ELLIPTIC ASYMPTOTICS FOR THE COMPLETE THIRD PAINLEVÉ TRANSCENDENTS

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Abstract. For a general solution of the third Painlevé equation of complete type we show the Boutroux ansatz near the point at infinity. It admits an asymptotic representation in terms of the Jacobi sn-function in cheese-like strips along generic directions. The expression is derived by using isomonodromy deformation of a linear system governed by the third Painlevé equation of this type. In our calculation of the WKB analysis, the treated Stokes curve ranges on both upper and lower sheets of the two sheeted Riemann surface.

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1. Introduction

As in the study of the spaces of initial values for Painlevé equations by Sakai [30], the third Painlevé equations are classified into three types $P_{\text{III}}(D_{6})$, $P_{\text{III}}(D_{7})$ and $P_{\text{III}}(D_{8})$. For $P_{\text{III}}(D_{7})$ and $P_{\text{III}}(D_{8})$ Ohyama et al. [27] examined basic properties including τ-functions, irreducibility, the spaces of initial values. Equation $P_{\text{III}}(D_{7})$ is the degenerate third Painlevé equation written in the form

$$u_{\tau\tau} = \frac{u_{\tau}^2}{u} - \frac{u_{\tau}}{\tau} + \frac{1}{\tau} (-8\epsilon u^2 + 2ab) + \frac{b^2}{u},$$

with $\epsilon = \pm 1$, $a \in \mathbb{C}$, $b \in \mathbb{R} \setminus \{0\}$. For this equation, using isomonodromy deformation Kitaev and Vartanian [19], [20] obtained asymptotic solutions as $\tau \to \pm \infty$, $\pm i\infty$ and $\tau \to \pm 0$, $\pm i0$ with connection formulas among them. Furthermore, a special meromorphic solution is studied by [18], [21], and trans-series solutions with related monodromy data are discussed by [33].

The third Painlevé equation $P_{\text{III}}(D_{6})$ of the form

$$\frac{d^2y}{dx^2} = \frac{1}{y} \left( \frac{dy}{dx} \right)^2 - \frac{1}{x} \frac{dy}{dx} + \frac{4}{x} (\theta_0 y^2 + 1 - \theta_\infty) + 4y^3 - 4y$$

with $\theta_0$, $\theta_1 \in \mathbb{C}$ is equivalent to the degenerate fifth Painlevé equation [7], [8], and is called the complete third Painlevé equation $P_{\text{III}}$ by Kitaev [14], in which the asymptotics of solutions on the positive real axis are studied. Basic properties of $P_{\text{III}}(D_{6})$, say irreducibility, τ-functions, special solutions are studied in [22], [33], [28]. Equation $P_{\text{III}}$ of the Sine-Gordon type is obtained from (1.1) with $\theta_0 = 1 - \theta_\infty = 0$ by $y = e^{iu/2}$, $4ix = \xi$, and its real-valued solutions with connection formulas are found in [10, Chaps. 8, 10], [6, Chaps. 14, 15].

The linear approximation of Painlevé equations enables us to know only restricted solutions, say, exponentially decaying ones, or ones along special directions. To capture the global picture of a general solution as a meromorphic function it is necessary to consider nonlinear approximation such as by an elliptic function, which is expected to be available also in generic directions. Indeed, for a general solution of $P_{1}$, Boutroux [2],...
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first studied asymptotic behaviours by comparing with the Weierstrass \( \wp \)-function, the modulus of which is determined by transcendental equations being now called the Boutroux equations. Rigorous treatments of these asymptotics for \( P_I \) and \( P_{II} \) are made by multiscale expansions \([11]\) or by isomonodromy techniques \([13]\), \([16]\), \([23]\), \([24]\). Thus for Painlevé transcendents the asymptotic expression in terms of an elliptic function is called the Boutroux ansatz. Elliptic asymptotics for \( P_{III} \) of the Sine-Gordon type are expressed by the Jacobi sn-function \([25]\), \([26]\), \([6, \text{Chap. 16}]\), and those for \( P_{III}(D_7) \) by the Weierstrass \( \wp \)-function \([32]\).

In this paper we show the Boutroux ansatz for the complete \( P_{III} \) with arbitrary parameters \( \theta_0, \theta_\infty \in \mathbb{C} \), that is, present an asymptotic representation of a general solution in terms of the Jacobi sn-function along generic directions near the point at infinity.

The main results are described in Section 2. The complete third Painlevé equation (1.1) governs isomonodromy deformation of the linear system

\[
\frac{dU}{d\lambda} = A(x, \lambda) U,
\]

\[
A(x, \lambda) = \frac{ix}{2} \sigma_3 \left( \frac{1}{x} \frac{1}{\theta_0 x^3 v^{-1}} - v^{-1} \left( \frac{1}{x} \frac{1}{\theta_\infty (2 - x^3 v^{-1}) + y(3 - x)} \right) - \frac{y^3 v}{2 \theta_\infty} \right) \lambda^{-1}
\]

which is equivalent to (1.1). As shown in Section 3 system (1.2) is a result of transformation of system (3.3). The isomonodromy deformation of system (3.3) is governed by equation (1.1), and solutions of (1.1) correspond to the monodromy data on the monodromy manifold for (3.3), which is defined by Stokes matrices \( S_{\infty}^j, S_0^j \) and connection matrices \( G = (g_{ij}), \hat{G} = (\hat{g}_{ij}) \in SL_2(\mathbb{C}) \) for solutions around \( \mu = 0 \) and \( \mu = \infty \). System (1.2) admits the same monodromy manifold as of (3.3), which is described by the same matrices \( S_{\infty}^j, S_0^j \) and \( G, \hat{G} \) for suitably chosen matrix solutions (cf. Proposition 3.8). Thus we may treat isomonodromy system (3.3) instead of (1.2). Applying WKB analysis under condition (5.1), we solve the direct monodromy problem for linear system (3.3) in Section 5 and obtain key relations (Corollary 5.2) containing the monodromy data and certain integrals. In its procedure we calculate analytic continuations along a Stokes curve ranging on both upper and lower sheets of the two-sheeted Riemann surface. Basic necessary materials for this calculation are summarised in Section 4. Asymptotic properties of these integrals are examined in Section 6 by the use of the \( \vartheta \)-function, and in Section 7 from these formulas, asymptotic forms in main theorems are found by solving the inverse monodromy problem for the prescribed monodromy data. These forms are given as a necessary condition to be a required solution. The justification as a solution of (1.1) is made along the line of Kitaev \([15]\), \([17]\). The final
section is devoted to the Boutroux equations, which determine a modulus contained in the elliptic representation of solutions.

Throughout this paper we use the following symbols:

(1) $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

(2) for complex-valued functions $f$ and $g$, we write $f \ll g$ or $g \gg f$ if $f = O(|g|)$,

2. Main results

To state our main results we give some explanations on necessary facts.

2.1. Monodromy data. Isomonodromy system (1.2) admits the matrix solutions

$$U_k^\infty(\lambda) = (I + O(\lambda^{-1})) \exp\left(\frac{1}{2}ix\lambda \sigma_3\right) \lambda^{-\frac{i}{2} \theta_0 \sigma_3}$$

as $\lambda \to \infty$ through the sector $|\arg \lambda + \arg x - k\pi| < \pi$, and

$$U_k^0(\lambda) = \Delta_0(I + O(\lambda)) \exp\left(-\frac{1}{2}ix\lambda^{-1} \sigma_3\right) \lambda^{\frac{i}{2} \theta_0 \sigma_3}$$

as $\lambda \to 0$ through the sector $|\arg \lambda - \arg x - k\pi| < \pi$, where $k \in \mathbb{Z}$, and

$$\Delta_0 = \begin{pmatrix} v^{1/2} f_1 & v^{1/2} \xi f_2 \\ v^{-1/2} f_1 & v^{-1/2}(3-x)f_2 \end{pmatrix},$$

$f_1, f_2$ being such that $xf_1 f_2 \equiv -1$. Let $(G, \hat{G}, S_0^\infty, S_1^\infty, S_0^0, S_1^0)$ be invariant monodromy data defined by $U_{k+1}^\infty(\lambda) = U_k^\infty(\lambda)S_k^\infty$, $U_{k+1}^0(\lambda) = U_k^0(\lambda)S_k^0$ with $k = 0, 1$, and by $U_0^\infty(\lambda) = U_0^0(\lambda)G$, $U_1^\infty(\lambda) = U_0^0(\lambda)\hat{G}$ with $G = (g_{ij})$, $\hat{G} = (\hat{g}_{ij}) \in SL_2(\mathbb{C})$. These Stokes matrices are written as

$$S_0^\infty = \begin{pmatrix} 1 & S_0^0 \\ 0 & 1 \end{pmatrix}, \quad S_1^\infty = \begin{pmatrix} 1 & 0 \\ S_1^0 & 1 \end{pmatrix}, \quad S_0^0 = \begin{pmatrix} 1 & S_0^0 \\ 0 & 1 \end{pmatrix}, \quad S_1^0 = \begin{pmatrix} 1 & 0 \\ S_1^0 & 1 \end{pmatrix}.$$

Then the monodromy manifold $\mathcal{M} \subset \mathbb{C}^{12}$ is given by

$$S_0^0 \hat{G} = GS_0^\infty, \quad G^{-1}S_0^0 S_1^0 e^{-\pi i \theta_0 \sigma_3} G = S_0^\infty S_1^\infty e^{\pi i \theta_0 \sigma_3}$$

(see Section 3.1 and 14). A generic point on the monodromy manifold $\mathcal{M}$ is determined by $(G, \hat{G})$ (cf. Propositions 3.6 and 3.7). As described in Section 3.1 a change of the matrix solution basis induces an action on the monodromy data on $\mathcal{M}$, and each solution of (1.1) corresponds to an orbit, or equivalence class, yielded by dividing $\mathcal{M}$ by this action. Then an orbit passing through $(G, \hat{G})$ parametrises a general solution, which will be simply called a solution labelled by $(G, \hat{G})$.

2.2. Elliptic curve and Boutroux equations. As shown in Lemma 8.1 as long as $A \in \mathbb{C} \setminus \{c < -2\}$, the polynomial $\lambda^4 - A\lambda^2 + 1$ has roots $\pm \lambda_1 = \pm \lambda_1(A), \pm \lambda_2 = \pm \lambda_2(A)$ such that $\lambda_1, \lambda_2 = 2, \lambda_1, \lambda_2 \in \{\Re \lambda \geq 0\}$ and $-\lambda_1, -\lambda_2 \in \{\Re \lambda \leq 0\}$. Let $\Pi_+$ and $\Pi_-$ be the copies of $P^1(\mathbb{C}) \setminus \{[-\lambda_2, -\lambda_1] \cup [\lambda_1, \lambda_2]\}$ and set $\Pi_A = \Pi_+ \cup \Pi_-$ glued along the cuts $[-\lambda_2, -\lambda_1]$ and $[\lambda_1, \lambda_2]$. The Riemann surface $\Pi_A$ is the elliptic curve given by

$$w(A, \lambda)^2 = \lambda^4 - A\lambda^2 + 1,$$

where the branch of

$$w(A, \lambda) = \sqrt{\lambda^4 - A\lambda^2 + 1} := \sqrt{(1 + \lambda_1^{-1}\lambda)(1 - \lambda_1^{-1}\lambda)(1 + \lambda_2^{-1}\lambda)(1 - \lambda_2^{-1}\lambda)}$$
\[ = \sqrt{1 + \lambda_1^{-1} \lambda} \sqrt{1 - \lambda_1^{-1} \lambda} \sqrt{1 + \lambda_2^{-1} \lambda} \sqrt{1 - \lambda_2^{-1} \lambda} \]

is chosen in such a way that \( \sqrt{1 \pm \lambda_j^{-1} \lambda} \to 1 \) as \( \lambda \to 0 \) on the upper plane \( \Pi_+ \). Then \( \lambda^{-2}w(A, \lambda) \to -1 \) as \( \lambda \to \infty \), and \( w(A, \lambda) \to 1 \) as \( \lambda \to 0 \) on the upper plane \( \Pi_+ \). The elliptic curve does not degenerate as long as \( A \neq \pm 2 \), and \( \Pi_A \) may be defined continuously.

As will be shown in Section 8, for each \( \phi \in \mathbb{R} \), there exists \( A_\phi \in \mathbb{C} \setminus \{c < -2\} \) with \( \Pi_{A_\phi} \) such that, for every cycle \( c \) on \( \Pi_{A_\phi} \)

\[ \text{Im} e^{i\phi} \int_a w(A_\phi, \lambda) \d\lambda = 0, \]

and that \( A_\phi \) has the properties (Proposition 8.16):
1. for every \( \phi \), \( A_\phi \) is uniquely determined;
2. \( A_\phi \) is continuous in \( \phi \in \mathbb{R} \), and is smooth in \( \phi \in \mathbb{R} \setminus \{m\pi/2 \mid m \in \mathbb{Z}\} \);
3. \( A_{\phi+\pi/2} = -A_{\phi}, \ A_{\phi+\pi} = A_{\phi}, \ A_{-\phi} = A_{\phi} \);
4. \( \Pi_{A_\phi} \) degenerates if and only if \( \phi = m\pi/2 \) with \( m \in \mathbb{Z} \), and \( A_0 = 2, \ A_{\pm\pi/2} = -2 \).

By Proposition 8.17 the roots \( \pm \lambda_1 = \pm \lambda_1(A_\phi), \ \pm \lambda_2 = \pm \lambda_2(A_\phi) \) of \( w(A_\phi, \lambda)^2 \) may be numbered in such a way that, for \( \phi \) close to 0,

\[ \text{Re} \lambda_1(A_\phi) < \text{Re} \lambda_2(A_\phi), \ \text{Im} \lambda_1(A_\phi) < 0 < \text{Im} \lambda_2(A_\phi) \quad \text{if} \ \phi < 0, \]

\[ \text{Re} \lambda_1(A_\phi) < \text{Re} \lambda_2(A_\phi), \ \text{Im} \lambda_1(A_\phi) > 0 > \text{Im} \lambda_2(A_\phi) \quad \text{if} \ \phi > 0 \]

(cf. Figure 2.1), and that the numbering is retained for \( 0 < |\phi| < \pi/2 \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{cycles}
\caption{(a) \( \phi < 0 \) (b) \( \phi > 0 \)}
\end{figure}

For small \( |\phi| \) we define basic cycles \( a \) and \( b \) on \( \Pi_{A_\phi} = \Pi_+ \cup \Pi_- \) drawn as in Figure 2.1 and, for \( 0 < |\phi| < \pi/2 \), cycles \( a \) and \( b \) are modified continuously on \( \Pi_{A_\phi} \). The Boutroux equations are given by

\[ \text{Im} e^{i\phi} \int_a w(A_\phi, \lambda) \d\lambda = 0, \quad \text{Im} e^{i\phi} \int_b w(A_\phi, \lambda) \d\lambda = 0 \]

admitting a unique solution \( A_\phi \). For \( |\phi| < \pi/2 \) the periods of \( \Pi_{A_\phi} \) along \( a \) and \( b \) are

\[ \Omega^a_\phi = \Omega_a = \int_a \frac{d\lambda}{w(A_\phi, \lambda)}, \quad \Omega^b_\phi = \Omega_b = \int_b \frac{d\lambda}{w(A_\phi, \lambda)}, \]

which satisfy \( \text{Im} \Omega_b / \Omega_a > 0 \).
2.3. **Main theorems.** Let $y(x) = y(G, \hat{G}, x)$ be a solution of (1.1) labelled by the monodromy data $(G, \hat{G}) \in SL_2(\mathbb{C})^2$ with $G = (g_{ij}), \hat{G} = (\hat{g}_{ij})$. The roots $\lambda_1 = \lambda_1(A_\phi), \lambda_2 = \lambda_2(A_\phi)$ of $w(A_\phi, \lambda)^2$ for $0 < |\phi| < \pi/2$ are given by

$$
\lambda_1 = \sqrt{A_\phi/2 - A_\phi^2/4 - 1}, \quad \lambda_2 = \sqrt{A_\phi/2 + A_\phi^2/4 - 1},
$$

where the branches are chosen in such a way that $\text{Re} \sqrt{A_\phi^2/4 - 1} \geq 0$ as $A_\phi \to 2 (\phi \to 0)$, and that $\lambda_{1,2}(A_\phi) \to 1$ as $A_\phi \to 2$ (cf. Proposition 8.17 and Figure 8.2). Then we have the following, in which $\text{sn}(u; k)$ denotes the sn-function solving $(z_u)^2 = (1 - z^2)/(1 - k^2 z^2)$.

**Theorem 2.1.** Suppose that $0 < \phi < \pi/2$ and that $g_{11} g_{12} g_{22} \hat{g}_{11} \hat{g}_{21} \neq 0$. Then

$$
y(x)^{-1} = i\lambda_1 \text{sn}(2i\lambda_2(x - x_0^+) + O(x^{-\delta}); \lambda_1/\lambda_2)
$$
as $2x = te^{i\phi} \to \infty$ through the cheese-like strip

$$
S(\phi, t_{\infty}, \kappa_0, \delta_0) = \{2x = te^{i\phi} \mid \text{Re} t > t_{\infty}, |\text{Im} t| < \kappa_0\} \setminus \bigcup_{\sigma \in \mathcal{P}(x_0^+)} \{te^{i\phi} - \sigma < \delta_0\}
$$

with

$$
\mathcal{P}(x_0^+) = \{\sigma \mid \text{sn}(i\lambda_2(\sigma - 2x_0^+); \lambda_1/\lambda_2) = \infty\} = 2x_0^+ - i(\frac{1}{2} \Omega_a + \frac{1}{2} \Omega_b \mathbb{Z} + \Omega_b \mathbb{Z}).
$$

Here $\delta$ is some positive number, $\kappa_0$ a given positive number, $\delta_0$ a given small positive number, $t_{\infty} = t_{\infty}(\kappa_0, \delta_0)$ a sufficiently large number depending on $(\kappa_0, \delta_0)$; and

$$
2ix_0^+ = 2ix_0^-(\Omega_a, \Omega_b, G, \hat{G}) = \frac{1}{2\pi i} \left(\Omega_a \log(g_{11} g_{22}) + \Omega_b \log \frac{g_{12} g_{21}}{g_{22} g_{11}}\right) - \frac{\Omega_a}{4}(\theta_0 - \theta_{\infty} + 2) - \frac{\Omega_b}{2} \mod \Omega_a \mathbb{Z} + \Omega_b \mathbb{Z}.
$$

**Theorem 2.2.** Suppose that $-\pi/2 < \phi < 0$ and that $g_{11} g_{12} g_{22} \hat{g}_{12} \hat{g}_{21} \neq 0$. Then $y(x)$ admits an asymptotic representation of the same form as in Theorem 2.1 with the phase shift given by

$$
2ix_0^- = 2ix_0^-(\Omega_a, \Omega_b, G, \hat{G}) = \frac{1}{2\pi i} \left(\Omega_a \log(g_{11} g_{22}) + \Omega_b \log \frac{g_{12} g_{21}}{g_{22} g_{11}}\right) - \frac{\Omega_a}{4}(\theta_0 - \theta_{\infty} + 2) - \frac{\Omega_b}{2} \mod \Omega_a \mathbb{Z} + \Omega_b \mathbb{Z}
$$
in $S(\phi, t_{\infty}, \kappa_0, \delta_0)$ with $\mathcal{P}(x_0^-)$.

**Remark 2.1.** By Corollary 3.3.3 and Proposition 3.5.

$$
\frac{g_{12} \hat{g}_{11}}{g_{12} g_{21}} = \frac{g_{22} (g_{11} - g_{12}^0)}{1 - g_{11} g_{22}} = \frac{g_{11} g_{22} (1 - g_{21}^0) g_{11}}{1 - g_{11} g_{22}},
$$

that is, the constants consisting of $g_{ij}$ and $\hat{g}_{ij}$ in $x_0^\pm$ are represented in terms of $g_{11} g_{22}$ and $\hat{g}_{11} g_{11}$, which are invariants under an action on monodromy data as in Proposition 3.6.

For $\phi$ such that $|\phi - m\pi| < \pi/2$ ($m \in \mathbb{Z}$), set $\Omega_{a,b}^\phi = e^{m\pi i} \Omega_{a,b}^{\phi - m\pi}$ with $\Omega_{a,b}^\phi = \Omega_{a,b}$ for $|\phi| < \pi/2$, and $\lambda_{1,2} = \lambda_{1,2}(A_{\phi - m\pi})$. Then we have the following.
Theorem 2.3. Suppose that $0 < \phi - m\pi < \pi/2$ (respectively, $-\pi/2 < \phi - m\pi < 0$) with $m \in \mathbb{Z} \setminus \{0\}$. Then $y(x)$ admits the expression

$$y(x)^{-1} = i\lambda_1 \sin(2i\lambda_2(x - x_0^{(m)}) + O(x^{-\delta}); \lambda_1/\lambda_2)$$

as $2x = t e^{i\phi} \to \infty$ through $S(\phi, t_\infty, k_0, d_0)$ with $P(x_0^{(m)}, g_{11}^{(m)} g_{12}^{(m)} g_{11}^{(m)} g_{22}^{(m)} \neq 0$ (respectively, $g_{11}^{(m)} g_{12}^{(m)} g_{22}^{(m)} g_{21}^{(m)} \neq 0$). Here, for $0 < \pm(\phi - m\pi) < \pi/2$, $m \in \mathbb{Z} \setminus \{0\}$,

$$x_0^{(m)} = x_0^{\pm}(\Omega^\phi_0, \Omega_b^\phi, G_m, \hat{G}_m)$$

with

$$G_m = (g_{ij}^{(m)}) = e^{2\pi i m \theta \sigma_3} (S_0^\infty S_1^\infty e^{\pi i \theta \sigma_3})^m e^{-\frac{1}{2} \pi i m \theta \sigma_3},$$

$$\hat{G}_m = (\hat{g}_{ij}^{(m)}) = e^{2\pi i m \theta \sigma_3} (S_0^\infty S_1^\infty e^{\pi i \theta \sigma_3})^m S_0^\infty e^{-\frac{1}{2} \pi i m \theta \sigma_3}.$$  

3. ISOMONODROMY DEFORMATION AND MONODROMY DATA

3.1. Monodromy data. We examine the monodromy data for system (1.2) (see also [14] Section 1). System (1.2) admits the matrix solutions

$$U_k^\infty(\lambda) = (I + O(\lambda^{-1})) \exp\left(\frac{1}{2}i x \lambda \sigma_3\right) \lambda^{-\frac{1}{2} \theta \sigma_3}$$

as $\lambda \to \infty$ through the sector $\{ \arg \lambda + \arg x - k\pi < \pi, \}$

$$U_k^0(\lambda) = \Delta_0(I + O(\lambda)) \exp(-\frac{1}{2}i x \lambda^{-1} \sigma_3) \lambda^{\frac{1}{2} \theta \sigma_3}$$

as $\lambda \to 0$ through the sector $\{ \arg \lambda - \arg x - k\pi < \pi, \}$ where

$$\Delta_0 = \left(\begin{array}{cc} v^{1/2} f_1 & v^{1/2} f_2 \\ v^{-1/2} f_1 & v^{-1/2} (3-x) f_2 \end{array}\right),$$

$f_1, f_2$ being such that $xf_1 f_2 \equiv -1$. Let $S_k^\infty, S_k^0$ be Stokes matrices such that

$$U_{k+1}^\infty(\lambda) = U_k^\infty(\lambda) S_k^\infty, \ U_{k+1}^0(\lambda) = U_k^0(\lambda) S_k^0.$$

Then

$$S_0^\infty = \left(\begin{array}{cc} 1 & s_0^\infty \\ 0 & 1 \end{array}\right), \ S_1^\infty = \left(\begin{array}{cc} 1 & 0 \\ s_1^\infty & 1 \end{array}\right), \ S_0^0 = \left(\begin{array}{cc} 1 & s_0^0 \\ 0 & 1 \end{array}\right), \ S_1^0 = \left(\begin{array}{cc} 1 & 0 \\ s_1^0 & 1 \end{array}\right).$$

Let $G = (g_{ij}), \hat{G} = (\hat{g}_{ij}) \in SL_2(\mathbb{C})$ be connection matrices defined by

$$U_0^\infty(\lambda) = U_0^0(\lambda) G, \ U_1^\infty(\lambda) = U_1^0(\lambda) \hat{G}.$$  

By the uniqueness of the asymptotic expression in each sector, for every $m \in \mathbb{Z},$

$$U_k^\infty(e^{-2\pi i m \lambda}) e^{-\pi i \theta \sigma_3} = U_{k+2m}^\infty(\lambda),$$

$$U_k^0(e^{-2\pi i m \lambda}) e^{-\pi i \theta \sigma_3} = U_{k+2m}^0(\lambda).$$

Then we have the following.

Proposition 3.1. For $k \in \mathbb{Z}$, $S_k^\infty = e^{\pi i \theta \sigma_3} S_k^0 e^{-\pi i \theta \sigma_3}, \ S_{k+2}^0 = e^{-\pi i \theta \sigma_3} S_k^0 e^{\pi i \theta \sigma_3}.$

Proposition 3.2. $S_0^0 S_1^0 e^{-\pi i \theta \sigma_3} = G S_0^\infty S_1^\infty, \ GS_0^\infty = S_0^\infty \hat{G}.$

Corollary 3.3. The entries of $S_k^\infty, S_k^0, G$ and $\hat{G}$ fulfill

$$s_0^0 s_1^0 e^{-\pi i \theta} + 2 \cos \pi \theta = s_0^\infty s_1^\infty e^{\pi i \theta} + 2 \cos \pi \theta,$$

$$g_{11} = \hat{g}_{11} + s_0^0 g_{22}, \ g_{12} + s_0^\infty g_{11} = \hat{g}_{12} + s_0^0 \hat{g}_{22}, \ g_{21} = \hat{g}_{21}, \ g_{22} + s_0^\infty g_{21} = \hat{g}_{22}.$$


Comparison of the entries of $G^{-1} S_0^0 S_1^0 e^{-\pi i \theta_0 \sigma_3} G = S_0^\infty S_1^\infty e^{\pi i \theta_\infty \sigma_3}$ leads us to the following.

**Proposition 3.4.** The entries $s_0^\infty$, $s_1^\infty$, $s_0^0$, $s_1^0$ and $g_{ij}$ fulfil

$$e^{-\pi i \theta_0} \left( s_0^\infty + \frac{g_{22}}{g_{21}} \right) = \frac{g_{12}}{g_{21}} e^{-\pi i \theta_0} \left( s_1^\infty + \frac{g_{22}}{g_{12}} e^{2\pi i \theta_0} \right),$$

$$e^{\pi i \theta_0} \left( s_1^\infty - \frac{g_{11}}{g_{12}} e^{-2\pi i \theta_0} \right) = \frac{g_{21}}{g_{12}} e^{\pi i \theta_0} \left( s_0^\infty - \frac{g_{11}}{g_{21}} \right),$$

$$\left( s_0^\infty + \frac{g_{22}}{g_{21}} \right) \left( s_1^\infty - \frac{g_{11}}{g_{12}} e^{-2\pi i \theta_0} \right) = \left( s_0^\infty - \frac{g_{11}}{g_{21}} \right) \left( s_1^\infty + \frac{g_{22}}{g_{12}} e^{2\pi i \theta_0} \right) = -1 - \frac{e^{\pi i (\theta_0 - \theta_\infty)}}{g_{12} g_{21}}.$$  

**Proposition 3.5.** The entries $g_{ij}$ and $\hat{g}_{ij}$ fulfil

$$\frac{\hat{g}_{11}}{g_{21}} = \frac{g_{11}}{g_{21}} - s_0^0, \quad \frac{\hat{g}_{12}}{g_{22}} = \frac{g_{12} g_{21}}{e^{\pi i (\theta_0 - \theta_\infty)}} + g_{12} g_{21} \left( \frac{g_{11}}{g_{21}} - s_0^0 \right).$$

**Proof.** By Corollary 3.3,

$$\frac{\hat{g}_{12}}{g_{22}} = \frac{g_{12} + s_0^0 g_{11}}{g_{22} + s_0^0 g_{21}} - s_0^0 = \frac{1}{g_{21}} - \frac{1}{g_{22} + s_0^0 g_{21}} = s_0^0,$$

and by the first and third relations in Proposition 3.4,

$$e^{-\pi i \theta_0} (g_{22} + s_0^\infty g_{21}) = g_{12} e^{-\pi i \theta_0} \left( s_0^\infty + \frac{g_{22}}{g_{12}} e^{2\pi i \theta_0} \right) = g_{12} e^{-\pi i \theta_0} \left( s_0^\infty - \frac{g_{11}}{g_{21}} \right)^{-1} \left( -1 - \frac{e^{\pi i (\theta_0 - \theta_\infty)}}{g_{12} g_{21}} \right).$$

From these equations the second relation of the proposition follows. \[\square\]

The monodromy manifold $\mathcal{M}$ is given by the equations $S_0^0 S_1^0 e^{-\pi i \theta_0 \sigma_3} e^{-\pi i \theta_\infty \sigma_3} = G S_0^\infty S_1^\infty$ and $G S_0^\infty = S_0^0 G$ in $\mathbb{C}^{12}$. These equations contain six independent relations and $g_{11} g_{22} - g_{12} g_{21} = g_{11} g_{22} - g_{12} g_{21} = 1$. Hence dim$_{\mathbb{C}} \mathcal{M} = 4$, and a generic point on $\mathcal{M}$ is denoted by, say $(G, s_0^0)$, or $(G, \hat{G})$ by Propositions 3.4 and 3.5.

Instead of $(U_j^\infty (\lambda), U_j^0 (\lambda), U_j^0 (\lambda; v, f_1, f_2))$, we may take

$$(\tilde{U}_j^\infty (\lambda), U_j^0 (\lambda; cv, f_1, f_2)), \text{ or } (U_j^0 (\lambda), U_j^0 (\lambda; v, \tilde{c} f_1, \tilde{c}^{-1} f_2))$$

for any $c, \tilde{c} \in \mathbb{C} \setminus \{0\}$, where $\tilde{U}_j^\infty (\lambda) := e^{c \sigma_3 / 2} U_j^\infty (\lambda) e^{-c \sigma_3 / 2} = (I + O(\lambda^{-1})) U_j^\infty (\lambda), U_j^0 (\lambda; cv, f_1, f_2) = c \sigma_3 / 2 U_j^0 (\lambda)$, and $U_j^0 (\lambda; v, \tilde{c} f_1, \tilde{c}^{-1} f_2) = U_j^0 (\lambda) e^{c \sigma_3}$. Then the connection formula $U_j^\infty (\lambda) = U_j^0 (\lambda) G$ becomes $\tilde{U}_j^\infty (\lambda) = U_j^0 (\lambda; cv, f_1, f_2) e^{-c \sigma_3 / 2},$ or $U_j^\infty (\lambda) = U_j^0 (\lambda; v, \tilde{c} f_1, \tilde{c}^{-1} f_2) e^{c \sigma_3} G$, respectively, and the same relation holds for $\hat{G}$. This fact induces the action $[\ast] : (G, \hat{G}) \mapsto (\tilde{c}^{-1} G \hat{c}^{-1} G e^{-c \sigma_3 / 2}, \tilde{c}^{-1} \sigma_3 \hat{G} e^{-c \sigma_3 / 2})$, and $(S_j^\infty, S_j^0) \mapsto (c \sigma_3 / 2 S_j^\infty e^{-c \sigma_3 / 2}, \tilde{c}^{-1} \sigma_3 S_j^0 e^{c \sigma_3})$ with any $c, \tilde{c} \in \mathbb{C}$. As shown in [29] Section 3.4], dividing $\mathcal{M}$ by this action $[\ast]$ yields orbits constituting an affine cubic surface $V(\mathcal{M}) \subset \mathbb{C}^3$ parametrised by $(\theta_0, \theta_\infty)$, and each solution of (1.1) corresponds to a point on $V(\mathcal{M})$. It is easy to see the following.

**Proposition 3.6.** The quantities

$$g_{11} g_{22}, \quad \frac{\hat{g}_{11}}{g_{11}} = 1 - s_0^0 \frac{g_{21}}{g_{11}}, \quad \frac{\hat{g}_{22}}{g_{22}} = 1 + s_0^\infty \frac{g_{21}}{g_{22}}, \quad \frac{\hat{g}_{12}}{g_{12}} = 1 - s_0^0 \frac{g_{22}}{g_{12}} + s_0^\infty \frac{g_{11}}{g_{12}}, \quad \frac{\hat{g}_{21}}{g_{21}} = 1$$

are invariants under the action $[\ast]$. Note that $e^{\pi i \theta_0 \sigma_3} S_j^\infty S_k^\infty e^{-\pi i \theta_\infty \sigma_3} = S_j^{k+2} S_k^{j+2}$. By induction on $m$, $S_j^\infty S_k^\infty = e^{(m-1) \pi i \theta_0 \sigma_3} S_j^\infty S_k^\infty e^{-(m-1) \pi i \theta_\infty \sigma_3}$. Thus we have the following.
Proposition 3.7. Let \( m \in \mathbb{Z} \setminus \{0\} \). Then
\[
S_{0}^{\infty}S_{0}^{\infty} \cdots S_{2m-2}^{\infty}S_{2m-1} = (S_{0}^{\infty}S_{0}^{\infty}e^{\pi i\delta_{0}S_{0}^2})^{m}e^{-m\pi i\delta_{0}S_{0}^2} \quad \text{for} \ m \geq 1, \quad \text{and} \\
S_{2m}S_{2m-2}^{\infty} \cdots S_{2}^{\infty}S_{1} = e^{m\pi i\delta_{0}S_{0}^2}(S_{0}^{\infty}S_{0}^{\infty}e^{\pi i\delta_{0}S_{0}^2})^{-m} \quad \text{for} \ m \leq -1.
\]

Comparison with the monodromy data of Kitaev. Kitaev [13] considered the isomonodromy deformation of the linear system
\[
(3.1) \quad \frac{d\Phi}{d\lambda} = \tau \left(-i\sigma_{3} - \frac{ai}{2\tau\lambda}\sigma_{3} - \frac{1}{\lambda} \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix} + \frac{i}{2\lambda^2} \begin{pmatrix} \sqrt{\theta^2 - AB} & A \\ -\sqrt{\theta^2 - AB} & -B \end{pmatrix} \right) \Phi
\]
with \( \theta^2 = 1 \) to treat the complete \( P_{III} \). For \( \tau > 0 \) system (3.1) has the matrix solutions
\[
X_k(\tilde{\lambda}) = (I + O(\tilde{\lambda}^{-1})) \exp(-i\tau \tilde{\lambda}\sigma_{3} + T_{\infty}\log(\tilde{\lambda}/\tau)) \quad \text{as} \ \tilde{\lambda} \to \infty, \\
Y_k(\tilde{\lambda}) = (\Phi_0 + O(\tilde{\lambda})) \exp(-\frac{i}{2}\tau \tilde{\lambda}^{-1}\sigma_{3} + T_{0}\log(\tilde{\lambda}/\tau)) \quad \text{as} \ \tilde{\lambda} \to 0
\]
through the sector \( |\arg \tilde{\lambda} + \arg \tau - (k-1)\pi| < \pi \) for \( k = 1, 2 \), where \( T_{\infty} = -\frac{1}{2}ai\sigma_{3}, \)
\( T_{0} = -\frac{1}{2}ci\sigma_{3} \), and det \( \Phi_0 \neq 0 \). Then the monodromy data \( G_1, G_2, S_{j} \) \((1 \leq j \leq 4)\) are given by
\[
X_2 = X_1S_1, \quad X_1(\tilde{\lambda}e^{-2\pi i})e^{2\pi iT_{\infty}} = X_2S_2, \quad Y_2 = Y_1S_3, \quad Y_1(\tilde{\lambda}e^{-2\pi i})e^{2\pi iT_{0}} = Y_2S_4, \\
Y_2 = X_2G_2, \quad Y_1 = X_1G_1.
\]
Let us set \( x = \sqrt{2i}\tau, \lambda = \sqrt{2i}\tilde{\lambda}, y = \sqrt{2i}w, \delta = \sqrt{2i}z \) in our system (1.2). Then we have
\[
(3.2) \quad \frac{dV}{d\lambda} = \left(-i\tau\sigma_{3} + \frac{1}{\lambda} \left(\frac{i}{2}\theta_{0}(\tau zv)^{-1} - v^{-1}(\frac{i}{2}\theta_{0}(2 - \tau z^{-1}) - 2w(z - \tau)) \right) \right) \frac{2wzv}{\frac{1}{2}\theta_{0}}
\]
which admits the matrix solutions
\[
V_k(\tilde{\lambda}) = (I + O(\tilde{\lambda}^{-1})) \exp(-i\tau \tilde{\lambda}\sigma_{3})\tilde{\lambda}^{-\frac{j}{2}\theta_{0}S_{0}^2}
\]
as \( \tilde{\lambda} \to \infty \) through the sector \( |\arg(\tilde{\lambda}/\tau) - (k-1)\pi| < \pi \), and
\[
V_k^0(\tilde{\lambda}) = \Delta_0(I + O(\tilde{\lambda})) \exp(-\frac{i}{2}i\tau \tilde{\lambda}^{-1}\sigma_{3})\tilde{\lambda}^{-\frac{j}{2}\theta_{0}S_{0}^2}
\]
as \( \tilde{\lambda} \to 0 \) through the sector \( |\arg(\tilde{\lambda}/\tau) - k\pi| < \pi \) for \( k \in \mathbb{Z} \). Note that \( V_k(\tilde{\lambda}) = U_k(\tilde{\lambda}/\tau)\tilde{\lambda}^{-\frac{j}{2}\theta_{0}S_{0}^2}, \)
\( V_k^0(\tilde{\lambda}) = U_k^0(\tilde{\lambda}/\tau)\tilde{\lambda}^{-\frac{j}{2}\theta_{0}S_{0}^2} \). If \( V_k(\tilde{\lambda}) \) and \( V_k^0(\tilde{\lambda}) \) solving (3.2) are identified with, respectively, \( X_2(\tilde{\lambda}e^{-2\pi i})e^{2\pi iT_{\infty}} \) and \( Y_2(\tilde{\lambda}e^{-2\pi i})e^{2\pi iT_{0}} \) solving (3.1) with suitable \( a, c \), then the monodromy data \( (S_{0}^\infty, S_{0}^0, G, \tilde{G}) \) and \( (S_{j}, G_{1}, G_{2}) \) fulfill
\[
(\sqrt{2i})^{-\frac{j}{2}\theta_{0}S_{0}^2}S_{1}(\sqrt{2i})^{-\frac{j}{2}\theta_{0}S_{0}^2} = \tau^{-T_{\infty}}S_{0}^{-1}T_{\infty}, \quad (\sqrt{2i})^{-\frac{j}{2}\theta_{0}S_{0}^2}S_{2}^{-1}T_{\infty} = \tau^{-T_{\infty}}S_{0}^{-1}T_{\infty}, \\
(\sqrt{2i})^{-\frac{j}{2}\theta_{0}S_{0}^2}S_{0}^{-1}T_{\infty} = \tau^{-T_{\infty}}S_{3}^{-1}T_{0}, \quad (\sqrt{2i})^{-\frac{j}{2}\theta_{0}S_{0}^2}S_{1}^{-1}T_{\infty} = \tau^{-T_{\infty}}S_{4}^{-1}T_{0}, \\
(\sqrt{2i})^{-\frac{j}{2}\theta_{0}S_{0}^2}G_{1}S_{0}^{-1}T_{\infty} = \tau^{-T_{0}}G_{1}^{-1}T_{\infty}, \quad (\sqrt{2i})^{-\frac{j}{2}\theta_{0}S_{0}^2}G_{2}S_{0}^{-1}T_{\infty} = \tau^{-T_{0}}G_{2}^{-1}T_{\infty}.
\]
3.2. Isomonodromy deformation. By $U = v^{1/2}Y$, $2x = \xi$, system (1.2) is taken into
\[\frac{dY}{d\lambda} = B(\xi, \lambda)Y\]
with
\[B(\xi, \lambda) = \frac{i\xi}{4} + \left(\frac{1}{8} \theta_0 \xi^2 - \frac{1}{2} \theta_\infty (2 - \frac{1}{2} \xi) - y(3 - \frac{1}{2} \xi) \frac{-y_3}{2 \theta_\infty}\right) \lambda^{-1}\]
\[\quad - i \left(3 - \frac{1}{2} \xi \frac{-y_3}{2 \theta_\infty}\right) \lambda^{-2}.
\]
Set $\xi = t e^{i\phi}$ and write the right-hand side in the form $B(\xi, \lambda) = (t/4)B(t, \lambda)$. In what follows instead of (1.2) we are concerned with the linear system
\[
\frac{dY}{d\lambda} = \frac{t}{4} B(t, \lambda)Y, \quad B(t, \lambda) = b_1 \sigma_1 + b_2 \sigma_2 + b_3 \sigma_3, \\
b_1 = (\Gamma(t, 3, y) - 2y_3 t^{-1}) \lambda^{-1} + i e^{i\phi} \lambda^{-2}, \\
b_2 = -i(\Gamma(t, 3, y) + 2y_3 t^{-1}) \lambda^{-1} - (4t^{-1} - e^{i\phi}) \lambda^{-2}, \\
b_3 = \frac{1}{2} e^{i\phi} - 2\theta_\infty t^{-1} \lambda^{-1} - i(4t^{-1} - e^{i\phi}) \lambda^{-2}
\]
with
\[\Gamma(t, 3, y) = -y(2t^{-1} - e^{i\phi}) + \frac{1}{2} e^{i\phi} \theta_3^{-1} - \frac{1}{2} \theta_\infty \lambda^{-1} (4t^{-1} - e^{i\phi}),
\]
\[
4 \lambda = ty^* y^{-2} + e^{i\phi} t(1 - y^{-2}) - (2\theta_\infty - 1)y^{-1},
\]
where $y$ and $y^*$ are arbitrary complex parameters.

System (3.3) admits the matrix solutions
\[Y_k^\infty(\lambda) = v^{-\sigma_3/2} U_k^\infty(\lambda) = v^{-\sigma_3/2} (I + O(\lambda^{-1})) \exp\left(\frac{1}{4} i e^{i\phi} t \lambda \sigma_3\right) \lambda^{-\frac{1}{2} \theta_\infty \sigma_3}\]
as $\lambda \to \infty$ through the sector $|\arg \lambda + \phi - k\pi| < \pi$, and
\[Y_k^0(\lambda) = v^{-\sigma_3/2} U_k^0(\lambda) = \Delta^0(\lambda + O(\lambda)) \exp\left(-\frac{1}{4} i e^{i\phi} t \lambda^{-1} \sigma_3\right) \lambda^{\frac{1}{2} \theta_0 \sigma_3}\]
as $\lambda \to 0$ through the sector $|\arg \lambda - \phi - k\pi| < \pi$, where
\[\Delta^0 = \begin{pmatrix} f_1 & f_2 \\ f_1 & f_2 \end{pmatrix}, \quad e^{i\phi} t f_1 f_2 \equiv -2.
\]
Then
\[Y_0^\infty(\lambda) = Y_0^0(\lambda) G, \quad Y_1^\infty(\lambda) = Y_1^0(\lambda) \hat{G}, \quad Y_{k+1}^\infty(\lambda) = Y_k^\infty(\lambda) S_k^\infty, \quad Y_k^0(\lambda) = Y_k^0(\lambda) S_k^0.
\]

Proposition 3.8. For $(Y_k^\infty(\lambda), Y_k^0(\lambda))$, system (3.3) has the same monodromy data $(G, \hat{G}, S_\infty^g, S_\infty^\infty, S_0^g, S_0^\infty)$ and the same monodromy manifold as of (1.2).

Then isomonodromy deformation of (3.3) may also be described as follows.

Proposition 3.9. The monodromy data of (3.3) for $(Y_k^\infty(\lambda), Y_k^0(\lambda))$ remain invariant under a small change of $t e^{i\phi}$ if and only if $y^* = (d/dt)y$ holds in (3.4) and
\[\frac{d \lambda}{dt} = -4y_3^2 + (2e^{i\phi} t - 1) \lambda - 2\theta_\infty \lambda + \frac{1}{2} (\theta_0 + \theta_\infty) e^{i\phi} t,
\]
which is equivalent to (1.1).
Let $Y_k^{∞,r}(λ)$ be the matrix solution of (3.3) such that

$$Y_k^{∞,r}(λ) = (I + O(λ^{-1})) \exp(\frac{1}{4}ie^{iφ}tλσ_3)λ^{-\frac{1}{2}θσ_3}$$

as $λ \to ∞$ in the sector $|arg λ + φ - kπ| < π$, and set

$$Y_0^{∞,r}(λ) = Y_0(λ)G^*, \quad Y_1^{∞,r}(λ) = Y_1(λ)\hat{G}^*.$$ 

Then $Y_k^{∞,r}(λ) = Y_k(λ)v^{σ3/2}$, and $Y_0^{∞,r}(λ) = Y_0(λ)Gv^{σ3/2}$, $Y_1^{∞,r}(λ) = Y_1(λ)\hat{G}v^{σ3/2}$.

**Proposition 3.10.** For $(Y_k^{∞,r}(λ), Y_0(λ))$, system (3.3) has the monodromy data

$$(G^*, \hat{G}^*, S_0^{∞,*}, S_1^{∞,*}, S_0^0, S_1^0)$$

Then

$$g_{11}^∗g_{22}^∗ = g_{11}g_{22}, \quad g_{11}^∗/g_{21}^∗ = g_{11}/g_{21}, \quad g_{12}^*/g_{22}^* = g_{12}/g_{22}, \quad \hat{g}_{11}^∗\hat{g}_{22}^* = \hat{g}_{11}\hat{g}_{22}, \quad \hat{g}_{11}^*/\hat{g}_{21}^* = \hat{g}_{11}/\hat{g}_{21}, \quad \hat{g}_{12}^*/\hat{g}_{22}^* = \hat{g}_{12}/\hat{g}_{22}.$$ 



**Corollary 3.11.** Let $G^* = (g_{ij}^*)$ and $\hat{G}^* = (\hat{g}_{ij}^*)$. Then

$$g_{11}g_{22} = g_{11}g_{22}, \quad g_{11}/g_{21} = g_{11}/g_{21}, \quad g_{12}/g_{22} = g_{12}/g_{22}, \quad \hat{g}_{11}\hat{g}_{22} = \hat{g}_{11}\hat{g}_{22}, \quad \hat{g}_{11}/\hat{g}_{21} = \hat{g}_{11}/\hat{g}_{21}, \quad \hat{g}_{12}/\hat{g}_{22} = \hat{g}_{12}/\hat{g}_{22}.$$ 



**4. WKB Analysis**

**4.1. Turning points and Stokes graphs.** Let us examine the characteristic roots $±μ = ±μ(t, λ)$ of $B(t, λ)$, the turning points, i.e. the roots of $μ$, and the Stokes graph, which are used in calculating monodromy data for system (3.3). The characteristic roots are given by

$$μ^2 = b_1^2 + b_2^2 + b_3^2$$

$$= -e^{2iφ} - 4ie^{iφ}θσ_3t^{-1}λ^{-1} + e^{2iφ}a_φλ^{-2} + 4ie^{iφ}θ_0t^{-1}λ^{-3} - e^{2iφ}λ^{-4}$$

with

$$a_φ = a_φ(t) = e^{-2iφ}(y^*y^{-1} + t^{-1}) - y^2 - y^{-2} - 4e^{-iφ}t^{-1}(θ_0y + θ_∞y^{-1}).$$

First suppose that $φ = 0$. The Boutroux equations (2.1) admit a unique solution $A_0 = 2$. Equation (1.1) has a two-parameter family of solutions such that $y(x) = i + O(x^{-1/2})$ as $x \to +∞$ along the positive real axis. Taking these facts into account and viewing the supposition [5,1], we set $a_0(t) = 2 + o(1)$, and then $μ(∞, λ)$ is equal to $λ^{-4}(λ^2 - 1)^2 = -λ^{-4}w(A_0, λ)^2$. This means that, around $φ = 0$, $μ(t, λ)$ admits four turning points $λ_1$, $λ_{-1}$, $λ_2$, $λ_{-2}$ such that, for $φ = 0$, $λ_1$ and $λ_2$ (respectively, $λ_{-1}$ and $λ_{-2}$) coalesce at 1 (respectively, at $-1$) as $t \to ∞$. Then $Re λ_1 ≤ Re λ_2$ for $φ$ close to 0, and that $λ_1 → -λ_1$, $λ_{-2} → -λ_{-2}$ as $t → ∞$. Here we use the same letters $λ_1$, $λ_2$ to denote the turning points and the limit turning points as $t → ∞$. By Proposition 8.16 the Boutroux equations admit unique solutions $A_{±π/2} = -2$, and $μ(∞, λ) = e^{2iφ}(1 - a_φλ^{-2} + λ^{-4})$ does not degenerate for $0 < |φ| < π/2$.

Suppose that $0 < |φ| < π/2$. The algebraic function $μ(t, λ)$ is considered on the two-sheeted Riemann surface $R_φ$ glued along the cuts $[λ_{-2}, λ_{-1}]$ and $[λ_1, λ_2]$. Let

$$μ(t, λ) = ie^{iφ}λ^{-2} \sqrt{1 - a_φλ^2 + λ^4 - 4ie^{-iφ}θ_0t^{-1} + 4ie^{-iφ}θ_∞λ^3t^{-1}}$$

$$= ie^{iφ}λ^{-2} \sqrt{(1 - λ_1^{-1}λ)(1 - λ_2^{-1}λ)(1 - λ_{-1}^{-1}λ)(1 - λ_{-2}^{-1}λ)}$$

$$= ie^{iφ}λ^{-2} \sqrt{1 - λ_1^{-1}λ} \sqrt{1 - λ_2^{-1}λ} \sqrt{1 - λ_{-1}^{-1}λ} \sqrt{1 - λ_{-2}^{-1}λ},$$
in which the branches of the square roots are fixed in such a way that \( \sqrt{1 - \lambda^2} \lambda \to 1 \) as \( \lambda \to 0 \) on the upper sheet of \( \mathcal{R}_\phi \). Then \( \lambda^2 \mu(t, \lambda) \to i e^{i \phi} \) as \( \lambda \to 0 \), and \( \mu(t, \lambda) \to -i e^{i \phi} \) as \( \lambda \to \infty \) on the upper sheet of \( \mathcal{R}_\phi \).

\[
\sqrt{1 - \lambda^2} \lambda \to 1
\]

Then \( \lambda^2 \mu(t, \lambda) \to i e^{i \phi} \) as \( \lambda \to 0 \), and \( \mu(t, \lambda) \to -i e^{i \phi} \) as \( \lambda \to \infty \) on the upper sheet of \( \mathcal{R}_\phi \).

**Figure 4.1.** Limit Stokes graphs for \(|\phi| \leq \pi/2\)

The Stokes graph consists of the Stokes curves and the vertices: each Stokes curve is defined by \( \text{Re} \int_{\lambda_*}^\lambda \mu(\lambda)d\lambda = 0 \) with a turning point \( \lambda_* \), and the vertices are turning points or singular points \( \lambda = 0, \infty \). The Stokes graph lies on \( \mathcal{R}_\phi \). The limit Stokes graph with \( t = \infty \) for the isomonodromy system (3.3) is considered to reflect the Boutroux equations (2.1). When \( \phi \) increases or decreases, the limit turning points for \( \lambda_1 \) and \( \lambda_2 \) move according to the solution \( A_\phi \) of the Boutroux equations (2.1). By Proposition 8.17, for \( \phi \) close to 0, the double turning point at \( \lambda_1 = \lambda_2 = 1 \) is resolved into two simple turning points \( \lambda_1 < 1 < \lambda_2 \) if \( \phi > 0 \), and such that \( \text{Im} \lambda_1 < 0 < \text{Im} \lambda_2 \), \( \text{Re} \lambda_1 < 1 < \text{Re} \lambda_2 \) if \( \phi < 0 \). For \( 0 < |\phi| < \pi/2 \) coalescence of turning points does not occur, and then the topological figure of the limit Stokes graph remains invariant. For \( -\pi/2 < \phi < 0 \) and \( 0 < \phi < \pi/2 \), the limit Stokes graphs are as in Figures 4.1(b) and (d), in which each limit turning point with \( t = \infty \) is also denoted by \( \lambda_1 \) or \( \lambda_{-i} = -\lambda_i \). In our calculation, for \( 0 < |\phi| < \pi/2 \), we use the Stokes curves from \( 0^+ \) to \( \infty^- \) passing through \( \lambda_1 \) and \( \lambda_2 \) and passing through \( -\lambda_1 \) to \( -\lambda_2 \), where \( 0^+ \) and \( \infty^- \) denote 0 and \( \infty \) on the upper and lower sheets, respectively. This ranges on both upper and lower sheets of \( \mathcal{R}_\phi \). For a technical reason, the cut \([\lambda_1, \lambda_2]\) on the upper sheet is made in such a way that the Stokes curve \((\lambda_1, \lambda_2)^-\) is located along the lower shore (respectively, the upper shore) of the cut if \( 0 < \phi < \pi/2 \) (respectively, \( -\pi/2 < \phi < 0 \)), and the cut \([\lambda_2, \lambda_1]\) in such a way that the Stokes curve \((\lambda_1, \lambda_2)^-\) is located along the upper shore (respectively, the lower shore) of the cut if \( 0 < \phi < \pi/2 \) (respectively, \( -\pi/2 < \phi < 0 \)) (cf. Figures 5.1 5.2 5.3).
An unbounded domain $D \subset \mathcal{R}_\phi$ is called a canonical domain if, for each $\lambda \in D$, there exist contours $C_{+}(\lambda) \subset D$ terminating in $\lambda$ such that
\[
\text{Re} \int_{\lambda_{0}^{-}}^{\lambda} \mu(\lambda)d\lambda \to -\infty \quad \left(\text{respectively, } \text{Re} \int_{\lambda_{0}^{+}}^{\lambda} \mu(\lambda)d\lambda \to +\infty\right)
\]
as $\lambda_{0}^{-} \to \infty$ along $C_{-}(\lambda)$ (respectively, as $\lambda_{0}^{+} \to \infty$ along $C_{+}(\lambda)$) (see [5], [6, p. 242]). The interior of a canonical domain contains exactly one Stokes curve, and its boundary consists of Stokes curves.

4.2. WKB solution. The following WKB solution will be used in our calculus.

Proposition 4.1. In the canonical domain $(\subset \mathcal{R}_\phi)$ whose interior contains a Stokes curve issuing from the turning point $\lambda_{\pm 1}$ or $\lambda_{\pm 2}$, system (3.3) with $B(\lambda, t) = b_{1}\sigma_{1} + b_{2}\sigma_{2} + b_{3}\sigma_{3}$ admits an asymptotic solution expressed as
\[
\Psi_{\text{WKB}}(\lambda) = T(I + O(t^{-\delta})) \exp\left(\int_{\lambda_{0}}^{\lambda} \Lambda(\tau)d\tau\right), \quad T = \left(\frac{1}{b_{1} - ib_{2}}, \frac{b_{3} - \mu}{b_{1} + ib_{2}}, 1\right)
\]
outside suitable neighbourhoods of zeros of $b_{1} \pm ib_{2}$ as long as $|\lambda - \lambda_{i}| \gg t^{-2/3 + (2/3)\delta}$ ($i = \pm 1, \pm 2$). Here $\delta$ is an arbitrary number such that $0 < \delta < 1$, $\hat{\lambda}$ is a base point near $\lambda_{\pm 1}$ or $\lambda_{\pm 2}$, and
\[
\Lambda(\lambda) = \frac{t}{4}\mu(t, \lambda)\sigma_{3} - \text{diag} T^{-1}T_{\lambda}.
\]

Remark 4.1. In the proposition above
\[
\text{diag} T^{-1}T_{\lambda} = \frac{1}{2\mu(\mu + b_{3})} (i(b_{1}b_{2}' - b_{1}'b_{2})\sigma_{3} + (b_{3}\mu' - b_{3}'\mu)I)
\]
\[
= \frac{1}{4} \left(1 - \frac{b_{3}}{\mu}\right) \frac{\partial}{\partial \lambda} \log \frac{b_{1} + ib_{2}}{b_{1} - ib_{2}} \sigma_{3} + \frac{1}{2} \frac{\partial}{\partial \lambda} \log \frac{\mu}{\mu + b_{3}} I,
\]
where $b_{1}' = (\partial/\partial \lambda) b_{1}$.

For the Schrödinger equation $(-\hbar^{2}(d/d\lambda)^{2} + p(\lambda))y = 0$ with Planck’s constant $\hbar$, the WKB solution has the form $(1 + O(\hbar))p(\lambda)^{-1/4}\exp(\pm \hbar^{1/2} \int_{0}^{\lambda} p(\lambda)d\lambda)$. On the other hand, by $Y = T\tilde{Y}$ system (3.3) is taken to $\tilde{Y}_{\lambda} = (\frac{1}{4}t\mu(t, \lambda)\sigma_{3} - T^{-1}T_{\lambda})\tilde{Y}$ formally admitting a matrix solution with the leading term $\exp(\pm \hbar^{1/2} \int_{0}^{\lambda} \mu(t, \lambda)d\lambda\sigma_{3})$ as $t \to +\infty$ (arg $t \to 0$). Hence our solution in Proposition 4.1 is the WKB solution with the perturbation variable $4t^{-1}$ in place of Planck’s constant $\hbar$. We make a further transform of the form $Y = (I + T_{1})(I + X_{1})Z$ with $T_{1}$ such that $[\frac{1}{4}t\mu\sigma_{3}, T_{1}] = T^{-1}T_{\lambda} - \text{diag} T^{-1}T_{\lambda}$ as in the proof of [31, Proposition 3.8]. Suitable choice of $X_{1}$ reduces system (3.3) to $Z_{\lambda} = (\frac{1}{4}t\mu(t, \lambda)\sigma_{3} - \text{diag} T^{-1}T_{\lambda})Z$, from which our WKB solution follows. Then we additionally use the fact: $\mu = \mp ie^{ib} + O(\lambda^{-2})$ near $\lambda = \infty^{\pm}$, and $= \pm ie^{ib}\lambda^{-2} + O(1)$ near $\lambda = 0^{\pm}$ on $\mathcal{R}_\phi$, where $\infty^{+}$, $0^{+}$ (respectively, $\infty^{-}$, $0^{-}$) denote $\infty$, 0 on the upper sheet (respectively, lower sheet) of $\mathcal{R}_\phi$ (for details see the proof of [31, Proposition 3.8] or [6, Theorem 7.2]), see also [5]).

4.3. Local solution around a turning point. Near turning points the WKB solution above fails in expressing its asymptotic behaviour. In the neighbourhood of $\lambda$, system (3.3) is reduced to
\[
\frac{dW}{d\zeta} = \begin{pmatrix} 0 & 1 \\ \xi & 0 \end{pmatrix} W,
\]
which has the solutions $T(Ai(\zeta), Ai_1(\zeta))$, $T(Bi(\zeta), Bi_1(\zeta))$ with the Airy function $Ai(\zeta)$ and $Bi(\zeta) = e^{-\pi i/6}Ai(e^{-2\pi i/3}\zeta)$ [1]. Then we have the following solution near each simple turning point [3] Theorem 7.3, [31] Proposition 3.9.

**Proposition 4.2.** For each simple turning point $\lambda_i$ ($i = \pm 1, \pm 2$) write $c_k = b_k(\lambda_i)$, $c'_k = (b'_k)(\lambda_i)$ ($k = 1, 2, 3$), and suppose that $c_k$, $c'_k$ are bounded and $c_1 \pm i c_2 \neq 0$. Let $t = 2(2\kappa_i)^{-1/3}(c_1 - ic_2)(t/4)^{1/3}$ with $\kappa_c = c_1 c'_1 + c_2 c'_2 + c_3 c'_3$. Then system (3.3) admits a matrix solution of the form

$$\Phi_i(\lambda) = T_i(I + O(t^{-\delta}))(1 \quad 0 \quad 0) \quad \sigma_t(\zeta), \quad T_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

in which $\lambda - \lambda_i = (2\kappa_c)^{-1/3}(t/4)^{-2/3}(\zeta + \zeta_0)$ with $|\zeta_0| \ll t^{-1/3}$, as long as $|\zeta| \ll t^{2/3-\delta'/3}$, that is, $|\lambda - \lambda_i| \ll t^{-2/3+(2\delta' - 3)/3}$. Here $\delta'$ is an arbitrary number such that $0 < \delta' < 2/3$, and $W(\zeta)$ solves system (4.3), which admits canonical solutions $W_\nu(\zeta)$ ($\nu \in \mathbb{Z}$) such that

$$W_\nu(\zeta) = \zeta^{-(1/4)3}\sigma_3(\sigma_3 + \sigma_1)(I + O(\zeta^{-3/2})) \exp((2/3)\zeta^{3/2}\sigma_3)$$

as $\zeta \to \infty$ through the sector $|\arg \zeta - (2\nu - 1)/3| < 2\pi/3$, and that $W_{\nu+1}(\zeta) = W_\nu(\zeta)S_\nu$ with

$$S_1 = \begin{pmatrix} 1 & 0 & -i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S_{\nu+1} = \sigma_1 S_\nu \sigma_1.$$

**Remark 4.2.** Putting $\lambda - \lambda_i = (2\kappa_c)^{-1/3}(e^{2\pi i/3})^j(t/4)^{-2/3}(\zeta + \zeta_0)$, $j \in \{0, \pm 1\}$, we have an expression of $\Phi_i(\lambda)$ with $t = 2(2\kappa_c)^{-1/3}(e^{2\pi i/3})^j(c_1 - ic_2)(t/4)^{1/3}$.

5. **Calculation of the connection matrices**

To get necessary information on the connection matrices $G = (y_{ij})$ and $\hat{G} = (\hat{y}_{ij})$ (cf. Sections 2.1 and 3.1) we calculate $G^*$ ($\tilde{y}_{ij}$) and $\hat{G}^*$ ($\tilde{\hat{y}}_{ij}$) such that

$$Y_0^{\infty,*}(\lambda) = Y_0^0(\lambda)G^*, \quad Y_1^{\infty,*}(\lambda) = Y_1^0(\lambda)\hat{G}^*$$

(cf. (3.7), Proposition 3.10) as a solution of the direct monodromy problem by applying WKB analysis to system (3.3). Suppose that $a_\phi(t)$ is given by (1.2) with a pair of arbitrary functions $(y, y^*) = (y(t), y^*(t))$ not necessarily solving (1.1) with $2x = \xi = te^{i\phi}$, and that

$$a_\phi(t) = A_\phi + \frac{B_\phi(t)}{t}, \quad B_\phi(t) \ll 1$$

for $t \in S_\phi(t_\infty, \kappa_1, \delta_1)$ with given $\kappa_1 > 0$, given small $\delta_1 > 0$ and sufficiently large $t_\infty > 0$. Here $A_\phi$ is a solution of the Boutroux equations (2.1), and

$$S_\phi(t_\infty, \kappa_1, \delta_1) = \{ t \mid \Re t > t'_\infty, \quad |\Im t| < \kappa_1, \quad |y(t)| + |y^*(t)| + |y(t)|^{-1} < \delta_1^{-1} \}.$$ 

Let $0 < \phi < \pi/2$. We calculate the analytic continuation of the matrix solution $Y_0^{\infty,*}(\lambda)$ near $\lambda = \infty$ along the Stokes curve consisting of

$$c_\infty = (\infty, \lambda_2)^\sim, \quad c_1 = (\lambda_2, \lambda_1)^\sim, \quad c_0 = (\lambda_1, 0^+)^\sim$$

starting from $\infty^-$ and terminating in $0^+$ as in Figure 5.1 to get the connection matrix $G^*$, where $\infty^-$ and $0^+$ denote $\infty$ and $0$ on the lower and upper sheets of $\mathcal{R}_\phi$, respectively. Recall that the Stokes curve is considered on the two-sheeted Riemann surface $\mathcal{R}_\phi$ of $\mu(t, \lambda)$, and that the curve $c_1$ is located along the lower shore of the cut $[\lambda_1, \lambda_2]$. The curve $c_\infty$ is on the lower sheet of $\mathcal{R}_\phi$, and $c_0$ and $c_1$ are on the upper sheet of $\mathcal{R}_\phi$. Under supposition (5.1) these curves $c_0$, $c_1$, $c_\infty$ lie within the distance $\ll t^{-1}$ from the limit Stokes graph in Figure 4.1.
In the WKB solution, write $\Lambda(\lambda)$ in the component-wise form $\Lambda(\lambda) = \Lambda_3(\lambda) + \Lambda_I(\lambda)$ with
\[
\Lambda_3(\lambda) = \frac{t}{4} \mu(t, \lambda) \sigma_3 - \text{diag} T^{-1} T_\lambda \sigma_3, \quad \Lambda_I(\lambda) = -\text{diag} T^{-1} T_\lambda I,
\]
in which $\text{diag} T^{-1} T_\lambda \sigma_3 \in \mathbb{C} \sigma_3$, $\text{diag} T^{-1} T_\lambda I \in \mathbb{C} I$. Denote by $\mu_-(t, \lambda)$ the branch of $\mu(t, \lambda)$ on the lower sheet of $R_\phi$ and by $\Lambda_-(\lambda)$, $\Lambda_3-(\lambda)$, $\Lambda_I-(\lambda)$ and $T_-$ those related to $\mu_-(t, \lambda)$. In Propositions 4.1 and 4.2 if $\delta = \delta' = 2/9 - \varepsilon$ with any $\varepsilon$ such that $0 < \varepsilon < 2/9$, then both propositions are applicable in the annulus
\[
A'_\varepsilon : \quad t^{-2/3 + (2/3)(2/9 - \varepsilon)} \ll |\lambda - \lambda_1| \ll t^{-2/3 + (2/3)(2/9 + \varepsilon/2)}
\]
($\varepsilon = \pm 1, \pm 2$). In what follows we set $\delta = 2/9 - \varepsilon$, and write $c_k = b_k(\lambda_1)$, $d_k = b_k(\lambda_2)$ ($k = 1, 2, 3$).

**Figure 5.1.** Curves $c_0, c_1, c_\infty$ for $0 < \phi < \pi/2$

**1.** Let $\Psi_\infty(\lambda)$ along $c_\infty = (\infty^-, \lambda_2^-)$ be a WKB solution by Proposition 4.1 and let $Y_0^{\infty,*}(\lambda)$ be given by (3.6). Set $Y_0^{\infty,*}(\lambda) = \Psi_\infty(\lambda) \Gamma_\infty$. Note that the branch of $\mu(t, \lambda)$ along $c_\infty$ is $\mu_-(t, \lambda) = ie^{i\phi}(1 + 2ie^{-i\phi} \theta_\infty t^{-1} \lambda^{-1} + O(\lambda^{-2}))$, and $\mu_--b_3 \ll \lambda^{-1}$ as $\lambda \to \infty^-$. Then
\[
\Gamma_\infty = \Psi_\infty(\lambda)^{-1} Y_0^{\infty,*}(\lambda)
= \exp \left( - \int_{\lambda_2}^\lambda \Lambda_-(\tau) d\tau \right) T^{-1} (I + O(t^{-\delta} + |\lambda|^{-1}))
\times \exp \left( \frac{1}{4} (ie^{i\phi} t \lambda - 2 \theta_\infty \log \lambda) \sigma_3 \right)
= C_3(\tilde{\lambda}_2) c_1(\tilde{\lambda}_2)(I + O(t^{-\delta}))
\times \exp \left( - \lim_{\lambda \to \infty^-} \left( \int_{\lambda_2}^\lambda \Lambda_-(\tau) d\tau - \frac{1}{4} (ie^{i\phi} t \lambda - 2 \theta_\infty \log \lambda) \sigma_3 \right) \right),
\]
in which $C_3(\tilde{\lambda}_2) = \exp(\int_{\lambda_2}^{\tilde{\lambda}_2} \Lambda_-(\tau) d\tau)$, $c_1(\tilde{\lambda}_2) = \exp(-\int_{\lambda_2}^{\tilde{\lambda}_2} \Lambda_-(\tau) d\tau)$, and $\tilde{\lambda}_2 \in c_\infty$, $\tilde{\lambda}_2 - \lambda_2 \asymp t^{-1}$.  

**2.** For $\Psi_\infty(\lambda)$ and for $\Phi_2^+(\lambda)$ given by Proposition 4.2 in the annulus $A'_\varepsilon$ around $\lambda_2$, set $\Psi_\infty(\lambda) = \Phi_2^+(\lambda) \Gamma_{2+}$ along $c_\infty$. Suppose that the curve $(2\kappa_d)^{1/3}(\lambda - \lambda_2) = (t/4)^{-2/3}(\zeta + O(t^{-1/3}))$, $\kappa_d = d_1 d'_1 + d_2 d'_2 + d_3 d'_3$ with $\lambda \in c_\infty$ enters the sector $|\arg \zeta - \pi/3| < 2\pi/3$ (the other cases are similarly treated by Remark 4.2). Write $K^{-1} = 2(2\kappa_d)^{-1/3}(d_1 - id_2)$. Then, by Propositions 4.1 and 4.2
\[
\Gamma_{2+} = \Phi_2^+(\lambda)^{-1} \Psi_\infty(\lambda).
\]
\[= \text{W}(\zeta)^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} (t/4)^{-1/3}K \\ (t/4)^{-1/3}K \end{pmatrix}^{-1} \left( I + O(t^{-\delta}) \right) \left( \frac{1}{-\frac{d_1}{d_1-id_2}} \right)^{-1} \]
\[
\times \left( \frac{1}{\left( \frac{b_1-b_2}{b_1+ib_2} \right)^{1/3}} \right) \left( I + O(t^{-\delta}) \right) \exp \left( \int_{\lambda_2}^{\lambda} \Lambda_-(\tau)d\tau \right) \]
\[
= \text{W}(\zeta)^{-1} \left( \frac{1}{(t/4)^{1/3}K} \frac{d_3}{d_3 + id_2} \right) \left( I + O(t^{-\delta}) \right) \exp \left( \int_{\lambda_2}^{\lambda} \Lambda_-(\tau)d\tau \right) \]

for \( \lambda \in \mathcal{A}_c^2 \cap c_{\infty} \), where \((\mu_--b_3)/(b_1 \pm ib_2) = (\mu_--d_3)/(d_1 \pm id_2) + O(\eta), \eta = \lambda - \lambda_2 \).

Since \( \mu_- = -(2\kappa_d)^{1/2}(1 + O(\eta)) = -2K(d_1 - id_2)(t/4)^{-1/3}(1 + O(\eta)) \), we have
\[
\Gamma_{2+} = \exp \left( -\frac{2}{3} \zeta^{3/2}\sigma_3 \right) \left( \sigma_3 + \sigma_1 \right)^{-1} \zeta^{(1/4)\sigma_3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]
\[
\times \left( \begin{array}{cc} 1 & -\frac{d_1-id_2}{d_3} \\ -\frac{d_1-id_2}{d_3} & 1 \end{array} \right) \left( I + O(t^{-\delta}) \right) \exp \left( \int_{\lambda_2}^{\lambda} \Lambda_-(\tau)d\tau \right) \]
\[
= \exp \left( \int_{\lambda_2}^{\lambda} \sigma_1 \Lambda_-(\tau) \sigma_1 d\tau - \frac{2}{3} \zeta^{3/2}\sigma_3 \right) \zeta^{1/4} \left( I + O(t^{-\delta}) \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \left( \begin{array}{cc} 0 & -\frac{d_1-id_2}{d_3} \\ \frac{d_1-id_2}{d_3} & 0 \end{array} \right) \]

By \( \sigma_1 \Lambda_-(\lambda)\sigma_1 = ((2\kappa_d)^{1/2}(t/4)^{1/2}(1 + O(\eta)) + O(\eta^{-1/2}))\sigma_3 \) and \( \Lambda_-(\lambda) = (-\eta^{-1/4} + O(\eta^{-1/2}))I \) (cf. Remark 4.1) for \( \eta = \lambda - \lambda_2, \lambda \in \mathcal{A}_c^2 \cap c_{\infty} \), it follows that
\[
\Gamma_{2+} = (\zeta_2)^{1/4} \left( I + O(t^{-\delta}) \right) C_3(\bar{\lambda}_2)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \left( \begin{array}{cc} \frac{d_1-id_2}{d_3} \\ 0 \end{array} \right) \]

with suitably chosen \( \zeta_2 \approx \lambda_2 - \lambda_2 \).

(3) Let \( \Phi_2^-(\lambda) \) be the solution by Proposition 4.2 near \( \mathbf{c}_1 = (\lambda_2, \lambda_1)^-, \) and set \( \Phi_2^+(\lambda) = \Phi_2^-(\lambda)\Gamma_{2+}, \) where \( \Phi_2^+(\lambda) \) is the analytic continuation along an arc in \( \mathcal{A}_c^2 \) in the anticlockwise direction. Then by Proposition 4.2
\[
\Gamma_{2+} = \Phi_2^-(\lambda)^{-1}\Phi_2^+(\lambda) = S_{1-}^{-1} = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}. \]

(4) For \( \Phi_2^- (\lambda) \) and the WKB solution \( \Psi_1^- (\lambda) \) along \( \mathbf{c}_1, \) set \( \Phi_2^- (\lambda) = \Psi_1^- (\lambda)\Gamma_{2-}. \) Note that \( \mathbf{c}_1 \) is on the upper sheet of \( \mathcal{R}_\Phi. \) Then, supposing the curve \((2\kappa_d)^{1/2}(\zeta_2 + 1/3) = (t/4)^{-2/3}(\zeta + O(t^{-1/3})) \) with \( \lambda \in \mathbf{c}_1 \) is in the sector \(| \arg \zeta - \pi | < 2\pi/3, \) we have, for \( \lambda_2 \in \mathbf{c}_1, \) \( | \lambda_2 - \lambda_2 | \approx t^{-1}, \)
\[
\Gamma_{2-} = \Psi_1^- (\lambda)^{-1}\Phi_2^- (\lambda)
\]
\[
= \exp \left( -\int_{\lambda_2}^{\lambda} \Lambda(\tau)d\tau \right) \left( I + O(t^{-\delta}) \right) \left( \frac{1}{-\frac{d_1}{d_1-id_2}} \right)^{-1} \]
\[
\times \left( \frac{1}{\left( \frac{b_1-b_2}{b_1+ib_2} \right)^{1/3}} \right) \left( I + O(t^{-\delta}) \right) \left( \begin{array}{cc} 0 \\ 1 \end{array} \right) \left( \begin{array}{cc} \frac{d_1-id_2}{d_3} \\ 0 \end{array} \right) \left( \frac{d_1-id_2}{d_3} \right) \left( I + O(t^{-\delta}) \right) \]
\[
= \exp \left( \frac{2}{3} \zeta^{3/2}\sigma_3 - \int_{\lambda_2}^{\lambda} \Lambda(\tau)d\tau \right) \zeta^{-1/4} \left( I + O(t^{-\delta}) \right) \left( \begin{array}{cc} 0 \\ 1 \end{array} \right) \left( \begin{array}{cc} 0 \\ \frac{d_1-id_2}{d_3} \end{array} \right), \]

where \( \tilde{K}^{-1} = 2(2\kappa_d)^{-1/3}(d_1 - id_2). \) This yields
\[
\Gamma_{2-} = (\zeta_2)^{-1/4} \left( I + O(t^{-\delta}) \right) C_3(\bar{\lambda}_2) \left( \begin{array}{cc} 0 \\ 1 \end{array} \right) \left( \begin{array}{cc} 0 \\ -\frac{d_1-id_2}{d_3} \end{array} \right) \]
with \( C'_3(\tilde\lambda_2) = \exp(\int_{\lambda_1}^{\tilde\lambda_2} \Lambda_3(\tau)d\tau) \) for some \( \tilde\zeta_2 \approx \tilde\lambda_2 - \lambda_2 \).

5. For \( \Psi_1^-(\lambda) \) and the WKB solution \( \Psi_1^+(\lambda) \) along \( c_1 \) near \( \lambda_1 \), set \( \Psi_1^-(\lambda) = \Psi_1^+(\lambda)\Gamma_{12} \). Then, for \( \lambda_1 \in c_1 \), \( \lambda_1 - \lambda_1 \approx t^{-1} \),

\[
\Gamma_{12} = \Psi_1^+(\lambda)^{-1}\Psi_1^-(\lambda) = C_3'(\tilde\lambda_2)^{-1}C_3''(\tilde\lambda_1)\hat c_I(\tilde\lambda_2, \tilde\lambda_1) \exp\left(-\int_{\lambda_1}^{\tilde\lambda_2} \Lambda_3(\tau)d\tau\right),
\]

where \( C_3''(\tilde\lambda_1) = \exp(\int_{\lambda_1}^{\tilde\lambda_1} \Lambda_3(\tau)d\tau) \), \( \hat c_I(\tilde\lambda_2, \tilde\lambda_1) = \exp(-\int_{\lambda_1}^{\tilde\lambda_2} \Lambda_I(\tau)d\tau) \).

6. For \( \Psi_1^+(\lambda) \) and for \( \Phi_1^+(\lambda) \) given by Proposition 4.2 in the annulus \( \mathcal{A}_\lambda^\circ \) around \( \lambda_1 \), set \( \Psi_1^+(\lambda) = \Phi_1^+(\lambda)\Gamma_{1+} \). Then, by the same argument as in (2) above with \( \mu \) in place of \( \mu_- \), we have

\[
\Gamma_{1+} = \Phi_1^+(\lambda)^{-1}\Psi_1^+(\lambda) = (\tilde\zeta_1)^{1/4}(I + O(t^{-\delta}))C_3''(\tilde\lambda_1)^{-1}\left(\begin{array}{cc} 1 & 0 \\ -i & 1 \end{array}\right)
\]

for some \( \tilde\zeta_1 \approx \tilde\lambda_1 - \lambda_1 \).

7. Let \( \Phi_1^-(\lambda) \) be the solution by Proposition 4.2 near \( c_0 = (\lambda_1, 0^+) \), and set \( \Phi_1^+(\lambda) = \Phi_1^-(\lambda)\Gamma_{1s} \), where \( \Phi_1^+(\lambda) \) is the analytic continuation along an arc in \( \mathcal{A}_\lambda^\circ \) in the clockwise direction. Then by Proposition 4.2,

\[
\Gamma_{1s} = \Phi_1^-(\lambda)^{-1}\Phi_1^+(\lambda) = S_0 = \left(\begin{array}{cc} 1 & 0 \\ -i & 1 \end{array}\right).
\]

8. For \( \Phi_1^-(\lambda) \) and the WKB solution \( \Psi_0(\lambda) \) along \( c_0 \), set \( \Phi_1^-(\lambda) = \Psi_0(\lambda)\Gamma_{1-} \). By the same argument as in (4), we have

\[
\Gamma_{1-} = \Psi_0(\lambda)^{-1}\Phi_1^-(\lambda) = (\tilde\zeta_2)^{1/4}(I + O(t^{-\delta}))\hat C_3(\tilde\lambda_1)^{-1}\left(\begin{array}{cc} 1 & 0 \\ -i & 1 \end{array}\right)
\]

with \( \hat C_3(\tilde\lambda_1) = \exp(\int_{\lambda_1}^{\tilde\lambda_1} \Lambda_3(\tau)d\tau) \) for some \( \tilde\zeta_2 \approx \tilde\lambda_1 - \lambda_1 \).

9. For \( \Psi_0(\lambda) \) and \( Y_0^0(\lambda) \) given by (3.5), set \( \Psi_0(\lambda) = Y_0^0(\lambda)\Gamma_0 \). Note that \( \mu(t, \lambda) = ie^{i\phi}\lambda^{-2} + O(1) \) as \( \lambda \to 0^+ \). Then

\[
\Gamma_0 = Y_0^0(\lambda)^{-1}\Psi_0(\lambda) = \exp\left(\frac{1}{4}(ie^{i\phi}t\lambda^{-1} - 2\theta_0 \log \lambda)\sigma_3\right)(\Delta_0^*)^{-1}T_0(I + O(t^{-\delta} + |\lambda|)) \exp\left(\int_{\lambda_1}^{\lambda} \Lambda(\tau)d\tau\right)
\]

with \( T_0 = T|_{\lambda=0} \). Choosing \( f_1 = c_0^* \), \( f_2 = c_0^*(\tilde{\lambda}_2 - e^{i\phi}t/2)^{-1} \) with \( c_0^* = \sqrt{2}e^{-i\phi/2}t^{-1/2}(3 - e^{i\phi}t/2)^{1/2} \), we have

\[
\Delta_0^* = c_0^*\left(\begin{array}{cc} 1 & 3(3 - e^{i\phi}t/2)^{-1} \\ 1 & 1 \end{array}\right).
\]

The (1, 2)- and (2, 1)-entries of \( T - I \) satisfy

\[
\frac{b_3 - \mu}{b_1 + ib_2} \sim \frac{-4ie^{i\phi}t^{-1}\lambda^{-2}}{-2it^{-1}(4\tilde{\lambda}_2 - e^{i\phi}t)\lambda^{-2}} \to \frac{3}{3 - e^{i\phi}t/2},
\]

\[
\frac{\mu - b_3}{b_1 - ib_2} \sim \frac{ie^{i\phi}\lambda^{-2} + it^{-1}(4\tilde{\lambda}_2 - e^{i\phi}t)\lambda^{-2}}{4it^{-1}\lambda^{-2}} \to 1
\]

as \( \lambda \to 0^+ \). This implies \( (c_0^*)^{-1}\Delta_0^* = T_0 \). Hence we have

\[
\Gamma_0 = \hat C_3(\tilde\lambda_1)^{-1}\hat c_I(\tilde\lambda_1)(c_0^*)^{-1}(I + O(t^{-\delta}))
\]
\[
x \times \exp \left( \lim_{\lambda \to 0^+} \left( \int_{\lambda_1}^{\lambda} \Lambda_3(\tau) d\tau + \frac{1}{4} (ie^{i\phi} t^{\lambda^{-1}} - 2\theta_0 \log \lambda) \sigma_3 \right) \right)
\]
with \(\hat{c}_1(\hat{\lambda}_1^i) = \exp(\int_{\hat{\lambda}_1^i}^{\hat{\lambda}_1} \Lambda_1(\tau) d\tau)\).

Collecting the matrices above, we have the connection matrix
\[
G^* = Y_0^0(\lambda)^{-1} Y_0^{\infty,*} (\lambda)
= \Gamma_0 \Gamma_{1-} \Gamma_1 \Gamma_{12} \Gamma_{2-} \Gamma_2 \Gamma_{2+} \Gamma_{\infty}
= \epsilon_+ i (I + O(t^{-\delta})) \exp(J_0 \sigma_3) \begin{pmatrix} 1 & 0 \\ 0 & -c_0^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -c_0 \end{pmatrix}
\times \exp(-J_1 \sigma_3) \begin{pmatrix} 1 & 0 \\ 0 & -d_0 \end{pmatrix} \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -d_0 \end{pmatrix} \exp(-J_{\infty} \sigma_3)
= \epsilon_+ (I + O(t^{-\delta}))
\times \begin{pmatrix} -e^{-J_{0-j_1-j_{\infty}} e^{-J_{0-j_1-j_{\infty}}} & -id_0 e^{-J_{0-j_1-j_{\infty}}} \\ -i(c_0^{-1} e^{-J_i + d_0^{-1} e^J_i} e^{-J_{0-j_1-j_{\infty}}} & e_0^{-1} d_0 e^{-J_{0-j_1-j_{\infty}}} \end{pmatrix}
\]
if \(0 < \phi < \pi/2\), where \(\epsilon_+^2 = 1\), \(c_0 = (c_1 - ic_2)/c_3\), \(d_0 = (d_1 - id_2)/d_3\), and

\[
J_0 \sigma_3 = \lim_{\lambda \to 0^+} \int_{\lambda_1}^{\lambda} \Lambda_3(\tau) d\tau + \frac{1}{4} (ie^{i\phi} t^{\lambda^{-1}} - 2\theta_0 \log \lambda) \sigma_3)
(5.2)
\]

\[
J_1 \sigma_3 = \int_{\lambda_1}^{\lambda_2} \Lambda_3(\tau) d\tau \text{ (along } c_1)\]
(5.3)

\[
J_{\infty} \sigma_3 = \lim_{\lambda \to \infty} \int_{\lambda_2}^{\lambda} \Lambda_3(\tau) d\tau - \frac{1}{4} (ie^{i\phi} t^{\lambda} - 2\theta_\infty \log \lambda) \sigma_3)
(5.4)
\]

Note that the curve \(c_1\) joining \(\lambda_1\) to \(\lambda_2\) is located along the lower shore of the cut on the upper sheet of \(\mathcal{R}_\phi\).

The connection matrix \(\hat{G}^*\) is obtained by calculating the matrix solution \(Y_1^{\infty,*}(\lambda)\) along the Stokes curve consisting of \(\hat{c}_\infty = (\infty^-, -\lambda_2)^\infty, \hat{c}_1 = (-\lambda_2, -\lambda_1)^\infty, \hat{c}_0 = (-\lambda_1, 0^*)^\infty\), the union of which joins \(\infty^-\) to \(0^+\) as in Figure 5.2. The curve \(\hat{c}_\infty\) lies on the lower sheet of \(\mathcal{R}_\phi\), and \(\hat{c}_1\) and \(\hat{c}_0\) on the upper sheet of \(\mathcal{R}_\phi\). Then we have

\[
\hat{G}^* = Y_0^0(\lambda)^{-1} Y_1^{\infty,*}(\lambda)
= \hat{c}_1 (I + O(t^{-\delta})) \exp(\hat{J}_0 \sigma_3) \begin{pmatrix} 1 & 0 \\ 0 & -\hat{c}_0^{-1} \end{pmatrix} \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -\hat{c}_0 \end{pmatrix}
\]

![Figure 5.2. Curves \(\hat{c}_0, \hat{c}_1, \hat{c}_\infty\) for \(0 < \phi < \pi/2\)](image-url)
\[ J_1 \sigma_3 = \int_{\lambda_1}^{\lambda_2} \Lambda_3(\tau) d\tau \quad \text{(along } c^-_1), \]

\[ J^{-1}_1 = J_1|_{c^-_1 \rightarrow c^-_1}, \]

in which \( c^-_1 \) is a curve joining \( \lambda_1 \) to \( \lambda_2 \) located along the upper shore of the cut on the upper sheet of \( \mathcal{R}_\phi \). Thus we have the following.

**Figure 5.3.** Stokes curve for \(-\pi/2 < \phi < 0\)
Proposition 5.1. Let $c_0 = (c_1 - ic_2)/c_3$, $d_0 = (d_1 - id_2)/d_3$, $\hat{c}_0 = (\hat{c}_1 - i\hat{c}_2)/\hat{c}_3$, $\hat{d}_0 = (\hat{d}_1 - i\hat{d}_2)/\hat{d}_3$ with $c_k = b_k(\lambda_1)$, $d_k = b_k(\lambda_2)$ $\hat{c}_k = b_k(\lambda_{-1})$, $\hat{d}_k = b_k(\lambda_{-2})$ for $k = 1, 2, 3$. If $0 < \phi < \pi/2$ and $g_{11}^{*}g_{22}^{*}g_{11}^{*}g_{22}^{*} \neq 0$, then
\[
 g_{11}^{*}g_{22}^{*} = -c_0^{-1}d_0(1 + O(t^{-2}))e^{-2J_0}, \quad g_{12}^{*}g_{21}^{*} = -ic_0(1 + O(t^{-2}))e^{2J_0},
\]
\[
 \hat{g}_{11}^{*}\hat{g}_{22}^{*} = -\hat{c}_0\hat{d}_0^{-1}(1 + O(t^{-2}))e^{2\hat{J}_0}, \quad \hat{g}_{12}^{*}\hat{g}_{21}^{*} = i\hat{c}_0(1 + O(t^{-2}))e^{2\hat{J}_0}.
\]
If $-\pi/2 < \phi < 0$ and $g_{11}^{*}g_{22}^{*}g_{11}^{*}g_{22}^{*} \neq 0$, then
\[
 g_{11}^{*}g_{22}^{*} = -c_0^{-1}d_0(1 + O(t^{-2}))e^{2J_0}, \quad g_{11}^{*}g_{22}^{*} = -ic_0(1 + O(t^{-2}))e^{2J_0},
\]
\[
 \hat{g}_{11}^{*}\hat{g}_{22}^{*} = -\hat{c}_0\hat{d}_0^{-1}(1 + O(t^{-2}))e^{-2\hat{J}_0}, \quad \hat{g}_{12}^{*}\hat{g}_{21}^{*} = i\hat{c}_0(1 + O(t^{-2}))e^{-2\hat{J}_0}.
\]

Here $J_0$, $J_1$, $\hat{J}_1$ are integrals given by (5.2), (5.3), (5.5), and $\hat{J}_0 = J_0|_{c_0 \to \hat{c}_0}$, $\hat{J}_1 = J_1|_{c_1 \to \hat{c}_1}$, $\hat{J}_1 = J_1|_{c_1 \to \hat{c}_1}$.

From the proposition above with entries of $G^{*}$ and $\hat{G}^{*}$ combined with Corollary 3.11 we derive the following key relations.

Corollary 5.2. If $0 < \phi < \pi/2$ and $g_{11}g_{22}g_{11}g_{22} \neq 0$, then
\[
 g_{11}g_{22} = -c_0^{-1}d_0(1 + O(t^{-2}))e^{-2J_0}, \quad g_{12}g_{21} = -c_0^{-1}d_0(1 + O(t^{-2}))e^{2\hat{J}_0}.
\]
If $-\pi/2 < \phi < 0$ and $g_{11}g_{22}g_{11}g_{22} \neq 0$, then
\[
 g_{11}g_{22} = -c_0^{-1}d_0(1 + O(t^{-2}))e^{2\hat{J}_0}, \quad g_{12}g_{21} = -c_0^{-1}d_0(1 + O(t^{-2}))e^{-2\hat{J}_0}.
\]

6. Asymptotic properties of monodromy data

6.1. Expressions of integrals. By (5.1), for sufficiently large $t$, consider the elliptic curve $\Pi_{a_\phi} = \Pi_\phi \cup \Pi_-$ for $w(a_\phi, \lambda) = \sqrt{1 - a_\phi^2\lambda^2 + \lambda^4}$, the branch of which is consistent with that of $w(A_\phi, \lambda)$ in Section 2. On $\Pi_{a_\phi}$ we may define cycles $a$ and $b$ which are identified with those on $\Pi_{A_\phi}$ as given as in Figure 2.1. Then the cycles $a$ and $b$ on $\Pi_{a_\phi}$ are independent of $t$, and are also regarded to be on $\mathcal{R}_\phi$. On $\mathcal{R}_\phi$ around these cycles,
\[
 \mu(t, \lambda) = ie^{i\phi}w(a_\phi, \lambda)/\lambda^2 + 2\theta_0 t^{-1}/w(a_\phi, \lambda) - 2\theta_\infty t^{-1}/w(a_\phi, \lambda) + O((|\lambda^4w^{-3}| + 1) t^{-2}).
\]

For integrals appearing in Corollary 5.2 we have
\[
 J_1(a_{(e)}) = \int_{\lambda_1, (e)}^{\lambda_2} \Lambda_3(\lambda) d\lambda, \quad J_{-1}(a_{(e)}) = \int_{\lambda_{-1}, (e_{-})}^{\lambda_{-2}} \Lambda_3(\lambda) d\lambda = \frac{1}{2} \int_b \Lambda_3(\lambda) d\lambda
\]
with $\Lambda_3(\lambda) = (t/4) \mu(t, \lambda) \sigma_3 - \text{diag } T^{-1}T_{\lambda\sigma_3} \sigma_3$. Note that
\[
 \int \frac{w}{\lambda^2} d\lambda = \frac{-w}{\lambda} + 2 \int \frac{\lambda^2}{w} d\lambda - a_\phi \int \frac{d\lambda}{w} = \frac{w}{\lambda} - a_\phi \int \frac{d\lambda}{w} + 2 \int \frac{d\lambda}{\lambda^2 w},
\]
\[
 \int \frac{d\lambda}{\lambda w} = \frac{1}{2} \log(\lambda^2 - a_\phi/2 + \lambda^2 w), \quad \int \frac{\lambda d\lambda}{w} = \frac{1}{2} \log(\lambda^2 - a_\phi/2 + w).
\]

Lemma 6.1. We have
\[
 \int_a \mu(t, \lambda) d\lambda = ie^{i\phi} \int_a \frac{w(a_\phi, \lambda)}{\lambda^2} d\lambda + 2\pi i(\theta_0 - \theta_\infty) t^{-1} + O(t^{-2}).
\]
Furthermore we have
\[
\lim_{\lambda \to 0^+} \left( \frac{t}{4} \int_{\lambda} \mu(t, \lambda) d\lambda + \frac{i e^{i\phi} t}{4\lambda} - \frac{\theta_0}{2} \log \lambda \right)
\]
\[
= \lim_{\lambda \to 0^+} \left( \frac{i e^{i\phi} t}{4} \left( \int_{\lambda} \frac{w}{\lambda^2} d\lambda + \frac{1}{\lambda} \right) + \frac{\theta_0}{2} \left( \int_{\lambda} \frac{d\lambda}{\lambda w} - \log \lambda \right) - \int_{\lambda} \frac{\theta_\infty \lambda}{2w} d\lambda \right) + O(t^{-1}),
\]
in which
\[
\int_{\lambda} \frac{w}{\lambda^2} d\lambda + \frac{1}{\lambda} = 2 \int_{\lambda} \frac{\lambda^2}{w} d\lambda - a_\phi \int_{\lambda} \frac{d\lambda}{w} + O(\lambda), \quad \int_{\lambda} \frac{d\lambda}{\lambda w} - \log \lambda = C_{\lambda_1, a_\phi} + O(\lambda^2)
\]
with \(C_{\lambda_1, a_\phi} = \frac{1}{2} \log(\lambda^{-2} - \frac{1}{2} a_\phi) - \frac{1}{2} \log 2\) as \(\lambda \to 0^+\), and a similar formula is obtained as \(\lambda \to 0^+\) along \(c_0\). Hence, in \(J_0 - J_0^i\),

\[
\lim_{\lambda \to 0^+} \left( \frac{t}{4} \int_{\lambda} \mu(t, \lambda) d\lambda + \frac{i e^{i\phi} t}{4\lambda} - \frac{\theta_0}{2} \log \lambda \right)
\]
\[
= \lim_{\lambda \to 0^+} \left( \frac{i e^{i\phi} t}{4} \left( 2 \int_{\lambda} \frac{\lambda^2}{w} d\lambda - a_\phi \int_{\lambda} \frac{d\lambda}{w} \right) - \int_{\lambda} \frac{\theta_\infty \lambda}{2w} d\lambda + O(t^{-1}) \right)
\]
\[
= - \frac{i e^{i\phi} t}{8} \int_{\lambda} \frac{w}{\lambda^2} d\lambda + O(t^{-1}).
\]

By Remark 4.1

\[
\lim_{\lambda \to 0^+} \left( \frac{t}{4} \int_{\lambda} \mu(t, \lambda) d\lambda + \frac{i e^{i\phi} t}{4\lambda} - \frac{\theta_0}{2} \log \lambda \right)
\]

\[
= \frac{1}{4} \left( 1 - \frac{b_3}{\mu} \frac{\partial}{\partial \lambda} \log \frac{b_1 + ib_2}{b_1 - ib_2} \right).
\]

To calculate our integrals, it is necessary to know diag \(T^{-1}T_\lambda|_{\sigma_3}\) in addition to Lemma 6.1 and (6.1). By (3.1) and (3.2),

\[
b_1 - ib_2 = - (\lambda - iy^{-1}) y \lambda^{-2} (y^* y^{-2} + e^{i\phi} - e^{i\phi} y^{-2}) + O(t^{-1}),
\]
\[
b_1 + ib_2 = - (\lambda + iy^{-1}) y \lambda^{-2} (y^* y^{-2} - e^{i\phi} - e^{i\phi} y^{-2}) + O(t^{-1}),
\]
\[
b_3 = ie^{i\phi} - i (y^* - e^{i\phi}) y^{-2} \lambda^{-2} + O(t^{-1}),
\]
\[
\mu = ie^{i\phi} \lambda^{-2} w + O(t^{-1}).
\]

Set

\[
\lambda_+ = iy^{-1} + O(t^{-1}), \quad \lambda_- = - iy^{-1} + O(t^{-1})
\]
such that \((b_1 - ib_2)(\lambda_+) = (b_1 + ib_2)(\lambda_-) = 0\). Note that

\[
\frac{b_3}{\mu} = - ie^{-i\phi} \lambda^2 b_3 \left( \frac{1}{w(\lambda)} + O(t^{-1}) \right),
\]

where \(\lambda^2 b_3 \sim ie^{i\phi} \lambda^2\) satisfies \(\lambda_+^2 b_3(\lambda_+) = - \lambda_-^2 \mu(\lambda_-) = - ie^{i\phi} w(\lambda_+) + O(t^{-1})\) as \(\lambda_+ \to \infty^+\) on the upper sheet, since \(\mu(\lambda_+)^2 = (b_1 - ib_2)(b_1 + ib_2)(\lambda_+) + b_3(\lambda_+)^2 = b_3(\lambda_+)^2\). These facts combined with (6.2) yield

\[
\text{diag } T^{-1}T_\lambda|_{\sigma_3} = \frac{1}{4} \left( 1 + ie^{-i\phi} \lambda^2 b_3 \left( \frac{1}{w(\lambda)} + O(t^{-1}) \right) \right) \left( \frac{1}{\lambda - \lambda_-} - \frac{1}{\lambda - \lambda_+} \right)
\]
where sn(u) with \(d\lambda/du\) (cf. Section 2). The then Corollary 5.2 and Lemma 6.1 lead to the following.

6.2. Expressions by the \(\theta\)-function. For \(w(\lambda)^2 = w(\phi, \lambda)^2 = \lambda^4 - a_\phi \lambda^2 + 1\), the differential equation \((d\lambda/du)^2 = w(\phi, \lambda)^2\) defines the doubly periodic function \(\lambda = sn(u) = \lambda_1 sn(\lambda_2 u; \lambda_1/\lambda_2)\), where sn(u; \(k\)) is the Jacobi sn-function, and \(\lambda_1\) and \(\lambda_2\) satisfy \(\lambda^4 - a_\phi \lambda^2 + 1 = (1 - \lambda_1^{-2} \lambda_2^2)(1 - \lambda_2^{-2} \lambda_1^2)\) and \(0 < \text{Re} \lambda_1 \leq \text{Re} \lambda_2\) if \(a_\phi\) is close to 2. The periods of the elliptic curve \(\Pi_{a_\phi}\) for \(w(\phi, \lambda)\) along the cycles \(a\) and \(b\) are

\[
\omega_a = \int_a \frac{d\lambda}{w(\phi, \lambda)}, \quad \omega_b = \int_b \frac{d\lambda}{w(\phi, \lambda)}, \quad \tau = \frac{\omega_b}{\omega_a}, \quad \text{Im} \tau > 0
\]

(cf. Section 2). The \(\theta\)-function \(\theta(z, \tau) = \theta(z)\) is defined by

\[
\theta(z, \tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 + 2\pi i n z}.
\]
and we set
\[ \nu = \frac{1 + \tau}{2} \]
(cf. [9], [35]). For \( \lambda, \tilde{\lambda} \in \Pi_{a_0} = \Pi_+ \cup \Pi_- \), let
\[ F(\tilde{\lambda}, \lambda) = \frac{1}{\omega_a} \int_\lambda^{\tilde{\lambda}} \frac{d\lambda}{w(\lambda)} . \]
For any \( \lambda_0 \in \Pi_{a_0} \) denote the projections of \( \lambda_0 \) on the respective sheets by
\[ \lambda_0^+ = (\lambda_0, w(\lambda_0)) = (\lambda_0, w(\lambda_0^+)) , \quad \lambda_0^- = (\lambda_0, -w(\lambda_0)) = (\lambda_0, -w(\lambda_0^+)) \].
If \( \lambda_0 \in \Pi_+ \) (respectively, \( \lambda_0 \in \Pi_- \), then \( \lambda_0^+ \in \Pi_\pm \) (respectively, \( \lambda_0^+ \in \Pi_\mp \)). Do not confuse this usage of the superscripts \( \pm \) with that in \( \infty^\pm \) (respectively, \( 0^\pm \)) denoting \( \infty \) (respectively, \( 0 \)) on \( \Pi_\pm \).

**Proposition 6.3.** For any \( \lambda_0 \in \Pi_{a_0} \)
\[ \frac{d\lambda}{(\lambda - \lambda_0)w(\lambda)} = \frac{1}{w(\lambda_0^+)} d\log \frac{\vartheta(F(\lambda_0^+, \lambda) + \nu, \tau)}{\vartheta(F(\lambda_0^-, \lambda) + \nu, \tau)} - g_0(\lambda_0) \frac{d\lambda}{w(\lambda)} , \]
\[ g_0(\lambda_0) = \frac{w'(\lambda_0^+)}{2w(\lambda_0^+)} - \frac{1}{\omega_a} \int_{\lambda_0^+}^{\lambda_0^-} \left( \frac{\pi i}{\omega_a} + \frac{\vartheta'}{\vartheta}(F(\lambda_0^-, \lambda_0^+) + \nu, \tau) \right) d\lambda . \]

**Proof.** For \( \lambda_0 = \text{sn}_*(u_0) \in \Pi_{a_0} \) let \( u_0^+ \) be such that \( \lambda_0^+ = \text{sn}_*(u_0^+) \). Then
\[ \frac{d\lambda}{(\lambda - \lambda_0)w(\lambda)} = \frac{d\lambda}{\text{sn}_*(u) - \text{sn}_*(u_0)} \]
\[ = \frac{1}{w(\lambda_0^+)} \left( \zeta(u - u_0^+) - \zeta(u - u_0^-) + \zeta(u_0^+ - u_0^-) - \frac{1}{2} w'(\lambda_0^+) \right) du \]
\[ = \frac{1}{w(\lambda_0^+)} d\log \frac{\sigma(u - u_0^+)}{\sigma(u - u_0^-)} + \frac{1}{w(\lambda_0^+)} \left( \zeta(u_0^+ - u_0^-) - \frac{1}{2} w'(\lambda_0^+) \right) du . \]
From
\[ d\log \frac{\sigma(u - u_0^+)}{\sigma(u - u_0^-)} = - \frac{2n_0}{\omega_a}(u_0^+ - u_0^-) du + d\log \frac{\vartheta(F(\lambda_0^+, \lambda) + \nu, \tau)}{\vartheta(F(\lambda_0^-, \lambda) + \nu, \tau)} \]
\[ = \frac{\vartheta'}{\vartheta}(F(\lambda_0^-, \lambda_0^+) + \nu, \tau) \]
\[ = \frac{\vartheta'}{\vartheta}(F(\lambda_0^-, \lambda_0^+) + \nu, \tau) \]
\[ \zeta(u_0^+ - u_0^-) = \frac{\vartheta'}{\vartheta}(u_0^+ - u_0^-) = \frac{2n_0}{\omega_a}(u_0^+ - u_0^-) + \frac{\pi i}{\omega_a} + \frac{\vartheta'}{\vartheta}(F(\lambda_0^-, \lambda_0^+) + \nu, \tau) \]
with \( F(\lambda_0^+, \lambda) = \omega_a^{-1} \int_{\lambda_0^+}^{\lambda} d\lambda/w(\lambda) \), the desired formula follows. \( \square \)

Observe that
\[ \log \vartheta(F(\lambda_0^+, \lambda) + \nu, \tau)|_a = 0 , \]
\[ \log \vartheta(F(\lambda_0^+, \lambda) + \nu, \tau) \]
\[ = \log \vartheta(F(\lambda_0^+, \lambda_0^+) + \nu, \tau) \]
\[ = \log \vartheta(F(\lambda_0^-, \lambda_0^+) + \nu, \tau) \]
\[ = \exp(-\pi i(2(F(\lambda_0^+, \lambda_0^+) + \nu) + \tau)) \]
\[ = \exp(\pi i(2(F(\lambda_0^-, \lambda_0^+) + \nu) + \tau)) \]
\[ = 2\pi i F(\lambda_0^-, \lambda_0^+) \]
for \( \lambda_0 \in b \cap (\Pi_+)^c \cap (\Pi_-)^c \), where \( (\Pi_+)^c \) denotes the closure of \( \Pi_+ \), since \( \vartheta(z \pm \tau, \tau) = e^{-\pi i(\tau \pm 2\pi z)} \vartheta(z, \tau) \). Then
\[ \int_a \frac{d\lambda}{(\lambda - \lambda_0)w(\lambda)} = -g_0(\lambda_0)\omega_a , \]
Differentiation of both sides with respect to $\lambda_0$ at $\lambda_0 = 0$ yields

$$
\int_b^a \frac{d\lambda}{(\lambda - \lambda_0)w(\lambda)} = \frac{2\pi i}{w(\lambda_0^+)} F(\lambda_0^-, \lambda_0^+) + \tau \int_a^b \frac{d\lambda}{(\lambda - \lambda_0)w(\lambda)}.
$$

In what follows let us adopt the convention that the path of the integral $\int_{\lambda_0}^{\lambda_1} w(\lambda)^{-1} d\lambda$ passes through $\lambda_1$, i.e. the left end of the cut $[\lambda_1, \lambda_2]$. Then

$$
\int_{\lambda_0}^{\lambda_1} \frac{d\lambda}{w(\lambda)} = 2 \int_{\lambda_1}^{\lambda_1} \frac{d\lambda}{w(\lambda)} = -2 \int_{-\lambda_1}^{-\lambda_1} \frac{d\lambda}{w(\lambda)} = \int_{-\lambda_0}^{-\lambda_0} \frac{d\lambda}{w(\lambda)} - \omega_a,
$$

which implies $-F(-\lambda_0^-, -\lambda_0^+) = F(\lambda_0^-, \lambda_0^+) + 1$. Using these formulas we have

**Proposition 6.4.** For $W(\lambda)$ as in Proposition 6.2 and for $\lambda_\pm$ by (6.3),

$$
\int_a^b W(\lambda) d\lambda = - (w(\lambda_+) g_0(\lambda_+) - w(\lambda_-) g_0(\lambda_-) + \lambda_+ - \lambda_-) \omega_a
$$

$$
= - (\lambda_+ + \frac{1}{2} w'(\lambda_+) - \lambda_- - \frac{1}{2} w'(\lambda_-)) \omega_a
$$

$$
+ \frac{\partial'}{\partial(\lambda_+)} (F(\lambda_-, \lambda_+^+) - \lambda_-) + \nu, \tau)
$$

$$
\int_{-\tau}^\tau \int_a^b W(\lambda) d\lambda = 2\pi i (F(\lambda_+, \lambda_+^+) - F(\lambda_-, \lambda_+^+)) = 4\pi i F(\lambda_-, \lambda_+^+) + 2\pi i + O(t^{-1}),
$$

and

$$
\int_a^b \frac{d\lambda}{\lambda w(\lambda)} = - g_0(0^+) \omega_a, \quad g_0(0^+) = - \frac{1}{\omega_a} \left( \frac{\pi i + \frac{\partial'}{\partial(F(\lambda_+, \lambda_+^+), \nu, \tau)} }{4\pi i} \right),
$$

$$
\left( \int_{-\tau}^\tau \int_a^b \frac{d\lambda}{\lambda w(\lambda)} = 2\pi i F(0^+, 0^+) \right),
$$

$$
\left( \int_{-\tau}^\tau \int_a^b \frac{d\lambda}{\lambda^2 w(\lambda)} = 4\pi i \omega_a \right)
$$

**Remark 6.2.** In the proposition above the first formula is rewritten in the form

$$
\int_a^b W(\lambda) d\lambda = - 2 \left( \frac{\partial'}{\partial(F(\lambda_+, \lambda_+^+) + \frac{1}{2}, \tau) - \frac{\partial'}{\partial(F(\lambda_-, \lambda_+^+), 0^+, \tau)} }{4\pi i} \right)
$$

$$
= - \frac{4\pi i}{\omega_a} \left( \frac{\partial'}{\partial(F(\lambda_+, \lambda_+^+) + \frac{1}{2}, \tau) \right),
$$

in which the right-hand side is found by comparing the poles in both expressions. Note that $Y = - (\lambda_+ + w'(\lambda_+)/2) \omega_a + (\partial'/\partial)(F(\lambda_-, \lambda_+^+), \nu, \tau)$ is meromorphic in $\lambda_+$ on $\Pi_{a\phi} = \Pi_+ \cup \Pi_-$. The part $\lambda_+ + w'(\lambda_+)/2$ has a pole $\sim 2\lambda_+$ at $\lambda_+ = \infty^+ \in \Pi_-$ and is analytic at $\lambda_+ = \infty^+ \in \Pi_+$. Since, for $k = \lambda_1/\lambda_2$,

$$
\int_{0}^{1} \frac{dz}{\sqrt{(1 - z^2)(1 - k^2z^2)}} = K = \lambda_2 \omega_a \frac{4}{2}, \quad \int_{0}^{\infty} \frac{dz}{\sqrt{(1 - z^2)(1 - k^2z^2)}} = iK' = \lambda_2 \omega_b \frac{2}{2}
$$

\[EQUATION\]
(cf. Section 7.1), we have
\[
\frac{1}{2} F((\infty)^-, \infty^+) = \frac{1}{2} \omega_a^{-1} \int_{(\infty)^-}^{(\infty)^+} \frac{d\lambda}{w} = \omega_a^{-1} \int_{\lambda_1}^{\infty} \frac{d\lambda}{w} = \frac{1}{4} - \frac{\tau}{2} = -\frac{1}{4} + \nu,
\]
and then we may write \( \Upsilon = -2(\partial' / \partial t)^{1/2} F(\lambda_1^+, \lambda_2^+ + 1, \tau) + C \). Thus the equality above follows (see also [17, pp. 117–119]).

6.3. Expression of \( B_\phi(t) \). Let us write the quantity \( B_\phi(t) \) in terms of
\[
\Omega_a = \int_a \frac{d\lambda}{w(A_\phi, \lambda)}, \quad \Omega_b = \int_b \frac{d\lambda}{w(A_\phi, \lambda)},
\]
\[
\mathcal{J}_a = \int_a \frac{w(A_\phi, \lambda)}{\lambda^2} d\lambda, \quad \mathcal{J}_b = \int_b \frac{w(A_\phi, \lambda)}{\lambda^2} d\lambda
\]
with \( w(A_\phi, \lambda) = \sqrt{1 - A_\phi \lambda^2 + \lambda^2} \) and \( a, b \) on \( \Pi_{A_\phi} = \Pi_+ \cup \Pi_- = \lim_{A_\phi(t) \to A_\phi} \Pi_{A_\phi} \).

Let \( 0 < \phi < \pi / 2 \). By Proposition 6.3, the integral \( \int_a W(\lambda) d\lambda \) is expressed in terms of \( \vartheta_\ast = (\partial' / \partial t)^{1/2} (\frac{1}{2} F(\lambda_1^+, \lambda_2^+) + \frac{1}{4}, \tau) \) in Remark 6.2 or \( w'(\lambda_2) \) and \( F(\lambda_2, \lambda_2^+) + \nu, \tau \), in which
\[
F(\lambda_2^+, \lambda_2^+) = \frac{1}{\omega_a} \int_{\lambda_2^+}^{\lambda_2^+} \frac{d\lambda}{w(a, \lambda)} = \frac{2}{\omega_a} \int_{\lambda_1}^{\lambda_2^+} \frac{d\lambda}{w(a, \lambda)}.
\]
Note that \( \int_a W(\lambda) d\lambda \) has no poles or zeros in \( S_\phi(t_{\infty}, \kappa_1, \delta_1) \). Indeed, if, say \( \vartheta_\ast = \infty \) at \( t = t_\ast \), then \( \lambda_+ = \infty \), and hence \( t_\ast \) is a zero of \( y(t) \), which is excluded from \( S_\phi(t_{\infty}, \kappa_1, \delta_1) \). Consider \( \lambda_+ = \lambda_+(t) \) (cf. (6.3)) moving on the elliptic curve \( \Pi_{A_\phi} \) crossing a- and b-cycles, and then \( F(\lambda_2^+, \lambda_2^+) = 2p(t) + 2q(t)\tau + O(1) \) with \( p(t), q(t) \in \mathbb{Z} \). This implies the boundedness of \( \Re (\partial' / \partial t)^{1/2} (\frac{1}{2} F(\lambda_2^+, \lambda_2^+) + \frac{1}{4}, \tau) \) in \( S_\phi(t_{\infty}, \kappa_1, \delta_1) \), and hence the modulus of \( \Re \int_a W(\lambda) d\lambda \) is uniformly bounded in \( S_\phi(t_{\infty}, \kappa_1, \delta_1) \). Note that, by (5.1),
\[
\frac{1}{\lambda^2}(w(a, \lambda) - w(A_\phi, \lambda)) = \frac{1}{\lambda^2}(\sqrt{\lambda^4 - a_\phi \lambda^2 + 1} - \sqrt{\lambda^4 - A_\phi \lambda^2 + 1}) = -\frac{t^{-1} B_\phi(t)}{2w(A_\phi, \lambda)} (1 + O(t^{-1} B_\phi(t))).
\]
By using this with \( B_\phi(t) \ll 1 \) and Proposition 6.4, the second formula in Proposition 6.2 (i) is written in the form
\[
\log \frac{g_{12} g_{21}}{g_{22} g_{11}} = -\frac{i e^{\phi} t}{4} \int_a \left( \frac{w(A_\phi, \lambda)}{\lambda^2} - \frac{t^{-1} B_\phi(t)}{2w(A_\phi, \lambda)} \right) d\lambda - \frac{1}{4} \int_a W(\lambda) d\lambda + \pi i + O(t^{-\delta}),
\]
which implies
\[
ie^{\phi} \left( t \mathcal{J}_a - \frac{\Omega_a}{2} B_\phi(t) \right) = -\int_a W(\lambda) d\lambda + 4 \pi i - 4 \log \frac{g_{12} g_{21}}{g_{22} g_{11}} + O(t^{-\delta}).
\]
Note that \( (G, \tilde{G}) = (G^* v^{-\sigma_2/2}, \tilde{G}^* v^{-\sigma_2/2}) = ((g_{ij}), (\tilde{g}_{ij})) \) with \( g_{ij} = g_{ij}(t), \tilde{g}_{ij} = \tilde{g}_{ij}(t) \) is a solution of the direct monodromy problem. Suppose that
\[
(6.4) \quad \left\lvert \log \frac{g_{12} g_{21}}{g_{22} g_{11}} \right\rvert \ll 1, \quad \left\lvert \log (g_{11} g_{22}) \right\rvert \ll 1 \quad \text{in } S_\phi(t_{\infty}, \kappa_1, \delta_1).
\]
By the Boutroux equations (2.1), \( \Im e^{\phi} \Omega_a B_\phi(t) \) is bounded as \( e^{\phi t} \to \infty \) through \( S_\phi(t_{\infty}, \kappa_1, \delta_1) \). By using the first formula of Proposition 6.2 (i), we have
\[
ie^{\phi} \left( t \mathcal{J}_b - \frac{\Omega_b}{2} B_\phi(t) \right) = -\int_b W(\lambda) d\lambda - 2 \pi i (\theta_0 - \theta_\infty) + 4 \log (g_{11} g_{22}) - 4 \pi i + O(t^{-\delta}),
\]
Thus, under (6.4), the boundedness of $B - O$ for some $t > 0$, so that $\Im e^{i\phi} \Omega_0 B_\phi(t) \ll 1$. From these facts it follows that $|B_\phi(t)| \leq C_0$ in $S_\phi(t'_\infty, \kappa_1, \delta_1)$ for some $C_0 > 0$. The implied constant of $B_\phi(t) \ll 1$ in (5.1) may be supposed to be $2C_0$, which causes no changes in the subsequent equations by choosing $t'_\infty$ larger if necessary. Thus, under (6.4), the boundedness of $B_\phi(t)$ has been derived independently of (5.1). The case $-\pi/2 < \phi < 0$ is discussed in the same way.

**Remark 6.3.** The argument above works also under a weaker condition, say $B_\phi(t) \ll t^{(1-\delta)/2}$. The supposition $B_\phi(t) \ll 1$ in (5.1) also guarantees that each turning point is located within the distance $O(t^{-1})$ from its limit one, which enables us to employ the limit Stokes graph in the WKB analysis.

**Proposition 6.5.** Suppose that $0 < |\phi| < \pi/2$ and that $|\log(g_{11}g_{22})|, |\log g| \ll 1$ in $S_\phi(t'_\infty, \kappa_1, \delta_1)$, where

$$g = \frac{g_{12}g_{21}}{g_{22}g_{11}} \quad \text{if} \quad 0 < \phi < \pi/2, \quad \text{and} \quad g = \frac{g_{11}g_{22}}{g_{21}g_{12}} \quad \text{if} \quad -\pi/2 < \phi < 0.$$ 

Then, in $S_\phi(t'_\infty, \kappa_1, \delta_1)$, we have $B_\phi(t) \ll 1$ and

$$ie^{i\phi}(\Omega_a - \frac{1}{2}B_\phi(t)) = 4\frac{\partial}{\partial \phi}(\frac{1}{2}F(\lambda_-, \lambda_+^+) + \frac{1}{3}, \tau) + 4\pi i - 4\log g + O(t^{-\delta}).$$

**Remark 6.4.** Conversely, (5.1), i.e. $B_\phi(t) \ll 1$ in $S_\phi(t'_\infty, \kappa_1, \delta_1)$ implies $|\log(g_{11}g_{22})|, |\log g| \ll 1$.

The following fact guarantees the possibility of limitation with respect to $a_\phi$.

**Proposition 6.6.** Under the same supposition as in Proposition 6.5 we have

$$\int_{\lambda_+^-}^{\lambda_+^+} \frac{d\lambda}{w(a_\phi, \lambda)} = \int_{\lambda_+^-}^{\lambda_+^+} \frac{d\lambda}{w(A_\phi, \lambda)} + O(t^{-1})$$

uniformly in $\lambda_+^\pm$ as $te^{i\phi} \to \infty$ through $S_\phi(t'_\infty, \kappa_1, \delta_1)$.

**Proof.** To show this proposition we note the lemma below, which follows from the relations

$$\int \frac{w}{\lambda^2} d\lambda = -\frac{6}{A_\phi} \int wd\lambda + \left(\frac{4}{A_\phi} - A_\phi\right) \int \frac{d\lambda}{w} - \frac{w}{\lambda} + \frac{2}{A_\phi} \lambda w,$$

$$\Omega_0 - \Omega_0 \Omega_0 = \frac{4A_\phi}{3} \pi i, \quad J_{a, b} = \int_{a, b} wd\lambda$$

with $w = w(A_\phi, \lambda)$, the latter being obtained by the same way as in the proof of Legendre’s relation [9], [35].

**Lemma 6.7.** $\Omega_a J_{b} - \Omega_b J_{a} = 8\pi i$.

From the boundedness of $B_\phi(t)$ it follows that $\omega_{a, b} = \Omega_{a, b} + O(t^{-1})$. By Propositions 6.2, 6.4 and Remark 6.1 in the case $0 < \phi < \pi/2$,

$$\log(g_{11}g_{22}) + \tau \log g_{12}g_{21} \frac{g_{22}g_{11}}{g_{21}g_{12}} = \left(\int_{b}^{a} - \tau \int_{a}^{b} \left(\frac{ie^{i\phi}t}{4}, \frac{w(a_\phi, \lambda)}{\lambda^2} + \frac{1}{4} W(\lambda)\right)d\lambda + \frac{\pi i}{2} (\theta_0 - \theta_\infty) + (1 + \tau)i \right) + O(t^{-\delta})$$

$$= -\frac{2\pi e^{i\phi}t}{\omega_a} + \pi i F(\lambda_-, \lambda_+^+) + O(1)$$
Hence, by Proposition 6.6, 

\[ \omega_a Y = -2\pi e^{i\phi} t - i e^{i\phi} (\Omega_a \mathcal{J}_b - \Omega_b \mathcal{J}_a) / 4 + \omega_b X - \omega_a Y + O(1) \]

\[ = -i (\omega_b X - \omega_a Y) + O(1) \ll 1\]

with \( \text{Im} (\omega_b / \omega_a) > 0 \), which implies \(|X|, |Y| \ll 1\), and hence

\[ \pi p(t) = -e^{i\phi} \mathcal{J}_a t / 4 + O(1), \quad \pi q(t) = e^{i\phi} \mathcal{J}_a t / 4 + O(1). \]

Since \( w(a, \lambda)^{-1} - w(A, \lambda)^{-1} = (\lambda^2 / 2) w(A, \lambda)^{-3} B_\phi(t) t^{-1} + O(t^{-2}) \), we obtain

\[
\left| \int_{\lambda_-}^{\lambda_+} \left( \frac{1}{w(a, \lambda)} - \frac{1}{w(A, \lambda)} \right) d\lambda \right| \ll \int_{\lambda_-}^{\lambda_+} \left| \frac{\lambda^2 B_\phi(t) t^{-1}}{w(A, \lambda)^3} \right| d\lambda + O(t^{-1})
\]

\[
\ll t^{-1} \int_{\lambda_-}^{\lambda_+} \frac{\lambda^2 d\lambda}{w(A, \lambda)^3} + O(t^{-1}) \ll t^{-1} |p(t) j_a + q(t) j_b| + O(t^{-1})
\]

\[
\ll |\mathcal{J}_b j_a - \mathcal{J}_a j_b| + O(t^{-1}) = 2 |(\partial / \partial A_\phi)(\mathcal{J}_b \Omega_a - \mathcal{J}_a \Omega_b)| + O(t^{-1}) \ll t^{-1},
\]

where \( j_{a,b} = \int_{a,b} \lambda^2 w(A, \lambda)^{-3} d\lambda \). This completes the proof of the proposition. \( \Box \)

7. Proofs of the main theorems

7.1. Proofs of Theorems [2.1] and [2.2] Suppose that \( 0 < \phi < \pi / 2 \). Let us consider the inverse monodromy problem for the prescribed matrices \( G = (g_{ij}), \hat{G} = (\hat{g}_{ij}) \in SL_2(\mathbb{C}) \) with \( g_{11} g_{22} \hat{g}_{11} \hat{g}_{22} \neq 0 \). Set

\[
\log(g_{11} g_{22}) + \tau \log \frac{g_{12} \hat{g}_{21}}{g_{22} \hat{g}_{11}} = -\frac{2\pi \epsilon^{i\phi} t}{\omega_a} + \pi i F(\lambda_-^+, \lambda_+^+) + \frac{\pi i}{2} (\theta_0 - \theta_\infty + 1) + (1 + \tau) \pi i + O(t^{-\delta})
\]

(cf. Proof of Proposition 4.6), in which

\[
F(\lambda_-^+, \lambda_+^+) = \frac{2}{\omega_a} \left( \int_{\lambda_0}^{0+} + \int_{0+}^{\lambda_0^+} \right) \frac{d\lambda}{w(a, \lambda)} = -\frac{1}{2} + \frac{2}{\omega_a} \int_{0+}^{\lambda_0^+} \frac{d\lambda}{w(a, \lambda)}.
\]

Hence, by Proposition 6.6,

\[
\log(g_{11} g_{22}) + \tau \log \frac{g_{12} \hat{g}_{21}}{g_{22} \hat{g}_{11}} = -\frac{2\pi \epsilon^{i\phi} t}{\omega_a} + \pi i \int_{0+}^{\lambda_0^+} \frac{d\lambda}{w(a, \lambda)} + \frac{\pi i}{2} (\theta_0 - \theta_\infty) + (1 + \tau) \pi i + O(t^{-\delta})
\]

\[
= -\frac{2\pi \epsilon^{i\phi} t}{\Omega_a} + \pi i \int_{0+}^{\lambda_0^+} \frac{d\lambda}{w(A, \lambda)} + \frac{\pi i}{2} (\theta_0 - \theta_\infty) + \frac{\Omega_b}{\Omega_a} \pi i + O(t^{-\delta}),
\]

which yields

\[
-\int_{0+}^{\lambda_0^+} \frac{d\lambda}{w(A, \lambda)} = i(e^{i\phi} t - 2x_0^+ (G, \hat{G}, \Omega_a, \Omega_b)) + O(t^{-\delta}).
\]
with
\[
2ix_0^+(G, \hat{G}, \Omega_a, \Omega_b) = \frac{1}{2\pi i} \left( \frac{\Omega_a \log(g_{11}g_{22}) + \Omega_b \log \frac{g_{12}g_{21}}{g_{22}g_{11}}}{g_{12}g_{21}} - \frac{\Omega_a}{4}(\theta_0 - \theta_\infty + 2) - \frac{\Omega_b}{2} \right).
\]
Observing that \( \lambda_+ = iy(x)^{-1} \), \( e^{i\phi}t = \xi = 2x \), and that
\[
\int_{0^+}^{\lambda_+} \frac{d\lambda}{w(\lambda)} = \frac{1}{\lambda_2} \int_{0^+}^{\lambda_+} \frac{dz}{\sqrt{(1 - z^2)(1 - (\lambda_1/\lambda_2)^2z^2)}}
\]
in which \( k = \lambda_1/\lambda_2 \), we find
\[
y^{-1} = i\lambda_1 \sin(i\lambda_2(e^{i\phi}t - 2x_0^+(G, \hat{G}, \Omega_a, \Omega_b)) + O(t^{-\delta}); \lambda_1/\lambda_2)
\]
with \( 2ix_0^+ \mod \Omega_a\mathbb{Z} + \Omega_b\mathbb{Z} \). For \(-\pi/2 < \phi < 0\), in a similar way we derive
\[
2ix_0^+(G, \hat{G}, \Omega_a, \Omega_b) = \frac{1}{2\pi i} \left( \frac{\Omega_a \log(g_{11}g_{22}) + \Omega_b \log \frac{g_{11}g_{22}}{g_{22}g_{11}}}{g_{11}g_{22}} - \frac{\Omega_a}{4}(\theta_0 - \theta_\infty + 2) - \frac{\Omega_b}{2} \right).
\]
These asymptotic forms of \( y(x) \) coincide with those in Theorems 2.1 and 2.2. The expression of \( B_\phi(t) \) follows from Proposition 6.5.

**Proposition 7.1.** In \( S_\phi(t', \kappa_1, \delta_1) \), we have \( B_\phi(t) \ll 1 \) and
\[
ie^{i\phi} \left( tJ_a - \frac{\Omega_a}{2} B_\phi(t) \right) = -\frac{4y}{\vartheta} \left( \Omega^{-1} i(e^{i\phi}t - 2x_0^+(G, \hat{G}, \Omega_a, \Omega_b)), \Omega_b \Omega_a^{-1} \right)
+ 4\pi i - 4\log g + O(t^{-\delta}).
\]

**Justification.** In the argument above each derived asymptotic form of \( y(x) \) is a necessary condition to represent a solution corresponding to the prescribed monodromy data. The justification of \( y(x) \) as a solution of (1.1) is made along the line in [17] pp. 105–106, pp. 120–121]. Suppose \( 0 < \phi < \pi/2 \). Let \( \mathcal{G} = \{g_{11}g_{22}, g_{12}g_{21}(g_{22}g_{11})^{-1}\} \) be a given point such that \( g_{11}g_{12}g_{22}g_{11}g_{21} \neq 0 \) on the monodromy manifold for system (1.2) or (3.3). Let
\[
y_{\text{as}}^{-1} = y_{\text{as}}(G, t)^{-1} = i\lambda_1 \sin(i\lambda_2(e^{i\phi}t - 2x_0^+(G, \hat{G}, \Omega_a, \Omega_b)); \lambda_1/\lambda_2)
\]
and \( (B_\phi)_{\text{as}} = (B_\phi)_{\text{as}}(G, t) \) be the leading term expressions of \( y(t)^{-1} \) and \( B_\phi(t) \) without \( O(t^{-\delta}) \) found in the argument above and Proposition 7.1. Viewing (4.2) and (5.1) we set
\[
y^*_{\text{as}} = -y_{\text{as}}^{-1} + e^{i\phi} \sqrt{y^2_{\text{as}} + A\phi y'_{\text{as}} + 1} + (4e^{-i\phi}(\theta_0 y_{\text{as}}^2 + \theta_\infty y_{\text{as}}) + (B_\phi)_{\text{as}}y_{\text{as}}^2)t^{-1},
\]
in which the branch of the square root is chosen in such a way that \( y^*_{\text{as}} \) is compatible with \( (d/dt)y_{\text{as}} \). Then \( (y_{\text{as}}, y^*_{\text{as}}) = (y_{\text{as}}(G, t), y^*_{\text{as}}(G, t)) \) fulfils (5.1) with \( B_\phi(t) = (B_\phi)_{\text{as}}(G, t) \) in the domain
\[
\tilde{S}(\phi, t_\infty, \kappa_0, \delta_2) = \{ t \mid \text{Re } t > t_\infty, |\text{Im } t| < \kappa_0 \} \setminus \bigcup_{i\sigma \in Z_0} \{ t - e^{-i\phi}\sigma | < \delta_2 \}
\]
with \( Z_0 = 2x_0^+i + (\frac{1}{2}\Omega_a\mathbb{Z} + \frac{1}{2}\Omega_b\mathbb{Z}) \). Let \( \mathcal{G}_{\text{as}}(t) \) be the monodromy data for system (3.3) containing \( (y_{\text{as}}, y'_{\text{as}}) \). As a result of the direct monodromy problem by the WKB analysis we have \( \|G_{\text{as}}(t) - \mathcal{G}\| \leq Ct^{-\delta} \) for \( |t| \geq t_\infty(G) \) in a neighbourhood of \( \mathcal{G} \), where \( C \) and \( \delta \) are independent of \( \mathcal{G} \). Then the justification scheme of Kitaev [15] applies to our case combined with Proposition 3.9. Using the maximal modulus principle in each
neighbourhood of $i\sigma = 2x_0^+ + \Omega_\Phi Z + \Omega_\phi Z$, we obtain Theorem 2.1. Theorem 2.2 is similarly proved.

7.2. Proof of Theorem 2.3. Let $\Omega_\Phi$ be an isomonodromy system. Equation (1.2) and system (1.2) remain invariant under the substitution $y = e^{i\pi \tau_0} \bar{y}$, $x = e^{i\pi \tau_0} \bar{x}$, $\lambda = e^{i\pi \tau_0} \lambda$, $\phi = m \pi + \phi$, with $\phi = \arg x$, $\bar{\phi} = \arg \bar{x}$. To show the theorem we use this symmetry (cf. [16]). Let $\phi$ be such that $0 < |\phi - m \pi| < \pi/2$. Then a new system with respect to $\bar{\lambda}, \bar{\gamma}, \bar{x}$ is an isomonodromy system for $0 < |\phi - m \pi| < \pi/2$. Denote by $G_m$ and $\hat{G}_m$ connection matrices corresponding to $G$ and $\hat{G}$ as the matrix monodromy data for the system governed by $\bar{y}(\bar{x}) = e^{i\pi \tau_0} \bar{y}(e^{i\pi \tau_0} \bar{x})$. We would like to know the relation between $(G_m, \hat{G}_m)$ and $(G, \hat{G})$. The matrix solutions of the new system are

$$\hat{U}_j(\bar{\lambda}) \sim \exp(\frac{1}{2} i \bar{x} \bar{\lambda} \sigma_3)$$

as $\bar{\lambda} \to \infty$ through the sector $|\arg \bar{\lambda} + \phi - j \pi| < \pi$, and

$$\hat{U}_j(\bar{\lambda}) \sim \Delta_0 \exp(-\frac{1}{2} i \bar{x} \bar{\lambda}^{-1} \sigma_3) \bar{\lambda}^{1/2 \theta_\Phi \sigma_3}$$

as $\bar{\lambda} \to 0$ through the sector $|\arg \bar{\lambda} - \bar{\phi} - j \pi| < \pi$. The connection matrices are defined by $\hat{U}_0(\bar{\lambda}) = \hat{U}_0(\lambda) G_m$ and $\hat{U}_1(\bar{\lambda}) = \hat{U}_1(\lambda) \hat{G}_m$. Note that $\hat{U}_\infty(\bar{\lambda})$ and $\hat{U}_0(\lambda)$ are also expressed as

$$\hat{U}_\infty(\bar{\lambda}) = \hat{U}_\infty(\bar{\lambda}) \sim \hat{U}_\infty(\bar{\lambda} \sim \exp(\frac{1}{2} i \bar{x} \lambda \sigma_3) \lambda^{1/2 \theta_\Phi \sigma_3} e^{i \pi \tau_0 \sigma_3}$$

in the sector $|\arg \lambda + \phi - 2m \pi| < \pi$, and that

$$\hat{U}_0(\bar{\lambda}) \sim \hat{U}_0(\bar{\lambda}) \sim \Delta_0 \exp(-\frac{1}{2} i \bar{x} \lambda^{-1} \sigma_3) \lambda^{1/2 \theta_\Phi \sigma_3} e^{-i \pi \tau_0 \sigma_3}$$

in the sector $|\arg \lambda - \phi| < \pi$. Then we have $\hat{U}_0(\bar{\lambda}) = U_0(\lambda) e^{-\frac{1}{2} i \pi \tau_0 \sigma_3}$ and $\hat{U}_\infty(\bar{\lambda}) = U_\infty(\lambda) e^{\frac{1}{2} i \pi \tau_0 \sigma_3}$. Similarly, $\tilde{U}_1(\bar{\lambda}) = \tilde{U}_1(\lambda) e^{-\frac{1}{2} i \pi \tau_0 \sigma_3}$ and $\tilde{U}_1(\bar{\lambda}) = U_\infty(\lambda) e^{\frac{1}{2} i \pi \tau_0 \sigma_3}$. Then, for $m \geq 1$, we have

$$G_m = e^{\frac{1}{2} i \pi \tau_0 \sigma_3} G \infty \sigma_3 \cdots \sigma_3 \infty e^{\frac{1}{2} i \pi \tau_0 \sigma_3},$$

$$G_m = e^{\frac{1}{2} i \pi \tau_0 \sigma_3} G \infty \sigma_3 \cdots \sigma_3 \infty e^{\frac{1}{2} i \pi \tau_0 \sigma_3}.$$
8. Modulus $A_{\phi}$ and the Boutroux equations

For a given $\phi \in \mathbb{R}$ we seek the modulus $A = A_{\phi} \in \mathbb{C}$ such that for every cycle $c \subset \Pi_\lambda$

$$\text{Im} e^{i\phi} \int_c \frac{w(A, \lambda)}{\lambda^2} d\lambda = 0$$

for $w(A, \lambda)^2 = \lambda^4 - A\lambda^2 + 1$. We note the following.

**Lemma 8.1.** As long as $A \in \mathbb{C} \setminus \{c < -2\}$, the polynomial $\lambda^4 - A\lambda^2 + 1$ has zeros $\lambda_1 = \lambda_2(A)$ and $\lambda_2 = \lambda_2(A)$ continuous in $A$ and satisfying $\lambda_1\lambda_2 = 1$ and $\lambda_1, \lambda_2 \in \{\text{Re } \lambda \geq 0\}$.

**Proof.** For $A$, say, close to 2 there exist such zeros. If $\lambda_1$ or $\lambda_2 \in i\mathbb{R}$, then $A/2 \pm \sqrt{A^2/4 - 1} = -r_0 > 0$, implying $A = -(r_0^2 + 1)/r_0 \leq -2$. \qed

By Lemma 8.1, for each $a \in \mathbb{C}$, there exist zeros $\lambda_1, \lambda_2 \in \{\text{Re } \lambda \geq 0\}$. Define the elliptic curve $\Pi_\lambda = \Pi_+ \cup \Pi_-$ glued along the cuts $[-\lambda_1, -\lambda_2]$ and $[\lambda_1, \lambda_2]$ as in Section 2.2. Let $a$ and $b$ be basic cycles on $\Pi_\lambda$ such that $\frac{1}{2}a = ( -\lambda_1, \lambda_1 )$, $\frac{1}{2}b = ( \lambda_1, \lambda_2 )^\ast$ on the upper sheet $\Pi_+$, where $(\lambda_1, \lambda_2)^\ast$ is located to the left of the cut $[\lambda_1, \lambda_2]$. The cycles thus defined may be supposed to coincide with $a$ and $b$ in Figure 2.1 if $A = A_{\phi}$. For $|\phi| \leq \pi/2$, let us consider the Boutroux equations

$$(\text{BE})_{\phi} : \quad \text{Im} e^{i\phi} I_a(A) = 0, \quad \text{Im} e^{i\phi} I_b(A) = 0,$$

where

$$I_{a,b}(A) := \int_{a,b} \frac{w(A, \lambda)}{\lambda^2} d\lambda = \int_{a,b} \frac{1}{\lambda^2} \sqrt{\lambda^4 - A\lambda^2 + 1} d\lambda.$$

It is easy to see that $w(A, \lambda)^2 = \lambda^4 - A\lambda^2 + 1$ has double roots if and only if

$$A = 2, \quad \lambda_1 = 1, -\lambda_1, -\lambda_2 = -1; \quad A = -2, -\lambda_1, -\lambda_2 = \pm i, -\lambda_1, \lambda_2 = \mp i.$$

**Example 8.1.** We have $I_a(2) = -8$, $I_b(2) = 0$, and $I_a(-2) = 0$, $I_b(-2) = 8i$. Indeed, say,

$$I_a(2) = 2 \int_{-1}^{1} \frac{1 - \lambda^2}{\lambda^2} d\lambda = -8,$$

in which the residue of the integrand at $\lambda = 0$ vanishes. Hence $A_0 = 2$ (respectively, $A_{\pm \pi/2} = -2$) fulfills $(\text{BE})_0$ (respectively, $(\text{BE})_{\pm \pi/2}$).

In accordance with [16, Section 7] we begin with the following:

**Proposition 8.2.** Suppose that $\text{Im} I_a(A) = 0$. Then $A \in \mathbb{R}$.

**Proof.** First we treat the case where $0 \leq \text{Re } \lambda_1 < \text{Re } \lambda_2$, or where $0 \leq \text{Re } \lambda_1 = \text{Re } \lambda_2$ and $|\text{Im } \lambda_1| < |\text{Im } \lambda_2|$. Note that $\frac{1}{2}I_a(A)$ is the integral along the segment joining $-\lambda_1$ to $\lambda_1$ on $\Pi_+$. Let $\Pi_+ = \Pi_+ \cup \Pi_+^\ast$ be the two-sheeted Riemann surface glued along the cuts $[-\lambda_1, \lambda_1]$ and $[-\lambda_2, \lambda_2] = [-\lambda_2, \infty] \cup [\infty, \lambda_2]$, and let $w_\ast(A, \lambda) = \sqrt{\lambda^4 - A\lambda^2 + 1}$ be the function on $\Pi_+^\ast$ such that, around $\lambda = \varepsilon i \in \Pi_+^\ast$ with small $\varepsilon > 0$, $w_\ast(A, \lambda)$ coincides with $w(A, \lambda)$. Then

$$I_a(A) = I_{a,\ast}(A) := \int_{a,\ast} \frac{w_\ast(A, \lambda)}{\lambda^2} d\lambda,$$

where $a, \ast$ is a cycle on $\Pi_+^\ast$ surrounding the cut $[-\lambda_1, \lambda_1]$ in the clockwise direction. Set

$$J_{a,\ast}(A) = \int_{a,\ast} \frac{v(A, \lambda)}{\lambda^2} d\lambda, \quad v(A, \lambda) = \sqrt{-\lambda^4 + A\lambda^2 - 1},$$

with small
and suppose that the cycle $a_*$ surrounds $\pm \lambda_1$ as well. Then the supposition $\text{Im } I_0(A) = 0$ means $J_{a_*}(A) = iI_{a_*}(A) = iI_{a_*}(A) \in i\mathbb{R}$, and

$$0 = J_{a_*}(A) + J_{a_*}(A) = J_{a_*}(A) + J_{\overline{A}}(A) = J_{a_*}(A) - J_{a_*}(A)$$

$$= \int_{a_*} \frac{1}{2\lambda^2} (v(A, \lambda) - v(\overline{A}, \lambda)) d\lambda = (A - \overline{A}) \int_{a_*} \frac{d\lambda}{v(A, \lambda) + v(\overline{A}, \lambda)}.$$  

In this proof, to simplify the description, we write $v(A, \lambda) = v_A(\lambda)$, $v(\overline{A}, \lambda) = v_{\overline{A}}(\lambda)$ and $v(\lambda, \lambda) + v(\overline{A}, \lambda) = (v_A + v_{\overline{A}})(\lambda)$. The polynomial $v_{\overline{A}}(\lambda)^2 = -w(\overline{A}, \lambda)^2$ has the roots $\pm \lambda_1$, $\pm \lambda_2$. The algebraic functions $(v_A + v_{\overline{A}})(z)$ may be considered on $\Pi_*$ with the additional cuts $[-\lambda_1, \lambda_1]$ and $[-\lambda_2, \lambda_2] = [-\lambda_2, \infty] \cup [-\lambda_1, \lambda_1]$, that is, the two-sheeted Riemann surface glued along the cuts $[-\lambda_1, \lambda_1], [-\lambda_1, \lambda_1], [-\lambda_2, \infty] \cup [\infty, \lambda_2]$ and $[-\lambda_2, \infty] \cup [\infty, \lambda_2]$. The cycle $a_*$ may be supposed to enclose both cuts $[-\lambda_1, \lambda_1]$ and $[-\lambda_1, \lambda_1]$. These related cuts, say as in Figure 8.1 (1), are modified into $[-\lambda_1, -\lambda_1] \cup [-\lambda_1, \lambda_1] \cup [-\lambda_1, \lambda_1]$ and then the cycle $a_*$ as in Figure 8.1 (1) may be deformed into a contour consisting of horizontal and vertical lines as in Figure 8.1 (2). To show $A \in \mathbb{R}$ it is sufficient to verify that, under the supposition $A - \overline{A} \neq 0$,

$$J = \int_{a_*} \frac{d\lambda}{(v_A + v_{\overline{A}})(\lambda)} \neq 0.$$  

Let us compute this integral along the contour $a_*$ as in Figure 8.1 (2) with vertices $\alpha \pm i\beta, -\alpha \pm i\beta$, in which $\alpha = \text{Re } \lambda_1$, $\beta = |\text{Im } \lambda_1|$.

![Figure 8.1](image-url)  

**Figure 8.1.** Modification of the cycle $a_*$.  

The integral $J$ is decomposed into three parts: $J = 2J_0 + J_{\text{rver}} + J_{\text{iver}}$ with the real line part

$$J_0 = \int_{-\alpha}^{\alpha} \frac{dt}{(v_A + v_{\overline{A}})(t)};$$

the right vertical part $J_{\text{rver}} = J_{\text{rver}}^+ + J_{\text{rver}}^-$, in which

$$J_{\text{rver}}^+ = \int_{0}^{\beta} \frac{idt}{(v_A + v_{\overline{A}})(\alpha + it)} + \int_{0}^{0} \frac{idt}{(v_A - v_{\overline{A}})(\alpha + it)};$$

$$J_{\text{rver}}^- = \int_{0}^{-\beta} \frac{idt}{(v_A + v_{\overline{A}})(\alpha + it)} + \int_{0}^{0} \frac{idt}{(v_A - v_{\overline{A}})(\alpha + it)};$$
and the left vertical part \( J_{\text{iver}} = J_{\text{iver}}^+ + J_{\text{iver}}^- \), in which

\[
J_{\text{iver}}^+ = \int_0^{-\beta} \frac{it}{-v_A - v_A} \left(-\alpha + it\right) dt + \int_{-\beta}^0 \frac{it}{-v_A + v_A} \left(-\alpha + it\right) dt,
\]

\[
J_{\text{iver}}^- = \int_0^{\beta} \frac{it}{-v_A + v_A} \left(-\alpha + it\right) dt + \int_{\beta}^0 \frac{it}{-v_A - v_A} \left(-\alpha + it\right) dt.
\]

Then we have

\[
J_{\text{iver}} = -\frac{2i}{A - A} \left( \int_0^{\beta} \frac{v_A(\alpha + it)}{\alpha + it^2} dt + \int_{-\beta}^0 \frac{v_A(\alpha - it)}{\alpha - it^2} dt \right) \in \mathbb{R}
\]

and

\[
J_{\text{iver}} = -\frac{2i}{A - A} \left( \int_0^{\beta} \frac{v_A(\alpha - it)}{\alpha + it^2} dt + \int_{-\beta}^0 \frac{v_A(\alpha + it)}{\alpha - it^2} dt \right) \in \mathbb{R},
\]

and hence \( J_{\text{iver}} + J_{\text{iver}} \in \mathbb{R} \). Furthermore, observing

\[
v_A(t) = i\sqrt{\sigma(t) + i\tau(t)}, \quad v_A(t) = i\sqrt{\sigma(t) - i\tau(t)},
\]

\[
\sigma(t) = t^4 - Re A \cdot t^2 + 1, \quad \tau(t) = -\text{Im} A \cdot t^2
\]

and \( v_A(0) = v_A(0) = i \), we have

\[
J_0 = -\frac{i}{\sqrt{2}} \int_{-\alpha}^{\alpha} \frac{dt}{\sqrt{\sigma(t) + \sqrt{\sigma(t)^2 + \tau(t)^2}}},
\]

which implies \( J_0 \in \mathbb{R} \) and \( J_0 \neq 0 \), provided that \( \alpha \neq 0 \). Thus in the case \( \text{Re} \lambda_1 \neq 0 \) it is shown that \( A \in \mathbb{R} \). If \( \text{Re} \lambda_1 = 0 \), then from \( \lambda_1 \lambda_2 = 1 \) it follows that \( A = \lambda_1^2 + \lambda_2^2 \in \mathbb{R} \).

In the case where \( 0 \leq \text{Re} \lambda_2 < \text{Re} \lambda_1 \), or where \( 0 \leq \text{Re} \lambda_2 = \text{Re} \lambda_1 \) and \( |\text{Im} \lambda_2| < |\text{Im} \lambda_1| \), the same argument as above applies. The remaining case \( \lambda_2 = \lambda_1 \) or \( \lambda_2 = \lambda_1 \) occurs only when \( A = \pm 2 \). Thus we have the proposition.

Let us examine \( I_b(A) \) for \( A \in \mathbb{R} \). It is easy to see that, \( w(A, z)^2 \) has real roots

\[
-\lambda_2 < -\lambda_1 < \lambda_1 < \lambda_2 \quad \text{if} \quad A^2 > 4.
\]

Then \( I_b(A) = 2 \int_{\lambda_1, \lambda_2} \lambda^{-2} w(A, \lambda) d\lambda \in i\mathbb{R} \setminus \{0\} \). If \( A = \pm 2 \), then \( \lambda_1 = \lambda_2 = 1 \) or \( \lambda_1 = -\lambda_2 = \pm i \), and \( I_b(2) = 0 \) and \( I_b(-2) = 8i \) as in Example 8.1.

Suppose that \( 0 \leq A^2 < 4 \). The roots of \( w(A, z)^2 \) are \( \alpha \pm i\beta, -\alpha \pm i\beta \) with \( \alpha, \beta \geq 0 \). Then \( b \) is a cycle enclosing the cut \( [\alpha - i\beta, \alpha + i\beta] \). Since \( I_b(A) = -I_b(A) \), we have \( I_b(A) \in i\mathbb{R} \). Since \( w(A, \alpha \pm i\beta) = 0 \), the integral

\[
I_b(A) = 2i \int_{-\beta}^{\beta} \frac{w(A, \alpha + it)}{(\alpha + it)^2} dt = 4i \int_0^{\beta} \text{Re} \frac{w(A, \alpha + it)}{(\alpha + it)^2} dt
\]

satisfies, for \(-2 < A < 2\),

\[
\frac{\partial}{\partial A} \left( \frac{1}{i} I_b(A) \right) = 2 \int_0^\beta \text{Re} w(A, \alpha + it)^{-1} dt = \sqrt{2} \int_0^\beta \frac{\sqrt{g + \sqrt{g^2 + h^2}}}{\sqrt{g^2 + h^2}} dt > 0,
\]

where

\[
g = g(t) = \text{Re} w(A, \alpha + it)^2 = t^4 - 6A^2t^2 + A\alpha^4 - A^2t^2 + 1,
\]

\[
h = h(t) = \text{Im} w(A, \alpha + it)^2 = -4A\alpha t^3 + 4A\alpha^3t - 2AAt.
\]

This implies \( I_b(A) \in i\mathbb{R} \setminus \{0\} \) for \(-2 < A < 2\).

The fact above combined with Proposition 8.2 and Example 8.1 implies the following.
Proposition 8.3. If \( \phi = 0 \), then the Boutroux equations (BE)\(_0\) admit a unique solution \( A_0 = 2 \).

Corollary 8.4. For every \( A \in \mathbb{C} \), \((I_A(A), I_{ib}(A)) \neq (0, 0)\).

Proposition 8.5. Suppose that, for \( A_\phi \) solving (BE)\(_\phi\) with \( 0 < |\phi| \leq \pi/2 \), the elliptic curve \( \Pi_{A_\phi} \) degenerates. Then \( \phi = \pm \pi/2 \) and \( A_{\pm \pi/2} = -2 \).

Proof. When \( \Pi_{A_\phi} \) degenerates, \( A_\phi = \pm 2 \). Suppose that \( A_\phi = -2 \). Then the roots of \( w(A_\phi, \lambda)^2 \) are \( \lambda_1 = -\lambda_2 = \pm i \), \( -\lambda_1 = \lambda_2 = \mp i \), and

\[
\Re \ni e^{i\phi} \int_{-\frac{\lambda}{\lambda}}^{\frac{\lambda}{\lambda}} \frac{1}{\lambda^2} \sqrt{\lambda^4 - A_\phi \lambda^2 + 1} \, d\lambda = e^{i(\phi - \pi/2)} \int_{-\frac{\pi}{\lambda}}^{\frac{\pi}{\lambda}} \frac{1}{\zeta^2} \sqrt{\zeta^4 + 2e^{\pi i} \zeta^2 + 1} \, d\zeta \neq 0,
\]

which implies \( \phi = \pm \pi/2 \). Similarly, if \( A_\phi = 2 \), then \( \phi = 0 \).

Proposition 8.6. If \( \phi = \pm \pi/2 \), then the Boutroux equations (BE)\(_{\pm \pi/2}\) admit a unique solution \( A_{\pm \pi/2} = -2 \).

Proof. For \( \phi = \pi/2 \), (BE)\(_{\pi/2}\) are equivalent to

\[
e^{\pi i/2} \int_{c}^{1} \frac{1}{\lambda^2} \sqrt{\lambda^4 - A_{\pi/2} \lambda^2 + 1} \, d\lambda \in \Re
\]

for every cycle \( c \) on \( \Pi_{A_{\pi/2}} \), which is written as (BE)\(_0\) with \( \phi = 0 \)

\[
\int_{e^{-\pi i/2}}^{1} \frac{1}{\zeta^2} \sqrt{\zeta^4 - e^{\pi i} A_{\pi/2} \zeta^2 + 1} \, d\zeta \in \Re \quad (\lambda = e^{\pi i/2} \zeta).
\]

Then by Proposition 8.3, \( e^{\pi i} A_{\pi/2} = 2 \), i.e. \( A_{\pi/2} = -2 \) is a unique solution of (BE)\(_{\pi/2}\).

The quotient \( h(A) = I_{ib}(A)/I_{a}(A) \) [24 Appendix I] is useful in examining \( A_\phi \).

Proposition 8.7. Suppose that \( A \in \mathbb{C} \).

1. If \( A \) solves (BE)\(_\phi\) for some \( \phi \in \Re \) and \( I_{a}(A) \neq 0 \), then \( h(A) \in \Re \).
2. If \( h(A) \in \Re \setminus \{0\} \), then, for some \( \phi \in \Re \), \( A \) solves (BE)\(_\phi\).

Proof. Suppose that \( h(A) = \rho \in \Re \setminus \{0\} \), and write \( I_{a}(A) = u + iv, I_{ib}(A) = U + iV \) with \( u, v, U, V \in \Re \). Then \( U = pu, V = pv \), and hence \( v/u = V/U = -\tan \phi \) for some \( \phi \in [-\pi/2, \pi/2] \). This implies \( \Im e^{i\phi} I_{ib}(A) = \Im e^{i\phi} I_{a}(A) = 0 \).

Proposition 8.8. The set \( \{A \in \mathbb{C} \mid A \) solves (BE)\(_\phi\) for some \( \phi \in \Re \} \) is bounded.

Proof. Note that \( w(A, z) \) admits the roots \( \lambda_{\pm 1} \sim \pm A^{-1/2}, \lambda_{\pm 2} \sim \pm A^{1/2} \), if \( A \) is large. Then

\[
\int_{\lambda_{1}}^{\lambda_{2}} \frac{w(A, \lambda)}{\lambda^2} \, d\lambda \sim \int_{A_{-1/2}}^{A_{1/2}} \frac{1}{\lambda^2} \sqrt{\lambda^4 - A \lambda^2 + 1} \, d\lambda \sim i A_{1/2} \int_{A_{-1}}^{A_{1}} \frac{1}{t} \sqrt{1 - t^2} \, dt \sim i A^{1/2} \log A,
\]

and

\[
\int_{\lambda_{1}}^{\lambda_{2}} \frac{w(A, \lambda)}{\lambda^2} \, d\lambda \sim \int_{A_{-1/2}}^{A_{1/2}} \frac{1}{\lambda^2} \sqrt{\lambda^4 - A \lambda^2 + 1} \, d\lambda \sim -A^{1/2} \int_{-1}^{1} \frac{1}{t^2} \sqrt{1 - t^2} \, dt \sim \pi A^{1/2}.
\]

This implies \( h(A) \notin \Re \) if \( A \) is sufficiently large, which completes the proof.

The following fact is used in discussing solutions of (BE)\(_\phi\).

Let \( 0 < |\phi| < \pi/2 \), and write

\[
I_{a}(A) = u(A) + iv(A), \quad I_{ib}(A) = U(A) + iV(A).
\]
Note that $A$ solves $(\text{BE})_\phi$ if and only if
$$\text{Im}e^{i\phi}I_a(A) = u(A) \sin \phi + v(A) \cos \phi = 0, \quad \text{Im}e^{i\phi}I_b(A) = U(A) \sin \phi + V(A) \cos \phi = 0,$$
that is,

$$(8.1) \quad u(A) \tan \phi + v(A) = 0, \quad U(A) \tan \phi + V(A) = 0.$$  

Then, by the Cauchy-Riemann equations the Jacobian for (8.1) with $A = x + iy$ is written as

$$(8.2) \quad \det J(\phi, A) = \det \begin{pmatrix} u_x \tan \phi + v_x & u_y \tan \phi + v_y \\ U_x \tan \phi + V_x & U_y \tan \phi + V_y \end{pmatrix}$$
$$= (1 + \tan^2 \phi)(v_x V_y - v_y V_x)$$
$$= -\frac{1}{4}(1 + \tan^2 \phi)|\Omega_a(A)|^2 \text{Im} \frac{\Omega_b(A)}{\Omega_a(A)}.$$

where $\Omega_a(A)$ and $\Omega_b(A)$ are periods of the elliptic curve $w(A, z)$. For $0 < |\phi| < \pi/2$ the derivatives of (8.1) with $t = \tan \phi$ are written as

$$J(\phi, A) \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} + \begin{pmatrix} u(A) \\ U(A) \end{pmatrix} \equiv 0.$$  

Then we have
$$\quad (x'(t), y'(t)) \neq (0, 0) \quad \text{and} \quad (d/d\phi)A = (x'(t) + iy'(t)) \cos^2 \phi \neq 0$$
for $0 < |\phi| < \pi/2$.

**Proposition 8.9.** Suppose that, for some $\phi_0$ such that $0 < |\phi_0| < \pi/2$, $A_{\phi_0}$ solves $(\text{BE})_{\phi_0}$. Then there exists a trajectory $T_0 : A = \chi(\phi_0, \phi)$ for $0 < |\phi| < \pi/2$ with the properties:

1. $\chi(\phi_0, 0) = A_{\phi_0};$
2. for each $\phi$, $A = \chi(\phi_0, \phi)$ solves $(\text{BE})_{\phi};$
3. $\chi(\phi_0, \phi)$ is smooth for $0 < |\phi| < \pi/2$.

**Proof.** Since the Jacobian (8.2) satisfies $\det J(\phi_0, A_{\phi_0}) \in \mathbb{R} \setminus \{0\}$, there exists a local trajectory $A = \chi_{\text{loc}}(\phi_0, \phi)$ having the properties (1), (2) and (3) above for $|\phi - \phi_0| < \delta$, where $\delta$ is sufficiently small. Since (8.2) does not vanish for $0 < |\phi| < \pi/2$, $\chi_{\text{loc}}(\phi_0, \phi)$ may be extended to the interval $0 < |\phi| < \pi/2$. \hfill $\square$

**Proposition 8.10.** The trajectory $T_0 : A = \chi(\phi_0, \phi)$ given above may be extended to $|\phi| \leq \pi/2$ in such a way that $\chi(\phi_0, \phi)$ is continuous in $\phi$ and that $\chi(\phi_0, 0) = A_0 = 2$, $\chi(\phi_0, \pm \pi/2) = A_{\pm \pi/2} = -2$.

**Proof.** By Proposition 8.8 the trajectory $T_0$ for $0 < |\phi| < \pi/2$ is bounded. To show that $\chi(\phi_0, \phi) \to A_0$ as $\phi \to 0$, suppose to the contrary that there exists a sequence $\{\phi_0, n\}$ such that $\phi_0 \to 0$ and that $\chi(\phi_0, \phi_0)$ does not converge to $A_0$. By the boundedness of $T_0$ there exists a subsequence $\{\phi_{\nu(n)}\}$ such that $\chi(\phi_{\nu(n)}, \phi_{\nu(n)}) \to A_0'$ for some $A_0' \neq A_0$. Then we have $\text{Im} I_a(A_0') = \text{Im} I_b(A_0') = 0$, which contradicts the uniqueness of a solution of $(\text{BE})_0$. Hence $\chi(\phi_0, \phi)$ is extended to $\phi = 0$ and is continuous in a neighbourhood of $\phi = 0$. By Proposition 8.6 it is possible to extend $\chi(\phi_0, \phi)$ to $\phi = \pm \pi/2$ by the same argument. \hfill $\square$

**Lemma 8.11.** $h'(A) = 4\pi i I_a(A)^{-2}$.

**Proof.** From $I_{a, b}'(A) = -\Omega_{a, b}/2$ and Lemma 6.7 the conclusion follows. \hfill $\square$
Corollary 8.12. If $I_n(A) \neq 0, \infty$, then $h(A)$ is conformal around $A$.

By Example 8.1, $h(A)$ is conformal at $A_0 = 2$ and $h(A_0) = 0$. Then, by Lemma 8.11,

$$h(A) = h'(A_0)(A - A_0) + o(A - A_0) = \frac{\pi i}{16}(A - A_0) + o(A - A_0)$$

around $A = A_0$. By Proposition 8.7, for a sufficiently small $\varepsilon > 0$, the inverse image of $(-\varepsilon, 0) \cup (0, \varepsilon)$ under $h(A)$ is a trajectory $T_{0-} \cup T_{0+}$: $A = \chi_0^+(\phi)$ solving (BE)$_0$, and is expressed as

$$(8.4) \quad \chi_0^+(\phi) = A_0 + \gamma_0(\phi)i + o(\gamma_0(\phi)),$$

near $\phi = 0$, where $\gamma_0(\phi) \in \mathbb{R}$ is continuous in $\phi$ and $\gamma_0(0) = 0$.

The fact above implies that there exists a local trajectory close to $A_0$ solving (BE)$_0$ for $\phi$ near $0$. From this with Proposition 8.9, a trajectory for $|\phi| \leq \pi/2$ as in Proposition 8.10 may be obtained. Furthermore, if two trajectories $\chi_1(\phi)$ and $\chi_2(\phi)$ solving (BE)$_0$ satisfy $\chi_1(\phi_0) = \chi_2(\phi_0)$ for some $\phi_0$ such that $0 < |\phi_0| < \pi/2$, then $\chi_0(\phi)$ or the conformality of $h(A)$ at $A = A_0$ implies $\chi_1(\phi) \equiv \chi_2(\phi)$. Thus we have the following.

Proposition 8.13. There exists a trajectory $A = A_0$ for $|\phi| \leq \pi/2$ with the properties:

1. for each $\phi$, $A_0$ is a unique solution of (BE)$_0$;
2. $A_0$ is smooth in $\phi$ for $0 < |\phi| < \pi/2$ and continuous in $\phi$ for $|\phi| \leq \pi/2$.

For any cycle $c$, it is easy to see that

$$e^{i\phi} \int_c \frac{1}{\lambda^2} w(A_0, \lambda) d\lambda = e^{i(\phi+\pi)} \int_{c+\pi i c} \frac{1}{\lambda^2} w(A_0, \lambda) d\lambda,$$

$$e^{i\phi} \int_c \frac{1}{\lambda^2} w(A_0, \lambda) d\lambda = e^{i(\phi+\pi/2)} \int_{c+\pi/2 i c} \frac{1}{\lambda^2} w(-A_0, \lambda) d\lambda,$$

which yields the following.

Proposition 8.14. The function $A_0$ with $|\phi| \leq \pi/2$ may be extended to $\phi \in \mathbb{R}$ by setting $A_{\phi+\pi} = A_0$, and solves the Boutroux equations (BE)$_0$. Furthermore $A_{\phi+\pi/2} = -A_0$, $A_{-\phi} = \overline{A_0}$.

Let us examine the properties of $A_0$ in more detail. Note that the trajectory $A = A_0 = x + iy$ with $|\phi| < \pi/2$ satisfies $h(A_0) \in \mathbb{R}$. Then by Lemma 8.11 and 8.3,

$$\frac{d}{dt} h(A_0) = (x'(t) + iy'(t))I_n(A_0)^{-2} \in \mathbb{R} \setminus \{0\}$$

with $t = \tan \phi$ for $0 < |\phi| < \pi/2$. Setting $I_n(A_0)^{-1} = P + iQ$, we have

$$\frac{1}{4\pi} \text{Im} \frac{d}{dt} h(A_0) = x'(t)(P^2 - Q^2) - 2y'(t)PQ = 0.$$

If $x'(t_0) = 0$ for some $t_0 = \tan(\phi_0) \neq 0, \pm \infty$, then $PQ = \text{Re} I_n(A_0)^{-1} \cdot \text{Im} I_n(A_0)^{-1} = 0$, and hence $I_n(A_{\phi_0}) \in i\mathbb{R} \setminus \{0\}$ or $\mathbb{R} \setminus \{0\}$. This is impossible for $0 < |\phi| < \pi/2$, which implies $x'(t) \neq 0$ for $0 < |\phi| < \pi/2$. Since $A_{\pm \pi/2} = -2$, we have $x'(t) < 0$ for $0 < \phi < \pi/2$ and $x'(t) > 0$ for $-\pi/2 < \phi < 0$. It is easy to see that $x'(0) = x'(\pm \pi/2) = 0$. If $y'(t_0) = 0$ for some $t_0$ with $0 < |\phi_0| < \pi/2$, then $P^2 - Q^2 = 0$, i.e. $I_n(A_{\phi_0})^{-1} = P(1 \pm i)$, implying $\phi_0 = \pm \pi/4$. Note that $PQ < 0$, $|P| > |Q|$ for $-\pi/4 < \phi < 0$ and that $PQ > 0$, $|P| > |Q|$ for $0 < \phi < \pi/4$. It follows that $y'(t) < 0$ for $0 < |\phi| < \pi/4$.
**Proposition 8.15.** The trajectory \( A_\phi = x(t) + iy(t) \) with \( t = \tan \phi \) has the properties:

1. \( x'(t) > 0 \) for \(-\pi/2 < \phi < 0\), and \( x'(t) < 0 \) for \( 0 < \phi < \pi/2\);
2. \( x'(0) = x'(\tan(\pm \pi/2)) = 0\);
3. \( y'(t) < 0 \) for \( 0 < |\phi| < \pi/4 \) and \( y'(\tan(\pm \pi/4)) = 0 \).

Thus we have

**Proposition 8.16.** For \( \phi \in \mathbb{R} \) there exists a trajectory \( A = A_\phi \) with the properties:

1. for each \( \phi \), \( A_\phi \) is a unique solution of (BE)\(_\phi\);
2. \( A_{\phi + \pi} = -A_\phi \), \( A_{\phi + \pi} = A_\phi \), \( A_{-\phi} = A_{\bar{\phi}} \);
3. \( A_0 = 2 \), \( A_{\pi/2} = -2 \);
4. \( A_\phi \) is continuous in \( \phi \in \mathbb{R} \), and smooth in \( \phi \in \mathbb{R} \setminus \{m\pi/2 \mid m \in \mathbb{Z}\} \).

The trajectory of \( A_\phi \) is roughly drawn in Figure 8.2.

![Figure 8.2. Trajectory of A_\phi for |\phi| ≤ \pi](image)

By Proposition 8.15 when \(|\phi| \) is sufficiently small, the location of the turning points may be examined. Small variance of \( A_\phi \) around \( \phi = 0 \) is given by \( A_\phi = 2 + \delta_\phi \) with \( \delta_\phi \) having the properties: (1) \( \delta_\phi \to 0 \) as \( \phi \to 0 \); (2) \( \text{Re} \delta_\phi \leq 0 \); (3) \( \text{Im} \delta_\phi \geq 0 \) if \( \phi \leq 0 \) and \( \text{Im} \delta_\phi \leq 0 \) if \( \phi \geq 0 \). Let the turning points \( \lambda_1 \), \( \lambda_2 \) with \( \text{Re} \lambda_1 \leq \text{Re} \lambda_2 \) be such that \( \lambda_1 = \sqrt{1 - \rho_1}, \lambda_2 = \sqrt{1 + \rho_2} \). Note that \( \lambda_1^2 \lambda_2^2 = 1 \) and \( \lambda_1^2 + \lambda_2^2 = 2 + \delta_\phi \), which yield \( \rho_2 - \rho_1 = \delta_\phi, \rho_1 \rho_2 = \delta_\phi \). Since \( \delta_\phi \) is small, \( \rho_1 \sim \delta_\phi^{1/2} - \frac{1}{2} \delta_\phi, \rho_2 \sim \delta_\phi^{1/2} + \frac{1}{2} \delta_\phi \). Thus we have the following.

**Proposition 8.17.** If \(|\phi| \) is sufficiently small, the turning points \( \lambda_1 \) and \( \lambda_2 \) are represented as

\[
\lambda_1 = 1 - \varepsilon_\phi e^{i\theta_\phi} + O(\varepsilon_\phi^2), \quad \lambda_2 = 1 + \varepsilon_\phi e^{i\theta_\phi} + O(\varepsilon_\phi^2),
\]

where \( \varepsilon_\phi \) and \( \theta_\phi \) fulfil

1. \( \varepsilon_\phi > 0 \) and \( \varepsilon_\phi \to 0 \) as \( \phi \to 0 \); and
2. \( \theta_\phi \to \pi/4 \) as \( \phi \to 0 \) with \( \phi < 0 \), and \( \theta_\phi \to -\pi/4 \) as \( \phi \to 0 \) with \( \phi > 0 \).

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