A LOTKA-VOLterra COMPETITION MODEL WITH NONLOCAL DIFFUSION AND FREE BOUNDARIES

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Abstract. This paper is concerned with a nonlocal diffusion Lotka-Volterra type competition model consisting of a native species and an invasive species in a one-dimensional habitat with free boundaries. We prove the well-posedness of the system and get a spreading-vanishing dichotomy for the invasive species. We also provide some sufficient conditions to ensure spreading success or spreading failure for the case that the invasive species is an inferior competitor or a superior competitor, respectively.

Keywords: Nonlocal diffusion; Free boundary; Lotka-Volterra type; Spreading-vanishing dichotomy; Sharp criteria

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1. Introduction

In this paper we are interested in the dynamics of the solution \((u(t, x), v(t, x), g(t), h(t))\) which is governed by the following nonlocal dispersal model with free boundaries in the one space dimension

\[
\begin{aligned}
&u_t = d_1 \left[ \int_{g(t)}^{h(t)} J(x-y)u(t, y)dy - u \right] + u(a_1 - b_1u - c_1v), \quad t > 0, \quad x \in (g(t), h(t)), \\
v_t = d_2 \left[ \int_{\mathbb{R}} J(x-y)v(t, y)dy - v \right] + v(a_2 - b_2u - c_2v), \quad t > 0, \quad x \in \mathbb{R}, \\
u(t, g(t)) = u(t, h(t)) = 0, \quad t > 0, \\
h'(t) = \mu \int_{g(t)}^{h(t)} \int_{h(t)}^{+\infty} J(x-y)u(t, x)dydx, \quad t > 0, \\
g'(t) = -\mu \int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J(x-y)u(t, x)dydx, \quad t > 0, \\
v(0, x) = v_0(x), \quad x \in \mathbb{R}, \\
u(0, x) = u_0(x), \quad h(0) = -g(0) = h_0, \quad x \in [-h_0, h_0]
\end{aligned}
\]

where \(u(t, x)\) represents the density of an invasive species and \(v(t, x)\) denotes the density of a native species. It is imposed that \(u\) exists in the initial region \([-h_0, h_0]\) and extends into the habitat by the spreading fronts \(x = g(t)\) and \(x = h(t)\), which to be determined together with \(u(t, x)\) and \(v(t, x)\), see Cao et al. [5] for the detailed derivation of the free boundary conditions \(h'(t)\) and \(g'(t)\). The constants \(d_1\) and \(d_2\) represent the dispersal rate of species \(u\).
and $v$, respectively; $\mu$ is a positive constant accounting for the expanding ability of $u$. The initial data satisfy

$$
\begin{align*}
&v_0(x) \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad v_0(x) > 0 \text{ in } \mathbb{R}, \\
&u_0(x) \in C([-h_0, h_0]), \quad u_0(-h_0) = u_0(h_0) = 0 \text{ and } u_0(x) > 0 \text{ in } (-h_0, h_0),
\end{align*}
$$

and the kernel $J : \mathbb{R} \to \mathbb{R}$ is a non-negative continuous function on $\mathbb{R}$. More precisely, we assume in what follows that

$$(J): \ J(0) > 0, \int_\mathbb{R} J(x)dx = 1, \ J \text{ is symmetric and } \sup_{\mathbb{R}} J \leq \infty.$$ 

The problem (1.1) is a nonlocal variation of a Lotka-Volterra model in which the diffusion is described by the Laplace operator, and the free boundary is denoted by the Stefan condition. For example, Du and Lin [11] considered the long time behavior of the following problem

$$
\begin{align*}
&u_t - d_1 \Delta u = u(a_1 - b_1 u - c_1 v), \quad t > 0, 0 \leq x < h(t), \\
v_t - d_2 \Delta v = v(a_2 - b_2 u - c_2 v), \quad t > 0, 0 \leq x < \infty, \\
u_x(t, 0) = v_x(t, 0) = 0, \ u(t, x) = 0, \quad t > 0, h(t) \leq x < \infty, \\
h'(t) = -\mu u_x(t, h(t)), \quad t > 0, \\
u(0, x) = u_0(x), \ h(0) = h_0, \quad 0 \leq x \leq h_0, \\
v(0, x) = v_0(x), \quad 0 \leq x < \infty.
\end{align*}
$$

(1.3)

In the case that $u$ is a superior competitor in the sense that $\frac{a_1}{a_2} > \max \left\{ \frac{b_1}{b_2}, \frac{c_1}{c_2} \right\}$, they established a spreading vanishing dichotomy for species $u$, that is, either $h(t) \to +\infty$ and $(u, v) \to \left( \frac{a_1}{b_1}, 0 \right)$ as $t \to +\infty$, or $h(t) < +\infty$ and $(u, v) \to \left( 0, \frac{a_2}{c_2} \right)$ as $t \to +\infty$; while in the case that $u$ is an inferior competitor, namely, $\frac{a_1}{a_2} < \min \left\{ \frac{b_1}{b_2}, \frac{c_1}{c_2} \right\}$, they showed that $(u, v) \to \left( 0, \frac{a_2}{c_2} \right)$ as $t \to +\infty$, which means that the native species $v$ win the competition. When the spreading of $u$ happens, they further showed a rough estimate for the spreading speed. These results have been extended to many Lotka-Volterra two species models, one can refer to Du et al. [13], Guo and Wu [14,15], Wang [31], Wang and Zhao [34] and references cited therein.

It is well-known that the invasion and spreading of nonlocal diffusion Lotka-Volterra type competition models have been studied intensively. In 2003, Hutson et al. [17] studied the competitive advantages and disadvantages of diffusive rate and diffusive distance in a spatially heterogeneous environment, namely, the following problem

$$
\begin{align*}
&u_t = d_1 \left[ \frac{1}{(L_u)^N} \int_\Omega J \left( \frac{x - y}{L_u} \right) u(t, y)dy - u(t, x) \right] + uf(u + v, x), \quad (t, x) \in (0, \infty) \times \Omega, \\
v_t = d_2 \left[ \frac{1}{(L_v)^N} \int_\Omega J \left( \frac{x - y}{L_v} \right) v(t, y)dy - v(t, x) \right] + vf(u + v, x), \quad (t, x) \in (0, \infty) \times \Omega,
\end{align*}
$$

(1.4)

where $d_i(i = 1, 2)$ and $J(\cdot)$ are the same as them in (1.1); the constants $L_u, L_v > 0$ characterize the diffusive distance (interpreted as spreads in $\Omega$). They showed that as in the case of reaction-diffusion models, for fixed spread slower rates of diffusion are always optimal, that is, for any non-trivial, non-negative initial conditions, if $L_u = L_v$ and $d_1 < d_2$, then the semi-trivial equilibrium $(u^*, 0)$ is globally asymptotically stable. While fixing the diffusion rate ($d_1 = d_2$)
and varying the spread, in the case of small spread, the smaller spread is selected (the semi-trivial equilibrium in the presence of the species with the smaller spread is the global attractor) and in the case of large spread the larger spread is selected.

We also mention that traveling wave solutions of nonlocal diffusive competition systems have been studied intensively. See e.g. Bao et al. [12], Pan et al. [24], Zhao and Ruan [42] for the existence of traveling wave fronts, and Du et al. [8,10], Li et al. [22] and Wang and Lv [33] for the existence of invasion entire solutions. Also, there are many works concerned with the spectral theory of nonlocal dispersal operators and entire solutions of nonlocal dispersal equations, see Coville [7], Hetzer and Shen [16], Li et al. [19,21], Shen and Zhang [29,30], Sun et al. [25,26], Sun et al. [28], Yang et al. [38], Zhang et al. [39] and Zhang et al. [40,41].

The main purpose of this paper is extend the above results into the free boundary problem with nonlocal diffusion. It must be emphasized that our approach to deal with the nonlocal diffusion problem (1.1) is totally different from these of the responding random (local) diffusion equations, including the well-posedness as well as the long-term behaviors. In particular, we establish a comparison principle (see Theorem 2.3) in a suitable parabolic domain to consider the global asymptotic stability of the semi-trivial equilibriums $(\frac{a_1}{b_1}, 0)$ and $(0, \frac{a_2}{c_2})$ with the different initial datas. Moreover, we give a precise classification of the dynamics for the invasion species $u$ in the case that $u$ is a superior competitor or an inferior one.

This paper is organized as follows. Section 2 is concerned with the existence and uniqueness of positive solutions of (1.1), which was established by two times using of the contraction mapping theorem. Section 3 is devoted to the dynamics of the solutions that obtained in Section 2. We first collect some essential results among the principal eigenvalues, and then establish a spreading vanishing dichotomy for species $u$ in the case that $u$ is a superior competitor. We further obtain a sharp criteria of expanding ability $\mu$ to ensure spreading or vanishing in the end.

In the end of this section, we must mention that after the completion of this article, we received the preparation paper of Du et al. [12]. They considered the equation (1.1) with the two species both located in the same growth domain and obtained some interesting asymptotical behavior of solutions to (1.1) with general reaction term. We also would like to mention the article [35]. Wang and Wang considered a class of free boundary problems of ecological models with nonlocal and local diffusions that can be regarded as extensions of free boundary problems for reaction diffusion systems.

### 2. The well-posedness of (1.1)

This section is focused on the global existence and uniqueness of solutions to the problem (1.1). For convenient, we introduce some notations first. For given $h_0, T > 0$, we define

$$\mathbb{H}_{h_0,T} = \left\{ h \in C([0,T]) \mid h(0) = h_0, \inf_{0 \leq t_1 < t_2 \leq T} \frac{h(t_2) - h(t_1)}{t_2 - t_1} > 0 \right\},$$

$$\mathbb{G}_{h_0,T} = \left\{ g \in C([0,T]) \mid -g(t) \in \mathbb{H}_{h_0,T} \right\},$$

$$C_0([-h_0, h_0]) = \left\{ u \in C([0,T]) \mid u(-h_0) = u(h_0) = 0 \text{ and } u_0(x) > 0 \text{ in } (-h_0, h_0) \right\}.$$  

For $g \in \mathbb{G}_{h_0,T}$, $h \in \mathbb{H}_{h_0,T}$ and $u_0(x) \in C_0([-h_0, h_0])$, define

$$\Omega_{g,h} = \{(t,x) \in \mathbb{R}^2 : 0 < t \leq T, g(t) < x < h(t)\},$$
\( \Omega_\infty = \{(t, x) \in \mathbb{R}^2 : 0 < t \leq T, \ x \in \mathbb{R} \}, \)
\( \mathbb{X}_{\Omega_0, \infty} = \left\{ \phi \in C(\Omega_\infty) \mid \phi(0, x) = v_0(x) \text{ for } x \in \mathbb{R} \text{ and } 0 < \phi \leq M_1 \text{ in } \Omega_\infty \right\}, \)
\( \mathbb{X}_{u_0, g, h} = \left\{ \psi \in C(\bar{\Omega}_{g, h}) \mid \psi \geq 0 \text{ in } \Omega_{g, h}, \ \psi(0, x) = u_0(x) \text{ for } x \in [-h_0, h_0] \right. \)
\( \left. \quad \text{and } \psi(t, g(t)) = \psi(t, h(t)) = 0 \text{ for } 0 \leq t \leq T \right\} \)

with \( M_1 := \max \{a_1, K_0, \|v_0\|_{L^\infty} \}. \) Following is Maximum Principle that will be frequently used later.

**Lemma 2.1.** (Maximum Principle [5]) Assume that (J) holds and for some given \( h_0, T > 0, \)
let \( g \in G_{h_0, T} \) and \( h \in \mathbb{H}_{h_0, T}. \) Assume that for all \((t, x) \in \Omega_{g, h}, \) functions \( u(t, x) \) and \( u_t(t, x) \) are continuous, and there exists a function \( c(t, x) \in L^\infty(\Omega_{g, h}) \) such that

\[
\begin{cases}
  u_t(t, x) \geq d \int_{h(t)}^{h(t)} J(x - y, u(t, y))dy - cu(t, x)u, \quad (t, x) \in \Omega_{g, h}, \\
  u(t, g(t)) \geq 0, \quad u(t, h(t)) \geq 0, \quad t > 0, \\
  u(0, x) \geq 0, \quad x \in [-h_0, h_0].
\end{cases}
\]

Then \( u(t, x) \geq 0 \) for all \( 0 \leq t \leq T \) and \( x \in [g(t), h(t)]. \) Further more, if \( u(0, x) \neq 0 \) in \([-h_0, h_0], \)
then \( u(t, x) > 0 \) for all \((t, x) \in \Omega_{g, h}. \)

**Proof.** See the proof of [5] Lemma 2.2. \qed

We first prove a existence and uniqueness result for a general free boundary problem. Consider the following free boundary problem

\[
\begin{aligned}
  u_t &= d_1 \left[ \int_{g(t)}^{h(t)} J(x - y, u(t, y))dy - u(t, x) \right] + f_1(u, v), \quad t > 0, \ x \in (g(t), h(t)), \\
  v_t &= d_2 \int_{\mathbb{R}} J(x - y, v(t, y))dy - v(t, x), \quad t > 0, \ x \in \mathbb{R}, \\
  u(t, g(t)) &= u(t, h(t)) = 0, \quad t > 0, \\
  h'(t) &= \mu \int_{g(t)}^{h(t)} J(x - y, u(t, x))dydx, \quad t > 0, \\
  g'(t) &= -\mu \int_{g(t)}^{h(t)} J(x - y, u(t, x))dydx, \quad t > 0, \\
  v(0, x) &= v_0(x), \quad x \in \mathbb{R}, \\
  u(0, x) &= u_0(x), \ h(0) = -g(0) = h_0, \quad x \in [-h_0, h_0]
\end{aligned}
\]

with \( f_1(0, v) = f_2(u, 0) = 0 \) for any \( u, v \in \mathbb{R}. \) Following are some imposed assumptions on reaction terms \( f_1(u, v) \) and \( f_2(u, v): \)

**A1:** there is constant \( K_0 > 0 \) such that \( f_1(u, v) < 0 \) for \( u > K_0 \) and \( f_2(u, v) < 0 \) for \( v > K_0; \)

**A2:** \( f_i(u, v), \ i = 1, 2 \) is locally Lipschitz continuous in \( \mathbb{R}_+^2, \) i.e., For any \( L_i > 0, \ i = 1, 2, \)
there exists constant \( K_1 = K_1(L_i) > 0 \) such that

\[
|f_i(u_1, v_1) - f_i(u_2, v_2)| \leq K_1 (|u_1 - u_2| + |v_1 - v_2|) \text{ for } u_i \in [0, L_1], \ v_i \in [0, L_2].
\]

The following theorem is the main result of this section.
Theorem 2.2. Assume that (J) and (A1)-(A2) hold, then for any given \( u_0(x) \) and \( v_0(x) \) satisfying (1.2) and \( h_0 > 0 \), problem (2.2) admits a unique positive solution \((u, v, g, h)\) defined for all \( t > 0 \). Moreover, for any given \( T > 0 \), \( u \in X_{u_0, g, h}, v \in X_{v_0, \infty}, g \in G_{h_0, T}, h \in H_{h_0, T}. \)

Proof. We divide the proof into two steps.

Step 1: Local existence and uniqueness.

For any given \( v^*(t, x) \in X_{v_0, \infty} \), it follows from Theorem 2.1 in [5] that for \( h_0 > 0 \) and \( u_0(x) \) satisfying (1.2) the following problem

\[
\begin{align*}
  u_t &= d_1 \left[ \int_{g(t)}^{h(t)} J(x-y) u(t,y) dy - u(t,x) \right] + f_1(u, v^*), \quad t > 0, \quad x \in (g(t), h(t)), \\
  u(t, g(t)) &= u(t, h(t)) = 0, \quad t > 0, \\
  h'(t) &= \mu \int_{g(t)}^{h(t)} \int_{h(t)}^{+\infty} J(x-y) u(t,x) dy dx, \quad t > 0, \\
  g'(t) &= -\mu \int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J(x-y) u(t,x) dy dx, \quad t > 0, \\
  u(0, x) &= u_0(x), \quad h(0) = -g(0) = h_0, \quad x \in [-h_0, h_0]
\end{align*}
\]

admits a unique solution, denoted by \((\tilde{u}(t, x), \tilde{g}(t), \tilde{h}(t))\), which is defined for all \( t > 0 \). Moreover, for any \( T > 0 \), there hold \( \tilde{g}(t) \in G_{h_0, T}, \tilde{h}(t) \in H_{h_0, T} \) and \( \tilde{u}(t, x) \in X_{u_0, g, h} \). Further, we have

\[
0 < \tilde{u}(t, x) \leq M := \max \left\{ \max_{-h_0 \leq x \leq h_0} u_0(x), \ K_0 \right\} \quad \text{for } 0 < t < T, \ x \in (\tilde{g}(t), \tilde{h}(t)).
\]

For the above known \( \tilde{u}(t, x) \), we first define

\[
 u^*(t, x) = \begin{cases} 
  \tilde{u}(t, x), & x \in [\tilde{g}(t), \tilde{h}(t)], \\
  0, & x \not\in [\tilde{g}(t), \tilde{h}(t)]
\end{cases}
\]

and then consider the following

\[
\begin{align*}
  v_t &= d_2 \left[ \int_{\mathbb{R}} J(x-y) v(t,y) dy - v(t,x) \right] + f_2(u^*, v), \quad 0 < t < T, \ x \in \mathbb{R}, \\
  v(0, x) &= v_0(x), \quad x \in \mathbb{R}.
\end{align*}
\]

The existence and uniqueness of the local solution of (2.6) is well known, we denote it by \( \tilde{v}(t, x) \), and \( \tilde{v}(t, x) \in C(\Omega_{\infty}). \)

Next, we want to show that \( v^* \) and \( \tilde{v} \) is coincide with each other in the sense that the map \( \Gamma \) defined by \( \Gamma v^* = \tilde{v} \) admits a unique fixed point in \( X_{v_0, \infty} \). We prove this conclusion by the contraction mapping theorem. Clearly, \( \Gamma \) maps \( X_{v_0, \infty} \) into itself. It is remained to prove that for small \( T > 0 \), \( \Gamma \) is contract.

Firstly we note that \( X_{v_0, \infty} \) is a complete metric space with the metric

\[
d(\phi_1, \phi_2) = \|\phi_1 - \phi_2\|_{L^\infty(\Omega_{\infty})}.
\]

Choose \( v_i^* \in X_{v_0, \infty} \) (i = 1, 2), we use \( \tilde{v}_i \) to denote the solution of (2.3) associate to \( v_i^* \), and then we use \( \tilde{v}_i \) to denote the solution of (2.6) associate to \( u_i^* \). Hence there is

\[
\tilde{v}_1(t, x) - \tilde{v}_2(t, x) = \int_0^t d_2 \int_{\mathbb{R}} J(x-y) [\tilde{v}_1(s, y) - \tilde{v}_2(s, y)] dy ds - \int_0^t d_2 [\tilde{v}_1 - \tilde{v}_2](s, x) ds
\]
where
\[ t, x \]
and
\[ C(\Omega_\infty) \]
and
\[ \| v_1 - v_2 \|_{C(\Omega_\infty)} \leq 2d_2 \| v_1 - v_2 \|_{C(\Omega_\infty)} t + K_1 \left[ \| v_1 - v_2 \|_{C(\Omega_\infty)} + \| u_1^* - u_2^* \|_{C(\Omega_\infty)} \right] t \]
\[ \leq (2d_2 + K_1) \| v_1 - v_2 \|_{C(\Omega_\infty)} + K_1 \| u_1 - u_2 \|_{C(\Omega_{\tilde{g}, \tilde{h}})} T, \]
Taking \( T \) sufficiently small such that
\[ (2d_2 + K_1) T \leq \frac{1}{2}, \] that is \( T \leq T_1 := \frac{1}{2 (2d_2 + K_1)}, \]
then we obtain that
\[ \| v_1 - v_2 \|_{C(\Omega_\infty)} \leq 2K_1 \| u_1 - u_2 \|_{C(\Omega_{\tilde{g}, \tilde{h}})} T. \]
We are now in a position to give an estimate to \( \| u_1 - u_2 \|_{C(\Omega_{\tilde{g}, \tilde{h}})} \). For any given \( (t^*, x^*) \in \Omega_{\tilde{g}, \tilde{h}} \) and \( t \in (0, T] \), define
\[ U(t, x^*) = u_1(t, x^*) - u_2(t, x^*), \quad V(t, x^*) = v_1(t, x^*) - v_2(t, x^*) \]
and
\[ t_x = \begin{cases} 
   t_{x, \tilde{g}} & \text{if } x \in [\tilde{g}(T), -h_0) \text{ and } x = \tilde{g}(t_{x, \tilde{g}}), \\
   0 & \text{if } x \in (-h_0, h_0], \\
   t_{x, \tilde{h}} & \text{if } x \in (h_0, \tilde{h}(T)) \text{ and } x = \tilde{h}(t_{x, \tilde{h}}).
\end{cases} \]
We also define
\[ H_1(t) = \min \left\{ \tilde{h}_1(t), \tilde{h}_2(t) \right\}, \quad H_2(t) = \max \left\{ \tilde{h}_1(t), \tilde{h}_2(t) \right\}, \]
\[ G_1(t) = \min \left\{ \tilde{g}_1(t), \tilde{g}_2(t) \right\}, \quad G_2(t) = \max \left\{ \tilde{g}_1(t), \tilde{g}_2(t) \right\}, \]
\[ \Omega_T = \Omega_{G_1, H_2} = \Omega_{\tilde{g}_1, \tilde{h}_1} \cup \Omega_{\tilde{g}_2, \tilde{h}_2}. \]
Three cases will be handled separately.

**Case 1**: \( x^* \in [-h_0, h_0] \).

By the equations of \( \tilde{u}_i(t, x^*) \) we have that
\[ \begin{cases} 
   U_t(t, x^*) + c_1(t, x^*) U(t, x^*) = c_2(t, x^*) V^*(t, x^*) + I(t, x^*), \\
   U(0, x^*) = 0,
\end{cases} \]
where \( V^*(t, x^*) = v_1^*(t, x^*) - v_2^*(t, x^*) \),
\[ c_1(t, x^*) = d_1 - \frac{f_1(\tilde{u}_1, v_1^*) - f_1(\tilde{u}_2, v_1^*)}{\tilde{u}_1 - \tilde{u}_2}, \quad c_2(t, x^*) = \frac{f_1(\tilde{u}_2, v_2^*) - f_1(\tilde{u}_2, v_1^*)}{v_1^* - v_2^*} \]
and
\[ I(t, x^*) = d_1 \int_{\tilde{g}_1(t)}^{\tilde{h}_1(t)} J(x^* - y) \tilde{u}_1(t, y) dy - d_1 \int_{\tilde{g}_2(t)}^{\tilde{h}_2(t)} J(x^* - y) \tilde{u}_2(t, y) dy \]
Clearly, at \( (t, x^*) \) there hold
\[ \| c_1 \|_{\infty}, \| c_2 \|_{\infty} \leq d_1 + K_1(M) := K^*_1, \]

With
\[ \| H^* \|_{C([0,T])} = \| \tilde{h}_1 - \tilde{h}_2 \|_{C([0,T])} + \| \tilde{g}_1 - \tilde{g}_2 \|_{C([0,T])}, \quad M_0 = \max \{ \| u_0 \|_{\infty}, \| v_0 \|_{\infty}, K_0 \}, \]
we also have
\[ |I(t, x^*)| \leq d_1 \|U\|_{C(\Omega_T)} + d_1 M_0 \|J\|_\infty \|H^*\|_{C([0,T])}. \]

Then we can find constant \( C_1 = C_1(d_1, u_0, M_0, J) \) such that
\[
\max_{t \in [0,T]} |I(t, x^*)| \leq C_1 \left[ \|U\|_{C(\Omega_T)} + \|H^*\|_{C([0,T])} \right].
\]

In addition, it follows from \([2.9]\) that
\[
\left[ e^{t_0^T c_1(\tau,x^*)d\tau} U \right]_t (t, x^*) = e^{t_0^T c_1(\tau,x^*)d\tau} [c_2(t, x^*)V^*(t, x^*) + I(t, x^*)].
\]

Then integration from 0 to \( t^* \) immediately leads to
\[
U(t^*, x^*) = e^{-\int_0^{t^*} c_1(\tau,x^*)d\tau} \int_0^{t^*} e^{t_0^T c_1(\tau,x^*)d\tau} [c_2(t, x^*)V^*(t, x^*) + I(t, x^*)] dt.
\]

Hence
\[
|U(t^*, x^*)| \leq e^{2K^*T} \left[ C_1 \|U\|_{C(\Omega_T)} + \|H^*\|_{C([0,T])} + K_1^* \|V^*\|_{C(\Omega_{\infty})} \right].
\]

Then for constant \( \tilde{C}_1 = \tilde{C}_1(C_1, K_1^*) > 0 \) there is
\[
\|U\|_{C(\Omega_T)} \leq \tilde{C}_1 e^{2K^*T} \left[ \|U\|_{C(\Omega_T)} + \|H^*\|_{C([0,T])} + \|V^*\|_{C(\Omega_{\infty})} \right].
\]

**Case 2:** \( x^* \in (h_0, H_1(s)) \).

In such a case, there exist \( t_1^*, t_2^* \in (0, t^*) \) such that \( x^* = h_1(t_1^*) = h_2(t_2^*) \). Without loss of generality, suppose that \( 0 < t_1^* \leq t_2^* \). According to \([2.9]\), it is routine to check that
\[
U(t^*, x^*) = e^{-\int_{t_1^*}^{t_2^*} c_1(\tau,x^*)d\tau} \int_{t_1^*}^{t_2^*} e^{t_0^T c_1(\tau,x^*)d\tau} [c_2(t, x^*)V^*(t, x^*) + I(t, x^*)] dt.
\]

Then we have
\[
|U(t^*, x^*)| \leq e^{K^*T} \left[ \|U(t_2^*, x^*)\| + \int_{t_1^*}^{t_2^*} e^{K^*T} \|K_1^*V^*(t, x^*) + I(t, x^*)\| dt \right]
\]
\[
\leq e^{K^*T} \left[ \|U(t_2^*, x^*)\| + T e^{2K^*T} \left[ \max_{t \in [0,T]} |I(t, x^*)| + K_1^* \|V^*\|_{C(\Omega_{\infty})} \right] \right]
\]
\[
\leq e^{K^*T} \left[ \|U(t_2^*, x^*)\| + e^{2K^*T} C_1 \|H^*\|_{C([0,T])} \right] + e^{2K^*T} \left[ K_1^* \|V^*\|_{C(\Omega_{\infty})} + C_1 \|U\|_{C(\Omega_T)} \right] T.
\]

Notice that
\[
U(t_2^*, x^*) = \tilde{u}_1(t_2^*, x^*) - \tilde{u}_1(t_1^*, x^*) + \tilde{u}_1(t_2^*, x^*) - \tilde{u}_2(t_2^*, x^*)
\]
\[
= \tilde{u}_1(t_2^*, x^*) - \tilde{u}_1(t_1^*, x^*) + \tilde{u}_1(t_2^*, h_1(t_1^*)) - \tilde{u}_2(t_2^*, h_2(t_2^*))
\]
\[
= \tilde{u}_1(t_2^*, x^*) - \tilde{u}_1(t_1^*, x^*) = \int_{t_1^*}^{t_2^*} (\tilde{u}_1)_t(t, x^*) dt.
\]

Using (A2) to conclude that there exists a constant \( C_3 = C_3(d_1, M_0, f_1) \) such that
\[
|U(t_2^*, x^*)| \leq \int_{t_1^*}^{t_2^*} d_1 \int_{y_1(t)}^{\tilde{y}_1(t)} J(x^* - y, \tilde{u}_1(t, y)dy - d_1 \tilde{u}_1(t, x^*) + f_1(\tilde{u}_1(t, x^*), \tilde{v}_1(t, x^*)) dt
\]
\[
\leq C_3(t_2^* - t_1^*).
Obviously, we have $U(t_2^*, x^*) = 0$ if $t_2^* = t_2^*$. If $t_1^* < t_2^*$, it then follows from $\frac{\tilde{h}_1(t_2^*) - \tilde{h}_1(t_1^*)}{t_2^* - t_1^*} \geq \mu c_1^*$ (please see the proof of Theorem 2.1 for the detailed formula of $c_1^*$) that

$$t_2^* - t_1^* \leq \left| \tilde{h}_1(t_2^*) - \tilde{h}_1(t_1^*) \right| (\mu c_1^*)^{-1}.$$ 

In addition, there hold

$$0 = \tilde{h}_1(t_1^*) - \tilde{h}_2(t_2^*) = \tilde{h}_1(t_1^*) - \tilde{h}_1(t_2^*) + \tilde{h}_1(t_2^*) - h_2(t_2^*).$$

And hence $\tilde{h}_1(t_2^*) - \tilde{h}_1(t_1^*) = \tilde{h}_1(t_2^*) - h_2(t_2^*)$.

$$t_2^* - t_1^* \leq \left| \tilde{h}_1(t_2^*) - \tilde{h}_1(t_1^*) \right| (\mu c_1^*)^{-1} = \left| \tilde{h}_1(t_2^*) - h_2(t_2^*) \right| (\mu c_1^*)^{-1},$$

which further indicates that we can find a constant $C_4 = C_4(\mu c_1^*, C_3)$ such that

$$(2.14) \quad \left| U(t_2^*, x^*) \right| \leq C_4\|\tilde{h}_1 - \tilde{h}_2\|_{C([0, T])}.$$ 

Now, $(2.13)$ coupled with $(2.14)$ deduces that

$$\left| U(t^*, x^*) \right| \leq e^{K_1 T} C_4\|\tilde{h}_1 - \tilde{h}_2\|_{C([0, T])} + e^{2K_1 T} C_1\|H^*\|_{C([0, T])} T + e^{2K_1 T} \left[ K_1\|V^*\|_{C(\Omega_{\infty})} + C_1\|U\|_{C(\Omega_T)} \right] T.$$ 

Again we can choose constant $\tilde{C}_4 = \tilde{C}_4(K_1^*, C_1, C_4) > 0$ such that

$$(2.15) \quad \|U\|_{C(\Omega_T)} \leq \tilde{C}_4 e^{2K_1 T} \left[ \|\tilde{h}_1 - \tilde{h}_2\|_{C([0, T])} + \|H^*\|_{C([0, T])} T + \|V^*\|_{C(\Omega_{\infty})} T + \|U\|_{C(\Omega_T)} T \right].$$

**Case 3:** $x^* \in \{H_1(s), H_2(s)\}$.

Without loss of generality, assume that $h_1(s) < h_2(s)$, then $H_1(s) = h_1(s)$, $H_2(s) = h_2(s)$, $\tilde{u}_1(t, x^*) = 0$ for $t \in [t_2^*, t^*]$, and

$$0 < h_2(t^*) - h_2(t_2^*) \leq h_2(t^*) - h_1(t^*).$$

Then we have

$$\tilde{u}_2(t^*, x^*) = \int_{t_2^*}^{t^*} \left[ d_1 \int_{g_1(t)}^{h_2(t)} J(x^* - y)\tilde{u}_2(t, y) dy - d_1 \tilde{u}_2(t, x^*) + f_1(\tilde{u}_2(t, x^*), v_2(t, x^*)) \right] dt$$

$$\leq M_0 [d_1 + K(M_0)] (t^* - t_2^*)$$

$$\leq M_0 [d_1 + K(M_0)] \left[ h_2(t^*) - h_2(t_2^*) \right] (\mu c_1^*)^{-1}$$

$$\leq M_0 [d_1 + K(M_0)] \left[ h_2(t^*) - \tilde{h}_1(t^*) \right] (\mu c_1^*)^{-1}$$

$$\leq C_5\|\tilde{h}_1 - \tilde{h}_2\|_{C([0, T])},$$

here constant $C_5 = M_0 [d_1 + K(M_0)] (\mu c_1^*)^{-1}$. And hence

$$(2.16) \quad \left| U(t^*, x^*) \right| = \left| \tilde{u}_2(t^*, x^*) \right| \leq C_5\|\tilde{h}_1 - \tilde{h}_2\|_{C([0, T])}.$$ 

Thus, there is

$$(2.17) \quad \|U\|_{C(\Omega_T)} \leq C_5\|\tilde{h}_1 - \tilde{h}_2\|_{C([0, T])}.$$ 

Therefore, by $(2.12)$, $(2.15)$ and $(2.17)$, we can find a constant $C_6$ that depends on $(u_0, v_0, d_1, \mu c_1^*, f_1, J)$ such that, whether we are in Case 1, 2 or 3, for $T \leq 1$, we always have

$$(2.18) \quad \|U\|_{C(\Omega_T)} \leq C_6 e^{2K_1 T} \left[ \|H^*\|_{C([0, T])} + \|U\|_{C(\Omega_T)} T + \|V^*\|_{C(\Omega_{\infty})} T \right].$$
Therefore, 

\[ C^* \leq C_6, \tilde{C}_6 \] 

Similarly, we have, for 

\[ C^* \leq 2(1 + 3h_0\|J\|\infty)\|h_1 - h_2\|_{C([0,T])} + \mu M_0\|\tilde{g}_1 - \tilde{g}_2\|_{C([0,T])} + T \]

We are now in a position to give an estimate for \( \|H^*\|_{C([0,T])} \), that is \( \|h_1 - \tilde{h}_2\|_{C([0,T])} + \tilde{g}_1 - \tilde{g}_2\|_{C([0,T])} \). Recalling that for \( i = 1, 2 \),

\[
\begin{align*}
\tilde{h}_i'(t) &= \mu \int_{\tilde{g}_i(t)}^{\tilde{h}_i(t)} \int_{\tilde{h}_i(t)}^{\infty} J(x - y) dy \tilde{u}_i(t,x) dx, \\
\tilde{g}_i'(t) &= -\mu \int_{\tilde{g}_i(t)}^{\tilde{h}_i(t)} \int_{-\infty}^{\tilde{g}_i(t)} J(x - y) dy \tilde{u}_i(t,x) dx.
\end{align*}
\]

Following the routine of the proof of Theorem 2.1 in [5], we obtain that

\[
\begin{align*}
\left| \tilde{h}_1(t) - \tilde{h}_2(t) \right| &\leq \mu \int_0^t \int_{\tilde{g}_1(\tau)}^{\tilde{h}_1(\tau)} \int_{\tilde{h}_1(\tau)}^{+\infty} J(x - y) \tilde{u}_1(\tau,x) dy dx d\tau - \int_0^t \int_{\tilde{g}_2(\tau)}^{\tilde{h}_2(\tau)} \int_{\tilde{h}_2(\tau)}^{+\infty} J(x - y) \tilde{u}_2(\tau,x) dy dx d\tau \\
&\leq \mu \int_0^t \int_{\tilde{g}_1(\tau)}^{\tilde{h}_1(\tau)} \int_{\tilde{h}_1(\tau)}^{+\infty} J(x - y) \tilde{u}_1(\tau,x) - \tilde{u}_2(\tau,x) dy dx \\
&\quad + \mu T \left[ \int_{\tilde{g}_1(\tau)}^{\tilde{h}_1(\tau)} \int_{\tilde{h}_1(\tau)}^{+\infty} \int_{\tilde{g}_2(\tau)}^{\tilde{h}_2(\tau)} J(x - y) \tilde{u}_1(\tau,x) dy dx d\tau \right] + \mu M_0\|\tilde{g}_1 - \tilde{g}_2\|_{C([0,T])} \\
&\leq 3h_0\mu\|\tilde{u}_1 - \tilde{u}_2\|_{C([0,T])} + \mu M_0\|\tilde{g}_1 - \tilde{g}_2\|_{C([0,T])} \\
&\leq C_0 T \left[ \|\tilde{u}_1 - \tilde{u}_2\|_{C([0,T])} + \|\tilde{g}_1 - \tilde{g}_2\|_{C([0,T])} \right],
\end{align*}
\]

where \( C_0 \) depends only on \( (h_0, \mu, u_0, J, M_0) \). Let us recall that \( \tilde{u}_i \) is always extended by 0 in \((0, \infty) \times \mathbb{R}\) \( \setminus \) \( \Omega_{\tilde{g}_i, \tilde{h}_i} \) for \( i = 1, 2 \).

Similarly, we have, for \( t \in [0,T] \),

\[
\left| \tilde{g}_1(t) - \tilde{g}_2(t) \right| \leq C_0 T \left[ \|\tilde{u}_1 - \tilde{u}_2\|_{C([0,T])} + \|\tilde{h}_1 - \tilde{h}_2\|_{C([0,T])} + \|\tilde{g}_1 - \tilde{g}_2\|_{C([0,T])} \right].
\]

Therefore,

\[
\|\tilde{h}_1 - \tilde{h}_2\|_{C([0,T])} + \|\tilde{g}_1 - \tilde{g}_2\|_{C([0,T])} \leq 2C_0 T \left[ \|\tilde{u}_1 - \tilde{u}_2\|_{C([0,T])} + \|\tilde{h}_1 - \tilde{h}_2\|_{C([0,T])} + \|\tilde{g}_1 - \tilde{g}_2\|_{C([0,T])} \right].
\]

If we choose \( \delta_3 > 0 \) small such that \( 2C_0\delta_3 \leq \frac{1}{2} \), then there is

\[
\|\tilde{h}_1 - \tilde{h}_2\|_{C([0,T])} + \|\tilde{g}_1 - \tilde{g}_2\|_{C([0,T])} \leq 4C_0 T \|\tilde{u}_1 - \tilde{u}_2\|_{C([0,T])} \quad \text{for} \quad T \leq \delta_3.
\]

Substituting (2.21) into inequality (2.19) leads to the following

\[
\|U\|_{C([0,T])} \leq C^* e^{2K_1 T} \left[ (4C_0 T + 1)\|U\|_{C([0,T])} + \|V^*\|_{C([\Omega_T])} \right] T
\]

If we choose \( \delta_4 > 0 \) small such that

\[
C^* e^{2K_1 \delta_4} (2C_0\delta_4 + 1)\delta_4 \leq \frac{1}{2},
\]

then for \( T \leq \delta_4 \), there is

\[
\|U\|_{C([0,T])} \leq \|V^*\|_{C([\Omega_T])}.
\]
Back to (2.7), we obtain that
\[ \|\tilde{v}_1 - \tilde{v}_2\|_{C(\Omega_{\infty})} \leq 2K_1 \|\tilde{u}_1 - \tilde{u}_2\|_{C(\Omega_T)}T \leq 2K_1 \|v_x^* - v_y^*\|_{C(\Omega_{\infty})}T. \]

Therefore, for \( T < \min \left\{ \frac{1}{4K_1}, T_1, \delta_1, \delta_2, \delta_3, \delta_4, 1 \right\} \), \( \Gamma \) is a contraction mapping.

**Step 2:** Global existence and uniqueness.

It follows from Step 1 that problem (2.2) admits a unique solution \((\tilde{u}, \tilde{v}, \tilde{g}, \tilde{h})\) that define for \( t \in (0, T) \), and for any given \( s \in (0, T) \), there is \( u(s, x) > 0 \) for all \( x \in (g(s), h(s)) \), and \( v(s, x) > 0 \) for \( x \in \mathbb{R} \). Also, \( u(s, \cdot) \) and \( v(s, \cdot) \) are continuous in \([g(s), h(s)]\) and \( \mathbb{R} \) respectively. So if we use \( u(s, \cdot), v(s, \cdot) \) as the initial functions and then repeat the above Step 1, the solution of (2.2) can be extended from \( t = s \) to some \( T' \geq T \). Through this extension procedure, we assume that \((0, T_{\max})\) is the maximum existence interval of which that \((\tilde{u}, \tilde{v}, \tilde{g}, \tilde{h})\) can be defined, below we will prove that \( T_{\max} = \infty \).

We will derive this by a contradiction. Suppose on the contrary that \( T_{\max} \in (0, \infty) \), note that
\[ \tilde{h}'(t) - \tilde{g}'(t) = \mu \int_{\tilde{g}(t)}^{\tilde{h}(t)} \left[ \int_{-\infty}^{+\infty} J(x - y)\tilde{u}(t, x)dy \right]dx \leq \mu M_0 \left[ \tilde{h}(t) - \tilde{g}(t) \right], \]
which in turn deduces that
\[ \tilde{h}(t) - \tilde{g}(t) \leq 2h_0e^{\mu M_0 t} \leq 2h_0e^{\mu M_0 T_{\max}}. \]

Since \( \tilde{h}(t) \) and \( \tilde{g}(t) \) are monotone for \( t \in [0, T_{\max}) \), so we define
\[ \tilde{h}(T_{\max}) = \lim_{t \to T_{\max}^-} \tilde{h}(t), \quad \tilde{g}(T_{\max}) = \lim_{t \to T_{\max}^-} \tilde{g}(t), \quad \text{then} \quad \tilde{h}(T_{\max}) - \tilde{g}(T_{\max}) \leq 2h_0e^{\mu M_0 T_{\max}}. \]

The free boundary conditions (resp. the third and forth equations) in problem (1.1), together with the conclusion \( 0 < \tilde{u}(t, x), \tilde{v}(t, x) \leq M_0 \) implies that \( \tilde{h}'(t), \tilde{g}'(t) \in L^\infty([0, T_{\max}) \). And hence the definitions of \( \tilde{g}(T_{\max}) \) and \( \tilde{h}(T_{\max}) \) show that \( \tilde{h}(t), \tilde{g}(t) \in C([0, T_{\max}]). \)

Also, we know that
\[ d_1 \left[ \int_{\tilde{g}(t)}^{\tilde{h}(t)} J(x - y)u(t, y)dy - u(t, x) \right] + f_1(u, v) \in L^\infty(\Omega_{\max}), \]
\[ d_2 \left[ \int_{\mathbb{R}} J(x - y)v(t, y)dy - v(t, x) \right] + f_2(u, v) \in L^\infty(\Omega_{\max}), \]
where \( \Omega_{\max} = \{(t, x) \in \mathbb{R}^2 : t \in [0, T_{\max}], x \in (\tilde{g}(t), \tilde{h}(t))\} \) and \( \Omega_{\max} = \{(t, x) \in \mathbb{R}^2 : t \in [0, T_{\max}], x \in \mathbb{R}\}. \) Then \( \tilde{u}(t, x) \in L^\infty(\Omega_{\max}) \) and \( \tilde{v}(t, x) \in L^\infty(\Omega_{\max}). \) Hence for each \( x \in (\tilde{g}(T_{\max}), \tilde{h}(T_{\max})) \), \( \tilde{u}(T_{\max}, x) := \lim_{t \to T_{\max}} \tilde{u}(t, x) \) exists, and for each \( x \in \mathbb{R}, \tilde{v}(T_{\max}, x) = \lim_{t \to T_{\max}} \tilde{v}(t, x) \) exists. In addition \( \tilde{u}(\cdot, x) \) and \( \tilde{v}(\cdot, x) \) are continuous for \( t = T_{\max} \).

Now for the known \( (\tilde{v}, \tilde{g}, \tilde{h}) \) and \( t_x \) defined in (2.8) with \( T \) replaced by \( T_{\max} \), if we regard \( \tilde{u} \) as the unique solution of the following ODEs
\[
\begin{aligned}
\tilde{u}_t &= d_1 \int_{\tilde{g}(t)}^{\tilde{h}(t)} J(x - y)\phi(t, y)dy - d_1 \tilde{u}(t, x) + f_1(u, \tilde{v}), \quad t_x < t \leq T_{\max}, \\
\tilde{u}(t_x, x) &= \tilde{u}_0(x), 
\end{aligned}
\]
with \( \phi = \tilde{u} \), and \( \tilde{u}_0(x) = u_0(x) \) if \( x \in [-h_0, h_0] \) and \( \tilde{u}_0(x) = 0 \) if \( x \notin [-h_0, h_0] \), since \( t_x \), \( J(\cdot) \) and \( f_1(u, \tilde{v}) \) all are continuous functions of the spatial variable \( x \), then it follows from the continuous dependence of the ODE solution on the initial function and the parameters involved.
in the equation that the pair \( \tilde{u} \) is continuous in \( \Omega^{\max}_{T} \). Hence for any \( s \in (0, T_{\max}) \), we get that \( \tilde{u} \in C(\Omega^s_T) \). For such a \( s \), we can get that \( \tilde{v} \in C(\Omega^s_{\infty}) \) in a paralle way.

Next, we will prove that \((\tilde{u}, \tilde{v})\) is continuous at \( t = T_{\max} \). For the continuity of \( \tilde{u} \), it is suffices to prove that \( \tilde{u}(t, x) \to 0 \) for \( (t, x) \to (T_{\max}, g(T_{\max})) \) and \( (t, x) \to (T_{\max}, h(T_{\max})) \). We just show the case that \( (t, x) \to (T_{\max}, g(T_{\max})) \), the remained one can be get analogously. As we see below, if \( (t, x) \to (T_{\max}, g(T_{\max})) \), there hold
\[
|\tilde{u}(t, x)| = \left| \int_{t_x}^{t} \left[ d_1 \int_{\tilde{g}(\tau)}^{\tilde{h}(\tau)} J(x - y)\tilde{u}(\tau, y)dy - \int_{\tilde{g}(\tau)}^{\tilde{h}(\tau)} f_1(\tilde{u}, \tilde{v}) \right] d\tau \right|
\leq (t - t_x) [2d_1 + K_1] M_0 \to 0
\]
since \( t_x \to T_{\max} \) if \( x \to g(T_{\max}) \). In addition, as \( t_x \to T_{\max} \), there holds, for each \( x \in \mathbb{R}, \)
\[
|\tilde{v}(t, x) - \tilde{v}(t_x, x)| \leq (t - t_x) [2d_2 + K_1] M_0 \to 0.
\]
Then \((\tilde{u}, \tilde{v}) \in C(\Omega_{T_{\max}}^{\max}) \times C(\Omega_{\infty}^{\max})\), and \((\tilde{u}, \tilde{v}, \tilde{g}, \tilde{h})\) verifies problem (1.1) for \( t \in (0, T_{\max}) \). Again, Lemma 2.1 shows that \( \tilde{u}(T_{\max}, x) > 0, \tilde{v}(T_{\max}, x) > 0 \) in \((\tilde{g}(T_{\max}), \tilde{h}(T_{\max}))\). Now if we use \((\tilde{u}(T_{\max}, x), \tilde{v}(T_{\max}, x))\) as the initial function and then take Step 1, so the solution of (1.1) can be extended to interval \((0, \tilde{T})\) with \( \tilde{T} > T_{\max} \), which contradicts to the definition of \( T_{\max} \). Then \( T_{\max} = \infty \) follows. This completes the proof.

Following is a comparison principle for the competitive model.

**Theorem 2.3.** (Comparison principle) Assume that (J) and (A1)-(A2) hold. For \( T \in (0, \infty) \), suppose that \( g, h, \tilde{g}, \tilde{h} \in C((0, T]), g \in C(D^g_T) \cap C(D^g_T) \) with \( D^g_T = \{0 < t \leq T, g(t) < x < h(t)\} \), \( u \in C(D^u_T) \cap C(D^u_T) \) with \( D^u_T = \{0 < t \leq T, g(t) < x < h(t)\} \), and \( \tilde{\sigma}, \tilde{\nu} \in (C \cap L^\infty)([0, T] \times \mathbb{R}) \) satisfying
\[
\begin{align*}
\begin{cases}
\overline{u}_t &\geq d_1 \left[ \int_{\overline{g}(t)}^{\overline{h}(t)} J(x - y)\overline{u}(t, y)dy - \overline{u} \right] + f_1(\overline{u}, \overline{v}), &0 < t \leq T, &x \in (\overline{g}(t), \overline{h}(t))_t, \\
\overline{\sigma}_t &\geq d_2 \left[ \int_{\overline{g}(t)}^{\overline{h}(t)} J(x - y)\overline{\sigma}(t, y)dy - \overline{\sigma} \right] + f_2(\overline{u}, \overline{\sigma}), &0 < t \leq T, &x \in \mathbb{R}, \\
\overline{u}_t &\leq d_1 \left[ \int_{\overline{g}(t)}^{\overline{h}(t)} J(x - y)\overline{u}(t, y)dy - \overline{u} \right] + f_1(\overline{u}, \overline{\sigma}), &0 < t \leq T, &x \in (g(t), h(t)), \\
\overline{\sigma}_t &\leq d_2 \left[ \int_{\overline{g}(t)}^{\overline{h}(t)} J(x - y)\overline{\sigma}(t, y)dy - \overline{\sigma} \right] + f_2(\overline{u}, \overline{\sigma}), &0 < t \leq T, &x \in \mathbb{R}
\end{cases}
\end{align*}
\]
with \( \overline{u}(0, x) \leq v_0(x) \leq \overline{v}(0, x) \) in \( \mathbb{R}, \overline{u}(0, x) \geq u_0(x) \) in \([-h_0, h_0], \overline{u}(0, x) \leq u_0(x) \) in \([g(0), \tilde{h}(0)]\), and
\[
\begin{align*}
\begin{cases}
\overline{h}'(t) &\geq \mu \int_{\overline{g}(t)}^{\overline{h}(t)} \int_{-\infty}^{+\infty} J(x - y)\overline{u}(t, x)dydx, &0 < t \leq T, \\
\overline{g}'(t) &\leq -\mu \int_{\overline{g}(t)}^{\overline{h}(t)} \int_{-\infty}^{\overline{h}(t)} J(x - y)\overline{u}(t, x)dydx, &0 < t \leq T, \\
\overline{u}_t &\leq \mu \int_{\overline{g}(t)}^{\overline{h}(t)} \int_{-\infty}^{+\infty} J(x - y)\overline{u}(t, x)dydx, &0 < t \leq T, \\
\overline{\sigma}_t &\leq -\mu \int_{\overline{g}(t)}^{\overline{h}(t)} \int_{-\infty}^{\overline{h}(t)} J(x - y)\overline{\sigma}(t, x)dydx, &0 < t \leq T
\end{cases}
\end{align*}
\]
with $\overline{h}(0) \geq h_0 \geq h(0)$ and $\overline{g}(0) \leq -h_0 \leq g(0)$. Further, we assume that $\overline{u}(t,x) = 0$ if $x \notin (\overline{g}(t), \overline{h}(t))$ and $u(t,x) = 0$ if $x \notin (g(t), h(t))$. Then the unique solution $(u,v,g,h)$ of problem (1.1) satisfies

$$
(2.24) \begin{cases}
u \leq \overline{u}, \ v \geq u, \ g \geq \overline{g} \text{ and } h \leq \overline{h} \text{ for } 0 < t \leq T \text{ and } x \in \mathbb{R}, \\
u \geq \overline{u}, \ v \leq \overline{u}, \ g \leq g \text{ and } h \geq h \text{ for } 0 < t \leq T \text{ and } x \in \mathbb{R}.
\end{cases}
$$

**Proof.** The idea of the proof comes from [11] Lemma 2.6. Since the results involving $(\overline{u}, \overline{v}, \overline{g}, \overline{h})$ and $(u, v, g, h)$ can be obtained in a similar manner, so we only show the proof of $u \leq \overline{u}, \ v \geq u, \ g \geq \overline{g}$ and $h \leq \overline{h}$. Assume that $\overline{u}$ and $v$ are bounded above by $\overline{M}$ in $[0,T] \times \mathbb{R}$, letting $w = \overline{M} - v$ and $\overline{w} = \overline{M} - u$, then $(\overline{w}, \overline{w}, \overline{g}, \overline{h})$ satisfies

$$
(2.25) \begin{cases}
u \left[ \int_{\overline{u}(t)}^{\overline{h}(t)} J(x-y)\overline{w}(t,y)dy - \overline{u} \right] \geq f_1(\overline{u}, \overline{M} - \overline{w}), \ 0 < t \leq T, \ x \in (\overline{g}(t), \overline{h}(t)), \\
u \left[ \int_{\overline{u}(t)}^{\overline{h}(t)} J(x-y)\overline{w}(t,y)dy - \overline{w} \right] \geq -f_2(\overline{w}, \overline{M} - \overline{w}), \ 0 < t \leq T, \ x \in \mathbb{R}, \\
z(0,x) \geq u_0(x), \ x \in [-h_0, h_0], \\
z(0,x) = \overline{M} - v(0,x) \geq \overline{M} - v_0(x), \ x \in \mathbb{R}.
\end{cases}
$$

We now state that $\overline{u} \geq 0$ over the region $D^*_T$ and $\overline{w} \geq 0$ over the region $[0,T] \times \mathbb{R}$. We only give the proof of $\overline{u} \geq 0$ in $D^*_T$ since the proof for $\overline{w} \geq 0$ is parallel.

Let $u^1(t,x) = \overline{u}(t,x)e^{k_1t}$, in which $k_1 > 0$ is a constant to be determined later. Then for all $(t,x) \in D^*_T$, there is

$$
(2.26) u^1_t \geq d_1 \int_{\overline{g}(t)}^{\overline{h}(t)} J(x-y)u^1(t,y)dy + \left[ k_1 - d_1 + f_1(\eta, \overline{M} - \overline{w}) \right] u^1(t,x),
$$

where $\eta(t,x)$ is between $\overline{u}$ and 0. Now, we choose $k_1$ is large such that $p(t,x) = k_1 - d_1 + f_1(\eta, \overline{M} - \overline{w}) > 0$ for all $(t,x) \in D^*_T$.

Taking $p_0 = \sup_{(t,x) \in D^*_T} p(t,x)$ and $T_1 = \min \left\{ T, \ \frac{1}{d_1 + p_0} \right\}$. Suppose on the contrary that there are $\tilde{t} \in (0, T_1]$ and $\tilde{x} \in (\overline{g}(\tilde{t}), \overline{h}(\tilde{t}))$ such that $u^1(\tilde{t}, \tilde{x}) < 0$. Then

$$
\frac{u_{\min}}{0 < t \leq T_1, \ x \in (\overline{g}(0), \overline{u}(t))} u^1(t,x) < 0.
$$

Assume that $u_{\min}$ is attained at $(t_1, x_1)$ for $t_1 \in (0, T_1]$ and $x_1 \in (g(t_1), h(t_1))$. For $0 < t \leq t_1$ and $x \in [g(t), h(t)]$, define

$$
u^1(t,x) = \begin{cases} 0, & x \notin [-h_0, h_0], \\
u_0(x), & x \in [-h_0, h_0], \text{ and } t = \begin{cases} t_{x,g}, & x \in [g(t), -h_0), \\
t_{x,h}, & x \in (h_0, h(t)], \\
t_{x,h}, & x \in (-h_0, h_0], \end{cases}
\end{cases}
$$

where $0 < t_{x,g} < t_1$ and $0 < t_{x,h} < t_1$ have the same meaning as them in (2.8). Integrating (2.26) from $t_{x_1}$ to $t_1$ yields that

$$
u^1(t_1, x_1) - \nu^1(t_{x_1}, x_{x_1}) \geq d_1 \int_{t_{x_1}}^{t_1} \int_{\overline{g}(t)}^{\overline{h}(t)} J(x-y)\nu^1(t,y)dydt + \int_{t_{x_1}}^{t_1} p(t,x_1)\nu^1(t,x_1)dt

\geq d_1 \int_{t_{x_1}}^{t_1} \int_{\overline{g}(t)}^{\overline{h}(t)} J(x-y)\nu_{\min}dydt + \int_{t_{x_1}}^{t_1} p(t,x_1)\nu_{\min}dt.
$$
\[ \geq (t_1 - t_x) (d_1 + p_0) u_{\min}^1 \]

Since \( u^1(t_{x1}, x_1) = e^{kt_x} \bar{u}(t_{x1}, x_1) \geq 0 \), then
\[ u^1(t_1, x_1) = u_{\min}^1 > t_1 (d_1 + p_0) u_{\min}^1. \]

And hence \( t_1 (d_1 + p_0) > 1 \), that is \( t_1 > \frac{1}{d_1 + p_0} \), which contradicts to our choice of \( t_1 \). It then follows that \( u^1(t, x) \geq 0 \) for all \((t, x) \in D_{T_1}^*\), and then \( \bar{w}(t, x) \geq 0 \) for \((t, x) \in D_T^*\) by repeating this process and each time, the time interval can be extended by \( T_1 \) units, and then \( \bar{u}(t, x) \geq 0 \) for all \((t, x) \in D_T^*\). Analogously, we have \( \bar{w}(t, x) \geq 0 \) for all \((t, x) \in [0, T] \times \mathbb{R} \).

Suppose that \( \bar{h}(0) < h_0 < \bar{h}(0) \) and \( \bar{g}(0) > -h_0 > \bar{g}(0) \), and claim that \( h(t) < \bar{h}(t) \) and \( g(t) > \bar{g}(t) \) (resp. \( h(t) > \bar{h}(t) \) and \( g(t) < \bar{g}(t) \)) for all \( t \in (0, T] \).

Clearly, it is true for small \( t > 0 \). If our claim does not hold, then we can find a first \( t^* \in (0, T] \) such that \( h(t) < \bar{h}(t) \), \( g(t) > \bar{g}(t) \) for all \( t \in (0, t^*) \), and \( h(t^*) = \bar{h}(t^*) \), \( g(t^*) = \bar{g}(t^*) \) hold (resp. \( h(t^*) > \bar{h}(t^*) \), \( g(t^*) < \bar{g}(t^*) \) hold).

Letting \( U(t, x) = (\bar{w} - u) e^{-k_2 t} \) and \( W(t, x) = (\bar{w} - w) e^{-k_2 t} \), then we get
\[
\begin{align*}
U_t - d_1 \left[ \int_{\bar{g}(t)}^{h(t)} J(x - y) U(t, y) dy - U \right] & \geq (a - k_2) U + b W, \quad 0 < t < t^*, \ x \in (g(t), h(t)), \\
W_t - d_2 \left[ \int_{\bar{g}(t)}^{h(t)} J(x - y) W(t, y) - W \right] & \geq c U + (d - k_2) W, \quad 0 < t < t^*, \ x \in \mathbb{R}, \\
U(0, x) = \bar{w}(0, x) - u_0(x) & \geq 0, \quad x \in [-h_0, h_0], \\
W(0, x) = \bar{w}(0, x) - w(0, x) = v_0(x) - \bar{v}(0, x) & \geq 0, \quad x \in \mathbb{R}
\end{align*}
\]

with
\[
\begin{align*}
a & = a(t, x) = f_{1,u} \left( \theta_1 \bar{w} + (1 - \theta_1) u, \bar{M} - \bar{w} \right) \quad \text{for} \ 0 < \theta_1 < 1, \\
b & = b(t, x) = -f_{1,u}(u, \theta_2 (\bar{M} - \bar{w}) + (1 - \theta_2) (\bar{M} - w)) \quad \text{for} \ 0 < \theta_2 < 1, \\
c & = c(t, x) = -f_{2,u}(u, \theta_3 \bar{w} + (1 - \theta_3) (\bar{M} - \bar{w})) \quad \text{for} \ 0 < \theta_3 < 1, \\
d & = d(t, x) = f_{2,u}(u, \theta_4 (\bar{M} - \bar{w}) + (1 - \theta_4) (\bar{M} - w)) \quad \text{for} \ 0 < \theta_4 < 1.
\end{align*}
\]

And \( k_2 > 0 \) is sufficiently large such that
\[
k_2 \geq 1 + |a(t, x)| + b(t, x) + c(t, x) + |d(t, x)| \quad \text{for} \ 0 < t < t^* \text{ and } x \in \mathbb{R}.
\]

Note that \( a(t, x), b(t, x), c(t, x) \) and \( d(t, x) \) all are bounded and \( b(t, x), c(t, x) \geq 0 \) for \( 0 < t < t^* \) and \( x \in \mathbb{R} \).

For given \( l \) with \( l > h(t^*) \) and \(-l < g(t^*)\), by setting
\[
\bar{U}(t, x) = U(t, x) + \frac{\bar{M}(x^2 + t)}{l^2}, \quad \bar{W}(t, x) = W(t, x) + \frac{\bar{M}(x^2 + t)}{l^2},
\]
then we obtain that
\[
\begin{align*}
U_t &- d_1 \int_{g(t)}^{h(t)} J(x-y) U(t,y) dy - U \geq (a-k_2) U + b W, \quad 0 < t \leq t^*, \ x \in (g(t), h(t)), \\
W_t &- d_2 \int_{\mathbb{R}} J(x-y) W(t,y) dy - W \geq c U + (d-k_2) W, \quad 0 < t \leq t^*, \ x \in \mathbb{R}, \\
U(0,x) &= U(0,x) + \frac{Mx^2}{l^2} > 0, \quad x \in [-h_0, h_0], \\
W(0,x) &= W(0,x) + \frac{Mx^2}{l^2} > 0, \quad -l < x < l.
\end{align*}
\]

Now we will show that
\[
\min \left\{ \min_{(t,x) \in [0,t^*] \times [-l,l]} U(t,x), \min_{(t,x) \in [0,t^*] \times [-l,l]} W(t,x) \right\} =: \tau^* \geq 0.
\]

If \( \tau^* < 0 \), then there exist \( 0 < t_1 \leq t^* \) and \( g(t_1) < x_1 < h(t_1) \) such that \( U(t_1, x_1) = \tau^* < 0 \), or there exist \( 0 < t_2 \leq t^* \) and \(-l < x_2 < l \) such that \( W(t_2, x_2) = \tau^* < 0 \). Assume that the former case occurs, then we can find \( 0 \in (0, t^*) \times (-l, l) \).

As the proof of \( u^*(t, x) \geq 0 \), taking
\[
q_0 = \sup_{(t,x) \in [0,t^*] \times [-l,l]} q(t,x) \quad \text{and} \quad T_2 = \min \left\{ t_1, \frac{1}{d_1 + q_0 + \max_{[0,t^*] \times [-l,l]} b(t,x)} \right\}.
\]

Since \( U(t_1, x_1) = \tau^* < 0 \), then we can find \( t_3 \in (0, T_2) \) such that
\[
U_{inf}^* = \inf_{0 \leq t \leq t_3, \ x \in [-l,l]} U^*(t,x) < 0.
\]

It is noticed we can find sequences \( t_n \in (0, t_3] \) and \( x_n \in [-l, l] \) such that \( U^*(t_n, x_n) \to U_{inf}^* \) as \( n \to \infty \). Integrating (2.27) from 0 to \( t_n \) yields
\[
U^*(t_n, x_n) - U^*(0, x_n) \\
\geq d_1 \int_0^{t_n} \int_{\mathbb{R}} J(x-y) U^*(t,y) dy dt + \int_0^{t_n} \left( q(t,x_n) U^*(t,x_n) + b(t,x_n) W^*(t,x_n) \right) dt \\
\geq d_1 t_n U_{inf}^* + q_0 t_n U_{inf}^* + b(t,x) t_n U_{inf}^*
\]

Since \( U^*(0, x_1) = U(0, x_1) > 0 \), then as \( n \to \infty \), there holds
\[
U_{inf}^* > d_1 t_n U_{inf}^* + q_0 t_n U_{inf}^* + b(t,x) t_n U_{inf}^* \geq t_n \left( d_1 + q_0 + \max_{[0,t^*] \times [-l,l]} b(t,x) \right) U_{inf}^*.
\]

That is
\[
t_n \left( d_1 + q_0 + \max_{[0,t^*] \times [-l,l]} b(t,x) \right) \geq 1,
\]

and hence
\[
t_3 \geq \frac{1}{d_1 + q_0 + \max_{[0,t^*] \times [-l,l]} b(t,x)},
\]

which contradicts to our choice of \( t_3 \). Then there must hold \( U^* \geq 0 \) and \( U \geq 0 \) in \([0, t^*] \times [-l,l] \).
For the second case that there exist $0 < t_2 \leq t^*$ and $-l < x < l$ such that $\overline{W}(t_2, x_2) = \tau^* < 0$, we also can get the same conclusion. From now on, we have obtained that $\overline{U} \geq 0$ and $\overline{W} \geq 0$ in $[0, t^*) \times [-l, l]$. It then follows that

$$U(t, x) \geq -\frac{\overline{M}(x^2 + l)}{l^2} \quad \text{and} \quad W(t, x) \geq -\frac{\overline{M}(x^2 + l)}{l^2}$$

in $[0, t^*) \times [-l, l]$. By taking $l \to \infty$ immediately yields that $U(t, x) \geq 0$ and $W(t, x) \geq 0$ for all $(t, x) \in [0, t^*) \times \mathbb{R}$, and therefore $\overline{u} \geq u$ and $\overline{w} \geq w$ in $[0, t^*) \times \mathbb{R}$. By applying the above argument over $[0, T] \times \mathbb{R}$, we have $\overline{u} \geq u$ and $\overline{w} \geq w$ in $[0, T] \times \mathbb{R}$.

For $(t, x) \in \Omega_{t^*} := \{(t, x) \in \mathbb{R}^2 : 0 < t \leq t^*, g(t) < x < h(t)\}$, letting $Z = (\overline{u} - u)e^{k_0t}$, then as in the proof of $\overline{u} \geq 0$ we get $Z(t, x) \geq 0$ in $\Omega_{t^*}$. Also, there is $Z(0, x) = \overline{u}(0, x) - u(0, x) \geq \not= 0$, then $Z(t, x) > 0$ and in turn $\overline{u}(t, x) > u(t, x)$ in $\Omega_{t^*}$.

On the other hand, we have

$$0 \geq \overline{h}'(t^*) - h'(t^*) \geq \mu \int_{\overline{u}(t^*)}^{u(t^*)} \int_{\overline{u}(t^*)}^{u(t^*)} J(x - y) (\overline{u} - u) (t^*, x)dydx > 0,$$

$$0 \leq \overline{g}'(t^*) - g'(t^*) \leq -\mu \int_{\overline{u}(t^*)}^{u(t^*)} \int_{\overline{u}(t^*)}^{u(t^*)} J(x - y) (\overline{u} - u) (t^*, x)dydx < 0,$$

contradictions happen. Then $h(t) < \overline{h}(t)$ and $g(t) > \overline{g}(t)$ for all $t \in (0, T]$ (resp. $h(t) > \overline{h}(t)$, $g(t) < \overline{g}(t)$ for all $t \in (0, T]$) hold), so the claim is true.

For the case that $h_0 = \overline{h}(0)$ and $-h_0 = \overline{g}(0)$. Let $(u_\epsilon, v_\epsilon, g_\epsilon, h_\epsilon)$ with $\epsilon > 0$ small be the unique positive solution to (1.1) with $h_0$ and $-h_0$ are respectively replaced by $h_0(1 - \epsilon)$ and $-h_0(1 + \epsilon)$. Using the continuous dependence of solutions on the parameters, we find that $(u_\epsilon, v_\epsilon, g_\epsilon, h_\epsilon) \to (u, v, g, h)$ as $\epsilon \to 0$, and $(u, v, g, h)$ is the unique solution of (1.1). Then the results can be obtained by letting $\epsilon \to 0$ in the inequalities $u_\epsilon \leq \overline{u}, v_\epsilon \geq \overline{v}, g_\epsilon > \overline{g}$ and $h_\epsilon < \overline{h}$. □

Following are two essential conclusions to be used later.

**Lemma 2.4.** ( [2] Lemma 3.3) Assume that (J) holds, and $h_0, T > 0$. Suppose that $u(t, x)$ as well as $u_t(t, x)$ are continuous in $\Omega_0 := [0, T] \times [-h_0, h_0]$, and for some $c \in L^\infty(\Omega_0)$,

$$\begin{cases}
 u_t(t, x) \geq d \int_{-h_0}^{h_0} J(x - y)u(t, y)dy - du + c(t, x)u, & t \in (0, T], \ x \in [-h_0, h_0], \\
 u(0, x) \geq 0, & x \in [-h_0, h_0].
\end{cases}$$

Then $u(t, x) \geq 0$ for all $0 \leq t \leq T$ and $x \in [-h_0, h_0]$. Moreover, if $u(0, x) \not= 0$ in $[-h_0, h_0]$, then $u(t, x) > 0$ in $(0, T] \times [-h_0, h_0]$.

### 3. Dynamics of the two species

In this section, we will devote to the long-term dynamics of the two species in (1.1) in the case that the population dynamics of the two species are not identical.

**3.1. Spectrum and principal eigenvalue.**

In this subsection we collect some essential results that regarding the following linear dispersal equation

$$u_t(t, x) = d \left[ \int_{\mathbb{R}} J(x - y) u(t, y)dy - u(t, x) + au(t, x) \right] \text{ for } t > 0 \text{ and } x \in \Omega,$$
where \( \Omega \in \mathbb{R} \) is an open set, parameter \( a > 0 \) is a constant. Let \( X = C(\overline{\Omega}, \mathbb{R}) \) be equipped with the maximum norm, \( X^+ = \{ u \in X \mid u(x) \geq 0, x \in \overline{\Omega} \} \) and \( X^{++} = \{ u \in X^+ \mid u(x) > 0, x \in \overline{\Omega} \} \).

For any given \( u_1, u_2 \in X \), define

\[
\begin{align*}
&u_1 \leq u_2 \text{ or } u_2 \geq u_1 \text{ if } u_2 - u_1 \in X^+, \\
&u_1 \ll u_2 \text{ or } u_2 \gg u_1 \text{ if } u_2 - u_1 \in X^{++}.
\end{align*}
\]

Following are some well known results.

**Theorem 3.1.** (Coville et al. \[6\]) Assume that \((J)\) holds. Let \( \Omega \subset \mathbb{R} \) is a bounded open interval. Then there exists a smallest \( \lambda_1 = \lambda_1(d, a, \Omega) \) such that problem

\[
\begin{align*}
\begin{cases}
\frac{d}{dt} \left( J(x - y)\phi(y)dy - \phi(x) \right) + a\phi(x) = -\lambda_1 \phi(x) & \text{in } \Omega, \\
\phi = 0 & \text{for all } x \notin \Omega \text{ and } \phi|_{\partial \Omega} \text{ is continuous}
\end{cases}
\end{align*}
\]

has a nontrivial solution. This eigenvalue is simple and the eigenfunctions are of constant sign in \( \Omega \). Moreover,

\[
\lambda_1(d, a, \Omega) = \min_{\phi \in C(\overline{\Omega}), \phi \neq 0} \frac{- d \int_{\Omega} \int_{\mathbb{R}} J(x - y)\phi(y)dy\phi(x)dx + a \int_{\Omega} \tilde{\phi}^2(x)dx}{\int_{\Omega} \tilde{\phi}^2(x)dx},
\]

where \( \tilde{\phi} \) denotes the extension by 0 of \( \phi \) to \( \mathbb{R} \) and the minimum is attained.

The asymptotic behavior both for small and larger domains read as

**Theorem 3.2.** (\[5, Proposition 3.4\]) Assume that \((J)\) holds and \( \Omega = [l_1, l_2] \) with \(-\infty < l_1 < l_2 < \infty \). Then the principle eigenvalue \( \lambda_1(d, a, \Omega) \) of \((3.2)\) is strictly decreasing and continuous in \( l = l_2 - l_1 \), and

1. \( \lim_{l_2 - l_1 \to 0} \lambda_1(d, a, \Omega) = d - a; \)
2. \( \lim_{l_2 - l_1 \to \infty} \lambda_1(d, a, \Omega) = -a; \)

**Theorem 3.3.** For the principle eigenvalue \( \lambda_1(d, a, \Omega) \) of \((3.2)\) with \( \Omega = [l_1, l_2] \), if \( 0 < a < d \), there exists a unique positive \( R^* \) such that \( \lambda_1(d, a, \Omega) = 0 \) if \( l_2 - l_1 = R^* \), \( \lambda_1(d, a, \Omega) > 0 \) if \( l_2 - l_1 < R^* \) and \( \lambda_1(d, a, \Omega) < 0 \) if \( l_2 - l_1 > R^* \).

The following results concerns with the asymptotic behavior of the solution of the evolution problem

\[
(3.3) \quad u_t(t, x) = d(J * u - u) + f(t, u) \text{ in } \mathbb{R}^+ \times \Omega \text{ and } u(0, x) = u_0(x) \text{ in } \Omega,
\]

where the reaction term \( f(x, u) \) satisfying the following assumption:

**\(A3\):** \( f(x, u) \in C(\mathbb{R} \times [0, +\infty)) \) is differential with respect to \( u \) and \( f_u(x, 0) \) is Lipshitz continuous in \( \mathbb{R} \), \( f(x, 0) = 0 \) and \( \frac{d}{du} f(x, u) \) is strictly decreasing in \( u \in \mathbb{R}^+ \); there exists a constant \( \tilde{K} > 0 \) such that \( f(x, u) < 0 \) for all \( x \in \mathbb{R} \) and \( u \geq \tilde{K} \).

**Theorem 3.4.** (\[4,7\]) Assume that \((J)\) and \((A3)\) hold, and \( \Omega \subset \mathbb{R} \) is bounded. Let \( u_0 \) be an arbitrary bounded and continuous function in \( \Omega \) such that \( u_0 \geq \neq 0 \). Let \( u(t, x) \) be the solution of \((3.3)\) with initial datum \( u(0, x) = u_0(x) \). Then problem \((3.3)\) admits a unique positive steady state \( u_\Omega \) if and only if \( \lambda_1(d, f_u(x, 0), \Omega) < 0 \).

1. \( u(t, x) \to u_\Omega \) uniformly in \( x \in \Omega \) as \( t \to +\infty \) if \( \lambda_1(d, a, \Omega) < 0 \);
2. \( u(t, x) \to 0 \) uniformly in \( x \in \Omega \) as \( t \to +\infty \) if \( \lambda_1(d, a, \Omega) \geq 0 \).
3.2. Vanishing Case \((h_\infty - g_\infty < \infty)\).

It follows from Theorem 2.2 that \(h(t)\) and \(-g(t)\) are monotone increasing. Then there exist \(h_\infty\) and \(g_\infty\) such that \(h_\infty = \lim_{t \to \infty} h(t)\) and \(g_\infty = \lim_{t \to \infty} g(t)\). To establish the long time behavior of \((u,v)\), we first derive an estimate.

**Theorem 3.5.** Let \((u,v,g,h)\) be the unique solution of (1.1). If \(h_\infty - g_\infty < \infty\), then \(\lim_{t \to \infty} g'(t) = \lim_{t \to \infty} h'(t) = 0\).

**Proof.** Choose constant \(K^*\) with \(K^* \geq \max\{\|u\|_{C([g(t),h(t))]}, \|u_t\|_{C([g(t),h(t))]}\}\), then for any \(\tau, s \geq 0\) and \(\theta\) between \(\tau\) and \(s\), we have

\[
h'(\tau) - h'(s) = \mu \int_{g(\tau)}^{h(\tau)} \int_{h(\tau)}^{+\infty} J(x - y)u(\tau, x)dydx - \mu \int_{g(s)}^{h(s)} \int_{h(s)}^{+\infty} J(x - y)u(s, x)dydx
\]

\[
= \mu \left( \int_{g(\tau)}^{h(s)} + \int_{g(s)}^{h(\tau)} \right) \int_{h(\tau)}^{+\infty} J(x - y)u(\tau, x)dydx
\]

\[
- \mu \int_{g(\tau)}^{h(s)} \left( \int_{h(s)}^{+\infty} J(x - y)u(s, x)dydx + \int_{h(s)}^{h(\tau)} J(x - y)u(\tau, x)dydx + \int_{g(s)}^{g(\tau)} J(x - y)u(\tau, x)dydx\right).
\]

And then for \(\xi_1, \xi_2\) between \(s\) and \(\tau\),

\[
|h'(\tau) - h'(s)| \leq \mu K^* |\tau - s| \cdot \left( |h(\tau) - g(\tau)| + |g(\tau) - g(s)| + 2\mu K^* |h(\tau) - h(s)|\right)
\]

along with the condition that \(h_\infty - g_\infty < \infty\) indicates that \(h'(t)\) is Lipschitz continuous in \([0, \infty)\). We use the condition \(h_\infty - g_\infty < \infty\) again to obtain that \(\lim_{t \to +\infty} h'(t) = 0\). Analogously, there is \(\lim_{t \to +\infty} g'(t) = 0\). \(\square\)

**Theorem 3.6.** Let \((u,v,g,h)\) be the solution of problem (1.1) with \(h_\infty - g_\infty < \infty\), then \(\lim_{t \to +\infty} \|u\|_{C([g(t),h(t))]} = 0\).

**Proof.** Since \(h_\infty - g_\infty < \infty\), it follows from Theorem 3.5 that \(h'(t), -g'(t) \to 0\) as \(t \to +\infty\). Also, by \(\|u, v\|_{\infty} \leq M_0\), there holds

\[
\begin{aligned}
    u_t & \geq d_1 \left( \int_{g(t)}^{h(t)} J(x - y)u(t, y)dy - u \right) + u(a_1 - b_1 u - c_1 M_0), \quad t > 0, \quad g(t) < x < h(t), \\
v_t & \geq d_2 \left( \int_{\mathbb{R}} J(x - y)v(t, y)dy - v \right) + v(a_2 - b_2 u - c_2 M_0), \quad t > 0, \quad x \in \mathbb{R}.
\end{aligned}
\]

Assume on the contrary that \(\lim_{t \to +\infty} \|u\|_{C([g(t),h(t))]} > 0\), then there exist \(\varepsilon_1 > 0\) and sequence \(\{(t_k, x_k)\}_{k=1}^{\infty}\) with \(x_k \in (g(t), h(t))\) and \(t_k \to +\infty\) as \(k \to \infty\) such that \(u(t_k, x_k) \geq \varepsilon_1\) for all \(k \in \mathbb{N}\).

Since \(g_\infty < g(t) < h(t) < h_\infty\), passing to a subsequence if necessary, we then have \(x_k \to x_0 \in (g_\infty, h_\infty)\) as \(k \to \infty\). For \(t \in (-t_k, +\infty)\) and \(x \in (g(t + t_k), h(t + t_k))\), define

\[
U_k(t, x) = u(t + t_k, x).
\]

A NONLOCAL DIFFUSION MODEL WITH FREE BOUNDARIES 17
By Theorem 2.2, we see that $u(t, x)$ is bounded, it then follows that (passing to a subsequence if necessary) $U'_k(t, x) \to \bar{U}_t(t, x)$ as $k \to \infty$, for $x \in (g_\infty, h_\infty)$, $\bar{U}_t(t, x)$ satisfies

$$
\begin{align*}
\bar{U}_t & \geq d_1 \int_{g_\infty}^{h_\infty} \int_{y_0}^{x} J(x-y) \bar{U}_t(y)dy - d_1 \bar{U}_t(t, x) + \bar{U}_t(a_1 - b_1 u - c_1 M_0), \\
\bar{U}_0(x_0) = \lim_{k \to \infty} U_k(0, x_k) &= \lim_{k \to \infty} u(t_k, x_k) \geq \frac{\epsilon_1}{2} > 0.
\end{align*}
$$

The Maximum Principle yields that $\bar{U}_t(t, x) > 0$ in $\mathbb{R} \times (g_\infty, h_\infty)$.

Further, since $h'(t), -g'(t) \to 0$ as $t \to \infty$, then there hold

$$
0 = \lim_{k \to \infty} h'(t + t_k) = \mu \lim_{k \to \infty} \int_{g(t) + t_k}^{h(t) + t_k} \int_{h(t) + t_k}^{+\infty} J(x-y)U_k(t, x)dydx
$$

$$
= \mu \int_{g_\infty}^{h_\infty} \int_{h_\infty}^{+\infty} J(x-y)\bar{U}_t(t, x)dydx > 0
$$

and

$$
0 = \lim_{k \to \infty} g'(t + t_k) = -\mu \lim_{k \to \infty} \int_{g(t) + t_k}^{h(t) + t_k} \int_{-\infty}^{g(t) + t_k} J(x-y)U_k(t, x)dydx
$$

$$
= -\mu \int_{g_\infty}^{h_\infty} \int_{-\infty}^{g_\infty} J(x-y)\bar{U}_t(t, x)dydx < 0,
$$

contradictions. Hence there holds $\lim_{t \to +\infty} \|u\|_{C((g(t), h(t))} = 0$. This completes the proof. \boxed{}

3.3. **Spreading Case ($h_\infty - g_\infty = \infty$) with $a_1 \over a_2 < \min \left\{ \frac{b_1}{b_2}, \frac{c_1}{c_2} \right\}$.**

The condition $a_1 \over a_2 < \min \left\{ \frac{b_1}{b_2}, \frac{c_1}{c_2} \right\}$ here means that when compared with the species $v$, the species $u$ is an inferior competitor. Further, we assume that

(F1): $a_2 < d_2$.

**Theorem 3.7.** Assume that $(u, v, g, h)$ is the unique positive solution of (1.1) with $a_1 \over a_2 < \min \left\{ \frac{b_1}{b_2}, \frac{c_1}{c_2} \right\}$. If $h_\infty - g_\infty = \infty$, then $\lim_{t \to +\infty} (u(t, x), v(t, x)) = (0, \frac{a_1}{a_2})$ holds uniformly in any compact subset of $\mathbb{R}$.

**Proof.** Note that for $t > 0$ and $x \in \mathbb{R}$, we have $u(t, x) \leq \bar{u}(t)$ and $v(t, x) \leq \bar{v}(t)$, here $\bar{u}(t)$ and $\bar{v}(t)$ are separately defined by

$$
\begin{align*}
\bar{u}(t) &= \frac{a_1}{b_1} e^{\frac{a_1}{b_1}t} \left( e^{\frac{a_1}{b_1}t} - 1 + \frac{a_1}{b_1\|u_0\|_{L^\infty}} \right)^{-1}, \\
\bar{v}(t) &= \frac{a_2}{c_2} e^{\frac{a_2}{c_2}t} \left( e^{\frac{a_2}{c_2}t} - 1 + \frac{a_2}{c_2\|v_0\|_{L^\infty}} \right)^{-1}.
\end{align*}
$$

Then it is obvious that

$$
\limsup_{t \to +\infty} u(t, x) \leq \frac{a_1}{b_1} \limsup_{t \to +\infty} v(t, x) \leq \frac{a_2}{c_2} \text{ uniformly for } x \in \mathbb{R}.
$$

And hence for $0 < \epsilon_1 < \frac{1}{2} \left( \frac{a_2}{b_2} - \frac{a_1}{b_1} \right)$, we can find some $t_0 > 0$ such that $u(t, x) \leq \frac{a_1}{b_1} + \epsilon_1$ for $t \geq t_0$ and $x \in \mathbb{R}$. Then by defining $A_1 = b_2 \left( \frac{a_2}{b_2} - \frac{a_1}{b_1} \epsilon_1 \right)$, we have

$$
\begin{align*}
v_t - d_2 \int_{\mathbb{R}} J(x-y)v(t, y)dy - v \geq v \left[ A_1 - c_2v \right], \quad t \geq t_0, \quad x \in \mathbb{R}, \\
v(t_0, x) > 0,
\end{align*}
$$

(3.4)
It follows from the comparison principal that \( v(t, x) \geq v^*(t, x) \) for all \( t \geq t_0 \) and \( x \in \mathbb{R} \), where \( v^*(t, x) \) is the solution to
\[
\begin{aligned}
&\left\{ \begin{array}{l}
v_t - d_2 \left[ \int_{-L}^{L} J(x-y)v^*(t,y)dy - v^* \right] = v^* \left[ A_1 - c_2 v^* \right], \quad t \geq t_0, \ x \in \mathbb{R}, \\
v^*(t_0, x) = v(t_0, x) > 0,
\end{array} \right. \\
&x \in \mathbb{R}.
\end{aligned}
\] (3.5)

For given \( L > \frac{R^*}{2} \), we can find some \( t_L > t_0 \) such that \( h(t_L) - g(t_L) \geq 2L \), and \( v^*(t, x) \geq v_L(t, x) \) for \( t \geq t_L \) and \( x \in (-L, L) \), where \( v_L(t, x) \) verifies
\[
\begin{aligned}
&\left\{ \begin{array}{l}
v_t - d_2 \left[ \int_{-L}^{L} J(x-y)v(t,y)dy - v \right] = v \left[ A_1 - c_2 v \right], \quad t \geq t_L, \ x \in (-L, L), \\
v(t_L, x) = v^*(t_L, x) > 0,
\end{array} \right. \\
&x \in (-L, L).
\end{aligned}
\] (3.6)

Denoting \( \lambda_1(d_2, A_1, \Omega_L) \) with \( \Omega_L = [-L, L] \) by the principal eigenvalue of problem (3.3), it then follows from assumption (F1) and Theorem 3.2 that
\[\lambda_1(d_2, A_1, \Omega_L) < \lambda_1(d_2, A_1, \Omega_{R^*}) = 0 \quad \text{with} \quad |\Omega_{R^*}| = R^*.
\]

And hence it follows from Theorem 3.4 that
\[
\lim_{t \to +\infty} v_L(t, x) = \frac{A_1}{c_2} = \frac{b_2}{c_2} \left( \frac{a_2}{b_2} - \frac{a_1}{b_1} - \epsilon_1 \right) \quad \text{uniformly in any bounded subset of} \quad \mathbb{R}.
\]

So for the given \( L > 0 \), we can find some \( t_L > t_1 \) such that
\[
v(t, x) \geq v^*(t, x) \geq \tilde{A} := \frac{A_1}{2c_2} \quad \text{for} \quad t \geq t_L \quad \text{and} \quad -L \leq x \leq L.
\] (3.7)

Check the equation of \( u \), note that \( u \) now satisfies
\[
\begin{aligned}
&\left\{ \begin{array}{l}
u_t - d_1 \left[ \int_{g(t)}^{b(t)} (x-y)u(t,y)dy - u \right] = u(a_1 - b_1 u - c_1 v), \quad t > t_L, \ x \in (g(t), h(t)), \\
u(t, x) = 0, \quad t > t_L, \ x \notin (g(t), h(t)), \\
u(t, x) \leq \frac{a_1}{b_1} + \epsilon_1, \quad t > t_L, \ x \in (g(t), h(t)).
\end{array} \right.
\end{aligned}
\] (3.8)

Then it follows from Comparison Principle that \( u \leq \overline{u} \) and \( v \geq \underline{v} \) for \( t \geq t_L \) and \( x \in [-L, L] \), where \( (\overline{u}, \underline{v}) \) is the solution of
\[
\begin{aligned}
&\left\{ \begin{array}{l}
\overline{u}_t - d_1 \left[ \int_{-L}^{L} J(x-y)\overline{u}(t,y)dy - \overline{u} \right] = \overline{u} \left( a_1 - b_1 \overline{u} - c_1 \underline{v} \right), \quad t > t_L, \ -L < x < L, \\
\underline{v}_t - d_2 \left[ \int_{-L}^{L} J(x-y)\underline{v}(t,y)dy - \underline{v} \right] = \underline{v} \left( a_2 - b_2 \overline{u} - c_2 \underline{v} \right), \quad t > t_L, \ -L < x < L, \\
\overline{u}(t, L) = \overline{u}(t, L) = \frac{a_1}{b_1} + \epsilon_1, \quad \underline{v}(t, L) = \underline{v}(t, L) = \tilde{A}, \quad -L \leq x \leq L,
\end{array} \right.
\end{aligned}
\] (3.9)

with \( \left( \frac{a_1}{b_1} + \epsilon_1, \tilde{A} \right) \) a pair of upper solution. In view of the dependence of solutions on initial data, we denote \((u(t, x; u_0, v_0), v(t, x; u_0, v_0)) \) (resp. \((\overline{u}(t, x; \overline{u}_L, \underline{v}_L), \underline{v}(t, x; \overline{u}_L, \underline{v}_L)) \)) by the solution of problem (1.1) (resp. (3.3)). Note that \( f_1^* := \overline{u}(a_1 - b_1 \overline{u} - c_1 \underline{v}) \) is nonincreasing in \( \underline{v} \) and \( f_2^* := \underline{v}(a_2 - b_2 \overline{u} - c_2 \underline{v}) \) is nonincreasing in \( \overline{u} \), then (3.9) generates a monotone dynamical system with respect to the order
\[
(u_1, v_1) \leq (u_2, v_2) \quad \text{if} \quad u_1 \leq u_2 \quad \text{and} \quad v_1 \geq v_2.
\]
This implies that for \( t_2 > t_1 \geq t_L \) and \( x \in [-L, L], \)
\[
\left( \overline{u}(t_2, x; \overline{u}_L, \overline{v}_L), \underline{u}(t_2, x; \underline{u}_L, \underline{v}_L) \right) \leq 2 \left( \overline{u}(t_1, x; \overline{u}_L, \overline{v}_L), \underline{u}(t_1, x; \underline{u}_L, \underline{v}_L) \right) \leq 2 \left( \frac{a_1}{b_1} + \epsilon_1, \tilde{A} \right).
\]
Hence \( \lim_{t \to +\infty} \left( \overline{u}(t, x; \overline{u}_L, \overline{v}_L), \underline{u}(t, x; \underline{u}_L, \underline{v}_L) \right) = (\overline{u}_L(x), \underline{v}_L(x)) \) uniformly in \([-L, L], \) where \((\overline{u}_L, \underline{v}_L)\) satisfies
\[
\begin{cases}
- d_1 \left[ \int_{-L}^{L} J(x-y)\overline{u}_L(y)dy - \overline{u}_L \right] = \overline{u}_L(a_1 - b_1 \overline{u}_L - c_1 \underline{v}_L), & -L < x < L, \\
- d_2 \left[ \int_{-L}^{L} J(x-y)\overline{u}_L(y)dy - \overline{u}_L \right] = \overline{u}_L(a_2 - b_2 \overline{u}_L - c_2 \underline{v}_L), & -L < x < L, \\
\overline{u}(-L) = \overline{u}(L) = \frac{a_1}{b_1} + \epsilon_1, & \overline{u}(L) = \overline{v}(-L) = \tilde{A}.
\end{cases}
\tag{3.10}
\]
By comparing the boundary conditions in (3.10) we then observe that for \( 0 < L_1 < L_2, \overline{u}_L(x) \geq \overline{u}_{L_2}(x) \) and \( \underline{v}_{L_1}(x) \leq \underline{v}_{L_2}(x) \) in \([-L_1, L_1], \) Hence, by letting \( L \to \infty \) and then a diagonal procedure, there is \((\overline{u}_L(x), \underline{v}_L(x)) \to (\overline{u}^*(x), \underline{v}^*(x))\) uniformly on any compact subset of \( \mathbb{R}, \) where \((\overline{u}^*, \underline{v}^*)\) satisfies
\[
\begin{cases}
- d_1 \left[ \int_{-L}^{L} J(x-y)\overline{u}^*(y)dy - \overline{u}^* \right] = \overline{u}^*(a_1 - b_1 \overline{u}^* - c_1 \underline{v}^*), & x \in \mathbb{R}, \\
- d_2 \left[ \int_{-L}^{L} J(x-y)\overline{u}^*(y)dy - \overline{u}^* \right] = \overline{u}^*(a_2 - b_2 \overline{u}^* - c_2 \underline{v}^*), & x \in \mathbb{R}, \\
\overline{u}^*(x) \leq \frac{a_1}{b_1} + \epsilon_1, & \overline{v}^*(x) \geq \tilde{A},
\end{cases}
\tag{3.11}
\]
On the other hand, since \( \frac{a_1}{a_2} < \min \left\{ \frac{b_1}{b_2}, \frac{c_1}{c_2} \right\}, \) then the solution \((u_1, v_1)\) of the following problem
\[
\begin{cases}
(u_1)_t = u_1(a_1 - b_1 u_1 - c_1 v_1), & t > 0, \\
(v_1)_t = v_1(a_2 - b_2 u_1 - c_2 v_1), & t > 0, \\
u_1(0) = \frac{a_1}{b_1} + \epsilon_1, & v_1(0) = \tilde{A},
\end{cases}
\tag{3.12}
\]
satisfies \( \lim_{t \to +\infty} (u_1(t), v_1(t)) = \left( 0, \frac{a_2}{c_2} \right), \) see Morita et al. [24]. This further implies that solution \((U(t, x), V(t, x))\) of the problem
\[
\begin{align*}
U_t - d_1(J \ast U - U) & = U(a_1 - b_1 U - c_1 V), & t > 0, & x \in \mathbb{R}, \\
V_t - d_2(J \ast V - V) & = V(a_2 - b_2 U - c_2 V), & t > 0, & x \in \mathbb{R}, \\
U(0, x) & = \frac{a_1}{b_1} + \epsilon_1, & V(0, x) = \tilde{A}, & x \in \mathbb{R}
\end{align*}
\tag{3.13}
\]
satisfies \( \lim_{t \to +\infty} (U(t, x), V(t, x)) = \left( 0, \frac{a_2}{c_2} \right) \) uniformly in any bounded subset of \( \mathbb{R}. \) Meanwhile, by using the comparison principle to problems (3.11) and (3.13) we obtain that
\[
\overline{u}^*(x) \leq U(t, x) \text{ and } \underline{v}^*(x) \geq V(t, x) \text{ for all } x \in \mathbb{R},
\]
which indicates that \( \overline{u}^*(x) = 0 \) and \( \underline{v}^*(x) \geq \frac{a_2}{c_2} \) for all \( x \in \mathbb{R}, \) and then \( \overline{u}_L(x) = 0 \) and \( \underline{v}_L(x) \geq \frac{a_2}{c_2} \) for \( x \in (-L, L), \) and hence
\[
\overline{u}(t, x; \overline{u}_L, \underline{v}_L) \to 0 \text{ and } \underline{v}(t, x; \overline{u}_L, \underline{v}_L) \geq \frac{a_2}{c_2} \text{ as } t \to +\infty.
\]
Further, we get \( u(t, x) = 0 \) and \( v(t, x) \geq \frac{a_2}{c_2} \) as \( t \to +\infty \). Then
\[
\lim_{t \to +\infty} \| u(t, \cdot) \|_{C([g(t), h(t)])} = 0 \quad \text{and} \quad \lim_{t \to +\infty} \| v(t, \cdot) \|_{C(\mathbb{R})} = \frac{a_2}{c_2}.
\]
This completes the proof. \( \square \)

3.4. **Spreading Case** \((h_\infty - g_\infty = \infty)\) **with** \( \frac{a_1}{a_2} > \max \left\{ \frac{b_1}{b_2}, \frac{c_1}{c_2} \right\} \).

The condition \( \frac{a_1}{a_2} > \max \left\{ \frac{b_1}{b_2}, \frac{c_1}{c_2} \right\} \) here means that when compared with the species \( v \), the species \( u \) is an superior competitor.

It is stated in section 3.1 that the eigenvalue problem
\[
d_1 (J - I) \phi(x) + a_1 \phi(x) = -\lambda_1 \phi(x) \text{ in } \Omega, \quad \phi = 0 \text{ for all } x \not\in \Omega \quad \text{and} \quad \phi |_{\partial \Omega} \text{ is continuous}
\]
admits an eigen pair \((\lambda_1(d_1, a_1, \Omega), \phi_1(x))\). And if we assume further that \( a_1 < d_1 \), then there exists a unique \( R^* \) such that \( \lambda_1(d_1, a_1, \Omega) = 0 \) when \(|\Omega| = R^*\). In what follows, we assume that

\[(F2): \; a_1 < d_1.\]

**Theorem 3.8.** Assume that \( \frac{a_1}{a_2} > \max \left\{ \frac{b_1}{b_2}, \frac{c_1}{c_2} \right\} \). If \( h_\infty - g_\infty < \infty \), then \( h_\infty - g_\infty \leq R^* \).

**Proof.** We first prove that \( h_\infty - g_\infty \leq R^* \). Otherwise \( h_\infty - g_\infty > R^* \) and there exists \( T_1 > 0 \) such that \( h(t) > h_\infty - \epsilon_2 \), \( g(t) < g_\infty + \epsilon_2 \) and \( h(t) - g(t) > h_\infty - g_\infty - 2\epsilon_2 > R^* \) for all \( t \geq T_1 \) and some small \( \epsilon_2 \) with \( 0 < \epsilon_2 < \frac{1}{2} \left( \frac{a_1 - a_2}{c_2} \right) \).

Since \( \limsup_{t \to +\infty} v(t, x) \leq \frac{a_2}{c_2} \), for the above \( \epsilon_2 \) we can find \( T_2 \geq T_1 \) such that \( v(t, x) \leq \frac{a_2}{c_2} + \epsilon_2 := A_2 \) for all \( t \geq T_2 \) and \( x \in \mathbb{R} \).

Denote \( \Omega^2_{\infty} = (g_\infty + \epsilon_2, h_\infty - \epsilon_2) \), and consider the following problem

\[(3.14) \quad \begin{cases} w_t - d_1 \left[ \int_{\Omega^2_{\infty}} J(x-y)w(t,y)dy - w \right] = w \left[ a_1 - c_1A_2 - b_1w \right], & t \geq T_2, \; x \in \Omega^2_{\infty}, \\
w(t, x) = 0, & t \geq T_2, \; x \not\in \Omega^2_{\infty}, \\
w(T_2, x) = u(T_2, x), & x \in \Omega^2_{\infty}. \end{cases}\]

It is well-known that (see [17][18]) problem \((3.14)\) admits a unique positive solution denoted by \( \overline{w}(t, x) = \overline{w}_2(t, x) \). It then follows from the comparison principle that
\[
u(t, x) \geq \overline{w}(t, x) \quad \text{for} \; t > T_2 \quad \text{and} \; x \in [g_\infty + \epsilon_2, h_\infty - \epsilon_2].
\]

In addition, if we use \( \lambda^2_1(\infty) \) to denote the principal eigenvalue of problem \((3.14)\), then \( \lambda^2_1(\infty) < \lambda_1(R^*) = 0 \). It then follows from Theorems 3.4 (see also Hutson et al. [17] Theorem 3.6]) that
\[
\overline{w}(t, x) \to \frac{a_1}{b_1} - \frac{c_1}{b_1}A_2 \text{ in } C([g_\infty + \epsilon_2, h_\infty - \epsilon_2]) \text{ as } t \to +\infty.
\]

It turns out that \( \liminf_{t \to +\infty} u(t, x) \geq \frac{a_1}{b_1} - \frac{c_1}{b_1}A_2 > 0 \) uniformly in \([g_\infty + \epsilon_2, h_\infty - \epsilon_2]\).

Similarly, the following problem
\[(3.15) \quad \begin{cases} w_t - d_1(J \ast w - w) = w \left[ a_1 - c_1A_2 - b_1w \right], & t \geq T_2, \; x \in (g_\infty, h_\infty), \\
w(t, x) = 0, & t \geq T_2, \; x \not\in (g_\infty, h_\infty), \\
w(T_2, x) = \tilde{u}(T_2, x), & x \in (g_\infty, h_\infty). \end{cases}\]

with \( \tilde{u}(T_2, x) = u(T_2, x) \) for \( x \in [g(T_2), h(T_2)] \) and \( \tilde{u}(T_2, x) = 0 \) if \( x \in (g_\infty, g(T_2)) \cup (h(T_2), h_\infty) \) admits a unique positive solution \( \tilde{w}(t, x) \) such that \( u(t, x) \leq \tilde{w}(t, x) \) for \( t > T_2 \) and \( x \in [g(t), h(t)] \).
Thus, there holds \(\limsup_{t \to +\infty} u(t, x) \leq \frac{a_1}{b_1} - \frac{c_1}{b_1} A_2\) for \(x \in [g_\infty, h_\infty]\). By taking \(\epsilon_2 \to 0\) deduces that \(\lim_{t \to +\infty} u(t, x) = \frac{a_1}{b_1} - \frac{c_1}{b_1} A_2 > 0\) for \(x \in [g_\infty, h_\infty]\). Combining this with Theorem 3.6 immediately deduces that \(h_\infty - g_\infty = \infty\), this contradiction proves that \(h_\infty - g_\infty < R^*\). \(\square\)

Theorem 3.8 also implies that if \(2h_0 \geq R^*\), then \(h_\infty - g_\infty = \infty\).

**Theorem 3.9.** Assume that \(\frac{a_1}{a_2} > \max\left\{\frac{b_1}{b_2}, \frac{c_1}{c_2}\right\}\) holds. Let \((u, v, g, h)\) be the unique positive solution of (1.1) with \(h_\infty - g_\infty = \infty\), then \(\lim_{t \to +\infty} (u, v)(t, x) = \left(\frac{a_1}{b_1}, 0\right)\) uniformly in any bounded subset of \(\mathbb{R}\).

**Proof.** The method here is similar as that in Theorem 3.7. Since \(\limsup_{t \to +\infty} v(t, x) \leq \frac{a_2}{c_2}\), then we can find \(T_4 > 0\) large such that \(v(t, x) \leq \frac{a_2}{c_2} + \epsilon_3\) for \(t \geq T_4\), where \(0 < \epsilon_3 \ll \frac{1}{2} \left(\frac{a_1}{c_1} - \frac{a_2}{c_2}\right)\).

Meanwhile, by \(h_\infty - g_\infty = \infty\), we can find \(T_5\) with \(T_3 \geq T_4\) such that \(h(t) - g(t) > R^*\) for \(t \geq T_5\).

Denoting \(B_1 = \frac{a_2}{c_2} + \epsilon_3\), using \((u, g, h)\) to denote the positive solution of the following problem (3.16)

\[
\begin{cases}
    u_t - d_1 \left[ \int_{g(t)}^{h(t)} J(x-y)u(t,y)dy - u \right] = u [a_1 - c_1 B_1 - b_1 u], & t \geq T_5, \ x \in (g(t), h(t)), \\
    u(t, x) = 0, & t \geq T_5, \ x \notin (g(t), h(t)), \\
    \frac{h'(t)}{\mu} = \int_{g(t)}^{h(t)} \int_{h(t)}^{+\infty} J(x-y)u(t,x)dydx, & t \geq T_5, \\
    \frac{g'(t)}{\mu} = -\int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J(x-y)u(t,x)dydx, & t \geq T_5, \\
    u(T_5, x) = u(T_5, x), \ h(T_5) = h(T_5), \ g(T_5) = g(T_5), & x \in (g(T_5), h(T_5)).
\end{cases}
\]

Then \(u(t, x) \geq u(t, x), \ g(t) \leq g(t)\) and \(h(t) \geq h(t)\) for \(t \geq T_5\) and \(x \in [g(t), h(t)]\) by the comparison principle.

In addition, we have \(h(T_5) - g(T_5) > R^*\). Then for \(t \geq T_5\), the principle eigenvalue \(\lambda_1^* = \lambda_1^*(d_1, a_1 - c_1 B_1, \tilde{\Omega}(t))\) of (3.16) satisfies \(\lambda_1^* \leq \lambda_1^*(d_1, a_1 - c_1 B_1, \tilde{\Omega}(T_5)) < 0\), where \(\tilde{\Omega}(t) = (g(t), h(t))\).

Since \(\lambda_1^* < 0\), then it follows from Theorem 3.4 that

\[
\lim_{t \to +\infty} u(t, x) = \frac{a_1}{b_1} \left(\frac{a_1}{a_2} - \frac{c_1}{c_2} \epsilon_3\right) := \tilde{B} > 0.
\]

Hence \(\liminf_{t \to +\infty} u(t, x) \geq \tilde{B}\). Then we can find \(T_l\) with \(T_l \geq T_5\) and \(l\) with \(l \geq R^*\) such that \(u(t, x) \geq \tilde{B}\) in \([T_l, \infty) \times [-l, l]\). Therefore, for our choices of \(T_l\) and \(l\), we arrive at

\[
u \geq u \text{ and } v \leq \overline{v} \text{ for } t \geq T_l \text{ and } x \in [-l, l],
\]
where \((u, \overline{v})\) denote the solution of the following problem

\[
\begin{align*}
\frac{\partial u}{\partial t} - d_1 \int_{-l}^{l} J(x-y) \frac{\partial u}{\partial y} \, dy - u &= u(a_1 - b_1 u - c_1 \overline{v}), \quad t \geq T_1, \, x \in (-l, l), \\
\frac{\partial \overline{v}}{\partial t} - d_2 \int_{-l}^{l} J(x-y) \frac{\partial \overline{v}}{\partial y} \, dy - \overline{v} &= \overline{v}(a_2 - b_2 u - c_2 \overline{v}), \quad t \geq T_1, \, x \in (-l, l), \\
\overline{v}(T_1, x) &= \overline{v}_{T_1}(x) = \tilde{B}, \quad \overline{v}(T_1, x) = \overline{v}_{T_1}(x) = \frac{a_2}{c_2} + \epsilon_3, \quad x \in (-l, l), \\
u(t, \pm l) &= \tilde{B}, \quad \overline{v}(t, \pm l) = \frac{a_2}{c_2} + \epsilon_3, \quad t \geq T_1
\end{align*}
\]

with \((\tilde{B}, \frac{a_2}{c_2} + \epsilon_3)\) a pair of lower solution. In view of the dependence of solutions on initial data, we denote \((u(t, x; u_{T_1}, \overline{v}_{T_1}), \overline{v}(t, x; u_{T_1}, \overline{v}_{T_1}))\) by the solution of problem (3.17). Note that \(f^{**}_1 := u(a_1 - b_1 u - c_1 \overline{v})\) is nonincreasing in \(\overline{v}\) and \(f^{**}_2 := \overline{v}(a_2 - b_2 u - c_2 \overline{v})\) is nonincreasing in \(u\), then (3.17) generates a monotone dynamical system with respect to the order \(\leq_2\). This implies that for \(t_2 > t_1 \geq T_1\) and \(x \in [-l, l]\),

\[
\left(\frac{\tilde{B}, a_2}{c_2} + \epsilon_3\right) \leq_2 \left(u(t_2, x; u_{T_1}, \overline{v}_{T_1}), \overline{v}(t_2, x; u_{T_1}, \overline{v}_{T_1})\right) \leq_2 \left(u(t_1, x; u_{T_1}, \overline{v}_{T_1}), \overline{v}(t_1, x; u_{T_1}, \overline{v}_{T_1})\right)
\]

Hence \(\lim_{t \to +\infty} (u(t, x; u_{T_1}, \overline{v}_{T_1}), \overline{v}(t, x; u_{T_1}, \overline{v}_{T_1})) = (\overline{v}_l(x), v_l(x))\) uniformly in \([-l, l]\), where \((u_l(x), \overline{v}_l(x))\) satisfies

\[
\begin{align*}
-d_1 \int_{-l}^{l} J(x-y) u_l(y) \, dy - u_l(x) &= u_l(a_1 - b_1 u_l - c_1 \overline{v}_l), \quad -l < x < l, \\
-d_2 \int_{-l}^{l} J(x-y) \overline{v}_l(y) \, dy - \overline{v}_l(x) &= \overline{v}_l(a_2 - b_2 u_l - c_2 \overline{v}_l), \quad -l < x < l, \\
u_l(-l) &= u_l(l) = \tilde{B}, \quad \overline{v}_l(l) = \overline{v}_l(-l) = \frac{a_2}{c_2} + \epsilon_3
\end{align*}
\]

and \(\lim_{t \to +\infty} (u_l(x), \overline{v}_l(x)) = (u^*(x), \overline{v}^*(x))\) with \((u^*(x), \overline{v}^*(x))\) satisfies

\[
\begin{align*}
-d_1 (J \ast u^* - u^*) &= u^*(a_1 - b_1 u^* - c_1 \overline{v}^*), \quad x \in \mathbb{R}, \\
-d_2 (J \ast \overline{v}^* - \overline{v}^*) &= \overline{v}^*(a_2 - b_2 u^* - c_2 \overline{v}^*), \quad x \in \mathbb{R}, \\
u^*(x) &\geq \tilde{B}, \quad \overline{v}^*(l) \leq \frac{a_2}{c_2} + \epsilon_3, \quad x \in \mathbb{R}.
\end{align*}
\]

Moreover, since \(\frac{a_2}{a_1} > \max\left\{\frac{b_1}{b_2}, \frac{c_1}{c_2}\right\}\), then by Morita et al. [23] that the solution \((u_2, v_2)\) of the following

\[
\begin{align*}
(u_2)_t &= u_2(a_1 - b_1 u_2 - c_1 v_2), \quad t > 0, \\
(v_2)_t &= v_2(a_2 - b_2 u_2 - c_2 v_2), \quad t > 0, \\
u_2(0) &= \tilde{B}, \quad v_2(0) = \frac{a_2}{c_2} + \epsilon_3
\end{align*}
\]

satisfies \(\lim_{t \to +\infty} (u_2(t), v_2(t)) = \left(\frac{a_2}{a_1}, 0\right)\), which implies that the solution \((U^*(t, x), V^*(t, x))\) of

\[
\begin{align*}
U^*_t - d_1 (J \ast U^* - U^*) &= U^*(a_1 - b_1 U^* - c_1 V^*), \quad t > 0, \quad x \in \mathbb{R}, \\
V^*_t - d_2 (J \ast V^* - V^*) &= V^*(a_2 - b_2 U^* - c_2 V^*), \quad t > 0, \quad x \in \mathbb{R}, \\
U^*(0, x) &= \tilde{B}, \quad V^*(0, x) = \frac{a_2}{c_2} + \epsilon_3, \quad x \in \mathbb{R}
\end{align*}
\]

satisfies \(\lim_{t \to +\infty} (U^*(t, x), V^*(t, x)) = \left(\frac{a_2}{a_1}, 0\right)\) uniformly in any bounded subset of \(\mathbb{R}\).
By using the comparison principle to problems (3.19) and (3.21) we obtain that \( u^*(x) \geq \frac{a_1}{b_1} \) and \( v^*(x) \leq V^*(t,x) \) for all \( x \in \mathbb{R} \), which indicates that \( u^*(x) \geq \frac{a_1}{b_1} \) and \( v^*(x) = 0 \) for all \( x \in \mathbb{R} \), and then \( u(x) \geq \frac{a_1}{b_1} \) and \( v(x) = 0 \) for \( x \in (-l,l) \), and hence \( u(t,x;u_{T_1},v_{T_1}) \geq \frac{a_1}{b_1} \) and \( v(t,x;u_{T_1},v_{T_1}) = 0 \) as \( t \to +\infty \). Further, we get \( u(t,x) \geq \frac{a_1}{b_1} \) and \( v(t,x) = 0 \) as \( t \to +\infty \).

Recall the proof of Theorem 3.8 that \( \limsup_{t \to +\infty} u(t,x) \leq \frac{a_1}{b_1} \) uniformly for \( x \in \mathbb{R} \). And then \( \lim_{t \to +\infty} \|u(t,\cdot)\|_{C([g(t),h(t))]} = \frac{a_1}{b_1} \) and \( \lim_{t \to +\infty} \|v(t,\cdot)\|_{L^\infty(\mathbb{R})} = 0 \). This completes the proof. \( \square \)

From now on, we can establish the spreading-vanishing dichotomy for problem (1.1).

**Theorem 3.10.** Assume that \( \frac{a_1}{a_2} > \max \left\{ \frac{b_1}{b_2}, \frac{c_1}{c_2} \right\} \) holds. Let \((u, v, g, h)\) be the unique positive solution of (1.7) with \( v_0(x) \neq 0 \). Then the following alternatives holds: Either

(i) spreading of \( u \): \( h_\infty - g_\infty = \infty \) and \( \lim_{t \to +\infty} (u,v) = \left( \frac{a_1}{b_1}, 0 \right) \) uniformly in any bounded subset of \( \mathbb{R} \); or

(ii) vanishing of \( u \): \( h_\infty - g_\infty \leq R^* \) and \( \lim_{t \to +\infty} (u,v) = \left( 0, \frac{a_2}{c_2} \right) \) uniformly in any bounded subset of \( \mathbb{R} \).

We obtained that spreading will always happens as long as \( 2h_0 \geq R^* \). Following is a criteria that devoted to the expanding ability \( \mu \) to govern the spreading alternative if \( 2h_0 < R^* \).

**Theorem 3.11.** Assume that \( \frac{a_1}{a_2} > \max \left\{ \frac{b_1}{b_2}, \frac{c_1}{c_2} \right\} \) holds. If \( 2h_0 < R^* \), then there exists \( \mu > 0 \) such that \( h_\infty - g_\infty = \infty \) if \( \mu \geq \mu_0 \).

**Proof.** Suppose on the contrary that \( h_\infty - g_\infty < \infty \) for all \( \mu > 0 \) if \( 2h_0 < R^* \). It then follows from Theorem 3.8 that \( h_\infty - g_\infty \leq R^* \). And hence we can find a large \( T \) such that \( 2h_0 < h(T) - g(T) \leq R^* \).

The free boundary conditions \( h'(t) \) and \( g'(t) \) yields that

\[
\begin{align*}
h(t) &= h(T) + \mu \int_T^t \int_{g(\tau)}^{h(\tau)} J(x-y)u(\tau,x)dydx, \\
g(t) &= g(T) - \mu \int_T^t \int_{g(\tau)}^{h(\tau)} J(x-y)u(\tau,x)dydx.
\end{align*}
\]

By letting \( t \to +\infty \) deduce that

\[
R^* - 2h_0 > (h_\infty - g_\infty) - (h(T) - g(T)) \geq \mu \int_T^\infty \left( \int_{g(\tau)}^{h(\tau)} J(x-y)u(\tau,x)dydx + \int_{g(\tau)}^{h(\tau)} J(x-y)u(\tau,x)dydx \right)
\]

Moreover, it follows from assumption (A1) that there exist constants \( \epsilon_0 > 0, \delta_0 > 0 \) such that \( J(x-y) \geq \delta_0 \) if \( |x-y| \leq \epsilon_0 \). Therefore,

\[
\mu \int_{g(\tau)}^{h(\tau)} J(x-y)u(\tau,x)dydx \geq \mu \int_{g(\tau)}^{h(\tau)} \left[ \int_{g(\tau)}^{h(\tau)+\frac{a_2}{c_2}} J(x-y)u(\tau,x)dydx \geq \frac{1}{2} \mu \epsilon_0 \delta_0 \int_{g(\tau)}^{h(\tau)} u(\tau,x)dx \right.
\]

and

\[
\mu \int_{g(\tau)}^{h(\tau)} J(x-y)u(\tau,x)dydx \geq \frac{1}{2} \mu \epsilon_0 \delta_0 \int_{g(\tau)}^{h(\tau)} u(\tau,x)dx.
\]
Therefore, we have
\[
R^* - 2h_0 > \mu \int_T^\infty \left( \int_{g(\tau)}^{h(\tau)} + \int_{g(\tau)}^{-\infty} \right) J(x - y)u(\tau, x)dydx\tau \\
\geq \frac{1}{2} \mu \varepsilon_0 \delta_0 \int_T^{\hat{T}} \left( \int_{g(\tau)}^{h(\tau)} + \int_{h(\tau) - \varepsilon_0}^{-\infty} \right) u(\tau, x)dxdr,
\]
where \( \hat{T} \in (T, \infty) \). Since \( u(t, x) > 0 \), then \( \Delta(\delta_0, \varepsilon_0, T, \hat{T}) \) defined by
\[
\Delta(\delta_0, \varepsilon_0, T, \hat{T}) = \frac{1}{2} \varepsilon_0 \delta_0 \int_T^{\hat{T}} \left( \int_{g(\tau)}^{h(\tau)} + \int_{h(\tau) - \varepsilon_0}^{-\infty} \right) u(\tau, x)dxdr > 0
\]
and thus
\[
0 < \mu < (R^* - 2h_0) \left\{ \Delta(\delta_0, \varepsilon_0, T, \hat{T}) \right\}^{-1} := \underline{\mu},
\]
which in turn indicates that we can find \( \underline{\mu} > 0 \) such that for all \( \mu \geq \underline{\mu} \), there holds \( h_\infty - g_\infty = \infty \) even though \( 2h_0 < h^* \).

**Theorem 3.11** states that the superior competitor \( u \) will spread eventually if the expanding ability \( \mu \geq \underline{\mu} > 0 \) even though the initial occupied stage is small. Below is a criteria on \( \mu \) that govern the vanishing case.

**Theorem 3.12.** Assume that \( \frac{a_1}{a_2} > \max \left\{ \frac{g_0}{h_0}, \frac{g_1}{h_0} \right\} \) holds. If \( 2h_0 < R^* \), then there exists \( \overline{\mu} \geq 0 \) such that \( h_\infty - g_\infty < \infty \) if \( 0 < \mu \leq \overline{\mu} \).

**Proof.** It is easy to find that \( u(t, x) \) in problem (1.1) satisfies the following
\[
\begin{cases}
    u_t \leq d_1 \left[ \int_{g(t)}^{h(t)} J(x - y)u(t, y)dy - u \right] + u(a_1 - b_1 u), \quad t > 0, x \in (g(t), h(t)), \\
    u(t, g(t)) = u(t, h(t)) = 0, \quad t > 0, \\
    h'(t) = \mu \int_{g(t)}^{h(t)} \int_{h(t)}^{+\infty} J(x - y)u(t, x)dydx, \quad t > 0, \\
    g'(t) = -\mu \int_{g(t)}^{h(t)} \int_{g(t)}^{h(t)} J(x - y)u(t, x)dydx, \quad t > 0, \\
    u(0, x) = u_0(x), \quad x \in [-h_0, h_0],
\end{cases}
\]
which immediately deduces that \( (u, g, h) \) is a lower solution of
\[
\begin{cases}
    \hat{u}_t = d_1 \left[ \int_{\hat{g}(t)}^{\hat{h}(t)} J(x - y)\hat{u}(t, y)dy - \hat{u} \right] + \hat{u}(a_1 - b_1 \hat{u}), \quad t > 0, x \in (\hat{g}(t), \hat{h}(t)), \\
    \hat{u}(t, \hat{g}(t)) = \hat{u}(t, \hat{h}(t)) = 0, \quad t > 0, \\
    \hat{h}'(t) = \mu \int_{\hat{g}(t)}^{\hat{h}(t)} \int_{\hat{h}(t)}^{+\infty} J(x - y)\hat{u}(t, x)dydx, \quad t > 0, \\
    \hat{g}'(t) = -\mu \int_{\hat{g}(t)}^{\hat{h}(t)} \int_{\hat{g}(t)}^{\hat{h}(t)} J(x - y)\hat{u}(t, x)dydx, \quad t > 0, \\
    \hat{u}(0, x) = u_0(x), \quad \hat{h}(0) = -\hat{g}(0) = h_0, \quad x \in [-h_0, h_0].
\end{cases}
\]
(3.22)

Note that problem (3.22) is the model that studied in [5], and it follows from [5, Theorem 3.12] that there exists \( \overline{\mu} \geq 0 \) such that vanishing of \( \hat{u} \) happens if \( 0 < \mu \leq \overline{\mu} \) since \( 2h_0 < R^* \). Therefore,
we obtain that \( h(t) - g(t) \leq \hat{h}(t) - \hat{g}(t) < \infty \) and \( u \leq \hat{u} \to 0 \) as \( t \to \infty \) if \( 0 < \mu \leq \gamma \). This completes the proof.

\[ \square \]

**Theorem 3.13.** Assume that \( \frac{a_1}{a_2} > \max \left\{ \frac{b_1}{b_2}, \frac{c_1}{c_2} \right\} \) and \( 2h_0 < R^* \). Then there exists \( \mu^* \geq 0 \) such that \( h_\infty - g_\infty = \infty \) if \( 0 < \mu \leq \mu^* \) and \( h_\infty - g_\infty < \infty \) if \( \mu > \mu^* \).

**Proof.** See [5, Theorem 3.14] for the detailed proof.

\[ \square \]

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