A complete classification of the spaces of compact operators
on $C([1, \alpha], l_p)$ spaces, $1 < p < \infty$

Dale E. Alspach and Elói Medina Galego

Abstract. We complete the classification, up to isomorphism, of the spaces of compact operators on $C([1, \gamma], l_p)$ spaces, $1 < p < \infty$. In order to do this, we classify, up to isomorphism, the spaces of compact operators $K(E, F)$, where $E = C([1, \lambda], l_p)$ and $F = C([1, \xi], l_q)$ for arbitrary ordinals $\lambda$ and $\xi$ and $1 < p < q < \infty$.

More precisely, we prove that it is relatively consistent with ZFC that for any infinite ordinals $\lambda$, $\mu$, $\xi$ and $\eta$ the following statements are equivalent:

(a) $K(C([1, \lambda], l_p), C([1, \xi], l_q))$ is isomorphic to $K(C([1, \mu], l_p), C([1, \eta], l_q))$.

(b) $\lambda$ and $\mu$ have the same cardinality and $C([1, \xi])$ is isomorphic to $C([1, \eta])$ or there exists an uncountable regular ordinal $\alpha$ and $1 \leq m, n < \omega$ such that $C([1, \xi])$ is isomorphic to $C([1, om])$ and $C([1, \eta])$ is isomorphic to $C([1, \alpha m])$.

Moreover, in ZFC, if $\lambda$ and $\mu$ are finite ordinals and $\xi$ and $\eta$ are infinite ordinals then the statements (a) and (b') are equivalent.

(b') $C([1, \xi])$ is isomorphic to $C([1, \eta])$ or there exists an uncountable regular ordinal $\alpha$ and $1 \leq m, n < \omega$ such that $C([1, \xi])$ is isomorphic to $C([1, om])$ and $C([1, \eta])$ is isomorphic to $C([1, \alpha n])$.

1. Introduction

We use standard set theory and Banach space theory terminology and notions as can be found in [14] and [15], respectively. Let $K$ be a compact Hausdorff space and $X$ a Banach space. The space $C(K, X)$ denotes the Banach space of all continuous $X$-valued functions defined on $K$ equipped with the supremum norm, i.e., $\|f\| = \sup_{k \in K} \|f(k)\|_X$. We will write $C([1, \alpha], X) = C(\alpha, X)$ when $K$ is the interval of ordinals $[1, \alpha] = \{\xi : 1 \leq \xi \leq \alpha\}$ endowed with the order topology. These spaces will be denoted by $C(K)$ and $C(\alpha)$, respectively, in the case $X = \mathbb{R}$.

For a set $\Gamma$, $c_0(\Gamma, X)$ is the Banach space of all $X$-valued functions $f$ on $\Gamma$ with the property that for every $\epsilon > 0$, the set $\{\gamma \in \Gamma : \|f(\gamma)\| \geq \epsilon\}$ is finite, and equipped with the sup norm. This space will be denoted by $c_0(\Gamma)$ in the case $X = \mathbb{R}$ and by $c_0$ when, in addition, the cardinality of $\Gamma$ (denoted by $|\Gamma|$) is $\aleph_0$.

Given Banach spaces $X$ and $Y$, $K(X, Y)$ denotes the Banach space of compact operators from $X$ to $Y$. As usual, when $X = Y$, this space will be denoted by $K(X)$.

2010 Mathematics Subject Classification. Primary 46B03; Secondary 46B25.

Key words and phrases. $C([1, \alpha])$ separable spaces, $l_p$ spaces, spaces of compact operators, isomorphic classifications.
\( \mathcal{K}(X) \). We write \( X \sim Y \) when \( X \) and \( Y \) are isomorphic and \( X \leftrightarrow Y \) when \( Y \) contains a copy of \( X \), that is, a subspace isomorphic to \( X \).

In this paper, we are mainly interested in completing the isomorphic classification of the spaces \( \mathcal{K}(C(\lambda, l_p), C(\xi, l_q)) \) obtained partially in \([10],[11]\) and \([12]\), where \( \lambda \) and \( \xi \) are arbitrary ordinals and \( l_p \) and \( l_q \) are the classical Banach spaces of scalar sequences with \( 1 < p, q < \infty \).

For \( p > q \), an isomorphic classification of these spaces was accomplished in \([10]\) Remark 4.1.3] for \( \lambda < \omega \) and \( \xi \geq \omega \), and in \([11]\) Remark 1.7] for \( \lambda \geq \omega \) and \( \xi \geq \omega \). We pointed out that, in both these cases, it was crucial the following geometric property of the spaces of compact operators involving the spaces \( l_p \), \( l_q \) and \( \ell_0 \).

Thus, in the present work, we turn our attention to the case \( 1 < p \leq q < \infty \). In this case the situation in quite different. Indeed, in contrast to (1), by \([1]\) Theorem 5.3, \( \mathcal{K}(l_p, l_q) \) is isomorphic to its \( \ell_0 \)-sum or, what is the same, that

\[ \mathcal{K}(l_p, l_q) \sim \mathcal{K}(l_p, \ell_0(l_q)) \]  

Nevertheless, in \([12]\) it was shown that the following cancellation law holds.

\[ \mathcal{K}(C(\lambda, l_p), C(\xi, l_q)) \sim \mathcal{K}(C(\mu, l_p), C(\eta, l_q)) \iff C(\xi) \sim C(\eta), \]

whenever \( \lambda, \mu, \xi \) and \( \eta \) are infinite countable ordinals or \( \lambda \mu < \omega \).

The main goal of this paper is to extend the above cancellation law by giving a complete isomorphic classification of the spaces \( \mathcal{K}(C(\lambda, l_p), C(\xi, l_q)) \), where \( \lambda \) and \( \xi \) are arbitrary ordinals.

To do this, first in section 3 we state a new geometric property of the spaces of compact operators involving the spaces \( l_p, l_q \) and \( \ell_0 \). Namely,

\[ \mathcal{K}(\ell_0) \not\sim \mathcal{K}(l_p, \ell_0(l_q)). \]

Moreover, in order to use the isomorphic classification of some \( C(\xi, X) \) spaces obtained in \([9]\), in section 4 we prove a stability property of certain spaces of compact operators containing copies of some \( \ell_0(\Gamma) \) spaces. Specifically, we show that if \( X \) is a Banach space and \( \Gamma \) is a set of cardinality \( \aleph_1 \), then

\[ \ell_0(\Gamma) \leftrightarrow \mathcal{K}(l_p, X) \iff \ell_0(\Gamma) \leftrightarrow X. \]

In section 5 we present our main result (Theorem 5.2). It provides a necessary and sufficient condition for two spaces \( \mathcal{K}(C(\lambda, l_p), C(\xi, l_q)) \) to be isomorphic, whenever \( \lambda \) and \( \xi \) are infinite and the cardinality of \( \lambda \) is strictly less than the least real-valued measurable cardinal \( m_r \).

We recall that a cardinal number \( m \) is a real-valued measurable cardinal if there exists a non-trivial real-valued measure defined on all subsets of a set of cardinal \( m \) for which points have measure \( 0 \) [6, page 560].

Theorem 5.2 is a complete isomorphic classification of the \( \mathcal{K}(C(\lambda, l_p), C(\xi, l_q)) \) spaces, where \( \lambda \) and \( \xi \) are infinite. Indeed, it is well known that the existence of real-valued measurable cardinals cannot be proved in ZFC [16] pages 106 and 108]. On the other hand, it is relatively consistent with ZFC that real-valued measurable cardinals do not exist [7, Theorem 4.14, page 972]. So it is relatively consistent with ZFC that we obtain a complete isomorphic classification of the spaces \( \mathcal{K}(C(\lambda, l_p), C(\xi, l_q)) \), where \( \lambda \) and \( \xi \) infinite.
Finally, in Theorem 5.3 we consider the remaining case \( \lambda \) is finite. Then, by using (2) we complete the proof of the isomorphic classification mentioned in the abstract.

### 2. Preliminaries

In this section we recall the isomorphic classification of the \( C(\xi) \) spaces, the generalization proved in [9], and some properties of the spaces of compact operators.

From now on \( |\xi| \) denotes the cardinality of the ordinal \( \xi \). Recall that an ordinal \( \alpha \) is said to be regular if its cofinality is itself. Otherwise \( \alpha \) is a singular ordinal. The smallest two infinite regular ordinals are \( \omega \) and \( \omega_1 \). We denote by \( X \hat{\otimes} Y \) the injective tensor product of the Banach spaces \( X \) and \( Y \).

Bessaga and Pelczynski [2] showed that for infinite countable ordinals, \( \xi \leq \eta \), \( C(\xi) \) is isomorphic to \( C(\eta) \) if and only if \( \eta < \xi^\omega \). In 1976 the classification was extended to nondenumerable ordinals \( \xi \) and \( \eta \). [13] and [18], independently. The general criterion is the same as that obtained by Bessaga and Pelczynski except in the case that the initial ordinal \( \alpha \) of cardinality \( |\xi| = |\eta| \) is a nondenumerable, regular ordinal, and \( \alpha \leq \xi \leq \eta \leq \alpha^2 \). For this case, equality of cardinalities of the ordinal quotients, \( \xi', \eta' \), where \( \xi = \alpha \xi' + \delta \), \( \eta = \alpha \eta' + \gamma \), and \( \delta, \gamma < \alpha \) is the requirement for isomorphism of \( C(\xi) \) and \( C(\eta) \).

For the spaces \( C(\xi, X) \) and \( C(\eta, X) \), where \( X \) is an infinite dimensional Banach space some additional difficulties arise in determining the isomorphic classification. Indeed, since that \( C(\xi, X) \) is isomorphic to \( C(\xi) \hat{\otimes} X \), it follows that

\[
C(\xi) \sim C(\eta) \implies C(\xi, X) \sim C(\eta, X).
\]

Moreover,

\[
X \sim X \oplus X \implies C(\alpha n, X) \sim C(\alpha, X)
\]

for all positive finite ordinals \( n \). Thus for a nondenumerable regular ordinal \( \alpha \) the classification is less fine than for the spaces of real-valued functions. However, for some Banach spaces it is possible to determine the isomorphic classification of the spaces \( C(\xi, X) \).

To state this result recall that a Banach space \( X \) is said to have the Mazur Property, MP in short, if every element of \( X^{**} \) which is sequentially weak* continuous is weak* continuous and thus is an element of \( X \). Such spaces were investigated in [6] and [17] and sometimes they are also called \( d \)-complete [19] or \( \mu B \)-spaces [24]. In [17] it was shown (Theorem 4.1) that \( C(\xi) \) for \( \xi < \omega_1 \), \( \left( \sum_{\gamma \in \Gamma} X_{\gamma} \right)_{l_1} \), where \( |\Gamma| \) is non-measurable and \( X \) has MP for all \( \gamma \), (Theorem 3.1), and \( X \hat{\otimes} Y \), where \( X \) has MP and \( Y \) is separable, (Corollary 5.2.3) have MP.

Let \( \mathcal{F} \) be the class of Banach spaces containing no copy of \( c_0(\Gamma) \), where \( |\Gamma| = \aleph_1 \), and having the Mazur property. For this class the main theorem of [9] gives a classification of the spaces \( C(\xi, X) \).

**Theorem 2.1.** Let \( X \in \mathcal{F} \) and \( \alpha \) an initial ordinal and \( \xi \leq \eta \) infinite ordinals. Then

1. If \( C(\xi, X) \) is isomorphic to \( C(\eta, X) \) then \( |\xi| = |\eta| \).
2. Suppose that \( |\xi| = |\eta| = |\alpha| \) and assume that \( \alpha \) is a singular ordinal, or \( \alpha \) is a nondenumerable regular ordinal with \( \alpha^2 \leq \xi \). Then

\[
C(\xi, X) \sim C(\eta, X) \iff \eta < \xi^\omega.
\]
Moreover, notice that $C_4$ and $\lambda |\text{cardinality represented by } K$ thus in the case $Y X$ theorem. As noted above this case can occur when $\text{order and } 2\gamma$.

In $[1]$ it was shown that $C(C(\lambda, Y), C(\xi, Z))$ is isomorphic to $C(\lambda, Y)^* \hat{\otimes} C(\xi, Z)$. Moreover, notice that $C(\lambda, Y)^*$ is isomorphic to $l_1(\Gamma, Y^*)$, where $\Gamma$ is a set with cardinality $|\lambda|$. Thus the isomorphism class of the spaces $K(C(\lambda, Y), C(\xi, Z))$ can be represented by $K(C(\lambda_0, Y), C(\xi, Z))$ where $\lambda_0$ is the smallest ordinal with $|\lambda_0| = |\lambda|$ or if $\lambda$ is finite, then $\lambda_0 = 1$.

In $[1]$ it was shown that $C(\omega, K(l_p, l_q)) \sim K(l_p, l_q)$, for $1 < p \leq q < \infty$. Because $C(\omega, K(l_p, l_q)) \sim K(l_p, C(\omega, l_q))$, thus in the case $Y = l_p$ and $Z = l_q$ there is the possibility of additional collapsing of the isomorphism classes. Indeed, if $\alpha$ is either a finite or uncountable regular ordinal and $\xi \leq \omega$. Then the spaces $C(\alpha \xi)$ are isomorphically distinct. However, if $\lambda \leq \omega$, then

$K(C(\lambda, l_p), C(\alpha \xi, l_q)) \sim l_{p'} \hat{\otimes} C(\alpha \xi) \hat{\otimes} l_q \sim C(\alpha, C(\xi, l_{p'} \hat{\otimes} l_q)) \sim C(\alpha, l_{p'} \hat{\otimes} l_q)$,

and therefore

$K(C(\lambda, l_p), C(\alpha \xi, l_q)) \sim K(l_p, C(\alpha, l_q))$.

The next proposition summarizes these remarks and the consequences of Theorem 2.1. Below, if $\gamma$ is an ordinal, $\gamma_0 = 1$ if $\gamma < \omega$ and otherwise $\gamma_0$ is the smallest ordinal with the same cardinality as $\gamma$.

Proposition 2.3. Suppose that $Y$ and $Z$ are Banach spaces in class $F$, have the approximation property, are isomorphic to their squares, and $K(Y, Z)$ is isomorphic to $C(\omega, K(Y, Z))$. Then

$K(C(\lambda, Y), C(\xi, Z)) \sim K(C(\lambda_0, Y), C(\psi(\lambda, \xi), Z))$,

where
(1) if \( \lambda \geq \omega \), \( \xi_0 \) is an uncountable regular ordinal, \( \xi \leq \xi_0^\omega \), and \( \xi = \xi_0 \delta + \xi \), then \( \psi(\lambda, \xi) = \xi_0 \xi_0^\omega \) if \( \xi^\omega \geq \omega \) and \( \psi(\lambda, \xi) = \xi_0 \) if \( \xi^\omega < \omega \).

(2) if \( \lambda < \omega \), \( \xi_0 \) is an uncountable regular ordinal, \( \xi \leq \xi_0^\omega \), and \( \xi = \xi_0 \delta + \xi \), then \( \psi(\lambda, \xi) = \xi_0 \xi_0^\omega \) if \( \xi^\omega > \omega \) and \( \psi(\lambda, \xi) = \xi_0 \) if \( \xi^\omega \leq \omega \).

(3) \( \xi_0 \) is an uncountable regular ordinal, and \( \xi_0^\omega < \xi \), then \( \psi(\lambda, \xi) = \max\{\xi_0^\omega, \gamma\} \) where \( \gamma \) is the smallest ordinal such that \( \gamma^\omega > \xi \).

(4) if \( \xi_0 < \omega \) or \( \lambda < \omega \) and \( \xi < \omega^\omega \), \( \psi(\lambda, \xi) = 1 \).

(5) if \( \xi_0 = \omega \) and \( \lambda \geq \omega \) or \( \xi_0 \) is an infinite singular ordinal, \( \psi(\lambda, \xi) \) is the smallest ordinal \( \psi \) such that \( \psi^\omega > \xi \).

Our goal in the next few sections is to show that the \( K(C(\lambda_0, Y), C(\psi(\lambda, \xi), Z)) \) spaces are from distinct isomorphism classes if \( Y \) is \( l_\rho \), and \( Z \) is \( l_q \) with \( 1 < \rho \leq q < \infty \).

Remark 2.4. Observe that if \( \lambda \) and \( \xi \) are infinite ordinals, then \( |\lambda| = |\lambda_0| \) and \( |\xi| = |\psi(\lambda, \xi)| \).

3. On the subspaces of \( K(c_0(X), l_q(Y)) \) spaces, \( 1 \leq q < \infty \)

The main aim of this section is to prove Theorem 3.2 which is the key ingredient for completing the isomorphic classification of the spaces of compact operators considered in this work. To do this, we need an auxiliary result. Let \( X \) and \( Y \) be Banach spaces and \( T \in K(c_0(X), l_q(Y)) \), with \( 1 \leq q < \infty \). Represent \( T \) as a matrix with entries in \( K(X, Y) \). By [20] Proposition 1.c.8 and following Remarks for any blocking of the \( c_0 \)-sum and \( l_q \)-sum the operator given by the block diagonal of the matrix is a bounded linear operator with norm no larger than \( \|T\| \). The next result gives a little more information about \( \|T\| \).

Lemma 3.1. Let \( X \) and \( Y \) be Banach spaces and let \( 1 \leq q < \infty \). Then the norm of a block diagonal operator of an operator in \( K(c_0(X), l_q(Y)) \) is the \( l_q \) norm of the norms of the operators on the blocks and the block diagonal operator is compact.

Proof. Let \( D_j \) be the \( j \)th block of the block diagonal operator \( D \) and \( (x_j) \in c_0(X) \) such that for each \( j \), \( x_j \) is supported block corresponding to \( D_j \). Then

\[
\|D_j x_j\|_{l_q(Y)} = \left( \sum_j \|D_j x_j\|_{l_q(Y)}^q \right)^{1/q} \leq \left( \sum_j \|D_j\|^q \|x_j\|_{l_q(Y)}^q \right)^{1/q} \leq (\sup_j \|x_j\|_{l_q(Y)}) \|D_j\|_{l_q}.
\]

Suppose \( 1 > \epsilon > 0 \), \( x_j \in X \), supported in the \( j \)th block, with \( \|x_j\|_{l_q(Y)} = 1 \) and \( \|D_j x_j\| \geq (1 - \epsilon) \|D_j\| \), \( j = 1, 2, \ldots \). For each \( N \), let \( z_N = (x_1, x_2, \ldots, x_N, 0, 0, \ldots) \). Then

\[
\|D z_N\| = \|(D_j x_j)_{j \leq N}\|_{l_q(Y)} = \left( \sum_{j \leq N} \|D_j x_j\|^q_{l_q(Y)} \right)^{1/q} \geq \left( \sum_{j \leq N} (1 - \epsilon) \|D_j\|^q \right)^{1/q}
\]

Taking the supremum over \( \epsilon \) and \( N \), gives

\[
\|D\| \geq \|(D_j)\|_{l_q}.
\]
It remains to show that the diagonal of a compact operator is compact. Let $Q_n$, respectively, $R_n$, be the projection onto the first $n$ coordinates of $I_q(\mathcal{Y})$, respectively, $c_0(X)$. A gliding hump argument shows that for any $T \in \mathcal{K}(c_0(X), I_q(\mathcal{Y}))$ and $\epsilon > 0$ there is an $N$ such that for all $n \geq N$,

$$||(I - Q_n)T|| < \epsilon.$$ 

Observe that for $j > n$,

$$(Q_j - Q_{j-1})T(R_j - R_{j-1}) = (Q_j - Q_{j-1})(I - Q_n)T(R_j - R_{j-1}).$$

Thus for $j > n$, the diagonal element of $T$, $D_j$, is also the diagonal element of $(I - Q_n)T$. By another application of Tong’s result [20] Proposition 1.c.8 it follows that

$$||(D_j)_{j < \infty} - (D_1, D_2, \ldots, D_N, 0, 0, \ldots)|| < \epsilon.$$ 

Thus the diagonal of $T$ is the limit of compact operators and therefore is compact.

□

The lemma will be applied near the end of the proof of the next result with $X = l_p$ and $Y = l_q$.

**Theorem 3.2.** Suppose that $1 < p < q < \infty$. Then

$$\mathcal{K}(c_0) \not\leftrightarrow \mathcal{K}(c_0(l_p), l_q).$$

**Proof.** First of all notice that $\mathcal{K}(c_0)$ is isomorphic to

$$l_q \otimes c_0 = c_0(l_1) = \left(\sum_{n=1}^{\infty} l_1\right)_{c_0}.$$ 

Let $P_n$ denote the standard projection from $c_0(l_p)$ onto $(\sum_{k=1}^{n} l_p)_n$ and let $(Q_n)$ be the basis projections for $l_q$.

Suppose that $(T_{s,n})$ is a normalized sequence in $\mathcal{K}(c_0(l_p), l_q)$ which is equivalent to the unit vector basis of $(\sum_{n=1}^{\infty} l_1)_{c_0}$, i.e., there are $\delta > 0$ and $K < \infty$ such that

$$\delta \max_s \sum_n |a_{s,n}| \leq \| \sum_s \sum_n a_{s,n} T_{s,n} \| \leq K \max_s \sum_n |a_{s,n}|$$

for all finitely non-zero sequences $(a_{s,n})$.

For fixed $s, r$, $(T_{s,n}P_r)$ is (equivalent to) a sequence in $\mathcal{K}(\sum_{n=1}^{\infty} l_p, l_q)$, which is isomorphic to $\mathcal{K}(l_p, l_q)$, and thus by [22] Theorem 3.1 cannot be equivalent to the unit vector basis of $l_1$. Hence there exists a normalized blocking of $(T_{s,n})_n, (U_{s,m})$, and an increasing sequence $(r_{s,m})_m$ such that

$$\| U_{s,m} P_{r_{s,m}} \| \to 0.$$ 

Further, because $U_{s,m}$ is compact, for fixed $s, m$,

$$\| U_{s,m}(I - P_r) \| \to 0.$$ 

Thus by a perturbation argument and passing to subsequences we may assume that $(r_{s,m})$ is strictly increasing in some order and that

$$U_{s,m}(P_{r_{s,m}} - P_{r'_{s,m'}}) = U_{s,m},$$

for all $s, m$, where $r'_{s,m'}$ is the maximal element strictly smaller than $r_{s,m}$.

For fixed $s, m$

$$\| (I - Q_k) U_{s,m} \| \to 0,$$
because $q < \infty$ and $U_{s,m}$ is compact. If $(s_t, m_t)$ is a sequence with $(s_t)$ strictly increasing $(U_{s_t,m_t})$ is equivalent to the usual unit vector basis of $c_0$. For fixed $k$ we claim that 

$$\|Q_k U_{s_t,m_t}\| \to 0.$$ 

Suppose not and let $x_t \in c_0(l_p)$, $\|x_t\| = 1$ and 

$$\|Q_k U_{s_t,m_t}x_t\| > \|Q_k U_{s_t,m_t}\|/2,$$ 

for all $t$. By passing to a subsequence and a perturbation argument because $k$ is fixed, we may assume that $(Q_k U_{s_t,m_t}x_t)$ is independent of $t$. We may also assume by the choice of $r_{s,m}$ that 

$$(P_{r_{s,t}} - P_{r_{s,t}^{'},m_t})x_t = x_t,$$ 

for all $t$. Thus for any finite $j$, $\|\sum_{j=1}^{t} x_i\| = 1$ but 

$$K \geq \|\sum_{j=1}^{t} Q_k U_{s_t,m_t}x_t\| = \|\sum_{j=1}^{t} Q_k U_{s_t,m_t}x_t\| > j \|Q_k U_{s_t,m_t}\|/2.$$ 

This contradicts $K < \infty$ and establishes the claim.

By again passing to a suitable subsequence and a perturbation we can assume that we have $(U_{s_t,m_t})$ with $(s_t)$ strictly increasing and $(k_t)$ strictly increasing such that 

$$U_{s_t,m_t} = (Q_{k_t} - Q_{k_{t-1}})U_{s_t,m_t}.$$ 

However by Lemma 3.3 this implies that $(U_{s_t,m_t})_t$ is equivalent to the basis of $l_q$.

\[ \Box \]

\[ \textit{Remark 3.3.} \textit{The proof of Theorem 3.2 uses } l_p \textit{ and } l_q \textit{ in a very minor way. } l_p \textit{ could be replaced by a space } Y \textit{ which isomorphic to its square, and } l_q \textit{ by a space with a basis that has a non-trivial lower estimate such that } l_1 \textit{ is isomorphic to no subspace of } K(Y, Z). \]

\[ \textit{Corollary 3.4.} \textit{If } \lambda, \xi \geq \omega, \textit{ and } 1 < p \leq q < \infty, \textit{ then } K(C(\lambda, l_p), C(\xi, l_q)) \textit{ is not isomorphic to a subspace of } K(C(\lambda, l_p), l_q). \]

\[ \textit{Proof.} \textit{Observe that } K(C(\lambda, l_p), C(\xi, l_q)) \textit{ is isomorphic to} \]

$$l_1(\lambda, l_p') \hat{\otimes} C(\xi, l_q) \sim l_1(\lambda, l_{p'}) \hat{\otimes} C(\xi, l_q) \sim C(\xi, l_1(\lambda, l_{p'}), \hat{l}_{l_q}).$$

Thus, it follows that $c_0(l_1)$ is isomorphic to a subspace of $K(C(\lambda, l_p), C(\xi, l_q))$. If $K(C(\lambda, l_p), C(\xi, l_q))$ is isomorphic to a subspace of $K(C(\lambda, l_p), l_q)$ then $c_0(l_1)$ is also. Because $c_0(l_1)$ is separable, this would imply that $c_0(l_1)$ is isomorphic to a subspace of $K(C(\mu, l_p), l_q)$ for some countable ordinal $\mu$ and hence to $K(C(\omega, l_p), l_q)$, contradicting Theorem 3.2. \[ \Box \]

\[ \section{4. Copies of } c_0(\Gamma) \textit{ in } K(l_p, X) \textit{ spaces, } 1 < p < \infty \]

Before proving our main theorem we need another preliminary result which will yield additional spaces that are in the class $\mathcal{F}$ and thus Theorem 2.1 will be applicable. In particular we will need to know that the spaces $K(l_p, X)$, with $1 < p < \infty$, contain a copy of $c_0(\Gamma)$ with $|\Gamma| = \aleph_1$ if and only if the Banach space $X$ has the same property. This is an immediate consequence of the following proposition. Below $B(Y, X)$ is the space of bounded linear operators from $Y$ to $X$. 


Proposition 4.1. If $X$ and $Y$ are Banach spaces and $Y$ is separable, then $c_0(\Gamma)$, where $|\Gamma| = \aleph_1$, is isomorphic to a subspace of $B(Y, X)$ if and only if $c_0(\Gamma)$ is isomorphic to a subspace of $X$.

Proof. Suppose that $(T_\gamma)_{\gamma \in \Gamma} \subset B(Y, X)$ is the image of the coordinate vectors $(e_\gamma)$ in $c_0(\Gamma)$ under some isomorphism $\mathcal{S}$. Let $(y_j)_{j=1}^\infty$ be a sequence which is dense in the unit sphere of $Y$. For $n_0 \in \mathbb{N}$ let

$$N_{j,n} = \{ \gamma \in \Gamma : ||T_\gamma(y_j)|| > 1/n \}.$$ 

Since $\Gamma = \bigcup_{j,n} N_{j,n}$, there is some $j_0, n_0$ so that $|N_{j_0,n_0}| = \aleph_1$. For each $\gamma \in N_{j_0,n_0}$ let $x^*_\gamma \in X^*$ such that $||x^*_\gamma|| = 1$ and

$$S^*(y_{j_0} \otimes x^*_\gamma)(e_\gamma) = T^*_\gamma x^*_\gamma(y_{j_0}) = x^*_\gamma(T_\gamma(y_{j_0})) > 1/n_0.$$ 

By Rosenthal’s disjointness lemma [21, Theorem 3.4] and the remark following, there is a subset $N'$ of $N_{j_0,n_0}$ of the same cardinality such that $(S^*(y_{j_0} \otimes x^*_\gamma))_{\gamma \in N'}$ is equivalent to the standard unit vectors in $l_1(N)$ and norms $[e_\gamma : \gamma \in N']$. Therefore the mapping $S_1 : [e_\gamma : \gamma \in N'] \to X$ defined by $S_1(e_\gamma) = T_\gamma(y_{j_0})$ extends to an isomorphism into $X$.

The converse is obvious. \qed

Corollary 4.2. Let $V$ and $W$ be Banach spaces such that $c_0$ is isomorphic to no subspace of $V$, $V$ has MP, and $W^*$ is separable and has the approximation property. If $|\lambda|$ is non-measurable then $\mathcal{K}(W, l_1(\lambda, V))$ is in the class $F$.

Proof. Because $V$ does not contain subspace isomorphic to $c_0$, it follows from a gliding hump argument that $l_1(\lambda, V)$ has the same property. By Proposition 4.1, $c_0(\Gamma)$ is not isomorphic to a subspace of $\mathcal{K}(W, l_1(\lambda, V))$. To see that $\mathcal{K}(W, l_1(\lambda, V))$ has MP we note that $\mathcal{K}(W, l_1(\lambda, V))$ is isomorphic to $W^* \hat{\otimes} l_1(\lambda, V)$ by [4, Proposition 4.3]. By the results of Kappeler [17], $l_1(\lambda, V)$ has MP and consequently $W^* \hat{\otimes} l_1(\lambda, V)$ has MP. \qed

5. A classification of $\mathcal{K}(C(\lambda, l_p), C(\xi, l_q))$ spaces, $1 < p \leq q < \infty$

In this section we classify, up to isomorphism, the spaces of compact operators $\mathcal{K}(C(\lambda, l_p), C(\xi, l_q))$, with $1 < p \leq q < \infty$.

The following theorem gives necessary conditions for two $\mathcal{K}(C(\lambda, l_p), C(\xi, l_q))$ spaces to be isomorphic in terms of the notation introduced in Proposition 2.3 and provides a converse to that result for $Y = l_p$ and $Z = l_q$. Recall that the density character of a Banach space $X$ is the smallest cardinal number $\delta$ such that there exists a set of cardinality $\delta$ dense in $X$.

Theorem 5.1. Suppose that $1 < p \leq q < \infty$ and $\xi, \eta, \lambda$ and $\mu$ are nonzero ordinals. If $\mathcal{K}(C(\lambda, l_p), C(\xi, l_q))$ is isomorphic to $\mathcal{K}(C(\mu, l_p), C(\eta, l_q))$, then $\lambda_0 = \mu_0$. Moreover, if $|\lambda| < m_\eta$ then $\psi(\lambda, \xi) = \psi(\mu, \eta)$.

Proof. Assume that $\mathcal{K}(C(\lambda, l_p), C(\xi, l_q))$ is isomorphic to $\mathcal{K}(C(\mu, l_p), C(\eta, l_q))$ for some nonzero finite ordinals $\xi, \eta, \lambda$ and $\mu$. Without loss of generality we can assume $|\lambda| \leq |\mu|$. Let $p'$ be the real number satisfying $1/p + 1/p' = 1$. We know that $C(\beta, l_p)$ has the approximation property for every ordinal $\beta$. So according to [4, Proposition 5.3] we see that

$$\mathcal{K}(C(\lambda, l_p), C(\xi, l_q)) \sim l_1(\lambda, l_{p'}) \hat{\otimes} C(\xi, l_q) \sim C(\xi, l_1(\lambda, l_{p'})) \hat{\otimes} l_q),$$

(3)
and
\[ K(C(\mu, l_p), C(\eta, l_q)) \sim l_3(\mu, l_{p'}) \hat{\otimes} C(\eta, l_q) \sim C(\eta, l_1(\mu, l_{p'}) \hat{\otimes} l_q) \]  \hspace{1cm} (4)

Consequently
\[ l_1(\mu) \hookrightarrow K(C(\mu, l_p), C(\eta, l_q)) \sim C(\xi, l_1(\lambda, l_{p'}) \hat{\otimes} l_q). \]  \hspace{1cm} (5)

Since \( l_1(\mu) \) contains no subspace isomorphic to \( c_0 \) and \( l_1(\lambda, l_{p'}) \hat{\otimes} l_q \) is isomorphic to its square, we deduce by (5) and [8, Theorem 2.3] that
\[ l_1(\mu) \hookrightarrow l_1(\lambda, l_{p'}) \hat{\otimes} l_q. \]  \hspace{1cm} (6)

Therefore if \( \lambda < \omega \) and \( \mu \geq \omega \), then by (6) we would conclude that \( l_{p'} \hat{\otimes} l_q \) contains a copy of \( l_1 \), which is absurd by [22, Theorem 3.1]. Hence either \( \lambda \mu < \omega \) and \( \lambda_0 = \mu_0 \) or \( \lambda \geq \omega \) and \( \mu \geq \omega \). In the last case, observe that the density character of \( l_1(\mu) \) is \( |\mu| \) and the density character of \( l_1(\lambda, l_{p'}) \hat{\otimes} l_q \) is \( |\lambda| \). Thus, it follows from (6) that \( |\mu| \leq |\lambda| \). So \( |\lambda| = |\mu| \). Thus we have that \( \lambda_0 = \mu_0 \).

If \( \xi < \omega \) and \( \lambda_0 \geq \omega \), then by Corollary 3.4, \( \eta < \omega \), and \( \psi(\lambda, \xi) = \psi(\mu, \eta) \).

If \( \lambda_0 = \mu_0 = 1 \), then if \( \xi \) or \( \eta \) is finite, we can replace it by \( \omega \), [1, Theorem 5.3]. (Observe that \( \psi(m, n) = \psi(m, \omega) \) if \( n < \omega \)) Thus we may assume for the remaining cases that \( \xi \) and \( \eta \) are infinite and that \( \lambda = \lambda_0 = \mu \).

Next, let
\[ X = l_1(\lambda_0, l_{p'}) \hat{\otimes} l_q = l_1(\lambda, l_{p'}) \hat{\otimes} l_q = l_1(\mu, l_{p'}) \hat{\otimes} l_q. \]

By Corollary 4.2, \( X \) has the Mazur Property. Hence by (3) and (4) we infer that
\[ C(\xi, X) \sim K(C(\lambda, l_p), C(\xi, l_q)) \sim K(C(\mu, l_p), C(\eta, l_q)) \sim C(\eta, X). \]  \hspace{1cm} (7)

Thus by (1) of Theorem 2.1 we conclude that \( |\xi| = |\eta| \). The other assertions of that theorem allow us to complete the argument once we deal with Theorem 2.1.13 (a).

Suppose that \( |\xi| = \xi_0 \) is an uncountable regular cardinal. Because \( X \) is isomorphic to its square, if the ordinal quotient \( \xi' \) is finite, \( C(\xi, X) \) is isomorphic to \( C(\xi_0, X) \). If \( \lambda_0 = 1 \) and \( |\xi'| = \omega \), then
\[ C(\xi_0, X) \sim C(\xi_0, C(\omega, X)) \sim C(\xi_0, X). \]

However, if \( |\xi'| = \omega \) and \( \lambda \geq \omega \), then by Corollary 3.4, \( C(\omega, X) \) is not isomorphic to \( X \). Thus \( C(\xi_0, X) \) is not isomorphic to \( C(\xi_0, X) \). Thus \( \psi(\lambda, \xi) = \psi(\mu, \eta) \) in all cases.

The results claimed in the abstract are immediate consequences.

**Theorem 5.2.** Suppose that \( 1 < p \leq q < \infty \) and \( \xi, \eta, \lambda \) and \( \mu \) are infinite ordinals with \( |\lambda| = |\mu| < m_r \). Then the following statements are equivalent

(a) \( K(C(\lambda, l_p), C(\xi, l_q)) \) is isomorphic to \( K(C(\mu, l_p), C(\eta, l_q)) \).

(b) \( C(\xi) \) is isomorphic to \( C(\eta) \) or there exists a finite or an uncountable regular ordinal \( \alpha \) and \( 1 \leq m, n < \omega \) such that \( C(\xi) \) is isomorphic to \( C(\alpha m) \) and \( C(\eta) \) is isomorphic to \( C(\alpha n) \).

Finally, the next theorem completes the isomorphic classification, up to isomorphism, of the spaces of compact operators on \( C(\xi, l_p) \) spaces, with \( 1 < p < \infty \).
Theorem 5.3. Suppose that $1 < p \leq q < \infty$ and $\xi$, $\eta$, $\lambda$ and $\mu$ are ordinals with $\omega \leq \xi \leq \eta$, $|\xi| = |\eta|$ and $\lambda \mu < \omega$. Then the following statements are equivalent

(a) $\mathcal{K}(C(\lambda, l_p), C(\xi, l_q))$ is isomorphic to $\mathcal{K}(C(\mu, l_p), C(\eta, l_q))$.

(b) $C(\xi)$ is isomorphic to $C(\eta)$ or there exists an uncountable initial regular ordinal $\alpha$ and $1 \leq m, n \leq \omega$ such that $C(\xi)$ is isomorphic to $C(\alpha m)$ and $C(\eta)$ is isomorphic to $C(\alpha n)$.

In view of Theorems 5.2 and 5.3 the following question arises naturally.

Problem 5.4. Does the above isomorphic classification of the spaces of compact operator $\mathcal{K}(C(\lambda, l_p), C(\xi, l_q))$, with $\lambda \geq \omega$, remain true in ZFC?

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