A Stress Tensor For Anti-de Sitter Gravity

Vijay Balasubramanian\textsuperscript{1,2}\footnote{vijayb@pauli.harvard.edu} and Per Kraus\textsuperscript{3}\footnote{pkraus@theory.uchicago.edu}

\textsuperscript{1}Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138, USA

\textsuperscript{2}Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106, USA

\textsuperscript{3}Enrico Fermi Institute, University of Chicago, Chicago, IL 60637, USA

Abstract

We propose a procedure for computing the boundary stress tensor associated with a gravitating system in asymptotically anti-de Sitter space. Our definition is free of ambiguities encountered by previous attempts, and correctly reproduces the masses and angular momenta of various spacetimes. Via the AdS/CFT correspondence, our classical result is interpretable as the expectation value of the stress tensor in a quantum conformal field theory. We demonstrate that the conformal anomalies in two and four dimensions are recovered. The two dimensional stress tensor transforms with a Schwarzian derivative and the expected central charge. We also find a nonzero ground state energy for global AdS$_5$, and show that it exactly matches the Casimir energy of the dual $\mathcal{N} = 4$ super Yang-Mills theory on $S^3 \times R$.

1 Introduction

In a generally covariant theory it is unnatural to assign a local energy-momentum density to the gravitational field. For instance, candidate expressions depending only on the metric and its first derivatives will always vanish at a given point in locally flat coordinates. Instead, we can consider a so-called “quasilocal stress tensor”, defined...
locally on the boundary of a given spacetime region. Consider the gravitational action thought of as a functional of the boundary metric $\gamma_{\mu\nu}$. The quasilocal stress tensor associated with a spacetime region has been defined by Brown and York to be \[^1\]:

$$T^{\mu\nu} = \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{\text{grav}}}{\delta \gamma^{\mu\nu}}$$ \hfill (1)

The resulting stress tensor typically diverges as the boundary is taken to infinity. However, one is always free to add a boundary term to the action without disturbing the bulk equations of motion. To obtain a finite stress tensor, Brown and York propose a subtraction derived by embedding a boundary with the same intrinsic metric $\gamma_{\mu\nu}$ in some reference spacetime, such as flat space. This prescription suffers from an important drawback: it is not possible to embed a boundary with an arbitrary intrinsic metric in the reference spacetime. Therefore, the Brown-York procedure is generally not well defined.

For asymptotically anti-de Sitter (AdS) spacetimes, there is an attractive resolution to this difficulty. A duality has been proposed which equates the gravitational action of the bulk viewed as a functional of boundary data, with the quantum effective action of a conformal field theory (CFT) defined on the AdS boundary \[^2, 3, 4\]. According to this correspondence, (1) can be interpreted as giving the expectation value of the stress tensor in the CFT:

$$\langle T^{\mu\nu} \rangle = \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{\text{eff}}}{\delta \gamma^{\mu\nu}}.$$ \hfill (2)

The divergences which appear as the boundary is moved to infinity are then simply the standard ultraviolet divergences of quantum field theory, and may be removed by adding local counterterms to the action. These subtractions depend only on the intrinsic geometry of the boundary and are defined once and for all, in contrast to the ambiguous prescription involving embedding the boundary in a reference spacetime. This interpretation of divergences was first discussed in \[^4\], and has been applied to various computations in, e.g., \[^8, 9, 10, 11\].

Inspired by the proposed correspondence, we develop a new procedure for defining the stress tensor of asymptotically locally anti-de Sitter spacetimes. We renormalize the stress-energy of gravity by adding a finite series in boundary curvature invariants to the action. The required terms are fixed essentially uniquely by requiring finiteness of the stress tensor. We then show that we correctly reproduce the masses and angular momenta of various asymptotically AdS spacetimes See, e.g., \[^12, 13, 14, 15, 16, 17\] for previous studies of energy in AdS.

\[^1\]See \[^5, 6, 7\] for some interesting examples.
According to (2), our definition should also exhibit the properties of a stress tensor in a quantum CFT. The boundary stress tensor of AdS3 is expected to transform under diffeomorphisms as a tensor plus a Schwarzian derivative. We verify this transformation rule, and so derive the existence of a Virasoro algebra with central charge $c = 3\ell/2G$, in agreement with the result of Brown and Henneaux [15]. We also demonstrate that the stress tensor acquires the correct trace anomaly $T^\mu_\mu = -\frac{c}{24\pi}R$.

The candidate dual to AdS5 gravity is four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory. Our procedure for computing the spacetime stress tensor (1) reproduces the expected trace anomaly of the gauge theory. An interesting — and at first surprising — feature of our stress tensor is that it is generally non-vanishing even when the bulk geometry is exactly AdS. In particular, global AdS5, with an $S^3 \times R$ boundary, has a positive mass. In contrast, the reference spacetime approach, by construction, gives pure AdS a vanishing mass. Our result is beautifully explained via the proposed duality with a boundary CFT. The dual super Yang-Mills theory on a sphere has a Casimir energy that precisely matches our computed spacetime mass.

We conclude by discussing prospects for defining an analogous quasilocal stress tensor in asymptotically flat spacetimes.

## 2 Defining The Stress Tensor

Brown and York’s definition of the quasilocal stress tensor is motivated by Hamilton-Jacobi theory [1]. The energy of a point particle is the variation of the action with respect to time: $E = -\partial S/\partial t$. In gravity, lengths are measured by the metric, so time is naturally replaced by the boundary metric $\gamma_{\mu\nu}$, yielding a full stress tensor $T^{\mu\nu}$:

$$T^{\mu\nu} = \frac{2}{\sqrt{-\gamma}} \frac{\delta S}{\delta \gamma_{\mu\nu}}.$$  \hspace{1cm} (3)

Here $S = S_{\text{grav}}(\gamma_{\mu\nu})$ is the gravitational action viewed as a functional of $\gamma_{\mu\nu}$. Of course, this is also the standard formula for the stress tensor of a field theory with action $S$ defined on a surface with metric $\gamma_{\mu\nu}$.

The gravitational action with cosmological constant $\Lambda = -d(d-1)/2\ell^2$ is:

$$S = -\frac{1}{16\pi G} \int_M d^{d+1}x \sqrt{g} \left( R - \frac{d(d-1)}{\ell^2} \right) - \frac{1}{8\pi G} \int_{\partial M} d^d x \sqrt{-\gamma} \Theta + \frac{1}{8\pi G} S_{\text{ct}}(\gamma_{\mu\nu}).$$  \hspace{1cm} (4)

The second term is required for a well defined variational principle (see, e.g., [20]), and $S_{\text{ct}}$ is the counterterm action that we will add in order to obtain a finite stress tensor. $\Theta$ is the trace of the extrinsic curvature of the boundary, and is defined below.

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\footnotetext{2}{Our conventions are those of [19]. Comparison with other references and certain symbolic manipulation packages may require a flip in the sign of the Riemann tensor.}
Consider foliating the $d + 1$ dimensional spacetime $M$ by a series of $d$ dimensional timelike surfaces homeomorphic to the boundary $\partial M$. We let $x^\mu$ be coordinates spanning a given timelike surface, and let $r$ be the remaining coordinate. It is convenient to write the spacetime metric in an ADM-like decomposition \[20\]:

$$ds^2 = N^2 dr^2 + \gamma_{\mu\nu}(dx^\mu + N^\mu dr)(dx^\nu + N^\nu dr).$$

Here $\gamma_{\mu\nu}$ is a function of all the coordinates, including $r$. We will refer to the surface at fixed $r$ as the boundary $\partial M_r$ to the interior region $M_r$. The metric on $\partial M_r$ is $\gamma_{\mu\nu}$ evaluated at the boundary value of $r$. In AdS, the boundary metric acquires an infinite Weyl factor as we take $r$ to infinity. So we will more properly think of the AdS boundary as a conformal class of boundaries (see, e.g., \[4\]).

To compute the quasilocal stress tensor for the region $M_r$ we need to know the variation of the gravitational action with respect to the boundary metric $\gamma_{\mu\nu}$. In general, varying the action produces a bulk term proportional to the equations of motion plus a boundary term. Since we will always consider solutions to the equations of motion, only the boundary term contributes:

$$\delta S = \int_{\partial M_r} d^d x \pi^{\mu\nu} \delta \gamma_{\mu\nu} + \frac{1}{8\pi G} \int_{\partial M_r} d^d x \frac{\delta S_{ct}}{\delta \gamma_{\mu\nu}} \delta \gamma_{\mu\nu},$$

where $\pi^{\mu\nu}$ is the momentum conjugate to $\gamma_{\mu\nu}$ evaluated at the boundary:

$$\pi^{\mu\nu} = \frac{1}{16\pi G} \sqrt{-\gamma}(\Theta^{\mu\nu} - \Theta^{\mu\nu}).$$

Here the extrinsic curvature is

$$\Theta^{\mu\nu} = -\frac{1}{2}(\nabla^\mu \hat{n}\nu + \nabla^\nu \hat{n}\mu),$$

where $\hat{n}\nu$ is the outward pointing normal vector to the boundary $\partial M_r$. The quasilocal stress tensor is thus

$$T^{\mu\nu} = \frac{1}{8\pi G} \left[ \Theta^{\mu\nu} - \Theta^{\mu\nu} + \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{ct}}{\delta \gamma_{\mu\nu}} \right].$$

$S_{ct}$ must be chosen to cancel divergences that arise as $\partial M_r$ tends to the AdS boundary $\partial M$. In this limit we expect to reproduce standard computations of the mass of asymptotically AdS spacetimes \[12, 15, 13, 16, 17\]. Brown and York propose to embed $\partial M_r$ in a pure AdS background and to let $S_{ct}$ be the action of the resulting spacetime region. A similar reference spacetime approach is taken by the authors of \[15, 16, 17\]. However, as noted by all these authors, it is not always possible to find

\[3\] See \[1\] for a detailed development of the formalism.
such an embedding, and so the prescription is not generally well-defined. A reference spacetime is also implicitly present in the treatment of Abbott and Deser [12] which constructs a Noether current for fluctuations around pure AdS. Finally, Ashtekar and Magnon [13] exploit the conformal structure of asymptotically AdS spaces to directly compute finite conserved charges. It would be interesting to understand the relation of our work to their approach.

We propose an alternative procedure: take $S_{ct}$ to be a local functional of the intrinsic geometry of the boundary, chosen to cancel the $\partial \mathcal{M}_r \rightarrow \partial \mathcal{M}$ divergences in (9). Here we set $S_{ct} = \int_{\partial \mathcal{M}_r} L_{ct}$, and state our results for $\text{AdS}_3$, $\text{AdS}_4$, and $\text{AdS}_5$:

\begin{align*}
\text{AdS}_3 : \quad L_{ct} &= -\frac{1}{\ell} \sqrt{-\gamma} \\
\Rightarrow \quad T^{\mu\nu} &= \frac{1}{8\pi G} \left[ \Theta^{\mu\nu} - \Theta \gamma^{\mu\nu} - \frac{1}{\ell} \gamma^{\mu\nu} \right] \\
\text{AdS}_4 : \quad L_{ct} &= -\frac{2}{\ell} \sqrt{-\gamma} \left( 1 - \frac{\ell^2}{4} R \right) \\
\Rightarrow \quad T^{\mu\nu} &= \frac{1}{8\pi G} \left[ \Theta^{\mu\nu} - \Theta \gamma^{\mu\nu} - \frac{2}{\ell} \gamma^{\mu\nu} - \ell G^{\mu\nu} \right] \\
\text{AdS}_5 : \quad L_{ct} &= -\frac{3}{\ell} \sqrt{-\gamma} \left( 1 - \frac{\ell^2}{12} R \right) \\
\Rightarrow \quad T^{\mu\nu} &= \frac{1}{8\pi G} \left[ \Theta^{\mu\nu} - \Theta \gamma^{\mu\nu} - \frac{3}{\ell} \gamma^{\mu\nu} - \frac{\ell}{2} G^{\mu\nu} \right]
\end{align*}

(10)

All tensors above refer to the boundary metric $\gamma_{\mu\nu}$, and $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R \gamma_{\mu\nu}$ is the Einstein tensor of $\gamma_{\mu\nu}$.

As we will see, the terms appearing in $S_{ct}$ are fixed essentially uniquely by requiring cancellation of divergences. The number of counterterms required grows with the dimension of AdS space. In general, we are also free to add terms of higher mass dimension to the counterterm action for $\text{AdS}_{d+1}$. But when $d$ is odd, dimensional analysis shows that these terms make no contribution to $T^{\mu\nu}$ as the boundary is taken to infinity. For $d$ even there is one potential ambiguity which we will explain and exorcise in later sections. The addition of $S_{ct}$ does not affect the bulk equations of motion or the Gibbons-Hawking black hole entropy calculations because the new terms are intrinsic invariants of the boundary.

After adding the counterterms (10), the stress tensor (9) has a well defined limit as $\partial \mathcal{M}_r \rightarrow \partial \mathcal{M}$. (More precisely, dimensional analysis determines the scaling of the stress tensor with the diverging Weyl factor of the boundary metric. However, observables like mass and angular momentum will be $r$ independent.) To assign a mass to an asymptotically AdS geometry, choose a spacelike surface $\Sigma$ in $\partial \mathcal{M}$ with metric $\sigma_{ab}$, and write the boundary metric in ADM form:

$$
\gamma_{\mu\nu} dx^{\mu} dx^{\nu} = -N_{\Sigma}^2 dt^2 + \sigma_{ab}(dx^{a} + N_{\Sigma}^{a}dt)(dx^{b} + N_{\Sigma}^{b}dt).
$$

(11)

Then let $u^{\mu}$ be the timelike unit normal to $\Sigma$. $u^{\mu}$ defines the local flow of time in $\partial \mathcal{M}$. If $\xi^{\mu}$ is a Killing vector generating an isometry of the boundary geometry, there
should be an associated conserved charge. Following Brown and York [1], this charge is:

\[ Q_\xi = \int_\Sigma d^{d-1}x \sqrt{\sigma} (u^\mu T_{\mu \nu} \xi^\nu) \] (12)

The conserved charge associated with time translation is then the mass of spacetime. Alternatively, we can define a proper energy density

\[ \epsilon = u^\mu u'^\nu T_{\mu \nu}. \] (13)

To convert to mass, multiply by the lapse \( N_\Sigma \) appearing in (11) and integrate:

\[ M = \int_\Sigma d^{d-1}x \sqrt{\sigma} N_\Sigma \epsilon. \] (14)

This definition of mass coincides with the conserved quantity in (12) when the timelike Killing vector is \( \xi^\mu = N_\Sigma u^\mu \). Similarly, we can define a momentum

\[ P_a = \int_\Sigma d^{d-1}x \sqrt{\sigma} j_a, \] (15)

where

\[ j_a = \sigma_{ab} u^\mu T^{a\mu}. \] (16)

When \( a \) is an angular direction, \( P_a \) is the corresponding angular momentum.

Although we have only written the gravitational action in (4), our formulae are equally valid in the presence of matter. In particular, (14) and (13) give the total mass and momentum of the entire matter plus gravity system.

3 AdS3

We begin with the relatively simple case of AdS3. We will show that our prescription correctly computes the mass and angular momentum of BTZ black holes, and reproduces the transformation law and conformal anomaly of the stress tensor in the dual CFT.

The Poincaré patch of AdS3 can be written as:

\[ ds^2 = \frac{\ell^2}{r^2} dr^2 + \frac{r^2}{\ell^2} (-dt^2 + dx^2). \] (17)

A boundary at fixed \( r \) is conformal to \( R^{1,1} \): \( -\gamma_{tt} = \gamma_{xx} = r^2/\ell^2 \). The normal vector to surfaces of constant \( r \) is

\[ \hat{n}^\mu = \frac{r}{\ell} \delta^{\mu r}. \] (18)

\[ ^4 \text{See, e.g., [22] for the embedding of the Poincaré patch in global AdS3.} \]
Applying (9) we find

\[8\pi G T_{tt} = -\frac{r^2}{\ell^3} + \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{ct}}{\delta \gamma^{tt}}\]
\[8\pi G T_{xx} = \frac{r^2}{\ell} + \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{ct}}{\delta \gamma^{xx}}\]
\[8\pi G T_{tx} = \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{ct}}{\delta \gamma^{tx}}.\]  

(19)

Neglecting \(S_{ct}\), one would obtain divergent results for physical observables such as the mass

\[M = \int dx \sqrt{|g_{xx}|} u^u u^t T_{tt} = \int dx T_{tt} \sim r^2 \rightarrow \infty.\]  

(20)

So \(T_{tt}\) must be independent of \(r\) for large \(r\) in order for the spacetime to have a finite mass density.

\(S_{ct}\) is defined essentially uniquely by the requirement that it be a local, covariant function of the intrinsic geometry of the boundary. It is readily shown that the only such term that can cancel the divergence in (20) is \(S_{ct} = (-1/\ell) \int \sqrt{-\gamma}.\) This then yields \(T_{\mu\nu} = 0\), which is clearly free of divergences. In general, we could have added further higher dimensional counterterms such as \(R\) and \(R^2\). Dimensional analysis shows that terms higher than \(R\) vanish too rapidly at infinity to contribute to the stress tensor. The potential contribution from the metric variation of \(R\) is \(G^{\mu\nu}\), the Einstein tensor, which vanishes identically in two dimensions. So the minimal counterterm in \((10)\) completely defines the \(AdS_3\) stress tensor.

Since the stress tensor is now fully specified, it must reproduce the mass and angular momentum of a known solution. To check this, we study spacetimes of the form:

\[ds^2 = \frac{\ell^2}{r^2}dr^2 + \frac{r^2}{\ell^2}(-dt^2 + dx^2) + \delta g_{MN} dx^M dx^N.\]  

(21)

Working to first order in \(\delta g_{MN}\), we find

\[8\pi G T_{tt} = \frac{r^4}{2\ell^5} \delta g_{rr} + \frac{\delta g_{xx}}{\ell} - \frac{r}{2\ell} \partial_r \delta g_{xx}\]
\[8\pi G T_{xx} = \frac{\delta g_{tt}}{\ell} - \frac{r}{2\ell} \partial_r \delta g_{tt} - \frac{r^4}{2\ell^5} \delta g_{rr}\]
\[8\pi G T_{tx} = \frac{1}{\ell} \delta g_{tx} - \frac{r}{2\ell} \partial_r \delta g_{tx}.\]  

(22)

The mass and momentum are:

\[M = \frac{1}{8\pi G} \int dx \left[ \frac{r^4}{2\ell^5} \delta g_{rr} + \frac{\delta g_{xx}}{\ell} - \frac{r}{2\ell} \partial_r \delta g_{xx} \right]\]
\[P_x = -\frac{1}{8\pi G} \int dx \left[ \frac{1}{\ell} \delta g_{tx} - \frac{r}{2\ell} \partial_r \delta g_{tx} \right].\]  

(23)
We can apply these formulae to the spinning BTZ solution \[21, 22\]:

\[
ds^2 = -N^2 dt^2 + \rho^2 (d\phi + N^\phi dt)^2 + \frac{r^2}{N^2 \rho^2} dr^2
\]  

(24)

with

\[
N^2 = \frac{r^2 (r^2 - r_+^2)}{\ell^2 \rho^2}, \quad N^\phi = -\frac{4GJ}{\rho^2},
\]

\[
\rho^2 = r^2 + 4GM\ell^2 - \frac{1}{2} r_+^2, \quad r_+ = 8G\ell \sqrt{M^2 \ell^2 - J^2},
\]  

(25)

where \(\phi\) has period 2\(\pi\). Expanding the metric for large \(r\) we find

\[
\delta g_{rr} = \frac{8GM\ell^4}{r^4}, \quad \delta g_{tt} = 8GM, \quad \delta g_{\phi\phi} = -4GJ.
\]  

(26)

Inserting these into (23) with \(x \to \ell \phi\) and \(\int dx \to \ell \int_0^{2\pi} d\phi\) gives the correct relations \(M = M\) and \(P_\phi = J\) in agreement with conventional techniques. When \(M = -1/8G\) and \(J = 0\), the BTZ metric reproduces global AdS, while the \(M = 0, J = 0\) black hole looks like Poincaré AdS with an identification of the boundary. It may seem surprising that global AdS apparently differs in mass from the Poincaré patch. The difference arises because the time directions of these coordinates do not agree, giving rise to different definitions of energy.

### 3.1 Conformal Symmetry of AdS

Brown and Henneaux [18] have shown that gravity in asymptotically AdS spacetime is a conformal field theory with central charge \(c = 3\ell/2G\). Both as a check of our approach, and because our covariant method will offer an alternative to the Hamiltonian formalism adopted in [18] and the Chern-Simons methods of [23], we would like to reproduce this result.\footnote{5Related work has been done by Hyun et.al. [9]}  

In light of the AdS/CFT correspondence, we can think of the conformal symmetry group as arising from a 1+1 dimensional non-gravitational quantum field theory living (loosely speaking) on the boundary of AdS. On a plane with metric \(ds^2 = -dx^+ dx^-\), diffeomorphisms of the form

\[
x^+ \to x^+ - \xi^+(x^+), \quad x^- \to x^- - \xi^-(x^-)
\]  

(27)

transform the stress tensor as:

\[
T_{++} \to T_{++} + (2\partial_+ \xi^+ T_{++} + \xi^+ \partial_+ T_{++}) - \frac{c}{24\pi} \partial_+^3 \xi^+,
\]

\[
T_{--} \to T_{--} + (2\partial_- \xi^- T_{--} + \xi^- \partial_- T_{--}) - \frac{c}{24\pi} \partial_-^3 \xi^-.
\]  

(28)
The terms in parenthesis are just the classical tensor transformation rules, while the last term is a quantum effect. Let us briefly recall the origin of the latter. Although (27) is classically a symmetry of the CFT, it is quantum mechanically anomalous since we must specify a renormalization scale $\mu$. To obtain a symmetry under (27), $\mu$ must also be rescaled to have the same measured value in the new coordinates as in the original coordinates. Equivalently, the metric should be Weyl rescaled to preserve the form $ds^2 = -dx^+dx^-$. Such a rescaling of lengths acts non-trivially in the quantum theory and produces the extra terms in (28).

We will focus on obtaining the final terms in (28) by starting from $\text{AdS}_3$ in the form
\begin{equation}
 ds^2 = \frac{\ell^2}{r^2}dr^2 - r^2dx^+dx^-,
\end{equation}
for which $T_{\mu\nu} = 0$. We think of the dual CFT as living on the surface $ds^2 = -r^2dx^+dx^-$ with $r$ eventually taken to infinity. Now consider the diffeomorphism (27). As above, this is not a symmetry since it introduces a Weyl factor into the boundary metric. To obtain a symmetry one must leave the asymptotic form of the metric invariant, and the precise conditions for doing so have been given by Brown and Henneaux [18]:
\begin{align}
 g_{++} &= -\frac{r^2}{2} + O(1), & g_{++} &= O(1), & g_{--} &= O(1),
 g_{rr} &= \frac{\ell^2}{r^2} + O\left(\frac{1}{r^2}\right), & g_{+r} &= O\left(\frac{1}{r^3}\right), & g_{-r} &= O\left(\frac{1}{r^3}\right).
\end{align}
The diffeomorphisms which respect these conditions are:
\begin{align}
 x^+ &\to x^+ - \xi^+ - \frac{\ell^2}{2r^2}\partial^2\xi^- \\
x^- &\to x^- - \xi^- - \frac{\ell^2}{2r^2}\partial^2\xi^+ \\
r &\to r + \frac{r}{2}(\partial_+\xi^+ + \partial_-\xi^-).
\end{align}
For large $r$, the corrections to the $x^\pm$ transformations are subleading, and we recover (27). The metric then transforms as
\begin{equation}
 ds^2 \rightarrow \frac{\ell^2}{r^2}dr^2 - r^2dx^+dx^- - \frac{\ell^2}{2}(\partial^3\xi^+)(dx^+)^2 - \frac{\ell^2}{2}(\partial^3\xi^-)(dx^-)^2.
\end{equation}
Since the asymptotic metric retains its form, this transformation is a symmetry. Using (32) we compute the stress tensor to be
\begin{align}
 T_{++} &= -\frac{\ell}{16\pi G}\partial^3\xi^+, & T_{--} &= -\frac{\ell}{16\pi G}\partial^3\xi^-.
\end{align}
This agrees with (28) if
\[ c = \frac{3\ell}{2G}. \]  
(34)

Thus we have verified the result of Brown and Henneaux \[18\].

In the CFT the full transformation law arose from doing a renormalization group rescaling of $\mu$, while on the gravity side it arose from a diffeomorphism which rescaled the radial position of the boundary. This fits very nicely with the general feature of the AdS/CFT correspondence that scale size in the CFT is dual to the radial position in AdS. According to \[24\], $r$ specifies an effective UV cutoff in the CFT; by rescaling $r$ before taking it to infinity we are changing the way in which the cutoff is removed — but this is just the definition of a renormalization group transformation.

We restricted attention to the diffeomorphism (31) because we were interested in symmetries which preserved the form of the boundary metric. More general diffeomorphisms may be studied, but these will modify the form of the CFT and so are not symmetries.

### 3.2 Conformal Anomaly for AdS$_3$

The stress tensor of a 1 + 1 dimensional CFT has a trace anomaly
\[ T_\mu^\mu = -\frac{c}{24\pi}R. \]  
(35)

We will now verify that our quasilocal stress tensor has a trace of precisely this form. The mechanism for obtaining a conformal anomaly from the AdS/CFT correspondence was outlined by Witten \[4\] and studied in detail by Henningson and Skenderis \[8\]. Our approach is somewhat different from that of \[8\].

Taking the trace of the AdS$_3$ stress tensor appearing in (10) we find
\[ T_\mu^\mu = -\frac{1}{8\pi G}(\Theta + 2/\ell). \]  
(36)

(36) gives the trace in terms of the extrinsic curvature; to compare with (33) we need to express the result in terms of the intrinsic curvature of the boundary.

Since (36) is manifestly covariant, we may compute the right hand side in any convenient coordinate system. We write
\[ ds^2 = \frac{\ell^2}{r^2}dr^2 + \gamma_{\mu\nu}dx^\mu dx^\nu. \]  
(37)

The extrinsic curvature in these coordinates is
\[ \Theta_{\mu\nu} = -\frac{r}{2\ell} \partial_t \gamma_{\mu\nu}. \]  
(38)
So in this coordinate system (36) becomes

\[ T^\mu_\mu = -\frac{1}{8\pi G} \left[ \frac{r}{2\ell} \gamma^{\mu\nu} \partial_r \gamma_{\mu\nu} + 2 \gamma_{\mu\nu} \right] \]  

(39)

To complete the calculation we need \( \gamma_{\mu\nu} \) as a power series in \( 1/r \). Einstein’s equations show [25] that only even powers appear and that the leading term goes as \( r^2 \). So we write

\[ \gamma_{\mu\nu} = r^2 \gamma^{(0)}_{\mu\nu} + \gamma^{(2)}_{\mu\nu} + \cdots. \]  

(40)

There are additional higher powers of \( 1/r \) as well as logarithmic terms [25], but these will not be needed. We now have

\[ T^\mu_\mu = -\frac{1}{8\pi G} \frac{1}{\ell r^2} \text{Tr} \left[ (\gamma^{(0)})^{-1} \gamma^{(2)} \right] + \cdots. \]  

(41)

Solving Einstein’s equations perturbatively gives [8]

\[ \text{Tr} \left[ (\gamma^{(0)})^{-1} \gamma^{(2)} \right] = \frac{\ell^2 r^2}{2} R \]  

(42)

where \( R \) is the curvature of the metric \( \gamma_{\mu\nu} \). Finally, inserting this into (41) and taking \( r \) to infinity we obtain

\[ T^\mu_\mu = -\frac{\ell}{16\pi G} R, \]  

(43)

which agrees with (35) when \( c = 3\ell/2G \).

4 AdS₄

The only difference between the AdS₄ and AdS₃ stress tensor derivations is the need for an extra term in \( S_{ct} \) to cancel divergences. Again, start with AdS₄ in Poincaré form:

\[ ds^2 = \frac{\ell^2}{r^2} dr^2 + \frac{r^2}{\ell^2} (-dt^2 + dx_i dx_i) \quad i = 1, 2. \]  

(44)

Following Sec. 3, we compute the mass of the spacetime and demand that it be finite:

\[ M = \int d^2x \sqrt{g_{xx}} N \Sigma^u u^i T_{tu} = \int d^2x \frac{r}{\ell} T_{tt}. \]  

(45)

A finite mass density requires \( T_{tt} \sim r^{-1} \) for large \( r \). Evaluating the stress tensor for the metric (44), we find

\[
8\pi G T_{tt} = -2\frac{r^2}{\ell^3} + \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{ct}}{\delta \gamma^{tu}}
\]

\[
8\pi G T_{x_i x_j} = 2\frac{r^2}{\ell} \delta_{ij} + \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{ct}}{\delta \gamma^{x_i x_j}}
\]

\[
8\pi G T_{tx_i} = \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{ct}}{\delta \gamma^{tx_i}}.
\]  

(46)
The divergences are cancelled by choosing $S_{ct} = -\frac{2}{\ell} \int \sqrt{-\gamma}$; in particular we find that $T_{\mu\nu} = 0$.

Now consider AdS$_4$ in global coordinates:

$$ds^2 = -\left(1 + \frac{r^2}{\ell^2}\right)dt^2 + \frac{dr^2}{\left(1 + \frac{r^2}{\ell^2}\right)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2).$$  \hspace{1cm} (47)

It is easy to show that the mass is still given by \eqref{eq:mass} in the limit $r \to \infty$, after replacing $d^2x$ by $\sin \theta d\theta d\phi$. We find that the counterterm introduced above correctly removes the $r^2$ divergence in $T_{\mu\nu}$, but there remains a $r^0$ behaviour (leading to a divergent mass which can be cancelled by adding $\int \ell \sqrt{-\gamma} R/2$ to $S_{ct}$. Altogether, this gives the counterterm action written in \eqref{eq:ct_action}. We are free to add higher dimensional objects like $R^2$ to $S_{ct}$, but they vanish too quickly at the AdS$_4$ boundary to contribute to the stress tensor. In total, the stress tensor for the metric \eqref{eq:global_metric} is:

$$8\pi G T_{tt} = \frac{\ell}{4r^2} + \cdots$$
$$8\pi G T_{\theta\theta} = \frac{\ell^3}{4r^2} + \cdots$$
$$8\pi G T_{\phi\phi} = \frac{\ell^3}{4r^2} \sin^2 \theta + \cdots$$

We test our definition on the AdS$_4$-Schwarzschild solution:

$$ds^2 = -\left[r^2 + 1 - \frac{r_0}{r}\right] dt^2 + \left[r^2 + 1 - \frac{r_0}{r}\right]^{-1} dr^2 + r^2 d\Omega_2^2.$$  \hspace{1cm} (49)

We find

$$8\pi G T_{tt} = \frac{r_0}{\ell r} + \cdots,$$  \hspace{1cm} (50)

leading to a mass

$$M = \frac{r_0}{2G}$$  \hspace{1cm} (51)

This agrees with the standard definition of the AdS$_4$ black hole mass.

### 4.1 Conformal Anomaly for AdS$_d$

Direct computation shows that the stress tensor for AdS$_4$ is traceless. There is also a general argument that the trace vanishes for any even dimensional AdS, which we give instead.

The stress tensor for AdS$_{d+1}$ has length dimension $-d$. Since for large $r$ the Weyl factor multiplying the boundary metric is proportional to $r^2$, it must be the case that

$$T^\mu_\mu \sim \frac{1}{r^d}.$$  \hspace{1cm} (52)
Working in coordinates like \( \text{\text{(37)}} \), the trace has the structure

\[ T_{\mu}^{\nu} \sim r^{\gamma_{\mu\nu}} \partial_{r} \gamma_{\mu\nu} + (\text{curvature invariants of } \gamma_{\mu\nu}). \]  

(53)

Now, \( \gamma_{\mu\nu} \) has an expansion in even powers of \( r \): \( \gamma_{\mu\nu} = r^{2} \sum_{n=0}^{\infty} \frac{\gamma^{(2n)}_{\mu\nu}}{r^{2n}} \).  

(54)

Using this in \( \text{(53)} \), and the fact that scalar curvature invariants always involve even powers of the metric, we find that only even powers of \( r \) can appear in the trace. Comparing with \( \text{(52)} \), shows that the stress tensor must vanish for odd \( d \).

This result is expected from the AdS/CFT correspondence, since even dimensional AdS bulk theories are dual to odd dimensional CFTs, which have a vanishing trace anomaly.

5 AdS\(_{5}\)

The AdS\(_{5}\) counterterms are derived in parallel with AdS\(_{4}\), so we can be brief. The expression for the spacetime mass is now:

\[ M = \int d^{3}x \sqrt{g_{xx}} N_{\Sigma} u^{i} u^{i} T_{tt} = \int d^{3}x \frac{r^{2}}{\ell^{2}} T_{tt}. \]  

(55)

A finite mass density therefore requires \( T_{tt} \sim r^{-2} \) for large \( r \). Upon evaluating the stress tensor in Poincaré and global coordinates and imposing finiteness, we arrive at the counterterms written in \( \text{(10)} \). By dimensional analysis, the only possible higher dimensional terms in \( S_{ct} \) that could make a finite contribution to the stress tensor are the squares of the Riemann tensor, the Ricci tensor and the Ricci scalar of the boundary metric. We will discuss these potential ambiguities in Sec. 5.1.

We now check our definition against the known mass of particular solutions. Consider the metric

\[ ds^{2} = \frac{r^{2}}{\ell^{2}} \left[ - \left( 1 - \frac{r_{0}^{4}}{r^{4}} \right) dt^{2} + (dx_{i})^{2} \right] + \left( 1 - \frac{r_{0}^{4}}{r^{4}} \right)^{-1} \frac{\ell^{2}}{r^{2}} dr^{2} \]  

(56)

that arises in the near-horizon limit of the D3-brane (see, e.g., \[ \text{(17)} \]). The stress tensor is

\[ 8\pi G T_{tt} = \frac{3r_{0}^{4}}{2\ell^{3}r^{2}} + \cdots \]

\[ 8\pi G T_{x_{i}x_{i}} = \frac{r_{0}^{4}}{2\ell^{3}r^{2}} + \cdots. \]  

(57)
Using (55) gives

\[ M = \frac{3r_0^4}{16\pi G\ell^5} \int d^3 x. \]  
(58)

This agrees with the standard formula for the mass density of this solution \[17\].

Next, consider the AdS-Schwarzschild black hole solution,

\[ ds^2 = - \left[ \frac{r^2}{\ell^2} + 1 - \left( \frac{r_0}{r} \right)^2 \right] dt^2 + \frac{dr^2}{\left[ \frac{r^2}{\ell^2} + 1 - \left( \frac{r_0}{r} \right)^2 \right]} + r^2(d\theta^2 + \sin^2 \theta d\phi^2 + \cos^2 \theta d\psi^2). \]  
(59)

Note that \( r_0 = 0 \) gives the global AdS\(_5\) metric. We find

\[ 8\pi GT_{tt} = \frac{3\ell}{8r^2} + \frac{3r_0^2}{2\ell r^2} + \cdots, \]
\[ 8\pi GT_{\theta\theta} = \frac{\ell^3}{8r^2} + \frac{\ell r_0^2}{2r^2} + \cdots, \]
\[ 8\pi GT_{\phi\phi} = \left( \frac{\ell^3}{8r^2} + \frac{\ell r_0^2}{2r^2} \right) \sin^2 \theta + \cdots, \]
\[ 8\pi GT_{\psi\psi} = \left( \frac{\ell^3}{8r^2} + \frac{\ell r_0^2}{2r^2} \right) \cos^2 \theta + \cdots. \]  
(60)

The mass is

\[ M = \frac{3\pi \ell^2}{32G} + \frac{3\pi r_0^2}{8G}. \]  
(61)

The standard mass of this solution is \( 3\pi r_0^2/8G \) [17], which is the second term of our result (61). We have the additional constant \( 3\pi \ell^2/32G \) which is then the mass of pure global AdS\(_5\) when \( r_0 = 0 \). It seems unusual from the gravitational point of view to have a mass for a solution that is a natural vacuum, but we will show that this is precisely correct from the perspective of the AdS/CFT correspondence.

**Casimir Energy**

String theory on AdS\(_5\) \( \times \) S\(_5\) is expected to be dual to four dimensional \( N = 4, SU(N) \) super Yang-Mills [2]. We use the conversion formula to gauge theory variables:

\[ \frac{\ell^3}{G} = \frac{2N^2}{\pi}. \]  
(62)

Then, setting \( r_0 = 0 \), the mass of global AdS\(_5\) is:

\[ M = \frac{3N^2}{16\ell}. \]  
(63)

---

\(^6\)We thank Gary Horowitz for pointing out the relevance of the CFT Casimir energy to our result, and for discussing his related work with Hirosi Ooguri.
The Yang-Mills dual of AdS$_5$ is defined on the global AdS$_5$ boundary with topology $S^3 \times R$. A quantum field theory on such a manifold can have a nonvanishing vacuum energy — the Casimir effect. In the free field limit, the Casimir energy on $S^3 \times R$ is:

$$E_{\text{casimir}} = \frac{1}{960r}(4n_0 + 17n_{1/2} + 88n_1),$$

(64)

where $n_0$ is the number of real scalars, $n_{1/2}$ is the number of Weyl fermions, $n_1$ is the number of gauge bosons, and $r$ is the radius of $S^3$. For SU($N$), $N = 4$ super Yang-Mills $n_0 = 6(N^2 - 1)$, $n_{1/2} = 4(N^2 - 1)$ and $n_1 = N^2 - 1$ giving:

$$E_{\text{casimir}} = \frac{3(N^2 - 1)}{16r},$$

(65)

To compare with (63), remember that $M$ is measured with respect to coordinate time while the Casimir energy is defined with respect to proper boundary time. Converting to coordinate time by multiplying by $\sqrt{-g_{tt}} = r/\ell$ gives the Casimir “mass”:

$$M_{\text{casimir}} = \frac{3(N^2 - 1)}{16\ell}.$$

(66)

In the large $N$ limit we find precise agreement with the gravitational mass (63) of global AdS$_5$.

In related work, Horowitz and Myers [17] compared the mass of an analytically continued non-extremal D3-brane solution to the corresponding free-field Casimir energy in the gauge theory, and found agreement up to an overall factor of $3/4$. They argued that the mathematical origin of the discrepancy was the same as for a $3/4$ factor [27] relating the gravitational entropy of the system to a free field entropy computation in the CFT dual. In both cases, the gravitational result is valid at strong gauge coupling and, apparently, the extrapolation from the free limit of the gauge theory involves a factor of $3/4$.

In our case, however, the coefficients match precisely. In general, gravity calculations may not be extrapolated to the weakly coupled gauge theory, because large string theoretic corrections can deform the bulk geometry in this regime. This is the origin of the $3/4$ factor discussed above. In our case, pure AdS$_5$ is protected from stringy corrections because all tensors which might modify Einstein’s equation actually vanish when evaluated in this background [28]. This is why the Casimir energy in the weakly coupled, large $N$ Yang-Mills exactly matches the gravitational mass of spacetime.

7Noting that $S^3 \times R$ is the Einstein static universe, we can adopt the results of [26].
5.1 Conformal Anomaly for AdS\(^5\)

The AdS\(^5\) conformal anomaly computation is a more laborious version of the AdS\(^3\) result in Sec. 3.2. The trace of the AdS\(^5\) stress tensor in (10) is

\[
T_\mu^\mu = -\frac{1}{8\pi G} (3\Theta + 12/\ell - \ell R/2). \tag{67}
\]

Again, write the bulk metric in the form (37) so that (38) gives the extrinsic curvature, yielding

\[
T_\mu^\mu = -\frac{1}{8\pi G} \left[ -\frac{3r}{2\ell} \gamma_{\mu\nu} \gamma^{\mu\nu} + \frac{12}{\ell} - \frac{\ell}{2} R(\gamma_{\mu\nu}) \right]. \tag{68}
\]

To identify the anomaly we must compute \(\gamma_{\mu\nu}\) to order \(r^{-2}\):

\[
\gamma_{\mu\nu} = r^2 \gamma_{\mu\nu}^{(0)} + \gamma_{\mu\nu}^{(2)} + r^{-2} \gamma_{\mu\nu}^{(4)} + \cdots. \tag{69}
\]

The coefficients are found to be \(^8\)

\[
\gamma_{\mu\nu}^{(2)} = \frac{\ell^2}{2} \left( R_{\mu\nu}^{(0)} - \frac{1}{6} R^{(0)} \gamma_{\mu\nu}^{(0)} \right)
\]

\[
\text{Tr} \left[ (\gamma^{(0)})^{-1} \gamma^{(4)} \right] = \frac{1}{4} \text{Tr} \left[ (\gamma^{(0)})^{-1} \gamma^{(2)} \right]^2. \tag{70}
\]

We also need the expansion of \(\mathcal{R}(\gamma_{\mu\nu})\):

\[
\mathcal{R}(\gamma_{\mu\nu}) = \frac{1}{r^2} R^{(0)} + \frac{\delta R}{\delta \gamma_{\mu\nu}} \bigg|_{r^2 \gamma_{\mu\nu}^{(0)}} \gamma_{\mu\nu}^{(2)} = \frac{1}{r^2} R^{(0)} - \frac{\ell^2}{2r^4} \left( R_{\mu\nu}^{(0)} R_{\mu\nu}^{(0)} - \frac{1}{6} R^{(0)} \right). \tag{71}
\]

Inserting these results into (68) and doing some algebra, one finds

\[
T_\mu^\mu = -\frac{\ell^3}{8\pi G} \left[ -\frac{1}{8} R^{\mu\nu} R_{\mu\nu} + \frac{1}{24} R^2 \right]. \tag{72}
\]

This result for the trace agrees with the work of Henningson and Skenderis \(^3\). These authors also show that upon using (62), precise agreement is obtained with the conformal anomaly of \(\mathcal{N} = 4\) super Yang-Mills.

An Ambiguity

The minimal AdS\(^5\) counterterm action in (10) can be augmented by the addition of terms quadratic in the Riemann tensor, Ricci tensor and Ricci scalar of the boundary metric. \(^\text{[9]}\) A convenient basis for this ambiguity is provided by:

\[
\Delta S_{ct} = \ell^3 \int_{\partial M} d^4 x \sqrt{-\gamma} \left[ a E + b C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + c R^2 \right]. \tag{73}
\]

\(^8\)Higher dimensional invariants give a vanishing contribution to the stress tensor at the AdS boundary.
The first term is the Euler invariant $E = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ and vanishes under variation, so we can omit it without loss of generality. $C^{\mu\nu\rho\sigma}$ is the Weyl tensor. Varying $\Delta S_{ct}$ with respect to the boundary metric produces an ambiguity in the stress tensor:

$$\Delta T_{\mu\nu} = \left(\frac{\ell^3}{16\pi G}\right)(bH^b_{\mu\nu} + cH^c_{\mu\nu}).$$  

(74)

The tensors $H^b$ and $H^c$ are computed in [26]; their trace gives a contribution to the anomaly

$$\Delta T^\mu_{\mu} \propto \Box R.$$  

(75)

For general boundary metrics there is therefore a two parameter set of possible stress tensors, whose anomalies have varying coefficients for $\Box R$. Exactly the same ambiguity is present in the definition of the renormalized stress tensor of the dual field theory on the curved boundary [26]. Our gravitational result can only be matched to field theory computations after the ambiguous parameters are matched. For conformally flat boundaries the tensor $H^b_{\mu\nu}$ vanishes leaving a one parameter ambiguity, which is fully specified by the coefficient of $\Box R$ in the anomaly. So we learn from (72) that gravitational energies computed with the minimal counterterm action $\Delta S_{ct} = 0$ should be compared with a field theory regularization which produces a vanishing $\Box R$ anomaly coefficient. Precisely this was done in the above comparison of Casimir energies for global AdS$_5$. The boundary $S^3 \times R$ is conformally flat, and we have checked that the field theory computation that produces (14) yields no $\Box R$ term in the anomaly. This explains the agreement between the gravity and field theory results, despite the apparent ambiguity in choosing $\Delta S_{ct}$.

6 Discussion

We have formulated a stress tensor which gives a well-defined meaning to the notions of energy and momentum in AdS. Through the AdS/CFT correspondence, we have also found results for the expectation value of the stress tensor in the dual CFT. Our proposal exhibits the desired features of a stress tensor, both from the gravitational and CFT points of view.

The procedure we have followed for defining the stress tensor is a particular example of the ideas developed in [26]. There it was shown how to associate the asymptotic behavior of each bulk field with the expectation value of a CFT operator. The relation studied here between the gravitational field and the stress tensor is an example of this correspondence.

It would be desirable to formulate an analogous stress tensor in asymptotically flat spacetimes. It is not immediately clear how to define counterterms, since there
is no longer a dimensionful parameter like $\ell$ allowing one to form a dimensionless counterterm action. On the other hand, flat spacetime is recovered from AdS by taking $\ell \to \infty$, so we might expect that applying this limit to our formulae would yield the appropriate stress tensor. However, the situation is complicated by the fact that we must keep $r$ finite while applying the limit, taking $r \to \infty$ afterwards. The stress tensor at finite $r$ should be interpreted in a CFT with an ultraviolet cutoff [24]. This implies that the limits $\ell \to \infty$, $r \to \infty$ can be understood in renormalization group terms [30].

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