text{spin,scal}}(\beta)$ to exponentiate in the following way,

$$\exp \left[ -\frac{k\pi m^2}{eE} \right] = \frac{1}{k^2} \exp \left[ -\frac{k\pi m^2(k, E)}{eE} \right]$$

$$\left(3.7\right)$$

It would thus be possible to absorb their effect completely into a field-dependent shift of the electron mass:

$$m_*(k, E) = m + \frac{1}{2} \alpha k c_k \sqrt{eE} - \frac{1}{2} \alpha k eE/m \quad (3.8)$$

Moreover, these contributions to the mass shift have a simple meaning in the coherent tunneling picture [17]: The negative term can be interpreted as the total Coulomb energy of attraction between opposite charges in a coherent group; the positive term, which is present only in the case $k \geq 2$, represents the energy of repulsion between like charges. For the attractive term, a completely different derivation of the same exponentiation was given in [18].

4 The imaginary part via Borel dispersion relations

Clearly, it would be of interest to understand in greater detail this prefactor series $K_{k}^{\text{spin,scal}}(\beta)$. In an effort to learn more about it, in [16] we used Borel techniques to study the imaginary part of the two-loop effective Lagrangian for a constant electric field background.
Let us review some basic facts on Borel summation. Consider an asymptotic series expansion of some function $f(g)$

$$f(g) \sim \sum_{n=0}^{\infty} a_n g^n$$  \hspace{1cm} (4.9)

where $g \to 0^+$ is a small dimensionless perturbation expansion parameter. In many physics applications perturbation theory leads to a divergent series in which the expansion coefficients $a_n$ have leading large-order behaviour

$$a_n \sim (-1)^n \rho^n \Gamma(\mu n + \nu) \hspace{1cm} (n \to \infty)$$  \hspace{1cm} (4.10)

for some real constants $\rho$, $\mu > 0$, and $\nu$. When $\rho > 0$, the perturbative expansion coefficients $a_n$ alternate in sign and their magnitude grows factorially, just as in the Euler-Heisenberg case. Borel summation is a useful approach to this case of a divergent, but alternating series. The leading Borel approximation is

$$f(g) \sim \frac{1}{\mu} \int_0^\infty \frac{ds}{s} \left( \frac{1}{1 + s} \right) \left( \frac{s}{\rho g} \right)^{\nu/\mu} \exp \left[- \left( \frac{s}{\rho g} \right)^{1/\mu} \right]$$  \hspace{1cm} (4.11)

For a non-alternating series, we need $f(-g)$. The Borel integral (4.11) is an analytic function of $g$ in the cut $g$ plane: $|\arg(g)| < \pi$. So a dispersion relation (using the discontinuity across the cut along the negative $g$ axis) can be used to define the imaginary part of $f(g)$ for negative values of the expansion parameter:

$$\text{Im}f(-g) \sim \pi \frac{1}{\mu} \left( \frac{1}{\rho g} \right)^{\nu/\mu} \exp \left[- \left( \frac{1}{\rho g} \right)^{1/\mu} \right]$$  \hspace{1cm} (4.12)

Returning to the Euler-Heisenberg Lagrangian, for a uniform magnetic background the weak-field expansions of the one-loop Lagrangians (2.2) are precisely of the form (4.9), (4.10) with $g = (eB/m^2)^2$. For example, in the spinor QED case one has

$$a_n^{(1)} = -\frac{2^{2n}B_{2n+4}}{(2n+4)(2n+3)(2n+2)} \sim (-1)^n \frac{1}{8\pi^4} \frac{\Gamma(2n+2)}{\pi^{2n}} \left( 1 + \frac{1}{2^{2n+4}} + \ldots \right)$$  \hspace{1cm} (4.13)
where the $B_n$ are Bernoulli numbers. For a uniform electric background, the only difference perturbatively is that $B^2$ is replaced by $-E^2$; that is, $g = \left(\frac{e B}{m^2}\right)^2$ is replaced by $-g = -\left(\frac{e E}{m^2}\right)^2$. So the perturbative one-loop Euler-Heisenberg series becomes non-alternating. Then from (4.12), with $\rho = \frac{1}{\pi^2}$ and $\mu = \nu = 2$, we immediately deduce the leading behaviour of the imaginary part of the one-loop Euler-Heisenberg effective Lagrangian:

$$\text{Im} \mathcal{L}^{(1)}_{\text{spin}}(E) \sim \frac{m^4}{8\pi^3} \left(\frac{e E}{m^2}\right)^2 \exp \left[-\frac{m^2 \pi}{e E}\right]$$

(4.14)

Taking into account also the sub-leading corrections (4.13) to the leading large-order behaviour of the expansion coefficients $a_n^{(1)}$, one can apply (4.12) successively to reconstruct the full Schwinger series (2.3).

At the two-loop level, no closed-form expression is known for the corresponding expansion coefficients $a_n^{(2)}$, so that the corresponding analysis becomes much more involved. In [16], first the integral representation for the purely magnetic case obtained in [11] was used to compute these coefficients up to $n = 15$. It was then established by a numerical analysis that

$$a_n^{(2)} \sim (-1)^n \frac{16}{\pi^2} \frac{\Gamma(2n + 2)}{\pi^{2n}} \left[1 - \frac{0.44}{\sqrt{n}} + \ldots \right]$$

(4.15)

Thus the leading large-order growth corresponds to the form (4.10), and moreover differs from the one-loop case (4.13) only by a global prefactor. To the contrary, the subleading correction term given in (4.15) shows a much weaker $n$ dependence than is found for the first correction in the one-loop case (4.13). This means that in the two-loop case the dominant corrections are to the prefactor in the leading behaviour. This is in contrast to the one-loop case where the first correction to the leading behaviour is exponentially suppressed. Indeed, applying the Borel relations, the correction term (4.13) leads to

$$\text{Im} \left( \mathcal{L}^{(1)}_{\text{spin}}(E) + \mathcal{L}^{(2)}_{\text{spin}}(E) \right) \sim \left(1 + \alpha \pi \left[1 - \left(0.44\right)\sqrt{\frac{2e E}{\pi m^2}} + \ldots \right]\right)$$

$$\times \frac{m^4}{8\pi^3} \left(\frac{e E}{m^2}\right)^2 \exp \left[-\frac{m^2 \pi}{e E}\right]$$

(4.16)

Thus the structure of (4.16) conforms already to the form (3.5), (3.6).
5 The self-dual case

It seems very difficult to make further progress along these lines for the purely magnetic/electric field cases, let alone the general constant field case. To the contrary, in the self-dual case this analysis can be carried much further \[6\]. Technically, this is because the self-duality condition (1.1) implies that

\[ F^2 = -f^2 \mathbb{1} \]  

(5.17)

where \( f^2 = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \), and \( \mathbb{1} \) denotes the identity matrix in Lorentz space. This leads to enormous simplifications for this type of calculations.

In the self-dual (‘SD’) case, the integral representation (2.2) for the renormalized one-loop scalar effective Lagrangian becomes

\[
L^{(1)(SD)}_{\text{scal}}(\kappa) = \frac{m^4}{(4\pi)^2} \frac{1}{4\kappa^2} \int_0^\infty \frac{dt}{t^3} e^{-2\kappa t} \left[ \frac{t^2}{\sinh^2(t)} - 1 + \frac{t^2}{3} \right]
\]  

(5.18)

where \( \kappa \equiv \frac{m^2}{2\epsilon \sqrt{f}} \) is the natural dimensionless parameter. The Lagrangian for the spinor QED case differs from this only by the standard global factor of \(-2\) for statistics and degrees of freedom:

\[
L^{(1)(SD)}_{\text{spin}}(\kappa) = -2 L^{(1)(SD)}_{\text{scal}}(\kappa)
\]  

(5.19)

Note that this relation holds for the renormalized effective Lagrangians, not for the unrenormalized ones. It is due to a supersymmetry of the self-dual background \[19\]. For real \( \kappa \), this self-dual Lagrangian is real and has properties very similar to the magnetic Lagrangian. Similarly, for purely imaginary \( \kappa \) the self-dual Lagrangian has an imaginary part, and provides a good analogue of the electric field case. These cases will therefore be called ‘magnetic’ and ‘electric’ in the following. In particular, the imaginary part of the ‘electric’ Lagrangian has a Schwinger-type expansion

\[
\text{Im} \left[ L^{(1)(SD)}_{\text{scal}}(i\kappa) \right] = \frac{m^4}{(4\pi)^2} \frac{1}{\kappa^2} \sum_{k=1}^{\infty} \left( \frac{2\pi \kappa}{k} + \frac{1}{k^2} \right) e^{-2\pi k \kappa}
\]  

(5.20)

Surprisingly, for the self-dual case even at two loops all parameter integrals can be done in closed form, leading to the following explicit formulas \[6\]:

7
$$\mathcal{L}^{(2)(SD)}_{\text{spin}}(\kappa) = -2\alpha \frac{m^4}{(4\pi)^3} \frac{1}{\kappa^2} \left[ 3\xi^2(\kappa) - \xi'(\kappa) \right]$$

$$\mathcal{L}^{(2)(SD)}_{\text{scal}}(\kappa) = \alpha \frac{m^4}{(4\pi)^3} \frac{1}{\kappa^2} \left[ \frac{3}{2} \xi^2(\kappa) - \xi'(\kappa) \right]$$

(5.21)

Here we have introduced the function $\xi$,

$$\xi(x) \equiv -x \left( \psi(x) - \ln(x) + \frac{1}{2x} \right)$$

(5.22)

and $\psi(\kappa) = \frac{d}{d\kappa} \ln \Gamma(\kappa)$ is the digamma function. This function has very simple expansions at zero as well as at infinity,

$$\psi(x) \sim -\frac{1}{x} - \gamma + \sum_{k=2}^{\infty} (-1)^k \zeta(k) x^{k-1}$$

(5.23)

$$\psi(x) \sim \ln x - \frac{1}{2x} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k x^{2k}}$$

(5.24)

so that the formulas (5.21) directly yield closed-form expressions for the coefficients of both the weak-field and strong-field expansions of the self-dual Lagrangians. Moreover, it is easy to obtain also a closed representation for their imaginary parts which exist in the ‘electric’ case:

$$\text{Im} \left[ \mathcal{L}^{(2)}_{\text{scal}}(i\kappa) \right] = \alpha \pi \frac{m^4}{(4\pi)^2} \frac{1}{2\kappa} \sum_{k=1}^{\infty} \left[ k - \frac{1}{2\pi \kappa} - \frac{3\kappa}{2\pi} \sum_{l=1}^{\infty} \frac{(-1)^l B_{2l}}{2l\kappa^{2l}} \right] e^{-2\pi \kappa k}$$

$$\text{Im} \left[ \mathcal{L}^{(2)}_{\text{spin}}(i\kappa) \right] = -2\alpha \pi \frac{m^4}{(4\pi)^2} \frac{1}{2\kappa} \sum_{k=1}^{\infty} \left[ k - \frac{1}{2\pi \kappa} - \frac{3\kappa}{2\pi} \sum_{l=1}^{\infty} \frac{(-1)^l B_{2l}}{2l\kappa^{2l}} \right] e^{-2\pi \kappa k}$$

(5.25)

Note that these formulas display the same structure found by Ritus and Lebedev for the physical electric case, (3.3), (3.6), but that in the self-dual case we have simple closed formulas for all the prefactors of the Schwinger exponentials. In (5.25) we have written these prefactors as series in $\frac{1}{\kappa}$, which corresponds
to the $\beta$-expansion (3.6). It is easy to see that this expansion is asymptotic, rather than convergent, for all $k$. For the scalar QED case, we have also verified [6] that the first ($k = 1$) exponential in (5.25), including its complete prefactor, can be correctly reproduced by an analysis of the asymptotic behaviour of the weak field expansion coefficients and the application of the Borel dispersion relations. It is also immediately evident that, for any given value of $\kappa$, the two-loop contribution to the prefactor of the $k$-th exponential will dominate over the corresponding one-loop quantity in (5.20) if $k$ is taken large enough. Given the close similarity between the electric and ‘electric’ cases at the one-loop level it is natural to assume that these properties hold also true for the electric case.

6 The $N$-photon amplitudes

As is well-known, the Euler-Heisenberg Lagrangians (2.2) can be used to obtain the one-loop QED $N$-photon amplitudes in the low-energy approximation, i.e. for photon momenta $k_i$ such that $m^2$ is much larger than all $k_i \cdot k_j$'s. After specializing to a self-dual background it still contains the information on the component of the $N$-photon amplitude with all helicities equal, say, all ‘+’. Using (2.2), (5.21),(5.24) together with the standard spinor helicity formalism (see, e.g., [20]), one obtains for this low-energy limit the following explicit formulas: At one loop,

$$\Gamma^{(1)(EH)}_{\text{scal}}[k_1, \epsilon_1^+; \ldots; k_N, \epsilon_N^+] = \frac{(2e)^N}{(4\pi)^2 m^{2N-4}} c^{(1)}_{\text{scal}}(\mathcal{N}) \chi_N$$

$$c^{(1)}_{\text{scal}}(n) = -\frac{B_{2n}}{2n(2n-2)}$$

$$\Gamma^{(1)(EH)}_{\text{spin}}[k_1, \epsilon_1^+; \ldots; k_N, \epsilon_N^+] = -2\Gamma^{(1)(EH)}_{\text{scal}}[k_1, \epsilon_1^+; \ldots; k_N, \epsilon_N^+]$$

and at two loops

$$\Gamma^{(2)(EH)}_{\text{scal}}[k_1, \epsilon_1^+; \ldots; k_N, \epsilon_N^+] = \alpha \pi \frac{(2e)^N}{(4\pi)^2 m^{2N-4}} c^{(2)}_{\text{scal}}(\mathcal{N}) \chi_N$$

$$c^{(2)}_{\text{scal}}(n) = \frac{1}{(2\pi)^2} \left\{ \frac{2n-3}{2n-2} B_{2n-2} + \frac{3}{2} \sum_{k=1}^{n-1} \frac{B_{2k} B_{2n-2k}}{2k (2n-2k)} \right\}$$

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\( \Gamma_{\text{spin}}^{(2)(EH)}[k_1, \varepsilon_1^+; \ldots; k_N, \varepsilon_N^+] = -2\alpha \pi \frac{(2e)^N}{(4\pi)^2 m^{2N-4}} c_{\text{spin}}^{(2)}(\frac{N}{2}) \chi_N \)

\[ c_{\text{spin}}^{(2)}(n) = \frac{1}{(2\pi)^2} \left\{ \frac{2n-3}{2n-2} B_{2n-2} + 3 \sum_{k=1}^{n-1} \frac{B_{2k} B_{2n-2k}}{2n} \right\} \]

\[(6.27)\]

In these formulas, the information on the external momenta is all contained in the invariant \( \chi_N \),

\[ \chi_N = \frac{(-1)^N}{2\pi^2} \left\{ [12]^2[34]^2 \cdots [(N-1)N]^2 + \text{all permutations} \right\} \]

Here \([ij] = \langle k_i^+|k_j^- \rangle \) denote the basic spinor products \([20]\).

### 7 Conclusions

The existence of the simple closed-form expressions \([5,21]\) for the self-dual two-loop Euler-Heisenberg Lagrangians is surprising, and we do not know of any comparable result in gauge theory. These results have allowed us to perform a more complete analysis of these effective Lagrangians than is possible for other backgrounds \([9]\), including the complete elucidation of the structure of the imaginary parts. Moreover, we have used them to obtain explicit formulas for the low-energy limits of the ‘all +’ components of the two-loop \( N \)-photon amplitudes. Let us close with a remark on the behaviour of these amplitudes in the limit where the number of photons becomes large. It is easy to show that

\[ \lim_{N \to \infty} \frac{\Gamma_{\text{spin,scal}}^{(2)(EH)}[k_1, \varepsilon_1^+; \ldots; k_N, \varepsilon_N^+]}{\Gamma_{\text{spin,scal}}^{(1)(EH)}[k_1, \varepsilon_1^+; \ldots; k_N, \varepsilon_N^+]} = \alpha \pi \quad (7.28) \]

The Borel dispersion relations allow one to relate this quantity to the \( \kappa \to \infty \) limit of the corresponding ratio of the imaginary parts \([5,20],[5,25]\), and thus to the factor \( \alpha \pi \) appearing in the Lebedev-Ritus exponentiation formula \([3,7]\). Based on the expectation that this exponentiation occurs also in the self-dual case, we conjecture that at loop order \( l \), the weak-field expansion will take the form

\[ L^{(l)(SD)}(\kappa) = \frac{(\alpha \pi)^{l-1}}{(l-1)!} \frac{m^4}{(4\pi)^2} \sum_{n=2}^{\infty} \frac{c_n^{(l)}}{\kappa^{2n}} \]

\[(7.29)\]
where for each $l$, the expansion coefficients $c_n^{(l)}$ have the same *leading* large $n$ growth rate as $c_n^{(1)}$. This would imply that

$$
\lim_{N \to \infty} \frac{\Gamma^{(l)(EH)}_{\text{spin,scal}}[k_1, \varepsilon_{k_1}^+; \ldots; k_N, \varepsilon_{k_N}^+]}{\Gamma^{(1)(EH)}_{\text{spin,scal}}[k_1, \varepsilon_{k_1}^+; \ldots; k_N, \varepsilon_{k_N}^+]} = \frac{(\alpha \pi)^{l-1}}{(l-1)!}
$$

(7.30)

It would be very interesting to verify this identity at the three-loop level.

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