Let $R$ be a Noetherian local ring, $I$ and $J$ two ideals of $R$, $M$ an $R$-module and $s$ and $t$ two integers. We study the relationship between the Bass numbers of $M$ and $H^i_{I,J}(M)$. We show that $\mu^t(M) \leq \sum_{i=0}^{t} \mu^{s+i}(H^i_{I,J}(M))$ and $\mu^s(H^t_{I,J}(M)) \leq \sum_{i=0}^{s-1} \mu^{s+t+1-i}(H^i_{I,J}(M)) + \mu^{s+t}(M) + \sum_{i=0}^{t-1} \mu^{s+t-1-i}(H^i_{I,J}(M))$.

As a consequence, it follows that if $I$ is a principal ideal of $R$ and $M$ is a minimax $R$-module, then $\mu^j(H^i_{I,J}(M))$ is finite for all $i \in \mathbb{N}_0$ and all $j \in \mathbb{N}_0$.

1. Introduction

Throughout this paper, $R$ is a commutative Noetherian ring with non-zero identity, $I$ and $J$ are two ideals of $R$, $M$ is an $R$-module and $s$ and $t$ are two integers. For notations and terminologies not given in this paper, the reader is referred to [3], [4] and [16] if necessary.

The theory of local cohomology, which was introduced by Grothendieck [7], is a useful tool for attacking problems in commutative algebra and algebraic geometry. Bijan-Zadeh [2] introduced the local cohomology modules with respect to a system of ideals, which is a generalization of ordinary local cohomology modules. As a special case of these extend modules, Takahashi, Yoshino and Yoshizawa [16] defined the local cohomology modules with respect to a pair of ideals. To be more precise, let $W(I,J) = \{ p \in \text{Spec}(R) : I'' \subseteq J + p \text{ for some positive integer } t \}$. The set of elements $x$ of $M$ such that $\text{Supp}_R Rx \subseteq W(I,J)$, is said to be $(I,J)$-torsion submodule of $M$ and is denoted by $\Gamma_{I,J}(M)$. $\Gamma_{I,J}(-)$ is a covariant, $R$-linear functor from the category of $R$-modules to itself. For an integer $i$, the local cohomology functor $H^i_{I,J}(-)$ with respect to $(I,J)$, is defined to be the $i$-th right derived functor of $\Gamma_{I,J}(-)$. Also $H^i_{I,J}(M)$ is called the $i$-th local cohomology module of $M$ with respect to $(I,J)$. If $J = 0$, then $H^i_{I,J}(-)$ coincides with the ordinary local cohomology functor $H^i_I(-)$. Let $\tilde{W}(I,J) = \{ a \subseteq R : I' \subseteq J + a \text{ for some positive integer } t \}$. It is

2010 Mathematics Subject Classification. 13D45, 13D07.

Key words and Phrases. Bass Number, Extension functor, Local cohomology.
easy to see that
\[ \Gamma_{I,J}(M) = \{x \in M : \exists a \in \tilde{W}(I,J), ax = 0\} = \bigcup_{a \in \tilde{W}(I,J)} (0:_M a). \]

An important problem in commutative Algebra is to determine when the Bass numbers of the \(i\)-th local cohomology module is finite. In [9] Huneke conjectured that if \((R, m, k)\) is a regular local ring, then for any prime ideal \(p\) of \(R\) the Bass numbers \(\mu^j(p, H^i_I(R)) = \dim k(p) \operatorname{Ext}^j_{R_p}(k(p), H^i_I(R_p))\) are finite for all \(i \in \mathbb{N}_0\) and all \(j \in \mathbb{N}_0\). There are some evidences that this conjecture is true; see [10], [12] and [13]. On the other hand, there is a negative answer to the conjecture (over a non-regular ring) that is due to Hartshorne, see [8]. However the conjecture does not hold over a non-regular ring, Kawasaki [11] proved that if \(R\) is a local ring, \(I\) is a principal ideal of \(R\), and \(M\) is a finitely generated \(R\)-module, then \(\mu^j(p, H^i_I(M))\) is finite for all \(p \in \text{Spec}(R)\), all \(i \in \mathbb{N}_0\) and all \(j \in \mathbb{N}_0\).

Dibaei and Yassemi [5] studied the relationship between the Bass numbers of an \(R\)-module and those of its local cohomology modules. They show that if \(R\) is a local ring, then
\[ \mu^t(M) \leq \sum_{i=0}^t \mu^{t-i}(H^i_I(M)) \]
and
\[ \mu^s(H^i_I(M)) \leq \sum_{i=0}^{t-1} \mu^{s+t+1-i}(H^i_I(M)) + \mu^{s+t}(M) + \sum_{i=t+1}^{s+t-1} \mu^{s+t-1-i}(H^i_I(M)). \]

In [2.4], by a different method, we generalize this result for local cohomology modules with respect to a pair of ideals. As a consequence, it follows that if \(R\) is a local ring and \(I\) is a principal ideal of \(R\), then \(\mu^j(H^i_{I,J}(M))\) is finite for all \(i \in \mathbb{N}_0\) and all \(j \in \mathbb{N}_0\), where \(M\) is a minimax \(R\)-module; see [2.7].

In section 3, we get some isomorphisms about the extension functors of local cohomology modules, which imply some equalities about the Bass numbers of local cohomology modules.

2. Bass numbers

Recall that \(R\) is a Noetherian ring, \(I\) and \(J\) are ideals of \(R\) and \(M\) is an \(R\)-module. The following theorem is the main result of this paper.

\textbf{Theorem 2.1.} Let \(N\) be an \((I,J)\)-torsion \(R\)-module. Then
(i) \[ \dim_R \text{Ext}_R^t(N, M) \leq \sum_{i=0}^{t} \dim_R \text{Ext}_R^{t-i}(N, H_{t,j}^i(M)). \]

(ii) \[ \dim_R \text{Ext}_R^t(N, H_{t,j}^i(M)) \leq \sum_{i=0}^{t-1} \dim_R \text{Ext}_R^{s+t+1-i}(N, H_{t,j}^i(M)) \]
\[ + \dim_R \text{Ext}_R^{s+t}(N, M) \]
\[ + \sum_{i=t+1}^{s+t-1} \dim_R \text{Ext}_R^{s+t-1-i}(N, H_{t,j}^i(M)). \]

Proof. Let \( F(\cdot) = \text{Hom}_R(N, \cdot) \) and \( G(\cdot) = \Gamma_{t,j}(\cdot) \). Then we have \( FG(M) = \text{Hom}_R(N, M) \). By [15, Theorem 11.38], there is the Grothendieck spectral sequence
\[ E_2^{p,q} := \text{Ext}_R^p(N, H_t^q(M)) \Rightarrow \text{Ext}_R^{p+q}(N, M). \]

There is a finite filtration
\[ 0 = \varphi^{p+q+1}H^{p+q} \subseteq \varphi^{p+q}H^{p+q} \subseteq \cdots \subseteq \varphi^1H^{p+q} \subseteq \varphi^0H^{p+q} = \text{Ext}_R^{p+q}(N, M) \]
such that \( E_{s-i,j}^{p+q} \cong \varphi^{p+q-i}H^{p+q}/\varphi^{p+q+1-i}H^{p+q} \) for all \( i \leq p+q \).
(i) We have to show that \( \dim_R \varphi^0H^t \leq \sum_{i=0}^{t} \dim_R E_2^{t-i,i} \). The sequence
\[ 0 \to \varphi^{t+1-i}H^t \to \varphi^{t-i}H^t \to E_\infty^{t-i,i} \to 0 \]
is exact for all \( i \leq t \). It follows that
\[ \dim_R \varphi^0H^t \leq \dim_R \varphi^1H^t + \dim_R E_\infty^{0,t} \]
\[ \leq \dim_R \varphi^2H^t + \dim_R E_\infty^{1,t-1} + \dim_R E_\infty^{0,t} \]
\[ \leq \cdots \]
\[ \leq \sum_{i=0}^{t} \dim_R E_\infty^{t-i,i}. \]
Since \( E_\infty^{t-i,i} \) is a subquotient of \( E_2^{t-i,i} \) for all \( i \leq t \), thus \( \dim_R E_\infty^{t-i,i} \leq \dim_R E_2^{t-i,i} \) and the claim holds.

(ii) We have to show that
\[ \dim_R E_2^{s,t} \leq \sum_{i=0}^{t-1} \dim_R E_2^{s+t+1-i,i} + \dim_R \varphi^0H^{s+t} + \sum_{i=t+1}^{s+t-1} \dim_R E_2^{s+t-1-i,i}. \]
The sequences

\[ 0 \longrightarrow \text{Ker} d_{t+1-i}^{s,t} \longrightarrow E_{t+1-i}^{s,t} \xrightarrow{d_{t+1-i}^{s,t}} E_{t+1-i}^{s+t+1-i,i} \]

and

\[ 0 \longrightarrow \text{Im} d_{t+1-i}^{s-t-1+i,2i-i} \longrightarrow \text{Ker} d_{t+1-i}^{s,t} \longrightarrow E_{t+2-i}^{s,t} \longrightarrow 0 \]

are exact for any integer \( i \). It follows that

\[
\dim_R E_{t+1-i}^{s,t} \leq \dim_R E_{2}^{s+2,t-1} + \dim_R \text{Ker} d_{2}^{s,t} \\
\leq \dim_R E_{2}^{s+2,t-1} + \dim_R E_{3}^{s,t} + \dim_R \text{Im} d_{2}^{s-2,t+1} \\
\leq \dim_R E_{2}^{s+2,t-1} + \dim_R E_{3}^{s+3,t-2} + \dim_R \text{Ker} d_{3}^{s,t} + \dim_R \text{Im} d_{2}^{s-2,t+1} \\
\leq \dim_R E_{2}^{s+2,t-1} + \dim_R E_{3}^{s+3,t-2} + \dim_R E_{4}^{s,t} + \dim_R \text{Im} d_{3}^{s-3,t+2} \\
\quad + \dim_R \text{Im} d_{2}^{s-2,t+1} \\
\leq \cdots \\
\leq \sum_{i=0}^{s-t-1} \dim_R E_{t+1-i}^{s+t+1-i,i} + \sum_{i=t+1}^{s-t} \dim_R \text{Im} d_{t+1-i}^{s+t+1-i,i}.
\]

Since \( E_{t+1-i}^{s+t+1-i,i} \) is a subquotient of \( E_{2}^{s+t+1-i,i} \), \( E_{s+t+2}^{s,t} = E_{\infty}^{s,t} \) is a subquotient of \( \phi^0 H^{s+t} \), and \( \text{Im} d_{t+1-i}^{s+t-1-i,i} \) is a subquotient of \( E_{2}^{s+t-1-i,i} \), the claim follows. \( \square \)

**Corollary 2.2.** Let \( N \) be an \((I,J)\)-torsion \( R \)-module. Let \( \text{Ext}^{s+t-i}_R(N, H^i_{J,J}(M)) = 0 \) for all \( i \neq t \) with \( i \leq s + t \), \( \text{Ext}^{s+t+1-i}_R(N, H^i_{J,J}(M)) = 0 \) for all \( i < t \), and let \( \text{Ext}^{s+t-1-i}_R(N, H^i_{J,J}(M)) = 0 \) for all \( t < i < s + t \). Then

\[
\dim_R \text{Ext}^{s}_R(N, H^i_{J,J}(M)) = \dim_R \text{Ext}^{s+t}_R(N, M).
\]

**Corollary 2.3.** Suppose that \( N \) is a finitely generated \( a \)-torsion \( R \)-module for some \( a \in \mathbb{W}(I,J) \). Then

(i) \[
\dim_R H^i_a(N, M) \leq \sum_{i=0}^{t} \dim_R \text{Ext}^{-i}_R(N, H^i_{J,J}(M)).
\]
\[ \dim_R \text{Ext}_R^i(N, H^i_{I,J}(M)) \leq \sum_{i=0}^{t-1} \dim_R \text{Ext}_R^{s+t+1-i}(N, H^i_{I,J}(M)) + \dim_R H^s_{a,j}(N, M) + \sum_{i=t+1}^{s+t-1} \dim_R \text{Ext}_R^{s+t-1-i}(N, H^i_{I,J}(M)). \]

**Proof.** Note that \( \Gamma_a(N) \subseteq \Gamma_{I,J}(N) \), and by [6, Lemma 2.1] we have \( \text{Ext}_R^i(N, M) \cong H^i_a(N, M) \) for any integer \( i \).

When \((R, m)\) is a local ring, we put \( \mu^i(M) := \mu^i(m, M) \). The following result is a generalization of the main results of [5].

**Corollary 2.4.** If \((R, m)\) is a local ring, then

(i) \( \mu^i(M) \leq \sum_{i=0}^{t} \mu^{t-i}(H^i_{I,J}(M)). \)

(ii) \( \mu^s(H^i_{I,J}(M)) \leq \sum_{i=0}^{t-1} \mu^{s+t+1-i}(H^i_{I,J}(M)) + \mu^{s+t}(M) + \sum_{i=t+1}^{s+t-1} \mu^{s+t-1-i}(H^i_{I,J}(M)). \)

**Proof.** In 2.1 put \( N = R/m \).

**Corollary 2.5.** Let \((R, m)\) be a local ring. Let \( \mu^{s+t-i}(H^i_{I,J}(M)) = 0 \) for all \( i \neq t \) with \( i \leq s + t \), \( \mu^{s+t+1-i}(H^i_{I,J}(M)) = 0 \) for all \( i < t \), and \( \mu^{s+t-1-i}(H^i_{I,J}(M)) = 0 \) for all \( t < i < s + t \). Then \( \mu^s(H^i_{I,J}(M)) = \mu^{s+t}(M). \)

**Corollary 2.6.** Let \((R, m)\) be a local ring and \( I = (a_1, a_2, \ldots, a_t) \). Then

\[ \mu^s(H^i_{I,J}(M)) \leq \sum_{i=0}^{t-1} \mu^{s+t+1-i}(H^i_{I,J}(M)) + \mu^{s+t}(M). \]

**Proof.** The claim follows by 2.4(ii) and [16, Proposition 4.11].

**Corollary 2.7.** Let \((R, m)\) be a local ring and \( I \) a principal ideal of \( R \). Let \( M \) be a minimax \( R \)-module. Then \( \mu^i(H^i_{I,J}(M)) \) is finite for all \( i \in \mathbb{N}_0 \) and all \( j \in \mathbb{N}_0 \).

**Proof.** Since \( I \) is principal, it follows by [16, Proposition 4.11] that \( H^i_{I,J}(M) = 0 \) for all \( i > 1 \). Therefore \( \mu^s(H^i_{I,J}(M)) \leq \mu^{s+2}(H^i_{I,J}(M)) + \mu^{s+1}(M) \), by 2.6. Now the claim follows by this fact that any minimax module has finite Bass numbers.
Proposition 2.8. Let $N$ be an $(I,J)$-torsion $R$-module. Then the following are true for all $p \in \text{Spec}(R)$ and all $j \in \mathbb{N}_0$:

(i) $$\mu^j(p, \text{Ext}^t_R(N,M)) \leq \sum_{i=0}^t \mu^i(p, \text{Ext}^{s+i+1-i}_R(N,H^i_{t,J}(M))).$$

(ii) $$\mu^j(p, \text{Ext}^t_R(N,H^i_{t,J}(M))) \leq \sum_{i=0}^{t-1} \mu^i(p, \text{Ext}^{s+i+1-i}_R(N,H^i_{t,J}(M)))$$

\begin{align*}
+ \mu^j(p, \text{Ext}^{s+t}_R(N,M)) \\
+ \sum_{i=t+1}^{s+t-1} \mu^i(p, \text{Ext}^{s+t-1-i}_R(N,H^i_{t,J}(M))).
\end{align*}

\textbf{Proof.} The proof is similar to that of [2,1]. \qed

3. SOME ISOMORPHISMS

In this section, we get some isomorphisms and equalities about the extension functors and the Bass numbers of local cohomology modules, respectively. The following result is a generalization of [1, Theorem 3.5].

Theorem 3.1. Let $N$ be an $(I,J)$-torsion $R$-module. Let $\text{Ext}^{s+t-i}_R(N,H^i_{t,J}(M)) = 0$ for all $i \neq t$ with $i \leq s+t$, $\text{Ext}^{s+t-1-i}_R(N,H^i_{t,J}(M)) = 0$ for all $i < t$, and let $\text{Ext}^{s+t-1-i}_R(N,H^i_{t,J}(M)) = 0$ for all $t < i < s+t$. Then $\text{Ext}^s_R(N,H^i_{t,J}(M)) \cong \text{Ext}^{s+t}_R(N,M)$.

\textbf{Proof.} Let $F(-) = \text{Hom}_R(N,-)$ and $G(-) = \Gamma_{I,J}(-)$. Then we have $FG(M) = \text{Hom}_R(N,M)$. By [15, Theorem 11.38], there is the Grothendieck spectral sequence $E^{p,q}_2 := \text{Ext}^p_R(N,H^q_{t,J}(M)) \Rightarrow \text{Ext}^{p+q}_R(N,M)$.

There is a finite filtration

$$0 = \varphi^{t+1}H^t \subseteq \varphi^tH^t \subseteq \cdots \subseteq \varphi^iH^t \subseteq \varphi^0H^t = \text{Ext}^t_R(N,M)$$

such that $E^{s+t-1,i}_\infty \cong \varphi^{t-i}H^t/\varphi^{t+1-i}H^t$ for all $i \leq t$. We have to show that $\varphi^0H^{s+t} \cong E^s_2$. Our hypothesis imply that $E^s_2 = 0$ for all $i \neq t$ with $i \leq s+t$. So $E^{s+t-1,i}_\infty = 0$ for all $i \neq t$ with $i \leq s+t$. The sequence

$$0 \rightarrow \varphi^{s+t+1-i}H^{s+t} \rightarrow \varphi^{s+t-i}H^{s+t} \rightarrow E^{s+t-1,i}_\infty \rightarrow 0$$

\textbf{Proof.} The proof is similar to that of [2,1]. \qed
Proof. \( R \) Hom \( \phi^i H^{s+i} \cong E_{s,t}^t \) and \( \phi^i H^{s+t} \cong s_i \phi^0 H^{s+t} \), and so that \( \phi^0 H^{s+t} \cong E_{s,t}^t \). Therefore it is enough to show that \( E_{s,t}^t \cong E_{s,t}^t \). Our hypothesis imply that \( E_{s,t}^{s+i+1} = 0 \) for all \( i < t \), and \( E_{s,t}^{s+t+1} = 0 \) for all \( t < i < s+t \). So \( E_{s,t}^{s-t+1} = 0 \) for all \( t-s < i < t \). Note that if \( i \leq t-s \), then \( E_{s,t}^{s-t+1+i,2t} = 0 \). Therefore \( E_{s,t}^{s-t-i,2t+i} = 0 \) for all \( i < t \), and so that \( \text{Im} d_{s,t}^{s-t+1,i,2t} = 0 \) for all \( i < t \). The sequences

\[
0 \rightarrow \text{Ker} d_{s,t}^{s-t+1,i,2t} \rightarrow E_{s,t}^{s-t+1,i,2t} \rightarrow E_{s,t}^{s-t+1,i,2t} \rightarrow 0
\]

are exact for any integer \( i \). It follows that \( E_{s,t}^{s,t} \cong E_{s,t}^{s,t} \), and the claim follows. 

**Corollary 3.2.** Let \( \mathfrak{p} \in W(I,J) \). Let \( \text{Ext}_{R}^{s+i}(R/\mathfrak{p},H_{I,J}^i(M)) = 0 \) for all \( i \neq t \) with \( i \leq s+t \), \( \text{Ext}_{R}^{s+i}(R/\mathfrak{p},H_{I,J}^i(M)) = 0 \) for all \( i < t \), and \( \text{Ext}_{R}^{s+t-1-i}(R/\mathfrak{p},H_{I,J}^i(M)) = 0 \) for all \( t < i < s+t \). Then \( \mu^i(\mathfrak{p},H_{I,J}^i(M)) = \mu^{s+t}(\mathfrak{p},M) \).

**Proof.** We note that \( \text{Ext}_{R}^{s}(R_{\mathfrak{p}}/pR_{\mathfrak{p}},H_{I,J}^i(M)) \cong \text{Ext}_{R_{\mathfrak{p}}}^{s+t}(R_{\mathfrak{p}}/pR_{\mathfrak{p}},M) \), by [5.4].

**Corollary 3.3.** Suppose that \( N \) is a finitely generated \( \mathfrak{a} \)-torsion \( R \)-module for some \( \mathfrak{a} \in \mathcal{W}(I,J) \). Suppose that \( \text{Ext}_{R}^{s+t-i}(N,H_{I,J}^i(M)) = 0 \) for all \( i \neq t \) with \( i \leq s+t \), \( \text{Ext}_{R}^{s+i-1}(N,H_{I,J}^i(M)) = 0 \) for all \( i < t \), and \( \text{Ext}_{R}^{s+t-1-i}(N,H_{I,J}^i(M)) = 0 \) for all \( t < i < s+t \). Then \( \text{Ext}_{R}^{s}(N,H_{I,J}^i(M)) \cong \text{Ext}_{R}^{s+t}(N,M) \cong H_{\mathfrak{a}}^{s+t}(N,M) \).

**Proof.** The result follows by [3.1] and [6] Lemma 2.1.

**Corollary 3.4.** Suppose that \( N \) is a finitely generated \( \mathfrak{a} \)-torsion \( R \)-module for some \( \mathfrak{a} \in \mathcal{W}(I,J) \). Suppose that \( \text{Ext}_{R}^{s+t-i}(N,H_{I,J}^i(M)) = 0 \) for \( j = t, t+1 \) and all \( i < t \). Then \( \text{Hom}_{R}(N,H_{I,J}^i(M)) \cong \text{Ext}_{R}^{i}(N,M) \cong H_{\mathfrak{a}}^{i}(N,M) \).

**Proof.** In [3.3] put \( s = 0 \).

**Corollary 3.5.** Let \( N \) be a finitely generated \( R \)-module with \( \text{Supp}_{R}N = V(\mathfrak{a}) \) for some \( \mathfrak{a} \in \mathcal{W}(I,J) \). If \( \text{Ext}_{R}^{s+t-i}(N,H_{I,J}^i(M)) = 0 \) for all \( i < t \) and all \( j \leq t+1-i \), then \( \text{Hom}_{R}(N,H_{I,J}^i(M)) \cong \text{Ext}_{R}^{i}(N,M) \cong H_{\mathfrak{a}}^{i}(N,M) \cong \text{Hom}_{R}(N,H_{\mathfrak{a}}^{i}(M)) \).

**Proof.** The result follows by [3.4] and [14] Corollary 2.5.
REFERENCES

[1] M. Aghapournahr, A. J. Taherizadeh and A. Vahidi, Extension functors of local cohomology modules, Bull. Iranian Math. Soc. 37 (2011), 117-134.

[2] M. H. Bijan-Zadeh, Torsion theories and local cohomology over commutative Noetherian rings, J. London Math. Soc. (2) 19 (1979), 402-410.

[3] M. P. Brodmann and R. Y. Sharp, Local Cohomology: An Algebraic Introduction with Geometric Applications, Cambridge University Press, Cambridge, 1998.

[4] W. Bruns and J. Herzog, Cohen-Macaulay Rings, Cambridge University Press, Cambridge, 1993.

[5] M. T. Dibaei and S. Yassemi, Bass Numbers of local cohomology modules with respect to an ideal, Algebr. Represent. Theory 11 (2008), 299-306.

[6] K. Divaani-Aazar, R. Sazeedeh and M. Tousi, On vanishing of generalized local cohomology modules, Algebra Colloq. 12 (2005), 213-218.

[7] A. Grothendieck (notes by R. Hartshorne), Local Cohomology, Springer Lecture Notes in Math., 41, Springer-Verlag, 1966.

[8] R. Hartshorne, Affine duality and cofiniteness, Invent. Math. 9 (1970), 145-164

[9] C. Huneke, Problems on local cohomology, Free resolutions in commutative algebra and algebraic geometry, Res. Notes Math. 2 (1992), 93-108.

[10] C. Huneke and R. Y. Sharp, Bass numbers of local cohomology modules, Trans. Amer. Math. Soc. 339 (1993), 765-779.

[11] K. I. Kawasaki, On the finiteness of Bass numbers of local cohomology modules, Proc. Amer. Math. Soc. 124 (1996), 3275-3279.

[12] G. Lyubezink, Finiteness properties of local cohomology modules (an application of $D$-modules to commutative Algebra), Invent. Math. 113 (1993), 41-55.

[13] G. Lyubezink, Finiteness properties of local cohomology modules for regular local rings of mixed characteristic: the unramified case, Comm. Algebra 28 (2000), 5867-5882.

[14] Sh. Payrovi and M. Lotfi Parsa, Regular sequences and local cohomology modules with respect to a pair of ideals, Eprint arXiv: 1305.0429v1.

[15] J. Rotman, An Introduction to Homological Algebra, Academic Press, Orlando, FL, 1979.

[16] R. Takahashi, Y. Yoshino and T. Yoshizawa, Local cohomology based on a nonclosed support defined by a pair of ideals, J. Pure Appl. Algebra 213 (2009), 582-600.

I. K. INTERNATIONAL UNIVERSITY, POSTAL CODE: 34149-1-6818 QAZVIN - IRAN

E-mail address: shpayrovi@ikiu.ac.ir
E-mail address: lotfi.parsa@yahoo.com
E-mail address: sakine-babaei@yahoo.com