The singular geometry of the sliver

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ABSTRACT: We consider “sliver” states which act as projection operators in the matter star product of Witten’s cubic string field theory. These sliver states, which might be associated with $D_p$-branes, are not finite norm states in the matter string Hilbert space. We describe the singularities of these states, and demonstrate that the sliver states are composed of strings having singular geometric features. These singularities take a particularly simple form in the zero slope limit $\alpha' \to 0$, where the star algebra factorizes into a product of the algebra of functions on space-time and the noncommutative star product of fields associated with higher string modes. An analogy to the sliver geometry suggests a natural mechanism for describing closed string states in open string field theory.

KEYWORDS: $D$-branes, String field theory.
1. Introduction

The celebrated Sen conjectures on open string tachyon condensation [1] have led to a resurgence of interest in string field theory, particularly in its applications to the physics of D-branes. Recently, certain solutions of the matter string field equations, known as sliver states [2, 3, 4], have received much attention [5-15]. In this paper we make some simple observations regarding the structure of these sliver states. We first consider the “D-instanton sliver” in the low-energy limit of string field theory. In this limit the matter string field star algebra factorizes into a space-time algebra and an algebra associated with massive string modes. The space-time part of the D-instanton sliver becomes a function which acts as a projection operator in an algebra of functions on space-time. While the appearance of a projection operator is expected from a number of viewpoints, a novel point is that this function...
is discontinuous. This discontinuity can be associated with a geometrical localization of the state to strings whose midpoints lie at the origin of space-time—a condition which can be represented algebraically as the constraint

\[ \hat{x}_0 |\tilde{\Xi}_0 \rangle = 0, \]  

(1.1)

where \( \hat{x}_0 \) is the operator measuring the center of mass of the string and \( |\tilde{\Xi}_0 \rangle \) is the \( \alpha' \rightarrow 0 \) limit of the D-instanton sliver state \( |\Xi_0 \rangle \). This discontinuity takes the state \( |\tilde{\Xi}_0 \rangle \) outside the string Hilbert space, as this state formally has vanishing norm. A similar type of discontinuity occurs in the full D-instanton sliver state \footnote{The existence of a singularity in the sliver state was first determined by Rastelli, Sen, and Zwiebach \[17\].}. We show that in this case the discontinuity can be associated with the algebraic condition

\[ \hat{x}(\pi/2) |\Xi_0 \rangle = 0, \]  

(1.2)

where \( \hat{x}(\pi/2) \) is the position of the string midpoint. This condition has the geometrical meaning that the functional on the space of string configurations associated with \( |\Xi_0 \rangle \) has support only on strings whose midpoint is fixed to live at the origin (i.e., on the locus of points defining the D-instanton). When the sliver state is taken to be extended in some space-time dimensions (such as the original D25-brane sliver), a related but distinct singular structure arises in the directions along which the associated brane is extended. We find that the geometrical significance of this singularity is that the strings composing the higher-dimensional sliver states can essentially “split” into two pieces at the midpoint, giving a pair of independent strings on the right and left halves of the string world-sheet. Algebraically, this condition is given by

\[ \frac{\delta}{\delta x_{\lambda}(\sigma)} \langle x(\sigma) |\Xi \rangle = 0, \]  

(1.3)

where \( x_{\lambda} = \lambda \theta(\sigma - \pi/2) \) is a step function with a jump discontinuity at the string midpoint. This result suggests that the sliver states may actually be related to a pair of Dp-branes. Recent numerical evidence also may support this possibility \[16\].

The observations we make here are of relevance in developing further the analogies between string field algebras and operator algebras such as \( C^* \) algebras, and for understanding the relationship between the the vacuum string field theory proposed by Rastelli, Sen and Zwiebach, and Witten’s cubic string field theory. Furthermore, a similar mechanism to that discussed here for the sliver states can give rise to singular squeezed open string states which have periodic boundary conditions on the open string, and which thus represent closed string states.

In Section 2, we review the 3-string vertex of Witten’s open bosonic string field theory and the sliver states. In Section 3, we consider the zero-slope limit of the sliver, and discuss the singular nature of this state. In Section 4, we describe the singular features of the D-instanton and higher-dimensional Dp-brane slivers. In Section 5 we propose a class of states
with similar singular structure to the sliver, which we believe represent closed string states. Section 6 contains conclusions.

Related work on the singular structure of the sliver has been done by Rastelli, Sen, and Zwiebach, and will appear in a publication by those authors [18].

2. D-Branes and the Sliver in String Field Theory

2.1 String field theory and the star product

In this subsection we briefly review the star product of matter fields in Witten’s cubic string field theory [19] and fix notation. We primarily follow here the conventions of [4], so when comparing with [20, 6, 10] factors of \( p(x) \) should be multiplied (divided) by \( \sqrt{2} \).

In this paper we are concerned with the matter sector of string field theory. The matter fields \( x^\mu \) on the string, \( (\mu \in \{1, \ldots, 26\}) \), can be decomposed in Fourier modes through

\[
x(\sigma) = x_0 + \sqrt{2} \sum_{n=1}^{\infty} x_n \cos(n\sigma), \quad 0 \leq \sigma \leq \pi.
\]  

(2.1)

(We will drop most spatial indices in this paper for clarity.) The modes in (2.1) can be related to creation and annihilation operators \( a_n^\dagger, a_n \) satisfying \([a_n, a_m^\dagger] = \delta_{n,m}\) through

\[
\hat{x}_n = \frac{i}{\sqrt{2n}} (a_n - a_n^\dagger) \quad \hat{p}_n = -i \frac{\partial}{\partial x_n} = \sqrt{\frac{n}{2}} (a_n + a_n^\dagger)
\]  

(2.2)

for \( n \neq 0 \), and through

\[
\hat{x}_0 = \frac{i}{\sqrt{2}} (a_0 - a_0^\dagger) \quad \hat{p}_0 = -i \frac{\partial}{\partial x_0} = \frac{1}{\sqrt{2}} (a_0 + a_0^\dagger)
\]  

(2.3)

for the zero modes.

A matter string field \( \Psi \) is a state in the string Hilbert space \( \mathcal{H} \), a basis for which is given by the set of states produced by acting with a finite number of matter oscillators \( a_n^\dagger, n \geq 0 \), on the matter vacuum \( |0\rangle \) annihilated by \( a_n \) for all \( n \geq 0 \). It is often convenient to describe matter string fields in terms of a momentum basis of states

\[
|p\rangle = \frac{1}{(\pi)^{1/4}} \exp \left[ -\frac{1}{2} p^2 + \sqrt{2} a_0^\dagger p - \frac{1}{2} (a_0^\dagger)^2 \right] |0\rangle
\]  

(2.4)

satisfying \( \hat{p}_0 |k\rangle = k |k\rangle \). The matter string field \( \Psi \) can be thought of as a functional \( \Psi[x(\sigma)] \). For well-behaved states in the Fock space (such as those given by acting on the vacuum with a finite number of raising operators \( a_n^\dagger \)), this functional corresponds to a well-behaved function on the countable set of string modes \( \{x_n\} \).

The star product of matter string fields corresponds to a map \( * : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \). For well-behaved Fock space states this product is associative. The star product is schematically
defined by “gluing” the right half of one string to the left half of another with a delta function interaction through

\[
(\Psi \star \Phi) [z(\sigma)] \equiv \int \prod_{0 \leq \sigma \leq \frac{\pi}{2}} dy(\sigma) \ dx(\pi - \sigma) \ \prod_{\frac{\pi}{2} \leq \sigma \leq \pi} \delta[x(\sigma) - y(\pi - \sigma)] \ \Psi[x(\sigma)] \Phi[y(\sigma)] \tag{2.5}
\]

\[
z(\sigma) = x(\sigma) \quad \text{for} \quad 0 \leq \sigma \leq \frac{\pi}{2},
\]

\[
z(\sigma) = y(\sigma) \quad \text{for} \quad \frac{\pi}{2} \leq \sigma \leq \pi.
\]

This definition can be made more precise using a mode decomposition of the string. An explicit calculation of the three-string vertex \(|V_3\rangle \in \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}\) satisfying

\[
|\Psi \star \Phi\rangle = (\cdot \otimes \langle \Psi \otimes \langle \Phi |) |V_3\rangle \tag{2.6}
\]

in terms of the string modes was given in [20, 21, 22, 23]. In the matter sector, this three-string vertex is given by

\[
|V_3\rangle = \int dp^{(1)} dp^{(2)} dp^{(3)} \ \delta^{26}(p^{(1)} + p^{(2)} + p^{(3)}) \ \exp \left( -\frac{1}{2} \sum_{r,s \leq 3} \left( \sum_{m,n \geq 0} V^r_{mn} V^s_{nm} (a^\dagger_m \cdot a^\dagger_n) + 2 \sum_{m \geq 1} V^r_{m0} V^s_{0m} (a^\dagger_m \cdot p) + V^r_{00} V^s_{00} (p \cdot p) \right) \right) \ |0\rangle \otimes |0\rangle \otimes |0\rangle \tag{2.7}
\]

where the Neumann coefficients \(V^r_{mn}\) are computable constants (see, e.g. [4]). Using momentum conservation, these coefficients can be chosen such that \(V^r_{00} = 0\) when \(r \neq s\), and \(\sum_{r=1}^3 V^r_{0n} = 0\). These coefficients have the symmetry \(V^r_{mn} = V^s_{nm}\) and cyclic symmetry under \((r, s) \rightarrow (r \mod 3 + 1, s \mod 3 + 1)\).

Using (2.4) and performing the Gaussian integrals on \(p\), the vertex (2.7) can be rewritten as [20, 4] (using \(b = 2\) in the notation of [4])

\[
|V_3\rangle = \left( \frac{2 \pi^{1/4}}{\sqrt{3(1 + V^r_{00})}} \right)^{26} \exp \left( -\frac{1}{2} \sum_{r,s \leq 3} \sum_{m,n \geq 0} V^r_{mn} V^s_{nm} (a^\dagger_m \cdot a^\dagger_n) \right) \ (|0\rangle \otimes |0\rangle \otimes |0\rangle) \tag{2.8}
\]

where \(V^r_{00} = V^r_{rr}\) and

\[
V^r_{mn} = V^r_{mn} - \frac{1}{1 + V^r_{00}} \sum_t V^{rt}_{m0} V^{ts}_{0n}.
\]

\[
V^r_{m0} = V^{sr}_{0m} = \sqrt{2} \frac{V^r_{00}}{1 + V^r_{00}} V^r_{ms}.
\]

\[
V^r_{00} = \frac{2}{3(1 + V^r_{00})} + \delta^{rs} (1 - 2/(1 + V^r_{00})).
\]

The expression (2.7) is useful for states of fixed momentum, while the expression (2.8) is useful for states associated with objects localized in space-time (such as D-instantons). By using (2.4) in only a subset of \(k\) dimensions, the vertex can be written in a form natural for objects of codimension \(k\) in space-time.
2.2 Matter projectors and the sliver state

The three-string vertex defines an algebra structure on the space of string fields. Given the close relations between D-branes, noncommutative solitons, and K-theory, and the central role of projection operators in the latter two subjects, it is natural to search for projection operators in the string field algebra. Indeed, projection operators in the matter string field star algebra seem to be naturally related to single and multiple D-brane configurations in target space \([5, 6, 7]\), although the details of this correspondence in the ghost sector are not yet fully understood.

One particularly natural set of projection operators of the string field star algebra are those related to the “sliver state” found in \([2]\). It was shown in \([3, 4]\) that the matter projection equation

\[
\Psi = \Psi \star \Psi \tag{2.10}
\]

is satisfied by the zero-momentum sliver state, which is, by definition:

\[
|\Xi\rangle = \left[\det(1 - Z) \det(1 + T)\right]^{13} \exp \left[-\frac{1}{2} \sum_{n,m \geq 1} a_m^\dagger S_{mn} a_n^\dagger\right] |p = 0\rangle \tag{2.11}
\]

where

\[
T = \frac{1}{2Z} \left(1 + Z - \sqrt{(1 + 3Z)(1 - Z)}\right),\tag{2.12}
\]

\[
Z = CV, \tag{2.13}
\]

\[
S = CT, \tag{2.14}
\]

\(V = V^{rr}\) and \(C_{mn} = \delta_{mn}(-1)^m\) are all infinite matrices indexed by \(m, n \geq 1\). The state \((2.11)\) is thus defined in terms of the Neumann coefficients \(V_{mn}^{rs}\) appearing in the three-string vertex \((2.7)\). Replacing these coefficients with the coefficients \(V'\) appearing in \((2.8)\) leads to a localized (“D-instanton”) form of the sliver state

\[
|\Xi_0\rangle = \left(\frac{\sqrt{3}}{2\pi^{1/4}}(1 + V_{00})\right)^{26} \left[\det(1 - Z') \det(1 + T')\right]^{13} \exp \left[-\frac{1}{2} \sum_{n,m \geq 0} a_m^\dagger S'_m a_n^\dagger\right] |0\rangle, \tag{2.15}
\]

where \(T', Z', S'\) are defined as above, but using \(C'\) and \(V'\) with indices \(m, n \geq 0\) instead of \(C\) and \(V\) with \(m, n \geq 1\). If we carry out the replacement of \(V\) with \(V'\) in a subset of \(25 - p\) of the spatial dimensions, we arrive at a hybrid sliver given by \((2.11)\) in \(p + 1\) space-time dimensions and \((2.15)\) in the remaining \(25 - p\) dimensions. This sliver is - hypothetically - associated with a Dp-brane.

The states \(|\Xi\rangle, |\Xi_0\rangle\) (and their Dp-brane relatives) are not finite norm states in the string Fock space, as we will discuss in Section 4. We will first find it instructive, however, to consider a simple limit of these states in which the massive string modes decouple.
3. Zero-slope slivers

3.1 The zero-slope limit

In [24] Witten pointed out that in a certain low energy limit the string field algebra should, in some sense, factorize

$$\mathcal{A} \to \mathcal{A}_0 \otimes \mathcal{A}_1$$  (3.1)

where, for open strings on $R^{26}$ with Neumann boundary conditions there is a well-defined subalgebra $\mathcal{A}_0$ of $\mathcal{A}$ defined by the $p = 0$ sector while $\mathcal{A}_1$ is the $C^\ast$ algebra of functions on space-time. When $B_{\mu\nu} = 0$ this algebra is commutative, and when $B_{\mu\nu}$ is nonzero (and the limit is taken appropriately as in [23, 26, 24]) $\mathcal{A}_1$ is related to the noncommutative Moyal algebra of functions on $R^{26}$.

There are several conceptual issues raised by trying to assign a precise meaning to (3.1). We will take the following pragmatic route. (See also the paper by Schnabl [27] for related remarks.)

Let us consider the 3-string vertex for Chern-Simons bosonic open string field theory with Neumann boundary conditions in a family of closed string backgrounds parametrized by constant fields $G_{\mu\nu}, B_{\mu\nu}$ on $R^{1,25}$. The string field vertices have been worked out in [28, 29]. These expressions are generally written in terms of oscillators that depend on the metric:

$$[a^\mu_n, a^\nu_m] = G^{\mu\nu}_{\delta n + m, 0}.$$  (3.2)

In the absence of a $B$ field, the three-string vertex $|V_3\rangle$ takes the form (2.7), where contractions are understood to be done using the metric $G_{\mu\nu}$. When thinking about the deformation of algebras in terms of explicit structure constants it is useful to keep the basis fixed and to let the structure constants alone vary. For a family of metrics with open string metric $G_{\mu\nu} = t^2 G^{(0)}_{\mu\nu}$ we define $\beta^\mu_n = ta^\mu_n$, so that

$$[\beta^\mu_n, \beta^\nu_m] = (G^{(0)})^{\mu\nu}_{\delta n + m, 0}.$$  (3.3)

We construct a basis for the Fock space using the oscillators $\beta^\mu_n$, ($n \neq 0$); our statements about limits are statements about the $t$-dependence of matrix elements in this basis.

There are three kinds of terms in the string field vertex that we must discuss:

1. Type I:

$$\alpha^{\mu}(r)G_{\mu\nu}a^{(s)}_m V^{rs}_{nm} = \beta^{\mu}(r)G^{(0)}_{\mu\nu}\beta^{(s)}_m V^{rs}_{nm} \quad n, m \geq 1$$

Since the Neumann coefficients $V^{rs}_{nm}$ are $t$-independent, these terms clearly lead to $t$-independent matrix elements in the Fock space basis of $\beta$’s.

2. Type II: The cross terms between the zero modes and the oscillators scale like

$$\sqrt{\alpha} p^{(r)}_{\mu} a^{(s)}_n \gamma V^{rs}_{0n} = t^{-1} \sqrt{\alpha} p^{(r)}_{\mu} \beta^{(s)}_n \gamma V^{rs}_{0n}$$  (3.4)
These go to zero for $t \to \infty$ at fixed $\sqrt{\alpha p_\mu}$, and even go to zero if we take $p_\mu = \epsilon^\theta q_\mu$ for $\theta < 1$, and $q_\mu$ fixed.

3. Type III:

$$\alpha' p_\mu G^{\mu
u} p_\nu^{(s)} V_{00}^{rs} \sim \frac{1}{t^{2-2\theta}} \to 0$$  \hspace{1cm} (3.5)

In particular, if we define $\epsilon = t^{-1+\theta}$ then our scaling limit has the same effect as having the terms $V_{00}^{rs}$ in the vertex scale with $\epsilon^2$ for $\epsilon \to 0$ while $V_{0n}$, $n \geq 1$ scale like $\epsilon$. (If we include a noncommutativity parameter and take an appropriate limit \cite{25,26} limit then $\theta = 1/2$ is preferred to get a nontrivial limit for the phase prefactor.)

3.2 The star product in the zero-slope limit

Let us now replace

$$V_{mn} \to V_{mn}$$
$$V_{m0} \to \epsilon V_{m0}$$
$$V_{00} \to \epsilon^2 V_{00}$$  \hspace{1cm} (3.6)

and take the limit $\epsilon \to 0$. One finds from (2.9) that the vertex factorizes

$$|V_3 \rangle \to |V_3^{(0)} \rangle \otimes |V_3^{(1)} \rangle.$$  \hspace{1cm} (3.7)

We will now give explicit formulae for the vertices $|V_3^{(0)} \rangle$ and $|V_3^{(1)} \rangle$, thereby defining the algebras $A_0$ and $A_1$ in \cite{3.1}.

The $p = 0$ algebra $A_0$ is defined through the three-string vertex

$$|V_3^{(0)} \rangle = (\sqrt{2\pi})^{26} \exp\left(-\frac{1}{2} \sum_{r,s \leq 3} \sum_{m,n \geq 1} V_{mn}^{rs} (a_m^{(r)\dagger} \cdot a_n^{(s)\dagger})\right) (|0\rangle \otimes |0\rangle \otimes |0\rangle).$$

The space-time algebra $A_1$ is defined through

$$(f \star g)(p^{(3)}) = \frac{1}{(2\pi)^{13}} \int dp^{(1)} dp^{(2)} \delta^{26}(p^{(1)} + p^{(2)} - p^{(3)}) f(p^{(1)}) g(p^{(2)})$$  \hspace{1cm} (3.8)

which is the expression in Fourier space of the pointwise product

$$(f \star_{\text{point}} g)(x) := f(x)g(x).$$  \hspace{1cm} (3.9)

Let us now consider the representation (2.8) of the three-string vertex for states localized in space-time. In the limit $\epsilon \to 0$, the coefficients $V'$ found by inserting (3.6) in (2.3) go to

$$V_{mn}^{rs} \to V_m^{rs} + \mathcal{O}(\epsilon^2)$$
$$V_{m0}^{rs} \to 0 + \mathcal{O}(\epsilon)$$
$$V_{00}^{rs} \to \frac{2}{3} - \delta^{rs} + \mathcal{O}(\epsilon^2).$$  \hspace{1cm} (3.10)
Again, the algebra factorizes. The space-time independent part of the algebra, $A_0$, is unchanged since $V_{mn}'' \to V_{mn}$. The space-time algebra $A_1$ is now represented by the three-string vertex

$$|V_3^{(1)}\rangle = \exp\left[\frac{1}{6} \left( (a_{(1)}^\dagger)^2 + (a_{(2)}^\dagger)^2 + (a_{(3)}^\dagger)^2 \right) - \frac{2}{3} \left[ a_{(1)}^\dagger a_{(2)}^\dagger + a_{(2)}^\dagger a_{(3)}^\dagger + a_{(3)}^\dagger a_{(1)}^\dagger \right] \right]$$

$$\times \left( \frac{\sqrt{2}}{\sqrt{3} \pi^{1/4}} \right)^{26} (|0\rangle \otimes |0\rangle \otimes |0\rangle)$$

(3.11)

This vertex encodes the same $C^*$ algebra as (3.9). This may be proved by straightforward manipulation of harmonic oscillators as follows. The generating function for Hermite functions is

$$\langle x | e^{\sqrt{2}t a^\dagger} | 0 \rangle = \sum \frac{(\sqrt{2t})^n}{\sqrt{n!}} \psi_n(x) = \pi^{-1/4} e^{-\frac{1}{2}x^2 + t^2 - 2itx}$$

(3.12)

We may multiply these pointwise in $x$ and re-express the result in terms of harmonic oscillators. The result is

$$e^{\sqrt{2}t a^\dagger} |0\rangle \ast_{\text{point}} e^{\sqrt{2}t_2 a^\dagger} |0\rangle = \pi^{-1/4} \sqrt{\frac{2}{3}} e^{\frac{1}{2}t^2 + t_1 t_2 - \frac{2}{3} t_1 t_2 - \frac{2}{3} t_1 t_2 a^\dagger a^\dagger + \frac{1}{3} (a^\dagger)^2} |0\rangle.$$

(3.13)

The right hand side of (3.13) can also be written as

$$1 \langle 0 | e^{\sqrt{2}t_1 a^\dagger_1} |0\rangle \langle 0 | e^{\sqrt{2}t_2 a^\dagger_2} |V\rangle$$

(3.14)

where

$$|V\rangle = \pi^{-1/4} \sqrt{\frac{2}{3}} e^{\frac{1}{6} ((a_1^\dagger)^2 + (a_2^\dagger)^2 + (a_3^\dagger)^2) - \frac{2}{3} [a_1^\dagger a_2^\dagger + a_2^\dagger a_3^\dagger + a_3^\dagger a_1^\dagger]] |0\rangle \otimes |0\rangle \otimes |0\rangle.$$  (3.15)

3.3 The limit of the space-time filling sliver

The projector associated with a space-filling D25-brane is given by the zero-momentum matter sliver state (2.11). Since the indices $m, n$ in (2.11) range from 1 to infinity, this state is space-time independent, and in the limit $\epsilon \to 0$ can be thought of as a product of a nontrivial state in $A_0$, given again by (2.11), and the function $f(x) = 1$ (i.e. the state $\sqrt{2\pi} |p_0 = 0\rangle$) in $A_1$, which clearly acts as a projection operator under (3.9).

3.4 The limit of the lower-dimensional slivers

In the limit $\epsilon \to 0$, we see from (2.9) that

$$V_{00}'' \to -1/3, \quad V_{0n}'' \to 0, \quad V_{n0}'' \to 0$$

$$Z_{00}'' \to -1/3, \quad Z_{0n}'' \to 0, \quad Z_{n0}'' \to 0$$

$$T_{00}'' \to -1, \quad T_{0n}'' \to 0, \quad T_{n0}'' \to 0$$

$$S_{00}'' \to -1, \quad S_{0n}'' \to 0, \quad S_{n0}'' \to 0.$$  (3.16-3.19)
The advantage of the low energy limit now becomes apparent. It separates out the space-time dependence of the sliver states associated with lower-dimensional Dp-branes, and makes manifest the existence of an eigenvector of $S'$ and of $T'$ of eigenvalue $-1$. This eigenvector is associated with the zero-mode raising operator $a_0^\dagger$, and gives the sliver a singular structure which we now discuss.

Separating out the space-time dependence encoded in $S'_{00}$, etc., we have

$$|\tilde{\Xi}_0\rangle = \lim_{t \to 0} |\Xi_0\rangle = A_1 \otimes A_0$$

(3.20)

where $A_0 = |\Xi\rangle$ and $A_1$ is given by

$$A_1 = \lim_{s \to -1} \left[ \pi^{1/4} \sqrt{1+s} \right]^{26} \exp \left( -\frac{s}{2}(a_0^\dagger)^2 \right) |0\rangle.$$  

(3.21)

The overall normalization of $A_1$ is determined by normalizing $A_0$ so that it becomes a projector with respect to $V^{(0)}$. Since the product is a projector, the normalization of $A_0$ fixes $A_1$ to be a projector.

We can translate from the squeezed state representation to a function of $x_0$ using (for each coordinate direction)

$$\langle x | \exp \left( -\frac{1}{2} s (a_0^\dagger)^2 \right) | 0 \rangle = \pi^{-1/4} \frac{1}{\sqrt{1+s}} \exp \left( -\frac{\lambda}{2} x_0^2 \right)$$

where

$$\lambda = \frac{1-s}{1+s}.$$  

(3.22)

Thus, we have

$$A_1(x_0) = \lim_{\lambda \to \infty} \exp \left( -\frac{\lambda x_0^2}{2} \right)$$

$$= \begin{cases} 0, & x_0 \neq 0, \\ 1, & x_0 = 0 \end{cases}$$

(3.23)

(3.24)

The function (3.24) is indeed a projection operator under the pointwise product (3.9) in the sense that $A_1^2 = A_1$. The surprising point here is that the limit is not a continuous projector. The singular nature of this sliver state can be expressed algebraically by the simple condition that

$$x_0^\mu |\tilde{\Xi}_0\rangle = 0$$  

(3.25)

in the directions transverse to the sliver. This condition states geometrically that the strings of which the zero-slope limit of the sliver are comprised have a center of mass which is confined to the $(p+1)$-dimensional hypersurface of the associated Dp-brane.

4. Singular geometry of Dp-brane slivers

In this section we show that, even without taking the zero-slope limit, the sliver states associated with Dp-branes have singularities which can be interpreted geometrically.
4.1 Formal eigenvalues of \( S \) and \( S' \)

The sliver states \(|\Xi\rangle\) and \(|\Xi_0\rangle\) are described as squeezed states in (2.11, 2.15). These squeezed states fail to have finite norm if the associated matrices \( S, S' \) have eigenvalues of \( \pm 1 \). It was found numerically by Rastelli, Sen, and Zwiebach that such eigenvalues indeed seem to occur \([17]\). In this subsection we give a simple analytic derivation of the eigenvectors associated with these problematic eigenvalues; another approach to determining these eigenvectors will be discussed in \([18]\).

Let us begin with the D-instanton sliver \(|\Xi_0\rangle\). It is convenient to recall some notation from \([20, 6]\). Note that in those references slightly different notation is used—in particular, the matrices we refer to as \( U', V' \) here are referred to as \( U, V \) in those references.

The D-instanton sliver is constructed from the matrix \( V'_{mn} \) of Neumann coefficients with indices \( m, n \geq 0 \). It was shown in \([20]\) that this matrix can be written as

\[
V' = \frac{1}{3} \left( C' + U' + \bar{U}' \right)
\]

in terms of a matrix \( U \) implicitly defined through the overlap conditions

\[
(1 - Y') E'(1 + U') = 0 \quad (4.2)
\]

\[
(1 + Y') E'^{-1}(1 - U') = 0 .
\]

These conditions are given in terms of the matrices

\[
E'^{-1}_{mn} = \delta_{mn}(\sqrt{n} + \delta_{m0}) \quad (4.3)
\]

and

\[
Y' = -\frac{1}{2} C' - \frac{\sqrt{3}}{2} i C' X', \quad (4.4)
\]

which is in turn defined through the matrix \( X' \) with nonzero matrix elements

\[
X'_{2k+1,2n} = X'_{2n,2k+1} = \frac{4(-1)^{k+n}(2k+1)}{\pi ((2k+1)^2 - 4n^2)} \quad n \geq 1, \quad (4.5)
\]

\[
X'_{2k+1,0} = X'_{0,2k+1} = \frac{2\sqrt{2}(-1)^k}{\pi (2k+1)} .
\]

It is useful to understand these matrices more conceptually. Note that the Hilbert space \( L^2[0, \frac{1}{2}\pi] \) has two distinct orthonormal bases:

\[
\varphi_0 := \sqrt{\frac{2}{\pi}} \quad \varphi_n := \sqrt{\frac{4}{\pi}} \cos(2n\sigma) \quad n = 1, \ldots
\]

\[\text{Note that this matrix, like equation (2.3), differs from the notation of [20, 3] by a factor of } \sqrt{2} \text{ in the 0-index component.}\]
and
\[ \psi_k := \sqrt{\frac{4}{\pi}} \cos[(2k + 1)\sigma] \quad k = 0, 1, \ldots \]  
(4.7)

The matrix \( X' \) is simply the unitary transformation between these bases:
\[ \psi_k = \sum_{n=0}^{\infty} X'_{2k+1,2n}\varphi_n \]  
(4.8)
\[ \varphi_n = \sum_{k=0}^{\infty} X'_{2n,2k+1}\psi_k \]

On the other hand, the Hilbert space \( \mathcal{H} = L^2[0, \pi] \) has a natural decomposition into \( \pm 1 \) eigenspaces of the flip operator \( f(\sigma) \rightarrow f(\pi - \sigma) \), namely \( \mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^- \). We refer to these as the even and odd sectors. Now, \( \frac{1}{\sqrt{2}}\varphi_n \) and \( \frac{1}{\sqrt{2}}\psi_k \) extend to form orthonormal bases for \( \mathcal{H}^\pm \), respectively. Therefore, the unitary transformation (4.8) defines a unitary isomorphism \( \mathcal{H}^+ \rightarrow \mathcal{H}^- \). This map is just the map
\[ f \rightarrow f\chi_{[0,\frac{1}{2}\pi]} - f\chi_{[\frac{1}{2}\pi,\pi]}, \]  
(4.9)
where \( \chi_A \) is the characteristic function of a measureable set \( A \). Indeed the main utility of \( X' \) in [6], was that it related modes on the full string to modes on the left and right halves of the string, and it was used extensively to develop a “split string” formalism for the matter sector of string field theory.

Now, formally, the map (4.9) annihilates the \( \delta \)-function \( f(\sigma) = \delta(\sigma - \frac{1}{2}\pi) \), so we might expect \( X' \) to have a null eigenvector. Indeed, one can check that
\[ \sum_n X'_{mn}w'_n = 0 \]  
(4.10)
where \( w'_n \) is given by
\[ w'_n = \begin{cases} \frac{1}{\sqrt{2}}, & n = 0 \\ (-1)^{n/2}, & n \text{ even, } n > 0 \\ 0, & n \text{ odd} \end{cases} \]  
(4.11)
We stress that the sum in (4.10) is convergent. While it might be alarming to find a zero eigenvalue of a unitary operator, there is really no mathematical contradiction: \( w'_n \) is not normalizable, indeed \( f(\sigma) = \delta(\sigma - \frac{1}{2}\pi) \) is not an \( L^2 \)-function, but rather a tempered distribution. Nevertheless, we will now argue that this formal eigenvector plays an important role in understanding the meaning of the sliver state.

It is convenient to decompose the various matrices into two by two block form associated with even and odd sectors, so for example
\[ X' = \begin{pmatrix} 0 & X'_{oo} \\ X'_{oe} & 0 \end{pmatrix}, \]  
(4.12)
\[ U' = \begin{pmatrix} U'_{oo} & U'_{oe} \\ U'_{eo} & U'_{ee} \end{pmatrix}, \quad \bar{U}' = \begin{pmatrix} U'_{oo} & -U'_{oe} \\ -U'_{eo} & U'_{ee} \end{pmatrix}. \tag{4.13} \]

In terms of these blocks, the even-even sector of the first equation in (4.2) gives
\[ E' (1 + U'_{ee}) = -\frac{i}{\sqrt{3}} X'_{eo} E' U'_{oe}. \tag{4.14} \]

Since the RHS vanishes when we act on the left with \( w' \), it follows immediately that the vector \( \nu' = E' w' \) is an eigenvector of the symmetric matrix \( U'_{ee} \) with eigenvalue \(-1\). From (4.1) this implies that \( \nu' \) is an eigenvector of \( V' \) with eigenvalue \(-1/3\), and hence an eigenvector of \( S' \) of eigenvalue \(-1\). This conclusion is also supported by a numerical analysis of the spectrum of \( V' \) in level truncation. Thus, we have identified an eigenvector \( \nu' \) of \( S' \) with components
\[ \nu'_n = \begin{cases} \frac{1}{\sqrt{2}}, & n = 0 \\ \frac{(-1)^{n/2}}{\sqrt{n}}, & n \text{ even, } n > 0 \\ 0, & n \text{ odd} \end{cases} \tag{4.15} \]

The existence of this (nonnormalizable) eigenvector of eigenvalue \(-1\) indicates that the norm of the D-instanton sliver state vanishes. We will discuss the geometrical meaning of this vanishing norm in the following subsection.

It is interesting to consider the behavior of the eigenvector \( \nu' \) in the limit \( t \to \infty \) considered in the previous section. Changing the metric to \( G_{\mu
u} = t^2 G_{\mu
u}^{(0)} \), we see that the essential change in the relationship between the modes \( x_n \) and the raising and lowering operators \( a_n, a_n^\dagger \) is that instead of (2.2) we have for \( n > 0 \)
\[ \hat{x}^n = \frac{i}{t^2 \sqrt{2}} (a_n - a_n^\dagger). \tag{4.16} \]

Note that the relation (2.3) is unchanged as we are keeping the definitions of \( x_0, p_0, a_0 \) fixed. This change means that in the first overlap equation of (1.2), the matrix \( E' \) must be replaced by the matrix
\[ E_{mn}^{(t)} = \begin{cases} \delta_{mn} t^2 \sqrt{n}, & m > 0 \\ \delta_{nn}, & m = 0 \end{cases} \tag{4.17} \]

This change must also be made in the matrix \( E' \) appearing on the left hand side of (4.14). This means that for an arbitrary value of \( t \), the vector \( \nu^{(t)} = E^{(t)} w' \) is an eigenvector of \( U'_{ee}^{(t)} \) with eigenvalue \(-1\). It is clear that in the limit this vector approaches
\[ \lim_{t \to \infty} \nu^{(t)}_n = \frac{1}{\sqrt{2}} \delta_{n0}. \tag{4.18} \]

This is precisely the eigenvector we identified in Section 3 in the \( t \to \infty \) limit.

Now let us turn to the D25-brane sliver \( |\Xi\rangle \). The matrix \( S \) appearing in the squeezed state description of this sliver (2.13) is related to the matrix \( V_{mn}(m, n > 0) \) of Neumann
coefficients in the same way that $S'$ is related to $V'$. Just as $V'$ can be related to the matrix $U'$ satisfying the overlap equations (4.2), the matrix $V$ can be related to a matrix $U$ satisfying similar overlap equations through

$$V = \frac{1}{3} \left( C + U + \bar{U} \right)$$  \hspace{1cm} (4.19)$$

Just as $U'$ is related to the matrix $X'$ given by (4.3), $U$ is related to the matrix $X$ which is given by simply restricting $X'$ to nonzero indices ($X_{mn} = X'_{mn}, m, n > 0$). The details of this relationship were worked out by Moeller in [15]. It follows immediately from the overlap equations for $U$ that

$$1 - U_{oo} = \frac{i}{\sqrt{3}} U_{oe} E^{-1} X_{eo} E.$$  \hspace{1cm} (4.20)$$

(This is equivalent to equation (2.31) in [15]). Just as $X'_{oe}$ has a null vector $w'$, the matrix $X_{eo}$ has the null vector $w$ with components $w_{2k+1} = (-1)^k/(2k + 1)$. This null vector can easily be understood geometrically, as these are the coefficients in an expansion of the constant function on the half string in terms of odd cosines. Thus, $X'_{eo} w$ is naturally a vector whose only nonzero component is in the 0-index direction, so that it follows immediately that $X_{eo} w = 0$. From this null vector and the equation (4.20) it follows that $\nu = E^{-1} w$ is an eigenvector of $U_{oo}$ with eigenvalue $+1$, and hence an eigenvector of $V$ with eigenvalue $+1/3$. In turn, $\nu$ must be an eigenvector of $S$ with eigenvector $+1$. Thus, we have identified an eigenvector $\nu$ of $S$ with components

$$\nu_n = \begin{cases} \frac{(-1)^{n-1}/2}{\sqrt{n}}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$  \hspace{1cm} (4.21)$$

### 4.2 Geometrical interpretation of sliver singularities

Let us now turn to a discussion of the geometrical significance of the singular eigenvectors $\nu', \nu$ we found for the matrices $S', S$ in Section 4.1. We begin by recalling that the zero-slope limit of the sliver $|\tilde{\Xi}\rangle$ is described by a squeezed state proportional to

$$e^{-\frac{s}{2}(a_0^d)^2}|0\rangle,$$  \hspace{1cm} (4.22)$$

with $s \to -1$. Acting on this state with $a_0$ gives

$$a_0 e^{-\frac{s}{2}(a_0^d)^2}|0\rangle = -sa_0^\dagger e^{-\frac{s}{2}(a_0^d)^2}|0\rangle$$  \hspace{1cm} (4.23)$$

so that when $s \to -1$ the state (4.22) is annihilated by the operator

$$\hat{x}_0 = i(a_0 - a_0^\dagger)/\sqrt{2},$$  \hspace{1cm} (4.24)$$

as stated in (3.25).
Similar relations hold for the Dp-brane sliver states, following from the existence of the eigenvectors $\nu'$ and $\nu$. First consider the D-instanton sliver. We have in general for any $\sigma$

$$\hat{x}(\sigma)\langle \Xi_0 | = -\frac{i}{\sqrt{2}} \left( \sum_{n=0}^{\infty} \xi_n(\sigma) a_n^\dagger \right)\langle \Xi_0 |$$

(4.25)

where

$$\xi_n(\sigma) = \zeta_n(\sigma) + \sum_{m=0}^{\infty} \zeta_m(\sigma) S_{mn}'$$

(4.26)

and $\zeta_0(\sigma) = 1$, $\zeta_m(\sigma) = \sqrt{\frac{2}{m}} \cos(m\sigma)$. Since $\nu'$ is an eigenvector of $S'$ with eigenvalue -1, and $\nu' = \frac{1}{\sqrt{2}} \zeta(\pi/2)$, it follows that

$$\hat{x}(\pi/2)\langle \Xi_0 | = \sqrt{2} \left( \nu'_0 \hat{x}_0 + \sum_{n>0} \nu'_n \sqrt{n} \hat{x}_n \right)\langle \Xi_0 | = 0.$$ 

(4.27)

Similar to the geometric condition on the zero-slope limit of the sliver, this condition states that the strings comprising the D-instanton sliver have a midpoint which is constrained to live at the origin of space-time. For a general Dp-brane sliver, this condition holds in the transverse dimensions, so that the string midpoints are constrained to live on the Dp-brane hypersurface.

Now let us consider the D25-brane sliver. We have in this case

$$\sum_n \nu_n a_n \langle \Xi | = -\sum_{n,m} \nu_n S_{nm} a_m^\dagger \langle \Xi | = -\sum_n \nu_n a_n^\dagger \langle \Xi |.$$ 

(4.28)

This leads to the condition

$$0 = \frac{i}{\sqrt{2}} \sum_n \nu_n (a_n + a_n^\dagger) \langle \Xi |$$

$$= \sum_{k=0}^{\infty} \left( \frac{(-1)^k}{2k+1} \right) \frac{\partial}{\partial x_{2k+1}} \langle x(\sigma) | \Xi |.$$ 

(4.29)

where the derivatives are interpreted as acting on the state $\langle x(\sigma) | \Xi |$ represented as a function of the string modes $\{x_n\}$. Just as the condition $0 = p_0 | p = 0 \rangle$ indicates that the string functional associated with the zero momentum ground state is flat under translations by a constant function $x(\sigma) = \lambda$, the condition (4.29) indicates that the D25-brane sliver state is described by a string functional invariant under translations by a function of the form

$$\delta x(\sigma) = \lambda \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cos(2k+1) \sigma = \begin{cases} +\lambda \pi/2, & 0 < \sigma < \pi/2 \\ -\lambda \pi/2, & \pi/2 < \sigma < \pi \end{cases}$$

(4.30)

This indicates that the strings comprising the D25-brane sliver can “split” into separate right and left half-strings which have different boundary conditions at the point $\sigma = \pi/2$.

We have found in this section that the Dp-brane sliver states are comprised of strings whose midpoints are bound to the $(p+1)$-dimensional hypersurface associated with the
relevant Dp-brane, but that these strings also seem to split naturally into two independent parts whose endpoints originally associated with the string midpoint can live at arbitrary points on the relevant Dp-brane world-volume. It is not clear what the physical interpretation of this geometrical condition should be, but it seems likely that this picture will play a part in clarifying the role of Dp-brane matter sliver projection operators in Witten’s string field theory and the simplified RSZ vacuum string field theory proposal. From the splitting of the strings involved in the D25-brane sliver state, it is tempting to postulate that this state may be closely related to a state in string field theory with two D25-branes rather than the original one of the perturbative $U(1)$ vacuum of Witten’s cubic string field theory. Such a two-brane state should exist in the $U(1)$ cubic string field theory, although as yet there is no convincing evidence for this state, either numerical or analytic. Note that it is the fluctuations around a two-brane state, rather than the two-brane state itself, in which the strings should naturally split. Nonetheless, this geometric picture of the sliver as a condensate of strings connected to a pair of Dp-branes is very suggestive. In the context of the superstring theory, we might expect a similar picture giving a brane-antibrane pair.

The geometrical picture we have found of the sliver singularities ties in neatly with previous calculations which have shown that the sliver states factorize into a product of Gaussians of the left and right string oscillators $r_{2k+1}, l_{2k+1}$ of the form

$$\exp\left(-\frac{1}{2} l \cdot M \cdot l - \frac{1}{2} r \cdot M \cdot r\right).$$

(4.31)

In the case of the D-instanton sliver, this factorization is natural since each half of the string is associated with a state in an ND string Hilbert space. For higher-dimensional branes, each half of the string is described by a state with NN boundary conditions in the longitudinal directions. In both cases the two half strings are completely decoupled, so it is natural for the full-string state to take a product form

### 4.3 Some mathematical embarrassments

As we have mentioned, the “eigenvectors” discussed above are not normalizable vectors in the Hilbert space $\ell_2$ of square-summable sequences. Nevertheless, as we have shown, they have very natural geometrical interpretations. What should we do about this? We do not really know the answer to this important question, but in this section we will take the opportunity to point out a few related awkward mathematical facts which further suggest that subtleties of functional analysis may be important in understanding this picture.

First, the string field product does not take Fock space states to well-defined states in the string field theory Hilbert space. Indeed, even the state $|0\rangle \ast |0\rangle$ does not actually exist as a rigorously well-defined state in the string field theory Fock space. The reason for this

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3As this paper was being completed, an interesting paper appeared by Hata and Moriyama, giving numerical evidence that indeed the sliver state should be associated with the two D25-brane states based on the energy of the solution of VSFT proposed in [12].
is that in rigorous treatments of Bogoliubov transforms \cite{30, 32} it is shown that the state $\exp[\frac{i}{2}V_{nm}a_n^\dagger a_m^\dagger] |0\rangle$ only exists if the operator $V_{nm}$ is Hilbert-Schmidt. On the other hand, it is easy to establish the asymptotic formula\footnote{Related results appeared in \cite{31}.}

\begin{equation}
V_{n,m}^{1,1} \sim \frac{8}{\pi \sqrt{27}} (-1)^{\frac{n+m}{2}} \frac{(nm)^{1/6}}{n^{4/3} + (nm)^{2/3} + m^{4/3}} (1 + O(1/n, 1/m))
\end{equation}

for $n, m$ both even, while for $n, m$ both odd we have the same result with an overall $-1$ in front. It follows that $\sum_{nm} |V_{n,m}|^2 \sim \int_0^\infty \frac{dr}{\sqrt{r}} = \infty$ and hence it cannot be Hilbert-Schmidt.\footnote{Similar allegations can be made against boundary states in boundary conformal field theory. In this context the nonexistence of the state is not physically important because one always works in practice with the state $q^{L_0} |B\rangle$. One can attempt to use a similar cure in string field theory, but this alters the algebra structure in an important way, replacing an associative algebra by a homotopy associative $A_\infty$ algebra. See, for example \cite{33}.}

It is clearly of interest to know if $S_{nm}$ is Hilbert-Schmidt. We can argue that this is not the case by contradiction as follows. If $S$ were Hilbert-Schmidt then $T = CS$ would be too. Meanwhile, $T$ satisfies

\begin{equation}
Z = CV = T \frac{i}{\sqrt{3}} \left( \frac{1}{T - e^{-i\pi/3}} - \frac{1}{T - e^{i\pi/3}} \right)
\end{equation}

If $T$ were HS then since it is symmetric it would be self-adjoint, and in particular $\frac{1}{T - e^{\pm i\pi/3}}$ would be bounded. Since Hilbert-Schmidt operators form an ideal in the algebra of bounded operators, if $T$ were Hilbert-Schmidt then $Z = CV$ would be also, but we have just seen that this is not the case.

We should also mention that reasoning similar to what we have used above can lead to apparent mathematical absurdities. Let us consider the identity on matrices:

\begin{equation}
X_{eo} E^{-4}_e X_{eo} = E^{-4}_o
\end{equation}

This equation only involves convergent sums, yet it is rather alarming because $E^{-4}_e$ and $E^{-4}_o$ are both diagonal matrices with disjoint sets of eigenvalues, and yet the unitary matrix $X_{eo}$ has transformed one into the other! As with our null eigenvalue for $X_{eo}$ there is no real mathematical contradiction here. The unbounded symmetric operator $-\frac{d^2}{d\sigma^2}$ acting on smooth functions on $[0, \pi]$ has inequivalent, self-adjoint extensions; the spectral theorem only applies to self-adjoint operators.

Finally, one might hope that although expressions in the matter theory are ill-defined due to facts such as those we have just quoted, nevertheless, the full matter $\otimes$ ghost theory will be better behaved. While this might prove to be the case the following example should serve as a cautionary tale. Consider a state in $L^2(R^2) = L^2(R) \otimes L^2(R)$ given by $\psi_a(x, y) = e^{-\frac{1}{2}ax^2} e^{-\frac{1}{2}ay^2}$. For all $a$ this state has norm $\sqrt{\pi}$. Nevertheless, we do not get a Cauchy sequence of states as $a \to 0$ or $a \to \infty$. Hence the limiting state $a \to \infty$ does not exist as a state in the Hilbert space. (it is “trying” to approach $\delta(x) \cdot 0$.)
4.4 Relation to skyscraper sheaves?

A useful perspective on the condition (4.27) for the D-instanton sliver, and the limit (3.25) of this condition as $t \to \infty$ can be gained from the split string formalism [6, 10]. Considering the string field as an operator on the half-string Hilbert space, the space-time dependence of the string field can be separated out using the string midpoint, giving a description of the string field as an operator-valued function on space time

$$\Psi \to \tilde{\Psi}(\bar{x}), \quad (4.35)$$

where $\bar{x} = x(\pi/2)$ is the position of the string midpoint and the operator $\tilde{\Psi}(\bar{x})$ acts on the usual Hilbert space of states of an ND string for every value of $\bar{x}$. The operator resulting from the D-instanton sliver is in this picture an operator which projects onto the Hilbert space over the point $\bar{x} = 0$. Thus, this operator is essentially projecting a general section of an $\mathcal{H}_{\text{ND}}$-bundle over space-time onto a “skyscraper sheaf” with a single $\mathcal{H}_{\text{ND}}$ fiber at the point 0.

There have been several indications over the years that D-branes localized in spacetime should be formulated in terms of sheaves. (See, for example, [34] and references therein for one such discussion.) It would be interesting to use string field theory projectors in combination with the zero-slope limit to derive the connection between D-branes and coherent sheaves from a more fundamental starting point.

5. Closed strings

A crucial question on which the considerations of this paper seem to shed some light is the question of whether and how closed strings can arise in the language of open string field theory. We found that the D25-brane sliver satisfies a nontrivial condition which essentially splits the string into two parts by allowing an arbitrary jump discontinuity in the position of the string at the midpoint. Due to this condition, the D25-brane matter sliver state has an infinite norm, analogous to the infinite norm of a state with fixed momentum. By choosing an alternative algebraic condition, we can select states which geometrically behave as closed strings. Namely, we can consider a class of squeezed states

$$|\Gamma\rangle = e^{-\frac{i}{2}a_n^K a_m^\dagger}|0\rangle \quad (5.1)$$

built on the usual open string vacuum which satisfy the algebraic condition

$$[\hat{x}(0) - \hat{x}(\pi)]|\Gamma\rangle = 0. \quad (5.2)$$

This condition is satisfied when the vector $\tilde{\nu} = [\zeta(0) - \zeta(\pi)]/2^{3/2}$ (using the notation of (4.26)) is an eigenvector of $K$ with eigenvalue -1. This vector is given by $\tilde{\nu}_n = 1/\sqrt{n}$ when $n$.

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6The utility of this picture of string field theory was suggested to us by I.M. Singer.
$n$ is odd, and $\tilde{\nu}_m = 0$ when $m$ is even. This vector is the same as the vector $\nu$, except for the signs.

We believe it is possible to construct a large family of states satisfying (5.2). We further propose that these are precisely the states given by writing the closed string Fock space vacuum and excited states as functionals of the closed string modes, translating to open string modes, and rewriting the resulting states in terms of open string oscillators. We now present two pieces of evidence for this proposal.

For a closed string on the interval $0 \leq \sigma \leq \pi$, we can perform a mode decomposition analogous to (2.1) through

$$x(\sigma) = z_0 + \sqrt{2} \sum_{n=1}^{\infty} z_n \cos(2n\sigma) + \sqrt{2} \sum_{m=1}^{\infty} y_m \sin(2m\sigma).$$

(5.3)

Just as the matrices $X'$ relate the NN and ND bases of the Hilbert space of $L^2$ functions on the interval $[0, \pi/2]$, a closely related matrix relates the basis $\cos n\sigma$ of functions on the interval $[0, \pi]$ with NN boundary conditions and the basis $\cos 2n\sigma, \sin 2m\sigma$ on which one could impose periodic boundary conditions. The transformation between these bases is given by

$$z_n = x_{2n}, \quad n = 1, 2, \ldots$$

$$y_m = \sum_{k=0}^{\infty} Y_{2m,2k+1} x_{2k+1} \quad m = 1, 2, \ldots$$

(5.4)

$$x_{2k+1} = \sum_{m=1}^{\infty} Y_{2k+1,2m} y_m \quad k = 0, 1, \ldots$$

where

$$Y_{2m,2k+1} = Y_{2k+1,2m} = (-1)^{m+k+1} \frac{2m}{2k+1}.$$  

$$X_{2m,2k+1} = -\frac{8m}{\pi((2k+1)^2 - 4m^2)}. \quad (5.5)$$

is the transformation matrix between two bases on $\mathcal{H}^-$, as follows from

$$\sin(2m\sigma) = \sum_{k=0}^{\infty} Y_{2m,2k+1} \cos(2k+1)\sigma \quad 0 \leq \sigma < \pi$$

(5.6)

$$\cos(2k+1)\sigma = \sum_{m=1}^{\infty} Y_{2k+1,2m} \sin(2m\sigma) \quad 0 < \sigma < \pi$$

(5.7)

Here we have indicated the intervals of pointwise convergence. It follows from pointwise convergence of (5.0) at $\sigma = 0$ that $x_{2k+1} = 1$ is a zeromode of $Y$. Put differently, since the vector $w$ is annihilated by $X_{co}$, it follows immediately that the vector with $x_{2k+1} = \tilde{w}_{2k+1} = (2k+1)w_{2k+1} = 1$ is annihilated by $Y$. This vector is associated with the singular distribution $x(\sigma) = \sum_k \cos(2k+1)\sigma$, which behaves like the derivative of a delta function at $\sigma = 0$ when this point is identified with $\sigma = \pi$. 

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Using the transformation (5.4), we can attempt to write a closed string state in terms of the open string oscillators acting on the open string ground state. For example, consider the closed string vacuum $|0\rangle_c$ associated with the closed string tachyon. This state is described in closed string coordinates by the functional (neglecting normalization factors)

$$|0\rangle_c \sim \exp \left( - \sum_{m=1}^{\infty} m(y_m^2 + z_m^2) \right).$$

(5.8)

This expression might seem unfamiliar to some readers. The standard oscillators for the closed string (with $\alpha' = \frac{1}{2}$) have nonvanishing commutators $[\alpha_n, \alpha^+_m] = \delta_{n,m}$, and $[\tilde{\alpha}_n, \tilde{\alpha}^+_m] = \delta_{n,m}$, $n, m = 1, \ldots, \infty$. The oscillators $\alpha^+_n, \tilde{\alpha}^+_n$ create left and right-moving modes, respectively. (The more conventional normalization is $\alpha_n = \sqrt{n} \alpha_n, \tilde{\alpha}_n = \sqrt{n} \tilde{\alpha}_n$.) In standard closed string quantization the oscillators are expressed in terms of $R$-valued coordinates $\xi_n$ and momenta $\eta_n$. These are canonically conjugate: $[\eta_n, \xi_m] = -i \delta_{n,m}$. The oscillators are then $\alpha_n = \frac{1}{\sqrt{2n}}(\eta_n - i n \xi_n)$ and can be written as

$$\alpha_n = -i \frac{\partial}{\sqrt{2n}} (\partial \xi_n + n \xi_n) = \frac{1}{\sqrt{2n}}(n \partial \eta_n + \eta_n)$$

(5.9)

depending on the choice of polarization. Together with their right-moving analogues $\tilde{\xi}_n, \tilde{\eta}_n$ we have

$$X(\sigma) = \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} (\xi_n + \tilde{\xi}_n) \cos(2n \sigma) - \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} (\eta_n + \tilde{\eta}_n) \sin(2n \sigma)$$

(5.10)

while

$$P(\sigma) = \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} (\eta_n + \tilde{\eta}_n) \cos(2n \sigma) + \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} n(\xi_n - \tilde{\xi}_n) \sin(2n \sigma)$$

(5.11)

Here $0 \leq \sigma \leq \pi$, and we have omitted the zeromodes. We thus identify $z_n = \frac{1}{2}(\xi_n + \tilde{\xi}_n)$ while $y_n = \frac{1}{2}(\eta_n - \tilde{\eta}_n)/n$. Now, the standard closed string vacuum is usually written as

$$\prod_{n=1}^{\infty} (\frac{\pi}{n})^{1/2} \exp[-\frac{1}{2}n(\xi_n^2 + \tilde{\xi}_n^2)]$$

(5.12)

We may write $\xi_n^2 + \tilde{\xi}_n^2 = \frac{1}{2}(\xi_n + \tilde{\xi}_n)^2 + \frac{1}{2}(\xi_n - \tilde{\xi}_n)^2$ and do a Fourier transform on the variable $y_n$ conjugate to $\frac{1}{2}(\xi_n - \tilde{\xi}_n)$ to produce the expression $\exp[-n(y_n^2 + z_n^2)]$.

Translating the closed string ground state (5.8) into open string coordinates using (5.4), gives

$$\exp \left( - \sum_{m=1}^{\infty} m x_{2m}^2 - \sum_{k,l=0,m=1}^{\infty} x_{2k+1} Y_{2k+1,2m} m Y_{2m,2l+1} x_{2l+1} \right).$$

(5.13)

We would now like to rewrite this functional as a squeezed state proportional to

$$|0\rangle_c \sim \exp \left[ -\frac{1}{2} \sum_{n,m \geq 1} a^+_m K_{mn} a^+_n \right] |p = 0\rangle.$$

(5.14)
Unfortunately, the summation over \( m \) in the last term of (5.13) is divergent, so a direct calculation of the corresponding matrix \( K \) is difficult. Note, however, that the sum is convergent whenever \( \sum_k x_{2k+1} = 0 \) and \( \sum_k |x_{2k+1}| < \infty \), as long as we first carry out the summations over \( k, l \). This is because the terms of order \( 1/m \) in the matrix element \( Y_{2m,2k+1} \) are independent of \( k \), and therefore cancel when the \( x_{2k+1} \)'s sum to 0. The divergence of the sum over \( m \) when \( \sum_k x_{2k+1} \neq 0 \) is a manifestation of the fact that the functional (5.13) vanishes whenever \( x_n \) has a nonzero component in the direction \( \tilde{\nu} \), which is exactly what we expect for a squeezed state where \( K \) has a -1 eigenvalue associated with the vector \( \tilde{\nu} \).

While the preceding argument suggests that \( K \) indeed has the claimed eigenvalue, we will now carry out a second check on this claim by the following indirect procedure: We will assume that the closed string ground state can be written in the form (5.14). We then find the linear condition satisfied by the open string creation and annihilation operators \( a_n, a_n^\dagger \) acting on the closed string vacuum. This gives a condition on the matrix \( K \) which must be satisfied in order for (5.13) to represent the closed string vacuum. We then show that this condition is compatible in a highly nontrivial fashion with the condition that \( \tilde{\nu} \) is an eigenvector of \( K \) with eigenvalue -1.

In order to carry out this procedure we must first write the open string creation and annihilation operators in terms of the closed string coordinates \( z_n, y_n \) and their derivatives. We have

\[
a_{2k+1} = -i \frac{1}{\sqrt{2}} \left[ \sqrt{2k+1} Y_{2k+1,2m} y_m + \frac{1}{\sqrt{2k+1}} Y_{2m,2k+1} \frac{\partial}{\partial y_m} \right] \tag{5.15}
\]

\[
a_{2k+1}^\dagger = -i \frac{1}{\sqrt{2}} \left[ -\sqrt{2k+1} Y_{2k+1,2m} y_m + \frac{1}{\sqrt{2k+1}} Y_{2m,2k+1} \frac{\partial}{\partial y_m} \right]
\]

For the squeezed state form of the closed string vacuum (5.14) we have

\[
(a_{2k+1} + K_{2k+1,2l+1} a_{2l+1}^\dagger) |0\rangle_c = 0.
\]  

(5.16)

Acting on the closed string vacuum functional (5.8) using (5.15) we find that for the condition (5.16) to be satisfied on the closed string vacuum we must have, for all values of \( m \),

\[
\sum_l K_{2k+1,2l+1} \left[ \sqrt{2l+1} Y_{2l+1,2m} + Y_{2m,2l+1} \frac{2m}{\sqrt{2l+1}} \right] = \left[ \sqrt{2k+1} Y_{2k+1,2m} - Y_{2m,2k+1} \frac{2m}{\sqrt{2k+1}} \right].
\]  

(5.17)

Formally, this equation determines the matrix \( K \) in the odd sector to be

\[
K = (E^{-1}YE - EYE^{-1})(E^{-1}YE + EYE^{-1})^{-1}
\]  

(5.18)

while \( K \) vanishes in the even sector. We are primarily interested in verifying that the matrix \( K \) has a -1 eigenvalue, associated with the vector \( \tilde{\nu} \). We can check this condition by acting
on the left of (5.17) with the vector \( \tilde{\nu}_k \) and summing over \( k \). Since \( \tilde{w}_k = \sqrt{2k + 1} \tilde{\nu}_k \) is annihilated by \( Y_{2k+1,2m} \), the first term on the right hand side of (5.17) cancels and we are left with the condition

\[
\sum_{k,l} \tilde{\nu}_k K_{kl} \left[ \sqrt{2l+1} Y_{2l+1,2m} + Y_{2m,2l+1} \frac{2m}{\sqrt{2l+1}} \right] = -\sum_k \tilde{\nu}_k Y_{2m,2k+1} \frac{2m}{\sqrt{2k+1}}. \quad (5.19)
\]

If \( \tilde{\nu} \) is an eigenvector of \( K \) with eigenvalue \(-1\), we again see that the first term on the left hand side cancels, and we are left with a simple identity. This shows that the defining condition (5.17) on the open string representation of (the matter part of) the closed string vacuum is indeed compatible with the desired role of \( \tilde{\nu} \) as an eigenvector of \( K \) with eigenvalue \(-1\) in a highly nontrivial fashion. This gives a second piece of evidence that the representation of the closed string vacuum as a squeezed state (5.14) constructed from open string raising operators is sensible, and that this squeezed state obeys the relation

\[
[\hat{x}(0) - \hat{x}(\pi)] |0\rangle_c = 0. \quad (5.20)
\]

Once we have identified the closed string vacuum as a state of the form (5.14), it is straightforward to construct all the excited closed string states in a similar fashion by acting with closed string raising operators, reexpressed as linear combinations of open string raising and lowering operators through (5.4). The resulting states give a representation of the full (matter) closed string Hilbert space in terms of states satisfying (5.2) which are described by acting with open string raising operators on the open string vacuum. While these states are nonnormalizable states with respect to the open string Hilbert space, they are no worse behaved than the sliver states we have discussed in previous sections, the Dirac position and momentum basis states in finite-dimensional quantum mechanics, or the boundary states used frequently to discuss D-branes in closed string theory.

There has been much debate about whether closed strings can be seen in a natural fashion in the classically stable vacuum of Witten’s cubic string field theory which arises after the open string tachyon condenses. While it is known that closed string poles can be seen in the one-loop two point function of the open string field theory \([35]\), it would be more satisfying if these closed string states could be explicitly constructed as asymptotic states in the open string field theory. Some approaches to describing closed string states in terms of open string field theory were developed in \([36, 37, 38, 39, 40, 41, 42]\). The discussion we have just given indicates the form that these closed string states should take. While these states are presumably nonnormalizable poles of the one-loop two-point function, a sequence of finite matrix approximations to the full two-point function given by successive level truncations should give a sequence of approximate poles approaching the desired closed string state. (This situation is analogous to considering a sequence of matrix approximations to the operator \( x \) acting on \( L^2(R) \) using a finite set of harmonic oscillator eigenstates–while the Dirac delta function \( \delta(x) \) is not a normalizable state in the Hilbert space, the sequence of lowest eigenvalues of this sequence of matrices approaches 0, and the associated eigenvectors
give successive approximations to the squeezed state representation of the delta function, \( e^{-a^{\dagger}a^{1/2}}|0\rangle \). It would be very desirable, although technically challenging, to check this level-truncated string field theory calculation explicitly.

If it is indeed possible to sensibly define all closed string states in the language of open string oscillators, it will be strong evidence that the open string field theory in the tachyonic vacuum in fact contains all information about closed string diagrams needed to construct a consistent closed string field theory. Indeed, it was shown by Giddings, Martinec, and Witten [43] that the set of open string diagrams produced by the cubic open string field theory precisely covers the moduli space of Riemann surfaces of arbitrary genus with a nonzero number of boundary components. By expressing the closed string states in the open string language, we are essentially contracting these boundary components to pointlike punctures, so that the moduli space of all closed string diagrams is naturally covered. Issues which may be related to this picture are discussed in [18].

Finally, the present discussion has some similarity with the theory of boundary states in conformal field theory. Here one can view open string oscillators as the quantization of the constrained subspace of the closed string phase space defined by the second class constraints \( (\alpha_n - \tilde{\alpha}_n) = 0, \forall n \in \mathbb{Z}, \) (NN conditions) or \( (\alpha_n + \tilde{\alpha}_n) = 0, \forall n \in \mathbb{Z}, \) (DD conditions). The boundary state \( |B\rangle \rangle = \exp[\pm \sum_{n=1}^{\infty} \frac{1}{n} \alpha_n \tilde{\alpha}_n]|0\rangle_{closed} \) is the squeezed state satisfying \( (\alpha_n \mp \tilde{\alpha}_n)|B\rangle \rangle = 0. \) In terms of the open string oscillators it is the singular squeezed state \( \exp[\pm \frac{1}{2} \sum_{n=1}^{\infty} \alpha_n a^{\dagger}_n]|0\rangle_{open} \) and the analog of the K-matrix has eigenvalue \( \pm 1. \) It might be interesting to pursue further the analogies with boundary states.

6. Discussion

It should be clear to the reader that the considerations of this paper can and should be generalized in several ways. First, we expect that it is a fairly straightforward exercise to include the \( B \) field in the above analysis, so we can take the Seiberg-Witten limit to recover the relation between D-branes and noncommutative geometry [14, 43, 44]. This should make contact with some of the considerations of [17]. To be slightly more explicit, it should be possible, starting with the 3-string vertex in a general background metric and \( B \) field as calculated in [28, 29] to calculate the general analogue of the sliver state by taking the limit of an infinite product of ground states \( |0\rangle \). It should then be possible to take the \( t \to \infty \) limit in which the star product factorizes into a space-time Moyal product algebra and the usual \( p = 0 \) star product algebra, as in [24]. In this limit the sliver should become the solitonic projection operator of [18]. This can be seen directly by noting that the noncommutative soliton projector arises as the limit of an infinite Moyal product of a space-time Gaussian with itself. If we then take the \( B \) field to vanish we return to the singular projection operator found in Section 3. It is interesting to note that in the \( t \to \infty \) limit the singular structure of the projection operator should be smooth out by the presence of the \( B \) field. Since the singularity arises from the -1 eigenvalue of \( S' \) for any \( t \), this singularity must also be smoothed
by the introduction of a $B$ field at arbitrary $t$, suggesting that there is a nontrivial nonsingular analogue of the D-instanton projection operator in nonzero $B$ field. Note, however, that the string splitting behavior we have found for the sliver in the longitudinal directions (all directions for the D25-brane) will not be affected by the introduction of a $B$ field as it arises purely from the odd sector of $V$, which is not affected by a $B$ field.

Another interesting question arising from this work is how one describes the zero-slope limit of a configuration of multiple D-instanton slivers. The most natural way to make multiple D-branes is to consider two branes at relative separation $y$ and let these approach one another. The translated sliver is $\exp[y\partial_{x_0}]|\Xi_0\rangle$. The expectation value

$$\langle \Xi_0 | \exp[y\partial_{x_0}]|\Xi_0\rangle = \exp[\frac{1}{2}y^2 - \langle 0 | \frac{1}{1 + S'} | 0 \rangle y^2]$$

(6.1)

is ill-defined because $S'$ has an eigenvalue $-1$. One should probably not conclude from this that the sliver and its translate are orthogonal. Rather one should probably introduce a regularization. We believe (but have not proved) that if we regularize $S'_{nm} \to q^{n+m}S'_{nm}$ then the zero eigenvalue of $1 + S'$ is lifted to $\epsilon'(\epsilon, q)$. We believe that $\epsilon'(\epsilon, q)$ goes to 0 if $q \to 1$, for any $\epsilon$. Moreover, for $\epsilon \to 0$ the null eigenvector of $1 + S'$ becomes parallel with $| 0 \rangle$. Therefore, in the limit $y \to 0, \epsilon \to 0, q \to 1$ (6.1) becomes $\exp[-y^2/\epsilon'(\epsilon, q)]$ and hence we can take a scaling limit such that two infinitesimally separated slivers are not orthogonal. The situation is now completely parallel to that of multiple noncommutative solitons [44, 45, 46], and based on our experience with these we should be able to recover higher rank projectors. But many details of this proposal remain to be filled in.

One of the main applications of sliver states in previous work has been the construction of D$p$-brane-like solutions in the RSZ vacuum string field theory model [4, 5, 6]. The results described here on the singular structure of sliver states may help to understand some aspects of how the RSZ model fits into the framework given by Witten’s original cubic string field theory. All of the sliver states we have discussed here have norms which are formally either vanishing or infinite. The problems with the norms of these states arise from the $\pm 1$ eigenvalue of the matrices $S, S'$ controlling the squeezed state representations of the sliver. Numerical evidence on the stable vacuum of Witten’s string field theory [49] suggests that this state is better behaved than the sliver states discussed in this paper. Defining a norm on the ghost sector using the operator $c_0$, we find that the contribution at level $L$ to the norm of the stable vacuum state decreases faster than $1/L$, even when the absolute value of each contributing term is added. This suggests, but does not prove, that the stable vacuum state of Witten’s cubic string field theory in Feynman-Siegel gauge is normalizable with respect to an appropriate inner product. The fact that the sliver states are not normalizable states in the Hilbert space suggests that the RSZ model is a somewhat singular limit of Witten’s cubic string field theory around the stable vacuum (assuming that the sliver states are relevant for describing D$p$-branes in the RSZ vacuum string field theory). Indeed, the results of [10] on the ghost sector of the RSZ model, which show that the action vanishes...
for any solution lying in the Hilbert space, give another indication of this singular nature of the RSZ model. It is not surprising, since the RSZ model completely decouples the matter and ghost sectors of the theory, that the separate anomalies which arise in the two sectors cause this model to be singular in some features. The interesting question is whether these singularities can be dealt with in a useful way to achieve new insight into the theory. It may be that the role of the ghosts and the more complicated BRST operator of the Witten theory is precisely to regulate these singularities in a consistent and well-defined way so that the D-brane states, which naively seem singular from the point of view of the matter theory, become well-behaved states in the open string Hilbert space.

A fundamental question about open string field theory, which has troubled workers in this area since the early days of the subject, is precisely what set of states should be allowed in the string field theory star algebra. As discussed in section 4.3, the matter star product is not closed on the open string matter Hilbert space. In this paper we have described a number of interesting states which have a similar behavior to Dirac’s position and momentum basis states for quantum mechanics. These states do not lie in the Hilbert space, but are “very close” to lying in the Hilbert space, in the sense that they can be described as suitable limits of well-behaved states. We have shown that this category of states which are “almost” in the open string Hilbert space includes the matter sliver states corresponding to $Dp$-branes of all dimensions, as well as the spectrum of closed string states described in terms of open string oscillators. One conservative approach would be to mandate that no states outside the Hilbert space are allowed at all. In this case we must either reject the sliver and closed string states from the theory, or hope that the inclusion of ghosts in Witten’s theory serves to regulate the singularities associated with these states. Another possibility, however, is that certain states outside the conventional open string Hilbert space must be included in the star algebra. Indeed, several arguments suggest that such states must be included for the theory to be consistent and to contain all the desired physics. The first of these arguments arises simply from the natural suggestion that the open string star algebra be closed, and the observation that this algebra does not close on the Hilbert space. Another argument arises from considering string states on a $Dp$-brane which has been translated in a transverse direction. These string states satisfy Dirichlet boundary conditions with different values than those associated with the original $Dp$-brane, and thus lie outside the Hilbert space of DD strings on the original D-brane. We would clearly like to be able to describe translation of D-branes in open string field theory. If this physical requirement on the theory is to be met it, is therefore necessary to include states in the star algebra which are not in the original open string Hilbert space of the conformal field theory used to define the Witten cubic SFT. Some progress in this direction was made in [50, 52], where an attempt was made to explicitly construct (the T-dual of) a translated $Dp$-brane in a level truncation of Witten’s cubic SFT in Feynman-Siegel gauge. It was found that the solution associated with the translated $Dp$-brane encounters a singularity at a distance on the order of the string length. This is where we would expect a state to leave the normalizable string Hilbert
space, so this result is perhaps not surprising. These arguments, however, suggest that it is necessary to include certain states somewhat outside the original normalizable open string Hilbert space to make Witten’s cubic string field theory consistent. If such states are added, it seems clear from the argument of the previous section that we should naturally expect closed string states to be incorporated into the theory. It may be that the most natural way to understand the appearance of all these singular states is by using the level-truncated theory. The regulator imposed by level truncation has the effect of naturally rendering finite the divergences associated with states like the sliver state and closed string states, so that we would hope to see some indication of these states in a finite approximation to the full open string field theory.

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