SU(1, 1) RANDOM POLYNOMIALS

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OCTOBER 30, 2018

Abstract

We study statistical properties of zeros of random polynomials and random analytic functions associated with the pseudoeuclidean group of symmetries SU(1, 1), by utilizing both analytical and numerical techniques. We first show that zeros of the SU(1, 1) random polynomial of degree $N$ are concentrated in a narrow annulus of the order of $N^{-1}$ around the unit circle on the complex plane, and we find an explicit formula for the scaled density of the zeros distribution along the radius in the limit $N \to \infty$. Our results are supported through various numerical simulations. We then extend results of Hannay [Han] and Bleher, Shiffman, Zelditch [BSZ2] to derive different formulae for correlations between zeros of the SU(1, 1) random analytic functions, by applying the generalized Kac-Rice formula. We express the correlation functions in terms of some Gaussian integrals, which can be evaluated combinatorially as a finite sum over Feynman diagrams or as a supersymmetric integral. Due to the SU(1, 1) symmetry, the correlation functions depend only on the hyperbolic distances between the points on the unit disk, and we obtain an explicit formula for the two point correlation function. It displays quadratic repulsion at small distances and fast decay of correlations at infinity. In an appendix to the paper we evaluate correlations between the outer zeros $|z_j| > 1$ of the SU(1, 1) random polynomial, and we prove that the inner and outer zeros are independent in the limit when the degree of the polynomial goes to infinity.

1 Introduction

In this paper we are interested in statistical properties of zeros of random polynomials and random analytic functions associated with the pseudoeuclidean group of symmetries SU(1, 1). The motivation for the study of zeros of random polynomials and random analytic functions comes from different applications, most importantly from the theory of quantum chaos (see papers of Bogomolny, Bohigas, and Leboeuf [BBF], Leboeuf and Shukla [LS], Hannay [Han], Korsch, Miller, and Wiescher [KMW], Nonnenmacher and Voros [NV], Forrester and Honner [FH], Leboeuf [L], Shiffman and Zelditch [SZ1], and others). There are different ensembles of random polynomials associated with different groups of symmetries. In this respect we can make a parallel to the theory of random matrices. In the latter, the basic ensembles include orthogonal, unitary, symplectic, circular, and some others (see [M]). Making the parallel, we may think of the $O(n + 1)$ ensemble of real polynomials as of an analogue of the orthogonal ensemble of random matrices. The $O(n + 1)$ ensemble of random polynomials consists of multivariate homogeneous real polynomials of $(n + 1)$ variable $z = (z_0, z_1, \ldots, z_n)$ of the form

$$\psi(z) = \sum_{|m| = N} \sqrt{C_N^m} \, a_m z^m,$$  \hspace{1cm} (1)
where \( m = (m_0, m_1, \ldots, m_n) \) is a multiindex, \( |m| = m_0 + m_1 + \cdots + m_n \), \( z^m = z_0^{m_0} z_1^{m_1} \cdots z_n^{m_n} \), \( C_N^m \) is the multinomial coefficient,

\[
C_N^m = \frac{N!}{m_0! m_1! \cdots m_n!},
\]

and \( a_m \) are independent standard, \( N(0, 1) \), real Gaussian random variables. Consider a set \( Z \) of joint real zeros of \( k, k \leq n \), independent copies of the \( O(n+1) \) random polynomial in the projective space \( \mathbb{R}P^n \). This is a random real algebraic variety of dimension \( n - k \). It is nondegenerate almost surely and it possesses a natural volume element induced by the standard metric in \( \mathbb{R}P^n \). The joint distribution functions of the zeros are invariant with respect to the action of the group \( O(n+1) \), and, in particular, the density function of the zeros is constant (cf. \([\text{ShSm}], [\text{Han}], [\text{BSZ1}] - [\text{BSZ4}]\)).

Similarly, the \( SU(n+1) \) ensemble of random polynomials consists of multivariate homogeneous complex polynomials of \( (n+1) \) complex variable \( z = (z_0, z_1, \ldots, z_n) \) of form \((\psi)\) where \( a_m \) are independent standard complex Gaussian random variables (cf. \([\text{BB}], [\text{Han}], [\text{BSZ1}] - [\text{BSZ4}]\)). It corresponds to the unitary ensemble of random matrices. The distribution of joint zeros of \( k \) independent copies of the \( SU(n+1) \) random polynomial is invariant with respect to the action of the group \( SU(n+1) \) on the complex projective space \( \mathbb{C}P^n \). As the degree \( N \) of the \( SU(n+1) \) random polynomial goes to infinity, the scaled correlation functions of zeros approach a limit, which is represented by the correlation functions of the \( W_n \) ensemble of random analytic functions (see \([\text{Han}], [\text{BSZ1}] - [\text{BSZ4}], [1]\)). The latter consists of random functions of the form

\[
\psi(z) = \sum_m \sqrt{\frac{1}{m!}} a_m z^m, \tag{3}
\]

where the sum runs over multiindices \( m = (m_1, \ldots, m_n) \) with \( m_j \geq 0, m! = m_1! \cdots m_n! \), and \( a_m \) are independent standard complex Gaussian random variables. The existence of the limit for the scaled correlation functions of zeros is valid in a very general framework of random sections of powers of line bundles over compact manifolds and the limit is universal (see \([\text{BSZ1}] - [\text{BSZ4}]\)). The explicit combinatorial formulae for the limit of two-point correlation functions are obtained in \([\text{BSZ2}], [\text{BSZ4}]\). They demonstrate a quadratic repulsion if \( k = n = 1 \), neutrality if \( k = n = 2 \), and attraction if \( k = n > 2 \).

In this paper we are interested in the pseudoeuclidean, \( SU(1,1) \) ensemble of random analytic functions. The general \( SU(n,1) \) ensemble consists of multivariate analytic functions of the form

\[
\psi(z) = \sum_m \sqrt{C^m_{|m|+L-1}} a_m z^m, \quad z = (z_1, \ldots, z_n), \tag{4}
\]

where \( L \geq 1 \) is a fixed integer, a parameter of the ensemble, the sum runs over multiindices \( m = (m_1, \ldots, m_n) \) with \( m_j \geq 0, z^m = z_1^{m_1} \cdots z_n^{m_n} \),

\[
C^m_{|m|+L-1} = \frac{(|m|+L-1)!}{(L-1)! m_1! \cdots m_n!}, \tag{5}
\]

and \( a_m \) are independent standard complex Gaussian random variables. We will restrict our study to the case \( n = 1 \). The basic calculations are extended to the case of any \( n \), and we are going to return to this extension in subsequent publications. Another possible extension concerns the universality result like in \([\text{BSZ2}], [\text{BSZ4}], [\text{BSZ3}]\).

The plan of the paper is the following. In Section 2 we consider the ensemble of \( SU(1,1) \) random polynomials, which is obtained by restricting \( m \) in \((4)\) to be bounded by \( N \) (with \( n = 1 \)). We calculate the density function for the distribution of zeros of the \( SU(1,1) \) random polynomial and we show that the zeros are concentrated in a narrow annulus around the unit circle, of the width of the order of \( 1/N \). We find the scaled profile of the density function along the radius in the limit \( N \to \infty \). In Subsection 2.2 we rederive the result of Leboeuf \([1]\) for the density of the \( SU(1,1) \) random analytic function. In Section 3 we derive different formulae for the correlation functions between zeros of the \( SU(1,1) \) random analytic function. First, we apply the general result of \([\text{BSZ2}]\) to get the Kac-Rice
type expression for the \( n \)-point correlation function. We then specify it for the case of a Gaussian random field (cf. [Han], [BD], [BSZ2]). This expresses the \( n \)-point correlation function in terms of some Gaussian averages, which is further represented as a sum over Feynman diagrams, or in a different approach developed in [BSZ4], as a supersymmetric integral. In Subsection 3.3 we carry out concrete calculations for the two point correlation function in the \( SU(1,1) \) ensemble. They demonstrate a quadratic repulsion at small distances and fast decay at infinity. In the limit when the main parameter \( L \) of the \( SU(1,1) \) polynomial goes to infinity we recover the \( W_1 \) correlation function of Hannay. There is an appendix at the end of the paper, in which we consider correlations between outer zeros \( z_j, |z_j| > 1 \), of the \( SU(1,1) \) random polynomial. We show that in the limit \( N \to \infty \), the outer zeros are independent of the inner zeros \( |z_j| < 1 \), and after changing the variable \( z \) to \( 1/z \), the correlations between outer zeros coincide with the correlations between zeros in the \( SU(1,1) \) ensemble with \( L = 1 \).

### 2 Basic Statistical Properties

#### 2.1 Scaled Density Function

Consider the following random polynomial, associated with the pseudoeuclidean group \( SU(1,1) \):

\[
\psi(z) = \sum_{m=0}^{N} \sqrt{C_{m+L-1}^m} a_m z^m,
\]

where \( L \) is a fixed positive integer, a parameter of the problem, \( C_{m+L-1}^m \) are Newton’s binomial coefficients, and \( a_m \) are independent standard complex Gaussian random variables, so that

\[
Ea_m = 0, \quad Ea_m \bar{a}_n = \delta_{mn}, \quad Ea_m a_n = 0.
\]

When \( L = 1 \), (6) reduces to the classical form

\[
\psi(z) = \sum_{m=0}^{N} a_m z^m.
\]

We wish to investigate the density of zeros \( p_N(z) \) for the \( SU(1,1) \) polynomial (6) as \( N \to \infty \). The density \( p_N(z) \) is determined by the condition that for any test function \( \varphi(z) \), which is infinitely differentiable and compactly supported,

\[
E \left( \sum_{j=1}^{N} \varphi(z_j) \right) = \int_{\mathbb{C}} p_N(z) \varphi(z) \, dz, \quad dz \equiv dx dy,
\]

where \( z_j \) are zeros of the random polynomial \( \psi(z) \). Since the total number of zeros is equal to \( N \), we have that

\[
\int_{\mathbb{C}} p_N(z) \, dz = N.
\]

Observe that the polynomial

\[
\psi(e^{i\theta} z) = \sum_{m=0}^{N} \sqrt{C_{m+L-1}^m} a_m e^{im\theta} z^m
\]

has the same probability distribution as \( \psi(z) \), hence

\[
p_N(re^{i\theta}) = p_N(r).
\]
The general formula for $p_N(z)$ is given by the Poincaré-Lelong type expression

$$p_N(z) = \frac{1}{\pi} \frac{\partial^2}{\partial z \partial \bar{z}} [\ln E(\psi(z)\bar{\psi}(z))]$$

(13)

(see [BSZ1], [L]). From (6),

$$E(\psi(z)\bar{\psi}(z)) = \sum_{m=0}^{N} C_{m+L-1}^m (zz')^m = \frac{1}{(L-1)!} \sum_{m=0}^{N} (m+1) \cdots (m+L-1)(zz')^m.$$  

(14)

In particular,

$$E(\psi(z)\bar{\psi}(z)) = F_{N,L}(x) \equiv \sum_{m=0}^{N} x^m = \frac{1}{(L-1)!} \sum_{m=0}^{N} (m+1) \cdots (m+L-1)x^m.$$  

(15)

where here and in what follows we use the notation

$$x = z\bar{z} = |z|^2.$$  

(16)

It is obvious that

$$F_{N,L}(x) > 0, \quad \forall x \geq 0.$$  

(17)

Substituting (15) into (13) gives that

$$p_N(z) = \frac{1}{\pi} \left[ x \left( \frac{F_{N,L}(x)}{F_{N,L}(x)} \right)' + \frac{F_{N,L}(x)}{F_{N,L}(x)} \right].$$  

(18)

We further define

$$G_{N,L}(x) = \sum_{m=0}^{N} x^{m+L-1} = x^{L-1} \frac{x^{N+1} - 1}{x - 1}.$$  

(19)

Then from (15),

$$F_{N,L}(x) = \frac{1}{(L-1)!} \frac{d^{L-1} G_{N,L}(x)}{dx^{L-1}}.$$  

(20)

Numerical simulations show that most of the roots of $SU(1,1)$ polynomials (6) are concentrated in a small annulus (of the width of the order of $1/N$) near the unit circle. This property is illustrated in Fig. 1, which contains a point-plot of the zeros of 150 $SU(1,1)$ polynomials of degree $N = 200$, with $L = 30$.

To get the asymptotics of the density $p_N(z)$ near the unit circle, we introduce a scaling of the variable $x$ in the form

$$x = 1 + \frac{s}{N}.$$  

(21)

Then,

$$\frac{d}{dx} = N \frac{d}{ds}.$$  

(22)

In subsequent calculations, we will assume that

$$-A \leq s \leq A,$$  

(23)

for some arbitrary fixed $A > 0$. The notation $R(s) = O(N^{-1})$ used below means that there exists some $C(A) > 0$ so that $|R(s)| \leq C(A)N^{-1}$ for all $s \in [-A, A]$. The main result of this section is the following theorem.
Figure 1: Zeros of $SU(1, 1)$ Random Polynomials

**Theorem 2.1.** As $N \to \infty$,

$$N^{-2} p_N \left( \left( 1 + \frac{s}{N} \right)^{1/2} e^{i \theta} \right) = \frac{1}{\pi} \left[ \frac{g^{(L)}(s)}{g^{(L-1)}(s)} \right]' + O(N^{-1}) .$$

where

$$g^{(j)}(s) = \frac{d^j}{ds^j} \left( \frac{e^s - 1}{s} \right) .$$

**Remark.** Observe that one $N$ in the normalization of $p_N(z)$ on the left is due to the rescaling (21) and one is due to the integral condition (10).

**Proof.** By substituting (21) into (19) we obtain that as $N \to \infty$,

$$G_{N,L}(x) = \left( 1 + \frac{s}{N} \right)^{L-1} \left( 1 + \frac{s}{N} \right)^{N+1} - 1 = N \frac{e^s - 1}{s} \left( 1 + O(N^{-1}) \right) = Ng(s) \left( 1 + O(N^{-1}) \right) ,$$

where we define

$$g(s) = \frac{e^s - 1}{s} = \sum_{j=0}^{\infty} \frac{s^j}{(j+1)!} .$$

Observe that for $j = 0, 1, 2, \ldots$,

$$g^{(j)}(s) = \frac{d^j g(s)}{ds^j} > 0 , \quad - \infty < s < \infty .$$

It is obvious from (27) for $s \geq 0$. For negative $s$, use the identity

$$g^{(j)}(-s) = \frac{j!}{(-s)^{j+1}} \left[ 1 - e^{-s} \left( 1 + \frac{s}{1!} + \cdots + \frac{s^j}{j!} \right) \right] .$$

Formula (26) holds obviously in a small complex neighborhood of the segment $-A \leq s \leq A$. Therefore, we can differentiate it in $s$, so that

$$\frac{d^j G_{N,L}(x)}{dx^j} = N^{j+1} g^{(j)}(s) \left( 1 + O(N^{-1}) \right) ,$$
Thus, from (20), we obtain that
\[ F_{N,L}(x) = N L \left( \frac{g^{(L-1)}(s)}{(L-1)!} \right) (1 + O(N^{-1})) , \quad g^{(L-1)}(s) \equiv \frac{d^{L-1}g(s)}{ds^{L-1}} , \] (31)
and
\[ \mathcal{F}_{N,L}(x) = \frac{N^{L+1} g^{(L)}(s)}{(L-1)!} (1 + O(N^{-1})) , \] (32)
hence
\[ \frac{\mathcal{F}'_{N,L}(x)}{\mathcal{F}_{N,L}(x)} = \frac{N g^{(L)}(s)}{g^{(L-1)}(s)} (1 + O(N^{-1})) . \] (33)

Let us go back to formula (18). In this formula, \( x = 1 + O(N^{-1}) \), and the first term in the brackets has an extra derivative so this is the leading term, and we can neglect the second term. This gives that
\[ N^{-2} p_N(z) = \frac{1}{\pi} \left[ \log g^{(L-1)}(s) \right]' + O(N^{-1}), \quad z \bar{z} = 1 + \frac{s}{N} , \] (34)
which was stated. Theorem 2.1 is proved.

Equation (24) can also be expressed in the following form:
\[ N^{-2} p_N(z) = \frac{1}{\pi} \left[ \log g^{(L-1)}(s) \right]'' + O(N^{-1}). \] (35)

From (24),
\[ \lim_{N \to \infty} N^{-2} p_N \left( \left( 1 + \frac{s}{2N} \right) e^{i \theta} \right) = p(s) \equiv \frac{1}{\pi} \left[ \frac{g^{(L)}(s)}{g^{(L-1)}(s)} \right]' = \frac{1}{\pi} \left[ \log g^{(L-1)}(s) \right]'' . \] (36)

Consider now the distribution function of zeros. Define
\[ P_N(x) = N^{-1} E \# \{ j : |z_j|^2 \leq x \} , \] (37)
which gives the expected value of the fraction of zeros in the disk of radius \( \sqrt{x} \). Then
\[ P'_N(x) = N^{-1} p_N(z), \quad x = |z|^2 . \] (38)

Hence (24) implies that
\[ P_N \left( 1 + \frac{s}{N} \right) = \frac{g^{(L)}(s)}{g^{(L-1)}(s)} + O(N^{-1}) . \] (39)

As follows from (29), the limiting distribution function,
\[ \lim_{N \to \infty} P_N \left( 1 + \frac{s}{N} \right) = P(s) \equiv \frac{g^{(L)}(s)}{g^{(L-1)}(s)} , \] (40)
has the following asymptotics at \( -\infty \):
\[ P(s) = -\frac{L}{s} + O \left( (-s)^{L-1} e^s \right) , \quad s \to -\infty . \] (41)

At \( +\infty \) we use the formula
\[ g^{(j)}(s) = \frac{e^s}{s} \left[ 1 - \frac{j}{s} + \frac{j(j-1)}{s^2} - \cdots + \frac{(-1)^j j!}{s^j} \right] + \frac{(-1)^{j+1} j!}{s^{j+1}} , \] (42)
which gives that

$$P(s) = \frac{1}{s} - \frac{L - 1}{s^2} + O(s^{-3}), \quad s \to \infty. \quad (43)$$

For $L = 1$, (40) reduces to the well-known result (see [BS])

$$P(s) = \frac{se^s - e^s + 1}{s(e^s - 1)}. \quad (44)$$

For $L = 1$, the limiting distribution is symmetric, so that

$$P(1 - s) = 1 - P(s) \quad (45)$$

(it is related to the symmetry of the zeros of $\psi(z)$ in (8) with respect to the transformation $z \to 1/z$).

Since

$$g(s) = 1 + \frac{s}{2!} + \frac{s^2}{3!} + \frac{s^3}{4!} + \ldots, \quad (46)$$

we have that

$$g^{(j)}(s) = \frac{1}{j + 1} + \frac{s}{1!(j + 2)} + \frac{s^2}{2!(j + 3)} + \frac{s^3}{3!(j + 4)} + \ldots. \quad (47)$$

Thus,

$$P_N\left(1 + \frac{s}{N}\right) = \frac{1}{L + 1} + \frac{s}{1!(L + 2)} + \frac{s^2}{2!(L + 3)} + \ldots + O(N^{-1}). \quad (48)$$

In particular,

$$P_N(1) = \frac{L}{L + 1} + O(N^{-1}), \quad (49)$$

which means that the expected value of the fraction of zeros inside the unit disk is asymptotically equal to $L/(L + 1)$.

In our numerical simulations we generated a large number of $SU(1,1)$ random polynomials of degree $N$, calculated the zeros using standard techniques, and counted the number of zeros in annuli of fixed width, which are concentrically spread around the origin, per one polynomial. The scaled density was obtained by dividing each of these numbers by the area of the corresponding annulus, as well as by adjusting it with respect to $N$ (according to the previously established result). Figure 2 shows the results of this procedure for $L = 4$ and $N = 150$, in comparison with the theoretical value as given by Eq. (24).

### 2.2 SU(1,1) Ensemble of Random Analytic Functions

In this section we consider the $SU(1,1)$ random analytic function,

$$\psi(z) = \sum_{m=0}^{\infty} \sqrt{C_{m+L-1}^m} a_m z^m, \quad (50)$$

which is obtained from (8) by setting $N = \infty$. Here again $C_{m+L-1}^m$ are Newton’s binomial coefficients, and $a_m$ are independent standard complex Gaussian random variables.
PROPOSITION 2.2. Random series (50) converges almost surely for all \(|z| < 1\). For all mutually distinct \(z_1, \ldots, z_n\), in the disk \(|z| < 1\) the covariance matrix

\[
A_n = \left( E \left( \psi(z_p) \psi(z_{p'}) \right) \right) _{p, p'=1, \ldots, n}
\]

is positive definite.

Proof. Almost sure convergence. Observe that \((m + j)/m \leq j + 1\) hence

\[
\frac{(m + 1) \ldots (m + L - 1)}{m^{L-1}} \leq L!
\]

and

\[
C_{m+L-1}^m = \frac{(m + L - 1) \ldots (m + 1)}{(L - 1)!} \leq Lm^{L-1}.
\]

Consider the set

\[
\Lambda_N = \{ a = (a_0, a_1, \ldots) : |a_m| \leq m \quad \forall m \geq N \}.
\]

Then for all \(a \in \Lambda_N\) series (50) converges and

\[
\lim_{N \to \infty} \text{Prob} \Lambda_N = 1,
\]

which implies the almost sure convergence.

Positive definiteness. Consider the quadratic form

\[
A_n(\mu) = \sum_{p, p'=1}^n E \left( \psi(z_p) \overline{\psi(z_{p'})} \right) \mu_p \overline{\mu_{p'}} = E \left| \sum_{p=1}^n \mu_p \psi(z_p) \right|^2.
\]

Assume that \(A_n\) is not positive definite. Then, for some nonzero vector \(\mu = (\mu_1, \ldots, \mu_n)\),

\[
\sum_{p=1}^n \mu_p \psi(z_p) = 0
\]
for almost all $\psi$. Hence, (57) holds for almost $\psi$ with coefficients vector $a \in \Lambda_N$. Assume that $N > n$. There exists a polynomial

$$\psi_0(z) = \sum_{m=0}^{n-1} \sqrt{C_{m+L-1}} a_m^0 z^m, \quad (58)$$

such that

$$\psi_0(z_p) = \overline{\mu_p}, \quad (59)$$

so that

$$\sum_{p=1}^n \mu_p \psi_0(z_p) = \sum_{p=1}^n |\mu_p|^2 \neq 0. \quad (60)$$

Let $a^0 = (a^0_0, a^0_1, \ldots, a^0_{n-1}, 0, 0, \ldots)$. From (54), it is obvious that if $a \in \Lambda_N$ then $(a + a^0) \in \Lambda_N$. The shift $a \rightarrow a + a^0$ is a measurable one-to-one transformation in $\Lambda_N$, hence for almost all vectors $a \in \Lambda_N$ we have that (57) holds along with

$$\sum_{p=1}^n \mu_p [\psi(z_p) + \psi_0(z_p)] = 0. \quad (61)$$

But this implies that

$$\sum_{p=1}^n \mu_p \psi_0(z_p) = 0, \quad (62)$$

which is in contradiction with (60). This contradiction proves the positive definiteness of $A_n$. Proposition 2.2 is proved.

**Remark.** The above proof gives also the positive definiteness of the covariance matrix $A_n$ corresponding to the random polynomial (3), provided $N > n$.

In this section, we will be interested in the density function $\rho(z)$ of the distribution of zeros of (50) in the disk $|z| < 1$. We have from (8) that

$$E(\psi(z)\overline{\psi(z')}) = \sum_{m=0}^{\infty} C_{m+L-1}^m (zz')^m = \frac{1}{(1-zz')^L}. \quad (63)$$

In particular,

$$E(\psi(z)\overline{\psi(z)}) = f(x) \equiv \frac{1}{(1-x)^L}, \quad x \equiv z\bar{z} = |z|^2. \quad (64)$$

In terms of $f$ the density function $\rho(z)$ of the zeros of (50) is given by:

$$\rho(z) = \frac{1}{\pi} \left[ x \left( \frac{f'(x)}{f(x)} \right)' + \frac{f'(x)}{f(x)} \right]. \quad (65)$$

Substituting (64) into this formula gives that

$$\rho(z) = \frac{1}{\pi} \left[ x \left( \frac{L}{1-x} \right)' + \frac{L}{1-x} \right] = \frac{L}{\pi (1-x)^2}, \quad (66)$$

or going back to the complex variable $z$, we obtain the result of Leboeuf [1],

$$\rho(z) = \frac{L}{\pi (1-|z|^2)^2}. \quad (67)$$

Numerical simulations were similar to those of the previous section. The density was simply obtained by dividing the number of counted zeros by the area of the corresponding annulus. Figure 3 shows the results of this procedure for the degree $N$ equal to 50, 75, 100, and 150, in comparison with the theoretical value as given by Eq. (67).
3 Correlations between Zeros

3.1 The $SU(1, 1)$ Symmetry

We have the following important property:

**Theorem 3.1.** The distribution of zeros of the random analytic function $\psi(z)$ is invariant with respect to the action of the group $SU(1, 1)$,

$$z \rightarrow \frac{az + b}{cz + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(1, 1).$$

(68)

**Remark.** This implies that all joint distribution functions of zeros of $\psi(z)$ are $SU(1, 1)$ invariant.

**Proof.** Consider the following homogeneous analytic function of two variables:

$$\Psi(z_0, z_1) = z_0^{-L} \sum_{m=0}^{\infty} \sqrt{C_m^{m+L-1}} a_m \left( \frac{z_1}{z_0} \right)^m, \quad |z_1| < |z_0|.$$ 

(69)

Then

$$\Psi(1, z_1) = \psi(z_1).$$

(70)

Let us find the covariance function of $\Psi(z_0, z_1)$:

$$E \left( \Psi(z_0, z_1) \overline{\Psi(z_0', z_1')} \right) = \left( \frac{z_0}{\overline{z_0}} \right)^{-L} \sum_{m=0}^{\infty} C_m^{m+L-1} \left( \frac{z_1}{\overline{z_0}} \right)^m.$$ 

(71)

$$= \left( \frac{z_0}{\overline{z_0}} \right)^{-L} \left( 1 - \frac{z_1 \overline{z_1}}{z_0 \overline{z_0}} \right)^{-L} = \left( \frac{z_0}{\overline{z_0}} - z_1 \overline{z_1} \right)^{-L}. \quad (72)

The action of a matrix $A \in SU(1, 1)$ preserves the (1,1) indefinite scalar product $z_0 \overline{z_0} - z_1 \overline{z_1}$. Thus, the covariance function of the Gaussian random analytic function $\Psi(z_0, z_1)$ is $SU(1, 1)$ invariant. In addition,

$$E \Psi(z_0, z_1) = 0, \quad E \Psi(z_0, z_1) \Psi(z_0', z_1') = 0.$$ 

(73)
For the Gaussian random function, the first two moments determine it uniquely. This implies that the distribution of zeros of $\Psi(z_0, z_1)$ is $SU(1,1)$ invariant. Restricting this to $z_0 = 1$ we get that the distribution of zeros of $\psi(z)$ is $SU(1,1)$ invariant as well. Theorem 3.1 is proved.

Recall that a general formula for a matrix $A \in SU(1,1)$ is the following:

$$A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \quad |b|^2 - |a|^2 = 1,$$

so that it depends on three real parameters. An $SU(1,1)$-invariant metric is

$$ds^2 = \frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2},$$

and the corresponding distance $\tau(z_1, z_2)$ on the disk $\{|z| < 1\}$ is determined by the equation

$$\tanh(\tau/2) = \frac{|z_1 - z_2|}{1 - z_1 \bar{z_2}}.$$  

The corresponding $SU(1,1)$-invariant volume element is

$$\frac{4dx dy}{(1 - x^2 - y^2)^2}.$$  

Theorem 3.1 implies the $SU(1,1)$ invariance for the normalized correlation functions.

### 3.2 Correlation Functions - Preliminaries

The $n$-point correlation function $K_n(z_1, \ldots, z_n)$ is determined by the condition that for any test functions $\varphi_1(z), \ldots, \varphi_n(z)$, which are infinitely differentiable and compactly supported in the disk $\{|z| < 1\}$, such that their supports do not intersect,

$$E \prod_{p=1}^n \left( \sum_{j=1}^{\infty} \varphi_p(\zeta_j) \right) = \int_{\mathbb{C}^n} K_n(z_1, \ldots, z_n) \prod_{p=1}^n \varphi(z_p) dz_p,$$

where $\zeta_j$ are zeros of the $SU(1,1)$ random analytical function $\psi(z)$ (we denote them by $\zeta_j$ to distinguish them from the variables $z_j$). The sum over $j$ is, in fact, finite because $\varphi_p(z)$ has a compact support. The general formula for $K_n(z_1, \ldots, z_n)$ is given by the Kac-Rice expression

$$K_n(z) = \int d\xi D_n(0, \xi; z) \prod_{p=1}^n (\xi_p \xi_p^*), \quad z = (z_1, \ldots, z_n), \quad \xi = (\xi_1, \ldots, \xi_n),$$

where $D_n(x, \xi; z)$, $x, \xi \in C^n$, is the distribution density of the two random vectors

$$X = (\psi(z_1), \ldots, \psi(z_n)), \quad \Xi = (\psi'(z_1), \ldots, \psi'(z_n)).$$

Formula (79) is derived in [BSZ2] in a much more general situation of sections of powers of a line bundle over a complex manifold, as a generalization of the original formula by Kac and Rice [Ka, Ri] (see also [Han] and [BD]).

For the Gaussian random functions, formula (79) can be specialized as follows (see [Han], [BL], and [BSZ2]):

$$K_n(z) = \frac{1}{\pi^n \det A_n} \left( \prod_{p=1}^n (\xi_p \xi_p^*) \right)^{\Lambda_n},$$

11
where
\[ A_n = \left( E \psi(z_p) \overline{\psi(z_{p'})} \right)_{p,p'=1,...,n}, \] (82)
and \( \langle \cdot \rangle_n \) stands for averaging with respect to Gaussian complex random variables \( \xi_1, ..., \xi_n \), such that
\[ \langle E \xi_p \xi_{p'} \rangle_{p,p'=1,...,n} = A_n, \quad E \xi_p = 0, \quad E \xi_p \xi_{p'} = 0, \quad p, p' = 1, ..., n, \] (83)
where
\[ A_n = C_n - B_n^* A_n^{-1} B_n \] (84)
and
\[ B_n = \left( E \psi(z_p) \psi'(z_{p'}) \right)_{p,p'=1,...,n}, \quad C_n = \left( E \psi'(z_p) \psi'(z_{p'}) \right)_{p,p'=1,...,n}. \] (85)

For \( n = 1 \), formula (81) reduces to
\[ K_1(z) = \frac{1}{\pi} \frac{A_n(z) C_n(z) - B_n(z) B_n(z)}{A_n^2(z)}, \] (86)
where
\[ A_n(z) = E \psi(z) \overline{\psi(z)}, \quad B_n(z) = \frac{\partial A_n(z)}{\partial z}, \quad C_n(z) = \frac{\partial^2 A_n(z)}{\partial z \partial \overline{z}}. \] (87)

The 1-point correlation function \( K_1(z) \) is nothing else than the density function of the distribution of zeros and formula (85) is equivalent to the Poincaré-Lelong type formula (13).

The normalized correlation function \( k_n(z_1, ..., z_n) \) is defined as
\[ k_n(z_1, ..., z_n) = \frac{K_n(z_1, ..., z_n)}{K_1(z_1) \cdots K_1(z_n)}. \] (88)

It satisfies the following theorem.

**Theorem 3.2.** The function \( k_n(z_1, ..., z_n) \) is invariant with respect to the action of the group \( SU(1,1) \),
\[ k_n(Az_1, ..., Az_n) = k_n(z_1, ..., z_n), \quad \forall A \in SU(1,1), \] (89)
where
\[ Az = \frac{az + b}{cz + d}. \] (90)

**Remark.** This implies that \( k_n(z_1, ..., z_n) \) is a function of pairwise distances \( \tau(z_p, z_q) \) defined by (73).

**Proof.** By Theorem 3.1, the distribution \( K_n(z_1, ..., z_n)dz_1 \cdots dz_n \) is \( SU(1,1) \) invariant. Also, the distribution \( K_1(z_1) \cdots K_1(z_n)dz_1 \cdots dz_n \) is \( SU(1,1) \) invariant. Hence their quotient (the Radon-Nikodim derivative),
\[ \frac{K_n(z_1, ..., z_n)dz_1 \cdots dz_n}{K_1(z_1) \cdots K_1(z_n)dz_1 \cdots dz_n} = k_n(z_1, ..., z_n) \] (91)
is \( SU(1,1) \) invariant, which was stated. Theorem 3.2 is proved.

The normalized correlation function can be expressed as a supersymmetric (Berezin) integral,
\[ k_n(z_1, ..., z_n) = \frac{1}{\det A_n} \int \frac{1}{\det [I + \Lambda_n(z)\Omega]} \, d\eta, \] (92)
where \( \Omega \) is an \( n \times n \) matrix,
\[ \Omega = (\delta_{pp'} \overline{\eta}_p \overline{\eta}_{p'})_{p,p'=1,...,n}, \] (93)
and the \( \eta_p, \overline{\eta}_p \) are anti-commuting (fermionic) variables, with \( d\eta = \prod_p d\eta_p d\overline{\eta}_p \). The integral in (92) is a Berezin integral, which is evaluated by simply taking the coefficient of the top degree form of the integrand \( 1/\det [I + \Lambda_n(z)\Omega] \). For a derivation of formula (92), see [BSZ].
3.3 The Two-Point Correlation Function

From (58), (59), (62), and (63), we can directly obtain the following expressions for $A_n$, $B_n$, and $C_n$ in the case of a two-point correlation function $k_2(z_1, z_2)$:

\[
A_2 = \begin{pmatrix}
(1 - |z_1|^2)^{-L} & (1 - z_1 \overline{z}_2)^{-L} \\
(1 - z_1 \overline{z}_2)^{-L} & (1 - |z_2|^2)^{-L}
\end{pmatrix}
\]

\[
B_2 = \begin{pmatrix}
L_1(1 - |z_1|^2)^{-L-1} & L_2(1 - z_1 \overline{z}_2)^{-L-1} \\
L_2(1 - z_1 \overline{z}_2)^{-L-1} & L_3(1 - |z_2|^2)^{-L-1}
\end{pmatrix}
\]

\[
C_2 = \begin{pmatrix}
L(1 + L|z_1|^2)(1 - |z_1|^2)^{-L-2} & L(1 + Lz_1 \overline{z}_2)(1 - z_1 \overline{z}_2)^{-L-2} \\
L(1 + Lz_1 \overline{z}_2)(1 - z_1 \overline{z}_2)^{-L-2} & L(1 + L|z_2|^2)(1 - |z_2|^2)^{-L-2}
\end{pmatrix}
\]

By Theorem 3.2, $k_2(z_1, z_2)$ is $SU(1, 1)$ invariant, that is

\[
k_2(Az_1, Az_2) = k_2(z_1, z_2), \quad \forall A \in SU(1, 1).
\]

The action of $SU(1, 1)$ is transitive, hence we can move $z_1$ to the origin, and by rotation, we can then move $z_2$ to the positive half-axis. Therefore, we will assume that $z_1 = 0$ and $z_2 = r > 0$. In that case, equations (94) become

\[
A_2 = \begin{pmatrix}
1 & 1 \\
1 & (1 - r^2)^{-L}
\end{pmatrix}
\]

\[
B_2 = \begin{pmatrix}
0 & 0 \\
Lr & Lr(1 - r^2)^{-L-1}
\end{pmatrix}
\]

\[
C_2 = \begin{pmatrix}
L & L \\
L & L(1 + Lr^2)(1 - r^2)^{-L-2}
\end{pmatrix}
\]

From (96) and (97), we further obtain

\[
A_2 = \begin{pmatrix}
L - L^2 r^2 \frac{1}{(1 - r^2)^{-L-1}} & L - L^2 r^2 \frac{(1 - r^2)^{-L-1}}{(1 - r^2)^{-L-1}} \\
L - L^2 r^2 \frac{(1 - r^2)^{-L-1}}{(1 - r^2)^{-L-1}} & L(1 + Lr^2)(1 - r^2)^{-L-2} - L^2 r^2 \frac{(1 - r^2)^{-L-2}}{(1 - r^2)^{-L-1}}
\end{pmatrix}
\]

For the two-point case, formula (98) reduces to the following expression:

\[
K_2(z_1, z_2) = \frac{1}{\pi^2 \det A_2} \left( \lambda_{2,11} \lambda_{2,22} + \lambda_{2,12} \lambda_{2,21} \right),
\]

which, when combined with (88), (86), and (67), results in the following formula for the normalized two-point correlation function:

\[
k_2(z_1, z_2) = \left( (1 - r^2)^{3L+2} + ((L^2 - 2L - 2)r^4 + (4L + 4)r^2 - 1) (1 - r^2)^{2L} + ((L + 1)^2r^4 - (4L + 2)r^2 - 1) (1 - r^2)^L + 1 \right) \left| \frac{z_1 - z_2}{|1 - z_1 \overline{z}_2|} \right|^3,
\]

where according to (70),

\[
r = \tanh \left( \frac{T}{2} \right) = \frac{|z_1 - z_2|}{|1 - z_1 \overline{z}_2|}.
\]
When $L = 1$, formula (99) simplifies to

$$k_2(z_1, z_2) = r^2(2 - r^2), \quad L = 1. \tag{101}$$

Plots of $k_2(\tanh(\frac{\tau}{2}))$ for $L = 1, 5$ and 50 are shown in Fig. 4.

As it can be seen from the plot, the two-point correlation function goes to 0 as $\tau \to 0$ (or, in other words, as $r \to 0$). The limiting behavior can be obtained through a series expansion of (99).

The following expression was obtained using Maple$^\text{TM}$:

$$k_2(r) = \frac{1}{2} \left(\frac{L + 1}{L}\right)^2 r^2 - 1 \left(\frac{L + 1}{L}\right)^2 r^4 - \frac{1}{36} \left(\frac{L^2 - 1}{L}\right)^2 r^6 - \frac{1}{72} \left(\frac{L^2 - 1}{L}\right)^2 r^8 + O(r^{10}) \tag{102}.$$ 

which shows dominating quadratic behavior in the neighborhood of $r = 0$. Thus, there is a quadratic repulsion between zeros.

We are also interested in the asymptotic behavior of the correlation function as $L \to \infty$. For this purpose, we introduce the scaling

$$r = \frac{u}{\sqrt{L}}. \tag{103}$$

Substituting this into (99) and taking the limit $L \to \infty$, we obtain the following expression:

$$k_2(u) = \frac{e^{3u^2} + (u^4 - 4u^2 - 1)e^{2u^2} + (u^4 + 4u^2 - 1)e^u^2 + 1}{(e^{u^2} - 1)^3}, \tag{104}$$

which can also be written as

$$k_2(u) = \frac{(\sinh^2 t + t^2) \cosh t - 2t \sinh t}{\sinh^3 t}, \quad t = \frac{u^2}{2}, \tag{105}$$

and agrees with the result obtained by Hannay [Han]. A plot of (105) is shown in Fig. 5.

In fact, under the scaling (103), the $SU(1, 1)$ random analytic function converges, as $L \to \infty$, to the $W_1$ random analytic function (cf. [11]). Indeed,

$$\psi\left(\frac{u}{\sqrt{L}}\right) = \sum_{m=0}^{\infty} \sqrt{\frac{L(L+1)\ldots(L+m-1)}{L^m m!}} a_m u^m. \tag{106}$$
As \( L \to \infty \), the expression under the radical approaches \( 1/m! \), so that \( \psi(u/\sqrt{L}) \) approaches

\[
\psi_0(u) = \sum_{m=0}^{\infty} \frac{1}{m!} a_m u^m,
\]

which is the \( W_1 \) random analytic function.

### Appendix. Correlations between Inner and Outer Zeros

In this appendix, we consider the limit of correlations between zeros of the random polynomial \( \Re \) as \( N \to \infty \). In the open disk \( \{|z| < 1\} \), the random polynomial \( \Re \) approaches the \( SU(1,1) \) random analytic function \( \Re \), and the correlations between zeros of the random polynomial approach the ones of the random analytic function. However, according to formula \( \Re \), there is a \( 1/(L+1) \) fraction of the zeros outside of the unit disk. The limiting correlations between those, outer zeros, and between inner and outer zeros are described by the following theorem. Let \( k_{n,N}^L(z_1, \ldots, z_n) \) denote the normalized \( n \)-point correlation function for the zeros of polynomial \( \Re \).

**Theorem A.1.** Assume that \( |z_1|, \ldots, |z_m| < 1 \) and \( |z_{m+1}|, \ldots, |z_n| > 1 \). Then

\[
\lim_{N \to \infty} k_{n,N}^L(z_1, \ldots, z_n) = k_m^L(z_1, \ldots, z_m) k_{n-m}^L(z_{m+1}, \ldots, z_n^{-1}),
\]

so that in the limit \( N \to \infty \), the inner and outer zeros become independent, and the limiting correlations between the outer zeros coincide, after the change of variable \( z \to 1/z \) in the argument, with the correlations between zeros of the \( SU(1,1) \) random analytic function with the parameter \( L = 1 \).

**Proof.** We will first find correlations between the outer zeros and then prove the independence of the inner and outer zeros.

**Correlations between outer zeros.** Consider the random polynomial \( \Re \) and define another random polynomial,

\[
\varphi(z) = \frac{1}{\sqrt{C_{N+L-1}}} z^N \psi(z^{-1}).
\]
Observe that if $z_j, |z_j| > 1$, is a zero of $\psi(z)$ then $z_j^{-1}$ is a zero of $\varphi(z)$ and $|z_j^{-1}| < 1$. Consider $\varphi(z)$ in the disk $|z| < 1$. From (110),

$$\varphi(z) = a_N + \sqrt{\frac{N \ldots (N + L - 2)}{(N + 1) \ldots (N + L - 1)}} a_{N-1}z + \sqrt{\frac{(N - 1) \ldots (N + L - 3)}{(N + 1) \ldots (N + L - 1)}} a_{N-2}z^2 + \ldots. \quad (\text{110})$$

As $N \to \infty$, the expressions under the radical approaches 1 from below. In addition, we can replace $a_{N-m}$ by $a_m$, because they are the same standard random variables. Thus, as $N \to \infty$, $\varphi(z)$ approaches the random function

$$\varphi(z) = a_0 + a_1z + a_2z^2 + \ldots, \quad |z| < 1, \quad (\text{111})$$

which is the $SU(1,1)$ random analytic function with the parameter $L = 1$. Observe that $\varphi(z)$ is a Gaussian random polynomial and for its correlations we have formula (113). From this formula, we obtain the convergence of correlations between zeros of $\varphi(z)$ as $N \to \infty$ to the ones of the $SU(1,1)$ random analytic function with $L = 1$.

**Independence.** We introduce the random function $\eta(z)$ such that $\eta(z) = \psi(z)$ for $|z| \leq 1$ and

$$\eta(z) = \frac{1}{\sqrt{C^N_{N+L-1}}} z^{-N} \psi(z), \quad |z| > 1. \quad (\text{112})$$

Then the zeros of $\eta(z)$ and $\psi(z)$ coincide. In addition, $\eta(z)$ is a Gaussian random field and its covariance function $E(\eta(z)\overline{\eta(z')})$ coincides with $E(\psi(z)\overline{\psi(z')})$ when $|z|, |z'| < 1$, and hence, as $N \to \infty$, it approaches the correlation function (113) of the $SU(1,1)$ random analytic function with the parameter $L$. Similarly, as we saw above in this appendix, when $|z|, |z'| > 1$, $E(\eta(z)\overline{\eta(z')})$ approaches the $SU(1,1)$ covariance function with $L = 1$, if we replace $z, z'$ by $z^{-1}, z'^{-1}$, respectively. Consider now the correlation function $E(\eta(z)\overline{\eta(z')})$ when $|z| > 1$ and $|z'| < 1$. From (114),

$$E(\eta(z)\overline{\eta(z')}) = \frac{1}{z^N} \sqrt{C^N_{N+L-1}} \sum_{m=0}^{N} C^m_{L+m-1} (z\overline{z'})^m. \quad (\text{113})$$

Assume that $|zz'| \leq 1$. Then we can estimate the correlation function as follows:

$$\left| E(\eta(z)\overline{\eta(z')}) \right| \leq \frac{1}{|z|^N} \sqrt{C^N_{N+L-1}} \sum_{m=0}^{N} C^m_{L+m-1} \leq \frac{N^L}{|z|^N}. \quad (\text{114})$$

If $|zz'| > 1$, then we similarly obtain that

$$\left| E(\eta(z)\overline{\eta(z')}) \right| = \frac{z^N}{\sqrt{C^N_{N+L-1}}} \sum_{m=0}^{N} C^m_{L+m-1} (z\overline{z'})^{m-N} \leq N^L |z'|^N. \quad (\text{115})$$

Combining the two cases we can write

$$\left| E(\eta(z)\overline{\eta(z')}) \right| \leq N^L \left( \max\{|z|^{-1}, |z'|\} \right)^N. \quad (\text{116})$$

This shows that the values of $\eta(z)$ inside and outside of the unit disk become independent as $N \to \infty$. Hence their zeros become independent. Explicit estimates for the correlations between the inner and outer zeros follow from formula (113) applied to $\eta(z)$.

**Acknowledgement.** Research of P.B. is partially supported by NSF grant #DMS-9970625 and this support is gratefully acknowledged.
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