ON THE LINEARITY PROPERTY FOR ALLOCATION PROBLEMS AND BANKRUPTCY PROBLEMS

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Abstract. This work provides an analysis of linear rules for bankruptcy problems and allocation problems from an axiomatic point of view and we extend the study of the additivity property presented in Bergantiños and Méndez-Naya [1] and Bergantiños and Vidal-Puga [2]. We offer a decomposition for the space of allocation problems into direct sum of subspaces that are relevant to the study of linear rules and obtain characterizations of certain classes of rules. Furthermore, for bankruptcy problems we propose an alternative version of the additivity property.

1. Introduction. A variety of economic problems can be modelled as a problem of how to allocate a resource among a set of agents who have claims on it. In this paper we consider problems where a estate \( E \) must be divided among a finite group of agents \( N \), \( c_i \) being the claim of agent \( i \).

In particular, we study two kinds of problems. A bankruptcy problem (O’Neill [4]) is a situation in which a set of agents claim a fraction of some good or estate, and where the sum of these claims exceeds the value of the estate. Whereas, in allocation problems (Herrero et al. [3]) no restriction is established on the parameters of the model (that is, the aggregate claim may exceed or fall short of the amount of the good and agents’ claims may be positive or negative).

Also, these problems differ in the way an estate must be divided. In bankruptcy problems an agent must receive at least 0 and at most his claim. On the other hand, in allocation problems agents can receive anything.

One of the most important topics of these problems is the axiomatic characterizations of rules. The idea is to propose desirable properties and find out which of them characterize every rule. Properties often help agents to compare different rules and to decide which rule is preferred for a particular situation.

In this paper we adopt such approach and provide an analysis of linear rules\(^1\) for these problems. Linearity is a standard property and it has been used in many situations. Although the justification of linearity is not as clear as with other properties (e.g., efficiency or symmetry), in most cases it produces very interesting classes of rules.

\(^1\)The precise definitions will be provided in Sec. 3 and 4.
A decomposition of the space of allocation problems is proposed, where a global
description of linear symmetric rules is derived. We then add additional properties
or restrictions and characterize those subclasses of solutions. Furthermore, we ob-
tain a new characterization for the rights-egalitarian solution proposed by Herrero
et al. [3].

With respect to bankruptcy rules, we analyse linear rules by means of an additiv-
ity axiom and provide a weak version of it since there exist no additive bankruptcy
rules (see Bergantiños and Méndez-Naya [1]). We then define additional properties
of rules and relate them with weak additivity.

Related topics have been analyzed in the literature from different viewpoints. In
dealing with the analysis of bankruptcy-like situations, see Thomson [5] for a survey
of this literature. Bergantiños and Vidal-Puga [2] study the property of additivity in
each of four kinds of problems, namely, bankruptcy problems, allocation problems,
surplus problems and loss problems.

The paper is organized as follows. We first recall the main basic features of bank-
ruptcy problems and allocation problems. In Sec. 3 we provide characterizations of
several subclasses of linear symmetric allocation rules. Finally in Sec. 4 we discuss
an alternative definition of an additivity property for bankruptcy rules.

2. Preliminaries. In this section we give some concepts and notations related to
allocation problems, as well as a decomposition of the space of such problems under
the action of the symmetric group, since it will be used in subsequent developments.

We study the problem involving the distribution of a given amount of *money*
among a number of *agents*, each of which is characterized by a monetary *claim*.
The money being distributed will be called the *estate* and it represents the worth
jointly owned by the agents. Its origin can stem from one of many circumstances
(e.g., administrative decisions, an enterprise to be liquidated, inheritance, etc.).
The vector of claims represents the agents’ individual rights (e.g. needs, claims,
benefits or private loans, shares, inheritance wills and others). By agents we mean
people or more general instances, such as expenditure categories, departments or
institutions.

In particular, we face up problems where a estate $E$ must be divided among a
finite group of agents $N$, $c_i$ being the claim of agent $i$. $c = (c_i)_{i \in N}$ denotes the
vector of claims and $C = \sum_{i \in N} c_i$ is the sum of all claims.

**Definition 2.1 (Problems).** Let $N = \{1, 2, \ldots, n\}$ represents a fixed set of agents.

- An allocation problem is a pair $(c, E) \in \mathbb{R}^n \times \mathbb{R}$.
- A bankruptcy problem is a pair $(c, E) \in \mathbb{R}^n_+ \times \mathbb{R}_+$, such that $C \geq E$.

Note that an allocation problem belongs to a wider than usual domain of prob-
lems (e.g. bankruptcy problems). In this case no restriction will be imposed on the
values that the parameters of the model take. In particular, the estate can be either
positive or negative, the claims may well have negative components, and the estate
may exceed or fall short of the aggregate claims. A negative estate simply means a
cost to be shared. A negative claim corresponds to a debt. An estate greater (resp.
smaller) than the aggregate claims represents a problem of distributing a surplus
(resp. sharing a deficit).

Let us call $A = \mathbb{R}^n \times \mathbb{R}$ the family of all allocation problems and $B = \mathbb{R}^n_+ \times \mathbb{R}_+$ the set of all bankruptcy problems.

The question that arises is: how to divide the estate among agents? This question
is answered by means of defining rules. A rule, say $f$, is a function which assigns
Definition 2.2 (Rules). Let $N = \{1, 2, ..., n\}$ be a fixed set of agents.

- An allocation rule on $A^{(n)}$ is a map $\varphi : A^{(n)} \to \mathbb{R}^n$.
- A bankruptcy rule on $B^{(n)}$ is a map $\psi : B^{(n)} \to \mathbb{R}^n$ satisfying that, for all $(c, E) \in B^{(n)}$:
  \[
  \sum_{j \in N} \psi_j(c, E) = E \quad \text{and} \quad 0 \leq \psi_i(c, E) \leq c_i \quad \text{for all} \quad i \in N
  \]

In the definition of a bankruptcy rule, we have established that each agent must be awarded a non-negative amount that does not exceed his claim, while the sum of the payoffs should equal the estate. Notice that the definition of an allocation rule does not impose any restriction on the map $\varphi$.

According to the previous definition, in any case we can see $\mathbb{R}^n$ as an space of payoff vectors.

Given $(c, E), (c', E') \in A^{(n)}$ (or $B^{(n)}$) and $\lambda \in \mathbb{R}$, we define the sum $(c, E) + (c', E')$ and the product $\lambda(c, E)$, in $A^{(n)}$ (or $B^{(n)}$), in the usual form, i.e.,
\[
(c, E) + (c', E') = (c + c', E + E') \quad \text{and} \quad \lambda(c, E) = (\lambda c, \lambda E)
\]
respectively. It is well known that $A^{(n)}$ is a vector space with these operations, and dim $A^{(n)} = n + 1$.

Now, the group of permutations of $N$, $S_n = \{\pi : N \to N \mid \pi \text{ is bijective}\}$, acts on $\mathbb{R}^n$, on $A^{(n)}$ and on $B^{(n)}$ in the natural way; i.e., for $\pi \in S_n$:
\[
\pi \cdot (x_1, x_2, ..., x_n) = (x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)})
\]
and
\[
\pi \cdot (c, E) = (\pi \cdot c, E)
\]
Here, $S_n$ is acting trivially on $\mathbb{R}$.

For $x \in \mathbb{R}^n$, let $\overline{x} \in \mathbb{R}$ denotes the mean of the values of $x$; i.e., $\overline{x} = \frac{1}{n} \sum_{j=1}^{n} x_j$.

The following subspaces of the space of payoff vectors, $\mathbb{R}^n$, will be used in the sequel:
\[
\Delta_n = \{(t, t, ..., t) \in \mathbb{R}^n\} \quad \text{and} \quad \Delta_n^\perp = \left\{ x \in \mathbb{R}^n \mid \sum_{j=1}^{n} x_j = 0 \right\}
\]
$\Delta_n$ represents the subspace of payoff vectors where every agent would receive the same amount. Whereas, $\Delta_n^\perp$ contains payoffs where the sum of amounts received by all agents is exactly zero.

Notice that $\Delta_n$ and $\Delta_n^\perp$ are irreducible subspaces; i.e., they are invariant subspaces under the action of $S_n$ and they do not contain any invariant subspace smaller than themselves (and different to $\{0\}$). Here, $0 = (0, 0, ..., 0) \in \mathbb{R}^n$.

There is a similar description for the space of allocation problems. Two important subspaces of $A^{(n)}$ are:
\[
\Delta_n \times \mathbb{R} \quad \text{and} \quad \Delta_n^\perp \times \{0\}
\]
$\Delta_n \times \mathbb{R}$ contains allocation problems where all agents have identical claims and $\Delta_n^\perp \times \{0\}$ represents the subspace of problems where the sum of individual claims is zero and estate $E = 0$.

\[2\]In general, a subspace $W$ of $\mathbb{R}^n$ or $P^{(n)}$ is invariant (for the action of $S_n$) if for every $w \in W$ and every permutation $\theta \in S_n$ we have that $\theta \cdot w \in W$. 

As the previous case, $\Delta_n \times \mathbb{R}$ and $\Delta_n^+ \times \{0\}$ are irreducible subspaces of $A^{(n)}$ (under all permutations of the agents).

**Proposition 1.** The decomposition of $A^{(n)}$ and $\mathbb{R}^n$, under $S_n$, into irreducible subspaces are:

$$A^{(n)} = \Delta_n \times \mathbb{R} \oplus \Delta_n^+ \times \{0\}$$

and

$$\mathbb{R}^n = \Delta_n \oplus \Delta_n^+$$

**Proof.** It is easy to check that $\Delta_n \cap \Delta_n^+ = \{0\}$ and $(\Delta_n \times \mathbb{R}) \cap (\Delta_n^+ \times \{0\}) = \{0, 0\}$.

The result follows from the fact that:

i: For $x \in \mathbb{R}^n$,

$$x = (\overline{x}, \overline{x}, \ldots, \overline{x}) + (x_1 - \overline{x}, x_2 - \overline{x}, \ldots, x_n - \overline{x})$$

ii: For $(c, E) \in A^{(n)}$,

$$(c, E) = ((\overline{c}, \overline{c}, \ldots, \overline{c}), E) + ((c_1 - \overline{c}, c_2 - \overline{c}, \ldots, c_n - \overline{c}), 0)$$

$\square$

Thus, Proposition 1 tells us that the space of allocation problems and the space of payoff vectors, with group of symmetry $S_n$, can be written each as an orthogonal direct sum of two subspaces which are invariant under permutations and which can no longer be further decomposed.

### 3. Allocation rules

In this section we present a brief description of some well-known properties for an allocation rule $\varphi : A^{(n)} \to \mathbb{R}^n$ used in this work and provide characterizations of certain linear classes of allocation rules.

**Axiom 1 (Linearity).** An allocation rule $\varphi$ is linear if

$$\varphi[(c, E) + (c', E')] = \varphi(c, E) + \varphi(c', E')$$

and

$$\varphi[\lambda(c, E)] = \lambda \varphi(c, E)$$

for all $(c, E), (c', E') \in A^{(n)}$ and $\lambda \in \mathbb{R}$.

The idea of this property is the following: suppose that the product sold by a firm depends on several parts (quality and marketing, for instance). The total revenue of the firm, $E + E'$, can be divided into two parts: one motivated by quality ($E$) and the other by marketing ($E'$). We can also determine the contribution of every agent of the firm to quality ($c$) and marketing ($c'$). Now we can allocate the total revenue according to two procedures. First, we allocate the total revenue ($E + E'$) according to the total contribution ($c + c'$). Second, we allocate the revenue motivated by quality ($E$) according to the contribution to quality ($c$), and the revenue of marketing ($E'$) according to the contribution to marketing ($c'$). This axiom guarantees that both procedures coincide.

The other part deals with scales; that is, scaling the claims $c_i$ and the estate $E$ by a constant factor causes the same scaling to the allocation rule of the original problem. So, if the claims and estate are measured in other currencies, then the allocation rule remains the same currency measure.
Axiom 2 (Symmetry). An allocation rule \( \varphi \) is said to be symmetric if and only if
\[
\varphi[\pi \cdot (c, E)] = \pi \cdot \varphi(c, E)
\]
for every \( \pi \in S_n \) and \( (c, E) \in A^{(n)} \), where the problem \( \pi \cdot (c, E) \) is defined as \( \pi \cdot (c, E) = (\pi \cdot c, E) \).

The interpretation of the previous axiom is that the payoffs from a symmetric allocation rule are independent of the way the agents are named; i.e., they are covariant under permutations of the set of agents.

Axiom 3 (Efficiency). An allocation rule \( \varphi \) is efficient if
\[
\sum_{j \in N} \varphi_j(c, E) = E
\]
for all \( (c, E) \in A^{(n)} \).

This axiom says that the allocation rule will exhaust the estate (the payoffs of all agents add up the estate).

Axiom 4 (Compatibility). An allocation rule \( \varphi \) satisfies the compatibility axiom if
\[
\varphi(c, C) = c
\]
for every \( (c, C) \in A^{(n)} \).

This property is an obvious restriction on an allocation rule. It establishes that if the claims are feasible, then the allocation rule should give each agent her claim. It is regarded this as a fundamental property of social justice.

Now, we characterize the set of linear symmetric allocation rules for allocation problems and a couple of subclasses of such set by adding other restrictions.

**Proposition 2.** If the allocation rule \( \varphi : A^{(n)} \to \mathbb{R}^n \) satisfies the linearity and symmetry axioms, then there exist unique real numbers \( \alpha, \beta, \gamma \) such that
\[
\varphi_i(c, E) = \alpha c_i + \beta E + \gamma C \tag{1}
\]

Conversely, for any real numbers \( \alpha, \beta, \gamma \), the allocation rule given by (1) is linear and symmetric.

**Proof.** Let \( \varphi : A^{(n)} \to \mathbb{R}^n \) be a linear and symmetric allocation rule. Since the allocation problems \( \{(e_k, 0)\}_{k \in N} \cup \{(0, 1)\} \) form a basis of \( A^{(n)} \), then for every problem \((c, E) \in A^{(n)}, (c, E) = \sum_{j \in N} c_j \cdot (e_j, 0) + E \cdot (0, 1) \). Set \( a_{i,j} = \varphi_i(e_j, 0) \) if \( i \neq j \), \( b = \varphi_i(e_i, 0) \) and \( \beta = \varphi_i(0, 1) \), then
\[
\varphi_i(c, E) = \sum_{j \in N} c_j \cdot \varphi_i(e_j, 0) + E \cdot \varphi_i(0, 1)
\]
for every \( i \in N \).

Now, let \( k, l \in N \setminus \{i\} \) and \( \pi \in S_n \) be such that \( \pi(k) = l \) and \( \pi(i) = i \). Since \( \pi \cdot (e_k, 0) = (e_l, 0) \) then, by symmetry:
\[
\varphi_i(e_k, 0) = \varphi_{\pi(i)}(e_k, 0) = \varphi_i(\pi \cdot (e_k, 0)) = \varphi_i(e_l, 0)
\]
\( ^3 \{e_k\}_{k=1}^n \) denotes the standard basis for \( \mathbb{R}^n \).
Therefore, \( a_{i,k} = a_{i,l} \) if \( k, l \in N \setminus \{ i \} \) are such that \( k \neq l \). Thus, if we set \( \gamma = a_{i,j} \) for each \( j \in N \setminus \{ i \} \) and \( \alpha = b - \gamma \), we obtain
\[
\varphi_i(c, E) = \gamma \sum_{j \in N \setminus \{ i \}} c_j + bc_i + \beta E = \alpha c_i + \beta E + \gamma C
\]

Uniqueness: to check uniqueness it is enough to prove that if
\[
0 = \alpha c_i + \beta E + \gamma C
\]
for every allocation problem \((c, E)\) and for every creditor \( i \), then the numbers \( \alpha, \beta, \gamma \) vanish.

Thus, for given \( \beta \in \mathbb{R} \) let \( i \in N \) and \((c, E) = (0, 1)\). Then the above sum reduces to
\[
0 = \beta
\]
Similarly, given \( \gamma \in \mathbb{R} \) let \( i, j \in N \) such that \( i \neq j \) and \((c, E) = (e_j, 0)\). In this case the sum is just
\[
0 = \gamma
\]
And for given \( \alpha \in \mathbb{R} \) let \( i \in N \) and \((c, E) = (e_i, 0)\). Then we get
\[
0 = \alpha
\]

Finally, it is straightforward to check that formula (1) defines a linear and symmetric allocation rule for any choice of coefficients.

Corollary 1. Any linear symmetric allocation rule
\[
\varphi : A^{(n)} = \Delta_n \times \mathbb{R} \oplus \Delta_n^\perp \times \{0\} \to \mathbb{R}^n = \Delta_n \oplus \Delta_n^\perp
\]
satisfies:

i): \( \varphi(\Delta_n \times \mathbb{R}) \subset \Delta_n \)
ii): \( \varphi(\Delta_n^\perp \times \{0\}) \subset \Delta_n^\perp \)

Moreover, for every \( (c, E) \in \Delta_n^\perp \times \{0\} \), there is a constant \( \lambda \in \mathbb{R} \) such that:
\[
\varphi(c, E) = \lambda c
\]

The next Theorem characterizes all allocation rules satisfying the linearity, symmetry and efficiency axioms.

**Theorem 3.1.** The allocation rule \( \varphi : A^{(n)} \to \mathbb{R}^n \) satisfies linearity, symmetry and efficiency axioms if and only if it is of the form
\[
\varphi_i(c, E) = \frac{E}{n} + \delta (c_i - \overline{c})
\]
for any real number \( \delta \).

Moreover, such representation is unique.

**Proof.** By Proposition 2,
\[
\varphi_i(c, E) = \alpha c_i + \beta E + \gamma C
\]
for some constants \( \alpha, \beta, \gamma \).

Efficiency implies:
\[
0 = \sum_{i \in N} \varphi_i(e_i, 0) = \alpha + n \gamma
\]
and

\[ 1 = \sum_{i \in N} \varphi_i(0, 1) = n\beta \]

Hence,

\[ \alpha = -n\gamma \quad \text{and} \quad \beta = \frac{1}{n} \]

Therefore

\[ \varphi_i(c, E) = -n\gamma c_i + \frac{E}{n} + \gamma C \]

Set \( \delta = -n\gamma \), then

\[ \varphi_i(c, E) = \frac{E}{n} + \delta (c_i - c) \]

The converse is a straightforward computation, and uniqueness follows from the uniqueness part of Proposition 2.

**Remark 1.** Notice that we obtain the same result from the decomposition of \( A^{(n)} \) (Proposition 1) and Corollary 1, as follows:

Let \( \varphi : A^{(n)} \to \mathbb{R}^n \) be a linear symmetric allocation rule. First of all, \( \Delta_n^+ \times \{0\} \) is the subspace of allocation problems \((c, E)\) for which \( E = 0 \) and trivially satisfy \( \sum_{j \in N} \varphi_j(c, E) = 0 \), since (by Corollary 1) their image lies in \( \Delta_n^+ \).

Therefore, efficiency need only be checked on \( \Delta_n \times \mathbb{R} \). Let \((c, E) \in \Delta_n \times \mathbb{R} \) and \( i \in N \). Since \((c, E)\) is fixed by all permutations in \( S_n \),

\[ \sum_{j \in N} \varphi_j(c, E) = n\varphi_i(c, E) = E \]

and so, \( \varphi_i(c, E) = \frac{E}{n} \).

Finally, let \( \varphi : A^{(n)} \to \mathbb{R}^n \) be a linear symmetric and efficient allocation rule and \((c, E) \in A^{(n)} \). From the proof of Proposition 3, the allocation problem decomposes as:

\((c, E) = ((\overline{c}, \overline{c}, \ldots, \overline{c}), E) + ((c_1 - \overline{c}, c_2 - \overline{c}, \ldots, c_n - \overline{c}), 0)\)

then by Corollary 1 and the above discussion,

\[ \varphi_i(c, E) = \varphi_i((\overline{c}, \overline{c}, \ldots, \overline{c}), E) + \varphi_i((c_1 - \overline{c}, c_2 - \overline{c}, \ldots, c_n - \overline{c}), 0) \]

\[ = \frac{E}{n} + \delta (c_i - \overline{c}) \]

for arbitrary \( \delta \in \mathbb{R} \).

**Corollary 2.** The space of all linear and symmetric allocation rules on \( A^{(n)} \) is 3-dimensional. Likewise, the (affine) space of all linear, symmetric and efficient allocation rules is 1-dimensional.

Now, recall that \( \Delta_n \times \mathbb{R} \) is the subspace of allocation problems where individual claims are all the same. The next Corollary characterizes the allocation rules to these problems in terms of linearity, symmetry and efficiency.

**Corollary 3.** Among all linear symmetric allocation rules \( \varphi : A^{(n)} \to \mathbb{R}^n \), the equal-sharing allocation rule

\[ \varphi_i(c, E) = \frac{E}{n} \quad \text{for all} \; i \in N \]

is characterized as the unique efficient allocation rule on \( \Delta_n \times \mathbb{R} \).

Next, we state a formula for the set of all allocation rules satisfying the linearity, symmetry and compatibility axioms.
Theorem 3.2. If the allocation rule $\varphi : A^{(n)} \to \mathbb{R}^n$ satisfies the linearity, symmetry and compatibility axioms, then there exist a unique real number $\gamma$ such that

$$\varphi_i(c, E) = c_i + \gamma (C - E)$$  (3)

Conversely, for any real number $\gamma$, the allocation rule given by (3) is linear, symmetric and compatible.

Proof. First of all, it is easy to verify that the allocation problems $\{(e_k, 1)\}_{k \in \mathbb{N}} \cup \{(0, 1)\}$ form a basis for $A^{(n)}$.

By Proposition 2,

$$\varphi_i(c, E) = \alpha c_i + \beta E + \gamma C$$

for some constants $\alpha, \beta, \gamma$.

Compatibility implies:

$$1 = \varphi_i(e_i, 1) = \alpha + \beta + \gamma$$

and

$$0 = \varphi_j(e_i, 1) = \beta + \gamma$$

Therefore, $\alpha = 1$ and $\beta = -\gamma$. Then

$$\varphi_i(c, E) = c_i + \gamma (C - E)$$

The proof in the other direction is straightforward, and once again the uniqueness follows from the uniqueness part of Proposition 2.

Corollary 4. The space of all allocation rules on $A^{(n)}$ satisfying linearity, symmetry and compatibility axioms is 1-dimensional.

The next Corollary characterizes a allocation rule where each agent receives the amount she claims, for allocation problems in $\Delta_n^+ \times \{0\}$.

Corollary 5. The allocation rule $\varphi : A^{(n)} \to \mathbb{R}^n$ given by

$$\varphi_i(c, E) = c_i$$

is the unique one on $\Delta_n^+ \times \{0\}$ satisfying linearity, symmetry and compatibility axioms.

Herrero et al. [3] introduced the rights-egalitarian allocation rule $F^{RE}$, which is defined for all $(c, E) \in A^{(n)}$ and all $i \in N$ as

$$F^{RE}_i(c, E) = c_i + \frac{1}{n} (E - C)$$  (4)

This allocation rule is regarded as conceding agents all their claims, and then dividing equally whatever is left (be it positive or negative). They called this allocation rule the “rights-egalitarian solution” to express the fact that rights are honored and further, the remainder is divided equally.

Such allocation rule divides equally the net worth $E - C$ among the $n$ agents. When $E > C$, the resulting allocation has a equal-gains form. And if $E < C$, then the allocation rule corresponds an equal-loss form.

Finally, using the properties of linearity, symmetry, efficiency and compatibility, we obtain a new characterization of the rights-egalitarian solution.

Theorem 3.3. The allocation rule $F^{RE} : A^{(n)} \to \mathbb{R}^n$ given by (4), is the unique allocation rule satisfying linearity, symmetry, efficiency and compatibility axioms.
Proof. Recall that the collection of allocation problems \{((e_k, 1))_{k \in N} \cup \{(0, 1)\}\} constitutes a basis of \(A^{(n)}\).

It is easy to check that the allocation rule given by (4) is of the form (2), so it satisfies linearity, symmetry and efficiency axioms; and it is straightforward to prove that it also satisfies the compatibility axiom. We prove uniqueness. Let \(\varphi\) be a allocation rule satisfying the four axioms. For any \((c, E) \in A^{(n)}\), there exist unique real numbers \(\{\lambda_k\}_{k=1}^{n+1}\) such that \((c, E) = \sum_{k \in N} \lambda_k(e_k, 1) + \lambda_{n+1}(0, 1)\). Then, by linearity \(\varphi(c, E) = \sum_{k \in N} \lambda_k \varphi(e_k, 1) + \lambda_{n+1} \varphi(0, 1)\). We will show that \(\varphi(c, E)\) is determined for all \((c, E) \in A^{(n)}\) and by the previous discussion, it is therefore sufficient to show that \(\varphi(e_k, 1)\) is determined for all \(k \in N\) and also to determine \(\varphi(0, 1)\). In this way, \(\varphi\) is unique for each \((e_k, 1), (0, 1)\) and so for \((c, E)\).

Notice that,

i: By the compatibility axiom,

\[\varphi_i(e_i, 1) = 1\]

for each \(i \in N\).

ii: If \(i, j, l \in N\) and \(\pi \in S_n\) is such that \(\pi(i) = j\) and \(\pi(j) = l\), then \(\pi \cdot (e_i, 1) = (e_j, 1)\) and by symmetry: \(\varphi_i(e_i, 1) = \varphi_{\pi(j)}(e_j, 1) = \varphi_j(\pi \cdot (e_i, 1)) = \varphi_j(e_i, 1)\).

Hence, by the efficiency axiom,

\[\varphi_i(e_j, 1) = 0\]

for each \(i \in N\) \(\backslash\{j\}\).

iii: If \(j, l \in N\) and \(\pi \in S_n\) is such that \(\pi(j) = l\), then by symmetry: \(\varphi_l(0, 1) = \varphi_{\pi(j)}(0, 1) = \varphi_j(\pi \cdot (0, 1)) = \varphi_j(0, 1)\).

Thus, by the efficiency axiom,

\[\varphi_i(0, 1) = \frac{1}{n}\]

for each \(i \in N\).

iv: In the decomposition of \((c, E)\), the precise values of the numbers \(\{\lambda_k\}_{k=1}^{n+1}\) are

\[\lambda_k = \begin{cases} c_k & \text{if } k \in N \\ E-C & \text{if } k = n+1 \end{cases}\]

And so,

\[\varphi_i(c, E) = \sum_{k \in N} \lambda_k \varphi_i(e_k, 1) + \lambda_{n+1} \varphi_i(0, 1)\]

\[= c_i + \frac{1}{n} (E-C) = F^{RE}_i(c, E)\]

\[\square\]

Remark 2. The axioms used to characterize these classes of allocation rules are independent:

- The equal-sharing rule \(\varphi(c, E) = (\frac{E}{n}, \frac{E}{n}, \ldots, \frac{E}{n})\) satisfies linearity, symmetry and efficiency, but fails to satisfy compatibility.
- The allocation rule \(\varphi(c, E) = (c_1, c_2, \ldots, c_{n-1}, E-C+c_n)\) satisfies linearity, efficiency and compatibility, but fails to satisfy symmetry.
- The proportional rule \(\varphi(c, E) = (\frac{c_1}{n} E, \frac{c_2}{n} E, \ldots, \frac{c_n}{n} E)\) satisfies symmetry, efficiency and compatibility, but fails to satisfy linearity.
4. Bankruptcy rules. In this section we deal with bankruptcy problems and their linear rules by means of an additivity axiom.

For a given \((c, E) \in B^{(n)}\), we denote by \(S(c, E)\) the set of all possible payoff vectors resulting from applying any bankruptcy rule; i.e.,

\[
S(c, E) = \left\{ z \in \mathbb{R}^n \mid \sum_{j=1}^{n} z_j = E \quad \text{and} \quad 0 \leq z_i \leq c_i \quad \forall i = 1, \ldots, n \right\}
\]

Now we turn to present some well-known properties used in bankruptcy rules.

**Axiom 5** (Additivity). A bankruptcy rule \(\psi\) satisfies additivity if

\[
\psi((c, E) + (c', E')) = \psi(c, E) + \psi(c', E')
\]

for all \((c, E), (c', E') \in B^{(n)}\).

The interpretation of this property is analogous to the one in allocation problems.

**Axiom 6** (Independence of irrelevant claims). A bankruptcy rule \(\psi\) satisfies independence of irrelevant claims if for every \((c, E) \in B^{(n)}\):

\[
\psi(c, E) = \psi(c^*, E)
\]

where \(c^* = \min\{c_i, E\}\).

This property states that agents having a claim larger than the estate receive the same amount as they would if their claim was equal to the estate. In other words, it is not possible for an agent to get more than the estate value of the bankruptcy firm, so it is irrelevant to have a claim larger than the estate of the firm.

**Axiom 7** (Equal treatment of equals). A bankruptcy rule \(\psi\) satisfies equal treatment of equals if for every \((c, E) \in B^{(n)}\) and for every agents \(i, j \in N\) such that \(c_i = c_j\):

\[
\psi_i(c, E) = \psi_j(c, E)
\]

The previous property requires that two agents having the same claim will receive the same amount as a payoff. It is expected that a bankruptcy rule treats all agents involved a dispute as equals in some sense.

It turns out that in bankruptcy problems, there are no rules satisfying the additivity axiom.

**Proposition 3** (Bergantiños and Méndez-Naya [1]). There exist no additive bankruptcy rules.

**Proof.** We proceed by contradiction. Let \(\psi\) be an additive bankruptcy rule and suppose that \(\psi((5, 15), 10) = (10 - x, x)\). If we take \(y \neq x\) such that \(5 \leq y \leq 10\), then by the additivity of \(\psi\):

\[
\begin{align*}
\psi((5, 15), 10) &= \psi((0, 15), y) + \psi((5, 0), 10 - y) \\
&= (0, y) + (10 - y, 0) = (10 - y, y) \\
&= (10 - x, x)
\end{align*}
\]

Therefore, \(x = y\). \(\square\)

Now, the next task is to relax the restriction imposed by the additivity axiom and so, provide a weaker version of it.
Axiom 8 (Weak additivity). A bankruptcy rule $\psi$ satisfies weak additivity if for any pair of bankruptcy problems $(c, E), (c', E') \in B^{(n)}$ such that $S(c, E) + S(c', E') = S(c + c', E + E')$, it holds that

$$\psi((c, E) + (c', E')) = \psi(c, E) + \psi(c', E')$$

On the contrary of the additivity axiom, there are several bankruptcy rules that satisfy this weaker version. One example of it is the serial dictator rule $SD^{\pi}$:

$$SD^{\pi}_i(c, E) = \min \left\{ c_i, \max \left\{ E - \sum_{j \in N: \pi(j) < \pi(i)} c_j, 0 \right\} \right\}$$

where $\pi \in S_n$ is any permutation of the set of agents.

Taking the arithmetic average over all orders $5$ of arrival of the awards vectors calculated in this way, we get the random arrival rule, $RA$:

$$RA_i(c, E) = \frac{1}{n!} \sum_{\pi \in S_n} SD^{\pi}_i(c, E)$$

which also satisfy the axiom of weak additivity.

Proposition 4. Any bankruptcy rule that satisfies the property of weak additivity, it also satisfies the property of independence of irrelevant claims.

Proof. First, we show that $S(c, E) = S(c^*, E)$ for every $(c, E) \in B^{(n)}$.

Let $(c, E) \in B^{(n)}$ and take any $z \in S(c, E)$, then it holds that $z_i \leq E$ and $0 \leq z_i \leq c_i^* \forall i$. Thus, $z \in S(c^*, E)$ and so, $S(c, E) \subseteq S(c^*, E)$. On the other hand, it is easy to see that $S(c^*, E) \subseteq S(c, E)$, since $c_i^* \leq c_i \forall i$. Therefore, $S(c, E) = S(c^*, E)$.

Now, notice that

$$S(c, E) = S(c^*, E) + S(c - c^*, 0)$$

since $(c, E) = (c^*, E) + (c - c^*, 0)$ and $S(c - c^*, 0) = \{0\}$.

If $\psi$ is a bankruptcy rule satisfying weak additivity, then

$$\psi(c, E) = \psi(c^*, E) + \psi(c - c^*, 0) = \psi(c^*, E) + 0$$

Thus, $\psi$ satisfies independence of irrelevant claims.

We can use the previous Proposition in two ways. First, it allows us to conclude that if a bankruptcy rule does not satisfy independence of irrelevant claims, then it does not satisfy weak additivity. Also, in proofs involving bankruptcy rules that satisfies weak additivity, we can assume without loss of generality that a problem $(c, E) \in B^{(n)}$ satisfies $c^* = 0$.

Remark 3. It is not difficult to show that not every bankruptcy rule satisfying weak additivity satisfies equal treatment of equals. For instance, the serial dictator rule $(SD^{\pi})$ satisfies weak additivity, while it does not satisfy equal treatment of equals.

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$^4$Imagine claimants arriving one at a time, and compensate them fully until money runs out. The resulting awards vector of course depends on the order ($\theta$) in which claimants arrive.

$^5$In order to remove the unfairness associated with a particular order.
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