QUASIXCELLENCE IMPLIES STRONG GENERATION

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Abstract. We prove that the bounded derived category of coherent sheaves on a quasicompact separated quasiexcellent scheme of finite dimension has a strong generator in the sense of Bondal–Van den Bergh. This extends a recent result of Neeman and is new even in the affine case. The main ingredient includes Gabber’s weak local uniformization theorem and the notions of boundedness and descendability of a morphism of schemes.

1. Introduction

In [3], Bondal and Van den Bergh introduced the notion of strong generator of a triangulated category. It is useful because under the existence of a strong generator and the properness assumption, a certain appealing form of the Brown representability theorem holds; see [3, Theorem 1.3] for the precise statement.

We wish to know that many of the naturally arising triangulated categories have strong generators. First, they proved in [3, Theorem 3.1.4] that if $X$ is a quasicompact separated scheme smooth over a field, the bounded derived category of coherent sheaves $D^b_{coh}(X)$, which is equal to $D^{perf}(X)$ in this case, admits a strong generator. Recently, in [9], Neeman generalized this result to the case where $X$ is a separated noetherian scheme essentially of finite type over an excellent scheme of dimension $\leq 2$.

The main result of this paper is the following further generalization of Neeman’s result, which we demonstrate in Section 5.

Main Theorem. If $X$ is a quasicompact separated quasiexcellent scheme of finite dimension, $D^b_{coh}(X)$ has a strong generator.

Neeman used de Jong’s theorem on alterations to prove his result. Our strategy is to rather use weak local uniformizations, whose existence for a quasiexcellent scheme is already known due to Gabber. In order to do so, we should contemplate on how $h$ covers (or alteration covers) and strong generators interact with each other. We found the two notions of boundedness and descendability of a morphism, which we treat in Section 3 and Section 4 respectively, useful when considering that problem.

Convention. To simplify the exposition, we always regard $D_{qcoh}$ and $D^{perf}$ of a quasicompact quasiseparated scheme and $D^{b}_{coh}$ of a noetherian scheme as stable $\infty$-categories. Moreover, all pullback, pushforward, and tensor product functors in this paper mean the derived ones, so we put neither $L$ nor $R$ to indicate how they are derived.

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2. Basic definitions

We first review some basic notions. Our notation and terminology may slightly differ from the common ones.

Definition 2.1. Let \( C \) be a stable \( \infty \)-category. We call a collection of objects of \( C \) closed if it is closed under finite coproducts and direct summands. For a collection \( S \subset C \), let \( \langle S \rangle \) denote the smallest closed subcollection containing \( S \). For two closed subcollections \( S, T \), we let \( S \ast T \) denote the smallest closed subcollection containing an object \( C \) such that there exists \( C', C'' \in S, C'' \in T \), and a cofiber sequence \( C' \to C \to C'' \).

We often omit curly braces to simplify the notation; for example, \( \langle C \rangle \) for an object \( C \) means what should be denoted by \( \langle \{C\} \rangle \), to be exact.

Remark 2.2. The operation \( \ast \) was considered for example in [1, Section 1.3], where they proved its associativity. (Note that \( \ast \) differs from what was denoted by \( \ast \) there in that we apply the closure operation.) Therefore, following the usual pattern, we write \( S^{\ast n} \) for the “\( n \)th power” of a closed subcollection \( S \) for \( n > 0 \) and \( S^{\ast 0} \) for the collection consisting of zero objects.

Definition 2.3. An object \( C \in C \) of a stable \( \infty \)-category \( C \) is called a strong generator if there exists an integer \( n \geq 0 \) such that the equality \( \langle \Sigma^i C \mid i \in \mathbb{Z} \rangle^{\ast n} = C \) holds.

Then we introduce the following “big” variant:

Definition 2.4. Let \( C \) be a stable \( \infty \)-category admitting small coproducts. We call a collection of objects of \( C \) big closed if it is closed under small coproducts and direct summands. For a collection \( S \subset C \), let \( \langle \langle S \rangle \rangle \) denote the smallest big closed subcollection containing \( S \). Note that if \( S \) and \( T \) are big closed, \( S \ast T \) is also big closed.

Example 2.5 (G. M. Kelly). For a commutative ring \( R \) of global dimension \( n \), we have \( \langle \Sigma^i R \mid i \in \mathbb{Z} \rangle^{\ast (n+1)} = D_{qcoh}(Spec R) \) by using arguments made in [3]. See [4, Section 8] for a detailed account.

The following result, which was proven in [3, Section 2], explains why we care about the big variant.

Theorem 2.6 (Neeman). Let \( X \) be a noetherian scheme. Suppose that an object \( F \in D_{coh}^b(X) \) satisfies \( \langle \Sigma^i F \mid i \in \mathbb{Z} \rangle^{\ast n} = D_{qcoh}(X) \) for some integer \( n \geq 0 \). Then \( F \) is a strong generator of \( D_{coh}^b(X) \).

3. Boundedness

We introduce the notion of boundedness of a morphism between schemes.

Definition 3.1. Let \( f : Y \to X \) be a morphism between noetherian schemes. It is called coherently bounded if for every \( G \in D_{coh}^b(Y) \), there exists an object \( F \in D_{coh}^b(X) \) and an integer \( n \geq 0 \) such that \( f_* G \in \langle \langle F \rangle \rangle^{\ast n} \) holds.
In this section, we prove that many morphisms are coherently bounded.

**Example 3.2.** Any proper morphism \( Y \to X \) between noetherian schemes is coherently bounded since the direct image functor sends an object of \( D^b_{\text{coh}}(Y) \) into \( D^b_{\text{coh}}(X) \).

**Proposition 3.3.** Any composition of two coherently bounded morphisms is coherently bounded.

**Proof.** Suppose that \( f : Y \to X \) and \( g : Z \to Y \) are coherently bounded morphisms between noetherian schemes. For \( H \in D^b_{\text{coh}}(Z) \), we can take an object \( G \in D^b_{\text{coh}}(Y) \) and an integer \( n \geq 0 \) satisfying \( g_* H \in \langle \langle G \rangle \rangle^{*n} \). Similarly, we can take an object \( F \in D^b_{\text{coh}}(Y) \) and an integer \( m \geq 0 \) satisfying \( f_* G \in \langle \langle F \rangle \rangle^{*m} \). Then we have \((f \circ g)_* H \simeq f_*(g_* H) \in f_*(\langle \langle G \rangle \rangle^{*n}) \subset \langle \langle f_* G \rangle \rangle^{*n} \subset \langle \langle F \rangle \rangle^{*mn} \). \( \square \)

**Lemma 3.4.** Any open immersion between separated noetherian schemes is coherently bounded.

**Proof.** Let \( j : U \to X \) be an open immersion between separated noetherian schemes. Consider an object \( G \in D^b_{\text{coh}}(U) \). We wish to find an object \( F \in D^b_{\text{coh}}(X) \) and an integer \( n \geq 0 \) satisfying \( j_* G \in \langle \langle F \rangle \rangle^{*n} \). Since \( G \) is a direct summand of some objects of the form \( j^* F' \) with \( F' \in D^b_{\text{coh}}(X) \), we may assume that \( G = j^* F' \) holds for some \( F' \in D^b_{\text{coh}}(X) \). According to \([8, \text{Theorem 6.2}]\), there exists an object \( F'' \in \text{Pr}^b_{\text{perf}}(X) \) and an integer \( n \geq 0 \) such that \( j_* \mathcal{O}_U \in \langle \langle F'' \rangle \rangle^{*n} \) holds. Then the pair consisting of \( F = F' \otimes F'' \in D^b_{\text{coh}}(X) \) and this \( n \) work since we have \( j_* G \simeq F' \otimes j_* \mathcal{O}_U \in F' \otimes \langle \langle F'' \rangle \rangle^{*n} \subset \langle \langle F' \otimes F'' \rangle \rangle^{*n} = \langle \langle F \rangle \rangle^{*n} \). \( \square \)

**Theorem 3.5.** Any morphism of finite type between separated noetherian schemes is coherently bounded.

**Proof.** Nagata’s compactification theorem says that such a morphism is factored into an open immersion followed by a proper morphism. So the desired result follows from Example 3.2, Proposition 3.3, and Lemma 3.4. \( \square \)

4. Descendability

In this section, the notion of descendability, which is introduced in \([8, \text{Section 3}]\), and see how it is related to our problem.

We let \( \text{Pr}^{\text{St}} \) denote the \( \infty \)-category whose objects are presentable \( \infty \)-categories and whose morphisms are colimit preserving functors. We equip it with the symmetric monoidal structure given in \([7, \text{Section 4.8.2}]\).

**Definition 4.1** (Mathew). Let \( \mathcal{C} \) be an object of \( \text{CAlg}(\text{Pr}^{\text{St}}) \); concretely, \( \mathcal{C} \) is a stable presentable \( \infty \)-category equipped with a symmetric monoidal structure whose tensor product operations preserve small colimits in each variable. A commutative algebra object \( A \in \text{CAlg}(\mathcal{C}) \) is called **descendable** if \( \mathcal{C} \) is the smallest thick tensor ideal containing \( A \).

**Example 4.2** (Bhatt–Scholze). Recall that a morphism \( f : Y \to X \) between noetherian schemes is called an h cover if it is of finite type and every base change is (topologically) submersive. According to \([2, \text{Proposition 11.25}]\), for such a morphism \( f \), the direct image \( f_* \mathcal{O}_Y \) is descendable when viewed as a commutative algebra object of \( D_{\text{qcoh}}(X) \).
The following characterization is standard; see [2, Lemma 11.20] for a proof.

**Proposition 4.3.** For $C \in \text{CAlg}(\text{Pr}^\text{st})$ and $A \in \text{CAlg}(C)$, let $K$ denote the fiber of the canonical morphism $1_C \to A$. Then $A$ is descendable if and only if $K^{\otimes k} \to 1_C$ is zero for some $k \geq 0$.

The following observation explains why this notion is useful for us; it says that a descendable commutative algebra object generates the given stable $\infty$-category using a finite number of steps in some sense.

**Proposition 4.4.** Consider $C \in \text{CAlg}(\text{Pr}^\text{st})$ and $A \in \text{CAlg}(C)$ and suppose that $A$ is descendable. Then $(A \otimes C \mid C \in C)^*k = C$ holds for some integer $k \geq 0$.

**Proof.** Let $S$ denote the collection $(A \otimes C \mid C \in C)$ and $K$ the fiber of the map $1_C \to A$. By Proposition 4.3 there is an integer $k \geq 0$ such that the canonical morphism $K^{\otimes k} \to 1_C$ is zero. Now we can check by induction on $0 \leq i \leq k$ that $\text{cofib}(K^{\otimes i} \to 1_C) \otimes C \in S^{*i}$ for every object $C \in C$; this is trivial when $i = 0$ and the inductive step follows from the cofiber sequence

$$A \otimes (K^{\otimes i} \otimes C) \to \text{cofib}(K^{\otimes(i+1)} \to 1_C) \otimes C \to \text{cofib}(K^{\otimes 1} \to 1_C) \otimes C.$$

Now we have $S^{*k} = C$ from the case when $i = k$ since $1_C$ is a direct summand of $\text{cofib}(K^{\otimes k} \to 1_C)$.

**Corollary 4.5.** Let $L: C \to \mathcal{D}$ be a morphism in $\text{CAlg}(\text{Pr}^\text{st})$. Assume that the right adjoint $R$ of $L$ preserves small colimits and the pair $(L, R)$ satisfies the projection formula; that is, the canonical morphism

$$C \otimes R(D) \to R(L(C) \otimes D)$$

is an equivalence for any $C \in C$ and $D \in \mathcal{D}$.

Suppose that $S \subset \mathcal{D}$ is a subcollection satisfying $\langle \langle S \rangle \rangle^n = \mathcal{D}$ for some integer $n \geq 0$. Then if $R(1_\mathcal{D})$ is descendable, $\langle \langle R(S) \rangle \rangle^m = C$ holds for some integer $m \geq 0$, where $R(S)$ denotes the (set theoretic) direct image of $S$ under $R$.

**Proof.** By the projection formula, we see that $R(1_\mathcal{D})$ contains $R(1_\mathcal{D}) \otimes C \simeq R(L(C))$ for every object $C \in C$. Hence by Proposition 4.3, there is an integer $k \geq 0$ satisfying $\langle \langle R(D) \rangle \rangle^{*k} = C$. Combining this with $R(D) = R(\langle \langle S \rangle \rangle^n) \subset \langle \langle R(S) \rangle \rangle^{*n}$, we have $\langle \langle R(S) \rangle \rangle^{*nk} = C$.

5. **Proof of Main Theorem**

By using the observations made so far, we obtain the following:

**Theorem 5.1.** Let $f: Y \to X$ be an $h$ cover between quasicompact separated noetherian schemes. Suppose that there is an object $G \in \text{D}_{\text{coh}}(Y)$ satisfying $\langle \langle \Sigma^i G \mid i \in \mathbb{Z} \rangle \rangle^n = \text{D}_{\text{qcoh}}(Y)$ for some integer $n \geq 0$. Then there exists an object $F \in \text{D}_{\text{coh}}(X)$ and an integer $m \geq 0$ such that $\langle \langle \Sigma^i F \mid i \in \mathbb{Z} \rangle \rangle^{*m} = \text{D}_{\text{qcoh}}(X)$ holds.

**Proof.** By Theorem 5.1, we can take an object $F \in \text{D}_{\text{coh}}(X)$ and an integer $k \geq 0$ such that $f_* G \in \langle \langle F \rangle \rangle^{*k}$ holds. On the other hand, by Example 4.2 we can apply Corollary 4.5 to see that there is an integer $l \geq 0$ satisfying $\langle \langle \Sigma^i f_* G \mid i \in \mathbb{Z} \rangle \rangle^l = \text{D}_{\text{qcoh}}(X)$. From these observations, we have $\langle \langle \Sigma^i F \mid i \in \mathbb{Z} \rangle \rangle^{*l} = \text{D}_{\text{qcoh}}(X)$.

We conclude this paper by proving Main Theorem, which we stated in Section 1.
Proof of Main Theorem. Gabber’s weak local uniformization theorem says that there exists an h cover $Y \to X$ where $Y$ is the spectrum of a regular ring, which automatically has finite (global) dimension; see [5] for a proof. Hence from Example 2.5 we can apply Theorem 5.1 to see that there exists an object $F \in D_{coh}^b(X)$ satisfying $\langle \Sigma F \mid i \in \mathbb{Z} \rangle^n = D_{coh}^b(X)$ for some integer $n \geq 0$. According to Theorem 2.6 this implies the desired result. □

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