Baby Skyrme models for a class of potentials

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Abstract

We consider a class of (2+1) dimensional baby Skyrme models with potentials that have more than one vacuum. These potentials are generalisations of old and new baby Skyrme models; they involve more complicated dependence on $\phi_3$. We find that when the potential is invariant under $\phi_3 \to -\phi_3$ the configurations corresponding to the baby skyrmions lying “on top of each other” are the minima of the energy. However, when the potential breaks this symmetry the lowest field configurations correspond to separated baby skyrmions. We compute the energy distributions for skyrmions of degrees between one and eight and discuss their geometrical shapes and binding energies. We also compare the 2-skyrmion states for these potentials. Most of our work has been performed numerically with the model being formulated in terms of three real scalar fields (satisfying one constraint).

1 Introduction

The Skyrme model describes a non-linear theory for SU(2) valued fields which has soliton solutions. Although the potential term is optional in the (3+1) dimensional nuclear Skyrme model its presence is necessary in (2+1) dimensions to ensure the stability of these solitonic solutions. In this paper we discuss multisolitons in a two-dimensional version of the model for a class of potentials which generalises the previously studied cases. In the literature we can find three specific potentials that have been studied in some detail.

The holomorphic model has only one stable solution (describing one skyrmion) and this solution has a simple analytical form. The old baby Skyrme model has stable solutions for any number of skyrmions. In they were shown to lead to a crystalline lattice of skyrmions. The new baby

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1this name was introduced in
Skyrme model was studied in detail in [9] where it was shown that its solutions corresponded to field configurations with radially symmetrical energy densities - which correspond to many skyrmions lying “on top of each other”. We generalise these studies by considering potentials with a more general dependence on $\phi_3$ thus leading to models with more vacua.

2 A Skyrme model for a class of potentials

The Skyrme model was first proposed by T.H.R. Skyrme [5] in 1960. Its classical solutions fall into various classes characterised by a topological number to be identified with the baryon number. Thus the model can describe mesons as well as various baryon configurations. The baby Skyrme models [6] are $(2+1)$ dimensional versions of the Skyrme model and its topologically nontrivial solutions are called baby skyrmions.

The Lagrangian density of the model contains three terms, from left to right: the pure $S^2$ sigma model, the Skyrme and the potential terms:

$$
\mathcal{L} = \partial_{\mu} \vec{\phi} \cdot \partial^{\mu} \vec{\phi} - \theta_S \left[ (\partial_{\mu} \vec{\phi} \cdot \partial^{\mu} \vec{\phi})^2 - (\partial_{\mu} \vec{\phi} \cdot \partial_{\nu} \vec{\phi}) (\partial^{\mu} \vec{\phi} \cdot \partial^{\nu} \vec{\phi}) \right] - \theta_V V(\vec{\phi}).
$$

The vector $\vec{\phi}$ is restricted to lie on a unit sphere $S^2$ hence $\vec{\phi} \cdot \vec{\phi} = 1$.

Note that to have finite potential energy the field at spatial infinity cannot depend on the polar angle $\theta$.

$$
\lim_{r \to \infty} \vec{\phi}(r, \theta) = \vec{\phi}(0).
$$

Hence this boundary condition defines an one-point compatification of $R_2$, allowing us to consider $\vec{\phi}$ on the extended plane $R_2 \bigcup \infty$ topologically equivalent to $S^2$. In consequence, the field configurations are maps

$$
\mathcal{M} : S^2 \to S^2.
$$

which can be labeled by an integer valued topological index $Q$:

$$
Q = \frac{1}{4\pi} \epsilon^{abc} \int dxdy \phi_a (\partial_b \phi_b) (\partial_c \phi_c).
$$

As a result of this non-trivial mapping the model has topologically nontrivial solutions which describe “extended structures”, namely, baby skyrmions. The different choices of the potential term lead to various shapes of the energy density of these baby skyrmions.

The equation of motion for a general potential that depends on $\phi_3$ takes the form:

$$
\partial_{\mu} \partial^\mu \phi_a - (\vec{\partial} \cdot \partial_{\nu} \vec{\phi}) \phi_a - 2\theta_S [(\partial_{\nu} \vec{\phi} \cdot \partial^\nu \vec{\phi}) \partial_{\mu} \partial^\mu \phi_a + (\partial_{\mu} \partial^\nu \vec{\phi} \cdot \partial^{\nu} \vec{\phi}) \partial_{\nu} \phi_a]
$$

$$
- (\partial_{\nu} \vec{\phi} \cdot \partial^\nu \vec{\phi}) \partial_{\nu} \partial_{\mu} \phi_a - (\partial_{\nu} \vec{\partial} \cdot \partial^{\nu} \vec{\phi}) \partial_{\nu} \phi_a + (\partial_{\mu} \vec{\partial} \cdot \partial^{\mu} \vec{\phi}) \partial_{\nu} \vec{\phi} \phi_a
$$

$$
- (\partial_{\nu} \vec{\phi} \cdot \partial_{\mu} \vec{\phi}) (\partial^{\nu} \vec{\phi} \cdot \partial^{\nu} \vec{\phi}) \phi_a + \frac{1}{2} \theta_V \frac{dV}{d\phi_3} (\delta_{a3} - \phi_a \phi_3) = 0.
$$

It is convenient to rewrite this equation as:

$$
\partial_{tt} \phi_a = K_{ab}^{-1} \mathcal{F}_b \left( \vec{\phi}, \partial_t \vec{\phi}, \partial_t \vec{\phi} \right)
$$

(6)

with

$$
K_{ab} = (1 + 2\theta_S \delta_{a3} - \partial_t \phi_a \partial_t \phi_b).
$$

(7)
and a rather complicated expression for $F_b$.

Then to find a solution to this equation we invert the matrix $K$ and simulate the time evolution by a 4th order Runge Kutta method supplemented by the imposition of a correction due to the constraint $\vec{\phi} \cdot \vec{\phi} = 1$.

The kinetic and potential energy densities of the baby Skyrme model are:

$$K = \left(\partial_t \vec{\phi} \cdot \partial_t \vec{\phi}\right) \left(1 + 2\theta_S \left(\partial_i \vec{\phi} \cdot \partial_i \vec{\phi}\right)\right) - 2\theta_S \left(\partial_i \vec{\phi} \cdot \partial_i \vec{\phi}\right)^2$$

and

$$V = \left(\partial_i \vec{\phi} \cdot \partial_i \vec{\phi}\right) + \theta_S \left[\left(\partial_i \vec{\phi} \cdot \partial_i \vec{\phi}\right)^2 - \left(\partial_i \vec{\phi} \cdot \partial_j \vec{\phi}\right) \left(\partial_i \vec{\phi} \cdot \partial_j \vec{\phi}\right)\right] + \theta V(\phi_3).$$

The potentials we want to consider in this paper are of the form

$$V(\phi_3) = (1 - \phi_3)G(\phi_3),$$

where $G(\phi_3) = 1$ or $(1 + \phi_3)$ or $\phi_3^2$ or $(1 + \phi_3)\phi_3^2$. Thus this potential vanishes at $\phi_3 = 1$ and also at $\phi_3 = -1$ and/or $\phi_3 = 0$.

The total energy takes the form:

$$E = \int dxdy \left(V + K\right).$$

### 3 Static Solutions

In this paper we are primarily interested in finding minimal energy configurations corresponding to many skyrmions. We want to see, for different forms of the potential, what such configurations are like and what properties they have.

We will consider a numerically found configuration to be a multiskyrmion if it satisfies certain numerical checks for a local minimum and if, moreover, its energy satisfies

$$E_n < E_k + E_\ell \quad \text{for all integers } 1 < \ell \text{ and } k < n \text{ such that } k + \ell = n$$

We have included this condition in our definition because we want multisoliton to be stable with respect to decay into multiskyrmions of smaller degree.

There are many ways of finding static solutions. First, as the potential is only a function of $\phi_3$, there is a symmetry corresponding to rotations around the $\phi_3$ axis. So choosing the spatial dependence conveniently it is clear that fields which correspond to a generalised multihedgehog ansatz will be static solutions of the equations of motion \[ B\]. Such fields are given by

$$\vec{\phi} = \begin{pmatrix} \sin[f(r)] \cos(n\theta - \chi) \\ \sin[f(r)] \sin(n\theta - \chi) \\ \cos[f(r)] \end{pmatrix}.$$  

where $(r, \theta)$ are polar coordinates in the $x-y$-plane and the function $f(r)$, called the profile function, is required to satisfy certain boundary conditions to be specified below. The angle $\chi$ is arbitrary, but fields with different $\chi$ are related by an iso-rotation and are therefore degenerate in energy. $n$ is a non-zero integer and equals to the topological charge.

For the $\phi$ field be regular at the origin the profile function $f(r)$ has to be satisfy:

$$f(0) = m\pi,$$

where $m$ is an integer.
We choose the vacuum at infinity to be $\vec{\phi}^0 = (0, 0, 1)$ and this results in another boundary condition, namely:

$$
\lim_{r \to \infty} f(r) = 0.
$$

(13)

The total energy of the field configuration then takes the form:

$$
E = (4\pi)^{1/2} \int_0^{\infty} r dr \left( f'^2 + n^2 \sin^2 f \left(1 + 2\theta s f''^2\right) + \theta V \tilde{V}(f)\right),
$$

(14)

where $f' = \frac{df}{dr}$ and $\tilde{V}(f) = V(\phi_3)$. To determine the profile function $f(r)$ we treat (14) as a variational problem and we get a second-order ODE for $f$:

$$
\left( r + \frac{2\theta s n^2 \sin^2 f}{r} \right) f'' + \left( 1 - \frac{2\theta s n^2 \sin^2 f}{r^2} + \frac{2\theta s n^2 \sin f f'}{r} \right) f' - \frac{n^2 \sin f \cos f}{r} = 0.
$$

(15)

This equation then has to be solved numerically (ie via the shooting method).

The topological charge takes the form:

$$
Q = -\frac{n}{2} \int_0^{\infty} r dr \left( \frac{f' \sin f}{r} \right) = \frac{n}{2} [\cos f(\infty) - \cos f(0)].
$$

(16)

This equation shows that to have an integer value for the topological charge, $m$ in (12) must be an odd number. In this paper we consider the solutions with $m = 1$.

The behaviour of solutions of this equation near the origin and for large $r$ can be deduced analytically and was also discussed in [3]. The result is that

- For small $r$, the profile function behaves as
  $$
  f \simeq \pi + C_n r^n
  $$
  (17)
  and so
  $$
  f' \simeq nC_n r^{n-1}
  $$
  (18)
  as long as $\frac{dV(f)}{df}$ tends to zero at this point.

- At large $r$, the ODE reduces to
  $$
  f'' + \frac{1}{r} f' - \frac{n^2}{r^2} f - \frac{\theta V}{2} \frac{d\tilde{V}(f)}{df} \big|_{\text{large } r} = 0.
  $$
  (19)

Using the boundary conditions, we see that if $\frac{d\tilde{V}(f)}{df} \big|_{\text{large } r} \to 1$ the profile function decreases exponentially

$$
f(r) \rightarrow \frac{1}{\sqrt{\theta V r}} \exp(-\theta V r).
$$

(20)

This means that the potential localizes the skyrmion exponentially.

Of course we do not know whether such fields are the global minima of the energy. To check this we can perform a numerical simulation of (6) taking our derived field configurations (11) as initial conditions. We perturb them a little and then evolve them according to (6) with an extra dissipative term - $\gamma \partial_t \phi_a$ added to the right hand side of (6). We vary our perturbation and look at the configurations and their energies that our fields finally settle at. When the original configurations
are the global minima of the energy the perturbed fields evolved back to them; otherwise (for sufficiently large perturbations) the fields evolve to other static solutions and, hopefully, the global minima. Another possibility, in particular to check whether these solutions are global minima, is to start with a more general field configuration corresponding to \( n \) skyrmions, say, \( n \) skyrmions located on a circle, and then evolve it using dissipative dynamics. In this case the circular set-up of \( n \) 1-skyrmions with relative iso-orientation \( \delta \chi = \frac{2\pi}{n} \) was obtained by using the combination of 1-skyrmions determined by the hedghog field \( (\Pi) \) and combined together as discussed in the appendix.

The actual outcome (i.e. which fields are the global minima) depends on the form of the potential. In the next section we discuss the potentials that we have used in our simulations.

4 Different potentials

4.1 General comments

A skyrmion is a \( S^2 \rightarrow S^2 \) map and so as the field at spatial \( \infty \) corresponds to \( \phi_3 = +1 \) (i.e. the “North pole” of the field \( S^2 \)) and the skyrmion’s position is at the point for which \( \phi_3 = -1 \). If \( V(\phi_3 = -1) \) does not vanish it costs energy for the field to take this value; hence in this case we expect the skyrmions to repel when they are brought “on top of each other”. Thus the minimal energy multiskyrmion configurations should be different when \( V(\phi_3 = -1) = 0 \) and \( \neq 0 \). In fact, this was seen in the earlier studies; in the “new baby Skyrme” model \( (V(\phi_3) = 1 - \phi_3^2) \) the skyrmions lie on “top of each other” while for the “old baby Skyrme” model \( (V(\phi_3) = 1 - \phi_3) \) they are separated from each other by finite distances. We have repeated these simulations and have extended them to a larger number of skyrmions. Below, we present our results:

4.2 \( V = (1 + \phi_3)^4 \)

The first potential studied in the baby Skyrme model was \( V = (1 + \phi_3)^4 \) \( (\Pi, \Pi, \Pi) \). This potential was chosen because the equations of motion have an analytic static solution (for \( n = 1 \)) of the form \( W = \lambda (x + iy) \), where \( \lambda \) is related to \( \theta_S \) and \( \theta_V \). The resultant soliton is only polynomially localised. Two such skyrmions repel each other and scatter at 90 degrees when sent towards each other with sufficient speed. The model has further solutions corresponding to several solitons “on top of each other”; but these solutions are unstable. When perturbed the solitons separate and move away to infinity. Thus this model have no stable multi-skyrmion solutions.

4.3 \( V = 1 - \phi_3 \)

This potential was studied in some detail in \( \Pi \) and \( \Pi \). As the value of the potential at \( r = 0 (\phi_3 = -1) \) is \( V(\phi_3) = 2 \) we expect some repulsion of the skyrmions when they are close together. Thus a field configuration of \( n \) skyrmions on top of each other is unstable. This was studied a little in \( \Pi \) where it was shown that as a result of this repulsion the minimal energy multi-skyrmion field configurations produce nice lattice-like patterns. We have repeated these studies and extended them further to larger number of skyrmions. As there appear to be many solutions we have performed many simulations starting with different initial conditions and different perturbations.

Our results show a rather complicated pattern of minimal energy field configurations. The results are presented in table 1. We use the coefficients of \( \Pi \) i.e add a factor \( \frac{1}{2} \) to the sigma model term, \( \theta_S = 0.25 \) and \( \theta_V = 0.1 \).

All configurations are built out of 1, 2 and 3 skyrmions. In fig. 1 we show the energy densities of a pair and a triple.
A pair of skyrmions is very bound, more bound than a triple, and the pairs and triples are further bound for larger values of $n$. Thus the binding per skyrmion of a configuration of 4 and 5 skyrmions is less that of a single pair or a single triple, respectively.

The first more interesting case is of 6 skyrmions which seems to have several bound states. Of these, the state corresponding to 3 pairs is the most bound, the other two (2 triples and 6 individual ones) are at best only local minima. For 7 skyrmions the lowest energy configuration corresponds to 2 pairs and a triple, which can be thought of a bound state of 4 and 3. The other state, corresponding to a triple sandwiched between two pairs has a higher energy and is at most a local minimum. In fig 2, we present energy densities of these two configurations.

For higher $n$ the situation becomes even more complicated; the lowest energy states always involve sets of pairs, but there seem to be other states involving triples and even singles.

One question one can ask is whether the $n = 2$ state involves two skyrmions “on top of each other” or slightly displaced. When started from the original configuration of two skyrmions “on top of each other” slightly perturbed the system evolves a little, but this evolution is comparable to the original perturbation. Thus we believe the two skyrmions are slightly displaced, but this displacement is almost infinitesimal.

### 4.4 $V = 1 - \phi_3^2$

The model with this potential was investigated in great detail by Weidig [9]. As both $\phi_3 = \pm 1$ correspond to the vacuum there is no energy argument which stops the skyrmions from “lying on top of each other”. This is in fact what Weidig saw in his simulations; the states found by the shooting method (and so corresponding to many skyrmions “on top of each other”) are the minima of the energy. Thus the energy densities of lowest energy multiskyrmion configurations have a ring-like structure.
Table 2: Multi-skyrmions of the baby Skyrme model with \( V = \phi_3^2(1 - \phi_3^2) \)

| Charge | Energy | Energy per skyrmion | Break-up modes | Ionisation Energy |
|--------|--------|---------------------|----------------|------------------|
| 1      | 1.281  | 1.281               | -              | -                |
| 2      | 2.332  | 1.166               | 1 + 1          | 0.231            |
| 3      | 3.430  | 1.143               | 2 + 1          | 0.184            |
| 4      | 4.540  | 1.135               | 2 + 2          | 0.125            |
|        |        |                     | 3 + 1          | 0.172            |
| 5      | 5.654  | 1.131               | 3 + 2          | 0.107            |
|        |        |                     | 4 + 1          | 0.166            |
| 6      | 6.771  | 1.129               | 3 + 3          | 0.087            |
|        |        |                     | 4 + 2          | 0.099            |
|        |        |                     | 5 + 1          | 0.164            |
| 7      | 7.891  | 1.127               | 4 + 3          | 0.078            |
|        |        |                     | 5 + 2          | 0.095            |
|        |        |                     | 6 + 1          | 0.162            |
| 8      | 9.011  | 1.126               | 4 + 4          | 0.067            |
|        |        |                     | 5 + 3          | 0.072            |
|        |        |                     | 6 + 2          | 0.092            |
|        |        |                     | 7 + 1          | 0.161            |

4.5 \( V = \phi_3^2(1 - \phi_3^2) \)

The extra factor \( \phi_3^2 \) in this potential causes it to have an additional zero and so the vacuum structure is even richer. However, like for the \( 1 - \phi_3^2 \) potential, there are no energetic arguments which would disfavour the skyrmions lying “on top of each other”. Indeed, this is what the simulations have shown; and we have checked this by starting with the hedgehog ansatz and \( n \) solitons on a circle. The simulations have shown that the hedgehog ansatz configurations are the minimal-energy solutions. The shapes of energy density have radial symmetry and resemble the configurations of Weidig. The difference is in the energy levels (in this potential the energy per skyrmion decreases monotonically with \( n \)). Our results are presented in table 2. In this (and the next table) we have added a factor \( \frac{1}{2} \) to the sigma term and chose the \( \theta \) coefficients to be \( \theta_S = 0.2 \) and \( \theta_V = 0.05 \). The values of the energies are divided by \( 4\pi \).

4.6 \( V = \phi_3^2(1 - \phi_3) \)

Like in the previous case the extra factor \( \phi_3^2 \) changes the nature of the solutions. In a way the effects due to this potential are closer to those of the “old baby Skyrme model”, in that \( V(\phi_3 = -1) \neq 0 \).

We have found that for this potential only the hedgehog ansatz with \( n = 1 \) has minimal-energy and that for \( 1 < n \leq 8 \) the skyrmions “on top of each other”, when perturbed, evolve into configurations corresponding to \( n \)-skyrmions that lie on regular polygons. When starting from a circular set-up for this potential we have calculated the total energy of \( n \) 1-skyrmions as a function of the radius of the circle to get the initial state with minimum energy. This way the initial state settles down to the stable state sooner. We have confirmed this by performing also some simulations with different radii. We have also added to the initial configuration a non-symmetrical exponentially decaying perturbation to check whether the resultant multi-soliton states are stable, and whether,
Table 3: Multi-skyrmions of the baby Skyrme model with $V = \phi_3^2(1 - \phi_3)$ after dissipation, we reproduce the previous solutions.

In fig. 3 we present picture of the energy density of the lowest energy field configurations involving 6 skyrmions.

Our results show that this time (when we compare this case with $V = 1 - \phi_3$) the bindings are more comparable; in fact three skyrmions are more bound than two, and so the pattern is very different. It is interesting to note that the most bound system involves 6 skyrmions.

Our results on the binding energies etc are presented in table 3.

## 5 Comparison of potentials between 2 skyrmions

We have also looked at the “potentials” between two skyrmions in our models. To do this we had to decide how to construct field configurations involving two skyrmions. We decided to do this as follows: We have taken $f(r)$ for $n = 1$ and computed the field $W$ from [31]. Then we combined two such fields with $\delta \chi = \pi$ and varied the distance between them computing the energy as a function of the distance. We examined the following ways of combining 2 skyrmions:

\[
W = W_1 + W_2 \quad (21)
\]
\[
W = W_1 - W_2 \quad (22)
\]
\[
W = W_1 + W_2 - W_1 W_2 \quad (23)
\]
\[
W = W_1 - W_2 + W_1 W_2 . \quad (24)
\]

As could be expected, we have found that for very small $r$ none of them gives reliable results. However, when two skyrmions are well separated the combination (23) gives us the lower energy.
Table 4: Comparison of energy per skyrmion between $V = 1 - \phi_3^2$ and $V = \phi_3^2(1 - \phi_3^2)$

| Charge | Energy per skyrmion for $V = 1 - \phi_3^2$ | Energy per skyrmion for $V = \phi_3^2(1 - \phi_3^2)$ |
|--------|------------------------------------------|---------------------------------|
| 1      | 1.564                                    | 1.564                           |
| 2      | 1.405                                    | 1.384                           |
| 3      | 1.371                                    | 1.344                           |
| 4      | 1.358                                    | 1.329                           |
| 5      | 1.352                                    | 1.322                           |
| 6      | 1.349                                    | 1.319                           |

Table 4 shows the energy per skyrmion for the new baby skyrmion model and for the model with the potential $V = \phi_3^2(1 - \phi_3^2)$ respectively. The results indicate that a further minimum in the potential increases the binding of the skyrmions (i.e., the multiskyrmions have lower energies). The same seems to be true for the other two potentials. We have also tried to understand the difference between the potentials $V = \phi_3^2(1 - \phi_3^2)$ (in the $\phi_3^2$) the minimum of the potential is very close to $r = 0$ while for $V = \phi_3^2(1 - \phi_3^2)$ it is clearly much further

6 Further Comments and Conclusions

The extra coefficient $\phi_3^2$ causes the potential to have a further minimum (at $\phi_3 = 0$). The question then arises of how this change affects the properties of the solutions. To perform such a comparison of multi-skyrmions of potentials $V = \phi_3^2(1 - \phi_3^2)$ and $V = \phi_3^2(1 - \phi_3^2)$, we have chosen the parameter $\theta$’s in both models to be such that the one skyrmion solutions of these models have the same energy.

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We have also tried to understand the difference between the potentials $V = \phi_3^2(1 - \phi_3^2)$ and $V = \phi_3^2(1 - \phi_3^2)$ (in the $\phi_3^2$) the minimum of the potential is very close to $r = 0$ while for $V = \phi_3^2(1 - \phi_3^2)$ it is clearly much further
Figure 1: The total energy of 2 baby skyrmions as a function of $r$ for different potentials ($G$, defined in (10) given by respectively, $G(\phi_3) = 1$, $G(\phi_3) = (1 + \phi_3)$, $G(\phi_3) = (1 + \phi_3)\phi_3^2$ and $G(\phi_3) = \phi_3^2$)
out). A possible suggestion is that, in the latter case, the skyrmions move to a distance between them at which $\phi_3 = 0$. Unfortunately this idea is not supported by the results of our numerical simulations. We have looked at the plots of the energy density and of the $\phi_3$ at different distances for 2 baby skyrmions and have found $\phi_3 \neq 0$. When we repeated this study for larger $\theta V$ (to have a more effective potential) the value of $\phi_3$ decreased but was still nonzero. So the behaviour of skyrmions is more complicated and it depends on the properties of the potential in a more global way.

We have also calculated the ionisation energies and various break-up modes of multiskyrmions for potentials (27) and (28). Comparing potentials (25) and (28) we see some further similarity (in each case the system of 6 skyrmions seems to be the most bound) and in each case, eg, the state of 8 skyrmions needs very little energy to break up into (2+6). This is more pronounced in the case of the potential (28) for which the binding is stronger.

In this paper we have looked at baby Skyrme models with more general potentials. We have found that as the skyrmions involve mappings between $S^2 \rightarrow S^2$ their properties depend crucially on whether the potential vanishes at the positions of the skyrmions (ie at $\phi_3 = -1$) (we have assumed that at spatial infinity $\phi_3 = 1$). When the potential vanishes the skyrmions “lie on top of each other”, when it does not - they separate and form interesting lattice like patterns. The shape of these patterns depends on the details of the potential. The same holds for the binding energies of skyrmions in all models. When the potential is more complicated (ie it has further zero) the skyrmions are more bound and in their patterns are more spread out - however, the actual positions and distortions depend on the details of the potential.

When the skyrmions are spread out the system of skyrmions has many local minima with some, larger or smaller, potential barriers between them. Thus for instance, for $V = 1 - \phi_3$, a system of 6 skyrmions has at least 3 local minima, and depending on the initial configuration the system can land in any of them. This is not unexpected and it suggests that similar models in, physically more relevant, 3 spatial dimensions may also have many local minima. Thus, the problem of finding multiskyrmion solutions of models in higher dimensions is clearly very complicated.

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7 Appendix

Here we make some remarks about our numerical procedures

**Profile functions** We have used the shooting method to determine $f(r)$ and have integrated (14) by a fourth-order Runge-Kutta method for any $n$. To avoid a singularity at $r = 0$ we have considered $r \approx \epsilon$ with $\epsilon$ small. We have used the formulae (17) and (18) with 10000 lattice points and with the spacing $dr = 0.003$. 

11
Hedgehog static solutions In most of our simulations we have used a 201 × 201 lattice with lattice spacing $\delta x = \delta y = 0.3$. For each $(x_i, y_j)$, $f(x_i, y_j)$ was determined by a linear interpolation using the values determined by the shooting method. Given $f$ we calculated $\vec{\phi}$ using the hedgehog field expression (11).

A linear superposition for static solutions with $n > 1$ In an alternative formulation we have used a single complex field $W$, which is related to $\vec{\phi}$ by

$$
\phi_1 = \frac{W + W^*}{1 + WW^*} \quad \phi_2 = i \frac{W - W^*}{1 + WW^*} \quad \phi_3 = \frac{1 - WW^*}{1 + WW^*}
$$

(29)

where * denotes the complex conjugation. Hence the complex field $W$ when expressed in terms of the profile function $f(r)$ and $\theta$ takes the form

$$
W = \tan \left( \frac{f(r)}{2} \right) e^{-i\theta}.
$$

(30)

Static initial field configurations with $Q = n$ were formed by a linear superposition. When the baby skyrmions are far from each other (but not too close to the boundary) we can construct a configuration with charge $n$ from $W$ with $n = 1$ by a linear superposition

$$
W(x, y) = \sum_\alpha W_\alpha(x - x_\alpha, y - y_\alpha),
$$

(31)

where $(x_\alpha, y_\alpha)$ is the location of the centre of the $\alpha$th skyrmion and $W_\alpha$ is the field (30) with $n = 1$ and appropriate $f(r)$. We have used this method of constructing initial configurations in some of our simulations. Namely, we have placed $n$ baby skyrmions at equal distances from the origin with relative phase shifts $\delta \chi = \frac{2\pi}{n}$ between them (for maximal attraction) and then used (29) to get $\vec{\phi}$.

Time evolution of the static solutions We integrated the equation of motion for each component of $\vec{\phi}$ independently. In this manner, during the simulations the field $\vec{\phi}$ would gradually move away from the unit sphere $S^2$. To correct this, every few iterations, we kept on rescaling the field as follows:

$$
\phi_a \rightarrow \frac{\phi_a}{\sqrt{\vec{\phi} \cdot \vec{\phi}}}
$$

(32)

and

$$
\partial_t \phi_a \rightarrow \partial_t \phi_a - \frac{\partial_t \vec{\phi} \cdot \vec{\phi}}{\vec{\phi} \cdot \vec{\phi}} \phi_a.
$$

(33)

Another problem we have had to face involved using a finite lattice. Thus we had to make sure that the boundary effects did not alter our results. To be certain of this we varied lattice spacing and the number of lattice points.

When the fields were not at the minimum of the energy we allowed them to flow to this minimum - reducing the energy by a damping term.

$$
\partial_t \phi_a = K^{-1}_{ab} F_b\left(\vec{\phi}, \partial_t \vec{\phi}, \partial_i \vec{\phi}\right) - \gamma \partial_t \phi_a,
$$

(34)

where $\gamma$ is the damping coefficient. We set $\gamma$ to 0.1. This term takes the energy out of the system.
We have used a fourth-order Runge-Kutta method to simulate the evolution of the field in time working in double precision. The time step was $\delta t = 0.1$ throughout. We have performed our simulation using the $\vec{\phi}$ formulation with the derivatives replaced by finite differences\(^2\) (as explained in [5], see also [6]).

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\(^2\)we have used the 9-point laplacian