Construction of one-loop $\mathcal{N} = 4$ SYM effective action on the mixed branch in the harmonic superspace approach

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Abstract

We develop the generic approach to the construction of the one-loop $\mathcal{N} = 4$ SYM effective action on the mixed branch, depending both on $\mathcal{N} = 2$ vector multiplet and on hypermultiplet background fields. Beginning with the formulation of $\mathcal{N} = 4$ SYM theory in terms of $\mathcal{N} = 2$ harmonic superfields, we construct the one-loop effective action using the covariant $\mathcal{N} = 2$ harmonic supergraphs and calculate this effective action in $\mathcal{N} = 2$ harmonic superfield form for constant Abelian strength $F_{mn}$ and corresponding constant hypermultiplet fields. The hypermultiplet-dependent effective action is derived and given by integral over analytic subspace of harmonic superspace. We show that each term in Schwinger-De Witt expansion of the low-energy effective action is written as integral over full $\mathcal{N} = 2$ superspace.
1 Introduction

$\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory attracts much attention due to its unique properties in the quantum domain such as finiteness and superconformal invariance and the remarkable links with string/brane theory (see e.g. the [1], [2] for review). Discovery of AdS/CFT correspondence arises a new significant interest to study the various aspects of $\mathcal{N} = 4$ SYM theory. AdS/CFT correspondence [3] states a duality between IIB superstring theory compactified on $AdS_5 \times S^5$ and four-dimensional $\mathcal{N} = 4$ SYM theory in 't Hooft limit. It is turned out, the low-energy properties of bulk string theory are related with $\mathcal{N} = 4$ supersymmetric gauge quantum field theory. To be more precise, in the limit under consideration, the bulk theory reduces to the higher dimensional classical supergravity encoded by correlation functions of gauge invariant composite operators in $D = 4, \mathcal{N} = 4$ SYM theory. Another implementation of $\mathcal{N} = 4$ SYM model to string theory related to conjecture that $D3$-brane interactions in static limit are completely described in terms of low-energy $D = 4, \mathcal{N} = 4$ SYM effective action on Coulomb branch [4], [5], [6]. All this allows to treat $\mathcal{N} = 4$ SYM quantum field model as an element of superstring theory.

Formulation of $\mathcal{N} = 4$ SYM theory possessing manifest off-shell $\mathcal{N} = 4$ supersymmetry is unknown so far. Superfield description of the $D = 4, \mathcal{N} = 4$ SYM theory is realized in terms of scalar superfield $W_{AB} = -W_{BA}, A, B = 1, ..., 4$ which transforms under the six-dimensional representation of the $SU(4)$ internal symmetry group and which is real: $\bar{W}^{AB} = \frac{1}{2} \epsilon^{ABCD} W_{CD}$. This superfield obeys the constrains

$$\mathcal{D}_{A} W_{BC} = \mathcal{D}_{[A} W_{BC]}, \quad \bar{\mathcal{D}}^{A}_{\dot{A}} W_{BC} = -\frac{2}{3} \delta^{[A}_{[\dot{B}} \bar{\mathcal{D}}^{E]} W_{C]}_{E}. \tag{1}$$

The constraints imply that the component contents of the superfield $W_{AB}$ corresponds to on-shell vector multiplet and includes 6 real scalar fields, 4 Majorana spinor fields and 1 vector field$^1$.

From the $\mathcal{N} = 2$ supersymmetric point of view, the $\mathcal{N} = 4$ vector multiplet consists of the $\mathcal{N} = 2$ vector multiplet and hypermultiplet [8]. Therefore the $\mathcal{N} = 4$ SYM action can be treated as some special $\mathcal{N} = 2$ supersymmetric theory, with action containing the action of $\mathcal{N} = 2$ SYM theory plus action of hypermultiplet in adjoint representation coupled to $\mathcal{N} = 2$ vector multiplet. In addition, this model possesses the hidden $\mathcal{N} = 2$ symmetry and as a result it actually is $\mathcal{N} = 4$ supersymmetric. Such a theory is naturally formulated in $\mathcal{N} = 2$ harmonic superspace [12], [8]. This formulation simplifies a quantum consideration due to manifest $\mathcal{N} = 2$ supersymmetry. Analysis of effective action is more simplified with help of harmonic superfield background method [14], [15].$^2$

At present, it is well known that the exact low-energy quantum dynamics of $\mathcal{N} = 4$ SYM theory in $\mathcal{N} = 2$ vector multiplet sector for $SU(N)$ gauge group spontaneously broken down to its maximal torus $U(1)^{N-1}$ is described by the non-holomorphic effective potential $\mathcal{H}(\mathcal{W}, \bar{\mathcal{W}})$, depending on $\mathcal{N} = 2$ strengths $\mathcal{W}, \bar{\mathcal{W}}$ [17], [18], [15]. We emphasize that the

$^1$The same multiplet can be described off shell in $\mathcal{N} = 3$ harmonic superspace [7], [8]. Quantum aspects of $\mathcal{N} = 3$ SYM theory have been discussed in [9]. Structure of effective action of such a theory was studied in [11], [10].

$^2$Straightforward calculation of higher-loop contributions to effective action in closed form is a very complicated technical problem. Therefore, a study of general structure of possible higher order corrections to the effective action would be very useful (see e.g. [16]).
structure of $\mathcal{H}(\mathcal{W}, \bar{\mathcal{W}})$ is so unique that it can be obtained entirely on the symmetry grounds of scale independence and R-invariance up to a numerical factor \cite{17, 19}. Moreover, the potential $\mathcal{H}(\mathcal{W}, \bar{\mathcal{W}})$ gets neither perturbative quantum corrections beyond one-loop nor instanton corrections \cite{17}, \cite{19}, (see also discussion of non-holomorphic potential in $\mathcal{N} = 2$ SYM theories \cite{19}, \cite{20} and structure of next-to-leading two-loop contributions to effective action \cite{21}, \cite{6}, \cite{22}). All these properties are very important for understanding of the low-energy quantum dynamics in $\mathcal{N} = 4$ SYM theory and its applications. In particular, the effective potential $\mathcal{H}(\mathcal{W}, \bar{\mathcal{W}})$ describes the leading terms in the interaction between parallel D3-branes in the superstring theory.

To clarify the restrictions on an effective action, stipulated by $\mathcal{N} = 4$ supersymmetry, to describe complete structure of effective action and to gain a deeper understanding of the $\mathcal{N} = 4$ SYM/supergravity correspondence, we must find an effective action not only in $\mathcal{N} = 2$ vector multiplet sector but its dependence on all the fields of $\mathcal{N} = 4$ vector multiplet. The problem of the $\mathcal{N} = 4$ SYM effective action on so called mixed branch remained unstudied for a long time. Recently, the complete exact low-energy effective potential $\mathcal{L}_q(X) (X = -q^a q^a_{W\bar{W}})$ containing the dependence both on $\mathcal{N} = 2$ gauge superfields and hypermultiplets has been discovered \cite{23}. It has been shown that the algebraic restrictions imposed by the hidden $\mathcal{N} = 2$ supersymmetry on a structure of the low-energy effective action in $\mathcal{N} = 2$ harmonic superspace approach turn out to be so strong that they allow to restore the dependence of the low-energy effective action on the hypermultiplets on basis of the known non-holomorphic effective potential $\mathcal{H}(\mathcal{W}, \bar{\mathcal{W}})$. As a result, the additional hypermultiplet-dependent contribution to low-energy effective action containing the on-shell $\mathcal{W}, \bar{\mathcal{W}}$ and the hypermultiplet $q^a$ superfields has been obtained. The leading low-energy effective Lagrangian $\mathcal{L}_q(X)$ was found in Ref. \cite{23} on a purely algebraic ground. Then, it was shown in paper \cite{24}, that the effective Lagrangian $\mathcal{L}_q(X)$ can be derived using the harmonic supergraph techniques and harmonic superspace background field method. Some later, the structure of the one-loop effective action beyond leading approximation has been found in \cite{26} on the base of formulation of $\mathcal{N} = 4$ SYM theory in terms of $\mathcal{N} = 1$ superfields and exploring the derivative expansion techniques in $\mathcal{N} = 1$ superspace \cite{27}. Although such a formulation preserves lesser number of manifest supersymmetries then in the harmonic superspace approach, the supersymmetric $R_\xi$-gauges and a special prescription on restoration of the $\mathcal{N} = 2$ supersymmetric form make possible to construct the effective action including dependence on arbitrary powers of Abelian strength $F_{mn}$ and special $R$-symmetry invariant combination of the constant component fields $\phi, f^a$ of hypermultiplet.

The present paper is devoted to analysis of hypermultiplet dependence of $\mathcal{N} = 4$ SYM low-energy effective action in $\mathcal{N} = 2$ harmonic superspace. In compare with paper \cite{24}, we consider the effective action beyond leading low-energy approximation and take into account all powers of Abelian strength $F_{mn}$. In compare with paper \cite{26}, we work completely in terms of $\mathcal{N} = 2$ harmonic superspace on all steps of consideration and justify the special heuristic prescription concerning a restoration of manifest $\mathcal{N} = 2$ supersymmetric form of effective action, which has been used in \cite{26}. As a result we obtain a proper-time representation of low-energy effective action written as integral over analytic subspace of harmonic superspace.

The work is organized as follows. In Section 2 we discuss the construction of $\mathcal{N} = 4$ SYM model in $\mathcal{N} = 2$ harmonic superspace and structure of perturbation theory in such a model. Section 3 is devoted to generic procedure of finding the one-loop effective action.
in hypermultiplet sector. In Section 4 we show how one can sum up an infinite sequence of covariant harmonic supergraphs with arbitrary number of hypermultiplet legs on non-trivial $\mathcal{N} = 2$ vector multiplet background to get a form of effective action which can be studied on the base of proper-time method. Section 5 is devoted to calculation of one-loop effective action with help of symbol operator techniques we develop in $\mathcal{N} = 2$ harmonic superspace. In Section 6, we derive the final result for one-loop effective action in form integral over analytic subspace of harmonic superspace. We also show that this result leads to spinor covariant derivative expansion of the effective action and find two first terms of this expansion in explicit form. We demonstrate that each term of the expansion can be rewritten as integral over full $\mathcal{N} = 2$ superspace. Then we summarize the main results.

\section{$\mathcal{N} = 4$ SYM theory in $\mathcal{N} = 2$ harmonic superspace}

The harmonic superspace is an universal construction for formulating the arbitrary $\mathcal{N} = 2$ supersymmetric theories in a manifestly $\mathcal{N} = 2$ supersymmetric way. The $\mathcal{N} = 2$ harmonic superspace was introduced in Ref. \cite{12} to construct the off-shell $\mathcal{N} = 2$ models in terms of unconstrained superfields. This approach extends the conventional $\mathcal{N} = 2$ superspace $z = (x^m, \theta^i, \dot{\theta}^\alpha)$ by the two-sphere $S^2 = SU(2)/U(1)$ parameterized by harmonics $u^{+i}$ and their conjugate $u^{-i}$. Throughout this paper we follow the conventions of the book \cite{8}. The harmonic superspace $z, u$ contains the analytic subspace parameterized by the variables $\zeta = (x^m, \theta^\alpha, \dot{\theta}^\alpha, u^+_{\alpha}, u^-_{\alpha})$, where the analytic basis is defined by

$$x^m_A = x^m - 2i \theta^{(i} \sigma^m \bar{\theta}^{j)} u^+_i u^-_j, \theta^\pm_{\alpha} = u^{\pm}_{\alpha}, \bar{\theta}^\pm_{\alpha} = u^{\pm}_{\bar{\alpha}}.$$  

The spinor covariant derivatives in the central and the analytic bases (see discussion of these bases in \cite{8}) are related by $D^\pm_\alpha = u^{\pm}_{\alpha} D^i_\alpha, \bar{D}^\pm_\alpha = u^{\pm}_{\bar{\alpha}} \bar{D}^i_\alpha$. A very important feature is that the operators $D^+_\alpha, \bar{D}^+_\alpha$ strictly anticommuting. A covariantly analytic superfield $\Phi^{(p)}(z, u)$\footnote{Here the superscript $p$ refers to the harmonic $U(1)$ charge, $D^0 \Phi^{(p)} = p \Phi^{(p)}$.} is defined to be annihilated by these operators, $D^+_\alpha \Phi^{(p)} = \bar{D}^+_\alpha \Phi^{(p)} = 0$.

The two basic $\mathcal{N} = 2$ ingredients of the $\mathcal{N} = 4$ SYM theory are a hypermultiplet and a $\mathcal{N} = 2$ vector multiplet. In harmonic superspace the hypermultiplet is described by an analytic superfield of $U(1)$ charge equal to $+1$,

$$D^+_\alpha q^+ = \bar{D}^+_\alpha q^+ = 0 \Rightarrow q^+ = q^+(x_A, \theta^+, \bar{\theta}^+, u).$$

In the on-shell case the hypermultiplet satisfies the condition of harmonic analyticity $D^{++} q^+ = 0$ and forms so called an ultrashort superfield. Here

$$D^{++} = u^{+i} \frac{\partial}{\partial u^{-i}} - 2i (\theta^+ \sigma^m \bar{\theta}^+) \frac{\partial}{\partial x^m_A},$$

is the covariant harmonic derivative in the analytic basis, which also plays the role of raising operator in the $su(2)$ algebra: $[D^{++}, D^{--}] = D^0$, $[D^0, D^{\pm \pm}] = \pm 2D^{\pm \pm}$, $[D^{++}, D^-] = D^+$. The classical action of the hypermultiplet is:

$$S_{hyper} = - \int d\zeta^{(-4)} du^+_i D^{++} q^+, \quad (1)$$
where the integration is carried out over the analytic subspace. The rules of harmonic
integration are given in [8].

The \( \mathcal{N} = 2 \) vector multiplet is described by real analytic superfield
\( V^{++}(x, \theta^+, \bar{\theta}^+, u) \) with \( U(1) \) charge equal to +2. It plays the role of the gauge connection in the covariantized
harmonic derivative
\[
\mathcal{D}^{++} = D^{++} + igV^{++},
\]
so that the “flat” commutation relations with \( D^{+(a,\bar{a})} \) are preserved: \([\mathcal{D}^{++}, D^{+(a,\bar{a})}] = 0\). In the
Wess-Zumino gauge the component contents of \( V^{++} \) is reduced to the off-shell \( \mathcal{N} = 2 \) vector multiplet. Namely this superfield
\( V^{++} \) is an unconstrained prepotential for \( \mathcal{N} = 2 \) SYM theory and all other objects, for example the superfield strength \( \mathcal{W} \), are expressed in its terms [12, 8]. The superfield strength is expressed through non analytic superfield
\( V^{--} \) satisfying the equation
\( D^{++}V^{--} - D^{--}V^{++} + i[V^{++}, V^{--}] = 0 \). This equation has the
solution in form of the power series
\[
V^{--} = \sum_{n=1}^{\infty} \int d^{2n}zu_{1}...du_{n}(-i)^{n+1}\frac{V^{++}(z,u_{1})...V^{++}(z,u_{n})}{(u^{+}u_{1}^{+})(u_{1}^{+}u_{2}^{+})...(u_{n}^{+}u^{+})},
\]
which is a non-local functional of \( V^{++} \) in harmonic sector. The conditions of \( u \) independence of \( \mathcal{W} \), \( \mathcal{D}^{\pm\pm}\mathcal{W} = 0 \) lead to
\[
\mathcal{W} = -\frac{1}{4}(\mathcal{D}^{+})^{2}\mathcal{W}^{--}, \bar{\mathcal{W}} = -\frac{1}{4}(\mathcal{D}^{+})^{2}\mathcal{W}^{--}.
\]
The remaining properties of \( \mathcal{W}, \bar{\mathcal{W}} \) are covariant chirality (antichirality): \( \mathcal{D}^{a}_{\alpha}\bar{\mathcal{W}} = \mathcal{D}^{\bar{a}}_{\bar{\alpha}}\mathcal{W} = 0 \)
and the Bianchi identity: \( \mathcal{D}^{\alpha}\mathcal{D}^{\beta}\mathcal{W} = \mathcal{D}^{\alpha}\mathcal{D}^{\beta}\mathcal{W} \). For further use ones write down also the algebra of covariant derivatives:
\[
\{\mathcal{D}^{a}_{\alpha}, \mathcal{D}^{b}_{\beta}\} = \{\mathcal{D}^{a}_{\alpha}, \mathcal{D}^{b}_{\beta}\} = \{\mathcal{D}^{a}_{\alpha}, \mathcal{D}^{b}_{\beta}\} = 0, \tag{4}
\]
\[
\{\mathcal{D}^{a}_{\alpha}, \mathcal{D}^{b}_{\beta}\} = -2i\varepsilon_{\alpha\beta}\bar{\mathcal{W}}, \quad \{\mathcal{D}^{a}_{\alpha}, \mathcal{D}^{b}_{\beta}\} = 2i\varepsilon_{\alpha\beta}\mathcal{W},
\]
\[
\{\mathcal{D}^{a}_{\alpha}, \mathcal{D}^{b}_{\beta}\} = -\{\mathcal{D}^{a}_{\alpha}, \mathcal{D}^{b}_{\beta}\} = 2i\mathcal{D}_{\alpha\beta},
\]
\[
[\mathcal{D}^{a}_{\alpha}, \mathcal{D}^{b}_{\beta}] = \mathcal{D}^{a}_{\alpha}, \quad [\mathcal{D}^{a}_{\alpha}, \mathcal{D}^{b}_{\beta}] = \mathcal{D}^{a}_{\alpha},
\]
\[
[\mathcal{D}^{a}_{\alpha}, \mathcal{D}^{b}_{\beta}] = \varepsilon_{\alpha\beta}\mathcal{D}^{+}_{\beta}\mathcal{W}, \quad [\mathcal{D}^{a}_{\alpha}, \mathcal{D}^{b}_{\beta}] = \varepsilon_{\alpha\beta}\mathcal{D}^{+}_{\beta}\mathcal{W}, \tag{5}
\]
\[
[\mathcal{D}^{a}_{\alpha\beta}, \mathcal{D}^{b}_{\beta\gamma}] = \frac{1}{2i}\{\varepsilon_{\alpha\beta\gamma}\mathcal{D}^{+}_{\beta}\mathcal{D}^{-}_{\gamma}\mathcal{W} + \varepsilon_{\gamma\beta\alpha}\mathcal{D}^{-}_{\alpha}\mathcal{D}^{+}_{\beta}\mathcal{W}\}.
\]

Description of the \( \mathcal{N} = 4 \) SYM theory in \( \mathcal{N} = 2 \) harmonic superspace includes hypermultiplte and vector multiplet superfields. The hypermultiplet superfield \( q^{+} \) belongs to the
adjoint representation of the gauge group and is minimally coupled to the vector multiplet. Action of \( \mathcal{N} = 4 \) SYM theory in the approach under consideration has the form:
\[
S = \frac{1}{2g^{2}}\operatorname{tr} \int d^{8}z\mathcal{W}^{2} - \frac{1}{2}\operatorname{tr} \int d\zeta(-4)q^{+a}(\mathcal{D}^{++} + iV^{++})q^{+}_{a}. \tag{6}
\]
Here \( a = 1, 2 \) is the index of the Pauli-Gürsey rigid \( SU(2) \) symmetry: \( q^{+}_{a} = (q^{+}, \bar{q}^{+}) \), \( q^{+a} = \varepsilon^{ab}q^{+}_{a} = (\bar{q}^{+}, -q^{+}) \), and
\( d^{8}z = d^{4}x \theta^{+} d^{2}\theta^{-} du \) is the chiral superspace integration measure
as well as \( d\zeta(-4) = d^3x d^2\theta^+ d^2\bar{\theta}^+ du \) is the analytic measure. The off-shell action (6) allows to develop the manifest \( \mathcal{N} = 2 \) supersymmetric quantization. Moreover, this action is invariant under hidden on-shell extra \( \mathcal{N} = 2 \) supersymmetry transformation [8] realized in terms of the vector and hypermultiplet superfields as follows:

\[
\delta V^{++} = (\varepsilon^a \theta^+_a + \bar{\varepsilon}^{\bar{a}} \bar{\theta}^{\bar{a}}) q^+_a, \quad \delta q^+ = -\frac{1}{2} (D^+)^4 [(\varepsilon^a \theta^+_a + \bar{\varepsilon}^{\bar{a}} \bar{\theta}^{\bar{a}}) V^{--}].
\]

As a result, the model is (on-shell) \( \mathcal{N} = 4 \) supersymmetric.

The on-shell structure of model is defined in terms of solutions to the corresponding equation of motion

\[
D^{++} q^+ + ig[V^{++}, q^+] = 0, \quad (D^+)^2 \mathcal{W} = [q^+, q^-].
\]

The simplest solution to these equations in Abelian case forms a set of constant background fields, which transform linearly through each other mixing up the \( \mathcal{W}, \bar{\mathcal{W}} \) with \( q^+ \) under hidden \( \mathcal{N} = 2 \) supersymmetry transformation [18], [15]:

\[
\delta \mathcal{W} = \frac{1}{2} \varepsilon^a D^- a^- q^+_a, \quad \delta \bar{\mathcal{W}} = \frac{1}{2} \bar{\varepsilon}^{\bar{a}} D^+ \bar{\mathcal{W}} = \frac{1}{4} (\varepsilon^a D^+_a \mathcal{W} + \bar{\varepsilon}^{\bar{a}} D^+ \bar{\mathcal{W}}) \mathcal{W}.
\]

Generic vacuum of any \( \mathcal{N} = 2 \) superconformal models, like for example \( \mathcal{N} = 4 \) SYM theory, includes only massless \( U(1) \) vector multiplets and massless neutral hypermultiplets, since charged hypermultiplets get masses by means of Higgs mechanism. The manifold of vacua is determined by the conditions of vanishing scalar potential (F-flatness plus D-flatness). The set of vacua with only massless neutral hypermultiplets forms the “Higgs branch” of the theory, the set with only \( U(1) \) vector multiplets forms the “Coulomb branch”, and the set of vacua with both kinds of the multiplets is forms the “mixed branch”. Thus, the low energy fields propagating on the mixed branch are massless neutral scalars, spinors and \( U(1) \) vectors which form the on shell superfields \( \mathcal{W}, \bar{\mathcal{W}}, q^+, q^- \) possessing the properties

\[
(D^+)^2 \mathcal{W} = (D^-)^2 \bar{\mathcal{W}} = 0,
\]

\[
D^{++} q^+ = D^{--} q^- = 0, \quad q^- = D^{-+} q^+, \quad D^- q^- = D^- q^- = 0.
\]

Further we will consider the low-energy effective action in \( \mathcal{N} = 4 \) SYM theory just on mixed branch.

The manifestly \( \mathcal{N} = 2 \) supersymmetric Feynman rules in harmonic superspace have been developed in [12] (see also [8], [13]). The calculations of the quantum corrections may contain the potentially dangerous harmonic singularities, that is the harmonic distributions at coinciding points. The problem of coinciding harmonic singularities in the framework of harmonic supergraph Feynman rules was first discussed in [28] where some solution to the problem was considered. The background field method for constructing the effective action in harmonic superspace has been developed in Refs. [14], [15] (see also [29] for construction of the background field method in standard \( N = 2 \) superspace). This method allows to find the effective action for arbitrary \( \mathcal{N} = 2 \) supersymmetric gauge model in a form preserving
the manifest $\mathcal{N} = 2$ supersymmetry and classical gauge invariance in quantum theory. In framework of background field method the fields $V^{++}, q^+$ are splitted into classical $V^{++}, q^+$ and quantum $v^{++}, Q^+$ fields with imposing the gauge conditions only on quantum fields. The Feynman rules are based on quantum action $S_{\text{quant}}$ of the form $S_{\text{quant}} = S_2 + S_{\text{int}}$ where action $S_2$ is quadratic in quantum fields and ghosts and $S_{\text{int}}$ describes interaction. Both action $S_2$ and action $S_{\text{int}}$ depend on background fields. All details are given in [13], [15].

Action $S_2$ defines the propagators depending on background fields. Further we use the background covariant gauge $\mathcal{D}^{++}v^{++} = 0$. In this case the propagator of quantum gauge superfield has the form

\[
G^{(2,2)}(1, 2) = \frac{1}{2\Box_1 \Box_2} (\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4\{\delta^{12}(z_1 - z_2)(\mathcal{D}_2^{---})^2\delta^{(-2,2)}(u_1, u_2)\}. \tag{10}
\]

We emphasize that this propagator is analytic superfield in each argument. The propagator of the Faddeev-Popov ghosts $b$ and $c$ is written as follows

\[
G^{(0,0)}(1, 2) = i < b(1)c^T(2) > = \frac{1}{\Box_1} (\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4\{\delta^{12}(z_1 - z_2)\frac{u_1^{-}u_2^{-}}{(u_1^+u_2^+)^3}\}. \tag{11}
\]

The $q^+$ hypermultiplet propagator associated with the action (6) for external $V^{++}$ has the form

\[
G^{(1,1)}(1, 2) = i < q^{+\alpha}(1)\bar{q}^{+\bar{\alpha}}(2) > = -\delta_\alpha^\beta \frac{1}{\Box_1} (\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4\{\delta^{12}(z_1 - z_2)\frac{1}{(u_1^+u_2^+)^3}\}. \tag{12}
\]

The propagators contain the operator $\widehat{\Box} = -\frac{1}{2}(\mathcal{D}^+)^4(\mathcal{D}^{---})^2$ which transforms each covariantly analytic superfield into a covariantly analytic one. On space of such superfields, the operator $\widehat{\Box}$ is equivalent to the second-order Laplace like differential operator

\[
\widehat{\Box} = \frac{1}{2}\mathcal{D}^{\alpha\beta}\mathcal{D}_{\alpha\beta} + \frac{i}{2}(\mathcal{D}^{+\alpha}\mathcal{W})\mathcal{D}_{\alpha} - \frac{i}{2}(\mathcal{D}_{\alpha}^+\mathcal{W})\mathcal{D}^{\alpha} - \frac{i}{4}(\mathcal{D}_{\alpha}^+\mathcal{D}^{\alpha}\mathcal{W})\mathcal{D}^{++} \tag{13}
\]

\[
+ \frac{i}{8}[\mathcal{D}^{++}, \mathcal{D}_{\alpha}^+]\mathcal{W} + \frac{1}{2}\{\mathcal{W}, \bar{\mathcal{W}}\},
\]

as a consequence of algebra of covariant derivatives [11]. It is remarkable that the differential part of $\widehat{\Box}$ is uniquely determined from the requirements that (i) $\widehat{\Box}$ a constructed in terms of the covariant derivatives only; (ii) $\widehat{\Box}$ moves every covariantly analytic superfield into a covariantly analytic one; (iii) this operator is said to be the analytic d’Alambertian. Among the important properties of $\widehat{\Box}$ is [14] : $[\mathcal{D}_{(\alpha, \bar{\alpha})}^+, \widehat{\Box}] = 0$. The $\mathcal{N} = 2$ propagators have a complicated structure due to nontrivial dependence on harmonics. Fortunately, as is shown recently [30] the harmonic dependence of the $\mathcal{N} = 2$ propagators simplifies drastically if the background vector multiplet satisfies the classical equations of motion $\mathcal{D}^{ij}\mathcal{W} = \mathcal{D}^{ij}\bar{\mathcal{W}} = 0$. In this case the harmonic dependence of the propagators is completely factorized what helps to keep the harmonic dependence of $\mathcal{N} = 2$ supergraphs under a control.

Evaluation of effective action within background field method is accompanied often by use of proper time or heat kernel techniques. This techniques allows efficiently to sum up
an infinite set of Feynman diagrams with increasing number of insertions of the background fields and to develop the background field derivative expansion of the effective action in manifestly gauge covariant way. The background field method and heat kernel techniques for \( \mathcal{N} = 1 \) SYM theories are well-developed (see [31], [32] for reviews). The background field method in harmonic superspace was elaborated in Refs. [14], some of its important applications were reviewed in [15]. However, until recently, the many aspects of heat kernel techniques in \( \mathcal{N} = 2 \) harmonic superspace and extend the one-loop results of [23], [24] for the leading low-energy quantum corrections to next-to leading contributions to the effective action depending both on \( \mathcal{N} = 2 \) vector multiplet and on \( \mathcal{N} = 2 \) hypermultiplet.

3 One-loop effective action in hypermultiplet sector

Our purpose is studying the hypermultiplet dependence of one-loop effective action. Before to begin with direct calculations, we discuss in this section a proper definition of such an effective action and an efficient way of the calculations.

We consider the \( \mathcal{N} = 4 \) SYM theory with gauge group \( SU(2) \) formulated in \( \mathcal{N} = 2 \) harmonic superspace. Action of the model has the form \( (6) \). On the mixed branch the gauge group \( SU(2) \) is broken down to \( U(1) \) so that background fields \( V^{++}, q^+ \) lie in the Cartan subalgebra of the gauge group.

We start, like in [24], with carrying out background-quantum splitting by the rule \( q^a \rightarrow q^a + Q^a, \quad V^{++} \rightarrow V^{++} + g v^{++} \), here \( q^a, V^{++} \) are the background and \( Q^a, v^{++} \) are the quantum ones. For one-loop calculations it is sufficient to consider only part of quantum action \( S_{\text{quant}} \) which is quadratic in quantum superfields:

\[
S_2 = -\frac{1}{2} \int d\zeta^{(-4)} [v^{++}\Box v^{++} + Q^{a\dagger}(D^{++} + iV^{++})Q^a + Q^{a\dagger}(ig v^{++})q^a + q^{a\dagger}(ig v^{++})Q^a]\]  \( (14) \)

Where \( \ldots \) mean the ghost contribution. The operator \( \Box \) includes the background \( \mathcal{N} = 2 \) superfield strengths \( \mathcal{W}, \bar{\mathcal{W}} \) which act in the adjoint representation of the gauge group and has the form \( (13) \). The set of the background superfields is on-shell, \( (D^+)^2 \mathcal{W} = 0, \quad D^{++} Q^a = 0 \) and satisfying the relations \( D^{++} D_\alpha^a \mathcal{W} = 0, \quad D^{\pm a} D_\alpha^a \mathcal{W} = 0 \). We redefine \( g Q^+ \rightarrow Q^+ \) and write the background superfields \( V^{++} = \tau_3 V_3^{++}, \quad q^+ = \tau_3 q_3^+ \) and hence \( \mathcal{W} = \tau_3 \mathcal{W}_3 \). Here \( \tau_i = \frac{1}{\sqrt{2}} \sigma_i \) are generators of \( su(2) \) algebra:

\[
[\tau_i, \tau_j] = i\sqrt{2} \epsilon_{ijk} \tau_k, \quad \text{tr}(\tau_i\tau_j) = \delta_{ij}.
\]

For the background belonging to Abelian subgroup we have also the further restrictions:

\( D^{\pm a} \mathcal{W} = D^{\pm a} \mathcal{W} \), and similarly for \( \mathcal{W} \) with \( D, \bar{D} \) being “flat” derivatives. Then taking into account the on-shell conditions and that all quantum superfields are in the adjoint representation \( v^{++} = v^{++}_i \tau_i, \quad Q^{a\dagger} = Q^{a\dagger}_i \tau_i \) ones get

\[
\Box v^{++} = \Box_{\text{cov}} v^{++} + \frac{i}{2} [D^{++} \mathcal{W}, D_\alpha^a v^{++}] + \frac{i}{2} [D^{a\dagger} \bar{\mathcal{W}}, \bar{D}^\alpha v^{++}] + [\mathcal{W}, [\bar{\mathcal{W}}, v^{++}]]. \]  \( (15) \)

\(^4\)Some new results were presented in [33].
As a result, the action (16) takes the form

\[ S_2 = \frac{-1}{2} \int d\zeta (-\bar{\nabla} \nabla \chi) + \bar{\nabla} \nabla \chi = q^{+}[v^{++}, Q^+_a]. \]

We also have \( D^{++}q^1 = D^{++}q^1 + \sqrt{2}V^{++}q^2, \) \( D^{++}q^2 = D^{++}q^2 - \sqrt{2}V^{++}q^1, \) \( D^{++}q^3 = D^{++}q^3, \) \( D_m = D_m + \sqrt{2}A_m. \)

Thus, the quadratic part of the action is written as

\[ S_2 = \frac{-1}{2} \int d\zeta (-\bar{\nabla} \nabla \chi) + \bar{\nabla} \nabla \chi. \]

For gauge still use on shell the notation (13) for (18). Now it is easy to read off the Feynman rules

\[ + \bar{\nabla} \nabla \chi. \]

Here \( V \) is taken directly from the

\[ S \]

This form of action is very convenient for perturbative calculations. It has a diagonal part

\[ \text{We define the new complex quantum superfields} \]

\[ \chi^{++} = \frac{1}{\sqrt{2}}(v_1^{++} + i v_2^{++}), \quad \bar{\chi}^{++} = \frac{1}{\sqrt{2}}(v_1^{++} - i v_2^{++}), \]

\[ \eta^{+a} = \frac{1}{\sqrt{2}}(Q_1^{+a} + i Q_2^{+a}), \quad \bar{\eta}^{+a} = \frac{1}{\sqrt{2}}(Q_1^{+a} - i Q_2^{+a}). \]

As a result, the action (16) takes the form

\[ S_2 = -\int d\zeta (-\bar{\nabla} \nabla \chi) + \bar{\nabla} \nabla \chi. \]

The form of action is very convenient for perturbative calculations. It has a diagonal part \( S_0 \) defying the propagators and non-diagonal part \( V \) responsible for interaction. The noninteracting fields \( Q_3^{+a}, v_3^{++} \) will be omitted further. As the next step we introduce the operator

\[ \hat{\Box}_{\text{short}} = \Box_{\text{cov}} + 2W\bar{W} - \frac{i}{\sqrt{2}}((D^{+a}W)D^{-a} + (D^{+a}\bar{W})\bar{D}^{-a}) \]

which is obtained from (13) by staying there only the superfields \( W, \bar{W} \) associated with the unbroken \( U(1) \) subgroup and putting their on shell. Note that on shell the form of the operator (13) and (18) related to each other by \( W \to \frac{W}{-\sqrt{2}}, \bar{W} \to \frac{\bar{W}}{-\sqrt{2}}. \) Further we still use on shell the notation (13) for (18). Now it is easy to read off the Feynman rules for calculating the effective action, which are derived in a similar manner to [12], [8], [14]. For gauge \( \chi^{++}, \bar{\chi}^{++} \) and hypermultiplet \( \eta^{+a}, \bar{\eta}^{+a} \) propagators we use [10], [12] respectively. Vertices are taken directly from the \( V \) in the form

\[ V = -i\chi^{++}\sqrt{2}q^{+a}\bar{\eta}_a^{+} + i\bar{\chi}^{++}\sqrt{2}q^{+a}\eta_a^{+}. \]
The Feynman rules look standard. We emphasize only the important point. In each vertex which includes the integral over analytic subspace one can use the factor \((D^+)^4\) from one propagator and convert the integral over \(d\zeta^{(-4)}\) to integral over full \(N=2\) measure \(d^{12}z\).

We point out that the functional change of variables in (17):\[\chi^{++}(1) \rightarrow \chi^{++}(1) - i \int d\zeta^{(-4)} G^{(2,2)}(1|2)q^a(2)\eta^+_a(2),\]
\[\bar{\chi}^{++}(1) \rightarrow \bar{\chi}^{++}(1) + i \int d\zeta^{(-4)} G^{(2,2)}(1|2)q^+_a(2)\eta^+_a(2),\]
with \(G^{2,2}(1|2)\) leads to diagonalization of the operator
\[
\int d\zeta^{(-4)} d\bar{\zeta}^{(-4)} \bar{\eta}^+(1) \{ \delta^b_a D_1^{++} \delta^{(1,3)}(1|2) + q^+_a(1)G^{(2,2)}(1|2)q^+(2) \} \eta^+_b(2).
\]

Then, the one-loop effective action \(\Gamma[V^{++}, q^+]\) defined by the path integral
\[
e^{i\Gamma[V^{++}, q^+]} = \int D\bar{\eta}^+ D\eta^+ D\chi^{++} D\bar{\chi}^{++} e^{iS_2[\eta^+, \bar{\eta}^+, \chi^{++}, \bar{\chi}^{++}, V^{++}, q^+]}\]
can be formally written as
\[
\Gamma[V^{++}, q^+, \bar{q}^+] = i \text{Tr} \ln \{ \delta^b_a D_1^{++} \delta^{(1,3)}(1|2) + q^+_a(1)G^{(2,2)}(1|2)q^+(2) \} + \Gamma[V^{++}] .
\] (19)

Where the last term \(\Gamma[V^{++}]\) is part of the full one-loop effective action which depends only on \(N=2\) gauge superfield. We will study mostly the first term there since it includes all the hypermultiplet dependence. The expression (19) written as an analytic nonlocal superfunctional will be a starting point for calculations of one-loop effective action in hypermultiplet sector. The Eq. (19) shows that the effective action is well defined within perturbation theory in powers of the non local interaction \(q^+_a(1)G^{(2,2)}(1|2)q^+(2)\). It leads to effective action in the form \(\Gamma[V^{++}, \bar{q}^+, q^+] = \sum_{n=1}^{\infty} \Gamma_{2n}[V^{++}, \bar{q}^+, q^+]\). Here the 2\(n\)-th term is given by a supergraph with 2\(n\) external \(\bar{q}^+, q^+\)-legs and any number \(V^{++}\)-legs. Since \(\Gamma[V^{++}, \bar{q}^+, q^+]\) is gauge invariant by construction, one can expect that each coefficient \(\Gamma_{2n}\) depends on background superfield \(V^{++}\) only via the strengths \(\mathcal{W}, \bar{\mathcal{W}}\) and their covariant derivatives. We emphasize that the supergraphs associated with this procedure contain the background dependent superpropagators.
4 Analysis of supergraphs for hypermultiplet dependent contributions to effective action

The hypermultiplet dependent contributions to the one-loop effective action are presented by following infinite sequence of the supergraphs:

Here the wavy line stands for $\mathcal{N} = 2$ gauge superfield propagator and solid external and internal lines stand for background hypermultiplet superfields and quantum hypermultiplet propagators respectively. The numbers 1, 2, ... mark the arguments $\zeta_a, u$ of the external hypermultiplets lines. As we emphasized above the whole dependence of the contributions on the background gauge superfield is included into background-dependent propagators.

An arbitrary supergraph with $2n$ external hypermultiplet lines looks a ring consisting of $n$ links of the form $< \tilde{\eta}^+ \eta^+ > < \chi^{++} \tilde{\chi}^{++} >$ or $n$ links of the form $< \eta^+ \tilde{\eta}^+ > < \tilde{\chi}^{++} \chi^{++} >$. The total contribution of these two kinds of the $2n$-point supergraph is given by following general expression

$$i\Gamma_{2n} = \frac{4}{n} \int d\zeta_1^{(-4)} d\zeta_2^{(-4)} \ldots d\zeta_{2n}^{(-4)}$$

$$\times \frac{(D^+)^4(D^-)^4}{(u_1^+ u_2^+)^3} \{ \frac{1}{\Box_1} \delta_{12}^{12} \} \{ \frac{1}{\Box_3} \delta_{34}^{12} \} \times \frac{(D^+_3)^4(D^-_3)^4}{(u_3^+ u_4^+)^3} \{ \frac{1}{\Box_3} \delta_{34}^{12} \} \times \frac{(D^+_4)^4(D^-_4)^4}{(u_4^+ u_5^+)^3} \{ \frac{1}{\Box_3} \delta_{34}^{12} \} \times \frac{(D^+_5)^4(D^-_5)^4}{(u_5^+ u_6^+)^3} \{ \frac{1}{\Box_3} \delta_{34}^{12} \} \times \frac{(D^+_6)^4(D^-_6)^4}{(u_6^+ u_7^+)^3} \{ \frac{1}{\Box_3} \delta_{34}^{12} \} \times \frac{(D^+_7)^4(D^-_7)^4}{(u_7^+ u_8^+)^3} \{ \frac{1}{\Box_3} \delta_{34}^{12} \} \times \frac{(D^+_8)^4(D^-_8)^4}{(u_8^+ u_9^+)^3} \{ \frac{1}{\Box_3} \delta_{34}^{12} \}$$

Here and further to avoid the coincident harmonic singularities we keep the $\mathcal{N} = 2$ gauge superfield propagator in the form which is manifestly analytic in both argument [28, 8].

The factor $4/n$ has the following origin (see [24]). The contribution from the ring type supergraph composed from $n$ repeating links $< \tilde{\eta}^+ \eta^+ > < \chi^{++} \tilde{\chi}^{++} >$ appears with the symmetry factor $2/n$. The same factor $2/n$ arises from the supergraph composed from $n$ repeating links $< \eta^+ \tilde{\eta}^+ > < \tilde{\chi}^{++} \chi^{++} >$. Further, each vertex brings the factor $-i$, every $< \eta^+ \tilde{\eta}^+ >$ and $< \tilde{\chi}^{++} \chi^{++} >$ propagators contribute the factor $i$ and $i/2$ respectively. Hence total of $n$ links contributes $2^{-n}$. Any vertex also carries the coefficient $\sqrt{2}$. This leads to the total factor $2^n$. Substituting all these contributions together, we obtain just the coefficient $4/n$. 

We begin with direct calculation of the term $\Gamma_2[V^{++},q^{+\alpha}]$ which in the analytic basis reads

$$i\Gamma_2 = \int d\zeta_1^{-4} d\zeta_2^{-4} du_1 du_2 \left\{ \frac{(D_+^+_1)^4(D_2^+)^4}{(u_1^+ u_2^+)^3} \frac{1}{\Box_1} \delta^{12}(1|2) \right\} \times \frac{(D_2^+)^4(D_1^+)^4}{\Box_2\Box_1} \delta^{12}(2|1)(D_1^{--})^2 \delta^{-2,2}(u_2, u_1) \right\} q_+^a(z_1, u_1) q^{+\alpha}(z_2, u_2).$$

(21)

According to the general strategy of handling such supergraphs we should first to restore the full Grassmann integration measure at the vertices by rule $d^{12}z_1 d^{12}z_2 = d^{-4}\zeta_1 d^{-4}\zeta_2 (D_1^+)^4 (D_2^+)^4$. Since we are interested in contributions which do not depend on space-time or Grassmann derivatives of background hypermultiplets, it is sufficient to treat them as constants. Then, integrating by parts with the help of delta function we shrink a loop into a point in superspace. However in harmonic space it remains still non-local expression

$$i\Gamma_2 = \int dz du_1 du_2 \frac{(D_2^+)^4(D_1^+)^4}{(u_1^+ u_2^+)^3} \delta^{12}(z) \left\{ (D_1^{--})^2 \delta^{-2,2}(u_2, u_1) \right\} q_+^a(z_1, u_1) q^{+\alpha}(z_2, u_2).$$

(22)

We point out the presence of the harmonic distribution $(u_1^+ u_2^+)^{-3}$ with coincident singularities. If we would have the flat covariant derivatives we could use the important identity: $(D_1^+)^4 (D_2^+)^4 \delta^8(\theta_1 - \theta_2) = (u_1^+ u_2^+)^4$ and get simplification. However the expression under consideration [22] contains the covariant spinor derivatives and further analysis becomes much more complicated. In principle, one can use the idea of paper [33] to express the covariant derivatives $D_2^{+,\alpha,\dot{\alpha}}$ through the covariant derivatives $D^{+,\alpha,\dot{\alpha}}$ to evaluate the two-point function of the form $(D_1^+)^4 (D_2^+)^4 \delta^{12}(z_1 - z_2) \frac{1}{(u_1^+ u_2^+)^3}$. It can help to calculate the $\Gamma_2$ [22], however a procedure of such a calculation still looks very complicated technically. From our point of view, it is more convenient to start with the representation [21] and work in the analytic subspace [22]. Then instead of computing the contribution [22] in full $\mathcal{N} = 2$ superspace we should actually look for an equivalent expression of the form

$$\int d\zeta^{-4}(D^+)^4 \mathcal{L}(\mathcal{W}, \mathcal{W}, q^+)$$

with some $\mathcal{L}$. To do that we return in [22] to analytic subspace, use twice the relation [33] and apply the harmonic identities $(u_1^+ u_2^+)|_{1=2} = 0$, $D_1^{--}(u_1^+ u_2^+) = (u_1^- u_2^+)$, $(u_1^+ u_2^-)|_{1=2} = 1$

$$\frac{1}{(u_1^+ u_2^+)^3} (D_1^+)^4 (D_2^+)^4 (D_1^+)^4 = (D_1^+)^4 \left\{ ... + (u_1^+ u_2^+)(u_2^- u_1^+)^2(u_1^- u_2^+)^2 \delta^{12}(z_1 - z_2) + \right\}.$$

(23)

As a result ones get

$$i\Gamma_2 = \int d\zeta^{-4} du_1 du_2 (u_1^+ u_2^+) (q_+^a(u_1) q^{+\alpha}(u_2))(D_1^{--})^2 \delta^{-2,2}(u_2 u_1) (D_1^+)^4 \frac{\delta^{12}(z)}{\Box_2}. $$

(24)

Then we take the factor $(D^{--})^2$ off the harmonic delta function and apply the harmonic identities

$$(D_1^{--})^2 \delta^{-2,2}(u_2 u_1) = (D_2^{--})^2 \delta^{2,2}(u_2 u_1), \quad D^{--} q^+ = q^-,$$

$(u_1^+ u_2^+)|_{1=2} = 0, \ldots$
Integration over one set of harmonics leads to final result for $\Gamma_2$

$$i\Gamma_2 = \int d\zeta^{(-4)} du(-4q^- q^a) \frac{(D^+){^4}}{\Box} \delta^{12}(z).$$

(25)

The next step is the calculation of the four-leg contribution $\Gamma_4[q^+]$. We start with general relation (20) for $n = 2$ and perform the same manipulations as in preliminary case. It gives

$$i\Gamma_4 = \int d\zeta_{(-4)}^{(-4)} du_1 du_2 du_3 du_4 \frac{(D^+)^4}{(u_1^+ u_2^+)^3} \frac{(D_2^+)^4}{(u_3^+ u_4^+)^3} \{ \frac{1}{\Box_1^{12}} \delta^{12}(1|2) \}
$$

\[
\times \left( \frac{(D_2^+)^4}{\Box_2^{3}} \right) \frac{\delta^{12}(2|3)(D_3^+)^2 \delta^{(-2,2)}(u_2, u_3)}{(u_4^+)^3} \left( \frac{(D_3^+)^4}{(u_3^+ u_4^+)^3} \right) \left( \frac{(D_3^-)^2}{(u_3^+ u_4^+)^3} \right) \left( \frac{1}{\Box_3^{12}} \delta^{12}(3|4) \right)
\]

(26)

\[
\times \left( \frac{(D_2^+)^4}{\Box_4^{1}} \right) \delta^{12}(4|1)(D_3^{(-2,2)}(u_4, u_1)]q^a(z_1, u_1)q^{+a}(z_2, u_2)q^b(z_3, u_3)q^{+b}(z_4, u_4) =
\]

\[
= \int \frac{dz du_1 du_2 du_3 du_4}{(u_1^+ u_2^+)^3 (u_3^+ u_4^+)^3} \frac{(D_2^+)^4}{(u_1^+ u_2^+)^3} \frac{(D_3^+)^4}{(u_1^+ u_2^+)^3} \frac{(D_3^{(-2,2)})^2}{(u_1^+ u_2^+)^3} \delta^{12}(z)(D_3^-)^2 \delta^{(-2,2)}(u_2, u_3)
\]

\[
\times \{ \frac{1}{\Box_1^{12}} \delta^{12}(1|2) \}
\]

\[
\times \left( \frac{(D_2^+)^4}{\Box_2^{3}} \right) \frac{\delta^{12}(2|3)(D_3^+)^2 \delta^{(-2,2)}(u_2, u_3)}{(u_4^+)^3} \left( \frac{(D_3^+)^4}{(u_3^+ u_4^+)^3} \right) \left( \frac{(D_3^-)^2}{(u_3^+ u_4^+)^3} \right) \left( \frac{1}{\Box_3^{12}} \delta^{12}(3|4) \right)
\]

Using the identities $(D_3^{(-2,2)}(u_2, u_3) = (D_2^{(-2,2)}(u_2, u_3),$ and $(D_2^+)^4(D_3^-)^2(D_3^+)^4 \delta^{(-2,2)}(u_2, u_3) = -2\Box_2(D_2^2)^4$ ones obtain

$$i\Gamma_4 = \int d\zeta^{(-4)} du_1 du_2 \frac{(D_2^+)^4}{(u_1^+ u_2^+)^3} \frac{(D_2^+)^4}{(u_1^+ u_2^+)^3} \delta^{12}(z) |q^a(z_1, u_1)q^{+a}(z_2, u_2)q^b(z_3, u_3)q^{+b}(z_4, u_4) |
$$

\[
= \int d\zeta^{(-4)} du_1 du_2 D_1^{-}\delta^{(-2,2)}(u_1 u_2) \frac{(D_2^+)^4}{(u_1^+ u_2^+)^3} \delta^{12}(z) q^a(z_1, u_1)q^{+a}(z_2, u_2)q^b(z_3, u_3)q^{+b}(z_4, u_4) |
\]

Writing $D_1^{-}\delta^{(-2,2)}(u_1 u_2) = D_2^{(-2,2)}(u_1 u_2)$ and integrating over $u_2$ ones get finally

$$i\Gamma_4 = \frac{1}{2} \int d\zeta^{(-4)} du(-4q^- q^a)^2 \frac{(D_2^+)^4}{(u_1^+ u_2^+)^3} \delta^{12}(z).$$

(27)

Analysis of general term $\Gamma_{2n}$ is carried out analogously. First of all we transform all analytic subspace integrals into ones over the full superspace taking the factors $(D_k^+)^4(D_{k+1}^+)^4$ from the hypermultiplet propagators. Then ones integrate over sets of Grassmann and space-time coordinates using the corresponding delta-functions in the integrand. It leads to

$$\int d^2z du_1 \ldots du_{2n} \frac{\delta^{(-2,2)}(u_2, u_3) \delta^{(-2,2)}(u_1) \delta^{(-2,2)}(u_4, u_5) \ldots \delta^{(-2,2)}(u_{2n}, u_1)}{(u_1^+ u_2^+)^3 (u_3^+ u_4^+)^3 \ldots (u_{2n-1}^+ u_{2n}^+)^3}$$

(27)
\[ \times \frac{(D_2^+)^4(D_4^+)^4...(D_{2n}^+)^4}{\square_1\square_2 \ldots \square_{2n}} \{ \delta^{12}(z - z') | q_0^+(u_1)q_0^+(u_2)q_0^+(u_3) \ldots q_0^+(u_{2n-1})q_0^+(u_{2n}) \}. \]

Then ones integrate over \( u_2, u_4, \ldots, u_{2n} \) using the harmonic delta functions and obtain

\[ \int du_1 du_3 \ldots du_{2n-1} \frac{(D_1^+)^4(D_2^+)^4 \ldots (D_{2n-1}^+)^4}{(u_1^+ u_2^+)^3(u_3^+ u_4^+)^3 \ldots (u_{2n-1}^+ u_1^+)^3} \square_2 \square_3 \ldots \square_{2n-1} \times \{ \delta^{12}(z - z') | q_0^+(u_1)q_0^+(u_3)q_0^+(u_3)q_0^+(u_5) \ldots q_0^+(u_{2n-1})q_0^+(u_{2n}) \}. \]

After some relabelling the indices \( c \to a, a \to b, \ldots; \ 3 \to 2, \ldots, (2n - 1) \to n \) we get the expression

\[ i\Gamma_{2n} = \frac{4(-1)^{n-2}n}{n} \int d^{12}zu_1 \ldots du_n \frac{(D_1^+)^4(D_2^+)^4 \ldots (D_n^+)^4}{(u_1^+ u_2^+)^3(u_3^+ u_4^+)^3 \ldots (u_{n+1}^+ u_1^+)^3} \square_{1,2} \square_3 \ldots \square_n \times \{ \delta^{12}(z - z') | q_0^+(u_1)q_0^+(u_3)q_0^+(u_5) \ldots q_0^+(u_{2n}) \}. \]

To simplify the expression (29) ones represent \( q_0^+(u_1) \) as \( q_0^+(u_1) = D_1^{++}q_0^+(u_1) \) (since \( q_0^+ \) sits on its mass shell). We note that \( D_1^{++} \) when acts on \( \square \) gives rise to the structures like \( (D_1^{++})^5 = 0 \). Now we integrate by parts and remove the harmonic derivative on the harmonic distributions

\[-D_1^{++} \frac{1}{(u_1^+ u_2^+)^3(u_3^+ u_4^+)^3} = \frac{1}{2} \{(D_1^{--})^2\delta^{(3, -3)}(u_1, u_2) \frac{1}{(u_1^+ u_2^+)^3} + (2 \leftrightarrow n)\} \]

and use the identity [12, 8]

\[(D_1^{--})^2\delta^{(3, -3)}(1|2) = (D_2^{--})^2\delta^{(-1, 1)}(1|2). \]

Then we take the factor \((D_2^{--})^2\) off the harmonic delta function. It is easy to see that this factor can give a non-vanishing result only when acts on \((D^+(u_2))^4\). We get

\[ \int du_1 \ldots \left( -\frac{1}{2} \right) \delta^{(-1, 1)}(1|2) \frac{(D_1^{+})^4(D_2^{--})^2(D_2^{+})^4 \ldots (D_n^+)^4}{(u_2^+ u_3^+)^3 \ldots (u_{n+1}^+ u_1^+)^3} \square_{1,2} \square_3 \ldots \square_n \times \{ \delta^{12}(z) | q_0^+(u_1)q_0^-(u_1)q_0^+(u_2)q_0^+(u_n) \} + (2 \leftrightarrow n). \]

We replace \( u_2 \leftrightarrow u_n \) in the second term here, after that it becomes identical to the first one. Doing integral over \( u_1 \) we obtain the expression \((-1)(-2\square_2)\frac{1}{\square_2}\). On the second step we repeat above procedure for \( q_0^-(u_2) \), i.e. represent it in the form \( q_0^-(u_2) = D_2^{++}q_0^- \) and integrate by parts with respect to \( D_2^{++} \). Performing the same manipulations as above ones obtain the factor \((-1)^2(2\square_3)^2\frac{1}{\square_3}\).

After \( n - 4 \) analogous steps \( q_3^+(u_3) = D_3^{++}q_3^- \) and so on we reduce the harmonic integral in expression (29) to that over three sets of harmonic:

\[ \frac{(D_u^+)^4(D_{u_{n-1}}^+)^4(D_{u_n}^+)^4}{\square_u \square_{u_{n-1}} \square_{u_n}} \left( -2^{n-3} \right) \left( u_{n-1}^+ u_n^+ \right)^3 \left( u_n^+ u_u^+ \right)^3 \{ \delta^{12}(z - z') \} \]

13
It is evident that the second term in (36) is (up to sign) the representation of the one-loop action in hypermultiplet sector (36). This representation is free of harmonic singularities. Using the techniques of the harmonic supergraphs we obtained the representation of the one-loop effective action contains also hypermultiplet independent part $\Gamma(36)$ which essentially depends on hypermultiplets. The full $\Gamma(36)$ is only a part of full effective action $\Gamma$ which essentially depends on hypermultiplets. The full effective action contains also hypermultiplet independent part $\Gamma(V^{++})$.

Now sum up all contributions $\Gamma_{2n}$ (35). The result is given in terms of functional determinant of special differential operator:

$$i\Gamma_{2n} = \frac{1}{n} \int d\zeta(-4) du \left( \frac{\mathcal{D}^+}{u+u_1} \right)^4 \delta^{12}(z-z')|q^{+a}(u)q_b^+(u)q_b^-(u)q_c^+(u_1)q_a^+(u_1)|.$$ (35)

On last step we write $q_c^+(u) = D_u^+ q_c^-$ and remove the $D_u^{++}$ on the harmonic factor. Repeating the same manipulations we perform the $u_{n-1}$-integration and get the expression

$$i\Gamma_{2n} = -\frac{(-2)^{n+2}}{n} \int d^2z dudu_1 \left( \frac{\mathcal{D}_u^+}{u+u_1} \right)^4 \delta^{12}(z-z')|q^{+a}(u)q_b^+(u)q_b^-(u)q_c^+(u_1)q_a^+(u_1)|.$$ (34)

Now ones return to analytic subspace applying the identities (23)

$$\int \frac{d\zeta(-4) du}{(u+u_1)^2} \mathcal{D}_u^+ \delta^{12}(z-z')|q^{+a}(u)q_b^+(u)q_b^-(u)q_c^+(u_1)q_a^+(u_1)|.$$ Then ones use the identities $q^+_1(u) = (u_1^+u^-)q^+_1(u) - (u_1^-u^+)q^-$, $q_a^-q^{-a} = 0$ leading to the factor $(u_1^+u_2^-)^2$. We get finally

$$i\Gamma_{2n} = \frac{1}{n} \int d\zeta(-4) du \left( \frac{\mathcal{D}^+}{u} \right)^4 \delta^{12}(z-z')|(-4q^-q^+)^n. $$

It is evident that the second term in (36) is (up to sign) the representation of the one-loop effective action for $N = 4$ SYM theory in sector the $N = 2$ vector multiplet [33], [24]. We see the $\Gamma$ (36) vanishes when the hypermultiplets vanish. We point out that $\Gamma$ (36) is only a part of full effective action (19) which essentially depends on hypermultiplets. The full effective action contains also hypermultiplet independent part $\Gamma(V^{++})$.

Thus, using the $N = 2$ harmonic superspace formulation of $N = 4$ SYM theory and techniques of the harmonic supergraphs we obtained the representation of the one-loop effective action in hypermultiplet sector [36]. This representation is free of harmonic singularities and, as we will see, it admits a straightforward evaluation with help of $N = 2$ superfield heat kernel method. We show in the next section, the general expression (36) allows us to obtain the exact proper-time representation of effective action and its expansion in covariant spinor derivatives of the $N = 2$ superfield Abelian strengths $\mathcal{W}, \mathcal{W}$ corresponding to the constant space-time background

$$\mathcal{W}|_{\theta = 0} = \text{const}, \mathcal{D}^\pm_{\alpha, \alpha} \mathcal{W}|_{\theta = 0} = \text{const}, \mathcal{D}^\pm_{\alpha} \mathcal{W}|_{\theta = 0} = \text{const}, \mathcal{D}_{\tilde{\alpha}} \mathcal{D}^\pm_{\beta} \mathcal{W}|_{\theta = 0} = \text{const}$$ (37)
and constant space-time background hypermultiplet \( q^{+a} |_{\theta=0} = \text{const} \). The Eq. (36) immediately leads to one of the main results concerning the hypermultiplet dependence of one-loop effective action: the hypermultiplet enters to the effective action in the combination \( \mathcal{W} \mathcal{W} + 2q_{a}^{-} q^{+a} \) which is invariant of \( R \) symmetry of \( \mathcal{N} = 4 \) supersymmetry. To see that ones consider the expression \( \hat{\square} + 4q_{a}^{-} q^{+a} \) and use the on-shell form of \( \hat{\square} \) (18). Then
\[
\hat{\square} + 4q_{a}^{-} q^{+a} = \frac{1}{2} \mathcal{D}_{\alpha a}^{a} \mathcal{D}_{\alpha a} - \frac{1}{\sqrt{2}} (\mathcal{D}^{+\alpha} \mathcal{W}) \mathcal{D}_{\alpha}^{-} + (\mathcal{D}_{\dot{\alpha}}^{\dot{a}} \tilde{\mathcal{W}}) \mathcal{D}^{-\dot{\alpha}} + 2\mathcal{W} \mathcal{W} + 4q_{a}^{-} q^{+a}.
\]
Of course, this result is correct only for the constant background hypermultiplets.

## 5 Proper-time representation of effective action

Relation (36) possesses remarkable features. First, we started with the model of two interacting fields \( V^{++}, q^{+} \) and resummed the supergraphs by such a way that the effective action is expressed in terms of a differential operator acting only in the sector of the vector multiplet. Whole dependence on hypermultiplets is included into this operator. Second, the effective action is written as an integral over an analytic subspace of the harmonic superspace. It is co-ordinated with the classical action of the theory which is also written as an integral over the analytic subspace. Third, the relation (36) has the form \( \text{Tr} \ln \hat{\mathcal{A}} \) with the operator \( \hat{\mathcal{A}} = \hat{\square} + 4q_{a}^{-} q^{+a} \) acting on the analytic superfields. We emphasize that the above simple form of the one-loop effective action is not evident from the very beginning, it is the result of resummation of the infinite sequence of the one-loop harmonic supergraphs with arbitrary number hypermultiplet external legs. The form of the effective action (36) is basic for using of the proper-time representation:

\[
\Gamma = i \int d\zeta (-4) du \int_{0}^{\infty} \frac{ds}{s} e^{-s(\hat{\square} + 4q_{a}^{-} q^{+a})} (\mathcal{D}^{+})^{4} \delta^{12}(z - z') |_{z'=z}
\]

\[
= i \int_{0}^{\infty} \frac{ds}{s} \text{Tr} \{ K(s) e^{-s(4q_{a}^{-} q^{+a})} \},
\]

(38)

Here \( K(s) \) is a superfield heat kernel, the operation \( \text{Tr} \) means the functional trace in the analytic subspace of the harmonic superspace \( \text{Tr} K(s) = \text{tr} \int d\zeta (-4) \mathcal{K}(\zeta, \zeta | s) \), where \( \text{tr} \) denotes the trace over the discrete indices. As a result, the problem of finding the one-loop effective action is reduced to evaluation of the kernel \( K(s) = e^{-s \hat{\square}} \). Further it is convenient to deal with \( \hat{\square} \) written by a definition as

\[
\hat{\square} = \frac{1}{2} \mathcal{D}_{\alpha a}^{a} \mathcal{D}_{\alpha a} + \frac{i}{2} (\mathcal{D}^{+\alpha} \mathcal{W}) \mathcal{D}_{\alpha}^{-} + \frac{i}{2} (\mathcal{D}_{\dot{\alpha}}^{\dot{a}} \tilde{\mathcal{W}}) \mathcal{D}^{-\dot{\alpha}} + \mathcal{W} \tilde{\mathcal{W}}.
\]

(39)

For the effective action calculation (38) we will use the techniques of symbols of operators in the analytic subspace of \( \mathcal{N} = 2 \) harmonic superspace 6. We begin with the Fourier representation of the delta-function in \( \mathcal{N} = 2 \) superspace

\[
\delta^{12}(z - z') = \int \frac{d^{4}p}{(2\pi)^{4}} \int d^{4}\psi^{+} d^{4}\psi^{-} e^{i p m (z - z') m}.
\]

(40)

5Note that the replacements \( \mathcal{W} \rightarrow \frac{\mathcal{W}}{\sqrt{2}}, \tilde{\mathcal{W}} \rightarrow \frac{\tilde{\mathcal{W}}}{\sqrt{2}} \) reduces the expression (40) to (18).

6Applications of techniques of symbols of operators for calculating the effective action in \( \mathcal{N} = 1 \) superspace are given in [27].
for example, as follows. We extract in (41) the exponential of the leading symbol that, all differential operators can lie only on $W$. Then, we have to act by the operators in the exponential to the right on the unit. After a straightforward calculations to get a final result in manifest covariant form.

A method generating the manifest supersymmetric asymptotical expansion of heat kernel in $\mathcal{N} = 1$ superspace has been developed in [25], [27], [28]. We generalize this method for the heat kernel in $\mathcal{N} = 2$ harmonic superspace. In each superspace point we introduce a tangent space forming the normal-coordinate system and a fiber frame obtained by a parallel transport from the base point. The pseudodifferential operators can be reexpressed in this local representation of the vector bundle. We consider the heat kernel using these operators and derive an algorithm for the asymptotic expansions of the heat kernel.

Let us introduce the following notions

$$A^{+\alpha} = \frac{i}{2} [D^{+\alpha}, \mathcal{W}], \quad \bar{A}^{+\dot{\alpha}} = -\frac{i}{2} [\bar{D}^{+\dot{\alpha}}, \bar{\mathcal{W}}], \quad A^{-\alpha} = [D^{-\alpha}, \mathcal{W}], \quad \bar{A}^{-\dot{\alpha}} = [\bar{D}^{-\dot{\alpha}}, \bar{\mathcal{W}}].$$

$$\{D^-_{\alpha}, A^+_{\beta}\} = N_{\alpha\beta} = N_{\beta\alpha} = \frac{i}{2} D^-_{\alpha} D^+_{\beta} \mathcal{W}, \quad \{\bar{D}^-_{\dot{\alpha}}, \bar{A}^+_{\dot{\beta}}\} = \bar{N}_{\dot{\alpha}\dot{\beta}} = \bar{N}_{\dot{\beta}\dot{\alpha}} = -\frac{i}{2} \bar{D}^-_{\dot{\alpha}} \bar{D}^+_{\dot{\beta}} \bar{\mathcal{W}}.$$ 

In this terms the algebra of covariant derivatives (11) takes the form:

$$[D_{\alpha\dot{\alpha}}, x^{\beta\dot{\beta}}] = 2\delta^{\beta\dot{\beta}}_{\alpha\dot{\alpha}}, \quad \{D_{\alpha}, \theta^{+\beta}\} = \delta_{\alpha}^{\beta}, \quad \{\bar{D}_{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\} = \delta_{\dot{\alpha}}^{\dot{\beta}},$$

$$[D^{\alpha\dot{\alpha}}, D_{\beta\dot{\beta}}] = -\delta^{\alpha\dot{\alpha}}_{\beta\dot{\beta}} N_{\beta\dot{\beta}} + \delta_{\alpha\dot{\alpha}}^\beta \bar{N}^{\dot{\beta}}, \quad [\bar{D}^{\dot{\alpha}}, D_{\dot{\beta}\beta}] = \varepsilon_{\dot{\alpha}\dot{\beta}} A^{+\beta},$$

$$D_{m} A^{\pm}_{(\alpha, \dot{\alpha})} = D_{(\delta, \dot{\delta})} N_{\alpha\beta} = D_{(\delta, \dot{\delta})} \bar{N}_{\dot{\alpha}\dot{\beta}} = D_{m} N_{\alpha\beta} = D_{m} \bar{N}_{\dot{\alpha}\dot{\beta}} = 0.$$ 

We see that the set of covariant derivatives together with the on-shell background superfields $\mathcal{W}, \bar{\mathcal{W}}$, corresponding to the constant space-time configurations (37), and their low-order derivatives generates a finite dimensional Lie superalgebra (42)-(44).
On the next step we lift the shifted operators \( X_m = D_m + ip_m, X^\pm_{(\alpha, \dot{\alpha})} = D^\pm_{(\alpha, \dot{\alpha})} + \psi^\pm_{(\alpha, \dot{\alpha})} \) satisfying the same algebra (42)-(44) in the tangent space by the "exponential map"

\[
X(p_M, \partial/\partial p_M) = U^{-1}X_M U^+ \text{ with }
\]

\[
U = e^{-\dot{\psi}^+D^\pm_{\alpha} - \dot{\psi}^\pm_{\dot{\alpha}}}, e^{2p_\alpha p_\dot{\alpha}\dot{\psi}^\pm_{\alpha} - 2\dot{\psi}^\pm_{\alpha}p_\dot{\alpha} \theta^{-\alpha}}, e^{\dot{\psi}^\pm_{(\alpha, \dot{\alpha})}D^\pm_{(\alpha, \dot{\alpha})} - \dot{\psi}^+_{(\alpha, \dot{\alpha})}}, e^{-\frac{1}{2}p_\alpha p_\dot{\alpha}D^\pm_{(\alpha, \dot{\alpha})}},
\]

where the role of a tangent vectors forming a normal-coordinate frame plays a set of derivatives

\[
\partial^\pm_{\psi} \equiv \frac{\partial}{\partial \psi^\pm_{\alpha}}, \partial^\pm_{\dot{\psi}} \equiv \frac{\partial}{\partial \dot{\psi}^\pm_{\dot{\alpha}}}, \partial^\alpha_p \equiv \frac{\partial}{\partial p_\alpha}, \partial^{\alpha\dot{\alpha}}_p \equiv \frac{\partial}{\partial p_{\alpha\dot{\alpha}}}, \delta^M_N.
\]

The action of the operator \( U \) on the shifted operators \( X_M \) is given by following expressions

\[
U^{-1}(X^+_\alpha)U = \psi^+_\alpha, \quad U^{-1}(\bar{X}^+_\dot{\alpha})U = -\dot{\psi}^+_\dot{\alpha},
\]

\[
U^{-1}(X^-_\alpha)U = -\psi^-_\alpha + 2\dot{\psi}^-_{\dot{\alpha}}p_{a\dot{\alpha}} + O(\partial^-_\alpha, \partial^-_p), \quad U^{-1}(\bar{X}^-_{\dot{\alpha}})U = \bar{\psi}^-_{\dot{\alpha}} + 2\dot{\psi}^-_{\dot{\alpha}}p_{\alpha\dot{\alpha}} + O(\partial^-_\dot{\alpha}, \partial^-_p),
\]

\[
\{X^+_\alpha, \bar{X}^+_\dot{\alpha}\} = 2X_{\alpha\dot{\alpha}} = -\{X^-_\alpha, \bar{X}^-_{\dot{\alpha}}\},
\]

\[
X_{\alpha\dot{\alpha}} = ip_{a\dot{\alpha}} + \partial^{\alpha}_{\psi}(\bar{D}^-_{\alpha}\bar{W}) - \partial^{\alpha}_{\dot{\psi}}(D^-_{\dot{\alpha}}\bar{W}) - \frac{1}{8}\{\partial^{\beta}_{pa}(\bar{D}^+_{\beta}\bar{D}^-_{\alpha}\bar{W}) + \partial^{\beta}_{p\dot{\alpha}}(D^-_{\beta}D^+_{\dot{\alpha}}\bar{W})\} + O(\partial^-_\alpha, \partial^-_p).
\]

The map of the function on the superspace in the point \( z \) into the tangent superspace is given by

\[
W \rightarrow W - \partial^\alpha_{\psi}(D^-_{\alpha}W) + O(\partial^-_\alpha, \partial^-_p), \quad \bar{W} \rightarrow \bar{W} + \partial^{\alpha}_{\dot{\psi}}(\bar{D}^-_{\alpha}\bar{W}) + O(\partial^-_\dot{\alpha}, \partial^-_p),
\]

\[
D^+_{\alpha}W \rightarrow D^+_{\alpha}W - \partial^\beta_{\psi}(D^-_{\beta}D^+_{\alpha}W) + O(\partial^-_\alpha, \partial^-_p), \quad \bar{D}^+_{\dot{\alpha}}\bar{W} \rightarrow \bar{D}^+_{\dot{\alpha}}\bar{W} + \partial^\beta_{\dot{\psi}}(D^-_{\beta}D^+_{\dot{\alpha}}\bar{W}) + O(\partial^-_\dot{\alpha}, \partial^-_p),
\]

\[
D^-_{\alpha}W \rightarrow D^-_{\alpha}W + O(\partial^-_\alpha, \partial^-_p), \quad \bar{D}^-_{\dot{\alpha}}\bar{W} \rightarrow \bar{D}^-_{\dot{\alpha}}\bar{W} + O(\partial^-_\dot{\alpha}, \partial^-_p).
\]

We point out that in this representation the operators \( X \), superfields and their derivatives satisfy the same algebra (13), (44). Generally speaking, if the background superfields are arbitrary, all above quantities are presented by an infinite series over \( \partial^i_p \) and a finite series over Grassmann derivatives \( \partial^i_{\psi} \) with the coefficients in a fixed point \( z^A \). But for the considered background (on-shell and space-time constant background fields) the representation (15) is exact.

The actual calculation of the effective action (38) with the kernel (41) is based on the following observation. The operator \( e^{-\frac{1}{2}D^\pm_{\alpha}W} \), where the operator \( \widehat{D}^\pm_{\alpha} \) is given in terms of shifted variables \( X \) (15), can be considered as an evolution operator for a Bose-Fermi quantum system with a Hamiltonian \( \hat{H} = \widehat{D}^\pm_{\alpha} \). The Eqs. (39) show that the Hamiltonian \( \hat{H} \) is a quadratic form in the operators \( p, \partial^i_p, \psi, \partial^i_{\psi} \) with constant coefficients (due to background under consideration). Therefore, calculation of \( TrK(s) = \int d\zeta^{-1}K(\zeta, \zeta|s) \) with \( K(\zeta, \zeta|s) \) given by (41), is an exactly solvable problem.

\(^7\text{Main property of this transformation is to eliminate the operators }D_m, D^+_{\alpha}, D^-_{\alpha} \text{ in } X_M \text{. Further we use the same denotations } X_M \text{ for transformed and initial quantities.} \)
Let us return to Eq. (41) where all operators and fields along with their derivatives are written in the representation (15). This is equivalent to an extension of the heat kernel to the tangent bundle on the superspace point $z$ where the coordinate $z$ is considered as a constant parameter. According to (11) the evolution operator should act on the unit. It is evident that a result of such action is obtained if we recombine all derivatives with $p$ and $\psi$ to the right and omit them. This procedure can be realized on the base of the Baker-Campbell-Hausdorff formula corresponding to the algebra (15), (13) (see Appendix A). The result we obtain is called a symbol of the evolution operator. This symbol has to be integrated over bosonic $p$ and fermionic $\psi$ variables, what leads us to the heat kernel (11).

Next useful observation is based on a fact that the exponential in the evolution operator contains only $D^- \sim \psi^-$. All $D^+ \sim \psi^+$ are in preexponential factor $(D^+)^4$ and saturate the integral over $d^4\psi^+$. Therefore we must omit all $O(\partial_\psi)$ in the operator $\hat{H}$ that gives us a more simple expression for (15). Besides, for doing the Berezin integral over $\psi^-$ sector we must extract a "projector" $(\psi^-)^4$ from the exponential under consideration.

This program can be effectively realized if we present the exponent $K(s) = e^{-s\Box}$ as a products of several operator exponents. Such a construction allows to overcome the difficulties which arise in previous attempts of computing the effective action of $\mathcal{N} = 2$ SYM theories directly in $\mathcal{N} = 2$ superspace. We write the operator $K(s) = e^{-s\Box}$ in the form

$$K(s) = \exp(-s\{A^+ D^- + \bar{A}^+ \bar{D}^- + \frac{1}{2} D^\alpha D_{\alpha\dot{\alpha}} + \mathcal{W} \}) \tag{47}$$

$$= \exp\{-f_{a\dot{a}}(s)D^{a\dot{a}}\} \exp\{-s\frac{1}{2} D^a D_{a\dot{a}}\} \exp\{-\Omega(s)\} \exp\{-s(A^{+a} D^-_a + \bar{A}^{+\dot{a}} \bar{D}^-_{\dot{a}})\}.$$ 

with some unknown coefficients in the right hand side. These coefficients can be found directly, i.e. using the Baker-Campbell-Hausdorff formula (representation of the Baker-Campbell-Hausdorff formula we use is given in Appendix A), and by solution to the system of a differential equation on the coefficients. Both ways lead to the same results. To find the mentioned system of equations we consider $(\frac{d}{ds} K)K^{-1}$ and substitute for $K$ first and second lines in (17) subsequently.

Equations for the functions $f^{a\dot{a}}(s)$ have the form

$$\dot{f}_{a\dot{a}}(s) = -f_{\beta\dot{\beta}}^{\dot{b}b} f^{b\dot{b}}_{a\dot{a}} - A^{+b}(D^-_b f_{a\dot{a}}) - \bar{A}^{+\dot{b}}(D^-_{\dot{b}} f_{a\dot{a}}) \tag{48}$$

$$+ A^{+b}_{\dot{b}} \bar{A}_{\dot{b}} (\int_0^s d\tau e^{\tau F})^{\dot{b}b}_{a\dot{a}} + \bar{A}^{+\dot{b}} A_{\beta} (\int_0^s d\tau e^{\tau F})^{\dot{b}b}_{a\dot{a}},$$

Analogously, equation for the function $\Omega$ is

$$\dot{\Omega}(s) - \mathcal{W} \mathcal{W} = -A^{+a}(D^-_a \Omega) - \bar{A}^{+\dot{a}}(D^-_{\dot{a}} \Omega) + A^{+a} f^{a\dot{a}} \bar{A}_{\dot{a}} + \bar{A}^{+\dot{a}} f^{\dot{a}a} A_a \tag{49}$$

$$-\frac{1}{2} A^{+\dot{b}} \bar{A}_{\dot{b}} (\int_0^s d\tau e^{\tau F})^{\dot{b}b}_{a\dot{a}} = f^{a\dot{a}} \mathcal{F}_{a\dot{a}}^\rho \rho + \frac{1}{2} A^{+\dot{b}} \bar{A}_{\dot{b}} (\int_0^s d\tau e^{\tau F})^{\dot{b}b}_{a\dot{a}} = f^{a\dot{a}} \mathcal{F}_{a\dot{a}}^\rho \rho.$$

It is easy to show that the solution to equation (48) is written as

$$f_{a\dot{a}} = -A^+ A^{\delta\dot{\delta}} N^{\delta\dot{\delta}} (s) \bar{A}_{\dot{\delta}} - \bar{A}^+ N^{\dot{\delta}\delta} (s) A^- \tag{50}$$
where the functions $N(N, \bar{N}), \bar{N}(N, \bar{N})$ are listed in the Appendix B. Solution of the equation (49) has the form

$$
\Omega(s) = sW\bar{W} + A^+\alpha\Omega_\alpha(s) + \bar{A}^+\bar{\alpha}\bar{\Omega}_{\bar{\alpha}}(s) + (A^+)^2\Psi(-2)(s) + (\bar{A}^+)^2\bar{\Psi}(-2)(s) + A^+\alpha\bar{A}^+\bar{\alpha}\bar{\Psi}(-2)(s) \tag{51}
$$

We point out that this solution is a finite order polynomials in powers of the Grassmann elements $A^\pm, \bar{A}^\pm$. The coefficients $\Omega_\alpha(s), \bar{\Omega}_{\bar{\alpha}}(s), \Psi(-2)(s), \bar{\Psi}(-2)(s), \bar{\Psi}(-2)(\bar{\alpha})$ are given in the Appendix B.

Now it is useful to write the last exponential in (47) in the form

$$
e^{-s(A^+D^+ + A^+\bar{D}^-)} = 1 + a^+\alpha D^- + \bar{a}^+\bar{\alpha} \bar{D}^- + f^{+2}(D^-)^2 + f^{+2}(\bar{D}^-)^2 + f^{+2}\bar{\alpha}D^-\bar{D}^- \tag{52}
$$

The coefficients of this expansion can be exactly found and are given in the Appendix B. We point out especially that $\Omega^+4(s) \sim (A^+)^4$. The integrand of the kernel (41) is represented as a product of the Schwinger type kernel $e^{-s\frac{1}{2}D^\alpha D_{\alpha\bar{\alpha}}} + \text{terms of expansion over powers in } D^-$ as well as of expansion $e^{-f_{\alpha\bar{\alpha}}D^{\alpha\bar{\alpha}}}, e^{-\Omega(s)}$ over powers of Grassmann variables $A^+, \bar{A}^+$. It leads to

$$
K(s) = \int \frac{d^4p}{(2\pi)^4} d^8\psi e^{-s\frac{1}{2}X^{\alpha\alpha}X_{\alpha\alpha}} \{ 1 + \frac{1}{2} f_{\alpha\bar{\alpha}}(s)X^{\alpha\bar{\alpha}} f_{\beta\bar{\beta}}(s)X^{\beta\bar{\beta}} + \ldots \}
$$

$$
\times e^{-\Omega(s)} \{ 1 + \ldots + \Omega^+4(s)(\psi^-)^4 \} (\psi^+)^4 \times 1 \tag{53}
$$

This very complicated expression can be significantly simplified using remarkable properties of the Berezin integral. Only last term in the last braces given a nontrivial contribution. Since $\Omega^+4 \sim (A^+)^4$ and all $f_{\alpha\bar{\alpha}}, \Omega$ are constructed from the elements $A^+, \bar{A}^+$ we must omit in the above expression for $K(s)$ all this functions except $e^{-sW\bar{W}}$. As a result we have

$$
K(s) = \int \frac{d^4p}{(2\pi)^4} K_{Sch}(s)e^{-sW\bar{W}}\Omega^+4(s) \tag{54}
$$

Last step of the consideration is computing the Schwinger type kernel for the operator $\Box_{\text{cov}}(X_m) = \frac{1}{2}X^{\alpha\alpha}X_{\alpha\alpha}, K_{Sch}(s) = \int \frac{d^4p}{(2\pi)^4} \cdot e^{-s\Box_{\text{cov}}(X_m)}$ where the operators $X_{\alpha\bar{\alpha}}$ are given in (55). Such a computation is standard now (see e.g. (25) for details). We write down only the final result

$$
K_{Sch}(s) = \frac{i}{(4\pi)^2} \frac{s^2(N^2 - \bar{N}^2)}{\cosh(sN) - \cosh(s\bar{N})} \tag{55}
$$

Here the value $N = \sqrt{-\frac{1}{2}D^4W^2}$ can be expressed in terms of the two invariants of the Abelian vector field $F = \frac{1}{4}F_{mn}F_{mn}$ and $G = \frac{1}{4}F_{\bar{m}\bar{n}}F_{\bar{m}\bar{n}}$ as $N = \sqrt{2(\mathcal{F} + i\mathcal{G})}$. In context of $N = 4$ SYM theory this kernel has been found in Eqs. (31), (33) on the base of various approaches. Here, we derived the kernel (53) completely in terms of $N = 2$ harmonic superspace.

---

8We note that taking the operation $tr$ in the Lorentz indices is trivial due to the identity

$$
N_{\alpha}^\beta N_{\bar{\beta}} \bar{\beta} = -\frac{1}{4}D^-\alpha D^+\beta WD_{\beta}D^+\delta W = -\frac{1}{8}D_{\alpha}^+D^+\beta D^-\bar{D}_{\beta}D^+\delta W^2 = -\frac{1}{2}\delta_\alpha^\beta(s)(D)^4W^2 = \delta_\alpha^\beta N^2
$$
6 Effective action and its spinor covariant derivative expansion

In the previous section we developed the proper-time techniques in the harmonic superspace. Now we apply this techniques to construction of the effective action. Effective action is written in the form (38) where the heat kernel at coïdent points is given by (41). We apply the decomposition (54), Eq. (55) for $K_{Sch}(s)$, Eq. (B.13) for $\Omega^{(+)}$ and take into account: (i) Eq. (41) already contains $(\psi^+)^4$ and hence we can use immediately $\int d^4\psi^+ (\psi^+)^4 = 1$, (ii) to saturate the integration over $\psi^-$ it is sufficient to keep in Eq. (52) only the last term $\Omega^{(+)4}(\psi^-)^4$. (iii) Since $\Omega^{(+)4} \sim (A^+)^4$ (B.13) we must omit in (47) all terms which dependent on $A^+$. It leads to the effective action in the final form

$$\Gamma = \frac{1}{(4\pi)^2} \int d\zeta d(-4) du \int_0^\infty ds \frac{s^2(N^2 - \bar{N}^2)}{\cosh(sN) - \cosh(s\bar{N})}$$

$$\times \frac{1}{16}(D^+W)^2(\bar{D}^+\bar{W})^2 \cosh(sN) - 1 \cosh(s\bar{N}) - 1$$

One can show that the integrand in (56) can be expended in power series in the quantities $s^2N^2, s^2\bar{N}^2$. After change of proper time $s$ to $s' = s2W\bar{W}$ we get the expansion in powers of $s^2/(4W^2\bar{W})$ and their conjugate. Besides, we point out, since the integrand of (56) is already $\sim (A^+)^4$ we can change in each term of expansion the quantities $N^2, \bar{N}^2$ by superconformal invariants $\Psi^2$ and $\bar{\Psi}^2$ expressing these quantities from

$$\Psi^2 = \frac{1}{W^2} D^4\ln W = \frac{1}{2W^2} \left\{ N^2_{\alpha\beta} N_{\alpha\beta} W^2 + 4A^+_{\alpha\beta} A^-_{\alpha\beta} W^3 + 3(A^+)^2(A^-)^2 W^4 \right\}$$

and its conjugate. After that, one can show that each term of the expansion can be rewritten as an integral over general $N = 2$ superspace.

The expression (56) at vanishing hypermultiplets has been obtained in Refs. [5], [33] by other methods. Hypermultiplet dependent effective action was derived in [26] in terms $N = 1$ superfields, its transformation to $N = 2$ harmonic form has been done in [26] on the base of a heuristic prescription how to reconstruct the effective action, given in terms of $N = 1$ superfields, in manifest $N = 2$ supersymmetric form. Now we justify the prescription used in [26] and derive the hypermultiplet depended effective action (56) completely in terms of the harmonic superspace.

Now we give a few first terms of the effective action expansion (56) in a power series in $N^2, \bar{N}^2$ and compare them with the results of $N = 1$ superspace calculations under the special prescription about reconstruction of manifest $N = 2$ supersymmetric form of the effective written in terms of $N = 1$ superfields. It was done in the work [26] on the base of a spinor covariant derivative expansion of the effective action. We will see that spinor derivative expansion of (56) actually reproduces the expansion in [26]. Each term of the spinor covariant derivative expansion of the effective action contains a definite power of the Abelian strength $F_{mn}$. Such an expansion allows to extract an explicitly dependence of $q^- q^+$. Since $N^2, \bar{N}^2$ includes the spinor covariant derivatives of superstrengths (see [42], this is just expression in spinor covariant derivatives of $W, \bar{W}$ . We use the expansions of $K_{Sch}(s)$ and...
$\Omega^+ (s)$. It leads to

$$\Gamma = \frac{1}{(8\pi)^2} \int d\zeta (-4) du \frac{1}{16} \frac{D^+ W D^+ W \bar{D}^+ \bar{W} D^+ \bar{W}}{W^2} \frac{1}{(1 - (-2q^{-q^+})^2)}$$

$$\times \{1 + \frac{s^4}{2 \cdot 5!} D^4 W^2 \bar{D}^4 \bar{W}^2 + \ldots\}.$$ 

Leading low-energy correction corresponds to $F^4$-term. Performing the integral over $s$ we have

$$\Gamma_{F^4} = \frac{1}{(4\pi)^2} \int d\zeta (-4) du \frac{1}{16} \frac{D^+ W D^+ W \bar{D}^+ \bar{W} D^+ \bar{W}}{W^2} \frac{1}{(1 - (-2q^{-q^+})^2)}$$

$$= \sum_{k=0}^{\infty} \frac{1}{(4\pi)^2} \int d\zeta (-4) du \frac{1}{16} \frac{D^+ W \ldots \bar{D}^+ \bar{W}}{(W W)^{k+2}} (k+1)(-2q^{-q^+})^k$$

$$= \frac{1}{(4\pi)^2} \int d\zeta (-4) du \frac{1}{16} \{D^{+2} \ln W \bar{D}^{+2} \ln \bar{W} + \sum_{k=1}^{\infty} \frac{1}{k^2(k+1)} D^{+2} \frac{1}{W^k} \bar{D}^{+2} \frac{1}{W^k} (-2q^{-q^+})^k\},$$

$$\Gamma_{F^4} = \frac{1}{(4\pi)^2} \int d^{12}z \{\ln W \ln \bar{W} + \sum_{k=1}^{\infty} \frac{1}{k^2(k+1)} (-2q^{-q^+})^k\}. \quad (59)$$

That exactly coincides with the earlier results$^9$ $^{[23, 24, 26]}$.

$$\Gamma_{F^4} = \frac{1}{(4\pi)^2} \int d^{12}z \{\ln W \ln \bar{W} + \mathrm{Li}_2(X) + \ln(1-X) - \frac{1}{X} \ln(1-X)\},$$

where $X = \frac{-2q^a q_{ai}}{WW}$ and $\mathrm{Li}_2(X)$ is the Euler dilogarithm function.

Next-to-leading correction corresponds to $F^8$-term$^{10}$. We have

$$\Gamma_{F^8} = \sum_{k=0}^{\infty} \frac{1}{2(4\pi)^2} \int d\zeta (-4) du \frac{1}{16} \frac{D^+ W \ldots \bar{D}^+ \bar{W}}{(W W)^6} D^4 W^2 \bar{D}^4 \bar{W}^4$$

$$= \frac{1}{5!} (k+1)(k+2)(k+3)(k+4)(k+5)(-2q^{-q^+})^k$$

$$\times \frac{1}{(W W)^{k+6}} D^+ W D^+ W D^+ W D^+ W D^+ W D^+ W D^+ W D^+ W D^+ W D^+ W \ldots$$

$$\frac{1}{(W W)^{k+4}} \frac{1}{W^{k+4}} D^{+2} \frac{1}{W^k} \bar{D}^{+2} \frac{1}{W^k} (-2q^{-q^+})^k$$

Using the transformations

$$D^4 W^2 = 4D^{+\alpha} \bar{D}^{-\beta} W D^{+\alpha} \bar{D}^{-\beta} W,$$

and

$$D^{+\alpha} \bar{D}^{-\beta} (D^{+\delta} W D^{+\delta} \bar{W} D^{-\beta} \bar{W} D^{-\beta} \bar{W}) = 2D^{+\delta} W D^{+\delta} \bar{W} D^{+\delta} \bar{W} D^{-\beta} W D^{-\beta} \bar{W} D^{-\beta} \bar{W}$$

ones get the chain of identities

$^9$We use $\int d\zeta (-4)(D^+)^4 = \int d^{12}z$ and $\int du = 1$ since in the central basis of the $\mathcal{N} = 2$ harmonic superspace the hypermultiplet superfields are expressed on shell as $q^{a\alpha} = q^{ia} u^{i+}$ and $\mathcal{W}$ is harmonic independent.

$^{10}$No $F^6$ quantum correction occurs in the one-loop effective action for $\mathcal{N} = 4$ SYM $^{[34, 5]}$. 

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21
\[= -2D^+ D^+ \left\{ \frac{1}{(k+2)(k+3)} D^+ D^+ \frac{1}{\mathcal{W}^{k+2}} D^- D^- \ln \mathcal{W} \right\} \]
\[= -2D^+ D^+ \left\{ \frac{1}{(k+2)(k+3)} \frac{1}{\mathcal{W}^{k+2}} D^4 \ln \mathcal{W} \right\}.\]

Similar transformations are done for the complex conjugate term. After restoration of the full measure \(d^{12}z = d\zeta \zeta^{(-4)}(D^+)^4\) we have the factor
\[\frac{1}{(k+2)^2(k+3)^2} \frac{1}{\mathcal{W}^2} D^4 \ln \mathcal{W} \frac{1}{\mathcal{W}^{2}} \bar{D}^4 \ln \mathcal{W} \left( \frac{-2q_a q_{a+}^+}{\mathcal{W} \bar{\mathcal{W}}} \right)^k.\]

This allows to write \(\Gamma_{FS}\) in the form \[26\]:
\[\Gamma_{FS} = \frac{1}{2(4\pi)^{25/2}} \int d^{12}z \sum_{k=0}^{\infty} \frac{(k+1)(k+4)(k+5)}{(k+2)(k+3)} \psi_2 \bar{\psi}_2 \left( \frac{-2q_a q_{a+}^+}{\mathcal{W} \bar{\mathcal{W}}} \right)^k.\]  \(61\)

It coincides with result of \(\mathcal{N} = 1\) calculations derived on the base of above prescription \[26\].

The analogous consideration allows in principle to get any term \(\Gamma_{FS}\) of derivative expansion of the effective action \(56\). We pay attention that the integrals over analytic subspace are transformed to the integrals over full \(\mathcal{N} = 2\) superspace in each term of expansion.

7 Summary

We have studied the problem of one-loop low-energy effective action in \(\mathcal{N} = 4\), SU(2) SYM theory. The theory is formulated in \(\mathcal{N} = 2\) harmonic superspace and possesses the manifest off-shell \(\mathcal{N} = 2\) supersymmetry and extra on-shell hidden \(\mathcal{N} = 2\) supersymmetry. We developed a new approach to derivations the effective action depending on all fields of \(\mathcal{N} = 4\) vector multiplet keeping the manifest \(\mathcal{N} = 2\) supersymmetry on all steps of calculations.

From \(\mathcal{N} = 2\) supersymmetric point of view, the effective action under consideration depends on fields of \(\mathcal{N} = 2\) vector multiplet and hypermultiplet and corresponds to \(\mathcal{N} = 2\) quantum gauge theory on mixed branch. The theory is quantized in framework of \(\mathcal{N} = 2\) background field method which allows to obtain the effective action in manifestly gauge invariant form. We carried out the calculations of effective action in low-energy approximation assuming the \(\mathcal{N} = 2\) superstrengths \(\mathcal{W}, \bar{\mathcal{W}}\) are on-shell and space-time independent and the hypermultiplet superfields \(q^a\) are constant. The effective action is given by integral over analytic subspace of harmonic superspace of function depending on \(\mathcal{W}, \bar{\mathcal{W}}\), their spinor covariant derivatives and hypermultiplet superfields. This dependence is exactly found under low-energy assumption. The effective action obtained is extension of the results obtained in the first papers \[23, 24\] where only lowest powers of the spinor covariant derivatives of superfield strengths have been taken into account. We also check and justify the results of Ref. \[26\] where the one-loop effective action in hypermultiplet sector has been found in terms of \(\mathcal{N} = 1\) superfields with help of special gauge fixing and some heuristic prescription.
about reconstruction of manifestly $\mathcal{N} = 2$ supersymmetric form of effective action. In the given work this problem is solved automatically.

We have developed a general method for derivations the one-loop, low-energy effective action of the theory under consideration completely in terms of $\mathcal{N} = 2$ harmonic superspace. Starting point of the consideration is a sequence of covariant harmonic supergraphs with arbitrary number of external hypermultiplet legs. Each of such supergraphs is written as an integral over analytic subspace and all contributions are summed up. The result is given by expression (36). This expression is analyzed on the base of proper-time method and techniques of operator symbols (the basic aspects of this techniques in $\mathcal{N} = 2$ harmonic superspace are developed in Section 5) using the key relation (47) (this new relation is derived in Appendix B). As a result we obtain the final expression (56) for the effective action.

At present there are at least two vast open problems concerning the hypermultiplet dependence of effective action in $\mathcal{N} = 4$ SYM theory. First, this is a problem of effective action where the spinor covariant derivatives of hypermultiplet superfields are not vanishing. In this case, the low-energy effective action could be constructed in form of expansion in spinor derivative of superstrengths and hypermultiplet superfields. In particular, such an expansion would help to clarify an answer the question if the effective action is invariant under hidden on-shell $\mathcal{N} = 2$ supersymmetry like the classical action or no. Second, this is a problem of hypermultiplet dependence of effective action at higher loops. We hope that the methods developed in this work will be useful for study both of above problems.

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Appendix A. Useful representation of Baker - Campbell - Hausdorff formula

Heat kernel associated with the operator $\mathcal{O}$ is given as matrix element of the operator $e^{\mathcal{O}}$. In many cases the operator $\mathcal{O}$ is linear combination of basis operators which form some Lie (super)algebra. Then evaluation of the heat kernel is simplified drastically on the base of Baker-Campbell-Hausdorf (BCH) fromula$^{11}$. This formula states that an exponent of sum of noncommuting operators $A$ and $B$ can be presented as a series in powers of commutator $[A, B]$.

$^{11}$N.P. would like to thank L.L. Salcedo for enlightening and useful discussions on various aspects of the BCH formula.
We derive another representation of BCH-formula
\[ e^{A+B} = e^{C_1 + C_2 + C_3 + \ldots} e^A, \]  
(A.1)

where the operators \( C_k \) are expressed through the commutators of the operators \( A \) and \( B \). We show here that the operators \( C_k \) can be defined such a way that each \( C_k \) has \( k \)-th power in operator \( B \) and infinite power in operator \( A \). This representation of BCH-formula (A.1) can be useful in the case when the operators \( A \) and \( B \) associated with some Lie (super)algebra what allows to summarize operator series \( C_1 + C_2 + C_3 + \ldots \) in explicit form.

We introduce in (A.1) the variables \( t \) as follows:
\[ e^{A} + tB = e^{tC_1 + t^2C_2 + t^3C_3 + \ldots} e^A. \]  
(A.2)

Next, let us define the function
\[ F = e^{A+tB} e^{-A} = e^{\sum_{k=1}^{\infty} t^k C_k(A,B)} \]  
(A.3)

and find the proper operators \( C_k \) in (A.2). It is obvious that at \( t = 0 \) the identity (A.2) takes place and at \( t = 1 \) ones get the initial relation (A.1). We calculate a logarithmic derivative of function (A.2) with respect \( t \) From one hand we have
\[ \dot{F} F^{-1} = \int_0^1 d\tau e^{\tau A + \tau t B} B e^{-\tau A - \tau t B}, \]  
(A.4)

while from the other hand
\[ F = e^{tC_1 + t^2C_2 + \ldots}, \quad \dot{F} F^{-1} = C_1 + 2tC_2 + \ldots \]  
(A.5)

If we put \( t = 0 \), we find
\[ C_1 = \int_0^1 d\tau e^{\tau A} B e^{-\tau A} = \int_0^1 d\tau B(\tau). \]  
(A.6)

In order to obtain \( C_2 \) we have to calculate first derivative with respect \( t \) of the logarithmic derivation:
\[ \frac{d}{dt}(\dot{F} F^{-1}) = \int_0^1 \int_0^1 d\tau' d\tau \{ e^{\tau' A + \tau' t B} (\tau B) e^{-\tau' A - \tau' t B} \} \{ e^{\tau A + \tau t B} B e^{-\tau A - \tau t B} \} \]  
(A.7)

\[ + \int_0^1 \int_0^1 d\tau' d\tau \{ e^{\tau A + \tau t B} B e^{-\tau A - \tau t B} \} \{ e^{\tau' A + \tau' t B} (-\tau B) e^{-\tau' A - \tau' t B} \}. \]

At point \( t = 0 \) we get the expression for \( C_2 \):
\[ 2C_2 = \int_0^1 \int_0^1 d\tau d\tau' \cdot \tau \cdot [B(\tau' \tau), B(\tau)]. \]  
(A.8)

Following the same way one can obtain all operators \( C_k \). For example, to get the operator \( C_3 \) we have to calculate second derivation of above logarithmic derivative. It leads to
\[ 6C_3 + [C_1, C_2] = \int_0^1 d\tau'' d\tau' d\tau \cdot \tau^2 \{ [B(\tau' \tau), [B(\tau'' \tau), B(\tau)]] - \tau'[B(\tau), [B(\tau'' \tau), B(\tau')]] \}. \]  
(A.9)
where $C_1$ and $C_2$ have been obtained above. The operator $C_4$ is found from

$$24C_4 + 6[C_1, C_3] + [C_1, [C_1, C_2]] = \int_0^1 d\tau'' d\tau' d\tau$$  \hspace{1cm} (A.10)

$$\times \tau^2 \{\tau' \tau[[B(\tau'' \tau'), B(\tau'), [B(\tau''), B(\tau)] - \tau'' \tau[B(\tau') [B(\tau''), B(\tau'')]] + \tau[B(\tau') [B(\tau''), B(\tau)]]}$$

$$\tau^2 \tau' \{\tau'' \tau[[B(\tau'' \tau'), B(\tau'')]B(\tau') [B(\tau'')] + \tau' \tau[[B(\tau'') B(\tau'), B(\tau') [B(\tau'')] + \tau[B(\tau'') B(\tau'), [B(\tau''), B(\tau)]]\}$$

And etc. Following the same procedure we can obtain, in principle, all terms of the BCH series in (A.1). Note that when Lie (super)algebra under consideration contains a central charge, this series terminates at some finite order and the representation (A.1) takes a simple form.

In the case under consideration, the problem consists in rewriting the exponent (47) of sum of the operators satisfying the algebraic relations (42), (44) as a product of the exponents of individual operators. This problem is solved in two steps. First, we take $A$ in (A.1) in the form $A^+ D^- + \tilde{A}^+ \tilde{D}^-$ and obtain for a $C_1$ a linear combination of operators $\frac{1}{2} D^{a\dot{a}} D_\alpha \overline{a\dot{a}}$, $f^{a\dot{a}}(A^\pm, \tilde{A}^\pm, N, \tilde{N}) D_{a\dot{a}}$ with some definite coefficients$^{12}$ plus some definite function of arguments $\mathcal{W}, \tilde{\mathcal{W}}, A^\pm, \tilde{A}^\pm, N, \tilde{N}$ as the central element. It means, that all other operators $C_2, C_3, ...$ will be proportional to the operator $D_{a\dot{a}}$ with some coefficient functions. Therefore, the series $C_1 + C_2 + C_3 + ...$ is reduced to summing up these coefficients functions what can be done in explicit form. The result looks $-s \frac{1}{2} D^{a\dot{a}} D_\alpha \overline{a\dot{a}}$ plus $f^{a\dot{a}}(A^\pm, \tilde{A}^\pm, N, \tilde{N}) D_{a\dot{a}}$ plus some central element. Second, we apply again the formula (A.1) to the expression $\exp(-s \frac{1}{2} D^{a\dot{a}} D_\alpha \overline{a\dot{a}} + f^{a\dot{a}} D_{a\dot{a}})$ and take $A$ as $-s \frac{1}{2} D^{a\dot{a}} D_\alpha \overline{a\dot{a}}$. All operators $C_k$ will be again proportional to the single operator $D_{a\dot{a}}$ and series of coefficient functions can be summed up in explicit form. As a result we get the right had side of expression (47). All coefficient functions are given in Appendix B. These coefficient functions can also be found from differential equations (48), (49). The results obtained both from BCH-formula and from the above differential equations coincide.

**Appendix B. Solution to equations (48), (49)**

The linear differential equations (48) and (49) for $f_{a\dot{a}}(s)$ and $\Omega(s)$ can be solved exactly in the form (50), (51). Coefficients of the power expansion $f_{a\dot{a}}(s)$ over basis of Grassmann elements $A^+, \tilde{A}^+$ are given as follows:

\[
N^{\delta\delta}_{a\dot{a}} = \int_0^s d\tau \left( e^{-\gamma N} \cdot e^{-s \tilde{N}} \right)_{a\dot{a}} = -\frac{e^{-sF} - 1}{NF} + \frac{e^{-s\tilde{N}} - 1}{NN}.
\]  \hspace{1cm} (B.1)

\[
\tilde{N}^{\delta\delta}_{a\dot{a}} = \int_0^s d\tau \left( e^{-\gamma N} \cdot e^{-s \tilde{N}} \right)_{a\dot{a}} = -\frac{e^{-sF} - 1}{NF} + \frac{e^{-s\tilde{N}} - 1}{NN}.
\]  \hspace{1cm} (B.2)

\[\text{12indeed, e.g. } [A^+ D^-, \frac{1}{2} D^{a\dot{a}} D_{a\dot{a}}] = -A^{+a} \tilde{A}^{-\dot{a}} D_{a\dot{a}}\]
Coefficients of the power expansion $\Omega(s)$ over basis of Grassmann elements $A^+_{\alpha}, \tilde{A}^+_{\bar{\alpha}}$ are given as follow:

\[
\Omega^\alpha_{\beta} = -\mathcal{W}\{\frac{e^{-sN} + sN - 1}{N^2}\}^\alpha_{\beta} A^+_{\beta}, \quad (B.3)
\]

\[
\tilde{\Omega}^\alpha_{\bar{\beta}} = -\mathcal{W}\tilde{A}^+_{\bar{\beta}}\{\frac{e^{-s\bar{N}} + s\bar{N} - 1}{\bar{N}^2}\}^\bar{\alpha}_{\bar{\beta}}, \quad (B.4)
\]

\[
\Psi^{(-2)} = \frac{1}{8}(A^-)^2\text{tr}\sum_{n=0}^{\infty}\sum_{p=1}^{n}\frac{s^{n+2}}{(n+2)!}C^p_n(-F)^{n-p}\{N^{p-1} - (-1)^n\bar{N}^{p-1}\} \quad (B.5)
\]

\[
\Psi^{(-2)} = (A^-)^2\{\frac{s^3}{6} + \frac{s^5}{5!}(N^2 + \bar{N}^2) + \ldots\},
\]

\[
\Psi^{\alpha\bar{\beta}} = \Psi^{\alpha\bar{\beta}} A^+_{\alpha} A^+_{\bar{\beta}}, \quad (B.6)
\]

\[
\psi_{\alpha\bar{\beta}}^{\alpha\bar{\beta}} = \frac{1}{NN(N-N)} + \frac{1}{NN}\{\frac{e^{-sN}}{N} - \frac{e^{-sN}}{N}\} + \frac{N^2 + \bar{N}^2}{2N^2(\bar{N}-N)}\frac{e^{s(N-N)}}{2N^2(\bar{N}+N)} + \frac{\bar{N}e^{s\bar{N}} - N^2e^{-sN}}{2N^2(\bar{N}+N)} \quad (B.7)
\]

\[
\psi_{\alpha\bar{\beta}}^{\alpha\bar{\beta}} = \frac{s^3}{3} + \frac{s^4}{8}(\bar{N}-N) + \frac{7s^5}{5!}(N^2 + \bar{N}^2) + \ldots.
\]

Coefficients of the derivative expansion $\exp\{-s(A^+D^- + \tilde{A}^+\tilde{D}^-)\}$ defined in (52) are given as follows:

\[
a^{+\alpha} = A^{+\beta}(\frac{e^{-sN}}{N} - \frac{1}{N})^\alpha_{\beta}, \quad \tilde{a}^{+\bar{\alpha}} = \tilde{A}^{+\bar{\beta}}(\frac{e^{-s\bar{N}}}{\bar{N}} - \frac{1}{\bar{N}})^{\bar{\alpha}}_{\bar{\beta}}, \quad (B.8)
\]

\[
f^{+2} = -\frac{1}{4}(A^{+})^2\text{tr}\{\frac{\cosh(sN) - 1}{N^2}\}, \quad \bar{f}^{+2} = -\frac{1}{4}(\tilde{A}^{+})^2\text{tr}\{\frac{\cosh(s\bar{N}) - 1}{\bar{N}^2}\}, \quad (B.9)
\]

\[
f^{+2\alpha\bar{\alpha}} = -A^{+\beta} A^{+\bar{\beta}}(\frac{e^{-sN}}{N} - \frac{1}{N})^{\alpha}_{\beta}(\frac{e^{-s\bar{N}}}{\bar{N}} - \frac{1}{\bar{N}})^{\bar{\alpha}}_{\bar{\beta}}, \quad (B.10)
\]

\[
\bar{\Xi}^{+3\alpha} = -\frac{1}{4}(A^{+})^2 A^{+\bar{\beta}}(\frac{e^{-sN}}{N})^{\alpha}_{\beta}\text{tr}\{\frac{\cosh(sN) - 1}{N^2}\}, \quad (B.11)
\]

\[
\bar{\Xi}^{+3\bar{\alpha}} = -\frac{1}{4}(\tilde{A}^{+})^2 A^{+\beta}(\frac{e^{-s\bar{N}}}{\bar{N}})^{\bar{\alpha}}_{\bar{\beta}}\text{tr}\{\frac{\cosh(s\bar{N}) - 1}{\bar{N}^2}\}, \quad (B.12)
\]

\[
\Omega^{+4} = -\frac{1}{16}(A^{+})^2(\tilde{A}^{+})^2\text{tr}\{\frac{\cosh(sN) - 1}{N^2}\}\text{tr}\{\frac{\cosh(s\bar{N}) - 1}{\bar{N}^2}\}, \quad (B.13)
\]
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