Hodge-type self(antiself)-duality for general p-form fields in arbitrary dimensions.

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Abstract

It is often claimed [1] that the (Hodge type) duality operation is defined only in even dimensional
spacetimes and that self-duality is further restricted to twice-odd dimensional spacetime theories.
The purpose of this paper is to extend the notion of both duality symmetry as well as self-duality.

By considering tensorial doublets, we introduce a novel well-defined notion of self-duality based
on a duality Hodge-type operation in arbitrary dimension and for any rank of these tensors. Thus,
a generalized Self-Dual Action is defined such that equations of motion are the claimed generalized
self-duality relations. We observe in addition, that taking the proper limit on the parameters of this
action, it always provides us with a master-action, which interpolates models well-studied in physics;
by considering a particular limit, we find an action which describes an interesting type of relation,
referred to as semi-self-duality, which results to be the parent action between Maxwell-type actions.

Finally, we apply these ideas to construct manifest Hodge-type self-dual solutions in a (2+1)-
dimensional version of the Maxwell’s theory.

1 Introduction

The purpose of this paper is to explore a certain freedom in the definition of the duality operation in
order to construct self and anti-self-dual models[2].

This issue is nowadays well-motivated [3]. The role of duality in the investigation of physical systems
is by now well-appreciated. This duality symmetry, that is fundamental in the current understanding
of quantum field theory, statistical mechanics and string theory, is a general concept relating physical
quantities in different regions of the parameter space. It relates a model in a strong coupling regime
to a distinct one in a weak coupling regime, providing a valuable mechanism for investigating strongly
interacting models.

One currently defines the Hodge-Duality (HD) operation by the contraction with the totally anti-
symmetric $\epsilon$-symbol. There is also an extension of this operation used for instance, in 2+1-dimensions,
which is basically a functional curl (rotational operator):

$$ ^* f_\mu = \frac{\chi}{m} \epsilon_{\mu\nu\lambda} \partial^\nu f^\lambda, \quad (1) $$

where $m$ is a constant to render the $^*$-operation dimensionless. We call this Differential Hodge Duality
(DHD), and in all cases, we name self(anti-self)-duality, when the relations $^* f = \pm f$ are (respectively)
satisfied.

The so-called Self-Dual Model [4, 5, 6] is given by the following action,

$$ S(f) = \int d^3x \left( \frac{\chi}{2m} \epsilon_{\mu\nu\lambda} f^\mu \partial^\nu f^\lambda - \frac{1}{2} f_\mu f^\mu \right). \quad (2) $$

The equation of motion is the self-duality relation:

$$ f_\mu = \frac{\lambda}{m} \epsilon_{\mu\nu\lambda} \partial^\nu f^\lambda, \quad (3) $$

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This model is claimed to be chiral, and the chirality $\chi = \pm 1$ results defined precisely from this self-duality.

A (Hodge-type) well-defined self (and anti-self-)duality between tensor fields with different ranks is lacking in the literature. This concept is important by itself; this can be applied, as we have suggested, in a formal generalization of the notion of chirality.

The possibility of defining self-duality and non merely duality between two systems is critically important also in topological configurations; in particular, it can provide us with a generalized notion of instantons.

The question we would like to address regards the possibility of imposing self-duality in general dimensions. This will be answered affirmatively, as will be shown below.

Let us describe better the problem we are concerned with:

Consider a general $q$-form

$$F_{\mu_1 \cdots \mu_q}. \quad (4)$$

The Hodge-dual field is then defined as,

$$\ast F^{\mu_{q+1} \cdots \mu_d} = \frac{1}{q!} \epsilon^{\mu_1 \cdots \mu_d} F_{\mu_1 \cdots \mu_q} \quad (5)$$

Note that, in principle, only in $d = 2q$ dimensions one could define self-duality, since the $q$-form field $F$ would be of the same rank as its dual.

This is the type of duality present in Maxwell-type theories, where $F$ is a exact differential, i.e $F = df$ for a $q - 1$-form field $f$.

In this context, the field equation and Bianchi identity for a source free field are

$$0 = \partial_{\mu_1} F^{\mu_1 \cdots \mu_q}$$

$$0 = \partial_{\mu_1} \ast F^{\mu_1 \cdots \mu_q} \quad (6)$$

The field equation and the Bianchi identity are of the same form so that the duality transformation $F \leftrightarrow \ast F$ is a symmetry but in general, is not present at level of the tensor-fields.

The dependence with dimensionality appears to be crucial.

It is well-known the problem of defining the Hodge duality for all dimensions; for instance, in Lorentzian four-dimensional spacetime, the main obstruction to self-duality comes from the relation of double-dualization for a rank-two tensor

$$**F = (-1)^s F$$

where $s$ is the signature of the Minkowski metric. For the case of the Lorentzian metric, where $s$ is an odd number, the self-duality concept seems inconsistent with the double dualization operation due to the minus sign in $(-1)^s$. This problem remains for dimensionality $d = 4m$ ($m \in \mathbb{Z}^+$), in contrast, it is absent for $d = 4m - 2$. Thus, self-duality is claimed to be well defined (only) in such dimensionality.

The self-duality present in $d = 4m - 2$ dimensions has attracted much attention because it seems to play an important role in many theoretical models. The possibility we are discussing in this work should allow us to extend these applications.

This work is organized according to the following outline:

In Section 2, these difficulties and its resolution are clarified, the notion of duality is generalized, allowing to construct a generalized manifestly self-dual action; next, in Section 3, we discuss the relation of this action with the so-called parent action, which interpolates between several couples of (dual) equivalent models. Finally, in Section 4, as an application, we build Hodge duality in 2+1-dimensions in a similar fashion with the Maxwell’s theory of the electromagnetism. This construction is new.
2 Generalization of HD (-DHD) and self(anti-self)-duality.

In order to remove the obstruction for consistency between duality and self-duality posed by the presence of the minus sign in (7), we start by exploring the existence of an ambiguity in the double dual operation that comes around from the fact that the duality operation is a mapping from the space of $d/2$ forms to its co-space, $*: \Lambda_{(0,d/2)} \rightarrow \Lambda_{(d/2,0)}$, so that the inverse mapping is not automatically defined. This leaves some room for distinct alternatives with interesting consequences. First, let us recall that (7) has led to the prejudice that the (Abelian) Maxwell theory would not possess manifest self-duality solutions. The resolution for this obstacle came with the recognition of an internal two-dimensional structure hidden in the space of potentials. Transformations in this internal duality space extends the self-duality concept to this case and is currently known under the names of Schwarz and Sen[15], but this deep unifying concept has also been appreciated by others[16, 17]. The actions obtained corresponds to self-dual and anti-self-dual representation of a given theory and make use of the internal space concept. The duality operation is now defined to include the internal (two dimensional) index $(\alpha, \beta)$ in the fashion

$$\hat{F}^\alpha = e^{\alpha\beta} * F^\beta$$

where the $2 \times 2$ matrix, $e$, depends on the signature and dimension of the spacetime in the form:

$$e^{\alpha\beta} = \begin{cases} 
\sigma_1^{\alpha\beta}, & \text{if } d = 4m - 2 \\
\epsilon^{\alpha\beta}, & \text{if } d = 4m.
\end{cases}$$

with $\sigma_1^{\alpha\beta}$ being the first of the Pauli matrices and $e^{\alpha\beta}$ is the totally antisymmetric $2 \times 2$ matrix with $\epsilon^{1,2} = 1$.

The double duality operation

$$\hat{\hat{F}} = F$$

generalizes (8) to allow consistency with self-duality. In [19] we show how the prescription (8) works in the construction of self-dual Maxwell actions.

Most of the usual discussion about duality transformations as a symmetry for the actions and the existence of self-duality are based on these concepts.

Remark: Up to now, this structure has always been defined only for tensorial objects where the field has the same tensorial rank that its corresponding dual.

Now, we generalize further these ideas, introducing more general doublets. This concept is new and constitutes our main contribution.

The matters described above can be avoided by defining Hodge duality (HD) and differential Hodge duality (DHD) for tensorial doublets in a matricial form. Let a p-form $(a$ totally antisymmetric tensor type $(0;p))$ on a $d$-dimensional space-time with signature $s$ $f_{\mu_1,...,\mu_p}$, one natural partner shall be any $((d - p);0)$-tensor, $g_{\mu_1,...,\mu_{d-p}}$. We build the tensor doublet $F := (f, g)$.

**Hodge Duality:**

Let us define now the generalized Hodge-operation $\mathcal{F}((i^*)$ for this object by means of

$$^*F := (^*g, S^*f),$$

where

$$(^*f)_{\mu_{p+1}...\mu_d} = \frac{1}{p!} \epsilon^{\mu_1...\mu_d} f_{\mu_1...\mu_p},$$

\[^2\text{i.e, this is the number of minuses occurring in the metric}
\[^3\text{For notational convenience, in order to avoid explicit indices, in the following lines we forget that the duality operation must be defined as a map from the field-space into its co-space.}

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and $S$ is a number defined by the double dualization operation:

$$^{(*f)} = Sf. \quad (13)$$

This depends on the signature $(s)$ and spacetime dimension in the form $S = (-1)^{s+p[d-p]}$, which clearly includes the case $p = d/2$ described above.

Notice that this Hodge-type self (anti-self)-duality is well-defined, since

$$^{*F} = \pm F, \quad (14)$$

is consistent with the double dualization requirement, $^{*(^{*F})} = F$.

**Differential Hodge Duality:**

Furthermore, DHD can be also generalized in this fashion: let us consider doublets $F = (f_{\mu_1,...,\mu_p}; g_{\mu_1,...,\mu_d-p-1})$

then, in terms of forms, DHD is defined as

$$^F \equiv K^{-1} * dF, \quad (15)$$

where $d(f, g) = (df, dg)$ and the matrix $K = \text{diag}[k_f; k_g]$, that is taken to be diagonal for simplicity, is introduced for dimensional reasons\(^4\).

Thus, self (anti-self)-duality is also well-defined in this case, since the relations

$$^{*F} = \pm F \quad (16)$$

may be consistent with the double dualisation requirement, $^{*(^{*F})} = F$.

The self-duality relation in this case reads

$$F = (\chi K^{-1}) * dF, \quad (17)$$

where $\chi = \pm 1$. As it can be trivially verified, the consistency of self-duality requires that $F$ satisfies the Proca equation with mass $m = \sqrt{k_f k_g}$. In fact, applying once more the operator $^*$ to (17), we have:

$$[\partial^2 + m^2] F = 0. \quad (18)$$

The next step is to obtain an action which expresses self-duality in this generalized sense. Then, we write down this one, which is a generalized Self-Dual Model (GSDM) in $d$-dimensions\(^5\):

$$S_{GSD}[F] = \int d^d x \left( \frac{\chi}{2} F \cdot * dF + \frac{1}{2} F K F \right). \quad (19)$$

This is the central object of this work. It is straightforward to verify that the equations of motion are precisely the self-duality relations (17). Notice that this action looks like SD-action in three dimensions, Eq. (2), and here is its main importance since the structure of these theories (in $2 + 1$) can naturally be extended to arbitrary dimensions. Results based on this issue, in the context of topologically massive theories\(^4\) and bosonization in general dimensions\(^5\), are being reported elsewhere.

Furthermore, as we show below, one can obtain, by taking or not, appropriate limits of the constants $(k_f; k_g)$, this corresponds to different Parent Actions, and so describing different dualities between models whose fields are precisely the components of the general doublet $F$ (see\(^4\) and references therein).

\(^4\) i.e. $k_f$ and $k_g$ must have dimension of mass.

\(^5\) The doublet internal product of pairs "s" is naturally given by $(f, g) \cdot (f', g') = f_{\mu_1,...,\mu_p} f'^{\nu_1,...,\nu_p} + g_{\mu_1,...,\mu_q} g'^{\nu_1,...,\nu_q}$, where $f, f'$ are $p$-forms, while $g, g'$ are any $q$-forms.
3 GSDM and Parent Actions.

As it has been motivated, in this section we argue that the parent actions describing duality between physically relevant models can be obtained from the GSDM-action by a different fixing of the tensor-doublet, the duality operation, and the coupling matrix $K$.

A very nice structure of dualities arises from this elegant analysis. Depending on the $K$-parameters we take, interesting consequences are obtained. For instance, if $k_g = k_f = 0$, we get a topological theory; if in contrast, $K$ is non-singular, this is the master action between two Proca models for both $f$ and $g$. We show this later.

The main result is obtained by considering only $k_g = 0$. This model reflects a sort of semi-self-duality as we will show, and provides us with the master action in general. We will illustrate this by means of a simple example and in the following subsection we shall prove this in general.

3.1 Example: Scalar-Tensor Duality

This is an example of duality between two systems of different tensorial ranks, where the relations of self-duality cannot be written. According to our prescription, we shall show that self-duality could be defined for this system, via an action of the type (19). With a proper limit to obtain a singular $K$-matrix, we recover the familiar parent actions of this problem. This example was discussed in a very illuminating way by Hjelmeland and Linstrøm[13].

Consider the action for a massless free Klein-Gordon field $\phi$ in $d = 4$

$$S(\phi) = \frac{1}{2} \int d^4x \partial_\mu \phi \partial^\mu \phi,$$

(20)

the field equation and the Bianchi identities for the free Klein-Gordon field are respectively:

$$\partial_\mu \partial^\mu \phi = 0$$

(21)

$$\partial_\mu * \phi^{\mu\nu\rho}(\phi) = 0$$

(22)

where $* \phi^{\mu\nu\rho} \equiv \epsilon^{\mu\nu\rho\sigma} \partial_\sigma \phi$.

On the other hand, the action for a free massless two-form field $A_{\mu\nu}$ is

$$S(A) = \frac{1}{3!} \int d^4x \partial_\mu A_{\nu\rho} \partial^{[\mu} A^{\nu\rho]},$$

(23)

Now, we write down the Bianchi identity and the field equations, respectively:

$$\partial_\mu * A^\mu = 0$$

(24)

$$\partial_\mu \partial^{[\mu} A^{\nu\rho]} = 0$$

(25)

where $* A^\mu \equiv \frac{1}{3!} \epsilon^{\mu\nu\rho\sigma} \partial_\nu A_{\rho\sigma}$.

The main point is that the field equation for the free Klein-Gordon field looks like the Bianchi identity for the free anti-symmetric field, and vice-versa. The change from one description to the other interchanges the roles of the field equations and the Bianchi identities. At the classical level, they are two models representing the same physics. To show this explicitly, it is currently introduced the so-called parent action:

$$S_p(F_{\mu\nu\rho}, \phi) = \frac{1}{3!} \int d^4x \left( F_{\mu\nu\rho} F^{\mu\nu\rho} + \sqrt{2} \phi \partial_\mu * F^{\mu} \right),$$

(26)

where $\phi$ and $F_{\mu\nu\rho}$ are independent field. Varying this action with respect to $\phi$ gives:

$$0 = \epsilon^{\mu\nu\rho\sigma} \partial_\mu F_{\nu\rho\sigma}.$$  

(27)
Hence, there exists a two-form field, $A_{\rho\sigma}$, such that $F_{\nu\rho\sigma} = \partial_{[\nu} A_{\rho\sigma]}$. Putting this back into the action (26), we recover the action (23). We have thus shown that (26) is equivalent to (23). Substitution of solutions of field equations into the parent action requires that consistency at level of field equations be verified; however, in this case, this is not a serious matter and the (on-shell) equivalence between the parent action and (23) is verified.

Now, in order to show that (26) is also equivalent to (20), we vary (26) with respect to $F_{\mu\nu\rho}$, thus:

$$F_{\mu\nu\rho} = -\frac{1}{\sqrt{2}} \epsilon^{\mu\nu\rho\sigma} \partial_\sigma \phi.$$  

Replacing this into $S_p(F, \phi)$ we obtain

$$S_p(F[\phi], \phi) = \frac{1}{2} \int d^4x \partial_\mu \phi \partial^\mu \phi.$$  

We have shown, using the parent action, that $S(\phi)$ and $S(A)$ are dual to each other; the two actions describe the same physical system, but the physical representation is given using different fields. A remarkable feature of this construction is that the field equations and the Bianchi identities are exchanged.

According to the doublets structure presented in the preceding section, we define the doublet $F = (\phi, F)$, and self-duality relations for this system can be written as two simultaneous equations:

$$F_{\mu\nu\rho} = -\frac{1}{k_F} \epsilon^{\mu\nu\rho\sigma} \partial_\sigma \phi,$$

$$\phi = \frac{1}{k_\phi} \epsilon^{\mu\nu\rho\sigma} \partial_\rho F_{\nu\mu\sigma},$$  

which may be derived of an theory with the form GSDM (Equation (19)),

$$S_F(F) = \int d^4x F[.^* d + K] F$$  

where $K = diag(k_F; k_\phi)$. However, if we take $k_\phi = 0$ and $k_F = \sqrt{2}$, we clearly obtain relations, instead of (30) and (31), which can be interpreted as a sort of semi-self-duality. In fact, the action (32) dictates the form of the parent action:

$$S_F(F) = \int d^4x \left( F_{\mu\nu\rho} \sqrt{2} F_{\nu\rho\mu} + \phi \epsilon^{\mu\nu\rho\sigma} \partial_\rho F_{\nu\mu\sigma} - F_{\nu\rho\sigma} \epsilon^{\mu\nu\rho\sigma} \partial_\mu F \right) = \frac{3}{\sqrt{2}} S_p(F_{\mu\nu\rho}, \phi) + \text{Total divergence}.$$  

This structure is illustrated in the Fig.4 below.

On the other hand, we obtain remarkably the parent action (26). Parent actions are not unique, as well as doublets in a given dimension.

Below, we understand that, in dimension four, another doublet can be chosen $G = (A_{\nu\rho}, f_{\mu})$, resulting in the same duality.

Thus, taking $k_f = \frac{1}{\sqrt{2}} k_A = 0$, the other parent action, which also shows that $S_G$ and $S_A$ are dual to one another, reads as follows:

$$S_G(G) = \int d^4x G[.^* d + K] G$$

$$= \int d^4x \left( G[.^* dG + \frac{1}{\sqrt{2}} f_{\mu} f^\mu] \right)$$

$$= \frac{1}{\sqrt{2}} \int d^4x \left( \frac{1}{2} f_{\mu} f^\mu + \sqrt{2} A_{\mu\rho} \epsilon^{\mu\nu\rho\sigma} \partial_\rho f_\sigma \right) + \text{Total divergence},$$  

where $f$ and $A$ are independent fields.
Varying this action with respect to $A_{\mu\nu}$, we obtain

$$0 = \epsilon^{\rho\mu\nu\sigma} \partial_{\rho} f_{\sigma}. \quad (35)$$

Again, this implies that there exists a scalar field $\phi$ (at least locally) such that $f_{\mu} = \partial_{\mu} \phi$.

Replacing this expression in (34), we have:

$$S_G = \frac{1}{\sqrt{2}} S(\phi) \quad (36)$$

Varying now the parent action with respect to $f_{\mu}$, we obtain

$$f_{\mu} = -\sqrt{2} \epsilon^{\mu\nu\rho\sigma} \partial_{\nu} A_{\rho\sigma}. \quad (37)$$

Plugging this back into $S_{F,A}$, we get

$$S_G(f[A], A) = \frac{1}{2} S(A) = \frac{1}{3!\sqrt{2}} \int d^4x \partial_{\mu} A_{\nu\rho} [\partial^{[\mu} A^{\nu\rho]}]. \quad (38)$$

This dual equivalence is shown in Fig.2.

We refer to these models as Maxwell-type, because their actions have the form $S(f) \sim \int (df)^2$. As we shall see below in detail, semi-self-dual actions, with duality DHD, describe dualities between this type of theories in general.

Another important duality for the doublet $G$ shall be mentioned. The action (34), with $K$ non-singular, namely

$$S(G) = \int d^4x \left( \frac{m}{6} f_{\sigma} \epsilon^{\sigma\rho\mu\nu} \partial_{\rho} A_{\mu\nu} + \frac{m^2}{2} f_{\sigma} f^{\sigma} - \frac{m^2}{4} A_{\mu\nu} A^{\mu\nu} \right), \quad (39)$$

also reveals the connection between the Proca model,

$$S_{Proca} = \int d^4x \left( \frac{1}{2} \partial_{[\mu} f_{\nu]} \partial^{[\mu} f^{\nu]} + m^2 f_{\mu} f^{\mu} \right), \quad (40)$$

Figure 1:
and the Kalb-Ramond model \([14]\) whose action is given by,

\[
S_{K-R} = \int d^4x \left( \frac{1}{6} \partial_\rho A_{\mu\nu} \partial^\rho A^{\mu\nu} - \frac{1}{6} A_{\mu\nu} A^{\mu\nu} \right). \tag{41}
\]

Notice finally, that the action (39) is manifestly self-dual for the doublet field, \(G\).

### 3.2 Semi-self-duality and Maxwell-type theories.

A system will be said to be Semi-Self-Dual iff

\[
^*dF = \chi KP_i F, \tag{42}
\]

where \(P_i; i = 1, 2\) represents the two projectors on the internal two-dimensional space. These relations can be derived from a GSDM action when the mass matrix is singular in the form \(k_j \neq i = 0\). Consider the doublet \((f, g)\), thus, this action is \([^6]\)

\[
L_p = F \cdot ^*dF + k_2 g^2. \tag{43}
\]

We shall show here that this action constitutes a parent action which interpolates two Maxwell type theories in general. For simplicity, let’s take \(k_2 = 1\).

The equations of motions for (43) will be, if we vary with respect to \(g\):

\[
^*df = g. \tag{44}
\]

Integrating the parent action by parts leads to:

\[
L_p = g^*df + Sf^*dg + g^2 = |1 - S|g^*df + g^2 + \text{Total divergence.} \tag{45}
\]

Substituting by (44), we have finally:

\[
L_p(F) = [2S - 1](df)^2 + \text{Total divergence.} \tag{46}
\]

[^6]: We take for instance, \(k_1 = k_f = 0\).
On the other hand, vary now with respect to $f$:

\[ *d g = 0; \]  

as consequence of this, there exists $\bar{g}$ such that

\[ g = d\bar{g}. \]  

Integrating once by parts (in the opposite sense):

\[ L_p = g *df - Sf *dg + g^2 = [S - 1]f *dg + g^2 + \text{Total divergence}. \]  

One can express the master action as a function of $\bar{g}$; substituting here by the equation of motion (47) and (48), we obtain:

\[ L_p = (d\bar{g})^2 + \text{Total divergence}. \]  

This proves the duality between two Maxwell-type models, (44) and (51), obtaining, on-shell, the two dual actions:

\[ L_p(F(f)) = [2S - 1][df]^2 \]  

and

\[ L_p(F(\bar{g})) = [d\bar{g}]^2 \]  

If we take the $k_g$ to be zero (and not $k_f$), the result is another duality between two Maxwell-type models for the fields $g$ and $\bar{f}$, where $f = df$

### 3.3 Dual Equivalence between Proca models.

Consider our master action given by (19), with non-singular matrix $K$. By taking the variation with respect to $f$, we obtain:

\[ *df = k_g g, \]  

thus, applying $*$ to both sides of this equation, we get

\[ df = Sk_g g. \]  

Substituting (53) and (54) in the action, the final result is

\[ L(f, g[f]) = \frac{2S - 1}{k_g}df^2 + k_f f^2. \]  

Variation with respect to $g$ results in

\[ L(g, f[g]) = \frac{2S - 1}{k_f}dg^2 + k_g^2 g^2. \]  

This constitutes the proof that the dual equivalence between two Proca theories is described by the action (19).

### 4 (2+1)-dimensions: Maxwell Theory and manifest HD.

In order to show one of the numerous possible applications of the doublets structures, we briefly describe HD in 3d, which would be the 3d-version of a electric-magnetic duality. This has also been discussed in another contexts [19, 9].

As we have pointed in relation to equation (6) in the Introduction, HD in (2+1)-dimensions is ill defined unless we use doublets of different tensor rank as we show below; otherwise, we only can work merely with the already known DHD described in the introduction by the action (2).
Let us consider the Maxwell-type theory in three dimensions:

\[ S = \int dx^3 F^2 \]  

(57)

where \( F \) is a doublet,

\[ F = (F_{\mu \nu}; f_\rho), \]  

(58)

where \( F = dA \) and \( A = (A_\mu; a) \).

Notice that this is the unique doublet that can be chosen such that its components are a differential of some potential-field.

Now, the self-duality (HD) is well-defined \(^7\):

\[ F = *F, \]  

(59)

this means the simultaneous relations:

\[ f_\rho = \epsilon_{\mu \nu \rho} F^{\mu \nu}, \]  

(60)

\[ F_{\mu \nu} = \epsilon_{\mu \nu \rho} f_\rho. \]  

(61)

The Electric and Magnetic part, for both \( f, F \), can be defined in the usual way and the relations (60) and (61) relate each other.

The equations of motion for (57) read

\[ \text{div} F = 0. \]  

(62)

As a consequence of the HD-relations (60) and (61), one of these (two) equations reduces to be an identity (the Bianchi identity). However, these HD-relations do not arise from the action as consequence of the equations of motion.

In order to render this duality manifest, we again can use our GSDM and redefine the doublet structure to be potential-field type. Let us take

\[ B = (a, F_{\mu \nu}) \]  

(63)

we shall show that this theory can be described by a semiself-dual action (one more time), including self-duality relations. The semi-self-dual action proposed is

\[ S = \int d^3x (B, *dB + F_{\mu \nu} F^{\mu \nu}). \]  

(64)

The equations of motion are

\[ F_{\mu \nu} = \epsilon_{\mu \nu \rho} \partial^\rho a, \]  

(65)

and

\[ \epsilon_{\mu \nu \rho} \partial^\rho F^{\mu \nu} = 0. \]  

(66)

This implies that \( \text{div} F = 0 \) and that there exists \( A_\mu \) such that \( F_{\mu \nu} = \partial_{[\mu} A_{\nu]} \), thus, defining \( f_\mu = \partial_\mu a \), we obtain from (65):

\[ f_\rho = \epsilon_{\mu \nu \rho} \partial^\nu A^\mu, \]  

(67)

and then \( \text{div} f = 0 \). Equations of motion (64) are verified and HD-relations are recovered (Equations (67) and (65)).

Finally, notice that in four dimensions it is more clearer to build this up, since it is possible construct a doublet of Maxwell field strengths and the corresponding Hodge type self-duality (HD) \(^7\); and in contrast, DHD can be defined in four dimensions too. Realizations in 4d with manifest DHD are precisely the actions \( S_F \) and \( S_G \) (with \( K \) generic) discussed in Section 3.1. In \(^7\), the equivalence of these models to Proca and massive Kalb-Ramond theories is discussed.

\(^7\)In (2+1)-dimensions the metric has two minuses, thus: \( S = 1 \).
5 Concluding remarks.

In this work, we have extended the notion of duality for all type of tensors in arbitrary dimensions in order to allow, a well-defined notion of self-duality. We have also built a general action describing this fact, which is shown to give (by taking proper limits in its parameters) the parent actions in the most of the case of dual models. Interesting possibilities that we open up as an application of the results presented here shall be explored: this action (GSDM) is related [23] to the topologically massive, the so-called BF-theories [20], via the manifest connection with the SD-model in three dimensions [11] presented here; furthermore, the bosonization technique in arbitrary dimensions, mainly in higher dimensions \( (d \geq 4) \), thanks to this connection, comes out related to a topologically massive model that mixes different gauge forms. These results shall soon be reported [?].

that we open up as an application of the results presented here is the study of bosonization in arbitrary dimensions, mainly in higher dimensions. This is not a trivial matter [?, ?, ?], but with the help of the technique suggested here, \( d \geq 4 \) bosonization comes out in connection with a topologically massive model that mixes different gauge forms. Results on this issue shall soon be reported elsewhere [2].

, which found important applications[20].

A novel definition of HD, and non merely the already known DHD, has been given in \((2+1)\)-dimensions. Reciprocally, the approach presented here, allows us write DHD in even dimensional spacetime.

There are several potential applications of this structure, for saying supersymmetric extensions of these general self-dual actions. Other approaches on this matter have already been made in the past [22, 23, 24].

Furthermore, as has been pointed in [24, 25, 26], a self-duality relating massive scalar and vector fields may be relevant for string theory in the context of massive type IIA Sugra, however the self-duality presented here is new.

We suggest, finally, a hypothesis motivated in all these examples: " All parent action interpolating between two dual models comes from (in the sense discussed in this work) a GSDM"; in other words, every duality at the level of the classical actions comes from some manifest duality between the fields involved in these actions.

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