An extension of a theorem and errata for “A Class of Representations of Hecke Algebras”

Dean Alvis

Abstract

By Theorem 1.12 of the paper “A Class of Representations of Hecke Algebras”, if \( W \) is a Coxeter group whose proper parabolic subgroups are finite, and if the module of a finite \( W \)-digraph \( \Gamma \) is isomorphic to the module of a \( W \)-graph, then \( \Gamma \) must be acyclic. Here we extend this result to Coxeter groups with finite dihedral parabolic subgroups and \( W \)-graphs with arbitrary scalar edge labels. Also, errata for the paper are listed in the last section.

1 An extension of Theorem 1.12 of [1]

Let \((W, S)\) be a Coxeter system with presentation

\[
W = \left\langle s \in S \mid (rs)^{n(r,s)} = e \text{ for } r, s \in S \text{ whenever } n(r,s) < \infty \right\rangle,
\]

where \(n(s,s) = 1\) and \(1 < n(r,s) = n(s,r) \leq \infty\) for \(r, s \in S, r \neq s\). Let \(\ell\) be the length function of \((W, S)\). Let \(u\) be an indeterminate over \(\mathbb{C}\), and let \(H\) be the Hecke algebra of \((W, S)\) over \(\mathbb{Q}(u)\).

See [1] for the definition of the notion of \(W\)-digraph. For \(\Gamma\) a \(W\)-digraph and \(\beta \in V(\Gamma)\), define \(\text{In}(\beta)\) to be the set of all \(s \in S\) such that \(\Gamma\) has an edge of the form \(\alpha \xrightarrow{s} \beta\) or \(\alpha \xleftarrow{s} \beta\) for some \(\alpha \in V(\Gamma)\). Observe \(\beta\) is a source (sink) in \(\Gamma\) if and only if \(\text{In}(\beta) = \emptyset\) (\(\text{In}(\beta) = S\), respectively). For \(J \subseteq S\), put

\[
N_{\Gamma}(J) = |\{\beta \in V(\Gamma) \mid \text{In}(\beta) = J\}|.
\]

This definition will only be applied when \(\Gamma\) is finite, i.e. when \(V(\Gamma)\) and \(E(\Gamma)\) are finite.

Now, let \(\Psi\) be a \(W\)-graph over the subfield \(F\) of \(\mathbb{C}\), in the sense of [2], Definition 2.1, with vertex-labeling function \(x \mapsto I_x \subseteq S, x \in X\), edge-labeling function \(\mu : X \times X \to F\), and with the indeterminate \(u\) here playing the role of \(q^{1/2}\) in [2]. For \(J \subseteq S\), put

\[
N_{\Psi}(J) = |\{x \in V(\Psi) \mid I_x = J\}|.
\]

The goal of this section is to prove the following.

**Theorem 1.1.** If \(n(s,t) < \infty\) for \(s,t \in S\), \(\Gamma\) is a finite \(W\)-digraph, \(\Psi\) is a \(W\)-graph over a subfield \(F\) of \(\mathbb{C}\), and \(M(\Gamma)^F = F(u) \otimes_{\mathbb{Q}(u)} M(\Gamma)\) is isomorphic to \(M(\Psi)\) as \(H^F\)-modules, then the following hold:

(i) \(N_{\Gamma}(J) = N_{\Psi}(J)\) for all \(J \subseteq S\).

(ii) \(\Gamma\) is acyclic.
We require the following results. For $F$ a subfield of $\mathbb{C}$, $\lambda$ a linear character of $H^F$, and $V$ an $H^F$-module, define

$$V_\lambda = \{v \in V \mid hv = \lambda(h)v \text{ for all } h \in H^F\}$$

and

$$\langle V, \lambda \rangle_{H^F} = \dim_{F(u)} V_\lambda.$$

Also, let $\text{ind}^F$ and $\text{sgn}^F$ be the linear characters of $H^F$ determined by $\text{ind}^F(T_w) = u_{w} = u^{2\ell(w)}$ and $\text{sgn}^F(T_w) = \varepsilon_w = (-1)^{\ell(w)}$ for $w \in W$.

**Lemma 1.2.** Let $F$ a subfield of $\mathbb{C}$. Let $\Gamma$ be a $W$-digraph, and put $V = M(\Gamma)^F$. Suppose $s \in S$ and $v = \sum_{\gamma \in \mathcal{X}} \lambda\gamma \in V$. Then the following hold:

(i) $T_s v = u^2 v$ if and only if $\lambda_\beta = \lambda_\alpha$ whenever $\alpha \overset{s}{\rightarrow} \beta$ or $\alpha \overset{s}{\Rightarrow} \beta$ is an edge of $\Gamma$.

(ii) $T_s v = -v$ if and only if

$$\lambda_\beta = \begin{cases} -u^{-2}\lambda_\alpha & \text{whenever } \alpha \overset{s}{\rightarrow} \beta \in \mathcal{E}(\Gamma), \\ -(u+1)(u^2-u)^{-1}\lambda_\alpha & \text{whenever } \alpha \overset{s}{\Rightarrow} \beta \in \mathcal{E}(\Gamma). \end{cases}$$

**Proof.** The argument given for Lemma 2.4 in [1] applies, with $F$, $H^F$, and $V = M(\Gamma)^F$ replacing $\mathbb{Q}$, $H$, and $M = M(\Gamma)$, respectively. \qed

**Lemma 1.3.** If $\Gamma$ is a $W$-digraph, $\mathcal{V}(\Gamma)$ is finite, and $F$ is a subfield of $\mathbb{C}$, then the following hold:

(i) The number of connected components of $\Gamma$ is equal to $\langle M(\Gamma)^F, \text{ind}^F \rangle_{H^F}$.

(ii) If $n(s,t) < \infty$ for all $s, t \in S$, then the number of acyclic connected components of $\Gamma$ is equal to $\langle M(\Gamma)^F, \text{sgn}^F \rangle_{H^F}$.

**Proof.** The proof for Theorem 1.7 in [1] applies here if $\mathbb{Q}$, $H$, and $M(\Gamma)$ are replaced by by $F$, $H^F$, $M(\Gamma)^F$, respectively, using Lemma 1.2 in place of [1], Lemma 2.4. \qed

**Lemma 1.4.** If $F$ is a subfield of $\mathbb{C}$, $\Psi$ is a $W$-graph over $F$, $x \mapsto I_x \subseteq S$ is the vertex-labeling function for $\Psi$, and $M(\Psi)^F_{\text{ind}^F} \neq \{0\}$, then there is some $x_0 \in \mathcal{V}(\Psi)$ such that $I_{x_0} = \emptyset$.

**Proof.** Let $X = \mathcal{V}(\Psi)$, and let $\mu$ be the edge-labeling function of $\Psi$. Suppose $v = \sum_{x \in X} \gamma_x x \in M(\Psi)^F_{\text{ind}^F}$ and $v \neq 0$. Replacing $v$ by a scalar multiple if necessary, we can assume $\gamma_x \in F[u]$ for all $x \in X$ and $\gcd\{\gamma_x \mid x \in X\} = 1$. Choose $x_0 \in X$ such that $\gamma_{x_0} \not\in uF[u]$. Suppose $I_{x_0} \neq \emptyset$, so that $s \in I_{x_0}$ for some $s \in S$. Since

$$u^2 v = T_s v = -\sum_{x \in X \atop s \in I_x} \gamma_x x + \sum_{y \in X \atop s \not\in I_y} \gamma_y \left( u^2 y + u \sum_{z \in X \atop s \not\in I_z} \mu(z, x) z \right),$$

comparing coefficients of $x_0$ shows $\gamma_{x_0} \in uF[u]$, and so a contradiction is reached. Therefore $I_{x_0} = \emptyset$. \qed
Proof of Theorem 1.1. For the remainder of this proof it is assumed that \( n(s,t) < \infty \) for \( s,t \in S \), \( \Gamma \) is a finite \( W \)-digraph, \( \Psi \) is a \( W \)-graph over the subfield \( F \) of \( \mathbb{C} \), and \( M(\Gamma)^F \) is isomorphic to \( M(\Psi) \) as \( H^F \)-modules.

Any connected component of \( \Gamma \) that contains a sink is acyclic by [1], Theorem 1.5(ii). On the other hand, any acyclic connected component \( C \) of \( \Gamma \) contains some sink \( \sigma \) because \( \Gamma \) is finite, and \( \sigma \) is the unique sink in \( C \) by [1], Theorem 1.5(i). Thus the number of sinks in \( \Gamma \), that is, \( N(\Gamma)(S) \), is equal to the number of acyclic connected components of \( \Gamma \). Thus by Lemma 1.3(ii), \( N(\Gamma)(S) \) is equal to \( \langle M(\Gamma)^F, \text{sgn}^F \rangle_{H^F} \). Also, \( \langle M(\Psi), \text{sgn}^F \rangle_{H^F} = N(\Psi)(S) \) because \( M(\Psi)_{\text{sgn}^F} \) has basis \( \{ x \in \mathcal{V}(\Psi) \mid I_x = S \} \) over \( F(u) \). Hence

\[
N(\Gamma)(S) = \langle M(\Gamma)^F, \text{sgn}^F \rangle_{H^F} = \langle M(\Psi), \text{sgn}^F \rangle_{H^F} = N(\Psi)(S).
\]

Now suppose \( J \subseteq S \). Let \( \Psi_J \) be the \( W_J \)-graph obtained from \( \Psi \) by replacing \( I_x \) by \( I_x \cap J \) for \( x \in \mathcal{V}(\Psi) \). Also, let \( \Gamma_J \) be the \( W_J \)-digraph obtained from \( \Gamma \) by removing all edges with labels in \( S \setminus J \). Then \( M(\Gamma_J)^F \cong M(\Gamma)^F |_{H^F_J} \cong M(\Psi) |_{H^F_J} \cong M(\Psi_J) \) as \( H^F \)-modules, and so the reasoning above gives \( N(\Gamma_J)(J) = N(\Psi_J)(J) \).

Thus

\[
\sum_{J \subseteq K \subseteq S} N(\Gamma)(K) = N(\Psi)(J) = N(\Psi_J)(J) = \sum_{J \subseteq K \subseteq S} N(\Psi)(K).
\]

(Note \( S \) itself is finite because \( \mathcal{E}(\Gamma) \) is finite, so the sums above are finite.) Thus part (i) of the theorem holds by induction on \( |S \setminus J| \).

Let \( M_0 \) be the \( H^F \)-submodule of \( M(\Psi) \) with basis \( \{ x \in \mathcal{V}(\Psi) \mid I_x \neq \emptyset \} \). By Lemma 1.3

\[
M(\Psi)_{\text{ind}^F} \cap M_0 = (M_0)_{\text{ind}^F} = \{ 0 \}.
\]

Thus

\[
\langle M(\Gamma)^F, \text{ind}^F \rangle_{H^F} = \langle M(\Psi), \text{ind}^F \rangle_{H^F} = \dim_{F(u)} M(\Psi)_{\text{ind}^F} \\
\leq \dim_{F(u)} (M(\Psi)/M_0) = |\{ x \in \mathcal{V}(\Psi) \mid I_x = \emptyset \}| \\
= N(\Psi)(\emptyset) = N(\Gamma)(\emptyset),
\]

with the last equality holding by part (i) of the theorem. Now, \( \langle M(\Gamma)^F, \text{ind}^F \rangle_{H^F} \) is equal to the number of connected components of \( \Gamma \) by Lemma 1.3(i), while \( N(\Gamma)(\emptyset) \) is equal to the number of sources of \( \Gamma \). Therefore \( \Gamma \) has at least as many sources as connected components. Because each connected component contains at most one source by [1], Theorem 1.5(i), it follows that every connected component of \( \Gamma \) contains a (unique) source. Hence every connected component of \( \Gamma \) is acyclic by [1], Theorem 1.5(ii). Therefore \( \Gamma \) itself is acyclic, so part (ii) of the theorem holds and the proof of the theorem is complete.

\[ \square \]

2 Errata from [1]

None of the errata listed here has an effect on the results of [1].

Page 315, line 4: The index of summation in the last sum should be \( i \), not \( \ell \). The displayed formula containing this line should read as follows:

\[
T_{s_{\ell-1}}[\bar{\eta}_{\ell}] = T_{s_{\ell-1}}[\bar{\varphi}_{\ell} + u\bar{\eta}_{\ell-1}] = T_{s_{\ell-1}}[\bar{\varphi}_{\ell}] + uT_{s_{\ell-1}}[\bar{\eta}_{\ell-1}]
\]

\[
= u^{2\ell}T_e + T_{s_{2\ell-1}} + uT_s \sum_{i=0}^{2\ell-2} u^i T_{s_{2\ell-i-2}}
\]

\[
= u^{2\ell}T_e + T_{s_{2\ell-1}} + \sum_{i=0}^{2\ell-2} u^{i+1} T_{s_{2\ell-i-1}} = \sum_{i=0}^{2\ell} u^i T_{s_{2\ell-i}}
\]
Page 316, line 3: Replace \( \mu_m \) by \( \mu'_m \). The displayed equations containing this line should read as follows:

\[
\begin{align*}
\mu_1 &= T_s \mu_0, \mu_2 = T_t \mu_1, \ldots, \mu_{m-1} = T_s' \mu_{m-2}, \mu_m = T_t' \mu_{m-1}, \\
\mu'_1 &= T_t \mu_0, \mu'_2 = T_s \mu'_1, \ldots, \mu'_{m-1} = T_t' \mu'_{m-2}, \mu'_m = T_s' \mu'_{m-1}.
\end{align*}
\]

Page 321, lines 7 and 9 from bottom: The edges in each of these lines should be reversed. The displayed formula containing these lines should read as follows:

\[
\zeta_i = \begin{cases} 
\frac{1}{u} & \text{if } \gamma_{i-1} \xrightarrow{s} \gamma_i \text{ or } \gamma_{i-1} \xrightarrow{t} \gamma_i \text{ is an edge of } \Gamma, \\
-u^2 & \text{if } \gamma_{i-1} \xleftarrow{s} \gamma_i \text{ or } \gamma_{i-1} \xleftarrow{t} \gamma_i \text{ is an edge of } \Gamma, \\
-u^2 - u & \text{if } \gamma_{i-1} \xrightarrow{s} \gamma_i \text{ or } \gamma_{i-1} \xrightarrow{t} \gamma_i \text{ is an edge of } \Gamma, \\
-u^2 + u & \text{if } \gamma_{i-1} \xleftarrow{s} \gamma_i \text{ or } \gamma_{i-1} \xleftarrow{t} \gamma_i \text{ is an edge of } \Gamma.
\end{cases}
\]

Page 322, lines 2–4: The direction of second and fourth edges should be reversed. The sentence containing these lines should read as follows:

It follows that the number of edges of type \( \gamma_{i-1} \rightarrow \gamma_i \) (labeled either \( s \) or \( t \)) is equal to the number of edges of type \( \gamma_{i-1} \leftarrow \gamma_i \), \( 1 \leq i \leq 2m \), and the number of edges of type \( \gamma_{i-1} \rightarrow \gamma_i \) is equal to the number of edges of type \( \gamma_{i-1} \leftarrow \gamma_i \), \( 1 \leq i \leq 2m \).

Page 323, lines 4–6: The direction of second and fourth edges should be reversed, and the label should be removed from the first edge. The sentence containing these lines should read as follows:

Since \( \Gamma \) has a unique sink \( \beta \) and the number of edges of type \( \gamma_{i-1} \rightarrow \gamma_i \) is equal to the number of edges of type \( \gamma_{i-1} \leftarrow \gamma_i \), \( 1 \leq i \leq 2m \), and the number of edges of type \( \gamma_{i-1} \rightarrow \gamma_i \) is equal to the number of edges of type \( \gamma_{i-1} \leftarrow \gamma_i \), \( 1 \leq i \leq 2m \), it follows that \( \beta = \gamma_m \) is opposite to \( \alpha \).

Page 327, line 3 from bottom: Replace \( \Gamma_s \) by \( \Gamma_{\{s\}} \).

Page 332, line 8: Replace \( \alpha \xrightarrow{s} \beta \) by \( \alpha \xleftarrow{s} \beta \). The relevant sentence reads as follows:

If \( \alpha \xleftarrow{s} \beta \in \mathcal{E}(\Gamma_{\rightarrow}) \) for some \( s \in S \), then \( \alpha \in [\sigma, \infty) \) because \( \beta \in [\sigma, \infty) \) and \( \alpha \in [\beta, \infty) \).

References

[1] D. Alvis. A Class of Representations of Hecke Algebras. Bull. Inst. Math. Acad. Sinica (N.S.), 11(2):301–342, 2016.

[2] Akihiko Gyoja. On the existence of a \( W \)-graph for an irreducible representation of a Coxeter group. J. Alg., 86:422–438, 1984.