Tradeoffs in multiple-parameter postselection measurements

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Abstract. We analyze simultaneous estimations of multiple parameters in postselection measurements in terms of a tradeoff relation. A system, or a sensor, is characterized by a set of parameters, interacts with a measurement apparatus (MA), and then is postselected onto a final state. Measurements of the MA yield an estimation of the parameters. We first derive classical and quantum Cramér-Rao lower bounds and discuss the tradeoffs in the postselection measurements in general. Then, we discuss simultaneous measurements of phase and its fluctuation as an example. We found that the quantum Cramér-Rao bound can be attained and the quantum tradeoff can be saturated and thus all the parameters can, in principle, attain the ultimate precision simultaneously.

Keywords: Quantum metrology, Postselection measurements, Tradeoff, Phase and phase fluctuation

1. Introduction

Quantum metrology is promising: It is applicable for a wide range of fields, such as quantum sensing [1, 2], quantum imaging [3], and detecting gravitational waves [4, 5]. The estimation of a single parameter has already been established [6–15]. Therein, several studies demonstrated the quantum-enhanced metrology by using entangled resources [6–13], quantum memory [14], or teleportation [15]. Note, however, that it is often demanded to estimate multiple parameters simultaneously in many practical applications. For example, estimations of phases are always invariably affected by environmental noise and thus, a simultaneous measurement of the phase and its fluctuation is necessary. Such joint estimations have been discussed recently [16–23]. Furthermore, various practical applications have been discussed, including damping and temperature [24], two-phase spin rotation [25], waveform [26], operators [27, 28], phase-space displacements [29,30]. The estimations of multiple phases [31,32] and parameters in multidimensional fields [33] have been discussed, too.
Typically, a sensor characterized by multiple parameters will be measured by a set of POVMs to estimate the parameters. We term it as “direct sensing” because no ancillary systems are required. The limit of the estimation precision is imposed by quantum mechanics and bounded by a so-called quantum Cramér-Rao bound (QCRB) [34, 35]. For a single-parameter estimation, the QCRB can be achieved by projecting the states of the sensor on the basis determined by eigenvectors of its symmetric logarithmic derivative (SLD) operator [16, 19, 36]. However, for a multiple-parameter estimation, SLD operators of different parameters may not commute and thus the basis determined by them may not be orthogonal. Such a case leads to a tradeoff in the estimation of different parameters [16–19]. This tradeoff is a kind of “competition” among parameters [37]. Although several theoretical and numerical studies of the optimal POVMs that saturate the QCRB as well as the tradeoff relations have been reported [18, 38, 39], achieving the QCRB is still a challenging task in the multiple-parameter estimations.

In contrast to the “direct sensing,” an “indirect sensing” with postselection is also possible for parameter estimations, hereafter referred to as a postselection measurement. The sensor interacts with an ancillary system, referred to as a measurement apparatus (MA). After the interaction, the sensor will be postselected while the final MA state will be measured to provide the estimation. Postselection measurements have been used to estimate single parameters [40–61] and various methods have been proposed, including the optimal choices of the system and MA states [49, 50], entangled sensors [47, 48], photon recycling [51–53], nonclassical MA [54] to improve the precision. There are, however, ongoing debates over the merit of postselection measurements if it defeats the ultimate-limit precision or not. On the one hand, most studies consent that postselection measurements cannot provide any advantages for the estimations [40–48]. In fact, the maximum attainable precision obtained from direct sensing is an ultimate limit. Thus, postselection measurements alone cannot beat it, even though all the data (from the success and failure postselections) are taken into account [40, 43–46]. For example, Knee et al. [40, 42], Tanaka and Yamamoto [41] claimed that quantum Fisher information obtained from the success postselection alone could not overcome the QCRB. On the other hand, there are still some benefits to use of postselection measurements. There are reports on achieving Heisenberg scale of the single parameter estimation using postselection measurements [45–48, 55, 56]. There may also be some advantages in suppressing certain types of technical noise [48, 54, 55, 57], systematic errors [58]. Especially, Jordan et al. claimed that for some special technical noise, indirect sensing gives higher Fisher information than direct one [55]. Their work has been verified experimentally [59]. The same advantage has been found when the correlation between success and failed postselections were taken into account [60].

In this work, we discuss the multiple-parameter estimations under postselection measurements. We take into account both the success and failed postselections, hereafter referred to as success and failure modes for short, respectively. Our work is significantly different from Tanaka and Yamamoto’s work [41], in which only a single parameter estimation in a success mode is considered. We compare the total (the success and failure
modes) quantum Fisher information matrix (QFIM) obtained from the postselection measurement (named as $Q$) and that determined by the sensor state only ($H$): $H$ corresponds to the maximum information that the sensor has. We prove that $Q$ can be achieved and its maximum can reach $H$. Moreover, the postselection measurements may be possible to extract the maximum information of the sensor ($H$) more easily (via $Q$) than direct measurements. It is because extra freedom of the MA may be used to control the saturation of the QCRB. We illustrate our idea in the estimation of phase and phase fluctuation, as an example [16–18, 20, 62].

This paper is organized as follows. § 2 introduces a measurement framework with postselection, and we formulate the Cramér-Rao bounds and tradeoff relations. The application of our framework to the estimation of a phase and its fluctuation is presented in § 3. We summarize the results and point out the benefit of measurements with postselections in § 4.

2. Postselected estimation process

2.1. Measurement process

We consider a quantum channel $\Lambda_\phi$ that is characterized by a set of $d$ parameters given as a vector $\phi = \{\phi_1, \phi_2, ..., \phi_d\}$ that we want to estimate. We perform the following process: (i) A state $\rho_{s,i}$ of the sensor is prepared. (ii) It evolves to $\rho_{s,f} = \Lambda_\phi(\rho_{s,i})$ after passing through the quantum channel $\Lambda_\phi$, which now contains the information of $\phi$. (iii) The sensor-MA interaction leads to a joint state

$$\rho_{sm} = \hat{U}_{sm}(\rho_{s,f} \otimes |\xi\rangle\langle\xi|)\hat{U}_{sm}^\dagger,$$

where $|\xi\rangle$ is an initial MA state and

$$\hat{U}_{sm} = \exp(-ig\hat{A}_s \otimes \hat{M}_m)$$

is the unitary evolution caused by the sensor-MA interaction. $g$ is the interaction strength and can be controlled. $\hat{A}_s$ and $\hat{M}_m$ are operators on the sensor and the MA, respectively. The role of the interaction is to transfer the information of $\phi$ from the sensor to the MA. (iv) The sensor is postselected onto a final state $\rho_{s,f} = |\psi_{s,f}\rangle\langle\psi_{s,f}|$. We consider two modes, a success mode (✔) which corresponds to the successful postselection and a failure mode (✗) which contains all the failed postselections, i.e., $\mathcal{X} = \{X_1, X_2, \ldots\}$. The sample state of the $i$’th failed postselection is represented as $\rho_{s,f}^{X_i}$. The probability of successful postselection is

$$w^\dagger = \text{Tr}[(\rho_{s,f}^{\dagger} \otimes I_m)\rho_{sm}],$$

where $I_m$ is the identity matrix on the MA space. The probability of the $i$’th failed mode, $w^{X_i}$, is similarly defined and $w^\dagger + \sum_i w^{X_i} = 1$. The MA state after the postselection reads

$$\rho_m^{\dagger} = \frac{\text{Tr}[(\rho_{s,f}^{\dagger} \otimes I_m)\rho_{sm}]}{w^\dagger},$$

(4)
for the success mode, and
\[ \rho_{m}^{s} = \frac{\text{Tr}_s[(\rho_{s,t}^{s} \otimes I_m)\rho_{sm}]}{w^{s}}, \]  
(5)
for the failure mode. \( \rho_{s,t}^{s} + \sum_i \rho_{s,t}^{i} = I_s \) by definition of the probability, where \( I_s \) is the identity matrix on the sensor space. Assume that we employ a POVM \( \hat{\Pi}_k \) on the final MA state for getting a measurement result \( k \), the corresponding probability distributions are given as
\[ P(k|\checkmark) = \text{Tr}_m[\rho_{m}^{s} \hat{\Pi}_k], \text{ and } P(k|\checkmark) = \text{Tr}_m[\rho_{m}^{s} \hat{\Pi}_k]. \]  
(6)
\( \phi \) are estimated from these.

2.2. Cramér-Rao bounds

Let us define \( F \), a postselected classical Fisher information matrix (pCFIM) which is given by the probability distributions when measuring the final MA state in both success and failure modes. Its elements are given as
\[ F_{\alpha\beta} = w^{\checkmark} \sum_k \frac{1}{P(k|\checkmark)} \frac{\partial P(k|\checkmark)}{\partial \phi_\alpha} \frac{\partial P(k|\checkmark)}{\partial \phi_\beta} + \sum_i w^{\checkmark} \sum_i \frac{1}{P(l|\checkmark_i)} \frac{\partial P(l|\checkmark_i)}{\partial \phi_\alpha} \frac{\partial P(l|\checkmark_i)}{\partial \phi_\beta}. \]  
(7)
We also define a postselected quantum Fisher information matrix (pQFIM) whose elements are
\[ Q_{\alpha\beta} = w^{\checkmark} \text{Tr}_m\left[\rho_{m}^{s} \frac{\hat{L}_{\alpha}^{s} \hat{L}_{\beta}^{s} + \hat{L}_{\beta}^{s} \hat{L}_{\alpha}^{s}}{2}\right] + \sum_i w^{\checkmark} \text{Tr}_m\left[\rho_{m}^{i} \frac{\hat{L}_{\alpha}^{i} \hat{L}_{\beta}^{i} + \hat{L}_{\beta}^{i} \hat{L}_{\alpha}^{i}}{2}\right], \]  
(8)
associating with the final MA states \( \rho_{m}^{s} \) and \( \rho_{m}^{i} \). \( \hat{L}_{k}^{\alpha} \) are symmetric logarithmic derivatives (SLDs) defined as
\[ \hat{L}_{k}^{\alpha} \rho_{m}^{\checkmark} + \rho_{m}^{\checkmark} \hat{L}_{k}^{\alpha} = 2 \frac{\partial \rho_{m}^{\checkmark}}{\partial \phi_k} \]  
(9)
for \( \Delta = \{ \checkmark, \checkmark_i \} \) [35, 63]. It is also worth mentioning that the general quantum Fisher information matrix (QFIM) of the sensor have elements [35, 64]
\[ H_{\alpha\beta} = \text{Tr}_s\left[\rho_{s,i}^{s} \frac{\hat{L}_{\alpha}^{s} \hat{L}_{\beta}^{s} + \hat{L}_{\beta}^{s} \hat{L}_{\alpha}^{s}}{2}\right], \]  
(10)
where \( \hat{L}_{k}^{i} \)’s are similarly defined with \( \rho_{s,i}^{s} \) as Eq. (9). The diagonal elements of \( H \) provide the ultimate achievable precision of the estimations and are limited by quantum mechanics [63]. The off-diagonal elements of \( H \) provide the correlation between parameters. In this work, we compare \( Q \) and \( H \) via a quantum tradeoff as we will introduce below.

The precision of the estimation of \( \phi \) is evaluated by its covariance matrix \( C \), where
\[ C_{\alpha\beta} = \langle \phi_\alpha \phi_\beta \rangle - \langle \phi_\alpha \rangle \langle \phi_\beta \rangle, \text{ where } \langle \ast \rangle \text{ denotes the average of } \ast. \text{ The diagonal element } C_{\alpha\alpha} \text{ is the variance } (\delta \phi_\alpha)^2. \text{ We obtain the lower bounds for the covariance matrix as}
\[ M \cdot C \geq F^{-1} \geq Q^{-1} \geq H^{-1}, \]  
(11)
where $M$ is the number of repeated measurements. See the proof in Appendix A. The inequality $M \cdot C \geq F^{-1}$ is a postselected classical Cramér-Rao bound (pCCRB). It may be saturated by using a maximum likelihood estimator [65]: The saturation of pQCRB means that $M \cdot C = F^{-1}$. The inequality $F^{-1} \geq Q^{-1}$ is referred to a postselected quantum Cramér-Rao bound (pQCRB). POVMs that saturate pQCRB or achieve $F = Q$ are defined as optimal. Although optimal POVMs can be constructed for single parameter estimation [16, 19, 36], it is being actively studied for multi-parameter estimations. However, the pQCRB can be satisfied (or $Q$ can be achieved) when $[\hat{L}_\alpha^\Delta, \hat{L}_\beta^\Delta] = 0$ or a weaker condition $\text{Tr}_m[\rho_m^\Delta[\hat{L}_\alpha^\Delta, \hat{L}_\beta^\Delta]] = 0$ for $\Delta = \{\check{\Diamond}, \check{\heartsuit}\}$ is held [66]. The inequality $Q^{-1} \geq H^{-1}$ is due to the fact that the maximum of $Q$ depends on the choice of the sensor and the MA states. As we will see in § 3, the pQCRB is saturated: $Q$ can be obtained, and $Q = H$ is achieved for a proper choice of the sensor and the MA states.

### 2.3. Tradeoff relations

From the inequality $F \leq H$ in Eq. (11), we define a classical tradeoff $\text{Tr}[F H^{-1}]$, which quantifies how $F$ can be close to $H$. This tradeoff is a kind of “competition” between the estimations of parameters. Similarly, the inequality $Q \leq H$ leads to a tradeoff $\text{Tr}[Q H^{-1}]$, which we refer as a quantum tradeoff.

In § 3, we will investigate these tradeoff relations in more concrete examples.

### 3. Simultaneous estimation of phase and fluctuation

A phase fluctuation of a sensor, as well as its phase, may provide dynamical information of the environment surrounding the sensor. Therefore, the phase fluctuation can be a parameter of interest. There are several reports on the simultaneous phase and phase fluctuation estimations, but they focused only on the direct measurements [16–18, 20, 62], where the maximum classical tradeoff reached only one and could not reach the maximum of two: Note that the number of parameters, i.e., $d = 2$ [16–18].

We re-examine the tradeoff relations in the case of postselection measurements with two common MA’s: a continuous Gaussian MA and a discrete qubit MA. We achieve $\text{Tr}[Q H^{-1}] = 2$ in the practical sensor and initial MA states: We can extract all information from the sensor. For comparison, we also analyze the classical tradeoff $\text{Tr}[F H^{-1}]$ for these two cases.

We assume the initial sensor state is $\rho_{s,i} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. After passing through the phase ($\phi$) and phase fluctuation ($\Gamma$) channels, it evolves to [17]

$$\rho'_{s,i} = \frac{1}{2} \begin{pmatrix} 1 & e^{-i\phi - \Gamma^2} \\ e^{i\phi - \Gamma^2} & 1 \end{pmatrix}. \tag{12}$$

In this case, $\phi = \{\phi, \Gamma\}$. The QFIM, $H$, related to this state is a diagonal matrix and
can be calculated from Eq. (10), as follows

$$H = \begin{pmatrix} H_{\phi\phi} & H_{\phi\Gamma} \\ H_{\Gamma\phi} & H_{\Gamma\Gamma} \end{pmatrix} = \begin{pmatrix} e^{-2\Gamma^2} & 0 \\ 0 & 4\Gamma^2 \frac{e^{2\Gamma^2-1}}{1} \end{pmatrix}. \tag{13}$$

Note that $\hat{L}_\phi$ and $\hat{L}_\Gamma$ can be easily calculated according to Eq. (9) [35]. Although the classical tradeoff $[FH^{-1}]$ cannot reach two [16, 17], the quantum tradeoff $[QH^{-1}]$ can reach two, as we will show in the following sections: The ultimate precision in the simultaneous estimations of $\phi$ and $\Gamma$ can be, in principle, achieved in the case of postselection measurements.

3.1. Continuous Gaussian MA

We first consider the continuous Gaussian MA with a zero-mean in position $x$, or

$$|\xi\rangle = \int dx \frac{1}{(2\pi\sigma^2)^{1/4}} \exp\left(-\frac{x^2}{4\sigma^2}\right) |x\rangle,$$

where we assume $\hbar = 1$. This MA is widely used in weak measurement studies [67–73] and is a prototype for discussing the postselection measurements. $|\xi\rangle$ is equivalently given by

$$|\xi\rangle = \int dp \left(\frac{2\sigma^2}{\pi}\right)^{1/4} \exp(-p^2\sigma^2) |p\rangle,$$

where $p$ is momentum.

We consider the unitary evolution $\hat{U}_{sm} = \exp(-ig\sigma_z \hat{p})$ as a sensor-MA interaction, where $\hat{p}$ is momentum operator, $\hat{p} = -i\partial/\partial x$. Throughout this paper, we fix $g = \pi/2$ for simplicity. The postselected state is chosen to be $\rho_{s,f}' = |\psi_{s,f}'\rangle \langle \psi_{s,f}'|$, where $|\psi_{s,f}'\rangle = \sin(\gamma/2)|0\rangle + \cos(\gamma/2)|1\rangle$. The probability of successful postselection, $w'$, is calculated according to Eq. (3) where $I_m$ is replaced with $\int_{-\infty}^{\infty} dp' |p'\rangle \langle p'|$.

$$w' = \left(\frac{2\sigma}{\pi}\right)^{1/2} \int dp e^{-2p^2\sigma^2} \rho_{s,f}' \hat{U}_s \rho_{s,f}' \hat{U}_s^\dagger = \frac{1}{2} \left(1 + e^{-\Gamma^2 - \frac{2}{\pi}\sigma \cos \phi \sin \gamma}\right), \tag{15}$$

where $\hat{U}_s = \exp(-i\frac{\pi}{2}p\sigma_z)$. Note that $p$ in $\hat{U}_s$ is not an operator.

We next decompose the sensor state as

$$\rho_{s,i}' = \sum_k \lambda_k |\psi_k\rangle \langle \psi_k| = \lambda_1 |\psi_1\rangle \langle \psi_1| + \lambda_2 |\psi_2\rangle \langle \psi_2|,$$

where $\lambda_k$ and $|\psi_k\rangle$ $(k = 1,2)$, are eigenvalues and eigenstates of $\rho_{s,i}'$, respectively. Substituting $\rho_{s,i}'$ into Eq. (4) or Eq. (5), we have

$$\rho_{m}^\phi = \sum_{k=1}^{2} \lambda_k \hat{B}_k |\xi\rangle \langle \xi| \hat{B}_k^\dagger = \sum_{k=1}^{2} \lambda_k |\xi_k^\phi\rangle \langle \xi_k^\phi|,$$

where $\Delta = \{\chi, \xi\}$, $\hat{B}_k' = (\langle \psi_{s,f}'| \otimes \hat{I}_m) \hat{U}_{sm} (|\psi_k\rangle \otimes \hat{I}_m)$ and $\hat{B}_k' = (\langle \psi_{s,f}'| \otimes \hat{I}_m) \hat{U}_{sm} (|\psi_k\rangle \otimes \hat{I}_m)$ for the success and failure modes, respectively. Note that there is now only one failed mode $|\psi_{s,f}'\rangle$: $\langle \psi_{s,f}'| \langle \psi_{s,f}'| = 0$, or $|\psi_{s,f}'\rangle = \cos(\gamma/2)|0\rangle - \sin(\gamma/2)|1\rangle$. We also define
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\[ |\xi_k^\Delta\rangle \equiv \hat{B}_k^\Delta |\xi\rangle / \sqrt{w_k^\Delta}. \]

We emphasize that, in general, \{\{\xi_k^\Delta\}\} are not orthogonal. In the case of \( \gamma = \pi/2 \) and \( \phi \to 0 \), however, we have \( \langle \xi_k^\Delta | \xi_2^\Delta \rangle = \langle \xi_2^\Delta | \xi_\delta \rangle = 0 \) (see Appendix B). We define \( N_k^\Delta \equiv \langle \xi_k^\Delta | \xi_k^\Delta \rangle, k = 1, 2. \)

It is well established that the pQFIM (\( Q \)) can be experimentally measured if \( \text{Tr}_m[\rho_m^{\Delta} | \hat{L}_\phi^\Delta, \hat{L}_\Gamma^\Delta \rangle] = 0 (\Delta = \{\check{\gamma}, \check{\xi}\}) \) [66]. We first prove that \( \text{Tr}_m[\rho_m^{\Delta} | \hat{L}_\phi^\Delta, \hat{L}_\Gamma^\Delta \rangle] \) is given as

\[
\text{Tr}_m[\rho_m^{\Delta} | \hat{L}_\phi^\Delta, \hat{L}_\Gamma^\Delta \rangle] = 4 \sum_{k,l=1}^{2} \left( \frac{\lambda_k}{N_k^\Delta} - \frac{\lambda_l}{N_l^\Delta} \right) \frac{\langle \xi_k^\Delta | \partial_\phi \rho_m^{\Delta} | \xi_l^\Delta \rangle \langle \xi_l^\Delta | \partial_\Gamma \rho_m^{\Delta} | \xi_k^\Delta \rangle}{(\lambda_k N_k^\Delta + \lambda_l N_l^\Delta)^2}. \tag{18}
\]

in the Gaussian MA case. Then, show \( \text{Tr}_m[\rho_m^{\Delta} | \hat{L}_\phi^\Delta, \hat{L}_\Gamma^\Delta \rangle] = 0 \) when \( \gamma = \pi/2 \) and \( \phi \to 0 \). It implies that \( Q \) can be achieved or the pQCRB is satisfied for \( \gamma = \pi/2 \) and \( \phi \to 0 \). We then analytically obtain \( Q \) with the conditions that \( \gamma = \pi/2 \) and \( \phi \to 0 \), as follows.

\[
Q_{\alpha\beta}^\Delta = 4 w_\Delta^\phi \sum_{k,l=1}^{2} \left( \frac{\lambda_k}{N_k^\Delta} \right)^2 \frac{\langle \xi_k^\Delta | \partial_\alpha \rho_m^{\Delta} | \xi_l^\Delta \rangle \langle \xi_l^\Delta | \partial_\beta \rho_m^{\Delta} | \xi_k^\Delta \rangle}{(\lambda_k N_k^\Delta + \lambda_l N_l^\Delta)^2}. \tag{19}
\]

where we have defined \( Q_{\alpha\beta}^\Delta, \Delta = \{\check{\gamma}, \check{\xi}\} \) are the pQFIMs for the success and failure modes, respectively. See Appendix B for detailed calculations. Finally, we have the total pQFIM, \( Q = Q^\gamma + Q^\phi \). Straightforward calculations give its \( Q \) as

\[
Q_{\phi\phi} = \frac{e^{-\Gamma^2}}{\cosh \Gamma^2 + \coth \left( \frac{\pi^2}{8\sigma^2} \right) \sinh \Gamma^2}, \tag{20}
\]

\[
Q_{\Gamma\Gamma} = 2 \Gamma^2 \csch \Gamma^2 \csch \left( \Gamma^2 + \frac{\pi^2}{8\sigma^2} \right) \sinh \left( \frac{\pi^2}{8\sigma^2} \right), \tag{21}
\]

\[
Q_{\phi\Gamma} = Q_{\Gamma\phi} = 0. \tag{22}
\]

where \( \csch x = 2/(e^x - e^{-x}) \).

We now calculate the quantum tradeoff. By using Eqs. (20, 21) and Eq. (13), the quantum tradeoff, \( \text{Tr}[QH^{-1}] \), in the simultaneous estimation of \( \phi \) and \( \Gamma \) reads

\[
\text{Tr}[QH^{-1}] = \frac{Q_{\phi\phi}}{H_{\phi\phi}} + \frac{Q_{\Gamma\Gamma}}{H_{\Gamma\Gamma}} = 2 e^{-\Gamma^2} \csch \left( \Gamma^2 + \frac{\pi^2}{8\sigma^2} \right) \sinh \left( \frac{\pi^2}{8\sigma^2} \right). \tag{23}
\]

The result is summarized in Fig. 1(a). The quantum tradeoff can reach the maximum of two for small \( \sigma \) regardless of \( \Gamma \), which is consistent with \( \text{Tr}_m[\rho_m^{\Delta} | \hat{L}_\phi^\Delta, \hat{L}_\Gamma^\Delta \rangle] = 0 \). This result implies that \( Q = H \) for a suitable choice of the MA state via \( \sigma \). It also implies that we can simultaneously estimate both the phase and its fluctuation with the quantum-limit precision. For a large \( \sigma \), the quantum tradeoff decreases when \( \Gamma \) increases. We also observe that \( Q_{\phi\phi}/H_{\phi\phi} = Q_{\Gamma\Gamma}/H_{\Gamma\Gamma} \), or they can always attain the same precision: It is known as a Fisher-symmetric informationally complete (FSIC) [74].

Now we discuss the classical tradeoff, \( \text{Tr}[FH^{-1}] \). Let us measure the moment of the final MA state. (We note that this measurement is not an optimal one; thus its tradeoff may not meet the maximum of 2). We do not need to assume \( \gamma = \pi/2 \) in this case. The probability distribution in the success mode can be derived by substituting
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Figure 1. (Color online) Continuous Gaussian MA: (a) the quantum tradeoff $\text{Tr}[QH^{-1}]$ and (b) the classical tradeoff $\text{Tr}[F\tilde{H}^{-1}]$ as functions of $\sigma$ and $\Gamma$ are shown at $\gamma = \pi/2$. The inset shows the dependence of $F_{\phi\phi}/H_{\phi\phi}$ and $F_{\Gamma\Gamma}/H_{\Gamma\Gamma}$ as a function of $\Gamma$ at $(\gamma, \sigma) = (\pi/2, 0.2)$. Their sum is invariant of $\Gamma$.

Eq. (17) into Eq. (6), which yields

$$P(p|\check{X}) = \frac{\sqrt{2\sigma^2} e^{-2\sigma^2 p^2} \left( e^{\Gamma^2} + \sin \gamma \cos (\pi p + \phi) \right)}{e^{\Gamma^2} + e^{-\pi^2/8} \sin \gamma \cos \phi}. \quad (24)$$

The probability distribution in the failure mode, $P(p|\check{X})$, is calculated similarly. The pCFIM, $F$, related to these probabilities can be derived numerically from Eq. (7), where $\sum_k$ is replaced by $\int dp$. After that, we evaluate $\text{Tr}[F\tilde{H}^{-1}]$. The result is shown in Fig. 1(b) when $\gamma = \pi/2$. The result shows that the maximum value of the classical tradeoff is one at small $\sigma$ regardless of $\Gamma$. This result is in agreement with the previous reports [16, 17]. The inset of Fig. 1 shows the $\Gamma$ dependence of $F_{\phi\phi}/H_{\phi\phi}$ and $F_{\Gamma\Gamma}/H_{\Gamma\Gamma}$ at $\sigma = 0.2$. In the small phase fluctuation ($\Gamma$) region, the ratio $F_{\phi\phi}/H_{\phi\phi}$ reaches the maximum, which implies that the $\phi$-estimation is optimal while $\Gamma$ is not accurately estimated. In the large $\Gamma$ region, the precisions of both the estimations are equal. These observations are quite reasonable.

3.2. Qubit MA

We now consider a qubit MA [75]. In this case, we choose the initial MA state as $|\xi\rangle = \sin(\theta/2)|0\rangle + \cos(\theta/2)|1\rangle$. The pre- and postselected state of the sensor are the same as in § 3.1. We choose the evolution of $\hat{U}_{\text{sm}} = \exp(-ig\sigma_z \otimes |1\rangle\langle 1|)$ by the sensor-MA interaction which is a prototype one in modular-value-based measurements [61, 76–81] and which is easy to realize [79–81]. We take $g = \pi/2$ again. The probability of
successful (failed) postselection, \( w' (w^x) \), is calculated as

\[
w' = \frac{1}{2} (1 - e^{-r^2} \cos \theta \sin \gamma \cos \phi),
\]

\[
w^x = 1 - w'.
\]

according to Eq. (3). Similarly, \( \rho^\phi_m, L^\phi_{\Gamma} \) and \( L^\phi_{\Gamma} \) are easily calculated according to Eqs. (1, 2, 4, 5), see Appendix C. By substituting these into \( \text{Tr}[\rho^\phi_m [\hat{L}^\phi_{\phi}, \hat{L}^\phi_{\Gamma}]] \), we obtain the following:

\[
\text{Tr}[\rho^\phi_m [\hat{L}^\phi_{\phi}, \hat{L}^\phi_{\Gamma}]] = \frac{-4i \Gamma e^{r^2} \sin^2 \theta \sin^2 \gamma \cos \gamma}{(e^{r^2} - \cos \theta \sin \phi \cos \phi)^3},
\]

\[
\text{Tr}[\rho^\phi_m [\hat{L}^\phi_{\phi}, \hat{L}^\phi_{\Gamma}]] = \frac{4i \Gamma e^{r^2} \sin^2 \theta \sin^2 \gamma \cos \gamma}{(e^{r^2} + \cos \theta \sin \phi \cos \phi)^3}.
\]

Obviously, \( \text{Tr}[\rho^\phi_m [\hat{L}^\phi_{\phi}, \hat{L}^\phi_{\Gamma}]] = 0 \) when \( \gamma = \pi/2 \). So we choose \( \gamma = \pi/2 \) from now. Next, we calculate the pQFIM. By substituting \( \rho^\phi_m, L^\phi_{\phi} \) and \( L^\phi_{\Gamma} \) at \( \gamma = \pi/2 \) into Eq. (8), we obtain \( Q \) as

\[
Q_{\phi\phi} = \frac{1}{e^{2r^2} \csc^2 \theta - \cot^2 \theta},
\]

\[
Q_{\Gamma\Gamma} = \frac{2\Gamma^2 (1 + \cot \Gamma^2) \sin^2 \theta}{e^{2r^2} - \cos^2 \theta},
\]

\[
Q_{\phi\Gamma} = Q_{\Gamma\phi} = 0.
\]

We now calculate the quantum tradeoff as

\[
\text{Tr}[QH^{-1}] = \frac{Q_{\phi\phi}}{H_{\phi\phi}} + \frac{Q_{\Gamma\Gamma}}{H_{\Gamma\Gamma}} = \frac{2}{\csc^2 \theta - e^{-2r^2} \cot^2 \theta},
\]

and show it as a function of \( \theta \) and \( \Gamma \) in Fig. 2(a). For a suitable choice of MA state via \( \theta \) (\( \theta = \pi/2 \)), the quantum tradeoff \( \text{Tr}[QH^{-1}] \) can reach two regardless of \( \Gamma \). It implies that \( Q \) approaches \( H \): again, we can simultaneously estimate both the phase and its fluctuation with the quantum-limit precision. A Fisher-symmetric informationally complete (FSIC) is again observed as in the case of continuous Gaussian MA. When \( \theta \) deviates from the value \( \pi/2 \), the quantum tradeoff will decrease as \( \Gamma \) increases. This effect is the same as § 3.1 when \( \sigma \) is large.

Let us measure the final MA state. We choose a set of projective measurements onto the bases \( |\xi'\rangle = \cos(\theta'/2)|0\rangle + \sin(\theta'/2)|1\rangle \) and \( |\xi''\rangle = \sin(\theta'/2)|0\rangle - \cos(\theta'/2)|1\rangle \), where \( \langle \xi' | \xi'' \rangle = 0 \). Again, this measurement is not an optimal one; we choose these bases for the sake of illustration and comparison. The probabilities \( P(\xi' | \triangle) \) and \( P(\xi'' | \triangle) \), for \( \triangle = \{\check{\text{V}}, \text{X}\} \), can be derived by substituting \( \rho^\phi_m \) into Eq. (6). The pCFIM, \( F \), related to these probabilities can be calculated analytically from Eq. (7). Then, we calculate the classical tradeoff, \( \text{Tr}[FH^{-1}] \), as

\[
\text{Tr}[FH^{-1}] = \frac{1}{\csc^2 \theta - e^{-2r^2} \cot^2 \theta}.
\]

We show \( \text{Tr}[FH^{-1}] \) as functions of \( \theta \) and \( \Gamma \) in Fig. 2(b), which is exactly half of \( \text{Tr}[QH^{-1}] \). The maximum of \( \text{Tr}[FH^{-1}] \) is one: When the precision of one parameter
Figure 2. (Color online) Qubit MA: (a) the quantum tradeoff $\text{Tr}[QH^{-1}]$ and (b) the classical tradeoff $\text{Tr}[FH^{-1}]$ as functions of $\theta$ and $\Gamma$ are shown at $\gamma = \pi/2$. The inset shows the ratios $\frac{F_{\phi\phi}}{H_{\phi\phi}}$, $\frac{F_{\Gamma\Gamma}}{H_{\Gamma\Gamma}}$, and their sum.

reaches the quantum limit, that of the other parameter should reduce to zero. We illustrate this competition in the inset of Fig. 2 at $\Gamma = 0.1$ and $\theta = \pi/2$. This result is in agreement with previous studies [16, 17].

4. Conclusion and Discussion

We analyze simultaneous estimations of multiple parameters in postselection measurements in terms of tradeoff relations. We first derive classical and quantum Cramér-Rao lower bounds and discuss the tradeoffs in the postselection measurements in general. Then, we discuss simultaneous measurements of phase and its fluctuation with two measurement apparatus (MA): a continuous Gaussian MA and a discrete qubit MA. These examples confirm our general results. We found that the quantum Cramér-Rao lower bound can be achieved and the quantum tradeoff can be saturated and thus all the parameters can, in principle, attain the ultimate precision simultaneously.

To achieve the quantum Cramér-Rao bound we need to construct such SLDs that satisfy $\text{Tr}_m[\rho_m^\Delta[\hat{L}_\alpha, \hat{L}_\beta]] = 0$, $\Delta = \{\check{\bigvee}, \mathcal{X}\}$. The additional freedom of an MA provides the possibility that one can construct such SLDs in the case of postselection measurements.

We conclude our paper by pointing out that postselection measurements are useful for multiple-parameter estimations.

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Appendix A. Proof of Cramér-Rao bounds

Appendix A.1. Proof of $F \leq Q$

We will first prove $F \leq Q$ in Eq. (11) in the main text. To do this, let us recast $F$ and $Q$ as $F = F' + \sum_{k} F^k$ and $Q = Q' + \sum_{k} Q^k$ for the success and failure modes. Then, we can prove that $F' \leq Q'$ and $F^k \leq Q^k$. Let us discuss the first case in detail. The latter cases are calculated similarly.

To prove $F' \leq Q'$, we need to show that $u^{\dagger} [F'] u \leq u^{\dagger} [Q'] u$ for arbitrary $d$-dimensional real vectors $u$ [39]. We first rewrite the elements of $F'$ as $F^{\alpha\beta} = w \sum_{\mu} [F(\mu|\sqrt{\cdot})]_{\alpha\beta}$, where $[F(\mu|\sqrt{\cdot})]_{\alpha\beta}$ is defined by

$$[F(\mu|\sqrt{\cdot})]_{\alpha\beta} \equiv \frac{1}{P(\mu|\sqrt{\cdot})} \frac{\partial P(\mu|\sqrt{\cdot})}{\partial \phi_{\alpha}} \frac{\partial P(\mu|\sqrt{\cdot})}{\partial \phi_{\beta}},$$

where we have used $P(\mu|\sqrt{\cdot}) = \text{Tr} [\rho_{m}^\alpha \hat{\Pi}_{\mu}]$. Using the SLD, $\partial_{k} \rho_{m}^\alpha = (\hat{L}_{k} \rho_{m}^\alpha + \rho_{m}^\alpha \hat{L}_{k}^\dagger)/2$, where $\partial_{k} \equiv \partial/\partial \phi_{k}$ and $k = \alpha, \beta$, we have

$$\frac{\partial \text{Tr}[\rho_{m}^\alpha \hat{\Pi}_{\mu}^\dagger \rho_{m}^\beta \hat{\Pi}_{\mu}^\beta]}{\partial \phi_{k}} = \text{Tr}[(\frac{\partial \rho_{m}^\alpha \hat{\Pi}_{\mu}}{\partial \phi_{k}})]$$

$$= \frac{1}{2} \left\{ \text{Tr}[\hat{L}_{k} \rho_{m}^\alpha \hat{\Pi}_{\mu}] + \text{Tr}[\rho_{m}^\alpha \hat{L}_{k}^\dagger \hat{\Pi}_{\mu}] \right\}$$

$$= \text{Re} \left[ \text{Tr}[\rho_{m}^\alpha \hat{\Pi}_{\mu} \hat{L}_{k}] \right],$$

wherein the last equality, we have used the cyclic property of trace, such that

$$\text{Tr}[\rho_{m}^\alpha \hat{L}_{k} \hat{\Pi}_{\mu}^\dagger \rho_{m}^\beta \hat{\Pi}_{\mu}^\beta] = \text{Tr}[\hat{L}_{k} \hat{\Pi}_{\mu}^\dagger \hat{\Pi}_{\mu}^\beta \rho_{m}^\alpha] = [\text{Tr}[\rho_{m}^\alpha \hat{\Pi}_{\mu} \hat{L}_{k}]]^*$$

(A.2)

Following [38], we calculate

$$\sum_{\alpha\beta} u_{\alpha}[F(\mu|\sqrt{\cdot})]_{\alpha\beta} u_{\beta} = \frac{\text{Re} \text{Tr}[\rho_{m}^\alpha \hat{\Pi}_{\mu} \sum_{\alpha} u_{\alpha} \hat{L}_{\alpha}^\dagger]^2}{\text{Tr}[\rho_{m}^\alpha \hat{\Pi}_{\mu}]}$$

(a) $\leq \frac{\text{Tr}[\rho_{m}^\alpha \hat{\Pi}_{\mu} \sum_{\alpha} u_{\alpha} \hat{L}_{\alpha}^\dagger]^2}{\text{Tr}[\rho_{m}^\alpha \hat{\Pi}_{\mu}]}$

(b) $\leq \sum_{\alpha\beta} u_{\alpha} u_{\beta} \text{Tr}[\rho_{m}^\alpha \hat{L}_{\alpha}^\dagger \hat{\Pi}_{\mu} \rho_{m}^\beta \hat{L}_{\beta}^\dagger \hat{\Pi}_{\mu}^\dagger \hat{L}_{\beta}^\dagger \hat{\Pi}_{\mu}^\beta]$

$$\leq \frac{1}{2} \sum_{\alpha\beta} u_{\alpha} \text{Tr}[\rho_{m}^\alpha (\hat{L}_{\alpha}^\dagger \hat{\Pi}_{\mu} \hat{L}_{\beta}^\dagger \hat{\Pi}_{\mu}^\beta + \hat{L}_{\beta} \hat{\Pi}_{\mu}^d \hat{L}_{\beta}^\dagger \hat{\Pi}_{\mu}^\beta)] u_{\beta}$$

$$\leq \sum_{\alpha\beta} u_{\alpha} [Q(\mu|\sqrt{\cdot})]_{\alpha\beta} u_{\beta},$$

(A.4)

(a) We have used an inequality $[\text{Re}(z)]^2 \leq |z|^2$ for a complex number $z$. (b) We have applied the Cauchy-Swartz inequality $|\text{Tr}(\hat{A}^{\dagger} \hat{B})|^2 \leq \text{Tr}(\hat{A}^{\dagger} \hat{A}) \text{Tr}(\hat{B}^{\dagger} \hat{B})$, where
\[ \hat{A} \equiv \sqrt{\hat{\Pi}_\mu} \sqrt{\rho_m(\phi)} \] and \[ \hat{B} \equiv \sum_\alpha \sqrt{\hat{\Pi}_\mu} \hat{L}_\alpha \sqrt{\rho_m(\phi)}. \] (c) We have used the symmetry of the indices \( \alpha \) and \( \beta \). (d) We have defined the elements of \( Q' \) as \( [Q(\mu | \check{v})]_{\alpha \beta} \equiv \text{Tr}_m [\frac{L_\mu L_\alpha + L_\beta L_\mu}{2}] \). By taking the sum over \( \mu \) and multiplying by \( w \) both sides of Eq. (A.4), we obtain

\[ \sum_{\alpha \beta} u_\alpha [F'_{\alpha \beta}] u_\beta \leq \sum_{\alpha \beta} u_\alpha [Q'_{\alpha \beta}] u_\beta. \quad (A.5) \]

Or, we have \( F' \leq Q' \). Similarly, we can obtain, \( F^{x_i} \leq Q^{x_i} \), and finally, we have \( F \leq Q \). \( \square \)

**Appendix A.2. Proof of \( Q \leq H \)**

We next prove the inequality \( Q \leq H \). Let us denote \( Q_{sm} \) the Fisher information matrix obtained from the joint measurements of the sensor-MA. We note that \( Q_{sm} = H \) since no information can be gained or lose after the sensor-MA interaction. We, therefore, will prove that \( Q \leq Q_{sm} \). Remind that \( Q = Q' + \sum_i Q^{x_i} \). Assume that the existence of optimal measurement in the success mode is a set of POVMs \( \{\Pi_k\} \) for \( k \) outcomes in the MA and the corresponding probability is \( P(k | \check{v}) \), then we have

\[ Q'_{\alpha \beta} = w' \sum_k \frac{1}{P(k | \check{v})} \frac{\partial P(k | \check{v})}{\partial \phi_\alpha} \frac{\partial P(k | \check{v})}{\partial \phi_\beta}. \quad (A.6) \]

Similarly, for the failure mode, we have

\[ Q^{x_i}_{\alpha \beta} = w^{x_i} \sum_{l_i} \frac{1}{P(l_i | x_i)} \frac{\partial P(l_i | x_i)}{\partial \phi_\alpha} \frac{\partial P(l_i | x_i)}{\partial \phi_\beta}. \quad (A.7) \]

The corresponding optimal POVM measurement of the joint state is given by a set \( \{\rho_{s_f} \otimes \Pi_k\}, \{(\rho_{s_{x_i}}^{l_i}) \otimes \Pi_k^{x_i}\}, \{(\rho_{s_1})^{l_1} \otimes \Pi_1^{x_1}\}, ... \}. The probabilities are \( w' P(k | \check{v}), w^{x_i} P(l_i | x_i) \), for \( i = 1, 2, ... \). The Fisher information corresponds to the joint sensor-MA state is

\[
Q_{sm} = \sum_k \frac{1}{w P(k | \check{v})} \left[ \frac{\partial [w' P(k | \check{v})]}{\partial \phi_\alpha} \frac{\partial [w' P(k | \check{v})]}{\partial \phi_\beta} \right] \\
+ \sum_i \sum_{l_i} \frac{1}{w^{x_i} P(l_i | x_i)} \left[ \frac{\partial [w^{x_i} P(l_i | x_i)]}{\partial \phi_\alpha} \frac{\partial [w^{x_i} P(l_i | x_i)]}{\partial \phi_\beta} \right] \\
= \sum_k \frac{1}{w P(k | \check{v})} \left[ \frac{\partial [w' P(k | \check{v})]}{\partial \phi_\alpha} + w' \frac{\partial P(k | \check{v})}{\partial \phi_\alpha} \right] \left[ \frac{\partial [w' P(k | \check{v})]}{\partial \phi_\beta} + w' \frac{\partial P(k | \check{v})}{\partial \phi_\beta} \right] \\
+ \sum_i \sum_{l_i} \frac{1}{w^{x_i} P(l_i | x_i)} \left[ \frac{\partial [w^{x_i} P(l_i | x_i)]}{\partial \phi_\alpha} + w^{x_i} \frac{\partial P(l_i | x_i)}{\partial \phi_\alpha} \right] \left[ \frac{\partial [w^{x_i} P(l_i | x_i)]}{\partial \phi_\beta} + w^{x_i} \frac{\partial P(l_i | x_i)}{\partial \phi_\beta} \right] \\
= w' \sum_k \frac{\partial [w P(k | \check{v})]}{\partial \phi_\alpha} \frac{\partial [w P(k | \check{v})]}{\partial \phi_\beta} + \sum_i w^{x_i} \sum_{l_i} \frac{\partial [w^{x_i} P(l_i | x_i)]}{\partial \phi_\alpha} \frac{\partial [w^{x_i} P(l_i | x_i)]}{\partial \phi_\beta} + \mathcal{F} \\
\geq Q \quad (A.8) \]

\( \square \)
where $\mathcal{F}$ is given by
\[
\mathcal{F} = \frac{1}{w^r} \frac{\partial w^r}{\partial \phi_\alpha} \frac{\partial w^r}{\partial \phi_\beta} + \sum_i \frac{1}{w^i} \frac{\partial w^i}{\partial \phi_\alpha} \frac{\partial w^i}{\partial \phi_\beta} + \sum_k \left( \frac{\partial w^r}{\partial \phi_\alpha} \frac{\partial P(k|\sqrt{\lambda})}{\partial \phi_\beta} + \frac{\partial w^r}{\partial \phi_\beta} \frac{\partial P(k|\sqrt{\lambda})}{\partial \phi_\alpha} \right) + \sum_{i,l} \left( \frac{\partial w^i}{\partial \phi_\alpha} \frac{\partial P(l_i|x_i)}{\partial \phi_\beta} + \frac{\partial w^i}{\partial \phi_\beta} \frac{\partial P(l_i|x_i)}{\partial \phi_\alpha} \right). \tag{A.9}
\]

Notable that $\mathcal{F}$ is the classical Fisher information contributed by all the success and failure postselection probabilities, hence it is non-negative. As a result, we have $Q \leq H$. □

Appendix B. Continuous Gaussian MA

Appendix B.1. Calculation of $\text{Tr} [\rho [L_\alpha, L_\beta]]$ in general

Let us first calculate the SLD operator corresponding to an arbitrary $\rho$ with the following Lyapunov representation [35]
\[
\hat{L}_\alpha = 2 \int_0^\infty dt e^{-t\rho} \left( \partial_\alpha \rho \right) e^{-t\rho}, \tag{B.1}
\]
with $\rho = \sum_k \lambda_k |\xi_k\rangle \langle \xi_k|$, where $\langle \xi_k | \xi_l \rangle = 0$ for $k \neq l$, and $\langle \xi_k | \xi_k \rangle = N_k$. Evaluating the exponential $e^{-t\rho}$, we have
\[
e^{-t\rho} = \sum_k \frac{e^{-t\lambda_k N_k}}{N_k} |\xi_k\rangle \langle \xi_k|. \tag{B.2}
\]
Substituting Eq. (B.2) into Eq. (B.1), we obtain
\[
\hat{L}_\alpha = 2 \sum_{kl} \frac{1}{N_k N_l} \frac{\langle \xi_k | \partial_\alpha \rho | \xi_l \rangle}{\lambda_k N_k + \lambda_l N_l} |\xi_k\rangle \langle \xi_l|. \tag{B.3}
\]
We next evaluate the term $\hat{L}_\alpha \hat{L}_\beta$ as
\[
\hat{L}_\alpha \hat{L}_\beta = 4 \sum_{kl} \frac{1}{N_k N_l N_l'} \frac{\langle \xi_k | \partial_\alpha \rho | \xi_l \rangle}{\lambda_k N_k + \lambda_l N_l} \frac{\langle \xi_l | \partial_\beta \rho | \xi_l' \rangle}{\lambda_l N_l + \lambda_{l'} N_{l'}} |\xi_k\rangle \langle \xi_l| \langle \xi_l'|'. \tag{B.4}
\]
Using $\rho = \sum_n \lambda_n |\xi_n\rangle \langle \xi_n|$, we have
\[
\rho \hat{L}_\alpha \hat{L}_\beta = 4 \sum_{kl} \frac{\lambda_k}{N_k N_l N_l'} \frac{\langle \xi_k | \partial_\alpha \rho | \xi_l \rangle}{\lambda_k N_k + \lambda_l N_l} \frac{\langle \xi_l | \partial_\beta \rho | \xi_l' \rangle}{\lambda_l N_l + \lambda_{l'} N_{l'}} |\xi_k\rangle \langle \xi_l| \langle \xi_l'|'. \tag{B.5}
\]
Taking the trace, we obtain
\[
\text{Tr} [\rho \hat{L}_\alpha \hat{L}_\beta] = 4 \sum_{kl} \frac{\lambda_k}{N_l} \frac{\langle \xi_k | \partial_\alpha \rho | \xi_l \rangle \langle \xi_l | \partial_\beta \rho | \xi_k \rangle}{(\lambda_k N_k + \lambda_l N_l)^2}. \tag{B.6}
\]
Similarly, we have
\[
\text{Tr} [\rho \hat{L}_\beta \hat{L}_\alpha] = 4 \sum_{kl} \frac{\lambda_l}{N_k} \frac{\langle \xi_k | \partial_\alpha \rho | \xi_l \rangle \langle \xi_l | \partial_\beta \rho | \xi_k \rangle}{(\lambda_k N_k + \lambda_l N_l)^2}. \tag{B.7}
\]
Finally, we obtain
\[
\text{Tr} [\rho [\hat{L}_\alpha, \hat{L}_\beta]] = 4 \sum_{kl} \left( \frac{\lambda_k}{N_l} - \frac{\lambda_l}{N_k} \right) \frac{\langle \xi_k | \partial_\alpha \rho | \xi_l \rangle \langle \xi_l | \partial_\beta \rho | \xi_k \rangle}{(\lambda_k N_k + \lambda_l N_l)^2}. \tag{B.8}
\]
where

$$\partial_\alpha \rho = \sum_n \left( \partial_\alpha \lambda_n \langle \xi_n | + \lambda_n | \partial_\alpha \xi_n \rangle \langle \xi_n | + \lambda_n | \xi_n \rangle \langle \partial_\alpha \xi_n | \right).$$

(B.9)

### Appendix B.2. Calculation of $\text{Tr}[\rho[L_\alpha, L_\beta]]$ in our example

Let us now apply the above calculations to our case of the continuous Gaussian MA where $\text{Tr}[\rho[L_\alpha, L_\beta]]$ now becomes $\text{Tr}[\rho^\alpha_m[L_\phi^\alpha, L_1^\phi]]$ for $\Delta = \{\check{\sigma}, \check{\sigma}\}$. We start from the sensor state given in Eq. (12) and decompose it into eigenvalues and eigenstates as in Eq. (16) in the main text. We obtain

$$\lambda_1 = \frac{1}{2} (1 - e^{-r^2}) \quad \text{and} \quad \lambda_2 = \frac{1}{2} (e^{-r^2} + 1),$$

(B.10)

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -e^{-i\phi} \\ 1 \end{pmatrix} \quad \text{and} \quad |\psi_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\phi} \\ 1 \end{pmatrix}.$$  

(B.11)

We next calculate $|\xi^\Delta_k\rangle$ which is defined by:

$$|\xi^\Delta_k\rangle = \frac{\hat{B}^\Delta_k}{\sqrt{w^\Delta}} |\xi\rangle,$$  

(B.12)

where $\hat{B}^\sigma_k = (\langle \psi_{s,f}| \otimes \hat{I}_m) \hat{U}_{sm} (|\psi_k\rangle \otimes \hat{I}_m)$ and $\hat{B}^\check{\sigma}_k = (\langle \psi_{s,f}'| \otimes \hat{I}_m) \hat{U}_{sm} (|\psi_k\rangle \otimes \hat{I}_m)$. We show explicitly

$$\hat{B}^\sigma_k = \left( \langle \psi_{s,f}| \otimes \int dp |p\rangle \langle p| \right) e^{-i\sigma z_\Delta \hat{p}} \left( |\psi_k\rangle \otimes \int dp' |p'\rangle \langle p'| \right)$$

$$= \int dp \langle \psi_{s,f}| e^{-i\sigma z_\Delta \hat{p}} |\psi_k\rangle |p\rangle \langle p|.$$  

(B.13)

Then we have

$$\hat{B}^\sigma_1 = \int dp \left[ \frac{e^{-i(p_{s,f}^2+2\varphi_\Delta)/2} (e^{i(p_{s,f} \varphi_\Delta) \cos \frac{\gamma_\Delta}{2} - \sin \frac{\gamma_\Delta}{2}})}{\sqrt{2}} \right] |p\rangle \langle p|,$$  

(B.14)

$$\hat{B}^\sigma_2 = \int dp \left[ \frac{e^{-i(p_{s,f}^2+2\varphi_\Delta)/2} (e^{i(p_{s,f} \varphi_\Delta) \cos \frac{\gamma_\Delta}{2} + \sin \frac{\gamma_\Delta}{2}})}{\sqrt{2}} \right] |p\rangle \langle p|,$$  

(B.15)

$$\hat{B}^\check{\sigma}_1 = \int dp \left[ \frac{e^{-i(p_{s,f}^2+2\varphi_\Delta)/2} (\cos \frac{\gamma_\Delta}{2} + e^{i(p_{s,f} \varphi_\Delta) \sin \frac{\gamma_\Delta}{2}})}{\sqrt{2}} \right] |p\rangle \langle p|,$$  

(B.16)

$$\hat{B}^\check{\sigma}_2 = \int dp \left[ \frac{e^{-i(p_{s,f}^2+2\varphi_\Delta)/2} (\cos \frac{\gamma_\Delta}{2} - e^{i(p_{s,f} \varphi_\Delta) \sin \frac{\gamma_\Delta}{2}})}{\sqrt{2}} \right] |p\rangle \langle p|.$$  

(B.17)

Substituting into Eq. (B.12) we obtain

$$|\xi^\Delta_k\rangle = \left( \frac{2\sigma^2}{\pi} \right)^{1/4} \frac{1}{\sqrt{w^\Delta}} \int dp \, e^{-p^2\sigma^2 \hat{B}^\Delta_k} |p\rangle,$$  

(B.18)

where we have used $\langle p|\xi\rangle = (2\sigma^2/\pi)^{1/4} \exp(-p^2\sigma^2)$ and $\hat{B}^\Delta_k$ are given by [*] in Eqs. (B.14-B.17) above.
We calculate \( \langle \xi^\Delta_k | \xi^\Delta_l \rangle \). For \( \phi \to 0 \), we have
\[
\langle \xi^\Delta | \xi^\Delta \rangle = \frac{e^{r^2 + \frac{s^2}{8\sigma^2}} \cos \gamma}{e^{r^2 + \frac{s^2}{8\sigma^2}} + \sin \gamma}, \quad \text{and} \quad \langle \xi^\Delta | \xi^\Delta \rangle = \frac{-e^{r^2 + \frac{s^2}{8\sigma^2}} \cos \gamma}{e^{r^2 + \frac{s^2}{8\sigma^2}} - \sin \gamma}.
\]

Then, \( |\xi^\Delta_k \rangle \) and \( |\xi^\Delta_l \rangle \) are orthogonal when \( \gamma = \pi/2 \). We select \( \gamma = \pi/2 \) hereafter. We next calculate the normalized constants \( N^\Delta_k = \langle \xi^\Delta_k | \xi^\Delta_k \rangle \) which are given as
\[
N^\Delta_1 = 1 - \frac{1 + e^{r^2}}{1 + e^{r^2 + \frac{s^2}{8\sigma^2}}}, \quad \text{and} \quad N^\Delta_2 = 1 + \frac{-1 + e^{r^2}}{1 + e^{r^2 + \frac{s^2}{8\sigma^2}}}.
\]

Finally, we calculate Eq. (B.8) which is recast as
\[
\text{Tr} \left[ \rho_m \hat{L}_{\phi}, \hat{L}_{\Gamma} \right] = 4 \sum_{k,l=1}^{2} \left( \frac{\lambda_k}{N^\Delta_k} - \frac{\lambda_l}{N^\Delta_l} \right) \frac{\langle \xi^\Delta_k | \partial_{\phi} \rho^\Delta_m | \xi^\Delta_l \rangle \langle \xi^\Delta_l | \partial_{\Gamma} \rho^\Delta_m | \xi^\Delta_k \rangle}{(\lambda_k N^\Delta_k + \lambda_l N^\Delta_l)^2}.
\]

First we derive Eq. (B.9):
\[
\partial_{\phi} \rho^\Delta_m = \sum_{n=1}^{2} \left( \partial_{\phi} \lambda_n |\xi^\Delta_n \rangle \langle \xi^\Delta_n | + \lambda_n |\partial_{\phi} \xi^\Delta_n \rangle \langle \xi^\Delta_n | + \lambda_n |\xi^\Delta_n \rangle \langle \partial_{\phi} \xi^\Delta_n | \right).
\]

Next we calculate the term \( \langle \xi^\Delta_k | \partial_{\phi} \rho^\Delta_m | \xi^\Delta_l \rangle \) in Eq. (B.22):
\[
\langle \xi^\Delta_k | \partial_{\phi} \rho^\Delta_m | \xi^\Delta_l \rangle \quad \text{(B.24)}
\]
\[
= \sum_{n=1}^{2} \left( \langle \xi^\Delta_k | \partial_{\phi} \lambda_n |\xi^\Delta_n \rangle \langle \xi^\Delta_n | \xi^\Delta_l \rangle + \langle \xi^\Delta_k | \lambda_n |\partial_{\phi} \xi^\Delta_n \rangle \langle \xi^\Delta_n | \xi^\Delta_l \rangle + \langle \xi^\Delta_k | \lambda_n |\xi^\Delta_n \rangle \langle \partial_{\phi} \xi^\Delta_n | \xi^\Delta_l \rangle \right)
\]
\[
= \sum_{n=1}^{2} \left( \partial_{\phi} \lambda_n \langle \xi^\Delta_k | \xi^\Delta_n \rangle \langle \xi^\Delta_n | \xi^\Delta_l \rangle + \lambda_n \langle \xi^\Delta_k | \partial_{\phi} \xi^\Delta_n \rangle \langle \xi^\Delta_n | \xi^\Delta_l \rangle + \lambda_n \langle \xi^\Delta_k | \xi^\Delta_n \rangle \langle \partial_{\phi} \xi^\Delta_n | \xi^\Delta_l \rangle \right)
\]
\[
= \sum_{n=1}^{2} \left( \partial_{\phi} \lambda_n N^\Delta_k \delta_{k,n} N^\Delta_l \delta_{l,n} + \lambda_n \langle \xi^\Delta_k | \partial_{\phi} \xi^\Delta_n \rangle N^\Delta_l \delta_{l,n} + \lambda_n N^\Delta_k \delta_{k,n} \langle \partial_{\phi} \xi^\Delta_n | \xi^\Delta_l \rangle \right).
\]

Equation (B.22) is explicitly given
\[
\text{Tr} \left[ \rho_m \hat{L}_{\phi}, \hat{L}_{\Gamma} \right] = 4 \left( \frac{\lambda_1}{N^\Delta_1} - \frac{\lambda_2}{N^\Delta_2} \right) \times
\]
\[
\frac{(\lambda_1 N^\Delta_1 \langle \partial_{\phi} \xi^\Delta_1 | \xi^\Delta_2 \rangle + \lambda_2 N^\Delta_2 \langle \xi^\Delta_2 | \partial_{\phi} \xi^\Delta_1 \rangle) \langle \lambda_1 N^\Delta_1 \langle \xi^\Delta_1 | \partial_{\Gamma} \xi^\Delta_1 \rangle + \lambda_2 N^\Delta_2 \langle \xi^\Delta_1 | \partial_{\Gamma} \xi^\Delta_2 \rangle \rangle}{(\lambda_1 N^\Delta_1 + \lambda_2 N^\Delta_2)^2}
\]
\[
+ 4 \left( \frac{\lambda_2}{N^\Delta_1} - \frac{\lambda_1}{N^\Delta_2} \right) \times
\]
\[
\frac{(\lambda_1 N^\Delta_1 \langle \xi^\Delta_2 | \partial_{\phi} \xi^\Delta_1 \rangle + \lambda_2 N^\Delta_2 \langle \partial_{\phi} \xi^\Delta_2 | \xi^\Delta_1 \rangle) \langle \lambda_1 N^\Delta_1 \langle \partial_{\Gamma} \xi^\Delta_1 | \xi^\Delta_1 \rangle + \lambda_2 N^\Delta_2 \langle \xi^\Delta_1 | \partial_{\Gamma} \xi^\Delta_2 \rangle \rangle}{(\lambda_1 N^\Delta_1 + \lambda_2 N^\Delta_2)^2}
\]
Indeed, we also calculate all inner products $\langle \xi_m^\alpha | \partial_\alpha \xi_n^\alpha \rangle$ and their complex conjugations similar as we did in Eqs. (B.19 - B.21). We list them here:

$$
\langle \xi_1^\alpha | \partial_\alpha \xi_1^\alpha \rangle = \langle \partial_\alpha \xi_1^\alpha | \xi_1^\alpha \rangle^\dagger = -\frac{i}{2} \left( 1 - \frac{1 + e^{r^2}}{1 + e^{r^2 + \frac{x^2}{8a^2}}} \right)
$$

(B.26)

$$
\langle \xi_2^\alpha | \partial_\alpha \xi_2^\alpha \rangle = \langle \partial_\alpha \xi_2^\alpha | \xi_2^\alpha \rangle^\dagger = \frac{i}{2} \left( 1 - \frac{1 + e^{r^2}}{1 + e^{r^2 + \frac{x^2}{8a^2}}} \right)
$$

(B.27)

$$
\langle \xi_2^\alpha | \partial_\alpha \xi_1^\alpha \rangle = \langle \partial_\alpha \xi_1^\alpha | \xi_2^\alpha \rangle^\dagger = \frac{i}{2} \left( 1 + \frac{-1 + e^{r^2}}{1 + e^{r^2 + \frac{x^2}{8a^2}}} \right)
$$

(B.28)

$$
\langle \xi_2^\alpha | \partial_\alpha \xi_2^\alpha \rangle = \langle \partial_\alpha \xi_2^\alpha | \xi_2^\alpha \rangle^\dagger = -\frac{i}{2} \left( 1 + \frac{-1 + e^{r^2}}{1 + e^{r^2 + \frac{x^2}{8a^2}}} \right)
$$

(B.29)

$$
\langle \xi_1^\alpha | \partial_\alpha \xi_1^\alpha \rangle = \langle \partial_\alpha \xi_1^\alpha | \xi_1^\alpha \rangle^\dagger = \frac{-1 + e^{\frac{x^2}{8a^2}}}{(1 + e^{r^2 + \frac{x^2}{8a^2}})^2} e^{r^2} \Gamma
$$

(B.30)

$$
\langle \xi_1^\alpha | \partial_\alpha \xi_2^\alpha \rangle = \langle \partial_\alpha \xi_2^\alpha | \xi_1^\alpha \rangle^\dagger = 0
$$

(B.31)

$$
\langle \xi_2^\alpha | \partial_\alpha \xi_1^\alpha \rangle = \langle \partial_\alpha \xi_1^\alpha | \xi_2^\alpha \rangle^\dagger = 0
$$

(B.32)

$$
\langle \xi_2^\alpha | \partial_\alpha \xi_2^\alpha \rangle = \langle \partial_\alpha \xi_2^\alpha | \xi_2^\alpha \rangle^\dagger = \frac{1 + e^{\frac{x^2}{8a^2}}}{(1 + e^{r^2 + \frac{x^2}{8a^2}})^2} e^{r^2} \Gamma
$$

(B.33)

$$
\langle \xi_1^\alpha | \partial_\alpha \xi_1^\alpha \rangle = \langle \partial_\alpha \xi_1^\alpha | \xi_1^\alpha \rangle^\dagger = -\frac{i}{2} \left( 1 - \frac{1 + e^{r^2}}{1 - e^{r^2 + \frac{x^2}{8a^2}}} \right)
$$

(B.34)

$$
\langle \xi_1^\alpha | \partial_\alpha \xi_2^\alpha \rangle = \langle \partial_\alpha \xi_2^\alpha | \xi_1^\alpha \rangle^\dagger = \frac{i}{2} \left( 1 - \frac{1 + e^{r^2}}{1 - e^{r^2 + \frac{x^2}{8a^2}}} \right)
$$

(B.35)

$$
\langle \xi_2^\alpha | \partial_\alpha \xi_1^\alpha \rangle = \langle \partial_\alpha \xi_1^\alpha | \xi_2^\alpha \rangle^\dagger = \frac{i}{2} \left( 1 + \frac{-1 + e^{r^2}}{1 - e^{r^2 + \frac{x^2}{8a^2}}} \right)
$$

(B.36)

$$
\langle \xi_2^\alpha | \partial_\alpha \xi_2^\alpha \rangle = \langle \partial_\alpha \xi_2^\alpha | \xi_2^\alpha \rangle^\dagger = -\frac{i}{2} \left( 1 + \frac{-1 + e^{r^2}}{1 - e^{r^2 + \frac{x^2}{8a^2}}} \right)
$$

(B.37)

$$
\langle \xi_1^\alpha | \partial_\alpha \xi_1^\alpha \rangle = \langle \partial_\alpha \xi_1^\alpha | \xi_1^\alpha \rangle^\dagger = -\frac{1 + e^{\frac{x^2}{8a^2}}}{(1 - e^{r^2 + \frac{x^2}{8a^2}})^2} e^{r^2} \Gamma
$$

(B.38)

$$
\langle \xi_1^\alpha | \partial_\alpha \xi_2^\alpha \rangle = \langle \partial_\alpha \xi_2^\alpha | \xi_1^\alpha \rangle^\dagger = 0
$$

(B.39)

$$
\langle \xi_2^\alpha | \partial_\alpha \xi_1^\alpha \rangle = \langle \partial_\alpha \xi_1^\alpha | \xi_2^\alpha \rangle^\dagger = 0
$$

(B.40)

$$
\langle \xi_2^\alpha | \partial_\alpha \xi_2^\alpha \rangle = \langle \partial_\alpha \xi_2^\alpha | \xi_2^\alpha \rangle^\dagger = \frac{1 - e^{\frac{x^2}{8a^2}}}{(1 - e^{r^2 + \frac{x^2}{8a^2}})^2} e^{r^2} \Gamma
$$

(B.41)

Finally, by substituting all into Eq. (B.25) and doing some calculations, we obtain

$$
\text{Tr}_m \left[ \rho_m^\alpha [\hat{L}_\alpha^\alpha, \hat{L}_\Gamma^\Gamma] \right] = 0,
$$

(B.42)

which implies that the pQCRB is satisfied: pQFIM $Q$ can be saturated.
Appendix B.3. The pQFIM

Similarly as in the case of § Appendix B.2, we obtain

\[ Q_{\alpha\beta} = 4w^2 \sum_{k,l=1}^{2} \frac{\lambda_k \langle \xi_k^\dagger \rho_m^\dagger \xi_l^\dagger \rangle \langle \xi_l^\dagger \rho_m^\dagger \xi_k^\dagger \rangle}{(\lambda_k N_k^\dagger + \lambda_l N_l^\dagger)^2}. \] (B.43)

By substituting \( \langle \xi_k^\dagger | \partial_\alpha \rho_m^\dagger | \xi_l^\dagger \rangle \)'s calculated in § Appendix B.2 into Eq. (B.43), we obtain \( Q_{\alpha\beta} \) and then Eqs. (20), (21) and (22).

Appendix C. Qubit MA

According to the definition (Eq. (1)), \( \rho_{sm} \) (4 \times 4 matrix) can be obtained. Then, \( \rho'_m \) is obtained as follows:

\[
\rho'_m = \begin{pmatrix}
\sin^2 \frac{\theta}{2} (e^{r^2} + \sin \gamma \cos \phi) & \sin \theta (ie^{r^2} \cos \gamma + \sin \gamma \sin \phi) \\
\frac{e^{r^2} - \cos \theta \sin \gamma \cos \phi}{\sin \theta (-i e^{r^2} \cos \gamma + \sin \gamma \sin \phi)} & 2(e^{r^2} - \cos \theta \sin \gamma \cos \phi)
\end{pmatrix},
\]

Then,

\[
L'_\phi = \begin{pmatrix}
-\frac{2 \cos^2 \frac{\theta}{2} \sin \gamma \sin \phi}{e^{r^2} - \cos \theta \sin \gamma \cos \phi} & 1 \\
\cot \theta - e^{r^2} \csc \theta \csc \gamma \sec \phi & \frac{2 \sin^2 \frac{\theta}{2} \sin \gamma \sin \phi}{e^{r^2} - \cos \theta \sin \gamma \cos \phi}
\end{pmatrix}
\]

\[
L'_\Gamma = \frac{e^{r^2} \Gamma (\coth \Gamma^2 - 1)}{e^{r^2} - \cos \theta \sin \gamma \cos \phi} \times
\begin{pmatrix}
2 \cos^2 \frac{\theta}{2} (1 - e^{r^2} \sin \gamma \cos \phi) & \sin \theta (i \cos \gamma + e^{r^2} \sin \gamma \sin \phi) \\
\sin \theta (-i \cos \gamma + e^{r^2} \sin \gamma \sin \phi) & 2 \sin^2 \frac{\theta}{2} (1 + e^{r^2} \sin \gamma \cos \phi)
\end{pmatrix}
\]

are easily obtained. Similarly, we obtain as follows:

\[
\rho'_m = \begin{pmatrix}
\sin^2 \frac{\theta}{2} (e^{r^2} - \sin \gamma \cos \phi) & \sin \theta (i e^{r^2} \cos \gamma + \sin \gamma \sin \phi) \\
\frac{e^{r^2} + \cos \theta \sin \gamma \cos \phi}{\sin \theta (-i e^{r^2} \cos \gamma + \sin \gamma \sin \phi)} & 2(e^{r^2} + \cos \theta \sin \gamma \cos \phi)
\end{pmatrix},
\]

(C.4)
\[ L^κ_φ = \begin{pmatrix} 2 \cos^2 \frac{θ}{2} \sin γ \sin φ \frac{1}{e^{r^2} + \cos θ \sin γ \cos φ} \\ \cot θ + e^{r^2} \csc θ \csc γ \sec φ \end{pmatrix} \begin{pmatrix} \frac{1}{2} \sin \frac{θ}{2} \frac{1}{\sin γ \sin φ} \\ \frac{1}{2} \cos \frac{θ}{2} \frac{\cot θ + e^{r^2} \csc θ \csc γ \sec φ}{\csc θ \csc γ \sec φ} \end{pmatrix}, \quad (C.5) \]

\[ L^Γ_Γ = \frac{e^{Γ^2} \Gamma (\coth Γ^2 - 1)}{e^{r^2} + \cos θ \sin γ \cos φ} \times \begin{pmatrix} 2 \cos^2 \frac{θ}{2} \frac{1 + e^{r^2} \sin γ \cos φ}{2} - \sin θ \frac{(i \cos γ + e^{r^2} \sin γ \sin φ)}{2} \\ - \sin θ \frac{(i \cos γ + e^{r^2} \sin γ \sin φ)}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \cos \frac{θ}{2} \frac{1}{\sin γ \sin φ} \\ \frac{1}{2} \sin \frac{θ}{2} \frac{\cot θ + e^{r^2} \csc θ \csc γ \sec φ}{\csc θ \csc γ \sec φ} \end{pmatrix}. \quad (C.6) \]

(C.7)

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