GAMES CHARACTERIZING CERTAIN FAMILIES OF FUNCTIONS

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Abstract. We obtain several game characterizations of Baire 1 functions between Polish spaces $X$, $Y$ which extends the recent result of V. Kiss. Then we propose similar characterizations for equi-Baire 1 families of functions. Also, using related ideas, we give game characterizations of Baire measurable and Lebesgue measurable functions.

1. Introduction

The game approach plays an important role in descriptive set theory. Let us recall Choquet games and the Banach-Mazur game in the studies of the Baire category problems [14 Sec. 8], and Wadge games with their influence on investigations in the Borel hierarchy [11 Sec. 21]. It is commonly known that Borel and projective determinacy provide a strong tool in set-theoretical investigations, cf. [14 Sec. 20, 38]. Note that various kinds of topological games make fruitful inspirations in topology and analysis, cf. [3], [12], [7]. They can distinguish new kinds of topological objects, cf. [12], [3].

In the recent decades, several nice characterizations for some classes of regular functions were obtained. Duparc [9] and Carroy [8] characterized Baire 1 functions from $\mathbb{N}^\mathbb{N}$ into itself by using the so-called eraser game (for more applications of this game, see [6]). Other significant results for different classes of functions between Polish zero-dimensional spaces are due to Andretta [2] (a game characterization of $\Delta_2^0$-measurable functions), Semmes [22] (Borel functions), Nobrega [19] (Baire class $\xi$ functions) and Motto Ros [18] (piecewise defined functions).

This note is motivated by Kiss [15], who introduced a game characterizing Baire class 1 functions between arbitrary two Polish spaces. This improved the results by Duparc [9] and Carroy [8] that have been mentioned above. Another idea characterizing Baire 1, real-valued functions, has been presented in [10]. Last but not least, a game that characterizes Baire class 1 functions between arbitrary separable metrizable spaces has been defined by Notaro in very recent paper [20].

Our first aim in this paper is to extend the result by Kiss. We simplify the proof of a harder implication of his result by the use of $\varepsilon$-$\delta$ characterization of Baire 1 functions. Then we modify the game defined by Kiss in two other manners, one in which Player II plays points in a space, and another in which Player II plays sets. Whereas in the earlier versions of the game, considered by Kiss, Player II was

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playing in a space containing the range of a function, here we let Player II play in the domain. This allows us to give strong game-theoretical characterizations of equi-Baire 1 families with both a point-based and a set-based game, and finally, characterizations of Baire-measurable and Lebesgue-measurable functions with set-based games.

We will use the following reasoning scheme throughout this work.

**Lemma 1.** Let $G(f)$ be a game with a parameter function $f \in Y^X$. For a given class of functions $\mathcal{F} \subseteq Y^X$ assume that:

1. if $f \in \mathcal{F}$ then Player II has a winning strategy in the game $G(f)$, and
2. if $f \notin \mathcal{F}$ then Player I has a winning strategy in $G(f)$.

Then the game $G(f)$ is determined and the class $\mathcal{F}$ can be characterized by $G_f$:

1. $f \in \mathcal{F}$ if and only if Player II has a winning strategy in the game $G(f)$, and
2. $f \notin \mathcal{F}$ if and only if Player I has a winning strategy in $G(f)$.

Assume that $X$ and $Y$ are Polish spaces. Through the paper, we assume that $d_X$ and $d_Y$ are the respective metrics in $X$ and $Y$.

Let us state preliminary facts on Baire 1 functions. A function $f: X \rightarrow Y$ between Polish spaces $X, Y$ is called Baire class 1 whenever the preimage $f^{-1}[U]$ is $F_\sigma$ in $X$ for any open set $U$ in $Y$. If $Y = \mathbb{R}$, this is equivalent to the property that $f$ is the limit of a pointwise convergent sequence of continuous functions, see e.g. [14] Theorem 24.10.

In the literature, we encounter various conditions which characterize the class of Baire 1 functions. The classical characterization given by Baire says that $f$ is Baire 1 if and only if $f \upharpoonright P$ has a point of continuity for every non-empty closed set $P \subseteq X$. This is the so-called Pointwise Continuity Property, in short (PCP), see e.g. [4]. An $\varepsilon$-$\delta$ characterization of Baire 1 functions, obtained in [17], says the following. A function $f: X \rightarrow Y$ is Baire 1 whenever, for any positive number $\varepsilon$, there is a positive function $\delta_\varepsilon: X \rightarrow \mathbb{R}$ such that for any $x_0, x_1 \in X$,

1. $d_X(x_0, x_1) < \min \{ \delta_\varepsilon(x_0), \delta_\varepsilon(x_1) \}$ implies $d_Y(f(x_0), f(x_1)) < \varepsilon.$

We will call such a $\delta_\varepsilon$ an $\varepsilon$-gauge for $f$.

We say that a family $\mathcal{F} \subseteq Y^X$ is equi-continuous at a point $x \in X$ whenever

2. $\forall_{\varepsilon > 0} \exists_{\delta > 0} \forall_{f \in \mathcal{F}} \ (d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \varepsilon).$

$\mathcal{F}$ is equi-continuous if it is equi-continuous at every $x \in X$.

A family $\mathcal{F} \subseteq Y^X$ is said to fulfill the Point of Equicontinuity Property ($\mathcal{F}$ has (PECP), in short) if for every non-empty closed $P \subseteq X$, the family

$$\mathcal{F} \upharpoonright P := \{ f \upharpoonright P : f \in \mathcal{F} \}$$

has a point of equicontinuity.

We say that a family $\mathcal{F} \subseteq Y^X$ is equi-Baire 1 if for any positive number $\varepsilon$ there is a positive function $\delta_\varepsilon: X \rightarrow \mathbb{R}_+$ such that for any $x_0, x_1 \in X$ and $f \in \mathcal{F}$ the condition (1) holds (i.e. all $f \in \mathcal{F}$ have a family of common $\varepsilon$-gauges). Clearly, every equi-continuous family is equi-Baire 1 and has (PECP), and the opposite implications do not hold. (In fact, if $\mathcal{F}$ is an equi-continuous family and $\varepsilon > 0$ then there is $\delta > 0$ which satisfies condition (2). Then the constant function $\delta_\varepsilon := \delta$ satisfies (1). On the other hand, if $f \in \mathbb{R}^X$ is Baire 1 function that is not continuous, then the family $\{ f \}$ is equi-Baire 1 but not equi-continuous, see [1]).
Both definitions were introduced by D. Lecomte in [10]. He proved the following equivalence.

**Theorem 2 ([10], Prop. 32]).** $\mathcal{F}$ has (PECP) if and only if $\mathcal{F}$ is equi-Baire 1.

Let us mention that, since $X$ is a Polish space, a non-empty closed set in conditions (PCP) and (PECP) can be equivalently replaced by a perfect set (that is, a non-empty closed set without isolated points).

Note that the definition of equi-Baire 1 family of functions was rediscovered later by A. Alikhani-Koopaei in [1]. The definition of families with (PECP) was used by E. Glasner and M. Megrelishvili in the context of dynamical systems in [11] (under the name “barely continuous family”).

Through the paper we assume the Axiom of Choice AC.

## 2. Game characterizations of Baire 1 functions

Recall the game defined by Kiss [15]. Let $X$ and $Y$ be Polish spaces. Let $f: X \to Y$ be an arbitrary function. At the $n$th step of the game $G_f$, Player I plays $x_n$, then Player II plays $y_n$:

| Player I | $x_0$ | $x_1$ | $x_2$ | \ldots |
|----------|-------|-------|-------|-------|
| Player II | $y_0$ | $y_1$ | $y_2$ | \ldots |

with the rules that for each $n \in \mathbb{N}$:

- $x_n \in X$ and $d_X(x_n, x_{n+1}) \leq 2^{-n}$;
- $y_n \in Y$.

Since $X$ is complete, $x_n \to x$ for some $x \in X$. Player II wins if and only if $\langle y_n \rangle_{n \in \mathbb{N}}$ is convergent and $y_n \to f(x)$. Recall the main result of Kiss:

**Theorem 3 ([15], Theorem 1]).** The game $G_f$ is determined, and

- Player I has a winning strategy in $G_f$ if and only if $f$ is not of Baire class 1.
- Player II has a winning strategy in $G_f$ if and only if $f$ is of Baire class 1.

The longest part of the original proof is the implication (1) from Lemma [11] “if $f$ is of Baire class 1 then Player II has a winning strategy”. We show that it can be significantly shortened by the use of $\varepsilon$-$\delta$ characterization of Baire 1 functions. We describe it in Lemma 3 which will be preceded by the following fact.

**Lemma 4.** A function $f: X \to Y$ is Baire 1 if and only if it possesses a family of gauges $\{\delta_\varepsilon: \varepsilon > 0\}$ such that for every $x \in X$ the map $\varepsilon \mapsto \delta_\varepsilon(x)$ is non-decreasing.

**Proof.** Only the implication “$\Rightarrow$” has to be proved. Assume that $f$ is Baire 1 and $\{\delta_\varepsilon: \varepsilon > 0\}$ is a family of gauges for $f$. For every $\varepsilon > 0$ fix $N_\varepsilon \in \mathbb{N}$ such that $N_\varepsilon = 1$ if $\varepsilon \geq 1$ and $\frac{1}{N_\varepsilon} \leq \varepsilon < \frac{1}{N_{\varepsilon-1}}$ for $\varepsilon < 1$. For $x \in X$ define

$$\delta'_\varepsilon(x) := \min\left\{\delta'_{\frac{n}{N_\varepsilon}}(x): n \leq N_\varepsilon\right\}.$$ 

Clearly, if $\varepsilon \leq \varepsilon_1$ then $N_\varepsilon \geq N_{\varepsilon_1}$, hence for any $x \in X$ we have $\delta'_\varepsilon(x) \leq \delta'_1(x)$. We will show that $\{\delta'_\varepsilon: \varepsilon > 0\}$ is a family of gauges for $f$. Indeed, assume that $d_X(x_0, x_1) < \min(\delta'_{\varepsilon}(x_0), \delta'_{\varepsilon}(x_1))$ for some $\varepsilon > 0$ and $x_0, x_1 \in X$. Then $\frac{1}{N_\varepsilon} \leq \varepsilon$ (by definition of $N_\varepsilon$) and $\delta'_\varepsilon(x_i) \leq \delta'_{\frac{1}{N_\varepsilon}}(x_i)$ for $i = 0, 1$, so

$$d_X(x_0, x_1) < \min(\delta'_{\frac{1}{N_\varepsilon}}(x_0), \delta'_{\frac{1}{N_\varepsilon}}(x_1)).$$
Hence

\[ d_Y(f(x_0), f(x_1)) < \frac{1}{N_\varepsilon} \leq \varepsilon. \]

\[ \square \]

**Lemma 5.** Let \( \Delta := \{ \varepsilon \in (0, 2) \} \) be a family of positive functions from \( X \) into \( \mathbb{R} \) such that, for every \( x \in X \), the map \( \varepsilon \mapsto \delta_\varepsilon(x) \) is non-decreasing. Then there is a function \( S_\Delta \) : \( X < \omega \to X \) such that, for every sequence \( (x_n) \) with \( d_X(x_n, x_{n+1}) \leq 2^{-n} \)

for each \( n \), and for every \( \varepsilon > 0 \) there exists \( N_\varepsilon \in \mathbb{N} \) with the property

- for every Baire 1 function \( f : X \to Y \), if \( \delta_\varepsilon \) is a family of \( \varepsilon \)-gauge for \( f \), then

\[ \forall n > N_\varepsilon \quad d_Y(f(S_\Delta^n(x_0, x_1, \ldots, x_n)), f(\lim_{n \to \infty} x_n)) < \varepsilon. \]

In particular,

\[ \lim_{n \to \infty} f(S_\Delta^n(x_0, x_1, \ldots, x_n)) = f(\lim_{n \to \infty} x_n), \]

so the function \( S_\Delta \) is a winning strategy for Player II in the game \( G_f \).

**Proof.** Fix \( \varepsilon > 0 \) and a sequence \( (x_n) \subseteq X \) such that \( d_X(x_n, x_{n+1}) \leq 2^{-n} \) for each \( n \). We may assume that, for each \( n \in \mathbb{N} \), Player I plays \( x_n \) in the \( n \)th move of the game \( G_f \). For each \( n \in \mathbb{N} \) let \( K_n := \overline{B}(x_n, 2^{-n+1}) \) be the closed ball around \( x_n \). Note that this is the smallest closed ball around \( x_n \) which ensures that \( x := \lim_{j \to \infty} x_j \in K_n \). Denote by \( M_n \) the greatest index \( m < n \) for which there exists a point \( x'_n \in K_n \) such that \( K_n \subseteq B\left( x'_n, \delta_m(x'_n) \right) \); then pick one of them and call it \( x'_n \). If such an index \( m \) does not exist, put \( M_n := -\infty \). Define

\[ S_\Delta^n(x_0, \ldots, x_n) := \begin{cases} x'_n & \text{if } M_n > -\infty, \\ x_n & \text{otherwise}. \end{cases} \]

It is enough to show that \( \lim_{n \to \infty} f(S_\Delta^n(x_0, \ldots, x_n)) = f(x) \) for each Baire 1 function \( f : X \to Y \) with the family of \( \varepsilon \)-gauge equal to \( \Delta \). Fix \( \varepsilon > 0 \) and find a positive integer \( M \) such that \( 1/M < \varepsilon \). There exists \( N \in \mathbb{N} \) such that for each \( n \geq N \),

(3)

\[ x \in K_n \subseteq B \left( x, \delta_{M_n}(x) \right). \]

Since \( x \in K_n \) for all \( n \), it follows that \( M_n \geq M > -\infty \) for all \( n > \max\{N, M\} \). Then \( \delta_{M_n}(x') \leq \delta_{M_n}(x') \) for all \( x' \in X \), hence

(4)

\[ S_\Delta^n(x_0, \ldots, x_n) = x'_n \in K_n \subseteq B \left( x'_n, \delta_{M_n}(x'_n) \right) \subseteq B \left( x'_n, \delta_{M_n}(x'_n) \right). \]

From (3) and (4) we get

\[ x, x'_n \in K_n \subseteq B \left( x, \delta_{M_n}(x) \right) \cap B \left( x'_n, \delta_{M_n}(x'_n) \right), \]

and so

\[ d_X(x'_n, x) < \min \left\{ \delta_{M_n}(x'_n), \delta_{M_n}(x) \right\}. \]

To finish the proof it is enough to observe that, since \( \delta_{M_n} \) is an \( \frac{1}{M} \)-gauge for \( f \), so

\[ d_Y(f(S_\Delta^n(x_0, \ldots, x_n)), f(x)) < \frac{1}{M} < \varepsilon. \]

\[ \square \]
Remark. In the original proof, Kiss noted that “the idea of the proof is to pick $y_n$ as the image of a point in $B(x_n, 2^{-n+1})$ at which $f$ behaves badly”. In fact, we are able to shorten his argument, since the family of $\varepsilon$-gauges encodes the “bad” behaviour of $f$.

Remark. From Lemma 5 (see also [15, Theorem 1]) it follows that Player II has a winning strategy in the game $G_f$ if and only if he has a winning strategy of the form $S(x_0, x_1, \ldots, x_n) = f(S'(x_0, x_1, \ldots, x_n))$. This is a motivation for introducing the games $G'_f$ and $G''_f$.

2.1. The games $G'_f$ and $G''_f$. Let $X$ and $Y$ be Polish spaces, $f : X \to Y$ be an arbitrary function. At the $n$th step of the game $G'_f$, Player I plays $x_n$, then Player II plays $x'_n$,

| Player I | $x_0$ | $x_1$ | $x_2$ | ... |
|--------|-------|-------|-------|-----|
| Player II | $x'_0$ | $x'_1$ | $x'_2$ | ... |

with the rules that for each $n \in \mathbb{N}$:

- $x_n \in X$ and $d_X(x_n, x_{n+1}) \leq 2^{-n}$;
- $x'_n \in X$.

Since $X$ is complete, $x_n \to x$ for some $x \in X$. Player II wins if $f(x'_n)$ is convergent to $f(x)$. Otherwise, Player I wins.

As a consequence of Lemma 5 we obtain the following result.

**Theorem 6.** The game $G'_f$ is determined, and

- Player I has a winning strategy in $G'_f$ if and only if $f$ is not of Baire class 1.
- Player II has a winning strategy in $G'_f$ if and only if $f$ is of Baire class 1.

**Proof.** We will apply Lemma 1 hence it is enough to show that:

(i) if $f$ is Baire class 1 then Player II has a winning strategy, and
(ii) if $f$ is not of Baire class 1 then Player I has a winning strategy.

To prove (i) observe that the function $S'_\Delta$ from Lemma 5 for $\Delta$ being a family of gauges of $f$, is a winning strategy for Player II.

To see (ii), observe that the winning strategy for Player I in $G_f$ is also a winning strategy for him in $G'_f$. Thus (ii) follows from Theorem 3. □

Now, we propose a further modification of the game to obtain a similar effect. This time, we will define a point-open game $G''_f$. Let $X$ and $Y$ be Polish spaces, $f : X \to Y$ be an arbitrary function. At the first step of the game $G''_f$, Player I plays $x_0 \in X$ and then Player II plays an open set $U_0 \ni x_0$. At the $n$th step of the game $G''_f$ ($n > 0$), Player I plays $x_n \in U_{n-1}$, then Player II plays an open set $U_n \ni x_n$:

| Player I | $x_0$ | $x_1$ | $x_2$ | ... |
|--------|-------|-------|-------|-----|
| Player II | $U_0$ | $U_1$ | $U_2$ | ... |

with the rules that for each $n \in \mathbb{N}$:

- $x_0 \in X$, and $x_n \in U_{n-1}$ for $n > 0$;
- $U_n \ni x_n$.

If $\langle x_n \rangle$ is convergent and $\lim_{n \to \infty} f(x_n) = f(\lim_{n \to \infty} x_n)$ then Player II wins. Otherwise, Player I wins.

**Theorem 7.** The game $G''_f$ is determined, and

- Player I has a winning strategy in $G''_f$ if and only if $f$ is not of Baire class 1.
Player II has a winning strategy in $G'_f$ if and only if $f$ is of Baire class 1.

Proof. It is enough to prove implications (1) and (2) from Lemma 1.

To see the first implication, assume that $f$ is Baire 1 and let $\{\delta_\varepsilon : \varepsilon > 0\}$ be a family of $\varepsilon$-gauges for $f$. Without loss of generality (see Lemma 1) we may assume that, for any fixed $x \in X$, the sequence $\langle \delta_\varepsilon(x) \rangle$ is decreasing, and $\delta_\varepsilon(x) < 2^{-n}$ for every $n > 0$. In the $n$th move, Player II plays $U_n := B(x_n, \delta_\varepsilon(x_n)/2)$. This is a winning strategy for Player II. Indeed, since $x_{n+1} \in U_n$, so $d_X(x_n, x_{n+1}) < \text{diam}(U_n) \leq 2^{-n}$ for every $n$. Hence $\langle x_n \rangle$ is a Cauchy sequence in a complete space $X$, so it converges. Let $x := \lim_{n \to \infty} x_n$. Fix $\varepsilon > 0$ and $N > 1/\varepsilon$. Then $x \in B(x_n, \delta_\varepsilon(x_n))$ for each $n \geq N$, and for all $n$ with $d_X(x, x_n) < \delta_\varepsilon(x)$, we have $d_Y(f(x), f(x_n)) < \varepsilon$. Thus $\langle f(x_n) \rangle$ is convergent to $f(x)$.

Now assume that $f$ is not Baire class 1. Then there are a perfect set $P \subseteq X$, $y_0 \in Y$ and $\varepsilon > 0$ such that both sets $A := \{x \in P : d_Y(f(x), y_0) < \varepsilon\}$ and $B := \{x \in P : d_Y(f(x), y_0) \geq 2\varepsilon\}$ are dense in $P$. The winning strategy for Player I in the game $G'_f$ consists in choosing $x_n \in A$ for odd $n$ and $x_n \in B$ for even $n$. In fact, if Player I plays this strategy then the sequence $\langle f(x_n) \rangle$ is not a Cauchy sequence.

3. Games for equi-Baire 1 families of functions

In this section, we modify games $G'_f$ and $G''_f$ to obtain characterizations of equi-Baire 1 families of functions.

Let $X$ and $Y$ be Polish spaces, let $F \subseteq Y^X$. At the $n$th step of the game $G'_F$, Player I plays $x_n$, then Player II plays $x'_n$,

- Player I: $x_0, x_1, x_2, \ldots$
- Player II: $x'_0, x'_1, x'_2, \ldots$

with the rules that for each $n \in \mathbb{N}$:
- $x_n \in X$ and $d_X(x_n, x_{n+1}) \leq 2^{-n}$;
- $x'_n \in X$.

Since $X$ is complete, $x_n \to x$ for some $x \in X$. Player II wins if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n \geq N \forall f \in F \ d_Y(f(x'_n), f(x)) < \varepsilon. \tag{5}$$

(Then we say that the indexed family of sequences $\{(f(x'_n)) : f \in F\}$ is equi-convergent to the indexed family $\{f(x) : f \in F\}$.) Otherwise, Player I wins.

We will use the fact that (5) implies the following Cauchy-type condition. (A proof of this fact is left to the reader.)

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n, m \geq N \forall f \in F \ d_Y(f(x'_n), f(x'_m)) < \varepsilon. \tag{6}$$

Theorem 8. The game $G'_F$ is determined, and

- Player I has a winning strategy in $G'_F$ if and only if $F$ is not equi-Baire 1.
- Player II has a winning strategy in $G'_F$ if and only if $F$ is equi-Baire 1.

Proof. We use the scheme of Lemma 1 so it is enough to show that:

(i) if $F$ is equi-Baire 1 then Player II has a winning strategy, and
(ii) if $F$ is not of equi-Baire 1 then Player I has a winning strategy.

To prove (1) assume that $F$ is equi-Baire 1, fix a family $\Delta := \{\delta_\varepsilon : \varepsilon > 0\}$ of positive functions from $X$ into $\mathbb{R}$ such that, for every $x \in X$, the map $\varepsilon \mapsto \delta_\varepsilon(x)$ is non-decreasing being the family of common $\varepsilon$-gauges for $F$, the sequence $\langle \delta_\varepsilon(x) \rangle$ is
Such a choice is possible because the $P$ where
\[(7)\]
define the equi-oscillation of $\mathcal{F}$ at $x$ as
\[
\omega_{\mathcal{F}}(x) := \inf \{ \omega_{\mathcal{F}}(B(x, h) \cap P): h > 0 \}.
\]
It is easy to observe that, for any integer $n > 0$, the set
\[
P_n := \left\{ x \in P: \omega_{\mathcal{F}}(x) \geq \frac{1}{n} \right\}
\]
is closed, and $x$ is a point of equicontinuity for $\mathcal{F}$ if and only if $\omega_{\mathcal{F}}(x) = 0$. Since $\mathcal{F}$ has no point of equicontinuity, $\bigcup_{n \in \mathbb{N}} P_n = P$. Since $P$ is a Polish space and all $P_n$'s are closed, by the Baire Category Theorem there exists $P_n$ with non-empty interior. Thus, without loss of generality, we may assume that for some $\varepsilon > 0$, $\omega_{\mathcal{F}}(x) \geq \varepsilon$ for each $x \in P$.

We are ready to provide a strategy $\mathcal{S}$ for Player I. In the first move he picks $x_0 \in P$. For $n > 0$, in the $n$th move Player I takes $\mathcal{S}(x'_0, \ldots, x'_{n-1}) = x_n \in P$ with
\[
x_n := \left\{ \begin{array}{ll}
    x_{n-1} & \text{if there exists } f \in \mathcal{F} \text{ such that } d_Y(f(x_{n-1}), f(x_{n-1})) \geq \varepsilon/3;
    a & \text{otherwise},
  \end{array} \right.
\]
where $a \in P$ and
\[
\begin{align*}
\text{(j)} & \quad d_X(x_{n-1}, a) < 1/2^n; \\
\text{(ii)} & \quad d_Y(f(x'_{n-1}), f(a)) \geq 2\varepsilon/3 \text{ for some } f \in \mathcal{F}.
\end{align*}
\]
Such a choice is possible because the $P$ is dense-in-itself, $x_{n-1} \in P$, and $\omega_{\mathcal{F}}(x_{n-1}) \geq \varepsilon$.

If the family of sequences $\{(f(x'_n)): f \in \mathcal{F}\}$ is not equi-convergent then Player I wins. Otherwise, we use (4). So, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$ and all $f \in \mathcal{F}$,
\[(7)\]
\[
d_Y(f(x'_n), f(x'_m)) < \frac{\varepsilon}{6}.
\]
We claim that there exists $M \geq N$ with $x_n = x_M$ for all $n \geq M$.

We have two possibilities: either $x_m = x_{m-1}$ for all $m > N$, or there exists $m_1 > N$ such that $x_{m_1} \neq x_{m_1-1}$. Since in the first case we are done, we assume the second one. Then, by the formula defining $x_n$,
\[
d_Y(f(x'_{m_1-1}), f(x_{m_1})) < \frac{\varepsilon}{3} \text{ for each } f \in \mathcal{F}.
\]
It follows from (ii) that
\[(8)\]
\[
d_Y(f_1(x'_{m_1-1}), f_1(x_{m_1})) \geq \frac{2\varepsilon}{3} \text{ for some } f_1 \in \mathcal{F}.
\]
Thus, by (7) and (8), for all $m \geq N$ we have
\[
d_Y(f_1(x'_m), f_1(x_{m_1})) \geq d_Y(f_1(x'_{m_1-1}), f_1(x_{m_1})) - d_Y(f_1(x'_m), f_1(x'_{m_1-1}))
\]
Hence, by the definition of \( x_n \) for \( n = m_1 + 1 \) we obtain the equality \( x_n = x_{m_1} \), so \( f_1(x_n) = f_1(x_{m_1}) \). Therefore, \( d_Y(f_1(x_{n+1}), f_1(x_n)) \geq \frac{\varepsilon}{2} \), so by the definition of \( x_{n+1} \) we get \( x_{n+1} = x_n = x_{m_1} \). In this way we show, by induction, that \( x_n = x_{m_1} \) for all \( n \geq m_1 \). This finishes the proof of the claim.

Since the sequence constructed by Player I is eventually constant, i.e. \( x_m = x_M \) for all \( m \geq M \), so \( \lim_{n \to \infty} x_n = x_M \). Recall that in both variants of the formula defining \( x_n \),

\[
\forall n > 0 \exists f \in F \quad d_Y(f(x'_n), f(x_n)) \geq \frac{\varepsilon}{3},
\]

Therefore, since \( x_m = x_M = \lim_{n \to \infty} x_n \) for all \( m \geq M \),

\[
\forall m \geq M \exists f \in F \quad d_Y(f(x'_m), f(\lim_{n \to \infty} x_n)) \geq \frac{\varepsilon}{3}.
\]

Fix \( f_2 \in F \) with

\[
d_Y\left(f_2(x'_M), f_2\left(\lim_{n \to \infty} x_n\right)\right) \geq \frac{\varepsilon}{3}.
\]

By (7), for every \( n \geq N \) we have

\[
d_Y\left(f_2(x'_n), f_2\left(\lim_{n \to \infty} x_n\right)\right) \geq \frac{\varepsilon}{6},
\]

thus \( \langle f_2(x'_n) \rangle \) does not converge to \( f_2(\lim_{n \to \infty} x_n) \), so Player I wins. \( \square \)

Now, we will describe the game \( G''_F \) which is a modification of \( G''_f \) for equi-Baire 1 families.

Let \( X \) and \( Y \) be Polish spaces, let \( F \subseteq Y^X \). At the first step of the game \( G''_F \), Player I plays \( x_0 \in X \) and then Player II plays an open set \( U_0 \ni x_0 \). At the \( n \)th step of the game \( G''_F \), Player I plays \( x_n \in U_{n-1} \), then Player II plays an open set \( U_n \ni x_n \):

Player I \( \quad x_0 \quad x_1 \quad x_2 \quad \cdots \)
Player II \( \quad U_0 \quad U_1 \quad U_2 \quad \cdots \)

with the rules that for each \( n \in \mathbb{N} \):

\begin{itemize}
  \item \( x_0 \in X \), and \( x_n \in U_{n-1} \) for \( n > 0 \);
  \item \( U_n \ni x_n \).
\end{itemize}

Player II wins if the sequence \( \langle x_n \rangle \) converges to some \( x \in X \), and the indexed family \( \{ (f(x_n)) : f \in F \} \) is equi-convergent to \( \{ f(x) : f \in F \} \). Otherwise, Player I wins.

**Theorem 9.** The game \( G''_F \) is determined, and

\begin{itemize}
  \item Player I has a winning strategy in \( G''_F \) if and only if \( F \) is not equi-Baire 1.
  \item Player II has a winning strategy in \( G''_F \) if and only if \( F \) is equi-Baire 1.
\end{itemize}

**Proof.** Firstly, we show that, if \( F \) is equi-Baire 1, then Player II has a winning strategy. We follow proof of Theorem [8] Let \( F \) be equi-Baire 1 and \( \Delta = \{ \delta_\varepsilon : \varepsilon > 0 \} \) be the family of common gauges for \( F \). Then for every \( x \in X \), \( \delta_\varepsilon(x) \) does not depend on \( f \in F \). We may assume that, for any fixed \( x \in X \), the sequence \( \langle \delta_\varepsilon(x) \rangle \) is decreasing, and \( \delta_\varepsilon(x) < 2^{-n} \) for every \( n > 0 \). So, we choose \( U_n := B(x_n, \delta_\varepsilon(x_n)/2) \). Then \( x_n \to x \) and note that the index \( N \) such that \( d_Y(f(x), f(x_n)) < \varepsilon \) for all \( n > N \) does not depend on \( f \in F \). Hence the family of sequences \( \{ (f(x_n)) : f \in F \} \) is equi-convergent to \( \{ f(x) : f \in F \} \) and we are done.

Secondly, assuming that \( F \) is not equi-Baire 1, we will show that Player I has a winning strategy. We follow the respective part in the proof of Theorem [8].
can assume that there exist a perfect set $P \subseteq X$ and $\varepsilon > 0$ such that $\omega_{\mathcal{F}(P)}(x) \geq \varepsilon$ for each $x \in P$. Initially, Player I picks $x_0 \in P$. Let $n > 0$. Since $x_{n-1} \in P$, we have $\omega_{\mathcal{F}(P)}(x_{n-1}) \geq \varepsilon$. Thus, knowing that $U_{n-1}$ is an open neighbourhood of $x_{n-1}$, Player I can choose $x_n \in U_{n-1}$ and $f_n \in \mathcal{F}$ such that

$$dy(f_n(x_{n-1}), f_n(x_n)) \geq \frac{\varepsilon}{3}.$$ 

This, by condition (iii), shows that the family of sequences $\{\langle f(x_n) \rangle : f \in \mathcal{F}\}$ is not equi-convergent. So, we have a winning strategy for Player I.

4. Game characterization of measurable functions

In this section, we propose another modification of the game $G_2^Y$ to obtain characterizations of Baire measurable and Lebesgue measurable functions.

Let $\Sigma$ be a $\sigma$-algebra of subsets of a set $Z \neq \emptyset$. A function $f : Z \rightarrow Y$, where $Y$ denotes a topological space, is called $\Sigma$-measurable if the preimage $f^{-1}(U)$ of any open set $U$ in $Y$ belongs to $\Sigma$. Note that if $Y$ is a separable metric space, then $f$ is measurable whenever the preimage $f^{-1}(B) \subseteq \Sigma$ for any open ball $B \subseteq Y$.

Let $H(\Sigma)$ be the $\sigma$-ideal given by $H(\Sigma) := \{A \subseteq Z : \forall B \subseteq A, B \in \Sigma\}$. Denote $\Sigma^+ := \Sigma \setminus H(\Sigma)$. We say that the $\sigma$-algebra $\Sigma$ satisfies condition ccc if every family of pairwise disjoint sets in $\Sigma^+$ is countable.

We will use the following lemma.

Lemma 10. Let $(Y, d)$ be a separable metric space and $\Sigma$ be a $\sigma$-algebra of subsets of $Z$ that satisfies condition ccc. A function $f : Z \rightarrow Y$ is not $\Sigma$-measurable if and only if there exist a set $W \in \Sigma^+$, a point $y \in Y$ and $\varepsilon > 0$ such that the sets 

$$\{z \in W : d(f(z), y) < \varepsilon\} \quad \text{and} \quad \{z \in W : d(f(z), y) \geq 2\varepsilon\}$$

intersect every subset of $W$ that belongs to $\Sigma^+$.

Proof. “$\Rightarrow$” Take the open ball $B := B(y, \varepsilon)$ in $Y$. The assumed condition implies that $W \cap f^{-1}(B) \notin \Sigma$, hence $f^{-1}(B) \notin \Sigma$ and consequently, $f$ is not $\Sigma$-measurable.

“$\Leftarrow$” Assume that $f$ is not $\Sigma$-measurable. Then there exist $y \in Y$ and $\varepsilon > 0$ such that $f^{-1}(B(y, \varepsilon)) \notin \Sigma$. Thus $A := \{z \in Z : d(f(z), y) < \varepsilon\} \notin \Sigma$ and $B := \{z \in Z : d(f(z), y) \geq \varepsilon\} \notin \Sigma$. Let $A$ (respectively, $B$) be a maximal family of pairwise disjoint $\Sigma^+$-subsets of $A$ (respectively, $B$). By condition ccc, the sets $\bigcup A$ and $\bigcup B$ belong to $\Sigma$. Let $A_0 := A \setminus \bigcup A$, $B_0 := B \setminus \bigcup B$, and $V := A_0 \cup B_0$. Then $A_0, B_0 \notin \Sigma$, and $V = Z \setminus (\bigcup A \cup \bigcup B)$, hence $V \in \Sigma$. Notice that for every $C \subseteq V$, if $C \in \Sigma^+$ then $A \cap C \neq \emptyset \neq B \cap C$.

Since the space $Y$ is separable, there exists a sequence of open balls $B(y_n, \varepsilon_n)$, $n \in \mathbb{N}$, such that $\bigcup_{n \in \mathbb{N}} B(y_n, \varepsilon_n) = B(y, \varepsilon)$ and $d(y, y_n) + 2\varepsilon_n < \varepsilon$ for all $n$. Since $V \cap A = A_0 \notin \Sigma$, there is $m \in \mathbb{N}$ such that the sets $C := V \cap f^{-1}[B(y_m, \varepsilon_m)]$ and $D := V \setminus f^{-1}[B(y_m, \varepsilon_m)]$ are not in $\Sigma$. Again, let $C$ and $D$ be maximal families of pairwise disjoint $\Sigma^+$-sets contained in $C$ and $D$, respectively. Define $C_0 = C \setminus \bigcup C$, $D_0 = D \setminus \bigcup D$, and $W = C_0 \cup D_0 = V \setminus (\bigcup C \cup \bigcup D)$. Then for any $S \subseteq W$, if $S \in \Sigma^+$ then $S \cap C_0 \neq \emptyset$ and $d(f(z), y_m) < \varepsilon_m$ for each $z \in S$. On the other hand, $S \subseteq V$, hence $S \cap B \neq \emptyset$ and $d(f(z), y) \geq \varepsilon$ for $z \in S \cap B$, and then $d(f(z), y) \geq d(f(z), y) - d(y, y_m) \geq \varepsilon + 2\varepsilon_m - \varepsilon = 2\varepsilon_m$. □

Let $X$ and $Y$ be topological Hausdorff spaces, $\Sigma$ be a $\sigma$-algebra on $X$, and $f : X \rightarrow Y$ be an arbitrary function. We define the following game $G_2^X$. At the first
step of the game $G_f^\Sigma$, Player I plays $W \in \Sigma^+$, then Player II plays a set $U_0$. At the $n$th step, where $n > 0$, Player I plays $x_n$ and Player II plays a $U_n$:

\[
\begin{array}{c|c|c|c|c|c|c|c}
\text{Player I} & W & x_1 & x_2 & \cdots \\
\text{Player II} & U_0 & U_1 & U_2 & \cdots \\
\end{array}
\]

with the rules that for each $n \in \mathbb{N}$:

- $x_n \in U_{n-1}$ for $n > 0$;
- $U_n \in \Sigma^+$ and $U_n \subseteq W$.

Player II wins the game $G_f^\Sigma$ if $\langle x_n \rangle$ is convergent and $\lim_{n \to \infty} f(x_n) = f(\lim_{n \to \infty} x_n)$. Otherwise Player I wins.

**Lemma 11.** Assume that $X$ is a topological Hausdorff space and $\Sigma$ is a $\sigma$-algebra of subsets of $X$ that satisfies condition ccc. Let $(Y,d)$ be a separable metric space. If $f: X \to Y$ is not $\Sigma$-measurable then Player I has a winning strategy in the game $G_f^\Sigma$.

**Proof.** Assume that $f$ is not $\Sigma$-measurable. By Lemma 10 there exist a set $W \subseteq X$, $y \in Y$ and $\varepsilon > 0$ such that $W \in \Sigma^+$ and both sets $A := \{x \in W : d_Y(f(x), y) < \varepsilon\}$ and $B := \{x \in W : d_Y(f(x), y) \geq 2\varepsilon\}$ intersect every subset of $W$ which is in $\Sigma^+$. Let Player I play the following strategy. At the first step, he chooses the set $W \in \Sigma^+$ obtained above. If $n > 0$, then $U_{n-1} \subseteq W$ and he chooses $x_n \in A \cap U_{n-1}$ when $n$ is even and $x_n \in B \cap U_{n-1}$ when $n$ is odd. Then $d(f(x_{2n}), y) \leq \varepsilon$ and $d(f(x_{2n+1}), y) \geq 2\varepsilon$, so $d(f(x_{2n}), f(x_{2n+1})) \geq \varepsilon$ for every $n$, and therefore $\langle f(x_n) \rangle$ is not convergent.

First, we will characterize Baire measurable functions, that is $\Sigma$-measurable functions, where $\Sigma = \text{Baire}$ denotes the $\sigma$-algebra of sets with the Baire property in a topological space, cf. [14, 8.21]. Note that the ideal $H(\text{Baire})$ is equal to the family of all meager sets, cf. [21, Theorem 5.5] (recall that we assume AC) and if $X$ is second countable then the algebra Baire satisfies condition ccc, cf. [5, Theorem 7.5].

**Theorem 12.** Let $X$ be a Polish space, $Y$ be a separable metric space, and $f: X \to Y$ be a function. Then the game $G_f^\text{Baire}$ is determined, and

- Player I has a winning strategy in $G_f^\text{Baire}$ if and only if $f$ is not Baire measurable;
- Player II has a winning strategy in $G_f^\text{Baire}$ if and only if $f$ is Baire measurable.

**Proof.** We apply Lemma 11 thus we have to prove two implications:

(i) if $f$ is Baire measurable then Player II has a winning strategy in the game $G_f^\text{Baire}$;

(ii) if $f$ is not Baire measurable then Player I has a winning strategy in $G_f^\text{Baire}$.

To prove (i) assume that $f$ is Baire measurable. We will describe a winning strategy for Player II in the game $G_f^\text{Baire}$. Let $G \subseteq X$ be a dense $G_\delta$ set such that $f \upharpoonright G$ is continuous. (See [14, Theorem 8.38].) Let $W \in \text{Baire}^+$ be chosen by Player I at the first move. Then Player II fixes any point $a \in G \cap W$ and picks $U_0 := B(a, 1) \cap G \cap W$. At the $(n+1)$-th move, Player I chooses $x_{n+1} \in U_n$. Then Player II plays $U_{n+1} := \{x_{n+1}\} \cup (B(a, \frac{1}{n+1}) \cap G \cap W)$. When the game is finished, one of the two cases is possible: either $x_n = x_N$ for some $N \in \mathbb{N}$ and all $n > N$,
or for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $x_n \in B(a, \varepsilon) \cap G$ for all $n > N$, which implies $\lim_{n \to \infty} x_n = a$. In both cases, $\lim_{n \to \infty} f(x_n) = f(\lim_{n \to \infty} x_n)$.

The implication (ii) follows from Lemma 11 with $\Sigma = \text{Baire}$. \qed

A similar idea can be used to characterize Lebesgue measurable functions from $X := \mathbb{R}^k$ to a separable metric space $Y$. Let $\text{Leb}$ denote the $\sigma$-algebra of Lebesgue measurable subsets of $\mathbb{R}^k$. Note that the ideal $H(\text{Leb})$ consists exactly of Lebesgue null sets in $\mathbb{R}^k$, cf. [21, Theorem 5.5], and $\text{Leb}$ satisfies condition ccc, cf. [5, Theorem 7.5].

**Theorem 13.** Let $X = \mathbb{R}^k$, $Y$ be a separable metric space, and $f : X \to Y$ be a function. Then the game $G_f^\text{Leb}$ is determined, and

- Player I has a winning strategy in $G_f^\text{Leb}$ if and only if $f$ is not measurable;
- Player II has a winning strategy in $G_f^\text{Leb}$ if and only if $f$ is measurable.

**Proof.** We use Lemma 1. First assume that $f$ is measurable. We will describe a winning strategy for Player II. Let $W \subseteq \text{Leb}^+$ be chosen at the initial move by Player I. Let $F \subseteq W$ be a compact set with positive measure. By the Luzin theorem (applied to the space $F$ with the restricted Lebesgue measure), see [14, Theorem 17.12], there exists a closed set $F_0 \subseteq F$ such that $f \upharpoonright F_0$ is continuous and the Lebesgue measure of $F_0$ is finite and positive. Then Player II picks $U_0 := F_0$. At the $(n+1)$-th move, Player I chooses $x_{n+1} \in U_n$. Then Player II plays $U_{n+1} := \{x_{n+1}\} \cup F_{n+1}$ where $F_{n+1} \subseteq F_n$ is a closed set of a positive measure with the diameter less than $\frac{1}{n+1}$. When the game is finished, we have $\bigcap_{n \geq 0} F_n = \{a\}$ for some $a \in X$. As in the previous proof, we infer that $\langle x_n \rangle$ is eventually constant or $x_n \to a \in F_0$, and so $\lim_{n \to \infty} f(x_n) = f(\lim_{n \to \infty} x_n)$.

The second implication follows from Lemma 11 with $\Sigma = \text{Leb}$. \qed

**Remark.** By [14, Theorem 17.12], see also [23, Theorem 18] can be extended to the case when $X$ is a Polish space equipped with a $\sigma$-finite Borel regular measure.

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