Normal subgroups of the group of column-finite infinite matrices

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Abstract

The classical result due to Jordan, Burnside, Dickson, says that every normal subgroup of $GL(n, K)$ ($K$ - a field, $n \geq 3$) which is not contained in the center, contains $SL(n, K)$. A. Rosenberg gave description of normal subgroups of $GL(V)$, where $V$ is a vector space of any infinite cardinality dimension. However, in countable case his result is incomplete. We fill this gap giving description of the lattice of normal subgroups of the group of infinite column-finite matrices indexed by positive integers over any field.

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1 Introduction

Description of normal subgroups is a fundamental problem in group theory. The classical result due to Jordan, Burnside, Dickson, says that every normal subgroup of $GL(n, K)$ ($K$ - a field, $n \geq 3$) which is not contained in the center, contains $SL(n, K)$. We extend this result, giving description of normal subgroups in the group $GL_{cf}(\mathbb{N}, K)$ of invertible column-finite infinite matrices over $K$ indexed by $\mathbb{N}$. Let $GL(n, \mathbb{N}, K)$ denote the subgroup

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of $GL_{cf}(\mathbb{N}, K)$ consisting of all matrices which differ from identity matrix $E$ only in first $n$ rows and by $SL(n, \mathbb{N}, K)$ the subgroup of $GL(n, \mathbb{N}, K)$ containing matrices which in left upper $n \times n$ corner have submatrix with determinant 1. We have natural embeddings:

$$GL(n, \mathbb{N}, K) \subseteq GL(n + 1, \mathbb{N}, K) \quad \text{and} \quad SL(n, \mathbb{N}, K) \subseteq SL(n + 1, \mathbb{N}, K)$$

and so we obtain two new subgroups of $GL_{cf}(\mathbb{N}, K)$:

$$GL_{fr}(\mathbb{N}, K) = \cup_{n>0} GL(n, \mathbb{N}, K) \quad \text{and} \quad SL_{fr}(\mathbb{N}, K) = \cup_{n>0} SL(n, \mathbb{N}, K).$$

By $D_{sc}(\mathbb{N}, K)$ we denote the subgroup of all scalar matrices in $GL_{cf}(\mathbb{N}, K)$, by $\langle H, H_1 \rangle$ we denote the subgroup generated by $H$ and $H_1$ and by $H \times H_1$ their internal direct product.

We prove the following results:

**Theorem 1.1.** The subgroups

$$D_{sc}(\mathbb{N}, K),$$

$$SL_{fr}(\mathbb{N}, K),$$

$$GL_{fr}(\mathbb{N}, K),$$

$$D_{sc}(\mathbb{N}, K) \times SL_{fr}(\mathbb{N}, K),$$

$$D_{sc}(\mathbb{N}, K) \times GL_{fr}(\mathbb{N}, K)$$

are normal subgroups of $GL_{cf}(\mathbb{N}, K)$.

**Theorem 1.2.** The group $SL_{fr}(\mathbb{N}, K)$ and the factor group $GL_{cf}(\mathbb{N}, K)/(D_{sc}(\mathbb{N}, K) \times GL_{fr}(\mathbb{N}, K))$ are simple. The group $D_{sc}(\mathbb{N}, K)$ and the factor groups

$$(D_{sc}(\mathbb{N}, K) \times SL_{fr}(\mathbb{N}, K))/SL_{fr}(\mathbb{N}, K),$$

$$(D_{sc}(\mathbb{N}, K) \times GL_{fr}(\mathbb{N}, K))/GL_{fr}(\mathbb{N}, K),$$

$$GL_{fr}(\mathbb{N}, K)/SL_{fr}(\mathbb{N}, K),$$

$$(D_{sc}(\mathbb{N}, K) \times GL_{fr}(\mathbb{N}, K))/(D_{sc}(\mathbb{N}, K) \times SL_{fr}(\mathbb{N}, K))$$

are isomorphic to $K^*.$

Since $D_{sc}(\mathbb{N}, K)$ is the center of $GL_{cf}(\mathbb{N}, K)$, every subgroup of $D_{sc}(\mathbb{N}, K)$ is normal in $GL_{cf}(\mathbb{N}, K)$. If $H_1, H_2$ are normal in $G$ and $H_2/H_1$ is abelian, then all subgroups $H$ of $G$ such that $H_1 \leq H \leq H_2$ are normal in $G$. This shows that both Theorems give the complete description of normal subgroups of $GL_{cf}(\mathbb{N}, K).$
The lattice of normal subgroups of $GL_{cf}(\mathbb{N}, K)$ “modulo the center” is shown in the figure below (we abbreviate notation for convenience, see the beginning of next section.) The thin line between subgroups $H_1$ and $H_2$ ($H_1 \leq H_2$) means that the factor group $H_2/H_1$ is simple, the thick line means that the factor group $H_2/H_1$ is isomorphic to $K^*$. 

Alex Rosenberg in [4] gave description of normal subgroups of $GL(V)$, where $V$ is a vector space of any infinite cardinality dimension $\aleph_\delta$. For any infinite cardinal $\aleph_\nu \leq \aleph_\delta$, by $L_\nu(V)$ we denote the set of all linear transformations with ranges of dimension $< \aleph_\nu$ and let $GL_\nu(V)$ be a subgroup of $GL(V)$ consisting of all elements of the form $id + A$, $A \in L_\nu(V)$. The main result (Theorem B) of [4] shows that if $N$ is a normal subgroup of $GL(V)$, then either $N = H \cdot id \times GL_\nu(V)$, where $0 < \nu \leq \delta$ and $H$ is a subgroup of $K^*$, or $N \leq K^* \cdot id \times GL_0(V)$. It is clear that in countable case this result is incomplete. In our terms, he proved only that $D_{sc}(\mathbb{N}, K) \times GL_{fr}(\mathbb{N}, K)$ is maximal normal subgroup of $GL_{cf}(\mathbb{N}, K)$ and thus $GL_{cf}(\mathbb{N}, K)/(D_{sc}(\mathbb{N}, K) \times GL_{fr}(\mathbb{N}, K))$ is simple.

Our results fill this gap giving the full description of the lattice of normal subgroups of the group of infinite column-finite matrices indexed by positive integers over any field. We note that similar description of ideals of the Lie algebra of infinite column-finite matrices over any field was obtained in [3].

2 Proofs of main results

Let $K$ be any field. We start with the following Lemma
Lemma 2.1. $D_{sc}(\mathbb{N}, K)$ is the center of $GL_{cf}(\mathbb{N}, K)$.

The proof is similar to that for finite dimensional matrices and we will omit it.

Lemma 2.2. The group $GL_{cf}(\mathbb{N}, K)$ is generated by the row and column-finite matrices and upper triangular matrices.

This lemma follows easily from considerations in chapter 2 of [4]. Another proof one can find in [6].

An infinite block diagonal matrix with finite blocks of sizes $n_1 \times n_1$, $n_2 \times n_2$, \ldots is called a string with the shape $(n_1, n_2, \ldots)$. Of course, the string with shape $(1, 1, \ldots)$ is a diagonal matrix.

The following Lemma is Theorem 3.3 of [5]. Its proof is contained in chapter 5 of [5]. P. Vermes in [6] gave another proof for $K = \mathbb{C}$, however it can be easily adopted to arbitrary field $K$ (see also [2]). By $GL_{rcf}(\mathbb{N}, K)$ we denote the subgroup of $GL_{cf}(\mathbb{N}, K)$ consisting of the row and column-finite matrices.

Lemma 2.3. The group $GL_{rcf}(\mathbb{N}, K)$ is generated by strings.

Proof of Theorem 1.1.

Now we prove normality of groups listed in Theorem 1. The normality of $D_{sc}(\mathbb{N}, K)$ follows from Lemma 1. Let $g \in GL_{fr}(\mathbb{N}, K)$ and $s$ be a string with shape $(n_1, n_2, \ldots)$. Then $g$ has a form

$$
\begin{pmatrix}
\hat{g} & * \\
0 & e
\end{pmatrix}
$$

where $\hat{g} \in GL(n, K)$ for some $n \in \mathbb{N}$ and $e$ is the infinite identity matrix. We choose minimal $t$ such that $m = \sum_{i=1}^{t} n_i \geq n$. We can extend block decomposition of $g$ to

$$
\begin{pmatrix}
\tilde{g} & * \\
0 & e
\end{pmatrix}
$$

where $\tilde{g} \in GL(m, K)$. Then $s^{-1}gs$ has a form

$$
\begin{pmatrix}
* & * \\
0 & e
\end{pmatrix}
$$
and belongs to $GL(m, \mathbb{N}, K)$. If $g \in SL_{fr}(\mathbb{N}, K)$, then similar arguments show that $s^{-1}gs \in SL_{fr}(\mathbb{N}, K)$. Now, if $u$ is any upper triangular matrix, then we can use on $u$ the same block structure as on $g$. Simple calculations show that $u^{-1}gu \in GL_{fr}(\mathbb{N}, K)$. The normality of $GL_{fr}(\mathbb{N}, K)$ (and $SL_{fr}(\mathbb{N}, K)$) follows from Lemma 2.2 and 2.3. The normality of other subgroups from Theorem 1.1 follows from remark after Theorem 1.2. \qed

Proof of Theorem 1.2.

Simplicity of $SL_{fr}(\mathbb{N}, K)$ was proved by Clowes and Hirsch in [1]. The fact that the factor group $GL_{cf}(\mathbb{N}, K)/(D_{sc}(\mathbb{N}, K) \times GL_{fr}(\mathbb{N}, K))$ is simple follows from Theorem B of [4].

Let $d(\alpha) = (\alpha - 1)e_{11} + e$, where $e$ has 1 in $(1,1)$ entry and zero elsewhere. For every matrix $g \in GL_{fr}(\mathbb{N}, K)$ we have a unique decomposition

$$g = d(\alpha) \cdot (d(\alpha^{-1}) \cdot g)$$

where $\alpha = \det(\hat{g})$ and $d(\alpha^{-1}) \cdot g \in SL_{fr}(\mathbb{N}, K)$. This shows that $GL_{fr}(\mathbb{N}, K)/SL_{fr}(\mathbb{N}, K)$ is isomorphic to $K^*$. Now other statements of Theorem 1.2 are obvious. \qed

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