Improving flavor symmetry in the Kogut-Susskind hadron spectrum

Tom Blum  
Department of Physics, Brookhaven National Lab, Upton, NY 11973, USA

Carleton DeTar  
Physics Department, University of Utah, Salt Lake City, UT 84112, USA

Steven Gottlieb, Kari Rummukainen  
Department of Physics, Indiana University, Bloomington, IN 47405, USA

Urs M. Heller  
SCRI, Florida State University, Tallahassee, FL 32306-4052, USA

James E. Hetrick, Doug Toussaint  
Department of Physics, University of Arizona, Tucson, AZ 85721, USA

R.L. Sugar  
Department of Physics, University of California, Santa Barbara, CA 93106, USA

Matthew Wingate  
Physics Department, University of Colorado, Boulder, CO 80309, USA

We study the effect of modifying the coupling of Kogut-Susskind quarks to the gauge field by replacing the link matrix in the quark action by a “fat link”, or sum of link plus three-link paths. Flavor symmetry breaking, determined by the mass difference between the Goldstone and non-Goldstone local pions, is reduced by approximately a factor of two by this modification.
Introduction

One of the major technical challenges facing lattice QCD is the extraction of continuum physics from lattices with few enough lattice sites that computations are feasible. Recently there has been progress in developing “improved actions”, which give better approximations to continuum physics with large lattice spacings\(^1\). Starting points for improving the gauge field action include cancellation of lattice artifacts in an expansion in powers of \(a\) (Symanzik actions)\(^2, 3\) or renormalization group ideas (perfect actions)\(^4\). Improved fermion actions which improve the quark dispersion relation have been introduced for Wilson quarks (“clover” action)\(^5\) and for Kogut-Susskind quarks\(^6\). In particular, the Naik action for Kogut-Susskind quarks adds a third-nearest-neighbor term to remove an unwanted term of order \(a^2\), relative to the desired term, from the dispersion relation. This produces improvements such as energy and baryon number susceptibilities at high temperature that quickly approach the free field (Stefan-Boltzmann) limits, and has been applied to QCD thermodynamics\(^7\). Another of the major problems with Kogut-Susskind quarks is the breaking of flavor symmetry. In particular, only one of the pions is a true Goldstone boson at nonzero lattice spacing, and at the lattice spacings where simulations are done the differences in the pion masses are large. Preliminary results suggest that improving the gauge action helps this problem, but the Naik improvement of the fermion action does not\(^8\). This is not surprising, since the motivation for the Naik action is the improvement of the free quark dispersion relation rather than the flavor symmetry.

Here we explore a simple modification of the Kogut-Susskind fermion action. In particular, we replace the link matrices in the fermion matrix by weighted sums of the simple link plus three link paths, or “staples”, connecting the points. This is a simple modification that is consistent with gauge invariance and the hypercube symmetries of the lattice. Modifications such as this are likely to arise in any renormalization group motivated improved fermion action — this is just the simplest possible addition. In simulations with dynamical fermions, simplicity will be important since the computation of the fermion force in the molecular dynamics integration could become impossibly complicated. Of course, this modification of the quarks’ parallel transport can, and probably should, be used in concert with improvements in the gauge field action and the Naik (third nearest neighbor) improvement to the quark dispersion relation.

In this paper we report on a study of the quenched meson spectrum using the “fat link” quark action. Quenched spectrum calculations are easy to do, especially when stored configurations are available, and there are many results in the literature using the conventional fermion action for comparison. We measure masses of the “local” mesons, which include the Goldstone pion and one non-Goldstone pion, the “SC” pion, or “\(\pi_2\)”\(^8\). The splitting between these two pions is a good indicator of flavor symmetry breaking, since the calculations that have measured other pion masses find that all of the non-Goldstone pions are nearly degenerate\(^10\).
Formalism

We modify the standard Kogut-Susskind Dirac operator by replacing each link matrix \( U_\mu(x) \) by a "fat link"

\[
U_{\text{fat,}\mu}(x) = U_\mu(x) + w \sum_{\nu \neq \mu} \left( U_\nu(x)U_\mu(x + \hat{\nu})U_\nu^\dagger(x + \hat{\mu}) + U_\nu^\dagger(x - \hat{\nu})U_\mu(x - \hat{\nu})U_\nu(x - \hat{\nu} + \hat{\mu}) \right) / \left( 1 + 6w \right)
\]

where \( w \) is an adjustable weight for the staples.

Symbolically, this is just:

\[
\begin{array}{c}
\times w \\
\times 1 \\
\times w
\end{array} = \begin{array}{c}
\times w \\
\times 1 \\
\times 1 \\
\times w
\end{array}
\]

Replacement of a link by a weighted average of links displaced in directions perpendicular to the direction of the link amounts to including second derivatives, \( \partial^2 A_\mu / \partial^2 \nu \), with \( \nu \neq \mu \). Expressing this in a gauge and Lorentz covariant form, to lowest order in \( a \) this modifies the action by including a term (expressed in four-component notation, before the Kogut-Susskind spin diagonalization):

\[
\bar{\psi} \left( \gamma_\mu \left[ D_\mu + \frac{a^2}{6} D_\mu^3 + a^2 \frac{w}{1 + 6w} (D_\nu F_{\nu\mu}) \right] + m \right) \psi
\]

where \( w \) is the staple weight. The factor of \( a^2 \), required by the dimensions of second derivative, is the expected power of the lattice spacing for effects of lattice artifacts with staggered fermions. (In contrast, the clover term for Wilson quarks is an order \( a \) modification.) The \( D_\mu^3 \) term, which violates rotational invariance, is unchanged by the replacement of the ordinary link with a fat link. It is this term that is canceled in the "Naik" (third-nearest-neighbor) derivative.

Another way to think about the use of fat links in a spectrum computation is to consider it as the result of a modified action for the gauge fields. That is, we can ask what gauge
action would produce links with the probability distribution of the fat links. In general, this
gauge action will be nonlocal, involving loops of all sizes. Also, the fat links are not unitary
matrices, so this will be an unconventional gauge action. For small \( w \) we could construct
this gauge action as a power series in \( w \). To first order in \( w \) the effect is the same as using a
gauge action consisting of the sum of the plaquettes minus \( 2w \) times the \( 2 \times 1 \) planar loops
and the \( 2 \times 1 \) bent loops. (The 2 is a combinatorial factor arising because each two-plaquette
loop can be generated in two ways, by adding a staple to either of the two plaquettes in the
loop.) These considerations indicate that the “fat link” modification of the quark action will
interact with improvements in the gauge action, and it would be dangerous to assume that
the same fattening parameter that is optimal for the Wilson gauge action will be optimal
for an improved action.

Results

As a dimensionless measure of flavor symmetry breaking we use the quantity
\[ \Delta_\pi = \frac{(m_{\pi_2} - m_\pi)}{m_\rho}. \]
We will also use the quantity \( am_\rho \) to define the lattice spacing, and the
dimensionless quantity \( m_\pi/m_\rho \) as a measure of the quark mass. Ideally, to compare with the
conventional quark action we would compare simulations at the same quark mass and lattice
spacing. This requires tuning of parameters or interpolation among various data points.

We began with a series of tests using a set of quenched lattices with the standard Wilson
gauge action at \( 6/g^2 = 5.85 \). The lattice size is \( 20^3 \times 48 \). Local meson propagators were
calculated from wall sources, using four sources in each lattice. Because we are interested
in surveying various masses and smearing weights, only a small fraction (30 lattices) of our
stored lattices were used. The resulting masses and mass ratios are tabulated in Table I.
To better expose the effect of fattening the link, we have generally chosen the same fitting
range for all the values of \( w \) at a given quark mass. (There are some exceptions where the
fitting program did not converge for the desired fit range.) The final column of this table
is the number of conjugate gradient iterations required to converge the quark propagator
calculation to a residual of 0.00005. Table I also contains a selection of masses with the
conventional fermion action for comparison. We also include a result at \( 6/g^2 = 6.5 \) to point
out that at this lattice spacing the flavor symmetry breaking has become very small[12].

It is clear from this table that the use of fat links reduces the flavor symmetry breaking,
mostly by making the \( \pi_2 \) lighter. As an added benefit, the smoother fat links require fewer
iterations of the conjugate gradient algorithm. There are no obvious effects on the nucleon
to rho mass ratio. More accurately, any change in the nucleon to rho mass ratio is of the
same order as the flavor symmetry breaking in the rho masses, and therefore cannot be
disentangled from changes in the flavor symmetry breaking for the \( \rho \)'s. It can also be seen
that the improvement is quite insensitive to the exact value of \( w \). Studies of the dependence
of the flavor symmetry breaking of the local pions on the lattice spacing with the conventional
action can be found in Refs. [13, 14]. In Fig. I we plot the squared masses of these two pions
as a function of quark mass at this fixed lattice spacing, and show linear extrapolations of
the $\pi_2$ mass to zero quark mass. Note that the intercept as well as the slope of the $\pi_2$ squared
masses is lower with the fat links, showing that improvement in flavor symmetry persists in
the chiral limit $m_q \to 0$.

Because the rho and nucleon masses are reduced when $w$ is turned on, part of the improve-
ment in flavor symmetry breaking can be attributed to a smaller effective lattice spacing. To
separate this effect from a “real” improvement, we want to compare calculations with the
same lattice spacing, which we define by the rho mass, and the same physical quark mass,
defined by $m_\pi/m_\rho$. In looking through the available quenched spectrum calculations with
the conventional action, we find simulations at $6/g^2 = 5.95$ and 6.0 with $m_\pi/m_\rho \approx 0.65$.
We therefore chose a quark mass, 0.033, which gives a similar ratio using fat links. We can
then compare the fat link spectrum with $m = 0.033$ and $w = 0.4$ to these two conventional
calculations in the box below. Here we see that even though $m_\pi/m_\rho$ is slightly smaller for the
fat link calculation ($\Delta_\pi$ increases as $m_q \to 0$) and the lattice spacing (from $am_\rho$) is larger,
$\Delta_\pi$ with the fat link action is about half that for the conventional action.

| $6/g^2$ | $am_q$ | $w$ | $am_\rho$ | $m_\pi/m_\rho$ | $\Delta_\pi$ |
|---------|--------|-----|-----------|----------------|-------------|
| 5.85    | 0.033  | 0.40| 0.678(11) | 0.621(10)      | 0.052(3)    |
| 5.95    | 0.025  | 0   | 0.5954(28)| 0.651(3)       | 0.107(4)    |
| 6.00    | 0.02   | 0   | 0.520(3)  | 0.648(5)       | 0.090(7)    |

Mass ratios for the fat link action at $6/g^2 = 5.85$ and approximately
matched conventional calculations. The conventional calculations are
at similar $m_\pi/m_\rho$, but at smaller lattice spacing as defined by the $\rho$
mass. However, the dimensionless flavor symmetry breaking parameter
is considerably smaller with the fat link action.

Motivated by the interaction of fat links and improvement of the gauge action discussed
above, as well as the fact that our studies of quenched Kogut-Susskind spectroscopy indicate
that improvement of the gauge action results in some reduction of flavor symmetry breaking,
we calculated the fat link spectrum on a set of stored configurations with an improved gauge
action $S_g$.

$$S_g = \frac{\beta}{3} \left\{ \sum \text{(plaquettes)} - \frac{1}{20u_0} (1 + 0.4805\alpha_s) \sum \text{(2x1 loops)} - \frac{1}{u_0} 0.03325\alpha_s \sum \text{(1x1x1 loops)} \right\}$$

(3)

Results are in table $\mathbb{F}$. Again we see a dramatic improvement in the flavor symmetry breaking.
Curiously, in this case the Goldstone pion mass increases with fattening, while it decreased
in the $6/g^2 = 5.85$ calculation.
Fat link masses

| $w$ | $m_\pi$   | $m_{\pi^2}$ | $m_{VT}$ | $m_{PV}$ | $m_N$   | $m_{\pi}/m_\rho$ | $m_N/m_\rho$ | $\Delta_\pi$ | CG   |
|-----|-----------|-------------|----------|----------|---------|-------------------|---------------|--------------|------|
| 0.00| .273(1)   | .435(15)    | .60(2)   | .61(3)   | .88(2)  | .448(22)          | 1.44(8)       | .266(28)     | 1413 |
| 0.10| .246(1)   | .316(10)    | .57(2)   | .56(1)   | .80(2)  | .439(14)          | 1.43(4)       | .125(18)     | 1008 |
| 0.20| .239(1)   | .292(7)     | .57(2)   | .56(1)   | .80(2)  | .427(8)           | 1.43(4)       | .095(13)     | 892  |
| 0.30| .236(1)   | .290(5)     | .55(1)   | .56(2)   | .79(2)  | .421(15)          | 1.41(6)       | .096(10)     | 871  |
| 0.40| .237(1)   | .284(4)     | .56(2)   | .56(2)   | .80(2)  | .423(15)          | 1.43(6)       | .091(8)      | 867  |

Table 1: Masses and mass ratios with fat link fermion action, and comparable spectrum results with the conventional fermion action. The simple plaquette gauge action was used. All of the fat link masses were run on the same set of configurations, so all of the masses are strongly correlated. The $6/g^2 = 6.0$ and 6.5 masses are from Ref. [12].
Figure 1: The squared pion masses versus quark mass for the Wilson gauge action at $6/g^2 = 5.85$. The plusses and crosses are the $\pi_2$ and Goldstone $\pi$ masses with the conventional fermion action, and the squares and octagons are the $\pi_2$ and Goldstone $\pi$ masses with a staple weight $w = 0.4$. The lines are linear fits to the squared non-Goldstone pion masses.

$6/g^2 = 7.40$, $am_q = 0.04$, improved gauge action

| $w$ | $m_\pi$ | $m_{\pi_2}$ | $m_{VT}$ | $m_{PV}$ | $m_N$ | $m_\pi/m_\rho$ | $m_N/m_\rho$ | $\Delta_\pi$ | CG |
|-----|---------|-------------|---------|---------|-------|-----------------|-------------|------------|----|
| 0.00 | 0.5347(3) | 1.06(3)    | 1.23(1) | 1.37(2) | 1.81(4) | 0.435(4)       | 1.47(3)     | 0.427(25)  | 298|
| 0.10 | 0.5627(3) | 0.894(13)  | 1.17(1) | 1.22(1) | 1.73(2) | 0.481(4)       | 1.48(2)     | 0.283(11)  | 225|
| 0.20 | 0.5719(3) | 0.842(7)   | 1.16(1) | 1.19(1) | 1.70(1) | 0.493(4)       | 1.47(2)     | 0.233(6)   | 206|
| 0.30 | 0.5776(4) | 0.824(6)   | 1.15(1) | 1.18(1) | 1.68(1) | 0.502(4)       | 1.46(2)     | 0.214(6)   | 201|
| 0.40 | 0.5818(4) | 0.817(5)   | 1.15(1) | 1.18(1) | 1.68(1) | 0.506(4)       | 1.46(2)     | 0.205(5)   | 198|

Table 2: Masses and mass ratios for fat link quarks with an improved gauge action.
\[ 6/g^2 = 5.85, \ am_q = 0.020, \] with Naik derivative

\[ w \ m_\pi \ m_{\pi^2} \ m_{\pi V} \ m_P \ m_N \ m_\pi/m_\rho \ m_N/m_\rho \ \Delta_\pi \ \text{CG} \]

0.00 0.3652(7) 0.487(8) 0.683(20) 0.70(4) 0.983(17) 0.535(15) 1.44(5) 0.733(17) 733
0.10 0.328(1) 0.387(3) 0.611(13) 0.65(5) 0.890(14) 0.537(12) 1.46(4) 0.97(6) 529
0.20 0.319(1) 0.368(3) 0.600(13) 0.66(5) 0.872(14) 0.532(12) 1.45(4) 0.82(6) 488
0.30 0.316(1) 0.362(3) 0.597(13) 0.66(4) 0.867(14) 0.529(12) 1.45(4) 0.77(6) 476

Table 3: Masses and mass ratios for a fat link action including the Naik third-nearest-neighbor term. The gauge configurations are the same as in table 1.

It is also interesting to combine the fattened links with Naik’s improvement to the fermion action. The Naik improvement removes the order \( a^2 \) (relative to the leading term) error in the quark dispersion relation. It simply consists of replacing the nearest neighbor term in \( D, U_\mu(x)\psi(x + \hat{\mu}) \) with a combination of nearest and third nearest neighbor terms:

\[
\frac{9}{8} U_\mu(x)\psi(x + \hat{\mu}) - \frac{1}{24} U_\mu(x)U_\mu(x + \hat{\mu})U_\mu(x + 2\hat{\mu})\psi(x + 3\hat{\mu}) \ .
\] (4)

This produces a rapid convergence of quantities like the free field energy and baryon number susceptibility as a function of the number of time slices, and has been used in a high temperature QCD simulation by the Bielefeld group. However, in our quenched spectrum calculations addition of the third neighbor term had little effect on the flavor symmetry breaking. Since high temperature calculations are typically done with larger lattice spacings than zero temperature calculations, it is especially important to improve the actions here. We may hope that a combination of the Naik improvement with fat links or similar improvement in the coupling of the quarks to the gauge fields could produce a simulation with an accurate continuum free field behavior at high temperature and a gas of the right number of light pions at low temperature. As a first step, we calculated the spectrum at \( am_q = 0.02 \) using a third nearest term. There are several decisions to be made here. How should tadpole improvement be applied to the third neighbor term when fat links are used? Should the fat link in the third neighbor term be the product of three of the single fat links, or some other weighted combination of paths? We began with a spectrum calculation using just the product of three fattened links, with the unimproved coefficient \(-1/24\) for the third nearest neighbor, using the same \( 6/g^2 = 5.85 \) lattices. Results are in table 1.

Extensions and Conclusions

We find that the replacement of the simple gauge link by a fattened link in the fermion action reduces the flavor symmetry breaking of the local pions by roughly a factor of two for the parameters used here. Since flavor symmetry breaking is expected to be proportional to \( a^2 \), this corresponds to a modest increase in the lattice spacing at which simulations of a prescribed quality can be carried out. However, the computer time required is proportional to a large power of the lattice spacing, so a small increase in lattice spacing can translate into a large gain in computer time.
The action considered here was primarily motivated by its simplicity and consistency with the lattice symmetries. It is plausible that this works because using the fat links in the fermion action smooths out the effects on the quarks of ultraviolet fluctuations in the gluon field, and the flavor symmetry breaking is less severe on the smoother configurations. (For comments on the effects of smoothing the gluon field seen by the quarks on the tadpole contributions, see section 3.2 in Ref. [15].) Clearly a better theoretical understanding is wanted. In particular, a computation of the optimum staple weight is needed. However, it is likely that for the relatively large lattice spacings at which one would like to perform simulations with improved actions a nonperturbative (i.e. empirical) determination of the coefficients will be necessary. It would also be very interesting to see how this action affects rotational invariance.

We expect that an improvement of flavor symmetry in quenched spectrum calculations is a strong indication that dynamical simulations with this action will better reproduce the physics of a pion cloud. This needs to be tested.

Acknowledgments

This work was supported by NSF grants NSF–PHY96–01227, NSF–PHY91–16964 and DOE contracts DE-2FG02–91ER–40628, DE-AC02–86ER–40253, DE-FG03–95ER–40906, DE-FG05–85ER250000, DE-FG05-92ER40742, and DE-FG02–91ER–40661. Calculations were carried out on the Intel Paragon at the San Diego Supercomputer Center. We thank Tom DeGrand and Craig McNeile for useful suggestions.

References

[1] For a recent summary, see F. Niedermayer, hep-lat/9608097, to be published in Nucl. Phys. B (Proc. Suppl.).

[2] M. Lüscher and P. Weisz, Phys. Lett. B, 158, 250 (1985); G.P. Lepage and P.B. Mackenzie, Phys. Rev. D 48, 2250 (1993).

[3] M. Alford et. al, Nucl. Phys. B (Proc. Suppl.), 42, 787 (1995).

[4] P. Hasenfratz and F. Niedermayer, Nucl. Phys. B414, 785 (1994);

[5] B. Sheikholeslami and R. Wohlert, Nucl. Phys. B259, 572 (1985); M. Alford, T. Klassen and P. Lepage, Nucl. Phys. B (Proc. Suppl.) 47, 370 (1996) (hep-lat/9509087).

[6] S. Naik, Nucl. Phys. B316, 239 (1989).
[7] F. Karsch et al., hep-lat/9608047, to be published in Nucl. Phys. B (Proc. Suppl.).

[8] K.C. Bowler et al., Nucl. Phys. B284, 299 (1987).

[9] C. Bernard et al., hep-lat/9608102, to be published in Nucl. Phys. B (Proc. Suppl.).

[10] N. Ishizuka, M. Fukugita, H. Mino, M. Okawa and A. Ukawa, Nucl. Phys. B(Proc. Suppl.) 26, 284 (1992); E. Laermann et al., Nucl. Phys. B(Proc. Suppl.) 26, 268 (1992).

[11] C. Bernard et al., Nucl. Phys. B (Proc. Suppl.) 47 345 (1996) (hep-lat/9509076); in preparation.

[12] S. Kim and D.K. Sinclair, Nucl. Phys. B(Proc. Suppl.) 30, 381 (1993); Nucl. Phys. B(Proc. Suppl.) 34, 347 (1994).

[13] S.R. Sharpe, Nucl. Phys. B(Proc. Suppl.) 26, 197 (1992).

[14] S. Aoki et al. (JLQCD collaboration), hep-lat/9608144, to be published in Nucl. Phys. B(Proc. Suppl.).

[15] G.P. Lepage, Lectures at the 1996 Schladming Winter School on Perturbative and Nonperturbative Aspects of Quantum Field Theory (Schladming, Austria, March 1996), hep-lat/9607076. See the exercise containing Eqs. 79 and 79.