INDUCTION AND RESTRICTION OF TWO VARIABLE HECKE ALGEBRAS

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Abstract. The purpose of this paper is to study induction and restriction of two-variable Hecke algebras. First, we provide the explicit form of the Mackey decomposition formula. And then, we elucidate how (anti-)involutions interact with induction product and restriction.

1. Introduction

Two-variable Hecke algebras are introduced as specializations of Iwahori-Hecke algebras (see [11, Section 4.4] or [12, Section 7]). Let $R$ be a unital commutative ring and $(W, S)$ a finite Coxeter system. Suppose we are given parameters $a_s, b_s \in R$ ($s \in S$) subject only to the requirement that $a_s = a_t$ and $b_s = b_t$ whenever $s$ and $t$ are conjugate in $W$. Following Geck and Pfeiffer [11, Definition 4.4.1], one can associate to $(W, S)$ the Iwahori-Hecke algebra $H_R(W, S, \{a_s, b_s : s \in S\})$ over $R$. For $a, b \in R$, the two variable Hecke algebra $H_W(a, b)$ over $R$ is defined to be the generic algebra $H_R(W, S, \{a_s, b_s : s \in S\})$ with the parameters $a_s = a, b_s = b$ for all $s \in S$. When $R$ is the field of complex numbers, it was proven in [1, Proposition 6.1] that $H_W(a, b)$ is isomorphic to one of the following families of algebras:

- Hecke algebras at a generic value,
- Hecke algebras at a root of unity,
- 0-Hecke algebras,
- nil-Coxeter algebras.

This result was originally stated only for finite Coxeter groups of type $A$, but one can extend it to arbitrary finite type in the exactly same way as in [1].

The first objective of this paper is to provide the Mackey decomposition formula of two variable Hecke algebras. Let us explain the motivation of this problem. In 2009, Bergeron and Li [2, Section 3.1] presented an axiomatic definition of a tower of algebras, which is a graded $\mathbb{C}$-algebra $A = (\oplus_{n \geq 0} A_n, \rho)$ satisfying certain five axioms. Then they proved that $G_0(A) := \oplus_{n \geq 0} G_0(A_n)$ and $K_0(A) := \oplus_{n \geq 0} K_0(A_n)$ have a bialgebra structure and are

\begin{itemize}
  \item Hecke algebras at a generic value,
  \item Hecke algebras at a root of unity,
  \item 0-Hecke algebras,
  \item nil-Coxeter algebras.
\end{itemize}

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dual to each other as connected graded bialgebras, where $G_0(A_n)$ (resp. $K_0(A_n)$) is the Grothendieck group of the category of all finitely generated $A_n$-modules (resp. projective $A_n$-modules). In their proof, the fifth axiom, which is an analogue of Mackey’s formula for $G_0(A)$ and $K_0(A)$, plays an important role. At this point, one might naturally ask if Bergeron and Li’s analogue of Mackey’s formula can be lifted to the module level in the same format, which is nontrivial unless the category of finitely generated $A_n$-modules is not semisimple for all $n \geq 0$. For instance, the category of finitely generated $H_n(0)$-modules is known to be not semisimple for $n \geq 3$ (see [8, Section 4.3] or [7, Theorem 3.1]). In [11, Proposition 9.1.8], the Mackey decomposition formula was stated in the case where given parameters $a_s, b_s \in R$ satisfy that $b_s = 1 - a_s$ for all $s \in S$ and the $H_R(W, S, \{a_s, 1 - a_s : s \in S\}$)-module in consideration is free as an $R$-module.

In [13, Theorem 3], this formula was provided for the family of $0$-Hecke modules called weak Bruhat interval modules. In either case, the question posed above turns out to be affirmative. The present paper deals with this question for arbitrary modules of arbitrary two-variable Hecke algebras. As stated before, when $R = \mathbb{C}$, our algebras in consideration cover Hecke algebras at a generic value, Hecke algebras at a root of unity, $0$-Hecke algebras, and nil-Coxeter algebras. Our Mackey decomposition formula is stated for parabolic subalgebras of $H_W(a, b)$ and its explicit form is provided (Theorem 3.4 and Corollary 3.6).

The second objective of this paper is to show how (anti-)involutions interact with induction product and restriction. In [9, Proposition 3.2], Fayers introduced involutions $\phi, \theta$ and an anti-involution $\chi$ on the $0$-Hecke algebra $H_n(0)$ defined by

$$\phi(\pi_s) = \pi_{w_0sw_0}, \quad \theta(\pi_s) = 1 - \pi_s, \quad \chi(\pi_s) = \pi_s \quad (s \in S),$$

where $w_0$ is the longest element of the symmetric group $S_n$, and used them to derive a branching rule that describes the submodule lattice of $M \uparrow^{H_n(0)}_{H_{n-1}(0)}$ for any simple module $M$ of $H_{n-1}(0)$. In [13, Section 3.4], the (anti-)involution-twists of weak and negative weak Bruhat interval modules are intensively investigated for the (anti-)involutions obtained by composing $\phi, \theta$ and $\chi$. It is quite interesting to note that they exhibit very strong symmetry (see [13, Table 1 and Table 2]). Here, mimicking [9], we introduce two involutions and one anti-involution on $H_W(a, b)$ (Lemma 4.2). Then we show how $H_W(a, b)$-modules behave under the process of (anti-)involution-twist for all the (anti-)involutions obtained by composing these (anti-)involutions (Theorem 4.4 and Theorem 4.6). In particular, Theorem 4.6 can be viewed as a generalization of Fayers’ lemma [9, Lemma 6.4].

This paper is organized as follows. In Section 2, we introduce the prerequisites on the two variable Hecke algebras including induction, restriction, and (anti-)automorphism twists. In Section 3, we provide the Mackey decomposition formula for two variable Hecke algebras. Section 4 is devoted to elucidating the comparability of (anti-)automorphism twists with induction product and restriction.
2. Two variable Hecke algebras

Let \((W, S)\) be a finite Coxeter system with \(S = \{s_1, s_2, \ldots, s_n\}\). That is,

\[
W = \langle s_1, s_2, \ldots, s_n \mid (s_is_j)^{m_{ij}} \rangle
\]

for some symmetric \(n\) by \(n\) matrix \((m_{ij})\) with entries \(\{2, 3, 4, 6\}\) with \(m_{ii} = 1\) and \(m_{ij} > 1\) for \(i \neq j\). For \(I \subseteq S\), we write \(W_I\) for the subgroup of \(W\) generated by \(I\), which is also a Coxeter group and called a parabolic subgroup of \(W\).

Throughout this paper, \(R\) denotes a commutative ring with 1.

**Definition 2.1.** For \(a, b \in R\), the two variable Hecke algebra \(H_W(a, b)\) of \(W\) over \(R\) is an \(R\)-algebra with 1 generated by the elements \(\pi_s\) (\(s \in S\)) subjects to the following defining relations:

\[
\pi_s \pi_s = a\pi_s + b, \quad (\pi_i \pi_j \pi_i \ldots)^{m_{ij}} = (\pi_j \pi_i \pi_j \ldots)^{m_{ij}} \quad \text{for all } i, j \in S \text{ with } i \neq j,
\]

where the notation \((aba \cdots)_m\) denotes the product of \(m\) terms following the order in the parenthesis.

When it is not necessary to specify \(a\) and \(b\), we frequently write \(H_W\) for \(H_W(a, b)\) for simplicity. For any reduced expression \(s_{i_1}s_{i_2} \ldots s_{i_l} = w \in W\), let \(\pi_w := \pi_{i_1} \pi_{i_2} \ldots \pi_{i_l}\). It is well known that \(\pi_w\) is independent of the choices of reduced expressions (Matsumoto's theorem). Given \(I \subseteq S\), let \(W_I\) be the parabolic subgroup of \(W\) corresponding to \(I\). The two variable Hecke algebra \(H_{W_I}(a, b)\) associated to the Coxeter system \((W_I, I)\) is a subalgebra of \(H_W(a, b)\), which is called the parabolic subalgebra of \(H_W(a, b)\) corresponding to \(I\).

**Theorem 2.2.** ([12, Section 7.1]) For \(a, b \in R\), the two variable Hecke algebra \(H_W(a, b)\) over \(R\) is a free \(R\)-module with basis elements \(\pi_w\) \((w \in W)\), and for all \(s \in S, w \in W\), multiplication is given by

\[
\pi_s \pi_w = \begin{cases} 
\pi_{sw} & \text{if } \ell(sw) > \ell(w), \\
(a\pi_w + b\pi_{sw}) & \text{if } \ell(sw) < \ell(w).
\end{cases}
\]

For later use, we note that

\[
\pi_w \pi_s = \begin{cases} 
\pi_{ws} & \text{if } \ell(ws) > \ell(w), \\
(a\pi_w + b\pi_{ws}) & \text{if } \ell(ws) < \ell(w)
\end{cases}
\]

and \(\pi_w \pi_w = \pi_{ww}\) if \(\ell(v) + \ell(w) = \ell(vw)\) for \(v, w \in W\).

The two variable Hecke algebras were studied intensively in [1, Section 6] in the case where \(R = \mathbb{C}\) and \(W = \mathfrak{S}_n\). Replacing \(\mathfrak{S}_n\) with \(W\) in the proof of [1, Proposition 6.1] gives the following classification.

**Theorem 2.3.** (cf. [1, Proposition 6.1]) Let \(R = \mathbb{C}\) and \(a, b \in \mathbb{C}\). Then \(H_W(a, b)\) is isomorphic to one of the following four families of algebras:
a Hecke algebra $H_W(\nu)$ at a generic value of $\nu$, or
- a Hecke algebra $H_W(\xi)$ at a root $\xi$ of unity, or
- the 0-Hecke algebra $H_W(0)$ (when $a \neq 0$ but $b = 0$), or
- the nil-Coxeter algebra $N_W$ (when $a = b = 0$).

For an $R$-algebra $A$, let $\text{mod } (A)$ be the category of left $A$-modules. Let $M \in \text{mod } (H_W)$ for some $I \subseteq S$. Then the induction of $M$ to $H_W$ is defined by

$$M \uparrow_{H_{W_I}}^{H_W} := H_W \otimes_{H_I} M,$$

where we are viewing $H_W$ as an $(H_W, H_{W_I})$-bimodule. If $N \in \text{mod } (H_W)$, we write the corresponding $H_{W_I}$-module as $N \downarrow_{H_{W_I}}^{H_W}$ and call it the restriction of $N$ to $H_{W_I}$.

For an algebra homomorphism $f: A \rightarrow B$ and $M \in \text{mod } (B)$, the following defines a left $A$-module structure on $M$.

$$a \cdot f m := f(a) \cdot m$$

Thus, $f$ induces a functor $F: \text{mod } (B) \rightarrow \text{mod } (A)$. For example, the functor induced by the inclusion $\iota: H_{W_I} \rightarrow H_W$ is the restriction functor.

We close this section by introducing the notion of (anti-)automorphism twists. Let $\mu: B \rightarrow A$ be a morphism of associative algebras. Given a $A$-module $M$, we define $\mu[M]$ be a $B$-module with the same underlying vector space as $M$ and with the action $\cdot^\mu$ twisted by $\mu$ in such a way that

$$b^\mu v := \mu(b) \cdot v \quad \text{for } a \in A \text{ and } v \in M.$$

This induces a covariant functor

$$F_\mu: \text{mod } A \rightarrow \text{mod } B, \quad M \mapsto \mu[M],$$

where $F_\mu(h): \mu[M] \rightarrow \mu[N], m \mapsto h(m)$ for every $A$-module homomorphism $h: M \rightarrow N$. We call $F_\mu$ the $\mu$-twist. For example, the functor induced by the inclusion $\iota: H_{W_I} \rightarrow H_W$ is the restriction functor.

Similarly, given an anti-homomorphism $\nu: B \rightarrow A$ of associative algebras, we define $\nu[M]$ to be the $B$-module with $M^*$, the dual space of $M$, as the underlying space and with the action $\cdot^\nu$ defined by

$$(a^\nu \delta)(v) := \delta(\nu(a) \cdot v) \quad \text{for } a \in A, \delta \in M^*, \text{ and } v \in M.$$  

(2.2)

Any anti-automorphism $\nu$ of $A$ induces a contravariant functor

$$G_\nu: \text{mod } A \rightarrow \text{mod } B, \quad M \mapsto \nu[M],$$

where $G_\nu(h): \nu[N] \rightarrow \nu[M], \delta \mapsto \delta \circ h$ for every $A$-module homomorphism $h: M \rightarrow N$. We call $G_\nu$ the $\nu$-twist.
3. The Mackey decomposition formula for two variable Hecke algebras

Let \((W, S)\) be a finite Coxeter system. We denote by \(W^I\) (resp. \(W^J\)) the set of minimal length left (resp. right) coset representatives of \(W_I\) in \(W\).

For any \(w \in W\), it is well known that \(w\) can be uniquely factorized as \(xy\) for \(x \in W^I\) and \(y \in W_I\) (for instance, see [4, Proposition 2.4.4]). In this case, it holds that \(\ell(w) = \ell(x) + \ell(y)\). Let us call this factorization as the reduced factorization of \(w\) with respect to \(W/W_I\). The reduced factorization with respect to \(W_I\) can be defined in a similar manner.

For \(I, J \subseteq S\), let \((W_J, W_I)\)-double coset in \(W_J \backslash W/W_I\). It is well known that there exists a unique minimal length element \(\tau\) in \(C\). Let \(\mathcal{W}^{I, J}\) be the set of minimal length representatives of all the double cosets in \(W_I \backslash W/W_I\), which has the following nice characterization.

**Lemma 3.1.** ([5, Chapter 4]) For \(I, J \subseteq S\),

\[ \mathcal{W}^{I, J} = J \cap W = \{w \in W | J \subseteq \text{Asc}_L(w) \text{ and } I \subseteq \text{Asc}_R(w)\} \]

where \(\text{Asc}_L(w) := \{s \in S | \ell(sw) = \ell(w) + 1\}\) and \(\text{Asc}_R(w) := \{s \in S | \ell(ws) = \ell(w) + 1\}\).

For \(I, J \subseteq S\) and \(w \in W\), let

\[ K(w) := J \cap (wIw^{-1}), \]
\[ w^{-1}K(w)w := (w^{-1}Jw) \cap I. \]

**Lemma 3.2.** ([10, Proposition 8.3], [6, Theorem 1.2], [3, Corollary 3], [12, Section 1.10]) Let \(I, J \subseteq S\) and \(\tau \in \mathcal{W}^{I, J}\). Then, for any element \(u \in W_J\), \(w \tau \in W^J\) if and only if \(u \in W_J^{K(\tau)}\). Consequently, every element of \(W_J \tau W_I\) can be written uniquely as \(w \tau v\), where \(w \in W_J^{K(\tau)}\), \(v \in W_I\), and \(\ell(w \tau v) = \ell(u) + \ell(\tau) + \ell(v)\).

From now on, when we are dealing with two variable Hecke algebras associated with parabolic subalgebras, we simply write \(H_I\) for \(H_{W_I}(a, b)\) for each \(I \subseteq S\). The subsequent Lemma plays a key role in the proof of our main result.

**Lemma 3.3.** Let \(I, J \subseteq S\) and \(w \in W\). Then the map

\[ c_w : H_{K(w)} \rightarrow H_{w^{-1}K(w)w}, \quad \pi_k \mapsto \pi_{w^{-1}kw} \quad (k \in K(w)) \]

is an \(R\)-algebra isomorphism. In particular, if \(w \in \mathcal{W}^{I, J}\) and \(\kappa \in W_{K(w)}\), then

\[ \pi_{\kappa} c_w = c_w \pi_{\kappa}. \]

**Proof.** To begin with, we note that \(c_w\) restricts to a bijection from \(\{\pi_k : k \in K(w)\}\) to \(\{w^{-1}\pi_k w : k \in K(w)\}\). Suppose that \(wsu^{-1}, ws'w^{-1} \in K(w)\) for some \(s, s' \in I\). By the definition of \(K(w)\), the elements \(t := wiw^{-1}, t' := wiw^{-1}\) are contained in \(J\). For any \(m \in \mathbb{N} \cup \{\infty\}\), it holds that

\[ w^{-1}(tt't\ldots)_m = (ss's\ldots)_m \quad \text{and} \quad w^{-1}(t't't\ldots)_m = (ss's's\ldots)_m. \]
hence the relation \((tt't\ldots)_m = (t't't\ldots)_m\) gives rise to the relation \((ss's\ldots)_m = (s's's\ldots)_m\).

In a direct way, one can also check that \(c_w(\pi_s)c_w(\pi_s) = ac_w(\pi_s) + b\). Furthermore, the inverse of \(c_w\) can be obtained from the mapping \(\pi_{\sigma} \mapsto \pi_{w\sigma w^{-1}}\). Putting the above discussions together, we can conclude that \(c_w\) is an \(R\)-algebra isomorphism.

For the second assertion, note that \(J \subseteq \text{Asc}_L(w)\) and \(\pi_l \pi_w = \pi_{lw}\) for each \(l \in K(w)\). The former follows from Lemma 3.1 and the latter from the inclusion \(K(w) \subseteq J\). Therefore, one has that \(\pi_k \pi_w = \pi_{kw}\) for all \(k \in W_K(w)\). On the other hand, since \(w \in W^I\) and \(w^{-1}kw \in W_I\), \(w(w^{-1}kw)\) is the reduced factorization of \(kw \in W\) with respect to \(W/W_I\), which induces the following equality:

\[
\pi_{kw} = \pi_w \pi_{w^{-1}kw}.
\]

Hence, letting \(s_1s_2\ldots s_l\) be any reduced expression for \(k\), it holds that

\[
\pi^{-1}_{w^{-1}kw} = \pi^{-1}_{w^{-1}s_1w} \pi^{-1}_{w^{-1}s_2w} \cdots \pi^{-1}_{w^{-1}s_lw} = c_w(\pi_k),
\]

as required.

Now we are in the position to state our main result.

**Theorem 3.4.** Let \((W, S)\) be a finite Coxeter system, and \(I, J \subseteq S\), and \(M \in \text{mod}(H_I)\). Then

\[
M \uparrow_{H_I} \downarrow_{H_J} \cong \bigoplus_{\tau \in J^W} c_\tau[M \downarrow_{H_{J_k^{-1}K(\tau)r}}] \uparrow_{H_J}
\]

as \(H_J\)-modules where \(c_\tau : H_{K(r)} \to H_{J_k^{-1}K(\tau)r}\) is the algebra homomorphism defined by \(\pi_k \mapsto \pi_{\tau^{-1}k\tau}\).

**Proof.** For any \(w \in W\), by Lemma 3.2, there exists a unique triple \((z, \hat{\tau}, y)\) with \(z \in W_{\hat{\tau}}^K, \hat{\tau} \in J^W, y \in W_I\) such that

\[
w = z\hat{\tau}y \quad \text{and} \quad \ell(w) = \ell(z) + \ell(\hat{\tau}) + \ell(y).
\]

(3.1)

For this factorization, it is clear that \(\pi_w = \pi_z \pi_{\hat{\tau}} \pi_y\). Consider the map

\[
\Phi : H_S \otimes M \rightarrow \bigoplus_{\tau \in J^W} H_J \otimes_{H_{K(\tau)}} c_\tau[M \downarrow_{H_{J_k^{-1}K(\tau)r}}] \uparrow_{H_J}
\]

\[
\pi_w \otimes m \mapsto E_{\tau}(\pi_z \otimes \pi_{\hat{\tau}} \cdot y \cdot m) \quad (w \in W, m \in M),
\]

where \(w = z\hat{\tau}y\) is the decomposition given by (3.1) and, for any \(\tau \in J^W\) and \(p \in H_J \otimes_{H_{K(\tau)}} c_\tau[M \downarrow_{H_{J_k^{-1}K(\tau)r}}]\), the notation \(E_{\tau}(p)\) denotes the element in the codomain of \(\Phi\) such that the \(\tau\)th entry is \(p\) and the other entries are all zero, i.e.,

\[
E_{\tau}(p) := (0, 0, \ldots, p \underbrace{0 \ldots 0}_{\tau\text{th}}, 0).
\]

We claim that \(\Phi\) is well-defined, more precisely,

\[
\Phi(\pi_w \pi_i \otimes m) = \Phi(\pi_w \otimes (\pi_i \cdot m)) \quad \text{for all } i \in I.
\]
By Theorem 2.2,
\[
\pi_y \pi_i = \begin{cases} 
\pi_{yi} & \text{if } \ell(yi) > \ell(y), \\
 a \pi_y + b \pi_{yi} & \text{if } \ell(yi) < \ell(y)
\end{cases}
\]
for all \( i \in I \). If \( \ell(yi) > \ell(y) \), then
\[
\pi_w \pi_i \otimes m = \pi_z \pi_y \pi_i \otimes m = \pi_z \pi_y \pi_i \otimes m = \pi_z \pi_y \otimes (\pi_{yi} \cdot m).
\]
Hence
\[
\Phi(\pi_w \pi_i \otimes m) = E_{\hat{\tau}}(\pi_z \otimes (\pi_{yi} \cdot m)) \\
= E_{\hat{\tau}}(\pi_z \otimes (\pi_y \cdot (\pi_i \cdot m))) \\
= \Phi(\pi_z \pi_y \pi_i \otimes (\pi_i \cdot m)) \\
= \Phi(\pi_w \otimes (\pi_i \cdot m)).
\]
In a similar way as above, it can be seen that the same result holds when \( \ell(yi) < \ell(y) \). Thus the claim is verified.

Next, we consider the map
\[
\Psi : \bigoplus_{\tau \in J W_l} H_{J \tau(K)} \otimes c_{\tau} [ M \big| H_{J \tau(K \tau \tau^{-1})} ] \rightarrow H_S \otimes M \\
E_{\hat{\tau}}(\pi_\xi \otimes m) \mapsto \pi_\xi \pi_\tau \otimes m \quad (\xi \in W_J, m \in M).
\]
This map is also well defined since, for every \( \kappa \in W_{K(\tau)} \),
\[
\Psi(E_{\tau}(\pi_\xi \pi_\kappa \otimes m)) = \pi_\xi \pi_\kappa \pi_\tau \otimes m \\
= \pi_\xi \pi_\tau c_{\tau}(\pi_\kappa) \otimes m \\
= \pi_\xi \pi_\tau \cdot c_{\tau}(\pi_\kappa) \cdot m \\
= \Psi(E_{\tau}(\pi_\xi \otimes \pi_\kappa \cdot c_{\tau} m)).
\]
Here the second equality follows from Lemma 3.3. For the definition of \( c_{\tau} \) in the final term, see (2.1).

We will show that \( \phi \) and \( \Psi \) are inverses to each other. To do this, choose an arbitrary element \( \xi \) in \( W_J \). Let
\[
\xi = z \kappa, \quad \text{where } z \in W_J^{K(\tau)} \text{ and } \kappa \in W_{K(\tau)},
\]
be the reduced factorization of \( \xi \) with respect to \( W_J / W_{K(\tau)} \). Then \( \pi_\xi = \pi_z \pi_\kappa \). Using this factorization, we can show that, for any \( m \in M \) and \( \tau \in J W_I \),
\[
\Psi(E_{\tau}(\pi_\xi \otimes m)) = \pi_\xi \pi_\tau \otimes m = \pi_z \pi_\kappa \pi_\tau \otimes m = \pi_z \pi_\tau c_{\tau}(\pi_\kappa) \otimes m.
\]
Here the third equality follows from Lemma 3.3. Hence,
\[
\Phi \circ \Psi(E_{\tau}(\pi_\xi \otimes m)) = E_{\tau}(\pi_\xi \otimes (c_{\tau}(\pi_\kappa) \cdot m)) \\
= E_{\tau}(\pi_\xi \otimes (\pi_\kappa \cdot c_{\tau} m)) \\
= E_{\tau}(\pi_\xi \pi_\kappa \otimes m) \\
= E_{\tau}(\pi_\xi \otimes m).
\]
In a similar manner as above, one can show that, for any \( w \in W \) and \( m \in M \),
\[
\Psi \circ \Phi(\pi_w \otimes m) = \Psi \circ \Phi(\pi_z \pi_{\tilde{\tau}} \pi_{\tilde{y}} \otimes m) \\
= \Psi(E_{\tilde{\tau}}(\pi_z \otimes (\pi_{\tilde{y}} \cdot m))) \\
= \pi_z \pi_{\tilde{\tau}} \otimes \pi_{\tilde{y}} \cdot m \\
= \pi_z \pi_{\tilde{\tau}} \pi_y \otimes m \\
= \pi_w \otimes m,
\]
where \( w = z_{\tilde{\tau}}y \) is given by (3.1). So we are done.

Finally, let us show that \( \Psi \) is an \( H_J \)-module homomorphism. Let \( \xi \in W_J, m \in M \), and \( \tau \in J^W \). And let \( \xi = z_{\kappa} \) be the decomposition given by (3.2). By Theorem 2.2,
\[
\pi_j \pi_z = \begin{cases} \pi_{jz} & \text{if } \ell(jz) > \ell(z), \\ a\pi_z + b\pi_{jz} & \text{if } \ell(jz) < \ell(z) \end{cases}
\]
for \( j \in J \). Assume that \( \ell(jz) > \ell(z) \). Let \( jz = z'\kappa' \), where \( z' \in W_J^{K(\tau)} \) and \( \kappa' \in W_K(\tau) \),
be the reduced factorization of \( jz \) with respect to \( W_J/W_K(\tau) \). Then \( \pi_{jz} = \pi_{z'}\pi_{\kappa'} \). Using this factorization, we can show that
\[
\pi_j E_{\tau}(\pi_\xi \otimes m) = E_{\tau}(\pi_j \pi_z \pi_\kappa \otimes m) \\
= E_{\tau}(\pi_{jz} \otimes \pi_\kappa \cdot c_{\tau} m) \\
= E_{\tau}(\pi_{z'} \pi_{\kappa'} \otimes \pi_\kappa \cdot c_{\tau} m).
\]
Hence,
\[
\Psi(\pi_j E_{\tau}(\pi_\xi \otimes m)) = \Psi(E_{\tau}(\pi_{z'} \otimes \pi_{\kappa'} \cdot c_{\tau} (\pi_\kappa \cdot c_{\tau} m))) \\
= \pi_{z'} \pi_{\tau} \otimes c_{\tau}(\pi_\kappa) \cdot (c_{\tau}(\pi_\kappa) \cdot m) \\
= \pi_{z'} \pi_{\tau} c_{\tau} \pi_\kappa \otimes (c_{\tau}(\pi_\kappa) \cdot m).
\]
On the other hand, we can show that
\[
\pi_j \Psi(E_{\tau}(\pi_\xi \otimes m)) = \pi_j \Psi(E_{\tau}(\pi_z \otimes \pi_\kappa \cdot c_{\tau} m)) \\
= \pi_j(\pi_z \pi_{\tau} \otimes (c_{\tau}(\pi_\kappa) \cdot m)) \\
= \pi_{z'} \pi_{\kappa'} \pi_{\tau} \otimes (c_{\tau}(\pi_\kappa) \cdot m).
\]
By Lemma 3.3, we conclude that $\Psi(\pi_1 E_\tau(\pi_\xi \otimes m)) = \pi_1 \Psi(E_\tau(\pi_\xi \otimes m))$. In the case of $\ell(jz) < \ell(z)$, we have $\pi_j \pi_z = a \pi_z + b \pi_{jz}$. Let $\pi_z = \pi_{z'} \pi_{z''}$ and $\pi_{jz} = \pi_{z'} \pi_{z''}$ be the reduced factorizations of $z$ and $jz$ with respect to $W_J/W_K(\tau)$, respectively. In a similar way as above, it can be seen that the same result holds when $\ell(jz) < \ell(z)$.

This completes the proof. \qed

Remark 3.5. Lemma 3.3 plays a key role in the proof of Theorem 3.4. By extending this lemma to Iwahori-Hecke algebras, one can state Theorem 3.4 for Iwahori-Hecke algebras without difficulty. The Mackey decomposition formula thus obtained recovers [11, Proposition 9.1.8] since the latter is stated only in the case where given parameters $a_s, b_s \in R$ satisfy that $b_s = 1 - a_s$ for all $s \in S$ and the $H_R(W, S, \{ a_s, 1 - a_s : s \in S \})$-module in consideration is free as an $R$-module.

Let’s take a closer look at the case where $R = \mathbb{C}$, $W = \mathcal{S}_{m+n}$, $W_I$ and $W_J$ are maximal parabolic subgroups of $W$, and $M$ is a tensor product of two modules. In the following, we simply write $H_n$ for $H_{\mathcal{S}_n}(a, b)$. Then Theorem 3.4 can be rewritten in the following simple form.

Corollary 3.6. For all $a, b \in \mathbb{C}$ and $1 \leq k \leq m+n-1$, we have the following isomorphism of $H_k \otimes H_{m+n-k}$-modules: for $M \in \text{mod } H_m$ and $N \in \text{mod } H_n$,

$$(M \boxtimes N)\downarrow_{H_k \otimes H_{m+n-k}}^{H_{m+n}} \cong \bigoplus_{t+s=k, \ell \leq m, s \leq n} T_{t,s} \left( (M \downarrow_{H_t \otimes H_{m-t}}^{H_m}) \otimes (N \downarrow_{H_s \otimes H_{n-s}}^{H_n}) \right) \uparrow_{H_k \otimes H_{m+n-k}}^{H_k \otimes H_{m+n-k}}$$

where $M \boxtimes N = M \otimes N \uparrow_{H_m \otimes H_n}^{H_{m+n}}$ and

$T_{t,s} : \text{mod } (H_t \otimes H_{m-t} \otimes H_s \otimes H_{n-s}) \to \text{mod } (H_t \otimes H_s \otimes H_{m-t} \otimes H_{n-s})$

is the functor sending $M_1 \otimes M_2 \otimes N_1 \otimes N_2 \mapsto M_1 \otimes N_1 \otimes M_2 \otimes N_2$.

Proof. Let $I = \{1, 2, \ldots, m-1, m+1, m+2, \ldots, m+n-1\}$ and $J = \{1, 2, \ldots, k-1, k+1, k+2, \ldots, m+n-1\}$. It is well known that

$W^I = \{ w \in \mathcal{S}_{m+n} | w(1) < \cdots < w(m) \text{ and } w(m+1) < \cdots < w(m+n) \}$

(for instance, see [4]). Combining $JW = \{ w | w^{-1} \in W^J \}$ with Lemma 3.1, we derive that

$JW^I = \{ w_t | 0 \leq t \leq m, 0 \leq k-t \leq n \}$

where

$$w_t(i) = \begin{cases} i & \text{if } 1 \leq i \leq t, \\ k-t+i & \text{if } t+1 \leq i \leq m, \\ t-m+i & \text{if } m+1 \leq i \leq m+k-t, \\ i & \text{if } m+k-t+1 \leq i \leq m+n, \end{cases}$$

i.e.,

$$w_t = 1 \ldots t | k+1 \ldots k+m-t | t+1 \ldots k | m+k-t+1 \ldots m+n$$
in one-line notation. It says that the functor $F_{c_w}$ induced by $c_w$ and the functor $T_{t,s}$ are the same, thus the assertion follows.

**Remark 3.7.** (1) Recall that $H_n(1,0) = H_n(0)$, the 0-Hecke algebra of $S_n$. Hence Corollary 3.6 is a generalization of [13, Theorem 3]. In fact, the latter is stated only for the 0-Hecke modules called Weak Bruhat interval modules.

(2) Consider tower of algebras $A = \bigoplus_{n \geq 0} A_n$ with $\dim(A_1) = 1$ such that Grothendieck groups $G(A)$ and $K(A)$ form graded dual Hopf algebras. Bergeron, Lam, and Li conjectured in [1, Conjecture 6.2] that such a tower of algebra is isomorphic to a tower $H(a,b) = \bigoplus_{n \geq 0} H_n(a,b)$ of algebras for some $a, b \in \mathbb{C}$. Thus, under the validity of the conjecture, we can state the following isomorphism of $A_k \otimes A_{m+n-k}$-modules: for $M \in \text{mod } A_m$ and $N \in \text{mod } A_n$,

$$
(M \boxtimes N)_{A_k \otimes A_{m+n-k}} \cong \bigoplus_{t+s=k, t \leq m, s \leq n} T_{t,s} \left( M_{A_t \otimes A_{m-t}} \otimes N_{A_s \otimes A_{n-s}} \right),
$$

where $M \boxtimes N = M \otimes N_{A_m \otimes A_n}$ and

$$
T_{t,s} : \text{mod } (A_t \otimes A_{m-t} \otimes A_s \otimes A_{n-s}) \to \text{mod } (A_t \otimes A_s \otimes A_{m-t} \otimes A_{n-s})
$$

is the functor sending $M_1 \otimes M_2 \otimes N_1 \otimes N_2$ to $M_1 \otimes N_1 \otimes M_2 \otimes N_2$.

4. **(Anti-)involutions of two variable Hecke algebras and their interaction with induction product and restriction**

Recall that the induced product of modules is given by $M \boxtimes N = M \otimes N_{H_{m+n}}^{H_{m+n} \otimes H_{m+n}}$. In this section, we investigate how the induction product $\boxtimes$ or the restriction $\downarrow_{H_{m+n} \otimes H_{m+n}}$ are intertwined with several (anti-)automorphisms twists of the two variable Hecke algebra $H_{m+n}$. Recently, this subject was considered in [13, Corollary 1] only for a class of modules of 0-Hecke algebras called Weak Bruhat interval modules. The content of this section can be seen as a broad generalization of [13, Corollary 1].

In [9], Fayers introduced the involutions $\theta$ and $\phi$ and the anti-involution $\chi$ of the 0-Hecke algebra $H_W(0)$ defined in the following manner:

$$
\phi : H_W(0) \to H_W(0), \quad \pi_s \mapsto \pi_w^0 s w_0 \quad \text{for } s \in S,
$$

$$
\theta : H_W(0) \to H_W(0), \quad \pi_s \mapsto 1 - \pi_s \quad \text{for } s \in S,
$$

$$
\chi : H_W(0) \to H_W(0), \quad \pi_s \mapsto \pi_s \quad \text{for } s \in S.
$$

We are going to extend these to the morphisms of the two variable Hecke algebras.

**Lemma 4.1.** For fixed $a, b \in R$, let $(y_n)_{n \geq 0}$ be the sequence in $R$ determined by the following recurrence

$$
y_n = -a y_{n-1} + b y_{n-2}, \quad y_0 = 0, \quad y_1 = -a.
$$
Then the following equality holds.

\[(\pi_i - a)(\pi_j - a)(\pi_i - a)\ldots)_n = (\pi_i \pi_j \pi_i \ldots)_n + \sum_{m=1}^{n-1} y_{n-m}(\pi_i \pi_j \pi_i \ldots)_m + (\pi_j \pi_i \pi_j \ldots)_m + y_n.\]

**Proof.** Use induction on \(n\).

**Lemma 4.2.** Let \((W, S)\) be a Coxeter system. There are automorphisms \(\phi, \theta\) and an anti-automorphism \(\chi\) on \(H_W\) defined by

\[
\phi : H_W \rightarrow H_W, \quad \pi_s \mapsto \pi_{\text{w} \text{w} \text{w} \text{w}} \quad \text{for } s \in S, \\
\theta : H_W \rightarrow H_W, \quad \pi_s \mapsto a - \pi_s \quad \text{for } s \in S, \\
\chi : H_W \rightarrow H_W, \quad \pi_s \mapsto \pi_s \quad \text{for } s \in S.
\]

Furthermore, they commute with each other and are involutions.

**Proof.** It is clear that \(\phi^2, \theta^2,\) and \(\chi^2\) are identity maps and they commute with each other, and \(\chi\) is an anti-automorphism. The map \(\phi\) is an automorphism due to Lemma 3.3. To show \(\theta\) is an automorphism, observe \((a - \pi_i)^2 = a^2 - 2a\pi_i + (\pi_i)^2 = a(a - \pi_i) + b.\) The braid relation follows from Lemma 4.1. \(\square\)

Let \(\pi_i := \pi_i - a.\) Then for any reduced expression \(s_{i_1} s_{i_2} \ldots s_{i_l}\) of \(w \in W,\) \(\pi_w := \pi_{i_1} \pi_{i_2} \ldots \pi_{i_l}\) is independent of choices of reduced expressions since \(\pi_i\) satisfy the braid relations. Similar to Theorem 2.2, one can show that two variable Hecke algebra \(H_W(a, b)\) over \(R\) is a free \(R\)-module with basis elements \(\pi_w(w \in W),\) and for all \(s \in S, w \in W,\) multiplication is given by

\[
\pi_s \pi_w = \begin{cases} 
\pi_{sw} & \text{if } \ell(sw) > \ell(w), \\
-a\pi_w + b\pi_{sw} & \text{if } \ell(sw) < \ell(w).
\end{cases}
\]

**4.1. Formulas for the involution twists and the induction product and formulas for the restriction.** In the following, let us investigate how (anti-)involution twists behave with respect to induction products and restrictions.

**Lemma 4.3.** Let \(B\) be a subalgebra of \(A\) and \(\alpha\) be an automorphism of \(A.\)

1. For a \(B\)-module \(K,\) \(\alpha[K]_{\uparrow B}^A \cong \alpha[K]_{\uparrow A}^{\alpha^{-1}(B)}.\)
2. For an \(A\)-module \(L,\) \(\alpha[L]_{\downarrow B}^A \cong \alpha[L]_{\downarrow A}^{\alpha^{-1}(B)}.\)

**Proof.** (1) Note that \(\alpha|_{\alpha^{-1}(B)} : \alpha^{-1}(B) \rightarrow B\) is an isomorphism, so \(\alpha[K]\) is an \(\alpha^{-1}(B)\)-module. Consider the bijection \(\Phi : \alpha[A \otimes_B K] \rightarrow \alpha[A] \otimes_{\alpha^{-1}(B)} \alpha[K]\) given by \(a \otimes k \mapsto a \otimes k\) for any \(a \in A\) and \(k \in K.\) Since \(\Phi(a \cdot b \otimes k) = a.\alpha^{-1}(b) \otimes k = a \otimes \alpha^{-1}(b).\alpha k = a \otimes b \cdot k = \Phi(a \otimes b \cdot k)\) for any \(b \in B,\) \(\Phi\) is well-defined. The \(A(= \alpha(A))\)-module structure on the left hand side is given by \(a'.\alpha(a \otimes k) = (\alpha(a') \cdot a) \otimes k\) for any \(a', a \in A\) and \(k \in K.\) And, \(a'.\alpha \Phi(a \otimes k) = a'.\alpha(a \otimes k) = (\alpha(a') \cdot a) \otimes k,\) which shows that \(\Phi\) is an \(A\)-module.
homomorphism. (2) Similarly, the identity map on the vector space $L$ is an isomorphism between two $\alpha^{-1}(B)$-modules.

Let $\omega := \phi \circ \theta$. Then we can derive the following relations.

**Theorem 4.4.** Let $M \in \text{mod } H_m(0)$, $N \in \text{mod } H_n(0)$. Then we have following isomorphisms of $H_{m+n}(0)$-modules:

1. $\phi[M \otimes N] \cong \phi[N] \otimes \phi[M]$
2. $\theta[M \otimes N] \cong \theta[M] \otimes \theta[N]$
3. $\omega[M \otimes N] \cong \omega[N] \otimes \omega[M]$

For $L \in \text{mod } H_{m+n}$, we have following isomorphisms of $H_m \otimes H_n$-modules:

4. $\phi[L] \downarrow_{H^{m+n}_{m+n} \otimes H_{m+n}} \cong \phi[L] \downarrow_{H_{m+n} \otimes H_{m+n}}$
5. $\theta[L] \downarrow_{H^{m+n}_{m+n} \otimes H_{m+n}} \cong \theta[L] \downarrow_{H_{m+n} \otimes H_{m+n}}$
6. If $L$ is a free $R$-module of finite rank, then $\chi[L] \downarrow_{H^{m+n}_{m+n} \otimes H_{m+n}} \cong \chi[L] \downarrow_{H_{m+n} \otimes H_{m+n}}$.

**Proof.** (1) Put $A = H_{m+n}$, $B = H_m \otimes H_n$ and $\alpha = \phi$ in Lemma 4.3 (1), then we have following isomorphisms:

$\phi[M \otimes N] \cong \phi[M \otimes N] \downarrow_{H_{m+n} \otimes H_{m+n}}$

$\cong \phi[M \otimes N] \downarrow_{H_{m+n} \otimes H_{m+n}}$

$\cong (\phi[M] \otimes \phi[N]) \downarrow_{H_{m+n} \otimes H_{m+n}}$

$\cong (\phi[M] \otimes \phi[N]) \downarrow_{H_{m+n} \otimes H_{m+n}}$

$\cong \phi[N] \otimes \phi[M]$

(2) Replace $\phi$ with $\theta$ in the above.

(3) Combine (1) and (2).

(4) & (5) Immediately follows from Lemma 4.3 (2).

(6) Let $\{e_i\}$ and $\{e_j\}$ be dual basis for $L$ and $\chi[L]$ so that if $\langle \cdot, \cdot \rangle$ is the bilinear form given by $\langle e_i, e_j \rangle = \delta_{ij}$, then $\langle \pi_i \cdot m, \mu \rangle = \langle m, \pi_i \cdot \mu \rangle$ for all $m \in M, \mu \in \chi[M]$ and $i \in \{1, 2, \ldots, m+n\}$. Then we automatically have $\langle \pi_i \cdot m, \mu \rangle = \langle m, \pi_i \cdot \mu \rangle$ for all $i \in \{1, 2, \ldots, m+n\} \setminus \{m\}$, which proves our assertion.

4.2. Formulas for the anti-involution twists and the induction product. In this subsection, we will show how the anti-involution twists interact with induction product. For $W = \mathfrak{S}_{m+n}$ and $W_I = \mathfrak{S}_m \times \mathfrak{S}_n$, let us denote by $\Gamma$ the set of minimal length left coset representatives $W_I$.

**Lemma 4.5.** Let $M \in \text{mod } (H_m \otimes H_n)$ be a free $R$-module of finite rank and $\beta$ be a basis for $M$. Then $\{\pi_\gamma \otimes \beta_i \mid \gamma \in \Gamma, \beta_i \in \beta\}$ and $\{\bar{\pi}_\gamma \otimes \beta_i \mid \gamma \in \Gamma, \beta_i \in \beta\}$ are bases for $M \downarrow_{H_{m+n} \otimes H_{m+n}}$. 

Proof. By the reduced factorization of each element in $\mathfrak{S}_{m+n}$ with respect to $\mathfrak{S}_{m+n}/(\mathfrak{S}_m \times \mathfrak{S}_n)$, $\{\pi_\gamma \otimes \beta_i\}$ spans $M \uparrow_{H_{m+n}}^{H_{m+n}}$. Since $\dim(M \uparrow_{H_{m+n}}^{H_{m+n}}) = |(m+n)| \cdot \dim(M)$ and $|(m+n)| = |\Gamma|$, $\{\pi_\gamma \otimes \beta_i\}$ is a basis for $M \uparrow_{H_{m+n}}^{H_{m+n}}$. Using the same argument, we can see that $\{\pi_\gamma \otimes \beta_i\}$ is also a basis for $M \uparrow_{H_{m+n}}^{H_{m+n}}$.

It is well known that

$$\Gamma = \{\gamma \in \mathfrak{S}_{m+n} \mid \gamma(1) < \cdots < \gamma(m) \text{ and } \gamma(m+1) < \cdots < \gamma(m+n)\}.$$ 

For later use, we simply write $\gamma \in \Gamma$ as

$$\gamma(1) < \cdots < \gamma(m) \mid \gamma(m+1) < \cdots < \gamma(m+n)$$

(refer to [4, Lemma 2.4.7]). Dividing cases according to where $i, i+1$ appear in the one-line notation of $\gamma$, we obtain the following equalities for any $i \in \{1, 2, \ldots, m+n-1\}$ and $\gamma \in \Gamma$:

$$\pi_i \pi_\gamma = \begin{cases} 
\pi_\gamma \pi_{\gamma-1}(i) & \text{if } \gamma^{-1}(i) \leq m, \gamma^{-1}(i+1) \leq m, \\
\pi_\gamma \pi_{\gamma-1}(i) & \text{if } \gamma^{-1}(i) > m, \gamma^{-1}(i+1) > m, \\
\pi_{s_i \gamma} & \text{if } \gamma^{-1}(i) \leq m, \gamma^{-1}(i+1) > m, \\
\pi_{s_i \gamma} + b \pi_{s_\gamma} & \text{if } \gamma^{-1}(i) > m, \gamma^{-1}(i+1) \leq m.
\end{cases}$$  \hspace{1cm} (4.1)

Similarly, we obtain the equalities for any $i \in \{m+n-1\}$ and $\gamma \in \Gamma$:

$$\pi_i \pi_\gamma = \begin{cases} 
\pi_\gamma \pi_{\gamma-1}(i) & \text{if } \gamma^{-1}(i) \leq m, \gamma^{-1}(i+1) \leq m, \\
\pi_\gamma \pi_{\gamma-1}(i) & \text{if } \gamma^{-1}(i) > m, \gamma^{-1}(i+1) > m, \\
\pi_{s_i \gamma} + a \pi_\gamma & \text{if } \gamma^{-1}(i) \leq m, \gamma^{-1}(i+1) > m, \\
b \pi_{s_\gamma} & \text{if } \gamma^{-1}(i) > m, \gamma^{-1}(i+1) \leq m.
\end{cases}$$  \hspace{1cm} (4.2)

Note that all $s_\gamma$'s appearing in (4.1) and (4.2) are in $\Gamma$.

Let $\hat{\Phi} := \Phi \circ \chi, \hat{\Theta} := \Theta \circ \chi, \hat{\omega} := \omega \circ \chi$. With this notation, we can state the main result of this subsection.

**Theorem 4.6.** Let $M \in \text{mod} \, H_m$ and $N \in \text{mod} \, H_n$ be free $R$-modules of finite rank. Then we have following isomorphisms of $H_{m+n}$-modules:

1. $\hat{\Phi}[M \otimes N] \cong \hat{\Phi}[M] \otimes \hat{\Phi}[N]$
2. $\chi[M \otimes N] \cong \chi[N] \otimes \chi[M]$
3. $\hat{\Theta}[M \otimes N] \cong \hat{\Theta}[N] \otimes \hat{\Theta}[M]$
4. $\hat{\omega}[M \otimes N] \cong \hat{\omega}[M] \otimes \hat{\omega}[N]$

Proof. (1) There exist bases $\{e_k^M : k = 1, 2, \ldots, \text{rank}(M)\}$ and $\{e_l^M : l = 1, 2, \ldots, \text{rank}(M)\}$ for $M$ and $\chi[\Phi[M]]$, respectively, and a bilinear pairing $\langle , \rangle_M$ such that $\langle e_k^M, e_l^M \rangle_M = \delta_{kl}$ and $\langle \pi_i \cdot x, y \rangle_M = \langle x, \pi_{m-i} \cdot y \rangle_M$ for $x \in M, y \in \chi[\Phi[N]]$, and $1 \leq i \leq m - 1$. Similarly, there exist bases $\{e_k^N : k = 1, 2, \ldots, \text{rank}(N)\}$ and $\{e_l^N : l = 1, 2, \ldots, \text{rank}(N)\}$ for $N$ and $\chi[\Phi[N]]$, respectively, and a bilinear pairing $\langle , \rangle_N$ such that $\langle e_k^N, e_l^N \rangle_N = \delta_{kl}$ and
\langle \pi_i \cdot x, y \rangle_N = \langle x, \pi_{m+n-i} \cdot y \rangle_N \text{ for all } x \in N, y \in \chi[\phi[N]] \text{ and } m+1 \leq i \leq m+n-1. \text{ Using this, one can deduce that there exist bases } \{e_k\} \text{ and } \{\epsilon_l\} \text{ for } M \otimes N \text{ and } \chi[\phi[M]] \otimes \chi[\phi[N]], \text{ respectively, and a bilinear pairing } \langle , \rangle : M \otimes N \times \chi[\phi[M]] \otimes \chi[\phi[N]] \rightarrow \mathbb{C} \text{ such that } \langle e_k, \epsilon_l \rangle = \delta_{kl} \text{ and }

\langle \pi_i \cdot z, \zeta \rangle = \begin{cases} \langle z, \pi_{m-i} \cdot \zeta \rangle, & \text{if } 1 \leq i \leq m-1 \\ \langle z, \pi_{m+n-i} \cdot \zeta \rangle, & \text{if } m+1 \leq i \leq m+n-1 \end{cases}

\text{for all } k, l \text{ and } z \in M \otimes N, \zeta \in \chi[\phi[M]] \otimes \chi[\phi[N]] \text{ for } i \in \{1, 2, \ldots, m + n - 1\}. \text{ From Lemma 4.5 it follows that } \{\pi_\gamma \otimes e_k\} \text{ and } \{\overline{\pi}_\lambda \otimes \epsilon_l\} \text{ are bases for } M \otimes N \uparrow_{H_m \otimes H_n} \chi[\phi[M]] \otimes \chi[\phi[N]] \uparrow_{H_m \otimes H_n}, \text{ respectively. For }

\gamma = \gamma(1) < \cdots < \gamma(m) | \gamma(m + 1) < \ldots < \gamma(m + n) \in \Gamma,

\text{we let }

\gamma' := \gamma'(1) < \cdots < \gamma'(m) | \gamma'(m + 1) < \ldots < \gamma'(m + n) \in \Gamma,

\text{where }

\gamma'(i) = \begin{cases} (m + n + 1) - \gamma(m + 1 - i), & \text{if } 1 \leq i \leq m, \\ (m + n + 1) - \gamma(2m + n + 1 - i), & \text{if } m + 1 \leq i \leq m + n. \end{cases}

\text{Clearly the assignment } \gamma \mapsto \gamma' \text{ induces an involution on the set } \Gamma. \text{ We now define a bilinear pairing } \langle , \rangle : M \otimes N \times \chi[\phi[M]] \otimes \chi[\phi[N]] \rightarrow \mathbb{C} \text{ by letting }

\langle \pi_\gamma \otimes z, \overline{\pi}_\lambda \otimes \zeta \rangle := \delta_{\gamma\lambda} \langle z, \zeta \rangle

\text{for } z \in M \otimes N, \zeta \in \chi[\phi[M]] \otimes \chi[\phi[N]], \text{ and } \gamma, \lambda \in \Gamma. \text{ For the assertion, we have only to show that }

\langle \pi_i \pi_\gamma \otimes e_k, \overline{\pi}_\lambda \otimes \epsilon_l \rangle = \langle \pi_\gamma \otimes e_k, \pi_{m+n-i} \overline{\pi}_\lambda \otimes \epsilon_l \rangle \quad (4.3)

\text{for all } k, l \text{ and } i \in \{1, 2, \ldots, m + n - 1\}, \text{ and } \gamma, \lambda \in \Gamma. \text{ Due to (4.1), the left hand side of (4.3) is given as follows: }

\text{(A1) If } \gamma^{-1}(i) \leq m \text{ and } \gamma^{-1}(i+1) \leq m, \text{ then }

\begin{cases} \langle \pi_{\gamma^{-1}(i)} \cdot e_k, \epsilon_l \rangle & \text{if } \lambda = \gamma', \\ 0 & \text{otherwise.} \end{cases}

\text{(A2) If } \gamma^{-1}(i) > m \text{ and } \gamma^{-1}(i+1) > m, \text{ then }

\begin{cases} \langle \pi_{\gamma^{-1}(i)} \cdot e_k, \epsilon_l \rangle & \text{if } \lambda = \gamma', \\ 0 & \text{otherwise.} \end{cases}

\text{(A3) If } \gamma^{-1}(i) \leq m \text{ and } \gamma^{-1}(i+1) > m, \text{ then }

\begin{cases} \langle e_k, \epsilon_l \rangle & \text{if } \lambda = (s_i \gamma)', \\ 0 & \text{otherwise.} \end{cases}
(A4) If $\gamma^{-1}(i) > m$ and $\gamma^{-1}(i + 1) \leq m$, then

\[
\begin{cases}
    a\langle e_k, \epsilon_l \rangle & \text{if } \lambda = \gamma', \\
    b\langle e_k, \epsilon_l \rangle & \text{if } \lambda = (s_i \cdot \gamma)', \\
    0 & \text{otherwise}.
\end{cases}
\]

On the other hand, due to (4.2), the right hand side of (4.3) is given as follows:

(B1) If $\lambda^{-1}(m + n - i) \leq m$ and $\lambda^{-1}(m + n - i + 1) \leq m$, then

\[
\begin{cases}
    \langle e_k, \pi_{\lambda^{-1}(m+n+i)} \cdot \epsilon_l \rangle & \text{if } \gamma = \lambda', \\
    0 & \text{otherwise}.
\end{cases}
\]

(B2) If $\lambda^{-1}(m + n - i) > m$ and $\lambda^{-1}(m + n - i + 1) > m$, then

\[
\begin{cases}
    \langle e_k, \pi_{\lambda^{-1}(m+n+i)} \cdot \epsilon_l \rangle & \text{if } \lambda = \gamma', \\
    0 & \text{otherwise}.
\end{cases}
\]

(B3) If $\lambda^{-1}(m + n - i) \leq m$ and $\lambda^{-1}(m + n - i + 1) > m$, then

\[
\begin{cases}
    a\langle e_k, \epsilon_l \rangle & \text{if } \lambda = \gamma', \\
    \langle e_k, \epsilon_l \rangle & \text{if } s_{m+n-i} \cdot \lambda = \gamma', \\
    0 & \text{otherwise}.
\end{cases}
\]

(B4) If $\lambda^{-1}(m + n - i) > m$ and $\lambda^{-1}(m + n - i + 1) \leq m$, then

\[
\begin{cases}
    b\langle e_k, \epsilon_l \rangle & \text{if } s_{m+n-i} \cdot \lambda = \gamma', \\
    0 & \text{otherwise}.
\end{cases}
\]

Comparing these calculations, one can deduce the equality (4.3). For instance, in the case where $\gamma^{-1}(i) \leq m$, $\gamma^{-1}(i + 1) > m$, and $\lambda = (s_i \cdot \gamma)'$, one has that $\lambda^{-1}(m + n - i) \leq m$ and $\lambda^{-1}(m + n - i + 1) > m$ and $s_{m+n-i} \cdot \lambda = \gamma'$. So, in this case, both sides of (4.3) are equal to $\langle e_k, \epsilon_l \rangle$ due to (A3) and (B4).

(2) Replace $M$ by $\Phi(M)$ and $N$ by $\Phi(N)$ in (1). Then the assertion follows from Theorem 4.4 (1).

(3) & (4) Combine (2) with Theorem 4.4 (1) and (2). □

**Remark 4.7.** The isomorphism

\[
\hat{\Phi}[M] \uparrow_{H_n(0)}^{H_{n-1}(0)} \cong \hat{\Phi}[M] \uparrow_{H_{n-1}(0)}^{H_n(0)}
\]

has already appeared in [9, Lemma 6.4]. The proof of Theorem 4.6 can be viewed as a generalization of Fayers’ proof.


References

[1] N. Bergeron, T. Lam, and H. Li. Combinatorial Hopf algebras and towers of algebras-dimension, quantization and functorality. *Algebr. Represent. Theory*, 15(4):675–696, 2012.

[2] N. Bergeron and H. Li. Algebraic structures on Grothendieck groups of a tower of algebras. *J. Algebra*, 321(8):2068–2084, 2009.

[3] S. C. Billey, M. Konvalinka, T. K. Petersen, W. Slofstra, and B. E. Tenner. Parabolic double cosets in Coxeter groups. *Electron. J. Combin.*, 25(1):Paper No. 1.23, 66, 2018.

[4] A. Björner and F. Brenti. *Combinatorics of Coxeter groups*, volume 231 of *Graduate Texts in Mathematics*. Springer, New York, 2005.

[5] N. Bourbaki. *Lie groups and Lie algebras. Chapters 4–6*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2002. Translated from the 1968 French original by Andrew Pressley.

[6] C. W. Curtis. On Lusztig’s isomorphism theorem for Hecke algebras. *J. Algebra*, 92(2):348–365, 1985.

[7] B. Deng and G. Yang. Representation type of 0-Hecke algebras. *Sci. China Math.*, 54(3):411–420, 2011.

[8] G. Duchamp, F. Hivert, and J.-Y. Thibon. Noncommutative symmetric functions. VI. Free quasi-symmetric functions and related algebras. *Internat. J. Algebra Comput.*, 12(5):671–717, 2002.

[9] M. Fayers. 0-Hecke algebras of finite Coxeter groups. *J. Pure Appl. Algebra*, 199(1-3):27–41, 2005.

[10] A. M. Garsia and D. Stanton. Group actions of Stanley-Reisner rings and invariants of permutation groups. *Adv. in Math.*, 51(2):107–201, 1984.

[11] M. Geck and G. Pfeiffer. *Characters of finite Coxeter groups and Iwahori-Hecke algebras*, volume 21 of *London Mathematical Society Monographs. New Series*. The Clarendon Press, Oxford University Press, New York, 2000.

[12] J. E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[13] W.-S. Jung, Y.-H. Kim, S.-Y. Lee, and Y.-T. Oh. Weak Bruhat interval modules of the 0-Hecke algebra. *Math. Z.*, 301(4):3755–3786, 2022.

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