Quantum incommensurate skyrmion crystals and commensurate to in-commensurate transitions in cold atoms and materials with spin–orbit couplings in a Zeeman field

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Keywords: Rashba spin orbit coupling on a lattice, quantum commensurate to in-commensurate transitions, non-coplanar incommensurate skyrmion crystal, bosonic Lifshitz transition, in-commensurate magnons

Abstract

In this work, we study strongly interacting spinor atoms in a lattice subject to a two dimensional (2d) anisotropic Rashba type of spin orbital coupling (SOC) and an Zeeman field. We find the interplay between the Zeeman field and the SOC provides a new platform to host rich and novel classes of quantum commensurate and in-commensurate phases, excitations and phase transitions. These commensurate phases include two collinear states at low and high Zeeman field, two co-planar canted states at mirror reflected SOC parameters respectively. Most importantly, there are non-coplanar incommensurate Skyrmion (IC-SkX) crystal phases surrounded by the four commensurate phases. New excitation spectra above all the five phases, especially on the IC-SkX phase are computed. Three different classes of quantum commensurate to in-commensurate transitions from the IC-SkX to its four neighboring commensurate phases are identified. Finite temperature behaviors and transitions are discussed. The critical temperatures of all the phases can be raised above that reachable by current cold atom cooling techniques simply by tuning the number of atoms \( N \) per site. In view of recent impressive experimental advances in generating 2d SOC for cold atoms in optical lattices, these new many-body phenomena can be explored in the current and near future cold atom experiments. Applications to various materials such as MnSi, Fe0.5Co0.5Si, especially the complex incommensurate magnetic ordering in Li2IrO3 are given.

1. Introduction

Quantum phases and phase transitions in quantum spin systems have been an important and vigorous research field in material science for many decades [1–3]. However, so far, most of these quantum phases are collinear phases in a bipartite lattice or co-planar commensurate phases in a geometrically frustrated lattice. The associated quantum phase transitions are commensurate to commensurate (C-C) ones. During the last decade, the investigation and control of spin–orbital coupling (SOC) [4] have become the subjects of intensive research in both condensed matter and cold atom systems after the discovery of the topological insulators [5, 6]. In the condensed matter side, there are increasing number of new quantum materials with significant SOC, including several new 4d or 5d transition metal oxides and heterostructures of transition metal systems [7]. In the cold atom side, there were series advances to generate Abelian gauge flux and quantum spin Hall effects in optical lattices [8–17]. Several groups worldwide [18–20] have also successfully generated a 1D (SOC) to neutral atoms. However, one of the main limitations to extend 1D SOC to a 2D SOC is the associated heating rates. Recently, there are also some advances [21–24] to overcome this difficulty in generating 2D Rashba SOC for cold atoms in both continuum and optical lattices and also in a Zeeman field. Most recently, a long-lived SOC gas of the high
magnetic fermionic element dysprosium to eliminate the heating due to the spontaneous emission, has been created in \[25\]. In view of these recent experimental advances, novel superfluid or magnetic phenomena due to the interplay among tunable interactions, SOC and a Zeeman field are ready to be investigated in near future experiments on both fermion and spinor BEC. It becomes topical and important to investigate what would be new phenomena due to such an interplay in both cold atoms and condensed matter systems.

In a recent work \[26\], we studied interacting spinor bosons at integer fillings loaded in a square optical lattice in the presence of non-Abelian gauge fields, in the strong coupling limit, it leads to the spin \(S = N/2\) rotated ferromagnetic Heisenberg model (RFHM) (equation (1) with \(\vec{H} = 0\)) which is a new class of quantum spin models to describe quantum magnetisms in cold atom systems or some materials with strong SOC \[26\]. Along the anisotropic one \((\alpha = \pi/2, 0 < \beta < \pi/2)\) of the 2d SOC, we identified a new spin–orbital entangled commensurate ground state: the Y-x state. It supports not only commensurate magnons \((\text{C}-\text{C}-\text{C}, \text{C}-\text{C})\), but also a new gapped elementary excitation: in-commensurate magnon \((\text{C}-\text{IC})\). The C-IC magnons may become the seeds to drive possible new classes of quantum C-IC transitions under various external probes. In this paper, we study possible dramatic effects of an external Zeeman field \(H\) applied to the RFHM equation (1). We find that the interplay among the strong interactions, SOC and the Zeeman field leads to a whole new classes of magnetic phenomena in quantum phases (especially the non-coplanar incommensurate Skyrmion crystals (IC-SkX)), excitation spectra (especially inside the IC-SkX), quantum phase transitions (especially the quantum Commensurate to incommensurate transitions (C-IC) transitions), which have wide and important applications in both cold atoms and various materials with SOC. Our main results are summarized in figures 1 and 2. We also discuss the finite temperature behaviors and finite temperature phase transitions above the \(T = 0\) quantum phases in figures 1 and 2. Particularly, we point out that any spin \(S = N/2\) of the RFHM can be simply achieved by tuning the number of atoms \(N\) per site, the critical temperatures of all the phases \(T_{c}\) are \(2S = N\) in 2 dimension can be easily increased above that reachable by the current cold atom cooling techniques. In view of recent impressive experimental advances in generating 2d SOC for cold atoms in optical lattices, these new many-body phenomena can be explored in the current and near future cold atom experiments. On the other hands, the SOC materials with a total spin \(J = 1/2\) automatically fall into the strong coupling regime. So we also discuss the applications of figures 1 and 2 to various materials such as MnSi, Fe_{0.5}Co_{0.5}Si, especially the complex incommensurate magnetic ordering in Li2IrO3.
The spin $S = N/2$ RFHM at a generic SOC parameters ($\alpha$, $\beta$) in a Zeeman field $H$ is [26]:

$$
\mathcal{H}_{\text{RFHM}} = -J \sum_i [S_i \cdot R(\hat{x}, 2\alpha) S_{i+\hat{x}} + S_i \cdot R(\hat{y}, 2\beta) S_{i+\hat{y}}] - H \cdot \sum_i S_i,
$$

where the $R(\hat{x}, 2\alpha)$, $R(\hat{y}, 2\beta)$ are two SO(3) rotation matrices around the $\hat{x}$, $\hat{y}$ spin axes by angle $2\alpha$, $2\beta$ putting along the two bonds $x$, $y$ respectively, $H$ is the Zeeman field which could be induced by the Raman laser in the cold atom set-ups [9–17]. See section 8 for its implications on SOC materials and cold atoms.

As shown in [26], the RFHM along the line ($\alpha = \pi/2$, $0 < \beta < \pi/2$) at $H = 0$ has the translational symmetry, the time reversal $T$, the three spin–orbital coupled $Z_i$ symmetries $P_T$, $P_x$, $P_z$. Most importantly, it also owns a hidden spin–orbital coupled $U(1)_{\text{loc}}$ symmetry generated by $U_1(\phi) = e^{i\phi \sum \delta_i S_i^z}$. In fact, at the two ends of the line $\beta = 0$ and $\beta = \pi/2$, we reach the AFM Heisenberg model in the rotated basis 

$$
\mathcal{H} = \sum_{i<j} S_i \cdot S_j,
$$

where $S_i = R(\hat{x}, \pi i\alpha) S_i R(\hat{y}, \pi i\beta) S_i$ respectively. So the Hamiltonian has the SU(2) symmetry in the rotated basis $SU(2)$ and $SU(2)$ at $\beta = 0$ and $\beta = \pi/2$ respectively. Transferring back to the original basis, the SU(2) symmetry is generated by $\sum_i (\alpha + \beta) S_i^z$, $\sum_i (1 - \beta) S_i^z$ at $\beta = 0$ and by $\sum_i (\alpha + \beta) S_i^z$, $\sum_i (1 - \beta) S_i^z$, $\sum_i (-1)^{\pi \alpha} S_i^z$ at $\beta = \pi/2$ respectively. Both contains the conserved quantity $\sum_i (-1)^{\pi \alpha} S_i^z$. In fact, the spin–orbital coupled $U(1)_{\text{loc}}$ symmetry $[\mathcal{H}, \sum_i (\alpha + \beta) S_i^z] = 0$ extends along the whole line ($\alpha = \pi/2$, $\beta$) connecting the two Abelian points. The $Y$-x state is the exact ground state along the solvable line with 2-fold degeneracy [3].

In this paper, we focus on studying the phenomena in the Zeeman field along the longitudinal $y$ direction. The $H$ breaks the $T$, $P_x$, $P_z$ symmetries, but still keeps the translation, $P_y$, the combinations $TP_x$, $TP_z$ and the hidden $U(1)_{\text{loc}}$ symmetry. It can be shown that under the Mirror transformation $M$ which consists of the local rotation $S_i = R(\hat{x}, \pi) R(\hat{y}, \pi) S_i$ followed by a time reversal transformation $T$, $(\beta, h) \rightarrow (\pi/2 - \beta, h)$. After rotating spin $Y$ axis to $Z$ axis by the global rotation $R(\hat{x}, \pi/2)$, the Hamiltonian equation (1) along the line ($\alpha = \pi/2$, $0 < \beta < \pi/2$) in the $H$ field along $y$ direction can be written as:

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5 Here we still use the same notation used in [26]. In the Y-(0, 0) called Y-y state, the first letter indicates the spin polarization, the second letter indicates the orbital order. After rotating spin $Y$ axis to $Z$ axis by the global rotation, the $Y$-x state becomes the $Z$-x state shown in figure 1. Then the Zeeman field along the y direction moves along the z direction as shown in equation (2). Equivalently, one can just go along the y bonds in the square lattice and the applied Zeeman field is along the z direction.

6 If one adds a staggered Zeeman field coupled to the conserved quantity $\sum (-1)^{\pi \alpha} S_i^z$, then it will pick one of the two degenerate Y-x state at $h = 0$. There are no phase transitions in this case.
\[ \mathcal{H} = - J \sum_i \left[ \frac{1}{2} (S_i^+ S_{i+x}^- + S_i^- S_{i+x}^+) - S_i^z S_{i+x}^z \right. \\
+ \left. \frac{1}{2} (e^{i2\beta S_i^z} S_{i+y}^z + e^{-i2\beta S_i^z} S_{i+y}^z) + S_i^z S_{i+y}^z \right] - H \sum_i S_i^z, \]

where the Zeeman field \( H \) is along the \( z \) direction after the global rotation (see footnote 5).

In the following sections, after introducing the Holstein–Primakoff (HP) bosons for suitably choosing sublattices on different phases in figure 1, we will perform a systematic \( 1/S \) expansion in terms of the HP bosons [26]:

\[ \mathcal{H}_{\text{spin}} = \mathcal{H}_0 + 2JS \left( \mathcal{H}_2 + \left( \frac{1}{\sqrt{S}} \right) \mathcal{H}_3 + \left( \frac{1}{S} \right) \mathcal{H}_4 + \cdots \right), \]

where \( \mathcal{H}_0 \) is the classical ground state energy of a given state. In this paper, we will only performed the calculations at the linear spin wave order (LSW) using \( \mathcal{H}_2 \). The \( 1/S \) corrections from the \( \mathcal{H}_3 \) and \( \mathcal{H}_4 \) terms to any physical quantities achieved in the LSW order are shown to be very small at \( H = 0 \) [27]. Due to the polarization effects of the Zeeman field, we expect these corrections become even smaller at a Zeeman field. The calculation details at the LSW order will be given in the appendices A–D. We will take \( 2JS \) as the energy unit, so all the physical quantities such as the Zeeman field \( h \), the magnon dispersion \( \omega_k \) and the gap \( \Delta \) will be dimensionless after taking their ratios over \( 2JS \). We will first focus on the left half of figure 1 with \( 0 < \beta < \pi/4 \), then study the right half using the Mirror transformation \( \mathcal{M} \). The mirror center \( \beta = \pi/4 \) respects the Mirror symmetry.

2. Z-x phase and C-IC transition at the low critical field \( h_{c1} \)

It was shown in [26] that at \( h = 0 \), the Z-x state (see footnote 5) is the exact ground state with an excitation gap. It remains the exact ground state at a small \( h \) until the gap closes at \( h_{c1} \). Any \( h > 0 \) will turn all the C-C0, C-C and C-IC at \( h = 0 \) into the C-IC magnons located only at one minimum \( k_0 = (0, 0 < k_y^0(\beta, h) < \pi) \) whose constant contour was shown in figure 2. The spin wave spectrum \( \omega_k(k) \) in the reduced Brillouin zone (RBZ) is worked out in the appendix A. Expanding the lower branch of the spin wave spectrum \( \omega_k(k) \) near its minimum \( k = k_0 + \tilde{q} \) leads to the non-relativistic dispersion:

\[ \omega_{\text{Z}}(\tilde{q}) = \Delta_Z + \frac{q_x^2}{2m_{Z,x}} + \frac{q_y^2}{2m_{Z,y}}, \]

where \( \Delta_Z(\beta, h) \) is the gap and \( m_{Z,x}(\beta, h) \), \( m_{Z,y}(\beta, h) \) are the two effective masses.

The lower critical Zeeman field \( h_{c1} \) is determined by \( \Delta_Z(\beta, h_{c1}) = 0 \). Its expression is given in the appendix A and shown in figure 1. Near \( h \sim h_{c1}, \Delta_Z \sim (h_{c1} - h)^2 \). The two effective masses remain non-critical at \( h_{c1} \). The condensation of the C-IC magnons indicates a transition from the Z-x state into a C-IC-SkX phase with the orbital ordering wavevectors \( (0, k_y^0) \) (figure 2) which has the dynamic exponent \( z = 2 \). The nature of this transition and the C-IC-SkX phase will be explored further from \( h_{c1} < h < h_{c2} \) below.

3. FM phase, C-C and C-IC transitions at the upper critical field \( h_{c2} \)

At a strong Zeeman field \( H \gg J > 0 \), the system is in a FM state subject to quantum fluctuations shown in figure 1. Its spin wave spectrum \( \omega_k(k) \) is worked out in the appendix B. It always has two degenerate minima located at \( (0, k_y^0) \) and \( (\pi, k_y^0) \) where \( 0 \leq k_y^0 \leq \pi \) shown in figure 2. The upper critical field \( h_{c2} \), determined by the vanishing gap at the two minima, takes a piece-wise form:

\[ h_{c2} = \begin{cases} 
1 + |\cos 2\beta|, & \beta \in I = (0, \beta_1) \cup (\beta_2, \pi/2) \\
\frac{1 - \cos 4\beta}{1 - \cos 4\beta}, & \beta \in II = [\beta_1, \beta_2],
\end{cases} \]

where the two different pieces indicate transitions to two different states. At \( \beta = \beta_1, \beta_2 \), the two expressions coincide. The two values \( \beta_1, \beta_2 \) coincides with the boundaries between C-C0, C-C and C-IC in the Z-x state at \( h = 0 \) at the leading order of linear SWE [26]. The physical reason for the coincidence is not known.

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7 The 1/S corrections to all the physical quantities achieved by the Linear spin wave order calculations were found to be very small. In fact, it is quite difficult to get a higher spin rotated anti-ferromagnet model (RAFM) studied in this reference. So it is quite difficult to raise the critical temperatures of RAFHM. On the cold atom experimental side, the heating issue with fermions in the Raman scheme is also more serious than that of spinor bosons. The two factors combine to make the observation of low temperature phases of RAFHM more difficult than those of RFHM studied in this paper.
When $0 < \beta < \beta_1$, expanding $\omega(k)$ around $(0, 0)$ or $(\pi, 0)$ leads to:

$$\omega_{F}(q) = \sqrt{\Delta_{F}^2 + v_{F,x}^2 q_x^2 + v_{F,y}^2 q_y^2} - c_{F} q_{F},$$

where $\Delta_{F} \sim (h - h_c) \gamma$ and $z = 1$. At $h = h_c$, $v_{F,x} = 1, v_{F,y} = \sqrt{\cos 2\beta}, c_{F} = \sin 2\beta$. At $\beta = 0, c_{F} = 0$, equation (6) recovers the relativistic form. The simultaneous condensations of the C-magnons at $k^0 = (0, 0)$ and $(\pi, 0)$ indicates a transition from the FM state into a canted phase with the two orbital ordering wavevectors shown in figure 2. The nature of the canted phase will be explored further from the IC-SkX phase with the orbital ordering wavevectors $(0, k_{y}^{0})$ and $(\pi, k_{y}^{0})$ shown in figure 2. The spin structure of this IC-SkX phase will be explored further from $h_{c1} < h < h_{c2}$. The most general form of the canted state can be obtained by applying the $U(1)_{soc}$ symmetry operator [26] $U_{1}(\phi) = e^{i\phi \sum_{\alpha=0}^{1} S_{\alpha}^z}$ to the FM state in the XZ plane:

$$S^z = \cos \theta, \quad S^+ = [S^-]^* = \sin \theta [\cos \phi + i \sin \phi (-1)^{H}].$$

It contains the two ordering wavevectors $Q_1 = (0, 0)$ and $Q_2 = (\pi, 0)$ which match those of the condensed C-magnons coming from $h > h_{c2}$. Setting $\phi = 0$, $\pi$ recovers the two FM states which do not break the translational symmetry, but breaks the $P_2$ and the $U(1)_{soc}$ symmetry. However, when $\phi = 0, \pi$, equation (8) also breaks the translational symmetry. For example, setting $\phi = \pi/2$ leads to a canted (co-planar) state $S^z = \cos \theta, S^+ = \sin \theta e^{i\theta/2}$ in the YZ plane which breaks the translational symmetry. Naively, if two states break different symmetries of the Hamiltonian, they belong to different states. However, here, they belong to the same family of states related by the $U(1)_{soc}$ symmetry of the Hamiltonian. This counter-intuitive result is a salient feature of the SOC.

The $U(1)_{soc}$ symmetry breaking leads to a gapless Goldstone mode $\phi$ located at $k^0 = (0, 0)$ and takes the rather peculiar form:

$$\omega_{k}(q) = \sqrt{v_{k,x}^2 q_x^2 + v_{k,y}^2 q_y^2} - c_{k} q_{k},$$

where $v_{k,x}, v_{k,y}, c_{k} \equiv h \tan \beta$ are listed in the appendix.

### 4.1. The C-C transition from the canted state to the FM phase at $h_{c2}$ driven by the roton dropping

Fixing $0 < \beta < \beta_1$, as one increases to $h_{c2}(\beta)$ from below, a roton minimum develops at $(0, 0)$ shown in figure 3(a). Its tilted angle with the $xz$-plane which breaks the translational symmetry. Naively, if two states

When $\beta_1 < \beta < \beta_2$, expanding $\omega(k)$ around the two minima $(0, k_{y}^{0})$ or $(\pi, k_{y}^{0})$, we obtain a similar non-relativistic form as equation (4):

$$\omega_{F}(q) = \Delta_{F} + \frac{q_x^2}{2m_{F,x}} + \frac{q_y^2}{2m_{F,y}},$$

where $\Delta_{F} \sim (h - h_c) \gamma$ and $z = 2$. At $h = h_c$, $m_{F,x} = \sqrt{\sin^4 2\beta - \cos^2 2\beta}, m_{F,y} = \sin^2 2\beta/m_{F,x}$. At $\beta = \beta_1$, $h_{c1} = 1 + \cos 2\beta, m_{F,x} = 0$ and $m_{F,y} = \infty$ which match the anisotropic $(z_x = 1, z_y = 3)$ dynamic behaviors at the M point in equation (6). The condensation of the C-IC magnons indicates a transition from the FM state with the orbital ordering wavevectors $(0, k_{y}^{0})$ and $(\pi, k_{y}^{0})$ shown in figure 2. The spin structure of this IC-SkX phase will be explored further from $h_{c1} < h < h_{c2}$ below.
As shown above, using the FM state in the XZ plane, the Goldstone mode equation (9) and the roton mode equation (10) are located at \((\pi, 0)\) and \((0, 0)\) respectively in the full BZ. However, as said above, choosing \(\phi = \pi/2\) leads to the canted (co-planar) state in the YZ plane, then one need to introduce two HP bosons \(a\) and \(b\) for the two sublattices A/B respectively, there are two modes \(\omega_a(k)\) inside the RBZ. Both the Goldstone mode \(\omega_r(k)\) and the roton mode \(\omega_c(k)\) are located at \((0, 0)\) in the RBZ. Obviously, when \(h\) gets close to \(h_{c2}\) from below, the roton mode \(\omega_r(k)\) becomes degenerate with the Goldstone mode \(\omega_c(k)\), so can not be dropped.

4.2. Bosonic Lifshitz type of C-IC transition from the canted state to the IC-SkX phase at the left critical field \(h_1\).

At fixed \(h < h_{c2}(\beta_i)\), as the SOC strength increases, the slope of the Goldstone mode \(v_{g,y} - c_x\) in equation (9) along \(q_y \geq 0\) decreases. Setting the slope vanishing, we obtain the left critical field \(h_1\):

\[
h_1 = \frac{2 \sin \beta \sqrt{\cos 2\beta}}{\sqrt{2 + \sec^2 \beta - 2 \sec^4 \beta}}
\]

which is shown in figure 1. Setting \(h_{c2} = h_1\) leads to \(\beta = \beta_i\) and \(h^y = h_{c2}(\beta_i) = \frac{\sqrt{5} + 1}{2}\) (Golden ratio). This is the multi-critical (M) point of the three phases: FM (collinear), canted (co-planar) and IC-SkX (non-coplanar) phase. It has the anisotropic dynamic exponents \((z_x = 1, z_y = 3)\). Near \(h_1\), by expanding the Goldstone mode equation (9) to higher orders, \(\omega(q_x, q_y = 0, q_y > 0) = (v_{g,y} - c_x) q_y + c_3 q_y^3 + \cdots\) where \(c_3 > 0\) is given in the appendix. When \(v_{g,y} - c_x < 0\), the minimum position is at \(q_y^0 = 0\), so it is in the canted state. When \(v_{g,y} - c_x > 0\), the minimum position is at \(q_y^0 = (\frac{v_{g,y} - c_x}{c_3})^{1/2}\), it is in the IC-SkX state equation (12) with the orbital order at \((\pi, q_y)\) (figure 2). Indeed, this infinitesimal small orbital order connects the one at \(h_{c2}\), \(\beta = \beta_i^1\) smoothly to the one at \(h_1\), \(\beta = 0^+\) due to the condensations of C-IC at \(h_{c2}\) and \(h_1\) respectively. This is a bosonic type of quantum Lifshitz transition [28, 29], however, with the odd power of terms such as \(q_y^3, q_y^5\), … which is a salient feature due to the SOC. So it is a completely new class of bosonic type of Lifshitz transition [30–32] with the anisotropic dynamic exponent \((z_x = 1, z_y = 3)\).

One can also show that \(\frac{\partial h_1}{\partial \beta}_{|\beta = 0} = \frac{\partial h_{c2}}{\partial \beta}_{|\beta = 0} = 2\). However, \(h_1\) is always above \(h_{c1}\), so there is always a narrow window of IC-SkX phase sandwiched between the collinear Z-x phase and the co-planar canted phase. There is NO direct transition between the two. This is consistent with the contour \((0, k_x^0 \to 0^+)\) in the \(\beta \to 0\) limit from \(h < h_{c1}\) in figure 1.

5. IC-SkX phases and C-IC transitions at \(h_{c1}\) and \(h_{c2}\)

It was shown in [26] and also mentioned in the introduction that the \(U(1)_{soc}\) symmetry \(\sum_i (-1)^i S_i^y = \sum_i \tilde{S}_i^y\) becomes explicit in the rotated basis \(SU(2)\) where \(\tilde{S}_i = R(\hat{x}, \pi \hat{i}) S_i\). This \(SU(2)\) basis is also called the \(U(1)_{soc}\) basis in [26]. So the anomalous spin correlation functions in the FM state at \(h > h_{c2}\) in figure 1 vanish. So the classical state due to the condensation of the IC-magnons at \((0, k_x^0)\) and \((\pi, k_x^0)\) in figure 2 can be determined in the \(U(1)_{soc}\) basis first. Transforming back to the original basis, then acting on it by the \(U(1)_{soc}\) leads to

8 For scaling functions with the anisotropic dynamic exponents \((z_x = 2, z_y = 1)\) where \(q_i\) is the colliding direction across a fermionic Lifshitz type of transitions, see [28].

9 For a classical bosonic Lifshitz type of transitions, see [29].
The critical behaviors at the Z-x state at \( \beta = \pi/4 \), so \( z = 2 \). Similarly, at \( h_{c3}(\beta = \pi/4) = \sqrt{3} - 1 \), \( \nu_{Gx} = \nu_{Gy} = 0 \). Pushing the expansion to \( k^4 \) in both equations (13) and (14), we find the effective masses of both the Goldstone mode and the roton mode coincide with the \( m_{Gx}, m_{Gy} \) achieved from the Z-x state equation (4), so \( z = 2 \). The critical behaviors at \( h = h_{c3} \) and \( h = h_{c4} \) are shown in figures 3(b1) and (b2) respectively.

\[ S^z = A + B(-1)^y, \]
\[ S^x = [S^-]_y = [C + D(-1)^y]e^{i(\phi - k_y y + \phi)} \]

which breaks the translational, \( P_2 \) and the \( U(1)_{soc} \) symmetry and has a non-vanishing Skyrmion density \( Q_{ijk} = S_i \langle (S_j \times S_k) \rangle \) where \( i, j, k \) are 3 lattice sites in a square lattice. However, from equation (12), one can see the IC-SkX still keeps the combination of the translation along the \( y \) axis and \( U(1)_{soc} \): \( y \to y + 1, \phi \to \phi - k_y^0 \) denoted as \( [U(1)_{soc}]_{y = 0 \to 1} \times (y \to y + 1) \). This remaining combined symmetry is important for the calculations of the spin wave spectrum above the IC-SkX phase. In the following, we will first focus on \( \beta = \pi/4 \) where \( k_y^0 = \pi/2 \) leads to the commensurate \( 2 \times 4 \) skyrmion crystal phase shown in figure 1. Then we will discuss the generic IC-SkX phases when \( \beta \approx \pi/4 \).

5.1. The commensurate \( 2 \times 4 \) skyrmion crystal phase at \( k_y^0 = \pi/2 \)

Minimization of the classical ground state energy leads to the two independent polar angles \( \theta_A, \theta_B \) in the two sublattices shown in figure 4(a). As shown in the appendix, after making suitable local rotations to align spin quantization axis along the \( Z \) axis, we find the spin wave spectrum \( \omega_{ic}(k) \) in the RBZ. Expanding \( \omega(\mathbf{k}) \) around the \( \Gamma = (0, 0) \) point leading to the expected gapless Goldstone mode \( \phi \):

\[ \omega_{ic}(k) = \sqrt{\nu_{Gx}^2 k_x^2 + \nu_{Gy}^2 k_y^2}, \]

where \( \nu_{Gx}, \nu_{Gy} \) are shown in figure 4(b).

As \( h \to h_{c2} \), there is also roton mode developing at \( (0, \pi) \) which takes the relativistic form:

\[ \omega_{ik}(q) = \sqrt{\nu_{Gx}^2 q_x^2 + \nu_{Gy}^2 q_y^2}, \]

where \( k = q + (0, \pi) \) and \( \Delta q = h_{c2} - h \).

In fact, as shown in figure 4(a), putting \( \theta_A = 0, \theta_B = \pi \) and \( \theta_A = \theta_B = 0 \), one can also push the calculations to the Z-x state at \( h < h_{c1} \) and the FM state at \( h > h_{c2} \) respectively, of course, in a different gauge than the original one used in previous sections. The gaps along the whole central line \( \beta \approx \pi/4 \) are shown in figure 4(c). As expected, the gaps are gauge invariant, but the minimum positions of excitations may shift at different gauges [26, 28] (see footnote 8). Indeed, in the original basis, both the Goldstone mode equation (13) and the roton mode equation (14) will shift to \( (0, \pi/2) \) in the RBZ, \( -\pi/2 < k_y < \pi/2 \), \( -\pi/4 < k_x < \pi/4 \).

At \( h_{c3}(\beta = \pi/4) = \sqrt{3} - 1 \), \( \nu_{Gx} = \nu_{Gy} = 0 \) and also \( \Delta q = 0, \nu_{Gx} = \nu_{Gy} = 0 \). Pushing the expansion to \( k^4 \) in both equations (13) and (14), we find the effective masses of both the Goldstone mode and the roton mode coincide with the \( m_{Gx}, m_{Gy} \) achieved from the Z-x state equation (4), so \( z = 2 \). The critical behaviors at \( h = h_{c3} \) and \( h = h_{c4} \) are shown in figures 3(b1) and (b2) respectively.

There may be some common insights shared between the symmetry breaking analysis and finite temperature phase transitions in this paper and those in [33].
5.2. The IC-SkX phase when $\beta = \pi/4$

The minimization of the classical energy leads to the two independent angles $\theta_A$, $\theta_B$ and the orbital order $k_0$ of the classical IC-SkX state as functions of $\beta$. At $h < h_l$, $\theta_A = \theta_B$ and $k_0 = 0$, it is the canted state in the left in figure 2. At $h > h_u$, $\theta_A = \theta_B$ and $k_0 = \pi$, it is the canted state in the right in figure 2. There is a quantum Lifshitz type C-IC transition at $h_l$ and $h_u$. At a fixed $\beta = \pi/5$, the three velocities $v_{G,x}$, $v_{G,y}$ and $c_G$ as a function of $h$. At $h_{c1}$ and $h_{c2}$, $v_{G,x} = v_{G,y} = c_G = 0$, the dispersions become quadratic indicating $z = 2$. There are quantum C-IC transitions at $h_{c1}$ and $h_{c2}$ due to the condensations of C-IC magnons.

After making suitably chosen rotations to align the spin quantization axis along the Z-axis, one need only introduce two HP bosons $a/b$ for the two sublattices A/B respectively and perform a Bogoliubov transformation to obtain the spin wave spectrum $\omega_\beta(k)$. After lengthy manipulations and very careful long wavelength expansion, we find the Goldstone mode $\phi$ at $\Gamma = (0,0)$:

$$\omega_\beta(k) = \sqrt{v_{G,x}^2 k_x^2 + v_{G,y}^2 k_y^2} - c_G k_y,$$

where $c_G(\beta, H) = -c_G(\pi/2 - \beta, H)$, so $c_G > 0$ when $\beta < \pi/4$, $c_G < 0$ when $\beta > \pi/4$ and $c_G = 0$ when $\beta = \pi/4$ recovering equation (13). How the three velocities $v_{G,x}$, $v_{G,y}$ and $c_G$ change from $h_{c1}$ to $h_{c2}$ at a fixed $\beta = \pi/5 < \pi/4$ are shown in figure 5(b).

Similarly, we find a roton mode developing near $(0, \pi)$ as $h \rightarrow h_{c2}$:

$$\omega_R(q) = \sqrt{\Delta_k + v_{G,x}^2 q_x^2 + v_{G,y}^2 q_y^2} - c_R q_y,$$

where $c_R(\beta, H) = -c_R(\pi/2 - \beta, H)$, so $c_R > 0$ when $\beta < \pi/4$, $c_R < 0$ when $\beta > \pi/4$ and $c_R = 0$ when $\beta = \pi/4$ recovering equation (14).

Comparing with equations (9) and (10), we find the Goldstone mode and the roton mode in the IC-SkX phase take similar forms as those in the canted phase. At a fixed $h$ in figure 2, we find that as $h \rightarrow h_{c1}^+$ (or $h \rightarrow h_{c2}$), $v_{G,y} = c_G \rightarrow 0$ (or $v_{G,x} + c_G \rightarrow 0$), it is a bosonic Lifshitz transition with the anisotropic dynamic exponent $z_x = 1$, $z_y = 3$. This picture is completely consistent as that achieved from the canted phase to the IC-SkX. These facts suggest some sort of duality between the cant phase and the IC-SkX phase on the two side of $h_{c1}$ (or $h_{c2}$) in figure 2.
Taking \( h \to h^+_c, v_{G,x} = v_{G,y} = 0 \) and \( c_G = 0 \) in equation (15), expanding it to the order \( k^4 \), we find it matches equation (4) reached from Z-x state below \( h_c \). Taking \( h \to h^-_c, v_{G,x} = v_{G,y} = 0 \) and \( c_G = 0 \) in equation (15) and \( \Delta_R = 0, v_x = v_{y} = 0 \) and \( \epsilon_R = 0 \) in equation (16), expanding both equations to order \( k^4 \), we find both matches \( m_{p,x} \) and \( m_{p,y} \) in equation (7) reached from FM state above \( h_c \).

6. Mirror reflection about \( \beta = \pi/4 \)

One can define a Mirror transformation \( \mathcal{M} \) which consists the local rotation \( \hat{S}_i = R(\hat{x}, \pi) R(\hat{z}, \pi_j) S_i \) followed by a time reversal transformation \( T \). Under \( \mathcal{M}, (\beta, h) \to (\pi/2 - \beta, h) \). Only \( \beta = \pi/4 \) is invariant under \( \mathcal{M} \). Note that this anti-unitary Mirror transformation is defined in SOC parameter space instead of position space.

When \( 0 < h < h_{1,1} \) under \( \mathcal{M} \), the gap minimum \( (0, k_y^0) \) at \((\beta, h)\) is mapped to the gap minimum \((0, \pi - k_y^0) \) at \((\pi/2 - \beta, h)\). This mapping also applies to \( h > h_{1,2} \) where there is one extra minimum at \((\pi, k_y^0) \). So the identity \( k_y^0 + (\pi - k_y^0) = \pi \) explains the reflection symmetry in the minimum positions about \( \beta = \pi/4 \) in the \((\beta, H)\) plane in figures 1 and 2. When \( h_L < h < h_{1,2} \) in the left hand side of figure 2, the two minima at \((0, 0)\) and \((\pi, 0)\) are mirror reflected to those at \((0, \pi)\) and \((\pi, \pi)\) when \( h_R < h < h_{1,2} \) in the right hand side. The right critical field \( h_{1,2} \) is given by equation (11) by setting \( \beta \to \pi/2 - \beta \) which is the mirror reflected image with respect to \( \beta = \pi/4 \) in figure 1.

For any state \(|\psi_l\rangle\) on the left \( \beta < \pi/4 \), one can get the state on the right by the Mirror transformation \(|\psi_R\rangle = \mathcal{M}|\psi_l\rangle \) where \( \mathcal{M} = T e^{2i\pi} e^{iS_\phi} \). Indeed, applying the operation on the canted state equation (8) on the left side leads to the canted state on the right hand side:

\[
S^\pm = \cos \theta, \\
S^+ = [S^-]' = -\sin \theta (-1)^i e^{-i(1)^{1/2}}
\]

which contains two ordering wavevectors \( \vec{Q}_1 = (0, \pi) \) and \( \vec{Q}_2 = (\pi, \pi) \) shown in figure 2. Setting \( \phi = \pi \) gives the state shown on the right in figure 1. Applying \( \mathcal{M} \) on the IC-SkX state equation (12) leads to \( k_y^0 \to \pi - k_y^0 \). Applying it on the state \( 2 \times 4 \) SkX state at \( k_y^0 = \pi/2 \) leads back to itself as expected.

7. Finite temperature properties

Any experiments are performed at finite temperatures which are controlled by the quantum phases and phase transitions at \( T = 0 \) in figures 1 and 2. Here, we discuss the effects of finite temperatures.

7.1. Physical quantities at Low temperatures

One can work out the thermodynamic quantities such as magnetization, uniform and staggered susceptibilities, specific heat and Wilson ratio at the low temperatures in all the 5 phases in figure 1. For example, the specific heat in the Z-x state at \( h < h_{1,1} \) and the FM state at \( h > h_{1,2} \) take the same form as that at the \( h = 0 \) achieved in [26] by just using the two \((\beta, h)\) dependent effective masses \( m_x = m_x(\beta, h), m_y = m_y(\beta, h) \). Of course, due to the Goldstone modes in the canted and the IC-SkX phases, the specific heat in the two phases takes the power law \( C_v \sim T^2 \). Similarly, one can work out various kinds of spin correlation functions at the low temperatures. Following the procedures [3, 28] (see footnote 8), one can also derive the scaling functions of these physical quantities at finite temperatures across the three C-IC quantum transitions in figures 6(a), (b) and also the C-C transition from the canted phase to the FM at the left or right segment of \( h_{1,2} \) in figure 6(c).

7.2. Finite temperature phase transitions

As argued in [26], there is only one finite temperature phase transition in the Ising universality class [34] above the Z-x phase. The FM state breaks no symmetries of the Hamiltonian, so no transitions above it. So we only need to discuss the finite temperature transitions above the canted phase and IC-SkX state as shown in figure 6(a).

(a) The canted phases: In the canted phase, from equation (7), one can see that at any \( T > 0 \), the Goldstone mode fluctuations equation (8) lead to \( \langle S^+ \rangle = 0 \), so the transverse spin correlation functions display algebraic orders at the two ordering wavevectors \( \vec{Q}_1 = (0, 0) \) and \( \vec{Q}_2 = (\pi, 0) \). So there is only one finite temperature phase transition \( T_{c}^C \) driven by the topological defects in the phase \( \phi \) in equation (8) above the canted phase to destroy the algebraic order (figures 6(b), (c)). In view of the rather peculiar anisotropic form of the Goldstone mode inequality (8), we expect that the \( T_{c}^C \) belongs to a new universality class different than the conventional KT one.
The transverse Bragg spectroscopy in the canted phase at $T = 0$ will display sharp peaks at $\vec{Q}_1 = (0, 0)$ and $\vec{Q}_2 = (\pi, 0)$. However at $0 < T < T_{*}^{IC}$, the transverse peaks at $\vec{Q}_1$ and $\vec{Q}_2$ will be replaced by some power law singularities [33] (see footnote 10). At $T > T_{*}^{IC}$, the power law singularities disappear.

(b) The IC-SkX phase: In the IC-SkX phase, from equation (11), one can see that at any $T > 0$, the Goldstone mode fluctuations equation (1) also lead to $\langle S^y \rangle = 0$, so the transverse spin correlation functions also display algebraic orders at the four in-commensurate ordering wavevectors $(0, \pm k_y^0)$ and $(\pi, \pm k_y^0)$. So there are two finite temperature phase transitions above the IC-SkX state: one transition $T_{*}^{IC}$ in the transverse spin sector to destroy the algebraic order, then another Ising $Z_2$ transition in the longitudinal spin sector $T_2$ to destroy the A and B sublattice $Z_2$ symmetry breaking as shown in figure 6(a). In view of the similar form of equation (15) as equation (8), we expect the $T_{*}^{IC}$ is in the same universality class as $T_{*}^{AB}$ above the canting phase. We also expect $T_{*}^{IC} < T_2$. Of course, at all the quantum phase transition boundaries in figure 1, $T_{*}^{IC} = T_2 = 0$.

The elastic longitudinal Bragg spectroscopy in the IC-SkX at $T = 0$ will display a sharp peak at $(\pi, 0)$, while the transverse Bragg spectroscopy will display sharp peaks at the four in-commensurate ordering wavevectors $(0, \pm k_y^0)$ and $(\pi, \pm k_y^0)$. However at $0 < T < T_{*}^{IC}$, the transverse peaks at $(0, \pm k_y^0)$ and $(\pi, \pm k_y^0)$ will be replaced by some power law singularities [33] (see footnote 10), the longitudinal peak remains sharp. At $T_{*}^{IC} < T < T_2$, the power law singularities disappear, but the longitudinal peak remains sharp. When $T > T_2$, the longitudinal peak disappears.

7.3 High temperature expansion

When the temperature is well above the critical temperatures of all the quantum phases in figure 1, we may perform a high temperature expansion which is complementary to the spin wave expansion at temperature well below the critical temperatures in J above. When $T \gg J, h_{c1}$, we may also perform a high temperature expansion [26] in terms of $J/T$ and $h/T$ where mixing terms in $J/T$ and $h/T$ are expected. The connections between the Wilson loop and specific heat can also be established.

8. Implications on recent cold atom experiments and materials with SOC

We will discuss the implications on cold atoms and SOC materials respectively. The two experimental systems have different advantages and limitations to explore different aspects of the rich many body phenomena in figures 1 and 2.

8.1. Experimental realizations and detections in the original and the $U(1)_{soc}$ basis in cold atoms

In the original basis equation (2), the gauge field configuration is achieved by putting $\pi/2\sigma_0$ in the x-bond, $\beta\sigma_z$ in the y-bond and the Raman laser induced Zeeman field $H$ along the $z$ direction. The two Abelian points $\beta = 0$, $\pi/2$ have been realized in previous experiments [8–17]. As pointed out in [15], the $\beta\sigma_z$ in the y-bond can be achieved by adding spin-flip Raman lasers or by driving the spin-flip transition with RF or microwave fields.
As discussed in [28] (see footnote 8), one of the big advantages of cold atom experiments over condensed matter systems is that different gauges can be realized in cold atoms, so both gauge non-invariant and gauge invariant quantities can be measured in cold atom experiments. As shown in [26], the $U(1)_{\text{so}}$ basis $S_\mu = R(\mathbf{x}, \pi n_\mathbf{x}) S_\mu$ may be more easily realized experimentally. In the $U(1)_{\text{so}}$ basis, the RFHM in the Zeeman field equation (2) becomes $H_{U(1)_{\text{so}}} = H_{U(1)_{\text{so}}} - H \sum \mathbf{S}_\mu$ where $H_{U(1)_{\text{so}}}$ is the RFHM in the $U(1)_{\text{so}}$ basis at $H = 0$ given in [26] and the Zeeman field becomes a staggered along $\mathbf{x}$ direction. Then applying the $R(\mathbf{x}, n_\mathbf{x})$ on all the states shown in figure 1 leads to the corresponding states in the $U(1)_{\text{so}}$ basis. The thermodynamic quantities are gauge invariant, so are the same in both basis. But the spin-correlation functions are gauge dependent, need to be re-evaluated in the $U(1)_{\text{so}}$ basis at both low and high temperatures.

Recently, using the optical Raman lattice scheme, the authors in the experiment [24] indeed realized the SOC with tunable $(\alpha, \beta)$ in a square lattice and the direction and magnitude of the Zeeman field $H$ are tunable. An optical lattice clock scheme [23] was proposed to suppress the heatings issue and generate a 2d SOC in an optical lattice. Most recently, by using the most magnetic fermionic element dysprosium to eliminate the heating due to the spontaneous emission, the authors in [25] created a long-lived SOC gas of quantum degenerate atoms. The long lifetime of this weakly interacting SOC degenerate Fermi gas will facilitate the experimental study of quantum many-body phenomena manifest at longer time scales, So the novel phases and phase transitions in figures 1 and 2 are ready to be explored in near future cold atom experiments.

As noted in [26] and repeated at the very beginning, the RFHM model equation (1) is for spin $S = N/2$ where $N$ is the number of atoms per site. As estimated in [26], taking some typical values of cold atoms in the strong coupling limit, $t \sim 3$ nK, $U \sim 50$ nK, the critical temperatures in figure 6 at 2d would be $T_c \sim J / t^2 / U \sim 0.2$ nK for a spin $S = 1/2$ RFHM at $U \gg 0$. It remains experimentally quite challenging to reach such a low temperature. However, because the critical temperatures scale as $T_c / J \sim 2S$, so if even taking $S = 5$, then $T_c \sim 2$ nK. In view of new cooling techniques [35, 36] to reach 0.35 nK, this enhanced critical temperature $T_c$ should be reachable with the current cold atom experimental cooling techniques [27] (see footnote 7). In fact, the $T_c$ can be enhanced further by going to a cubic lattice, but with no SOC along the $\mathbf{z}$ direction. Adding the $\mathbf{z}$ direction without putting the SOC along it will not increase the experimental difficulties [24], but will certainly increase the critical temperatures. In fact, there have been extensive experimental efforts to investigate the AFM correlations [37] in SOC free fermionic systems. However, the AFM is for spin $S = 1/2$ and gapless, so $T_c = 0$ at 2d. It is a remarkable property of the $S = N/2$ RFHM: its suppressed critical temperature $T_c \sim J / t^2 / U$ can be compensated by increasing the atom number $N$ per site. Unfortunately, the RAFHM may not share such nice properties [27] (see footnote 7). So in the aspect of temperature requirements, it would be easier to study the IC-SkX correlations in figures 1 and 2 than to study the AFM correlations [27] (see footnote 7).

As argued in [26], all the physical quantities calculated in section 7.1 can be precisely determined by various experimental techniques such as dynamic or elastic, energy or momentum resolved, longitudinal or transverse atom or light Bragg spectroscopies [38–43], specific heat measurements [44, 45] and in-situ measurements [46].

8.2. Implications to materials with strong SOC

Although the RFHM was derived as the strong coupling model of interacting spinor boson Hubbard model at integer fillings in the presence of SOC, we may just treat it as an effective lattice quantum spin model which incorporate competitions among AFM Heisenberg physics, FM Kitaev physics and DM physics. The Zeeman field adds a new dimension to these competitions. So RFHM + H can be used to not only to describe cold atom systems, but also the universal features of some strongly correlated materials which host some of these interactions.

The IC-SkX phase in figure 1 can be realized in some materials with a strong Dzyaloshinskii–Moriya (DM) interaction [47]. Indeed, a 2D skyrmion lattice has been observed between $h_{11} = 50$ mT and $h_{12} = 70$ mT in some chiral magnets [48] MnSi or a thin film of Fe$_{0.5}$Co$_{0.5}$Si [48].

The 3d hyperhoneycomb iridates $\alpha$, $\beta$, $\gamma$-Li$_2$IrO$_3$ was previously considered to be a promising candidate to realize Kitaev spin liquid phases. Unfortunately, so far, no sign of any spin liquids was detected in this so called Kitaev materials. Instead, an incommensurate, counter-rotating (in A/B sublattice), non-coplanar magnetic orders with the ordering wavevector $q = (0, 0, q)$, $q = \pi + \delta$, $\delta \sim 0.14\pi$ lying along the orthorhombic $\mathbf{a}$ axis was detected on the iridates [49–51]. Most remarkably, the IC-SkX phase equation (12) is strikingly similar to the this state. In the following, we provide some insights and explanations on the magnetic orderings in iridates $\alpha$, $\beta$, $\gamma$-Li$_2$IrO$_3$ from the RFHM + H perspective.

As shown in [26], when expanding the two $R$ matrices in equation (1), one can see that it leads to a Heisenberg + Kitaev (or quantum compass model in a square lattice)$^{11}$ + DM interaction

11 Note that in this paper, we only studied the RFHM + H on a square lattice, it is important to extend it to a honeycomb lattice [28, 52] (see footnote 8) with generic gauge parameters $(\alpha, \beta, \gamma)$ put on the three bonds. Then the IC-SkX phase may be turned into a quantum spin liquid phase which breaks no symmetries of the Hamiltonian, but with some intrinsic topological orders with associated edge states.
$H_e = -J \sum_{\langle i,j \rangle} \hat{J}_{ij} \hat{S}_i \cdot \hat{S}_j + J \sum_{\langle i,j \rangle} \hat{J}_{ij}^z \hat{S}_i^z \hat{S}_j^z + \sum_{\langle i,j \rangle} \hat{J}_{ij}^z \hat{a} \cdot \hat{S}_i \times \hat{S}_j$ where $\hat{a} = \hat{x}, \hat{y}$, $J_{ij}^x = \cos 2\alpha$, $J_{ij}^y = \cos 2\beta$; $J_{ij}^z = 2 \sin^2 \alpha$, $I_{ij}^x = 2 \sin^2 \beta$ and $I_{ij}^z = \sin 2\alpha, I_{ij}^y = \sin 2\beta$.

Obviously, at $\alpha = \beta = 0$, the Hamiltonian becomes the usual FM Heisenberg model $H = -J \sum_{\langle i,j \rangle} \hat{S}_i \cdot \hat{S}_j$.

At each end of the solvable line ($\alpha = \pi/2, \beta = 0$), we get the FM Heisenberg model in one rotated basis $H = -J \sum_{\langle i,j \rangle} R(\hat{x}, \pi_i) \hat{S}_i \cdot R(\hat{x}, \pi_j) \hat{S}_j$. At the other end of the solvable line ($\alpha = \pi/2, \beta = \pi/2$), we get the FM Heisenberg model in another rotated basis $H = -J \sum_{\langle i,j \rangle} R(\hat{y}, \pi_i) \hat{S}_i \cdot R(\hat{y}, \pi_j) \hat{S}_j$.

Along the whole solvable line ($\alpha = \pi/2, \beta = 0$), we can write: $J_{ij}^x = -1$, $J_{ij}^y = \cos 2\beta, J_{ij}^z = 2$, $I_{ij}^x = 2 \sin^2 \beta$, $I_{ij}^y = 0, I_{ij}^z = \sin 2\beta$. It is easy to see $J_{ij}^y > 0$ when $\beta < \pi/4$, $J_{ij}^x < 0$ when $\beta > \pi/4$ and vanishes at $\beta = \pi/4$. While $I_{ij}^x = 1$ and $I_{ij}^y = 1$ at $\beta = \pi/4$. Obviously, the FM Kitaev term dominates, plus a AFM Heisenberg term in both bond when $\beta > \pi/4$, plus a DM term in XZ plane $J \sin 2\beta (S_x S_z - S_y S_y)$. In the presence of the Zeeman term $H$, the RFHM + H leads to the IC-SkX state in the center regime in figures 1 and 2. The IC-SkX matches well the in-commensurate, counter-rotating (in A/B sublattice), non-coplanar magnetic orders detected by neutron and x-ray diffractions on iridates [49-51], $\alpha, \beta, \gamma$-Li$_2$IrO$_3$. So our RFHM-H could be an alternative to the minimal $(J, K, \Gamma)$ model used in [53, 54] or to the minimal $(J, K, \Gamma)$ model used in [55, 56] to fit the experimental data phenomenologically. Of course, both the $(J, K, \Gamma)$ and $(J, K, \Gamma)$ model were directly extracted from the spin, orbital and crystal structures of the material itself. One common thing among all the three models is that it is dominated by FM Kitaev term, plus a small AFM Heisenberg term. However, our RFHM + H has only two independent parameters $(\beta, H)$.

We reach the global phase diagram figure 1 in $(\beta, H)$ by the well controlled 1/S quantum fluctuations calculations. We also achieved the magnon spectra equations (15), (16) above the IC-SkX phase. While the solutions of the minimal $(J, K, \Gamma)$ model have to involve un-controlled Luttinger–Tisza approximation, those of minimal $(J, K, \Gamma)$ model involve analytical or numerical calculations only at the classical level.

These SOC materials are automatically in the strong coupling regime. So far, there is no experimental data on the magnon spectrum above the IC-SkX phase in the Iridates $\alpha, \beta, \gamma$-Li$_2$IrO$_3$. However, in contrast to the cold atom systems in (a), its total angular momentum $J = 1/2$ is fixed due to the spin crystal splitting, it is difficult to tune various parameters to study the three classes of quantum C-IC transitions in figures 1 and 2.

9. Discussions and conclusions

The classical commensurate to in-commensurate (C-IC) transitions are discussed in the context of adatom adsorption on periodic substrates such as graphite [35, 56]. However, it seems there are very little works on quantum in-commensurate phases and associated quantum C-IC transitions. There are previous theoretical works on In-commensurate spin density waves (IC-SDW) in the $J_1-J_2-J_3$ frustrated Heisenberg model [57]. The in-electron neutron scattering experiments [58] on the high $T_c$ cuprate La$_{2-x}$Sr$_x$CuO$_4$ indeed found that the magnetic peak at momentum $(\pi, \pi)$ in the AFM state near half filling splits into four incommensurate peaks at $(\pi \pm \delta, \pi \pm \delta)$ in the underdoped and superconducting regime. The incommensurability $\delta$ scales as the doping concentration $x$. It was known that this IC-SDW is co-planar and is due to the geometric frustrations in the quantum Heisenberg model with the spin SU(2) symmetry. Our theoretical work discovered that it is the combination of the SOC and the Zeeman field in a bipartite lattice which leads to the IC-SkX state in a large parameter space in figures 1 and 2. The IC-SkX is non-coplanar with non-vanishing skyrmion density instead of collinear. So the geometric frustrations and the SOC are two completely different mechanisms leading to the in-commensurate phases which also own very different properties in the two cases.

Usually, there could just be a direct second order transition between two compatible commensurate phases. However, between any two in-compatible commensurate phases, there can only be four possible routes: (1) a direct first order transition (2) through some in-commensurate phases (3) through a quantum spin liquid phase (4) through a de-confined quantum critical point. Our main results well crafted in figures 1 and 2 show that the case (2) is happening here for the RFHM in a Zeeman field. There are one quantum C-C transition from the Z-FM state to the canted state at $(h_{z,2}, 0 < \beta < \beta_1)$. Most importantly, there are three quantum C-IC transitions: the $X$-state to the IC-SkX at $h_{z,1}$, the $Z$-FM state to the IC-SkX at $(h_{z,2}, \beta_1 < \beta < \beta_2)$, the canted state to the IC-SkX at $(h_1, h_2, \beta_1 < \beta < \beta_2)$. The first two are due to the condensation of C-IC magnons driven by the external Zeeman field $h$, while the third is due to the bosonic quantum Lifshitz transition driven by the SOC. All are second order quantum phase transitions, but in different universality classes. It is the interplay between the SOC and the Zeeman field which leads to the spin-orbital correlated collinear, co-planar (canted), non-coplanar (skyrmion crystal) phases in a square (which is a bipartite) lattice. Figures 1 and 2 can be contrasted with the collinear magnetic phases in a bipartite lattice, spiral or non-coplanar magnetic phases found in geometrically frustrated lattices [1]. The effective actions and renormalization group analysis of all the quantum phase
transitions (especially the quantum C-IC transitions) with the dynamic exponents \( z = 1, z = 2 \) and the anisotropic ones \( (z_x = 1, z_y = 3) \) in figure 1 will be studied, the finite temperature transitions above all the five phases will be investigated, the RFHM + H model at generic SOC parameters \( (\alpha, \beta) \) will be presented in separate publications.

Obviously, due to the lacking of the spin SU(2) symmetry in equation (1), applying the Zeeman field along the two transverse directions \( H_x \) and \( H_y \) lead to quite different phenomena and will be presented in separate publications [34]. Rotated anti-ferromagnetic Heisenberg model (RAFM) will show quite different behaviors [27] (see footnote 7) and will also be presented in separate publications.

It is instructive to compare the \( (z_x = 1, z_y = 3) \) bosonic Lifshitz C-IC transition at \( h = h_{1, h_0} \) in figure 1 with the quantum dimer model (QDM) used to describe the transition from one valence bond solid (VBS) to another VBS state with possible intervening in-commensurate VBS [30–32]. It was known that near the solvable Rokhsar–Kivelson (RK) point, in the height representation, the transition can be described by a low energy effective quantum bosonic Lifshitz action \( \mathcal{L}_{RK} = \frac{1}{2}((\partial_t h)^2 + K_2(\nabla h)^4 + K_3(h) + \cdots) \) where \( K_2 = 0 \) at the RK point with the dynamic exponent \( z = 2 \). It maybe interesting to explore the possible interesting connections between the \( (z_x = 1, z_y = 3) \) bosonic Lifshitz C-IC transition at \( h = h_1 \), the multi-critical (M) point and the IC-SkX phases in figure 1 with the \( z = 2 \) bosonic Lifshitz transition, the RK point and the incommensurate tilted VBS phases in the QDM. However, the degree of freedoms in the QDM is dimers instead of quantum spins. Our model is a quantum spin model in the presence of both SOC and Zeeman field, so maybe more experimentally accessible than the QDM.

It is necessary to point out some very recent new experimental techniques developed in the cold atoms to realize synthetic gauge fields. Based on the proposal to use internal atomic states as effective ‘synthetic dimensions’ [23], the fermionic optical lattice clock scheme was just successfully implemented for both \(^{87}\text{Sr}\) clock in [59] and \(^{172}\text{Yb}\) clock in [60], where the heating and atom loss from spontaneous emissions are eliminated, the exceptionally long lifetime \( \sim 100 \text{ s} \) of the excited clock state have been achieved. The ‘synthetic dimensions’ idea has also been used [61] to generate strong SOC in an effective two-dimensional manifold of discrete atomic momentum states of \(^{87}\text{Rb}\). As advocated by all the three experimental groups [59–61], these ground-breaking experiments set-up a very promising platform to observe novel many-body phenomena due to interplay between SOC and interaction in optical lattices. They also open up a new frontier of combining clock precision measurement, metrology and many body phenomena unique to SOC. It is quite promising that the novel many body phenomena in figures 1 and 2 could be observed in these cold atom experiments also in the near future.

In short, quantum spin systems with SOC subject to a Zeeman field opens a new platform to display rich and novel class of quantum commensurate (C) and in-commensurate (IC) phases, excitations and quantum C-C and C-IC phase transitions, which can be observed in both cold atoms and materials with SOC. The results achieved in this paper just reveals a tip of an iceberg.

**Acknowledgments**

We thank W Ketterle and Ruquan Wang for helpful discussions on current and future experimental status. We acknowledge AFOSR FA9550-16-1-0412 for supports. The work at KITP was supported by NSF PHY11-25915. W M Liu is supported by NSFC under Grants No. 10934010 and No. 60978019, the NKBRSFC under Grants No. 2012CB821300.

**Appendix**

In this appendix, we provide some technical details on the results achieved in the main text: (1) the Z-x state below \( h_{c1} \), (2) the FM state above \( h_{c2} \), (3) the canted state on the left side \( h_1 < h < h_{c2} \), (4) the incommensurate Skyrmion (IC-SkX) states at generic \( \beta \) which reduces to the \( 2 \times 4 \) SkX state at \( \beta = \pi / 4 \).

**Appendix A. Lower critical field \( h_{c1} \) in the Z-x state**

We perform SWE [26] on equation M1 to leading order in \( 1/S \). Introducing the two HP bosons for the two A/B sublattices respectively and introducing a unitary transformation:
\[
\begin{pmatrix}
  a_k \\
  b_k
\end{pmatrix} = \begin{pmatrix}
  \sin \frac{\theta_{k,h}}{2} & \cos \frac{\theta_{k,h}}{2} \\
  -\cos \frac{\theta_{k,h}}{2} & \sin \frac{\theta_{k,h}}{2}
\end{pmatrix}\begin{pmatrix}
  \alpha_k \\
  \beta_k
\end{pmatrix},
\]

(A1)

where the matrix elements are given by:

\[
\sin \theta_{k,h} = \frac{\cos k_x}{\sqrt{\cos^2 k_x + (\sin 2\beta \sin k_y - h)^2}}, \quad \cos \theta_{k,h} = \frac{\sin 2\beta \sin k_y - h}{\sqrt{\cos^2 k_x + (\sin 2\beta \sin k_y - h)^2}}.
\]

(A2)

Setting \( h = 0 \) reduces to the unitary transformation in [26].

One can put the Hamiltonian into the diagonal form:

\[ H_L = E_0 + 4JS \sum_k [\omega_z(k) \alpha_k + \omega_x(k) \beta_k], \]

(A3)

where \( E_0 = -2NJS^2 \) is the ground state energy (the same as that at \( h = 0 \)), \( \vec{k} \) belongs to the RBZ and the spin wave spectrum is:

\[ \omega_z(k) = 1 - \frac{1}{2} \cos 2\beta \cos k_y + \frac{1}{2} \sqrt{\cos^2 k_x + (\sin 2\beta \sin k_y - h)^2}, \]

(A4)

whose minima location \((0, k_y^0)\) is one of the roots of following quartic equation

\[ \sin^2 2\beta \sin^4 k_y - 2h \sin 2\beta \sin^3 k_y + (1 + h^2 - \sin^2 2\beta - \sin^4 2\beta) \sin^2 k_y + 2h \sin^3 2\beta \sin k_y - h^2 \sin^2 2\beta = 0. \]

(A5)

It turns out that there is always one and only one physical root. The constant contour of \( k_y^0 \) is shown in figure 2.

Expanding \( \omega_z(k) \) near the minimum \( \vec{k} = k^0 + \vec{q} \) leads to:

\[ \omega_z(\vec{q}) = \Delta_z + \frac{q_x^2}{2m_{z,x}} + \frac{q_y^2}{2m_{z,y}}, \]

(A6)

where the \( \Delta_z(\beta, h) \) is the gap of the C-IC magnons and \( m_{z,x}(\beta, h), m_{z,y}(\beta, h) \) are their two effective masses. By comparing with the expansion \( \omega_z(\vec{q}) = \Delta_z + cq_z + \cdots \), one can identify the dynamic exponent \( z = 2 \).

The lower critical magnetic field \( h_{c1} \) is determined by \( \Delta_z(\beta, h_{c1}) = 0 \) and is given by:

\[
\begin{align*}
  h^0 \cos^4 2\beta - 2h^0(3 - 10 \sin^2 2\beta + 6 \sin^4 2\beta + \sin^6 2\beta) \\
  -h^0(15 + 36 \sin^2 2\beta - 31 \sin^4 2\beta - 28 \sin^6 2\beta - \sin^8 2\beta) \\
  -2h^2 (4 - 29 \sin^2 2\beta + 6 \sin^4 2\beta + 5 \sin^6 2\beta + 5 \sin^8 2\beta) + \sin^2 2\beta \sin^2 2\beta (8 + \sin^2 2\beta) (1 - 3 \sin^2 2\beta)^2 = 0.
\end{align*}
\]

(A7)

When \( \beta = \pi/4 \), it simplifies to:

\[ 9h^4 - 72h^2 + 36 = 0 \implies h_n = \sqrt{3} - 1. \]

(A8)

Expanding \( \Delta_z \) around \( h_{c1} \), we obtain:

\[ \Delta_z = \frac{\partial \Delta_z}{\partial h} \bigg|_{h = h_{c1}} (h - h_{c1}) = \frac{\sin 2\beta \sin k_y}{2(2 - \cos 2\beta \cos k_y)} (h - h_{c1}). \]

(A9)

The coefficient is nonzero for \( \beta = 0, \pi/2 \), thus we obtain \( \Delta \sim (h_{c1} - h)^1 \) whose slope is given in figure A1(a).

The values of the two effective masses at \( h_{c1} \) are shown in figure A1(b).

At \( \beta = \pi/4 \), the minimum is at \( k_y^0 = \pi/2 \), expanding around the minimum \( k = q + (0, \pi/2) \) leads to:

\[ \omega_z(\vec{q}) = 2 - \sqrt{2 + 2h + h^2} + \frac{q_x^2}{2\sqrt{2 + 2h + h^2}} + \frac{(1 + h)q_y^2}{2\sqrt{2 + 2h + h^2}} \]

(A10)

which gives the dynamic exponent \( z = 2 \) and the critical mass at \( h_{c1} = \sqrt{3} - 1 \) shown in figure A1(b):

\[ m_{z,x} = \sqrt{2 + 2h_{c1} + h_{c1}^2} = 2 \text{ and } m_{z,y} = \sqrt{2 + 2h_{c1} + h_{c1}^2}/(1 + h_{c1}) = 2/\sqrt{3} \]

which will be used to compare with those achieved from \( h_{c1} < h < h_{c2} \) in section 4.

**Appendix B. Determinations of three segments of \( h_{c2} \) from high field FM state**

The crucial difference of the upper critical field \( h_{c2} \) from the lower critical field \( h_{c1} \) is that one has to split \( h_{c2} \) into three different segments (namely, piece-wise) shown in figure 1. It indicates transitions to 3 different class of states: two canted states and one IC-SKX state.
The FM state in the high field $h > h_{c2}$ breaks no symmetry of the Hamiltonian. The $U(1)_{soc}$ symmetry dictates there must be at least two degenerate minima in the excitations above the FM state. So one only need to introduce one HP boson. After performing a Bogliubov transformation, we obtain:

$$\mathcal{H}_H = -NH \left( S + \frac{1}{2} \right) + JS \sum_k \omega_k + 2JS \sum_k \omega_k \alpha_k^{\dagger} \alpha_k,$$

where the spin wave dispersion is

$$\omega_k = \sqrt{(h - \cos 2\beta \cos \gamma)^2 - \cos^2 k_x - \sin 2\beta \sin k_y},$$

Due to the reflection symmetry about $\beta = \pi / 4$, we only need to focus on $0 < \beta < \pi / 4$. It is easy to see that as dictated by the $U(1)_{soc}$ symmetry, there are always two degenerate minima located at $k_x = 0$, $\pi$. The minimization in $k_y$ leads to:

$$0 = \frac{\partial \omega_k}{\partial k_y} = \frac{\cos 2\beta \sin k_y (h - \cos 2\beta \cos \gamma)}{\sqrt{(h - \cos 2\beta \cos \gamma)^2 - 1}} - \sin 2\beta \cos k_y.$$  \hspace{1cm} (B3)

If equation (B3) has a real solution $k_y^0$, then plugging it back into equation (B2) leads to:

$$\omega_{\text{min}} = \frac{\sin k_y^0 (h \cos 2\beta - \cos k_y^0)}{\sin 2\beta \cos k_y^0}.$$  \hspace{1cm} (B4)

The gap vanishing condition is:

$$\cos k_y^0 = h \cos 2\beta.$$  \hspace{1cm} (B5)

In fact, equation (B3) is a quartic equation of $\cos k_y$

$$\sin^2 2\beta \cos^2 k_y + (\cos^2 2\beta - \cos^2 k_y)(h - \cos 2\beta \cos k_y)^2 = 0.$$  \hspace{1cm} (B6)

If there exists one root with $\cos k_y^0 \leq 1$ in equation (B6), substituting equation (B5) into (B6) leads to:

$$\frac{1}{8} h_{c2}^2 \sin^2 4\beta [3 - h_{c2}^2 + (h_{c2}^2 - 1) \cos 4\beta] = 0 \implies h_{c2} = \frac{3 - \cos 4\beta}{\sqrt{1 - \cos 4\beta}}.$$  \hspace{1cm} (B7)

If all positive roots of equation (B6) require $\cos k_y^0 > 1$, then the minimum is located at $\cos k_y^0 = 1$, namely, $k_y^0 = 0$. Then substituting $\cos k_y^0 = 1$ into equation (B2) leads to:

$$0 = \omega_{\text{min}} = \sqrt{(h_{c2} - \cos 2\beta)^2 - 1} \implies h_{c2} = 1 + \cos 2\beta$$  \hspace{1cm} (B8)

which is also the condition ensuring a real spectrum.

Combining the two piece-wise $h_{c2}$ equations leads to:

$$1 + \cos 2\beta = \frac{3 - \cos 4\beta}{\sqrt{1 - \cos 4\beta}} \implies \beta = \beta_1$$  \hspace{1cm} (B9)

which is shown in figure 1.
After extending to $\beta \in (0, \pi/2)$, we obtain:

$$h_{\beta} = \begin{cases} 1 + |\cos 2\beta|, & \beta \in I = (0, \beta_1) \cup (\beta_2, \pi/2) \\ \frac{\beta - \cos 4\beta}{1 - \cos 4\beta}, & \beta \in II = [\beta_1, \beta_2]. \end{cases}$$

(B10)

The two different piece-wise forms of $h_{\beta}$ in the regime I and II indicates transitions to two different states: canted state and IC-SkX state respectively with the dynamic exponents $z = 1$ and $z = 2$ respectively.

At $\beta = \pi/4$, $k_{\beta}^2 = \pi/2$, expanding around the two minima $(0, \pi/2)$ or $(\pi, \pi/2)$, equation (B2) becomes:

$$\omega_{fi}(\bar{q}) = \sqrt{h^2 - 1} \left[ 1 + \frac{q_x^2}{2h^2 - 1} \right],$$

(B11)

where $\vec{k} = (0, \pi/2) + \bar{q}$ or $\vec{k} = (\pi, \pi/2) + \bar{q}$. It gives the two masses at $h_{\beta} = \sqrt{2}$:

$m_{F,x} = \sqrt{h_{\beta}^2 - 1} = 1, m_{F,y} = 1$ and $z = 2$ which will be compared to those achieved from $h < h_{\beta}$.

Appendix C. The Goldstone and roton mode in the canted state

It is most convenient to perform SWE on the simplest FM state in the XZ plane with $\phi = 0$ in equation M5. We first make a global rotation $R_x(\theta)$ to align the spin quantized axis along the $Z$ axis then only need to introduce one HP boson to perform the SWE $H = H_0 + H_1 + H_2 + \cdots$. We obtain $H_0 = -2NJS\left(\cos^2 \beta + \frac{h^2}{4\cos^2 \beta}\right)$, $H_1 = 0$ which is dictated by the correct classical ground state equation M5 and

$$H_C = -2NJS\cos^2 \beta + JS\sum_k \omega_k + 2JS\sum_k \omega_k \alpha_k^\dagger \alpha_k,$$

(C1)

where the spin wave excitation spectrum is

$$\omega_k = \sqrt{A_k^2 - B_k^2} - C_k,$$

(C2)

where

$$A_k = 2\cos^2 \beta \left(1 - \frac{h^2}{4\cos^4 \beta}\right) \cos_k + \frac{h^2\sin 2\beta}{4\cos^4 \beta} \cos_k,$$

$$B_k = \frac{h^2}{4\cos^4 \beta} \cos_k + \sin^2 \beta \left(1 - \frac{h^2}{4\cos^4 \beta}\right) \cos_k,$$

$$C_k = h \tan \beta \sin k,$$

(C3)

The excitation spectrum is always gapless at $k^0 = (\pi, 0)$.

Expanding around $\vec{k} = k^0 + \bar{q}$, we obtain the Goldstone mode equation M6

$$\omega_k(\bar{q}) = \sqrt{v_{g,x}^2 q_x^2 + v_{g,y}^2 q_y^2 - c_q q_y},$$

(C4)

where

$$v_{g,x}^2 = \frac{h^2(1 + \sin^2 \beta)}{4\cos^4 \beta} - \sin^2 \beta, \quad v_{g,y}^2 = \frac{h^2(1 + \sin^2 \beta)}{4\cos^4 \beta} - \sin^2 \beta \cos 2\beta, \quad c_q = h \tan \beta.$$  

(C5)

As $h$ increases to $h_{\beta_2}$, $0 < \beta < \beta_1$, there is also a roton minimum developing at $(0, 0)$ shown in figure 3(a):

$$\omega_k(\vec{k}) = \sqrt{\Delta_k^2 + v_{r,x}^2 k_x^2 + v_{r,y}^2 k_y^2 - c_r k_y},$$

(C6)

where

$$\Delta_k^2 = 4\cos^2 \beta - \frac{h^2}{\cos^2 \beta} = 4(1 - (h/h_{\beta_2})^2 \cos^2 \beta), \quad c_r = h \tan \beta,$$

$$v_{r,x}^2 = \frac{h^2(2 + \cos^2 \beta)}{4\cos^4 \beta} - \cos^2 \beta - 1, \quad v_{r,y}^2 = 1 + \cos 2\beta \cos^2 \beta + \frac{h^2(2\cos 2\beta \sin^2 \beta - 1)}{4\cos^4 \beta}.$$  

(C7)

At $h = h_{\beta_2} = 1 + \cos 2\beta$, we find the Goldstone mode at $(\pi, 0)$ and the roton mode at $(0, 0)$ achieved from below $h \leq h_{\beta_2}$, $0 < \beta < \beta_1$ coincide with those achieved from above $h \geq h_{\beta_2}$, $0 < \beta < \beta_1$, namely:

$v_{g,x} = v_{r,x} = 1, v_{g,y} = v_{r,y} = \sqrt{\cos 2\beta}$ and $c_q = c_r = \sin 2\beta$. This indicates the transition from the FM to the canted state maybe a second order transition with $z = 1$.

Now we look at the bosonic Lifshitz type of transition at $h = h_L$ from the canted state to the IC-SkX state. We need to perform higher-order gradient expansion around $(\pi, 0)$ in the canted state near $h = h_L$ to see the
nature of the transition:

\[ \omega_k(q') = \sqrt{v_{xx}^2 q_x'^2 + v_{yy}^2 q_y'^2 + v_{xc} q_x'^4 + v_{yc} q_y'^4 + v_{xy} q_x'^2 q_y'^2 - c_\xi q_x' + c' q_y'^3}, \quad (C8) \]

where

\begin{align*}
    v_{xx} &= \frac{1}{4} - \frac{h^2(7 + \sin^2 \beta)}{48 \cos^2 \beta} + \frac{1}{12} \sin^2 \beta, \\
    v_{yy} &= \frac{1}{4} - \frac{h^2(1 + 7 \sin^2 \beta)}{48 \cos^2 \beta} + \frac{1}{12} \sin^2 \beta \cos 2\beta, \\
    v_{xy} &= \frac{1}{2} \left(1 - \frac{h^2}{4 \cos^2 \beta}\right) \cos^2 \beta, \quad c' = (h/6) \tan \beta. \quad (C9)
\end{align*}

Note that \( v_{xx} \) is not positive-defined, but it will not cause any instabilities due to \( v_{xx,x} > 0 \). Both \( v_{xy,x} \) and \( v_{xy,y} \) are positive.

Without losing the physics of the transition from the canted state to the IC-SkX, setting \( q_x = 0 \) simplifies equation (C8) to:

\[ \omega_k(q_x = 0, q_y) = \sqrt{v_{xx} q_y'^2 + v_{yy} q_y'^4 - c_\xi q_y' + c' q_y'^3} = v_{xx} |q_y'| - c_\xi q_y' + c' q_y'^3. \quad (C10) \]

Because the transition first happens in \( q_y \) direction, one can see that when \( v_{xy,y} - c_\xi > 0 \), the minimum position is at \( q_y^0 = 0 \), so it is in the canted state. When \( v_{xy,y} - c_\xi < 0 \), the minimum position is at \( q_y^0 = (\frac{\pi - v_{xy,y}}{c_\xi})^{1/2} \) where \( c_\xi = \frac{v_{xx} v_{yy}}{2 v_{xy}} + c' \), then it is in the IC-SkX state with the orbital order at \((\pi, q_y^0)\). Note that the IC-SkX states breaks the \( U(1)_{loc} \) symmetry, so acting on the state with the orbital order at \((\pi, q_y^0)\) will generate a generic IC-SkX state with the two orbital orders \((0, q_y^0)\) and \((\pi, q_y^0)\) shown in figure 1(a). Indeed, this infinitesimal small orbital order connects the one at \( h_{22}, \beta = \beta_1^+ \) smoothly to the one at \( h_{11}, \beta = 0^+ \). This is a bosonic type of Lifshitz transition, however, with the odd power of terms such as \( q_y^0, q_y'^3, \ldots \) which is due to the SOC. So it is completely new class of bosonic type of Lifshitz transition with the anisotropic dynamic exponents \( (z_x = 1, z_y = 3) \).

**Appendix D. The Goldstone and roton mode in the IC-SkX states**

We follow the procedures used in the main text. We first discuss the two modes in the \( 2 \times 4 \) commensurate SkX at \( \beta = \pi/4 \). Then we extend the calculations to the generic IC-SkX states when \( \beta \approx \pi/4 \).

**D.1. 2 \times 4 commensurate SkX at \( \beta = \pi/4 \)**

At \( \beta = \pi/4 \), putting \( k_0^2 = \pi/4 \) in equation M11 leads to the explicit form of the \( 2 \times 4 \) SkX state:

\begin{align*}
    S_i &= S(\sin \theta_A \cos \phi_A - i_y \pi/2, \sin \theta_A \sin (\phi_A - i_y \pi/2, \cos \theta_A), i \in A; \\
    S_j &= S(\sin \theta_B \cos (\phi_B + j_x \pi/2, \sin \theta_B \sin (\phi_B + j_x \pi/2, \cos \theta_B), j \in B. \quad (D1)\]
\end{align*}

Using equation (12), we find the classic ground state energy as

\[ E_{\text{classic}} = -NJ^2 [\cos^2 \theta_A - \cos \theta_A \cos \theta_B + \cos \phi_A + \phi_B] \sin \theta_A \sin \theta_B + h(\cos \theta_A + \cos \theta_B)]. \quad (D2) \]

Because \( \sin \theta_A \sin \theta_B > 0 \), the minimization requires \( \phi_A + \phi_B = 0 \). For \( \theta_A \) and \( \theta_B \), the minimization leads to:

\begin{align*}
    - \frac{1}{NJ^2} \frac{\partial E_{\text{classic}}}{\partial \theta_A} &= - \sin 2\theta_A + \sin (\theta_A + \theta_B) - h \sin \theta_A = 0, \\
    - \frac{1}{NJ^2} \frac{\partial E_{\text{classic}}}{\partial \theta_B} &= \sin (\theta_A + \theta_B) - h \sin \theta_B = 0 \quad (D3)\]
\end{align*}

whos solutions lead to the two polar angles shown in figure 4(a).

We have also taken a general \( 4 \times 4 \) structure, then numerically minimize the classic ground state energy with respect to the 32 parameters (16 polar angles + 16 azimuthal angles). Numerical results always find the configuration shown in figure 4(a).

In equation (12), setting \( \phi_A = -\phi_B = 0 \) which is nothing but the Goldstone mode, equation (12) can be written in the form of equation M11. We find\(^\text{12}\) an oscillating skyrmion density \( Q_{ik} = \mathbf{S}_i \cdot (\mathbf{S}_k - \mathbf{S}_i) = (-1)^{i-j} \sin \theta_A \sin 2\theta_B (\cos \phi - \sin \phi) - \cos \phi \sin^2 \theta_B \) where \( i, j, k \) are taken as three lattice points around a square. At \( \phi = 0 \), it reduces to \( (-1)^{i-j} \sin \theta_A (\cos \theta_A - (-1)^{i-j} \theta_B) \sin \theta_B \)

\(^{12}\)Obviously, the skyrmion density \( Q_{ik} \) is not gauge invariant. In the \( U(1)_{loc} \) basis, it becomes a constant \( Q_{ik} = -\sin (\theta_A - \theta_B) \sin \theta_B \) independent of the latticisite and the angle \( \phi \).
From equation (12), we make suitably chosen rotations $\vec{S}_i = R_{ij}(\theta_i) R_z(\phi_i) \vec{S}_i$, where $\phi_i = \phi - i y \pi / 2$, $\theta_i = \theta_j$, $i \in A$; $\phi_i = -\phi + j y \pi / 2$, $\theta_i = \theta_j$, $j \in B$ to align the spin quantization axis along the Z axis. Then one need only introduce two HP bosons $a / b$ for the two sublattices A/B respectively and perform a Bogoliubov transformation to obtain:

$$\mathcal{H}_S = E_0 + 2JS \sum_k [\omega_+(k) \alpha_k^\dagger \alpha_k + \omega_-(k) \beta_k^\dagger \beta_k]$$  \hspace{1cm} \text{ (D4)}$$

with the spin wave spectrum

$$\omega_\pm(k) = \frac{1}{\sqrt{2}} \sqrt{C_k \pm \sqrt{C_k^2 - 4D_k^2}}$$  \hspace{1cm} \text{ (D5)}$$

where

$$C_k = A_0^2 + B_0^2 - 2 \cos(\theta_A + \theta_B) \cos^2 k_x + 2(A_0 \cos \theta_A - B_0) \cos k_x + 2 \cos^2 \theta_A \cos^2 k_x,$$

$$D_k^2 = [(A_0 + \cos k_y)(B_0 - \cos k_y) - \cos^2 k_y][(A_0 + \cos \theta_A \cos k_y)(B_0 - \cos k_y) - \cos^2(\theta_A + \theta_B) \cos^2 k_y]$$,  \hspace{1cm} \text{ (D6)}$$

where

$$A_0 = h \cos \theta_A + \cos 2\theta_A - \cos(\theta_A + \theta_B) > 0;$$

$$B_0 = h \cos \theta_B + 1 - \cos(\theta_A + \theta_B) > 0.$$  \hspace{1cm} \text{ (D7)}$$

We now evaluate $D_k^2$ at the $\Gamma = (0,0)$ point,

$$D_k^2 |_{k=\Gamma} = [(A_0 + 1)(B_0 - 1) - 1][(A_0 + \cos 2\theta_A)(B_0 - 1) - \cos^2(\theta_A + \theta_B)].$$  \hspace{1cm} \text{ (D8)}$$

Note that equation (D8) leads to $(A_0 + 1)(B_0 - 1) = 0$, therefore $D_k^2 |_{k=\Gamma} = 0$. It indicates $\omega(k) = 0$ which is the gapless Goldstone mode at $k = 0$.

Now we perform a long wavelength expansion around the $\Gamma$ point,

$$C_k = C_0 + C_k k_x^2 + C_k k_y^2 + \cdots = C_0 + 2 \cos(\theta_A + \theta_B) k_x^2 + (B_0 - A_0 \cos^2 \theta_A - 2 \cos^2 \theta_B) k_y^2 + \cdots$$

$$D_k^2 = D_k k_x^2 + D_k k_y^2 + \cdots,$$  \hspace{1cm} \text{ (D9)}$$

where we have introduced

$$C_0 = C_0 |_{k=0} = A_0^2 + B_0^2 - 2 \cos(\theta_A + \theta_B) + 2(A_0 \cos \theta_A - B_0) + 2 \cos^2 \theta_A > 0,$$

$$D_k = (A_0 + \cos 2\theta_A)(B_0 - 1) - \cos^2(\theta_A + \theta_B),$$

$$D_0 = (A_0 + B_0) \cos^2(\theta_A + \theta_B) / 2.$$  \hspace{1cm} \text{ (D10)}$$

thus we can extract the Goldstone mode from $\omega(k)$ in the long wavelength limit at the $\Gamma = (0,0)$ point:

$$\omega_G(\vec{k}) = \sqrt{\nu_{Gx}^2 k_x^2 + \nu_{Gy}^2 k_y^2},$$  \hspace{1cm} \text{ (D11)}$$

where its velocity $\nu_{Gx} = \frac{D_k}{C_0}$, $\nu_{Gy} = \frac{D_0}{C_0}$ are shown in figure 4(b).

In fact, as shown in figure 4(a), putting $\theta_A = 0$, $\theta_B = \pi$ and $\theta_A = \theta_B = 0$, one can also push the calculations to the Z-x state at $h < h_{GZ}$ and the FM state at $h > h_{GZ}$ but in a different gauges than the original one used in the previous sections. As expected, the minimum positions of excitations may shift at different gauges. The gaps along the whole line $\beta = \pi/4$ are shown in figure 4(c).

At the lower critical field $h = h_{GZ} = \sqrt{3} - 1$, $\theta_A = 0$, $\theta_B = \pi$, $A_0 = 2$, $B_0 = 2 - h_{GZ}$, we find $\nu_{Gx}(h_{GZ}) = \nu_{Gy}(h_{GZ}) = 0$. Then we expand equation (D5) to next leading order $k^2$:

$$\omega_G(\vec{k}) = \sqrt{(k_x^2 + k_y^2)^2 / 16 + O(k_x^3, k_y^3)} \approx \frac{k_x^2}{2m_{G1x}} + \frac{k_y^2}{2m_{G1y}},$$  \hspace{1cm} \text{ (D12)}$$

where we identify the two effective masses $m_{G1x} = 2$, $m_{G1y} = 2/\sqrt{3}$ which match $m_{Zx}$, $m_{Zy}$ in equation (A10) achieved from below $h_{GZ}$. This match indicates the transition from the Z-state to the IC-SkX at $h = h_{GZ}$ is a second order transition with $z = 2$. As shown in figure 3(b2), in contrast to near $h_{GZ}$ to be discussed below, there is no extra roton mode near $h_{GZ}$.

At $h = h_{GZ} = \sqrt{2}$, $\theta_A = \theta_B = 0$, $A_0 = B_0 = h_{GZ}$, we find $\nu_{Gx}(h_{GZ}) = \nu_{Gy}(h_{GZ}) = 0$. Then we expand equation (D5) to next leading order $k^3$:

$$\omega_G(\vec{k}) = \sqrt{(k_x^2 + k_y^2)^2 / 4 + O(k_x^3, k_y^3)} \approx \frac{k_x^2}{2m_{G2x}} + \frac{k_y^2}{2m_{G2y}},$$  \hspace{1cm} \text{ (D13)}$$

where we identify the two effective masses $m_{G2x} = m_{G2y} = 1$ and $z = 2$.

As shown in the figure 3(b1), as $h \rightarrow h_{GZ}$, there is also roton mode developing at $(0, \pi)$. Expanding equation (D5) near $k = q + (0, \pi)$ leads to:
where
\[
C' = A_0^2 + B_0^2 - 2 \cos(\theta_A + \theta_B) \cos^2 q_x - 2 (A_0 \cos^2 \theta_A - B_0) \cos q_y + 2 \cos^2 \theta_A \cos^2 q_y,
\]
\[
D'^2 = [ (A_0 - \cos q_y)(B_0 + \cos q_y) - \cos^2 q_y ][ (A_0 - \cos 2\theta_A \cos q_y)(B_0 + \cos q_y) - \cos^2(\theta_A + \theta_B) \cos^2 q_y].
\]

Now we perform a long wavelength expansion around \((0, \pi)\):
\[
C' = C'_{0} + C'_x q_x^2 + C'_y q_y^2 + \cdots
\]
\[
D'^2 = D'_{0} + D'_x q_x^2 + D'_y q_y^2 + \cdots,
\]
where we have introduced:
\[
C' = A_0^2 + B_0^2 - 2 \cos(\theta_A + \theta_B) - 2(A_0 \cos^2 \theta_A - B_0) \cos^2 \theta_A),
\]
\[
C'_x = 2 \cos(\theta_A + \theta_B),
\]
\[
C'_y = A_0 \cos^2 \theta_A - B_0 - 2 \cos^2 \theta_A
\]

and
\[
D'_{0} = [(A_0 - 1)(B_0 + 1) - 1][(A_0 - \cos 2\theta_A)(B_0 + 1) - \cos^2(\theta_A + \theta_B)],
\]
\[
D'_x = (A_0 - \cos 2\theta_A)(B_0 + 1) - \cos(\theta_A + \theta_B),
\]
\[
D'_y = (2 - A_0 + B_0)[(A_0 - \cos 2\theta_A)(B_0 + 1) - \cos^2(\theta_A + \theta_B)]/2.
\]

Thus the roton mode in \(\omega(q)\) takes the form:
\[
\omega_R(q) = \sqrt{\Delta_R + v_{R,x}^2 q_x^2 + v_{R,y}^2 q_y^2 + \frac{q_x^4}{4m_{R,x}^2} + \frac{q_y^4}{4m_{R,y}^2} + \frac{q_x^2 q_y^2}{2m_{R,xy}^2}},
\]
where
\[
\Delta_R = \sqrt{(C'_{0} - \sqrt{C'^2_{0} - 4D'^2_{0}})/2},
\]
\[
v_{R,x}^2 = \left[ C'_x - \frac{C'_0 C'_{0} - 2D'_x}{\sqrt{C'^2_{0} - 4D'^2_{0}}} \right]/2,
\]
\[
v_{R,y}^2 = \left[ C'_y - \frac{C'_0 C'_{0} - 2D'_y}{\sqrt{C'^2_{0} - 4D'^2_{0}}} \right]/2.
\]

Near the upper critical field \(h \to h_{c2}\), we find \(\Delta_R = \frac{4}{7} \sqrt{10 + \sqrt{2}} (h_{c2} - h) + O[(h_{c2} - h)^2]\). At \(h = h_{c2} = \sqrt{2}\), \(\Delta_R = 0, \theta_A = \theta_B = 0, v_{R,x} = v_{R,y} = 0\), then the roton mode becomes critical:
\[
\omega_R(q) = \sqrt{(q_x^4 + q_y^4)/4 + O(q_x^4, q_y^4)} \sim \frac{q_x^2}{2m_{R,x}} + \frac{q_y^2}{2m_{R,y}},
\]
where we also identify the two effective masses \(m_{R,xx} = m_{R,yy} = 1\) and \(z = 2\).

The effective masses of both the Goldstone mode equation \((D13)\) and the roton mode equation \((D21)\) coincide with the \(m_{F,xx}, m_{F,y} \) achieved from the FM state equation \((B11)\), or equivalently \(h \to h_{c2}, \beta = \pi/4\):
\[
\begin{align*}
\beta &= \pi/4. \\
m_{F,xx} &= \sin^4 2\beta \cos^2 2\beta = m_{F,y} = \sin^2 2\beta/m_{F,x} = 1 \text{ at } \beta = \pi/4.
\end{align*}
\]

D.2. In-commensurate SkX when \(\beta \neq \pi/4\)

Equation \((12)\) can be easily generalized to the most general IC-SkX at general \(\beta\):
\[
S_i = S(\sin \theta_A \cos(\phi_{i} + j_k^0)), \sin \theta_A \sin(\phi_{i} + j_k^0), \cos \theta_A), \quad i \in A;
S_j = S(\sin \theta_B \cos(\phi_{j} + j_k^0)), \sin \theta_B \sin(\phi_{j} + j_k^0), \cos \theta_B), \quad j \in B.
\]

At \(\beta = \pi/4, k_x^0 = \pi/2\), equation \((D22)\) reduces to equation \((12)\).

Because \(\phi_{A} + \phi_{B} = 0\), one can set \(\phi_{A} = -\phi_{B} = \phi\) which is nothing but the gapless Goldstone mode. Then equation \((D22)\) can be cast into the form in equation M11:
where $A$, $B$ and $C$, $D$ can be expressed in terms of $\theta_b$ and $\theta_h$.

The classical ground state energy of the IC-SkX becomes

$$E_{\text{classic}} = -N J_S^2 [(\cos^2 \theta_A + \cos^2 \theta_B)/2 + (\cos(2\beta + k_y^0)\sin^2 \theta_A + \cos(2\beta - k_y^0)\sin^2 \theta_B)/2$$

$$\quad - \cos(\theta_A + \theta_B) + h(\cos(\theta_A + \theta_B))]$$

which reduces to equation (D2) at $\beta = \pi/4$. The classical $\theta_A$, $\theta_B$ and $k_y^0$ are determined by the minimization condition:

$$\frac{1}{N J_S^2} \frac{\partial E_{\text{classic}}}{\partial \theta_A} = \frac{1}{N J_S^2} \frac{\partial E_{\text{classic}}}{\partial \theta_B} = \frac{1}{N J_S^2} \frac{\partial E_{\text{classic}}}{\partial k_y^0} = 0$$

which reduces to equation (D3) at $\beta = \pi/4$. Along the horizontal line $h = 1$, they are shown in figure 5(a). At $h < h_1$, $\theta_A = \theta_B$, $k_y^0 = 0$, it is in the canted phase in the left of figure 2. At $h > h_2$, $\theta_A = \theta_B$, $k_y^0 = \pi$, it is in the canted phase in the right of figure 2. Obviously, there is a mirror symmetry about $\beta = \pi/4$ in figure 5(a).

Following similar procedures as those at $\beta = \pi/4$ outlined in 4.1: from equation (D22), we make suitably chosen rotations $\hat{S}_i = R_x(\theta)R_y(\phi)S_i$, where $\phi = \phi_i - i k_y^0$, $\theta_i = \theta_i$, $i \in A$; $\phi_j = -\phi + j k_y^0$, $\theta_j = \theta_b$, $j \in B$ to align the spin quantization axis along the Z axis. Then one need only introduce two HP bosons $a/b$ for the two sublattices A/B respectively and perform a Bogoliubov transformation to obtain the spin wave spectrum $\omega_k(k)$. After very lengthy manipulations and very careful long wavelength expansion, we find the Goldstone mode at $\Gamma = (0, 0)$ in equation (D11) at $\beta = \pi/4$ is replaced by:

$$\omega_G(\vec{k}) = \sqrt{v_{G,x}^2 k_x^2 + v_{G,y}^2 k_y^2} - c_G k_y,$$

where $c_G(\beta, H) = -c_G(\pi/2 - \beta, H)$, so $c_G > 0$ when $\beta < \pi/4$, $c_G < 0$ when $\beta > \pi/4$ and $c_G = 0$ when $\beta = \pi/4$ recovering equation (D11). How the three velocities $v_{G,x}$, $v_{G,y}$ and $c_G$ changes from $h_c1$ to $h_c2$ at a fixed $\beta = \pi/5 < \pi/4$ is shown in figure 5(b).

Similarly, the roton mode equation (D19) at $\beta = \pi/4$ developed as $h \rightarrow h_c2$ near $(0, \pi)$ is replaced by:

$$\omega_R(\vec{q}) = \sqrt{\Delta_R^2 + v_{R,x}^2 q_x^2 + v_{R,y}^2 q_y^2} - c_R q_y,$$

where $c_R(\beta, H) = -c_R(\pi/2 - \beta, H)$, so $c_R > 0$ when $\beta < \pi/4$, $c_R < 0$ when $\beta > \pi/4$ and $c_R = 0$ when $\beta = \pi/4$ recovering equation M11.

Compared with equations M6 and M7, we find the Goldstone mode and the Roton mode take similar forms as those in the canted phase. At a fixed $h$ in figure 2 (for fixed $h = 1$, see figure 5(a)), we find that as $h \rightarrow h_c1$ (or $h \rightarrow h_c2$), $v_{G,y} = c_G = 0$ (or $v_{R,y} + c_R = 0$), it is a bosonic Lifshitz transition with the anisotropic dynamic exponent $z_\perp = 1$, $z_\parallel = 3$. This picture is completely consistent as that achieved from the canted phase to the IC-SkX. These facts suggest some sort of duality between the cant phase and the IC-SkX phase on the two side of $h_c1$ in figure 1.

Taking $h \rightarrow h_c1$, $v_{G,x} = v_{G,y} = 0$ and $c_G = 0$ in equation (15), expanding it to the order $k^4$, we find it matches equation M2 reached from $Z$-x state below $h_c1$.

Taking $h \rightarrow h_c2$, $v_{G,x} = v_{G,y} = 0$ and $c_R = 0$ in equation (15) and $\Delta_R = 0$, $v_{R,x} = v_{R,y} = 0$ and $c_R = 0$ in equation (16), expanding both equations to order $k^4$, we find both matches $m_{E,x}$ and $m_{E,y}$ reached from FM state above $h_c2$.

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