An Extension of Birkhoff–James Orthogonality Relations in Semi-Hilbertian Space Operators

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Abstract. Let \( \mathcal{B}(\mathcal{H}) \) denote the \( \mathcal{C}^* \)-algebra of all bounded linear operators on a Hilbert space \( (\mathcal{H}, \langle \cdot, \cdot \rangle) \). Given a positive operator \( A \in \mathcal{B}(\mathcal{H}) \), and a number \( \lambda \in [0, 1] \), a seminorm \( \| \cdot \|_{(A,\lambda)} \) is defined on the set \( \mathcal{B}_{A^{1/2}}(\mathcal{H}) \) of all operators in \( \mathcal{B}(\mathcal{H}) \) having an \( A^{1/2} \)-adjoint. The seminorm \( \| \cdot \|_{(A,\lambda)} \) is a combination of the sesquilinear form \( \langle \cdot, \cdot \rangle_A \) and its induced seminorm \( \| \cdot \|_A \). A characterization of Birkhoff–James orthogonality for operators with respect to the discussed seminorm is given. Moving \( \lambda \) along the interval \([0, 1]\), a wide spectrum of seminorms are obtained, having the \( A \)-numerical radius \( w_A(\cdot) \) at the beginning (associated with \( \lambda = 0 \)) and the \( A \)-operator seminorm \( \| \cdot \|_A \) at the end (associated with \( \lambda = 1 \)). Moreover, if \( A = I \) the identity operator, the classical operator norm and numerical radius are obtained. Therefore, the results in this paper are significant extensions and generalizations of known results in this area.

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1. Introduction and Preliminaries

Let \( \mathcal{B}(\mathcal{H}) \) be the algebra of all bounded linear operators on a complex Hilbert space \( \mathcal{H} \) with an inner product \( \langle \cdot, \cdot \rangle \) and the corresponding norm \( \| \cdot \| \). Let \( I \) stand for the identity operator on \( \mathcal{H} \). Throughout this paper, we assume that \( A \in \mathcal{B}(\mathcal{H}) \) is a positive operator, which induces a positive semidefinite sesquilinear form \( \langle \cdot, \cdot \rangle_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C} \) defined by \( \langle x, y \rangle_A = \langle Ax, y \rangle \). We denote by \( \| \cdot \|_A \) the seminorm induced by \( \langle \cdot, \cdot \rangle_A \). For the semi-Hilbertian space \( (\mathcal{H}, \| \cdot \|_A) \) the \( A \)-Cauchy–Schwartz inequality holds, that is, \( |\langle x, y \rangle_A| \leq \|x\|_A \|y\|_A \) for all \( x, y \in \mathcal{H} \). For \( T \in \mathcal{B}(\mathcal{H}) \), an operator \( S \in \mathcal{B}(\mathcal{H}) \) is called an \( A \)-adjoint operator of \( T \) if \( \langle Tx, y \rangle_A = \langle x, Sy \rangle_A \), for every \( x, y \in \mathcal{H} \). The
existence of an $A$-adjoint operator is not guaranteed. The set of all operators admitting $A^{1/2}$-adjoints is denoted by $\mathcal{B}_{A^{1/2}}(\mathcal{H})$. Clearly, $\langle \cdot, \cdot \rangle_A$ induces a seminorm on $\mathcal{B}_{A^{1/2}}(\mathcal{H})$. Indeed, if $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$, then
\[\|T\|_A = \sup \{ \|Tx\|_A : x \in \mathcal{H}, \|x\|_A = 1 \} < +\infty.\]

Notice that it may happen that $\|T\|_A = +\infty$ for some $T \in \mathcal{B}(\mathcal{H}) \setminus \mathcal{B}_{A^{1/2}}(\mathcal{H})$. It can be verified that, for $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$, we have $\|Tx\|_A \leq \|T\|_A \|x\|_A$ for all $x \in \mathcal{H}$.

Furthermore, the $A$-minimum modulus of $T$, denoted by $[T]_A$, is defined as
\[ [T]_A = \inf \{ \|Tx\|_A : x \in \mathcal{H}, \|x\|_A = 1 \}. \]

More details on semi-Hilbertian space operators can be found in [2,9].

The $A$-numerical radius and the $A$-Crawford number of $T \in \mathcal{B}(\mathcal{H})$ are defined, respectively, by
\[ w_A(T) = \sup \{ \|Tx\|_A : x \in \mathcal{H}, \|x\|_A = 1 \}, \]
\[ c_A(T) = \inf \{ \|Tx\|_A : x \in \mathcal{H}, \|x\|_A = 1 \}. \]

In particular, if we consider $A = I$ in the definitions of $A$-operator seminorm, $A$-minimum modulus, $A$-numerical radius and $A$-Crawford number of $T$ then we get the classical operator norm, minimum modulus, numerical radius and Crawford number, respectively, i.e., $\|T\|_A = \|T\|$, $[T]_A = [T]$, $w_A(T) = w(T)$ and $c_A(T) = c(T)$. It is known that $w_A(\cdot)$ defines a seminorm on $\mathcal{B}_{A^{1/2}}(\mathcal{H})$, and that
\[ \frac{1}{2} \|T\|_A \leq w_A(T) \leq \|T\|_A, \quad T \in \mathcal{B}_{A^{1/2}}(\mathcal{H}). \]

For other related information on the numerical radius of operators in semi-Hilbertian spaces, we refer the reader to [12,14,16] and the references therein.

In normed spaces, there are several notions of orthogonality, all of which are generalizations of orthogonality in a Hilbert space. Among them, the Birkhoff–James orthogonality is one of the most important. Given two elements $x, y$ in a normed space $(X, \| \cdot \|)$, it is said that $x$ is orthogonal to $y$, in the Birkhoff–James sense [6,8], denoted by $x \perp_B y$, if
\[ \|x + \xi y\| \geq \|x\|, \quad \xi \in \mathbb{C}. \]

The Birkhoff–James orthogonality plays a central role in approximation theory. On a Hilbert space $\mathcal{H}$, an operator $T \in \mathcal{B}(\mathcal{H})$ is a best approximation of $S \in \mathcal{B}(\mathcal{H})$ from a linear subspace $\mathbb{M}$ of $\mathcal{B}(\mathcal{H})$ if, and only if, $T$ is a Birkhoff–James orthogonal projection of $S$ onto $\mathbb{M}$; see [7] and references therein. Bhatia and Šemrl in [4, Remark 3.1] and Paul in [13, Lemma 2] independently proved that $T \perp_B S$ if and only if there exists a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ such that
\[ \lim_{n \to \infty} \|Tx_n\| = \|T\| \quad \text{and} \quad \lim_{n \to \infty} \langle Tx_n, Sx_n \rangle = 0. \]
It follows that if the Hilbert space $\mathcal{H}$ is finite-dimensional, then $T \perp_B S$ if and only if there is a unit vector $x \in \mathcal{H}$ such that $\|Tx\| = \|T\|$ and $\langle Tx, Sx \rangle = 0$. A number of authors have recently extended the well-known result of Bhatia and Šemrl; see, e.g., [5,7,10,15,17].

In this paper, we define a new seminorm on $\mathbb{B}_{A^{1/2}}(\mathcal{H})$, which generalizes simultaneously the $A$-operator seminorm and the $A$-numerical radius. We give a necessary and sufficient condition to hold that the seminorm of the sum of elements in $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ is equal to the sum of their seminorms. We also characterize Birkhoff–James orthogonality of semi-Hilbertian space operators with respect to this seminorm. Our results cover and extend some theorems in [1,3,4,11,17]. In particular, related to a result due to Bhatia and Šemrl [4], we give another equivalent condition of the Birkhoff–James orthogonality for Hilbert space operators.

2. Main Results

We start the section with the following definition.

**Definition 2.1.** Let $(\mathcal{H}, \langle \cdot , \cdot \rangle)$ be a Hilbert space, $A \in \mathbb{B}(\mathcal{H})$ be a positive operator and $\lambda \in [0,1]$. For every $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, define

$$\| T \|_{(A,\lambda)} = \sup \left\{ \sqrt{\lambda \| Tx \|_A^2 + (1-\lambda) |\langle Tx, x \rangle_A|^2} : x \in \mathcal{H}, \| x \|_A = 1 \right\}.$$  

**Remark 1.** For $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ we have $\| T \|_{(A,0)} = w_A(T)$ and $\| T \|_{(A,1)} = \| T \|_A$.

First of all, let us prove that $\| \cdot \|_{(A,\lambda)}$ is a seminorm on $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ sitting between the $A$-numerical radius and $A$-operator seminorm.

**Proposition 2.2.** The function $\| \cdot \|_{(A,\lambda)}$ is a seminorm on $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ and the following inequality holds for every $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$:

$$w_A(T) \leq \| T \|_{(A,\lambda)} \leq \| T \|_A.$$  

**Proof.** Let $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. It is trivial that $\| \alpha T \|_{(A,\lambda)} = |\alpha| \| T \|_{(A,\lambda)}$ for every $\alpha \in \mathbb{C}$. Therefore, to show that $\| \cdot \|_{(A,\lambda)}$ is a seminorm on $\mathbb{B}_{A^{1/2}}(\mathcal{H})$, it suffices to show that $\| \cdot \|_{(A,\lambda)}$ is subadditive.

Let $x \in \mathcal{H}$ with $\| x \|_A = 1$. We have

$$\lambda \| (T + S)x \|_A^2 + (1-\lambda) |\langle (T + S)x, x \rangle_A|^2$$

$$= \lambda \| Tx + Sx \|_A^2 + (1-\lambda) |\langle Tx, x \rangle_A + \langle Sx, x \rangle_A|^2$$

$$\leq \lambda (\| Tx \|_A + \| Sx \|_A)^2 + (1-\lambda) (|\langle Tx, x \rangle_A| + |\langle Sx, x \rangle_A|)^2$$

$$\leq (\lambda \| Tx \|_A^2 + (1-\lambda) |\langle Tx, x \rangle_A|^2)$$

$$+ 2 (\lambda \| Tx \|_A \| Sx \|_A + (1-\lambda) |\langle Tx, x \rangle_A| |\langle Sx, x \rangle_A|)$$

$$+ (\lambda \| Sx \|_A^2 + (1-\lambda) |\langle Sx, x \rangle_A|^2)$$

$$\leq (\lambda \| Tx \|_A^2 + (1-\lambda) |\langle Tx, x \rangle_A|^2)$$
+ 2\sqrt{\lambda\|Tx\|_A^2 + (1 - \lambda)|\langle Tx, x \rangle_A|^2} \sqrt{\lambda\|Sx\|_A^2 + (1 - \lambda)|\langle Sx, x \rangle_A|^2}

(by the Cauchy–Bunyakovsky–Schwarz inequality)

\leq \|T\|_{(A,\lambda)}^2 + 2\|T\|_{(A,\lambda)} \|S\|_{(A,\lambda)} + \|S\|_{(A,\lambda)}^2 = (\|T\|_{(A,\lambda)} + \|S\|_{(A,\lambda)})^2.

Therefore,

\sqrt{\lambda}\|(T + S)x\|_A^2 + (1 - \lambda)|\langle (T + S)x, x \rangle_A|^2 \leq \|T\|_{(A,\lambda)} + \|S\|_{(A,\lambda)}.

Taking supremum over all \(x \in \mathcal{H}\) with \(\|x\|_A = 1\) in the above inequality, we get

\|T + S\|_{(A,\lambda)} \leq \|T\|_{(A,\lambda)} + \|S\|_{(A,\lambda)}.

Furthermore, by the A-Cauchy–Schwarz inequality, for every A-unit vector \(x \in \mathcal{H}\) we have

\(|\langle Tx, x \rangle_A| = \sqrt{\lambda\|Tx\|_A^2 + (1 - \lambda)|\langle Tx, x \rangle_A|^2}

\leq \lambda\|Tx\|_A^2 + (1 - \lambda)|\langle Tx, x \rangle_A|^2

\leq \lambda\|Tx\|_A^2 + (1 - \lambda)\|Tx\|_A^2 = \|Tx\|_A^2,

and hence

\(|\langle Tx, x \rangle_A| \leq \sqrt{\lambda\|Tx\|_A^2 + (1 - \lambda)|\langle Tx, x \rangle_A|^2} \leq \|Tx\|_A.

This implies that \(w_A(T) \leq \|T\|_{(A,\lambda)} \leq \|T\|_A\) which completes the proof.

\[\square\]

Remark 2. Another lower and upper bound for the seminorm \(\| \cdot \|_{(A,\lambda)}\) of bounded linear operators can be presented as follows. Let \(T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})\). For every \(x \in \mathcal{H}\) with \(\|x\|_A = 1\), by the arithmetic geometric mean inequality, we have

\[2\sqrt{\lambda(1 - \lambda)}c_A(T)\|Tx\|_A \leq 2\sqrt{\lambda(1 - \lambda)}|\langle Tx, x \rangle_A|\|Tx\|_A \leq \lambda\|Tx\|_A^2 + (1 - \lambda)|\langle Tx, x \rangle_A|^2 \leq \|T\|_{(A,\lambda)}^2.

Hence

\[2\sqrt{\lambda(1 - \lambda)}c_A(T)\|Tx\|_A \leq \|T\|_{(A,\lambda)}^2.

Taking supremum over all \(x \in \mathcal{H}\) with \(\|x\|_A = 1\) in the above inequality, we arrive at

\[2\sqrt{\lambda(1 - \lambda)}c_A(T)\|T\|_A \leq \|T\|_{(A,\lambda)}^2.

On the other hand, for every A-unit vector \(x \in \mathcal{H}\), by [16, p. 172], we have

\[\|Tx\|_A^2 \leq 2w_A(T) \left( w_A(T) + \sqrt{w_A^2(T) - c_A^2(T)} \right).

Therefore,
\[ \lambda \|Tx\|^2 + (1 - \lambda) \langle Tx, x \rangle_A^2 \]
\[ \leq 2\lambda w_A(T) \left( w_A(T) + \sqrt{w_A^2(T) - c_A^2(T)} \right) + (1 - \lambda) w_A^2(T) \]
\[ = (1 + \lambda) w_A^2(T) + 2\lambda w_A(T) \sqrt{w_A^2(T) - c_A^2(T)}, \]

which yields
\[ \|T\|^2_{(A,\lambda)} \leq (1 + \lambda) w_A^2(T) + 2\lambda w_A(T) \sqrt{w_A^2(T) - c_A^2(T)}. \]

In the following theorem, we give a necessary and sufficient condition for the equality \( \|T + S\|_{(A,\lambda)} = \|T\|_{(A,\lambda)} + \|S\|_{(A,\lambda)} \) to hold in \( \mathbb{B}_{A^{1/2}}(\mathcal{H}) \).

**Theorem 2.3.** Let \( T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H}) \). The following statements are equivalent.

(i) \( \|T + S\|_{(A,\lambda)} = \|T\|_{(A,\lambda)} + \|S\|_{(A,\lambda)} \).

(ii) There exists a sequence \( \{x_n\} \) of \( A \)-unit vectors in \( \mathcal{H} \) such that
\[ \lim_{n \to \infty} \left( \lambda \langle Sx_n, Tx_n \rangle_A + (1 - \lambda) \langle x_n, Tx_n \rangle_A \langle Sx_n, x_n \rangle_A \right) = \|T\|_{(A,\lambda)} \|S\|_{(A,\lambda)}. \]

**Proof.** (i)\(\Rightarrow\)(ii) Let \( \|T + S\|_{(A,\lambda)} = \|T\|_{(A,\lambda)} + \|S\|_{(A,\lambda)} \). Then, there exists a sequence of \( A \)-unit vectors \( \{x_n\} \) in \( \mathcal{H} \) such that
\[ \lim_{n \to \infty} \left( \lambda \|T + S\|_{(A,\lambda)}^2 + (1 - \lambda) \|\langle T + S, x_n \rangle_A \|^2 \right) = (\|T\|_{(A,\lambda)} + \|S\|_{(A,\lambda)})^2. \]

For every \( n \in \mathbb{N} \), we have
\[ \lambda \|\langle T + S, x_n \rangle_A \|^2 + (1 - \lambda) \|\langle T + S, x_n \rangle_A \|^2 \]
\[ \leq \lambda \|Tx_n\|^2 + 2\text{Re} \left( \lambda \langle Sx_n, Tx_n \rangle_A \right) + \lambda \|Sx_n\|^2 + (1 - \lambda) \|Tx_n\|^2 \]
\[ + 2\text{Re} \left( (1 - \lambda) \langle x_n, Tx_n \rangle_A \langle Sx_n, x_n \rangle_A \right) + (1 - \lambda) \|Sx_n, x_n \rangle_A \|^2 \]
\[ = \lambda \|Tx_n\|^2 + (1 - \lambda) \|\langle Tx_n, x_n \rangle_A \|^2 + \lambda \|Sx_n\|^2 + (1 - \lambda) \|\langle Sx_n, x_n \rangle_A \|^2 \]
\[ + 2\text{Re} \left( \lambda \langle Sx_n, Tx_n \rangle_A + (1 - \lambda) \langle x_n, Tx_n \rangle_A \langle Sx_n, x_n \rangle_A \right) \]
\[ \leq \|T\|^2_{(A,\lambda)} + \|S\|^2_{(A,\lambda)} + 2 \left( \lambda \|Sx_n\|_A \|\langle Tx_n, x_n \rangle_A \| + (1 - \lambda) \|\langle Tx_n, x_n \rangle_A \| \|\langle Sx_n, x_n \rangle_A \| \right) \]
\[ \leq \|T\|^2_{(A,\lambda)} + \|S\|^2_{(A,\lambda)} + 2 \lambda \|Tx_n\|_A \|Sx_n\|_A + (1 - \lambda) \|\langle Tx_n, x_n \rangle_A \| \|\langle Sx_n, x_n \rangle_A \| \]
\[ \text{(by the A-Cauchy–Schwarz inequality)} \]
\[ \leq \|T\|^2_{(A,\lambda)} + \|S\|^2_{(A,\lambda)} + 2 \|T\|_{(A,\lambda)} \|S\|_{(A,\lambda)} = (\|T\|_{(A,\lambda)} + \|S\|_{(A,\lambda)})^2, \]

and therefore, from (1), we obtain
\[ \lim_{n \to \infty} \text{Re} \left( \lambda \langle Sx_n, Tx_n \rangle_A + (1 - \lambda) \langle x_n, Tx_n \rangle_A \langle Sx_n, x_n \rangle_A \right) = \|T\|_{(A,\lambda)} \|S\|_{(A,\lambda)}. \]
In addition, for every \( n \in \mathbb{N} \), we have
\[
\Re \left( \lambda \langle Sx_n, Tx_n \rangle_A + (1 - \lambda) \langle x_n, Tx_n \rangle_A \langle Sx_n, x_n \rangle_A \right)
+ \Im \left( \lambda \langle Sx_n, Tx_n \rangle_A + (1 - \lambda) \langle x_n, Tx_n \rangle_A \langle Sx_n, x_n \rangle_A \right)
= \left| \lambda \langle Sx_n, Tx_n \rangle_A + (1 - \lambda) \langle x_n, Tx_n \rangle_A \langle Sx_n, x_n \rangle_A \right|^2
\leq \|T\|_{(A, \lambda)}^2 \|S\|_{(A, \lambda)}^2,
\]
and so by (2), we conclude that
\[
\lim_{n \to \infty} \Im \left( \lambda \langle Sx_n, Tx_n \rangle_A + (1 - \lambda) \langle x_n, Tx_n \rangle_A \langle Sx_n, x_n \rangle_A \right) = 0.
\]
It follows from (2) that
\[
\lim_{n \to \infty} \left( \lambda \langle Sx_n, Tx_n \rangle_A + (1 - \lambda) \langle x_n, Tx_n \rangle_A \langle Sx_n, x_n \rangle_A \right) = \|T\|_{(A, \lambda)} \|S\|_{(A, \lambda)}.
\]
(ii) \( \Rightarrow \) (i) Suppose that for a sequence of \( A \)-unit vectors \( \{x_n\} \) in \( \mathcal{H} \) we have
\[
\lim_{n \to \infty} \left( \lambda \langle Sx_n, Tx_n \rangle_A + (1 - \lambda) \langle x_n, Tx_n \rangle_A \langle Sx_n, x_n \rangle_A \right) = \|T\|_{(A, \lambda)} \|S\|_{(A, \lambda)}.
\]
Hence
\[
\lim_{n \to \infty} \Re \left( \lambda \langle Sx_n, Tx_n \rangle_A + (1 - \lambda) \langle x_n, Tx_n \rangle_A \langle Sx_n, x_n \rangle_A \right) = \|T\|_{(A, \lambda)} \|S\|_{(A, \lambda)}.
\]
Since, for every \( n \in \mathbb{N} \),
\[
\left| \lambda \langle Sx_n, Tx_n \rangle_A + (1 - \lambda) \langle x_n, Tx_n \rangle_A \langle Sx_n, x_n \rangle_A \right|^2
\leq \left( \lambda \|Tx_n\|_A^2 + (1 - \lambda) \|Tx_n, x_n\|_A^2 \right) \left( \lambda \|Sx_n\|_A^2 + (1 - \lambda) \|Sx_n, x_n\|_A^2 \right)
\leq \left( \lambda \|Tx_n\|_A^2 + (1 - \lambda) \|Tx_n, x_n\|_A^2 \right) \|S\|_{(A, \lambda)}^2
\leq \|T\|_{(A, \lambda)}^2 \|S\|_{(A, \lambda)}^2,
\]
we obtain
\[
\lim_{n \to \infty} \left( \lambda \|Tx_n\|_A^2 + (1 - \lambda) \|Tx_n, x_n\|_A^2 \right) = \|T\|_{(A, \lambda)}^2.
\]
By a similar argument, we get
\[
\lim_{n \to \infty} \left( \lambda \|Sx_n\|_A^2 + (1 - \lambda) \|Sx_n, x_n\|_A^2 \right) = \|S\|_{(A, \lambda)}^2.
\]
Therefore,
\[
\left( \|T\|_{(A, \lambda)} + \|S\|_{(A, \lambda)} \right)^2
= \lim_{n \to \infty} \left( \lambda \|Tx_n\|_A^2 + (1 - \lambda) \|Tx_n, x_n\|_A^2 \right)
+ 2 \lim_{n \to \infty} \Re \left( \lambda \langle Sx_n, Tx_n \rangle_A + (1 - \lambda) \langle x_n, Tx_n \rangle_A \langle Sx_n, x_n \rangle_A \right)
+ \lim_{n \to \infty} \left( \lambda \|Sx_n\|_A^2 + (1 - \lambda) \|Sx_n, x_n\|_A^2 \right)
= \lim_{n \to \infty} \left( \lambda \|(T + S)x_n\|_A^2 + (1 - \lambda) \|(T + S)x_n, x_n\|_A^2 \right)
\leq \|T + S\|_{(A, \lambda)}^2 \leq \left( \|T\|_{(A, \lambda)} + \|S\|_{(A, \lambda)} \right)^2.
\]
Hence, $\|T + S\|_{(A, \lambda)} = \|T\|_{(A, \lambda)} + \|S\|_{(A, \lambda)}$. \qed

As an immediate consequence of the preceding theorem, we obtain the following result due to Barraa and Boumazgour [3].

**Corollary 2.4.** [3, Theorem 2.1] Let $T, S \in \mathcal{B}(\mathcal{H})$. Then $\|T + S\| = \|T\| + \|S\|$ if and only if there exists a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ such that

$$\lim_{n \to \infty} \langle Sx_n, Tx_n \rangle = \|T\|\|S\|.$$

**Proof.** Let $A = I$ and $\lambda = 1$, and apply Theorem 2.3. \qed

As another application of Theorem 2.3, by letting $A = I$ and $\lambda = 0$, we get a characterization of the equality $w(T + S) = w(T) + w(S)$ for Hilbert space operators (see [1]).

**Corollary 2.5.** ([1, Proposition 3.6]) For $T, S \in \mathcal{B}(\mathcal{H})$, the equality $w(T + S) = w(T) + w(S)$ holds if and only if there exists a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ such that

$$\lim_{n \to \infty} \langle x_n, Tx_n \rangle \langle Sx_n, x_n \rangle = w(T)w(S).$$

An operator $T \in \mathcal{B}(\mathcal{H})$ is called **Birkhoff–James numerical radius orthogonal** to $S \in \mathcal{B}(\mathcal{H})$, denoted by $T \perp_{w} S$, if $w(T + \xi S) \geq w(T)$, for all $\xi \in \mathbb{C}$. See [11] for characterization of the Birkhoff-James orthogonality with respect to numerical radius for Hilbert space operators. Analogously, we introduce a concept of $(A, \lambda)$-Birkhoff–James orthogonality for semi-Hilbertian space operators.

**Definition 2.6.** Let $T, S \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$. The operator $T$ is called **$(A, \lambda)$-Birkhoff–James orthogonal** to $S$, in short, $T \perp_{(A, \lambda)} S$, if

$$\|T + \xi S\|_{(A, \lambda)} \geq \|T\|_{(A, \lambda)}, \quad \xi \in \mathbb{C}.$$

Obviously, this is a generalization of both the concept of Birkhoff–James orthogonality and the concept of Birkhoff–James numerical radius orthogonality of Hilbert space operators.

In the next theorem, some characterizations of $(A, \lambda)$-Birkhoff–James orthogonality for bounded linear operators in semi-Hilbertian spaces are presented.

**Theorem 2.7.** Let $T, S \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$. Then the following statements are equivalent.

(i) For each $\theta \in [0, 2\pi)$ there exists a sequence $\{x_n\}$ of $A$-unit vectors in $\mathcal{H}$ such that the following two conditions hold.

(i-1) $\lim_{n \to \infty} \left( \lambda \|Tx_n\|^2_A + (1 - \lambda)\|\langle Tx_n, x_n \rangle_A \| \right) = \|T\|^2_{(A, \lambda)},$

(ii) $\lim_{n \to \infty} \Re(e^{i\theta} \lambda \langle Sx_n, Tx_n \rangle_A + e^{i\theta} (1 - \lambda)\langle x_n, Tx_n \rangle_A \langle Sx_n, x_n \rangle_A) \geq 0.$

(ii) For all $\xi \in \mathbb{C}$, $\|T + \xi S\|^2_{(A, \lambda)} \geq \|T\|^2_{(A, \lambda)} + |\xi|^2m^2_{(A, \lambda)}(S)$, where

$$m_{(A, \lambda)}(S) = \inf \left\{ \sqrt{\lambda \|Sx\|^2_A + (1 - \lambda)\|\langle Sx, x \rangle_A \|^2} : x \in \mathcal{H}, \|x\|_A = 1 \right\}. $$
(iii) \( T \perp_{(A, \lambda)} S \).

Proof. (i)⇒(ii) Suppose that (i) holds and let \( \xi \in \mathbb{C} \). Then there exists \( \theta \in [0, 2\pi) \) such that \( \xi = |\xi|e^{i\theta} \). Let \( \{x_n\} \) be a sequence of \( A \)-unit vectors in \( \mathcal{H} \) such that (i-1) and (i-2) hold. For \( n \in \mathbb{N} \) we have

\[
\|T + \xi S\|^2_{(A, \lambda)} \geq \lambda \|Tx_n + |\xi|e^{i\theta} Sx_n\|^2_A + (1 - \lambda)\langle(T + |\xi|e^{i\theta} S)x_n, x_n\rangle_A^2
\]

\[
= \lambda \|Tx_n\|^2_A + (1 - \lambda)\|Tx_n, x_n\|_A^2
\]

\[
+ |\xi|^2\left(\lambda\|Sx_n\|^2_A + (1 - \lambda)\langle Sx_n, x_n\rangle_A^2\right)
\]

\[
+ 2|\xi|Re\left(e^{i\theta}\lambda\langle Sx_n, Tx_n\rangle_A + e^{i\theta}(1 - \lambda)\langle x_n, Tx_n\rangle_A\langle Sx_n, x_n\rangle_A\right),
\]

and so

\[
\|T + \xi S\|^2_{(A, \lambda)} \geq \|T\|^2_{(A, \lambda)} + |\xi|^2 \limsup_{n \to \infty} \left(\lambda\|Sx_n\|^2_A + (1 - \lambda)\langle Sx_n, x_n\rangle_A^2\right)
\]

\[
\geq \|T\|^2_{(A, \lambda)} + |\xi|^2 m^2_{(A, \lambda)}(S).
\]

Hence \( \|T + \xi S\|^2_{(A, \lambda)} \geq \|T\|^2_{(A, \lambda)} + |\xi|^2 m^2_{(A, \lambda)}(S) \).

(ii) ⇒ (iii) This implication is trivial.

(iii) ⇒ (i) Let \( T \perp_{(A, \lambda)} S \). Then \( \|T + \xi S\|^2_{(A, \lambda)} \geq \|T\|^2_{(A, \lambda)} \) for every \( \xi \in \mathbb{C} \). We may assume that \( \|T\|_{(A, \lambda)} \neq 0 \) otherwise (i) trivially holds. Let \( \theta \in [0, 2\pi) \). Thus \( \|T\|_{(A, \lambda)} \leq \|T + \frac{e^{i\theta}}{n} S\|_{(A, \lambda)} \) for all \( n \in \mathbb{N} \). Since \( \|T\|_{(A, \lambda)} > 0 \), for sufficiently large \( n \), we have

\[
0 < \|T\|_{(A, \lambda)} - \frac{1}{n^2} < \|T\|_{(A, \lambda)} \leq \|T + \frac{e^{i\theta}}{n} S\|_{(A, \lambda)}.
\]

Therefore, there exists a sequence \( \{x_n\} \) of \( A \)-unit vectors in \( \mathcal{H} \) such that

\[
\left(\|T\|_{(A, \lambda)} - \frac{1}{n^2}\right)^2 < \lambda \|T + \frac{e^{i\theta}}{n} S\|_{A}^2 + (1 - \lambda)\langle(T + \frac{e^{i\theta}}{n} S)x_n, x_n\rangle_A^2.
\]

It follows from (3) that

\[
\|T\|^2_{(A, \lambda)} - \frac{2}{n^2}\|T\|_{(A, \lambda)} + \frac{1}{n^4}
\]

\[
< \lambda \|Tx_n\|^2_A + (1 - \lambda)\|Tx_n, x_n\|_A^2
\]

\[
+ \frac{1}{n^2}\left(\lambda\|Sx_n\|^2_A + (1 - \lambda)\langle Sx_n, x_n\rangle_A^2\right)
\]

\[
+ \frac{2}{n}Re\left(e^{i\theta}\lambda\langle Sx_n, Tx_n\rangle_A + e^{i\theta}(1 - \lambda)\langle x_n, Tx_n\rangle_A\langle Sx_n, x_n\rangle_A\right),
\]

and hence

\[
\frac{n}{2}\left(\|T\|^2_{(A, \lambda)} - \lambda \|Tx_n\|^2_A - (1 - \lambda)\|Tx_n, x_n\|_A^2\right)
\]

\[
< \frac{1}{n}\|T\|_{(A, \lambda)} - \frac{1}{2n^3} + \frac{1}{2n}\left(\lambda\|Sx_n\|^2_A + (1 - \lambda)\|Sx_n, x_n\|^2\right)
\]
\[ + \text{Re} \left( e^{i\theta} \lambda \langle Sx_n, Tx_n \rangle_A + e^{i\theta} (1 - \lambda) \langle x_n, Tx_n \rangle_A \langle Sx_n, x_n \rangle_A \right). \]

Since \( \|T\|_{(A, \lambda)}^2 - \lambda \|Tx_n\|_A^2 - (1 - \lambda) \|Tx_n, x_n\|_A^2 \geq 0 \), we obtain
\[ 0 < \frac{1}{n} \|T\|_{(A, \lambda)}^2 - \frac{1}{2n^2} + \frac{1}{2n} \left( \lambda \|Sx_n\|_A^2 + (1 - \lambda) \|Sx_n, x_n\|_A^2 \right) \]
\[ + \text{Re} \left( e^{i\theta} \lambda \langle Sx_n, Tx_n \rangle_A + e^{i\theta} (1 - \lambda) \langle x_n, Tx_n \rangle_A \langle Sx_n, x_n \rangle_A \right). \] (4)

By letting \( n \to \infty \) in (4) and passing through a subsequence if necessary, we get
\[ \lim_{n \to \infty} \text{Re} \left( e^{i\theta} \lambda \langle Sx_n, Tx_n \rangle_A + e^{i\theta} (1 - \lambda) \langle x_n, Tx_n \rangle_A \langle Sx_n, x_n \rangle_A \right) \geq 0. \]

Further, by (3), we have
\[ \lambda \|Tx_n\|_A^2 + (1 - \lambda) \|Tx_n, x_n\|_A^2 \]
\[ > \left( \|T\|_{(A, \lambda)}^2 - \frac{1}{n^2} \right)^2 - \frac{1}{n^2} \left( \lambda \|Sx_n\|_A^2 + (1 - \lambda) \|Sx_n, x_n\|_A^2 \right) \]
\[ - \frac{2}{n} \text{Re} \left( e^{i\theta} \lambda \langle Sx_n, Tx_n \rangle_A + e^{i\theta} (1 - \lambda) \langle x_n, Tx_n \rangle_A \langle Sx_n, x_n \rangle_A \right), \]
and therefore, by letting \( n \to \infty \), we obtain
\[ \lim_{n \to \infty} \left( \lambda \|Tx_n\|_A^2 + (1 - \lambda) \|Tx_n, x_n\|_A^2 \right) \geq \|T\|_{(A, \lambda)}^2. \]

Since \( \lim_{n \to \infty} \left( \lambda \|Tx_n\|_A^2 + (1 - \lambda) \|Tx_n, x_n\|_A^2 \right) \leq \|T\|_{(A, \lambda)}^2 \), we conclude that
\[ \lim_{n \to \infty} \left( \lambda \|Tx_n\|_A^2 + (1 - \lambda) \|Tx_n, x_n\|_A^2 \right) = \|T\|_{(A, \lambda)}^2. \]

Hence, (i-1) and (i-2) hold. \( \square \)

The following corollary is a direct consequence of Theorem 2.7. It gives a characterization of the Birkhoff–James orthogonality for Hilbert space operators.

**Corollary 2.8.** Let \( T, S \in \mathcal{B}(\mathcal{H}) \). The following statements are equivalent.

1. For each \( \theta \in [0, 2\pi] \) there exists a sequence \( \{x_n\} \) of unit vectors in \( \mathcal{H} \) such that \( \lim_{n \to \infty} \|Tx_n\| = \|T\| \) and \( \lim_{n \to \infty} \text{Re} (e^{i\theta} \langle Sx_n, Tx_n \rangle) \geq 0 \).
2. For all \( \xi \in \mathbb{C} \), \( \|T + \xi S\|^2 \geq \|T\|^2 + |\xi|^2 |S|^2 \).
3. \( T \perp_B S \).

**Proof.** Apply Theorem 2.7 with \( A = I \) and \( \lambda = 1 \). \( \square \)

Finally, we get the following result due to Mal et al. [11].

**Corollary 2.9.** ([11, Theorem 2.3]) Let \( T, S \in \mathcal{B}(\mathcal{H}) \). The following statements are equivalent.

1. For each \( \theta \in [0, 2\pi] \) there exists a sequence \( \{x_n\} \) of unit vectors in \( \mathcal{H} \) such that \( \lim_{n \to \infty} \|Tx_n, x_n\| = w(T) \) and \( \lim_{n \to \infty} \text{Re} (e^{i\theta} \langle x_n, Tx_n \rangle \langle Sx_n, x_n \rangle) \geq 0 \).
2. For all \( \xi \in \mathbb{C} \), \( w^2(T + \xi S) \geq w^2(T) + |\xi|^2 c^2(S) \).
(iii) $T \perp_{B} S$.

**Proof.** It follows immediately from Theorem 2.7 with $A = I$ and $\lambda = 0$. □

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