K-HARMONIC CURVES INTO A RIEMANNIAN MANIFOLD WITH CONSTANT SECTIONAL CURVATURE

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Abstract. In [5], J. Eells and L. Lemaire introduced $k$-harmonic maps, and T. Ichiyama, J. Inoguchi and H. Urakawa [1] showed the first variation formula. In this paper, we describe the ordinary differential equations of 3-harmonic curves into a Riemannian manifold with constant sectional curvature, and show biharmonic curve is $k$-harmonic curve ($k \geq 2$).

Introduction

Theory of harmonic maps has been applied into various fields in differential geometry. The harmonic maps between two Riemannian manifolds are critical maps of the energy functional

$$E(\phi) = \frac{1}{2} \int_M \|d\phi\|^2 v_g,$$

for smooth maps $\phi : M \to N$.

On the other hand, in 1981, J. Eells and L. Lemaire [5] proposed the problem to consider the $k$-harmonic maps: they are critical maps of the functional

$$E_k(\phi) = \int_M e_k(\phi)v_g, \quad (k = 1, 2, \cdots),$$

where $e_k(\phi) = \frac{1}{2} \|(d + d^*)^k \phi\|^2$ for smooth maps $\phi : M \to N$. G.Y. Jiang [4] studied the first and second variation formulas of the bi-energy $E_2$, and critical maps of $E_2$ are called biharmonic maps. There have been extensive studies on biharmonic maps.

Recently, in 2009, T. Ichiyama, J. Inoguchi and H. Urakawa [1] studied the first variation formula of the $k$-energy $E_k$, whose critical maps are called $k$-harmonic maps. Harmonic maps are always $k$-harmonic maps by definition. In this paper, we study $k$-harmonic curves.

In [1] we introduce notation and fundamental formulas of the tension field.

In [2] we show biharmonic curves into a Riemannian manifold with constant sectional curvature is always $k$-harmonic curves.

1. Preliminaries

Let $(M, g)$ be an $m$ dimensional Riemannian manifold, $(N, h)$ an $n$ dimensional one, and $\phi : M \to N$, a smooth map. We use the following notation. The second fundamental form $B(\phi)$ of $\phi$ is a covariant differentiation $\nabla d\phi$ of 1-form $d\phi$, which is a section of $\otimes^2 T^* M \otimes \phi^{-1} T N$. For every $X, Y \in \Gamma(TM)$, let

$$B(X, Y) = (\nabla d\phi)(X, Y) = (\nabla_X d\phi)(Y)$$

(1)

$$= \nabla_X d\phi(Y) - d\phi(\nabla_X Y) = \nabla^N_{d\phi(X)} d\phi(Y) - d\phi(\nabla_X Y).$$

Here, $\nabla, \nabla^N, \nabla, \bar{\nabla}$ are the induced connections on the bundles $TM, TN, \phi^{-1} TN$ and $T^* M \otimes \phi^{-1} TN$, respectively.

If $M$ is compact, we consider critical maps of the energy functional

$$E(\phi) = \int_M e(\phi)v_g,$$

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where \( e(\phi) = \frac{1}{2} \| d\phi \|^2 = \sum_{i=1}^{m} \frac{1}{2} (d\phi(e_i), d\phi(e_i)) \) which is called the energy density of \( \phi \), and the inner product \( \langle \cdot, \cdot \rangle \) is a Riemannian metric \( h \). The tension field \( \tau(\phi) \) of \( \phi \) is defined by

\[
\tau(\phi) = \sum_{i=1}^{m} \left( \nabla d\phi(e_i), e_i \right) = \sum_{i=1}^{m} \left( \nabla e_i d\phi(e_i) \right).
\]

Then, \( \phi \) is a harmonic map if \( \tau(\phi) = 0 \).

And we define

\[
\Delta = \nabla^* \nabla = - \sum_{k=1}^{m} \left( \nabla e_k \nabla e_k - \nabla \nabla e_k e_k \right),
\]

is the rough Laplacian.

And we define \( R \) as follows:

\[
R(V) := \sum_{i=1}^{m} R^N(V, d\phi(e_i))d\phi(e_i), \quad V \in \Gamma(\phi^{-1}TN),
\]

where,

\[
R^N(U, V) = \nabla^N_U \nabla^N_V - \nabla^N_V \nabla^N_U - \nabla^N_{[U,V]}, \quad U, V \in \Gamma(TN),
\]

is the curvature tensor of \((N, h)\).

2. \( k \)-harmonic curves into a Riemannian manifold with constant sectional curvature

In this section, we consider curves into a Riemannian manifold with constant sectional curvature. Then, we show the necessary and sufficient condition of 3-harmonic curve, and biharmonic curve is \( k \)-harmonic curve.

J. Eells and L. Lemaire [5] proposed the notation of \( k \)-harmonic maps. The Euler-Lagrange equations for the \( k \)-harmonic maps was shown by T. Ichiyama, J. Inoguchi and H. Urakawa [1]. We first recall it briefly.

**Theorem 2.1** ([1]). Let \( k = 2, 3, \cdots \). Then, we have

\[
\left. \frac{d}{dt} \right|_{t=0} E_k(\phi_t) = - \int_M \langle \tau_k(\phi), V \rangle v_g,
\]

where

\[
\tau_k(\phi) := J \left( \Delta^{(k-2)} \tau(\phi) \right) = \Delta \left( \Delta^{(k-2)} \tau(\phi) \right) - R \left( \Delta^{(k-2)} \tau(\phi) \right),
\]

and

\[
\Delta^{(k-2)} \tau(\phi) = \underbrace{\Delta \cdots \Delta}_{k-2} \tau(\phi).
\]

As a corollary of this theorem, we have

**Corollary 2.2** ([1]). \( \phi : (M, g) \to (N, h) \) is a \( k \)-harmonic map if

\[
\tau_k(\phi) := J \left( \Delta^{(k-2)} \tau(\phi) \right) = \Delta \left( \Delta^{(k-2)} \tau(\phi) \right) - R \left( \Delta^{(k-2)} \tau(\phi) \right) = 0.
\]

We say for a \( k \)-harmonic map to be proper if it is not harmonic.
Let us recall the definition of the Frenet frame.

Definition 2.3. The Frenet frame \( \{e_1\}_{i=1 \cdots n} \) associated to a curve \( \gamma : I \in \mathbb{R} \to (N^n, \langle \cdot, \cdot \rangle) \), parametrized by arc length, is the orthonormalisation of the \( (n+1) \)-uple \( \{\nabla_N \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \}_{k=1 \cdots n} \), described by

\[
\begin{align*}
    e_1 &= \frac{d\gamma}{dt}, \\
    \nabla_N e_1 &= \kappa_1 e_2, \\
    \nabla_N e_i &= -\kappa_{i-1} e_{i-1} + \kappa_i e_{i+1} \quad (i = 2, \cdots, n - 1), \\
    \nabla_N e_n &= -\kappa_{n-1} e_{n-1},
\end{align*}
\]

where the functions \( \kappa_1, \kappa_2, \cdots, \kappa_{n-1} \) are called the curvatures of \( \gamma \). Note that \( e_1 = \gamma' \) is the unit tangent vector field along the curve.

First, we show the necessary and sufficient condition of \( k \)-harmonic curves into a Riemannian manifold with constant sectional curvature.

Proposition 2.4. Let \( \gamma : I \to (N^n, \langle \cdot, \cdot \rangle) \) be a smooth curve parametrized by arc length from an open interval of \( \mathbb{R} \) into a Riemannian manifold \( (N^n, \langle \cdot, \cdot \rangle) \) with constant sectional curvature \( K \). Then, \( \gamma \) is \( k \)-harmonic if and only if,

\[
(\nabla_N^N \nabla_N^N)^{k-1} \tau(\gamma) - K \{(\nabla_N^N \nabla_N^N)^{k-2} \tau(\gamma) - \langle \gamma', (\nabla_N^N \nabla_N^N)^{k-2} \tau(\gamma) \rangle \gamma' \} = 0.
\]

Proof. \( \triangle \tau(\gamma) = (-1)^{k-1}(\nabla_N^N \nabla_N^N)^{k-1} \tau(\gamma), \)

\[
\mathcal{R}(\nabla_N^N \nabla_N^N)^{k-2} \tau(\gamma)) = K \{(\nabla_N^N \nabla_N^N)^{k-2} \tau(\gamma) - \langle \gamma', (\nabla_N^N \nabla_N^N)^{k-2} \tau(\gamma) \rangle \gamma' \}.
\]

Therefore, we have Proposition 2.4. \( \square \)

Using Proposition 2.4, we show the necessary and sufficient condition of biharmonic curve and \( 3 \)-harmonic curve, respectively.

Proposition 2.5. Let \( \gamma : I \to (N^n, \langle \cdot, \cdot \rangle) \) be a smooth curve parametrized by arc length from an open interval of \( \mathbb{R} \) into a Riemannian manifold \( (N^n, \langle \cdot, \cdot \rangle) \) with constant sectional curvature \( K \). Then, \( \gamma \) is proper biharmonic if and only if,

\[
\begin{align*}
    \kappa_1^2 + \kappa_2^2 &= K, \\
    \kappa_1 &= \text{constant} \neq 0, \\
    \kappa_2 &= \text{constant}, \\
    \kappa_2 \kappa_3 &= 0.
\end{align*}
\]

Proof. \( \tau(\gamma) = \kappa_1 e_2 \). So we caluclate \( (\nabla_N^N \nabla_N^N)(\kappa_1 e_2) \) as follows.

\[
(\nabla_N^N \nabla_N^N)(\kappa_1 e_2) = -3\kappa_1 \kappa_3 e_1 + (\kappa_1^2 - \kappa_1^2 - \kappa_1 \kappa_3^2) e_2 + (2\kappa_1 \kappa_2 + \kappa_1 \kappa_3) e_3 + \kappa_1 \kappa_2 \kappa_3 e_4.
\]

Using Proposition 2.4 and \( \kappa_1 \neq 0 \), we have Proposition 2.5. \( \square \)
Proposition 2.6. Let \( \gamma : I \to (N^n, \langle \cdot, \cdot \rangle) \) be a smooth curve parametrized by arc length from an open interval of \( \mathbb{R} \) into a Riemannian manifold \((N^n, \langle \cdot, \cdot \rangle)\) with constant sectional curvature \(K\). Then, \(\gamma\) is \(3\)-harmonic if and only if,

\[
\begin{aligned}
&-2\kappa_1'\kappa_1'' - \kappa_1\kappa_1^{(3)} + 2\kappa_1'^3\kappa_2'' + \kappa_1\kappa_1'^3\kappa_2 + \kappa_1^2\kappa_2\kappa_1'' = 0 \\
&-15\kappa_1(\kappa_1')^2 - 10\kappa_1'^3\kappa_1'' + 5\kappa_1^5 + 2\kappa_1'^3\kappa_2^2 + \kappa_1^{(4)} - 6\kappa_1'\kappa_2\kappa_2'' - 12\kappa_1'\kappa_2\kappa_2'' = 0 \\
&-3\kappa_1(\kappa_1')^2 - 4\kappa_1\kappa_1'^3\kappa_2'' + \kappa_1\kappa_1'^3\kappa_2 + \kappa_1^2\kappa_2\kappa_1'' + K(\kappa_1'' - \kappa_1'\kappa_2') = 0 \\
&4\kappa_1^2\kappa_2 - 9\kappa_1'^2\kappa_1 - 4\kappa_1'\kappa_2 - 6\kappa_1\kappa_2\kappa_2'' + 6\kappa_1'^2\kappa_1'' - 5\kappa_1'^2\kappa_1' \\
&+ 4\kappa_1'\kappa_1'' - 4\kappa_1'\kappa_2\kappa_1' - 3\kappa_1\kappa_2\kappa_2'' - 3\kappa_1\kappa_2\kappa_2'' + K\{2\kappa_1\kappa_2 + \kappa_1\kappa_2''\} = 0 \\
&6\kappa_1''\kappa_2\kappa_3 - \kappa_1^2\kappa_2\kappa_3 + 6\kappa_1'\kappa_2\kappa_3 + 3\kappa_1\kappa_2\kappa_3 - 3\kappa_1\kappa_2\kappa_3'' + K\{\kappa_1\kappa_2\kappa_3\} = 0 \\
&4\kappa_1'\kappa_2\kappa_3\kappa_4 + 3\kappa_1\kappa_2\kappa_3\kappa_4 + 2\kappa_1\kappa_2\kappa_3\kappa_4 + \kappa_1\kappa_2\kappa_3\kappa_4'' = 0 \\
&\kappa_1\kappa_2\kappa_3\kappa_4\kappa_5 = 0
\end{aligned}
\]

Proof. We calculate \((\nabla^N_\gamma \nabla^N_\gamma)^2 \tau(\gamma)\) as follows.

\[
(\nabla^N_\gamma \nabla^N_\gamma)^2 \tau(\gamma) = (-10\kappa_1'\kappa_1'' - 5\kappa_1\kappa_1^{(3)} + 10\kappa_1'^3\kappa_1'' + 5\kappa_1'\kappa_2\kappa_1'' + 5\kappa_1^2\kappa_2\kappa_1'')e_1 \\
+ (-15\kappa_1(\kappa_1')^2 - 10\kappa_1'^3\kappa_1'' + \kappa_1^5 + 2\kappa_1'^3\kappa_2^2 + \kappa_1^{(4)} - 6\kappa_1'\kappa_2\kappa_2'' - 12\kappa_1'\kappa_2\kappa_2'')
\]

Using Proposition 2.4 and 2.3, we have Proposition 2.6. \(\square\)

We showed biharmonic curve is \(k\)-harmonic curve into 2-dimensional unit sphere \(S^2\). We generalize this as following.

Theorem 2.7. Let \( \gamma : I \to (N^n, \langle \cdot, \cdot \rangle) \) be a smooth curve parametrized by arc length from an open interval of \( \mathbb{R} \) into a Riemannian manifold \((N^n, \langle \cdot, \cdot \rangle)\) with constant sectional curvature \(K\). Then, biharmonic is \(k\)-harmonic \((k \geq 2)\).

Proof. By Proposition 2.6, \(\gamma\) is proper biharmonic if and only if

\[
\begin{align*}
\kappa_1^2 + \kappa_2^2 &= K, \\
\kappa_1 &= \text{constant} \neq 0, \\
\kappa_2 &= \text{constant}, \\
\kappa_2\kappa_3 &= 0.
\end{align*}
\]
Then, we calculate $(\nabla^N \nabla^N_{\gamma'})^k \tau(\gamma)$.

$$(\nabla^N \nabla^N_{\gamma'})^k \tau(\gamma) = (-1)^k \kappa_1 (\kappa_1^2 + \kappa_2^2)^k e_2$$

$$(\nabla^N \nabla^N_{\gamma'})^k \tau(\gamma) = (-1)^k \kappa_1 K^k e_2.$$ 

So, we have

$$[(\nabla^N \nabla^N_{\gamma'})^{k-1} \tau(\gamma) + K ((\nabla^N \nabla^N_{\gamma'})^{k-2} \tau(\gamma) - (\gamma', (\nabla^N \nabla^N_{\gamma'})^{k-2} \tau(\gamma)) \gamma')]$$

$$(\nabla^N \nabla^N_{\gamma'})^{k-1} \tau(\gamma) + K ((\nabla^N \nabla^N_{\gamma'})^{k-2} \tau(\gamma) - (\gamma', (\nabla^N \nabla^N_{\gamma'})^{k-2} \tau(\gamma)) \gamma')$$

$$= (-1)^{k-1} \kappa_1 K^{k-1} e_2 + K (-1)^{k-2} \kappa_1 K^{k-2} e_2$$

$$= 0.$$ 

And harmonic is always $k$-harmonic. So we have Theorem 2.7.

\[\square\]

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