On $K$-eccentric and $K$-hyper eccentric indices of Benzenoid $H_k$ system

M. Bhanumathi$^1$, R. Rohini$^2$ and G. Srividhya$^3$

Abstract
Let $G$ be a connected graph with vertex set $V(G)$ and edge set $E(G)$. Bhanumathi and Easu Julia Rani introduced the first $K$-Eccentric index $B_1E(G)$ and the second $K$-Eccentric index $B_2E(G)$ of a graph $G$ as $B_1E(G) = \sum_{ue} [e_G(u) + e_{L(G)}(e)]$, $B_2E(G) = \sum_{ue} [e_G(u)e_{L(G)}(e)]$. They also defined the first $K$-Hyper eccentric index $HB_1E(G)$ and the second $K$-Hyper eccentric index $HB_2E(G)$ of a graph $G$ as $HB_1E(G) = \sum_{ue} [e_G(u) + e_{L(G)}(e)]^2$, $HB_2E(G) = \sum_{ue} [e_G(u)e_{L(G)}(e)]^2$ where in all the cases $ue$ means that the vertex $u$ and edge $e$ are incident in $G$ and $e_{L(G)}(e)$ is the eccentricity of $e$ in the line graph $L(G)$ of $G$. They have defined the multiplicative version of these indices also. In this paper, we calculate the first and second $K$-eccentric, the first and second $K$-hyper eccentric indices and their multiplicative versions of benzenoid $H_k$ system.

Keywords
$K$-eccentric index, $K$-hyper eccentric index, Multiplicative $K$-eccentric index, Multiplicative $K$-hyper eccentric index, Circo.

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1. Introduction

A topological index is a real number associated with chemical constitution. It correlates the chemical structure with various physical and chemical properties and biological activity.

All graphs in this paper are simple, finite and undirected. A graph $G$ is a finite nonempty set $V(G)$ together with a prescribed set $E(G)$ of unordered pair of distinct elements of $V$. The cardinality of $V(G)$ and $E(G)$ are represented by $|V(G)|$ and $|E(G)|$, respectively. Let, $d_G(v)$ be the degree of a vertex $v$ of $G$ and $N_G(v)$ be the neighborhood of a vertex $v$ of $G$. The distance between the vertices $u$ and $v$ of a connected graph $G$ is represented by $d_G(u,v)$. It is defined as the number of edges in a shortest path connects the vertices $u$ and $v$. The eccentricity $e_G(v)$ of a vertex $v$ in $G$ is the largest distance between $v$ and any other vertices $u$ of $G$.

To take an account on contributions of pairs of incident elements, Kulli [5] introduced the first and second $K$ Banhatti indices. In [4], Bhanumathi and Easu Julia Rani introduced the first $K$-Eccentric index $B_1E(G)$ and the second $K$-Eccentric index $B_2E(G)$ of a graph $G$ as

$$B_1E(G) = \sum_{ue} [e_G(u) + e_{L(G)}(e)], B_2E(G) = \sum_{ue} [e_G(u)e_{L(G)}(e)]$$

and also defined the first $K$-Hyper eccentric index $HB_1E(G)$ and the second $K$-Hyper eccentric index $HB_2E(G)$ of a graph $G$ as $HB_1E(G) = \sum_{ue} [e_G(u) + e_{L(G)}(e)]^2$, $HB_2E(G) = \sum_{ue} [e_G(u)e_{L(G)}(e)]^2$ where in all the cases $ue$ means that the vertex $u$ and edge $e$ are incident in $G$ and $e_{L(G)}(e)$ is the eccentricity of $e$ in the line graph $L(G)$ of $G$ [4].
2. First and second $K$-Eccentric indices, First and Second $K$-Hyper Eccentric indices of Benzenoid $H_k$ system:

The circumcoronene homologous series of benzenoid also belongs to the family of molecular graphs that has several copy of benzene $C_6$ on its circumference. The terms of this series are represented as, $H_1$-benzene, $H_2$-coronene, $H_3$-circumcoronene and $H_4$ circumcircumcoronene etc. A benzenoid system is a connected geometric figure. It is obtained by arranging congruent regular hexagons in a plane. Consequently two hexagons are either disjoint or have a common edge.

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The eccentricities of $u, v \in V(G)$ are denoted by $e(u), e(v)$ respectively and for $e = uv \in E(G)$, denote the eccentricities of the end vertices of the edge $e$ by $(e(u), e(v))$.

Let $V$ be the vertex set of $H_k$ and $E$ be the edge set in $H_k$, then $|V| = 6k^2$ and $|E| = 9k^2 - 3k$ for the structure of $H_k$. First, we shall determine the number of edges $e = uv \in E(G)$ with the eccentricity of the end vertices $e(u), e(v)$ and eccentricity of the edge $e$ in $L(G)$. We give these values in the following Table 1.

Theorem 2.1. For any positive integer number $k$, let $H_k$ be the general form of circumcoronene series of benzenoid system, then

(i) $B_1E(H_k) = 6 \sum_{i=1}^{k-1} \left[8k + 4(2i - 1)\right] + \sum_{i=1}^{k-1} [8k + 4(2i - 1)] + 1 + 12 \sum_{i=1}^{k-1} i [8k + 4(2i)]$

(ii) $B_2E(H_k) = 6 \sum_{i=1}^{k-1} \left[2(2k + 2i - 1)\right]^2$

(iii) $HB_2E(H_k) = 6 \sum_{i=1}^{k-1} \left[(2(2k + 2i - 1))^2 + (2(2k + 2i - 1))\right]$

(iv) $HEB_2E(H_k) = 6 \sum_{i=1}^{k-1} [2(2k + 2i - 1)]^2 + \left((2k + 2i) + (2k + 2i - 1)\right)^2$

The eccentric indices, $H$-coronene, $H$-circumcoronene etc. A benzenoid system is a connected geometric figure. It is obtained by arranging congruent regular hexagons in a plane. Consequently two hexagons are either disjoint or have a common edge.
(iv) \( HB_2E (H_k) = 6 \sum_{i=1}^{k} \left( (2k + 2i - 1)^2 + (2k + 2i - 1)^2 \right) \)
+ \( 6 \sum_{i=1}^{k-1} \left( (2k + 2i - 1)^2 + (2k + 2i)(2k + 2i - 1) \right) \)
+ \( 12 \sum_{i=1}^{k-1} \left( (2k + 2i)^2 + ((2k + 2i)(2k + 2i)) \right) \)

**Proof.** Consider the General form of \( H_k \)-Circumcoronene graph.

(i) \( B_1E (H_k) = \sum_{uv \in E(G)} \left[ e_{H_k}(u) + e_{L(H_k)}(e) \right] \)

\( = \sum_{uv \in E(G)} \left[ e_G(u) + e_G(v) + e_{L(G)}(e) \right] + \ldots \)

\( + \sum_{uv \in E_{E_{(k-1)}(G)}} \left[ e_G(u) + e_G(v) + e_{L(G)}(e) \right] \)

\( = 6 \sum_{i=1}^{k} \left( 2(2k + 2i - 1) \right) \)
+ \( 6 \sum_{i=1}^{k-1} \left( 2k + 2i - 1 \right)^2 \)
+ \( 12 \sum_{i=1}^{k-1} \left( (2k + 2i)^2 + ((2k + 2i)(2k + 2i)) \right) \)

(ii) \( B_2E (H_k) = \sum_{uv \in E(G)} \left[ e_{H_k}(u) \times e_{L(H_k)}(e) \right] \)

\( = \sum_{uv \in E(G)} \left[ e_G(u) + e_G(v) + e_{L(G)}(e) \right] + \ldots \)

\( + \sum_{uv \in E_{E_{(k-1)}(G)}} \left[ e_G(u) + e_G(v) + e_{L(G)}(e) \right] \)

\( = 6 \sum_{i=1}^{k} \left( (2k + 2i - 1)^2 \right) \)
+ \( 6 \sum_{i=1}^{k-1} \left( (2k + 2i - 1)^2 + (2k + 2i)(2k + 2i - 1) \right) \)
+ \( 12 \sum_{i=1}^{k-1} \left( (2k + 2i)^2 + ((2k + 2i)(2k + 2i)) \right) \)

(iii) \( HB_1E (H_k) = \sum_{uv \in E(G)} \left[ e_{H_k}(u) + e_{L(H_k)}(e) \right]^2 \)

\( = \sum_{uv \in E(G)} \left[ e_G(u) + e_G(v) + e_{L(G)}(e) \right]^2 + \ldots \)

\( + \sum_{uv \in E_{E_{(k-1)}(G)}} \left[ e_G(u) + e_G(v) + e_{L(G)}(e) \right]^2 \)

\( = 6 \sum_{i=1}^{k} \left( (2k + 2i - 1)^2 + (2k + 2i - 1)^2 \right) \)
+ \( 6 \sum_{i=1}^{k-1} \left( (2k + 2i - 1)^2 + (2k + 2i)(2k + 2i - 1)^2 \right) \)
+ \( 12 \sum_{i=1}^{k-1} \left( (2k + 2i)^2 + ((2k + 2i)(2k + 2i)) \right) \)

(iv) \( HB_2E (H_k) = \sum_{uv \in E(G)} \left[ e_{H_k}(u) \times e_{L(H_k)}(e) \right]^2 \)

\( = \sum_{uv \in E(G)} \left[ e_G(u) \times e_G(v) + e_{L(G)}(e) \right]^2 + \ldots \)

\( + \sum_{uv \in E_{E_{(k-1)}(G)}} \left[ e_G(u) \times e_G(v) + e_{L(G)}(e) \right]^2 \)

\( = 6 \sum_{i=1}^{k} \left( (2k + 2i - 1)^2 + (2k + 2i)(2k + 2i - 1)^2 \right) \)
+ \( 6 \sum_{i=1}^{k-1} \left( (2k + 2i - 1)^2 + (2k + 2i)(2k + 2i - 1) \right) \)
+ \( 12 \sum_{i=1}^{k-1} \left( (2k + 2i)^2 + ((2k + 2i)(2k + 2i)) \right) \)

For example, let us evaluate the indices for \( H_4 \). Consider the \( H_4 \)-Circumcircumcoronene graph.

![Figure 2]

Let \( V \) be the vertex set and \( E \) be the edge set in \( H_4 \) = Circumcircumcoronene, then \( |V| = 96 \) and \( |E| = 132 \). Also, the number of edges with eccentricities of end vertices \( e = uv \in E(G) \) and \( e \in L(G) \) are given as follows:

| Table 2 |
|---|---|---|---|
| Edge set | No. of edges | Eccentricity of end vertices \((e(u), e(v))\) | Eccentricity of \(e\) in \(L(G)e_{L(G)}(e)\) |
| \(E_1\) | 6 | (9,9) | 9 |
| \(E_2\) | 6 | (9,10) | 9 |
| \(E_3\) | 12 | (10,11) | 10 |
| \(E_4\) | 6 | (11,11) | 11 |
| \(E_5\) | 12 | (11,12) | 11 |
| \(E_6\) | 24 | (12,13) | 12 |
| \(E_7\) | 6 | (13,13) | 13 |
| \(E_8\) | 18 | (13,14) | 13 |
| \(E_9\) | 36 | (14,15) | 14 |
| \(E_{10}\) | 6 | (15,15) | 15 |
Corollary 2.2. \( H_1 \) be the first terms of this Circumcoronene series of Benzene \( H_k \). Then

(i) \( B_1 E (H_1) = 72 \)

(ii) \( B_2 E (H_1) = 108 \)

(iii) \( HB_1 E (H_1) = 432 \)

(iv) \( HB_2 E (H_1) = 972 \).

Corollary 2.3. \( H_2 \) be the second terms of this Circumcoronene series of Benzene \( H_k \). Then

(i) \( B_1 E (H_2) = 714 \)

(ii) \( B_2 E (H_2) = 2154 \)

(iii) \( HB_1 E (H_2) = 8634 \)

(iv) \( HB_2 E (H_2) = 82182 \)

Corollary 2.4. \( H_3 \) be the third terms of this Circumcoronene series of Benzene \( H_k \). Then

(i) \( B_1 E (H_3) = 2646 \)

(ii) \( B_2 E (H_3) = 12366 \)

(iii) \( HB_1 E (H_3) = 49770 \)

(iv) \( HB_2 E (H_3) = 1134150 \)

3. Multiplicative First and Second \( \kappa \)-Eccentric indices, Multiplicative First and Second \( \kappa \) Hyper Eccentric indices of Benzenoid \( H_k \) system:

Theorem 3.1. For any positive integer number \( k \), let \( H_k \) be the general form of circumcoronene series of benzenoid system, then

(i) \( \text{BPI}_1 (H_k) = 6 \prod_{i=1}^{k} [4(2k + 2i - 1)^2] \)

\( \times 6 \prod_{i=1}^{k-1} i [(2k + 2i - 1)(4k + 4i - 1)] \)

\( \times 12 \prod_{i=1}^{k-1} i [(2k + 2i)(4k + 4i + 1)] \)

(ii) \( \text{BPI}_2 (H_k) = 6 \prod_{i=1}^{k} [(2k + 2i - 1)^4] \)

\( \times 6 \prod_{i=1}^{k-1} i [(2k + 2i - 1)^3(2k + 2i)] \)

\( \times 12 \prod_{i=1}^{k-1} i [(2k + 2i)^3(2k + 2i + 1)] \)

(iii) \( HB\text{PI}_1 (H_k) = 6 \prod_{i=1}^{k} [16(2k + 2i - 1)^4] \)

\( \times 6 \prod_{i=1}^{k-1} i [(4(2k + 2i - 1))^2] \)

\( \times [(2k + 2i) + (2k + 2i - 1)^2] \)

\( \times 12 \prod_{i=1}^{k-1} i [(4(2k + 2i)^2) + ((2k + 2i + 1) + (2k + 2i))^2] \)

(iv) \( \text{HBPI}_2 (H_k) = 6 \prod_{i=1}^{k} [(2k + 2i - 1)^8] \)

\( \times 6 \prod_{i=1}^{k-1} i [(2k + 2i - 1)^6] \times [(2k + 2i)] \)

\( \times 12 \prod_{i=1}^{k-1} i [(2k + 2i)^6] [(2k + 2i + 1)] \)

Proof. Consider the General form of \( H_k \) - Circumcoronene
graph. Using Table 1, we obtain the following:

(i) \( B_{	ext{E}}(H_k) = \prod_{u \in E(G)} \left( e_G(u) + e_G(v) \right) \times \ldots \times \prod_{u \in E(G)} \left( e_G(u) + e_G(v) \right) \)

\[ = \prod_{u \in E(G)} \left[ e_G(u) + e_G(v) \right] \times \prod_{u \in E(G)} \left[ e_G(u) + e_G(v) \right] \times \ldots \times \prod_{u \in E(G)} \left[ e_G(u) + e_G(v) \right] \]

\[ = 6 \sum_{k=1}^{k-1} \left( 4(2k+2i-1)^2 \right) \times 6 \sum_{i=1}^{k-1} (4(2k+2i-1)^2) \times \ldots \times \prod_{u \in E(G)} \left[ e_G(u) + e_G(v) \right] \]

(ii) \( B_{\text{E}}(H_k) = \prod_{u \in E(G)} \left( e_G(u) \times e_G(v) \right) \times \ldots \times \prod_{u \in E(G)} \left( e_G(u) \times e_G(v) \right) \)

\[ = \prod_{u \in E(G)} \left[ e_G(u) \times e_G(v) \right] \times \prod_{u \in E(G)} \left[ e_G(u) \times e_G(v) \right] \times \ldots \times \prod_{u \in E(G)} \left[ e_G(u) \times e_G(v) \right] \]

\[ = 6 \sum_{k=1}^{k-1} \left( 4(2k+2i-1)^2 \right) \times 6 \sum_{i=1}^{k-1} (4(2k+2i-1)^2) \times \ldots \times \prod_{u \in E(G)} \left[ e_G(u) \times e_G(v) \right] \]

(iii) \( H_{\text{E}}(H_k) = \Pi_{u \in E(G)} \left[ e_G(u) + e_G(v) \right] \times \ldots \times \Pi_{u \in E(G)} \left[ e_G(u) + e_G(v) \right] \)

\[ = \prod_{u \in E(G)} \left[ e_G(u) + e_G(v) \right] \times \prod_{u \in E(G)} \left[ e_G(u) + e_G(v) \right] \times \ldots \times \prod_{u \in E(G)} \left[ e_G(u) + e_G(v) \right] \]

\[ = 6 \sum_{k=1}^{k-1} \left( 4(2k+2i-1)^2 \right) \times 6 \sum_{i=1}^{k-1} (4(2k+2i-1)^2) \times \ldots \times \prod_{u \in E(G)} \left[ e_G(u) + e_G(v) \right] \]

(iv) \( H_{\text{E}}(H_k) = \prod_{u \in E(G)} \left[ e_G(u) \times e_G(v) \right] \times \ldots \times \prod_{u \in E(G)} \left[ e_G(u) \times e_G(v) \right] \)

\[ = \prod_{u \in E(G)} \left[ e_G(u) \times e_G(v) \right] \times \prod_{u \in E(G)} \left[ e_G(u) \times e_G(v) \right] \times \ldots \times \prod_{u \in E(G)} \left[ e_G(u) \times e_G(v) \right] \]

\[ = 6 \sum_{k=1}^{k-1} \left( 4(2k+2i-1)^2 \right) \times 6 \sum_{i=1}^{k-1} (4(2k+2i-1)^2) \times \ldots \times \prod_{u \in E(G)} \left[ e_G(u) \times e_G(v) \right] \]

Using MATLAB programme, we have calculated these indices for \( H_1, H_2 \) and \( H_3 \). Those values are given below corollaries.

\[ \square \]

4. Conclusion

In chemical graph theory a topological index of a molecular graph characterizes its topology. Here, we have computed the first, second \( K \)-eccentric indices, \( K \)-hyper eccentric indices and multiplicative first, second \( K \)-eccentric and \( K \)-hyper eccentric indices of benzenoid \( H_k \) system.

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