Quasi-deterministic dynamics, memory effects, and lack of self-averaging in the relaxation of quenched ferromagnets

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We discuss the interplay between the degree of dynamical stochasticity, memory persistence and violation of the self-averaging property in the aging kinetics of quenched ferromagnets. We show that, in general, the longest possible memory effects, which correspond to the slowest possible temporal decay of the correlation function, are accompanied by the largest possible violation of self-averaging and a quasi-deterministic descent into the ergodic components. This phenomenon is observed in different systems, such as the Ising model with long-range interactions, including mean-field, and the short-range random field Ising model.

When computing thermodynamic properties one must, in principle, consider the full statistical-mechanical average, namely over the realizations of the stochastic trajectories, the initial conditions and, if present, over the quenched disorder distribution. However, if the sample has specific self-averaging properties, the latter two averages are not necessary because they are realised by the system itself in the thermodynamic limit. Restricting for the moment the discussion to clean samples, i.e. without quenched disorder, this occurs when the system is ergodic. After some time a large part of phase space is visited, and the memory of the initial condition is fully lost. In this case the fate of a thermodynamical transformation does not depend on the specific initial microstate belonging to the same macrostate.

The situation is more subtle when phase space breaks into ergodic components. In this case, if the initial state is well inside one of such components its memory cannot be deleted because the other cannot be accessed. For instance, considering a uniaxial ferromagnet, below the critical temperature $T_c$ there are two symmetry related components $\Omega_\pm$ and the associated order parameter, the magnetisation $M$, takes at equilibrium the two possible values $M_{\pm} = \pm M_{eq}$. A sample prepared with a positive $M(t = 0)$ will evolve inside $\Omega_+$, and $M(t)$ will always stay close to $M_+$. A different situation occurs when the system is initially on the boundary $\mathcal{B}$ between ergodic components. In ferromagnets, $\mathcal{B}$ is the set of configurations with $M \approx 0$, and this happens when the initial state is sampled from a high temperature ($T \geq T_c$) equilibrium state. Aging phenomena are often associated to systems whose dynamics remains on $\mathcal{B}$ for ever [1]. This is strictly true if the thermodynamic limit is taken from the onset, namely before taking time $t$ large. However, in all physical situations, one deals with a large but finite system. Therefore the initial state, due to thermal fluctuations, will have some offset $M(0)$ from $\mathcal{B}$ and one can ask how this may change the destiny of the system.

As we will discuss below, the answer is not unique and depends on the character of the dynamical evolution. Basically three possibilities exist. In the first case the system keeps staying close to $\mathcal{B}$ for ever [2]. Namely the representative point is not attracted by ergodic components, and the offset $M(0)$ does not amplify: the magnetisation stays close to zero, $M(t) \approx (M(t)) = 0$, hence self-averaging is at work and memory of the initial condition is washed out as fast as possible. In the opposite situation the system deterministically falls in the ergodic component selected by the sign of $M(0)$. In this case the offset $M(0)$ is strongly amplified and $M(t)$ grows as fast as possible. This process is associated to the longest possible memory of the initial condition and to the strongest violation of self-averaging. In between these two extreme cases there are intermediate situations, where the sample is slightly drifted towards the ergodic component. Here $M(0)$ is amplified, memory is retained and self-averaging is spoiled, but more softly than in the previous case.

In this paper we show that one can toggle between the cases above by considering different ferromagnetic models. In particular, we show that strong memory effects and self-averaging violations are found in the presence of long-range interactions even when they fall off sufficiently rapidly so that extensivity and additivity are preserved. A similar effect can be obtained in systems with short-range interactions by adding quenched disorder.

In order to set the stage with a specific example, let us start our discussion by considering the one-dimensional clean ferromagnet described by the Hamiltonian

$$H = -\frac{1}{2} \sum_{i,j} J(|i - j|) s_i s_j,$$

(1)
where \( s_i = \pm 1 \) are \( N \) Ising variables, and \( J(r) = \delta_{r,1} \) for nearest neighbors (nn) couplings, and \( J(r) = 1/r^{1+\sigma} \) in the case of long-range interactions. We will focus on the case \( \sigma > 0 \) where additivity and extensivity hold. The model has a ferromagnetic phase below a finite critical temperature \( T_c(\sigma) > 0 \) for \( \sigma < 1 \); it has a Kosterlitz-Thouless transition \( \lambda \) for \( \sigma = 1 \); finally, \( T_c = 0 \) for \( \sigma > 1 \). We will study the relaxation of the model with a non-conserved order parameter after a quench from \( T_i = \infty \) to a low \( T \). We consider Glauber dynamics where a random spin is reversed with probability \( w = (1+\exp(\Delta E/T))^{-1} \), where \( \Delta E \) is the energy difference due to the spin-flip.

Not only the static properties, also the non-equilibrium kinetics change crossing \( \sigma = 1 \). The process is characterised by coarsening\( \square \) with a domains size \( L(t) \) growing with a dynamical exponent \( \square \), \( \xi \equiv \langle x^2 \rangle \), for large \( X \) we have \( F(X) \sim -1/X^\sigma \). Therefore the closure time of a domain of initial size \( X(0) = L \) is \( t = L^2 \) with \( z = 1 + \sigma \) for \( \sigma \leq 1 \) and \( z = 2 \) for \( \sigma > 1 \). The difference between these two regimes is due to the deterministic force \( F(X) \), that affects the coarsening process in the former while it is irrelevant in the latter. For this reason they will be called convective and diffusive regimes, respectively.

These two regimes can be clearly distingushed by considering the fluctuating magnetisation \( \xi(t) = \sum_{i=1}^{n} s_i \), which is shown in Fig. 1 for systems prepared with a fixed condition \( \xi(0) / \sqrt{N} \) equal for all \( \sigma \)-values. In the convective regime \( \xi(t) \) asymptotically diverges and it typically has the same sign as \( \xi(0) \). In the diffusive regime it fluctuates around \( \xi(0) \). This means that the convective regime keeps memory of the initial condition, while the diffusive does not. This implies that decorrelation is slower in the first case and, actually, we will show in a moment that it occurs in the slowest possible way. Self-averaging with respect to initial conditions is therefore broken for \( 0 < \sigma \leq 1 \), while it holds for \( \sigma > 1 \).

With this example in mind, we now turn to a more general discussion. Let us consider the correlation function which, using a continuous picture for a scalar field \( \phi(x, t) \) in \( d \) dimensions, reads \( S(r; t_1, t_2) \equiv \langle \phi(x+r, t_1)\phi(x, t_2) \rangle \). Here \( t_2 > t_1 \) and \( \langle \cdots \rangle \) is the full non-equilibrium statistical average, namely taken over dynamical trajectories, initial conditions and quenched noise, if present. We focus on the scaling regime where the autocorrelation function \( C(t_1, t_2) = S(r=0; t_1, t_2) \) obeys \[ \xi(t_1, t_2) \propto |L(t_1)/L(t_2)|^\lambda \].

Under certain conditions, the exponent \( \lambda \) is known \( \square \) to satisfy the Fisher-Huse bounds, \( \frac{d}{2} \leq \lambda \leq d \).

The right inequality, \( \lambda \leq d \), is defined in \[ \square \] as a “suggestive bound” since it cannot be derived from first principles without resorting to some additional hypotheses which are difficult to control. A derivation of Eq. \( 1 \) is provided below. We indicate with \( u_t = \phi_l(\mathbf{q}, t_2) \) the Fourier transform of the field \( \phi(l, t_1) \) evaluated at the time \( t_1 \) during the \( l \)-th realization of the dynamics. Similarly we define \( \bar{u}_t = \phi_l(\mathbf{q}, t_2) \) at the time \( t_2 \). We can therefore define the scalar product as \[ \bar{u} \cdot \bar{v} \equiv \frac{1}{2N} \sum_{l} (u_l v_l^* + c.c.) = \frac{1}{2} \left[ S(q, t_1, t_2) + S^*(q, t_1, t_2) \right] \], where \( \bar{u} \cdot \bar{v} \equiv \frac{1}{2N} \sum_{l} (u_l v_l^* + c.c.) = \frac{1}{2} \left[ S(q, t_1, t_2) + S^*(q, t_1, t_2) \right] \]

FIG. 1. The fluctuating magnetization \( M(t) \) for a single realization starting from the same initial condition. The system size is \( N = 10^5 \) and the quench temperature is \( T = 0.1 \).

FIG. 2. \( C(t_1, t_2) \) is plotted against \( L(t_2) / L(t_1) - 1 \) for different \( \sigma \) after a quench (to \( T = 10^{-1} \) for \( \sigma = 0.8 \), \( T = 10^{-2} \) for \( \sigma = 1 \) and \( \sigma = 3 \)). System size is \( N = 4 \cdot 10^6 \). For \( \sigma = 3 \) and \( \sigma = 0.8 \) equal symbols with different colours correspond to different of \( t_1 \) (see key) to show collapse. The exact result for nn is shown with a dotted orange curve. The dashed straight lines are the decays \( x^{-\lambda} \) with \( \lambda = 1 \) and \( \lambda = 1/2 \).

\[ \square \]

\[ \square \]
\[ \langle \phi(q, t_1) \phi(-q, t_2) \rangle = \text{the Fourier transform of } S(r; t_1, t_2). \]

We can now apply the Cauchy-Schwarz inequality, \( |\vec{u} \cdot \vec{v}| \leq |u||v| \) and obtain

\[ \frac{1}{2} |S(q, t_1, t_2) + S^*(q, t_1, t_2)| \leq \sqrt{S(q, t_1)S(q, t_2)}, \tag{5} \]

where, for ease of notation, \( S(q, t) \equiv S(q, t, t) \). If we integrate over \( q \) we find

\[ C(t_1, t_2) \leq \frac{1}{V(2\pi)^d} \int dq \sqrt{S(q, t_1)S(q, t_2)}. \tag{6} \]

Using Eq. \( 2 \) and the scaling form \( S(q, t) = L^d(t)f(qL) \), with \( f(x) \approx 1 \) for \( x < 1 \) and \( f(x) \) negligibly small for \( x \gg 1 \), we find the lower bound of Eq. \( 3 \).

We now prove that the same inequality can be derived from the term \( q = 0 \) only of Eq. \( 5 \),

\[ S(0, t_1, t_2) \leq \sqrt{S(0, t_1)S(0, t_2)}. \tag{7} \]

Using the scaling form for \( S(q, t) \) (see below Eq. \( 11 \)) it is straightforward to rewrite the previous equation as

\[ S(0, t_1, t_2) \leq f(0)(L_1L_2)^{d/2}, \tag{8} \]

where we used the shorthand \( L_1 \equiv L_1(t) \), and similarly for \( L_2 \). The left-hand side of Eq.\( 6 \) can be worked out expressing the two-time correlation function as follows, \( C(t_1, t_2) = \frac{1}{V(2\pi)^d} \int dq S(q, t_1, t_2) = \frac{L_1^d}{V(2\pi)^d} \int dq F(qL_2, L_1/L_2) \), where we have used the scaling hypothesis \( S(q, t_1, t_2) = L_1^d F(qL_2, L_1/L_2) \), valid when both times \( t_1 \) and \( t_2 \) are in the scaling regime.

In the limit of large \( L_2 \) (i.e., of large \( t_2 \)) only wavevectors \( q < 1/L_2 \) contribute to the integral. If \( S(q \rightarrow 0, t_1, t_2) \) goes to a constant, which is the case for quenches below \( T_c \) or to \( T = 0 \), we can finally write

\[ C(t_1, t_2) \simeq \frac{1}{V(2\pi)^d} \frac{S(0, t_1, t_2)}{L_2^d}. \tag{9} \]

Using this relation and Eq. \( 5 \), we find \( C(t_1, t_2) \leq \text{const} \times \left( \frac{L_1}{L_2} \right)^{d/2} \) and the scaling form \( 3 \) gives \( \lambda \geq d/2 \). Therefore Eq. \( 7 \) is equivalent to the lower bound \( 3 \).

We remark that the lower limit in the Cauchy-Schwarz inequality is achieved when the vectors \( u_1 \) and \( v_1 \) are parallel. This can occur when, during the same realization of the dynamics, the configuration at time \( t_1 \) determines the subsequent one as also highlighted by the behavior of the magnetization. In fact, it is straightforward to obtain

\[ \langle M(t_1)M(t_2) \rangle = S(0, t_1, t_2), \]

and using Eqs. \( 10 \) and \( 11 \),

\[ \langle M(t_1)M(t_2) \rangle = \text{const} \times L_1^\lambda L_2^{d-\lambda}. \tag{10} \]

If, following Fisher and Huse \( 14 \), we assume that “forgetting of an initial bias appears unlikely” we can read out of Eq. \( 11 \) the upper bound of Eq. \( 3 \).

We should now consider the role of the different statistical averages. The full average \( \langle \cdots \rangle \) is taken over the stochastic trajectories, \( \langle \cdots \rangle_{tr} \); the initial condition, \( \langle \cdots \rangle_i \); and, if present, over the quenched disorder, \( \langle \cdots \rangle_q \). Let us consider, to begin with, a clean system. We can split the fluctuating magnetisation as \( M(t) = \langle M(t) \rangle_{tr} + \psi(t) \), where \( \psi(t) \) is the stochasticity left over after taking the partial averaging \( \langle M(t) \rangle_{tr} \), so that \( \langle \psi(t) \rangle_{tr} = 0 \). Then we have

\[ \langle M(t_1)M(t_2) \rangle = \langle \langle M(t_1) \rangle_{tr} \langle M(t_2) \rangle_{tr} \rangle_i + \langle \psi(t_1)\psi(t_2) \rangle_i \tag{11} \]

If we now fix \( t_1 \) and let \( t_2 \) diverge, \( \langle \psi(t_1)\psi(t_2) \rangle = \langle \psi(t_1)\psi(t_2) \rangle = 0 \) and Eq. \( 11 \) we obtain

\[ \langle \langle M(t_1) \rangle_{tr} \langle M(t_2) \rangle_{tr} \rangle_i \simeq L_2^{d-\lambda}. \tag{12} \]

Next we argue that, if the quench is made in a ferromagnetic phase, due to the presence of two ergodic components, for large \( t_1 \) it is sign \( M(t_1) = \text{sign} \langle M(t_1) \rangle \). This is well very observed for \( \sigma < 1 \), see Fig. \( 1 \) and indeed it is \( \text{sign} \langle M(t_1) \rangle_{tr} = \text{sign} \langle M(t_1) \rangle_{tr} \), therefore Eq. \( 12 \) (valid for \( t_1 \) fixed) amounts to

\[ \langle M(t) \rangle_{tr} \simeq L(t)^{d-\lambda}, \tag{13} \]

where we have denoted \( t_2 \) as \( t \) to ease the notation. Notice that the equation above is more general and applies to systems without a proper ferromagnetic phase, such as the 1d Ising model with \( \sigma > 1 \) or with nn, because in this case there is no development of magnetisation starting from a given state, see Fig. \( 1 \) and indeed it is \( \lambda = d \).

Equation \( 13 \) shows that the lowest possible decorrelation, \( \lambda = d/2 \), is accompanied by the fastest possible growth of the magnetization developed from an initial condition \( 17 \). Let us observe that such maximum growth is the one expected upon assuming a random arrangements of a number \( L^{-d} \) of domains of size \( L \) each contributing a magnetisation \( \sim L^d \). Eq. \( 13 \) for \( \lambda = d/2 \) then derives from the central limit theorem.

The result \( 13 \) implies also that there is breaking of self-averaging with respect to initial conditions if \( \lambda < d \), as reflected by the fact that, for large \( N \), the observable magnetisation does not attain its average value \( \lim_{N \rightarrow \infty} \langle M(t) \rangle = 0 \) unless the average over initial conditions is performed. The most severe self-averaging breakdown occurs when \( \lambda \) is at the lower bound in \( 3 \), whilst it is fully restored when it is at the upper bound.

Let us put these arguments to the test in different models, starting from the 1d model of Eq. \( 1 \). Let us recap what is known about \( \lambda \). For nn there is the exact result \( 17 \) \( \lambda = 1 \), namely the upper bound of Eq. \( 3 \) is saturated and self-averaging holds. For the long-range case studied in Ref. \( 20 \), there are two universality classes: the diffusive one associated to \( \lambda = 1 \), valid for \( \sigma > 1 \), and a convective one, valid for \( 0 < \sigma < 1 \) and characterized by \( \lambda = 1/2 \). This is shown in Fig. \( 2 \) where one sees that there are two distinct scaling functions: all models with \( \sigma > 1 \) collapse onto the nn (diffusive) case, while all those with \( \sigma < 1 \) superimpose on a \( \sigma \)-independent mastercurve identifying another universality class, the convective one.
different clearly that in the convective regime ($0 < \sigma \leq 1$) where $\lambda = d/2$ it is $\langle M(t) \rangle_{tr} \sim \sqrt{L(t)}$ while in the diffusive case ($\sigma > 1$ or nn) it is $\langle M(t) \rangle_{tr} \sim M(0)$, as expected after Eq. (13). Hence $\sigma = 1$ separates the two opposite situations in which the dynamics occurs on the boundary $\mathcal{B}$ of the ergodic components (for $\sigma > 1$) from the one where it deterministically sinks into such components (for $\sigma \leq 1$). We should stress that $T_c = 0$ is not a sufficient condition to have $\lambda = d$, as attested by the 2d XY-model where $\lambda \approx 1.17 < d = 2$ even if $T_c = 0$.

In our model, determinism can be ascribed to the presence of the force $F(X)$ in the convective case, whose effects can be understood from the following example. Suppose to have two close domains of sizes $t_1, t_2$, with $t_2$ slightly larger than $t_1$. In the diffusive case the average closure time of $t_1, t_2$, is slightly smaller than the one of $t_2$, $t_2$, but the probability that $t_1 < t_2$ is only slightly larger than 1/2. In the convective regime, instead, the dominance of the deterministic force makes a domain wall always move towards the closest one, so that $t_1$ is always smaller than $t_2$. This induces a memory effect, since domains which are eliminated have a larger probability to be anti-aligned with $M(t)$ and their removal further increase $M(t)$. Summarising, in the convective regime there is a reduced degree of stochasticity and an increased memory with respect to the diffusive one, and this is the physical origin of the saturation of $\lambda$ to the lower bound.

Let us now discuss the short-range ferromagnetic model in $d > 1$. In this case we have strict inequalities for any $d, d/2 < \lambda < d$ [22]. Hence self-averaging is spoiled, $\lambda < d$, in opposition to $d = 1$. This is because in $d > 1$ interfaces do not freely diffuse, there is a deterministic drift induced by the curvature. However the fate of the system is not fully determined by such deterministic force because the shape of the percolating cluster plays a major role in the subsequent dynamics [22]. Hence there is only a weak drift from $\mathcal{B}$ towards the ergodic components and $\lambda$ stays larger than $d/2$.

When long-range interactions are present, results in $d > 1$ are rare [24] and studies of $\lambda$ are almost absent [23]. However it is interesting that for the nn case in the limit $d \to \infty$, which corresponds to the, so to say, *longest possible range of interactions*, the mean-field, one has $\lambda \to d/2$ [22] and $M(t) \sim L(t)^{d/2}$ [11,12], as expected on the basis of our previous argument. In this limit there are no interfaces and, therefore such strong memory effects leading to $\lambda = d/2$ cannot be associated to the determinism of their motion, as in finite dimension. Instead, it can be observed that the mean-field amounts to an averaging procedure which makes the evolution, in a sense, more deterministic. Again, this reduction of the stochastic degree is perhaps the physical origin of the saturation of $\lambda$ to the lower bound of Eq. (3).

There is another well known limit in which phase-ordering has a similar character. This is the case of a vectorial order parameter $\phi(x,t)$ with a large number $\mathcal{N}$ of components and short-range interactions. In the $\mathcal{N} \to \infty$ limit (a model sometimes denoted also as *spherical model*) one finds [24] $\lambda = d/2$ for any $d$. By choosing an initially magnetised state it can be shown [27] that the magnetisation evolves deterministically as $M(t) \sim L(t)^{d/2}$, as expected after Eq. (13). It must be recalled that the large-$\mathcal{N}$ limit effectively amounts to replace $\phi^2$ with its mean value [26]. Then, similarly to mean-field, the model realises a sort of averaging where it smears the stochasticity and sets $\lambda$ to the minimum possible value.

Up to now we have only considered clean systems. It is now interesting to discuss the case with quenched disorder focusing, as a paradigm, on the Random Field Ising Model (RFIM). The RFIM Hamiltonian is given by Eq. (11), plus a contribution $-\sum_i h_i s_i$ due to a quenched random external field that in the following we will consider with zero average and bimodal distribution $h_i = \pm h$. We will focus on the nn case. In order to discuss the role of the different averages, as done before, we must now take into account that in this case also the quenched one $\langle \cdots \rangle_q$ comes into the game. Splitting the magnetisation as $M(t) = \langle M(t) \rangle_{tr,i} + \psi(t)$, similarly to what done previously for the clean case but where now $\langle \cdots \rangle_{tr,i}$ is a partial average taken over both dynamical trajectories and initial conditions, one can follow the same line of reasoning as before, arriving at the same results, replacing everywhere $\langle M(t) \rangle_{tr}$ with $\langle M(t) \rangle_{tr,i}$.

Let us start discussing the case with $d = 1$, for which some analytical arguments are available. The model is characterised [28] by a value of $\lambda$ at its minimum, $\lambda = 1/2$. Hence, one should expect $\langle M(t) \rangle_{tr,i} \sim L(t)^{1/2}$. In the inset of Fig. 3 we plot $\langle M(t) \rangle_{tr,i}$ versus the average size of domains $L(t)$ (which grows as $(\ln t)^2$). The result nicely confirms our expectation. In this case the growth
of $\langle M(t) \rangle_{tr,i}$ can be traced back to the fact that the sum of the random fields in a given quenched realisation is of order $N^{-1/2}$ and, hence, there is an explicit breaking of the up-down spin symmetry. Hence, here it is the random field which causes the deterministic fall into the ergodic components. Interestingly, this effect seems not to be limited to one dimension. For $d > 1$ the RFIM can only be studied numerically. For $d = 2$ one observes [28] that $\lambda = d/2 = 1$ is still at the lowest possible value, as for $d = 1$. This suggests that the mechanism found in $d = 1$ might be a general feature with random fields.

In conclusion, we have discussed the interplay between stochasticity, memory effects, ergodicity breaking and self-averaging, in the context of aging ferromagnetic systems quenched to a low temperature. Memory, encoded in the two-time correlation function in terms of the $\lambda$ exponent, is lost as fast as possible – compatibly with basic principles – when the upper bound of the Fisher-Huse inequality is met. In this case magnetisation does not develop regardless of the initial preparation of the system, $\langle M(t) \rangle_{tr} \simeq M(0)$, and there is no breaking of self-averaging namely, in a clean system, $\langle M(t) \rangle_{tr} = \langle M(t) \rangle$. Averaging over initial conditions is, in these cases, pointless. This occurs, for instance in the 1d Ising model with nn, or in the 2d, $O(2)$ model E [21]. In all the other cases, namely when $\lambda < d$, there is a violation of self-averaging which gets more severe as $\lambda$ approaches the lower bound $\lambda = d/2$ and memory extends in time. This, of course, does not necessarily implies that other observables may not self-average, but this cannot be taken for granted.

The arguments presented in this paper are rather general for systems quenched to a phase with ergodicity breaking. Therefore we expect them to apply also to long-range systems in $d > 1$. To our knowledge, the only study in this case is a preprint [24] where, however, the authors claim that in $d = 2$ the Fisher-Huse inequality is violated, a fact worth of further investigations.

The case of aging without ergodicity breaking, as in the case of a ferromagnet quenched to the critical temperature, is also another test bench where the relation between stochasticity, memory effects and self-averaging ought to be considered. In this case the Fisher-Huse lower bound generalises [15] to $\lambda \geq (d + \beta)/2$, where $\beta$ is an exponent characterizing the small $q$ behavior of the structure factor. It would be interesting to test if in this case it is still possible to relate the bounds on $\lambda$ to specific features of the dynamics.

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