A DESCRIPTION OF THE ASSEMBLY MAP FOR THE BAUM-CONNES CONJECTURE WITH COEFFICIENTS

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Abstract. In this note we set a configuration space description of the equivariant connective K-homology groups with coefficients in a unital C*-algebra for proper actions. Over this model we define a connective assembly map and prove that in this setting is possible to recover the analytic assembly map.

1. Introduction

Let $G$ be a discrete group and $B$ a separable $G$-C*-algebra. The purpose of this note is to give a configuration space description of $G$-equivariant connective K-homology groups with coefficients in $B$ on the category of proper $G$-CW-complex. We use that model to give a description of the analytic assembly map for the Baum-Connes conjecture with coefficients. This work is a continuation of [17], and most of the results and the proofs in Section 2 are generalizations of this paper.

The Baum-Connes conjecture with coefficients predicts that the assembly map

$$
\mu^G: RKK^G_i(C_0(\mathbb{F}G), B) \rightarrow KK^i(C, B \rtimes_r G)
$$

is an isomorphism. The space $\mathbb{F}G$ is the classifying space for proper actions (see [1], Def. 1.6). The group $RKK^G_i(C_0(\mathbb{F}G), B)$ can be defined as

$$
RKK^G_i(C_0(\mathbb{F}G), B) = \lim_{\rightarrow} KK^G_i(C_0(X), B).
$$

And $B \rtimes_r G$ is the reduced crossed product.

We give a description of the group $KK^G_i(C_0(X), B)$ in terms of configuration spaces. It is described as a limit of $G$-equivariant connective K-homology groups of $X$ with coefficients in $B$, see Definition 2.5.

The idea to use configuration spaces to describe homology theories appears in [6], there the authors prove that the reduced singular homology groups can be described as the homotopy groups of the symmetric product, moreover the symmetric product can be described as the configuration space with labels on natural numbers, later Graeme Segal in [14] extend this idea to describe connective K-homology. In this case one should consider configuration spaces with labels on the set of mutually orthogonal finite dimensional subspaces of a fixed Hilbert space. We generalize this idea taking coefficients in a separable unital $G$-C*-algebra.

Results obtained here are related with descriptions of the assembly map using controlled categories as in [7], where we use configuration spaces instead of geometric modules.

This note is organized as follows:

In Section 2 we introduce the configuration space and relate it with some space of operators. In Section 3 we prove that equivariant connective K-homology groups with coefficients can be represented as the homotopy groups of the orbits of the

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configuration space defined in Section 2. In Section 4 we reformulate the analytic assembly map for the Baum-Connes conjecture with coefficients in terms of configurations spaces.

Preliminaries

Let $G$ be a discrete group. Let $X$ and $Y$ be (left) $G$-spaces. There is a canonical (left) $G$-action on the set of continuous maps from $X$ to $Y$ defined by

$$G \times \text{Maps}(X,Y) \longrightarrow \text{Maps}(X,Y)$$

$$(g,f) \mapsto (x \mapsto g(f(g^{-1}x))).$$

A $G$-CW-pair $(X,Y)$ is a pair of $G$-CW-complexes. It is called proper if all isotropy groups of $X$ are finite. Information about $G$-CW-pairs can be found in [10, Section 1 and 2].

Given a $G$-CW-pair $(X,Y)$ we denote by $C_0(X,Y)$ the $C^*$-algebra of continuous maps from $X$ to $\mathbb{C}$ that vanish at $Y$ and at infinity. When $Y = \emptyset$ we set $C_0(X,\emptyset) = C_0(X \coprod \{+,\{-\})$ where $G$ acts trivially on $+$. We denote by $\Sigma X$ the reduced suspension of $X$, and define $\Sigma \emptyset$ as $S^0$ with the trivial $G$-action.

**Definition 1.1.** A $G$-$C^*$-algebra is a $\mathbb{Z}/2\mathbb{Z}$-graded $C^*$-algebra equipped with a $G$-action by $^*$-automorphisms.

Let $B$ be a $G$-$C^*$-algebra. A ($\mathbb{Z}/2\mathbb{Z}$-graded) pre-Hilbert $G$-module over $B$ is a right $B$-module $E$ with a continuous $G$-action compatible with the $G$-action over $B$, together with a $B$-valued inner product $\langle , \rangle : E \times E \rightarrow B$. If $E$ is complete respect to the norm $\|x\| = ||(x,x)||^{1/2}$ we say that $E$ is a Hilbert $G$-module over $B$. Details about Hilbert $G$-modules can be found in [12].

$B$ is itself a Hilbert $G$-module over $B$, we can of course also form $B^n$. We denote by $B^\infty$ the pre-Hilbert $G$-module over $B$

$$B^\infty = \bigoplus_{n=0}^{\infty} B^n.$$ Let $\mathcal{H}_B$ be the completion of $B^\infty$. We denote by $M_n(B)$ the $C^*$-algebra of endomorphism of $B^n$.

On the other hand let $E$ be a pre-Hilbert $B$-module, we denote by $\mathfrak{B}(E)$ to the set of all continuous module homomorphisms $T : E \rightarrow E$ for which there is an adjoint continuous module homomorphism $T^* : E \rightarrow E$ with $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x,y \in E$. $\mathfrak{H}(E)$ is defined as the closure of the pre-Hilbert $B$-module of finite-rank operators on $E$, when $E = \mathcal{H}_B$ we denote $\mathfrak{H}(\mathcal{H}_B)$ simply by $\mathfrak{H}_B$. Let $B$ be a $G$-$C^*$-algebra, then

$$C_c(G,B) = \{ f : G \rightarrow B \mid f \text{ is continuous with compact support} \}$$

becomes a $^*$-algebra with respect to convolution and the usual involution. Similarly one can define

$$l^2(G,B) = \left\{ \chi : G \rightarrow B \mid \sum_{g \in G} \chi(g)^*\chi(g) \text{ converges in } B \right\}.$$

Endowed with the norm

$$\|\chi\| = \|\sum_{g \in G} \chi(g)^*\chi(g)\|$$

$l^2(G,B)$ is a Banach space. The left regular representation $\lambda_{G,B}$ of $C_c(G,B)$ on $l^2(G,B)$ is given by
\[(\lambda_{G,B}(f)\chi)(h) = \sum_{g \in G} h^{-1} f(g)\chi(g^{-1}h)\]

for each \(f \in C_c(G,B)\) and \(\chi \in l^2(G,B)\)

Definition 1.2. The reduced crossed product \(B \rtimes_r G\) is the operator norm closure of \(\lambda_{G,B}(C_c(G,B))\) in \(\mathcal{B}(l^2(G,B))\).

If \(B = \mathbb{C}\) we denote \(\mathbb{C} \rtimes_r G\) also by \(C^*_r G\).

In order to define the configuration space we need to recall the symmetric product.

Definition 1.3. Let \((X, x_0)\) be a based CW-complex. Consider the natural action of the symmetric group \(S_n\) over \(X^n\). The orbit space of this action
\[SP^n(X) = X^n/S_n\]
provided with the quotient topology is called the n-th symmetric product of \(X\). We denote elements in \(SP^n(X)\) as formal sums
\[\sum_{i=1}^n x_i,\]
where \(x_i \in X\).

2. Equivariant connective K-homology and configuration spaces

Let \((X,Y)\) be a proper \(G\)-CW-pair and \(B\) a unital separable \(G\)-C*-algebra. In this section we construct a configuration space \(D_G(X,Y,B)\) representing the equivariant connective K-homology groups with coefficients in \(B\). First we will prove that the homotopy groups of \(D_G(\cdot,\cdot,B)/G\) form an equivariant homology theory in the sense of [11], then we define a natural transformation from this functor to equivariant KK-theory groups that is an isomorphism in proper orbits over positive indexes.

2.1. Configuration space.

Definition 2.1. Let \((X,Y)\) be a \(G\)-CW-pair (non-necessarily proper). Let \(B\) be a separable unital \(G\)-C*-algebra. We say that a bounded *-homomorphism
\[F : C_0(X,Y) \to M_n(B)\]
is strongly diagonalizable if there is an orthonormal base \(\{v_1,\ldots,v_n\}\) of \(B^n\) over \(B\) and characters \(x_i : C_0(X,Y) \to \mathbb{C},\) for \(i = 1,\ldots,n\), such that for every \(f \in C_0(X,Y)\) and \(b \in B^n\)
\[F(f)(b) = \sum_{i=1}^n x_i(f)(b) v_i.\]

The space of the strongly diagonalizable operators from \(C_0(X,Y)\) to \(M_n(B)\) with the compact-open topology is denoted by
\[C^n_G(X,Y;B)\]

Definition 2.2. Let \(G\) be a discrete group, \(B\) be a separable \(G\)-C*-algebra and \((X,Y)\) be a \(G\)-connected \(G\)-CW-pair. There is a natural inclusion
\[C^n_G(X,Y;B) \subseteq C^{n+1}_G(X,Y;B).\]

Let \(C_G(X,Y;B)\) be the \(G\)-space defined as
\[C_G(X,Y;B) = \bigcup_{n \geq 0} C^n_G(X,Y;B),\]
with the compact open topology. The $G$-action is defined as follows
\[ G \times C_G(X, Y; B) \to C_G(X, Y; B) \]
\[ (g, F) \mapsto g \cdot F = (f \mapsto g[F(gf)]g^{-1}), \]
for every $f \in C_0(X, Y)$.

Now we will describe the space $C_G(X, Y; B)$ as a configuration space.

**Definition 2.3.** Let $B$ be a separable $G$-$C^*$-algebra and $L$ be a pre-Hilbert module over $B$, define the topological partial monoid $\text{MOD}_B L$ whose elements are closed, finitely generated projective $B$-submodules of $L$, with the operation $\oplus$ defined only when the $B$-modules are orthogonal in $L$. We have a natural topology on $\text{MOD}_B L$, considering it as a as a subspace of $\mathfrak{B}(L)$ (viewed as the space of projections). The canonical base point is the zero operator $0$.

Let $(X, Y)$ be a $G$-CW-pair, let $B$ be a separable $G$-$C^*$-algebra and $L$ a pre-Hilbert module over $B$, define the configuration space $D_{G,n}(X, Y; B, L)$ with labels in $\text{MOD}_B L$ as follows. Here we follows ideas of [13].

Let $W_n \subseteq SP^n((X/Y) \wedge \text{MOD}_B L, (\{Y\}, 0))$ whose elements are sums $\sum_{i=1}^n (x_i, M_i)$ such that every pair of elements in $\{M_1, \ldots, M_n\}$ are composable. $W_0$ is defined as a point.

The space $D_{G,n}(X, Y; B, L)$ is the quotient of $W_n$ by the relations $(x, M'_1) + (x, M'_2) + W = (x, M'_1 \oplus M'_2) + W,$

for every $W \in W_n$. And

$(x, 0) = (+, 0) = (\{Y\}, M),$ for every $x \in X/Y$ and for every $M \in \text{MOD}_B L$.

There is a natural inclusion of $D_{G,n}(X, Y; B, L)$ on $D_{G,n+1}(X, Y; B, L)$ given by add $(+, 0)$.

Over the space $D_{G,n}(X, Y; B, L)$ we have a $G$-action induced from the $G$-actions of $G$ over $X$ and $B$, defined as follows:

$g \cdot \left( \sum_{i=1}^n (x_i, M_i) \right) = \sum_{i=1}^n (gx_i, gM_i),$

for every $g \in G$ and $\sum_{i=1}^n (x_i, M_i) \in D_{G,n}(X, Y; B)$.

Endowed with that action $D_{G,n}(X, Y; B, L)$ is a $G$-invariant closed subspace of $D_{G,n+1}(X, Y; B, L)$.

Define the configuration space as the increasing union

$D_G(X, Y; B, L) = \bigcup_{n \geq 0} D_{G,n}(X, Y; B, L).$

When $L = B^\infty$ we denote $D_G(X, Y; B, L)$ just by $D_G(X, Y; B)$.

As the elements in $C_G(X, Y; B)$ can be diagonalized we have the following result

**Theorem 2.4.** Let $(X, Y)$ be a $G$-connected, $G$-CW-pair, then there is a natural $G$-homeomorphism

$C_G(X, Y; B) \cong D_G(X, Y; B).$
Proof. Let $F \in C^n_G(X, Y; B)$, then there are characters $x_i : C_0(X) \to \mathbb{C}$ for $i = 1, \ldots, n$ and a base $\{v_1, \ldots, v_n\}$ such that

$$F(f)(b) = \sum_{i=1}^{n} x_i(f)(b, v_i) \cdot v_i.$$  

Define a $G$-map

$$C^n_G(X, Y; B) \xrightarrow{\Phi_n} \mathcal{D}_G(X, Y; B)$$

$$F \mapsto \sum_{i=1}^{n} (x_i, \langle v_i \rangle).$$  

With a similar argument as in Thm. VI.1.4 in [2] we obtain that the map $\Phi_n$ is continuous.

On the other hand, let

$$\sum_{i=1}^{n} (x_i, M_i) \in \mathcal{D}_{G,n}(X, Y; B)$$

one can associate a unique operator $F \in C^N_G(X, Y; B)$ for $N$ large enough, such that its eigenvalues are given by the corresponding characters $x_i : C_0(X) \to \mathbb{C}$, and where each $x_i$ has associated the eigenspace $M_i$.

We define

$$\Xi_n : \mathcal{D}_{G,n}(X, Y; B) \to C^N_G(X, Y; B)$$

$$\sum_{i=1}^{n} (x_i, M_i) \mapsto F.$$  

This map is continuous because the map

$$\mathcal{W}_n : C^N_G(X, Y; B) \to \mathcal{D}_{G,n}(X, Y; B)$$

$$\sum_{i=1}^{n} (x_i, M_i) \mapsto F$$

is continuous and satisfy the following conditions

- $\chi((x, M_1') + (x, M_2') + W) = \chi((x, M_1 \oplus M_2) + W)$
- $\chi(x, 0) = \chi(+, M)$.

Then $\Xi_n$ is continuous being the quotient of $\chi$.

It is clear that when we take colimits $\bigcup_n \Phi$ and $\bigcup_n \Xi$ are inverse maps, then $\bigcup_n \Phi$ is a homeomorphism.

From now on we identify $C_G(X, Y; B)$ and $\mathcal{D}_G(X, Y; B)$.

Note that there is a canonical base point in $\mathcal{D}_G(X, Y; B)$ associated to the zero map, we denote it by $0$.

2.2. Connective K-homology.

Definition 2.5. We define a covariant functor from $G-CW$-pairs to the category of $\mathbb{Z}$-graded abelian groups.

$$k^n_G(X, Y, B) = \pi_{n+1}(\mathcal{D}_G(\Sigma X, \Sigma Y; B)/G, 0).$$

In particular

$$k^n_G(X, B) = \pi_{n+1}(\mathcal{D}_G(\Sigma(X_+), +; B)/G, 0).$$
We denote the elements in \( \mathcal{D}_G(X, Y; M)/G \) by \( \sum_{i=1}^n (x_i, M_i) \).

Now we will prove that \( k^G(-; B) \) satisfies the axioms for a \( G \)-homology theory in the sense of [11].

**Theorem 2.6.** The functor \( k^G(-; B) \) is a \( G \)-homology theory.

**Proof.**

1. **Homotopy axiom**
   Let \( f_t : (X, Y) \to (X', Y') \) \( (t \in [0, 1]) \) be \( G \)-homotopy, then the map \( f_t : \mathcal{D}_G(X, Y; B) \to \mathcal{D}_G(X', Y'; B) \) is a \( G \)-homotopy (because the topology is the compact-open topology). Hence the functor \( k^G(-; B) \) is \( G \)-homotopy invariant.

2. **Long exact sequence axiom**
   To prove the long exact sequence axiom for \( k^G \) we will show that \( p_* : \mathcal{D}_G(X; B)/G \to \mathcal{D}_G(X, Y; B)/G \) is a quasifibration.

   For a proper \( G \)-CW pair \( (X, Y) \) we have an inclusion
   \( \mathcal{D}_G(Y; B) \to \mathcal{D}_G(X; B) \),
   and a canonical projection
   \( p_* : \mathcal{D}_G(X; B) \to \mathcal{D}_G(X, Y; B) \)
given by neglecting the points in \( Y \).

**Theorem 2.7.** The map
\( p_* : \mathcal{D}_G(X; B)/G \to \mathcal{D}_G(X, Y; B)/G \)
is a quasifibration.

**Proof.** The proof is similar to given in [17] and then we give only a sketch. For this proof we need to recall the following lemma.

**Lemma 2.8 ([6]).** A map \( p : E \to B \) is a quasifibration if any one of the following conditions is satisfied:

(a) The space \( B \) can be decomposed as the union of open sets \( V_1 \) and \( V_2 \) such that each of the restrictions \( p^{-1}(V_1) \to V_1 \), \( p^{-1}(V_2) \to V_2 \), and \( p^{-1}(V_1 \cap V_2) \to V_1 \cap V_2 \) are quasifibrations.

(b) The space \( B \) is the union of an increasing sequence of subspaces \( B_1 \subseteq B_2 \subseteq \cdots \) with the property that each compact set in \( B \) lies in some \( B_n \), and such that each restriction \( p^{-1}(B_n) \to B_n \) is a quasifibration.

(c) There is a deformation \( \Gamma_t \) of \( E \) into a subspace \( E_0 \), covering a deformation \( \Gamma_t \) of \( B \) into a subspace \( B_0 \), such that the restriction \( E_0 \to B_0 \) is a quasifibration and \( \Gamma_1 : p^{-1}(b) \to p^{-1}(\Gamma_1(b)) \) is a weak homotopy equivalence for each \( b \in B \).

Note that we have a filtration of \( \mathcal{D}_G(X, Y; B) \) by closed \( G \)-spaces in the following way
\[
\mathcal{D}_G^n(X; Y; B) = \left\{ \sum_{i=1}^m (x_i, M_i) \mid \sum_{i=1}^m \text{rank}_{B}(M_i) \leq n \right\}
\]

The idea is to proceed by induction on \( n \), using property (b) in Lemma 2.8.
First we want to find an open set $U \subseteq D^n_G(X, Y; B)/G$ containing $D^n_G(X, Y; B)/G$ such that $U$ is a deformation retract of $D^n_G(X, Y; B)/G$ satisfying the condition (c) in Lemma 2.8. In other words we have to find an open set $U$ such that we have a commutative diagram

\[
\begin{array}{ccc}
D^n_G(X; B)/G & \xrightarrow{p} & D^n_G(X, Y; B)/G \\
\downarrow i & & \downarrow i \\
D^n_G(X; B)/G & \xrightarrow{p} & D^n_G(X, Y; B)/G
\end{array}
\]

where $i$ denotes the inclusion, such that $U$ satisfy condition (c) on Lemma 2.8.

Let $f_t : (X, Y) \rightarrow (X, Y)$ a $G$-homotopy such that $f_0 = id_X$ and $N \subseteq f_1^{-1}(Y)$ is an open neighborhood of $Y$ in $X$. Let $\mathcal{U} \subseteq D^n+1_G(X; B)/G$ be the set of configurations with at least one point in $N$, let $U = p(\mathcal{U})$. Both sets are open. Consider the induced map

\[
f_1^* : D^{n+1}_G(X; B)/G \rightarrow D^{n+1}_G(X, Y; B)/G,
\]

The homotopy $f_1^*$ is a weak deformation of $U$ into $D^n_G(X; B)/G$ covering a weak deformation of $U$ into $D^n_G(X, Y; B)/G$. To apply Lemma 2.8 we only need to verify that

\[
f_1^* : p^{-1}(b) \rightarrow p^{-1}(\mathcal{T}_1^*(b))
\]

is a weak homotopy equivalence for every $b \in p(\mathcal{U})$.

The set $\mathcal{U}$ can be described as the orbit set of configurations in $D^{n+1}_G(X; B)$ such that they have at least one point in $N$.

Let $b \in \mathcal{U}$, with

\[
b = \sum_{i=1}^n (x_i, M_i)
\]

and $\mathcal{T}_1^*(b) = \sum_{k=1}^l (f_1(x_{i_k}), M_{i_k})$ with $l \leq n$

then set $p^{-1}(b)$ can be described as the set whose elements have the form

\[
b + \sum_{j=1}^m (y_j, M'_j)
\]

where $y_j \in Y$ and $M'_j$ are composable elements in $\mathcal{MOD}_B(\oplus M_i)^\perp$. Then we have a homeomorphism

\[
p^{-1}(b) \xrightarrow{b \mapsto} D_G(Y; B, (\oplus M_i)^\perp)/G
\]

\[
b + \sum_{j=1}^m (y_j, M'_j) \mapsto \sum_{j=1}^m (y_j, M'_j)
\]

On the other hand the map $f_1^*$ is defined as

\[
f_1^* : p^{-1}(b) \rightarrow p^{-1}(\mathcal{T}_1^*(b))
\]

\[
b + \sum_{j=1}^m (y_j, N_j) \mapsto \mathcal{T}_1^*(b) + \sum_{j=1}^m (f_1(y_j), N_j)
\]
We have the following commutative diagram

\[
p^{-1}(b) \xrightarrow{h_o} D_G(Y; B, \oplus (M_i)_i) / G \\
p^{-1}(T_{1*}(b)) \xrightarrow{h_{T_{1*}}(o)} D_G(Y; B, \oplus (M_i)_i) / G.
\]

The map \( \chi \) can be described as sending

\[
\sum_j (y_j, N_j) \mapsto \sum_j (f_1(y_j), N_j) + b',
\]

where \( b' \) is the part of \( T_{1*}(b) \) contained in \( N - Y \). As \( f_1 \) is \( G \)-homotopic to the identity and one can deform \( b' \) to 0 using a continuous path, the map \( \chi \) is a homotopy equivalence and then the same is true for \( f_1 \). By part (c) in Lemma 2.8 we have \( p: U \to p(U) \) is a quasifibration.

The second part consist to prove that \( p|_{Q_{n+1}} \) and \( p|_{Q_{n+1} \cap p^{-1}(U)} \) are quasifibrations, where \( Q_{n+1} = p^{-1}(D_{n+1}^G(X, A; G) / G - D_n^G(X, A; G) / G) \), and then use part (a) in Lemma 2.8. The argument is similar to the given in Thm. 3.15 in [17].

(3) **Excision**

It is a consequence of the isomorphism of \( G \)-C*-algebras between \( C_0(X_1 \cup f X_2, X_2) \) and \( C_0(X_1, Y_1) \) induced by the inclusion

\[
i : (X_1, Y_1) \to (X_1 \cup f X_2, X_2).
\]

(4) **Disjoint union axiom**

We have a natural isomorphism of \( G \)-C*-algebras

\[
C_0(\coprod_{i \in I} X_i) \cong \bigoplus_{i \in I} C_0(X_i),
\]

then we have a \( G \)-homeomorphism

\[
D_G(\coprod_{i \in I} X_i, B) \cong \coprod_{i \in I} D_G(X_i, B)
\]

taking homotopy groups on orbit spaces we have the desired isomorphism.

In order to relate the homology theory \( E^G(\cdot; B) \) with \( G \)-equivariant K-homology groups with coefficients in \( B \) we will use the machinery of equivariant KK-theory, let us recall some necessary notions for our work, we follow the treatment in [3].

**Definition 2.9.** Let \( C \) and \( B \) be \( \mathbb{Z}/2 \)-graded \( G \)-C*-algebras. The set of Kasparov \( G \)-modules for \((C, B)\), that is the set of triples \((E, \phi, F)\) such that

1. \( E \) is a graded countably generated Hilbert \( B \)-module with a continuous \( G \)-action.
2. \( \phi : C \to \mathcal{B}(E) \) is a \( G \)-equivariant graded *-homomorphism.
3. \( F \) is a \( G \)-continuous operator in \( \mathcal{B}(E) \) of degree 1, such that for every \( c \in C \) and \( g \in G \)
   - \( F \phi(c) - \phi(c)F \),
   - \( (F^2 - Id)\phi(c) \),
   - \( (F - F^* \phi(c) \) and
   - \( (g \cdot F - F)\phi(c) \)
   are all in \( \mathfrak{H}(E) \).
There is a very general homotopy relation defined over Kasparov $G$-modules (see for example Def. 17.2.2 in [3]). We denote a homotopy class of a Kasparov $G$-module by $[E, \phi, F]$ and by $KK_G(C, B)$ to the set of equivalence classes of Kasparov $G$-modules for $(C, B)$ under the homotopy relation.

We will define a natural transformation $\mathfrak{A}(\cdot)$ from $KK_G^\ast(-; B)$ to the equivariant $KK$-theory groups $KK_G(G/H, B)$ such that

$$KK_G^G(G/H, B) = [S^{i+1}, D_G(\Sigma(G/H_+), +; B)/G] \xrightarrow{\mathfrak{A}^i(G/H)} KK_G^i(C_0(G/H), B)$$

is an isomorphism for $i \geq 0$ when $H$ is a finite subgroup of $G$. The crucial step is to assign to the $G$-orbit of a configuration over $\Sigma(X_+)$ a $G$-equivariant $*$-homomorphism

$$C_0(\Sigma(X_+), +) \to \mathfrak{A}_B.$$ This result is proved in the following lemmas that are inspired in Sections 2.2 and 2.3 in [16].

Lemma 2.10. Let $b \in D_G(\Sigma(X_+), +; B)/G$, then if $F$ is a representing of the orbit $b$, we define a $*$-homomorphism

$$\mathfrak{A}(F) : C_0(\Sigma(X_+), +) \to \mathfrak{A}_B$$

$$f \mapsto \sum_{g \in G}(g \cdot F)(f).$$

Then

1. The sum $\sum_{g \in G}(g \cdot F)(f)$ converges in the norm topology.
2. $\mathfrak{A}(F)$ is continuous.
3. $\mathfrak{A}(F)$ only depends on the orbit $b$ and $\mathfrak{A}(F)$ is $G$-equivariant.

Proof. Let $f \in C_0(\Sigma(X_+), +)$ with support $A \subseteq \Sigma(X_+) = (I \times X_+)/\sim$. If $F$ has eigenvalues given by characters

$$+, (t_1, x_1), \ldots, (t_n, x_n) \in \Sigma(X_+),$$

then for every $g \in G$, $g \cdot F$ has eigenvalues

$$+, (t_1, gx_1), \ldots, (t_n, gx_n) \in \Sigma(X_+).$$

As $X$ is $G$-proper, the set $\{gx_i \mid g \in G\}$ is discrete for every $i$, then $A \cap \{gx_i \mid g \in G\}$ is finite, it implies that the sum $\sum_{g \in G}(g \cdot F)(f)$ only has finite terms for each $f \in C_0(\Sigma(X_+, +))$, it implies (1) and (2), on the other hand statement (3) is obvious.

As $\mathfrak{A}_B$ is complete there is a continuous extension of $\mathfrak{A}(F)$ to $C_0(\Sigma(X_+, +)$, as $\mathfrak{A}(F)$ only depends on $b$ we can denote by $\mathfrak{A}(b)$ the extension of $\mathfrak{A}(F)$ to $C_0(\Sigma(X_+, +)$.

Given a $G$-equivariant $*$-homomorphism

$$\phi : C_0(\Sigma(X_+, +) \to \mathfrak{A}_B,$$ one can assign an element in $KK_G(C_0(\Sigma(X_+, +), B)$, namely $[\mathfrak{A}_B, \phi, 0]$. Then we have a map

$$\mathfrak{A} : \pi_0(D_G(X_+, +; B)/G, 0) \to KK_G^0(C_0(\Sigma(X_+, +)), B)$$

$$[b] \mapsto [\mathfrak{A}_B, \mathfrak{A}(b), 0].$$

This map is well defined because homotopy of $*$-homomorphism is a special case of the homotopy relation of the Kasparov cycles. Moreover, this association is really a natural transformation $\mathfrak{A}$ from $\pi_0(D_G(-; B)/G)$ to $KK_G^0(C_0(-), B)$. To extend
this natural transformation to all $n \geq 0$ we need the following form of the Bott periodicity theorem. For a proof consult [3, Corol. 19.2.2].

**Theorem 2.11.** For any $G$-$C^*$-algebras $A$ and $B$, we have natural isomorphisms

$$KK_0^G(A, B) \xrightarrow{\beta} KK_0^G(A, SB)$$

$$KK_0^G(SA, B) \xrightarrow{\alpha} KK_0^G(A, B)$$

where the suspension of $B$ is

$$SB := \{f : S^1 \to B \mid f \text{ is continuous and } f(1) = 0\}.$$

with the supremum norm.

Given a based continuous map $f : S^1 \to D_G(\Sigma(X_+), +; B)/G$, and $f \in C_0(\Sigma(X_+, +))$, we can induce a continuous map

$$\mathfrak{A}(f)(-)(f) : S^1 \to \mathfrak{R}_B$$

$$\theta \mapsto \mathfrak{A}(f(\theta))(f),$$

it is an element of $S(\mathfrak{R}_B)$, that means that we have a map

$$\text{Map}_0(S^1, D_G(\Sigma(X_+), +; B)/G) \to \text{Hom}^*(C_0(\Sigma(X_+, +)), S(\mathfrak{R}_B))^G,$$

and to every element in $\phi \in \text{Hom}^*(C_0(X), S(\mathfrak{R}_B))^G$ we can associate the Kasparov module $(S(\mathfrak{R}_B), \phi, 0)$. Taking homotopy classes we have a homomorphism

$$k^n_G(X, B) = \pi_1(D_G(\Sigma(X_+), +; B), 0) \to KK_0^G(C_0(\Sigma(X_+, +)), SB) \cong KK_0^G(C_0(X), B),$$

where the last isomorphism is given by Theorem 2.11 identifying $C_0(\Sigma(X_+, +))$ with $S(C_0(X)))$. The transformation $\mathfrak{A}^0(X)$ is defined as the composition of the above maps.

For every $n \geq 1$ the transformation $\mathfrak{A}^n(A)$ is defined in an analogue way using Theorem 2.11 repeatedly.

**Remark 2.12.** As $k^n_G$ is a $G$-homology theory it satisfies the suspension axiom, moreover the same argument proves that we have a canonical identification

$$k^n_G(pt; B) \cong \text{Groth}(\pi_0(C_G(pt; B)/G)),$$

where Groth denotes the Grothendieck group associated to the monoid $\pi_0(C_G(X, Y; B)/G)$ with direct sum.

**Theorem 2.13.** Let $H$ be a finite group. The homomorphism

$$\mathfrak{A}^n(pt) : k^n_H(pt, B) \to KK^n_H(\mathbb{C}, B)$$

is an isomorphism for every $n \geq 0$.

**Proof.** The argument is similar to given in [17], we will give here for completeness. Let $\alpha = (E, \phi, F)$ be a $(\mathbb{C}, B)$-Kasparov $H$-module, for $n = 0$ we will prove that $\alpha$ is homotopic to a module with the form $(B^n, \mathbf{1}, 0)$ with $\mathbf{1} : \mathbb{C} \to M_{n}(B)$ the diagonal inclusion. The Hilbert $B$-module $E$ is $\mathbb{Z}/2\mathbb{Z}$-graded and the map $\phi$ is a projection of degree 0, which means that $E = E_0 \oplus E_1$ and $\phi(1) = \text{diag}(P, Q)$ for projections $P$ and $Q$. The operator $F$ has the form $F = \begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix}$. The Kasparov module is $KK$-equivalent to $\beta = \begin{pmatrix} \text{Im}(P) \oplus \text{Im}(Q), 1, \tilde{F} = \begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix} \end{pmatrix}$. By Prop. 3.27 in [5] we can suppose that $\beta$ is homotopic to $(\ker(\tilde{F}) \oplus \ker(\tilde{F}^*), 1, 0)$, where $\ker(\tilde{F}) \oplus \ker(\tilde{F}^*)$ is a finitely generated $H$-equivariant Hilbert $B$-submodule of $B^n$. The map
A DESCRIPTION OF THE ASSEMBLY MAP

\[ [\alpha] \mapsto [\ker(\tilde{F})] - [\ker(\tilde{F}^*)] \]
gives us an inverse for \( \mathfrak{A}^0(pt) \) (here we are using the identification on Remark 2.12).

If \( n > 0 \) note that the same argument to prove Corol. 19.2.2 in [3] proves that we have an isomorphism

\[ \pi_{n+1}(D_H(S^1, 1; B)/H, 0) \xrightarrow{\text{Bott}} k^H_0(pt, S^n(B)) \]
given by Bott periodicity, we have the following commutative diagram

\[ \pi_{n+1}(D_H(S^1, 1; B)/H, 0) \xrightarrow{\text{Bott}} k^H_0(pt, S^n(B)) \]
\[ \xrightarrow{\mathfrak{A}^n(pt)} \]
\[ \xrightarrow{\text{KK}_H(C, S^n(B))} \]

As \( \mathfrak{A}^0(pt) \) is an isomorphism and Bott is an isomorphism then \( \mathfrak{A}^n(pt) \) is an isomorphism also. \( \square \)

In a similar way as in Thm. 5.5 in [17] it can be proved that \( k^G_i(\cdot, B) \) has an induction structure in the sense of [11], moreover using Theorem 2.13 and cellular induction it can be proved the following result.

**Theorem 2.14.** The functor \( k^G_i(\cdot, B) \) is naturally equivalent to the G-equivariant connective homology theory associated to the functor \( KK^G_i(C_0(\cdot), B) \) over the category of proper G-CW pairs.

**2.3. Recovering K-homology.** Now we know that \( k^G_\ast \) is a homology theory is represented by the connective cover \( k^G_0(B) \) of the proper G-spectrum associated to equivariant K-theory (with coefficients in \( B \)) as is defined for example in Section 2 in [5].

Using the Bott periodicity is possible to define a natural transformation

\[ \beta : k^G_i(\cdot, B) \to k^G_{i+2}(\cdot, B) \]
(in this case it is not an equivalence). With this \( \beta \) it is possible to recover the (non-connective) equivariant K-homology from its connective version.

**Proposition 2.15.** For every proper G-CW-complex \( X \) there is a natural isomorphism

\[ \lim_{\to n} k^G_{i+2n}(X, B) \cong KK^G_i(C_0(X), B) \]

The direct limit is taken over the maps \( \beta \) defined above.

**Proof.** We already know that \( k^G_0(B) \) is a proper G-spectrum representing equivariant connective K-homology, as the periodicity maps \( \beta \) commutes with the structure maps of \( k^G_0(B) \) then \( \lim_{\to n} k^G_0(B) \) is a proper G-spectrum, then \( \lim_{\to n} k^G_{i+2n}(\cdot, B) \) is a G-homology theory, moreover \( \lim_{\to n} \mathfrak{A}^n_{i+2n} \) is a natural transformation from \( \lim_{\to n} k^G_{i+2n}(\cdot, B) \) to \( KK^G_{i+2n}(C_0(\cdot), B) \cong KK^G_{i+2n}(C_0(\cdot), B) \), such that is an isomorphism on proper orbits \( G/H \), then the natural transformation is an equivalence. \( \square \)

3. THE ANALYTIC ASSEMBLY MAP

In this section we will describe a version of the assembly map for the Baum-Connes conjecture with coefficients, in terms of configuration spaces.

First we briefly recall the descent morphism of Kasparov. For details the reader can consult [9] Lemma 3.9.\[ \]
Let \((E, \phi, F)\) be a \(G\)-equivariant \((A, B)\)-Kasparov module. We can consider \(C_c(G, E)\) as a pre-Hilbert \(C_c(G, B)\)-module with a norm defined using the left regular representation in \(\mathfrak{B}(l^2(G, E))\) in a similar way as in Section 1.

The operator norm closure of the left regular representation of \(C_c(G, E)\) in \(\mathfrak{B}(l^2(G, E))\) is denoted by \(E \rtimes_r G\). It is a Hilbert \(B \rtimes_r G\)-module.

On the other hand, the natural \(\mathbb{Z}/2\mathbb{Z}\)-graded \(*\)-homomorphism \(\phi^* : C_c(G, A) \to C_c(G, E)\) can be extended to a \(\mathbb{Z}/2\mathbb{Z}\)-graded \(*\)-homomorphism \(\tilde{\phi} : A \rtimes_r G \to \mathfrak{B}(E \rtimes_r G)\).

Finally we define \(\tilde{F} \in \mathfrak{B}(E \rtimes_r G)\) by
\[
\tilde{F}(\alpha)(g) = F(\alpha)(g) \quad \text{for} \quad \alpha \in C_c(G, E).
\]

Let us denote by \(j_G([E, \phi, F]) = [E \rtimes_r G, \tilde{\phi}, \tilde{F}]\).

**Lemma 3.1.** For any \(G\)-C*-algebras \(A\) and \(B\) there is a functorial morphism \(j_G : KK_*^G(A, B) \to KK_*^G(A \rtimes_r G, B \rtimes_r G)\).

The map \(j_G\) is called the descent morphism.

As \(X\) is proper and \(G\)-compact there is a non-negative \(h \in C_c(X)\) such that \(\sum_{g \in G} h(g^{-1}x) = 1\) for all \(x \in X\).

Define \(p \in C_c(G, C_c(X))\) by
\[
p(g, x) = \sqrt{h(x)}h(g^{-1}x),
\]
\(p\) is a projection in \(C_c(G, C_c(X))\) and hence in \(C_0(X) \rtimes_r G\). Consider the homomorphism
\[
\theta : \mathbb{C} \to C_0(X) \rtimes_r G \quad \lambda \mapsto \lambda p,
\]
it induces a morphism
\[
\theta^* : KK^i(C_0(X) \rtimes_r G, B \rtimes_r G) \to KK^i(\mathbb{C}, B \rtimes_r G).
\]

We define the analytic assembly map as the composition
\[
\mu^G = \theta^* \circ j_G : KK^i_G(C_0(X), B) \to KK^i(\mathbb{C}, B \rtimes_r G).
\]

We proceed to define a version of the assembly map for the configuration space description of equivariant K-homology.

Note that when \(G = 1\) the natural transformation \(\mathfrak{A}\) defined on Lemma 2.10 can be described as
\[
\mathfrak{A}(b) = [[b(1), 1, 0]],
\]
where \(1 : \mathbb{C} \to M_n(B)\) is the diagonal inclusion.

**Definition 3.2.** Let \(X\) be a proper, co-compact \(G\)-CW-complex, define the connective assembly map \(\mu^G_P\), as the map that complete the following commutative diagram
\[
\begin{array}{ccc}
K^i_G(X, B) & \xrightarrow{\mathfrak{A}_0(\cdot)} & KK^i_G(C_0(X), B) \\
\pi^i_G \downarrow & & \downarrow \mu^G_P \\
K^i(\{pt\}, B \rtimes_r G) & \xrightarrow{\mathfrak{A}^i(\cdot)} & KK^i(\mathbb{C}, B \rtimes_r G)
\end{array}
\]
By Theorem 2.13 we know that $\mathcal{A}'(\{pt\})$ is an isomorphism, then

$$\overline{\mu}_i^G = (\mathcal{A}'(\{pt\}))^{-1} \circ \mu_i^G \circ \mathcal{A}'_G(X).$$

The map $\overline{\mu}_i^G$ can be described in the configuration space picture of K-homology. Let $b \in D_G(X, B)/G$, as we know $b$ can be identified with the $G$-orbit of a configuration $\sum (x_i, M_i)$ where each $x_i$ corresponds to eigenvalues and each $M_i$ corresponding to eigenspaces. On the other hand

$$\mu_0^G(\mathcal{A}(b)) : C \rightarrow K_{B \rtimes rG},$$

is completely determined by the image of $1 \in C$, that in this case is the reduced crossed product of the norm closure of the pre-Hilbert $B$-module $\bigoplus_i (G \cdot M_i) \subseteq H_B$ with the natural $G$-action.

Then one can define a version of the assembly map for configuration spaces as

$$D_G(X, B)/G \xrightarrow{\overline{\mu}_i^G} D(pt, B \rtimes rG) \xrightarrow{\sum_n (x_i, M_i) \mapsto \left( pt, \bigoplus_{i=1}^{n} (G \cdot M_i) \rtimes rG \right).}$$

We have proved the following theorem

**Theorem 3.3** (Description of the assembly map for configuration spaces). Let $X$ be a proper, co-compact $G$-CW-complex, there is a commutative diagram

$$
\begin{array}{ccc}
\kappa_i^G(X, B) & \xrightarrow{\mathcal{A}'_G(X)} & K K_i^G(C_0(X), B) \\
\pi_i^G \downarrow & & \downarrow \mathcal{A}'(\{pt\}) \\
\kappa_i(\{pt\}, B \rtimes rG) & \xrightarrow{\mathcal{A}'(\{pt\})} & K K_i(C, B \rtimes rG)
\end{array}
$$

Where the right vertical arrow is the analytic assembly map and the left vertical arrow is the induced in homotopy groups of the map defined above.

**Remark 3.4.** The map $\overline{\mu}_i^G$ can be described as the map sending a configuration to the reduced crossed product of the norm closure of the direct sum of the orbits of the labels.

It is clear that $\overline{\mu}_i^G$ commutes with the periodicity map $\beta$, then applying Theorem 3.3 and Prop. 2.15 we have that the analytic assembly map $\mu_i^G$ can be recovered in the following way.

**Theorem 3.5.** Let $X$ be a proper, co-compact $G$-CW-complex, there is a commutative diagram where the horizontal arrows are isomorphism

$$
\begin{array}{ccc}
\lim_{n \to \infty} k_i^G(X, B) & \xrightarrow{\mathcal{A}'_G(X)} & \lim_{n \to \infty} K K_i^G(C_0(X), B) \\
\lim_{n \to \infty} \pi_i^G \downarrow & & \downarrow \mathcal{A}'(\{pt\}) \\
\lim_{n \to \infty} k_i(\{pt\}, B \rtimes rG) & \xrightarrow{\mathcal{A}'(\{pt\})} & \lim_{n \to \infty} K K_i(C, B \rtimes rG)
\end{array}
$$

Finally we can define an assembly map equivalent to the analytic assembly map as follows.
Theorem 3.6. Let $G$ be a discrete group, and let $B$ be a separable $G$-$C^*$-algebra, there is a commutative diagram

$$
\begin{array}{ccc}
\lim_{\rightarrow n} k_{i+2n}^G (X, B) & \xrightarrow{\lim_{\rightarrow n} \rho_n^G} & \lim_{\rightarrow n} \mathbb{A}_n^{i+2n} (X) \\
\lim_{\rightarrow n} \lim_{\rightarrow n} k_{i+2n}^G (X, B) & \xrightarrow{\mathbb{A}_n^{i+2n}} & \lim_{\rightarrow n} \mathbb{A}_n^{i+2n} (X) \\
\lim_{\rightarrow n} k_{i+2n}^G (\{pt\}, B \rtimes_r G) & \xrightarrow{\lim_{\rightarrow n} \rho_n^G} & K_{i+2n}^G (\mathbb{C}, B \rtimes_r G).
\end{array}
$$

Where $X$ varies over the co-compact subsets of $EG$.

4. Final remarks

The above description of the assembly map is similar to the obtained in [7], it will be good to explore how to use techniques of controlled categories in the context of configuration spaces.

We hope to extend this results for the case of generalized actions of Lie groupoids, and following ideas from [15] and [4], extend the configuration space description to the twisted Baum-Connes assembly map.

References

[1] P. Baum, A. Connes, and N. Higson. Classifying space for proper actions and $K$-theory of group $C^*$-algebras. In $C^*$-algebras: 1943–1993 (San Antonio, TX, 1993), volume 167 of Contemp. Math., pages 240–291. Amer. Math. Soc., Providence, RI, 1994.
[2] R. Bhatia. Matrix analysis, volume 169 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1997.
[3] B. Blackadar. $K$-theory for operator algebras, volume 5 of Mathematical Sciences Research Institute Publications. Cambridge University Press, Cambridge, second edition, 1998.
[4] P. Carrillo Rouse and B.-L. Wang. Geometric Baum-Connes assembly map for twisted dif- ferentiable stacks. Ann. Sci. Éc. Norm. Supér. (4), 49(2):277–323, 2016.
[5] J. F. Davis and W. Lück. Spaces over a category and assembly maps in isomorphism conjectures in $K$- and $L$-theory. K-Theory, 15(3):201–252, 1998.
[6] A. Dold and R. Thom. Quasifaserungen und unendliche symmetrische Produkte. Ann. of Math. (2), 67:239–281, 1958.
[7] I. Hambleton and E. K. Pedersen. Identifying assembly maps in $K$- and $L$-theory. Math. Ann., 328(1-2):27–57, 2004.
[8] N. Higson. A primer on $KK$-theory. In Operator theory: operator algebras and applications, Part 1 (Durham, NH, 1988), volume 51 of Proc. Sympos. Pure Math., pages 239–283. Amer. Math. Soc., Providence, RI, 1990.
[9] G. G. Kasparov. Equivariant $KK$-theory and the Novikov conjecture. Invent. Math., 91(1):147–201, 1988.
[10] W. Lück. Transformation groups and algebraic $K$-theory, volume 1408 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1989. Mathematische Gottingensis.
[11] W. Lück. Chern characters for proper equivariant homology theories and applications to $K$- and $L$-theory. J. Reine Angew. Math., 543:193–234, 2002.
[12] V. M. Manuilov and E. V. Troitsky. Hilbert $C^*$-modules, volume 226 of Translations of Mathematical Monographs. American Mathematical Society, Providence, RI, 2005. Translated from the 2001 Russian original by the authors.
[13] J. Mostovoy. Partial monoids and Dold-Thom functors. In The influence of Solomon Lefschetz in geometry and topology, volume 621 of Contemp. Math., pages 89–100. Amer. Math. Soc., Providence, RI, 2014.
[14] G. Segal. $K$-homology theory and algebraic $K$-theory. In $K$-theory and operator algebras (Proc. Conf., Univ. Georgia, Athens, Ga., 1975), pages 113–127. Lecture Notes in Math., Vol. 575. Springer, Berlin, 1977.
[15] J.-L. Tu, P. Xu, and C. Laurent-Gengoux. Twisted $K$-theory of differentiable stacks. Ann. Sci. École Norm. Sup. (4), 37(6):841–910, 2004.
[16] A. Valette. On the Baum-Connes assembly map for discrete groups. In Proper group actions and the Baum-Connes conjecture, Adv. Courses Math. CRM Barcelona, pages 79–124. Birkhäuser, Basel, 2003. With an appendix by Dan Kucerovsky.
[17] M. Velásquez. A configuration space for equivariant connective K-homology. *J. Noncommut. Geom.*, 9(4):1343–1382, 2015.

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