MORITA ENVELOPING FELL BUNDLES

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Abstract. We introduce notions of weak and strong equivalence for non-saturated Fell bundles over locally compact groups and show that every Fell bundle is strongly (resp. weakly) equivalent to a semidirect product Fell bundle for a partial (resp. global) action. Equivalences preserve cross-sectional C*-algebras and amenability. We use this to show that previous results on crossed products and amenability of group actions carry over to Fell bundles.

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1. Introduction

A Fell bundle over a locally compact group \( G \) is a continuous bundle \( A \to G \) of Banach spaces \( (A_t)_{t \in G} \) together with continuous multiplications \( A_s \times A_t \to A_{st} \), \((a,b) \mapsto a \cdot b\), and involutions \( A_s \to A_{s^{-1}} \), \( a \mapsto a^* \), satisfying properties similar to those valid for a C*-algebra, like the positivity \( a^* \cdot a \geq 0 \) and the C*-axiom \( \|a^* \cdot a\| = \|a\|^2 \). Fell bundles generalise partial actions of groups. Indeed, in [12] Exel defines twisted partial actions of groups on C*-algebras and to each such action, a Fell bundle is constructed, the so-called semidirect product of the twisted partial action. This is, of course, a generalisation of the semidirect product construction for ordinary (global) actions already introduced by Fell in his first papers on the subject, see [10, 11, 15–17]. In [12] only continuous twisted partial actions are considered, but even the measurable twisted actions of Busby-Smith [6] can be turned into (continuous) Fell bundles, see [14].

The main result of [12] already indicates that Fell bundles are very close to partial actions: it asserts that every regular Fell bundle is isomorphic to a semidirect product by a twisted partial action. It is moreover shown that every separable Fell bundle can be “regularised” via stabilisation. The stabilisation procedure should

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be viewed as a form of producing an “equivalent” Fell bundle in the spirit of Morita equivalence of C*-algebras. Hence we may say that the results in [12] show that every separable Fell bundle is equivalent to one associated to a twisted partial action. A very basic question appears: can the twist be “removed”, that is, is every Fell bundle equivalent to a partial action semidirect product Fell bundle? For saturated Fell bundles this, indeed, follows from the famous Packer-Raeburn Stabilisation Trick which asserts that every twisted (global) action is stably isomorphic to an untwisted action. As a result, every saturated separable Fell bundle is equivalent to one coming from an ordinary action. This version is also known for (separable) saturated Fell bundles over groupoids as proved in [9, 18], where precise notions of equivalence of saturated Fell bundles are introduced. A version of the stabilisation trick for non-saturated Fell bundles is only known for discrete groups: it is proved in the master thesis of Sehnem [21] and reproduced in Exel’s book [13] that, after stabilisation, every separable Fell bundle becomes isomorphic to one coming from an (untwisted) partial action.

In [9] a new point of view is introduced from which saturated Fell bundles are interpreted as actions of the underlying group(oid) in the bicategory of C*-correspondences. Indeed, the algebraic structure of the Fell bundle can be used to turn each fibre $A_t$ into a Hilbert bimodule over the unit fiber C*-algebra $A := A_e$, and these bimodules are imprimitivity (or equivalence) bimodules if (and only if) the Fell bundle is saturated. Hence we may view a saturated Fell bundle as an action of $G$ on $A$ by equivalences. A non-saturated Fell bundle should be viewed as a partial action of $G$ on $A$ by (partial) equivalences.

Although the notion of equivalence between saturated Fell bundles over groups is already well established nowadays, little is known for non-saturated Fell bundles. Only recently a more general notion of equivalence has been introduced in [4], which in the present work we call weak equivalence. This notion originates in [3]: the relationship between the Fell bundles of a partial action and its enveloping action is precisely that of weak equivalence. We introduce yet another notion of equivalence, the strong equivalence. As the name suggests, strong equivalence is stronger than weak equivalence. Strong equivalence is the natural extension of the notion of (Morita) equivalence for partial actions as introduced by the first named author in [3]. Indeed, we are going to extend one of the main results in [3] and prove that every (not necessarily saturated or separable) Fell bundle over $G$ is strongly equivalent to a semidirect product Fell bundle by a partial action of $G$ (Theorem 3.5). On the other hand we will prove that, as long as saturated Fell bundles are concerned, there is no difference between weak equivalence and strong equivalence of Fell bundles (Corollary 4.10); they extend the usual notion of equivalence for global actions.

In the recent paper [19] by Kwaśniewski and Meyer the notion of Morita globalization of a Fell bundle over a discrete group is introduced, and it is shown that every Fell bundle (over a discrete group) has a Morita globalization. It can be shown that the Fell bundle associated to the action involved in the definition of a Morita globalization of a Fell bundle is weakly equivalent, in our sense, to the original Fell bundle. As a result a Morita globalization of a Fell bundle is an instance of what we call here a Morita enveloping Fell bundle (see Definition 2.11).

The notion of weak equivalence will allow us to show that every partial action of $G$, once viewed as a Fell bundle, is weakly equivalent to a global action. As
a conclusion, every Fell bundle is weakly equivalent to one associated to a global action. We shall prove this in one step, showing directly that the Fell bundle is weakly equivalent to the semidirect product Fell bundle of a global action. This global action is directly constructed from the Fell bundle. Indeed, it is the same action $\alpha$ appearing in [3] which takes place on the $C^*$-algebra of kernels $k(A)$ of the Fell bundle $A$. As already shown in [3], $k(A)$ can be canonically identified with the crossed product $C^*_r(A) \rtimes_{\delta_A} G$ by the dual coaction $\delta_A$ of $G$ on the full cross-sectional $C^*$-algebra $C^*_r(A)$ of $A$; one can also use the reduced $C^*$-algebra $C^*_r(A)$ together with its dual coaction $\delta^*_A$ of $G$, which is a normalisation of $\delta_A$. Since the dual coaction $\delta_A$ is maximal, $k(A) \rtimes_{\alpha} G \cong C^*_r(A) \otimes \mathbb{K}(L^2(G))$ and similarly $k(A) \rtimes_{\alpha,r} G \cong C^*_r(A) \otimes C^*(G)$. As a consequence of this, the notion of weak (hence strong) equivalence preserves full and reduced cross-sectional $C^*$-algebras, that is, weakly equivalent Fell bundles have (strongly Morita) equivalent full and reduced cross-sectional $C^*$-algebras. Using the same idea, we also derive a version of this result for certain exotic completions $C^*_\mu(A)$ introduced in [7] (some of the latter results were also obtained in [4], though with different methods). Moreover, we show that two Fell bundles $A$ and $B$ are weakly equivalent if and only if the corresponding actions on their $C^*$-algebras of kernels $k(A)$ and $k(B)$ are equivariantly Morita equivalent, if and only if their dual coactions on $C^*(A)$ and $C^*(B)$ are equivariantly Morita equivalent. Strong equivalence of Fell bundles can also be characterised in a similar fashion by the restriction of the global actions on $A$ and $B$ to the partial actions on the $C^*$-algebras of compact operators $\mathbb{K}(L^2(A))$ and $\mathbb{K}(L^2(B))$.

Section 5 of the paper can be viewed as a sample of the potential applications of our main results from the previous sections. We study the partial action associated to Fell bundles on spectra level: a Fell bundle $A$ over $G$ induces a partial action of $G$ on the spectrum (both primitive and irreducible representations) of the unit fibre $A_e$. This partial actions have already been introduced in [1] for discrete groups. We extend the construction to all locally compact groups, proving that the partial action is always continuous. We then show that the enveloping action of the spectral partial action associated to a Fell bundle $A$ is precisely the global action on the spectrum of $k(A)$ induced by its canonical action $\alpha$. In particular, many results of [1] can be obtained from the already existing results for crossed products by ordinary actions. In the same spirit we extend some results about amenability and nuclearity of crossed products to the realm of Fell bundles.

We also add an appendix where we use the tensor product construction of equivalence bundles from [4] to prove that strong equivalence of Fell bundles is an equivalence relation.

2. Fell bundles and globalization of weak group partial actions

Let $B$ be a Fell bundle over a locally compact group $G$, both fixed for the rest of this article. We denote by $dt$ the integration with respect to a fixed left invariant Haar measure on $G$ and write $\Delta$ for the modular function of $G$.

Notation 2.1. Given two sets $X$ and $Y$ for which a product $xy$ between elements $x \in X$ and $y \in Y$ is defined and is contained in a normed vector space, we write $XY$ to mean the closed linear space of all such products, that is,

$$XY := \overline{\text{span}} \{xy : x \in X, y \in Y\}.$$
Moreover,\( B \) and implements a Morita equivalence between\( J \).

Definition 2.3. We say that\( D \) and\( G \) are\( \alpha \)-equivariant\( J \)-bundle if it satisfies:\n\begin{align*}
(1R) \quad &\text{For all } r, s \in G, X_r B_s \subseteq X_{r s} \text{ and } \langle X_r, X_s \rangle_B \subseteq B_{r^{-1} s}.
(2R) \quad &\text{For all } r, s \in G \text{ and } x \in X_r \text{ the function } X_r \times B_s \rightarrow X_{r s}, (x, b) \mapsto x b, \text{ is bilinear and } X_r \rightarrow B_{r^{-1} s}, y \mapsto \langle x, y \rangle_B, \text{ is linear.}
(3R) \quad &\text{For all } x, y \in X \text{ and } b \in B, \langle x, y \rangle_B^* = \langle y, x \rangle_B, \langle x, y b \rangle_B = \langle x, y \rangle_B b, \langle x, b \rangle_B^* \geq 0 \text{ (in } B_c) \text{ and } \|x\|^2 = \|\langle x, x \rangle_B\|.
\end{align*}

We say that\( X \) is full if
\[
\sup \{ \langle X_r, X_r \rangle_B : r \in G \} = B_c. \tag{2.2}
\]
We say that\( X \) is strongly full if
\[
\sup \{ \langle X_r, X_r \rangle_B = B_r^* B_r : r \in G \} \quad \text{for all } r \in G. \tag{2.3}
\]
Remark 2.4. (1) By a Banach bundle we mean a continuous Banach bundle in the sense of Doran-Fell, see [10]. In particular, a Fell bundle is a continuous Banach bundle, by definition. However, the main axiom concerning the continuity of the bundle, namely, the continuity of the norm function $B \rightarrow [0, \infty)$, $b \mapsto \|b\|$, is somehow automatic, see [9, Lemma 3.16]. A similar observation holds for every Hilbert $B$-bundle: the continuity of the norm function $x \mapsto \|x\|$ on $X$ follows from the continuity of the norm function on $B$ because $\|x\| = \|\langle x, x \rangle_B\|^{1/2}$.

(2) The fullness condition (2.2) is equivalent to the condition that

$$B_r = \text{span} \{ (X_s, X_{sr})_B : s \in G \} = \text{span} \{ (X_s, X_t)_B : s^{-1}t = r \}$$

for all $r \in G$ because if (2.2) holds, then

$$B_r = B_e B_r = \text{span} \{ (X_s, X_s)_B B_r : s \in G \} \subseteq \text{span} \{ (X_s, X_{sr})_B : s \in G \} \subseteq B_r.$$ 

In general, $(X_r, X_r)_B$ is only contained in $B_r$, not necessarily in the ideal $B^*_e B_r \subseteq B_e$. Hence the strong fullness condition (2.3) requires that $(X_r, X_r)_B$ is contained and is linearly dense in the ideal $B^*_e B_r$. Moreover, if $X$ is strongly full, then $\text{span} (X_r, X_s)_B = B^*_e B_s$ for all $r, s \in G$ because:

$$\text{span} (X_r, X_s)_B = \text{span} (X_r (X_r, X_r)_B X_s (X_s, X_s)_B)_B$$

$$\subseteq \text{span} (X_r, X_r)_B (X_r, X_s)_B (X_s, X_s)_B$$

$$\subseteq \text{span} B^*_e (B_r (X_r, X_s)_B) B^*_s B_s \subseteq \text{span} B^*_e B_s B^*_s B_s = B^*_e B_s$$

and

$$B^*_e B_s = B^*_e (B_r^{-1})^* B_{r^{-1}} B_r B_{r^{-1}} B_s = \text{span} B^*_e (X_r^{-1}, X_r^{-1})_B B_s (X_s, X_s)_B$$

$$= \text{span} (X_r^{-1} B_r, X_r^{-1})_B (X_s B_s^*, X_s)_B \subseteq \text{span} (X_r, X_r^{-1})_B (X_e, X_s)_B$$

$$\subseteq \text{span} (X_e (X_r^{-1}, X_s)_B X_s)_B \subseteq \text{span} (X_r, X_s)_B.$$ 

Left Hilbert bundles are similarly defined. We spell out the complete definition for convenience.

Definition 2.5. Let $\mathcal{A}$ be a Fell bundle over $G$. A left Hilbert $\mathcal{A}$-bundle is a Banach bundle $X$ over $G$ with continuous functions

$$\mathcal{A}(\cdot, \cdot) : X \times X \rightarrow \mathcal{A}, (x, y) \mapsto \mathcal{A}(x, y), \quad \mathcal{A} \times X \rightarrow X, (a, x) \mapsto a x$$

(2.4)

such that:

(1L) For all $r, s \in G$, $A_r X_s \subseteq X_{rs}$ and $\mathcal{A}(X_r, X_s) \subseteq A_{rs^{-1}}$.

(2L) For all $r, s \in G$ and $x \in X_r$ the function $A_r \times X_s \rightarrow X_{rs}$, $(a, x) \mapsto a x$, is bilinear and $X_s \rightarrow A_{rs^{-1}}$, $y \mapsto \mathcal{A}(y, x)$, is linear.

(3L) For all $x, y \in X$ and $a \in A$, $\mathcal{A}(x, y)^* = \mathcal{A}(y, x)$, $\mathcal{A}(a x, y) = a \mathcal{A}(x, y)$, $\mathcal{A}(x, ax) \geq 0$ (in $A_c$) and $\|x\|^2 = \|\mathcal{A}(x, x)\|$.

If

$$A_e = \text{span} \{ \mathcal{A}(X_r, X_r) : r \in G \},$$

(2.5)

$X$ is called full, and if

$$A_r A_r^* = \text{span} \mathcal{A}(X_r, X_r) \quad \text{for all } r \in G,$$

(2.6)

$X$ is called strongly full.
Definition 2.6. Let $\mathcal{A}$ and $\mathcal{B}$ be Fell bundles over $G$. A weak $\mathcal{A} - \mathcal{B}$-equivalence bundle is a Banach bundle $\mathcal{X}$ which is a full left Hilbert $\mathcal{A}$-bundle, a full right Hilbert $\mathcal{B}$-bundle and $\mathcal{A}(x,y)z = x(y,z)_{\mathcal{B}}$ for all $x, y, z \in \mathcal{X}$. In this case we say that $\mathcal{A}$ and $\mathcal{B}$ are weakly equivalent. If, in addition, $\mathcal{X}$ is strongly full, both as a left and right bundle, we say that $\mathcal{X}$ is a strong $\mathcal{A} - \mathcal{B}$-equivalence and that $\mathcal{A}$ and $\mathcal{B}$ are strongly equivalent.

We have included an appendix where we show several properties regarding tensor products of equivalence bundles. For example we show, in Theorem 1.1, that strong equivalence is an equivalence relation. Weak equivalence was shown to be an equivalence relation in [1].

Example 2.7. Every equivalence of partial actions (see [9, Section 4.2] and [13, Definition 15.7]) can be turned into a strong equivalence between the associated Fell bundles. Suppose $\alpha = \{I_t \rightarrow \alpha(t) I_t \}_{t \in G}$ and $\beta = \{J_t \rightarrow \beta(t) J_t \}_{t \in G}$ are partial actions on the $C^*$-algebras $\mathcal{A}$ and $\mathcal{B}$ respectively, and suppose $\mathcal{X}$ is an $\mathcal{A} - \mathcal{B}$-equivalence bimodule such that $I_t \mathcal{X} = \mathcal{X} J_t$ for all $t \in G$. For $t \in G$, define $X_t := I_t \mathcal{X} = \mathcal{X} J_t$ and suppose $\gamma = \{X_t \rightarrow \gamma(t) X_t \}_{t \in G}$ is a partial action of $G$ on $\mathcal{X}$ such that

$$\alpha_t((x,y)_{\mathcal{A}}) \gamma_t(z) = \gamma_t((x,y)_{\mathcal{A}} z) = \gamma_t(x(y,z)_{\mathcal{B}}) = \gamma_t(x) \beta_t((y,z)_{\mathcal{B}})$$

for all $t \in G$ and $x, y, z \in X_t$. Then $\alpha$ and $\beta$ are said to be Morita equivalent, and the following notations are used: $X^1 := \mathcal{A}$, $X^r := \mathcal{B}$, $\gamma^l = \alpha$, and $\gamma^r = \beta$ (in fact $X$ determines $\mathcal{A}$ and $\mathcal{B}$ up to isomorphism, and then $\gamma^l$ and $\gamma^r$ are determined by $\gamma$; see [3] for details). Let $L(\gamma)$ be the linking partial action of $\gamma$ (see the proof of Proposition 4.5 of [2]) and let $\mathcal{B}_{L(\gamma)}$ be the Fell bundle associated with $L(\gamma)$. Define $\mathcal{X}_{\gamma}$ as the Banach subbundle of $\mathcal{B}_{L(\gamma)}$

$$\mathcal{X}_{\gamma} := \left\{ \left( \begin{array}{cc} 0 & x \\ 0 & 0 \end{array} \right) : x \in X_t, \ t \in G \right\}.$$ 

With the structure inherited from the identity $\mathcal{B}_{L(\gamma)} - \mathcal{B}_{L(\gamma)}$-bundle structure of $\mathcal{B}_{L(\gamma)}$, $\mathcal{X}_{\gamma}$ is a strong $\mathcal{B}_x - \mathcal{B}_x$-equivalence bundle, where $\mathcal{B}_x$ and $\mathcal{B}_x$ denote the Fell bundles associated with $\alpha$ and $\beta$, respectively.

The notion of weak equivalence allows us to "identify" partial actions with the corresponding enveloping actions, in case these exist. This is explained in the following example. In particular this shows that a non-saturated Fell bundle may be weakly equivalent to a saturated one.

Example 2.8. Let $\beta$ be a global action of $G$ on the $C^*$-algebra $B$ and assume that $A$ is a $C^*$-ideal of $B$ such that $B = \overline{\mathbb{M}}_{\gamma} \{\beta_t(A) : t \in G\}$. This means that $\beta$ is the enveloping (global) action of the partial action $\alpha$ given as the restriction of $\beta$ to $A$ (see [3]). In this situation, $\mathcal{B}_x$ is weakly equivalent to $\mathcal{B}_x$. The equivalence is implemented by the bundle $\mathcal{X} = A \times G$, considered as a Banach subbundle of $\mathcal{B}_x$ and viewing $\mathcal{B}_x$ as a Fell subbundle of $\mathcal{B}_x$. The operations are the ones inherited from the identity $\mathcal{B}_x - \mathcal{B}_x$-bundle. Notice that $\mathcal{B}_x$ is, in general, not strongly equivalent to $\mathcal{B}_x$ because a strong equivalence between Fell bundles implies in a (strong) Morita equivalence between their unit fibers $A$ and $B$. And it is easy to produce examples where this is not the case. For instance, one may take a commutative $C^*$-algebra $B = C_0(X)$ and an ideal $A \subseteq B$ which is not isomorphic to $B$, like $B = C_0(\mathbb{R})$ and $A = C_0((0,1) \cup (1,2))$ with $G = \mathbb{R}$ acting by translation.
With notation as in Example 2.8, we have $B_0 \mathcal{X} \subseteq \mathcal{X}$, $\mathcal{X}B_\delta = \mathcal{X}$, $B_\delta = \mathcal{X}\mathcal{X}^*$ and $\mathcal{X}^* \mathcal{X} = B_\delta$ (where, for example, the equality $B_\delta = \mathcal{X}\mathcal{X}^*$ means that the $t$-fiber of $B_\delta$ is the closed linear span of all $\mathcal{X}_s\mathcal{X}_r^*$ with $sr^{-1} = t$). This motivates the following.

**Definition 2.9.** An enveloping bundle of a Fell bundle $\mathcal{A}$ is a saturated Fell bundle $\mathcal{B}$ for which there exists a Fell subbundle $\mathcal{C} \subseteq \mathcal{B}$ and an isomorphism of Fell bundles $\pi: \mathcal{A} \to \mathcal{C}$ such that for $\mathcal{X} := \mathcal{C}B$, we have $\mathcal{X}\mathcal{X}^* = \mathcal{C}$ and $\mathcal{X}^* \mathcal{X} = B$.

**Remark 2.10.** With notation as above, the bundle $\mathcal{X}$ above is a weak equivalence $\mathcal{A}-\mathcal{B}$-bundle with the operations $\mathcal{A}(x, y) = \pi^{-1}(xy^*)$, $(a, x) \mapsto \pi(a)x$, $(x, y)\mathcal{B} = x^*y$ and $(x, b) \mapsto xb$. Hence every Fell bundle is weakly equivalent to its enveloping bundle (if it admits one). The equivalence is, however, not strong in general (see Example 2.8).

Imitating the notion of Morita enveloping action from [3] we state the following.

**Definition 2.11.** A Morita enveloping bundle of a Fell bundle $\mathcal{A}$ is a saturated Fell bundle $\mathcal{B}$ which is the enveloping bundle of a Fell bundle strongly equivalent to $\mathcal{A}$.

It is shown in [3] that every partial action on a C*-algebra has a Morita enveloping action. In the next section we show that every Fell bundle admits a Morita enveloping Fell bundle. Moreover, we show that this Morita enveloping Fell bundle can be realised as a semidirect product bundle of a global action. This global action is unique up to Morita equivalence of actions on C*-algebras.

**Remark 2.12.** Since weak equivalence of Fell bundles is an equivalence relation, the Morita enveloping bundle of a Fell bundle is unique up to weak equivalence. In fact, we will show in Corollary 4.11 that it is unique up to strong equivalence.

**The bundle of generalized compact operators.** Given a full right Hilbert $\mathcal{B}$-bundle $\mathcal{X}$ there exists, up to isomorphism, a unique Fell bundle $\mathcal{K}(\mathcal{X})$ such that $\mathcal{X}$ is a weak $\mathcal{K}(\mathcal{X})-\mathcal{B}$ equivalence bundle. We recall next the main lines of the construction of $\mathcal{K}(\mathcal{X})$, and we refer to [4] for complete details.

To describe the fiber over $t \in G$ of the bundle $\mathcal{K}(\mathcal{X})$, note first that, given $x, y \in \mathcal{X}$, say $x \in \mathcal{X}_t$s and $y \in \mathcal{X}_s$ for some $s, t \in G$, we have a map $[x, y]: \mathcal{X} \to \mathcal{X}$ such that $[x, y]z := x(y, z)$ for all $z \in \mathcal{X}$. The map $[x, y]$ has the following properties:

1. $[x, y]X_r \subseteq X_{tr}$ for all $r \in G$.
2. $[x, y]$ is linear when restricted to each fiber $X_r$ of $\mathcal{X}$.
3. $[x, y]$ is continuous.
4. $[x, y]$ is bounded: its norm $\| [x, y] \| := \sup\{z \in X : \|z\| \leq 1\} \| [x, y]z \|$ is finite with $\| [x, y]z \| \leq \| x \| \| y \|$.
5. $[x, y]$ is adjointable: there exists a (necessarily unique) adjoint operator $[x, y]^* : \mathcal{X} \to \mathcal{X}$ such that $\langle [x, y]z, z' \rangle = \langle z, [x, y]^*z' \rangle$ for all $z, z' \in \mathcal{X}$. Moreover, we have $[x, y]^* = [y, x]$.

It is not hard to check that the vector space $\mathcal{B}_t(\mathcal{X})$ of maps $S : \mathcal{X} \to \mathcal{X}$ that satisfy properties (1)–(5) above is a Banach space, in fact a C*-ternary ring with the operation $(S_1, S_2, S_3) := S_1S_2S_3$. The elements of $\mathcal{B}_t(\mathcal{X})$ are called adjointable operators of order $t$. If $G_d$ is the group $G$ with the discrete topology, it follows that the family $(\mathcal{B}_t(\mathcal{X}))_{t \in G}$ is a Fell bundle over $G_d$, where the product is given by composition. Now define $\mathcal{K}_t(\mathcal{X}) := \text{span}\{[x, y] : x \in \mathcal{X}_t, y \in \mathcal{X}_s, s \in G\}$. It is...
easy to check that \( \mathbb{K}(\mathcal{X}) := (\mathbb{K}_t(\mathcal{X}))_{t \in G} \) is a Fell subbundle of \((\mathbb{B}_t(\mathcal{X}))_{t \in G}\). Finally, there is a suitable topology on \( \mathbb{K}(\mathcal{X}) \) making it a Fell bundle over \( G \), and \( \mathcal{X} \) is a weak \( \mathbb{K}(\mathcal{X}) \)-\( \mathcal{B} \) equivalence bundle with the obvious operations and inner products, see [3] for details.

The Fell bundle \( \mathbb{K}(\mathcal{X}) \) is unique in the following sense: if \( \mathcal{X} \) is a weak \( \mathcal{A} \)-\( \mathcal{B} \) equivalence, then there exists an isomorphism \( \pi : \mathcal{A} \to \mathbb{K}(\mathcal{X}) \) such that \( \pi(\mathcal{A}(x,y)) = [x,y] \) for all \( x,y \in \mathcal{X} \) (see [3, Corollary 3.10]).

The linking Fell bundle of an equivalence bundle. Given a weak \( \mathcal{A} \)-\( \mathcal{B} \) equivalence \( \mathcal{X} \), it is possible to define a Fell bundle \( \mathcal{L}(\mathcal{X}) = (L_t)_{t \in G} \) which plays a role similar to that of the linking algebra of an imprimitivity bimodules. The fiber \( L_t \) over \( t \in G \) is defined to be \( L_t := \left( \begin{array}{cc} A_t & X_t \\ X_{t^{-1}} & B_t \end{array} \right) \) with entrywise vector space operations (here, given an \( \mathcal{A} \)-\( \mathcal{B} \) Hilbert bimodule \( X \), \( \tilde{X} \) denotes its dual \( \mathcal{B} \)-\( \mathcal{A} \) Hilbert bimodule). The operations and topology on \( \mathcal{L}(\mathcal{X}) \) are defined as follows:

1. **Product and involution on \( \mathcal{L}(\mathcal{X}) \)** are given by
   \[
   \left( \begin{array}{cc} a & x \\ y & b \end{array} \right) \left( \begin{array}{cc} c & u \\ v & d \end{array} \right) = \left( \begin{array}{cc} ac + \mathcal{A}(x,v) & au + xd \\ \langle y,u \rangle_\mathcal{B} + bd & \langle y,u \rangle_\mathcal{B} \end{array} \right)
   \]
   and
   \[
   \left( \begin{array}{cc} a & x \\ y & b \end{array} \right)^* = \left( \begin{array}{cc} a^* & y \\ \tilde{x} & b^* \end{array} \right).
   \]

2. **Given** \( \xi \in C_c(\mathcal{A}) \), \( \eta \in C_c(\mathcal{B}) \) and \( f,g \in C_c(\mathcal{X}) \) the function
   \[
   \left( \begin{array}{cc} \xi & f \\ g & \eta \end{array} \right) : t \mapsto \left( \begin{array}{cc} \xi(t) & f(t) \\ g(t^{-1}) & \eta(t) \end{array} \right)
   \]
   is a continuous section (see [10, 13.18]).

The subbundle \( \mathcal{A} \oplus \mathcal{X} \) of \( \mathcal{L}(\mathcal{X}) \) is then a weak \( \mathcal{A} \)-\( \mathcal{B} \) equivalence bundle, and the subbundle \( \mathcal{X} \oplus \mathcal{B} \) is a weak \( \mathcal{L}(\mathcal{X}) \)-\( \mathcal{B} \) equivalence bundle. We refer the reader to the third section of [3] for details.

### 3. Canonical action on the kernels and Morita equivalence

Recall that \( L^2(\mathcal{B}) \) is the (full) right Hilbert \( B_e \)-module obtained as the completion of \( C_c(\mathcal{B}) \) with respect to the pre-Hilbert \( B_e \)-module structure given by the operations
\[
\langle f,g \rangle_{L^2} := \int_G f(t)^* g(t) \, dt, \quad (f \cdot b)(t) := f(t)b,
\]
for \( f, g \in C_c(\mathcal{B}) \) and \( b \in B_e \).

The Banach bundle \( L^2(\mathcal{B}) \) is, as a topological bundle, the constant fiber bundle
\[
L^2(\mathcal{B}) \times G \to G, \quad f \delta_t \mapsto t.
\]
The norm is given by \( \|f\delta_t\| := \Delta(r)^{-1/2} \|f\|_{L^2} \).

**Proposition 3.1.** Given \( r,s,t,p \in G \), \( f,g \in C_c(\mathcal{B}) \) and \( b \in B_t \), define
\[
\langle f \delta_r, g \delta_s \rangle_\mathcal{B} := \int_G f(pr)^* g(ps) \, dp \quad \text{(3.1)}
\]
\[
f \delta_r b := fb \delta_{rt}, \quad \text{with } fb(p) := f(pt^{-1})b. \quad \text{(3.2)}
\]

With these operations, \( L^2(\mathcal{B}) \) becomes a full right Hilbert \( \mathcal{B} \)-bundle.
Proof. To simplify the notation we define \( L_r := C_c(B) \times \{ r \} \subseteq L^2 B \). It is clear that the function \( L_r \times B_t \to L_{rt}, (f \delta_t, b) \mapsto f \delta_t b \), is bilinear and that \( L_r \times L_s \to B_{rs^{-1}}, g \delta_s \mapsto \langle f \delta_r, g \delta_s \rangle_B \), is linear. Straightforward computations show that \( \langle f \delta_r, g \delta_s \rangle_B = \langle f \delta_r, g \delta_s \rangle_B \). The Cauchy-Schwarz inequality in \( L^2 \parallel \cdot \parallel \) \( F \ell \) operators to do this take a representation \( f, g \in B \), \( \langle f \delta_r, g \delta_s \rangle_B = \langle g \delta_s, f \delta_r \rangle_B \) and \( \langle f \delta_r, f \delta_r \rangle_B = \Delta(r)^{-1} \langle f, f \rangle_{L^2} \). In particular, \( \langle f \delta_r, f \delta_r \rangle_B \geq 0 \).

The canonical pre-Hilbert \( B \)-module structure of \( L_r \) induces the norm of \( L^2 B \), because
\[
\| \langle f \delta_r, f \delta_r \rangle_B \| = \| \Delta(r)^{-1} \langle f, f \rangle_{L^2} \| = \Delta(r)^{-1} \| f \|_{L^2}^2.
\]

The action of \( B \) on \( C_c(B) \times G \) can be extended in a unique way to \( L^2 B \) because
\[
\| f \delta_r \| = \| f \delta_r \|_B = \| f \delta_r \|_B \leq \| f \|_{L^2} \| f \delta_r \|_B = \| f \|_{L^2} \| f \delta_r \|_B.
\]

To see that the inner product defined on \( C_c(B) \times G \) extends to \( L^2 B \) it suffices to prove that
\[
\| \langle f \delta_r, g \delta_s \rangle_B \| \leq \| f \delta_r \| \| g \delta_s \|.
\]
To do this take a representation \( T : B \to B(\mathcal{H}) \) with \( T_{|B^c} \) faithful. Then \( \| T_b \| = \| T_{b|B^c} \|^{1/2} = \| b \|^{1/2} = \| b \| \) for all \( b \in B \). Let \( T f(t) \in C_c(G, B(\mathcal{H})) \) be defined as \( T f(t) = T f(t \delta_r) \) and consider \( C_c(G, B(\mathcal{H})) \) as a subspace of \( L^2(G, B(\mathcal{H})) \). The Cauchy-Schwarz inequality in \( L^2(G, B(\mathcal{H})) \) implies
\[
\| \langle f \delta_r, g \delta_s \rangle_B \| = \| \int_G \langle T f(t) \delta_r, g(t) \delta_s \rangle_B dt \| = \| \langle T f(t) \delta_r, g(t) \delta_s \rangle_B \|
\]
\[
\leq \| \langle T f(t) \delta_r, g(t) \delta_s \rangle_B \|^{1/2} \| \langle T g(t) \delta_r, f(t) \delta_s \rangle_B \|^{1/2}
\]
\[
\leq \| T(f \delta_r, g \delta_s) \|^{1/2} \| T(g \delta_s, f \delta_r) \|^{1/2} = \| f \delta_r \| \| g \delta_s \|.
\]
This implies inequality (3.3).

With respect to the density of inner products note that
\[
\mathcal{S} \mathcal{P} \mathcal{A} \mathcal{N}(L_r, L_r)_B = \mathcal{S} \mathcal{P} \mathcal{A} \mathcal{N} \Delta(r)^{-1} \langle C_c(B), C_c(B) \rangle_B = B_c,
\]
for all \( r \in G \).

The constant section associated to \( f \in L^2(B) \) is \( f \delta : G \to L^2 B, t \mapsto f \delta_t \). Since for all \( r \in G \), \( \{ \delta_r : f \in C_c(B) \} = L_r \), to show that the inner product and action are continuous it suffices to prove that the functions
\[
G \times G \to B, (r, s) \mapsto \int_G f(t r)^* g(t s) dt, \quad \text{and} \quad G \times G \to L^2(B), (r, s) \mapsto f | g[r(t)] |
\]
are continuous for all \( f, g \in C_c(B) \). The continuity of the first function follows adapting [10] II 15.19. The other function has range in \( C_c(B) \) and is continuous in the inductive limit topology, so it is continuous as a function with codomain \( L^2(B) \).

Definition 3.2. The canonical \( L^2 \)-bundle of the Fell bundle \( B \) is the Hilbert \( B \)-bundle \( L^2 B \) described in the last Proposition.

We are interested in the identification of the Fell bundle of generalized compact operators \( \mathbb{K}(L^2 B) \) of \( L^2 B \) (see end of Section 2), up to isomorphism of Fell bundles, because this Fell bundle is weakly equivalent to \( B \). We will show that \( \mathbb{K}(L^2 B) \) is a semidirect product Fell bundle associated to an action of \( G \) on a \( C^* \)-algebra.

Following [3], we write \( k_c(B) \) for the space of compactly supported continuous functions \( k : G \times G \to B \) with \( k(r, s) \in B_{rs^{-1}} \) for all \( r, s \in G \). In other words, \( k_c(B) \)
is the space of compactly supported continuous sections of the pullback of $\mathcal{B}$ along the map $G \times G \to G, (r, s) \mapsto rs^{-1}$. It is a normed $^\ast$-algebra with

$$h \ast k(r, s) = \int_G h(r, t)k(t, s) \, dt \quad k^\ast(r, s) = k(s, r)^\ast$$

$$\|k\|_2 := \left( \int_{G^2} \|k(r, s)\|^2 \, dr \, ds \right)^{1/2}.$$  

We may also endow $\mathcal{k}_c(\mathcal{B})$ with the inductive limit topology and in this way it becomes a topological $^\ast$-algebra.

Completing $\mathcal{k}_c(\mathcal{B})$ with respect to $\| \cdot \|_2$ we obtain the Banach $^\ast$-algebra $\mathcal{HS}(\mathcal{B})$ of Hilbert-Schmidt operators of $\mathcal{B}$. The $C^\ast$-algebra of kernels of $\mathcal{B}$ is the enveloping $C^\ast$-algebra of $\mathcal{HS}(\mathcal{B})$; it is denoted by $\mathcal{k}(\mathcal{B})$. There is a canonical action of $G$ on $\mathcal{k}(\mathcal{B})$ given by the formula $\beta_t(k)(r, s) = \Delta(t)k(tr, st)$ for $k \in \mathcal{k}_c(\mathcal{B})$ and $r, s, t \in G$.

The $C^\ast$-algebra $\mathcal{K}(L^2(\mathcal{B}))$ of (generalised) compact operators of the Hilbert $B_r$-module $L^2(\mathcal{B})$ can be canonically identified with an ideal in $\mathcal{k}(\mathcal{B})$: for $f, g \in \mathcal{C}_c(\mathcal{B})$, the usual operator $\theta_{f, g} \in \mathcal{K}(L^2(\mathcal{B}))$ given by $\theta_{f, g}(h) = f(g[h])$ is identified with the element $\varphi(f, g) \in \mathcal{k}_c(\mathcal{B})$ defined by $\varphi(f, g)(r, s) = f(r)g(s)^\ast$. These elements span an ideal $I_c(\mathcal{B}) := \text{span}\{\varphi(f, g) : f, g \in \mathcal{C}_c(\mathcal{B})\}$ in $\mathcal{k}_c(\mathcal{B})$. Its closure $I(\mathcal{B})$ is therefore a $C^\ast$-ideal of $\mathcal{k}(\mathcal{B})$. The $\beta$-orbit of $I_c(\mathcal{B})$ is dense in $\mathcal{k}_c(\mathcal{B})$ in the inductive limit topology. Moreover, $I_c(\mathcal{B})$ is dense in $\mathcal{k}_c(\mathcal{B})$ in the inductive limit topology if and only if $\mathcal{B}$ is saturated.

Remark 3.3. There is a canonical coaction $\delta_G$ of $G$ on $C^\ast(\mathcal{B})$, the so-called dual coaction, and it is shown in [3] that $\mathcal{k}(\mathcal{B})$ is canonically isomorphic to the crossed product $C^\ast(\mathcal{B}) \rtimes_{\delta_G} G$ by this coaction. Moreover, this isomorphism carries the canonical action of $G$ on $\mathcal{k}(\mathcal{B})$ to the dual action of $G$ on $C^\ast(\mathcal{B}) \rtimes_{\delta_G} G$. Thus $\mathcal{k}(\mathcal{B}) \cong C^\ast(\mathcal{B}) \rtimes_{\delta_G} G$ as $G$-$C^\ast$-algebras. The dual coaction on $C^\ast(\mathcal{B})$ is maximal and its normalisation is the dual coaction $\delta^\ast_G$ on $C^\ast_r(\mathcal{B})$ (see [7]). This means that there exists a natural isomorphism

$$\mathcal{k}(\mathcal{B}) \rtimes_{\beta} G \cong C^\ast_r(\mathcal{B}) \rtimes_{\delta_G} G \rtimes_{\delta_G} G \cong C^\ast_r(\mathcal{B}) \otimes \mathcal{K}(L^2(G))$$  \hspace{1cm} (3.4)

which factors through an isomorphism

$$\mathcal{k}(\mathcal{B}) \rtimes_{\beta, r} G \cong C^\ast_r(\mathcal{B}) \rtimes_{\delta_G} G \rtimes_{\delta_G} G \cong C^\ast_r(\mathcal{B}) \otimes \mathcal{K}(L^2(G)).$$

Before we state our next result we introduce some notation. We shall denote by $\mathcal{B}_\beta = \mathcal{k}(\mathcal{B}) \rtimes_{\beta} G$ the semidirect product Fell bundle associated to $\beta$ (as defined in [11], page 708) for ordinary actions or, more generally, for twisted partial actions in [12]).

Recall that $\mathcal{K}(L^2(\mathcal{B}))$ can be identified with an ideal of $\mathcal{k}(\mathcal{B})$, so we have a (possibly non-faithful) representation of $\mathcal{k}(\mathcal{B})$ as adjointable operators of $L^2(\mathcal{B})$. We use the notation $Tf$ to represent the action of $T \in \mathcal{k}(\mathcal{B})$ on $f \in L^2(\mathcal{B})$. For every $k \in \mathcal{k}_c(\mathcal{B})$ and $f \in \mathcal{C}_c(\mathcal{B})$ we have $kf \in \mathcal{C}_c(\mathcal{B})$ and $kf(r) = \int_G k(r, s)f(s) \, ds$.

Theorem 3.4. Let $\mathcal{B}$ be a Fell bundle and denote by $L^2(\mathcal{B})$ its canonical $L^2$-bundle, which is a full right Hilbert $\mathcal{B}$-bundle. Then $L^2(\mathcal{B})$ is a full left Hilbert $\mathcal{B}_\beta$-bundle with the action and inner product given by

$$T\delta_r f\delta_r = \Delta(t)^{1/2} \beta_t^{-1}(T)f\delta_r \quad \mathcal{B}_\beta(\langle f\delta_r, g\delta_s \rangle) = \Delta(rs)^{-1/2} \beta_r(\langle f, g \rangle)\delta_{rs^{-1}},$$
where $T\delta_i \in B_\beta$ and $f\delta_r, g\delta_s \in L^2B$. Moreover, the left and right Hilbert bundles structures of $L^2B$ are compatible and therefore $L^2B$ is a weak equivalence $B_\beta - B$-bundle.

**Proof.** The left action is clearly bilinear and the left inner product is linear in the first variable because the inner product $\langle \ , \ \rangle$ is linear in the first variable. Moreover,

$$B_\beta\langle f\delta_r, g\delta_s \rangle^* = \Delta(r)^{-1/2}\Delta(s)^{-1/2}\beta_{s^{-1}}(\beta_r(\kappa(f,g)^*))\delta_{s^{-1}r} = B_\beta(g\delta_s, f\delta_r).$$

To show that the left operations are compatible, we compute

$$B_\beta\langle T\delta_t f\delta_r, g\delta_s \rangle = \Delta(rs)^{-1/2}\beta_{tr}(\kappa(\beta_{tr}^{-1}(T)f,g))\delta_{tr^{-1}}.$$  

and

$$T\delta_t B_\beta\langle f\delta_r, g\delta_s \rangle = \Delta(rs)^{-1/2}T\beta_{tr}(\kappa(f,g))\delta_{tr^{-1}}.$$  

Then the compatibility of the left operations will follow once we show that

$$\beta_{tr}(\kappa(\beta_{tr}^{-1}(T)f,g)) = T\beta_{tr}(\kappa(f,g)),$$  

for all $T \in k(B)$ and $f, g \in L^2(B)$. But using linearity and continuity it suffices to consider $T \in k_c(B)$ and $f, g \in C_c(B)$. Then we can make all the computations in $k_c(B)$. With this assumption, the left hand side of (3.5) evaluated at $(x, y) \in G^2$ is

$$\beta_{tr}(\kappa(\beta_{tr}^{-1}(T)h,g))(x,y) = \Delta(tr)[\beta_{tr}^{-1}(T)f](xtr)g(ytr)^* = \Delta(tr)\int_G \beta_{tr}^{-1}(T)(xtr,z)f(z)\,dz\,g(ytr)^* = \int_G T(x, zr^{-1}t^{-1})f(z)\,dz\,g(ytr)^*.$$  

The right hand side of (3.5) evaluated at $(x, y) \in G^2$ is

$$T\beta_{tr}(\kappa(f,g))(x,y) = \int_G T(x, z)\Delta(tr)f(ztr)g(ytr)^*\,dz = \Delta(tr)\int_G T(x, ztr^{-1}t^{-1})f(ztr)\,dz\,g(ytr)^* = \int_G T(x, zr^{-1}t^{-1})f(z)\,dz\,g(ytr)^*.$$  

Thus we have shown that (3.5) holds and this implies that the left operations are compatible.

Now note that $B_\beta\langle f\delta_r, f\delta_r \rangle = \Delta(r)^{-1}\beta_r(\kappa(f,f))\delta_e \geq 0$ and

$$\|B_\beta\langle f\delta_r, f\delta_r \rangle\|^2 = \Delta(r)^{-1}\|\beta_r(\kappa(f,f))\|^2 = \Delta(r)^{-1}\|f\|^2_{L^2} = \|\langle f\delta_r, f\delta_r \rangle\|,$$

so the norms given by the left and right inner products agree.

The continuity of the left operations follows directly from their definition and from the fact that the topologies of $B_\beta = k(B) \times G$ and $L^2B = L^2(B) \times G$ are the product topologies.

Since the linear $\beta$-orbit of $k(L^2(B))$ is dense in $k(B)$, it follows from the definition of the left inner product that

$$k(B)\delta_e = \overline{\operatorname{span}} \{B_\beta\langle f\delta_r, g\delta_r \rangle : f, g \in L^2(B), \ r \in G\}.$$
At this point we know that $L^2\mathcal{B}$ is a full left Hilbert $B_r$-bundle. The proof will be completed once we show that the left and right operations are compatible, that is,

$$g_s\langle f \delta_r, g \delta_s \rangle h \delta_t = f \delta_r(g \delta_s, h \delta_t) g_s.$$ 

It suffices to consider $f, g, h \in C_c(\mathcal{B})$ and by computing the left and right sides we obtain the following equivalent equation (without the place marker $\delta_{rs-1}$):

$$\Delta(s)^{-1} \beta_{t-1}(z(f, g)) h = f \int_G g(zs)^* h(zt) \, dz.$$ 

The left hand side evaluated at $x \in G$ is

$$\Delta(t)^{-1} \int_G f(xt^{-1}s) g(zt^{-1}s)^* h(z) \, dz = f(xt^{-1}s) \int_G g(zs)^* h(zt) \, dz,$$

which is exactly $f \int_G g(zs)^* h(zt) \, dz$ evaluated at $x$. \qed

The following result shows that every Fell bundle is strongly equivalent to the semidirect product Fell bundle of a partial action.

**Theorem 3.5.** Let $\mathcal{B}$ be a Fell bundle and denote by $L^2\mathcal{B} = \{ L_t \}_{t \in G}$ its canonical $L^2$-bundle. If $X = \{ L_tB_t^*B_t \}_{t \in G}$ and $\alpha$ is the restriction to $\mathcal{K} := \mathcal{K}(L^2(\mathcal{B}))$ of the canonical action on the $C^*$-algebra of kernels of $\mathcal{B}$, then $X$ is a Banach subbundle of $L^2\mathcal{B}$ and it is a strong equivalence $B_{x, \alpha}$-bundle with the structure inherited from $L^2\mathcal{B}$.

**Proof.** Since $\{ B_t^*B_t \}_{t \in G}$ is a continuous family of ideals of $B_{x, \alpha}$, $X$ is a Banach subbundle of $L^2\mathcal{B}$.

To simplify our notation we define $\mathcal{K}_t := \beta_t(\mathcal{K}) \cap \mathcal{K} = \beta_t(\mathcal{K}) \cdot \mathcal{K}$. Recall that the fiber over $t$ of $\mathcal{B}_s$ is $\mathcal{K}_t \delta_t$.

To continue we identify the ideal of $\mathcal{K}$ corresponding to $\mathcal{B}_t^*\mathcal{B}_t$ through $L^2(\mathcal{B})$; we claim this ideal is $\mathcal{K}_{t-1}$. Given $f, g \in C_c(\mathcal{B})$ and $a, b, c, d \in \mathcal{B}_t$ we define $u, v \in C_c(\mathcal{B})$ by $u(r) := f(rt)a^*bd^*$ and $v(r) := g(rt)c^*$. Then $
abla(fa^*b, gc^*) = \beta_{t-1}(z(u, v)) \in \mathcal{K}_{t-1}$. This implies $L^2(\mathcal{B})B_t^*B_t \subseteq \mathcal{K}_{t-1}L^2(\mathcal{B})$.

To prove $\mathcal{K}_{t-1}L^2(\mathcal{B}) \subseteq L^2(\mathcal{B})B_t^*B_t$ it suffices to show

$$\langle \mathcal{K}_{t-1}L^2(\mathcal{B}), L^2(\mathcal{B}) \rangle \subseteq B_t^*B_t.$$

Note that $\mathcal{K}_{t-1}L^2(\mathcal{B}) = \beta_{t-1}(\mathcal{K})L^2(\mathcal{B}) = \beta_{t-1}(\mathcal{K})L^2(\mathcal{B})$. If $k \in \mathcal{K}(\mathcal{B})$ and $f, g \in C_c(\mathcal{B})$ then

$$\langle \beta_{t-1}(k) f, g \rangle_{\mathcal{B}_s} = \int_G \int f(s)k(t^{-1}, st^{-1})^* g(r) \, ds \, dr.$$ 

Now, if $k$ represents an element of the ideal $\mathcal{K}$, then $k(s, t) \in \mathcal{B}_s \mathcal{B}_t^*$ for all $s, t \in G$. Since $f^*(s)k(rt^{-1}, st^{-1})^* g(r) \in B_s \mathcal{B}_{s-1} \mathcal{B}_{t-1} \mathcal{B}_t \subseteq B_{t-1} \mathcal{B}_t = B_t^*B_t$ for all $r, s \in G$, we conclude that $\mathcal{K}_{t-1}L^2(\mathcal{B}) = L^2(\mathcal{B})B_t^*B_t$.

The operations of $X$ are the ones inherited from $L^2\mathcal{B}$. To show they are well defined and satisfy (1R-3R) and (1L-3L), it suffices to show

$$B_s X \subseteq X, \ \mathcal{B}_s \subseteq B \ \text{and} \ \mathcal{B}_s \langle X, X \rangle \subseteq B_{x, \alpha}.$$

(3.6)

To prove the first inclusion take $r, s \in G$, $T \in \mathcal{K}_r$ and $f \in L^2(\mathcal{B})B_r^*B_s$. By Cohen’s factorisation theorem, we may decompose $f$ as $f = T' f'$ with $T' \in \mathcal{K}$ and $f' \in L^2(\mathcal{B})$. Since $L_{rs}B_r^*B_r = \beta_{rs}^{-1}(\mathcal{K})\mathcal{K}L^2(\mathcal{B})\delta_{rs}$ we have

$$T\delta_r f \delta_s = \Delta(r)^{1/2} \beta_{rs}^{-1}(T) f \delta_s = \Delta(r)^{1/2} \beta_{rs}^{-1}(T)T' f' \delta_{rs} \in L_{rs}B_r^*B_r.$$
This shows that $B_\alpha X \subseteq X$.

The second inclusion in (4.3) follows from $L_\alpha B_\alpha^* B_\alpha B_r \subseteq L_{sr} B_{sr}^* B_{sr}$, which we now show. Since $B_\alpha^* B_\alpha$ and $B_r B_r^*$ are ideals of $B_\alpha$ and $B_r = B_r^* B_r$, we have

$$L_\alpha B_\alpha^* B_\alpha B_r = L_\alpha B_\alpha^* B_\alpha B_r B_r B_r^* B_\alpha B_r \subseteq L_{sr} (B_s B_r)^* (B_s B_r) \subseteq L_{sr} B_{sr}^* B_{sr}.$$

The inclusion $B_\beta \langle X, X \rangle \subseteq B_\alpha$ follows from the fact that

$$B_\beta \langle L_r B_r^* B_r, L_s B_s^* B_s \rangle = \beta_r (K_{r-1} L^2 (B) K_{s-1} L^2 (B)) \delta_{rs-1} \subseteq K_r \cap K_{rs-1} \delta_{rs-1} \subseteq K_{rs-1} \delta_{rs-1}. $$

To finish we need to show $X$ is strongly full on both sides. The strong fullness on the right follows from the computation

$$\ker \langle L_t B_t^* B_t, L_t B_t^* B_t \rangle_B = \ker \langle L_t^2 (B) B_t^* B_t \delta_t, L^2 (B) B_t^* B_t \delta_t \rangle_B = \ker \{ B_t^* B_t (L^2 (B) B_t^* B_t) B_t^* B_t \} = B_t^* B_t.$$

And the strong fullness on the left is implied by

$$\ker B_\beta \langle L_t B_t^* B_t, L_t B_t^* B_t \rangle_B = \ker B_\beta \langle L_t^2 (B) B_t^* B_t \delta_t, L^2 (B) B_t^* B_t \delta_t \rangle_B = \ker \{ \beta_t (T f, T f) \delta_{t_0}: T \in K_{t-1}, f \in L^2 (B) \} = K_t \delta_{t_0} = K_t (K_t \delta_{t_0})^*.$$

As an immediate consequence of the last result and Example 2.8, we get that every Fell bundle has a Morita enveloping Fell bundle which is the semidirect product bundle of an action on a $C^*$-algebra. We will see later (Corollary 4.11) that the Morita enveloping Fell bundle is unique up to strong equivalence.

### 4. Morita equivalence of actions and Fell bundles

From the previous sections we know that the canonical action on the $C^*$-algebra of kernels of a Fell bundle determines the Morita equivalence class of that bundle. But what can we say about the canonical actions on the kernels of two Morita equivalent Fell bundles? Of course these actions, when viewed as Fell bundles are weakly equivalent. Our goal is to show that they are (Morita) equivalent as actions (hence strongly equivalent as Fell bundles).

Assume $A$ and $B$ are Fell bundles over $G$ and let $\alpha$ and $\beta$ stand for the canonical actions on the $C^*$-algebras of kernels of $A$ and $B$, respectively. If $\alpha$ is Morita equivalent to $\beta$ (as actions on $C^*$-algebras) then $B_\alpha$ and $B_\beta$ are Morita equivalent as Fell bundles (via a strong equivalence: Example 2.7) and, by transitivity, $A$ is strongly (hence weakly) equivalent to $B$. Before proving the converse we prove the following.

**Lemma 4.1.** If $A$ is a Fell subbundle of $B$ then the natural inclusion $\iota: k_c(A) \to k_c(B)$ extends to an injective (hence isometric) $\ast$-homomorphism $\pi: k(A) \to k(B)$.

**Proof.** First note that $\iota$ has a unique extension to a $\ast$-homomorphism $\mathcal{H}S(A) \to \mathcal{H}S(B)$, which induces the $\ast$-homomorphism $\pi: k(A) \to k(B)$ that we want to show is injective. Since $A$ is included in $B$, $L^2(A)$ is contained in $L^2(B)$ as a
Since and denote by symmetry, we also have over, \( \pi \). 

Proof. We may think of the enveloping action \([3, \text{Theorem } 2.1]\) implies the unique \( \alpha = (\gamma|_{k(X)})^\gamma \) and \( \beta = (\gamma|_{k(X)})^\gamma \) (recall the notation from Example \([2,7]\)).

**Theorem 4.2.** Let \( A \) and \( B \) be Fell bundles over \( G \) and let \( \alpha \) and \( \beta \) be the canonical actions on the \( C^* \)-algebras of kernels of \( A \) and \( B \), respectively. Then \( A \) is weakly equivalent to \( B \) if and only if \( \alpha \) is Morita equivalent to \( \beta \).

**Proof.** The converse follows by the comments we made at the beginning of the present section. For the direct implication assume that \( \mathcal{X} \) is an \( A \rightarrow B \) equivalence bundle. We may think of \( A \) and \( B \) as full hereditary Fell subbundles of their linking Fell bundle \( L(\mathcal{X}) \), so \( k(A) \) and \( k(B) \) are \( C^* \)-subalgebras of \( k(L(\mathcal{X})) \) by Lemma \([4,1]\).

Note that \( k_c(A)k_c(L(\mathcal{X}))k_c(A) \subseteq k_c(A) \) and \( k_c(B)k_c(L(\mathcal{X}))k_c(B) \subseteq k_c(B) \), so by \( C^* \)-algebras of kernels of \( B \) and \( C \). We may think of \( V \) and \( U \) with support contained in \( U \) and such that \( \int_G \phi^2_u(p) \, dp = 1 \). Define \( k^U_f \in k_c(\mathcal{X}) \) as \( k^U_f(p,q) = \phi(t^{-1})f(pq^{-1}) \). Then

\[
\lim_U (k^U_f)*k^U_g(r,s) = \lim_U \int_G \phi^U(t^{-1})^2(f(pr^{-1}), g(ps^{-1})) \, dp = \langle f(t^{-1}), g(t^{-1}) \rangle \in B.
\]

We conclude that the closure \( C \) of \( \{u(r,s) : u \in \Gamma\} \) contains \( X_{tr^{-1}} \subseteq X_{ts^{-1}} \) for all \( t \in G \). By Remark \([2,3]\) this implies \( C = B_{tr^{-1}} \).

Now we know that \( k(X) \) is a \( k(A) - k(B) \)-equivalence bimodule. To finish the proof note that if \( \gamma \) is the canonical action of \( G \) on \( k(L(\mathcal{X})) \) then \( k(X) \) is \( \gamma \)-invariant, \( \alpha = (\gamma|_{k(X)})^\gamma \) and \( \beta = (\gamma|_{k(X)})^\gamma \) (recall the notation from Example \([2,7]\)).
Corollary 4.3. (cf. [4, Proposition 4.13]). If \( \mathcal{A} \) and \( \mathcal{B} \) are weakly equivalent Fell bundles over \( G \), then their full and reduced cross-sectional C*\(^{\star}\)-algebras are (strongly) Morita equivalent. This equivalence respects the dual coactions. Conversely, if the dual coactions on the (full or reduced) cross-sectional C*\(^{\star}\)-algebras of \( \mathcal{A} \) and \( \mathcal{B} \) are equivalent (as coactions), then \( \mathcal{A} \) and \( \mathcal{B} \) are weakly equivalent as Fell bundles.

Proof. By Theorem 4.2, \( \mathcal{A} \) and \( \mathcal{B} \) are weakly equivalent if and only if their C*\(^{\star}\)-algebras of kernels \( k(\mathcal{A}) \) and \( k(\mathcal{B}) \) are \( G \)-equivariantly Morita equivalent. And by Remark 3.3 we have canonical isomorphisms of \( G \)-C*\(^{\star}\)-algebras \( k(\mathcal{A}) \cong C^{\star}(\mathcal{A}) \rtimes_{\delta} G \cong C^{\star}_{r}(\mathcal{A}) \rtimes_{\delta} G \) and \( k(\mathcal{B}) \cong C^{\star}(\mathcal{B}) \rtimes_{\delta} G \cong C^{\star}_{r}(\mathcal{B}) \rtimes_{\delta} G \), where \( \delta_{\mathcal{A}}^{(r)} \) and \( \delta_{\mathcal{B}}^{(r)} \) denote the dual coactions on \( C^{\star}_{r}(\mathcal{A}) \) and \( C^{\star}_{r}(\mathcal{B}) \), respectively. The coaction \( \delta_{\mathcal{A}} \) is maximal, and the coaction \( \delta_{\mathcal{B}}^{(r)} \) is normal (the normalisation of \( \delta_{\mathcal{A}} \)), see [7]. This means that \( \delta_{\mathcal{A}} \) (resp. \( \delta_{\mathcal{B}}^{(r)} \)) is Morita equivalent to the dual coaction on \( k(\mathcal{A}) \rtimes_{\alpha} G \) (resp. \( k(\mathcal{A}) \rtimes_{\alpha,r} G \)), and a similar assertion holds for \( \mathcal{B} \) in place of \( \mathcal{A} \). Combining all this and the standard result that equivalent actions or coactions have equivalent (full or reduced) crossed products, the desired result now follows. \( \square \)

Remark 4.4. The above result extends to the exotic cross-sectional C*\(^{\star}\)-algebras \( C^{\star}_{\mu}(\mathcal{B}) \) associated to Morita compatible cross-product functors \( \times_{\mu} \) as defined in [6]. This is because, by definition, \( C^{\star}_{\mu}(\mathcal{B}) \) is the quotient of \( C^{\star}(\mathcal{B}) \) that turns the isomorphism 3.4 into an isomorphism

\[
k(\mathcal{B}) \rtimes_{\beta,\mu} G \cong C^{\star}_{\mu}(\mathcal{B}) \otimes \mathbb{K}(L^{2}(G)).
\]

And this isomorphism preserves the dual coactions whenever the exotic cross-product admits such a coaction; this means that \( \times_{\mu} \) is a duality cross-product functor in the language of [8] (see also [4]). This is a big class of functors and, in particular, includes all correspondence functors (see [7, Corollary 4.6]).

Recall that a Fell bundle \( \mathcal{A} \) is amenable if the regular representation \( \lambda_{\mathcal{A}} : C^{\star}(\mathcal{A}) \rightarrow C^{\star}_{r}(\mathcal{A}) \) is faithful.

Corollary 4.5. Let \( \mathcal{A} \) and \( \mathcal{B} \) be two weakly equivalent Fell bundles. Then \( \mathcal{A} \) is amenable if and only if \( \mathcal{B} \) is amenable.

Proof. As already explained in Remark 3.3, the canonical isomorphism \( C^{\star}(\mathcal{A}) \otimes \mathbb{K}(L^{2}(G)) \cong k(\mathcal{A}) \rtimes_{\alpha} G \) factors (via the regular representations) through an isomorphism \( C^{\star}_{r}(\mathcal{A}) \otimes \mathbb{K}(L^{2}(G)) \cong k(\mathcal{A}) \rtimes_{\alpha,r} G \). This means that \( \mathcal{A} \) is amenable if and only if the \( G \)-action \( \alpha \) on \( k(\mathcal{A}) \) is amenable in the sense that the regular representation \( k(\mathcal{A}) \rtimes_{\alpha} G \rightarrow k(\mathcal{A}) \rtimes_{\alpha,r} G \) is faithful. Since \( \mathcal{A} \) is weakly equivalent to \( \mathcal{B} \) if and only if the \( G \)-actions on \( k(\mathcal{A}) \) and \( k(\mathcal{B}) \) are equivalent by Theorem 4.2, the assertion follows from the well-known result that amenability of actions is preserved by equivalence of actions. \( \square \)

We will have more to say about amenability in Section 8.

Corollary 4.6. Let \( \alpha \) and \( \beta \) be partial actions of \( G \) on C*\(^{\star}\)-algebras. Then \( \mathcal{B}_{\alpha} \) is weakly equivalent to \( \mathcal{B}_{\beta} \) if and only if \( \alpha \) and \( \beta \) have equivalent Morita enveloping actions.

Proof. Recall from [3] that the canonical action on the C*\(^{\star}\)-algebra of kernels of \( \mathcal{B}_{\alpha} \) is a Morita enveloping action for \( \alpha \) and that Morita enveloping actions are unique up to Morita equivalence of actions. Then the proof follows directly using transitivity of Morita equivalence of Fell bundles and the last theorem. \( \square \)
Our next theorem will show that strong equivalence of Fell bundles corresponds to Morita equivalence of partial actions in the ordinary sense (recall Example 2.7 and the notation used there). First we need the following auxiliary result.

**Lemma 4.7.** Suppose $X$ is an $A - B$-equivalence bimodule, $\gamma$ is an action of $G$ on $X$ and that $I$ and $J$ are $C^*$-ideals of $A$ and $B$ (respectively) such that $IX = JX$. Then $\gamma^{|I} = (\gamma|_{IX})^I$ and $\gamma^{|J} = (\gamma|_{XJ})^r$. In particular $\gamma^{|I}$ is Morita equivalent to $\gamma^{|J}$ (as partial actions).

**Proof.** To simplify the notation, we denote $\alpha := \gamma^I$ and $\beta = \gamma^r$. Since for all $t \in G$ we have $\alpha_t(I)X = \gamma_t(IX)$, the ideal of $X$ corresponding to $I_t = I \cap \alpha_t(I)$ is $IX \cap \gamma_t(IX)$. By symmetry we obtain $JX = XI_t$. Since $I_{t^{-1}} = \text{span}_A(X \cap \gamma_t(IX)), X \cap \gamma_t(IX))$ and for $x, y \in X \cap \gamma_t(IX)$ we have

$$\alpha_t(x, y) = \alpha_t(x) = \alpha_t(y) = (\gamma|_{IX})^I(\alpha_t(x, y)).$$

We conclude that $\alpha^{|I} = (\gamma|_{IX})^I$. The rest follows by symmetry. \[\square\]

**Theorem 4.8.** Let $A$ and $B$ be Fell bundles over $G$ and denote by $\alpha$ and $\beta$ the restrictions of the canonical action on the $C^*$-algebras of kernels, $k(A)$ and $k(B)$, to $k_A := k(L^2(A))$ and $k_B := k(L^2(B))$, respectively. Then $A$ is strongly equivalent to $B$ if and only if $\alpha$ is Morita equivalent to $\beta$.

**Proof.** If $\alpha$ is equivalent to $\beta$ in the usual sense (as defined in 3), then their associated Fell bundles $B_{\alpha}$ and $B_{\beta}$ are strongly equivalent, as shown in Example 2.7. Since $A$ (resp. $B$) is strongly equivalent to $B_{\alpha}$ (resp. $B_{\beta}$) by Theorem 4.3, the strong equivalence between $A$ and $B$ follows by transitivity (Theorem A.1).

For the converse assume that $X$ is a strong equivalence $A - B$-bundle. Let $k(X)$ be the $k(A) - k(B)$-equivalence bimodule constructed in the proof of Theorem 4.7, which was constructed inside $k(L(X))$.

By the previous Lemma and the proof of Theorem 4.2 it suffices to show that $k_A k(X) = k(X) k_B$. Let $C$ be the pullback of $B$ along the map $\rho: G^2 \to G$, $(r, s) \mapsto rs^{-1}$. We think of $k_{\rho}(B)$ as $C_c(C)$.

Note that the bundle $D^B := \{B(r, B, s^{-1})_{(r, s)} \in G^2 \}$ is a Banach subbundle of $C$ (recall Notation 2.24). Moreover, using 3 Lemma 5.1 [as in the proof of Theorem 4.2] one shows that $\text{span} \{k(f, g) : f, g \in C_c(B)\}$ is dense in $C_c(D^B)$ for the inductive limit topology. Thus $k_B$ is the closure of $C_c(D^B)$ in $k(B)$.

In a similar way define $D^{X_B} := \{X(r, B, s^{-1})_{(r, s)} \in G^2 \}$, which is a Banach subbundle of the pullback of $L(X)$ along $\rho$. Note that $k_c(X) \text{span}_A(C_c(B), C_c(B)) \subset C_c(D^X)$ and that, by arguments similar to those in the proof of Theorem 4.2, $k_c(X) \text{span}_A(C_c(B), C_c(B))$ is dense in $C_c(D^{X_B})$ for the inductive limit topology. Thus $k_A k(X)$ is the closure of $C_c(D^{X_B})$ in $k(L(X))$.

By symmetry we obtain that $k(A) k(B)$ is the closure in $k(L(X))$ of the space of compactly supported continuous sections of the bundle $D^{X_A} := \{A, X^{-1} \}_{(r, s)} \in G^2 \}$.

Thus all we need to show is that $D^{X_A} = D^{X_B}$. It suffices to prove that $A_r X_s = X_r B_s$ for all $(r, s) \in G^2$. If we think of $B_s$ as a right full $B^* B$-Hilbert module and we use that $B^* B_s = \text{span}_A \langle X_r, X_r \rangle_B$, then we conclude that

$$X_r B_s = \text{span}_A \langle X_r, B_s \rangle_B (X_r, X_s) = \text{span}_A \langle X_r, X_s \rangle_B X_r B_s \subseteq X_r B_s.$$ 

Conversely,

$$A_r X_s = \text{span}_A \langle X_r, X_r \rangle A_r X_s = \text{span}_A \langle X_r, A_r X_s \rangle \subseteq X_r B_s. \quad \square$$
The next result shows that our notion of strong equivalence of Fell bundles recovers exactly the notion of equivalence of partial actions (introduced in [3]).

**Corollary 4.9.** Let $\alpha$ and $\beta$ be partial actions of $G$ on $C^*$-algebras. Then $\alpha$ is Morita equivalent to $\beta$ if and only if $\mathcal{B}_\alpha$ is strongly equivalent to $\mathcal{B}_\beta$.

**Proof.** This follows from Theorem 4.8 together with the fact (proved in [3]) that the restricted partial action to $\mathbb{K}(L^2(\mathcal{B}_\alpha))$ of the canonical global action on $\mathfrak{k}(\mathcal{B}_\alpha)$ is Morita equivalent to $\alpha$. $\square$

**Corollary 4.10.** Two saturated Fell bundles are weakly equivalent if and only if strongly equivalent.

**Proof.** If $\mathcal{A}$ is a saturated Fell bundle, then its $C^*$-algebra of kernels $\mathfrak{k}(\mathcal{A})$ coincides with $\mathbb{K}(L^2(\mathcal{A}))$ and the canonical partial action is already global. The result now follows as a direct combination of Theorems 4.2 and 4.8 $\square$

Now, combining Corollary 4.10, Example 2.8, Remark 2.12, and Theorem 3.5, we get:

**Corollary 4.11.** Every Fell bundle has a Morita enveloping Fell bundle which is the semidirect product bundle of an action on a $C^*$-algebra. This action is unique up to (strong) Morita equivalence of actions on $C^*$-algebras. Any two Morita enveloping Fell bundles of a Fell bundle are strongly equivalent.

5. Partial actions associated with Fell bundles.

By the *spectrum* of a $C^*$-algebra $A$ we mean the space $\hat{A}$ of unitary equivalence classes $[\pi]$ of irreducible representations $\pi$ of $A$ with the Jacobson topology induced from the primitive ideal space $\text{Prim}(A) = \{\ker(\pi) : [\pi] \in \hat{A}\}$. Open subsets of $\hat{A}$ or $\text{Prim}(A)$ correspond bijectively to ideals of $A$: the open subset of $\text{Prim}(A)$ (resp. $\hat{A}$) associated with an ideal $I \subseteq A$ is $\{p \in \text{Prim}(A) : I \not\subseteq p\}$ (resp. $\{[\pi] : [\pi]_I \neq 0\}$).

As shown in [1], every Fell bundle $\mathcal{B} = (\mathcal{B}_t)_{t \in G}$ over a discrete group $G$ induces a partial action $\hat{\alpha}^\mathcal{B}$ of $G$ on the spectrum $\hat{\mathcal{B}}_e$ of $\mathcal{B}_e$. We briefly recall how $\hat{\alpha}^\mathcal{B}$ is defined. For each $t \in G$ we let $D_t^\mathcal{B} := B_t B_t^*$; then $D_t^\mathcal{B}$ is an ideal of $\mathcal{B}_e$ and $B_t$ can be viewed as a $D_t^\mathcal{B} - D_{t^{-1}}^\mathcal{B}$ imprimitivity bimodule. Let $\nu_t^\mathcal{B} := \{[\pi] \in \hat{\mathcal{B}}_e : [\pi]_{D_t^\mathcal{B}} \neq 0\}$ be the open subset of $\hat{\mathcal{B}}_e$ associated with $D_t^\mathcal{B}$. If $[\pi] \in \nu_t^\mathcal{B}$, then $[\pi]_{|_{D_t^\mathcal{B}}} \in \hat{\mathcal{B}}_{t^{-1}}^\mathcal{B}$, and therefore $[\text{Ind}_{\mathcal{B}_t}([\pi]_{|_{D_t^\mathcal{B}}})] \in \hat{\mathcal{B}}_{t^{-1}}^\mathcal{B}$. Then $\hat{\alpha}_t([\pi]) \in \nu_t^\mathcal{B}$ is defined to be the class of the unique extension of $\text{Ind}_{\mathcal{B}_t}([\pi]_{|_{D_t^\mathcal{B}}})$ to all of $\mathcal{B}_e$ (see the second statement of [5.2]). In fact one can give a more direct definition: $\hat{\alpha}_t([\pi]) = [\text{Ind}_{\mathcal{B}_e} \pi]$ (this is a consequence of our Lemma [5.2]3). Here and throughout, if $E$ is a Hilbert $C' - C$-bimodule for $C^*$-algebras $C', C$, we write $\text{Ind}_E(\pi)$ for the representation of $C'$ induced via $E$ from a representation $\pi: C \to \mathfrak{B}(H)$. Recall that $\text{Ind}_E(\pi)$ acts on the (balanced tensor product) Hilbert space $E \otimes_x H$ by the formula $\text{Ind}_E \pi(x)(y \otimes_x h) := x \cdot y \otimes_x h$ for all $x \in C'$, $y \in E$ and $h \in H$.

Only discrete groups are considered in [1]. But in this section we prove that the partial action $\hat{\alpha}^\mathcal{B}$ is always continuous if $G$ is a locally compact group. We also show that strongly equivalent Fell bundles have isomorphic partial actions, and that the action of a saturated Fell bundle is the enveloping action of the partial action of any Fell bundle weakly equivalent with the former.
We begin with some preliminary results about induced representations via Hilbert bimodules; most of them are certainly well known, but we include the proofs here for convenience. Let $A$ be a $C^*$-subalgebra of the $C^*$-algebra $C$, and suppose $\pi : C \to B(H)$ is a nondegenerate representation of $C$. We denote by $\pi_A$ the nondegenerate part of the restriction $\pi|_A$, that is, $\pi_A : A \to B(H_A)$ is given by $\pi_A(a)h := \pi(a)h$ for all $a \in A$ and $h \in H_A$, where $H_A := \text{span}\{\pi(a)h : a \in A, h \in H\}$, the essential space of $\pi|_A$.

**Lemma 5.1.** Let $E$ be a Hilbert $C^*-C$-bimodule, $A'$ and $A$ $C^*$-subalgebras of $C'$ and $C$ respectively, and $F \subseteq E$ such that $F$ is a Hilbert $A'-A$-bimodule with the structure inherited from $E$. Suppose $\pi : C \to B(H)$ is a representation and $K \subseteq H$ is a closed subspace which is invariant under $\pi_A$. Then:

1. There exists a unique isometry $V : F \otimes_{\pi_A} K \to E \otimes_{\pi} H$ that satisfies $V(x \otimes_{\pi_A} k) = x \otimes_{\pi} k$ for all $x \in F$, $k \in K$.
2. The isometry $V$ intertwines $(\text{Ind}_E \pi)|_{A'}$ and $\text{Ind}_F \pi_A$, that is,

$$\text{Ind}_E \pi(a') V = V \text{Ind}_F \pi_A (a') \quad \text{for all } a' \in A'.$$

**Proof.** For a finite sum of elementary tensors $\sum_i x_i \otimes_{\pi_A} k_i \in F \otimes_{\pi_A} K$ we compute:

$$\left\| \sum_i x_i \otimes_{\pi_A} k_i \right\|^2 = \sum_{i,j} \langle x_i \otimes_{\pi_A} k_i, x_j \otimes_{\pi_A} k_j \rangle_K = \sum_{i,j} \langle k_i, \pi_A((x_i, x_j)_C) k_j \rangle_K$$

$$= \sum_{i,j} \langle k_i, \pi((x_i, x_j)_C) k_j \rangle_H = \left\| \sum_i x_i \otimes_{\pi} k_i \right\|^2$$

Then there exists an isometry $V : F \otimes_{\pi_A} K \to E \otimes_{\pi} H$ such that $V(x \otimes_{\pi_A} k) = x \otimes_{\pi} k$ for all $x \in F$, $k \in K$, as claimed in (1). Now, if $a' \in A'$, $x \in F$ and $k \in K$:

$$\text{Ind}_E \pi(a') V(x \otimes_{\pi_A} k) = a' x \otimes_{\pi} k = V(a' x \otimes_{\pi_A} k) = V \text{Ind}_F \pi_A (a') (x \otimes_{\pi_A} k),$$

which proves (2). \qed

**Lemma 5.2.** Let $\pi : C \to B(H)$ be a nondegenerate representation of a $C^*$-algebra $C$.

1. Let $Y$ be a closed right ideal of $C$ and $A := YY^*$ (recall Notation 3.1), which is a hereditary $C^*$-subalgebra of $C$. Consider $Y$ as a Hilbert $A-C$-bimodule. Then $\text{Ind}_Y \pi$ is equivalent to $\pi_A$.
2. Let $I$ be a closed two-sided ideal of $C$ and let $F_I$ be $I$ with its natural structure of Hilbert $C-I$-bimodule. If $\rho : I \to B(K)$ is a nondegenerate representation, let $\tilde{\rho} : C \to B(K)$ be the unique extension of $\rho$ to a representation of $C$ on $K$, which is determined by $\tilde{\rho}(c)(\rho(x)k) = \rho(cx)k$ for all $c \in C$, $x \in I$ and $k \in K$. Then $\text{Ind}_{F_I} \rho$ is equivalent to $\tilde{\rho}$.
3. Suppose $\pi$ is irreducible. If $C'$ is a $C^*$-algebra and $E$ is a Hilbert $C'-C$-bimodule such that $\pi$ does not vanish on the ideal $I := \text{span}_{C'}(E, E)_C$, then $\text{Ind}_E \pi$ is irreducible and equivalent to $\text{Ind}_{F_I} (\text{Ind}_E \pi_I)$, where $I := \text{span}_{C'}(E, E)$.

**Proof.** Let $K' := \text{span}_{C'}(\pi(y)h : y \in Y, h \in H)$. Note that $K'$ agrees with the essential space $H_A$ of $\pi|_A$. In fact we have $H_A \subseteq K'$ because $A \subseteq Y$, and the reverse inclusion follows from the fact that $y = \lim_{\lambda} e_{\lambda} y$ and hence $\pi(y) = \lim_{\lambda} \pi(e_{\lambda}) \pi(y)$
for every \( y \in Y \) and every approximate unit \((e_\lambda)\) of \( A \). Now, if \( \sum_i y_i \otimes h_i \) is a finite sum of elementary tensors in \( Y \otimes \pi H \), then

\[
\left\| \sum_i y_i \otimes h_i \right\|^2 = \sum_{i,j} \langle y_i \otimes h_i, y_j \otimes h_j \rangle = \sum_{i,j} \langle h_i, \pi(y_i^* y_j) h_j \rangle_H = \left\| \sum_i \pi(y_i) h_i \right\|^2.
\]

Thus we have an isometry \( U : Y \otimes \pi H \to H_A \) such that \( y \otimes h \mapsto \pi(y)h \). This isometry is surjective by the previous observation, hence \( U \) is a unitary operator. Finally, if \( a \in A \), \( y \otimes h \in Y \otimes \pi H \):

\[
U\text{Ind}_Y \pi(a)(y \otimes h) = U(ay \otimes h) = \pi(ay)h = \pi(a)\pi(y)h = \pi(a)(y \otimes h),
\]

which proves our first statement.

To prove (2) we observe that exactly the same argument used in the proof of (1) shows that there is a unitary operator \( U : F_I \otimes \rho K \to K \) that intertwines \( \text{Ind}_{F_I} \rho \) with \( \hat{\rho} \).

As for (3), since \( I \) and \( J \) are the ideals generated by the left and right inner products of the bimodule \( E \), we may view \( E \) as an \( I - J \)-imprimitivity bimodule. By (1), \( \pi_J \) is irreducible. And the essential space of \( \pi_J \) is \( H \) because \( \pi(J)H \) is a non-zero \( \pi \)-invariant subspace of \( H \), and \( \pi \) is irreducible. Since \( E \) is an \( I - J \)-imprimitivity bimodule, \( \text{Ind}_E \pi_J : I \to \mathcal{B}(E \otimes \pi_J H) \) also is irreducible. Now let \( V : E \otimes \pi_J H \to E \otimes \pi H \) be the isometry provided by (1) of Proposition 5.1 which in this case is obviously surjective, thus a unitary operator. Since, according to Lemma 5.3, \( V \) intertwines \( \text{Ind}_E \pi_I \) and \( \text{Ind}_E \pi_J \), and the latter is irreducible, then so is \( \text{Ind}_E \pi_J \). Therefore \( \text{Ind}_E \pi \) is irreducible.

Moreover, if \( \rho := \text{Ind}_{E} \pi_J \), it is an easy task to show that \( V \pi(c') = V \text{Ind}_E \pi(c') \) for all \( c' \in C' \). This ends the proof, for \( \text{Ind}_{F_I} (\text{Ind}_E \pi_J) \) and \( \hat{\rho} \) are equivalent by (2).

Lemma 5.3. Suppose, in the conditions of Lemma 5.1, that \( \pi \) is irreducible, \( A' \) is a hereditary \( C^* \)-subalgebra of \( C' \), and \( \pi|_{\langle E, E \rangle_{C'}} \neq 0 \). Then the isometry \( V \) is a unitary operator, and \( \left| (\text{Ind}_E \pi)_{A'} \right| = \left| \text{Ind}_E \pi_{A'} \right| \).

Proof. Since \( \pi \) is irreducible, then so is \( \text{Ind}_E \pi \) by Lemma 5.2 (3). Moreover, if \( A' \) is a hereditary \( C^* \)-subalgebra of \( C' \), then \( (\text{Ind}_E \pi)_{A'} \) is either zero or an irreducible representation of \( A' \) (Proposition 5.5.2). But \( (\text{Ind}_E \pi)_{A'} \) cannot be zero because of Lemma 5.1 (2) and the fact that \( \pi_A \), and therefore \( \text{Ind}_E \pi_A \), are non-zero representations. Now it follows from Lemma 5.1 (2) that \( V(F \otimes \pi_A K) \) is a non-zero \( (\text{Ind}_E \pi)_{A'} \)-invariant subspace of \( E \otimes \pi H \) and, since \( (\text{Ind}_E \pi)_{A'} \) is irreducible, we must have \( V(F \otimes \pi_A K) = E \otimes \pi H \). That is, \( V \) is a surjective isometry, which ends the proof.

We show next that two strongly equivalent Fell bundles give rise to isomorphic partial actions on spectra level.

Theorem 5.4. Let \( A \) and \( B \) be Fell bundles over a discrete group \( G \), and suppose that \( \mathcal{X} \) is a strong \( A - B \) equivalence. Let \( h_{X_e} : B_e \to \hat{A}_e \) be the Rieffel homeomorphism associated to the \( A_e - B_e \) imprimitivity bimodule \( X_e \). Then \( h_{X_e} : \hat{\alpha}^B \to \hat{\alpha}^A \) is an isomorphism of partial actions.

Proof. Let \( \mathcal{C} = (C_t)_{t \in G} \) stand for the linking bundle of \( \mathcal{X} \). Then \( C_e = \mathbb{L}(X_e) \) (the linking algebra of \( X_e \)), and \( Y = \left( \begin{array}{cc} A_e & X_e \\ 0 & 0 \end{array} \right) \) is an \( A_e - C_e \) imprimitivity bimodule, so it defines the Rieffel homeomorphism \( h_Y : \hat{C}_e \to \hat{A}_e \), which in our case is given by
We claim that $h_{Y}(\gamma) = [\pi_{A_{e}}]$. Therefore the Rieffel correspondence $R: \mathcal{I}(C_{e}) \rightarrow \mathcal{I}(A_{e})$ between the ideals of $C_{e}$ and $A_{e}$, defined by $Y \cong A_{e} \oplus X_{e}$, is given by $R(I) = A_{e} \cap I$. 

Given the previous Corollary we obtain the following result.

The previous result allows us to associate a partial action to every Fell bundle, not only to those over discrete groups:

**Proposition 5.5.** Let $\mathcal{B}$ be a Fell bundle over $G$, and let $G_{d}$ and $\mathcal{B}_{d}$ as above. Then the partial action $\hat{\alpha}^{\mathcal{B}_{d}}$ of $G_{d}$ on $\hat{\mathcal{B}}_{e}$ is a continuous partial action of $G$ on $\hat{\mathcal{B}}_{e}$.

**Proof.** Consider the canonical action $\beta$ of $G$ on $\mathbb{k}(\mathcal{B})$, and let $\gamma$ be its restriction to $\mathbb{k}(L^{2}(\mathcal{B}))$. Let $\mathcal{A} := \mathcal{B}_{d}$. Since, by Theorem 5.3, $\mathcal{A}$ is strongly equivalent to $\mathcal{B}$, then also $\mathcal{A}_{d}$ is strongly equivalent to $\mathcal{B}_{d}$. Note that if we forget the topology of $G$, then $\gamma = \hat{\alpha}^{\mathcal{A}_{d}}$, so the latter is a continuous partial action of $G$ on the spectrum of $\mathbb{k}(L^{2}(\mathcal{B}))$. On the other hand $\hat{\alpha}^{\mathcal{A}_{d}}$ and $\hat{\alpha}^{\mathcal{B}_{d}}$ are isomorphic partial actions by Theorem 5.4, and, since $\hat{\alpha}^{\mathcal{A}_{d}}$ is continuous, so must be $\hat{\alpha}^{\mathcal{B}_{d}}$.

The previous result allows us to associate a partial action to every Fell bundle, not only to those over discrete groups:

**Definition 5.6.** Let $\mathcal{B}$ be a Fell bundle over $G$, and denote by $\hat{\alpha}_{\mathcal{B}}$ the partial action $\hat{\alpha}^{\mathcal{B}_{d}}$ considered as a partial action of $G$ on $\hat{\mathcal{B}}_{e}$. We say that $\hat{\alpha}^{\mathcal{B}}$ is the partial action associated to $\mathcal{B}$.

Now Theorem 5.4 can be stated for Fell bundles over arbitrary groups:

**Corollary 5.7.** Suppose $\mathcal{X}$ is a strong $\mathcal{A} - \mathcal{B}$-equivalence bundle. If $h: \hat{\mathcal{B}}_{e} \rightarrow \hat{\mathcal{A}}_{e}$ is the Rieffel homeomorphism induced by the $A_{e} - B_{e}$-equivalence bimodule $X_{e}$, then $h$ is an isomorphism between $\hat{\alpha}^{\mathcal{A}}$ and $\hat{\alpha}^{\mathcal{B}}$.

**Corollary 5.8.** Let $\mathcal{B}$ be a Fell bundle, and $\beta: G \times \mathbb{k}(\mathcal{B}) \rightarrow \mathbb{k}(\mathcal{B})$ the canonical action. Then $\hat{\beta}$ is the enveloping action of $\hat{\alpha}^{\mathcal{B}}$.

In previous sections we have decomposed a weak equivalence between Fell bundles as strong equivalence followed by globalization of partial actions and Morita equivalence of enveloping actions 5.3, 5.5 and 1.22. Combining this decomposition with the previous Corollary we obtain the following result.
Corollary 5.9. If $A$ and $B$ are weakly equivalent Fell bundles, then $\hat{\alpha}^A$ and $\hat{\alpha}^B$ have the same enveolving action.

Proof. Let $\mu$ and $\nu$ be the canonical actions on $k(A)$ and $k(B)$, respectively. The Fell bundle associated to $\mu|_{K(L^2(A))}$ is strongly Morita equivalent to $A$. Hence the partial action on the spectrum of $K(L^2(A))$ induced by $\mu$ is isomorphic to $\alpha$ and its enveolving action is the one induced by $\mu$ on $k(A)$, $\hat{\mu}$. For the same reasons $\hat{\nu}$ is an enveolving action of $\beta$. We also know $\mu$ and $\nu$ are Morita equivalent, so $\hat{\mu}$ is isomorphic to $\hat{\nu}$ and this implies $\hat{\mu}$ is an enveolving action of $\beta$. \hfill \Box

Proposition 5.10. A Fell bundle $B$ is saturated if and only if its associated partial action $\hat{\alpha}^B$ is global.

Proof. If $B$ is saturated then $D_t^B = B_tB_{t^{-1}} = B_e$ for all $t \in G$. Thus the open set of $\hat{B}_e$ corresponding to $D_t^B$, $U_t$, is $\hat{B}_e$ itself for all $t \in G$. In other words, $\hat{\alpha}^B$ is global.

Conversely, in case $\hat{\alpha}^B$ is global we have $U_t = \hat{B}_e$ for all $t \in G$. Since the correspondence between $C^*$-ideals of $B_e$ and open sets of $\hat{B}_e$ is bijective, we conclude that $B_tB_{t^{-1}} = D_t^B = B_e$ for all $t \in G$. Then for every $r, s \in G$, considering $B_{rs}$ as a left $B_e$-module, we deduce that $B_{rs} = B_eB_{rs} = B_tB_{t^{-1}}B_{rs} \subseteq B_tB_s \subseteq B_{rs}$, so $B$ is saturated. \hfill \Box

The last proposition implies that saturation is an invariant of strong equivalence:

Corollary 5.11. Let $\mathcal{X}$ be an $A-B$ strong equivalence bundle. Then $A$ is saturated if and only if $B$ is saturated.

Proof. If two partial actions are isomorphic, then one of them is global if and only if so is the other. Therefore our claim follows from 5.10. \hfill \Box

5.1. Partial actions on primitive ideal spaces. Consider a Fell bundle $B = (B_t)_{t \in G}$, and let $\beta$ be the canonical action of $G$ on $k(B)$. Let $A$ be the Fell bundle associated to the partial action $\alpha := \beta|_{K(L^2(B))}$. In particular $A_e = K(L^2(B))$, and $\hat{\alpha} = \hat{\alpha}^A$. By Theorem 3.5 we know that $L^2(B)$ is a strong $A-B$ equivalence. In particular we have the Rieffel homeomorphisms $h : \hat{B}_e \rightarrow A_e$ and $\hat{h} : \text{Prim}(B_e) \rightarrow \text{Prim}(A_e)$. Given a $C^*$-algebra $A$, let $\kappa : \hat{A} \rightarrow \text{Prim}(A)$ be the map given by $\kappa([\pi]) = \ker \pi$. Then the Rieffel homeomorphisms satisfy $\hat{h}\kappa = \kappa h$. According to [3], $\alpha$ induces a partial action $\tilde{\alpha}$ of $G$ on $\text{Prim}(A_e)$, which is determined by $\tilde{\alpha}(\kappa([\pi])) = \kappa(\tilde{\alpha}_t([\pi]))$. Here $\tilde{\alpha}_t : \mathcal{O}_{t^{-1}} \rightarrow \mathcal{O}_t$, where $\mathcal{O}_t := \{P \in \text{Prim}(A_e) : P \not\supseteq D_t^A\}$. This partial action $\tilde{\alpha}$ is continuous, because it is a restriction of the global action $\beta$ induced by $\beta$ on $\text{Prim}(k(B))$. Now, conjugating $\tilde{\alpha}$ by $\hat{h}$, we obtain a continuous partial action $\hat{\alpha}^B$ of $G$ on $\text{Prim}(B_e)$, which satisfies $\kappa\hat{\alpha}^B([\pi]) = \hat{\alpha}_t^B(\kappa([\pi]))$ for all $[\pi] \in \mathcal{V}_{t-1}$. Thus we have:

Theorem 5.12. Every Fell bundle $B$ over the locally compact Hausdorff group $G$ induces a continuous partial action $\hat{\alpha}^B = (\{\tilde{\alpha}_t\}_{t \in G}, \{\tilde{\alpha}_t^B\}_{t \in G})$ of $G$ on $\text{Prim}(B_e)$, which is given by $\tilde{\alpha}_t(P) = B_tPB_t^*$ for all $P \in \mathcal{O}_{t^{-1}}$; hence the following diagram is commutative for all $t \in G$:

\begin{equation*}
\begin{array}{ccc}
\mathcal{V}_{t-1} & \xrightarrow{\hat{\alpha}_t^B} & \mathcal{V}_t \\
\kappa \downarrow & & \kappa \downarrow \\
\mathcal{O}_{t^{-1}} & \xrightarrow{\tilde{\alpha}_t} & \mathcal{O}_t
\end{array}
\end{equation*}
Moreover,

(1) If $X = (X,t)$ is a strong $A - B$ equivalence bundle, the Rieffel homeomorphism $\tilde{h}_X : \text{Prim}(B_e) \to \text{Prim}(A_e)$ is an isomorphism between $\hat{\alpha}^B_t$ and $\hat{\sigma}^A_t$.

(2) If $\beta$ is the canonical action of $G$ on $k(B)$, then $\tilde{\beta} : G \times \text{Prim}(k(B)) \to \text{Prim}(k(B))$ is the enveloping action of $\hat{\alpha}^B_t$.

Proof. We only need to prove (2), since the remaining statements follow at once from the definition of $\hat{\alpha}^B_t$. Now assertion (2) is a direct consequence of (1) and [3, Proposition 7.4].

The moral of the preceding section is that a Fell bundle is essentially the same object that a semidirect product Fell bundle for a partial action, in the sense that it is always strongly equivalent to such a product. With this in mind, one should be able to translate results from semidirect product Fell bundles to arbitrary Fell bundles.

As an example, we have the following generalization of [3, Corollary 7.2]:

Corollary 5.13. Let $B$ be a Fell bundle over the locally compact Hausdorff group $G$. If $\text{Prim}(B_e)$ is compact, then there exists an open subgroup $H$ of $G$ such that the reduction of $B$ to $H$ is a saturated Fell bundle. In particular, if $G$ is a connected group, then $B$ is a saturated Fell bundle.

Proof. Since $\text{Prim}(B_e)$ is compact, [3, Proposition 1.1] shows there exists an open subgroup $H$ of $G$ for which the restriction of $\hat{\alpha}^B_t$ to $H$ is a global action, that is $D^B_t = B_e$ for all $t \in H$. Thus the reduction of $B$ to $H$ is a saturated Fell bundle. Since the only open subgroup of a connected group is the group itself, the proof is finished. □

6. $C_0(X)$-Fell bundles and amenability

Given a locally compact Hausdorff space $X$, a $C^*$-algebra $C$ is a $C_0(X)$-algebra if there exists a nondegenerate $^*$-homomorphism $\phi : C_0(X) \to ZM(C)$. In this situation there exists a unique continuous function $f_\phi : \tilde{C} \to X$ such that

$$\pi(a) = a(f_\phi([\pi]))1_\pi$$

for all $a \in C_0(X)$ and $[\pi] \in \tilde{C}$, where $\pi$ is the natural extension of the irreducible representation $\pi : C \to \mathbb{B}(\mathcal{H})$ to $M(C)$ and $1_\pi$ is the identity operator of $\mathcal{H}$.

Assume now that $\theta$ is an action of $G$ on $C_0(X)$, $\beta$ an action of $G$ on $C$ and that $\phi$ is equivariant in the sense that, for all $t \in G$, $a \in C_0(X)$ and $c \in C : \beta_t(\phi(a)c) = \phi(\theta_t(a))\beta_t(c)$. In this situation $f_\phi$ is $\tilde{\beta} - \tilde{\theta}$-equivariant and the Fell bundle $B_\beta$ is a $\theta$-Fell bundle in the following sense.

Definition 6.1. Let $\sigma$ be an action of the locally compact Hausdorff group $G$ on the locally compact Hausdorff space $X$. A $\sigma$-Fell bundle is a Fell bundle over $G$, $B$, for which there exists a continuous function $f : B_e \to X$ which is a morphism of partial actions between $\hat{\sigma}$ and $\sigma$.

The example that motivated this definition has a converse. Suppose $\beta$ is an action of $G$ on the $C^*$-algebra $B$ and that $B_\beta$ is a $\sigma$-Fell bundle. Then the unit fiber of $B$ is $B$ and the action defined by $B_\beta$ on $\tilde{B}$ is the action defined by $\beta, \tilde{\beta}$. By hypothesis there exists a $\tilde{\beta} - \sigma$-equivariant continuous function $f : \tilde{B} \to X$. Since
the points of $X$ are closed, there exists (by [22 Lemma C.6]) a unique continuous function $g: \text{Prim}(B) \to X$ such that $g \circ \kappa = f$, where $\kappa: \hat{B} \to \text{Prim}(B)$ is given by $\kappa(\pi) = \ker(\pi)$, as in the preceding section. The condition $g \circ \kappa = f$ ensures that $g: \text{Prim}(B) \to X$ is equivariant, considering on $\text{Prim}(B)$ the action induced by $\beta$. Using Dauns-Hofmann Theorem we conclude that there exists a unique nondegenerate and equivariant $^*$-homomorphism $\phi: C_0(X) \to ZM(B)$, where the action considered on $C_0(X)$ is the one defined by $\sigma$.

**Theorem 6.2.** Let $B$ be a Fell bundle over $G$ and $\sigma$ an action of $G$ on the locally compact Hausdorff space $X$. If $\beta$ is the canonical action of $G$ on $k(B)$ and $\theta$ is the action on $C_0(X)$ defined by $\sigma$, then the following are equivalent:

1. $B$ is a $\sigma$-Fell bundle.
2. There exists a nondegenerate $^*$-homomorphism $\phi: C_0(X) \to ZM(k(B))$ such that for all $t \in G$, $a \in C_0(X)$ and $k \in k(B)$, $\beta_t(\phi(a)k) = \phi(\theta_t(a))\beta(k)$.

**Proof.** By the comments preceding the statement, the implication (1) $\Rightarrow$ (2) will follow after we show that $B_{\beta}$ is a $\sigma$-Fell bundle. Assume that $f: \hat{B}_c \to X$ is an equivariant continuous function. If we denote by $\alpha$ the restriction of $\beta$ to $A := \mathbb{K}(L^2(B))$ and $h: \hat{B}_c \to \hat{A}$ is the Rieffel homeomorphism given by the equivalence bimodule $L^2(B)$, then $f \circ h^{-1}: \hat{A} \to X$ is equivariant. By [3 Proposition 7.4.] $\hat{\beta}$ is the enveloping action of $\hat{\alpha}$ and by [3 Theorem 1.1.] there exists a unique $\hat{\beta} - \sigma$-equivariant continuous extension of $f \circ h^{-1}$.

The next result is an extension of [3 Theorem 5.3.] to Fell bundles.

**Theorem 6.3.** Let $\sigma$ be an action of $G$ on the locally compact Hausdorff space $X$. Consider the conditions:

1. $\sigma$ is amenable.
2. Every $\sigma$-Fell bundle is amenable, that is, $C^*(B) = C^*_r(B)$.
3. For every $\sigma$-Fell bundle $B$ with $B_e$ nuclear, $C^*_r(B)$ is nuclear.
4. $C_0(X) \times_r G$ is nuclear.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) and if $G$ is discrete (4) $\Rightarrow$ (1).

**Proof.** Name $\beta$ the canonical action on $k(B)$. Since $B$ is equivalent to $B_{\beta}$, $B$ is amenable if and only if $B_{\beta}$ is amenable. Moreover, since $C^*_r(B)$ and $C^*_r(B_{\beta})$ are Morita equivalent, one is nuclear if and only if the other one is.

Assume (1) holds. By [5 Theorem 5.3.] and Theorem [6.2] $B_{\beta}$ is amenable and so $B$ is amenable. Now assume that (2) holds, then (2) from [5 Theorem 5.3.] holds and it suffices to show that $k(B)$ is nuclear. We know $\mathbb{K}(L^2(B))$ is nuclear because $B_e$ is nuclear. Then [3 Proposition 2.2.] implies that $k(B)$ is nuclear.

The rest of the proof follows directly from [5] Example (3) of 4.4. together with Theorem 5.8.

**Appendix A. Tensor products of equivalence bundles**

Throughout this section we use the construction of adjoint and tensor product of equivalence bundles of [3].

**Theorem A.1.** If $X$ and $Y$ are $A - B$ and $B - C$-strong equivalence bundles, respectively, then the tensor product bundle $Z := X \otimes_B Y$ is (left and right) strongly full. In particular, strong equivalence of Fell bundles is an equivalence relation.
Although the tensor products (\(X\)) actions by equivalences. The composition of arrows is given by inner tensor product. With our convention at the beginning of Section 2 where we view \(F\) ell bundles as objects. To proceed analogously with \(A\)-equivalence bundles we need a notion of unitary operator between equivalence bundles.

Fix \(t \in G\), \(c \in C_t^* C_t\) and \(\varepsilon > 0\). Since \(Y\) is strongly full there exists \(y_{j,k} \in Y_t\) \((j = 1, 2\) and \(k = 1, \ldots, n\)) such that \(\|c - \sum_{k=1}^{n} \langle y_{1,k}, y_{2,k} \rangle c\| < \varepsilon\). We also know that \(X\) is strongly full, then \(X_x\) is a \(A_e\) \(-\) equivalence bimodule and we can find \(x_{j,k} \in X_x\) \((j = 1, 2\) and \(k = 1, \ldots, m\)) such that

\[
\delta := \left\| c - \sum_{k=1}^{n} \sum_{l=1}^{m} \langle y_{1,k}, (x_{1,l}, x_{2,l}) y_{2,k} \rangle c \right\| < \varepsilon.
\]

From the first paragraph of this proof it follows that we may find \(\xi_{p,k,l} \in Z_t\) \((p = 1, 2, k = 1, \ldots, n,\) and \(l = 1, \ldots, m\)) such that \(\|\langle y_{1,k}, (x_{1,l}, x_{2,l}) y_{2,k} \rangle c - \langle \xi_{1,k,l}, \xi_{2,k,l} \rangle c\| < \frac{\varepsilon}{nm}\). Thus

\[
\left\| c - \sum_{k=1}^{n} \sum_{l=1}^{m} \langle \xi_{k,l,1}, \xi_{k,l,2} \rangle c \right\| < \varepsilon.
\]

By symmetry, \(Z\) is also left strongly full. Hence strong equivalence is a transitive relation. Regarding the symmetric and reflexive properties of strong equivalence, we leave to the reader the verification of the fact that the adjoint of \(X\), \(X^*\) is a full \(B - A\)-equivalence bundle and that \(B\), considered as a \(B - B\)-equivalence bundle in the natural way, are left and right strongly full.

An equivalence module \(X_B\) between the \(C^*\)-algebras \(A\) and \(B\) is usually viewed as an arrow from \(A\) to \(B\); here we view it as an arrow from \(B\) to \(A\) to be consistent with our convention at the beginning of Section 2 where we view Fell bundles as actions by equivalences. The composition of arrows is given by inner tensor product. Although the tensor products \((X \otimes Y) \otimes Z\) and \(X \otimes B Y \otimes C Z\) are not the same object, but they are naturally isomorphic (via a unitary); this gives the associativity of composition. Then we obtain a category with \(C^*\)-algebras as objects and unitary equivalence classes of equivalence modules as objects. To proceed analogously with Fell bundles we need a notion of unitary operator between equivalence bundles.

**Definition A.2.** Let \(X\) and \(Y\) be two \(A - B\)-equivalence bundles. A unitary from \(X\) to \(Y\) is an isomorphism of equivalence bundles \(\rho: X \rightarrow Y\) such that \(A(\rho(x), \rho(y)) = A(x, y)\) and \(B(\rho(x), \rho(y)) = (x, y)\) (this means that \(\rho^l = \text{id}_A\) and \(\rho^r = \text{id}_B\) in the notation of [4]). If such an isomorphism exists, we say that \(X\) is unitarily equivalent (or just isomorphic) to \(Y\).

To obtain a category with Fell bundles (over a fixed group \(G\)) as objects, isomorphism classes of equivalence bundles as morphisms and the tensor product of [4] as composition we need to show that the composition is well defined on the of isomorphism classes and that it is associative. This boils down to the following results.

**Proposition A.3.** Suppose \(\pi: X_1 \rightarrow X_1\) is a unitary between \(A - B\)-equivalence bundles and \(\rho: Y_1 \rightarrow Y_2\) is a unitary between \(B - C\)-equivalence bundles. Then \(X_1 \otimes_B Y_1\) is unitarily equivalent to \(X_2 \otimes_B Y_2\).

**Proof.** The way the tensor product is constructed is one of the key factors of this proof, so it will be necessary to recall it here. Start by considering the bundle \(Z_j := \{X_{j_r} \otimes_{B_r} Y_{j_s}\}_{(r,s) \in G \times G}, j = 1, 2\). The topology of that bundle is determined...
by the set of sections \( \Gamma_j := \text{span}\{f \otimes g : f \in C_c(X_j), g \in C_c(Y_j)\} \) where \( f \otimes g(r, s) = f(r) \otimes g(s) \). Note [10] II 13.16 implies the existence of a unique isomorphism of Banach bundles \( \mu : Z_1 \rightarrow Z_2 \) such that \( \mu(x \otimes y) = \pi(x) \otimes \rho(y) \). Recall that the construction of \( X_j \otimes_B Y_j \) is performed using actions of \( A \) and \( C \) on \( Z_j \) and operations \( \alpha_j : Z_j \times Z_j \rightarrow C \) and \( \beta_j : Z_j \times Z_j \rightarrow A \) uniquely determined by the identities

\[
\alpha(x \otimes y) = (ax) \otimes y \quad \quad \quad \quad \quad (x \otimes y)c = x \otimes yc
\]

The reader can easily check that \( a\mu(z) = \mu(az), \mu(z)c = \mu(z), \mu(z) \otimes \mu(z') = z \odot z' \) and \( \mu(z) \triangleright \mu(z') = z \triangleright z' \) for all \( z, z' \in Z_1, a \in A \) and \( c \in C \).

If we think of \( Z_1 \) and \( Z_2 \) as the same object, then there is nothing else to prove and \( X_1 \otimes_B Y_1 \) is in fact the same as \( X_2 \otimes_B Y_2 \). In other case the next step is to redefine, for every \( t \in G, U_{j,t} \) as the reduction of \( Z_j \) to \( H^t := \{(r, s) \in G \times G : rs = t\} \). Then we get an untupologized bundle \( U_j := \{U_{j,t}\}_{t \in G} \) and define pre-inner product and actions in the following way:

\[
\langle u, v \rangle_A := \int_{G \times G} u(p, p^{-1} r) \odot_j v(q, q^{-1} s) \, dp \, dq; \tag{A.1}
\]

\[
\langle u, v \rangle_C := \int_{G \times G} u(p, p^{-1} r) \triangleright_j v(q, q^{-1} s) \, dp \, dq; \tag{A.2}
\]

\[
au \in U_{j,tt} \quad \text{by the formula} \quad (au)(p, p^{-1} tr) := au(t^{-1} p, p^{-1} tr) \quad \text{and} \tag{A.3}
\]

\[
uc \in U_{j,tt} \quad \text{by the formula} \quad (uc)(p, p^{-1} r) := u(p, p^{-1} r)c. \tag{A.4}
\]

It is then clear that the composition with \( \mu \) identifies the pre-inner products an actions of \( U_1 \) and \( U_2 \), for example \( \langle u, v \rangle_A = \langle \mu \circ u, \mu \circ v \rangle_A \). Each fiber \( U_{j,t} \) is a seminormed space when considered with the seminorm \( \|u\| := \|\langle u, u \rangle_A\|^{1/2} = \|\langle u, u \rangle_C\|^{1/2} \). The space \( [U_{j,t}] \) is defined as the quotient of \( U_{j,t} \) by the subspace of zero length vectors, where square brackets are used to represent equivalence classes.

The tensor product \( X_1 \otimes_B Y_1 \) is obtained by completing each fiber of \( \{U_{j,t}\} = \{[U_{j,t}]\}_{t \in G} \) and a set of continuous sections of this tensor product is given by those of the form \( \xi \), for \( \xi \in C_c(Z_1) \), where \( \xi(t) = [\xi |_{H^t}] \) and \( [\xi |_{H^t}] \) represents the restriction of \( \xi \) to \( H^t \).

Note there exists a unique bijective isometry \( \mu^* : [U_{2,t}] \rightarrow [U_{1,t}] \) such that \([u] \mapsto [\mu \circ u]\). Then there exists a unique function \( \mu^* : X_1 \otimes_B Y_1 \rightarrow X_2 \otimes_B Y_2 \) which is linear and bounded on each fiber and extends each \( \mu^*_t \). Clearly, \( \mu^* \) is an isometry and \( \mu^* \circ [\xi] = [\mu \circ \xi] \) for all \( \xi \in C_c(Z_1) \). In this situation [10] II 13.16 implies \( \mu^* \) is an isomorphism of Banach bundles. Moreover, it preserves the left and right inner products because \( \mu \) transforms the inner products and actions of the bundle \( \{U_{1,t}\}_{t \in G} \) to those of \( \{U_{2,t}\}_{t \in G} \).

### Proposition A.4

Let \( X, Y \) and \( Z \) be \( A-B, B-C \) and \( C-D \)-equivalence bundles, respectively. Then

(a) \( A \otimes A X \) is unitarily equivalent to \( X \) and \( X \otimes A B \rightarrow B \).

(b) \( \tilde{X} \otimes A X' \) is unitarily equivalent to \( B \) and \( X \otimes \tilde{X} \rightarrow \tilde{X} \) to \( A \).

(c) The tensor products \( (X \otimes_B Y) \otimes_C Z \) and \( X \otimes_B (Y \otimes_C Z) \) are unitarily equivalent.

**Proof.** The proofs of the two claims in (a) are analogous, thus we just prove the first one; the same comment holds for (b).


Let $Z$ be the bundle constructed from $\mathcal{A}$ and $\mathcal{X}$ ($Z_{(r,s)} = A_r \otimes_{A_e} X_s$) as in the proof of A.3. We claim that there exists a unique continuous map $\pi: Z \to \mathcal{X}$ such that: $\pi(Z_{(r,s)}) \subset X_{rs}$, $\pi|_{Z_{r,s}}$ is linear and $\pi(a \otimes x) = ax$ for all $r, s \in G$, $a \in \mathcal{A}$ and $x \in \mathcal{X}$. First note there exists a unique linear isometry $\pi_{r,s}: Z_{r,s} \to X_{rs}$ sending $a \otimes x$ to $ax$ because, for every $a_1, \ldots, a_n \in A_r$ and $x_1, \ldots, x_n \in X$, we have
\[
\langle \sum_{i=1}^n a_i \otimes x_i, \sum_{i=1}^n a_i \otimes x_i \rangle_B = \sum_{i,j=1}^n \langle x_i, a_i^* a_j x_j \rangle_B = \langle \sum_{i=1}^n a_i x_i, \sum_{i=1}^n a_i x_i \rangle_B.
\]
If $\pi: Z \to \mathcal{A}$ is the unique map extending all the $\pi_{r,s}$, then, using [10, II 13.16], we conclude that $\pi$ is continuous for all $f \in C_c(\mathcal{A})$ and $g \in C_c(\mathcal{X})$, $\pi \circ (f \boxtimes g)$ is continuous.

Now let $\mathcal{U}$ be constructed from $Z$ as in the proof of A.3. For $f, u \in C_c(\mathcal{A})$ and $g, v \in C_c(\mathcal{X})$ we have
\[
\langle f \boxtimes g |_{H^r}, u \boxtimes v |_{H^s} \rangle_B^G = \left( \int_G \pi(f \boxtimes g(p, p^{-1}r)) \, dp, \int_G \pi(u \boxtimes v(q, q^{-1}s)) \, dq \right)_B.
\]
Since $\text{span}\{f \boxtimes g |_{H^r}: f \in C_c(\mathcal{A}), g \in C_c(\mathcal{X})\}$ is dense in the inductive limit topology of $U_t$ and $\langle \cdot, \cdot \rangle_B^G$ is continuous on each variable (separately) with respect to this topology, we conclude that
\[
\langle \xi, \eta \rangle_B^G = \left( \int_G \pi(\xi(p, p^{-1}r)) \, dp, \int_G \pi(\eta(q, q^{-1}s)) \, dq \right)_B
\]
for all $\xi \in U_r$, $\eta \in U_s$ and $r, s \in G$.

Then we can define a map $\mu: [\mathcal{U}] \to \mathcal{X}$ such that, for $\xi \in U_r$,
\[
\mu([\xi]) = \int_G \pi(\xi(p, p^{-1}r)) \, dp.
\]
This map is linear and isometric on each fiber, so it can be (continuously) extended to the closure of each fiber. The resulting extension is a map $\mu: \mathcal{A} \otimes_{\mathcal{A}} \mathcal{X} \to \mathcal{X}$. Recall from [11] that the topology of $\mathcal{A} \otimes_{\mathcal{A}} \mathcal{X}$ is constructed using the sections of the form $r \mapsto [\eta |_{H^r}]$, where $\eta \in C_c(\mathcal{Z})$. Besides, for every $\eta \in C_c(\mathcal{Z})$, the section $r \mapsto \mu([\eta |_{H^r}])$ is continuous (this is not immediate but can be proved by standard arguments, see for example the ideas developed in [10, II 15.19]). Then [10, II 13.16] implies $\mu$ is continuous. Note Equation A.5 implies $\mu'' = \text{id}_B$ and the reader can show that $\mu' = \text{id}_A$ with analogous computations. Then all we need to do is to show $\mu$ is surjective or, alternatively, that $\mu([U_r])$ is dense in $X_r$ for all $r \in G$.

Fix $x \in X_r$ and take $a \in A_e$ and $x' \in X_r$ such that $ax' = x$. Now take $f \in C_c(\mathcal{A})$ and $g \in C_c(\mathcal{X})$ such that $f(e) = a$ and $g(r) = x'$. Denote $I$ the directed set of compact neighbourhoods of $e$ in $G$ with respect the usual order: $i \leq j$ if $j \subset i$. For each $i \in I$, take a function $\varphi_i \in C_c(G)^+$ with $\int_G \varphi_i(t) \, dt = 1$ and $\text{supp} \varphi_i \subset i$. Then $\lim_{i} \mu((\varphi_i f) \boxtimes g |_{H^r}) = f(e)g(r) = x$; we conclude that $\mu$ is surjective. This implies that $\mu$ is unitary.

The proof of [11] is very similar to that of A3. We start by constructing a surjective isometry $\pi: Z \to B$, where $Z = \overline{\{X_r^{-1} \otimes_{A_e} X_s\}}_{(r,s) \in G \times G}$. Take $r, s \in G$,
$x^i_1, \ldots, x^i_n \in X_{r-1}$ and $y^i_1, \ldots, y^i_n \in X_s$ \((j = 1, 2)\). Then note that

$$\sum_{i=1}^n x^i_1 \otimes y^i_1 = \sum_{i,j=1}^n x^i_j \otimes y^i_j = \sum_{i=1}^n (y^i_1, A(x^i_1 x^i_2 y^i_2) \beta) = \sum_{i=1}^n (y^i_1, x^i_1 x^i_2 y^i_2 \beta) \beta$$

Besides, the restriction of $\triangleright$ to $Z_{(r,s)} \times Z_{(r,s)}$ is the inner product of $Z_{(r,s)}$. Then we conclude there exists a unique map $\pi: Z \to B$ such that: $\pi(x \otimes y) = (x, y) \beta$, $\pi(Z_{(r,s)}) \subset B_{(r,s)}$ and $\pi|Z_{(r,s)}$ is linear for all $x, y \in X$ and $r, s \in G$. Moreover, $\pi \triangleright w = \pi(z)^* \pi(w)$, $\pi(zb) = \pi(zb)$ and $\pi(bz) = b \pi(z)$ for all $z, w \in Z$ and $b \in B$. Note also that $\pi \triangleright w$ is continuous because, for $f, g \in C_c(X)$, we have $\pi \circ \tilde{f} \otimes g(r, s) = \langle f(r-1), g(s) \rangle \beta$ and so $\pi \circ \tilde{f} \otimes g$ is continuous.

To complete the proof of (b) it suffices to follow the steps of the proof of (a), using the map $\pi$ we have just constructed instead of the map $\pi$ we used to prove (a).

We now deal with (c). Let $[U]$ and $[V]$ be the bundles whose fiber-wise completion gives $\mathcal{X} \otimes_{\mathcal{C}} \mathcal{Y}$ and $\mathcal{Y} \otimes_{\mathcal{C}} Z$, respectively (see the proof of Proposition 3A). For every pair $(f, g) \in C_c(\mathcal{X}) \times C_c(\mathcal{Y})$ we have a section $[f \otimes g] \in C_c(\mathcal{X} \otimes_{\mathcal{C}} \mathcal{Y})$ such that $[f \otimes g](t) = [f \otimes g](H^t) \in [U]$. Define $\Gamma_U$ as the linear span of the sections $[f \otimes g]$. Part of the construction of $\mathcal{X} \otimes_{\mathcal{C}} \mathcal{Y}$ is based on the fact that $\{\xi(t): \xi \in \Gamma_U\}$ is dense in $[U]$ and so in the fiber over $t$ of $\mathcal{X} \otimes_{\mathcal{C}} \mathcal{Y}$. Of course that the same holds for $\Gamma_V$.

Fix $r, s \in G$, $f, u \in C_c(\mathcal{X})$, $g, v \in C_c(\mathcal{Y})$ and $h, w \in C_c(Z)$. We want to prove that

$$\langle [\tilde{f} \otimes [g \otimes h]](r), [u \otimes [v \otimes w]](s) \rangle_{\mathcal{D}} = \langle [f \otimes [g \otimes h]](r), [u \otimes [v \otimes w]](s) \rangle_{\mathcal{D}}. \quad (A.6)$$

To do this first note that, using the definitions of the inner product of tensor products bundles, we obtain

$$\langle [f \otimes [g \otimes h]](r), [u \otimes [v \otimes w]](s) \rangle_{\mathcal{D}}$$

Using the substitutions $p \mapsto xp$ and $q \mapsto xq$ in the integrals, we obtain

$$\langle [f \otimes [g \otimes h]](r), [u \otimes [v \otimes w]](s) \rangle_{\mathcal{D}}$$
Using equation \[A.6\] and \[10\] II 13.16 we can justify the existence of a unique isometric isomorphism of Banach bundles \(\mu: (\mathcal{X} \otimes_B \mathcal{Y}) \otimes_G Z \rightarrow \mathcal{X} \otimes_B (\mathcal{Y} \otimes_G Z)\) such that \(\mu([(f \otimes g) \otimes h](r)) = [f \otimes (g \otimes h)](r)\). We leave to the reader the verification of the fact that \(\mu([f \otimes g] \otimes h)(r)d = \mu([f \otimes g] \otimes h)[r])d\). After this it is immediate that \(\mu(\xi d) = \mu(\xi)d\) for all \(\xi \in (\mathcal{X} \otimes_B \mathcal{Y}) \otimes_G Z\) and \(d \in D\). Note equation \[A.6\] implies \(\langle \mu(\xi), \mu(\eta) \rangle_D = \langle \xi, \eta \rangle_D\) for all \(\xi, \eta \in (\mathcal{X} \otimes_B \mathcal{Y}) \otimes_G Z\). Then \(\mu\) is a morphism of equivalence bundles because

\[
\mu(\xi(\eta, \zeta)) = \mu(\xi)\langle \eta, \zeta \rangle_D = \mu(\xi)\langle \mu(\eta), \mu(\zeta) \rangle_D.
\]

The identity \(\langle \mu(\xi), \mu(\eta) \rangle_D = \langle \xi, \eta \rangle_D\) tells us that \(\mu^* = \text{id}_A\) (the proof of which is analogous to the proof of \[A.6\]).

Using the last two propositions one can construct a category \(\mathcal{E}_G^\text{pr}\) (resp. \(\mathcal{E}_G^a\)) of Fell bundles over \(G\) as objects and isomorphism classes of weak (resp. strong) equivalence bundles as arrows. The identity morphism associated to the Fell bundle \(A\) is the isomorphism class of \(A\), \([A]\). The composition of the arrows \([X] \rightarrow B\) and \([Y] \rightarrow C\) is \([X \otimes_B Y] \rightarrow C\). Propositions \[A.3\] and \[A.4\] tell us that we indeed obtain a category with these definitions and, moreover, every arrow is invertible in this category. Only a weak 2-category (or bicategory) can be formed if we do not take isomorphism class, but just the equivalence bundles as arrows. The unitaries introduced in Definition \[A.2\] can be used as 2-arrows for this weak 2-category. This works similarly to the 2-category of \(C^*\)-algebras with correspondences as arrows introduced in \[9\].

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