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Computing the hull number in toll convexity

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Abstract A walk $W$ between vertices $u$ and $v$ of a graph $G$ is called a tolled walk between $u$ and $v$ if $u$, as well as $v$, has exactly one neighbour in $W$. A set $S \subseteq V(G)$ is toll convex if the vertices contained in any tolled walk between two vertices of $S$ are contained in $S$. The toll convex hull of $S$ is the minimum toll convex set containing $S$. The toll hull number of $G$ is the minimum cardinality of a set $S$ such that the toll convex hull of $S$ is $V(G)$. The main contribution of this work is an algorithm for computing the toll hull number of a general graph in polynomial time.

Keywords Extreme vertex · hull number · toll convexity

Mathematics Subject Classification (2010) 05C85

1 Introduction

We consider finite, simple, and undirected graphs. For a graph $G$, its vertex and edge sets are denoted by $V(G)$ and $E(G)$, respectively, while the open and the closed neighborhoods of a vertex $w \in V(G)$ are denoted by $N_G(w)$ and $N_G[w]$, respectively. Recall that a walk between vertices $u$ and $v$ of a graph $G$ is a sequence of vertices $w_1 \ldots w_k$ such that $k \geq 1$, $w_iw_{i+1} \in E(G)$ for $1 \leq i < k$, $u = w_1$, and $v = w_k$. As a motivation, consider that a graph $G$ models a space containing two points with huge gravitational force, represented by vertices $u, v \in V(G)$. Thus, a valid trajectory of a spacecraft $S$ launched from $u$ with destination to $v$ is represented by a walk that contains exactly two vertices of $N_G[u]$ and of $N_G[v]$, since the necessary energy for $S$ to move away from $u$...
is all wasted in the take off and once $S$ reaches the neighborhood of $v$, it is immediately absorbed by $v$. This scenario prevents $S$ from passing through the neighborhood of $u$ a second time, because in this case $S$ would be absorbed by $u$ and the mission will be failed. Path convexities has gained attention in the last decades (Duchet 1988, Gimbel 2003, Henning et al. 2013, Pelayo 2013), and this kind of relaxation of path originated the toll convexity (Alcón et al. 2015, Gologranc and Repolusk 2017). A tolled walk between $u$ and $v$, or a tolled $(u,v)$-walk, is a walk $W = w_1 \ldots w_k$ in which $u = w_1$, $v = w_k$, and if $k \geq 2$, then $w_2$ is the only neighbor of $u$ and $w_{k-1}$ is the only neighbor of $v$ in $W$.

A family $\mathcal{C}$ of subsets of a finite set $X$ is a convexity on $X$ if $\varnothing, X \in \mathcal{C}$ and $\mathcal{C}$ is closed under intersections (van de Vel 1993). Given a graph $G$, a set $S \subseteq V(G)$ is toll convex if the vertices contained in any tolled walk between two vertices of $S$ are contained in $S$; and $S$ is toll concave if $V(G) \setminus S$ is toll convex. The toll interval of $u, v \in V(G)$ is $[u,v]^G = \{ w : w \text{ belongs to some tolled } (u,v)-\text{walk} \}$. The toll interval set of $S$ is $[S]^G = \bigcup_{u,v \in S} [u,v]^G$ if $|S| \geq 2$ and $[S]^G = S$ otherwise. If $[S]^G = V(G)$, then $S$ is said to be a toll interval set of $G$ and the minimum cardinality of a toll interval set of $G$ is the toll number of $G$. The toll convex hull of $S$, denoted by $(S)^G$, is the minimum toll convex set containing $S$. If $(S)^G = V(G)$, then $S$ is said to be a toll hull set of $G$ and the minimum cardinality of a toll hull set of $G$ is the toll hull number of $G$. Note that if $G'$ is an induced subgraph of $G$ and $W$ is a tolled $(u,v)$-walk of $G'$, then $W$ is also a tolled $(u,v)$-walk in $G$. Hence, $(S)^G \subseteq (S')^G$ and $(S)^G \subseteq (S)^G$. For shortness, we will drop the superscript and subscript indicating the graph and the convexity when there is no ambiguity.

For $S \subseteq V(G)$, denote by $G - S$ the graph obtained by the deletion of the vertices of $S$; and by $G[S]$ the subgraph of $G$ induced by $S$. If every two vertices of $S$ are adjacent, then $S$ is a clique of $G$. Vertices $u,v \in V(G)$ are (true) twins if $N_G[u] = N_G[v]$. Vertex $u$ is simplicial in $G$ if $N(u)$ is a clique. If $V(G)$ is a clique, then $G$ is said to be a complete graph. The neighborhood of $S$ is $N(S) = ( \bigcup_{u \in S} N(u)) \setminus S$ and the border of $S$ is $S = \{ u : u \in S$ and $N(u) \cap (V(G) \setminus S) \neq \varnothing \}$. We will also use $\overrightarrow{S} = S \setminus \overleftarrow{S}$ and, for a family os sets $S$, $\overrightarrow{S}$ will stand for $\{ \overrightarrow{S} : S \in S \}$. A vertex $u$ of a toll convex set $S$ is extreme in $S$ if $S \setminus \{ u \}$ is also a toll convex set. Denote the set of toll extreme vertices of $V(G)$ by $Extt(G)$. It is clear that $Extt(G)$ is subset of every toll interval set and of every toll hull set of $G$ and the every toll extreme vertex is a simplicial vertex but the converse is not always true.

In the well-known geodetic convexity (Farber and Jamison 1986, Pelayo 2013), monophonic convexity (Duchet 1988, Edelman and Jamison 1985), and $P_k$ convexity (Dourado et al. 2012, Henning et al. 2013) all above concepts are analogously defined by replacing “tolled walk” by “shortest path”, “minimal path”, and “path of order three”, respectively. In the geodetic convexity, determining whether the hull number is at most $k$ is APX-hard for general graphs (Coelho et al. 2015), NP-complete for partial cube graphs (Albenque
and Knauer 2016) and chordal graphs (Bessey et al. 2018), and solvable in polynomial time for unit interval graphs, cographs, split graphs (Dourado et al. 2009), cactus graphs, \( P_2 \)-sparse graphs (Araujo et al. 2013), distance hereditary graphs (Kante and Nourine 2016), \( P_5 \)-triangle-free graphs (Araujo et al. 2016). In the \( P_3 \) convexity, this problem is APX-hard even for bipartite graphs with maximum degree \( \Delta \leq 4 \) (Coelho et al. 2015), and can be solved in polynomial time for block graphs and chordal graphs (Centene et al. 2011). However, the monophonic hull number can be computed in polynomial time for general graphs (Dourado et al. 2010). In the toll convexity, it is known that the hull number of every tree different of a caterpillar is equal to 2 (Alcón et al. 2015).

A graph \( G \) is an interval graph if every vertex of \( G \) can be associated with an interval of a straight line such that two vertices of \( G \) are neighbors if and only if the corresponding intervals intersect. Given a convexity \( C \) on the vertex set of \( G \), we say that \( G \) is a convex geometry under \( C \) if every \( C \)-convex set of \( G \) is equal to the \( C \)-convex hull of its \( C \)-extreme vertices. In (Alcón et al. 2015), it was shown that the interval graphs are precisely the graphs which are convex geometries in the toll convexity. They also characterized the toll convex sets of a general graph and of some graph products. In (Gologranc and Repolusk 2017), the toll number of the Cartesian and the lexicographic product of graphs are studied, where some characterizations are presented.

The text is organized as follows. In the next section, we present the notion of hull representing family, which plays an important role in the proposed algorithm and can be an useful tool for further works dealing with the hull number. In Section 3, we present a polynomial-time algorithm for computing the toll hull number of a general graph. In the conclusions, we discuss that this result leads to an algorithm for generating all minimum toll hull sets of a general graph with polynomial delay and to a characterization of the toll extreme vertices of a graph.

2 Hull characteristic family

We begin this section proving useful properties of tolled walks.

**Lemma 1** Let \( G \) be a graph, let \( S \subset V(G) \), let \( C \subseteq \overrightarrow{S} \) such that \( G[C] \) is connected, and let \( x, y \notin S \). The following sentences are equivalent.

1. There is a tolled \((x, y)\)-walk containing vertices of \( C \);
2. There is a tolled \((x, y)\)-walk containing vertices \( x', y' \in \overrightarrow{S} \) such that \( xy' \notin E(G) \), \( x'y \notin E(G) \), \( N(x') \cap C \neq \emptyset \), and \( N(y') \cap C \neq \emptyset \);
3. \( C \subset [x, y] \).

**Proof** \((1) \Rightarrow (2) \Rightarrow (3)\) Let \( W \) be a tolled \((x, y)\)-walk containing vertex \( v \in C \). Since \( S \) separates \( \overrightarrow{S} \) from \( V(G) \setminus S \), \( W \) contains at least two occurrences
\(x'\) and \(y'\) of vertices in \(\overrightarrow{S}\) such that \(v\) appears between \(x'\) and \(y'\) in \(W\). By definition, \(x'y \not\in E(G)\) and \(y'x \not\in E(G)\). Furthermore, we can write \(W = x \ldots x'' \ldots v \ldots y'' \ldots y\) such that \(x'', y'' \in C\). Now, the assumption that \(G[C]\) is connected guarantees that there is a \((v, v)\)-walk \(W'\) containing all vertices of \(C\). Since \(N[x] \cap \overrightarrow{S} = \emptyset\) and \(N[y] \cap \overrightarrow{S} = \emptyset\), the walks \(W\) and \(W'\) can be combined to form a tolled \((x, y)\)-walk containing all vertices of \(C\) as desired.

(3) \(\Rightarrow\) (1) is direct from definition.

The following interesting consequence of Lemma 1 does not work in general for other path convexities.

**Corollary 1** If \(S\) induces a connected graph and is toll concave, then any set that induces a connected graph and contains \(S\) is toll concave.

Before introducing the hull characteristic families, we recall an useful result.

**Lemma 2** (Alcón et al. 2015) A vertex \(v\) is in some tolled walk between two non-adjacent vertices \(x\) and \(y\) if and only if \(N[x] \setminus \{v\}\) does not separate \(v\) from \(y\) and \(N[y] \setminus \{v\}\) does not separate \(v\) from \(x\).

Observe that Lemma 2 can be used to test whether a vertex is toll extreme, a set is toll concave, and to show that \(N(F)\) is a clique for every toll concave set \(F\).

If \(F\) is a concave set of a convexity \(C\) on a set \(X\), then every hull set of \(C\) has at least one vertex of \(F\). We define the granularity of \(F\) under \(C\) as the maximum integer \(g_c(F)\) such that every hull set of \(C\) has at least \(g_c(F)\) vertices of \(F\). Let \(F\) be a family of pairwise disjoint concave sets of \(C\). The granularity of \(F\) is the sum of the granularities of its members. We say that \(F\) is a hull characteristic family of \(C\) if the hull number of \(C\) is equal to the granularity of \(F\).

The problem of computing the hull number of \(C\) can be reduced to the one of finding a hull characteristic family of \(C\) and computing the granularity of each of its members. The family formed only by \(X\) is itself a trivial hull characteristic family of \(C\), but it brings no advantage of the use of this notion for determining the hull number of \(C\). The number of hull characteristic families of \(C\) can be an exponential on the cardinality of \(V\). For instance, every partition of the vertex set \(V(G)\), where \(G\) is a complete graph, is a hull characteristic family of the toll convexity of \(G\), since the toll hull number of \(G\) is \(|V(G)|\) if \(G\) is a complete graph. An example of a non-trivial hull characteristic family in toll convexity is the family \(C = \{S_1 = \{v_1, v_2, v_3\}, S_2 = \{v_4, v_5, v_6\}\}\) of vertices of the graph \(G\) of Figure 1. One can use Lemma 2 to see that the members of \(C\) are really toll concave sets. In fact, this lemma can be used to show that all vertices of \(S_1\) are extreme vertices, then \(g(S_1) = 3\). Since \(S_1\) is not a toll hull set of \(G\), the toll hull number of \(G\) is at least 4. Now, one can use Lemma 2 again to prove that \(S_1 \cup \{v_5\}\) is a toll hull set of \(G\) concluding that \(g(S_2) = 1\) and also that the toll hull number of \(G\) is 4.
3 The algorithm

The central idea of the proposed algorithm is to find a toll hull characteristic family $C$ of the input graph such that the granularity of each member of $C$ can be determined in polynomial time. In order to get this, initially, one family of sets $F$ is constructed such that, during the algorithm, its member, that are not toll concave, are getting bigger, possibly concatenating with other members of $F$ so that, at the end, the toll concave sets of $F$ form the desired family.

The following classification of the toll concave sets $F$ of a graph is useful to accomplish this task.

$$
\text{Type of } F = \begin{cases} 
1, & \text{if there is a vertex } u \in F \text{ non-adjacent to some vertex of } N(F) \\
2, & \text{otherwise and if } F \text{ is not a clique} \\
3, & \text{otherwise}
\end{cases}
$$

**Lemma 3** If $F$ is a toll concave set of a graph $G$, then $g(F) \geq i$ if the type of $F$ is $i \in \{1, 2\}$ and $g(F) = |F|$ if the type of $F$ is 3.

**Proof** Let $F$ be a toll concave set of $G$ with type $t$. The case $t = 1$ is trivial.

For the case $t = 2$, suppose for contradiction that $S$ is a toll hull set of $G$ such that $\{x\} = S \cap F$. Since $F$ is toll concave and $F - \{x\}$ is not, for some $y \notin F$, there is a tolled $(y, x)$-walk containing some vertex $v \in F \setminus \{x\}$. However, since $N(F) \setminus \{x\} \subset N[x]$ because $t = 2$, $N[x] \setminus \{v\}$ separates $v$ from $y$, which contradicts Lemma 2.

Finally consider $t = 3$. We claim that all vertices of $F$ are extreme vertices. Suppose the contrary and let $W$ be a tolled $(x, y)$-walk containing some vertex $v \in F \setminus \{x, y\}$. Since $F$ is toll concave, at least one, say $x$, belongs to $F$. But $x$ and $v$ are twins, because $F$ is a clique and every vertex of $N(F)$ is universal to $F$ by definition of type 3. This contradicts Lemma 2 because $N[x] \setminus \{v\}$ separates $v$ from $y$.

An example of a toll concave set with granularity strictly bigger than its type is the set $F = \{v_1, v_2, v_3, v_7, v_8, v_9, v_{10}\}$ of Figure 1, since the type of $F$ is 1 and $g(F) \geq 3$ because vertices $v_1, v_2, v_3$ are toll extreme vertices of the graph.
We need some additional definitions. Consider a graph $G$. We say that $S \subset V(G)$ separates vertices $u, v \in V(G)$ if there is a $(u, v)$-path in $G$ but there is no one in $G - S$; that $S$ is a separator of $G$ if $S$ separates some pair of vertices of $G$; and that $S \subset V(G)$ is a clique separator of $G$ if $S$ is a clique and a separator of $G$. We say that $G$ is reducible if it contains a clique separator, otherwise it is prime. A maximal prime subgraph of $G$, or mp-subgraph of $G$, is a maximal induced subgraph of $G$ that is prime. An mp-subgraph $F$ of a reducible graph $G$ is called extremal if there is an mp-subgraph $F'$ different of $F$ such that, for every mp-subgraph $F''$ different of $F$, it holds $F \cap F'' \subseteq F \cap F'$. As an example, consider the graph $G$ of Figure 1. The mp-subgraphs of $G$ are induced by the following sets \{v_1, v_2, v_3, v_7, v_8\}, \{v_7, v_8, v_9, v_{10}\}, \{v_9, v_{10}, v_{11}, v_{12}\}, and \{v_{11}, v_{12}, v_4, v_5, v_6\}. The following result states an useful property of reducible graphs.

**Lemma 4** (Leimer 1993) Every reducible graph has at least two extremal mp-subgraphs.

**Lemma 5** If $M$ is a non-extremal mp-subgraph of $G$, then $G - M$ is disconnected.

**Proof** Let $M$ be a non-extremal mp-subgraph of $G$ and let $M_1$ and $M_2$ be mp-subgraphs of $G$ such that there is no mp-subgraph of $G$ different of $M$ containing $(M \cap M_1) \cup (M \cap M_2)$. Then, there are vertices $v_1 \in (M \cap M_1) \setminus M_2$ and $v_2 \in (M \cap M_2) \setminus M_1$. There are also $u_1 \in M_1$ and $u_2 \in M_2$ such that $M \cap M_1$ separates $u_1$ from $v_2$ and $M \cap M_2$ separates $u_2$ from $v_1$. Therefore $M$ separates $u_1$ from $u_2$.

The following result on the monophonic convexity solves the problem when the input graph is prime.

**Theorem 1** (Dourado et al. 2010) If $G$ is a prime graph that is not a complete graph, then every pair of non-adjacent vertices is a monophonic hull set of $G$.

**Corollary 2** Let $G$ be a prime graph. If $V(G)$ is a clique, then $\text{th}(G) = |V(G)|$; otherwise every two non-adjacent vertices form a toll hull set of $G$.

**Proof** If $G$ is a complete graph, it is clear that $V(G)$ is the only toll hull set of $G$. If $G$ is a not a complete graph, the result follows from Theorem 1 because $S \subseteq \overrightarrow{S}$ for any set $S \subseteq V(G)$.

Once a toll concave set $F^*$ is found by the algorithm, it is added to $\mathcal{F}$ and keep this way until the end of the algorithm. Therefore, it will be a member of the toll hull characteristic family constructed for the input graph. Therefore, one can determine its type and choose the vertices of $F^*$ that compose the minimum toll hull set that will be returned. The possible selections appear as numbered choices in the algorithm and are detailed in the sequel.

**Choice 1** add $u$ to $S$ such that $u \in F^*$ and $u$ has a non-neighbor in $F^*$. 
Algorithm 1: Minimum toll hull set

input: A graph $G$
output: A minimum toll hull set of $G$

1. if $V(G)$ is a clique then
   return $V(G)$
2. if $G$ is prime then
   return two non-adjacent vertices of $G$
3. compute the mp-subgraphs of $G$
4. $\mathcal{M} \leftarrow \{ F : F$ is the vertex set of a non-extremal mp-subgraph of $G \}$
5. $\mathcal{F} \leftarrow \{ F : F$ is the vertex set of an extremal mp-subgraph of $G \}$
6. for $F \in \mathcal{F}$ do
5.1. if $F$ is toll concave of type 1 then
          apply Choice 1
7. for $F \in \mathcal{F}$ do
5.2. if $F$ is toll concave of type 2 then
          apply Choice 4
8. if $F$ is toll concave of type 3 then
          add $F$ to $S$
9. $S \leftarrow \emptyset$
10. while there is $\tilde{F}' \in \tilde{F}$ that is not toll concave and $\tilde{F}' \subseteq F$ for some $F \in \mathcal{M} \cup \mathcal{F} \setminus \{ F' \}$ do
11.     $\mathcal{M}' \leftarrow \{ F : F \in \mathcal{M}$ and $\tilde{F}' \subseteq F \}$
12.     $\mathcal{M} \leftarrow \mathcal{M} \setminus \mathcal{M}'$
13.     $\mathcal{F}' \leftarrow \{ F : F \in \mathcal{F}$ and $\tilde{F}' \subseteq F \}$
14.     $\mathcal{F}^* \leftarrow \bigcup_{F \in \mathcal{M}' \cup \mathcal{F}'} F$
15.     $\tilde{F} \leftarrow (\tilde{F} \setminus \tilde{F}^*) \cup \{ \mathcal{F}^* \}$
16.     if $\tilde{F}^*$ is toll concave of type $i$ then
          let $k$ be the number of members $F$ of $\mathcal{F}'$ such that $\tilde{F}$ is toll concave
          if $i = 1$ and $k = 0$ then
            if possible, apply Choice 2; else apply Choice 3
          if $i = 2$ and $k = 0$ then
            if possible, apply Choice 5; else apply Choice 6
          if $i = 2$ and $k = 1$ then
            if possible, apply Choice 7; else apply Choice 8
30. $\mathcal{C} \leftarrow \{ F : F$ is a toll concave set of $\mathcal{F} \}$
31. return $S$

Choice 2 add $u$ to $S$ for which there are $F_1, F_2 \in \mathcal{M} \cup \mathcal{F}'$ with $F_1 \neq F_2$ such that $\tilde{F}^* \subseteq F_2$, $u \in F_1$, there is $u' \in \tilde{F}' \setminus N(u)$, and there is $u'' \in \tilde{F}^* \setminus N(u)$.

Choice 3 add $u$ to $S$ for which there are $F_1, F_2 \in \mathcal{M} \cup \mathcal{F}'$ with $F_1 \neq F_2$ such that $\tilde{F}^* \subseteq F_2$, $u \in F_1$, and there is $u' \in \tilde{F}' \setminus N(u)$.

Choice 4 add $u_1$ and $u_2$ to $S$ such that $u_1$ and $u_2$ are non-adjacent vertices of $\tilde{F}^*$. 
Choice 5 add \( u_1 \) and \( u_2 \) to \( S \) such that there is \( F_1 \in \mathcal{M}' \cup \mathcal{F}' \) with \( u_1, u_2 \in \overrightarrow{F_1} \) and both, \( u_1 \) and \( u_2 \), have non-neighbors in \( \overrightarrow{F} \).

Choice 6 for \( i \in \{1, 2\} \), add \( u_i \) to \( S \) such that there is \( F_i \in \mathcal{M}' \cup \mathcal{F}' \) with \( u_i \in \overrightarrow{F_i} \), \( u_i \) has a non-neighbor in \( \overrightarrow{F_i} \), and \( F_1 \neq F_2 \).

Choice 7 add \( u_2 \) to \( S \) such that there is \( F_2 \in \mathcal{M}' \cup \mathcal{F}' \setminus \{F_1\} \) with \( u_2 \in \overrightarrow{F_2} \) and \( u_2 \) has a non-neighbor in \( \overrightarrow{F} \).

Choice 8 add \( u_2 \) to \( S \) such that there is \( F_2 \in \mathcal{M}' \cup \mathcal{F}' \setminus \{F_1\} \) with \( u_2 \in \overrightarrow{F} \).

Lemma 6 At any moment of Algorithm 1, the family \( \mathcal{F} \) satisfies the following sentences.

1. if \( F \in \mathcal{F} \), then \( \overrightarrow{F} \) is non-empty and \( G[\overrightarrow{F}] \) is connected;
2. the members of \( \mathcal{F} \) are pairwise disjoint;
3. if \( F \in \mathcal{F} \) and \( \overrightarrow{F} \) is not a clique, then \( G[V \setminus F] \) is disconnected.

Proof After line 7, each member of \( \mathcal{F} \) is a different extremal mp-subgraph of \( G \). Then items (1) and (2) hold at this moment. After line 21 of each iteration of the While loop, one member \( F \) is added to \( \mathcal{F} \) which is the union of some members removed from \( \mathcal{F} \) plus some members of \( \mathcal{M} \), which are mp-subgraphs of \( G \) do not belonging to any other member of \( \mathcal{F} \). It is clear that this operation preserves the property that the members of \( \mathcal{F} \) form a partition of a subfamily of the mp-subgraphs of \( G \) each one containing at least one extremal mp-subgraph and that \( G[\overrightarrow{F}] \) is a connected graph.

Since an extremal mp-subgraph contains a vertex not belonging to any other mp-subgraph, item (1) holds and the fact that the intersection between two mp-subgraphs \( M \) and \( M' \) is a subset of \( \overrightarrow{M} \) implies item (2).

For item (3), let \( F \in \mathcal{F} \) be such that \( \overrightarrow{F} \) is not a clique and let \( u_1, u_2 \in \overrightarrow{F} \) be two non-adjacent vertices. Recall that \( F \) is the union of some mp-subgraphs of \( G \) and that, if \( M_1 \) is an mp-subgraph of \( G \) containing \( u_1 \), then there is an mp-subgraph \( M'_1 \) of \( G \) not contained in \( F \) such that \( u_1 \in C_1 = M_1 \cap M'_1 \subseteq \overrightarrow{F} \).

Analogously, there are \( C_2, M_2, \) and \( M'_2 \) for \( u_2 \). Note that \( M_2 \) can be equal to \( M_1 \), but \( C_2 \neq C_1 \) and \( M'_2 \neq M'_1 \). Now, observe that \( C_1 \) separates \( u_2 \) from \( u'_1 \) but does not separate \( u_2 \) from \( u'_2 \). Since \( u'_1, u'_2 \notin F \), it follows that \( u'_1 \) and \( u'_2 \) belong to different connected components of \( G - F \).

The following result guarantees that if \( F^* \) is a toll concave set constructed in Algorithm 1 by the union of other sets, then at most two of these sets \( F \) are such that \( \overrightarrow{F} \) is toll concave. Furthermore, the type of \( \overrightarrow{F} \) is 1.

Lemma 7 Let \( \mathcal{F}' \) and \( F^* \) be obtained in lines 19 and 20 of the same iteration of the While loop of Algorithm 1, respectively. If \( \overrightarrow{F^*} \) is toll concave, then \( \overrightarrow{F'} \) has at most two toll concave sets and each of them has type 1.
Proof First, suppose for contradiction that \( \tilde{F}' \) has three toll concave sets \( \tilde{F}_1, \tilde{F}_2, \) and \( \tilde{F}_3 \). Since \( \tilde{F}' \subseteq \tilde{F}_1, \tilde{F}_1 \) is toll concave, and the border of every toll concave set is a clique, we conclude that \( \tilde{F}' \) is a clique. Hence, every pair \( \{u, v\} \) for which there is a tolled \((u, v)\)-walk \( W \) containing some vertex of \( \tilde{F}' \) satisfies \( u, v \notin \tilde{F}' \).

By Lemma 6, the sets \( \tilde{F}_1, \tilde{F}_2, \) and \( \tilde{F}_3 \) are pairwise disjoint. This implies that for at least one of them, say \( \tilde{F}_3 \), it holds \( u, v \notin \tilde{F}_3 \). Furthermore, we have that \( u, v \notin \tilde{F}_i \) for \( i \in \{1, 2, 3\} \), because each of \( u \) and \( v \) has at least one non-neighbor in \( \tilde{F}' \), \( \tilde{F}' \subseteq \tilde{F}_i \) for \( i \in \{1, 2, 3\} \), and \( \tilde{F}_i \) is a clique. Observe that \( W \) has at least two occurrences \( u' \) and \( v' \) of vertices of \( \tilde{F}' \) such that \( u'v \notin E(G) \) and \( v'u \notin E(G) \). Now, since \( \tilde{F}' \subseteq \tilde{F}_3 \), every vertex of \( \tilde{F}_3 \) has at least one neighbor in \( \tilde{F}_3 \), and \( G[\tilde{F}_3] \) is connected by Lemma 6, it holds, by Lemma 1(2), that \( \tilde{F}_3 \subseteq [u, v] \), which contradicts the assumption that \( \tilde{F}_3 \) is toll concave. Therefore, \( \tilde{F}' \) has at most 2 toll concave sets.

Now, suppose for contradiction that \( \tilde{F}_1 \) is a toll concave set of type 2 or 3. This means that every vertex of \( \tilde{F}_1 \) is universal to \( \tilde{F}_1 \). Therefore, we have that \( u, v \notin F_1 \) because each of \( u \) and \( v \) has at least one non-neighbor in \( \tilde{F}' \), \( \tilde{F}' \subseteq \tilde{F}_1 \), and \( \tilde{F}_1 \) is a clique. As in the previous case, this implies that \( \tilde{F}_1 \subseteq [u, v] \), which means that \( \tilde{F}_1 \) is not a toll concave set, a contradiction.

Now, we show that all choices done by the algorithm are possible in the specific situations that they are done.

**Lemma 8** Consider Algorithm 1. Choices 1, 4, 3, and 8 are always possible in lines 10, 12, 25, and 29, respectively. Choice 5 or 6 is always possible in line 27.

Proof Choice 1 is always possible for a toll concave set having type 1 and Choices 4 and 8 are always possible for a toll concave set having type 2.

Then consider line 25. By definition of type 1, there is \( u \in \tilde{F}' \) with a non-neighbor in \( \tilde{F}' \). If \( \tilde{F}' \subseteq \tilde{F}_1 \), it is clear that \( u \in \tilde{F}_1 \) for some \( F \in \mathcal{M}' \cup \mathcal{F}' \). Then, set \( F_i = F \), and set any other member of \( \mathcal{M}' \cup \mathcal{F}' \) as \( F_j \). If \( \tilde{F}' \subseteq \tilde{F}_3 \), then there is only one member of \( \mathcal{M}' \cup \mathcal{F}' \) containing \( \tilde{F}' \). Set such member as \( F_j \). Since, for any member \( F \in \mathcal{M}' \cup \mathcal{F}' \setminus \{F_j\} \), it holds that \( \tilde{F} \) and \( \tilde{F}_j \) are disjoint by Lemma 6, any member of \( \mathcal{M}' \cup \mathcal{F}' \setminus \{F_j\} \) can be chosen as \( F_j \) and any member of \( \tilde{F}_j \) as \( u \).

Finally, consider line 27. Since \( \tilde{F}' \) is not toll concave and \( \tilde{F}' \) is a clique, there are vertices \( u_i, u_j \notin F' \) for which there is a tolled \((u_i, u_j)\)-walk \( W \) containing
vertices of $\vec{F}'$. Since $\vec{F}'$ is toll concave of type 2, every vertex of $\vec{F}'$ is universal to $\vec{F}'$. Therefore, $u_i, u_j \in \vec{F}'$ and we can write $u_i \in F_i \setminus (\vec{F}' \cup \vec{F}^*)$ and $u_j \in F_j \setminus (\vec{F}' \cup \vec{F}^*)$, for $F_i, F_j \in M' \cup F' \setminus \{F^*\}$, which matches with Choice 5 or 6.

The next result is essential to show that only one vertex suffices for every set $F$ belonging to the toll hull characteristic family constructed by the algorithm such that $\vec{F}$ has type 1. We need one more definition. If $S \subset V(G)$ is such that there is a maximal set $S'$ containing $S$ that induces a connected graph, then we denote by $\bar{S}$ the vertex set of the connected component of $G - S$ containing $S' \setminus S$.

**Lemma 9** If $F \in \mathcal{C}$ of Algorithm 1 is such that $\vec{F}$ has type 1, then $S \cap \vec{F} = \{u\}$ and, for every $v \in \vec{F}$, it holds $\vec{F} \subset \langle u, v \rangle_t$.

**Proof** Sets are added to $\mathcal{F}$ in lines 7 and 21. First, consider that $F^*$ was added to $\mathcal{F}$ in line 7. Let $v \in \vec{F}^*$. Since $\vec{F}^*$ is an extremal mp-subgraph, there is an mp-subgraph $F$ in $G$ such that $\vec{F} \subset F$. Let $H$ be the connected component of $G - (\vec{F}^* - u')$ containing $\vec{F} \setminus \vec{F}^*$. Then there is a $(v, u')$-path $P$ in $H$. The concatenation of $P$ with a $(u, u')$-path of $G[F^*]$ is a tolled $(u, v)$-walk containing $u'$. Since $F^* \subset \langle u, u' \rangle_t$, it holds that $\vec{F}^* \subset \langle u, v \rangle_t$.

Now, we consider that $F^*$ was added to $\mathcal{F}$ in line 21. Let $\vec{F}'$, $F' = \{F', F_1, \ldots, F_{k'}\}$, and $M' = \{F_{k+1}, \ldots, F_{k'}\}$ for $k' \geq 1$ be obtained in lines 16, 17, and 19, respectively, of the same iteration that $F^*$ was obtained. Observe that every vertex $w \notin F^*$ has a non-neighbour in $\vec{F}'$ because otherwise $\{w\} \cup F'$ would be a clique, which would mean that $\{w\} \cup F'$ is contained in some mp-subgraph of $G$, and then $w$ would belong to $F^*$ by the construction of $F^*$. For any vertex $v \notin \vec{F}^*$, we will denote by $v'$ a vertex of $\vec{F}'$ that is not adjacent to $v$. Observe that $\vec{F}' \setminus \vec{F}^* \subset \vec{F}^*$ because otherwise it would exist an mp-subgraph outside $\vec{F}^*$ containing $\vec{F}'$. This implies that $\vec{F}' \setminus \vec{F}^* \neq \emptyset$.

For every toll concave set $S$ of type 1 added to $\mathcal{F}$, we associate a natural number $\ell(S)$. It is clear that a set is added to $\mathcal{F}$ at most once and this occurs in lines 7 or 21. If $S$ is added to $\mathcal{F}$ in line 7, set $\ell(S) = 1$. If $S$ is added to $\mathcal{F}$ in line 21 and all members of $\mathcal{F}'$, of the same iteration, are not toll concave, set $\ell(S) = 1$. Otherwise, define $\ell(S) = 1 + \max\{\ell(F) : F \in \mathcal{F}' \text{ and } \vec{F} \text{ is toll concave}\}$.

It is clear that $\ell$ is well defined. We use induction on $\ell(F^*)$ to prove that $\vec{F} \subset \langle u, v \rangle_t$ for every $F \in M' \cup \mathcal{F}'$. For the basis, consider $\ell(F^*) = 1$. One can choose $v' \in \vec{F}' \setminus \vec{F}^*$ for any $v \notin \vec{F}^*$ because, for any set $S$, there is no edge
between a vertex of $\bar{S}$ and a vertex of $V(G) \setminus S$. We will show that we can always do at least one of the following choices for $F^*$.

Now, suppose that Choice 2 is possible. Let $P_1$ be a $(v, u'')$-path of $G - N[u]$, let $P_2$ be a $(u, v')$-path of $G - N[v]$, and let $P_3 = w''v$. It is clear that the paths $P_1$, $P_2$, and $P_3$ form a tolled $(u, v)$-walk $W$. For every $F_p \in \mathcal{F}' \cup \mathcal{F}'' \setminus \{F_1, F_2\}$, it holds that $u'', v' \in F_1$, because $F'' \subseteq F_2$. Since $F_p$ induces a connected graph by Lemma 6 and every vertex of $F_p$ has a neighbor in $F_1$, Lemma 1(2) implies that $F_p \cap \langle u, v \rangle_t$ for every $p \notin \{i, j\}$.

Next, we show that $F_2 \subset \langle u, v \rangle_t$. Let $C_1, \ldots, C_s$ be the vertex sets of the connected components of $G[F_2] \setminus (F^* \cup F_1)$. Since $G[F_2] \setminus F^*$ and $G[F_2] \setminus F_1$ are connected graphs, for $1 \leq z \leq s$, every $C_z$ contains a neighbor $w_z \in F^* \setminus F_1$ and a neighbor $w_z' F_1 \setminus F^*$. Now, let $P_z$ be a $(v, w_z)$-path of $G[V' \cup \{w_z\}]$ and $P_z'$ be a $(v, w_z')$-path of $G[F_1 \cup \{w_z', F_1\}]$. Now, for each set $C_z$, the paths $P_z$ and $P_z'$ can be used to find a tolled $(u, v)$-walk such that, using Lemma 1(2), we can conclude that $F_z \subset \langle u, v \rangle_t$.

For this case, it remains to show that $F_1 \subset \langle u, v \rangle_t$. Since $F_1$ is not toll concave, there are vertices $w_p, w_q \notin F_1$ for which there is a tolled $(w_p, w_q)$-walk $W'$ containing some vertex of $F_1$. At least one of $\{w_p, w_q\}$ belongs to $F^* \setminus F_1$ because $F^*$ is toll concave. On one hand, both belong to $F^* \setminus F_1$ and we can write $w_p \in F_p \setminus F^*$ and $w_q \in F_q \setminus F^*$ for $i \notin \{p, q\}$. We claim that if $w_p \in F_p$, then $p \neq q$. Then, suppose that $w_p \in F_p$ and $p = q$. This means that $F_p$ is not a clique. Hence, $F_p \cap F_1$ contains $F^*$ but not contains $w_p$, i.e., $w_p$ has a non-neighbor in $F^*$. This implies that $p = j$ and $w_p \in F^*$, a contradiction, because $w_p \in F^*$. Therefore, either $w_p \notin F_p$ and $w_q \in F_q$ or $p \neq q$. In the latter case, since every vertex of $F_p$ contains a neighbor in $F_1$, we can assume that $w_p \notin F_p$ and, analogously, that $w_q \in F_q$. On the other hand, we can write $w_p \notin F^*$ and $w_q \in F_q$ for $q \neq i$. As in the previous case, we can assume that $w_q \in F_q$ and, since every vertex of $F^*$ has a neighbor in $V(G) \setminus F^*$, we can assume that $w_p = v \notin F^*$. In both hands, using Lemma 1(2), we have that $F_1 \subset \langle u, v \rangle_t$.

We now consider that only Choice 3 can be done. This implies that the only member of $\mathcal{F}' \cup \mathcal{F}''$ containing $F^*$ is $F_2$ and all vertices of the other members are universal to $F^*$. The proof that $F_2 \subset \langle u, v \rangle_t$ is the same of the previous case. Since any $F_p \in \mathcal{F}' \cup \mathcal{F}'' \setminus \{F_2\}$ is not toll concave, there are vertices $w, w' \notin F_2$ for which $F_2 \cap \{w, w'\} \neq \emptyset$. At least one of $\{w, w'\}$, say $w$, belongs to $F_2$ because $F^*$ is toll concave. Hence, there is a tolled $(w, w')$-walk
or a toled \((w, v)\)-walk containing vertices of \(\overrightarrow{F}_p\). Since \(\overrightarrow{F}_p\) induces a connected graph by Lemma 6, by Lemma 1(1), we have \(\overrightarrow{F}_p \subseteq (u, v)_1\).

Now, consider \(\ell(F^*) \geq 2\) and that the result holds for every toll concave set \(F\) added to \(F\) such that \(\ell(F) < \ell(F^*)\). This means that \(\overrightarrow{F} \setminus \{\overrightarrow{F'}\}\) contains at least one member that is toll concave of type 1, say \(F_1\). By the induction hypothesis, there is \(u_1 \in \overrightarrow{F}_1\) such that \(\overrightarrow{F_1} \subseteq (u_1, v')_1\) for every vertex of \(w \notin F_1\), in particular for \(w = v \notin F^*\).

We claim that some vertex of \(\overrightarrow{F}_1\) has a non-neighbor in \(\overrightarrow{F'}\). Suppose the contrary. Since \(\overrightarrow{F'}\) is not toll concave, there exist vertices \(z, z' \notin F'\) such that \(\overrightarrow{F'} \cap [z, z'] \neq \emptyset\). Since both \(z\) and \(z'\) have at least one non-neighbor in \(\overrightarrow{F'}\) and, by assumption, \(\overrightarrow{F'}\) is toll concave, at least one, say \(z\) belongs to \(\overrightarrow{F'} \setminus \overrightarrow{F}_1\). Since every vertex of \(F^*\) has a neighbor in \(\overrightarrow{F}_1\), we conclude that a toled \((z, z')\)-walk containing some vertex of \(\overrightarrow{F'}\) can be modified to contain a vertex of \(\overrightarrow{F}_1\), which is not possible because \(\overrightarrow{F}_1\) is toll concave. Therefore, let \(u \in \overrightarrow{F}_1\) having a non-neighbor \(u'' \in \overrightarrow{F'}\). Recall that \(v' \in \overrightarrow{F'}\) is a vertex non-adjacent to \(v\). Therefore, there is a toled \((u, v')\)-walk containing vertices \(u''\) and \(v'\) that imply, using Lemma 1(2), that \(\overrightarrow{F}_p \subseteq (u, v)_1\) for every \(F_p \in \mathcal{M}' \cup \mathcal{F}' \setminus \{F_1\}\) because every vertex of \(\overrightarrow{F}^*\) has a neighbor in \(\overrightarrow{F}_p\) and \(\overrightarrow{F}_p \setminus \overrightarrow{F}^*\) induces a connected graph.

It remains to show that \(\overrightarrow{F}_j \setminus \overrightarrow{F}^* \subseteq (u, v)_1\) for \(0 \leq j \leq k'\). Let \(w \in \overrightarrow{F}_j \setminus \overrightarrow{F}^*\). This means that \(w\) has a neighbor in \(\overrightarrow{F}_j\) and a neighbor in \(\overrightarrow{F}_{j'}\) for some \(j' \neq j\). Therefore \(\overrightarrow{F}^* \subseteq (u, v)_1\).

The above proof has the following consequence.

**Corollary 3** If \(\overrightarrow{F}^*\) was obtained in line 20 of Algorithm 1 such that \(\overrightarrow{F}^*\) has type 1, then the family \(F\) obtained in line 19 of the same iteration has at most one member \(F\) such that \(\overrightarrow{F}\) is toll concave.

Now, we show that the toll hull number of \(\overrightarrow{F}\) is 2 for every \(F\) belonging to the toll hull characteristic family constructed by the algorithm such that \(\overrightarrow{F}\) has type 2.

**Lemma 10** If \(F \in \mathcal{C}\) of Algorithm 1 is such that \(\overrightarrow{F}\) has type 2, then \(S \cap \overrightarrow{F} = \{u_1, u_2\}\) and \(\overrightarrow{F} \subseteq (u_1, u_2)_1\).

**Proof** Sets are added to \(F\) in lines 7 and 21. For the case that \(\overrightarrow{F}^*\) was added to \(F\) in line 7, the result follows due Corollary 2 because \(\overrightarrow{F}^*\) is an \(mp\)-subgraph.

Now, we consider that \(\overrightarrow{F}^*\) was added to \(F\) in line 21. Let \(\overrightarrow{F'}, \mathcal{M}'\), and \(\mathcal{F}'\) be obtained in lines 16, 17, and 19, respectively, of the same iteration that
$F^*$ was obtained. Let $\{F_1, \ldots, F_k\}$ be the members $F$ of $F'$ such that $\vec{F}$ is toll concave. By Lemma 7, $k \leq 2$.

If it was done Choice 5, then, by Lemma 1(1), it holds $\vec{F}_k \subseteq \langle u_1, u_2 \rangle_t$ for every $F_k \in \mathcal{M}' \cup F' \setminus \{F_1\}$ because every vertex of $F'$ has a neighbor in $\vec{F}_k$. If $k = 2$ or it was done Choice 6 or 7, then, by Lemma 1(1), it holds $\vec{F}_k \subseteq \langle u_1, u_2 \rangle_t$ for every $F_k \in \mathcal{M}' \cup F' \setminus \{F_1, F_2\}$ because every vertex of $F'$ has a neighbor in $\vec{F}_k$. Therefore, we have to show that $\vec{F}_1 \subseteq \langle u_1, u_2 \rangle_t$ for Choice 5 and that $\vec{F}_1 \cup \vec{F}_2 \subseteq \langle u_1, u_2 \rangle_t$ for $k = 2$ or Choice 6 or 7.

First consider $k = 0$. For Choice 5, since no vertex outside $F^*$ belongs to a toll walk containing vertices of $\vec{F}^*$, there are vertices $w, w' \in \vec{F}^* \setminus F_1$ such that $\vec{F}_1 \subseteq \langle w, w' \rangle_t$. Then, $\vec{F}_1 \subset \langle u_1, u_2 \rangle_t$. For Choice 6, we can assume that Choice 5 is not possible. This means that, for some $F_k \in \mathcal{M}' \cup F' \setminus \{F_1, F_2\}$, there is $u_k \in \vec{F}_k$ such that $\vec{F}_1 \subset \langle u_1, u_k \rangle_t$ or there are $u_k, u_k' \in \vec{F}_k$ such that $\vec{F}_1 \subset \langle u_k, u_k' \rangle_t$. Therefore, $\vec{F}_1 \subset \langle u_1, u_2 \rangle_t$. Which implies by symmetry that $\vec{F}_2 \subset \langle u_1, u_2 \rangle_t$.

For $k = 1$, consider first that Choice 7. From Lemma 9, it holds that $\vec{F}_1 \subseteq \langle u_1, u_2 \rangle_t$. Now, since no vertex outside $F^*$ belongs to a toll walk containing vertices of $\vec{F}_2$, it holds $\vec{F}_2 \subset \langle u_1, u_2 \rangle_t$. Now consider that Choice 8 was done. Note that we can assume that Choice 7 is not possible. Again Lemma 9 implies that $\vec{F}_1 \subseteq \langle u_1, u_2 \rangle_t$. Now, since every vertex of $F \in \mathcal{M}' \cup F' \setminus \{F_1\}$ is universal to $F'$ and both $\vec{F}$ and $\vec{F}^*$ are toll concave, there are vertices in $\vec{F}_1$ whose toll interval contain vertices of $F$, which means, by Lemma 1(1), that $\vec{F}_2 \subseteq \langle u_1, u_2 \rangle_t$.

For $k = 2$, the result follows directly from Lemma 9.

It remains to show that $\vec{F}_p \setminus F^* \subseteq \langle u_i, u_j \rangle_t$ for $F_p \in \mathcal{M}' \cup F'$. Let $w \in \vec{F}_p \setminus F^*$. This means that $w$ has a neighbor in $\vec{F}_p$ and a neighbor in $\vec{F}_q$ for $F_q \in \mathcal{M}' \cup F' \setminus \{F_p\}$. Therefore $\vec{F}^* \subseteq \langle u_i, u_j \rangle_t$.

**Theorem 2** Algorithm 1 is correct.

**Proof** If $G$ is prime, let $\mathcal{C} = \{V(G)\}$; otherwise, let $\mathcal{C}$ be the family of line 30. Observe that, for every member $\vec{F}$ of $\mathcal{C}$ with type $i$, it holds that if $i \in \{1, 2\}$, then $|S \cap F| = i$; and if $i = 3$, then $\vec{F} \subseteq S$. Furthermore, Lemma 1 implies that every vertex of $S$ matches exactly one of these three possibilities. Since the members of $\mathcal{C}$ are pairwise disjoint and toll concave, by Lemma 3, it suffices to show that $S$ is a toll hull set of $G$. We also have as consequence that $\mathcal{C}$ is a hull characteristic family of $G$. 
First, consider $C = \{F\}$. Then, the type of $\vec{F}$ is 2 or 3. In the former case, the result follows by Lemma 10 and, in the latter case, $G$ is a complete graph and the result follows by Corollary 2.

Now, consider $|C| \geq 2$ and let $F \in C$. First, we show that $\vec{F} \subset \langle S \rangle_t$ for every $F \in C$. If $F$ has type 3, $\vec{F} \subset S$. If $F$ has type 2, then $\vec{F} \subset \langle S \rangle_t$ by Lemma 10. In the last case, $\vec{F}$ has type 1. By Lemma 9, it suffices to show that $\vec{F}$ contains a set of $\vec{C}$. Suppose the contrary. Since $\vec{F}$ is toll concave, $\vec{F}$ is a clique separator of $G$. This implies that there is at least one mp-subgraph $M$ of $G$ containing $\vec{F}$. Denote by $H$ the subgraph of $G$ induced by $\vec{F}$ union the vertex set of the connected components of $G - \vec{F}$ containing $M \setminus \vec{F}$. Let $M_1, \ldots, M_p$ be the extremal mp-subgraphs of $G$ contained in $H$. Each $M_i$ for $1 \leq i \leq p$ was added to $F$ in line 7. Furthermore, each $M_i$ belongs to a member of a subfamily $\mathcal{F}'' = \{F_1, \ldots, F_k\}$ of $\mathcal{F}$ at the end of the While loop such that $\vec{F}_1$ is not toll concave and $F_1$ is not a clique because of the halt condition of the While loop. Denote by $\mathcal{M}''$ the mp-subgraphs of $\mathcal{M}'$ contained in $V(H)$.

We can say that $M$ is contained only in $F_1$. Then, since $\vec{F}_1$ is not a clique, $F_1$ shares vertices with $F$ and a set $F'_1 \subseteq \mathcal{M}'' \cup \mathcal{F}''$. Therefore, $F'_1$ shares vertices with some $F''_2 \subseteq \mathcal{M}'' \cup \mathcal{F}'' \setminus \{F'_1\}$. Using Lemmas 5 and 6(3), we conclude that there is an ordering for the members of $\mathcal{M}'' \cup \mathcal{F}''$ such that $F''_j$ shares vertices with some $F''_{j+1} \subseteq \mathcal{M}'' \cup \mathcal{F}'' \setminus \{F''_1, \ldots, F''_j\}$. Since $H$ is finite, we have a contradiction. Therefore, $\vec{F}$ contains a set of $\vec{C}$.

It remains to show that every $v \notin \bigcup_{F \in C} \vec{F}$ belongs to $\langle S \rangle_t$. Suppose the contrary and let $v \in B \subseteq V(G) \setminus \langle S \rangle_t$, such that $G[B]$ is connected and $B$ is maximal. Denote $G' = G - B$. We have two possibilities for $v$, either $v$ belongs to a member of the family $\mathcal{M}'$ at the end of the While loop, or to a member of $\mathcal{F}$ at the end of the While loop that is not toll concave. For both cases, $G'$ is disconnected because otherwise $B$ would contain an extremal mp-subgraph, which would mean that $B$ is a member of $\mathcal{F}$ that is not toll concave, contradicting the assumption on $\mathcal{F}$. Then, let $u_1$ and $u_2$ be vertices of different connected components of $G'$. It is clear that there is a tolled $(u_1, u_2)$-walk containing $v$. By the maximality of $B$, $u_1, u_2 \in \langle S \rangle_t$, which is a contradiction.

**Theorem 3** For an input graph of order $n$ and size $m$, Algorithm 1 runs in $O(n^3m)$ steps.

**Proof** We begin observing that time complexity of each Choice $i$ for $i \in \{1, \ldots, 8\}$ is clearly $O(n^2)$. Furthermore, since each loop has $O(n)$ iterations, the costs of all choices is $O(n^3)$.

Lines 3 and 5 can done in $O(nm)$ using the algorithm in (Leimer 1993). Lines 6 and 7 can be done in $O(n^3)$. The number of iterations of the While loop is $O(n)$. Using Lemma 2, one can test whether a set is toll concave in $O(n^3m)$ steps. Then lines 8 to 14 can be done in $O(n^3m)$ steps.
Every time that line 16 is reached, we already know, for each member of $F \in \mathcal{F}$, whether $\mathcal{F}$ is not toll concave. Then the conditions of line 16 can be tested in $O(n^3)$. Each operation from line 17 to 21 can be done in $O(n^3m)$. Since lines 22 to 29 can be done in $O(n^2m)$, the While loop costs $O(n^3m)$ which is the overall time complexity of Algorithm 1 because the cost of line 30 is $O(n^3)$.

4 Concluding remarks

We conclude discussing some consequences of Algorithm 1. First, we observe that the number of minimum toll hull sets can be exponential on the size of the graph. However, using the toll hull characteristic family constructed by Algorithm 1, one can enumerate all minimum toll hull sets of $G$ with polynomial time delay. For this, it suffices to change the choices used by the algorithm so that they find all possible selections for a concave set $S$ accordingly to the appropriate choice, i.e., if $S$ has type 1, let $t(S)$ be formed by all vertices $x$ such that $x$ satisfies the appropriate choice for $S$; and if $S$ has type 2, let $t(S)$ be formed by all pairs $\{x,y\}$ such that $\{x,y\}$ satisfies the appropriate choice for $S$. Therefore, the algorithm of enumeration consists of finding all combinations considering the possible choices for each concave set of the toll hull characteristic family.

Another consequence of Algorithm 1 together with the notion of granularity is a characterization of toll extreme vertices of a graph. As discussed in (Alcón et al. 2015), the property of a vertex being an extreme vertex is not well-behaved in toll convexity as in other well-studied convexities, such as geodetic, nonophonic, and $P_3$ convexities, where the neighborhood of the vertex has all information to answer the question. Using the toll hull characteristic family of Algorithm 1, we have the following characterization of the toll extreme vertices of a graph.

**Corollary 4** Let $\mathcal{C}$ be the family constructed in line 30 of Algorithm 1 run over a graph $G$. The set of extreme vertices of $G$ is formed by the vertices belonging to concave sets of $\mathcal{C}$ having type 3.

**Proof** By Theorem 2, it suffices to show that there are no extremal vertices in every set $\mathcal{F}$ for $S \in \mathcal{C}$ of type 1 or 2. If $t(S) > 1$, we are done. Then assume $t(S) = 1$. First consider that $\mathcal{F}$ has type 1. This means that there are $u \in \mathcal{F}$ and $u' \in \mathcal{F}$ such that $uu' \not\in E(G)$. If $N[u]$ is a clique, it holds that $N[u] \setminus \mathcal{F}$ is a toll concave of type 3, because $\mathcal{F} \setminus \{u\}$ toll concave of type 2, Which contradicts Algorithm 1. Then, $u$ is not a simplicial vertex, and therefore, $u$ is not a toll extreme vertex.

Now, consider that $\mathcal{F}$ has type 2 and let $t(S) = \{\{u,v\}\}$. We have that $N[u] \setminus \mathcal{F}$ and $N[u] \setminus \mathcal{F}$ are cliques and that $\{u\}$ and $\{v\}$ are toll concave sets of type 3, which contradicts Algorithm 1.
A direct application of Lemma 2 leads to an algorithm for finding the toll extreme vertices of a graph in $O(n^2 m)$. Using the following characterization, this can be done in $O(n^3)$ using lines 5, 7, and 14 of Algorithm 1.

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