LECTURE NOTES ON TURBULENT INSTABILITY OF ANTI-DE SITTER SPACETIME *

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In these lecture notes we discuss recently conjectured instability of anti-de Sitter space, resulting in gravitational collapse of a large class of arbitrarily small initial perturbations. We uncover the technical details used in the numerical study of spherically symmetric Einstein-massless scalar field system with negative cosmological constant that led to the conjectured instability.

Keywords: Anti-de Sitter Space; General Relativity; Gravitational Collapse; Numerical Methods.

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1. Introduction

Anti-de Sitter (AdS) spacetime is the unique maximally symmetric solution of the vacuum Einstein equations \( G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 0 \) with negative cosmological constant \( \Lambda \).

Geometrically, \( \text{AdS}_{d+1} \) can be thought of as being wrapped infinitely many times around hyperboloid

\[- (X^0)^2 + \sum_{k=1}^{d} (X^k)^2 - (X^{d+1})^2 = -\ell^2, \]

embedded in flat, \( O(d, 2) \) invariant space, with a line element

\[ds^2 = - (dX^0)^2 + \sum_{k=1}^{d} (dX^k)^2 - (dX^{d+1})^2.\]

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Parametrization

\[ X^0 = \ell \sec x \cos t, \quad X^{d+1} = \ell \sec x \sin t, \quad X^k = \ell \tan x n^k, \quad (3) \]

with

\[ -\infty < t < +\infty, \quad 0 \leq x < \pi/2, \quad \sum_{k=1}^{d} (n^k)^2 = 1, \quad (4) \]

yields induced metric on the hyperboloid in the form

\[ ds^2 = \frac{\ell^2}{\cos^2 x} \left( -dt^2 + dx^2 + \sin^2 x d\Omega^2_{S^{d-1}} \right). \quad (5) \]

This metric is indeed a solution to the vacuum Einstein equations with \( \Lambda = -d(d-1)/(2\ell^2) \). Conformal infinity for this metric, located at \( x = \pi/2 \) is the timelike cylinder \( I = \mathbb{R} \times S^{d-1} \) with the boundary metric \( ds_I^2 = -dt^2 + \sin^2 x d\Omega^2_{S^{d-1}} \).

Asymptotically anti-de Sitter (AAdS) spacetimes (that is spacetimes which share the conformal boundary with AdS but may be very different in the bulk, in particular may contain horizons) have come to play a central role in theoretical physics, prominently due to the AdS/CFT correspondence which conjectures a duality between gravity in the AdS bulk and a quantum conformal field theory with a large number of strongly interacting degrees of freedom living in the spacetime corresponding to the AdS conformal boundary.\(^2,3\) By the positive energy theorem, AdS spacetime is a ground state among AAdS spacetimes, much as Minkowski spacetime is a ground state among asymptotically flat spacetimes. However, the evolution of small perturbations of these ground states are different. In the case of Minkowski, small perturbations disperse to infinity and the spacetime is asymptotically stable.\(^4\) In contrast, asymptotic stability of AdS is precluded because the conformal boundary acts like a mirror at which perturbations propagating outwards bounce off and return to the bulk that results in complex nonlinear wave interactions in an effectively bounded domain. Understanding of these interactions is the key to the problem of stability of AdS spacetime.

A recent numerical and analytic study of the four dimensional spherically symmetric Einstein-massless scalar field equations with negative cosmological constant indicated that AdS space may be unstable against the formation of a black hole under large class of arbitrarily small perturbations.\(^5\) Qualitatively the same results were obtained later in higher dimensions.\(^6,7\) Although gravitational collapse seems to be a generic fate of a small perturbation of AdS, it was suggested in\(^5\) that there may also exist a set of initial data for which the evolution remains globally regular in time. This conjecture was substantiated in,\(^8\) where the evidence for the existence of globally regular, nonlinearly stable, time-periodic solutions in the Einstein AdS - massless scalar field system was given. Similar class of globally regular, nonlinearly stable, asymptotically AdS solutions, namely boson stars was studied in.\(^9\) The similar behavior is expected for the pure vacuum case with simplifying symmetry assumptions\(^10\) and with no symmetry assumptions.\(^11,12\) The outcome of all
these studies suggests that the structure of phase space for gravity with negative cosmological constant is complicated and probably for still some time numerical simulations will be the key tool to investigate this problem and to assist analytic attempts like. Thus in this short lecture notes we will concentrate on numerical tricks and details used in numerical codes of\textsuperscript{5,6} that are usually omitted in research papers, however crucial they are for stable long-time numerical simulations of AAdS spacetimes.

2. The model

To make the long-time numerical investigation of AdS stability tractable we assume spherical symmetry. Since by Birkhoff’s theorem spherically symmetric vacuum solutions are static, we need to add matter to generate dynamics. A simple matter model is the minimally coupled massless scalar field in $d+1$ spacetime dimensions:

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi G \left( \partial_\alpha \phi \partial_\beta \phi - \frac{1}{2} g_{\alpha\beta} \partial_\mu \phi \partial^\mu \phi \right),$$

$$g^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi = 0.$$  

Let us recall that in the asymptotically flat case ($\Lambda = 0$) this model has led to important insights, such as the proof of the weak cosmic censorship by Christodoulou\textsuperscript{14,15} and the discovery of critical phenomena at the threshold for black hole formation by Choptuik.\textsuperscript{16} We parametrize the $(d+1)$–dimensional asymptotically AdS metric by the ansatz

$$ds^2 = \frac{\ell^2}{\cos^2 x} \left( -Ae^{-\delta} dt^2 + A^{-1} dx^2 + \sin^2 x d\Omega_{d-1}^2 \right),$$

where $\ell^2 = -d(d-1)/(2\Lambda)$, $d\Omega_{d-1}^2$ is the round metric on $S^{d-1}$, $-\infty < t < \infty$, $0 \leq x < \pi/2$, and $A, \delta$ are functions of $(t, x)$. For this ansatz the evolution of a self-gravitating massless scalar field $\phi(t, x)$ is governed (see the accompanying Mathematica notebook equations.nb\textsuperscript{1}) by the following system (using units in which $8\pi G = d - 1$)

$$\dot{\Phi} = (Ae^{-\delta} \Pi)' , \quad \Pi = \frac{1}{\tan^{d-1} x} \left( \tan^{d-1} x A e^{-\delta} \Phi \right)' ,$$

$$A' = \frac{d - 2 + 2 \sin^2 x}{\sin x \cos x} \left( 1 - A \right) - \sin x \cos x A \left( \Phi^2 + \Pi^2 \right) ,$$

$$\delta' = -\sin x \cos x \left( \Phi^2 + \Pi^2 \right) ,$$

where $' = \partial_t$, $' = \partial_x$, and

$$\Phi = \phi', \quad \Pi = A^{-1} e^{\delta} \phi.'$$

Note that the length scale $\ell$ drops out from the equations. This system has a one-parameter family of static solutions ($\phi = 0$, $\delta = \text{const}$, $A = 1 - M \cos^2 x/(\tan x)^{d-2}$) which are Schwarzschild-AdS black holes for $M > 0$ and the pure AdS for $M = 0$. In
analogy to the Schwarzschild-AdS black hole the mass function for the system can be defined as \( m(t, x) = (1 - A(t, x)) \sec^2 x (\tan x)^{d-2} \). We restrict our attention to smooth solutions with a finite mass

\[
M(t) := \lim_{x \to \pi/2} m(t, x) = \int_0^{\pi/2} (A \Phi^2 + A \Pi^2) (\tan x)^{d-1} \, dx < \infty.
\]

(13)

Smoothness at spatial infinity and the requirement of the total mass (13) to be finite, imply that near \( x = \pi/2 \) (using \( z = \pi/2 - x \)) (see the accompanying MATHEMATICA notebook boundary.nb)

\[
A(t, x) = 1 - M z^d + O(z^{d+2}), \quad \delta(t, x) = \delta_0(t) + O(z^d),
\]

\[
\phi(t, x) = f_0 + f_d(t) z^d + O(z^{d+2}),
\]

(14)

where the subsequent terms in the expansions are expressed by constant \( M \) and the functions \( f_0(t), \delta_0(t) \) and their derivatives. These are in turn uniquely determined by the evolution of initial data \(^a\). Thus in this particular model if we require the total mass to be finite it implies the mass to be conserved as well. The local well-posedness of the above initial-boundary value problem was proved in.\(^{17}\) It is important to stress that for odd \( d \) higher even terms in the expansion for \( \phi \) do not vanish (they vanish identically only in the limit \( M \to 0 \)). Exercise 1: verify this statement using boundary.nb notebook. Thus for odd \( d \) the boundary behavior of the scalar field is not compatible with eigenfunctions of the linear self-adjoint operator, that governs the evolution of a massless scalar field on a fixed AdS\(_{d+1} \) background. This operator reads

\[
L = - (\tan x)^{1-d} \partial_x \left( (\tan x)^{d-1} \partial_x \right).
\]

Its spectrum is given by \( \omega^2_j = (d + 2j)^2, \ j = 0, 1, ... \) and the eigenfunctions read

\[
e_j(x) = 2 \frac{\sqrt{j! (j + d - 1)!}}{\Gamma(j + d/2)} \cos^d x P_j^{(d/2-1,d/2)}(\cos 2x),
\]

(15)

where \( P_j^{(\alpha,\beta)}(x) \) are the Jacobi polynomials. These eigenfunctions form an orthonormal base in the Hilbert space of functions \( L^2([0, \pi/2], (\tan x)^{d-1} \, dx) \). Below we denote the inner product on this Hilbert space by \( (f | g) := \int_0^{\pi/2} f(x)g(x)(\tan x)^{d-1} \, dx \).

3. Numerical method

We solve the system (9)-(11) numerically using a fourth-order accurate finite-difference approximation for spatial derivatives in evolution equations. For the general introduction to finite difference approximation see Hirota Okawa’s lecture in this volume. To integrate the evolution equations in time we use the method of lines and a 4th-order Runge-Kutta (RK4) scheme, where at each time step the metric functions are updated by solving constraint equations; the Hamiltonian constraint

\(^a\)In particular, for compactly supported initial data \( f_0 = 0 \).
(10) and the slicing condition (11) with RK4 in space. The (adjustable) time step \( \Delta t \) is kept \( 1/6 \leq e^{-\delta(t,\pi/2)} \Delta t/\Delta x \leq 1/3 \) for the constant spatial grid spacing \( \Delta x \). Preservation of the momentum constraint \( \dot{\Pi} + 2 \sin x \cos x A^2 e^{-\Phi} \Pi = 0 \) can be monitored to check the accuracy of the code. This (fully constrained) scheme allows for stable long-time evolution. Another crucial ingredient of stable evolution of non-linear evolution equations on compact domains is careful treatment of boundaries of numerical domain. Here as method of thumb we avoid nonsymmetric stencils at the boundary. Instead we try to put as much analytic information as we have into the code. That is we use:

1. Symmetry of the function. If we know that the function is symmetric or anti-symmetric with respect to the boundary of numerical domain we use this information in finite-difference approximation for the derivatives of this function at the boundary. For example in the 4th order finite difference approximation for the 1st derivative of the function \( f(x) \) at \( x = 0 \)
\[
f'(0) = (f(-2\Delta x) - 8f(-\Delta x) + 8f(\Delta x) - f(2\Delta x))/(12\Delta x) + \mathcal{O}(\Delta x^4)
\]
we can put \( f(-k\Delta x) = \pm f(k\Delta x) \) for symmetric and antisymmetric functions respectively.

2. Taylor expansion of the function. If we know the Taylor expansion of the function at the boundary of numerical domain we can use this information to express the function at the boundary with its values inside the domain. For example if the function \( f(x) \) has the following Taylor expansion at \( x = 0 \)
\[
f(x) = f_0 + f_2x^2 + f_4x^4 + \mathcal{O}(x^6),
\]
then from the set of linear equations \( f(k\Delta x) = f_0 + f_2(k\Delta x)^2 + f_4(k\Delta x)^4 \) for \( k = 1, 2, 3 \) we can get the coefficients \( f_0, f_2, f_4 \) and in particular \( f(0) = f_0 \).

3. De l’Hôpital’s rule. It often happens that the right hand side of evolution equations at the boundary is of the form \( 0/0 \) (as happens here for \( \dot{\Pi} \) in (9) at \( x = 0 \) and \( x = \pi/2 \)). A useful numerical trick to deal in such case is to use de l’Hôpital’s rule. For example suppose that \( f(x) \xrightarrow{z \to 0} 0 \). Then to deal with numerical instabilities produced by \( f(x)/x \) at small \( x \) one can use the identity \( f/x = f'(x) - x(f'/x)' \), where the instabilities coming from small denominators are suppressed by the small factor \( x \) comparing to the leading term \( f'(x) \). We have learned this useful trick from Anıl Zenginoğlu during his stay in Cracow in 2010.

In particular, in our code numerical boundaries \( (x = 0 \) and \( x = \pi/2) \) are treated as follows. We put Eq. (9) for \( \dot{\Pi} \) in the form
\[
\dot{\Pi} = (Ae^{-\Phi})' + (d-1)Ae^{-\Phi} \frac{\Phi'}{\sin x \cos x},
\]
where close to the boundaries we use de l’Hôpital’s rule:
\[
\frac{Ae^{-\Phi}}{\sin x \cos x} = \frac{(Ae^{-\Phi})'}{\cos 2x} - \frac{1}{2} \tan 2x \left( \frac{Ae^{-\Phi}}{\sin x \cos x} \right)'.
\]
This is crucial for the stable evolution at $x = \pi/2$ and convenient in the center $x = 0$: if de l’Hôpital rule (18) is used in the center, numerical dissipation can be turned off. If the de l’Hôpital trick is not applied at $x = 0$ then one should use the equation for $\Pi$ in the form (9), as the form (17) is numerically unstable(!) in the symmetry center. The values $\Phi(t, 0)$, $\Phi(t, \pi/2)$, $\Pi(t, \pi/2)$ are set to zero due to boundary behavior. The values $\Pi(t, 0)$, $\Phi(t, \pi/2 - \Delta x)$, $\Pi(t, \pi/2 - \Delta x)$ are set from the Taylor expansion at the boundaries. Symmetry properties of $\Phi$, $\Pi$, $A$ and $\delta$ in the center $x = 0$ are used when needed in evaluating spatial derivatives on the right hand side of evolution equations near the center. In the on-line material\footnote{we include the Mathematica notebook \texttt{evolution.nb} that allows for testing of what was stated above.} we include the Mathematica notebook \texttt{evolution.nb} that allows for testing of what was stated above.\footnote{Exercise 2: experiment with \texttt{evolution.nb} switching de l’Hôpital trick at $x = \pi/2$ and/or $x = 0$ on and off; see what happens. Exercise 3: turn de l’Hôpital trick at $x = 0$ off and implement the form (17) in the center; see what happens.} 

At each time step we check for the minimum of $A(t, x)$. If it drops below a certain threshold (that we set to be $2^{-k+7}$ on the grid with $n = 2^k + 1$ points, with $k \geq 10$) we say that an apparent horizon forms and we stop the evolution.

If transfer of energy to higher modes of linearized problem is present during the evolution of initial data, then higher spatial resolution may be needed. If that happens we perform global mesh refinement from $\Delta x$ to $\Delta x/2$ grid spacing. This brute force approach requires a lot of computational power and should be replaced with some more subtle algorithm in the future. At this point the brute force is supplied with parallel computing. We use MPI library (for a concise introduction to MPI programming see\textsuperscript{18}). While stepping in time in evolution equations (9) is naturally suited for parallelization, solving the constraints (10, 11) can be a thin throat for parallel computing. We solve this problem in the way, that processors are not synchronized on the grid. That is, while solving a constraint equation (what is being done in sequence from $x = 0$ to $x = \pi/2$) a processor waits only for its nearest left-hand neighbor on the grid to complete the task. As a result processors are not synchronized and these starting solving the constraints (at $x = 0$ in our case) can be a few steps in time ahead comparing to those finishing a given time step (on a part of the grid close to $x = \pi/2$ in our case). In this setting, time step and horizon formation control that requires exchange of information between non-neighboring processors is realized with non-blocking MPI library procedures \texttt{MPI\_isend} and \texttt{MPI\_irecv}. In this way we get a scaling with the factor of two with the number of cores.
4. Numerical results and their origin

Our key numerical argument for AdS instability is depicted in Fig. 1 (quoted from\(^5\)). We numerically evolve initial data of the form

\[
\Phi(0, x) = 0, \quad \Pi(0, x) = \varepsilon \exp \left( -\frac{4 \tan^2 x}{\pi^2 \sigma^2} \right),
\]

with fixed width \(\sigma = 1/16\) and varying amplitude \(\varepsilon\). For such data the scalar field is well localized in space and propagates in time as a narrow wave packet. We look at the Ricci curvature in the center, \(\ell^2 R(t, 0) = -2\Pi^2(t, 0) - 12\), that can serve as a good indicator for the onset of instability. This quantity oscillates with frequency \(\approx 2\) (as it takes time \(\approx \pi\) for the wave packet to make the round trip from and back to the center). An upper envelope of the oscillations of \(\Pi^2(t, 0)\) is shown in Fig. 1a. We see that after some time, when the amplitude remains approximately constant, there begins a subsequent phases of (roughly) exponential growth, until finally the solution collapses. It can be seen that the time of onset of the subsequent phases scales as \(\varepsilon^{-2}\) (see Fig. 1b). Indeed, after the rescaling \(\Pi^2(t, 0) \rightarrow \varepsilon^{-2}\Pi^2(\varepsilon^2 t, 0)\) all curves of Fig. 1a seem to converge to some universal curve (for this family of initial data with \(\sigma\) fixed to 1/16), that means that arbitrarily small perturbations eventually start growing. Note that this behavior is morally tantamount to instability of AdS space, regardless of what happens later, in particular whether the solution will collapse or not. In the second part of\(^5\) we sketched the mechanism behind this numerically discovered instability of AdS in the framework of weakly nonlinear perturbation theory (that we previously extensively used to predict nonlinear tails of dispersing solution in asymptotically flat case). To summarize there are two key ingredients to the mechanism of the instability of AdS:

\[\text{In the evolution.nb notebook, for the clarity of the code, we use nonsymmetric stencils for evaluating } \Phi(t, \pi/2 - \Delta x), \Pi(t, \pi/2 - \Delta x) \text{ instead of setting } \Phi(t, \pi/2 - \Delta x), \Pi(t, \pi/2 - \Delta x) \text{ from the Taylor expansion at the boundary.}\]
Conservation of the total mass of the system. The first crucial ingredient is conservation of the total mass of the system, that (opposite to a flat case) can not be dispersed to infinity. Conformal structure of AdS space makes it effectively bounded and to make the initial-boundary value problem well posed setting boundary conditions may be needed. In the model at hand reflecting boundary conditions at conformal infinity follow directly from the regularity of solutions of Einstein equations at the boundary and the requirement of the total mass (13) to be finite. More general models, like conformally coupled scalar field or AdS-Einstein-Yang-Mills system, allow for more freedom in setting boundary conditions, where the conservation of the total mass at conformal infinity can be picked up as physically natural boundary condition. However one should keep in mind that other choices are possible as well. We are grateful to Helmut Friedrich for stressing this point.

Resonant spectrum. The second crucial ingredient for instability is the resonant spectrum of the wave operator on the fixed AdS background (see the previous section). That is the spectrum consists of equidistantly spaced real numbers. Once the modes of the corresponding wave operator are coupled through gravity (or nonlinearity) it results in resonant coupling between the modes that generically, efficiently transfers (conserved) energy into higher and higher modes. We call this process a turbulent transfer of energy, in analogy to Navier-Stokes equations that transfer the eddies to smaller and smaller spatial scales. In the case of Einstein equations this process is ultimately cut by a black hole formation, but mind the special case of $d = 2$. Interestingly, it seems that asymptotically resonant spectrum is also good enough for such turbulent transfer of energy to take place, see.\(^{20}\)

To demonstrate the transfer of energy to higher frequencies we define the Fourier coefficients $\Phi_j := \left( e_j \left| \sqrt{A} \Phi \right| \right)$ and $\Pi_j := \left( e_j \left| \sqrt{A} \Pi \right| \right)$ and express the total (conserved) mass as the Parseval sum $M = \sum_{j=0}^{\infty} E_j(t)$, where $E_j := \Pi_j^2 + \omega_j^{-2} \Phi_j^2$ is the $j$-mode energy. The evolution of the energy spectrum, that is the distribution of mass among the modes, is depicted in Fig. 2 for gaussian initial data (19) with $\varepsilon = 6/\pi$. Initially, the energy is concentrated in low modes; the exponential cutoff of the spectrum expresses the smoothness of initial data. During the evolution the range of excited modes increases and the spectrum becomes broader. Just before horizon formation the spectrum exhibits the power-law scaling $E_j \sim j^{-\alpha}$ with exponent $\alpha \approx 6/5$. This value seems to be universal, i.e., the same for all initial data, but it changes with dimension $d$. Our preliminary guess based on studying the instability in $d = 3, 4, 5$ is that

$$\alpha(d) = 6/5 + 4(d - 3)/5.$$  \((20)\)

Note that the formation of a black hole provides a cutoff for the turbulent energy cascade (in amusing analogy to viscosity for the turbulent cascade in fluids). Clearly, the formation of the power-law spectrum reflects the loss of smoothness of the
solution during collapse; it would be very interesting to compute $\alpha(d)$ analytically.

The plot in Fig. 2 was made for the instants of time when the scalar field was imploding through the center, and decaying exponentially fast at infinity (because we started with compactly supported initial data). Mind that, in odd number of spatial dimensions $d$, if we expressed the total mass as the Parseval sum over the modes at instants of time when the scalar pulse is concentrated close to infinity (with a polynomial decay according to (14)) we would get steep but still polynomial tail of energy spectrum just because of the fact that the basis of the eigenfunctions of the wave operator on a fixed AdS background is incompatible (in odd $d$) with the boundary behavior of the massless scalar field in the self-gravitating case. Of course, such polynomial decay would have nothing to do with the effect depicted in Fig. 2 and would just illustrate the fact that the $e_j(x)$ base is not well adapted to self-gravitating case in odd number of spatial dimensions.

Fig. 2. Log-log plot of the energy spectrum at three moments of time: initial, intermediate, and just before collapse for the solution evolving from gaussian initial data (19) with $\epsilon = 6/\pi$. The fit of the power law $E_j \sim j^{-\alpha}$ at time $t = 950 \times \pi/2$ gives the slope $\alpha \approx 6/5$.

5. Time-periodic solutions

This section is in fact only a short overview of the paper, covering the second lecture delivered by one of the authors at NR/HEP2 school.

However weakly nonlinear perturbation theory provides a good hint for explanation of the mechanism of AdS instability it fails to provide quantitative predictions for a given initial data, like time of the black hole formation, its initial radius, etc. Still it proved to be extremely useful in perturbative construction of time-periodic solutions.

It turns out that negative cosmological constant allows for stable, globally regular time-periodic solutions. In that paper we constructed time-periodic solutions
with two independent methods. Perturbative method consists in representing the
time-periodic solution as a perturbative series in the amplitude of a mode dominat-
ing perturbative solution. The subsequent terms in this expansion are determined
by removing resonant terms that generically arise. Numerical construction is build
upon new spectral code well suited to study the solutions that do not collapse. Con-
vergence of results of these two independent methods makes us feel confident in the
results. Moreover numerical evolution of the initial data for a time-periodic solu-
tions found with those methods reproduces this solution. This is a strong evidence
that those time-periodic solutions are nonlinearly stable.

After these lectures had been delivered it was shown in,\textsuperscript{9} that the initial data
(19) with $\sigma \approx 0.5$ are immune to the turbulent instability. We are building-up
the evidence\textsuperscript{21} that the explanation to this immunity comes from the existence of
nonlinearly stable time-periodic solutions in the system. Namely, for the initial data
(19) there is a window in $\sigma$, where they belong to the stability island of time-periodic
solution bifurcating from the $e_0(x)$ mode of the massless scalar field propagating on
a \textit{fixed} AdS$_{3+1}$ background.

6. Summary and outlook

In these lecture notes we discussed recently conjectured instability of anti-de Sitter
space, highlighting numerical techniques used in the study of spherically symmetric
Einstein - massless scalar field system with negative cosmological constant.\textsuperscript{5,6}
This case is a neat example of the fact that from numerical explorations of Ein-
stein’s equations there can grow understanding, conjectures, roads to proofs and
phenomena that would not have been imaginable in the pre-computer era. The role
of computation in general relativity seems destined to expand in future, playing a
role of a telescope for a theoretical physicist.

The dynamics of AAdS turns out to be an exceptional meeting point of funda-
mental problems in general relativity, PDE theory, theory of turbulence, and high
energy physics. Understanding of these connections is still at its infancy. The issues
discussed here raise probably more questions than give answers.

• The proofs of instability of AdS and the existence of time-periodic solutions are
still lacking. There is a natural question here: what makes the AdS spacetime
unstable, while making some other AAdS solutions\textsuperscript{8,9,12} stable. Probably it
would be rewarding to understand the resonant coupling between the modes and
the energy spectrum (20) in more detail. There is also some tension between
numerical\textsuperscript{20} and heuristic results\textsuperscript{12} to what extent only \textit{asymptotically}
(that is for high wave numbers) resonant spectrum is good enough for a turbulent
transfer of energy to take place.

• So far, our studies have been restricted to spherical symmetry (or other effec-
tively $1 + 1$ dimensional settings). Thus relaxing symmetry assumptions should
be the next step. The heuristic arguments\textsuperscript{11,12} suggest that the similar picture
of instability, with some basins of attraction of some stable AAdS solutions

...
emerges also outside spherical symmetry. The numerical studies of axially sym-
metric AAdS spacetimes containing a black hole have been initiated in but to
build a code capable of evolving small perturbations of AdS and detecting
 eventual black hole formation there is still some way to go.

• The next question (closely related to the previous point) is about the end state
of evolution. The AdS-Schwarzschild solution, being stable, is a natural can-
didate in spherical symmetry. However outside spherical symmetry, the problem
is more delicate as small AdS-Kerr black holes are expected to be unstable due
to superradiant instability.

• There are some more general models like conformaly coupled scalar field or
Einstein-Yang-Mills system, allowing for more freedom in prescribing boundary
conditions at the timelike boundary of AAdS. We are currently studying these
more general boundary conditions.

• It might be rewarding to understand better AdS/CFT counterpart of stable
AAdS solutions.

Hopefully we will see some advances at least on the part of these problems in not
too far future.

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Appendix. List of publicly available codes
There are three publicly available MATHEMATICA notebooks, to be downloaded
from NR/HEP2 Spring School web page that are integral part of these lecture
notes:

• equations.nb - derivation of the system of equations (9)-(11).
• boundary.nb - derivation of the boundary behavior (14).
• evolution.nb - schematic code for time evolution.

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