A Note on the Convergence of Mirrored Stein Variational Gradient Descent under \((L_0, L_1)\)−Smoothness Condition

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Abstract

In this note, we establish a descent lemma for the population limit Mirrored Stein Variational Gradient Method (MSVGD). This descent lemma does not rely on the path information of MSVGD but rather on a simple assumption for the mirrored distribution \(\nabla \Psi \propto \exp(-V)\). Our analysis demonstrates that MSVGD can be applied to a broader class of constrained sampling problems with non-smooth \(V\). We also investigate the complexity of the population limit MSVGD in terms of dimension \(d\).

1 Introduction

The constrained optimization problem

\[
\min_{\theta \in \Omega} F(\theta),
\]

where \(\Omega\) is some constrained convex region in \(\mathbb{R}^d\), is of importance in practice. The above constrained optimization problem can be solved by constructing a differentiable one to one map \(T : \mathbb{R}^d \to \Omega\), then by compositing \(T\), we get the unconstrained optimization problem

\[
\min_{x \in \mathbb{R}^d} F \circ T(x),
\]

which can be solved efficiently by gradient based approaches. One method based on such kind of transformation is the renowned mirror descent algorithm \cite{Nemirovskij and Yudin 1983}:

\[
\nabla \Psi(\theta_{n+1}) = \nabla \Psi(\theta_n) - \gamma \nabla F(\theta_n),
\]

where \(\Psi(\cdot) : \Omega \to \mathbb{R}\) is the mirror function, \(\nabla \Psi(\cdot) : \Omega \to \mathbb{R}^d\) is called the mirror map and \(\gamma\) is the step-size. This algorithm is based on the discretization of the gradient flow of \(F \circ \nabla^{-1}(x)\):

\[
\frac{d}{dt} x_t = -\nabla F \circ \nabla^{-1}(x_t) = -\nabla^2 \Psi^{-1}(x_t) \nabla F(\nabla^{-1}(x_t)).
\]

Inspired by this, most recently \cite{Shi et al. 2021} proposed a variant of Stein Variational Gradient Descent (SVGD) called Mirrored Stein Variational Gradient Descent (MSVGD) to solve the constrained sampling problem: to sample points from distribution \(\pi \propto \exp(-F)\) with support on \(\Omega\). This method is

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equivalent to SVGD applied on the mirrored distribution $\pi = \nabla \Psi \tilde{\pi}$, however it doesn’t require the computation related to $\bar{\pi}$. In Shi et al. [2021], under the $L$-smoothness condition of $V$ they showed that MSVGD decreases the KL divergence following the method from Liu [2017]. In this note, inspired by Sun et al. [2022] we reanalyze MSVGD under a weaker smoothness condition called $(L_0, L_1)$-smoothness originally from Zhang et al. [2019]. This condition allows $V$ to grow like a polynomial with degree larger than 2, so it include a large class of non-smooth functions into it. One more advantage of our analysis is that we don’t require the path information of MSVGD like in Shi et al. [2021], so we can establish a complexity analysis of the population limit MSVGD in terms of the desired accuracy $\varepsilon$ and dimension $d$.

1.1 Related work

A parallel approach to solve the constrained sampling problem is by the Mirrored Langevin algorithm Hsieh et al. [2018], Ahn and Chewi [2021], Zhang et al. [2020]. The work of Liu and Wang [2016] introduced SVGD as a sampling method, since then several variants of SVGD have been considered, these include: random batch method SVGD (RBM-SVGD) Li et al. [2020], matrix kernel SVGD Wang et al. [2019], Newton version SVGD Detommaso et al. [2018], Stochastic SVGD Gorham et al. [2020], Mirrored SVGD Shi et al. [2021] etc. However, most of the theoretical understanding of SVGD is still limited to continuous time approximation of SVGD or population limit SVGD: Lu et al. [2019] studied the scaling limit of SVGD in continuous time, Duncan et al. [2019] studied the geometry related to SVGD, Liu [2017] first built a convergence result of SVGD in the population limit, Korba et al. [2020] established a descent lemma for SVGD in population limit in terms of Kullback-Leibler divergence however their analysis relied on the path information of SVGD which is unknown beforehand, Salim et al. [2021] provided a clean analysis based on the work of Korba et al. [2020], they assumed $\pi$ satisfies Talagrand’s $T_1$ inequality then they resolved the problem of relying on path information and got a complexity bound for SVGD in terms of the desired accuracy $\varepsilon$ and dimension $d$ which is first shown up in the analysis of SVGD. Sun et al. [2022] introduced the $(L_0, L_1)$-smoothness condition to analyze SVGD and under a generalized $T_p$ inequality they also got a complexity bound for SVGD in terms of $\varepsilon$ and $d$.

1.2 Contributions

The contributions of this work can be listed in three folds:

- We study the population limit MSVGD under a novel smoothness criterion that was utilized to analyze SVGD in Sun et al. [2022]. This smoothness condition originally comes from the optimization literature and allows the local smoothness constant to increase with the gradient norm and so it is strictly weaker than the original Lipschitz gradient assumption.

- Under this smoothness assumption, we build a descent lemma for population limit MSVGD. Our result is complete in that we only make assumptions on the mirrored target distribution $\pi$ but not on the path information like in Shi et al. [2021].

- We provide a complexity bound for the population limit MSVGD under this smoothness condition.

1.3 Paper structure

The paper is organized as follows. Section 2 introduces the background needed on optimal transport, reproducing kernel Hilbert space (RKHS) and Mirrored Stein variational gradient descent (MSVGD); Section 3 presents the assumptions needed in our analysis; Section 4 shows theoretical results on the population limit

\footnote{1 see Equation (16) with $p = 1$.}
MSVGD under assumptions from Section 3. We conclude this work in section 5 and for the missing proofs and explanations, please refer to Appendix 5.

2 Preliminaries

2.1 Notations

For every \( x = (x_1, \ldots, x_d) \mathsf{T}, y = (y_1, \ldots, y_d) \mathsf{T} \in \mathbb{R}^d \), we will denote \( \|x\|_p := \left( \sum_{i=1}^{d} |x_i|^p \right)^{\frac{1}{p}} \), \( \forall p \geq 1 \), \( \|x\| = \sqrt{\sum_{i=1}^{d} |x_i|^2} \) and \( \langle x, y \rangle = \sum_{i=1}^{d} x_i y_i \). We will use \( \nabla^2 V(x) \) to denote the Hessian of potential function \( V(\cdot) : \mathbb{R}^d \to \mathbb{R} \) at point \( x \) and \( \|\nabla^2 V(x)\|_{op} \) to denote the operator norm of matrix \( \nabla^2 V(x) \). The Jacobian matrix of a vector valued function \( h(\cdot) = (h_1(\cdot), \ldots, h_d(\cdot))^\mathsf{T} \) with each \( h_i(\cdot) : \mathbb{R}^d \to \mathbb{R} \) is a \( d \times d \) matrix defined as

\[
J_h(x) := \left( \frac{\partial h_i}{\partial x_j}(x) \right)_{i=1, j=1}^{d, d}.
\]

We will use \( \|J_h(x)\|_{HS} \) to denote the Hilbert-Schmidt norm of matrix \( J_h(x) \), that is

\[
\|J_h(x)\|_{HS} = \sqrt{\text{tr}(J_h(x)^\mathsf{T} J_h(x))}.
\]

The notations \( \Omega \) and \( \mathcal{O} \) follow the convention in complexity analysis, i.e., \( f = \Omega(g) \) means \( f \gtrsim g \), and \( f = \mathcal{O}(g) \) means \( f \lesssim g \). Notations \( \Omega, \mathcal{O} \) are used in a similar way, but they don’t take into account the dependence of parameter except \( p, d, \lambda, C_{\mathbb{R}^p} \) and \( \varepsilon \). \( I_d \) is denoted as the \( d \times d \) identity matrix.

2.2 Optimal Transport

Let \( \mathcal{X}_1, \mathcal{X}_2 \) be two measurable spaces, and denote \( \mathcal{P}(\mathcal{X}_1), \mathcal{P}(\mathcal{X}_2) \) as the sets of all Borel probability measures on \( \mathcal{X}_1, \mathcal{X}_2 \) respectively. Given a measurable map \( T : \mathcal{X}_1 \to \mathcal{X}_2 \) and \( \mu \in \mathcal{P}(\mathcal{X}_1) \), we denote by \( T\#\mu \in \mathcal{P}(\mathcal{X}_2) \) the pushforward measure of \( \mu \) by \( T \), characterized by the transfer lemma \( \int_{\mathcal{X}_1} \phi(T(x))d\mu(x) = \int_{\mathcal{X}_2} \phi(y)d(T\#\mu)(y) \), for any measurable and bounded function \( \phi \) defined on \( \mathcal{X}_2 \). We denote by \( \mathcal{P}_p(\mathbb{R}^d) \) with \( p \geq 1 \) the set of all Borel measures \( \mu \) on \( \mathbb{R}^d \) with finite \( p \)-th absolute moment, that is \( \int \|x\|^p d\mu(x) < +\infty \). For every \( \mu, \nu \in \mathcal{P}_p(\mathbb{R}^d) \), we denote \( \Gamma(\mu, \nu) \) as the set of all the coupling measures between \( \mu \) and \( \nu \) on \( \mathbb{R}^d \times \mathbb{R}^d \), that is for any \( \gamma \in \Gamma(\mu, \nu) \), we have \( \mu = T_{1\#}\gamma, \nu = T_{2\#}\gamma \) with \( T_1(x, y) = x, T_2(x, y) = y \). The Wasserstein distance \( W_p \) between \( \mu \) and \( \nu \) is defined by

\[
W_p(\mu, \nu) = \left( \inf_{\gamma \in \Gamma(\mu, \nu)} \int \|x - y\|^p d\gamma(x, y) \right)^\frac{1}{p}, \tag{5}
\]

we can show that \( W_p(\cdot, \cdot) \) is a metric on \( \mathcal{P}_p(\mathbb{R}^d) \) and so \( (\mathcal{P}_p(\mathbb{R}^d), W_p) \) is a metric space, see [Ambrosio et al., 2005]. We define the Kullback-Leibler (KL) divergence of \( \mu \) with respect to \( \pi \) as

\[
\text{KL}(\mu \mid \pi) := \begin{cases} \int \log \left( \frac{\mu}{\pi} \right)(x)d\mu(x) & \mu \text{ is absolutely continuous with respect to } \pi \\ \infty & \text{else} \end{cases}, \tag{6}
\]

2.3 Mirrored Stein Variational Gradient Descent (MSVGD)

We first introduce the background on the Reproducing Kernel Hilbert Space (RKHS). Let \( \mathcal{H}_0 \) denote a Reproducing Kernel Hilbert Space (RKHS) with kernel \( k(\cdot, \cdot) : \Omega \times \Omega \to \mathbb{R} \). For every \( f, g \in \mathcal{H}_0 \), denote the inner product between \( f \) and \( g \) on \( \mathcal{H}_0 \) as \( \langle f, g \rangle_{\mathcal{H}_0} \) and the norm of \( f \) as \( \|f\|_{\mathcal{H}_0} := \sqrt{\langle f, f \rangle_{\mathcal{H}_0}} \). The
Algorithm 1 Mirrored Stein Variational Gradient Descent [Shi et al. [2021]]

1: **Input:** density \( p \) on \( \Omega \), kernel \( k(\cdot, \cdot) \), mirror function \( \Psi \), particles \( \theta^i_n \in \Omega \), step sizes \( \gamma \)
2: **Init:** \( x^i_0 = \nabla \Psi(\theta^i_0) \) for \( i \in [N] \)
3: for \( k = 0, 1, \ldots, n \) do
4: \( x^i_{k+1} \leftarrow x^i_k + \gamma \frac{1}{N} \sum_{j=1}^{N} k(\theta^j_k, \theta^i_k) \nabla^2 \Psi(\theta^j_k)^{-1} \nabla \theta \log(\pi) \theta^j_k + \nabla \theta \cdot (\nabla^2 \Psi(\theta)^{-1} k(\theta^j_k, \theta)) |_{\theta=\theta^i_k} \)
5: for \( i \in [N] \)
6: \( \theta^i_{k+1} \leftarrow \nabla \Psi^*(x^i_{k+1}) \) for \( i \in [N] \)
7: end for
8: **Return:** \( \theta^i_{n+1} \) for \( i \in [N] \).

most important property on \( \mathcal{H}_0 \) is the so called reproducing property: \( f(\theta) = \langle f(\cdot), k(\theta, \cdot) \rangle_{\mathcal{H}_0}, \forall f \in \mathcal{H}_0 \). The \( d \)-fold Cartesian product of \( \mathcal{H}_0 \) is denoted by \( \mathcal{H} \), then the inner product on \( \mathcal{H} \) is given by \( \langle f, g \rangle_{\mathcal{H}} := \sum_{i=1}^{d} \langle f_i, g_i \rangle_{\mathcal{H}_0} \), here \( f = (f_1, \ldots, f_d)^\top, g = (g_1, \ldots, g_d)^\top \in \mathcal{H} \) (that is each \( f_i, g_i \in \mathcal{H}_0 \)).

Let \( \Omega \) be some closed convex domain in \( \mathbb{R}^d \) and \( \Psi(\cdot) : \Omega \to (-\infty, +\infty) \) be some proper, closed and strongly \( K \)-convex function defined on \( \Omega \). Define \( \Psi^*(x) := \sup_{\theta \in \Omega} x^\top \theta - \Psi(\theta) \) the convex conjugate of \( \Psi \), we always assume the domain of \( \Psi^* \) is \( \mathbb{R}^d \). It is well known that \( \Psi^* \) is convex on \( \mathbb{R}^d \) and the inverse map of \( \nabla \Psi^* \) is \( \nabla \Psi \), that is \( \nabla \Psi^*(x) |_{x=\nabla \Psi(\theta)} = \theta \), so it is easy to verify that \( \| \nabla^2 \Psi^* \|_{op} = \| \nabla^2 \Psi^{-1} \|_{op} \leq \frac{1}{K} \) (see [Beck [2017]])

One important example in application is \( \Omega = \{ \theta \in \mathbb{R}^d : \sum_{i=1}^{d} \theta_i \leq 1, \theta_i \geq 0, \forall i \in [d] \} \) and \( \Psi(\theta) = \sum_{i=1}^{d} \theta_i \log(\theta_i) + (1 - \sum_{i=1}^{d} \theta_i) \log(1 - \sum_{i=1}^{d} \theta_i) \). It is easy to verify that \( \Psi \) is 1-strongly convex and \( \text{dom}(\nabla \Psi^*) = \mathbb{R}^d \).

With the map \( \Psi \), we can now turn the constrained sampling problem \( \pi(\theta) := \exp(-F(\theta)), \theta \in \Omega \) into unconstrained sampling problem \( \bar{\pi}(x) := \exp(-V(x)), x \in \mathbb{R}^d \) with \( \bar{\pi} = \nabla \Psi_{\#} \pi \). The idea behind the MSVGD is first to sample point \( x \sim \bar{\pi} \) through ordinary SVGD method, then map \( \nabla \Psi^* \) we get point \( \theta = \nabla \Psi^*(x) \sim \pi \).

For any \( \mu \in \mathcal{P}(\Omega) \), we have

\[
\int_{\theta \in \Omega} k(\theta, \cdot) \nabla^2 \Psi(\theta)^{-1} \nabla \theta \log(\frac{\mu}{\pi})(\theta) d\mu(\theta) \in \mathcal{H},
\]

we can now give the definition of the Mirrored Stein Fisher information:

**Definition 1.** The Mirrored Stein Fisher information of \( \mu \) relative to \( \pi \) with mirror map \( \Psi \) is given by

\[
I_{MStein}(\mu | \pi) := \int_{\theta \in \Omega} \int_{\theta' \in \Omega} k(\theta, \theta') \left< \nabla^2 \Psi(\theta)^{-1} \nabla \theta \log(\frac{\mu}{\pi})(\theta), \nabla^2 \Psi(\theta')^{-1} \nabla \theta' \log(\frac{\mu}{\pi})(\theta') \right> d\mu(\theta)d\mu(\theta').
\]

On the one hand, when \( \Omega = \mathbb{R}^d \) and \( \Psi(\theta) = \| \theta \|^2 \), this Mirrored Stein Fisher information will be reduced to Stein Fisher information [Duncan et al. [2019]]; on the other hand, it can be seen as the Stein Fisher information of \( \nabla \Psi_{\#} \mu \) relative to \( \nabla \Psi_{\#} \pi \) with kernel \( k(\nabla \Psi^*(\cdot), \nabla \Psi^*(\cdot)) \), see Lemma 1.

**Lemma 1.** Let \( \bar{\mu} := \nabla \Psi_{\#} \mu \) and \( \bar{\pi} := \nabla \Psi_{\#} \pi \), then we have

\[
g_{\bar{\mu}}(\cdot) := P_{\bar{\mu}} \nabla \log(\frac{\bar{\mu}}{\pi})(\cdot) = \int_{x \in \mathbb{R}^d} k(\nabla \Psi^*(x), \cdot) \nabla x \log(\frac{\bar{\mu}}{\pi})(x)d\bar{\mu}(x) = \int_{\theta \in \Omega} k(\theta, \cdot) \nabla^2 \Psi(\theta)^{-1} \nabla \theta \log(\frac{\mu}{\pi})(\theta) d\mu(\theta)
\]

and

\[
I_{MStein}(\mu | \pi) = \int_{x \in \mathbb{R}^d} \int_{y \in \mathbb{R}^d} k(\nabla \Psi^*(x), \nabla \Psi^*(y)) \left< \nabla x \log(\frac{\bar{\mu}}{\pi})(x), \nabla y \log(\frac{\bar{\mu}}{\pi})(y) \right> d\bar{\mu}(x)d\bar{\mu}(y) = \| g_{\bar{\mu}} \|_H^2.
\]
We first state an assumption on the mirror function $\Psi$.

**Assumption 2.** The mirror function $\Psi(\cdot) : \Omega \rightarrow (-\infty, +\infty]$ is strongly $K$-convex and the range of the mirror map $\nabla \Psi$ is $\mathbb{R}^d$.

Next, we state the standard assumption on the kernel $k(\cdot, \cdot) : \Omega \times \Omega \rightarrow \mathbb{R}$.

**Assumption 2.** There exists $B_1, B_2 > 0$ s.t. for all $\theta \in \Omega$, $k(\theta, \theta) \leq B_1^2$ and $\partial_{\theta_i} \partial_{\theta_j} k(\theta, \theta') |_{\theta' = \theta} \leq B_2^2$, $\forall i \in [d], j \in [d]$.
By reproducing property, this is equivalent to say \( \langle k(\theta, \cdot), k(\theta, \cdot) \rangle_{H_0} \leq B_1^2 \) and \( \langle \partial_{\theta} k(\theta, \cdot), \partial_{\theta} k(\theta, \cdot) \rangle_{H_0} \leq B_2^2 \) for all \( \theta \in \Omega \). If \( k(\theta, \theta') \) is in the form \( f(\theta - \theta') \) and \( f \) is smooth at point \( 0 \in \mathbb{R}^d \), then it satisfies Assumption 2. One can also normalize \( k(\theta, \theta') \) by \( k(\theta/d, \theta'/d) \) to make \( B_2 = \mathcal{O}(d^{1/2}) \).

Shi et al. [2021] assumed \( L \)-smoothness of \( V \), where \( V \) is the potential function of the mirrored distribution \( \pi = \exp(-V) \), that is

\[
\|\nabla^2 V(x)\|_{op} \leq L,
\]

where \( L > 0 \) is a bounded constant. Different from the \( L \)-smoothness assumption on \( V \), We introduce a weaker smoothness assumption called \((L_0, L_1)\)-smoothness from Zhang et al. [2019]. The assumption reads as in the following:

**Assumption 3.** \( \exists L_0, L_1 \geq 0 \) s.t.

\[
\|\nabla^2 V(x)\|_{op} \leq L_0 + L_1\|\nabla V(x)\|,
\]

for any \( x \in \mathbb{R}^d \).

For instance \( g(x) = |x|^{2+\delta} \) with \( x \in \mathbb{R} \) and \( \delta > 0 \) satisfies \((L_0, L_1)\)-smoothness condition with \( L_0 = (2 + \delta)(1 + \delta)^{1+\delta} \), \( L_1 = 1 \), while it is not a \( L \)-smooth function. We also need the assumption on \( \bar{\pi} \).

**Assumption 4.** There exists \( p \geq 1 \), \( x_0 \in \mathbb{R}^d \) and \( s > 0 \) such that

\[
\int \exp(s \|x - x_0\|^p)d\bar{\pi}(x) < +\infty.
\]

If Assumption 4 holds, then based on results from Bolley and Villani [2005] (see Lemma 8 in the Appendix), we have

\[
W_p(\mu, \bar{\pi}) \leq C_{\bar{\pi},p} \left[ \text{KL}(\mu \mid \bar{\pi})^{1/p} + \left( \frac{\text{KL}(\mu \mid \bar{\pi})}{2} \right)^{1/p} \right], \quad \forall \mu \in \mathcal{P}_p(\mathbb{R}^d),
\]

where

\[
C_{\bar{\pi},p} := 2 \inf_{x_0 \in \mathbb{R}^d, s > 0} \left( \frac{1}{s} \left( \frac{3}{2} + \log \int \exp(s \|x - x_0\|^p)d\bar{\pi}(x) \right) \right)^{1/p} < +\infty.
\]

Note constant \( C_{\bar{\pi},p} \) may depend on dimension, for instance if \( \bar{\pi}(x) \propto \exp(-\|x\|^p) \), we have \( C_{\bar{\pi},p} = \mathcal{O}\left(d^{\bar{\pi}}\right) \) by simple calculation. Assumption 4 is used to guarantee inequality (14) hold, we can instead directly assume Talagrand’s \( T_p \) inequality.

**Assumption 5.** For any \( \mu \in \mathcal{P}_p(\mathbb{R}^d) \), we have

\[
W_p(\mu, \bar{\pi}) \leq \sqrt{\frac{2 \text{KL}(\mu \mid \bar{\pi})}{\lambda}}.
\]

It is not clear the condition on \( \bar{\pi} \) such that \( T_p \) inequality holds except for a few cases. When \( p = 1 \), a sufficient and necessary condition for Equation (16) to be hold is there exists \( x_0 \in \mathbb{R}^d, s > 0 \) such that \( \int \exp(s \|x - x_0\|^2)d\bar{\pi}(x) < +\infty \) (see Villani [2009, Theorem 22.10]). When \( V \) is strongly \( K \)-convex, then Equation (16) holds with \( p = 2, \lambda = K \) (Bakry-Émery criterion, see Bakry and Émery [1985]), also \( T_2 \) is preserved under bounded perturbation (Holley-Strook criterion, see Holley and Stroock [1986] or Steiner [2021]).

To bound the Stein Fisher information along trajectory of MSVDG, we also need an assumption on the growth rate of \( \nabla V \).
Assumption 6. There exists $p \geq 1$ and a constant $C_p$ such that
\[ \| \nabla V(x) \| \leq C_p (\| x \|^p + 1), \]
for any $x \in \mathbb{R}^d$.

One simple example that satisfies both Assumption $\Box$ and Assumption $\Box$ is $V(x) = \| x \|^p$.

4 Main Theory

The following proposition is close to the descent lemma, however the step-size $\gamma$ in this proposition depends on the path information. In the following, we always denote $\bar{\pi} = \nabla \Psi \# \pi$, $\mu_n = \nabla \Psi \# \mu_n$ for $n = 0, 1, 2, \ldots$.

Proposition 1. Suppose that Assumptions $\Box \Box \Box \Box$ are satisfied. Let $\alpha > 1$ and choose
\[
\gamma \leq \min \left\{ \frac{(\alpha - 1)K}{\alpha B_2 d I_{MStein}(\mu_n | \pi)^{\frac{1}{2}}} + 1, \frac{K^2}{B_1 I_{MStein}(\mu_n | \pi)^{\frac{1}{2}} L_1 (\alpha^2 B_2^2 d^2 + 2(e - 1)B_1^2 A_n)} \right\},
\]
where $A_n := L_0 + L_1 \mathbb{E}_{X \sim \bar{\mu}_n} \| \nabla V(X) \|$. Then
\[
KL(\mu_{n+1} | \pi) - KL(\mu_n | \pi) \leq -\frac{\gamma}{2} I_{MStein}(\mu_n | \pi),
\]
(18)

In the above proposition, the upper bound for the step-size $\gamma$ depend on $I_{MStein}(\mu_n | \pi)^{\frac{1}{2}}$ and $A_n$, these values can be further bounded by $KL(\mu_n | \pi)$ by the following two lemmas.

Lemma 3. If Assumptions $\Box \Box$ are satisfied, then
\[
I_{MStein}(\mu_n | \pi)^{\frac{1}{2}} \leq B_1 \mathbb{E}_{X \sim \bar{\mu}_n} \| \nabla V(X) \| + \frac{B_2 d}{K}
\]
for all $n \in \mathbb{N}$.

Let us denote function $G_p(x) := x^{\frac{1}{p}} + (\frac{x}{2})^{\frac{1}{p}}$ in the following.

Lemma 4. Suppose Assumptions $\Box \Box \Box \Box$ hold. Then,
\[
\mathbb{E}_{X \sim \bar{\mu}_n} \| \nabla V(X) \| \leq C_p (C_{\pi,p} (G_p(KL(\mu_n | \pi)) + G_p(KL(\mu_0 | \pi))) + W_p(\bar{\mu}_0, \delta_0))^p + C_p
\]
(20)

where $C_p$ from Assumption $\Box$ $C_{\pi,p}$ from (19).

Now, using Equations (19) and (20) we can give an upper bound for $\gamma$ such that it satisfies condition (17), then based on Proposition 1 we have the following descent lemma which only depends on the initial distribution $\mu_0, \bar{\mu}_0$ and the target distribution $\pi, \bar{\pi}$.

Theorem 1 (Descent lemma). Suppose Assumptions $\Box \Box \Box \Box$ are satisfied. Also, suppose that the step-size $\gamma$ satisfies
\[
\gamma \leq M (C_p (2C_{\pi,p} G_p(KL(\mu_0 | \pi)) + W_p(\bar{\mu}_0, \delta_0))^p + C_p),
\]
where $M(x) := \min \left\{ \min \left\{ \frac{1}{B_1 L_1}, \frac{(\alpha - 1)K}{\alpha B_2 d}, \frac{K}{B_1 L_1 + B_2 d}, \frac{K}{\alpha^2 B_2^2 d^2 + 2(e - 1)B_1^2 (L_1 + L_0)} \right\} \right\}$, then the following bound is true:
\[
KL(\mu_{n+1} | \pi) - KL(\mu_n | \pi) \leq -\frac{\gamma}{2} I_{MStein}(\mu_n | \pi).
\]
(22)
Now, based on the above descent lemma, we can derive the complexity bound for MSVGD in population limit.

**Corollary 1** (Convergence). Suppose Assumptions 1, 2, 3, and 6 are satisfied. Also, suppose that the step-size \(\bar{\gamma}\) satisfies
\[
\frac{1}{n} \sum_{k=0}^{n-1} I_{\text{Stein}}(\mu_k \mid \pi) \leq \frac{2\text{KL}(\mu_0 \mid \pi)}{n\gamma}.
\]
If \(\bar{\pi}(x) = \exp(-V(x))\) and \(\bar{\mu}_0 = N(0, I_d)\), then we have
\[
\frac{1}{n} \sum_{k=0}^{n-1} I_{\text{Stein}}(\mu_k \mid \pi) \leq \varepsilon,
\]
we need
\[
n = \Omega\left(\frac{8pC_p^p\Gamma((p+d+1)/2)^2}{(p+1)^2\Gamma((d+1)/2)^2\varepsilon}\right).
\]

If \(B_2 = O(1)\), we can replace the kernel \(k(\cdot, \cdot)\) by \(k(\cdot/d, \cdot/d)\), then it satisfies \(B_2 = O(1/d)\). By Stirling formula, we know the dimension dependency of \(\Gamma((p+d+1)/2)\) is of order \(1 + p\), so if \(p\) is not big, there should be
\[
n = \tilde{\Omega}\left(\frac{C_p^p\Gamma(d+1)^2}{\varepsilon}\right).
\]

### 4.1 Better Results under \(T_p\) Inequality when \(1 \leq p \leq 2\)

In the next, we assume \(\bar{\pi}\) satisfies Talagrand’s \(T_p\) inequality instead of Assumption 4 and keep the other assumptions the same. When \(p \in [1, 2]\), Equation (16) is more preferable in our analysis since \(W_p(\bar{\mu}, \bar{\pi}) = O(\text{KL}(\bar{\mu} \mid \bar{\pi})^{1/2})\) while Equation (14) gives the bigger bound \(W_p(\bar{\mu}, \bar{\pi}) = O(\text{KL}(\bar{\mu} \mid \bar{\pi})^{1/8})\). We focus on the cases when \(p \in [1, 2]\) and we give a modified version of Lemma 4, Theorem 1 and Corollary 1.

**Lemma 5.** Suppose Assumptions 3, 5 and 6 hold. Then,
\[
\mathbb{E}_{X \sim \bar{\mu}_n} \left[ \| \nabla V(X) \| \right] \leq C_p \left( \sqrt{\frac{2\text{KL}(\mu_n \mid \pi)}{\lambda}} + \sqrt{\frac{2\text{KL}(\mu_0 \mid \pi)}{\lambda}} + W_p(\bar{\mu}_0, \delta_0) \right)^p + C_p,
\]
where \(C_p\) from Assumption 6.

**Theorem 2.** Suppose Assumptions 1, 2, 4 and 6 are satisfied. Also, suppose that the step-size \(\gamma\) satisfies
\[
\gamma \leq M \left( C_p \left( \sqrt{\frac{2\text{KL}(\mu_n \mid \pi)}{\lambda}} + \sqrt{\frac{2\text{KL}(\mu_0 \mid \pi)}{\lambda}} + W_p(\bar{\mu}_0, \delta_0) \right)^p + C_p \right),
\]
where \(M(x) := \min \left\{ \min \left\{ \frac{1}{B_1}, \frac{(\alpha-1)K}{\alpha B_2 d} \right\}, \frac{K}{KB_1 x + B_2 d + \alpha^2 B_2^d + K^2 B_2^d (e-1)(L_1 x + L_2)} \right\} \}, \) then the following bound is true:
\[
\text{KL}(\mu_{n+1} \mid \pi) - \text{KL}(\mu_n \mid \pi) \leq -\frac{\gamma}{2} I_{\text{Stein}}(\mu_n \mid \pi).
\]

Now, based on the above modified descent lemma, we can derive the complexity bound for MSVGD in population limit.

**Corollary 2.** Suppose Assumptions 1, 2, 3, 5 and 6 hold. If \(\gamma\) satisfies condition (26) and \(B_2 = O(1/d)\), then we have
\[
\frac{1}{n} \sum_{k=0}^{n-1} I_{\text{Stein}}(\mu_k \mid \pi) \leq \frac{2\text{KL}(\mu_0 \mid \pi)}{n\gamma}.
\]
If \( \bar{\pi}(x) = \exp(-V(x)) \) and \( \bar{\mu}_0 = N(0, I_d) \), then to make \( \frac{1}{n} \sum_{k=0}^{n-1} \mathcal{I}_{MStein}(\mu_k | \pi) \leq \varepsilon \), we need

\[
n = \tilde{\Omega}\left(\frac{d^{(p+2)(p+1)}}{4 \lambda \varepsilon^2}\right).
\]

When \( 1 \leq p \leq 2 \), if we assume \( \pi \) satisfies Talagrand’s \( T_p \) inequality (16) instead of Assumption 4, \( n \) will have a slightly better dimension dependency, that is of order \( (p+2)(p+1) \).

**Remark 1.** Here we provide a sufficient condition on which \( \lim_{n \to \infty} \mathcal{I}_{MStein}(\mu_n | \pi) \) implies \( \mu_n \to \pi \) weakly, this condition can be found in [Gorham and Mackey 2017]. Since Mirrored Stein Fisher information depends on the target distribution \( \pi \), mirror function \( \Psi \) and kernel \( k(\cdot, \cdot) \), we need the following two properties on \( \bar{\pi} \) and \( k(\nabla \Psi^*(\cdot), \nabla \Psi^*(\cdot)) \) respectively:

1. \( \bar{\pi} \) is distant dissipative, that is \( \kappa_0 \triangleq \lim \inf_{r \to \infty} \kappa(r) > 0 \) with
   \[
   \kappa(r) = \inf \left\{ \frac{2 \langle \nabla V(x) - \nabla V(y), x - y \rangle}{\|x - y\|^2} : \|x - y\| = r \right\}.
   \]
   If \( V \) is strongly convex outside a compact set, then \( \bar{\pi} \) is distant dissipative, for instance \( V(x) = \|x\|^{2+\delta} \) with \( \delta \geq 0 \).

2. \( k(\nabla \Psi^*(\cdot), \nabla \Psi^*(\cdot)) \) is an inverse multiquadratic kernel, i.e.,
   \[
   k(\nabla \Psi^*(x), \nabla \Psi^*(y)) = (c^2 + \|x - y\|^2)^{\beta}
   \]
   for some \( c > 0 \) and \( \beta \in (-1, 0) \). It is easy to check that Assumption 2 is satisfied.

**5 Conclusion**

We proved a descent lemma for minimizing the KL divergence with the MSVGD algorithm under \((L_0, L_1)\)-smoothness condition. Though in this paper we assume \( k(\cdot, \cdot) \) is scalar, our analysis can also be applied on matrix kernel. The result implies \( O(1/n) \) convergence under Mirrored Stein Fisher information. The results in this note match the ones from [Sun et al. 2022], this is not surprising since MSVGD is equivalent to SVGD for target distribution \( \nabla \Psi \). The proved results remain on the theoretical level, the algorithm we analyzed in the infinite particle regime which is not implementable on a computer. There is paper analysing the finite particle regime [Shi et al. 2021], however they assumed a \( L \)-smooth \( V \) and their result is asymptotic. A relevant analysis of the finite particle regime still lacks certain clarity and is yet to be developed.
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Appendix

For simplicity, in this appendix we will denote $k_{\Psi}(. , \cdot) := k(\nabla \Psi^*(\cdot), \nabla \Psi^*(\cdot))$. Since $\text{KL} (\nabla \Psi_{\# \mu} \ | \ \nabla \Psi_{\# \pi}) = \text{KL} (\mu \ | \ \pi)$ for diffeomorphism $\nabla \Psi$, so it is enough to prove the results using $\text{KL}(\bar{\mu} \ | \ \bar{\pi})$, where $\bar{\mu} = \nabla \Psi_{\# \mu}, \bar{\pi} = \nabla \Psi_{\# \pi}$.

A Proof of Proposition 1

We follow the proof procedure in Sun et al. [2022]. First due to Lemma 1, we have the following

$$
\varphi(\gamma) = \varphi(0) + \gamma \varphi'(0) + \int_0^\gamma (\gamma - t) \varphi''(t)dt.
$$

(31)

By the definition of the MSVGD iteration, we have that $g_{\mu_n}$ and $\varphi(\gamma) = \text{KL} (\bar{\mu}_n \ | \ \bar{\pi})$. Let us now compute the term of (31) corresponding to the first order derivative. In order to do that we show that $\phi_{t}$ is a diffeomorphism.

**Lemma 6.** Suppose that Assumption 1 holds. Then, for any $x \in \mathbb{R}^d$ and $h \in \mathcal{H}$,

$$
\|Jh(x)\|_{\text{HS}} \leq \frac{B_2}{K} \|h\|_{\mathcal{H}}.
$$

Applying the lemma to the function $g$, we obtain the following:

$$
\|tJg(x)\|_{\text{op}} \leq \|tJg(x)\|_{\text{HS}} \leq t \frac{B_2}{K} \|g\|_{\mathcal{H}} < 1.
$$

(32)

The latter inequality is due to the condition on the step-size $\gamma (\gamma \leq \frac{(\alpha - 1)K}{\alpha B_2 d \|g\|_{\mathcal{H}}})$. The inequality (32) implies that $\phi_{t}$ is a diffeomorphism. Therefore, $\bar{\mu}_t$ admits a density given by the formula of transformation of probability densities:

$$
\bar{\mu}_t(x) = |J\phi_t (\phi_t^{-1}(x))|^{-1} \bar{\mu}_n (\phi_t^{-1}(x)).
$$

Changing the variable of integration and applying the transfer lemma we get the following formula for $\varphi(t)$:

$$
\varphi(t) = \int \log \left( \frac{\bar{\mu}_t(y)}{\bar{\pi}(y)} \right) \bar{\mu}_t(y)dy
$$

$$
= \int \log \left( \frac{\bar{\mu}_n(x) |J\phi_t(x)|^{-1}}{\bar{\pi}(\phi_t(x))} \right) \bar{\mu}_n(x)dx
$$

$$
= \int \left[ \log (\bar{\mu}_n(x)) + \log (|J\phi_t(x)|^{-1}) - \log (\bar{\pi}(\phi_t(x))) \right] \bar{\mu}_n(x)dx
$$

Let us then compute the time derivative of $\varphi(t)$. Taking the derivative inside and applying Jacobi’s formula for matrix determinant differentiation we obtain the following equality:

$$
\varphi'(t) = -\int \text{tr} \left( J\phi_t(x)^{-1} \frac{dJ\phi_t(x)}{dt} \right) \bar{\mu}_n(x)dx - \int \left( \nabla \log \bar{\pi}(\phi_t(x)) \cdot \frac{d\phi_t(x)}{dt} \right) \bar{\mu}_n(x)dx.
$$
By definition, $d\phi_t/dt = g$. Therefore, we can use the explicit expression of $\phi_t$ to write:

$$\varphi'(t) = \int \text{tr} \left( J\phi_t(x)^{-1} Jg(x) \right) \tilde{\mu}_n(x) dx + \int \langle \nabla V(\phi_t(x)), g(x) \rangle \tilde{\mu}_n(x) dx.$$  

The Jacobian at time $t = 0$ is simply equal to the identity matrix since $\phi_0 = I_d$. It follows that $\text{tr} \left( J\phi_0(x)^{-1} Jg(x) \right) = \text{tr}(Jg(x)) = \text{div}(g(x))$ by the definition of the divergence operator. Using integration by parts:

$$\varphi'(0) = -\int [- \text{div}(g(x)) - \langle \nabla \log \tilde{\pi}(x), g(x) \rangle] \tilde{\mu}_n(x) dx$$

$$= -\int \left[ \nabla \log \left( \frac{\mu_n}{\pi} \right) \right] (x), g(x) \right] \tilde{\mu}_n(x) dx$$

$$= -\left( P_\mu \nabla \log \left( \frac{\mu_n}{\pi} \right) \right) (x), g(x) \right] \mathcal{H}$$

$$= -\|g\|^2_H.$$  

Next, we calculate the term of (31) that contains the second derivative. First,  

$$\varphi''(t) = \int \left[ \text{tr} \left( \left( Jg(x) (J\phi_t(x))^{-1} \right)^2 \right) + \langle g(x), \nabla^2 V(\phi_t(x)) g(x) \rangle \right] \tilde{\mu}_n(x) dx.$$  

From the definition of the function $\phi_t$, we know that $J\phi_t(x) = (I_d + tJg)(x)$. Thus, $J\phi_t^{-1}$ and $Jg$ commute and $\|Jg(x) (J\phi_t(x))^{-1} \|^2_{HS}$.  

Overall we have the following:

$$\varphi''(t) = \int \|Jg(x) (J\phi_t(x))^{-1} \|^2_{HS} \tilde{\mu}_n(x) dx + \int \langle g(x), H V(\phi_t(x)) g(x) \rangle \tilde{\mu}_n(x) dx.$$  

First, we bound $\psi_1(t)$. Cauchy-Schwarz implies that

$$\|Jg(x) (J\phi_t(x))^{-1} \|^2_{HS} \leq \|Jg(x)\|^2_{HS} \|J\phi_t(x)^{-1}\|^2_{op}.$$  

From Lemma 6 we have $\|Jg(x)\|_{HS} \leq \frac{B \kappa d}{K} \|g\|_{\mathcal{H}}$. To bound the second term, let us recall that $\phi_t = I - t g$ and that $t \leq \gamma$. Thus, the following bound is true:

$$\|J\phi_t(x)^{-1}\|_{op} = \|((I_d - t Jg)(x))^{-1}\|_{op} \leq \sum_{k=0}^{\infty} \|t Jg(x)\|^k_{op} \leq \sum_{k=0}^{\infty} \|Jg(x)\|^k_{HS}.$$  

Recalling (17) and combining it with Lemma 6 we obtain

$$\|J\phi_t(x)^{-1}\|_{op} \leq \sum_{k=0}^{\infty} \left( \frac{\gamma B \kappa d}{K} \right)^k \leq \sum_{k=0}^{\infty} \left( \frac{\alpha - 1}{\alpha} \right)^k = \alpha.$$  

Summing up, we have that

$$\psi_1(t) \leq \frac{\alpha^2 B^2 \kappa d^2}{K^2} \|g\|^2_H.$$  

Next, we bound $\psi_2$. By definition

$$\psi_2(t) = \mathbb{E}_{X \sim \tilde{\mu}_n} \left[ \langle g(X), \nabla^2 V(\phi_t(X)) g(X) \rangle \right] \leq \mathbb{E}_{X \sim \tilde{\mu}_n} \left[ \|\nabla^2 V(\phi_t(X))\|_{op} \|g(X)\|^2 \right].$$
Let us bound the norm of \( g(x) \). The reproduction property of the RKHS yields the following:

\[
\|g(x)\|_H^2 = \sum_{i=1}^{d} \langle k(x, .), g_i \rangle_{\mathcal{H}_0}^2 \leq \|k(\cdot, .)\|_{\mathcal{H}_0}^2 \|g\|_{\mathcal{H}}^2 \leq B_1^2 \|g\|_{\mathcal{H}}^2. \tag{35}
\]

Therefore,

\[
\psi_2(t) \leq B_1^2 \|g\|_{\mathcal{H}}^2 \mathbb{E}_{X \sim \mu_n} \left[ \|\nabla^2 V(\phi_t(X))\|_{op} \right]. \tag{36}
\]

Let us bound \( \mathbb{E}_{X \sim \mu_n} \left[ \|H V(\phi_t(X))\|_{op} \right] \). Assumption \(^3\) implies the following inequality:

\[
\|\nabla^2 V(\phi_t(x))\|_{op} \leq L_0 + L_1 \|\nabla V(\phi_t(x))\|,
\]

for every \( x \in \mathbb{R}^d \).

To bound the term \( \|\nabla V(\phi_t(x))\| \), we introduce the following lemma from Sun et al. \( ^2022 \) without proof.

**Lemma 7.** Let \( V \) be an \((L_0, L_1)\)-smooth function and \( \Delta > 0 \) be a constant. For any \( x, x^+ \in \mathbb{R}^d \) such that \( \|x^+ - x\| \leq \Delta \), we have

\[
\|\nabla V(x^+)\| \leq \frac{L_0}{L_1} (\exp(\Delta L_1) - 1) + \|\nabla V(x)\| \exp(\Delta L_1).
\]

We will apply Lemma 7 to \( \phi_t(x) \) and \( \phi_0(x) \). By definition, \( \phi_t(x) - \phi_0(x) = tg(x) \) and according to inequality (35),

\[
\|\phi_t(x) - \phi_0(x)\| \leq tB_1 \|g\|_{\mathcal{H}}.
\]

Thus, using Lemma 7 for \( x = \phi_0(x), x^+ = \phi_t(x) \) and \( \Delta = tB_1 \|g\|_{\mathcal{H}} \), we obtain the following:

\[
\|\nabla^2 V(\phi_t(x))\|_{op} \leq L_0 + L_1 \left( \frac{L_0}{L_1} (\exp(tB_1 \|g\|_H L_1) - 1) + \|\nabla V(\phi_0(x))\| \exp(tB_1 \|g\|_H L_1) \right)
= (L_0 + L_1 \|\nabla V(x)\|) \exp(tB_1 \|g\|_H L_1), \tag{37}
\]

where the last equality is due to \( \phi_0(x) = x \). Combining (36) and (37) we obtain

\[
\psi_2(t) \leq B_1^2 \|g\|_{\mathcal{H}} (L_0 + L_1 \mathbb{E}_{X \sim \mu_n} \left[ \|\nabla V(X)\| \right]) \exp(tB_1 \|g\|_H L_1).
\]

Summing up, the bounds on \( \psi_1 \) and \( \psi_2 \) yield the following inequality:

\[
\varphi''(t) \leq \frac{1}{2} \left| \begin{array}{c} \|g\|^2_{\mathcal{H}} \left[ \frac{\alpha^2 B_3^2 d^2}{K^2} + B_1^2 \left( L_0 + L_1 \mathbb{E}_{X \sim \mu_n} \left[ \|\nabla^2 V(X)\| \right] \right) \exp(tB_1 \|g\|_H L_1) \right] \\
\end{array} \right| dt
\]

\[
= \frac{\gamma^2}{2} \|g\|^2_{\mathcal{H}} \left[ \frac{\alpha^2 B_3^2 d^2}{K^2} + B_1^2 \|g\|^2_{\mathcal{H}} A_n \exp(\gamma B_1 \|g\|_H L_1) - \gamma B_1 \|g\|_H L_1 - 1 \right] B_1^2 \|g\|^2_{\mathcal{H}} L_1^2 + \frac{B_1^2 \|g\|^2_{\mathcal{H}} L_1^2}{2}.
\]

One can check that \( \exp(t) - t - 1 \leq (e - 1)t^2/2 \), when \( t \in [0, 1] \), since \( \gamma B_1 \|g\|_H L_1 < 1 \), we deduce that

\[
\int_0^\gamma (\gamma - t) \varphi''(t) dt \leq \|g\|^2_{\mathcal{H}} \left( \frac{\gamma^2}{2} + \frac{e - 1}{2} B_1^2 A_n \gamma^2 \right), \tag{38}
\]
combine with (33), we finally have
\[
\varphi(\gamma) - \varphi(0) \leq -\gamma \|g\|_H^2 + \gamma^2 \|g\|_H^2 \left( \frac{\alpha^2 B^2 d^2}{2 K^2} + \frac{e - 1}{2} B^2 A_n \right)
\]
\[
= -\gamma \left( 1 - \frac{1}{2} \gamma \left( \frac{\alpha^2 B^2 d^2}{K^2} + (e - 1) B^2 A_n \right) \right) \|g\|_H^2
\]
\[
\leq \frac{i}{\gamma} \|g\|_H^2.
\]

(i) is since we choose \( \gamma \leq \frac{K^2}{(\alpha^2 B^2 d^2 + K^2(e - 1) B^2 A_n)} \).

Finally, by Lemma \[ \|g\|_H^2 = I_{MStein}(\mu_n | \pi) \]. This concludes the proof of the Proposition 1.

**Remark 2.** One may notice that we did not use the exact formula of \( g \) except for (33). In fact the proposition remains true for a general \( g \in H \) with a slight change. If we skip the last equality in (33) and repeat the other steps, then we get the following:
\[
\varphi(\gamma) - \varphi(0) \leq -\gamma \left( P_{\bar{\mu}} \nabla \log \left( \frac{\bar{\mu}_n}{\pi} \right)(x, g(x)) \right)_H + \frac{\gamma^2}{2} \|g\|_H^2 \left( \frac{\alpha^2 B^2 d^2}{K^2} + (e - 1) B^2 A_n \right)
\]
(39)

**B Lemmas**

The following lemma is from [Bolley and Villani, 2005, Corollary 2.3.].

**Lemma 8.** Let \( \mathcal{X} \) be a measurable space equipped with a measurable distance \( d \), let \( p \geq 1 \) and let \( \nu \) be a probability measure on \( \mathcal{X} \). Assume that there exist \( x_0 \in \mathcal{X} \) and \( s > 0 \) such that \( \int \exp(s d(x_0, x)^p) d\nu(x) \) is finite. Then
\[
\forall \mu \in \mathcal{P}(\mathcal{X}), \quad W_p(\mu, \nu) \leq C_{\nu, p} \left[ \text{KL}(\mu | \nu)^{\frac{1}{p}} + \left( \frac{\text{KL}(\mu | \nu)}{2} \right)^{\frac{1}{p}} \right],
\]
where
\[
C_{\nu, p} : = 2 \inf_{s \geq 0} \left( \frac{3}{2} + \log \int \exp(s d(x_0, x)^p) d\nu(x) \right)^{\frac{1}{p}} < +\infty.
\]

**B.1 Proof of Lemmas**

**Proof of Lemma[1]** By the formula of transformation of probability densities, we first have
\[
\bar{\mu}(x) = \mu(\nabla \Psi(x)) | \det \nabla^2 \Psi(\nabla \Psi(x)) |^{-1} \quad \text{and} \quad \bar{\pi}(x) = \pi(\nabla \Psi(x)) | \det \nabla^2 \Psi(\nabla \Psi(x)) |^{-1},
\]
so
\[
\log(\frac{\bar{\mu}}{\bar{\pi}})(\theta) = \log(\frac{\mu}{\pi})(x),
\]
(41)
where \( x = \nabla \Psi(\theta) \). Since \( \mu(\theta) = \left( \nabla \Psi^{*}(\bar{\mu}) \right)(\theta) \), we have

\[
\int_{\theta \in \Omega} k(\theta, \cdot) \nabla^{2} \Psi(\theta)^{-1} \nabla \theta \log(\frac{\mu}{\pi})(\theta) d\mu(\theta) \\
= \int_{\theta \in \Omega} k(\theta, \cdot) \nabla^{2} \Psi(\theta)^{-1} \nabla \theta \log(\frac{\mu}{\pi})(\theta) d\left( \nabla \Psi^{*}(\bar{\mu}) \right)(\theta) \\
= \int_{\theta \in \Omega} k(\nabla \Psi^{*}(x), \cdot) \nabla^{2} \Psi(\nabla \Psi^{*}(x))^{-1} \nabla \theta \log(\frac{\mu}{\pi})(\nabla \Psi^{*}(x)) d\bar{\mu}(x) \\
= \int_{\theta \in \Omega} k(\nabla \Psi^{*}(x), \cdot) \nabla^{2} \Psi(\nabla \Psi^{*}(x))^{-1} \nabla \theta \log(\frac{\mu}{\pi})(\nabla \Psi^{*}(x)) d\bar{\mu}(x) \\
= \int_{\theta \in \Omega} k(\nabla \Psi^{*}(x), \cdot) \nabla x \log(\frac{\mu}{\pi})(\nabla \Psi^{*}(x)) d\bar{\mu}(x). \\
\]

Similarly, we have

\[
\int_{\theta \in \Omega} \int_{\theta' \in \Omega} k(\theta, \theta') \left( \nabla^{2} \Psi(\theta)^{-1} \nabla \theta \log(\frac{\mu}{\pi})(\theta), \nabla^{2} \Psi(\theta')^{-1} \nabla \theta' \log(\frac{\mu}{\pi})(\theta') \right) d\mu(\theta) d\mu(\theta') \\
= \int_{\theta \in \Omega} \int_{\theta' \in \Omega} k(\theta, \theta') \left( \nabla^{2} \Psi(\theta)^{-1} \nabla \theta \log(\frac{\mu}{\pi})(\theta), \nabla^{2} \Psi(\theta')^{-1} \nabla \theta' \log(\frac{\mu}{\pi})(\theta') \right) d\left( \nabla \Psi^{*}(\bar{\mu}) \right)(\theta) d\left( \nabla \Psi^{*}(\bar{\mu}) \right)(\theta') \\
= \int_{\theta \in \Omega} \int_{\theta' \in \Omega} k(\theta, \theta') \left( \nabla^{2} \Psi(\nabla \Psi^{*}(x))^\dagger \nabla \theta \log(\frac{\mu}{\pi})(\nabla \Psi^{*}(x)), \nabla^{2} \Psi(\nabla \Psi^{*}(y))^\dagger \nabla \theta' \log(\frac{\mu}{\pi})(\nabla \Psi^{*}(y)) \right) d\bar{\mu}(x) d\bar{\mu}(y) \\
= \int_{\theta \in \Omega} \int_{\theta' \in \Omega} k(\theta, \theta') \left( \nabla^{2} \Psi(\nabla \Psi^{*}(x))^\dagger \nabla \theta \log(\frac{\mu}{\pi})(\nabla \Psi^{*}(x)), \nabla^{2} \Psi(\nabla \Psi^{*}(y))^\dagger \nabla \theta' \log(\frac{\mu}{\pi})(\nabla \Psi^{*}(y)) \right) d\bar{\mu}(x) d\bar{\mu}(y) \\
= \int_{\theta \in \Omega} \int_{\theta' \in \Omega} k(\theta, \theta') \left( \nabla^{2} \Psi(\nabla \Psi^{*}(x))^\dagger \nabla \theta \log(\frac{\mu}{\pi})(\nabla \Psi^{*}(x)), \nabla^{2} \Psi(\nabla \Psi^{*}(y))^\dagger \nabla \theta' \log(\frac{\mu}{\pi})(\nabla \Psi^{*}(y)) \right) d\bar{\mu}(x) d\bar{\mu}(y) \\
= \left\| g_{\bar{\mu}} \right\|_{\mathcal{H}}^2. \\
\]

where \( x = \nabla \Psi(\theta), y = \nabla \Psi(\theta') \).

**Proof of Lemma**

The proof is direct.

\[
\int_{\theta \in \Omega} k(\theta, \cdot) \nabla^{2} \Psi(\theta)^{-1} \nabla \theta \log(\frac{\mu}{\pi})(\theta) d\mu(\theta) \\
= \int_{\theta \in \Omega} k(\theta, \cdot) \nabla^{2} \Psi(\theta)^{-1} \nabla \theta \log(\mu)(\theta) \mu(\theta) d\theta - \int_{\theta \in \Omega} k(\theta, \cdot) \nabla^{2} \Psi(\theta)^{-1} \nabla \theta \log(\pi)(\theta) \mu(\theta) d\theta \\
= \int_{\theta \in \Omega} k(\theta, \cdot) \nabla^{2} \Psi(\theta)^{-1} \nabla \theta \mu(\theta) d\theta - \int_{\theta \in \Omega} k(\theta, \cdot) \nabla^{2} \Psi(\theta)^{-1} \nabla \theta \log(\pi)(\theta) \mu(\theta) d\theta \\
= \int_{\theta \in \Omega} \nabla \theta \cdot (k(\theta, \cdot) \nabla^{2} \Psi(\theta)^{-1}) d\mu(\theta) - \int_{\theta \in \Omega} k(\theta, \cdot) \nabla^{2} \Psi(\theta)^{-1} \nabla \theta \log(\pi)(\theta) \mu(\theta) d\theta, \\
(i) \text{ is due to integration by parts and the assumption that } k(\theta, \cdot) \nabla^{2} \Psi(\theta)^{-1} \mu(\theta) \rightarrow 0 \text{ as } \theta \rightarrow \partial \Omega. \\
\]
Before we proceed to prove the left lemmas, we need to do some calculation first. Let \( e_i \) be the \( i \)-th standard basis of \( \mathbb{R}^d \), then

\[
\begin{aligned}
\langle \partial_{x_i} k(\nabla \Psi^*(x), \cdot), \partial_{y_j} k(\nabla \Psi^*(y), \cdot) \rangle_{\mathcal{H}_0} &= \lim_{\epsilon_1 \to 0} \lim_{\epsilon_2 \to 0} \epsilon_1 \epsilon_2 \left\{ \frac{k(\nabla \Psi^*(x + \epsilon_1 e_i), \nabla \Psi^*(y + \epsilon_2 e_i)) - k(\nabla \Psi^*(x + \epsilon_1 e_i), \nabla \Psi^*(y))}{\epsilon_1} \right. \\
&\quad - \left. \frac{k(\nabla \Psi^*(x), \nabla \Psi^*(y + \epsilon_2 e_i)) - k(\nabla \Psi^*(x), \nabla \Psi^*(y))}{\epsilon_2} \right\} \\
&= \lim_{\epsilon_1 \to 0} \frac{1}{\epsilon_1} \left\{ \sum_{j=1}^d \partial_i \partial_j \Psi^*(y) \left( \partial_{\theta_j} k(\nabla \Psi^*(x + \epsilon_1 e_i), \theta') - \partial_{\theta_j'} k(\nabla \Psi^*(x), \theta') \right) \right|_{\theta = \nabla \Psi^*(y)} \\
&= \sum_{j,j'} \partial_i \partial_j \Psi^*(x) \partial_i \partial_{j'} \Psi^*(y) \partial_{\theta_j} \partial_{\theta_{j'}} k(\theta, \theta') \left|_{\theta = \nabla \Psi^*(x), \theta' = \nabla \Psi^*(y)} \right.
\end{aligned}
\]

set \( x = y \) and denote \( A(x) = (a_{ij}(x))_{i,j=1}^{d,d} \), with \( a_{ij}(x) := \sum_{k=1}^d \partial_k \partial_i \Psi^*(x) \partial_k \partial_j \Psi^*(x) \). Since \( \Psi^* \) is convex, then \( \nabla^2 \Psi^* \) is non-negative definite and so is \( A(x) = \nabla^2 \Psi^*(x) \nabla^2 \Psi^*(x) \), and then \( |a_{ij}(x)| \leq \frac{a_{ii} + a_{jj}}{2} \) (since \( a_{ii}a_{jj} - a_{ij}^2 \geq 0 \), so we have

\[
\| \nabla k_{\Psi}(x, \cdot) \|_{\mathcal{H}}^2 = \sum_{i=1}^d \langle \partial_{x_i} k(\nabla \Psi^*(x), \cdot), \partial_{x_i} k(\nabla \Psi^*(x), \cdot) \rangle_{\mathcal{H}_0} \\
\leq \frac{1}{2} B^2 \sum_{j,j'} (a_{jj}(x) + a_{jj'}(x)) \\
= B^2 d \text{Tr}(A(x)) \\
= B^2 d \text{Tr}(\nabla^2 \Psi^*(x) \nabla^2 \Psi^*(x)) \\
\leq \frac{B^2 d^2}{K^2}.
\]

**Proof of Lemma** Denote \( \Phi(x) := k_{\Psi}(x, \cdot) \in \mathcal{H} \). Then by definition of the Mirrored Stein Fisher information and integration by parts, we have the following:

\[
I_{MStein}(\mu_n \mid \pi) \frac{B}{2} = \left\| \mathbb{E}_{X \sim \mu_n} \left[ (\nabla V(X) \Phi(X) - \nabla \Phi(X)) \right] \right\|_{\mathcal{H}} \\
\leq \mathbb{E}_{X \sim \mu_n} \left[ \left\| \nabla V(X) \Phi(X) \right\|_{\mathcal{H}} \right] + \mathbb{E}_{X \sim \mu_n} \left[ \left\| \nabla \Phi(X) \right\|_{\mathcal{H}} \right] \\
= \mathbb{E}_{X \sim \mu_n} \left[ \left\| \nabla V(X) \right\|_{\mathcal{H}} \left\| \Phi(X) \right\|_{\mathcal{H}} \right] + \mathbb{E}_{X \sim \mu_n} \left[ \left\| \nabla \Phi(X) \right\|_{\mathcal{H}} \right] \\
\leq B_1 \mathbb{E}_{X \sim \mu_n} \left[ \left\| \nabla V(X) \right\|_{\mathcal{H}} \right] + \frac{B_2 d}{K}.
\]
Proof of Lemma 4

\[ \mathbb{E}_{X \sim \tilde{\mu}_n}[\|\nabla V(X)\|] \leq C_p \mathbb{E}_{X \sim \tilde{\mu}_n}[\|X\|^p] + C_p \]
\[ = C_p W_p^p(\tilde{\mu}_n, \delta_0) + C_p \]
\[ \leq C_p (W_p(\tilde{\mu}_n, \tilde{\pi}) + W_p(\tilde{\pi}, \mu_0) + W_p(\mu_0, \delta_0))^p + C_p \]
\[ \leq C_p \left( C_{\tilde{\pi}, p} \left( \text{KL}(\tilde{\mu}_n | \tilde{\pi}) \right)^{\frac{1}{p}} + \left( \frac{\text{KL}(\tilde{\mu}_n | \tilde{\pi})}{2} \right)^{\frac{1}{np}} \right) \]
\[ + C_{\tilde{\pi}, p} \left( \text{KL}(\mu_0 | \tilde{\pi}) \right)^{\frac{1}{p}} + \left( \frac{\text{KL}(\mu_0 | \tilde{\pi})}{2} \right)^{\frac{1}{np}} + W_p(\mu_0, \delta_0)^p + C_p \]
\[ = C_p \left( C_{\tilde{\pi}, p} (G_p(\text{KL}(\tilde{\mu}_n | \tilde{\pi})) + G_p(\text{KL}(\mu_0 | \tilde{\pi}))) + W_p(\mu_0, \delta_0)^p \right) + C_p. \]

This concludes the proof.

Proof of Lemma 6

The proof is based on the reproducing property and Cauchy-Schwarz inequality in the RKHS space. Indeed,

\[ \|Jh(x)\|^2_{\mathcal{H}} = \sum_{i,j=1}^{d} \left| \frac{\partial h_i(x)}{\partial x_j} \right|^2 \]
\[ = \sum_{i,j=1}^{d} \langle \partial x_j k\Phi(x, \cdot), h_i \rangle_{\mathcal{H}_0} \]
\[ \leq \sum_{i,j=1}^{d} \left\| \partial x_j k\Phi(x, \cdot) \right\|^2_{\mathcal{H}_0} \|h_i\|^2_{\mathcal{H}_0} \]
\[ = \left\| \nabla k\Phi(x, \cdot) \right\|^2_{\mathcal{H}} \|h\|^2_{\mathcal{H}} \]
\[ \leq \frac{B^2 d^2}{K^2} \|h\|^2_{\mathcal{H}}. \]

This concludes the proof.

C Main Theory

Proof of Theorem 7

From Equation (17) and Lemma 3 we have

\[ \min \left\{ \frac{(\alpha - 1) K}{\alpha B_2 d \|g_{\mu_\alpha}\|_{\mathcal{H}}}, \frac{1}{B_1 \|g_{\mu_\alpha}\|_{\mathcal{H} L_1}}, \frac{B_1}{(\alpha^2 B_2^2 d^2 + K^2(e - 1) B_1^2 A_1) \frac{K^2}{K}} \right\} \]
\[ \geq \min \left\{ \frac{1}{B_1 L_1}, \frac{(\alpha - 1) K}{\alpha B_2 d} \right\} \frac{K}{K B_1 \mathbb{E}_{X \sim \tilde{\mu}_n}[\|\nabla V(X)\|]} + B_2 d^2 \alpha^2 B_2^2 d^2 + K^2 B_1^2 (e - 1) (L_1 \mathbb{E}_{X \sim \tilde{\mu}_n}[\|\nabla V(X)\|] + L_0) \]
\[ =: M \left( \mathbb{E}_{X \sim \tilde{\mu}_n}[\|\nabla V(X)\|] \right), \]

so \( M (\cdot) \) is a non-increasing function. By Lemma 4 we further have

\[ M \left( \mathbb{E}_{X \sim \tilde{\mu}_n}[\|\nabla V(X)\|] \right) \geq M \left( C_p (C_{\tilde{\pi}, p} (G_p(\text{KL}(\tilde{\mu}_n | \tilde{\pi}))) + G_p(\text{KL}(\mu_0 | \tilde{\pi}))) + W_p(\mu_0, \delta_0)^p \right) + C_p. \]

(47)

(48)
From now on, we will use mathematical induction. First,

$$\gamma \leq M\left(C_p \left(2C_{\pi,p}G_p KL(\bar{\mu}_0 \mid \bar{\pi}) + W_p(\bar{\mu}_0, \delta_0)\right)^p + C_p\right)$$

$$\leq \min \left\{ \frac{(\alpha - 1)K}{\alpha B_2 d\|g_{\bar{\mu}_n}\|_H}, \frac{1}{B_1\|g_{\bar{\mu}_n}\|_H L_1}, \frac{K^2}{(\alpha^2 B_2^2 d^2 + K^2(e - 1)B_1^2 A_n)} \right\}$$

so it satisfies the condition of Proposition[1] and we will have $KL(\bar{\mu}_1 \mid \bar{\pi}) \leq KL(\bar{\mu}_0 \mid \bar{\pi})$. Next, suppose for integer $n \geq 1$, we have $\gamma \leq M\left(C_p \left(2C_{\pi,p}G_p KL(\bar{\mu}_0 \mid \bar{\pi}) + W_p(\bar{\mu}_0, \delta_0)\right)^p + C_p\right)$ satisfying the condition of Proposition[1] for $(\pi_k)_{k=0}^{n}$ (and so the sequence $(KL(\bar{\mu}_k \mid \bar{\pi}))_{k=0}^{n}$ non-increasing), we need to prove $\gamma \leq M\left(C_p \left(2C_{\pi,p}G_p KL(\bar{\mu}_0 \mid \bar{\pi}) + W_p(\bar{\mu}_0, \delta_0)\right)^p + C_p\right)$ also satisfies the condition of Proposition[1] that is

$$\gamma \leq M\left(C_p \left(\bar{C}_{\pi,p} \left(2C_{\pi,p}G_p KL(\bar{\mu}_n \mid \bar{\pi}) + G_p KL(\bar{\mu}_0 \mid \bar{\pi})\right) + W_p(\bar{\mu}_0, \delta_0)\right)^p + C_p\right)$$

$$\leq \min \left\{ \frac{(\alpha - 1)K}{\alpha B_2 d\|g_{\bar{\mu}_n}\|_H}, \frac{1}{B_1\|g_{\bar{\mu}_n}\|_H L_1}, \frac{K^2}{(\alpha^2 B_2^2 d^2 + K^2(e - 1)B_1^2 A_n)} \right\}$$

(50) is due to the non-increasing property of sequence $(KL(\bar{\mu}_k \mid \bar{\pi}))_{k=0}^{n}$,

$$\gamma \leq M\left(C_p \left(2C_{\pi,p}G_p KL(\bar{\mu}_n \mid \bar{\pi}) + W_p(\bar{\mu}_0, \delta_0)\right)^p + C_p\right)$$

$$\leq M\left(C_p \left(\bar{C}_{\pi,p} \left(2C_{\pi,p}G_p KL(\bar{\mu}_0 \mid \bar{\pi}) + G_p KL(\bar{\mu}_0 \mid \bar{\pi})\right) + W_p(\bar{\mu}_0, \delta_0)\right)^p + C_p\right)$$

$$\leq \vdots$$

$$\gamma \leq M\left(C_p \left(\bar{C}_{\pi,p} \left(2C_{\pi,p}G_p KL(\bar{\mu}_n \mid \bar{\pi}) + G_p KL(\bar{\mu}_0 \mid \bar{\pi})\right) + W_p(\bar{\mu}_0, \delta_0)\right)^p + C_p\right)$$

$$\leq \min \left\{ \frac{(\alpha - 1)K}{\alpha B_2 d\|g_{\bar{\mu}_n}\|_H}, \frac{1}{B_1\|g_{\bar{\mu}_n}\|_H L_1}, \frac{K^2}{(\alpha^2 B_2^2 d^2 + K^2(e - 1)B_1^2 A_n)} \right\}$$

So by Proposition[1] we have $(KL(\bar{\mu}_k \mid \bar{\pi}))_{k=0}^{n+1}$ non-increasing. By mathematical induction, we proved the theorem.

**C.1 Complexity Analysis**

In this section, we denote $\Pi$ as the area of unit radius circle. Assume $\bar{\mu}_0 = \mathcal{N}(0, I_d)$, we need first calculate the value of $\int_{x \in \mathbb{R}^d} \|x\|^{p+1} \tilde{\mu}_0(x) dx$.

**Lemma 9.** Assume $\bar{\mu}_0 = \mathcal{N}(0, I_d)$, then for any $p > -d - 1$ we have

$$\int_{x \in \mathbb{R}^d} \|x\|^{p+1} \tilde{\mu}_0(x) dx = 2^{(p+1)/2} \Gamma\left(\frac{p+d+1}{2}\right) \frac{1}{\Gamma\left(\frac{d}{2}\right)}.$$

(52)
Proof. We integrate this in polar coordinate \((r, \varphi_1, \ldots, \varphi_{d-1})\).

\[
\int_{x \in \mathbb{R}^d} \|x\|^{p+1} \tilde{\mu}_0(x) dx \\
= (2\pi)^{-d/2} \int_{x \in \mathbb{R}^d} \|x\|^{p+1} \exp(-\|x\|^2/2) dx \\
= (2\pi)^{-d/2} \int_0^\infty \cdots \int_0^\infty \int_0^{2\pi} r^{p+1} \exp(-r^2/2) r^{d-1} \sin^{d-2} (\varphi_1) \sin^{d-3} (\varphi_2) \cdots \sin (\varphi_{d-2}) dr d\varphi_1 d\varphi_2 \cdots d\varphi_{d-1} \\
= (2\pi)^{-d/2} \sqrt{2\pi} \frac{1}{2} \int_0^\infty \cdots \int_0^\infty \int_0^{2\pi} r^{p+d} \exp(-r^2/2) dr \cdots \int_0^{2\pi} \sin^{d-2} (\varphi_1) \sin^{d-3} (\varphi_2) \cdots \sin (\varphi_{d-2}) d\varphi_1 d\varphi_2 \cdots d\varphi_{d-1} \\
= \frac{(2\pi)^{-(d-1)/2}}{2} M_{p+d} |S_{d-1}| \\
= \frac{(2\pi)^{-(d-1)/2}}{2} 2\pi^{d/2} \frac{2\pi}{\Gamma(d/2)} \\
= \frac{(2\pi)^{-(d-1)/2}}{2} \frac{2\pi^{d/2}}{\sqrt{\pi}} \frac{2\pi^{d/2}}{\Gamma(d/2)} \\
= 2^{(p+1)/2} \frac{\Gamma(p+d+1)}{\Gamma(d/2)},
\]

where \(|S_{d-1}|\) is the \(d-1\)-dimensional volume of the sphere \(S_{d-1}\), \(M_{p+1} := \int_{x \in \mathbb{R}} |x|^{p+1} \frac{1}{\sqrt{2\pi \sigma^2}} e^{-x^2/2\sigma^2} dx\) and from [Winkelbauer, 2012], we know the \(p+1\)-th central absolute moment is \(M_{p+1} = \frac{\sqrt{\pi \Gamma(\frac{p+d+1}{2})}}{\Gamma(d/2)} = O(\sqrt{\pi \Gamma(\frac{p+d+1}{2})})\).

Assume \(\bar{\pi}(x) = e^{-V(x)}\), then we have the following lemma.

Lemma 10. Let Assumption 3 hold and \(\tilde{\mu}_0 = \mathcal{N}(0, I_d)\), we then have

\[
\text{KL}(\tilde{\mu}_0 \mid \bar{\pi}) \leq \frac{d}{2} \log\left(\frac{1}{2\pi e}\right) + V(0) + \frac{C_p}{p+1} \left(2^{(p+1)/2} \frac{\Gamma(p+d+1)}{\Gamma(d/2)} + (p+1) \sqrt{\frac{\pi \Gamma(d+1)}{\Gamma(d/2)}}\right).
\]

Proof. By Assumption 5, we know

\[
\|\nabla V(x)\| \leq C_p \left(\|x\|^p + 1\right),
\]

so

\[
V(x) = \int_0^1 \langle \nabla V(tx), x \rangle dt + V(0) \\
\leq \int_0^1 \|\nabla V(tx)\| \|x\| dt + V(0) \\
\leq \int_0^1 C_p \left(\|tx\|^p + 1\right) \|x\| dt + V(0) \\
\leq C_p \left(\frac{\|x\|^{p+1}}{p+1} + \|x\|^p\right) + V(0).
\]
We proceed to calculate \( KL(\bar{\mu}_0 \mid \bar{\pi}) \),

\[
KL(\bar{\mu}_0 \mid \bar{\pi}) = \int_{x \in \mathbb{R}^d} \log\left(\frac{\bar{\mu}_0}{\bar{\pi}}(x)\right) \bar{\mu}_0(x) dx
\]

\[
= \int_{x \in \mathbb{R}^d} \log(\bar{\mu}_0)(x) \bar{\mu}_0(x) dx + \int_{x \in \mathbb{R}^d} V(x) \bar{\mu}_0(x) dx
\]

\[
\leq \frac{d}{2} \log\left(\frac{1}{2\pi e}\right) + \int_{x \in \mathbb{R}^d} \left(C_p \left(\frac{\|x\|^{p+1}}{p+1} + \|x\|\right) + V(0)\right) \bar{\mu}_0(x) dx
\]

\[
= \frac{d}{2} \log\left(\frac{1}{2\pi e}\right) + V(0) + \frac{C_p}{p+1} \int_{x \in \mathbb{R}^d} \|x\|^{p+1} + (p+1) \|x\| \bar{\mu}_0(x) dx
\]

\[
= \frac{d}{2} \log\left(\frac{1}{2\pi e}\right) + V(0) + \frac{C_p}{p+1} \left(2^{(p+1)/2} \frac{\Gamma\left(\frac{p+d+1}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)} + (p+1) \sqrt{\frac{2}{\pi}} \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\right)
\]

where we use Lemma \(^9\).

According to Lemma \(^10\) the order of \( KL(\bar{\mu}_0 \mid \bar{\pi}) \) is

\[
KL(\bar{\mu}_0 \mid \bar{\pi}) = \tilde{O}\left(\frac{2^{(p+1)/2} \Gamma\left(\frac{p+d+1}{2}\right)}{(p+1) \Gamma\left(\frac{d+1}{2}\right)}\right).
\]

**Proof of Corollary \(^1\)** By Theorem \(^1\) we know that

\[
\frac{1}{n} \sum_{k=0}^{n-1} I_{M,Stein}(\bar{\mu}_k \mid \bar{\pi}) \leq \frac{2 KL(\bar{\mu}_0 \mid \bar{\pi})}{n\gamma},
\]

we need to estimate the order of \( \frac{KL(\bar{\mu}_0 \mid \bar{\pi})}{\gamma} \). Firstly, from Lemma \(^10\) we know \( KL(\bar{\mu}_0 \mid \bar{\pi}) = O((C_p+1)dM) \), so

\[
G_p(KL(\bar{\mu}_0 \mid \bar{\pi})) = \left(KL(\bar{\mu}_0 \mid \bar{\pi})\right)^{\frac{1}{p}} + \left(\frac{KL(\bar{\mu}_0 \mid \bar{\pi})}{2}\right)^{\frac{1}{2p}}
\]

\[
= O(KL(\bar{\mu}_0 \mid \bar{\pi})^{\frac{1}{p}}).
\]

Next, we estimate the order of \( \frac{1}{\gamma} \), we have

\[
\frac{1}{\gamma} = M \left( C_p (2C_p G_p(KL(\bar{\mu}_0 \mid \bar{\pi})) + W_p(\bar{\mu}_0, \delta_0))^p + C_p\right)
\]

\[
= \tilde{O}((2C_p G_p(KL(\bar{\mu}_0 \mid \bar{\pi})) + W_p(\bar{\mu}_0, \delta_0))^p)
\]

\[
= \tilde{O}\left(2C_p G_p \left(\frac{2^{(p+1)/2} \Gamma\left(\frac{p+d+1}{2}\right)}{(p+1) \Gamma\left(\frac{d+1}{2}\right)}\right) + \left(\frac{2\sqrt{\frac{2}{\pi}} \Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\right)^{\frac{1}{p}}\right)^p
\]

\[
= \tilde{O}\left(\frac{32^p C_p G_p \Gamma\left(\frac{p+d+1}{2}\right)}{(p+1) \Gamma\left(\frac{d+1}{2}\right)^2}\right).
\]

So we finally have

\[
\frac{KL(\bar{\mu}_0 \mid \bar{\pi})}{\gamma} = \tilde{O}\left(\frac{8^p C_p G_p \Gamma\left(\frac{p+d+1}{2}\right)^2}{(p+1)^2 \Gamma\left(\frac{d}{2}\right)^2}\right),
\]

22
to get $\frac{2 \text{KL}(\hat{\mu}_0|\bar{\pi})}{n^\gamma} \leq \varepsilon$, we need $n \geq \frac{2 \text{KL}(\hat{\mu}_0|\bar{\pi})}{\gamma \varepsilon}$, that is

$$n = \Omega \left( \frac{8^p C_{\lambda, p}^{\gamma} \Gamma(\frac{p+\gamma+1}{2})^2}{(p+1)^2 \Gamma(\frac{4}{2})^2 \varepsilon} \right).$$  \hspace{1cm} (62)

By the Stirling formula $\Gamma(z+1) \sim \sqrt{2\pi z} \left(\frac{z}{e}\right)^z$, we know

$$\frac{\Gamma(\frac{p+\gamma+1}{2})^2}{\Gamma(\frac{4}{2})^2} = \Omega \left( \frac{\Pi(d+p-1) \left(\frac{d+p-1}{2e}\right)^{d+p-1}}{\Pi(d-2) \left(\frac{d-2}{2e}\right)^{d-2}} \right)$$

$$= \Omega \left( \left(\frac{d+p-1}{d-2}\right)^{d-2} \left(\frac{d+p-1}{2e}\right)^{p+1} \right)$$

$$= \Omega (d^{p+1}), \text{ where we only consider the order of } d,$n so the iteration complexity has dimension dependency of order $p+1$.

D The Cases under $T_p$ inequality with $1 \leq p \leq 2$

Proof of Lemma $\number{5}$

$$\mathbb{E}_{X \sim \hat{\mu}_n}[||\nabla V(X)||] \leq C_p \mathbb{E}_{X \sim \hat{\mu}_n}[||X||^p] + C_p$$

$$= C_p W_p^p(\hat{\mu}_n, \delta_0) + C_p$$

$$\leq C_p (W_p(\hat{\mu}_n, \bar{\pi}) + W_p(\bar{\pi}, \hat{\mu}_0) + W_p(\hat{\mu}_0, \delta_0))^p + C_p$$

$$\leq C_p \left( \frac{2 \text{KL}(\hat{\mu}_n | \pi)}{\lambda} + \frac{2 \text{KL}(\hat{\mu}_0 | \bar{\pi})}{\lambda} + W_p(\hat{\mu}_0, \delta_0) \right)^p + C_p$$  \hspace{1cm} (64)

\[\square\]

Proof of Theorem $\number{2}$ The proof is the same as the one for Theorem $\number{1}$ but with this new bound $\frac{\text{KL}(\hat{\mu}_0|\bar{\pi})}{\gamma}$. \hspace{1cm} \[\square\]

Proof of Corollary $\number{2}$ Similar to the proof of Corollary $\number{1}$, we need to estimate the order of $\frac{\text{KL}(\hat{\mu}_0|\bar{\pi})}{\gamma}$. By Equation (58) and the Stirling formula $\Gamma(z+1) \sim \sqrt{2\pi z} \left(\frac{z}{e}\right)^z$, we know

$$\text{KL}(\hat{\mu}_0 | \bar{\pi}) = \hat{O} \left( \frac{\Gamma(\frac{p+\gamma+1}{2})}{\Gamma(\frac{4}{2})} \right)$$

$$= \hat{O} \left( \sqrt{\Pi(d+p-1) \left(\frac{d+p-1}{2e}\right)^{d+p-1} \frac{1}{\Pi(d-2) \left(\frac{d-2}{2e}\right)^{d-2}}} \right)$$

$$= \hat{O} \left( \left(\frac{d+p-1}{d-2}\right)^{d-2} \left(\frac{d+p-1}{2e}\right)^{\frac{p+1}{2}} \right)$$

$$= \hat{O} (d^{p+1})$$

we also know

$$W_p^p(\hat{\mu}_0, \delta_0) = \int_{x \in \mathbb{R}^d} ||x||^p d\hat{\mu}_0(x) = 2^p \frac{\Gamma(\frac{p+1}{2})}{\Gamma(d/2)} = \mathcal{O} \left( d^{\frac{p}{2}} \right).$$  \hspace{1cm} (66)
So we have

\[
\left( 2 \sqrt{\frac{2 \text{KL}(\bar{\mu}_0 \mid \bar{\pi})}{\lambda}} + W_p(\bar{\mu}_0, \delta_0) \right)^p
\]

\[
= \tilde{O} \left( \left( \sqrt[4]{\frac{1}{\lambda} d^{\frac{p+1}{4}} + d^2} \right)^p \right)
\]

\[
= \tilde{O} \left( \max \left\{ \frac{1}{\lambda^2} d^{\frac{p(p+1)}{4}}, d^2 \right\} \right)
\]

\[
= \tilde{O} \left( \frac{d^{\frac{p(p+1)}{4}}}{\lambda^2} \right),
\]

and then

\[
\text{KL}(\bar{\mu}_0 \mid \bar{\pi}) \gamma = \tilde{O} \left( \frac{d^{\frac{d(p+1)}{4}}}{\lambda^2} d^{\frac{p+1}{2}} \right)
\]

\[
= \tilde{O} \left( \frac{d^{\frac{(p+2)(p+1)}{4}}}{\lambda^2} \right).
\]

(67)

To get \( \frac{2 \text{KL}(\bar{\mu}_0 \mid \bar{\pi})}{n \gamma} \leq \varepsilon \), we need \( n \geq \frac{2 \text{KL}(\bar{\mu}_0 \mid \bar{\pi})}{\gamma \varepsilon} \), that is

\[
n = \tilde{\Omega} \left( \frac{d^{\frac{(p+2)(p+1)}{4}}}{\lambda^2 \varepsilon} \right).
\]

(68)