BRANCHING RULES FOR FINITE-DIMENSIONAL 
$\mathcal{U}_q(\mathfrak{su}(3))$-REPRESENTATIONS WITH RESPECT TO A RIGHT COIDEAL SUBALGEBRA

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Abstract. We consider the quantum symmetric pair $(\mathcal{U}_q(\mathfrak{su}(3)), \mathcal{B})$ where $\mathcal{B}$ is a right coideal subalgebra. We prove that all finite-dimensional irreducible representations of $\mathcal{B}$ are weight representations and are characterised by their highest weight and dimension.

We show that the restriction of a finite-dimensional irreducible representation of $\mathcal{U}_q(\mathfrak{su}(3))$ to $\mathcal{B}$ decomposes multiplicity free into irreducible representations of $\mathcal{B}$. Furthermore we give explicit expressions for the highest weight vectors in this decomposition in terms of dual $q$-Krawtchouk polynomials.

1. Introduction

The theory of quantum symmetric pairs of Lie groups has been developed by Koornwinder, Dijkhuizen, Noumi and Sugitani and others [2, 3, 22, 20, 21, 24] for classical Lie groups and by G. Letzter [13, 15, 16, 17, 18] for all semisimple Lie algebras, see also [10]. The motivating example for the development for this theory was given by Koornwinder [11], who studied scalar-valued spherical functions on the quantum analogue of $(\text{SU}(2), \text{U}(1))$ considering twisted primitive elements in the quantised universal enveloping algebra of $\mathcal{U}_q(\mathfrak{sl}(2))$. Koornwinder identified all scalar-valued spherical functions with Askey-Wilson polynomials in two free parameters. Dijkhuizen and Noumi [2] extended the work of Koornwinder to quantum analogues of $(\text{SU}(n + 1), \text{U}(n))$ considering two sided coideals of the quantised universal enveloping algebra of $\mathcal{U}_q(\mathfrak{gl}(n + 1))$. More generally, Letzter considered the quantised universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ with a right coideal subalgebra $\mathcal{B}$, which is the quantum analogue of $\mathcal{U}(\mathfrak{k})$ for a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. In [17] all scalar-valued spherical functions for quantum symmetric pairs with reduced restricted root systems are identified with Macdonald polynomials. However, the requirement of having a reduced restricted root system excludes the quantum analogue of $(\text{SU}(3), \text{U}(2))$.

One recent extension of this situation [1] arises with the study of matrix-valued spherical functions of the quantum analogue of $(\text{SU}(2) \times \text{SU}(2), \text{diag})$ where higher-dimensional representations of coideal subalgebra $\mathcal{B}$ are involved. The quantum symmetric pair is given by the quantised universal enveloping algebra of $\mathcal{U}_q(\mathfrak{g})$, where $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, and a right coideal subalgebra $\mathcal{B}$ than can be identified with $\mathcal{U}_q(\mathfrak{su}(2))$. As in the Lie group setting [3, 9, 25, 5], the explicit knowledge of the branching rules plays a fundamental role in the explicit determination of the matrix-valued spherical functions. In this first case, the branching rules for the irreducible representations of $\mathcal{U}_q(\mathfrak{g})$ with respect to $\mathcal{B}$ follow using the standard Clebsch-Gordan decomposition.

One of the first technical difficulties that one finds to extend the results of [1] to more general quantum symmetric pairs is the lack of the explicit branching rules for finite-dimensional...
\( \mathcal{U}_q(\mathfrak{g}) \)-representations with respect to a right coideal subalgebra. In this paper we deal with this problem for the quantised universal enveloping algebra \( \mathcal{U}_q(\mathfrak{su}(3)) \) with a right coideal subalgebra \( \mathcal{B} \) as in Kolb \[10\]. We study the problem of describing all irreducible representations that occur in the restriction to \( \mathcal{B} \) of finite-dimensional irreducible representations of \( \mathcal{U}_q(\mathfrak{su}(3)) \). In general, information about branching rules for quantum symmetric pairs \((\mathcal{U}_q(\mathfrak{g}), \mathcal{B})\) as in Kolb \[10\] and Letzter \[13, 15\] is relatively scarce in particular in case the coideal subalgebra depends on an additional parameter as in this paper. However see Oblomkov and Stokman \[23\] for partial information on the branching rules for the quantum analogue of \((\mathfrak{gl}(2n), \mathfrak{gl}(n) \oplus \mathfrak{gl}(n))\).

This paper is organised as follows. In Section 2 we review the construction of the quantised universal enveloping algebra \( \mathcal{U}_q(\mathfrak{su}(3)) \) and its finite dimensional irreducible representations. Then we collect a series of commutation identities for the generators of \( \mathcal{U}_q(\mathfrak{su}(3)) \) and introduce an orthogonal basis for finite-dimensional \( \mathcal{U}_q(\mathfrak{su}(3)) \)-representations which is an analogue of Mudrov \[19\]. We also describe the action of the generators of \( \mathcal{U}_q(\mathfrak{su}(3)) \) on this basis. In Section 3 we fix a right coideal subalgebra \( \mathcal{B} \) of the quantised universal enveloping algebra which depends on two complex parameters \( c_1, c_2 \). We describe the generators of the Cartan subalgebra of \( \mathcal{B} \) and we use them to classify all finite-dimensional irreducible representations of \( \mathcal{B} \) under a mild genericity condition on the parameters. More precisely we prove that every finite-dimensional irreducible representation of \( \mathcal{B} \) is completely characterised by its highest weight and dimension. In Section 4 we prove the main theorem of the paper. We show that any irreducible finite-dimensional representation of \( \mathcal{U}_q(\mathfrak{su}(3)) \) decomposes multiplicity free into irreducible representations of the \( \mathcal{B} \) and we characterise the representations that occur in the decomposition by their highest weight and dimension. The highest weight vectors of the coideal subalgebra \( \mathcal{B} \)-representations are obtained by diagonalising an element of the Cartan subalgebra of \( \mathcal{B} \) restricted to a certain subspace where it acts tridiagonally. The eigenvectors can be then identified explicitly in terms of dual \( q \)-Krawtchouk polynomials.

2. The quantised universal enveloping algebra \( \mathcal{U}_q(\mathfrak{su}(3)) \)

Let \( \mathfrak{g} = \mathfrak{sl}(3) = \{ X \in \mathfrak{gl}(3, \mathbb{C}) : \text{tr}(X) = 0 \} \). We fix the Cartan subalgebra \( \mathfrak{h} \) of diagonal matrices. Let \( A = (a_{i,j})_{i,j} \) be the Cartan matrix for \( \mathfrak{g} \), i.e. \( a_{i,j} = 2, i = 1, 2, \) and \( a_{i,j} = -1 \) for \( i \neq j \). Let \( R \subset \mathfrak{h} \) denote the root system of \( \mathfrak{g} \). We denote by \( R^+ \) the subset of positive roots, so that we have the decomposition \( \mathfrak{g} = n^- \oplus \mathfrak{h} \oplus n^+ \). We denote by \( (,.) \) the canonical inner product on \( \mathfrak{h} \) and by \( \Pi = \{ \alpha_1, \alpha_2 \} \) the simple roots so that \( (\alpha_i, \alpha_j) = a_{i,j} \). The fundamental weights are given by \( \varpi_1 = \frac{2}{3} \alpha_1 + \frac{1}{3} \alpha_2 \) and \( \varpi_2 = \frac{1}{3} \alpha_1 + \frac{2}{3} \alpha_2 \).

The quantised universal enveloping algebra \( \mathcal{U}_q(\mathfrak{su}(3)) \) is the unital associative algebra generated by \( E_i, F_i \) and \( K_i^{\pm 1} \), where \( i = 1, 2 \), subject to the relations

\[
\begin{align*}
K_i^{\pm 1} K_j^{\pm 1} &= K_j^{\pm 1} K_i^{\pm 1}, \\
K_i^{\pm 1} K_j^{\mp 1} &= K_j^{\mp 1} K_i^{\pm 1}, \quad K_i K_i^{-1} = 1 = K_i^{-1} K_i, \\
K_i E_j &= q^{(a_{i,j})} E_j K_i, \quad K_i F_j = q^{-(a_{i,j})} F_j K_i, \quad [E_i, F_j] = \frac{K_i - K_i^{-1}}{q - q^{-1}} \delta_{i,j},
\end{align*}
\]

for \( i, j = 1, 2 \) and, for \( i \neq j \), the quantum Serre’s relations

\[
E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0 = F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2.
\]
We assume that $q \in [0, 1]$. The quantised universal enveloping algebra $U_q(\mathfrak{su}(3))$ has a Hopf algebra structure with comultiplication $\Delta$, counit $\epsilon$ and antipode $S$ defined by

$$
\Delta : E_i, F_i, K_i^{\pm 1} \mapsto E_i \otimes 1 + K_i \otimes E_i, F_i \otimes K_i^{-1} + 1 \otimes F_i, K_i^{\pm 1} \otimes K_i^{\pm 1},
$$

$$
\epsilon : E_i, F_i, K_i^{\pm 1} \mapsto 0, 0, 1, \quad S : E_i, F_i, K_i^{\pm 1} \mapsto -K_i^{-1}E_i, -F_iK_i, K_i^{\pm 1},
$$

with $i = 1, 2$. The $*$-structure on $U_q(\mathfrak{su}(3))$ is given by

$$
( K_i^{\pm 1} )^* = K_i^{\mp 1}, \quad ( E_i, F_i )^* = ( E_i, F_i )^*, \quad ( K_i )^* = K_i^{-1}, \quad i = 1, 2,
$$

so that $U_q(\mathfrak{su}(3))$ is a Hopf $*$-algebra. Following Mudrov [19] we define for $a \in \mathbb{R}$

$$
F_3 = [ F_1, F_2 ]_q = F_1 F_2 - q F_2 F_1, \quad E_3 = [ E_2, E_1 ]_q = E_2 E_1 - q E_1 E_2,
$$

$$
\hat{F}_3[a] = F_1 F_2 \left( \frac{q^{a+1} K_2 - q^{-a-1} K_2^{-1}}{q - q^{-1}} \right) - F_2 F_1 \left( \frac{q^a K_2 - q^{-a} K_2^{-1}}{q - q^{-1}} \right),
$$

$$
\hat{E}_3[a] = \left( \frac{q^{a+1} K_2 - q^{-a-1} K_2^{-1}}{q - q^{-1}} \right) E_2 E_1 - \left( \frac{q^a K_2 - q^{-a} K_2^{-1}}{q - q^{-1}} \right) E_1 E_2,
$$

and $\hat{F}_3 = \hat{F}_3[0]$, $\hat{E}_3 = \hat{E}_3[0]$.

**Lemma 2.1.** The following relations hold in $U_q(\mathfrak{su}(3))$:

(i) $F_1 \hat{F}_3[a] = \hat{F}_3[a] F_1$,

(ii) $E_2 \hat{F}_3[a] = \hat{F}_3[a - 2] E_2 - \frac{q^a - q^{-a}}{q - q^{-1}} F_1$,

(iii) $K_i \hat{F}_3[a] = q^{-a} \hat{F}_3[a] K_i$, $K_i \hat{E}_3[a] = q \hat{E}_3[a] K_i$, $i = 1, 2$.

*Proof.* Straightforward verifications using (2.1) and (2.3). 

**Lemma 2.2.** For $i = 1, 2$:

(i) $E_i F_i^k = F_i^k E_i + kq^{-1} (q^{-k} K_i - q^{k-1} K_i^{-1})$,

(ii) $E_i^k F_i = \frac{q^k (q^2; q^2)_k (q^{2-k} K_i^2; q^2)_k K_i^{-k}}{(1 - q^2)^{2k}} (1 - q^{k-2}) (K_i^{-2}; q^2)_k K_i^k + U_q(\mathfrak{su}(3)) E_i$.

*Proof.* Straightforward verifications using (2.1) and (2.3) and induction.

### 2.1. The finite-dimensional representations of $U_q(\mathfrak{su}(3))$

Finite-dimensional representations of $U_q(\mathfrak{su}(3))$ are weight representations and are uniquely determined, up to equivalence, by their highest weights. Let $(\pi, V_\lambda)$ be an irreducible finite-dimensional representation with highest weight $\lambda = \lambda_1 \varpi_1 + \lambda_2 \varpi_2$, $\lambda_1, \lambda_2 \in \mathbb{N}$, and $v_\lambda$ a highest weight vector such that

$$
E_i v_\lambda = 0, \quad K_i v_\lambda = q^{(\lambda, a_i)} v_\lambda = q^\lambda v_\lambda.
$$

Then the dimension of $V_\lambda$ is the same as the dimension of the corresponding irreducible representation $\pi_\lambda$ of $\mathfrak{su}(3)$, namely

$$
\dim(V_\lambda) = \frac{1}{2}(\lambda_1 + 1)(\lambda_2 + 1)(\lambda_1 + \lambda_2 + 2).
$$
Furthermore for a weight $\nu = \nu_1 \varpi_1 + \nu_2 \varpi_2$, the dimension of the weight space

$$V_\lambda(\nu) = \{ v \in V_\lambda : K_i v = q^{(\nu_1 \alpha_i, v)} v, \ i = 1, 2 \},$$

and the dimension of the weight space corresponding to the weight $\nu$ in the representation of $\mathfrak{su}(3)$ coincide, see [6, Ch. 7]. In particular, $\dim(V_\lambda(\lambda)) = 1$. The vector space $V_\lambda$ is generated by the vectors $v_\lambda$ and $F_1 F_2 \ldots F_m v_\lambda$, $i_j \in \{ 1, 2 \}$ and is equipped with an inner product $\langle \cdot, \cdot \rangle$ that satisfies

$$\langle v_\lambda, v_\lambda \rangle = 1, \quad \langle X v, w \rangle = \langle v, X^* w \rangle, \quad \forall X \in \mathcal{U}_q(\mathfrak{su}(3)), \quad \forall v, w \in V_\lambda.$$

Mudrov [19] describes the Shapovalov basis for the Verma modules of $\mathcal{U}_q(\mathfrak{su}(3))$, and we have adapted his proof and construction to an orthonormal basis for the finite-dimensional unitary representations of $\mathcal{U}_q(\mathfrak{su}(3))$. For completeness, we have sketched the proof in Appendix A. It is essentially due to Mudrov [19, §8].

**Theorem 2.3.** The set of vectors

$$\mathcal{B} = \{ F_2^m F_3^l v_\lambda | 0 \leq m \leq \lambda_1, 0 \leq l \leq \lambda_2, 0 \leq k \leq \lambda_2 + m - l \}$$

forms an orthogonal basis for $V_\lambda$. Explicitly,

$$\langle F_2^k F_3^l F_1^m v_\lambda, F_2^{k'} F_3^{l'} F_1^{m'} v_\lambda \rangle = \delta_{k,k'} \delta_{l,l'} \delta_{m,m'} H_{k,l,m},$$

where

$$H_{k,l,m} = (q^2, q^{-2(\lambda_2 - l + m)}; q^2)_k (q^2, q^{-2\lambda_1}; q^2)_m (q^2, q^{-2\lambda_2}, q^{-2(\lambda_2 + 1 + m)}, q^{-2(\lambda_1 + \lambda_2 + 1)}; q^2)_l \times (1 - q^2)^{-2(k + 2l + m)} (-1)^{k + l + m} q^{3(k + 3l + m)} q^{-l(l - 2m)} q^{-2\lambda_2}.$$ 

In Theorem 2.3 we use the standard notation in [4] for $q$-shifted factorials

$$(q^n; q)_n = (1 - q^n)(1 - q^{n+1}) \ldots (1 - q^{n+n-1}),$$

$$(q^{a_1}, q^{a_2}, \ldots, q^{a_n}; q)_n = (q^{a_1}, q^a, q^{a_2})_n \ldots (q^{a_i}, q)_n.$$ 

Note that $H_{k,l,m}$ is indeed positive. In the following proposition we calculate the action of the generators of $\mathcal{U}_q(\mathfrak{su}(3))$ in the basis $\mathcal{B}$ of Theorem 2.3.

**Proposition 2.4.** In the basis $\mathcal{B}$ of $V_\lambda$ as in Theorem 2.3 we have

(i) $K_1 F_2^k F_3^l F_1^m v_\lambda = q^{\lambda_1 + k - l - 2m} F_2^k F_3^l F_1^m v_\lambda,$

(ii) $K_2 F_2^k F_3^l F_1^m v_\lambda = q^{\lambda_2 - 2k + l + m} F_2^k F_3^l F_1^m v_\lambda,$

(iii) $F_1 F_2^k F_3^l F_1^m v_\lambda = a_k(l, m) F_2^k F_3^l F_1^{m+1} v_\lambda + b_k(l, m) F_2^{k-1} F_3^l F_1^m v_\lambda,$

(iv) $E_1 F_2^k F_3^l F_1^m v_\lambda = a_k(l, m) F_2^k F_3^l F_1^{m-1} v_\lambda + b_k(l, m) F_2^{k+1} F_3^l F_1^m v_\lambda,$

(v) $F_2 F_2^k F_3^l F_1^m v_\lambda = F_2^{k+1} F_3^l F_1^m v_\lambda,$

(vi) $E_2 F_2^k F_3^l F_1^m v_\lambda = \eta_k(l, m) F_2^{k-1} F_3^l F_1^m v_\lambda,$
with coefficients

\[ a_k(l, m) = \frac{(q^{\lambda_2+m+1-k-l} - q^{-\lambda_2-m-1+k+l})}{(q^{\lambda_2+m+1-k-l} - q^{-\lambda_2-m-1+k+l})}, \quad b_k(l, m) = \frac{(q^k - q^{-k})}{(q^{\lambda_2+m+1-k-l} - q^{-\lambda_2-m-1+k+l})}, \]

\[ \eta_k(l, m) = \frac{q^k - q^{-k}}{q - q^{-1}}q^{-1-k+n-k-l+m} - q^{-k-1-l+m}, \]

\[ \alpha_k(l, m) = \frac{(q^m - q^{-m})(q^{\lambda_1-m+1} - q^{-\lambda_1-m+1})(q^{\lambda_2+m+1} - q^{-\lambda_2-m-1})}{(q - q^{-1})^2(q^{\lambda_2-m+1} - q^{-\lambda_2-m-1})}, \]

\[ \beta_k(l, m) = \frac{(q^k - q^{-k})}{(q - q^{-1})^2(q^{\lambda_2-l+1} - q^{-\lambda_2-l+1})}. \]

**Remark 2.5.** Note that the denominators in \(a_k(l, m), b_k(l, m), \eta_k(l, m), \alpha_k(l, m)\) and \(\beta_k(l, m)\) are non-zero by the ranges of \(k, l, m\) as in Theorem 2.3.

**Proof.** The action of \(K_i\), \(i = 1, 2\), follows from (2.4), (2.1) and Lemma 2.1 iii. The action of \(E_2\) is trivial. The action of \(E_3\) follows from Lemma 2.1 ii, Lemma 2.1 i and (2.1) and the established actions of \(K_2\). This completes the proof of (ii), (i) and (vi).

In order to establish the action of \(F_1\), we first show that there exist constants \(a_k\) and \(b_k\) so that

\[ F_1F_2F_3^kF_1^{m_1}v_\lambda = a_kF_2F_3^kF_1^{m_1}v_\lambda + b_kF_2^{k-1}F_3^{l+1}F_1^{m}v_\lambda \]

by induction with respect to \(k\). The case \(k = 0\) with \(a_0 = 1, b_0 = 0\) is immediate from Lemma 2.1 ii. In case \(k = 1\), we write

\[ F_1F_2F_3F_1^{m_1}v_\lambda = F_1F_2\frac{qK_2 - q^{-1}K_2^{-1}}{q - q^{-1}} q^{-1} q^l q^{\lambda_2-l+m} - q^{-\lambda_2-l+m} \frac{\hat{F}_3^{l}F_1^{m_1}v_\lambda}{\hat{F}_3^{l}F_1^{m_1}v_\lambda} \]

\[ = \frac{q - q^{-1}}{q^{1+\lambda_2-l+m} - q^{-1-\lambda_2-l+m}} \left( \frac{\hat{F}_3 + F_2F_1K_2^{-1}}{q - q^{-1}} \right) \hat{F}_3^{l}F_1^{m_1}v_\lambda \]

\[ = \frac{q - q^{-1}}{q^{1+\lambda_2-l+m} - q^{-1-\lambda_2-l+m}} F_2F_3^{l}F_1^{m_1+1}v_\lambda \]

\[ + \frac{q - q^{-1}}{q^{1+\lambda_2-l+m} - q^{-1-\lambda_2-l+m}} \hat{F}_3^{l+1}F_1^{m_1}v_\lambda \]

again using Lemma 2.1 ii. So the case \(k = 1\) is proved with

\[ a_1 = \frac{q^{\lambda_2-l+m} - q^{-\lambda_2-l+m}}{q^{1+\lambda_2-l+m} - q^{-1-\lambda_2-l+m}}, \quad b_1 = \frac{q - q^{-1}}{q^{1+\lambda_2-l+m} - q^{-1-\lambda_2-l+m}}. \]

For the induction we assume \(k \geq 2\), so that

\[ F_1F_2F_3^kF_1^{m_1}v_\lambda = F_1F_2F_2^{k-2}\hat{F}_3^{l}F_1^{m_1}v_\lambda = \left( -F_2^2F_1 + (q + q^{-1})F_2F_1F_2 \right) F_2^{k-3}\hat{F}_3^{l}F_1^{m_1}v_\lambda \]

by the \(q\)-Serre relation (2.2). Using the induction hypothesis, we find

\[ F_1F_2F_3^kF_1^{m_1}v_\lambda = -F_2^2 \left( a_{k-2}F_2^{k-2}\hat{F}_3^{l}F_1^{m_1}v_\lambda + b_{k-2}F_2^{k-3}\hat{F}_3^{l+1}F_1^{m_1}v_\lambda \right) + (q + q^{-1})F_2 \left( a_{k-1}F_2^{k-1}\hat{F}_3^{l}F_1^{m_1+1}v_\lambda + b_{k-1}F_2^{k-2}\hat{F}_3^{l+1}F_1^{m_1+1}v_\lambda \right) \]

\[ = \left( -a_{k-2} + (q + q^{-1})a_{k-1} \right) F_2^{k-2}\hat{F}_3^{l}F_1^{m_1+1}v_\lambda + \left( -b_{k-2} + (q + q^{-1})b_{k-1} \right) F_2^{k-1}\hat{F}_3^{l+1}F_1^{m_1}v_\lambda \]

which proves the induction step as well as the recurrence

\[ a_k + a_{k-2} = (q + q^{-1})a_{k-1}, \quad b_k + b_{k-2} = (q + q^{-1})b_{k-1}, \quad k \geq 2. \]
This recursion is solved by the Chebyshev polynomials (of the second kind) at \( \frac{1}{2}(q + q^{-1}) \) as well as by the associated Chebyshev polynomials. This gives the solution for the recurrences and proves \( \Box \).

The action of \( E_1 \) follows from that of \( F_1 \), considering the adjoint. Note that
\[
\langle E_1 F_2^k \hat{F}_3^d F_1^m v_\lambda, F_2^l \hat{F}_3^e F_1^{m'} v_\lambda \rangle = \langle F_2^k \hat{F}_3^d F_1^m v_\lambda, E_1^* F_2^l \hat{F}_3^e F_1^{m'} v_\lambda \rangle = \langle F_2^k \hat{F}_3^d F_1^m v_\lambda, K_1 F_1 F_2^l \hat{F}_3^e F_1^{m'} v_\lambda \rangle,
\]
equals zero if \((k', l', m') \neq (k, l, m + 1), (k + 1, l - 1, m)\). Moreover we have
\[
\alpha_k(l, m) H_{k, l, m-1} = \langle E_1 F_2^k \hat{F}_3^d F_1^m v_\lambda, F_2^l \hat{F}_3^e F_1^{m-1} v_\lambda \rangle = \langle F_2^k \hat{F}_3^d F_1^m v_\lambda, K_1 F_1 F_2^l \hat{F}_3^e F_1^{m-1} v_\lambda \rangle = q^{k-l-2m+\lambda_1} a_k(l, m-1) H_{k, l, m},
\]
and
\[
\beta_k(l, m) H_{k+1, l-1, m} = \langle E_1 F_2^k \hat{F}_3^d F_1^m v_\lambda, F_2^{k+1} \hat{F}_3^d F_1^{m-1} v_\lambda \rangle = \langle F_2^k \hat{F}_3^d F_1^m v_\lambda, K_1 F_1 F_2^{k+1} \hat{F}_3^d F_1^{m-1} v_\lambda \rangle = q^{k-l-2m+\lambda_1+2} b_{k+1}(l - 1, m) H_{k, l, m}.
\]

Now the expressions of \( \alpha_k(l, m) \) and \( \beta_k(l, m) \) follow from the explicit expression of \( H_{k, l, m} \) Theorem 2.3 by a straightforward computation. \( \Box \)

3. The coideal subalgebra

In this section we follow Kolb [10] and introduce a right coideal subalgebra \( \mathcal{B} \) of \( \mathcal{U}_q(\mathfrak{su}(3)) \) which is the quantum analogue of \( \mathfrak{U}(\mathfrak{u}(2)) \) with \( \mathfrak{u} = \mathfrak{u}(2) \) embedded in \( \mathfrak{g} = \mathfrak{su}(3) \). Let \( c_1, c_2 \in \mathbb{C}^\times \) and write \( c = (c_1, c_2) \). Following [10, Example 9.4], \( \mathcal{B}_c = \mathcal{B} \) is the right coideal subalgebra of \( \mathcal{U}_q(\mathfrak{su}(3)) \), i.e. \( \Delta(\mathcal{B}) \subset \mathcal{B} \otimes \mathcal{U}_q(\mathfrak{su}(3)) \), generated by

\[
(3.1) \quad K = \left( K_1 K_2^{-1} \right)^{\pm 1}, \quad \mathcal{B}_c^1 = B_1 = F_1 - c_1 E_2 K_1^{-1}, \quad \mathcal{B}_c^2 = B_2 = F_2 - c_2 E_1 K_2^{-1}.
\]

Throughout Sections 3 and 4 we omit the subscript and superscript \( c \) in \( \mathcal{B}_c \) and \( \mathcal{B}_c^i \) since the coideal subalgebra \( \mathcal{B} \) will be fixed.

If we assume \( c_1 c_2 = q^3 = c_1 c_2 \) then it follows that \( \mathcal{B}_c^1 = -c_1 K^{-1} B_2, \mathcal{B}_c^2 = -c_2 K B_1 \) and \( K^* = K \), so that \( \mathcal{B}^* = \mathcal{B} \). By a straightforward computation we have
\[
\Delta(\mathcal{B}_1) = B_1 \otimes K_1^{-1} + 1 \otimes F_1 - c_1 K_1^{-1} \otimes E_2 K_1^{-1},
\]
\[
\Delta(\mathcal{B}_2) = B_2 \otimes K_2^{-1} + 1 \otimes F_2 - c_2 K \otimes E_1 K_2^{-1}.
\]

The Serre relations for \( \mathcal{B} \) follow from from [10, Lemma 7.2, Theorem 7.4] taking \( \mathcal{Z}_1 = -K^{-1} \) and \( \mathcal{Z}_2 = -K \)

\[
(3.2) \quad B_1^2 B_2 - [2]_q B_1 B_2 B_1 + B_2 B_1^2 = [2]_q (qc_2 K + q^{-2} c_1 K^{-1}) B_1,
\]
\[
B_2^2 B_1 - [2]_q B_2 B_1 B_2 + B_1 B_2^2 = [2]_q (qc_1 K^{-1} + q^{-2} c_2 K) B_2.
\]

Alternatively [3.2] can be verified directly from the definitions of \( B_1, B_2 \) and \( K \).
The Cartan subalgebra of $B$ is generated by $K^{\pm 1}$, $C_1$ and $C_2$, where
\begin{equation}
C_1 = B_1 B_2 - q B_2 B_1 - \frac{1}{q - q^{-1}} c_2 K + \frac{q + q^{-1}}{q - q^{-1}} c_1 K^{-1},
\end{equation}
\begin{equation}
C_2 = B_2 B_1 - q B_1 B_2 - \frac{1}{q - q^{-1}} c_1 K^{-1} + \frac{q + q^{-1}}{q - q^{-1}} c_2 K.
\end{equation}
Moreover if $c_1, c_2 \in \mathbb{R}^\times$, then $C_1$ and $C_2$ are self-adjoint. The generators of the Cartan subalgebra of $B$ satisfy the relations $[K, C_i] = 0$ for $i = 1, 2$, $[C_1, C_2] = 0$ and
\begin{equation}
KB_1 = q^{-3} B_1 K, \quad C_1 B_1 = q B_1 C_1, \quad C_2 B_1 = q^{-1} B_1 C_2,
\end{equation}
\begin{equation}
KB_2 = q^3 B_2 K, \quad C_1 B_2 = q^{-1} B_2 C_1, \quad C_2 B_2 = q B_2 C_2.
\end{equation}
Note that by [12] Theorem 8.5] the center of $B$ is of rank 2. Hence the center of $B$ is generated by $K^{\frac{1}{3}} C_1$ and $K^{-\frac{1}{3}} C_2$, extending $B$ by cube roots of $K$. Then the central elements are self-adjoint for $c_1, c_2 \in \mathbb{R}^\times$.

3.1. Representation theory of $B$. Let $(\tau, W)$ be a finite-dimensional representation of $B$. Since $W$ is a finite-dimensional complex vector space, there exists a vector $w \in W$ such that $\tau(K)w = \nu w$ for some $\nu \in \mathbb{C}$. Then it follows from (3.3) that
\[
\tau(K) \tau(B_i)^i w = q^{-3i} \tau(B_i)^i \tau(K)w = q^{-3i} \nu \tau(B_i)^i w, \quad i \in \mathbb{N},
\]
so that the vectors $(\tau(B_i)^i w)_i$ are eigenvectors of $\tau(K)$ with different eigenvalues. Since $W$ is finite-dimensional, there exists $j \in \mathbb{N}$ such that $\tau(B_i^{j+1})w = 0$ and $\tau(B_i^j)w \neq 0$. Therefore $w_0 = \tau(B_1^j)w$ is a highest weight vector, i.e.
\[
\tau(B_1) w_0 = 0, \quad \tau(K) w_0 = \kappa w_0, \quad q^{-3} \kappa \notin \sigma(K),
\]
where $\kappa$ is the weight of $w_0$ and $\sigma(K)$ is the spectrum of $K$. Note that $\kappa \in \mathbb{C}^\times$ since it is the eigenvalue of an invertible operator.

**Proposition 3.1.** Let $\tau$ be a finite-dimensional irreducible representation of $B$ on the vector space $W$. Then $\tau$ is determined by the dimension of $W$ and the action of $K$ on a highest weight vector.

**Proof.** Let $\kappa \in \mathbb{C}^\times$ be the highest weight of $\tau$ and let $w_0$ be a highest weight vector, i.e. $\tau(K) w_0 = \kappa w_0$ and $\tau(B_1) w_0 = 0$. Since $\tau(K), \tau(C_1)$ and $\tau(C_2)$ form a commuting family of operators, we can assume that $\tau(C_1) w_0 = \eta_1 w_0$ and $\tau(C_2) w_0 = \eta_2 w_0$. For every $i \in \mathbb{N}$, we define the vector $w_i = \tau(B_2)^i w_0 \in W$. Since $W$ is finite-dimensional, there exists $n \in \mathbb{N}$ such that $w_i \neq 0$ for $0 \leq i \leq n$ and $w_{n+1} = 0$. It follows from (3.3) that $\tau(K) w_i = q^{3i} \kappa w_i$, so that $(w_i)_{i=0}^n$ is a set of linearly independent vectors since they are eigenvectors of $\tau(K)$ for different eigenvalues. Moreover (3.4) implies
\[
\tau(C_1) w_i = \tau(C_1) \tau(B_2)^i w_0 = q^{-i} \tau(B_2)^i \tau(C_1) w_0 = \eta_1 q^{-i} w_i,
\]
and similarly $\tau(C_2) w_i = \eta_2 q^i w_i$. We will show that it is indeed a basis of $W$.

We prove by induction in $i$ that there exist $b_i \in \mathbb{C}$ such that $\tau(B_1) w_i = b_i w_{i-1}$ for $i = 0, \ldots, n$. The statement holds for $i = 0$ taking $b_0 = 0$ since $w_0$ is a highest weight vector. Let $i > 0$ and assume that $\tau(B_1) w_j = b_j w_{j-1}$ for all $j < i$. Using (3.3) we find the recurrence

\[
\tau(B_1) w_i = \tau(B_1) \tau(B_2)^i w_0 = q^{-i} \tau(B_2)^i \tau(B_1) w_0 = \eta_1 q^{-i} b_i w_{i-1} = b_i w_{i-1},
\]
Corollary 3.3. \( \text{irreducibility of } B \)

Hence (3.5) follows by a straightforward computation.

\[ \tau(B_1)w_i = \tau(B_2)w_i = \tau(B_2B_1)w_i = \tau\left(C_1 + qB_2B_1 + \frac{c_2}{(q-q^{-1})} - \frac{(q+q^{-1})}{(q-q^{-1})}c_1K^{-1}\right)w_i, \]

By the inductive hypothesis, \( \tau(B_2B_1)w_i = b_{i-1} \tau(B_2)w_{i-2} = b_{i-1}w_{i-1}, \) so that

\[ \tau(B_1)w_i = (qb_{i-1} + q^{1-i}b_i + q^{3i-3}q^{2j+2n-1} - \frac{(q+q^{-1})}{(q-q^{-1})}q^{3-3i}c_1 \tau(B_1))w_i. \]

Hence \( \tau(B_1)w_i = b_{i}w_{i-1}. \) Since \( \tau \) is an irreducible representation we have that \( W = \tau(B)w_0 = \langle \{w_0, w_1, \ldots, w_n\} \rangle, \) and therefore \( \{w_i\}_{i=0}^n \) is a basis of \( W. \) This completes the proof of the proposition.

Remark 3.2. Since we assume \((\tau, W)\) irreducible, the coefficients \( b_i \) in the proof of Proposition 3.1 are non-zero for \( i = 1, \ldots, n. \) This follows from the fact that if \( b_{i_0} = 0 \) for some \( 1 \leq i_0 \leq n, \) then \( \{w_{i_0}, w_{i_0+1}, \ldots, w_n\} \) is an invariant subspace and this contradicts the irreducibility of \( \tau. \)

Corollary 3.3. Let \((\tau, W)\) be a finite-dimensional irreducible representation of \( B \) of dimension \( n + 1 \) and highest weight \( \kappa. \) Let \( w_0 \) be a highest weight vector and let \( w_i = (B_2)^iw_0 \) for \( i = 1, \ldots, n. \) Then \( \{w_i\}_{i=0}^n \) is a basis of \( W. \) The action of the generators of \( B \) on this basis is given by

\[ \tau(K)w_j = q^{3j}w_j, \quad \tau(B_2)w_j = w_{j+1}, \quad \tau(B_1)w_j = b_jw_{j-1} \]

where

\[ b_0 = 0, \quad b_j = c_1 \kappa^{-1} q^{-2n-1} \left[ j \right]_q \frac{(1 - q^{2n-2j+2})(1 + c_2c_1^{-1}q^{2j+2n-1})}{(q-q^{-1})}. \]

Moreover, \( \tau(C_1)w_j = q^{-j} \eta_1 w_j \) and \( \tau(C_2)w_j = q^j \eta_2 w_j, \) where

\[ \eta_1 = \frac{c_1 \kappa^{-1} q(1 + q^{2n-2}) - c_2 \kappa q^{2n}}{q-q^{-1}}, \quad \eta_2 = \frac{c_2 \kappa q^{-1} \left( 1 + q^{2n} \right) - c_1 \kappa^{-1} q^{-2n}}{q-q^{-1}}. \]

Proof. The fact that \( \{w_i\}_{i=0}^n \) is a basis of \( W \) and the action of \( \tau(K) \) on \( w_j \) follow directly from the proof of Proposition 3.1. It is clear that \( b_0 = 0. \) We now show that

\[ b_j = \left[ j \right]_q \left( \eta_1 + \frac{c_2 \kappa q^{2j-2} - c_1 \kappa^{-1} q^{2j-2}}{(q-q^{-1})} \right), \]

for all \( j = 1, \ldots, n. \) We proceed by induction on \( i. \) If \( i = 1, \) then the statement follows directly from (3.5). Now we assume that (3.6) is true for some \( j, 1 < j \leq n. \) Then it follows from (3.5) and the inductive hypothesis that

\[ b_j = q \left[ j - 1 \right]_q \left( \eta_1 + \frac{c_2 \kappa q^{2j-4} - c_1 \kappa^{-1} q^{3j-2j}(1 + q^{2j})}{(q-q^{-1})} \right) + q^{1-j} \eta_1 + \frac{q^{3j-3} \kappa c_1}{(q-q^{-1})} \left( \frac{q+q^{-1}}{q-q^{-1}} \right)^{3-3j} c_1. \]

Now (3.6) follows by a straightforward computation.
It follows from the proof of Proposition 3.1 that \( \tau(C_1)w_j = q^{-j} \eta_1 w_j \) where \( \eta_1 \) is the eigenvalue for the highest weight vector \( w_0 \). From the construction of the vectors \( w_i \) in Proposition 3.1, it follows that \( \tau(B_2)w_n = 0 \). Hence (3.3) and (3.6) yield

\[
q^{-n} \eta_1 w_n = \tau(C_1)w_n = q\tau(B_2B_1)w_n - \frac{1}{q-q^{-1}} c_2 \tau(K)w_n + \frac{q+q^{-1}}{q-q^{-1}} c_1 \tau(K^{-1})w_n
\]

\[
= -\frac{q^{n+1} - q^{-n+1}}{q-q^{-1}} \eta_1 - \frac{q^{n+1} - q^{-n+1}}{q-q^{-1}} \left( \frac{c_2 \kappa q^{2n-2} - c_2 \kappa^{-1} q^{1-2n} (1 + q^{2n})}{q-q^{-1}} \right)
\]

Now the expression of \( \eta_1 \) follows by a straightforward computation. The expression of \( \eta_2 \) can be obtained similarly from the action of \( C_2 \) on \( w_n \).

\[\square\]

**Remark 3.4.** If \( \tau \) is an irreducible representation with highest weight \( \kappa \) and dimension \( n+1 \), it follows from Remark 3.2 and the explicit expression of the coefficient \( b_i \) in Corollary 3.3 that \( c_2 c_i^{-1} \kappa^2 \neq -q^{-2j-2n+1} \) for all \( j = 1, \ldots, n \).

**Remark 3.5.** It follows from Proposition 3.1 and Corollary 3.3 that a finite-dimensional irreducible representation \( (\tau, W) \) of \( B \) is completely determined by the highest weight \( \kappa \) and the eigenvalue of \( \eta_1 \) of the highest weight vector as eigenvector of \( \tau(C_1) \).

**Corollary 3.6.** Every irreducible finite-dimensional representation of \( B \) is determined by a pair \( (\kappa, n) \) where \( \kappa \) is the highest weight and the dimension is \( n+1 \). Conversely, to each pair \( (\kappa, n) \) with \( \kappa \in \mathbb{C}^\times \), \( n \in \mathbb{N} \) and \( \kappa^2 \neq -c_2 c_i^{-1} q^{1-n} \), there corresponds an irreducible representation \( (\tau_{(\kappa, n)}, \mathcal{W}_{(\kappa, n)}) \) with highest weight \( \kappa \) and dimension \( n+1 \).

**Proof.** It follows directly from Proposition 3.1, Corollary 3.3 and Remark 3.4. \[\square\]

**Proposition 3.7.** Assume that \( \kappa \in \mathbb{R}^\times \) and \( c_2 \tau_2 = q^3 \). Let \( (\tau, W) \) be an irreducible finite-dimensional representation of \( B \). Then \( \tau \) is unitarizable.

**Proof.** Since \( c_2 \tau_2 = q^3 \), we have that \( B^* = B \). More precisely \( B_1^* = -\overline{\tau} K^{-1} B_2, B_2^* = -\overline{\tau} K B_1 \) and \( K^* = K \). Let \( \{w_i\}_{i=0}^n \) be the basis of \( W \) given in Corollary 3.3 and let \( \langle \cdot, \cdot \rangle \) be the hermitian bilinear form defined on the basis elements by \( \langle w_0, w_0 \rangle = 1 \),

\[
\langle w_k, w_k \rangle = \langle \tau(((B_2)^k)^* (B_2)^k)w_0, w_0 \rangle, \quad \langle w_i, w_j \rangle = 0, \quad i \neq j.
\]

Observe that

\[
\langle w_k, w_k \rangle = \langle \tau(((B_2)^k)^* (B_2)^k)w_0, w_0 \rangle
\]

\[= (-1)^k c_2 k q^{3(\frac{k}{2})} \langle \tau(K^k (B_1)^k(B_2)^k)w_0, w_0 \rangle = (-1)^k c_2 k q^{3(\frac{k}{2})} \langle \tau(K^k)w_0, w_0 \rangle \prod_{i=1}^k b_i
\]

\[= \frac{q^{3(\frac{k}{2})} - q^{2(2n-1)}}{(1 - q^2)^k} [k]! (q^{2n} - q^{-2})_k (-c_2 c^{-1} \kappa^2 q^{2n-1}; q^2)_k \langle w_0, w_0 \rangle.
\]

Since \( q^{2} c_2 c_i^{-1} = c_1 \tau_2 c_2 c_i^{-1} = |c_2|^2 > 0 \), it follows that \( c_2 c_i^{-1} > 0 \) and thus (3.7) is positive. Therefore \( \langle \cdot, \cdot \rangle \) is a positive definite bilinear form. Moreover, \( \langle \tau(X)w_i, w_j \rangle = \langle w_i, \tau(X^*) w_j \rangle \) for all \( X \in B \). This follows from a straightforward verification on the generators of \( B \). \[\square\]
Remark 3.8. Let $\kappa \in \mathbb{R}^+$ and $n \in \mathbb{N}$. Let $(w_i)_{i=0}^n$ be the orthogonal basis for $W^{(n)}$ as in Corollary 3.3. We define an orthonormal basis $(\tilde{w}_i)_{i=0}^n$ by $\tilde{w}_i = w_i/||w_i||$. The actions of $C_1$, $C_2$ and $K$ on the orthonormal basis are the same. For $B_1$ and $B_2$ we have

$$
\tau_{(\kappa,n)}(B_1)\tilde{w}_i = -c_1\kappa^{-1}q^{-2i-n+1}\sqrt{(1-q^{2i})(1-q^{2n-2i+2})(q+c_2c_1^{-1}\kappa^2q^{2n+2i})}\tilde{w}_{i-1},$

$$
\tau_{(\kappa,n)}(B_2)\tilde{w}_i = q^{i-n+1}\sqrt{(1-q^{2i+2})(1-q^{2n-2i})(q+c_2c_1^{-1}\kappa^2q^{2n+2i+2})}\tilde{w}_{i+1}.
$$

4. The branching rule

In this section we prove the main theorem of the paper. We fix a coideal subalgebra $\mathcal{B}$ and show that any finite-dimensional representation of $\mathcal{U}_q(\mathfrak{su}(3))$ restricted to $\mathcal{B}$ decomposes multiplicity free as finite-dimensional representations of $\mathcal{B}$ and we characterise the $\mathcal{B}$-representations that occur in this decomposition. In case $\mathcal{B}$ is $\ast$-invariant, every finite-dimensional irreducible representation of $\mathcal{U}_q(\mathfrak{su}(3))$ restricted to $\mathcal{B}$ obviously decomposes into finite-dimensional irreducible representations. This fact is also noted by Letzter [14, Theorem 3.3].

Theorem 4.1. Let $\lambda \in P^+$ such that $\lambda = \lambda_1\varpi_1 + \lambda_2\varpi_2$ and fix the finite-dimensional irreducible representation $\pi_\lambda$ of $\mathcal{U}_q(\mathfrak{su}(3))$ on the vector space $V_\lambda$. Let $\mathcal{B}$ be a coideal subalgebra with $c_2c_1^{-1} \notin q^{2\lambda_1+2\lambda_2+1-N}$. The representation $\pi_\lambda$ restricted to $\mathcal{B}$ decomposes multiplicity free into irreducible representations;

$$
\pi_\lambda|_\mathcal{B} \simeq \bigoplus_{(\kappa,n)} \tau_{(\kappa,n)}, \quad V_\lambda = \bigoplus_{(\kappa,n)} W_{(\kappa,n)},
$$

where the sum is taken over $(\kappa,n) = (q^{\lambda_1+\lambda_2}, i + x)$, with $0 \leq i \leq \lambda_1$ and $0 \leq x \leq \lambda_2$.

The proof of Theorem 4.1 will be carried out in the next subsections. If $(\tau_{(\kappa,n)}, W_{(\kappa,n)})$ is a representation of $\mathcal{B}$ that occurs in the representation $\pi_\lambda$ upon restriction to $\mathcal{B}$ then a highest weight vector $w_{(\mu,n)}^{(\nu)}$ for $\tau_{(\kappa,n)}$ is completely determined by the highest weight $\kappa$ and the eigenvalue $\eta_1$, see Remark 3.5. Hence, highest weight vectors for $\mathcal{B}$-representations in $V_\lambda$ are the eigenvectors of $\pi_\lambda(C_1)$ that belong to the kernel of $\pi_\lambda(B_1)$. In Subsection 4.1 we determine the kernel of $\pi_\lambda(B_1)$.

Remark 4.2. Observe that the Serre relations for $\mathcal{B}$ imply that the kernel of $\pi_\lambda(B_1)$ is invariant under the action of $B_1B_2$ and thus under the action of $C_1$.

In Subsection 4.2 we diagonalize the restriction of $\pi_\lambda(C_1)$ to $\ker(\pi_\lambda(B_1))$. In most of the proofs we identify $\pi_\lambda(X)$, $X \in \mathcal{U}_q(\mathfrak{su}(3))$, with $X$.

Remark 4.3. The restriction on $c_1$ and $c_2$ in Theorem 4.1 is assumed in order to ensure the complete reducibility of $\pi_\lambda$ upon restriction to $\mathcal{B}$. This is not always true for the excluded values of $c_1$ and $c_2$. For example let $\lambda = \varpi_1$. Then $V_\lambda$ is a three dimensional vector space. Mudrov’s basis in Theorem 2.3 is given by

$$
\mathcal{B} = \{v_\lambda, F_1v_\lambda, F_2F_1v_\lambda\}.
$$
In this basis, the operator $C_1$ is given by the $3 \times 3$ matrix

$$C_1 = \begin{pmatrix}
\frac{c_1 q^2 + c_1 - q c_2}{q(q^2 - 1)} & 0 & -c_1 c_2 \\
0 & \frac{x_1 q^3 + c_1 - q c_2}{q^2 - 1} & 0 \\
-q & 0 & \frac{c_1 q^4 + c_1 - q^2 c_2}{q(q^2 - 1)}
\end{pmatrix}. $$

The eigenvectors of $C_1$ are (multiples of) the vectors

$$\rho_1 = \begin{pmatrix} c_1 \\ 0 \\ 1 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} -c_2/q \\ 0 \\ 1 \end{pmatrix}. $$

If $c_1 \neq -c_2/q$, then $V_\lambda$ decomposes as a sum of a two-dimensional and a one-dimensional irreducible representations of $W$:

$$V_\lambda = W_{(q,0)} \oplus W_{(q-2,1)}, $$

where $W_{(q,0)} = \langle \{\rho_1\} \rangle$ and $W_{(q-2,1)} = \langle \{\rho_2, \rho_3\} \rangle$. Moreover, the highest weight vectors of $W_{(q,0)}$ and $W_{(q-2,1)}$ are $\rho_1$ and $\rho_2$ respectively. If we let $c_1 = -c_2/q$ then the matrix $C_1$ degenerates into a non-diagonalizable matrix. The only eigenvectors are the multiples of $\rho_2$ and $\rho_3$ and therefore, although $W_{(q-2,1)}$ is a $B$-invariant subspace of $V_\lambda$, there is no one-dimensional $B$-invariant subspace in $V_\lambda$.

4.1. **The kernel of $B_1$.** The goal of this subsection is to describe the structure of the kernel of $\pi_\lambda(B_1)$ by introducing a particular basis. For each $i = 0, \ldots, \lambda_1$, we introduce the following subspaces of $V_\lambda$:

$$U_i = \langle B_i \rangle, \quad B_i = \{F_2^k F_3^l F_1^{k+l} v_\lambda : 0 \leq l \leq \lambda_2, 0 \leq k \leq \lambda_1 - i\}. $$

It follows from weight space considerations, that $F_1, E_2 : U_i \to U_{i+1}$ and $F_2, E_1 : U_{i+1} \to U_i$ so that $B_1 : U_i \to U_{i+1}$ and $B_2 : U_{i+1} \to U_i$. This is shown in Figure [3] for the highest weight $\lambda = 2\varpi_1 + 5\varpi_2$.

**Remark 4.4.** For each $i = 0, \ldots, \lambda_1$, the basis $B_i$ consists on $\lambda_1 - i + 1$ layers of $\lambda_2 + 1$ vectors. More precisely, for $k = 0, \ldots, \lambda_1 - i$, the $k$-th layer is given by the vectors

$$F_2^k F_3^l F_1^{k+l} v_\lambda, \quad l = 0, \ldots, \lambda_2. $$

This structure is indicated in the Figure [2] for the representation $\lambda = 2\varpi_1 + 5\varpi_2$. The layers appear as circled numbers.

**Remark 4.5.** The dimension of $U_i$ is $(\lambda_2 + 1)(\lambda_1 - i + 1)$. Therefore, the dimension of $\ker(B_1)|_{U_i}$ is, at least, $\lambda_2 + 1$. In particular, $U_{\lambda_1} \subset \ker(B_1)$.

**Proposition 4.6.** The kernel of $\pi_\lambda(B_1)|_{U_i}$ has dimension $\lambda_2 + 1$. Moreover, a basis of $\ker \pi_\lambda(B_1)|_{U_i}$ is given by $(u_n^i)_{n=0}^{\lambda_1-i}$, where

$$u_n^i = \sum_{k=0}^{\lambda_1-i} \sum_{l=0}^{\lambda_2} \gamma_{k,l}^n F_2^k F_3^l F_1^{k+l} v_\lambda, $$

and the coefficients $\gamma_{k,l}^n$ are given by the recurrence relation

$$a_k(l, k + i) \gamma_{k,l}^n + b_{k+1}(l - 1, k + i + 1) \gamma_{k+1,l-1}^n - c_1 q^{l+i+k+1-\lambda_1} \eta_{k+1}(l, k + i + 1) \gamma_{k+1,l}^n = 0, $$

where $a_k(l, k + i)$, $b_{k+1}(l - 1, k + i + 1)$, and $c_1$ are constants dependent on $\lambda$.
for $k = 1, \ldots, \lambda_1 - i - 1$, $l = 0, \ldots, \lambda_2$, with initial values $\gamma_{\lambda_1-i,l}^n = \delta_{n,l}$.

Proof. Let $u = \sum_{k=0}^{\lambda_1-i} \sum_{l=0}^{\lambda_2} \gamma_{k,l} f_2^k \hat{f}_3^l f_1^{k+i} v_\lambda$ be a vector in the kernel of $B_1$. Then
\[
B_1 u = \sum_{k=0}^{\lambda_1-i} \sum_{l=0}^{\lambda_2} \gamma_{k,l} (f_1 f_2^k \hat{f}_3^l f_1^{k+i} v_\lambda - c_1 E_1^{-1} f_2^k \hat{f}_3^l f_1^{k+i} v_\lambda)
\]
\[
= \sum_{k=0}^{\lambda_1-i} \sum_{l=0}^{\lambda_2} \gamma_{k,l} (a_k(l, k+i) f_2^k \hat{f}_3^l f_1^{k+i+1} v_\lambda + b_k(l, k+i) f_2^{k-1} \hat{f}_3^{l+1} f_1^{k+i} v_\lambda
\]
\[- c_1 q^{l+2i+k-\lambda_1} \eta_k(l, k+i) f_2^{k-1} \hat{f}_3^{l+1} f_1^{k+i} v_\lambda
\]
\[
= \sum_{k=0}^{\lambda_1-i} \sum_{l=0}^{\lambda_2} (a_k(l, k+i) \gamma_{k,l} + b_{k+1}(l-1, k+i+1) \gamma_{k+1,l-1}
\]
\[- c_1 q^{l+2i+k+1-\lambda_1} \eta_{k+1}(l, k+i+1) f_2^k \hat{f}_3^l f_1^{k+i+1} v_\lambda.
\]
Since the elements $f_2^k \hat{f}_3^l f_1^{k+i+1}$, $0 \leq k \leq \lambda_1 - i$, $0 \leq l \leq \lambda_2$, are linearly independent it follows that the coefficients $\gamma_{k,l}$ satisfy the following recurrence relation.
\[
(4.2) a_k(l, k+i) \gamma_{k,l} + b_{k+1}(l-1, k+i+1) \gamma_{k+1,l-1} - c_1 q^{l+2i+k+1-\lambda_1} \eta_{k+1}(l, k+i+1) \gamma_{k+1,l} = 0.
\]
For each \( n = 0, 1, \ldots, \lambda_2 \), if we set \( \gamma_{\lambda_1-i}^n = \delta_{n,i} \), then (4.2) determines uniquely a vector \( u_n \) in the kernel of \( B_1 \). The vectors \( u_n \) are clearly linearly independent and span the kernel of \( B_1 \) restricted to \( U_i \). This completes the proof of the proposition. \( \square \)

**Remark 4.7.** According to the layer structure of \( B_i \) described in Remark 4.4, the vector \( u_n^i \) has a single non-zero contribution from the vectors of the upper layer, namely from \( F_2^{\lambda_1-i} F_3^n F_1^{\lambda_1} \), and two contributions from the one but upper layer. Therefore, we have

\[
(4.3) \quad u_n^i = F_2^{\lambda_1-i} F_3^n F_1^{\lambda_1} v_\lambda + \gamma_{\lambda_1-i-1,n}^n F_2^{\lambda_1-i-1} F_3^n F_1^{\lambda_1} v_\lambda + \gamma_{\lambda_1-i-1,n+1}^n F_2^{\lambda_1-i-1} F_3^{n-1} F_1^{\lambda_1} v_\lambda + \sum_{k=0}^{\lambda_1-i-2} \sum_{l=0}^{\lambda_2} \gamma_{k,l}^n F_2^k F_3^l F_1^{i+k} v_\lambda.
\]

The coefficients \( \gamma_{\lambda_1-i-1,n}^n \) and \( \gamma_{\lambda_1-i-1,n+1}^n \) corresponding to the vectors of the one but last layer are given by

\[
(4.4) \quad \gamma_{\lambda_1-i-1,n}^n = \frac{c_1 q^{n+i} (q^{\lambda_1-i} - q^{-\lambda_1+i})(q^{\lambda_2+\lambda_1-n} - q^{-\lambda_2-\lambda_1+n})}{(q - q^{-1})^2},
\]

\[
\gamma_{\lambda_1-i-1,n+1}^n = -\frac{(q^{\lambda_1-i} - q^{-\lambda_1+i})(q^{\lambda_2+\lambda_1-n-1} - q^{-\lambda_2-\lambda_1+n+1})}{(q^{\lambda_2+\lambda_1+1-n} - q^{-\lambda_2-\lambda_1-1+n})(q^{\lambda_2+1-i-n} - q^{-\lambda_2-i+n})}.
\]

The structure of the vectors \( u_n^i \) for \( U_{\lambda_1-2} \) is depicted in Figure 3.

**Remark 4.8.** The basis \( \{ u_n^i \}_n \) of the kernel of \( \pi_\lambda(B_1) \) is not an orthogonal basis. In fact, it follows from Remark 4.7 that

\[
u_{0i}^{\lambda_1-1} = F_2^{\lambda_1-1} F_1^{\lambda_1} v_\lambda + \gamma_{\lambda_1-2,0}^0 F_2^{\lambda_1-2} F_1^{\lambda_1} v_\lambda,
\]

\[
u_{1i}^{\lambda_1-1} = F_2^{\lambda_1-1} F_3 F_1^{\lambda_1} v_\lambda + \gamma_{\lambda_1-2,1}^1 F_2^{\lambda_1-2} F_3 F_1^{\lambda_1} v_\lambda + \gamma_{\lambda_1-2,2}^1 F_2^{\lambda_1-2} F_1^{\lambda_1} v_\lambda,
\]

and therefore

\[
\langle u_{0i}^{\lambda_1-1}, u_{1i}^{\lambda_1-1} \rangle = \gamma_{\lambda_1-2,0}^0 \gamma_{\lambda_1-2,2}^1 H_{\lambda_1-2,0,\lambda_1-1} \neq 0,
\]

using the explicit expressions (4.4).
The lemma is a direct consequence of Proposition 2.4.

4.2. The action of \( C_1 \). In Remark 4.2 we observed that the kernel of \( B_1 \) is stable under the action of \( C_1 \). Furthermore for each \( i = 0, \ldots, \lambda_1 \), \( U_i \) is stable under \( C_1 \). The goal of this subsection is to compute the action of \( C_1 \) in the basis of \( \pi_\lambda(B_1) \) given in Proposition 4.6.

Lemma 4.9. In the basis \( \mathcal{B} \) of Theorem 2.3 we have

\[
\begin{align*}
F_1 F_2 F_3^{k+i} F_1 F_3^{k+i} v_\lambda &= a_{k+1}(l, k + i) F_2^{k+1} F_3^{k+i+1} F_1 v_\lambda + b_{k+1}(l, k + i) F_2^k F_3^{k+i+1} F_1 v_\lambda, \\
E_2 F_2 F_3^{k+i} F_1 v_\lambda &= \eta_{k+1}(l, k + i) F_2^k F_3^{k+i} F_1 v_\lambda, \\
F_1 E_1 F_2 F_3^{k+i} F_1 v_\lambda &= \alpha_k(l, k + i) a_k(l, k + i - 1) F_2^k F_3^{k+i} F_1 v_\lambda \\
&+ \alpha_k(l, k + i) b_k(l, k + i - 1) F_2^{k-1} F_3^{k+i-1} F_1 v_\lambda \\
&+ \beta_k(l, k + i) a_{k+1}(l, k + i) F_2^k F_3^{k+i-1} F_1 v_\lambda \\
&+ \beta_k(l, k + i) b_{k+1}(l, k + i) F_2^k F_3^{k+i} F_1 v_\lambda, \\
E_2 E_1 F_2 F_3^{k+i} F_1 v_\lambda &= \alpha_k(l, k + i) \eta_{k}(l, k + i - 1) F_2^{k-1} F_3^{k+i-1} F_1 v_\lambda \\
&+ \beta_k(l, k + i) \eta_{k+1}(l, k + i) F_2^k F_3^{k+i} F_1 v_\lambda, \\
K F_2^{\lambda_1-i} F_3^{\lambda_1} F_1 v_\lambda &= q^{\lambda_1-\lambda_2-3i} F_2^{\lambda_1-i} F_3^{\lambda_1} F_1 v_\lambda, \\
K^{-1} F_2^{\lambda_1-i} F_3^{\lambda_1} F_1 v_\lambda &= q^{\lambda_2-\lambda_1+3i} F_2^{\lambda_2-i} F_3^{\lambda_1} F_1 v_\lambda.
\end{align*}
\]

Proof. The lemma is a direct consequence of Proposition 2.4. \( \square \)

Since \( K \) acts as a multiple of the identity on each \( U_i \), it suffices to determine the action of \( B_1 B_2 \) on \( U_i \).

Lemma 4.10. For \( i = 0, \ldots, \lambda_1 \), in the basis \( (u_n^i)_n \) of \( \ker(B_1) \), we have

\[
B_1 B_2 u_n^i = A(n) u_{n+1}^i + B(n) u_n^i + C(n) u_{n-1}^i, \quad n = 0, \ldots, \lambda_2,
\]

where

\[
\begin{align*}
A(n) &= q^{\lambda_2+1-n}(1 - q^2)(1 - q^{2\lambda_1+2\lambda_2-2n}) \\
&\quad (1 - q^{2\lambda_2+2\lambda_1-2n}) (1 - q^{2\lambda_2+2n-2n}), \\
B(n) &= -c_1 q^{2n+1-\lambda_1-\lambda_2} (1 - q^{2\lambda_2-2n+2}) + c_2 q^{\lambda_1-\lambda_2+2n-i+1}(1 - q^{-2n-2i}), \\
C(n) &= c_1 c_2 q^{3n-3\lambda_2-i-2}(1 - q^{2n}) (1 - q^{2\lambda_2-2n+2}) (1 - q^{2\lambda_1+2\lambda_2-2n+4}) (1 - q^{2\lambda_2+2n-2n}) \\
&\quad (1 - q^{2\lambda_2+2\lambda_1+2-2n}).
\end{align*}
\]
Proof. Since \( U_i \) is stable under \( B_1B_2 \) and \((u^n_i)_n \) is a basis of \( U_i \), we have

\[
B_1B_2 u^n_i = \sum_{j=0}^{\lambda_2} \nu_j u^n_j,
\]

for certain coefficients \( \nu_j \). Since \( \mathcal{B}_i \) is an orthogonal basis and \( u^n_i \) has a single contribution from the vectors in the upper layer of \( \mathcal{B}_i \), see Remark 4.7, we obtain that

\[
\langle B_1B_2 u^n_i, F_2^{\lambda_i - i} \hat{F}_3^{\bar{s}} F_1^{\lambda_1} v_\lambda \rangle = \sum_{j=0}^{\lambda_2} \nu_j \langle u^n_j, F_2^{\lambda_i - i} \hat{F}_3^{\bar{s}} F_1^{\lambda_1} v_\lambda \rangle = \nu_s H^2_{\lambda_i - i, s, \lambda_1}.
\]

On the other hand, from (3.1) we have

\[
(4.5) \quad B_1B_2 F_2^k \hat{F}_3^{i} F_1^{k+i} v_\lambda = F_1 F_2 F_2^k \hat{F}_3^{i} F_1^{k+i} v_\lambda - c_1 q^{l+k+2i-\lambda_1} E_2 F_2 F_2^k \hat{F}_3^{i} F_1^{k+i} v_\lambda - c_2 q^{k+l-i-\lambda_2} F_1 F_1 F_2^k \hat{F}_3^{i} F_1^{k+i} v_\lambda + c_1 c_2 q^{2l+2k+i-\lambda_1-\lambda_2 - 2} E_2 E_2 F_2 F_2^k \hat{F}_3^{i} F_1^{k+i} v_\lambda.
\]

Applying Lemma 4.9 to (4.5), we verify that the action of \( B_1B_2 \) on the vector of the \( k \)-th layer \( F_2^k \hat{F}_3^{i} F_1^{k+i} v_\lambda \) has contributions from the \( (k-1) \)-th, \( k \)-th and \( (k+1) \)-th layer. Hence, Remark 4.7 implies

\[
(4.6) \quad \langle B_1B_2 u^n_i, F_2^{\lambda_i - i} \hat{F}_3^{\bar{s}} F_1^{\lambda_1} v_\lambda \rangle = \langle B_1B_2 F_2^{\lambda_i - i} \hat{F}_3^{\bar{s}} F_1^{\lambda_1} v_\lambda, F_2^{\lambda_i - i} \hat{F}_3^{\bar{s}} F_1^{\lambda_1} v_\lambda \rangle + \gamma_{\lambda_1 - i, n} \langle B_1B_2 F_2^{\lambda_i - i-1} \hat{F}_3^{\bar{s}} F_1^{\lambda_1 - 1} v_\lambda, F_2^{\lambda_i - i} \hat{F}_3^{\bar{s}} F_1^{\lambda_1} v_\lambda \rangle + \gamma_{\lambda_1 - i, n+1} \langle B_1B_2 F_2^{\lambda_i - i-1} \hat{F}_3^{\bar{s}} F_1^{\lambda_1 - 1} v_\lambda, F_2^{\lambda_i - i} \hat{F}_3^{\bar{s}} F_1^{\lambda_1} v_\lambda \rangle.
\]

From Lemma 4.9 we obtain that (4.6) is zero unless \( s = n - 1, n, n + 1 \). Moreover, we have

\[
\langle B_1B_2 u^n_i, F_2^{\lambda_i - i} \hat{F}_3^{\bar{s}} F_1^{\lambda_1} v_\lambda \rangle = [b_{\lambda_i - i+1}(n, \lambda_1) + \gamma_{\lambda_i - i-1, n+1} a_{\lambda_i - i}(n + 1, \lambda_1 - 1)] H^2_{\lambda_i - i, n+1, \lambda_1},
\]

\[
\langle B_1B_2 u^n_i, F_2^{\lambda_i - i} \hat{F}_3^{\bar{s}} F_1^{\lambda_1} v_\lambda \rangle = [-c_1 q^{n+i-1} \eta_{\lambda_i - i+1}(l, \lambda_1) - c_2 q^{\lambda_i + n-2i-\lambda_2} a_{\lambda_i - i}(n, \lambda_1) a_{\lambda_i - i}(n, \lambda_1 - 1) - c_2 q^{\lambda_i - 2i + n - \lambda_2} \beta_{\lambda_i - i-1}(n, \lambda_1) b_{\lambda_i - i+1}(n - 1, \lambda_1) + \gamma_{\lambda_i - i-1, n} \eta_{\lambda_i - i}(n, \lambda_1 - 1) - c_2 q^{\lambda_i + n - 2i - \lambda_2} \alpha_{\lambda_i - i}(n, \lambda_1) a_{\lambda_i - i}(n, \lambda_1 - 1)] H^2_{\lambda_i - i, n+1, \lambda_1},
\]

\[
\langle B_1B_2 u^n_i, F_2^{\lambda_i - i} \hat{F}_3^{\bar{s}} F_1^{\lambda_1} v_\lambda \rangle = [c_1 q^{\lambda_i - 2i + 2n - i - 2} \beta_{\lambda_i - i-1}(n, \lambda_1) \eta_{\lambda_i - i+1}(n - 1, \lambda_1) - c_2 q^{\lambda_i - 2i + n - 2i - 1} \gamma_{\lambda_i - i-1, n} b_{\lambda_i - i-1}(n, \lambda_1 - 1) a_{\lambda_i - i}(n - 1, \lambda_1 - 1) - c_2 q^{\lambda_i - 2i + n - 2i - 1} \gamma_{\lambda_i - i-1, n} \beta_{\lambda_i - i-1}(n, \lambda_1 - 1) a_{\lambda_i - i}(n - 1, \lambda_1 - 1)] H^2_{\lambda_i - i, n+1, \lambda_1}.
\]

Now the lemma follows from Propositions 2.3 and (4.3). \( \Box \)

Lemma 4.11. For \( i = 0, \ldots, \lambda_1 \), in the basis \((u^n_i)_n \) of \( \ker(B_1) \), we have

\[
C_1 u^n_i = A(n) u^n_{i+1} + (B(n) + D) u^n_i + C(n) u^n_{i-1}, \quad D = -c_2 q^{\lambda_1 - 2i - 3i q - q^{-1} + c_1 q^{\lambda_2 - \lambda_1 + 3i q - q^{-1}}.
\]

Proof. Lemma 4.10 (3.3) and \( K \) acting as a multiple of the identity give the result. \( \Box \)

We are now ready to find the eigenvectors of \( C_1 \) restricted to \( \ker(B_1)|_{U_i} \). We will describe these eigenvectors as a linear combination of the vectors \( u^n_i \) with explicit coefficients given
in terms of dual $q$-Krawtchouk polynomials. For $N \in \mathbb{N}$ and $n = 0,1,\ldots,N$, the dual $q$-Krawtchouk polynomials are given explicitly by

$$K_n(\lambda(x); c, N|q) = \frac{(q^{x-N}; q)_n}{(q^{-N}; q)_n q^{n^2}} 2\phi_1 \left( \frac{q^{-n}, q^{-x}}{q^{x-N-n+1}} \right),$$

where $\lambda(x) = q^{-x} + cq^{x-N}$, see [3, (3.17.1)]. We follow the standard notation of [4] for basic hypergeometric series. The polynomials

$$(4.7) \quad r_l(\lambda(x)) = (q^{-2N}; q)_l K_l(\lambda(x); c, N|q^2),$$

satisfy the three term recurrence relation

$$(4.8) \quad x r_l(x) = r_{l+1}(x) + (1 + c)q^{2l-2N} r_l(x) + c q^{-2N}(1 - q^{2l})(1 - q^{2l-2N-2}) r_{l-1}(x).$$

**Proposition 4.12.** For $i = 0, \ldots, \lambda_1$, the set $\{\psi_x^{i\lambda_2}\}_{x=0}^{\lambda_2}$ where

$$\psi_x^i = \sum_{l=0}^{\lambda_2} c_l^i q^{-l(\lambda_1+2)+(l-1)/2} \frac{q^{-2l\lambda_2}, q^{-2l-2\lambda_2}; q^2)_l}{(q^{-2l\lambda_2-2\lambda_1}, q^{-2l-2\lambda_2}; q^2)_l} K_l(\lambda(x), -c^{-1} c_2 q^{2\lambda_1-2l+1}, \lambda_2, q^2) u^i_l,$$

is a basis of eigenvectors of $C_1$ restricted to $\ker(B_1)|_{U_i}$. The eigenvalue of $\psi_x^i$ is

$$\eta_i = \frac{c_1 \kappa^{-1} q(1 + q^{-2n-2}) - c_2 \kappa q^{2n}}{q - q^{-1}},$$

for $\kappa = q^{\lambda_1-3i-\lambda_2}$ and $n = x + i$.

**Remark 4.13.** As we pointed out in Remark 4.8, the basis $(u^i_n)_n$ is not orthogonal. Still the operator $C_1$ acts tridiagonally. Moreover, if $B$ is $*$-invariant then the basis $\{\psi_x^{i\lambda_2}\}_{x=0}^{\lambda_2}$ in Proposition 4.12 is orthogonal although, because of the non-orthogonality of $(u^i_n)_n$, this does not follow directly from the orthogonality of the dual $q$-Krawtchouk polynomials.

**Proof.** Assume there exist polynomials $p_l(x)$ such that $v = \sum_{l=0}^{\lambda_2} p_l(x) u^i_l$ is an eigenvector of $C_1$ with eigenvalue $\eta_1$, i.e. $C_1 v = \eta_1 v$. From Lemma 4.11 we have

$$C_1 v = \sum_{l=0}^{\lambda_2} p_l(x)(A(l) u^i_{l+1} + (B(l) + D) u^i_{l} + C(l) u^i_{l-1}) = \sum_{l=0}^{\lambda_2} \eta_l p_l(x) u^i_l.$$

Since $(u^i_l)_l$ is a basis of $\ker(B_1)|_{U_i}$ the vectors $u^i_l$ are linearly independent and hence the polynomials $p_l$ satisfy the following three term recurrence relation

$$\eta_l p_l(x) = C(l+1) p_{l+1}(x) + (B(l) + D) p_l(x) + A(l-1) p_{l-1}(x).$$

If $k_l$ is the leading coefficient of $p_l$, then $P_l = k_l^{-1} p_l$ is a sequence of monic polynomials satisfying the recurrence relation

$$(4.9) \quad \eta_l P_l(x) = P_{l+1}(x) + (B(l) + D) P_l(x) + C(l) A(l-1) P_{l-1}(x),$$

where

$$B(l) + D = -\frac{c_1 q^{2l+1+i-\lambda_1-\lambda_2}(1 - c^{-1}_1 c_2 q^{2\lambda_1-2l+1})}{(1 - q^2)} = \frac{c_1 q^{3l-i+\lambda_1+\lambda_2+2}}{(1 - q^2)},$$

$$C(l) A(l-1) = -\frac{c_1 c_2 q(1 - q^2)(1 - q^{2l-2\lambda_2-2})}{(1 - q^2)^2},$$

and

$$\eta_l = \frac{c_1 \kappa^{-1} q(1 + q^{-2n-2}) - c_2 \kappa q^{2n}}{q - q^{-1}}.$$
Let $W$ be a $\kappa$-dimensional vector space. We will identify the polynomials $P_l$ with the dual $q$-Krawtchouk polynomials. If we let
\[ c = -c_1^{-1} c_2 q^{2\lambda_1 - 2i + 1}, \quad N = \lambda_2, \]
the recurrence relation (4.3) is given by
\[ x r_i(x) = r_{i+1}(x) + (1 + c_1^{-1} c_2 q^{2\lambda_1 - 2i + 1}) q^{2l - 2\lambda_2} r_i(x) + c_1^{-1} c_2 q^{2\lambda_1 - 2\lambda_2 - 2i + 1}(1 - q^2)(1 - q^{2l - 2\lambda_2 - 2}) r_{i-1}(x). \]

If we let $\tilde{r}_i(x) = a^{-i} r_i(ax)$ with $a = -c_1^{-1} q^{\lambda_1 - \lambda_2 - i}(1 - q^2)$, by a straightforward computation we obtain
\[ (4.10) \quad (x - \frac{c_1 q^{3i - \lambda_1 + \lambda_2 + 2}}{1 - q^2}) \tilde{r}_i(x) = \tilde{r}_{i+1}(x) + (B(l) + D) \tilde{r}_i(x) + C(l) A(l - 1) \tilde{r}_{i-1}(x). \]

If we evaluate (4.10) in $\lambda(x)a^{-1}$, the eigenvalue is given by
\[ \frac{\lambda(x)}{a} - \frac{c_1 q^{3i - \lambda_1 + \lambda_2 + 2}}{1 - q^2} = \frac{c_1 q^{3i - \lambda_1 + \lambda_2 + 2} (1 + q^{-2x - 2i - 2}) + c_2 q^{\lambda_1 - \lambda_2 - i + 2x}}{q - q^{-1}}. \]

Therefore the polynomials $P_l(x) = \tilde{r}(\lambda(x)a^{-1}) = a^{-i} r_i(\lambda(x))$ satisfy the recurrence (4.9) with eigenvalue
\[ \eta_l = \frac{c_1 \kappa^{-1} q(1 + q^{-2n - 2}) - c_2 \kappa q^{2n}}{q - q^{-1}}, \]
with $\kappa = q^{\lambda_1 - 3i - \lambda_2}$ and $n = x + i$, for $x = 0, \ldots, \lambda_2$. Finally, $p_l(x) = k_l a^{-1} r_l(\lambda(x))$. The explicit expression of $p_l$ follows from (4.7) and Lemma 4.10.

\[ \square \]

**Proof of Theorem 4.12** From Proposition 4.12 we obtain vectors $\psi_x^i$ for $i = 0, \ldots, \lambda_1$, $x = 0, \ldots, \lambda_2$ such that
\[ \pi_\lambda(B_1) \psi_x^i = 0, \quad \text{and} \quad C_1 \psi_x^i = \frac{c_1 \kappa^{-1} q(1 + q^{-2n - 2}) - c_2 \kappa q^{2n}}{q - q^{-1}} \psi_x^i = \eta_1 \psi_x^i, \]
where $\kappa = q^{\lambda_1 - 3i - \lambda_2}$ and $n = x + i$, so that $\psi_x^i$ is a highest weight vector. It follows from Corollary 4.6 that the highest weight vector $\psi_x^i$ defines an irreducible representation of $B$ of dimension $x + i + 1$
\[ W_{q^{\lambda_1 - \lambda_2 - 3i}, x+i} = \{ \psi_x^i, \pi_\lambda(B_2) \psi_x^i, \pi_\lambda(B_2)^2 \psi_x^i, \ldots, \pi_\lambda(B_2)^{x+i} \psi_x^i \}. \]

Let $W = \oplus_{(\kappa,n)} W_{(\kappa,n)}$ where the sum is taken over $(\kappa,n) = (q^{\lambda_1 - 3i - \lambda_2}, x + i)$ for $i = 0, \ldots, \lambda_1$, $x = 0, \ldots, \lambda_2$. We have that $W \subset V_\lambda$ and
\[ \dim W = \sum_{i,x} \dim W_{q^{\lambda_1 - \lambda_2 - 3i}, x+i} = \frac{1}{2} (\lambda_1 + 1)(\lambda_2 + 1)(\lambda_1 + \lambda_2 + 2) = \dim V_\lambda. \]

Therefore $W = V_\lambda$ and this completes the proof of the theorem. \[ \square \]
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Appendix A. Proof of Theorem 2.3

Lemma A.1. The following relations hold in $U_q(\mathfrak{su}(3))$:

\begin{itemize}
  \item[(i)] $F_2 \hat{F}_3[a] = \hat{F}_3[a + 1] F_2,$
  \item[(ii)] $E_1 \hat{F}_3[a] = \hat{F}_3[a + 1] E_1 + F_2 \frac{q^{a+1} K_1 K_2 - q^{-a-1}(K_1 K_2)^{-1}}{(q - q^{-1})},$
  \item[(iii)] $F_2 F_3 = q F_3 F_2,$
  \item[(iv)] $(\hat{F}_3[a])^* = q E_3[a] (K_1 K_2)^{-1},$ $F_3^* = q E_3(K_1 K_2)^{-1},$
  \item[(v)] $\hat{F}_3 = F_3^* \frac{q K_2 q^{-1} K_1^{-1}}{q - q^{-1}} + q F_2 F_1 K_2,$
  \item[(vi)] $E_1 \hat{F}_3 = F_3 E_1 + F_2 K_1.$
\end{itemize}

Proof. Straightforward verifications using (2.1) and (2.3). □

Corollary A.2. For $l \in \mathbb{N}$ and $a \in \mathbb{R}$ we have

$$E_1 (\hat{F}_3[a])^l = (\hat{F}_3[a + 1])^l E_1 + \frac{q^l - q^{-l}}{q - q^{-1}} F_2 (\hat{F}_3[a])^{l-1} \frac{(q^{a+2-l} K_1 K_2 - q^{-a-2+l}(K_1 K_2)^{-1})}{(q - q^{-1})}.$$

Proof. By induction on $l$ using Lemma [A.1][i] and [i][i]. □

Proof of Theorem 2.3 By the PBW-theorem, $F_2^k \hat{F}_3 F_1^m v_\lambda$ for $k, l, m \in \mathbb{N}$ spans $V_\lambda$. By Proposition 2.4

\begin{equation}
K_1 F_2^k \hat{F}_3 F_1^m v_\lambda = q^{\lambda_1 + k - l - 2m} F_2^k \hat{F}_3 F_1^m v_\lambda,
\end{equation}

\begin{equation}
K_2 F_2^k \hat{F}_3 F_1^m v_\lambda = q^{\lambda_2 - 2k - l + m} F_2^k \hat{F}_3 F_1^m v_\lambda.
\end{equation}

Since $K_i$, $i = 1, 2$, are self-adjoint, we find that $\langle F_2^k \hat{F}_3 F_1^m v_\lambda, F_2^{k'} \hat{F}_3 F_1^{m'} v_\lambda \rangle = 0$ in case $k - l - 2m \neq k' - l' - 2m'$ or $-2k - l + m \neq -2k' - l' + m'$. For $k' > k$ we find

\begin{equation}
\langle F_2^k \hat{F}_3 F_1^m v_\lambda, F_2^{k'} \hat{F}_3 F_1^{m'} v_\lambda \rangle = \langle (E_2 K_2^{-1})^{k'} F_2^{k'} \hat{F}_3 F_1^{m'} v_\lambda, \hat{F}_3^{m'} F_1^{m'} v_\lambda \rangle
\end{equation}

\begin{equation}
= q^{k' - k + m} \langle F_2 F_3 E_2 F_2^{k'} \hat{F}_3 F_1^{m'} v_\lambda, \hat{F}_3^{m'} F_1^{m'} v_\lambda \rangle = 0,
\end{equation}

since $E_2^{k'} F_2^{k'} \in U_q(\mathfrak{su}(3))$ $E_2^{k'} \neq k$ for $k, k' \in \mathbb{N}, k' > k$, using also Lemma [2.1][ii] for $a = 0, (2.1)$ and [2.4]. Because of the symmetry between $k$ and $k'$, we see that the inner product [A.2] is 0 for $k \neq k'$. With the above remark, we find

$$\langle F_2^k \hat{F}_3 F_1^m v_\lambda, F_2^{k'} \hat{F}_3 F_1^{m'} v_\lambda \rangle = 0$$

in case $k \neq k'$ or $l \neq l'$ or $m \neq m'$.
So it suffices to calculate the norm of the vectors, and see that this is non-zero precisely for the range mentioned. First, using the case \( k = k' \) of the first part of (A.2) and that \( K_2 \) acts on \( E_{1,2}^k F_3^l F_3^m v_\lambda \) by the scalar \( q^{\lambda_2-l+m} \), we find
\[
\langle F_2^k F_3^l F_3^m v_\lambda, F_2^k F_3^l F_3^m v_\lambda \rangle = q^{k(k+1)-k(\lambda_2-l+m)} \langle E_2^k F_3^l F_3^m v_\lambda, F_3^l F_3^m v_\lambda \rangle.
\]
Now use Lemma 2.2(iii) for \( i = 2 \) and next the commutation relations of Lemma 2.1(ii) and (2.1) to see that the \( \mathcal{U}_q(\mathfrak{su}(3))E_2\)-part of Lemma 2.2(ii) gives zero contribution. Because of the action of \( K_2 \) being diagonal, we find
\[
\langle F_2^k F_3^l F_3^m v_\lambda, F_2^k F_3^l F_3^m v_\lambda \rangle = \frac{(q^2;q^2)_k}{(1-q^2)^k} (q^{-2(\lambda_2-l+m)}; q^2)_k (-1)^k q^{3k} \langle F_3^l F_3^m v_\lambda, F_3^l F_3^m v_\lambda \rangle
\]
Next we write
\[
\langle F_2^k F_3^l F_3^m v_\lambda, F_2^k F_3^l F_3^m v_\lambda \rangle = \langle F_3^l F_3^m v_\lambda, F_3^l F_3^m v_\lambda \rangle = q^{m(m+1)} q^{-m(\lambda_1-l)} \langle E_1^m F_3^l F_3^m v_\lambda, F_3^l F_3^m v_\lambda \rangle
\]
using Lemma 2.1(ii), the \( \ast \)-structure (2.3), (2.1) and (A.1). Following Mudrov [19] [8] we replace \( F_3^l \) on the left by \( F_3^l \). First use Lemma A.1(iii)
\[
\langle E_1^m F_3^l F_3^m v_\lambda, F_3^l F_3^m v_\lambda \rangle = \frac{q^{2+\lambda_2-l+m} q^{-2-\lambda_2+l-m}}{q - q^{-1}} \langle E_1^m F_3^l F_3^m F_3^m v_\lambda, F_3^l F_3^m v_\lambda \rangle
\]
In the second term, move \( F_2 \) to the left using (2.1), and then the other side so that essentially an \( E_2 \) which we can move through, by Lemma 2.1(iii), to the highest weight vector, and hence gives zero. This we can repeat, since \( F_2 \) also \( q \)-commutes with \( F_3 \) by Lemma A.1(iii). This yields
\[
\langle F_3^l F_3^m v_\lambda, F_3^l F_3^m v_\lambda \rangle = (-1)^l q^{2(l+\lambda_2+m)} q^{-l(\lambda_2-l)} (q^{-2\lambda_2} q^2)_l \langle F_3^l F_3^m F_3^m v_\lambda, F_3^l F_3^m v_\lambda \rangle
\]
Using Lemma A.1(iv), and moving \( F_2 \) to the other side, where \( F_2^\ast \) kills \( F_3^l v_\lambda \), we see
\[
\langle E_1^m F_3^l F_3^m v_\lambda, F_3^l F_3^m v_\lambda \rangle = (-1)^m q^{-m(m-2)+m\lambda_1} (q^2; q^2)_m (q^{-2\lambda_2} q^2)_m \langle F_3^l F_3^m v_\lambda, F_3^l F_3^m v_\lambda \rangle
\]
by Lemma 2.2(ii). Assume \( l \geq 1 \), so it remains to calculate
\[
\langle F_3^l v_\lambda, F_3^l v_\lambda \rangle = \langle F_3^l F_3^m v_\lambda, (F_3^m)^\ast F_3^l v_\lambda \rangle = q^1-(\lambda_1+\lambda_2-2l) (F_3^l F_3^m v_\lambda, (E_2 E_1 - E_1 E_2) F_3^l v_\lambda)
\]
where we use Lemma A.1(iv), the diagonal action of \( K_1 \), and the fact that the action of \( E_1 E_2 \) is zero by Lemma 2.1(ii) and (2.4). By Corollary A.2 for \( a = 0 \) and (2.4) we find
\[
E_1 F_3^l v_\lambda = \frac{q^l - q^{-l}}{q - q^{-1}} q^{2\lambda_1+\lambda_2-l} - \frac{q^{-2\lambda_2+2l}}{q - q^{-1}} F_2 F_3^l v_\lambda
\]
and next applying \( E_2 \), using (2.1), (2.4) and Lemma 2.1(iii) we find
\[
E_2 E_1 F_3^l v_\lambda = \frac{q^l - q^{-l}}{q - q^{-1}} q^{2\lambda_1+\lambda_2-l} - \frac{q^{-2\lambda_2+2l}}{q - q^{-1}} q^{2\lambda_2-l+1} - \frac{q^{-2\lambda_2+l}}{q - q^{-1}} F_3^l v_\lambda.
\]
so that
\[
\langle F_3^l v_\lambda, \tilde{F}_3^l v_\lambda \rangle = q^{1-(\lambda_1+\lambda_2-2l)} \frac{q^l - q^{-l}}{q - q^{-1}} \times \frac{q^{2+\lambda_1+\lambda_2-l} - q^{-2-\lambda_1-\lambda_2+l} q^{\lambda_1-\lambda_2+1} - q^{-\lambda_2+l-1}}{q - q^{-1}} \langle F_3^{l-1} v_\lambda, \tilde{F}_3^{l-1} v_\lambda \rangle.
\]

Iterating, since we normalize \(\langle v_\lambda, v_\lambda \rangle = 1\), we find
\[
\langle F_3^l v_\lambda, \tilde{F}_3^l v_\lambda \rangle = q^{l(\lambda_2+7)} q^{-\frac{1}{2}l(l+1)} \frac{(q^2;q^2)_l}{(1-q^2)_l^3} (q^{-2}\lambda_2; q^2)_l(q^{-2(\lambda_1+\lambda_2+1)}; q^2)_l.
\]
Note that this expression is positive for \(0 \leq l \leq \lambda_2\) and equals zero for \(l > \lambda_2\). Collecting all the intermediate results gives the explicit expression for the norm of the basis elements. □

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