STEFAN PROBLEM AS A VANISHING VISCOSITY LIMIT∗

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Abstract. The one phase Stefan free boundary problem is derived as a vanishing viscosity limit of the nonlinear parabolic equation
\[ \partial_t u_\varepsilon = \Delta \left( u_\varepsilon^+ - \varepsilon u_\varepsilon^- \right) + f(u_\varepsilon), \]
where \( u_\varepsilon^+ := \max\{u_\varepsilon, 0\} \) and \( u_\varepsilon^- := \max\{-u_\varepsilon, 0\} \) and \( f \) is a Lipschitz nonlinearity.

1. Introduction

In this paper we study the convergence as \( \varepsilon \to 0 \) of solutions to the following nonlinear parabolic problem:
\[ \partial_t u_\varepsilon = \Delta \alpha_\varepsilon(u_\varepsilon) + f(u_\varepsilon). \]
The function \( \alpha_\varepsilon \) is
\[ \alpha_\varepsilon(u) := \begin{cases} u, & \text{if } u \geq 0 \\ \varepsilon u, & \text{if } u \leq 0. \end{cases} \]
The nonlinearity \( f \) is assumed to be Lipschitz and satisfies \( f(0) = 0 \).

This equation arises in chemical reaction models (see Cannon and Hill [3], Evans [10]) and the spatial segregated limit of competing systems for two species in ecology models, that is, the \( k \to +\infty \) limit of the following system (here we omit intra-species terms):
\[ \begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -k u_1 u_2, \\ \partial_t u_2 - d_2 \Delta u_2 = -k u_1 u_2, \end{cases} \]

see Evans [9], Dancer et. al. [5] and Crooks et. al. [4]. See also [19] for the case with more than two species.

For (1.1), \( u^+ := \max\{u, 0\} \) and \( u^- := \max\{-u, 0\} \) represent the density of two species, while the nodal set \( \{u = 0\} \) is the segregated interface between them. Regularity properties of the solution and of the interface have been studied by Cannon and Hill in [3] and Tonegawa in [18].

In this paper we are interested in the situation where the diffusion rate of one species is so small that negligible. For example, in the liquid-solid phase transition model, sometimes we make the ideal assumption that the heat only conducts inside...
the liquid phase and across the phase interface, but not inside the solid phase. For (1.1) this corresponds to letting \( \varepsilon \to 0 \) in (1.1), hence a kind of vanishing viscosity problem. We will show the limit is a one phase Stefan free boundary problem.

The nonlinear Stefan problem describing biological spreading has been studied by Du and his collaborators in a series of works, see [7] and the survey paper [6], as well as [1] where the Stefan free boundary condition is derived based on an ecology consideration. The derivation of a similar one phase Stefan problem from a system similar to (1.2) has been conducted by Hilhorst et. al. in a series of works [12, 13, 14], where they considered the spatial segregation limit \( k \to +\infty \) of the system

\[
\begin{aligned}
\partial_t u_1 - d_1 \Delta u_1 &= -ku_1 u_2, \\
\partial_t u_2 &= -ku_1 u_2,
\end{aligned}
\]

In the same spirit, a two phase Stefan free boundary problem is derived from a modified system of (1.2) by Hilhorst et. al. in [15, 16].

Our result, combined with the ones from [5] and [19], gives a derivation of this Stefan problem in two steps: first (1.1) is derived from (1.2) as a spatial segregation limit, then the Stefan problem is derived from (1.1) as a vanishing viscosity limit. Note that (1.3) can also be viewed as a vanishing viscosity limit of (1.2). Therefore the derivation in [12, 13, 14] differs from ours in the order of these two steps.

2. Setting and main result

We are only interested in the interior case. Hence we work in the following setting: suppose \( u_{\varepsilon_i} \) is a sequence of solutions to (1.1) in \( Q_1^+ := B_1 \times (0, 1) \subset \mathbb{R}^n \times \mathbb{R} \) with \( \varepsilon_i \to 0 \), and there exists a constant \( \Lambda < +\infty \) such that

\begin{equation}
\|u_{\varepsilon_i}\|_{L^\infty(Q_1^+)} \leq \Lambda.
\end{equation}

Since \( L^\infty(Q_1^+) \) is the dual space of \( L^1(Q_1^+) \), after passing to a subsequence, we may assume \( u_{\varepsilon_i} \) converges to a limit \( u \), and \( f(u_{\varepsilon_i}) \) to \( \bar{f} \), both \(*\)-weakly in \( L^\infty(Q_1^+) \). Similarly, we will also assume \( u_{\varepsilon_i}(0) \) converges \(*\)-weakly to a limit \( u_0 \) in \( L^\infty(B_1) \).

The main result of this paper is

**Theorem 2.1.** Under the above assumptions, we have

(i) \( u_{\varepsilon_i}^+ \) converges to \( u^+ \) in \( C_{loc}(Q_1^+) \), and \( u_{\varepsilon_i}^- \) converges to \( u^- \) \(*\)-weakly in \( L^\infty(Q_1^+) \);

(ii) \( \Omega(t) := \{u(t) > 0\} \) is open and increasing in \( t \) in the sense that

\[
\Omega(t_1) \subset \Omega(t_2), \quad \forall \ 0 < t_1 < t_2 < 1;
\]

in particular, there exists an upper semi-continuous function \( T : B_1 \mapsto [0, 1] \)

such that

\[
\Omega = \{(x, t) \in Q_1^+ : t > T(x)\};
\]

(iii) for a.e. \( 0 < s < t < 1 \),

\begin{equation}
(2.2) \quad u(x, t) = u(x, s) + \int_s^t \bar{f}(x, \tau)d\tau, \quad \text{for a.e. } x \in B_1 \setminus \Omega(t).
\end{equation}
(iv) in $Q_1^+$, $u^+$ satisfies

$$\partial_t \beta(u^+) = \Delta u^+ + f(u^+) \quad \text{in the distributional sense},$$

where

$$\beta(u^+)(x, t) = \begin{cases} u^+(x, t), & \text{if } (x, t) \in \Omega \\
-u_0(x) - \int_0^{T(x)} \bar{f}(x, s) ds, & \text{if } (x, t) \in Q_1^+ \setminus \Omega; \end{cases}$$

(v) $\nabla u^+\epsilon$ converges to $\nabla u^+$ strongly in $L^2_\text{loc}(Q_1^+)$. 

**Remark 2.2.**

- The $L^\infty$ bound usually comes from global consideration, e.g. if there is a suitable initial-boundary value condition.
- It is possible that $\bar{f}\neq f(u)$, which is a common phenomena about nonlinear functions of $*$-weakly convergent sequences. But we do have the corresponding weak-$*$ convergence from $u_\epsilon^+$ to $u^+$.
- The equation (2.3) is the weak formulation of one phase Stefan problem, see Duvaut [8], Friedman and Kinderlehrer [11].
- For each $\epsilon > 0$, $u_\epsilon$ is continuous (in fact, Hölder continuous, see [3]), but the limit $u$ may exhibit a jump across $\partial \Omega$. Therefore a transition layer is created near $\partial \Omega$ when we take the above vanishing viscosity limit. For example, take a point $(x_0, t_0) \in \partial \Omega$ and assume in a neighborhood of it the free boundary is a smooth hypersurface, given by the graph \{t = T(x)\}. Then by taking a hyperbolic scaling

$$\tilde{u}_\epsilon(x, t) := u_\epsilon(x_0 + \epsilon x, t_0 + \epsilon t),$$

we have

$$\tilde{u}_\epsilon(x, t) \to \tilde{u}(x, t) := A \left[ 1 - e^{\frac{t + \xi \cdot x}{|\xi|^2}} \right] \text{ locally smoothly in } \{t < \xi \cdot x\}.$$  

Here $\xi := \nabla T(x_0)$ and

$$A = u_0(x_0) + \int_0^{t_0} \bar{f}(x_0, s) ds.$$  

In fact, $\tilde{u}$ is understood as the solution of

$$\begin{cases} \partial_t \tilde{u} - \Delta \tilde{u} = 0 & \text{in } \{t < \xi \cdot x\}, \\
\tilde{u} = 0 & \text{on } \{t = \xi \cdot x\}. \end{cases}$$

Note that we have

$$|\nabla \tilde{u}|^2 = A^2 |\xi|^{-2} = -A \partial_t \tilde{u}.$$  

Combining this equation with the relations

$$|\nabla u_\epsilon^+|^2 = \epsilon^2 |\nabla u_\epsilon^-|^2, \quad \partial_t u_\epsilon^+ = \partial_t u_\epsilon^- \text{ on } \{u_\epsilon = 0\},$$

and letting $\epsilon \to 0$ we formally get

$$|\nabla u^+(x_0, t_0)|^2 = |A| \partial_t u^+(x_0, t_0).$$

This is the classical Stefan free boundary condition (with a possibly non-constant thermal conductivity coefficient $A$).
Let $\beta_{\varepsilon}$ be the inverse of $\alpha_{\varepsilon}$, that is,

$$\beta_{\varepsilon}(v) = \begin{cases} v, & \text{if } v \geq 0 \\ v/\varepsilon, & \text{if } v \leq 0. \end{cases}$$

Then $v_{\varepsilon} := \alpha_{\varepsilon}(u_{\varepsilon})$ satisfies

$$\partial_t \beta_{\varepsilon}(v_{\varepsilon}) = \Delta v_{\varepsilon} + f(\beta_{\varepsilon}(v_{\varepsilon})). \quad (2.5)$$

We will mainly use $v_{\varepsilon}$ to study the convergence as $\varepsilon \to 0$. From (1.1) it is seen that there is a hope to get a good a priori estimates for $u_{\varepsilon}^+$, while it may be not so for $u_{\varepsilon}^-$. This loss of estimates is remedied by using $v_{\varepsilon}$. Indeed, a main step in our proof is the uniform continuity of $v_{\varepsilon}$, or equivalently, of $u_{\varepsilon}^+$. This uniform continuity is an $\varepsilon$-version of the continuity of solutions to one phase Stefan problem proved by Caffarelli and Friedman [2].

As in Duvaut [8] and Friedman and Kinderlehrer [11], we will also use a parabolic variational inequality formulation to study the convergence, by considering

$$w_{\varepsilon}(x, t) := \int_0^t v_{\varepsilon}(x, s)ds.$$ 

It turns out the equation for $w_{\varepsilon}$ is very good and $w_{\varepsilon}$ enjoys better regularities than $u_{\varepsilon}$ or $v_{\varepsilon}$.

Finally, in any open set $U \subset Q_1^+ \setminus \Omega$, (2.2) says

$$\partial_t u = \bar{f} \quad \text{in the distributional sense.} \quad (2.6)$$

It will be seen that we do not use any estimate on

$$\partial_t u_{\varepsilon} - \varepsilon \Delta u_{\varepsilon} = f(u_{\varepsilon}) \quad \text{in } U.$$ 

The equation (2.2) is derived solely from the one for $w_{\varepsilon}$. Note that (2.2) could be stronger than (2.6) because we do not know if $\partial \Omega$ has zero Lebesgue measure. The defect of this approach is our failure of the determination of the form of $\bar{f}$. We believe this is not achievable in general, unless the convergence from $u_{\varepsilon}(0)$ to $u_0$ is better than the weak-* convergence in $L^\infty(B_1)$.

**Notations:** The following notations will be employed in this paper.

- We will omit the subscript $i$ and just write $\varepsilon \to 0$ for notational simplicity.
- An open ball is denoted by $B_r(x)$. If $x = 0$, we simply write it as $B_r$.
- The forward parabolic cylinder $Q_1^+(x, t) := B_r(x) \times (t, t + r^2)$, the backward parabolic cylinder $Q_r^-(x, t) := B_r(x) \times (t - r^2, t)$, and the parabolic cylinder $Q_1^+(x, t) := B_r(x) \times (t - r^2, t + r^2)$.
- The notion of almost every property is always understood with respect to the standard Lebesgue measure.
- The space $V_2(Q_1^+)$ consists of functions satisfying

$$\|u\|_{V_2(Q_1^+)} := \sup_{t \in (0, 1)} \|u(t)\|_{L^2(B_1)} + \int_{Q_1^+} |\nabla u(x, t)|^2dxdt < +\infty.$$ 

- We use $L$ to denote the Lipschitz constant of the nonlinearity $f$. 

• We use $C$ to denote a constant which does not depend on $\varepsilon$. If we want to emphasize its dependence on some quantities, it is written as $C(\cdot)$.

The remaining part of the paper is organized as follows. In Section 3 we derive some basic uniform regularity of $u_\varepsilon$ and $v_\varepsilon$ in Sobolev spaces. In Section 4 we introduce the parabolic variational inequality formulation and study the convergence in this framework. Section 5 is devoted to the proof of uniform convergence of $u_\varepsilon^+$ (equivalently, $v_\varepsilon$). Finally we prove Theorem 2.1 in Section 6.

3. Uniform Sobolev regularity

In this section we show uniform boundedness of $v_\varepsilon$ in some Sobolev spaces. For any $\eta \in C_0^\infty(B_1)$, multiplying (1.1) by $u_\varepsilon \eta^2$ and integrating by parts we obtain

\[
\frac{d}{dt} \int_{B_1} u_\varepsilon^2 \eta^2 = - \int_{B_1} |\nabla (u_\varepsilon^+ \eta)|^2 + \int_{B_1} |u_\varepsilon^+|^2 |\nabla \eta|^2 - \varepsilon \int_{B_1} |\nabla (u_\varepsilon^- \eta)|^2 + \varepsilon \int_{B_1} |u_\varepsilon^-|^2 |\nabla \eta|^2 + \int_{B_1} f(u_\varepsilon) u_\varepsilon \eta^2.
\]

Substituting standard cut-off function we see that for any $r \in (0, 1)$, there exists a constant $C(r)$ such that

\[
\int_0^1 \int_{B_r} |\nabla u_\varepsilon^+|^2 + |\partial_t u_\varepsilon^+|^2 + \varepsilon (|\nabla u_\varepsilon^-|^2 + |\partial_t u_\varepsilon^-|^2) \leq C(r) \Lambda^2.
\]

Using $v_\varepsilon$ this is rewritten as

\[
\begin{cases}
\int_0^1 \int_{B_r} |\nabla v_\varepsilon^+|^2 + |\partial_t v_\varepsilon^+|^2 \leq C(r) \Lambda^2, \\
\int_0^1 \int_{B_r} |\nabla v_\varepsilon^-|^2 + |\partial_t v_\varepsilon^-|^2 \leq C(r) \Lambda^2 \varepsilon.
\end{cases}
\]

Combining these estimates with the $L^\infty$ bound on $v_\varepsilon$, after passing to a subsequence, we may assume $v_\varepsilon \to u_1$ weakly in $V^2(Q_1^r)$ (for any $r \in (0, 1)$) and strongly in $L^2_{loc}(Q_1^r)$. Since

\[
\|u_\varepsilon^+ - v_\varepsilon\|_{L^\infty(Q_1)} \leq \Lambda \varepsilon,
\]

$u_\varepsilon^+$ also converges strongly to $u_1$ in $L^2_{loc}(Q_1^r)$. After passing to a further subsequence, we also assume $u_\varepsilon^+$ and $v_\varepsilon$ converge a.e. to $u_1$ in $Q_1^r$.

By these convergence and the weak-* convergence of $u_\varepsilon$, $u_\varepsilon^-$ converges *-weakly to $u_2 := u_1 - u$ in $L^\infty_{loc}(Q_1)$.

We emphasize that at this stage we do not claim that $u_1 = u^+$, although this will be shown to be indeed the case later. (As a consequence, the above operation of passing to a subsequence does not lead to different limits.) What we have now is only $u_1 \geq u^+$ (because weak-* convergence preserves the order relation), and consequently $u_2 \geq u^-$. 

\[
\]
4. Parabolic variational inequality

Take an arbitrary $h \in [0, 1)$. If $h > 0$, after passing to a subsequence, assume $u_\varepsilon(h)$ converges to a limit $u_h$ *-weakly in $L^\infty(B_1)$.

Define

$$w_{h, \varepsilon}(x, t) := \int_h^t v_\varepsilon(x, s)ds, \quad \forall (x, t) \in B_1 \times [h, 1].$$

Direct calculation using (2.5) gives

(4.1) \[ \Delta w_{h, \varepsilon} - \partial_t w_{h, \varepsilon} = g_{h, \varepsilon}, \]

where

$$g_{h, \varepsilon}(x, t) := \beta_\varepsilon(v_\varepsilon(x, t)) - v_\varepsilon(x, t) - \beta_\varepsilon(v_\varepsilon(x, h)) - \int_h^t f(\beta_\varepsilon(v_\varepsilon)(x, s))ds$$

(4.2) \[ = -(1 - \varepsilon)u_\varepsilon(x, t) - u_\varepsilon(x, h) - \int_h^t f(u_\varepsilon(x, s))ds. \]

Because $u_\varepsilon$ and $v_\varepsilon$ are uniformly bounded in $L^\infty(Q_1)$, there exists a constant $C$ such that

$$\|g_{h, \varepsilon}\|_{L^\infty(B_1 \times (h, 1))} \leq C.$$

By definition,

$$-\Lambda \varepsilon \leq w_{h, \varepsilon} \leq \Lambda, \quad \text{in } B_1 \times (h, 1).$$

Then standard $W^{2,p}$ estimates imply that both $\nabla^2 w_{h, \varepsilon}$ and $\partial_t w_{h, \varepsilon}$ are uniformly bounded in $L^p_{loc}(B_1 \times (h, 1))$, for any $p < +\infty$. By Sobolev embedding theorems, $w_{h, \varepsilon}$ are uniformly bounded in $C^{1+\alpha, \frac{1+\alpha}{2}}_{loc}(B_1 \times [h, 1))$ for any $\alpha \in (0, 1)$.

After passing to a subsequence of $\varepsilon \to 0$, we may assume

- $w_{h, \varepsilon}$ converges to $w_h$ in $C^{1+\alpha, \frac{1+\alpha}{2}}_{loc}(B_1 \times [h, 1))$ for any $\alpha \in (0, 1)$;
- $\nabla^2 w_{h, \varepsilon}$ and $\partial_t w_{h, \varepsilon}$ converge to $\nabla^2 w_h$ and $\partial_t w_h$ respectively, with respect to the weak topology in $L^p_{loc}(B_1 \times (h, 1))$ for any $p < +\infty$;
- $g_{h, \varepsilon}$ converges to $g_h$ *-weakly in $L^\infty(B_1 \times (h, 1))$.

Note that by the convergence of $v_\varepsilon$ established in Section 3, $\partial_t w_h = u_1$ in the distributional sense. Later we will show that all of these limits are independent of the choice of subsequences of $\varepsilon \to 0$, and $g_h$ is independent of the choice of $h$.

Passing to the limit in (4.1) we get

(4.3) \[ \Delta w_h - \partial_t w_h = g_h, \quad \text{in } B_1 \times (h, 1). \]

Concerning $w_h$, we observe the following facts:

1. Since $\partial_t w_h = u_1 \geq 0$, $\{w_h(t) > 0\}$ (which is an open set by the continuity of $w_h$) is increasing in $t$ in the sense that
   \[ \{w_h(t_1) > 0\} \subset \{w_h(t_2) > 0\}, \quad \forall h < t_1 < t_2 < 1; \]
2. $w_h \geq 0$, which follows by combining (1) with the fact that $w_h(x, h) \equiv 0$;
3. $g_h = 0$ a.e. in $\{w_h = 0\}$, which is a consequence of the fact that $w_h \in W^{2,1}_{p, loc}(B_1 \times (h, 1))$. 

5. Uniform convergence

In this section we prove the uniform convergence of \( u_\varepsilon^+ \). In the following we denote, for any \( r > 0 \),

\[
Q_r^* := B_r \times \left( -\frac{r^2}{4n}, 0 \right).
\]

**Lemma 5.1.** For any \( M \geq 0 \), there exists a constant \( \sigma(M) > 0 \) so that the following holds. Suppose \( v_\varepsilon \) is a continuous solution of (2.5) (with a possibly different nonlinearity \( f \), but its Lipschitz constant is still bounded by \( L \)) in \( Q_1^- \) satisfying \( -M\varepsilon \leq v_\varepsilon \leq 1 \) and

\[
\frac{1}{|Q_1^-|} \int_{Q_1^-} (1 - v_\varepsilon) \leq \sigma.
\]

Then

\[
v_\varepsilon \geq \frac{1}{2} \quad \text{in} \quad Q_{1/2}^*.
\]

**Proof.** The proof is divided into two steps.

**Step 1. Caccioppoli inequality.** Given \( -1/(4n) < t_1 < t_2 < 0 \) and \( k \in (0, 1) \), for any \( \eta \in C_0^\infty(B_1 \times (t_1, t_2)) \) and \( t \in (t_1, t_2) \), multiplying (2.5) by \( (k - v_\varepsilon)_+ \eta^2 \) and integrating in \( B_1 \times (t_1, t) \) leads to

\[
\int_{t_1}^t \int_{B_1} |\nabla [(k - v_\varepsilon)_+ \eta]|^2 + \int_{B_1} B_{k, \varepsilon}(v_\varepsilon(t)) \eta^2
\]

\[
= \int_{t_1}^t \int_{B_1} [(k - v_\varepsilon)_+^2 |\nabla \eta|^2 + 2B_{k, \varepsilon}(v_\varepsilon(t)) \eta \partial_t \eta - f(u_\varepsilon)(k - v_\varepsilon)_+ \eta^2],
\]

where

\[
B_{k, \varepsilon}(v) := \begin{cases} 0, & \text{if } v \geq k \\ \frac{k^2}{2} + \frac{1}{2}v^2 - \frac{k}{\varepsilon}v, & \text{otherwise.} \end{cases}
\]

Since \( v_\varepsilon \geq -M\varepsilon \), there exists a constant \( C(M) \) depending only on \( M \) such that

\[
\frac{1}{2}(k - v_\varepsilon)_+^2 \leq B_{k, \varepsilon}(v_\varepsilon) \leq C(M)(k - v_\varepsilon)_+^2.
\]

With this estimate (5.2) is transformed into

\[
\int_{t_1}^t \int_{B_1} |\nabla [(k - v_\varepsilon)_+ \eta]|^2 + \int_{B_1} (k - v_\varepsilon(t))_+^2 \eta^2
\]

\[
\leq C(M) \int_{t_1}^t \int_{B_1} [(k - v_\varepsilon)_+^2 (|\nabla \eta|^2 + |\eta \partial_t \eta|) + (k - v_\varepsilon)_+ \eta^2].
\]

Since \( t \) is arbitrary, by this inequality and Sobolev embedding theorem we get two constants \( p > 1 \) and \( C(M) \) such that

\[
\left( \int_{t_1}^{t_2} \int_{B_1} (k - v_\varepsilon)_+^{2p} |\eta|^{2p} \right)^{\frac{1}{p}} \leq C(M) \int_{t_1}^{t_2} \int_{B_1} (k - v_\varepsilon)_+ [|\nabla \eta|^2 + |\eta \partial_t \eta| + \eta^2].
\]
Step 2. De Giorgi iteration. For any \( m \geq 1 \), set
\[
k_m := 2^{-1} + 2^{-m}, \quad r_m = 2^{-1} + 2^{-m}
\]
and
\[
a_m := \int_{Q^*_m} (k_m - v_\varepsilon)_+. \]
Take a function \( \eta_m \in C_0^\infty(Q^*_m) \) such that \( \eta_m \equiv 1 \) in \( Q^*_{r_{m+1}} \), \( 0 \leq \eta_m \leq 1 \) and \( |\nabla \eta_m|^2 + |\partial_t \eta_m| \leq 16(r_m - r_{m+1})^{-2} \). Substituting \( \eta_m \) into (5.4) leads to
\[
\left( \int_{Q^*_{r_{m+1}}}(k_m - v_\varepsilon)^{2p}_+ \right)^{\frac{1}{p}} \leq C(M)4^ma_m. \tag{5.5}
\]
In \( \{(k_{m+1} - v_\varepsilon)_+ \neq 0\} \), we have
\[
(k_m - v_\varepsilon)_+ \geq 2^{-m-1}.
\]
Therefore
\[
\int_{Q^*_{r_{m+1}}}(k_m - v_\varepsilon)^{2p}_+ \geq 2^{-(m+1)(2p-1)}\int_{Q^*_{r_{m+1}}}(k_{m+1} - v_\varepsilon)_+.
\]
Substituting this into (5.5) we get a constant \( A(M) > 1 \) such that
\[
a_{m+1} \leq A(M)^ma^p_m. \tag{5.6}
\]
By our assumptions on \( v_\varepsilon \), we have
\[
a_1 = \int_{Q^*_1}(1 - v_\varepsilon)_+ \leq \int_{Q^*_1}(1 - v_\varepsilon) \leq \sigma|Q^*_1|.
\]
By [17], if \( \sigma \) is small enough (depending only on \( A(M) \)), then \( \lim_{m \to +\infty} a_m = 0 \). Hence
\[
\int_{Q^*_1/2}\left(\frac{1}{2} - v_\varepsilon\right)_+ = 0.
\]
Since \( v_\varepsilon \) is continuous, this implies that \( v_\varepsilon \geq 1/2 \) in \( Q^*_1/2 \). \( \square \)

Lemma 5.2. For any \( \varepsilon, \sigma, M > 0 \), there exist two constants \( \theta := \theta(\varepsilon, \sigma, M) \in (0, 1) \) and \( \rho := \rho(\varepsilon, \sigma, M) \in (0, 1) \) so that the following holds. Suppose \( v_\varepsilon \) is a continuous solution of (2.5) in \( Q^*_1 \) (with a possibly different nonlinearity \( f \), but its Lipschitz constant is still bounded by \( L \)) satisfying \( v_\varepsilon \geq -M\varepsilon \), \( \sup_{Q^*_1} v_\varepsilon = 1 \) and
\[
\frac{1}{|Q^*_1|} \int_{Q^*_1} v_\varepsilon \leq 1 - \sigma, \]
then
\[
v_\varepsilon \leq \theta \quad \text{in} \quad Q^*_\rho.
\]
Proof. Take
\[
\gamma := 1 - \frac{\sigma}{2}, \quad \beta := 1 - \frac{1 - \sigma}{1 - \sigma/2}.
\]
Then it is readily verified that
\[
|\{v_\varepsilon < \gamma\} \cap Q^*_1| \geq \beta|Q^*_1|.
\]
Since
\[ \partial_t v^+_\varepsilon - \Delta v^+_\varepsilon \leq L v^+_\varepsilon, \]
the conclusion follows by applying [2, Lemma 3.1].

**Lemma 5.3.** For any \( \delta > 0 \), there exists an \( r(\delta) < \delta \) (independent of \( \varepsilon \)) such that, for any \((x, t) \in \{v_\varepsilon \leq 0\} \cap Q_{1-\delta}(0, 1)\),
\[
\sup_{Q_{r(\delta)}^*(x, t)} v_\varepsilon \leq \delta.
\]

**Proof.** Assume by the contrary, there exists a \( \delta > 0 \), a sequence of points \((x_\varepsilon, t_\varepsilon) \in Q_{1-\delta}(0, 1)\) satisfying \( v_\varepsilon(x_\varepsilon, t_\varepsilon) \leq 0 \), and a sequence of \( r_\varepsilon \to 0 \) such that
\[
(5.7) \quad \sup_{Q_{r(\delta)}^*(x_\varepsilon, t_\varepsilon)} v_\varepsilon \geq \delta.
\]

For any \( r > 0 \), denote
\[
M_\varepsilon(r) := \sup_{Q_1^*(x_\varepsilon, t_\varepsilon)} v_\varepsilon.
\]

For any \( \varepsilon > 0 \) and \( r \in (r_\varepsilon, \delta) \), take the rescaling
\[
v_\varepsilon^r(x, t) := \frac{1}{M_\varepsilon(r)} v_\varepsilon(x_\varepsilon + rx, t_\varepsilon + r^2 t).
\]

It is a continuous solution of
\[
\partial_t \beta_\varepsilon(v^r_\varepsilon) = \Delta v^r_\varepsilon - \frac{r^2}{M_\varepsilon(r)} f(M_\varepsilon(r) \beta_\varepsilon(v^r_\varepsilon)).
\]

Note that the Lipschitz constant of the nonlinearity is still bounded by \( L \). Furthermore, \( v^r_\varepsilon \geq -M \varepsilon \) in \( Q_1^* \) and \( \sup_{Q_1^*} v^r_\varepsilon = 1 \), where \( M := \Lambda/\delta \). Since \( v^r_\varepsilon(0, 0) \leq 0 \), byLemma 5.1 we must have
\[
\frac{1}{|Q_1^*|} \int_{Q_1^*} (1 - v^r_\varepsilon) \geq \sigma(M).
\]
Therefore Lemma 5.2 is applicable, which gives
\[
\sup_{Q_{r(\delta)}^*} v^r_\varepsilon \leq \theta := \theta(\sigma(M), M) < 1.
\]

Rescaling back this is
\[
M_\varepsilon(\rho r) \leq \theta M_\varepsilon(r), \quad \forall r \in (2r_\varepsilon, \delta/2).
\]

An iteration of this estimate leads to
\[
M_\varepsilon(r_\varepsilon) \leq \theta^{|\log(\frac{r_\varepsilon}{2\varepsilon})|/|\log \rho|} M_\varepsilon(\rho \delta).
\]

Since \( \lim_{\varepsilon \to 0} r_\varepsilon = 0 \) and \( M_\varepsilon(\rho \delta) \leq \Lambda \), we obtain
\[
\lim_{\varepsilon \to 0} M_\varepsilon(r_\varepsilon) = 0,
\]
which is a contradiction with (5.7). \( \square \)
Lemma 5.4. For any \( \lambda > 0 \), there exists a \( \rho(\lambda) > 0 \) (independent of \( \varepsilon \)) so that the following holds. If \((x_\varepsilon, t_\varepsilon) \in Q^*_{r_\varepsilon} \) and \( v_\varepsilon(x_\varepsilon, t_\varepsilon) \geq \lambda \), then
\[
v_\varepsilon \geq \frac{\lambda}{2} \text{ in } Q^*_{\rho(\lambda)}(x_\varepsilon, t_\varepsilon).
\]

Proof. Define \( M_\varepsilon(r) \) and \( v_\varepsilon \) as in the proof of the previous lemma. Note that \( M_\varepsilon(r) \geq \lambda \) for any \( r \). We still have \( v_\varepsilon \geq -M_\varepsilon \) in \( Q^*_1 \) and \( \sup_{Q^*_1} v_\varepsilon = 1 \), with \( M := \Lambda/\lambda \).

Lemma 5.1 says
\[
\frac{1}{|Q^*_1|} \int_{Q^*_1} (1 - v_\varepsilon^r) \leq \sigma(M) \implies \inf_{Q^*_{\rho(\lambda)}(x_\varepsilon, t_\varepsilon)} v_\varepsilon \geq \frac{1}{2} M_\varepsilon(r),
\]
while Lemma 5.2 says
\[
\frac{1}{|Q^*_1|} \int_{Q^*_1} (1 - v_\varepsilon^r) \geq \sigma(M) \implies M_\varepsilon(\rho r) \leq \theta M_\varepsilon(r).
\]

Let \( k_0 \) be the minimal positive integer such that (5.8) holds for \( r = \rho^{k_0} \). Then for any \( k < k_0 \), by (5.9) we have
\[
M_\varepsilon(\rho^{k+1}) \leq \theta M_\varepsilon(\rho^k).
\]
Since for any \( r, \lambda \leq M_\varepsilon(r) \leq \Lambda \), we must have
\[
\frac{\log (\Lambda/\lambda)}{\log \theta} \leq k_0.
\]
Take \( \rho(\lambda) := \frac{\log (\Lambda/\lambda)}{\log \theta} \). By the definition of \( k_0 \), it satisfies the assumption in (5.8). Because \( M_\varepsilon(\rho^{k_0}) \geq \lambda \), we get
\[
\inf_{Q^*_{\rho(\lambda)}(x_\varepsilon, t_\varepsilon)} v_\varepsilon \geq \inf_{Q^*_{\rho^{k_0}}(x_\varepsilon, t_\varepsilon)} v_\varepsilon \geq \frac{1}{2} M_\varepsilon(\rho^{k_0}) \geq \frac{\lambda}{2}.
\]

Remark 5.5. With more work it is possible to obtain a modulus of continuity for \( v_\varepsilon \), which holds uniformly in \( \varepsilon \). We will not prove this because a qualitative result is sufficient for our purpose.

The following lemma shows that for solutions of (1.1), the positivity set is almost non-decreasing in time. This will be passed to the \( \varepsilon \to 0 \) limit as a real monotonicity property.

Lemma 5.6. Given \( \lambda > 0 \) and a cylinder \( Q^r := B_\varepsilon(x) \times (t, t + T) \), suppose \( u_\varepsilon \) is a continuous solution of (1.1) in this cylinder, satisfying
\[
\begin{align*}
\bullet & u_\varepsilon \geq -\Lambda \text{ in } Q^r; \\
\bullet & u_\varepsilon \geq \lambda \text{ in } B_{2r/3}(x) \times \{t\}.
\end{align*}
\]
If \( \varepsilon \) is sufficiently small, then \( u_\varepsilon > 0 \) in \( B_{r/4}(x) \times (t + r^2/8, t + T) \).

Proof. Without loss of generality, assume \((x, t) = (0, 0)\) and \( r = 1 \). We divide the proof into three steps.

Step 1. Comparison functions. Take a function \( \varphi \in C^\infty(\mathbb{R}^n) \), satisfying \( \varphi = \lambda \) in \( B_{1/2}(0) \), \( \varphi \leq \lambda \) everywhere and \( \varphi \equiv -\Lambda \) outside \( B_{2/3}(0) \). Let \( u_{\varepsilon,*} \) be the
solution of (1.1) in $Q^T$ with initial-boundary value $\varphi$. By the comparison principle for (1.1) (see [3, Theorem 6]),

$$u_\varepsilon \geq u_{\varepsilon,*} \text{ in } Q^T.$$  

Therefore we only need to show $u_{\varepsilon,*} > 0$ in $B_{1/4} \times (1/8, T)$.

Set $v_{\varepsilon,*} := \alpha_\varepsilon(u_{\varepsilon,*})$ and

$$w_{\varepsilon,*}(x, t) := \int_0^t v_{\varepsilon,*}(x, s)ds.$$  

As in Section 3 and Section 4, we assume $w_{\varepsilon,*} \to w_*\text{ and } v_{\varepsilon,*} \to v_*$ in the corresponding sense.

Since

$$\partial_t u_{\varepsilon,*} - \varepsilon \Delta u_{\varepsilon,*} \leq Lu_{\varepsilon,*},$$

by the comparison principle

$$u_{\varepsilon,*}^- \leq e^{Lt}\tilde{u}_\varepsilon^- \text{ in } Q^T.$$  

Here $\tilde{u}_\varepsilon$ is the solution of

$$\begin{cases}
\partial_t \tilde{u}_\varepsilon = \varepsilon \Delta \tilde{u}_\varepsilon, & \text{in } Q^T, \\
\tilde{u}_\varepsilon = \varphi^-, & \text{on } \partial Q^T.
\end{cases}$$

**Step 2. A positive lower bound.** Let $G(x, y, t)$ be the heat kernel in $B_1$ with Dirichlet boundary condition for the standard heat operator $\partial_t - \Delta$. Then the heat kernel for $\partial_t - \varepsilon \Delta$ is $G(x, y, \varepsilon t)$. Hence we have the representation formula

$$\tilde{u}_\varepsilon(x, t) = \Lambda - \int_{B_1} G(x, y, \varepsilon t) \left[ \Lambda - \varphi^-(y) \right]dy.$$  

From this representation and the continuity of $\varphi^-$, we get

$$\lim_{\varepsilon \to 0} \tilde{u}_\varepsilon(x, t) = \varphi^-(x), \text{ uniformly in } B_{2/3} \times [0, T].$$

Combining (5.11) and (5.14), we see as $\varepsilon \to 0$, $u_{\varepsilon,*}^- \to 0$ uniformly in $B_{1/2} \times [0, T]$. Substituting this into (4.2) gives

$$\Delta w_{\varepsilon,*}(x, t) - \partial_t w_{\varepsilon,*}(x, t) = \varphi(x, 0) - \int_0^t f(u_{\varepsilon,*}^+) + o(1), \text{ in } B_{1/2} \times (0, T),$$

where $o(1)$ is measured in $L^\infty$. Passing to the limit we obtain

$$\Delta w_*(x, t) - \partial_t w_*(x, t) = \varphi(x, 0) - \int_0^t f(v_*), \text{ in } B_{1/2} \times (0, T).$$

Taking derivative in $t$ and noting that $v_* = \partial_t w_*$, we get

$$\partial_t v_* - \Delta v_* = f(v_*) \text{ in } B_{1/2} \times (0, T).$$

The above three equations are all understood in the distributional sense. However, by standard parabolic theory, $v_*$ is smooth in $B_{1/2} \times (0, T)$. As before we still have $v_* \geq 0$. By the strong maximum principle and Harnack inequality, there exists a constant $\gamma > 0$ such that

$$v_* > \gamma \text{ strictly in } B_{1/3} \times [1/16, T].$$
Step 3. Completion of the proof. By Section 3, as $\varepsilon \to 0$, $v_{\varepsilon,*}$ converges strongly to $v_*$ in $L^1_{\text{loc}}(Q^+_1)$. Thus for any $(x,t) \in B_{1/4} \times (1/8, T)$,

$$\lim_{\varepsilon \to 0} \int_{Q_{1/16}^+(x,t)} v_{\varepsilon,*} = \int_{Q_{1/16}^+(x,t)} v_* \geq \gamma |Q_{1/16}^*|.$$

Combining this estimate with Lemma 5.3, we get an $\varepsilon_*$ such that

$$v_{\varepsilon,*} > 0 \quad \text{in} \ B_{1/4} \times [1/8, T], \quad \text{if} \ \varepsilon \leq \varepsilon_*.$$

The proof is complete by noting that $v_\varepsilon \geq v_{\varepsilon,*}$ in $Q^T$.

Proposition 5.7. As $\varepsilon \to 0$, $v_\varepsilon \to u_1$ in $C_{\text{loc}}(Q^+_1)$.

Proof. For any $(x,t) \in Q^+_1$, let

$$v^*(x,t) := \limsup_{\varepsilon \to 0} v_\varepsilon(x,t,\varepsilon),$$

$$v_*(x,t) := \liminf_{\varepsilon \to 0} v_\varepsilon(x,t,\varepsilon).$$

To prove the uniform convergence of $v_\varepsilon$ to $u_1$, in view of the a.e. convergence of $v_\varepsilon$, it is sufficient to show that $v_* = v^*$ everywhere. Since we always have $0 \leq v_* \leq v^* \leq \Lambda$, this is trivially true if $v^*(x,t) = 0$.

It remains to consider the case when $\lambda := v^*(x,t) > 0$. Take a subsequence $\varepsilon_i \to 0$ and a sequence of points $(x_i, t_i) \to (x,t)$ to attain the limsup in (5.16). By Lemma 5.4, there exists a $\rho := \rho(\lambda/2)$ such that $v_{\varepsilon_i} \geq \lambda/4$ in $Q_\rho(x_{\varepsilon_i}, t_{\varepsilon_i})$. Then Lemma 5.6, applied to $v_{\varepsilon_i}$ in the cylinder $B_\rho(x_{\varepsilon_i}) \times [t_{\varepsilon_i} - \rho^2, \rho^2 + \rho^2]$, implies that $v_{\varepsilon_i} > 0$ strictly in $Q_{\rho/4}(x_{\varepsilon_i}, t_{\varepsilon_i})$. Hence for all $\varepsilon_i$ small,

$$\partial_t v_{\varepsilon_i} - \Delta v_{\varepsilon_i} = f(v_{\varepsilon_i}), \quad \text{in} \ Q_{\rho/6}(x,t).$$

Since $0 < v_{\varepsilon_i} < \Lambda$ in $Q_{\rho/6}(x,t)$, standard parabolic regularity theory and Arzela-Ascoli theorem imply that $v_{\varepsilon_i}$ converge to $u_1$ in a smooth way in $Q_{\rho/7}(x,t)$. As a consequence,

$$\partial_t u_1 - \Delta u_1 = f(u_1), \quad \text{in} \ Q_{\rho/7}(x,t).$$

By the strong maximum principle, there exists a constant $\gamma > 0$ such that $u_1 \geq \gamma$ in $Q_{\rho/8}(x,t)$.

Since $v_\varepsilon \to u_1$ in $L^1(Q_{\rho/8}(x,t))$, similar to Step 3 in the proof of Lemma 5.6, we deduce that for all $\varepsilon$ small (before passing to the subsequence), $v_\varepsilon > 0$ and converges uniformly to $u_1$ in $Q_{\rho/10}(x,t)$. In particular, $v_*(x,t) = v^*(x,t)$.

Because $\|u_\varepsilon^+ - v_\varepsilon\|_{L^\infty(Q_1)} \leq \Lambda \varepsilon$, we get

Corollary 5.8. As $\varepsilon \to 0$, $u_\varepsilon^+ \to u_1$ in $C_{\text{loc}}(Q^+_1)$.

6. Completion of the proof

Since $v_\varepsilon \to u_1$ in $C_{\text{loc}}(Q^+_1)$, $u_1 \in C(Q^+_1)$. In particular, $\Omega := \{u_1 > 0\}$ is an open subset of $Q^+_1$, and for each $t \in (0,1)$, $\Omega(t) := \{u_1(t) > 0\}$ is an open subset of $B_1$.

Property 1. $u_1 = u^+$ and $u_2 = u^-$. 
Proof. For any domain $\Omega'$ compactly contained in $\Omega$, $u_1$ has a positive lower bound in $\Omega'$. Then by Corollary 5.8, there exists an $\varepsilon(\Omega')$ such that for all $\varepsilon < \varepsilon(\Omega')$, $u_\varepsilon > 0$, or equivalently, $u_\varepsilon^- \equiv 0$ in $\Omega'$. Taking the weak-* limit, we see $u_2 = 0$ a.e. in $\Omega$. Because $u_1 = 0$ outside $\Omega$, we obtain $u_1u_2 = 0$ a.e. in $Q_1^+$. The conclusion follows by noting that $u = u_1 - u_2$. □

**Property 2.** $\partial_t u - \Delta u = f(u)$ and $\bar{f} = f(u)$ in $\Omega$.

Proof. This follows from the local uniform convergence of $u_\varepsilon^+$ to $u$ in $\Omega$. □

**Property 3.** $\Omega(t)$ is increasing in $t$.

Proof. Take a point $x \in \Omega(t)$. By the continuity of $u^+$, there exists a cylinder $Q_\varepsilon(x)$ and a positive constant $\lambda > 0$ such that $u \geq 2\lambda$ in $Q_\varepsilon(x)$. By the uniform convergence of $u_\varepsilon^+$, for all $\varepsilon$ small, $u_\varepsilon \geq \lambda$ in $Q_\varepsilon(x)$. Applying Lemma 5.6 to $u_\varepsilon$ in the cylinder $B_\varepsilon(x) \times (t - r^2, 1)$, we see for these $\varepsilon$, $u_\varepsilon > 0$ in $B_\varepsilon(x) \times (t - r^2/4, 1)$.

Hence

$$\partial_t u_\varepsilon - \Delta u_\varepsilon = f(u_\varepsilon), \quad \text{in} \quad B_\varepsilon(x) \times (t - r^2/4, 1).$$

Then Harnack inequality gives a constant $\gamma > 0$ such that $u_\varepsilon \geq \gamma$ in $\bar{B}_{\varepsilon/2}(x) \times [t, 1]$. Letting $\varepsilon \to 0$ and using the uniform convergence of $u_\varepsilon^+$, we deduce that $B_{\varepsilon/2}(x) \times [t, 1] \subset \Omega$. □

By this monotonicity of domain, we obtain a function $T : B_1 \mapsto [0, 1]$ such that

$$\Omega = \{(x, t) \in Q_1^+ : t > T(x)\},$$

which is the waiting time of $x$. By the continuity of $u^+$, $T$ is an upper semi-continuous function.

**Property 4.** In $B_1 \times (h, 1)$, $\{w_h > 0\} = \Omega$.

Proof. For any $(x, t) \in Q_1^+ \setminus \Omega$, by Property 3,

$$u(x, s) \leq 0, \quad \text{a.e. in} \quad (0, t).$$

Therefore Property 1 and Proposition 5.7 imply that

$$v_\varepsilon(x, s) \to 0, \quad \text{uniformly in} \quad s \in [0, t].$$

Then by the definition of $w_{h,\varepsilon}$, we get

$$w_h(x, s) = 0, \quad \forall s \in [h, t].$$

In other words, $w_h = 0$ outside $\Omega$.

Next we claim that $w_h > 0$ in $\Omega$. Indeed, for any $(x, t) \in \Omega$, there exists an open neighborhood of it where $u > 0$ strictly. The claim then follows from the fact that $\partial_t w_h = u^+$. □

Before proceeding further, we need a lemma about the form of $g_h$.

**Lemma 6.1.** In $B_1 \times (h, 1)$,

$$g_h(x, t) = -u(x, t)^- - w_h(x) - \int_h^t \bar{f}(x, s)ds \quad \text{a.e..}$$
Proof. Because \( u^- \) converges to \( u^- \) *-weakly in \( L^\infty(Q^+_1) \), by (4.2) and our assumption on the weak-* convergence of \( u^\varepsilon(h) \) in Section 4, we conclude the proof. \( \square \)

**Property 5.** For any \( 0 \leq h < 1, \)
\[
(6.1) \quad u(x, t) = u_h(x) + \int^t_h \tilde{f}(x, s) ds, \quad \text{for a.e.} \quad (x, t) \in [B_1 \times (h, 1)] \setminus \Omega.
\]
This property roughly means \( \partial_t u = \tilde{f} \) in \( Q^+_1 \setminus \Omega \) (understood in the distributional sense). But note that \( Q^+_1 \setminus \Omega \) could be a very bad set.

**Proof.** By Property 4, \( [B_1 \times (h, 1)] \setminus \Omega \subset \{ w_h = 0 \} \). By results in Section 4, \( g_h = 0 \) a.e. in \( [B_1 \times (h, 1)] \setminus \Omega \). The conclusion then follows from Lemma 6.1. \( \square \)

**Property 6.** For a.e. \( h \in (0, 1), \)
\[
(6.2) \quad u(x, h) = u_h(x), \quad \text{for a.e.} \quad x \in B_1 \setminus \Omega(h).
\]
**Proof.** Taking \( h = 0 \) in (6.1) we obtain
\[
(6.3) \quad \Delta w_0(x, t) - \partial_t w_0(x, t) = -u_0(x) - \int^{T(x)}_0 \tilde{f}(x, s) ds - \int^t_{T(x)} f(u(x, s)) ds.
\]
**Proof.** Since \( g_{0,\varepsilon}(x, 0) = -u_\varepsilon(x, 0) \) converges to \( -u_0(x) \) *-weakly in \( L^\infty(B_1) \), \( \partial_t g_{0,\varepsilon} = -f(u_\varepsilon) \) converges to \( -\tilde{f} \) *-weakly in \( L^\infty(Q^+_1) \), we deduce that \( g_{0,\varepsilon} \) converges *-weakly in \( L^\infty(Q^+_1) \) to
\[
-u_0(x) - \int^{T(x)}_0 \tilde{f}(x, s) ds.
\]
Since \( \{ x \} \times (T(x), t] \subset \Omega \), by Property 2,
\[
\tilde{f}(x, s) = f(u(x, s)), \quad \forall s \in (T(x), t).
\]
Therefore
\[
\int^t_{T(x)} \tilde{f}(x, s) ds = \int^{T(x)}_0 \tilde{f}(x, s) ds + \int^t_{T(x)} f(u(x, s)) ds.
\]
This finishes the proof. \( \square \)

Differentiating (6.3) in \( t \) gives the equation for \( u^+ \), (2.3).

**Property 8.** \( \nabla u^- \rightharpoonup \nabla u^+ \) strongly in \( L^2_{loc}(Q^+_1) \).
Proof. Since $u^+$ is continuous and $\partial_t u - \Delta u = f(u)$ in $\{u > 0\}$, for any $\eta \in C_0^\infty(Q_1^+)$ and $k > 0$, we have
\begin{equation}
\int_{Q_1^+} -(u-k)^+|\eta\partial_t \eta + |\nabla (u-k)^+|^2 \eta^2 + 2\eta (u-k)^+ \nabla (u-k)^+ \cdot \nabla \eta - f(u)(u-k)_+ \eta^2 = 0.
\end{equation}
Letting $k \to 0$ we get
\begin{equation}
\int_{Q_1^+} -|u^+|^2 \eta \partial_t \eta + |\nabla u^+|^2 \eta^2 + 2\eta u^+ \nabla u^+ \cdot \nabla \eta - f(u^+) u^+ \eta^2 = 0.
\end{equation}
On the other hand, testing (1.1) with $u^+ \eta^2$ we obtain
\begin{equation}
\int_{Q_1^+} -|u^+|^2 \eta \partial_t \eta + |\nabla u^+|^2 \eta^2 + 2\eta u^+ \nabla u^+ \cdot \nabla \eta - f(u^+) u^+ \eta^2 = 0.
\end{equation}
Letting $\varepsilon \to 0$ in (6.6), by the strong convergence of $u^+ \varepsilon$ and weak convergence of $\nabla u^+ \varepsilon$ in $L^2_{loc}(Q_1^+)$, we obtain
\begin{align*}
\lim_{\varepsilon \to 0} \int_{Q_1^+} |\nabla u^+ \varepsilon|^2 \eta^2 &= \int_{Q_1^+} |u^+|^2 \eta \partial_t \eta - 2\eta u^+ \nabla u^+ \cdot \nabla \eta + f(u^+) u^+ \eta^2 \\
&= \int_{Q_1^+} |\nabla u^+|^2 \eta^2.
\end{align*}
This gives the strong convergence of $\nabla u^+ \varepsilon$ in $L^2_{loc}(Q_1^+)$. \hfill \Box

The proof of Theorem 2.1 is complete.

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