The tree of quadratic transforms of a regular local ring of dimension two

William Heinzer\textsuperscript{a}, K. Alan Loper\textsuperscript{b}, Bruce Olberding\textsuperscript{c}

\textsuperscript{a}Department of Mathematics, Purdue University, West Lafayette, Indiana 47907-1395 U.S.A.
\textsuperscript{b}Department of Mathematics, Ohio State University – Newark, Newark, OH 43055
\textsuperscript{c}Department of Mathematical Sciences, New Mexico State University, Las Cruces, NM 88003-8001 U.S.A.

Abstract

Let $D$ be a 2-dimensional regular local ring and let $Q(D)$ denote the quadratic tree of 2-dimensional regular local overrings of $D$. We explore the topology of the tree $Q(D)$ and the family $R(D)$ of rings obtained as intersections of rings in $Q(D)$. If $A$ is a finite intersection of rings in $Q(D)$, then $A$ is Noetherian and the structure of $A$ is well understood. However, other rings in $R(D)$ need not be Noetherian. The two main goals of this paper are to examine topological properties of the quadratic tree $Q(D)$, and to examine the structure of rings in the set $R(D)$.

Keywords: regular local ring, quadratic transform, quadratic tree, Zariski topology, patch topology

2000 MSC: 13A15, 13C05, 13E05, 13H15

1. Introduction

Let $D$ be a 2-dimensional regular local ring. Among the overrings of $D$ inside the field of fractions of $D$, the rings that are 2-dimensional regular local rings form a partially ordered set $Q(D)$ with respect to inclusion. For rings $\alpha$ and $\beta$ in $Q(D)$ with $\alpha \subseteq \beta$, it is known from work of Abhyankar, that the regular local rings dominating $\alpha$ and dominated by $\beta$ form a finite linearly ordered chain. Thus $Q(D)$ is a tree with respect to inclusion.

The tree $Q(D)$ reflects ideal-theoretic properties of the complete ideals of $D$ that are primary for the maximal ideal $m_D$. Zariski’s theory of complete ideals implies the existence of a one-to-one correspondence between the elements in each of the following 3 sets:
1. The simple complete $m_D$-primary ideals.
2. The rings in the quadratic tree $Q(D)$.
3. The order valuation rings of the rings in $Q(D)$.

Motivation to examine properties of overrings of $D$ that are the intersection of elements in the quadratic tree $Q(D)$ arises from two sources:

1. The beautiful structure of the tree $Q(D)$.
2. Similarities between the intersections of elements in $Q(D)$ and the representation of a Krull domain as the intersection of its essential valuation rings.

Let $R(D)$ denote the family of rings obtained as intersections of rings in $Q(D)$.

**Remark 1.1.** The Noetherian rings in $R(D)$ are all Krull domains. Associated to a Krull domain $A$ is a unique set of DVRs, the set $E(A)$ of essential valuation rings of $A$; $E(A) = \{ A_p \}$, where $p$ varies over the height 1 prime ideals of $A$. Two useful properties related to $E(A)$ are:

1. $A = \bigcap \{ A_p \mid A_p \in E(A) \}$ and the intersection is irredundant.
2. The set $E(A)$ defines an essential representation of $A$.

Since $D$ is a Noetherian ring of Krull dimension 2, every Krull domain between $D$ and its quotient field is a Noetherian ring [9, Theorem 9].

A goal in this paper is to examine topological properties of the quadratic tree $Q(D)$, and the structure of rings in the set $R(D)$. In the paper [14] a description is given of the Noetherian rings in $R(D)$. Subsets $U$ of $Q(D)$ that are closed points of nonsingular projective models over $D$ are examined. The rings obtained are Noetherian rings that can be described in detail.

For example, it is shown in [14, Corollary 6.5] that if $n$ is a positive integer and $R$ is an irredundant intersection of $n$ elements in $Q(D)$, then $R$ is a Noetherian regular domain with precisely $n$ maximal ideals, each maximal ideal of $R$ is of height 2, and the localizations of $R$ at its maximal ideals are the $n$ elements in $Q(D)$ that intersect irredundantly to define $R$.

A focus in the current paper is to describe non-Noetherian rings in $R(D)$. Related to this we examine the topological structure of the quadratic tree $Q(D)$. In Sections 4 and 5 we investigate the topology of Noetherian subsets of $Q(D)$, find the patch limits points and prove that Noetherian subsets of $Q(D)$ are precisely the subsets that are bounded in the sense that they are the points of $Q(D)$ contained in finitely many dominating valuation overrings of $D$. Examples in Section 6 show that for Noetherian subsets $U$ of $Q(D)$, the structure of rings $R = O_U$ is more complicated than the situation with projective models.
Theorem 5.6 establishes that the subsets \( \mathcal{U} \) of \( Q(D) \) that are Noetherian subspaces of \( Q(D) \) are precisely the ones for which there exist finitely many valuation overrings \( V_i \) of \( D \) that dominate \( D \) and are such that each ring in \( \mathcal{U} \) is contained in one of the \( V_i \). The closed irreducible subspaces of \( Q(D) \) are all Noetherian, and are in one-to-one correspondence with the dominating valuation overrings of \( D \). By Lemma 5.5 an irreducible component of \( Q(D) \) is either:

1. the set of rings of \( Q(D) \) contained in a prime divisor of the second kind for \( D \), or
2. the set of rings of \( Q(D) \) contained in a minimal valuation overring \( V \) of \( D \), where \( V \) is not a subring of a prime divisor of the second kind on \( D \).

For \( \alpha \in Q(D) \), let \( P(\alpha) \) denote the set of points proximate to \( \alpha \). A finite union of sets of the form \( P(\alpha_i) \) is Noetherian, and so is the set of the closed points of a nonsingular projective model over \( D \). Also, an infinite sequence of iterated local quadratic transforms defines a Noetherian subspace of \( Q(D) \). In this paper we are moving beyond projective models and focusing on what can be said about intersections of rings in Noetherian subsets of \( Q(D) \) with special emphasis on sets of points proximate to finitely many elements of \( Q(D) \).

Theorem 6.15 describes an infinite subset \( \mathcal{U} \) of \( Q(D) \) such that \( \mathcal{U} \) defines an irredundant essential representation of \( B = \mathcal{O}_\mathcal{U} \), and \( B \) is an almost Krull domain\(^1\) that is not Noetherian. Corollary 4.14 implies that the constructed ring \( \mathcal{O}_\mathcal{U} \) is the intersection of an almost Dedekind domain\(^2\) and a principal ideal domain.

Let \( V \) be a minimal valuation overring of \( D \). Theorem 6.8 establishes the existence of a subset \( \mathcal{U} \) of \( Q(D) \) such that the ring \( \mathcal{O}_\mathcal{U} = C \) has the property that \( V \) is a localization of \( C \). If \( V \) is chosen not to be a DVR, then \( C \) is not an almost Krull domain.

Theorem 6.17 describes a non-Noetherian local domain in \( \mathcal{R}(D) \). This is the most intricate example constructed in Section 6.

2. Notation and Terminology

We mainly follow the notation used by Matsumura in [19]. Thus a local ring need not be Noetherian. An extension ring \( B \) of an integral domain \( A \) is said to be an overring of \( A \) if \( B \) is a subring of the field of fractions of \( A \).

---

\(^1\) An integral domain \( B \) is said to be an almost Krull domain if \( B_P \) is a Krull domain for each prime ideal \( P \) of \( B \).

\(^2\) An integral domain \( A \) is an almost Dedekind domain if \( A_P \) is a Dedekind domain for each maximal ideal \( P \) of \( A \).
For the definition of a quadratic transform, also called a local quadratic transform, we refer to [2, pp. 569-577], [32, p. 367] and [31, p. 263]. The powers of the maximal ideal of a regular local ring \( R \) define a rank one discrete valuation ring denoted \( \text{ord}_R \). If \( \dim R = d \), then the residue field of \( \text{ord}_R \) is a pure transcendental extension of the residue field of \( R \) of transcendence degree \( d - 1 \).

Complete ideals, also called integrally closed ideals, are defined and studied in the book of Swanson and Huneke [31]. Let \( R \) be a Noetherian integral domain, and let \( V \), with maximal ideal \( m_V \), be a valuation overring of \( R \). Following notation as in [31, Definition 9.3.1], \( V \) is said to be a divisorial valuation ring with respect to \( R \) if the transcendence degree of \( V/m_V \) over the field \( R_p/pR_p \) is \( \text{ht } p - 1 \). Every divisorial valuation ring with respect to \( R \) is Noetherian [31, Theorem 9.3.2]. Divisorial valuation rings are classically called prime divisors on \( R \) [32, p. 95]. \( V \) is a prime divisor of the first kind if \( p = m_V \cap R \) has height one. If \( \text{ht } p > 1 \), then \( V \) is said to be a prime divisor of the second kind.

**Notation and Remarks 2.1.** Let \( R \) be an integral domain and let \( S \) be a local overring of \( R \).

1. The center of \( S \) on \( R \) is the prime ideal \( m_S \cap R \), where \( m_S \) denotes the maximal ideal of \( S \).
2. If \( R \) is a local ring, then \( S \) is said to dominate \( R \) if the center of \( S \) on \( R \) is the maximal ideal of \( R \), that is, \( m_S \cap R = m_R \), where \( m_R \) is the maximal ideal of \( R \).
3. A valuation overring \( V \) of \( R \) is said to be a minimal valuation overring of \( R \) if \( V \) is minimal with respect to set-theoretic inclusion in the set of valuation overrings of \( R \).
4. If \( W \) is a valuation overring of \( R \) and the center of \( W \) on \( R \) is a nonmaximal prime ideal, then by composite construction [20, p. 43], there exists a valuation overring \( V \) of \( R \) such that \( V \subset W \) and \( V \) is centered on a maximal ideal of \( R \).
5. Assume that \( R \) is local. Every valuation overring of \( R \) contains a valuation overring of \( R \) that dominates \( R \). If \( W \) is a valuation overring of \( R \) that dominates \( R \) and the field \( W/m_W \) is not algebraic over \( R/m_R \), then by composite construction, there exists a valuation overring \( V \) of \( R \) such that \( V \) dominates \( R \) and \( V \subsetneq W \).
6. Assume that \( R \) is Noetherian and local. A valuation overring \( V \) of \( R \) is a minimal valuation overring of \( R \) if and only if \( V \) dominates \( R \) and the field \( V/m_V \) is algebraic over the field \( R/m_R \). Let \( X(R) \) denote the set of minimal valuation overrings of the Noetherian local domain \( R \).

---

What are called quadratic transforms in [2] and [32] are called local quadratic transforms in [31, p. 263].
7. Assume that $R$ is a regular local ring and $V$ is a prime divisor on $R$ that dominates $R$. Abhyankar proves in [1, Prop. 3] that there exists a unique finite sequence

$$R = R_0 \subset R_1 \subset \cdots \subset R_h \subset R_{h+1} = V$$

of regular local rings $R_j$, where $\dim R_h \geq 2$ and $R_{j+1}$ is a local quadratic transform of $R_j$ along $V$ for each $j \in \{0, \ldots, h\}$, and $\ord_{R_h} = V$. The association of the prime divisor $V$ with the regular local ring $R_h$ in Equation 1 and the uniqueness of the sequence in Equation 1 establishes a one-to-one correspondence between the prime divisors $V$ dominating $R$ and the regular local rings $S$ of dimension at least 2 that dominate $R$ and are obtained from $R$ by a finite sequence of local quadratic transforms as in Equation 1. The regular local rings $R_j$ with $j \leq h$ in Equation 1 are called the infinitely near points to $R$ along $V$. In general, a regular local ring $S$ of dimension at least 2 is called an infinitely near point to $R$ of level $h$ if there exists a sequence

$$R = R_0 \subset R_1 \subset \cdots \subset R_h = S, \quad h \geq 0$$

of regular local rings $R_j$ of dimension at least 2, where $R_{j+1}$ is a local quadratic transform of $R_j$ for each $j$ with $0 \leq j \leq h - 1$ [18, Definition 1.6].

8. Assume that $R$ is a regular local ring with $\dim R = 2$, and $S$ is a regular local overring of $R$.

(a) If $\dim S \geq 2$, then $\dim S = 2$ and $S$ dominates $R$.

(b) If $\dim S = 2$ and $S \neq R$, then $m_RS$ is a proper principal ideal of $S$. It follows that $S$ dominates a unique local quadratic transform $R_1$ of $R$.

Moreover, there exists for some positive integer $h$ a sequence

$$R = R_0 \subset R_1 \subset \cdots \subset R_h = S,$$

where $R_j$ is a local quadratic transform of $R_{j-1}$ for each $j \in \{1, \ldots, h\}$. The rings $R_j$ are precisely the regular local domains that are subrings of $S$ and contain $R$ [1, Theorem 3]. Every 2-dimensional regular local overring $S$ of $R$ is an infinitely near point to $R$. Each $V \in X(R)$ is the union of the infinite quadratic sequence of $R$ along $V$ [1].

(c) The Zariski theory about the unique factorization of the complete ideals of the 2-dimensional regular local ring $R$ yields a one-to-one correspondence between the following 3 sets:

i. the simple complete $m_R$-primary ideals,

ii. the infinitely near points to $R$, and

iii. the prime divisors that dominate $R$.  

3. The quadratic tree of a 2-dimensional regular local ring

The following notation will be used throughout the rest of the article.

**Notation 3.1.** Let \( D \) be a 2-dimensional regular local ring with quotient field \( F \) and maximal ideal \( \mathfrak{m}_D = (x, y)D \).

1. Let \( Q(D) \) denote the set of all 2-dimensional regular local overrings of \( D \). As noted in Remark \( \ref{remark:quadratic-tree} \), the rings in \( Q(D) \) are infinitely near points to \( D \). We call \( Q(D) \) the **quadratic tree** determined by \( D \).

2. Regarded as a partially ordered set with respect to inclusion, \( Q(D) \) is a tree, and is the disjoint union of subsets \( Q_j(D) \), \( j \geq 0 \), where \( Q_0(D) = \{ D \} \), and \( Q_j(D) \) for \( j \geq 1 \) is the set of infinitely near points to \( D \) of level \( j \) as in Remark \( \ref{remark:quadratic-tree} \).

3. As in Lipman [\cite{18}], it is often convenient to denote rings in \( Q(D) \) with lower case Greek letters. For \( \alpha \in Q(D) \), let \( Q(\alpha) \) denote the quadratic tree determined by \( \alpha \). For each subset \( U \) of \( Q(D) \), let \( \mathcal{O}_U = \bigcap_{\alpha \in U} \alpha \). In case \( U \) is the empty set, we define \( \mathcal{O}_U = F \).

4. If \( \alpha \subseteq \beta \) in \( Q(D) \), then \( \beta \) is **proximate** to \( \alpha \) if \( \beta \) is a subring of the order valuation ring, \( \text{ord}_\alpha \), of \( \alpha \). Let \( P(\alpha) \) denote the set of all \( \beta \) proximate to \( \alpha \).

5. Let \( \mathcal{R}(D) \) denote the set of rings of the form \( \mathcal{O}_U \) for some subset \( U \) of \( Q(D) \).

Proposition \( \ref{prop:quadratic-tree-properties} \) records properties of the quadratic tree \( Q(D) \) and the set \( X(D) \) of minimal valuation overrings of \( D \).

**Proposition 3.2.** Assume Notation \( \ref{notation:quadratic-tree} \).

1. If \( \alpha \in Q(D) \) properly contains \( D \), then there exists a unique positive integer \( j \) such that \( \alpha \in Q_j(D) \). The regular local rings between \( D \) and \( \alpha \) are linearly ordered with respect to inclusion and form a chain of length \( j \).

2. Each local ring \( \alpha \in Q(D) \) is essentially finitely generated over \( D \), that is, there exist finitely many elements \( x_1, \ldots, x_n \in \alpha \) such that \( D[x_1, \ldots, x_n]_p = \alpha \), where \( p \) is the center of \( \alpha \) on \( D[x_1, \ldots, x_n] \).

3. If \( \alpha, \beta \in Q(D) \), then the following are equivalent:
   (a) \( \alpha \) is a subring of \( \beta \).
   (b) \( \alpha \) is in the chain of regular local rings from \( D \) to \( \beta \).
(c) As points on the quadratic tree $Q(D)$, $\alpha \leq \beta$.

4. The valuation domains in $X(D)$ are paired in a one-to-one correspondence with the branches of the quadratic tree $Q(D)$. Each $V \in X(D)$ is the union of the quadratic sequence of $D$ along $V$.

Proof. These assertions follow from Remark 2.1.

Remark 3.3. Let $R \in \mathcal{R}(D)$ and let $\mathcal{U}' = \{\alpha \in Q(D) \mid R \subseteq \alpha\}$. Let $\mathcal{U}$ denote the subset of $\mathcal{U}'$ of minimal points of $\mathcal{U}'$. Then the points in $\mathcal{U}$ are incomparable and $\mathcal{O}_\mathcal{U} = \mathcal{O}_\mathcal{U}'$. Therefore:

1. Each ring $R \in \mathcal{R}(D)$ has the form $R = \mathcal{O}_\mathcal{U}$, where the $\alpha \in \mathcal{U}$ are incomparable and are minimal among points of $Q(D)$ that contain $R$.
2. If $D$ is Henselian and $\mathcal{U}$ is as defined in item 1, then it is shown in [14] that the representation $R = \bigcap_{\alpha \in \mathcal{U}} \alpha$ is irredundant.

Remark 3.4. For $\alpha \in Q(D)$ the set $P(\alpha)$ of points proximate to $\alpha$ is by definition

$$P(\alpha) = \{\beta \in Q(D) \mid \alpha \subseteq \beta \text{ and } \beta \subseteq \text{ord}_\alpha =: V\}.$$ 

The points in $Q_1(\alpha)$ are all proximate to $\alpha$. For each $\beta \in Q_1(\alpha)$, there exists a unique chain in $Q(\beta)$ of points in $P(\alpha)$ that may be described as follows:

1. The center of $V$ on $\beta$ is a height-one regular prime $p = z\beta$, for some $z \in \beta$. Let $w \in m_\beta$ be such that $m_\beta = (z, w)\beta$. Then $\beta[z/w]$ and $\beta[w/z]$ are affine components that define the blowup of $m_\beta$. Since $\beta[w/z]$ is not contained in $V$, the only point of $Q_1(\beta)$ that is contained in $V$ is $\beta[z/w](w, z/w)\beta$, and $V$ is centered on the height-one regular prime of this ring generated by $z/w$.
2. The process iterates to give as the union the rank 2 valuation domain contained in $V$ obtained by composite center construction with respect to the residue class ring $\beta/z\beta$.
3. In summary, the set $P(\alpha)$ consists of all the points in the first neighborhood $Q_1(\alpha)$ of $\alpha$ together with an infinite ascending ray emanating from each point $\beta \in Q_1(\alpha)$. The rays are in one-to-one correspondence with the points in $Q_1(\alpha)$.

Corollary 5.6 implies that the set $P(\alpha)$ of points proximate to $\alpha$ is a Noetherian subspace of $Q(D)$ in the Zariski topology.

4. It is a classical fact that an infinitely near point $\gamma$ to $D$ is proximate to at most 2 points of $Q(D)$. An argument for this in a more general setting is given in the proof of [13, Lemma 2.7].

4In the terminology of [13], $V$ is a Shannon extension of $D$. 

7
5. Let $\gamma \in Q(D)$ with $\gamma \neq D$. Then there exists a unique point $\alpha \in Q(D)$ such that $\gamma \in Q_1(\alpha)$. The set $E(\gamma)$ of essential valuation rings for $\gamma$ contains $\text{ord}_\alpha$ and at most one other element that is not in $E(D)$ the set of essential valuation rings of $D$.

6. Let $V \in E(D)$, then $V = D_p$, where $p$ is a height 1 prime of $D$. The rings $\gamma$ in $Q(D)$ that are contained in $V$ are determined by the transform of $p$ in $\gamma$ as defined by Lipman in [18]. The transform of $p$ in $\gamma$ is either $\gamma$ or a height 1 prime $q$, where $q \cap D = p$ and $D_p = \gamma_q = V$. If the transform of $p$ in $\gamma$ is a prime ideal $q$, then the finite sequence of quadratic transforms from $D$ to $\gamma$ induces a finite sequence of local quadratic transforms of the local domain $D/p$ to the local domain $\gamma/q$, cf. [3, Corollary II.7.15, p. 165] and [7, Proposition 4.1].

7. The integral closure of $D/p$ is a finite intersection of DVRs. Associated to each of these DVRs, by composite construction, there exists a rank 2 valuation overring $W$ of $D$. Then $W \in X(D)$, the set of minimal valuation overrings of $D$. Hence $W$ determines a unique branch in the quadratic tree $Q(D)$. The points $\gamma \in Q(D)$ such that $\gamma \subset D_p$ are the points on these branches.

8. In summary, if the integral closure of $D/p$ has $t$ maximal ideals, then there exists for each positive integer $n$ at most $t$ points in $Q_n(D)$ that are contained in $V = D_p$. There are $t$ branches in $Q(D)$ uniquely determined by $D_p$, and the points on these branches are precisely the points of $Q(D)$ that are contained in $V = D_p$.

Discussion 3.5. For nonzero elements $f$ and $g$ in $D$, the rational function $f/g$ has a position in each element of the quadratic tree $Q(D)$. We use the following terminology:

1. $f/g$ has a zero at $\alpha \in Q(D)$ if $f/g \in m_\alpha$, the maximal ideal of $\alpha$.
2. $f/g$ has a pole at $\alpha \in Q(D)$ if $g/f \in m_\alpha$.
3. $f/g$ is a unit at $\alpha$ if both $f/g \in \alpha$ and $g/f \in \alpha$.
4. $f/g$ is undetermined at $\alpha$ if both $f/g \notin \alpha$ and $g/f \notin \alpha$.

It is clear that if $f/g$ has a zero or pole, or is a unit at $\alpha$, then $f/g$ has, respectively, a zero or pole or is a unit in each $\beta \in Q(D)$ that dominates $\alpha$.

Since the elements in $Q(D)$ are not valuation rings, many rational functions $f/g$ are such that $f/g$ is undetermined at $\alpha$. The rational function $f/g$ is undetermined

---

If $D$ is not pseudo-geometric, the integral closure of $D/p$ may not be finite, but if the integral closure of $D/p$ has $t$ maximal ideals, then there is a finite integral extension of $D/p$ that has $t$ maximal ideals.
at $\alpha$ if and only if $g/f$ is undetermined at $\alpha$. In this case, it is natural to consider $f/g$ in the points of $Q_1(\alpha)$.

Since $\alpha$ is a UFD, we may assume that $f$ and $g$ have no common prime factors in $\alpha$. A prime factor of $f$ in $\alpha$ generates a height 1 prime $q$ of $\alpha$. Thus, there exists a nonempty finite set of cardinality at most 2 of height 1 primes of $\alpha$ that contain $\mathfrak{m}_D$. If $q \cap D = \mathfrak{m}_D$, then $\alpha_q$ is the order valuation ring of one of the points $\beta$ in the finite sequence from $D$ to $\alpha$, and $\alpha$ is proximate to $\beta$.

If $q \cap D = \mathfrak{p}$ is a height 1 prime of $D$, then $D_\mathfrak{p} = \alpha_q$. As noted in Remark 3.4, the finite sequence of quadratic transforms from $D$ to $\alpha$ induces a finite sequence of local quadratic transforms of the local domain $D_\mathfrak{p}$ to the local domain $\alpha_q$. This can be helpful in describing the zeros and poles with respect to the points of $Q(D)$ of a rational function $f/g$.

Example 3.6 illustrates how to compute the position of a specific rational function in the quadratic tree. Transforms of the ideal $(f, g)_D$ as defined in [18] are useful for such computations.

Example 3.6. Assume terminology as in Discussion 3.5. Let

$$f = xy \quad \text{and} \quad g = y^2 + x^3.$$  

Then $f/g$ is undetermined at $D$. Every point of $Q_1(D)$ is a localization of either $D[y/x]$ or of $D[x/y]$, and all the points of $Q_1(D)$ other than $D[y/x](x,y/x)$ and $D[x/y](y,x/y)$ are localizations of both $D[y/x](x,y/x)$ and $D[x/y](y,x/y)$.

Consider $D[y/z]$ and let $y_1 = \frac{y}{z}$. Then $y = xy_1$ and

$$f = \frac{xy}{y^2 + x^3} = \frac{x^2y_1}{x^2y_1^2 + x^3} = \frac{y_1}{y_1^2 + x}.$$  

It follows that $f/g$ is undetermined at the point $\alpha := D[y/x](x,y/x)$, the point in $Q_1(D)$ with maximal ideal generated by $(x, y_1)$.

Consider $\beta := \alpha[y/x^2](x,y/x^2)$, the point in $Q_2(D)$ with maximal ideal generated by $(x, y_2)$, where $y_2 = \frac{y_1}{x}$. Then $y_1 = xy_2$ and

$$f = \frac{xy}{y^2 + x^3} = \frac{y_1}{y_1^2 + x} = \frac{xy_2}{x^2y_2^2 + x} = \frac{y_2}{xy_2^2 + 1}.$$  

It follows that $f/g$ has a zero at $\beta$.

Consider $D[x/y]$ and let $x_1 = x/y$. Then $x = xy_1$ and

$$f = \frac{xy}{y^2 + x^3} = \frac{y^2x_1}{y^2 + y^3x_1} = \frac{x_1}{1 + yx_1^2}.$$  

9
If follows that \( f/g \) has a zero at \( \gamma := D[y/x](y,x) \), the point in \( Q_1(D) \) with maximal ideal generated by \( (y, x^2) \).

We conclude that \( \frac{f}{g} = \frac{xy}{y^2 + x^3} \) has two “distinguished zeros”, namely \( \beta \in Q_2(D) \) and \( \gamma \in Q_1(D) \). The points \( \beta \) and \( \gamma \) are incomparable and the points in the quadratic tree \( Q(D) \) at which \( f/g \) has a zero are precisely the points that dominate either \( \beta \) or \( \gamma \).

The local ring \( R := D[f/g](x,y,f/g) \) is dominated by \( \beta \) and \( \gamma \). The integral closure of the ideal \( I = (xy, y^2 + x^3)D \) is the product \( J \) of the simple complete ideals \( (x, y)D \) and \( (y, x^2)D \). The complete ideal \( J = (xy, y^2 + x^3)D \) has a saturated factorization as defined in [11, Definition 5.11], and the projective model \( \text{Proj } D[Jt] \) is nonsingular with \( \beta \) and \( \gamma \) points on this model.

The affine component \( A := D[y^2, x^3]/(x^2, xy) \) of \( \text{Proj } D[Jt] \) contains \( D[y^2 + x^3]/(x^2, xy) \) as a subring. The points in the quadratic tree \( Q(D) \) at which \( f/g \) has a pole are precisely the points that dominate \( \delta := D[y^2]/(x^2, x^3) \). Notice that \( \delta \in Q_2(D) \) and \( \delta \) is the integral closure of \( S \). Thus \( \delta \) is the unique “distinguished pole” of \( f/g \).

4. The patch topology of \( Q^*(D) \)

In this section and the next we examine topological properties of the tree \( Q(D) \) and the partially ordered set \( Q^*(D) \) consisting of \( Q(D) \) and the valuation rings birationally dominating \( D \). We first describe the patch topology of \( Q^*(D) \), a topology that is finer than the Zariski topology. As follows in Remark 4.4, properties of the Zariski topology, which are the focus of the next section, can be derived from the patch topology, so our approach in this section is to focus on the patch limit points of subsets of \( Q^*(D) \) and use this description in the next section to describe properties of the Zariski topology of \( Q^*(D) \). The patch topology is a common tool for studying the Zariski-Riemann space of valuation rings of a field; see for example [3, 14, 17, 25, 26, 27, 28].

Our methods in this section require us to work occasionally not just in \( Q^*(D) \) but also in the set of all local rings birationally dominating \( D \). We formalize our notation for this section and the next as follows.

Notation 4.1. Let \( D \) be a regular local ring of dimension 2 and let \( F \) denote the quotient field of \( D \).
1. Let $L(D)$ denote the set of all local rings that birationally dominate $D$. Similarly, for $\alpha \in Q(D)$, $L(\alpha)$ is the set of local rings that birationally dominate $\alpha$.

2. For each $x_1, \ldots, x_n \in F$, we let
   \[ U(x_1, \ldots, x_n) = \{ R \in L(D) \mid x_1, \ldots, x_n \in R \}, \]
   and
   \[ V(x_1, \ldots, x_n) := \{ R \in L(D) \mid x_1, \ldots, x_n \text{ are not all in } R \}. \]
   Thus $V(x_1, \ldots, x_n) = L(D) \setminus U(x_1, \ldots, x_n)$.

3. The Zariski topology on $L(D)$ is the topology having a basis of open sets given by the sets $U(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n \in F$. Thus a basis of closed sets is given by sets $V(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n \in F$.

4. As in Notation 2.1.6, we denote by $X(D)$ the subset of $L(D)$ consisting of the minimal valuation overrings of $D$.

5. We let $Q^*(D)$ be the union of $Q(D)$ and the set of valuation rings in $L(D)$. Thus $Q^*(D)$ is the union of the set $Q(D)$, the set $X(D)$ and the set of prime divisors of $D$ of the second kind.

6. We extend the definition of $O$ given in Notation 3.1.3 to $Q^*(D)$ by defining, for each subset $\mathcal{U}$ of $Q^*(D)$, $O_{\mathcal{U}}$ to be the intersection of the rings in $\mathcal{U}$. If $\mathcal{U}$ is empty, then we set $O_{\emptyset} = F$.

The set $Q^*(D)$ is partially ordered by inclusion. While the subset $Q(D) \cup X(D)$ of $Q^*(D)$ is a tree by Proposition 3.2.4, the set $Q^*(D)$ is not. This is because a prime divisor of $D$ of the second kind contains infinitely many valuation rings in $X(D)$.

We prove in Corollary 4.9 that $Q^*(D)$ with the Zariski topology is a spectral space. This means that it is $T_0$ and quasicompact, it has a basis of quasicompact open sets closed under finite intersections, and every irreducible closed set has a (unique) generic point. This is useful not only because it situates the topology of $Q^*(D)$ in an appropriate context, but because it also allows us to use some of the tools for working with spectral spaces.

The proof that $Q^*(D)$ is a spectral space involves proving that it is a patch closed subset of a larger spectral space, namely $L(D)$. We recall that the patch topology of a spectral space $X$ has as a basis of open sets the sets that are an intersection of a

---

6 A theorem of Hochster [15, Corollary, p. 45] shows that a topological space is spectral if and only if it is homeomorphic to the prime spectrum of a ring.

7 The patch topology is sometimes referred to as the constructible or Hausdorff topology of a spectral space.
Proposition 4.2. The space \( L(D) \) with the Zariski topology is a spectral space having as a basis of quasicompact open sets the sets of the form \( U(x_1, \ldots, x_n) \), where \( x_1, \ldots, x_n \in F \).

Proof. For the purpose of this proof, we introduce additional notation that will not be needed later. We denote by \( L' \) the set of local overrings of \( D \). Thus \( L' \) consists of the rings in \( L(D) \) and the local overrings of \( D \) that do not dominate \( D \); these latter rings are precisely the essential valuation rings of \( D \). For \( x_1, \ldots, x_n \in F \), we let

\[
U'(x_1, \ldots, x_n) = \{ R \in L(D) \mid x_1, \ldots, x_n \in R \}.
\]

Thus \( U(x_1, \ldots, x_n) = U'(x_1, \ldots, x_n) \cap L(D) \). The sets of the form \( U'(x_1, \ldots, x_n) \) yield a basis for a topology on \( L'(D) \), which we refer to as the Zariski topology on \( L'(D) \). Viewed as a subspace of \( L'(D) \), \( U'(x_1, \ldots, x_n) \) is a spectral space (see [3, Corollary 2.14] or [26, Example 2.2(7)]). In particular \( U'(x_1, \ldots, x_n) \) is quasicompact. Since \( L'(D) = U'(1) \), we have also that \( L'(D) \) is a spectral space. We make use of these facts in what follows.

We claim that for \( x_1, \ldots, x_n \in F \), the subspace \( U(x_1, \ldots, x_n) \) of \( L(D) \) is a patch closed subset of the spectral space \( L'(D) \). To prove this, it suffices to show that each patch limit point of \( U(x_1, \ldots, x_n) \) in \( L'(D) \) is in \( U(x_1, \ldots, x_n) \). Let \( R \) be a patch limit point of \( U(x_1, \ldots, x_n) \) in \( L'(D) \). To prove that \( R \) is in \( U(x_1, \ldots, x_n) \), it suffices to show \( R \) dominates \( D \) and \( x_1, \ldots, x_n \in R \).

To prove that \( R \) dominates \( D \), let \( R^\times \) denote the set of units in \( R \). Then the set

\[
U' := \bigcup_{u \in R^\times} (U'(u) \cap U'(u^{-1}))
\]

is open in the patch topology of \( L'(D) \) and contains \( R \). Since \( R \) is a patch limit point of \( U(x_1, \ldots, x_n) \) in \( L'(D) \), there exists \( \alpha \in U(x_1, \ldots, x_n) \) such that \( \alpha \in U' \). Thus every unit in \( R \) is a unit in \( \alpha \). If \( R \) does not dominate \( D \), then there exists a nonunit \( d \in D \) such that \( d \) is a unit in \( R \), hence also in \( \alpha \). But \( \alpha \) dominates \( D \), so this is impossible. We conclude that \( R \) dominates \( D \).

To see that \( x_1, \ldots, x_n \in R \), suppose to the contrary that \( x_i \not\in R \) for some \( i \). Then \( R \in \mathcal{V}' := \{ S \in L'(D) \mid x_i \not\in S \} \). Since \( R \) is a patch limit point of \( U(x_1, \ldots, x_n) \) in \( L'(D) \) and \( \mathcal{V}' \) is a patch open set in \( L'(D) \), the intersection \( U(x_1, \ldots, x_n) \cap \mathcal{V}' \) is nonempty, a contradiction. Therefore \( x_1, \ldots, x_n \in R \), which proves that \( U(x_1, \ldots, x_n) \) is patch closed in \( L'(D) \). A patch closed subspace of a spectral space is a spectral space [34, Tag 0902], so \( U(x_1, \ldots, x_n) \) is a spectral space and hence is quasicompact. Since \( L(D) = U(1) \), Proposition 4.2 now follows. \( \Box \)
Corollary 4.3. The patch topology of the spectral space $L(D)$ has as a basis of closed and open sets the sets of the form

$$(U(S_1) \cup \cdots \cup U(S_n)) \cap V(T_1) \cap \cdots \cap V(T_m),$$

where $S_1, \ldots, S_n, T_1, \ldots, T_m$ are finite subsets of $F$.

Proof. Since the patch topology on a spectral space has a basis of open sets the sets that are an intersection of a quasicompact open set and the complement of a quasicompact open set, this follows from Proposition 4.2.

Remark 4.4. Although the patch topology on $L(D)$ is finer than the Zariski topology, the Zariski topology can be recovered from it: The Zariski closed sets $V$ of $L(D)$ are precisely the downsets (with respect to set inclusion) of the patch closed subsets of $L(D)$. This follows for example from [15, Corollary, p. 45]. The upsets of patch closed subsets of $L(D)$ form a basis for the inverse topology on $L(D)$.

Viewing $Q(D)$ as a subspace of $L(D)$, we describe in Theorem 4.8 the patch closure of $Q(D)$ in $L(D)$, i.e., the closure of $Q(D)$ in $L(D)$ with respect to the patch topology. To do this, we first describe in the next lemmas additional topological properties of $Q(D)$ and the limit points of $Q(D)$ in the patch topology.

Lemma 4.5. For each $\alpha \in Q(D)$, the subspace $L(\alpha)$ is patch clopen in $L(D)$.

Proof. By Proposition 3.2.2, there exist $x_1, \ldots, x_n \in \alpha$ such that $\alpha = D[x_1, \ldots, x_n]_p$ for some prime ideal $p$ of $D[x_1, \ldots, x_n]$. Write $p = (y_1, \ldots, y_m)D[x_1, \ldots, x_n]$, where $y_1, \ldots, y_m$ are nonzero. Since $\alpha$ has dimension 2, $p$ is a maximal ideal of $D[x_1, \ldots, x_n]$. Since $L(\alpha)$ is the set of rings in $L(D)$ that dominate $\alpha$, it follows that

$$L(\alpha) = U(x_1, \ldots, x_n) \cap V(1/y_1) \cap \cdots \cap V(1/y_m).$$

Thus $L(\alpha)$ is patch clopen in $L(D)$.

In light of Remark 4.3, the following notation will be useful in this section and the next.

Notation 4.6. For a nonempty subset $S$ of $Q^*(D)$, we denote by $\downarrow S$ the downset of $S$ in $Q(D)$:

$$\downarrow S = \{ \alpha \in Q(D) \mid \alpha \subseteq \beta \text{ for some } \beta \in S \}.$$

When $S$ consists of a single ring $R$, we write $\downarrow R$ for $\downarrow \{ R \}$. Note that while $S$ may be a subset of $Q^*(D)$, in our notation the members of the downset are restricted to $Q(D)$. To differentiate between these two cases, we set
\[ \downarrow^* \mathcal{S} = \{ \alpha \in Q^*(D) \mid \alpha \subseteq \beta \text{ for some } \beta \in \mathcal{S} \}. \]

Theorem 4.7 describes the valuation rings that are patch limit points in \( L(D) \) of a subset \( \mathcal{S} \) of \( Q(D) \). Theorem 4.3 implies patch limits points of subsets of \( Q(D) \) are necessarily valuation rings in \( Q^*(D) \). This description shows that the patch limit points of \( \mathcal{S} \) are determined by the properties of the partially ordered set \( Q^*(D) \) rather than the nature of the underlying objects that comprise the set \( Q^*(D) \).

**Theorem 4.7.** Let \( \mathcal{S} \) be a subset of \( Q(D) \), and let \( V \) be a valuation ring in \( Q^*(D) \).

1. Suppose \( V \in X(D) \). Then the following are equivalent.
   (a) \( V \) is a patch limit point of \( \mathcal{S} \) in \( L(D) \).
   (b) \( \mathcal{S} \cap Q(\alpha) \) is nonempty for each \( \alpha \in Q(D) \) dominated by \( V \).

2. Suppose \( V \not\in X(D) \). Then \( V \) is a prime divisor of the second kind and an order valuation ring of some \( \alpha \in Q(D) \). In this case, the following are equivalent.
   (a) \( V \) is a patch limit point of \( \mathcal{S} \) in \( L(D) \).
   (b) \( Q_1(\alpha) \cap \downarrow \mathcal{S} \) is infinite.
   (c) There are infinitely many incomparable rings in \( \mathcal{S} \) contained in \( V \).
   (d) There are infinitely many incomparable rings in \( \mathcal{S} \) proximate to \( \alpha \).

**Proof.** To prove item 1, suppose that \( V \in X(D) \) is a patch limit point of \( \mathcal{S} \) in \( L(D) \) and that \( \alpha \in Q(D) \) is dominated by \( V \). Since \( V \in X(D) \), Remark 2.1.8.b implies that there exists \( \alpha_1 \in Q_1(\alpha) \) such that \( V \) dominates \( \alpha_1 \). Let \( U = L(\alpha) \setminus L(\alpha_1) \). By Lemma 4.3, \( U \) is a patch open set of \( L(D) \) containing \( V \). As \( V \) is a patch limit point of \( \mathcal{S} \) in \( L(D) \), the set \( U \cap \mathcal{S} \) is nonempty. Since \( \mathcal{S} \subseteq Q(D) \) and \( Q(\alpha) = Q(D) \cap L(\alpha) \), we conclude that \( Q(\alpha) \cap \mathcal{S} \) is nonempty.

Conversely, suppose \( Q(\alpha) \cap \mathcal{S} \) is nonempty for each \( \alpha \) dominated by \( V \). Since \( V \in X(D) \), Remark 2.1.8.b implies there is an infinite sequence

\[
\alpha_0 \subseteq \alpha_1 \subseteq \cdots \subseteq \alpha_i \subseteq \cdots
\]

of elements of \( Q(D) \) dominated by \( V \) such that \( V = \bigcup_i \alpha_i \). By assumption, \( \mathcal{S} \cap Q(\alpha_i) \) is nonempty for each \( i \geq 1 \). Let \( U \) be a patch open subset of \( L(D) \) containing \( V \). By Corollary 4.3 there are nonzero \( x_1, \ldots, x_n, y_1, \ldots, y_m \in F \) such that

\[
V \in U(x_1, \ldots, x_n) \cap V(y_1) \cap \cdots \cap V(y_m) \subseteq U.
\]

Since \( V \) is a valuation ring, we have \( y_1^{-1}, \ldots, y_m^{-1} \in \mathfrak{M}_V \). Since \( V \) is a directed union of the \( \alpha_i \), there is \( i \geq 0 \) such that \( x_1, \ldots, x_n, y_1^{-1}, \ldots, y_m^{-1} \in \alpha_i \). Moreover, since \( V \) dominates \( \alpha_i \), the maximal ideal of \( \alpha_i \) contains \( y_1^{-1}, \ldots, y_m^{-1} \). Since the rings in \( Q(\alpha_i) \) dominate \( \alpha_i \), it follows that

\[
Q(\alpha_i) \subseteq U(x_1, \ldots, x_n) \cap V(y_1) \cap \cdots \cap V(y_m) \subseteq U.
\]
Thus the assumption that $S \cap Q(\alpha_i)$ is nonempty implies that $S \cap U$ is nonempty, which proves that $V$ is a patch limit point of $S$ in $L(D)$.

It remains to prove item 2. Since $V \notin X(D)$, $V$ is a prime divisor of $D$ of the second kind. Therefore, by Remark 2.1.7, there is $\alpha \in Q(D)$ such that $V$ is the order valuation ring of $\alpha$.

(a) $\Rightarrow$ (b) Suppose $V$ is a patch limit point of $S$. We show that there are infinitely many local rings in $Q_1(\alpha)$ that are dominated by local rings in $S$. To this end, let $\beta_1, \ldots, \beta_n \in Q_1(\alpha)$. By Lemma 2.1.5 and the fact that points are closed in the patch topology, we have that

$$U := L(\alpha) \setminus (L(\beta_1) \cup \cdots \cup L(\beta_n) \cup \{\alpha\})$$

is a nonempty patch open subset of $L(D)$. Since $V$ is the order valuation ring of $\alpha$, $V$ dominates $\alpha$ but does not dominate any local ring in $Q_1(\alpha)$. Consequently, $V \in U$. Since $V$ is a patch limit point of $S$, there exists $\gamma \in U \cap S$. Because $\gamma$ properly dominates $\alpha$ and does not dominate any of the $\beta_i$, it follows that $\gamma$ dominates some member of $Q_1(\alpha)$ distinct from $\beta_1, \ldots, \beta_n$. Since this is true for any finite subset $\{\beta_1, \ldots, \beta_n\}$ of $Q_1(\alpha)$, we conclude there are infinitely many local rings in $Q_1(\alpha)$ dominated by local rings in $S$.

(b) $\Rightarrow$ (c) This is clear since the rings in $Q_1(\alpha)$ are incomparable and contained in $V$.

(c) $\Rightarrow$ (d) The rings $\beta$ in $Q(D)$ contained in $V$ are either proximate to $\alpha$ or are dominated by $V$ (see Remark 2.1.7). In the latter case, $\alpha$ dominates $\beta$, and hence there are only finitely many rings in $Q(D)$ dominated by $V$. Item d now follows from item c.

(d) $\Rightarrow$ (b) This follows from Remark 3.1.4.

(b) $\Rightarrow$ (a) Suppose that the set of local rings in $Q_1(\alpha)$ that are dominated by local rings in $S$ is infinite. We show $V$ is a patch limit point of $S$. Let $X$ be the projective model of $\text{Spec} \ \alpha$ obtained by blowing up the maximal ideal of $\alpha$. Then $Q_1(\alpha)$ is the set of closed points of $X$ and $V$ is the generic point of the irreducible Zariski closed subset $\mathcal{C} := \{V\} \cup Q_1(\alpha)$ of $X$. Since $\mathcal{C}$ is a Noetherian space in the Zariski topology, every closed subset of $\mathcal{C}$ is a finite union of irreducible closed sets. The only irreducible proper closed subsets of $\mathcal{C}$ are the singleton subsets of $Q_1(\alpha)$, so it follows that any infinite subset of $\mathcal{C}$ is dense in $\mathcal{C}$. By assumption, there is an infinite subset $\mathcal{T}$ of $Q_1(\alpha)$ such that each local ring in $\mathcal{T}$ is dominated by a local ring in $S$. Since $\mathcal{T}$ is Zariski dense in $\mathcal{C}$ and $V$ is the generic point for $\mathcal{C}$, it follows that $V$ is a patch limit point of $\mathcal{T}$ [25, Proposition 2.6(1)].

To see now that $V$ is a patch limit point of $S$, let

$$S' = \{\sigma \in S \mid \tau \subseteq \sigma \text{ for some } \tau \in \mathcal{T}\}.$$
Fix $y \neq 0$ in the maximal ideal of $V$. For all nonzero $x_1, \ldots, x_n \in V$, we have

$$V \in \mathcal{U}(x_1, \ldots, x_n, y) \cap \mathcal{V}(1/y).$$

Since $\mathcal{U}(x_1, \ldots, x_n, y) \cap \mathcal{V}(1/y)$ is patch open and $V$ is a patch limit point of $\mathcal{T}$, there exists $\tau \in \mathcal{T} \cap \mathcal{U}(x_1, \ldots, x_n, y) \cap \mathcal{V}(1/y)$. Let $\sigma \in S'$ such that $\tau \subseteq \sigma$. Since $\sigma$ dominates $\tau$, we have $\sigma \in \mathcal{U}(x_1, \ldots, x_n, y) \cap \mathcal{V}(1/y)$. This shows that for all nonzero $x_1, \ldots, x_n \in V$, we have

$$S' \cap \mathcal{U}(x_1, \ldots, x_n, y) \cap \mathcal{V}(1/y) \neq \emptyset.$$

Since the patch closure $\overline{S'}$ of $S'$ in $L(D)$ is compact in the patch topology and the collection of patch closed sets of the form $\overline{S'} \cap \mathcal{U}(x_1, \ldots, x_n, y) \cap \mathcal{V}(1/y)$ has the finite intersection property, there is a ring $W \in \overline{S'}$ such that $W \in \mathcal{U}(x_1, \ldots, x_n, y) \cap \mathcal{V}(1/y)$ for all $x_1, \ldots, x_n \in V$. Thus $V \subseteq W \subset F$. Since $V$ is a DVR, we conclude that $V = W$, which proves that $V \in \overline{S'}$. Since $V$ is not in $S'$, it must be that $V$ is a patch limit point of $S'$, hence a patch limit point of $S$.

Using the description of patch limit points in Theorem 4.7, we next show that the patch limit points of the set $Q(D)$ in $L(D)$ is the set of all valuation rings that birationally dominate $D$.

**Theorem 4.8.** The set of patch limit points of $Q(D)$ in $L(D)$ is precisely the set of valuation rings in $L(D)$. Thus the patch closure of $Q(D)$ in $L(D)$ is $Q^*(D)$.

**Proof.** By Proposition 4.2, $L(D)$ is a spectral space. First, we prove that every patch limit point of $Q(D)$ in $L(D)$ is a valuation ring. Let $R \in L(D)$ be a patch limit point of $Q(D)$, and let $V$ be a valuation ring in $L(D)$ dominating $R$. By Remarks 2.1.8, the rings in $Q(D)$ dominated by $R$ form a chain under inclusion. If this chain is infinite, then by Remarks 2.1.8, we conclude that $R = V$, and the proof is complete.

Suppose there are only finitely many rings in $Q(D)$ dominated by $R$, and choose $\alpha \in Q(D)$ such that no member of $Q(D)$ properly dominates $\alpha$ and is dominated by $R$. Write $m_\alpha = (x, y)\alpha$. We claim that $x/y \in R$ or $y/x \in R$. Suppose to the contrary that $x/y \notin R$ and $y/x \notin R$. Then $R \subset L(\alpha) \cap \mathcal{V}(x/y) \cap \mathcal{V}(y/x)$. By Lemmas 4.3 and 4.5, the set $L(\alpha) \cap \mathcal{V}(x/y) \cap \mathcal{V}(y/x)$ is patch open in $L(D)$. Since $R$ is a patch limit point of $Q(D)$ and the patch topology is Hausdorff, this implies that the set $L(\alpha) \cap \mathcal{V}(x/y) \cap \mathcal{V}(y/x) \cap Q(D)$ is infinite. Since $Q(\alpha) = L(\alpha) \cap Q(D)$, we have then that $Q(\alpha) \cap \mathcal{V}(x/y) \cap \mathcal{V}(y/x)$ is infinite. But since $x, y$ is a system of regular parameters for $\alpha$, it follows that $Q(\alpha) \cap \mathcal{V}(x/y) \cap \mathcal{V}(y/x) = \{\alpha\}$, a contradiction. Therefore $x/y \in R$ or $y/x \in R$. 

16
Assume \( x/y \in R \). The center of \( R \) in the ring \( \alpha[x/y] \) contains the prime ideal \( y\alpha[x/y] \). If \( R \) is centered on a prime ideal properly containing \( y\alpha[x/y] \), then \( R \) is centered on \( y\alpha[x/y] \). Since the localization of \( \alpha[x/y] \) at this height one prime ideal is the order valuation ring of \( \alpha \), we conclude that \( R \) contains the order valuation of \( \alpha \). Therefore the order valuation ring is a DVR birationally dominated by \( R \), so it follows that \( R \) is the order valuation ring of \( \alpha \), proving that \( R \) is a valuation ring. This shows that the patch limit points of \( Q(D) \) in \( L(D) \) are valuation rings.

Finally, every valuation ring \( V \) in \( L(D) \) is a patch limit point of \( Q(D) \), since if \( V \) is a minimal valuation ring of \( D \), then \( V \) is a patch limit point of \( Q(D) \) by Theorem 4.7.1, while if \( V \) is a prime divisor of \( D \) of the second kind, then \( V \) is a patch limit point of \( Q(D) \) by Theorem 4.7.2.

**Corollary 4.9.** \( Q^*(D) \) is a spectral space with respect to the Zariski topology. The patch topology of this spectral space is the subspace topology induced by the patch topology of \( L(D) \).

*Proof.* As a patch closed subset of the spectral space \( L(D) \), \( Q^*(D) \) is a spectral space with respect to the Zariski topology [30, Tag 0902]. That the patch topology of the spectral space \( Q^*(D) \) is the subspace topology of \( Q^*(D) \) with respect to the patch topology of \( Q^*(D) \) follows from the discussion after Theorem 1 in [13, p. 45].

**Corollary 4.10.** Every infinite subset of \( Q^*(D) \) has a patch limit point in \( Q^*(D) \).

*Proof.* Let \( S \) be an infinite subset of \( Q^*(D) \). By Corollary 4.9, \( Q^*(D) \) is a spectral space, so \( Q^*(D) \) is quasicompact in the patch topology [13, Theorem 1]. Therefore the patch closure of \( S \) in \( Q^*(D) \) is infinite and quasicompact in the patch topology, hence not discrete. Consequently, the patch closure of \( S \) contains a patch limit point.

**Corollary 4.11.** The set of patch isolated points of \( Q^*(D) \) is \( Q(D) \). Thus \( Q(D) \) is discrete in the subspace topology induced by the patch topology of \( Q^*(D) \).

*Proof.* Let \( \alpha \in Q(D) \). By Lemma 4.5, \( L(\alpha) \) is patch clopen in \( L(D) \). Since \( Q^*(\alpha) = Q^*(D) \cap L(\alpha) \), we have that \( Q^*(\alpha) \) is clopen in the subspace topology of \( Q^*(D) \) induced by the patch topology on \( L(D) \). To see now that \( \alpha \) is a patch isolated point in \( Q^*(D) \), let \( s, t \) be a system of regular parameters for \( \alpha \). Since every local ring in \( Q^*(\alpha) \setminus \{\alpha\} \) contains either \( \alpha[s/t] \) or \( \alpha[t/s] \), we have

\[
\{\alpha\} = Q^*(\alpha) \cap \mathcal{V}(s/t) \cap \mathcal{V}(t/s).
\]

Therefore \( \{\alpha\} \) is open in the subspace topology of \( Q^*(D) \) induced by the patch topology of \( L(D) \), and hence by Corollary 4.9 \( \alpha \) is a patch isolated point of \( Q^*(D) \).
Since every valuation ring in $Q^*(D)$ is by Theorem 4.8 a patch limit point of $Q(D)$, Corollary 4.11 now follows.

**Discussion 4.12.** We summarize some of the key features of the patch topology on $Q(D)$ and $Q^*(D)$. The patch topology on $Q^*(D)$ is Hausdorff and has a basis of clopens (i.e., the patch topology is zero-dimensional). With the subspace topology induced by the patch topology, $Q(D)$ is a discrete space (Corollary 4.11), and hence the patch topology is not interesting intrinsically for $Q(D)$. With respect to the patch topology, it is only when viewed as a subspace of $Q^*(D)$ that $Q(D)$ becomes topologically interesting. Indeed, the set of patch limit points of any infinite subset $S$ of $Q(D)$ must consist exclusively of valuation rings in $Q^*(D)$ (Corollaries 4.10 and 4.11). Which valuation rings are patch limit points of $S$ is determined by the configuration of $S$ within the tree $Q(D)$; see Theorem 4.7. The calculation of these limit points is an important tool in the next section when we work with the Zariski topology on $Q(D)$ and $Q^*(D)$.

As an application of some of the ideas in this section, we describe the structure of an intersection of order valuation rings of rings below a fixed level in $Q(D)$. Recall that an integral domain $R$ is an almost Dedekind domain if for each maximal ideal $M$ of $R$, $R_M$ is a DVR.

**Corollary 4.13.** Let $n > 0$, and let $S$ be a nonempty subset of $Q(D)$ consisting of rings of level at most $n$. Then the intersection of the order valuation rings of the rings in $S$ is an almost Dedekind domain.

**Proof.** Since every ring properly between an almost Dedekind domain and its quotient field is almost Dedekind, it suffices to prove the lemma in the case in which $S$ is the set of all of rings in $Q(D)$ of level at most $n$. By Theorems 4.7 and 4.8, the patch closure of $S$ consists of $S$ and the set $T$ of all prime divisors $V$ of $D$ of the second kind such that $V$ is the order valuation ring of a ring in $Q(D)$ of level at most $n - 1$. As the set of patch limit points of $S$, $T$ is a patch closed subset of $Q^*(D)$. By Corollary 4.9, $Q^*(D)$ is a spectral space and hence quasicompact in the patch topology [15, Theorem 1]. Therefore, as a patch closed subspace of $Q^*(D)$, $T$ is quasicompact in the patch topology. Since the patch topology is finer than the Zariski topology, $T$ is quasicompact in the Zariski topology. The intersection of rings in a Zariski quasicompact set of DVRs of a field having the property that the intersection of the maximal ideals of these DVRs is nonzero is an almost Dedekind domain [21, Corollary 5.8]. Therefore the intersection of the order valuation rings of the rings in $S$ is an almost Dedekind domain.

Corollary 4.13 need not remain valid if the restriction on the levels of the rings in $Q(D)$ is removed. For example, the intersection of the order valuations rings of
all the rings in \(Q(D)\) (i.e., the intersection of all prime divisors of the second kind) is \(D\). To see this, suppose that \(x \in F \setminus D\). If \(x^{-1} \in D\), then no prime divisor of the second kind contains \(x\). Otherwise, if \(x^{-1} \not\in D\), then no prime divisor of the second kind of \(D[x^{-1}]_{(\mathfrak{m}_D, x^{-1})D}\) contains \(x\). In either case, we find a prime divisor of the second kind not containing \(x\), from which it follows that \(D\) is the intersection of the prime divisors of the second kind. In contrast with the situation in Corollary 4.13, the set of prime divisors of the second kind of \(D\) (with no restriction on level) is not quasicompact, and hence we may not appeal to [27, Corollary 5.8] as we did in the proof of Corollary 4.13.

From Corollary 4.13 we deduce a representation theorem for rings in \(R(D)\) obtained as an intersection of rings of level at most \(n\) in \(Q(D)\).

**Corollary 4.14.** Let \(n > 0\), and let \(U\) be a nonempty subset of \(Q(D)\) consisting of rings of level at most \(n\). Then \(O_U\) is the intersection of a PID that is a localization of \(D\) and an almost Dedekind overring.

**Proof.** If \(O_U = D\), then for a prime element \(z\) of \(D\), \(O_U = D[1/z] \cap DzD\) is the intersection of a PID that is a localization of \(D\) and a DVR. Suppose that \(O_U \neq D\). Let \(A\) be the intersection of all the prime divisors of the first kind that contain \(O_U\). Since \(A \neq D\), \(A\) is a PID that is a localization of \(D\) obtained by inverting the prime elements associated to prime divisors of the first kind that do not contain \(A\). Each \(\alpha \in U\) is an intersection of prime divisors of \(D\) of the first kind and at most 2 prime divisors of \(D\) of the second kind, each of which is an order valuation ring of some \(\beta \in Q(D)\) of level less than the level of \(\alpha\), see the discussion after Remark 3.4. Let \(B\) be the intersection of these valuation rings. By Corollary 4.13, \(B\) is an almost Dedekind domain. Since \(O_U = A \cap B\), this proves Corollary 4.14. \(\square\)

### 5. The Zariski topology of \(Q(D)\)

We describe in this section the Zariski topology of \(Q(D)\) and \(Q^*(D)\) using the fact from Remark 4.4 that the Zariski closure of a subset \(S\) of \(Q^*(D)\) is the downset of the patch closure of \(S\). For the purposes of several results in this section, we need the following notation to distinguish among the patch limit points of a subset \(S\) of \(Q(D)\) those that are prime divisors of the second kind.

**Notation 5.1.** Let \(S\) be a nonempty subset of \(Q(D)\). We let \(S_\infty\) be the set of prime divisors \(V\) of the second kind of \(D\) such that \(V\) contains infinitely many incomparable rings in \(S\). By Theorem 4.7, \(S_\infty\) is precisely the set of patch limit points of \(S\) that are prime divisors of \(D\) of the second kind.

Theorem 5.2 implies that the Zariski closure of a nonempty subset \(S\) of \(Q(D)\) can be calculated from how \(S\) is situated in the partially ordered set \(Q^*(D)\).
Theorem 5.2. If $S$ is a nonempty subset of $Q(D)$, then the Zariski closure of $S$ in $Q(D)$ is $\downarrow(S \cup S_\infty)$.

Proof. By Remark 4.4, the Zariski closure of $S$ in $Q^*(D)$ is the downset of the patch closure $\overline{S}$ of $S$ in $Q^*(D)$, and so $\alpha \in Q(D)$ is in the Zariski closure of $S$ in $Q(D)$ if and only if $\alpha$ is in the downset of $\overline{S}$. Therefore to prove the theorem it suffices to show that $\downarrow\overline{S} = \downarrow(S \cup S_\infty)$. By Theorem 4.7.2, $S \cup S_\infty \subseteq \overline{S}$, and so $\downarrow(S \cup S_\infty) \subseteq \downarrow\overline{S}$.

To prove the reverse inclusion, let $\alpha \in \downarrow\overline{S}$. Then there exists $R \in \overline{S}$ such that $\alpha \subseteq R$. Since $\overline{S}$ is the union of $S$ and the set of patch limit points of $S$ in $Q^*(D)$, we have either $R \in S$, and hence $\alpha \in \downarrow S$, or $R$ is a patch limit point of $S$ in $Q^*(D)$. In the former case, we have $\alpha \in \downarrow(S \cup S_\infty)$. To see that this is also true in the latter case, suppose $R$ is a patch limit point of $S$ in $Q^*(D)$. By Theorem 4.8, $R$ is a valuation ring. If $R \in X(D)$, then by Theorem 4.7.1, the fact that $\alpha$ is a subring of $R$ implies that the local ring $\alpha$ is dominated by some member of $S$. In this case, $\alpha \in \downarrow S$. On the other hand, if $R \not\in X(D)$, then Theorem 4.7.2 implies $R$ contains infinitely many incomparable rings in $S$ and since $R \in S_\infty$. Thus $\alpha \in \downarrow(S \cup S_\infty)$. This proves that $\downarrow\overline{S} \subseteq \downarrow(S \cup S_\infty)$, which completes the proof.

Remark 5.3. A downset $S$ in $Q(D)$ need not be Zariski closed. For example, let $S = Q_1(D) \cup \{D\}$. By Theorem 4.7, the order valuation ring $V$ of $D$ is the only patch limit point of $S$. Therefore, by Theorem 5.2, the Zariski closure $\overline{S}$ of $S$ in $Q(D)$ is the set $P(D)$ of all points in $Q(D)$ that are proximate to $D$. Choose $W$ to be any rank two valuation overring of $D$ contained in $V$. By Remark 2.1.8.b there is an infinite chain of rings in $Q(D)$ contained in $W$ and hence in $\overline{S}$. Only two of these rings are in $S$, so $\overline{S} = P(D)$ is larger than the downset $S$.

We focus next on the Noetherian subspaces of $Q^*(D)$. To simplify terminology, we use the following definition.

Definition 5.4. A subset $S$ of $Q^*(D)$ is Noetherian if it is a Noetherian space in the Zariski topology.

In Theorem 5.6 we characterize the Noetherian subspaces of $Q^*(D)$. As a step in doing so, we show that the Zariski closure in $Q^*(D)$ of a Noetherian subspace of $Q^*(D)$ is also a Noetherian space. Proving that this is the case involves analyzing the irreducible components of subsets of $Q(D)$ in terms of downsets as defined in Notation 4.6.

Lemma 5.5. A nonempty Zariski closed subset $S$ of $Q^*(D)$ is irreducible if and only if $S = \downarrow^* R$ for some $R \in Q^*(D)$. Moreover, the following are equivalent for a nonempty Zariski closed subset $S$ of $Q(D)$.

1. $S$ is irreducible.
2. The Zariski closure of $S$ in $Q^*(D)$ is irreducible.
3. $S = \downarrow R$ for some $R \in Q^*(D)$. \[\text{Proof.} \text{The first assertion is a consequence of the fact that } Q^*(D) \text{ is a spectral space with respect to the Zariski topology (Corollary 149, and hence an irreducible closed set has a unique generic point.}

(1) $\Rightarrow$ (2) Suppose $S$ is an irreducible closed subset of $Q(D)$, and let $\overline{S}$ denote the Zariski closure of $S$ in $Q^*(D)$. To prove that $\overline{S}$ is irreducible, it suffices to show that the Zariski closure of each nonempty Zariski open set in $\overline{S}$ is $\overline{S}$. Let $U$ be a nonempty Zariski open set in $\overline{S}$. Since $S$ is dense in $\overline{S}$, the set $U \cap S$ is nonempty. Thus, since $S$ is irreducible, the Zariski closure of $U \cap S$ in $S$ is $S$. Consequently, the Zariski closure of $U$ in $Q^*(D)$ is $\overline{S}$.

(2) $\Rightarrow$ (3) Suppose $\overline{S}$ is irreducible. The first assertion of the theorem shows that $\overline{S} = \downarrow^* R$ for some $R \in Q^*(D)$. Now $S = \overline{S} \cap Q(D) = (\downarrow^* R) \cap Q(D) = \downarrow R$, which verifies (3).

(3) $\Rightarrow$ (1) Suppose that $S = \downarrow R$ for some ring $R \in Q^*(D)$. Let $\overline{S}$ denote Zariski closure in $Q^*(D)$. We claim that $R \in \overline{S}$. If $R \in Q(D)$, then $R \in \downarrow R \subseteq S$, and the claim is clear. Suppose $R \notin Q(D)$. Then $R$ is a valuation ring, and to show that $R \in \overline{S}$, it suffices to prove that every Zariski basic open subset of $Q^*(D)$ that contains $R$ has nonempty intersection with $S$. Let $x_1, \ldots, x_n$ be elements of the quotient field $F$ of $D$ such that $R \in U(x_1, \ldots, x_n)$. Then $x_1, \ldots, x_n \in R$. Since $R$ is a valuation ring in $Q^*(D)$, there are two cases to consider, that in which $R$ is a minimal valuation ring of $D$ and that in which $R$ is a prime divisor of $D$ of the second kind. If $R$ is a minimal valuation ring of $D$, then $R$ is a directed union of rings from $Q(D)$ by Remark 2.118.b. Thus the set $S \cap U(x_1, \ldots, x_n)$ is nonempty since $x_1, \ldots, x_n \in R$.

Suppose that $R$ is a prime divisor of $D$ of the second kind. Let $\mathfrak{M}$ denote the maximal ideal of $R$. Then $D/\mathfrak{m}_D = (D + \mathfrak{M})/\mathfrak{M} \subseteq R/\mathfrak{M}$, and $R/\mathfrak{M}$ is a finitely generated field extension $D/\mathfrak{m}_D$ of transcendence degree one. Therefore $(D[x_1, \ldots, x_n] + \mathfrak{M})/\mathfrak{M}$ is a proper subring of $R/\mathfrak{M}$, so there exists a rank one valuation of $R/\mathfrak{M}$ that contains $D/\mathfrak{m}_D$. By composite construction, there exists a minimal valuation ring $V$ of $D$ with $V \subseteq R$ such that $x_1, \ldots, x_n \in V$. Since $V$ is a directed union of rings from $Q(D)$, we conclude again that the set $S \cap U(x_1, \ldots, x_n)$ is nonempty. In all cases, $S \cap U(x_1, \ldots, x_n)$ is nonempty, which proves that $R$ is in the Zariski closure of $S$ in $Q^*(D)$.

To prove now that $S$ is irreducible, suppose $T_1$ and $T_2$ are Zariski closed subsets of $S$ such that $S = T_1 \cup T_2$, then $\overline{S} = \overline{T_1} \cup \overline{T_2}$, where $\overline{()}$ denotes Zariski closure in $Q^*(D)$. We have proved that $R \in \overline{S}$, so without loss of generality $R \in \overline{T_1}$. Thus

\[\text{If the set } S \text{ is finite, then } R \in Q(D).\]
\[ S = \downarrow R \subseteq T_1. \] Since \( T_1 \) is closed in \( S \), we have then that \( S \subseteq S \cap T_1 = T_1 \), proving that \( S = T_1 \). This shows that \( S \) is irreducible. \[ \square \]

A consequence of Theorem 5.6 is that whether a subset \( S \) of \( Q(D) \) is Noetherian can be detected from order-theoretic properties of \( Q^*(D) \).

**Theorem 5.6.** The following are equivalent for a nonempty subset \( S \) of \( Q^*(D) \).

1. \( S \) is Noetherian.
2. \( S \) has only finitely many irreducible components.
3. There are valuation rings \( V_1, \ldots, V_n \) in \( Q^*(D) \) such that \( S \subseteq \downarrow^* \{ V_1, \ldots, V_n \} \).
4. The Zariski closure of \( S \) in \( Q^*(D) \) is Noetherian.

If also \( S \subseteq Q(D) \), then items 1–4 are equivalent to

5. The Zariski closure of \( S \) in \( Q(D) \) is Noetherian.

**Proof.** (1) \( \Rightarrow \) (2) This is clear since a Noetherian space has finitely many irreducible components [30, Tag 0052].

(2) \( \Rightarrow \) (3) Let \( S_1, \ldots, S_n \) be the irreducible components of \( S \). Let \( \overline{\cdot} \) denote Zariski closure in \( Q^*(D) \). By Lemma 5.5, there exist rings \( R_1, \ldots, R_n \) in \( Q^*(D) \) such that \( \overline{S_i} = \downarrow^* R_i \) for each \( i \). Thus \( S \subseteq \downarrow \{ R_1, \ldots, R_n \} \). For each \( i \), let \( V_i \) be a valuation ring in \( Q^*(D) \) that dominates \( R_i \). Then \( S \subseteq \downarrow \{ V_1, \ldots, V_n \} \), which verifies item 3.

(3) \( \Rightarrow \) (4) Suppose there are valuation rings \( V_1, \ldots, V_n \) in \( Q^*(D) \) such that \( S \subseteq \downarrow^* \{ V_1, \ldots, V_n \} \). Then the Zariski closure of \( S \) in \( Q^*(D) \) is also a subset of the Zariski closed set \( \downarrow^* \{ V_1, \ldots, V_n \} \). Since a subspace of a Noetherian space is Noetherian, it suffices to prove that \( \downarrow^* \{ V_1, \ldots, V_n \} \) is Noetherian. Moreover, since a finite union of Noetherian subspaces is Noetherian, it suffices to show that every set of the form \( C = \downarrow^* V \), where \( V \) is a valuation ring in \( Q^*(D) \), is Noetherian.

Consider first the case that \( V \) is a minimal valuation ring of \( D \). Then \( \downarrow V \) consists of the quadratic sequence \( D = \alpha_0 \subseteq \alpha_1 \subseteq \cdots \) along \( V \). The proper Zariski closed subsets of \( C \) are precisely the sets \( \downarrow \alpha_i \). Since these sets are finite, the closed sets of \( C \) satisfy the descending chain condition. Therefore \( C \) is a Noetherian space in the Zariski topology.

Next suppose that \( V \) is not a minimal valuation ring of \( D \). To deal with this case, we prove first the following claim.

**Claim:** If \( V \) is not a minimal valuation ring, then no proper Zariski closed subset of \( C = \downarrow^* V \) contains infinitely many incomparable rings.

Let \( B \) be a Zariski closed set of \( C \) that contains infinitely many incomparable rings. We prove \( B = C \). Either \( B \) contains infinitely many incomparable rings in \( Q(D) \) or \( B \) contains infinitely many valuation rings. By Theorem 4.7.2, \( V \) is a prime
divisor of $D$ of the second kind and $V$ is a patch limit point of every infinite set of incomparable rings in $\downarrow V$. If $B$ contains an infinite set of incomparable rings in $Q(D)$, then $V$ is a patch limit point of $B$. Since $B$ is Zariski closed in $Q^*(D)$, Remark [4.4] implies $B = C$ and the claim is proved.

To complete the proof of the claim, it remains to consider the case where $B$ contains an infinite set $S$ of valuation rings in $Q^*(D)$. We may assume that each valuation ring in $S$ is a proper subring of $V$, since otherwise the claim is clear from the fact that $C = \downarrow^* V$. Since $B$ is a Zariski closed subset of $C$ containing $S$, to show that $B = C$ it suffices to show that the Zariski closure of $S$ is $C$. Let $U$ be a nonempty open subset of $C$. Since $C = \downarrow^* V$, there exist $x_1, \ldots, x_n \in V$ such that $U = C \cap U(x_1, \ldots, x_n)$. Let $\mathfrak{M}$ denote the maximal ideal of $V$. Since $V$ is a prime divisor of the second kind of $D$, $D/\mathfrak{m}_D = (D + \mathfrak{M})/\mathfrak{M} \subsetneq V/\mathfrak{M}$, and $V/\mathfrak{M}$ is a finitely generated field extension of transcendence degree one of $D/\mathfrak{m}_D$. It follows that each nonzero element in the field $V/\mathfrak{M}$ is contained in all but at most finitely many of the rank one valuation rings of the field $V/\mathfrak{M}$ that contain $(D + \mathfrak{M})/\mathfrak{M}$.

For each valuation ring $W$ in $S$, we have $(D + \mathfrak{M})/\mathfrak{M} \subseteq W/\mathfrak{M} \subseteq V/\mathfrak{M}$ and $W/\mathfrak{M}$ is a rank one valuation ring of the field $V/\mathfrak{M}$. Thus there exists a valuation ring $W \in S$ such that $x_1, \ldots, x_n \in W$. Therefore $W \in S \cap U(x_1, \ldots, x_n)$, which proves that $S$ is dense in $C$ in the Zariski topology. Hence $B = C$, and this completes the proof of the claim.

We use the claim to prove that $C = \downarrow^* V$ is a Noetherian space. It suffices to show that every proper Zariski closed subset $B$ of $C$ is a Noetherian space. If $B$ is finite, this is clear, so suppose $B$ is infinite. Since $B$ is proper and $V$ is the generic point of $C$, we cannot have $V \in B$. As a closed subset of the spectral space $Q^*(D)$ (with respect to the Zariski topology), $B$ has minimal elements with respect to the partial order $R \leq S$ iff $S$ is in the Zariski closure of $\{R\}$. The elements in $B$ minimal with respect to this partial order are precisely the rings that are maximal in $B$ with respect to set inclusion. Thus every ring in $B$ is contained in a ring in $B$ that is maximal with respect to set inclusion. Since the rings in $B$ that are maximal with respect to set inclusion are incomparable, the claim implies that there are only finitely many of them, say $R_1, \ldots, R_n$. Therefore $B = \downarrow^* \{R_1, \ldots, R_n\}$. We have already established that each Zariski closed set $\downarrow^* R_i$ is a Noetherian space since as a proper subring of $V$ each $R_i$ is contained in a minimal valuation ring. Thus $B$ is a Noetherian space since it is a finite union of Noetherian subspaces. This proves that every proper closed subset of $C$ is Noetherian. It follows that $C$ satisfies the descending chain condition on closed sets and hence $C$ is Noetherian. This proves that item 3 implies item 4.

That item 4 implies items 1 and 5 follows from the fact that a subspace of a Noetherian space is Noetherian. If $S$ is a subset of $Q(D)$, item 5 similarly implies
Item 1.

Remark 5.7. Noetherian subspaces of the Zariski-Riemann space of valuation overrings of a two-dimensional Noetherian domain are the subject of [22, 24]. See also [23, 28].

Remark 5.8. If $S$ is a nonempty Noetherian subset of $Q^*(D)$ with respect to the Zariski topology and $R$ is a flat $D$-submodule of $F$, then $(\bigcap_{\alpha \in S} \alpha)R = \bigcap_{\alpha \in S}(\alpha R)$. This is because flat submodules of the quotient field of a domain commute with intersections of rings from quasicompact sets of overrings [6, Theorem 3].

Examples presented in Theorems 6.15 and 6.17 in Section 6 show that even for Noetherian subsets $U$ of $Q^*(D)$, $O_U$ need not be a Noetherian ring. We consider in Lemma 5.9 and Theorem 5.10 restrictions on $U$ that guarantee $O_U$ is a Noetherian ring.

Lemma 5.9. If a subset $S$ of $Q^*(D)$ is an intersection of a downset and a Zariski quasicompact open set in $Q^*(D)$, then $O_S$ is a normal Noetherian domain.

Proof. Write $S = U \cap V$, where $U$ is a quasicompact open set in $Q^*(D)$ and $V$ is a downset in $Q^*(D)$. By Corollaries 4.9 and 4.9 the quasicompactness of $U$ implies there exist finite subsets $A_1, \ldots, A_n$ of $F$ such that $S = (U(A_1) \cup \cdots \cup U(A_n)) \cap V$. Let $T$ be the set of rings that are minimal with respect to inclusion in $S$. Since $V$ is a downset, the rings in $T$ are minimal in the set $U(A_1) \cup \cdots \cup U(A_n)$. Thus $O_T = O_S$. The intersection of finitely many normal Noetherian overrings of $D$ is a Krull domain and hence by Remark 1.1 is a normal Noetherian ring. Therefore to verify the theorem it suffices to prove that $O_S$ is a Noetherian ring in the case where $S = U(A_1)$.

Assuming $S = U(A_1)$, the rings in $T$ are minimal in $Q(D)$ with respect to containing $D[A_1]$. Write $A_1 = \{x_1, \ldots, x_t\}$. Then $D[A_1]$ is the coordinate ring of an affine component of the projective model $X$ defined by $x_1, \ldots, x_t$. Since the rings in $T$ are elements of $Q(D)$ that are minimal with respect to dominating closed points in $X$, these points are also closed points in the minimal desingularization $X'$ of $X$ [14, Theorem 5.3]. This proves that $T$ is a set of closed points in a nonsingular projective model over $D$. By [14, Theorem 7.4], $O_T$ is a Noetherian normal ring.

Theorem 5.10. Let $V$ be a Noetherian downset in $Q^*(D)$. Then for every Zariski open subset $U$ of $V$, the ring $O_U$ is a Noetherian normal domain.

Proof. Let $U$ be a Zariski open subset of $V$, and let $U'$ be a Zariski open subset of $Q^*(D)$ such that $U = U' \cap V$. By Corollary 4.9 $Q^*(D)$ is a spectral space, so $U'$ is a union of Zariski quasicompact open subsets of $Q^*(D)$, say $U' = \bigcup_i U_i$, where each
\( U_i \) is a Zariski quasicompact open subset of \( Q^*(D) \). Then \( \{ V \cap U_i \} \) is an open cover of \( U \). Since \( V \) is a Noetherian subset of \( Q^*(D) \), every subset of \( V \) is quasicompact in the Zariski topology, and so \( U \) has a finite subcover in \( \{ V \cap U_i \} \). Since the union of finitely many quasicompact subspaces is quasicompact, we conclude that \( U = V \cap W \) for some quasicompact open subset \( W \) of \( Q^*(D) \). By Lemma 5.9 \( \mathcal{O}_U \) is a Noetherian normal domain.

**Corollary 5.11.** Let \( \alpha \in Q(D) \). The subset \( P(\alpha) \) of \( Q(D) \) of points proximate to \( \alpha \) is a Noetherian subset of \( Q(D) \) such that for each Zariski open subset \( U \) of \( P(\alpha) \), \( \mathcal{O}_U \) is a Noetherian normal domain.

**Proof.** If \( V \) is the order valuation ring of \( \alpha \), then \( P(\alpha) \subseteq \downarrow V \), so \( P(\alpha) \) is Noetherian by Theorem 5.6. Since \( P(\alpha) \) is a downset of \( Q^*(\alpha) \), the second claim now follows from Theorem 5.10 applied to \( Q^*(\alpha) \).

**Remark 5.12.** We may view \( \mathcal{O} \) as a presheaf on \( Q^*(D) \) with respect to either the patch or Zariski topologies. In either topology, the stalks of the presheaf \( \mathcal{O} \) are the rings in \( Q^*(D) \).

1. Working with the patch topology on \( Q^*(D) \), we have by Lemma 5.9 that for each patch clopen set \( U \), the ring of sections \( \mathcal{O}_U \) of \( U \) is Noetherian.

2. Since \( \mathcal{O} \) is defined on the empty set to be the field \( F \), \( \mathcal{O} \) is not a sheaf with respect to either the patch or Zariski topologies of \( Q^*(D) \) because for \( \mathcal{O} \) to be a sheaf would require \( \mathcal{O} \) to be the zero ring on the empty set. Even with the modification that \( \mathcal{O} \) is defined on the empty set to be the zero ring, \( \mathcal{O} \) is not a sheaf. This follows from the fact that \( Q^*(D) \) is not an irreducible space in either the Zariski or patch topologies.

3. By restricting to \( P(\alpha) \), where \( \alpha \in Q(D) \), we obtain that the modification to the definition of \( \mathcal{O} \) as discussed in item 2 produces a sheaf on \( P(\alpha) \) with respect to the Zariski topology, since by Lemma 5.5 \( P(\alpha) \) is irreducible in the Zariski topology. Therefore \( P(\alpha) \) with structure sheaf \( \mathcal{O} \) is a locally ringed Noetherian space for which the ring of sections of each open set is a Noetherian ring. By appending the order valuation of \( \alpha \) to the set \( P(\alpha) \), we obtain a locally ringed spectral Noetherian space whose rings of sections are Noetherian. This locally ringed space is not a scheme since the stalks of the structure sheaf have localizations that do not appear as other stalks.

**6. Examples**

The purpose of this section is to present examples that illustrate the range of behavior of intersections of rings in subsets of \( Q(D) \). We use the following terminology.
Definition 6.1. A subset $U$ of $Q(D)$ is said to be complete if $O_U = \bigcap_{R \in U} R = D$. For a point $\alpha \in Q(D)$, a subset $U$ of $Q(D)$ of points that dominate $\alpha$ is said to be complete over $\alpha$ if $O_U = \alpha$.

Remark 6.2. Let $J$ be an integrally closed ideal of $D$. The set $U$ of closed points of the projective model $X = \text{Proj } D[Jt]$ over $D$ is complete over $D$, since every valuation ring $V \in X(D)$ is centered on a closed point of $X$, and $D$ is the intersection of the valuation rings in $X(D)$. We are identifying $\text{Proj } D[Jt]$ with the set of local rings obtained by homogeneous localization of the graded domain $D[Jt]$ at its relevant homogeneous prime ideals. Therefore $D = \bigcap_{\alpha \in U} \alpha$. It is natural to ask if $D$ is the intersection of the rings in a proper subset of $U$. This is equivalent to asking if the representation $D = \bigcap_{\alpha \in U} \alpha$ is irredundant. If $D$ is Henselian and $X$ is nonsingular, then this representation is irredundant; see Remark 6.2.

Example 6.3. Assume Notation 3.1. There exist 2-dimensional normal Noetherian local overrings $R$ and $S$ of $D$ such that $D = R \cap S$ and $R$ and $S$ both properly contain $D$. Let $R = D[y^2/x](x,y,y^2/x)D[y^2/x]$ and $S = D[x^2/y](x,y,x^2/y)D[x^2/y]$. Then $R$ and $S$ both properly contain $D$. Since $D$ is the intersection of valuation domains $V$ centered on $m_D$, to prove that $D = R \cap S$, it suffices to prove that each such valuation domain $V$ has the property that either $R \subseteq V$ or $S \subseteq V$. Since $V$ has center $m_D$ on $D$, both $xV$ and $yV$ are contained in $m_V$. If $y^2V \subseteq xV$, then $R \subseteq V$. On the other hand, if $xV \subseteq y^2V$, then $x^2V \subseteq xV \subseteq y^2V \subseteq yV$ implies that $x^2V \subseteq yV$ and $S \subseteq V$. Therefore $D = R \cap S$.

Discussion 6.4. Assume Notation 3.1 and let $\gamma \in Q_1(D)$. It observed in [14, Theorem 8.3] that the set $Q_1(D) \setminus \{\gamma\}$ is not complete. Let $Q_1(\gamma)$ denote the points in $Q_2(D)$ that dominate $\gamma$. Then $\gamma$ is the intersection of the points in $Q_1(\gamma)$. Hence $U = Q_1(\gamma) \cup (Q_1(D) \setminus \{\gamma\})$ is a complete subset of $Q(D)$. We are interested in describing subsets $\Lambda$ of $Q_1(\gamma)$ such that $\Lambda \cup (Q_1(D) \setminus \{\gamma\})$ is complete.

Example 6.5. Assume Notation 3.1 and assume that $D$ has an algebraically closed coefficient field $k$. Let

$$\beta = D[x, y]/(y^3 - D[y^2/x]) \in Q_1(D),$$

and consider the nonsingular projective model $X$ over $\text{Spec } D$ gotten by blowing up $m_D$ and then blowing up $\beta$. The Zariski theory [32, Appendix 5] or [16] implies that $X = \text{Proj } D[Jt]$, where $J = (x, y)(x, y^2) = (x^2, xy, y^3)D$. The model $X$ is obtained

\[\text{This is precisely the infinitely near points in the first neighborhood of } \gamma.\]
from $\text{Proj } D[xt, yt]$ by removing the point $\beta$ and adding the projective line $Q_1(\beta)$. The set of closed points of $\text{Proj } D[Jt]$ is

$$
U := (Q_1(D) \setminus \{\beta\}) \cup Q_1(\beta).
$$

As in Discussion 6.4, the set $U$ is complete and so $\mathcal{O}_U = D$. To show this representation is irredundant, we use:

**Fact 6.6.** To show an element $\delta \in U$ is irredundant in the representation $D = \bigcap_{R \in U} R$, it suffices to show that there exists an essential valuation ring $V$ of $D$ such that $\delta \subset V$ and no other element of $U$ is contained in $V$ [14, Lemma 8.5].

The points in $Q_1(D) \setminus \{\beta\}$ are the localizations of $D[\frac{y}{x}]$ at maximal ideals containing $xD[\frac{y}{x}]$. Since $k$ is an algebraically closed coefficient field for $D$, these maximal ideals have the form

$$
m_b = (x, \frac{y}{x} - b)D[\frac{y}{x}], \quad \text{where } b \in k.
$$

For each $b \in k$, let $p_b = (y - bx)D$, let $V_b = D_{p_b}$, and let $R_b = D[\frac{y}{x}]_{m_b}$. Then the set

$$
\{R_b\}_{b \in k} = Q_1(D) \setminus \{\beta\}
$$

and for each $b \in k$, we have $R_b \subset V_b$, and $R_b$ is the unique point in $Q_1(D)$ that is contained in $V_b$. Since $\beta$ is not contained in $V_b$, no point of $Q_1(\beta)$ is contained in $V_b$. Hence $R_b$ is the unique point of $U$ that is contained in $V_b$.

Let $x_1 = \frac{y}{x}$. Then $(x_1, y)\beta$ is the maximal ideal of $\beta$. The points in $Q_1(\beta)$ are the localizations of $\beta[\frac{x_1}{y}]$ at maximal ideals containing $y\beta[\frac{x_1^2}{y}]$, and the localization of $\beta[\frac{y}{x_1}]$ at the maximal ideal $(x_1, \frac{y}{x_1})\beta[\frac{x_1^2}{y}]$. Let $\gamma$ denote the localization of $\beta[\frac{y}{x_1}]$ at the maximal ideal $(x_1, \frac{y}{x_1})\beta[\frac{x_1^2}{y}]$.

The maximal ideals of $\beta[\frac{x_1}{y}]$ containing $y$ have the form

$$
n_b = (y, \frac{x_1}{y} - b)\beta[\frac{x_1}{y}], \quad \text{where } b \in k.
$$

For each $b \in k$, let $q_b = (x - by^2)D$, let $W_b = D_{q_b}$, and let $S_b = \beta[\frac{x_1}{y}]_{m_b}$. Then the set

$$
\{S_b\}_{b \in k} = Q_1(\beta) \setminus \{\gamma\}
$$

and for each $b \in k$, we have $S_b \subset W_b$. Since $D/q_b$ is a DVR, $S_b$ is the unique point of $U$ contained in $W_b$.

To show the representation $D = \mathcal{O}_U$ is irredundant, by Fact 6.6, it remains to show there exists an essential valuation ring $V$ of $D$ such that $\gamma \subset V$ and $\gamma$ is the only point of $U$ contained in $V$. Let $p = (x^2 - y^3)D$ and let $V = D_p$. Since the integral closure of $D/p$ is local, $V$ contains at most one point of $U$.

Since $x = yx_1$, the transform of $p$ in $D[x_1]$ is $(x_1^2 - y)D[x_1]$. Let $y_1 = \frac{y}{x_1}$, then $y = x_1y_1$ and the transform of $(x_1^2 - y)D[x_1]$ in $\beta[y_1]$ is $(x_1 - y_1)\beta[y_1]$. Since $(x_1, y_1)\gamma = m_\gamma$, we have $\gamma \subset V$. Therefore the representation $D = \mathcal{O}_U$ is irredundant.
Remark 6.7. With the notation of Example 6.5 and so \( y_1 = \frac{x^2}{x} \), it is shown in [14, Example 7.6] that \( R = D[y_1(x,y,y_1)]D[y_1] \) is a 2-dimensional normal Noetherian local domain, and \( \text{Proj } D[\mathcal{I}] \) is the minimal desingularization of \( R \).

Theorem 6.8 establishes that every minimal valuation overring of \( D \) is a localization of a ring in \( \mathcal{R}(D) \). Theorem 6.17 and Theorem 6.8 both describe subsets \( \mathcal{U} \) of \( Q(D) \) for which the rings in \( \mathcal{U} \) are incomparable and \( \mathcal{O}_\mathcal{U} \) is not an almost Krull domain.

**Theorem 6.8.** Let \( V \) be a minimal valuation overring of \( D \). There exists a subset \( \mathcal{U} \) of \( Q(D) \) such that that \( V \) is a localization of \( C = \mathcal{O}_\mathcal{U} \). In particular, if \( V \) is chosen not to be a DVR, then \( C \) is not an almost Krull domain.

**Proof.** Let \( D = \alpha_0 \subset \alpha_1 \subset \alpha_2 \subset \cdots \) be the quadratic sequence determined by \( V \) as in Remark 2.18. For each integer \( i \geq 1 \) let \( \beta_i \) be a first level quadratic extension of \( \alpha_i \), different from \( \alpha_{i+1} \), and let \( C = \bigcap_{i=1}^\infty \beta_i \). Thus \( C = \mathcal{O}_\mathcal{U} \), where \( \mathcal{U} = \{ \beta_i \}_{i=1}^\infty \). We observe that \( C \) is the following directed union. Since \( \alpha_i \subset \beta_i \) for each \( i \), we have

\[
\alpha_1 \subseteq (\beta_1 \cap \alpha_2) \subseteq (\beta_1 \cap \beta_2 \cap \alpha_3) \subseteq \cdots \subseteq C.
\]

For each integer \( n \geq 1 \), let \( C_n := (\bigcap_{i=1}^n \beta_i) \cap \alpha_{n+1} \). Since \( \alpha_{n+1} \subseteq \bigcap_{j>n} \beta_j \), we have \( \bigcup_{n=1}^\infty C_n = C \). Corollary 6.5 of [14] implies that \( C_n \) is a Noetherian regular domain with precisely \( n+1 \) maximal ideals all of height 2, and the localizations of \( C_n \) at its maximal ideals are \( \beta_1, \ldots, \beta_n \) and \( \alpha_{n+1} \).

Let \( \mathfrak{m}_V \) denote the maximal ideal of \( V \). Then \( \mathfrak{m}_V \cap \alpha_i = \mathfrak{m}_i \) is the maximal ideal of \( \alpha_i \), and \( \mathfrak{m}_V = \bigcup_{i=1}^\infty \mathfrak{m}_i \).

Let \( \mathfrak{p}_n = \mathfrak{m}_V \cap C_n \). Then \( (C_n)_{\mathfrak{p}_n} = \alpha_{n+1} \). Since this holds for all \( n \), we have \( V = \bigcup \alpha_n = \bigcup (C_n)_{\mathfrak{p}_n} \). Let \( \mathfrak{p} = \mathfrak{m}_V \cap C \) denote the center of \( V \) on \( C \). It follows that \( C_\mathfrak{p} = V \). If \( V \) is not a DVR, then \( C \) is not an almost Krull domain.

We observe in Remark 6.9 that the set \( \mathcal{U} \) in Theorem 6.8 is not Noetherian.

**Remark 6.9.** Let \( \mathcal{U} = \{ \beta_i \}_{i=1}^\infty \), and the valuation ring \( V = \bigcup_{n \geq 0} \alpha_n \) be as in Theorem 6.8. Each \( \beta_n \) is a point distinct from \( \alpha_{n+1} \) in the first neighborhood \( Q_1(\alpha_n) \) of \( \alpha_n \). Since \( V \) dominates \( \alpha_{n+1} \), \( V \) does not dominate \( \beta_n \).

Each \( \beta_i \) is a maximal element of \( \mathcal{U} \) and a maximal element of the down set \( \downarrow \mathcal{U} \). By construction, \( \alpha_n \in \downarrow \mathcal{U} \) for all \( n \). Theorem 4.7.1 implies that \( V \) is a patch limit point of \( \mathcal{U} \).

The rings \( \beta_n \) are incomparable, that is \( \beta_i \subseteq \beta_j \) implies \( i = j \). If \( W \) is a minimal valuation overring of \( D \), then at most one of the \( \beta_n \) is dominated by \( W \).

Notice that \( \downarrow \mathcal{U} = \mathcal{U} \cup \{ \alpha_n \}_{n=0}^\infty \). Let \( W \in X(D) \) with \( W \neq V \). Then only finitely many of the \( \alpha_n \) are contained in \( W \). Since the \( \beta_i \) are incomparable, no more than
one of the $\beta_i$ is dominated by $W$. Since $W \in X(D)$, for each integer $n \geq 1$ there
is a unique ring $\gamma_n \in Q_n(D)$ that is contained in $W$ and $W$ then dominates $\gamma_n$.
Therefore no more than one of the $\beta_i$ is contained in $W$. By Theorem 4.7.1, $W$ is
not a patch limit point of $U$.

Let $W$ be a prime divisor of the second kind on $D$, and let $\gamma \in Q(D)$ be such
that $W = \text{ord}_\gamma$. We consider two cases:

1. Assume $\gamma$ is contained in $V$. Then $\gamma$ is dominated by $V$ and $\gamma = \alpha_n$ for some
   $n$. Since $Q_1(\alpha_n) \cap \downarrow U$ is finite, Theorem 4.7.2 implies that $W$ is not a patch
   limit point of $U$.

2. Assume $\gamma$ is not contained in $V$. Then $\alpha_n$ does not dominate $\gamma$, and $\beta_n \in
   Q_1(\alpha_n)$ implies $\beta_n$ does not dominate $\gamma$. Hence $\downarrow U \cap Q_1(\gamma)$ is empty and by
   Theorem 4.7.2, $W$ is not a patch limit point of $U$.

We conclude that $V$ is the unique patch limit point of $U$. Since the $\beta_n$ are
maximal elements of $U$ and are not in $V$, the set $U$ has infinitely many irreducible
components and is not Noetherian.

**Remark 6.10.** Corollary 4.10 implies that every infinite subset of $Q(D)$ has at least
one patch limit point in $Q^*(D)$. Remark 6.9 illustrates that a subset $U$ of $Q(D)$ with
one patch limit point may also have infinitely many finite irreducible components.

**Example 6.11.** Assume Notation 3.1. With $k$ an algebraically closed field of char-
acteristic zero and $D$ the localized polynomial ring $k[x, y](x, y)$, Shannon in [29, Ex-
ample 3.9, p.306] describes an infinite strictly ascending chain

$$D \subset R_2 \subset R_3 \subset \cdots \subset R_n \subset \cdots$$

of 2-dimensional normal Noetherian local domains, where

$$R_2 = D[y^2/x, y^3/x^2],$$

$R_3$ is the localization of $D[y^2/x, y^3/x^2]$ at the maximal ideal

$$(x, y, y^2/x, y^3/x^2)D[y^2/x, y^3/x^2],$$

and $R_n$ is the localization of $D[y^2/x, y^3/x^2, \ldots, y^n/x^n-1]$ at the maximal ideal

$$(x, y, y^2/x, \ldots, y^n/x^n-1, x^n-1),$$

for $n \geq 3$. Each of the rings $R_n$ is dominated by $\alpha = D[y/x]D[y/x] \in Q_1(D)$.

Since each of the rings $R_n$ is a 2-dimensional normal Noetherian local domain, each $R_n \in R(D)$ by [14, Theorem 7.4]. Let $S = \bigcup_{n=2}^\infty R_n$. Then $S$ is a 2-dimensional
normal local domain dominated by $\alpha$. Since $y/x \notin S$, we have $S \subseteq \alpha$. Since 
$y(x/y)^n \in S$ for all $n \geq 1$, the element $y/x$ is almost integral over $S$, and $S$ is not 
completely integrally closed. However, a ring in $R(D)$ is an intersection of 
completely integrally closed domains and is therefore completely integrally closed. 
This implies that $S$ is not in $R(D)$.

We use Setting 6.12 in Theorems 6.15 and 6.17.

Setting 6.12. Assume Notation 3.1, and assume that $D$ has an algebraically closed 
coefficient field $k$. Let $\text{ord}_D$ denote the order valuation ring of $D$, and define 

$$A := D[\{\frac{y^2}{x + ay} \mid a \in k\} \cup \{\frac{x^2}{y}\}].$$

Then $A \subset \text{ord}_D$ and $\text{ord}_D$ is centered on the maximal ideal $m$ of $A$, where 

$$m := (x, y, \frac{y^2}{x + ay} \mid a \in k) \cup \{\frac{x^2}{y}\}A.$$ 

Define $C := A_m$. We prove in Proposition 6.13 that $C$ is an infinite directed union 
of 2-dimensional normal Noetherian local domains. Therefore $C$ is integrally closed. 

For each $a \in k$, define 

$$\alpha_a := D[\frac{x + ay}{y}]_{(y, x^2, y^2, y^2)}^n \subset Q_2(D) \quad \text{and} \quad \alpha' := D[\frac{y}{x}]_{(x, \frac{y}{x})}^n \subset Q_2(D).$$

For each $a \in k$, define 

$$\beta_a := \alpha_a[\frac{y^2}{x + ay} \mid (x + ay, y^2, x^2 + ay)] \subset Q_3(D) \quad \text{and} \quad \beta' := \alpha'[\frac{y}{x}]_{(x, \frac{y}{x})}^n \subset Q_3(D).$$

Let $B = \bigcap_{a \in k} \beta_a$, and let $C' = B \cap \beta'$

Proposition 6.13. The local ring $C$ defined in Setting 6.12 is an infinite directed union 
of normal Noetherian local domains.

Proof. Let $n$ be a positive integer and let $S = \{a_1, \ldots, a_n\}$ be a set of $n$ distinct 
elements of $k$. Define the ring $R_S$ to be the localization of $D[\frac{x^2}{y}, \{\frac{y^2}{x + a_iy}\}_{i=1}^n]$ at 
the maximal ideal generated by $(x, y, \frac{x^2}{y}, \{\frac{y^2}{x + a_iy}\}_{i=1}^n)$. Each of the ideals $(y, x^2)D,$ 
$(x + a_1y, y^2)D, \ldots, (x + a_ny, y^2)D$ is a simple complete ideal. Let $J$ denote the 
product of these $n + 1$ complete ideals. Since $D$ is a 2-dimensional regular local ring, $J$ is a complete ideal, and $R_S$ is the local ring on $\text{Proj} D[Jt]$ dominated by $\text{ord}_D$. Therefore $R_S$ is a normal Noetherian local domain that is dominated by $C$.

The rings $R_S$ obtained as we vary over finite sets of distinct elements of $k$ form 
a directed family of normal Noetherian local domains dominated by $C$, and $C$ is the 
directed union of this family of rings. Remark 6.14 implies that for $S_1 \subsetneq S_2$, the 
associated ring $R_{S_1} \subsetneq R_{S_2}$. 

30
**Remark 6.14.** Let \( R := R_S \) be as defined in the proof of Proposition 6.13. The minimal desingularization of \( \text{Spec} R \) is \( \text{Proj} R[\mathfrak{m}_R] \). All but \( n + 1 \) of the 2-dimensional regular local rings in \( \text{Proj} R[\mathfrak{m}_R] \) are local rings in \( Q_1(D) \), the elements in \( Q_1(D) \) that are missing are \( \alpha' = D_[(x,y)](x,y) \) and \( \alpha_i = D_[(x+a)y^{-i}](y,x+a,y) \), \( 1 \leq i \leq n \).

The 2-dimensional regular local rings in \( \text{Proj} R[\mathfrak{m}_R] \) that are not in \( Q_1(D) \) are in \( Q_2(D) \) and are \( \beta' = \alpha'[(2, x^2, y)](x, \sqrt{a}) \) and \( \beta_i = \alpha_i[(2, x^2, y)](y, x+a, y) \), \( 1 \leq i \leq n \).

**Theorem 6.15.** Assume Setting 6.12, and let \( U = \{ \beta_a \mid a \in k \} \). Then \( B = \mathcal{O}_U = \bigcap_{a \in k} \beta_a \) is a non-Noetherian almost Krull domain, and \( U \) defines an irredundant essential representation of \( B \).

**Proof.** Observe that \( D_[(x,y)] \subseteq B \). Let \( a, b \in k \). Then \( x^2 + ay \in \beta_b \iff x^2 \in (x+ay)\beta_b \).

If \( a = b \) this follows because \( x^2 + ay \in \beta_a \). Assume \( a \neq b \). To compute the transform of \( x + ay \) in \( \alpha_b = D_[(x,y)](x, \frac{x}{y}, b) \), let \( x = \frac{x}{y} \). Then \( x = yx_1 \) and \( x + ay = yx_1 + ay = y(x_1 + a) \). Since \( a \neq b \), \( x_1 + a \) is a unit of \( \alpha_b \) and hence also a unit of \( \beta_b \). Therefore \( x^2 \in (x + ay)\beta_b \) also in this case. It follows that \( D_[(x,y)](\{ (x^2) \mid a \in k \}) \subseteq B \).

Let \( q_a = (\frac{x+ay}{y})\beta_a \). Then \( q_a \) is a height 1 prime of \( \beta_a \) and \( (\beta_a)_{q_a} \) is the order valuation ring of \( \alpha_a \). The image of \( \frac{x^2}{x+ay} \) in the field of fractions of \( \beta_a/q_a \) is transcendental over \( k \). Hence \( p_a := q_a \cap B \) is a height 1 prime ideal of \( B \) and \( B_{p_a} = (\beta_a)_{q_a} \).

Since \( p_a \cap D = \mathfrak{m}_D \) for every \( a \in k \), the elements in \( \mathfrak{m}_D \) are in infinitely many height 1 primes of \( B \). It follows that \( B \) is not a Krull domain and is not Noetherian.

The center of \( \beta_a \) on \( B \) is the maximal ideal \( (\frac{x+ay}{y}, \frac{x^2}{x+ay})B \). Since \( \beta_a \) dominates \( \alpha_a \) for each \( a \in k \), the element \( \frac{x+ay}{y} \) is a unit in \( \beta_b \) for each \( b \in k \) with \( b \neq a \). It follows that the representation \( B = \bigcap_{a \in k} \beta_a \) is irredundant.

The center on \( B \) of the order valuation ring \( \text{ord}_D \) of \( D \) is the ideal

\[
p = (x, y, \{ \frac{y^2}{x+ay} \mid a \in k \})B.
\]

Then \( p \) is a nonmaximal prime of \( B \) and \( B_p = \text{ord}_D \).

Since \( \beta_a = B(\frac{x+ay}{y}, \frac{x^2}{x+ay})B \) for each \( a \in k \), each \( \beta_a \) is essential in the representation \( B = \bigcap_{a \in k} \beta_a \). \( \square \)

**Remark 6.16.** The prime divisors of the 2nd kind for \( D \) that are centered on height 1 primes of \( C \) are the order valuation rings of the \( \alpha_a \) and the order valuation ring of \( \alpha' \).

Let \( p_a = (\frac{x+ay}{y})\beta_a \). Then \( p_a \) is a height 1 prime of \( \beta_a \) and \( (\beta_a)p_a \) is the order valuation ring of \( \alpha_a \). The image of \( \frac{y^2}{x+ay} \) in the field of fractions of \( \beta_a/p_a \) is transcendental over \( k \). Hence \( p_a \cap C \) is a height 1 prime ideal of \( C \) and \( C_{p_a \cap C} = (\beta_a)p_a \).
Let $p' = (\frac{y}{x})\beta'$. Then $p'$ is a height 1 prime of $\beta'$ and $\beta'_p$, is the order valuation ring of $\alpha'$. The image of $\frac{x^2}{y}$ in the field of fractions of $\beta'/p'$ is transcendental over $k$. Hence $p' \cap C$ is a height 1 prime ideal of $C$ and $C_{p' \cap C} = (\beta'_p)$.

The height 1 prime $p$ of $B$ such that $B_p = \text{ord}_D$ has the property that $p \cap C = m_C$, the maximal ideal of $C$. If $q$ is a height 1 prime of $B$ such that $q \neq p$, then $q \cap C$ is a height 1 prime of $C$ and $C_{(q \cap C)} = B_q$.

Theorem 6.17 describes a non-Noetherian local domain in $\mathcal{R}(D)$.

**Theorem 6.17.** Assume notation as in Setting 6.12 and in Theorem 6.15. Then $C$ is a non-Noetherian local domain in $\mathcal{R}(D)$. Indeed, $C = C' = \bigcap_{a \in k} \beta_a \cap \beta' = B \cap \beta'$.

**Proof.** It is established in Theorem 6.15 that $C \subseteq B$. To see that $C \subseteq C'$, we show that $\frac{x^2}{x+ay} \in \beta'$. Consider $(x+ay)\alpha'$ and set $y_1 = \frac{y}{x}$. Then $x+ay = x+axy_1 = x(1+ay_1)$. Since $1+ay_1$ is a unit of $\alpha'$ and $y^2 \in x\alpha'$, it follows that $y^2 \in (x+ay)\beta'$.

Therefore, for each $a \in k$, $\frac{x^2}{x+ay} \in B \cap \beta'$.

It is clear that $\frac{x^2}{y} \in \beta'$, and $\frac{x^2}{y} \in \beta_a \iff x^2 \in y\beta_a$. Since $y_1 = \frac{y}{x} \in \alpha_a$, we have $x^2 \in y\alpha_a = xy_1\alpha_a$ because $x \in y_1\alpha_a$. Then $\alpha_a \subseteq \beta_a$ implies that $C \subseteq \bigcap_{a \in k} \beta_a \cap \beta'$.

It remains to show that $C' = B \cap \beta' \subseteq C$. To prove this, it suffices to prove that each minimal valuation overring of $C$ contains $C'$. A minimal valuation overring $V$ of $C$ dominates $C$ by Remark 2.1.5. Therefore $V$ also dominates $D$.

Let $V$ be a minimal valuation overring of $C$, and let $v$ denote a valuation associated to $V$. Then $v(x) > 0$ and $v(y) > 0$. Since $x^2/y$ and $y^2/x$ are in the maximal ideal of $C$, we also have $v(x^2) > v(y)$ and $v(y^2) > v(x)$.

If $v(x) > v(y)$, then $D[\frac{x}{y}, \frac{y^2}{x}]_{[x,y]} \subseteq V$. Since $y^2/x \in C$, it follows that $D[\frac{x}{y}, \frac{y^2}{x}]_{[x,y]} \subseteq V$. Therefore $C' \subseteq V$ in this case.

If $v(y) > v(x)$, then $D[\frac{y}{x}, \frac{x^2}{y}]_{[y,x]} \subseteq V$. Since $x^2/y \in C$, it follows that $D[\frac{y}{x}, \frac{x^2}{y}]_{[y,x]} \subseteq V$. Therefore $C' \subseteq V$ in this case.

If $v(x) = v(y)$, then $D[\frac{y}{x}] \subseteq V$. Since $D[\frac{y}{x}]xD[\frac{y}{x}] = \text{ord}_D$ and $\text{ord}_D$ contains $C'$, we may assume that $V \neq \text{ord}_D$. Then $V$ is centered on a maximal ideal of $D[\frac{y}{x}]$ that lies over $m_D$. Since $D$ has an algebraically closed coefficient field $k$, it follows that there exists $a \in k$ such that $v(x+ay) > v(x) = v(y)$. Therefore

$$\alpha_a = D[\frac{x+ay}{y}]_{(y, \frac{x+ay}{y})} \subseteq V$$

and $V$ dominates $\alpha_a$. Then $y^2/(x+ay) \in m_C \subseteq m_V$ implies that

$$\beta_a = \alpha_a[\frac{y^2}{x+ay}]_{(\frac{x^2+ay^2}{x+ay})} \subseteq V.$$  

Therefore $C' \subseteq V$ also in this case. This completes the proof that $C = C'$. \qed
Remark 6.18. The infinitely near points $\beta$ and $\beta'$ of Setting 6.12 are all proximate to $D$. Hence the ring $B$ of Theorem 6.15 and the ring $C$ of Theorem 6.17 are defined by Noetherian subsets of $Q(D)$.

The non-Noetherian rings $R \in \mathcal{R}(D)$ that we have described have infinitely many height one primes that contain $m_D$. This motivates us to ask:

Question 6.19. Let $R \in \mathcal{R}(D)$. If Spec $R$ is a Noetherian topological space, does it follow that $R$ is a Noetherian ring?

Comments 6.20. Let $R = \mathcal{O}_U$, where $U \subset Q(D)$ is as in Remark 3.3 that is, the rings $\alpha \in U$ are incomparable and are minimal in $Q(D)$ with respect to containing $R$.

Let $\mathcal{E} = \{ V \mid V$ is an essential valuation ring for some $\alpha \in U \}$. It is clear that $R = \bigcap \{ V \mid V \in \mathcal{E} \}$. Each $V \in \mathcal{E}$ is either an essential valuation for $D$ or a prime divisor of the second kind on $D$. If $V$ is a prime divisor of the second kind on $D$, then $m_D$ is contained in the center of $V$ on $R$. Related to Question 6.19 we have:

1. There exist Noetherian subspaces $U$ of $Q(D)$ for which the ring $R = \mathcal{O}_U$ fails to have Noetherian spectrum. The ring $B$ in Theorem 6.15 and the ring $C$ in Theorem 6.17 both fail to have Noetherian spectrum, and both are defined by Noetherian subspaces of $Q(D)$.

2. If there are only finitely many $V \in \mathcal{E}$ that are of the second kind on $D$, then the set $\mathcal{E}$ has finite character and $R$ is a Krull domain, so $R$ is Noetherian.

3. If Spec $R$ is Noetherian, then $m_D$ is contained in only finitely many height one primes of $R$.

4. Each $W \in \mathcal{E}$ that is irredundant in the representation $R = \bigcap \{ V \mid V \in \mathcal{E} \}$ is a localization of $R$ and thus is centered on a height one prime of $R$, cf. [21, Theorem 5.1, p. 330]. Therefore Spec $R$ is Noetherian implies only finitely many $W \in \mathcal{E}$ of the second kind on $D$ are irredundant in the representation $R = \bigcap \{ V \mid V \in \mathcal{E} \}$.

Remark 6.21. As a generalization of the situation considered in this paper and in [14], let $A$ be a 2-dimensional Noetherian regular integral domain\footnote{For example, $A$ could be a polynomial ring in 2 variables over a field.} and let $Q(A)$ denote the “quadratic tree” of 2-dimensional regular local overrings of $A$. The set $Q(A)$ is partially ordered with respect to inclusion and has properties similar to those used in this paper and in [14].

Let $\mathcal{R}(A)$ denote the family of rings obtained as intersections of rings in $Q(A)$. It would be interesting to examine topological properties of $Q(A)$, and to examine the structure of rings in the set $\mathcal{R}(A)$.
References

[1] S. S. Abhyankar, *On the valuations centered in a local domain*, Amer. J. Math., vol 78 (1956) 321-348.

[2] S. S. Abhyankar, *Lectures on Algebra I*, World Scientific, 2006.

[3] C. Finocchiaro, M. Fontana and K. A. Loper, Ultrafilter and constructible topologies on spaces of valuation domains, Comm. Alg. 41 (2013), 1825–1835.

[4] C. Finocchiaro, M. Fontana and K. A. Loper, The constructible topology on spaces of valuation domains, Trans. Amer. Math. Soc. 365 (2013), 6199–6216.

[5] C. Finocchiaro, M. Fontana and D. Spirito, New distinguished classes of spectral spaces: a survey. Multiplicative ideal theory and factorization theory, 117–143, Springer Proc. Math. Stat., 170, Springer, 2016.

[6] C. Finocchiaro and D. Spirito, Topology, intersections and flat modules, Proc. Amer. Math. Soc. 144 (2016), 4125–4133.

[7] L. Guerrieri, W. Heinzer, B. Olberding and M. Toeniskoetter, Directed Unions of Local Quadratic Transforms of Regular Local Rings and Pullbacks, in Rings, Polynomials, and Modules, Springer, 2017, 257–280.

[8] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag New York, 1977.

[9] W. Heinzer, On Krull overrings of a Noetherian domain, Proc. Amer. Math. Soc. 22 (1969), 217–222.

[10] W. Heinzer, Noetherian intersections of integral domains II, Conference in Commutative Algebra, *Lecture Notes in Mathematics* 311, Springer-Verlag, New York (1973), 107-119.

[11] W. Heinzer, B. Johnston, D. Lantz, and K. Shah, Coefficient ideal in and blowups of a commutative Noetherian domain, J. Algebra, 162 (1993), 355–391.

[12] W. Heinzer, Y. Kim, M Toeniskoetter, Blowing up finitely supported complete ideals in a regular local ring, J. Algebra 458 (2016), 364–386.

[13] W. Heinzer, K. A. Loper, B. Olberding, H. Schoutens and M. Toeniskoetter, Ideal theory of infinite directed unions of local quadratic transforms, J. Algebra 474 (2017), 213–239.
[14] W. Heinzer and B. Olberding, Noetherian intersections of regular local rings of dimension two, submitted, arXiv:1709.00486.

[15] M. Hochster, Prime ideal structure in commutative rings, Trans. Amer. Math. Soc. 142 (1969), 43–60.

[16] C. Huneke, Complete ideals in two dimensional regular local rings, Commutative Algebra, Berkeley, California 1987, Math. Sci. Res. Inst. Publ., 15, Springer, New York, 1989.

[17] F.-V. Kuhlmann, Places of algebraic function fields in arbitrary characteristic, Adv. Math. 188 (2004), 399–424.

[18] J. Lipman, On complete ideals in regular local rings, Algebraic Geometry and Commutative Algebra in Honor of Masayoshi Nagata, (1986) 203-231.

[19] H. Matsumura, Commutative Ring Theory Cambridge Univ. Press, Cambridge, 1986.

[20] M. Nagata, Local Rings, Wiley, 1962.

[21] J. Ohm, Some counterexamples related to integral closure in $D[[x]]$, Trans. Amer. Math. Soc. 122 (1966), 321-333.

[22] B. Olberding, Overrings of 2-dimensional Noetherian domains representable by Noetherian spaces of valuation rings, J. Pure Appl. Algebra 212 (2008), 1797-1821.

[23] B. Olberding, Noetherian spaces of integrally closed rings with an application to intersections of valuation rings, Comm. Alg. 38 (2010), 3318–3332.

[24] B. Olberding, Intersections of valuation overrings of two-dimensional Noetherian domains, in Commutative Algebra: Noetherian and non-Noetherian perspectives, Springer, 2010, 335–362.

[25] B. Olberding, Affine schemes and topological closures in the Zariski-Riemann space of valuation rings, J. Pure Appl. Algebra 219 (2015), 1720–1741.

[26] B. Olberding, Topological aspects of irredundant intersections of ideals and valuation rings. Multiplicative ideal theory and factorization theory, 277–307, Springer Proc. Math. Stat., 170, Springer, 2016.

[27] B. Olberding, A principal ideal theorem for compact sets of rank one valuation rings, J. Algebra 489 (2017), 399–426.
[28] B. Olberding, On the topology of valuation-theoretic representations of integrally closed domains, J. Pure Appl. Alg., 222 (2018), 2267–2287.

[29] D. Shannon, Monoidal transforms of regular local rings, Amer. J. Math. 95 (1973), 294-320.

[30] The Stacks Project Authors, Stacks Project, http://stacks.math.columbia.edu, 2018.

[31] I. Swanson and C. Huneke, Integral Closure of Ideals, Rings, and Modules, London Math. Soc. Lecture Note Series 336, Cambridge Univ. Press, Cambridge, 2006.

[32] O. Zariski and P. Samuel, Commutative Algebra, Vol. 2, Van Nostrand, New York, 1960.