Quasihomogeneous Infinite Systems of Linear Algebraic Equations

F. M. Fedorov¹, N. N. Pavlov², O. F. Ivanova³ and S. V. Potapova⁴
North-Eastern Federal University, 58 Belinsky str., Yakutsk 677000, Russia
E-mail: ¹foma_46@mail.ru; ²pnn10@mail.ru; ³o_buskarova@mail.ru; ⁴sargyp@mail.ru

Abstract. The broad practical application of an infinite systems of linear algebraic equation is greatly limited by the insufficient development of the theory of these systems. In particular, the question about uniqueness of infinite systems solutions remains open. This is because that solving a corresponding homogeneous system is much more difficult task than solving an inhomogeneous system. The solving a homogeneous infinite systems is connected with the study of special infinite systems the so-called quasihomogeneous systems. Such systems are found, for example, in the theory of electrical circuits. An infinite system of linear algebraic equations is quasihomogeneous, if the finite amount of numbers on the right-hand side are not equal to zero, and the infinite amount of them are equal to zero. In this paper, we develop a numerical algorithm for finding the solution of quasihomogeneous infinite systems. In particular, it has been shown that such systems can behave both as “purely” inhomogeneous, and as almost homogeneous systems, it depends on coefficients of the system.

1. Introduction

As noted in [1, 2], the history of the study of infinite systems of linear algebraic equations with an infinite number of unknowns is more than 200 years. Fourier (1807) obtained a solution to the Dirichlet problem for the case of an infinite strip with the use of infinite systems, although he came to the right result in a somewhat slippery way. Much later, the astronomer G. Hill (1877) successfully applied an infinite system for integrating one ordinary second-order differential equation obtained by study the motion of moon. Also, the practical solving a ODE compelled A. Poincare to turn to infinite systems. The solving an integral equations led A. Fredholm and D. Hilbert to the idea of investigating infinite systems.

To develop an applications of the Fourier method, L.V. Kantorovich proposed a method based on reducing the boundary problem to an infinite system of linear equations, and he expounded an original theory of regular (completely regular, quasiregular) infinite systems [3]. Until now, this theory is the only theory of infinite systems successfully applied to solve various problems of the statistical theory of elasticity. The second most developed theory is the theory of periodic infinite systems [1], including systems with difference indices [4, 5]. In developing the theory of wave diffraction, the periodic systems are widely used by V. Shestopalov and his students [6, 7], in doing so they significantly developed the theory of solving these systems. At the same time, the insufficient development of theory and methods for solving general infinite systems makes it very difficult to apply these systems in mathematical modeling of various physical, chemical, biological, technological, etc. processes [8]. Nevertheless, attempts at practical application of infinite systems for solving many problems of mechanics, engineering, physics, and science do
not subside, for example, for solving problems of spectral theory and parametric circuits [9] and waveguide theory [10], as well as for radio circuits [11]. To solve many physical and technical problems, special types of infinite systems can be successfully used [12], although there is still no complete clarity about the properties of these systems [13]. Note here that the concept of principal solution of an infinite system widely used in a scientific literature does not stand up to criticism [13].

Thus, the theory of infinite systems of linear algebraic equations arose and develops due to applications that it has the theory of integral equations and, especially, in solving boundary value problems of mathematical physics.

At the present time we have proposed and developed the theory of infinite systems from general positions [14], a brief account of which will be given in following sections. At the same time, the question about uniqueness of solution of infinite systems is still open, that is, finding a solution of homogeneous systems remains an open problem. The present paper is devoted to infinite systems the main part of which is a homogeneous system.

Basic information about infinite systems, matrices, determinants and minors can be found in papers [1, 2, 3, 15, 16].

2. Preliminaries

We recall some facts concerning infinite systems and refer to [1, 2], [10, 11, 12] for details. We consider an infinite system of linear algebraic equations with infinitely many unknowns

\[
\begin{align*}
&\begin{array}{c}
a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n + \ldots = b_1, \\
a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n + \ldots = b_2, \\
\vdots \\
a_{n,1}x_1 + a_{n,2}x_2 + \ldots + a_{n,n}x_n + \ldots = b_n,
\end{array}
\end{align*}
\]

where \(a_{i,k}\) are known and \(x_k\) are unknown in some field \(F\).

The set of the values \(x_1, x_2, \ldots\) is called a solution of the system (2.1) if after substitution of these values into (2.1), a series are convergent and an identities are satisfied.

An infinite system is consistent if it is solvable and inconsistent otherwise.

Let’s consider the sequence of principal minors of the matrix \(A\) of system (2.1) – \(|A_1| = a_{1,1}, |A_2| = \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} \ldots\). If there exists a limit of this sequence, then such a limit is called an infinite determinant of the system (2.1). Let the infinite determinant be nonzero, \(|A| \neq 0\).

Due to the R. G. Cooke [16] an infinite matrix \(A\) can be lower or upper triangular in accordance with analogous concepts for finite matrices.

**Definition 1.** If all elements of the main diagonal of the lower triangular matrix are not equal to zero, then such matrix is simply called triangular. If the main diagonal of the upper triangular matrix consists of non-zero elements, then such a matrix is called a Gaussian matrix.

Thus, infinite determinants (if they exist) of triangular and Gaussian matrices are always nonzero, this is the fundamental difference between the lower and upper triangular matrices from the triangular and Gaussian matrices, respectively.

If an infinite system has a Gaussian matrix, then we say that system is given in the Gaussian form or simply it is called the Gaussian system.

Since it is assumed that the infinite determinant of system (2.1) is nonzero, then the infinite matrix of system (2.1) has infinite rank [15]. Therefore, to generalize the Gauss elimination algorithm, we use the theorem [17]:

**Theorem 1.** Any matrix of infinite rank for which the sequence of principal minors are nonzero can be represented as a product of the triangular matrix \(B\) by the Gaussian matrix \(C\).
that is, \( A = BC \), and the matrix coefficients of \( B \) and \( C \) are recursively written in terms of coefficients of the matrix \( A \).

If we put \( b_{j,j} \equiv 1 \), then we obtain the Gauss algorithm for a given infinite system. Instead of system (2.1), the latter allows solving the following equivalent Gaussian system:

\[
\sum_{p=0}^{\infty} c_{j,j+p} x_{j+p} = \bar{b}_j, \quad j = 1, 2, 3, \ldots,
\]

(2.2)

where \( c_{j,j+p} \) are elements of the Gaussian matrix \( C \), and \( \bar{b}_j \) are elements of the matrix \( B^{-1}b \).

There have been attempts to generalize the Gauss algorithm to an infinite system [18, 19], but from the algorithmic point of view they turned out to be of little use for their practical implementation.

For simplicity, we write the Gaussian system (2.2) in more general form, without relating it to the system (2.1):

\[
\sum_{p=0}^{\infty} a_{j,j+p} x_{j+p} = b_j, \quad j = 1, 2, 3, \ldots,
\]

(2.3)

where \( a_{j,j+p} \) are elements of the Gaussian matrix \( A \), and \( b_j \) are free terms.

As the basic method for solving the Gaussian system (2.3), we propose to use the reduction method, but not in its classical sense.

**Definition 2.** If in reduction method for solving infinite systems of algebraic equations the number of unknowns and the number of equations remain the same in truncated system, then we can say that the reduction method *is understood in narrow sense*, and if the number of unknowns is greater than the number of equations, then we say that the reduction method *is understood in broad sense*.

In fact, the reduction method in narrow sense is a known simple reduction method. But a different understanding of the reduction method is of fundamental importance. The method of reduction in narrow sense is used only to solve an inhomogeneous infinite system in order to obtain its special particular solution, and in the broad sense only for solving a homogeneous system in order to obtain its particular nontrivial solution.

Here it should be emphasized that finding a solution of truncated system, for example, by the method of successive approximations, does not usually succeed [13].

To solve effectively certain Gaussian infinite systems, the concept of periodic Gaussian systems is used.

**Definition 3.** Gaussian infinite system (2.3) is said to be a periodic Gaussian system when coefficients satisfying conditions

\[
\frac{a_{j,j+p}}{a_{j+p,j+p}} = a_p, \quad j = 1, 2, 3, \ldots
\]

(2.4)

**3. New results obtained earlier**

Some new results for the general infinite system (2.1) we have received earlier [14, 20, 21] solving the system (2.3) by the simple method, we obtain

**Theorem 2.** Let the system (2.3) is truncated by the reduction method in narrow sense into the finite Gaussian system of the form

\[
\sum_{p=0}^{n-j} a_{j,j+p} x_{j+p} = b_j, \quad a_{j,j} \neq 0, \quad j = 1, \ldots, n.
\]

(3.1)

Then the solution of the finite system (3.1) is an expression:

\[
\bar{x}_j = B_{n-j}, \quad j = 1, 2, \ldots, n,
\]

(3.2)
where

\[
B_{n-j} = \frac{b_j}{a_{j,j}} - \sum_{p=0}^{n-j-1} \frac{a_{j,n-p}}{a_{j,j}} B_p, \quad B_0 = \frac{b_n}{a_{n,n}}, \quad j = 1, n-1.
\] (3.3)

The question is, when does the method of simple reduction converge to the solution of general system (2.1). In order to answer this question, suppose that the following two conditions are fulfilled:

1) Suppose that the limit \( \lim_{n \to \infty} B_{n-j}(j) = B(j) \) exists. This condition guarantees, as it can be seen from the expression (3.2), that the method of reduction in narrow sense converges;

2) Suppose that in (3.3) it is possible to pass term-by-term to the limit in the sense of formula

\[
\lim_{n \to \infty} \sum_{p=j+1}^{n} \frac{a_{j,p}}{a_{j,j}} B_{n-p} = \sum_{p=j+1}^{\infty} \frac{a_{j,p}}{a_{j,j}} \lim_{n \to \infty} B_{n-p}.
\] (3.4)

The condition 2) is a sufficient condition for numbers \( B(j) \) to be a particular solution of Gaussian system (2.3).

**Theorem 3.** Let conditions 1) and 2) hold, then the limit value \( \lim_{n \to \infty} B_{n-j} = B(j) \) is a particular solution of inhomogeneous infinite Gaussian system (2.3), and therefore for the system (2.1) too.

**Definition 4.** The particular solution \( x_j = B(j) \) of inhomogeneous infinite Gaussian system (2.3) is called a strictly particular solution of the system (2.1).

**Theorem 4.** If inhomogeneous Gaussian system (2.3) consistent, then it always has a strictly particular solution given by Cramer formula.

**Theorem 5.** The general inhomogeneous infinite system (2.1) with nonzero determinant is consistent if and only if it has a strictly partial solution.

**Theorem 6.** The consistent Gaussian system (2.3) has a strictly partial solution

\[
x_j = B(j) = \sum_{p=0}^{\infty} \frac{(-1)^p A_p(j) b_{j+p}}{a_{j+p,j+p}}, \quad j = 1, 2, \ldots
\] (3.5)

where

\[
A_p(j) = \sum_{k=0}^{p-1} \frac{(-1)^{p-1-k} a_{j+k,j+k} a_k(j)}{a_{j+k,j+k}} A_0(j), \quad A_0(j) = 1 \forall j.
\] (3.6)

In fact, the expression (3.6) is the value of the characteristic determinant

\[
A_{n-j}(j) = \begin{vmatrix}
\frac{a_{j,j+1}}{a_{j,j}} & 1 & 0 & \ldots & 0 & 0 \\
\frac{a_{j,j+2}}{a_{j,j}} & \frac{a_{j+1,j+2}}{a_{j+1,j+1}} & 1 & \ldots & 0 & 0 \\
\frac{a_{j,j+3}}{a_{j,j}} & \frac{a_{j+1,j+3}}{a_{j+1,j+1}} & \frac{a_{j+2,j+3}}{a_{j+2,j+2}} & \ldots & 0 & 0 \\
\frac{a_{j,j+4}}{a_{j,j}} & \frac{a_{j+1,j+4}}{a_{j+1,j+1}} & \frac{a_{j+2,j+4}}{a_{j+2,j+2}} & \ldots & \frac{a_{j+3,j+4}}{a_{j+3,j+3}} & \ldots \\
\frac{a_{j,j+n-2}}{a_{j,j}} & \frac{a_{j+1,j+n-2}}{a_{j+1,j+1}} & \frac{a_{j+2,j+n-2}}{a_{j+2,j+2}} & \ldots & \frac{a_{j+3,j+n-2}}{a_{j+3,j+3}} & \ldots & \frac{a_{j+n-2,j+n-2}}{a_{j+n-2,j+n-2}} & 1 \\
\frac{a_{j,j+n-1}}{a_{j,j}} & \frac{a_{j+1,j+n-1}}{a_{j+1,j+1}} & \frac{a_{j+2,j+n-1}}{a_{j+2,j+2}} & \ldots & \frac{a_{j+3,j+n-1}}{a_{j+3,j+3}} & \ldots & \frac{a_{j+n-2,j+n-1}}{a_{j+n-2,j+n-2}} & \frac{a_{j+n-1,j+n-1}}{a_{j+n-1,j+n-1}}
\end{vmatrix}
\] (3.7)
It follows from (3.5) that \( \lim_{n \to \infty} B_{n-j}(j) = B(j) \), where

\[
\frac{n}{x_j} = B_{n-j}(j) = \sum_{p=0}^{n-j-1} (-1)^p A_p(j) \frac{b_{j+p}}{a_{j+p,j+p}}, \quad j = 1, 2, \ldots, n. \quad (3.5')
\]

**Theorem 7.** Let the homogeneous Gaussian system (2.3) \( (b_j = 0) \) is truncated by the reduction method in broad sense into the homogeneous finite Gaussian system of the form

\[
\sum_{p=0}^{n-j} a_{j,j+p} x_{j+p} = 0, \quad a_{j,j} \neq 0, \quad j = 0, n - 1.
\]

Then the solution of (3.8) is an expression

\[
\frac{n+1}{x_j} = \frac{(-1)^j x_0 A_{n-j}(j)}{A_n(0)}, \quad j = 0, 1, \ldots, n, \quad (3.9)
\]

where \( A_{n-j}(j) \) (3.7), \( x_0 \) is an arbitrary real number, and note that \( n \) is the order of system (3.8) and the number of unknowns is \( (n + 1) \).

For convenience, the numbering of equations in (2.3) starts from zero: \( j = 0, 1, 2, \ldots \).

Let’s indicate the most important properties of a strictly particular solution that follow from Theorems 4–6.

**Property 1.** A consistent inhomogeneous system (2.1) always has a strictly particular solution, which is expressed by the Cramer formula.

**Property 2.** A strictly partial solution is not contained as an additive term in a nontrivial solution of corresponding homogeneous system. That is why this solution was called a strictly particular solution.

**Property 3.** A strictly particular solution is the principal solution of infinite system, if it exists. This principal solution is obtained when we combine the reduction method with the method of successive approximations whose convergence does not depend on reduction method convergence.

**Property 4.** A trivial solution of homogeneous infinite system is also its strictly particular solution. Hence, we can not obtain a trivial solution of homogeneous system by the method of reduction in narrow sense.

4. **The uniqueness of the solution**

Even if the infinite determinant is nonzero, the corresponding homogeneous infinite system may have a nontrivial solutions. That is the main difference between finite and infinite systems. Therefore, the study of corresponding homogeneous systems becomes actual from the point of view of the uniqueness of inhomogeneous infinite systems solutions.

There is another way to investigate the uniqueness of solution of infinite systems, namely, it can be determined by the uniqueness of solution of the original problem itself, as done, for example, in [6]. But this way is not always possible to implement, see [22].

Therefore, the study of homogeneous systems remains the most common approach.

To solve a homogeneous infinite systems is a much more difficult than to solve inhomogeneous systems, as Riesz [23] pointed out. The methods for solving inhomogeneous and homogeneous systems are fundamentally different (see Property 4). At present, there are occasional works on solving a homogeneous infinite systems, that is, some special systems can be solved. On the other hand, there are infinite system where only a finite amount of numbers on the right-hand side are not equal to zero, and the infinite amount of them are equal to zero. In practice, when studying such systems, one does not pay attention to its peculiarity.
For a clear understanding of the differences in methods of solving the same system in inhomogeneous and homogeneous cases, we give a detailed scheme for solving one special infinite system.

**Example 1.** Consider the periodical infinite system solving one mixed problem of mathematical physics [1, 24]

\[
\sum_{p=0}^{\infty} \frac{(2j + 2p)!}{(2p)!} x_{j+p} = b^j, \quad j = 0, 1, 2, ..., \quad b > 0, \tag{4.1}
\]

it is shown that it has exact analytic solution

\[
x_j = \frac{b^j}{(2j)! \sinh(\sqrt{b})}, \quad j = 0, 1, 2, .... \tag{4.2}
\]

If it is not possible to solve the given infinite system analytically, then one must resort to computational methods.

Therefore, without taking into account its specificity, using only Theorems 2–7, in particular, formulas (3.5), (3.6) and (3.5'), (3.9), we solve the system (4.1) as a Gaussian system obtained from the general system (2.1).

To calculate the series (3.5) using the recurrence formula (3.6), we calculate the absolute value of the difference between the last two terms of the sum (3.5') for each \( j \) for \( n \to \infty \), and then if this value does not exceed the specified accuracy \( \varepsilon \), we stop calculations (for some \( n = N_j \)). Of course, for each \( j \) there exists its own \( N_j \), and this allows us to follow the residual, i.e., for the difference between the left and right sides of the system (4.1) for the found values of \( x_j \) for each \( j \). If these residuals tend to zero, then the obtained numbers \( x_j \) are approximate solutions of system (4.1) with guaranteed accuracy in the case of sufficiently fast convergence of series in (3.5).

Let us explain the above. Indeed, if the numerical sequence \( x_j^n \) converges as \( n \) increases, then this means that a simple reduction converges. Here it is necessary to remember that in fact, the formula (3.5') with (3.6) gives the exact solution of truncated finite system of \( n \)-th order for each \( n \). But the convergence of a simple reduction does not mean that the reduction converges precisely to the solution of a Gaussian infinite system, for example (4.1), thus to the general system (2.1). But if it converges, then it converges to the value determined by the Cramer formula for the corresponding \( j \). In order to show that a simple reduction converges to the solution of Gaussian system (4.1), we need to follow the residual. From the formula (3.5') with regard to (3.6) it follows that

\[
\varepsilon = \left| x_j^n - x_{j+1}^n \right| = \left| A_{n+1-j}(j) \frac{b_{n+1}}{a_{n+1,n+1}} \right|,
\]

where \( A_{n+1-j}(j) \) are characteristic determinants (3.7) of the Gaussian matrix \( A \), \( a_{j,j+p} \in A \), \( b_j \) are free terms of the system (2.3).

Table 1 shows a results of calculations of the system (4.1) for \( b = 0.5 \), \( b = 2.3 \), \( \varepsilon = 10^{-8} \) and its exact solution (4.2).

In Table 2, the residuals of the first three equations of the system are given. They show that solutions satisfy the initial system with sufficient accuracy (Table 1 also confirms this assertion).

Thus, as shown in Table 1 and Table 2, the analytical solution (4.2) is in fact a strictly particular solution of the system (4.1).
Table 1.

| b   | x_0  | x_1   | x_2   | x_3   | x_4   |
|-----|------|-------|-------|-------|-------|
| 0.5 | 0.7933 | 0.1983 | 0.0083 | 0.0001 | 0.0000 |
| (4.2) | 0.7933 | 0.1983 | 0.0083 | 0.0001 | 0.0000 |
| 2.3 | 0.4193 | 0.4809 | 0.0924 | 0.0071 | 0.0003 |
| (4.2) | 0.4188 | 0.4816 | 0.0923 | 0.0071 | 0.0003 |

Table 2.

| b   | 1 | 2             | 3             |
|-----|---|---------------|---------------|
| 0.5 | 3.787 · 10^{-12}| -4.712 · 10^{-10} | -2.940 · 10^{-9} |
| 2.3 | 7.229 · 10^{-11}| 1.444 · 10^{-8}   | 2.172 · 10^{-6}  |

Since the system (4.1) is periodic, in homogeneous case it is possible to find its analytic solution in closed form, and in fact all its nontrivial solutions.

**Example 2.** Find all nontrivial solutions of the system (4.1) in homogeneous case

\[
\sum_{p=0}^{\infty} \frac{(2j + 2p)!}{(2p)!} x_{j+p} = 0, \quad j = 0, \infty. \tag{4.3}
\]

**Solution.** Although the solution of this example is given repeatedly in our earlier papers, for example in [24], due to the importance of the example, here we give the full scheme for solving the system (4.3).

Let the system

\[
\sum_{p=0}^{\infty} a_p x_{j+p} = 0, \quad j = 0, 1, \ldots, \tag{4.4}
\]

be homogeneous system with difference indices, where \(a_p\) is from (2.4). It is known [2] that between solutions of a homogeneous periodic system and the corresponding system (4.4) there exists an isomorphism by the relation \(y_j = \frac{x_j}{a_{j,j}}\), where \(y_j\) is a solution of periodic system, and \(x_j\) is a solution of systems with difference indices (4.4).

As shown in [1, 2], for systems with difference indices (hence for periodic systems) the main role in the question of existence of nontrivial solutions is played by the characteristic of system (or otherwise characteristic equation):

\[
f(x) = \sum_{p=0}^{\infty} (-1)^p a_p x^p = 0. \tag{4.5}
\]

It is important to understand that zeros of this algebraic equation determine a nontrivial solutions of infinite systems with difference indices, and thus on periodic ones.

If equation (4.5) does not have zeros, then the inhomogeneous system (2.3), in the case of its periodicity, has a unique solution. If \(x = \xi\) is a single zero of equation (4.5), then the expression \(x_j = \frac{(-\xi)^j C}{a_{j,j}}\) is a nontrivial solution of corresponding homogeneous periodic system, where \(C\) is an arbitrary number, [1, 2].
In our case, we have \( a_p = \frac{1}{(2p)!} \) and

\[
f(x) = \sum_{p=0}^{\infty} (-1)^p \frac{1}{(2p)!} x^p = \cos(\sqrt{x}) = 0,
\]

then this equation has single zeros \( x = \xi_k = \frac{\pi^2(2k+1)^2}{4} \), \( k = 0, 1, 2, \ldots \). Obviously, the function \( f(x) \) has no other zeros.

Consequently, all fundamental solutions of the homogeneous Gaussian system with difference indices (4.4) have the form

\[
x^{(k)}_j = \frac{(-1)^j \pi^{2j}(2k+1)^2 C_k}{2^j 2^j}, \quad j, k = 0, 1, 2, \ldots, C_0 = x_0, \tag{4.6}
\]

and the corresponding periodic system (4.3) have the form

\[
x^{(k)}_j = \frac{(-1)^j \pi^{2j}(2k+1)^2 C_k}{(2j)! 2^j}, \quad j, k = 0, 1, 2, \ldots, C_0 = x_0, \tag{4.7}
\]

where \( C_k \) are arbitrary constants.

An isomorphic system with difference indices to the periodic system (4.3) can be written as follows:

\[
\sum_{p=0}^{\infty} \frac{1}{(2p)!} x_{j+p} = 0, \quad j = 0, 1, 2, \ldots \tag{4.8}
\]

Obviously, the least magnitude of exact solution of the system (4.8) follows from (4.6) for \( k = 0 \) (\( x_0 = 1 \)):

\[
x_j = (-1)^j \frac{\pi^{2j}}{2^j}, \quad j = 0, 1, 2, \ldots \tag{4.9}
\]

Table 3 shows results of calculations using the formula (3.9) for \( n = 3, n = 5, n = 10 \) and the exact solution (4.9) of the infinite homogeneous system (4.8):

| \( n \) | \( x_0 \) | \( x_1 \) | \( x_2 \) | \( x_3 \) | \( x_4 \) |
|---|---|---|---|---|---|
| 3 | 1.0 | -2.45901639 | 5.90163934 | -11.80327868 |  |
| 5 | 1.0 | -2.46729082 | 6.08539023 | -14.96407434 | 35.91377842 |
| 10 | 1.0 | -2.46740109 | 6.08806814 | -15.02170510 | 37.06455106 |
| (4.9) | 1.0 | -2.46740110 | 6.08806818 | -15.02170614 | 37.06457428 |

Table 3 shows, the resulting numerical solution of the homogeneous system (4.8) converges very quickly to an exact solution.

Table 4 gives values of characteristic determinants \( A_{n-j}(j) \) that appear in formula (3.9) for the considered \( n \).

Hence it is clear that values of \( A_{n-j}(j) \) strongly decrease.

Now we consider the original homogeneous periodic system (4.3).

The least magnitude of the exact solution of system (4.3) follows from (4.7) for \( k = 0 \) (\( x_0 = 1 \))

\[
x_j = (-1)^j \frac{\pi^{2j}}{(2j)! 2^{2j}}, \quad j = 0, 1, 2, \ldots \tag{4.10}
\]
Table 4.

| j  | 0    | 1    | 2    | 3   | 4    | 5   |
|----|------|------|------|-----|------|-----|
| 3  | $8.5 \cdot 10^{-2}$ | $2.1 \cdot 10^{-1}$ | $5.0 \cdot 10^{-1}$ | 1   |       |     |
| 5  | $1.4 \cdot 10^{-2}$ | $3.4 \cdot 10^{-2}$ | $8.5 \cdot 10^{-2}$ | $2.1 \cdot 10^{-1}$ | $5.0 \cdot 10^{-1}$ | 1   |
| 10 | $1.5 \cdot 10^{-4}$ | $3.8 \cdot 10^{-4}$ | $9.3 \cdot 10^{-4}$ | $2.3 \cdot 10^{-3}$ | $5.6 \cdot 10^{-3}$ | $1.4 \cdot 10^{-2}$ |

Table 5.

| n  | $x_0$ | $x_1$       | $x_2$       | $x_3$      | $x_4$     |
|----|-------|-------------|-------------|------------|-----------|
| 3  | 1.0   | $-1.22950819$ | $0.24590163$ | $-0.01639344$ |          |
| 5  | 1.0   | $-1.23364541$ | $0.25355792$ | $-0.02078343$ | $0.00089071$ |
| 10 | 1.0   | $-1.23370054$ | $0.25366950$ | $-0.02086347$ | $0.00091925$ |
| (4.10) | 1.0 | $-1.23370055$ | $0.25366950$ | $-0.02086348$ | $0.00091926$ |

Table 5 presents results of numerical calculations of infinite periodic homogeneous system (4.3) for $n = 3$, $n = 5$, $n = 10$ by formula (3.9) and its exact solution (4.10). Table 5 also shows the rapid convergence of formula (3.9).

Table 6 gives a values of characteristic determinants $A_{n-j}(j)$ for systems under consideration.

Table 6.

| j  | 0    | 1    | 2    | 3    | 4    | 5    |
|----|------|------|------|------|------|------|
| 3  | $6.1 \cdot 10^4$ | $7.5 \cdot 10^4$ | $1.5 \cdot 10^4$ | 1    |       |     |
| 5  | $5.1 \cdot 10^4$ | $6.2 \cdot 10^4$ | $1.3 \cdot 10^4$ | $1.1 \cdot 10^4$ | $4.5 \cdot 10^4$ | 1    |
| 10 | $3.7 \cdot 10^{14}$ | $4.6 \cdot 10^{14}$ | $9.4 \cdot 10^{13}$ | $7.7 \cdot 10^{12}$ | $3.4 \cdot 10^{11}$ | $9.3 \cdot 10^9$ |

As Table 6 shows, a characteristic determinants of $A_{n-j}(j)$ increase without bound with increasing $n$, nevertheless their corresponding relations (3.9) give solutions of the system (4.3), as Table 5 indicates.

5. Quasihomogeneous infinite systems

Apparently, for the first time the prefix "quasi" was introduced with respect to regular infinite systems [3]. The system is said to be quasiregular if the regularity condition is satisfied starting with some $j = N+1$, that is, for a finite number $j = 1, 2, ..., N$ of system equations the regularity condition is not satisfied. As shown in [1], the question about existence of a solution of such system reduces to the question about existence of a solution of finite system, even if the regular part of an infinite system has a unique bounded solution.

It should be noted that there are as many "quasi" systems as there are special systems, for example, the concept of quasiperiodic systems [1, 2].

We consider a quasihomogeneous systems as an infinite systems in which all equations are homogeneous for $j \geq N+1$, and for the rest $j = 1, 2, ..., N$ equations are inhomogeneous.
Quasihomogeneous infinite systems can be referred to special infinite systems for two reasons. First, they, like all quasi systems, ultimately reduce to solving a finite systems, thereby they acquire property of finite systems. Secondly, the most important, a methods of solving an inhomogeneous and homogeneous systems are fundamentally different. Therefore, the solving a quasihomogeneous systems as inhomogeneous systems may not always be justified.

For simplicity, we investigate a quasihomogeneous system with only one inhomogeneous equation. The application of such systems in the theory of parametric radio circuits is reflected, for example, in [11, 25].

**Example 3.** First we consider the following infinite system given in [3] as an example of an approximate solution of a regular system:

\[ \sum_{i=1}^{\infty} \frac{x_i}{(2j + 1 - 2i)(2j - 1 - 2i)} = b_j, \quad j = 1, 2, \ldots, \]  

(5.1)

where \( b_1 = -1, b_j = 0, j = 2, 3, \ldots \).

In this paper, following constraints are given for unknowns:

\[ 1,211 \leq x_1 \leq 1,331; \quad 0,538 \leq x_2 \leq 0,726; \quad 0,346 \leq x_3 \leq 0,592; \]
\[ 0,231 \leq x_4 \leq 0,534; \quad 0,135 \leq x_5 \leq 0,506; \quad 0 \leq x_i \leq 0,492 \quad (i = 6, 7, \ldots). \]

We solve the infinite system (5.1) with the use of a simple reduction by using formulas (3.5), (3.6) and (3.5'), we call this approach the first method for solving the quasihomogeneous system (5.1). The results of calculations are given in Table 7, which shows that quantities \( \lim_{n \to \infty} x_j \) converge to some limit, that is, a simple reduction converges.

**Table 7.**

| \( \varepsilon \) | \( x_1 \)  | \( x_2 \)  | \( x_3 \)  | \( x_4 \)  | \( x_5 \)  | \( x_{10} \) |
|-------------------|-----------|-----------|-----------|-----------|-----------|-----------|
| \( 10^{-5} \)     | 1.27146   | 0.63445   | 0.47503   | 0.39526   | 0.34536   | 0.23281   |
| \( 10^{-6} \)     | 1.27268   | 0.63593   | 0.47669   | 0.39705   | 0.34727   | 0.23509   |
| \( 10^{-7} \)     | 1.27306   | 0.63640   | 0.47722   | 0.39762   | 0.34787   | 0.23581   |

Table 8 shows residuals of the first 5 equations. Hence it is clear that the residual remains stable with increasing number of equations of the system, that is, it does not decrease.

**Table 8.**

| \( n \)  | 1           | 2         | 3           | 4           | 5           |
|----------|-------------|-----------|-------------|-------------|-------------|
| \( 10^{-5} \) | \( -3.4 \cdot 10^{-3} \) | \( -4.2 \cdot 10^{-3} \) | \( -5.0 \cdot 10^{-3} \) | \( -6.1 \cdot 10^{-3} \) | \( -7.4 \cdot 10^{-3} \) |
| \( 10^{-6} \) | \( -3.8 \cdot 10^{-3} \) | \( -4.5 \cdot 10^{-3} \) | \( -5.3 \cdot 10^{-3} \) | \( -6.3 \cdot 10^{-3} \) | \( -7.6 \cdot 10^{-3} \) |
| \( 10^{-7} \) | \( -4.0 \cdot 10^{-3} \) | \( -4.6 \cdot 10^{-3} \) | \( -5.3 \cdot 10^{-3} \) | \( -6.3 \cdot 10^{-3} \) | \( -7.7 \cdot 10^{-3} \) |

If \( \varepsilon = 10^{-5} \), then the maximum number of equations \( \max(j + p) \) is equal to 338, if \( \varepsilon = 10^{-6} \) then \( \max(j + p) = 1062 \), and if \( \varepsilon = 10^{-7} \) then \( \max(j + p) = 3352 \).

The slow convergence of series in (3.5) is probably due to the fact that the infinite system (5.1) is in fact more homogeneous system than inhomogeneous system, since only the one equation
is inhomogeneous. But as already pointed out above, methods for solving inhomogeneous and homogeneous systems are fundamentally different, therefore such systems can be referred to special types of infinite systems [13]. The first method for solving system (5.1) assumes that it is "purely" inhomogeneous. Therefore, we introduce the second method of solving the system (5.1), that is, we will take into account its homogeneous part. In accordance with this, first we solve the homogeneous part of the system (5.1)

$$\sum_{i=1}^{\infty} \frac{x_i}{(2j+1-2i)(2j-1-2i)} = 0, \quad j = 2, 3, \ldots$$  (5.2)

We solve the homogeneous system (5.2) with the use of reduction in broad sense by formula (3.9) using expression (3.6). The results of calculations are given in Table 9.

Table 9.

| n   | x_1  | x_2  | x_3  | x_4  | x_5  | x_10 |
|-----|------|------|------|------|------|------|
| 338 | 1.00000 | 0.49926 | 0.37389 | 0.31111 | 0.27181 | 0.18298 |
| 1062 | 1.00000 | 0.49976 | 0.37465 | 0.31206 | 0.27292 | 0.18468 |
| 3352 | 1.00000 | 0.49993 | 0.37489 | 0.31236 | 0.27327 | 0.18522 |

We substitute solutions $x_i$ ($i = 1, 2, \ldots$) of the homogeneous system (5.2) into the first equation of the system and obtain a value $C$ of series on the left-hand side of the first equation. Then we divide these solutions $x_i$ ($i = 1, 2, \ldots$) by $C$. The latter values will be the solution of the original system (5.1) and they are given in Table 10.

Table 10.

| n   | x_1  | x_2  | x_3  | x_4  | x_5  | x_10 |
|-----|------|------|------|------|------|------|
| 338 | 1.27230 | 0.63521 | 0.47570 | 0.39582 | 0.34583 | 0.23281 |
| 1062 | 1.27294 | 0.63617 | 0.47690 | 0.39723 | 0.34741 | 0.23509 |
| 3352 | 1.27314 | 0.63648 | 0.47729 | 0.39768 | 0.34792 | 0.23581 |

Since the homogeneous system (5.2) is more accurately solved, as shown in previous examples, and the first equation of system (5.1) is practically exactly solved, then the Table 10 should reflect the most exact solution of the system (5.1). But in this case both methods practically coincide. In the Table 11, we compare results obtained by both methods for $\max(j + p) = 3352$. From this, it can be seen that both methods are almost identical.

Table 11.

|    | x_1    | x_2    | x_3    | x_4    | x_5    | x_10   |
|----|--------|--------|--------|--------|--------|--------|
| 1- | 1.27306 | 0.63640 | 0.47722 | 0.39762 | 0.34787 | 0.23581 |
| 2- | 1.27314 | 0.63648 | 0.47729 | 0.39768 | 0.34792 | 0.23581 |
Thus, the quasihomogeneous system (5.1) behaves as a "purely" inhomogeneous system. How is this explained? If we compare the last columns of Table 7 and Table 10, we see that they completely coincide. This indicates that the solution of the system (5.1) obtained by the first method from the tenth unknown coincides with the solution of the corresponding homogeneous system.

Although, as Table 11 shows, boundaries of unknowns are qualitatively correctly reflected by the limit method, quantitatively they turned out to be very rough, especially, large error is observed with an increase of number of unknowns. Apparently, such a significant error is due to the fact that the method does not take into account the specifics of the system (5.1), or, more precisely, does not take into account the singularity of the solution of homogeneous part of the system (5.1). In fact, the solution of quasihomogeneous system (5.1), as indicated above, very quickly coincides with the nontrivial solution of the homogeneous part.

Now we consider an example of a quasihomogeneous system when its homogeneous part has only a trivial solution. To do this, we take as a basis the system given in our paper [13], as an example of system having a unique solution.

**Example 4.** Let an infinite system (2.1) be given with the following coefficients and free terms

\[
a_{j,i} = \begin{cases} 
  i + 1 & j \geq i \\
  j + 1 & i < j 
\end{cases}, \quad b_j = \sum_{k=0}^{j} \frac{b^k}{(1-b)^2}, \quad b \neq 1, \quad j, i = 0, 1, 2, \ldots,
\]

which in expanded form will have the form

\[
\begin{align*}
x_0 + x_1 + x_2 + \cdots + x_n &= b_0, \\
x_0 + 2x_1 + 2x_2 + \cdots + 2x_n &= b_1, \\
x_0 + 2x_1 + 3x_2 + \cdots + 3x_n &= b_2, \\
\vdots \\
x_0 + 2x_1 + 3x_2 + \cdots + nx_n &= b_n, \\
\vdots \\
\end{align*}
\]

In the homogeneous case, system (5.4) has only a trivial solution, as shown in [13]. From (5.4) we form a quasihomogeneous system: we leave the first equation and assume others to be homogeneous:

\[
\begin{align*}
x_0 + x_1 + x_2 + \cdots + x_n &= \frac{1}{(1-b)^2}, \\
x_0 + 2x_1 + 2x_2 + \cdots + 2x_n &= 0, \\
x_0 + 2x_1 + 3x_2 + \cdots + 3x_n &= 0, \\
\vdots \\
x_0 + 2x_1 + 3x_2 + \cdots + nx_n &= 0, \\
\vdots \\
\end{align*}
\]

Setting \(b = 0.9\) and \(\varepsilon = 10^{-10}\), we compute the strictly particular solution of the system (5.5) by formulas (3.5), (3.6) and (3.5').

Numerical calculations show that numbers \(x_0 = 200, \quad x_1 = -100, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = 0, \ldots\) are the desired solution. Indeed, by a direct substitution of these numbers into the system (5.5) we see that it is satisfied.

The Gaussian form of system (5.5) in homogeneous case has the next form [13]

\[
\sum_{p=0}^{\infty} x_{j+p} = 0 \quad j = 0, 1, 2, \ldots
\]

Using the periodicity of the system (5.6), it is proved that the system (5.6) has only a trivial solution [13]. Here we prove this assertion without applying the periodicity of system (5.6), that is, we shall prove from general positions.

12
Therefore, we use formula (3.9) to determine nontrivial solutions of the system (5.6). To do this, using the relation (3.6), we calculate quantities $A_p(j)$: $A_1(j) = a_{j,j}A_0(j) = 1$, $A_2(j) = -A_0(j) + A_1(j) = 0$, then $A_p(j) = 0$ for all $p \geq 2$.

Hence it follows that quantities $A_{n-j}(j) = 0$, $A_n(0) = 0$ for $n \to \infty$. Then formula (3.9) does not give anything, which means that system (5.6) has only a trivial solution. One important fact should be noted. The homogeneous part of system (5.6) is satisfied due to the linear dependence of all homogeneous equations in the case $x_i = 0$ for all $i \geq 2$ and $x_i \neq 0$ for $x_i < 2$. This is easily seen from the system (5.5), that is, we have $x_0 + 2x_1 = 0$, solving it with the first equation $x_0 + x_1 = 100$ we get what we got.

**Example 5.** Investigate a quasihomogeneous system, the basis of which is the Gaussian system (4.1).

Multiplying the Gaussian matrix of system (4.1) and its column of free terms by a triangular matrix with unity on the main diagonal, and the remaining elements below the diagonal are considered in detail in Example 1 and it is shown that this homogeneous system has a nontrivial solution (see Table 3 and Table 5).

To do this, we multiply the matrix and the column of free terms of Gaussian system (4.1) by a matrix with unity on the main diagonal, and the remaining elements below the diagonal are assumed to be random numbers from 1 to 5.

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 5 & 1 & 0 & 0 & 0 \\
2 & 2 & 4 & 1 & 0 & 0 \\
2 & 1 & 2 & 3 & 1 & 0 \\
1 & 3 & 1 & 5 & 1 & 1 \\
\end{pmatrix}
$$

then we obtain a system with some right-hand side.
Hence we form the following quasihomogeneous system:

\[ \begin{align*}
    x_0 + x_1 + x_2 + x_3 + x_4 + \cdot &= 1, \\
    x_0 + 3x_1 + 13x_2 + 31x_3 + 57x_4 + \cdot &= 0, \\
    x_0 + 11x_1 + 85x_2 + 511x_3 + 1961x_4 + \cdot &= 0, \\
    2x_0 + 6x_1 + 122x_2 + 2222x_3 + 26994x_4 + \cdot &= 0, \\
    2x_0 + 4x_1 + 62x_2 + 2912x_3 + 104218x_4 + \cdot &= 0, \\
    x_0 + 7x_1 + 61x_2 + 4051x_3 + 142969x_4 + \cdot &= 0, \\
\end{align*} \]  
(5.9)

We solve the system (5.9) using the first method, that is, we find numerical values of the strictly particular solution of system (5.9) for \( b = 0.5 \) and \( \varepsilon = 10^{-8} \). To determine the first 10 solutions, a system with \( \max(j + p) = 13 \) was sufficient.

**Table 12.**

| \( x_0 \)   | \( x_1 \)   | \( x_2 \)   | \( x_3 \)   | \( x_4 \)   | \( x_5 \)   |
|------------|-------------|-------------|-------------|-------------|-------------|
| 1.24445    | -0.15035    | -0.121292   | 0.02959     | -0.00249    | 0.00009     |

Now we solve the system (5.9) using the second method, that is, we solve the homogeneous system

\[ \begin{align*}
    x_0 + 3x_1 + 13x_2 + 31x_3 + 57x_4 + \cdot &= 0, \\
    x_0 + 11x_1 + 85x_2 + 511x_3 + 1961x_4 + \cdot &= 0, \\
    2x_0 + 6x_1 + 122x_2 + 2222x_3 + 26994x_4 + \cdot &= 0, \\
    2x_0 + 4x_1 + 62x_2 + 2912x_3 + 104218x_4 + \cdot &= 0, \\
    x_0 + 7x_1 + 61x_2 + 4051x_3 + 142969x_4 + \cdot &= 0, \\
\end{align*} \]  
(5.10)

**Table 13.**

| \( x_0 \)   | \( x_1 \)   | \( x_2 \)   | \( x_3 \)   | \( x_4 \)   | \( x_5 \)   |
|------------|-------------|-------------|-------------|-------------|-------------|
| 1          | -0.72107    | 0.09170     | -0.03017    | 0.00133     | -0.00004    |

We substitute the solution of system (5.10) into the first equation of system (5.9) and obtain \( C = 0.34175 \). Then the solution of system (5.9) will be the solution of system (5.10) divided by \( C = 0.34175 \).

**Table 14.**

| \( x_0 \)   | \( x_1 \)   | \( x_2 \)   | \( x_3 \)   | \( x_4 \)   | \( x_5 \)   |
|------------|-------------|-------------|-------------|-------------|-------------|
| 2.92611    | -2.10995    | 0.26832     | -0.08828    | 0.00389     | -0.00011    |

As can be seen from Table 12 and Table 14, solutions differ significantly, and the solution in the second case is close to the solution of homogeneous system. As these tables show, both solutions quickly decrease to zero. Thus, the choice of solution depends strongly on the initial physical problem.
Acknowledgments
The research has been supported by the Ministry of Education and Science of the Russian Federation (Grant No. 1.6069.2017/8.9).

References
[1] Fedorov F M 2009 Periodic infinite systems of linear algebraic equations (Novosibirsk: Nauka) [in Russian]
[2] Fedorov F M 2011 Infinite systems of linear algebraic equations and their applications (Novosibirsk: Nauka) [in Russian]
[3] Kantorovich L V and Krylov V I 1958 Approximate Methods of Higher Analysis (Groningen: P. Nordhoff) [in Russian]
[4] Feld Ya N 1955 Transactions (Doklady) of the USSR Academy of Sciences 102 257
[5] Masalov S A 1981 USSR Computational Mathematics and Mathematical Physics 21 81
[6] Shestopalov V P, Kirilenko A A and Masalov S A 1984 Matrix Convolution Equations (Kiev: Naukova dumka) [in Russian]
[7] Shestopalov V P, Litvinenko L N, Masalov S A, Sologub V G 1973 Diffraction of waves in lattices (Kharkov: Publishing house of Kharkov University) [in Russian]
[8] Fedorov F M 2000 Boundary method for solving applied problems of mathematical physics (Novosibirsk: Nauka) [in Russian]
[9] Taft V A 1964 Fundamentals of spectral theory and calculation of chains with variable parameters (Moscow: Nauka) [in Russian]
[10] Mittra R and Lee S 1974 Analytical Methods of the Theory of Waveguides (Moscow: Mir) [in Russian]
[11] Birjuk N D, Gorbatenko V V, Gorbatenko S A and Pozdnyakov M V 2000 Electricity 5 55 [in Russian]
[12] Papernov E L 1978 USSR Computational Mathematics and Mathematical Physics 18 219
[13] Ivanova O F, Pavlov N N, Fedorov F M 2016 Computational Mathematics and Mathematical Physics 56 343
[14] Fedorov F M 2017 AIP Conference Proceedings 1907 030006
[15] Kagan V F 1922 Foundations of the Theory of Determinants (Kiev: Gos. Izd. Ukrainy) [in Russian]
[16] Cooke R G 1950 Infinite Matrices and Sequence Spaces Macmillan (London: Ltd)
[17] Fedorov F M 2012 Mat. Zamet. YAGU 19 133 [in Russian]
[18] Koch H 1910 Compte rendu of Scandinavian Congress of Mathematicians in Stockholm 1909 (Leipzig, Teubner)
[19] Finta B 2006 Petru Maior
[20] Fedorov F M 2015 J Generalized Lie Theory Appl 9 224
[21] Fedorov F M 2015 TWMS J. Pure Appl. Math. 6 202
[22] Fedorov F M, Ivanova O F and Pavlov N N 2015 Mathematical Notes of North-Eastern Federal University 22 62 [in Russian]
[23] Riesz F 1913 Les systemes d’équation lineaires a une infinite d’inconnues (Paris: Gauthier-Villars)
[24] Fedorov F M and Osipova T L 2012 Mat. Zamet. YAGU 19 133 [in Russian]
[25] Byelozlazov V V, Birjuk N D and Yurgelas V V 2010 Proceedings of Voronezh State University. Series: Physics. Mathematics 2 175 [in Russian]