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The interplay between topology and dynamics in complex networks is a fundamental but mostly unexplored problem. Here we study this phenomenon for a variable following extremal dynamics on a network which is in turn shaped by the variable itself. Each vertex is assigned a fitness, and the vertex with minimum fitness and its neighbours are updated as in the Bak–Sneppen model. On the other hand, the links are determined by the vertices’ variables as in the fitness network model. We show analytically and numerically that the system self–organizes to a nontrivial state which differs from what is obtained when the two processes are decoupled. A power–law decay of dynamical and topological quantities above a threshold emerges spontaneously, as well as a feedback between different dynamical regimes and the underlying network’s correlation and percolation properties.

The properties of dynamical processes defined on complex networks have been found to display a strong dependence on the topology [1, 2]. On the other hand, there is growing empirical evidence [3, 4] that many networks are in turn shaped by some variable associated to each vertex, an aspect captured by the ‘fitness’ or ‘hidden–variable’ model [5, 6]. Until now, these two facets of the same problem have been treated as separate, by considering on one hand dynamical processes on static networks [7, 8], and on the other hand network formation mechanisms driven by quenched variables [9, 10]. This may be perhaps justified for short time scales. However, in the long–term evolution it is crucial to understand the effects that these mechanisms have on each other, without ad hoc specifications of any fixed structure either in the topology or in the dynamical variables. Remarkably, the interplay of dynamics and topology can drive the network to a self–organized state that cannot be inferred by studying the two evolutionary processes as decoupled.

Here we explore explicitly the possibility that the network supports a dynamical process which in turn shapes its topology, with a continuous feedback between dynamics and structure. A few models where both dynamical and topological properties are continuously updated have been considered [10, 11, 12, 13]. In these cases, however, the rewiring of links is not driven by the dynamical variables. By contrast, our main interest here is the complete description of a self–organized process where the dynamical variable acts also as the ‘hidden variable’ shaping network topology explicitly, as in the fitness model. Due to the increased complexity of the problem, in this paper we choose the simplest possible dynamical rule for the hidden variable. We focus on the extremal dynamics defined in the Bak–Sneppen (BS) model [14], a traditional model of self–organized criticality (SOC) inspired by biological evolution. Consider a graph with N vertices, each regarded as a species having a fitness value \( x_i \) initially drawn from a uniform distribution between 0 and 1. At each timestep the species with lowest fitness and all its neighbours undergo a mutation, and their fitness values are drawn anew from the same uniform distribution. We couple this rule with the fitness model assumption that the network is formed by drawing a link between any two vertices \( i \) and \( j \) with fitness–dependent probability \( f(x_i, x_j) \). When the fitness \( x_i \) of a species \( i \) is updated to \( x_i' \), the links from \( i \) to all the other vertices \( j \) are drawn anew with probability \( f(x_i', x_j) \). The ensemble spanned by this process is equivalent to the one obtained if the whole network (including the links between vertices whose fitness is unchanged) is updated at each timestep. Therefore the results that we obtain here hold also if heterogeneous updating timescales are introduced across the network, making the model very general in this respect.

An important point to notice is that the main limitations of the two models disappear when they are coupled together. The BS model has been studied on regular lattices [14], random graphs [15], small–world [16] and scale–free [17] networks. However, a fundamental problem remains in all static cases. After a mutation, the new species always inherits exactly all the links of the previous one. This is hard to justify, since it is precisely the structure of ecological connections among species which is thought to be both the origin and the outcome of macroevolution [10]. Here the fitness–driven link updating overcomes this problem. Similarly, here the fitness distribution self–organizes spontaneously to a stationary probability density, thus removing the need to specify an ad hoc distribution as in the static fitness model. As we discuss below, a proper interpretation of the fitness also allows us to remove the remaining arbitrariness in the choice of \( f(x_i, x_j) \). However, in order to keep our approach as general as possible, we first study the model analytically for a generic form of \( f \), and focus on particular choices only later.

The master equation for the time–dependent fitness distribution \( \rho(x, t) \) can be written as \( \partial \rho(x, t)/\partial t = r^{in}(x, t) - r^{out}(x, t) \), where \( r^{in}(x, t) \) and \( r^{out}(x, t) \) represent the average fractions of vertices with fitness \( x \) which are added and removed at time \( t \) respectively. If we look for the long–term stationary distribution \( \rho(x) \) we must require that \( \partial \rho(x, t)/\partial t = 0 \), or \( r^{in}(x) = r^{out}(x) \). These quantities are conditional on the value of the minimum fitness \( m \equiv x_{\min} \). When the minimum value is \( m \) the number of updated values is \( 1 + k(m) = \phi(x_{\min}) \).
\[ 1 + N \int_0^1 \rho(x) f(x,m) dx, \] where \( k(m) \) is the degree of the minimum–fitness vertex, and each of them is uniformly drawn between 0 and 1. Therefore \( r^{in}[x|m] = [1 + k(m)/N] \). By contrast, \( r^{out}[x|m] \) depends on \( x \) according to the replacement rule. It is convenient to define \( \tau \) as the value below which one fitness value (the minimum one) falls in the steady state, that is \( N \int_0^\tau \rho(x) dx \equiv 1 \). As happens for the random–neighbour variant of the BS model \[12\], the annealed nature of the links is such that, since \( m \) is continuously replaced by uniformly distributed values, it is uniformly distributed between 0 and \( \tau \). Thus the distribution for \( m \) in the stationary state is \( q(m) = \Theta(\tau - m)/\tau \). This also means that

\[
\rho(x) = \begin{cases} 
q(x)/N = \frac{1}{\tau} & x < \tau \\
\rho_1(x) & x > \tau 
\end{cases}
\] (1)

with \( \rho_1(x) \) to be determined. Now, \( r^{out}[x|m] = 1/N \) when \( x = m \) since the minimum is always replaced, while \( r^{out}[x|m] = \rho(x)f(x,m) \) when \( x > m \) since the probability that a vertex with fitness \( x > m \) is updated is equal to the probability \( f(x,m) \) that it is connected to the vertex with fitness \( m \), no matter how old the link is. This is the key consideration that makes our model exactly solvable. Turning to the unconditional quantities \( r^{in}(x) = \int q(m)r^{in}[x|m] dm \), \( r^{out}(x) = \int q(m)r^{out}[x|m] dm \) and equating them yields

\[
\frac{1 + \langle k_{\text{min}} \rangle}{N} = \begin{cases} 
\frac{1}{\tau} & x < \tau \\
\rho_1(x)r^{-1} \int_0^\tau f(x,m) dm & x > \tau 
\end{cases}
\] (2)

where \( \langle k_{\text{min}} \rangle \equiv \int q(m)k(m) dm = \tau^{-1} \int_0^\tau k(m) dm \) is the average degree of the vertex with minimum fitness. Note that both terms appearing on the r.h.s. must be equal to the one on the l.h.s., which also means that they must be equal to each other. This allows to determine \( \rho_1(x) \) and put it back into eq. (1), which finally yields

\[
\rho(x) = \begin{cases} 
\frac{1}{\tau} & x < \tau \\
\frac{1}{N \int_0^\tau f(x,m) dm} & x > \tau 
\end{cases}
\] (3)

The value of \( \tau \) can be determined through the normalization condition \( \int_0^1 \rho(x) dx = 1 \) which reads

\[
\int_\tau^1 dx \int_0^\tau f(x,m) dm = N - 1
\] (4)

This characterizes the stationary state completely. Once \( \rho(x) \) is known, all the expected topological quantities can be determined as in the static fitness model \[13,14\]. Note that if \( \tau \) is nonzero the model preserves the step–like behaviour displayed on static networks \[14,15,16,17\], but here we find the novel feature that \( \rho(x) \) is in general not uniform for \( x > \tau \). Therefore the system spontaneously evolves from a random structure to a complex network with nontrivial dynamical and topological properties.

The above analytical solution holds for any form of \( f(x,y) \). We now consider the choice of this function.

First note that the null choice is \( f(x,y) = p \), the network being a random graph. It is nonetheless an instructive simple case, and we briefly discuss it. This choice is asymptotically equivalent to the random neighbour variant \[13\] of the BS model, the average degree of each vertex being \( d = pN \). Our analytical results read

\[
\rho(x) = \begin{cases} 
\frac{(\tau N)^{-1}}{(pN)^{-1}} & x < \tau \\
\frac{1}{1 + d^{-1}} & x > \tau
\end{cases}
\] (5)

and, depending on how \( p \) scales with \( N \), eq. (4) implies

\[
\tau = \frac{1}{1 + pN} \rightarrow \begin{cases} 
\frac{1}{(1 + d)^{-1}} & pN \rightarrow 0 \\
0 & pN \rightarrow \infty
\end{cases}
\] (6)

We note that these three dynamical regimes are tightly related to an underlying topological phase transition. As \( p \) decreases, the whole system splits up into a number of smaller subsets or clusters. Such process displays a critical behaviour near the threshold \( p_c \approx 1/N \) \[1,2\]. Below \( p_c \), each node is isolated or linked to a small number of peers. Above \( p_c \), a large giant component emerges including a number of nodes of order \( O(N) \), whose fraction tends to 1 as \( p \rightarrow 1 \). This explains the dynamical regimes in eq. (6). If \( pN \rightarrow \infty \) (dense regime), then \( \tau \rightarrow 0 \) and \( \rho(x) \) is uniform between 0 and 1 as in the initial state, since an infinite number \( \langle k_{\text{min}} \rangle \approx pN \) of farnesses is continuously updated as on a complete graph. If \( pN \rightarrow d \) with finite \( d > 1 \) (sparse regime), then \( \tau \) remains finite as \( N \rightarrow \infty \), and this is the case considered in ref. \[13\] that we recover correctly. Finally, if \( p \) falls faster than \( 1/N \) the graph is below the percolation threshold (subcritical regime); the updates cannot propagate and \( \tau \rightarrow 1 \), as for \( N \) isolated vertices. Therefore the dynamical transition is rooted in an underlying topological phase transition. This previously unrecognized property is fundamental and, as we show below, is also general.

A nontrivial form of \( f(x,y) \) must be chosen carefully. On static or fitness–independent networks \( x_i \) is usually interpreted as the fitness barrier against further mutation, and the links are interpreted as feeding relations \[14\]. However, once the topology depends on \( x \) these two interpretations are difficult to reconcile. The coupling we have introduced requires consistent interpretations of \( x \) and of the links. Also, the form of \( f(x,y) \) must be consistent with the feature that the updates of \( x \) propagate through the network determined by it. This very instructive aspect must characterize any model with coupled topology and dynamics, and reduces significantly the arbitrariness introduced in the static case. Here we suggest that the simplest self–consistent interpretation is the following. Since there is no external world in the model, the environment experienced by a species is simply the set of its ecological interactions. Now let \( x_i \) represent the fitness (rather than the barrier) of \( i \), and let a link between two species mean ‘being fit to coexist with each other’ (i.e. it represents an undirected, non–feeding interaction beneficial to both). The more a species is connected to
other species, the more it is fit to the environment. This picture is self-consistent provided that the larger $x$ and $y$, the larger $f(x,y)$. Following the results of refs.\[18,19\], the simplest unbiased \[19\] choice for such a function is

$$f(x,y) = \frac{zy}{1+zy} \quad (7)$$

where $z$ is a positive parameter controlling the number of links. With the above choice, $\rho(x)$ can be computed analytically through eq.\[4\]. However we write it in approximated form, exact when $\tau \to \infty$, in order to solve also more complicated integrals involving it. We assume $(f(x,m)) \approx f(x,\langle m \rangle)$, that is $\tau^{-1}\int_{0}^{\tau} f(x,m)dm \approx (zz\tau/2)/(1+z\tau\tau/2)$. Then eq.\[4\] yields

$$\rho(x) = \begin{cases} (\tau N)^{-1} & x < \tau \\ (\tau N)^{-1} + 2/(zN\tau^2) & x > \tau \end{cases} \quad (8)$$

and eq.\[4\] reads $1/\tau - 2z^{-1}\tau^{-2} \log \tau = N$, solved by

$$\tau = \sqrt{\frac{\phi(zN)}{zN}} \rightarrow \begin{cases} 1 & zN \to 0 \\ \sqrt{\phi(d)/d} & zN = d \\ 0 & zN \to \infty \end{cases} \quad (9)$$

for large $N$, where $\phi(x)$ is the ProductLog function defined as the solution of $\phi e^{\phi} = x$. As for random graphs, we find a marked transition as the scaling of $z$ changes from $N^{-1}$ to more rapidly decaying. This suggests an analogous underlying percolation transition. As we show below, this is indeed the case. We can therefore still refer to the subcritical, sparse and dense regimes.

The main panel of fig.\[1\] shows the cumulative density function (CDF) of the fitness $\rho_{\tau}(x) \equiv \int_{0}^{\tau} \rho(x')dx'$, while the inset shows a plot of $\tau(zN)$. The theoretical results are in excellent agreement with large numerical simulations. As predicted by eq.\[5\], $\rho(x)$ is the superposition of a uniform distribution and a power–law with exponent $-1$. For $z \ll 1$ we have $f(x,y) \approx zxy$ and $\rho(x) \propto x^{-1}$ for $x > \tau$. This purely power–law behaviour, that becomes exact in the sparse regime $z = d/N$ for $N \to \infty$, results in a logarithmic CDF looking like a straight line in log–linear axes. Note that, despite the value of the exponent, the presence of a nonzero lower threshold ensures that $\rho(x)$ is normalizable. This mechanism may provide a natural explanation for the onset of Pareto distributions with a finite minimum value in real systems. By contrast, for large $z$ the uniform part is nonvanishing and $\rho(x)$ deviates from the purely power–law behaviour. The decay of $\rho(x)$ for $x > \tau$ is a completely novel outcome of the extremal dynamics due to the feedback with the topology: now the fittest species at a given time is also the most likely to be connected to the least fit species and to mutate at the following timestep. Being more connected also means being more subject to changes. This enriches the coexistence patterns displayed on static networks.

We now check the conjectured percolation transition. For different system sizes, we find that the cluster size distribution $P(s)$ displays power–law tails when the control parameter $d \equiv zN$ approaches a critical value $d_c = 1.32 \pm 0.05$ (corresponding to $z_c = d_c/N$), suggesting the onset of a second–order percolation–like phase transition. As shown in figure\[2\] $P(s) \propto s^{-\gamma}$ with $\gamma = 2.45 \pm 0.05$ at the phase transition. Fig.\[3\] shows that the average fraction of nodes in the largest component remains
negligible for $d < d_c$, whereas it takes increasing finite values above $d_c$. As an additional check, following the method adopted in ref. [20], we have plotted the average size fraction of non-giant components, which diverges (in the infinite volume limit) when $P(s)$ decays algebraically as reported in the inset of fig. 8.

![Graph](image)

**FIG. 4:** Left: $k(x)$ ($N = 5000$; from right to left, $z = 0.01$, $z = 0.1$, $z = 1$, $z = 10$, $z = 100$, $z = 1000$). Right: $P_x(k)$ (same parameter values, inverse order from left to right). Solid lines: theoretical curves, points: simulation results.

Even if one of the most studied properties of the BS model on regular lattices is the statistics of avalanches characterizing the SOC behaviour [14], we do not consider it here. This is because, as shown in ref. [21], when considering long-range [13] instead of nearest-neighbour connections, it can lead to a wrong assessment of the SOC state, which is put into question by the absence of spatial correlations even if the avalanches are power-law distributed. Rather, we further characterize the topology at the stationary state by considering the degree distribution $P(k)$ and the degree correlations. We do this by studying, as a function of $x$, the average degree $k(x) = N \int f(x, y) \rho(y) dy$ of a vertex with fitness $x$:

$$k(x) = \frac{2}{z \tau^2} \ln \frac{1 + z x}{1 + z x^x} + \frac{z x - \ln(1 + z x)}{z \tau x} \quad (10)$$

The inverse function $x(k)$ can be used to obtain the cumulative degree distribution $P_x(k) = \int_k^{k(1)} P(k') dk' = \rho_x(x(k))$. As shown in fig. 4, $k(x)$ is linear for small $z$ since $f(x, z) \approx zxy$, while for large $z$ it saturates to the maximum value $k_{\text{max}} = k(1)$. This implies that in the sparse regime $P(k)$ mimics $\rho(x)$ and is characterized by the threshold value $k(\tau)$ and by a power-law decay $P(k) \propto k^{-1}$ above it (see fig. 4). Note that here $\tau$ remains finite even if $P(k) \propto k^{-\gamma}$ with $\gamma < 3$, in striking contrast with what obtained on static scale-free networks [17]. Differently, for large $z$ the saturation $k \rightarrow k_{\text{max}}$ translates into a cut-off that makes $P(k)$ deviate from the pure power-law behavior for $k > k(\tau)$. As shown in refs. [4], [18], this saturation determines disassortative degree correlations and a hierarchy of degree-dependent clustering coefficients as observed in many real-world networks (this is not shown here for brevity). As $N \rightarrow \infty$, these correlations vanish in the sparse regime ($\tau > 0$), while they survive in the dense regime ($\tau \rightarrow 0$). Structural correlations and a nonzero threshold $\tau$ are then mutually excluding in this model, which is another interesting effect of the feedback we have introduced.

Our results represent a first step into the unexplored domain of systems with generic self-organized coupling between dynamics and topology. A huge class of such processes needs to be studied in the future, to further understand the unexpected effects of this coupling.

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