On Repeated-Root Constacyclic Codes of Length $2^a mp^r$ over Finite Fields.

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Abstract

In this paper we investigate the structure of repeated root constacyclic codes of length $2^a mp^r$ over $\mathbb{F}_{p^s}$ with $a \geq 1$ and $(m, p) = 1$. We characterize the codes in terms of their generator polynomials. This provides simple conditions on the existence of self-dual negacyclic codes. Further, we gave cases where the constacyclic codes are equivalent to cyclic codes.

Keywords: Repeated-root Constacyclic codes, Negacyclic Codes, Self-dual Codes.

1 Introduction

Constacyclic codes over finite fields form a remarkable class of linear codes, as they include the important family of cyclic codes. Constacyclic codes also have practical applications as they can be efficiently encoded using simple shift registers. They have rich algebraic structures for efficient error detection and correction. This explains their preferred role in engineering. Repeated-root constacyclic codes, were first studied in 1967 by Berman [3], then by several authors such as Falkner et al [11] and Salagean [16]. Repeated-root cyclic codes were first investigated in the most generality in the 1990s by Castagnoli et al [4], and van Lint [17], where they showed that repeated-root cyclic codes have a concatenated construction, and are not asymptotically good. However, it turns out that optimal repeated-root constacyclic codes still exist. These motivate researchers to further study this class of codes.

Recently, Dinh, in a series of papers [8], [9] and [10], determined the generator polynomials of all constacyclic codes over $\mathbb{F}_q$, of lengths $2p^r$, $3p^r$ and $6p^r$. These results have been

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extended to more general code lengths. The generator polynomials of all constacyclic codes of length \( 2^a p^r \) over \( \mathbb{F}_{p^s} \) were given in [1]. The generator polynomials of all constacyclic codes of length \( lp^r \) over \( \mathbb{F}_q \) were characterized in [5], where \( l \) is a prime different from \( p \).

In this paper, we extend the main results of Batoul et al given in [2] and of Guenda and Gulliver given in [12] to constacyclic codes of length \( 2^a mp^r \) over \( \mathbb{F}_{p^s} \), where \( a \geq 1 \) and \( m \) is an odd integer with \( (m, p) = 1 \). The remainder of the paper is organized as follows. Some preliminary results are given in Section 2. In Section 3, the structure of generator of constacyclic codes of length \( mp^r \) is given using the generator polynomial of constacyclic codes of length \( m \). In Section 4, the structure of generator of constacyclic codes of length \( 2^a mp^r \) is given. Further, we gave cases where the constacyclic codes are equivalent to cyclic codes. It is well known that the only self-dual constacyclic codes over finite fields are either cyclic codes over fields with even characteristic or negacyclic codes. For that in Section 5, we give conditions on the existence of self-dual negacyclic codes of length \( 2^a mp^r \) over \( \mathbb{F}_{p^s} \), where \( p \) is odd.

2 Preliminaries

Let \( p \) be a prime number and \( \mathbb{F}_q \) the finite field with \( q = p^s \) elements. We note its group of units \( \mathbb{F}_q^* \).

The order of an element \( a \) in the multiplicative group \( \mathbb{F}_q^* \) is the least integer \( b \) such that \( a^b = 1 \) in \( \mathbb{F}_q^* \), we note \( b \) by \( \text{ord}_q(a) \). Let \( i \) be an integer with \( 0 \leq i < n \), the \( q \)-cyclotomic coset of \( i \) modulo \( n \) is the set

\[
C_i = \{i, iq, \ldots, iq^{l-1}\} \pmod{n},
\]

where \( l \) is the smallest positive integer such that \( iq^l \equiv i \pmod{n} \).

The minimal polynomial of \( \beta^i \) over \( \mathbb{F}_q \) is

\[
M_{\beta^i}(x) = \prod_{j \in C_i} (x - \beta^j),
\]

where \( C_i \) is the \( q \)-cyclotomic coset modulo \( n \) and \( \beta \) is a primitive element of \( \mathbb{F}_q \).

An \([n, k]\) linear code \( C \) over \( \mathbb{F}_{p^s} \) is a \( k \)-dimensional subspace of \( \mathbb{F}_{p^s}^n \). For \( \lambda \) in \( \mathbb{F}_q^* \), a linear code \( C \) of length \( n \) over \( \mathbb{F}_q \) is said to be constacyclic if it satisfies

\[
(\lambda c_{n-1}, c_0, \ldots, c_{n-2}) \in C, \text{ whenever } (c_0, c_1, \ldots, c_{n-1}) \in C.
\]

When \( \lambda = 1 \) the code is called cyclic, and when \( \lambda = -1 \) the code is called negacyclic. The Euclidean dual code \( C^\perp \) of \( C \) is defined as \( C^\perp = \{x \in \mathbb{F}_q^n; \sum_{i=1}^{n} x_i y_i = 0 \forall y \in C\} \). An interesting class of codes is the so-called self-dual codes. A code is called Euclidean self-dual if it satisfies \( C = C^\perp \). Note that the dual of a \( \lambda \)-constacyclic code is a \( \lambda^{-1} \)-constacyclic
code. A monomial linear transformation of $\mathbb{F}_q^n$ is an $\mathbb{F}_q$-linear transformation $\tau$ such that there exists scalars $\lambda_1, \ldots, \lambda_n$ in $\mathbb{F}_q^*$ and a permutation $\sigma \in S_n$ (the group of permutation of the set $\{1, 2, \ldots, n\}$) such that, for all $(x_1, x_2, \ldots, x_n) \in \mathbb{F}_q^n$, we have

$$\tau(x_1, \ldots, x_n) = (\lambda_1x_{\sigma(1)}, \lambda_2x_{\sigma(2)}, \ldots, \lambda_nx_{\sigma(n)}).$$

Two linear codes $C$ and $C'$ of length $n$ are called monomially equivalent if there exists a monomial transformation of $\mathbb{F}_q^n$, such that $\tau(C) = C'$. Here, whenever two codes are said to be equivalent it is meant that they are monomially equivalent. Let $C$ be a $\lambda$-constacyclic, then $C$ is an ideal of the quotient ring $R_n = \mathbb{F}_{p^s}[x]/(x^n - \lambda)$. It is well known that every $\lambda$-constacyclic code is generated by a unique polynomial of least degree. Such a polynomial is called the generator of the code and it is a divisor of $x^n - \lambda$. Therefore, there is a one-to-one correspondence between constacyclic codes of length $n$ over $\mathbb{F}_q$, and divisors of $x^n - \lambda$.

### 3 Constacyclic Codes of Length $mp^r$ over $\mathbb{F}_{p^s}$

Throughout this section $p$ is an odd prime number and $n = mp^r$, with $m$ an integer such that $(m, p) = 1$.

This section provides the structure of constacyclic codes of length $mp^r$ over $\mathbb{F}_q$. We give this important lemma which we need after.

**Lemma 3.1** Let $q = p^s$ be a prime power. Then for all $\lambda \in \mathbb{F}_q^*$ there exist $\lambda_0 \in \mathbb{F}_q^*$ such that $\lambda = \lambda_0^{p^r}$, for $r \in \mathbb{N}$.

**Proof.** Let $\lambda \in \mathbb{F}_q^*$, $q = p^s$ and $s$ be a positive integer. If $s < r$ then there exists integers $k$ and $m$ such that $r - m = ks$ with $0 \leq m \leq s - 1$, $s - m = s - (r - ks) = s(k + 1) - r$. Let $\lambda_0 = \lambda^{p^r-m}$, then $\lambda_0^{p^r} = (\lambda^{p^{(k+1)-r}})^{p^r} = (\lambda^{p^{(k+1)}})^{p^r} = \lambda$. If $r < s$, then $(\lambda^{p^{s-r}})^{p^r} = \lambda$.

It is well know that $\lambda$-constacyclic codes over $\mathbb{F}_q$ are principal ideals generated by factors of $x^{mp^r} - \lambda$. Since $\mathbb{F}_{p^s}$ has characteristic $p$, and by Lemma 3.1 the polynomial $x^{mp^r} - \lambda$ can be factored as

$$x^{mp^r} - \lambda = x^{mp^r} - \lambda_0^{p^r} = (x^m - \lambda_0)^{p^r}. \quad (1)$$

The polynomial $x^m - \lambda_0$ is a monic square free polynomial. Hence from [6 Proposition 2.7] it factors uniquely as a product of pairwise coprime monic irreducible polynomials $f_0(x), \ldots, f_l(x)$. Thus from (1) we obtain the following factorization of $x^{mp^r} - \lambda_0$.

$$x^{mp^r} - \lambda_0^{p^r} = f_0(x)^{p^r} \ldots f_l(x)^{p^r}. \quad (2)$$
A \( \lambda \)-constacyclic code of length \( n = mp^r \) over \( \mathbb{F}_p \) is then generated by a polynomial of the form
\[
A(x) = \prod f_i^{k_i},
\]
where \( f_i(x), 0 \leq i \leq l, \) are the polynomials given in [2] and \( 0 \leq k_i \leq p^r \).

## 4 Constacyclic Codes of Length \( 2^a mp^r \) over \( \mathbb{F}_p \)

In this section we first recall the following important result of Batoul et al. given in [2]

**Proposition 4.1** [2, Proposition 3.2] Let \( q \) be a prime power, \( n \) a positive integer and \( \lambda \) an element in \( \mathbb{F}_q^* \). If \( \mathbb{F}_q^* \) contains an element \( \delta \), where \( \delta \) is an \( n \)-th root of \( \lambda \), then a \( \lambda \)-constacyclic code of length \( n \) is equivalent to a cyclic code of length \( n \).

And we give the structure of repeated-root constacyclic codes over \( \mathbb{F}_q, q = p^s \) of length \( 2^a mp^r \), \( a \geq 1 \). But before that and in the goal of using the isomorphism between cyclic codes and constacyclic codes of the same length, given in Proposition 4.1, we give the structure of cyclic codes of length \( 2^a mp^r \) over \( \mathbb{F}_q \), for that, we need the following Lemma:

**Lemma 4.2** Let \( a \geq 1 \) and \( \alpha \) a primitive \( 2^a \)-th root of the unity in \( \mathbb{F}_q^* \), the following holds:

1) \( \alpha^{2^i} \) is a primitive \( 2^{a-i} \)-th root of the unity in \( \mathbb{F}_q^* \) for all \( i, i \leq a \).

2) \( \alpha^m \) is a primitive \( 2^a \)-th root of the unity in \( \mathbb{F}_q^* \) for all odd integer \( m \).

3) \( \prod_{k=1}^{2^a} \alpha^k = -1 \).

**Proof.**

1) Let \( i, i \leq a \), in the cyclic group \( \mathbb{F}_q^* \), we have that \( \text{ord}(\alpha^{2^i}) = \frac{\text{ord}(\alpha)}{(2^i, \text{ord}(\alpha))} = \frac{2^a}{(2^i, 2^a)} = \frac{2^a}{2^a} = 2^{a-i} \).

2) Since \( (2^a, m) = 1 \), so \( \text{ord}(\alpha^m) = \frac{\text{ord}(\alpha)}{(m, \text{ord}(\alpha))} = \frac{2^a}{(m, 2^a)} = 2^a \).

3) \( (x^{2^a} - 1) = \prod_{k=1}^{2^a} (x - \alpha^k) \) then \( \prod_{k=1}^{2^a} \alpha^k = (-1)^{2^a} (-1)^{2^a} = -1 \).

**Proposition 4.3** Let \( q \) be a power of an odd prime \( p \) and \( n = 2^a m \) a positive integer such that \( m \) is an odd integer and \( (m, p) = 1, a \geq 1 \). Then if \( \mathbb{F}_q^* \) contains a primitive \( 2^a \)-root of unity \( \alpha \) and the \( f_i(x), 0 \leq i \leq l \) are the monic irreducible factors of \( x^m - 1 \) in \( \mathbb{F}_q[x] \), then:
\[
x^{2^a m} - 1 = \prod_{k=1}^{2^a} \left( \prod_{i=0}^{l} f_i(\alpha^{-k} x) \right). \tag{4}
\]
Proof. Assume that $x^m - 1 = \prod_{i=0}^{l} f_i(x)$ is the factorization of $x^m - 1$ into monic factors over $\mathbb{F}_q$. This factorization is unique since it is over a unique factorization domain (UFD). Let $\alpha \in \mathbb{F}_q^*$ be a primitive $2^a$-th root of unity and let $1 \leq k \leq 2^a$.

$$
\begin{align*}
(\alpha^{-k}x)^m - 1 &= \prod_{i=0}^{l} f_i(\alpha^{-k}x) \\
(\alpha^{-k})^m(x^m - (\alpha^k)^m) &= \alpha^{-k} \prod_{i=0}^{l} f_i(\alpha^{-k}x) \\
(x^m - \alpha^{km}) &= \alpha^{k(m-1)} \prod_{i=0}^{l} f_i(\alpha^{-k}x) \\
(x^m - (\alpha^m)^k) &= \alpha^{k(m-1)} \prod_{i=0}^{l} f_i(\alpha^{-k}x)
\end{align*}
$$

Then by Lemma 4.2, $\alpha^m$ is also a primitive $2^a$-th root of unity, we obtain:

$$
\prod_{k=1}^{2^a} (x^m - (\alpha^m)^k) = \prod_{k=1}^{2^a} \alpha^{k(m-1)} \prod_{i=0}^{l} f_i(\alpha^{-k}x) = \left( \prod_{k=1}^{2^a} \alpha^{k(m-1)} \right) \left( \prod_{k=1}^{2^a} \prod_{i=0}^{l} f_i(\alpha^{-k}x) \right) = \left( \prod_{k=1}^{2^a} \alpha^{km} \right) \left( \prod_{k=1}^{2^a} \prod_{i=0}^{l} f_i(\alpha^{-k}x) \right)
$$

Since $(x^{2^am} - 1) = ((x^m)^{2^a} - (\alpha^m)^{2^a}) = \prod_{k=1}^{2^a} (x^m - \alpha^{km})$ we obtain the result:

$$
x^{2^am} - 1 = \left( \prod_{k=1}^{2^a} \prod_{i=0}^{l} f_i(\alpha^{-k}x) \right).
$$

\[\blacksquare\]

Corollary 4.4 Let $q$ be a power of an odd prime $p$ and $n = 2^a mp^r$ a positive integer such that $m$ is an odd integer and $(m, p) = 1$, $a \geq 1$. Then if $\mathbb{F}_q^*$ contains a primitive $2^a$-root of unity $\alpha$ and the $f_i(x)$, $0 \leq i \leq l$ are the monic irreducible factors of $x^m - 1$ in $\mathbb{F}_q$ then:

$$
(x^{2^amp^r} - 1) = (x^{2^am} - 1)p^r = \prod_{k=1}^{2^a} \prod_{i=0}^{l} f_i p^r(\alpha^{-k}x).
$$

Proof. Since the characteristic of $\mathbb{F}_q$ is $p$, the proof follows from Proposition 4.3 \[\blacksquare\]

In the following we give the structure of cyclic codes of length $2^a mp^r$ over $\mathbb{F}_q$

Corollary 4.5 Let $q$ be a power of an odd prime $p$, $n = 2^a mp^r$ be a positive integer such that $m$ is odd integer, with $a \geq 1$ and $(m, p) = 1$. Then if $\mathbb{F}_q^*$ contains a primitive $2^a$-root of unity $\alpha$ and the $f_i(x)$, $0 \leq i \leq l$ are the monic irreducible factors of $x^m - 1$ in $\mathbb{F}_q[x]$ then any cyclic code of length $n = 2^a mp^r$ is generated by $\prod_{k=1}^{2^a} \prod_{i=0}^{l} f_i j_i(\alpha^{-k}x)$ where $0 \leq j_i \leq p^r$.

Proof. Since any cyclic code of length $n = 2^a mp^r$ is generated by a divisor of $(x^{2^amp^r} - 1)$, hence by Corollary 4.4 we have that

$$
(x^{2^amp^r} - 1) = (x^{2^am} - 1)p^r = \prod_{k=1}^{2^a} \prod_{i=0}^{l} f_i p^r(\alpha^{-k}x)
$$

5
So we deduce the result.

Now we generalize Proposition 4.1.

**Theorem 4.6** Let \( q \) be a power of an odd prime \( p \) and \( m \) an odd integer such that \((m, p) = 1\). Let \( \lambda \) and \( \delta \) in the multiplicative group \( \mathbb{F}_q^* \) such that \( \delta^m = \lambda \), if \( \delta = \beta^{2^a} \) in \( \mathbb{F}_q^* \), then the following hold:

(i) The \( \lambda \)-constacyclic codes of length \( 2^{a}mp^r \) over \( \mathbb{F}_q \) are equivalent to cyclic codes of length \( 2^{a}mp^r \) over \( \mathbb{F}_q \).

(ii) If \( q \equiv 1 \mod 2^{a+1} \), then \( -\lambda \)-constacyclic codes of length \( 2^{a}mp^r \) over \( \mathbb{F}_q \) are equivalent to cyclic codes of length \( 2^{a}mp^r \) over \( \mathbb{F}_q \).

**Proof.** For the part (i), let \( \lambda \in \mathbb{F}_q^* \) such that there exists \( \delta \in \mathbb{F}_q^* \), \( \delta^m = \lambda \), and \( \delta = \beta^{2^a} \) in \( \mathbb{F}_q^* \). Then \( \lambda = \beta^{2^am} \). So by Lemma 3.1 there exists \( \beta_0 \in \mathbb{F}_q^* \) such that \( \beta = \beta_0^{p^r} \). Hence by Proposition 4.1, \( \lambda \)-constacyclic codes of length \( 2^{a}mp^r \) over \( \mathbb{F}_q \) are equivalent to cyclic codes over \( \mathbb{F}_q \).

For the part (ii), since \( q \equiv 1 \mod 2^{a+1} \) and by Lemma 4.2 there exists a primitive \( 2^{a+1} \)-root of unity \( \alpha \in \mathbb{F}_q^* \). So \( \alpha^{2^{a}} = -1 \) and

\[
-\lambda = (\alpha^{2^{a}})^{mp^r} \beta_0^{2^ammp^r} = (\alpha \beta_0)^{2^ammp^r}.
\]

Then by Proposition 4.1, \( -\lambda \)-constacyclic codes of length \( 2^{a}mp^r \) over \( \mathbb{F}_q \) are equivalent to cyclic codes over \( \mathbb{F}_q \). ■

**Corollary 4.7** Let \( \lambda = \beta_0^{2^ammp^r} \), \( \alpha \) a primitive \( 2^{a} \)-th root of unity in \( \mathbb{F}_q^* \) and let \( C \) be a \( \lambda \)-constacyclic code of length \( 2^{a}mp^r \), then:

\[
C = \langle \prod_{k=1}^{2^a} \left( \prod_{i=0}^{l} f_i^{j_i} (\beta_0^{-1} \alpha^{-k} x) \right) \rangle,
\]

where \( 0 \leq j_i \leq p^r \).

**Proof.** By Lemma 4.2 if \( q \equiv 1 \mod 2^a \), then there exists a primitive \( 2^{a} \)-th root \( \alpha \) of unity in \( \mathbb{F}_q^* \). Thus by Corollary 4.4,

\[
(x^{2^ammp^r} - 1) = (x^{2^am} - 1)^{p^r} = \prod_{k=1}^{2^a} \prod_{i=0}^{l} f_i^{p^r} (\alpha^{-k} x),
\]

then

\[
((\beta_0^{-1} x)^{2^ammp^r} - 1) = ((\beta_0^{-1} x)^{2^am} - 1)^{p^r} = \prod_{k=1}^{2^a} \prod_{i=0}^{l} f_i^{p^r} (\beta_0^{-1} \alpha^{-k} x),
\]

so

\[
(x^{2^ammp^r} - \lambda) = \lambda((\beta_0^{-1} x)^{2^am} - 1)^{p^r} = \prod_{k=1}^{2^a} \prod_{i=0}^{l} f_i^{p^r} (\beta_0^{-1} \alpha^{-k} x).
\]
Since any $\lambda$-constacyclic codes of length $2^a mp^r$ is generated by a divisor of $(x^{2^a mp^r} - \lambda)$ then we have the result.

**Corollary 4.8** Let $\lambda = \beta_0^{2^m}$ and let $C$ be a $-\lambda$-constacyclic code of length $2^a mp^r$. If $q \equiv 1 \mod 2^a + 1$ then

$$C = \langle \prod_{k=1}^{2^a} \prod_{i=0}^{l} f_i^j (\beta_0^{-1} \alpha^{-2k+1} x) \rangle$$

where $0 \leq j_i \leq p^r$.

**Proof.** By Lemma 4.2 if $q \equiv 1 \mod 2^a + 1$ there exists a primitive $2^a + 1$-th root $\alpha$ of unity in $\mathbb{F}_q$. Thus by Corollary 4.4

$$(x^{2^a + 1} mp^r - 1) = (x^{2^a m} - 1)^{p^r} (x^{2^a m} + 1)^{p^r} = \prod_{k=1}^{2^a} \prod_{i=0}^{l} f_i^j (\alpha^{-2k} x) \prod_{k=1}^{2^a} \prod_{i=0}^{l} f_i^j (\alpha^{-2k+1} x).$$

Then

$$((\beta_0^{-1} x)^{2^a mp^r} + 1) = ((\beta_0^{-1} x)^{2^a m} + 1)^{p^r} = \prod_{k=1}^{2^a} \prod_{i=0}^{l} f_i^j (\beta_0^{-1} \alpha^{-2k+1} x).$$

Then

$$(x^{2^a mp^r} + \lambda) = \lambda ((\beta_0^{-1} x)^{2^a m} - 1)^{p^r} = \prod_{k=1}^{2^a} \prod_{i=0}^{l} f_i^j (\beta_0^{-1} \alpha^{-2k+1} x).$$

Since any $-\lambda$-constacyclic codes of length $2^a mp^r$ is generated by a divisor of $(x^{2^a mp^r} + \lambda)$ we obtain the result.

**Example 4.9** Let $n = 2 \cdot 7 \cdot 5^3 = 1750$, $q = 5^2$ and $\beta$ be a primitive element of $\mathbb{F}_{25}^*$. Further, let $\lambda \in \{\beta^2, 1 \leq i \leq 12\}$. Since $(7, 24) = 1$, $\beta^2 = \beta^{2(7 \cdot 7 - 24)} = (\beta^7)^2 \cdot 7$. Then in $\mathbb{F}_{25}$ we have that $x^7 - 1 = (x + 4)(x^3 + 3x^2 + 3x + 1)(x^3 + 3x^2 + 3x + 1) = f_1(x) f_2(x) f_3(x)$. So then $x^{14} - 1 = f_1(x) f_2(x) f_3(x) f_1(-x) f_2(-x) f_3(-x)$.

Since cyclic codes of length $2 \cdot 7 \cdot 5^3$ over $\mathbb{F}_{25}$ are ideals of $\mathbb{F}_{25}[x]_{x^{1750} - \lambda}$ which is a principal ideal ring. Then these codes are generated by

$$\langle f_1^s(x) f_2^t(x) f_3^k(x) \rangle, s, j, k, l, m, s \in \{0, \ldots, 5^3\}.$$

Therefore, $\lambda$-constacyclic codes of length $2 \cdot 7 \cdot 5^3 = 1750$ over $\mathbb{F}_{25}$ are ideals of $\mathbb{F}_{25}[x]_{x^{1750} - \lambda}$ which is a principal ideal ring, and these codes are generated by

$$\langle f_1^s(\beta^{-11i} x) f_2^t(\beta^{-11i} x) f_3^k(\beta^{-11i} x) \rangle, s, j, k, l, m, t \in \{0, \ldots, 5^3\}, 1 \leq i \leq 12.$$
In $\mathbb{F}_{25}$ we have $7^2 = 49 = -1$, so for $\lambda \in \{\beta^{2i}, 1 \leq i \leq 12\}$, $-\lambda \in \{(7\alpha)^{2i}, 1 \leq i \leq 12\}$. Thus $-\lambda$-constacyclic codes of length $2 \cdot 7 \cdot 5^3 = 1750$ over $\mathbb{F}_{25}$ are ideals of $\frac{\mathbb{F}_{25}[x]}{x^{1750} + \lambda}$ which is a principal ideal ring, and these codes are generated by $F(x)$ ideals generated by the factors of $x^{1750} + 1$. Hence we obtain Table 1.

5 Self-Dual Negacyclic Codes of Length $2^a mp^r$ over $\mathbb{F}_{p^s}$

Let $p$ be an odd prime number and $n = mp^r$, with $m$ an integer (odd or even) such that $(m, p) = 1$. This section provides conditions on the existence of self-dual negacyclic codes of length $n = 2^a mp^r$ over $\mathbb{F}_{p^s}$. It is well known that negacyclic codes over $\mathbb{F}_{p^s}$ are principal ideals generated by the factors of $x^{mp^r} + 1$. Since $\mathbb{F}_{p^s}$ has characteristic $p$, and negacyclic codes are a particular case of constacyclic codes, so by results of Section 4, the polynomial $x^{mp^r} + 1$ can be factored as

$$x^{mp^r} + 1 = (x^m + 1)^{p^r}. \quad (5)$$

The polynomial $x^m + 1$ is a monic square free polynomial. Hence from [6, Proposition 2.7] it factors uniquely as a product of pairwise coprime monic irreducible polynomials $f_0(x), \ldots, f_l(x)$. Thus from (5) we obtain the following factorization of $x^{mp^r} + 1$

$$x^{mp^r} + 1 = f_0(x)^{p^r} \cdots f_l(x)^{p^r}. \quad (6)$$

A negacyclic code of length $n = mp^r$ over $\mathbb{F}_{p^s}$ is then generated by a polynomial of the form

$$A(x) = \prod f_i^{k_i}, \quad (7)$$

where $f_i(x), 0 \leq i \leq l$, are the polynomials given in (6) and $0 \leq k_i \leq p^r$.

For a polynomial $f(x) = a_0 + a_1 x + \ldots + a_r x^r$, with $a_0 \neq 0$ and degree $r$ (hence $a_r \neq 0$), the reciprocal of $f$ is the polynomial denoted by $f^*$ and defined as

$$f^*(x) = x^r f(x^{-1}) = a_r + a_{r-1} x + \ldots + a_0 x^r. \quad (8)$$

If a polynomial $f(x)$ is equal to its reciprocal, then $f(x)$ is called self-reciprocal. We can easily verify the following equalities

$$(f(x)^*)^* = f(x) \text{ and } (fg(x))^* = f(x)^*g(x)^*. \quad (9)$$

It is well known (see [8, Proposition 2.4]), that the dual of the negacyclic code generated by $A(x)$ is the negacyclic code generated by $B^*(x)$ where

$$B(x) = \frac{x^m + 1}{A(x)}. \quad (10)$$

Hence we have the following lemma.
Table 1: The Generators Polynomials of $\lambda$-Constacyclic Codes of Length 1750 over $\mathbb{F}_{25}$.

| $\beta^{11i}$ | $\lambda = (\beta^{11i})^{27^3}$ | $f_1^{*}(-7\beta^{-11i}x)f_2^{*}(-7\beta^{-11i}x)f_3^{k}(-7\beta^{-11i}x)$ |
|---------------|-------------------------------|-------------------------------------------------|
| $\beta^{11}$  | $(\beta^{11})^{27^3}$        | $f_1^{*}(-7\beta^{-11i}x)f_2^{*}(-7\beta^{-11i}x)f_3^{k}(-7\beta^{-11i}x)$ |
| $\beta^{11-2}$| $(\beta^{11-2})^{27^3}$      | $f_1^{*}(-7\beta^{-11i-2}x)f_2^{*}(-7\beta^{-11i-2}x)f_3^{k}(-7\beta^{-11i-2}x)$ |
| $\beta^{11-3}$| $(\beta^{11-3})^{27^3}$      | $f_1^{*}(-7\beta^{-11i-3}x)f_2^{*}(-7\beta^{-11i-3}x)f_3^{k}(-7\beta^{-11i-3}x)$ |
| $\beta^{11-4}$| $(\beta^{11-4})^{27^3}$      | $f_1^{*}(-7\beta^{-11i-4}x)f_2^{*}(-7\beta^{-11i-4}x)f_3^{k}(-7\beta^{-11i-4}x)$ |
| $\alpha^{11-5}$| $(\alpha^{11-5})^{27^3}$     | $f_1^{*}(-7\alpha^{-11i-5}x)f_2^{*}(-7\alpha^{-11i-5}x)f_3^{k}(-7\alpha^{-11i-5}x)$ |
| $\alpha^{11-6}$| $(\alpha^{11-6})^{27^3}$     | $f_1^{*}(-7\alpha^{-11i-6}x)f_2^{*}(-7\alpha^{-11i-6}x)f_3^{k}(-7\alpha^{-11i-6}x)$ |
| $\alpha^{11-7}$| $(\alpha^{11-7})^{27^3}$     | $f_1^{*}(-7\alpha^{-11i-7}x)f_2^{*}(-7\alpha^{-11i-7}x)f_3^{k}(-7\alpha^{-11i-7}x)$ |
| $\alpha^{11-8}$| $(\alpha^{11-8})^{27^3}$     | $f_1^{*}(-7\alpha^{-11i-8}x)f_2^{*}(-7\alpha^{-11i-8}x)f_3^{k}(-7\alpha^{-11i-8}x)$ |
| $\alpha^{11-9}$| $(\alpha^{11-9})^{27^3}$     | $f_1^{*}(-7\alpha^{-11i-9}x)f_2^{*}(-7\alpha^{-11i-9}x)f_3^{k}(-7\alpha^{-11i-9}x)$ |
| $\alpha^{11-10}$| $(\alpha^{11-10})^{27^3}$    | $f_1^{*}(-7\alpha^{-11i-10}x)f_2^{*}(-7\alpha^{-11i-10}x)f_3^{k}(-7\alpha^{-11i-10}x)$ |
| $\alpha^{11-11}$| $(\alpha^{11-11})^{27^3}$    | $f_1^{*}(-7\alpha^{-11i-11}x)f_2^{*}(-7\alpha^{-11i-11}x)f_3^{k}(-7\alpha^{-11i-11}x)$ |
| $\alpha^{11-12}$| $(\alpha^{11-12})^{27^3}$    | $f_1^{*}(-7\alpha^{-11i-12}x)f_2^{*}(-7\alpha^{-11i-12}x)f_3^{k}(-7\alpha^{-11i-12}x)$ |
Lemma 5.1 A negacyclic code $C$ of length $n$ generated by a polynomial $A(x)$ is self-dual if and only if
$$A(x) = B^*(x).$$

Denote the factors $f_i$ in the factorization of $x^n + 1$ which are self-reciprocal by $g_1, \ldots, g_s$, and the remaining $f_j$ grouped in pairs by $h_1, h_1^*, \ldots, h_t, h_t^*$. Hence $l = s + 2t$ and the factorization given in (3) becomes
$$x^n + 1 = (x^n + 1)^{p^r} = g_1^{p^r}(x) \cdots g_s^{p^r}(x) \times h_1^{p^r}(x) h_1^{*p^r}(x) \cdots h_t^{p^r}(x) h_t^{*p^r}(x). \tag{11}$$

In the following we give the structure of negacyclic codes over $\mathbb{F}_{p^r}$ of length $2^aq^{mp^r}$, $a \geq 1$.

We begin with this useful lemma.

Lemma 5.2 Let $q = p^s$ be an odd prime power such that $q \equiv 1 \mod 2^{a+1}$. Then there is a ring isomorphism between the ring $\mathbb{F}_{q^s}[x]/x^{2mp^r} - 1$ and the ring $\mathbb{F}_{q^s}[x]/x^{mp^r} + 1$.

Proof. If $q \equiv 1 \mod 2^{a+1}$ then by Lemma 4.2 there exists a primitive $2^{a+1}$-th root $\alpha$ of unity in $\mathbb{F}_{q^s}^\times$. So $-1 = (-1)^{mp^r} = (\alpha^{2^{a+1}})^{p^r}$ and then by Proposition 4.1 negacyclic codes of length $2^aq^{mp^r}$ over $\mathbb{F}_q$ are equivalent to cyclic codes of length $2^aq^{mp^r}$ over $\mathbb{F}_q$. 

Corollary 5.3 Let $q = p^s$ be an odd prime power such that $q \equiv 1 \mod 2^{a+1}$ and $n = 2^aq^{mp^r}$ with $m$ an odd integer such that $(m, p) = 1$. Then a negacyclic code of length $n$ over $\mathbb{F}_{p^s}$ is a principal ideal of $\mathbb{F}_{p^s}[x]/(x^n + 1)$ generated by a polynomial of the following form
$$\prod_{k=1}^{2^a}(\prod_{i=0}^{t} f_i(\alpha^{-2k+1}x))$$
where $0 \leq j_i \leq p^r$ and $f_i(x)$ are monic irreducible factors of $x^n - 1$.

Proof. It suffices to find the factors of $x^{2mp^r} + 1$. Since $q \equiv 1 \mod 2^{a+1}$ and from Lemma 4.2, there exist $\alpha \in \mathbb{F}_{q^s}^\times$, a primitive $2^{a+1}$-th root of unity. So $x^{2mp^r} + 1$ can be decomposed as $(x^{2m} + 1)^{p^r} = (\prod_{k=1}^{2^a} \prod_{i=0}^{t} f_i(\alpha^{-2k+1}x))^{p^r}$. The result then follows from the isomorphisms given in Lemma 5.2.

We recall the most important result of [12].

Theorem 5.4 ([12, Theorem 2.2]) There exists a self-dual negacyclic code of length $mp^r$ over $\mathbb{F}_{p^r}$ if and only if there is no $g_i$ (self-reciprocal polynomial) in the factorization of $x^{mp^r} + 1$ given in (14). Furthermore, a self-dual negacyclic code $C$ is generated by a polynomial of the following form
$$h_1^{b_1}(x)h_1^{sp^r-b_1}(x) \cdots h_t^{b_t}(x)h_t^{*p^r-b_t}(x). \tag{12}$$

In the following we generalize [12, Theorem 3.7] for the length $2^aq^{mp^r}$. But before that we need the following lemmas.
Lemma 5.5 (\cite{12}, Lemma 3.5) Let $m$ be an odd integer and $Cl_m(i)$ the $p^s$ cyclotomic class of $i$ modulo $m$. The polynomial $f_i(x)$ is the minimal polynomial associated with $Cl_m(i)$, hence we have $Cl_m(i) = Cl_m(-i)$ if and only if $f_i(x) = f_i^*(x)$.

Lemma 5.6 (\cite{12}, Lemma 3.6) Let $m$ be an odd integer and $p$ a prime number. Then $\text{ord}_m(p^s)$ is even if and only if there exists a cyclotomic class $Cl_m(i)$ which satisfies $Cl_m(i) = Cl_m(-i)$.

Theorem 5.7 Let $q = p^s$ be an odd prime power such that $q \equiv 1 \mod 2^{a+1}$, and $n = 2^a mp^r$ be an integer with $(m, p) = 1$ and $a > 1$. Then there exists a negacyclic self-dual code of length $2^a mp^r$ over $\mathbb{F}_{p^s}$ if and only if $\text{ord}_m(q)$ is odd.

Proof. Under the hypothesis on $q$, and $m$ we have from Corollary 5.3 that the polynomial $x^{2^a mp^r} + 1 = \prod_{i=1}^{2^a} \prod_{j=0}^{2^a} f_{i,j}(\alpha^{2k+1}x)$, where $f_i(x)$ are the monic irreducible factors of $x^m - 1$ in $\mathbb{F}_{p^s}$. By Lemma 5.6, $\text{ord}_m(p^s)$ is odd if and only if there is no cyclotomic class such that $Cl_m(i) = Cl_m(-i)$. From Lemma 5.5, this is equivalent to saying that there are no irreducible nontrivial factors of $x^m - 1$ such that $f_i(x) = f_i^*(x)$. From corollary Corollary 5.3, we obtain that $f_i(x) \neq f_i^*(x)$ for all $i \neq 0$ $(f_0(x) = (x - 1))$ is true if and only if $f_i(\alpha^k x) \neq f_i^*(\alpha^k x)$ are true for all $1 \leq k \leq 2^{a+1}$. Then from Theorem 5.4, self-dual negacyclic codes exist.

Example 5.8 A self-dual negacyclic code of length 70 over $\mathbb{F}_5$ does not exist. There is no self-dual negacyclic code of length 30 over $\mathbb{F}_9$, but there is a self-dual code over $\mathbb{F}_9$ of length 126.

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