MLE’S BIAS PATHOLOGY MOTIVATES MCMLE
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Summary

Maximum likelihood estimates are often biased. It is shown that this pathology is inherent to the traditional ML estimation method for two or more parameters, thus motivating from a different angle the use of MCMLE.

1 MLE and Bias

Various methods have been proposed to reduce the $O(n^{-1})$ term of the asymptotic bias of maximum likelihood estimate (MLE). Firth (1993) observed that most methods are “corrective” in character rather than “preventive”, i.e. the MLE is first calculated and then corrected, and proposed a preventive approach with systematic correction of the score equations. Bias reduction of MLE’s continues to be a topic of interest as the current literature indicates; see, for example, Giles (2012), Zhang (2013) and the references therein.

In this work it is shown that bias is inherent to the traditional ML estimation method when two or more parameters of a semi-regular model are estimated. This result is confirmed in several examples and motivates the use of the preventive Model Corrected MLE (Yatracos, 2013), thus achieving in these same examples either partial or total bias reduction. For the Pareto distribution in particular with both parameters unknown, the MCMLE $\hat{\psi}_{MC}$ of the scale parameter $\psi$ improves not only the bias but also the variance of the MLE $\hat{\psi}$.

2 The Result-Examples

Let $X$ be a random vector from a parametric model with density $f(x|\theta, \psi)$, unknown parameters $\theta \in R$, $\psi \in R$ and $f$ semi-regular, i.e. at least the $\psi$-score

$$U_\psi(x, \theta, \psi) = \frac{\partial \log f(x|\theta, \psi)}{\partial \psi}$$

is well defined (and used to obtain MLE $\hat{\psi}$) and

$$EU_\psi(x, \theta, \psi) = 0. \quad (1)$$
In the next proposition it is seen that (1) may cause bias of the MLE \( \hat{\psi} \) because it implies often that \( EU_\psi(x, \hat{\theta}, \psi) \neq 0; \hat{\theta} \) is the MLE of \( \theta \). Using instead the score for the data \( Y \) in \( U_\psi(x, \hat{\theta}, \psi) \) this drawback is avoided for some models thus motivating from a different angle the use of MCMLE.

**Proposition 2.1 (Bias pathology of MLE)** Let \( X \) be a random vector from the semi-regular parametric model \( f(x|\theta, \psi) \) with \( \theta, \psi \) both unknown and with score \( U_\psi \) satisfying (1). Obtain MLE \( \hat{\theta} \) either by direct maximization of the likelihood of \( X \) or by solving, if it exists, the \( \theta \)-score equation

\[
U_\theta(x, \hat{\theta}, \psi) = 0.
\]

a) If \( \frac{\partial U_\psi(x, \hat{\theta}, \psi)}{\partial \psi} = C \) is fixed constant, \( C \neq 0 \), then \( \hat{\psi} \) is biased estimate of \( \psi \) if and only if

\[
EU_\psi(x, \hat{\theta}, \psi) \neq 0.
\]  
(2)

Since (1) holds \( \hat{\psi} \) is expected to be biased.

b) If \( \frac{\partial U_\psi(x, \hat{\theta}, \psi)}{\partial \psi} = C(x, \hat{\theta}, \psi) \), \( \hat{\psi} \) is expected more often to be biased.

**Proof: a)** Obtain \( \hat{\psi} \) by solving the score equation

\[
U_\psi(x, \hat{\theta}, \psi) = 0.
\]

Make a Taylor expansion of \( U_\psi(x, \hat{\theta}, \hat{\psi}) \) around \( \psi \),

\[
U_\psi(x, \hat{\theta}, \hat{\psi}) = U_\psi(x, \hat{\theta}, \psi) + (\hat{\psi} - \psi)C. \tag{3}
\]

It follows that

\[
E(\hat{\psi} - \psi) = -C^{-1}EU_\psi(x, \hat{\theta}, \psi) \neq 0
\]
if and only if \( EU_\psi(x, \hat{\theta}, \psi) \neq 0 \).

b) Equation (3) remains valid with \( C = C(x, \hat{\theta}, \psi) \) evaluated at \( \psi = \psi^* \) between \( \psi \) and \( \hat{\psi} \). Then \( \hat{\psi} \) is biased if and only if

\[
EU_\psi(x, \hat{\theta}, \psi^*)C^{-1}(x, \hat{\theta}, \psi^*) \neq 0. \tag{4}
\]

Make a second order Taylor approximation of the left side in (4) around \( EU_\psi = EU_\psi(x, \hat{\theta}, \psi), \ EC = EC(x, \hat{\theta}, \psi^*) \),

\[
E \frac{U_\psi}{C} \approx \frac{EU_\psi}{EC} - \frac{Cov(U_\psi, C)}{E^2C} + \frac{Var(C)EU_\psi}{E^3C}. \tag{5}
\]
Whether or not $EU_{\psi} = 0$, (5) is not expected to vanish.

Proposition 2.1(a) shows MLE’s inherent bias pathology since for an “ideal” regular model (2) is expected to hold and thus $\hat{\psi}$ is biased. Proposition 2.1(a) holds in all the examples that follow and MCMLE $\hat{\psi}_{MC}$ reduces $\hat{\psi}$’s bias.

**Example 2.1** Let $X_1, \ldots, X_n$ be independent random variables from a normal distribution with mean $\theta$ and variance $\sigma^2$ both unknown. To use Proposition 2.1(a) for $\sigma^2$ w.l.o.g. re-parametrize taking $\psi = \sigma^2$.

The log-likelihood of the sample is

$$-\frac{n}{2} \log \sigma^2 - \frac{\sum_{i=1}^{n} (X_i - \theta)^2}{2\sigma^2} = -\frac{n}{2} \log \psi - \frac{\sum_{i=1}^{n} (X_i - \theta)^2}{2\psi},$$

the score equations excluding the constants are

$$U_\theta(X, \theta, \psi) = \sum_{i=1}^{n} (X_i - \theta) = 0 \rightarrow \hat{\theta} = \bar{X},$$

$$U_\psi(X, \hat{\theta}, \psi) = -n\psi + \sum_{i=1}^{n} (X_i - \bar{X})^2 = 0 \rightarrow \hat{\psi} = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

and $\hat{\psi}$ is biased estimate of $\psi = \sigma^2$ since

$$\frac{\partial U_\psi(X, \hat{\theta}, \psi)}{\partial \psi} = -n$$

and $EU_{\psi}(X, \hat{\theta}, \psi) \neq 0$.

**Example 2.2** Let $X_1, \ldots, X_n$ be independent random variables from a shifted exponential distribution with parameters $\theta$ and $\psi(> 0)$ both unknown and density

$$f(x, \theta, \psi) = \psi^{-1} e^{-(x-\theta)/\psi} I_{[\theta, \infty)}(x);$$

$I$ denotes the indicator function. Let $X_{(i)}$ denote the $i$-th order statistic, $i = 1, \ldots, n$.

MLE $\hat{\theta} = X_{(1)}$ and the score equation for $\psi$, after replacing $\theta$ by $X_{(1)}$, is

$$U_\psi(X, \hat{\theta}, \psi) = -n\psi + \sum_{i=1}^{n} (X_{(i)} - X_{(1)}) = 0 \rightarrow \hat{\psi} = \frac{1}{n} \sum_{i=1}^{n} (X_{(i)} - X_{(1)}).$$

Since

$$\frac{\partial U_\psi(X, \hat{\theta}, \psi)}{\partial \psi} = -n$$

and $EU_{\psi}(X, \hat{\theta}, \psi) \neq 0$, $\hat{\psi}$ is biased for $\psi$. 3
In the Pareto family example that follows with parameters $\psi$ and $\theta$ both unknown, the model corrected MLE, $\hat{\psi}_{MC}$, of the shape $\psi$ reduces by 50% the bias of the MLE $\hat{\psi}$ and has also smaller variance. With this parametrization $\hat{\psi}$ is not unbiased even when $\theta$ is known. Using the parametrization $\psi = 1/\psi^*$, MLE $\hat{\psi}^*$ is unbiased for $\psi^*$ when $\theta$ is known but when $\theta$ is unknown MCMLE $\hat{\psi}^*_{MC}$ is unbiased.

**Example 2.3** Let $X_1, \ldots, X_n$ be independent random variables from a Pareto distribution with density

$$f(x|\theta, \psi) = \psi\theta x^{-(\psi+1)}I_{[\theta,\infty)}(x), \; \psi > 0, \; \theta > 0;$$

$I$ denotes the indicator function, $n > 3$. The log-likelihood function of the sample is

$$n \log \psi + n\psi \log \theta - (\psi + 1) \sum_{i=1}^{n} \log X_i + \sum_{i=1}^{n} \log I_{[\theta,\infty)}(X_i)$$

and the MLE estimate of $\theta$ is the smallest observation, $\hat{\theta} = X_{(1)}$. The score

$$U_\psi(X, \hat{\theta}, \psi) = n - \psi \sum_{i=2}^{n} \log \frac{X_i}{X_{(1)}}$$

and the MLE

$$\hat{\psi} = \frac{n}{\sum_{i=2}^{n} \log \frac{X_i}{X_{(1)}}}.$$ 

Since $Y = \sum_{i=2}^{n} \log \frac{X_i}{X_{(1)}}$ has a $\Gamma(n-1, \psi)$ distribution (see, e.g, Baxter, 1980 and references therein) it follows that $\hat{\psi}$ is biased and

$$\mathbb{E} \hat{\psi} - \psi = \frac{2}{n-2}\psi, \; \text{Var}(\hat{\psi}) = \frac{n^2}{(n-2)^2(n-3)}\psi^2.$$ 

The corrected score based on the data $Y$ is

$$(n-1) - \psi Y$$

and the MCMLE is

$$\hat{\psi}_{MC} = \frac{n-1}{\sum_{i=2}^{n} \log \frac{X_i}{X_{(1)}}},$$

with

$$\mathbb{E} \hat{\psi}_{MC} - \psi = \frac{1}{n-2}\psi, \; \text{Var}(\hat{\psi}_{MC}) = \frac{(n-1)^2}{(n-2)^2(n-3)}\psi^2.$$
Observe that $\hat{\psi}_{MC}$ improves both the bias and the variance of $\hat{\psi}$.

Consider the re-parametrization $\psi = \frac{1}{\psi^*}$. The score

$$U_{\psi^*}(X, \hat{\theta}, \psi^*) = -n\psi^* + \sum_{i=2}^{n} \log \frac{X_i}{X_{(1)}}$$

and the MLE

$$\hat{\psi}^* = \frac{\sum_{i=2}^{n} \log \frac{X_i}{X_{(1)}}}{n}.$$ 

Proposition [2,1] holds for $U_{\psi^*}(X, \hat{\theta}, \psi^*)$ with $C = -n$ and $EU_{\psi^*}(X, \hat{\theta}, \psi^*) \neq 0$ indicating that $\hat{\psi}^*$ is biased. Using the model from data $Y = \sum_{i=2}^{n} \log \frac{X_i}{X_{(1)}}$ the corrected score is

$$-(n-1)\psi^* + Y$$

and the MCMLE

$$\hat{\psi}^{*}_{MC} = \frac{\sum_{i=2}^{n} \log \frac{X_i}{X_{(1)}}}{n-1}$$

is unbiased for $\psi^*$.

References

[1] Baxter, M. A. (1980) Minimum variance unbiased estimation of the parameters of the Pareto distribution. *Metrika*, 27, 133-138.

[2] Firth, D. (1993). Bias reduction of maximum likelihood estimates. *Biometrika*, 80, 27-38.

[3] Giles, D. E. (2012). Bias reduction for the maximum likelihood estimators of the parameters in the half-logistic distribution. *Communications in Statistics Theory and Methods*, 41, 212-222.

[4] Yatracos, Y. G. (2013) Fishers Specification Problem and Model Corrected Maximum Likelihood Estimates (MCMLE). Submitted for publication.

[5] Zhang, J. (2013) Reducing bias of the maximum-likelihood estimation for the truncated Pareto distribution. *Statistics*, 47, 792-799.