COHEN-MACAULAY HOMOLOGICAL DIMENSIONS

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Abstract. We introduce new homological dimensions, namely the Cohen-Macaulay projective, injective and flat dimensions for homologically bounded complexes. Among other things we show that (a) these invariants characterize the Cohen-Macaulay property for local rings, (b) Cohen-Macaulay flat dimension fits between the Gorenstein flat dimension and the large restricted flat dimension, and (c) Cohen-Macaulay injective dimension fits between the Gorenstein injective dimension and the Chouinard invariant.

1. Introduction

A commutative Noetherian local ring $R$ is regular if the residue field $k$ has finite projective dimension and only if all $R$-modules have finite projective dimension [25]. This theorem of Auslander, Buchsbaum and Serre is a main motivation of studying homological dimensions. The injective and flat dimensions have similar behavior.

Auslander and Bridger [1], introduced a homological dimension for finitely generated modules designed to single out modules with properties similar to those of modules over Gorenstein rings. They called it G-dimension and it is a refinement of the projective dimension and showed that a local Noetherian ring $(R, m, k)$ is Gorenstein if the residue field $k$ has finite G-dimension and only if all finitely generated $R$-modules have finite G-dimension.

To extend the G-dimension beyond the realm of finitely generated modules over Noetherian rings, Enochs and Jenda [12] introduced the notion of Gorenstein projective module. Then the notion of Gorenstein projective dimension was studied in [7].

The notion of Gorenstein injective module is dual to that of Gorenstein projective module and were introduced in the same paper by Enochs and Jenda [12]. Then the notion of Gorenstein injective dimension was studied in [7].

Another extension of the G-dimension is based on Gorenstein flat modules, a notion due to Enochs, Jenda, and Torrecillas [13]. Then the notion of Gorenstein flat dimension was studied in [7].

More recently, the complete intersection dimension has been introduced for finitely generated $R$-modules, using quasi-deformations and projective dimension, to characterize the complete intersection property of local rings [9]. Parallel to Gorenstein projective, injective and flat dimensions, the complete intersection projective, injective and flat dimensions have been introduced and studied in [22], [23], [24] and [21].

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The **Cohen-Macaulay dimension** of a finitely generated $R$-module $M$, as defined by Gerko [16] is

$$\text{CM-dim}_R(M) := \inf \left\{ \text{G-dim}_Q (M \otimes_R R') - \text{G-dim}_Q (R') \mid R \to R' \leftarrow Q \text{ is a CM-quasi-deformation} \right\}$$

(see Section 2 for the definition of CM-quasi-deformation).

The purpose of this paper is to develop a similar theory of projective, injective and flat analogue for Cohen-Macaulay case. Thus we introduce Cohen-Macaulay projective dimension (CM$_{\text{pd}}$), Cohen-Macaulay injective dimension (CM$_{\text{id}}$) and Cohen-Macaulay flat dimension (CM$_{\text{fd}}$) for homologically bounded complexes over commutative Noetherian local rings $(R, \mathfrak{m}, k)$ with identity (see Definition 3.1). In particular CM-dim$_R(M) = \text{CM}_{\text{pd}}_R(M) = \text{CM}_{\text{fd}}_R(M)$, for a finitely generated $R$-module $M$. Among other things, we show that these invariants characterize the Cohen-Macaulay property for local rings. We also show that if $M$ is a homologically bounded $R$-complex, then we have the inequalities

$$\text{Rfd}_R(M) \leq \text{CM}_{\text{fd}}_R(M) \leq \text{Gfd}_R(M),$$

with equality to the left of any finite value. In particular if Gfd$_R(M) < \infty$, then CM$_{\text{fd}}_R(M) = \text{Gfd}_R(M)$, and if CM$_{\text{fd}}_R(M) < \infty$, then

$$\text{CM}_{\text{fd}}_R(M) = \sup \{ \text{depth}_p - \text{depth}_{R_p}(M_p) \mid p \in \text{Spec } (R) \},$$

where Rfd$_R(M)$ is the large restricted flat dimension. Also, we show that there are inequalities

$$\sup \{ \text{depth}_p - \text{width}_{R_p} M_p \mid p \in \text{Spec } (R) \} \leq \text{CM}_{\text{id}}_R(M) \leq \text{Gid}_R(M),$$

such that if Gid$_R(M) < \infty$, then CM$_{\text{id}}_R(M) = \text{Gid}_R(M)$, and if CM$_{\text{id}}_R(M) < \infty$ for a homologically finite $R$-complex $M$, then

$$\text{CM}_{\text{id}}_R(M) = \sup \{ \text{depth}_p - \text{width}_{R_p}(M_p) \mid p \in \text{Spec } (R) \} = \text{depth } R - \text{inf}(M).$$

Finally we compare our Cohen-Macaulay homological dimensions with the homological dimensions of Holm and Jørgenson [17].

## 2. Definitions and Notations

Let $(R, \mathfrak{m}, k)$ and $(S, \mathfrak{n}, l)$ be commutative local Noetherian rings.

We work in the derived category $D(R)$ of complexes of $R$-modules, indexed homologically. A complex $M$ is **homologically bounded** if $H_i(M) = 0$ for all $|i| \gg 0$; and it is **homologically finite** if $\oplus_i H_i(M)$ is finitely generated.

Fix $R$-complexes $M$ and $N$. Let $M \otimes^L_R N$ and $R\text{Hom}_R(M, N)$ denote the left-derived tensor product and right-derived homomorphism complexes, respectively. Let inf$(M)$ and sup$(M)$ denote the infimum and supremum, respectively, of the set \{ $n \in \mathbb{Z}$ | $H_n(M) \neq 0$ \}.

**Definition/Notation 2.1.** A homologically finite $R$-complex $M$ is **reflexive** if the complex $R\text{Hom}_R(M, R)$ is homologically bounded and the biduality morphism $\delta_M : M \to R\text{Hom}_R(R\text{Hom}_R(M, R), R)$ is an isomorphism in $D(R)$. Set

$$\text{G-dim}_R(M) := - \text{inf}(R\text{Hom}_R(M, R)),$$
if $M$ is reflexive, and $\text{G-dim}_R(M) := \infty$ otherwise. Set also $\text{G-dim}_R(0) = -\infty$. This is the $G$-dimension of Auslander and Bridger \[1\] and Yassemi \[27\].

**Definition/Notation 2.2.** An $R$-module $G$ is $G$-projective if there exists an exact sequence of $R$-modules

$$X = \cdots \to P_1 \xrightarrow{\partial X_1} P_0 \xrightarrow{\partial X_0} P_{-1} \xrightarrow{\partial X_{-1}} P_{-2} \xrightarrow{\partial X_{-2}} \cdots$$

such that $G \cong \text{Coker}(\partial X_1)$, each $P_i$ is projective, and $\text{Hom}_R(X, Q)$ is exact for each projective $R$-module $Q$.

An $R$-module $G$ is $G$-flat if there exists an exact sequence of $R$-modules

$$Y = \cdots \to F_1 \xrightarrow{\partial Y_1} F_0 \xrightarrow{\partial Y_0} F_{-1} \xrightarrow{\partial Y_{-1}} F_{-2} \xrightarrow{\partial Y_{-2}} \cdots$$

such that $G \cong \text{Coker}(\partial Y_1)$, each $F_i$ is flat, and $I \otimes_R Y$ is exact for each injective $R$-module $I$.

An $R$-module $G$ is $G$-injective if there exists an exact sequence of $R$-modules

$$Z = \cdots \to I_1 \xrightarrow{\partial Z_1} I_0 \xrightarrow{\partial Z_0} I_{-1} \xrightarrow{\partial Z_{-1}} I_{-2} \xrightarrow{\partial Z_{-2}} \cdots$$

such that $G \cong \text{Coker}(\partial Z_1)$, each $I_i$ is injective, and $\text{Hom}_R(I, Z)$ is exact for each injective $R$-module $I$.

Let $M$ be a homologically bounded $R$-complex. A $G$-projective resolution of $M$ is an isomorphism $H \simeq M$ in $\mathcal{D}(R)$ where $H$ is a complex of $G$-projective $R$-modules such that $H_i = 0$ for all $i < 0$. The $G$-projective dimension of $M$ is

$$\text{Gpd}_R(M) := \inf \{\sup \{n \mid H_n \neq 0\} \mid H \simeq M \text{ is a } G\text{-projective resolution}\}.$$  

The $G$-flat dimension of $M$ is defined similarly and denoted $\text{Gfd}_R(M)$, while the $G$-injective dimension $\text{Gid}_R(M)$ is dual \[7\]. These are the $G$-projective, $G$-flat, and $G$-injective dimensions of Enochs, Jenda and Torrecillas (which they consider only in the case of modules) \[12\] and \[13\].

**Remark 2.3.** (1) It is known that, for a homologically bounded $R$-complex $M$, $\text{Gpd}_R(M)$ and $\text{Gfd}_R(M)$ are simultaneously finite \[21\] Proposition 4.3].

(2) Let $R \to S$ be a flat local homomorphism and $M$ a finitely generated $R$-module. Then it is well-known that, $\text{G-dim}_R(M) = \text{G-dim}_S(M \otimes_R S)$ and $\text{G-dim}_R(M) = \text{Gfd}_R(M) = \text{Gpd}_R(M)$ \[9\].

(3) The finiteness of $G$-projective, $G$-flat, and $G$-injective dimensions characterize the Gorenstein property of local rings \[7\].

**Definition/Notation 2.4.** A finitely generated $R$-module $M$ is called $G$-perfect if $G$-dim$_R M = \text{grade}_R M := \inf \{i \mid \text{Ext}^i_R(M, R) \neq 0\}$. Let $Q$ be a local ring and $J$ an ideal of $Q$. By abuse of language we say that $J$ is $G$-perfect if the $Q$-module $Q/J$ has the corresponding property.

A $CM$-deformation of $R$ is a surjective local homomorphism $Q \to R$ such that $J = \ker(Q \to R)$ is a $G$-perfect ideal in $Q$. A $CM$-quasi-deformation of $R$ is a diagram of local homomorphisms $R \to R' \leftarrow Q$, with $R \to R'$ a flat extension and $R' \leftarrow Q$ a $CM$-deformation.
The Cohen-Macaulay dimension of a nonzero finitely generated $R$-module $M$, as defined by Gerko [16] is

\[ \text{CM-dim}_R(M) := \inf \left\{ \text{G-dim}_Q(M \otimes_R R') - \text{G-dim}_Q(R') \left| \begin{array}{c} R \rightarrow R' \leftarrow Q \text{ is a} \\ \text{CM-quasi-deformation} \end{array} \right. \right\}, \]

and set $\text{CM-dim}_R(0) = -\infty$.

**Remark 2.5.** By [16] Theorems 3.8 and 3.9, and Proposition 3.10 we have

1. $R$ is Cohen-Macaulay if and only if $\text{CM-dim}_R(k) < \infty$.
2. If $M$ is a finitely generated $R$-module such that $\text{CM-dim}_R(M) < \infty$, then

\[ \text{CM-dim}_R(M) = \text{depth } R - \text{depth } R(M). \]

3. For each prime ideal $p$ of $R$, $\text{CM-dim}_{R_p}(M_p) \leq \text{CM-dim}_R(M)$.

**Definition/Notation 2.6.** A finitely generated $R$-module $C$ is semidualizing if the homothety morphism $\chi_C^R : R \rightarrow R\text{Hom}_R(C,C)$ is an isomorphism in $\mathcal{D}(R)$. A finitely generated $R$-module $D$ is canonical if it is semidualizing and $\text{id}_R(D)$ is finite.

Let $\varphi : R \rightarrow S$ be a local ring homomorphism. We denote $\hat{R}$ the completion of $R$ at its maximal ideal and let $\varepsilon_R : R \rightarrow \hat{R}$ denote the natural map. The completion of $\varphi$ is the unique local ring homomorphism $\hat{\varphi} : \hat{R} \rightarrow \hat{S}$ such that $\hat{\varphi} \circ \varepsilon_R = \varepsilon_S \circ \varphi$. The semi-completion of $\varphi$ is the composition $\varepsilon_S \circ \varphi : R \rightarrow \hat{S}$.

### 3. Cohen-Macaulay Projective, Flat and Injective Dimensions

In this section we introduce a Cohen-Macaulay projective dimension, Cohen-Macaulay flat dimension, and Cohen-Macaulay injective dimension for homologically bounded $R$-complexes and derive their basic properties. When $M$ is a module, Definition 3.1 is from [22], which is in turn modeled on [3] and [16].

**Definition 3.1.** Let $(R, m)$ be a local ring. For each homologically bounded $R$-complex $M$, define the Cohen-Macaulay projective dimension, Cohen-Macaulay flat dimension and Cohen-Macaulay injective dimension of $M$ as,

- $\text{CM}_*\text{-pd}_R(M) := \inf \left\{ \text{Gpd}_Q(M \otimes_R R') - \text{Gfd}_Q(R') \left| \begin{array}{c} R \rightarrow R' \leftarrow Q \text{ is a} \\ \text{CM-quasi-deformation} \end{array} \right. \right\}$
- $\text{CM}_*\text{-fd}_R(M) := \inf \left\{ \text{Gfd}_Q(M \otimes_R R') - \text{Gfd}_Q(R') \left| \begin{array}{c} R \rightarrow R' \leftarrow Q \text{ is a} \\ \text{CM-quasi-deformation} \end{array} \right. \right\}$
- $\text{CM}_*\text{-id}_R(M) := \inf \left\{ \text{Gfd}_Q(M \otimes_R R') - \text{Gfd}_Q(R') \left| \begin{array}{c} R \rightarrow R' \leftarrow Q \text{ is a} \\ \text{CM-quasi-deformation} \end{array} \right. \right\}$

respectively.

**Remark 3.2.**

1. It is known that $\text{Gpd}_R(M)$ and $\text{Gfd}_R(M)$ are simultaneously finite by Remark 2.3(1). Hence $\text{CM}_*\text{-pd}_R(M)$ and $\text{CM}_*\text{-fd}_R(M)$ are simultaneously finite.

2. By taking the trivial CM-quasi-deformation $R \rightarrow R \leftarrow R$, one has

\[ \text{CM}_*\text{-pd}_R(M) \leq \text{Gpd}_R(M), \]

\[ \text{CM}_*\text{-fd}_R(M) \leq \text{Gfd}_R(M), \]
\[ \CM_*-\id_R(M) \leq \Gid_R(M). \]

(3) By Remark 2.3(2) it can be seen that if \( M \) is a finitely generated \( R \)-module then, \( \CM_*-\pd_R(M) = \CM_*-\fd_R(M) = \CM\dim_R(M) \).

The following two theorems show that the finiteness of these dimensions characterize the Cohen-Macaulay rings.

**Theorem 3.3.** The following conditions are equivalent:

1. The ring \( R \) is Cohen-Macaulay.
2. \( \CM_*-\pd_R(M) < \infty \) for every homologically bounded \( R \)-complex \( M \).
3. \( \CM_*-\pd_R(k) < \infty \).
4. \( \CM_*-\fd_R(M) < \infty \) for every homologically bounded \( R \)-complex \( M \).
5. \( \CM_*-\fd_R(k) < \infty \).

**Proof.** (1)\( \Rightarrow \) (2) Let \( \hat{R} \) be the \( \m \)-adic completion of \( R \). Since \( R \) is Cohen-Macaulay, so is \( \hat{R} \). Therefore by Cohen’s structure theorem, \( \hat{R} \) is isomorphic to \( Q/J \), where \( Q \) is a regular local ring. By Cohen-Macaulay-ness of \( \hat{R} \) and regularity of \( Q \), the ideal \( J \) is G-perfect. Thus \( R \to \hat{R} \leftarrow Q \) is a CM -quasi-deformation. Since \( Q \) is regular \( \Gpd_Q(M \otimes_R \hat{R}) < \infty \) for every homologically bounded \( R \)-complex \( M \). Thus \( \CM_*-\pd_R(M) \) is finite.

(2)\( \Rightarrow \) (3) and (4)\( \Rightarrow \) (5) are trivial.

(2)\( \Rightarrow \) (4) and (3)\( \Rightarrow \) (5) are trivial since \( \CM_*-\fd_R(M) \leq \CM_*-\pd_R(M) \).

(5)\( \Rightarrow \) (1) It follows from Remark 3.2(3) that \( \CM\dim_R(k) = \CM_*-\fd_R(k) < \infty \). Now Remark 2.5(1), completes the proof. \( \square \)

**Theorem 3.4.** The following conditions are equivalent.

1. The ring \( R \) is Cohen-Macaulay.
2. \( \CM_*-\id_R(M) < \infty \) for every homologically bounded \( R \)-complex \( M \).
3. \( \CM_*-\id_R(k) < \infty \).

**Proof.** (1)\( \Rightarrow \) (2) is the same as proof of part (1)\( \Rightarrow \) (2) of Theorem 3.3.

(2)\( \Rightarrow \) (3) is trivial.

(3)\( \Rightarrow \) (1) Suppose \( \CM_*-\id_R(k) < \infty \). So that there is a CM -quasi-deformation \( R \to \hat{R} \leftarrow Q \), such that \( \Gid_Q(k \otimes_R \hat{R}) \) is finite. It is clear that \( k \otimes_R \hat{R} \) is a cyclic \( Q \)-module. Consequently \( Q \) is a Gorenstein ring by [15, Theorem 4.5]. We plan to show that \( \hat{R} \) is a Cohen-Macaulay ring. Let \( I = \ker(Q \to \hat{R}) \) which is G-perfect by definition. We have

\[ \text{ht } I = \text{grade } (I, Q) \]
\[ = \text{G-dim}_Q \hat{R} \]
\[ = \text{depth } Q - \text{depth } Q \hat{R} \]
\[ = \text{depth } Q - \text{depth } R' \]
\[ = \text{dim } Q - \text{depth } R' \]
\[ = \text{ht } I + \dim R' - \text{depth } R', \]

in which the equalities follow from Cohen-Macaulay-ness of \( Q \); \text{G-perfectness of} \( I \); Auslander-Buchsbaum formula; [4, Exercise 1.2.26]; Cohen-Macaulay-ness of \( Q \); and [4, Corollary 2.1.4] respectively. Therefore we obtain that \( \dim R' - \text{depth } R' = \)
0, that is \( R' \) is Cohen-Macaulay. Now [4, Theorem 2.1.7] gives us the desired result. \[ \Box \]

The proof of the above theorem says something more, viz., a local ring \( R \) is Cohen-Macaulay if and only if there exists a cyclic \( R \)-module of finite Cohen-Macaulay injective dimension.

**Corollary 3.5.** Assume that \( C \neq 0 \) is a cyclic \( R \)-module. Then \( R \) is a Cohen-Macaulay ring if and only if \( \text{CM}_* \text{id}_R C < \infty \).

**Remark 3.6.** Let \( M \) be a homologically finite \( R \)-complex such that \( \text{Gid}_R(M) < \infty \). Then by [15, Theorem 3.6], we obtain that \( \text{Gid}_{\hat{R}}(M \otimes_R \hat{R}) < \infty \). Hence using [11, Corollary 2.3], we have

\[
\text{Gid}_{\hat{R}}(M \otimes_R \hat{R}) = \text{depth} \hat{R} - \inf(M \otimes_R \hat{R}) = \text{depth} R - \inf(M) = \text{Gid}_R(M).
\]

**Proposition 3.7.** Let \( M \) be a homologically finite \( R \)-complex. Then

\[
\text{CM}_* \text{id}_R(M) = \inf \left\{ \text{Gid}_Q(M \otimes_R R') - \text{Gfd}_Q(R') \mid \begin{array}{c}
R \to R' \leftarrow Q \text{ is a CM-quasi-deformation such that } Q \text{ is complete}
\end{array} \right\}.
\]

**Proof.** It is clear that the left hand side is less than or equal to the right hand side. Now let \( R \to R' \leftarrow Q \) be a CM-quasi-deformation. Then note that \( R \to \hat{R} \leftarrow \hat{Q} \) is also a CM-quasi-deformation such that

\[
\text{Gid}_Q(M \otimes_R R') = \text{Gid}_{\hat{Q}}(M \otimes_R R' \otimes_Q \hat{Q}) = \text{Gid}_{\hat{Q}}(M \otimes_R \hat{R'}),
\]

and \( \text{Gfd}_Q(R') = \text{Gfd}_{\hat{Q}}(\hat{R'}) \), where the first equality holds by Remark 3.6. So we can assume in the CM-quasi-deformation \( R \to R' \leftarrow Q \) that, \( Q \) is a complete local ring. This shows the equality. \[ \Box \]

**Proposition 3.8.** Let \( M \) be a homologically bounded \( R \)-complex. Then

\[
\text{CM}_* \text{fd}_R(M) = \inf \left\{ \text{Gfd}_Q(M \otimes_R R') - \text{Gfd}_Q(R') \mid \begin{array}{c}
R \to R' \leftarrow Q \text{ is a CM-quasi-deformation such that } Q \text{ is complete}
\end{array} \right\}.
\]

**Proof.** The proof is the same as proof of Proposition 3.7, but here use [19, Corollary 8.9] instead of Remark 3.6. \[ \Box \]

Let \( M \) be homologically bounded \( R \)-complex. Then Foxby showed that

\[
\text{Gpd}_{\hat{R}}(M \otimes_R \hat{R}) \leq \text{Gpd}_R(M)
\]
(see [10, Ascent table II(b)]).

**Proposition 3.9.** Let \( M \) be a homologically bounded \( R \)-complex. Then

\[
\text{CM}_* \text{pd}_R(M) = \inf \left\{ \text{Gpd}_Q(M \otimes_R R') - \text{Gfd}_Q(R') \mid \begin{array}{c}
R \to R' \leftarrow Q \text{ is a CM-quasi-deformation such that } Q \text{ is complete}
\end{array} \right\}.
\]

**Proof.** The proof is the same as proof of Proposition 3.7, but here use the comment just before the proposition instead of Remark 3.6. \[ \Box \]
A homological dimension should not grow under localization. Let \( p \) be a prime ideal of \( R \) and \( M \) a homologically bounded \( R \)-complex. It is well known that
\[
\text{Gfd}_R(M_p) \leq \text{Gfd}_R(M),
\]
and Foxby showed that (when \( R \) has finite Krull dimension)
\[
\text{Gpd}_R(M_p) \leq \text{Gpd}_R(M)
\]
(see [9, Page 262]). On the other hand if \( R \) has a dualizing complex then,
\[
\text{Gid}_R(M_p) \leq \text{Gid}_R(M)
\]
by [9, Proposition 5.5].

**Theorem 3.10.** Let \( M \) be a homologically finite \( R \)-complex. For each prime ideal \( p \in \text{Spec}(R) \) there is an inequality
\[
\text{CM}_* \text{-id}_{R_p}(M_p) \leq \text{CM}_* \text{-id}_R(M).
\]

**Proof.** Assume that \( \text{CM}_{*} \text{-id}_R(M) < \infty \). Let \( R \to R' \leftarrow Q \) be a CM-quasi-deformation with \( Q \) a complete local ring, such that \( \text{Gid}_Q(M \otimes R') < \infty \) and
\[
\text{CM}_{*} \text{-id}_R(M) = \text{Gid}_Q(M \otimes R') - \text{Gfd}_Q(R') \text{ by Proposition 3.7.}
\]
Hence \( Q \) admits a dualizing complex.

Let \( p \) be a prime ideal of \( R \). Since \( R \to R' \) is a faithfully flat extension of rings, there is a prime ideal \( \mathfrak{p}' \) in \( R' \) lying over \( p \). Let \( q \) be the inverse image of \( \mathfrak{p}' \) in \( Q \). The map \( R_p \to R'_p \) is flat, and \( R'_p \leftarrow Q_q \) is a CM-deformation and note that \( \text{Gfd}_{Q_q}(R'_p) = \text{Gfd}_{Q}(R') \). Therefore the diagram \( R_p \to R'_p \leftarrow Q_q \) is a CM-quasi-deformation with
\[
\text{Gid}_{Q_q}(M_p \otimes R'_p) = \text{Gid}_{Q_q}((M \otimes R') \otimes Q Q_q) \leq \text{Gid}_Q(M \otimes R') < \infty,
\]
where the inequality holds by [9, Proposition 5.5]. Hence \( \text{CM}_{*} \text{-id}_{R_p}(M_p) < \infty \). So we obtain
\[
\text{CM}_{*} \text{-id}_{R_p}(M_p) \leq \text{Gid}_{Q_q}(M_p \otimes R'_p) - \text{Gfd}_{Q_q}(R'_p)
\leq \text{Gid}_Q(M \otimes R') - \text{Gfd}_Q(R')
= \text{CM}_{*} \text{-id}_R(M).
\]
Thus the desired inequality follows. \( \square \)

We do not know when the inequality \( \text{CM}_{*} \text{-id}_{R_p}(M_p) \leq \text{CM}_{*} \text{-id}_R(M) \) holds in general. However for \( \text{CM}_{*} \text{-pd}_R(M) \) and \( \text{CM}_{*} \text{-fd}_R(M) \) we have

**Theorem 3.11.** Let \( M \) be a homologically bounded \( R \)-complex. For each prime ideal \( p \in \text{Spec}(R) \) there is an inequality
\begin{enumerate}
\item \( \text{CM}_{*} \text{-pd}_{R_p}(M_p) \leq \text{CM}_{*} \text{-pd}_R(M) \).
\item \( \text{CM}_{*} \text{-fd}_{R_p}(M_p) \leq \text{CM}_{*} \text{-fd}_R(M) \).
\end{enumerate}

**Proof.** The proof is the same as proof of Theorem 3.10, but here we do not need \( Q \) is a complete local ring. \( \square \)
Proposition 3.12. Let $M$ be a homologically finite $R$-complex. Then there is an equality

$$CM_\ast\text{-id}_R(M) = \inf \left\{ Gid_Q(M \otimes_R R') - Gfd_Q(R') \mid R \to R' \xleftarrow{Q} \text{ is a CM-quasi-deformation such that the closed fibre of } R \to R' \text{ is Artinian} \right\}.$$ 

Proof. It is clear that the left hand side is less than or equal to the right hand side. Let $R \to R' \xleftarrow{Q}$ be a CM-quasi-deformation with $Q$ a complete local ring, such that $CM_\ast\text{-id}_R(M) = Gid_Q(M \otimes R') - Gfd_Q(R')$ by Proposition 3.13. Hence $Q$ admits a dualizing complex. Now choose $p' \in \text{Spec}(R')$ such that it is a minimal prime ideal containing $mR'$; thus $m = p' \cap R$ and $p' = q/J$ for some $q \in \text{Spec}(Q)$, where $J = \ker(Q \to R')$. Now the diagram $R \to R'_p \xleftarrow{Q_p}$ is a CM-quasi-deformation such that the closed fiber of $R \to R'_p$ is Artinian. It is clear that $Gfd_Q R' = Gfd_{Q_p} R'_p$. Also we have

$$Gid_{Q_p} (M \otimes_R R'_p) = Gid_{Q_p} (M \otimes_R (R'_p \otimes_Q Q_p)) = Gid_{Q_p} ((M \otimes_R R'_p) \otimes_Q Q_p) \leq Gid_Q (M \otimes_R R'_p),$$

where the inequality holds by [9 Proposition 5.5]. Hence $Gid_{Q_p} (M \otimes_R R'_p) - Gfd_{Q_p} (R'_p) \leq CM_\ast\text{-id}_R(M)$. So the proof is complete. □

Proposition 3.13. Let $M$ be a homologically bounded $R$-complex. Then there are equalities

$$CM_\ast\text{-pd}_R(M) = \inf \left\{ Gpd_Q(M \otimes_R R') - Gfd_Q(R') \mid R \to R' \xleftarrow{Q} \text{ is a CM-quasi-deformation such that the closed fibre of } R \to R' \text{ is Artinian} \right\},$$

$$CM_\ast\text{-fd}_R(M) = \inf \left\{ Gfd_Q(M \otimes_R R') - Gfd_Q(R') \mid R \to R' \xleftarrow{Q} \text{ is a CM-quasi-deformation such that the closed fibre of } R \to R' \text{ is Artinian} \right\}.$$ 

Proof. The proof is the same as proof of Proposition 3.12 but here we do not need $Q$ is a complete local ring. □

Remark 3.14. (1) Let $M$ be a homologically finite $R$-complex. Then, one can combine the proofs of Propositions 3.12 and 3.13 to obtain an equality

$$CM_\ast\text{-id}_R(M) = \inf \left\{ Gid_Q(M \otimes_R R') - Gfd_Q(R') \mid R \to R' \xleftarrow{Q} \text{ is a CM-quasi-deformation such that } Q \text{ is complete and the closed fibre of } R \to R' \text{ is Artinian} \right\}.$$
(2) Likewise for a homologically bounded $R$-complex $M$, one can combine the proofs of Propositions 3.8, 3.9 and 3.12 to obtain the equalities

$$\text{CM}_*\text{-fd}_R(M) = \inf \left\{ \text{Gfd}_Q(M \otimes_R R') - \text{Gfd}_Q(R') : \begin{array}{l} R \to R' \leftarrow Q \text{ is a } \text{CM-}\text{-quasi-deformation} \\ \text{such that } Q \text{ is complete} \\ \text{and the closed fibre of } R \to R' \text{ is Artinian} \end{array} \right\},$$

$$\text{CM}_*\text{-pd}_R(M) = \inf \left\{ \text{Gpd}_Q(M \otimes_R R') - \text{Gfd}_Q(R') : \begin{array}{l} R \to R' \leftarrow Q \text{ is a } \text{CM-}\text{-quasi-deformation} \\ \text{such that } Q \text{ is complete} \\ \text{and the closed fibre of } R \to R' \text{ is Artinian} \end{array} \right\}.$$ 

4. LARGE RESTRICTED FLAT DIMENSION AND CHOINUARD’S INVARIANT

Recall from [8], that the large restricted flat dimension is defined by

$$\text{Rfd}_R(M) := \sup \{ \sup(F \otimes_R^L M) : F \text{ an } R\text{-module with } \text{fd}_R(F) < \infty \}.$$ 

This number is finite, as long as $H(M)$ is nonzero and the Krull dimension of $R$ is finite; see [8, Proposition 2.2]. It is useful to keep in mind an alternative formula [8, Theorem 2.4] for computing this invariant:

$$\text{Rfd}_R(M) = \sup \{ \text{depth } R_p - \text{depth } R_{M_p} : p \in \text{Spec } (R) \}.$$

Recall here that the depth of a homologically bounded $R$-complex $M$ is defined by

$$\text{depth}_R(M) = -\sup(\text{RHom}_R(k, M)),$$

and it is shown that $\text{depth}_R(M) \geq -\sup(M)$.

It is proved in [19, Theorem 8.8] that for an $R$-complex $M$, $\text{Rfd}_R(M)$ is a refinement of $\text{Gfd}_R(M)$, that is

$$\text{Rfd}_R(M) \leq \text{Gfd}_R(M),$$

with equality if $\text{Gfd}_R(M)$ is finite.

First, we plan to show that, when the Cohen-Macaulay flat dimension of a homologically bounded $R$-complex $M$ is finite, then it is equal to the large restricted flat dimension of $M$. The following proposition is the main tool.

**Proposition 4.1.** Let $R \to S \leftarrow Q$ be a CM-quasi-deformation, and let $M$ be a homologically bounded $R$-complex. Then

$$R\text{fd}_R(M) = R\text{fd}_Q(M \otimes_R S) - R\text{fd}_Q(S).$$

**Proof.** First we prove the equality

$$\text{Rfd}_S(Y) = \text{Rfd}_Q(Y) - \text{G-dim}_Q(S),$$

for a homologically bounded $S$-complex $Y$. To this end, choose by [8, Theorem 2.4(b)] a prime ideal $p$ of $S$ such that the first equality below holds. Let $q$ be the
inverse image of $p$ in $Q$. Therefore there is an isomorphism $Y_p \cong Y_q$ of $Q_q$-modules and a CM-deformation $Q_q \to S_p$. Hence

$$\text{Rfd}_S(Y) = \text{depth}_{S_p} - \text{depth}_{Q_q} Y_p$$

$$= \text{depth}_{Q_q} S_p - \text{depth}_{Q_q} Y_p$$

$$= \text{depth}_{Q_q} - \text{G-dim}_{Q_q} S_p - \text{depth}_{Q_q} Y_p$$

$$\leq \text{Rfd}_Q(Y) - \text{G-dim}_{Q_q}(S_p)$$

$$= \text{Rfd}_Q(Y) - \text{G-dim}_Q(S).$$

The second equality holds since $Q_q \to S_p$ is surjective and [18, Proposition 5.2(1)]; the third equality holds by Auslander-Bridger formula [1]; the fourth equality is due to the G-perfectness assumption of $S$ over $Q$; while the inequality follows from [8, Theorem 2.4(b)]. Now by [26, Proposition 3.5] we have

$$\text{Rfd}_Q(Y) \leq \text{Rfd}_S(Y) + \text{Rfd}_Q(S) \leq \text{Rfd}_Q(Y) - \text{G-dim}_Q(S) + \text{Rfd}_Q(S) = \text{Rfd}_Q(Y),$$

which is the desired equality.

Now we have

$$\text{Rfd}_Q(M \otimes_R S) \leq \text{Rfd}_S(M \otimes_R S) + \text{Rfd}_Q(S)$$

$$= \text{Rfd}_S(M \otimes_R S) + \text{G-dim}_Q(S)$$

$$= \text{Rfd}_Q(M \otimes_R S),$$

where the inequality is in [26, Proposition 3.5], the first equality follows from the hypotheses, and the second equality follows from the above observation. Hence

$$\text{Rfd}_Q(M \otimes_R S) - \text{Rfd}_Q(S) = \text{Rfd}_S(M \otimes_R S) = \text{Rfd}_R(M)$$

where the second equality holds by [19, Lemma 8.5(1)].

**Corollary 4.2.** Let $M$ be a homologically bounded $R$-complex. Then we have the inequalities

$$\text{Rfd}_R(M) \leq \text{CM}_* - \text{fd}_R(M) \leq \text{Gfd}_R(M),$$

with equality to the left of any finite value. In particular if $\text{CM}_* - \text{fd}_R(M) < \infty$, then

$$\text{CM}_* - \text{fd}_R(M) = \sup\{\text{depth}_{R_p} - \text{depth}_{R_p}(M_p) \mid p \in \text{Spec}(R)\}$$

$$\leq \dim R + \sup(M).$$

Now using Corollary [172] we investigate the effect of change of ring on Cohen-Macaulay flat dimension.

**Proposition 4.3.** Let $M$ be a homologically bounded $R$-complex. Let $R \to R'$ be a local flat extension, and $M' = M \otimes_R R'$. Then

$$\text{CM}_* - \text{fd}_R(M) \leq \text{CM}_* - \text{fd}_{R'}(M')$$

with equality when $\text{CM}_* - \text{fd}_{R'}(M')$ is finite.

**Proof.** Suppose that $\text{CM}_* - \text{fd}_{R'}(M') < \infty$, and let $R' \to R'' \leftarrow Q$ be a CM-quasi-deformation with $\text{Gfd}_Q(M' \otimes_R R'') < \infty$. Since $R \to R'$ and $R' \to R''$ are flat extensions, the local homomorphism $R \to R''$ is also flat. Hence $R \to R'' \leftarrow Q$ is a
CM-quasi-deformation with $Gfd_Q(M \otimes_R R') < \infty$. It follows that $CM_*-fd_R(M)$ is finite. Now by Corollary 4.2 and [19, Lemma 8.5(1)], we have

$$CM_*-fd_R(M) = Rfd_R(M) = Rfd_{R'}(M') = CM_*-fd_{R'}(M'),$$

to complete the proof. □

**Proposition 4.4.** For every homologically bounded $R$-complex $M$

$$CM_*-fd_R(M) = CM_*-fd_{\hat{R}}(M \otimes_R \hat{R}).$$

**Proof.** If $CM_*-fd_R(M) = \infty$, then we obtain that $CM_*-fd_{\hat{R}}(M \otimes_R \hat{R}) = \infty$ by Proposition 4.3. Now assume that $CM_*-fd_R(M) < \infty$. Using Proposition 4.3, it is sufficient to prove that $CM_*-fd_{\hat{R}}(M \otimes_R \hat{R})$ is finite. To this end, choose a CM-quasi-deformation $R \to \hat{R} \leftarrow Q$ of $R$ such that $Gfd_Q(M \otimes_R \hat{R}) < \infty$. So we have $\hat{R} \to \hat{R}' \leftarrow \hat{Q}$ is a CM-quasi-deformation of $\hat{R}$ with respect to their maximal ideal-adic completions. Now using [19, Corollary 8.9] we obtain

$$Gfd_{\hat{Q}}((M \otimes_R \hat{R}) \otimes_R \hat{R}') = Gfd_Q(M \otimes_R R') < \infty.$$ Hence $Gfd_{\hat{Q}}((M \otimes_R \hat{R}) \otimes_R \hat{R}')$ is finite which in turn implies that $CM_*-fd_{\hat{R}}(M \otimes_R \hat{R})$ is finite. □

Next, recall that the width of an $R$-complex $M$ is defined by

$$width_R(M) = \inf(M \otimes_R k),$$

and that $width_R(M) \geq \inf(M)$. Also, if $M$ is homologically finite, then

$$width_R(M) = \inf(M).$$

It is the dual notion for depth $R(M)$. In particular by [3] Proposition 4.8, we have

$$width_R(M) = \text{depth}_R(R\text{Hom}_R(M, E_R(k))),$$

where $E_R(k)$ denotes the injective envelope of $k$ over $R$.

The Chouinard invariant [6, Corollary 3.1] is denoted by $Ch_R(M)$ and

$$Ch_R(M) := \sup\{\text{depth}_p - width_{R_p}(M_p) | p \in \text{Spec}(R)\}.$$ It is proved in [11] Theorem 2.2] that for an $R$-complex $M$, $Ch_R(M)$ is a refinement of $Gid_R(M)$, that is

$$Ch_R(M) \leq Gid_R(M),$$

with equality if $Gid_R(M)$ is finite. Now we want to show that the Cohen-Macaulay injective dimension is bounded below by the Chouinard’s invariant.

**Lemma 4.5.** Suppose that $Q \to S$ is a surjective local homomorphism and $Y$ is an $S$-complex. Then we have

$$width_S(Y) = width_Q(Y).$$
Proof. We have the following equalities:

\[
\text{width}_S(Y) = \text{depth}_S \text{RHom}_S(Y, E_S(k)) \\
= \text{depth}_S \text{RHom}_S(Y, \text{Hom}_Q(S, E_Q(k))) \\
= \text{depth}_Q \text{RHom}_Q(Y, E_Q(k)) \\
= \text{width}_Q(Y),
\]

where the first one is by [8, Proposition 4.8]; the second one is by [5, Lemma 10.1.15]; the third one is by adjointness of Hom and tensor; the fourth one is true since \( Q \to S \) is surjective and [18, Proposition 5.2(1)]; while the last one is again by [8, Proposition 4.8]. Here we used \( k \) for the residue fields of \( Q \) and \( S \), and \( E_Q(k) \) and \( E_S(k) \) for the injective envelopes of \( k \) over respectively \( Q \) and \( S \).

\[\square\]

Lemma 4.6. Suppose that \( R \to S \) is a flat local ring homomorphism, and \( M \) is a homologically bounded \( R \)-complex. Then we have

\[
\text{width}_S(M \otimes_R S) = \text{width}_R(M).
\]

Proof. A standard application of the Künneth formula yields the equality. \[\square\]

Proposition 4.7. Let \( R \to S \) be a flat local homomorphism and let \( M \) be a homologically bounded \( R \)-complex. Then

\[
\text{Ch}_R(M) \leq \text{Ch}_S(M \otimes_R S).
\]

Proof. Let \( p \in \text{Spec}(R) \) such that the first equality below holds. Let \( q \in \text{Spec}(S) \) contain \( p \) minimally. Since \( R \to S \) is a flat local homomorphism we have \( p = q \cap R \). Hence:

\[
\text{Ch}_R(M) = \text{depth}_R p - \text{width}_R (M_p) \\
= \text{depth}_R q - \text{width}_S (M_p \otimes_R S_q) \\
= \text{depth}_S q - \text{width}_S (M \otimes_R S)_q \\
\leq \text{Ch}_S(M \otimes_R S),
\]

in which the second equality holds by Lemma 4.6 and the fact that \( R_p \to S_q \) has Artinian closed fibre. \[\square\]

Proposition 4.8. Let \( Q \to S \) be a CM-deformation, and \( Y \) be a homologically bounded \( S \)-complex. Then

\[
\text{Ch}_S(Y) \leq \text{Ch}_Q(Y) - \text{Gfd}_Q(S).
\]

Proof. Choose a prime ideal \( p \) of \( S \) such that the first equality below holds. Let \( q \) be the inverse image of \( p \) in \( Q \). Therefore there is an isomorphism \( Y_p \cong Y_q \) of
$Q_q$-complexes and a CM-deformation $Q_q \to S_p$. Hence
\[
\text{Ch}_S(Y) = \text{depth } S_p - \text{width } s_q(Y_p)
\]
\[
= \text{depth } Q_s S_p - \text{width } Q_q(Y_p)
\]
\[
\leq \text{Ch}_Q(Y) - \text{Gfd } Q_q(S_p)
\]
\[
= \text{Ch}_Q(Y) - \text{Gfd } Q(S).
\]

The second equality holds since $Q_q \to S_p$ is surjective; the third equality holds by Auslander-Bridger formula [1]; the fourth equality is due to the G-perfectness assumption of $S$ over $Q$. □

**Theorem 4.9.** Let $M$ be a homologically bounded $R$-complex. Then there is the inequality
\[
\text{Ch}_R(M) \leq \text{CM}_* - \text{id}_R(M).
\]

**Proof.** We can assume that $\text{CM}_* - \text{id}_R(M) < \infty$. Choose a CM-quasi-deformation $R \to R' \leftarrow Q$, such that $\text{CM}_* - \text{id}_R(M) = \text{Gid}_R(M \otimes_R R') - \text{Gfd } Q(R')$. Hence we have
\[
\text{CM}_* - \text{id}_R(M) = \text{Gid}_R(M \otimes_R R') - \text{Gfd } Q(R')
\]
\[
= \text{Ch}_R(M \otimes_R R') - \text{Gfd } Q(R')
\]
\[
\geq \text{Ch}_R(M \otimes_R R') \geq \text{Ch}_R(M),
\]
in which the second equality comes by [11, Theorem 2.2], and inequalities follow Propositions 4.8 and 4.7 respectively. □

**Corollary 4.10.** Let $M$ be a homologically bounded $R$-complex. Then there are inequalities
\[
\text{Ch}_R(M) \leq \text{CM}_* - \text{id}_R(M) \leq \text{Gid}_R(M),
\]
such that if $\text{Gid}_R(M) < \infty$, then $\text{Gid}_R(M) = \text{CM}_* - \text{id}_R(M)$.

**Proof.** The inequalities hold by Theorem 4.9 and Remark 3.2(2). And if $\text{Gid}_R(M) < \infty$, then the equality holds by [11, Theorem 2.2]. □

**Corollary 4.11.** Let $M$ be a homologically finite $R$-complex such that $\text{CM}_* - \text{id}_R(M)$ is finite. Then
\[
\text{CM}_* - \text{id}_R(M) = \text{Ch}_R(M) = \text{depth } R - \inf(M)
\]
\[
\leq \text{dim } R - \inf(M).
\]

**Proof.** By Proposition 4.12 there is a CM-quasi-deformation $R \to R' \leftarrow Q$ such that the closed fibre of $R \to R'$ is Artinian and the first equality below holds. So
that
\[ CM_*-\text{id}_R(M) = \text{Gid}_Q(M \otimes_R R') - \text{Gfd}_Q(R') \]
\[ = \text{depth} Q - \inf(M \otimes_R R') - \text{depth} Q + \text{depth} R' \]
\[ = \text{depth} R' - \inf(M \otimes_R R') \]
\[ = \text{depth} R - \inf(M). \]

The second equality holds by [11, Corollary 2.3] and the Auslander-Bridger formula [1], while the last equality holds, because the closed fiber of \( R \to R' \) is Artinian and [4, Proposition 1.2.16].

Now by Theorem 4.9, \( \text{depth} R - \inf(M) \leq \text{Ch}_R(M) \leq CM_*-\text{id}_R(M) = \text{depth} R - \inf(M). \) Therefore \( CM_*-\text{id}_R(M) = \text{Ch}_R(M) = \text{depth} R - \inf(M). \)

\[ \square \]

In concluding, recall that there are notions of Cohen-Macaulay projective dimension, Cohen-Macaulay flat dimension and Cohen-Macaulay injective dimension of Holm and Jørgensen, which are different with our Definition 3.1.

**Definition 4.12.** (cf., [17, Definition 2.3]) Let \((R, m)\) be a local ring. For each homologically bounded \( R \)-complex \( M \), the Cohen-Macaulay projective, flat and injective dimension, of \( M \) is defined as, respectively,

\[ CM_{pd} R(M) := \inf \{ \text{Gpd}_{R \otimes C}(M) | C \text{ is a semidualizing module} \} \]
\[ CM_{fd} R(M) := \inf \{ \text{Gfd}_{R \otimes C}(M) | C \text{ is a semidualizing module} \} \]
\[ CM_{id} R(M) := \inf \{ \text{Gid}_{R \otimes C}(M) | C \text{ is a semidualizing module} \} \]

Here \( R \otimes C \) denotes the trivial extension ring of \( R \) by \( C \); it is the \( R \)-module \( R \oplus C \) equipped with the multiplication \( (r, c)(r', c') = (rr', rc' + r'c) \).

**Remark 4.13.** (1) For each homologically bounded \( R \)-complex \( M \), we have

\[ CM_*-pd_R(M) \leq CM_{pd} R(M) \]
\[ CM_*-fd_R(M) \leq CM_{fd} R(M) \]
\[ CM_*-id_R(M) \leq CM_{id} R(M). \]

More precisely, assume that \( CM_{pd} R(M) < \infty \) and choose a semidualizing \( R \)-module \( C \) such that \( CM_{pd} R(M) = \text{Gpd}_{R \otimes C}(M) \). Then by [16, Lemma 3.6], we have the CM-quasi-deformation \( R \to R \xleftarrow{\tau} Q \) where \( Q := R \otimes C \) and \( \tau(r, c) = r \), such that \( \text{Gdim}_Q(R) = 0 \). Thus we obtain

\[ \text{Gpd}_Q(M \otimes_R R) - \text{Gfd}_Q(R) = \text{Gpd}_{R \otimes C}(M). \]

This shows the first inequality. The proof of the other two inequalities are the same as the first one.

(2) The finiteness of the Cohen-Macaulay homological dimensions in Definition 4.12 characterize Cohen-Macaulay rings admitting a canonical module [17, Theorem 5.1].

(3) Assume that \((R, m, k)\) is a Cohen-Macaulay ring, not admitting a canonical module (e.g., see [14] for such an example). Then \( CM_*-pd_R(k) < \infty \) (and, \( CM_*-fd_R(k) < \infty \), \( CM_*-id_R(k) < \infty \)) but \( CM_{pd} R(k) = \infty \) (and, \( CM_{fd} R(k) = \infty \), \( CM_{id} R(k) = \infty \)).
Lemma 4.14. Assume that $C$ is a semidualizing $R$-module and let $M$ be a homologically bounded $R$-complex. Consider $M$ as a $R \otimes C$-complex via the natural surjection $\tau : R \otimes C \to R$.

1. If $\Gfd_{R \otimes C}(M) < \infty$, then $\Gfd_{R \otimes C}(M) = \Rfd_{R}(M)$.
2. If $\Gid_{R \otimes C}(M) < \infty$, then $\Gid_{R \otimes C}(M) = \Ch_{R}(M)$.

Proof. Note that $\Spec(R \otimes C) = \{p \otimes C \mid p \in \Spec(R)\}$ and $(R \otimes C)_{p \otimes C} \cong R_{p} \otimes C_{p}$ by [5] Exercise 6.2.12. Let $L$ be an $R$-module which is an $R \otimes C$-module via the surjection $\tau : R \otimes C \to R$, and let $p$ be a prime ideal of $R$. Then $\varphi : L_{p \otimes C} \to L_{p}$ sending $l/(r,c)$ to $l/r$ is an $R_{p}$-isomorphism. By [19] Theorem 8.8 we have the first equality below.

$$\Gfd_{R \otimes C}(M) = \sup\{\depth R_{p} \otimes C_{p} - \depth (R \otimes C)_{p \otimes C}(M_{p \otimes C}) \mid p \in \Spec(R)\} = \sup\{\depth R_{p} \otimes C_{p} - \depth R_{p \otimes C}(M_{p \otimes C}) \mid p \in \Spec(R)\} = \sup\{\depth R_{p} - \depth R_{p}(M_{p}) \mid p \in \Spec(R)\} = \Rfd_{R}(M).$$

The third equality holds since there is a surjection $R_{p} \otimes C_{p} \to R_{p}$ and [18] Proposition 5.2(1). The fourth equality uses

$$\depth R_{p} \otimes C_{p} = \min\{\depth R_{p}, \depth R_{p}(C_{p})\} = \depth R_{p}.$$ The proof of (2) is the same as (1) using [11] Theorem 2.2 instead of [19] Theorem 8.8, and Lemma 4.5 instead of [18] Proposition 5.2(1).

Corollary 4.15. Let $M$ be a homologically bounded $R$-complex.

1. If $\CMfd_{R}(M) < \infty$, then $\CMfd_{R}(M) = \CM_{e}-fd_{R}(M)$.
2. If $\CMid_{R}(M) < \infty$, then $\CMid_{R}(M) = \CM_{e}-id_{R}(M)$.

Proof. Note that there are the inequalities

$$\Rfd_{R}(M) \leq \CM_{e}-fd_{R}(M) \leq \CMfd_{R}(M) = \Rfd_{R}(M)$$

(resp., $\Ch_{R}(M) \leq \CM_{e}-id_{R}(M) \leq \CMid_{R}(M) = \Ch_{R}(M)$) by Corollary 4.2 (resp., Theorem 4.3), and Lemma 4.14.

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