Kei modules and unoriented link invariants

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Abstract

We define invariants of unoriented knots and links by enhancing the integral kei counting invariant \( \Phi_X(K) \) for a finite kei \( X \) using representations of the kei algebra, \( Z_K[X] \), a quotient of the quandle algebra \( Z[X] \) defined by Andruskiewitsch and Graña. We give an example that demonstrates that the enhanced invariant is stronger than the unenhanced kei counting invariant. As an application, we use a quandle module over the Takasaki kei on \( Z_3 \) which is not a \( Z_K[X] \)-module to detect the non-invertibility of a virtual knot.

Keywords: Kei algebra, kei modules, involutory quandles, enhancements of counting invariants

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1 Introduction

In [9], Mituhisa Takasaki introduced an algebraic structure known as kei (or \( \equiv \) in the original kanji). In [6] this same structure was reintroduced under the name involutory quandle, a special case of a more general algebraic structure related to oriented knots known as quandles. These algebraic structures can be understood as arising from the unoriented and oriented Reidemeister moves respectively via a certain labeling scheme, encoding knot structures in algebra.

In [1], for every finite quandle \( X \) an associative algebra \( Z[X] \) was defined with generators representing coefficients of “beads” indexed by quandle labelings of arcs, with relations defined from the Reidemeister moves. Representations of \( Z[X] \), known as quandle modules, were used to define new invariants of oriented knots and links in [3]. In [7] a modification of \( Z[X] \) for finite racks (a generalization of quandles to the case of blackboard-framed isotopy) was used to define invariants of framed and unframed oriented knots and links.

In this paper we define a modification of the quandle algebra we call the kei algebra \( Z_K[X] \) and use it to extend the invariants defined in [7] to unoriented knots and links. The paper is organized as follows. In section 2 we review the basics of kei and the kei counting invariant. In section 3 we define the kei algebra and kei modules. In section 4 we define the kei module enhanced counting invariant. As an application, we use a module over \( Z[X] \) for a kei \( X \) which is not a \( Z_K[X] \)-module to detect the non-invertibility of a virtual knot. In section 5 we collect a few questions for future research.

2 Kei

Kei or involutory quandles were introduced by Mituhisa Takasaki in 1945 [9] and later reintroduced independently by David Joyce and S.V. Matveev in the early 1980s [6, 8].

Definition 1 A kei or involutory quandle is a set \( X \) with a binary operation \( \searrow \) satisfying for all \( x, y, z \in X \)

(i) \( x \searrow x = x \),

(ii) \( (x \searrow y) \searrow y = x \), and

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(iii) \((x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)\).

**Example 1** Let \(X\) be any abelian group regarded as a \(\mathbb{Z}\)-module. Then \(X\) is a kei under the operation
\[ x \triangleright y = 2y - x. \]

Such a kei is known as a *Takasaki kei*. If \(X \cong \mathbb{Z}_n\) then \(X\) is often denoted as \(R_n\) in the knot theory literature, known as the *dihedral quandle* on \(n\) elements. \(R_n\) can also be understood as the set of reflections of a regular \(n\)-gon.

**Example 2** Let \(X\) be any module over \(\mathbb{Z}[t]/(t^2 - 1)\). Then \(X\) is a kei known as an *Alexander kei* under the operation
\[ x \triangleright y = tx + (1 - t)y. \]

A Takasaki kei is an Alexander kei with \(t = -1\).

**Example 3** Let \(L\) be an unoriented link diagram and let \(A = \{a_1, \ldots, a_n\}\) be a set of generators corresponding bijectively with the set of arcs of \(L\). The *Fundamental Kei* of \(L\), \(FK(L)\), is defined in the following way. First, let \(W(L)\) be the set of *kei words* in \(A\), defined recursively by the rules

1. \(a \in A \Rightarrow a \in W(L)\) and
2. \(x, y \in W(L) \Rightarrow x \triangleright y \in W(L)\).

Then the *free kei* on \(A\) is the set of equivalence classes of kei words in \(A\) under the equivalence relation generated by relations of the forms

1. \(x \triangleright x \sim x\),
2. \((x \triangleright y) \triangleright y \sim x\), and
3. \((x \triangleright y) \triangleright z \sim (x \triangleright z) \triangleright (y \triangleright z)\)

for all \(x, y, z \in W(L)\). The free kei is a kei under the operation \([x] \triangleright [y] = [x \triangleright y]\). Now, at each crossing in \(L\), we have a *crossing relation* given by \(z = x \triangleright y\) where \(y\) is the overcrossing arc and \(x\) and \(z\) are the undercrossing arcs. That is, we have

\[
\begin{array}{c}
  \rule{2cm}{.5mm} \\
  x \triangleright y
\end{array}
\]

Then the *fundamental kei* of \(L\), \(FK(L)\), is the set of equivalence classes of free kei elements modulo the crossing relations of \(L\), or equivalently \(FK(L)\) is the set of equivalence classes of kei words in \(A\) modulo the equivalence relation determined by the crossing relations together with the free kei relations.

It is convenient to describe a finite kei \(X = \{x_1, \ldots, x_n\}\) with a matrix encoding the operation table of \(X\), i.e. a matrix \(M_X\) whose \((i, j)\) entry is \(k\) where \(x_k = x_i \triangleright x_j\). For example, the Takasaki kei on \(\mathbb{Z}_3\) has matrix
\[
M_X = \begin{bmatrix}
  1 & 3 & 2 \\
  3 & 2 & 1 \\
  2 & 1 & 3
\end{bmatrix}
\]

where we set \(x_1 = 0\), \(x_2 = 1\) and \(x_3 = 2\).

As with groups and other algebraic structures, we have the following standard notions:
Definition 2  Let $X$ and $Y$ be kei.

- A map $f : X \to Y$ is a kei homomorphism if for all $x, x' \in X$ we have $f(x \triangleright x') = f(x) \triangleright f(x')$;
- A subset $Y \subset X$ which is itself a kei under the kei operation $\triangleright$ of $X$ is a subkei of $X$. It is easy to check that $Y \subset X$ is a subkei if and only if $Y$ is closed under $\triangleright$.

For defining invariants of unoriented links, we have the following well-known result:

Theorem 1  If $L$ and $L'$ are ambient isotopic unoriented links, then there is an isomorphism of kei $\phi : FK(L) \to FK(L')$. For any finite kei $X$, the sets of homomorphisms $\text{Hom}(FK(L), X)$ and $\text{Hom}(FK(L'), X)$ are finite and there is an induced bijection $\phi^* : \text{Hom}(FK(L), X) \to \text{Hom}(FK(L'), X)$. In particular, the cardinality $\Phi^Z_X(L) = |\text{Hom}(FK(L), X)|$ is a non-negative integer-valued invariant of unoriented links known as the integral kei counting invariant.

A kei homomorphism $f : FK(L) \to X$ can be represented as a labeling of the arcs of $L$ with elements of $X$ satisfying the crossing relations at every crossing – such a labeling defines a unique homomorphism, and every $f \in \text{Hom}(FK(L), X)$ can be so represented.

Example 4  We can use the kei counting invariant to see that the trefoil knot $3_1$ is nontrivially knotted. Let $X$ be the Takasaki kei on $\mathbb{Z}_3$; we have $x \triangleright y = 2y - x = 2y + 2x$. The crossing relations in $3_1$ give us the system of linear equations:

\[
\begin{align*}
    z &= 2x + 2y \\
    y &= 2z + 2x \\
    x &= 2x + 2y
\end{align*}
\]

and the solution space is two-dimensional, giving us a total of $\Phi^Z_X(3_1) = 9$ solutions. Since $\Phi^Z_X(\text{Unknot}) = 3$, the integral kei counting invariant detects the knottedness of the trefoil.

Remark 1  Replacing the second kei axiom with the alternative axiom

(ii') There exists a second operation $\triangleright^{-1}$ satisfying $(x \triangleright y) \triangleright^{-1} y = x = (x \triangleright^{-1} y) \triangleright y$ for all $x, y \in X$

yields an algebraic object known as a quandle, which is the oriented analog of kei. Labeling oriented links according to the signed crossing conditions

\[
\begin{align*}
    x \triangleright y & \quad \quad \quad x \triangleright^{-1} y \\
    y & \quad \quad \quad x
\end{align*}
\]

defines homomorphisms from the fundamental quandle of the link $L$ into $X$; the integral quandle counting invariant $\Phi^Z_X(L)$ is then an invariant of oriented links.

3  Kei algebras and modules

Let $X$ be a finite kei. We would like to define an associative algebra on $X$ generated by “beads” such that secondary labelings of $X$-labeled link diagrams by beads are preserved by Reidemeister moves. Specifically,
at a crossing in a link diagram with arcs labeled $x, y$ and $x \triangledown y$, we define the following relationship between the beads $a, b$ and $c$:

\[ c = t_{x,y}a + s_{x,y}b. \]

The *kei algebra* of $X$, $\mathbb{Z}_K[X]$, will be the quotient of the polynomial algebra $\mathbb{Z}[t_{x,y}, s_{x,y}]$ by the ideal $I$ required to obtain invariance under unoriented Reidemeister moves.

First, we note that the bead relationship above also requires that $a = t_{x\triangledown y,y}c + s_{x\triangledown y,y}b$; together these imply

\[ a = t_{x,y}t_{x\triangledown y,y}a + (t_{x,y}s_{x\triangledown y,y} + s_{x,y})b, \]

which yields

\[ t_{x,y}t_{x\triangledown y,y} = 1 \quad \text{and} \quad t_{x,y}s_{x\triangledown y,y} + s_{x,y} = 0. \]

(1)

From the Reidemeister I move, we must have $t_{x,x} + s_{x,x} = 1$:

\[ a = t_{x,x}a + s_{x,x}a \]

(2)

The Reidemeister II move yields conditions equivalent to equation (1):

\[ c = t_{x,y}a + s_{x,y}b \]
\[ a = t_{x\triangledown y,y}c + s_{x\triangledown y,y}b \]
\[ \Rightarrow a = t_{x\triangledown y,y}t_{x,y}a + (t_{x\triangledown y,y}s_{x,y} + s_{x\triangledown y,y})b \]
The Reidemeister III move yields the defining equations for the original rack algebra $\mathbb{Z}[X]$ from \[1\]:

\[
\begin{align*}
t_{x\triangleright y,z}t_{x,y} & = t_{x\triangleright z,y\triangleright z}t_{x,z}, & t_{x\triangleright y,z}s_{x,y} & = s_{x\triangleright z,y\triangleright z}t_{x,z}, & & \text{and} & s_{x\triangleright y,z} & = s_{x\triangleright z,y\triangleright z}s_{y,z} - t_{x\triangleright z,y\triangleright z}s_{x,z}.
\end{align*}
\]

We can now define the kei algebra of a finite kei $X$.

**Definition 3** Let $X$ be a finite kei. The *kei algebra* $\mathbb{Z}[K][X]$ of $X$ is the quotient of the polynomial algebra $\mathbb{Z}[t_{x,y}, s_{x,y}]$ for all $x, y \in X$ by the ideal $I$ generated by all elements of the forms

- $t_{x,y}s_{x\triangleright y,y} - s_{x,y}$,
- $t_{x,y}t_{x\triangleright y,y} - 1$,
- $t_{x,x} + s_{x,x} - 1$,
- $t_{x\triangleright y,z}t_{x,y} - t_{x\triangleright z,y\triangleright z}t_{x,z}$,
- $t_{x\triangleright y,z}s_{x,y} - s_{x\triangleright z,y\triangleright z}t_{y,z}$, and
- $s_{x\triangleright y,z} - s_{x\triangleright z,y\triangleright z}s_{y,z} - t_{x\triangleright z,y\triangleright z}s_{x,z}$

for all $x, y, z \in X$. A $\mathbb{Z}[K][X]$-module or just an $X$-module is a representation of $\mathbb{Z}[K][X]$, i.e. an abelian group $A$ with a family of automorphisms $t_{x,y} : A \to A$ and endomorphisms $s_{x,y} : A \to A$ satisfying the conditions (1), (2) and (3) above.

**Example 5** Let $X$ be a kei. Any ring $R$ becomes a $\mathbb{Z}[K][X]$-module by choosing invertible elements $t_{x,y}$ and elements $s_{x,y}$ for $x, y \in X$ satisfying the conditions (1), (2) and (3). In particular, if $X = \{x_1, x_2, \ldots, x_n\}$ is a finite kei, we can specify a $\mathbb{Z}[K][X]$-module structure on $R$ with a $n \times 2n$ block matrix $M_R = [T|S]$ where $T(i,j) = t_{x_i,x_j}$ and $S(i,j) = s_{x_i,x_j}$.

**Remark 2** The *quandle algebra* defined in \[1\] is the quotient of the polynomial algebra $\mathbb{Z}[t_{x,y}^{\pm 1}, s_{x,y}]$ by the ideal generated by the relations coming from the Reidemeister I and III moves, i.e.,

- $t_{x,x} + s_{x,x} - 1$
- $t_{x\triangleright y,z}t_{x,y} - t_{x\triangleright z,y\triangleright z}t_{x,z}$
- $t_{x\triangleright y,z}s_{x,y} - s_{x\triangleright z,y\triangleright z}t_{y,z}$
- $s_{x\triangleright y,z} - s_{x\triangleright z,y\triangleright z}s_{y,z} - t_{x\triangleright z,y\triangleright z}s_{x,z}$. 

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with the type II move condition handled by the bead labeling rule below.

\[
x \triangleright y = 3, \quad x \triangleright^{-1} y = 2
\]

\[c = t_{x,y}a + s_{x,y}b\]

The kei algebra \( Z_K[X] \) is a quotient of the quandle algebra by the additional relations

\[t_{x,y}s_{x\triangleright y,y} + s_{x,y} \quad \text{and} \quad 1 - t_{x,y}t_{x\triangleright y,y}.\]

**Example 6** For a specific instance of the type of kei module defined in example 5, let \( X \) be the 3-element Takasaki kei with kei matrix

\[
M_X = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}
\]

and let \( R = \mathbb{Z}_5 \). Our **python** computations indicate that there are 48 \( Z_K[X] \)-module structures on \( R \), including for instance

\[
M_R = \begin{bmatrix} 4 & 1 & 3 & 2 & 4 & 1 \\ 3 & 4 & 2 & 3 & 2 & 3 \\ 2 & 1 & 4 & 4 & 1 & 2 \end{bmatrix}.
\]

**Remark 3** For a given kei \( X \), the set of \( Z_K[X] \)-modules over a given ring \( R \) is a subset of the set of \( \mathbb{Z}[X] \)-modules, and can be a proper subset depending on \( R \), since a \( \mathbb{Z}[X] \)-module satisfies the conditions in equation (1) and (3) but not necessarily those of equation (2). For instance, our **python** computations reveal a total of 32 \( \mathbb{Z}[X] \)-modules on the kei \( X \) and ring \( R \) in example 6 which are not \( Z_K[X] \)-modules, including for instance

\[
M_R = \begin{bmatrix} 2 & 1 & 2 & 4 & 2 & 3 \\ 1 & 2 & 2 & 2 & 4 & 3 \\ 4 & 4 & 2 & 4 & 4 & 4 \end{bmatrix}.
\]

The invariants defined in the next section associated with such modules are invariants of oriented links but not invariants of unoriented links.

**Example 7** Another important example of a \( Z_K[X] \) module is the fundamental \( Z_K[X] \)-module of an \( X \)-labeled link. Let \( L \) be an unoriented link with a labeling \( f : FK(L) \to X \) by a kei \( X \). On each arc of \( L \), we place a bead; the set of crossing relations then determines a presentation for a \( Z_K[X] \)-module, denoted \( Z_f[X] \), which we can represent concretely with a coefficient matrix of the resulting homogeneous system of linear equations. For instance, let \( X \) be the kei with matrix

\[
M_X = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \\ 3 & 3 & 3 \end{bmatrix};
\]

then the (4,2)-torus link with the \( X \)-labeling below has fundamental \( Z_K[X] \)-module presented by the matrix \( M_{Z_f[X]} \):

\[
M_{Z_f[X]} = \begin{bmatrix} t_{13} & s_{13} & -1 & 0 \\ 0 & t_{32} & s_{32} & -1 \\ -1 & 0 & t_{23} & s_{23} \\ s_{31} & -1 & 0 & t_{31} \end{bmatrix}.
\]
4 Kei module enhancements of the counting invariant

We can now define invariants of unoriented knots and links using kei modules. The idea is to use the set of homomorphisms $g : Z_f[X] → R$ from the fundamental kei module of an $X$-labeled diagram $L$ to the kei module $R$ as a signature for each kei homomorphism $f : FK(L) → X$.

**Definition 4** Let $L$ be an unoriented knot or link, $X$ a finite kei and $R$ a finite $Z_K[X]$-module. The *kei module enhanced multiset* invariant of $L$ associated to $X$ and $R$ is the multiset of cardinalities of the sets of $Z_k[X]$-module homomorphisms, i.e.,

$$Φ_{X,R}^{K,M}(L) = \left\{ |\text{Hom}_{Z_K[X]}(Z_f[X], M)| : f ∈ \text{Hom}(FK(L), X) \right\}.$$  

Taking the generating function of this multiset gives us a polynomial-form invariant for easy comparison: the *kei module enhanced invariant* of $L$ with respect to $X$ and $M$ is

$$Φ_{X,R}^K(L) = \sum_{f ∈ \text{Hom}(FK(L), X)} u^{\text{Hom}_{Z_K[X]}(Z_f[X], M)}.$$  

By construction, we have the following:

**Theorem 2** If $L$ and $L'$ are ambient isotopic unoriented links, $X$ is a finite kei and $R$ is a $Z_K[X]$-module, then $Φ_{X,R}^{K,M}(L) = Φ_{X,R}^{K,M}(L')$ and $Φ_{X,R}^K(L) = Φ_{X,R}^K(L')$.

**Remark 4** If $R$ is not a finite ring, we can replace the infinite cardinality $|\text{Hom}_{Z_K[X]}(Z_f[X], M)|$ with the rank of the set $\text{Hom}_{Z_K[X]}(Z_f[X], M)$ as a $Z_K[X]$-module.

To compute $Φ_{X,R}^K$, for each kei labeling $f : FK(L) → X$ of $L$ by $X$, we first obtain the matrix for $Z_f[X]$, replace each $t_{x,y}$ and $s_{x,y}$ with its value in $R$, and solve the resulting system of equations to obtain the contributions to $Φ_{X,R}^K$ for $f$.

**Example 8** Let $L$ be the figure eight knot 41 and let $X$ and $R$ be the kei and kei module on $Z_5$ from example 6. The set of $X$-labelings of $L$ includes only constant labelings, i.e. every arc is labeled with a 1, 2 or 3. For example, the constant labeling with every arc labeled 1 yields the listed $Z_f[X]$-presentation matrix:

$$M_{Z_f[X]} = \begin{bmatrix} t_{11} & -1 & s_{11} & 0 \\ 0 & s_{11} & t_{11} & -1 \\ -1 & 0 & t_{11} & s_{11} \\ s_{11} & t_{11} & 0 & -1 \end{bmatrix}$$

Replacing the $t_{xy}$ and $s_{xy}$ with their values in $R$ and row-reducing over $Z_5$, we obtain

$$\begin{bmatrix} 4 & 4 & 2 & 0 \\ 0 & 2 & 4 & 4 \\ 4 & 0 & 4 & 2 \\ 2 & 4 & 0 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and this $X$-labeling contributes a $u^{25}$ to the invariant $Φ_{X,R}^K(41)$. Summing these contributions over the complete set of $X$-labelings gives us $Φ_{X,R}^K(41) = 3u^{25}$. Comparing this to the unknot, which has $Φ_{X,R}^K(\text{Unknot}) = 3u^5$, we see that $Φ_{X,R}^K$ distinguishes the unoriented figure eight from the unoriented unknot despite the two having equal kei counting invariant values. In particular, since $Φ_{X}^K(k)$ is obtained from $Φ_{X,R}^K$ by evaluating at $u = 1$, $Φ_{X,R}^K$ is a strictly stronger invariant than $Φ_{X}^K(k)$.
Example 9 Our python computations yield the listed values for $\Phi^K_{X,M}$ with $X$ the 3-element Takasaki kei and the randomly selected $Z_K[X]$-module over $Z_7$ below for the prime knots with up to eight crossings and prime links with up to seven crossings as listed in the knot atlas [2]:

$$M_R = \begin{bmatrix} 6 & 3 & 5 & 2 & 5 & 3 \\ 5 & 6 & 3 & 3 & 2 & 5 \\ 3 & 5 & 6 & 5 & 3 & 2 \end{bmatrix}$$

| $\Phi^K_{X,M}(L)$ | $L$ |
|--------------------|-----|
| $3u^r$             | unknot, 41, 51, 62, 63, 72, 73, 75, 76, 81, 82, 83, 84, 85, 86, 87, 88, 89, 812, 813, 814, 817, $L2a1, La1a1, L6a2, L6a4, L681, L7a2, L7a3, L7a4, L7a7, L7n1, L7n2 |
| $3u^7 + 6u^{49}$   | 31, 61, 74, 810, 811, 815, 819, 820, 821, $L6a1, L6a3, L6a5, L7a1, L7a5$ |
| $3u^7 + 24u^{49}$  | 52, 71, 816, $L7a6$ |
| $9a^{49}$          | 77, 85 |

Remark 5 As with most enhancements of quandle-related counting invariants, $\Phi^K_{X,M}$ is well-defined for unoriented virtual links as well as classical links.

In our final example, we use a quandle module which is not a kei module to detect the non-invertibility of a virtual knot.

Example 10 Let $X$ be the kei from example [6] and $M$ the quandle module from remark [3] Since $M$ is not a kei module, $\Phi^M_X$ is an invariant of oriented knots and links, but not unoriented knots and links. Thus, we can potentially use $\Phi^M_X$ to compare the two orientations of a non-invertible knot. In particular, consider the virtual knot numbered 4.97 in the Knot Atlas [2]; it is the closure of the virtual braid below. Let us denote 4.97 with the upward orientation by 4.97+ and 4.97 with the downward orientation as 4.97−. The only labelings of 4.97 by $X$ are constant labelings, of which there are three for both orientations, the unenhanced integral kei counting invariant $\Phi^K_X (4.97+)$ = 3 = $\Phi^K_X (4.97−)$, and $\Phi^M_X$ does not distinguish $4.97+ \neq 4.97−$. However, the constant labeling with every arc labeled with a 1 ∈ $X$ yields the listed fundamental kei module presentation matrices. Replacing $t_{1,1}$ and $s_{1,1}$ with their values from $M$ yields the listed matrices, which we row-reduce over $Z_5$ to obtain the cardinalities of the solution spaces which form the signature of the constant labeling by the element 1 ∈ $X$.

$$M_{2[4]}(4.97−) : \begin{bmatrix} s_{11} & -1 & t_{1,1} & 0 \\ s_{11} & 0 & -1 & t_{1,1} \\ t_{1,1} & -1 & 0 & s_{11} \\ -1 & 0 & s_{11} & t_{1,1} \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 4 & 2 & 0 \\ 4 & 0 & 4 & 2 \\ 2 & 4 & 0 & 4 \\ 4 & 0 & 4 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{2[4]}(4.97+) : \begin{bmatrix} s_{1,1} & t_{1,1} & -1 & 0 \\ s_{1,1} & 0 & t_{1,1} & -1 \\ -1 & t_{1,1} & 0 & s_{1,1} \\ t_{1,1} & 0 & s_{1,1} & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 2 & 4 & 0 \\ 4 & 0 & 2 & 4 \\ 4 & 2 & 0 & 4 \\ 2 & 0 & 4 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since $t_{1,1} = t_{2,2} = t_{3,3} = 2$ and $s_{1,1} = s_{2,2} = s_{3,3} = 4$, we get the same signatures for all three labelings for each knot, respectively $u^{25}$ and $u^9$, and thus we have

$$\Phi^M_X(4.97+) = 3u^{25} \neq 3u^9 = \Phi^M_X(4.97−)$$

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and for non-kei module quandle modules $M$ over a finite kei, $X$, the quandle module enhanced counting invariant $\Phi_X^M$ is capable of detecting invertibility of virtual (and hence classical) knots.

5 Questions

In this section we collect a few open questions for future research.

In our computations we have only considered the simplest type of $\mathbb{Z}_K[X]$ modules, namely $\mathbb{Z}_k[X]$-module structures on $\mathbb{Z}_n$ with the action of $t_{x,y}$ and $s_{x,y}$ given by multiplication by fixed elements of $\mathbb{Z}_K[X]$. Expanding to other abelian groups and other automorphisms $t_{x,y} : X \to X$ and endomorphisms $s_{x,y} : X \to X$ should give interesting results. We are particularly interested in the case of non-commuting $t_{x,y}$ and $s_{x,y}$ values.

We have generalized the rack module bead counting invariant from [7], but several other oriented link invariants using the quandle algebra were defined in [3]; these invariants should have generalizations to the unoriented case using the kei algebra.

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