Persistent current in small superconducting rings

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We study theoretically the contribution of fluctuating Cooper pairs to the persistent current in superconducting rings threaded by a magnetic flux. For sufficiently small rings, in which the coherence length $\xi$ exceeds the radius $R$, mean field theory predicts a full reduction of the transition temperature to zero near half-integer flux. We find that nevertheless a very large current is expected to persist in the ring as a consequence of Cooper pair fluctuations that do not condense. For larger rings with $R \gg \xi$ we calculate analytically the susceptibility in the critical region of strong fluctuations and show that it reflects competition of two interacting complex order parameters.

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Introduction and main results - Superconducting fluctuations have been the subject of intense research during the last decades [1]. At temperature above the transition temperature $T_c$ to the superconducting state, the system is still metallic, pairs of electrons are formed for a limited time. These superconducting fluctuations affect both transport and thermodynamic properties.

In bulk superconductors $T_c$ can be reduced or even completely suppressed by various phase-breaking mechanisms, for example by applying a magnetic field or introducing magnetic impurities. A special situation occurs for superconducting rings and cylinders threaded by a magnetic flux $\phi$. $T_c$ is periodically reduced as a function of $\phi$, a phenomenon known as Little-Parks oscillations [2]. The period of the oscillations is equal to 1 as a function of the reduced flux $\phi = \phi/\phi_0$, where the superconducting flux quantum is $\phi_0 = 2\pi hc/2e = \pi/e$ [3], see Fig. 1.

The magnitude of the maximal reduction in $T_c$ is size-dependent. As we see in Fig. 1 mean field (MF) theory predicts that for small rings or cylinders with $r \equiv R/\xi < 0.6$ the transition temperature is equal to zero in a finite interval close to half-integer flux, giving rise to a flux-tuned quantum phase transition, see also Eq. 1 below. In this Letter we show that the pair fluctuations give a large contribution to the persistent current (PC) $I$ even at fluxes for which $T_c$ is reduced to zero and the system has a finite resistance.

Recent experiments added significantly to our understanding of fluctuation phenomena in superconductors with doubly-connected geometry. Strong Little-Parks oscillations in the region where $\xi > R$, where $T_c$ is reduced to zero, have been observed in a transport measurement on superconducting cylinders [3]. Koshnick et al. [5] measured the PC in small superconducting rings in the regime where $R > \xi$, for the smallest rings under study $T_c$ was reduced by $\approx 6\%$. In this Letter we discuss the PC both in the regime of moderate $T_c$ suppression for $r = R/\xi \gtrsim 1$ as well as the strong Little-Parks oscillations for $r < 0.6$. Before presenting details of our approach, we summarize the main results of our analysis.

I. Regime with $r = R/\xi < 0.6$: For $r < 0.6$ the mean field $T_c$ vanishes and one would naively expect a small normal state PC. We find, however, that close to the critical mean field line (see Fig. 1) there is a parametrically large enhancement of the PC due to quantum fluctuations that decay only slowly away from that line. The magnitude for the normal PC is $I_N \sim \frac{P_{FL}}{\Delta g}$, where $g$ is the dimensionless ring conductance [6, 7]. Our calculations show that the PC due to pair fluctuations near the critical flux $\varphi_c$ is parametrically larger and at low $T$ given by

$$I_{FL} \approx -\frac{T_c}{\phi_0} \frac{1}{\varphi_c} \frac{\xi}{R} \log \left( \frac{1}{\Delta \varphi} \right),$$

where $\Delta \varphi \equiv (\varphi - \varphi_c)/\varphi_c$ measures the distance to the critical flux $\varphi_c$. When increasing $T$ the PC initially grows before going through a maximum at finite $T$, where it can considerably exceed the result of Eq. 1 (see Fig. 1). Since $r^{-1} = \xi/R$ is a number of order 1 and $\frac{P_{FL}}{\Delta g} = \frac{1}{\xi R}$ for a weakly disordered superconductor, we find an enhancement factor of $\log(g) \log(1/\Delta \varphi)$.

Our results are obtained for the case when the flux...
acts as a pair breaking mechanism. Other pair breaking mechanisms, e.g. magnetic impurities or a magnetic field penetrating the ring itself will lead to similar results. They cause a reduction of $T_c$ to zero, the pair fluctuations, however, lead to a parametric enhancement of the PC in the normal state. Ref. [8] suggests that a similar mechanism due to magnetic impurities is related to the unexpectedly large PC in noble metal rings [9, 10].

A metallic state with small but finite resistance was observed experimentally in superconducting cylinders [4, 11] with $\phi \approx 1/2$. Further theoretical and experimental studies will be needed in order to clarify the relation to our findings, where a large PC is caused by pair fluctuations that are unable to condense.

II. Regime with $r > 1$: The case $r > 1$ is suitable for the description of the experiments on persistent currents by Koschnick et al. [5]. Previously the experiment has been interpreted using a one-dimensional Ginzburg-Landau theory to describe the order parameter fluctuations [12]. Following these lines one has to resort to numerical methods [13] in order to describe the critical region close to $T_c$, where fluctuations proliferate.

Our key observation is that part of the rings in the experiment allow for a description using a suitable generalization [14] of the 0d Ginzburg-Landau theory. Indeed, following an expansion of the order parameter field $\psi(\phi)$ in terms of angular momentum modes $\psi_n$, a simple physical picture arises in the limit $\sqrt{\alpha} \gg r$. Two of the modes compete with each other close to half-integer flux, while at the same time both of them strongly fluctuate in the critical regime close to $T_c$.

Formally, the competition arises due to the quartic term in the GL functional that induces an interaction between the modes [13] and reveals itself in the experiment mostly in the “slope” of the PC, the susceptibility $\chi = -\frac{\partial^2}{\partial \phi^2}$. With this insight $\chi$ can be calculated analytically even in the critical fluctuation regime.

As an example, denoting the susceptibility at $T_c$ and zero flux by $\chi_0$ and at $T_c, \varphi = 1/2$ by $\chi_{1/2}$, we find

$$\chi_{1/2}/\chi_0 \approx -2.7 \sqrt{g/r}. \quad (2)$$

Experimentally, a strong enhancement of the magnetic susceptibility near $\varphi = 1/2$ compared to $\varphi \approx 0$ was observed and Eq. (2) demonstrates that it is controlled by the parameter $\sqrt{g/r}$. If it is large, the current will rapidly change sign as a function of the flux at half-integer flux, leading to a saw-tooth like shape of $i_\varphi$. The full $T$ dependence of $\chi_{1/2}$ is given in Eq. (7). For the smallest rings in Ref. [5, 16] $\sqrt{g} \approx 33r$.

Classical GL functional - After presenting the main results in Eqs. (1) and (2) we now give more details of our approach starting with the description of rings with only a moderate suppression of $T_c$ (i.e. $r \gtrsim 1$).

When the superconducting coherence length $\xi(T)$ and the magnetic penetration depth $\lambda(T)$ are much larger than the ring thickness, the system is well described by a one-dimensional order parameter field $\psi$ [17]. The partition function can be written as a weighted average over configurations of the order parameter $\psi$, $\mathcal{Z} = \int D\psi \exp[-\mathcal{F}/T]$. Introducing angular momentum modes as $\psi(\phi) = \frac{1}{\sqrt{F}} \sum_n \psi_n e^{in\phi}$, where $V$ is the volume of the ring, the free energy functional takes the form

$$\mathcal{F} = \sum_n a_{n\varphi} |\psi_n|^2 + \frac{b}{2V} \sum_{nmkl} \delta_{n+k,l+m} \psi_m \psi_n \psi_l \psi_k^* \quad (3)$$

Here we wrote $a_{n\varphi} = a T_c \varepsilon_{n\varphi}$, where $\varepsilon_{n\varphi} = \frac{(T - T_{nc}/T_c}$ is the reduced temperature and $\varepsilon_{n\varphi} = T_c(1 - (n - \varphi)^2/r^2)$ is determined by the sign change of the coefficient $a_n(\varphi)$ and can thus loosely be interpreted as the transition temperature of mode $\psi_n$. The mean field transition occurs at $T_{\varphi}$ that is equal to the maximal $T_n$ for given $\varphi$, i.e. at the point where the first mode becomes superconducting when lowering the temperature (cf. Fig. [1]). The 0d Ginzburg parameter $G = (2b/\alpha^2 T_c V)^{1/2}$ is an estimate for the width of the critical regime in the variable $\varepsilon_n$. The parameter $\sqrt{g/r} \approx 1/5r^2 G_i$ has been used when stating our results. Its relevance is now easily understood. $1/r^2$ is a measure for the typical spacing between the transition temperatures $T_n$ for different modes, since $(T_0 - T_1)/T_c = (1 - 2\varphi)/r^2$. This spacing should be compared to the typical width of the non-Gaussian fluctuation region, $G_i$. If it is large, a theory including only one or two angular momentum modes is applicable.

Persistent current - The persistent current $I$ is found from the free energy $F = -T \ln \mathcal{Z}$ by differentiation $I = -\partial F/\partial \phi$. The normalized current is given by

$$i = I/(T_c/\phi_0) = \sum_{n = -\infty}^{\infty} \frac{2\alpha}{r^2} \left( n - \varphi \right) \langle |\psi_n|^2 \rangle. \quad (4)$$

The averaging is performed with respect to the functional $\mathcal{F}$ in Eq. (3). $i_\varphi$ is periodic in the flux $\varphi$ with period one. Since it is also an odd function of the flux, it vanishes when the flux takes integer or half-integer values.

Case $\varphi \approx 0$: The most important contribution in the regime of non-Gaussian fluctuations close to integer fluxes comes from the angular momentum mode $\psi_n$ with the highest transition temperature $T_{nc}$. One may then approximate Eq. (3) by a single-mode and calculate with $\mathcal{F}_n = a_n|\psi_n|^2 + \frac{\alpha g}{\sqrt{\alpha G}} |\psi_n|^4$ [18]. This is the 0d limit of the GL functional [21] where the functional integral becomes a conventional integral. Indeed, performing the integral in polar coordinates, one finds $Z = \frac{2\alpha g}{\sqrt{\alpha G}} \exp (x_0^2) \text{erf}(x_0)$, where $x_n = \varepsilon_n/G_i$ [21]. Using now Eq. (4) with one mode only we find

$$i_n = 4\Lambda (n - \varphi) f(x_n) \quad \text{for } \varphi \approx n. \quad (5)$$

Here $\Lambda \equiv 1/r^2 G_i \approx 5 \sqrt{g/r}$ and $f(x) = \frac{\exp(-x^2)}{\sqrt{\text{erf}(x)}} - x$ [21]. We note in passing the high degree of universality
implied by this result: All PC measurements will fall on the same curve, if the PC — measured in suitable units $i = I / (T_c / \phi_0)$ — and the reduced temperature $\varepsilon_n = (T - T_c) / T_c$ are scaled as $i \to \sqrt{\varepsilon_n} \varepsilon_n / r \sqrt{\beta}$. The scaling function $f$ was given above. This relation is a valuable guide in characterizing different rings in experiments.

Far above $T_c$ one obtains as a limiting case the Gaussian result for a single mode $i_n \approx 2(n - \varphi) / r^2 \varepsilon_n$, that can also be obtained directly by neglecting the quartic term in the GL functional. It is known, however, that as soon as temperatures are too high, $\varepsilon_n \gg 1 / r^2$, it is important to sum the contribution of all modes [22]. Far below $T_c$ one recovers the mean field result $i_{MF} \equiv -\frac{2}{\sqrt{\pi} \varepsilon} \varepsilon_{n=\varphi} (n - \varphi)$ for the PC in the superconducting regime. An alternative route to finding the mean field result would be to minimize the full single mode functional, which leads to the condition $|\psi_n|^2 = -a_n V / b$ and then to use Eq. (4).

The PC $i_n$ in Eq. (5) interpolates smoothly between the Gaussian and the mean field result.

**Case $\varphi \approx 1/2$:** A very interesting situation occurs at half integer values of $\varphi$. The transition temperatures for two modes become equal, their coupling becomes crucial ($\varphi \approx 1/2$ for definiteness), and we approximate [14]

$$\mathcal{F} = \sum_{i=0,1} \alpha_i |\psi_i|^2 + \frac{b}{2 \sqrt{\pi} |\psi_0|^4 + |\psi_1|^4 + 4 |\psi_0|^2 |\psi_1|^2}$$

Calculation of the PC in the presence of the coupling requires a generalization of the approach used for the single mode case [14, 22]. In Fig. 2 we display the PC $i_2$ as calculated from Eq. (6) for three different temperatures, $T < T_c$, $T = T_c$, and $T > T_c$. We compare it to the MF result as well as to $i_{20}$ obtained by neglecting the coupling $|\psi_0|^2 |\psi_1|^2$ in Eq. (6).

Above $T_c$ ($\varepsilon = -0.05$ in Fig. 2), in the region where the mean field result vanishes near half-integer flux, the PC is purely fluctuation. We deduce from Fig. 2 that the coupling of the modes is crucial for $\chi(1/2)$, but not for the overall shape when $T > T_c$. However, just below $T_c$ ($\varepsilon = -0.11$ in Fig. 2) the coupling is essential. The mean field result is not applicable as it gives an infinitely sharp jump in the PC at half-integer flux. The result without coupling of the modes, $i_{20}$, gives a finite slope, but it is far from the full current $i_2$ that includes the mode coupling. The coupling drives the current $i_2$ towards the mean field approximation $i_{MF}$ which includes only one mode. This occurs because for a repulsive coupling the dominant mode suppresses the subdominant one. Indeed, if mode $n = 0$ is dominant then the coupling adds a mass term $2b / V (|\psi_0|^2) |\psi_1|^2$ to mode $n = 1$ and reduces its $T_c$.

**Susceptibility** — We will now discuss in more detail the slope at half-integer flux, which is most sensitive to the coupling between the modes below, and to the non-Gaussian fluctuations close to $T_c$. Differentiating the expression [22] for $i_2$ we obtain

$$\chi_{\varphi=1/2} = \chi / (T_c / \phi_0) = 4 \Lambda ^2 g_1 / 4 \Lambda ^2 g_2$$

where $\alpha_n (\varphi) = 1 / \sqrt{\pi} (n - \varphi)^2$ and $\Omega_k = 2 \pi k T$ are bosonic Matsubara frequencies [26, 27]. Following the standard approach, we first find the critical line $\varphi(T)$

$$x = \frac{\varphi}{1/2} + \frac{1}{2} \Lambda$$

The dimensionless smooth functions $g_1 (x) = \frac{1}{2} \sqrt{x} e^{-x^2} \text{erfc}(x) - \frac{x}{3}$ and $g_2 (x) = \frac{2}{\sqrt{\pi} \sqrt{x}} e^{-x^2} \text{erfc}(x) - 1$, where $J(x) = \int_x^\infty e^{t^2} \text{erfc}(t)$, obey $g_1 (0) \approx 0.78$ and $g_2 (0) \approx 0.315$.

For large $\Lambda = 1 / r^2 G_i$ one should neglect the first term in Eq. (7). Then one obtains $\chi_{1/2} = -4 \Lambda ^2 g_2$. For the susceptibility close to integer flux one easily obtains $\chi_0 = 4 \Lambda f (x_0)$ from Eq. (5). Comparing to the expression for $\chi_{1/2}$, we find Eq. (6).

This is the strong enhancement of $\chi_{1/2}$ compared to $\chi_0$ observed in the experiment [7]. Comparison with the numerical calculation of Ref. [8] shows that our analytical results are accurate to within a few percent already for $\sqrt{\beta} / r \gtrsim 8 [12, 16]$.

**Quantum critical regime** — So far we have discussed the limit $r = R / \xi > 1$, where the suppression of $T_c$ is small and a finite temperature phase transition occurs. We will now discuss the case where $r = R / \xi < 0.6$ and $T_c$ is reduced to zero at a critical flux $\varphi_c$ near $\varphi = 1/2$, see Fig. 4. Near the quantum critical point (QCP) it is no longer legitimate to use the classical GL functional, in which only the static component of the order parameter field is considered. Instead, all Matsubara frequencies should be taken into account in the imaginary time formalism. The full fluctuation propagator is given by:

$$(\nu L)^{-1} = \ln \left[ \frac{T}{T_c} \right] + \psi \left[ \frac{1}{2} + \frac{\alpha_n + |\Omega_k|/2}{2 \pi T} \right] - \psi \left[ \frac{1}{2} \right]$$

where $\alpha_n (\varphi) = 1 / \sqrt{\pi} (n - \varphi)^2$ and $\Omega_k = 2 \pi k T$ are bosonic Matsubara frequencies [26, 27]. Following the standard approach, we first find the critical line $\varphi(T)$.

![FIG. 2: The PC $i = I / (T_c / \phi_0)$ as a function of the flux $\varphi$. Parameters are $r = R / \xi = 1.66$, $\Lambda = 1 / (r^2 G_i) \approx 5 \sqrt{\beta} / r = 50$, $\varepsilon = (T - T_c) / T_c$. The transition for $\varphi = 1/2$ occurs at $\varepsilon = -0.09$. Full lines: $i_{20}$ calculated with $\mathcal{F}$ of Eq. (6), it takes into account two modes and the interaction between them. We compare $i_{20}$ to two approximations, which neglect this interaction. Dotted lines: The mean field approximation $i_{MF}$ discussed before Eq. (6) and dashed line: $i_{20}$ calculated with $\mathcal{F}$ of Eq. (6) without coupling [22]. Inset: MF phase diagram, superconducting region in grey.](image-url)
in the temperature-flux plane by equating $\mathcal{L}^{-1}_{00} = 0$. For the QCP at $T = 0$ one obtains the critical flux $\phi_c = \pi r/(2\sqrt{2}\gamma E)$, $\gamma E \approx 1.75$. Due to the flux-periodicity of the phase diagram, the QCP can only be observed in the ring geometry if $\varphi_c < 1/2$ which implies $r < \sqrt{2}\gamma E/\pi \approx 0.6$. Notice that this critical value of $r = R/\xi$ is 20% larger than a naive application of the quadratic approximation valid for $r \gg 1$ would suggest.

Restricting ourselves to the interval $\varphi \in (0, 0.5)$ we find that near the QCP it is sufficient to consider the $n = 0$ mode. In the Gaussian regime we obtain (cf. Fig.1) the following fluctuation contribution to the PC $i_G = -\frac{2\varphi_c}{\pi\varepsilon^2(T)} T \sum_k \mathcal{L}_{0k}$. Expanding $\mathcal{L}_0^{-1}$ in small $\Delta \varphi = [\alpha_0(\varphi) - \alpha(\varphi(T))] / \alpha(\varphi_c)$ we find with logarithmic accuracy $i_G = -\frac{\Delta \varphi}{\pi\varepsilon^2} h(\Delta \alpha, t) \approx -s \frac{\gamma}{\varepsilon^2}, h(\Delta \alpha, t) = \ln \frac{\Delta \alpha}{\sqrt{\gamma}t} + s(1 + s), s = \frac{\Delta \alpha}{\sqrt{\gamma}t}$ and $t = T/T_c$.

A few remarks are in order concerning this result. The second term in the expression for $h$ is the classical $\Omega = 0$ contribution to the sum. The upper cutoff for the frequency summation has been chosen as $\Omega = 2\alpha_0(\varphi(T))$. The function $h$ has the asymptotic form $h \approx \gamma t/\Delta \alpha + \ln(1/2\gamma t)$ for $\Delta \alpha \ll t$ and $h \approx \ln 1/\Delta \alpha$ for $t \ll \Delta \alpha$. It is important that $\Delta \alpha$ is $T$-dependent and in order to reveal the full T-dependence of $i_G$ one should first find the transition line $\alpha_0(\varphi(T))$. $h(T)$ is displayed in Fig.3. The maximum of $h$ at finite $T$ is a result of two competing mechanisms. As $T$ grows from zero, thermal fluctuations become stronger. At the same time the distance to the critical line becomes larger for fixed $\varphi$, which eventually leads to a decrease of $i_G$.

**Conclusion** - In conclusion, we showed that on the normal side of the flux-tuned superconductor normal-metal transition in small rings the fluctuation PC can be very large compared to the normal case and decays only logarithmically away from the critical point. For larger rings as studied in recent experiments we obtained detailed analytical predictions for the strong fluctuation region.

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An analogous formula has been given in Ref. [14].

The condition $x = 0$ defines $T_{c,1/2} = T_c(1 - 1/4\pi^2)$.

$i_2$ and $i_{20}$ include the (very small) contribution of $\psi_n$ with $n \neq 0, 1$ in a modified Gaussian approximation [14].

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In general: $i_G = -T/T_c \sum_n L_n \partial L^{-1}_n / \partial \varphi$. Further away from the QCP $i_G$ should be evaluated numerically.

The divergence is cured by the contribution of all non-singular modes with $n \neq 0$.

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