Corrections to the Emergent Canonical Commutation Relations
Arising in the Statistical Mechanics of Matrix Models

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ABSTRACT

We study the leading corrections to the emergent canonical commutation relations arising in the statistical mechanics of matrix models, by deriving several related Ward identities, and give conditions for these corrections to be small. We show that emergent canonical commutators are possible only in matrix models in complex Hilbert space for which the numbers of fermionic and bosonic fundamental degrees of freedom are equal, suggesting that supersymmetry will play a crucial role. Our results simplify, and sharpen, those obtained earlier by Adler and Millard.
1. Introduction

It is widely believed that at distances of order the Planck length $\ell_P \sim 10^{-33}$ cm our conventional notions of the geometry of spacetime break down, as a result of quantum gravity effects. One indication of the modifications in physics that might be expected is provided by string theory models of quantum gravity, in which several studies suggest a modification of the uncertainty relation of the form [1]

$$\Delta x \Delta p \geq \frac{\hbar}{2}[1 + \beta(\Delta p)^2 + ...], \quad \beta > 0$$

(1a)

implying a finite minimum uncertainty $\Delta x_0 = \hbar \beta^{1/2}$ in the vicinity of the Planck length.

As discussed by Kempf and collaborators [2], Eq. (1a) corresponds to a correction to the Heisenberg canonical commutation relations of the form

$$[x, p] = i\hbar(1 + \beta p^2 + ...).$$

(1b)

We wish in this paper to discuss modifications of the Heisenberg algebra arising in another context, that of the statistical mechanics of matrix models, and to compare them with Eqs. (1a, b). Several years ago, Adler proposed a set of rules for a generalized quantum or trace dynamics, which is a Lagrangian and Hamiltonian mechanics with arbitrary non-commutative phase space variables $q, p$, and this was developed in a series of papers with various collaborators [3]. For theories in which the action is constructed as the trace of a sum of matrix products of $N \times N$ matrix variables, trace dynamics gives a powerful, basis independent, way of representing the same dynamics that can also be described in terms of the $N^2$ individual matrix elements. A significant new result emerging from this point of view was obtained by Adler and Millard [4], who argued that the statistical mechanics of trace dynamics takes the form of conventional quantum field theory, with the Heisenberg
commutation relations holding for statistical averages over certain effective canonical variables obtained by projection from the original operator canonical variables. Recently, it has become clear [5] that the underlying assumptions of trace dynamics are satisfied by matrix models, for which the methods of trace dynamics provide a very convenient calculational tool. Hence the results of Adler and Millard can be reinterpreted as providing a statistical mechanics of matrix models, and showing that thermal averages in this statistical mechanics can behave as Wightman functions in an emergent local quantum field theory. These results, together with recent work [6] suggesting that the underlying dynamics for string theory may be a form of matrix model, raise in turn the question of determining the form of the leading corrections to the Heisenberg algebra implied by the statistics of matrix models, formulating conditions for these corrections to be small, and seeing whether they can be related to the string theory result of Eqs. (1a, 1b).

An investigation of these questions is the focus of this paper, which is organized as follows. In Sec. 2 we give a brief synopsis of the rules of trace dynamics in the context of matrix models. We show that the conservation of the operator [4, 7] $\tilde{C}$ can be understood as a simple consequence of unitary invariance. We also remark that, with Grassmann fermions, $\tilde{C}$ is independent of the classical parts of the matrix phase space variables, and review the statistical mechanics [4, 8] of matrix models. In Sec. 3 we consider the simple case of a bosonic matrix model with Hamiltonian quadratic in the canonical momenta, and, making no approximations, derive a simplified form of the Ward identity used in Ref. [4] to obtain the effective canonical algebra. This analysis shows that there are corrections to the canonical commutator quadratic in the canonical momentum. In Sec. 4 we use the symplectic formalism of Ref. [4] to repeat this calculation in the case of a general commutator/anticommutator of
canonical variables in a generic matrix model, that can include fermions, and we generalize
the treatment of [4] to allow nonzero sources for the classical parts of the matrix variables.
From the analyses of Secs. 3 and 4, we formulate conditions for the corrections to the
emergent canonical algebra to be small. We show that these conditions require $\tilde{C}$ to be
an intensive rather than extensive thermodynamic quantity, and that they can be satisfied
in complex Hilbert space (if at all) only in matrix models with precisely equal numbers of
bosonic and fermionic degrees of freedom. This result strongly suggests that candidate matrix
models for prequantum mechanics should be supersymmetric. We conclude by generalizing
the conditions to ones that permit the recovery of the full emergent quantum field theory
structure derived in Ref. [4]. We also compare the prequantum corrections to the canonical
algebra derived in Secs. 3 and 4, in which (as is usual in field theories) the spatial coordinate
is simply a label, and the field variables are the dynamical canonical variables, to the string
theory inspired expression of Eq. (1b) in which $x$ is a coordinate operator.

2. The Statistical Mechanics of Matrix Models

We begin by reviewing the statistical mechanics of trace dynamics, taking into ac-
count the simplifications [5] that become possible when Grassmann algebras are employed
to represent the fermion/boson distinction. Let $B_1$ and $B_2$ be two $N \times N$ matrices with
matrix elements that are even grade elements of a complex Grassmann algebra, and Tr the
ordinary matrix trace, which obeys the cyclic property

$$\text{Tr} B_1 B_2 = \sum_{m,n} (B_1)_{mn} (B_2)_{nm} = \sum_{m,n} (B_2)_{nm} (B_1)_{mn} = \text{Tr} B_2 B_1 \quad . \quad (2a)$$

Similarly, let $\chi_1$ and $\chi_2$ be two $N \times N$ matrices with matrix elements that are odd grade
elements of a complex Grassmann algebra, which anticommute rather than commute, so that
the cyclic property for these takes the form

\[ \text{Tr} \chi_1 \chi_2 = \sum_{m,n} (\chi_1)_{mn} (\chi_2)_{nm} = - \sum_{m,n} (\chi_2)_{nm} (\chi_1)_{mn} = -\text{Tr} \chi_2 \chi_1 \]  

(2b)

The cyclic/anticyclic properties of Eqs. (2a, 2b) are just those assumed for the trace operation \( \text{Tr} \) of trace dynamics.* From Eqs. (2a, b), one immediately derives the trilinear cyclic identities

\[ \text{Tr} B_1 [B_2, B_3] = \text{Tr} B_2 [B_3, B_1] = \text{Tr} B_3 [B_1, B_2] \]

\[ \text{Tr} B_1 \{ B_2, B_3 \} = \text{Tr} B_2 \{ B_3, B_1 \} = \text{Tr} B_3 \{ B_1, B_2 \} \]  

(2c)

\[ \text{Tr} B \{ \chi_1, \chi_2 \} = \text{Tr} \chi_1 \{ \chi_2, B \} = \text{Tr} \chi_2 \{ \chi_1, B \} \]

\[ \text{Tr} \chi_1 \{ B, \chi_2 \} = \text{Tr} \{ \chi_1, B \} \chi_2 = \text{Tr} [\chi_1, \chi_2] B \]

which are used repeatedly in trace dynamics calculations.

The basic observation of trace dynamics is that given the trace of a polynomial \( P \) constructed from noncommuting matrix or operator variables (we shall use the terms “matrix” and “operator” interchangeably in the following discussion), one can define a derivative of the \( c \)-number \( \text{Tr} P \) with respect to an operator variable \( O \) by varying and then cyclically permuting so that in each term the factor \( \delta O \) stands on the right, giving the fundamental definition

\[ \delta \text{Tr} P = \text{Tr} \frac{\delta \text{Tr} P}{\delta O} \delta O \]  

(3a)

---

* In Refs. [3, 4] the fermionic operators were realized as ordinary matrices with complex matrix elements, all of which anticommute with a grading operator \((-1)^F\) which formed part of the definition of the graded trace \( \text{Tr} \), for which fermions then obeyed Eq. (2b) while bosons obeyed Eq. (2a). Since the use of Grassmann odd fermions eliminates the need for the inclusion of the \((-1)^F\) factor, the graded trace obeying Eqs. (2a, b) is here just the usual matrix trace, for which we use the customary notation \( \text{Tr} \).
or in the condensed notation that we shall use throughout this paper, in which \( P \equiv \text{Tr}P \),

\[
\delta P = \text{Tr} \frac{\delta P}{\delta O} \delta O \quad .
\]  

(3b)

Letting \( L[q_r, \dot{q}_r] \) be a trace Lagrangian that is a function of the bosonic or fermionic operators \( \{q_r\} \) and their time derivatives (which are all assumed to obey the cyclic relations of Eqs. (2a-c) under the trace), and requiring that the trace action \( S = \int dt L \) be stationary with respect to variations of the \( q_r \)'s that preserve their bosonic or fermionic type, one finds [3] the operator Euler-Lagrange equations

\[
\frac{\delta L}{\delta q_r} - \frac{d}{dt} \frac{\delta L}{\delta \dot{q}_r} = 0 \quad .
\]  

(3c)

Because, by the definition of Eq. (3a), we have

\[
\left( \frac{\delta L}{\delta q_r} \right)_{ij} = \frac{\partial L}{\partial (q_r)_{ji}} \quad ,
\]  

(3d)

for each \( r \) the single Euler-Lagrange equation of Eq. (3c) is equivalent to the \( N^2 \) Euler-Lagrange equations obtained by regarding \( L \) as a function of the \( N^2 \) matrix element variables \( (q_r)_{ji} \). Defining the momentum operator \( p_r \) conjugate to \( q_r \), which is of the same bosonic or fermionic type as \( q_r \), by

\[
p_r \equiv \frac{\delta L}{\delta \dot{q}_r} \quad ,
\]  

(4a)

the trace Hamiltonian \( H \) is defined by

\[
H = \text{Tr} \sum_r p_r \dot{q}_r - L \quad .
\]  

(4b)

In correspondence with Eq. (3d), the matrix elements \( (p_r)_{ij} \) of the momentum operator \( p_r \) just correspond to the momenta canonical to the matrix element variables \( (q_r)_{ji} \). Performing
general same-type operator variations, and using the Euler-Lagrange equations, we find from Eq. (4b) that the trace Hamiltonian $H$ is a trace functional of the operators $\{q_r\}$ and $\{p_r\}$,

$$H = H[\{q_r\}, \{p_r\}], \quad (5a)$$

with the operator derivatives

$$\frac{\delta H}{\delta q_r} = -\dot{p}_r, \quad \frac{\delta H}{\delta p_r} = \epsilon_r \dot{q}_r, \quad (5b)$$

with $\epsilon_r = 1(-1)$ according to whether $q_r, p_r$ are bosonic (fermionic). Letting $A$ and $B$ be two trace functions of the operators $\{q_r\}$ and $\{p_r\}$, it is convenient to define the generalized Poisson bracket

$$\{A, B\} = \text{Tr} \sum_r \epsilon_r \left( \frac{\delta A}{\delta q_r} \frac{\delta B}{\delta p_r} - \frac{\delta B}{\delta q_r} \frac{\delta A}{\delta p_r} \right). \quad (6a)$$

Then using the Hamiltonian form of the equations of motion, one readily finds that for a general trace functional $A[\{q_r\}, \{p_r\}]$, the time derivative is given by

$$\frac{d}{dt} A = \{A, H\} ; \quad (6b)$$

in particular, letting $A$ be the trace Hamiltonian $H$, and using the fact that the generalized Poisson bracket is antisymmetric in its arguments, it follows that the time derivative of $H$ vanishes. An important property of the generalized Poisson bracket is that it satisfies [3] the Jacobi identity,

$$\{A, \{B, C\} + \{C, \{A, B\}\} + \{B, \{C, A\}\} = 0 \quad . \quad (6c)$$

As a consequence, if $Q_1$ and $Q_2$ are two conserved charges, that is if

$$0 = \frac{d}{dt} Q_1 = \{Q_1, H\} , \quad 0 = \frac{d}{dt} Q_2 = \{Q_2, H\} , \quad (6d)$$
then their generalized Poisson bracket \{Q_1, Q_2\} also has a vanishing generalized Poisson bracket with \(\mathbf{H}\), and is conserved. This has the consequence that Lie algebras of symmetries can be represented as Lie algebras of trace functions under the generalized Poisson bracket operation.

A significant feature of trace dynamics is that, as discovered by Millard [7], the anti-self-adjoint operator [7, 4]

\[
\tilde{C} \equiv \sum_{r \text{ bosons}} [q_r, p_r] - \sum_{r \text{ fermions}} \{q_r, p_r\}
\]  

is conserved by the dynamics. As we shall now show, conservation of \(\tilde{C}\) holds whenever the trace dynamics has a global unitary invariance, that is, whenever the trace Hamiltonian obeys

\[
H[\{U^\dagger q_r U\}, \{U^\dagger p_r U\}] = H[\{q_r\}, \{p_r\}]
\]  

(8a)

for a constant unitary \(N \times N\) matrix \(U\), or equivalently, by Eq. (4b), whenever the trace Lagrangian obeys

\[
L[\{U^\dagger q_r U\}, \{U^\dagger \dot{q}_r U\}] = L[\{q_r\}, \{\dot{q}_r\}]
\]  

(8b)

Letting \(U = \exp \Lambda\), with \(\Lambda\) an anti-self-adjoint bosonic generator matrix, and expanding to first order in \(\Lambda\), Eq. (8a) implies that

\[
H[\{q_r - [\Lambda, q_r]\}, \{p_r - [\Lambda, p_r]\}] = H[\{q_r\}, \{p_r\}]
\]  

(9a)

But applying the definition of the variation of a trace functional given in Eq. (3b), Eq. (9a) becomes

\[
\text{Tr} \sum_r \left( -\frac{\delta H}{\delta q_r} [\Lambda, q_r] - \frac{\delta H}{\delta p_r} [\Lambda, p_r] \right) = 0
\]  

(9b)
which by use of the trilinear cyclic identities of Eq. (2c) yields

\[ \text{Tr} \Lambda \sum_r \left( \frac{\delta H}{\delta q_r} q_r - \epsilon_r q_r p_r + \frac{\delta H}{\delta p_r} p_r - \epsilon_r p_r \frac{\delta H}{\delta p_r} \right) = 0. \]  

(9c)

Since the generator \( \Lambda \) is an arbitrary anti-self-adjoint \( N \times N \) matrix, the anti-self-adjoint matrix that multiplies it in Eq. (9c) must vanish, giving the matrix identity

\[ \sum_r \left( \frac{\delta H}{\delta q_r} q_r - \epsilon_r q_r p_r + \frac{\delta H}{\delta p_r} p_r - \epsilon_r p_r \frac{\delta H}{\delta p_r} \right) = 0. \]  

(10a)

But now substituting the Hamilton equations of Eq. (5b), Eq. (10a) takes the form

\[ 0 = \sum_r \left( -\dot{p}_r q_r + \epsilon_r q_r \dot{p}_r - \epsilon_r \dot{q}_r p_r - p_r \dot{q}_r \right) \]

\[ = \frac{d}{dt} \sum_r \left( -p_r q_r + \epsilon_r q_r \dot{p}_r \right) \]  

(10b)

\[ = \frac{d}{dt} \left( \sum_r [q_r, p_r] - \sum_r \{ q_r, p_r \} \right), \]

completing the demonstration of the conservation of \( \tilde{C} \).

Corresponding to the fact that \( \tilde{C} \) is conserved in any matrix model with a global unitary invariance, it is easy to see \([4, 8]\) that \( \tilde{C} \) can be used to construct the generator of global unitary transformations of the Hilbert space basis. Consider the trace functional

\[ G_\Lambda = -\text{Tr} \Lambda \tilde{C}, \]  

(11a)

with \( \Lambda \) a fixed bosonic anti-self-adjoint operator, which can be rewritten, using cyclic invariance of the trace, as

\[ G_\Lambda = \text{Tr} \sum_r [\Lambda, p_r] q_r = -\text{Tr} \sum_r p_r [\Lambda, q_r]. \]  

(11b)

Hence for the variations of \( p_r \) and \( q_r \) induced by \( G_\Lambda \) as canonical generator, which have a
structure analogous to the Hamilton equations of Eq. (5b), we get

\[ \delta p_r = - \frac{\delta G_{\Lambda}}{\delta q_r} = -[\Lambda, p_r] \quad , \]
\[ \delta q_r = \epsilon_r \frac{\delta G_{\Lambda}}{\delta p_r} = -[\Lambda, q_r] \quad . \]

Comparing with Eqs. (8a) and (9a), we see that these have just the form of an infinitesimal global unitary transformation.

For each phase space variable \( q_r, p_r \), let us define the classical part \( q^c_r, p^c_r \) and the noncommutative remainder \( q'_r, p'_r \), by

\[ q^c_r = \frac{1}{N} \text{Tr} q_r \quad , \quad p^c_r = \frac{1}{N} \text{Tr} p_r \quad , \]
\[ q'_r = q_r - q^c_r \quad , \quad p'_r = p_r - p^c_r \quad , \]

so that bosonic \( q^c_r, p^c_r \) are \( c \)-numbers, fermionic \( q^c_r, p^c_r \) are Grassmann \( c \)-numbers (where by a \( c \)-number we mean a multiple of \( 1_N \), the \( N \times N \) unit matrix), and the remainders are traceless,

\[ \text{Tr} q'_r = \text{Tr} p'_r = 0 \quad . \]

Then since \( q^c_r, p^c_r \) commute (anticommute) with \( q'_s, p'_s \) for \( r, s \) both bosonic (fermionic), we see that the classical parts of the phase space variables make no contribution to \( \tilde{\mathcal{C}} \), and Eq. (7) can be rewritten as

\[ \tilde{\mathcal{C}} = \sum_{r \text{ bosons}} [q'_r, p'_r] - \sum_{r \text{ fermons}} \{q'_r, p'_r\} \quad . \]

Thus \( \tilde{\mathcal{C}} \) is completely independent of the values of the classical parts of the matrix phase space variables.

Making the assumption that trace dynamics is ergodic (which undoubtedly requires an interacting as opposed to a free theory, and may presuppose taking the \( N \to \infty \) limit), one can then analyze [4] the statistical mechanics of trace dynamics for the generic case in which
the conserved quantities are the trace Hamiltonian $H$ and the operator $\tilde{C}$. As discussed in
detail in the second paper cited in Ref. [5], the analysis of [4] carries over to the case in which
the fermions are represented by Grassmann matrices; the demonstration of a generalized
Liouville theorem still holds, and the requirements for convergence of the partition function
are much less stringent, eliminating the complexities addressed in Appendix F of Ref. [4].
With Grassmann fermions, for the typical models we are studying the bosonic part of $H$ is a
positive operator, from which $H$ inherits good positivity properties. The canonical ensemble
then takes the simple form given in Eq. (48c) of [4],

$$
\rho = Z^{-1} \exp(-\tau H - \text{Tr} \tilde{\lambda} \tilde{C})
$$

$$
Z = \int d\mu \exp(-\tau H - \text{Tr} \tilde{\lambda} \tilde{C})
$$

(13)

with $d\mu$ the invariant matrix (or operator) phase space measure provided by Liouville’s the-
orem, with $\tau$ a real number, and with $\tilde{\lambda}$ an anti-self-adjoint matrix that in the generic case
(which we assume) has no zero eigenvalues. (Equation (13) can be derived directly [4] by
maximizing the entropy subject to the constraints imposed by the conservation of $H$ and $\tilde{C}$,
or indirectly [8] by first calculating the corresponding microcanonical ensemble correspond-
ing to these conserved quantities, and then using standard statistical physics methods to
calculate the canonical ensemble from the microcanonical one.) We wish to make two points
about the partition function defined in Eq. (13). First of all, it is not invariant under the
unitary transformation of Eq. (8a) for fixed $\tilde{\lambda}$, but is invariant when $\tilde{\lambda}$ is simultaneoulsy
transformed to $U^\dagger \tilde{\lambda} U$; hence the partition function breaks unitary invariance, but has a
specific form of unitary covariance. Second, the partition function contains a weighted sum
over all possible commutators $[q_r, p_s]$ for bosonic variables and all possible anticommutators
$\{q_r, p_s\}$ for fermionic variables; there is no restriction to the classical or quantum mechanical
evaluation of these commutators/anticommutators as 0 or $i\delta_{rs}$ respectively. However, statistical integrals like Eq. (13) are typically dominated by specific regions of the integration domain, and we will see, by a study of the Ward identities following from Eq. (13), that this can lead to *effective* quantum mechanical commutators inside statistical averages. The structure of the Ward identities or equipartition theorems following from Eq. (13) will be reanalyzed in the next two sections without making approximations used in [4], so as to determine the leading corrections to the emergent canonical commutation relations. From this analysis we will infer a set of conditions for obtaining the full emergent quantum field theory structure of [4].
3. Corrections to the Bosonic Commutator \([q_s, p_r]\) in a Simplified Unitary Invariant Matrix Model

We consider in this section the simplified bosonic matrix model with trace Hamiltonian

\[
H = \text{Tr} \left[ \sum_r \frac{1}{2} p_r^2 + V(\{q_r\}) \right],
\]

with the \(q_r\) self-adjoint \(N \times N\) complex matrix variables and with \(V\) a global unitary invariant potential. This form is general enough to include the matrix model forms of the bosonic field theories of greatest interest, including the Goldstone model, non-Abelian gauge models, and the Higgs model. As we saw in the previous section, the Hamiltonian dynamics for this model conserves both the real number \(H\) and the matrix \(\tilde{C}\), which in this case is given simply by

\[
\tilde{C} = \sum_r [q_r, p_r].
\]

Letting \(\rho\) and \(Z\) be respectively the canonical ensemble and partition function given in terms of \(H\) and \(\tilde{C}\) by Eq. (13), we define the ensemble average of an arbitrary function \(O\) of the dynamical variables by

\[
\langle O \rangle_{AV} = \int d\mu \rho O = Z^{-1} \int d\mu e^{-\tau H - \text{Tr}(\tilde{C})} O.
\]

Letting \(O\) be the conserved operator \(\tilde{C}\), and noting that the right hand side of Eq. (13) can be a function only of the ensemble parameters \(\tilde{\lambda}\) and \(\tau\), we have

\[
\langle \tilde{C} \rangle_{AV} = f(\tilde{\lambda}, \tau),
\]

with \(f\) an anti-self-adjoint matrix, which in general can be written as a phase matrix \(i_{\text{eff}}\) times a commuting magnitude matrix \(|f|\),

\[
f = i_{\text{eff}} |f|, \quad i_{\text{eff}}^2 = -1, \quad i_{\text{eff}}^\dagger = -i_{\text{eff}}, \quad [i_{\text{eff}}, |f|] = 0.
\]
We shall now specialize to an ensemble for which the magnitude matrix $|f|$ makes no distinction among the different bases in Hilbert space, and so takes the form of a positive real multiple (which we shall call $\hbar$) of the unit matrix. [As discussed in Appendix B of Ref. [4], when $H$ can be expressed in terms of the phase space operators $\{q_r, p_r\}$ using only real number coefficients, this assumption implies that we are restricting attention to the special class of ensembles for which $\tilde{\lambda} = i_{\text{eff}} \lambda$, with $\lambda$ a real multiple of the unit matrix.] Equations (16a, b) then become

$$\langle \tilde{C} \rangle_{AV} = i_{\text{eff}} \hbar. \quad (16c)$$

Since for finite $N$ we necessarily have $\text{Tr} \tilde{C} = 0$, the phase matrix $i_{\text{eff}}$ must have vanishing trace,

$$\text{Tr} i_{\text{eff}} = 0, \quad (16d)$$

which implies that $i_{\text{eff}}$ has $N/2$ eigenvalues $i$ and $N/2$ eigenvalues $-i$. Thus, we are making a choice of ensemble for which the $U(N)$ symmetry of $H$ is broken, by the term $\text{Tr} \tilde{\lambda} \tilde{C}$, to $U(N/2) \times U(N/2) \times R$, with $R$ the discrete reflection symmetry that interchanges the eigenvalues $\pm i$ of $i_{\text{eff}}$. This is clearly the largest symmetry group of the ensemble for which one can have $\langle \tilde{C} \rangle_{AV} \neq 0$; if one were to attempt to preserve the full $U(N)$ symmetry by taking an ensemble with $\tilde{\lambda} = i\lambda$, with $\lambda$ a $c$-number, then in the canonical ensemble the term $\text{Tr} \tilde{\lambda} \tilde{C}$ would vanish by virtue of the tracelessness of $\tilde{C}$, and the resulting ensemble would have $\langle \tilde{C} \rangle_{AV} = 0$. Requiring the largest possible nontrivial symmetry group plays the role in our derivation of giving a single Planck constant for all pairs of canonical variables; if on the other hand, we were to sacrifice all of the $U(N)$ symmetry by allowing generic $\tilde{\lambda}$, then the emergent canonical commutation relations derived below would generically yield $N/2$
different $h$’s for the $N/2$ pairs of canonical variables. It would clearly be desirable to have a deeper justification from first principles of our choice of ensemble, perhaps based on a more detailed understanding of the underlying dynamics, but at present we must simply introduce it as a postulate.

For this choice of ensemble, let us now consider the Ward identity obtained from

$$ Z\langle \text{Tr} \tilde{C} p_r \rangle_{AV} = \int d\mu e^{-\tau H - \text{Tr}\tilde{C}\tilde{C} p_r}, \quad (17a) $$

by using invariance of the measure $d\mu$ under a constant shift of $p_s$, which implies

$$ 0 = \int d\mu \delta_{p_s} \left[ e^{-\tau H - \text{Tr}\tilde{C}\tilde{C} p_r} \right] 
= \int d\mu e^{-\tau H - \text{Tr}\tilde{C}\tilde{C}}
\times \left[ (-\tau \delta_{p_s} H - \text{Tr}\tilde{C}\delta_{p_s} + \text{Tr}\delta_{p_s} \tilde{C} p_r) + \text{Tr} (\delta_{p_s} \tilde{C}) p_r + \text{Tr} \tilde{C} \delta_{rs} \delta p_s \right] \quad (17b) $$

Now from Eqs. (14a, b) we have

$$ \delta_{p_s} H = \text{Tr} p_s \delta p_s, \quad \delta_{p_s} \tilde{C} = [q_s, \delta_{p_s}] \quad (18) $$

Substituting these into Eq. (17b), multiplying by $Z^{-1}$, and using the trilinear cyclic identities of Eq. (2c), we get

$$ 0 = \langle (-\tau \text{Tr} p_s \delta p_s - \text{Tr}[\tilde{C}, q_s] \delta p_s) \text{Tr} \tilde{C} p_r + \text{Tr}[p_r, q_s] \delta p_s + \text{Tr} \tilde{C} \delta_{rs} \delta p_s \rangle_{AV} \quad (19a) $$

which since $\delta p_s$ is an arbitrary self-adjoint matrix, implies that the operator multiplying $\delta p_s$ inside the trace must vanish,

$$ 0 = \langle (-\tau p_s - [\tilde{C}, q_s]) \text{Tr} \tilde{C} p_r + [p_r, q_s] + \tilde{C} \delta_{rs} \rangle_{AV} \quad (19b) $$

Since $\tilde{C}$ is a constant matrix, it can be taken outside the ensemble average, and so the second term in Eq. (19b) takes the form

$$ -[\tilde{C}, \langle q_s \text{Tr} \tilde{C} p_r \rangle_{AV}] \quad (19c) $$
which vanishes since the ensemble average inside the commutator in Eq. (19c) is a matrix function only of \( \tilde{\lambda} \) and \( \tau \), and hence commutes with \( \tilde{\lambda} \). Substituting \( p_r = p_r^c + p_r' \) into the term multiplied by \( \tau \) in Eq. (19b), and using \( \text{Tr} \tilde{C} p_r^c = p_r^c \text{Tr} \tilde{C} = 0 \), the traceless part of Eq. (19b) reduces to

\[
0 = \langle -\tau p'_s \text{Tr} \tilde{C} p'_r + [p_r, q_s] + \tilde{C} \delta_{rs} \rangle_{AV} .
\]  

(20a)

Since \( \langle \tilde{C} \rangle_{AV} = i_{\text{eff}} \hbar \), this equation can be rewritten as the exact relation

\[
\langle [q_s, p_r] \rangle_{AV} = i_{\text{eff}} \hbar \delta_{rs} - \tau \langle p'_s \text{Tr} \tilde{C} p'_r \rangle_{AV} ,
\]

(20b)

showing that the ensemble averaged commutator of the canonical coordinate and momentum operators has the form of the usual quantum mechanical canonical commutator, with \( i_{\text{eff}} \) playing the role of the imaginary unit, and with a correction term proportional to \( \tau \) that is quadratic in the non-classical parts of the canonical momenta. Using Eqs. (14b) and (16c), Eq. (20b) can also be written in the form

\[
\sum_{t,u} \langle [q_t, p_u] \rangle_{AV} (\delta_{tr} \delta_{us} - \delta_{tu} \delta_{rs}) = -\tau \langle p'_s \text{Tr} \tilde{C} p'_r \rangle_{AV} .
\]

(20c)

The derivation given here sharpens that given in Adler and Millard [4], both in that here no approximations have been made, and that because we are working in complex Hilbert space, we have not had to first project out the parts of \( q_s \) and \( p_r \) that commute with \( i_{\text{eff}} \).*

We conclude this section with several remarks on the principal result of Eq. (20b). First of all, the fluctuations about the ensemble average are fundamental to the possibility of emergent quantum behavior. The average of the commutator on the left hand side of

* As discussed in the Appendix, this projection is needed in real and quaternionic Hilbert space.
Eq. (20b) is not the same as the commutator of the averages, which in fact vanishes since \( \langle q_s \rangle_{AV} \) and \( \langle p_r \rangle_{AV} \) are both functions only of \( \tau \) and \( \tilde{\lambda} \),

\[
\langle [q_s, p_r] \rangle_{AV} \neq \langle \{q_s, p_r\} \rangle_{AV} = 0 .
\]  
(21a)

Secondly, in order to have emergent quantum behavior, the dynamics must be such that the second term on the right hand side of Eq. (20b) is much smaller than the first term on the right hand side, that is, one must have

\[
h^{-1} \tau |\langle p'_s \text{Tr} \tilde{\lambda} \tilde{C} \text{Tr} \tilde{C} p'_r \rangle_{AV}| \ll 1 .
\]  
(21b)

We shall now show that this condition cannot be satisfied if \( \tilde{C} \) is an extensive thermodynamic quantity that grows linearly with the size of the system. To see this, we note that a second Ward identity, similar in form to Eq. (20b), can be derived by starting from

\[
0 = \int d\mu \delta_{p_s} \left[ e^{-\tau \mathcal{H} - \text{Tr} \tilde{\lambda} \tilde{C} \text{Tr} i_{eff} \hbar p_r} \right] ,
\]  
(22a)

in which the factor of \( \tilde{C} \) multiplying \( p_r \) has been replaced by its ensemble average \( i_{eff} \hbar \), and then proceeding as in Eqs. (17b-20b) above. The resulting Ward identity is

\[
0 = i_{eff} \hbar \delta_{rs} - \tau \langle p'_s \text{Tr} i_{eff} \hbar p'_r \rangle_{AV} ,
\]  
(22b)

and is an analog in our context of the usual equipartition theorem of classical statistical mechanics. Now if \( \tilde{C} \) were an extensive quantity, the difference between \( \tilde{C} \) and its ensemble average \( i_{eff} \hbar \) would be a fluctuation that vanishes as the system size becomes infinite, which would make it permissible to accurately approximate the right hand side of Eq. (20b) by the right hand side of Eq. (22b). This would lead to the conclusion \( \langle [q_s, p_r] \rangle_{AV} = 0 \), that is, the thermodynamics would give emergent classical, rather than quantum mechanical, behavior.
Thus, the inequality of Eq. (21b) can hold only if $\bar{C}$ is not extensive; this conclusion is consistent with the observation that to have the average of $\bar{C}$ play the role of the intensive quantity $i_{\text{eff}}\hbar$, one would expect that $\bar{C}$ should behave as a thermodynamically intensive quantity. We shall give further evidence for this conclusion in the next section, where we consider systems containing fermions as well as bosons.

Although we have first discussed the purely bosonic model of this section for expository reasons, it is easy to see from Eqs. (20b, c) that the inequality of Eq. (21b) cannot hold in a purely bosonic system. When the right hand side of Eq. (20c) can be neglected, multiplying by $\delta_{rs}$ and summing over $r, s$ gives the relation

$$\sum_{t,u}\langle[q_t, p_u]\rangle_{AV}\delta_{tu}(1 - N) = 0,$$

(22c)

which for $N > 1$ implies that $\langle[q_s, p_r]\rangle_{AV} = 0$, again giving classical behavior. Thus, a purely bosonic matrix dynamics system in complex Hilbert space cannot have emergent quantum behavior. However, we shall see in the next section that in the interesting case in which the numbers of bosonic and fermionic degrees of freedom are equal, a condition that holds for supersymmetric theories, the relation of Eq. (22c) is modified, and emergent quantum behavior becomes possible.

4. Corrections to the Full Canonical Algebra in a General Unitary Invariant Matrix Model with Classical Sources

Although one can give derivations similar to that of the previous section for other canonical commutators (e.g., the $[p_r, p_s]$ and $[q_r, q_s]$ commutators, or the $[p_s, q_r]$ commutator obtained by interchanging the roles of $p$ and $q$ in the above derivation), and their fermionic anticommutator analogs, it is most efficient in the general case to use the sym-
plectic formalism introduced by Adler and Millard in Ref. [4]. In this notation one defines
\( x_1 = q_1, x_2 = p_1, x_3 = q_2, x_4 = p_2, \ldots, x_{2R-1} = q_R, x_{2R} = p_R \) for the matrix phase space
variables; in terms of these, the Hamilton equations of Eq. (5b), the generalized Poisson
bracket of Eq. (6a), and the conserved operator \( \tilde{C} \) of Eq. (7) take the form
\[
\dot{x}_r = \sum_{s=1}^{2R} \omega_{rs} \frac{\delta H}{\delta x_s},
\]
\[
\{A, B\} = \text{Tr} \sum_{r,s=1}^{2R} \frac{\delta A}{\delta x_r} \omega_{rs} \frac{\delta B}{\delta x_s},
\]
\[
\tilde{C} = \sum_{r,s} x_r \omega_{rs} x_s,
\]
with \( \omega \) a numerical symplectic metric given (in terms of standard Pauli matrices \( \tau_{1,2,3} \)) by
\( i\tau_2 \) for a bosonic pair of canonical variables, and by \( -\tau_1 \) for a fermionic canonical pair (see
Eqs. (10a-c) on p. 202, and Eqs. (10a'-c') on p. 224, of Ref. [4].) This symplectic metric
obeys [4] the useful identities
\[
\omega_{sr} = -\epsilon_r \omega_{rs} = -\epsilon_s \omega_{rs},
\]
\[
\sum_r \omega_{rs} \omega_{rt} = \sum_r \omega_{sr} \omega_{tr} = \delta_{st}.
\]

We shall now consider a matrix dynamics generated by a general trace Hamiltonian
\( H \), that can contain fermionic as well as bosonic degrees of freedom, and in the statistical
partition function shall allow the presence of nonvanishing classical sources \( J_r^c \) for the classical
parts \( x_r^c \) of the phase space variables [cf. Eqs. (12a-c) above.] Thus we start from the ensemble
\[
\rho = Z^{-1} \exp(-\tau H - \text{Tr} \tilde{\lambda} \tilde{C} - \sum_r J_r^c x_r^c),
\]
\[
Z = \int d\mu \exp(-\tau H - \text{Tr} \tilde{\lambda} \tilde{C} - \sum_r J_r^c x_r^c),
\]
which is now used in place of the ensemble of Eq. (13) in the definition of \( \langle O \rangle_{AV} \) given in
Eq. (15). For this ensemble, we consider the Ward identity obtained by using shift invariance
of the integration measure $d\mu$ starting from

$$Z(\text{Tr}\tilde{C}\sigma_t x'_t)_{AV} = \int d\mu e^{-\tau \mathbf{H} - \text{Tr} \tilde{\lambda} \tilde{C} - \sum_r J'_r x'_r} \text{Tr}\tilde{C}\sigma_t x'_t,$$  \hspace{1cm} (25a)

where $\sigma_t$ are a set of $c$-number auxiliary parameters that are complex for bosonic $x_t$, and complex Grassmann for fermionic $x_t$. As in Eqs. (12a, b) above, we use the notation $x'_t$ to denote the noncommutative part of $x_t$ that remains when the classical part is subtracted away, that is, $x'_t = x_t - x^c_t$.

The Ward identity derivation now proceeds exactly as in Eqs. (17b-20b). Making a constant shift of the noncommutative part $x'_s$ of the phase space variable $x_s$, we have

$$0 = \int d\mu \delta_{x'_s} \left[ e^{-\tau \mathbf{H} - \text{Tr} \tilde{\lambda} \tilde{C} - \sum_r J'_r x'_r} \text{Tr}\tilde{C}\sigma_t x'_t \right]$$

$$= \int d\mu e^{-\tau \mathbf{H} - \text{Tr} \tilde{\lambda} \tilde{C} - \sum_r J'_r x'_r} \times \left[ (-\tau \delta_{x'_s} \mathbf{H} - \text{Tr}\tilde{\lambda} \delta_{x'_s} \tilde{C}) \text{Tr}\tilde{C}\sigma_t x'_t + \text{Tr}(\delta_{x'_s} \tilde{C})\sigma_t x'_t + \text{Tr}\tilde{C}\sigma_t \delta_{st} \delta x'_s \right] \hspace{1cm} (25b)$$

Now from Eqs. (23a, b) we have

$$\delta_{x'_s} \mathbf{H} = \text{Tr} \left( \frac{\delta \mathbf{H}}{\delta x'_s} \right)' = \text{Tr} \sum_r \dot{x}'_r \omega_{r s} \delta x'_s,$$

$$\text{Tr}\tilde{\lambda} \delta_{x'_s} \tilde{C} = \text{Tr} \tilde{\lambda} \sum_r \omega_{r s} (x'_r \delta x'_s - \epsilon_r \delta x'_s x'_r) = \text{Tr} \tilde{\lambda} \sum_r \omega_{r s} x'_r \delta x'_s \hspace{1cm} (26)$$

$$\text{Tr}(\delta_{x'_s} \tilde{C})\sigma_t x'_t = \text{Tr}[\sigma_t x'_t, \sum_r \omega_{r s} x'_r] \delta x'_s.$$

Substituting these into Eq. (25b) and multiplying by $Z^{-1}$, we get

$$0 = \langle (-\tau \text{Tr} \sum_r \dot{x}'_r \omega_{r s} \delta x'_s - \text{Tr} \tilde{\lambda} \sum_r \omega_{r s} x'_r \delta x'_s) \text{Tr}\tilde{C}\sigma_t x'_t$$

$$+ \text{Tr}[\sigma_t x'_t, \sum_r \omega_{r s} x'_r] \delta x'_s + \text{Tr}\tilde{C}\sigma_t \delta_{st} \delta x'_s \rangle_{AV} \hspace{1cm} (27a)$$

which since $\delta x'_s$ is an arbitrary traceless matrix (with the adjointness properties of $x'_s$), implies that the traceless part of the operator multiplying $\delta x'_s$ inside the trace must vanish,

$$0 = \langle (-\tau \sum_r \dot{x}'_r \omega_{r s} - \tilde{\lambda} \sum_r \omega_{r s} x'_r) \text{Tr}\tilde{C}\sigma_t x'_t$$

$$+ \sigma_t x'_t \sum_r \omega_{r s} x'_r + \text{Tr}\tilde{C}\sigma_t \delta_{st} \rangle_{AV} \hspace{1cm} (27b)$$
Since $\tilde{\lambda}$ is a constant matrix, as before it can be taken outside the ensemble average, and so the second term in Eq. (27b) takes the form

$$- [\tilde{\lambda}, \sum_r \omega_{rs} \langle x'_r \text{Tr} \tilde{C} \sigma_t x'_t \rangle_{AV}] ,$$

(27c)

which again vanishes since the ensemble average inside the commutator is a matrix function only of $\tilde{\lambda}$ and $\tau$. Contracting the remainder of Eq. (27b) with $\sum_s \omega_{us}$, using Eqs. (16c) and (23b), and noting that $[x'_u, \sigma_t x'_t] = [x_u, \sigma_t x_t]$ because the classical parts do not contribute to the commutator, we get as our final result

$$\langle [x_u, \sigma_t x_t] \rangle_{AV} = i_{\text{eff}} \bar{\hbar} \omega_{ut} \sigma_t - \tau \langle \dot{x}'_u \text{Tr} \tilde{C} \sigma_t x'_t \rangle_{AV} .$$

(28a)

Equation (28a), like Eq. (20b) of the preceding section that it generalizes, is exact.

In order to have emergent quantum behavior, it is necessary that the second term on the right hand side of Eq. (28a) be much smaller than the first term, that is, we require

$$\frac{1}{\hbar \tau} |\langle \dot{x}'_u \text{Tr} \tilde{C} \sigma_t x'_t \rangle_{AV}| << 1 .$$

(28b)

Again, by replacing $\tilde{C}$ by its expectation value at the start of the derivation leading to Eq. (28a), we get a second Ward identity

$$0 = i_{\text{eff}} \hbar \omega_{ut} \sigma_t - \tau \langle \dot{x}'_u \text{Tr} i_{\text{eff}} \hbar \sigma_t x'_t \rangle_{AV} .$$

(28c)

Hence the inequality of Eq. (28b) can be satisfied only if $\tilde{C}$ is not an extensive quantity, since if $\tilde{C}$ were extensive one could, in the large system limit, approximate it in Eq. (28b) by its expectation $i_{\text{eff}} \hbar$, giving an expression that, by Eq. (28c), cannot be small.

Because the derivation of this section is valid for fermions as well as bosons, one can in fact make a stronger statement about the conditions for emergent quantum behavior.
Letting the indices $t$ and $u$ in Eq. (28a) be either both bosonic or both fermionic, we get the respective relations

$$
\langle \{q_r, p_r\} \rangle_{AV} = i_{\text{eff}} \hbar - \tau \langle \dot{q}_r' \text{Tr} \tilde{C} p'_r \rangle_{AV} \quad r \text{ bosonic}
$$
$$
\langle \{q_r, p_r\} \rangle_{AV} = i_{\text{eff}} \hbar - \tau \langle \dot{q}_r' \text{Tr} \tilde{C} p'_r \rangle_{AV} \quad r \text{ fermionic}
$$

(29a)

Substituting this into Eq. (7) for $\tilde{C}$, taking the ensemble average, and using Eq. (16c), we get

$$
i_{\text{eff}} \hbar = \langle \tilde{C} \rangle_{AV} = \langle \sum_{r \text{ bosons}} q_r, p_r \rangle - \langle \sum_{r \text{ fermions}} \{q_r, p_r\} \rangle_{AV}
$$

$$
= \left( \sum_{r \text{ bosons}} - \sum_{r \text{ fermions}} \right) i_{\text{eff}} \hbar - \tau \left( \sum_{r \text{ bosons}} - \sum_{r \text{ fermions}} \right) \langle \dot{q}_r' \text{Tr} \tilde{C} p'_r \rangle_{AV}
$$

(29b)

which on division by $\hbar$ and transposition of terms gives

$$
\left( \sum_{r \text{ bosons}} - \sum_{r \text{ fermions}} \right) \hbar^{-1} \tau \langle \dot{q}_r' \text{Tr} \tilde{C} p'_r \rangle_{AV} = i_{\text{eff}} \left( \sum_{r \text{ bosons}} - \sum_{r \text{ fermions}} -1 \right)
$$

(29c)

When the condition of Eq. (28b) for emergent canonical behavior is satisfied, the left hand side of Eq. (29c) is a sum of very small terms. Assuming that this sum yields at most a finite, bounded total, let us consider the case in which $r$ includes the spatial label of a translation invariant field theory. Then the number of bosonic and fermionic modes per unit volume contributing on the right hand side of Eq. (29c) must be equal, since if not, the right hand side of Eq. (29c) would become infinite as the spatial volume grows to infinity, contradicting the boundedness of the left hand side. Therefore, in a complex Hilbert space*, a candidate pre-quantum mechanics theory must have equal numbers of bosonic and fermionic degrees of freedom, making it plausible that such a candidate theory should be supersymmetric. When

* For a discussion of how our arguments must be modified in real and quaternionic Hilbert space, see the Appendix.
the numbers of bosonic and fermionic modes are in balance, Eq. (29c) simplifies to

\[
\left( \sum_{r \text{ bosons}} - \sum_{r \text{ fermions}} \right) \hbar^{-1} \tau \langle \hat{q}_r' \text{Tr} \tilde{C} \hat{p}_r' \rangle_{AV} = -i_{\text{eff}} , \tag{29d}
\]

showing that the remainder terms in Eq. (28a), that are neglected when Eq. (28b) is satisfied, sum in Eq. (29c) to give a total of unit magnitude.

Corresponding to the Ward identity of Eq. (28a), we can derive a class of more general Ward identities by replacing \( \sigma_t x'_t \) in Eq. (25b) by a general \( U \), constructed as a Weyl ordered (i.e., symmetrized) polynomial in the products \( \{ \sigma_r x_r \} \), with coefficients that are \( c \)-number functions of 1 and of \( i_{\text{eff}} \). In place of Eq. (27b), we now get

\[
0 = \langle ( -\tau \sum_r \hat{x}_r' \omega_{rs} - [\tilde{\lambda}, \sum_r \omega_{rs} x'_r]) \text{Tr} \tilde{C} U 
+ [U, \sum_r \omega_{rs} x'_r] + \sum_{\text{each } x_s \text{ in } U} U(\text{one } x_s \rightarrow \tilde{C} )' \rangle_{AV} . \tag{30a}
\]

As long as \( U \) has coefficients that depend only on the matrices (or operators) 1 and \( i_{\text{eff}} \), the second term on the right in Eq. (30a), which involves a commutator with \( \tilde{\lambda} \), vanishes by the same arguments as before. Contracting the remainder with \( \omega_{us} \), but making no approximations, we get as the exact general Ward identity analogous to Eq. (28a),

\[
\langle [x_u, U] \rangle_{AV} = \langle \sum_s \omega_{us} \sum_{\text{each } x_s \text{ in } U} U(\text{one } x_s \rightarrow \tilde{C} )' \rangle_{AV} - \tau \langle \hat{x}_u' \text{Tr} \tilde{C} U \rangle_{AV} . \tag{30b}
\]

Suppose now that we can make the following two approximations, (i) we replace \( \tilde{C} \) in the first term on the right hand side of Eq. (30b) by its ensemble average \( i_{\text{eff}} \hbar \), and (ii) we neglect the \( \tau \) term in Eq. (30b). We then are left with the relation

\[
\langle [x_u, U] \rangle_{AV} = \langle \sum_s \omega_{us} \sum_{\text{each } x_s \text{ in } U} U(\text{one } x_s \rightarrow i_{\text{eff}} \hbar') \rangle_{AV} , \tag{30c}
\]

which extends the effective canonical algebra inside ensemble averages to the commutator of \( x_u \) with a general Weyl ordered polynomial \( U \).
By the methods of Appendix E of [4], Equation (30c) can be extended to include sources for the remainder parts $x'_r$ of the phase space variables. Specializing the relations obtained this way to the case $U = i_{\text{eff}}$ implies that $i_{\text{eff}}$ can be freely commuted with phase space variables inside ensemble averages. As argued in [4], the resulting set of Ward identities then yields the canonical generator structure, including the time evolution relations, of Heisenberg picture quantum field theory. The only assertion in [4] that cannot be derived this way is the claim that the time evolution equation is more exact than the other generator relations; this claim used the assumption that $\tilde{C}$ can be replaced by its ensemble average inside the $\tau$ term, which we have seen is not correct. The remainder of the conclusions of [4] rest on the two approximations that we made above, which can be rephrased as the assumptions that, (i) in the terms of Eq. (30b) that involve the unvaried canonical ensemble $\rho$ of Eq. (13) with a factor of $\tilde{C}$ in the integrand, the fact that the ensemble is sharply peaked around the mean allows us to replace the integrand factor $\tilde{C}$ by its ensemble average $i_{\text{eff}}\bar{\hbar}$, and (ii) the canonical ensemble displays a certain rigidity, in that terms of the form $\int d\mu \delta \rho \text{Tr} \tilde{C} U$ can be dropped. On the other hand, we have seen that terms of the form $\int d\mu \delta \rho \text{Tr} i_{\text{eff}} \bar{\hbar} U$ cannot be dropped; this does not contradict our assumptions because $\delta \rho$ can be rapidly varying around the peak of the ensemble.

5. Discussion

We have seen that a statistical mechanics can be formulated for a wide class of matrix models with a global unitary invariance, and that within this statistical mechanics, the ensemble averages of canonical variables obey the exact relations of Eq. (20b), (28a), and (29a), and (30a). When these can be approximated by dropping the $\tau$ terms [and, in Eq. (30b), replacing the $\tilde{C}$ insertions in $U'$ by their ensemble averages], the result is
emergent quantum mechanical behavior for the statistical ensemble averages. The condition for validity of the approximation of neglecting the $\tau$ terms is rather delicate: we have argued that it requires that $\tilde{C}$ should be an intensive thermodynamic quantity, and that the numbers of bosonic and fermionic degrees of freedom should balance.

We conclude with some brief remarks:

[1] In the first two references of [3], it is shown that one can readily formulate trace dynamics models in which global unitary operator invariance is gauged to give a local unitary operator invariance. Since global unitary invariance is a special case of local unitary invariance, the considerations of this paper apply to these models.

[2] In Refs. [5], it is shown that supersymmetric Yang Mills theory, and the related “matrix model for $M$ theory”, fit naturally into the trace dynamics framework analyzed in this paper. In these models, $\tilde{C}$ vanishes up to a surface term contribution, a behavior consistent with its being an intensive thermodynamic quantity.

[3] Although our final results of Eqs. (20b), (28a), and (29a) superficially resemble the string-inspired formula of Eq. (1b), there is an important difference. In Eq. (1b) the coordinate is a quantum operator, as is usual in nonrelativistic quantum mechanics, whereas in our results of Eqs. (20b), etc., the coordinate is merely a degree of freedom label $r$, as it always is in quantum field theory, and the coordinates and momenta are canonical field variables with label $r$. It may be possible to make a connection between the two types of modified commutation relations when the metric structure of the coordinate manifold is taken into account, using the fact that the proper distance is related to the coordinate interval by $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. In a field theoretic interpretation, $dx^\mu$ is just the change in the degree of freedom label, but the metric is
a dynamical variable, and hence so is $ds$. This suggests that there may be an analog of Eq. (1b) involving the relativistic proper distance, and that this is the relation to be compared with our results in this paper.

[4] As is well known, in a complex Hilbert space the canonical algebra $[q, p] = i$, $[q, i] = [p, i] = 0$ cannot have finite dimensional representations, since this algebra implies, for example, the relation $q^2 p^2 + p^2 q^2 - 2qp^2 q = -2$, which in a finite dimensional Hilbert space would have a left hand side with trace zero and a right hand side with trace nonzero. However, it is consistent for the canonical algebra to emerge as the limit $N \to \infty$ of an algebra in an $N$ dimensional Hilbert space, which is the behavior argued for in Ref. [4] and here. Because the emergent canonical algebra involves not the imaginary unit $i$ of the underlying complex Hilbert space, but rather the operator $i_{\text{eff}}$ with $\text{Tr}i_{\text{eff}} = 0$, a basis for the operator algebra in the emergent theory is provided by a set of operators that commute with $i_{\text{eff}}$, together with one additional operator that anticommutes with $i_{\text{eff}}$, and that plays the role of the time reversal operator in the emergent complex quantum mechanics. In fact, because the condition $\text{Tr}i_{\text{eff}} = 0$ implies that one can find a representation in which $i_{\text{eff}}$ is a real matrix (just as for Pauli matrices $\rho_{1,2,3}$ the matrix $i\rho_3$ can be given the real form $i\rho_2$ by a change of representation), the quantum mechanics emergent from matrix dynamics has the structure of a complexified real quantum mechanics, for which the operator algebra has the form just described (see, e.g., Sec. 2.6 of the second citation in Ref. [3]).

[5] We have seen that the emergence of quantum mechanical behavior from matrix model statistical mechanics requires a certain “rigidity” of the statistical ensemble.
It is easy to see [8, 9] that this rigidity is a sufficient condition for the canonical and microcanonical ensembles to give the same Ward identities, and hence the same emergent quantum behavior. The need for a rigid statistical ensemble in our context suggests a possible analogy with the concept of London rigidity in the theory of superconductivity [10]. In the presence of an applied vector potential $\vec{A}$, the induced current density in a metal is given by

$$\langle \vec{j} \rangle = -\frac{ne}{m} \langle \vec{p} + \frac{e}{c} \vec{A} \rangle .$$

(31)

In a normal metal the two terms on the right hand side of Eq. (31a) nearly cancel, leaving a small residual diamagnetism. However, in a superconductor the rigidity of the wave function leads to the vanishing of $\langle \vec{p} \rangle$, giving perfect diamagnetism and the Meissner effect. An analogy with the results of this paper would equate normal metal behavior with the case in which $\tilde{C}$ can be replaced by its ensemble average in the $\tau$ term; in this case the right hand side of Eq. (20b) is approximately equal to the right hand side of Eq. (22b), leading to cancellation of the emergent canonical commutator. Similarly, the analogy would equate superconducting behavior with the case in which the $\tau$ term containing $\tilde{C}$ can be dropped because of “rigidity” of $\delta \rho$, leading through Eqs. (20b), (28a), and (29a) to an emergent canonical commutator as an analog of the superconductive Meissner effect.

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**Appendix: Real and Quaternionic Hilbert Space**

For reasons that we now describe, in real and quaternionic Hilbert space the arguments of this paper must be modified and yield weaker conclusions. The underlying reason for this modification is that only in complex Hilbert space can one have a non-real trace that nonetheless obeys the cyclic property. In quaternionic Hilbert space, as a consequence of the noncommutativity of the quaternions, only the real part of the ordinary trace obeys the cyclic property. In real Hilbert space, the trace is necessarily real, and the trace of any anti-self-adjoint operator vanishes. Thus in these two cases, if one follows [3, 4] and defines the graded trace $\text{Tr}$ to be the one that obeys the cyclic property, then the definition must include taking the real part, and to get a nonzero result one must require the operator argument $V$ of $\text{Tr} V$ to be self-adjoint. As a consequence, in the general case derivation analogous to that starting from Eq. (25b), one must consider $\text{Tr}\{i_{\text{eff}}, \tilde{C}\} \sigma_t x'_t$ rather than $\text{Tr} \tilde{C} \sigma_t x'_t$. This gives the following analog of Eq. (28a),

$$
\langle [x_u, \sigma_t \{i_{\text{eff}}, x_t\}] \rangle_{AV} = \{i_{\text{eff}}, i_{\text{eff}} \hbar\}' \omega_{ut} \sigma_t - \tau \langle \dot{x}'_u \text{Tr}\{i_{\text{eff}}, \tilde{C}\} \sigma_t x'_t \rangle_{AV} , 
$$

(A1)

where the prime on the first term on the right hand side of Eq. (A1) indicates extraction of the traceless part. However, since the traceless part of $\{i_{\text{eff}}, i_{\text{eff}} \hbar\} = -2 \hbar$ is zero, Eq. (A1) becomes

$$
\langle [x_u, \sigma_t \{i_{\text{eff}}, x_t\}] \rangle_{AV} = -\tau \langle \dot{x}'_u \text{Tr}\{i_{\text{eff}}, \tilde{C}\} \sigma_t x'_t \rangle_{AV} , 
$$

(A2)

which has a structure more like the equation obtained by subtracting Eq. (28c) from Eq. (28a).
than like Eq. (28a) itself. In complex Hilbert space, Eqs. (A2) and Eqs. (28a, c) all hold, and conditions for emergent quantum behavior can be formulated from the latter as in the text; in the real and quaternionic Hilbert space cases, only Eq. (A2) holds, and we cannot proceed with the analysis of the text.

The remarks made here correct a subtle error made in Sec. 6 of Adler and Millard [4]. There, the classical sources introduced in this paper were set to zero, and Ward identities were derived by varying $x_s$, not just the noncommutative part $x'_s$ as we have done here. This gives as the Ward identity analogous to Eq. (A1)

$$\langle [x_u, \sigma_t \{ i_{\text{eff}}, x_t \}] \rangle_{AV} = \{ i_{\text{eff}}, i_{\text{eff}} \bar{h} \} \omega_{ut} \sigma_t - \tau \langle \dot{x}_u \text{Tr} \{ i_{\text{eff}}, \tilde{C} \} \sigma_t x_t \rangle_{AV} . \quad (A3)$$

Equation (A3) is a valid relation, but it actually implies two relations, quite different in structure, for its classical or $c$-number part and its primed or traceless part. Separating Eq. (A3) into these parts, and ignoring interference terms between $x^c$ and $x'$ in the $\tau$ term, one finds that the primed part of Eq. (A3) gives Eq. (A2), while the $c$-number part of Eq. (A3) becomes an equipartition identity for $x^c$ and gives no direct information about the expectation of the commutator. Hence the arguments of [4] do not, in their present form, provide evidence for emergent quantum behavior in the real and quaternionic Hilbert space cases.
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