STANDARD PAIRS FOR MONOMIAL IDEALS IN SEMIGROUP RINGS

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ABSTRACT. We extend the notion of standard pairs to the context of monomial ideals in semigroup rings. Standard pairs can be used as a data structure to encode such monomial ideals, providing an alternative to generating sets that is well suited to computing intersections, decompositions, and multiplicities. We give algorithms to compute standard pairs from generating sets and vice versa and make all of our results effective. We assume that the underlying semigroup ring is positively graded, but not necessarily normal. The lack of normality is at the root of most challenges, subtleties, and innovations in this work.

1. INTRODUCTION

The polynomial ring on \(d\) variables over a field is \(\mathbb{Z}^d\)-graded ring, where the degree of a monomial is defined to be its exponent vector. This is a fine grading: every graded piece is a vector space over the base field of dimension at most one. From this point of view, a monomial ideal is a \(\mathbb{Z}^d\)-homogeneous ideal. Affine semigroup rings are also finely graded, and it makes sense to talk about monomial ideals in this context as well. Monomial ideals in polynomial rings are a mainstay of combinatorial commutative algebra, and have been extensively studied (see for instance, the texts [22, 26]). In contrast, much less is known about monomial ideals in affine semigroup rings (but see [5, 12, 21]). This is not surprising, as general semigroup rings do not satisfy many properties the polynomial ring enjoys.

A monomial ideal \(I\) is determined by the monomials that belong to \(I\), but also by the monomials that do not belong to \(I\), which are known as the standard monomials of \(I\). Usually, we encode a monomial ideal through a (finite) monomial generating set, a description that is best suited to working with the monomials in \(I\). Standard pairs, introduced in [28], give a finite way of encoding the standard monomials of a monomial ideal in a polynomial ring. If \(I\) and \(J\) are monomial ideals, the set of standard monomials of \(I \cap J\) is the union of the standard monomials of \(I\) and \(J\). This makes standard pairs particularly well-adapted to tasks involving intersections and decompositions of monomial ideals. The main goal of this article is to extend this point of view to monomial ideals in semigroup rings.

Beyond their original use in [28] to give combinatorial bounds for degrees of projective schemes, standard pairs have been used to prove properties of initial ideals of toric ideals [14, 16, 24], in applications related to optimization [15], in combinatorial settings [2] and to compute series solutions of hypergeometric systems [25]. An algorithm for computing standard pairs is given in [8, 25], and is implemented in the computer algebra system Macaulay2 [11].

The standard pair definition given in [28] naturally extends to monomial ideals in semigroup rings. However, even in the normal case, standard pairs in this context exhibit behavior that is not present over the polynomial ring. Nevertheless, basic results about standard pairs still hold (with different

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proofs): A monomial ideal $I$ has finitely many standard pairs (Theorem 3.16). The associated primes of $I$ can be read off immediately from its standard pairs, and the standard pairs of $I$ can be used to give combinatorial primary and irreducible decompositions of $I$ (Theorem 3.8 and Proposition 3.12). Finally, counting (equivalence classes) of standard pairs yields multiplicities of associated primes (Proposition 3.14).

We are particularly concerned with the computational aspects of standard pairs. This is motivated by the difficulty of computing combinatorial structures associated with general binomial ideals (ideals generated by polynomials with at most two terms). Since an affine semigroup ring is isomorphic to the quotient of a polynomial ring modulo a prime binomial ideal, the quotient of an affine semigroup ring by a monomial ideal is isomorphic to the quotient of a polynomial ring modulo the sum of a prime binomial ideal and a monomial ideal. In other words, monomial ideals in semigroup rings can be identified with special kinds of binomial ideals.

The general study of binomial ideals was initiated in [9], where it was shown that binomial ideals can have a primary decomposition consisting of binomial ideals (when the base field is algebraically closed). Specialized algorithms for finding such decompositions can be found in [9] (see also [10, 17, 23]). Combinatorial structures controlling the decompositions of binomial ideals were given in [7, 18], but there are currently no known algorithms to compute these structures, even if a primary decomposition is known by other means.

Standard pairs represent a different combinatorial approach to decompositions of our special of binomial ideals: they are not the specialization of the structures from [7, 18]. Moreover, one of our main results gives a method to compute the standard pairs of a monomial ideal from a generating set (Theorem 4.5). We also provide a method to compute a generating set of a monomial ideal given its standard pairs (Theorem 4.8). Using standard pairs, we describe a method to produce an irredundant irreducible decomposition of a monomial ideal in a semigroup ring (Theorem 4.11). An irreducible decomposition algorithm already existed in the case that the underlying semigroup ring is normal [12]. However, the general case given in Theorem 4.11 was indicated as an open problem in the notes of [22, Chapter 11]. Finally, computing intersections of monomial ideals using standard pairs is particularly straightforward (Remark 4.9).

As with most computations involving affine semigroup rings, our procedures for finding and using the standard pairs of a monomial ideal use ideas and techniques from convex discrete optimization. Loosely speaking, our algorithms require solving multiple integer linear programs (ILPs), which are famously known as NP-complete. However, our specific situation is not as bad as it sounds: if we fix the ambient affine semigroup ring, then finding standard pairs is not general integer programming, but integer programming in fixed dimension, which is famously solvable in polynomial time [20]. Even further, when the ambient ring is fixed, all the ILPs we need to solve arise from a single known matrix (the matrix of generators of the affine semigroup) in a finite number of possible ways. This means that, after some possibly costly pre-computations (that can be done once and stored), finding and using standard pairs in a fixed semigroup ring should not be too computationally intensive. Implementation of these ideas is done by the StdPairs [30] package on SageMath developed by the second author. Some more details on this work in progress can be found in Section 5.

Outline. This article is organized as follows. In Section 2 we set notation and review background material. Section 3 develops the theory of standard pairs and shows how to use standard pairs to give combinatorial descriptions of primary and irreducible decompositions of monomial ideals in
semigroup rings. In Section 4 we describe algorithms to compute and use standard pairs. Algorithms outlined in this section include: computation of standard pairs given the generators of a monomial ideal, computation of the generators of a monomial ideal given its standard pairs, computation of irreducible (and primary) decompositions, computation of multiplicities. Section 5 explains the authors’ project to implement these methods in the computer algebra system SageMath and Macaulay2.

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2. Preliminaries

We adopt the convention that \( \mathbb{N} = \{0, 1, 2, \ldots\} \) is the set of nonnegative integers and \( k \) is an infinite field.

Throughout this article, \( A = \{a_1, \ldots, a_n\} \subset \mathbb{Z}^d \setminus \{0\} \) is a fixed finite set of nonzero lattice points, called a configuration. We may abuse notation and use \( A \) to also denote the \( d \times n \) integer matrix whose columns are \( a_1, \ldots, a_n \). We let \( \mathbb{N}A \) be the monoid of nonnegative integer combinations of \( a_1, \ldots, a_n \); in the most commonly used terminology, \( \mathbb{N}A \) is called an affine semigroup. Similarly \( \mathbb{Z}A \) is the (free abelian) group of integer combinations of elements of \( A \), and \( \mathbb{R}_\geq 0A \) is the cone of nonnegative real combinations of elements of \( A \). To simplify notation, we assume that \( \mathbb{Z}A = \mathbb{Z}^d \).

(When \( \mathbb{Z}A \neq \mathbb{Z}^d \) we use \( \mathbb{Z}A \) as a ground lattice, and our proofs go through essentially unchanged.) We also assume that \( \mathbb{R}_\geq 0A \) is strongly convex cone, meaning that it contains no lines. We point out that Lemma 3.17, which is used in the proof of Theorem 3.16, fails without strong convexity.

We say that \( F \subseteq A \) is a face of \( A \) if \( \mathbb{R}_\geq 0F \) is a face of \( \mathbb{R}_\geq 0A \), and \( \mathbb{R}_\geq 0F \cap A = F \). In this case, we also abuse notation and use \( F \) to denote both a configuration and its corresponding matrix (whose columns are the elements of the configuration). It is known that \( \mathbb{R}_\geq 0A \) is strongly convex if and only if \( \{0\} \) is a face of \( \mathbb{R}_\geq 0A \). We use the convention that \( F = \emptyset \) refers to the origin as a face of \( A \).

**Definition 2.1.** If \( H \) is a facet (a codimension one face) of \( A \), we define its primitive integral support function \( \varphi_H : \mathbb{R}^d \to \mathbb{R} \) by the following properties:

1. \( \varphi_H \) is linear,
2. \( \varphi_H(\mathbb{Z}^d) = \mathbb{Z} \),
3. \( \varphi_H(a_i) \geq 0 \) for \( i = 1, \ldots, n \),
4. \( \varphi_H(a_i) = 0 \) if and only if \( a_i \in H \).

Primitive integral support functions give a measure of how far a point is from a facet of \( A \): if \( a \in \mathbb{Z}^d \), \( \varphi_H(a) \) is the number of hyperplanes parallel to \( RH \) that pass through integer points, and lie between \( a \) and \( RH \), with a sign to indicate whether \( a \) is on the side of \( RH \) that contains \( \mathbb{R}_\geq 0A \).

We denote by \( \mathbb{N}F \) the affine semigroup generated by a face \( F \), \( \mathbb{R}_\geq 0F \) the cone over \( F \), and \( RF \) the real linear span of \( F \). Since \( \mathbb{R}_\geq 0F \cap A = F \), we have that \( \mathbb{N}F = \mathbb{N}A \cap \mathbb{R}_\geq 0F \).
We work with the semigroup ring $k[NA] = k[t_1^{a_1}, \ldots, t_n^{a_n}]$, which is a subring of the Laurent polynomial ring $k[t^{\pm}] = k[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$. This ring has the presentation $k[NA] \cong k[x]/I_{A}$, where $k[x] = k[x_1, \ldots, x_n]$, and $I_{A} = \langle x^u - x^v | u, v \in \mathbb{N}^n, Au = Av \rangle$ (here we have used $A$ to denote a matrix). Since $\mathbb{R}_{\geq 0}A$ is strongly convex, the only multiplicative units in $k[NA]$ are the nonzero elements of the field $k$.

The semigroup $NA$ is saturated if $\mathbb{R}_{\geq 0}A \cap \mathbb{Z}A = NA$. When $NA$ is not saturated, $(\mathbb{R}_{\geq 0}A \cap \mathbb{Z}A) \setminus NA$ is called the set of holes of $NA$. It is a well-known result that $NA$ is saturated if and only if the domain $k[NA]$ is normal (meaning that it is integrally closed over its field of fractions).

The ring $k[NA]$ is $\mathbb{Z}A = \mathbb{Z}^d$-graded, via $\deg(t^a) = a$. In the presentation $k[NA] \cong k[x]/I_{A}$, the grading is induced by setting $\deg(x_i) = a_i$. Strong convexity $\mathbb{R}_{\geq 0}A$ means that this is a positive grading: the unique maximal $\mathbb{Z}^d$-homogeneous ideal of $k[NA]$ is $\langle t^{a_1}, \ldots, t^{a_n} \rangle$. A $\mathbb{Z}^d$-homogeneous ideal in $k[NA]$ is called a monomial ideal. Equivalently, a monomial ideal in $k[NA] \subset k[t^{\pm}]$ is an ideal generated by Laurent monomials.

There is a one to one inclusion reversing correspondence between the set of monomial prime ideals in $k[NA]$ and the set of faces of $\mathbb{R}_{\geq 0}A$, given in the following statement.

**Lemma 2.2** ([22, Lemma 7.10]). If $F$ is a face of $A$, the monomial ideal $p_F = \langle t^a | a \in NA \setminus NF \rangle \subset k[NA]$ is prime. All prime monomial ideals in $k[NA]$ are of this form.

**Notation 2.3.** We emphasize that, throughout this article, divisibility refers to the ring $k[NA]$, and not to $k[t^{\pm}]$. To be completely precise, $t^{a'} | t^a$ means that $a - a' \in NA$. We abuse terminology, and also state that $a'$ divides $a$ in this case.

Affine semigroup rings are Noetherian, as they are quotients of polynomial rings. This can be restated as a version of Dickson’s Lemma.

**Lemma 2.4.** Let $S$ be a nonempty subset of $NA$ such that no two elements of $S$ are comparable with respect to divisibility. Then $S$ is finite.

**Proof.** By contradiction, assume that $S$ contains an infinite sequence $\{b_i\}_{i=1}^\infty$. Consider $I_j = \langle b_1, \ldots, b_j \rangle$ for $j \geq 1$. Since $t^{b_i} \in I_j$ if and only if $t^{b_i} | t^{b_j}$ for some $1 \leq i \leq j$, we see that $I_1 \subset I_2 \subset I_3 \subset \cdots$ is an infinite ascending chain, which contradicts Noetherianity of $k[NA]$. □

We close out this section by providing some running examples. We represent semigroup rings and monomial ideals pictorially by plotting exponent vectors of monomials. In Figures [1a] and [2a] the (exponents of) standard monomials of the given ideal are colored blue, while the (exponents of) monomials in the ideal are colored black.

**Example 2.5.**

(1) Let $A$ be a $d \times d$ identity matrix. Then $NA = \mathbb{N}^d$ and consequently $k[NA] \cong k[x_1, \ldots, x_d]$. A face of $NA$ is a set of all nonnegative integral combinations of a subset of (the columns of) $A$. In Figure [1a] the shaded region represents the monomial ideal $I = \langle x^3y, xy^2 \rangle \subset k[x, y]$.

(2) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. Then $NA$ is a saturated semigroup, and $k[NA] \cong k[x, y, y^2]$. A normal semigroup ring, a subring of $k[x, y]$. Figure [1b] illustrates the ideal $\langle x^2y^2, x^3y \rangle \subset k[NA]$.

(3) Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. In this case, $k[NA] \cong k[z, xz, yz, xyz]$ is a saturated affine semigroup ring. We depict the ideal $\langle x^2z^2, x^2yz^2, x^2yz^2 \rangle \subset k[NA]$ in Figure [2a].
3. Standard Pairs, Decompositions, and Multiplicities

In this section we develop the theory of standard pairs in the context of monomial ideals in affine semigroup rings. We then use standard pairs to describe primary and irreducible decompositions of a monomial ideal, and to compute multiplicities of associated primes. Our first step is to introduce the general combinatorial framework for this section.

Definition 3.1. We call \((a, F)\), where \(a \in \mathbb{N}A\) and \(F\) is a face of \(A\), a pair of \(A\). In this case, \((a, F)\) belongs to the face \(F\).
(1) \((a, F) \preceq (b, G)\) denotes the containment \(a + NF \subseteq b + NG\). As the notation implies, this gives a partial order among pairs.

(2) We say that \((a, F)\) and \((b, F)\) overlap if \(a - b \in \mathbb{Z}F\), equivalently, if \((a + NF) \cap (b + NF) \neq \emptyset\).

Overlapping is an equivalence relation among pairs. We emphasize that overlapping is only defined for pairs that belong to the same face. The overlap class containing \((a, F)\) is denoted \([a, F]\).

(3) We say that \((a, F)\) divides \((b, G)\) if there is \(c \in NA\) such that \(a + c + NF \subseteq b + NG\). This extends the notion of divisibility in \(\mathbb{k}[NA]\) (see Notation 2.3) to the pairs of \(A\).

Overlapping is a special case of divisibility, which means that divisibility is not an antisymmetric relation, and therefore not a partial order on pairs. This difficulty is resolved if we extend the definition of divisibility to overlap classes of pairs.

**Lemma 3.2.** Suppose \((a, F)\) divides \((b, G)\). If \((a', F)\) overlaps \((a, F)\) and \((b', G)\) overlaps \((b, G)\), then \((a', F)\) divides \((b', G)\). In particular, divisibility is a partial order on overlap classes.

**Proof.** Since \((a, F)\) and \((a', F)\) overlap, we may choose \(c_1 \in NF\) such that \(a' + c_1 = a + NF\), which implies that \(a' + c_1 + NF \subseteq a + NF\). As \((a, F)\) divides \((b, G)\), there is \(c_2 \in NA\) such that \(a + c_2 + NF \subseteq b + NG\). But then \(a' + c_1 + c_2 + NF \subseteq b + NG\). Finally, select \(c_3 \in NG\) such that \(b + c_3 \in b' + NG\). Then \(a' + c_1 + c_2 + c_3 + NF \subseteq b' + NG\). It follows that divisibility is well defined on overlap classes of pairs of \(A\). Showing that divisibility is a partial order, in this case, is similarly straightforward.

3.1. **Standard pairs.** We are now ready to relate the combinatorial notion of pairs to the algebraic context of monomial ideals.

Let \(I\) be a monomial ideal in \(\mathbb{k}[NA]\). The standard monomials of \(I\) are the monomials in \(\mathbb{k}[NA]\) that do not belong to \(I\). We denote

\[
\text{std}(I) = \{a \in NA \mid t^a \notin I\}.
\]  

**Definition 3.3.** Let \(I\) be a monomial ideal in \(\mathbb{k}[NA]\). A proper pair of \(I\) is a pair \((a, F)\) of \(A\) such that \(a + NF \subseteq \text{std}(I)\). A standard pair of \(I\) is a proper pair which is maximal with respect to \(\preceq\) (Definition 3.11). The collection of standard pairs of a monomial ideal \(I\) is denoted \(\text{stdPairs}(I)\).

This is the natural extension of the original definition of standard pairs from \([28]\), although the partial order is reversed. On the other hand, standard pairs exhibit behaviors over semigroup rings that do not occur over polynomial rings, as can be seen in the following examples.

**Example 3.4** (Continuation of Example 2.5).

(1) Consider \((x^3y, xy^2)\) in \(\mathbb{k}[x, y]\). Denote \(F = \{(1, 0)\}, G = \{(0, 1)\}\), and \(O = \emptyset\). These subsets of the (columns of) \(A\) respectively span the nonnegative \(x\)-axis, the nonnegative \(y\)-axis, and the origin, which are the proper faces of \(\mathbb{R}_{\geq 0}A\). Our ideal has four standard pairs, \(((0, 0), F)\), \(((0, 0), G)\), \(((1, 1), O)\), and \(((2, 1), O)\), depicted in Figure 3a using thick lines. In this case, \(((1, 1), O)\) divides \(((2, 1), O)\). Thus there are three maximal standard pairs with respect to divisibility. There are no overlapping standard pairs. Consider again the ideal \((x^2y^2, x^3y)\) in \(\mathbb{k}[x, xy, y^3]\) from Example 2.5 (2). This ideal also has four standard pairs: \(((0, 0), G)\), \(((1, 1), G)\), \(((0, 0), F)\), and \(((2, 1), O)\) depicted in Figure 3b. Here \(F = \{(1, 0)\}, G = \{(1, 2)\}\), and \(O = \emptyset\) correspond to the proper faces of the cone \(\mathbb{R}_{\geq 0}A\). The standard pair \(((0, 0), G)\)
(A) Standard pairs of $I = \langle x^3y, xy^2 \rangle$ in $\mathbb{k}[x, y]$

(B) Standard pairs of $\langle x^2y^2, x^3y \rangle$ in $\mathbb{k}[x, xy, xy^2]$

FIGURE 3. Standard pairs in two-dimensional affine semigroup rings

divides $((1, 1), G)$, and we again have three maximal standard pairs with respect to divisibility. There are no overlapping standard pairs.

(2) This example illustrates that overlapping standard pairs can occur even if the semigroup ring is normal. Consider $\langle x^2z^2, xyz, x^2yz^2 \rangle$ in $\mathbb{k}[z, xz, yz, xyz]$. Let $F = \{(0, 0, 1), (0, 1, 1)\}$, which gives the face of $\mathbb{R}_{\geq 0}A$ whose linear span is the $yz$-plane. In this case, we have three standard pairs $((0, 0, 0), F)$ (a blue region in Figure 4a), $((1, 0, 1), F)$ (a yellow region in Figure 4a), and $((1, 1, 1), F)$ (a red region in Figure 4a). The standard pairs $((1, 0, 1), F)$ and $((1, 1, 1), F)$ overlap. In this case, there are two overlap classes of standard pairs. As the pair $((0, 0, 0), F)$ divides (the overlap class of) $((1, 0, 1), F)$, we have only one overlap class which is maximal with respect to divisibility.

(3) Recall the ideal $\langle x, xyz, xyz^2 \rangle$ in $\mathbb{k}[x, xy, xz, xyz, y^2, z^2]$ from Example 2.5(4). Note again that this semigroup ring is not normal. Let $F = \{(0, 0, 2), (0, 2, 0)\}$ be the face of $\mathbb{R}_{\geq 0}A$ whose linear span is the $yz$-plane, and let $G = \{(0, 2, 0)\}$ be the face whose linear span is the $y$-axis. In this case, our monomial ideal has three standard pairs: $((0, 0, 0), F)$ (yellow points in Figure 4b), $((1, 0, 1), F)$ (red points in Figure 4b), and $((1, 1, 0), G)$ (blue points in Figure 4b). Since $((1, 1, 0), G)$ cannot divide $((1, 0, 1), F)$, it follows that there are two standard pairs that are maximal with respect to divisibility. A feature of this example is that the Zariski closure of the set $(1, 0, 1) + \mathbb{N}F$ contains $(1, 1, 0) + \mathbb{N}G$, a situation that does not occur for standard pairs of monomial ideals in polynomial rings.

3.2. Primary Decomposition. Our goal now is to use standard pairs to give a primary decomposition of a monomial ideal $I$ in $\mathbb{k}[NA]$ with monomial primary components. This is achieved in Theorem 3.8 whose proof we break into several steps.

First, we give a sufficient condition for a monomial ideal to be primary.

**Proposition 3.5.** Let $I$ be a monomial ideal in $\mathbb{k}[NA]$. If all the standard pairs of $I$ belong to the same face $F$ of $A$, then $I$ is $p_F$-primary.
Recall that we denote $\langle a, F \rangle$ the overlap class containing $(a, F)$. We claim that if $b \in a + NF$, then $(I : t^b) = p_F$. To see this, let $c \in NA \setminus NF$. If $t^{b+c} \notin I$, then $b + c$ belongs to $a' + NF$ for some standard pair $(a', F)$ of $I$ (all standard pairs of $I$ belong to $F$). As $c \notin NF$, this contradicts the maximality of $(a, F)$. We conclude that if $c \in NA \setminus NF$, then $t^{b+c} \in I$, so that $(I : t^b) \supset p_F$. We already knew the reverse inclusion, therefore $(I : t^b) = p_F$, which shows that $p_F$ is associated to $I$.

To see that no other prime is associated we show that $(I : t^b)$ is not prime ideal in all other cases. If the overlap class of $(a, F)$ is not maximal with respect to divisibility and $b \in a + NF$, then $(I : t^b)$ is not prime. Since $[a, F]$ is not maximal, there is a standard pair $(a', F)$ of $I$, whose overlap class is maximal with respect to divisibility, and such that $(a, F)$ divides $(a', F)$. In particular, there is $c \in NA$ such that $b + c \in a' + NF$. Indeed, $c \notin NF$ as $(a, F)$ and $(a'F)$ are not in the same overlap class. Since $(a', F)$ is a standard pair of $I$, it follows that $t^c \notin (I : t^b)$. By the previous argument, however, since $c \in NA \setminus NF$ and $b + c \in a' + NF$, we have $t^c \in (I : t^{b+c})$, which implies $t^{2c} \in (I : t^b)$. We conclude that $(I : t^b)$ is not prime.

The converse of Proposition 3.5 holds and is proved by exhibiting a primary decomposition (Theorem 3.3). Before we can do that, we prove a finiteness result.

**Lemma 3.6.** Let $I$ be a monomial ideal in $k[NA]$, and let $F$ be a face of $A$ such that $I$ has a standard pair belonging to $F$. There are finitely many overlap classes of standard pairs of $I$ belonging to $F$ that are maximal with respect to divisibility.
Proof. If \((a, F)\) and \((b, F)\) are standard pairs of \(I\) that do not overlap, and whose overlap classes are maximal with respect to divisibility, then \(a - b \notin NA\) and \(b - a \notin NA\). Now apply Lemma 2.4. 

Our next step is to construct a \(p_F\)-primary ideal, which is later shown to be a valid choice for a \(p_F\)-primary component of \(I\).

**Proposition 3.7.** Let \(I\) be a monomial ideal in \(k[NA]\), and let \(F\) be a face of \(A\) such that \(I\) has a standard pair belonging to \(F\). Set

\[
S = \left\{ t^b \mid b \in NA \text{ divides some element of } a + NF \text{ for some standard pair } (a, F) \text{ of } I \text{ whose overlap class is maximal with respect to divisibility} \right\}.
\]

Then \(S\) is the set of standard monomials of a monomial ideal \(C_F\), \(C_F \supseteq I\), and \(C_F\) is \(p_F\)-primary.

Proof. The first assertion is equivalent to the following statement, whose proof is straightforward: if \(t^b\) does not divide any monomial arising from the maximal overlap classes, and \(c \in NA\), then \(t^{b+c}\) cannot divide any such monomial either. The second assertion follows from the fact that \(I\) is an ideal: no monomial in \(S\) can belong to \(I\).

It remains to be shown that \(C_F\) is \(p_F\)-primary. By Proposition 3.5 it is enough to show that all standard pairs of \(C_F\) belong to the face \(F\). To see this, we first observe that if \(t^b \in S\), then \(t^{b+c} \in S\) for all \(c \in NF\). This implies that \((b, F)\) is a proper pair for all \(t^b \in S\).

To finish the proof, we check that \(C_F\) has no proper pairs of the form \((b, G)\), where \(G\) strictly contains \(F\). This is a consequence of the following claim:

If \(t^b \in S\) and \(c \in NA \setminus NF\), there is a positive integer \(k\) such that \(t^{b+kc} \notin S\).

To prove the claim, as \(c \in NA \setminus NF\), there is a facet \(H\) of \(A\) such that \(H\) contains \(F\) and \(\varphi_H(c) > 0\) (see Definition 2.1).

Since \(H\) contains \(F\), \(\varphi_H\) is constant on each set \(a + NF\). Moreover, if \((a, F)\) and \((a', F)\) are overlapping standard pairs of \(I\), then the value of \(\varphi_H\) on \(a + NF\) equals the value of \(\varphi_H\) on \(a' + NF\). Now, by Lemma 3.6 there are finitely many maximal overlap classes of standard pairs. This implies that there is a positive integer \(N\) which is an upper bound for the values that \(\varphi_H\) attains on these classes. It follows that for any monomial in \(S\), the value of \(\varphi_H\) on its exponent is at most \(N\). In particular, \(\varphi_H(b) \leq N\). Since \(\varphi_H(c) > 0\), we may choose a sufficiently large \(k\) so that \(\varphi_H(b + kc) = \varphi_H(b) + k\varphi_H(c) > N\). It follows that \(b + kc \notin S\), as was claimed.

**Theorem 3.8.** Let \(I\) be a monomial ideal in \(k[NA]\). Let

\[
\mathcal{S} = \{ F \text{ face of } A \mid I \text{ has a standard pair belonging to } F \}.
\]

For \(F \in \mathcal{S}\), let \(C_F\) be as in Proposition 3.7. Then \(I = \bigcap_{F \in \mathcal{S}} C_F\) is an irredundant primary decomposition of \(I\). Consequently,

1. \(p_F\) is associated to \(I\) if and only if \(I\) has a standard pair that belongs to \(F\).
2. \(I\) is \(p_F\)-primary if and only if all standard pairs of \(I\) belong to \(F\).

Proof. By Proposition 3.7 it is enough to show that \(I = \bigcap_{F \in \mathcal{S}} C_F\). By construction, \(I \subseteq C_F\) for all \(F \in \mathcal{S}\), so that \(I \subseteq \bigcap_{F \in \mathcal{S}} C_F\). To see the reverse inclusion, we claim that if \(t^b \notin I\), then \(t^b \notin \bigcap_{F \in \mathcal{S}} C_F\), or equivalently, \(t^b \notin C_F\) for some \(F \in \mathcal{S}\). To see this, since \(t^b \notin I\), there is a standard pair \((a, F)\) of \(I\) such that \(b \in a + NF\). But then \(t^b \notin C_F\) by the construction of \(C_F\).
To show the irredundancy, suppose that the decomposition is not irredundant. Then, there exists a face $F$ such that $C_F \supseteq \bigcap_{G \neq F} C_G \supseteq I$. Let $(a, F)$ be a maximal overlap class of $I$. Denote $\bigcup[a, F]$ be the union of all monomials in $a' + NF$ where $(a', F)$ is in the overlap class $[a, F]$. Then, for any monomial $b \in \bigcup[a, F]$, there is a face $G$ such that $b \notin C_G$. Hence, $b$ divides some element of $c + NG$ such that $(c, G)$ form a standard pair of an overlap class which is maximal with respect to divisibility. This implies two facts; first of all, $F$ is not a vertex. If $F$ is a vertex, then $(a', F)$ is inside of the standard pair, contradicting the maximality of the standard pair. Next, $(b, G)$ is a proper pair of $I$, hence it is contained in $\bigcup[b, G]$. From Lemma 3.6, $\bigcup[a, F]$ is covered by by finitely many $\bigcup[d, G']$ for some faces $G'$ with monomials $d$.

Now we show that every overlap class covers finitely many monomials of $\bigcup[a, F]$. This contradicts pigeonhole principle since $\bigcup[a, F]$ is infinite. Suppose that $\bigcup[d, G']$ is such that $\bigcup[a, F] \cap \bigcup[d, G']$ is infinite. This implies $a + c \in d + NG'$ for some $c \in NF$. If $F \subseteq G'$, then $(a, G')$ is a proper pair containing $(a, F)$, a contradiction. Conversely, if $G' \subseteq F$, then there exists $e \in NF$ such that $d + e \in I$. Thus, $a + c + e = d + c' + e \in I$, a contradiction. Thus, there exist two hyperplanes $H_F$ and $H_G$ such that $\varphi_{HF}(NF)$ is bounded while $\varphi_{HG}(NG)$ is not, and vice versa. Since $\bigcup[a, F] \cap \bigcup[d, G']$ is infinite, let $(f_i) \subseteq NF$ be an ordered sequence such that $\varphi_{HG}(f_i) \varphi_{HG}(f_{i+1})$ for all $i$ and $a + f_i \in \bigcup[a, F] \cap \bigcup[d, G']$. Such a sequence exists, since $\{\varphi_{HG}(f) < n : f \in NF\}$ is always finite for any $n \in \mathbb{N}$. Then, $\varphi_{HG}(\{a + f_i : i \in \mathbb{N}\})$ is not bounded, contradicting the fact that $\varphi_{HG}(\bigcup[d, G'])$ is bounded.

Example 3.9 (Continuation of Examples 2.5 and 3.4).

1. In Example 2.5, $\langle x^3y, xy^2 \rangle \subseteq k[x, y]$ has three maximal overlap classes of standard pairs, $[(0, 0), F]$, $[(0, 0), G]$, and $[(2, 1), O]$. In the notation of Proposition 3.7 and Theorem 3.8, $C_F = \langle xy, xy^2 \rangle$, $C_G = \langle x \rangle$, and $C_O = \langle x^2, xy^2 \rangle$, yielding the primary decomposition

$$\langle x^3y, xy^2 \rangle = \langle xy, xy^2 \rangle \cap \langle x \rangle \cap \langle x^2, xy^2 \rangle.$$

2. Figure 5 depicts the primary decomposition of $\langle x^2y^2, x^3y \rangle \subseteq k[x, xy, xy^2]$. Ideals are indicated using shaded regions, standard pairs are indicated using thick lines and circles.

3. In Example 2.5, the ideal $\langle x^2z^2, x^2yz^2, x^2y^2z^2 \rangle \subseteq k[z, xz, yz, xyz]$ under consideration is $p_F$-primary.

4. A primary decomposition of $\langle x, xyz, xy^2z \rangle \subseteq k[x, xy, xz, xyz, y^2, z^2]$ is depicted in Figure 6.

Exponents of monomials in the ideal are colored black. Other colors are used to indicate monomials from the same standard pair.

3.3. Irreducible Decomposition. We now address the irreducible decomposition of monomial ideals in semigroup rings using standard pairs. While the existence of monomial irreducible decomposition of monomial ideals in semigroup rings is known [22, Corollary 11.5, Proposition 11.4], an effective combinatorial description of such a decomposition was missing from the literature before this work. As a side note, we recall that monomial ideals in semigroup rings can be viewed as binomial ideals in polynomial rings, and mention that binomial ideals do not in general have irreducible decompositions into binomial ideals [19].

In order to decide whether an ideal is irreducible, one must examine socles. That is the gist of the following result.

Theorem 3.10 ([29] Proposition 3.14]). Let $(R, m)$ be a local noetherian ring and let $M$ be a finitely generated $R$-module. Let $p$ be an associated prime of $M$, and denote its residue field
STANDARD PAIRS FOR MONOMIAL IDEALS IN SEMIGROUP RINGS

\[ x^2y^2, x^3y \]

(A) Standard pairs of \( \langle x^2y^2, x^3y \rangle \)

\[ \langle x \rangle \]

(B) Standard pairs of \( C_G = \langle x \rangle \)

\[ \langle xy, xy^2 \rangle \]

(C) Standard pairs of \( C_F = \langle xy, xy^2 \rangle \)

\[ \langle x^2, xy^2 \rangle \]

(D) Standard pairs of \( C_O = \langle x^2, xy^2 \rangle \)

**Figure 5.** A primary decomposition of \( \langle x^2y^2, x^3y \rangle \) in \( \mathbb{k}[x, xy, xy^2] \)

by \( K \). Let \( N \) be the submodule of \( M \) whose elements are annihilated by \( p \). The number of \( p \)-primary components in an irredundant irreducible decomposition of the null submodule of \( M \) is the dimension of the localization \( N_p \) as a \( K \)-vector space.

We are now able to determine whether a monomial ideal in \( \mathbb{k}[N,A] \) is irreducible.

**Corollary 3.11.** Let \( I \) be a \( p_F \)-primary monomial ideal in \( \mathbb{k}[N,A] \). The number of overlap classes of standard pairs of \( I \) that are maximal with respect to divisibility equals the number of components in an irredundant irreducible decomposition of \( I \). In particular, \( I \) is irreducible if and only if it has a single overlap class of standard pairs that is maximal with respect to divisibility.

**Proof.** Theorem 3.8 shows that all standard pairs of \( I \) belong to \( F \). The proof of Proposition 3.5 shows that, in this situation, the submodule of \( \mathbb{k}[N,A]/I \) whose elements are annihilated by \( p_F \) is spanned as a \( k \)-vector space by the monomials \( t^b \) such that \( b \in a + NF \) for some standard pair \((a, F)\) whose overlap class is maximal with respect to divisibility. After localization at \( p_F \),
this module becomes a vector space over the residue field whose dimension equals the number of overlap classes of standard pairs that are maximal with respect to divisibility. This assertion follows from the following observations:

Note also that a linear combination of monomials with coefficients in the residue field can only be zero if the pairwise differences of the exponents of the monomials belong to $\mathbb{Z}_F$. Now the desired result follows from Theorem 3.10.

By Theorem 3.8 in order to perform irreducible decompositions of monomial ideals, it is enough to do it for primary monomial ideals.

**Proposition 3.12.** Let $I$ be a $p_F$-primary monomial ideal in $\mathbb{k}[N_A]$, and let $[b_1, F], \ldots, [b_\ell, F]$ be the maximal overlap classes of standard pairs of $I$ with respect to divisibility. For each $1 \leq i \leq \ell$, let $T_i = \{c \in b + NF \mid (b, F) \text{ is a standard pair of } I \text{ whose overlap class divides } [b_i, F]\}$. Then $T_i$ is the set of standard monomials of a monomial ideal $J_i$. Moreover $J_i \supset I$, $J_i$ is irreducible, and $I = J_1 \cap \cdots \cap J_\ell$ is an irredundant irreducible decomposition of $I$.

**Proof.** The arguments that proved Proposition 3.7 show that $J_i$ is a monomial ideal all of whose standard pairs belong to $F$. By construction, $[b_i, F]$ is the unique overlap class of standard pairs of $J_i$ that is maximal with respect to divisibility. It follows that $J_i$ is irreducible by Corollary 3.11. The decomposition $I = \cap_{i=1}^\ell J_i$ is verified in the same way as the primary decomposition in Theorem 3.8.

We emphasize that Theorem 3.8 and Proposition 3.12 can be combined to produce an irredundant irreducible decomposition of a monomial ideal in $\mathbb{k}[N_A]$ in terms of its standard pairs.

**Example 3.13.** The primary decompositions in Example 3.9 are also irredundant irreducible decompositions. We now give two more examples for non-normal two-dimensional semigroup rings. In the first one, the primary decomposition of Theorem 3.8 is already irreducible, in the second one, the primary decomposition is not irreducible.
(i) Let $A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 0 & 1 & 0 \end{bmatrix}$, and consider $I = \langle x^3 y^2, x^5 y, x^6 y \rangle \subset S := \mathbb{k}[x y, x y^2, x^2, x^3] \cong \mathbb{k}[\mathcal{N}A]$. The irreducible decomposition arising from Proposition 3.12 is depicted in Figure 7. Blue points in Figure 7a including \{(4, 2), (5, 3)\} are standard monomials of $I$. Whereas, black points in the shaded region are monomials in the ideal. There is no way to generate \{(4, 2), (5, 3)\} from the generators of $I$ due to the holes \{(1, 0), (2, 1)\} of $\mathcal{N}A$. The socle of $(\mathbb{k}[\mathcal{N}A]/J_1)\langle x^2, x^3 \rangle$ is generated by any monomial whose degree is in $(3, 3) + \mathbb{N}(1, 2)$, since \[(3, 3), \{(1, 2)\}\] is an overlap class which is maximal with respect to divisibility. Similarly, the socle of $(\mathbb{k}[\mathcal{N}A]/J_2)\langle xy^2 \rangle$ is generated by any monomial whose degree is in $\mathbb{N}(1, 0)$ since \[(0, 0), \{(1, 0)\}\] is an overlap class which is maximal with respect to divisibility. Lastly, the socle of $(\mathbb{k}[\mathcal{N}A]/J_3)$ is $\mathbb{k}\{x^5 y^3\}$ since the overlap class which is maximal with respect to divisibility is $[(5, 3), 0]$.

(ii) Let $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, and consider $I = \langle y^2, xy^2 \rangle \subset \mathbb{k}[x^2, y, xy] \cong \mathbb{k}[\mathcal{N}A]$. The irreducible decomposition of $I$ arising from Proposition 3.12 is depicted in Figure 8. In this case, if $F = \{(2, 0)\}$, the color yellow is used for the standard pair $((0, 0), F)$, the color red for $((1, 1), F)$ and blue for $((0, 1), F)$. The pairs $((0, 0), F)$ and $((1, 1), F)$ are maximal with respect to divisibility and do not overlap because $(1, 0) \notin \mathcal{N}A$. The socle of $(\mathbb{k}[\mathcal{N}A]/I)\langle y \rangle$ is $\mathbb{k}\{x^2\} \{y, xy\}$ since $[(0, 1), F]$ and $[(1, 1), F]$ are all overlap classes which are maximal with respect to divisibility.
3.4. Multiplicities and Counting Standard Pairs. One of the main goals of this article is to provide an effective computation of irreducible and primary decomposition of monomial ideals in semigroup rings. Given our previous results, this can be achieved once we know how to compute standard pairs. A key question then is whether a monomial ideal $I$ always has finitely many standard pairs. We answer this question in the affirmative, by linking the number of overlap classes of standard pairs to the multiplicities of associated primes introduced in [28], which we now recall.

Let $I$ be a monomial ideal in $\mathbb{k}[NA]$, and let $p_F$ be an associated prime of $I$. Following [28], we define $\text{mult}_I(p_F)$ to be the length of a maximal strictly increasing chain of ideals

$$I = J_1 \subset J_2 \subset \cdots \subset J_\ell \subset J$$

where $J = \bigcup_{j > 0} (I : p_F^j)$ is the intersection of the primary components of $I$ with associated primes not containing $p_F$, and each $J_k$ is the intersection of $J$ with some $p_F$-primary ideal. Equivalently, $\text{mult}_I(p_F)$ is the length of the largest ideal of finite length in $\mathbb{k}[NA]_{p_F}/\mathbb{k}[NA]_{p_F}$. We emphasize that $\text{mult}_I(p_F)$ is finite.

The following statement generalizes [28] Lemma 3.3. Our argument here is based on the proof of that result.

**Proposition 3.14.** Let $I$ be a monomial ideal in $\mathbb{k}[NA]$ and let $p_F$ be an associated prime of $I$. Then $\text{mult}_I(p_F)$ equals the number of overlap classes of standard pairs of $I$ that belong to $F$.

**Proof.** Recall the decomposition $I = \cap_{G \subseteq \mathcal{J} \supseteq F} C_G$ from Theorem 3.8. Let $\bar{I} = \cap_{G \in \mathcal{J}, G \supseteq F} C_G$. Then $\mathbb{k}[NA]_{p_F}/\bar{I}\mathbb{k}[NA]_{p_F}$ is isomorphic to $\mathbb{k}[NA]_{p_F}/\mathbb{k}[NA]_{p_F}$, so that $\text{mult}_I(p_F) = \text{mult}_{\bar{I}}(p_F)$. Moreover, the standard pairs of $C_G$ are (essentially by construction) the standard pairs of $I$ belonging to $G$. This implies that $I$ and $\bar{I}$ have the same standard pairs belonging to $F$. We have thus reduced the proof to the case when $I = \bar{I}$, and we assume this henceforth. In particular, $J = \cap_{G \in \mathcal{J}, G \neq F} C_G$ is the ideal used in (2) in this case.

Set $J_1 = I$. Let $[a, F]$ be an overlap class of standard pairs of $I$ which is maximal with respect to divisibility. Let

$$T_2 = \bigcup_{(b, G) \in \text{stdPairs}(I) \text{ with } G \supseteq F \text{ and } (b, G) \notin [a, F]} (b + NG) \quad \text{and} \quad S_2 = \bigcup_{(b, F) \in \text{stdPairs}(C_F) \text{ with } (b, F) \notin [a, F]} (b + NF).$$

![Figure 8](image-url)  
(A) Standard pairs of $I = \langle y^2, xy^2 \rangle$  
(B) Standard pairs of $J_1 = \langle y \rangle$  
(C) Standard pairs of $J_2 = \langle xy, y^2 \rangle$

**Figure 8.** An irredundant irreducible decomposition of $I = J_1 \cap J_2$ in $\mathbb{k}[x^2, y, xy]$. 

Then $T_2$ is the set of standard monomials of an ideal $J_2 \supset I$, $S_2$ is the set of standard monomials of a $p_F$-primary ideal $C_2$ and $J_2 = J \cap C_2$.

To see that $T_2$ is the set of standard monomials of a monomial ideal $J_2$, let $b \notin T_2$ and $c \in NA$. If $b + c \in T_2$, then $b + c \in a' + NF$ for some standard pair $(a', G)$ of $I$ (not belonging to $[a, F]$), and therefore $t^{b+c} \notin I$. We conclude that $t^b \notin I$, which implies that $b \in \hat{a} + NF$, where $(\hat{a}, F)$ is a standard pair of $I$ belonging to $[a, F]$. By assumption on $I$, all of its standard pairs belong to faces containing $F$. It follows that $(\hat{a}, F)$ divides $(a', G)$, which implies that $G = F$. Hence, $[a, F]$ is not maximal with respect to divisibility, a contradiction.

The same argument shows that $S_2$ is the set of standard monomials of a monomial ideal $C_2$, which is primary by Proposition 3.5. The equality $J_2 = J \cap C_2$ holds by construction. The localization of $J_2/I$ at $p_F$ is a one-dimensional vector space over the residue field since monomials in overlapping standard pairs differ by a unit in the localization.

Applying this argument successively, one constructs a chain as in (2), whose length is the number of overlap classes of standard pairs of $I$ belonging to $F$. This chain is maximal since the successive quotients are one-dimensional after localization at $p_F$. □

**Corollary 3.15.** Let $I$ be a monomial ideal in $k[NA]$. There are finitely many overlap classes of standard pairs of $I$.

**Proof.** By Proposition 3.14, the number of overlap classes of standard pairs of $I$ is the sum of the multiplicities of its associated primes. Since these multiplicities are finite, and $I$ has only finitely many associated primes, the desired result follows. □

The following is the main result of this section.

**Theorem 3.16.** Let $I$ be a monomial ideal in $k[NA]$. Then $I$ has finitely many standard pairs.

In order to prove Theorem 3.16, we need an auxiliary result.

**Lemma 3.17.** Let $I$ be a monomial ideal in $k[NA]$. If $(a, F)$ is a standard pair of $I$, then $t^a$ is minimal with respect to divisibility in $\{t^b \mid b \in (a + RF) \cap NA\}$.

**Proof.** Let $a' \in (a + RF) \cap NA$ and assume that $t^{a'}$ divides $t^a$. Our goal is to show that $a' = a$.

We claim that $(a', F)$ is a proper pair of $I$. To see this, let $c \in NF$. If $a' + c \notin \text{std}(I)$, then $t^{a'+c} \in I$. Since $t^{a'}$ divides $t^a$, it follows that $t^{a'+c} \in I$, so that $a + c \notin \text{std}(I)$. But this contradicts the fact that $(a, F)$ is proper, and the claim follows. Moreover, since $t^{a'}$ divides $t^a$, we have that $a' + NF \supset a + NF$, in other words, $(a', F) \succ (a, F)$, and as $(a, F)$ is a standard pair, we see that $a' + NF = a + NF$, and so $a' - a \in NF$ and also $a - a' \in NF$. Hence $t^{a-a'}$ is a unit in $k[NA]$. By the strong convexity assumption, the only units in $k[NA]$ belong to $k$, from which we conclude that $a = a'$.

**Proof of Theorem 3.16** By Corollary 3.15 it is enough to show that the equivalence classes under the overlap relation are finite.

Let $(a, F)$ and $(a', F)$ be overlapping standard pairs of $I$. In this case, $a - a' \in ZF$, so that $a + RF = a' + RF$. By Lemma 3.17, this implies that $a$ and $a'$ are minimal with respect to divisibility in $(a + RF) \cap NA$. It follows that $a - a' \notin NA$ and $a' - a \notin NA$. Now apply Lemma 2.4.
4. Algorithms for Finding and Using Standard Pairs

We now turn to concrete methods to compute standard pairs and use standard pairs to produce primary and irreducible decompositions for monomial ideals in an affine semigroup ring. The algorithms outlined in this article are based on three important facts.

1. The complete face lattice of the cone $\mathbb{R}_{\geq 0}A$ can be computed if $A$ is given. This includes finding the primitive integral support functions (Definition 2.1) for all the facets of $\mathbb{R}_{\geq 0}A$.
2. A (homogeneous or inhomogeneous) system of linear equations and inequalities with integer coefficients can be solved, in the sense that there exist algorithms to find the coordinatewise minimal solutions and free variables.
3. There are algorithms to compute standard pairs for monomial ideals in polynomial rings.

We emphasize that solving linear systems over the integers is a fundamental problem in many areas and continues to be the focus of much research, especially in convex and discrete optimization; finding the faces of a cone is an important basic question in discrete geometry. There are many approaches to carry out the computational tasks mentioned above. We discuss specific implementations in Section 5.

Relevant questions that can be easily stated as systems of linear equations and inequalities include the following. Given $a \in \mathbb{Z}^d$, and $F$ a finite subset of $\mathbb{Z}^d$. Does $a$ belong to $\mathbb{Z}F$? Does $a$ belong to $\mathbb{N}F$? With these in hand and knowledge of the faces of $\mathbb{R}_{\geq 0}A$, we can determine, given two pairs $(a, F)$ and $(b, G)$ of $A$, whether $(a, F) \prec (b, G)$, whether $(a, F)$ divides $(b, G)$, and whether $(a, F)$ and $(b, F)$ overlap.

In what follows, for $F$ a face of $A$, we use $\mathbb{N}^F$ to denote $\mathbb{N}^{|F|}$ with coordinates indexed by the elements of $F$. If $G$ is another face of $A$, and $F \subseteq G$, then we consider the natural inclusion $\mathbb{N}^F \subseteq \mathbb{N}^G$ where elements of $\mathbb{N}^F$ are considered as elements of $\mathbb{N}^G$ whose coordinates indexed by $G \setminus F$ are zero.

The following algorithm is the main building block for computing standard pairs in Theorem 4.5. This algorithm decomposes the difference between translations of faces as a finite union of such translations. Its proof is inspired by ideas from [13].

**Theorem 4.1.** Let $b, b' \in \mathbb{N}A$ and let $G, G'$ be faces of $A$ such that $G \subseteq G'$. The procedure below generates a set $\bigcup_{(a, F) \in C}(a + \mathbb{N}F)$ equal to $(b + \mathbb{N}G) \setminus (b' + \mathbb{N}G')$.

1. Find the set $B = \{ (u, w) \in \mathbb{N}^G \times \mathbb{N}^{G'} \mid b + G \cdot u = b' + G' \cdot w \}$.
2. Collect all the minimal generators of the projection of $B$ onto the first component.
3. Construct the ideal $J$ generated by the collection of minimal generators found in the previous step.
4. Find the standard pairs of $J$.
5. Return $\bigcup_{(u, \sigma) \in \text{stdPairs}(J)} (b + G \cdot u + \mathbb{N}\{a_i \mid i \in \sigma\})$.

**Proof.** Consider the set

$$\{ u \in \mathbb{N}^G \mid b + G \cdot u \in (b' + \mathbb{N}G') \}.$$ (3)
Since \( G \subseteq G' \), this is the set of the exponents of the monomials in a monomial ideal \( J \) in \( \mathbb{k}[\mathbb{N}^G] = \mathbb{k}[y_j \mid a_i \in G] \). Observe that
\[
(b + \mathbb{N}G) \smallsetminus (b' + \mathbb{N}G') = \{ b + G \cdot v \mid v \in \mathbb{N}^G \text{ does not belong to the set (3)} \} = \{ b + G \cdot v \mid y^v \notin J \}
\]
Our goal is thus to find the standard monomials of \( J \). First we determine minimal generators for \( J \), which are the coordinatewise minimal elements of (3).

Now consider
\[
\{(u, w) \in \mathbb{N}^G \times \mathbb{N}^{G'} \mid b + G \cdot u = b' + G' \cdot w \}.
\]
The set (3) is the projection onto the first component of the set (5).

Let \( \bar{u} \) be a coordinatewise minimal element of (3). Then there is \( \bar{w} \in \mathbb{N}^{G'} \) such that \( (\bar{u}, \bar{w}) \) belongs to (5). Let \( (u, w) \) be a coordinatewise minimal element of (5) that is coordinatewise less than or equal to \( (\bar{u}, \bar{w}) \). It follows that \( u \) belongs to (3) and is coordinatewise less than or equal to \( \bar{u} \) so that \( u = \bar{u} \). This shows that the coordinatewise minimal elements of (3) are the projections of the coordinatewise minimal elements of (5). Since the set (5) is the set of integer solutions of a system of linear equations and inequalities defined over \( \mathbb{Z} \), its coordinatewise minimal elements can be computed. That there are finitely many such elements follows from Dickson’s Lemma.

Since we now know the generators of the monomial ideal \( J \), we can compute its standard pairs and write
\[
(b + \mathbb{N}G) \smallsetminus (b' + \mathbb{N}G') = \bigcup_{(u, \sigma) \in \text{stdPairs}(J)} \{ b + G \cdot u + \mathbb{N}\{a_i \mid i \in \sigma \} \}
\]
We use the convention that the standard pairs of \( J \subset \mathbb{k}[\mathbb{N}^G] \) are of the form \( (u, \sigma) \) where \( u \in \mathbb{N}^G \) and \( \sigma \subset \{ i \mid a_i \in G \} \). By definition, the fact that \( (u, \sigma) \) is a standard pair of \( J \) implies that \( y^u \prod_{i \in \sigma} y_i^{\lambda_i} \notin J \) for all \( \lambda_i \in \mathbb{N}, i \in \sigma \).

It only remains to be proven that if \( (u, \sigma) \) is a standard pair of \( J \) then \( \{a_i \mid i \in \sigma \} \) is a face of \( G \).

Let \( (u, \sigma) \) be a standard pair of \( J \), and let \( F \) be the smallest face of \( G \) such that \( \mathbb{N}\{a_i \mid i \in \sigma \} \) meets the relative interior of \( \mathbb{R}_{\geq 0} F \). Let \( \sum_{i \in \sigma} \lambda_i a_i \) be an element of the relative interior of \( F \) with \( \lambda_i \in \mathbb{N} \) for \( i \in \sigma \), and set \( \lambda \in \mathbb{N}^G \) whose \( i \)th coordinate is \( \lambda_i \) if \( i \in \sigma \) and 0 otherwise. Then \( y^{u+N\lambda} \notin J \) for all \( N \in \mathbb{N} \). Now let \( a = \sum_{i \in F} \mu_i a_i \) be an element of \( \mathbb{N}^F \), with the \( \mu_i \in \mathbb{N} \), and let \( \mu \in \mathbb{N}^G \) whose \( i \)th coordinate is \( \mu_i \) if \( a_i \in F \) and 0 otherwise, so that \( a = G \cdot \mu \). Since \( \sum_{i \in \sigma} \lambda_i a_i \) is in the relative interior of \( \mathbb{R}_{\geq 0} F \), we may choose \( N \) large enough that \( NG \cdot a - a \in \mathbb{N}^F \), and we may write \( G \cdot (N\lambda - \mu) = G \cdot \nu \) with \( \nu \in \mathbb{N}^F \subset \mathbb{N}^G \). But then \( G \cdot (\nu + \mu) = G \cdot (N\lambda) \), and as \( b + G \cdot (u + N\lambda) \notin b' + \mathbb{N}G' \) (because \( y^{u+N\lambda} \notin J \)), we have \( b + G \cdot (u + \nu + \mu) \notin b' + \mathbb{N}G' \), which in turn implies that \( y^{u+\mu} \notin J \). It follows that \( (u, \{i \mid a_i \in F\}) \) is a proper pair of \( J \). Since \( (u, \sigma) \) is a standard pair of \( J \), we conclude that \( \sigma = \{i \mid a_i \in F\} \).

We need two more auxiliary results for the computation of standard pairs in Theorem 4.5. Here is the first one.

**Lemma 4.2.** Let \( F \) be a face of \( A \) and let \( a \in \mathbb{N}A \). The output of the algorithm below is the minimal elements (with respect to divisibility) of the set \((a + \mathbb{R}F) \cap \mathbb{N}A\).

1. Find \( \{u \in \mathbb{N}^n \mid \varphi_H(A \cdot u) = \varphi_H(a) \text{ for all } H \text{ facet of } A, H \supseteq F\} \) where \( \varphi_H \) is the primitive integral support function of the facet \( H \) of \( A \).

**Proof.** The primitive integral support functions of the facets of \( A \) (Definition 2.1) are linear forms with integer coefficients, and which can be computed. Then \( b \in (a + \mathbb{R}F) \) if and only if \( \varphi_H(b) =
strictly contained in is a finite union of sets.

If such a pair exists, \( (a, F) \) is not proper. Otherwise, find whether there is a pair \( (b, G) \) that is not strictly contained in \( F \). If no such pair exists, \( (a, G) \) is not proper. This is because, when \( (a, G) \) is proper, the elements of \( a + NG \) are exponents of standard monomials. Since \( C_1 \) is a cover of standard monomials, \( a + NG = (a + NG) \cap (\cup_{(a,F) \in C_1} (b + NF)) = \cup_{(b,F) \in C_1} ((a + NG) \cap (b + NF)). \) Each intersection \( (a + NG) \cap (b + NF) \) is a finite union of sets \( c + NF' \) where \( F' \subseteq G \cap F \subseteq G \). If all the intersections involve faces that are strictly contained in \( G \), then we have written \( a + NG \) as a finite union of sets \( c + NF' \) with \( F' \) strictly contained in \( G \), which is impossible for dimension reasons.

If such a pair exists, \( (a + NG) \setminus (b + NG') \) is a union of sets of the form \( a' + NG'' \) where \( G'' \) is a proper face of \( G \), so we reduce to verifying whether the pairs \( (a', G'') \) in the union are proper pairs of \( I \). This yields an iterative procedure to determine whether \( (a, G) \) is proper.

If \( (a, F) \in C_1 \), replace \( (a, F) \) by all pairs of the form \( (a, G) \), where \( G \) is not strictly contained in \( F \). If \( (a, G) \) is proper for \( I \), and \( G \) is maximal with this property. We obtain a finite collection of pairs \( C_2 \), which is still a cover for the standard monomials of \( I \). Now apply to \( C_2 \) the same procedure.

Definition 4.3. Let \( I \) be a monomial ideal in \( k[NA] \) whose set of standard monomials is \( \text{std}(I) \). A cover of \( \text{std}(I) \) is a finite collection \( C \) of pairs of \( A \) such that

\[
\text{std}(I) = \bigcup_{(a,F) \in C} (a + NF).
\]

Proposition 4.4. Let \( I \) be a monomial ideal in \( k[NA] \). The algorithm below has a cover of standard monomials of \( I \) as its input. Its output is the set of standard pairs of \( I \).

1. For each \( (a, F) \in C \), find \( \{(b, F) : b \in (a + RF) \cap NA\} \) using Lemma 4.2.
2. Let \( C_1 \) be the union all the sets from the previous step.
3. For each \( (a, F) \in C_1 \) and \( G \) a face that is not strictly contained in \( F \),
   \( a \) If there is a pair \( (b, G') \in C_1 \) for \( G' \supseteq G \),
   \( i \) If \( (a + NG) \setminus (b + NG') \not\subseteq a + NG \)
   \( (A) \) Delete \( (a, F) \) in \( C_1 \) and append all maximal pairs of the decomposition of \( (a + NG) \setminus (b + NG') \) by Theorem 4.1.
4. Suppose \( C_2 \) is the changed collection from \( C_1 \) by the previous step.
5. If \( C_2 \neq C \), set \( C := C_2 \) and go to (1). Otherwise, return \( C_2 \).

Proof. Let \( C_0 \) be a cover of the standard monomials of \( I \). Then all the elements of \( C_0 \) are proper pairs of \( I \). For \( (a, F) \in C \), if \( b \in NA \) divides \( a \), then \( (b, F) \) is also a proper pair of \( I \).

For each \( (a, F) \in C_0 \), use Lemma 4.2 to compute the minimal elements with respect to divisibility of \( (a + RF) \cap NA \), and replace \( (a, F) \) by the collection of pairs \( (b, F) \), where \( b \) is a minimal element of \( (a + RF) \cap NA \) that divides \( a \). In this way we obtain another collection of pairs \( C_1 \), which is also a cover of the standard monomials of \( I \).

Next, given \( (a, F) \) in \( C_1 \), and \( G \) a face of \( A \) that is not strictly contained in \( F \), we can determine algorithmically whether \( (a, G) \) is a proper pair of \( I \), as follows. First, if \( C_1 \) contains no pairs of the form \( (b, G') \in C_1 \) with \( G' \supseteq G \), then \( (a, G) \) is not proper. Otherwise, find whether there is a pair \( (b, G') \in C_1 \) with \( G' \supseteq G \) such that \( (a + NG) \setminus (b + NG') \not\subseteq a + NG \). If no such pair exists, \( (a, G) \) is also not proper. This is because, when \( (a, G) \) is proper, the elements of \( a + NG \) are exponents of standard monomials. Since \( C_1 \) is a cover of standard monomials, \( a + NG = (a + NG) \cap \bigcup_{(a,F) \in C_1} (b + NF) = \bigcup_{(b,F) \in C_1} ((a + NG) \cap (b + NF)). \) Each intersection \( (a + NG) \cap (b + NF) \) is a finite union of sets \( c + NF' \) where \( F' \subseteq G \cap F \subseteq G \). If all the intersections involve faces that are strictly contained in \( G \), then we have written \( a + NG \) as a finite union of sets \( c + NF' \) with \( F' \) strictly contained in \( G \), which is impossible for dimension reasons.
we used to go from $C_0$ to $C_1$ to construct a new cover $C_3$, and apply to $C_3$ the same procedure we applied to $C_1$, to get a new cover $C_4$.

We claim that repeating this process yields, after finitely many iterations, a cover $C$ which is stable under the given operations. To see this, first, our procedure replaces a proper pair by a collection of proper pairs, all of which are greater than or equal to the original pair with respect to the partial order $\prec$ (see Definitions 3.1 and 3.3). Next, the set of proper pairs of $I$ has finitely many elements that are maximal with respect to $\prec$, namely the standard pairs (Theorem 3.16). Finally, $\prec$-chains that are bounded above are finite, because of the strong convexity assumption on $\mathbb{R}_{\geq 0} A$. From these observations it follows that our procedure arrives at a stable cover $C$ after finitely many steps.

The stable cover $C$ has the following properties

- If $(a, F) \in C$ and $F'$ is a face of $A$ that strictly contains $F$, then $(a, F')$ is not a proper pair of $I$.
- If $(a, F) \in C$ and $(a, G)$ is a proper pair of $I$, then $C$ contains a pair $(a, G')$ such that $G' \supseteq G$.

We claim that $C$ contains the standard pairs of $I$.

Let $(b, G)$ be a standard pair of $I$, and let $(a, F) \in C$ such that $b \in a + \mathbb{N} F$. Every element of $a + \mathbb{N} G$ divides some element of $b + \mathbb{N} G$. This implies that $(a, G)$ is a proper pair of $I$ since $(b, G)$ is proper. By construction of $C$, $C$ contains a pair $(a, F')$ such that $F'$ contains $G$. Then $b + \mathbb{N} G \subseteq a + \mathbb{N} F'$. Since $(a, F')$ is proper, and $(b, G)$ is standard, we must have $(a, F') = (b, G)$, which means that $(b, G) \in C$.

Thus, in order to obtain the standard pairs of $I$, we select the elements of $C$ that are maximal with respect to $\prec$.

We are now ready to compute standard pairs.

**Theorem 4.5.** The algorithm below has the set of (monomial) generators of a monomial ideal $I = \langle t^{b_1}, t^{b_2}, \ldots, t^{b_n} \rangle$ in $k[N_A]$ as its input. Its output is the set of standard pairs of $I$.

1. Find a cover $C$ of $I_1 := \langle t^{b_1} \rangle$ by refining the cover, obtained from the decomposition of $N_A \setminus (b + N_A)$ using Theorem 4.1 and Proposition 4.4.
2. If $n = 1$, return the cover. Otherwise, let $C$ be the cover found by the previous step. Set $i = 2$.
   a. For $(a, F) \in C$,
      i. Compute the decomposition $(a + \mathbb{N} F) \setminus (b_i + N_A)$ from Theorem 4.1 and save them on $C'$.
      ii. Refine $C'$ using Proposition 4.4. If $i = n$, return the refined $C'$. Otherwise, set $C := C'$ and go to (a) with increased $i$ by 1.

**Proof.** Suppose that $I = \langle t^{b} \rangle$. Then the set of standard monomials of $I$ is $N_A \setminus (b + N_A)$, and we can compute the standard pairs of $I$ using Theorem 4.1 and Proposition 4.4.

If $I = \langle t^{b_1}, t^{b_2} \rangle$, first compute the standard pairs of $\langle t^{b_1} \rangle$ and for each such standard pair $(a, F)$, compute $(a + \mathbb{N} F) \setminus (b_2 + N_A)$. This yields a cover of the standard monomials of $I$, which can be massaged using Proposition 4.4 to obtain the standard pairs of $I$.

In general, if the standard pairs of $\langle t^{b_1}, \ldots, t^{b_n} \rangle$ are known, then we may use the same idea to compute the standard pairs of $\langle t^{b_1}, \ldots, t^{b_n}, t^{b_{n+1}} \rangle$.

Finally, finding the overlap classes and determining the maximal ones with respect to divisibility can be done by finding whether certain linear systems of equations and inequalities have integer solutions.


**Remark 4.6.** Having computed the (overlap classes of) standard pairs of \(I\), the associated primes of \(I\) and their corresponding multiplicities can be computed by inspection.

**Example 4.7** (Continuation of Example 2.5(2)). We return to \(I = \langle x^2y^2, x^3y \rangle \subset \mathbb{k}[x, xy, xy^2] \cong \mathbb{k}[NA]\) for \(A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}\). In this case, we illustrate how to compute the standard pairs of \(I\) using the method described in Theorem 4.5.

First we apply Theorem 4.1 to \(NA \setminus ((2, 2) + NA)\), to obtain a cover of standard monomials for \(I_0 = \langle x^2y^2 \rangle\). In this case, the set (5) turns out to be

\[
\{(u, w) \in NA \times NA \ | \ b + A \cdot u = (2, 2)^t + A \cdot w\} = \{(u, w) \in NA \times NA \ | \ A(u - w) = (2, 2)^t\}.
\]

A straightforward calculation shows that this set is the same as

\[
\{(u, w) : u - w = (0, 2, 0) \text{ or } (1, 0, 1)\} = \{(u + w, w) : u = (0, 2, 0) \text{ or } (1, 0, 1), w \in NA\}.
\]

It follows that the minimal solutions we are looking for are \((0, 2, 0)\) and \((1, 0, 1)\). We see that the ideal \(J \subset \mathbb{k}[NA] = \mathbb{k}[z_1, z_2, z_3]\) in the proof of Theorem 4.1 equals \(\langle z_2^2, z_1 z_3 \rangle\) and its standard pairs are

\[
((0, 0, 0), \{(0, 0, 1)\}), \quad ((0, 1, 0), \{(0, 0, 1)\}), \quad ((0, 0, 0), \{(1, 0, 0)\}), \quad \text{and} \quad ((0, 1, 0), \{(1, 0, 0)\}).
\]

Thus,

\[
NA \setminus (2, 2)^t + NA = N\{(1, 2)\} \cup ((1, 1) + N\{(1, 2)\}) \cup N\{(1, 0)\} \cup ((1, 1) + N\{(1, 0)\})
\]

These form a cover of the standard monomials of \(\langle x^2y^2 \rangle\), and it is easily checked that this is the cover by standard pairs of the standard monomials of \(\langle x^2y^2 \rangle\).

Now, to find the standard pairs of \(I\), we compute \(b + NF \setminus ((3, 1) + NA)\) for each \(b + NF\) where \((b, F) \in \text{stdPairs}(\langle x^2y^2 \rangle)\). The sets \(N\{(1, 2)\}, (1, 1) + N\{(1, 2)\}\), and \(N\{(1, 0)\}\) have an empty intersection with \((3, 1) + NA\). Hence we only need to compute \((1, 1) + N\{(1, 0)\}) \setminus ((3, 1) + NA)\).

We apply Theorem 4.1 to do this, yielding \(J = \langle z_1^2 \rangle\) in \(\mathbb{k}[z_1]\). This ideal has two standard pairs \((0, \emptyset)\) and \((1, \emptyset)\). It follows that

\[
((1, 1) + N\{(1, 0)\}) \setminus ((3, 1) + NA) = \{(0, 0), (1, 1)\}
\]

The following is a cover of standard monomials of \(I\).

\[
\{((0, 0), \{(1, 2)\}), ((0, 1), \{(1, 2)\}), ((0, 0), \{(0, 0, 0)\}), ((0, 0), \emptyset), ((1, 1), \emptyset)\}
\]

We next use Proposition 4.4 to remove \(((0, 0), \emptyset)\). Finally, the set of standard pairs of \(I\) is

\[
((0, 0), \{(1, 2)\}), ((0, 1), \{(1, 2)\}), ((0, 0), \{(1, 0)\}), \quad \text{and} \quad ((1, 1), \emptyset),
\]

as depicted in Figure 3b.

We now know how to compute the standard pairs of a monomial ideal \(I\) in \(\mathbb{k}[NA]\). It is natural to try to reverse the process and find generators of a monomial ideal whose standard pairs are given.

**Theorem 4.8.** The algorithm below has the set \(C\) of standard pairs of a monomial ideal in \(\mathbb{k}[NA]\) as its input. Its output is the set of generators for this ideal.

1. **For each** \((a, F) \in C\)
   a. Compute the decomposition \(D\) of \(a + NF \setminus (\cup_{(b, G) \in \text{stdPairs}(I)\mid G \neq F} b + NG)\) using Theorem 4.7.
   b. Set \(J = \emptyset\).
Theorem 4.11. The algorithm below has the set $C$ of standard pairs of a monomial ideal $I$ in $\mathbb{k}[N A]$ as its input. Its output is the set of irreducible decomposition for $I$.

1. Find an overlap class in $C$ which is maximal with respect to divisibility.
2. For such maximal overlap classes $[a, F]$, 
   (a) Find $U = \bigcup_{[a, F] \subset [a, F]} \{ (u, v, w) \in N^A \times N^A \times N^F \mid A \cdot u + A \cdot v = a + F \cdot w \}$. 
   (b) Project the set above as $\pi(U) := \{ u \in N^A : (u, v, w) \in U \}$. 
   (c) For each $u \in \pi(U)$,
      (i) Append all components of the decomposition $N A \cdot u + N A$ by Theorem 4.7 to the collection $C$. 
   (d) Generate the ideal $p_F$ whose standard cover is $C$ by Theorem 4.8.

Proof. If the standard pairs of a monomial ideal $I$ are known, we can determine which overlap classes are maximal with respect to divisibility (among all pairs belonging to the same face). For each standard pair $(a, F)$ in such an overlap class, we can compute $a + N F \setminus \bigcup_{(b, G) \in \text{stdPairs}(I) \setminus G \neq F \setminus B} b + N G$ as a union over pairs $(a, F')$ of sets $a + N F'$. For each such pair $(a, F')$ and each $a_i \mid F'$, check whether $t^a_i t^\alpha \in I$ using the given standard pair. Let $J_1$ be the ideal generated by all monomials $t^a_i t^\alpha$ obtained in this way not in any standard pair. Then $J_1 \subset I$.

Compute the standard pairs of $J_1$. If they coincide with the standard pairs of $I$, then $J_1 = I$ and we are done.

Otherwise, pick maximal overlap classes of standard pairs of $J_1$ that are not standard pairs of $I$, remove all standard pairs of $I$, and use this to find elements of $I$ that do not belong to $J_1$. Obtain an ideal $J_2 \supseteq J_1$.

Repeat this procedure. Since $\mathbb{k}[N A]$ is Noetherian, this process must arrive at $I$ in a finite number of steps. 

Remark 4.9. We observe that standard pairs can be used to compute intersections of monomial ideals. If $I$ and $J$ are monomial ideals in $\mathbb{k}[N A]$, then the union of the collections of standard pairs of $I$ and $J$ is a cover for the standard monomials of $I \cap J$. Applying Proposition 4.4 yields the standard pairs of $I \cap J$, and we can compute generators using Theorem 4.8.

Example 4.10. We return to Example 3.13(ii). In this case, the standard pairs are $\{ (0, 0), \{ (2, 0) \} \}$, $\{ (0, 1), \{ (2, 0) \} \}$, and $\{ (1, 1), \{ (2, 0) \} \}$; we wish to recover the generators of the ideal from this information, using the method from Theorem 4.8. If we start with the standard pair $\{ (0, 1), \{ (2, 0) \} \}$, we obtain the ideal $J_1 = \langle xy^2, x^2y^2 \rangle$. Using $\{ (0, 0), \{ (2, 0) \} \}$ next, we find the monomials $y^2, xy^2$, which generate $I$. 

We can now compute irreducible decompositions of monomial ideals in $\mathbb{k}[N A]$. 

Theorem 4.11. The algorithm below has the set $C$ of standard pairs of a monomial ideal $I$ in $\mathbb{k}[N A]$. Its output is the set of irreducible decomposition for $I$.

1. Find an overlap class in $C$ which is maximal with respect to divisibility.
2. For such maximal overlap classes $[a, F]$:
   (a) Find $U = \bigcup_{[a, F] \subset [a, F]} \{ (u, v, w) \in N^A \times N^A \times N^F \mid A \cdot u + A \cdot v = a + F \cdot w \}$. 
   (b) Project the set above as $\pi(U) := \{ u \in N^A : (u, v, w) \in U \}$. 
   (c) For each $u \in \pi(U)$,
      (i) Append all components of the decomposition $N A \cdot u + N A$ by Theorem 4.7 to the collection $C$. 
   (d) Generate the ideal $p_F$ whose standard cover is $C$ by Theorem 4.8.
Proof. Given the standard pairs of \( I \), we can determine the associated primes of \( I \). If \( p_F \) is associated to \( I \), let \([\bar{a}, F]\) be an overlap class of standard pairs of \( I \) that is maximal with respect to divisibility (among overlap classes belonging to \( F \)).

Let \((a, F)\) be a standard pair of \( I \) whose overlap class is \([\bar{a}, F]\). Define

\[
\bigcup_{(a, F) \in [\bar{a}, F]} \{(u, v, w) \in \mathbb{N}^A \times \mathbb{N}^A \times \mathbb{N}^F \mid A \cdot u + A \cdot v = a + F \cdot w\}.
\]

We see that \( u \) belongs to the projection of (7) onto the first factor if and only if \( A \cdot u \) divides an element of \( a + \mathbb{N}^F \) where \((a, F) \in [\bar{a}, F]\). Using Theorem 3.8 and Proposition 3.12, we see that these are (exponents of) the standard monomials in a valid irreducible component of \( I \).

Adapting the method from Theorem 4.1, we can find a cover of the standard monomials of this irreducible component. Proposition 4.4 yields the corresponding standard pairs, and Theorem 4.8 provides generators.

\[\square\]

Remark 4.12. We can adapt the proof of Theorem 4.11 to compute primary components. Alternatively, we can compute the irreducible components first, and then use Remark 4.9 to intersect all irreducible components associated to the same prime, yielding the corresponding primary component.

5. Implementation

In this section, we briefly discuss ongoing work towards implementation of the algorithms presented in Section 4 in the computer algebra system SageMath [31] and Macaulay2 [11].

All algorithms in Section 4 were implemented in the SageMath called StdPairs [30]. The three key ingredients in our algorithms are computing the face lattice of a cone, solving linear systems of equations and inequalities over \( \mathbb{Z} \), and finding standard monomials of standard pairs in polynomial rings. The first can be done in StdPairs using the Polyhedra module of SageMath. StdPairs calculates all solution sets of integer linear equations for algorithms in Section 4 using a command zsolve from the software 4ti2 [1]. Lastly, StdPairs uses the command standardPairs in Macaulay2 (for details see [8]) to compute standard pairs over polynomial rings. All computational results of StdPairs can be archived as a binary file so that users may avoid repeated calculations.

Since Macaulay2 also has access to the software Polyhedra [3, 4] and 4ti2, the StdPairs package can be translated into a package for Macaulay2. The author of the SageMath version StdPairs package is currently translating StdPairs as a C++ library. This will provide a Macaulay2 interface for this package similar to the one for 4ti2 and we hope it will also increase the speed of computation.

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