Mechanics of incompressible test bodies moving in Riemannian spaces

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Communicated by: P. M. Mariano

In the present paper, we have discussed the mechanics of incompressible test bodies moving in Riemannian spaces with nontrivial curvature tensors. For Hamilton's equations of motion, the solutions have been obtained in the parametrical form and the special case of the purely gyroscopic motion on the sphere has been discussed. For the geodetic case when the potential is equal to zero, the comparison between the geodetic and geodesic solutions has been done and illustrated in details for the case of a particular choice of the constants of motion of the problem. The obtained results could be applied, among others, in geophysical problems, for example, for description of the movement of continental plates or the motion of a drop of fat or a spot of oil on the surface of the ocean (e.g., produced during some “ecological disaster”), or generally in biomechanical problems, for example, for description of the motion of objects with internal structure on different curved two-dimensional surfaces (e.g., transport of proteins along the curved biological membranes).

KEYWORDS
geodesics, geodetics, gyroscopic motion, incompressibility constraints, mechanics of infinitesimal test bodies, Riemannian spaces

MSC CLASSIFICATION
53A05; 53A10; 70E05; 70H06

1 | INTRODUCTION

In our previous papers,1-4 we have discussed in details different aspects of the mechanics of infinitesimal test bodies moving in Riemannian spaces with nontrivial curvature tensors. It is generally well known that for a given $n$-dimensional Riemannian space, its isometry group has dimension $k \leq n(n+1)/2$, whereas the maximal possible dimension is attained for spaces of constant curvature.5-7 That is mainly the reason why we have chosen as an instructive example in the present paper the most symmetrical case of the spherical surface which has simultaneously constant positive Gaussian and mean curvatures.

Let us also mention that there are plenty of microscopic models of continua that are dealing with material points with attached to them geometric objects (collective or internal degrees of freedom), for example, liquid crystals that can be described as continua of infinitesimal rods.8-10 Therefore, our infinitesimal test bodies are quite legitimate objects that can appear in mathematical modelling of many real-world situations. For instance, in description of the motion of some pollution regions (drops of fat or spots of oil spilled from damaged tankers during some “ecological disasters”) on the oceanic surface (modelled as relatively small two-dimensional bodies on the spherical surface) or in description of the
motion of objects with internal structure on different curved two-dimensional surfaces (e.g., transport of proteins along the curved biological membranes).

This work is a continuation of our recent research where we have investigated the motion of infinitesimal gyroscopes (rigid bodies) on such very interesting and instructive two-dimensional surfaces as Delaunay surfaces (spheres and cylinders as limiting cases of unduloids) of constant mean curvature\textsuperscript{11} or Mylar balloons.\textsuperscript{12} In the present article, we are generalizing the description to the situation of the incompressible test bodies for which apart from rotations also some deformation is allowed. Let us also describe the subject of our interest in the two-fold manner, that is, from the very beginning let us introduce some general formulation (independent of the particular form of the two-dimensional surface on which the test body is moving) and simultaneously illustrate the general procedure on the example of an incompressible test body moving on a two-dimensional spherical surface embedded into some three-dimensional Euclidean space.

Let us start with the introduction of the tripple \((M, \Gamma, g)\), where \(M\) is denoting a differential manifold (e.g., a two-dimensional sphere) endowed with some affine connection \(\Gamma\) and metric tensor \(g\) (they can be interrelated or not). For an infinitesimal affinely rigid (homogeneously deformable) test body, we have that \(x \in M\) represents the spatial position of the body “as a whole” (it is a remnant of the centre of mass position in the flat-space theory), whereas the internal configuration (additional variables attached at the spatial position of the body) of such a homogeneously deformable body is injected into the tangent space \(T_x M\) (microphysical space) where it can be identified with linear frames (ordered bases) \(e_A \in T_x M\). In this way, the configuration space of the infinitesimal homogeneously deformable test body moving in the physical space \(M\) is given by the manifold \(FM\) of linear frames in \(M\)

\[
Q = FM = \bigcup_{x \in M} F_x M,
\]

where \(F_x M\) denotes the manifold of linear frames in the tangent space \(T_x M\). Then, any system of local coordinates \(x^i\) introduced on \(M\) induces local coordinates \((x^i, e^A)\) on \(FM\) and local coordinates \((x^i, e^A)\) on \(FM\), which is the manifold of all linear coframes in \(M\).

From the mechanical point of view, we have that the action of the structural group \(GL(n, \mathbb{R})\) for our infinitesimal homogeneously deformable test body on \(Q = FM\) and \(Q^* = FM\) corresponds to “micromaterial” transformations that can be seen as the infinitesimal limit of the usual material transformations. This means that \(e \in F_x M\) is canonically identical with some linear isomorphism of \(\mathbb{R}^n\) onto \(T_x M\), whereas \(\tilde{e} \in F_x^* M\) is canonically identical with a linear isomorphism of \(T_x M\) onto \(\mathbb{R}^n\). In this way, \(\mathbb{R}^n\) (additionally equipped with the metric tensor \(\eta\)) plays the role of the micromaterial space (corresponding Lagrange variables) and \(T_x M\) plays the role of the “microphysical” space (corresponding Euler variables).

In the general case of unconstrained infinitesimal homogeneously deformable test bodies, \((x^i, e^A)\) are “good” independent (unconstrained) generalized coordinates. However, after some constraints are imposed, the quantities \(e^A\) are no longer independent and cannot be used as generalized coordinates. For example, in the case of incompressibility constraints, we have that\textsuperscript{4,13}

\[
\det [e_A] = \sqrt{\det [g_{ij}]/\det [\eta_{AB}]}.
\]

When the metric tensors \(g\) and \(\eta\) are both given with the help of identity matrices (the flat-space situation), then in the right-hand side of (2), we have simply 1. In our situation, the right-hand side can be simplified to \(\sqrt{\det [g_{ij}]}\) because the micromaterial space is chosen to be some Euclidean space \(\mathbb{R}^n\) with \(\eta_{AB} = \delta_{AB}\).

Nevertheless, some geometric techniques based on the use of orthonormal aholonomic reference frames may be developed. In this situation, we can choose on \(M\) some preestablished, fixed once and for all fields of linear orthonormal aholonomic frames \(E\) for which we have the following relation:

\[
g(E_A, E_B) = \eta_{ij} E_i^A E_j^B = \eta_{AB}.
\]

The dual coframes \(E = (\ldots, E^A, \ldots)\) are orthonormal with respect to the inverse (contravariant) metrics \(\tilde{g}\) and \(\tilde{\eta}\) defined on the differential manifold \(M\) and micromaterial space \(\mathbb{R}^n\) respectively, that is,

\[
\tilde{g}(E^A, E^B) = E^A_i E^B_j g_{ij} = \eta^{AB}.
\]
Their particular choice, both technically convenient and geometrically lucid, in most cases, is dictated by the structure of a given Riemannian space \((M, g)\). So, in this way, we obtain that in the very description of the considered problem is to some extent incorporated the information about the intrinsic geometry of the differential manifold \(M\) (defined by its first fundamental form in the situation when it is embedded into the higher-dimensional space), including the information about its curvature.

For example, for the considered situation of the two-dimensional sphere embedded into the three-dimensional Euclidean space, that is, \(S^2(0, R) \subset \mathbb{R}^3\), we can use the following parameterization (a particular choice of local coordinates \(X = (x, y, z)\) in the physical space \(M\):

\[
x = R \sin u \cos v, \quad y = R \sin u \sin v, \quad z = R \cos u, \quad x^2 + y^2 + z^2 = R^2,
\]

where \(R\) is the fixed radius of the sphere and \(u \in [0, \pi], v \in [0, 2\pi]\) are the Euler angles in \(\mathbb{R}^3\) (when \(u = 0\) corresponds to the “North Pole” and \(u = \pi\) corresponds to the “South Pole” of our sphere and the condition \(v = \text{const}\) defines the meridians). For the parameterization (5), we obtain that the first fundamental form \(I = \{E, F, G\}\) has the following components:

\[
E = g_{uu} = (X_u, X_u) = R^2, \quad F = g_{uv} = (X_u, X_v) = (X_v, X_u) = 0, \quad G = g_{vv} = (X_v, X_v) = R^2 \sin^2 u,
\]

where \(X_i\) denotes the derivative of \(X\) with respect to the local coordinate \(x^i\).

From the other side, the orthonormal aholonomic reference frames \(E_A\) can be chosen in the following way (dependent on the above-defined particular choice of local coordinates in the physical space \(M\)):

\[
E^\nu_{\ u} = \frac{1}{\sqrt{g_{uu}}} \frac{\partial}{\partial u} = \frac{1}{R} \frac{\partial}{\partial u}, \quad E^\nu_{\ v} = E^\nu_{\ u} = 0, \quad E^\nu_{\ v} = \frac{1}{\sqrt{g_{vv}}} \frac{\partial}{\partial v} = \frac{1}{R \sin u} \frac{\partial}{\partial v}.
\]

So, dynamically speaking, at any time instant \(t \in \mathbb{R}\), the body is instantaneously placed at the geometric point \(X(t) \in M\) and its internal configuration is described by linear frames \(e(t)_A = E(X(t)B \varphi(t)^B_A).\) In this way, instead of describing the motion in terms of time-dependent quantities \((x(t)^i, e(t)^j_A)\), we can describe it in terms of quantities \((x(t)^i, \varphi(t)^A_B)\).

### 2 | AFFINE VELOCITIES FOR INTERNAL MOTION

In our previous papers on the affine bodies,\(^4\)\(^{13-15}\) we have shown that apart from the quantities \(\varphi(t)\), it is also convenient to introduce the so-called affine velocities \(\Omega\) that are corresponding to the internal motion of our infinitesimal homogeneously deformable test body. They are defined in the comoving representation by the following equation:

\[
\frac{DE_A}{Dt} = e_B \hat{\varepsilon}^B_A.
\]

In order to write them explicitly, we must first find formulas for the covariant derivatives in terms of the quantities \(E\) and \(\varphi\), that is,

\[
\frac{DE_A}{Dt} = \frac{D}{Dt} (E_B \varphi^B_A) = \frac{DE_B}{Dt} \varphi^B_A + E_B \frac{d\varphi^B_A}{dt}.
\]

Let us now express the along-curve differentiation of \(E_B\) through its field-differentiation (which is well-defined because \(E\) is given as a global vector field on \(M\)), that is,

\[
\frac{DE_B}{Dt} = (\nabla_{E_B} \frac{dx^i}{dt}) \frac{dx^i}{dt} = (\nabla_{E_B} E^C_i) \frac{dx^i}{dt}.
\]

It is more convenient to use auxiliary aholonomic coefficients of our affine connection \(\Gamma\) with respect to the fields \(E\), that is,

\[
\nabla_{E_B} E_B = \Gamma_{BC}^A E_A.
\]

In this way, the usual holonomic coefficients of \(\Gamma\) with respect to coordinates \(x^i\) are given as

\[
\Gamma_{jk}^i = E^i_A \Gamma_{BC}^A E^B_j E^C_k + E^i_A E^A_{j,k} = E^i_A \Gamma_{BC}^A E^B_j E^C_k + \Gamma [E]^i_{jk}.
\]
where the second term denotes the teleparallelism connection induced by the fields \( E \), that is, \( \nabla_{[E]} E_A = 0 \). In other words, it is the only affine connection with respect to which all the fields \( E_A \) and their dual cofields \( E^A \) are parallel, whereas its curvature tensor vanishes and the corresponding torsion tensor is

\[
S[ E^i_j k ] = E^A_{[j,k]} = \frac{1}{2} E^A ( E_A^{j,k} - E_A^{k,j} ) .
\] (13)

For the sphere’s case, when the metric tensor’s components \( g_{ij} \) are defined by formulas in (6), we can calculate explicitly the Levi-Civita affine connection’s coefficients \( \Gamma_{ijk} \) with respect to coordinates \( x^i \) given by (5) as follows:

\[
\Gamma_{ijk} = \frac{1}{2} g^{im} ( g_{mj,k} + g_{mk,j} - g_{jk,m} ) ,
\] (14)

with only three of the eight components being nonzero, that is,

\[
\Gamma_{wv} = - \sin u \cos u , \quad \Gamma_{vw} = \Gamma_{uv} = \cot u .
\] (15)

Next, for the reference frames \( E_A \) defined by formulas in (7), the aholonomic coefficients \( \Gamma^{ABC} \) from (11) can be explicitly calculated as follows:

\[
\Gamma^{ABC} = \Lambda^E_{ijF} E^F_{[i,k]} E^E_{j,k} ,
\] (16)

with only two of the eight components being nonzero, that is,

\[
- \Gamma^{uv} = \Gamma^{vu} = \frac{1}{R} \cot u .
\] (17)

Finally, inserting expressions (10) to (12), (16), and (17) into expression (9), we obtain that the co-moving affine velocity \( \hat{\Omega} \) may be given as the sum of the “drift” (or “drive”) term \( \hat{\Omega}_{dr} \) which describes the time rate of the part of internal motion that is contained in the very fields \( E_A \) themselves, that is,

\[
\hat{\Omega}_{dr}^{A_B} = \varphi^{-1} C \hat{\chi}_{dr}^{C_D} \hat{\varphi}^{D_B} ,
\] (18)

where the matrix \( \hat{\chi}_{dr} \) can be defined with the help of the two-dimensional nonsingular skew-symmetric matrix \( S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \) for which \( \det S = 1 \) as follows:

\[
\hat{\chi}_{dr}^{C_D} = \Gamma^{C_DF} E^F \frac{dx^F}{dt} \cos u \frac{dv}{dt} S^C_D ,
\] (19)

and the “relative” term \( \hat{\Omega}_{rl} \) which refers to the part of internal motion performed with respect to the just passed prescribed reference frames \( E_A \), that is,

\[
\hat{\Omega}_{rl}^{A_B} = \varphi^{-1} C \frac{d\varphi^{B}}{dt} .
\] (20)

The shifting of indices in \( \hat{\Omega}_{rl}^{A_B} = \hat{\Omega}_{dr}^{A_B} + \hat{\Omega}_{rl}^{A_B} \) can be done with the help of the metric tensor \( \eta_{AB} \). We can also introduce its spatial representation, that is, \( \Omega^i_j = e^i_A \hat{\Omega}_{rl}^{A_B} e^B_j \), in which we can shift indices using the metric tensor \( g_{ij} \).

3 | TWO-POLAR DECOMPOSITION OF CONFIGURATIONS

In some situations (e.g., when the body is isotropic in the micromaterial space), it is convenient to use the two-polar decomposition of the configuration matrix, that is, for the incompressibility constraints, we have that \( \varphi = LDR^{-1} \), where \( L \) and \( R \) are orthogonal matrices and \( D \) is a diagonal one of the following form:

\[
L = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} , \quad D = \begin{bmatrix} e^i & 0 \\ 0 & e^{-i} \end{bmatrix} , \quad R = \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix} .
\] (21)
Of course, the incompressibility constraints may be described analytically as the requirement that the affine velocity should be traceless, that is,

\[ \text{Tr} \hat{\Omega} = \hat{\Omega}^A_A = \hat{\Omega}_{\text{dr}}^A_A + \hat{\Omega}_{\text{nl}}^A_A = 0, \]  

(22)

and really we can easily check that in the two-polar decomposition (21), we have

\[ \hat{\Omega}_{\text{dr}}^A_A = \varphi^{-1} C \hat{\chi}_{\text{dr}}^C D \varphi^D_A = \hat{\chi}_{\text{dr}}^C = \cos u \, \dot{v} \, S^C_C = 0, \]  

(23)

\[ \hat{\Omega}_{\text{nl}}^A_A = \varphi^{-1} C \varphi^C_A = \hat{\chi}_{\text{nl}}^A_A + D^{-1} C \hat{\chi}_{\text{nl}}^A_A - \hat{\theta}_{\text{nl}}^A_A = 0, \]  

(24)

because the comoving angular velocities of the left and right fictitious gyroscopes in the two-polar decomposition (21)

\[ \hat{\chi}_{\text{dr}} = L^{-1} \dot{L} = \dot{a} S^T, \quad \hat{\theta}_{\text{nl}} = R^{-1} \dot{R} = \dot{\beta} S^T \]  

(25)

are given by the skew-symmetric (therefore, also traceless) matrices \( S^T \), whereas

\[ D^{-1} \dot{D} = \begin{bmatrix} e^{-i} & 0 \\ 0 & e^{i} \end{bmatrix} \begin{bmatrix} e^{i} & 0 \\ 0 & -e^{-i} \end{bmatrix} \dot{\lambda} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \dot{\lambda} \]  

(26)

is given by the traceless matrix as well.

Explicitly in the two-polar decomposition (21), we have that the components of the total affine velocity \( \hat{\Omega} \) are given by the following expression:

\[ \hat{\Omega} = \hat{\Omega}_{\text{dr}} + \hat{\Omega}_{\text{nl}} = R \left( D^{-1} \hat{\chi} D + D^{-1} \dot{D} - \hat{\theta}_{\text{nl}} \right) R^{-1}, \]  

(27)

where for the sphere’s case (5), the expression \( \hat{\chi} = \hat{\chi}_{\text{dr}} + \hat{\chi}_{\text{nl}} = (\dot{a} - \cos u \, \dot{v}) S^T \) contains the drift (19) and relative (25) terms of the \( L \)-rotation, while the \( R \)-rotation has no drift term \( \hat{\theta}_{\text{dr}} = 0 \) and only relative one \( \hat{\theta}_{\text{nl}} \). In this way, we see that the left fictitious gyroscope \( L \) alone absorbs the whole information about the geometry of the physical space \( M \) and leaves the right fictitious gyroscope \( R \) to be geometry-independent.

Let us note that from the incompressibility constraints, we can easily obtain the special case of the purely gyroscopic motion when in the two-polar decomposition (21), we suppose that \( \lambda = \beta = 0 \). In this case, the comoving affine velocity \( \hat{\Omega} \) becomes an angular velocity \( \hat{\omega} \) (which is given by the skew-symmetric matrix \( S \)), and for the sphere’s case (5), it is given as follows:

\[ \hat{\omega} = \hat{\omega}_{\text{dr}} + \hat{\omega}_{\text{nl}} = (\dot{a} - \cos u \, \dot{v}) S^T. \]  

(28)

## 4 D’ALEMBERT MODELS OF KINETIC ENERGY

Let us now consider the traditional d’Alembert method of deriving the kinetic energy for which the Lagrangian of our infinitesimal homogeneously deformable test body moving in a Riemannian space (e.g., sphere) can be given as follows:

\[ L = \frac{m}{2} \dot{g}_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} + \frac{1}{2} \dot{g}_{ij} \frac{D e^j_A}{D t} \frac{D e^i_B}{D t} J^{AB} - V(u, \lambda), \]  

(29)

where \( m \) denotes the mass of the infinitesimal test body, \( J \in \mathbb{R}^n \otimes \mathbb{R}^n \) is the symmetric and positively definite micromaterial inertial tensor (postulated as something primary) describing the internal properties of our infinitesimal test body, whereas \( V \) is some potential term well-suited to the geometry of the physical space \( M \) (e.g., sphere) and it is dependent only on the translational degree of freedom \( u \) and the deformation invariant \( \lambda \) from the two-polar decomposition (21) (therefore, the variables \( v, \alpha, \) and \( \beta \) are cyclic coordinates, i.e., they do not occur explicitly in the corresponding equations of motion).

In the above expression, the total kinetic energy \( T \) is postulated as the sum of the translational part which for the sphere’s case (5) has the following form:

\[ T_\text{tr} = \frac{m R^2}{2} \left[ \ddot{u}^2 + \sin^2 u \, \dot{v}^2 \right], \]  

(30)
and the internal part which using (8) can be rewritten as follows:

\[ T_{\text{int}} = \frac{1}{2} G[e]_{AB} \hat{\Omega}^A_C \hat{\Omega}^B_D J^{CD}, \]  

(31)

where

\[ G[e]_{AB} = g_{ij} e^i_A e^j_B = g_{ij} E^i_C E^j_D \varphi^C_A \varphi^D_B = \eta_{CD} \varphi^C_A \varphi^D_B \]  

(32)
is the Green deformation tensor defined in the micromaterial space \( \mathbb{R}^n \).

In the two-polar decomposition (21), we would have that

\[ G[e] = \varphi^T \varphi = R D^2 R^{-1} = \text{Id}_2 \cosh(2\lambda) + \Lambda(2\beta) \sinh(2\lambda), \]  

(33)

where \( \text{Id}_2 \) is the identity matrix and

\[ \Lambda(\cdot) = \begin{bmatrix} \cos(\cdot) & \sin(\cdot) \\ \sin(\cdot) & -\cos(\cdot) \end{bmatrix} \]

is traceless.

Therefore, using (27), (33), and supposing that our infinitesimal test body is isotropic in the micromaterial space, that is, in two dimensions, we have that \( J^{AB} = (I/2) \eta^{AB} \); therefore, \( \text{Tr}(\eta I) = (I/2) \text{Tr}(\text{Id}_2) = I \), we obtain that the expression for the internal kinetic energy in the sphere’s case (5) is as follows:

\[ T_{\text{int}} = \frac{L}{4} \text{Tr} \left( \hat{\Omega}^T G[e] \hat{\Omega} \right) = \frac{L}{4} \text{Tr} \left( \left( \dot{D} - D \ddot{\varphi} + \hat{\varphi} \right) \left( \dot{D} + D \ddot{\varphi} - D \hat{\varphi} \right) \right) 
\]

\[ = \frac{L}{4} \text{Tr} \left( \dot{D} \right) - \frac{L}{4} \text{Tr} \left( D \ddot{\varphi} \right) + \frac{L}{2} \text{Tr} \left( D \ddot{\varphi} \right) \]  

(34)

For the special case of the purely gyroscopic motion (28), we obtain that the Green deformation tensor is equal to the metric tensor in the micromaterial space, that is, \( G[e] = \eta \); the internal kinetic energy (34) is reduced to the expression \((I/2)(\dot{a} - \cos u \dot{v})^2\); therefore, the Lagrangian (29) can be written as follows:

\[ L = \frac{m R^2}{2} [\dot{u}^2 + \sin^2 u \dot{v}^2] + \frac{L}{2} (\dot{a} - \cos u \dot{v})^2 - V(u). \]  

(35)

Equivalently, we can rewrite the total kinetic energy in the following form:

\[ T = T_u + T_{\text{int}} = \frac{m}{2} G(q)_{ij} \frac{dq^i}{dt} \frac{dq^j}{dt}, \]  

(36)

where the generalized coordinates are ordered as \( \{q^i\} = \{u, v, \alpha, \beta, \lambda\} \) and the metric-like matrix \( G(q)_{ij} \) for the sphere’s case (5) is given as follows:

\[ G(q)_{ij} = \begin{bmatrix} R^2 & 0 \\ 0 & M_3 \end{bmatrix}, \]  

(37)

with \( 0_3 \) being the three-dimensional zero vector and the matrix \( M_3 \) is given as

\[ \begin{bmatrix} R^2 \sin^2 u + \frac{L}{m} \cosh(2\lambda) \cos^2 u & -\frac{L}{m} \cosh(2\lambda) \cos u & \frac{L}{m} \cos u \\ -\frac{L}{m} \cosh(2\lambda) \cos u & \frac{L}{m} \cosh(2\lambda) & -\frac{L}{m} \cos u \\ \frac{L}{m} \cos u & -\frac{L}{m} & \frac{L}{m} \cosh(2\lambda) \end{bmatrix}. \]  

(38)

The inverse matrix to (37) has the following form:

\[ G(q)^{ij} = \begin{bmatrix} \frac{1}{R^2} & 0_3 & 0 \\ 0_3 & M_3^{-1} & 0_3 \\ 0 & 0_3 & \frac{m}{\cosh(2\lambda)} \end{bmatrix}, \]  

(39)
with the matrix $M_3^{-1}$ given as

$$
\left[
\begin{array}{ccc}
\frac{1}{R^2 \sin^2 u} & \cos u/R^2 \sin^2 u & 0 \\
\cos u/R^2 \sin^2 u & \frac{\cos^2 u}{R^2 \sin^2 u} + \frac{m \cosh(2\lambda)}{I \sinh^2(2\lambda)} & \frac{m}{I} \frac{1}{\sinh^2(2\lambda)} \\
0 & \frac{m}{I} \frac{1}{\sinh^2(2\lambda)} & \frac{m \cosh(2\lambda)}{I \sinh^2(2\lambda)}
\end{array}
\right].
$$

(40)

From expressions (37) and (39), we can easily obtain the form of the Legendre transformation $p_i = \partial L/\partial \dot{q}_i = mG(q)_i \dot{q}_i^i$, where

$$
p_u = mR^2 \dot{u}, \quad p_v = \left[mR^2 \sin^2 u + I \cosh(2\lambda) \cos^2 u\right] \dot{v} - I \cosh(2\lambda) \cos u \dot{a} + I \cos u \dot{\beta},
$$

(41)

$$
p_a = -I \cosh(2\lambda) \cos u \dot{v} + I \cosh(2\lambda) \dot{a} - I \dot{\beta}, \quad p_\beta = I \cos u \dot{v} - I \dot{a} + I \cosh(2\lambda) \dot{\beta}, \quad p_\lambda = I \cosh(2\lambda) \dot{\lambda},
$$

(42)

and the inverse Legendre transformation $\dot{q}_i^i = (1/m)G(q)^{ij}p_j$, where

$$
\dot{u} = \frac{p_u}{mR^2}, \quad \dot{v} = \frac{p_v + \cos u \ p_a}{mR^2 \sin^2 u},
$$

(43)

$$
\dot{a} = \frac{\cos u (p_v + \cos u \ p_a)}{mR^2 \sin^2 u} + \frac{\cosh(2\lambda)p_a + \rho}{I \sinh^2(2\lambda)}, \quad \dot{\beta} = \frac{p_a + \cosh(2\lambda) \rho}{I \sinh^2(2\lambda)}, \quad \dot{\lambda} = \frac{p_\lambda}{I \cosh(2\lambda)}.
$$

(44)

Finally, denoting that

$$
\dot{a} - \cos u \dot{v} = \frac{\cosh(2\lambda)p_a + \rho}{I \sinh^2(2\lambda)}
$$

(45)

and substituting (43) to (45) into the expression for the Lagrangian with the translational (30) and internal (34) kinetic energies, we obtain the Hamiltonian $H(q, p) = T(q, p) + V(u, \lambda)$ with the total kinetic energy given as follows:

$$
T(q, p) = \frac{p_u^2}{2mR^2} + \frac{(p_v + \cos u \ p_a)^2}{2mR^2 \sin^2 u} + \frac{p_\beta^2}{2I \cosh(2\lambda)} + \frac{(p_a + \rho)^2}{8I \cosh^2 \lambda} + \frac{(p_a - \rho)^2}{8I \cosh^2 \lambda}.
$$

(46)

Again, for the purely gyroscopic motion ($\lambda = \beta = 0$), we have that

$$
T(q, p) = \frac{p_u^2}{2mR^2} + \frac{(p_v + \cos u \ p_a)^2}{2mR^2 \sin^2 u} + \frac{p_\beta^2}{2I}.
$$

(47)

### 5. SOME CONVENIENT MODELS OF POTENTIALS

Let us consider the following convenient choices of the separable potentials $V(u, \lambda) = V_u(u) + V_\lambda(\lambda)$. We suggest the first term $V_u(u)$ to be well suited to the geometry of the considered problem, for example, for the sphere’s case, we have that

$$
V_u(u) = f(u) \det \left[ g^{ij} \right] = \frac{f(u)}{R^2 \sin^2 u},
$$

(48)

where $f(u)$ is some function of the variable $u$. Therefore, for the different choices of the function $f(u)$,

$$
f_1(u) = \frac{R^2 x_1}{2m}, \quad f_2(u) = \frac{R^2 x_2 \cos u}{m}, \quad f_3(u) = \frac{R^2 x_3 \cos^2 u}{2m},
$$

(49)

where $x_i$ are some constants, we obtain a class of potentials $V_u(u)$ built of trigonometric functions, that is,

$$
V_1(u) = \frac{x_1}{2mR^2 \sin^2 u}, \quad V_2(u) = \frac{x_2 \cos u}{mR^2 \sin^2 u}, \quad V_3(u) = \frac{x_3 \cos^2 u}{2mR^2 \sin^2 u}.
$$

(50)
Similarly, we can also consider a certain class of potentials $V_i(\lambda)$ built of hyperbolic functions ($\sigma_i$ are some constants), that is,

$$V_1(\lambda) = \frac{\sigma_1}{2I \cosh(2\lambda)}, \quad V_2(\lambda) = \frac{\sigma_2}{8I \sinh^2 \lambda}, \quad V_3(\lambda) = \frac{\sigma_3}{8I \cosh^2 \lambda}.$$  

(51)

Finally, for both classes (50) and (51), we obtain the resulting effective shifting in the canonical momenta in the expression for the Hamiltonian function as follows:

$$H(q,p) = \frac{p_u^2}{2mR^2} + \frac{p_\lambda^2 + \sigma_1}{2I \cosh(2\lambda)} + \frac{(p_a + p_{\beta})^2 + \sigma_2}{8I \sinh^2 \lambda} + \frac{(p_a - p_{\beta})^2 + \sigma_3}{8I \cosh^2 \lambda}.$$  

(52)

6 | HAMILTON’S EQUATIONS OF MOTION

In Hamiltonian mechanics, the time evolution of the classical physical system is defined by the Hamilton’s equations, that is, for a given Hamiltonian function (52), we obtain that the equations of motion can be calculated as follows:

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad \dot{q}^i = \frac{\partial H}{\partial p_i}.$$  

(53)

The second part of (53) is essentially equivalent to the inverse Legendre transformations (43) and (44), but the first part produces the following expressions:

$$\dot{p}_u = -\frac{\partial}{\partial u} \left[ \frac{p_u^2 + \sigma_1}{2mR^2} + \frac{(p_a + p_{\beta})^2 + \sigma_2}{8I \sinh^2 \lambda} + \frac{(p_a - p_{\beta})^2 + \sigma_3}{8I \cosh^2 \lambda} \right],$$  

(54)

$$\dot{p}_\lambda + \frac{p_u^2}{2I} \frac{\partial}{\partial \lambda} \left[ \frac{1}{\cosh(2\lambda)} \right] = -\frac{\partial}{\partial \lambda} \left[ \frac{\sigma_1}{2I \cosh(2\lambda)} + \frac{(p_a + p_{\beta})^2 + \sigma_2}{8I \sinh^2 \lambda} + \frac{(p_a - p_{\beta})^2 + \sigma_3}{8I \cosh^2 \lambda} \right],$$  

(55)

where $\dot{p}_u = 0$, $\dot{p}_\lambda = 0$, $\dot{p}_{\beta} = 0$, that is, $p_u = mC_2$, $p_\lambda = IC_3$, and $p_{\beta} = IC_4$ are constants of motion and $C_i$ are some integration constants.

Now, the left-hand sides of (54) and (55) (understood as some functions of $u$ and $\lambda$) can be rewritten with the help of (43) and (44) as follows:

$$\frac{p_u \partial p_u}{mR^2 \partial u} = \frac{\partial}{\partial u} \left[ \frac{p_u^2}{2mR^2} \right], \quad \frac{p_\lambda \partial p_\lambda}{I \cosh(2\lambda) \partial \lambda} + \frac{p_u^2}{2I} \frac{\partial}{\partial \lambda} \left[ \frac{1}{\cosh(2\lambda)} \right] = \frac{\partial}{\partial \lambda} \left[ \frac{p_u^2}{2I \cosh(2\lambda)} \right].$$  

(56)

Combining the above expressions with the right-hand sides of (54) and (55) and performing the integration, we obtain $p_u$ and $p_\lambda$ as the some functions of the variables $u$ and $\lambda$ respectively, that is,

$$p_u^2(u) = C_1^2 - \frac{p_u^2 + \sigma_1 + 2 \cos u (p_a p_a + \kappa_2) + \cos^2 u (p_a^2 + \kappa_3)}{\sin^2 u},$$  

(57)

$$\frac{p_\lambda^2(\lambda)}{\cosh(2\lambda)} = C_5^2 - \frac{\sigma_1}{\cosh(2\lambda)} - \frac{(p_a + p_{\beta})^2 + \sigma_2}{4 \sinh^2 \lambda} - \frac{(p_a - p_{\beta})^2 + \sigma_3}{4 \cosh^2 \lambda},$$  

(58)

where $C_1$ and $C_5$ are integration constants. Substituting (57) and (58) into (52), we obtain that the energy of the considered system is also a constant of motion, that is,

$$E = \frac{C_1^2}{2mR^2} + \frac{C_5^2}{2I}.$$  

(59)
Next, let us express the first and last expressions in (43) and (44) as follows:

\[
\frac{mR^2 du}{p_u(u)} = \frac{I \cosh(2\lambda) d\lambda}{p_\lambda(\lambda)} = \frac{dt}{1}.
\]

(60)

Substituting (57) and (58) into the above expression (60), we obtain that

\[
t(u) = mR^2 \int \frac{\sin u \ du}{\sqrt{A_1 - 2A_2 \cos u - A_3 \cos^2 u}}.
\]

(61)

\[
t(\lambda) = I \int \frac{\cosh(2\lambda) \sinh(2\lambda) \ d\lambda}{\sqrt{C_5^2 \cosh^3(2\lambda) - B_1 \cosh^2(2\lambda) - B_2 \cosh(2\lambda) + \sigma_1}}.
\]

(62)

where

\[
A_1 = C_1^2 - p_\alpha^2 - \kappa_1, \ A_2 = p_\alpha \kappa_2 + \kappa_2, \ A_3 = C_1^2 + p_\alpha^2 + \kappa_2
\]

(63)

\[
B_1 = p_\alpha^2 + p_\beta^2 + \sigma_1 + \frac{\sigma_2 + \sigma_3}{2}, \ B_2 = C_5^2 + 2p_\alpha p_\beta + \frac{\sigma_2 - \sigma_3}{2}.
\]

(64)

Changing the variables in the first integral to \(x = -\cos u\) and in the second one to \(y = \cosh(2\lambda)\), we can rewrite (61) and (62) as follows:

\[
t(x) = mR^2 \int \frac{dx}{\sqrt{A_1 + 2A_2 x - A_3 x^2}}, \quad t(y) = I \int \frac{y dy}{\sqrt{C_5^2 y^3 - B_1 y^2 - B_2 y + \sigma_1}}.
\]

(65)

In this way, we parametrically represent the time variable \(t\) through the translational variable \(x\) (respectively \(u\)) or deformational one \(y\) (respectively \(\lambda\)). Therefore, inverting the obtained functions \(t(x), t(y)\) after the proper integration of the expressions in (65), we finally obtain two solutions \(x(t)\) and \(y(t)\) (or equivalently \(u(t)\) and \(\lambda(t)\)) of our equations of motion (53).

The time dependency of the other three angle variables \(v, \alpha, \beta\) can be obtain after substituting of (60) into the corresponding expressions for their time derivatives from (43) and (44) and integrating them with respect to \(x\) and \(y\) as follows:

\[
v(x) = \int \frac{p_v - p_\alpha x}{1 - x^2} \ dx \sqrt{A_1 + 2A_2 x - A_3 x^2}.
\]

(66)

\[
\alpha(x, y) = \int \frac{p_\alpha x - p_v}{1 - x^2} \ dx \sqrt{A_1 + 2A_2 x - A_3 x^2} + \int \frac{p_\alpha y + p_\beta}{2(y^2 - 1)} \ y dy \sqrt{C_5^2 y^3 - B_1 y^2 - B_2 y + \sigma_1}.
\]

(67)

\[
\beta(y) = \int \frac{p_\beta y + p_\alpha}{2(y^2 - 1)} \ y dy \sqrt{C_5^2 y^3 - B_1 y^2 - B_2 y + \sigma_1}.
\]

(68)

Then, after the proper integration of the expressions (66) to (68) and substituting the previously obtained functions \(x(t)\) and \(y(t)\) into the resulting expressions, we obtain the other three solutions \(v(t), \alpha(t), \beta(t)\) of the equations of motion.

Let us note that even in the simplest case of the spherical surface for the internal dynamics, we are obtaining that the second of the solutions in (65), (67), and (68) are expressed through the special functions (i.e., incomplete elliptic integrals and Jacobi elliptic functions). The detailed analysis of the obtained solutions will be done in the following paper on this subject.

Nevertheless, we can notice that the translational part of the motion, that is, the first of the solutions in (65) and (66), is influenced only by the degrees of freedom connected to the left fictitious gyroscope in our two-polar decomposition, that is, the coefficients in the corresponding integrals are depending only on the constant of motion \(p_\alpha\) and are independent of the internal deformation \(\lambda\) and the constant of motion \(p_\beta\). Therefore, in order to study qualitatively the translational motion of our incompressible test body on the spherical surface, we can consider the very interesting special case of the purely gyroscopic motion described in the next section.
7 | SPECIAL CASE OF PURELY GYROSCOPIC MOTION

For the particular situation of the purely gyroscopic motion \((\lambda = \beta = 0)\), we obtain that for the sphere’s case using (47) the Hamiltonian is given as follows:

\[
H(q, p) = \frac{p_v^2}{2mR^2} + \frac{(p_v + \cos u p_a)^2}{2mR^2 \sin^2 u} + \frac{p_a^2}{2I} + \frac{x_1 + 2x_2 \cos u + x_3 \cos^2 u}{2mR^2 \sin^2 u}.
\]  

Then, Hamilton’s equations of motion (53) can be written as follows:

\[
\dot{p}_v = -\frac{\partial}{\partial u} \left[ \frac{p_v^2 + x_1 + 2 \cos u (p_v p_a + x_2) + \cos^2 u (p_a^2 + x_3)}{2mR^2 \sin^2 u} \right],
\]

\[
\dot{p}_a = \frac{p_a}{mR^2}, \quad \dot{v} = \frac{p_v + \cos u p_a}{mR^2 \sin^2 u}, \quad \dot{u} = \frac{\cos u (p_v + \cos u p_a)}{mR^2 \sin^2 u} + \frac{p_a}{I},
\]

where \(\dot{p}_v = \dot{p}_a = 0\), that is, \(p_v = mC_2, p_a = IC_3\). Performing the corresponding integration of (70), we obtain that

\[
p_a^2(u) = C_1^2 - \frac{p_v^2 + x_1 + 2 \cos u (p_v p_a + x_2) + \cos^2 u (p_a^2 + x_3)}{\sin^2 u}
\]

and therefore, instead of (59), we have that

\[
E = \frac{C_1^2}{2mR^2} + \frac{I}{2} C_3^2,
\]

whereas the parametric solutions of the equations of motion are written as follows:

\[
t(x) = mR^2 \int \frac{dx}{\sqrt{A_1 + 2A_2 x - A_3 x^2}},
\]

\[
v(x) = \int \frac{p_v - p_a x}{1 - x^2} \frac{dx}{\sqrt{A_1 + 2A_2 x - A_3 x^2}},
\]

\[
a(x) = C_3 t(x) + \int \frac{p_a x - p_v}{1 - x^2} \frac{x dx}{\sqrt{A_1 + 2A_2 x - A_3 x^2}},
\]

where \(A_i\) are given by the expressions (63).

8 | COMPARISON OF GEODETICS AND GEODESICS

In order to illustrate the general scheme on the example of the geodetic gyroscopic motion on a sphere and compare the obtained solutions with the corresponding geodesics \((\alpha = 0)\) on the sphere, let us make some additional assumption about the considered system.

Let us suppose that we have the infinitesimal test body of the unit mass \(m = 1\) moving on the unit sphere \(R = 1\) and consider the special physical case when its internal inertia is interrelated with its mass, that is, \(I = mR^2 = 1\). Apart from that, let us also suppose that the translational motion of such an infinitesimal gyroscope on the sphere is performed with the unit speed, that is, \(g_{uu} \dot{u}^2 + g_{uu} \dot{v}^2 = 1\); therefore, the constant \(C_1 = 1\). Moreover, let us consider the particular value of the constant \(C_2 = 0\) that is corresponding to the geodesics that are the great circles on the sphere with \(v = \text{const}\), that is, \(\dot{v} = 0\) and \(\dot{u} = 1\).

For the above-described special case of the purely gyroscopic motion on the sphere, we have that the constants (63) are expressed as follows: \(A_1 = 1, A_2 = 0, A_3 = 1 + C_3^2\), where \(C_3^2\) is describing the internal part of the motion of our infinitesimal gyroscope and can be related to its total energy \(E\) by the expression (73). For illustrative properties, let us also suppose that we are considering the situation when 25% of the total energy of the gyroscope are allocated into
the translational motion and 75% of the total energy are allocated into the internal motion, that is, \( C_3 = \pm \sqrt{3} \) and then \( E = 1/2 + 3/2 = 2 \); therefore, \( A_3 = 4 \).

Taking into account all the above assumptions, the parametric solutions (74) and (75) of the geodesic solutions of the equations of motion (when \( C_3 = 0, A_3 = 1 \)) are now given as follows:

\[
t(x) = \int \frac{dx}{\sqrt{1-x^2}} = \arcsin(x) + t_0, \quad v(x) = v_0, \tag{77}
\]

where \( t_0 \) and \( v_0 \) are constants of integration. From the graph of the above function (for \( t_0 = 0 \)) shown on the Figure 1, we see that for the discussed particular kind of geodesics with \( v = \text{const} \) (i.e., the meridians that are given as great circles on the sphere) the whole range of the variable \( x \) from \( x_{\text{min}} = -1 \) to \( x_{\text{max}} = 1 \) (the ultimate values are corresponding to the North and South Poles respectively) is realized during the motion when time is going from \( t_{\text{min}} = -\pi/2 \) to \( t_{\text{max}} = \pi/2 \) (half of the period of the corresponding function \( x(t) = \sin(t) \)). Therefore, within the time interval \( t \in [-\pi/2, \pi/2] \), we can invert the function \( t(x) \) and obtain that the considered geodesics (meridians) are defined by the following functions:

\[
x(t) = -\cos u(t) = \sin (t - t_0), \quad v(t) = v_0, \quad t \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]. \tag{78}
\]

From the other side, the corresponding parametric solution (74) for the geodetic equations of motion is given as follows:

\[
t(x) = \int \frac{dx}{\sqrt{1-4x^2}} = \frac{1}{2} \arcsin(2x) + t_0. \tag{79}
\]

From the graph of the solution (for \( t_0 = 0 \)) presented on the Figure 2, we can see that the range of the variable \( x \) is now restricted to the interval \([-0.5, 0.5]\), which corresponds to the range of the time variable from \( t_{\text{min}} = -\pi/4 \) to \( t_{\text{max}} = \pi/4 \), that is, again we consider the half of the period of the following solution:

\[
x(t) = -\cos u(t) = \frac{1}{2} \sin [2(t - t_0)], \quad v(t) = v_0, \quad t \in \left[ -\frac{\pi}{4}, \frac{\pi}{4} \right]. \tag{80}
\]

**FIGURE 1** The values of function \( t(x) \) for geodesics on sphere for \( t_0 = 0 \): the whole range of variable \( x \in [-1, 1] \) is allowed [Colour figure can be viewed at wileyonlinelibrary.com]

**FIGURE 2** The values of function \( t(x) \) for geodetics on sphere for \( t_0 = 0 \): the range of variable \( x \) is restricted to \( x \in [-0.5, 0.5] \) [Colour figure can be viewed at wileyonlinelibrary.com]
In order to obtain the complete picture of the translational part of the geodetic motion, we need to study also the dependency of the other translational variable \( x \) which is given as follows (for \( C_3 = \pm \sqrt{3} \)):

\[
v_±(x) = \pm \sqrt{3} \int \frac{x}{x^2 - 1} \frac{dx}{\sqrt{1 - 4x^2}} = \arctan \left( \frac{\sqrt{1 - 4x^2}}{\pm \sqrt{3}} \right) + v_0. \tag{81}\n\]

From the graph of the obtained solution \( v_+(x) \) (for \( v_0 = 0 \)) shown on the Figure 3, we can deduce that the geodetic motion is restricted not only in the variable \( x \) but also in the variable \( v \), that is, it is realized between the corresponding parallels \( x_{\min} = -0.5 \) (\( u_{\min} = \pi/3 \)) and \( x_{\max} = 0.5 \) (\( u_{\max} = 2\pi/3 \)) and corresponding meridians \( v_{\min} = 0 \) and \( v_{\max} = \pi/6 \) (half of the period).

In other words, the geodetic motion (for the particular values of the integration constants \( t_0 = 0 \) and \( v_0 = 0 \)) starts for the time instant \( t_{\min} = -\pi/4 \) on the geodesic (meridian) corresponding to \( v_{\min} = 0 \) on the level of the parallel \( u_{\min} = \pi/3 \), then deviates from the original geodesic in the direction of the higher values of the variable \( v \) up to the maximal deviation for the time instant \( t = 0 \) when it lands on the meridian \( v_{\max} = \pi/6 \) on the level of the equator (\( u = \pi/2 \)), and then it returns in the symmetrical way to the original geodesic (\( v_{\min} = 0 \)) on which it lands again for the time instant \( t_{\max} = \pi/4 \) on the level of the parallel \( u_{\max} = 2\pi/3 \).

The other half of the period is obtained when we consider the solution \( v_-(x) \) for \( C_3 = -\sqrt{3} \) shown on the Figure 4 that is taken from \( x_{\max} = 0.5 \) to \( x_{\min} = -0.5 \) for the time interval \( t \in [\pi/4, 3\pi/4] \). Glued together, both solutions \( v_+(x) \) produce a small circle on the sphere (its graph is shown on the Figure 5, where \( \Delta u = 2\pi/3 - \pi/3 = \pi/3 \) and \( \Delta v = \pi/6 + \pi/6 = \pi/3 \)). Therefore, our geodetics are given as planar curves (small circles) on the sphere corresponding to the geodesics that are given as great circles (meridians).

And finally, for the internal (rotational) variable \( \alpha \), we have that

\[
\alpha_±(x) = \pm \sqrt{3} \int \frac{1}{1-x^2} \frac{dx}{\sqrt{1-4x^2}} = \alpha_0 + \frac{1}{2} \left[ \arctan \left( \frac{(1+4x)}{\pm \sqrt{3 \left( 1 - 4x^2 \right)}} \right) - \arctan \left( \frac{(1-4x)}{\pm \sqrt{3 \left( 1 - 4x^2 \right)}} \right) \right], \tag{82}\n\]
where $a_0$ is a constant of integration. From the graph of the solution $a_+ (x)$ (for $a_0 = 0$) presented on the Figure 6, we see that during the geodesic motion, our infinitesimal gyroscope is rotating all the time in the same direction taking values from $a_{\text{min}} = -\pi/2$ for the time instant $t_{\text{min}} = -\pi/4$ up to $a_{\text{max}} = \pi/2$ for the time instant $t_{\text{max}} = \pi/4$ (half of the period). The other half of the period is obtained when we consider the solution $a_- (x)$ for $C_3 = -\sqrt{3}$ shown on the Figure 7 that is taken from $x_{\text{max}} = 0.5$ to $x_{\text{min}} = -0.5$ for the time interval $t \in [\pi/4, 3\pi/4]$. Glued together, both solutions $a_{\pm} (x)$ produce one complete revolution in the internal variable $\alpha$ (from $-\pi/2$ to $3\pi/2$) per one complete revolution in the translational variables $u$ and $v$ (shown on the Figure 5).

Key details of the above discussion on the geodesic motion of the infinitesimal gyroscope on the sphere are summarized in the following table (where in (80) to (82) for $t \in [-\pi/4, \pi/4]$, we take the constants $(t_0, v_0, a_0) =$...
(0, 0, 0) and \( C_3 = +\sqrt{3} \), whereas for \( t \in [\pi/4, 3\pi/4] \), we have \( (t_0, v_0, a_0) = (0, 0, \pi) \) and \( C_3 = -\sqrt{3} \):

| \( t \) | \( -\pi/4 \) | \( -\pi/8 \) | \( 0 \) | \( \pi/8 \) | \( \pi/4 \) | \( 3\pi/8 \) | \( \pi/2 \) | \( 5\pi/8 \) | \( 3\pi/4 \) |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| \( x \) | -0.5 | -0.35 | 0 | 0.35 | 0.5 | 0.35 | 0 | -0.35 | -0.5 |
| \( v \) | 0° | 22.2° | 30° | 22.2° | 0° | -22.2° | -30° | -22.2° | 0° |
| \( a \) | -90° | -40.9° | 0° | 40.9° | 90° | 139.1° | 180° | 220.9° | 270° |

9 | CONCLUSIONS

In the present paper, we have discussed the mechanics of incompressible test bodies moving in Riemannian spaces with nontrivial curvature tensors. From the very beginning, the general scheme has been illustrated with the help of the quite simple but very instructive example of a two-dimensional surface with constant positive Gaussian and mean curvatures, that is, a sphere, which is embedded into the three-dimensional Euclidean space. Next we have discussed the D’Alembert model of the kinetic energy in the two-polar decomposition of the configuration matrix with some convenient choices of the potential energy. As a result, Hamilton’s equations of motion have been formulated and their solutions in the parametric form have been obtained for the incompressible test bodies as well as for the special case of the purely gyroscopic motion. In order to illustrate the obtained solutions, the comparison of geodetics and geodesics has been performed and the influence of the internal (gyroscopic) degrees of freedom on the translational ones has been analyzed. We have shown that for geodesics given as great circles (meridians) on the sphere, the corresponding geodetics are given as small circles (planar curves) on the sphere (therefore, the motion is restricted in the plane of the translational variables \( u \) and \( v \)).

As a continuation of the presented work, we are planning to analyse in more detail in the following papers the internal part of the motion for which the second of the solutions in (65), (67), and (68) are expressed with the help of incomplete elliptic integrals and Jacobi elliptic functions, as well as consider the motion of incompressible test bodies on different and more irregular two-dimensional surfaces embedded into the three-dimensional Euclidean space, for example, other (apart from spheres) Delaunay and minimal surfaces of constant (including zero) mean curvature (cylinders, catenoids, helicoids, unduloids, nodoids, gyroids, etc), other (apart from spheres) algebraic surfaces of the second and fourth orders (ellipsoids, pseudo-spheres, tori, etc), or quite specific but very interesting from the geometrical point of view surface which is called the Mylar balloon.\(^{12,16,17}\)

ACKNOWLEDGEMENTS

This paper presents part of the results obtained during the realization of the joint research project 32 under the title “Mechanics of Test Bodies Moving on Curved Membranes” accomplished within the realm of the scientific cooperation between Bulgarian and Polish Academies of Sciences for the period 2018 to 2020.

CONFLICTS OF INTEREST

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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**How to cite this article:** Kovalchuk V, Gołubowska B, Rożko EE. Mechanics of incompressible test bodies moving in Riemannian spaces. *Math Meth Appl Sci*. 2020;43:9790–9804. https://doi.org/10.1002/mma.6651