Abstract

The low-energy dynamics of pions and magnons — the Goldstone bosons of the strong interactions and of magnetism — are analogous in many ways. The electroweak interactions of pions result from gauging an $SU(2)_L \otimes U(1)_Y$ symmetry which then breaks to the $U(1)_{em}$ gauge symmetry of electromagnetism. The electromagnetic interactions of magnons arise from gauging not only $U(1)_{em}$ but also the $SU(2)_s$ spin rotational symmetry, with the electromagnetic fields $\vec{E}$ and $\vec{B}$ appearing as non-Abelian vector potentials. Pions couple to electromagnetism through a Goldstone-Wilczek current that represents the baryon number of Skyrmions and gives rise to the decay $\pi^0 \rightarrow \gamma \gamma$. Similarly, magnons may couple to an analogue of the Goldstone-Wilczek current for baby-Skyrmions which induces a magnon-two-photon vertex. The corresponding analogue of photon-axion conversion is photon-magnon conversion in an external magnetic field. The baryon number violating decay of Skyrmions can be catalyzed by a magnetic monopole via the Callan-Rubakov effect. Similarly, baby-Skyrmion decay can be catalyzed by a charged wire. For more than two flavors, the Wess-Zumino-Witten term enters the low-energy pion theory with a quantized prefactor $N_c$ — the number of quark colors. The magnon analogue of this prefactor is the anyon statistics angle $\theta$ which need not be quantized.

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1 Introduction

The concept of a spontaneously broken continuous global symmetry [1] is important in many areas of physics. Irrespective of the details of the dynamics, Goldstone’s theorem [2] predicts the existence of massless excitations just based upon the symmetry group $G$ and its unbroken subgroup $H$. At low energies the physics of the strong interaction is dominated by the lightest particles of QCD — the pions [3]. In the chiral limit of zero quark masses, the pions are exactly massless Nambu-Goldstone bosons which stem from the $G = SU(2)_L \otimes SU(2)_R \otimes U(1)_B$ chiral symmetry of QCD that is spontaneously broken to $H = SU(2)_{L=R} \otimes U(1)_B$ at low temperatures. In the real world, on the other hand, the small non-zero quark masses give rise to a small pion mass and the pions are only pseudo-Goldstone bosons. The spontaneous breaking of the $G = SU(2) \cong SO(3)$ spin rotational symmetry group down to $H = U(1) \cong SO(2)$ plays an important role in the condensed matter physics of ferro- and antiferromagnets. In particular, the low-energy physics of these systems is governed by associated massless Goldstone bosons — the magnons or spin waves.

Since Goldstone bosons are massless, their dynamics dominate the low-energy physics. These dynamics can be described by a low-energy effective theory involving only the Goldstone boson degrees of freedom. Irrespective of a concrete physical system, the Goldstone bosons are described by fields in the coset space $G/H$ [4] whose dimension equals the number of spontaneously broken generators. The low-energy effective theory must respect all symmetries of the underlying microscopic theory — in particular those of the global symmetry group $G$. Spontaneous symmetry breaking implies that the Goldstone bosons interact weakly at low momenta. As a consequence, the Lagrangian of the effective theory can be constructed perturbatively as a derivative expansion, since terms with few derivatives dominate at low energies. A systematic effective field theory approach for describing Goldstone boson physics — chiral perturbation theory — was developed for the pions in QCD [5], but is generally applicable to all Goldstone boson phenomena. In condensed matter physics chiral perturbation theory has been applied to both ferromagnetic [6, 7] and antiferromagnetic magnons [8–11].

In this paper we compare the low-energy physics of apparently quite different systems — pions in QCD and magnons in ferro- and antiferromagnets. Since both pions and magnons are Goldstone bosons, it is not surprising that they share a number of common features [12]. However, we find a correspondence between numerous pion and magnon phenomena that goes beyond what one might have expected. In the standard model of particle physics the global $SU(2)_L$ symmetry of the pion effective theory turns into a local gauge symmetry once the weak interactions are taken into account. Similarly, the global $SU(2)_s$ spin symmetry of a magnon effective Lagrangian becomes local when electromagnetic interactions are included. This is quite surprising because electromagnetism, of course, results from gauging $U(1)_{em}$. Still, as pointed out by Fröhlich and Studer [13], for non-relativistic systems the
electromagnetic fields $\vec{E}$ and $\vec{B}$ appear as non-Abelian vector potentials of a local $SU(2)_s$ symmetry.

The topological Skyrmion excitations of the pion field [14] correspond to the baby-Skyrmions in ferro- and antiferromagnets [15]. When pions are coupled to the electromagnetic field, the Skyrme current, which describes baryon number, is generalized to the Goldstone-Wilczek current [16]. The Goldstone-Wilczek term in the low-energy effective Lagrangian gives rise to the anomalous decay $\pi^0 \rightarrow \gamma\gamma$ of the neutral pion into two photons. When magnons are coupled to electromagnetism, the baby-Skyrmion current is replaced by an analogue of the Goldstone-Wilczek current. As we will see, this term induces a magnon-two-photon vertex if the baby-Skyrmion carries electric charge.

Baby-Skyrmions have been established experimentally in quantum Hall ferromagnets [17]. They are well understood theoretically [18], and, in particular, they do carry electric charge. Hence, magnon decay into two photons should indeed occur in quantum Hall ferromagnets. Another particle that can decay into two photons is the hypothetical axion [19, 20] — the Goldstone boson of a spontaneously broken Peccei-Quinn symmetry [21]. In order to make the axion visible in the laboratory, it has been proposed to use the axion-two-photon vertex in order to convert laser photons into axions inside a strong magnetic field [22, 23]. As we will see, there is a condensed matter analogue of this effect — namely photon-magnon conversion in an external magnetic field.

The importance of baby-Skyrmions in the undoped antiferromagnetic precursors of high-temperature superconductors [24–27] as well as topological mechanisms for superconductivity [28] have also been discussed. Effective theories for holes doped into antiferromagnets were constructed in [29, 30]. It is possible that baby-Skyrmions in layered cuprate antiferromagnets carry hole quantum numbers. This hypothesis can, at least in principle, be tested because it implies the decay of antiferromagnetic magnons into two photons. Again, photon-magnon conversion in an external magnetic field may make this process experimentally accessible. If baby-Skyrmions represent holes, the hedgehog structure of their staggered magnetization field may explain why antiferromagnetism is destroyed by doping [31]. If, in addition, baby-Skyrmions have an attractive interaction that can overcome their Coulomb repulsion, pairs of baby-Skyrmions may form already in the antiferromagnetic phase. It should be pointed out that the electromagnetic interactions of baby-Skyrmions are particularly interesting because the electromagnetic fields $\vec{E}$ and $\vec{B}$ appear as $SU(2)_s$ non-Abelian vector potentials. If baby-Skyrmion pairs form, it is conceivable that hole-doping of the antiferromagnet leads to their condensation and hence to superconductivity. It remains to be seen if this basic picture can be turned into a more quantitative mechanism for high-temperature superconductivity. Our hope is that an effective field theory framework will be useful in this context.

There are further analogies between pions and magnons. The decay of both
Skyrmions and baby-Skyrmions can be catalyzed by electromagnetic interactions. In particle physics this requires the presence of a magnetic monopole which can catalyze baryon decay by the Callan-Rubakov effect [32, 33]. Similarly, baby-Skyrmion decay can be catalyzed by a charged wire sticking out of a magnet. Furthermore, when several flavors \( N_f \geq 3 \) of quarks are considered, the Wess-Zumino-Witten term [34, 35] enters the low-energy pion effective theory. Interestingly, the prefactor of the Wess-Zumino-Witten term is quantized and corresponds to the number of quark colors \( N_c \). In multi-layer quantum Hall ferromagnets the layer index plays the role of flavor [36]. An analogue of the Wess-Zumino-Witten term arises also in the corresponding generalized magnon effective theories. In this case the corresponding prefactor is the anyon statistics angle \( \theta \) for baby-Skyrmions which need not be quantized.

In this paper we concentrate on the general features of pion and magnon effective theories with special emphasis on topological properties. In particular, in the magnon context we do not limit ourselves to one specific microscopic condensed matter system, but rather characterize the universal properties of their low-energy description. Our framework is sufficiently general to incorporate condensed matter systems as different as single- or multi-layer quantum Hall ferromagnets and antiferromagnetic precursors of high-temperature superconductors. Many phenomena investigated here are well understood in condensed matter or particle physics, respectively. In fact, one purpose of this work is to underscore the numerous common features of apparently quite different condensed matter and particle physics systems and to describe them in a universal effective field theory framework. Still, several condensed matter phenomena discussed here — namely magnon decay into two photons, photon-magnon conversion in a magnetic field, baby-Skyrmion decay catalyzed by a charged wire, or the effects caused by the Wess-Zumino-Witten term for magnons — have, as far as we know, not been discussed before.

The outline of this paper is as follows. In section 2 we compare the basic low-energy effective theories of pions and magnons and we discuss their Skyrmion and baby-Skyrmion topological excitations. Section 3 contains a discussion of the electromagnetic interactions of pions and magnons, in particular, pion decay into two photons and photon-magnon conversion in an external magnetic field. In section 4 the decay of Skyrmions and baby-Skyrmions catalyzed by a magnetic monopole or a charged wire, respectively, is investigated. Section 5 generalizes the pion and magnon systems to several flavors and contains a discussion of the corresponding Wess-Zumino-Witten terms. Finally, section 6 contains a summary and our conclusions.
2 Low-Energy Effective Theories for Pions and Magnons

In this section we review the low-energy description of the dynamics of pions in 3+1 dimensions as well as of ferro- and antiferromagnetic magnons in 2+1 dimensions. We also discuss the topological solitons of these theories — Skyrmions and baby-Skyrmions — which can be quantized as bosons or fermions in 3+1 dimensions, and as anyons in 2+1 dimensions.

2.1 Pions

QCD with \( N_c \geq 3 \) colors and two massless quark flavors has a global chiral symmetry group \( G = SU(2)_L \otimes SU(2)_R \otimes U(1)_B \) that is spontaneously broken down to the subgroup \( H = SU(2)_{L=R} \otimes U(1)_B \) at low temperatures. Consequently, there are three massless Goldstone bosons — the pions \( \pi^+ \), \( \pi^0 \), \( \pi^- \) — which are described by fields

\[
U(x) = \exp(2i\pi^a(x)T^a/F_\pi)
\]

in the coset space \( G/H = SU(2)_L \otimes SU(2)_R \otimes U(1)_B/SU(2)_{L=R} \otimes U(1)_B = SU(2) \).

We have introduced the generators of \( SU(2) \) such that \( \text{Tr}(T^aT^b) = \frac{1}{2}\delta_{ab} \). At low energies the pion dynamics are described by chiral perturbation theory. To lowest order, the corresponding Euclidean action is given by [5]

\[
S[U] = \int d^4x \frac{F_\pi^2}{4} \text{Tr}[\partial_\mu U^{\dagger} \partial_\mu U].
\]

To leading order, \( F_\pi \) is the pion decay constant. For simplicity, we have neglected an explicit symmetry breaking term due to non-zero quark masses. Indeed, the above action is invariant under global \( G = SU(2)_L \otimes SU(2)_R \) transformations

\[
U'(x) = L^\dagger U(x)R.
\]

The spontaneously selected constant vacuum field configuration \( U(x) = 1 \) is invariant only under simultaneous transformations \( L = R \in H = SU(2)_{L=R} \) on the left and on the right.

2.2 Skyrmions

Chiral perturbation theory is an expansion in small field fluctuations around the vacuum configuration \( U(x) = 1 \). While such perturbative fields can always be continuously deformed into the vacuum configuration, general pion field configurations have non-trivial topological properties. In particular, the homotopy group
\[ \Pi_3[\text{SU}(2)] = \Pi_3[S^3] = \mathbb{Z} \] implies that, at every instant in time, the pion field is characterized by an integer winding number

\[ B = \frac{1}{24\pi^2} \int d^3x \varepsilon_{ijk} \text{Tr} \left[ (U^\dagger \partial_i U)(U^\dagger \partial_j U)(U^\dagger \partial_k U) \right]. \tag{2.4} \]

It was first realized by Skyrme that \( B \) can be identified with the baryon number [14]. Hence, despite the fact that the pions themselves do not carry baryon number, the topological solitons of the pion field — the Skyrmions — can be regarded as baryons [35]. The baryon current

\[ j_\mu = \frac{1}{24\pi^2} \varepsilon_{\mu\nu\rho\sigma} \text{Tr} \left[ (U^\dagger \partial_\nu U)(U^\dagger \partial_\rho U)(U^\dagger \partial_\sigma U) \right] \tag{2.5} \]

is topologically conserved, i.e. \( \partial_\mu j_\mu = 0 \), independent of the equations of motion. The dynamical properties of Skyrmions are not accessible in chiral perturbation theory because higher order terms in the action are then equally important as the lowest-order term. Still, as we will see later, the fact that topologically non-trivial pion fields carry baryon number has profound consequences for the low-energy electromagnetic properties of pions.

In addition to \( \Pi_3[\text{SU}(2)] \), the homotopy group \( \Pi_4[\text{SU}(2)] = \mathbb{Z}(2) \) is also non-trivial. Consequently, pion field configurations (now depending not only on space but also on time) fall in two distinct classes. Those that can be continuously deformed into the vacuum configuration \( U(x) = 1 \) have a “winding” number \( \text{Sign}[U] = 1 \), while all other configurations have \( \text{Sign}[U] = -1 \). The topological number \( \text{Sign}[U] \) can be identified as the fermion permutation sign of the Skyrme soliton. For an odd number of colors (such as in the real world with \( N_c = 3 \)) Skyrmions should be quantized as fermions because baryons then consist of an odd number of quarks. For even \( N_c \), on the other hand, Skyrmions must be quantized as bosons. A pion field configuration \( U(x) \) in which two Skyrmions interchange their positions as they evolve in time has \( \text{Sign}[U] = -1 \) [35]. For odd \( N_c \), i.e. for fermionic Skyrmions, the Pauli principle demands that such configurations contribute to the path integral with a negative sign. For even \( N_c \), on the other hand, they should contribute with a positive sign. Hence, a factor \( \text{Sign}[U]^{N_c} \) appears in the pion path integral which takes the form

\[ Z = \int \mathcal{D}U \exp(-S[U]) \text{Sign}[U]^{N_c}. \tag{2.6} \]

A configuration \( U(x) \) in which a single Skyrmion rotates by \( 2\pi \) during its time evolution also has \( \text{Sign}[U] = -1 \). The inclusion of \( \text{Sign}[U]^{N_c} \) in the path integral is necessary to ensure that the Skyrmion has half-integer spin for odd \( N_c \) and integer spin for even \( N_c \).

### 2.3 Antiferromagnetic Magnons

Antiferromagnets are interesting condensed matter systems. In particular, the undoped precursors of high-temperature layered cuprate superconductors are quantum
antiferromagnets [37]. Microscopic models for these systems are, for example, the quantum Heisenberg model or the Hubbard model on a square lattice at half-filling. Here we concentrate on models in 2 + 1 dimensions which describe a single spatially 2-dimensional cuprate layer. In such systems at zero temperature the spin rotational symmetry group \( G = SU(2)_s \) is spontaneously broken down to the unbroken subgroup \( H = U(1)_s \) by the formation of a staggered magnetization. As a consequence, there are two massless Goldstone bosons — the antiferromagnetic spin waves or magnons — which are described by a unit-vector field

\[
\vec{e}(x) = (e_1(x), e_2(x), e_3(x)), \quad \vec{e}(x)^2 = 1,
\]

in the coset space \( G/H = SU(2)_s/U(1)_s = S^2 \). Again, chiral perturbation theory describes the low-energy magnon dynamics by a Euclidean effective action \([38, 39]\), which to lowest order reads

\[
S[\vec{e}] = \int d^2 x \int_{\mathbb{R}} dt \frac{\rho_s}{2} \left[ \partial_i \vec{e} \cdot \partial_i \vec{e} + \frac{1}{c^2} \partial_t \vec{e} \cdot \partial_t \vec{e} \right].
\]

Here \( \rho_s \) is the spin stiffness — the analogue of \( F_s^2 \) in the pion case — and \( c \) is the spin wave velocity. The index \( i \) \( \in \{1, 2\} \) labels the two spatial directions, while the index \( t \) refers to the time direction. We have compactified the Euclidean time dimension to a circle \( S^1 \) of circumference \( \beta = 1/T \), which puts the system at a non-zero temperature \( T \). Indeed, the magnon action is invariant under global rotations \( O \in SO(3)_s \approx SU(2)_s = G \),

\[
\vec{e}'(x) = O\vec{e}(x).
\]

The spontaneously selected constant vacuum field configuration \( \vec{e}(x) = (0, 0, 1) \) is invariant only under transformations \( O \in SO(2)_s \approx U(1)_s = H \) in the unbroken subgroup. It is interesting that antiferromagnetic magnons have a “relativistic” energy-momentum dispersion relation

\[
E = |\vec{p}|c,
\]

despite the fact that the underlying electron dynamics is non-relativistic. Of course, the spin wave velocity \( c \) is smaller than the velocity of light.

### 2.4 Ferromagnetic Magnons

Ferromagnets are another class of interesting condensed matter systems which, for example, play a role in the context of the quantum Hall effect [18]. In particular, the Coulomb interaction in a spatially 2-dimensional quantum Hall system favors a totally antisymmetric orbital wave function. Consequently, the spin wave function is totally symmetric and the system is ferromagnetically ordered. A microscopic model for ferromagnets is the quantum Heisenberg model. Just as in an antiferromagnet, in a ferromagnet the global spin rotational symmetry group \( G = SU(2)_s \) is
spontaneously broken down to the subgroup \( H = U(1)_s \), now by the formation of a uniform magnetization. Again, the corresponding Goldstone bosons — in this case ferromagnetic magnons — are described by a unit-vector field \( \vec{e}(x) \) in the coset space \( G/H = SU(2)_s/U(1)_s = S^2 \). Unlike for antiferromagnets, the order parameter of a ferromagnet — the uniform magnetization — is a conserved quantity. This has interesting consequences for the low-energy physics of ferromagnetic magnons. In particular, their energy-momentum dispersion relation

\[
E = \frac{\rho_s}{m} |\vec{p}|^2, \tag{2.11}
\]
is non-relativistic. Here \( \rho_s \) is again the spin stiffness and \( m \) is the magnetization density. The lowest-order chiral perturbation theory Euclidean action for a ferromagnet is given by [6]

\[
S[\vec{e}] = \int d^2 x \left[ \int_{S^1} dt \frac{\rho_s}{2} \partial_t \vec{e} \cdot \partial_t \vec{e} - im \int_{H^2} dt \ d\tau \ \vec{e} \cdot (\partial_t \vec{e} \times \partial_\tau \vec{e}) \right]. \tag{2.12}
\]

The second term on the right-hand side of this equation is of topological nature. In order to write it in a manifestly \( SU(2)_s \) invariant form, the \( (2+1) \)-dimensional space-time has been extended into a fourth dimension with a coordinate \( \tau \in [0,1] \) which plays the role of a deformation parameter. The manifold \( H^2 \) is a 2-dimensional hemisphere with the compactified Euclidean time \( S^1 \) as its boundary. The magnon field \( \vec{e}(x) \) at physical space-time points \( x \in \mathbb{R}^2 \times S^1 \) is extended to a field \( \vec{e}(x, \tau) \) in the 4-dimensional space-time \( \mathbb{R}^2 \times H^2 \) such that \( \vec{e}(x, 1) = \vec{e}(x) \) and \( \vec{e}(x, 0) = (0, 0, 1) \). Of course, the \( (2+1) \)-dimensional physics should be independent of how the field \( \vec{e}(x, \tau) \) is deformed into the bulk of the fourth dimension. It should only depend on the boundary values \( \vec{e}(x) \), i.e. on the magnon field in the physical part of space-time. This is possible because the integrand in the second term of eq.(2.12) is a total divergence closely related to the winding number of \( \Pi_2[S^2] = \mathbb{Z} \). In fact, when the integration in eq.(2.12) over the hemisphere \( H^2 \) is replaced by an integration over a sphere \( S^2 \), the term

\[
n = \frac{1}{4\pi} \int_{S^2} dt \ d\tau \ \vec{e} \cdot (\partial_t \vec{e} \times \partial_\tau \vec{e}) \in \Pi_2[S^2] = \mathbb{Z} \tag{2.13}
\]
is an integer winding number. Hence, modulo an integer \( n \), \( S[\vec{e}] \) gets contributions only from the boundary of \( \mathbb{R}^2 \times H^2 \), i.e. from the \( (2+1) \)-dimensional physical space-time \( \mathbb{R}^2 \times S^1 \). Of course, one must still ensure that the integer contribution \( n \) from the 4-dimensional bulk cancels. This is indeed the case, because the topological term \( \frac{1}{4\pi} \int_{H^2} dt \ d\tau \ \vec{e} \cdot (\partial_t \vec{e} \times \partial_\tau \vec{e}) \) enters the action with a prefactor

\[
\int d^2 x \ 4\pi im = 4\pi i M. \tag{2.14}
\]

Here \( m \) is the magnetization density and

\[
M = \int d^2 x \ m \tag{2.15}
\]
is the total spin of the entire magnet and hence an integer or a half-integer. The 4-dimensional bulk ambiguity $4\pi iMn$ in the action $S[\vec{e}]$ cancels in the path integral. Due to the fact that $\exp(4\pi iMn) = 1$, the factor $\exp(-S[\vec{e}])$ that enters the path integral is unambiguously defined. It is remarkable that consistency of the low-energy magnon path integral requires the quantization of the total spin in integer or half-integer units.

### 2.5 Baby-Skyrmions

Just as pion fields support Skyrmions, both ferro- and antiferromagnetic magnon fields support baby-Skyrmions. Baby-Skyrmions are solitons whose topological charge

$$B = \frac{1}{8\pi} \int d^2 x \ v_{ij} \cdot (\partial_i \vec{e} \times \partial_j \vec{e}),$$  \hspace{1cm} (2.16)

again defined at every instant in time, is an element of the homotopy group $\Pi_2[S^2] = \mathbb{Z}$. In QCD the Skyrmion topological charge has been identified with baryon number. What physical quantity the baby-Skyrmion number $B$ represents depends on the specific ferro- or antiferromagnetic system in question. At this point we keep the discussion general and do not identify $B$ with a specific physical quantity. As before, the topological current

$$j_\mu = \frac{1}{8\pi} \varepsilon_{\mu\nu\rho} \vec{e} \cdot (\partial_\nu \vec{e} \times \partial_\rho \vec{e})$$  \hspace{1cm} (2.17)

is conserved, i.e. $\partial_\mu j_\mu = 0$, independent of the equations of motion. As in the pion case, the detailed properties of baby-Skyrmions are not accessible in magnon chiral perturbation theory. Still, as we will discuss below, in cases where $B$ is related to the electric charge, the fact that baby-Skyrmions exist has profound consequences for the low-energy electromagnetic properties of magnons.

As in the pion case, there is another non-trivial homotopy group, $\Pi_3[S^2] = \mathbb{Z}$, which is relevant for baby-Skyrmions. It implies that magnon fields (which now depend on both space and time) fall into distinct topological classes. The corresponding winding number is the Hopf number $H[\vec{e}] \in \Pi_3[S^2] = \mathbb{Z}$ which characterizes the braiding of baby-Skyrmion paths in time. In $2+1$ dimensions particles can not only be quantized as bosons or fermions, but may have any spin and statistics. In particular, baby-Skyrmions can be quantized as anyons characterized by a statistics angle $\theta$ [15]. The cases $\theta = 0$ and $\theta = \pi$ correspond to bosons and fermions, respectively. Including the Hopf term, the magnon path integral (both for ferro- and antiferromagnets) takes the form

$$Z = \int D\vec{e} \ \exp(-S[\vec{e}]) \ \exp(i\theta H[\vec{e}]).$$  \hspace{1cm} (2.18)

The angle $\theta$ enters the magnon effective theory in a similar way as the number of colors $N_c$ enters the pion effective theory. The value of $\theta$ must be determined for each
individual underlying microscopic system. For example, for the antiferromagnetic quantum Heisenberg model it has been argued that no Hopf term is generated [40–44], i.e. $\theta/2\pi \in \mathbb{Z}$. Hence, in that case the baby-Skyrmions should be bosons. It should be noted that Skyrmions in $3+1$ dimensions cannot be quantized as anyons. The homotopy group $\Pi_4[SU(2)] = \mathbb{Z}(2)$ allows only two cases — bosons or fermions.

3 Electromagnetism of Pions and Magnons

In this section we couple the low-energy effective theories for pions and magnons to electromagnetism. In both cases, there are topological effects due to Skyrmions or baby-Skyrmions. In the pion case these effects are described by a Goldstone-Wilczek term which contains the vertex for the anomalous decay $\pi^0 \to \gamma\gamma$ of a neutral pion into two photons. In cases where the baby-Skyrmion topological charge $B$ is related to the electric charge there is an analogue of the Goldstone-Wilczek term for magnons which then gives rise to a magnon-two-photon vertex. This vertex can be used to convert photons into magnons in an external magnetic field.

3.1 Pions and Photons

At the quark level, the electric charge is given by

$$Q = T^3_L + T^3_R + \frac{1}{2}B,$$

where $T^3_L$ and $T^3_R$ are the diagonal generators of $SU(2)_L$ and $SU(2)_R$ and $B$ is the baryon number. Since $U(1)_B$ is not a subgroup of $SU(2)_L \otimes SU(2)_R$, it is not entirely straightforward to gauge the $U(1)_{em}$ symmetry of electromagnetism at the level of the pion effective theory. Naively, one would just replace ordinary derivatives with covariant ones. The electromagnetic covariant derivative of the pion field takes the form

$$D_\mu U(x) = \partial_\mu U(x) + ieA_\mu[T^3, U(x)],$$

where $A_\mu$ is the electromagnetic vector potential and $e$ is the electric charge. The action is then given by

$$S[U, A_\mu] = \int d^4x \frac{F^2_{\mu\nu}}{4} \text{Tr}[D_\mu U^\dagger D_\mu U].$$

However, incorporating the covariant derivatives alone is not sufficient in order to gauge $U(1)_{em}$ correctly. Although the pions themselves do not carry baryon number, it is crucial to incorporate the baryon current in the effective theory since the quark charge $Q$ of eq.(3.1) contains the baryon number $B$. In particular, if one does not
include the baryon current, the decay $\pi^0 \to \gamma\gamma$ does not happen in the effective theory. The baryon current of eq.(2.5) is no longer conserved when ordinary derivatives are replaced with covariant ones, and the correct conserved baryon current is the Goldstone-Wilczek current \cite{16,45}

$$j_{\mu}^{GW} = \frac{1}{24\pi^2}\varepsilon_{\mu\nu\rho\sigma}\text{Tr}\left[(U^\dagger D_\nu U)(U^\dagger D_\rho U)(U^\dagger D_\sigma U)\right]$$

$$- \frac{i e}{16\pi^2}\varepsilon_{\mu\nu\rho\sigma}F_{\nu\rho}\text{Tr}\left[T^3(D_\sigma U U^\dagger + U^\dagger D_\sigma U)\right]. \quad (3.4)$$

Since the quark charge $Q$ contains the baryon number $B$ with a prefactor $1/2$, the Goldstone-Wilczek current should be coupled to the electromagnetic field through an additional contribution to the action

$$S_{GW}[U, A_\mu] = \frac{e}{2}\int d^4x\; A_\mu j_{\mu}^{GW}.$$ \quad (3.5)

The path integral of pions coupled to an external electromagnetic field then takes the form

$$Z[A_\mu] = \int \mathcal{D}U \exp(-S[U, A_\mu]) \text{Sign}[U]^{N_c}\exp(iS_{GW}[U, A_\mu]). \quad (3.6)$$

One can now identify the vertex responsible for the decay $\pi^0 \to \gamma\gamma$. Putting $U(x) \approx 1+2i\pi^0(x)T^3/F_\pi$, after partial integration the second term in the Goldstone-Wilczek current of eq.(3.4) indeed yields the vertex

$$L_{\pi^0\gamma\gamma}(x) = -i\frac{e^2}{32\pi^2F_\pi}\pi^0(x)\varepsilon_{\mu\nu\rho\sigma}F_{\mu\nu}(x)F_{\rho\sigma}(x), \quad (3.7)$$

where

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x), \quad (3.8)$$

is the electromagnetic field strength tensor. The above vertex is independent of the number of colors $N_c$. Indeed, in \cite{46} it was shown that, in contrast to textbook knowledge, the $\pi^0 \to \gamma\gamma$ decay width does not depend on $N_c$ explicitly.

### 3.2 Local SU(2)$_s$ Spin Symmetry of the Pauli Equation

In order to make our paper self-contained, we include this subsection which is based on work of Fröhlich and Studer \cite{13}. They realized that, up to order $1/M^3$ corrections (where $M$ is the electron mass), the non-relativistic Pauli equation (which results from reducing the Dirac equation to its upper components) has a local $SU(2)_s$ spin symmetry. In the next section, we will use this symmetry to construct an effective theory describing the electromagnetic interactions of magnons and photons. The Pauli equation for electrons interacting with the electromagnetic field, combined with the Pauli equation for atomic nuclei, can be viewed as a condensed matter
analogue of the standard model of particle physics. Indeed, the electrodynamics of non-relativistic electrons and atomic nuclei interacting with photons, as complicated as it may be to solve, should, at least in principle, capture all phenomena in condensed matter. In practice it is impossible to derive emergent phenomena like the quantum Hall effect or high-temperature superconductivity from first principles of the underlying Pauli equation. Still, considering the underlying microscopic theory is useful, because its symmetries are inherited by the low-energy effective theories that are used to describe the various phenomena in question.

Up to corrections of order $1/M^3$ (and putting $\hbar = c = 1$), the Pauli equation describing the interaction of electrons with an external electromagnetic field $\Phi, \vec{A}$ can be cast into the form [13]

$$\begin{align*}
  i(\partial_t - ie\Phi + ie \frac{\vec{E} \cdot \vec{E}}{8M^2} + i e \frac{\vec{B} \cdot \vec{\sigma}}{2M})\Psi &= -\frac{1}{2M} (\vec{\nabla} + ie\vec{A} - i \frac{e}{4M} \vec{E} \times \vec{\sigma})^2 \Psi. \\
  (3.9)
\end{align*}$$

Here $\Psi$ is a 2-component Pauli spinor, and $\vec{E} = -\vec{\nabla}\Phi - \partial_t \vec{A}$ and $\vec{B} = \vec{\nabla} \times \vec{A}$ are the usual electromagnetic field strengths. The first two terms on the left-hand side form the $U(1)$ covariant derivative familiar from electrodynamics. The third and fourth term on the left-hand side represent relativistic effects: the Darwin and Zeeman term, respectively. The first two terms on the right-hand side again form an ordinary $U(1)$ covariant derivative, while the third term represents the relativistic spin-orbit coupling. Fröhlich and Studer noticed a remarkable mathematical structure in the Pauli equation — a local $SU(2)_s$ spin symmetry. Indeed, the above equation can be written as

$$iD_t \Psi = -\frac{1}{2M} D_t D_\lambda \Psi, \quad (3.10)$$

with an $SU(2)_s \otimes U(1)_{em}$ covariant derivative given by

$$D_\mu = \partial_\mu + ieA_\mu(x) + W_\mu(x). \quad (3.11)$$

The components of the non-Abelian vector potential

$$W_\mu(x) = iW^a_\mu(x)T^a, \quad (3.12)$$

(with $T^a = \frac{1}{2}\sigma^a$) can be identified as

$$W^a_t(x) = \mu B^a(x), \quad W^a_i(x) = \frac{\mu}{2} \varepsilon_{iab} E^b(x). \quad (3.13)$$

Interestingly, the electromagnetic field strengths $\vec{E}$ and $\vec{B}$ enter the theory in the form of non-Abelian vector potentials $W_\mu$ of a local $SU(2)_s$ symmetry. The anomalous magnetic moment $\mu = ge/2M$ of the electron (where, up to QED corrections, $g = 2$) plays the role of the non-Abelian gauge coupling. The Abelian vector potential $A_\mu$ is the one familiar from electrodynamics, except for a small contribution to the scalar potential due to the Darwin term, $A_t = -\Phi + (e/8M^2)\vec{\nabla} \cdot \vec{E}$.
Fröhlich and Studer’s observation implies that in non-relativistic systems, at least up to corrections of order $1/M^3$, spin plays the role of an internal quantum number analogous to flavor in particle physics. It should be pointed out that $SU(2)_s$ is not a local symmetry of the full microscopic theory underlying condensed matter physics. This follows because the energy density $\vec{E}^2 + \vec{B}^2$ of the electromagnetic field is invariant only under global (and not under local) $SU(2)_s$ transformations. In order to still make use of the local symmetry, we separate the electromagnetic field into internal and external contributions. The internal contributions are responsible for the complicated dynamics that turn electrons and atomic nuclei into ferro- or antiferromagnets. The internal fields have been integrated out and thus do not appear explicitly in the low-energy effective theory. External electromagnetic fields, on the other hand, are used to probe the physics of the magnetic material and appear explicitly in the effective Lagrangian. Under these circumstances, the $\vec{E}^2 + \vec{B}^2$ contribution of the external field does not enter the dynamics, and the local $SU(2)_s$ symmetry is indeed realized. While it seems difficult to make these arguments quantitative, we think that they capture the essence of how magnets respond to external electromagnetic fields. Hence, although we expect the local $SU(2)_s$ symmetry to be only approximate in actual materials, in what follows we impose it as an exact symmetry. This allows us to identify the most important terms in the effective Lagrangian of magnons and photons. It should be noted that, despite the intriguing mathematical structure, $SU(2)_s$ is not a gauge symmetry in the usual sense. In particular, the non-Abelian vector potentials $W_\mu$ are nothing but the Abelian field strengths $\vec{E}$ and $\vec{B}$ and thus do not represent independent physical degrees of freedom. Furthermore, there is no $SU(2)_s$ Gauss law. Consequently, “gauge-variant” states that carry a non-zero spin certainly still belong to the physical Hilbert space. Also, since $SU(2)_s$ is not a true gauge symmetry, its spontaneous breakdown to $U(1)_s$ does not induce the Higgs mechanism. Since there are no $SU(2)_s$ gauge bosons as independent degrees of freedom, the Goldstone boson mode cannot be incorporated as a longitudinal polarization state. Still, as we will see later, some of the magnons pick up a mass due to their interactions with external electromagnetic fields.

### 3.3 Magnons and Photons

It is interesting to ask how magnons couple to photons. Despite the fact that magnons are electrically neutral this question is non-trivial. The crucial observation is that external electromagnetic fields couple to non-relativistic condensed matter in the form of $SU(2)_s$ non-Abelian vector potentials. For antiferromagnetic magnons coupled to external $\vec{E}$ and $\vec{B}$ fields the effective action takes the form

$$S[\vec{e}, W_\mu] = \int d^2 x \int_{S^1} dt \rho_s \left[ D_t \vec{e} \cdot D_t \vec{e} + \frac{1}{c^2} D_t \vec{e} \cdot D_t \vec{e} \right],$$

(3.14)
with the covariant derivative
\[ D_\mu \vec{e}(x) = \partial_\mu \vec{e}(x) + \vec{e}(x) \times \vec{W}_\mu(x). \quad (3.15) \]

Similarly, for ferromagnetic magnons
\[ S[\vec{e}, W_\mu] = \int d^2 x \left[ \int_{S^1} dt \frac{\rho_s}{2} D_i \vec{e} \cdot D_i \vec{e} \right. \\
\left. - \text{im} \int_{H^2} dt \, dt' \, \vec{e} \cdot (\partial_i \vec{e} \times \partial_{i'} \vec{e}) + \text{im} \int_{S^1} dt \, \vec{e} \cdot \vec{W}_l. \right] \\
\left. \right. (3.16) \]
The last term is necessary to cancel the $SU(2)$, “gauge” variation of the second term. Both terms together are invariant.

Since magnons are electrically neutral, one might think that they do not couple to the electromagnetic vector potential $A_\mu$. However, the case of pions in QCD has taught us that Goldstone bosons can couple indirectly to external fields through their topological excitations. For example, despite the fact that pions themselves do not carry baryon number, they couple to electromagnetism anomalously through the baryon number of their Skyrmion excitations. In fact, the decay $\pi^0 \to \gamma\gamma$ is entirely due to this coupling. Similarly, magnons may couple to electromagnetism indirectly through their baby-Skyrmion topological excitations. The magnon analogue of the Goldstone-Wilczek current is
\[ j^{GW}_\mu = \frac{1}{8\pi} \varepsilon_{\mu\nu\rho} \vec{e} \cdot (D_\nu \vec{e} \times D_\rho \vec{e} + \vec{W}_{\nu\rho}), \quad (3.17) \]
with the non-Abelian field strength given by
\[ \vec{W}_{\mu\nu}(x) = \partial_\mu \vec{W}_\nu(x) - \partial_\nu \vec{W}_\mu(x) - \vec{W}_\mu(x) \times \vec{W}_\nu(x). \quad (3.18) \]
The condensed matter analogue of the standard model relation $Q = T_3^L + T_3^R + \frac{1}{2} B$ is
\[ Q = -F, \quad (3.19) \]
i.e. the electric charge in magnets is carried by the fermion number $F$ of the electrons. In this case there is no contribution from $T_3^3$ because both spin up and spin down electrons carry the same charge $-e$, while quarks of flavor up and down have different electric charges. The fermion number of baby-Skyrmions is determined by the anyon angle as $F = B\theta/\pi = -Q$. For example, if $\theta = \pi$, the baby-Skyrmions have electron quantum numbers, while for $\theta = 2\pi$ they are bosons with charge $-2e$. If the baby-Skyrmions carry an electric charge (i.e. if $\theta \neq 0$) the Goldstone-Wilczek current (times $-e\theta/\pi$) is the electric current, which hence couples to the electromagnetic field. Consequently, in analogy to QCD, a Goldstone-Wilczek term
\[ S_{GW}[\vec{e}, A_\mu, W_\mu] = -\frac{e\theta}{\pi} \int d^3 x \, A_\mu j^{GW}_\mu, \quad (3.20) \]
arises. The value of $\theta$ depends on the specific microscopic system in question.
Besides the Goldstone-Wilczek term, also the Hopf term $i\theta H[\vec{e}]$ contributes to the magnon action. The Hopf number $H[\vec{e}]$ is not invariant under local $SU(2)_s$ transformations. In fact, under a local transformation $g(x)\in SU(2)_s$,

$$W'_\mu(x) = g(x)^\dagger(W_\mu(x) + \partial_\mu)g(x),$$

it changes by the winding number

$$n[g] = \frac{1}{24\pi^2} \int d^3x \varepsilon_{\mu\nu\rho} \text{Tr} \left[(g^\dagger \partial_\mu g)(g^\dagger \partial_\nu g)(g^\dagger \partial_\rho g)\right],$$

and turns into

$$H[\vec{e}'] = H[\vec{e}] + n[g].$$

This “gauge” variation indicates an anomaly in the baby-Skyrmion sector of the magnon effective theory, analogous to Witten’s global anomaly [47]. Since the local $SU(2)_s$ symmetry does not represent a true gauge symmetry of the underlying microscopic theory of electrons and atomic nuclei, this anomaly does not imply an inconsistency of the quantum theory, and thus need not necessarily be canceled. Still, the anomaly may be canceled by a Chern-Simons term

$$S_{CS}[W_\mu] = \frac{1}{8\pi^2} \int d^3x \varepsilon_{\mu\nu\rho} \text{Tr}[W_\mu(\partial_\nu W_\rho + \frac{2}{3}W_\nu W_\rho)].$$

Like the Hopf term, the Chern-Simons term is not gauge invariant, and its gauge variation is given by

$$S'_{CS}[W'_\mu] = S_{CS}[W_\mu] - n[g].$$

Hence, $H[\vec{e}] + S_{CS}[W_\mu]$ is indeed invariant even against topologically non-trivial gauge transformations.

The magnon partition function takes the form

$$Z[A_\mu, W_\mu] = \int D\vec{e} \exp(-S[\vec{e}, W_\mu]) \exp(i\theta H[\vec{e}]) \exp(iS_{GW}[\vec{e}, A_\mu, W_\mu]).$$

Introducing small magnon fluctuations $m^a(x) (a = 1, 2)$ around a (staggered) magnetization in the 3-direction

$$\vec{e}(x) \approx (0, 0, 1) + \frac{1}{\sqrt{\rho_s}}(m^1(x), m^2(x), -\frac{1}{2\sqrt{\rho_s}}(m^1(x)^2 + m^2(x)^2)),$$

one can identify the vertex responsible for the decay of a magnon into two photons

$$\mathcal{L}_{m\gamma\gamma}(x) = -i \frac{e\theta}{8\pi^2\sqrt{\rho_s}} m^a(x)\varepsilon_{\mu\nu\rho} F_{\mu\nu}(x) W_\rho^a(x).$$

One photon is represented by the field strength tensor $F_{\mu\nu}$, while the other one is contained in $W_\rho^a$. The experimental observation of magnon decay into two photons would unambiguously demonstrate that baby-Skyrmions indeed carry electric
charge. Detecting this process inside a magnetic material is certainly challenging, if not impossible. For example, for exactly massless magnons there is no phase space for the decay into photons. In the next subsection we discuss a set-up that may simplify the detection of the $L_{m\gamma\gamma}$ vertex.

In QCD the process $\pi^0 \rightarrow \gamma\gamma$ explicitly breaks the $G$-parity symmetry [48] through electromagnetic effects. At the level of the underlying microscopic standard model, the anomaly results from non-trivial transformation properties of the fermionic measure. In the low-energy effective theory, on the other hand, the measure is invariant under the symmetry, and the anomaly is represented by an explicit symmetry breaking term in the action.

It is interesting to ask what symmetry is anomalously broken by magnon decay into two photons. The magnon analogue of $G$-parity is the $Z(2)$ symmetry that turns $\vec{e}$ into $-\vec{e}$. Indeed, this symmetry is explicitly broken by the magnon analogue of the Goldstone-Wilczek term. For antiferromagnetic magnons this electromagnetic effect is the only source of explicit $Z(2)$ symmetry breaking. For ferromagnetic magnons, on the other hand, the topological term $\vec{e} \cdot (\partial_t \vec{e} \times \partial_\tau \vec{e})$ also breaks this symmetry. The microscopic origin of this anomaly is the spin commutation relation $[S_i, S_j] = i\varepsilon_{ijk}S_k$ which is satisfied for $\vec{S}$, but not for $-\vec{S}$. It is interesting that, in contrast to the standard model, microscopically the anomaly does not originate from a non-trivial fermionic measure, but from a non-trivial commutation relation. In both cases, the breaking of the symmetry originates from quantum effects.

### 3.4 Photon-Magnon Conversion in an External Magnetic Field

We have seen that, just like pions, magnons can turn into two photons, provided that baby-Skyrmions carry electric charge. Hence, by studying magnon-photon interactions, one can learn something non-trivial about baby-Skyrmions. Observing magnon decay into two photons in a condensed matter experiment is a challenging problem, and may even be impossible in practice. In this subsection we discuss a possible way of enhancing the magnon-two-photon process, which may make it more easily detectable. Again, the idea is inspired by particle physics — namely by the conversion of photons into axions in an external magnetic field.\(^1\) The axion [19, 20] is a hypothetical particle associated with the Peccei-Quinn mechanism [21] for solving the strong CP problem. Like pions and magnons, the axion is a Goldstone boson that can decay into two photons. Due to its very weak couplings, the axion is practically invisible and has indeed not yet been found. However, in order to enhance axion visibility, interesting conversion experiments have been proposed [22, 23]. In particular, if one shines a very intense laser beam into a strong magnetic

\(^1\)We thank W. Bernreuther for reminding us of this process.
field, one can convert the laser photons into axions (provided that axions exist at all). This process makes use of the vertex for axion decay into two photons. One (real) photon is provided by the laser beam and the second (virtual) photon stems from the external magnetic field. Here we consider the magnon analogue of this process — namely photon-magnon conversion in an external electromagnetic field. It remains to be seen if a laser beam shone into an antiferromagnetic precursor of a high-temperature superconductor or a quantum Hall ferromagnet will reveal the magnon-two-photon vertex experimentally. Here we provide some necessary theoretical background.

Let us first consider a ferromagnet in an external magnetic field $\vec{B} = B\vec{e}_z$. Obviously, in order to minimize the energy, the vector $\vec{e}$ describing the uniform magnetization then aligns with the field $\vec{B}$. This follows immediately from the term

$$m \vec{e}(x) \cdot \vec{W}_t(x) = m\mu \vec{e}(x) \cdot \vec{B},$$

in the effective magnon-photon Lagrangian of eq.(3.16). Expanding the magnon field as in eq.(3.27), the term from above modifies the dispersion relation for ferromagnetic magnons [6] to

$$E = M_m + \frac{\rho_s}{m}|\vec{p}|^2,$$

i.e. it leads to a magnon “rest mass” (or, more precisely, rest energy)

$$M_m = \mu B.$$ (3.31)

Under these circumstances, the magnon-two-photon vertex of eq.(3.28) can be written as

$$\mathcal{L}_{m\gamma\gamma}(x) = -i \frac{e\theta\mu B}{4\pi^2 \sqrt{\rho_s}} \left[ m^1(x)B^1(x) + m^2(x)B^2(x) \right].$$

Here $B^i (i = 1, 2)$ represents the (real) laser photons, while the factor $B$ represents the (virtual) photons of the external magnetic field. It is important to ensure that the magnetic field component of the injected laser field is perpendicular to the direction of the external magnetic field.

Let us now consider an antiferromagnet in an external magnetic field. In order to minimize their energy, the spins in an antiferromagnet point antiparallel to one another, but they should now also follow the external magnetic field. The best compromise to satisfy these competing requirements is achieved by a canted state in which the staggered magnetization points perpendicular to the field. We now choose $\vec{B} = B\vec{e}_x$ and we again use the expansion of eq.(3.27). For static fields, the contribution to the action that determines the canted state takes the form

$$D_t\vec{e}(x) \cdot D_t\vec{e}(x) = (\vec{e}(x) \times \vec{W}_t) \cdot (\vec{e}(x) \times \vec{W}_t) = \mu^2 B^2(1 - m^1(x)^2).$$

Hence, in a magnetic field the magnon $m^1$ again picks up a mass $M_m = \mu B$, while the magnon $m^2$ remains massless. The dispersion relation for the massive antiferromagnetic magnon is still “relativistic”, i.e.

$$E = \sqrt{M_m^2 + |\vec{p}|^2c^2}.$$ (3.34)
The magnon-two-photon vertex of eq.(3.28) now takes the form
\[ \mathcal{L}_{m\gamma\gamma}(x) = -i \frac{e\theta B}{4\pi^2\sqrt{\rho_s}} m_1(x) B^3(x). \]

(3.35)

In this case, \( B^3 \) represents the laser photons. Again, the magnetic field component of the injected laser field should be perpendicular to the direction of the external magnetic field. Note that the laser photons can be converted only into the massive magnon \( m^1 \).

4 Skyrmion and Baby-Skyrmion Decay

Skyrmions and baby-Skyrmions are topologically stable solitons. Still, when they interact with external gauge fields, Skyrmions as well as baby-Skyrmions can become unstable and decay. The decay of a Skyrmion baryon in the pion effective theory can be induced by baryon number violating electroweak instantons through the ’t Hooft anomaly. In addition, magnetic monopoles can catalyze Skyrmion decay. There is no analogue of the ’t Hooft anomaly for baby-Skyrmions. However, baby-Skyrmion decay can still be catalyzed by the condensed matter analogue of a magnetic monopole, an electrically charged wire.

4.1 Pions, Skyrmions, and \( W \)-Bosons

Since for magnons a local \( SU(2)_s \) spin symmetry emerged somewhat unexpectedly, we now ask if there is an analogue of this for pions. Indeed, the weak gauge interactions turn the global \( SU(2)_L \) symmetry into a local one by coupling the pions to the non-Abelian \( W \)-boson field. In addition, the \( U(1)_Y \) subgroup of \( SU(2)_R \) is also gauged by coupling the pions to the Abelian \( B \)-bosons. The \( SU(2)_L \otimes U(1)_Y \) symmetry then breaks spontaneously to the \( U(1)_{em} \) symmetry of electromagnetism. The photon emerges as a linear combination of \( W^3 \) and \( B \). In this subsection we concentrate on the \( W \)-bosons and thus we gauge only \( SU(2)_L \) but not \( U(1)_Y \) or \( U(1)_{em} \). A more detailed discussion of the electroweak interactions of pions is contained in [46].

Gauging \( SU(2)_L \) is straightforward. One just replaces ordinary derivatives by covariant derivatives
\[ D_\mu U(x) = (\partial_\mu + W_\mu(x)) U(x). \]

(4.1)

Here \( W_\mu = igW^a_\mu T^a \) is the \( SU(2)_L \) gauge field with gauge coupling \( g \) and field strength
\[ W_{\mu\nu}(x) = \partial_\mu W_\nu(x) - \partial_\nu W_\mu(x) + [W_\mu(x), W_\nu(x)]. \]

(4.2)
The action now takes the form

\[ S[U, W_\mu] = \int d^4x \frac{F^2}{4} \text{Tr}[D_\mu U^\dagger D_\mu U], \]

(4.3)

which is invariant under local transformations

\[ U'(x) = L^\dagger(x)U(x), \quad W'_\mu(x) = L^\dagger(x)(W_\mu(x) + \partial_\mu)L(x). \]

(4.4)

While the pion action \( S[U, W_\mu] \) of eq.(4.3) is gauge invariant, the path integral as a whole is not. This is because

\[ \text{Sign}[U'] = \text{Sign}[LU] = \text{Sign}[L]\text{Sign}[U]. \]

(4.5)

As pointed out by Witten [35] and by D’Hoker and Farhi [45], the \( SU(2)_L \) gauge variation of the fermion permutation sign of the Skyrmions is a manifestation of Witten’s global anomaly [47]. For odd \( N_c \) the gauged pion theory is inconsistent, unless the anomaly is canceled by additional fields. In the standard model the global anomaly is canceled by the left-handed lepton doublet of neutrino and electron. For even \( N_c \), on the other hand, the pure pion theory without leptons is anomaly-free and thus consistent at the quantum level.

When \( SU(2)_L \) is gauged, baryon number conservation is violated through the ’t Hooft anomaly by electroweak instantons [49]. In this case, the Goldstone-Wilczek baryon number current [16, 45] takes the form

\[ j^G_W \mu = \frac{1}{24\pi^2} \varepsilon_{\mu\rho\sigma} \text{Tr} \left[(U^\dagger D_\nu U)(U^\dagger D_\rho U)(U^\dagger D_\sigma U)\right] - \frac{1}{16\pi^2} \varepsilon_{\mu\rho\sigma} \text{Tr} \left[W_\nu(D_\rho UU^\dagger)\right]. \]

(4.6)

Its divergence is given by

\[ \partial_\mu j^G_W \mu = -\frac{1}{32\pi^2} \varepsilon_{\mu\rho\sigma} \text{Tr}[W_\mu W_\rho]. \]

(4.7)

Consequently, an electroweak gauge field with topological charge

\[ Q = -\frac{1}{32\pi^2} \int d^4x \varepsilon_{\mu\rho\sigma} \text{Tr}[W_\mu W_\rho] \in \Pi_3[SU(2)_L] = \mathbb{Z} \]

(4.8)

causes violation of baryon number conservation by \( Q \) units. There is no analogue of the ’t Hooft anomaly for magnons. The analogue of the Goldstone-Wilczek current for magnons is conserved independent of the form of the \( SU(2)_s \) spin gauge field.

### 4.2 Pions, Skyrmions, and Magnetic Monopoles

The existence of magnetic monopoles was contemplated by Dirac as early as 1931 [50]. The standard model of particle physics does not contain magnetically charged
particles and even Dirac did not believe in the existence of magnetic monopoles at the end of his life [51]. Still, some extensions of the standard model — for example, the \( SU(5) \) grand unified theory — contain very heavy \( 't \) Hooft-Polyakov monopoles which look like Dirac monopoles from large distances. In the monopole core the \( SU(5) \) symmetry is unbroken and quarks and leptons are indistinguishable there. As a consequence, baryons that enter the monopole core can reappear as leptons and thus the monopole itself can catalyze baryon decay. This is known as the Callan-Rubakov effect [32, 33]. In the \( SU(5) \) grand unified theory, \( B - L \) is conserved and thus baryon and lepton number are violated by the same amount. As a result, \( SU(5) \) monopoles also catalyze lepton decay.

The magnetic current of a monopole is given by

\[
m_{\sigma} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} \partial_{\mu} F_{\nu\rho},
\]

which measures the amount of violation of the Abelian Bianchi identity. In the presence of magnetic charge, the Goldstone-Wilczek current of eq.(3.4) is no longer conserved because

\[
\partial_{\mu} j_{\mu}^{GW} = -\frac{i e}{8\pi^2} m_{\sigma} \text{Tr} \left[ T^3 (D_{\sigma} U U^\dagger + U^\dagger D_{\sigma} U) \right].
\]

For a magnetic monopole at rest at \( \vec{x} = \vec{0} \) we have

\[
m_0(\vec{x}, t) = 4\pi g \delta(\vec{x}), \quad m_i(\vec{x}, t) = 0,
\]

where \( g \) is the magnetic charge. In spherical coordinates \((r, \theta, \varphi)\), a vector potential describing this situation is given by

\[
\vec{A}(\vec{x}) = g \frac{1 - \cos \theta}{r \sin \theta} \vec{e}_\varphi.
\]

This potential is singular along the negative \( z \)-axis, due to the Dirac string. Writing \( U(x) = \exp(2i\pi^0(x)T^3/F_\pi) \) and integrating eq.(4.10) over space we obtain the rate of change of the baryon number as

\[
\partial_t B(t) = \frac{e g}{\pi F_\pi} \partial_t \pi^0(\vec{0}, t).
\]

Using the Dirac quantization condition \( e g = 1/2 \) one obtains

\[
B(\infty) - B(-\infty) = \frac{1}{2\pi F_\pi} \left[ \pi^0(\vec{0}, \infty) - \pi^0(\vec{0}, -\infty) \right].
\]

Hence, if the neutral pion field \( \pi^0(\vec{0})/F_\pi \) at the location of the monopole rotates by \( 2\pi n \), baryon number is violated by \( n \) units.
4.3 Magnons, Baby-Skyrmions, and Charged Wires

The question arises if monopole catalyzed baryon decay has an analogue for magnons. The corresponding analogue of the Goldstone-Wilczek current takes the form of eq.(3.17). In analogy to the magnetic current we introduce

$$\vec{m} = \varepsilon_{\mu\nu\rho} D_\mu \vec{W}_{\nu\rho},$$

which measures the amount of violation of the non-Abelian Bianchi identity. In analogy to the QCD case, for non-vanishing $\vec{m}$ the Goldstone-Wilczek current is no longer conserved because

$$\partial_\mu j^{GW}_\mu = \frac{1}{8\pi} \vec{m} \cdot \vec{e}.$$ (4.16)

As in the monopole case, we consider a point-like violation of the Bianchi-identity. The simplest example is

$$m^a(x) = 4\pi g \delta^a_3 \delta(x).$$ (4.17)

In this case, the $\delta$-function includes time, i.e. the violation of the Bianchi-identity is event-like — not particle-like. Hence, in the magnon theory the analogue of the magnetic monopole is a 3-dimensional instanton. In complete analogy to the vector potential for a Dirac monopole one obtains

$$W^a_i(x) = g \delta^a_3 \frac{1 - \cos \theta}{r \sin \theta} e_{\varphi, i}.$$ (4.18)

Introducing cylindrical space-time coordinates $\rho = r \sin \theta$, $\varphi$, and $t = r \cos \theta$ and using eq.(3.13) this equation translates into

$$\vec{E}(\rho, t) = \frac{2g}{\mu \rho} (1 - \frac{t}{\sqrt{t^2 + \rho^2}}) \vec{e}_\rho.$$ (4.19)

In the far future the electric field vanishes, while in the distant past it takes the form $\vec{E} \sim 4g\vec{e}_\rho/\mu \rho$. This is the electric field of a thin charged wire perpendicular to the 2-dimensional spatial plane with charge $8\pi g/\mu$ per unit length. Hence, the instanton event describes discharging a wire that leads out of the plane of the magnetic material. The discharging wire is the condensed matter analogue of the magnetic monopole. Similarly, a static charged wire (with time-independent charge) is the analogue of the Dirac string. The resulting amount of baby-Skyrmion number violation is given by

$$B(\infty) - B(-\infty) = \frac{g}{2} e^3(0).$$ (4.20)

It is clear that a wire sticking out of a magnet can transport electric charge out of the system. From the point of view of a 2-dimensional observer confined to the inside of the magnet this process violates charge conservation.

The requirement that the Dirac string emanating from a monopole is invisible implies the Dirac quantization condition. In particular, an Aharonov-Bohm scattering experiment on the Dirac string does not yield an observable interference pattern.
In contrast to this, there is a non-trivial Aharonov-Casher effect, i.e. an observable interference pattern, when one scatters neutral particles with a non-zero magnetic moment off a static charged wire \([13, 52]\). Using the concept of a local \(SU(2)_s\) symmetry, this effect was also discussed by Anandan \([53]\).\(^2\) There is no physical analogue of the Dirac quantization condition for charged wires. In particular, there is no reason why the mathematical analogue of the quantization condition should be realized in physical systems. After all, the amount of charge per unit length in the wire is under experimental control and need not be quantized.

## 5 Generalization to Several Flavors

There are interesting modifications of the low-energy effective theory for the Goldstone bosons in QCD with more than two flavors. In particular, for \(N_f \geq 3\) flavors the Wess-Zumino-Witten term arises with a quantized prefactor \(N_c\). When one considers several coupled 2-dimensional layers of magnetic materials, the layer index may play the role of flavor. Then an analogue of the Wess-Zumino-Witten term may arise, however, its prefactor need no longer be quantized.

### 5.1 Pions, Kaons, and \(\eta\)-Mesons

For QCD with \(N_f \geq 3\) massless quarks the chiral symmetry group is \(G = SU(N_f)_L \otimes SU(N_f)_R \otimes U(1)_B\) which is spontaneously broken down to the subgroup \(H = SU(N_f)_{L=R} \otimes U(1)_B\). Hence, the Goldstone bosons are now described by fields in the coset space \(G/H = SU(N_f)\). As a result, there are \(N_f^2 - 1\) Goldstone bosons. For \(N_f = 3\) there are 8 Goldstone bosons: the 3 pions, 4 kaons, and the \(\eta\)-meson. The leading order chiral perturbation theory action is still given by eq.(2.2) as in the \(N_f = 2\) case. Since \(\Pi_3[SU(N_f)] = \mathbb{Z}\) for any \(N_f \geq 3\), the Skyrme and Goldstone-Wilczek currents of eqs.(2.5,3.4) also remain unchanged.

In contrast to the two flavor case, the homotopy group \(\Pi_4[SU(N_f)]\) is trivial for \(N_f \geq 3\). Hence, space-time-dependent Goldstone boson fields \(U(x) \in SU(N_f)\) can then always be continuously deformed into the trivial field \(U(x) = 1\). The question arises how the fermionic or bosonic nature of the Skyrmion manifests itself in the effective theory. Witten solved this problem by introducing a fifth coordinate \(x_5 \in [0,1]\) which plays the role of a deformation parameter \([35]\). He extended the 4-dimensional field \(U(x)\) to a field \(U(x, x_5)\) on a 5-dimensional hemisphere \(H^5\) whose boundary \(\partial H^5 = S^4\) is (compactified) space-time, such that \(U(x, 0) = 1\) and

\[^2\]We thank L. Stodolsky for bringing this work to our attention.
\( U(x, 1) = U(x) \). One can now construct the Wess-Zumino-Witten term [34, 35] as

\[
S_{WZW}[U] = \frac{1}{480\pi^4} \int_{H^5} d^5x \; \varepsilon_{\mu\nu\rho\sigma\lambda} \text{Tr} \left[ (U^\dagger \partial_\mu U)(U^\dagger \partial_\nu U)(U^\dagger \partial_\rho U)(U^\dagger \partial_\sigma U)(U^\dagger \partial_\lambda U) \right].
\]

(5.1)

In analogy to the case of ferromagnetic magnons, the 4-dimensional Goldstone boson physics should be independent of how the field \( U(x, x_5) \) is extended to the bulk of the fifth dimension. It should only depend on the boundary values \( U(x) \), i.e. on the Goldstone boson field in the physical part of space-time. Similar to the ferromagnetic magnon case, the integrand in eq.(5.1) is a total divergence, and it is closely related to the winding number \( \Pi_5[SU(N_f)] = Z \). If the integration in eq.(5.1) is performed over a sphere \( S^5 \) instead of the hemisphere \( H^5 \), the result is the integer winding number of \( U(x, x_5) \). Hence, modulo integers, \( S_{WZW}[U] \) gets contributions only from the boundary of \( H^5 \), i.e. from the 4-dimensional physical space-time \( S^4 \). In order to ensure that the integer contribution from the 5-dimensional bulk cancels, \( S_{WZW}[U] \) enters the path integral with the quantized prefactor \( N_c \), the number of colors,

\[
Z = \int \mathcal{D}U \exp(-S[U]) \exp(2\pi i N_c S_{WZW}[U]).
\]

(5.2)

It should be noted that eq.(5.2) is the natural extension of eq.(2.6) in the \( N_f = 2 \) case. Indeed, for \( U(x) \in SU(2) \),

\[
\exp(2\pi i N_c S_{WZW}[U]) = \text{Sign}[U]^{N_c}.
\]

(5.3)

The argument of the Wess-Zumino-Witten term is a 5-dimensional Goldstone boson field \( U(x, x_5) \in SU(N_f) \) which reduces to a 4-dimensional \( SU(2) \) field \( U(x) \) at the boundary of \( H^5 \). The argument of the sign factor, on the other hand, is just the 4-dimensional field \( U(x) \in SU(2) \). The Wess-Zumino-Witten term plays a similar role as \( \text{Sign}[U] \) in the \( N_f = 2 \) case. In particular, for odd \( N_c \) it ensures that the Skyrmion is quantized as a fermion with half-integer spin, while for even \( N_c \) it is quantized as a boson with integer spin [35].

### 5.2 Antiferromagnetic Magnons with Several Flavors

The question arises if the analogies between pions and magnons extend to several flavors. It is not clear, a priori, how to introduce additional flavors of magnons. Instead of starting from concrete condensed matter systems, we let mathematics be our guide. We will ask later if the theories that arise in this way are realized in condensed matter physics. In QCD the two flavor case is generalized to several flavors by replacing the pion field \( U(x) \in SU(2) \) by a Goldstone boson field \( U(x) \in SU(N_f) \). What should replace the magnon unit-vector field \( \vec{e}(x) \) in a generalization to several flavors? The goal is to generalize the \( SU(2)_s \) spin rotational symmetry to \( SU(N_f) \). Until now the magnon field \( \vec{e}(x) \) lived in the coset space \( S^2 = SU(2)/U(1) = CP(1) \).
This suggests the generalization to \( CP(N_f - 1) \) models. In particular, if a symmetry \( G = SU(N_f) \) gets spontaneously broken to the subgroup \( H = U(N_f - 1) \) the Goldstone bosons are described by fields in the coset space

\[
G/H = SU(N_f)/U(N_f - 1) = CP(N_f - 1).
\] (5.4)

We will now consider low-energy effective theories describing such Goldstone bosons.

Goldstone bosons of \( CP(N_f - 1) \) are described by \( N_f \times N_f \) Hermitian projection matrices \( P(x) \) that obey

\[
P(x)^\dagger = P(x), \quad \text{Tr}P(x) = 1, \quad P(x)^2 = P(x).
\] (5.5)

In the \( N_f = 2 \) (or \( CP(1) = O(3) \)) case the projection matrix is given by

\[
P(x) = \frac{1}{2}(\mathbb{1} + \vec{e}(x) \cdot \vec{\sigma}),
\] (5.6)

where \( \sigma^a = 2T^a \) are the Pauli matrices. The lowest-order chiral perturbation theory action for \( CP(N_f - 1) \) antiferromagnetic magnons is given by

\[
S[P] = \int d^2x \int_{S^1} dt \rho_s \left[ \text{Tr}(\partial_i P \partial_i P) + \frac{1}{c^2} \text{Tr}(\partial_t P \partial_t P) \right].
\] (5.7)

This action is invariant under global special unitary transformations \( g \in G = SU(N_f) \)

\[
P'(x) = g^\dagger P(x) g.
\] (5.8)

The spontaneously selected vacuum field configuration \( P(x) = \text{diag}(1,0,...,0) \) is invariant only under transformations \( g \) in the unbroken subgroup \( U(N_f - 1) \). Again, the magnons have a “relativistic” energy-momentum dispersion relation \( E = |\vec{p}|c \).

### 5.3 Ferromagnetic Magnons with Several Flavors

Ferromagnetic magnons with several flavors arise in multi-layer quantum Hall ferromagnets [36]. In these systems the layer index plays the role of flavor. Indeed, \( CP(N_f - 1) \) effective theories have already been used to describe these systems [54, 55]. For ferromagnetic magnons with several flavors the leading order chiral perturbation theory action is given by

\[
S[P] = \int d^2x \left[ \int_{S^1} dt \rho_s \text{Tr}(\partial_i P \partial_i P) - 4m \int_{H^2} dt d\tau \text{Tr}(P \partial_t P \partial_\tau P) \right].
\] (5.9)

The second term on the right-hand side of this equation is again of topological nature. The corresponding integrand is a total divergence closely related to the winding number of

\[
\Pi_2[CP(N_f-1)] = \Pi_2[SU(N_f)/U(N_f-1)] = \Pi_1[U(N_f-1)] = \Pi_1[U(1)] = \mathbb{Z}.
\] (5.10)
Again, when the integration in eq.(5.9) over the hemisphere \( H^2 \) is replaced by an integration over a sphere \( S^2 \), the term
\[
n = \frac{1}{\pi i} \int_{S^2} dt \, d\tau \, \text{Tr}(P \partial_t P \partial_\tau P)
\]
is an integer winding number. In order to ensure that the integer contribution \( n \) from the 4-dimensional bulk cancels, the prefactor
\[
\int d^2 x \ 4\pi i m = 4\pi i M
\]
must again be quantized, i.e. \( M \) is an integer or a half-integer.

### 5.4 Baby-Skyrmions with Several Flavors

Magnons with \( CP(N_f - 1) \) low-energy dynamics also support baby-Skyrmions because \( \Pi_2[CP(N_f - 1)] = \mathbb{Z} \). The corresponding integer valued topological charge
\[
B = \frac{1}{2\pi i} \int d^2 x \ \varepsilon_{ij} \text{Tr}(P \partial_i P \partial_j P)
\]
is constant in time because the topological current
\[
j_\mu = \frac{1}{2\pi i} \varepsilon_{\mu\nu\rho} \text{Tr}(P \partial_\nu P \partial_\rho P)
\]
is conserved, i.e. \( \partial_\mu j_\mu = 0 \).

In the QCD case we have seen that \( \Pi_4[SU(2)] = \mathbb{Z}(2) \) while \( \Pi_4[SU(N_f)] \) is trivial for \( N_f \geq 3 \), which gives rise to the Wess-Zumino-Witten term. In addition, \( \Pi_5[SU(N_f)] = \mathbb{Z} \) leads to the quantization condition for the prefactor \( N_c \). Similarly, for magnons \( \Pi_5[CP(1)] = \mathbb{Z} \) while \( \Pi_5[CP(N_f - 1)] \) is trivial for \( N_f \geq 3 \). This gives rise to an analogue of the Wess-Zumino-Witten term. However, since \( \Pi_4[CP(N_f - 1)] \) is trivial, the prefactor of this term needs not to be quantized. This is expected because the analogue of \( N_c \) for magnons is the anyon statistics angle \( \theta \) which is indeed not quantized.

Let us now construct the analogue of the Wess-Zumino-Witten term for magnons. Since \( \Pi_3[CP(N_f - 1)] = \{0\} \) for \( N_f \geq 3 \) space-time-dependent magnon fields \( P(x) \in CP(N_f - 1) \) are topologically trivial and can always be continuously deformed into the constant field \( P(x) = \text{diag}(1, 0, \ldots, 0) \). As before, we introduce a fourth coordinate \( \tau \in [0, 1] \) which plays the role of a deformation parameter. First, we extend the 3-dimensional field \( P(x) \) to a field \( P(x, \tau) \) on the 4-dimensional hemisphere \( H^4 \) whose boundary \( \partial H^4 = S^3 \) is (compactified) space-time, such that \( P(x, 0) = \text{diag}(1, 0, \ldots, 0) \) and \( P(x, 1) = P(x) \). The analogue of the Wess-Zumino-Witten term [56] takes the form
\[
S_{WZW}[P] = \frac{1}{4\pi^2} \int_{H^4} d^4 x \ \varepsilon_{\mu\nu\rho\sigma} \text{Tr}(P \partial_\mu P \partial_\nu P \partial_\rho P \partial_\sigma P).
\]
Again, the 3-dimensional magnon physics should be independent of how the field $P(x, \tau)$ is deformed into the bulk of the fourth dimension. It should only depend on the boundary values $P(x)$, i.e. on the magnon field in the physical part of space-time. In contrast to the QCD case where $\Pi_5[SU(N_f)] = Z$, in the magnon case $\Pi_4[CP(N_f - 1)]$ is trivial. Hence, if the integration in eq.(5.15) is performed over a sphere $S^4$ instead of the hemisphere $H^4$ the result simply vanishes. Thus, $S_{WZW}[P]$ gets contributions only from the boundary of $H^4$, i.e. from the 3-dimensional physical space-time $S^3$. In contrast to the QCD case, no bulk ambiguity arises and thus the prefactor of the Wess-Zumino-Witten term need not be quantized. The path integral then takes the form

$$Z = \int \mathcal{D}P \exp(-S[P]) \exp(i\theta S_{WZW}[U]),$$

(5.16)

where $\theta$ is again the (unquantized) anyon statistics angle. Indeed eq.(5.16) is the natural extension of eq.(2.18) in the $N_f = 2$ case. In particular, for $P(x)$ as in eq.(5.6) we find

$$\exp(i\theta S_{WZW}[P]) = \exp(i\theta H[\vec{e}]).$$

(5.17)

The argument of the Wess-Zumino-Witten term is a 4-dimensional magnon field $P(x, \tau) \in CP(N_f - 1)$ which reduces to a 3-dimensional $CP(1)$ field $P(x)$ at the boundary of $H^4$. The argument of the Hopf term, on the other hand, is just the 3-dimensional field $\vec{e}(x) \in S^2$. For $N_f \geq 3$ the Wess-Zumino-Witten term plays a similar role as the Hopf term for $N_f = 2$. In particular, it determines that the baby-Skyrmion is quantized as an anyon with statistics angle $\theta$.

It is interesting to consider the effects of gauging the $SU(N_f)$ symmetry. Under an $SU(N_f)$ gauge transformation the magnon field transforms as

$$P'(x) = g^\dagger(x)P(x)g(x),$$

(5.18)

and the corresponding non-Abelian gauge field transforms as

$$W'_\mu(x) = g^\dagger(x)(W_\mu(x) + \partial_\mu)g(x).$$

(5.19)

The gauged Goldstone-Wilczek current then takes the form

$$j^{GW}_\mu = \frac{1}{2\pi i} \varepsilon_{\mu\nu\rho} \text{Tr}(PD_\nu PD_\rho P + \frac{1}{2}PW_{\nu\rho}),$$

(5.20)

which is again conserved. Hence, just as in the $N_f = 2$ case, there is no analogue of the ’t Hooft anomaly in the baryon number current.

Let us also consider the modifications of the Wess-Zumino-Witten term when the $SU(N_f)$ symmetry is gauged. In the QCD context this has been done in [35, 57–59]. For magnons, the gauge variation of the Wess-Zumino-Witten term is given by

$$S_{WZW}[P] - S_{WZW}[P'] = \frac{1}{4\pi^2} \int d^3x \varepsilon_{\mu\nu\rho}[2(\partial_\mu gg^\dagger)\partial_\nu P\partial_\rho PP$$

$$+ 2(\partial_\mu gg^\dagger)P(\partial_\nu gg^\dagger)\partial_\rho PP + \frac{2}{3}(\partial_\mu gg^\dagger)P(\partial_\nu gg^\dagger)P(\partial_\rho gg^\dagger)P$$

$$- (\partial_\mu gg^\dagger)P(\partial_\nu gg^\dagger)(\partial_\rho gg^\dagger)P - (\partial_\mu gg^\dagger)\partial_\nu P(\partial_\rho gg^\dagger)P].$$

(5.21)
Note that here the integration extends over the ordinary (compactified) \((2 + 1)\)-dimensional space-time only. The gauge variation can be compensated by additional contributions to the Wess-Zumino-Witten term, which then takes the form

\[
S_{WZW}[P, W_\mu] = \frac{1}{4\pi^2} \int_\mathcal{M} d^4 x \varepsilon_{\mu\nu\rho\sigma} \text{Tr}(P \partial_\mu P \partial_\nu P \partial_\rho P)
+ \frac{1}{4\pi^2} \int d^3 x \varepsilon_{\mu\nu\rho}(P \partial_\mu PW_\nu PW_\rho - \partial_\mu PPW_\nu PW_\rho)
+ 2P \partial_\mu P \partial_\nu PW_\rho + \frac{2}{3} P \partial_\mu PW_\nu PW_\rho + PW_\mu P \partial_\nu W_\rho). \tag{5.22}
\]

In some multi-layer quantum Hall ferromagnets \(SU(2)_s\) is a subgroup of \(SU(N_f)\). In these cases, Fröhlich and Studer’s observations on the Pauli equation indeed imply that this subgroup should be turned into a local symmetry, with the electromagnetic field strengths \(\vec{E}\) and \(\vec{B}\) appearing as \(SU(2)_s\) non-Abelian vector potentials. This is analogous to the standard model of particle physics where not the full chiral symmetry, but only its \(SU(2)_L \otimes U(1)_Y\) subgroup, is gauged. Besides the Goldstone-Wilczek term, the gauged Wess-Zumino-Witten term contributes to anomalous electromagnetic effects. Again, this is analogous to QCD [46].

## 6 Summary and Conclusions

In this paper we have compared the low-energy physics of pions and magnons and we have found a surprising correspondence between various physical phenomena in these apparently quite different particle and condensed matter physics systems. For the two flavor case \((N_f = 2)\) the analogies between pion and magnon physics are summarized in table 1. Similarly, table 2 summarizes the correspondences in the multi-flavor case \(N_f \geq 3\). As we have seen, the topological structures in \((3 + 1)\)-dimensional pion and \((2 + 1)\)-dimensional magnon effective theories are very similar. Just based upon symmetry breaking patterns, low-energy effective field theory provides us with a universal framework for describing Goldstone boson physics, relating systems as different as pions in QCD, undoped antiferromagnetic precursors of high-temperature superconductors, and single- or multi-layer quantum Hall ferromagnets.

We have investigated low-energy effective theories for magnons without specifying a concrete magnetic material. The predictive power of the effective theory results from the fact that the details of the microscopic model enter the effective theory only through the values of some low-energy constants. For example, to leading order, the dynamics of magnons is determined by the spin stiffness \(\rho_s\), as well as by the spin wave velocity \(c\) (for antiferromagnets), or by the magnetization density \(m\) (for ferromagnets). At leading order, the electromagnetic properties of magnons are determined by the local \(SU(2)_s\) symmetry with the anomalous magnetic moment \(\mu\) being the only low-energy parameter. Anomalous electromagnetic processes of magnons are due to the Goldstone-Wilczek term which, due to the relation \(Q = -F\)
Table 1: Analogies between pion and magnon physics.

| Quantity                          | Pions                              | Magnons                           |
|-----------------------------------|------------------------------------|-----------------------------------|
| global symmetry $G$               | $SU(2)_L \otimes SU(2)_R$         | $SU(2)_s$                         |
| unbroken subgroup $H$             | $SU(2)_{L=R}$                      | $U(1)_s$                          |
| Goldstone field in $G/H$          | $U(x) \in SU(2)$                  | $\vec{e}(x) \in S^2$             |
| coupling strength                 | pion decay constant $F_\pi$       | spin stiffness $\rho_s$           |
| propagation speed                 | velocity of light                 | spin wave velocity                |
| topological solitons              | Skyrmions                         | baby-Skyrmions                    |
| topological charge                | baryon number $B$                 | electron number $F$               |
| soliton homotopy                  | $\Pi_3[SU(2)] = \mathbb{Z}$       | $\Pi_3[S^2] = \mathbb{Z}$        |
| soliton statistics                | bosons or fermions                | anyons                            |
| statistics homotopy               | $\Pi_4[SU(2)] = \mathbb{Z}(2)$    | $\Pi_3[S^2] = \mathbb{Z}$        |
| statistics factor                 | $\text{Sign}[U]^{N_c}$            | $\exp(i\theta H|\vec{e}|)$       |
| discrete symmetry                 | $G$-parity                        | $\vec{e} \to -\vec{e}$ symmetry  |
| local symmetry                    | electroweak $SU(2)_L \otimes U(1)_Y$ | local $SU(2)_s \otimes U(1)_{em}$ |
| electromagnetic decay             | $\pi^0 \to \gamma\gamma$         | magnon $\to \gamma\gamma$        |
| conversion in $B$ field           | photon-axion                      | photon-magnon                     |
| soliton decay catalyzer           | magnetic monopole                 | charged wire                      |

Table 2: Analogies between pion, kaon, and $\eta$-meson physics and the physics of magnons with several flavors.

| Quantity                          | Pions, Kaons, and $\eta$-Mesons | $N_f$ Magnon Flavors |
|-----------------------------------|----------------------------------|----------------------|
| global symmetry $G$               | $SU(N_f)_L \otimes SU(N_f)_R$   | $SU(N_f)$            |
| unbroken subgroup $H$             | $SU(N_f)_{L=R}$                  | $U(N_f - 1)$         |
| Goldstone field in $G/H$          | $U(x) \in SU(N_f)$               | $P(x) \in CP(N_f - 1)$ |
| $SU(2)$ doublets                  | flavors ($u, d$), ($c, s$)       | spins ($\uparrow, \downarrow$) in layers |
| additional label                  | generation index                 | layer index           |
| soliton homotopy                  | $\Pi_3[SU(N_f)] = \mathbb{Z}$    | $\Pi_2[CP(N_f - 1)] = \mathbb{Z}$ |
| soliton statistics                | bosons or fermions               | anyons                |
| statistics homotopy               | $\Pi_4[SU(N_f)] = \{0\}$        | $\Pi_3[CP(N_f - 1)] = \{0\}$  |
| WZW term                          | $\exp(2\pi i N_c S_{WZW}[U])$   | $\exp(i\theta S_{WZW}[P])$     |
| WZW homotopy                      | $\Pi_5[SU(N_f)] = \mathbb{Z}$    | $\Pi_4[CP(N_f - 1)] = \{0\}$  |
| statistics parameter              | quantized $N_c$                  | unquantized $\theta$         |

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between the electric charge and fermion number, is proportional to the anyon statistics parameter $\theta$. Determining the values of low-energy parameters (such as $\rho_s$, $c$, $m$, and $\theta$) for a concrete material is a non-trivial task. In general, one must use experiments in order to extract this information. For simple model systems, such as the Heisenberg antiferromagnet, $\rho_s$ and $c$ have been obtained in very accurate quantum Monte Carlo calculations [60, 61].

In the context of the anomalous electromagnetism of magnons, the most interesting parameter is $\theta$, which determines the electric charge of baby-Skyrmions. For quantum Hall ferromagnets it is known that baby-Skyrmions have $\theta/\pi = \nu$, where $\nu$ is the Landau level filling fraction [18]. Hence, for $\nu = 1$ baby-Skyrmions are fermions and carry the charge of one electron. At fractional fillings, on the other hand, baby-Skyrmions, just like Laughlin quasi-particles, are anyons with fractional statistics and with fractional electric charge.

For the antiferromagnetic precursor insulators of high-temperature superconductors the value of $\theta$ seems to be less clear. Let us discuss three different possible scenarios:

- Since no Hopf term seems to be generated in the antiferromagnetic quantum Heisenberg model [40–44], one might conclude that $\theta = 0$. In that case baby-Skyrmions are neutral bosons and there is no vertex for magnon decay into two photons. As a consequence, photon-magnon conversion in a magnetic field is then impossible. Since neutral bosonic baby-Skyrmions contain no net electrons, they are not directly affected by doping of the antiferromagnet. If $\theta$ indeed vanishes for the antiferromagnetic precursor insulators of high-temperature superconductors, one cannot hope to learn anything about the destruction of antiferromagnetism by doping or about the preformation of Cooper pairs from the effective theory.

- A more interesting scenario arises for $\theta = \pi$. Then the baby-Skyrmions are quasi-particles with electron quantum numbers. In that case, photon-magnon conversion in a magnetic field is possible. Doping forces a net number of baby-Skyrmions into the system, which may explain the destruction of antiferromagnetism due to the hedgehog form of the baby-Skyrmion’s staggered magnetization. Furthermore, investigating the forces between baby-Skyrmions mediated by magnon or photon exchange, one may hope to learn something about Cooper pair preformation within the effective theory. Still, in this case one must understand the apparent conflict with the results of [40–44].

- The case $\theta = 2\pi$ is also interesting. Then, in agreement with [40–44], no Hopf term is generated. However, the resulting bosonic baby-Skyrmions now have fermion number two and thus carry the electric charge $-2e$. In that case, the baby-Skyrmions themselves represent preformed Cooper pairs. Again, doping forces baby-Skyrmions into the system, thus providing an explanation
for the destruction of antiferromagnetism, and photon-magnon conversion is possible (even at a higher rate due to the larger value of $\theta$). Of course, if the baby-Skyrmions themselves are the preformed Cooper pairs, one cannot learn anything about the mechanism of their formation in the framework of the effective theory. Interestingly, this last scenario is analogous to QCD if one identifies electrons with quarks. In QCD $N_c$ quarks are confined inside a baryon which manifests itself as a Skyrmion. Although the prefactor $N_c$ of the Wess-Zumino-Witten term counts the number of baryon constituents, the effective theory does not shed any light on the mechanism by which quarks are confined inside baryons. Similarly, in the $\theta = 2\pi$ scenario, two electrons are “confined” inside a preformed Cooper pair. The analogue of $N_c$ is $\theta/\pi = 2\nu \neq 0$ which again counts the number of constituents inside a baby-Skyrmion and appears as the prefactor of the Wess-Zumino-Witten term.

To decide if one of the above scenarios is realized in the antiferromagnetic precursors of high-temperature superconductors requires non-trivial insight into these materials. On the one hand, one may hope to gain theoretical insight from numerical simulations of the Hubbard model. On the other hand, one may perform appropriate experiments. Motivated by the decay of the neutral pion into two photons, the analogy with magnons has led us to investigate if magnons can decay into photons. Indeed, if baby-Skyrmions carry electric charge, the effective field theory predicts such a decay. The condensed matter analogue of photon-axion conversion, namely photon-magnon conversion in an external magnetic field relies on the magnon-two-photon vertex and may perhaps be realizable in experiments. If so, it should be observable in quantum Hall ferromagnets for which $\theta/\pi = \nu \neq 0$. If photon-magnon conversion was also observed in the antiferromagnetic precursors of high-temperature superconductors, this would show unambiguously that baby-Skyrmions in these materials also carry electric charge. Even the value of $\theta$ can, at least in principle, be determined in this way. The practical feasibility of photon-magnon conversion experiments will be investigated in future studies.

Some condensed matter effects discussed in this paper were derived from analogies which are well understood in particle physics. One may also ask if we can learn something new about particle physics, based on phenomena that are well understood in condensed matter. In this context, the layer index of a multi-layer quantum Hall ferromagnet suggests itself as a condensed matter analogue of flavor, or (more precisely) of quark-lepton generation number. While the origin of flavor or generation number is a mystery in particle physics, the layer index in 2-dimensional quantum Hall systems obviously originates from the different locations of electrons in the third dimension. From this point of view, multi-layer quantum Hall systems seem to support the ideas behind brane worlds in which different flavors are localized in different positions of an additional fourth spatial dimension [62]. In this context, attempts to use condensed matter language in order to describe fundamental physics should also be mentioned. This includes D-theory in which classical fields emerge
dynamically by dimensional reduction of discrete variables [63], emergent relativity [64], effective gravity from quantum liquids [65], as well as higher-dimensional analogs of the quantum Hall effect [66, 67], artificial light, and other recent developments [68]. Apparently, exploring the relations between particle and condensed matter physics remains promising.

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