\( r \)-LOCAL UNLABELED SENSING: IMPROVED ALGORITHM AND APPLICATIONS

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ABSTRACT

The unlabeled sensing problem is to solve a noisy linear system of equations under unknown permutation of the measurements. We study a particular case of the problem where the permutations are restricted to be \( r \)-local, i.e. the permutation matrix is block diagonal with \( r \times r \) blocks. Assuming a Gaussian measurement matrix, we argue that the \( r \)-local permutation model is more challenging compared to a recent sparse permutation model. We propose a proximal alternating minimization algorithm for the general unlabeled sensing problem that provably converges to a first order stationary point. Applied to the \( r \)-local model, we show that the resulting algorithm is efficient. We validate the algorithm on synthetic and real datasets. We also formulate the 1-d unassigned distance geometry problem as an unlabeled sensing problem with a structured measurement matrix.

Index Terms— Unlabeled sensing, unassigned distance geometry problem, proximal alternating minimization.

1. INTRODUCTION

The standard least squares problem is to recover signal \( X^* \in \mathbb{R}^{d \times m} \) given measurements \( Y \in \mathbb{R}^{n \times m} \) and the measurement matrix \( B \). The signal matrix \( X^* \) is estimated by solving the least squares minimization problem \( \min \| Y - BX \|_F^2 \). This formulation assumes that there is one to one correspondence between measurements (the rows of \( Y \)) and the measurement vectors (the rows of \( B \)), i.e., we know which measurement corresponds to what measurement vector. However, in many problems of applied interest such as header free communication for mobile wireless networks [11], sampling in the presence of clock jitter [2], linear regression without correspondence [3] and point cloud registration [4], this mapping is not explicitly given. This motivates the study of the unlabeled sensing problem with the prototypical form given by

\[
Y = P^* B X^* + W,
\]

where \( P \in \mathbb{R}^{n \times n} \) denotes the unknown permutation matrix and \( W \in \mathbb{R}^{n \times m} \) denotes an additive Gaussian noise with \( W_{ij} \sim \mathcal{N}(0, \sigma^2) \). The problem is to estimate \( X^* \) and \( P^* \) given the measurement matrix \( B \) and measurements \( Y \).

Related work: The unlabeled sensing problem was first considered with a Gaussian measurement matrix \( B \) in the single-view setup, where \( m = 1 \), in [5]. There, the authors showed that \( n = 2d \) noiseless measurements are necessary and sufficient for recovery of \( x^* \). The same result was subsequently proven in [6] using an algebraic argument. For the case of Gaussian \( B \), the work in [7] showed that with \( \text{SNR} \triangleq \| X^* \|_F^2 / n \sigma^2 \), \( \text{SNR} = \Omega(n^2) \) is necessary for recovering \( P^* \) exactly. For the multi-view setup, where \( m > 1 \), the result in [8] shows that the necessary SNR for recovery decreases as \( m \) increases, with necessary SNR \( \text{SNR} = O(1) \) for \( m = O(\log n) \).

Several algorithms have been proposed for the single view unlabeled sensing problem [1][7][9]. Algorithms for the multi-view problem setup have been proposed in works [8][10][11][12][13]. As computing the maximum likelihood estimate of \( P^* \) may be NP-hard [11], the works in [8][10][12][13] impose a \( k \)-sparse structure on \( P^* \) (see Figure [1a]) where \( n - k \) elements of the permutation are on the diagonal. In [13], the authors introduced the \( r \)-local model (see section 2) along with an algorithm that is based on graph matching. A permutation matrix \( P^* \) of size \( n \times n \) is \( r \)-local if it is composed of \( n \times r \) blocks. Formally, \( P^* = \text{diag} \left( P_1^*, \ldots, P_{n/r}^* \right) \), where \( P_i^* \in \Pi_r \) denotes an \( r \times r \) permutation matrix (see Figure [1b]).

Applications: In this paper, we consider two applications of the \( r \)-local unlabeled sensing problem. The first application is the unassigned distance geometry problem (uDGP) [14] (see section 5). The second application is image unscrambling [15] (see section 5). Another application of the \( r \)-local permutation model is linear regression with blocking [16].

Contributions: (a) We compare the \( r \)-local model with the \( k \)-sparse model and argue that the \( r \)-local model is more challenging under the most widely studied case of Gaussian measurement matrix. (b) We formulate the 1-d unassigned distance geometry problem (uDGP) [14] as an unlabeled sensing problem with a structured measurement matrix. (c) We propose a proximal alternating minimization algorithm for the \( r \)-local unlabeled sensing problem. Simulation results show that the proposed algorithm outperforms the algorithms in [8][11][13][12].

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Fig. 1: (a). A sparse (or partially shuffled) permutation considered in [8, 10, 12, 17]. (b). The r-local permutation structure considered in this paper, with r = 20. (c). Figure plots the lower bound on the event \( \Pr\{\hat{P} \neq P^*\} \) against SNR for the two models (see equation (2)).

2. \( r \)-LOCAL MODEL VS \( k \)-SPARSE MODEL

A permutation matrix \( P \) is \( k \)-sparse if \( n-k \) of its elements are on the diagonal, i.e., \( (I, P) = n-k \). Figures 1a and 1b show a \( k \)-sparse and an \( r \)-local permutation matrix, respectively. We compare the two models by the difficulty of recovering \( P^* \) under each model when the matrix \( B \) is assumed to be random Gaussian. Under this assumption, for any estimator \( \hat{P} \), the probability of error \( \Pr\{\hat{P} \neq P^*\} \) is lower bounded by the result in [8]

\[
\Pr\{\hat{P} \neq P^*\} \geq 1 - \frac{1 + \frac{n}{2} \log_2 (1 + \text{SNR})}{H(P^*)}, \tag{2}
\]

where \( P^* \) is a random variable and \( H(P^*) \) denotes the entropy of \( P^* \). For uniformly distributed \( P^* \), the entropy is \( H(P^*) = \log_2 a \), where \( a \) is the number of possible permutations. Without any assumption on the permutation structure, the entropy of \( P^* \in \mathbb{R}^{n \times n} \) is \( H(P^*) = \log_2 (n!) \). The entropy for \( r \)-local permutations is \( H_r(P^*) = \log_2 r!n/r \). The entropy for \( k \)-sparse permutations is \( H_k(P^*) = \log_2 (n!/(n-k)!)) \). Performance guarantees for proposed algorithms are given for \( k = n/8 \) in [8] and \( k = n/4 \) in [12]. All simulation results for permutation recovery in [10, 17] are for \( k \leq n/2 \). Figure 1c shows that, for the same SNR, the lower bound in (2) is higher for the \( r \)-local model, \( r = \{n/10, n/20\} \), than for the \( k \)-sparse model, with \( k = n/2 \). This suggests that permutation recovery under the \( r \)-local model is challenging in the large \( n \) regime.

3. 1-D UNASSIGNED DISTANCE GEOMETRY AS AN UNLABELED SENSING PROBLEM

The one-dimensional unassigned distance geometry problem (1-d uDGP) (see Figure 2) is to recover point coordinates from their unlabeled pairwise distances [14, 18, 19]. The distances are unlabeled as the pair \((i, j)\) corresponding to the distance \(|x_i - x_j|\) is not known. Let \( \mathbf{x} \in \mathbb{R}^d = [x_1, \cdots, x_d]^\top \) be the vector of unknown coordinates. Without loss of generality, we assume that the unknown point coordinates are sorted in decreasing order. It follows that each pairwise distance \( x_i - x_j \) is a linear measurement of \( \mathbf{x} \). As the distances are translation invariant, we can also assume that the left most point is at the origin. The distance vector \( \mathbf{y} \in \mathbb{R}^{d(d-1)/2} \) is the matrix-vector product \( \mathbf{y} = \mathbf{B}_u \mathbf{x}^* \) where \( \mathbf{x}^* \in \mathbb{R}^{d-1} \). It can be verified that \( \mathbf{B}_u \) has the following form.

\[
\mathbf{B}_u = [I_{(d-1)\times(d-1)}; B_2; \cdots; B_{d-1}],
\]

\[
[B_i \in \mathbb{R}^{(d-i)\times(d-1)}]_{pq} = \begin{cases} 1 & \text{if } q = i - 1 \\ -1 & \text{if } q = p + i - 1 \\ 0 & \text{otherwise} \end{cases} \tag{3}
\]

where \([A_1; A_2]\) denotes vertical concatenation of the matrices \( A_1, A_2 \). The unlabeled sensing formulation for the example in Figure 2 with \( d = 4 \) points, is given below

\[
\begin{bmatrix} 3 \\ 2 \\ 4 \\ 6 \\ 7 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 3 \\ 7 \end{bmatrix}. \]

Why \( r \)-local model for uDGP? Imposing an \( r \)-local structure on \( P^* \) has a practical interpretation for the uDGP problem. The problem where, in addition to the distances \( |x_i - x_j| \), one of the two corresponding indices \((i, j)\) is known, is modelled by an \( r \)-local \( P^* \) as \( \text{blkdiag}(d-1, \cdots, 2) \). The \( r \)-local structure also models the problem where distance assignments are known up to a cluster of points. For example, for two clusters, each distance is known to be between a pair of points in cluster 1 or cluster 2 (intra-cluster) or a point in cluster 1 and a point in cluster 2 (inter-cluster) but the pair of points corresponding to each distance is still unknown.

4. PROPOSED APPROACH AND ALGORITHM

In this section, we present a new algorithm for the \( r \)-local unlabeled sensing problem. To motivate our proposed algorithm, we note that for i.i.d Gaussian noise \( \mathbf{W} \) in [4], the maximum likelihood estimate of \( P^* \) is given by

\[
\arg\min_{\mathbf{X}, \mathbf{P} \in \Pi_n} F(\mathbf{X}, \mathbf{P}) = \left\| \mathbf{Y} - \mathbf{P} \mathbf{B} \mathbf{X} \right\|_F^2, \tag{4}
\]
where $\Pi_n$ denotes the set of permutations of order $n$. The optimization problem in (4) is non-convex since the set of permutations is discrete. A natural optimization scheme for the problem in (4) is the alternating minimization algorithm which alternates between updating the signal matrix $X$ and the permutation $P$. The works in [10][11][12][17] propose one step estimators for $P^*$, $X^*$. We consider complete alternating minimization for $P$ yielding the following updates

$$P^{(t)} = \arg\min_{P \in \Pi_n} \langle -YX^{(t)}B^\top, P \rangle,$$  \quad (5)

$$X^{(t+1)} = \arg\min_{X} F(X, P^{(t)}) = (P^{(t)}B)^\top Y.$$  \quad (6)

The iterates $(P^{(t)}, X^{(t+1)})$ above are monotonically decreasing $F(P^{(t+1)}, X^{(t+1)}) \leq F(P^{(t)}, X^{(t+1)})$. However, this is not sufficient to establish convergence of the iterates. Specifically, we can not claim that $\lim_{t \to \infty} \{P^{(t)}, X^{(t)}\}$ exists. We propose to use proximal alternating minimization (PAM) [20] instead which, as we show in section 4.1, converges to a first order stationary point of the objective in (4). PAM adds a regularization term that encourages the new iterate to be close to the current iterate. Formally, for the objective in (4), the PAM updates are given by

$$P^{(t)} = \arg\min_{P \in \Pi_n} F(X^{(t)}, P) + \lambda^{(t)}\|P - P^{(t-1)}\|_F^2,$$  \quad (7)

$$X^{(t+1)} = \arg\min_{X} F(X, P^{(t)}) + \lambda^{(t)}\|X - X^{(t)}\|_F^2,$$  \quad (8)

where $\lambda^{(t)} > 0$. Our algorithm follows from the updates in (7) and (8). Similar to (7) and (8), the updates are a linear program and a regularized least squares problem, respectively. Assuming an $r$-local structure on $P^*$, the update in (7) is equivalent to the simpler update of $n/r$ blocks of size $r$. A summary of our algorithm is given in Algorithm 1.

**Algorithmic details:** We use the collapsed initialization, as also done in [13], to initialize $Y^{(0)} = B^\top X^{(0)}$. The initialization estimates $X$ from $n/r$ measurements given by summing $r$ consecutive rows in $B$ and $Y$. The convergence criteria is based on the relative change in the objective $(F^{(t+1)} - F^{(t)}) / F^{(t)}$. The total cost of updating all the local permutations by linear programs is $O(n^2r)$. In line 8, $\dagger$ denotes the Moore-Penrose pseudo-inverse. The regularized least squares problem can be solved in $O(2nd^2 - \frac{2}{3}d^3)$.

4.1. Convergence analysis

**Proposition 1.** The sequence of iterates $\{X^{(t)}, P^{(t)}\}$ generated by (7), (8) converges to a first order stationary point of the objective in (4).

**Proof.** The result follows from Theorem 9 in [20]. To use Theorem 9, we need to verify that the Kurdyka-Lojasiewicz (KL) inequality holds for the objective $F(X, P) = \|Y - PBX\|_F^2 + I_C(P)$ where $I_C(P)$ is the indicator function of the set of permutations. First, note that the set of permutations of order $n$ is semi-algebraic because each of the $n!$ permutations can be specified via $n^2$ linear equality constraints.

| Algorithm 1 Proximal Alt-Min for $r$-local unlabeled sensing |
|---------------------------------|
| **Input:** $B, Y, \hat{Y}^{(0)}, \hat{X}^{(0)}, r, \lambda, \epsilon$ |
| **Output:** $\hat{P}$ |
| 1: $t \leftarrow 0$ |
| 2: $\hat{P}_k^{(t)} \leftarrow 1_{r \times r}$ |
| 3: while relative change $> \epsilon$ do |
| 4: for $k \in 1 \cdots n/r$ do |
| 5: $\hat{P}_k^{(t)} = \arg\min_{P \in \Pi_n} \langle Y\hat{Y}^{(t)} + \lambda\hat{P}_k^{(t)}, P_k \rangle$ |
| 6: end for |
| 7: $\hat{B}^{(t+1)} \leftarrow \text{diag}(\hat{P}_1, \cdots, \hat{P}_{n/r})$ |
| 8: $\hat{X}^{(t+1)} \leftarrow \left[ \frac{B_{n \times d}}{\sqrt{M_{d \times d}}} \right]^\dagger [\hat{P}^{(t+1)r}Y]$ |
| 9: $\hat{Y}^{(t+1)} \leftarrow B\hat{X}^{(t+1)}$ |
| 10: $t \leftarrow t + 1$ |
| 11: end while |

The indicator function of a semi-algebraic function is a semi-algebraic function [20]. The first term in the objective is polynomial, and therefore real-analytic. The sum of a real-analytic function and a semi-algebraic function is sub-analytic, and sub-analytic functions satisfy the KL inequality [21].

5. EXPERIMENTS

For all experiments using Algorithm 1 we set the tolerance $\epsilon = 0.01$. The proximal regularization is set to $\lambda^{(1)} = 100$ and scaled as $\lambda^{(t+1)} = \lambda^{(t)}/10$. MATLAB code for the experiments is available at the first author’s github account: github.com/aabbus02/Proximal-Alt-Min-For-ULS-UDGP.

5.1. Synthetic simulations

**Data generation:** The entries of the signal matrix $X^*$, the sensing matrix $B$, and the noise $W$ are sampled i.i.d. from the normal distribution. The matrix $W$ is subsequently scaled by $\sigma$ to set a specific SNR. The permutation $P^*$ is picked uniformly at random from the set of $r$-local permutations. All results are averages of 75 Monte Carlo (MC) runs.

**Baselines:** We compare our algorithm to four algorithms proposed in [8][11][12][13]. First is a biconvex relaxation of (4) proposed in [8] and solved via the alternating direction method of multipliers (ADMM) algorithm. Second is the Levsort algorithm in [11]. For $m = d$ views and $\sigma^2 = 0$, the Levsort algorithm provably recovers $P^*$ exactly. The third algorithm in [12] approximates $X \simeq B^\top Y$ in (5) and estimates $P = \arg\min_{P \in \Pi_n} \langle Y\hat{Y}^* B^\top B^\top, P \rangle$. The approximation is justified when matrix $B$ is Gaussian i.i.d. and the rows of $B$ are partially shuffled. In order to ensure a fair comparison with the algorithms in [8][11][12], we specify an additional constraint to constrain the permutation solutions to be $r$-local. For each MC run, the ADMM penalty parameter $\rho$ for [8] was tuned in the range $10^{-3}$ to $10^5$ and the best results were retained. We also compare to the algorithm in [13] that solves a quadratic assignment problem to estimate $P^*$. We refer to the
PSNR is defined in terms of the mean square reconstruction error in the estimated point coordinates against increasing noise variance. The $d = 50$ points are sampled i.i.d. from the normal distribution with variances $\{1, 5, 10\}$, and the permutation $P_*$ is $r$-local with blocks of sizes $(d - 1, \ldots, 2)$.

5.3. 1-d uDGP

We consider the 1-dimensional uDGP problem (Section 3) where the distances are corrupted with i.i.d. Gaussian noise of variance $\sigma^2$. The results in Figure 3d show that the relative error in the recovered points is low even for high noise variance. The results are noteworthy because the general 1-d uDGP problem with noisy distances is NP-hard [24].

6. CONCLUSION

We propose a proximal alternating minimization algorithm for the unlabeled sensing problem and apply it to the setting where the unknown permutation is $r$-local. The resulting algorithm is efficient and theoretically converges to a first-order stationary point. Experiments on synthetic and real data show that the algorithm outperforms competing baselines. We formulate the 1-d unassigned distance geometry problem (uDGP) as an unlabeled sensing problem with a structured measurement matrix. Future work will explore information-theoretic inachievability results for the $r$-local permutation model and uDGP.
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