Conditions for Regional Frequency Stability in Power Systems—Part I: Theory

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Abstract—This paper considers the phenomenon of distinct regional frequencies recently observed in some power systems. We first study the causes through dynamic simulations and then develop the mathematical model describing this behaviour. Then, techniques to solve the model are discussed, demonstrating that the post-fault frequency evolution in any given region is equal to the frequency evolution of the Centre Of Inertia plus certain inter-area oscillations. This finding leads to the deduction of conditions for guaranteeing frequency stability in all regions of a power system, a deduction performed using a mixed analytical-numerical approach that combines mathematical analysis with regression methods on simulation samples. The proposed stability conditions are linear inequalities that can be implemented in any optimisation routine allowing the co-optimisation of all existing ancillary services for frequency support: inertia, multi-speed frequency response, load damping and an optimised largest power infeed. This is the first reported mathematical framework exhibiting inter-area oscillations in frequency.

Index Terms—Power system dynamics, inertia, frequency stability, unit commitment.

Indices and Sets

\( i, j, n \) All-purpose indices.
\( \mathcal{F}_i \) Set of neighbouring regions to region \( i \).

Constants and Parameters

\( \Delta f_{\text{max}} \) Maximum admissible frequency deviation (Hz).
\( \delta_{\text{max}} \) Maximum admissible frequency deviation at \( t_{\text{ss}} \) (Hz).
\( \phi_i \) Steady-state phase angle of voltage in bus \( i \) (rad).
\( \omega_i \) Angular frequency of inter-area oscillations for region \( i \) (rad/s).
\( a_i \) Attenuation of inter-area oscillations in region \( i \) (s\(^{-1}\)).
\( A_i \) Amplitude of inter-area oscillations for region \( i \) (Hz).
\( D_i \) Load damping factor in region \( i \) (%/Hz).
\( H_i \) System inertia in region \( i \) (MW·s).
\( P^D_i \) Total demand in region \( i \) (MW).
\( P^L_i \) Largest power infeed in region \( i \) (MW).
\( R_i \) Total PFR in region \( i \) (MW).
\( T_g \) Delivery time of PFR (s).
\( T_{i,j} \) Stiffness of transmission between buses \( i \) and \( j \) (MW).
\( V_i \) Voltage magnitude in bus \( i \) (kV).
\( X_{i,j} \) Reactance of transmission between buses \( i \) and \( j \) (Ω).

Functions and Operators

\( \Delta f_i(t) \) Time-evolution of post-fault frequency deviation from nominal state in region \( i \) (Hz).
\( \Delta F_i(s) \) Laplace transform of \( \Delta f_i(t) \) (Hz).
\( \Delta P^\text{import}_i(t) \) Deviation from steady-state power import to region \( i \), after an outage (MW).
\( \delta_i(t) \) Post-fault phase angle of voltage in bus \( i \) (rad).
\( \mathcal{L}\{\cdot\} \) Laplace transform operator.
\( \mathcal{L}^{-1}\{\cdot\} \) Inverse Laplace transform operator.
\( \sup\{\cdot\} \) Supremum of a set.
\( t^* \) Time when the frequency nadir occurs (s).

I. INTRODUCTION

MAINTAINING system frequency within acceptable limits is critical for the secure operation of a power grid. In the event of a generation outage, the subsequent frequency drop is contained by certain ancillary services: system inertia, load damping and Frequency Response (FR). While these ancillary services were widely available in grids dominated by thermal generators as a by-product of energy, the increasing penetration of non-synchronous Renewable Energy Sources (RES) has greatly reduced the level of system inertia, therefore increasing the risk of violating frequency stability.

Frequency stability is studied through comprehensive dynamic simulations of the power system, while closed-form stability conditions (which can be used by system operators to procure ancillary services) have been deduced in the literature from the swing equation [1]. The swing equation is a simplification of the actual frequency dynamics in a power grid, which considers a single-bus representation assuming that all of the system’s generators move coherently as a single lumped mass (Center Of Inertia concept) and load damping is modelled as a single constant. While the uniform frequency model has provided a precise representation of frequency dynamics in systems dominated by thermal units, recent studies have shown that the Centre Of Inertia (COI) representation can be inaccurate in modern grids, in which RES are typically located in remote areas far from load centres, creating a non-uniform distribution of inertia. This non-uniform inertia distribution causes distinct regional frequencies as can be observed in Fig. 1, obtained from the dynamic simulation of a two-region system where a generation outage occurs in region 2.

These geographical discrepancies in frequency have been observed after frequency events in recent years by utilities all over the world. Several recent publications and reports [2]–[5] highlight this issue. The authors in [6] propose a method for optimal placement of virtual inertia based on minimising deviation from nominal frequency after an outage. The work in [7] performs a probabilistic frequency-stability analysis from dynamic simulations of systems with heterogeneous distributions of inertia, while reference [8] developed a simple algebraic formula for accurately estimating frequency in any of the grid’s buses, which can significantly decrease the computation of dynamic simulations. However, to date no work has deduced conditions for regional frequency stability.

Fig. 1 illustrates the dangers of considering the Centre of Inertia (COI) model in systems with non-uniform inertia distribution: the actual need for frequency ancillary services would be underestimated, leading to higher regional Rate of Change of Frequency (RoCoF) and lower regional nadirs than expected (note that the frequency of the COI evolves in between the frequencies of the two areas, which exhibit
some inter-area oscillations). If the generation scheduling or ancillary-services market is not appropriately constrained to reflect the distinct regional frequencies, unexpected tripping of RoCoF relays and triggering of Under-Frequency Load Shedding (UFLS) could take place, which in turn could cause cascading outages potentially leading to a blackout.

In this context, the present paper focuses on generalising the currently available conditions for guaranteeing frequency stability to account for the spatial variations in post-fault frequency dynamics. In order to do so, mathematical inequalities representing the stability boundary are deduced from a spatial swing model, which generalises the swing equation by considering \( N \) different regions in the grid coupled by ac transmission lines. By solving this spatial swing model, constraints for guaranteeing regional frequency stability can be obtained, to be later implemented in optimisation routines. Nevertheless, solving this model analytically is a challenging task, even impossible for the simplest systems as will be demonstrated in coming sections of this paper. To overcome this difficulty, in this work we propose a mixed analytical-numerical method to obtain the constraints, using theoretical mathematical techniques combined with regression methods on simulation samples.

The key contributions of this work are three-fold:
1) To the best of our knowledge, this is the first work to deduce analytical conditions for regional frequency stability in a power grid.
2) A linear formulation of the frequency-security constraints is provided to allow a computationally-efficient optimisation. This formulation allows the co-optimisation of all existing frequency services, including inertia, multi-speed frequency response, load damping and an optimised largest power infeed.
3) Several case studies are run using a frequency-secured Stochastic Unit Commitment model, highlighting the applicability of the proposed frequency-stability conditions.

This paper is organised as follows: Section II analyses the causes of distinct regional frequencies, while the dynamic model and a discussion on how to solve it is presented in Section III. Sections IV and V introduce the proposed stability conditions for two-region and \( N \)-region systems. Finally, Section VI gives the conclusion.

II. CAUSES OF DISTINCT REGIONAL FREQUENCIES

Before undertaking the deduction of mathematical constraints that would guarantee regional frequency security, we analyse here the causes of distinct post-fault frequencies in a power system. Let’s consider the Great Britain (GB) system split in two areas, roughly corresponding to Scotland and England. Scotland has significant wind resources while most load centres are located in England, a fact that drives a non-uniform inertia distribution.

The two-region England-Scotland system was simulated in MATLAB/Simulink, using a mathematical model that will be discussed in Section III. Simulations were run by changing the system operating condition, i.e. changing the inertia, load damping and frequency response in each region, the location of the generation outage (placed alternatively in Scotland and England) and the strength of the electrical interconnection between the regions (driven by the impedance of the connecting transmission corridors). This last parameter was the first shown to significantly affect the post-fault regional frequencies, as illustrated in Fig. 2.

Both cases in Fig. 2, which consider an even split of inertia, load damping and frequency response among England and Scotland, show a roughly equal COI frequency. These cases consider a uniform inertia distribution, with both England and Scotland having the same inertia.
The answer is that there is a second cause for frequencies in recent times, as reported in works such as why are system operators observing distinct regional mission corridors have not significantly changed in recent network through, for example, parallel circuits is chosen, but be weak given its great length (unless reinforcement of the renewable generators and the rest of the grid will necessarily

This fact implies that the electrical connection between these renewable generators and the rest of the grid will necessarily be weak given its great length (unless reinforcement of the network through, for example, parallel circuits is chosen, but this approach involves high investment costs).

At this point a questions arises: given that existing transmission corridors have not significantly changed in recent years, why are system operators observing distinct regional frequencies in recent times, as reported in works such as [2]–[5]? The answer is that there is a second cause for this phenomenon: these renewables located in remote regions also drive a non-uniform inertia distribution, as inertia in the regions dominated by renewable generation can be very low.

Consider the examples in Fig. 3, as these simulations show that regional frequencies become significant when there are gradients of inertia in the system, particularly so when the generation outage occurs in the low-inertia region: the 0.6GW loss in Scotland displayed in Fig. 3b exhibits extreme RoCoF in that region during the first few seconds, while a 1.8GW loss in England shows a similar behaviour to the COI model. The observation of highest oscillations in the first instants after the fault, for a fault happening in the low-inertia region was also reported in [2] and [3].

In conclusion, the simulations conducted in this section have shown that two causes driving distinct regional frequencies are: 1) a weak electrical interconnection, as it increases the amplitude of inter-area oscillations; and 2) a non-uniform distribution of inertia (i.e. a geographical gradient of inertia) combined with a generation outage in the low-inertia region.

Two options can be immediately identified to limit the impact of regional frequencies, guaranteeing that RoCoF and nadir requirements are respected: 1) improving the ac connection between the regions (note that using dc connections would effectively decouple the regions for inertia purposes), although this would involve high investment costs; and 2) appropriately procuring ancillary services for frequency support in each region, notably inertia and the largest possible loss.

The findings reported in this section are the reason for undertaking the work of deducing frequency-stability constraints for multi-region systems, but no further assumption on the strength of the electrical connection or the inertia distribution is made in the remaining of this paper. The impedance of the interconnection can be chosen in our model to match that of any particular system under study, while the location of inertia will be considered as a decision variable in our constraints.

III. DYNAMICS OF POST-FAULT REGIONAL FREQUENCIES

This section describes the mathematical model of post-fault frequency evolution in multi-region systems, from which the conditions for regional frequency security will be obtained. For now let’s consider the simplest case of 2 regions, as the generalisation for $N$ regions follows the same logic. The post-fault frequency in each region is described by this set of differential equations:

$$2H_1 \frac{d\Delta f_1(t)}{dt} + D_1 P_1^D \Delta f_1(t) = FR_1(t) - P_1^L + \Delta P_{1,\text{import}}^\text{imp}(t) \quad (1.1)$$

$$2H_2 \frac{d\Delta f_2(t)}{dt} + D_2 P_2^D \Delta f_2(t) = FR_2(t) - P_2^L + \Delta P_{2,\text{import}}^\text{imp}(t) \quad (1.2)$$

Eqs. (1) can be described as two coupled swing equations, in which the coupling term is the ac power exchange between regions. The coupling term $\Delta P_{i,\text{import}}^\text{imp}(t)$ represents the deviation from the steady state power being imported to region $i$. That is, before the outage there was a certain amount of power being imported to region $i$; after the outage, the amount of power imported changes following the laws of energy conservation, so that a frequency equilibrium is restored in the whole network. $\Delta P_{i,\text{import}}^\text{imp}(t)$ is the difference between the power imported after-outage and before-outage.

To obtain the mathematical description of the term $\Delta P_{i,\text{import}}^\text{imp}(t)$, let’s first consider the steady-state power transfer between 2 neighbouring buses, $i$ and $j$. This steady-state power transfer is described by the following equation [1]:

$$P_{i,j}^{\text{transfer}} = \frac{V_i V_j}{X_{i,j}} \sin(\delta_i - \delta_j) \quad (2)$$

Note that the line resistance is neglected as we consider $X_{i,j} \gg R_{i,j}$ (a typical assumption in transmission lines), and that $\delta_i$, $\delta_j$ are bus voltage phase-angles referred to the rotating frame, which rotates at synchronous speed (50Hz in Europe). Therefore, these phase angles $\delta_i$, $\delta_j$ are time-invariant in steady state. The phase angles will however change in the event of a generation outage in the system, as the rotating speed of the generators deviates from 50Hz while they release the kinetic energy stored in their rotating masses.
The deviation from the steady state power transfer between busses \( i, j \) after a fault can be calculated as:

\[
\Delta P_{ij}^{\text{transfer}}(t) = V_{i,j} \cdot \sin\left(\delta_i(t) - \delta_j(t)\right) - V_{i,j} \cdot \sin\left(\delta_i^s - \delta_j^s\right)
\]

(3)

Linearising around the operating point \( \delta_i = \delta_i^s, \delta_j = \delta_j^s \) [1]:

\[
\Delta P_{ij}^{\text{transfer}}(t) = V_{i,j} \cdot \cos(\delta_i^s - \delta_j^s) \cdot \left[\Delta \delta_i(t) - \Delta \delta_j(t)\right]
\]

(4)

where \( \Delta \delta_i(t) = \delta_i(t) - \delta_i^s \). We have assumed that the average voltage variation in the grid is small after the generation outage, an assumption also made in the literature [9], [10].

Now, in order to express the power imported by any region in terms of electric frequency \( \Delta f(t) \) rather than phase angle \( \Delta \delta(t) \), it’s important to understand the relation between these two magnitudes:

\[
\frac{d\delta(t)}{dt} = \frac{d\delta_m(t)}{dt} - \omega_s = \omega_m(t) - \omega_s = \Delta \omega(t) = 2\pi \Delta f(t)
\]

(5)

Where \( \delta \) is assumed to be expressed in radians and \( \Delta f \) in Hz. \( \omega_m \) is the absolute angle of the synchronous generator, not referred to the rotating reference as \( \delta \) is; \( \omega_s \) is the synchronous rotational speed of the generators (equivalent to the 50Hz electrical frequency).

Note that this relationship still holds when considering phase-angle deviations from steady state, \( \Delta \delta(t) \):

\[
\frac{d\Delta \delta(t)}{dt} = \frac{d\delta(t)}{dt} - \frac{d\delta^s}{dt} = \Delta \omega(t) = 2\pi \Delta f(t)
\]

(6)

So the post-fault power imported by any region \( i \) can be expressed as:

\[
\Delta P_i^{\text{import}}(t) = -\sum_{j \in J_i} T_{i,j} \left[\int_0^t \Delta f_i(\tau)d\tau - \int_0^t \Delta f_j(\tau)d\tau\right]
\]

(7)

Where \( J_i \) is the set of neighbours areas of area \( i \). Note that a negative value of \( \Delta P_i^{\text{import}}(t) \) means that region \( i \) is exporting power to other regions. To make (7) and following expressions clearer, we have defined the electrical stiffness of the transmission line as [11]:

\[
T_{i,j} = 2\pi \cdot \frac{V_{i,j}}{X_{i,j}} \cdot \cos(\delta_i^s - \delta_j^s)
\]

(8)

The generalization for \( N \) regions of the coupled swing equations, including the inter-region power transfer term, follows immediately:

\[
\begin{align*}
2H_1 \frac{d\Delta f_1(t)}{dt} + D_1 P_1^D \Delta f_1(t) &= FR_1(t) - P_{1L} + \Delta P_1^{\text{import}}(t) \\
2H_2 \frac{d\Delta f_2(t)}{dt} + D_2 P_2^D \Delta f_2(t) &= FR_2(t) - P_{2L} + \Delta P_2^{\text{import}}(t) \\
&\vdots \\
2H_n \frac{d\Delta f_n(t)}{dt} + D_n P_n^D \Delta f_n(t) &= FR_n(t) - P_{nL} + \Delta P_n^{\text{import}}(t)
\end{align*}
\]

(9)

where \( \Delta P_i^{\text{import}}(t) \) is modelled as in (7).

The set of integro-differential equations (9) is the extension of the single-bus swing equation to consider distinct regions, and therefore allows to consider geographical gradients in frequency within a power system. The power system is modelled as a set of regions that do not swing coherently but are connected through synchronous ac transmission lines. This model generalizes the uniform frequency model, as (9) reduces to the single-bus swing equation if the impedance of the transmission lines tends to zero.

A. Solving the dynamic model: two-region case

This section discusses how to obtain a solution for \( \Delta f_1(t) \) and \( \Delta f_2(t) \) as defined by (1). Without loss of generality and for the sake of simplicity, we consider a single response service, Primary Frequency Response (PFR), modelled as:

\[
PFR_i(t) = \begin{cases} 
R_i/T_g \cdot t & \text{if } t \leq T_g \\
R_i & \text{if } t > T_g
\end{cases}
\]

(10a)

(10b)

In order to obtain an analytical solution for the set of integro-differential equations (1), we propose to first apply the Laplace transform so that a set of algebraic equations is obtained. Then, this set of algebraic equations can be solved analytically, and the inverse Laplace transform can be applied to obtain the solution in time domain.

Applying the Laplace transform to (1) we obtain:

\[
\begin{align*}
2H_1 s \Delta F_1(s) + D_1 P_1^D \Delta F_1(s) &= PFR_1(s) - \frac{P_{1L}}{s} + \Delta P_1^{\text{import}}(s) \\
2H_2 s \Delta F_2(s) + D_2 P_2^D \Delta F_2(s) &= PFR_2(s) - \frac{P_{2L}}{s} + \Delta P_2^{\text{import}}(s)
\end{align*}
\]

(11.1)

(11.2)

where \( \Delta F_i(s) = \mathcal{L}\{\Delta f_i(t)\} \) and \( \Delta P_i^{\text{import}}(s) \) is given by:

\[
\Delta P_i^{\text{import}}(s) = -\sum_{j \in J_i} \frac{\Delta F_j(s) - \Delta F_j(s)}{s}
\]

(12)

Considering that PFR is still ramping up, which is the period of interest for RoCoF and nadir purposes:

\[
PFR_i(s) = \frac{R_i}{T_g} \cdot \frac{1}{s^2}
\]

(13)

The set of algebraic equations (11) where the unknowns are \( \Delta F_i(s) \) can be solved analytically (e.g. using a computer algebra system such as MATLAB or Maple), which gives the following solution:

\[
\Delta F_i(s) = \frac{C_1 \cdot s^3 + C_2 \cdot s^2 + C_3 \cdot s + C_4}{C_5 \cdot s^3 + C_6 \cdot s^2 + C_7 \cdot s + C_8 \cdot s^2}
\]

(14)

Where the constants \( C_j \) are all real numbers, which are functions of the system parameters that appear in (11) such as \( H_1, H_2, P_1^D, \) etc. These functions of system parameters are significantly convoluted as shown later in this section, hence the use of these generic constants \( C_j \) to represent them in a simple way.

In order to apply the inverse Laplace transform and then obtain the time-domain solution \( \Delta f_i(t) \), (14) must be decomposed in partial fractions:

\[
\Delta F_i(s) = \frac{C'_1}{s^2} + \frac{C'_2}{s} + \frac{C'_3 \cdot s^3 + C'_4 \cdot s^2 + C'_7 \cdot s + 1}{C'_8 \cdot s^2 + \sum_{k=1}^{3} Z_k} = \frac{C'_1}{s^2} + \frac{C'_2}{s} + \sum_{k=1}^{3} \frac{Z_k}{s - Z_k}
\]

(15)

Note that these constants \( C'_j \) are again generic functions of the system parameters but different to the \( C_j \) used in (14). Terms \( Z_k \) and \( z_k \) are generic complex numbers. The factorisation performed in (15) makes use of the Fundamental Theorem of Algebra, which states that every polynomial of degree \( n \)
has \( n \) roots. The inverse Laplace transform of (15) gives the time-domain solution:

\[
\Delta f_1(t) = \mathcal{L}^{-1}\left\{\Delta F_1(s)\right\} = \mathcal{L}^{-1}\left\{\frac{C_1}{s^2} + \frac{C_2}{s} + \sum_{k=1}^{3} \frac{Z_k}{s - z_k}\right\}
\]

\[
= C_1 \cdot t + C_2 + \sum_{k=1}^{3} Z_k \cdot e^{z_k t} \quad (16)
\]

To fully understand (16), let’s consider the two possibilities for the roots of the last denominator in (15), \( z_k, k = 1 \) all \( z_k \) are real; or 2) \( z_2 \) is real while \( z_2 \) and \( z_3 \) are complex conjugates. This conclusion comes from the Complex Conjugate Root Theorem, which states that if a polynomial in one variable with real coefficients (as is the case in (15)) has a complex root, then the conjugate is also a root. **Proposition:** for the range of values in a realistic power system, the third-order denominator in (15) has one real root and two complex conjugate roots, i.e. \( z_1 \) is real while \( z_2 \) and \( z_3 \) are complex conjugates in (15) and (16). The validity of this proposition is supported by the evidence reported in the literature [2]–[5] and in the dynamic simulations in Section II, which show that the post-fault regional frequencies behave as the COI plus certain inter-area oscillations. As is shown in the next paragraphs, such dynamic behaviour corresponds to one real root and two complex conjugate roots for the denominator in (15).

Considering this proposition, the last term in (16) can be expressed as:

\[
\sum_{k=1}^{3} Z_k \cdot e^{z_k t} = Z_1 \cdot e^{z_1 t} + Z_2 \cdot e^{z_2 t} + Z_3 \cdot e^{z_3 t}
\]

\[
= Z_1 \cdot e^{z_1 t} + (a_2 + j b_2) \cdot e^{(a_2 + j b_2) t} + (a_2 - j b_2) \cdot e^{(a_2 - j b_2) t}
\]

\[
= Z_1 \cdot e^{z_1 t} + e^{a_2 t} [a_2 \cdot e^{j b_2 t} + e^{-j b_2 t}] + j b_2 \cdot e^{j b_2 t - e^{-j b_2 t}}
\]

\[
= Z_1 \cdot e^{z_1 t} + e^{a_2 t} [a_2 \cdot 2 \cos(b_2 t) - j b_2 \cdot 2 \sin(b_2 t)] \quad (17)
\]

where \( a_2 \) and \( b_2 \) are generic real numbers. Note that (17) shows that the last term in (16) corresponds to an exponential function plus certain oscillations in time-domain.

Since a third-order polynomial can be factorised algebraically (for example, using Cardano’s formula), we now describe the procedure to do so. First, let’s consider the explicit values for the generic constants \( C_j \) in (14) and \( C_i \) in (15) for region 1 (the expression for region 2 is omitted but equivalent): see eq. (18).

Eq. (18) has been simplified using: \( H = H_1 + H_2, R = R_1 + R_2, D_j = D_j \cdot P_j \) and \( D' = D_1 \cdot P_1 + D_1 \cdot P_2 \). Before applying the inverse Laplace transform and then obtain the time-domain solution \( \Delta f_1(t) \) (18) must be decomposed in partial fractions as was done to obtain (15) from (14). An algebraic partial-fraction decomposition can be done using a computer algebra system, giving eq. (19). The value of the terms ‘\( C \)’ in eq. (19) is:

\[
C_1'' = 4 H_1 H_2 (P_1^{L_1} + T_1 g T_1 D_1')^2 + 2 H R T_1 g - (D_2')^2 + R_2 D_2 D_2' \quad (20)
\]

\[
C_2'' = 2 (P_1^{L_1} T_1 g D_1') (H_1 - H_2) (D_1' + D_2') + 2 R T_1 g (D_2' H_1^2 + D_1 H_2^2) + D_2 (D_2')^2 (H_1 R_2 - H_2 R_1)
\]

\[
+ (D_1')^2 D_2 R_2 - (D_2')^2 H_1 R_1 \quad (21)
\]

\[
C_3'' = 4 H^2 R (T_1 g)^2 + 2 H D' P_1^{L_1} T_1 g (T_1 g)^2 - (D_2')^2 T_1 g D' + 2 H (2 R_1 + R_2) + 2 D_2' D_1 T_1 g (H_1 R_2 + 2 H_2 R_1 + H_2 R_2)
\]

\[
- 2 (D_1')^2 T_1 g H_2 R_2 + D_1 (D_2')^2 (R_2 D_1' - R_1 D_2') \quad (22)
\]

At this point, it is revealing to compare (19) with the partial-fraction decomposition of the uniform frequency model, which was deduced in [12]:

\[
\Delta F_{COI}(s) = \frac{R}{D' T_g} \cdot s^2 + \frac{2 H R + D' T_g P_1}{(D')^2 T_g} \left( \frac{1}{s} + \frac{D'}{2 T_g} \right) - \frac{1}{s} \quad (23)
\]

By applying the inverse Laplace transform to (23), the \( \frac{1}{s} \) term would give rise to a linear function in time-domain, the \( \frac{1}{s} \) term to a constant in time-domain and the \( \frac{1}{s + \frac{D'}{2 T_g}} \) term to an exponential function in time-domain:

\[
\Delta f_{COI}(t) = \frac{R}{D' T_g} \cdot t + \frac{2 H R + D' T_g P_1}{(D')^2 T_g} \left( e^{-\frac{D'}{2 T_g} t} - 1 \right) \quad (24)
\]

By comparing (19) to (23), one can notice that the \( \frac{1}{s} \) term is the same in both expressions, and the \( \frac{1}{s} \) in (19) reduces to the one in (23) as \( T_{1,2} \) tends to infinity, i.e. the impedance of the interconnecting line tends to zero and therefore the two regions become effectively one. The comparison of the \( \frac{1}{s + \frac{D'}{2 T_g}} \) term in (23) with the last fraction in (19) is not as straightforward, but the evidence reported in the literature and in the simulations conducted in Section II shed some light for the comparison: the post-fault frequency in a multi-region system exhibits the behaviour of the COI plus certain inter-area oscillations. This observation is consistent with (17), which demonstrates that the last term in (19) corresponds to an exponential function plus oscillations in time-domain. Therefore the exponential time-domain term rising from the \( \frac{1}{s + \frac{D'}{2 T_g}} \) term in (23) is roughly equal to the exponential term in (17).

In conclusion, the deductions in this section have shown that for practical purposes in any realistic power system, the post-fault frequency evolution in a two-region system is of the form:

\[
\Delta f_i(t) \approx \Delta f_{COI}(t) + e^{-\omega_i t} A_i \sin(\omega_i t + \phi_i) + C_i \quad (25)
\]

Note that the approximation in (25) lies in the COI term, as discussed in the paragraph above.
The deductions in this section have allowed to obtain the mathematical structure of the post-fault frequency evolution in a two-region system, given by eq. (25). This expression will be used as an initial step to obtain conditions that guarantee frequency security in a two-region system, in Section IV.

Finally, it is worth noting that since the third-order denominator in (19) can be factorised analytically, in principle the exact expression that has been approximated in (25) could be obtained analytically. However, this is not done in this paper for two reasons: 1) given that the expressions (19) through (22) are cumbersome, the solution would include highly non-linear functions which would make frequency-security constraints very inefficient for being implemented in optimisation routines (if even possible to obtain such constraints); and 2) deducing this analytical solution would only be possible for a two-region system, and not for a system with three or more regions, as is proved in the next section.

B. Solving the dynamic model: N-region case

This section discusses how to obtain a solution for every region’s post-fault frequency, \( \Delta f_i(t) \), in an \( N \)-region system as defined by (9). The Laplace transform of this set of differential eqs. (9) is given by:

\[
\begin{align*}
2H_1 s \Delta F_1(s) + D_1 P_0^1 \Delta F_1(s) &= \text{PFR}_1(s) - \frac{P_1^L}{s} + \Delta P_{\text{import}}^1(s) \quad (26.1) \\
2H_2 s \Delta F_2(s) + D_2 P_0^2 \Delta F_2(s) &= \text{PFR}_2(s) - \frac{P_2^L}{s} + \Delta P_{\text{import}}^2(s) \quad (26.2) \\
\vdots
2H_i s \Delta F_i(s) + D_i P_0^n \Delta F_i(s) &= \text{PFR}_n(s) - \frac{P_n^L}{s} + \Delta P_{\text{import}}^n(s) \quad (26.n)
\end{align*}
\]

where \( \Delta P_{\text{import}}^i(s) \) is given by (12). The solution for a generic region \( i \) of this set of algebraic eqs. is:

\[
\Delta F_i(s) = \frac{C_1}{s^2} - \frac{C_2}{s} + \text{polynomial in } s \text{ of degree } 2 \cdot N - 2
\]

In order to apply the inverse Laplace transform to (27) and then obtain the time-domain solution \( \Delta f_i(t) \), the last term in (27) must be decomposed in partial fractions. However, since it is not possible to algebraically factorise polynomials of degree higher than four as proved by the Abel-Ruffini Theorem, it is not possible to decompose in partial fractions the last term in (27). Therefore, it is not possible to obtain an analytical solution in time-domain for \( \Delta f_i(t) \) for a general power system with \( N \) regions, i.e. a system with three or more regions. Proposition: for the range of values in a realistic power system, the denominator of the last term in (27) always has one real root and \( 2N - 2 \) complex conjugate roots. This proposition makes use of the Fundamental Theorem of Algebra and the Complex Conjugate Roots theorem, see the previous Section III-A discussing two-region systems for more details. This proposition is consistent with the studies reported in the literature [2]–[5], which show post-fault frequencies behaving as the COI plus certain inter-area oscillations.

Therefore, although no analytical solution can be obtained for \( \Delta f_i(t) \) for a general system with \( N \) regions, the proposition mentioned above gives information about the mathematical structure of \( \Delta f_i(t) \), in a similar fashion as was deduced in Section III-A for a two-region system.

As discussed in Section III-A, the real root of the denominator in of the last term in (27) gives rise to an exponential in time-domain, which added to the first two terms in (27) corresponds to the post-fault frequency of the COI; the \( 2N - 2 \) complex conjugate roots give rise to \( N - 1 \) oscillations in time-domain, as was discussed for the two complex conjugate roots of the two-region system considered in Section III-A. Therefore, the solution for the post-fault frequency evolution of any region \( i \) in an \( N \)-region system is:

\[
\Delta f_i(t) \approx \Delta f_{\text{COI}}(t) + \sum_{j=1}^{N-1} e^{-a_j t} A_j \sin(\omega_j t + \phi_j) + C_j \quad (28)
\]

IV. CONDITIONS FOR FREQUENCY SECURITY: 2 REGIONS

In this section we propose a method to obtain conditions for respecting RoCoF, nadir and quasi-steady-state (q-s-s) limits in any region \( i \). To illustrate this method, let’s start by remembering that the time-evolution of frequency deviation in any of the two regions, \( \Delta f_i(t) \), is of the form:

\[
\Delta f_i(t) \approx \Delta f_{\text{COI}}(t) + e^{-a_i t} A_i \sin(\omega_i t + \phi_i) + C_i \quad (29)
\]

as we have demonstrated in Section III-A. In (29), \( \Delta f_{\text{COI}}(t) \) is the frequency deviation of the Centre Of Inertia, which was deduced in Section III-A:

\[
\Delta f_{\text{COI}}(t) = \left( \frac{P_L}{D \cdot P^D} + \frac{2H \cdot R}{T_g (D \cdot P^D)^2} \right) \left( e^{-\omega_R^D t} - 1 \right) + \frac{R \cdot t}{T_g D \cdot P^D} \quad (30)
\]

Eq. (29) therefore gives valuable insight on the mathematical structure of the solution for \( \Delta f_i(t) \), but it is not possible to analytically obtain the coefficients \( a_i \), \( A_i \) and \( \omega_i \) as functions of the system parameters (\( H_1 \), \( H_2 \), \( P^L \), etc.). Using the information provided by (29), we propose to use a numerical approach to estimate the coefficients \( a_i \), \( A_i \) and \( \omega_i \). This approach consists on numerically solving the set of differential eqs. (9) for several possible operating points of a power system (therefore considering only plausible values for \( H_1 \), \( H_2 \), \( P^L \) and the rest of the system variables), and using a regression technique on the solution samples to estimate the coefficients \( a_i \), \( A_i \) and \( \omega_i \).

In summary, this mixed analytical-numerical approach goes as far as possible using analytical techniques and completes the task of obtaining frequency-security constraints by using numerical techniques to estimate the remaining parameters that cannot be obtained analytically. This method is driven by the two following goals: 1) to always guarantee frequency security, therefore becoming conservative if necessary (although it will be shown that the conservativeness introduced is small); and 2) to obtain linear constraints that would provide computational efficiency for the optimisation problem in which they would eventually be implemented.

A. RoCoF constraint: analytical deduction

Taking into consideration (29) the RoCoF in region \( i \) is:

\[
\text{RoCoF}_i(t) = \text{RoCoF}_{\text{COI}}(t) + \text{RoCoF}_{\text{oscillations}}(t) \quad (31)
\]

where \( \text{RoCoF}_{\text{oscillations}}(t) \) is the derivative of the attenuated oscillations in (29):

\[
\text{RoCoF}_{\text{oscillations}}(t) = \pm e^{-a_i t} A_i \left[ \omega \cos(\omega t + \phi_i) - a_i \sin(\omega t + \phi_i) \right] \quad (32)
\]
Since this RoCoF \(_{\text{oscillations}}(t)\) is a non-convex function, then RoCoF \(_1(t)\) in eq. (31) is also a non-convex function and therefore it is not possible to obtain its global maximum analytically. We make use of the following inequality to come up with a conservative estimation of this maximum for RoCoF \(_1(t)\):

\[
\sup\{\text{RoCoF}_1(t) \mid t \geq 0\} \leq \sup\{\text{RoCoF}_\text{COF}(t) \mid t \geq 0\} + \sup\{\text{RoCoF}_{\text{oscillations}}(t) \mid t \geq 0\}
\]

The \(\sup\{\text{RoCoF}_{\text{oscillations}}(t) \mid t \geq 0\}\) cannot be obtained analytically, because RoCoF \(_{\text{oscillations}}(t)\) is a non-convex function as mentioned before, but it can be overestimated by:

\[
\sup\{\text{RoCoF}_{\text{oscillations}}(t) \mid t \geq 0\} \leq A_i \cdot \omega_i \quad (34)
\]

Expression (34) has been obtained by neglecting the attenuation term \(e^{-\alpha t}\), therefore assuming that the oscillations are not attenuated.

In conclusion, the constraints that guarantee the RoCoF to be within specified limits in a two-region system are:

\[
\text{RoCoF}_1 = \frac{P_L}{2(H_1 + H_2)} + A_1 \cdot \omega_1 \leq \text{RoCoF}_{\text{max}} \quad (35a)
\]

\[
\text{RoCoF}_2 = \frac{P_L}{2(H_1 + H_2)} + A_2 \cdot \omega_2 \leq \text{RoCoF}_{\text{max}} \quad (35b)
\]

where \(A_i\) and \(\omega_i\) are dependent on the operating condition of the system, i.e. are functions of \(H_i, P_i^L\), etc. These functions can be estimated by using a regression on samples obtained from dynamic simulations of the system described by (1). We propose to fit a function into these samples that yields linear RoCoF constraints, which is discussed in detail in Part II of this paper.

**B. Nadir constraint: analytical deduction**

In a similar fashion as for the RoCoF constraints discussed in the previous Section IV-A, it is not possible to deduce a purely analytical nadir constraint from the solution of \(\Delta f\), given by (29), as it would involve finding the global minimum of that non-convex function (29) (see Fig. 1 for graphical evidence of the existence of several local minima in the post-fault frequency evolution of a region). Deducing the mathematical expression for this global minimum would be necessary to obtain the expression for \(t^*\), which is needed to obtain an analytical nadir constraint \(|\Delta f_{\text{nadir}}| = |\Delta f(t = t^*)| \leq \Delta f_{\text{max}}\).

We then propose to use a mixed analytical-numerical approach for the nadir constraints. The analytical part of this methodology consists on formulating the energy equilibrium at the time of the frequency nadir, that yields linear nadir constraints.

Assuming that the generation outage occurs in region 2 as in the example shown in Fig. 1, the energy-based nadir constraint takes this form for region 1:

\[
T_{1,2} \int_0^{t^*} \int_0^t [\Delta f_1(\tau) - \Delta f_2(\tau)] \, d\tau \, dt \leq 2H_1 \Delta f_{\text{max}} + \frac{R_2}{T_g} \left( \frac{(t^*)^2}{2} + P_2^L \int_0^{t^*} \Delta f_1(t) \, dt \right) \quad (36)
\]

This expression has been obtained by integrating the swing equation for region 1, i.e. integrating (1.1). In plain words, constraint (36) enforces that the energy “lost” must be lower than the maximum admissible energy “injected”. This maximum admissible energy “injected” is limited by the maximum kinetic energy that can be extracted from the rotating masses in the system so that frequency does not drop below \(\Delta f_{\text{max}}\). It is clear from (36) that if the energy related to the inertia term is higher, the frequency deviation would be higher than \(\Delta f_{\text{max}}\). The energy “lost” corresponds to the energy sent to the outaged region 2 through the transmissions corridors.

For region 2, where the generation outage happens, the nadir constraint is as follows:

\[
P_2^L \cdot t^* \leq 2H_2 \Delta f_{\text{max}} + \frac{R_2}{T_g} \left( \frac{(t^*)^2}{2} + P_2^D \int_0^{t^*} \Delta f_2(t) \, dt \right) \quad (37)
\]

Which again states that the energy “lost” from the generation outage \(P_2^L\) cannot be higher than the maximum admissible energy that can be extracted from inertia (while considering the energy contribution from FR, damping and imports).

The integrals including terms \(\Delta f_1\) and \(\Delta f_2\) must be estimated in the above constraints, for which we propose to use a numerical methodology. This numerical methodology, explained in detail in Part II of this paper, estimates these terms using a linear function of the system parameters (\(H_1, H_2, P_2^L\), etc.) so that the final form of constraints (36) and (37) is linear.

Finally, since it is not possible to obtain the analytical expression for \(t^*\) as a function of the system parameters (as explained before, it would involve finding the global minimum of a non-convex function), we propose to discretise using several time-intervals and apply conditional nadir constraints. For \(n\) segments in the discretisation of time \(t \in [t_1, t_2, \ldots, t_{n-1}, T_g]\), the nadir constraints for a two-region system would take this shape:

- if \(\frac{R}{T_g} \cdot t_1 > \frac{P_2^L}{T_g} - D \cdot P_2 \cdot \Delta f_{\text{max}}\) then enforce:

\[
T_{1,2} \int_0^{t_1} \int_0^t [\Delta f_1(\tau) - \Delta f_2(\tau)] \, d\tau \, dt \leq 2H_1 \Delta f_{\text{max}} + \frac{R_2 (t_1)^2}{T_g} + D_1 \int_0^{t_1} \Delta f_1(t) \, dt \quad (38)
\]

\[
P_2^L \cdot t_1 \leq 2H_2 \Delta f_{\text{max}} + \frac{R_2 (t_1)^2}{T_g} + D_2 \int_0^{t_1} \Delta f_2(t) \, dt + T_{1,2} \int_0^{t_1} \int_0^t [\Delta f_1(\tau) - \Delta f_2(\tau)] \, d\tau \, dt \quad (39)
\]

- else if \(\frac{R}{T_g} \cdot t_2 > \frac{P_2^L}{T_g} - D \cdot P_2 \cdot \Delta f_{\text{max}}\) then enforce:

\[
T_{1,2} \int_0^{t_2} \int_0^t [\Delta f_1(\tau) - \Delta f_2(\tau)] \, d\tau \, dt \leq 2H_1 \Delta f_{\text{max}} + \frac{R_2 (t_2)^2}{T_g} + D_1 \int_0^{t_2} \Delta f_1(t) \, dt \quad (40)
\]
\[ P_L^2 \cdot t_2 \leq 2H_2\Delta f_{\text{max}} + \frac{R_2}{T_f} (\Delta f_2)^2 + D_2 p^D \int_0^{t_2} \Delta f_2(t) \, dt + T_{1,2} \int_0^{t_2} (\Delta f_1(\tau) - \Delta f_2(\tau)) \, d\tau \, dt \]  
\[ \cdots \] 
\]

(41)

While only PFR has been considered for the sake of simplicity, note that any other frequency response service with different speed and even an activation delay can be directly included in these energy-based nadir constraints. As an example, a fast service such as Enhanced Frequency Response in GB (which must be delivered by 1s after a fault, as compared to the 10s required for PFR) with a certain activation delay could be included in the above constraint by simply adding the following term:

\[ \text{EFR} \cdot \frac{(t^* - t_{\text{delay}})^2}{2} \]  

(42)

The analytical part of the methodology proposed in this section to obtain nadir constraints entails no approximation, while the only approximation resides in the linear regressions used to estimate the integrals of \( \Delta f_1 \) and \( \Delta f_2 \) that is explained in Part II of this paper. The deduced constraints allow not only to consider the contribution of load damping to support the nadir, neglected in many works due to the mathematical complexity it introduces, but notably also allow to consider the power sharing through transmission corridors between different regions of the power system. While certainly the constraints have some degree of conservativeness due to the approximations, it is nonetheless preferable to somewhat overestimate the need for frequency services in a multi-region system than to neglect the distinct regional post-fault frequencies altogether. Furthermore, the conservativeness introduced by this approximation is assessed in Part II, which demonstrates that the overestimation is small for practical purposes in a power system.

C. Quasi-steady-state constraint

Regarding the quasi-steady-state frequency in multi-region systems, this magnitude does not in practice show distinct values across the network, since the inter-area oscillations are attenuated by devices such as Power System Stabilizers [1] or even appropriately controlled wind farms [13]. While the action of these oscillation-damping devices is limited before the frequency nadir (which would happen in a sub-10seconds scale in GB) and therefore can be neglected, the areas are shown to swing back together by the q-s-s in the reported studies. Therefore, the q-s-s constraint aggregates the FR and load damping in all regions to bring frequency back to \( \Delta f_{\text{max}} \):

\[ \sum_{i=1}^{2} R_i \geq P_L^L - \Delta f_{\text{max}}^2 \sum_{i=1}^{2} D_i \cdot p^D \]  

(43)

V. CONDITIONS FOR FREQUENCY SECURITY: N REGIONS

It has been demonstrated in Section III-B that the post-fault frequency evolution in an \( N \)-region system is that of the COI plus certain inter-area oscillations:

\[ \Delta f_i(t) \approx \Delta f_{\text{COI}}(t) + \sum_{j=1}^{N-1} e^{-a_j t} A_j \sin(\omega_j t + \phi_j) + C_j \]  

(44)

Which is a generalisation of the two-region case described by eq. (29). Since the post-fault frequency evolution in any region of an \( N \)-region system also behaves as the COI frequency plus some oscillations, the same principles and procedures discussed in Section IV to obtain RoCoF and nadir constraints for a two-region system can be directly applied to the more general \( N \)-region case.

VI. CONCLUSION

This paper has shown that weak transmission corridors and non-uniform distributions of inertia are causes of distinct regional frequencies. Using a mathematical model for post-fault frequency dynamics in multi-region systems, it has been demonstrated that the post-fault frequency evolution in any given region is equal to the frequency of the Centre Of Inertia plus certain inter-area oscillations. This mathematical result is consistent with the evidence reported in the literature. Using this result, conditions for regional frequency security have been deduced for the first time, using a mixed analytical-numerical approach: while it is not possible to obtain purely analytical constraints, the proposed methodology uses analytical techniques to go as far as they allow, while the procedure is completed with numerical techniques on simulation samples.

Part II of this paper demonstrates the applicability of the proposed constraints, showing that the conservative stpdf taken for their deduction do not introduce a significant underestimation of the frequency-stable space for practical purposes in a power system. In addition, case studies using a Stochastic Unit Commitment are run to understand the implications of inter-area frequency oscillations in the need for procuring ancillary services.

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