Comments on \( s \)-rule violating configurations in field theory

William Cottrell, Akikazu Hashimoto, and Mohandas Pillai

Department of Physics, University of Wisconsin, Madison, WI 53706, USA

Abstract

We explicitly construct a configuration of \( \mathcal{N} = 4 \) supersymmetry Yang-Mills theory with gauge group \( U(N) \) on an interval on length \( L \) with a D5-like boundary condition on one end and an NS5-like boundary condition on the other. For \( N > 1 \), such a configuration violates the \( s \)-rule and is non-supersymmetric. We compute the energy relative to the BPS bound of these configurations and find that it is proportional to \( N(N^2 - 1)g_{YM}^2 L^{-3} \).
1 Introduction

In constructions involving branes in string theory, there is an important concept known as the s-rule. This concept was originally formulated in [1] and states that while an arbitrary number of D3-branes can generally end on NS5-branes and D5-branes, when a NS5-brane and a D5-branes are oriented so that they are linked, then not more than one D3-brane can stretch between the said NS5 and D5-branes while preserving supersymmetry. By linked we mean that the two brane can not exchange positions by going around each other. The setup considered in [1] consisted of an NS5-brane extended along the 012345 directions and a D5-brane extended along the 012789 directions, separated along the $x_6$ coordinate. A single D3-brane can stretch between the NS5 and the D5-brane. But when two or more D3-branes are forced to stretch between these 5-branes, it can not do so while preserving supersymmetry. The prototype configuration violating s-rule is illustrated in figure 1.a when $N > 1$.

![Figure 1: (a) The prototypical s-rule violating configuration for $N > 1$, (b) a configuration with two s-rule violating components, and (c) IR equivalent configuration via Hanany-Witten transition which clearly do not admit supersymmetric stationary state.](image)

The argument for why this configuration breaks supersymmetry presented originally in [1] is very simple. Consider a configuration illustrated in figure 1.b which consists of two s-rule violating components. Upon moving the D5-brane to the right in the $x_6$ direction, the configuration turns into the one illustrated in figure 1.c. But the configuration in figure 1.c is clearly non-supersymmetric.

Although the s-rule seemed mysterious at first, various reformulations that shortly followed made it much less so. For instance, one can apply a chain of dualities to map the suspended D3 branes to a fundamental string. In such a frame, the NS5 and the D5 branes are both mapped to D-branes with 8 relatively transverse coordinates. The massless strings stretched between D-branes oriented that way only consists of fermions. The s-rule then can be viewed as a manifestation of the Fermi exclusion principle [2,3]. Another manifestation of the s-rule can be inferred from the non-existence of supersymmetric brane embeddings when quantum numbers of the embeddings violate the s-rule. In these approaches, the dynamics
of Pauli exclusion principle is manifested classically, in the appropriate duality frame \[4–6\]. More recently, the classical manifestation of s-rule was illustrated in \[7\] in the zero slope defect field theory limit where the NS5 and the D5-branes on which the D3 brane ends are realized as BPS boundary conditions classified by Gaiotto and Witten \[8,9\]. In \[7\], it was argued that the Nahm pole on the D5-like boundary of \( \mathcal{N} = 4 \) SYM is incompatible with the NS5-like boundary on the other end while respecting supersymmetry.

The s-rule has interesting dynamical consequences. For instance, a \( \mathcal{N} \leq 3 \) Chern-Simons Yang-Mills theory with gauge group \( U(N) \) and level \( k \) can be engineered by suspending \( N \) D3-branes between an NS5-brane and a \((1,k)\) 5-brane\[1\] as was considered in \[5,10\]. Theories of this type are expected to exhibit dynamical supersymmetry breaking when \( N \) is taken to be large. For the \( \mathcal{N} = 1 \) minimal Chern-Simons Yang-Mills theory, Witten has computed the supersymmetric index and argued that supersymmetry is likely broken dynamically when \( N > 2k \) \[11\]. For the \( \mathcal{N} = 2 \) and \( \mathcal{N} = 3 \) theories arising from brane construction of \[5,10\], Ohta has computed the Witten index and argued that supersymmetry is likely broken for \( N > k \) \[12\]. This condition \( N > k \) is identical to the condition for the s-rule to be violated. This observation supports the expectation that supersymmetry is dynamically broken for \( \mathcal{N} = 2,3 \) Chern-Simons Yang-Mills theory of \[5,10\] in 2+1 dimensions.

Although these considerations offer considerable confidence that dynamical supersymmetry breaking is taking place, these systems have yet to offer intuitive understandings regarding the effective dynamics and the scale of symmetry breaking phenomena. One can envision taking a 't Hooft like large \( N \) limit keeping \( \lambda = N/k \) fixed, and one expects, on dimensional grounds, that the scale of dynamical supersymmetry breaking is parameterized as

\[
\epsilon_3 = \Lambda_{DSB}^3 = \alpha(\lambda - 1)^\beta(g_{YM}^2)^3
\]

but we do not have a reliable estimate of \( \alpha \) and \( \beta \), nor have we identified the effective order parameter characterizing the supersymmetry breaking vacuum. Some attempts to address these questions from the field theory point of view \[13\] as well as using gauge/gravity correspondence \[14–16\] has so far been inconclusive\[2\].

In the classical manifestation of s-rule discussed from the brane perspective in \[4–6\] and from the boundary field theory perspective \[7\], the absence of supersymmetric configurations violating the s-rule does not preclude the existence of a non-supersymmetric configuration solving the equation of motion and the boundary condition. The energy of the non-supersymmetric configuration is the vacuum energy associated with the dynamical

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1In our notion, \((p,q)\) 5-brane is a bound state of \( p \) NS5-branes and \( q \) D5-branes.

2These papers do offer some conjectures, which would be interesting to confirm in an independent field theory analysis.
supersymmetry breaking, and can be computed. This was left as an open exercise in [4]. We will first compute the profile and the energy of s-rule violating configuration for a $\mathcal{N} = 4$ SYM subjected to NS5 and D5 boundary conditions with gauge group $U(N)$. For $N > 2$, we expect the lowest energy configuration to be non-supersymmetric. We conclude with open issues and future directions.

2 Boundary Field Theory Analysis

In this section, we consider field configurations of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in 3+1 dimensions on $R^{1,2} \times I$, where $I$ is an interval, subjected to a D5-like boundary condition, in the terminology of [8,9] on one end, and an NS5-like boundary condition on the other end. In other words, this is the configuration illustrated in figure 1.a.

The D5-like boundary imposes a Nahm pole boundary condition. If the boundary condition on the other end were also D5-like, this problem reduces to the standard multi-monopole construction reviewed, for instance, in [17]. For $N = 2$, the solution takes on the standard form involving elliptic functions. When the interval is extended to take semi-infinite form, then one obtains the fuzzy funnels discussed in [18]. If the interval is of finite size with a D5-like boundary condition on one end and impose the $N$ NS5-like boundary condition on the other, then there is a BPS configuration which was worked out in section 3.6 of [7].

When the $U(N)$ on an interval is forced to respect D5-like boundary condition on one end but NS5-like boundary on the other, there is a tension between the Nahm pole blow up along the 789 coordinates along with the D5 is extended while the NS5 is localized and imposes a Dirichlet boundary condition. Because of this tension, solutions to the first order BPS equation with these boundary conditions are impossible unless $N = 1$.

One could, however, look for a non-BPS solution to the second order equation of motion. Consider the bosonic component of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in 3+1 dimensions viewed as a dimensional reduction of $\mathcal{N} = 1$ supersymmetric Yang-Mills theory in 9+1 dimensions.

The action of this theory can be written simply as

$$S = -\frac{1}{4g_{YM}^2} \int \text{Tr} F_{ij} F^{ij}$$  \hspace{1cm} (2.1)

where

$$F_{ij} = \partial_i A_j - \partial_j A_i + i[A_i, A_j]$$  \hspace{1cm} (2.2)

and

$$g_{YM}^2 = 2\pi g_s$$  \hspace{1cm} (2.3)
following the standard conventions in string theory (see e.g (212) and (275) of [19]). Note that it is somewhat unconventional to normalize the non-abelian gauge kinetic term in the trace form with a factor of 1/4. In order to relate to the standard convention used e.g. in [17] where the action is presented as

\[ S = -\frac{1}{2e^2} \int \text{Tr} F_{ij} F^{ij}, \]  

one must relate

\[ e^2 = 2g_{YM}^2. \]  

For static configurations, the energy density

\[ \epsilon_4 = -\mathcal{L} = \frac{1}{4g_{YM}^2} \text{Tr} F_{ij} F^{ij} \]  

The ansatz we consider is extremely simple, namely;

\[ A_{6+i} = f(z) T^i, \quad i = 1 \ldots 3 \]  

where \( z \) parameterizes the \( x_6 \) coordinate, and with other \( A_i \) set to zero, and \( T^i \) are the \( N \) dimensional generators of \( SU(2) \). For \( N = 2 \),

\[ T^i = \frac{1}{2} \sigma^i, \]  

and for general \( N \),

\[ \text{Tr} \sum_i (T^i)^2 = \frac{N(N^2 - 1)}{4}. \]  

The configuration we seek is a solution to the equations of motion with the boundary condition that at \( z = 0 \), the solution approaches the Nahm pole [8,9]

\[ f(z) = \frac{1}{z} \]  

with coefficient 1 and at \( z = L \) for some fixed \( L \),

\[ f(z) = 0 \]

to respect the Dirichlet boundary condition imposed by the NS5-brane.

Upon substituting the ansatz (2.7) to the Yang-Mills equation of motion, we simply obtain an equation of for \( f(z) \) which reads

\[ f''(z) - 2f^3(z) = 0 \]
This is essentially the equation of motion for $\phi^4$ theory dimensionally reduced to 0+1 dimension. It can also be viewed as the equation of motion for a non-linear spring.

This equation of motion can also be written in the form
\[
\frac{(f'(z)^2 - f(z)^4)^'}{2f'(z)} = 0.
\] (2.13)

which implies
\[
f'(z)^2 - f(z)^4 = c
\] (2.14)
is conserved, and this equation can further be integrated
\[- \frac{df}{\sqrt{c + f^4}} = dz
\] (2.15)
The two integration constants can be fixed by requiring $f = 0$ at $z = L$ and $f = \infty$ at $z = 0$, i.e.
\[L = \int_0^\infty \frac{df}{\sqrt{c + f^4}} = \frac{1}{4\sqrt{\pi}} c^{-1/4} \Gamma \left( \frac{1}{4} \right)^2
\] (2.16)
from which we read off that
\[c = \frac{\Gamma \left( \frac{1}{4} \right)^8}{256\pi^2 L^4}
\] (2.17)
It is clear, then, that near $z = 0$, the Nahm pole boundary condition (2.10) is satisfied.

The full solution consistent with these boundary conditions can be expressed in terms of a hyper-geometric function
\[z = \frac{2F_1 \left( \frac{1}{4}, \frac{1}{2}, \frac{5}{4}, -\frac{\Gamma \left( \frac{1}{4} \right)^8}{256\pi^2 L^4} f^4 \right)}{f}
\] (2.18)
and has the form illustrated in figure 2. Note that the solution asymptotes to $f = 1/z$ near $z = 0$ while approaching $f = 0$ at $z = L$. The only exception is the case of $N = 1$ for which $T^i = 0$ and therefore the Nahm pole is absent and $f = 0$ is the trivial, BPS, solution.

Having found the stationary field configuration associated with a non-BPS state, it would be interesting to compute its energy. Substituting the ansatz (2.7) into the energy density, we find
\[\epsilon_4 = \frac{1}{2g_{YM}^2} \text{Tr} \sum_i (T^i)^2 \left( f'^2 + f^4 \right) = \frac{N(N^2 - 1)}{8g_{YM}^2} \left( c + 2f^4 \right)
\] (2.19)
having used (2.14) and (2.9).

An interesting quantity is the effective three dimensional energy density obtained by integrating
\[\epsilon_3 = \frac{N(N^2 - 1)}{8g_{YM}^2} \int_0^L dz \left( c + 2f^4 \right) = \int_0^\infty df \frac{c + 2f^4}{\sqrt{c + f^4}}
\] (2.20)
Figure 2: The solution $f(z)$ which reflects the profile of a non-abelian funnel-like structure for the $s$-rule violating configuration of $U(N)$ gauge theory on an interval $0 < z < L$ with a D5-like boundary at $z = 0$ and an NS5-like boundary at $z = L$ is illustrated by the blue curve. The green curve is the BPS solution corresponding to the Nahm pole/fuzzy funnel $f(z) = 1/z$.

which diverges as $z$ approaches 0 where $f$ goes to infinity.

We should recall, however, that the quantity of interest is the energy above the BPS bound. The energy density can be written in the form

$$\epsilon_4 = \epsilon_{4\text{non-BPS}} + \epsilon_{4\text{BPS}}$$

where

$$\epsilon_{4\text{non-BPS}} = \frac{1}{2g_{YM}^2} \text{Tr} \left( \frac{dA^i}{dz} + \frac{i}{2} \epsilon_{ijk}[A_j, A_k] \right)^2$$

is the positive definite non-extremal contribution, and

$$\epsilon_{4\text{BPS}} = -\frac{i}{3g_{YM}^2} \frac{d}{dz} \text{Tr}(\epsilon_{ijk}A_iA_jA_k)$$

is the contribution which one expects from a BPS configuration. To extract the non-extremal component, we should add $\epsilon_{3\text{BPS}}$ for our ansatz, which takes the form

$$\epsilon_{3\text{BPS}} = \int_0^z dz \epsilon_{4\text{BPS}} = -\frac{N(N^2 - 1)}{8g_{YM}^2} \int_0^\infty df \, 2f^2,$$

to (2.20). Similar consideration of separating the BPS component from the non-extremal part was discussed in (9) of [20]. The non-extremal contribution to the energy inferred this way is

$$\epsilon_{3\text{non-BPS}} = \frac{N(N^2 - 1)}{8g_{YM}^2} \int_0^\infty df \left( \sqrt{c + 2f^4} - 2f^2 \right) = \# \frac{N(N^2 - 1)}{g_{YM}^2 L^3}$$
where $#$ is a numerical factor of order one which is easily calculable.

Equation (2.25) is the main result of this paper. It is the leading small $g_{YM4}$ behavior of the non-extremal contribution to the energy of $\mathcal{N} = 4 U(N)$ supersymmetric Yang-Mills theory on an interval of length $L$ with a D5-like boundary condition on one end and an NS5-like boundary condition on the other. Although the equations of motion and the boundary condition respects half of the supersymmetries of the $\mathcal{N} = 4$ theory, the stationary solution is not supersymmetric. As such, this is an example of spontaneously broken supersymmetry. The dependence on $g_{YM4}^2$ and $L^3$ may have been anticipated from dimensional grounds, but the dependence on $N$ is somewhat non-trivial. The exercise of computing the non-extremal energy for these non-supersymmetric stationary states was suggested, for instance, at the end of [4]. In this paper, we reported on a simple ansatz which allowed this exercise to be carried out in a closed form by working in a context where s-rule is manifested in a strictly field theoretic, zero slope limit of string theory [7].

3 Discussion

The system we considered in this note, namely $\mathcal{N} = 4$ supersymmetric $U(N)$ Yang-Mills theory on an interval with NS5-like and D5-like boundaries on each ends, is a simple example of a theory exhibiting dynamical supersymmetry breaking in that the equation of motion and boundary conditions are supersymmetric but the solution to the equations are not. Being defined on an interval, the theory is effectively 2+1 dimensions at long distances, but the dynamics of supersymmetry breaking relied on full 3+1 dimensional physics. A case in point is that the scale of supersymmetry breaking scales like $L^{-1}$ and diverges in the small $L$ limit. Also, this system is empty in the deep IR limit and does not have a clean interpretation as a 2+1 dimensional system in the first place.

There are several generalizations to our exercise that one can consider. One which immediately comes to mind is to generalize the NS5-like boundary condition corresponding to the single NS5, to that of a stack of $k$ NS5-branes. Then, the s-rule will permit, as was demonstrated in [7], up to $N = k$ D3-branes with a Nahm pole on a D5-like boundary on the other end. Once the boundary condition are generalized this way, it is likely that multiple, possibly a continuous family, of solutions exist for a given quantum number $N$. It would be interesting to enumerate these possibilities explicitly.

A setup that would be extremely interesting to understand is the energy of s-rule violating configuration of $N$ D3-branes stretched between a NS5-like boundary and a $(1, k)$ 5-brane-like boundary on the other, oriented in such a way as to engineer $\mathcal{N} = 2$ or $\mathcal{N} = 3 U(N)$ Chern-Simons Yang-Mills theory in 2+1 dimensions with level $k$ [5, 10]. Unfortunately,
for this setup, it appears that one must analyze the quantum effects to demonstrate the spontaneous breaking of supersymmetry.

The dynamics of spontaneous supersymmetry breaking will manifest itself in the S-dual system consisting of a D5-like boundary on one end and a \((k, 1)\)-like boundary on the other. Unfortunately, as discussed in section 8.3 of [9], the \((k, 1)\)-like boundary appears to be somewhat subtle, and have neither a concrete understanding of the BPS configuration for \(N \leq k\) nor of the non-BPS configurations with \(N > k\). Even if we did manage to understand the manifestation of dynamical supersymmetry breaking classically in this duality frame, it will not be quantitatively reliable in the limit where the 2+1 dimensional Yang-Mills coupling is taken to be much smaller than the scale of the interval \(g_{YM}^2 \ll L^{-1}\), so some other approach would be required to address the problem of computing \(\alpha\) and \(\beta\) in (1.1).

Another possible extension of our work is to study the nonsupersymmetric \(s\)-rule violating brane embeddings which was posed in the conclusion of [4]. The BPS embeddings in various manifestations of the configurations respecting the \(s\)-rule have been presented in the literature [4,6]. Unfortunately, the embedding appears to be rather complicated even for the first order BPS equations, and it is not immediately clear how one can extend this exercise to solve the full second order equation of motion. We hope to present better understating of these issues in the near future [21].

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