TIME PERIODIC SOLUTIONS TO NAVIER-STOKES-KORTEWEG SYSTEM WITH FRICITION

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Abstract. In this paper, the compressible Navier-Stokes-Korteweg system with friction is considered in $\mathbb{R}^3$. Via the linear analysis, we show the existence, uniqueness and time-asymptotic stability of the time periodic solution when a time periodic external force is taken into account. Our proof is based on a combination of the energy method and the contraction mapping theorem. In particular, this is the first paper that a time periodic solution can be obtained in the whole space $\mathbb{R}^3$ only under the suitable smallness condition of $H^{N-1} \cap L^1$-norm($N \geq 5$) of time periodic external force.

1. Introduction. The isentropic compressible Navier-Stokes-Korteweg (NSK) system with friction that governs the motions of a general barotropic compressible viscous capillary fluid which reads as follows:

\[
\begin{aligned}
\rho_t + \nabla \cdot (\rho u) &= 0, \\
(pu)_t + \nabla \cdot (\rho u \otimes u) + \nabla P(\rho) - \mu \Delta u &= - (\nu + \mu) \nabla (\nabla \cdot u) + a \rho u = \kappa \rho \nabla \Delta \rho + \rho f(t,x).
\end{aligned}
\]

Here $\rho > 0$, $u = (u_1, u_2, u_3) \in \mathbb{R}^3$, $P = P(\rho)$ denote the density, the velocity and the pressure respectively. $a \rho u$ is a friction term with $a > 0$. Furthermore, the viscosity coefficients $\mu, \nu$ satisfy the usual physical conditions $\mu > 0$, $\nu + \frac{2}{3} \mu \geq 0$, and the constant $\kappa > 0$ represents capillary coefficient. In addition, $f(t, x)$ is a given external force.

In this paper, we consider the problem (1) for $(\rho, u)$ around a constant state $(\rho_\infty, 0)$ in $\mathbb{R}^3$, where $\rho_\infty$ is a positive constant. Meanwhile $P = P(\rho)$ is smooth in
a neighborhood of $\rho_\infty$ satisfying $P'(\rho_\infty) > 0$, and $f$ is time periodic with period $T > 0$.

The aim of this paper is to show that the problem (1) admits a time periodic solution around the constant state $(\rho_\infty, 0)$ in the whole space $\mathbb{R}^3$ which has the same period as $f$. With the energy method and the optimal decay estimates of the solution to the linearized system, we prove the existence and uniqueness of time periodic solution in some suitable function space by the contraction mapping theorem.

Precisely, let $N \geq 5$ be a positive integer, and define the solution space by

$$X_\h(0, T) = \left\{ (\rho, u)(t, x) \right\} _{\rho(t, x) \in C(0, T; H^N(\mathbb{R}^3)) \cap C^1(0, T; H^{N-2}(\mathbb{R}^3))},$$

$$u(t, x) \in C(0, T; H^{N-1}(\mathbb{R}^3)) \cap C^1(0, T; H^{N-3}(\mathbb{R}^3)),$$

$$\nabla \rho(t, x) \in L^2(0, T; H^{N+1}(\mathbb{R}^3)),$$

$$u(t, x) \in L^2(0, T; H^{N+1}(\mathbb{R}^3)), ||(\rho, u)|| \leq \h.$$

for some positive constant $\h$ and with the norm $||| \cdot |||$ given by

$$|||(\rho, u)|||^2 = \sup_{0 \leq t \leq T} \{ ||\rho(t)||^2_N + ||u(t)||^2_{N-1} \} + \int_0^T (||\nabla \rho(t)||^2_{N+1} + ||u(t)||^2_{N+1}) dt.$$

Then the main results of the present paper are the following.

**Theorem 1.1.** Let $N \geq 5$, we assume $f(t, x) \in C(0, T; H^{N-1}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3))$. Then there exists a constant $\h_0 > 0$, such that if

$$\sup_{0 \leq t \leq T} ||f(t)||_{H^{N-1} \cap L^1} \leq \h_0,$$

for some enough small constant $\h_0 > 0$, then the problem (1) admits a time periodic solution $(\rho_{\text{per}}, u_{\text{per}})$ with period $T$, satisfying

$$(\rho_{\text{per}} - \rho_\infty, u_{\text{per}}) \in X_{\h_0}(0, T).$$

Moreover, the periodic solution is unique in the following sense: if there is another time periodic solution $(\rho_{\text{per}}', u_{\text{per}}')$ satisfying (1) with the same $f(t, x)$, and $(\rho_{\text{per}}' - \rho_\infty, u_{\text{per}}') \in X_{\h_0}(0, T)$, then $(\rho_{\text{per}}', u_{\text{per}}') = (\rho_{\text{per}}', u_{\text{per}}').$

To study the stability of the time periodic solution $(\rho_{\text{per}}, u_{\text{per}})$ obtained in Theorem 1.1, we consider the following initial value problem

$$\begin{cases}
\rho_t + \nabla \cdot (\rho u) = 0, \\
(\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla P(\rho) - \mu \Delta u = -\nu \Delta u \rho + f(t, x), \\
(\rho, u)(t, x)|_{t=0} = (\rho_0, u_0)(x) \to (\rho_\infty, 0), \quad \text{as } |x| \to \infty.
\end{cases}$$

(2)

Here the initial data $(\rho_0, u_0)$ is a small perturbation of the time periodic solution $(\rho_{\text{per}}, u_{\text{per}})$. And we have the following stability result.

**Theorem 1.2.** Under the assumptions of Theorem 1.1, let $(\rho_{\text{per}}, u_{\text{per}})$ be the time periodic solution thus obtained. If $\|\rho_0 - \rho_{\text{per}}(0)\|_{N-1}, \|u_0 - u_{\text{per}}(0)\|_{N-2}$ are sufficiently small, then there exists a unique global classical solution $(\rho, u)$ to the initial value problem (2) satisfying

$$\rho - \rho_{\text{per}} \in C(0, \infty; H^{N-1}(\mathbb{R}^3)) \cap C^1(0, \infty; H^{N-3}(\mathbb{R}^3)),$$

$$u - u_{\text{per}} \in C(0, \infty; H^{N-2}(\mathbb{R}^3)) \cap C^1(0, \infty; H^{N-4}(\mathbb{R}^3)).$$
Furthermore, there exists a constant $C_0 > 0$ such that
\begin{equation}
\|(\rho - \rho^{per})(t)\|_{N^{-1}}^2 + \|(u - u^{per})(t)\|_{N^{-2}}^2 \\
+ \int_0^t \left( \|\nabla(\rho - \rho^{per})(\tau)\|_{N^{-1}}^2 + \|u - u^{per}(\tau)\|_{N^{-1}}^2 \right) d\tau \\
\leq C_0 \left( \|\rho_0 - \rho^{per}(0)\|_{N^{-1}}^2 + \|u_0 - u^{per}(0)\|_{N^{-2}}^2 \right)
\end{equation}
for any $t \geq 0$ and
\begin{equation}
\|(\rho - \rho^{per}, u - u^{per})(t)\|_{L^\infty} \to 0 \text{ as } t \to \infty.
\end{equation}

The work in this paper is motivated by some similar results which have been obtained for the compressible Navier-Stokes equations, the Boltzmann equation, the compressible Korteweg system and magnetohydrodynamic equations, cf. [4, 6, 14, 16, 18, 19]. While their studies need the space dimension $n \geq 5$. Compared to these results, the important difference in the paper is that we can investigate the similar problem in dimension three due to the good effect of the friction term. In Section 4, it is observed that the presence of friction speeds up the decay rate of the velocity of the NSK system with the factor $\frac{1}{2}$ compared to the NS system.

There are extensive studies on the existence, stability and convergence rates of solutions to the isentropic or non-isentropic compressible Navier-Stokes-Korteweg system. Here we only mention some of them related to our study. For the compressible Navier-Stokes-Korteweg system without the external force, the results obtained are rich. Hattori and Li [9, 10] proved the local existence and the global existence of smooth solutions in Sobolev space for the small initial data. Danchin and Desjardins [5] considered the existence of suitably smooth solutions in critical Besov space. Bresch, Desjardins and Lin [2] improved their results in [7]. The local existence of strong solutions was proven in [12]. Recently, Wang and Tan [20] established the optimal decay rates of global smooth solutions. Zhang and Tan [21] discussed the global existence and optimal $L^2$ decay rates of solutions on the non-isentropic case. However, for the system (NSK) with potential external force, the related study is very limited so far. Li [13] discussed the global existence and optimal $L^2$-decay rates of smooth solutions. And Haspot investigated the existence of global strong solution of Navier-Stokes-Korteweg system with friction in $\mathbb{R}^2$ for the viscosity coefficients and capillarity coefficient depending on density in [8].

The rest of the paper is arranged as follows. We will reformulate the problem and give some preliminaries for later use in Section 2. In Section 3, we give the energy estimates for the linearized system. And in Section 4, we apply the spectral analysis to the semigroup and establish the linear $L^2$ time-decay estimates. The proof of Theorem 1.1 and Theorem 1.2 is given in the last two sections respectively.

Notations. Throughout this paper, for simplicity, we will omit the variables $t, x$ of functions if it does not cause any confusion. $C$ denotes a generic positive constant which may vary in different estimates. $\langle \cdot , \cdot \rangle$ is the inner product in $L^2(\mathbb{R}^3)$. The norm in the usual Sobolev Space $H^s(\mathbb{R}^3)$ is denoted by $\|\cdot\|_s$ for $s \geq 0$. When $s=0$, we will simply use $\|\cdot\|$. And $\hat{f}$ is the Fourier transform of $f$. Moreover, we denote $\|\cdot\|_{H^s} + \|\cdot\|_{L^1}$ by $\|\cdot\|_{H^s \cap L^1}$. $\nabla = (\partial_1, \partial_2, \partial_3)$ with $\partial_i = \partial_{x_i}, i = 1, 2, 3$ and for any integer $l \geq 0$, $\nabla^l g$ denotes all $x$ derivatives of order $l$ of the function $g$. Finally, for
multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, it is standard that
\[ \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}, \quad |\alpha| = \sum_{i=1}^{3} \alpha_i. \]

2. Reformulated system and preliminaries. In order to simplify the coming calculation, we set

\[ \gamma = \sqrt{P'(\rho_\infty)}, \quad \kappa' = \frac{\rho_\infty}{\gamma} \kappa, \quad \mu' = \frac{\mu}{\rho_\infty}, \quad \nu' = \frac{\nu + \mu}{\rho_\infty}, \quad \nu_1 = \frac{\gamma}{\rho_\infty}, \quad \nu_2 = \frac{\rho_\infty}{\gamma}, \]

and denote

\[ q = \rho - \rho_\infty, \quad \nu = \nu_2 u, \]

then the system (1) is reformulated as

\[
\begin{aligned}
\varrho_t + \gamma \nabla \cdot v &= Q_1(\varrho, v), \\
v_t - \mu' \Delta v - \nu' \nabla (\nabla \cdot v) + a v + \gamma \nabla q - \kappa' \nabla \Delta q &= Q_2(\varrho, v) + \nu_2 f,
\end{aligned}
\]

where

\[
Q_1(\varrho, v) = -\nu_1 \nabla \cdot (q v),
\]

\[
Q_2(\varrho, v) = \left( \frac{\mu}{\varrho + \rho_\infty} - \frac{\mu}{\rho_\infty} \right) \Delta v + \left( \frac{\mu + \nu}{\varrho + \rho_\infty} - \frac{\mu + \nu}{\rho_\infty} \right) \nabla (\nabla \cdot v)
\]

\[
- \nu_1 (v \cdot \nabla) v - \nu_2 \left( \frac{P'(\varrho + \rho_\infty)}{\varrho + \rho_\infty} - \frac{P'(\rho_\infty)}{\rho_\infty} \right) \nabla q.
\]

Notice that $Q_1$ and $Q_2$ have the following properties:

\[
\begin{aligned}
Q_1(\varrho, v) &\sim \nabla q \cdot v + q \nabla \cdot v, \\
Q_2(\varrho, v) &\sim q \Delta v + q \nabla (\nabla \cdot v) + (v \cdot \nabla) v + q \nabla q.
\end{aligned}
\]

Here $\sim$ means that two sides are of same order.

Set $U = (\varrho, v), \quad Q = (Q_1, Q_2), \quad F = (0, \nu_2 f)$ and define

\[
A = \begin{pmatrix}
0 & 0 \\
\gamma \nabla - \kappa' \nabla \Delta & -\mu' \Delta - \nu' \nabla \div + a
\end{pmatrix},
\]

then the system (5) takes the form

\[ U_t + AU = Q(U) + F. \]

We first consider the linearized system of (5)

\[
\begin{aligned}
\varrho_t + \gamma \nabla \cdot v &= Q_1(\bar{U}), \\
v_t - \mu' \Delta v - \nu' \nabla (\nabla \cdot v) + a v + \gamma \nabla q - \kappa' \nabla \Delta q &= Q_2(\bar{U}) + \nu_2 f,
\end{aligned}
\]

for any given functions $\bar{U} = (\bar{\varrho}, \bar{v})$ satisfying

\[ \bar{\varrho} \in H^{N+2}(\mathbb{R}^3), \quad \bar{v} \in H^{N+1}(\mathbb{R}^3). \]

Notice that the system (8) can be written as

\[ U_t + AU = Q(\bar{U}) + F. \]

By the Duhamel’s principle, the solution to the system (8) can be written in the mild form as

\[ U(t) = S(t, s)U(s) + \int_s^t S(t, \tau) \left( Q(\bar{U}) + F \right)(\tau) d\tau, \quad t \geq s, \]
where $S(t, s)$ is the corresponding linearized solution operator defined by

$$S(t, s) = e^{(t-s)A}, \quad t \geq s.$$ 

In the next section, we will establish the energy estimates on $(\varrho, v)$, and in Section 4, the decay rates of the solution operator $S(t, s)$ will be obtained by the spectral analysis.

3. Energy estimates. Throughout this section, we assume that $f(t, x) \in H^{N-1}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ for all $t \geq 0$. To start with, we recall some known elementary inequalities which will be used frequently later, cf. [1, 17].

Lemma 3.1. Let $f \in H^2(\mathbb{R}^3)$, then we have

$$\|f\|_{L^\infty}^2 \leq C\|\nabla f\|_1 \leq C\|f\|_1.$$ 

Lemma 3.2. Let $f, g, h \in H^2(\mathbb{R}^3)$, then we have

$$(i) \int_{\mathbb{R}^3} f \cdot g \cdot h \, dx \leq \epsilon \|\nabla f\|_1^2 + C_{\epsilon}\|g\|_1^2 \|h\|_1^2,$$

$$(ii) \int_{\mathbb{R}^3} f \cdot g \cdot h \, dx \leq \epsilon \|f\|_1^2 + C_{\epsilon}\|\nabla g\|_1^2 \|h\|_1^2,$$

for any $\epsilon > 0$. Here and hereafter, $C_{\epsilon}$ denotes a positive constant depending only on $\epsilon$.

In what follows, two lemmas on the energy estimates are given. Firstly, the lower order energy estimate of $(\varrho, v)$ is obtained in the following lemma.

Lemma 3.3. There exist two suitably small constants $\delta_0 > 0$ and $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, it holds

$$\frac{d}{dt} \left( \|U(t)\|^2 + \|\nabla \varrho(t)\|^2 + C_{\epsilon}\|\varrho\|_1^2 + \|\nabla \varrho(t)\|_1^2 \right) \leq \epsilon C\|\nabla v(t)\|_1^2 + C_{\epsilon}C \left( \|U(t)\|_2^2 \|\nabla U(t)\|_1^2 + \|f(t)\|_{L^1}^2 \right),$$

where $C$ depends only on $\rho_\infty, \mu, \nu, a$ and $\kappa$.

Proof. Multiplying (8) and (9) by $\varrho$ and $v$, respectively, and integrating them over $\mathbb{R}^3$, we have by integrating that

$$\frac{1}{2} \frac{d}{dt} \|U\|^2 + \mu'\|\nabla v\|^2 + \nu'\|\nabla \cdot v\|^2 + a\|v\|^2,$$

$$= \langle Q_1(U), \varrho \rangle + \langle Q_2(U), v \rangle + \kappa'\langle \nabla \Delta \varrho, v \rangle + \nu_2 \langle f, v \rangle,$$

$$= I_0 + I_1 + I_2 + I_3.$$ (12)

From (6) and Lemma 3.2, we have

$$I_0 \leq \epsilon\|\nabla \varrho\|_1^2 + C_{\epsilon}C \left( \|\nabla \varrho\|_1^2 \|\varrho\|_1^2 \right) \leq \epsilon\|\nabla \varrho\|_1^2 + C_{\epsilon}C\|U\|_2^2 \|\nabla U\|_1^2,$$ (13)

and

$$I_1 \leq \epsilon\|\nabla v\|_1^2 + C_{\epsilon}C\|U\|_2^2 \|\nabla U\|_1^2.$$ (14)
For $I_2$, integrating by parts and using (8)$_1$, (6) and Lemma 3.2, we deduce that

$$I_2 = -\kappa' \langle \Delta \varrho, \nabla \cdot v \rangle = \frac{\kappa'}{\gamma} \langle \Delta \varrho, \varrho_t - Q_1(\bar{U}) \rangle$$

$$= -\frac{\kappa'}{2\gamma} \frac{d}{dt} \|\nabla \varrho\|^2 - \frac{\kappa'}{\gamma} \langle \Delta \varrho, Q_1(\bar{U}) \rangle$$

$$\leq -\frac{\kappa'}{2\gamma} \frac{d}{dt} \|\nabla \varrho\|^2 + \epsilon \|\nabla^2 \varrho\|^2 + C_\epsilon C \|\nabla \bar{U}\|^2 \|\nabla \bar{U}\|^2.$$  \hspace{1cm} (15)

For $I_3$, Cauchy-Schwartz inequality and Lemma 3.1 give

$$I_3 \leq \epsilon \|\nabla v\|^2 + C_\epsilon C \|f\|_{L^1}^2.$$  \hspace{1cm} (16)

Substituting (13)–(16) into (12) yields

$$\frac{d}{dt} \left( \|U\|^2 + \|\nabla \varrho\|^2 \right) + \|\nabla v\|^2 + \|\nabla \cdot v\|^2 + \|v\|^2$$

$$\leq \epsilon C \left( \|\nabla \varrho\|^2 + \|\nabla^2 \varrho\|^2 \right) + C \left( \|\bar{U}\|^2 + \|f\|_{L^1}^2 \right),$$  \hspace{1cm} (17)

provided that $\epsilon$ is small enough, where $C$ depends only on $\rho_\infty, \mu, \nu, a$ and $\kappa$.

Next, we estimate $\|\nabla \varrho\|^2$. Taking the $L^2$ inner product with $\nabla \varrho$ on both sides of (8)$_2$ and then integrating by parts, we have

$$\gamma \|\nabla \varrho\|^2 + \kappa'' \|\nabla^2 \varrho\|^2 = -\langle v_t, \nabla \varrho \rangle + \mu' \langle \Delta v, \nabla \varrho \rangle + \nu' \langle \nabla (\nabla \cdot v), \nabla \varrho \rangle$$

$$+ \langle Q_2(\bar{U}), \nabla \varrho \rangle + \nu \langle f, \nabla \varrho \rangle - \rho_a \langle v, \nabla \varrho \rangle$$

$$= I_4 + I_5 + I_6 + I_7 + I_8.$$  \hspace{1cm} (18)

Similar to (15), the term $I_4$ can be controlled by

$$I_4 = -\frac{d}{dt} \langle v, \nabla \varrho \rangle - \langle \nabla \cdot v, \varrho_t \rangle$$

$$= -\frac{d}{dt} \langle v, \nabla \varrho \rangle - \langle \nabla \cdot v, -\gamma \nabla \cdot v + Q_1(\bar{U}) \rangle$$

$$\leq -\frac{d}{dt} \langle v, \nabla \varrho \rangle + 2\gamma \|\nabla \cdot v\|^2 + C \|\nabla \bar{U}\|^2 \|\nabla \bar{U}\|^2.$$  \hspace{1cm} (19)

Integrating by parts and using the Cauchy-Schwartz inequality, it is easy to get

$$I_5 + I_6 \leq \frac{\kappa'}{4} \|\nabla^2 \varrho\|^2 + C \left( \|\nabla v\|^2 + \|\nabla \cdot v\|^2 \right).$$  \hspace{1cm} (20)

Finally, (6) and the Cauchy-Schwartz inequality imply that

$$I_7 \leq \frac{\gamma}{4} \|\nabla \varrho\|^2 + C \left( \|\nabla \bar{U}\|^4 + \|f\|^2 \right),$$  \hspace{1cm} (21)

$$I_8 \leq \frac{\gamma}{4} \|\nabla \varrho\|^2 + C \|v\|^2.$$  \hspace{1cm} (22)

Combining (18)–(22), we obtain

$$\frac{d}{dt} \langle v, \nabla \varrho \rangle + \|\nabla \varrho\|^2 + \|\nabla^2 \varrho\|^2$$

$$\leq C \left( \|\nabla v\|^2 + \|v\|^2 \right) + C \left( \|\nabla \bar{U}\|^4 + \|f\|^4 \right),$$  \hspace{1cm} (23)

where the constant $C$ depends only on $\rho_\infty, \mu, \nu, a$ and $\kappa$. Multiplying (23) with a small constant $\delta_0 > 0$ and then adding the resultant equation to (17), one can get (11) immediately by the smallness of $\delta_0$ and $\epsilon$. This completes the proof of Lemma 3.3. \hfill \Box
Lemma 3.4. Let $N \geq 5$, then there exist two suitably small constants $\delta_1 > 0$ and $\epsilon_1 > 0$ such that for $0 < \epsilon \leq \epsilon_1$, it holds

$$\frac{d}{dt} \left( \| \nabla \varphi(t) \|_X^2 + \| \nabla v(t) \|_X^{2N} + \delta_1 \sum_{|\alpha|=1}^N \langle \partial_x^\alpha \varphi, \partial_x^\alpha \nabla \varphi \rangle(t) \right) + \| \nabla^2 \varphi(t) \|_X^2 + \| \nabla v(t) \|_X^2 \leq \epsilon C \| \nabla \varphi(t) \|_X^2 + C \| U(t) \|_X^{2N} - 1 \| \nabla U(t) \|_X^2 + \| f(t) \|_X^{2N-1},$$

(24)

where $C$ is depending only on $\rho_\infty, \mu, \nu, a$ and $\kappa$.

Proof. For each multi-index $\alpha$ with $1 \leq |\alpha| \leq N$, applying $\partial_x^\alpha$ to (8)1 and (8)2 and then taking the $L^2$ inner product with $\partial_x^\alpha \varphi$ and $\partial_x^\alpha v$ on the two resultant equations respectively, we have by integrating that

$$\frac{1}{2} \frac{d}{dt} \left( \| \partial_x^\alpha \varphi \|_2^2 + \| \partial_x^\alpha v \|_2^2 \right) + \mu \| \partial_x^\alpha \nabla v \|_2^2 + \nu \| \partial_x^\alpha \nabla : v \|_2^2 + a \| \partial_x^\alpha v \|_2^2 = \langle \partial_x^\alpha \varphi, \partial_x^\alpha \varphi \rangle + \langle \partial_x^\alpha \varphi, \partial_x^\alpha \varphi \rangle + \kappa \| \partial_x^\alpha \nabla \varphi \|_2^2 \| \partial_x^\alpha v \|_2^2 + \nu \partial_x^\alpha f, \partial_x^\alpha v \rangle = J_0 + J_1 + J_2 + J_3.$$

(25)

Now, we estimate $J_0 - J_3$ term by term. For $J_0$, we deduce from (6) and the Cauchy-Schwarz inequality that

$$J_0 \leq \epsilon \| \partial_x^\alpha \varphi \|_2^2 + C(\| \partial_x^\alpha \varphi, \partial_x^\alpha \varphi \|_2^2) \leq \epsilon \| \partial_x^\alpha \varphi \|_2^2 + C \| \partial_x^\alpha (\nabla \varphi \cdot \bar{v}) \|_2^2 + \| \partial_x^\alpha (\bar{\varphi} \nabla \cdot \bar{v}) \|_2^2.$$

(26)

By using Leibniz’s formula and Minkowski’s inequality, we get

$$\| \partial_x^\alpha (\nabla \varphi \cdot \bar{v}) \|_2 \leq C \left( \| \partial_x^\alpha (\nabla \varphi \cdot \bar{v}) \|_2 \right) + C \left( \| \partial_x^\alpha (\nabla \varphi \cdot \bar{v}) \|_2 \right) + \| \partial_x^\alpha (\bar{\varphi} \nabla \cdot \bar{v}) \|_2.$$

(27)

Here $C_{\beta}^\alpha$ denotes the binomial coefficients corresponding to multi-indices. For $J_4$, Lemma 3.1 gives

$$J_4 \leq C \left( \| \bar{v} \|_{L^\infty}^2 \| \partial_x^\alpha \nabla \varphi \|_2^2 + \| \nabla \varphi \|_{L^\infty}^2 \| \partial_x^\alpha \varphi \|_2^2 \right) \leq C \left( \| \nabla \bar{v} \|_{L^\infty}^2 \| \nabla \varphi \|_{L^\infty}^2 \| \partial_x^\alpha \varphi \|_2^2 \right).$$

(28)

For the terms $J_5$ and $J_6$, notice that for any $\beta < \alpha$ with $|\beta| \leq N - 2$,

$$|\beta| + 3 \leq N + 1, \quad 1 \leq |\alpha - \beta| \leq N - 1,$$

and for any $\beta < \alpha$ with $|\beta| > N - 2$,

$$|\beta| + 1 < N + 1, \quad |\alpha - \beta| + 2 < N - (N - 2) + 2 = 4 \leq N - 1.$$

Hence, we deduce from Lemma 3.1 that

$$J_5 \leq C \sum_{0 < |\beta| < |\alpha|, |\beta| \leq N - 2} \| \partial_x^\beta \nabla \varphi \|_{L^\infty}^2 \| \partial_x^{\alpha - \beta} \bar{v} \|_2^2 \leq C \| \bar{U} \|_{L^\infty}^2 \| \nabla \bar{U} \|_X^2,$$

(29)
and \[ J_6 \leq C \sum_{0 < |\beta| < |\alpha|, |\alpha-\beta| > N-2} \| \partial_\beta^\alpha \nabla \varrho \|_2^2 \| \partial_\beta^\alpha \varrho \|_L^\infty \leq C \| \tilde{U} \|_{N-1}^2 \| \nabla \tilde{U} \|_N^2. \] (30)

Putting (28)–(30) into (27), we arrive at
\[ \| \partial_\beta^\alpha (\nabla \tilde{v} \cdot \varrho) \|_2^2 \leq C \| \tilde{U} \|_{N-1}^2 \| \nabla \tilde{U} \|_N^2. \] (31)

Similarly, it holds
\[ \| \partial_\beta^\alpha (\varrho \nabla \tilde{v}) \|_2^2 \leq C \| \tilde{U} \|_{N-1}^2 \| \nabla \tilde{U} \|_N^2. \] (32)

Combining (26), (31) and (32) yields
\[ J_6 \leq \epsilon \| \partial_\beta^\alpha \varrho \|_2^2 + C_\epsilon \| \tilde{U} \|_{N-1}^2 \| \nabla \tilde{U} \|_N^2. \] (33)

For the term \( J_1 \), let \( \alpha_0 \leq \alpha \) with \( |\alpha_0| = 1 \), then
\[ J_1 = -\langle \partial_\beta^\alpha \varrho \Delta \varrho, \partial_\beta^\alpha \varrho \rangle \leq \epsilon \| \partial_\beta^\alpha \varrho \|_2^2 + C_\epsilon \| \partial_\beta^\alpha \varrho \|_2^2. \] (34)

Similar to the estimate of (31), we have
\[ \| \partial_\beta^\alpha \varrho \|_2^2 \leq C \| \tilde{U} \|_{N-1}^2 \| \nabla \tilde{U} \|_N^2. \] (35)

Thus, it follows from (34) and (35) that
\[ J_1 \leq \epsilon \| \partial_\beta^\alpha \varrho \|_2^2 + C_\epsilon \| \tilde{U} \|_{N-1}^2 \| \nabla \tilde{U} \|_N^2. \] (36)

Notice that (31) and (32) imply
\[ \| \partial_\beta^\alpha \varrho \|_2^2 \leq C (\| \partial_\beta^\alpha (\nabla \tilde{v} \cdot \varrho) \|_2^2 + \| \partial_\beta^\alpha (\varrho \nabla \tilde{v}) \|_2^2) \leq C \| \tilde{U} \|_{N-1}^2 \| \nabla \tilde{U} \|_N^2. \] (37)

Therefore, we derive from (8)1, (37) and the Cauchy-Schwarz inequality that
\[ J_2 = -\frac{\kappa'}{\gamma} \langle \partial_\beta^\alpha \varrho \Delta \varrho, \partial_\beta^\alpha \varrho \rangle + \partial_\beta^\alpha Q_1 (\tilde{U}) \]
\[ = -\frac{\kappa'}{\gamma} \langle \partial_\beta^\alpha \nabla \varrho, \partial_\beta^\alpha \nabla \varrho \rangle - \frac{\kappa'}{\gamma} \langle \partial_\beta^\alpha \Delta \varrho, \partial_\beta^\alpha Q_1 (\tilde{U}) \rangle \]
\[ \leq -\frac{\kappa'}{2\gamma} \frac{d}{dt} \| \partial_\beta^\alpha \nabla \varrho \|_2^2 + \epsilon \| \partial_\beta^\alpha \varrho \|_2^2 + C_\epsilon \| \tilde{U} \|_{N-1}^2 \| \nabla \tilde{U} \|_N^2. \] (38)

Moreover, it holds that
\[ J_3 = -\nu_2 (\partial_\beta^\alpha \varrho \nabla v, \partial_\beta^\alpha f) \leq \epsilon \| \partial_\beta^\alpha \varrho \|_2^2 + C_\epsilon \| f \|_{N-1}^2, \] (39)

where \( \alpha_0 \) is defined in (34). Combining (25), (33), (36), (38) and (39), if \( \epsilon \) is small enough, we have
\[ \frac{d}{dt} (\| \partial_\beta^\alpha \varrho \|_2^2 + \| \partial_\beta^\alpha \varrho \|_2^2) + \| \partial_\beta^\alpha \nabla v \|_2^2 + \| \partial_\beta^\alpha \nabla \cdot v \|_2^2 + \| \partial_\beta^\alpha v \|_2^2 \]
\[ \leq \epsilon C \| \partial_\beta^\alpha \varrho \|_2^2 + C_\epsilon \| \partial_\beta^\alpha \varrho \|_2^2 + C_\epsilon C \left( \| \tilde{U} \|_{N-1}^2 \| \nabla \tilde{U} \|_N^2 + \| f \|_{N-1}^2 \right), \] (40)

where \( C \) depends only on \( \rho_\infty, \mu, \nu, a \) and \( \kappa \).

Now we turn to estimate \( \| \partial_\beta^\alpha \Delta \varrho \|_2^2 \) for \( 1 \leq |\alpha| \leq N \). As we did for the first order derivative estimate, applying \( \partial_\beta^\alpha \) to (8)2 and then taking the \( L^2 \) inner product with \( \partial_\beta^\alpha \nabla \varrho \) on the resultant equation, we have by integrating that
\[ \kappa' \| \partial_\beta^\alpha \Delta \varrho \|_2^2 + \gamma \| \partial_\beta^\alpha \nabla \varrho \|_2^2 \]
\[ = -(\partial_\beta^\alpha v, \partial_\beta^\alpha \nabla \varrho) + \mu' (\partial_\beta^\alpha \Delta \varrho, \partial_\beta^\alpha \nabla \varrho) + \nu' (\partial_\beta^\alpha \nabla (\varrho \cdot v), \partial_\beta^\alpha \nabla \varrho) \]
\[ + (\partial_\beta^\alpha G_2 (\tilde{U}), \partial_\beta^\alpha \nabla \varrho) + \nu_2 (\partial_\beta^\alpha f, \partial_\beta^\alpha \nabla \varrho) - a (\partial_\beta^\alpha v, \partial_\beta^\alpha \nabla \varrho) \]
\[ = J_7 + J_8 + J_9 + J_{10} + J_{11} + J_{12}. \] (41)
The first term $J_7$ is controlled by

\[
J_7 = - \frac{d}{dt} \langle \partial_x^\alpha v, \partial_x^\alpha \nabla \varrho \rangle + \langle \partial_x^\alpha v, \partial_x^\alpha \nabla \varrho_t \rangle \\
= - \frac{d}{dt} \langle \partial_x^\alpha v, \partial_x^\alpha \nabla \varrho \rangle - \langle \partial_x^\alpha \nabla \cdot v, \partial_x^\alpha (-\gamma \nabla \cdot v + Q_1(\tilde{U})) \rangle \\
\leq - \frac{d}{dt} \langle \partial_x^\alpha v, \partial_x^\alpha \nabla \varrho \rangle + 2\gamma \|\partial_x^\alpha \nabla \cdot v\|^2 + C \|\tilde{U}\|_{N-1}^2 \|\nabla \tilde{U}\|_N^2.
\]

Here, in the last inequality of (42), we have used (37). By integrating by parts and using the Cauchy-Schwartz inequality, the other terms $J_8$–$J_{12}$ can be estimated as follows

\[
J_8 + J_9 \leq \frac{\kappa'}{4} \|\partial_x^{\alpha+\alpha_0} \nabla \varrho\|^2 + C \left( \|\partial_x^\alpha \nabla v\|^2 + \|\partial_x^\alpha \nabla \cdot v\|^2 \right),
\]

\[
J_{10} \leq \frac{\kappa'}{4} \|\partial_x^{\alpha+\alpha_0} \nabla \varrho\|^2 + C \|\tilde{U}\|_{N-1}^2 \|\nabla \tilde{U}\|_N^2,
\]

\[
J_{11} \leq \frac{\kappa'}{4} \|\partial_x^{\alpha+\alpha_0} \nabla \varrho\|^2 + C \|f\|_{N-1}^2,
\]

\[
J_{12} \leq \frac{\gamma}{4} \|\partial_x^\alpha \nabla \varrho\|^2 + C \|\partial_x^\alpha v\|^2,
\]

where \(\alpha_0\) is given in (34). Combining (41)–(46), we obtain

\[
\frac{d}{dt} \langle \partial_x^\alpha v, \partial_x^\alpha \nabla \varrho \rangle + \|\partial_x^\alpha \nabla \varrho\|^2 + \|\partial_x^\alpha \nabla \varrho_t\|^2 \\
\leq C \left( \|\partial_x^\alpha \nabla v\|^2 + \|\partial_x^\alpha \nabla \cdot v\|^2 \right) + C \|\partial_x^\alpha v\|^2 + C \left( \|\tilde{U}\|_{N-1}^2 \|\nabla \tilde{U}\|_N^2 + \|f\|_{N-1}^2 \right).
\]

Multiplying (47) with a suitably small constant \(\delta_1 > 0\) and then adding the resultant equation to (40) gives

\[
\frac{d}{dt} \left( \|\partial_x^\alpha \varrho\|^2 + \|\partial_x^\alpha v\|^2 + \delta_1 \langle \partial_x^\alpha v, \partial_x^\alpha \nabla \varrho \rangle + \|\partial_x^\alpha \nabla \varrho\|^2 + \|\partial_x^\alpha v\|^2 \right) \\
\leq \epsilon C \|\partial_x^\alpha \varrho\|^2 + C \left( \|\tilde{U}\|_{N-1}^2 \|\nabla \tilde{U}\|_N^2 + \|f\|_{N-1}^2 \right),
\]

provided that \(\delta_1\) and \(\epsilon\) are small enough, where \(C\) depends only on \(\rho_\infty, \mu, \nu, a\) and \(\kappa\). Summing up \(\alpha\) with \(1 \leq |\alpha| \leq N\) in (48), then (24) follows immediately by the smallness of \(\epsilon\). This completes the proof of Lemma 3.4.

4. Linear decay estimates. Consider the Cauchy problem for the corresponding homogeneous linear system to (5)

\[
\begin{align*}
\varrho_t + \gamma \nabla \cdot v &= 0, \\
v_t - \mu^\prime \Delta v - \nu^\prime \nabla (\nabla \cdot v) + av + \gamma \nabla \varrho - \kappa' \nabla \Delta \varrho &= 0, \\
\varrho_{t=t_s} &= \varrho_s(x), \quad v_{t=t_s} = v_s(x).
\end{align*}
\]

In terms of the semigroup theory for evolutionary system, the solution \((\varrho, v)^t\) of the Cauchy problem (49) can be expressed as

\[
(\varrho, v)^t(t, x) = G(t, x) * (\varrho_s, v_s)^t, \quad t \geq s,
\]

where \(G(t, x)\) is the Green’s matrix for the system (49).

Applying the Fourier transform to the system (49), we make use of the similar argument of Lemma 2.1 in [20], then the explicit expression for the Fourier transform \(\hat{G}(\xi, t)\) is obtained by a direct computation.
Lemma 4.1. The Fourier transform $\hat{G}$ of the Green's matrix for the linear system (49) is given by

$$
\hat{G}(\xi,t) = \hat{G}_1(\xi,t) + \hat{G}_2(\xi,t)
$$

where

$$
\begin{align*}
\hat{G}_1(\xi,t) &= \begin{pmatrix}
\frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} & -i\gamma\left(\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\right) \\
-i\gamma\left(\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\right) & e^{-(\mu' + 4\eta^2 + 4\gamma^2) t} (I - \frac{\xi\xi'}{|\xi|^2} ) + \frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} \frac{\xi\xi'}{|\xi|^2}
\end{pmatrix},
\end{align*}
$$

and

$$
\hat{G}_2(\xi,t) = \begin{pmatrix}
0 & \xi \\
\xi & 0
\end{pmatrix},
$$

where $\eta = \mu' + \nu'$ and

$$
\lambda_\pm(\xi) = -\frac{1}{2}(\eta|\xi|^2 + a) \pm \frac{1}{2}\sqrt{(\eta|\xi|^2 + a)^2 - 4\kappa^2|\xi|^4 - 4\gamma^2|\xi|^2}.
$$

We simply denote the four components of $\hat{G}_1$ by $\hat{g}_{11}$, $\hat{g}_{12}$, $\hat{g}_{21}$, and $\hat{g}_{22}$ and the nonzero component of $\hat{G}_2$ by $\hat{g}$. And we shall estimate them term by term. Considering it's very difficult to determine that the eigenvalues $\lambda_\pm$ are real or complex, we divide the arguments into two cases in terms of the eigenvalues $\lambda_\pm$. It is also natural to observe that $\lambda_\pm$ are real for $|\xi| \ll 1$. Thus there exists a fixed constant $R > 0$ such that $\lambda_\pm$ are real for $|\xi| \leq R$.

Case 1. When $\lambda_\pm$ are complex numbers, we denote

$$
\lambda_\pm = -\frac{1}{2}(\eta|\xi|^2 + a) \pm bi,
$$

where

$$
b = \frac{1}{2}\sqrt{4\kappa^2|\xi|^4 + 4\gamma^2|\xi|^2 - (\eta|\xi|^2 + a)^2}.
$$

Hence we have

$$
\frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} = e^{-\frac{1}{2}(\eta|\xi|^2 + a)t} \left[ \cos(bt) + \frac{\sin(bt)}{2b} (\eta|\xi|^2 + a) \right],
$$

$$
\frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} = e^{-\frac{1}{2}(\eta|\xi|^2 + a)t} \left[ \cos(bt) - \frac{\sin(bt)}{2b} (\eta|\xi|^2 + a) \right],
$$

$$
\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} = e^{-\frac{1}{2}(\eta|\xi|^2 + a)t} \frac{\sin(bt)}{b}.
$$

In this case, we only need to deal with the high frequency $|\xi| \geq R$. Using the fact that

$$
\left| \frac{\sin(bt)}{2b} (\eta|\xi|^2 + a) e^{-\frac{1}{2}(\eta|\xi|^2 + a)t} \right| \leq \frac{1}{2} (\eta|\xi|^2 + a) e^{-\frac{1}{2}(\eta|\xi|^2 + a)t} \leq 2e^{-\frac{1}{2}(\eta|\xi|^2 + a)t},
$$

and substituting (51)–(53) into (50), we deduce that, setting $\eta' = \min \{ \eta, \mu' \}$

$$
\begin{align*}
|\hat{g}_{11}(\xi,t), \hat{g}_{12}(\xi,t)| &\leq Ce^{-\frac{1}{2}(\eta'|\xi|^2 + a)t}, \\
|\hat{g}_{21}(\xi,t), \hat{g}_{22}(\xi,t)| &\leq C|\xi|^{-1} e^{-\frac{1}{2}(\eta'|\xi|^2 + a)t}, \\
|\hat{g}(\xi,t)| &\leq C|\xi| e^{-\frac{1}{2}(\eta'|\xi|^2 + a)t}.
\end{align*}
$$
**Case 2.** \( \lambda_{\pm} \) are real eigenvalues. For \( |\xi| \geq R \), in terms of the definitions of \( \lambda_{\pm} \), it holds that

\[
\lambda_- \leq -\frac{1}{2}(\eta|\xi|^2 + a),
\]

and

\[
\lambda_+ = -\frac{|\xi|^2}{2} \cdot \frac{4\kappa'\gamma|\xi|^2 + 4\gamma^2}{\eta|\xi|^2 + a + \sqrt{(\eta|\xi|^2 + a)^2 - 4\kappa'\gamma|\xi|^4 - 4\gamma^2|\xi|^2}} \leq -\frac{\kappa'\gamma}{\eta + \frac{a}{|\xi|^2}} |\xi|^2.
\]

Since \( |\lambda_{\pm}|, |\lambda_+ - \lambda_-| = O(|\xi|^2) \), setting \( \alpha = \min\left\{ \frac{\kappa'\gamma}{\eta + \frac{a}{|\xi|^2}}, \frac{\eta}{2} \right\} \), we derive from the expression (50) and (55)–(56) that

\[
|\dot{g}_{11}(\xi, t), \dot{g}_{22}(\xi, t)| \leq Ce^{-\alpha|\xi|^2t},
|\dot{g}_{12}(\xi, t), \dot{g}_{21}(\xi, t)| \leq C|\xi|^{-1}e^{-\alpha|\xi|^2t},
|\dot{g}(\xi, t)| \leq C|\xi|e^{-\alpha|\xi|^2t},
\]

which together with the fact that \( |\lambda_+| = O(|\xi|^2) \), \( |\lambda_-| = O(1) \), \( |\lambda_+ - \lambda_-| = O(1) \) gives

\[
|\dot{g}_{11}(\xi, t)| \leq C(1 + |\xi|^2)e^{-\alpha'|\xi|^2t},
|\dot{g}_{12}(\xi, t), \dot{g}_{21}(\xi, t)| \leq C|\xi|e^{-\alpha'|\xi|^2t},
|\dot{g}_{22}(\xi, t)| \leq C|\xi|^2e^{-\alpha'|\xi|^2t} + Ce^{-\frac{1}{2}(\alpha' + a)t},
|\dot{g}(\xi, t)| \leq C|\xi|^3e^{-\alpha'|\xi|^2t},
\]

where \( \alpha' = \min\left\{ \frac{\gamma^2}{\eta R^2 + a}, \mu', \frac{\eta}{2} \right\} \).

In view of (54) and (57), we conclude that there exists some constant \( \beta = \min\left\{ \frac{1}{4}\eta', a \right\} \) such that

\[
|\dot{g}_{11}(\xi, t), \dot{g}_{22}(\xi, t)| \leq Ce^{-\beta|\xi|^2t},
|\dot{g}_{12}(\xi, t), \dot{g}_{21}(\xi, t)| \leq C|\xi|^{-1}e^{-\beta|\xi|^2t},
|\dot{g}(\xi, t)| \leq C|\xi|e^{-\beta|\xi|^2t},
\]

for \( |\xi| \geq R \).

With the help of the formula (50) for Green’s matrix in Fourier space and the asymptotical analysis on its elements, we are able to establish the following linear optimal decay estimates as \([3, 20]\).
Lemma 4.2. Let \( l \geq 0 \) be an integer. Assume that \((\varrho, v)\) is the solution of the problem (49) with the initial data \( \varrho_0 \in H^{l+1} \cap L^1 \) and \( v_0 \in H^l \cap L^1 \), then

\[
\|\varrho(t)\| \leq C(1 + t)^{-\frac{3}{2}} \left( \|\varrho(0)\|_{L^1} + \|v(0)\|_{L^1} \right),
\]

\[
\|\nabla^{k+1} \varrho(t)\| \leq C(1 + t)^{-\frac{3}{2} - \frac{k+1}{2}} \left( \|\varrho(0)\|_{L^1} + \|\nabla^{k+1} \varrho_0, \nabla^{k+1} v_0\| \right),
\]

\[
\|\nabla^k v(t)\| \leq C(1 + t)^{-\frac{3}{2} - \frac{k-1}{2}} \left( \|\varrho(0)\|_{L^1} + \|\nabla^{k+1} \varrho_0, \nabla^{k+1} v_0\| \right),
\]

where \( k \) is an integer satisfying \( 0 \leq k \leq l \).

Proof. Now by the expression (50), we have

\[
\varrho_1(\xi, t) = \hat{g}_1 \hat{\varrho}_s + \hat{g}_2 \hat{v}_s,
\]

\[
\nu_1(\xi, t) = \hat{g}_2 \hat{\varrho}_s + \hat{g}_3 \hat{v}_s + \hat{g}_4 \hat{\nu}_s.
\]

Let \( R > 0 \) be a fixed constant as before. By the pointwise estimates (58) and (59), together with the Parseval theorem and Hausdorff-Young’s inequality, we have the \( L^2 \)-decay rates for \((\varrho, v)\) and the derivatives of \((\varrho, v)\) as

\[
\|\varrho(t)\|^2 = \int_{\mathbb{R}^3} |\hat{g}_1(\xi, t)|^2 d\xi + \int_{\mathbb{R}^3} |\hat{g}_2(\xi, t)|^2 d\xi + \int_{\mathbb{R}^3} |\hat{g}_3(\xi, t)|^2 d\xi + \int_{\mathbb{R}^3} |\hat{g}_4(\xi, t)|^2 d\xi
\]

\[
\leq C \int_{|\xi| \leq R} e^{-\alpha |\xi|^2} (1 + |\xi|^2 + |\xi|^4) |\hat{\varrho}(\xi)|^2 + e^{-\alpha |\xi|^2} |\hat{v}(\xi)|^2 + e^{-|\xi|^2} |\hat{\varrho}(\xi)|^2 + e^{-|\xi|^2} |\hat{v}(\xi)|^2 d\xi
\]

\[
+ C \int_{|\xi| \geq R} e^{-\beta |\xi|^2} (|\hat{\varrho}(\xi)|^2 + |\hat{v}(\xi)|^2) d\xi
\]

\[
\leq C(1 + t)^{-\frac{3}{2}} \|\varrho(0)\|_{L^1}^2 + C e^{-\beta R^2} \|\varrho(0)\|_{L^1}^2,
\]

and

\[
\|\nabla^{k+1} \varrho(t)\|^2 = \int_{\mathbb{R}^3} |\xi|^{2(k+1)} |\hat{g}_1(\xi, t)|^2 d\xi + \int_{\mathbb{R}^3} |\xi|^{2(k+1)} |\hat{g}_2(\xi, t)|^2 d\xi + \int_{\mathbb{R}^3} |\xi|^{2(k+1)} |\hat{g}_3(\xi, t)|^2 d\xi
\]

\[
\leq C \int_{|\xi| \leq R} e^{-\alpha |\xi|^2} (1 + |\xi|^2 + |\xi|^4) |\hat{\varrho}(\xi)|^2 + e^{-\alpha |\xi|^2} |\hat{v}(\xi)|^2 + e^{-|\xi|^2} |\hat{\varrho}(\xi)|^2 + e^{-|\xi|^2} |\hat{v}(\xi)|^2 d\xi
\]

\[
+ C \int_{|\xi| \geq R} e^{-\beta |\xi|^2} (|\hat{\varrho}(\xi)|^2 + |\hat{v}(\xi)|^2) d\xi
\]

\[
\leq C(1 + t)^{-\frac{3}{2} - k} \|\varrho(0)\|_{L^1}^2 + C e^{-\beta R^2} \|\varrho(0)\|_{L^1}^2,
\]

for \( 0 \leq k \leq l \). The proof of Lemma 4.2 is completed. □
5. **Existence of time periodic solution.** Now, we are ready to prove Theorem 1.1 as follows.

**Proof of Theorem 1.1.** The proof is divided into two steps.

**Step 1.** For any $U = (\rho, v) \in X_{h_0}(0, T)$, we first define

$$
\varphi[U](t) = \int_{-\infty}^{t} S(t, \tau) \left( Q(U) + F(\tau) \right) d\tau.
$$

Suppose that there exists a time periodic solution $U^{\text{per}}(t) := (\rho^{\text{per}}(x, t), v^{\text{per}}(x, t))$, $t \in \mathbb{R}$ of the system (5) with period $T$, and $U^{\text{per}}(t) \in X_{h_0}(0, T)$ for some constant $h_0 > 0$. Then it solves (5) with initial date $U_s = U^{\text{per}}(s)$ for any given time $s \in \mathbb{R}$. Choosing $s = -kT$ for $k \in \mathbb{N}$. Clearly, $U^{\text{per}}(-kT) = U^{\text{per}}(0)$, thus (10) can be written in the mild form as

$$
U^{\text{per}}(t) = S(t, -kT)U^{\text{per}}(0) + \int_{-kT}^{t} S(t, \tau)(Q(U^{\text{per}})(\tau) + F(\tau)) d\tau. \quad (60)
$$

We denote $S(t, -kT)U^{\text{per}}(0) := (\rho_1^{\text{per}}(t), v_1^{\text{per}}(t))$, then applying Lemma 4.2 to $S(t, -kT)U^{\text{per}}(0)$, we have

$$
\|\rho_1^{\text{per}}(t)\|_{N} \leq (1 + t + kT)^{-\frac{1}{2}} \left( \|\rho_0^{\text{per}}(t), v_0^{\text{per}}(t)\|_{L^1} + \|\rho_0^{\text{per}}\|_{N} + \|v_0^{\text{per}}\|_{N-1} \right) \to 0 \quad \text{as} \quad k \to \infty, \quad (61)
$$

and

$$
\|v_1^{\text{per}}(t)\|_{N-1} \leq (1 + t + kT)^{-\frac{1}{2}} \left( \|\rho_0^{\text{per}}(t), v_0^{\text{per}}(t)\|_{L^1} + \|\rho_0^{\text{per}}\|_{N} + \|v_0^{\text{per}}\|_{N-1} \right) \to 0 \quad \text{as} \quad k \to \infty. \quad (62)
$$

Since $L^2 \cap L^1$ is dense in $L^2$, (61) and (62) still hold for $U^{\text{per}}(0) = (\rho_0^{\text{per}}, v_0^{\text{per}}) \in H^N(\mathbb{R}^n) \times H^{N-1}(\mathbb{R}^n)$. On the other hand, denote

$$
S(t, \tau)(Q(U^{\text{per}})(\tau) + F(\tau)) := (S_1(t, \tau), S_2(t, \tau)).
$$

Similar to the proof of Lemma 4.2, we get

$$
\|S_1(t, \tau)\|_{2}^2 = \int_{\mathbb{R}^3} |\tilde{g}_{11}(\xi, t - \tau)Q_1(U^{\text{per}})(\xi, \tau) \left( Q_2(U^{\text{per}}) + \nu_2 f \right)(\xi, \tau)|^2 d\xi
+ \int_{\mathbb{R}^3} |\tilde{g}_{12}(\xi, t - \tau)(Q_2(U^{\text{per}}) + \nu_2 f)(\xi, \tau)|^2 d\xi
= C \int_{\mathbb{R}^3} |\tilde{g}_{11}(\xi, t - \tau)|^2 \left( \|\rho^{\text{per}}_{\text{per}}(\xi, \tau)\|_{L^1} \right) \left( \|\rho^{\text{per}}_{\text{per}}(\xi, \tau)\|_{L^1} \right) d\xi
+ \int_{\mathbb{R}^3} \left( \|Q_2(U^{\text{per}}) + \nu_2 f\|_{L^1} \right) \left( \|\rho^{\text{per}}_{\text{per}}(\xi, \tau)\|_{L^1} \right) d\xi
\leq C \int_{|\xi| \leq R} e^{-\alpha' t^2} |\xi|^2 \left( \|\rho^{\text{per}}_{\text{per}}(\xi, \tau)\|_{L^1} \right) d\xi
+ \int_{|\xi| \geq R} e^{-\beta t^2} |\xi|^2 \left( \|Q_2(U^{\text{per}}) + \nu_2 f\|_{L^1} \right) d\xi
$$

(63)
we obtain

\[ (U_2(t) + \nu_2 f)(\tau) \] 

Then (65) shows that \( \phi \) is suitably small, then

Step 2. Throughout this step, we will show that if

\[ 0 \leq t \leq T \]

is suitably small, then \( \phi \) has a unique fixed point in the space \( X_{b_0}(0, T) \) for some appropriate constant \( b_0 > 0 \). The proof is divided into three parts.

(1) Assume that \( \bar{U} = (\bar{\varrho}, \bar{\nu}) \) in the system (8) is time periodic with period \( T \). Denote \( \bar{U} = \varphi[\bar{U}] \) with \( U = (\varrho, \nu) \). Then by the same argument as (66), one can
Thus, from Lemma 3.3–3.4, we obtain

\[
\frac{d}{dt} \left(\|\varphi(t)\|_{N+1}^2 + \|v(t)\|_{N+1}^2 + \delta_0 (v, \nabla \varphi)(t) + \delta_1 \sum_{|\alpha| = 1}^N \langle \partial_x^\alpha v, \partial_x^\alpha \nabla \varphi \rangle(t) \right) + \|\nabla \varphi(t)\|_{N+1}^2 + \|v(t)\|_{N+1}^2 \leq C \left(\|\tilde{U}(t)\|_{N-1}^2 \|\nabla \tilde{U}(t)\|_{N}^2 + \|f(t)\|_{H^{N-1} \cap L^1}^2 \right).
\]

Integrating (67) in \( t \) over \([0, T]\) gets

\[
\int_0^T \left(\|\nabla \varphi(t)\|_{N+1}^2 + \|v(t)\|_{N+1}^2 \right) dt \\
\leq C \int_0^T \left(\|\tilde{U}(t)\|_{N-1}^2 \|\nabla \tilde{U}(t)\|_{N}^2 + \|f(t)\|_{H^{N-1} \cap L^1}^2 \right) dt \\
\leq C \sup_{0 \leq t \leq T} \|\tilde{U}(t)\|_{N-1}^2 \int_0^T \|\nabla \tilde{U}(t)\|_{N}^2 dt + C \int_0^T \|f(t)\|_{H^{N-1} \cap L^1}^2 dt \\
\leq C \|\tilde{U}(t)\|^4 + CT \sup_{0 \leq t \leq T} \|f(t)\|_{H^{N-1} \cap L^1}^2.
\]

(2) Similar to the estimates of (64), we have

\[
\|\varphi(t)\|_{N} \leq \int_{-\infty}^t (1 + t - \tau)^{-\frac{1}{2}} \left(\|\tilde{\varphi}(\tau)\|_{L^1 \cap H^{N+1}}^2 \right) d\tau, \\
\|v(t)\|_{N-1} \leq \int_{-\infty}^t (1 + t - \tau)^{-\frac{1}{2}} \left(\|\tilde{Q}_2(\tilde{U}) + \nu_2 f(\tau)\|_{L^1 \cap H^{N-1}}^2 \right) d\tau.
\]

From (6), (35) and (37), we easily deduce that

\[
\|\tilde{\varphi}(\tau)\|_{L^1} \leq \|\tilde{\varphi}(\tau)\|_{L^1}, \\
\|\tilde{\varphi}(\tau)\|_{N+1} \leq \|\tilde{\varphi}(\tau)\|_{N+1} \|\nabla \tilde{U}(\tau)\|_{N}, \\
\|\tilde{Q}_1(\tilde{U})(\tau)\|_{L^1} \leq \|\nabla \tilde{U}(\tau)\|_{L^1}(\tilde{U}(\tau)), \\
\|\tilde{Q}_1(\tilde{U})(\tau)\|_{N} \leq \|\tilde{U}(\tau)\|_{N-1} \|\nabla \tilde{U}(\tau)\|_{N}, \\
\left\|\left(\tilde{Q}_2(\tilde{U}) + \nu_2 f(\tau)\right)(\tau)\right\|_{L^1} \leq C \|\nabla \tilde{U}(\tau)\|_{L^1}(\tilde{U}(\tau)) + C \|f(\tau)\|_{L^1}, \\
\left\|\left(\tilde{Q}_2(\tilde{U}) + \nu_2 f(\tau)\right)(\tau)\right\|_{N-1} \leq C \|\tilde{U}(\tau)\|_{N-1} \|\nabla \tilde{U}(\tau)\|_{N} + C \|f(\tau)\|_{N-1}.
\]

Combining (69) and (70), we obtain

\[
\|\varphi(t)\|_{N} \leq C \int_{-\infty}^t (1 + t - \tau)^{-\frac{1}{2}} \left(\|\tilde{\varphi}(\tau)\|_{L^1} \|\tilde{\varphi}(\tau)\|_{N} \right) d\tau
\]

(71)
where
\[ A_j = C \int_{t-(j+1)T}^{t-jT} (1 + t - \tau)^{-\frac{3}{2}} \left( \| \tilde{g}(\tau) \| + \| \tilde{U}(\tau) \|_{N-1} \| \nabla \tilde{U}(\tau) \|_{N} \right) d\tau \]

Substituting (72) into (71) gives
\[ \| \tilde{g}(t) \|_{N} \leq C \| \tilde{U} \|^2 + C \sup_{0 \leq t \leq T} \| f(t) \|_{H^{N-1}}. \]  

Similarly, it holds that
\[ \| \tilde{v}(t) \|_{N-1} \leq C \| \tilde{U} \|^2 + C \sup_{0 \leq t \leq T} \| f(t) \|_{H^{N-1}}. \]

Thus, we deduce from (68), (73) and (74) that
\[ \| \varphi(\tilde{U}) \| \leq C_1 \| \tilde{U} \|^2 + C_2 \sup_{0 \leq t \leq T} \| f(t) \|_{H^{N-1}}, \]

where \( C_1 \) and \( C_2 \) are some positive constants depending only on \( \rho_\infty, \mu, \nu, a, \kappa \) and \( T \).

(3) Let \( \tilde{U}_1 = (\tilde{g}_1, \tilde{v}_1) \) and \( \tilde{U}_2 = (\tilde{g}_2, \tilde{v}_2) \) be time periodic functions with period \( T \) in the space \( X_{h_0}(0, T) \), where \( h_0 > 0 \) will be determined below. Then similar to (1) and (2), we can get
\[ \| \varphi(\tilde{U}_1) - \varphi(\tilde{U}_2) \| \leq C_3 \left( \| \tilde{U}_1 \| + \| \tilde{U}_2 \| \right) \| \tilde{U}_1 - \tilde{U}_2 \|, \]

where \( C_3 \) is a positive constant depending only on \( \rho_\infty, \mu, \nu, a, \kappa \) and \( T \). Choose \( h_0 > 0 \) and a sufficiently small constant \( h > 0 \) such that
\[ C_1 h_0^2 + C_2 h \leq h_0 \quad \text{and} \quad 2C_3 h_0 < 1. \]

That is,
\[ \frac{1 - \sqrt{1 - 4C_1 C_2 h h_0}}{2C_1} \leq h_0 \leq \min \left\{ \frac{1 + \sqrt{1 - 4C_1 C_2 h h_0}}{2C_1}, 1 \right\}. \]  

Notice that
\[ \frac{1 - \sqrt{1 - 4C_1 C_2 h h_0}}{2C_1} \to 0 \quad \text{as} \quad h \to 0. \]

Then there exists a constant \( h_0 > 0 \) depending only on \( \rho_\infty, \mu, \nu, a, \kappa \) and \( T \) such that if \( 0 < h \leq h_0 \), the set of \( h_0 \) satisfying (75) is not empty. For \( 0 < h \leq h_0 \), when \( h_0 \) satisfies (75), \( \varphi \) is a contraction map in the complete space \( X_{h_0}(0, T) \), thus \( \varphi \) has a unique fixed point in \( X_{h_0}(0, T) \). This completes the proof of Theorem 1.1. \( \Box \)
6. Stability of time periodic solution. Now, we are in a position to prove the stability of the obtained time periodic solution of Theorem 1.2. We first consider the global existence of the solution to the Cauchy problem (2). Let \((\rho_{\text{per}}, u_{\text{per}})\) be the time periodic solution constructed in Theorem 1.1 and \((\rho, u)\) be the solution of (2). Set

\[
(\rho_{\text{per}}, v_{\text{per}}) = (\rho_{\text{per}} - \rho_{\infty}, \nu_2 u_{\text{per}}),
\]

\[
(\rho, v) = (\rho - \rho_{\infty}, \nu_2 u).
\]

Denote \((\bar{\rho}, \bar{v}) = (\rho - \rho_{\text{per}}, v - v_{\text{per}}) = (\rho - \rho_{\text{per}}, \nu_2 (u - u_{\text{per}}))\), so \((\bar{\rho}, \bar{v})\) is a solution to the Cauchy problem

\[
\begin{aligned}
\bar{\rho}_t + \gamma \nabla \cdot \bar{v} &= Q_1(\bar{\rho} + \rho_{\text{per}}, \bar{v} + v_{\text{per}}) - Q_1(\rho_{\text{per}}, v_{\text{per}}), \\
\bar{v}_t - \mu' \Delta \bar{v} - \nu' \nabla (\nabla \cdot \bar{v}) + a\bar{v} + \gamma \nabla \bar{\rho} - \kappa' \nabla \Delta \bar{\rho} &= Q_2(\bar{\rho} + \rho_{\text{per}}, \bar{v} + v_{\text{per}}) - Q_2(\rho_{\text{per}}, v_{\text{per}}), \\
(\bar{\rho}, \bar{v})|_{t=0} &= (\bar{\rho}_0, \bar{v}_0)(x) = (\rho_0(x) - \rho_{\text{per}}(0), \nu_2 (u_0(x) - u_{\text{per}}(0))).
\end{aligned}
\]

Let us define the solution space and the solution norm of the Cauchy problem (76) by

\[
\bar{X}(t_1, t_2) = \left\{ (\bar{\rho}, \bar{v})(t, x) \mid \bar{\rho} \in C(t_1, t_2; H^{N-1}(\mathbb{R}^3)) \cap C^1(t_1, t_2; H^{N-3}(\mathbb{R}^3)), \\
\bar{v} \in C(t_1, t_2; H^{N-2}(\mathbb{R}^3)) \cap C^1(t_1, t_2; H^{N-4}(\mathbb{R}^3)), \\
\nabla \bar{\rho} \in L^2(t_1, t_2; H^{N-1}(\mathbb{R}^3)), \bar{v} \in L^2(t_1, t_2; H^{N-1}(\mathbb{R}^3)), \right\}
\]

and

\[
\|((\bar{\rho}, \bar{v})(t)\|_2^2 = \sup_{t_1 \leq t \leq t_2} \left\{ \|\bar{\rho}(t)\|_{N-1}^2 + \|\bar{v}(t)\|_{N-2}^2 \right\} + \int_{t_1}^{t_2} \left( \|\nabla \bar{\rho}(t)\|_{N-1}^2 + \|\bar{v}(t)\|_{N-1}^2 \right) dt,
\]

for any \(0 \leq t_1 \leq t_2 \leq \infty\). Notice that \((\rho_{\text{per}}, v_{\text{per}}) \in \bar{X}(0, T)\).

Then, by the standard argument of the contracting map theorem on general hyperbolic-parabolic system as \([11, 15]\), one can obtain the local existence of a strong solution. The details are omitted.

**Lemma 6.1.** (Local existence) Under the assumptions of Theorem 1.1, suppose that \((\bar{\rho}_0, \bar{v}_0) \in H^{N-1}(\mathbb{R}^3) \times H^{N-2}(\mathbb{R}^3)\) and \(\inf \rho_0(x) > 0\). Then there exists a positive constant \(T_0\) depending only on \(||((\bar{\rho}_0, \bar{v}_0)\|_2\) such that the Cauchy problem (76) admits a unique classical solution \((\bar{\rho}, \bar{v}) \in \bar{X}(0, T_0)\), which satisfies

\[
\|((\bar{\rho}, \bar{v})(t)\|_2 \leq C_4 \|((\bar{\rho}_0, \bar{v}_0)\|_2,
\]

where \(C_4\) is a positive constant independent of \(||((\bar{\rho}_0, \bar{v}_0)\|_2\).

As usual, to extend the local solution to a global in time solution, we need to establish the following priori estimate.

**Lemma 6.2.** (A priori estimate) Suppose that \((\bar{\rho}_0, \bar{v}_0) \in H^{N-1}(\mathbb{R}^3) \times H^{N-2}(\mathbb{R}^3)\), and assume that the Cauchy problem (76) has a unique classical solution \((\bar{\rho}, \bar{v}) \in \bar{X}(0, T_1)\) for some positive constant \(T_1\), satisfying

\[
\sup_{0 \leq t \leq T_1} \|((\bar{\rho}, \bar{v})(t)\|_2 \leq \zeta,
\]
for a small constant $\zeta > 0$. Then there exists a constant $C_5 > 0$ which is independent of $T_1$ such that for any $t \in [0, T_1]$, it holds that

$$
\|\bar{\varrho}(t)\|_{N-1}^2 + \|\bar{v}(t)\|_{N-2}^2 + \int_0^t \left( \|\nabla \bar{\varrho}(\tau)\|_{N-1}^2 + \|\bar{v}(\tau)\|_{N-1}^2 \right) d\tau 
\leq C_5 \left( \|\bar{\varrho}_0\|_{N-1}^2 + \|\bar{v}_0\|_{N-2}^2 \right).
$$

(77)

Proof. Noticing that some smallness condition can be imposed on $(\varrho^{per}, v^{per})$, without loss of generality, we may assume $|||((\varrho^{per}, v^{per}))||| \leq \epsilon$ with $\epsilon > 0$ being sufficiently small. Then by the similar argument as in the proof of Lemma 3.3–3.4, we can obtain

$$
\frac{d}{dt} \left( \|\bar{\varrho}\|_{N-1}^2 + \|\bar{v}\|_{N-2}^2 + \delta_2 \langle \bar{v}, \nabla \bar{\varrho} \rangle + \|\nabla \bar{\varrho}\|_{N-1}^2 + \|\bar{v}\|_1^2 \right) 
\leq C \left( \|\nabla^3 \varrho\|_{N+1}^2 + \|\nabla^2 \bar{v}\|_2^2 \right),
$$

(78)

and

$$
\frac{d}{dt} \left( \|\nabla \bar{\varrho}\|_{N-2}^2 + \|\nabla \bar{v}\|_{N-3}^2 + \delta_3 \sum_{|\alpha|=1}^{N-2} \langle \partial_\alpha^2 \bar{v}, \partial_\alpha^2 \nabla \bar{\varrho} \rangle \right) 
+ \|\nabla^2 \bar{\varrho}\|_{N-2}^2 + \|\nabla \bar{v}\|_{N-2}^2 
\leq C \left( \|\nabla \bar{\varrho}\|_{N-2}^2 + \|\bar{v}\|_{N-2}^2 \right),
$$

(79)

where $\delta_2 > 0$ and $\delta_3 > 0$ are some suitably small constants, and $C$ is a constant depending only on $\rho_\infty, \mu, \nu, a$ and $\kappa$. Adding (78) to (79), it holds

$$
\frac{d}{dt} \left( \|\bar{\varrho}\|_{N-1}^2 + \|\bar{v}\|_{N-2}^2 + \delta_2 \langle \bar{v}, \nabla \bar{\varrho} \rangle + \delta_3 \sum_{|\alpha|=1}^{N-2} \langle \partial_\alpha^2 \bar{v}, \partial_\alpha^2 \nabla \bar{\varrho} \rangle \right) 
+ \|\nabla \bar{\varrho}\|_{N-2}^2 + \|\bar{v}\|_{N-2}^2 
\leq C \left( \|\nabla \bar{\varrho}\|_{N-2}^2 + \|\bar{v}\|_{N-2}^2 \right),
$$

(80)

provided that $\epsilon$ is sufficiently small. Integrating (80) in $t$ over $(0, t)$, one can immediately get (77), since

$$
\|\bar{\varrho}\|_{N-1}^2 + \|\bar{v}\|_{N-2}^2 + \delta_2 \langle \bar{v}, \nabla \bar{\varrho} \rangle + \delta_3 \sum_{|\alpha|=1}^{N-2} \langle \partial_\alpha^2 \bar{v}, \partial_\alpha^2 \nabla \bar{\varrho} \rangle \sim \|\bar{\varrho}\|_{N-1}^2 + \|\bar{v}\|_{N-2}^2
$$

by the smallness of $\delta_2$ and $\delta_3$. This completes the proof of Lemma 6.2. \qed

Proof of Theorem 1.2. The proof of Theorem 1.2 is based on Lemma 6.1–6.2 and the continuity argument, then the Cauchy problem (76) admits a unique solution $(\bar{\varrho}, \bar{v})$ globally in time, which satisfies (3) and (4). Then all the statements in Theorem 1.2 follow immediately. This completes the proof of Theorem 1.2. \qed

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