Continuous dependence estimates and homogenization of quasi-monotone systems of fully nonlinear second order parabolic equations

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Abstract

Aim of this paper is to extend the continuous dependence estimates proved in [27] to quasi-monotone systems of fully nonlinear second order parabolic equations. As by-product of these estimates, we get an Hölder estimate for bounded solutions of systems and a rate of convergence estimate for the vanishing viscosity approximation.

In the second part of the paper we employ similar techniques to study the periodic homogenization of quasi-monotone systems of fully nonlinear second order uniformly parabolic equations. Finally, some examples are discussed.

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1 Introduction

This paper is devoted to the weakly coupled system of parabolic equations

\[
\partial_t u_i + H_i(t,x,u,Du_i,D^2u_i) = 0 \quad \text{in } (0,T) \times \mathbb{R}^n, \quad i = 1,\ldots,m
\]

(1.1)

where \(\partial_t \equiv \partial/\partial t\), the operators \(H_i: (0,T) \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times S^n\) are given by

\[
H_i(t,x,r,p,X) = \min_{\zeta \in \mathbb{Z}_i} \max_{\theta \in \Theta_i} \left\{ - \text{tr} \left( A_i^{\theta \zeta}(t,x,p)X \right) + f_i^{\theta \zeta}(t,x,r,p,X) \right\}
\]

(1.2)

and \(u(x) = (u_1(x),\ldots,u_m(x))\). In fact, our techniques may be easily adapted to the case of systems of elliptic equations. Here, all (sub-, super-) solutions will always be in viscosity sense (see below for the precise definition; for the main properties, we refer the reader to [26] and also to [15] for a single equation).

Quasi-monotonicity is a basic assumption which guarantees the validity of the maximum principle for weakly coupled systems. In [26] this assumption has been exploited to prove general existence and uniqueness results for solutions of systems of fully nonlinear second order PDEs. Aim of this paper is to show that this assumption allows to extend to weakly coupled systems two well known properties of fully nonlinear equations: continuous dependence estimates and periodic homogenization.

Continuous dependence estimates (namely, an estimate of \(|u(t,x) - v(t,x)|\) where \(u\) and \(v\) are two solutions to (1.1)-(1.2) with different coefficients) are useful tools to obtain regularity results

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and rate of convergence estimates (e.g. for vanishing viscosity and numerical approximation). A general result for, possibly degenerate, scalar equations was proved in \cite{27,28} (see also \cite{31}) using techniques based on the maximum principle for semi-continuous solutions: doubling the variables and adding a penalization term. We show that the quasi-monotonicity assumption allows to extend the result in \cite{27} to weakly coupled systems at the same level of generality (see also \cite{7}, \cite{9} for related results). As an application of continuous dependence estimates, we obtain regularity estimates (a priori $L^\infty$ and Hölder bounds; we refer the reader to \cite{12,22} for Harnack type estimates for these systems) and a rate of convergence estimate for the vanishing viscosity approximation (in this direction, this paper extends the results in \cite{5} to the case of quasi-monotone systems).

We shall also illustrate our results for a class of systems which arises in optimal control and, as scalar equation, it encompasses the Hamilton-Jacobi-Bellman-Isaacs equation associated to stochastic differential games (see \cite{19,20}). In this case, taking advantage of the special form of the coefficients, we obtain a simpler expression of the estimates.

In the second part of the paper we are concerned with periodic homogenization of weakly coupled systems of uniformly parabolic equations. In this case, the coefficients in (1.2) have the form

$$A^{\theta \xi}_i = A^{\theta \xi}_i \left( x, \frac{x}{\varepsilon} \right), \quad f^{\theta \xi}_i = f^{\theta \xi}_i \left( x, \frac{x}{\varepsilon}, r, p \right)$$

and are $\mathbb{Z}^n$-periodic in the $x/\varepsilon$-variables. The parameter $\varepsilon$ is meant to tend to 0. This fact modelizes a medium displaying heterogeneities in a microscopic scale while one seeks a description of the macroscopic phenomena (which are the only relevant ones). At the limit, the solutions are expected to converge to the solution of a “homogenized” problem where the effective operator needs to be suitably defined.

The homogenization problem for a scalar equation has been studied e.g. in \cite{1,3,17,24,30} (see \cite{2} for a general review of the results). A homogenization result for quasi-monotone systems of first order Hamilton-Jacobi equations was obtained in \cite{13}. For systems of second order equations we refer the reader to \cite{6,4} for the quasi-linear case and to \cite{10,11} for homogenization via probabilistic techniques. Also for the homogenization we are able to generalize the result from the scalar case to the weakly coupled one making a crucial use of the quasi-monotonicity assumption. The proof relies on an appropriate modifications of the perturbed test function’s method introduced by Evans \cite{17}.

This paper is organized as follows: in the rest of this section, we introduce our notations, we list the standing assumptions and we recall the definition of viscosity solution of a system of PDEs. Section 2 is devoted to the continuous dependence estimate; in particular, we illustrate our results for a class of systems which arise in optimal control problems. Taking advantage of this estimate, in Section 3 we deduce a regularity result and a rate of convergence for the vanishing viscosity; moreover, we work out in detail the vanishing viscosity approximation of a first order system arising in optimal control. Section 4 is devoted to homogenization results. Finally, in the Appendix we give the proof of a technical Lemma and, for the sake of completeness, we quote some results yet established in the literature.

1.1 Notations and standing assumptions

**Notations:** We set $I := \{1, \ldots, m\}$ and $Q_t := (0, t) \times \mathbb{R}^n$. $S^n$ denotes the set of $n \times n$ real symmetric matrices; it is endowed with the Frobenius norm and the usual order, namely: $|X| = \text{tr}(XX^T)^{1/2}$ and $X \geq Y$ whenever $X - Y$ is a semidefinite positive matrix. For each function $h$ defined on $\mathbb{R}^n$, the parabolic super- and sub-solutions at the point $(\tau, \xi)$ and $\mathcal{P}_{\tau}^+ h(\tau, \xi)$ denote respectively the parabolic super- and sub-solutions at the point $(\tau, \xi)$ (see \cite{15} Section 8). For $f : \mathbb{R}^n \to \mathbb{R}^m$, we define the $C^0$-norm by $\|f\| := \sup_{\tau \in I, x \in \mathbb{R}^n} |f(x)|$ and, for $\mu \in (0,1]$, the Hölder seminorm by $\|f\|_\mu := \sup_{\tau \in I, x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\mu}$. For $\mu \in (0,1]$, $C^0(\mathbb{R}^n)$ denotes the Hölder space of functions $f$ such that: $\|f\| + \|f\|_\mu < +\infty$. Finally, $\text{BUC}(\mathbb{R}^n)$ denotes the space of uniformly continuous, bounded functions $f : \mathbb{R}^n \to \mathbb{R}^m$.

**Standing assumptions:** For $i \in I$ and $H_i$ defined as in (1.2), we assume

\begin{itemize}
    \item[(C0)] The sets $\Theta_i$ and $Z_i$ are compact metric spaces. Moreover, wlog, we assume $\Theta_i = \Theta$ and $Z_i = Z$.
\end{itemize}
more, let us recall that a wide class of nonlinear operators can be written in the form

\[ f_{i}^{\Theta}(t, x, r, p, X) \geq f_{i}^{\Theta}(t, x, r, p, Y) \quad \forall t, x, r, p, \theta, \zeta. \]

(C3) For every \( t, x, p, \theta, \zeta \), \( A_{i}^{\Theta}(t, x, p) = a_{i}^{\Theta}(t, x, p)A_{i}^{\Theta}(t, x, p)^{T} \) for some \( a_{i}^{\Theta} \in C([0, T] \times \mathbb{R}^{n} \times \mathbb{R}^{n}) \). Furthermore, for every \( R > 0 \), \( a_{i}^{\Theta} \) is uniformly continuous on \( [0, T] \times \mathbb{R}^{n} \times B_{\mathbb{R}^{n}}(0, R) \) uniformly in \( \theta, \zeta \).

(C4) For every \( R > 0 \), there is \( \gamma_{R} \in \mathbb{R} \) s.t. if \( r, s \in [-R, R]^{m} \) and \( r_{j} - s_{j} = \max_{k \in I} \{ r_{k} - s_{k} \} \geq 0 \), then

\[ f_{j}^{\Theta}(t, x, r, p, X) - f_{j}^{\Theta}(t, x, s, p, X) \geq \gamma_{R}(r_{j} - s_{j}) \quad \forall t, x, p, X, \theta, \zeta. \]

(C5) There exists \( \mu \in (0, 1] \) such that: for every \( R > 0 \) there exists a constant \( C_{f,R} \) such that

\[ |f_{j}^{\Theta}(t, x, r, p, X) - f_{j}^{\Theta}(t, y, r, p, X)| \leq C_{f,R} (|p||x - y| + |x - y|^\mu) \]

for every \( \theta, \zeta, \iota, t, x, y, p, r, X \) with \( |r| < R \).

(C6) There is a constant \( C_{a} \) such that

\[ |a_{i}^{\Theta}(t, x, p) - a_{i}^{\Theta}(t, y, p)| \leq C_{a} |x - y| \quad \forall \theta, \zeta, i, t, x, y, p. \]

(C7) There holds: \( C_{f} := \sup_{\theta, \zeta, i, t, x} |f_{i}^{\Theta}(t, x, 0, 0, 0)| \leq +\infty. \)

Remark 1.1 Assumption (C4) implies a quasi-monotonicity property of the system (1.1); namely, for every \( R > 0 \), there is \( \gamma_{R} \in \mathbb{R} \) s.t. if \( r, s \in [-R, R]^{m} \) and \( r_{j} - s_{j} = \max_{k \in I} \{ r_{k} - s_{k} \} \geq 0 \), then

\[ H_{j}(t, x, r, p, X) - H_{j}(t, x, s, p, X) \geq \gamma_{R}(r_{j} - s_{j}) \quad \forall t, x, p, X. \quad (1.3) \]

Remark 1.2 We refer the reader to Section [27] for a class of systems (arising in optimal control theory) which fulfills assumptions (C0)-(C4). Let us also observe that, when system (1.1)-(1.2) reduces to a single equation, the above assumptions are satisfied, e.g., by: the Hamilton-Jacobi-Bellman-Isaacs equation associated to a two-players zero-sum stochastic differential game, the equation of mean curvature flow of graphs, the Hamilton-Jacobi-Bellman-Isaacs equation associated to a two-players zero-sum stochastic differential game, the Hamilton-Jacobi-Bellman-Isaacs equation associated to a two-players zero-sum stochastic differential game, the equation of mean curvature flow of graphs, the Hamilton-Jacobi-Bellman-Isaacs equation associated to a two-players zero-sum stochastic differential game.

Definition of solution (26): (i) An USC function \( u : Q_{T} \to \mathbb{R}^{m} \) is a subsolution of (1.1) if: whenever \( \phi \in C^{2}(Q_{T}) \), \( i \in I \) and \( u_{i} - \phi \) attains a local maximum at \((t, x)\), then there holds

\[ \partial_{t}\phi(t, x) + H_{i}(t, x, u(t, x), D\phi(t, x), D^{2}\phi(t, x)) \leq 0. \]

(ii) A LSC function \( u : Q_{T} \to \mathbb{R}^{m} \) is a supersolution of (1.1) if: whenever \( \phi \in C^{2}(Q_{T}) \), \( i \in I \) and \( u_{i} - \phi \) attains a local minimum at \((t, x)\), then there holds

\[ \partial_{t}\phi(t, x) + H_{i}(t, x, u(t, x), D\phi(t, x), D^{2}\phi(t, x)) \geq 0. \]

(iii) A function \( u \) is a solution of (1.1) if it is both a sub- and a supersolution. In particular, it belongs to \( C(Q_{T}) \).
\section{The continuous dependence estimate}

In this section we prove the continuous dependence estimate for the problem (1.1)-(1.2).

**Theorem 2.1** Assume that, for \( k = 1, 2, \) \( H^k = \{ H^k_i \}_{i \in I} \) satisfies assumptions (C0)-(C4) with constant \( \gamma^k_R \). Let \( u^1 \) and \( u^2 \) be respectively a bounded subsolution to problem (1.1)-(1.2) with \( H^0 = H^1 \) and a bounded supersolution to problem (1.1)-(1.2) with \( H^0 = H^2 \). Set \( R := \max(\|u^1\|, \|u^2\|) \) and \( \gamma = \min(\gamma^k_R, \gamma^2_R) \). Then for each \( 0 \leq t \leq T, \gamma \geq 0 \) and \( \alpha > 0 \), we have

\[
\sup_{E^0_t} \left( e^{\gamma t} (u^1_t(r, x) - u^2_t(r, y)) - \frac{\alpha}{2} e^{\gamma t} |x - y|^2 \right) \leq \\
\sup_{E^0_t} \left( u^1_t(0, x) - u^2_t(0, y) - \frac{\alpha}{2} |x - y|^2 \right)^+ + t \sup_{D^0_t} \left( e^{\gamma t} [f^\theta(t, x, r, p, X) - f^\theta(t, x, r, p, X)] \right) \\
+ 3ae^\gamma \|u^2_t\|_{x, y} \leq \frac{\alpha}{2} |e^{\gamma t} |x - y|^2 \right)^+ 
\]

where

\[
\Delta^\alpha := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \leq 2 \frac{R^{1/2}}{\sqrt{\alpha}}\} 
\]

\[
E^\alpha_t := \{(r, x, y, i) : 0 \leq \tau < t, (x, y) \in \Delta^\alpha, i \in I\} 
\]

\[
D^\alpha_t := \{(r, x, y, i, r, p, X, \theta, \zeta) : p = \alpha(x - y)e^{(\gamma - \gamma)\tau}, (r, x, y, i) \in E^\alpha_t, \}
\]

\[
|\tau| \leq e^{-\gamma} \min(\|u^1\|, \|u^2\|), |X| \leq 3ae^{(\gamma - \gamma)\tau}, \theta \in \Theta, \zeta \in Z\}.
\]

**Proof** We first consider the case \( \gamma = 0 \). Without loss of generality, assume \( \|u^1\| \leq \|u^2\| \) (the other case can be dealt with in a similar manner and we shall omit it). Fix \( t \in (0, T] \), \( \alpha > 0 \) and \( \gamma \geq 0 \). For every \( 0 < \varepsilon \leq \alpha/5 \), we set

\[
\sigma^0 := \sup_{E^0_t} \left( u^1_t(0, x) - u^2_t(0, y) - \frac{\alpha}{2} |x - y|^2 \right)^+ - \alpha \int (\frac{\alpha}{2} e^{\gamma t} |x - y|^2) \]

\[
\sigma := -\sigma^0 + \sup_{E^0_t} \left( u^1_t(r, x) - u^2_t(r, y) - \frac{\alpha}{2} e^{\gamma t} |x - y|^2 + \frac{\varepsilon}{2} |x|^2 + |y|^2 + \frac{\varepsilon}{t - \tau} \right) \}
\]

Since we want to derive an upper bound of \( \sigma \), it is not restrictive to assume \( \sigma > 0 \). For \( \delta \in (0, 1) \), set

\[
\psi(\tau, x, y, i) := u^1_t(r, x) - u^2_t(r, y) - \frac{\delta \sigma}{t} - \left( \frac{\alpha}{2} e^{\gamma t} |x - y|^2 + \frac{\varepsilon}{2} |x|^2 + |y|^2 + \frac{\varepsilon}{t - \tau} \right)
\]

for every \( \tau \in (0, t), x, y \in \mathbb{R}^n \) and \( i \in I \). Since the functions \( u^1_t \) and \( u^2_t \) are bounded in \( Q_t \) and \( \psi \) tends to \(-\infty \) both as \( \tau \to t^- \) and as \( |x| + |y| \to +\infty \), we deduce that there exists a point \((\tau_0, x_0, y_0, i_0)\) where the function \( \psi \) attains its global maximum, i.e.

\[
\psi(\tau_0, x_0, y_0, i_0) \geq \psi(\tau, x, y, i) \quad \forall (\tau, x, y, i) \in [0, t] \times \mathbb{R}^n \times \mathbb{R}^n \times I.
\]

By its definition (2.2), the function \( \psi \) satisfies

\[
\sup_{E^\alpha_t} \psi \geq \sigma + \sigma^0 - \delta \sigma = (1 - \delta)\sigma + \sigma^0.
\]

**Lemma 2.1** Let \((\tau_0, x_0, y_0, i_0)\) be the point where the function \( \psi \) in (2.2) attains its maximum. Then

i) There holds

\[
|x_0 - y_0| \leq 2 \left( \frac{R}{\alpha} \right)^{1/2}, \quad |x_0|, |y_0| \leq 2 \left( \frac{R}{\varepsilon} \right)^{1/2}
\]

where \( R \) is the constant introduced in Theorem 2.1 in fact, there exists a modulus of continuity in such that

\[
|x_0|, |y_0| \leq \varepsilon^{-1/2} m(\varepsilon).
\]
ii) Assume that \(u^1\) and \(u^2\) are continuous in \(x\) uniformly in \(t\), namely, there exists a modulus of continuity \(\omega\) such that: 
\[ |u^j(t,x) - u^j(t,y)| \leq \omega(|x - y|) \quad (j = 1, 2). \]
Then, we have
\[ \alpha e^{\tau_0}|x_0 - y_0|^2 \leq \omega(|x_0 - y_0|). \]
(2.6)

iii) Assume that either \(u^1\) or \(u^2\) belongs to \(C^1\). Then, we have
\[ \alpha e^{\tau_0}|x_0 - y_0| \leq n \left[ \min_{j=1,2} \{ |w^j|_1 \} + \varepsilon^{1/2} \sqrt{2R} \right]. \]
(2.7)
The proof is postponed to the Appendix. We continue with the proof of Theorem 2.11.

By Lemma 2.11(i), we deduce that \(\tau_0 > 0\); actually, for \(\tau_0 = 0\), inequality (2.6) implies
\[ \sigma_0 + (1 - \delta)\sigma \leq \psi(0, x_0, y_0, \varphi) \leq \sigma_0 \]
and, in particular, \(\sigma \leq 0\), a contradiction.

We introduce the test function
\[ \phi(t, x, y) := \frac{\delta\sigma}{\alpha} t + \frac{\alpha}{2} e^{\tau t}|x - y|^2 + \frac{\varepsilon}{2} (|x|^2 + |y|^2) + \frac{\varepsilon}{2} t - \tau \]
and, for \(i = i_0\) fixed, we invoke [15] Thm. 8.3]: for every \(\nu > 0\), there exist values \(a, b \in \mathbb{R}\) and matrices \(X, Y \in S^n\) such that
\[ (a, p_{x0}, X) \in \mathcal{P}^{2,+}u^1_{t_0}(\tau_0, x_0), \quad (b, p_{y0}, Y) \in \mathcal{P}^{2,-}u^2_{t_0}(\tau_0, y_0), \]
(2.8)
\[ a - b = \partial_t \phi(\tau_0, x_0, y_0) \equiv \frac{\delta\sigma}{\alpha} t + \frac{\varepsilon}{(t - \tau_0)^2} + \frac{\alpha}{2} e^{\tau_0}|x_0 - y_0|^2 \]
(2.9)
\[ - (\nu^{-1} + \alpha + \varepsilon) \begin{pmatrix} 1 & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \Phi + \nu\Phi^2, \]
(2.10)
where
\[ \bar{\alpha} := e^{\tau_0}\alpha, \quad \Phi := \begin{pmatrix} \phi_{xx} & \phi_{xy} \\ \phi_{xy} & \phi_{yy} \end{pmatrix}_{(\tau_0, x_0, y_0)}, \quad p_{x0} := D_x\phi(\tau_0, x_0, y_0), \quad p_{y0} := -D_y\phi(\tau_0, x_0, y_0) \]
(note that, according to notations of [15], the norm of a symmetric matrix \(A\) is defined as follows: 
\[ |A| : = \sup \{ |\lambda| : \lambda \ is \ an \ eigenvalue \ of \ A \} = \sup \{ | < v, A, v > | : |v| \leq 1 \}; \] recall also that \(|A| \leq n|A|_s\). For \(\nu = (\alpha + 2\varepsilon)^{-1}\), relation (2.10) entails
\[ -2(\alpha + \varepsilon) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 2\varepsilon \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \]
(2.11)
From this inequality, one can deduce that, for every \((\theta, \xi) \in \Theta \times \mathbb{R}\), there holds
\[ \text{tr} \left( A_{t_0}^{\theta, 1}(\tau_0, x_0, p_{x0})X \right) - \text{tr} \left( A_{t_0}^{\theta, 2}(\tau_0, y_0, p_{y0})Y \right) \leq 3\alpha \left| a_{t_0}^{\theta, 1}(\tau_0, x_0, p_{x0}) - a_{t_0}^{\theta, 2}(\tau_0, y_0, p_{y0}) \right|^2 \]
\[ + 2\varepsilon \left| a_{t_0}^{\theta, 1}(\tau_0, x_0, p_{x0}) \right|^2 + \left| a_{t_0}^{\theta, 2}(\tau_0, y_0, p_{y0}) \right|^2 \]
(2.12)
In order to prove this inequality, we shall use the arguments by Ishii [25]. Multiplying the latter inequality in (2.11) by the matrix
\[ \begin{pmatrix} a_{t_0}^{\theta, 1}(\tau_0, x_0, p_{x0}) & a_{t_0}^{\theta, 2}(\tau_0, x_0, p_{x0})^T \\ a_{t_0}^{\theta, 1}(\tau_0, x_0, p_{x0})^T & a_{t_0}^{\theta, 2}(\tau_0, y_0, p_{y0}) \end{pmatrix} \]
(2.13)
(which is symmetric and nonnegative definite) and evaluating the trace, we obtain
\[
\begin{align*}
\text{tr} \left( A_{i_0}^{\theta,1}(\tau_0, x_0, p_{x_0}) X - A_{i_0}^{\theta,2}(\tau_0, y_0, p_{y_0}) Y \right) & \leq \\
3\alpha & \text{tr} \left[ \left( A_{i_0}^{\theta,1}(\tau_0, x_0, p_{x_0}) - A_{i_0}^{\theta,2}(\tau_0, y_0, p_{y_0}) \right) \left( A_{i_0}^{\theta,1}(\tau_0, x_0, p_{x_0}) - A_{i_0}^{\theta,2}(\tau_0, y_0, p_{y_0}) \right)^T \right] \\
+ 2\epsilon & \text{tr} \left( A_{i_0}^{\theta,1}(\tau_0, x_0, p_{x_0}) + A_{i_0}^{\theta,2}(\tau_0, y_0, p_{y_0}) \right)^T \\
+ \alpha & \gamma e^{\tau_0}|x_0 - y_0|^2.
\end{align*}
\]
and therefore, by using our choice of \( \epsilon \) and of \( \alpha \), we get relation (2.14). Since \( u^1 \) is a subsolution to problem (1.1), the former relation in (2.8) and (2.9) yield
\[
0 \geq a + \min_{\theta \in \Theta} \max_{\tau \in \mathbb{R}} \left\{ - \text{tr} \left( A_{i_0}^{\theta,1}(\tau_0, x_0, p_{x_0}) X \right) + f_{i_0}^{\theta,1}(\tau_0, x_0, u^1(\tau_0, x_0, p_{x_0}, X) \right) \right\}
\]
\[
\geq b + \frac{\| \delta \|}{\epsilon} \min_{\theta \in \Theta} \max_{\tau \in \mathbb{R}} \left\{ - \text{tr} \left( A_{i_0}^{\theta,2}(\tau_0, y_0, p_{y_0}) Y \right) + f_{i_0}^{\theta,2}(\tau_0, y_0, u^2(\tau_0, y_0, p_{y_0}, Y) \right) \right\} + \frac{\alpha}{2} \gamma e^{\tau_0}|x_0 - y_0|^2.
\]
Hence, since $u^2$ is a supersolution, we get
\[
\frac{\delta \sigma}{t} \leq \max_{\theta, \xi} \left\{ -\text{tr} (A^0_{\theta, \xi}(\tau_0, y_0, p_{y_0})Y - A^0_{\theta, \xi}(\tau_0, x_0, p_{x_0})X) - f^0_{\theta, \xi}(\tau_0, x_0, u^1(\tau_0, x_0), p_{x_0}, X) \\
+ f^0_{\theta, \xi}(\tau_0, y_0, u^1(\tau_0, x_0), p_{y_0}, X - 4\varepsilon I) \right\} + \frac{\alpha}{2} \gamma e^{\gamma \tau}|x_0 - y_0|^2 \\
\leq \max_{\theta, \xi} \left\{ 3\alpha |a^0_{\theta, \xi}(\tau_0, x_0, p_{x_0}) - a^0_{\theta, \xi}(\tau_0, y_0, p_{y_0})|^2 \\
+ f^0_{\theta, \xi}(\tau_0, y_0, u^1(\tau_0, x_0), p_{y_0}, X - 4\varepsilon I) - f^0_{\theta, \xi}(\tau_0, x_0, u^1(\tau_0, x_0), p_{x_0}, X) \\
+ 2\varepsilon(|a^0_{\theta, \xi}(\tau_0, x_0, p_{x_0})|^2 + |a^0_{\theta, \xi}(\tau_0, y_0, p_{y_0})|^2) \right\} + \frac{\alpha}{2} \gamma e^{\gamma \tau}|x_0 - y_0|^2 \\
\right. \\
(2.17)
\]
where the last inequality is due to (2.12).

Set $p := \alpha e^{\gamma \tau_0}(x_0 - y_0)$, $p^x := \varepsilon x_0$, $p^y := \varepsilon y_0$ and observe that $p_{x_0} = p + p^x$, $p_{y_0} = p - p^y$. We define
\[
F^0_{\tau_0, \varepsilon} := \left\{ (\tau, x, y, i, r, p, p^x, p^y, X, \theta, \xi) : X = X_1 + X_2, (\tau, x, y, i, r, p, X_1, \theta, \xi) \in D^0_\varepsilon, \right. \\
|X_2| \leq 2n\varepsilon, \varepsilon^{1/2}|x|, \varepsilon^{1/2}|y| \leq m(\varepsilon), |p^x|, |p^y| \leq (2\varepsilon)^{1/2} \right\} \\
(2.18)
\]
From (2.17), we have
\[
\frac{\delta \sigma}{t} \leq \sup_{F^0_{\tau_0, \varepsilon}} \left\{ (3\alpha + 2\varepsilon)|a^0_{i, \xi}(\tau, x, p + p^x) - a^0_{i, \xi}(\tau, y, p - p^y)|^2 \\
+ f^0_{i, \xi}(\tau, y, r, p - p^y, X - 4\varepsilon I) - f^0_{i, \xi}(\tau, x, r, p + p^x, X) \\
- \frac{\alpha}{2} \gamma e^{\gamma \tau}|x - y|^2 + 2\varepsilon(|a^0_{i, \xi}(\tau, x, p + p^x)|^2 + |a^0_{i, \xi}(\tau, y, p - p^y)|^2) \right\}^+. \\
\]
By definition of $F^0_{\tau_0, \varepsilon}$ and (C1) and (C3), we get that there exists a modulus of continuity $\omega$ such that
\[
\frac{\delta \sigma}{t} \leq \sup_{F^0_{\tau_0, \varepsilon}} \left\{ f^0_{i, \xi}(\tau, y, r, p, X) - f^0_{i, \xi}(\tau, x, r, p, X) + 3\alpha |a^0_{i, \xi}(\tau, x, p) - a^0_{i, \xi}(\tau, y, p)|^2 \\
- \frac{\alpha}{2} \gamma e^{\gamma \tau}|x - y|^2 + \omega(|p^x| + |p^y| + \varepsilon) + 2\varepsilon(|a^0_{i, \xi}(\tau, x, p + p^x)|^2 + |a^0_{i, \xi}(\tau, y, p - p^y)|^2) \right\}^+. \\
\]
If $(\tau, x, y, i) \in E^0_{\varepsilon}$, by definition of $\sigma$, we get
\[
u^0_1(\tau, x) - \nu^0_1(\tau, y) - \frac{\alpha}{2} \gamma e^{\gamma \tau}|x - y|^2 \leq \sigma + \sigma_0 + \varepsilon \left\{ \frac{1}{t - \tau} + \frac{1}{2}(|x|^2 + |y|^2) \right\} \\
\]
By the last two inequalities we get
\[
u^0_1(\tau, x) - \nu^0_1(\tau, y) - \frac{\alpha}{2} \gamma e^{\gamma \tau}|x - y|^2 \\
\leq \sigma_0 + \frac{1}{\delta} \sup_{F^0_{\tau_0, \varepsilon}} \left\{ f^0_{i, \xi}(\tau, y, r, p, X) - f^0_{i, \xi}(\tau, x, r, p, X) + 3\alpha |a^0_{i, \xi}(\tau, x, p) - a^0_{i, \xi}(\tau, y, p)|^2 \\
- \frac{\alpha}{2} \gamma e^{\gamma \tau}|x - y|^2 + \omega(|p^x| + |p^y| + \varepsilon) + 2\varepsilon(|a^0_{i, \xi}(\tau, x, p + p^x)|^2 + |a^0_{i, \xi}(\tau, y, p - p^y)|^2) \right\}^+ \\
+ \varepsilon \left\{ \frac{1}{t - \tau} + \frac{1}{2}(|x|^2 + |y|^2) \right\} \\
(2.19)
\]
Observe that, by (C3) and the definition of $F^0_{\tau_0, \varepsilon}$, we have
\[
\varepsilon \left( |a^0_{i, \xi}(\tau, x, p + p^x)|^2 + |a^0_{i, \xi}(\tau, x, p - p^y)|^2 \right) \leq C\varepsilon(1 + |x|^2) \leq C\varepsilon(1 + \frac{m(\varepsilon)^2}{\varepsilon}).
\]
Then sending $\varepsilon \to 0$ (note that, since $\|u_1\| = \|u_1\| \wedge \|u_2\|$, by (2.18), as $\varepsilon \to 0$, $F_t^{\alpha,\varepsilon}$ converges to $D_{\alpha t}$) and then $\delta \to 1^-$ in (2.19), we get the estimate.

The general case $\gamma \neq 0$ can be proved following the argument of the corresponding result in [27, Thm 3.1].

Remark 2.1 This result can be generalized (using the same proof) to the case of a bounded above subsolution $u_1$ and a bounded below supersolution $u_2$. Actually, in order to treat this case, it suffices to set $R := \max\{\|u_1\|, \|u_2\|\}$ and to replace $\min\{\|u_1\|, \|u_2\|\}$ with $\min\{\|u_1\|, \|u_2\|\}$ in the definition of $D_{\alpha t}$.

Remark 2.2 Theorem [27] can be improved when either $u_1$ or $u_2$ satisfies additional regularity properties. For instance, when they are both continuous functions with modulus of continuity $\omega_1$ and $\omega_2$, respectively, the result of Theorem [27] holds with $\Delta_\alpha$ in (2.1) defined by

$$\Delta^\alpha := \left\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \alpha|x - y|^2 - \omega_1(|x - y|) - \omega_2(|x - y|) \leq 0\right\}.$$ 

When either $u_1$ or $u_2$ belongs to $C^1(Q_T)$, then $\Delta_\alpha$ can be defined as

$$\Delta^\alpha := \left\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \leq n \min\{|u_1|, |u_2|\}^{-1} \right\}.$$ 

With the previous definitions of $\Delta_\alpha$, the proof of Theorem [27] can be easily adapted by using Lemma [27] (ii) and (iii) (see [27] for more details).

2.1 Systems arising in control theory

Weakly coupled systems are the dynamic programming equations of optimal control problem of Markov process with random switching (see [21]) and arise in many areas as in connection with the optimal control of hybrid systems ([8, 16, 23]). Consider the control problem with dynamics

$$dX(s) = b_{\nu_s}^{\theta_s, \zeta_s}(s, X(s))ds + a_{\nu_s}^{\theta_s, \zeta_s}(s, X(s))dW_s, \quad s \in [t, T]$$

$$X(t) = x$$

(2.20)

where $W_t$ is a standard Brownian motion, $\theta_t$, $\zeta_t$ are the controls and $\nu_t$ is a continuous time random process with state space $\{1, \ldots, m\}$ for which

$$\mathbb{P}\{X_{t+\Delta t} = j \mid \nu_t = i, X_t = x\} = \delta(t, x)\Delta t + O(\Delta t)$$

(2.21)

for $\Delta t \to 0$, $i, j \in I$, $i \neq j$. Let $v = (v_1, \ldots, v_m)$ be the value function defined by

$$v_i(x, t) = \inf_{\theta \in T} \sup_{\zeta \in \mathcal{Z}} \mathbb{E}_{x,i} \left\{ \int_t^T l_{\theta, \zeta}(s, X(s))ds + u_{0,i}(X(T)) \right\}$$

(2.22)

$$\mathbb{E}_{x,i}\{\}$$

for $i \in I$. Then the function $u(x, T-t) := v(x, t)$ is formally the solution of (1.1) with initial datum $u_i(x, 0) = u_{0,i}(x)$ where the operators $H_i$ are defined by

$$H_i(x, t, r, p, X) = \min_{\zeta \in \mathcal{Z}} \max_{\theta \in \mathcal{T}} \left\{ - tr \left( A_i^{\theta, \zeta}(t, x)X \right) + b_i^{\theta, \zeta}(t, x) \cdot p + l_i^{\theta, \zeta}(t, x) + \sum_{j \neq i} d_{ij}^{\theta, \zeta}(t, x)r_j \right\}$$

(2.23)

and $A_i^{\theta, \zeta}(t, x) = a_i^{\theta, \zeta}(t, x)a_i^{\theta, \zeta}(t, x)^T$, $d_{ij}^{\theta, \zeta} = -c_{ij}^{\theta, \zeta}$. Besides assumptions (C0)-(C6), we require the following assumptions.
Remark 2.3

Theorem 3.1 below guarantees that \( u \) and \( \gamma \) are both solutions and \( \gamma \) fulfills the above assumptions. Let \( u^1 \) and \( u^2 \) be respectively a bounded subsolution to problem \( 3.1 \) with \( H = H^1 \) and \( u_0 = u_0^1 \in C^\mu(\mathbb{R}^n) \) and a bounded supersolution to problem \( 3.1 \) with \( H = H^2 \) and \( u_0 = u_0^2 \in C^\mu(\mathbb{R}^n) \). Set \( R := \max(\|u^1\|, \|u^2\|) \) and \( \gamma := \min(\gamma_0, \gamma_1) \). Then there exists a constant \( K > 0 \) (depending only on \( T \) and \( R \)) and on the constants entering in our assumptions) such that, for every \( 0 \leq t \leq T \), there holds

\[
e^{-\gamma t} \|u^1(t, \cdot) - u^2(t, \cdot)\| \leq \|u_0^1 - u_0^2\| + Kt \sup_{i \in I, (x, t) \in Q_t, \theta, \zeta} \left[ |t^{\theta, \zeta, 1} - t^{\theta, \zeta, 2}| + |d^{\theta, \zeta} - d^{\theta, \zeta}| \right] + Kt^\gamma \sup_{i \in I, (x, t) \in Q_t, \theta, \zeta} \left[ |t^{\theta, \zeta, 1} - t^{\theta, \zeta, 2}| + |a_{i}^{\theta, \zeta, 1} - a_{i}^{\theta, \zeta, 2}| \right]
\]

where \( d^{\theta, \zeta} \) is the matrix \( (d^{\theta, \zeta}_{ij})_{i,j \in I} \) for \( k = 1, 2 \).

Remark 2.3 If \( u^1 \) and \( u^2 \) are both solutions and \( \sup_{\theta, \zeta, t, x, i} |a_{i}^{\theta, \zeta, k}(t, x)| < +\infty (k = 1, 2) \), then Theorem 2.1 below guarantees that \( u^1 \) and \( u^2 \) are bounded and it also provides an estimate of \( R \).

Proof By the arguments of [27] Thm 3.2 and 4.1, this result is a consequence of Theorem 2.1 and of the regularity of the coefficients and of the initial data.

For \( C_\mu := \min\{|u_0^1|, |u_0^2|\} \), we have

\[
e^{-\gamma t} \|u^1(t, \cdot) - u^2(t, \cdot)\| \leq \sup_{E_0^1} \left( e^{\gamma t} (u^1_0(t, \cdot) - u^2_0(t, \cdot)) - \frac{\alpha}{2} e^{\gamma t} |x - y|^2 \right)^+ \sup_{E_0^1} \left( u^1_0(t, \cdot) - u^2_0(t, \cdot) \right) \leq \|u_0^1 - u_0^2\| + \alpha^{\gamma t} |x - y|^2.
\]

By these inequalities, for

\[
f^{\theta, \zeta, k}_{ij} = t^{\theta, \zeta, k}_{ij}(t, x) \cdot p + t^{\theta, \zeta, k}_{ij}(t, x) \cdot a_{ij}^{\theta, \zeta, k}(t, x) \quad (k = 1, 2),
\]

Theorem 2.1 yields

\[
e^{-\gamma t} \|u^1(t, \cdot) - u^2(t, \cdot)\| \leq \|u_0^1 - u_0^2\| + \alpha^{\gamma t} |x - y|^2 + \sup_{D_{\gamma t}^1} \left( e^{\gamma t} \left[ |t^{\theta, \zeta, 1}(t, \cdot) - t^{\theta, \zeta, 2}(t, \cdot)| \right] \right) + \sup_{D_{\gamma t}^1} \left( e^{\gamma t} \left[ |d^{\theta, \zeta, 1}(t, \cdot) - d^{\theta, \zeta, 2}(t, \cdot)| \right] \right) + \sup_{D_{\gamma t}^1} \left( e^{\gamma t} \left[ |a_{i}^{\theta, \zeta, 1}(t, \cdot) - a_{i}^{\theta, \zeta, 2}(t, \cdot)| \right] \right)
\]
where the last inequality is due to the Young one and to the choice $p = \alpha(x - y)e^{(\gamma - \gamma)\tau}$.

We choose $\tilde{\gamma}$ sufficiently large such that

$$1 + L_0 + 6L_0^2 - \tilde{\gamma}/2 = -1.$$ 

Furthermore, by standard calculus, we get

$$(L_I + mL_dR)e^{\gamma T}|x - y|^\mu - \alpha e^{\gamma \tau}|x - y|^2 \leq K_1 \alpha^{-\mu/(2 - \mu)}$$

where $K_1$ is a constant depending only on $L_I$, $L_d$,$R$, $\gamma$, $\tilde{\gamma}$ and $T$. Taking into account the last three inequalities, for $K_2 := \alpha^{-2/\mu}/2 + tK_1$ we obtain

$$e^{\gamma \tau}\|u^1(t, \cdot) - u^2(t, \cdot)\|$$

$$\leq \|u_0^1 - u_0^2\| + t\sup_{D_{\gamma T}}\left\{ e^{\gamma \tau} \left[ |b_{i\theta}^{\kappa,1}(\tau, x) - b_{i\theta}^{\kappa,2}(\tau, x)| + R \sum_{j \in I} |d_{ij}^{\kappa,1}(\tau, x) - d_{ij}^{\kappa,2}(\tau, x)| \right] \right\}$$

$$+ K_2 \alpha^{-\mu/(2 - \mu)} + \alpha t \sup_{D_{\gamma T}}\left\{ e^{\gamma \tau} \left[ |a_{ij}^{\kappa,2}(\tau, x)|^2 \right] \right\}.$$ 

Minimizing the right-hand side by an adequate choice of $\alpha$, we have

$$e^{\gamma \tau}\|u^1(t, \cdot) - u^2(t, \cdot)\|$$

$$\leq \|u_0^1 - u_0^2\| + t\sup_{D_{\gamma T}}\left\{ e^{\gamma \tau} \left[ |b_{i\theta}^{\kappa,1}(\tau, x) - b_{i\theta}^{\kappa,2}(\tau, x)| + R \sum_{j \in I} |d_{ij}^{\kappa,1}(\tau, x) - d_{ij}^{\kappa,2}(\tau, x)| \right] \right\}$$

$$+ K_3 \alpha^{\mu \gamma T}/2 \left( \sup_{D_{\gamma T}}\left\{ e^{\gamma \tau} \left[ |a_{ij}^{\kappa,2}(\tau, x)|^2 \right] \right\} \right)^{\mu/2}.$$ 

where $K_3$ is a constant depending only on $T$, and on the constants entering in our assumptions. Finally, by the Young inequality, one can easily accomplish the proof. \hfill \Box

**Remark 2.4** By the proof above, this result is still true when, for each $i \in I$, either $u_{0i}^1$ or $u_{0i}^2$ belongs to $C^0(\mathbb{R}^n)$ and the assumptions on the regularity are fulfilled by either $a_{i1}^{\theta,1}$ or $a_{i1}^{\theta,2}$, either $b_{i1}^{\theta,1}$ or $b_{i1}^{\theta,2}$, or he $l_{i1}^{\theta,1}$ or $l_{i1}^{\theta,2}$.

## 3 Regularity estimates and vanishing viscosity

In this Section we collect some applications of Theorem 2.1. the first part is devoted to establish a regularity estimate for the solution to system (10) provided that the initial condition and the coefficients are Hölder continuous. In the second part we prove an estimate of the vanishing viscosity approximation.

### 3.1 Regularity estimates

In this section, we address the Cauchy problem

$$\begin{cases}
\partial_t u_i + H_i(t,x,u,Du_i, D^2 u_i) = 0 & \text{in } Q_T \\
u_i(0,x) = u_{0i}(x) & \text{on } \mathbb{R}^n, i \in I
\end{cases}$$

(3.1)

with $H_i$ of the form (1.2) and we establish two results for the solution $u$: an $L^\infty$-estimate and the Hölder continuity.

**Theorem 3.1** Assume conditions (C0)-C4 and (C7). For $u_{0i}$ continuous and bounded ($i \in I$), let $u$ be a bounded solution to problem (3.1). Then, for $\gamma := \gamma u$ (the constant $\gamma u$ is introduced in (C4)) there holds:

$$\|u(t, \cdot)\| \leq e^{\gamma T}\|u_0\| + T e^{\gamma T}C_f.$$
Proof. Assume $\gamma_{||u||} = 0$ in (C4). We shall proceed following the same arguments as those of Theorem 2.1 with $u_1 \equiv 0$ and $u_2 = u$ (clearly, $u_1$ is the solution to (1.1) with zero coefficients). Relations (2.5) and (2.9) guarantee
\[
0 \leq b + \min_{\zeta \in Z} \max_{\theta \in \Theta} \{- \text{tr} \left( A\zeta_{i_0}(\tau_0, y_0, p_{y_0})Y \right) + f^{\zeta_{i_0}}_{i_0}(\tau_0, y_0, u(\tau_0, y_0), p_{y_0}, Y) \} \\
\leq a - \frac{\delta \sigma}{\ell} + \min_{\zeta \in Z} \max_{\theta \in \Theta} \{- \text{tr} \left( A\zeta_{i_0}(\tau_0, y_0, p_{y_0})Y \right) + f^{\zeta_{i_0}}_{i_0}(\tau_0, y_0, u(\tau_0, y_0), p_{y_0}, Y) \}
\]
where $p_{y_0} = \alpha e^{\gamma_{t_0}}(x_0 - y_0) - \varepsilon y_0$. We observe that $(a, p_{x_0}, X) \in \mathcal{T}_2^{2+0}$ iff $a = 0$, $p_{x_0} = 0$ and $X \geq 0$; hence, by (2.14), we get $Y \geq X - 4\varepsilon I \geq -4\varepsilon I$. Therefore, the above estimate entails
\[
\sigma \leq t\delta^{-1} \min_{\zeta \in Z} \max_{\theta \in \Theta} \{ 4\varepsilon \text{tr} \left( A\zeta_{i_0}(\tau_0, y_0, p_{y_0}) \right) + f^{\zeta_{i_0}}_{i_0}(\tau_0, y_0, u(\tau_0, y_0), p_{y_0}, -4\varepsilon I) \} \\
\leq t\delta^{-1} \min_{\zeta \in Z} \max_{\theta \in \Theta} \{ 4\varepsilon \text{tr} \left( A\zeta_{i_0}(\tau_0, y_0, \text{const.}\varepsilon^{1/2}) \right) + f^{\zeta_{i_0}}_{i_0}(\tau_0, y_0, 0, \text{const.}\varepsilon^{1/2}, -4\varepsilon I) \}
\]
where the last inequality is due to the same arguments as in (2.10) and to Lemma 2.1 (i) and (iii). Observe that assumption (C3) and estimate (2.5) ensure
\[
\varepsilon \text{tr} \left( A_{i_0}\zeta(\tau_0, y_0, \text{const.}\varepsilon^{1/2}) \right) \leq \text{const.}m(\varepsilon).
\]
Letting $\varepsilon \to 0$ and $\delta \to 1$, we obtain
\[
-\|u_0\| - \inf_{x \in \mathbb{R}^n, t \in I} \{ u(t, \cdot) \} \leq \sigma \leq tC^f;
\]
namely, one side of the statement is established. Reversing the role of $u$ and 0, one can easily deduce the other inequality of the statement. The case $\gamma_{||u||} \neq 0$ will follow as for Theorem 2.1.

**Theorem 3.2** Assume (C0)-(C6) and $u_0 \in C^\mu(\mathbb{R}^n)$ for some $\mu \in (0, 1)$. Then any bounded continuous $u$ to problem (3.1) is Hölder continuous in $x$ and, for some positive constant $K$, it fulfills
\[
[u(t, \cdot)]_{\mu} \leq Ke^{\gamma t} \left( |u_0|_{\mu} + t^{1-\mu/2} \varepsilon^{\gamma t} C_{f,i_0}\|u\| \right)
\]
where $\gamma^+ := \max\{0, \gamma_{||u||}\}$ and $\bar{\gamma} := 2(C_{f,i_0}\|u\| + 3C_a^2 + 1) + \gamma^+$ (the constants $\gamma_R$, $C_{f,R}$ and $C_a$ are those introduced respectively in (C4), (C5) and (C6)).

**Proof.** This proof relies on the arguments of [17] Thm 3.3-(b)]: the application of Theorem 2.1 with $f^{\zeta_{i_0}}_1 = f^{\zeta_{i_0}}_2$, $a^{\zeta_{i_0}}_1 = a^{\zeta_{i_0}}_2$ and $u_1 = u^2$ with a careful estimates of the two sides. For the sake of completeness, let us sketch them. We observe that
\[
\sup_{x \in I, x \in \mathbb{R}^n} \left( e^{\gamma t}(u_1(t, x) - u_2(t, x)) \right) \leq \sup_{E^{\tau}} \left( e^{\gamma t}(u_1(t, \tau, x) - u_2(t, \tau, y)) - \frac{\alpha}{2} e^{\gamma t}|x - y|^2 \right)
\]
and
\[
\sup_{E^{\tau}} \left( u_1(0, x) - u_2(0, y) - \frac{\alpha}{2}|x - y|^2 \right) \leq [u_0]|_{\mu}|x - y|^\mu - \frac{\alpha}{2}|x - y|^2 \leq 2[u_0]|_{\mu}^{\frac{\alpha - \mu}{\alpha}} \frac{\alpha - \mu}{\alpha}
\]
where the last inequality is due to the Young inequality with exponents $2/\mu$ and $2/(2 - \mu)$. Moreover, by conditions (C5) and (C6) (recall $p = \alpha(x - y)e^{(\gamma - \gamma) t}$) and by our choice of $\bar{\gamma}$, we have
\[
e^{\gamma t}[f^{\zeta_{i_0}}_1(t, r, p, X) - f^{\zeta_{i_0}}_1(t, x, r, p, X)] + 3\alpha e^{\gamma t}[a^{\zeta_{i_0}}_1(t, x, p) - a^{\zeta_{i_0}}_2(t, y, p)]^2 - \frac{\alpha}{2} \bar{\gamma} e^{\gamma t}|x - y|^2 \\
\leq e^{\gamma t}C_{f,i_0}\|u\| |x - y|^\mu + \alpha e^{\gamma t}|x - y|^2 \left( C_{f,i_0}\|u\| + 3C_a^2 \right) \\
\leq e^{\gamma t} \left[ C_{f,i_0}\|u\| |x - y|^\mu - \alpha|x - y|^2 \right] \leq K_1 e^{\gamma t}(C_{f,i_0}\|u\|)^{2/(2-\mu)} \frac{\alpha - \mu}{\alpha}
\]
where last inequality is due to standard calculus and $K_1$ is a constant depending only on $\mu$. Therefore, taking into account the last two inequalities, Theorem [24] entails
\[
e^{\gamma \tau} (u^\varepsilon_i(\tau, x) - u^\varepsilon_i(\tau, y)) \leq \left[ 2|u_0|^2 + K_1 e^{\gamma \tau} (C_{f,\|u\|})^{2/(2-\mu)} \right] + 2 \eta \varepsilon \left( |f| + |\Delta u_0| \right)
\]
and the statement follows by a suitable choice of $\alpha$ (see [24] Thm 3.3-(b) for detailed calculations).

3.2 Vanishing viscosity

We consider the viscous approximation to (1.1)
\[
\partial_t u^\varepsilon_i + H_i(t, x, u, Du^\varepsilon_i, D^2 u^\varepsilon_i) = \varepsilon \Delta u^\varepsilon_i \quad \text{in } Q_T, \ i \in I
\]
where $H_i$ is as in (1.2). In the next proposition we establish an estimate on the rate of convergence of $u^\varepsilon$ to $u$.

**Proposition 3.1** Assume (C0)-(C7) and that, for any $\varepsilon > 0$, there exists a bounded solution $u^\varepsilon$ to (1.2). Then there exists a solution $u \in C^b(Q_T)$ to (1.1)-(1.2) and
\[
\|u(t, \cdot) - u^\varepsilon(t, \cdot)\| \leq C (\|u(0, \cdot) - u^\varepsilon(0, \cdot)\| + \varepsilon^{1/2}) \quad t \in [0, T]
\]
where $C$ is independent of $\varepsilon$.

**Proof** The existence of the solution $u$ to (1.1) and the local uniform convergence of the sequence $u^\varepsilon$ to $u$ can be obtained by employing the classical weak limit method introduced by Barles-Perthame, which can be easily adapted to systems. Moreover by Theorem 3.2, the functions $u^\varepsilon$ and $u$ belong to $C^b(\bar{Q}_T)$ for any $\varepsilon$. The proof of the rate of convergence is based on the estimate in Theorem 2.1 applied to problem (1.1) and (3.2) with
\[
J^\varepsilon(t, x, r, p, X) = f^\varepsilon(t, x, r, p, X) \quad A^\varepsilon(t, x, p) = A^\varepsilon(t, x, p) \quad A^\varepsilon(t, x, p) = A^\varepsilon(t, x, p)
\]
Since it is very similar to the proof of the corresponding result in [24], we omit it. $\Box$

**Remark 3.1** A similar estimate for the vanishing viscosity approximation of weakly coupled systems has been recently proved in [14] using different techniques and stronger assumptions.

3.2.1 Vanishing viscosity for a first order problem

Let us establish a rate of convergence for the vanishing viscosity approximation of a first order system arising in optimal control problem. Being a straightforward application of Proposition 3.1, the proof is omitted.

**Proposition 3.2** Assume the hypotheses of Section 2.2. Let $u^\varepsilon$ and $u$ be the solution of
\[
\partial_t u^\varepsilon_i + \min_{\zeta \in \mathbb{Z}} \max_{\theta \in \Theta} \left\{ -\varepsilon \text{ tr} (A^\varepsilon_i(t, x) D^2 u^\varepsilon_i) + b^\varepsilon_i(t, x) \cdot Du^\varepsilon_i + l^\varepsilon_i(t, x) + \sum_{j \in I} d^\varepsilon_{ij}(t, x) u_j \right\} = 0
\]
and respectively of
\[
\partial_t u_i + \min_{\zeta \in \mathbb{Z}} \max_{\theta \in \Theta} \left\{ b^\varepsilon_i(t, x) \cdot Du_i + l^\varepsilon_i(t, x) + \sum_{j \in I} d^\varepsilon_{ij}(t, x) u_j \right\} = 0.
\]
Then
\[
\|u(t, \cdot) - u^\varepsilon(t, \cdot)\| \leq C (\|u(0, \cdot) - u^\varepsilon(0, \cdot)\| + \varepsilon^{1/2}) \quad t \in [0, T].
\]
4 Periodic Homogenization of quasi-monotone systems

In this section we study the periodic homogenization of the fully nonlinear systems

$$\begin{align*}
\partial_t u_i^\varepsilon + H_i \left( x, \frac{x}{\varepsilon}, u_i^\varepsilon, Du_i^\varepsilon, D^2 u_i^\varepsilon \right) &= 0 & \text{in } Q_T \\
u_i^\varepsilon(0,x) &= u_{0i}(x) & \text{on } \mathbb{R}^n, \ i \in I
\end{align*}$$

(4.1)

where

$$H_i(x,y,r,p,X) = \min_{\zeta \in Z} \max_{\theta \in \Theta} \left\{ -\text{tr} \left( A_i^{0\varepsilon}(x,y) \right) + f^0_i(x,y,r,p) \right\}.$$  

For the sake of clarity, let us list the assumptions that will hold throughout this section.

(H0) The sets $\Theta$ and $Z$ are two compact metric spaces.

(H1) The functions $f^0_i$ are continuous and, for some constant $L_f$ and a modulus of continuity $\omega$, they satisfy

$$|f^0_i(x_1, y_1, r_1, p_1) - f^0_i(x_2, y_2, r_2, p_2)| \leq L_f |(x_1, y_1) - (x_2, y_2)| + \omega(|(x_1, y_1) - (x_2, y_2)| + L_f)$$

for every $x_1, y_1, r_1, \theta, \zeta, i$ $(k = 1, 2)$. Moreover, there exists a constant $C$ such that

$$|f^0_i(x, y, 0, 0)| \leq C \quad \forall x, y, r, \theta, \zeta. \quad (4.2)$$

(H2) $A_i^{0\varepsilon}(x,y) = a_i^{0\varepsilon}(x,y) a_i^{0\varepsilon}(x,y)^T$ for some bounded, continuous matrix $a_i^{0\varepsilon}$ satisfying

$$|a_i^{0\varepsilon}(x_1, y_1) - a_i^{0\varepsilon}(x_2, y_2)| \leq L_a |(x_1, y_1) - (x_2, y_2)| \quad \forall x_k, y_k, \theta, \zeta, i \in I \quad (k = 1, 2).$$

(H3) $f^0_i(x, \cdot, r, p)$ and $a_i^{0\varepsilon}(x, \cdot)$ are $\mathbb{Z}^n$-periodic in $y$ for any $x, r, p, \theta, \zeta, i$.

(H4) The matrix $A_i^{0\varepsilon}$ is uniformly elliptic, namely, for some positive constant $\nu$ there holds

$$a_i^{0\varepsilon}(x, y) \geq \nu I, \quad \forall x, y, \theta, \zeta, i.$$

(H5) There exists $\gamma \in \mathbb{R}$ such that if $r, s \in \mathbb{R}^m$ and $r_j - s_j = \max_{k \in I} \{r_k - s_k\} \geq 0$, then

$$f^0_i(x, y, r, p) - f^0_i(x, y, s, p) \geq \gamma (r - s) \quad \forall x, y, p, \theta, \zeta. \quad (4.3)$$

We consider the cell problem:

For any fixed $i \in I$ and $(x, r, p, X) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times S^n$, find a constant $\overline{H}_i = \overline{H}_i(x,r,p,X)$ such that the equation

$$H_i(x,y,r,p,X + D^2_y v(y)) = \overline{H}_i, \quad y \in \mathbb{R}^n \quad (4.3)$$

admits a periodic solution $v_i = v_i(\cdot; x, r, p, X)$.  

It is well known (see: [17 8 2 31]) that there exists exactly one value $\overline{H}_i$ such that (4.3) has a solution; moreover, $\overline{H}_i$ can be obtained as the (uniform) limit of $-\lambda v_{\lambda,i}$ as $\lambda \to 0$, where the approximated corrector $v_{\lambda,i} := v_{\lambda,i}(y; x, r, p, X)$ is the solution to

$$-\lambda v_{\lambda,i} + H_i(x, y, r, p, X + D^2_y v_{\lambda,i}) = 0, \quad y \in \mathbb{R}^n. \quad (4.4)$$

We associate to each Hamiltonian $H_i$ the corresponding effective Hamiltonian $\overline{H}_i$. Note that at this level the index $i$ is fixed, hence the definition of the effective Hamiltonians does not involve any coupling among the equations. Nevertheless, in view of existence and uniqueness results for the homogenized problem, we need to study the regularity of the effective Hamiltonians in particular with respect to the variable $r \in \mathbb{R}^m$.

In the next proposition we collect some useful properties of the approximated correctors $v_{\lambda,i}$ and of the effective operators $H_i$.  

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Proposition 4.1 The following properties hold:

i) For any $i, x, r, p, X$, the approximated equation (4.4) admits exactly one periodic continuous solution $u_{\lambda,i}$. Moreover, as $\lambda \to 0^+$, $\lambda u_{\lambda,i}$ and $(v_{\lambda,i} - v_{\lambda,i}(0))$ converge respectively to the ergodic constant $-\overline{\Pi}_i$ and to a solution $v_i$ of (1.3) with $v_i(0) = 0$.

ii) For any $i \in I$, the effective Hamiltonian $\overline{\Pi}_i$ is continuous in $(x, r, p, X)$ and

a) For some constant $C_1 > 0$ and a modulus of continuity $\omega_1$, there holds

$$|\overline{\Pi}_i(x, r_1, p_1, X_1) - \overline{\Pi}_i(x, r_2, p_2, X_2)| \leq C_1 (|r_1 - r_2| + |p_1 - p_2| + |X_1 - X_2|);$$

$$|\overline{\Pi}_i(x, r, p, X) - \overline{\Pi}_i(x_2, r, p, X_2)| \leq C_1 (1 + |p| + |r| + |X|)|x_1 - x_2| + \omega_1(|x_1 - x_2|);$$

$$|\overline{\Pi}_i(x, r, p, X)| \leq \max_{y \notin X} |-tr(A_i^{0c}(x, y)X) + f_i^{0c}(x, y, r, p)|$$

for every $x_k, p_k, r_k, X_k$ ($k = 1, 2$).

b) $\overline{\Pi}_i$ is uniformly elliptic. Moreover, if $H_i$ is convex, then $\overline{\Pi}_i$ is also convex.

c) $\{\overline{\Pi}_i\}_{i \in I}$ is quasi-monotone, namely, it satisfies (1.3).

Proof For statement (i), we refer to [17] (see also [2] and [3]). The estimates in (ii).a follow by the continuous dependance estimates in [31] Thm 3.1 (note that in the cell problem both $r$ and $p$ are fixed), while property (ii).b is proved for example in [2] and in [17]. We finally prove that $\overline{\Pi}_i$, $i \in I$, satisfy the quasi-monotonicity condition (1.3). Assume by contradiction that there exist $r$, $s \in \mathbb{R}^n$ such that $r_j - s_j = \max_{k \in I} \{r_k - s_k\} \geq 0$ and

$$\overline{\Pi}_j(x, r, p, X) < \overline{\Pi}_j(x, s, p, X)$$

for some $x \in \mathbb{R}^n$, $p \in \mathbb{R}^n$, $X \in S^n$. Let $u_r$ and $u_s$ be two periodic solutions respectively of

$$H_j(x, y, r, p, X + D^2u_r) = \overline{\Pi}_j(x, r, p, X) \quad y \in \mathbb{R}^n,$n

$$H_j(x, y, s, p, X + D^2u_s) = \overline{\Pi}_j(x, s, p, X) \quad y \in \mathbb{R}^n.$$n

(these functions exist by point (i)). Since $u_r$, $u_s$ are bounded, by adding a constant we can assume w.l.o.g. $u_r > u_s$ in $\mathbb{R}^n$. Since

$$H_j(x, y, r, p, X + D^2u_r) = \overline{\Pi}_j(x, r, p, X) < \overline{\Pi}_j(x, s, p, X) = H_j(x, y, s, p, X + D^2u_s) \leq H_j(x, y, r, p, X + D^2u_s)$$

(where the last inequality follows by (H5)), then for $\lambda$ sufficiently small

$$\lambda u_r + H_j(x, y, r, Du_r, X + D^2u_r) \leq \lambda u_s + H_j(x, y, r, Du_s, X + D^2u_s) \quad y \in \mathbb{R}^n.$$n

By the comparison principle for problem (4.4), we deduce $\lambda u_r \leq \lambda u_s$; as $\lambda \to 0^+$, we infer $\overline{\Pi}_i(x, r, p, X) \geq \overline{\Pi}_i(x, s, p, X)$ which gives the desired contradiction. □

Proposition 4.2 Let $u_0 \in BUC(\mathbb{R}^n)$. Then

- For any $\varepsilon > 0$ there exists a unique solution $u_\varepsilon \in BUC(\mathbb{Q}_T)$ to (4.1). Moreover $u_\varepsilon$ is bounded uniformly in $\varepsilon$.

- There exists a unique solution $u \in BUC(\mathbb{Q}_T)$ to the effective problem

$$\begin{cases} 
\partial_t u_i + \overline{\Pi}_i(x, u_i, Du_i, D^2u_i) = 0 & \text{in } Q_T \\
\overline{\Pi}_i(0, u) = u_{0i}(x) & \text{on } \mathbb{R}^n, i \in I 
\end{cases}$$

(4.5)

where the operators $\overline{\Pi}_i$ are defined by the cell problem (1.3).
Proof. By routine adaptation of the arguments in [26], [41] and [46] satisfy a comparison principle for sub and supersolution. In order to prove the existence of the solution, we note that assumption \((H1)\) ensures \(\|f^\vp(x,y,r,0)\| \leq C + L|r|\). We deduce that, for a constant \(\tilde{C}\) sufficiently large, the functions \(u^\pm(x,t) = \pm(\|u_0\| + e^{\tilde{C}t}, \ldots, \|u_0\| + e^{\tilde{C}t})\) are respectively a super- and a subsolution of (4.1). Actually, by this inequality, we have
\[
\partial_t u^+_t + H_i\left(x, \frac{x}{\varepsilon}, u^0, Du^+_t, D^2 u^+_t\right) = \tilde{C}e^{\tilde{C}t} + H_i\left(x, \frac{x}{\varepsilon}, u^+, 0, 0\right) \geq (\tilde{C} - L)e^{\tilde{C}t} - C - L\|u_0\| \geq 0
\]
provided that \(\tilde{C} = L + 1 + C + L\|u_0\|\); hence \(u^+\) is a supersolution. Being similar, the proof for \(u^-\) is omitted. By the Perron’s method for system, see [26], it follows the existence of a solution \(u_\varepsilon \in \text{BUC}(Q_T)\) to (4.1) such that
\[
-\|u_0\| - e^{\tilde{C}T} \leq u^+_\varepsilon(t,x) \leq \|u_0\| + e^{\tilde{C}T}, \quad (t,x) \in Q_T, i \in I.
\]
The existence of a bounded solution to (4.1) is proved in the same way. \(\square\)

**Theorem 4.1** The solution \(u^\varepsilon\) of (4.1) converges locally uniformly on \([0,T] \times \mathbb{R}^n\) to the solution \(u \in \text{BUC}(Q_T)\) of (4.5).

Proof. By Proposition 4.2 there exists a continuous solution \(u^\varepsilon\) of (4.1) which is bounded independently of \(\varepsilon\). We follow the argument in [41, Thm 3.5]. We introduce the half-relaxed limits
\[
\overline{v}(t,x) = \limsup_{\varepsilon \to 0, (t_\varepsilon,x_\varepsilon) \to (t,x)} u^\varepsilon(t_\varepsilon, x_\varepsilon) \quad \text{and} \quad \underline{v}(t,x) = \liminf_{\varepsilon \to 0, (t_\varepsilon,x_\varepsilon) \to (t,x)} u^\varepsilon(t_\varepsilon, x_\varepsilon).
\]
We first show that \(\overline{v}\) is a subsolution of the system (4.5). We assume there exists \(i \in I\) and \(\phi \in C^2\) such that \(\overline{v} - \phi\) has a strict maximum point at some \((\overline{t}, \overline{x}) \in (0,T) \times \mathbb{R}^n\) with \(\overline{v}(\overline{t}, \overline{x}) = \phi(\overline{t}, \overline{x})\). We assume wlog \(i = 1\) and we want show that
\[
\partial_t \phi(\overline{t}, \overline{x}) + H_1(\overline{t}, \overline{x}, \overline{v}(\overline{t}, \overline{x}), D\phi(\overline{t}, \overline{x}), D^2 \phi(\overline{t}, \overline{x})) \leq 0. \quad (4.6)
\]
Let \(v = v(y)\) be a periodic viscosity solution of
\[
H_1(\overline{t}, \overline{x}, \overline{v}(\overline{t}, \overline{x}), D\phi(\overline{t}, \overline{x}), D^2 \phi(\overline{t}, \overline{x}) + D^2 v(y)) = H_1(\overline{t}, \overline{x}, \overline{v}(\overline{t}, \overline{x}), D\phi(\overline{t}, \overline{x}), D^2 \phi(\overline{t}, \overline{x}));
\]
namely, \(v\) solves the cell problem (4.8) with \((x,p,r,X) = (\overline{t}, \overline{x}, \overline{v}(\overline{t}, \overline{x}), D\phi(\overline{t}, \overline{x}), D^2 \phi(\overline{t}, \overline{x}))\) (we recall that its existence is ensured by Proposition 4.1(i)). By [41, Lemma 2.7] (recalled in Lemma A.2 below) for each \(\eta > 0\), there exists a periodic supersolution \(w \in C(\mathbb{R}^n) \cap W^{2,\infty}(\mathbb{R}^n)\) of
\[
H_1(\overline{t}, \overline{x}, \overline{v}(\overline{t}, \overline{x}), D\phi(\overline{t}, \overline{x}), D^2 \phi(\overline{t}, \overline{x}) + D^2 w(y)) = H_1(\overline{t}, \overline{x}, w(\overline{t}, \overline{x}), D\phi(\overline{t}, \overline{x}), D^2 \phi(\overline{t}, \overline{x})) - \eta. \quad (4.7)
\]
Define the “perturbed test-function”
\[
\phi^\varepsilon(t,x) = \phi(t,x) + \varepsilon^2 w\left(\frac{x}{\varepsilon}\right).
\]
By standard results, we have that, up to extract subsequences, there exist \((t_\varepsilon, x_\varepsilon) \in Q_T, (t_\varepsilon, x_\varepsilon) \to (\overline{t}, \overline{x})\) for \(\varepsilon \to 0\) such that \((t_\varepsilon, x_\varepsilon)\) is a local maximum of \(u^\varepsilon_1(t,x) - \phi^\varepsilon(t,x)\) and \(\lim_{\varepsilon \to 0} u^\varepsilon_1(t_\varepsilon, x_\varepsilon) = \overline{v}(\overline{t}, \overline{x})\).

Assume for the moment that \(w \in C^2(\mathbb{R}^n)\) so that \(\phi^\varepsilon\) is an admissible test function for \(u^\varepsilon_1\) at \((t_\varepsilon, x_\varepsilon)\). Then
\[
\partial_t \phi(t_\varepsilon, x_\varepsilon) + H_1\left(x_\varepsilon, \frac{x_\varepsilon}{\varepsilon}, u^\varepsilon(t_\varepsilon, x_\varepsilon), D\phi(t_\varepsilon, x_\varepsilon) + \varepsilon Dw\left(\frac{x_\varepsilon}{\varepsilon}\right), D^2 \phi(t_\varepsilon, x_\varepsilon) + D^2 w\left(\frac{x_\varepsilon}{\varepsilon}\right)\right) \leq 0. \quad (4.8)
\]
Set $\delta_{\varepsilon} := u_{\varepsilon}^j(t_{\varepsilon}, x_{\varepsilon}) - \phi^j(t_{\varepsilon}, x_{\varepsilon})$. By the definition of $\overline{\w}$, up to a subsequence, for $j \neq 1$ $u_{\varepsilon}^j(t_{\varepsilon}, x_{\varepsilon}) \to \bar{r}_j$ with $\bar{r}_j \leq \overline{w_j}(t, \overline{x})$. By (H5),

$$0 \geq \partial_t \phi(t_{\varepsilon}, x_{\varepsilon}) + H_1 \left( x_{\varepsilon}, \frac{x_{\varepsilon}}{\varepsilon}, u_{\varepsilon}^c(t_{\varepsilon}, x_{\varepsilon}), D\phi(t_{\varepsilon}, x_{\varepsilon}) + \varepsilon D\overline{w} \left( \frac{x_{\varepsilon}}{\varepsilon} \right) \right),$$

$$\partial_t \phi(t_{\varepsilon}, x_{\varepsilon}) + H_1 \left( x_{\varepsilon}, \frac{x_{\varepsilon}}{\varepsilon}, u_{\varepsilon}^c(t_{\varepsilon}, x_{\varepsilon}) + \delta_{\varepsilon}, u_{\varepsilon}^c(t_{\varepsilon}, x_{\varepsilon}), \ldots, u_{\varepsilon}^c(t_{\varepsilon}, x_{\varepsilon}) \right), D\phi(t_{\varepsilon}, x_{\varepsilon}) + \varepsilon D\overline{w} \left( \frac{x_{\varepsilon}}{\varepsilon} \right),$$

$$H^2 \partial_t \phi(t_{\varepsilon}, x_{\varepsilon}) + D^2 \phi(t_{\varepsilon}, x_{\varepsilon}) + D^2 w \left( \frac{x_{\varepsilon}}{\varepsilon} \right)$$

$$\geq \partial_t \phi(t_{\varepsilon}, x_{\varepsilon}) + H_1 \left( x_{\varepsilon}, \frac{x_{\varepsilon}}{\varepsilon}, u_{\varepsilon}^c(t_{\varepsilon}, x_{\varepsilon}), \ldots, u_{\varepsilon}^c(t_{\varepsilon}, x_{\varepsilon}) \right), D\phi(t_{\varepsilon}, x_{\varepsilon}) + \varepsilon D\overline{w} \left( \frac{x_{\varepsilon}}{\varepsilon} \right),$$

$$H^2 \partial_t \phi(t_{\varepsilon}, x_{\varepsilon}) + D^2 \phi(t_{\varepsilon}, x_{\varepsilon}) + D^2 w \left( \frac{x_{\varepsilon}}{\varepsilon} \right) + \gamma_{\delta_{\varepsilon}}.$$

We denote by $\xi$ the limit in $\mathbb{R}^n/\mathbb{Z}^n$ of $x_{\varepsilon}/\varepsilon$ as $\varepsilon \to 0$. Passing to the limit for $\varepsilon \to 0$ in the previous inequality, by the periodicity of $H_1$ and $\overline{w}$, (4.7) and (H5) with $r = (\overline{w_1}(t, \overline{x}), \overline{w_2}, \ldots, \overline{w_m})$ and $s = (\overline{w_1}(t, \overline{x}), \overline{w_2}(t, \overline{x}), \ldots, \overline{w_m}(t, \overline{x}))$ we get

$$0 \geq \partial_t \phi(t_{\overline{x}}, \overline{x}) + H_1(\overline{x}, \xi, (\overline{w_1}(t, \overline{x}), \overline{w_2}, \ldots, \overline{w_m}(t, \overline{x})), D\phi(t_{\overline{x}}, \overline{x}), D^2 \phi(t_{\overline{x}}, \overline{x}) + D^2 w(\xi))$$

and, for the arbitraryness of $\eta$ we get (4.6). If $w$ is not smooth, using in (24) Lemma 3.6] (recalled in Lemma A.3) it is possible to find $X_{\varepsilon} \in S^n$ such that

$$(D\overline{w(\frac{x_{\varepsilon}}{\varepsilon}), X_{\varepsilon}}) \in J^2 w(\frac{x_{\varepsilon}}{\varepsilon}),$$

$$(D\phi(t_{\varepsilon}, x_{\varepsilon}) + \varepsilon D\overline{w(\frac{x_{\varepsilon}}{\varepsilon}), X_{\varepsilon}}) \in J^2 + \varepsilon u^c(t_{\varepsilon}, x_{\varepsilon})$$

hence the above arguments hold with $X_{\varepsilon}$ in place of $D^2 w(\frac{x_{\varepsilon}}{\varepsilon})$. The rest of the proof to obtain (4.6) is exactly the same.

We prove that $\overline{w}$ is a viscosity supersolution of (4.5) in a similar way. From Proposition 4.2 we then obtain $\overline{w} \leq \overline{w}$ in $Q_T$, hence $\overline{w} = \overline{w} := u$ where $u$ is the (local) uniform limit of the $u^c$'s. □

**Remark 4.1** Observe that in the previous proof we exploit three facts

- for each $i \in I$, $H_i$ is ergodic, i.e. the cell problem (4.8) admits a solution for any $(x, r, p, X)$.

- there exist “sufficiently regular” approximations to the solution to the cell problem (4.8)

- The effective Hamiltonian $\overline{H}_i$ satisfies the properties in Prop. 4.1.ii).

The uniform ellipticity of $H_i$ is a sufficient condition to ensure these properties (for the last one, some regularity assumptions on the coefficients is also needed). Let us stress that such properties still hold under different hypotheses as, for instance, for first order equations, the coercivity with respect to $p$ (in this case, the regular approximations of the solution to the cell problem will belong to $W^{1,\infty}$).

**Example 4.1** Consider the weakly coupled system

$$\partial_t u_i^c - \operatorname{tr} \left( a_i \left( x, \frac{x}{\varepsilon} \right) D^2 u_i^c \right) + F_i \left( x, \frac{x}{\varepsilon}, u, Du_i^c \right) = 0 \quad (t, x) \in Q_T, \quad i \in I \quad (4.9)$$

where $F_i(x, y, r, p) = \min_{\omega \in \Theta} \max_{\xi \in \mathbb{Z}} \left\{ -j_{i}^{\nu}(x, y) \cdot p - l_{i}^{\nu}(x, y) - \sum_{j} d_{ij}^{\nu}(x, y) r_j \right\}$.

For each $\overline{x}, \overline{w}, \overline{w}, \overline{X}$, the cell problem reads

$$-\operatorname{tr} \left( a_i(\overline{x}, y) D^2 v \right) - \operatorname{tr} \left( a_i(\overline{x}, y) \overline{X} \right) + F_i(\overline{x}, y, \overline{w}, \overline{p}) = \overline{H}(\overline{x}, \overline{w}, \overline{p}, \overline{X}).$$
By standard theory for linear ergodic problems (see \cite{6} and also \cite{2}), there holds
\[
\mathcal{P}(x, u, p, X) = -\text{tr} (\bar{a}(x)X) + \mathcal{F}_i (x, u, p)
\] (4.10)
where the effective diffusion $\bar{a}$ and the effective operator $\mathcal{F}_i$ have respectively the form
\[
\bar{a}_i(x) := \int_{[0,1]^n} a_i(x, y) \, d\mu_x(y), \quad \mathcal{F}_i (x, r, p) := \int_{[0,1]^n} F_i (x, y, r, p) \, d\mu_x(y).
\]
Here, for $\mathcal{T}$ fixed, the measure $\mu_\mathcal{T}$ is the unique invariant measure for the diffusion $a(\mathcal{T}, y)$, i.e. the solution in the sense of distributions of the equation
\[
\sum_{i,j=1}^n \frac{\partial^2}{\partial y_i \partial y_j} (a_{ij}(\mathcal{T},y)) = 0, \quad \mu_\mathcal{T} \text{ periodic.}
\]

As a straightforward application of Proposition 4.1(ii), Proposition 4.2 and Theorem 4.1, we have the following result

**Corollary 4.1** Let $u^e$ and $u$ be respectively the solution to system (4.11) with $H_z$ as in (4.17) and the solution to (4.5) with $H$ as in (4.10). Then $u^e$ converges locally uniformly to $u$ on $[0, T] \times \mathbb{R}^n$.

### Appendix

For the proof of Lemma 2.1, we need the following technical Lemma:

**Lemma A.1** Let $f \in USC(\mathbb{R}^N \times \mathbb{R}_+ \times I)$ be bounded from above and $g \in C(\mathbb{R}^N \times \mathbb{R}_+)$ be nonnegative. For $\varepsilon > 0$, set $\psi_\varepsilon(\xi, t, i) := f(\xi, t, i) - \varepsilon g(\xi, t)$ and assume that $\psi_\varepsilon$ attains its global maximum in some point $(\xi_0^\varepsilon, t^\varepsilon_0, i^\varepsilon_0)$. Then, as $\varepsilon \to 0$, max $\psi_\varepsilon \to$ sup $f$ and $\varepsilon g(\xi_0^\varepsilon, t^\varepsilon_0) \to 0$.

**Proof** Set $m_\varepsilon := \max \psi_\varepsilon$ and $m := \sup f$. For $\eta > 0$, let $(\xi', t', i')$ be such that: $f(\xi', t', i') \geq m - \eta$. For $\varepsilon'$ sufficiently small, we have: $\varepsilon' g(\xi', t') \leq \eta$. In particular, since $g$ is nonnegative, there holds
\[
m \geq m_\varepsilon - \varepsilon' g(\xi', t') \geq f(\xi', t', i') - \varepsilon' g(\xi', t') \geq m - 2\eta.
\]

Letting $\varepsilon' \to 0$, we get the first part of the statement.

For $\varepsilon$ sufficiently small, the above relations entail
\[
m_\varepsilon = f(\xi_0^\varepsilon, t^\varepsilon_0, i^\varepsilon_0) - \varepsilon g(\xi_0^\varepsilon, t^\varepsilon_0) \geq m - 2\eta;
\]
in particular, for $k_\varepsilon := \varepsilon g(\xi_0^\varepsilon, t^\varepsilon_0)$, we deduce that the sequence $\{k_\varepsilon\}$ is bounded. Let us pick a subsequence (still denoted $k_\varepsilon$) convergent to some value $k \geq 0$. Since $m_\varepsilon = f(\xi_0^\varepsilon, t^\varepsilon_0, i^\varepsilon_0) - k_\varepsilon \leq m - k_\varepsilon$, by the first part of the statement, as $\varepsilon \to 0$, we obtain $k \leq 0$. Hence $k = 0$ and the statement is completely proved.

**Proof of Lemma 2.1 (i).** Relations (2.2) and (2.3) entail
\[
0 \leq \psi(\tau_0, x_0, y_0, i_0) \leq 2R - \left(\frac{\alpha}{2} e^{\gamma \tau_0} |x_0| - |y_0|^2 + \frac{\varepsilon}{2} (|x_0|^2 + |y_0|^2)\right);
\]
therefore, inequalities (2.4) easily follows. The estimates (2.5) are an immediate consequence of (2.4) and Lemma A.1.

**Proof of Lemma 2.1 (ii).** The inequality $2\psi(\tau_0, x_0, y_0, i_0) \geq \psi(\tau_0, x_0, x_0, i_0) + \psi(\tau_0, y_0, y_0, i_0)$ yields
\[
\alpha e^{\gamma \tau_0} |x_0 - y_0|^2 \leq \left[ u^1_{i_0}(\tau_0, x_0) - u^1_{i_0}(\tau_0, y_0) \right] + \left[ u^2_{i_0}(\tau_0, x_0) - u^2_{i_0}(\tau_0, y_0) \right];
\]
therefore, inequality (2.6) is a consequence of the regularity assumption.

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(iii). Assume $u^i \in C^1$ (being similar, the other case will be omitted). Let $\{e_k\}$ be an orthogonal basis of $\mathbb{R}^n$. For $h \in \mathbb{R}$ sufficiently small, the inequality $\psi(\tau_0, x_0, y_0, i_0) \geq \psi(\tau_0, x_0 + he_1, y_0, i_0)$ yields

$$\alpha e^{\tau_0} \left( |x_0 - y_0 + he_1|^2 - |x_0 - y_0|^2 \right) + \frac{\varepsilon}{2} \left( |x_0 + he_1|^2 - |x_0|^2 \right) \leq u^i_{\psi}(\tau_0, x_0) - u^i_{\phi}(\tau_0, x_0 + he_1).$$

Dividing by $h$ and letting $h \to 0^\pm$, we obtain

$$|\alpha e^{\tau_0}(x_{0,k} - y_0,k) + \varepsilon x_{0,k}| \leq [u^i]_1.$$ 

Summing on $k$ and taking advantage of estimate (2.4), we conclude the proof. \qed

For the sake of completeness, let us now state two results established by Horie and Ishii in [24, Lemma 2.7 and 3.6]. For their proof, we refer the reader to the original paper.

**Lemma A.2** Assume conditions (H0)-(H4) and fix $x, p \in \mathbb{R}^n, r \in \mathbb{R}^m, X \in S^n, i \in I$. Let $v = v(y)$ be a bounded continuous solution to (4.3). Then

(a) $v$ is Lipschitz continuous in $\mathbb{R}^n$.

(b) Let $R > 0$ be a constant such that $\|Dv\| \leq R$. Then, for each $\varepsilon > 0$, there are functions $v^\pm \in C(\mathbb{R}^n) \cap W^{2,\infty}(\mathbb{R}^n)$ and a constant $C$ (depending on $R$ and on the constants entering in the assumptions) such that

$$\|v - v^\pm\| \leq \varepsilon, \quad \|v^\pm\| \leq \|v\|$$

$$\|Dv^\pm\| \leq \|Dv\|, \quad \|v^\pm\|_{W^{1,\infty}(\mathbb{R}^n)} \leq C,$$

and

$$H_i(x, y, r, p, X + D^2v^+(y)) \geq H_i(x, r, p, X) - \varepsilon \quad \text{in } \mathbb{R}^n$$

$$H_i(x, y, r, p, X + D^2v^-(y)) \leq H_i(x, r, p, X) + \varepsilon \quad \text{in } \mathbb{R}^n.$$ 

**Lemma A.3** Let $\Omega \subset \mathbb{R}^n$ be open, $u \in USC(\Omega)$ and $v \in C(\Omega) \cap W^{2,\infty}(\Omega)$. Let $\hat{x} \in \Omega$ and $(p, X) \in J^{2,+}(u - v)(\hat{x})$. Then there exists a $Y \in S^n$ such that

$$(Dv(\hat{x}), Y) \in J^2v(\hat{x}), \quad (p + Dv(\hat{x}), X + Y) \in J^{2,+}u(\hat{x})$$

where $J^{2,+}u(\hat{x})$ is the set of superjets of $u$ at the point $\hat{x}$ (see [15, Section 2]) while $J^2v(x)$ denotes the set of those points $(q, Y) \in \mathbb{R}^n \times S^n$ for which there is a sequence $x_i \to x$ such that $v$ is twice differentiable at $x_i$ and $(Dv(x_i), D^2v(x_i)) \to (q, Y)$ (see [15, Section 3]).

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References

[1] O. Alvarez and M. Bardi. Singular perturbations of nonlinear degenerate parabolic PDEs: a general convergence result. Arch. Ration. Mech. Anal. 170 (2003), 17 – 61.

[2] O. Alvarez and M. Bardi. Ergodicity, stabilization, and singular perturbations for Bellman-Isaacs equation. Mem. Amer. Math. Soc. 204 (2010), n. 960.

[3] M. Arisawa and P.L. Lions. On ergodic stochastic control. Comm. Partial Differential Equations 23 (1998), 2187 – 2217.

[4] A. Bensoussan. Homogenization of systems of partial differential equations. in Variational Analysis and applications. Nonconvex Optim. Appl. 79, Springer, 2005.
[5] M. Bardi and I. Capuzzo Dolcetta. Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations. Birkhäuser, Boston, 1997.

[6] A. Bensoussan, J.-L. Lions and G. Papanicolaou. Asymptotic analysis for periodic structures. North-Holland, Amsterdam, 1978.

[7] I.H. Biswas, E.R. Jakobsen, and K.H. Karlsen. Viscosity solutions for a system of integro-PDEs and connections to optimal switching and control of jump-diffusion processes. Appl. Math. Optim. 62 (2010), 47 – 80.

[8] M. Bladt, E. Méndez and P. Padilla. Pricing derivatives incorporating structural market changes and in time correlation. Stoch. Models 24 (2008), 164 – 183.

[9] A. Briani, F. Camilli and H. Zidani. Approximation schemes for monotone systems of nonlinear second order partial differential equations: convergence result and error estimate. Differ. Equ. Appl. (on print).

[10] R. Buckdahn and Y. Hu. Probabilistic approach to homogenizations of systems of quasilinear parabolic PDEs with periodic structures. Nonlinear Anal. 32 (1998), 609 – 619.

[11] R. Buckdahn, Y. Hu and S. Peng. Probabilistic approach to homogenization of viscosity solutions of parabolic PDEs. NoDEA Nonlinear Differ. Equ. Appl. 6 (1999), 395– 411.

[12] J. Busca and B. Sirakov. Harnack type estimates for nonlinear elliptic systems and applications. Ann. Inst. H. Poincaré Anal. Non Linéaire 21 (2004), 543 – 590.

[13] F. Camilli, O. Ley and P. Loreti. Homogenization of monotone systems of Hamilton-Jacobi equations. ESAIM Control Optim. Calc. Var. 16 (2010), 58 – 76.

[14] F. Cagnetti, D. Gomes and H.V. Tran. Adjoint methods for obstacle problems and weakly coupled systems of PDE. arXiv:1103.3226.

[15] M.G. Crandall, H. Ishii and P.-L. Lions. User’s guide to viscosity solutions of second order partial differential equations. Bull. Amer. Math. Soc. (N.S.) 27 (1992), 1 – 67.

[16] B. Djehiche, S. Hamadène and A. Popier. A finite horizon optimal multiple switching problem. SIAM J. Control Optim. 48 (2009), 2751 – 2770.

[17] L. Evans. Periodic homogenisation of certain fully nonlinear partial differential equations Proc. Roy. Soc. Edinburgh Sect. A 120 (1992), 245 – 265.

[18] L. Evans and P.E. Souganidis. Differential games and representation formulas for solutions of Hamilton-Jacobi-Isaacs equations. Indiana Univ. Math. J. 33 (1984), 773 – 797.

[19] W.H. Fleming and H.M. Soner. Controlled Markov Processes and Viscosity Solutions. Springer-Verlag, Berlin, 1993.

[20] W.H. Fleming and P.E. Souganidis. On the existence of value functions of two-players, zero-sum stochastic differential games. Indiana Univ. Math. J. 38 (1989), 293 – 314.

[21] W. H. Fleming and Q. Zhang. Risk-sensitive production planning of a stochastic manufacturing system. SIAM J. Control Optim. 36 (1998), 1147 – 1170.

[22] J. Földes and P. Poláčik. On cooperative parabolic systems: Harnack inequalities and asymptotic symmetry. Discrete Contin. Dyn. Syst. 25 (2009), 133 – 157.

[23] M.K. Ghosh, A. Arapostathis and S.I. Marcus. Optimal control of switching diffusions with application to flexible manufacturing systems. SIAM J. Control Optim. 31 (1993), 1183 – 1204.
[24] K. Horie and H. Ishii. Simultaneous effects of homogenization and vanishing viscosity in fully nonlinear elliptic equations. Funkcial. Ekvac. 46 (2003), no. 1, 63 – 88.

[25] H. Ishii. On uniqueness and existence of viscosity solutions of fully nonlinear second-order elliptic PDE’s. Comm. Pure Appl. Math. 42 (1989), 15 – 45.

[26] H. Ishii and S. Koike. Viscosity solutions for monotone systems of second-order elliptic PDEs. Comm. Partial Differential Equations 16 (1991), 1095 – 1128.

[27] E.R. Jakobsen and K.H. Karlsen. Continuous dependence estimates for viscosity solutions of fully nonlinear degenerate parabolic equations. J. Differential Equations 183 (2002), 497 – 525.

[28] E.R. Jakobsen and K.H. Karlsen. Continuous dependence estimates for viscosity solutions of fully nonlinear degenerate elliptic equations. Electron. J. Differential Equations (2002), n. 39, 1 – 10.

[29] M.A. Katsoulakis. A representation formula and regularizing properties for viscosity solutions of second-order fully nonlinear degenerate parabolic equations. Nonlinear Anal. 24 (1995), 147 – 158.

[30] P.-L. Lions, G. Papanicolaou, S.R.S. Varadhan. Homogenization of Hamilton-Jacobi Equations. Unpublished, 1986.

[31] C. Marchi. Continuous dependence estimates for the ergodic problem of Bellman-Isaacs operators via the parabolic Cauchy problem, ESAIM Control Optim. Calc. Var. (on print).