Ground-state phase diagram and excitation spectrum of a Bose-Einstein condensate with spin-orbital-angular-momentum coupling

Ke-Ji Chen, Fan Wu, Jianshen Hu, and Lianyi He

Department of Physics and State Key Laboratory of Low-Dimensional Quantum Physics, Tsinghua University, Beijing 100084, China

(Dated: November 5, 2019)

We investigate the ground-state phase diagram and excitation spectrum of an interacting spinor Bose-Einstein condensate with spin-orbital-angular-momentum (SOAM) coupling realized in recent experiments by introducing atomic Raman transition with a pair of copropagating Laguerre-Gaussian laser beams that carry different orbital angular momenta (OAM) [Chen et al., Phys. Rev. Lett. 121, 113204 (2018) and Zhang et al., Phys. Rev. Lett. 122, 110402 (2019)]. Because of the ground-state degeneracy of the single-particle Hamiltonian at vanishing detuning, several angular-stripe phases, which are superposition of states with different angular quantum numbers, appear in the phase diagram. However, these phases normally exist at small detuning, which makes them hard to be probed in experiments. We show that for a large OAM difference of the laser beams, an asymmetric kind of angular-stripe phase can exist even at large detuning. The excitation spectra in different phases exhibit distinct features: in the angular-stripe phase there exist two gapless bands corresponding to the broken U(1) and rotational symmetries, while in the half-skyrmion phase the gapless band exhibits a roton-like structure. Our predictions of the angular-stripe phases and the low-energy excitations can be examined in recently realized BECs with SOAM coupling.

Spin-orbit coupling (SOC), i.e., coupling between a particle’s spin and orbital motion, gives rise to many fascinating phenomena, such as quantum spin Hall effects [1-4], topological insulators, and topological superconductors [5,6]. Ultracold atoms with the high controllability of degrees of freedom provide an ideal platform to simulate and study those exotic physical effects by creating synthetic gauge potentials [7,9]. A prominent example is the so-called spin-linear-momentum (SLM) coupling, which is achieved by introducing Raman transitions among the internal states of atoms by two counterpropagating laser field [10,12]. A variety of exotic quantum states have been observed in quantum gases with SLM coupling [7,15].

In addition to the SLM coupling, another fundamental type of SOC, the spin-orbital-angular-momentum (SOAM) coupling, was recently realized in spin-1/2 and spin-1 $^{87}$Rb Bose gases by introducing atomic Raman transition with a pair of copropagating Laguerre-Gaussian (LG) optical fields that carry different orbital angular momentum (OAM) [16,17]. In this scheme, an OAM change is imparted to the atoms while the linear momentum change used for conventional SLM coupling is suppressed due to the use of copropagating beams. Different from SLM coupling, SOAM coupling preserves the rotational symmetry and possesses a discrete spectrum. Thus a spinor Bose-Einstein condensate (BEC) subjected to SOAM coupling may exhibit distinct properties compared to SLM coupling. Previous theoretical studies have predicted several exotic phases, such as the vortex-antivortex pair phase, half-skyrmion phase, and angular-stripe phase [18,23].

In this work, we present a systematic theoretical study of the ground-state phase diagram and excitation spectrum of a two-dimensional (2D) spinor BEC with SOAM coupling realized in recent experiments [16,17]. An important feature of SOAM coupling is that it leads to ground-state degeneracy of the single-particle spectrum (Fig. 1). Similar ground-state degeneracy was also found in spinor BEC with SLM coupling, leading to the exotic stripe phase [25,30]. Here we expect that the degeneracy leads to the angular-stripe phase, where the many-body ground state is a superposition of different states with definite angular quantum numbers. However in a realistic system, detuning and Raman coupling are unavoidable. They lift the degeneracy and hence kill the angular-stripe phase in a noninteracting spinor BEC.

The intra- and inter-species interactions are also unavoidable, leading to rich ground-state phases. At small interspecies interaction, there exists a room for the angular-stripe phase (Fig. 2). For the OAM difference $l = 1$ of the LG laser beams used in present experiments [16,17], the largest detuning for the angular-stripe phase is around $\hbar \omega_\perp$ with $\omega_\perp$ being the 2D trapping frequency. It is rather small to be realized in experiments. However, for larger OAM difference $l$, we find a new angular-stripe phase which exists even at large detuning [see Fig. 2(b) for $l = 2$]. This may provide a new possibility to observe the angular-stripe state in future experiments.
The confinement potential is given by beams and couples to the atom spin via the SOAM coupling term eigenvalue equation be obtained straightforwardly by solving the following interactions, the energy spectrum and eigenstates can understand the origin of stripe phase. In the absence of cients of the laser z axis, which couples to the atom spin via the SOAM coupling term \( L_z \sigma_z \). The spatial-dependent coupling of the laser beams is given by \( \Omega(r) = \Omega_R (r/w) |l_1|^2 |l_2|^2 e^{-2r^2/w^2} \) with the Rabi frequency \( \Omega_R \), waist \( w \), and detuning \( \delta \). \( l_1 \) and \( l_2 \) are winding numbers of a pair copropagating LG laser beams and \( l = (l_1 - l_2)/2 > 0 \). Without loss of generality, we consider the case \( l_1 = 0 \) and \( l_2 = -2l \). The external confinement potential is given by \( V_{\text{ext}}(r) = M\omega_r^2 r^2/2 \).

Before discussing the effects of interactions, an analysis of single-particle Hamiltonian will be helpful to understand the origin of stripe phase. In the absence of interactions, the energy spectrum and eigenstates can be obtained straightforwardly by solving the following eigenvalue equation \( \hat{H}_s \psi_s(r) = E_s \psi_s(r) \). Because of the rotational symmetry, \( \{ \hat{H}_s, L_z \} = 0 \), we can separate the angular part by writing \( \psi_s(r) = \varphi_s(r) \Theta_m(\theta) \), where \( \Theta_m(\theta) = e^{im\theta}/\sqrt{2\pi} \). The radial part \( \varphi_s(r) \) and the energy spectrum \( E_s(m) \) can be determined numerically. For vanishing detuning (\( \delta = 0 \)) and small Rabi frequency \( \Omega_R \), the ground state generally exists two-fold degeneracy for \( m = l \) and \( m = -l \), as shown in Fig.1(a). The ground-state degeneracy gives rise to the possibility of an angular-stripe state, since the superposition \( \psi_s^\Gamma(r) = \cos \phi \psi_s^{\uparrow \downarrow}(r) + \sin \phi \psi_s^{\downarrow \uparrow}(r) \), with \( \psi_s^{\uparrow \downarrow}(r) = \varphi_s(r)\Theta_{l\pm}(\theta) \) being the two degenerate ground states, also has the minimal energy. Here \( \gamma \) is an arbitrary real number and \( \phi \in [0, \pi] \). The superposition with \( \phi \neq 0 \) and \( \phi \neq \pi/2 \) is the so-called angular-stripe state, of which the density profile modulates along the angular direction, as shown in Fig.1(b). Therefore, for vanishing detuning (\( \delta = 0 \)) and small Rabi frequency \( \Omega_R \), the superposition with arbitrary \( \phi \) and \( \gamma \) can be the ground state of a noninteracting Bose gas with SOAM coupling.

In Fig.1(c), we show the phase diagram of a noninteracting Bose gas with SOAM coupling. In the presence of detuning (\( \delta \neq 0 \)), or at large Rabi frequency \( \Omega_R \), ground-state degeneracy is broken and the angular-stripe state is no longer the ground state. So far we have not yet considered the interaction effect. The interaction Hamiltonian \( H_{\text{int}} \) is given by

\[
H_{\text{int}} = \int \text{d}r \left[ \frac{g}{2} (\hat{n}_1^2 + \hat{n}_2^2) + g_{\uparrow \downarrow} \hat{n}_\uparrow \hat{n}_\downarrow \right],
\]

where \( \hat{n}_\sigma = \hat{c}_\sigma ^\dagger \hat{c}_\sigma \) \( (\sigma = \uparrow, \downarrow) \) is the density operator, and \( g \) and \( g_{\uparrow \downarrow} \) denote the contact intra- and inter-species interactions, respectively. The presence of interactions make the ground state of the system subtle. We note that the interaction Hamiltonian can be written as a more illustrative form \( H_{\text{int}} = \int \text{d}r \left[ c_0 (\hat{n}_\uparrow + \hat{n}_\downarrow)^2 + c_1 (\hat{n}_\uparrow - \hat{n}_\downarrow)^2 \right] \), where \( c_0 = (g + g_{\uparrow \downarrow})/4 \) and \( c_1 = (g - g_{\uparrow \downarrow})/4 \). When \( g < g_{\uparrow \downarrow} \), the system tends to be polarized to decrease the interaction energy, which disfavors the stripe phase. However, for \( g > g_{\uparrow \downarrow} \), the density profiles of two spin components tend to be balanced, which makes the stripe phase more favorable. As we will show below, for \( g > g_{\uparrow \downarrow} \), the interaction effect will make the angular-stripe state more stable against the detuning. The system thus has a rich phase structure due to the competition among the detuning, Raman coupling, and the interaction.

Ground-state phase diagram. The system can be conveniently studied using the imaginary-time functional path integral approach, where the action is given by \( S = \int_0^\beta \text{d}\tau \left[ \psi^\dagger \partial_\tau \psi + H(\psi, \psi^\dagger) \right] \), with \( \tau \) being the imaginary time and \( \beta = 1/(k_B T) \). In the BEC state, the field \( \psi(\tau, r) \) can be decomposed as

\[
\psi(\tau, r) = \phi_{0\sigma}(r) + \phi_{\sigma}(\tau, r),
\]

where \( \phi_{0\sigma}(r) \) is the condensate wave function and is independent of \( \tau \). We expand the action \( S \) in powers of the fluctuation field \( \phi_{\sigma}(\tau, r) \) and obtain \( S = S_0 + S_2 + \cdots \).
FIG. 2. (a) (b) Ground-state phase diagrams of an interacting Bose gas with SOAM coupling in the Ω_{R}−δ plane for \( l = 1 \) (a) and \( l = 2 \) (b). (c) Typical density profiles for different phases for the \( l = 2 \) case. The profiles shown in (A)-E correspond to the points A-E in the phase diagram (b). In (A) and (B), the last plots are the zoom-in version of \( \uparrow \) component.

The mean-field part \( \mathcal{S}_0 \) reads

\[
\mathcal{S}_0 = \int dx \left[ \phi_0^{\dagger} \left( \hat{H}_s - \mu \right) \phi_0 + \frac{g}{2} \left( n_0^2 + n_0^2 + g \tau_1 n_0^1 n_0^1 \right) \right],
\]

where \( \phi_0 = (\phi_{0\uparrow}, \phi_{0\downarrow})^T \) and \( n_{0\sigma} = \phi_{0\sigma}^\dagger \phi_{0\sigma} \) is the condensate density for the spin component \( \sigma \). Here \( x = (\tau, r) \) and \( \int dx = \int d\tau \int dr \). \( S_\sigma \) denotes the action which is quadratic in the fluctuation field \( \phi_\sigma(x) \). It can be expressed as \( S_\sigma = \frac{1}{2} \int dx \Phi^\dagger(x) \mathcal{M} \Phi(x) \), where \( \Phi = (\phi_{\uparrow}, \phi_{\uparrow}, \phi_{\downarrow}, \phi_{\downarrow})^T \) and the explicit form of the matrix \( \mathcal{M} \) can be found in the supplemental material.

The ground-state phase diagram of a BEC with SOAM coupling can be determined by minimizing \( \mathcal{S}_0 \) in Eq. (5). To this end, we express \( \phi_{0\sigma}(r) \) in a general form,

\[
\phi_{0\sigma}(r) = \sum_{nm} c_{nm\sigma} R_{n,m-st}^\dagger(r) \Theta_m(\theta),
\]

where \( s = +1 \) for \( \sigma = \uparrow \) and \( s = -1 \) for \( \sigma = \downarrow \). The basis for the radial part is chosen as the eigen radial wave function of the 2D harmonic potential, i.e.,

\[
R_{n,m}(r) = \frac{1}{a_\perp} \sqrt{\frac{2n!}{(n+|m|)!}} \left( \frac{r}{a_\perp} \right)^{|m|} e^{-r^2/(2a_\perp^2)} L_n^{m}(r) \left( r^2 \right),
\]

with \( a_\perp = \sqrt{\hbar/(M\omega_\perp)} \) being the characteristic length. The variational coefficients \( c_{nm\sigma} \) is to be determined by minimizing the ground-state energy under the normalization condition \( \sum_{nm\sigma} |c_{nm\sigma}|^2 = N \), where \( N \) is the total particle number. The ground-state energy reads

\[
E_G = \sum_{nm\sigma} \varepsilon_{nm\sigma} |c_{nm\sigma}|^2 + \sum_m \int \Omega(r) \left[ U_{m\sigma}^s \left( r \right) U_{m\sigma} \left( r \right) + h.c. \right] + \int dr \left[ \frac{g}{2} \left( n_0^2 \left( r \right) + n_0^2 \left( r \right) \right) + g \tau_1 n_0^1 \left( r \right) n_0^1 \left( r \right) \right],
\]

with the notations \( \varepsilon_{nm\sigma} = (2n + |m - s| + 1) \hbar \omega_\perp + s \delta/2 \) and \( U_{nm\sigma} \left( r \right) = \sum_n c_{nm\sigma} R_{n,m-st} \left( r \right) \). We consider a realistic quasi-2D system realized by imposing a very strong harmonic confinement with frequency \( \omega_\perp \) along the \( z \) direction. Thus the wave function along the \( z \) direction can be well approximated as the ground state of 1D harmonic potential, with its characteristic length \( a_z = \sqrt{\hbar/(M\omega_\perp)} \). The effective 2D couplings constants in (7) are given by \( g = \sqrt{8\pi \hbar} a_s/(M a_z) \) and \( g \tau_1 = \sqrt{8\pi \hbar} a^4 \tau_1/(M a_z) \), where \( a_s \) and \( a^4 \tau_1 \) are the 3D s-wave scattering lengths for intra- and inter-species interactions, respectively.

To study the ground-state phase diagram, we minimize the ground-state energy \( E_G \) with respect to the variational parameters \( c_{nm\sigma} \) numerically. Different phases are characterized by the angular distribution

\[
P_m = \frac{1}{N} \sum_{nm\sigma} |c_{nm\sigma}|^2,
\]

corresponding to the occupation number for a given quantum number \( m \). The ground-state phase diagrams for OAM transfer \( l = 1 \) and \( l = 2 \) at \( g / g_{\perp} \approx 2 \) are shown in Fig. 2(a) and (b). Rich phases are found in the \( \Omega_R \)−\( \delta \) plane, including the vortex-antivortex pair phase (P0), the half-skyrmion phase (P1), and three different angular-stripe phases (S1, S2, and T).

For large enough Raman coupling \( \Omega_R \), the ground state is the so-called vortex-antivortex pair phase denoted by P0, which is characterized by the angular distribution \( P_0 = 1 \) and \( P_{m \neq 0} = 0 \). This can be understood by the fact that ground state of single-particle Hamiltonian is located at \( m = 0 \). For small Raman coupling \( \Omega_R \) and small detuning \( \delta \), the ground state is the angular-stripe phase denoted by S1, which is characterized by a double occupation of the \( m = \pm l \) and \( m = -l \) states, i.e., \( P_{\pm l} \neq 0 \) and \( P_m \) is vanishingly small for \( m \neq \pm l \). While this angular-stripe state appears only at vanishing detuning for the noninteracting case, its regime in the phase diagram is broadened by the interaction effect. With increasing detuning \( \delta \), the difference between \( P_{\pm l} \) and \( P_{-l} \) becomes larger and larger. For sufficiently large detuning and small Raman coupling, the system finally enters the half-skyrmion phase denoted by P1, characterized by the
distribution $P_{\pm l} \neq 0$ (or $P_{-l} \neq 0$) and $P_m = 0$ all other values of $m$. At moderate values of the Raman coupling, two new angular-stripe phases, denoted by T and S2, appears in the phase diagram. These two phases are actually not expected from the single-particle physics. The T phase, which appears at small detuning, is characterized by a triple occupation of the $m = \pm l$, $m = 0$, and $m = -l$ states, i.e., $P_m$ is finite for $m = l, 0, -l$ and is vanishingly small for all other values of $m$. The S2 phase is more exotic. It is characterized by a double occupation of the $m = 0$ and $m = \pm l$ (or $m = -l$) states. For $l = 1$, it exists only for small detuning and its regime in the phase diagram is tiny as shown in Fig. 2(a). However, the situation is changed if we consider a larger OAM transfer, such as $l = 2$ considered in Fig. 2(b). In this case, the S2 phase exists for much larger detuning, compared to the other two angular-stripe phases S1 and T.

In Fig. 2(c), we show typical density profiles of different phases for the $l = 2$ case. In the angular-stripe phases (S1, S2, and T), the rotational symmetry is broken and thus the density profile becomes anisotropic. However, these phases still have a C4 symmetry, i.e., the symmetry under $\pi/4$ rotation. The C4 symmetry is most pronounced in the S1 phase. However, in the T and S2 phases, the C4 symmetry restores gradually.

The interaction effects on the ground-state phase diagram is shown in Fig. 3. It is evident that that for large interspecies interaction, i.e., large values of $g_{1\perp}/g$, the ground state favors the states with a single occupation of the $m$-states (P1 state for the Raman coupling we considered). While for small interspecies interaction, i.e., small values of $g_{1\perp}/g$, the ground state energetically favors the angular-stripe phase. To reach a large regime of the angular-stripe phase, one needs to tune the interspecies interaction $g_{1\perp}$ as small as possible, which may be realized by using the Feshbach resonances.

**Excitation spectrum.** The elementary excitations of the SOAM coupled BEC, or the so-called Bogoliubov excitations, can be obtained from the action $S_2$ on top of the ground state we have discussed above. For the sake of simplicity, we consider the angular-stripe phase S1 and the half-skyrmion phase P1 at vanishing detuning [see Fig. 3]. Generalization of our calculation to other phases and nonzero detuning is straightforward but rather lengthy and complicated.

For the S1 and P1 phases at $\delta = 0$, we can restrict $m = (2\alpha - 1)l$ ($\alpha \in \mathbb{Z}$) and the condensate wave function can be expressed as

$$\phi_{\alpha}(r) = \sum_{n\alpha} c_{n\alpha} R_{n,(2\alpha-1)l}(r) \Theta_{(2\alpha-1)l}(\theta).$$

The angular-stripe phase S1 corresponds to the superposition of the $\alpha = 0$ and $\alpha = 1$ states (other components are vanishingly small), while the half-skyrmion phase P1 corresponds to the single occupation of the $\alpha = 0$ or $\alpha = 1$ state. The ground-state energy $E_G$ is given by Eq. 7 by setting $m = (2\alpha - 1)l$. The chemical potential $\mu$ can be obtained from the relation $\mu = \partial E_G/\partial N$.

To study the elementary excitations, we consider the action $S_2$ and expand the fluctuation field $\phi_\sigma(r, \tau, \mathbf{r})$ using the radial and angular functions, i.e.,

$$\phi_\sigma(r, \tau, \mathbf{r}) = \sum_{m\sigma\alpha} \Phi_{n\alpha}(i\omega_\nu, m)e^{-i\omega_\nu\tau} R_{m,(2\alpha-1)l+m}(r) \Theta_{(2\alpha-1)l+m}(\theta),$$

where $\omega_\nu = 2\pi\nu/\beta$ ($\nu \in \mathbb{Z}$) is the boson Matsubara frequency and $m$ now denotes the angular quantum number of the excitation. The summation over $\alpha$ runs from $-\infty$ to $\infty$. Substituting this expression into $S_2$, we obtain

$$S_2 = \frac{\beta}{2} \sum_{m \nu} \sum_{mn'\alpha'\alpha} \Phi^*_{n\alpha}(i\omega_\nu, m)(G^{-1})_{n\alpha, n'\alpha'} \Phi_{n'\alpha'}(i\omega_\nu, m),$$

where the notation $\Phi_{n\alpha}$ is defined as

$$\Phi_{n\alpha}(i\omega_\nu, m) = \begin{pmatrix} \phi_{n\alpha1}(i\omega_\nu, m) \\ \phi_{n\alphaL}(i\omega_\nu, m) \\ \phi^*_{n\alpha1}(-i\omega_\nu, -m) \\ \phi^*_{n\alphaL}(-i\omega_\nu, -m) \end{pmatrix}. \quad (12)$$
The quantity $G^{-1}(i\omega_n, m)$, which is also a matrix in the spaces spanned by the quantum numbers $n$ and $\alpha$, characterizes the propagation of the excitation. The explicit form of its elements is given in the supplemental material.

The excitation spectrum $E(m)$ can be obtained by

$$\det \left[ G^{-1} (E, m) \right] = 0$$

(13)

after the analytical continuation $i\omega_n \to E + i0^+$, where the determinant is taken also in the spaces spanned by the quantum numbers $n$ and $\alpha$. The excitation spectra in the angular-stripe phase $S_1$ and the half-skyrmion phase $P_1$ for $l = 2$ are shown in Fig. 4(a) and (b), respectively. In both phases, there exists a gapless band, i.e., the Goldstone mode corresponding to the broken U(1) symmetry. In the angular-stripe phase $S_1$ [Fig. 4(a)], we also find an additional gapless band corresponding to the broken rotational symmetry. In the half-skyrmion phase $P_1$ [Fig. 4(b)], the excitation spectrum exhibits roton-like structure around $m = -4$.

In summary, we have studied the ground-state phase diagram and excitation spectrum of an interacting spinor BEC with SOAM coupling realized in recent experiments. For a large OAM difference $l = 2$, we predict a new angular-stripe phase which can exist at large detuning. The excitation spectra in the angular-stripe phase and the half-skyrmion phase have distinct features. These findings would be helpful to identify the interesting angular-stripe phase in future experiments.

We acknowledge fruitful discussions with Shi-Guo Peng and Xiao-Long Chen. This work was supported by the National Key Research and Development Program of China (Grant No. 2018YFA0306503) and National Natural Science Foundation of China (Grant Nos. 11775123 and 11890712).

* lianyi@mail.tsinghua.edu.cn

[1] Y. K. Kato, R. C. Myers, A. C. Gossard, and D. D. Awschalom, Science 306, 1910 (2004)

[2] C. L. Kane and E. J. Mele, Phys. Rev. Lett. 95, 146802 (2005)

[3] M. König, S. Wiedmann, C. Brüne, A. Roth, H. Buhmann, L. W. Molenkamp, X.-L. Qi, and S.-C. Zhang, Science 318, 766 (2007)

[4] B. A. Bernevig, T. L. Hughes, and S.-C. Zhang, Science 314, 1757 (2006)

[5] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. 82, 3045 (2010)

[6] X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. 83, 1057 (2011)

[7] J. Daïl bard, F. Gerbier, G. Juzeliūnas, and P. Öhberg, Rev. Mod. Phys. 83, 1523 (2011)

[8] V. Galitski and I. B. Spielman, Nature (London) 494, 49 (2013)

[9] H. Zhai, Rep. Prog. Phys., 78, 026001 (2015)

[10] Y. J. Lin, K. Jiménez-Garcia, and I. B. Spielman, Nature (London) 471, 83 (2011)

[11] P. Wang, Z.-Q. Yu, Z. Fu, J. Miao, L. Huang, S. Chai, H. Zhai, and J. Zhang, Phys. Rev. Lett. 109, 095301 (2012)

[12] L. W. Cheuk, A. T. Sommer, Z. Hadzibabic, T. Yefsah, W. S. Bakr, and M. W. Zwierlein, Phys. Rev. Lett. 109, 095302 (2012)

[13] Z. Wu, L. Zhang, W. Sun, X.-T. Xu, B.-Z. Wang, S.-C. Ji, Y. Deng, S. Chen, X.-J. Liu, and J.-W. Pan, Science 354, 83 (2016)

[14] M. A. Khamehchi, K. Hossain, M. E. Mossmann, Y. Zhang, T. Busch, M. M. Forbes, and P. Engels, Phys. Rev. Lett. 118, 155301 (2017)

[15] J.-R. Li, J. Lee, W. Huang, S. Burchesky, B. Shitarmas, F. Ç. Top, A. O. Jamison, and W. Ketterle, Nature (London) 543, 91 (2017)

[16] D. Zhang, T. Gao, P. Zou, L. Kong, R. Li, X. Shen, X.-L. Chen, S.-G. Peng, M. Zhan, H. Pu, and K. Jiang, Phys. Rev. Lett. 122, 110402 (2019)

[17] H.-R. Chen, K.-Y. Lin, P.-K. Chen, N.-C. Chiu, J.-B. Wang, C.-A. Chen, F.-P. Huang, S.-K. Yip, Y. Kawaguchi, and Y.-J. Lin, Phys. Rev. Lett. 121, 113204 (2018)

[18] M. DeMarco and H. Pu, Phys. Rev. A 91, 033630 (2015)

[19] K. Sun, C. Qu, and C. Zhang, Phys. Rev. A 91, 063627 (2015)

[20] C. Qu, K. Sun, and C. Zhang, Phys. Rev. A 91, 053630 (2015)

[21] L. Chen, H. Pu, and Y. Zhang, Phys. Rev. A 93, 013629 (2016)

[22] Y.-X. Hu, C. Miniatura, and B. Grémaud, Phys. Rev. A 92, 033615 (2015)

[23] J. Hou, X.-W. Luo, K. Sun, and C. Zhang, Phys. Rev. A 96, 011603 (2017)

[24] X.-L. Chen, S.-G. Peng, P. Zou, X.-J. Liu, and H. Hu, arXiv:1901.02595

[25] C. Wang, C. Gao, C.-M. Jian, and H. Zhai, Phys. Rev. Lett. 105, 160403 (2010)

[26] Z. X.-F. Wu Cong-Jun, Ian Mondragon-Shem, Chinese Physics Letters 28, 097102 (2011)

[27] T.-L. Ho and S. Zhang, Phys. Rev. Lett. 107, 150403 (2011)

[28] Y. Li, L. P. Pitaevskii, and S. Stringari, Phys. Rev. Lett. 108, 255301 (2012)

[29] Y. Li, G. I. Martone, L. P. Pitaevskii, and S. Stringari, Phys. Rev. Lett. 110, 235302 (2013)

[30] R. Liao, Phys. Rev. Lett. 120, 140403 (2018)
Supplemental material for "Ground-state phase diagram and excitation spectrum of a Bose-Einstein Condensate with spin-orbital-angular-momentum coupling"

The matrix $M$ appears in the expression of the action $S_2$ is given by

$$M = \begin{pmatrix} \partial_r + K_+ & \Omega(r) + g_{\uparrow\downarrow}\phi_0\phi_{\downarrow}^* & g_{\uparrow\downarrow}\phi_0^2 & g_{\uparrow\downarrow}\phi_0\phi_0^* \\ \Omega(r) + g_{\uparrow\downarrow}\phi_{\downarrow}\phi_0^* & \partial_r + K_+ & g_{\uparrow\downarrow}\phi_{\downarrow}\phi_0^* & g_{\uparrow\downarrow}\phi_{\downarrow}\phi_0^* \\ g_{\uparrow\downarrow}\phi_{\downarrow}\phi_0^* & g_{\uparrow\downarrow}\phi_{\downarrow}\phi_0^* & \partial_r + K_+ & \Omega(r) + g_{\uparrow\downarrow}\phi_{\downarrow}\phi_0^* \\ g_{\uparrow\downarrow}\phi_{\downarrow}\phi_0^* & g_{\uparrow\downarrow}\phi_{\downarrow}\phi_0^* & g_{\uparrow\downarrow}\phi_{\downarrow}\phi_0^* & \partial_r + K_+ \end{pmatrix}, \quad (S1)$$

where

$$K_+ = -\frac{\hbar^2}{2Mr} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{(L_z - i\hbar \sigma)^2}{2Mr^2} + \frac{\delta}{2} + V_{\text{ext}}(r) - \mu + 2gn_{\uparrow} + g_{\uparrow\downarrow}n_{\downarrow},$$

$$K_- = -\frac{\hbar^2}{2Mr} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{(L_z + i\hbar \sigma)^2}{2Mr^2} - \frac{\delta}{2} + V_{\text{ext}}(r) - \mu + 2gn_{\downarrow} + g_{\uparrow\downarrow}n_{\uparrow}. \quad (S2)$$

In the representation given in Eq. (11) of the main text, the Green's function $G^{-1}$ can be expressed as $G^{-1} = G_0^{-1} + G_1^{-1} + G_2^{-1}$, where $G_0^{-1}$, $G_1^{-1}$, and $G_2^{-1}$ are given by

$$\begin{align*}
(G_0^{-1})_{n\alpha, n'\alpha'} &= \begin{pmatrix} -i\omega_\nu + \xi_{n\alpha\uparrow}(m) & 0 & 0 & 0 \\
\phi_0 & 0 & -i\omega_\nu + \xi_{n\alpha\downarrow}(m) & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix} \delta_{n\alpha, n'\alpha'}, \\
(G_1^{-1})_{n\alpha, n'\alpha'} &= \begin{pmatrix} \Omega_m(n\alpha, n'\alpha') & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \Omega_m(n\alpha, n'\alpha') & 0 \\
0 & 0 & 0 & 0 \end{pmatrix} \delta_{n\alpha, n'\alpha'}, \\
(G_2^{-1})_{n\alpha, n'\alpha'} &= \begin{pmatrix} A_{n\alpha, n'\alpha'} & D_{n\alpha, n'\alpha'} & B_{n\alpha, n'\alpha'} & C_{n\alpha, n'\alpha'} \\
D_{n\alpha, n'\alpha'} & A_{n\alpha, n'\alpha'} & C_{n\alpha, n'\alpha'} & B_{n\alpha, n'\alpha'} \\
B_{n\alpha, n'\alpha'} & C_{n\alpha, n'\alpha'} & A_{n\alpha, n'\alpha'} & D_{n\alpha, n'\alpha'} \\
C_{n\alpha, n'\alpha'} & B_{n\alpha, n'\alpha'} & D_{n\alpha, n'\alpha'} & A_{n\alpha, n'\alpha'} \end{pmatrix}, \quad (S3)
\end{align*}$$

where $\xi_{n\alpha\sigma}(m) = (2n + |(2\alpha - 1)l + m - s| + 1)\hbar \omega - \mu$. The quantities $A_{n\sigma}(n\alpha, n'\alpha')$, $B_{n\sigma}(n\alpha, n'\alpha')$, $C_{n\sigma}(n\alpha, n'\alpha')$, $D_{n\sigma}(n\alpha, n'\alpha')$ are defined as

$$\begin{align*}
A_{n\sigma}(n\alpha, n'\alpha') &= \frac{1}{2\pi} \sum_{\alpha_1, \alpha_2} \int dr R_{n, (2\alpha - 1)l + m + s}(r) R_{n', (2\alpha' - 1)l + m - s}(r) \left( 2g_{\alpha_1\sigma} U_{\alpha_2\sigma} + g_{\alpha_1\uparrow} U_{\alpha_2\downarrow} + g_{\alpha_1\downarrow} U_{\alpha_2\uparrow} \right) \delta_{\alpha_1 + \alpha_2, n' + n'}, \\
B_{n\sigma}(n\alpha, n'\alpha') &= \frac{9}{2\pi} \sum_{\alpha_1, \alpha_2} \int dr R_{n, (2\alpha - 1)l + m + s}(r) R_{n', (2\alpha' - 1)l + m - s}(r) U_{\alpha_1\sigma} U_{\alpha_2\sigma} \delta_{\alpha_1 + \alpha_2, n' + n'}, \\
C_{n\sigma}(n\alpha, n'\alpha') &= \frac{9}{2\pi} \sum_{\alpha_1, \alpha_2} \int dr R_{n, (2\alpha - 1)l + m - s}(r) R_{n', (2\alpha' - 1)l + m + s}(r) U_{\alpha_1\uparrow} U_{\alpha_2\downarrow} \delta_{\alpha_1 + \alpha_2, n' + n'}, \\
D_{n\sigma}(n\alpha, n'\alpha') &= \frac{9}{2\pi} \sum_{\alpha_1, \alpha_2} \int dr R_{n, (2\alpha - 1)l + m + s}(r) R_{n', (2\alpha' - 1)l + m - s}(r) U_{\alpha_1\downarrow} U_{\alpha_2\uparrow} \delta_{\alpha_1 + \alpha_2, n' + n'}, \quad (S4)
\end{align*}$$

and $\Omega_m(n\alpha, n'\alpha')$ is given by

$$\Omega_m(n\alpha, n'\alpha') = \int dr R_{n, (2\alpha - 1)l + m - s}(r) \Omega(r) R_{n', (2\alpha' - 1)l + m + s}(r). \quad (S5)$$

Here $U_{\alpha\sigma}(r) = \sum_n c_{n\alpha\sigma} R_{n, (2\alpha - 1)l - m}(r)$. From the above definitions, we have $A_{n\sigma}(n\alpha, n'\alpha') = A_{n\sigma}(n'\alpha', n\alpha)$ and $B_{n\sigma}(n\alpha, n'\alpha') = B_{n\sigma}(n'\alpha', n\alpha)$. 