The Keller-Osserman type conditions for the study of existence of entire radial solutions to a semilinear elliptic system

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Abstract
We are concerned here with functions \((u_1, u_2)\) which are of class \(C^2\) and satisfy the systems of the form
\[
\begin{align*}
\Delta u_1 &= p_1 (|x|) f_1 (u_2) \text{ in } \mathbb{R}^N, \\
\Delta u_2 &= p_2 (|x|) f_2 (u_1) \text{ in } \mathbb{R}^N,
\end{align*}
\]
where \(p_1, f_1, p_2,\) and \(f_2\) are continuous functions satisfying certain new conditions. Our considerations center about the behavior of \((u_1, u_2)\) as \(|x| \to \infty\).

Keywords: Entire solution; Large solution; Elliptic system

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1 Introduction

Entire large and bounded solutions of semilinear elliptic systems have been received an increased interest in the past several decades. In this article we analyze semilinear elliptic systems of the type
\[
\begin{align*}
\Delta u_1 &= p_1 (|x|) f_1 (u_2) \text{ for } x \in \mathbb{R}^N \ (N \geq 3), \\
\Delta u_2 &= p_2 (|x|) f_2 (u_1)
\end{align*}
\]
in which the functions \(p_1, p_2, f_1\) and \(f_2\) takes various forms, which are mentioned later on. Such problems are referred in the literature as the Bieberbach and Rademacher problems type. The system (1.1) will be studied under three different types of boundary conditions:

- **Finite Case:** Both components \((u_1, u_2)\) are bounded, that is,
\[
\begin{align*}
\lim_{|x| \to \infty} u_1 (|x|) &< \infty, \\
\lim_{|x| \to \infty} u_2 (|x|) &< \infty.
\end{align*}
\]
Infinite Case: Both components \((u_1, u_2)\) are large, that is,
\[
\begin{align*}
\lim_{|x| \to \infty} u_1 (|x|) &= \infty, \\
\lim_{|x| \to \infty} u_2 (|x|) &= \infty.
\end{align*}
\] (1.3)

Semifinite Case: One of the components bounded while the other is large, that is,
\[
\begin{align*}
\lim_{|x| \to \infty} u_1 (|x|) &= < \infty, \\
\lim_{|x| \to \infty} u_2 (|x|) &= \infty,
\end{align*}
\] or
\[
\begin{align*}
\lim_{|x| \to \infty} u_1 (|x|) &= \infty, \\
\lim_{|x| \to \infty} u_2 (|x|) &= < \infty.
\end{align*}
\] (1.4) (1.5)

Definition 1.1. A solution \((u_1, u_2) \in C^2([0, \infty)) \times C^2([0, \infty))\) of the system (1.1) is called an entire bounded solution if the condition (1.2) holds; an entire large solution if the condition (1.3) holds; a semifinite entire large solution when (1.4) or (1.5) holds.

The study of existence of large solutions for semilinear elliptic systems of the form (1.1) goes back to the pioneering papers by Keller [12] and Osserman [21]. In 1957 Osserman [21] proved that, for a given positive, continuous and nondecreasing function \(f\), the semilinear elliptic partial differential inequality
\[
\Delta u \geq a(x) f(u) \text{ in } \mathbb{R}^N,
\] (1.6)
with \(a(x) = 1\), possesses an entire large solution \(u : \mathbb{R}^N \to \mathbb{R}\) if and only if
\[
\int_1^\infty \left( \int_0^t f(s) \, ds \right)^{-1/2} \, dt = +\infty,
\] (1.7)
(assumption known today as the Keller-Osserman condition).

Such problems are drawn by the mathematical modelling of many natural phenomena related to steady-state reaction-diffusion, subsonic fluid flows, electrostatic potential in a shiny metallic body inside or subsonic motion of a gas, automorphic functions theory, geometry and control theory (see, for example, L. Bieberbach [1], Diaz [2], Keller [13], Lasry and Lions [14], Matero [15], Marinescu and Varsan [16], Iftimie-Marinescu and Varsan [17], and Rademacher [24] for a more detailed discussion). For example, reading the work of Lasry and Lions [14], we can observe that such problems arise in stochastic control theory. The controls are to be designed so that the state of the system is constrained to some region. Finding optimal controls is then shown to be equivalent to finding large solutions for a second order nonlinear elliptic partial differential equation.

Even if these problems are treated directly or indirectly, in many papers from the specialized literature, they have not been completely clarified yet. This does not surprise us, because practical applications mainly reveal new horizons, complexity and aspects that allow new theoretical approaches.

Our objective in the present research, in short, is to complete and to find new ideas to treat the principal results of the author [2, 3, 4], Goyal [6, 7], Magliaro-Mari-Mastroia and Rigoli [17], Lieberman [16], Nehari [20], Redheffer [25], Rhee [23], Reichel-Walter [26] and other associated works.

More exactly, the present work is related to the first and last of the preceding questions:
1. The first one is the problem of existence of solution to (1.1) that satisfies (1.2), (1.3), (1.4) or (1.5) to be entire large.

2. The second one is to give a necessary and a sufficient condition for a positive radial solution of (1.1) to be entire large.

However, there is no results for systems (1.1), where $f_1$, $f_2$ satisfy a condition of the form (1.7). Other purpose of this paper is to fill this gap. To be more precise, we consider $p_1, p_2$ and the nonlinearities $f_1, f_2$ satisfy:

1. $p_1, p_2 : [0, \infty) \to [0, \infty)$ are spherically symmetric continuous functions (i.e., $p_1(x) = p_1(|x|)$ and $p_2(x) = p_2(|x|)$);
2. $f_1, f_2 : [0, \infty) \to [0, \infty)$ are continuous, non-decreasing, $f_1(0) = f_2(0) = 0$ and $f_1(s) > 0, f_2(s) > 0$ for all $s > 0$;
3. there exist positive constants $\tau_1, \tau_2$, the continuous and increasing functions $h_1, h_2, \omega_1, \omega_2 : [0, \infty) \to [0, \infty)$ such that

\[
\begin{align*}
  f_1(t_1 \cdot w_1) &\leq \tau_1 h_1(t_1) \cdot \omega_1(w_1) \quad \forall \ t_1 \geq 1 \text{ and } \forall \ t_1 \geq M_1 \cdot f_2(a), \\
  f_2(t_2 \cdot w_2) &\leq \tau_2 h_2(t_2) \cdot \omega_2(w_2) \quad \forall \ t_2 \geq 1 \text{ and } \forall \ t_2 \geq M_2 \cdot f_1(b),
\end{align*}
\]

where $M_1$ and $M_2$ are such that

\[
M_1 = \begin{cases} \frac{a}{f_2(a)} & \text{if } b > f_2(a), \\ 1 & \text{if } b \leq f_2(a), \end{cases} \quad \text{and} \quad M_2 = \begin{cases} \frac{a}{f_1(a)} & \text{if } a > f_1(b), \\ 1 & \text{if } a \leq f_1(b). \end{cases}
\]

To facilitate the presentation of the results we introduce some notations:

\[
\begin{align*}
G_1(z) &= \int_0^z s^{N-1} p_2(s) \, ds \\
G_2(z) &= \int_0^z s^{N-1} p_1(s) \, ds \\
P_1(r) &= \int_0^r y^{1-N} \int_0^y t^{N-1} p_1(t) f_1(b + f_2(a) \int_0^t z^{1-N} G_1(z) \, dz) \, dt \, dy, \\
Q_1(r) &= \int_0^r y^{1-N} \int_0^y t^{N-1} p_2(t) f_2(a + f_1(b) \int_0^t z^{1-N} G_2(z) \, dz) \, dt \, dy, \\
P_2(r) &= \int_0^r z^{1+\epsilon} p_1(z) \omega_1 \left( 1 + \int_0^z \frac{1}{t^{N-1}} \int_0^t s^{N-1} p_2(s) \, ds \, dt \right) \, dz, \\
Q_2(r) &= \int_0^r z^{1+\epsilon} p_2(z) \omega_2 \left( 1 + \int_0^z t^{1-N} \int_0^t s^{N-1} p_1(s) \, ds \, dt \right) \, dz, \\
P_3(r) &= \int_0^r \sqrt{2} \phi_1(z) \omega_1 \left( 1 + \int_0^z t^{1-N} \int_0^t s^{N-1} p_2(s) \, ds \, dt \right) \, dz \text{ where } \phi_1(z) = \max_{0 \leq t \leq z} p_1(t), \\
Q_3(r) &= \int_0^r \sqrt{2} \phi_2(z) \omega_2 \left( 1 + \int_0^z t^{1-N} \int_0^t s^{N-1} p_1(s) \, ds \, dt \right) \, dz \text{ where } \phi_2(z) = \max_{0 \leq t \leq z} p_2(t), \\
H_1(r) &= \int_a^r \frac{1}{\sqrt{\int_0^s h_1(M_1 f_2(t)) \, dt}} \, ds, \quad H_2(r) = \int_b^r \frac{1}{\sqrt{\int_0^s h_2(M_2 f_1(t)) \, dt}} \, ds,
\end{align*}
\]

\[
P_i(\infty) = \lim_{r \to \infty} P_i(r), \quad Q_i(\infty) = \lim_{r \to \infty} Q_i(r), \quad H_i(\infty) = \lim_{r \to \infty} H_i(r) \text{ for } i = 1, 2.
\]

Our main result are the following:
**Theorem 1.1.** Assume that $H_1(\infty) = H_2(\infty) = \infty$. If $p_1, p_2, f_1, f_2$ satisfy (P1), (C1) and (C2), then the problem (1.1) has a nonnegative entire radial solution $(u_1, u_2)$ with central value in $(a, b)$ (i.e. $(u_1, u_2) = (a, b)$). Moreover,

i.) if in addition $r^{2N-2}p_1(r), r^{2N-2}p_2(r)$ are nondecreasing for large $r$ and $p_1, p_2$ satisfy

$$P_2(\infty) < \infty \text{ and } Q_2(\infty) < \infty,$$

then for any nonnegative radial solution $(u_1, u_2)$ of (1.1) with central value in $(a, b)$ we have (1.2);

ii.) if $p_1$ and $p_2$ satisfy

$$P_1(\infty) = Q_1(\infty) = \infty,$$  \hspace{1cm} (1.10)

and $(u_1, u_2)$ is any nonnegative radial solution of (1.1) with central value in $(a, b)$ then (1.3) holds;

iii.) if in addition $r^{2N-2}p_1(r)$ is nondecreasing for large $r$ and $p_1, p_2$ satisfy

$$P_2(\infty) < \infty \text{ and } Q_1(\infty) = \infty,$$  \hspace{1cm} (1.11)

then for any nonnegative radial solution $(u_1, u_2)$ of (1.1) with central value in $(a, b)$ we have (1.2);

iv.) if in addition $r^{2N-2}p_2(r)$ is nondecreasing for large $r$ and $p_1, p_2$ satisfy

$$P_1(\infty) = \infty \text{ and } Q_2(\infty) < \infty,$$  \hspace{1cm} (1.12)

then for any nonnegative radial solution $(u_1, u_2)$ of (1.1) with central value in $(a, b)$ we have (1.2);

v.) if (1.1) has a nonnegative entire large solution $(u_1, u_2)$ with central value in $(a, b)$ and $r^{2N-2}p_1(r), r^{2N-2}p_2(r)$ are nondecreasing for large $r$, then $p_1$ and $p_2$ satisfy

$$P_2(\infty) = Q_2(\infty) = \infty,$$  \hspace{1cm} (1.13)

for every $\varepsilon > 0$.

**Theorem 1.2.** Assume that the hypotheses (P1), (C1) and (C2) are satisfied. The following hold:

i.) If $P_3(\infty) < H_1(\infty) < \infty$ and $Q_3(\infty) < H_2(\infty) < \infty$ then the system (1.1) has one positive bounded radial solution $(u_1, u_2) \in C^2([0, \infty)) \times C^2([0, \infty))$, with central value in $(a, b)$, such that

$$\begin{align*}
\left\{ \begin{array}{l}
a + P_1(r) \leq u_1(r) \leq H_1^{-1}(r)^{1/2}P_3(r), \\
b + Q_1(r) \leq u_2(r) \leq H_2^{-1}(r)^{1/2}Q_3(r).
\end{array} \right.
\end{align*}$$

ii.) If $H_1(\infty) = \infty, P_1(\infty) = \infty$ and $Q_3(\infty) < H_2(\infty) < \infty$ then the system (1.1) has one positive radial solution

$$(u_1, u_2) \in C^2([0, \infty)) \times C^2([0, \infty)),$$

with central value in $(a, b)$, such that (1.2) holds;

iii.) If $P_3(\infty) < H_1(\infty) < \infty$ and $H_2(\infty) = \infty, Q_1(\infty) = \infty$ then the system (1.1) has one positive radial solution

$$(u_1, u_2) \in C^2([0, \infty)) \times C^2([0, \infty)),$$

with central value in $(a, b)$, such that (1.2) holds;

iv.) If $r^{2N-2}p_1(r)$ is nondecreasing for large $r, H_1(\infty) = \infty, P_2(\infty) < \infty$ and $Q_3(\infty) < H_2(\infty) < \infty$ then the system (1.1) has one positive radial solution

$$(u_1, u_2) \in C^2([0, \infty)) \times C^2([0, \infty)),$$

for every $\varepsilon > 0$.}
with central value in \((a, b)\), such that (1.2) holds.

v) If \(r^{2N-2}p_2(r)\) is nondecreasing for large \(r\), \(P_1(\infty) < H_1(\infty) < \infty\) and \(H_2(\infty) = \infty\), \(Q_2(\infty) < \infty\) then the system \((1.1)\) has one positive radial solution

\[
(u_1, u_2) \in C^2([0, \infty)) \times C^2([0, \infty)),
\]

with central value in \((a, b)\), such that \((1.2)\) holds;

**Remark 1.1.** Our assumption \((C2)\) is further discussed in the famous book of Krasnosel’skii and Rutickii \([14]\) (see also Gustavsson, Maligranda and Peetre \([9]\)).

**Remark 1.2.** A simple example of nonlinearities \(f_1\) and \(f_2\) satisfying the assumptions in Theorem 1.1 and Theorem 1.2 are

\[
f_1(u_2) = h_1(u_2) = \omega_1(u_2) = u_2^\alpha \quad \text{and} \quad f_2(u_1) = h_2(u_1) = \omega_1(u_1) = u_1^\beta
\]

where \(\alpha, \beta \in \mathbb{R}\). The results in Theorem 1.1 work with \(\alpha \cdot \beta \leq 1\) and \(\bar{c}_1 = \bar{c}_2 = 1\). On the other hand, the results in Theorem 1.2 are proved for \(\alpha \cdot \beta > 1\) and \(\bar{c}_1 = \bar{c}_2 = 1\).

## 2 Proofs of the Theorems

We prove the existence of a solution \((u_1, u_2)\) for the system

\[
\begin{align*}
\Delta u_1 (r) &= p_1 (r) f_1 (u_2 (r)) \quad \text{for} \quad r := |x|, \\
\Delta u_2 (r) &= p_2 (r) f_2 (u_1 (r)) \quad \text{for} \quad r := |x|.
\end{align*}
\]

(2.1)

In the radial setting, the system \((2.1)\) becomes a system of differential equations of the form

\[
\begin{align*}
(r^{N-1}u'_1 (r))' &= r^{N-1}p_1(r) f_1 (u_2 (r)), \\
(r^{N-1}u'_2 (r))' &= r^{N-1}p_2(r) f_2 (u_1 (r)).
\end{align*}
\]

(2.2)

Then, a radial solution of \((2.2)\) is any solution \((u_1, u_2)\) for the integral equations

\[
\begin{align*}
u_1 (r) &= a + \int_0^r t^{1-N} \int_0^t s^{N-1} p_1 (s) f_1 (u_2 (s)) \, ds \, dt, \\
u_2 (r) &= b + \int_0^r t^{1-N} \int_0^t s^{N-1} p_2 (s) f_2 (u_1 (s)) \, ds \, dt.
\end{align*}
\]

To establish a solution to this system, we use successive approximation. Define sequences \(\{u_1^k\}_{k \geq 1}\) and \(\{u_2^k\}_{k \geq 1}\) on \([0, \infty)\) by

\[
\begin{align*}
u_1^0 &= a \quad \text{and} \quad u_2^0 = b, \quad r \geq 0, \\
u_1^k (r) &= a + \int_0^r t^{1-N} \int_0^t s^{N-1} p_1 (s) f_1 (u_2^{k-1} (s)) \, ds \, dt, \\
u_2^k (r) &= b + \int_0^r t^{1-N} \int_0^t s^{N-1} p_2 (s) f_2 (u_1^{k-1} (s)) \, ds \, dt.
\end{align*}
\]

We remark that, for all \(r \geq 0\) and \(k \in \mathbb{N}\)

\[
u_1^k (r) \geq a \quad \text{and} \quad u_2^k (r) \geq b.
\]
Moreover, proceeding by mathematical induction we conclude that \( \{u^k_1\} \) and \( \{u^k_2\} \) are non-decreasing sequences on \([0, \infty)\). We will next prove the "upper bounds". To do this, we note that \( \{u^k_1\} \) and \( \{u^k_2\} \) satisfy

\[
\begin{align*}
\left[ r^{N-1} (u^k_1 (r)) \right]' &= r^{N-1} p_1 (r) f_1 (u^{k-1}_2 (r)), \\
\left[ r^{N-1} (u^k_2 (r)) \right]' &= r^{N-1} p_2 (r) f_2 (u^{k-1}_1 (r)).
\end{align*}
\]

Using the monotonicity of \( \{u^k_1\} \) and \( \{u^k_2\} \) we find the inequalities

\[
\begin{align*}
\left[ r^{N-1} (u^k_1 (r)) \right]' &= r^{N-1} p_1 (r) f_1 (u^{k-1}_2 (r)) \leq r^{N-1} p_1 (r) f_1 (u^k_2 (r)), \\
\left[ r^{N-1} (u^k_2 (r)) \right]' &= r^{N-1} p_2 (r) f_2 (u^{k-1}_1 (r)) \leq r^{N-1} p_2 (r) f_2 (u^k_1 (r)).
\end{align*}
\]

Then, going back to the previous computation we have

\[
\begin{align*}
\left[ r^{N-1} (u^k_1 (r)) \right]' &= r^{N-1} p_1 (r) f_1 (u^{k-1}_2 (r)) \\
&\leq r^{N-1} p_1 (r) f_1 (u^k_2 (r)) \\
&\leq r^{N-1} p_1 (r) f_1 \left( b + \int_0^r t^{1-N} \int_0^t s^{N-1} P_2 (s) f_2 (u^{k-1}_1 (s)) \, ds \, dt \right) \\
&\leq r^{N-1} p_1 (r) f_1 \left( b + \int_0^r t^{1-N} \int_0^t s^{N-1} P_2 (s) f_1 (u^k_1 (s)) \, ds \, dt \right) \\
&\leq r^{N-1} p_1 (r) f_1 \left( b + f_2 (u^k_1 (r)) \int_0^r t^{1-N} \int_0^t s^{N-1} P_2 (s) \, ds \, dt \right) \tag{2.4}
\end{align*}
\]

and

\[
\begin{align*}
\left[ r^{N-1} (u^k_2 (r)) \right]' &= r^{N-1} p_2 (r) f_2 (u^{k-1}_1 (r)) \\
&\leq r^{N-1} p_2 (r) f_2 \left( M_1 f_1 (u^k_1 (r)) \left( 1 + \int_0^r t^{1-N} \int_0^t s^{N-1} P_1 (s) \, ds \, dt \right) \right) \tag{2.5}
\end{align*}
\]
By (2.4) and (2.5), we have
\[ r^{N-1} (u^k_1)'' \leq (N - 1) r^{N-2} (u^k_1)' + r^{N-1} (u^k_1)'' = r^{N-1} (u^k_1)' \]
\[ \leq r^{N-1} p_1(r) c_1 h_1 (M_1 f_2 (u^k_1 (r))) \omega_1 \left( 1 + \int_0^r t^{1-N} \int_0^t s^{N-1} p_2 (s) \, ds \, dt \right), \]
\[ r^{N-1} (u^k_2)'' \leq r^{N-1} (u^k_2)', \]
\[ \leq r^{N-1} p_2(r) c_2 h_2 (M_2 f_1 (u^k_2 (r))) \omega_2 \left( 1 + \int_0^r t^{1-N} \int_0^t s^{N-1} p_1 (s) \, ds \, dt \right), \]  
(2.6)

Multiplying the first inequality by \( (u^k_1)' \) and the second by \( (u^k_2)' \), we obtain
\[ \left\{ \begin{array}{l}
\left\lbrack (u^k_1 (r))' \right\rbrack^2 \leq 2 p_1 (r) c_1 h_1 (M_1 f_2 (u^k_1 (r))) (u^k_1 (r))' \omega_1 \left( 1 + \int_0^r t^{1-N} \int_0^t s^{N-1} p_2 (s) \, ds \, dt \right), \\
\left\lbrack (u^k_2 (r))' \right\rbrack^2 \leq 2 p_2 (r) c_2 h_2 (M_2 f_1 (u^k_2 (r))) (u^k_2 (r))' \omega_2 \left( 1 + \int_0^r t^{1-N} \int_0^t s^{N-1} p_1 (s) \, ds \, dt \right).
\end{array} \right. \]  
(2.7)

Integrating in (2.7) from 0 to \( r \) we also have
\[ \left\{ \begin{array}{l}
\left\lbrack (u^k_1 (r))' \right\rbrack^2 \leq \int_0^r 2 p_1 (z) c_1 h_1 (M_1 f_2 (u^k_1 (z))) (u^k_1 (z))' \omega_1 \left( 1 + \int_0^z t^{1-N} \int_0^t s^{N-1} p_2 (s) \, ds \, dt \right) \, dz, \\
\left\lbrack (u^k_2 (r))' \right\rbrack^2 \leq \int_0^r 2 p_2 (z) c_2 h_2 (M_2 f_1 (u^k_2 (z))) (u^k_2 (z))' \omega_2 \left( 1 + \int_0^z t^{1-N} \int_0^t s^{N-1} p_1 (s) \, ds \, dt \right) \, dz.
\end{array} \right. \]  
(2.8)

Set now
\[ \phi_1 (r) = \max \{ p_1 (z) | 0 \leq z \leq r \}, \]
\[ \phi_2 (r) = \max \{ p_2 (z) | 0 \leq z \leq r \}. \]  
(2.9)

Thanks to the definition of \( \phi_1 (r) \) and \( \phi_2 (r) \) we get from the inequalities (2.8) that
\[ \left\{ \begin{array}{l}
\left\lbrack (u^k_1 (r))' \right\rbrack^2 \leq 2 c_1 \phi_1 (r) \omega_1 \left( 1 + \int_0^r t^{1-N} \int_0^t s^{N-1} p_2 (s) \, ds \, dt \right) \int_0^r h_1 (M_1 f_2 (u^k_1 (z))) (u^k_1 (z))' \, dz, \\
\left\lbrack (u^k_2 (r))' \right\rbrack^2 \leq 2 c_2 \phi_2 (r) \omega_2 \left( 1 + \int_0^r t^{1-N} \int_0^t s^{N-1} p_1 (s) \, ds \, dt \right) \int_0^r h_2 (M_2 f_1 (u^k_2 (z))) (u^k_2 (z))' \, dz.
\end{array} \right. \]  
(2.10)

As a consequence of (2.10), we also have
\[ \left\{ \begin{array}{l}
(u^k_1 (r))' \leq \sqrt{2 \phi_1 (r) \omega_1 \left( 1 + \int_0^r t^{1-N} \int_0^t s^{N-1} p_2 (s) \, ds \, dt \right) \left( f^u_1 (r) \circ h_1 (M_1 f_2 (z)) \right) \, dz}, \\
(u^k_2 (r))' \leq \sqrt{2 \phi_2 (r) \omega_2 \left( 1 + \int_0^r t^{1-N} \int_0^t s^{N-1} p_1 (s) \, ds \, dt \right) \left( f^u_2 (r) \circ h_2 (M_2 f_1 (z)) \right) \, dz}.
\end{array} \right. \]  
(2.11)

and, thus
\[ \left\{ \begin{array}{l}
\frac{(u^k_1 (r))'}{\left( f^u_1 (r) \circ h_1 (M_1 f_2 (z)) \right) \, dz} \leq \frac{1}{2} \omega_1 \sqrt{2 \phi_1 (r) \omega_1 \left( 1 + \int_0^r t^{1-N} \int_0^t s^{N-1} p_2 (s) \, ds \, dt \right) \left( f^u_1 (r) \circ h_1 (M_1 f_2 (z)) \right) \, dz}, \\
\frac{(u^k_2 (r))'}{\left( f^u_2 (r) \circ h_2 (M_2 f_1 (z)) \right) \, dz} \leq \frac{1}{2} \omega_2 \sqrt{2 \phi_2 (r) \omega_2 \left( 1 + \int_0^r t^{1-N} \int_0^t s^{N-1} p_1 (s) \, ds \, dt \right) \left( f^u_2 (r) \circ h_2 (M_2 f_1 (z)) \right) \, dz}.
\end{array} \right. \]  
(2.12)
Integrating \((2.12)\) leads to
\[
\begin{align*}
\int_a^{u_1^k (r)} \frac{1}{\sqrt{\int_0^1 h_1 (M_1 f_2 (t)) dt}} \, dz & \leq \tau_1^{1/2} \int_0^t \sqrt{2 \phi_1 (z) \omega_1 \left( 1 + \int_0^1 t^{1-N} \int_0^t s^{N-1} p_2 (s) \, ds \, dt \right) dz}, \\
\int_b^{u_2^k (r)} \frac{1}{\sqrt{\int_0^1 h_2 (M_2 f_1 (t)) dt}} \, dz & \leq \tau_2^{1/2} \int_0^t \sqrt{2 \phi_2 (z) \omega_2 \left( 1 + \int_0^1 t^{1-N} \int_0^t s^{N-1} p_1 (s) \, ds \, dt \right) dz},
\end{align*}
\]
which is equivalent to
\[
\begin{align*}
H_1 \left( u_1^k (r) \right) & \leq \tau_1^{1/2} P_3 (r), \\
H_2 \left( u_2^k (r) \right) & \leq \tau_2^{1/2} Q_3 (r).
\end{align*}
\tag{2.13}
\]
Since \(H_1^{-1}\) and \(H_2^{-1}\) are strictly increasing on \([0, \infty)\), as previously discussed, we have that
\[
\begin{align*}
u_1^k (r) & \leq H_1^{-1} \left( \tau_1^{1/2} P_3 (r) \right), \\
u_2^k (r) & \leq H_2^{-1} \left( \tau_2^{1/2} Q_3 (r) \right).
\end{align*}
\tag{2.14}
These inequalities are independent of \(k\).

**Proof of Theorem 1.1 completed** Combining \((2.13)\) with
\[
H_1 (\infty) = H_2 (\infty) = \infty
\]
yields that the sequences \(\{u_1^k\}_{k \geq 1}\) and \(\{u_2^k\}_{k \geq 1}\) are bounded and equicontinuous on \([0, c_0]\) for arbitrary \(c_0 > 0\). Possibly after passing to a subsequence, we may assume that \(\{(u_1^k, u_2^k)\}_{k \geq 1}\) converges uniformly to \((u_1, u_2)\) on \([0, c_0]\). At the end of this process, we conclude by the arbitrariness of \(c_0 > 0\), that \((u_1, u_2)\) is a positive entire solution of system \((1.1)\). The solution constructed in this way will be radially symmetric. Since the radial solutions of \((1.1)\) are solutions of the ordinary differential equations system \((2.2)\) it follows that the radial solutions of \((1.1)\) with \(u_1(0) = a, u_2(0) = b\) satisfy:
\[
\begin{align*}
u_1 (r) & = a + \int_0^r \frac{1}{t^{N-1}} \int_0^t s^{N-1} p_1 (s) f_1 (u_2 (s)) \, ds \, dt, r \geq 0, \\
u_2 (r) & = b + \int_0^r \frac{1}{t^{N-1}} \int_0^t s^{N-1} p_2 (s) f_2 (u_1 (s)) \, ds \, dt, r \geq 0.
\end{align*}
\tag{2.15, 2.16}
\]
Choose \(R > 0\) so that \(r^{2N-2} p_1 (r)\) and \(r^{2N-2} p_2 (r)\) are non-decreasing for \(r \geq R\). In order to prove cases i.), ii.), iii.), iv.) and v.) above we intend to establish some inequalities. Using the same arguments as in \((2.4)\) and \((2.5)\) we can see that
\[
\begin{align*}
\left[ r^{N-1} (u_1 (r))' \right]' & \leq r^{N-1} p_1 (r) \tau_1 h_1 (M_1 f_2 (u_1 (r))) \omega_1 \left( 1 + \int_0^t t^{1-N} \int_0^t s^{N-1} p_2 (s) \, ds \, dt \right), \\
\left[ r^{N-1} (u_2 (r))' \right]' & \leq r^{N-1} p_2 (r) \tau_2 h_2 (M_2 f_1 (u_2 (r))) \omega_2 \left( 1 + \int_0^t t^{1-N} \int_0^t s^{N-1} p_1 (s) \, ds \, dt \right).
\end{align*}
\tag{2.17}
\]
Multiplying the first equation in \((2.17)\) by \(r^{N-1} (u_1 (r))'\) and the second by \(r^{N-1} (u_2 (r))'\) and integrating gives
\[
\begin{align*}
\left[ r^{N-1} (u_1 (r))' \right]^2 & \leq \left[ R^{N-1} (u_1 (R))' \right]^2, \\
+2 \int_R^r z^{2N-2} p_1 (z) \tau_1 \omega_1 \left( 1 + \int_0^t t^{1-N} \int_0^t s^{N-1} p_2 (s) \, ds \, dt \right) \frac{d}{dz} \int_0^{u_1 (z)} h_1 (M_1 f_2 (s)) \, ds \, dz, \\
\left[ r^{N-1} (u_2 (r))' \right]^2 & \leq \left[ R^{N-1} (u_2 (R))' \right]^2, \\
+2 \int_R^r z^{2N-2} p_1 (z) \tau_2 \omega_2 \left( 1 + \int_0^t t^{1-N} \int_0^t s^{N-1} p_1 (s) \, ds \, dt \right) \frac{d}{dz} \int_0^{u_2 (z)} h_2 (M_2 f_1 (z)) \, ds \, dz,
\end{align*}
\]
for \( r \geq R \). We get from the monotonicity of \( z^{2N-2} p_1(z) \) and \( z^{2N-2} p_2(z) \) for \( r \geq z \geq R \) that

\[
\begin{align*}
\left[ r^{N-1} (u_1(r)) \right]^2 & \leq C_1 + 2 \mathcal{C}_1 r^{2N-2} p_1(r) \omega_1 \left( 1 + \int_0^r t^{1-N} \int_0^t s^{N-1} p_2(s) \, ds dt \right) \mathcal{P}_1 (u_1(r)), \\
\left[ r^{N-1} (u_2(r)) \right]^2 & \leq C_2 + 2 \mathcal{C}_2 r^{2N-2} p_2(r) \omega_2 \left( 1 + \int_0^r t^{1-N} \int_0^t s^{N-1} p_1(s) \, ds dt \right) \mathcal{P}_2 (u_2(r)),
\end{align*}
\]

where \( C_1 = \left[ R^{N-1} (u_1(R)) \right]^2 \), \( C_2 = \left[ R^{N-1} (u_2(R)) \right]^2 \), \( \mathcal{P}_1 (u_1(r)) = \int_0^{u_1(r)} h_1 (M_1 f_2(s)) \, ds \) and \( \mathcal{P}_2 (u_2(r)) = \int_0^{u_2(r)} h_2 (M_2 f_1(s)) \, ds \). This implies that

\[
\begin{align*}
\left( \frac{u_1(r)}{\mathcal{H}_1 (u_1(r))} \right) & \leq \sqrt{\frac{C_1 r^{1-N}}{\mathcal{H}_1 (u_1(r))}} + \sqrt{\frac{2 \mathcal{C}_1 p_1(r)}{\mathcal{H}_1 (u_1(r))}} \sqrt{\omega_1 \left( 1 + \int_0^r t^{1-N} \int_0^t s^{N-1} p_2(s) \, ds dt \right)}, \\
\left( \frac{u_2(r)}{\mathcal{H}_2 (u_2(r))} \right) & \leq \sqrt{\frac{C_2 r^{1-N}}{\mathcal{H}_2 (u_2(r))}} + \sqrt{\frac{2 \mathcal{C}_2 p_2(r)}{\mathcal{H}_2 (u_2(r))}} \sqrt{\omega_2 \left( 1 + \int_0^r t^{1-N} \int_0^t s^{N-1} p_1(s) \, ds dt \right)}.
\end{align*}
\]

(2.18)

In particular, integrating (2.18) from \( R \) to \( r \) and using the fact that

\[
\sqrt{2 p_1(r) \mathcal{P}_1 (1 + \int_0^r t^{1-N} \int_0^t s^{N-1} p_2(s) \, ds dt)}
\]

\[
\leq r^{1+\varepsilon} p_1(r) \mathcal{P}_1 (1 + \int_0^r t^{1-N} \int_0^t s^{N-1} p_2(s) \, ds dt) + r^{-1-\varepsilon},
\]

lead to

\[
\int_{u_1(R)}^{u_2(r)} \left[ \int_0^t h_1 (M_1 f_2(z)) \, dz \right]^{-1/2} \, dt = H_1 (u_1(r)) - H_1 (u_1(R))
\]

\[
\leq \sqrt{\mathcal{C}_1} \int_R^r t^{1-N} \left( \int_0^{u_1(t)} h_1 (M_1 f_2(z)) \, dz \right)^{-1/2} \, dt + \int_R^r z^{1+\varepsilon} p_1(z) \mathcal{P}_1 (1 + \int_0^r t^{1-N} \int_0^t s^{N-1} p_2(s) \, ds dt) \, dz
\]

(2.19)

\[
\leq \frac{\sqrt{\mathcal{C}_1} \int_R^r t^{1-N} \, dt}{\left( \int_0^{u_1(R)} h_1 (M_1 f_2(z)) \, dz \right)^{1/2}} + \mathcal{P}_2 (r) + \frac{1}{\varepsilon R^e}.
\]

A special case of this inequality, is originally due to [2]. We next turn to estimating the second sequence. A similar calculation yields

\[
H_2 (u_2(r)) - H_2 (u_2(R)) \leq \frac{\sqrt{\mathcal{C}_2} \int_R^r t^{1-N} \, dt}{\left( \int_0^{u_2(R)} h_2 (M_2 f_1(z)) \, dz \right)^{1/2}} + \mathcal{P}_2 (r) + \frac{1}{\varepsilon R^e}.
\]

(2.20)

The inequalities (2.19) and (2.20) are needed in proving the "boundedness" of the functions \( u_1 \) and \( u_2 \). Indeed, they can be written as

\[
\begin{align*}
\left\{ \begin{array}{c}
u_1(r) \leq H_1^{-1} \left( H_1 (u_1(R)) + \frac{\sqrt{\mathcal{C}_1} \int_R^r t^{1-N} \, dt}{\left( \int_0^{u_1(R)} h_1 (M_1 f_2(z)) \, dz \right)^{1/2}} + \mathcal{P}_2 (r) + \frac{1}{\varepsilon R^e} \right) , \\
u_2(r) \leq H_2^{-1} \left( H_2 (u_2(R)) + \frac{\sqrt{\mathcal{C}_2} \int_R^r t^{1-N} \, dt}{\left( \int_0^{u_2(R)} h_2 (M_2 f_1(z)) \, dz \right)^{1/2}} + \mathcal{P}_2 (r) + \frac{1}{\varepsilon R^e} \right).
\end{array} \right.
\]

(2.21)
Having discussed the "bounded" case, we now turn to the Cases i.), ii.), iii.), iv.) and v.).

**Case i.):** When $P_2(\infty) < \infty$ and $Q_2(\infty) < \infty$ we find from (2.21) that

\[
\begin{cases}
\lim_{r \to \infty} u_1(r) < \infty \\
\lim_{r \to \infty} u_2(r) < \infty
\end{cases}
\]

for all $r \geq 0$

and so $(u_1, u_2)$ is bounded. We next consider:

**Case ii.):** The case $P_1(\infty) = Q_1(\infty) = \infty$ is proved in the following:

\[
u_1(r) = \begin{array}{c}
& a + \int_0^r \int_0^t t^{1-N} s^{N-1} p_1(s) f_1(u_2(s)) ds dt \\
& = \begin{cases}
& a + \int_0^r \int_0^y t^{1-N} p_1(t) f_1(b + \int_0^t z^{1-N} \int_0^s s^{N-1} p_2(s) f_2(u_1(s)) ds) dz dt dy \\
& \geq a + \int_0^r \int_0^y t^{1-N} p_1(t) f_1(b + f_2(a) \int_0^t z^{1-N} G_1(z) dz) dt dy \\
& = P_1(r).
\end{cases}
\]

Similar arguments show that 

\[u_2(r) \geq Q_1(r).
\]

Letting $r \to \infty$ in (2.22) and in the above inequality we conclude that

\[
\lim_{r \to \infty} u_1(r) = \lim_{r \to \infty} u_2(r) = \infty.
\]

**Case iii.):** In the spirit of Case i.) and Case ii.) above, we have

\[
\lim_{r \to \infty} u_1(r) \leq H_1^{-1} \left( \frac{H_1(u_1(R)) \varepsilon R^\varepsilon + 1}{\varepsilon R^\varepsilon} + \frac{\sqrt{C_1} \int_R^r t^{1-N} dt}{\left( \int_0^{u_1(R)} h_1(M_1 f_2(z)) dz \right)^{1/2} + \bar{c}_1 P_2(r)} \right)
\]

\[
\leq H_1^{-1} \left( \frac{H_1(u_1(R)) \varepsilon R^\varepsilon + 1}{\varepsilon R^\varepsilon} + \frac{R^{N-2} \sqrt{C_1}}{(N-2) \left( \int_0^{u_1(R)} h_1(M_1 f_2(z)) dz \right)^{1/2} + \bar{c}_1 P_2(\infty)} \right)
\]

\[
< \infty.
\]

Arguing as in [2] (see also [4]) we have

\[
\lim_{r \to \infty} u_2(r) = \infty.
\]

So, if 

\[P_2(\infty) < \infty \text{ and } Q_1(\infty) = \infty\]

we have that

\[
\lim_{r \to \infty} u_1(r) < \infty \text{ and } \lim_{r \to \infty} u_2(r) = \infty.
\]
Case iv.): By a straightforward modification of the proofs presented in the Case iii.) the results hold true since any statement about $P_2(\infty)$ can be translated into a statement about $Q_2(\infty)$.

Case v.): If $(u_1, u_2)$ is a nonnegative non-trivial entire large solution of (1.1), then $(u_1, u_2)$ satisfy

\[
 u_1(r) \leq H_1^{-1}\left( H_1(u_1(R)) + \frac{\sqrt{C_1} \int_R^\infty t^{1-N} dt}{\left( \int_0^{u_1(R)} h_1(M_1 f_2(z)) dz \right)^{1/2}} + \bar{c}_1P_2(r) + \frac{1}{\varepsilon R^\varepsilon} \right), 
\]

(2.23)

\[
 u_2(r) \leq H_2^{-1}\left( H_2(u_2(R)) + \frac{\sqrt{C_2} \int_R^\infty t^{1-N} dt}{\left( \int_0^{u_2(R)} h_2(M_2 f_1(z)) dz \right)^{1/2}} + \bar{c}_2Q_2(r) + \frac{1}{\varepsilon R^\varepsilon} \right), 
\]

(2.24)

where

\[ C_1 = [R^{N-1}(u_1(R))']^2 \quad \text{and} \quad C_2 = [R^{N-1}(u_2(R))']^2. \]

Next, assuming to the contrary that $P_2(\infty) < \infty$ and $Q_2(\infty) < \infty$, then (1.13) yields by taking $r \to \infty$ in (2.23) and (2.24).

Proof of Theorem 1.2 completed It follows from (2.13) and the conditions of the theorem that

\[ H_1\left(u_1^k(r)\right) \leq \bar{c}_1P_3(\infty) < \bar{c}_1H_1(\infty) < \infty, \]

\[ H_2\left(u_2^k(r)\right) \leq \bar{c}_2Q_3(\infty) < \bar{c}_2H_2(\infty) < \infty. \]

On the other hand, since $H_1^{-1}$ and $H_2^{-1}$ are strictly increasing on $[0, \infty)$, we find that

\[ u_1^k(r) \leq H_1^{-1}\left(\bar{c}_1^{1/2}P_3(\infty)\right) < \infty \quad \text{and} \quad u_2^k(r) \leq H_2^{-1}\left(\bar{c}_2^{1/2}Q_3(\infty)\right) < \infty, \]

and then the non-decreasing sequences $\{u_1^k(r)\}_{k \geq 1}$ and $\{u_2^k(r)\}_{k \geq 1}$ are bounded above for all $r \geq 0$ and all $k$. Combining these two facts, we conclude that $(u_1^k(r), u_2^k(r)) \to (u_1(r), u_2(r))$ as $k \to \infty$ and the limit functions $u_1$ and $u_2$ are positive entire bounded radial solutions of system (1.1).

Case ii.), iii.), iv) and v): For the proof, we follow the same steps and arguments as in the proof of Theorem 1.1.

Remark 2.1. Our proof actually gives a slightly stronger result, compared with other references. Since the conditions of the form

\[ H_1(\infty) \leq \infty \quad \text{and} \quad H_2(\infty) \leq \infty \]

are firstly introduced here.

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