QUANTIFIER FREE DEFINABLE RELATIONS ON
FINITE DIMENSIONAL SUBSPACE LATTICES WITH
INVOLUTION

CHRISTIAN HERRMANN AND MARTIN ZIEGLER

Abstract. For finite dimensional hermitean inner product spaces $V$, over $\ast$-fields $F$, and in the presence of orthogonal bases providing form elements in the prime subfield of $F$, we show that quantifier free definable relations in the subspace lattice $L(V)$, endowed with the involution induced by orthogonality, admit quantifier free descriptions within $F$, also in terms of Grassmann-Plücker coordinates. In the latter setting, homogeneous descriptions are obtained if one allows quantification type $\Sigma_1$. In absence of involution, these results remain valid.

1. Introduction

Translating geometric concepts into algebraic ones is a well established method since Descartes. For Projective Geometry, the approach motivated by computational proof methods, as described e.g. in [9, §3], first expresses geometric configurations within Grassmann-Cayley algebra, then within the algebra of bracket polynomials, and finally via algebraic expressions in coordinates. This is particular successful if the geometric concept can be captured by an equation $t = 0$ with a simple expression $t$: such can be translated into $p = 0$ with some bracket polynomial. The reverse direction is Cayley factorization: known to be possible, up to multiplying with a bracket associated to a tableau, for projective dimension $\geq 2$ and $p$ with integer coefficients [11] or for multilinear $p$ [10].

The present note deals with this back and forth translation between projective geometry and coordinates: considering the more abstract concept of the lattice $L(V)$ of linear subspaces of a vector space over a field $F$, on the side of geometry, and allowing more general expressions on both sides; to wit, any first order formulas: that is, we relate first
order definable $n$-ary relations on $L(V)$ with first order definable relations (of suitable arity) on $F$ - fields are just considered as rings with unit.

Since any first order definable relation on $L(V)$ is invariant under lattice automorphisms, invariance conditions for relations on $F$ will be required in that correspondence. Whiteley [12,13] gives a characterization of such invariant arithmetic formulas (via equivalence to formulas built from bracket expressions) and an associated Completeness Theorem.

Our approach also (and primarily) considers orthogonality on $V$, given by non-degenerated $*$-hermitean forms w.r.t. an involution on $F$, and admitting orthogonal bases. Thus, $L(V)$ carries an involution, too. In [3] we related $L(V)$ and $F$, using as an intermediate structure the $*$-ring of endomorphism of $V$. This allowed translations preserving quantification type $\Sigma_1$ in both directions. It remained open whether quantifier freeness can be preserved when translating from $L(V)$ to $F$ – for forms $\langle .| . \rangle$ admitting an orthogonal basis of vectors $v_i$ such that the $\langle v_i|v_i \rangle$ are in the prime subfield of $F$. In the present note we show, by an analysis of Gauss elimination, that this is indeed possible and also extends to Grassmann-Plücker coordinates. Though, requiring in the latter setting formulas built from homogeneous equations, only $\Sigma_1$- as well as $\Pi_1$-formulas have been achieved and it remains doubtful that one can arrive always at quantifier free such formulas. In any case, even for relations on $L(V)$ defined by a conjunction $t(\bar{x}) = 0 \land s(\bar{x}) = 1$ of lattice equations ($t(\bar{x}) = 0$ suffices in the presence of involution) the coordinates may fail to be definable by positive quantifier free formulas resp. by conjunctions of equations and negated equations.

In [3] it was also shown that preservation of quantifier freeness is not possible, in general, for translation in the converse direction. A sufficient condition for preservation is that coordinate systems (or, in the absence of involution, systems of elements in general position, cf. [1]) are implicitly given by the defining formula: in analogy to Cayley factorization.

All translations in this note are effective, a detailed discussion of complexity shall be postponed to subsequent work. The particular case, where $F$ is a $*$-subfield of $\mathbb{C}$ and $V$ admits an orthonormal basis, has been studied in [4] in the context of complexity of real computation.

2. Preliminaries

Statements presented as “Fact” are well known or obvious; proofs will be omitted or sketched. In the sequel, let $F$ be a field with involution
$r \mapsto r^*$ (and prime subfield $F_0$) and $V$ a (right) $F$-vector space of (fixed) $\dim V = d < \infty$ turned into an inner product space by a non-alternate non-degenerate $*$-hermitean form $\langle .|.,\rangle_V$ (we will speak just of a form and write $\langle .|.,\rangle$ if there is no confusion), that is: additive in both arguments and

$$\langle vr|ws \rangle = r^* \langle v|w \rangle s, \quad \langle w|v \rangle = \langle v|w \rangle^*$$

as well as $\langle v|v \rangle \neq 0$ for some $v$, and $\langle w|v \rangle = 0$ for all $w \in V$ only if $v = 0$ cf. [2] Chapter I. We write $|v| = \langle v|v \rangle$. A basis $\bar{v} = (v_1, \ldots, v_d)$ of $V$ is orthogonal if $\langle v_i|v_j \rangle = 0$ for $i \neq j$; we will speak of a $\perp$-basis. Recall that such always exist [2] II §2 Corollary 1: any $v_1 \neq 0$ can be completed to a $\perp$-basis. Given a $\perp$-basis $\bar{v}$ of $V$, forms are in 1-1-correspondence with $\delta = (\delta_1, \ldots, \delta_d) \in F^d$ such that $\delta_i = \delta_i^* \neq 0$ for all $i$: namely, $\delta_i = |v_i|$ and

$$\sum_i v_i r_i \sum_j v_j s_j = \sum_k r_k^* \delta_k s_k.$$

A scaled isometry (with factor $r \neq 0$ in $F$) between inner product spaces $V$ and $W$ over $F$ is a linear isomorphism $\omega : V \rightarrow W$ such that $\langle \omega x|\omega y \rangle_W = r \langle x|y \rangle_V$ for all $x, y \in V$. Given $\perp$-bases $\bar{v}$ and $\bar{w}$ of $V$ and $W$ the linear isomorphism matching $\bar{v}$ and $\bar{w}$ is a scaled isometry with factor $r$ iff $(|w_1|, \ldots, |w_d|) = r(|v_1|, \ldots, |v_d|)$. In particular, $|\bar{v}| := (1, |v_1|^{-1}|v_2|, \ldots, |v_1|^{-1}|v_d|)$ determines the isometry type of $(V, \bar{v})$ up to scaling. Call $\bar{\alpha} = (\alpha_1, \ldots, \alpha_d)$ admissible (w.r.t. $V$) if $\alpha_1 = 1$ and $\alpha_i = \alpha_i^* \neq 0$ for all $i$ and if there is a $\perp$-basis $\bar{v}$ of $V$ such that $\bar{\alpha} = |\bar{v}|$. $F^d_\bar{\alpha}$ will denote the space $F^d$ with canonical basis $\bar{v}$ such that $|v_i| = \alpha_i$.

The linear subspaces of $V$ form a lattice $L(V)$ with bounds $0, V$, joins given as $U_1 + U_2$, meets as $U_1 \cap U_2$, and involution

$$U \mapsto U^\perp = \{ x \in V \mid \forall u \in U. \langle x|u \rangle = 0 \}.$$

If $d \geq 3$, then any involution (that is, an order reversing map of order 2) on the lattice $L(V)$ is induced by some kind of inner product. Observe that

$$U_1 \cap U_2 = (U_1^\perp + U_2^\perp)^\perp$$

and $V^\perp = 0$. $L(V)$ remains unchanged under scaling the form on $V$. Moreover, any scaled isometry $\omega : V \rightarrow W$ induces an isomorphism $L(V) \rightarrow L(W)$: $U \mapsto \omega(U)$. In particular, given a $\perp$-basis $\bar{v}$ of $V$ one has an isomorphism $\Omega_{\bar{v}} : L(V) \rightarrow L(F^d_\bar{\alpha})$ where $\bar{\alpha} = |\bar{v}|$. If there is no confusion we write $L$ in place of $L(V)$.

We consider $F^d$ as a space of columns $u$ and, for $m \leq 2d$, $F^{d \times m}$ the space of $d \times m$-matrices $A = (a_{ij})_{ij}$ over $F$ with columns $a_j$. Let $\text{rk}(A)$
denote the rank of $A$ and $\text{Span}(A)$ the $F$-linear subspace of $F^d$ spanned by the columns of $A$; recall that $B \in \text{Span}(A)$ iff $B = AC$ for some $C \in F^{d \times d}$.

Write $M_m = \{1, \ldots, m\}$; let $d^\#$ consist of all strictly monotone maps $f : M_k \to M_d$ where $k = |f| \leq d$. The matrix $A$ is in column echelon normal form, shortly NF, with positions $f \in d^\#$ of pivots $a_{f(j)j}$ (shortly $f$-NF) if for all $j \leq |f|$, $h \leq m$, and $i \leq d$

\[
\begin{align*}
a_{f(j)j} &= 1 \\
a_{ij} &= 0 \quad \text{if } i < f(j) \\
a_{ih} &= 0 \quad \text{if } i < f(j+1) \text{ and } h > j < |f| \\
a_{f(j)h} &= 0 \quad \text{if } h \neq j.
\end{align*}
\]

$A$ is in weak normal form (shortly wNF resp. $f$-wNF) if $rA$ is in NF resp. $f$-NF for some $r \neq 0$.

$A^* = (a^*_{ji})_{ij}$ is the conjugate (w.r.t. the involution on $F$) transpose of $A = (a_{ij})_{ij}$, $\bar{\alpha} = (\alpha_1, \ldots, \alpha_d) \in F^d$ with $\alpha_1 = 1$ and $\alpha_i = \alpha_i^* \neq 0$, $D_\bar{\alpha}$ the diagonal matrix with diagonal entries $\alpha_i$, and $E_k$ the diagonal matrix with first $k$ diagonal entries 1, 0 else.

**Fact 1.**

(i) For each $A \in F^{d \times m}$ there is unique $A^\# \in F^{d \times d}$ in NF such that $\text{Span}(A) = \text{Span}(A^\#)$. Moreover, $|f| = \text{rk}(A)$ iff $A^\#$ in $f$-NF. For suitable $r \neq 0$, $rA^\#$ can be obtained from $A$ by Gaussian column transformations without inversion of scalars.

(ii) For idempotent $A \in F^{d \times d}$ one has $\text{Span}(A) = \text{Span}(I - A^\alpha)$ in $F_{\bar{\alpha}}$ where $A^\alpha_i = D_{\bar{\alpha}}^{-1}A^\alpha a_{ij}$.

(iii) For any $f \in d^\#$ there is a permutation matrix $P$ such that $\text{Span}(A) = \text{Span}(rI - (E_{|f|}PA)^\alpha)$, for any $A \in F^{d \times d}$ in $f$-wNF with $A = rA^\#$.

**Proof.**

(i) To avoid inversion, multiply any column, to be changed, first by a suitable scalar. Once echelon form is obtained with each pivot the only non-zero entry in its row, multiply each pivot with the product of the others.

(ii) This is well known in the context of the $*$-regular ring defined by the form on $F^d_{\bar{\alpha}}$. For convenience, we recall the proof. Observe that the linear map $\varphi$ defined by $A$ w.r.t. the canonical basis $\bar{v}$ has adjoint $\varphi^\dagger$ in $F^d_{\bar{\alpha}}$ defined by $A^\dagger_{\bar{\alpha}}$. Indeed, we have $\langle \varphi_j | v_k \rangle = a^\alpha_{kj} \alpha_k$ and $\langle v_j | A^\alpha_{\bar{\alpha}} v_k \rangle = \langle v_j | \alpha_j^{-1} a_{kj} \alpha_k v_j \rangle = \langle v_j | v_j \rangle \alpha_j^{-1} a_{kj} \alpha_k = a^\alpha_{kj} \alpha_k$. Now, since $\varphi$ is idempotent, so is $\varphi^\dagger$ whence $\langle \varphi x | y - \varphi^\dagger y \rangle = \langle x | \varphi^\dagger (y - \varphi^\dagger y) \rangle = \langle x | 0 \rangle = 0$ for all $x, y \in F^d$; that is $\text{im} \varphi = \text{Span}(A)$ orthogonal to $\text{im}(\text{id} - \varphi^\dagger) = \text{Span}(I - A^\alpha)$. Since $A^\alpha_{\bar{\alpha}}$ is idempotent, one has $\text{rk}(I - A^\alpha) = d - \text{rk}(A^\alpha) = \text{rk}(A)$ whence $\text{Span}(A) = \text{Span}(I - A^\alpha)$. 


(iii) Choose $P = P_\sigma$ with a permutation $\sigma$ of $M_d$ such that $\sigma \circ f = \text{id}_{M(f)}$. Then one has block matrices
\[
\frac{1}{r}PA = \begin{pmatrix} I_k & 0 \\ \frac{1}{r}K & 0 \end{pmatrix} \quad \text{and} \quad X := E_kP = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} P,
\]
where $k = |f|$ and $I_k$ the unit in $F^{k \times k}$. It follows that $\frac{1}{r}XA$ is idempotent and $\text{Span}(\frac{1}{r}XA) = \text{Span}(\frac{1}{r}A) = \text{Span}(A)$ and, by (ii)
$\text{Span}(A)^\perp = \text{Span}(I - (\frac{1}{r}XA)^\top) = \text{Span}(rI - (XA)^\top)$.

We consider first order languages with countably many variables: $\Lambda_L$ in the signature of $+,0,\leq$ of bounded lattices with involution (defining meet by $s \cap t = (s^\perp + t^\perp)^\perp$ and $1 = 0^\perp$), $\Lambda_F$ in the signature of $\ast$-rings, with constants $0,1$, and $\Lambda_F^+$ with additional constants $c_1, \ldots, c_d$ to be interpreted as $\alpha_1, \ldots, \alpha_d$ given any $\bar{a}$ admissible w.r.t. $V$, that is, considering $F$ with additional constants $\alpha_i$. A $\Lambda_F^+$-term $t(x_1, \ldots, x_n)$ is basic if its is a multivariate $\ast$-polynomial in variables $x_1, \ldots, x_n$ with coefficients from $R = \mathbb{Z}[c_1, \ldots, c_d]$, that is an $R$-linear combination of terms $\prod_i x_i^{k_i}(x_i^\ast)^{l_i}$. A $\Lambda_F^+$-formula is basic if it is a conjunction of formulas $p_i = 0$ and $q_j \neq 0$ where the $p_i$ and $q_j$ are basic terms.

An $n$-ary relation $R$ on $\mathbb{L}$ resp. $F$ is definable if there is a formula $\varphi(\bar{x})$ (in the relevant language) such that $R$ consists of all $\bar{a}$ such that $\varphi(\bar{a})$ holds in $\mathbb{L}$ resp. $F$. Here, finite strings of variables or elements are written e.g. as $\bar{x}$ and $\bar{a}$, the length being given by context. We also use matrices $X = (x_{ij})_{ij}$ of variables in an obvious way. $\Sigma_k$ ($\Pi_k$) consists of the prenex formulas with at most $k$ blocks, each consisting of quantifiers of the same type, the first block being of type $\exists$ ($\forall$).

**Fact 2.** Given a $\ast$-field $F$ and values $\alpha_i$ of constants $c_i$.

(i) If the $\alpha_i$ are $\Lambda_F^+$-definable within $F$ then any relation on $F$ which is definable within $\Lambda_F^+$ is also definable within $\Lambda_F$.

(ii) Any $\Lambda_F^+$-term is equivalent to a basic term (uniformly for all $F$ and $\alpha_i$).

(iii) Any relation defined in $F$ by a quantifier free $\Lambda_F^+$-formula is the disjoint union of relations defined by basic $\Lambda_F^+$-formulas. Moreover, if all $\alpha_i \in F_0$, then the latter can be chosen in $\Lambda_F$ (that is, with integer coefficients).

(iv) If $\ast$ is the identity involution of $F$, any multivariate $\ast$-polynomial can be considered a multivariate polynomial in the same coefficients and variables.

**Proof.** (i) Given a $\Lambda_F^+$-formula $\varphi$, if the $\alpha_i$ are defined by the $\alpha_i^\#(y_i)$, replace each occurrence of $c_i$ in $\varphi$ by $y_i$ and add the conjuncts $\exists y_i.\alpha_i^\#(y_i)$.
(as bounded quantifiers) to the formula so obtained. (ii) follows from commutativity of \( F \) and then the first claim in (iii) by disjoint normal form of Boolean expressions. Now assume that the \( \alpha_i \) are in \( F_0 \) whence defined within \( F \) by \( s_i \alpha_i = t_i \) with constant \( \Lambda_F \)-terms \( s_i, t_i \). In this case, multiply any equation \( p(\bar{x}, \bar{c}) = 0 \) with all the constant \( \Lambda_F \)-terms \( s_j^k \) where \( k_j \) is the maximum power to which \( c_j \) occurs in the equation. (iv) is obvious. \( \square \)

For the following compare [5, Fact 3.2]. Call an equation in \( \Lambda_L \) special if it is of the form \( x = 0 \), \( z = x + y \), or \( y = x \perp \).

**Fact 3.** For every quantifier free formula \( \varphi(\bar{x}) \) in \( \Lambda_L \) there is a conjunction \( \varphi'(\bar{x}, \bar{z}, \bar{y}) \) of special equations with new variables \( \bar{z}, \bar{y} \) and a boolean combination \( \varphi''(\bar{x}, \bar{y}) \) of equations between variables from \( \bar{x}, \bar{y} \) such that \( \varphi(\bar{x}) \) is equivalent within lattices with involution to both

\[
\varphi^3(\bar{x}, \bar{z}, \bar{y}) \equiv \exists \bar{z} \bar{y}. \varphi'(\bar{x}, \bar{z}, \bar{y}) \land \varphi''(\bar{x}, \bar{y}) \\
\varphi^5(\bar{x}, \bar{z}, \bar{y}) \equiv \forall \bar{z} \bar{y}. \varphi'(\bar{x}, \bar{z}, \bar{y}) \Rightarrow \varphi''(\bar{x}, \bar{y}).
\]

Moreover, for any \( \bar{u} \) in an involutive lattice \( L \) there are unique \( \bar{v}, \bar{w} \) such that \( L \models \varphi'(\bar{u}, \bar{w}, \bar{v}) \). Also, if \( \varphi \) is a conjunction (disjunction) of equations and negated equations then so is \( \varphi'' \).

### 3. Review of earlier work

Recalling some definitions and results from Sections 8–10 of [5], we shall relate \( L \) and \( F \), directly, without using endomorphism rings as intermediate step. Given a basis \( \bar{v} \) define the relation \( \theta_\bar{v} \) between \( L^n \) and \( (F^{d \times d})^n \) by

\[
\bar{u} \theta_\bar{v} \bar{A} \text{ iff } \bar{u} = (\text{Span}(A_1), \ldots, \text{Span}(A_n)) \text{ for } \bar{u} \in L^n, \bar{A} \in (F^{d \times d})^n.
\]

This gives rise to maps \( \theta_{L \leftrightarrow F} \) and \( \theta_{F \leftrightarrow L} \) mapping subsets \( M \) of \( L^n \) to subsets \( K \) of \( (F^{d \times d})^n \) and vice versa

\[
\theta_{L \leftrightarrow F}(M) = \{ \bar{A} \in (F^{d \times d})^n \mid \exists \bar{u} \in L^n. \bar{u} \theta_\bar{v} \bar{A} \} \\
\theta_{F \leftrightarrow L}(K) = \{ \bar{u} \in L^n \mid \exists \bar{A} \in (F^{d \times d})^n. \bar{u} \theta_\bar{v} \bar{A} \}
\]

Our objective is to match definable subsets of \( L^n \) with definable subsets of \( (F^{d \times d})^n \) via suitable translations – preserving quantification type as much as possible. Observe that \( F^{d \times d} \) is not considered a (\*(-)ring, but just a power of the field \( F \), formatted in a suitable fashion. Definable subsets have certain invariance properties which we now recall from [5], Section 9.
Let $F^+$ the multiplicative subgroup $\{ r \mid 0 \neq r = r^* \in F \}$ of $F$ and $O^+(V)$ consist of all scaled orthogonal maps $g$: for some $r \in F^+$ and orthogonal $h$

$$ gv = h(vr) \text{ for all } v \in V. $$

Observe that the group $Q^+(V)$ is not changed if the form on $V$ is scaled by an element of $F^+$.

For any $g \in O^+(V)$, the map $U \mapsto g^L(U) := g(U)$ is an automorphism of $L$ (actually, for $d \geq 3$ any automorphism of $L$ is of this form). We say that $M \subseteq L^n$ is invariant if it is invariant under the component wise action of the $g^L$, $g \in O^+(V)$. Clearly, definable $M$ are invariant.

Let $GL(F, d)$ denote the set of of invertible matrices in $F^{d \times d}$. We define $O^+_\alpha(F, d)$, to consist of the $T \in GL(F, d)$ such that $T^{*\alpha} = rT^{-1}$ for some $0 \neq r \in F^+$, and consider the action $A \mapsto TAT^{-1}$ on $F^{d \times d}$. We call $K \subseteq (F^{d \times d})^n$ $\bar{\alpha}$-invariant if it is invariant under the component wise action of $O^+_\alpha(F, d)$; right invariant if $(A_1T_1, \ldots, A_nT_n) \in K$ for all $(A_1, \ldots, A_n) \in K$ and $T_i \in GL(F, d)$; and $\bar{\alpha}$-bi-invariant if both conditions are satisfied, i.e. if $(TA_1T_1, \ldots, TA_nT_n) \in K$ for all $(A_1, \ldots, A_n) \in K$, $T \in O^+_\alpha(F, d)$, and $T_i \in GL(F, d)$.

Of course, given a first order formula, right invariance of the subset of $(F^{d \times d})^n$ it defines can be stated by a first order sentence; similarly $\bar{\alpha}$-invariance if $\bar{\alpha}$ is definable. The following is [5, Fact 9.2].

**Fact 4.** $\theta_{L\bar{\alpha}F}$ and $\theta_{F\bar{\alpha}L}$ induce mutually inverse bijections between the set of all subsets $M$ of $L^n$ and the set of all right invariant subsets $K$ of $(F^{d \times d})^n$.

Given $\bar{\alpha}$, define $\theta_{L\bar{\alpha}F}(M)$, and $\theta_{F\bar{\alpha}L}(K)$, respectively, as the union of the $\theta_{L\alpha F}(M)$, and $\theta_{F\alpha L}(K)$ where $\bar{v}$ ranges over all $\perp$-bases $\bar{v}$ with $|\bar{v}| = \bar{\alpha}$. The following is [5, Proposition 9.3].

**Proposition 5.** Fix a $\perp$-basis $\bar{v}$ and $\bar{\alpha} = |\bar{v}|$. Then $\theta_{L\alpha F}$ and $\theta_{F\bar{\alpha}L}$ induce mutually inverse bijections between the set of all invariant $M \subseteq L^n$ and the set of all $\bar{\alpha}$-bi-invariant $K \subseteq (F^{d \times d})^n$; moreover, for such $M$ and $K$, $\theta_{L\bar{\alpha}F}(M) = \theta_{L\alpha F}(M)$ and $\theta_{F\alpha L}(K) = \theta_{F\bar{\alpha}L}(K)$.

In Sections 8 and 10 (cf. Theorem 10.4(ii)) of [5] we have constructed translations $\tau_{L\alpha F}^2 : \Lambda_L \rightarrow \Lambda_F^+$ and $\tau_{F\alpha L}^2 : \Lambda_F^+ \rightarrow \Lambda_L$ which preserve quantification type $\Sigma_1$ and such that, for any admissible $\bar{\alpha}$, $\tau_{L\alpha F}^2(\varphi(\bar{x}))$ defines $\theta_{L\alpha F}(M)$ if $\varphi(\bar{x})$ defines $M$ in $L^n$ and, if in addition $\bar{\alpha}$ in $F_0$, then $\tau_{F\alpha L}^2(\psi(\bar{X}))$ defines $\theta_{F\alpha L}(K)$ if $\psi(\bar{X})$ defines $K$ in $(F^{d \times d})^n$.

### 4. Translation via Gauss

In [5] the translation from $L^+_{\alpha}$ to $F^+_{\bar{\alpha}}$ was constructed with the *-ring of endomorphism of $V$ as an intermediate structure. While this
translation preserves quantification type $\Sigma_1$, a translation from $\Lambda_L$ to $\Lambda_F^+$ preserving quantifier freeness can be constructed based on Gaussian elimination. Recall that $F_0$ denotes the prime subfield of $F$.

**Theorem 6.** For any fixed $d$ and admissible $\bar{\alpha}$, there is a map $\tau_{L,\alpha,F} : \Lambda_L \to \Lambda_F^+$ such that $\theta_{L,\alpha,F}(M) \subseteq (F^{d \times d})^n$ is defined by $\tau_{L,\alpha,F}(\varphi)(\bar{X})$ if $M \subseteq \mathbb{L}^n$ is defined by $\varphi(\bar{x})$ in $\Lambda_L$; $\tau_{L,\alpha,F}(\varphi)$ is quantifier free if so is $\varphi(\bar{x})$. Moreover, if $\bar{\alpha} \in F_0^d$, then $\tau_{L,\alpha,F}(\varphi)(\bar{X})$ is in $\Lambda_F$.

The proof needs some preparation.

**Fact 7.** Fix $m$ and a $d \times m$-matrix $X$ of variables.

(i) Normal form. For any $f \in d^\#$ and $m \leq d$, there is a quantifier free formula $\nu^{mf}(X)$ in $\Lambda_F$ such that for any $A \in F^{d \times m}$ one has $F \models \nu^{mf}(A)$ if and only if $A$ in $f$-wNF.

(ii) Computation of normal forms: distinction of cases. With a $d \times m$-matrix $X$ of variables, for each $f \in d^\#$, there is a finite set $\Sigma^{mf}$ of quantifier free formulas $\sigma(X)$ in $\Lambda_F$ and for each $\sigma \in \Sigma^{mf}$ there is a $d \times d$-matrix $P^{\sigma f}(X) = (p^{\sigma f}_{ij})_{ij}$ of $\Lambda_F$-terms such that for any $A \in F^{d \times m}$ one has

(a) $F \models \sigma(A)$ for some $f \in d^\#$ and $\sigma \in \Sigma^{mf}$

(b) for all $f \in d^\#$ and $\sigma \in \Sigma^{mf}$, if $F \models \sigma(A)$ then $P^{\sigma f}(A)$ is in $f$-wNF and $\text{Span}(P^{\sigma f}(A)) = \text{Span}(A)$.

(iii) Orthogonals. With a $d \times m$-matrix $X$ of variables, for each $f \in d^\#$ there is a matrix $Q^f(X) = (q^f_{ij})_{ij}$ of $\Lambda_F^+$-terms such that for each admissible $\bar{\alpha}$, substituted for $\bar{y}$, and each $A \in F^{d \times d}$ in $f$-wNF one has $\text{Span}(Q^f(A)) = \text{Span}(A)_{\perp}$ in $F_0^d$.

**Proof.** (i). The construction of $\nu^{mf}$ is obvious. (ii) Given $f \in d^\#$, the $\sigma \in \Sigma^{mf}$ capture the distinction of cases in (column) Gauss-elimination, applied to matrices $A$, to yield $f$-wNF, and each $\sigma$ grants that such exists. This is easily (and tediously) expressed via quantifier free formulas. The terms in $P^{\sigma f}$ then combine the elimination calculations, followed by multiplications with terms obtained for the pivots. (iii) is immediate by (iii) of Fact 7. \qed

Recall that $\theta_{L,\alpha,F} = \theta_{L,\bar{\epsilon},F}$ where $\bar{v}$ is any $\perp$-basis with $|\bar{v}| = \bar{\alpha}$. Thus, to verify that the translation $\varphi \to \tau_{L,\alpha,F}(\varphi)$ matches $M$ with $\theta_{L,\alpha,F}(M)$, we may argue based on an unspecified $\bar{v}$; this allows to identify $V$ with $F_0^d$. We now have to explain how to relate $\Lambda_L$-terms to terms and quantifier free formulas in $\Lambda_F^+$. We associate with each variable $x_k$ a matrix $X_k$ of variables. The following captures the matrix computations associated with the evaluation of $\Lambda_L$-terms.
Lemma 8. One can associate with each $\Lambda_L$-term $t(\bar{x})$ a finite set $\Gamma_t$ of pairs $(\pi(X), f)$, where $\pi(X)$ is a quantifier free formula in $\Lambda^+_F$ and $f \in d^\#$, and with each $(\pi(X), f) \in \Gamma_t$ a $d \times d$-matrix $P^{\pi f}(X) = (p^\pi_{ij} f(\bar{x}))_{ij}$ of $\Lambda^+_F$-terms, such that the following hold in $F_d$ for any $F$ and admissible $\bar{a}$:

1. For any $\bar{A} \in (F^{d \times d})^n$ there is $(\pi(\bar{x}), f) \in \Gamma_t$ such that $F \models \pi(\bar{A})$.
2. For any $(\pi(X), f) \in \Gamma_t$ and any $\bar{A} \in (F^{d \times d})^n$, if $F \models \pi(\bar{A})$ then $P^{\pi f}(\bar{A})$ is in $f$-NF and $\text{Span}(P^{\pi f}(\bar{A})) = t(U)$ where $U_k = \text{Span}(A_k)$.

Proof. For a variable $x$ we choose $\Gamma_x = \{((\sigma(X), f) \mid f \in d^\#, \sigma(X) \in \Sigma^d\}$ and $P^{\sigma f}(X)$ as in Fact 7(ii).

Given matrices $A, B$, one obtains $U = \text{Span}(A) + \text{Span}(B)$ as $\text{Span}(C)$ with $C$ derived from the compound matrix $(A|B)$, choosing dim $U$ independent columns and completing to a $d \times d$-matrix by adding zero columns. Let $m = 2d$ and denote by $(X|Y)$ the $d \times m$-matrix obtained from the $d \times d$-matrices $X, Y$ of variables. By Fact 7(ii) one obtains a finite set $\Gamma$ of pairs $(\sigma(X|Y), f)$, where $\sigma(X|Y)$ is a quantifier free formula in $\Lambda_F$ and $f \in d^\#$, and for each $(\sigma, f) \in \Gamma$ a $d \times d$-matrix $S^{\sigma f}(X|Y)$ of terms in $\Lambda_F$ such that for any $A, B \in F^{d \times d}$ there is $(\sigma, f) \in \Gamma$ with $F \models \sigma(A|B)$ and $S^{\sigma f}(A|B)$ in $f$-wNF and with $\text{Span}(A) + \text{Span}(B) = \text{Span}(S^{\sigma f}(A|B))$. Now, for $t(\bar{x}) = t_1(\bar{x}) + t_2(\bar{x})$ let $\Gamma_t$ consist of all pairs $(\pi(X), f)$ where $(\sigma, f) \in \Gamma$ and

$$\pi(X) \equiv \pi_1(X) \land \pi_2(X) \land \sigma(P^{t_1 \pi_1 f_1}(X)) P^{t_2 \pi_2 f_2}(X))$$

with $(\pi_i(X), f_i) \in \Gamma_{t_i}$. Put

$$P^{t \pi f} = S^{\sigma f}(P^{t_1 \pi_1 f_1}(X)) P^{t_2 \pi_2 f_2}(X)).$$

By Fact 7(iii), for any $A \in F^{d \times d}$ in $g$-wNF, one obtains $U = \text{Span}(A)^\perp$ as $\text{Span}(C)$ with $C = Q^\#(A)$ and can apply Fact 7(ii) to transform $C$ into wNF. Formally, this proceeds as follows. Again, Fact 7(ii) yields for each $g \in d^\#$ a finite set $\Gamma_g$ of pairs $(\sigma, f)$ and for each $(\sigma, f) \in \Gamma_g$ a matrix $P^{\sigma f g}(X)$ of $\Lambda_F$-terms such that for any $A \in F^{d \times d}$ in $g$-wNF there is $(\sigma, f) \in \Gamma_g$ with $F \models \sigma(Q^\#(A))$ and in that case $R^{\sigma f g}(A)$ is in $f$-wNF and $\text{Span}(R^{\sigma f g}(A)) = \text{Span}(A)^\perp$ where $R^{\sigma f g}(X) = P^{\sigma f}(Q^\#(X))$. Now, for $t(\bar{x}) = t_1(\bar{x})^\perp$ let $\Gamma_t$ consist of all $(\pi(X), f)$ where $(\sigma, f) \in \Gamma_g$ and

$$\pi(X) \equiv \pi_1(X) \land \sigma(P^{t_1 \pi_1 g}(X))$$

with $(\pi_1(X), g) \in \Gamma_{t_1}$. Put

$$P^{t \pi f} = R^{\sigma f g}(P^{t_1 \pi_1 g}(X)).$$
This provides the translation of $\Lambda_L$-terms $t(\bar{x})$.

Proof. of Thm 5. To deal with equations, in view of Lemma 8, define $\gamma_{t_1t_2}^{L}(\bar{X})$ as the conjunction of all implications

$$\pi_1(\bar{X}) \wedge \pi_2(\bar{X}) \Rightarrow p_0^{t_2\pi_2f}(\bar{X})P^{t_1\pi_1f}(\bar{X}) = p_0^{t_2\pi_2f}(\bar{X})P^{t_2\pi_2f}(\bar{X})$$

where $(\pi_i, f) \in \Gamma_t$, and $p_0^{t_1\pi_1f}$ the entry in position $(f(1), 1)$ of $P^{t_1\pi_1f}(\bar{X})$. That is, this formula expresses that for any substitution $\bar{X}$ where the first pivots. Thus, for all $\bar{X}$ yields the same matrix in wNF up to crosswise multiplying with the first pivots. Thus, for all $\bar{A}$ and $\bar{U}_k = \text{Span}(A_k)$

$$t_1(\bar{U}) = t_2(\bar{U}) \text{ if and only if } F \models \gamma_{t_1t_2}(\bar{A}).$$

The $\gamma_{t_1t_2}^{L}(\bar{X})$ give the required translations to $\Lambda_F^L$ for equations $t_1(\bar{x}) = t_2(\bar{x})$, that is, the atomic $\Lambda_L$-formulas. This then extends, canonically, to quantifier free formulas and further to prenex formulas. Fact 2 yields the last claim of the theorem.

5. THE GRASSMANN-PLÜCKER POINT OF VIEW

An alternative to describing subspaces via matrices is to use Plücker coordinates (cmp. [6, Ch. VII], [7, §14.1]). Fix $d$ and consider $0 < k \leq m \leq d$. Let $F_k$ be the set of all pivot positions $f$ for $d \times m$-matrices $A$ in wNF with $\text{rk}(A) = k$ — that is, the set of all strictly monotone maps $M_k \to M_d$. One has a quantifier free formula $\rho^k(X)$ in $\Lambda_F$ such that $F \models \rho^k(A)$ if and only if $\text{rk}(A) = k$: requiring all $\ell \times \ell$-subdeterminants to be zero if $\ell > k$ but some to be non-zero for $\ell = k$.

Let $I_k$ denote the set of all $k$-tuplets $\bar{i} = (i_1 < i_2 < \ldots i_k)$, ordered lexicographically. For a $d \times m$ matrix $X$ of variables and $\bar{j}, \bar{i} \in I_K$ let $X^\bar{i}$ the $d \times k$-matrix where the $h$-column is the $j_h$-column of $X$ and $X^\bar{i}$ the $k \times k$-matrix where the $\ell$-th row is the $i_{\ell}$-th row of $X^\bar{i}$. Define the following terms and $|I_k|$-tuplets of terms in $\Lambda_F$

$$D^\bar{\bar{j}}(X) = \text{det}(X^\bar{j}) \text{ and } D^\bar{i}(X) = (D^\bar{i}_{\bar{j}}(X) \mid \bar{j} \in I_k);$$

$$D_i(X) = D^\bar{i}_i(X) \text{ and } D_h(X) = D^\bar{i}_h(X) \text{ where } \bar{j} = (1, \ldots, k).$$

Then for $\text{rk}(A) = k$ the set of projective Grassmann-Plücker coordinates depends on $U = \text{Span}(A)$, only: for any $\lambda \neq 0$

$$P^d_k(U) := P^d_k(A) := \text{Span}\{D^\bar{i}(A) \mid \bar{j} \in I_k\} \setminus 0 = D_k(\lambda A^\#)F \setminus 0.$$

Fact 9. Let $A, B \in F^d \times m$ and $\text{rk}(A) = \text{rk}(B) = k > 0$. Then $P^d_k(A) \cap P^d_k(B) \neq 0$ if and only if $P^d_k(A) = P^d_k(B) \neq 0$ if and only if $B = AT$ for some $T \in \text{GL}(F, m)$. 

Thus, for a matrix one has \( F \) some \( \lambda \) is \( \mu_r \) let \( \pi \)
Considered a projective variety, the Grassmannian \( \tilde{\Gamma}_k^d(F) \). To deal with

Proof. Cf. the proof of Theorem II in Chapter VII of [6]. For

Lemma 10. Normal form from Grassmann-Plücker coordinates. For
each \( 0 < k \leq d \), and \( f \in \mathcal{F}_k \) there are a quantifier free \( \Lambda_F \)-formula \( \pi_f(y) \) in variables \( y_i \mid i \in I_k \) and terms \( p_0^f(y) \) and a matrix \( P^f(y) = (p_{ij}^f(y))_{ij} \) of terms, all in \( \Lambda_F \), such that for any field \( F \) and \( r \in \Gamma_k^d(F) \) one has \( F \models \pi_f(r) \) if and only if \( p_0^f(r) \neq 0 \) and \( \lambda_r = D_k(A) \) for some \( \lambda \neq 0 \) and matrix \( A \in F^{d \times k} \) in \( f \)-wNF, namely \( \lambda = p_0^f(r) \) and \( A = (p_{ij}^f(r))_{ij} \).

Proof. Cf. the proof of Theorem II in Chapter VII of [6]. For \( f \in \mathcal{F}_k \) let \( \pi_f(y) \) the formula with states that the first \( i \) in \( I_k \) such that \( y_i \neq 0 \) is

Thus, for a matrix \( A \) in \( wNF \) and \( r \in \mathcal{P}_k^d(A) \) one has \( F \models \pi_f(r) \) if and only if \( A \) has positions \( f \) of pivots.

To define the required terms, let \( I_f \) consist of all \((i, j), i, j \leq d \) such that \( f(h) < i < f(h+1) \) and \( j \leq h \) for some \( h \) (where \( f(k+1) := d+1 \)) and put

\[ f_{ij} = (f(1), f(j-1), f(j+1), \ldots, f(h), i, f(h+1), \ldots, f(k)) \] if \( j < h \)

\[ f_{ij} = (f(1), \ldots, f(j-1), i, f(h+1), \ldots, f(k)) \] if \( j = h \)

Now, put \( p_0^f(y) = y_{f_0}^{k-1} \)

\[ p_{ij}^f(y) = \begin{cases} y_{f_0} & \text{if } i = f(j) \\ (-1)^{h-j}y_{f_{ij}} & \text{if } (i, j) \in I_f \text{ and } f(h) < i < f(h+1) \\ 0 & \text{else} \end{cases} \]

If \( A \) is in \( NF \) with positions \( f \) of pivots then \( A \) is recovered from \( r = D_k(A) \) as \( P^f(r) \), as required, and \( \lambda = p_0^f(r) = 1 \).

Now, assume \( r \in \Gamma_k^d(F) \) and \( F \models \pi_f(r) \), in particular \( r_{f_0} \neq 0 \). Then \( \mu_r = \mathcal{O} := D_k(B) \) for some \( \mu \neq 0 \) and \( B \) which may be chosen in \( NF \). As \( \pi_f \) is “homogeneous”, one has \( F \models \pi_f(s) \) and \( B \) with pivot positions \( f \). It follows \( s_{f_0} = D_{f_0}(B) = 1 \) and \( \mu = r_{f_0}^{-1} \). As observed, above,
\( B = P^f(x) \). We are to determine \( \nu \) such that \( A = \nu B \) is as required in the Lemma. First we should have \( A = P^f(\nu s) \) since all terms \( p_{ij}^f(y) \) are linear in \( y \). Second, \( D(k)(A) = \nu^k D_k(A) = \nu^k s \) since we deal with \( k \times k \)-subdeterminants. Now, the required \( \lambda \) is \( \lambda = p_0^f(r) = r_{f_0}^{-1}; \) with \( \nu = r_{f_0} \) one obtains, indeed, \( \lambda \Sigma = \lambda r_{f_0} s = \nu^k s. \)

Given a dimension vector \( d \), define

\[
\Gamma_d^d(F) := \Gamma_{d_1}^d(F) \times \ldots \times \Gamma_{d_n}^d(F)
\]

and let \( F^{d \times d} \) consist of all \( \bar{A} \in (F^{d \times d})^n \) with \( \text{rk}(A_k) = d_k \) for \( k = 1, \ldots, n \). Observe that \( \Gamma_d^d(F) \) is defined within \( F^{d_1} \times \ldots \times F^{d_n} \) by the conjunction of the Plücker relations together with \( y_k \neq 0 \) for \( d_k \neq 0 \) while \( y_k \) is the constant 0 for \( d_k = 0 \). On the other hand, \( F^{d \times d} \) is defined within \( (F^{d \times d})^n \) by the (quantifier free) conjunction of the \( \rho^d_k(X_k) \). Call \( \Delta \subseteq \Gamma_d^d(F) \) scalar invariant if

\[
(\tau_1, \ldots, \tau_n) \in \Delta \Rightarrow (\lambda_1 \tau_1, \ldots, \lambda_n \tau_n) \in \Delta
\]

for all \( \lambda_k \neq 0 \). Such \( \Delta \) may be considered a subset of the product of Grassmannians \( \Gamma_d^d(A) \).

Call \( \Delta \) \( \bar{a} \)-bi-invariant if \( \Delta \) is scalar invariant and if, in addition, for all \( T \in O^+(F, d) \), \( \bar{A} \in F^{d \times d}, \tau_k \in P_d^d(A_k) \), and \( s_k \in P_d^d(T A_k) \), \( k = 1, \ldots, n, \) one has

\[
(\tau_1, \ldots, \tau_n) \in \Delta \Rightarrow (s_1, \ldots, s_n) \in \Delta.
\]

Define

\[
\theta_{F_d^d} : F^{d \times d} \to \Gamma_d^d(F), \quad \theta_{F_d^d}(\bar{A}) = P_{d_1}^d(A_1) \times \ldots \times P_{d_n}^d(A_n)
\]

\[
\theta_{d_d} : F_d^d \to F^{d \times d}, \quad \theta_{F_d^d}(\{\tau_1, \ldots, \tau_n\}) = \{\bar{A} \in F^{d \times d} \mid \forall k \tau_k \in P_{d_k}^d(A_k)\}
\]

Use the same notation for the associated maps from sets to sets – taking unions of the images.

To deal with definability, for \( d_k > 0 \) associate with \( d \times d \)-matrices \( X_k \) of \( \Lambda^+_k \)-variables a \( d_k \)-tuplet \( y_k = (y_{ki}) | i \in I_k \) of \( \Lambda^+_k \)-variables, and vice versa; in case \( d_k = 0 \) we match the constant zero matrix in \( F^{d \times d} \) with the constant 0 \( \in F \). Also, with a \( \Lambda^+_k \)-formula \( \psi(X_1, \ldots, X_n) \) we associate a \( \Lambda^+_k \)-formula \( \chi(y_1, \ldots, y_n) \), and vice versa. Namely, given \( \psi \)

choose \( \chi \equiv \tau_{F_d^d}(\psi) \) as follows, using the conjunction \( \eta_{d_k}^d(y_k) \) of Plücker relations defining \( \Gamma_d^d(F) \) and the formulas \( \pi_f \) and matrices \( P_f \) of terms from Lemma 10.

\[
\bigwedge_{k=1}^n \eta_{d_k}^d(y_k) \land \left( \bigvee_{f_1 \in F_{d_1}, \ldots, f_n \in F_{d_n}} \bigwedge_{k=1}^n \pi_{f_k}(y_k) \land \psi(P_{f_k}(y_1), \ldots, P_{f_n}(y_n)) \right).
\]
Given $\chi$ choose $\psi \equiv \tau_{\Gamma d F}(\chi)$ as
\[
\bigvee_{j_1 \in I_1, \ldots, j_n \in I_n} \bigwedge_{k=1}^n \theta^{d_k}(X_{k}^{j_k}) \wedge \chi(D_{d_1}(X_{1}^{j_1}), \ldots, D_{d_n}(X_{n}^{j_n})).
\]

Call $\chi$ scalar invariant (w.r.t. $F$) if for all $\vec{r} \in \Gamma^d_d(F)$, $i = 1, \ldots, n$, and all $\lambda_1, \ldots, \lambda_n$ in $F \setminus \{0\}$ one has
\[
F \models \chi(\vec{l}_1, \ldots, \vec{l}_n) \text{ iff } F \models \chi(\lambda_1\vec{l}_1, \ldots, \lambda_n\vec{l}_n);
\]
that is, iff $\chi$ defines scalar invariant $\Delta \subseteq \Gamma^d_d(F)$.

**Lemma 11.** $\theta_{\Gamma d F}$ and $\theta_{\Gamma d F}$ establish mutually inverse bijections between the set of all right invariant subsets $K$ of $F^{d \times L}$ and the set of all scalar invariant subsets $\Delta$ of $\Gamma^d_d(F)$. Under this correspondence,

(i) $K$ is $\bar{\alpha}$-bi-invariant if and only if $\Delta$ is $\bar{\alpha}$-bi-invariant.

(ii) If $K$ is $\bar{\alpha}$-bi-invariant and defined by $\psi$ (within $F$) then $\Delta$ is defined by the scalar invariant formula $\tau_{\Gamma d F}(\psi)$.

(iii) If $\Delta$ is scalar-invariant and defined by $\chi$ then $K$ is defined by $\tau_{\Gamma d F}(\chi)$.

Here, definability is within $F$ by $\Lambda^+_F$-formulas with constants $\bar{c}$ interpreted as $\bar{\alpha}$. Both translations are uniform for all $F$, preserve quantifier freeness and $\Sigma_1$ and leave $\Lambda_F$ invariant.

**Proof.** Obviously, any $\theta_{\Gamma d F}(K)$ is scalar invariant and any $\theta_{\Gamma d F}$ is right invariant (by Fact 9). Also, recall that $P^d_{d_k}(A_k)$ is either 1-dimensional or zero (if $d_k = 0$).

Assume that $\Delta$ is scalar invariant. Then one has $\bar{A} \in K := \theta_{\Gamma d F}(\Delta)$ if and only if $(\vec{l}_1, \ldots, \vec{l}_n) \in \Delta$ whenever $\vec{r}_k \in P^d_{d_k}(A_k)$ for all $k$. This applies to any $\bar{r} = (\vec{l}_1, \ldots, \vec{l}_n) \in \theta_{\Gamma d F}(K)$ with suitable $\bar{A} \in K$ to yield $\bar{r} \in \Delta$. This proves $\theta_{\Gamma d F}(\theta_{\Gamma d F}(\Delta)) = \Delta$.

Assume that $K$ is right invariant and $\Delta = \theta_{\Gamma d F}(K)$. Let $\bar{B} \in \theta_{\Gamma d F}(\Delta)$, that is, there is $(\vec{l}_1, \ldots, \vec{l}_n) \in \Delta$ such that $\vec{r}_k \in P^d_{d_k}(B_k)$ for all $k$, whence for some $\bar{A} \in K$ also $\vec{r}_k \in P^d_{d_k}(A_k)$ for all $k$. By Fact 9 there are $T_k \in \text{GL}(F, d)$ such that $B_k = A_k T_k$ whence $\bar{B} \in K$ by right invariance. Thus, $\theta_{\Gamma d F}(\theta_{\Gamma d F}(K)) = K$.

(i) is now obvious. (ii) and (iii) follow, immediately, in view of Lemma 10 and by inspection of the formulas. \[\square\]

We write $\Gamma^d_d(F_{\bar{\alpha}})$ to refer to the underlying space $F^d_{\bar{\alpha}}$. Now, assume $d \geq 3$ and define $\mathbb{L}^d = \{\bar{u} \in \mathbb{L}^n \mid \text{dim } u_k = d_k\}$ and
\[
\theta_{\mathbb{L}d \alpha \Gamma} : \mathbb{L}^d \to \mathcal{P}(\Gamma^d_d(F)), \quad \theta_{\mathbb{L}d \alpha \Gamma} = \theta_{\Gamma d F} \circ \theta_{\mathbb{L}d F}|\mathbb{L}^d
\]
Recall from [5, Fact 5.1] that $L^d$ is positive primitive definable within $L^n$. $M \subseteq L^d$ is defined by $\varphi(\bar{x}) \in \Lambda_L$ within $L^d$ if $M = \{ \bar{u} \in L^d \mid L \models \varphi(\bar{u}) \}$. Theorem 6, Lemma 11, and [5, Theorem 10.4(ii)] yield, immediately, the following where

$$\tau_{\bar{L}^d_{\bar{\alpha} \Gamma}} = \tau_{\bar{F}^d_{\bar{\alpha}}} \circ \tau_{\bar{L}_F}, \quad \tau_{\bar{L}^d_{\bar{\alpha} L}} = \tau_{\bar{L}_F} \circ \tau_{\bar{L}^d_{\bar{\alpha} F}}.$$  

**Theorem 12.** Assume admissible $\bar{\alpha}$ and consider a dimension vector $\bar{d}$. $\theta_{\bar{L}^d_{\bar{\alpha} \Gamma}}$ establishes a bijection from the set of invariant subsets $M$ of $L^d$ onto the set of $\bar{\alpha}$-bi-invariant subsets $\bar{\Delta}$ of $\Gamma^d_{\bar{\alpha} L}(F)$. Moreover

1. If $M$ is defined by $\varphi$ within $L^d$ then $\theta_{\bar{L}^d_{\bar{\alpha} \Gamma}}(M)$ is $\bar{\alpha}$-bi-invariant and defined within $\Gamma^d_{\bar{\alpha} d}(F_{\bar{\alpha}})$ by the scalar invariant formula $\tau_{\bar{L}_\alpha \Gamma}(\varphi)$ – which, is in $\Sigma_1$ resp. quantifier free if so is $\varphi$ and which is in $\Lambda_{\bar{F}}$ if $\bar{\alpha} \in F^d_0$.

2. If $\bar{\Delta}$ is $\bar{\alpha}$-bi-invariant and defined by $\chi$ within $\Gamma^d_{\bar{\alpha} d}(F_{\bar{\alpha}})$ then $\theta_{\bar{L}^d_{\bar{\alpha} L}}(\bar{\Delta})$ is defined by $\tau_{\bar{L}^d_{\bar{\alpha} L}}(\chi)$ within $L^d$ – which is $\Sigma_1$ if so is $\chi$.

**Corollary 13.** Given any $\ast$-field $F$, $\alpha_i = \alpha_i^* \neq 0$ in $F$, quantifier free definable $M \subseteq L(F^d_{\alpha})$ and $d_1, \ldots, d_n \leq d$, the Plücker coordinates $\{(P_{d_1}(u_1), \ldots, P_{d_n}(u_n)) \mid \bar{u} \in M, \dim u_i = d_i (i \leq n)\}$ form a subset of the product of the Grassmannians $\bar{\Gamma}^d_{d_k}(F)$ which is a disjoint union of sets defined by basic scalar invariant $\Lambda^+_{\bar{F}}$-formulas.

This follows by Fact 2.

### 6. Homogeneous Formulas

As mentioned, earlier, Plücker coordinates are projective coordinates; thus, one should look for “homogeneous” descriptions of definable sets. First, variables will be sorted according to dimensions $k \leq d$ and come as strings of pairwise distinct members of the same sort, corresponding to the dimension $\binom{d}{k}$ of Grassmann-Plücker coordinates in dimension $k$; we write $\bar{x} \in X_k$. A $\ast$-polynomial is *homogeneous* if, after replacing $x^*$ by $x$ for each variable $x$, one obtains for each $\bar{x} \in X_k$ a homogeneous polynomial in variables $\bar{x}$, considering the others as constants (in a suitable polynomial ring). An equation in $\Lambda_F$ is *homogeneous* if it is of the form $p = 0$ with homogeneous $\ast$-polynomial $p$ with integer coefficients; a formula in $\Lambda_F$ is *homogeneous* if each of its atomic subformulas is a homogeneous equation. ‘Contradiction’ $\bot$ and ‘tautology’ $\top$ will be considered homogeneous equations.
For \( r \in \Gamma_k^d(F) \) let \( \theta_k(r) \) denote the unique subspace of \( F^d \) such that \( r \in \mathcal{P}_k^d(U) \). Also, let \( f_r \in \mathcal{F}_k \) give the pivot positions of rank \( k \) matrices \( A \) in wNF, according to Lemma 10, such that \( D_k(A) \in \theta_k(r) \). From the proof of Lemma 10 we have

**Fact 14.** For each \( f \in \mathcal{F}_k \) there are a basic homogeneous formula \( \pi_f(\bar{x}) \), \( \bar{x} \in X_k \), and a \( d \times k \) matrix \( A_f(\bar{x}) \) of integer multiples of the \( x_i \) such that, for any \( r \in \Gamma_k^d(F) \), \( F \models \pi_f(r) \) iff \( f_r = f \) and then \( r = D_k(A_f(\bar{x})) \).

Considering interpretations of homogeneous formulas we require that \( \bar{x} \in X_k \) is mapped onto some \( r \in \Gamma_k^d(F) \). Fixing the values \( \alpha_i \in F \) for the constants \( c_i \), validity of a homogeneous formula \( \varphi(\bar{x}_1, \ldots, \bar{x}_n) \) under \( \bar{x}_i \mapsto r_i \) has an obvious meaning in the atomic case, and so in general: we write \( \Gamma^d(F_\alpha) \models \varphi(r_1, \ldots, r_n) \). To be more formal, \( \Gamma^d(F_\alpha) \) is the multi-sorted structure with sorts \( \Gamma_k^d(F) \) and multi-sorted relations given by the homogeneous equations.

For simplicity, in this section we assume \( V = F^d \) with canonical basis \( \bar{v} \) which is an \( \alpha \)-basis. Thus, \( L = L(F^d_\alpha) \). For \( M \subseteq \Lambda^n \) and a dimension vector \( \underline{d} = (d_1, \ldots, d_n) \) define

\[
\theta_{\underline{d}}(M) = \{ (r_1, \ldots, r_n) \mid r_i \in \Gamma_{d_i}^d(F), (\theta_{d_1}(r_1), \ldots, \theta_{d_n}(r_n)) \in M \}.
\]

Define \( \theta(M) \subseteq \Gamma^d(F_\alpha) \) as the union of the \( \theta_{\underline{d}}(M) \) where \( \underline{d} \) ranges over all dimension vectors of length \( n \).

**Theorem 15.** Assume admissible \( \bar{\alpha} \in F^d_0 \). If \( M \) is defined within \( L(F^d_\alpha)^d \) [within \( L(F^d_\alpha)^n \), respectively] by a formula \( \varphi(\bar{x}) \) in \( \Sigma_k (\Pi_k) \), \( k \geq 1 \), then \( \theta_{\underline{d}}(M) \) \( \{\theta(M)\} \) can be defined within \( \Gamma_{d_i}^d(F_\alpha) \) [within \( \Gamma^d(F_\alpha) \)] by a homogeneous \( \Lambda_F \)-formula in \( \Sigma_k (\Pi_k) \).

**Proof.** We prove special cases, first.

(i) For any \( d_0 = d_1 \leq d \) there is a conjunction \( \eta_{d_0d_1}(\bar{x}_0, \bar{x}_1) \) of homogeneous equations (where \( \bar{x}_i \in X_{d_i} \)) such that, for any \( \bar{r}_i \in \Gamma_{d_i}^d(F) \), \( \theta_{d_0}(\bar{r}_0) = \theta_{d_1}(\bar{r}_1) \) iff \( F \models \eta_{d_0d_1}(\bar{r}_0, \bar{r}_1) \).

(ii) For any \( d_1, d_2 \leq d_0 \leq d \), \( d_0 \leq d_1 + d_2 \), there is a conjunction \( \sigma_{d_0d_1d_2}(\bar{x}_0, \bar{x}_1, \bar{x}_2) \) of homogeneous equations (where \( \bar{x}_i \in X_{d_i} \)) such that, for any \( \bar{r}_i \in \Gamma_{d_i}^d(F) \), one has \( \theta_{d_0}(\bar{r}_0) = \theta_{d_1}(\bar{r}_1) + \theta_{d_2}(\bar{r}_2) \) iff \( F \models \sigma_{d_0d_1d_2}(\bar{r}_0, \bar{r}_1, \bar{r}_2) \).

(iii) Assume admissible \( \bar{\alpha} \in F^d_0 \), \( d_i \leq d \), \( d_0 + d_1 = d \). There is a conjunction \( \kappa_{d_0d_1}(\bar{x}_0, \bar{x}_1) \) of homogeneous equations (where \( \bar{x}_i \in X_{d_i} \)) such that, for any \( \bar{r}_i \in \Gamma_{d_i}^d(F) \), one has \( \theta_{d_0}(\bar{r}_0) = (\theta_{d_1}(\bar{r}_1))^{\perp} \) iff \( F \models \kappa_{d_0d_1}(\bar{r}_0, \bar{r}_1) \).

In all cases, we first consider \( f_i \in \mathcal{F}_{d_i} \) and construct formulas to be applied only to \( r_i \in \Gamma_{d_i}^d(F) \) with \( f_{\bar{r}_i} = f_i \). Form the matrices \( A_{f_i} = \).
\( A_{f_i}(\bar{x}_i) \) of integer multiples of variables form \( \bar{x}_i \), according to Fact [14].

First, assume \( d_0 < d \). In (i) and (ii) form the compound matrices \((A_{f_0}|A_{f_1})\) resp. \((A_{f_0}|A_{f_1}|A_{f_2})\); for \( d_0 < d \), the formula \( \eta_{f_0|f_1} \) in (i) states that any \( d_0 + 1 \times d_0 + 1 \)-subdeterminant is 0; \( \sigma_{f_0|f_1|f_2} \) in (ii) states that any \( d_0 + 1 \times d_0 + 1 \)-subdeterminant of \((A_{f_0}|A_{f_1}|A_{f_2})\) is 0 but some \( d_0 \times d_0 \)-subdeterminant of \((A_{f_1}|A_{f_2})\) is not 0. In (iii) form the compound matrix \((A_{f_0}|Q^{f_1}(A_{f_1})\) (where \( Q^{f_1}(X) \) is as in (iii) of Fact 7) and require by means of \( \kappa_{f_0|f_1} \) all \( d_0 + 1 \times d_0 + 1 \)-subdeterminants to be 0. Now, put

\[
\sigma_{d_0d_1d_2} \equiv \bigvee_{f_i \in F_{d_i}, i=0,1,2} \bigwedge_{i=0}^2 \sigma_{f_i}(\bar{x}_i) \land \sigma_{f_0f_1f_2}(\bar{x}_0, \bar{x}_1, \bar{x}_2)
\]

and similarly for \( \eta_{d_0d_1} \) and \( \kappa_{d_0d_1} \). In case \( d_0 = d \) nothing is required in (i) and (iii), only the second part in (ii).

For proving the theorem, it suffices to consider \( \varphi \) a conjunction (disjunction) of equations and negated equations and to derive a translation in \( \Sigma_1 \) (\( \Pi_1 \)). Then \( \varphi'' \) from Fact [3] has the same form as \( \varphi \). We use \( \xi \) and \( \kappa \) as names for variables occurring in \( \varphi' \) or \( \varphi'' \). Consider maps \( \delta \) associating with each \( \xi \) a dimension \( \delta(\xi) \in \{0, \ldots, d\} \). For each \( \delta \) and \( \xi \) choose a specific vector \( \xi^\delta \in X_{\delta(\xi)} \) of variables. Call \( \delta \) admissible for \( \varphi \) if \( \delta(x_i) = d_i \) for all \( i \) and if for each special equation in \( \varphi' \) the relevant dimension restrictions are satisfied:

\[
\begin{align*}
\delta(\xi_1), \delta(\xi_2) &\leq \delta(\xi_0) \leq \delta(\xi_1) + \delta(\xi_2) & \text{for } \xi_0 = \xi_1 + \xi_2 \\
\delta(\xi_0) = d - \delta(\xi_1) &\quad \text{for } \xi_0 = \xi_1^d \\
\delta(\xi_0) = \delta(\xi_1) &\quad \text{for } \xi_0 = \xi_1 \\
\delta(\xi_0) = 0 &\quad \text{for } \xi_0 = 0
\end{align*}
\]

Let \( D \) denote the set of all admissible \( \delta \). Observe that any assignment of values \( \bar{u} \in L^n \) to \( \bar{x} \) gives rise to \( \delta \in D \): the dimensions of values \( t(\bar{u}) \) of subterms \( t(\bar{x}) \) under the evaluation of \( \varphi \).

Given \( \delta \in D \), we define the translation \( \tau^{Q_\delta} \), \( Q \in \{\exists, \forall\} \), first for the special equations making up \( \varphi' \) and let \( \tau^{Q_\delta}(\varphi') \) denote the conjunction of all these.

\[
\begin{align*}
\tau^{Q_\delta}(\xi_0 = \xi_1 + \xi_2) &\equiv \sigma_{\delta(\xi_0),\delta(\xi_1),\delta(\xi_2)}(\xi_\delta^\delta, \xi_1^\delta, \xi_2^\delta) \\
\tau^{Q_\delta}(\xi_0 = \xi_1^d) &\equiv \kappa_{\delta(\xi_0),\delta(\xi_1)}(\xi_0^\delta, \xi_1^\delta) \\
\tau^{Q_\delta}(\xi_0 = 0) &\equiv \top.
\end{align*}
\]

Equalities in \( \varphi'' \) are translated as

\[
\tau^{Q_\delta}(\xi_0 = \xi_1) \equiv \begin{cases} 
\eta_{\delta(\xi_0),\delta(\xi_1)}(\xi_\delta^\delta, \xi_1^\delta) & \text{if } \delta(\xi_0) = \delta(\xi_1) \\
\bot & \text{if } \delta(x_0) \neq \delta(\xi_1).
\end{cases}
\]

For a negated equality \( \beta \equiv \neg(\xi_0 = \xi_1) \) occurring in \( \varphi'' \) we define \( \tau^{Q_\delta}(\beta) \) as \( \neg\eta_{\delta(\xi_0),\delta(\xi_1)}(\xi_\delta^\delta, \xi_1^\delta) \) if \( \delta(\xi_0) = \delta(\xi_1) \); otherwise, \( \tau^{Q_\delta}(\beta) \equiv \top \) and
Now, \( \tau^{\delta}(\varphi) \equiv \bot \). Now, \( \tau^{\delta}(\varphi'(\varphi'')) \) is the conjunction of all these, \( \tau^{\delta}(\varphi'(\varphi'')) \) the disjunction (recall the assumption on \( \varphi \) and \( \varphi'' \)). With \((x_1, x_2, \ldots)^{\delta} = (a_1^\delta, x_2^\delta, \ldots)\), and similarly for \( y^\delta \) and \( z^\delta \), we finally arrive at the translations

\[
\begin{align*}
\tau^\delta(\varphi(x)) &\equiv \bigvee_{\delta \in D} \exists z \exists y \left( \tau^\delta(\varphi(x))(x^\delta, z^\delta, y^\delta) \right) \\
\tau^\delta(\varphi(x)) &\equiv \bigwedge_{\delta \in D} \forall z \forall y \left( \tau^\delta(\varphi(x))(x^\delta, z^\delta, y^\delta) \right)
\end{align*}
\]

For the proof just observe that any substitution for the \( x_i \) in \( \Gamma^d(F_{\bar{\alpha}}) \) gives rise to a substitution for the \( z_j, y_k \) such that the corresponding assignment of subspaces satisfies \( \varphi' \), that is, with the associated dimensions given by \( \delta \in D \), the assignment in \( \Gamma^d(F_{\bar{\alpha}}) \) satisfies \( \tau^{\delta}(\varphi') \) and is, in the projective setting, uniquely determined by the values of the \( x_i \). This provides a translation \( \tau^{\delta}(\varphi) := \tau^{\delta}(\varphi') \) in case of fixed dimension vector \( \bar{d} \). To obtain a defining formula for \( \theta(M) \) within \( \Gamma^d(F_{\bar{\alpha}}) \), one just has to form the disjunction of the \( \tau^{\delta}(\varphi) \) with \( \bar{d} \) ranging over all \( \bar{d} = (d_1, \ldots, d_n) \), \( d_i \leq d \).

Recall that \( \leq \) may be considered a fundamental relation of \( \Lambda_L \) or defined by \( x \leq y \iff x + y = y \). Modifying (i) and (iii), allowing \( d_0 \leq d_1 \) respectively \( d_0 + d_1 \leq d \), one obtains the following.

**Corollary 16.** Assume admissible \( \bar{\alpha} \in F_{\bar{\alpha}} \). If \( M \) is defined within \( L(F_{\bar{\alpha}})^n \) by inequalities of the form \( x_i \leq x_j \) and \( x_i \leq x_j^k \) then \( \theta_{\bar{d}}(M) \) can be defined within \( \Gamma_{\bar{d}}^d(F_{\bar{\alpha}}) \) by a quantifier free homogeneous \( \Lambda_F \)-formula.

Given a partially ordered set \( P \) with involution and a space \( V \), an example of such \( M \) is obtained as the set of all representations of \( P \) in \( V \). Of course, the Corollary applies to \( M \) defined by a conjunction of basic equations. An extension to equations given by compound terms appears doubtful; anyway, the approach of Theorem 6 hardly can be modified to preserve homogeneity.

### 7. Translating lattice formulas

We show that on the lattice side quantifier free definability amounts to definability by equations \( t = 0 \) and \( s = 1 \). For this, we refer to the concept of frame which underlies the coordinatization of modular lattices (requiring \( d \geq 3 \)): A frame of \( L = L(V) \), \( \dim V = d \), is a system \( \bar{a} = (a_{ij} \mid 1 \leq i, j \leq d) \) of elements such that for pairwise distinct \( i, j, k \) (where \( a_i = a_{ii} \))

\[
\begin{align*}
1 &= \bigoplus_{\ell} a_{\ell}, \quad a_{ij} = a_{ji}, \quad a_i + a_j = a_i \oplus a_{ij}, \quad a_{ik} = (a_i + a_k) \cap (a_{ij} + a_{jk}).
\end{align*}
\]
Such are in correspondence with bases \( \bar{v} \) via \( a_{i i} = v_i F, a_{ij} = (v_j - v_i)F \). Given a frame, for any \( i \neq j \) there is an isomorphism \( \omega_{ij}^a \) of the field \( F \) onto \( R_{ij}(\bar{a}) = \{ u \in L \mid u \oplus a_i = a_i + a_j \} \) endowed with operations given by lattice terms with constants from \( \bar{a} \), including the operation \( r \to r^{-1} \) where \( 0^{-1} = 0 \). For a corresponding basis one has \( \omega_{ij}^a(r) = (v_j - rv_i)F \).

In [5] Lemma 10.2], frames \( \bar{a} \) associated with \( \bot \)-bases \( \bar{v} \) such that \( \bar{a} = [\bar{v}] \) have been characterized by additional relations. Such frames are called \( \alpha \)-frames; for such, there is also a term in \( \Lambda_F \) defining an involution on \( R_{11}(\bar{a}) \) such that \( \omega_{11}^a \) becomes an isomorphism of \( \ast \)-fields. For \( \bar{a} = (1, \ldots, 1) \), \( \alpha \)-frames are orthonormal and the additional relations amount to \( a_j \ominus a_j a_{ij} = (a_i + a_j) \cap a_{ij}^2 \) where \( \ominus \) is the term describing subtraction in \( R_{ij}(\bar{a}) \). (cf. proof of [4] Theorem 2.7).

Given \( d \geq 3 \), \( \varphi(\bar{x}) \in \Lambda_F \), \( \bar{x} = (x_1, \ldots, x_n) \), define \( M_{\varphi} = M_{\varphi}(F) \) to consist of all \( (\bar{a}, \bar{b}) \in L \) such that \( \bar{a} \) is an \( \alpha \)-frame and \( \bar{b} \in R_{11}^2 \) satisfying \( \varphi(\bar{x}) \). Observe that \( M_{\varphi} \) consists of tuplets of 1-dimensionals, only; let \( \vec{d} \) the corresponding dimension vector.

**Fact 17.** Given an \( \alpha \)-frame in \( L \) and quantifier free \( \varphi(\bar{x}) \in \Lambda_F \) there is a term \( t(\bar{z}, \bar{x}) \) in \( \Lambda_L \) such that \( M_{\varphi} = \{ (\bar{a}, \bar{b}) \mid t(\bar{a}, \bar{b}) = 0 \} \).

We may also assume \( F \) endowed with the operation \( r \mapsto r^{-1} \) where \( 0^{-1} = 0 \) and the symbol \( -1 \) included into the language \( \Lambda_F \).

**Proof.** Observe that a conjunction of equations \( t_i = 0 \) can be comprised to a single one \( \sum_i t_i = 0 \); similarly, \( t_i = 1 \to \bigcap_i t_i = 1 \). Also, \( t = 1 \) is equivalent to \( t^\perp = 0 \). Thus, it suffices to give a list of equations \( t_i = 0 \) and \( s_j = 1 \) which defines \( M_{\varphi} \).

First, we have to provide such equations in variables \( z_{ij} \) defining frames (writing \( z_i \) for \( z_{ii} \)). Put \( y_{ij} = \sum_{k \neq i, j} z_k \) and \( y_i = y_{ii} \). That \( 1 = \bigoplus a_k \) can be expressed by \( \sum_i z_i = 1 \) and \( \bigcap_i y_i = 0 \). Now, an equation \( a \oplus a_i = a_i + a_j \) can be captured by \( (x + z_i + z_j) \cap y_{ij} = 0 \), \( x \cap z_i = 0 \), and \( x + y_j = 1 \); and to require \( a = b \) where \( b \) is substituted for \( y \) such that \( b \oplus a_i = a_i + a_j \) we may use \( (x + y) \cap z_i = 0 \). Orthogonality of the frame is captured by \( y_i \cap (y_i^\perp + y_{ii}) = 0 \).

Recall that \( \varphi \) is equivalent in \( F \) to a disjunction of conjunctions of equations \( p_i(\bar{x}) = 0 \) and \( q_j(\bar{x}) \neq 0 \) for terms in \( \Lambda_F \) and that any disjunction of equations \( t_i = 0 \) in \( \Lambda_L \) can be comprised into a single one (cf. [5] Fact 6)). Now, for any term \( p(\bar{x}) \) in \( \Lambda_F \) there is a term \( \hat{p}(\bar{x}) \) in \( \Lambda_L \) having value \( p(\bar{b}) \) in \( R_{21} \) for any \( \alpha \)-frame \( \bar{a} \) and \( \bar{b} \in R_{21}(\bar{a}) \). By this, \( p_i(\bar{x}) = 0 \) can be expressed by \( (\hat{p}_i(\bar{x}) + z_1) \cap z_2 = 0 \) and \( q_i(\bar{x}) \neq 0 \) by \( \hat{q}(\bar{x}) \cap z_1 = 0 \). \( \square \)
In view of Theorem 15 one obtains the following, improving Theorem 10.4(ii) in [5] for this special case.

**Corollary 18.** Let \( d \geq 3, \mathbf{d} = (1, \ldots, 1) \), and \( \bar{\alpha} \in F_0^d \). Within \( \Gamma^d(F_0) \), a subset \( S \) is definable by a homogeneous \( \Sigma_1 \)-formula from \( \Lambda_F \) if and only if \( S = \theta_{\bar{\alpha}}(M) \) for some \( M \) definable within \((L(F)^d)_{\mathbf{d}} \) by some formula \( \exists y.t(\bar{x}, y) = 0 \).

8. COUNTEREXAMPLES

Now, we shall give examples where on the analytic side there are restrictions on the possible descriptions. For simplicity, we assume \( F \) a \( * \)-subfield of the complex number field with conjugation and \( V \) endowed with canonical scalar product w.r.t. some basis, in particular \( \bar{\alpha} = (1, \ldots, 1) \). In view of this, we may omit reference to \( \bar{\alpha} \) and identify \( V \) with \( F^d \) -turning the canonical basis \( \bar{v} \) into an \( \alpha \)-basis. Thus, \( L = L(F_\bar{\alpha}^d) \). Observe that \( \theta_{L_{d^d}} = \theta_{F_{d^d}} \circ \theta_{L_{dF}} \) by Proposition 5. Considering the multi-sorted structure \( \Gamma^d(F_\bar{\alpha}) \) as in Section 13, for \( M \subseteq L^n \) we have \( \theta(M) \) the union of the \( \theta_{L_{d^d}} \) where \( \mathbf{d} \) ranges over all dimension vectors of length \( n \).

**Fact 19.** Let \( \varphi \in \Lambda_L \) and \( \psi \in \Lambda_F \) quantifier free formulas such that \( \psi \) defines in \( \Gamma^d(F_\bar{\alpha}) \) the relation \( \theta(M) \) where \( M \) is defined in \( L(F_\bar{\alpha}^d) \) by \( \varphi \). Then, for any \( F' \) a \( * \)-subfield of \( F \), \( \varphi \) and \( \psi \) are related in the same way.

**Proof.** Observe that \( L(F_\mathbf{d}^d) \) is embedded into \( L(F_\bar{\alpha}^d) \) by tensoring with \( F \) and the same applies to \( \Gamma^d(F_\bar{\alpha}) \) and \( \Gamma^d(F_\bar{\alpha}) \). W.r.t. the canonical basis \( \bar{v} \) of \( F_\mathbf{d}^d \), the multi-sorted structure with sorts \( L(F_\mathbf{d}^d) \) and \( \Gamma^d(F_\bar{\alpha}) \) and relation \( \theta \) becomes a substructure of the analogous one over \( F \). Also, there is an obvious quantifier free formula relating the models of \( \varphi \) and \( \psi \) via \( \theta \) over \( F \). And validity of this formula is inherited by the substructure. \( \square \)

The examples use the fact that with a formula \( \varphi \) in \( \Lambda_F \) and defining formula \( \psi \) for \( M_\varphi \) one can associate a formula \( \psi' \) in \( \Lambda_F \) which is equivalent to \( \varphi \) and inherits structural properties from \( \psi \). This is done as follows. Put \( v_{ii} = v_i \) and \( v_{ij} = v_j - v_i \) for \( i \neq j \), and \( \bar{v} = (v_{ij}F|i, j \leq d) \); also, given \( \bar{r} \in F_\mathbf{n} \) put \( \bar{r} = ((v_1 - r_kv_2)F|k \leq n) \). In view of the isomorphism \( \omega_{21} \) it follows that \( F \models \varphi(\bar{r}) \) if and only if \( (\bar{v}, \bar{r}) \in \theta(M_\varphi) \). Considering any formula \( \psi(\bar{z}, \bar{y}) \) defining \( \theta(M_\varphi) \), the latter is equivalent to \( F \models \psi(\bar{v}, \bar{r}) \). Choosing the canonical basis \( \bar{v} \), substituting \( \bar{v} \) for \( \bar{z} \) in \( \psi \) and, simultaneously, \( \bar{w} \) for \( \bar{y} \) where \( w_k = (v_1 - x_kv_2)F \), one obtains a formula \( \psi'(\bar{x}) \) equivalent to \( \varphi(\bar{x}) \) in \( F \).
Example 20. Let $F$ a *-subfield of $\mathbb{C}$. $\theta(M_\varphi)$ cannot be defined in $\Gamma^d(F_\alpha)$ by any formula $\psi$ such that

(i) $\psi$ is quantifier free and positive; here $\varphi$ is $x \neq 0$.
(ii) $\psi$ is a conjunction of equations and negated equations; here $\varphi$ is $x_1 = 0 \Leftrightarrow x_2 \neq 0$.
(iii) $\psi$ does not not involve involution; here $\varphi$ is $x_2 = x_1^*$ and $F = \mathbb{C}$.

Though, for all these $\varphi$, $M_\varphi$ can be defined within $L$ by an equation $t = 0$.

Proof. Definability within $L$ follows from Fact [17]. For the negative claims consider the associated $\psi'(x)$ and derive contradictions. In (i) and (ii) we may assume $F = \mathbb{Q}$ and $\psi'(x)$ be of the same form as $\psi$ with atomic formulas $p_h(x) = 0$, $p_h(x) \in \mathbb{Q}[x]$. Thus, in (i) $\psi'(x)$ is built from equations $p_h(x) = 0$ by conjunction and disjunction; since $\mathbb{Q} \models \neg \psi'(0)$, $\psi'(x)$ can have only finitely many satisfying assignments in $\mathbb{Q}$, in contrast to $\varphi(x)$. In (ii) $\psi'(x_1, x_2)$ would be equivalent to some $\bigwedge_{h=1}^m p_h(x_1, x_2) = 0 \land q(x_1, x_2) \neq 0$. Observe that for any $r_1 \in \mathbb{Q}$ there are infinitely many $r_2$ such that $\mathbb{Q} \models \varphi(r_1, r_2)$ which implies that the $p_h$ are zero-polynomials. Thus, $\varphi(x_1, x_2)$ would be equivalent in $\mathbb{Q}$ to $q(x_1, x_2) \neq 0$. Contradiction, since $\neg \varphi(x_1, x_2)$ defines a set which is not closed.

(iii) Assuming $\theta(M_\varphi)$ definable over the field $\mathbb{C}$, any automorphism of $\mathbb{C}$ would leave $\theta(M_\varphi)$ invariant. To arrive at a contradiction, consider an irreducible $p(x) \in \mathbb{Q}[x]$ of odd degree. There is an automorphism $\omega$ of $\mathbb{C} \to \mathbb{C}$ mapping some zero $a \in \mathbb{R}$ of $p$ to a zero $b \not\in \mathbb{R}$, that is $\omega a^* \neq (\omega a)^*$. Now consider $\tilde{a}$ the frame given by the canonical basis $\tilde{v}$ and $\tilde{b} = ((v_1 - av_1)C,(v_1 - a^*v_2)C))$. Then $(\tilde{a},\tilde{b})$ is in $\theta(M_\varphi)$ but its image under $\omega$ is not. $\square$

In view of Lemma [11] observe the following. If $\psi(X)$ defines bi-invariant $K \subseteq F^{d \times d}$ where $d = (1, \ldots, 1) \in \mathbb{N}^m$ then $\theta_{F^{d \times d}}(K)$ is defined within $\Gamma^d_d(F) = (F^d \setminus \{0\})^m$ by (scalar invariant) $\chi \equiv \psi (\overline{P(y_1)}, \ldots, \overline{P(y_m)})$ where the $d \times d$-matrix $\overline{P(y)}$ has first column $y$, zero else. In particular, if $\psi$ defines $\theta_{L_0E}(M)$, where $M$ consists of $\bar{u}$ with all dim $u_i = 1$, then $\chi$ defines $\theta_{L_0E}(M)$ within $\Gamma^d_d(F) = (F^d \setminus \{0\})^m$, that is, within the $d - 1$-dimensional projective space over $F$. Thus, the above counterexamples apply to $\theta_{L_0E}$ as well, $\tilde{v}$ the canonical basis.

9. Absence of involution

We speak of absence of involution if $F$ is just a field, $V$ a vector space, $\bar{v}$ any basis, and $L$ the lattice of all linear subspaces of $V$, $\Lambda_L^*$
and $\Lambda'_F$ the languages of bounded lattices and rings, respectively, having multivariate polynomials in place of $\ast$-polynomials (with integer coefficients).

We consider $\Lambda'_L$ a subset of $\Lambda_L$, generated from variables by the operations $+, 0, t \cap s := (t^\perp + s^\perp)^\perp$, and $1 := 0^\perp$. Observe that for a lattice $L$ admitting some involution, terms from $\Lambda'_L$ can be evaluated and the value does not depend on the choice of involution. This allows to transfer results to this case, introducing an involution on $L$ endowing $F$ with the identity involution and $V$ with a form declaring some basis orthonormal: that is $\bar{\alpha} = (1, \ldots, 1)$. In this setting, one may read $\Lambda_F$ formulas as such in $\Lambda'_F$, just omitting $\ast$.

**Corollary 21.** In absence of involution, the following remain valid (mutatis mutandis): Theorem 6, Theorem 12, Theorem 15, and Corollary 18; moreover Fact 17 and Example 20 (i), (ii) with a conjunction of equations of the form $t_i = 0, s_j = 1$.

Alternatively, in order to deal with meets in Theorem 6 one may refer to the (column) Zassenhaus algorithm, that is, given $A \in F^{d \times k}$ and $B \in F^{d \times \ell}$ transforming the $2d \times (k + \ell)$-matrix

\[
\begin{pmatrix}
A & B \\
A' & 0 \\
A'' & 0
\end{pmatrix}
\]

into wNF

\[
\begin{pmatrix}
A' & 0 & 0 \\
A'' & B' & 0
\end{pmatrix}
\]

with $A'$ a $d \times m$-matrix, $m = \text{rk}(A)$, and $\text{Span}(B') = \text{Span}(A) \cap \text{Span}(B)$.

This gives translations as in Section 4 valid for any choice of basis. In particular, the formulas and terms in Lemma 8 are in $\Lambda_F$. In the proof for meets, we proceed as for joins, with $d \times d$-matrix $M^\sigma_f(X|Y)$ in $\Lambda_F$ yielding, under appropriate distinction of cases, $\text{Span}(A) \cap \text{Span}(B) = \text{Span}(M^\sigma_f(A|B))$; here $M^\sigma_f$ describes the calculations in the Zassenhaus algorithm.

Given a partially ordered set $P$, by Corollary 18 the set of all representations of $P$ within a given vector space $V$ (that is, order preserving maps $P \rightarrow L(V)$) gives rise to a subset of $\Gamma^d(F)$ defined by homogeneous quantifier free formulas. It remains to clarify how this is related to the quiver Grassmannians (cf. [8]) of quivers derived from partially ordered sets.

10. **Hidden coordinates**

Quantifier free translations from $F$ to $L$ can be obtained for quantifier free $\Lambda_F$-formulas which implicitly provide coordinate systems –
similarly to the version of Cayley factorization due to [11]. Assume that $V$ admits an ON-basis, that is $\bar{\alpha} = |\bar{v}| = (1, \ldots, 1)$.

We say that $\psi(\bar{X})$ in $\Lambda_F$ hides coordinate systems if there are finitely many tuplets $\bar{a}^k(\bar{x})$ and $\bar{t}^k(\bar{z})$ of $\Lambda_L$-terms such that, for any $\bar{u} \in L^n$, if $L \models \tau_{\bar{a} \bar{t}}(\psi)(\bar{u})$ then there is $k$ such that $\bar{a}^k = \bar{a}^k(\bar{u})$ is an orthonormal frame and $\bar{u} = \bar{t}^k(\bar{a}^k)$.

**Proposition 22.** Assume $d \geq 3$ and that $V$ admits ON-bases. If quantifier free $\psi(\bar{X})$ hides coordinate systems and defines $\bar{\alpha}$-bi-invariant $K \subseteq (F^{d \times d})^n$ then $M = \theta_{\bar{a} \bar{t}}(K)$ can be defined by quantifier free $\varphi(\bar{x})$.

**Proof.** We put $\varphi(\bar{x}) \equiv \bigvee_k \varphi_k(\bar{x})$ where, for any $k$, $\varphi_k(\bar{x})$ is the quantifier free formula
\[
\bigwedge_{i=1}^n x_i = t^k(\bar{a}^k(\bar{x})) \land \tau_{\bar{a} \bar{t}}(\psi)(\bar{x}, \bar{a}^k(\bar{x})).
\]

By hypothesis, if $L \models \tau_{\bar{a} \bar{t}}(\psi)(\bar{u})$ then $L \models \varphi(\bar{u})$. Conversely, if $L \models \varphi_k(\bar{u})$ then $\bar{a}^k(\bar{u})$ witnesses $\exists \bar{z}$ in $L \models \tau_{\bar{a} \bar{t}}(\psi)(\bar{u})$. \qed

In absence of involution (where ‘frames’ correspond to bases), one may find sufficient conditions for hidden coordinate systems using $m$-tuplets $\bar{w}$ in $L$ which are associated to frames via tuplets $\bar{a} (\bar{y})$ and $\bar{s} (\bar{z})$ of lattice terms, that is, $\bar{a} (\bar{w})$ is a frame of $L$ and $\bar{w} = \bar{s} (\bar{a} (\bar{w}))$; in particular, the sublattice generated by $\bar{w}$ is isomorphic to $L(F_0^d)$, $F_0$ the prime subfield of $F$. For $d = 3$ and $m = 4$, such quadruples are given by 4 points no 3 of which are collinear, or by 3 non-collinear points and a line incident with none of these, or the duals of such. More generally, for fixed $d \geq 3$ and $m \geq 4$ there are only finitely many isomorphism types of $m$-tuplets which are associated to frames and a finite collection of terms providing witnesses for these associations, uniformly for all $F$ (cf. [1, 3]). Using these terms, for any fixed $d \geq 3$, one obtains finitely many tuplets $\bar{w}^\ell (\bar{x})$ and $\bar{v}^\ell (\bar{z})$ of lattice terms such that $\psi(\bar{X}) \in \Lambda_F$ hides coordinate systems provided that, for any $\bar{u} \in L^n$, if $L \models \tau_{\bar{a} \bar{t}}(\psi)(\bar{u})$ then there is $\ell$ such that $\bar{w} = \bar{w}^\ell (\bar{u})$ is associated to a frame and $\bar{u} = \bar{v}^\ell (\bar{w})$.

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DEFINABLE RELATIONS

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(Christian Herrmann) Technische Universität Darmstadt FB4, Schloßgartenstr. 7, 64289 Darmstadt, Germany
E-mail address: herrmann@mathematik.tu-darmstadt.de

(Martin Ziegler) KAIST School of Computing, 291 Daehak-ro, Yuseong-gu, Daejeon, South Korea 34141
E-mail address: ziegler@cs.kaist.ac.kr