BOUNDDED MARTIN’S MAXIMUM WITH MANY WITNESSES

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Abstract. We study a strengthening of Bounded Martin’s Maximum which asserts that if a \(\Sigma_1\) fact holds of \(\omega_2^V\) in a stationary set preserving extension then it holds in \(V\) for a stationary set of ordinals less than \(\omega_2\). We show that this principle implies Global Projective Determinacy, and therefore does not hold in the \(P_{\text{max}}\) model for \(\text{BMM}\), but that the restriction of this principle to forcings which render \(\omega_2^V\) countably cofinal does hold in the \(\text{BMM}\) model, though it is not a consequence of \(\text{BMM}\).

Introduction

Bounded Martin’s Maximum, denoted \(\text{BMM}\), is the assertion that

\[
(H(\omega_2), \in) \prec_{\Sigma_1} (H(\omega_2), \in)^{V^P}
\]

whenever \(P\) is stationary set preserving. Because \(\Sigma_1\) facts are upward absolute and \(\text{Col}(\omega_1, \omega_2)\) can be appended to a given stationary set preserving forcing, the formulation below is equivalent.

\(\text{BMM}\) denotes the following assertion. Suppose \(a \in H(\omega_2), \varphi(x,a)\) is a \(\Sigma_1\) formula, \(P\) is stationary set preserving forcing notion and whenever \(G \subset P\) is generic

\[
(H(\omega_2), \in)^{V[G]} \models \varphi(\omega_2^V, a).
\]

Then

\[
(H(\omega_2), \in)^V \models \varphi(\delta, a)
\]

for some ordinal \(\delta < \omega_2\).

In this paper we will study a strengthening of \(\text{BMM}\) which asserts that if a \(\Sigma_1\) fact holds of \(\omega_2^V\) in a stationary set preserving extension then it holds in \(V\) for a stationary set of ordinals less than \(\omega_2\). In other words, we will strengthen \(\text{BMM}\) as above by replacing the phrase ”some ordinal \(\delta < \omega_2\)” with ”a stationary set of ordinals \(\delta < \omega_2\)”.

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BMM$^s$ denotes the following assertion. Suppose $a \in H(\omega_2)$, $\varphi(x,a)$ is a $\Sigma_1$ formula, $\mathbb{P}$ is stationary set preserving forcing notion and whenever $G \subset \mathbb{P}$ is generic

$$(H(\omega_2), \in)^{V[G]} \models \varphi(\omega_2^V, a).$$

Then

$$(H(\omega_2), \in)^V \models \varphi(\delta, a)$$

for a stationary set of ordinals $\delta < \omega_2$.

A definition of BMM$^{s,++}$ is obtained by replacing the occurrences of the structure $(H(\omega_2), \in)$ with the structure $(H(\omega_2), \in, \text{NS})$ in the definition above, and regarding $\varphi$ as $\Sigma_1$ in the expanded language.

Our main theorem is that BMM$^{s,+++}$ implies that Projective Determinacy holds in all set-generic extensions. To put this in perspective, it is open whether BMM implies $\Delta_3^1$ determinacy, and known that BMM does not imply $\Delta_3^1$ determinacy in the universe after $\omega_2$ is collapsed as Theorem 10.99 of [18] shows that

$$L^{M^\#_1}(\mathbb{R})|G] \models \text{BMM}^{++}$$

where $N = L^{M^\#_1}(\mathbb{R})$ is the minimal inner model closed under the $M^\#_1$ operation and containing $\mathbb{R}$, $N$ satisfies AD, and $G \subset \mathbb{P}_{max}$ is $N$-generic. Thus, it seems that BMM$^{s,+++}$ is a bit stronger than BMM and in particular does not hold in the model described above. We are not able to prove PD from BMM$^s$ alone, but we can if we assume in addition that the nonstationary ideal is saturated, and the proof produces for example a set of ordinals $E$ for which $M^\#_2(E)$ exists and

$$P(\omega_1) \subset M^\#_2(E),$$

which is enough to conclude that BMM$^s$ fails in the BMM model. We are able to show, however, that a special case of this forcing axiom does hold in the BMM model, namely BMM$^{s_0}$, which will denote the restriction of BMM$^s$ to stationary set preserving forcings $\mathbb{P}$ for which

$$V[G] \models cf(\omega_2^V) = \omega$$

whenever $G \subset \mathbb{P}$ is $V$-generic.

Theorem 1.

1. BMM$^{s,+++}$ implies PD in all generic extensions.
2. BMM$^s$ fails in the BMM model.
3. BMM$^{s_0,++}$ holds in the BMM model.
The proof of the theorem makes essential use of the well ordering of \( P(\omega_1) \) given under BPFA by Caicedo and Velickovic in [4]. Once one understands why their well-ordering is \( \Delta_1 \) over \( H(\omega_2) \) it is easy to construct from it a bijection

\[
W : \omega_2 \to P(\omega_1)
\]

whose initial segments are uniformly \( \Sigma_1 \) definable over \( H(\omega_2) \). That is, \( W \) is a bijection and there is a \( \Sigma_1 \) formula \( \psi \) and a parameter \( a \in H(\omega_2) \) such that for every \( x \in H(\omega_2) \) and \( \beta < \omega_2 \),

\[
(H(\omega_2), \epsilon) \models \psi(x, \beta, a) \text{ iff } x = W \upharpoonright \beta.
\]

Well-orderings due to Moore and Todorcevic do not seem to suit our purposes, though Woodin’s well-ordering from \( \psi_{AC} \) does give such a set \( W \) which is uniformly \( \Sigma_1 \) definable over the structure \( (H(\omega_2), \epsilon, NS) \). The reader curious about how the well-ordering is used to increase the expressive power of the \( \Sigma_1 \) formula in the definition of \( \text{BMM}^* \) could skip directly to Lemma 9 below.

We now give some further background information. \( \text{BMM} \) implies a bounded version of the strong reflection principle which we denote by \( \text{BSRP}(\omega_2) \), and which asserts that any projective stationary subset \( S \) of \( [\omega_2]^{\omega} \) which is \( \Sigma_1 \)-definable over the structure \( (H(\omega_2), \epsilon) \) reflects to a club in \( [\gamma]^{\omega} \) for some, equivalently unboundedly many, ordinals \( \gamma < \omega_2 \). The nucleus of this paper, now essentially the base case of the PD induction, was the observation that some open questions regarding \( \text{BMM} \) could be solved assuming in addition the following enhanced version of \( \text{BSRP}(\omega_2) \) giving a stationary set of club reflection points. The axiom \( \text{BMM}^* \) is the natural generalization of \( \text{BMM} \) which implies this stronger reflection principle.

\( \text{BSRP}^*(\omega_2) \) denotes the assertion that any projective stationary set \( S \subset [\omega_2]^{\omega} \) which is \( \Sigma_1 \)-definable in \( H(\omega_2) \) reflects to a club stationarily often in the sense that \( S \cap [\gamma]^{\omega} \) contains a club in \( [\gamma]^{\omega} \) for a stationary set of \( \gamma < \omega_2 \).

The particular questions which interest us ask whether certain consequences of Martin’s Maximum in fact follow from \( \text{BMM} \), for example those on the following list.

1. The nonstationary ideal is precipitous
2. Woodin’s principle \( \psi_{AC} \)
3. Canonical function bounding
4. \( \delta_2^1 = \omega_2 \)
5. \( \Delta_2^1 \) determinacy
Of course, the determinacy question is really just a question of consistency strength. The best result to date is due to Schindler who shows in [13] that BMM implies the existence of an inner model with a strong cardinal. Regarding $\delta_2^1 = \omega_2$, Woodin gets this from NS saturated with a measurable cardinal in [18], and hence from BMM together with a Woodin cardinal and a measurable above using a theorem of Shelah. Schindler and Claverie have recently proved $\delta_2^1 = \omega_2$ from BMM together with NS precipitous. Canonical function bounding follows outright from $\psi_{AC}$ by an argument of Aspero in [2], and Woodin obtains $\psi_{AC}$ as a consequence of BMM with either a measurable cardinal or NS precipitous assumed in addition (see [18]). Our initial observation went as follows.

Theorem 2. Assume BMM and BSRPS($\omega_2$). Then $\Delta^1_2$ determinacy, $\delta_2^1 = \omega_2$, and canonical function bounding holds.

The theorem is proved by showing that

\[ W^\ast \models \text{NS saturated} \]

where NS denotes the nonstationary ideal on $\omega_1$. This suggests a new entry for the above list of possible consequences of BMM.

Does BMM++ imply NS saturated in $L(P(\omega_1))$?

As one would suspect, BSRPS($\omega_2$) is not a consequence of BMM, and we establish this by way of the Tilde operation. Recall that $T$, for a subset $T$ of $\omega_1$, is defined to be the set of $\alpha < \omega_2$ for which there is a club of $\sigma \in [\alpha]^{\omega}$ whose order type belongs to $T$. We show that in a forcing extension of a model satisfying Martin’s Maximum, BMM holds and the nonstationary ideal is saturated, yet there exists a stationary set $T \subset \omega_1$ for which $T$ is nonstationary in $\omega_2$. An argument of Larson from [10] shows that such a set $T$ must be stationary under BSRPS($\omega_2$) together with NS saturated, and so BSRPS($\omega_2$) must fail in this model.

Theorem 3. BSRPS($\omega_2$) is not a consequence of BMM++ together with the saturation of the nonstationary ideal.

In a similar vein, arguments of Larson from [10] coupled with a Theorem of Woodin from [18] will produce models in which BMM^s fails but BMM as well as other hypotheses hold. Finally, we give an application of BMM^s which does not seem to have anything to do with stationary reflection but involves rather the notion of a disjoint club sequence on $\omega_2$, an invention of Krueger from [8] who derives one from MM(c). We observe here that BMM^s implies the existence of such a sequence, and this will allow us to separate BMM and some consequences of BMM^s used in the proof of Theorem 1, from BMM^s itself.
This paper is organized as follows. We start with a brief discussion of the $\Sigma_1$ well-ordering and then prove the two results (Theorems 2 and 3) concerning $\text{BSRP}^s(\omega_2)$. We then give the PD proofs, followed by the $\text{P}_{\text{max}}$ argument, and close with the other separation result. We would like to thank Paul Larson for directing us to the relevant results in [10], and for many enlightening conversations.

1. Results

We need that the wellordering of [4] gives rise to a uniformly $\Sigma_1$ enumeration of $P(\omega_1)$ as described in the introduction. For the reader’s convenience we describe how this is obtained.

Lemma 4. (Caicedo, Velickovic [4]) Assume BPFA. Then there is a bijection

$$W : \omega_2 \to P(\omega_1)$$

whose initial segments are uniformly $\Sigma_1$ definable over $H(\omega_2)$.

Proof. The parameter involved in the definition is a certain subset $c \subset \omega_1$. This parameter gives rise to an $\omega_1$-sequence $d$ of pairwise almost disjoint elements of $[\omega]^\omega$ which will be used as well. The authors of [4] define a notion of a triple $\alpha < \beta < \gamma < \omega_2$ coding a real $r$ which is a $\Sigma_1$ notion in the parameter $c$. Fixing a reasonable way of coding a triple by a single ordinal and composing, we thus have a $\Sigma_1$ formula which says that an ordinal $\delta < \omega_2$ codes a real $r$, and they prove that every real is so coded. Every subset $a$ of $\omega_1$ is coded by a real $r \subset \omega_1$ via $d$ in the standard way since $\text{MA}_{\omega_1}$ holds. Let $T$ be the theory described in [4] which includes the sentence asserting that every real is coded by an ordinal, as well as $\text{MA}_{\omega_1}$, among other axioms. Then $H(\omega_2) \models T$ and for transitive models $M, N$ of $T$

1. If $M \cap \omega_2 = N \cap \omega_2$ then $M = N$
2. If $M \cap \omega_2 < N \cap \omega_2$ then $M \in N$.

For a real $r$ let $N_r$ be the least model of $T$ with $r \in N_r$. Then a well-ordering of the reals is given by $r < s$ if $N_r \in N_s$ or $N_r = N_s = N$ and the least ordinal which codes $r$ in the sense of $N$ is less than the least ordinal which codes $s$ in the sense of $N$. Let $\{r_\delta \mid \delta < \omega_2\}$ be the enumeration of the reals according to this ordering, and set $W_0(\delta) = a$ if $r_\delta$ codes $a$ via $d$. Then $W_0$ is a uniformly $\Sigma_1$ definable surjection therefore gives rise to such a uniformly definable bijection $W$ in the obvious way. \qed

We now show how $\text{BSRP}^s(\omega_2)$ can be used to prove that $\text{NS}$ is saturated in an inner model with a measurable cardinal which contains $P(\omega_1)$. We
use Schindler’s theorem from [13] to produce the measurable, although this can be avoided if BPFA++ is assumed in place of BMM.

**Theorem 5.** Assume either BMM or BPFA++. Assume that BSRP^s(ω_2) holds in addition. Then

1. \( \Delta^2_2 \) determinacy holds
2. Every function from \( \omega_1 \) to \( \omega_1 \) is bounded by a canonical function
3. \( \delta^1_2 = \omega_2 \).

**Proof.** Since BPFA holds we can let \( W \) denote the uniformly \( \Sigma^1_1 \) enumeration of \( P(\omega_1) \) given by Lemma 4. For convenience we will think of \( W \) as a subset of \( \omega_2 \times \omega_1 \) with the property that \( P(\omega_1) = \{ W_\alpha \mid \alpha < \omega_2 \} \) where \( W_\alpha \) denotes the set \( \{ \gamma \mid (\alpha, \gamma) \in W \} \). Thus there is a \( \Sigma^1_1 \) formula \( \phi(x, y, z) \) and a parameter \( a \in H(\omega_2) \) such that for every \( x \in H(\omega_2) \) and \( \beta < \omega_2 \)

\[
(H(\omega_2), \in) \models \psi(x, \beta, a) \text{ iff } x = W \cap (\beta \times \omega_1).
\]

First assume BMM and BSRP^s(ω_2). Schindler has shown (see [12] and [13]) that \( X^\dagger \) exists for every set \( X \) under BMM so in particular \( W^\dagger \) exists. For any set \( X \) we let \( \mathcal{M}(X) \) denote \( X^\dagger \). We will use BSRP^s(ω_2) to seal a putative least bad antichain in \( \mathcal{M}(W) \) thereby showing that \( \mathcal{M}(W) \models NS \) is saturated.

Let us assume toward a contradiction that in \( \mathcal{M}(W) \) there is a maximal antichain in \( P(\omega_1)/NS \) of size \( \omega_2^{\mathcal{M}(W)} = \omega_2 \). Let \( A \) denote the least antichain in the canonical well-ordering of \( \mathcal{M}(W) \). Using \( W \) we may code \( A \) as a subset \( A \) of \( \omega_2 \) given by

\[
A = \{ \alpha < \omega_2 \mid W_\alpha \in A \}.
\]

For \( \sigma \in [\omega_2]^\omega \) let \( \pi_\sigma : \sigma \to otp(\sigma) \) be the collapse of \( \sigma \), let \( W_\sigma \) denote the image \( \pi_\sigma[W \cap \sigma] \), noting that \( \pi_\sigma \) acts on pairs in the obvious way, and let \( A_\sigma = \pi_\sigma[A \cap \sigma] \). For a club of \( \sigma \) it will be true that the code of the least antichain of \( \mathcal{M}(W_\sigma) \) is \( A_\sigma \). In every case, let us use \( A_\sigma \) to denote the code of the least maximal antichain of length \( \omega_2 \) in the sense of \( \mathcal{M}(W_\sigma) \) if it exists. Define the set \( S \subset [\omega_2]^\omega \) to consist of all \( \sigma \) satisfying

1. \( \mathcal{M}(W_\sigma) \) thinks that \( W_\sigma \subset \omega_2 \times \omega_1 \) enumerates \( P(\omega_1) \) in length \( \omega_2 \)
2. \( \mathcal{M}(W_\sigma) \) thinks that the least \( NS \) antichain exists and is coded as above by some \( A_\sigma \subset otp(\sigma) \)
3. There exists \( \alpha \in \sigma \) such that \( \pi_\sigma(\alpha) \in A_\sigma \) and \( \sigma \cap \omega_1 \in W_\alpha \).
This set is $\Sigma_1^1$ definable over $(H(\omega_2), \in)$. To verify $\sigma \in S$ it suffices to find an ordinal $\delta > \omega_1$ with $\text{sup}(\sigma) \subset \delta$ and a transitive set $N$ with $\delta \subset N$ which satisfies enough set theory, computes $W \cap \delta \times \omega$ correctly, contains $\sigma$, contains a $W_\sigma$ premouse $M$, thinks that $M = M(W_\sigma)$ and that the conditions above are satisfied. Note that any such structure is correct about $M = M(W_\sigma)$ since it is a $\Pi_1^1$ condition. We claim that $S$ is projective stationary. Since $P(\omega_1) \subset M(W)$ the antichain coded by $A$ is truly a maximal antichain in $P(\omega_1)/\text{NS}$. It is well known that the set of $\sigma \in [\omega_2]^\omega$ such that $\sigma \cap \omega_1 \in \bigcup_{\alpha \in \sigma \cap A} W_\alpha$ is projective stationary. Our set $S$ differs from this set on a nonstationary subset of $[\omega_2]^\omega$. Let $\theta$ be large enough and let $(H_\xi | \xi < \omega_2)$ be an increasing sequence of elementary submodels of $H(\theta)$, each of size $\omega_1$, so that $M(W) \in H_\xi$ and $H_\xi \cap \omega_2 \in \omega_2$ for each $\xi$. We may assume that the sequence $H_\xi \cap \omega_2$ is strictly increasing, continuous, and converges to $\omega_2$. Let $C \subset \omega_2$ be a club so that $H_\xi \cap \omega_2 = \xi$ for $\xi \in C$. Thus there exists a $\delta < \omega_2$ such that

(a) $\delta \in C$ and
(b) $S$ contains a club in $[\delta]^\omega$.

Let $\pi : H_\delta \to H$ be the transitive collapse. Thus

$$\pi(M(W)) = M(W \cap (\delta \times \omega_1))$$

so that the least antichain of $M(W \cap (\delta \times \omega_1))$ is coded by $\pi(A) = A \cap \delta$. Now we may let $(N_\gamma | \gamma < \omega_1)$ be an increasing sequence of countable, elementary submodels of $H(\theta)$ which contain $\delta$ so that $(\sigma_\gamma | \gamma < \omega_1)$ is continuous and exhaustive in $[\delta]^\omega$, where $\sigma_\gamma = N_\gamma \cap \delta$. It follows by (b) that there is a club $D \subset \omega_1$ so that $\sigma_\gamma \in S$ for $\gamma \in D$. This implies that the diagonal union of the sets coded in $A \cap \delta$ contains $D$, which is a contradiction. Thus in $M(W)$ there is a measurable cardinal and $\text{NS}$ is saturated. It follows $\Delta^1_2$-determinacy and $\delta^1_2 = \omega_2$ hold in $M(W)$ by 7.1 of [14] and 3.17 of [18] respectively. They therefore hold in $V$ as $P(\omega_1) \subset M(W)$. Canonical function bounding is a consequence of $\text{NS}$ saturated so it holds in $M(W)$ and hence in $V$. Now we assume $\text{BPFA}^{++}$ in place of $\text{BMM}$. Thus, we have to do without Schindler’s theorem. The argument above shows that

$L[W] \models \text{NS}$ is saturated

using only $\text{BPFA}$ and $\text{BSRP}^*(\omega_2)$. This is because if $L[W]$ thinks that there is a maximal antichain $A$ of size $\omega_2$ then we can define an operation $M(x)$ which associates to a set of ordinals $x$ the least level
of $L[x]$ which satisfies a sufficient fragment of set theory, thinks that $x \subset \omega_2 \times \omega_1$ codes $P(\omega_1)$ and that such an antichain exists. Virtually the same argument yields a contradiction. Working inside $L[W]$ we can argue that $x^\dagger$ exists for every bounded subset $x$ of $\omega_1$ and use the generic ultrapower to conclude that $P(\omega_1)$ is closed under daggers. We may now simply quote 10.108 of [18] to conclude that $W^\dagger$ exists, and in fact that every set has a dagger, but we will elaborate on this point as a generalization of this argument will play a role in the PD proof. Assume toward a contradiction that there is a set of ordinals $A$ so that $A^\dagger$ does not exist. Let $\theta$ be a cardinal containing $A$ and let $g \subset \text{Col}(\omega_1, \theta)$ be $V$-generic. In $V[g]$ let $a \subset \omega_1$ be such that $A \in L[a]$.

By standard arguments we must have that $V[g] \models a^\dagger$ does not exist, but that $V[g] \models (a \cap \alpha)^\dagger$ does exist for every $\alpha < \omega_1$ as no new reals were added. Thus in $V[g]$ there are stationary sets $S, T$ and $p$ such that $\alpha \in S$ implies $p \in (a \cap \alpha)^\dagger$ and $\alpha \in T$ implies $p \notin (a \cap \alpha)^\dagger$. This can be reformulated as a $\Sigma_1$ fact in the language of set theory together with a predicate for NS. Since $\text{Col}(\omega_1, \theta)$ is proper there are such sets $\bar{a}, \bar{S}, \bar{T}, \bar{p}$ in $V$ which implies that $\bar{a}^\dagger$ cannot exist in $V$, a contradiction. Another application of the sealing argument gives the saturation of NS in $W^\dagger$ from which the result follows. □

From BSRP$^s(\omega_2)$ together with a uniformly $\Sigma_1$ enumeration of $P(\omega_1)$ the argument above shows that $L(P(\omega_1))$ is a model of ZFC together with NS saturated, and hence bounding holds and there is an inner model with a Woodin cardinal by a recent result from [19]. We conjecture that BSRP$^s(\omega_2)$ alone implies an inner model with a Woodin cardinal. Before moving on to global principles we demonstrate that BSRP$^s(\omega_2)$ is not a consequence of BMM. Recall that $\tilde{T}$ is the set of $\alpha < \omega_2$ for which there is a club of $\sigma \in [\alpha]^{\omega_2}$ with $\text{otp}(\alpha) \in T$. Paul Larson pointed out to us that in the presence of NS saturated, the principle BSRP$^s(\omega_2)$ would imply that $\tilde{S}$ is stationary for every stationary set $S \subset \omega_1$ (see 3.9 of [10]), and that a model of his from [10] could be modified to produce a model of BMM$^{++}$ together with a stationary set $\tilde{S}$ so that

$$\{\alpha \in \tilde{S} \mid \text{cf}(\alpha) = \omega\}$$
is nonstationary. A subtle point, however, is that \( \text{MM}(c) \) holds in this model, and so \( \text{BSRP}^s(\omega_2) \) doesn’t always give a stationary set of cofinality \( \omega \) reflection points. It turns out that, assuming \( \text{MM} \), we can shoot a club through some \( \tilde{T} \) without adding subsets of \( \omega_1 \) while preserving the saturation of NS. This will be enough to separate \( \text{BMM} \) from \( \text{BSRP}^s(\omega_2) \).

**Lemma 6.** (Larson) Assume \( \text{BSRP}^s(\omega_2) \) holds and that the nonstationary ideal is saturated. Then \( \tilde{T} \subset \omega_2 \) is stationary for every stationary set \( T \subset \omega_1 \).

**Proof.** Fix stationary sets \( S, T \subset \omega_1 \). Fix a function \( f : [\omega_2]^\omega \to \omega_2 \) and let \( j : V \to M \subset V[G] \) be the NS generic ultrapower. Let \( H \subset j(P(\omega_1)/NS) \) be \( M \)-generic with ultrapower map \( k : M \to N \). We assume that \( S \in G \) and \( j(T) \in H \). Let \( \sigma \) denote \( (k \circ j)(\omega_2^S) \). Then \( \sigma \) is a countable subset of \( (k \circ j)(\omega_2) \) with the property that \( \sigma \cap \omega_1^N \in k \circ j(S) \) and \( \text{otp}(\sigma) \in k \circ j(S) \). Further, \( \sigma \) is closed under \( (k \circ j)(f) \). \( N \) must see such a countable set with these properties and so by elementarity and the fact that \( S \) was arbitrary we conclude that

\[
\{ \sigma \in [\omega_2]^\omega \mid \text{otp}(\sigma) \in T \}
\]

is projective stationary in \( V \). Since this set is \( \Sigma_1 \) definable from \( T \) as a parameter, we get the desired conclusion. \( \square \)

**Theorem 7.** Assume \( \text{MM} \). Then there is a forcing notion \( \mathbb{P} \) of size \( \omega_2 \) such that whenever \( G \subset \mathbb{P} \) is \( V \)-generic then \( V[G] \) satisfies

1. \( \text{BMM}^{++} \),
2. the nonstationary ideal is saturated, and
3. there is a stationary set \( T \subset \omega_1 \) with \( \tilde{T} \) nonstationary.

Thus \( \text{BSRP}^s(\omega_2) \) must fail in \( V[G] \).

**Proof.** Fix a stationary and costationary set \( T \subset \omega_1 \) and assume \( \text{MM} \). Let \( \mathbb{P} \) be the poset for shooting a club through \( \tilde{T} \). \( \mathbb{P} \) consists of closed subsets of \( \tilde{T} \) of size \( \omega_1 \) ordered by end extension. We first show that \( \mathbb{P} \) does not add new subsets of \( \omega_1 \). Let \( S \) denote

\[
\{ \sigma \in [\omega_2]^\omega \mid \sup(\sigma) \in \tilde{T} \text{ and } \text{otp}(\sigma) \in T \}
\]

**Claim 8.** \( S \) is projective stationary.

**Proof.** Fix a stationary set \( S \subset \omega_1 \). Let \( G \subset P(\omega_1)/NS \) be \( V \)-generic with \( S \in G \). Let \( \pi : V \to M \subset V[G] \) be the generic embedding with \( G \subset P(\omega_1)/NS \). Let \( f : [\omega_2]^{<\omega} \to \omega \) be arbitrary. Let \( C \) denote the set of \( \delta < \omega_2 \) with \( f([\delta]^{<\omega}) \subset \delta \). By Lemma 5.8 of [18],

\[
\{ \alpha \in \tilde{T} \mid \text{cf}(\alpha) = \omega \}
\]
is a stationary subset of $\omega_2$, and so there is $\delta \in \bar{T} \cap C$ such that $\delta$ has countable cofinality. Let $\sigma = \pi[\delta]$. Then $\sigma$ has the following properties in $V[G]$.

(1) $\delta = \text{otp}(\sigma) \in \pi(T)$
(2) $\sup(\sigma) = \pi(\delta)$ belongs to the tilde of $\pi(T)$
(3) $\sigma \cap \omega_1 \in \pi(S)$.
(4) $\sigma$ is closed under $\pi(f)$

Since $\sigma$ is countable in $V[G]$ we have $\sigma \in M$ and so by elementarity we find in $V$ such a set $\sigma$ in $S$ which is closed under $f$ and with $\sigma \cap \omega_1 \in S$ as desired. □

Let $\theta$ be large and let $S^*$ denote the set of countable elementary submodels $X \prec H(\theta)$ with $P \in X$ and $X \cap \omega_2 \in S$. Let $\tau$ be a term for a subset of $\omega_1$. Using MM let

$$\{X_\alpha \mid \alpha < \omega_1\}$$

be an increasing, continuous $\in$ chain with each

$$X_\alpha \cap \omega_2 \in S^*,$$

and $\tau \in X_0$. Inductively define a decreasing sequence of conditions $p_\alpha$ so that $p_\alpha \in X_{\alpha+1}$,

$$p_\alpha = q \cup \{\sup(X_\alpha \cap \omega_2)\}$$

where $q$ is an $X$-generic filter. We can assume that

$$\bigcup_{\alpha < \omega_1} X_\alpha \cap \omega_2 = \delta$$

for some $\delta < \omega_2$ and so $\delta \in \bar{T}$. It follows that

$$p = (\bigcup_{\alpha < \omega_1} p_\alpha) \cup \{\delta\}$$

decides $\tau$. Thus $P$ does not add new subsets of $\omega_1$. Now let $\tau$ be a term and $p_0$ a condition such that $p$ forces that $\tau$ is a function from $\omega_2$ to $P(\omega_1)/NS$ whose range is a maximal antichain. We suppress $p_0$. Let $S^{**}$ consist of all $X \in [H(\omega_2)]^{\omega}$ satisfying

(1) $X \prec H(\omega_2)$
(2) $X \cap \omega_2 \in S^*$
(3) For a dense set of $q \in P \cap X$ there is an ordinal $\gamma \in X$ so that

$$q \Vdash^Y_{P} X \cap \omega_1 \in \tau(\gamma).$$
We claim that $S^{**}$ is projective stationary. Otherwise by pressing down we find a condition $q_0$ and a stationary set $T$ of $X \prec H(\omega_2)$ with $X \cap \omega_2 \in S^*$ such that for all $q \in P \cap X$ with $q \leq q_0$ and all $\gamma \in X$,

$$\neg (q \Vdash^P X \cap \omega_1 \in \tau(\gamma)).$$

Since NS is saturated the set $T$ must be $A$-projective stationary for some stationary set $A \subset \omega_1$. Let $q_1$ be a condition below $q$, $\gamma < \omega_2$ and $B \subset A$ so that $q_1$ forces $\tau(\gamma) \cap A = B$. There are stationary many $X \in T$ such that $X \cap \omega_1 \in B$, $\gamma \in X$, and $q_1 \in X$. Any such $X$ gives the desired contradiction. Now let $G \subset P$ be $V$-generic. We have shown that $\text{BMM}^{++}$ holds and NS is saturated in $V[G]$. Moreover, $\tilde{T}$ contains a club so that

$$\tilde{S} \subset \omega_2 \setminus \tilde{T}$$

must be nonstationary where $S = \omega_1 \setminus T$. Since $S$ is stationary, $\text{BSRP}^s(\omega_2)$ must fail by Lemma 6.

The $\text{BPFA}^{++}$ part of the $\Delta^1_2$ determinacy proof illustrates the two main elements, **sealing** and **lifting**, of the proof of $\text{PD}$ from $\text{BMM}^s^{++}$. This proof is modeled on Woodin’s proof of $\text{PD}$ from $\text{MM}(c)$ in which the saturation of NS is the key hypothesis used in contradicting the existence of the core model. While it is unlikely that $\text{BMM}^s^{++,+}$ implies the saturation of NS, it will imply saturation inside the various inner models of interest as we proceed through the $\text{PD}$ induction. The other element of the $\text{MM}(c)$ proof involves lifting closure under fine structural operations from $P(\omega_1)$ to $P(\omega_2)$ using simultaneous reflection as in 9.78 of [18]. With $\text{BMM}^s^{++,+}$ we can lift closure from $P(\omega_1)$ to $V$ by an argument which is more in the spirit of 10.108 of [18], and this is where the ”++” seems unavoidable. We can however get by with $\text{BMM}^s$ in this connection if NS is saturated in addition, though we can only lift closure from $P(\omega_1)$ to definable subsets of $\omega_2$. This is enough to get $\text{PD}$ and is how we will show that $\text{BMM}^s$ fails in the $\text{BMM}$ model.

We first derive the following definable version of $\text{MM}(c)$ to illustrate how the definable wellordering is used to increase the expressive power of the $\Sigma_1$ formula appearing in the definition of $\text{BMM}^s$. As a corollary we get a version of $\text{BSRP}^s(\omega_2)$ for all definable projective stationary sets which will be used in Theorem 11 below.

**Lemma 9.** Assume $\text{BMM}^s$. Suppose $P$ and $D$ are first order definable over $H(\omega_2, \in)$, $P$ is a poset, and $D$ is a partial map from $H(\omega_2)$ to $H(\omega_2)$ with the property that $D(a) \subset P$ is dense where defined. Then there is a stationary set of $\delta < H(\omega_2)$ such that there exists $X$ and $G$ satisfying

$$\neg (q \Vdash^P X \cap \omega_1 \in \tau(\gamma)).$$
(1) $X$ is a transitive and fully elementary submodel of $H(\omega_2)$
(2) $X \cap \omega_2 = \delta$
(3) $G$ is a filter on $\mathbb{P} \cap X$
(4) $D(a) \cap G \neq \emptyset$ for any $a \in X \cap \text{dom}(D)$.

Proof. Let $\mathbb{P}$ and $D$ be as above. Let $G \subset \mathbb{P}$ be $V$-generic. Let $\psi(x, y)$ be the formula (with parameter suppressed) which defines the initial segments of $W$ uniformly over $H(\omega_2)$. Note that for any $\beta < \omega^V_{\omega_2}$ the set $W \upharpoonright \beta$ is the unique witness to $\psi(x, \beta)$ in $V$ as well as $V[G]$. Thus in $V[G]$, the set $W$ is the unique witness to the formula $\chi(w, \omega^V_{\omega_2})$ which asserts that there exists an increasing sequence $(\beta_\xi \mid \xi < \omega_1)$, which is cofinal in $\omega^V_{\omega_2}$, and a sequence $(w_\xi \mid \xi < \omega_1)$ with each $w_\xi$ satisfying $\psi(w_\xi, \beta_\xi)$ and

$$w = \bigcup_{\xi < \omega_1} w_\xi.$$  

With a $\Sigma_1$ formula involving $\omega^V_{\omega_2}$ as a parameter we can therefore identify $H(\omega_2)^V$ as

$$H = L_{\omega^V_{\omega_2}}[W],$$

and using the definitions of $\mathbb{P}$ and $D$ assert the existence of a filter $G$ on $\mathbb{P}$ which meets $D(a)$ for every $a \in H$. Indeed, all of this can be verified by a transitive structure $N$ of a sufficient fragment of set theory containing $G$ and $H$ which satisfied a formula involving $\omega^V_{\omega_2}$. Thus in $V$, by intersecting with the appropriate club, we get a stationary set of $\delta$ such that

$$L_\delta[W \upharpoonright \delta] \prec L_{\omega_2}[W] = H(\omega_2)$$

and a filter $\bar{G}$ on $\mathbb{P} \cap X$ where $X = L_\delta[W \upharpoonright \delta]$ which has the desired properties. \qed

Theorem 10. $\text{BMM}^{s++}$ implies $\text{PD}$ in all generic extensions.

Proof. Recall that $M^*_n(a)$ is the minimal sound $a$-mouse with active top extender which is closed under the $M^*_n$ operation. We show that $M^*_n(a)$ exists for every transitive set $a$ by induction on $n < \omega$. The base case is already accomplished by Theorem 5 so we just assume the induction hypothesis holds for some $n < \omega$. We need to see that

$$M^*_n(W) \models NS \text{ saturated}.$$  

Suppose toward a contradiction that there is a maximal antichain in $P(\omega_1)/NS$ which belongs to $M^*_n(W)$ and has size $\omega_2$. We assume that $A$ is the least such in the definable wellordering of $M^*_n(W)$. Since $P(\omega_1) \subset N$, the notion of being a maximal antichain is absolute. Let
$\mathbb{P}$ be the standard poset for sealing $\mathcal{A}$. Thus if $G \subset \mathbb{P}$ is $V$-generic then in $V[G]$ there is an enumeration
\[ \mathcal{A} = \{ A_\alpha \mid \alpha < \omega_1 \} \]
whose diagonal union contains a club $C$. Inside $V[G]$ let $N$ denote
the transitive collapse of an elementary submodel $X$ of a large enough
$H(\theta)$ so that $X$ contains $M^*_n(W)$ as well as the enumeration of $\mathcal{A}$ and
the club $C$. Then $N$ reflects the relevant properties of these objects
mentioned above. Under our closure assumptions, there is a formula $\chi$ so that for any countable and transitive set $a$ and countable structure
$M$,
\[ M = M^*_n(a) \iff (H(\omega_1), \in) \models \chi(a, M). \]
Since $\mathbb{P}$ does not add countable sets we know that
\[ H(\omega_1)^V = H(\omega_1)^{V[G]} = H(\omega_1)^{L[W]}. \]
Hence, inside $N$ there is a continuous sequence of substructures
\[ \{ X_\alpha \mid \alpha < \omega_1 \} \]
of some $H(\kappa)$, each of which contain $M^*_n(W)$ so that letting $M_\xi$ and $w_\xi$ be the image of $M^*_n(W)$ and $W$ respectively under the map which collapses $X_\xi$, the fact that $M_\xi = M^*_n(w_\xi)$ is certified by the formula $\chi$ and the structure $H(\omega_1)^{L[W]}$. Back in $V$ we get a model $\bar{N}$ as above whose version of $W$ is $W_\delta$ where $\delta$ is such that $M^*_n(W_\delta)$ is fully elementary in $M^*_n(W)$. It follows that the version of $M^*_n(W_\delta)$ that $\bar{N}$ sees is the true version, since it collapsed correctly on a club. Moreover, $\bar{N}$ sees that the least antichain $A_\delta$ of $M^*_n(W_\delta)$ is sealed. Since this antichain is a subset of $\mathcal{A}$ this gives a contradiction. Now, under these conditions, the argument of Lemmas 16 and 17 of [15] immediately give closure of $P(\omega_1)$ under the $M^*_{\# n+1}$ operation, and we turn toward the lifting portion of the induction step. This is modeled on 10.108 of [18] which shows that $\text{BMM}^+^+$ lifts closure under sharps from $P(\omega_1)$ to all of $V$. We claim that $M^*_{\# n+1}(a)$ exists for every set $a$. Otherwise we may pass to $V[g]$ where $g \subset \text{Col}(\omega, \kappa)$ is $V$-generic for a sufficiently large $\kappa$ and find a subset $b$ of $\omega_1$, a term $t$, and stationary sets $S, T$ such that
\[ \beta \in S \Rightarrow t \in M^*_{\# n+1}(b \cap \beta) \]
and
\[ \beta \in T \Rightarrow t \notin M^*_{\# n+1}(b \cap \beta). \]
Using the same trick as above to certify each $M^*_{\# n+1}(b \cap \beta)$, we find in sets $\bar{b}, \bar{S}, \bar{T}$ back in $V$ with the property above. This contradicts
the existence of $M_{n+1}^*(\bar{b})$. Repeating the two arguments finishes the proof. □

**Theorem 11.** BMM$^\#(c)$ fails in the $\mathbb{P}_{max}$ model for BMM.

**Proof.** Assume otherwise. Thus we have BMM$^\#(c)$, BMM and NS saturated at our disposal. We will prove PD from these assumptions and the proof will yield the desired contradiction. We first claim that if $S \subset [\omega_2]^\omega$ is projective stationary and first order definable over the structure $(H(\omega_2), \in)$. Then $S \cap [\delta]^\omega$ contains a club in $[\delta]^\omega$ for a stationary set of $\delta < \omega_2$. This principle, denote DSRP$^\#(\omega_2)$, can be deduced from Lemma 9 as follows. Suppose $S$ is such a set and let $\mathbb{P}$ be the standard poset for shooting a club through $S$. Thus elements of $\mathbb{P}$ are countable continuous increasing sequences

$$p = (\sigma_\xi \mid \xi \leq \gamma)$$

from $S$. For $\alpha < \omega_2$ let $D(\alpha)$ denote the set of conditions $p$ as above for which $\alpha \in \sigma_\xi$ for some $\xi$ in the domain of $p$. Lemma 9 gives the desired stationary set of club reflection points. We now claim that if $S, T \subset [\omega_2]^\omega$ are stationary and first order definable over the structure $(H(\omega_2), \in)$. Then

$$S \cap [\delta]^\omega \text{ and } T \cap [\delta]^\omega$$

are both stationary in $[\delta]^\omega$ for a stationary set of $\delta < \omega_2$. Given such a pair $S, T$ it follows from NS saturated that there are stationary sets $A_S, A_T \subset \omega_1$ such that $S$ is $A_S$-projective stationary, $T$ is $A_T$-projective stationary, and

$$A_S \cap A_T = \emptyset.$$ 

The set

$$P(S, T) = \{\sigma \mid (\sigma \cap \omega_1 \in A_S \rightarrow \sigma \in S) \land (\sigma \cap \omega_1 \in A_T \rightarrow \sigma \in T)\}$$

is projective stationary and so reflects to a club in $[\delta]^\omega$ for a stationary set of $\delta < \omega_2$ by DSRP$^\#(\omega_2)$, and this proves the claim. Woodin’s proof of PD from MM$^\#(c)$ only uses NS saturated and the simultaneous reflection principle WRP$^{(2)}(\omega_2)$. The definable version of this principle that we now have at our disposal suffices with the caveat that one can only show that $M_n^*(W)$ exists by induction on $n < \omega$, as opposed to closure of $P(\omega_2)$ under the $M_n^*$ operation. This however, is enough to implement the argument, and we refer the reader to [15] for more details. Since $M_1^*(W)$ does not exist in the BMM model we get the desired contradiction. □
Theorem 12. Let $N$ be the minimal inner model containing $\mathbb{R}$ and closed under the $M_1^\#$ operation, and assume $N \models \text{AD}$. Then

$$N[G] \models \text{BMM}^{\text{so}++}$$

whenever $G \subset P_{\text{max}}$ is $N$-generic.

Proof. Suppose $G \subset P_{\text{max}}$ is $N$-generic and $Q$ is a poset in $N[G]$ such that

$$\models_{N[G]} \phi(\omega_2^V, a^*)$$

holds where $\phi(x, a)$ is $\Sigma_1$ in the appropriate language with parameter $a^* \subset \omega_1$. We may assume that $a^* = a_G$. We also assume that

$$\models_{N[G]} \text{cf}(\omega_2^V) = \omega.$$

Fix a condition $\mathcal{M}_0 = ((M_0, I_0), a_0) \in G$ which forces this as well as that $\dot{C}$ is a club subset of $\omega_2$. Note that

$$H(\omega_1)^N = H(\omega_2)^{N[G]}.$$

Working in $N[G]$ we are going to produce a condition $\mathcal{M}_1$ below $\mathcal{M}_0$ so that $\mathcal{M}_1 \in N$ and

$$\mathcal{M}_1 \models_{N[P_{\text{max}}]} \exists \gamma \in \dot{C} \land \phi(\gamma, a_G).$$

This will prove the theorem. First let us introduce some notation. We think of $x^#$ for a real $x$ as an $x$-mouse $(L_\alpha[x], \mu)$. Let $\kappa$ be the critical point of the measure $\mu$, and $j$ the map obtained by iterating the measure $\omega_1$ times. We say that a pair $c = (x^#, \beta)$ with $\kappa < \beta < \alpha$ is a code for an ordinal $\gamma$ if $j(\beta) = \gamma$. We let $\gamma_c$ denote the ordinal just described. In our present situation, every ordinal less than $\omega_2^V$ has a code because $u_2 = \delta_2 = \omega_2$

in $N[G]$. Now, let $H \subset Q$ be $N[G]$ generic. Using our closure hypothesis, we can create a condition $\mathcal{M} = ((M, I), a^*)$ in a sufficiently large collapse over $N[G][H]$ with an ordinal $\delta \in M$ satisfying the following conditions. Note that $\mathcal{M}_0$ is iterable in all generic extensions as $N[G]$ is sufficiently correct.

(1) $((M, I), a^*) < ((M_0, I_0), a_0)$

(2) $M \models \phi(\delta, a^*)$

(3) $M \models f : \omega \to \delta$ is a cofinal

(4) for every $n < \omega$ there is a condition $\mathcal{P}_n = ((P_n, J_n), b_n)$ which is greater than $((M, I), a^*)$ and a code $c(n)$ such that

(a) $\mathcal{P}_n$ and $c(n)$ belong to $M$ and $M < \mathcal{P}_n$

(b) $\mathcal{P}_n \models_{N[P_{\text{max}}]} \gamma_c(n) \in \dot{C}$

(c) $M \models f(n) < \gamma_c(n) < \delta$
By < we mean of course the $\mathbb{P}_{\text{max}}$ ordering. To construct the condition let $f : \omega \to \delta$ be any cofinal map where $\delta = \omega^V_2$ and let $\theta$ be sufficiently large. Note that

$$H(\theta) \models \phi(\delta, a^*).$$

Let $E$ be set of ordinals so that $H(\theta) \in L[E]$. Let

$$Y = M^H_1(E)$$

and let $g$ be generic over $N[G][H]$ so that $Y$ is countable in $N[G][H][g]$. Let $\hat{g} \in N[G][H][g]$ be $Y$ generic for making NS presaturated and then forcing $\text{MA}$, and let

$$\mathcal{M} = ((Y[\hat{g}], NS^Y[\hat{g}]), a^*).$$

We claim that $\mathcal{M}$ satisfies the conditions above. Since $P(\omega_1)^{N[G][H]} \subset Y$ and $H$ is generic over $N[G]$ for stationary set preserving forcing we know that

$$NS^Y \cap N[G] = NS^{N[G][H]} \cap N[G] = NS^{N[G]},$$

and since $\hat{g}$ preserves stationary sets we have

$$NS^Y[\hat{g}] \cap N[G] = NS^{N[G]}.$$

It follows that (1) holds as witnessed by the iteration of $\mathcal{M}_0$ determined by the generic $G$. The next two conditions hold as they are upward absolute. For $n < \omega$ there will be a condition $P_n \in G \cap H(\theta)^{N[G][H]}$ and a code $c(n)$ with the properties above because $\dot{C}_G$ is cofinal in $\delta = \omega^V_2$, and by the reasoning used to establish (1). Now let us go back to $N[G]$. Let $X$ be an elementary submodel of a large enough rank initial segment of $N[G]$ which contains everything relevant and let $\pi : X \to N$ denote the transitivization map. Let $\bar{H} \subset \pi(\mathbb{Q})$ be $\mathcal{N}$ generic for the collapse of $\mathbb{Q}$ and let $\bar{g}$ be $\mathcal{N}[\bar{H}]$ generic for the sufficiently large collapse in the sense of $\mathcal{N}$. Then $\mathcal{N}[\bar{H}][\bar{g}]$ thinks there is a condition

$$\mathcal{M}_1 = ((M_1, I_1), a_1)$$

which satisfies the conditions above. Since $\mathcal{N}$ is closed under sharps, this condition is truly iterable in $N[G]$. Of course $a_1 = a^* \cap \mathcal{N}$ but the rest of the properties are upward absolute. The second clause of condition (4) holds by elementarity of $\pi$ and the fact that the conditions $P_n$ are countable. Moreover, this condition is in the ground model as $\mathbb{P}_{\text{max}}$ does not add reals, and has the properties there as well. Let us check that

$$\mathcal{M}_1 \models N[\mathbb{P}_{\text{max}}] \exists \gamma \in \dot{C} \land \phi(\gamma, \dot{a}_G).$$

Let $G \subset \mathbb{P}_{\text{max}}$ be $N$ generic below $\mathcal{M}_1$ and let

$$j : M_1 \rightarrow M^*$$
be the iteration determined by \( G \), and let \( \delta \) and \( f \) be as in the conditions enumerated above. Thus \( M^* \models \phi(j(\delta), a_G) \) and hence
\[
(H(\omega_2), \in, NS) \models \phi(j(\delta), a_G).
\]
Let \( C = \dot{C}_G \). For each \( n < \omega \) we have \( M_1 < \mathcal{P}_n \) and so \( \mathcal{P}_n \in G \) as well. Thus \( \gamma_{c(n)} \in C \) for each \( n < \omega \). Now, we may assume that the sequence
\[
(\mathcal{P}_n \mid n < \omega)
\]
is an element of \( M_1 \) although this is not necessary. Thus
\[
M_1 \models \delta = \bigcup_{n<\omega} \gamma_{c(n)}
\]
where each \( \gamma_{c(n)} \) is computed in \( M_1 \), and so
\[
M^* \models j(\delta) = \bigcup_{n<\omega} \gamma_{c(n)}
\]
and we conclude that \( j(\delta) \in C \) as desired. \( \square \)

The proof given above, with the extra moves required for the \( s_0 \) clause suitable excised, constitutes a reorganization of the proof of the following equivalent formulation of the consistency result for BMM from \[18\].

Assume \((*)\) and that \( M_1^\#(X) \) exists for every set. Suppose \( N \) is an inner model of ZFC containing \( P(\omega_1) \) and closed under the \( M_1^\# \) operation. Then \( N \models \text{BMM}^{++} \).

(10.99 of \[18\])

However, even though \( \text{BMM}^{s_0} \) holds in the BMM model, it is not a consequence of \((*)\) together with global closure under the \( M_1^\# \) operation. This phenomenon is well preceded in \[18\], for example in the case of the saturation of the nonstationary ideal, which holds in the \( P_{\max} \) extension of \( L(\mathbb{R}) \) but is not a consequence of the \( P_{\max} \) axiom \((*)\). Recall that \( \tilde{T} \), for a set \( T \subset \omega_1 \), is the set of \( \alpha < \omega_2 \) for which there is a club of \( \sigma \in [\alpha]^{\omega} \) with the order type of \( \sigma \) in \( T \). Theorem 5.8 of \[18\], which was used in the proof of Theorem 7, shows that under MM the set
\[
\tilde{T}^0 = \{ \alpha \in \tilde{T} \mid cf(\alpha) = \omega \},
\]
is stationary for every stationary set \( T \subset \omega_1 \). It is straightforward to check that \( \text{BMM}^{s_0} \) together with the saturation of the nonstationary

\[1\]If \( \omega_2^{s_0} \) were not countably cofinal in \( N[G][H] \) we could choose \( f : \omega_1 \to \delta \) to be a bijection and use conditions \( \mathcal{P}_\xi \) for \( \xi < \omega_1 \) as above. The problem occurs at the end of the argument as \( \delta \) is not a continuity point of the embedding \( j \).
ideal suffice for this result. Arguments of Larson from [10] can be used to show that the poset $\mathbb{P}$ for shooting an $\omega$-club through $\check{T}^0$ over the BMM model does not add new subsets of $\omega_1$ and hence preserves (*) together with global closure under the $M_1^#$ operation. Assuming $\mathcal{T}$ is costationary this yields the desired separation. Moreover, his arguments show that $\text{MM}^{++}(c)$ could be preserved as well if the ground model were taken to be the richer $\mathbb{P}_{\text{max}}$ model for $\text{MM}(c)$ together with BMM. For a proof that $\text{BMM}^+$ is not implied by $\text{BMM}^{++}$ the interested reader could just check that $\mathbb{P}$ does not add $\omega_1$ sequences under the assumption that $\text{MM}$ holds in the ground model.

Finally, we prove another separation result which does not seem to involve the consequences of $\text{BMM}^+$ for projective stationary sets. It involves rather the concept of a disjoint club sequence on $\omega_2$, which is a sequence

$$(C_\alpha \mid \alpha \in A)$$

of pairwise disjoint sets, with each $C_\alpha$ a club subset of $[\alpha]^\omega$ and $A$ a stationary subset of $\omega_2$ consisting of ordinals of uncountable cofinality. This is an invention of Krieger from [8] who derives one from $\text{MM}(c)$.

**Theorem 13.** $\text{BMM}^+$ implies the existence of a disjoint club sequence on $\omega_2$.

**Proof.** Let us fix a canonical way of coding sets like $C_\alpha$ above as subsets of $\omega_1$. Define $A_W \subset \omega_2$ and $\vec{C} = \{C_\alpha \mid \alpha \in A_W\}$ by induction as follows. Given $\vec{C} \upharpoonright \alpha$ and $A_W \cap \alpha$, for an ordinal $\alpha$ of uncountable cofinality, if

$$C = \bigcup_{\beta < \alpha} C_\beta \subset [\alpha]^\omega$$

is nonstationary in $[\alpha]^\omega$ then let $C_\alpha$ be a club disjoint from $C$ with the earliest index according to $W$, and put $\alpha$ in $A_W$. We need to see that $A_W$ is stationary. Theorem 4.4 of [8] shows that there is a stationary set preserving notion of forcing $\mathbb{P}$ such that whenever $G \subset \mathbb{P}$ is $V$-generic $\omega_2^V$ has uncountable cofinality and there is in $V[G]$ a club $C$ in $[\omega_2^V]^\omega$ which is disjoint from

$$D = \bigcup_{\beta < \omega_2^V} C_\beta \subset [\alpha]^\omega.$$ 

The key point is that for any $\sigma \in C$ it can be verifies that $\sigma \notin D$ by consulting $H = L_{\omega_2^{\alpha+1}}[W]$ of which $\vec{C}$ is an element. The existence of club $C$ and the structure $H$ witnessing that $C \cap D = \emptyset$ is a $\Sigma_1$ property of $\omega_2^V$ so we get a stationary set of witnesses $\delta < \omega_2$ in $V$, each of which such that

$$L_{\delta+1}[W] \prec L_{\omega_2+1}[W],$$
and each of these ordinals must therefore belong to $A_W$ as desired. □

The argument of 3.4 of [8] which shows that

$$A \cup \{ \gamma < \omega_2 \mid cf(\gamma) = \omega \}$$

does not contain a club whenever $A$ indexes a disjoint club sequence is used to show that any disjoint club sequence can be killed with a forcing that leaves $H(\omega_2)$ undisturbed. We are sure this would be known to the authors of [8] but we prove it here so we can observe that in the extension the analogue of $\text{BSRP}_s(\omega_2)$ from Theorem 11, which we denote by $\text{DSRP}_s(\omega_2)$, persists while the set $A_W$ becomes nonstationary so that $\text{BMM}^s(c)$ fails.

**Theorem 14.** Assume $\text{MM}$. Then there is a forcing notion $\mathbb{P}$ of size $\omega_2$ such that whenever $G \subset \mathbb{P}$ is $V$-generic,

$$V[G] \models \text{BMM}^{++} \land \text{DSRP}^s(\omega_2) \land \neg \text{BMM}^s(c).$$

**Proof.** Let $\{C_\alpha \mid \alpha \in A_W\}$ be the set produced in Theorem 7. Let $\mathbb{P}$ consist of closed subsets of $\omega_2 \setminus A_W$ of size $\omega_1$, ordered by end extension. We claim that forcing with $\mathbb{P}$ does not introduce new subsets of $\omega_1$. Note that $\mathbb{P}$ is $\sigma$-closed. Let $\tau$ be a $\mathbb{P}$ term which is forced by a condition $p$ to be a subset of $\omega_1$. Fix a large enough $\theta$, and consider sequences

$$( (N_\gamma, p_\gamma, s_\gamma) \mid \gamma < \omega_1 )$$

satisfying the following conditions:

1. $N_\gamma \prec H(\theta)$ and $N_\gamma \in N_{\gamma+1}$
2. $(N_\gamma \mid \gamma < \omega_1)$ is increasing and continuous
3. $p_0 = p$ and each $p_\gamma \in \mathbb{P} \cap N_\gamma$
4. $(p_\gamma \mid \gamma < \omega_1)$ is $<_{\mathbb{P}}$-decreasing
5. $p_\gamma \Vdash_{\mathbb{P}} \tau \cap \gamma = s_\gamma$
6. $\{N_\gamma \cap \omega_2 \mid \gamma < \omega_1\}$ is club in $[\alpha]^{\omega}$ for some $\alpha < \omega_2$.

We need find such a sequence with $\alpha \notin A_W$, for then

$$q = ( \bigcup_{\gamma < \omega_1} p_\gamma ) \cup \{ \alpha \} \in \mathbb{P}$$

and

$$q \Vdash_{\mathbb{P}} \tau = f$$

where $f = \bigcup_{\gamma < \omega_1} f_\gamma$.

Define $B$ to be the set of $\alpha$ for which there exists a sequence as above. It is easy to see that $B$ is stationary. So suppose toward a contradiction that $B \subset A_W$ and for $\alpha \in B$ and let $(N_\gamma^\alpha \mid \gamma < \omega_1)$ be the sequence as above. As in 3.4 of [8], let $c_\alpha \subset \omega_1$ be club so that

$$\{N_\gamma^\alpha \cap \omega_2 \mid \gamma \in c_\alpha\}$$
is club in \([\alpha]^{\omega}\) and contained in \(C_\alpha\). Let \(i_\alpha\) be the minimum element of \(c_\alpha\) and let \(d_\alpha = c_\alpha \setminus \{i_\alpha\}\). Let
\[
S = \{N_\gamma^\alpha \cap H(\omega_2) \mid \alpha \in B \land \gamma \in d_\alpha\}.
\]
Then \(S\) is stationary in \([H(\omega_2)]^{\omega}\) and by pressing down we get \(\alpha < \beta\) such that
\[
N_{i_\alpha}^\alpha \cap \omega_2 = N_{i_\beta}^\beta \cap \omega_2,
\]
a contradiction as \(C_\alpha \cap C_\beta\) is empty. Now let \(G \subset \mathbb{P}\) be \(V\)-generic. We have that
\[
H(\omega_2)^{V[G]} = H(\omega_2)^V
\]
and so \(V[G] \models \text{BMM}^{++}\) and
\[
(A_W)^{V[G]} = (A_W)^{V[G]}
\]
so \(\text{BMM}^{++(c)}\) must fail in \(V[G]\). Now fix a projective stationary set \(S \subset [\omega_2]^{\omega}\) which is first order definable over \(H(\omega_2)^{V[G]}\). Thus \(S\) is projective stationary in \(V\). Let \(C\) be a term for a club subset of \(\omega_2\). Note that the proof above shows that \(B^* = B \setminus A_W\) is stationary. We may assume that we have required that the condition produced at the end forces that \(\alpha \in C\). Using \(\text{MM}\) we can then find such an \(\alpha \in B^*\) which is a club reflection point for \(S\). Thus \(\text{DSRP}^*(\omega_2)\) continues to hold.
\[\square\]

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