Enhanced Gauge Symmetries on Elliptic $K3$

L. Bonora†, C. Reina, A. Zampa

*International School for Advanced Studies (SISSA/ISAS)*
*Via Beirut 2–4, 34014 Trieste, Italy*
† *and INFN, Sezione di Trieste*

**Abstract:** We show that the geometry of K3 surfaces with singularities of type A-D-E contains enough information to reconstruct a copy of the Lie algebra associated to the given Dynkin diagram. We apply this construction to explain the enhancement of symmetry in F and IIA theories compactified on singular K3’s.
1. Introduction

F–theory and IIA superstring theory compactified on a K3 are conjectured to be s–dual to heterotic string theory on a 2–torus and 4–torus, respectively. At generic points of the moduli space of the latter, the gauge symmetry is abelian while at special moduli the gauge group is non–abelian. F–theory and IIA theory do not possess non–abelian gauge fields in their perturbative spectrum. Therefore, if duality is to hold, something exceptional must happen corresponding to special moduli in such a way that a non–abelian gauge symmetry appears. This gauge symmetry enhancement is conjectured on the basis of duality and is mostly supported by the appearance of Dynkin diagrams in correspondence with the resolution of singular K3’s. However we would like to reverse this argument and ask ourself whether the K3 geometry, in a IIA or F–theory environment, contains enough information to allow us to retrieve a non–abelian gauge theory framework. Accordingly, symmetry enhancement is not just a consequence of the conjectured duality, but rather constitutes a piece of evidence of it.

In this paper we address exactly this problem. That is, we consider F–theory (or type IIA theory) compactified on an elliptic K3. We go to the limit in which the K3 becomes singular (where a non–abelian gauge symmetry is expected to arise) and we ask ourselves whether from these data we can reconstruct the framework of a non–abelian gauge theory. We shall see that, in the case of a singular elliptic K3, it is in fact possible to associate to singular fibres, whose singularity is of A-D-E type, a copy of the Lie algebra with the same Dynkin diagram 1.

1 In this paper we limit ourselves to the simplest possible examples of symmetry enhancement. We do not consider here, for example, the appearance of non–simply laced symmetry groups or of ‘frozen’ singularities, see [1] and references therein.
2. From duality to symmetry enhancement

In this section we recall a few standard facts about heterotic string compactified on tori and the symmetry enhancements that ensue in F–theory or IIA theory compactified on K3 as a consequence of the hypothesis of duality.

2.1 Gauge symmetry of the heterotic string compactified on tori

Let us consider for definiteness [2] the heterotic SO(32) superstring compactified on a torus $T^p$. The coordinates of the string are divided in three groups: the uncompactified coordinates will be denoted $X^\lambda$, the $X^I$ with $I = 1, ..., 16$ are left–moving scalars on the maximal torus of SO(32), while $X^i$ with $i = 1, ..., p$ are the compactified string coordinates. $n^I, m^I$ and $n^i, m_i$ will be the corresponding winding and KK numbers. The general constant background (moduli) involves a gauge field $A_I^I$, beside the metric $g_{ij}$ and the two–form potential $B_{ij}$. The lattice of the conjugate momenta, $\Gamma_{p,16+p}$, defined by

$$L^i = n^i - B_i^k n^k + \frac{1}{2} g^{ij} m_j + \frac{1}{2} g^{ij} A_j^I (m^I - \frac{1}{2} A_I^I n^k)$$

$$\tilde{L}^i = n^i + B_i^k n^k + \frac{1}{2} g^{ij} m_j - \frac{1}{2} g^{ij} A_j^I (m^I - \frac{1}{2} A_I^I n^k)$$

$$L^I = m^I - A_I^I n^i$$

has scalar product

$$L^i g_{ij} L^j - \tilde{L}^i g_{ij} \tilde{L}^j - L^I L^I = - n_i m^i - n'_i m^i - m^I m^I$$

and is unimodular, integral and even.

The spectrum of the compactified theory is determined by the physicality conditions in the different sectors of the theory. The moduli space of the theory contains $p^2$ parameters corresponding to $g_{ij}$ and $B_{ij}$, and $16p$ parameters $A_I^I$. For generic values of the parameters we have $16 + 2p$ massless vector states. They are obtained by simply choosing $n^i = m_j = 0$ and $m^I = 0$ for any $i, j$ and $I$ (i.e. we sit at the origin of the lattice) and forming the right

$$b_{-\frac{1}{2}}^{\lambda} |0 >_R \otimes \alpha_{-1}^i |0 >_L, \quad b_{-\frac{1}{2}}^{\lambda} |0 >_R \otimes \tilde{\alpha}_{-1}^I |0 >_L \}$$

and left states

$$b_{-\frac{1}{2}}^i |0 >_R \otimes \alpha_{-1}^\lambda |0 >_L$$

respectively. The $\alpha$ oscillators are the bosonic ones, the $b(d)$ are the NS(R) fermionic ones, as usual. Therefore, at a generic point of the moduli space one finds an abelian gauge group $U(1)^{16+2p}$. More massless states, and therefore possible enhancing of symmetries, can be found at particular points of the moduli space. We give an
explicit example in the Appendix. There we show that at the point of the moduli space determined by $g_{11} = g_{22} = 1/4, g_{12} = 0$ and $A_I^J = \delta_I^J$, the symmetry of the heteroric theory compactified on $T^2$ is enhanced to $SO(36) \times U(1)^2$. Choosing different backgrounds we can find an enormous variety of different gauge groups (of total rank 20).

One could have started from the $E_8 \times E_8$ heterotic string instead, but once compactified on a torus the two heterotic theories are equivalent [3]. Similar things can be repeated for the heterotic string compactified on $T^p$. In this case the total rank of the gauge group is $16 + 2p$. In general the moduli space of the theory is, apart from the dilaton, isomorphic to

$$\mathcal{M}_{h,p} = O(p, 16 + p, \mathbb{Z}) \setminus O(p, 16 + p, \mathbb{R})/O(p, \mathbb{R}) \times O(16 + p, \mathbb{R})$$

where $O(p, 16 + p, \mathbb{Z})$ represents the group of t–duality equivalences [4].

2.2 Gauge symmetry on $K3$

In this subsection we summarize how one can figure gauge symmetry enhancement on the IIA and F–theory compactified on a K3 surface, on the ground that these theories must be dual to the heterotic string theory on $T^4$ and $T^2$, respectively.

The basic observation behind duality is to identify part of the moduli space of the heterotic theory with the moduli space of suitable structures on the K3 surface. For example, in the case of IIA compactified on K3, the moduli space of Einstein metrics on K3 is embedded in the moduli space of the conformal non–linear $\sigma$–model on K3, which in turn can be identified with $\mathcal{M}_{h,4}$, [6]. The moduli space of the Einstein–Kähler metrics on K3 surface $X$ is isomorphic to the Grassmannian of time–like 3–planes in $H^2(X, \mathbb{R})$ modulo $O(3, 19, \mathbb{Z})$, up to a positive real parameter which represents the volume of $X$. Actually, since we are interested in the case in which the $B$ field vanishes $^2$, we can simply identify the two moduli spaces.

The phenomenon of enhancement of symmetry is based on the existence of the roots of length $-2$, which, according to the previous subsection, correspond to massless non–abelian gauge fields. These correspond to homology 2–spheres $C$ in $X$ of self–intersection $C \cdot C = -2$. Moreover, since they have components only in the space–like part of the lattice, they are orthogonal to the time–like 3–plane spanned by the holomorphic two–form $\Omega$ and by the Kähler form $\omega$ of $X$. Now $\omega \cdot C$ measures the area of $C$, and, due to orthogonality, $\omega \cdot C = 0$. Therefore the roots of length $-2$ correspond to spheres of shrinking area. That is, our K3 will contain orbifold points [5]. This suggests a physical picture of the origin of the enhanced symmetry: the zero mass states are generated by 2–branes of type IIA, wrapped around the shrinking cycles.

$^2$See [7, 6] for the subtle distinction between the role of the $B$ field in orbifold conformal field theories and in enhanced symmetry theories.
The same can be done for the heterotic string compactified on $T^2$, which is expected to be dual to F–theory compactified on an elliptically fibered K3, [8],[9]. With respect to the previous case, we have now a smaller moduli space $\mathcal{M}_{h,2}$ on the heterotic side. Looking at the F–theory side, we can be more concrete. In fact $\mathcal{M}_{h,2}$ is isomorphic to the moduli space $\mathcal{M}_U$ of algebraic K3’s whose Picard lattice contains the hyperbolic plane $U$. It can be shown (see e.g. [6], p. 77) that this condition on $\text{Pic}(X)$ implies that $X$ is an elliptic fibration with a section. Of course, in this case too we can repeat what we said above for the IIA theory compactified on K3. We expect the symmetry enhancement to occur in correspondence with collapsing 2–cycles, i.e. when two or more singular fibres of type $I_1$ collide. The locations of a singular fibre on the base represents the position of a D–7–brane of IIB theory (F–theory is by definition a realization of such a non–trivial configuration). This fact lends itself to a string theory interpretation of the enhancement of symmetry: the massless vector states correspond to the string modes that become massless when two or more D–7–branes collide. However, in the F–theory case, we are in the condition to say much more: we can show that the geometry of singular K3’s contains enough information to allow us to reconstruct, in correspondence with the singular points, the expected non–abelian data, i.e. a copy of the Lie algebra associated to the given Dynkin diagram. This will be the subject of the next section. As we will see, this construction actually extends also to the case of the IIA theory.

3. From elliptic K3 geometry to enhanced gauge symmetry

To start with, let us be more specific about the moduli space $\tilde{\mathcal{M}}_U$ of elliptic K3’s with a section. These are the K3’s in which the Picard lattice is constrained to contain the hyperbolic lattice $U$ spanned by a fibre $F$ and the section $\Sigma$. Notice that every such K3 admits a Weierstrass presentation. It is known that the locus of the the varieties $X$ with $\text{Pic}(X) = U$ is an open smooth subvariety $\mathcal{M}_U \subset \tilde{\mathcal{M}}_U$. The generators of $H^2(X, \mathbb{Z})$ are the first Chern classes of 22 smooth line bundles over $X$. Among these two, namely those living in $H^{2,0}(X, \mathbb{C}) \oplus H^{0,2}(X, \mathbb{C}) \subset H^2(X, \mathbb{Z}) \otimes \mathbb{C}$, are Chern classes of line bundles which are never algebraic, while the two line bundles corresponding to $F$ and $\Sigma$ are always algebraic. Notice also that $F$ cannot be contracted because it is not exceptional (i.e. $F \cdot F = 0$), while contracting $\Sigma$ one loses the elliptic fibration and then leaves the moduli space $\tilde{\mathcal{M}}_U$. The complement of these four generators gives us 18 line bundles which at generic moduli are only smooth and therefore belong to the transcendental lattice. These classes are our candidates to account for the (right–handed) abelian $U(1)^{18}$ gauge symmetry which is susceptible of non–abelian enhancement.

When we move to $\tilde{\mathcal{M}}_U - \mathcal{M}_U$, the Weierstrass presentation gives a singular $X$: some of the transcendental cycles vanish but become components of the exceptional divisor $E$ on the resolution $\pi : \tilde{X} \to X$ of $X$. As we saw in the previous section
physics tells us that the enhancement of symmetry occurs on $X$ (not on $\tilde{X}$), and that the line bundles corresponding to the cycles which vanish on $X$ should be actually identified with the Cartan generators of the larger symmetry group. Our basic observation $^3$ is that on $\tilde{X}$ each component $E_\alpha$, obtained from the blow-up of $X$, is a divisor and, since $-E_\alpha \cdot E_\alpha = 2$, the line bundle $\mathcal{O}_\tilde{X}(-E_\alpha)$ restricts to $\mathcal{O}(2)$ on $E_\alpha$. This is good news because $\mathcal{O}(2)$ is the tangent sheaf $T_{E_\alpha}$ of $E_\alpha$ and the space of its holomorphic sections $\mathbb{C}\{z^2 \partial_z, z \partial_z, \partial_z\}$ is isomorphic to the Lie algebra $sl(2, \mathbb{C})$.

The section $z \partial_z$, which corresponds to the standard Cartan generator, vanishes at the nodes where $E_\alpha$ meets the other components of the singular fibre, and can be extended on $\tilde{X}$ as a section of $\mathcal{O}_\tilde{X}(F - E_\alpha)$. The next observation is that the direct image $\pi_* \mathcal{O}_\tilde{X}(F - E_\alpha)$ is not locally free on $X$: it is a line bundle on $X - p$, $p$ being the singular point, while its stalk at $p$ is generated as an $\mathcal{O}_p$ module by the standard generators of $sl(2, \mathbb{C})$. This is the end of the story when we blow down only one $E_\alpha$.

To understand the general phenomenon of enhancement of symmetry we need the explicit realization of the exceptional divisor $E$ in the resolution of a singularity as a “Dynkin curve” in the complete flag variety $F$ associated to the A-D-E group. We will freely use below some aspects of this construction and refer to [10] for an expository account. The starting point is the fact that the intersection matrix of the components of the exceptional curve $E$ is indeed the opposite of the Cartan matrix associated to the singularity. Let us call $g$ the simple Lie algebra with such a Cartan matrix. Each component $E_\alpha$ of $E$ is actually a Riemann sphere which is expected to correspond to the subalgebra $sl_\alpha(2) \subset g$ generated by a triplet associated to a root $\alpha$.

The second step comes from the embedding of $E$ as a Dynkin curve. This goes as follows: let $r$ be the rank of $g$, $x \in g$ be a subregular (i.e. with a commutant $Z(x)$ of rank $r + 2$) nilpotent element, and $x, h, y$ an $sl(2)$ triplet associated to $x$. The Dynkin curve $E$ is the set of flags $f \in F$ stabilized by $\exp(tx)$, $\forall t \in \mathbb{C}$. As well known $^4$ $g$ is isomorphic to the Lie algebra of the holomorphic vector fields on $F$. A simple idea would be to restrict these vector fields to the Dynkin curve $E$. However, this restriction has a non-trivial kernel: by definition, at least the fundamental vector field associated to $x$ restricts to zero on $E$. By considering the infinitesimal action of an element $w \in g$ we see that the corresponding vector field is tangent to $E$ if and only if $[x, w] = \lambda x$ for some $\lambda \in \mathbb{C}$, therefore $w + (\lambda/2)h$ commutes with $x$ showing that the space of fundamental vector fields tangent to $E$ is isomorphic to $Z(x) \oplus \mathbb{C}\{h\}$. Notice moreover that $h$ does not vanish on $E$.

Our proposal to restore the entire algebra is to restrict the fundamental vector

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$^3$See [12] for definitions and notations concerning sheaf theory.

$^4$Indeed, $F$ is a homogeneous $G$-space so any element $w \in g$ gives rise to a non-trivial fundamental holomorphic vector field on $F$. Since the flag variety is compact and the fundamental vector fields span the tangent space to $F$ at every point, it follows that each holomorphic vector field is actually fundamental.
fields on $F$ to a family of Dynkin curves. Since all Dynkin curves in $F$ are conjugate, we look for a subvariety $E \to \Delta$ fibered in Dynkin curves $E_t$, $t \in \Delta$ (with $\Delta$ within the adjoint orbit through $x$) which is minimal with respect to the properties that:

**P1:** no fundamental vector field vanishes identically on $E$,

**P2:** the space of holomorphic sections of a subsheaf of $i^* TF$ is isomorphic to $\mathfrak{g}$, where we denote by $i : E \to F$ the embedding of the family.

An explicit construction of $E$ runs as follows:

**Proposition.** Let $x, h, y$ be an $\mathfrak{sl}(2)$ triplet associated to $x$ and let $E$ be the Dynkin curve stabilized by $x$. The infinitesimal family $E = \bigcup_{t \in \mathbb{C}} \exp(ty) \cdot E \pmod{t^2}$ satisfies the property $P1$.

**Proof.** By the above description, a fundamental vector field vanishing on $E$ vanishes on $E$ as well if and only if it commutes with the entire triplet, and hence it belongs to the "reductive centralizer" $c = Z(x) \cap Z(y)$. It is known [10] that $c$ is zero for the type $D$ and $E$ algebras, while it is one dimensional for the singularities of type $A$. In the latter case, taking $x$ to be the standard subregular nilpotent element (see p.87 of [10]), a generator for $c$ reads $c = \text{diag}(r, -1, ..., -1)$ and the corresponding vector field does not vanish on $E$. Indeed, the explicit realization of the Dynkin curve given in p.88 of [10] shows that the vector field associated to $c$ is nonzero at least on the component $E_\alpha$ with $\alpha = L_1 - L_2$ being the “first” simple root (see [11] for notations). □

Let $\tilde{F}$ be the sheaf of sections of $(i^* TF)(-E)$ on $E$. A direct computation shows that $H^0(E, \tilde{F}) = \mathfrak{g}$:

**Proposition.** The family $E$ satisfies $P2$.

**Proof.** The space $H^0(E, i^* TF)$ of holomorphic sections of the restriction of the tangent bundle to $E$ is generated over $\mathbb{C}[t]/t^2$ by the fundamental vector fields, hence it is isomorphic to $t\mathfrak{g} \oplus \mathfrak{z}$, $\mathfrak{z} = \mathfrak{g}/\ker(i^*: \mathfrak{g} \to H^0(E, i^* TF))$ being the space of fundamental vector fields not vanishing on $E$. □

We can get rid of $E$ by projecting $\varpi : E \to E$ on $E$ and taking the direct image sheaf $F = \varpi_\ast \tilde{F}$. To make contact with enhancement of symmetry we simply embed the Dynkin curve $E$ as the exceptional divisor of $\pi : \tilde{X} \to X$, consider the direct image $\tilde{G}$ of $F$ under the embedding, blow down $\tilde{X}$ to $X$ and take the direct image $G = \pi_\ast \tilde{G}$. This is a skyscraper sheaf with stalk the Lie algebra $\mathfrak{g}$ supported at the singular point $p \in X$.

### 4. Final comments

The results of the previous section refer to elliptically fibered K3’s with a section that is, specifically, to the F–theory case. However the construction is local around the
singularity and can be applied to a generic complex surface, in particular to the compactification of IIA theory on singular K3’s. A generic K3 is not elliptically fibered and therefore the line bundle $\mathcal{O}(F - E_\alpha)$ does not exist. However, the construction really depends only on the fact that the exceptional divisor associated to a singularity of type A-D-E is a Dynkin curve. Of course this does not rely on the presence of an elliptic fibration and continues to be true even if the surface is not algebraic. In both cases the construction above produces a skyscraper sheaf of Lie algebras on singular K3’s. From the point of view of space-time we have projections $q_1 : X \times \mathbb{R}^6 \to X$ and $q_2 : X \times \mathbb{R}^n \to \mathbb{R}^n$ ($n = 6, 8$) and $q_2 \ast q_1^* G$ is now a trivial sheaf of Lie algebras on the noncompact part of space-time. In conclusion, we can reconstruct out of the singular K3 a bundle of Lie algebras on $\mathbb{R}^n$ which is an essential ingredient to start the study of the duality with the heterotic string.

Of course one would like to retrieve, in correspondence with symmetry enhancement, the full non–abelian framework of a gauge theory, including the gauge bosons with values in $g$. This however requires some additional information, which is not encoded in the geometry of the compactification space: in particular we need the notion of one–form in the uncompactified space. This goes beyond the scope of this paper, so we limit ourselves to a few words in the case of IIA theory. Here the additional ingredient we need comes from physics: it is the IIA theory 3-form which, on $X \times \mathbb{R}^6$, has a Kunneth component of degree $(2, 1)$. This component can be written as a superposition of harmonic forms on $X$ whose coefficients are 1-forms on $\mathbb{R}^6$. When $X$ is singular one can resolve the singularities, work on the smooth model $\tilde{X}$ and take 3-forms with coefficients in $\tilde{F}$. This may be a suggestion to get the algebra-valued 1-forms on $\mathbb{R}^6$.

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Appendix

In this Appendix we consider the explicit example of compactification of the heterotic string on the torus $T^2$ with background $g_{11} = g_{22} = 1/4, g_{12} = 0, B_{12} = 0$ (see subsection 2.1.) and construct an elliptic K3 with the same symmetry enhancement. Let us introduce the notation: $n = (n^1, n^2), m = (m_1, m_2)$ and $A^I = (A_1^I, A_2^I)$. Then
the conditions for the bosonic massless states are

\[ 0 = -\frac{3}{2} + N_{NS} + \tilde{N}_A + \frac{1}{4} \mathbf{n} \cdot \mathbf{n} + (\mathbf{m} + \mathbf{A}'(m' - \frac{1}{2} \mathbf{A}' \cdot \mathbf{n}))^2 + \frac{1}{2}(m' - \mathbf{A}' \cdot \mathbf{n})^2 \]

\[ \frac{1}{2} = -N_{NS} + \tilde{N}_A + \mathbf{n} \cdot (\mathbf{m} + \mathbf{A}'(m' - \frac{1}{2} \mathbf{A}' \cdot \mathbf{n})) + \frac{1}{2}(m' - \mathbf{A}' \cdot \mathbf{n})^2, \quad (4.1) \]

where

\[ N_{NS} = \sum_{n=1}^{\infty} (\alpha^\lambda_n \alpha^\lambda_n + \alpha^i_n \alpha^i_n) + \sum_{r=1}^{\infty} (rb^\lambda_r b^\lambda_r + rb^i_r b^i_r) \]

\[ N_A = \sum_{n=1}^{\infty} (\alpha^\lambda_{-n} \alpha^\lambda_{-n} + \alpha^i_{-n} \alpha^i_{-n}) + \sum_{n=1}^{\infty} (nd^\lambda_n d^\lambda_n + nd^i_n d^i_n) \]

\[ \tilde{N}_A = \sum_{n=1}^{\infty} (\tilde{\alpha}^\lambda_n \tilde{\alpha}^\lambda_n + \tilde{\alpha}^i_n \tilde{\alpha}^i_n) + \sum_{n=1}^{\infty} \alpha^I_n \alpha^I_n. \quad (4.2) \]

Here \( \lambda \) denotes the uncompactified dimensions and \( i = 1, 2 \) the compact ones. These indices, when repeated, are supposed to be summed over.

Now let us restrict the values of the gauge fields to \( A'_I = \delta'_I. \) The massless vector states are 18 right states which come from the conditions

\[ N_{NS} = \frac{1}{2}, \quad \tilde{N}_A = 1, \quad L^i g_{ij} L^j = 0, \quad \tilde{L}^i g_{ij} \tilde{L}^j + L^i L^j = 0. \]

These are the states (2.3). More right massless vector states are given by tensoring \( b^\lambda_{-\frac{1}{2}} |0 >_R \) with the scalars corresponding to points of length \(-2\) in the left-handed lattice, i.e. states obtained by imposing the conditions

\[ N_{NS} = \frac{1}{2}, \quad \tilde{N}_A = 0, \quad L^i g_{ij} L^j = 0, \quad \tilde{L}^i g_{ij} \tilde{L}^j + L^i L^j = 2. \quad (4.3) \]

There are altogether 612 such states, which together with the 18 states (2.3), form the adjoint representation of SO(36). The 18 states are the Cartan subalgebra generators. Notice that the massless vector states not belonging to the Cartan subalgebra come from points of length \(-2\) in the lattice (2.1), due to (2.2) and (4.3).

There are also left massless vector states. Two of them come from the conditions

\[ N_{NS} = \frac{1}{2}, \quad \tilde{N}_A = 1, \quad L^i g_{ij} L^j = 0, \quad \tilde{L}^i g_{ij} \tilde{L}^j + L^i L^j = 0, \]

i.e. they correspond to the states (2.4). There are no more massless vector states. Therefore at the point of the moduli space determined by \( g_{11} = g_{22} = 1/4, g_{12} = 0 \) and \( A'_I = \delta'_I, \) the symmetry of the heterotic theory compactified on \( T^2 \) is enhanced to \( \text{SO}(36) \times \text{U}(1)^2. \)

Let us see the same enhancement of symmetry on the F–theory side. An elliptically fibered K3 surface with a singularity of type \( D_{18} \) is, for example, the one
explicitly given by the following Weierstrass presentation: \( y^2 = 4x^3 - g_2x - g_3 \), where

\[
g_2 = 4^{1/3}(18z_0^8 + 30z_0^6z_1^2 + 12z_0^4z_1^4 + 3z_0^2z_1^6),
\]

and

\[
g_3 = -(63z_0^{11}z_1 + 70z_0^9z_1^3 + 42z_0^7z_1^5 + 12z_0^5z_1^7 + 2z_0^3z_1^9).
\]

We have

\[
\delta = g_2^3 - 27g_3^2 = 23328z_0^{24} + 9477z_0^{22}z_1^2 + 2916z_0^{20}z_1^4,
\]

showing that this surface has a singularity of type \( D_{18} \) over \([z_0 : z_1] = [0 : 1] \in \mathbb{P}^1\) (and four fibres of type \( I_1 \) in the Kodaira classification).

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