RETURN OF $k$-BONACCI RANDOM WALKS

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Abstract. In this work, the probability of return for random walks on $\mathbb{Z}$, whose increment is given by the $k$-bonacci sequence, is determined. Also, the Hausdorff, packing and box-counting dimensions of the set of these walks that return an infinite number of times to the origin are given. As an application, we study the return for tribonacci random walks to the first term of the tribonacci sequence.

1. Introduction and main results

The Fibonacci sequence, commonly denoted by $(f_n)_{n \geq 0}$, is a sequence of integer numbers such that each of them is the sum of the two preceding ones, starting from 0 and 1, i.e.,

\[
\begin{align*}
  f_0 &= 0, & f_1 &= 1, \\
  f_n &= f_{n-1} + f_{n-2}, & \text{for } n \geq 2.
\end{align*}
\]

This sequence was first introduced by Leonardo Fibonacci and is tightly connected to the golden ratio $\varphi = (1 + \sqrt{5})/2 = 1.61803398\ldots$. Since then, many researchers have been interested in the study of the properties of this sequence and their applications. One can cite, for example, [7] where it was proved that $(f_n)_{n \geq 0}$ increases exponentially with $n$ at a rate given by $\varphi$. A more general case was treated in [15], where the author considered the random Fibonacci sequence $(t_n)_{n \geq 0}$ defined by $t_1 = t_2 = 1$ and for $n > 2$,

\[
t_n = \pm t_{n-1} \pm t_{n-2},
\]

where each $\pm$ sign is independent and either $+$ or $-$ with probability $1/2$. It is not even obvious that $|t_n|$ should increase with $n$. Although, it was proved that, almost surely

\[
\lim_{n \to +\infty} \sqrt{|t_n|} = 1.13198824\ldots
\]

Later, in [11], the author considered the Fibonacci random walk and determined the probability of its return to the origin. More precisely, he considered the random walk on $\mathbb{Z}$ whose increments are given by $(f_n)_{n \geq 0}$, i.e.,

\[
\hat{F}_n = \sum_{k=1}^{n} f_kw_k,
\]

where $(w_i)_{i \geq 1}$ is a sequence of independent, identically distributed random variables with $\mathbb{P}(w_1 = \pm 1) = 1/2$. 

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We denote by $\mathbb{N}$ the set of positive integers and consider the space of infinite sequences $\mathcal{A} = \{-1, 1\}^{\mathbb{N}}$. We define, for an elementary event $w \in \mathcal{A}$, the set
$$F(w) = \{n \geq 1, \hat{F}_n(w) = 0\}.$$ 
Denoting by "$\sharp B$" the cardinality of a given set $B$, we set,
$$R_i = \{w \in \mathcal{A} \mid \sharp F(w) = i\}, \quad i \in \mathbb{N} \cup \{0\}$$
and
$$N = \{w \in \mathcal{A} \mid \sharp F(w) = \infty\}.$$ 
It is known from [11] that the probability of $R_i$ is $3/4^{i+1}$. In particular, $\mathbb{P}(N) = 0$. The idea of studying such kind of problems comes from a classical result due to Pólya [13], who was interested to the simple random walk of length $n \geq 1$, on the integers, given by
$$S_n = \sum_{j=1}^{n} w_j.$$ 
This means that $S_n$ is seen as to be the position, after $n$ steps, of a walk on the integers of an individual, who is supposed to start its motion at the origin of the lattice, i.e., $S_0 = w_0 = 0$. Pólya [13] showed this walk to be recurrent, which means that it almost surely returns to the origin in a finite number of steps. The reader can also see, for example, [4, 6, 10] for more discussions on this problem.

In this paper, we are interested, for a given integer $k \geq 2$, to the $k$-bonacci random walk given by $\hat{F}_n$, where we take in consideration the $k$-bonacci sequence $(f_n)_{n \geq 0}$ defined by $f_0 = 0$,
$$f_n = \sum_{j=1}^{k} f_{n-j}, \quad \text{for} \quad n \geq k + 1. \quad (1.1)$$
and the $k$ initializing terms $(f_n)_{1 \leq n \leq k}$ are supposed to satisfy the following condition
$$\sum_{j=1}^{n} \pm f_j \neq 0 \quad \text{for} \quad 1 \leq n \leq k, \quad (AS)$$ 
The condition (AS) is guarantied, for example, in the following situation
$$f_n = 1 + \sum_{j=0}^{n-1} f_j, \quad \text{for} \quad 1 \leq n \leq k.$$ 
We study the probability of return of the $k$-bonacci random walk to the origin. For this, we establish a necessary and sufficient condition for the $k$-bonacci random walk to reach $0$ at least one time (Proposition 2.1). Our first main result is the following.

**Theorem 1.1.** For $i \in \mathbb{N}$, $\mathbb{P}(R_i) = \frac{1}{2^{(k+1)i}}$.

Next, we are interested to the set $N$ of walks returning infinitely many times to $0$. Since $\mathbb{P}(N) = 0$, it’s natural to ask the question about the size of $N$ as a subset of $\mathcal{A}$. Denoting by $\text{dim}_H, \text{dim}_P$ and $\text{dim}_B$ respectively the Hausdorff, packing and box-counting dimensions, we can state our second result as follows.
**Theorem 1.2.** \( \dim_H(N) = \dim_P(N) = \dim_B(N) = \frac{1}{k+1} \).

We refer the reader to the Appendix A and the reference therein for the definitions and more details about these fractal dimensions.

As an application, using the same technics, we study the return for the tribonacci random walks to the term \( f_1 \) of the tribonacci sequence.

2. **Proof of Main results**

We consider the \( k \)-bonacci sequence \( (f_i)_{i \geq 0} \) defined by (1.1) and (AS). Let \( n \geq k + 1 \). We easily obtain by induction,

\[
\sum_{j=1}^{n} f_j < f_{n+2}.
\] (2.1)

Moreover,

\[
\begin{align*}
 f_{n+1} &= \sum_{j=n-k+1}^{n} f_j = f_n + \sum_{j=n-k+1}^{n-1} f_j \\
 &= f_n - f_{n-k} \\
 &= 2f_n - f_{n-k}.
\end{align*}
\]

This means that

\[
2f_n \geq 1 + f_{n+1}.
\] (2.2)

Next, we give a necessary and sufficient condition to obtain \( \sum_{i=1}^{n} w_if_i = 0 \). For this, we consider, for any integer \( i \geq 2 \), the finite sequences,

\[
 v_i^+ = +1, +1, \ldots, +1, +1 \quad \text{and} \quad v_i^- = -1, -1, \ldots, -1, -1.
\]

We also consider, for a given integer \( p \geq 1 \), the condition

\[
(w_p, w_{p+1}, \ldots, w_{p+k}) \in \{v_k^+, v_k^-\} \quad C(k, p).
\]

It is clear that if the condition \( C(k, p) \) holds, then \( \sum_{j=p}^{p+k} w_j f_j = 0 \).

**Proposition 2.1.** Consider the \( k \)-bonacci sequence \( (f_i)_{i \geq 0} \) given by (AS) and (1.1) and let \( w = (w_i)_{i \geq 0} \in A \). We have,

\[
\hat{F}_n(w) = 0
\] (2.3)

if and only if \( n = (k+1)m \), for some integer \( m > 0 \), and

\[
C(k, (k+1)i+1) \quad \text{holds for all} \quad 0 \leq i < m.
\] (2.4)

To prove this proposition, we need to show the following result.

**Lemma 2.2.** Suppose that \( C(k, p) \) does not hold for an integer \( p \geq 1 \), then

\[
\begin{align*}
(1) \quad & \left| \sum_{j=p}^{p+k} w_j f_j \right| \geq 2f_p, \\
(2) \quad & |\hat{F}_{p+k}(w)| > 1.
\end{align*}
\]
Proof. (1) By (1.1), we have

\[ \left| \sum_{j=p}^{p+k} w_j f_j \right| = \left| \sum_{j=p}^{p+k-1} (w_j + w_{p+k}) f_j \right|. \]  

(2.5)

We supposes that \( w_{p+k} = 1 \) (the case \( w_{p+k} = -1 \) is analogous).

We consider the set

\[ A_{p,k} = \left\{ j, \ p \leq j \leq p + k - 1, \ w_j + w_{p+k} \neq 0 \right\}. \]

Since \( C(k, p) \) does not hold, then \( A_{p,k} \neq \emptyset \). So, equation (2.5) leads to

\[ \left| \sum_{j=p}^{p+k} w_j f_j \right| = 2 \sum_{j \in A_{p,k}} f_j \geq 2f_p. \]

(2) If \( p = 1 \) and \( C(k, 1) \) does not hold, then using Lemma 2.2 (1), we obtain

\[ |\hat{F}_{k+1}(w)| \geq 2f_1 > 1. \]

Otherwise,

\[ |\hat{F}_{p+k}(w)| \geq \left| \sum_{j=p}^{p+k} w_j f_j \right| - |\hat{F}_{p-1}(w)|. \]

Since \( C(k, p) \) does not hold, then again using Lemma 2.2 (1) leads to

\[ |\hat{F}_{p+k}(w)| = 2f_p - |\hat{F}_{p-1}(w)| \geq 2f_p - \sum_{j=1}^{p-1} f_j. \]

Applying successively (2.1) and (2.2), we obtain

\[ |\hat{F}_{p+k}(w)| > 2f_p - f_{p+1} \geq 1. \]

\( \square \)

Proof of Proposition 2.1 \( \Leftarrow \): Obviously, if (2.4) is realized, then by (1.1) we obtain (2.3).

\( \Rightarrow \): Conversely, suppose that (2.3) is insured. Then, using condition \( (AS) \), it becomes that \( n \geq k + 1 \). Let \( m \) be the unique positive integer such that \( n = (k + 1)m + t(n) \).

(1) if there exits \( p \in \{(k + 1)j + t(n), 0 \leq j < m\} \), such that \( C(k, p) \) is not satisfied, then we set

\[ p(n) = \sup \{ p = (k + 1)j + t(n) + 1, \ 0 \leq j < m, \ C(k, p) \text{ does not hold} \}. \]

Thanks to Lemma 2.2 we have \( \hat{F}_{p(n)}(w) = \hat{F}_{p(n)+k}(w) \neq 0 \).

(2) if \( t(n) \neq 0 \) and \( C(k, p) \) is satisfied for all \( p \in \{(k + 1)j + t(n), 0 \leq j < m\} \), then by condition \( (AS) \), we have

\[ \hat{F}_{n}(w) = \hat{F}_{t(n)}(w) + \sum_{i=0}^{m-1} \left( \sum_{t=1}^{k+1} w_{(k+1)i+t(n)+t} f_{(k+1)i+t(n)+t} \right) = \hat{F}_{t(n)}(w) \neq 0. \]

Consequently, we must have that \( t(n) = 0 \) and (2.4) satisfied. This ends the proof.
2.1. **Proof of Theorem 1.1.** Let \( i \geq 1 \), from Proposition 2.1 we deduce that \( \hat{F}_n \) reaches the origin exactly \( i \)-times if and only if \( n \geq (k+1)i \), with

\[
\hat{F}_{(k+1)i} = 0 \quad \text{and} \quad \hat{F}_{(k+1)(i+1)} \neq 0.
\]

Moreover, we have

\[
\mathbb{P}(\hat{F}_{(k+1)i} = 0) = \frac{2^i}{2(k+1)i} = \frac{1}{2k+1} \quad \text{and} \quad \mathbb{P}(\hat{F}_{(k+1)(i+1)} = 0) = \frac{1}{k+1}.
\]

It follows that

\[
\mathbb{P}(R_i) = \frac{1}{k+1} = \mathbb{P}(\hat{F}_{(k+1)i} = 0) \times \mathbb{P}(\hat{F}_{(k+1)(i+1)} = 0) \times \mathbb{P}(\hat{F}_{(k+1)i} = 0).
\]

2.2. **Proof of Theorem 1.2.** We consider the metric \( d \), defined for any couple \( (u_i)_i, (v_i)_i \) of \( A \times A \), by

\[
d((u_i)_i, (v_i)_i) = \sum_{i=1}^{\infty} \frac{|u_i - v_i|}{2^i}.
\]

Endowed with this metric, \((A, d)\) becomes a compact metric space. As a direct consequence of Proposition 2.1, we obtain the following lemma.

**Lemma 2.3.** We have,

\[
N = \{v_{k+}, v_{k-}\}^N.
\]

Now, we consider the mappings \( T_1 \) and \( T_2 \) defined for any \( w = (w_i)_i \in A \), by

\[
T_1(w) = (v_{k+}, w_1, w_2, \ldots) \quad \text{and} \quad T_2(w) = (v_{k-}, w_1, w_2, \ldots).
\]

For \( i \in \{1, 2\} \) and for any \( (u, v) = ((u_j)_j, (v_j)_j) \in A^2 \), we have

\[
d(T_i(u), T_i(v)) = \sum_{j=k+2}^{\infty} \frac{|u_j - v_j|}{2^j} = \frac{1}{2k+1} \quad \text{and} \quad d(u, v).
\]

This means that \( T_1 \) and \( T_2 \) are contracting similarities on the metric space \((A, d)\), with contraction rates

\[
r_1 = r_2 = \frac{1}{2k+1}.
\]

Coming back to [5], one deduces the existence of a unique compact self-similar subset \( F \) of \( A \), such that \( F = T_1(F) \cup T_2(F) \). Lemma 2.3 implies that \( F = N \). Furthermore, we have \( T_1(N) \cap T_2(N) = \emptyset \). Hence, \( N \) is a self-similar set satisfying the open set condition in the compact metric space \((A, d)\). Their fractal dimension is then given by Corollary 3.3

\[
dim_H(N) = \dim_P(N) = \dim_B(N) = \frac{\ln(2)}{\ln(2k+1)} = \frac{1}{k+1}.
\]
3. Application

We are interested in applying the ideas presented in the previous sections to a class of tribonacci sequences, defined by

\[ f_0 = 0, \quad f_1 = 1, \quad f_2 = 3, \quad f_3 = 6 \]

and

\[ f_i = \sum_{j=1}^{3} f_{i-j}, \quad \text{for} \quad i \geq 4. \]  

(3.1)

The return point on which we focus our attention is no longer the origin. Our ideas are still applicable when considering the number of visits of \( \hat{F}_n \) to \( f_1 \). We consider, for an elementary event \( w \in \mathcal{A} \), the set \( F_1(w) \) for which \( \hat{F}_n(w) \) reaches \( f_1 \) after \( n \) steps of the walk, i.e.,

\[ F_1(w) = \{ n \geq 1, \hat{F}_n(w) = f_1 \}. \]

For \( i \in \mathbb{N} \), we denote by \( R_{1,i} \) the event for which elements \( \hat{F}_n \) passes through \( f_1 \) exactly \( i \) times, i.e.,

\[ R_{1,i} = \left\{ w \in \mathcal{A} \mid \#F_1(w) = i \right\}. \]

**Theorem 3.1.** For \( i \in \mathbb{N} \), \( \mathbb{P}(R_{1,i}) = \frac{7}{2^{3(i+1)+1}}. \)

In a similar way, we consider the set \( N_1 \) consisting on elements of \( \mathcal{A} \) for which \( \hat{F}_n \) passes through \( f_1 \) an infinite number of times, i.e.,

\[ N_1 = \left\{ w \in \mathcal{A} \mid \#F_1(w) = \infty \right\}. \]

**Theorem 3.2.** We have,

\[ \dim_H(N_1) = \dim_P(N_1) = \dim_B(N_1) = \frac{1}{4}. \]

In the same spirit of Proposition 2.1, we have

**Proposition 3.3.** Consider the tribonacci sequence \( (f_i)_{i \geq 0} \) given by (3.1) and let \( (w_i)_{i \geq 0} \in \mathcal{A} \), with \( w_1 = 1 \). We have,

\[ \hat{F}_n(w) = f_1 \]  

if and only if either \( n = 1 \) or \( n = 4m + 1 \), for some integer \( m \geq 1 \), and

\[ C(3, 4j + 2) \quad \text{holds for all} \quad 0 \leq j < m. \]  

(3.3)

**Proof.** \( \Leftarrow \): Obviously, if either \( n = 1 \) or \( n = 4m + 1 \), for some integer \( m \geq 1 \), and (3.3) holds, then thanks to (3.1) we obtain (3.2).

\( \Rightarrow \): Conversely, supposing that \( n \geq 4 \) and that \( C(3, n - p) \) is not satisfied, we obtain a contradiction by Lemma 2.2. Otherwise, arguing by induction, we obtain

\[ \hat{F}_n(w) = \hat{F}_{t(n)}(w). \]

If \( t(n) \neq 1 \), then \( \hat{F}_{t(n)} \) is even and positive, so \( |\hat{F}_{t(n)}| > f_1 \). Again a contradiction. It follows that either \( n = 1 \) or \( t(n) = 1 \) and (3.3) holds. \( \square \)
3.1. **Proof of Theorem 3.1.** We take $i \geq 1$. From Proposition 3.3, we deduce:

$\hat{F}_n$ reaches $f_1$ exactly $i$-times if and only if $n \geq 4i + 1$, with

$\hat{F}_{4i+1} = 1$ and $\hat{F}_{4(i+1)+1} \neq 1$.

Moreover, we have

\[
\mathbb{P}(\hat{F}_{4i+1} = 1) = \frac{2i}{2^{4i+1}} = \frac{1}{2^{4i+1}} \quad \text{and} \quad \mathbb{P}\left(\hat{F}_{4(i+1)+1} = 1 \mid \hat{F}_{4i+1} = 1\right) = \frac{2}{2^4} = \frac{1}{8}.
\]

It follows that

\[
\mathbb{P}(R_{1,i}) = \mathbb{P}\left(\hat{F}_{4(i+1)+1} \neq 1 \mid \hat{F}_{4i+1} = 1\right) \times \mathbb{P}\left(\hat{F}_{4i+1} = 1\right) = \frac{7}{2^{3(i+1)+1}}.
\]

3.2. **Proof of Theorem 3.2.** We have that

$N_1 = \{1\} \times \{v^3_+, v^3_-\}^N$.

We consider the mapping $T$ defined for any $w = (w_i)_i \in \mathcal{A}$, by

$T(w) = 1, w$.

For $(u, v) = ((u_j)_j, (v_j)_j) \in \mathcal{A}^2$, we have

\[
d(T(u), T(v)) = \sum_{j=1}^{\infty} \frac{|T(u)_j - T(v)_j|}{2^j} = \sum_{j=2}^{\infty} \frac{|u_{j-1} - v_{j-1}|}{2^j} = \frac{1}{2} d(u, v).
\]

This means that $T$ is a bi-Lipschitz mapping. Since $N_1 = T(N)$, we have

$\dim_H(N_1) = \dim_H(T(N)) = \frac{1}{4}$.

Coming back to Corollary 3.3 we deduce

$\dim_H(N_1) = \dim_B(N_1) = \dim_P(N_1) = \frac{1}{4}$.

4. **Concluding remarks and perspectives**

We think it to be very interesting to make the point on some remarks and possible extensions of our work.

(1) The results given by Theorems 3.1 and 3.2 remain still valid if we take $w_1 = -1$. In other words, if we take the tribonacci sequence defined by (3.1) and consider the set

$F_{-1}(w) = \{ n \geq 1, \hat{F}_n(w) = -f_1 \}$

and if we denote, for $i \in \mathbb{N}$, by $R_{-1,i}$ the event for which elements $\hat{F}_n$ passes through $(-f_1)$ exactly $i$ times, i.e.,

$R_{-1,i} = \left\{ w \in \mathcal{A} \mid \sharp F_{-1}(w) = i \right\}$,

then, we have that $\mathbb{P}(R_{-1,i}) = \frac{7}{2^{3(i+1)+1}}$.

Moreover, if the set $N_{-1}$ consists on the elements of $\mathcal{A}$ for which $\hat{F}_n$ passes through $(-f_1)$ an infinite number of times, i.e.,

$N_{-1} = \left\{ w \in \mathcal{A} \mid \sharp F_{-1}(w) = \infty \right\}$,
then, \( N_{-1} \) is such that \( \dim_H(N_{-1}) = \dim_P(N_{-1}) = \dim_B(N_{-1}) = \frac{1}{4} \).

(2) The results obtained in this work depend strongly on the \( k \) initializing terms of the \( k \)-bonacci sequence: \((f_i)_{1 \leq i \leq k}\). Particularly, thanks to (AS), \( \hat{F}_n \) is allowed to visit 0 or \( \pm f_1 \) only one time in its first \( k \) steps of the walk: \((\hat{F}_i)_{1 \leq i \leq k}\). If (AS) is no longer satisfied, then \( \hat{F}_n \) can reach these values (0 or \( \pm f_1 \)) more than one time before its \((k + 1)\)th term: \( \hat{F}_{k+1} \).

(3) One can think of the possibility to reach other terms the \( k \)-bonacci sequence and the eventual necessary and/or sufficient conditions to realize this task by the \( \hat{F}_n \).

**Appendix A. Fractal dimensions**

For a non-empty subset \( U \) of the Euclidian space \( \mathbb{R}^m, m \geq 1 \), the diameter of \( U \) is defined as

\[ |U| = \sup\{|x - y|, x, y \in U\}. \]

Let \( I \) and \( F \) be respectively nonempty subsets of \( \mathbb{N} \) and \( \mathbb{R}^n \) (\( I \) may be either finite or countable). We say that \((U_i)_{i \in I}\) is a \( \delta \)-covering of \( F \) if

\[ F \subset \bigcup_{i \in I} U_i \quad \text{and} \quad 0 < |U_i| \leq \delta, \quad \forall i \in I. \]

The \( s \)-dimensional Hausdorff measure of \( F \) is defined as

\[ \mathcal{H}^s(F) = \lim_{\delta \to 0^+} \inf \left\{ \sum_{i \in \mathbb{N}} |U_i|^s \right\}, \]

where the infimum is taken over all the countable \( \delta \)-coverings \((U_i)_{i \in \mathbb{N}}\) of \( F \).

The Hausdorff dimension of \( F \) is defined as

\[ \dim_H F = \sup\{s > 0, \mathcal{H}^s(F) = \infty\} = \inf\{s > 0, \mathcal{H}^s(F) = 0\}, \]

with the convention \( \sup \emptyset = 0 \) and \( \inf \emptyset = \infty \).

The \( s \)-dimensional packing measure of \( F \) is defined as

\[ \mathcal{P}^s(F) = \lim_{\delta \to 0^+} \sup \left\{ \sum_{i \in \mathbb{N}} |B_i|^s \right\}, \]

where the supremum is taken over all the packings \( \{B_i\}_{i \in \mathbb{N}} \) of \( F \) by balls centered on \( F \) and with diameter smaller than or equal to \( \delta \). The packing dimension of \( F \) is defined as

\[ \dim_P F = \sup\{s > 0, \mathcal{P}^s(F) = \infty\} = \inf\{s > 0, \mathcal{P}^s(F) = 0\}, \]

with the convention \( \sup \emptyset = 0 \) and \( \inf \emptyset = \infty \).

Let \( N_\delta(F) \) be the smallest number of sets of diameter at most \( \delta \) which can cover \( F \). The lower and upper box counting dimensions of \( F \) are respectively defined as

\[ \dim_B(F) = \lim_{\delta \to 0} \inf \frac{\log N_\delta(F)}{\log(\delta)}, \]

and

\[ \overline{\dim}_B(F) = \lim_{\delta \to 0} \sup \frac{\log N_\delta(F)}{\log(\delta)}. \]
If $\text{dim}_B(F) = \overline{\text{dim}}_B(F)$, this common value is denoted $\text{dim}_B(F)$ and referred to as box counting dimension or simply the box dimension of $F$, i.e.,

$$\text{dim}_B(F) = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{\log(\delta)}.$$

For more details, the reader can be referred, for example, to [2] [8] [12].

**APPENDIX B. Fractal dimension of iterated function system (IFS)**

Let $m$ and $p$ be two positive integers, with $p \geq 2$, and $X$ be a nonempty closed set of $\mathbb{R}^m$. A family $\{S_i, i = 1, \ldots, p\}$ of contractive mappings on $X$ is called an iterated function system (IFS) on $X$. Hutchinson showed in [5] that there is a unique nonempty compact set $K$ of $X$, called the attractor of $\{S_i, i = 1, \ldots, p\}$, such that

$$K = \bigcup_{i=1}^{p} S_i(K).$$

The local dimension at a point $x \in \mathbb{R}^m$ is defined by

$$d(\mu, x) = \lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log(r)}.$$

where $B(x, r)$ denotes the closed ball of radius $r$ centered at $x$. A probability measure $\mu$ on $\mathbb{R}^m$ is said to be exactly dimensional if there is a constant $C$ such that

$$d(\mu, x) = C, \quad \mu - a.e. x \in \mathbb{R}^m.$$

It is proved that the Hausdorff dimension of the measure $\mu$ is

$$\text{dim}(\mu) = C.$$

This result was firstly shown by Young [16]. The reader can also be referred to [3] [8] [12] for the details.

**Definition B.1.** Let $m$ and $p \geq 2$ be two positive integers, $X$ a nonempty closed set of $\mathbb{R}^m$ and $\mathcal{S} = \{S_i\}_{1 \leq i \leq p}$ an IFS on $X$. Then, $\mathcal{S}$ is said to satisfy the open set condition (OSC) if there is a non-empty, bounded and open set $V$, such that

1. $\bigcup_{i=1}^{p} S_i(V) \subset V$.
2. $S_i(V) \cap S_j(V) = \emptyset$, if $i \neq j$.

This definition allows us to recall the following result, which will make us able to calculate the fractal dimension of $N$.

**Theorem B.2.** [Theorem 9.3 in 2] Let $m$ and $p$ be two positive integers, with $p \geq 2$, $X$ be a nonempty closed set of $\mathbb{R}^m$ and $\mathcal{S} = \{S_i\}_{1 \leq i \leq p}$ be an IFS on $X$. Suppose that, for $1 \leq i \leq p$, $S_i$ is a similarity with ratio $r_i$ and attractor $F$. Suppose, also that $\mathcal{S}$ satisfies the (OSC) and let $s$ be the unique value, such that

$$\sum_{i=1}^{p} r_i^s = 1.$$

Then, $\text{dim}_H(F) = \text{dim}_B(F) = s$. 

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\[ \text{RETURN OF } \phi \text{-Bonacci RANDOM WALKS} \]
In particular, if \( r_1 = \ldots = r_p = r \) for some \( r \), then
\[
\dim_H(F) = \dim_B(F) = -\frac{\log n}{\log r}.
\]

The reader can find a proof of the dimension formula for self-similar sets either in [2] or in [9]. It is well known [2] that
\[
\dim_H(F) \leq \dim_P(F) \leq \dim_B(F).
\]

**Corollary B.3.** [2] Suppose that the conditions of Theorem B.2 are satisfied. If we have, in addition, \( r_1 = \ldots = r_p = r \). Then,
\[
\dim_H(F) = \dim_P(F) = \dim_B(F) = -\frac{\log n}{\log r}.
\]

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