Inexact indefinite proximal ADMMs for 2-block separable convex programs and applications to 4-block DNNSDPs

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July 8, 2015

Abstract

This paper is concerned with two-block separable convex minimization problems with linear constraints, for which it is either impossible or too expensive to obtain the exact solutions of the subproblems involved in the proximal ADMM (alternating direction method of multipliers). Such structured convex minimization problems often arise from the two-block regroup of three or four-block separable convex optimization problems with linear constraints, or from the constrained total-variation superresolution image reconstruction problems in image processing. For them, we propose an inexact indefinite proximal ADMM of step-size \( \tau \in (0, \frac{\sqrt{5}+1}{2}) \) with two easily implementable inexactness criteria to control the solution accuracy of subproblems, and establish the convergence under a mild assumption on indefinite proximal terms. We apply the proposed inexact indefinite proximal ADMMs to the three or four-block separable convex minimization problems with linear constraints, which come from the important class of doubly nonnegative semidefinite programming (DNNSDP) problems with many linear equality and/or inequality constraints. Numerical results indicate that the inexact indefinite proximal ADMM with the absolute error criterion has a comparable performance with the directly extended multi-block ADMM of step-size \( \tau = 1.618 \) without convergence guarantee, whether in terms of the number of iterations or the computation time.

Keywords: Separable convex optimization, inexact proximal ADMM, DNNSDPs

1 Introduction

Let \( X, Y \) and \( Z \) be the finite dimensional vector spaces endowed with the inner product \( \langle \cdot, \cdot \rangle \) and its induced norm \( \| \cdot \| \). Given closed proper convex functions \( f: X \to (-\infty, +\infty] \)

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and $g: \mathbb{Y} \to (-\infty, +\infty]$, we are concerned with the separable convex optimization problem

$$
\min_{x \in X, y \in \mathbb{Y}} f(x) + g(y)
\text{ s.t. } A^*x + B^*y = c,
$$

where $A: \mathbb{Z} \to X$ and $B: \mathbb{Z} \to \mathbb{Y}$ are the given linear operators, $A^*$ and $B^*$ denote the adjoint operators of $A$ and $B$, respectively, and $c \in \mathbb{Z}$ is a given vector.

As well known, there are many important cases with the form of (1), which include the covariance selection problems and semidefinite least squares problems in statistics [1, 30, 39], the sparse plus low-rank recovery problem arising from the so-called robust PCA (principle component analysis) with noisy and incomplete data [34, 32], the constrained total-variation image restoration and reconstruction problems [22, 29], the simultaneous minimization of the nuclear norm and $\ell_1$-norm of a matrix arising from the low-rank and sparse representation for image classification and subspace clustering [40, 36], and so on.

For the structured convex minimization problem (1), the alternating direction method of multipliers (ADMM for short), first proposed by Glowinski and Marrocco [11] and Gabay and Mercier [12], is one of the most popular methods. For any given $\sigma > 0$, let $L_\sigma: \mathbb{X} \times \mathbb{Y} \times \mathbb{Z} \to (-\infty, +\infty]$ denote the augmented Lagrangian function of problem (1)

$$
L_\sigma(x, y, z) := f(x) + g(y) + \langle z, A^*x + B^*y - c \rangle + \frac{\sigma}{2}\|A^*x + B^*y - c\|^2.
$$

The ADMM, from an initial point $(x^0, y^0, z^0) \in \text{dom } f \times \text{dom } g \times \mathbb{Z}$, consists of the steps

$$
\begin{align}
x^{k+1} &\in \arg \min_{x \in \mathbb{X}} L_\sigma(x, y^k, z^k), \\
y^{k+1} &\in \arg \min_{y \in \mathbb{Y}} L_\sigma(x^{k+1}, y, z^k), \\
z^{k+1} &= z^k + \tau \sigma (A^*x^{k+1} + B^*y^{k+1} - c),
\end{align}
$$

where $\tau \in (0, \frac{1+\sqrt{5}}{2})$ is a constant to control the step-size in (2c). The iterative scheme of ADMM actually embeds a Gaussian-Seidel decomposition into each iteration of the classical augmented Lagrangian method of Hestenes-Powell-Rockafellar [14, 25, 28], so that the challenging task (i.e., the exact solution or the approximate solution with a high precision of the Lagrangian minimization problem) is relaxed to several easy ones.

Notice that the subproblems (2a) and (2b) in the ADMM may have no closed-form solutions or even be difficult to solve. When the functions $f$ and $g$ enjoy a closed-form Moreau envelope, one usually introduces the proximal terms $\frac{1}{2}\|x - x^k\|_{P_x}$ and $\frac{1}{2}\|y - y^k\|_{P_y}$ respectively into the subproblems (2a) and (2b) to cancel the operators $AA^*$ and $BB^*$ so as to get the exact solutions of proximal subproblems. This is the so-called proximal-
ADMM which, for a chosen initial point \((x^0, y^0, z^0) \in \text{dom } f \times \text{dom } g \times Z\), consists of

\[
\begin{align*}
    x^{k+1} &= \arg\min_{x \in X} L_\sigma(x, y^k, z^k) + \frac{1}{2}\|x - x^k\|_{P_f}^2, \\
    y^{k+1} &= \arg\min_{y \in Y} L_\sigma(x^{k+1}, y, z^k) + \frac{1}{2}\|y - y^k\|_{P_g}^2, \\
    z^{k+1} &= z^k + \tau\sigma (A^*x^{k+1} + B^*y^{k+1} - c).
\end{align*}
\]

The existing works on the proximal ADMM mostly focus on the positive definite proximal terms (see, e.g., [15, 35, 41]). It is easy to see that the proximal subproblems with the positive definite proximal terms will have a big difference from the original subproblems of ADMM. In fact, as pointed out in the conclusion remarks of [15], “large and positive definite proximal terms will lead to easy solution of subproblems, but the number of iterations will increase. Therefore, for subproblems which are not extremely ill-posed, the proximal parameters should be small.” In view of this, some researchers recently develop the semi-proximal or indefinite proximal ADMM [38, 9, 21] by using the positive semidefinite even indefinite proximal terms. The numerical experiments in [9] show that such tighter proximal terms display better numerical performance. In addition, it is worthwhile to emphasize that the ADMM itself is a semi-proximal (of course an indefinite proximal) ADMM, but is not in the family of positive definite proximal ADMMs.

In this paper we are interested in problem (1) in which the functions \(f\) and/or \(g\) may not have a closed-form Moreau envelope or the linear operators \(A\) and/or \(B\) have a large spectral norm (now the proximal subproblems with a positive definite proximal term are bad surrogates for those of the ADMM), for which it is impossible or too expensive to achieve the exact solutions of the proximal subproblems though they are unique. Such separable convex optimization problems arise directly from the constrained total-variation superresolution image reconstruction problems [4, 24] in image processing, and the two-block regroup of three or four-block separable convex minimization problems. Indeed, for the following four-block separable convex minimization problem

\[
\begin{align*}
    \min_{x_i \in X_i} \sum_{i=1}^4 f_i(x_i) \\
    \text{s.t. } \sum_{i=1}^4 A_i^*x_i = c
\end{align*}
\]

(4)

where \(f_i : X_i \rightarrow (-\infty, +\infty]\) for \(i = 1, 2, 3, 4\) are closed proper convex functions, and \(A_i : Z \rightarrow X_i\) for \(i = 1, 2, 3, 4\) are linear operators, since the directly extended multi-block ADMM does not have the convergence guarantee (see the counterexamples in [3]), one may rearrange it as the form of (1) by reorganizing any two groups of variables into one group, and then apply the classical ADMM for solving the two-block regrouped problem. Clearly, the exact solution of each subproblem of ADMM for the two-block regrouped problem is difficult to obtain due to the cross of two classes of variables. In particular, the two-block regroup resolving of multi-block separable convex optimization also has a separate study value.

To resolve this class of difficult two-block separable convex minimization problems, we propose an inexact indefinite proximal ADMM with a step-size \(\tau \in (0, \frac{\sqrt{5}+1}{2})\), in which
the proximal subproblems are solved to a certain accuracy with two easily implementable
inexactness criteria to control the accuracy. Here, an indefinite proximal term, instead
of a positive definite proximal term, is introduced into each subproblem of the ADMM
to guarantee that each proximal subproblem has a unique solution as well as becomes
a good surrogate for the original subproblem of the ADMM. For the proposed inexact
indefinite proximal ADMM, we establish its convergence under a mild assumption on
the indefinite proximal terms. To the best of our knowledge, this is the first convergent
inexact proximal ADMM in which step-size $\tau$ may take the value in the interval $(1, \frac{\sqrt{5}+1}{2})$.

We notice that a few existing research papers on inexact versions of the ADMM all focus
on the unit step-size; see [8, 15, 24, 13, 5], and moreover, only the papers [24, 13, 5]
develop truly implementable inexactness criteria in the exact solutions are not required.
Our inexact indefinite proximal ADMM is using the same absolute error criterion and
and a little different relative error from the one used in [24]. It is well known that the
ADMM with $\tau = 1.618$ requires less 20% to 50% iterations than the one with $\tau = 1$,
especially for those difficult SDP problems [33]. Thus, the proposed inexact indefinite
proximal ADMMs with a large step-size is expected to have better performance.

In this work, we apply the inexact indefinite proximal ADMMs to the three and four-
block separable convex minimization problems with linear constraints, coming from the
duality of the doubly nonnegative semidefinite programming (DNNSDP) problems with
many linear equality and/or inequality constraints. Specifically, we solve the two-block
regroupment for the dual problems of DNNSDPs with the inexact indefinite proximal
ADMM. Observe that the iterates yielded by solving each subproblem in an alternating
way can satisfy the optimality condition approximately. Hence, in the implementation of
the inexact indefinite proximal ADMMs, we get the inexact solution of each subproblem
by minimizing the two group of variables alternately. Numerical results indicate that
the inexact indefinite proximal ADMM with the absolute error criterion is comparable
with the directly extended multi-block ADMM with step-size $\tau = 1.618$ whether in
terms of the number of iterations or the computation time, while the one with the
relative error criterion requires less outer-iterations but more computation time since the
error criterion is more restrictive and requires more inner-iterations. Thus, the inexact
indefinite proximal ADMM with the absolute error criterion provides an efficient tool for
handling the three and four-block separable convex minimization problems.

We observe that there are several recent works [37, 18, 19, 10] to regroup the multi-
block separable convex minimization problems into two-block or several subblocks, and
then solve each subblock simultaneously by introducing a positive definite proximal term
related to the numbers of subproblems. Such procedures lead to easily solvable subpro-
blems, but their performance becomes worse due to larger proximal terms.

The rest of this paper is organized as follows. Section 2 gives some notations and the
main assumption. Section 3 describes the inexact indefinite proximal ADMMs and ana-
lyzes the properties of the sequence generated. The convergence of the inexact indefinite
proximal ADMMs is established in Section 4. Section 5 applies the inexact indefinite
proximal ADMMs for solving the duality of the doubly DNNSDPs with many linear
equality and/or inequality constraints. Some concluding remarks are given in Section 6.

2 Notations and assumption

Notice that the functions \( f : \mathbb{R} \to (-\infty, +\infty] \) and \( g : \mathbb{Y} \to (-\infty, +\infty] \) are closed proper convex, and the subdifferential mappings of closed proper convex functions are maximal monotone [26, Theorem 12.17]. Hence, there exist self-adjoint operators \( \Sigma_f \geq 0 \) and \( \Sigma_g \geq 0 \) such that for all \( x, \tilde{x} \in \text{dom } f, u, \tilde{u} \in \partial f(x) \) and \( \tilde{v} \in \partial f(\tilde{x}) \),

\[
f(x) \geq f(\tilde{x}) + \langle \tilde{u}, x - \tilde{x} \rangle + \frac{1}{2} \| x - \tilde{x} \|^2_{\Sigma_f} \quad \text{and} \quad \langle u - \tilde{u}, x - \tilde{x} \rangle \geq \| x - \tilde{x} \|^2_{\Sigma_f};
\]

and for all \( y, \tilde{y} \in \text{dom } g, v, \tilde{v} \in \partial g(y) \) and \( \tilde{v} \in \partial g(\tilde{y}) \),

\[
g(x) \geq g(\tilde{y}) + \langle \tilde{v}, y - \tilde{y} \rangle + \frac{1}{2} \| y - \tilde{y} \|^2_{\Sigma_g} \quad \text{and} \quad \langle v - \tilde{v}, y - \tilde{y} \rangle \geq \| y - \tilde{y} \|^2_{\Sigma_g}.
\]

For a self-adjoint linear operator \( T : \mathbb{X} \to \mathbb{X} \), the notation \( T \geq 0 \) (respectively, \( T > 0 \)) means that \( T \) is positive semidefinite (respectively, positive definite), that is, \( \langle x, T x \rangle \geq 0 \) for all \( x \in \mathbb{X} \) (respectively, \( \langle x, T x \rangle > 0 \) for all \( x \in \mathbb{X} \setminus \{0\} \)). Given a self-adjoint positive semidefinite linear operator \( T : \mathbb{X} \to \mathbb{X} \), we denote by \( \| \cdot \|_T \) the norm induced by \( T \), i.e.,

\[
\| x \|_T := \sqrt{\langle x, T x \rangle} \quad \forall x \in \mathbb{X}.
\]

Given a self-adjoint positive definite linear operator, we denote by \( \lambda_{\text{max}}(T) \) and \( \lambda_{\text{min}}(T) \) the largest eigenvalue and the smallest eigenvalue of \( T \), respectively, and by \( D_T(x, \Omega) \) the distance induced by \( T \) from \( x \) to a closed set \( \Omega \), that is, \( D_T(x, \Omega) := \min_{z \in \Omega} \| z - x \|_T \).

When \( T \) is the identity operator, we suppress the notation \( T \) in \( D_T(x, \Omega) \) and write simply \( D(x, \Omega) \). Clearly, for any positive definite linear operator \( T : \mathbb{X} \to \mathbb{X} \) and \( \gamma > 0 \),

\[
2|\langle u, v \rangle| \leq \gamma^{-1} \| u \|_T^2 + \| v \|_T^2 \quad \forall u, v \in \mathbb{X}.
\]

In addition, for any \( u, v \in \mathbb{X} \) and any self-adjoint linear operator \( T : \mathbb{X} \to \mathbb{X} \), the following two identities will be frequently used in the subsequent analysis:

\[
2\langle u, T v \rangle = \langle u, T u \rangle + \langle v, T v \rangle - \langle u - v, T(u - v) \rangle = \langle u + v, T(u + v) \rangle - \langle u, T u \rangle - \langle v, T v \rangle.
\]

Throughout this paper, we make the following assumption for problem (1):

**Assumption 2.1** Problem (1) has an optimal solution, to say \( (x^*, y^*) \in \text{dom } f \times \text{dom } g \), and there exists a point \( (\hat{x}, \hat{y}) \in \text{ri(dom } f \times \text{dom } g) \) such that \( A^* \hat{x} + B^* \hat{y} = b \).

Under Assumption 2.1, from [27, Corollary 28.2.2 & 28.3.1] and [27, Theorem 6.5 & 23.8], it follows that there exists a Lagrange multiplier \( z^* \in \mathbb{Z} \) such that

\[
-Az^* \in \partial f(x^*), \quad -Bz^* \in \partial g(x^*) \quad \text{and} \quad A^* x^* + B^* y^* - c = 0
\]

where \( \partial f \) and \( \partial g \) are the subdifferential mappings of \( f \) and \( g \), respectively. Moreover, any \( z^* \in \mathbb{Z} \) satisfying (9) is an optimal solution to the dual problem of (1). In the sequel, we call \( (x^*, y^*, z^*) \in \text{dom } f \times \text{dom } g \times \mathbb{Z} \) a primal-dual solution pair of problem (1).
3 Inexact indefinite proximal ADMMs

In this section, we describe the iteration steps of the inexact indefinite proximal ADMMs for solving problem (1), and then analyze the properties of the sequence generated.

The iteration steps of our inexact indefinite proximal ADMMs are stated as follows.

| IEIDP-ADMM (Inexact indefinite proximal ADMM for (1)) |
|-------------------------------------------------------|
| (S.0) Let $\sigma, \tau > 0$ be given. Choose self-adjoint linear operators $P_f : X \rightarrow X$ and $P_g : Y \rightarrow Y$ such that $T_f := P_f + \sigma \mathcal{A}^* \mathcal{A} > 0$ and $T_g := P_g + \sigma \mathcal{B}^* \mathcal{B} > 0$. Choose an initial point $(x^0, y^0, z^0) \in \text{dom} f \times \text{dom} g \times X$. Set $k := 0$. |
| (S.1) Find $x^{k+1} := \arg \min_{x \in X} \phi_k(x) := L_\sigma(x, y^k, z^k) + \frac{1}{2} \|x - x^k\|^2_{T_f}$. |
| (S.2) Find $y^{k+1} := \arg \min_{y \in Y} \psi_k(y) := L_\sigma(x^{k+1}, y, z^k) + \frac{1}{2} \|y - y^k\|^2_{T_g}$. |
| (S.3) Update the Lagrange multiplier $z^{k+1}$ via the following formula $z^{k+1} = z^k + \tau \sigma (\mathcal{A}^* x^{k+1} + \mathcal{B}^* y^{k+1} - c)$. |
| (S.4) Let $k \leftarrow k + 1$, and go to Step (S.1). |

The approximate optimality in (S.1) and (S.2) is measured by the following criteria:

(C1) $D(0, \partial \phi_k(x^{k+1})) \leq \mu_{k+1}, D(0, \partial \psi_k(y^{k+1})) \leq \nu_{k+1}$ and $\sum_{k=0}^{\infty} \max(\mu_{k+1}, \nu_{k+1}) < \infty$;

(C2) $D_\mathcal{F}(0, \partial \phi_k(x^{k+1})) \leq \mu_{k+1} \|x^{k+1} - x^k\|_{T_f}, D_\mathcal{G}(0, \partial \psi_k(y^{k+1})) \leq \nu_{k+1} \|y^{k+1} - y^k\|_{T_g}$ and $\sum_{k=0}^{\infty} \max(\mu_{k+1}, \nu_{k+1}) < \infty$, where $\mathcal{F} : X \rightarrow X$ and $\mathcal{G} : Y \rightarrow Y$ are self-adjoint positive definite linear operators with $\mathcal{F}^{-1} \leq T_f$ and $\mathcal{G}^{-1} \leq T_g$;

(C2’) $D_\mathcal{F}(0, \partial \phi_k(x^{k+1})) \leq \mu_{k+1} \|x^{k+1} - x^k\|_{T_f}, D_\mathcal{G}(0, \partial \psi_k(y^{k+1})) \leq \nu_{k+1} \|y^{k+1} - y^k\|_{T_g}$ and $\sum_{k=0}^{\infty} \max(\mu_{k+1}^2, \nu_{k+1}^2) < \infty$, where $\mathcal{F}$ and $\mathcal{G}$ are same as the one in (C2).

Notice that (C1) is an absolute error criterion, while (C2) and (C2’) are relative error criteria. Clearly, when the approximate optimality of $x^{k+1}$ and $y^{k+1}$ is measured by (C1), (S.1) and (S.2) are equivalent to finding $(x^{k+1}, \xi^{k+1})$ and $(y^{k+1}, \eta^{k+1})$ such that

\[
\begin{align*}
{\xi}^{k+1} &\in \partial \phi_k(x^{k+1}), \quad \|{\xi}^{k+1}\| \leq \mu_{k+1} \quad \text{with} \quad \sum_{k=0}^{\infty} \mu_{k+1} < \infty, \\
{\eta}^{k+1} &\in \partial \psi_k(y^{k+1}), \quad \|{\eta}^{k+1}\| \leq \nu_{k+1} \quad \text{with} \quad \sum_{k=0}^{\infty} \nu_{k+1} < \infty.
\end{align*}
\]

If the approximate optimality of $x^{k+1}$ and $y^{k+1}$ is measured by (C2) or (C2’), (S.1) and (S.2) are equivalent to finding $(x^{k+1}, \xi^{k+1})$ and $(y^{k+1}, \eta^{k+1})$ such that with $p = 1$ or 2,

\[
\begin{align*}
{\xi}^{k+1} &\in \partial \phi_k(x^{k+1}), \quad \|{\xi}^{k+1}\| \leq \mu_{k+1} \|x^{k+1} - x^k\|_{T_f} \quad \text{with} \quad \sum_{k=0}^{\infty} \mu_{k+1}^p < \infty, \\
{\eta}^{k+1} &\in \partial \psi_k(y^{k+1}), \quad \|{\eta}^{k+1}\| \leq \nu_{k+1} \|y^{k+1} - y^k\|_{T_g} \quad \text{with} \quad \sum_{k=0}^{\infty} \nu_{k+1}^p < \infty.
\end{align*}
\]

(10)
Remark 3.1 (a) When the proximal operators $P_F$ and $P_g$ are chosen as $\beta I$ for a constant $\beta > 0$ and the step-size $\tau$ is set to be $1$, the IEIDP-ADMM with (C1) reduces to the IADM1 in [24]. If, in addition, taking $F = G = \frac{1}{\beta}I$, the IEIDP-ADMM with (C2') requires
\[
\left\{ \begin{array}{ll}
\xi^{k+1} \in \partial \phi_k(x^{k+1}), & \|\xi^{k+1}\| \leq \mu_{k+1}\sqrt{\beta}\|x^{k+1} - x^k\|_{\sigma AA^* + \beta I} \\
\eta^{k+1} \in \partial \psi_k(y^{k+1}), & \|\eta^{k+1}\| \leq \nu_{k+1}\sqrt{\beta}\|y^{k+1} - y^k\|_{\sigma BB^* + \beta I}
\end{array} \right. \\
\text{with} \quad \sum_{k=0}^{\infty} \frac{\nu^2_{k+1}}{\nu^2_k} < \infty,
\]
whereas the LADM2 in [24] is actually requiring that $\xi^{k+1}$ and $\eta^{k+1}$ satisfy
\[
\left\{ \begin{array}{ll}
\xi^{k+1} \in \partial \phi_k(x^{k+1}), & \|\xi^{k+1}\| \leq \mu_{k+1}\beta\|x^{k+1} - x^k\| \\
\eta^{k+1} \in \partial \psi_k(y^{k+1}), & \|\eta^{k+1}\| \leq \nu_{k+1}\beta\|y^{k+1} - y^k\| \end{array} \right. \\
\text{with} \quad \sum_{k=0}^{\infty} \frac{\nu^2_{k+1}}{\nu^2_k} < \infty.
\]
Since $\sqrt{\beta}\|x^{k+1} - x^k\|_{\sigma AA^* + \beta I} \geq \beta\|x^{k+1} - x^k\|$ and $\sqrt{\beta}\|y^{k+1} - y^k\|_{\sigma BB^* + \beta I} \geq \beta\|y^{k+1} - y^k\|$, the above inexact criterion (C2') is looser than Criterion 2 used in [24].

(b) When $P_F$ and $P_g$ are chosen to be self-adjoint positive semidefinite operators, the IEIDP-ADMMs with $\mu_k \equiv \nu_k \equiv 0$ reduce to the semi-proximal ADMM in [38, 9].

(c) For the self-adjoint positive definite linear operators $F$ and $G$ in (C2) and (C2'), an immediate choice is $F = \frac{1}{\lambda_{\min}(F)}I$ and $G = \frac{1}{\lambda_{\min}(G)}I$. Since $\lambda_{\min}(F)$ and $\lambda_{\min}(G)$ are easy to estimate, such a choice is convenient for the numerical implementation.

Next we study the properties of the sequence generated by the IEIDP-ADMMs. For convenience, we let $h(x, y) := A^*x + B^*y - c$ for $(x, y) \in X \times Y$, and for each $k \geq 1$ write
\[
x^k := x^k - x^*, \quad y^k := y^k - y^*, \quad z^k := z^k - z^*;
\]
\[
\Delta y^k := y^k - y^{k-1}, \quad \Delta x^k := x^k - x^{k-1}, \quad \Delta z^k := z^k - z^{k-1}.
\]
Using these notations and noting that $h(x^*, y^*) = 0$, we can rewrite Step (S.3) as
\[
z^k = z^{k+1} - \tau \sigma h(x^{k+1}, y^{k+1}) = z^{k+1} - \tau \sigma (A^*x^k + B^*y^k).
\]
(12)

Lemma 3.1 Let $\{(x^k, y^k, z^k)\}$ be the sequence generated by the IEIDP-ADMMs with $(x^k, \xi^k)$ and $(y^k, \eta^k)$ satisfying equation (10) or (11). Suppose that Assumption 2.1 holds and the operator $P_g$ also satisfies $P_g + \frac{\beta}{2}I \succeq 0$. Then, for all $k \geq 0$ we have
\[
(2-\tau)\sigma\|h(x^{k+1}, y^{k+1})\|^2 + (\tau \sigma)^{-1}(\|x^{k+1}\|^2 - \|z^{k+1}\|^2) + \|y^{k+1}\|^2_{P_g} - \|y^{k+1}\|^2_{\frac{\beta}{2}I} + \|\Delta y^{k+1}\|^2_{\frac{\beta}{2}I} \leq 2(1-\tau)\sigma h(x^k, y^k, B^*\Delta y^{k+1}) + r^{k+1} - \|\Delta x^{k+1}\|^2_0 - \|\Delta y^{k+1}\|^2_{\frac{\beta}{2}I},
\]
where $r^{k+1} := 2\langle x^{k+1}, \xi^{k+1} \rangle + 2\langle y^{k+1}, \eta^{k+1} \rangle + 2\|\xi^{k+1} - \eta^{k, \Delta y^{k+1}}\|$.

Proof: From the expressions of $\phi_k$ and $\psi_k$ and equations (10) and (11), it follows that
\[
\xi^{k+1} = A^*z^k - \sigma A(x^k + B^*y^k - c) - P_F \Delta x^{k+1} = \partial f(x^{k+1}),
\]
\[
\eta^{k+1} = B^*z^k - \sigma B(x^k + B^*y^k - c) - P_g \Delta y^{k+1} = \partial g(y^{k+1}).
\]
(13)
(14)
Substituting the first identity in (12) into equations (13) and (14) respectively yields

\[(\tau - 1)\sigma Ah(x^{k+1}, y^{k+1}) - Az^{k+1} + \sigma AB^* \Delta y^{k+1} - P_\delta \Delta z^{k+1} + \xi^{k+1} \in \partial f(x^{k+1}),\]
\[(\tau - 1)\sigma Bh(x^{k+1}, y^{k+1}) - Bz^{k+1} - P_\delta \Delta y^{k+1} + \eta^{k+1} \in \partial g(y^{k+1}).\]

In view of inequalities (5) and (6), from the last two inclusions and equation (9) we have

\[\langle x_e^{k+1}, (\tau - 1)\sigma Ah(x^{k+1}, y^{k+1}) - Az^{k+1} + \sigma AB^* \Delta y^{k+1} - P_\delta \Delta z^{k+1} + \xi^{k+1} \rangle \geq \|x_e^{k+1}\|^2_{\Sigma_y},\]
\[\langle y_e^{k+1}, (\tau - 1)\sigma Bh(x^{k+1}, y^{k+1}) - Bz^{k+1} - P_\delta \Delta y^{k+1} + \eta^{k+1} \rangle \geq \|y_e^{k+1}\|^2_{\Sigma_y}.\]

Adding the last two inequalities together and using equation (12) yields that

\[\langle x_e^{k+1}, (\tau - 1)\|h(x^{k+1}, y^{k+1})\|^2 - (\tau \sigma)^{-1}\langle \Delta z^{k+1}, x_e^{k+1} \rangle + \sigma \langle h(x^{k+1}, y^{k+1}), B^* \Delta y^{k+1} \rangle \rangle - \langle x_e^{k+1}, P_\delta \Delta z^{k+1} - \xi^{k+1} \rangle - \langle y_e^{k+1}, (P_\delta + \sigma BB^*) \Delta y^{k+1} - \eta^{k+1} \rangle \geq \|x_e^{k+1}\|^2_{\Sigma_y} + \|y_e^{k+1}\|^2_{\Sigma_y}.\]

Next we deal with the term \(\sigma \langle h(x^{k+1}, y^{k+1}), B^* \Delta y^{k+1} \rangle\) in inequality (15). Notice that

\[\sigma \langle h(x^{k+1}, y^{k+1}), B^* \Delta y^{k+1} \rangle = (1 - \tau)\sigma \langle h(x^{k+1}, y^{k+1}) - h(x^k, y^k), B^* \Delta y^{k+1} \rangle + \langle \Delta z^{k+1}, B^* \Delta y^{k+1} \rangle + (1 - \tau)\sigma \langle h(x^k, y^k), B^* \Delta y^{k+1} \rangle.\]

We first bound the first two terms in (15). From equations (14) and (12), it follows that

\[-Bz^{k+1} + (\tau - 1)\sigma Bh(x^{k+1}, y^{k+1}) - P_\delta \Delta y^{k+1} + \eta^{k+1} \in \partial g(y^{k+1}),\]
\[-Bz^k + (\tau - 1)\sigma Bh(x^k, y^k) - P_\delta \Delta y^k + \eta^k \in \partial g(y^k).\]

Combining the last two inclusions with the second inequality in (6) yields that

\[\langle \Delta y^{k+1} - \Delta y^k, P_\delta \Delta y^{k+1} \rangle + \langle \eta^{k+1} - \eta^k, \Delta y^{k+1} \rangle \geq \|\Delta y^{k+1}\|^2_{\Sigma_y}.\]

Using equation (8) and the given assumption \(P_\delta + \frac{3}{4}\Sigma_y \geq 0\), we have that

\[\langle \Delta y^{k+1}, P_\delta \Delta y^{k+1} \rangle = \frac{1}{2}\|y^{k+1} - y^k\|^2_{P_\delta} - \frac{1}{2}\|\Delta y^k\|^2_{P_\delta} - \frac{1}{2}\|\Delta y^{k+1}\|^2_{P_\delta} - \|\Delta y^{k+1}\|^2_{\Sigma_y} \leq \frac{1}{2}\|y^{k+1} - y^k\|^2_{P_\delta} - \frac{1}{2}\|\Delta y^k\|^2_{P_\delta} - \frac{3}{2}\|\Delta y^{k+1}\|^2_{P_\delta} - \frac{3}{4}\|\Delta y^{k+1}\|^2_{\Sigma_y},\]

where the last inequality is using \(\frac{1}{2}\|y^{k+1} - y^k\|^2_{P_\delta} \leq \|\Delta y^k\|^2_{P_\delta} + \frac{3}{4}\Sigma_y \leq \|\Delta y^{k+1}\|^2_{P_\delta} + \frac{3}{4}\Sigma_y + \frac{3}{4}\|\Delta y^{k+1}\|^2_{\Sigma_y}.\)

Combining inequalities (18) and (17) with equation (16), we immediately obtain

\[\sigma \langle h(x^{k+1}, y^{k+1}), B^* \Delta y^{k+1} \rangle \leq (1 - \tau)\sigma \langle h(x^k, y^k), B^* \Delta y^{k+1} \rangle + \langle \eta^{k+1} - \eta^k, \Delta y^{k+1} \rangle + \frac{1}{2}\|\Delta y^k\|^2_{P_\delta} + \frac{3}{4}\Sigma_y - \frac{1}{2}\|\Delta y^{k+1}\|^2_{P_\delta} - \frac{3}{4}\|\Delta y^{k+1}\|^2_{\Sigma_y}.\]
Now substituting inequality \((19)\) into equation \((15)\), we immediately obtain that
\[
(\tau - 1)\sigma \| h(x^{k+1}, y^{k+1}) \|^2 - (\tau \sigma)^{-1} \langle \Delta z^{k+1}, z^{k+1} \rangle - \langle x^{k+1}, P_j \Delta x^{k+1} \rangle \\
+ (1 - \tau) \sigma \langle h(x^k, y^k), B^* \Delta y^{k+1} \rangle + \frac{1}{2} \| \Delta y^k \|^2_{P^*_\rho + \frac{3}{2} \Sigma_y} - \frac{1}{2} \| \Delta y^{k+1} \|^2_{P^*_\rho + \frac{3}{2} \Sigma_y} \\
- \langle y^{k+1}_e, (P_g + \sigma BB^*) \Delta y^{k+1} \rangle + \frac{1}{2} \| x^{k+1} \|^2_{\Sigma_y} + \frac{1}{2} \| y^{k+1}_e \|^2_{\Sigma_y} + \frac{1}{4} \| \Delta y^{k+1} \|^2_{\Sigma_y}. \tag{20}
\]
By the first equality of \((8)\) and equation \((12)\), the term \(\langle \Delta z^{k+1}, z^{k+1} \rangle\) can be written as
\[
\langle \Delta z^{k+1}, z^{k+1} \rangle = \frac{1}{2} \| z^{k+1} \|^2 - \frac{1}{2} \| z^{k} \|^2 + \frac{(\tau \sigma)^2}{2} \| h(x^{k+1}, y^{k+1}) \|^2.
\]
Applying equation \((8)\) to \(\langle x^{k+1}_e, P_j \Delta x^{k+1} \rangle\) and \(\langle y^{k+1}_e, (P_g + \sigma BB^*) \Delta y^{k+1} \rangle\) yields
\[
\langle x^{k+1}_e, P_j \Delta x^{k+1} \rangle = \frac{1}{2} \| x^{k+1}_e \|^2_{P^*_\rho} - \frac{1}{2} \| x^{k}_e \|^2_{P^*_\rho} + \frac{1}{2} \| \Delta x^{k+1} \|^2_{P^*_\rho},
\]
\[
\langle y^{k+1}_e, (P_g + \sigma BB^*) \Delta y^{k+1} \rangle = \frac{1}{2} \| y^{k+1}_e \|^2_{P^*_\rho + \sigma BB^*} - \frac{1}{2} \| y^{k}_e \|^2_{P^*_\rho + \sigma BB^*} + \frac{1}{2} \| \Delta y^{k+1} \|^2_{P^*_\rho + \sigma BB^*}.
\]
Substituting the last three equalities into inequality \((20)\), we have that
\[
(\tau - 2)\sigma \| h(x^{k+1}, y^{k+1}) \|^2 + (\tau \sigma)^{-1} (\| z^{k+1} \|^2 - \| z^{k} \|^2) + \frac{1}{2} \| \Delta y^k \|^2_{P^*_\rho + \frac{3}{2} \Sigma_y} - \frac{1}{2} \| \Delta y^{k+1} \|^2_{P^*_\rho + \frac{3}{2} \Sigma_y} \\
+ 2(1 - \tau) \sigma \langle h(x^k, y^k), B^* \Delta y^{k+1} \rangle + r^{k+1} - \| \Delta y^{k+1} \|^2_{P^*_\rho} \\
\geq \| x^{k+1}_e \|^2_{\Sigma_y} + \| y^{k+1}_e \|^2_{\Sigma_y} + \| \Delta x^{k+1} \|^2_{P^*_\rho} + \| y^{k+1}_e \|^2_{\Sigma_y} + \| y^{k}_e \|^2_{\Sigma_y} - \frac{1}{2} \| \Delta y^{k+1} \|^2_{\Sigma_y}.
\]
Notice that \(\| x^{k+1}_e \|^2_{\Sigma_y} + \| y^{k}_e \|^2_{\Sigma_y} \geq \frac{1}{2} \| \Delta x^{k+1} \|^2_{\Sigma_y}\) and \(\| y^{k+1}_e \|^2_{\Sigma_y} + \| y^{k}_e \|^2_{\Sigma_y} \geq \frac{1}{2} \| \Delta y^{k+1} \|^2_{\Sigma_y}\). The last inequality implies the desired result. The proof is completed.

The following lemma provides an upper bound for the term \(r^{k+1}\) in Lemma 3.1.

**Lemma 3.2** If \((C1)\) is used in \((S.1)\) and \((S.2)\), then for any given \(\gamma > 0\) we have
\[
| r^{k+1} | \leq \gamma^{-1} (\mu_{k+1} | x^{k+1}_e \|^2_{T_f} + \nu_{k+1} | y^{k+1}_e \|^2_{T_g}) + \gamma^{-1} (\nu_{k+1} + \nu_k) \| \Delta y^{k+1} \|^2_{T_g} \\
+ \gamma \| T_f^{-1} \| \mu_{k+1} + \gamma \| T_g^{-1} \| (2\nu_{k+1} + \nu_k) \quad \text{for } k \geq 1; \tag{21}
\]
if the criterion \((C2)\) is used for the minimization in \((S.1)\) and \((S.2)\), then
\[
| r^{k+1} | \leq \mu_{k+1} | x^{k+1}_e \|^2_{T_f} + \nu_{k+1} | y^{k+1}_e \|^2_{T_g} + \mu_{k+1} \| \Delta x^{k+1} \|^2_{T_f} \\
+ (\nu_k + 3\nu_{k+1}) \| \Delta y^{k+1} \|^2_{T_g} + \nu_k \| \Delta y^k \|^2_{T_g} \quad \text{for } k \geq 1; \tag{22}
\]
and if \((C2')\) is used for the minimization in \((S.1)\) and \((S.2)\), then for any given \(\gamma > 0\),
\[
| r^{k+1} | \leq \gamma(\mu_{k+1} | x^{k+1}_e \|^2_{T_f} + \nu_{k+1} | y^{k+1}_e \|^2_{T_g}) + \gamma^{-1} \| \Delta x^{k+1} \|^2_{T_f} \\
+ (2\gamma^{-1} + \nu_k^2 + \nu_{k+1}^2) \| \Delta y^{k+1} \|^2_{T_g} + \gamma^{-1} \| \Delta y^k \|^2_{T_g} \quad \text{for } k \geq 1. \tag{23}
\]
Proof: When the criterion (C1) is used in (S.1) and (S.2), for any given \( \gamma > 0 \) we have

\[
2 \langle x_e^{k+1}, \xi^{k+1} \rangle \leq \frac{\mu_{k+1}}{\gamma} \| x_e^{k+1} \|_T^2 + \gamma \| T_f \| \mu_{k+1},
\]

\[
2 \langle y_e^{k+1}, \eta^{k+1} \rangle \leq \frac{\nu_{k+1}}{\gamma} \| y_e^{k+1} \|_T^2 + \gamma \| T_g \| \nu_{k+1},
\]

\[
2 \langle \eta^{k+1} - \eta^{k}, \Delta y^{k+1} \rangle \leq \frac{\nu_{k+1} + \nu_k}{\gamma} \| \Delta y^{k+1} \|_T^2 + \gamma \| T_g \| (\nu_{k+1} + \nu_k)
\)

for all \( k \geq 1 \). Indeed, when \( \mu_{k+1} = 0 \), the first inequality in (24) holds since now \( \xi^{k+1} = 0 \); and when \( \mu_{k+1} > 0 \), from equations (7) and (10) we have

\[
2 \langle x_e^{k+1}, \xi^{k+1} \rangle \leq \frac{\mu_{k+1}}{\gamma} \| x_e^{k+1} \|_T^2 + \frac{\gamma}{\mu_{k+1}} \| T_f \| \| \xi^{k+1} \|_2^2 \leq \frac{\mu_{k+1}}{\gamma} \| x_e^{k+1} \|_T^2 + \gamma \| T_f \| \mu_{k+1},
\]

Similarly, we can prove that the last two inequalities hold for all \( k \geq 1 \). Adding the three inequalities in (24) yields (21). When (C2) is used in (S.1) and (S.2), for all \( k \geq 1 \)

\[
2 \langle x_e^{k+1}, \xi^{k+1} \rangle \leq \mu_{k+1} \| x_e^{k+1} \|_T^2 + \mu_{k+1} \| \Delta x^{k+1} \|_T^2,
\]

\[
2 \langle y_e^{k+1}, \eta^{k+1} \rangle \leq \nu_{k+1} \| y_e^{k+1} \|_T^2 + \nu_{k+1} \| \Delta y^{k+1} \|_T^2,
\]

\[
2 \langle \eta^{k+1} - \eta^{k}, \Delta y^{k+1} \rangle \leq \left( \nu_k + 2\nu_{k+1} \right) \| \Delta y^{k+1} \|_T^2 + \nu_k \| \Delta y^{k} \|_T^2
\]

Indeed, when \( \nu_{k+1} = 0 \), the second inequality in (25) holds since now \( \eta^{k+1} = 0 \); and when \( \nu_{k+1} \neq 0 \), from equation (7) and \( F^{-1} \leq T_f \) and \( G^{-1} \leq T_g \), it follows that

\[
2 \langle y_e^{k+1}, \eta^{k+1} \rangle \leq \nu_{k+1} \| y_e^{k+1} \|_G^{-1} + \frac{\gamma}{\nu_{k+1}} \| \eta^{k+1} \|_G \leq \nu_{k+1} \| y_e^{k+1} \|_T^2 + \nu_{k+1} \| \Delta y^{k+1} \|_T^2
\]

Similarly, we can prove that another two inequalities hold for all \( k \geq 1 \). Summing up the three inequalities in (25) yields (22). When the criterion (C2') is used, for any \( \gamma > 0 \),

\[
2 \langle x_e^{k+1}, \xi^{k+1} \rangle \leq \gamma \mu_{k+1} \| x_e^{k+1} \|_T^2 + \frac{1}{\gamma} \| \Delta x^{k+1} \|_T^2,
\]

\[
2 \langle y_e^{k+1}, \eta^{k+1} \rangle \leq \gamma \nu_{k+1} \| y_e^{k+1} \|_T^2 + \frac{1}{\gamma} \| \Delta y^{k+1} \|_T^2,
\]

\[
2 \langle \eta^{k+1} - \eta^{k}, \Delta y^{k+1} \rangle \leq (\gamma^{-1} + \gamma \nu_{k+1}^2) \| \Delta y^{k+1} \|_T^2 + \gamma^{-1} \| \Delta y^{k} \|_T^2
\]

for all \( k \geq 1 \). Indeed, when \( \nu_k = 0 \), the third inequality in (26) holds since now \( \eta^k = 0 \); and when \( \nu_k \neq 0 \), from equation (7) and \( F^{-1} \leq T_f \) and \( G^{-1} \leq T_g \), it follows that

\[
2 \langle \eta^{k+1} - \eta^k, \Delta y^{k+1} \rangle \leq (\gamma \nu_{k+1}^2 + \gamma \nu_{k+1}^2) \| \Delta y^{k+1} \|_T^2 + \frac{1}{\gamma} \| \eta^{k+1} \|_T^2 + \frac{1}{\gamma} \| \eta^{k+1} \|_T^2
\]

Similarly, one can prove that another two inequalities hold for all \( k \geq 1 \). Summing up the three inequalities in (26) yields (23). The proof is completed.

Based on the results of Lemmas 3.1 and 3.2, we obtain the following proposition.
Proposition 3.1 Let \( \{(x^k, y^k, z^k)\}\) be the sequence generated by the IEIPD-ADMMs. Suppose that Assumption 2.1 holds and the operator \( T_g \) also satisfies \( T_g + \frac{\sigma}{2} \Sigma_g \geq 0 \).

(a) If the criterion (C1) is used in (S.1) and (S.2), then for any given \( \gamma > 0 \) we have

\[
\begin{align*}
& (1 - \mu_{k+1}) \|x^{k+1}_e\|_{\Sigma_{\gamma}}^2 - \|x^k_e\|_{\Sigma_{\gamma}}^2 + (1 - \frac{\mu_{k+1} + \nu_{k+1}}{\gamma}) \|y^{k+1}_e\|_{\Sigma_{g}}^2 - \|y^k_e\|_{\Sigma_{g}}^2 \\
& - \|y^k\|_{\Sigma_{g}}^2 (\sigma - 1) \|z^{k+1}_e\|^2 + \|\Delta y^{k+1}_e\|_{\Sigma_{\gamma}}^2 - \|\Delta y^{k}_e\|_{\Sigma_{\gamma}}^2 - \|\Delta y^{k}_{\Sigma_{g}}\|_{\Sigma_{\gamma}}^2 \\
& + (2 - \tau - 2\gamma^{-1}\mu_{k+1}) \|h(x^{k+1}, y^{k+1})\|^2 \\
& \leq 2(1 - \tau)\|h(x^k, y^k), B^* \Delta y^{k+1}\| - (1 - \gamma^{-1}\nu_{k+1} - \gamma^{-1}\nu_{k}) \|\Delta y^{k+1}\|_g^2 \\
& - \|\Delta x^{k+1}_e\|_{\Sigma_{\gamma}} + \gamma \|T_f^{-1}\|\mu_{k+1} + \gamma \|T_g^{-1}\| (2\nu_{k+1} + \nu_k) \quad \text{for } k \geq 1.
\end{align*}
\]

(b) If the criterion (C2) is used for (S.1) and (S.2), then we have

\[
\begin{align*}
& (1 - \mu_{k+1}) \|x^{k+1}_e\|_{\Sigma_{\gamma}}^2 - \|x^k_e\|_{\Sigma_{\gamma}}^2 + (1 - \nu_{k+1} - 5\mu_{k+1}) \|y^{k+1}_e\|_{\Sigma_{g}}^2 - \|y^k_e\|_{\Sigma_g}^2 \\
& + (\sigma - 1) \|z^{k+1}_e\|^2 + \nu_{k+1} \|\Delta y^{k+1}_e\|_{\Sigma_{\gamma}}^2 - \|\nu_k\| \|\Delta y^{k}_e\|_{\Sigma_{\gamma}}^2 - 6\mu_{k+1}\|h(x^{k+1}, y^{k+1})\|^2 \\
& + (2 - \tau - 3\mu_{k+1}) \|h(x^{k+1}, y^{k+1})\|^2 + \|\Delta y^{k+1}_e\|_{\Sigma_{\gamma}}^2 - \|\Delta y^{k}_e\|_{\Sigma_{\gamma}}^2 - \|\Delta y^{k}_{\Sigma_{g}}\|_{\Sigma_{\gamma}}^2 \\
& \leq 2(1 - \tau)\|h(x^k, y^k), B^* \Delta y^{k+1}\| - (1 - \nu_{k} - 4\nu_{k+1} - 4\mu_{k+1}) \|\Delta y^{k+1}\|_g^2 \\
& - \|\Delta x^{k+1}_e\|_{\Sigma_{\gamma}} + \gamma \|T_f^{-1}\|\mu_{k+1} + \gamma \|T_g^{-1}\| (2\nu_{k+1} + \nu_k) \quad \text{for } k \geq 1.
\end{align*}
\]

(c) If the criterion (C2') is used for (S.1) and (S.2), then for any given \( \gamma > 0 \) we have

\[
\begin{align*}
& (1 - \gamma\mu_{k+1}^2) \|x^{k+1}_e\|_{\Sigma_{\gamma}}^2 - \|x^k_e\|_{\Sigma_{\gamma}}^2 + (1 - \nu_{k+1} - 3\gamma\mu_{k+1}^2) \|y^{k+1}_e\|_{\Sigma_{g}}^2 - \|y^k_e\|_{\Sigma_g}^2 \\
& + (\sigma - 1) \|z^{k+1}_e\|^2 + \nu_{k+1} \|\Delta y^{k+1}_e\|_{\Sigma_{\gamma}}^2 - \|\nu_k\| \|\Delta y^{k}_e\|_{\Sigma_{\gamma}}^2 - 1.5\gamma\mu_{k+1} \|h(x^{k+1}, y^{k+1})\|^2 \\
& + \|\Delta y^{k+1}_e\|_{\Sigma_{\gamma}}^2 - \|\Delta y^{k}_e\|_{\Sigma_{\gamma}}^2 - \|\Delta y^{k}_{\Sigma_{g}}\|_{\Sigma_{\gamma}}^2 \\
& \leq 2(1 - \tau)\|h(x^k, y^k), B^* \Delta y^{k+1}\| + \|\Delta x^{k+1}_e\|_{\Sigma_{\gamma}}^2 + \gamma \|T_f^{-1}\|\mu_{k+1} + \gamma \|T_g^{-1}\| (2\nu_{k+1} + \nu_k) \\
& - (1 - 6\gamma^{-1} - \gamma\nu_{k+1}^2 - \gamma\nu_{k+1}^2) \|\Delta y^{k+1}\|_g^2 + \gamma^{-1} \|h(x^k, y^k)\|^2 \quad \text{for } k \geq 1.
\end{align*}
\]

Proof: (a) From inequality (21) and the result of Lemma 3.1, it follows that

\[
\begin{align*}
& (2 - \tau)\|h(x^{k+1}, y^{k+1})\|^2 + \frac{1}{\sigma} \|z^{k+1}_e\|^2 - \|z^k_e\|^2 + (1 - \frac{\nu_{k+1}}{\sigma}) \|y^{k+1}_e\|_{\Sigma_{g}}^2 - \|y^{k}_e\|_{\Sigma_{g}}^2 \\
& + (1 - \frac{h_{k+1}}{\gamma}) \|x^{k+1}_e\|_{\Sigma_{\gamma}}^2 - \|x^k_e\|_{\Sigma_{\gamma}}^2 + \|\Delta y^{k+1}_e\|_{\Sigma_{\gamma}}^2 - \|\Delta y^{k}_e\|_{\Sigma_{\gamma}}^2 - \|\Delta y^{k}_{\Sigma_{g}}\|_{\Sigma_{\gamma}}^2 \\
& \leq 2(1 - \tau)\|h(x^k, y^k), B^* \Delta y^{k+1}\| - \|\Delta x^{k+1}_e\|_{\Sigma_{\gamma}} + \gamma \|T_f^{-1}\|\mu_{k+1} + \gamma \|T_g^{-1}\| (2\nu_{k+1} + \nu_k).
\end{align*}
\]

Since \( \frac{1}{2} \|x^{k+1}_e\|_{\Sigma_{\Sigma_{g}}}^2 \leq \sigma \|h(x^{k+1}, y^{k+1})\|^2 + \|y^{k+1}_e\|_{\Sigma_{g}}^2 \leq \sigma \|h(x^{k+1}, y^{k+1})\|^2 + \|y^{k+1}_e\|_{\Sigma_{g}}^2 \)
where the second inequality is due to \( \sigma B^* \leq T_g \), the last inequality implies part (a).
(b) From inequality (22) and the result of Lemma 3.1, it follows that
\[
(2 - \tau) \sigma \| h(x^{k+1}, y^{k+1}) \|^2 + (\tau - 1) \left( \| z^{k+1}_e \|^2 - \| z_e^{k+1} \|^2 \right) + (1 - \nu_{k+1}) \| y^{k+1} \|^2 \mathcal{T}_g - \| y_e^k \|^2_{\mathcal{T}_g} + (1 - \mu_{k+1}) \| x_e^{k+1} \|^2_{\mathcal{T}_f + \Sigma_f} - \| x_e^k \|^2_{\mathcal{T}_f + \Sigma_f} + \nu_{k+1} \| \Delta y^{k+1} \|^2_{\mathcal{T}_g} - \| \Delta y^k \|^2_{\mathcal{T}_g} + \| \Delta y^{k+1} \|^2_{\mathcal{T}_g} + \| \Delta y^k \|^2_{\mathcal{T}_g}
\leq 2(1 - \tau) \sigma \langle h(x^k, y^k), \mathcal{B}^*(y^{k+1} - y^k) \rangle - (1 - \nu_k - 4\nu_{k+1}) \| \Delta y^{k+1} \|^2_{\mathcal{T}_g} + \mu_{k+1} \| \Delta x^{k+1} \|^2_{\mathcal{A}_A^*} + \mu_k \| \Delta x^{k+1} \|^2_{\mathcal{A}_A^*} - \| \Delta x^{k+1} \|^2_{\mathcal{A}_A^*} - \nu_k \| \Delta x^{k+1} \|^2_{\mathcal{A}_A^*} (29)
\]

For the terms \( \| x_e^{k+1} \|^2_{\mathcal{A}_A^*} \) and \( \| \Delta x^{k+1} \|^2_{\mathcal{A}_A^*} \), using equation (7) yields that
\[
\| x_e^{k+1} \|^2_{\mathcal{A}_A^*} = \| h(x^{k+1}, y^{k+1}) \|^2 - \| x_e^k \|^2_{\mathcal{B}_A^*} - 2\langle h(x^k, y^k), \mathcal{B}_A^* y^{k+1} \rangle \leq \| h(x^{k+1}, y^{k+1}) \|^2 + \| x_e^{k+1} \|^2_{\mathcal{B}_A^*} + \frac{1}{4} \| h(x^{k+1}, y^{k+1}) \|^2 + 4 \| y^{k+1} \|^2_{\mathcal{B}_A^*},
\]
\[
\| \Delta x^{k+1} \|^2_{\mathcal{A}_A^*} = \| h(x^{k+1}, y^{k+1}) \|^2 + \| h(x^{k}, y^{k}) \|^2 + \| \Delta y^{k+1} \|^2_{\mathcal{B}_A^*} - 2(h(x^{k+1}, y^{k+1}), h(x^{k}, y^{k})) + 2(h(x^{k}, y^{k}) - h(x^{k+1}, y^{k+1}), \mathcal{B}_A^* \Delta y^{k+1}) \leq \| h(x^{k+1}, y^{k+1}) \|^2 + \| h(x^{k}, y^{k}) \|^2 + \| \Delta y^{k+1} \|^2_{\mathcal{B}_A^*} + \frac{1}{4} \| h(x^{k+1}, y^{k+1}) \|^2 + 4 \| h(x^{k}, y^{k}) \|^2 + \| \Delta y^{k+1} \|^2_{\mathcal{B}_A^*} + \frac{1}{2} \| h(x^{k+1}, y^{k+1}) \|^2 + 2 \| \Delta y^{k+1} \|^2_{\mathcal{B}_A^*}.
\]

Combining the last inequalities with (29) and using \( \sigma \mathcal{B}_A^* \preceq \mathcal{T}_g \) yields part (b).

The proof of Part (c) is similar to that of part (b), we here omit it. \( \Box \)

4 Convergence analysis of the IEIDP-ADMMs

In this section we analyze the convergence of the IEIDP-ADMMs with the approximation criterion (C1) and (C2) respectively chosen for the minimization in (S.1) and (S.2).

4.1 Convergence of the IEIDP-ADMM with (C1)

For convenience, we write \( w^k := (x^k; y^k; z^k; \Delta y^k; \Delta z^k) \) for \( k \geq 1 \), and let \( \mathcal{H} : X \times Y \times Z \rightarrow X \times Y \times Z \times Y \times Z \) denote the block diagonal operator defined by
\[
\mathcal{H} := \text{Diag} \left( (\mathcal{P}_f + \Sigma_f)^{1/2}, (\mathcal{T}_g)^{1/2}, \frac{1}{\sqrt{\tau \sigma}} \mathcal{T}, (\mathcal{P}_g + \frac{3}{4} \Sigma_g)^{1/2}, \frac{1}{\tau \sqrt{\sigma}} \mathcal{T} \right)
\]
with the proximal operators \( \mathcal{P}_f \) and \( \mathcal{T}_g \) satisfying \( \mathcal{P}_f + \Sigma_f \succeq 0 \) and \( \mathcal{P}_g + \frac{3}{4} \Sigma_g \succeq 0 \).

**Lemma 4.1** Let \( \{ (x^k, y^k, z^k) \}_{k \geq 1} \) be the sequence generated by the IEIDP-ADMM with the criterion (C1) and \( \max(\mu_k, \nu_k) \leq \gamma \min \left( \frac{1}{\delta}, \frac{2\gamma}{\tau} \right) \) for some constant \( \gamma > 0 \). Suppose
that Assumption 2.1 holds and \( \mathcal{P}_f \) and \( \mathcal{P}_g \) also satisfy \( \mathcal{P}_f + \frac{1}{\gamma} \mathcal{P}_f \succeq 0 \) and \( \mathcal{P}_g + \frac{3}{\gamma} \mathcal{P}_g \succeq 0 \). Then, when \( \tau \in (0, 2) \), there exists an absolute constant \( c > 0 \) such that for all \( k \geq 1 \)

\[
\|Hw^{k+1}\|_{\mathcal{W}_{k+1}}^2 \leq \left[ 1 + \frac{4\nu_k}{\gamma(2 - \tau)} \right] \left[ 1 + \frac{2(\nu_k + \nu_{k+1})}{\gamma} \right] \|Hw^k\|_{\mathcal{W}_k}^2 + c\gamma(\nu_k + \nu_{k+1} + \mu_{k+1}) \\
- \min\left( \frac{2}{3} \min(\tau, 1 + \tau - \tau^2), \min(\tau, 1 + \tau - \tau^2) \right) \left( \frac{3\sigma}{2\tau} \|h(x^k, y^k)\|^2 + \|\Delta y^{k+1}\|^2 \right),
\]

where \( \mathcal{W}_k : \mathbb{X} \times \mathbb{Y} \times \mathbb{Z} \times \mathbb{X} \times \mathbb{Y} \times \mathbb{Z} \times \mathbb{Y} \times \mathbb{Z} \) is the block diagonal linear operator

\[
\mathcal{W}_k := \text{Diag}(1 - \gamma^{-1}\mu_k), (1 - 2\gamma^{-1}\mu_k - \gamma^{-1}\nu_k)I, I, I, (2 - \tau - 2\gamma^{-1}\mu_k)I).
\]

**Proof:** For each \( k \geq 1 \), let \( \mathcal{V}_k : \mathbb{X} \times \mathbb{Y} \times \mathbb{Z} \times \mathbb{X} \times \mathbb{Y} \times \mathbb{Z} \rightarrow \mathbb{X} \times \mathbb{Y} \times \mathbb{Z} \times \mathbb{Y} \times \mathbb{Z} \) be defined by

\[
\mathcal{V}_k = \text{Diag}(I, I, I, I, \frac{2 - \tau}{1 - \gamma^{-1}\alpha_k}, I) \text{ with } \alpha_k = \nu_k + \nu_{k-1}.
\]

With the notations \( \mathcal{W}_k \) and \( \mathcal{V}_k \), we first establish the following important inequality:

\[
\|Hw^{k+1}\|_{\mathcal{W}_{k+1}}^2 - \|Hw^k\|_{\mathcal{W}_k}^2 \leq - \min(\tau, 1 + \tau - \tau^2) \left[ 1 - \gamma^{-1}\alpha_{k+1} \right] \|h(x^k, y^k)\|^2 + \left( 1 - \frac{\alpha_{k+1}}{\gamma} \right) \|\Delta y^{k+1}\|^2 \gamma \Bigg] \\
+ \max\left( \|T_f^{-1}\|, 2\|T_g^{-1}\| \right) \|\alpha_{k+1} + \mu_{k+1}\|.
\]

(32)

Indeed, when \( \tau \in (0, 1] \), since \( \gamma^{-1}\alpha_{k+1} = \gamma^{-1}(\nu_{k+1} + \nu_k) < 1 \), by equation (7) we have

\[
(1 - \tau)\sigma\|h(x^k, y^k), B^*\Delta y^{k+1}\| \leq \frac{\sigma(1 - \tau)}{1 - \gamma^{-1}\alpha_{k+1}} \|h(x^k, y^k)\|^2 + \left( 1 - \frac{\alpha_{k+1}}{\gamma} \right) \|\Delta y^{k+1}\|^2 \sigma(1 - \tau)BB^*.
\]

Substituting this inequality into Proposition 3.1(a) and using (12), we obtain that

\[
\left( 1 - \frac{\alpha_{k+1}}{\gamma} \right) \|x^{k+1}\|_{\mathcal{P}_{f,1} + \mathcal{P}_g}^2 - \|x^k\|_{\mathcal{P}_{f,1} + \mathcal{P}_g}^2 + \left( 1 - \frac{\nu_{k+1}}{\gamma} - 2\frac{\mu_{k+1}}{\gamma} \right) \|y^{k+1}\|_{\mathcal{P}_g}^2 - \|y^k\|_{\mathcal{P}_g}^2 \\
+ \frac{1}{\tau^2\sigma} \left( \|z^{k+1}\|^2 - \|z^k\|^2 \right) + \|\Delta y^{k+1}\|_{\mathcal{P}_{f,1} + \mathcal{P}_g}^2 - \|\Delta y^k\|_{\mathcal{P}_{f,1} + \mathcal{P}_g}^2 \\
+ \left( 2 - \tau - 2\frac{\alpha_{k+1}}{\gamma} \right) \frac{1}{\tau^2\sigma} \|\Delta z^{k+1}\|^2 - \left( 2 - \tau \right) \sigma \left( 1 - \frac{\alpha_{k+1}}{\gamma} \right) \|\Delta z^k\|^2 \\
\leq \left( 1 - \frac{\alpha_{k+1}}{\gamma} \right) \|h(x^k, y^k)\|^2 - \left( 1 - \frac{\alpha_{k+1}}{\gamma} \right) \|\Delta y^{k+1}\|^2 \|T_g^{-1}\|^{-1} \sigma(1 - \tau)BB^* \\
- \|\Delta y^{k+1}\|^2 \|T_f^{-1}\|^{-1} + \|\Delta y^{k+1}\|^2 \|T_g^{-1}\|^{-1} \|\Delta y^{k+1}\|^2 \|T_g^{-1}\|^{-1} \sigma(1 - \tau)BB^* \\
\leq \left( 1 - \frac{\alpha_{k+1}}{\gamma} \right) \|h(x^k, y^k)\|^2 - \tau \left( 1 - \frac{\alpha_{k+1}}{\gamma} \right) \|\Delta y^{k+1}\|^2 \|T_g^{-1}\|^{-1} \\
+ \max\left( \|T_f^{-1}\|, 2\|T_g^{-1}\| \right) \|\alpha_{k+1} + \mu_{k+1}\|.
\]

(33)
where the second inequality is using \( T_g - (1-\tau)\sigma BB^* \succeq \tau T_g \) and \( T_g + \frac{1}{\tau} \Sigma_f \succeq 0 \). For the case where \( \tau \in (1, 2) \), from \( \gamma^{-1} \alpha_{k+1} = \gamma^{-1}(\nu_k + \nu_k) < 1 \) and equation (7) we have
\[
2(1-\tau)\sigma \langle h(x^k, y^k), B^* \Delta y^{k+1} \rangle \\
\leq \frac{\sigma(\tau-1)}{\tau(1-\gamma^{-1}\alpha_{k+1})} \| h(x^k, y^k) \|^2 + \tau(1-\frac{\alpha_{k+1}}{\gamma}) \| \Delta y^{k+1} \|^2_{\sigma(\tau-1)BB^*}.
\]
Substituting this inequality into Proposition 3.1(a) and using (12) yields that
\[
(1 - \frac{\mu_{k+1}}{\gamma}) \| x^{k+1}_e \|^2_{T_f + \Sigma_f} - \| x^k_e \|^2_{T_f + \Sigma_f} + (1 - \frac{\nu_k + \nu_k}{\gamma}) \| y^{k+1}_e \|^2_{T_g} - \| y^{k}_e \|^2_{T_g} \\
+ (\tau \sigma)^{-1}(\| z^k_e \|^2 - \| z^{k+1}_e \|^2) + (\| \Delta y^{k+1} \|^2_{T_f + \frac{1}{\tau} \Sigma_f} - \| \Delta y^{k+1} \|^2_{T_f + \frac{1}{\tau} \Sigma_f}) \\
+ (2-\tau - \frac{2\mu_{k+1}}{\gamma}) \frac{1}{\tau \sigma^{\gamma}} \| \Delta z^{k+1} \|^2 - \frac{1}{1-\gamma^{-1}\alpha_{k+1}} \| h(x^k, y^k) \|^2
\]
\[
\leq -\frac{1+\tau-\tau^2-\tau}{1-\gamma^{-1}\alpha_{k+1}} \| h(x^k, y^k) \|^2 - (1 - \frac{\alpha_{k+1}}{\gamma}) \| \Delta y^{k+1} \|^2_{T_g} - (\gamma T_f)^{-1} \| \mu_{k+1} + \gamma T_g^{-1} \| (2\nu_k + \nu_k) \\
- \| \Delta x^{k+1} \|^2_{T_f + \frac{1}{\tau} \Sigma_f} + \gamma \| T_f^{-1} \| \| \mu_{k+1} + \gamma T_g^{-1} \| (2\nu_k + \nu_k) \\
\leq -\frac{1+\tau-\tau^2}{1-\gamma^{-1}\alpha_{k+1}} \| h(x^k, y^k) \|^2 - (1 - \tau^2 - \tau)(1 - \frac{\alpha_{k+1}}{\gamma}) \| \Delta y^{k+1} \|^2_{T_g} \\
+ \max(\| T_f^{-1} \|, 2 \| T_g^{-1} \|) \| (\gamma_{k+1} + \mu_{k+1}) \|, \quad (34)
\]
where the second inequality is using \( T_f + \frac{1}{\tau} \Sigma_f \succeq 0 \) and \( T_g - (\tau-\tau^2)T_g \). Notice that \( \mathcal{W}_k > 0 \) and \( \mathcal{V}_k > 0 \) for all \( k \geq 1 \) due to \( \max(\mu_k, \nu_k) \leq \gamma \min(\frac{1}{\gamma}, \frac{2-\tau}{\tau^2}) \). From the definition of \( w^{k+1} \) and the expressions of \( \mathcal{H}, \mathcal{W}_k \) and \( \mathcal{V}_k \), the left hand side of (33) and (34) equals \( \| \mathcal{H} w^{k+1} \|_{\mathcal{W}_{k+1}}^2 - \| \mathcal{H} w^k \|_{\mathcal{V}_{k+1}}^2 \). Along with (33) and (34), we get (32).

Now by the assumption \( \max(\mu_k, \nu_k) \leq \gamma \min(\frac{1}{\gamma}, \frac{2-\tau}{\tau^2}) \), it is not difficult to verify that
\[
\frac{1}{1-\frac{\nu_k}{\gamma}} - \frac{2\mu_k}{\gamma} \leq \frac{1}{1-\frac{\nu_k}{\gamma}} - \frac{2\mu_k}{\gamma} \leq \left(1 + \frac{2\alpha_{k+1}}{\gamma}\right) \left[1 + \frac{4\nu_k}{\gamma(2-\tau)}\right]
\]
By the expressions of \( \mathcal{V}_{k+1} \) and \( \mathcal{W}_k \), we have \( \mathcal{V}_{k+1} \mathcal{W}_k^{-1} \leq (1 + \frac{2\alpha_{k+1}}{\gamma}) \left[1 + \frac{4\nu_k}{\gamma(2-\tau)}\right] I \). Then
\[
\| \mathcal{H} w^k \|^2_{\mathcal{V}_{k+1}} = \langle \mathcal{H} w^k, \mathcal{V}_{k+1} \mathcal{W}_k^{-1} \mathcal{W}_k \mathcal{H} w^k \rangle \leq (1 + \frac{2\alpha_{k+1}}{\gamma}) \left[1 + \frac{4\nu_k}{\gamma(2-\tau)}\right] \| \mathcal{H} w^k \|^2_{\mathcal{W}_k}. \quad (35)
\]
In addition, since \( \max(\mu_k, \nu_k) \leq \gamma \min(\frac{1}{\gamma}, \frac{2-\tau}{\tau^2}) \) for all \( k \geq 1 \), we have \( \frac{2}{\tau} \leq 1 - \frac{\alpha_{k+1}}{\gamma} \leq 1 \). Let \( c = \max(\| T_g^{-1} \|, 2 \| T_f^{-1} \|) \). Combining (35) with (32) yields the desired result. \( \square \)

By Lemma 4.1, we may establish the convergence of the IEIDP-ADMM with (C1).

**Theorem 4.1** Let \( \{ (x^k, y^k, z^k) \}_{k \geq 1} \) be the sequence generated by the IEIDP-ADMM with (C1) and \( \max(\mu_k, \nu_k) \leq \gamma \min(\frac{1}{\gamma}, \frac{2-\tau}{\tau^2}) \), where \( \gamma \) is same as that of Lemma 4.1. Suppose
that Assumption 2.1 holds and \( \mathcal{P}_f \) and \( \mathcal{P}_g \) also satisfy \( \mathcal{P}_f + \frac{1}{2} \Sigma_f \geq 0 \) and \( \mathcal{P}_g + \frac{3}{2} \Sigma_g \geq 0 \). Then, for (a) \( \tau \in (0, \frac{1+\sqrt{5}}{2}) \) or (b) \( \tau \in [\frac{1+\sqrt{5}}{2}, 2) \) but \( \sum_{k=0}^{\infty} \left( \frac{3}{2\tau} \right)^k \| h(x^k, y^k) \|^2 + \| \Delta y^{k+1} \|^2_{\tau_g} < \infty \), the sequence \{\( (x^k, y^k) \)\} converges to an optimal solution of (1) and the sequence \{\( z_k \)\} converges to an optimal solution to the dual problem of (1).

Proof: We write \( \vartheta_k = \left[ 1 + \frac{4\nu_k}{\tau(2-\tau)} \right] \left[ 1 + \frac{2(\nu_k+\nu_{k+1})}{\gamma} \right] \), \( \omega_k = c_f^*(v_k + \nu_{k+1} + \mu_{k+1}) \) and \( R_k = \frac{3\sigma}{2\tau} \| h(x^k, y^k) \|^2 + \| \Delta y^{k+1} \|^2_{\tau_g} \) for \( k \geq 1 \). By Lemma 4.1 we have that

\[
\| \mathcal{H} w^{k+1} \| \omega_{k+1} \leq -\min \left[ \frac{2}{3}, \min(\tau, 1+\tau-\tau^2), \min(\tau, 1+\tau-\tau^2) \right] \left( \sum_{l=0}^{k-1} \prod_{j=l+1}^{k} \vartheta_j R_l + R_k \right) + \prod_{j=0}^{k} \vartheta_j \| \mathcal{H} w^0 \| \omega_0 + \sum_{l=0}^{k-1} \prod_{j=l+1}^{k} \vartheta_j \omega_l + \omega_k.
\]

Since \( \sum_{k=0}^{\infty} \| w_k \| < \infty \), we have \( 1 \leq \prod_{k=1}^{\infty} \vartheta_k \leq K_1 \) for some \( K_1 \geq 1 \). Hence, we have that

\[
\left\{ \begin{array}{ll}
\| \mathcal{H} w^{k+1} \|^2_{\omega_{k+1}} \leq K_1 (\| \mathcal{H} w^0 \|^2_{\omega_0} + \sum_{l=0}^{k} \| \omega_l \| R_l) - \frac{2}{3} \min(\tau, 1+\tau-\tau^2) \sum_{l=0}^{k} \| \omega_l \| R_l & \text{if } \tau \in (0, \frac{1+\sqrt{5}}{2}), \\
\| \mathcal{H} w^{k+1} \|^2_{\omega_{k+1}} \leq K_1 (\| \mathcal{H} w^0 \|^2_{\omega_0} + \sum_{l=0}^{k} \| \omega_l \| R_l) + K_1 (\tau^2 - 1 - \tau) \sum_{l=0}^{k} \| \omega_l \| R_l & \text{if } \tau \in [\frac{1+\sqrt{5}}{2}, 2).
\end{array} \right.
\]

Notice that \( \sum_{k=1}^{\infty} \max(\mu_k, \nu_k) < \infty \) implies \( \sum_{l=0}^{\infty} \| \omega_l \| < \infty \) for some \( K_2 \geq 2 \). Then,

\[
\left\{ \begin{array}{ll}
\| \mathcal{H} w^{k+1} \|^2_{\omega_{k+1}} + \frac{4}{2\max(3, \sqrt{\tau})} \min(\tau, 1+\tau-\tau^2) \sum_{l=0}^{k} \| \omega_l \| R_l \leq K_1 (\| \mathcal{H} w^0 \|^2_{\omega_0} + K_2) & \text{if } \tau \in (0, \frac{1+\sqrt{5}}{2}), \\
\| \mathcal{H} w^{k+1} \|^2_{\omega_{k+1}} \leq K_1 (\| \mathcal{H} w^0 \|^2_{\omega_0} + K_2) + K_1 (\tau^2 - 1 - \tau) \sum_{l=0}^{k} \| \omega_l \| R_l & \text{if } \tau \in [\frac{1+\sqrt{5}}{2}, 2).
\end{array} \right.
\]

Notice that \( \max(\mu_k, \nu_k) \leq \frac{\gamma}{2\max(3, \sqrt{\tau})} \) implies \( \mathcal{W}_k \geq \mathcal{W} \) for some \( \mathcal{W} > 0 \). Then, under conditions (a) and (b), the last two inequalities imply that the sequence \{\( \mathcal{H} w^k \)\} is bounded and \( \sum_{l=0}^{\infty} R_l < +\infty \). The latter implies that \( \lim_{k \to \infty} R_k = 0 \), and consequently,

\[
\lim_{k \to \infty} \| \Delta y^{k+1} \|_{\tau_g} = 0 \quad \text{and} \quad \lim_{k \to \infty} \| \Delta z^k \| = 0. \tag{36}
\]

By equality (12), the limits in (36) imply that the sequence \{\( \| x^{k+1} \|_{\mathcal{A}^*} \)\} is bounded. By the definition of \( \mathcal{H} \) and the boundedness of \{\( \mathcal{H} w^k \)\}, the sequence \{\( \| x^{k+1} \|_{\mathcal{A}_g} \)\} is also bounded. Thus, the sequence \{\( \| x^{k+1} \|_{\tau_g} \)\} is bounded. In addition, the boundedness of \{\( \mathcal{H} w^k \)\} also implies that the sequences \{\( \| y^{k+1} \|_{\tau_g} \)\} and \{\( \| z^{k+1} \| \)\} are bounded. Together with the positive definiteness of \( \mathcal{T}_g \) and \( \mathcal{T}_g \), it follows that \{\( (x^k, y^k, z^k) \)\} is bounded. So, there exists a convergent subsequence, to say \{\( (x^k, y^k, z^k) \)\} \( \mathcal{K} \). Without loss of generality, we assume \{\( (x^k, y^k, z^k) \)\} \( \mathcal{K} \to (x^\infty, y^\infty, z^\infty) \). Since \( \lim_{k \to \infty} \| z^k - z^{k-1} \| = 0 \), we have \( h(x^\infty, y^\infty) = 0 \). In addition, taking the limit \( k \to \infty \) with \( k \in \mathcal{K} \) on the both sides of (13) and (14), and using the closedness of the graphs of \( \partial f \) and \( \partial g \) (see [27]), we have

\[
-A^* z^\infty \in \partial f(x^\infty) \quad \text{and} \quad -B^* z^\infty \in \partial g(y^\infty).
\]

Along with (9), \( (x^\infty, y^\infty) \) is an optimal solution of (1) and \( z^\infty \) is the associated multiplier.
Finally, we argue that \((x^{\infty}, y^{\infty}, z^{\infty})\) is actually the unique limit point of \(\{(x^k, y^k, z^k)\}\). Recall that \((x^{\infty}, y^{\infty})\) is an optimal solution to (1) and \(z^{\infty}\) is the associated multiplier. Hence, we could replace \((x^{\ast}, y^{\ast}, z^{\ast})\) with \((x^{\infty}, y^{\infty}, z^{\infty})\) in the previous arguments, starting from (13) and (14). Thus, inequality (32) still holds with \((x^{\ast}, y^{\ast}, z^{\ast})\) replaced by \((x^{\infty}, y^{\infty}, z^{\infty})\). Hence, from the definition of \(w^k\), \(\sum_{k=1}^{\infty} \max(\mu_k, \nu_k) < \infty\) and equation (36), we have \(\lim_{k \to \infty, k \in K} \|Hw^k\|_{\mathcal{W}} \leq \lim_{k \to \infty, k \in K} \|Hw^k\|_{\overline{\mathcal{W}}} = 0\) where

\[
\overline{\mathcal{W}} = \text{Diag}(I, I, I, I, (2 - \tau)I) .
\]

This means that for any \(\varepsilon > 0\), there exists a sufficiently large \(k_0 \in K\) such that \(\|Hw^{k_0}\|_{\mathcal{W}} < \frac{\varepsilon}{2K_1}\) and \(\sum_{l=k_0}^{\infty} \omega_l < \frac{\varepsilon}{2K_1}\). By Lemma 4.1, for any \(k \geq k_0\) we have

\[
\|Hw^{k+1}\|^2_{\mathcal{W}_{k+1}} \leq \left(\frac{1}{k+1}\right) \sum_{j=k_0}^{k} \|d_j\|^2_{\mathcal{W}_{k_0}} + \sum_{l=k_0}^{k} \|P\|_{\mathcal{W}_{k_0}} + \sum_{l=k_0}^{k} \omega_l \leq \varepsilon .
\]

This, by the positive definiteness of \(\overline{\mathcal{W}}\), shows that \(\lim_{k \to \infty} Hw^{k+1} = 0\). Consequently,

\[
\lim_{k \to \infty} \|x^{k+1} - x^{\infty}\|_{\mathcal{P}_f + \Sigma_f} = 0, \quad \lim_{k \to \infty} \|y^{k+1} - y^{\infty}\|_{\mathcal{T}_g} = 0 \text{ and } \lim_{k \to \infty} \|z^{k+1} - z^{\infty}\| = 0 .
\]

Combining \(\lim_{k \to \infty} \|y^{k+1} - y^{\infty}\|_{\mathcal{T}_g} = 0\) with \(\mathcal{T}_g > 0\) yields \(\lim_{k \to \infty} H(x^k, y^k) = 0\). Together with

\[
\|x^{k+1} - x^{\infty}\|^2_{\mathcal{A}^*} \leq 2\|y^{k+1} - y^{\infty}\|_{\mathcal{B}^*}^2 + 2\|h(x^{k+1}, y^{k+1})\|^2 ,
\]

we obtain \(\lim_{k \to \infty} \|x^{k+1} - x^{\infty}\|_{\mathcal{A}^*} = 0\). Noting that \(\lim_{k \to \infty} \|x^{k+1} - x^{\infty}\|_{\mathcal{P}_f + \Sigma_f} = 0\), we have \(\lim_{k \to \infty} \|x^{k+1} - x^{\infty}\|_{\mathcal{T}_f} = 0\). By the positive definiteness of \(\mathcal{T}_f\), it follows that \(\lim_{k \to \infty} \|x^{k} - x^{\infty}\| = 0\). Thus, \(\lim_{k \to \infty} x^k = x^{\infty}\), \(\lim_{k \to \infty} y^k = y^{\infty}\) and \(\lim_{k \to \infty} z^k = z^{\infty}\). That is, \((x^{\infty}, y^{\infty}, z^{\infty})\) is the unique limit point of \(\{(x^k, y^k, z^k)\}\).

\textbf{Remark 4.1} Theorem 4.1 shows that one can establish the convergence of \(\{(x^k, y^k, z^k)\}\) generated by the IEIDP-ADMM with (C1) if \(\mathcal{P}_f\) and \(\mathcal{P}_g\) are chosen such that

\[
\mathcal{P}_f + \frac{1}{2} \Sigma_f \geq 0, \quad \mathcal{P}_g + \frac{3}{8} \Sigma_g \geq 0, \quad \Sigma_f + \mathcal{P}_f + \sigma \mathcal{A}^* \geq 0, \quad \Sigma_g + \mathcal{P}_g + \sigma \mathcal{B}^* \geq 0 .
\]

In fact, using the same arguments, one can get the convergence of \(\{(x^k, y^k, z^k)\}\) generated by the IEIDP-ADMM with (C1) if \(\mathcal{P}_f\) and \(\mathcal{P}_g\) are chosen such that for some \(a \in [\frac{1}{2}, 1]\),

\[
\mathcal{P}_f + \frac{1}{2} \Sigma_f \geq 0, \quad \mathcal{P}_g + \frac{1}{a} \Sigma_g \geq 0, \quad \Sigma_f + \mathcal{P}_f + \sigma \mathcal{A}^* \geq 0, \quad a \Sigma_g + \mathcal{P}_g + \sigma \mathcal{B}^* \geq 0 .
\]
4.2 Convergence of the IEIDP-ADMM with (C2)

For each $k \geq 1$, we write $w^k := (x^k; y^k; z^k; \Delta y^k; \Delta z^k)$, and let $\mathcal{H} : X \times Y \times Z \times Y \times Y \times Z \rightarrow X \times Y \times Z \times Y \times Y \times Z$ be the block diagonal linear operator defined by

$$
\mathcal{H} := \text{Diag} \left( (P_f + \Sigma_f)^{1/2}, (T_g)^{1/2}, \frac{1}{\sqrt{\tau \sigma}} I, (P_g + \frac{3}{4} \Sigma_g)^{1/2}, (T_f)^{1/2}, \frac{1}{\tau \sqrt{\sigma}} I \right)
$$

for the proximal operators $P_f$ and $P_g$ satisfying $P_f + \Sigma_f \succeq 0$ and $P_g + \frac{3}{4} \Sigma_g \succeq 0$. To establish the convergence of the IEIDP-ADMM with (C2), we need the following lemma.

**Lemma 4.2** Let $\{(x^k, y^k, z^k)\}_{k \geq 1}$ be the sequence given by the IEIDP-ADMM with (C2) and $\max(\mu_k, \nu_k) \leq \min(0.1, \frac{2}{\tau - 2})$. Suppose that Assumption 2.1 holds and $P_f$ and $P_g$ also satisfy $P_f + \frac{3}{4} \Sigma_f \succeq 0$ and $P_g + \frac{3}{4} \Sigma_g \succeq 0$. Then, when $\tau \in (0, 2)$, for all $k \geq 1$ we have

$$
\frac{\|\mathcal{H} w^{k+1}\|^2}{W_{k+1}} \leq \left[ 1 + 10 \nu_k + 40 (\mu_{k+1} + \nu_{k+1}) \right] \left[ 1 + \frac{12 (\mu_k + 2 \mu_{k+1})}{2 - \tau} \right] \frac{\|\mathcal{H} w^k\|^2}{W_k}
$$

$$
- \min(0.1 \min(\tau, 1 + \tau - \tau^2), \min(\tau, 1 + \tau - \tau^2)) \left( \frac{10 \sigma}{\tau} \|h(x^k, y^k)\|^2 + \|\Delta y^{k+1}\|^2_{T_f} \right)
$$

where the operator $W_k : X \times Y \times Z \times Y \times Y \times Z \rightarrow X \times Y \times Z \times Y \times Y \times Z$ is defined by

$$
W_k := \text{Diag}(1 - \mu_k I, 1 - 5 \mu_k - \nu_k I, \mathcal{I}, \mathcal{I}, \nu_k \mathcal{I}, (2 - \tau - 3 \mu_k) \mathcal{I}).
$$

**Proof:** Let $\mathcal{V}_k : X \times Y \times Z \times Y \times Y \times Z \rightarrow X \times Y \times Z \times Y \times Y \times Z$ for $k \geq 1$ be defined by

$$
\mathcal{V}_k := \text{Diag}(\mathcal{I}, \mathcal{I}, \mathcal{I}, \mathcal{I}, \nu_k \mathcal{I}, (\frac{2 - \tau}{1 - \alpha_k} + 6 \mu_k) \mathcal{I})
$$

with $\alpha_k = \nu_{k-1} + 4 (\nu_k + \mu_k)$.

With the notations $W_k$ and $V_k$, we first establish the following important inequality:

$$
\frac{\|\mathcal{H} w^{k+1}\|^2}{W_{k+1}} - \frac{\|\mathcal{H} w^k\|^2}{W_k} \leq - \frac{\sigma}{\tau (1 - \alpha_{k+1})} \|h(x^k, y^k)\|^2 + \frac{(1 - \alpha_k + 1) \|\Delta y^{k+1}\|^2_{T_f}}{2}
$$

Indeed, when $\tau \in (0, 1)$, since $\alpha_{k+1} \leq 0.9$ by $\max(\mu_k, \nu_k) \leq 0.1$, it follows from (7) that

$$
2(1 - \tau) \sigma \|h(x^k, y^k), \Delta^* \Delta y^{k+1}\| \leq \frac{\sigma (1 - \tau)}{1 - \alpha_k} \|h(x^k, y^k)\|^2 + \frac{(1 - \alpha_k + 1) \|\Delta y^{k+1}\|^2_{T_f}}{2 (1 - \tau) \sigma B^*}
$$

Substituting the last inequality into Proposition 3.1(b) and using (12), we obtain that

$$
\begin{align*}
&\left( 1 - \mu_k \right) \|z^{k+1}\|^2_{P_f + \Sigma_f} - \|z_k\|^2_{P_f + \Sigma_f} + \left( 1 - \nu_k - 5 \mu_k \right) \|y^{k+1}\|^2_{T_f} - \|y_k\|^2_{T_f} \\
&\quad + \frac{1}{(\tau \sigma)^{-1}} \left( \|z^{k+1}\|^2_{P_y + \frac{3}{4} \Sigma_y} - \|z^k\|^2_{P_y + \frac{3}{4} \Sigma_y} + \|\Delta y^{k+1}\|^2_{P_y + \frac{3}{4} \Sigma_y} - \|\Delta y^k\|^2_{P_y + \frac{3}{4} \Sigma_y} \right) \\
&\quad + \nu_{k+1} \|\Delta y^{k+1}\|^2_{T_f} - \nu_k \|\Delta y^k\|^2_{T_f} + (2 - 3 \mu_k) \sigma \|h(x^{k+1}, y^{k+1})\|^2 \\
&\quad - \frac{2 - \tau}{1 - \alpha_{k+1}} \left[ 1 + 6 \mu_k \right] \frac{1}{\tau^2 \sigma} \|\Delta z^{k+1}\|^2 \\
&\leq - \frac{\sigma}{1 - \alpha_k} \|h(x^k, y^k)\|^2 - (1 - \alpha_k + 1) \|\Delta y^{k+1}\|^2_{T_f} + \|\Delta x^{k+1}\|^2_{P_f + \frac{3}{4} \Sigma_g - \mu_{k+1} (P_y + \frac{3}{4} \Sigma_g)} \\
&\leq - \frac{\sigma}{1 - \alpha_k} \|h(x^k, y^k)\|^2 - \tau \left( 1 - \alpha_k + 1 \right) \|\Delta y^{k+1}\|^2_{T_f},
\end{align*}
$$

(38)
where the last inequality is using \( P_f + \frac{1}{2} \Sigma_f - \mu_{k+1}(P_f + \Sigma_f) \succeq 0 \) and \( T_g - (1 - \tau) \sigma BB^* \succeq \tau T_g \), implied by \( P_f + \frac{3}{8} \Sigma_f \succeq 0 \) and \( \mu_{k+1} \leq 0.1 \). When \( \tau \in (1, 2) \), by (7) and \( \max(\mu_k, \nu_k) \leq 0.1 \),

\[
2(1 - \tau) \sigma \langle h(x^k, y^k), B^* (y^{k+1} - y^k) \rangle \leq \frac{\sigma \| h(x^k, y^k) \|^2}{1 - \alpha_{k+1}} \frac{1 - \tau}{\tau} + (1 - \alpha_{k+1}) \| \Delta y^{k+1} \|^2_{\sigma^{(\tau - 1)} \sigma BB^*}.
\]

Substituting the last inequality into Proposition 3.1(b) and using (12), we obtain that

\[
\begin{align*}
(1 - \mu_{k+1}) \| x^{k+1} \|^2_{P_f + \Sigma_f} - \| x^k \|^2_{P_f + \Sigma_f} + (1 - \nu_{k+1} - 5\mu_{k+1}) \| y^{k+1} \|^2_{\Sigma_f} - \| y^k \|^2_{\Sigma_f} + (1 - \tau - 3\mu_{k+1}) \sigma \| h(x^{k+1}, y^{k+1}) \|^2 & - \left( \frac{2 - \tau}{1 - \alpha_{k+1}} + 6\mu_{k+1} \right) \frac{1}{\tau^2}\| \Delta z^k \|^2 \\
& \leq \left( \frac{1}{1 - \alpha_{k+1}} \right) \| \Delta y^{k+1} \|^2_{\Sigma_f} + 1 - \alpha_{k+1} \| \Delta y^k \|^2_{\Sigma_f} - (1 - \tau - \tau^2) (1 - \alpha_{k+1}) \| \Delta y^{k+1} \|^2_{\Sigma_f} \tag{39}
\end{align*}
\]

where the last inequality is due to \( P_f + \frac{1}{2} \Sigma_f - \mu_{k+1}(P_f + \Sigma_f) \succeq 0 \) by \( \mu_{k+1} \leq 0.1 \), and

\[
T_g - (\tau - 1) \sigma BB^* \succeq (1 + \tau - \tau^2) T_g.
\]

By the definitions of the vector \( w^{k+1} \) and the operators \( \mathcal{H}, \mathcal{W}_k \) and \( \mathcal{V}_k \), the left hand side of (38) and (39) is \( \| \mathcal{H} w^{k+1} \|^2_{\mathcal{V}_{k+1}} - \| \mathcal{H} w^k \|^2_{\mathcal{V}_{k+1}} \). Along with (38) and (39), we get (37).

Since \( 0 \leq \max(\mu_k, \nu_k) \leq \min(0.1, \frac{2}{2-\tau}) \) for all \( k \geq 1 \), it is not difficult to check that

\[
\frac{1}{1 - \alpha_{k+1}} \leq 1 + 10\nu_k + 40(\mu_{k+1} + \nu_{k+1}) \quad \text{and} \quad \frac{1}{1 - \frac{3\mu_k}{2 - \tau}} \leq 1 + \frac{12\mu_k}{2 - \tau},
\]

which in turn implies that

\[
\frac{2 - \tau}{(1 - \alpha_{k+1}) \left( 1 - \frac{3\mu_k}{2 - \tau} \right)} \leq \left[ 1 + 10\nu_k + 40(\mu_{k+1} + \nu_{k+1}) \right] \left( 1 + \frac{12\mu_k}{2 - \tau} \right)
\]

and

\[
\frac{6\mu_{k+1}}{2 - \tau - 3\mu_k} \leq \frac{2(\mu_{k+1})}{2 - \tau}. \quad \text{Together with the expression of } \mathcal{V}_{k+1} \mathcal{W}_k^{-1}, \text{ we obtain that}
\]

\[
\mathcal{V}_{k+1} \mathcal{W}_k^{-1} \leq \left[ 1 + 10\nu_k + 40(\mu_{k+1} + \nu_{k+1}) \right] \left( 1 + \frac{12(\mu_k + 2\mu_{k+1})}{2 - \tau} \right) I. \tag{40}
\]

Since \( \nu_k \leq 0.1 \) for all \( k \geq 1 \), we have \( 0.1 \leq 1 - \alpha_{k+1} \leq 1 \). Now combining (40) with inequality (37) yields the desired result. Thus, we complete the proof. \( \square \)

By Lemma 4.2 one may obtain the following convergence result of the IEIDP-ADMM with the criterion (C2). Since the proof is similar to that of Theorem 4.1, we omit it.
Theorem 4.2 Let \( \{x^k, y^k, z^k\}\) be the sequence generated by the IEIDP-ADMM with the criterion \((C2)\) and \(\max(\mu_k, v_k)\) \(\leq \min(0.1, \frac{2-\tau}{3})\). Suppose that Assumption 2.1 holds and the operators \(P_f\) and \(P_g\) also satisfy \(P_f + \frac{3}{8}\Sigma_f \succeq 0\) and \(P_g + \frac{3}{8}\Sigma_g \succeq 0\). Then, for \((a)\) \(\tau \in (0, \frac{1}{12})\) or \((b)\) \(\tau \in [\frac{1}{12}, 2)\) but \(k\sum_{k=0}^{\infty} (\frac{1}{2\tau})(\|h(x^k, y^k)\|^2 + \|\Delta y^{k+1}\|^2) < \infty\), the sequence \(\{(x^k, y^k, z^k)\}\) converges to an optimal solution of problem \((1)\) and the sequence \(\{z^k\}\) converges to an optimal solution to the dual problem of \((1)\).

Remark 4.2 Theorem 4.2 shows that one can establish the convergence of \(\{(x^k, y^k, z^k)\}\) generated by the IEIDP-ADMM with \((C2)\) if \(P_f\) and \(P_g\) are chosen such that
\[
\begin{align*}
P_f + \frac{3}{8}\Sigma_f &\succeq 0, \quad P_g + \frac{3}{8}\Sigma_g \succeq 0, \quad \Sigma_f + P_f + \sigma AA^* > 0, \quad \Sigma_g + P_g + \sigma BB^* > 0.
\end{align*}
\]
In fact, using the same arguments, one can get the convergence of \(\{(x^k, y^k, z^k)\}\) generated by the IEIDP-ADMM with \((C2)\) if \(P_f\) and \(P_g\) are chosen such that for some \(a_1, a_2 \in [\frac{1}{2}, 1),\)
\[
\begin{align*}
P_f + \frac{1}{2}\Sigma_f &\succeq 0, \quad P_g + \frac{1}{2}\Sigma_g \succeq 0, \quad a_1\Sigma_f + P_f + \sigma AA^* > 0, \quad a_2\Sigma_g + P_g + \sigma BB^* > 0.
\end{align*}
\]

4.3 Convergence of the IEIDP-ADMM with \((C2')\)

Let \(w^k\) for \(k \geq 1\) be same as the one of the last subsection. Define the block diagonal linear operator \(H: X \times Y \times Z \times X \times Y \times Z \to X \times Y \times Z \times Y \times Y \times Z\) by
\[
H := \text{Diag}((P_f + \Sigma_f)^{1/2}, (\Sigma_g)^{1/2}, (\tau\sigma)^{-1}I, (P_g + \frac{3}{8}\Sigma_g)^{1/2}, (\Sigma_g)^{1/2}, (\gamma^2\sigma)^{-1}I)
\]
with the proximal operators \(P_f\) and \(P_g\) satisfying \(P_f + \Sigma_f \succeq 0\) and \(P_g + \frac{3}{8}\Sigma_g \succeq 0\).

Lemma 4.3 Let \(\{x^k, y^k, z^k\}\) be the sequence generated by the IEIDP-ADMM with \((C2')\) and \(\max(\mu_k^2, v_k^2) \leq \min(\frac{1}{8\tau}, \frac{2-\tau-2.5\gamma^{-1}}{4\gamma})\) for some constant \(\gamma \geq 360\). Suppose that Assumption 2.1 holds and the operators \(P_f\) and \(P_g\) also satisfy \(P_f + \frac{3}{8}\Sigma_f \succeq 0\) and \(P_g + \frac{3}{8}\Sigma_g \succeq 0\). Then, when \(\tau \in (0, 2)\), the following inequality holds for all \(k \geq 1\)
\[
\|Hw^{k+1}\|_{W_{k+1}} \leq \max \left(1 + \frac{3\gamma\mu_k^2}{2 - \tau - 2.5\gamma^{-1}}, \frac{\gamma\nu_k^2 + 3\mu_k^2}{2 - \tau - 2.5\gamma^{-1}}\right) \|Hw^k\|_{W_k},
\]
where
\[
\begin{align*}
\frac{2}{2-\tau - 6.5\gamma^{-1}} - \frac{\max((1 - \gamma), (\tau - 1)/3)}{4\gamma}, \quad c_2 = |2 - \tau - 6.5\gamma^{-1} - \max((1 - \gamma), (\tau - 1)/3)|,
\end{align*}
\]
and \(W_k: X \times Y \times Z \times X \times Y \times Z \to X \times Y \times Z \times Y \times Y \times Z\) is the block diagonal operator
\[
W_k := \text{Diag}((1 - \gamma\mu_k^2)I, (1 - \gamma\nu_k^2 - 3\gamma\mu_k^2)I, I, I, (1 - \gamma^{-1}I, (2 - \tau - 2.5\gamma^{-1} - 1.5\gamma\mu_k^2)I).
\]

19
Proof: Let \( \Psi : \mathbb{X} \times \mathbb{Y} \times \mathbb{Z} \times \mathbb{X} \times \mathbb{Y} \times \mathbb{Z} \to \mathbb{X} \times \mathbb{Y} \times \mathbb{Z} \times \mathbb{Y} \times \mathbb{Z} \) be the block diagonal linear operator defined by \( \Psi := \text{Diag}(I, I, I, I, I, (2-\tau-2.5\gamma^{-1})I) \). With the notations \( \mathcal{W}_k \) and \( \mathcal{V} \), we first establish the following inequality

\[
\| \mathcal{H}w^{k+1} \|_{\mathcal{W}_{k+1}}^2 - \| \mathcal{H}w^k \|_{\mathcal{V}}^2 \leq -\min(\tau, 2.6-1.6\tau)[1-6\gamma^{-1}-\gamma(v_k^2 + v_{k+1}^2)]\|\Delta y^{k+1}\|_{T_g} - c_k(\gamma, \tau)\|h(x^k, y^k)\|_2^2,
\]

where

\[
c_k(\gamma, \tau) := 2-\tau - \frac{6.5}{\gamma} - \max(1-\tau, (\tau - 1)/1.6) / \gamma.
\]

Indeed, when \( \tau \in (0,1] \), from \( 1-6\gamma^{-1}-\gamma(v_k^2 + v_{k+1}^2) > 0 \) and equation (7) it follows that

\[
2\sigma(1-\tau)\|h(x^k, y^k), B^*\Delta y^{k+1}\| \leq \frac{\sigma(1-\tau)}{1-6\gamma^{-1}-\gamma(v_k^2 + v_{k+1}^2)}\|h(x^k, y^k)\|_2^2
\]

\[
+ [1-6\gamma^{-1}-\gamma(v_k^2 + v_{k+1}^2)]\|\Delta y^{k+1}\|_{(1-\tau)\sigma BB^*}^2.
\]

Substituting the last inequality into Proposition 3.1(c) then yields that

\[
\| \mathcal{H}w^{k+1} \|_{\mathcal{W}_{k+1}}^2 - \| \mathcal{H}w^k \|_{\mathcal{V}}^2 + c_k(\gamma, \tau)\|h(x^k, y^k)\|_2^2
\]

\[
\leq -[\Delta x^{k+1}]^2_{\mathcal{P}_g + \frac{1}{2}(\Sigma_f - \frac{1}{4}(\mathcal{P}_g + \Sigma_f))} \Delta x^{k+1} \|\Delta y^{k+1}\|_{\mathcal{T}_g-(1-\tau)\sigma BB^*}^2
\]

\[
\leq -(1-6\gamma^{-1}-\gamma(v_k^2 + v_{k+1}^2))\|\Delta y^{k+1}\|_{1.6(1-\tau)\sigma BB^*}^2 - \Delta x^{k+1} \|\Delta y^{k+1}\|_{\mathcal{T}_g-(1-\tau)\sigma BB^*}^2
\]

where the last inequality is using \( \mathcal{P}_g + \frac{1}{2}(\Sigma_f - \gamma^{-1}(\mathcal{P}_g + \Sigma_f) \geq 0 \) and \( \mathcal{T}_g - (1-\tau)\sigma BB^* \geq \tau \mathcal{T}_g \).

When \( \tau \in (1,2) \), from \( 1-6\gamma^{-1}-\gamma(v_k^2 + v_{k+1}^2) > 0 \) and equation (7) it follows that

\[
2\sigma(1-\tau)\|h(x^k, y^k), B^*(y^{k+1} - y^k)\| \leq \frac{\sigma(1-\tau)}{1-6\gamma^{-1}-\gamma(v_k^2 + v_{k+1}^2)}\|h(x^k, y^k)\|_2^2
\]

\[
+ (1-6\gamma^{-1}-\gamma(v_k^2 + v_{k+1}^2))\|\Delta y^{k+1}\|_{1.6(1-\tau)\sigma BB^*}^2.
\]

Substituting it into Proposition 3.1(c) and using the notations \( \mathcal{W}_k \) and \( \mathcal{V} \), we have

\[
\| \mathcal{H}w^{k+1} \|_{\mathcal{W}_{k+1}}^2 - \| \mathcal{H}w^k \|_{\mathcal{V}}^2 + c_k(\gamma, \tau)\|h(x^k, y^k)\|_2^2
\]

\[
\leq -(1-6\gamma^{-1}-\gamma(v_k^2 + v_{k+1}^2))\|\Delta y^{k+1}\|_{\mathcal{T}_g-1.6(1-\tau)\sigma BB^*}^2
\]

\[
\leq -(2.6-1.6\tau)(1-6\gamma^{-1}-\gamma(v_k^2 + v_{k+1}^2))\|\Delta y^{k+1}\|_{\mathcal{T}_g}^2,
\]

where the last inequality is using \( \mathcal{P}_g + \frac{1}{2}(\Sigma_f - \gamma^{-1}(\mathcal{P}_g + \Sigma_f) \geq 0 \) and \( \mathcal{T}_g - 1.6(1-\tau)\sigma BB^* \geq (1-1.6(1-\tau))\mathcal{T}_g \). From (42) and (43), we immediately obtain inequality (41).

Now by the given condition \( \max(v_k^2, v_{k+1}^2) \leq \min(\frac{1}{\gamma}, 2-2.5\gamma^{-1}) \), we can check that

\[
\frac{2-\tau-2.5\gamma^{-1}}{2-\tau-2.5\gamma^{-1}-1.5\gamma v_k^2} \leq 1 + \frac{3\gamma v_k^2}{2-\tau-2.5\gamma^{-1}} \text{ and } \frac{1}{1-\gamma v_k^2-3\gamma v_{k+1}^2} \leq 1 + 2\gamma(v_k^2 + v_{k+1}^2).
\]
Together with the expressions of $\mathcal{V}$ and $\mathcal{W}_k$, it is not difficult to verify that

$$\mathcal{V}\mathcal{W}_k^{-1} \leq \max\left(1 + \frac{3\gamma\mu_k^2}{2 - \tau - 2.5\gamma - \tau}, 1 + 2\gamma(\nu_k^2 + 3\mu_k^2)\right)I.$$ 

Combining this relation with (41) and the condition $\max(\mu_k^2, \nu_k^2) \leq \min\left(\frac{1}{8}, \frac{2 - \tau - 2.5\gamma - \tau}{3\gamma - \tau}\right)\frac{1}{2}$, we obtain the desired result. The proof is completed.

By Lemma 4.3 we can establish the following convergence result of the IEI DP-ADMM with the criterion (C2'). Since the proof is similar to that of Theorem 4.1, we omit it.

**Theorem 4.3** Let $\{(x^k, y^k, z^k)\}_{k \geq 1}$ be the sequence generated by the IEI DP-ADMM with the criterion (C2') and $\max(\mu_k^2, \nu_k^2) \leq \min\left(\frac{1}{8}, \frac{2 - \tau - 2.5\gamma - \tau}{3\gamma - \tau}\right)\frac{1}{2}$ for some constant $\gamma \geq 360$. Suppose that Assumption 2.1 holds and $\mathcal{P}_I$ and $\mathcal{P}_g$ also satisfy $\mathcal{P}_I + \frac{3}{8}\mathcal{I} \succeq 0$ and $\mathcal{P}_g + \frac{\delta}{2}\Sigma \succeq 0$. Then, for (a) $\tau \in (0, 1.6]$ or (b) $\tau \in (1.6, 2]$ but $\sum_{k=0}^{\infty}(2.6 - 1.6\tau\|h(x^k, y^k)\|^2 + c_2\|\Delta y^{k+1}\|_T^2) < \infty$, the sequence $\{(x^k, y^k)\}$ converges to an optimal solution of (1) and the sequence $\{z^k\}$ converges to an optimal solution to the dual problem of (1).

To close this section, we want to point out that the convergence of the inexact positive definite proximal ADMM [24] with (C1) and a special (C2') is only established for $\tau = 1$, while the convergence results of Theorem 4.1 and Theorem 4.3 extend it to the inexact indefinite proximal ADMM with $\tau \in (0, \frac{\sqrt{3} + 1}{2})$ and $\tau \in (0, 1.6]$, respectively.

## 5 Applications to doubly nonnegative SDPs

Let $\mathcal{S}^n_+$ be the cone of $n \times n$ positive semidefinite matrices in the vector space $\mathcal{S}^n$ of $n \times n$ real symmetric matrices, endowed with the Frobenius inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\|\cdot\|$. The doubly nonnegative SDP problem is described as follows:

$$\max \left\{ -\langle C, X \rangle \mid A_E X = b_E, A_I X \succeq b_I, X \in \mathcal{S}^n_+, X - M \in \mathcal{K} \right\}$$

(44)

where $A_E : \mathcal{S}^n \to \mathbb{R}^{m_E}$ and $A_I : \mathcal{S}^n \to \mathbb{R}^{m_I}$ are the linear operators, $b_E \in \mathbb{R}^{m_E}$ and $b_I \in \mathbb{R}^{m_I}$ are the given vectors, and $X - M \in \mathcal{K}$ means that every entry of $X - M$ is nonnegative. We always assume that $A_E$ is surjective. The dual of (44) has the form

$$\min \left(\delta_{\mathcal{S}_+^n}(y_I) - \langle b_I, y_I \rangle \right) + \langle \delta_{\mathcal{K}^*}(Z) - (M, Z) \rangle - \langle b_E, y_E \rangle + \delta_{\mathcal{S}_+^n}(S)$$

s.t. $A_I^* y_I + Z + A_E^* y_E + S = C$

(45)

where $\mathcal{K}^*$ is the positive dual cone of $\mathcal{K}$. For the four-block separable convex minimization problem (45), one may use the multi-block ADMM with Gaussian back substitution [16, 17] or the proximal ADMM [31] to solve. In this section, we apply the IEIDP-ADMMs for (45) by viewing $(y_I, Z)$ as a block and $(y_E, S)$ as a block (respectively,
viewing \((Z, y_E)\) as a block and \(S\) as a block when \(m_I = 0\). Notice that, by introducing a slack variable, problem (44) can be equivalently written as

\[
\max \left\{ -\langle C, X \rangle \mid A_E X = b_E, \ A_I X - x = b_I, \ X \in S^+_n, \ X - M \in \mathcal{K}, \ x \geq 0 \right\},
\]

and an elementary calculation yields the dual problem of (46) as follows

\[
\min -\langle b_I, y_I \rangle + \delta_{\mathbb{R}^m_+}(z) + \left( \delta_{\mathcal{K}^*}(Z) - \langle M, Z \rangle \right) - \langle b_E, y_E \rangle + \delta_{S^+_n}(S)
\]

s.t. \(A_I^* y_I + Z + A_E^* y_E + S = C, \ y_I - z = 0.\) (47)

Problem (47) is still a four-block separable convex minimization since \((z, Z)\) can be solved simultaneously. Hence, in this section we also apply the EIDP-ADMMs for solving (47) by viewing \((y_I, z, Z)\) as a block and \((y_E, S)\) as a block.

Throughout this section, instead of using the constraint qualification (CQ) in Assumption 2.1, we use the following more familiar Slater’s CQ for problem (45):

**Assumption 5.1**

(a) For problem (44), there exists a point \(\hat{X} \in S^n\) such that

\[
A_E \hat{X} = b_E, \ A_I \hat{X} \geq b_I, \ \hat{X} \in \text{int}(S^+_n), \ \hat{X} \in \mathcal{K}.
\]

(b) For problem (45), there exists a point \((\hat{y}_I, \hat{Z}, \hat{y}_E, \hat{S}) \in \mathbb{R}^{m_I} \times S^n \times \mathbb{R}^{m_E} \times S^n\) such that

\[
A_I^* \hat{y}_I + \hat{Z} + A_E^* \hat{y}_E + \hat{S} = c, \ \hat{Z} \in \mathcal{K}^*, \ \hat{y}_I \in \mathbb{R}^{m_I}, \ \hat{S} \in \text{int}(S^+_n).
\]

By [2, Corollary 5.3.6], under Assumption 5.1, the strong duality for (44) and (45) holds, and the following Karush-Kuhn-Tucker (KKT) condition has nonempty solutions:

\[
\begin{cases}
A_E X - b_E = 0, \\
A_I^* y_I + Z + A_E^* y_E + S - C = 0, \\
\langle X, S \rangle = 0, \ X \in S^+_n, \ S \in S^+_n, \\
\langle X, Z \rangle = 0, \ X \in \mathcal{K}, \ Z \in \mathcal{K}^*, \\
\langle y_I, A_I X - b_I \rangle = 0, \ A_I X - b_I \geq 0, \ y_I \in \mathbb{R}^{m_I}.
\end{cases}
\] (48)

### 5.1 Numerical results for the DNNSDPs without \(A_I X \geq b_I\)

In this case since the linear operator \([I, A_E^*] [I, A_I^*] \) is not positive definite, we impose a semi-proximal term \(\frac{\sigma}{2} (Z - Z^k, y_E - y^k_E) \text{Diag}(\varepsilon I, 0) (Z - Z^k, y_E - y^k_E)^T\) to guarantee that

\[
\mathcal{T}_I \succeq \sigma [I, A_E^*] [I, A_I^*] + \text{Diag}(\varepsilon I, 0) \left[ \frac{\sigma}{2} I \begin{bmatrix} A_E^* & A_E A_I^* \end{bmatrix} \right] \succ 0, \quad (49)
\]

and propose the following partial IEIDP-ADMM for problem (45) with three blocks, where for a given \(\sigma > 0\), the augmented Lagrangian function of (45) is defined as

\[
L_o(y_I, Z, y_E, S, X) := (\delta_{\mathbb{R}^m_+}(y_I) - \langle b_I, y_I \rangle) + (\delta_{\mathcal{K}^*}(Z) - \langle M, Z \rangle) - \langle b_E, y_E \rangle + \delta_{S^+_n}(S)
\]

\[
+ \langle X, A_I^* y_I + Z + A_E^* y_E + S - C \rangle + \frac{\sigma}{2} \left\| A_I^* y_I + Z + A_E^* y_E + S - C \right\|^2
\]

\forall (y_I, Z, y_E, S, X) \in \mathbb{R}^{m_I} \times S^n \times \mathbb{R}^{m_E} \times S^n \times S^n.
Comparing this system with the optimality condition of 
min \sigma \text{ by solving the problem (50)},
This means that \( F(0) \) from the expression of the function \( \phi \) with 
By using equation (49) and [20, Theorem 7.7.6], it is not difficult to verify that 
\[ \nu_k \equiv 0 \text{ when } \|Z^{k+1} - Z^k\| = \sigma \|A_E^* y^{k+1}_E - y^k_E + S_{k+1} - S_k\| \leq \mu_{k+1} \text{ and } \sum_{k=0}^{\infty} \mu_{k+1} \leq \infty, \]
Algorithm 5.1 (A partial IEIDP-ADMM for (45) with three blocks)
(S.0) Let \( \mathcal{T} = \varrho I - A_I A_I^* \) for \( \varrho > \lambda_{\text{max}}(A_I A_I^*) \). Let \( \sigma, \tau > 0 \) be given. Choose a 
small constant \( \varepsilon > 0 \) and a point \((Z^0, y^0_E, S^0, X^0) = (0, 0, 0, 0) \). Set \( k := 0 \).
(S.1) Compute the following problems by one of the criteria (C1) and (C2):
\[ (Z^{k+1}, y^{k+1}_E) = \arg \min_{Z, y_E} \phi_k(Z, y_E) := L(0, Z, y_E, S^k, X^k) + \frac{1}{2} \|Z - Z^k\|^2_{\mathcal{I}}; \] 
\[ S^{k+1} = \arg \min_S L(0, Z^{k+1}, y^{k+1}_E, S, X^k) = \Pi_S^+(C - A_E^* y^{k+1}_E - Z^{k+1} - \sigma^{-1} X^k). \]
(S.3) Update the Lagrange multiplier \( X^{k+1} \) via the following formula
\[ X^{k+1} = X^k + \tau \sigma (Z^{k+1} + A_E^* y^{k+1}_E + S^{k+1} - C); \]
(S.4) Let \( k \leftarrow k + 1 \), and go to Step (S.1).

For the approximate optimal solution \((Z^{k+1}, y^{k+1}_E)\) of subproblem (50), one may get 
it by solving the problem \( \min_{Z, y_E} \phi_k(Z, y_E) \) in an alternating way. Let \( k_0 = k \). The 
iterates \((Z^{k_j}, y^{k_j}_E)\) yielded by solving the problem \( \min_{Z, y_E} \phi_k(Z, y_E) \) alternately satisfy 
\[ Z^{k_j} = \arg \min_{Z \in S^n} \phi_k(Z, y^{k_{j-1}}_E) \quad \text{and} \quad y^{k_j}_E = \arg \min_{y_E \in R^{m_E}} \phi_k(Z^{k_j}, y_E) \quad \text{for} \quad j = 1, 2, \ldots. \]
From the expression of the function \( \phi_k(\cdot, \cdot) \), it is immediate to obtain that 
\[ \begin{cases} 
0 \in N_{K^*}(Z^{k_j}) - M + X^k + \sigma (Z^{k_j} + A_E^* y^{k_j-1}_E + S^k - C) + \varepsilon (Z^{k_j} - Z^k), \\
0 = A_E^* X^k - b_E + \sigma A_E (A_E^* y^{k_j}_E + Z^{k_j} + S^k - C). 
\end{cases} \]
Comparing this system with the optimality condition of \( \min_{Z, y_E} \phi_k(Z, y_E) \), with \( \xi^{k_j} = \sigma A_E^* (y^{k_j}_E - y^{k_{j-1}}_E) \) we have \((\xi^{k_j}, 0) \in \partial \phi_k(Z^{k_j}, y^{k_j}_E)\). This means that \((Z^{k_j}, y^{k_j}_E)\) satisfies 
the criterion (C1) with \( \nu_k \equiv 0 \) when \( \|\xi^{k_j}\| \leq \mu_{k+1} \) and \( \sum_{k=0}^{\infty} \mu_{k+1} < \infty \). In addition, let 
\[ \delta := (\sqrt{\sigma} + \varepsilon - \sqrt{\sigma}) \min \left( \sqrt{\sigma + \varepsilon}, \frac{\sigma}{\sqrt{\sigma + \varepsilon}} \right) \min \left( \sigma, \frac{\sigma}{\sqrt{\sigma + \varepsilon}} \right) \min \left( \|A_E A_E^*\| \right). \]
By using equation (49) and [20, Theorem 7.7.6], it is not difficult to verify that 
\[ T_j \geq \sigma \left[ \begin{array}{cc} \sqrt{\sigma + \varepsilon} - \sqrt{\sigma} & 0 \\
0 & \sqrt{\sigma + \varepsilon} - \sqrt{\sigma} \end{array} \right] \begin{array}{c} 0 \\
\frac{\sigma}{\sqrt{\sigma + \varepsilon}} A_E A_E^* \end{array} \right] \geq \delta I. \]
This means that \((Z^{k_j}, y^{k_j}_E)\) satisfies the criterion (C2) with \( F = \delta^{-1} I \) and \( \nu_k \equiv 0 \) once 
\[ \|\xi^{k_j}\| \leq \sqrt{\delta} \mu_{k+1} \sqrt{\|Z^{k_j} - Z^k + A_E^* (y^{k_j}_E - y^{k_{j-1}}_E)\|^2 + \frac{\varepsilon}{\sigma} \|Z^{k_j} - Z^k\|^2} \quad \text{and} \quad \sum_{k=0}^{\infty} \mu_{k+1} < \infty, \]
since the right hand side of the first inequality is less than $\sqrt{\delta \mu_{k+1}}\|\langle Z_{kj} - Z_k; y_{kj} - y_k \rangle \|_{T_j}$. In the sequel, we call Algorithm 5.1 with the subproblems in (50) solved alternately by the criteria (C1) and (C2) IEIDP-ADMM1 and IEIDP-ADMM2, respectively.

We apply IEIDP-ADMM1 and IEIDP-ADMM2 for the doubly nonnegative SDPs without inequality constraint $A_I X \geq b_I$, and compare their performance with that of the 3-block ADMM of step-size $\tau = 1.618$ (for short, ADMM3d). Among others, the doubly nonnegative SDP test examples can be found in [31]. We have implemented IEIDP-ADMM1, IEIDP-ADMM2 and ADMM3d in MATLAB, where $\varepsilon = 10^{-5}$ and $\mu_k = \min(0.1, \frac{1}{k+1})$ for $k \geq 1$ are used for IEIDP-ADMM1 and IEIDP-ADMM2. Notice that when $\|\langle Z_{kj} - Z_k; y_{kj} - y_k \rangle \|_{T_j} < 1$, the criterion (C2) is more restrictive than (C1). Moreover, the criterion (C2) will require much more inner iterations as the primal and dual infeasibility becomes smaller since $\|\langle Z_{kj} - Z_k; y_{kj} - y_k \rangle \|_{T_j}$ is close to 0. So, in the implementation of IEIDP-ADMM2, we modify the criterion (C2) into

$$\|\xi_{kj}\| \leq \max \left( \sqrt{\delta \mu_{k+1}}\|\langle Z_{kj} - Z_k; y_{kj} - y_k \rangle \|_{T_j}, 0.1 \max(\eta_P, \eta_D) \right),$$

where $\eta_P$ and $\eta_D$ are defined below. In addition, the implementation of ADMM3d here is different from that of [33] since the former uses the solution order $Z \rightarrow y_E \rightarrow S$, while the latter uses the order $y_E \rightarrow Z \rightarrow S$. The computational results for all DNNSDPs are obtained on a Windows system with Intel(R) Core(TM) i3-2120 CPU@3.30GHz.

We measure the accuracy of an approximate optimal solution $(Z, y_E, S, X)$ for (44) and (45) by using the relative residual $\eta = \max \{\eta_P, \eta_D, \eta_S, \eta_K, \eta_{S^*}, \eta_{K^*}, \eta_{C_1}, \eta_{C_2}\}$ where

$$\eta_P = \frac{\|A_E X - b_E\|}{1 + \|b_E\|}, \quad \eta_D = \frac{\|A^*_E y_E + S + Z - C\|}{1 + \|C\|}, \quad \eta_S = \frac{\|\Pi_{S^*}(X)\|}{1 + \|X\|}, \quad \eta_K = \frac{\|\Pi_K(X)\|}{1 + \|X\|},
$$

$$\eta_{S^*} = \frac{\|\Pi_{S^*}(-S)\|}{1 + \|S\|}, \quad \eta_{K^*} = \frac{\|\Pi_{K^*}(-Z)\|}{1 + \|Z\|}, \quad \eta_{C_1} = \frac{\langle X, S \rangle}{1 + \|X\| + \|S\|}, \quad \eta_{C_2} = \frac{\langle X, Z \rangle}{1 + \|X\| + \|Z\|}.$$

We terminated the three solvers IEIDP-ADMM1, IEIDP-ADMM2 and ADMM3d whenever $\eta < 10^{-6}$ or the number of iteration is over $k_{\text{max}} = 20000$.

In the implementation of the three solvers, the penalty parameter $\sigma$ is dynamically adjusted according to the progress of the algorithms, and the idea to adjust $\sigma$ is to balance the progress of primal feasibilities ($\eta_P, \eta_S, \eta_K$) and dual feasibilities ($\eta_D, \eta_{S^*}, \eta_{K^*}$). The exact details on the adjustment strategies are not given here. In addition, all the solvers also adopt some kind of restart strategies to ameliorate slow convergence. During the testing, we use the same adjustment strategy of $\sigma$ and restart strategy for all the solvers.

Figure 1 shows the performance profiles of IEIDP-ADMM1, IEIDP-ADMM2 and ADMM3d in terms of number of iterations and computing time, respectively, for the total 605 (including BIQ(165), RCP(120), $\theta_1(113)$, FAP(13) and QAP(95)) tested problems. We recall that a point $(x, y)$ is in the performance profiles curve of a method if and only if it can solve $100y\%$ of all tested problems no slower than $x$ times of any other methods. We see that IEIDP-ADMM1 and ADMM3d need the comparable iterations and
computing time. Among others, IEIDP-ADMM2 requires the least number of iterations for 60\% test problems, but it needs the most computing time which is about 1.5 times that of IEIDP-ADMM1 and ADMM3d for about 80\% test problems.

![Performance Profile (iter)](image1)

![Performance Profile (time)](image2)

Figure 1: Performance profiles of the number of iterations and computing time of solvers

5.2 Numerical results for the DNNSDPs with $A_I X \geq b_I$

For this case, we may apply the proposed IEIDP-ADMMs for solving (45) or (47). Firstly, we report the numerical results of the IEIDP-ADMMs for solving problem (45).

5.2.1 Numerical results of the IEIDP-ADMMs for problem (45)

Since $[I, A_I^*]^*[I, A_I^*]$ and $[A_E^*, I]^*[A_E^*, I]$ are not positive definite and $A_I$ is not surjective, we introduce the semi-proximal terms $\frac{1}{2}(y_I - y_I^k, Z - Z^k)\text{Diag}(\sigma T, 0)(y_I - y_I^k, Z - Z^k)^T$ and $\frac{1}{2}(y_E - y_E^k, S - S^k)\text{Diag}(0, \epsilon I)(y_E - y_E^k, S - S^k)^T$ to guarantee that

$$T_I \succeq \sigma [A_I^*, I]^*[A_I^*, I] + \text{Diag}(\sigma T, 0) = \sigma \begin{bmatrix} \sigma I & A_I^* \\ A_I & I \end{bmatrix} \succ 0 \quad (52)$$

and

$$T_g \succeq \sigma [A_E^*, I]^*[A_E^*, I] + \text{Diag}(0, \epsilon I) = \sigma \begin{bmatrix} A_E A_E^* & \sigma A_E \\ A_E^* & \sigma I \end{bmatrix} \succ 0, \quad (53)$$

and propose the following partial inexact indefinite proximal ADMMs for solving (45).
Algorithm 5.2 (An inexact indefinite-proximal ADMM for (45))

(S.0) Let $\mathcal{T} = \rho \mathcal{I} - A_I A_I^*$ for $\rho > \lambda_{\text{max}}(A_I A_I^*)$. Let $\sigma, \tau > 0$ be given. Choose a small constant $\varepsilon > 0$ and a point $(y_0^I, Z^0, y_0^E, S_0, X_0) = (0, 0, 0, 0, 0)$. Set $k := 0$.

(S.1) Compute the following problems by one of the criteria (C1)-(C2):

\[
(y_{k+1}^I, Z^{k+1}) \approx \arg \min_{y_I, Z} \phi_k(y_I, Z) := L_0(y_I, Z, y_k^I, S_k, X_k) + \frac{1}{2} \|y_I - y_k^I\|_T^2 \tag{54}
\]

\[
(y_{k+1}^E, S^{k+1}) \approx \arg \min_{y_E, S} \psi_k(y_E, S) := L_0(y_I^{k+1}, Z^{k+1}, y_E, S, X_k) + \frac{1}{2} \|S - S_k\|_T^2.
\]

(S.3) Update the Lagrange multiplier $X^{k+1}$ via the formula

\[
X^{k+1} = X^k + \tau \sigma (A_I^* y_{k+1}^I + Z^{k+1} + A_E^* y_{E}^{k+1} + S^{k+1} - C).
\]

(S.4) Let $k \leftarrow k + 1$, and go to Step (S.1).

One may obtain the approximate optimal solutions $(y_{k+1}^I, Z^{k+1})$ and $(y_{k+1}^E, S^{k+1})$ by computing $\min_{y_I, Z} \phi_k(y_I, Z)$ and $\min_{y_E, S} \psi_k(y_E, S)$ in an alternating way. Let $k_0 = k$. The iterates $(y_k^I, Z_k)$ for $j \geq 1$ yielded by minimizing $\phi_k(y_I, Z)$ alternately satisfy

\[
\begin{cases}
0 \in \mathcal{N}_{\mathcal{F}_k}^I(y_k^I) - b_I + A_I X_k + \sigma A_I (A_I^* y_k^I + Z_{k-1} + A_E^* y_E^k + S_k - C) + \sigma T (y_{k}^I - y_k^I), \\
0 \in \mathcal{N}_{\mathcal{F}_k}^E(Z_k) - M + X_k + \sigma (A_I^* y_k^I + Z_k + A_E^* y_E^k + S_k - C).
\end{cases}
\]

Let $\xi_k = \sigma A_I (Z_k - Z_{k-1})$. Comparing the last system with the optimality condition of $\min_{y_I, Z} \phi_k(y_I, Z)$, we have $(\xi_k, 0) \in \partial \phi_k(y_k^I, Z_k)$. This means that $(y_k^I, Z_k)$ satisfies the criterion (C1) when $\|\xi_k\| \leq \mu_{k+1}$ and $\sum_{k=0}^\infty \mu_{k+1} < \infty$. Notice that

\[
T_f \succeq \begin{bmatrix} \rho \mathcal{I} & A_I \\ -\phi^{-1} A_I^* & \mathcal{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{I} & 0 \\ 0 & \phi^{-1} A_I^* A_I \end{bmatrix} \begin{bmatrix} \mathcal{I} & \phi^{-1} A_I \\ 0 & \mathcal{I} \end{bmatrix}^{-1} \\
\succeq \min (\rho, \lambda_{\text{min}}(I - \phi^{-1} A_I^* A_I)) \begin{bmatrix} \mathcal{I} & 0 \\ 0 & -\phi^{-1} A_I^* \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{I} & -\phi^{-1} A_I \\ 0 & \mathcal{I} \end{bmatrix}^{-1} \mathcal{I} \\
= \partial \lambda_{\text{min}} \left( \begin{bmatrix} \mathcal{I} & -\phi^{-1} A_I \\ 0 & \mathcal{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{I} & 0 \\ 0 & \mathcal{I} \end{bmatrix} \right) \mathcal{I} \succeq \partial \mathcal{I}
\]

where $\partial : = \min (\rho, 1 - \phi^{-1} \lambda_{\text{max}}(A_I^* A_I))$. So, $(y_k^I, Z_k)$ satisfies (C2) with $\mathcal{F} = \frac{1}{\sigma \rho} \mathcal{I}$ when

\[
\|\xi_k\| \leq \mu_{k+1} \sqrt{\rho} \sqrt{\|y_{k}^I - y_{k}^I + A_I^* (Z_k - Z_k)\|^2 + \rho \|y_{k}^I - y_{k}^I\|^2 - \|A_I^* (Z_k - Z_k)\|^2}.
\]

26
The iterates \((y_k^j, S^j_k)\) for \(j \geq 1\) yielded by minimizing \(\psi_k(y_E, S)\) alternately satisfy
\[
\begin{align*}
0 &= -b_E + A_E X^k + \sigma A_E (A_I^* y_I^{k+1} + Z^{k+1} + A_E^* y_E^{k+1} + S^{k+1}_E - C), \\
0 &\in N\sigma(S^j_k) + X^k + \sigma(A_I^* y_I^{k+1} + Z^{k+1} + A_E^* y_E^{k+1} + S^{k-j}_j - C) + \varepsilon (S^{j_k} - S^k).
\end{align*}
\]
Let \(\eta^{k_j} = \sigma A_E (S^{k_j} - S^{k_j-1})\). Comparing the last system with the optimality conditions of problem \(\min_{y_E, S} \psi_k(y_E, S)\), we have \((\eta^{k_j}, 0) \in \partial \psi_k(y_E^{k_j}, S^{k_j})\). This means that \((y_E^{k_j}, S^{k_j})\) satisfies (C1) when \(\|\eta^{k_j}\| \leq \nu_{k+1} + \sum_{k=0}^{\infty} \nu_{k+1} < \infty\). In addition, let
\[
\delta := (\sqrt{\sigma + \varepsilon} - \sqrt{\sigma}) \min \left(\frac{\sigma}{\sqrt{\sigma + \varepsilon}}, \lambda_{\min}(A_E A_E^* E)\right).
\]
By using equation (56) and [20, Theorem 7.7.6], it is not difficult to verify that
\[
G_{\sigma} \geq \sigma \begin{bmatrix}
\frac{\sqrt{\sigma + \varepsilon} \sqrt{\sigma}}{\sqrt{\sigma + \varepsilon}} & 0 \\
0 & \frac{\sigma}{\sqrt{\sigma + \varepsilon}} - \sqrt{\sigma} + \varepsilon I
\end{bmatrix} \geq \delta I.
\]
This means that \((y_E^{k_j}, S^{k_j})\) satisfies the criterion (C2) with \(G = \delta^{-1} I\) once
\[
\|\eta^{k_j}\| \leq \sqrt{\delta \sigma \nu_{k+1}} \sqrt{\|A_E^* (y_E^{k_j} - y_E^k) + (S^{k_j} - S^k)\|^2 + \frac{\varepsilon}{\sigma} \|S^{k_j} - S^k\|^2}.
\]
We call Algorithm 5.2 with the two subproblems in (S.1) solved alternately by the criteria (C1) and (C2) IEIDP-ADMM1 and IEIDP-ADMM2, respectively.

We apply the IEIDP-ADMM1 and IEIDP-ADMM2 for solving the extended BIQ problems described in Section 4.2 of [31], and compare its performance with the four-block proximal ADMM of step-size \(\tau = 1.618\) (although without convergent guarantee) by adding a proximal term \(\frac{\delta \sigma}{2} \|y_I - y_I^k\|^2_I\) for the \(y_I\) part, where \(T = \|A_I A_I^*\| I - A_I A_I^*\). We call this method PADMM4d. The computational results for all the extended BIQ problems are obtained on the same desktop computer as before.

We measure the accuracy of an approximate optimal solution \((X, y_I, Z, y_E, S)\) for (44) and (45) by the relative residual \(\eta = \max \{\eta_P, \eta_D, \eta_S, \eta_K, \eta_{K^*}, \eta_{C_I}, \eta_{C_I^*}, \eta_{C_2}, \eta_{C_2^*}\}\), where \(\eta_P, \eta_S, \eta_K, \eta_{K^*}, \eta_{C_I}, \eta_{C_I^*}, \eta_{C_2}, \eta_{C_2^*}\) are defined as before, and \(\eta_D, \eta_I, \eta_I^*\) are given by
\[
\eta_D = \frac{\|A_I y_I + Z + A_E^* y_E + S - C\|}{\|C\|}, \quad \eta_I = \frac{\|\max(0, b_I - A_I X)\|}{\|b_I\|}, \quad \eta_I^* = \frac{\|\max(0, -y_I)\|}{\|y_I\|}.
\]
The three solvers IEIDP-ADMM1 and IEIDP-ADMM2 and PADMM4d were stopped whenever \(\eta < 10^{-6}\) or the number of iteration is over \(k_{\max} = 40000\).

Figure 2 plots the performance profiles of IEIDP-ADMM1, IEIDP-ADMM2 and PADMM4d in terms of the number of iterations and computing time, respectively, for the total 165 extended BIQ tested problems. It can be seen from this figure that IEIDP-ADMM1, IEIDP-ADMM2 and PADMM4d are comparable in terms of the iterations and computing time, IEIDP-ADMM1 and IEIDP-ADMM2 need the least number of iterations for at least 80% tested problems, which is about 90% that of PADMM4d, and PADMM4d requires the least computing time for about 70% tested problems, which is about 90% that of IEIDP-ADMM2.
5.2.2 Numerical results of the IEIDP-ADMMs for problem (47)

Since \([A_I^T 0 I][A_I^T 0 I] + [I - I 0][I - I 0]\) and \([A_E^T I][A_E^T I]\) are not positive definite, we introduce the semi-proximal terms \(\frac{1}{2}(y_I - y_I^k, z - z^k, Z - Z^k)^T \text{Diag}(\sigma \varepsilon I, 0, 0)(y_I - y_I^k, z - z^k, Z - Z^k)^T\) and \(\frac{1}{2}(y_E - y_E^k, S - S^k) \text{Diag}(0, \varepsilon I)(y_E - y_E^k, S - S^k)^T\) to ensure that

\[
\mathcal{T}_f \geq \sigma \left( [A_I^T 0 I][A_I^T 0 I] + [I - I 0][I - I 0] \right) + \text{Diag}(\sigma \varepsilon I, 0, 0) > 0 \tag{55}
\]

and

\[
\mathcal{T}_g \geq \sigma [A_E^T I][A_E^T I] + \text{Diag}(0, \varepsilon I) = \sigma \left[ \frac{A_E^* A_E^T}{\sigma + \varepsilon I} \right] > 0, \tag{56}
\]

and propose the following inexact indefinite proximal ADMMs for solving (47), where for a given \(\sigma > 0\), the augmented Lagrangian function of problem (47) is defined as:

\[
L_{\sigma}(y_I, z, y_E, S, X, x) := -\langle b_I, y_I \rangle + \delta_{R_+^{m_I}}(z) + (\delta_{K^c}(Z) - \langle M, Z \rangle) - \langle b_E, y_E \rangle + \delta_{S_+^m}(S) + \langle X, A_I^T y_I + Z + A_E^T y_E + S - C \rangle + \langle x, y_I - z \rangle + \sigma \left\| A_I^T y_I + Z + A_E^T y_E + S - C \right\|^2 + \frac{\sigma}{2} \| y_I - z \|^2 
\tag{57}
\]

\(\forall (y_I, z, y_E, S, X, x) \in \mathbb{R}^{m_I} \times \mathbb{R}^{m_I} \times \mathbb{S}^n \times \mathbb{R}^{m_E} \times \mathbb{S}^n \times \mathbb{S}^n \times \mathbb{R}^{m_I} \).
Algorithm 5.3 (An inexact indefinite proximal ADMM for (47))

(S.0) Let \( \sigma, \tau > 0 \) be given. Choose a sufficiently small constant \( \varepsilon > 0 \) and an initial point \( (y_I^0, z^0, Z^0, y_E^0, S^0, X^0, x^0) = (0, 0, 0, 0, 0, 0, 0) \). Set \( k := 0 \).

(S.1) Compute the following problems by one of the criteria (C1)-(C2):

\[
(y_{I}^{k+1}, (z^{k+1}, Z^{k+1})) \approx \arg \min_{y_I, z, Z} L_{\sigma}(y_I, z, Z, y_E^k, S^k, X^k, x^k) + \frac{\sigma \varepsilon}{2} \| y_I - y_I^k \|^2,
\]

\[
(y_{E}^{k+1}, S^{k+1}) \approx \arg \min_{y_E, S} L_{\sigma}(y_{I}^{k+1}, z^{k+1}, Z^{k+1}, y_E, S, X^k, x^k) + \frac{\varepsilon}{\sigma} \| S - S_k \|^2.
\]

(S.3) Update the Lagrange multipliers \((X^{k+1}, \zeta^{k+1})\) via the following formula

\[
X^{k+1} = X^k + \tau \sigma (Z^{k+1} + A_I^* v^{k+1} + A_E^* y_{E}^{k+1} + S^{k+1} - C),
\]

\[
x^{k+1} = x^k + \tau \sigma (y_{I}^{k+1} - z^{k+1}).
\]

(S.4) Let \( k \leftarrow k + 1, \) and go to Step (S.1).

For the approximate optimal solution \((y_{I}^{k+1}, z^{k+1}, Z^{k+1})\) in (S.1), one may get it by solving the problem \( \min_{y_I, z, Z} \phi_k(y_I, z, Z) \) in an alternating way, where

\[
\phi_k(y_I, z, Z) := L_{\sigma}(y_I, z, Z, y_E^k, S^k, X^k, x^k) + \frac{\sigma \varepsilon}{2} \| y_I - y_I^k \|^2.
\]

The iterates \((y_{I}^{k_j}, z^{k_j}, Z^{k_j})\) given by solving \( \min_{y_I, z, Z} \phi_k(y_I, z, Z) \) alternately satisfy

\[
y_{I}^{k_j} = \arg \min_{y_I \in \mathbb{R}^{m_I}} \phi_k(y_I, z^{k_{j-1}}, Z^{k_{j-1}}), \quad (z^{k_j}, Z^{k_j}) = \arg \min_{(z, Z) \in \mathbb{R}^{m_I} \times \mathbb{S}^n} \phi_k(y_{I}^{k_j}, z, Z) \quad \text{for} \quad j = 1, 2, \ldots
\]

with \( k_0 = k \). We apply the conjugate gradient method to the first minimization, i.e.,

\[
(A_I A_I^* + (1+\varepsilon) I)^{-1} y_{I}^{k_j} = [z^{k_{j-1}} + A_I (C - Z^{k_{j-1}} - S^k - A_E^* y_{E}^{k_j} - \frac{X^k}{\sigma}) - \frac{x^k - b_I}{\sigma} + \varepsilon y_{I}^{k_j}] + R^{k_j},
\]

where \( R^{k_j} \) denotes the error yielded by the conjugate gradient method. Let

\[
\xi^{k_j} = \sigma (Z^{k_j} - Z^{k_{j-1}}) + \sigma (z^{k_j} - z^{k_{j-1}}) - \sigma R^{k_j}.
\]

Then, together with the definition of \((z^{k_j}, Z^{k_j})\), we have \((\xi^{k_j}, 0, 0) \in \partial \phi_k(y_{I}^{k_j}, z^{k_j}, Z^{k_j})\).

This means that \((y_{I}^{k_j}, z^{k_j}, Z^{k_j})\) satisfies (C1) when \( \|\xi^{k_j}\| \leq \mu_{k+1} + \sum_{k=0}^{\infty} \mu_{k+1} < \infty \).

For the approximate optimal solution \((y_{E}^{k+1}, S^{k+1})\) in (S.1), one may obtain it by solving the corresponding minimization alternately. Also, from Subsection 5.2.1 it follows that \((y_{E}^{k_j}, S^{k_j})\) satisfies the criterion (C2) with \( G = \delta^{-1} I \) if \( \eta^{k_j} = \sigma A_E (S^{k_j} - S^{k_{j-1}}) \) satisfies

\[
\| \eta^{k_j} \| \leq \sqrt{\delta \sigma} \nu_{k+1} \sqrt{\| A_E^* (y_{E}^{k_j} - y_{E}^{k}) + (S^{k_j} - S^k) \|^2 + \frac{\varepsilon}{\sigma} \| S^{k_j} - S^k \|^2}.
\]

29
We call Algorithm 5.3 with the subproblems solved alternately by (C1) IEIDP-ADMM1.

We apply the IEIDP-ADMM1 for solving the extended BIQ problems described in Section 4.2 of [31], and compare its performance with the previous PADMM4d and the four-block ADMM of step-size $\tau = 1.618$ (although without convergent guarantee). We call the latter ADMM4d. The computational results for all the extended BIQ problems are obtained on the same desktop computer as before. We measure the accuracy of an approximate optimal solution $(X, y_I, z, Y, S)$ for (44) and (47) by the relative residual $\eta = \max \{ \eta_P, \eta_S, \eta_K, \eta_{S^*}, \eta_C, \eta_{C^*}, \eta_I, \eta_{I^*} \}$, where $\eta_P, \eta_S, \eta_K, \eta_{S^*}, \eta_C, \eta_{C^*}, \eta_I, \eta_{I^*}$ are defined as before. The solvers IEIDP-ADMM1 and PADMM4d and ADMM4d were terminated whenever $\eta < 10^{-6}$ or the number of iteration is over $k_{\max} = 40000$.

![Performance Profile (iter) (165 Extended BIQ problems)](image)

![Performance Profile (time) (165 Extended BIQ problems)](image)

Figure 3: Performance profiles of the number of iterations and computing time for EBIQ

Figure 3 plots the performance profiles of IEIDP-ADMM1, PADMM4d and ADMM4d in terms of the number of iterations and computing time, respectively, for the total 165 extended BIQ tested problems. We see that, when applying the IEIDP-ADMM for solving the dual problem (47), the number of iterations and the computing time of the IEIDP-ADMM1 are still comparable with those of ADMM4d, but PADMM4d requires more 4 times iterations than IEIDP-ADMM1 and ADMM4d as do for at least 80% test problems. This means that a small proximal term as possible is the key to the performance of proximal-type ADMMs. The computing time of PADMM4d is a little less than that of IEIDP-ADMM1 and ADMM4d since the latter solves an $m_I \times m_I$ linear system with the conjugate gradient method, where $m_I$ may attain 374250.
6 Conclusion

We developed an inexact indefinite proximal ADMM of step-size $\tau \in (0, \frac{\sqrt{5}+1}{2})$ with two easily implementable inexactness criteria for the two-block separable convex minimization problems with linear constraints, for which it is either impossible or too expensive to obtain the exact solutions of the subproblems involved in the proximal ADMM. Numerical results for the DNNSDPs with many linear equality and/or inequality constraints show that the inexact indefinite proximal ADMMs are effective for this class of difficult three or four block separable separable convex optimization problems with linear constraints. Among others, the inexact indefinite proximal ADMM with the absolute error criterion (C1) is comparable with the directly extended ADMM of step-size $\tau = 1.618$, whether in terms of the number of iterations or computing time, and is superior to the one with the relative error criterion (C2) by weighing the number of iterations and the computing time since the latter is very restrictive and requires too many iterations for the solution of subproblems. In our future research work, we will explore other easily implementable inexact criteria like relaxing $\mu_{k+1}$ and $\nu_{k+1}$ in (C2) to be a constant, and study the nonergodic convergence \[6, 7\] for the inexact indefinite proximal ADMMs.

Acknowledgements. The authors would like to thank Professor Kim-Chuan Toh from National University of Singapore for providing us the code, and Professor Defeng Sun from National University of Singapore for the suggestion on numerical comparison of (45) and (47).

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