Collapsing Sub-Critical Bubbles

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Abstract

In the standard scenario, the electroweak phase transition is a first order phase transition which completes by the nucleation of critical bubbles. Recently, there has been speculation that the standard picture of the electroweak phase transition is incorrect. Instead, it has been proposed that throughout the phase transition appreciable amounts of both broken and unbroken phases of $SU(2)$ coexist in equilibrium. I argue that this can not be the case. General principles insure that the universe will remain in a homogenous state of unbroken $SU(2)$ until the onset of critical bubble production.

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1 Introduction

Spurred by the interest in electroweak baryogenesis, a great deal of effort has been undertaken to describe and quantify many salient aspects of the electroweak phase transition (EWPT) in the minimal standard model and its extensions. It has become standard lore that the EWPT is a first order phase transition which proceeds by the nucleation and the subsequent growth of critical bubbles\cite{1,2,3}. Although the phase transition is weakly first order in the minimal standard model, the strength of the phase transition is model dependent, and the EWPT is more strongly first order in very simple extensions of the minimal standard model\cite{2}.

Recently however, there has been speculation that the electroweak phase transition is actually not a first order phase transition after all\cite{4}. Instead, Gleiser and Kolb have suggested that during the phase transition the two phases of broken and unbroken $SU(2) \times U(1)$ coexist simultaneously with equilibrium between the two phases being established and maintained by sub-critical bubbles. Similar arguments have been advanced by Tetradis\cite{5}. In particular, Gleiser and Kolb and Tetradis argue that at the critical temperature $T_C$ the universe is filled with equal parts of the broken and unbroken phases. $T_C$ is defined as the temperature of the universe when the free energy density of the system, plotted as a function of $\langle \phi \rangle$ has two degenerate minima. The basic contention of these authors is that as long as the expansion rate is slow compared to the rates of thermal processes (Gleiser and Kolb consider processes mediated by sub-critical bubbles), the universe will be driven into a state equally populated by both phases. If true, this argument would have important ramifications for scenarios of baryogenesis which invoke first order phase transitions. In addition, one might wonder if the standard picture of the universe trapped in a homogeneous state when the temperature reaches $T_C$ is an assumption which has should be checked case by case or if there are general dynamical and statistical effects which guarantee this. Below I will argue that the basic the picture advanced by Gleiser and Kolb is in contradiction with the second law of thermodynamics. Other criticisms of sub-critical bubbles have also been made\cite{3}. Similar remarks would apply to the analysis of Tetradis and previous studies of subcritical bubbles\cite{6}. Instead, very general properties of statistical mechanics guarantee that the equilibrium state at temperature $T_C$ is a homogeneous state.

2 Thermal Equilibrium at the Phase Transition

Gleiser and Kolb and Tetradis argue that at the critical temperature $T_C$, the universe is filled with equal parts of the broken and unbroken phases. The assertion that both wells are equally populated would be true for an ensemble of particles interacting with an external potential. However, metastability in one-dimensional mechanics is very different from metastability in a field theory. The disparate nature of these two cases is qualitative as well as quantitative. It is instructive to contrast these two cases
to see where some types of intuition can lead us astray. Let’s compare the symmetric double well in field theory and in one dimensional mechanics. First consider the one dimensional mechanical example at fixed temperature. For an ensemble of particles interacting with the external potential given in figure 1a, the thermal equilibrium state of the system is one where both wells are equally populated with particles. This situation is in sharp contrast to the case in quantum field theory. Under conditions present at the EWPT, the thermodynamic requirement that the total entropy of the universe can only increase is equivalent to demanding that the free energy only decreases. Thus, the equilibrium state of the system minimizes the free energy. Recall that the free energy of the system is:

\[ F = \int d^3x \frac{1}{2}(\nabla \phi)^2 + V(\phi, T). \] (1)

For convenience we can normalize the free energy density so that \( V(\phi_+) = V(\phi_0) = 0 \). A universe held at a temperature \( T_C \) and left to equilibrate will end up either in the homogeneous ground state \( \langle \phi \rangle = \phi_+ \) or \( \langle \phi \rangle = \phi_0 = 0 \). This is because both the gradient term and \( V(\phi, T) \) are positive and nonvanishing inside any boundary separating the two phases. So a universe filled with domains of both phases must have a larger free energy than either homogeneous state. According to the second law of thermodynamics, while an individual fluctuation can occasionally increase the systems free energy, the cumulative statistical effect of these fluctuations must decreases the free energy. Moreover, because the universe is cooling, there is no question which state the universe occupies when the temperature reaches \( T_C \). The state of lowest free energy at temperatures above \( T_C \) is \( \langle \phi \rangle = \phi_0 \), and when the temperature reaches \( T_C \) the newly degenerate, homogeneous vacua at \( \langle \phi \rangle = \phi_+ \) is separated from the state \( \langle \phi \rangle = \phi_0 \) by an infinite barrier. So when the temperature drops to \( T_C \) the universe finds itself in the homogeneous vacuum \( \langle \phi \rangle = \phi_0 \). This is in direct conflict with the analyses of Gleiser and Kolb and Tetradis.

Although it is clear from these general grounds that the universe is filled with the homogeneous state \( \langle \phi \rangle = 0 \) when the temperature cools to \( T_C \), it is useful to see the how this arises from the dynamical equations governing the evolution of the scalar field. Before discussing the statistical evolution of the scalar field it will be useful to recall a few basic properties of nucleated bubbles. Consider a bubble containing \( \langle \phi \rangle = \phi_+ \) in a sea of vacuum \( \langle \phi \rangle = 0 \) (see figure 2). By convention we will choose the

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\[ \text{At temperatures above } T_C, \text{ the metastable state which first appears at } \langle \phi \rangle = \phi_+ \text{ is separated from the homogeneous state } \langle \phi \rangle = \phi_0 \text{ by an infinite barrier. Any finite region of space containing the new phase is not a meta-stable state since it can be continuously deformed to the ground state without surmounting an energy barrier. At temperatures equal to and above } T_C \text{ finite regions of } \langle \phi \rangle = \phi_+ \text{ are completely unstable. Only when the temperature drops below } T_C, \text{ the can system can be continuously deformed from the state } \langle \phi \rangle = \phi_0 \text{ to the new equilibrium state } \langle \phi \rangle = \phi_+ \text{ by crossing a finite barrier (see figure 2). The height of this free energy barrier is the critical bubble free energy.} \]
state $\langle \phi \rangle = 0$ to have zero free energy. Then the surplus free energy of a nucleated bubble is
\[ F = \int d^3x \left\{ \frac{1}{2} \left( \vec{\nabla} \phi \right)^2 + V(\phi, T) \right\}. \tag{2} \]
The free energy of this bubble has two contributions: a surface free energy $F_S$, coming mostly from the derivative terms in Eq. (2.2), and a volume term $F_V$, which arises from the difference in free energy density inside and outside the bubble. These two contributions scale like
\[ F \sim F_S + F_V \sim 2\pi R^2 \left( \frac{\delta \phi}{\delta R} \right)^2 \delta R + \frac{4\pi}{3} R^3 \tilde{V}(\phi_+), \tag{3} \]
where $R$ is the radius of the bubble, $\delta R$ is the thickness of the bubble wall, $\delta \phi \sim \phi_+$, and $\tilde{V}(\phi_+)$ is the average value of the potential inside the bubble. For the bubbles we are interested in, it is energetically favorable to make the gradient term as small as possible, so the bubbles will be thick walled. For thick walled bubbles $\delta R \sim R$, and the surface energy of the bubble grows like $R$. In contrast, the volume term increases in magnitude like $R^3$. For temperatures below $T_C$ the volume term in equation Eq. (2.3) can be negative (See figure 2). At this temperature, although the homogeneous state $\langle \phi \rangle = \phi_+$ has a lower free energy, a thermal fluctuation producing a bubble of true vacuum which starts from a radius of zero and expands in radius to envelope the system, must have a free energy greater than or equal to some critical value. The radius of the critical bubble is found by differentiating Eq. (2.3), $R_c \sim \phi_+ / \sqrt{-2\tilde{V}(\phi_+)}$. Sub-critical bubbles, those bubbles with radii smaller than this critical size, will collapse under their surface tension. The free energy of a critical bubble is:
\[ F_c \sim \frac{\phi_+^3}{\sqrt{-\tilde{V}(\phi_+)}}. \tag{4} \]
Notice as the temperature approaches the critical temperature from below $\tilde{V} \to 0$, and both the radius and free energy of the critical bubble become infinite. So at $T \geq T_C$, all bubbles are subcritical.

Aided by this qualitative understanding of nucleated bubbles we can examine the dynamical equations describing the abundance of regions containing the $SU(2)$ broken phase. In a hot universe in the vacuum state $\langle \phi \rangle = \phi_0 = 0$, thermal fluctuations will produce bubbles inside of which $\langle \phi \rangle$ is nonvanishing. At temperatures below $T_C$, these thermal fluctuations produce critical bubbles at a rate per unit volume:
\[ \frac{\Gamma}{V} \approx T^4 e^{-\beta F_c}. \tag{5} \]
The exponential suppression $\exp(-\beta F_c)$ is the usual Boltzmann suppression for producing configurations close to the critical bubble, while the prefactor gives the rate of
typical processes which are not Boltzmann suppressed. The suppression in the rate for thermal vacuum change events arises because the system must cross a barrier in order to produce bubbles large enough to grow.

Consider the rate at which regions of $\langle \phi \rangle \sim \phi_+$ are populated at temperatures near $T_c$. Let $f$ denote the fraction of space filled regions of $\langle \phi \rangle \sim \phi_+$. The master equation for the evolution of $f$ is:

$$\frac{df}{dt} = (1 - f) \Gamma(\phi_0 \rightarrow \phi_+) - f \Gamma(\phi_+ \rightarrow \phi_0)$$  \hspace{1cm} (6)

Although the rate given in Eq. (2.5) has strictly only been motivated for critical bubble production, it is not unreasonable to assume that its generalization gives a good estimate of the rate other configurations are produced. Any fluctuation producing a region of $\langle \phi \rangle \sim \phi_+$, will be Boltzmann suppressed because energy is required to form the domain boundaries. So the regions of $\langle \phi \rangle \sim \phi_0$ with spatial extent $R$ will be converted to regions of $\langle \phi \rangle \sim \phi_+$ at a rate:

$$\Gamma(\phi_0 \rightarrow \phi_+) \simeq T(RT)^3 e^{-\beta F(R)},$$  \hspace{1cm} (7)

where $F(R)$ is the free energy of a subcritical bubble of radius $R$. From Eq. (2.3), $F \gtrsim 2\pi \phi_+^2 R$. Fluctuations which create energetically disfavored structures can also remove them. Even in a universe filled equally with domains of both phases, there will be fluctuations which decrease the abundance of domain walls and take the universe toward a homogeneous state. Fluctuations of this sort, which decrease the volume occupied by domain boundaries, do not cost energy so their rates are not Boltzmann suppressed. If anything they should be enhanced relative to rates which leave $\langle \phi \rangle$ unchanged. Regions of $\langle \phi \rangle \sim \phi_+$ with spatial extent $R$ are depleted by both fluctuations, and the dynamical collapse resulting from the region’s surface tension:

$$\Gamma(\phi_+ \rightarrow \phi_0) \gtrsim T(RT)^3 + 1/\tau.$$  \hspace{1cm} (8)

A simple estimate of the collapse time gives $\tau \sim R$. Although the self induced collapse is typically faster than the rate of bubble production, fluctuations are even more effective at removing regions of the unstable phase. In steady state, detailed balance requires that the fraction of space containing bubbles of broken phase is exponentially suppressed. From Eqs. (2.6) - (2.8),

$$f \leq \frac{e^{-F/T}}{1 + e^{-F/T}}.$$  \hspace{1cm} (9)

where $F$ is the free energy of a sub-critical bubble including the bubble walls. Since we are interested in bubbles which change the value of the scalar field condensate we can set a lower limit on the magnitude of a bubbles free energy. In order to produce a classical shift in the scalar field condensate, a bubble of scalar field must contain many quanta[7]. Since the wavelength of a typical quantum comprising a bubble is
order $R$, with $F \sim n_q/R$ and $n_q >> 1$ we must have $(F/T)(RT) >> 1$, where $n_q$ is the number of quanta. Using Eq. (2.3),

$$F/T \gtrsim 2\pi \left(\frac{\phi_+}{T}\right)^2 (RT) >> 1/RT.$$ (10)

In the standard model, the ratio $F/T$ satisfying Eq. (2.10) is not small (See Figure 3). Thus, until the onset of critical bubble nucleation, the universe finds itself in a homogeneous state with an exponentially suppressed number of ephemeral regions containing the $SU(2)$ broken phase.

Although Eq. (2.9) demonstrates that the unbroken phase is always favored before the phase transition occurs, one might wonder if there are models where Eq. (2.9) allows for a departure from the standard formalism of false vacuum decay. In models where $\phi_+/T << 1$ it is possible to have $FR >> 1$ with $F/T << 1$. Indeed, we know in the limit $\phi_+ \to 0$ we must have $f \to \frac{1}{2}$. However, even in the extreme case $\phi_+/T << 1$ the standard formalism of first order phase transitions should remain valid. The new ground state $\langle \phi \rangle = \phi_+$ will not dominate until fluctuations can produce regions large enough to grow, and this will not happen until the temperature drops below $T_C$. Whether such a region is produced all at once or by the coalescence of smaller regions, the rate for producing the saddle point solution is given by Eq. (2.5). The first order phase transition will occur at a temperature where fluctuations produce regions of the new phase large enough to grow at a rate which exceeds the expansion rate of the universe.

\[\text{‡} \text{ In simple extension of the standard model, virtual effects of additional particles can make the phase transition proceed as it would if the Higgs Boson mass was significantly smaller.} \]

\[\text{§} \text{ When calculating the thermodynamic probability of producing a critical bubble by a saddle point evaluation of the partition function no choice is made to include some histories at the expense of others.} \]
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Figure 1: The symmetric double well in a mechanical example and in field theory.

Figure 2: The effective potential at temperatures near $T_C$.

Figure 3: $\phi_+$ at the end of the phase transition verses the Higgs boson mass for $m_t = 120$. The dashed curve is the thin wall approximation, while the solid curve is the numerical result. The upper and lower dotted curves correspond to the values of $\phi_+/T$ at temperatures $T_2$ and $T_C$. 