A Classification of Entanglement in Multipartite States with Translation Symmetry

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I. INTRODUCTION

There are two fundamental issues for a given multipartite entangled state, which one has to identify in order to distinguish one from others, the pattern of entangled state and the measurement of entanglement. Since the finding of GHZ and W states in 3-qubit system, it is realized that there would exist distinct forms of entanglement in multipartite states, which cannot be converted into each other by stochastic local operations and classical communication (SLOCC). From point of quantum information, these entangled states may display different ability to handle different tasks. Consequently it becomes meaningful to present a general classification of multipartite entangled states by SLOCC.

However this task is so difficult that even for the simple 4-qubit entangled states, there exist many inconsistent classifications, based on different prerequisites. This difficulty comes from the fact that entanglement in multipartite states does not completely adhere to local features of state since the absence of Schmidt decomposition, and instead is a global character of state. Thus the local measurements can only present limited information of entanglement. Although this difficulty, some methods have been proposed for classification of multipartite entangled states, focused on different aspects of multipartite entanglement. A natural candidate is the generalization of Schmidt decomposition into multipartite case. Besides the so-called Schmidt tensor rank has also been introduced, of which the crucial idea is to find the minimal decomposition on product basis. An alternative method is to find the polynomial invariants under SLOCC, of which the distinct pattern can be used to classify the multipartite entanglement. Besides these works, there are distributed efforts, focused on different characters of multipartite entanglement, such as the generalized majorization, criteria of inequality, local unitary equivalence, graphical methods, and the relaxations.

A common feature in previous studies is that one has to introduce different local measurements as many as possible in order to obtain the full information of entanglement in multipartite states. Recently symmetry in multipartite entangled state has received extensive attention since it reflects the global feature of state. It then is an interesting issue what is the relation between the two global features of multipartite state. Some positive conclusions have been obtained. For example, a classification of multipartite entangled states with permutation invariance has been achieved by identifying distinct substructures of state or finding the roots of a Majorana polynomial. Since the natural conjecture that the overall feature of entangled state could only be manifested from global viewpoints, it is possible to find a classification of multipartite entangled states by imposing a certain global symmetry. This article is presented for this purpose.

In this article, a classification of multipartite entangled state with translational symmetry is proposed, by founding a translation-invariant entangled-state basis of state space. Our study shows that for arbitrary N-qubit system, the symmetric entangled basis states can be divided into several SLOCC inequivalent classes by two global features of state, periodic pattern and cyclic unit. In addition the concept of phase can be defined by identifying distinct topological features. We argue that the interconversion between different phases has to accompany with the conventional phase transition. However for a general state, which is the superposition of different basis states, is considered as symmetry-broken since it would be a mixture of at least two distinct global features. Then these superposition states are all classified as a single phase, symmetry-broken phase. Importantly these basis states can be realized readily in spin-chain systems. Thus this classification is meaningful from experiment.
II. TRANSLATIONAL SYMMETRY OF STATE AND ITS IMPLICATION FOR ENTANGLEMENT

We first present a general discussion about the meaning of symmetry of state. Symmetry of state is defined as

Definition 1 For a symmetry operator $\hat{S}$, if

$$\hat{S}|\psi\rangle = c|\psi\rangle,$$

then $|\psi\rangle$ is said to be with symmetry $S$ or invariant under $\hat{S}$.

The constant $c$ is complex with unit norm as for normalization. $|\psi\rangle$’s with different $c$ are obviously orthogonal to each other since Eq. (1) is an eigen equation of operator $\hat{S}$. Thus a symmetric basis of state space can be founded by eigenvectors of $\hat{S}$. In general, the eigenvector with $c$ is degenerate, which means that there are distinct forms for states with the same symmetry.

A. One-dimensional duo-lattice picture of entanglement of multipartite state

Symmetry manifests the global feature of state. Thus by this feature distinct multipartite entangled state can be identified. A crucial question is what symmetry is suitable for this purpose? Our answer is translation symmetry, including permutation symmetry as a special case. The reason comes from a picturesque understanding of entanglement: Entanglement actually describes the coherent correlation of single-party states belong to distinguished parties. Suppose every single-party state as a lattice site. Then one has a lattice model for $N$-qubit system, composed of two $N$-site sublattices, as exemplified in Fig. 1. In this picture, a qubit is represented by two sites belong to different sublattices with same labels. Then the entangled state up to superposition coefficients can be represented by the lines, which connect the lattice sites with distinct labels, whereas there is no line for fully separable state. As for this duo-lattice model, translational symmetry is much fundamental. Thus it is a reasonable choice for our discussion. Furthermore from experimental point, the lattice picture implies that entangled state with translational symmetry could be prepared readily in solid-state systems since its natural periodic structure. By the way the periodic boundary condition (PBC) is supposed automatically.

It should point out that this lattice model is only for quantum state, not related to any concrete system. The reason is simple that for a given state, one can prepare it in many different systems. The crucial point of entanglement is the coherent connection between distinguishable single-party states. Consequently our lattice model can be assumed to be one-dimensional for simplicity.

B. Translation symmetry of state

With this picture, translational operation $\hat{T}$ of multipartite state is defined schematically in Fig. 2, in which PBC implies $|\psi\rangle_{N+i} = |\psi\rangle_i (i = 1, 2, \ldots, N)$ for single-party state $|\psi\rangle_i$. The central point is for $\hat{T}$ that a single-party state does not be fixed to any definite qubit (lattice site), and can be transferred arbitrarily onto other qubits (lattice sites) by a global rotation of all single-
party states, as shown in Fig. 2. More importantly the relative distance of all single-party states must be kept invariant in this rotation. Thus the set of single-party states \{\|a\|, \|b\|, \cdots\} is named as cyclic unit in this article, which decides the form of multipartite state. For example \(|W\rangle_3\) is a superposition state of all possible combination of the cyclic unit \{\(\|1\rangle, \|0\rangle\}\} (More examples can be found in Appendix). This definition can be easily generalized to arbitrary n-level case. It should emphasize that \(T\) is applied only for single-party state, not for concrete particle.

C. Construction of symmetric basis of N-qubit system

We are ready to found the symmetric basis by translational operation \(T\). The first task is to find the possible values of \(c\) in Eq. (1). By several examples in Appendix, one can obtains

**Observation 1** For N-qubit state, the eigenvalues \(c\) of translational operation \(T\) equal to \(e^{i2\pi k/n}\) with \(n = 0, 1, 2, \cdots, N - 1\).

This observation is not strange since the translational operation \(T\), defined in Fig. 2 actually displays a cyclic rotation of all single-party states. Thus \(T\) behaviors a planar rotation \(C_N\), and the phase angles correspond to the character values of point group \(C_N\).

In order to construct a symmetric basis, the cyclic unit of every basis state necessarily is required unique. This requirement can be satisfied easily for qubit systems since there are only two single-party states \(\|1\rangle\) and \(\|0\rangle\). Thus the cyclic unit is just one of their possible combinations, only with restriction that the total number of single-party states must be \(N\). As exemplifications, we present the symmetric bases for \(N = 3, 4, 5, 6\) in Appendix. It is obvious, for instance that the unique cyclic unit in \(|W\rangle_3\) is three single-party states, \{\(\|1\rangle, \|0\rangle\}\}. In contrast, the cyclic unit in \(|W\rangle_4\) is four single-party states, \{\(\|1\rangle, \|0\rangle\}\}. The common feature of the two basis states is that there is only one cyclic unit, 100 or 1100 respectively, and the state is composed from all possible arrangements of the single-party states in cyclic unit after \(T\).

It should emphasize that the connection of single-party states in cyclic unit is crucial. For example the information in cyclic unit of \(|W\rangle_4\) is that there are two nearest neighbor states having the same state \(\|1\rangle\), the other two having another state \(\|0\rangle\). However, the cyclic unit of \(|GHZ\rangle_4\) provides the information that the next nearest neighbor single-party states always are same, but the nearest neighbor parties have different states. Another example is for \(|W\rangle_4\) and \(|W\rangle_4\), which are LOCC-equivalent. Although the forms of cyclic units are different, they have the same connection of single-party states; One and only one qubit has different state from the other. Thus the sole criterion for discriminating cyclic units is the connection in all single-party states. In the following discussion when we say two cyclic units different, it means the distinct connection of single-party states. In addition the difference between distinct cyclic units is also un-local since it is spread over all qubits because of translation symmetry. And the local operator \(\otimes_{i=1}^4 M_i\) does not exist, which could simultaneously remedy the discrepancy of items in two states. Consequently the concept of cyclic unit can be used as a criterion of classification of multipartite states.

Besides of cyclic unit, periodic pattern is another global character of basis state. For instance there are two types of symmetric basis states for 4-qubit case, so called 2-period, 4-period, as shown in Appendix; It is obvious that there are two items in \(|GHZ_1(2)\rangle_4\) since the cyclic unit is a double of 10, while there are four items in \(|W_1(2)\rangle_4\) since the cyclic unit has no repeated substructure as the former and has to be rotated three times by \(T\) in order to scan all its possible forms. It should point out that this definition is consistent with the idea of Schmit tensor rank, by which two SLOCC-equivalent states show the same tensor rank \(6\). As for the symmetric basis state, the periodic pattern limits completely the number of superposition items in state, which is inevitably minimal since any extra item or absence of any item would destroy translation symmetry of state. Thus \(n\), the number of items, is also a global feature of basis state, which is called \(n\)-period in this place.

A crucial observation is

**Observation 2** For \(N = n \times m\) qubit system, there exist \(n\)-period or \(m\)-period translation invariant basis state, except of the trivial cases \(n = 1\) or \(m = 1\).

This result can be understood from the point of group theory; For \(N = n \times m\), one has decomposition \(C_N = C_n \otimes C_m\). That is to say that the point group \(C_N\) is direct product of subgroups \(C_n\) and \(C_m\). For example, \(C_4 = C_2 \otimes C_2, C_5 = C_2 \otimes C_3\). As for symmetric basis states, it means that there are different translation invariant structures, labeled by the periodic pattern.

The construction of state-space basis is much simple for qubit systems, since there are only two single-party states \(\|1\rangle\) and \(\|0\rangle\). The only interest is what the number of single-party states \(\|1\rangle\) or \(\|0\rangle\) is respectively in the cyclic unit and their relative distance in duo-lattice model. Then a state with translation symmetry is composed of all possible forms of its cyclic unit after rotation \(T\), depicted in Fig. 2. Furthermore the phase difference between induced items is also important since it leads to different \(c\).

Two interesting issues in the symmetric basis are necessary to point out. First there is a parity effect for GHZ-like states. For example, \(|GHZ_1(2)\rangle_4\) act as independent role in the symmetric basis of 3-qubit case since they are SLOCC-inequivalent to the other basis states. In contrast, although one can still define \(|GHZ_1(2)\rangle_4\), their role are overlap with the 2-period symmetric states \(|GHZ_1(2)\rangle_4\) since the equivalence \(|GHZ_1(2)\rangle_4 = \sigma_2^\sigma\).
\[ \sigma_1^x |GHZ'_{1(2)}\rangle_4. \]  The similar phenomenon can be found for 5- or 6-qubit case. Thus it is a direct speculation that for even \( N \), GHZ-like state would behavior as 2-period symmetric state, whereas for odd \( N \) it has distinct features from the other symmetric basis states. This effect is obviously dependent on the parity of the qubit number \( N \), and thus is also a global feature of state space. In addition it means that the two types of states would have different ability to handle quantum information task. Second we argue in this article that \( W \)-like state is translational invariant rather than permutational. This reason can be manifested by imposing phase difference between items; although the permutational symmetry is destroyed in this case, translational symmetry can be maintained by choosing proper phase difference, for example \( \delta \phi = 2\pi/3 \) in \( |T_{1(2)}\rangle_3 \). One can easily verified that this phenomenon is absent for Dicke state, which has to keep all items be in phase in order to maintain the permutational symmetry. Interestingly the permutational symmetry of GHZ-like state is robust against this phase variation since all single-party states are the same one simultaneously.

Consequently one can identify a periodic pattern as a phase from point of phase transition. Thus the conversion of different phases have to be accompanied with phase transition, which corresponds to a discontinuous behavior. In addition this phase is also topological from the point that it may include several different cyclic units; This degeneracy is sometime robust since its breaking has to introduce the long-range interaction, as shown in the next subsection. Furthermore one can define quantum number to label different phases. In this article the quantum number is nothing but the common divisors of qubit number \( N \) by Observation

### III. CLASSIFICATION OF SYMMETRIC BASIS STATES: GENERAL VIEW

From the discussion above one can identify the different basis states by two global features, periodic pattern and cyclic unit. The periodic pattern is a geometrical property of state space as shown by Observation. Thus the periodic pattern of state cannot be changed by any local invertible operation, and can be used to distinguish different basis states. However, the cyclic unit presents the details of states. In general, there may be different cyclic units for a certain periodic pattern. But it does not means that the states with different cyclic units but same periodic pattern can be converted into each other by SLOCC since the difference between cyclic units is spread over all qubit because of translation invariance. Thus the two global features have to be combined together in order to classify basis states. The classifications of basis states for \( N = 3, 4, 5, 6 \) have been provided in Appendix. Instead this section is preferred to give a more general discussion.

#### A. Phase transition point of the classification of symmetric basis states

Mathematically SLOCC corresponds to continuous transformation since its local feature and invertibility. Thus two SLOCC-inequivalent states must have different characters such that one cannot transform one continuously into another; Geometrically the difference between two states are inevitably global or topological. By imposing certain symmetry, the difference can probably be identified by a variation of the symmetry.

As for translational symmetry, the symmetric basis state can be realized in concrete systems. Let consider a spin-half chain system with ferromagnetic interaction. The Hamiltonian is written as \((h = 1)\)

\[
H_F^N = -\sum_{n=1}^{N} \sigma_n^z \sigma_{n+1}^z, \tag{2}
\]

with boundary condition \( \sigma_N^z = \sigma_1^z \).

**3-qubit case** There are two degenerate ground states, \( |GHZ_1\rangle_3 \) and \( |GHZ_2\rangle_3 \), with energy \( E_0 = -3 \). When one spin or two spins is flipped, one obtain the degenerate first excited states, \( |W_{1(2)}\rangle_3, |T_{1(2)}\rangle_3 \) and \( |T_{1(2)}^*\rangle_3 \), with energy \( E_1 = 1 \). It is obviously that the two types of symmetric basis states are separated by a energy gap \( \Delta E = 4 \). Thus the conversion between them has to close the gap, i.e., a phase transition.

In addition \( |W_{1(2)}\rangle_3, |T_{1(2)}\rangle_3 \) and \( |T_{1(2)}^*\rangle_3 \) have the same cyclic unit and periodic pattern, except of the value of \( c \). Thus the three states is LOCC-equivalent and have the same physical property. Similar analysis can be applied for the other cases.

**4-qubit case** The two degenerate ground states are \( |GHZ_{1(0)}\rangle_4 \) with energy \( E_0 = -4 \). There are two different situation for first excited states; The first is that only one spin is flipped, \( |W_{1(2)}\rangle_4, |T_{1(2)}\rangle_4, |T_{1(2)}^*\rangle_4 \) and \( |T'_{1(2)}\rangle_4 \). The second is that two nearest neighbored spins are flipped simultaneously, \( |W_3\rangle_4, |T_3\rangle_4, |T'_3\rangle_4 \) and \( |T''_3\rangle_4 \). The two situations can be differentiated by introducing next nearest neighbored perturbation \( H'_1 = -\lambda (\sigma_1^z \sigma_2^z + \sigma_2^z \sigma_3^z) \) with \( \lambda \ll 1 \). Then one has the energy \( E_1' = 0 \) for the first case, \( E_1'' = 2\lambda \) for the second case. As for \( |W_2\rangle_4 \) and \( |W_2\rangle_4 \), the corresponding cyclic units have the same relative connection of single-party, and thus they are LOCC-equivalent. Similar analysis can be applied for other basis state.

The highest energy levels are \( |GHZ_1'\rangle_4 \) and \( |GHZ_2'\rangle_4 \) with energy \( E_2 = 4 \), which also are the degenerate ground states of \( H_	ext{AF}^N = -H_F^N \) at the same time. With the equivalence \( H_	ext{AF}^N = (\sigma_1^z \otimes \sigma_2^z) H_F^N (\sigma_1^z \otimes \sigma_2^z) \), \( |GHZ_{1(2)}\rangle_4 \)
is obviously LOCC equivalent to $|\text{GHZ}_{1(2)}\rangle_4 = \frac{1}{\sqrt{2}}(|1111\rangle \pm |0000\rangle)$.

5-qubit $|\text{GHZ}_1\rangle_5$ and $|\text{GHZ}_2\rangle_5$ are the ground states of $H_F^5$. Similar to 4-qubit case, the first excited state have two distinct structures, 
\[
\{ |W_{1(2)}\rangle_5, |T_{1(2)}\rangle_5, |T_{1(2)}\rangle_5/5/5/5/5/5 \} \quad \text{and} \quad \{ |W_{3(4)}\rangle_5, |T_{3(4)}\rangle_5, |T_{3(4)}\rangle_5/5/5/5/5/5 \}.
\]
One can also introduce the next nearest interaction $H' = -\lambda \sum_{n=0}^{5} \sigma_n \sigma_{n+2}$ in order to differentiate them. Then the energy are $E'_1 = -1 - \lambda$ for the former and $E'_1 = -1 + 3\lambda$ for the latter. Finally the class \[
\{ |W_{5(6)}\rangle_5, |T_{5(6)}\rangle_5, |T_{5(6)}\rangle_5, |T_{5(6)}\rangle_5, |T_{5(6)}\rangle_5 \}
\] composes the highest level.

6-qubit case

Except of the degenerate ground states $|\text{GHZ}_{1(2)}\rangle_6$, the situation becomes complex.

The first excited state with $E_1 = -2$ have three distinct structures, 
\[
\{ |W_{1(2)}\rangle_6, |T_{1(2)}\rangle_6, |T_{1(2)}\rangle_6, |T_{1(2)}\rangle_6, |T_{1(2)}\rangle_6, |T_{1(2)}\rangle_6 \},
\]
\[
\{ |W_{3(4)}\rangle_6, |T_{3(4)}\rangle_6, |T_{3(4)}\rangle_6, |T_{3(4)}\rangle_6, |T_{3(4)}\rangle_6, |T_{3(4)}\rangle_6 \},
\]
and \[
\{ |W_{7(8)}\rangle_6, |T_{7(8)}\rangle_6, |T_{7(8)}\rangle_6, |T_{7(8)}\rangle_6, |T_{7(8)}\rangle_6, |T_{7(8)}\rangle_6 \}.
\]
By introducing the next nearest interaction $H' = -\lambda \sum_{n=0}^{5} \sigma_n \sigma_{n+2}$, the energy becomes $E'_1 = -2 - 2\lambda$ for the first case and $E'_1 = -2 + 2\lambda$ for the other two case. Then one has to introduce the next nearest interaction in order to differentiate the latter two cases. Similar situation can be found for the second excited states with $E_2 = 2$. There are four distinct classes, 
\[
\{ |W_{5(6)}\rangle_6, |T_{5(6)}\rangle_6, |T_{5(6)}\rangle_6, |T_{5(6)}\rangle_6, |T_{5(6)}\rangle_6, |T_{5(6)}\rangle_6 \},
\]
\[
\{ |W_{8(9)}\rangle_6, |T_{8(9)}\rangle_6, |T_{8(9)}\rangle_6, |T_{8(9)}\rangle_6, |T_{8(9)}\rangle_6, |T_{8(9)}\rangle_6 \},
\]
and \[
\{ |W_{10(11)}\rangle_6, |T_{10(11)}\rangle_6, |T_{10(11)}\rangle_6, |T_{10(11)}\rangle_6, |T_{10(11)}\rangle_6, |T_{10(11)}\rangle_6 \}.
\]
By introducing the next nearest interaction, one can differentiate the first from the other, the last two cases can be differentiated by introducing the next nearest interaction.

The highest level corresponds to $|\text{GHZ}_{1(2)}\rangle_6$ with $E_3 = 6$. Similar to 4-qubit case, they also the ground states of $H_{AF}^6 = -H_F^6$, of which equivalence is $H_{AF}^6 = (\sigma_1^0 \otimes \sigma_2^0 \otimes \sigma_3^0) H_F^6 (\sigma_1^0 \otimes \sigma_2^0 \otimes \sigma_3^0)$. Thus $|\text{GHZ}_{1(2)}\rangle_6$ are LOCC equivalent to $|\text{GHZ}_{1(2)}\rangle_6$.

C. Generalization to arbitrary states

Two situations can be identified in this case. One is when a multipartite state is fully separable. It is known that all fully separable states are LOCC-equivalent via local unitary operation $U = \otimes_{n=1}^{N} u_n$. The fully separable states can be classified as the classical phase since there is no any entanglement.

The other is for a general entangled state, which can be decomposed under the symmetric basis. Since it is composed of different global entanglements, we call it hybrid. Then two cases can be found: One is for the hybrid state which is the superposition of the basis states with different $c$. Thus the translational invariance is inevitably broken in this situation. The other is for the hybrid state which is the superposition of the basis states with the same $c$. Although the translational invariance would be preserved, the state may include different periodic patterns or cyclic units. Then the global symmetry is also broken. Thus all hybrid states can be classified as the symmetry-broken phase.

In general, we cannot tell the equivalence or not between arbitrary two hybrid states only by their decomposition on the symmetric basis since the entanglement depends not on the basis states, but on the coefficients of superposition. Thus one has to rely on the case-by-case studies in this situation. From point of phase transition, hybrid state could be considered as the intermediated state, which is induced by perturbation. Then it would be possible for a judgement of the equivalence to impose into a concrete system.

IV. CONCLUSION

A phase diagram is constructed, as shown in Fig. 3. By the property of entanglement, two grand classes can be identified as classical phase and quantum phase. It should emphasize that the term of quantum phase is used to emphasize the difference between entangled and fully separable state in this place. In indifferent to the definition in condensed matter phase [12]. As for quantum phase, there are two subclasses. When translational invariance is imposed, a symmetric basis can be constructed, which is divided into several classed by periodic patter and cyclic unit. Whenever there is no symmetry, all states are classified as symmetry-broken phase. We have to admit that it is coarse since there are some interesting situations to identify case by case. Actually hybrid state could be considered as a intermediated state from the point of dynamics of phase transition. Thus its behavior would be dependent on concrete Hamiltonian.

In conclusion, a classification of $N$-qubit entangled states is presented for states with translational symmetry. By imposing the translational invariance, the symmetric basis can be constructed readily, which is the man-
ification of the global features of entangles states. By identifying periodic pattern and cyclic unit, the symmetric basis states can be divided into several inequivalent classes. We wish this discussion would helpful for the general understanding of multipartite entanglement.

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Appendix: Several examples of the construction of symmetric basis

This appendix provides several examples for an illustration of Observation 2.

1. 3-qubit case

The symmetric basis is

\[ |\text{GHZ}_1\rangle_3 = \frac{1}{\sqrt{2}} (|111\rangle + |000\rangle) \]
\[ |\text{GHZ}_2\rangle_3 = \frac{1}{\sqrt{2}} (|111\rangle - |000\rangle) \]

3-period:

\[ |W_1\rangle_3 = \frac{1}{\sqrt{3}} (|100\rangle + |010\rangle + |001\rangle) \]
\[ |W_2\rangle_3 = \frac{1}{\sqrt{3}} (|110\rangle + |011\rangle + |101\rangle) \]
\[ |T_1\rangle_3 = \frac{1}{\sqrt{3}} (|100\rangle + e^{i2\pi/3}|010\rangle + e^{i4\pi/3}|001\rangle) \]
\[ |T_1^*\rangle_3 = (|T_1\rangle_3)^\ast \]
\[ |T_2\rangle_3 = \frac{1}{\sqrt{3}} (|110\rangle + e^{i2\pi/3}|101\rangle + e^{i4\pi/3}|011\rangle) \]
\[ |T_2^*\rangle_3 = (|T_2\rangle_3)^\ast \]  \hspace{1cm} \text{(3)}

Thus \( c = 1, e^{\pm i2\pi/3}, \) and the orthogonal relation of basis states can be easily found by \( 1 + e^{i2\pi/3} + e^{i4\pi/3} = 0. \) Interestingly the values of \( c \) corresponds to the characters of point group \( C_3, \) as shown in Fig. 4. This feature is popular, as shown in the following subsections.

It is obvious

\[ |T_1\rangle_3 = I \otimes \begin{pmatrix} e^{i2\pi/3} & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} e^{i4\pi/3} & 0 \\ 0 & 1 \end{pmatrix} |W_1\rangle_3. \hspace{1cm} \text{(6)} \]

Similar relation can be found for \( |T_2\rangle_3. \) Thus by LOCC there are two inequivalent classes \( \{|\text{GHZ}_1\rangle_3, |\text{GHZ}_2\rangle_3\} \) and \( \{|W_{1(2)}\rangle_3, |T_{1(2)}\rangle_3, |T_{1(2)}^*\rangle_3\}. \)
2. 4-qubit case

The symmetric basis is

\[ |\text{GHZ}_1\rangle_4 = \frac{1}{\sqrt{2}}(|1111\rangle + |0000\rangle) \]
\[ |\text{GHZ}_2\rangle_4 = \frac{1}{\sqrt{2}}(|1111\rangle - |0000\rangle) \]

2-period:
\[ |\text{GHZ}'_1\rangle_4 = \frac{1}{\sqrt{2}}(|1010\rangle + |0101\rangle) \]
\[ |\text{GHZ}'_2\rangle_4 = \frac{1}{\sqrt{2}}(|1010\rangle - |0101\rangle) \] (7)

4-period:
\[ |W_1\rangle_4 = \frac{1}{2}(|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle) \]
\[ |T_1\rangle_4 = \frac{1}{2}(|1000\rangle + e^{i\pi/2}|0100\rangle + e^{i\pi}|0010\rangle + e^{i3\pi/2}|0001\rangle) \]
\[ |T_1^*\rangle_4 = (|T_1\rangle_4)^* \]
\[ |T_1'\rangle_4 = \frac{1}{2}(|1000\rangle - |0100\rangle + |0010\rangle - |0001\rangle) \]
\[ |W_2\rangle_4 = \frac{1}{2}(|1110\rangle + |1101\rangle + |1011\rangle + |0111\rangle) \]
\[ |T_2\rangle_4 = \frac{1}{2}(|1110\rangle + e^{i\pi/2}|1101\rangle + e^{i\pi}|1011\rangle + e^{i3\pi/2}|0111\rangle) \]
\[ |T_2^*\rangle_4 = (|T_2\rangle_4)^* \]
\[ |T_2'\rangle_4 = \frac{1}{2}(|1110\rangle - |1101\rangle + |1011\rangle - |0111\rangle) \]
\[ |W_3\rangle_4 = \frac{1}{2}(|1100\rangle + |0110\rangle + |0011\rangle + |1001\rangle) \]
\[ |T_3\rangle_4 = \frac{1}{2}(|1100\rangle + e^{i\pi/2}|0110\rangle + e^{i\pi}|0011\rangle + e^{i3\pi/2}|1001\rangle) \]
\[ |T_3^*\rangle_4 = (|T_3\rangle_4)^* \]
\[ |T_3'\rangle_4 = \frac{1}{2}(|1100\rangle - |0110\rangle + |0011\rangle - |1001\rangle) \] (8)

Besides of \(|\text{GHZ}_{1(2)}\rangle_4\) and \(|W_{1(2,3)}\rangle_4\), it is easy to verify

\[ \hat{T}|\text{GHZ}'_1\rangle_4 = |\text{GHZ}'_1\rangle_4 \]
\[ \hat{T}|\text{GHZ}'_2\rangle_4 = -|\text{GHZ}'_2\rangle_4 \]
\[ \hat{T}|T_1\rangle_4 = -i|T_1\rangle_4 \]
\[ \hat{T}|T_2\rangle_4 = i|T_2\rangle_4 \]
\[ \hat{T}|T_3\rangle_4 = -i|T_3\rangle_4 \]
\[ \hat{T}|T'_{1(2,3)}\rangle_4 = -|T'_{1(2,3)}\rangle_4. \] (9)

Thus \(c = \pm 1, \pm i\), which are the characters of point group \(C_4\) as shown in Fig[6].

FIG. 6: (Color online) A graphical illustration of the value of \(c\) in Eq.[I] for 5-qubit case.
3. 5-qubit case

The symmetric basis is

\[
|\text{GHZ}_1\rangle_5 = \frac{1}{\sqrt{2}} (|11111\rangle + |00000\rangle)
\]
\[
|\text{GHZ}_2\rangle_5 = \frac{1}{\sqrt{2}} (|11111\rangle - |00000\rangle)
\]
\[
|W_1\rangle_5 = \frac{1}{\sqrt{5}} (|10000\rangle + |01000\rangle + |00100\rangle + |00010\rangle + |00001\rangle)
\]
\[
|T_1\rangle_5 = \frac{1}{\sqrt{5}} \left( |10000\rangle + e^{i2\pi/5}|01000\rangle + e^{i4\pi/5}|00100\rangle + e^{i6\pi/5}|00010\rangle + e^{i8\pi/5}|00001\rangle \right)
\]
\[
|T_1^*\rangle_5 = \langle (T_1) |\langle T_1^* \rangle_5 = \frac{1}{\sqrt{5}} \left( |01000\rangle + e^{i2\pi/5}|10000\rangle + e^{i4\pi/5}|01000\rangle + e^{i6\pi/5}|00100\rangle + e^{i8\pi/5}|00010\rangle \right)
\]
\[
|T_2\rangle_5 = \frac{1}{\sqrt{5}} \left( |01111\rangle + e^{i2\pi/5}|10111\rangle + e^{i4\pi/5}|11011\rangle + e^{i6\pi/5}|11101\rangle + e^{i8\pi/5}|11110\rangle \right)
\]
\[
|T_2^*\rangle_5 = \langle (T_2) \rangle_5 \frac{1}{\sqrt{5}} \left( |01111\rangle + e^{i2\pi/5}|10111\rangle + e^{i4\pi/5}|11011\rangle + e^{i6\pi/5}|11101\rangle + e^{i8\pi/5}|11110\rangle \right)
\]
\[
|T_3\rangle_5 = \frac{1}{\sqrt{5}} \left( |11000\rangle + e^{i2\pi/5}|01100\rangle + e^{i4\pi/5}|00110\rangle \right)
\]
\[
|T_3^*\rangle_5 = \langle (T_3) \rangle_5 \frac{1}{\sqrt{5}} \left( |11000\rangle + e^{i2\pi/5}|01100\rangle + e^{i4\pi/5}|00110\rangle \right)
\]
\[
|T_4\rangle_5 = \frac{1}{\sqrt{5}} \left( |00111\rangle + e^{i4\pi/5}|01101\rangle + e^{i8\pi/5}|11001\rangle \right)
\]
\[
|T_4^*\rangle_5 = \langle (T_4) \rangle_5 \frac{1}{\sqrt{5}} \left( |00111\rangle + e^{i4\pi/5}|01101\rangle + e^{i8\pi/5}|11001\rangle \right)
\]
\[
|W_2\rangle_5 = \frac{1}{\sqrt{5}} \left( |01111\rangle + |10111\rangle + |11011\rangle + |11101\rangle \right)
\]
\[
|W_3\rangle_5 = \frac{1}{\sqrt{5}} \left( |11000\rangle + |01100\rangle + |00110\rangle + |00011\rangle + |10010\rangle \right)
\]
\[
|W_4\rangle_5 = \frac{1}{\sqrt{5}} \left( |00111\rangle + |10011\rangle + |11001\rangle + |11100\rangle \right)
\]
\[
|W_5\rangle_5 = \frac{1}{\sqrt{5}} \left( |10100\rangle + |01010\rangle + |00101\rangle + |00010\rangle + |00001\rangle \right)
\]
\[
|W_6\rangle_5 = \frac{1}{\sqrt{5}} \left( |01011\rangle + |10101\rangle + |11010\rangle + |11100\rangle \right)
\]
\[
|T_5\rangle_5 = \frac{1}{\sqrt{5}} \left( |01000\rangle + e^{i2\pi/5}|01010\rangle + e^{i4\pi/5}|00101\rangle + e^{i6\pi/5}|00100\rangle + e^{i8\pi/5}|00010\rangle \right)
\]
\[
|T_5^*\rangle_5 = \langle (T_5) \rangle_5 \frac{1}{\sqrt{5}} \left( |01000\rangle + e^{i2\pi/5}|01010\rangle + e^{i4\pi/5}|00101\rangle + e^{i6\pi/5}|00100\rangle + e^{i8\pi/5}|00010\rangle \right)
\]
\[
|T_6\rangle_5 = \frac{1}{\sqrt{5}} \left( |00111\rangle + e^{i4\pi/5}|01101\rangle + e^{i8\pi/5}|11001\rangle \right)
\]
\[
|T_6^*\rangle_5 = \langle (T_6) \rangle_5 \frac{1}{\sqrt{5}} \left( |00111\rangle + e^{i4\pi/5}|01101\rangle + e^{i8\pi/5}|11001\rangle \right)
\]
\[
|T_7\rangle_5 = \frac{1}{\sqrt{5}} \left( |01111\rangle + e^{i2\pi/5}|10111\rangle + e^{i4\pi/5}|10011\rangle + e^{i6\pi/5}|10001\rangle \right)
\]
\[
|T_7^*\rangle_5 = \langle (T_7) \rangle_5 \frac{1}{\sqrt{5}} \left( |01111\rangle + e^{i2\pi/5}|10111\rangle + e^{i4\pi/5}|10011\rangle + e^{i6\pi/5}|10001\rangle \right)
\]

(11)
It is easy to verify, for instance,

\[
\hat{T}_T \ket{1_5} = \frac{1}{\sqrt{5}} \left( (01000) + e^{i2\pi/5} |00100\rangle + e^{i4\pi/5} |00010\rangle \\
+ e^{i6\pi/5} |00001\rangle + e^{i8\pi/5} |10000\rangle \right) \\
= e^{i8\pi/5} \left( |10000\rangle + e^{i2\pi/5} |01000\rangle + e^{i4\pi/5} |00100\rangle \\
+ e^{i6\pi/5} |00010\rangle + e^{i8\pi/5} |00001\rangle \right) \\
= e^{-i2\pi/5} \ket{1_5}
\]

\[
\hat{T}_T \ket{1_6} = \frac{1}{\sqrt{6}} \left( (01000) + e^{i4\pi/5} |00100\rangle + e^{i8\pi/5} |00010\rangle \\
+ e^{i12\pi/5} |00001\rangle + e^{i16\pi/5} |10000\rangle \right) \\
= e^{-i4\pi/5} \ket{1_6}.
\]

(13)

The other values of \(c\) can also be obtained by similar method. Obviously the values of \(c\) are just the characters of point group \(C_5\), as shown in Fig.6.

There are 4 classes, dependent on the periodicity and the cyclic unit,

\[
\begin{align*}
\{ & |GHZ(1_2,5)\rangle \}, \\
\{ & |W(1_2,5)\rangle, |T_{1(2),5} \rangle, |T'_{1(2),5} \rangle |T''_{1(2),5} \rangle \}, \\
\{ & |W(3_4,5)\rangle, |T(3_4,5)\rangle, |T'_{3(4),5} \rangle, |T''_{3(4),5} \rangle \}, \\
\{ & |W(5_6,5)\rangle, |T(5_6,5)\rangle, |T'_{5(6),5} \rangle, |T''_{5(6),5} \rangle \}
\end{align*}
\]

(14)

4. 6-qubit case

The symmetric basis is

\[
\ket{GHZ_1}_6 = \frac{1}{\sqrt{2}} (|111111\rangle + |000000\rangle)
\]

\[
\ket{GHZ_2}_6 = \frac{1}{\sqrt{2}} (|111111\rangle - |000000\rangle)
\]

2-period:

\[
\ket{GHZ_1'}_6 = \frac{1}{\sqrt{2}} (|101010\rangle + |101010\rangle)
\]

\[
\ket{GHZ_2'}_6 = \frac{1}{\sqrt{2}} (|101010\rangle - |101010\rangle).
\]

3-period:

\[
\ket{W_0}_6 = \frac{1}{\sqrt{3}} (|100100\rangle + |010010\rangle + |001001\rangle)
\]

\[
\ket{T_0}_6 = \frac{1}{\sqrt{3}} (|100100\rangle + e^{i2\pi/3} |010010\rangle + e^{i4\pi/3} |001001\rangle)
\]

\[
\ket{T_6}_6 = \frac{1}{\sqrt{3}} (|101011\rangle + e^{i2\pi/3} |101101\rangle + e^{i4\pi/3} |110110\rangle)
\]

\[
\ket{T_0'}_6 = \frac{1}{\sqrt{3}} (|101011\rangle + e^{i2\pi/3} |101101\rangle + e^{i4\pi/3} |110110\rangle)
\]

\[
\ket{T_6'}_6 = \frac{1}{\sqrt{3}} (|101011\rangle + e^{i2\pi/3} |101101\rangle + e^{i4\pi/3} |110110\rangle)
\]

6-period:

\[
\ket{W_0'}_6 = \frac{1}{\sqrt{3}} (|110110\rangle + |011011\rangle + |101101\rangle)
\]

\[
\ket{T_0''}_6 = \frac{1}{\sqrt{3}} (|011011\rangle + e^{i2\pi/3} |101101\rangle + e^{i4\pi/3} |110110\rangle)
\]

\[
\ket{T_6''}_6 = \frac{1}{\sqrt{3}} (|011011\rangle + e^{i2\pi/3} |101101\rangle + e^{i4\pi/3} |110110\rangle)
\]

\[
\ket{T_0'''}_6 = \frac{1}{\sqrt{3}} (|110110\rangle + |011000\rangle + |001100\rangle)
\]

\[
\ket{T_6'''}_6 = \frac{1}{\sqrt{3}} (|110110\rangle - |011000\rangle + |001100\rangle)
\]

\[
\ket{T_0''''}_6 = \frac{1}{\sqrt{3}} (|110110\rangle - |011000\rangle - |001100\rangle)
\]
\[ |W_4\rangle = \frac{1}{\sqrt{6}} \left( |001111\rangle + |101111\rangle + |110011\rangle + |111001\rangle + |111100\rangle + |011110\rangle \right) \]

\[ |T_4\rangle = \frac{1}{\sqrt{6}} \left( |001111\rangle + e^{i\pi/3}|100111\rangle + e^{i2\pi/3}|110011\rangle + e^{i\pi}|111001\rangle + e^{i4\pi/3}|111100\rangle + e^{i\pi/3}|011110\rangle \right) \]

\[ |T'_4\rangle = (|T_4\rangle)^* \]

\[ |W_5\rangle = \frac{1}{\sqrt{6}} \left( |101000\rangle + |011010\rangle + |001010\rangle + |000101\rangle + |100010\rangle + |010001\rangle \right) \]

\[ |T_5\rangle = \frac{1}{\sqrt{6}} \left( |101000\rangle + e^{i\pi/3}|010100\rangle + e^{i2\pi/3}|001010\rangle + e^{i\pi}|000101\rangle + e^{i4\pi/3}|100010\rangle + e^{i\pi/3}|010001\rangle \right) \]

\[ |T'_5\rangle = (|T_5\rangle)^* \]

\[ |T_{w6}\rangle = \frac{1}{\sqrt{6}} \left( |010111\rangle + |101101\rangle + |110101\rangle + |011101\rangle + |100110\rangle + |010011\rangle \right) \]

\[ |T_{6}\rangle = \frac{1}{\sqrt{6}} \left( |010111\rangle + e^{i\pi/3}|101101\rangle + e^{i2\pi/3}|110101\rangle + e^{i\pi}|111010\rangle + e^{i4\pi/3}|011101\rangle + e^{i\pi/3}|100110\rangle \right) \]

\[ |T'_{6}\rangle = (|T_{6}\rangle)^* \]

\[ |T_{7}\rangle = \frac{1}{\sqrt{6}} \left( |011000\rangle + e^{i\pi/3}|011001\rangle + e^{i2\pi/3}|011011\rangle + e^{i\pi}|000111\rangle + e^{i4\pi/3}|001101\rangle \right) \]

\[ |T'_{7}\rangle = (|T_{7}\rangle)^* \]

\[ |T_{8}\rangle = \frac{1}{\sqrt{6}} \left( |111000\rangle + e^{i2\pi/3}|011000\rangle + e^{i4\pi/3}|001110\rangle + e^{i\pi}|000111\rangle + e^{i2\pi/3}|100111\rangle + e^{i4\pi/3}|110001\rangle \right) \]

\[ |T'_{8}\rangle = (|T_{8}\rangle)^* \]

\[ |T_{9}\rangle = \frac{1}{\sqrt{6}} \left( |011000\rangle + |011010\rangle + |001101\rangle + |000110\rangle + |100011\rangle + |010010\rangle \right) \]

\[ |T'_{9}\rangle = (|T_{9}\rangle)^* \]

\[ |T_{w10}\rangle = \frac{1}{\sqrt{6}} \left( |011000\rangle + e^{i\pi/3}|011001\rangle + e^{i2\pi/3}|011010\rangle + e^{i\pi}|011011\rangle + e^{i4\pi/3}|001100\rangle \right) \]

\[ |T'_{w10}\rangle = (|T_{w10}\rangle)^* \]

\[ |T_{11}\rangle = \frac{1}{\sqrt{6}} \left( |011000\rangle + e^{i\pi/3}|011001\rangle + e^{i2\pi/3}|011010\rangle + e^{i\pi}|011011\rangle + e^{i4\pi/3}|001100\rangle \right) \]

\[ |T'_{11}\rangle = (|T_{11}\rangle)^* \]

\[ |T_{w12}\rangle = \frac{1}{\sqrt{6}} \left( |111000\rangle + |011000\rangle + |001110\rangle + |000111\rangle + |100011\rangle + |110011\rangle \right) \]

\[ |T'_{w12}\rangle = (|T_{w12}\rangle)^* \]

\[ |W_{2}\rangle = \frac{1}{\sqrt{6}} \left( |111000\rangle - |011000\rangle + |001110\rangle + |000111\rangle + |100011\rangle + |110011\rangle \right) \]

\[ |W_{3}\rangle = \frac{1}{\sqrt{6}} \left( |111000\rangle + |011000\rangle + |001110\rangle + |000111\rangle + |100011\rangle + |110011\rangle \right) \]

\[ |W_{4}\rangle = \frac{1}{\sqrt{6}} \left( |111000\rangle - |011000\rangle + |001110\rangle + |000111\rangle + |100011\rangle + |110011\rangle \right) \]

\[ |T_{7}\rangle = \frac{1}{\sqrt{6}} \left( |111000\rangle + e^{i2\pi/3}|011000\rangle + e^{i4\pi/3}|001110\rangle + e^{i\pi}|000111\rangle + e^{i2\pi/3}|100111\rangle + e^{i4\pi/3}|110001\rangle \right) \]

It is not difficult to verify, for instance,

\[ T_i |T_{1}\rangle = e^{-i\pi/3} |T_{1}\rangle \]

\[ T_i |T'_{1}\rangle = e^{-i2\pi/3} |T'_{1}\rangle \]

\[ T_i |T'_{w10}\rangle = -|T'_{w10}\rangle \]

Interestingly there are two entangled state of 3-period, which are double replications of 3-qubit case. It is not strange since the values of \( e \) corresponds to the characters of point group \( C_6 \), which consists of two \( C_3 \) subgroups, as shown in Fig.\( \text{[7]} \).

By LOCC, there are 7 other classes determined by the
FIG. 7: (Color online) A graphical illustration of the value of $c$ in Eq. (1) for 6-qubit case.

periodicity and cyclic units,

\[
\begin{align*}
\{ |\text{GHZ}_{1(2)}⟩, |\text{GHZ}_{1(2)}′⟩, \\
|W_0⟩, |T_0⟩, |T_0′⟩, \\
|W_1⟩, |T_1⟩, |T_1′⟩, |T_1′′⟩, |T_1′′′⟩, \\
|W_2⟩, |T_2⟩, |T_2′⟩, |T_2′′⟩, |T_2′′′⟩, \\
|W_3⟩, |T_3⟩, |T_3′⟩, |T_3′′⟩, |T_3′′′⟩, \\
|W_4⟩, |T_4⟩, |T_4′⟩, |T_4′′⟩, |T_4′′′⟩, \\
|W_5⟩, |T_5⟩, |T_5′⟩, |T_5′′⟩, |T_5′′′⟩, \\
|W_6⟩, |T_6⟩, |T_6′⟩, |T_6′′⟩, |T_6′′′⟩, |T_6′′′′⟩, |T_6′′′′′⟩
\end{align*}
\]

The first class can be considered as 2-period, the second class as 3-period. The last 6 classes are all 6-period, distinguished by cyclic units.

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