ON LINEAR EXTENSION FOR INTERPOLATING SEQUENCES.

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Abstract. Let $A$ be a uniform algebra on the compact space $X$ and $\sigma$ a probability measure on $X$. We define the Hardy spaces $H^p(\sigma)$ and the $H^p(\sigma)$ interpolating sequences $S$ in the $p$-spectrum $\mathcal{M}_p$ of $\sigma$. We prove, under some structural hypotheses on $\sigma$ that "Carleson type" conditions on $S$ imply that $S$ is interpolating with a linear extension operator in $H^s(\sigma)$, $s < p$ provided that either $p = \infty$ or $p \leq 2$.

This gives new results on interpolating sequences for Hardy spaces of the ball and the polydisc. In particular in the case of the unit ball of $\mathbb{C}^n$ we get that if there is a sequence $\{\rho_a\}_{a \in S}$ bounded in $H^\infty(B)$ such that $\forall a, b \in S$, $\rho_a(b) = \delta_{ab}$, then $S$ is $H^p(B)$-interpolating with a linear extension operator for any $1 \leq p < \infty$.

1. Introduction

Let $B$ be the unit ball of $\mathbb{C}^n$; we denote as usual by $H^p(B)$ the Hardy spaces of holomorphic functions in $B$. Let $S$ a sequence of points in $B$ and $1 \leq p \leq \infty$ ; we say that $S$ is $H^p$-interpolating if

$$\forall \lambda \in \ell^p(S), \exists f \in H^p(B) \text{ s.t. } \forall a \in S, f(a) = \lambda_a (1 - |a|^2)^{n/p}.$$ 

Let $a \in B$ we set $k_a(z) := \frac{1}{(1 - \bar{a} \cdot z)^n}$ its reproducing kernel and $k_{p,a} := \frac{k_a}{\|k_a\|_p}$ the normalized reproducing kernel for $a$ in $H^p(B)$. Now if $S$ is $H^p$-interpolating, then we have, with $p'$ the conjugate exponent for $p$:

$$\exists C > 0, \forall a \in S, \exists \rho_a \in H^p(B) \text{ s.t. } \langle \rho_a, k_{p',a} \rangle = \delta_{ab}.$$ 

We shall say that $S$ is dual bounded in $H^p(B)$ if the dual system $\{\rho_a\}_{a \in S}$ to $\{k_{p,a}\}_{a \in S}$ exits and is bounded in $H^p(B)$. Hence if $S$ is $H^p$-interpolating then $S$ is dual bounded in $H^p(B)$.

Definition 1.1. We say that the $H^p(B)$ interpolating sequence $S$ has the linear extension property (LEP) if there is a bounded linear operator $E : \ell^p \to H^p(B)$ such that $\forall \lambda \in \ell^p$, $E\lambda$ interpolates the sequence $\lambda$ in $H^p(B)$ on $S$, i.e.

$$\exists C > 0, \forall \lambda \in \ell^p, E\lambda \in H^p(B), \|E\lambda\|_p \leq C \text{ s.t. } \forall a \in S, E\lambda(a) = \lambda_a \|k_a\|_{p'}.$$ 

Natural questions are the following:
If $S$ is dual bounded in $H^p(B)$, is $S \in IH^p(B)$ ?
If $S \in IH^p(B)$ has $S$ automatically the LEP ?

This is true in the classical case of the Hardy spaces of the unit disc $D$ : for $p = \infty$ this is the famous characterization of $H^\infty$ interpolating sequences by L. Carleson [7] and the LEP was given by P. Beurling [6].
for $p \in [1, \infty]$ this was done by H. Shapiro and A. Shields [16] and because the characterization is the same for all $p \in [1, \infty]$, the LEP is deduced easily from the $H^\infty$ case and was done explicitly with 7 methods in [2].

For the Bergman classes $A^p(D)$, it is no longer true that the interpolating sequences are the same for $A^p(D)$ and $A^q(D)$, $q \neq p$. But A.P. Schuster and K. Seip [15], [14] proved that $S$ dual bounded in $A^p(D)$ implies $S A^p(D)$-interpolating still with the LEP.

The first question is open, even in the ball $B$ of $\mathbb{C}^n$, $n \geq 2$, with $H^p(B)$, the usual Hardy spaces of the ball or in the polydisc $D^n$ of $\mathbb{C}^n$, $n \geq 2$ still with the usual Hardy spaces.

The second one is known only in the case $p = \infty$ as we shall see later.

Nevertheless in the case of the unit ball of $\mathbb{C}^n$, B. Berndtsson [4] proved that if the product of the Gleason distances of the points of $S$ is bounded below away from 0 then $S$ is $H^\infty(B)$. He also proved that this condition is not necessary for $n > 1$.

B. Berndtsson, A. S-Y. Chang and K-C. Lin [5] proved the same theorem in the polydisc of $\mathbb{C}^n$.

In this paper we shall prove that loosing a little bit on the value of $p$, $S$ dual bounded in $H^p(B)$ implies $\forall s < p$, $S \in IH^s(B)$ with the LEP, provided that $1 < p \leq 2$ or $p = \infty$. In particular:

**Theorem 1.2.** If $S \subset B$ is dual bounded in $H^p(B)$, then it is $H^s$-interpolating for any $1 \leq s < p$, provided that $p \in [1, 2]$ or $p = \infty$. Moreover $S$ has the property that there is a bounded linear operator from $\ell^s(S) \rightarrow H^s(B)$ doing the interpolation.

The methods we use being purely functional analytic, these results extend to the setting of uniform algebras.

2. **Uniform algebras.**

Let $A$ be a uniform algebra on the compact space $X$, i.e. $A$ is a sub-algebra of $C(X)$, the continuous functions on $X$, which separates the points of $X$ and contains 1.

Let $\sigma$ be a probability measure on $X$.

For $1 \leq p < \infty$ we define as usual the Hardy space $H^p(\sigma)$ as the closure of $A$ in $L^p(\sigma)$.

$H^\infty(\sigma)$ will be the weak-* closure of $A$ in $L^\infty(\sigma)$.

Let $M$ be the Guelfand spectrum of $A$, i.e. the multiplicative elements of $A'$. We note the same way an element of $A$ and its Guelfand transform:

$\forall a \in M \subset A'$, $\forall f \in A$, $f(a) := \hat{f}(a) = a(f)$.

We shall use the following notions, already introduced in [3].

**Definition 2.1.** Let $M$ be the spectrum of $A$ and $a \in M$; we call $k_a \in H^p(\sigma)$ a $p$-reproducing kernel for the point $a$ if $\forall f \in A$, $f(a) = \int_X f(\zeta) \overline{k_a}(\zeta) \, d\sigma(\zeta)$.

We define the $p$-spectrum of $\sigma$ as the subset $M_p$ of $M$ such that every element has a $p'$-reproducing kernel with $p'$ the conjugate exponent for $p$, $\frac{1}{p} + \frac{1}{p'} = 1$.

**Definition 2.2.** We say that $S \subset M_p$ is $H^p(\sigma)$ interpolating for $1 \leq p < \infty$, $S \in IH^p(\sigma)$ if $\forall \lambda \in \ell^p$, $\exists f \in H^p(\sigma)$ s.t. $\forall a \in S$, $f(a) = \lambda_a \|k_a\|_{p'}$.

We say that $S \subset M_{\infty}$ is $H^\infty(\sigma)$ interpolating, $S \in IH^\infty(\sigma)$ if $\forall \lambda \in \ell^\infty$, $\exists f \in H^\infty(\sigma)$ s.t. $\forall a \in S$, $f(a) = \lambda_a$.

**Remark 2.3.** If $S$ is $H^p(\sigma)$-interpolating then there is a constant $C_I$, the interpolating constant, such that [3]:

$\forall \lambda \in \ell^p$, $\exists f \in H^p(\sigma)$, $\|f\|_p \leq C_I \|\lambda\|_p$, s.t. $\forall a \in S$, $f(a) = \lambda_a \|k_a\|_{p'}$.  


Definition 2.4. We say that the $H^p(\sigma)$ interpolating sequence $S$ has the linear extension property (LEP) if there is a bounded linear operator $E : \ell^p \to H^p(\sigma)$ such that $\forall \lambda \in \ell^p$, $E\lambda$ interpolates the sequence $\lambda$ in $H^p(\sigma)$ on $S$, i.e.

$$\exists C > 0, \forall \lambda \in \ell^p, E\lambda \in H^p(\sigma), \|E\lambda\|_p \leq C \text{ s.t. } \forall a \in S, E\lambda(a) = \lambda_a \|a\|_p$$

Let $S \subset M_p$, so $k_{p,a} := \frac{a}{\|a\|_p}$, the normalized reproducing kernel, exits for any $a \in S$; let us consider a dual system $\{\rho_a\}_{a \in S} \subset H^p(\sigma)$, i.e. $\forall a, b \in S$, $\langle \rho_a, k_{p,b} \rangle = \delta_{a,b}$ when it exists.

Definition 2.5. We say that $S \subset M_p$ is dual bounded in $H^p(\sigma)$ if a dual system $\{\rho_a\}_{a \in S} \subset H^p(\sigma)$ exists and if this sequence is bounded in $H^p(\sigma)$, i.e. $\exists C > 0$ s.t. $\forall a \in S, \|\rho_a\|_p \leq C$.

We shall show that, under some structural hypotheses on $\sigma$ and the fact that $S$ is Carleson (the definition of Carleson sequences will be given later):

Theorem 2.6. If $1 \leq s < p$ and either $p < 2$ or $p = \infty$, $S \subset M_p \cap M_s$ is dual bounded in $H^p(\sigma)$ and $S$ is a Carleson sequence, then $S \in IH^s(\sigma)$ with the linear extension property.

The passage from $p = 2$ to $p \leq 2$ in the case of the ball is due to F. Bayart: he uses Khintchine’s inequalities which reveal to be very well fitted to this problem. In fact F. Lust-Piquart showed me a way not to use Khintchine’s inequalities: one can use the fact that $L^p$ spaces are of type $p$ in the part $p \leq 2$ in the proof of theorem 2.6.

I shall add this proof.

The case $p = \infty$ of this theorem is the best possible in this generality. There is no hope to have that dual boundedness in $H^\infty$ implies $H^\infty$-interpolation as L. Carleson proved for the unit disc.

In [10] and in [12] the authors proved that in the spectrum of the uniform algebra $H^\infty(\mathbb{D})$ there are sequences $S$ of points such that the product of the Gleason distances is bounded below away from 0, which implies that $S$ is dual bounded in $H^\infty(\mathbb{D})$, but $S$ is not $H^\infty$-interpolating.

The general theorem 2.6 implies a polydisc and a ball version.

In the polydisc $\mathbb{D}^n \subset \mathbb{C}^n$ the structural hypotheses are true [3], hence

Theorem 2.7. Let $S \subset \mathbb{D}^n$ be a Carleson sequence and dual bounded in $H^p(\mathbb{D}^n)$ with either $p = \infty$ or $1 < p \leq 2$, then $S$ is $H^s(\mathbb{D}^n)$ interpolating for any $1 \leq s < p$ with the LEP.

In the ball, the structural hypotheses are true [3] and moreover we know, by an easy corollary of a theorem of P. Thomas [18], that $S$ dual bounded in $H^p(\mathbb{B})$ implies $S$ Carleson, hence

Theorem 2.8. Let $S \subset \mathbb{B}$ be dual bounded in $H^p(\mathbb{B})$ with either $p = \infty$ or $1 < p \leq 2$, then $S$ is $H^s(\mathbb{B})$ interpolating for any $1 \leq s < p$ with the LEP.

As usual by use of the “subordination lemma” [1] we have the same result for the Bergman classes of the ball. Denote by $A^p(\mathbb{B})$ the holomorphic functions in $L^p(\mathbb{B})$ for the area measure of the ball then

Corollary 2.9. Let $S \subset \mathbb{B}$ be dual bounded in $A^p(\mathbb{B})$ with either $p = \infty$ or $1 < p \leq 2$, then $S$ is $A^s(\mathbb{B})$ interpolating for any $1 \leq s < p$ with the LEP.

In [2] it was proved:

Theorem 2.10. Let $p \geq 1$, $1 \leq s < p$ and $q$ be such that $\frac{1}{s} = \frac{1}{p} + \frac{1}{q}$. Suppose that $S \subset M_s \cap M_q$ is $H^p(\sigma)$ interpolating, $q$-Carleson and $\sigma$ verifies the structural hypotheses, then $S$ is $H^s(\sigma)$ interpolating.
The theorem 2.10 is better for \( p \in [1, 2] \) or \( p = \infty \); we have the LEP under the weaker assumption that \( S \) is dual bounded in \( H^p(\sigma) \).

But we have not the full range of \( p \) as in theorem 2.10.

3. Reproducing kernels.

Let us recall some facts about reproducing kernels and \( p \)-spectrum.

First the reproducing kernel for \( a \in \mathcal{M} \) if it exists is unique. Suppose there are 2 of them, \( k_a \in H^p(\sigma) \) and \( k'_a \in H^q(\sigma) \):

\[
\forall f \in A, \ 0 = f(a) - f(a) = \int_X f(k_a - k'_a) \, ds \implies k_a = k'_a \ \text{a.e.}
\]

because, by definition, \( A \) is dense in \( H^p(\sigma) \) with \( r := \min(p, q) \). Hence it is correct to denote it by \( k_a \) without reference to the \( H^p(\sigma) \) where it belongs.

Let \( a \in \mathcal{M}_p \) then \( k_a \in H^{p'}(\sigma) \); if \( p < q \rightarrow q' < p' \) hence \( k_a \in H^{q'}(\sigma) \) because \( \sigma \) is a probability measure so \( a \in \mathcal{M}_q \) and we have \( p < q \rightarrow \mathcal{M}_p \subset \mathcal{M}_q \).

To simplify the notation we shall use:

\[
\langle f, g \rangle := \int_X f \overline{g} \, d\sigma,
\]

whenever this is meaningful.

If \( a \in \mathcal{M}_2 \) we always have a ”Poisson kernel” associated to \( a \), \( P_a := \frac{|k_a|^2}{\|k_a\|^2} \) and the well known

**Lemma 3.1.** \( P_a \in L^1(\sigma) \), \( \|P_a\|_1 = 1 \) and

\[
\forall f \in A, \ f(a) = \langle f, P_a \rangle = \int_X f P_a \, d\sigma.
\]

Proof

\[
\int_X f P_a \, d\sigma = \int_X f \frac{k_a \overline{k_a}}{\|k_a\|^2} \, d\sigma = \frac{1}{\|k_a\|^2} f(a)k_a(a) = f(a),
\]

because \( fk_a \in H^2(\sigma) \) and \( k_a(a) = \int_X k_a \overline{k_a} \, d\sigma = \|k_a\|^2 \).

This allows us to define the Poisson integral of a bounded function on \( X \):

**Definition 3.2.** Let \( f \in L^\infty(\sigma) \) we set \( \forall a \in \mathcal{M}_2, \ \tilde{f}(a) := \langle f, P_a \rangle \) its Poisson integral.

If \( f \in L^2(\sigma) \) we set \( f^* := P_2 f \) its orthogonal projection on \( H^2(\sigma) \); we extend \( f^* \) on \( \mathcal{M}_2 \):

\[
\forall f \in L^2(\sigma), \ \forall a \in \mathcal{M}_2, \ f^*(a) := \langle f^*, k_a \rangle = \langle f, k_a \rangle.
\]

Of course if \( f \in A \) we have \( f^* = \tilde{f} = f \) and for any \( f \in L^\infty(\sigma), \ \tilde{(f^*)} = f^* \).

3.1. Structural hypotheses. We shall need some structural hypotheses on \( \sigma \) relative to the reproducing kernels.

**Definition 3.3.** Let \( q \in ]1, \infty[ \), we say that the measure \( \sigma \) verifies the structural hypothesis \( SH(q) \) if, with \( q' \) the conjugate of \( q \):

\[
(3.1) \quad \exists \alpha = \alpha_q > 0 \ s.t. \ \forall a \in \mathcal{M}_q \cap \mathcal{M}_{q'} \subset \mathcal{M}_2, \ \|k_a\|^2 \geq \alpha \|k_a\|_q \|k_a\|_{q'}.
\]
This is opposite to the Hölder inequalities.

Because \( a \in \mathcal{M}_q \cap \mathcal{M}_{q'} \subset \mathcal{M}_2 \), we have \( k_a(a) = \int_X k_a(\xi) \overline{f_a}(\xi) \, d\sigma = \|k_a\|_2^2 \) and the condition above is the same as \( \|k_a\|_q \|k_a\|_{q'} \leq \alpha_q^{-1} k_a(a) \).

**Definition 3.4.** Let \( p, s \in [1, \infty] \) and \( q \) such that \( \frac{1}{s} = \frac{1}{p} + \frac{1}{q} \). We say that the measure \( \sigma \) verifies the structural hypothesis \( \text{SH}(p,s) \) if

\[
\exists \beta = \beta_{p,q} > 0 \text{ s.t. } \forall a \in \mathcal{M}_s, \|k_a\|_{s'} \leq \beta \|k_a\|_p \|k_a\|_{q'}.
\]

(3.2)

This is meaningful because \( s < p, s < q \) hence \( \mathcal{M}_s \subset \mathcal{M}_p \cap \mathcal{M}_q \).

In the case of the unit ball \( \mathbb{B} \subset \mathbb{C}^n \) and \( \sigma \) the Lebesgue measure on \( X = \partial \mathbb{B} \) and in the case of the polydisc \( \mathbb{D}^n \subset \mathbb{C}^n \) and \( \sigma \) the Lebesgue measure on \( \mathbb{T}^n \), it is shown in [3] that these two hypotheses are verified for all \( p, s, q \).

### 3.2. Interpolating sequences.

We shall use the following facts proved in [3]:

**Theorem 3.5.** If, for a \( p \geq 1 \), \( S \subset \mathcal{M}_p \), if \( S \in \text{IH}^\infty(\sigma) \) and if \( \sigma \) verifies \( \text{SH}(p) \) then \( S \in \text{IH}^p(\sigma) \) with the L.E.P.

**Theorem 3.6.** If \( S \subset \mathcal{M}_1 \) and \( S \) is dual bounded in \( \text{H}^p(\sigma) \) for a \( p > 1 \), then \( S \in \text{IH}^1(\sigma) \).

We shall need to truncate \( S \) to its first \( N \) elements, say \( S_N \). Clearly if \( S \in \text{IH}^p(\sigma) \) then \( S_N \in \text{IH}^p(\sigma) \) with a smaller constant than \( C_I \). Let \( I_{S_N}^p := \{ f \in \text{H}^p(\sigma) \text{ s.t. } f|_{S_N} = 0 \} \) be the module over \( A \) of the functions zero on \( S_N \). We have then for \( \lambda \in \ell^p \), with \( \{ \rho_a \}_{a \in S} \) a bounded dual sequence, that the function \( f_N := \sum_{a \in S_N} \lambda_a \rho_a \) interpolates \( \lambda \) on \( S_N \) and we have \( \|f_N\|_{\text{H}^p(\sigma)/I_{S_N}^p} \leq C_I \|\lambda\|_p \).

We also have the converse for \( 1 < p \leq \infty \), which is all what we need [3]:

**Lemma 3.7.** If \( S \) is such that all its truncations \( S_N \) are in \( \text{IH}^p(\sigma) \) for a \( p > 1 \), with a uniform constant \( C_I \) then \( S \in \text{IH}^p(\sigma) \) with the same constant.

### 4. Carleson sequences.

As before we denote by \( k_{q,a} := \frac{k_a}{\|k_a\|_q} \) the normalized reproducing kernel in \( \text{H}^q(\sigma) \).

**Definition 4.1.** We say that the sequence \( S \subset \mathcal{M}_{q'} \) is a \( q \)-Carleson sequence if \( 1 \leq q < \infty \) and

\[
\exists D_q > 0, \forall \mu \in \ell^q, \left\| \sum_{a \in S} \mu_a k_{q,a} \right\|_q \leq D_q \|\mu\|_q.
\]

We say that the sequence \( S \subset \mathcal{M}_{q'} \) is a weakly \( q \)-Carleson sequence if \( 2 \leq q < \infty \) and

\[
\exists D_q > 0, \forall \mu \in \ell^q, \left\| \sum_{a \in S} |\mu_a|^2 |k_{q,a}|^2 \right\|_{q/2} \leq D_q \|\mu\|_{q/2}^2.
\]

We call "weakly" Carleson the second condition because

**Lemma 4.2.** If \( 2 \leq q < \infty \) and \( S \) is \( q \)-Carleson then it is weakly \( q \)-Carleson.
Proof
for a sequence $S$ we introduce a related sequence \{${\epsilon}_a\}$ of independent random variables with the same law $P(\epsilon_a = 1) = P(\epsilon_a = -1) = 1/2$. We shall denote by $E$ the associated expectation.

Let $S$ be a $q$-Carleson sequence, with the associated \{${\epsilon}_a\}$ we have
\[
\left\|\sum_{a \in S} \mu_a {\epsilon}_a k_{q,a} \right\|_q^q \lesssim \|\mu\|_q^q
\]
because $|{\epsilon}_a| = 1$. Taking expectation on both sides leads to
\[
\left\|E \left\|\sum_{a \in S} \mu_a {\epsilon}_a k_{q,a} \right\|^q \right\|_1 = E \left\|\sum_{a \in S} \mu_a {\epsilon}_a k_{q,a} \right\|_q^q \lesssim \|\mu\|_q^q.
\]
Now using Khintchine’s inequalities for the left expression
\[
\left\|E \left\|\sum_{a \in S} \mu_a {\epsilon}_a k_{q,a} \right\|^q \right\|_1 \lesssim \sum_{a \in S} |\mu_a|^q |k_{q,a}|^q \right\|_q^q.
\]
we get
\[
\left\|\sum_{a \in S} |\mu_a|^q |k_{q,a}|^q \right\|_{q/2} \lesssim \left\|E \left\|\sum_{a \in S} \mu_a {\epsilon}_a k_{q,a} \right\|_q^q \lesssim \|\mu\|_q^q,
\]
and the lemma. \[\square\]

Now if $S$ is weakly $p$-Carleson is $S$ weakly $q$-Carleson for other $q$ ?

Notice that any sequence $S$ is weakly 2-Carleson :
\[
\forall \nu \in \ell^1, \left\|\sum_{a \in S} \nu_a |k_{2,a}|^2 \right\|_1 \leq \sum_{a \in S} |\nu_a| \left\|\sum_{a \in S} |k_{2,a}|^2 \right\|_1 \leq \|\nu\|_1,
\]
because $\|k_{2,a}\|_2 = \left\|\sum_{a \in S} |k_{2,a}|^2 \right\|_1 = 1$.

Hence if $S$ is weakly $q$-Carleson with $q > 2$ we can try to use interpolation of linear operators. Let us define our operator $T$ :
\[
T : \ell^q(\omega_q) \rightarrow L_q^p(\sigma); \quad T \lambda := \sum_{a \in S} \lambda_a |k_a|^2,
\]
with the weight $\omega_q(a) := \|k_a\|_2^{-2q}$; this means that
\[
\lambda \in \ell^q(\omega_q) \Rightarrow \|\lambda\|^q_{\ell^q(\omega_q)} := \sum_{a \in S} |\lambda_a|^q \omega_q(a) < \infty.
\]

By a theorem of E. Stein and G. Weiss [17] we know that if $T$ is bounded from $\ell^q(\omega_q)$ to $L_q^p(\sigma)$ and from $\ell^1(\omega_1)$ to $L^1(\sigma)$ then $T$ is bounded from $\ell^p(\omega'_p)$ to $L^p(\sigma)$ with $1 \leq p \leq q$ provided that the weight satisfies the condition
\[
\text{if } \frac{1}{p} = \frac{1}{p} - \frac{\theta}{q} + \frac{\theta}{q} \text{ then } \omega'_p = \omega_1^{p(1-\theta)} \omega_q^{\theta/q}.
\]
Here this means
\[
\omega'_p(a) = \|k_a\|_2^{-2p(1-\theta)} \|k_a\|_2^{-2p\theta}.
\]
Then $\|\lambda\|_p^p \lesssim \|\lambda\|_{\ell^p(\omega'_p)} = \sum_{a \in S} |\lambda_a|^p \omega'_p(a)$.

Hence if $\omega'_p(a) \lesssim \omega_p(a)$ we shall have
\[ \| T \lambda \|_p^p \lesssim \| \lambda \|_{L^p(\omega_p)}^p = \sum_{a \in S} |\lambda_a|^p \omega_p(a) \lesssim \sum_{a \in S} |\lambda_a|^p \omega_p(a), \]

and this will be OK.

**Lemma 4.3.** Let \( q \geq 1 \) and \( \frac{1}{p} = \frac{1}{q} + \frac{\theta}{q} \) with \( 0 < \theta < 1 \), then

\[ \| k_a \|_{2p} \leq \| k_a \|_{2}^{(1-\theta)} \| k_a \|_{2q}^{\theta} , \]

Proof: let \( \frac{1}{p} = \frac{1}{s} + \frac{\theta}{r} \) with \( s = \frac{1}{1-\theta} \) and \( r = \frac{s}{\theta} \).

Hölder’s inequalities give, for \( f \in L^s(\sigma) \), \( g \in L^r(\sigma) \)

\[ \left( \int_X |fg|^p \, d\sigma \right)^{1/p} \leq \left( \int_X |f|^s \, d\sigma \right)^{1/s} \left( \int_X |g|^r \, d\sigma \right)^{1/r} . \]

Set \( f = |k_a|^{2-\theta} \), \( g := |k_a|^\theta \) we get

\[ \left( \int_X |k_a|^{2p} \, d\sigma \right)^{1/p} \leq \left( \int_X |k_a|^{2(1-\theta)} \, d\sigma \right)^{1/s} \left( \int_X |k_a|^{2\theta} \, d\sigma \right)^{1/r} , \]

hence replacing \( s, r \)

\[ \left( \int_X |k_a|^{2p} \, d\sigma \right)^{1/p} \leq \left( \int_X |k_a|^2 \, d\sigma \right)^{1-\theta} \left( \int_X |k_a|^{2q} \, d\sigma \right)^{\theta/q} , \]

hence

\[ \| k_a \|_{2p} \leq \| k_a \|_{2}^{(1-\theta)} \| k_a \|_{2q}^{\theta} , \]

and the lemma. \( \square \)

Back to our operator \( T \), we have \( \omega_p'(a) = \| k_a \|^{2-2p(1-\theta)} \| k_a \|^{2p} \) but the lemma above says

\[ \| k_a \|_{2p} \lesssim \| k_a \|_{2}^{1-\theta} \| k_a \|_{2q}^{\theta} \] with \( \frac{1}{p} = \frac{1}{s} + \frac{\theta}{q} \) which implies \( \omega_p'(a) \lesssim \| k_a \|_{2p} = \omega_p(a) \) and the condition of the Stein-Weiss theorem are fullfilled, so we proved

**Lemma 4.4.** If \( S \) is weakly \( q \)-Carleson, with \( q > 2 \) then \( S \) is weakly \( p \)-Carleson for any \( 2 \leq p \leq q \).

We notice too that any sequence \( S \) is 1-Carleson

\[ \forall \mu \in \ell^1, \left\| \sum_{a \in S} \mu_a k_{1,a} \right\|_1 \leq \sum_{a \in S} |\mu_a| \left\| k_{1,a} \right\|_1 \leq \| \mu \|_1 , \]

and the same proof as above gives

**Lemma 4.5.** If \( S \) is \( q \)-Carleson, with \( q > 1 \) then \( S \) is \( p \)-Carleson for any \( 1 \leq p \leq q \).

In the ball or in the polydisc, we have much better:

**Remark 4.6.** If \( S \) is \( q \)-Carleson for a \( q \in ]1, \infty[ \) then \( S \) is \( p \)-Carleson for any \( p \). Moreover \( S \) \( q \)-Carleson is equivalent to \( S \) weakly \( 2q \)-Carleson.

## 5. Main results

Now we are in position to state our main results.

**Theorem 5.1.** Let \( p \geq 1 \), \( 1 \leq s < p \) and \( q \) be such that \( \frac{1}{s} = \frac{1}{p} + \frac{1}{q} \). Suppose that \( S \subset \mathcal{M}_s \cap \mathcal{M}_q \),

that \( S \) is dual bounded in \( H^p(\sigma) \), \( p \leq 2 \), that \( S \) is weakly \( q \)-Carleson and \( \sigma \) verifies the structural hypotheses \( SH(q) \) and \( SH(p,s) \). Then \( S \) is \( H^s(\sigma) \) interpolating and has the L.E.P. in \( H^s(\sigma) \).
Using this time the fact that Kacchine’s inequalities also provide a way to put absolut values inside sums, we get the other extremity for the range of $p$’s:

**Theorem 5.2.** Let $1 \leq s < \infty$. Suppose that $S \subset \mathcal{M}_s \cap \mathcal{M}_s'$, that $S$ is dual bounded in $H^\infty(\sigma)$, $S$ is weakly $p$-Carleson for a $p > s$ and $(A, \sigma)$ verify the structural hypotheses $SH()$. Then $S$ is $H^s(\sigma)$ interpolating with the L.E.P.

These theorems will be consequence of the next lemma.

As above, if $S$ is a sequence of points in $\mathcal{M}$, we introduce the related sequence $\{\epsilon_a\}_{a \in S}$ of independent Bernoulli variables.

**Lemma 5.3.** Let $S \subset \mathcal{M}_p$ be a sequence of points such that a dual system $\{\rho_{p,a}\}_{a \in S}$ exists in $H^p(\sigma)$; let $1 \leq s < p$ and $q$ be such that $\frac{s}{p} = \frac{1}{p} + \frac{1}{q}$

if $\forall \lambda \in \ell^p(S)$, $E\left[ \sum_{a \in S} \lambda_\epsilon \rho_{p,a} \right]_{p}^{\leq} \|\lambda\|_{\ell^p}^p$, $S$ is $q$-weakly Carleson and $\sigma$ verifies $SH(q)$, $SH(p,s)$ then $S$ is $H^s(\sigma)$ interpolating and moreover $S$ has the L.E.P.

**Proof**

If $p = 1$ we have nothing to prove: the functions $\rho_{1,a}$ are uniformly bounded in $H^1(\sigma)$, just set $\forall \lambda \in \ell^1$, $T(\lambda) := \sum_{a \in S} \lambda_a \rho_{1,a}$, this function interpolates the sequence $\lambda$, is bounded in $H^1(\sigma)$, and clearly the operator $T$ is also linear and bounded.

If $p > 1$, we may suppose that $1 < s < p$ because if $S \in IH^s(\sigma)$ then by theorem 3.6 for $S \subset \mathcal{M}_1$ we also have that $S \in IH^1(\sigma)$.

First we truncate the sequence: $S_N$ is the first $N$ elements of $S$. We shall get estimates independent of $N$, i.e. for $s \in [1,p[\, and $\nu \in \ell^\infty_N$ we shall built a function $h \in H^s(\sigma)$ such that:

$\forall j = 0, ..., N - 1, h(a_j) = \nu_j \|k_{a_j}\|_{s'}$ and $\|h\|_{H^s} \leq C \|\nu\|_{\ell^\infty_N}$

with the constant $C$ independent of $N$. We conclude then by use of lemma 3.7

We choose $q$ such that $\frac{1}{s} = \frac{1}{p} + \frac{1}{q}$; then $q \in]p', \infty[\,$ with $p'$ the conjugate exponent of $p$ and we set $\nu_j = \lambda_j \mu_j$ with $\mu_j := |\nu_j|^{s/p} \in \ell^q$, $\lambda_j := |\nu_j|^{s/p} \in \ell^p$ then $\|\nu\|_s = \|\lambda\|_p \|\mu\|_q$.

Let $c_a := \|k_a\|_{s'} = \|k_a\|_{s'} \|k_a\|_q$. By $SH(q)$ we have $k_a(a) \geq \alpha \|k_a\|_q \|k_a\|_q$, hence

$c_a \leq \alpha^{-1} \|k_a\|_{s'}$ and by $SH(p,s)$ we get $c_a \leq \alpha^{-1} \beta$.

(i) Now set $h(z) := \sum_{a \in S} \nu_a \rho_a k_{q,a}(z)$ then:

$h(a) = \nu_a \|k_a\|_{s'}$ because $\rho_a(b) = \delta_{ab} \|k_a\|_{s'}$.

These are the good values, hence $h$ interpolates $\nu$ and moreover $h$ is clearly linear in $\nu$.

(ii) Estimate on the $H^s(\sigma)$ norm of $h$.

Set $f(\epsilon, z) := \sum_{a \in S} \lambda_a \epsilon_a \rho_a(z)$,

$g(\epsilon, z) := \sum_{a \in S} \mu_a \epsilon_a k_{q,a}(z)$.

Then $h(z) = \mathbb{E}(f(\epsilon, z) g(\epsilon, z))$ because $\mathbb{E}(\epsilon_j \epsilon_k) = \delta_{jk}$.

So we get

$|h(z)|^s = |\mathbb{E}(fg)|^s \leq (\mathbb{E}(|fg|))^s \leq \mathbb{E}(|fg|^s)$,
hence
\[ \|h\|_s = \left( \int_X |h(z)|^s \, d\sigma(z) \right)^{1/s} \leq \left( \int_X \mathbb{E}(|fg|^s) \, d\sigma(z) \right)^{1/s}. \]

But, using Hölder’s inequality, we get
\[ \int_X \mathbb{E}(|fg|^s) \, d\sigma(z) = \mathbb{E} \left[ \int_X |fg|^s \, d\sigma(z) \right] \leq \left( \mathbb{E} \left[ \int_X |f|^p \, d\sigma \right] \right)^{s/p} \left( \mathbb{E} \left[ \int_X |g|^q \, d\sigma \right] \right)^{s/q}. \]

Let \( \forall \alpha \in S, \ \check{\lambda}_a := c_a \lambda_a \implies \|\check{\lambda}\|_p \leq \alpha \beta \|\lambda\|_p \) and the first factor is controlled by the lemma hypothesis
\[ \mathbb{E} \left[ \int_X |f|^p \, d\sigma \right] = \mathbb{E} \left[ \left\| \sum_{a \in S} \lambda_a c_a \epsilon_a \rho_{p,a} \right\|^p \right] \lesssim \|\check{\lambda}\|_p^p \lesssim \|\lambda\|_{\ell^q}^p. \]

Fubini theorem gives for the second factor
\[ \mathbb{E} \left[ \int_X |g|^q \, d\sigma \right] = \int_X \mathbb{E} \left[ |g|^q \right] \, d\sigma. \]

We apply Khintchine’s inequalities to \( \mathbb{E} \left[ |g|^q \right] \)
\[ \mathbb{E} \left[ |g|^q \right] \simeq \left( \sum_{a \in S} |\mu_a|^2 |k_{q,a}|^2 \right)^{q/2}, \]
hence \( S \) being weak \( q \)-Carleson implies
\[ \int_X \mathbb{E} \left[ |g|^q \right] \, d\sigma \lesssim \int_X \left( \sum_{a \in S} |\mu_a|^2 |k_{q,a}|^2 \right)^{q/2} \, d\sigma \lesssim \|\mu\|_{\ell^q}^q. \]

So putting (5.2) and (5.3) in (5.1) we get the lemma. \[ \Box \]

5.1. **Proof of theorem 5.1** Let us recall the theorem we want to prove.

**Theorem 5.4.** Let \( p \geq 1, 1 \leq s < p \) and \( q \) be such that \( \frac{1}{s} = \frac{1}{p} + \frac{1}{q} \). Suppose that \( S \subset \mathcal{M}_s \cap \mathcal{M}_q \), that \( \{\rho_{p,a}\}_{a \in S} \) is a norm bounded sequence in \( H^p(\sigma) \), \( p \leq 2 \), that \( S \) is weakly \( q \)-Carleson and \( \sigma \) verifies the structural hypotheses \( SH(q), \ SH(p, s) \). Then \( S \) is \( H^s(\sigma) \)-interpolating with the L.E.P..

It remains to prove that the hypotheses of the theorem implies those of the lemma 5.3

We have to prove that
\[ \mathbb{E} \left[ \left\| \sum_{a \in S} \lambda_a c_a \rho_{p,a} \right\|^p \right] \lesssim \|\lambda\|_{\ell^q}^p, \]
knowing that the dual sequence \( \{\rho_{p,a}\}_{a \in S} \) is bounded in \( H^p(\sigma) \), i.e.
\[ \sup_{a \in S} \|\rho_{p,a}\|_p \leq C. \]

By Fubini’s theorem
\[ \mathbb{E} \left[ \left\| \sum_{a \in S} \lambda_a c_a \rho_{p,a} \right\|^p \right] = \int_X \mathbb{E} \left[ \left\| \sum_{a \in S} \lambda_a c_a \rho_{p,a} \right\|^p \right] \, d\sigma, \]
and by Khintchine’s inequalities we have
\[ \mathbb{E} \left[ \left( \sum_{a \in S} \lambda_a \epsilon_a \rho_{p,a} \right)^p \right] \simeq \left( \sum_{a \in S} |\lambda_a|^2 |\rho_{p,a}|^2 \right)^{p/2}. \]

Now \( p \leq 2 \), so
\[ \left( \sum_{a \in S} |\lambda_a|^2 |\rho_{p,a}|^2 \right)^{1/2} \leq \left( \sum_{a \in S} |\lambda_a|^p |\rho_{p,a}|^p \right)^{1/p} \]
hence
\[ \int_X \mathbb{E} \left[ \left( \sum_{a \in S} |\lambda_a \epsilon_a \rho_{p,a}|^p \right)^p \right] \, d\sigma \leq \int_X \left( \sum_{a \in S} |\lambda_a|^p |\rho_{p,a}|^p \right) \, d\sigma = \sum_{a \in S} |\lambda_a|^p \|\rho_{p,a}\|^p. \]
So, finally
\[ \mathbb{E} \left[ \left( \sum_{a \in S} |\lambda_a \epsilon_a \rho_{p,a}|^p \right)^p \right] \leq \sup_{a \in S} \|\rho_{p,a}\|^p \|\lambda\|^p, \]
and the theorem 5.1.

Suggested by F. Lust-Piquard, one can use that \( H^p(\sigma) \subset L^p(\sigma) \) hence, because \( p \leq 2 \), \( H^p(\sigma) \)
is of type \( p \) which means precisely (\cite{1922}, Th III.9) that
\[ \mathbb{E} \left[ \left( \sum_{a \in S} |\lambda_a \rho_{a}|^p \right)^p \right] \lesssim \sum_{a \in S} |\lambda_a \rho_{a}|^p, \]
hence integrating and using Fubini, we get
\[ \mathbb{E} \left[ \left( \sum_{a \in S} |\lambda_a \epsilon_a \rho_{p,a}|^p \right)^p \right] \lesssim \int \sum_{a \in S} |\lambda_a \rho_{a}|^p \, d\sigma \lesssim \left( \sup_{a \in S} \|\rho_{p,a}\|_p \|\lambda\|_p^p \right) \lesssim \|\lambda\|_p^p. \]
And again the theorem.

5.2. Proof of theorem 5.2

Let us recall the theorem we want to prove.

**Theorem 5.5.** Let \( 1 \leq s < \infty \). Suppose that \( S \subset \mathcal{M}_s \cap \mathcal{M}_{s'}, \) that \( \{\rho_a\}_{a \in S} \) is a norm bounded sequence in \( H^{\infty}(\sigma) \), weakly \( p \)-Carleson for \( a \) \( p > s \) and \( \sigma \) verifies the structural hypotheses \( SH(p,s) \), \( SH(q) \) for \( q \) such that \( \frac{1}{s} = \frac{1}{p} + \frac{1}{q} \). Then \( S \) is \( H^s(\sigma) \) interpolating with the L.E.P.

Proof

the idea is still to use lemma 5.3, but in two steps. Let \( s < \infty \) be given and take \( p \) such that \( s < p < \infty \) and \( S \) is weakly \( p \)-Carleson.

Set \( \forall a \in S, \rho_{p,a} := \rho_a k_{p,a}. \) We have \( \|\rho_{p,a}\|_p \leq \|\rho_a\|_{\infty} \|k_{p,a}\|_p = \|\rho_a\|_{\infty} \leq C \) by hypothesis. We want to prove that
\[ \mathbb{E} \left[ \left( \sum_{a \in S} |\lambda_a \epsilon_a \rho_{p,a}|^p \right)^p \right] = \int_X \mathbb{E} \left[ \left( \sum_{a \in S} |\lambda_a \epsilon_a \rho_{p,a}|^p \right)^p \right] \, d\sigma \lesssim \|\lambda\|_{\ell^p}^p, \]
in order to apply lemma 5.3.

By Khintchine’s inequalities we have
\[ \mathbb{E} \left[ \left( \sum_{a \in S} |\lambda_a \epsilon_a \rho_{p,a}|^p \right)^p \right] \simeq \left( \sum_{a \in S} |\lambda_a|^2 |\rho_{p,a}|^2 \right)^{p/2}, \]
but this time we use that \( |\rho_{p,a}| \leq \|\rho_{\infty,a}\| |k_{a,p}| \leq C |k_{a,p}| \) hence
Using that $S$ is weakly $p$-Carleson, we get
\[ \left\| \sum_{a \in S} |\lambda_a|^2 |k_{a,p}|^2 \right\|_{p/2} \leq D \|\lambda\|_p^p, \]
hence
\[ \mathbb{E} \left[ \left\| \sum_{a \in S} \lambda_a \epsilon_a \rho_{p,a} \right\|_p^p \right] \lesssim C^p \left( \sum_{a \in S} |\lambda_a|^2 |k_{a,p}|^2 \right)^{p/2}. \]

and we can apply the lemma 5.3 which gives the theorem because $p > s$ implies that $S$ is still weakly $s$-Carleson by lemma 4.4.

\[ \square \]

6. Application to the ball and to the polydisc.

In [3] it is proved that the structural hypotheses hold in the polydisc. Moreover the Carleson measures, hence the Carleson sequences, are characterized geometrically and they are the same for all $p \in ]1, \infty[$ (see [3], [4]). So it is enough to say “Carleson sequence” in the theorem:

**Theorem 6.1.** Let $S \subset \mathbb{D}^n$ be a Carleson sequence and dual bounded in $H^p(\mathbb{D}^n)$ with either $p = \infty$ or $p \leq 2$, then $S$ is $H^s(\mathbb{D}^n)$ interpolating for any $s < p$ with the LEP.

Still in [3] it is proved that the structural hypotheses hold in the ball. Again the Carleson measures, hence the Carleson sequences, are characterized geometrically and they are the same for all $p \in ]1, \infty[$ (see [11]) but moreover a theorem of P. Thomas [18] gives that $S$ dual bounded in $H^p(\mathbb{B})$ implies $S$ Carleson, hence

**Theorem 6.2.** Let $S \subset \mathbb{B}$ be dual bounded in $H^p(\mathbb{B})$ with either $p = \infty$ or $p \leq 2$, then $S$ is $H^s(\mathbb{B})$ interpolating for any $s < p$ with the LEP.

We have for free the same result for the Bergman classes of the ball by the ”subordination lemma” [12]:

to a function $f(z)$ defined on $z = (z_1, ..., z_n) \in \mathbb{B}_n \subset \mathbb{C}^n$ associate the function.
\[ \tilde{f}(z, w) := f(z) \text{ defined on } z = (z_1, ..., z_n, w) \in \mathbb{B}_{n+1} \subset \mathbb{C}^{n+1}. \]

Then we have that $f \in A^p(\mathbb{B}_n) \iff \tilde{f} \in H^p(\mathbb{B}_{n+1})$ with the same norm. Moreover if $F \in H^p(\mathbb{B}_{n+1})$ then
\[ f(z) := F(z, 0) \in A^p(\mathbb{B}_n) \text{ with } \|f\|_{A^p(\mathbb{B}_n)} \leq \|F\|_{H^p(\mathbb{B}_{n+1})}. \]

Suppose that $S \subset \mathbb{B}_n$ is dual bounded in $A^p(\mathbb{B}_n)$, this means that
\[ \exists \{\rho_a\}_{a \in S} \text{ s.t. } \forall a \in S, \|\rho_a\|_{A^p(\mathbb{B}_n)} \leq C \text{ and } \rho_a(b) = \delta_{ab}(1 - |a|^2)^{(n+1)/p}, \]
because the normalized reproducing kernel for $A^p(\mathbb{B}_n)$ is $b_a(z) := \frac{(1 - |a|^2)^{(n+1)/p'}}{(1 - \overline{a} \cdot z)^{n+1}}$.

Embed $S$ in $\mathbb{B}_{n+1}$ by $\tilde{S} := \{(a, 0), a \in S\}$ as in [12], then the sequence $\{\tilde{\rho}_a\}_{a \in S}$ is precisely a bounded dual sequence for $\tilde{S} \subset \mathbb{B}_{n+1}$ in $H^p(\mathbb{B}_{n+1})$ hence we can apply the previous theorem:
if $p = \infty$ or $p \leq 2$ then $\tilde{S}$ is $H^s(\mathbb{B}_{n+1})$ interpolating with the L.E.P.. If $T$ is the operator making the extension,
\[ \forall \lambda \in L^s \longrightarrow T\lambda \in H^s(\mathbb{B}_{n+1}), \quad (T\lambda)(a, 0) = \lambda_a \|k(a, 0)\|_{H^s(\mathbb{B}_{n+1})}, \quad \|T\lambda\|_{H^s(\mathbb{B}_{n+1})} \leq C_I \|\lambda\|_s. \]
then the operator $(U\lambda)(z) := (T\lambda)(z, 0)$ is a bounded linear operator from $\ell^s$ to $A^s(\mathbb{B}_n)$ making the extension because $\|k(a,0)\|_{H^s(\mathbb{B}_{n+1})} = \|b_k\|_{A^s(\mathbb{B}_n)}$ where $k$ is the kernel for $H^s(\mathbb{B}_{n+1})$ and $b$ is the kernel for $A^s(\mathbb{B}_n)$. Hence we proved

**Corollary 6.3.** Let $S \subset \mathbb{B}$ be dual bounded in $A^p(\mathbb{B})$ with either $p = \infty$ or $p \leq 2$, then $S$ is $A^s(\mathbb{B})$ interpolating for any $s < p$ with the LEP.

We also get the same result for the Bergman spaces with weight of the form $(1 - |z|^2)^k$, $k \in \mathbb{N}$ just by the same method, but considering $H^p(\mathbb{B}_{n+k+1})$ instead of $H^p(\mathbb{B}_{n+1})$.

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