Perturbations of Gauss-Bonnet Black Strings in Codimension-2 Braneworlds

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Abstract

We derive the Lichnerowicz equation in the presence of the Gauss-Bonnet term. Using the modified Lichnerowicz equation we study the metric perturbations of Gauss-Bonnet black strings in Codimension-2 Braneworlds.

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1 Introduction

Higher-dimensional black holes and black strings appear as classical solutions of gravity theories in more than four dimensions. One central issue concerning their behaviour is the stability issue. Recently, the stability of higher-dimensional black holes and in particular of black strings, is under intense investigation. The stability of higher-dimensional black holes has been addressed adequately in [1]. Also, the understanding of the stability of black rings and in general of higher-dimensional black objects has also been improved considerably (for a review see [2, 3]).

In the case of black holes that are solutions of gravity theories with high order curvature terms [4–9], the stability issue is an intriguing one. The Lovelock theory [10] is a non-trivial extension of General Relativity which apart from the Einstein-Hilbert term also includes higher order curvature terms. In fact, the Lovelock theory contains terms only up to the second order derivatives in the equations of motion which makes the theory more tractable. In spite of that, the black hole solutions of Lovelock theory could be found mostly because they are highly symmetric (for a review see [11], [12]). The study of their stability requires the application of linear perturbation theory, which however, confronts with the high complexity of the Lovelock equations, since the perturbative terms break the simplifying symmetries of the background metric.

In the simplest case of a Gauss-Bonnet theory which is a second order Lovelock theory, the classical stability of black hole solutions has been studied. The stability analysis under scalar, vector and tensor perturbations has been performed [13–15]. It was found that there exists a scalar mode instability in five dimensions, a tensor mode instability in six dimensions, and no instability in other dimensions. Recently the master equations of Lovelock black holes of scalar, vector and tensor perturbations were derived in [16–18] and it was shown that small Lovelock black holes are unstable in the asymptotically flat cases [19].

Higher order curvature terms are known to appear in string theory, generically introducing higher derivatives in the metric [20]. This will inevitably lead to perturbative ghosts. However, the second order curvature terms are known to take the form of the Gauss-Bonnet combination [21]. This corresponds to the non-trivial second order Lovelock theory which has no higher derivatives in the effective string action, so we expect no ghosts to appear in second order. However, it was found that spherically symmetric vacuum solutions in the Gauss-Bonnet theory suffer from ghost-like instabilities and it was conjectured that these instabilities persist in all vacuum Lovelock solutions [22]. In fact it was shown that there is a limit, the Chern-Simons limit, in which the theory becomes strongly coupled and linear perturbation theory breaks down. In this limit it was also shown, that if there is a fine-tuning between the parameters, the two branches of vacuum solutions coincide with the Chern-Simons black hole solution which has maximum symmetry. Scalar perturbations of the Chern-Simons black holes were studied in [23].

Higher order curvature terms also appear in braneworlds and especially in codimension-2 braneworld models which were mainly introduced because there is a mechanism of self-tuning of the effective cosmological constant to zero [24]. However, soon it was realized [25] that one can only find nonsingular solutions if the brane energy momentum tensor is
proportional to its induced metric. To reproduce the effective four-dimensional Einstein equation on the brane one has to modify the gravitational action by the inclusion of a Gauss-Bonnet term [26,27] (for a review see [28]) or to regularize the conical singularities (see [29] and references therein).

Recently black hole solutions of codimension-2 braneworlds have been found. A six-dimensional black hole localized on a 3-brane of codimension-2 was proposed in [30]. However, it is not clear how to realize these solutions in the thin brane limit where high order curvature terms are needed to accommodate matter on the brane. Generalizations to include rotations were presented in [31] and perturbative analysis of these black holes were carried out in [32–34].

In the case that there is a Gauss-Bonnet term in the bulk, black hole solutions were studied in a five-dimensional codimension-2 braneworld model [35]. Two classes of solutions were found. The first class consists of the familiar BTZ black hole [36] which solves the junction conditions on a conical 2-brane in vacuum. These solutions in the bulk are BTZ string-like objects with regular horizons and no pathologies. The second class of solutions consists of BTZ black holes with short distance corrections. These solutions correspond to a BTZ black hole conformally dressed with a scalar field [37,38] and they have black string-like extensions into the bulk. Generalizations to include angular momentum and charge were presented in [39]. Also four-dimensional Schwarzschild-AdS black hole solutions on the brane were found, which in the six-dimensional spacetime look like black string-like objects with regular horizons [40]. The warping to extra dimensions depends on the Gauss-Bonnet coupling which is fine-tuned to the six-dimensional cosmological constant. The presence of the Gauss-Bonnet term in codimension-2 braneworlds has important consequences. Its projection on the brane gives a consistency relation [27] that dictates the form of the solutions. It allows black string solutions in five dimensions, and in six dimensions it specifies the kind of matter which is needed in the bulk in order to support a black hole solution on the brane. We note here that black string solutions with a Gauss-Bonnet term in the bulk are not possible in codimension-1 braneworlds [41].

The stability analysis of these static black hole solutions of codimension-2 braneworlds is interesting. They solve N-dimensional Einstein field equations with the presence of a Gauss-Bonnet term in the action. The symmetries of the solutions are $M^d \times K^{n-d}$, where $M$ is a maximally symmetric space while the space $K$ is axially symmetric. Therefore, any stability analysis of these static solutions, unlike the Lovelock black hole solutions which in general are spherically symmetric, has also to confront with the particular symmetries of the solutions.

In this work we address the problem of stability of codimension-2 black strings with a Gauss-Bonnet term in the bulk and more generally of gravity theories with high order curvature terms that have time-independent solutions with symmetries other than spherical.\footnote{In codimension-1 braneworlds the Schwarzschild metric on the brane was considered and its black string extension into the bulk [44] was studied. It was found that this string is unstable to classical linear perturbations [45]. A lower dimensional version of a black hole}
living on a (2+1)-dimensional braneworld was considered in [46]. A BTZ black hole on the 
brane was found which can be extended as a BTZ black string in a four-dimensional AdS 
bulk. Their thermodynamical stability analysis showed that the black string remains a sta-
ble configuration when its transverse size is comparable to the four-dimensional AdS radius,
being destabilized by the Gregory-Laflamme instability [45] above that scale, breaking up
to a BTZ black hole on a 2-brane. The stability of BTZ black string was also discussed 
in [47].

One way to study linear stability of higher dimensional objects with curvature correc-
tions is to find the eigenvalues of the Lichnerowicz operator of a given perturbation [48].
This method has the advantage of being formulated in a gauge invariant way allowing the
study of metrics with various symmetries. We will derive the Lichnerowicz equation in
the presence of the Gauss-Bonnet term. We will show that a simple application of the
modified Lichnerowicz equation to the case of D=6 spherically symmetric black hole so-
lutions of Gauss-Bonnet theory can easily reproduce the known results of [13] for tensor
perturbations for these black holes.

We will subsequently use the modified Lichnerowicz equation to the study of linear
perturbations of five-dimensional black string solution of codimension-2 braneworld model 
of [35], away from the Chern-Simons limit. We will show that the knowledge of the Lich-
nerowicz equation can provide very important information on the stability analysis of a
complex system of coupled differential equations that has to be solved. In the case of
codimension-2 geometries, the black string can propagate in two transverse extra dimen-
sions so intuitively, one expects that metric perturbations to have more severe stability
problems than the conventional five-dimensional black strings propagating in one extra
transverse dimension.

Our analysis shows that for tensor perturbations the modified Lichnerowicz equation,
due to the symmetries of the codimension-2 black string solution, can not give any informa-
tion on the perturbed system, indicating that either there is no tensor propagating modes
or we are in a strong coupling regime. For the vector perturbations we find a degeneracy
of the modes on the bulk space having no dependence on the brane coordinates indicating
an instability on the modes generated on the brane and propagating into the bulk. To cal-
culate the scalar perturbations, we will apply the derived modified Lichnerowicz equation
to the scalar part of the metric perturbations. We will show that, due to the fact that the
metric of the black string solutions can be brought to a factorizable form, the results are
the same as having studied the propagation of a scalar field in the background metric of
the black string, by solving the Klein-Gordon equation. We find stability for the scalar
modes by both methods.

Another important information that the modified Lichnerowicz equation can give us is
the behaviour of the theory in the Chern-Simons limit. As we have already discussed, in the
Gauss-Bonnet theory there are two branches of spherically symmetric vacuum solutions:
The Schwarzschild-AdS branch (known as the Einstein limit) and the string branch (known
as the Gauss-Bonnet limit) [22]. Both branches coincide at the Chern-Simons limit where
the theory becomes strongly coupled. As we will discuss in the following, in our case the
Chern-Simons limit manifest itself as a prefactor in front of the modified Lichnerowicz
equation. The limit where this prefactor goes to zero is an indication of a strong coupling
problem, signaling that linear perturbation theory breaks down.

The paper is organized as follows. In section 2 we present the general formalism for linear perturbations and derive the Lichnerowicz equation. We further generalize this formalism by including the Gauss-Bonnet term and derive the modified Lichnerowicz equation where a source term appears, due to the presence of the Gauss-Bonnet term. In section 3 we apply these results to spherically symmetric solutions of the Gauss-Bonnet theory. In section 4 we calculate the scalar perturbations of the five-dimensional black hole solution of the codimension-2 braneworld model and we discuss its vector and tensor perturbations. Finally, in section 5 we conclude.

2 The Lichnerowicz Equation

In this section we will review the general formalism of linear metric perturbations and we will derive the D-dimensional Lichnerowicz equation [48] and the D-dimensional modified Lichnerowicz equation in the presence of a Gauss-Bonnet term.

2.1 General formalism for linear metric perturbations

Consider a $D$-dimensional Einstein manifold $(\mathcal{B}, \bar{g})$. The stability analysis can be reduced in finding a solution of an ordinary differential equation of a Schrödinger type, i.e., the Lichnerowicz equation. This amounts to determine the spectrum of the Lichnerowicz operator on transverse traceless symmetric tensor fields of the manifold $\mathcal{B}$. We will first expose the general formalism and then we will apply it to a manifold with a Gauss-Bonnet term.

We start from the definition for the Riemann Tensor as in [49],

$$\nabla_C \nabla_D T^A - \nabla_D \nabla_C T^A = R^A_{BCD} T^B \quad (2.1)$$

and consider the following linear perturbation of a $D$-dimensional metric background $\bar{g}_{MN}$

$$g_{MN} = \bar{g}_{MN} + \varepsilon h_{MN}, \quad (2.2)$$

where the Latin capital indices $M, N$ take values in the $D$-dimensional space, and all unperturbed quantities are written as $\bar{X}$. We also decompose the Ricci tensor and a $D$-dimensional energy momentum tensor $T_{MN}$ respectively as

$$R_{MN} = \bar{R}_{MN} + \varepsilon \delta R_{MN}, \quad (2.3)$$
$$T_{MN} = \bar{T}_{MN} + \varepsilon \delta T_{MN}. \quad (2.4)$$

The Einstein’s field equations are

$$G_{MN} = R_{MN} - \frac{1}{2} R g_{MN} = T_{MN}, \quad (2.5)$$

and using the trace they can be written as

$$R_{MN} = T_{MN} - \frac{1}{2} \frac{2}{D-2} g_{MN} T^L_L. \quad (2.6)$$
The zeroth order Einstein equations (or background solution) are
\[ \bar{\mathcal{R}}_{MN} = \bar{T}_{MN} - \frac{1}{2} \frac{2}{D-2} \bar{g}_{MN} \bar{T}_{L}, \] (2.7)
while the first order Einstein equations (or perturbed equations) are
\[ \delta R_{MN} = \delta S_{MN}, \] (2.8)
where
\[ \delta R_{MN} = -\frac{1}{2} \left( \Box h_{MN} + \nabla_N \nabla_M h - \nabla^K \nabla_M h_{KN} - \nabla^K \nabla_N h_{MK} \right), \] (2.9)
with \( \Box h_{MN} = \bar{g}^{AB} \nabla_A \nabla_B h_{MN}, \) and
\[ \delta S_{MN} = \delta T_{MN} - \frac{1}{2} \frac{2}{D-2} \left( \bar{g}_{MN} \bar{g}^{KL} \delta T_{KL} + \bar{g}_{MN} h^{PS} T_{PS} - h_{MN} T_{L} \right). \] (2.10)

It is easy to check that the Einstein equations (2.5) are satisfied for \( g_{MN} = \bar{g}_{MN} + \varepsilon h_{MN} \) and \( T_{MN} = \bar{T}_{MN} + \varepsilon \delta T_{MN}. \) To simplify the equations we choose the de Donder gauge, namely, the traceless and the transverse gauge conditions, respectively given by
\[ \bar{g}^{MN} h_{MN} = 0, \] (2.11)
\[ \nabla^M h_{MN} = 0. \] (2.12)

Then, in this gauge the first order Ricci tensor (2.9) can be written in the following form
\[ \delta R_{MN} = -\frac{1}{2} \left( \Box h_{MN} - 2 \bar{R}_{KNML} h^{KL} - \bar{R}^K_M h_{KN} - \bar{R}^K_N h_{MK} \right). \] (2.13)

In vacuum (\( T_{MN} = 0 \)), the Einstein equation reduces to \( R_{MN} = 0, \) whose zeroth order part, \( \bar{R}_{MN} = 0, \) can be used in the first order part (2.13) to obtain the Lichnerowicz equation for linear perturbations in vacuum
\[ \Box h_{MN} - 2 \bar{R}_{KMN} h^{KL} = 0. \] (2.14)

When a cosmological constant is present, i.e., \( T_{MN} = -\Lambda_D g_{MN}, \) the zeroth and first order parts of the Einstein equations (2.6) are respectively
\[ \bar{R}_{MN} = -\tilde{\Lambda}_D \bar{g}_{MN}, \] (2.15)
\[ \delta R_{MN} = -\tilde{\Lambda}_D h_{MN}, \] (2.16)
where \( \tilde{\Lambda}_D = \frac{D-3}{D-2} \Lambda_D. \) Using (2.15) in (2.13), the first order part (2.16) gives again the Lichnerowicz equation (2.14).
2.2 The modified Lichnerowicz equation

We want to establish the equivalent to the Lichnerowicz equation in the case where a Gauss-Bonnet term is included in the manifold $\mathcal{B}$. In five or in six dimensions the Gauss-Bonnet density term in the action is given by

$$ \alpha \mathcal{L}_{\text{GB}} = \alpha \left( R^2 - 4 R_{KL} R^{KL} + R_{KLPQ} R^{KLPQ} \right), $$

(2.17)

where $\alpha (\geq 0)$ is the Gauss-Bonnet coupling constant. Its variation includes the Gauss-Bonnet term $\alpha H_{MN}$ in the field equations with $H_{MN}$ being the Gauss-Bonnet tensor,

$$ G_{MN} - \alpha H_{MN} = -\Lambda_D g_{MN}. $$

(2.18)

For convenience, we split this tensor in five terms $H_{MN_i}(i = 1, \ldots, 5)$ which are

$$ H_{MN_i} = \frac{1}{2} g_{MN} L_{\text{GB}}, $$

(2.19)

$$ H_{MN_2} = -2 R R_{MN}, $$

(2.20)

$$ H_{MN_3} = +4 R_{MK} R^K_N, $$

(2.21)

$$ H_{MN_4} = +4 R_{MLN} R^K_L, $$

(2.22)

$$ H_{MN_5} = -2 R_{MKLP} R^K_{LP}. $$

(2.23)

As for the linear perturbations, we decompose the Gauss-Bonnet tensor as

$$ H_{MN} = \bar{H}_{MN} + \varepsilon \delta H_{MN} = \sum_{i=1}^{5} \left( \bar{H}_{MN_i} + \varepsilon \delta H_{MN_i} \right), $$

(2.24)

and after some lengthy calculations, where we used the gauge conditions (2.11) and (2.12), we find that the first order Gauss-Bonnet tensor decompositions are

$$ \delta H_{MN_1} = \frac{1}{2} h_{MN} L_{\text{GB}} - \bar{g}_{MN} h^{KL} \bar{R}_{KL} \bar{R} + 2 \bar{g}_{MN} \bar{R}^{KL} \bar{R}^{Q} h_{QP} - \bar{g}_{MN} R^{KLPQ} h_{KR} R_{LPQ} + 2 \bar{g}_{MN} R^{KLPQ} \nabla_Q \nabla_L h_{KP}, $$

(2.25)

$$ \delta H_{MN_2} = 2 h^{KL} \bar{R}_{KL} \bar{R}_{MN} + \bar{R} \square h_{MN} - 2 \bar{R} \bar{R}^{K} \bar{R}^{L} h_{LK}, $$

(2.26)

$$ \delta H_{MN_3} = 2 \bar{R}^{K} \left( \bar{R} \bar{R}_{MN} - \bar{R} \bar{R}_{MN} \right), $$

(2.27)

$$ \delta H_{MN_4} = 2 \bar{R}^{KL} \left( \bar{R}^{P} \bar{R}^{MQ} h_{MP} + \nabla_{L} \nabla_{M} h_{KL} - \nabla_{L} \nabla_{K} h_{MN} - \nabla_{N} \nabla_{K} h_{KL} + \nabla_{N} \nabla_{K} h_{ML} \right), $$

(2.28)

$$ \delta H_{MN_5} = 2 \bar{R}^{KL} \left( \bar{R}^{P} \bar{R}^{MQ} h_{MP} + \nabla_{L} \nabla_{M} h_{KL} - \nabla_{L} \nabla_{K} h_{MN} - \nabla_{N} \nabla_{K} h_{KL} + \nabla_{N} \nabla_{K} h_{ML} \right). $$

(2.29)

The Einstein’s field equations (2.18) can also be written as

$$ R_{MN} = \alpha H_{MN} - \Lambda_D g_{MN} - \frac{1}{D-2} g_{MN} \left( \alpha H - D \Lambda_D \right), $$

(2.30)
where the zeroth order equations are
\[ \bar{R}_{MN} = \alpha \bar{H}_{MN} - \Lambda_D \bar{g}_{MN} - \frac{1}{D-2} \bar{g}_{MN} (\alpha \bar{H} - D \Lambda_D). \] (2.31)

Using (2.31) together with (2.13) the first order field equations can be written as
\[ \Box h_{MN} - 2\bar{R}_{KMNL} h^{KL} = \alpha B_{MN}, \] (2.32)
where
\[ B_{MN} = -2\delta H_{MN} + \frac{2}{D-2} \bar{g}_{MN} \bar{g}^{KL} \delta H_{KL} - \frac{2}{D-2} \bar{g}_{MN} \bar{H}_{KL} h^{KL} + \bar{H}^K_M h_{KN} + \bar{H}^K_N h_{MK}. \] (2.33)

When we calculated the variation of the Gauss-Bonnet term, the harmonic gauge was used. However, there is still some gauge freedom which can be further gauged away.

The second term can be written as
\[ \bar{g}^{KL} \delta H_{KL} = (2 - D) h^{MN} \bar{R}_{MN} \bar{R} + (8 - 2D) \bar{R}^M N \Box h_{MN} + (4D - 10) \bar{R}^M S \bar{R}^P N h_{PN} \]
\[ - (D - 2) \bar{R}^{ABMS} \bar{R}^P_{BMS} h_{AP} + 4 \bar{R}^{KP} \bar{R}^P_{L} h_{LT} + (2D - 8) \bar{R}^{ABMS} \nabla_M \nabla_B h_{AS} \] (2.34)
while the third term becomes
\[ \bar{H}_{KL} h^{KL} = -\bar{R} \bar{R}_{KL} h^{KL} + 4 \bar{R}_{KP} \bar{R}^L_P h^{KL} + 4 \bar{R}^M_{KPL} \bar{R}^P_M h^{KL} - 2 \bar{R}_{KMNP} \bar{R}^{MNP}_L h^{KL} \] (2.35)

Equation (2.32) is the modified Lichnerowicz equation, the difference with (2.14) is the appearance of the source term \( \alpha B_{MN} \) due to the Gauss-Bonnet term in the action.

Another way of arriving to the same result is to vary directly the equation (2.18). However, the modified Lichnerowicz equation (2.32) is useful because it gives directly the perturbation equations once the form of the perturbation is known.

3 Perturbation analysis of spherically symmetric Gauss-Bonnet black hole solutions

As an application of the formalism developed in the previous section we will examine the perturbations of spherically symmetric solutions of Einstein-Gauss-Bonnet equations
\[ G_{MN} - \alpha H_{MN} = -\Lambda_D g_{MN}, \] (3.1)
where \( G_{MN} \) is the Einstein tensor and \( H_{MN} \) is the Gauss-Bonnet tensor
\[ H^N_M = \frac{1}{2} g^N_M \left( R^2 - 4 R_{KL} R^{KL} + R_{KLPQ} R^{KLPQ} \right) - 2 R^N_M \]
\[ + 4 R_{MK} R^{NK} + 4 R_{KMP} R^{KP} - 2 R_{MKLP} R^{NKL}. \] (3.2)
Namely, we will investigate the following equation

\[ \delta G^{B}_{A} - \alpha \delta H^{B}_{A} = -\Lambda_{D} \delta g^{B}_{A}. \] (3.3)

In the transverse traceless gauge the variation of the Einstein tensor gives

\[ \delta G^{AB} = -\frac{1}{2} \left( \Box h^{AB} + 2R^{L}_{A}K^{B}_{L}h_{KL} - R^{L}_{A}h_{BL} - R^{L}_{B}h_{AL} \right) - \frac{1}{2} h_{AB}R + \frac{1}{2} g_{AB}h^{KL}R_{KL} \] (3.4)

and

\[ \delta G^{B}_{A} = g^{BK} \delta G^{K}_{A} - h^{BK} G^{K}_{A}. \] (3.5)

Furthermore,

\[ \delta H^{B}_{A} = g^{BK} \delta H^{K}_{A} - h^{BK} H^{K}_{A}, \] (3.6)

where the variations \( \delta H^{K}_{A} \) are given by the equations (2.25)-(2.29) which already incorporate the transverse and traceless gauge. Additionally, it is easy to see that

\[ \delta g^{B}_{A} = g^{BK} h^{K}_{A} - h^{BK} g^{K}_{A} = 0. \] (3.7)

We will consider spherically symmetric black hole solutions of the Einstein-Gauss-Bonnet theory in \( D = 6 \) dimensions. The procedure is the same in the case of five dimensions. The resulting equations are

\[ \delta G'^{B}_{A} + \alpha' \delta H'^{B}_{A} = 0, \] (3.8)

where \( \delta H'^{B}_{A} \) are given by (2.25)-(2.29) and they coincide with the equations derived in [13] for tensor perturbations with the identifications \( -\frac{1}{2} H^{AB} = H'_{AB} \) and \( \alpha = 2\alpha' \).

Following [13] these equations can be solved transforming them to a Schrödinger-like equation. Consider the following spherically symmetric metric

\[ ds^{2} = -f(r)dt^{2} + \frac{1}{f(r)}dr^{2} + r^{2} \left[ d\theta^{2} + \sin^{2} \theta \left( d\varphi^{2} + \sin^{2} \varphi \left( d\chi^{2} + \sin^{2} \chi d\psi^{2} \right) \right) \right] \] (3.9)

which satisfies equations (3.1). Perturbations of the above metric read [13]

\[ g_{\alpha\beta} \rightarrow g_{\alpha\beta} + h_{\alpha\beta}, \] (3.10)

where

\[ h_{\alpha\beta} = r^{2} \phi (r,t) \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_{\theta\theta} & h_{\theta\varphi} & h_{\theta\chi} & h_{\theta\psi} \\ 0 & 0 & h_{\theta\varphi} & h_{\varphi\varphi} & h_{\varphi\chi} & h_{\varphi\psi} \\ 0 & 0 & h_{\theta\chi} & h_{\chi\varphi} & h_{\chi\chi} & h_{\chi\psi} \\ 0 & 0 & h_{\theta\psi} & h_{\psi\varphi} & h_{\psi\chi} & h_{\psi\psi} \end{pmatrix}. \] (3.11)

For every mode in the perturbation we have \( h_{ij} = h_{ij} (\theta, \varphi, \chi, \psi) \) with \( i, j = \theta, \varphi, \chi, \psi \). Furthermore, the transverse and traceless choice for the gauge alongside with the symmetries of the metric implies that the restriction of \( h_{\alpha\beta} \) to the sphere is transverse, traceless, and
so it can be expanded using a basis of eigentensors of the Laplacian on the sphere [50]. So with \( i, j \) running on the sphere we have

\[
h_{ij} = r^2 \phi(r, t) \bar{h}_{ij}(x),
\]

where

\[
\nabla^k \nabla_k \bar{h}_{ij} = \gamma \bar{h}_{ij}, \quad \nabla^i \bar{h}_{ij} = 0, \quad \bar{g}^{ij} \bar{h}_{ij} = 0.
\]

The bar refers to metric and tensors on the \( S^4 \).

Substituting (3.11) into (3.5) and using the expansion of \( h_{ij} \) according to the basis of eigentensors of the Laplacian on the sphere we find equation (11) of [13]. But somehow this is expected. In addition, we find the same equations as in [13] for the Gauss-Bonnet combination. Substituting again (3.11) to (3.6) and using both the expansion of \( h_{ij} \) according to the basis of eigentensors of the Laplacian on the sphere, alongside with the transverse and traceless gauge, we end up after a lot of algebra to the equation (12) of [13].

The crucial point in the above analysis comes from the symmetries of spacetime. Here spherical symmetry allows for the decomposition of \( h_{ij} \) according to the basis of eigentensors of the Laplacian on the sphere, which at the end is the key factor for obtaining the final master equation of the perturbation analysis. Such a simplification is more difficult and sometimes not possible in the cases where the spacetime allows for symmetries other than spherical as we will discuss in the following sections.

4 Perturbations of codimension-2 black strings

We consider the following gravitational action in five dimensions with a Gauss-Bonnet term in the bulk

\[
S_{\text{grav}} = \frac{M_5^3}{2} \left\{ \int d^5 x \sqrt{-g^{(5)}} \left[ R^{(5)} + \alpha \left( R^{(5)2} - 4 R^{(5)MN} R^{(5)MN} + R^{(5)MNKL} R^{(5)MNKL} \right) \right] \right\}.
\]

We are looking for solutions of the form

\[
ds_5^2 = g_{\mu\nu}(x, \rho) dx^\mu dx^\nu + a^2(x, \rho) d\rho^2 + L^2(x, \rho) d\theta^2.
\]

The topology of the two-dimensional space \((\rho, \theta)\) can be represented by a cone of deficit angle \(\beta\). Regularization of this space dictates the introduction of a brane located at the tip of the cone. As we will discuss, the presence of the brane has important consequences in the perturbative analysis of these spaces. Solutions of the form (4.2) have been obtained in [35]. In this section we will derive the general formalism of metric perturbations of these solutions and we will discuss the scalar, vector, and tensor perturbations.

4.1 General formalism of metric perturbations

Consider the metric

\[
ds^2 = f^2(\rho) \left[ -n^2(r) dt^2 + \frac{dr^2}{n^2(r)} + r^2 d\phi^2 \right] + d\rho^2 + b^2(\rho) d\theta^2,
\]
which is of the form (4.2). In what follows we will consider \( f(\rho) = \cosh (\rho/2\sqrt{\alpha}) \), \( b(\rho) = 2\beta \sqrt{\alpha} \sinh (\rho/2\sqrt{\alpha}) \), and \( n^2(r) = -M + r^2/l^2 \), which is a solution of the action (4.1) with the inclusion of a brane boundary term [35].

In order to investigate the stability of the solution we need to adopt an Ansatz for our perturbations. It is possible to see that the above metric has 3 Killing vectors. We can adopt a Fourier expansion of the perturbation with respect to the coordinates \( t, \phi, \theta \) and keep all the components of the perturbation as functions of \( r, \rho \). We are going to consider perturbations which retain the angular symmetry on the brane as well as on the transverse space. Keeping the axial symmetry both on the brane and in the transverse space means that everything must be Lie derived by \( \partial_\phi \) and \( \partial_\theta \) to zero [32]. In this way, we will work on the \( s-wave \) approximation of our system for the two angular coordinates.

As a result of the symmetries of the spacetime we can split the perturbation into a purely two-dimensional transverse piece, a mixed transverse and three-dimensional piece and a purely three-dimensional piece. This can be represented by

\[
\begin{pmatrix}
    h_{\mu\nu} & h_{\mu i} \\
    h_{ji} & h_{ij}
\end{pmatrix}
\]

(4.4)

where \( (\mu, \nu = t, r, \phi) \) and \( (i, j = \rho, \theta) \). In the Kaluza-Klein spirit, these perturbations can be interpreted as scalar, vector, and tensor, respectively, with respect to the three-dimensional spacetime. We will consider the following Ansatz for our perturbation

\[
h_{AB} = e^{\Omega t} \begin{pmatrix}
    h_{\mu\nu}(r, \rho) & h_{\mu i}(r, \rho) \\
    h_{ji}(r, \rho) & h_{ij}(r, \rho)
\end{pmatrix}
\]

(4.5)

The above ansatz contains the maximum information concerning the perturbation of our system, while at the same time is consistent with the aforementioned arguments.

Substituting (4.5) into the Lichnerowicz equation (2.32), we can see in a straight forward manner that, the \( h_{\phi\phi}(r, \rho) \) mode decouples from the rest of the modes straight away, while the \( h_{t\phi}(r, \rho) \), \( h_{r\phi}(r, \rho) \) and \( h_{\rho\phi}(r, \rho) \) modes are coupled together and the same happens also for \( h_{t\theta}(r, \rho) \), \( h_{r\theta}(r, \rho) \) and \( h_{\rho\theta}(r, \rho) \) modes. We could set them to zero in order to examine the rest of the system, but for the moment we will keep them, since in any case these modes, for the \( s-wave \) approximation that we are considering, do not interact with the other modes. As a first step we will examine the behaviour of the scalar modes \( h_{\theta\theta}(r, \rho) \), \( h_{\rho\rho}(r, \rho) \) and \( h_{\rho\theta}(r, \rho) \).

### 4.2 Scalar perturbations-solving the Lichnerowicz equation

We consider first the scalar perturbations. Our aim is to solve the modified Lichnerowicz equation (2.32) in the background metric (4.3) using the perturbation Ansatz (4.5). Substituting our Ansatz for the perturbation we can see that all the perturbation equations are governed by a prefactor, which has the form \( (l^2 - 4\alpha) \). We see that when we are close to \( l^2 \rightarrow 4\alpha \), we face a strong coupling problem. We will refer to this limit later on.

Using (2.34) and (2.35) we can calculate the two equations for the scalar modes. These equations couple the scalar modes of the perturbation with the tensor and the vector modes.
However, they can be decoupled. Here we will summarize the results while in the appendix we present the technical details. The two scalar modes are given by

\[
    h_{\rho\rho} = \frac{1}{\beta^2} \partial_\rho \left[ \frac{h_{\theta\theta}}{\sqrt{a} \sinh \left( \frac{\rho}{\sqrt{a}} \right)} \right], \quad (4.6)
\]

\[
    h_{\theta\theta} = \left[ 2 \beta \sqrt{\alpha} \sinh \left( \frac{\rho}{2\sqrt{\alpha}} \right) \right]^2 u(r, \rho). \quad (4.7)
\]

Separating the variables choosing \( u(r, \rho) = f(r) y(\rho) \), the functions \( f(r) \) and \( y(\rho) \) are given by the differential equations

\[
    \frac{4\alpha^{3/2} \coth \left( \frac{\rho}{2\sqrt{\alpha}} \right)}{l^2} \left[ 3 \cosh \left( \frac{\rho}{\sqrt{\alpha}} \right) \frac{\partial y(\rho)}{\partial \rho} + \sqrt{\alpha} \sinh \left( \frac{\rho}{\sqrt{\alpha}} \right) \frac{\partial^2 y(\rho)}{\partial \rho^2} \right] + my(\rho) = 0, \quad (4.8)
\]

with \( m \) being the separation constant

\[
    \left( m - \frac{8\alpha^2 \Omega^2}{l^2 M - r^2} \right) f(r) + \frac{8\alpha^2}{l^4 r} \left[ (l^2 M - 3r^2) \frac{\partial f(r)}{\partial r} + r (l^2 M - r^2) \frac{\partial^2 f(r)}{\partial r^2} \right] = 0. \quad (4.9)
\]

The equation (4.9) can be written in the following form

\[
    (-l^2 M + r^2) \frac{\partial^2 f(r)}{\partial r^2} + \frac{(-l^2 M + 3r^2)}{r} \frac{\partial f(r)}{\partial r} + \frac{l^4}{8\alpha^2} \left( \frac{8\alpha^2 \Omega^2}{l^2 M - r^2} - m \right) f(r) = 0. \quad (4.10)
\]

Setting \( f(r) = \psi(r)/\sqrt{r} \), and introducing the tortoise coordinate defined by

\[
    dr_* = \frac{dr}{-M + r^2/l^2}
\]

Eq.(4.10) can be written as a Schrödinger equation

\[
    \frac{d^2 \psi(r)}{dr_*^2} + \left[ V(r) - \Omega^2 \right] \psi(r) = 0, \quad (4.11)
\]

where \( V(r) \) reads

\[
    V(r) = \frac{M}{2l^2} + \frac{M^2}{4l^2} - \frac{3r^2}{4l^4} + \frac{l^2 m M}{8\alpha^2} - \frac{m r^2}{8\alpha^2}. \quad (4.12)
\]

Note that the above potential depends only on the separation constant and the Gauss-Bonnet coupling constant \( \alpha \). This is the only information that it has from the bulk. It does not depend on the deficit angle \( \beta \). This can be understood because \( b(\rho) \) expresses the response of the bulk geometry to the presence of the conical singularity, information that is entirely encoded in the decoupled equation (4.8), which describes the behaviour of the perturbation in the extra-dimensions. We will show in the following subsection that the deficit angle explicitly appears in this extra-dimensional equation only if we consider an
angular dependence in the total perturbation as can be seen in Eq. (4.28) together with its solution (4.47).

The above potential is similar to the one found in the calculation of quasinormal modes of BTZ black hole [51]. Furthermore, it is exactly the same as the potential we will find in the next subsection by solving the Klein-Gordon equation for the background metric, provided that we make the identification $\nu^2 \rightarrow \frac{\ell^2 m}{8\alpha^2}$. The Schrödinger equation (4.11) will be solved in the next subsection. The solution implies that

$$\Omega = -\sqrt{\frac{M}{l}} \left( 1 + 2N + \sqrt{1 + \frac{ml^4}{8\alpha^2}} \right), \quad (4.13)$$

where $N$ is a positive integer. As this quantity is always negative, it yields a decaying solution for the perturbation assuring the stability of the model under scalar metric perturbations. Although in general the mass may also take negative values, the spectrum of $m$ can be analysed by looking at the bulk equation (4.8), whose general solution is

$$y(\rho) = C_1 q^A {}_2F_1(A; B; C; q) + C_2 q^{A'} {}_2F_1(A'; B'; C'; q), \quad (4.14)$$

with

$$q = \cosh^2(\rho/2\sqrt{\alpha}), \quad (4.15)$$

$$A = -\frac{1}{2} - \frac{1}{4} \sqrt{4 + 2l^2m/\alpha}, \quad (4.16)$$

$$B = \frac{5}{2} - \frac{1}{4} \sqrt{4 + 2l^2m/\alpha}, \quad (4.17)$$

$$C = 1 - \frac{1}{2} \sqrt{4 + 2l^2m/\alpha}, \quad (4.18)$$

$$A' = -C/2, \quad (4.19)$$

$$B' = B - A - C/2, \quad (4.20)$$

$$C' = -2A. \quad (4.21)$$

For this solution let us choose $C_1$ and $C_2$ such that we can obtain a decaying behaviour which ascertains a smooth transition towards infinity. As the arguments of the hypergeometric functions need to be real, the first restriction on $m$ comes from the square root: $m \geq 4\alpha/l^2$. The equality in this equation when substituted in (4.13) corresponds to the strong coupling or Chern-Simons limit. Interesting enough, at this limit the extra term disappears and we recover the results of a three-dimensional quasinormal modes of the BTZ black hole [51]. However, the hypergeometric functions are not well-defined for all the values of $m$ above this limit. This strictly depends upon the parameters of each function. A careful analysis shows that $C_2$ needs to be zero since it cannot produce decaying solutions. Moreover, recalling the properties of these functions, we were able to find a special case by setting $B = 0$, which corresponds to $m = 48\alpha/l^2$, where the perturbation manifests a decaying behaviour.

We have one more scalar mode namely the $h_{\rho \phi}$ mode. As we have already mentioned, this mode is coupled with the $h_{tg}$ and $h_{r\phi}$ modes. Still using the transverse gauge, it can be easily seen, that this scalar mode also decouples from the other two modes and quite surprisingly it obeys the same Schrödinger equation as the $h_{tg}$ mode.
4.3 Scalar perturbations-solving the Klein-Gordon equation

In this subsection we will carry out the analysis of the scalar perturbations solving the Klein-Gordon equation. We will show that we get the same results as the ones we got by solving the Lichnerowicz equation, but the study of the Klein-Gordon equation gives us a better understanding of the stability of the codimension-2 black string.

Consider the massive Klein-Gordon equation
\[ \Box \Phi = \frac{1}{\sqrt{-g}} \partial_M (\sqrt{-g} g^{MN} \partial_N) \Phi = m^2 \Phi. \]  

(4.22)

By using the metric (4.3) and decomposing the scalar field as
\[ \Phi(t, r, \phi, \rho, \theta) = Z(t, r, \rho) \Xi(\phi) \Theta(\theta), \]  

(4.23)

we arrive to the following equation,
\[ -\frac{1}{f^2 n^2} \frac{\partial^2 Z}{Z} + \left( \frac{n^2}{f^2 r} + \frac{2m n}{f^2} \right) \frac{\partial_r Z}{Z} + n^2 \frac{\partial^2 Z}{Z} + \left( \frac{3f'}{f} + \frac{b'}{b} \right) \frac{\partial_\rho Z}{Z} + \frac{\partial^2 Z}{Z} \]
\[ + \frac{l^2}{f^2 r^2} \frac{\partial^2 \Xi}{\Xi} + \frac{1}{b^2} \frac{\partial^2 \Theta}{\Theta} - m^2 = 0. \]  

(4.24)

We can apply variable separation method to the \( \theta \)- and \( \phi \)-dependent parts of this equation. Thus, the solutions to \( \Theta(\theta) \) and \( \Xi(\phi) \) are given by
\[ \Theta(\theta) = Ae^{i\kappa \theta} + Be^{-i\kappa \theta} \]  

(4.25)
\[ \Xi(\phi) = Ce^{i\epsilon \phi} + De^{-i\epsilon \phi}, \]  

(4.26)

where \( A \), \( B \), \( C \), and \( D \) are integration constants while \( \kappa \) and \( \epsilon \) are constants generated through the variable separation. The remaining equation can be decoupled in two equations with the Ansatz \( Z(t, r, \rho) = \Psi(t, r) P(\rho) \) as follows,
\[ \partial_t^2 \Psi - n^2 \left( \frac{n^2}{r} + 2m n \right) \partial_r \Psi - n^2 \partial^2_r \Psi + \left( \frac{\epsilon^2 f n^2}{r^2} + \nu^2 n^2 \right) \Psi = 0 \]  

(4.27)
\[ \partial^2_\rho P + \left( \frac{3f'}{f} + \frac{b'}{b} \right) \partial_\rho P + \left( -\frac{\kappa^2}{b^2} - m^2 + \nu^2 f^2 \right) P = 0, \]  

(4.28)

being \( \nu^2 \) the constant of variable separation. Equation (4.27) describes the evolution of the scalar perturbation on the brane, while equation (4.28) says how this perturbation behaves at different distances away from the brane.

In order to solve Eq.(4.27) we should rewrite the equation in terms of the tortoise coordinate,
\[ r_* = \int \frac{dr}{n^2}. \]  

(4.29)

Thus, Eq.(4.27) adopts the Schrödinger form,
\[ -\frac{\partial^2 X(t, r_*)}{\partial t^2} + \frac{\partial^2 X(t, r_*)}{\partial r_*^2} = V[r(r_*)]X(t, r_*). \]  

(4.30)
Here $X(t, r_*)$ is a new function defined as

$$X(t, r_*) = \sqrt{r} \Psi(t, r),$$  \hfill (4.31)

and the so-called potential term can be written as

$$V(r) = \frac{n^2}{2r} \left( 2n\dot{n} - \frac{n^2}{2r} \right) + \frac{\epsilon^2 l^2 n^2}{r^2} + \nu^2 n^2. \hfill (4.32)$$

When we compare this potential with the pure 3-dimensional case studied in [51], we see that Eq.(4.32) has a $\nu^2 n^2$ additional correction. This new term connects brane to bulk perturbations through the constant $\nu$ and does not vanish, in general.

We choose the time dependence as $e^{-i\omega t}$, so that $X(t, r) = e^{-i\omega t} R(r)$, thus, Eq.(4.30) becomes

$$\frac{d^2 R}{dr_*^2} + [\omega^2 - V(r)] R = 0. \hfill (4.33)$$

When $n(r)$ is of the BTZ form, $n^2(r) = -M + r^2/l^2$, the potential (4.32) becomes

$$V(r) = \left( \frac{3}{4l^4} + \frac{\nu^2}{l^2} \right) r^2 - \left( \frac{M}{2l^2} + M\nu^2 - \epsilon^2 \right) + \left( -\frac{M^2}{4} - l^2 M\epsilon^2 \right) \frac{1}{r^2}. \hfill (4.34)$$

The tortoise coordinate can analytically be found from (4.29) as

$$r = -l\sqrt{M} \coth \left( \frac{r_* \sqrt{M}}{l} \right). \hfill (4.35)$$

Notice that $r_H = l\sqrt{M} \leq r < \infty$ corresponds to $-\infty < r_* \leq 0$.

Let us make the following variable transformation,

$$x = \frac{1}{\cosh^2 \left( \frac{r_* \sqrt{M}}{l} \right)}, \hfill (4.36)$$

with $0 \leq x \leq 1$, so that the potential turns into

$$V(x) = -\frac{x}{4} \left[ \frac{4M(1 + \nu^2 l^2) - Mx - 4l^2 \epsilon^2(x - 1)}{l^2(x - 1)} \right]. \hfill (4.37)$$

and Eq.(4.33) becomes

$$4x(1 - x) \frac{d^2 R}{dx^2} + (4 - 6x) \frac{dR}{dx} + \left[ \frac{\omega^2 l^2}{Mx} + \frac{1 + \nu^2 l^2}{x - 1} - \frac{x}{4(x - 1)} - \frac{l^2 \epsilon^2}{M} \right] R = 0. \hfill (4.38)$$

In order to write this equation in a more familiar way we make a new substitution as follows,

$$R = \frac{(x - 1)^{3/4}}{x^{i\omega l/2\sqrt{M}}} y(x). \hfill (4.39)$$
Then, Eq.(4.38) turns to be

\[
-x(x-1)y'' + \left[1 - 3x + \frac{i\omega l}{\sqrt{M}(x-1)}\right]y' + \left[\frac{\omega^2 l^2}{4M} + \frac{i\omega l}{\sqrt{M}} - \frac{l^2 \epsilon^2}{4M} - 1 + \frac{\nu^2 l^2}{4(x-1)}\right]y = 0.
\]

(4.40)

The general solution of this equation can be expressed in terms of hypergeometric functions of the second kind \(_2F_1(a, b; c; x)\). As we look for stable behaviour, we keep the decaying branch of the general solution and establish the restrictions for this stability,

\[
y(x) = (1-x)^{-\left(1+\sqrt{1+\nu^2l^2}/2\right)/2} \ _2F_1(a, b; c; x),
\]

(4.41)

where

\[
a = \frac{1}{2} \left(1 - \sqrt{1 + \nu^2 l^2} - \frac{i\omega l}{\sqrt{M}} + \frac{i\epsilon}{\sqrt{M}}\right),
\]

\[
b = \frac{1}{2} \left(1 - \sqrt{1 + \nu^2 l^2} - \frac{i\omega l}{\sqrt{M}} - \frac{i\epsilon}{\sqrt{M}}\right),
\]

\[
c = 1 - \frac{i\omega l}{\sqrt{M}}.
\]

(4.42)

Since the (2+1) spacetime is asymptotically AdS, the correct boundary condition to be taken in Eq.(4.41) is the flux condition, i.e.,

\[
\mathcal{F} \sim (y^* \partial_{\mu} y - y \partial_{\mu} y^*) \bigg|_{x=1} = 0.
\]

(4.43)

In order to evaluate this condition at \(x = 1\) we can use the following property of hypergeometric functions. Given a \(_2F_1(a', b'; c'; x)\), if \(c'\) is not a negative integer, the series converges when \(x = 1\) if \(\Re(c' - a' - b') > 0\), and we can write the hypergeometric function as

\[
_2F_1(a', b'; c'; 1) = \frac{\Gamma(c')\Gamma(c' - a' - b')}{\Gamma(c' - a')\Gamma(c' - b')}.
\]

(4.44)

The derivative of the hypergeometric functions we are dealing with has the following form,

\[
_2F_1'(a, b; c; x = 1) = \frac{ab}{c} \ _2F_1(a + 1, b + 1; c + 1; x = 1).
\]

(4.45)

Using (4.42) we verify that \(\Re(c - a - b - 1) = \sqrt{1 + \nu^2 l^2} - 1 > 0\) since \(\nu^2 > 0\). Thus, we can write (4.45) in terms of \(\Gamma\) functions as in (4.44) and put back in the flux condition (4.43). The new equation is fulfilled when \(c - a = -N\) or \(c - b = -N\), where \(N\) is a positive integer. In this way we obtain the quasinormal frequencies,

\[
\omega = \pm \epsilon - i\frac{\sqrt{M}}{l}(1 + 2N + \sqrt{1 + \nu^2l^2}).
\]

(4.46)

We observe that the imaginary part of these frequencies is negative, thus, displaying the stability of the model under scalar perturbations. Notice that in (4.46) a term proportional to \(\nu^2\) appears which encodes the information from the bulk and when \(\nu^2 = 0\) the pure three-dimensional BTZ results [51] are recovered.

To complete our analysis we must solve Eq.(4.28). We have two subcases.
1. When \( f(\rho) = \cosh(\rho/2\sqrt{\alpha}) \) and \( b(\rho) = 2\beta\sqrt{\alpha}\sinh(\rho/2\sqrt{\alpha}) \), the most general solution of Eq. (4.28) is given by

\[
P(z) = \frac{(z - 1)^{\kappa/2\beta}}{\sqrt{2z}} \left[ C_1 z^{-\sqrt{1+4\nu^2\alpha}/2} 2F_1(\hat{a}, \hat{b}; \hat{c}; z) + C_2 z^{\sqrt{1+4\nu^2\alpha}/2} 2F_1(\hat{a}', \hat{b}'; \hat{c}'; z) \right],
\]

(4.47)

where

\[
z = \cosh^2 \left( \frac{\rho}{2\sqrt{\alpha}} \right),
\]

(4.48)

\( C_1 \) and \( C_2 \) are constants, and

\[
\begin{align*}
\hat{a} & = \frac{1}{2} \left( 1 + \frac{\kappa}{\beta} - \sqrt{1+4\nu^2\alpha} + 2\sqrt{1+m^2\alpha} \right) \\
\hat{b} & = \frac{1}{2} \left( 1 + \frac{\kappa}{\beta} - \sqrt{1+4\nu^2\alpha} - 2\sqrt{1+m^2\alpha} \right) \\
\hat{c} & = 1 - \sqrt{1+4\nu^2\alpha} \\
\hat{a}' & = \frac{1}{2} \left( 1 + \frac{\kappa}{\beta} + \sqrt{1+4\nu^2\alpha} - 2\sqrt{1+m^2\alpha} \right) \\
\hat{b}' & = \frac{1}{2} \left( 1 + \frac{\kappa}{\beta} + \sqrt{1+4\nu^2\alpha} + 2\sqrt{1+m^2\alpha} \right) \\
\hat{c}' & = 1 + \sqrt{1+4\nu^2\alpha}.
\end{align*}
\]

(4.49)

A careful study of both hypergeometric functions shows that \( C_2 \) needs to be zero if we want to have a decaying solution. Given a suitable set of parameters the remaining hypergeometric function has the pursued behaviour. A special case appears when \( \nu^2 \) adopts the following form,

\[
\nu^2 = m^2 + \frac{1}{\alpha} + \frac{\kappa}{2\alpha\beta} \left( 1 + \frac{\kappa}{2\beta} \right) + \frac{\sqrt{m^2\alpha + 1}}{\alpha} \left( 1 + \frac{\kappa}{\beta} \right),
\]

(4.50)

which produces \( b = 0 \). This case corresponds to a damped oscillation in \( \rho \). Therefore, the model is well behaved under this kind of perturbation having the quasinormal frequencies given in (4.46).

2. When \( f(\rho) = \pm 1 \) and \( b(\rho) = \gamma\sinh(\rho/\gamma) \), with \( \gamma = \sqrt{(l^2 - 4\alpha)/2} \), we find the following solution for (4.28),

\[
P(w) = \sqrt{2}(w - 1)^{\kappa/2} \left[ C_3 2F_1(\tilde{a}, \tilde{b}; \tilde{c}; w) + C_4 \sqrt{2w} 2F_1(\tilde{a}', \tilde{b}'; \tilde{c}'; w) \right],
\]

(4.51)

where

\[
w = \cosh^2 \left( \frac{\sqrt{2}\rho}{\sqrt{l^2 - 4\alpha}} \right),
\]

(4.52)
\[ C_3 \text{ and } C_4 \text{ are constants, and} \]
\[
\begin{align*}
\tilde{a} &= \frac{1}{4} + \frac{\kappa}{2} - \frac{1}{4} \sqrt{1 + (l^2 - 4\alpha)(2m^2 - 2\nu^2)} \\
\tilde{b} &= \frac{1}{4} + \frac{\kappa}{2} + \frac{1}{4} \sqrt{1 + (l^2 - 4\alpha)(2m^2 - 2\nu^2)} \\
\tilde{c} &= \frac{1}{2} \\
\tilde{a}' &= \frac{3}{4} + \frac{\kappa}{2} + \frac{1}{4} \sqrt{1 + (l^2 - 4\alpha)(2m^2 - 2\nu^2)} \\
\tilde{b}' &= \frac{3}{4} + \frac{\kappa}{2} - \frac{1}{4} \sqrt{1 + (l^2 - 4\alpha)(2m^2 - 2\nu^2)} \\
\tilde{c}' &= \frac{3}{2}.
\end{align*}
\] (4.53)

In this case both hypergeometric functions display a growing behaviour. This means that the scalar perturbation when going into the bulk propagates without boundary, an analogous behaviour to the one found in the five-dimensional black string solution of [44].

In the last two subsections we studied the scalar perturbations of the five-dimensional black string solution of codimension-2. Using both the Lichnerowicz equation and the Klein-Gordon equation we found stability under scalar perturbations. For an alert reader we comment on the approximation we used. We have chosen the s-wave approximation for our ansatz, both for the angular coordinate on the brane and on the transverse space. However, on general grounds the Fourier decomposition should also include, the frequency modes of the two angular dimensions. Still it is not clear to us whether the deficit angle is playing a significant role in the stability analysis. The procedure we have followed in the quasi-normal mode analysis shows (see equations (4.27) and (4.28)) that each angular coordinate, affects the corresponding spatial dimension. The \( \phi \) dependence is reflected in the equation for the brane, while the \( \theta \) dependence and hence the deficit angle comes with the transverse space equation. Any effect of the deficit should be reflected on the solution of the transverse piece. However, the only information that equation (4.28) give us is to tell us how the perturbation created on the brane is transmitted into the bulk.

Should we had chosen a frequency mode in the Fourier decomposition for the \( \theta \) dimension, then we should examine whether the scalar part decouples from the rest of the system and even if the variables in the Klein-Gordon equation can be separated. This procedure is very involved, due to the presence of the Gauss-Bonnet’s quadratic terms and it is beyond the scope of the present work. This procedure may give different results, however, even in this case, the deficit angle due to its intrinsic geometrical nature, may still has an effect only on the transverse space.

### 4.4 Vector and tensor perturbations

We have seen that as far as we are away from the Chern-Simons limit we can always apply the transverse and traceless gauge and examine the stability behaviour. We saw that
for scalar perturbations the system declines exponentially, meaning that $\Omega$ is negative, even though the mass of the scalar mode takes negative values until a certain limit. This feature ensures that our system is stable under scalar metric perturbations. Of course, the quadratic nature of the Gauss-Bonnet combination does not give straightforward decoupled equations for the scalar part and the decoupling procedure is not easy, as it happens in the case of having just the Einstein tensor where the scalar part of the perturbation decouples immediately.

It is known that a category of spherically symmetric vacuum solutions with a Gauss-Bonnet term suffer from ghost-like instabilities, and furthermore, there is a strong coupling problem close to the Chern-Simons limit [22]. In our case we expect the strong coupling problem to be present also in the vector and tensor perturbations. Our analysis has shown that under the particular Ansatz that we have chosen for the perturbation there is a proportionality factor to every equation, namely, $(l^2 - 4\alpha)$. As $l^2 \to 4\alpha$, all equations are identically satisfied. This means that the first order parametrization of the perturbation is not significant, and we have to move to the second order.

We saw at the beginning of this section that the scalar part of our perturbation, can be decoupled from the rest of the system, with the help of the transverse and traceless gauge. Furthermore, a careful investigation of the perturbation equations unveiled that the scalar perturbations do not have any pathological behaviour. To discuss the other kind of perturbations we will set these modes equal to zero and examine the rest of the system.

The equation for the vector mode $h_{\phi\theta}$ is very simple and it is decoupled in a straightforward manner from the rest of the system

$$\coth\left(\frac{\rho}{\sqrt{\alpha}}\right) \frac{\partial h_{\phi\theta}}{\partial \rho} - \sqrt{\alpha} \frac{\partial^2 h_{\phi\theta}}{\partial \rho^2} = 0.$$ (4.54)

The solution of this equation is trivial and it gives information only about the transverse space. As we can see there is no information about the brane. Before making a comment on that let us see the equation for $h_{t\theta}$ and $h_{r\theta}$ modes. Making use of the transverse gauge, these modes decouple from each other and the result is that both of them obey the same equation as $h_{\phi\theta}$, namely equation (4.54). This is quite unnatural from many aspects. First of all since the brane dependence is arbitrary, this means that it could be satisfied by a mode that has positive $\Omega$, signaling this way departure from stability. Furthermore, the system is degenerate since these three modes behave exactly the same, while at the same time we can not make any solid prediction about its brane behaviour. This could be a signal of a strong coupling problem, but it is not clear whether this is exactly the case or not.

We still have the tensor part plus one more vector. It can be seen that choosing any of the remaining perturbation equations, with the help of the transverse and traceless gauge, we can set them identically equal to zero. This is again an unnatural result. Either it means that we have no propagating degrees of freedom, which indeed could be the case because in three dimensions we do not have a graviton, or again we are in a strong coupling regime, which restricts us from making any conclusion from the linear terms and we have to move forward to higher order ones \(^3\).

\(^3\)In [52] the holographic description of the Gauss-Bonnet theory in five dimensions was discussed and it
degrees of freedom seems more probable, since in the s-wave approximation that we have chosen, the brane can "see" the transverse space through the angular symmetry that is manifested both on the brane and on the transverse space. The absence of a graviton mode on the brane, due to its three dimensional nature is reflected also in transverse space due to the aforementioned symmetry.

5 Conclusions

We studied the metric perturbations of a five-dimensional black string in codimension-2 braneworlds with a Gauss-Bonnet term in the bulk. After reviewing the general formalism of linear metric perturbations, we derived the modified Lichnerowicz equation in the presence of the Gauss-Bonnet term. As an application, we considered the six-dimensional spherically symmetric Gauss-Bonnet black hole solutions and we applied the modified Lichnerowicz equation to these solutions to derive the known results for the tensor perturbations.

We considered the scalar perturbations of a five-dimensional black string solution of codimension-2 braneworlds with a Gauss-Bonnet term. We carried out the analysis using both the Lichnerowicz equation and solving explicitly the Klein-Gordon equation. Away from the Chern-Simons limit, we showed that the results from the two methods coincide. The behaviour of the black string under scalar perturbations can be described by two equations. One of them describes the evolution of the scalar perturbation on the brane while the other equation shows how this perturbation behaves at different distances from the brane. The evolution of the black string on the brane is controlled by the quasinormal modes which are similar to the quasinormal modes of the BTZ black hole with the addition of an extra term which has the information from the bulk. The stability analysis shows that the black string is stable under scalar perturbations.

We also studied the vector and tensor perturbations of the black string solution. We found that for the vector modes we can only get information on their behaviour in the bulk while for the tensor modes we did not find any physical propagating modes. We mainly attribute this behaviour to the specific symmetries of the considered black string solution but also may be an indication of a strong coupling problem signaling the need to go to the next order in perturbation theory.

It is interesting to extent the analysis to the six-dimensional black string solutions derived in [40]. In six dimensions there is also a limit where the theory becomes strongly coupled. Away from this limit, we expect the application of the modified Lichnerowicz equation to this solution to simplify significantly the calculations but the main problem which has to be addressed is the presence of matter in the bulk. The matter is necessary to support the black string solution on the brane and its extension into the bulk. However, there is a limit where the matter in the bulk decouples and the general formalism developed in this work can be applied. This issue is under investigation.

was found that there is a particular Weyl anomaly that prevents the Gauss-Bonnet theory to go smoothly to the Chern-Simons limit signaling the breakdown of perturbation theory at linear order.
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A The Lichnerowicz equation for the scalar modes

In this appendix we give some technical details for calculating the scalar modes. First, we take the traceless equation, solve it for the function $h_{rr}$, and then substitute into the $(\theta \theta)$ component of the Lichnerowicz equation (2.32). Also we substitute the function $h_{rr}$ into $\nabla^A h_{\rho \rho}$ equation, from which we solve for $\partial_r h_{\rho \rho}$. Then, we replace both $\partial_r h_{\rho \rho}$ and $\partial_r \partial_\rho h_{\rho \rho}$ into the $(\theta \theta)$ equation. We follow the same steps also for the $(\rho \rho)$ equation of (2.32). From the resulting two equations we take the combination $4 \alpha \beta^2 \frac{\tanh(\frac{\rho}{\sqrt{\alpha}})}{\text{sech}^2(\frac{\rho}{2 \sqrt{\alpha}})} (\rho \rho) - (\theta \theta)$ and we find the following equation for the two scalar modes

$$
\frac{\left(2 l^2 \alpha \Omega^2 - (l^2 M - r^2) \cosh \left(\frac{\rho}{\sqrt{\alpha}}\right) \text{sech}^4 \left(\frac{\rho}{2 \sqrt{\alpha}}\right)\right)}{4 l^2 (l^2 M - r^2) \alpha} h_{\theta \theta} + \beta^2 \frac{(3r^2 - l^2 (3M - \alpha \Omega^2)) \left(1 - 8 \cosh \left(\frac{\rho}{\sqrt{\alpha}}\right)\right) - 5 (l^2 M - r^2) \cosh \left(\frac{2 \rho}{\sqrt{\alpha}}\right)}{8 l^2 (l^2 M - r^2) \text{sech}^{-4} \left(\frac{\rho}{2 \sqrt{\alpha}}\right)} h_{\rho \rho} - \frac{6 \text{csch}^3 \left(\frac{\rho}{\sqrt{\alpha}}\right) \sinh^2 \left(\frac{\rho}{2 \sqrt{\alpha}}\right)}{l^2 \sqrt{\alpha}} \partial_\rho h_{\theta \theta} - \frac{\sqrt{\alpha} \beta^2 \left(-3 + 4 \cosh \left(\frac{2 \rho}{\sqrt{\alpha}}\right)\right) \sinh \left(\frac{\rho}{2 \sqrt{\alpha}}\right)}{l^2 \cosh^3 \left(\frac{\rho}{2 \sqrt{\alpha}}\right)} \partial_\rho h_{\rho \rho} - \frac{\partial_{\rho \rho} h_{\theta \theta}}{l^2 \left(1 + \cosh \left(\frac{\rho}{\sqrt{\alpha}}\right)\right)} - \frac{2 \alpha \beta^2 \tanh^2 \left(\frac{\rho}{2 \sqrt{\alpha}}\right) \partial_{\rho \rho} h_{\rho \rho}}{l^2} - \frac{(l^2 M - 3r^2) \text{sech}^4 \left(\frac{\rho}{2 \sqrt{\alpha}}\right) \partial_r h_{\theta \theta}}{2 l^4} - \frac{(l^2 M - r^2) \text{sech}^4 \left(\frac{\rho}{2 \sqrt{\alpha}}\right) \partial_r h_{\rho \rho}}{2 l^4} + \frac{32 (l^2 M - 3r^2) \alpha \beta^2 \text{csch}^4 \left(\frac{\rho}{\sqrt{\alpha}}\right) \sinh^6 \left(\frac{\rho}{2 \sqrt{\alpha}}\right) \partial_\rho h_{\rho \rho}}{l^{14}} + \frac{32 (l^2 M - r^2) \alpha \beta^2 \text{csch}^4 \left(\frac{\rho}{\sqrt{\alpha}}\right) \sinh^6 \left(\frac{\rho}{2 \sqrt{\alpha}}\right) \partial_r h_{\rho \rho}}{l^4} = 0
$$

We need one more equation in order to find a solution for the scalar perturbations. The second equation comes from the $(t \rho)$ component of (2.32), and it can be derived in a similar fashion as (A.1). Again we solve the traceless condition for $h_{rr}$ and substitute into the $(t \rho)$ equation. We also replace this equation into $\nabla^A h_{\lambda \rho}$ and solve for $\partial_\rho h_{\lambda \rho}$. We do the same
for $\nabla^2 h_{\alpha\kappa}$ and solve for $\partial_r h_{\rho\rho}$, from which we get $\partial_r \partial_\rho h_{\rho\rho}$. Finally, from the equation $(t\rho)$ we get the following

$$h_{\rho\rho} = \frac{1}{\beta^2} \partial_\rho \left( \frac{h_{\theta\theta}}{\sqrt{\alpha} \sinh \left( \frac{\rho}{\sqrt{\alpha}} \right)} \right). \quad (A.2)$$

Using equation (A.2), equation (A.1) becomes

$$-2\alpha \Omega^2 l^2 + (8 (r^2 - l^2 M) - 6\alpha \Omega^2 l^2) \cosh \left( \frac{\rho}{\sqrt{\alpha}} \right) + (r^2 - l^2 M) \left[ 7 + \cosh \left( \frac{2\rho}{\sqrt{\alpha}} \right) \right] h_{\theta\theta}$$

$$8l^2 (l^2 M - r^2) \alpha \sinh^{-6} \left( \frac{\rho}{2\sqrt{\alpha}} \right)$$

$$-8\alpha \Omega^2 l^2 + (6 (r^2 - l^2 M) + 4\alpha \Omega^2 l^2) \cosh \left( \frac{\rho}{\sqrt{\alpha}} \right) + (r^2 - l^2 M) \left[ 11 - 5 \cosh \left( \frac{2\rho}{\sqrt{\alpha}} \right) \right] \partial_\rho h_{\theta\theta}$$

$$- \frac{16 l^2 (l^2 M - r^2) \sqrt{\alpha} \csc^{-1} \left( \frac{\rho}{2\sqrt{\alpha}} \right) \sinh^{-5} \left( \frac{\rho}{2\sqrt{\alpha}} \right)}{4l^2}$$

$$- \frac{[ -7 + \cosh \left( \frac{\rho}{\sqrt{\alpha}} \right) ] \sech^4 \left( \frac{\rho}{2\sqrt{\alpha}} \right)}{4l^2} \partial_{\rho\rho} h_{\theta\theta} - \frac{2\sqrt{\alpha} \csc \left( \frac{\rho}{\sqrt{\alpha}} \right) \tanh^2 \left( \frac{\rho}{2\sqrt{\alpha}} \right)}{l^2} \partial_{\rho\rho} h_{\theta\theta}$$

$$- \frac{(l^2 M - 3r^2) \left[ 1 + 3 \cosh \left( \frac{\rho}{\sqrt{\alpha}} \right) \right] \sech^6 \left( \frac{\rho}{2\sqrt{\alpha}} \right)}{4l^4} \partial_r h_{\theta\theta}$$

$$+ \frac{1}{l^4} \frac{(l^2 M - 3r^2) \sqrt{\alpha} \sech^4 \left( \frac{\rho}{2\sqrt{\alpha}} \right) \tanh \left( \frac{\rho}{2\sqrt{\alpha}} \right)}{l^4} \partial_r \partial_\rho h_{\theta\theta}$$

$$- \frac{(l^2 M - r^2) \left[ 1 + 3 \cosh \left( \frac{\rho}{\sqrt{\alpha}} \right) \right] \sech^6 \left( \frac{\rho}{2\sqrt{\alpha}} \right)}{4l^4} \partial_{rr} h_{\theta\theta}$$

$$+ \frac{(l^2 M - r^2) \sqrt{\alpha} \sech^4 \left( \frac{\rho}{2\sqrt{\alpha}} \right) \tanh \left( \frac{\rho}{2\sqrt{\alpha}} \right)}{l^4} \partial_{rr} \partial_\rho h_{\theta\theta} = 0 \quad (A.3)$$

Inspecting equation (A.3) we can see that it has a third derivative with respect to $\rho$ plus mixed derivatives $\partial_r \partial_\rho$ and $\partial_{rr} \partial_\rho$. This makes the handling of this equation extremely difficult even for numerical methods. To proceed we take the following Ansatz for the function $h_{\theta\theta}$

$$h_{\theta\theta} = \left[ 2\beta \sqrt{\alpha} \sinh \left( \frac{\rho}{2\sqrt{\alpha}} \right) \right]^2 u (r, \rho). \quad (A.4)$$

Then, by substituting (A.4) into (A.3) we obtain

$$\frac{2\alpha \Omega^2 \sech^2 \left( \frac{\rho}{2\sqrt{\alpha}} \right) \tanh^4 \left( \frac{\rho}{2\sqrt{\alpha}} \right)}{l^2 M - r^2} u (r, \rho)$$

$$- \frac{[6 (r^2 - l^2 M) + 8\alpha \Omega^2 l^2] \left[ 1 + \cosh \left( \frac{\rho}{\sqrt{\alpha}} \right) \right] - 3 (r^2 - l^2 M) \left[ 1 + \cosh \left( \frac{2\rho}{\sqrt{\alpha}} \right) \right]}{4l^2 \alpha^{-\frac{1}{2}} \sech^{-4} \left( \frac{\rho}{2\sqrt{\alpha}} \right) \tanh^{-1} \left( \frac{\rho}{2\sqrt{\alpha}} \right) (l^2 M - r^2)} \partial_\rho u (r, \rho)$$
\[
\alpha \left[ -1 + 7 \cosh \left( \frac{\rho}{\sqrt{\alpha}} \right) \right] \sech^2 \left( \frac{\rho}{2 \sqrt{\alpha}} \right) \tanh^2 \left( \frac{\rho}{2 \sqrt{\alpha}} \right) \partial_{\rho \rho} u (r, \rho)
\]

\[
- \frac{4 \alpha^3}{l^2} \tanh^3 \left( \frac{\rho}{2 \sqrt{\alpha}} \right) \partial_{\rho \rho \rho} u (r, \rho)
\]

\[
- \frac{2 \left( l^2 M - 3r^2 \right) \alpha \sech^2 \left( \frac{\rho}{2 \sqrt{\alpha}} \right) \tanh^4 \left( \frac{\rho}{2 \sqrt{\alpha}} \right)}{l^4 r} \partial_r u (r, \rho)
\]

\[
- \frac{2 \left( l^2 M - 3r^2 \right) \alpha \sech^2 \left( \frac{\rho}{2 \sqrt{\alpha}} \right) \tanh^4 \left( \frac{\rho}{2 \sqrt{\alpha}} \right)}{l^4} \partial_{rr} u (r, \rho)
\]

\[
+ \frac{128 \left( l^2 M - 3r^2 \right) \alpha^3 \csch^5 \left( \frac{\rho}{\sqrt{\alpha}} \right) \sinh^8 \left( \frac{\rho}{2 \sqrt{\alpha}} \right)}{l^4} \partial_{\rho \rho} u (r, \rho)
\]

\[
+ \frac{128 \left( l^2 M - 3r^2 \right) \alpha^3 \csch^5 \left( \frac{\rho}{\sqrt{\alpha}} \right) \sinh^8 \left( \frac{\rho}{2 \sqrt{\alpha}} \right)}{l^4} \partial_{rr \rho} u (r, \rho) = 0
\]  

(A.5)

After some algebra the above equation takes the following elegant form,

\[
- \frac{\partial}{\partial \rho} \left\{ \frac{2 \Omega^2 l^2}{(l^2 M - r^2) \cosh \left( \frac{\rho}{2 \sqrt{\alpha}} \right)} u (r, \rho) + \frac{3 \left[ 1 - 2 \cosh^2 \left( \frac{\rho}{2 \sqrt{\alpha}} \right) \right]}{\sqrt{\alpha} \sinh \left( \frac{\rho}{\sqrt{\alpha}} \right)} \right\}
\]

\[
- \frac{2 \cosh \left( \frac{\rho}{2 \sqrt{\alpha}} \right) \frac{\partial^2 u (r, \rho)}{\partial \rho^2} + \frac{2 \frac{\partial}{\partial r} \left[ r (r^2 M - r^2) \frac{\partial u (r, \rho)}{\partial r} \right]}{l^2 r \cosh \left( \frac{\rho}{2 \sqrt{\alpha}} \right)} }{l^4 r} = 0
\]  

(A.6)

This equation is fully integrable in the \( \rho \) coordinate. By setting the integration constant to zero and separating the variables with the Ansatz \( u (r, \rho) = f (r) y (\rho) \), we get the following two differential equations,

\[
\frac{4 \alpha^{3/2} \coth \left( \frac{\rho}{2 \sqrt{\alpha}} \right)}{l^2} \left( 3 \cosh \left( \frac{\rho}{\sqrt{\alpha}} \right) \frac{\partial y (\rho)}{\partial \rho} + \sqrt{\alpha} \sinh \left( \frac{\rho}{\sqrt{\alpha}} \right) \frac{\partial^2 y (\rho)}{\partial \rho^2} \right) + m y (\rho) = 0,
\]  

(A.7)

with \( m \) being the variable separation constant.

\[
\left( m - \frac{8 \alpha^2 \Omega^2}{l^2 M - r^2} \right) f (r) + \frac{8 \alpha^2}{l^4 r} \left( (l^2 M - 3r^2) \frac{\partial f (r)}{\partial r} + r (l^2 M - r^2) \frac{\partial^2 f (r)}{\partial r^2} \right) = 0.
\]  

(A.8)

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