Bose-Einstein Condensation of Dilute Gases in Traps

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Abstract. The ground state of a gas of Bosons confined in an external trap potential and interacting via repulsive two-body forces has recently been shown to exhibit complete Bose-Einstein condensation in the dilute limit, yielding for the first time a rigorous proof of this phenomenon in a physically realistic setting. We give here an account of this work about the Gross-Pitaevskii limit where the particle number $N$ goes to infinity with $N\alpha$ fixed, where $\alpha$ is the scattering length of the interparticle interaction, measured in units of the trap size.

1. Introduction and Main Results

During the last few years it has become experimentally feasible to realize the long-predicted Bose-Einstein condensation (BEC) of gases by confining them in traps at very low temperatures. A rigorous theoretical demonstration of this phenomenon – starting from the basic many-body Hamiltonian of interacting particles – is, however, a very difficult task. Following [LS], we will provide in this paper such a rigorous derivation for the ground state of Bosons in an external trap interacting with repulsive pair potentials, and in the well-defined limit in which the Gross-Pitaevskii (GP) formula is applicable. It is the first proof of BEC for interacting particles in a continuum (as distinct from lattice) model and in a physically realistic situation. We present here a detailed version of the proof given in [LS]. We remark that an extension of the results presented here was recently obtained in [LSY4], where it was shown that the ground state is 100% superfluid in the GP limit.

The Gross-Pitaevskii limit under discussion here is a mathematically simpler limit than the usual thermodynamic limit in which the average density is held fixed as the particle number goes to infinity. A proof of BEC in this limit is still missing. The only available rigorous results concern the non-interacting gas, various mean-field and other toy models (see, e.g., [LVZ] for a recent preprint), or the hard core lattice gas [KLS].

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In the GP limit one also lets the range of the potential go to zero as $N$ goes to infinity, but in such a way that the overall effect is non-trivial. That is, the combined effect of the infinite particle limit and the zero range limit is such as to leave a measurable residue — the GP function. The limit in which the GP function can be expected to be equal to the condensate wave function should be chosen so that all three terms in the GP functional \[[1.2]\] make a contribution. This indicates that fixing $N\alpha$ as $N \to \infty$ is the right thing to do, and this is quite relevant since experimentally $N$ can be quite large, $10^9$ and more, and $N\alpha$ can range from about 1 to $10^4$ \cite{PeSm}. Fixing $N\alpha$ also means that we really are dealing with a dilute system, because the mean density $\bar{\rho}$ is then of the order $N$ and hence $a^3 \bar{\rho} \sim N^{-2}$, meaning that the range of the interaction is much shorter than the mean particle distance.

We shall now describe the setting more precisely. Let $H$ be the Hamiltonian for $N$ identical Bosons in a trap potential $V$ in $\mathbb{R}^3$, interacting via a pair potential $v$:

$$H = \sum_{i=1}^{N} \left( -\Delta_i + V(x_i) \right) + \sum_{1 \leq i < j \leq N} v(x_i - x_j).$$ \[1.1\]

It acts on the subspace of totally symmetric functions in $\bigotimes^N L^2(\mathbb{R}^3)$. Units are chosen such that $\hbar = 2m = 1$, where $m$ denotes the particle mass. We assume the trap potential $V$ to be a locally bounded function that tends to infinity as $|x| \to \infty$. The interaction potential $v$ is assumed to be nonnegative, spherically symmetric, and have a finite range $R_0$. Its scattering length will be denoted by $a$. (For the definition of scattering length, see \cite{LSY1} or \cite{LY2}.) The finiteness of the range will be assumed for simplicity, but this assumption can be relaxed. Note that we do not demand $v$ to be locally integrable; it is allowed to have a hard core, which forces the wave functions to vanish whenever two particles are close together. In the following, we want to let $a$ vary with $N$, and we do this by scaling, i.e., we write $v(x) = v_1(x/a)/a^2$, where $v_1$ has scattering length 1, and keep $v_1$ fixed when varying $a$. The ground state energy of $H$, denoted by $E_{QM}$, then depends only on $N$ and $a$ (for fixed $V$ and $v_1$), so the notation $E_{QM}(N,a)$ is justified. The results below are independent of the particular shape of the interaction potential $v_1$.

The Gross-Pitaevskii functional is given by

$$\mathcal{E}^{GP}[\phi] = \int \left( |\nabla \phi(x)|^2 + V(x)|\phi(x)|^2 + g|\phi(x)|^4 \right) d^3x.$$ \[1.2\]

Here $g$ is a positive parameter that is related to the particle number and the scattering length of the interaction potential appearing in \[[1.1]\] via

$$g = 4\pi N\alpha.$$ \[1.3\]

We denote by $\phi^{GP}$ the minimizer of $\mathcal{E}^{GP}$ under the normalization condition $\int |\phi|^2 = 1$. Existence, uniqueness, and some regularity properties of $\phi^{GP}$ were proved in the appendix of \cite{LSY1}. In particular, $\phi^{GP}$ is continuously differentiable and strictly positive. Of course $\phi^{GP}$ depends on $g$, but we omit this dependence for simplicity of notation. For later use, we define the projector

$$P^{GP} = |\phi^{GP}\rangle \langle \phi^{GP}|.$$ \[1.4\]

The minimizer $\phi^{GP}$ fulfills the variational equation

$$-\Delta \phi^{GP} + V\phi^{GP} + 2g(\phi^{GP})^3 = \mu^{GP} \phi^{GP},$$ \[1.5\]
which is called the GP equation. Here \( \mu^{GP} \) is the chemical potential, given by

\[
\mu^{GP} = E^{GP} + g \int |\phi^{GP}|^4,
\]

with \( E^{GP} = E^{GP}(g) \) the lowest energy of \( E^{GP} \) under the condition \( \int |\phi|^2 = 1 \).

It was shown in \[LSY1\] (see Theorem 2 below) that, for each fixed \( g \), the minimization of the GP functional correctly reproduces the large \( N \) asymptotics of the ground state energy and density of \( H \) – but no assertion about BEC in this limit was made. This was extended in \[LS\], where complete BEC was proved. The precise statement is as follows.

Let \( \Psi \) denote the nonnegative and normalized ground state of \( H \). BEC refers to the reduced one-particle density matrix

\[
\gamma(x, x') = N \int \Psi(x, X)\Psi(x', X)dX,
\]

where we denoted \( X = (x_2, \ldots, x_N) \) and \( dX = \prod_{j=2}^{N} dx_j \) for short. It is the kernel of a positive trace class operator \( \gamma \) on \( L^2(\mathbb{R}^3) \), with \( \text{Tr}[\gamma] = N \).

Complete (or 100%) BEC is defined to be the property that

\[
\frac{1}{N} \gamma(x, x') \text{ becomes a simple product } \phi(x')\phi(x') \text{ as } N \to \infty,
\]

In which case \( \phi \) is called the condensate wave function. I.e., \( \frac{1}{N} \gamma \) converges to a one-dimensional projection. In particular, there is one eigenvalue of \( \gamma \) of the order \( N \) (even equal to \( N \) in the limit), and all the others are of lower order. In the GP limit, i.e., \( N \to \infty \) with \( g = 4\pi Na \) fixed, we can show that this is the case, and the condensate wave function is, in fact, the GP minimizer \( \phi^{GP} \).

**Theorem 1 (Bose-Einstein Condensation).** For each fixed \( g \)

\[
\lim_{N \to \infty} \frac{1}{N} \gamma(x, x') = \phi^{GP}(x)\phi^{GP}(x')
\]

in trace class norm, i.e., \( \text{Tr}[\gamma/N - P^{GP}] \to 0 \).

**Remark 1.** Theorem 1 implies that there is 100% condensation for all \( n \)-particle reduced density matrices of \( \Psi \), i.e., they converge to the one-dimensional projector onto the corresponding \( n \)-fold product of \( \phi^{GP} \). See \[LS\] for details.

**Remark 2.** Convergence does not hold for the Sobolev norm \( \text{Tr}[(1-\Delta)(\gamma/N - P^{GP})] \). It can be shown \[LS\] that

\[
\lim_{N \to \infty} \frac{1}{N} \text{Tr}[\Delta \gamma] = \text{Tr}[\Delta P^{GP}] + gs \int |\phi^{GP}|^4
\]

for some parameter \( 0 < s \leq 1 \) depending only on the interaction potential \( v_1 \).

A corollary of Theorem 1, important for the interpretation of experiments, concerns the momentum distribution of the ground state.

**Corollary 1 (Convergence of momentum distribution).** Let

\[
\hat{\rho}(k) = \int \gamma(x, x') \exp[i(k \cdot (x - x'))]dx'dx
\]

denote the one-particle momentum density of \( \Psi \). Then, for fixed \( g \),

\[
\lim_{N \to \infty} \frac{1}{N} \hat{\rho}(k) = |\hat{\phi}^{GP}(k)|^2
\]

strongly in \( L^1(\mathbb{R}^3) \). Here, \( \hat{\phi}^{GP} \) denotes the Fourier transform of \( \phi^{GP} \).
PROOF. If $\mathcal{F}$ denotes the (unitary) operator ‘Fourier transform’ and if $h$ is an arbitrary $L^\infty$-function, then
\[
\left| \frac{1}{N} \int \hat{\rho} h - \int |\hat{\phi}^{\text{GP}}|^2 h \right| = |\text{Tr}[\mathcal{F}^{-1} (\gamma/N - P^{\text{GP}}) \mathcal{F} h]| \leq \|h\|_\infty \text{Tr}[|\gamma/N - P^{\text{GP}}|],
\]
from which we conclude that
\[
\|\hat{\rho}/N - |\hat{\phi}^{\text{GP}}|^2\|_1 \leq \text{Tr}[|\gamma/N - P^{\text{GP}}|].
\]

(1.12)

(1.13)

It is important to note that in the limit considered $v$ becomes a hard potential of short range. This is the opposite of the usual mean field limit, where the strength of the potential goes to zero while its range tends to infinity.

We also wish to emphasize that in this GP limit the fact that there is 100% condensation does not mean that no significant interactions occur. The kinetic and potential energies can differ markedly from those obtained with a simple variational function that is an $N$-fold product of one-body condensate wave functions. This assertion might seem paradoxical, and the explanation is that near the GP limit the region in which the wave function differs from the condensate function has a tiny volume that goes to zero as $N \to \infty$. Nevertheless, the interaction energy, which is proportional to $N$, resides in this tiny volume.

Before proving Theorem 1, let us state some prior results on which we shall build. These results concern the asymptotic behavior of the ground state energy and density of (1.1) in the limit $N \to \infty$ with $g = 4\pi Na$ fixed. The following Theorem 2 was proved in [LSY1].

**Theorem 2 (Asymptotics of Energy and Density).** Let $\rho(x) = \gamma(x, x)$ denote the density of the ground state of $H$. For fixed $g = 4\pi Na$,
\[
\lim_{N \to \infty} \frac{1}{N} E^{\text{QM}}(N, a) = E^{\text{GP}}(g)
\]
and
\[
\lim_{N \to \infty} \frac{1}{N} \rho(x) = |\phi^{\text{GP}}(x)|^2
\]

in the sense of weak convergence in $L^1(\mathbb{R}^3)$.

**Remark 3.** The convergence in (1.13) was shown in [LSY1] to be in the weak $L^1(\mathbb{R}^3)$ sense, but our result here implies strong convergence, in fact. This can easily be deduced from Theorem 2 (cf. the proof of Corollary 1).

We now give a brief outline of the proof of Theorem 1. There are two essential ingredients. The first is a proof that the part of the kinetic energy that is associated with the interaction $v$ is mostly located in small balls surrounding each particle. More precisely, these balls can be taken to have radius $N^{-7/17}$, which is much smaller than the mean particle spacing $N^{-1/3}$. This allows us to conclude that the function of $x$ defined for each fixed value of $X$ by
\[
f_X(x) = \frac{1}{\phi^{\text{GP}}(X)} \psi(x, X) \geq 0
\]
has the property that $\nabla_x f_X(x)$ is almost zero outside the small balls centered at points of $X$. This is made precise in Lemma 3.
The complement of the small balls has a large volume but it can be a weird set; it need not even be connected. Therefore, the smallness of \( \nabla_x f_X(x) \) in this set does not guarantee that \( f_X(x) \) is nearly constant (in \( x \)), or even that it is continuous.

We need \( f_X(x) \) to be nearly constant in order to conclude BEC. What saves the day is the knowledge that the total kinetic energy of \( f_X(x) \) (including the balls) is not huge. The result that allows us to combine these two pieces of information in order to deduce the almost constancy of \( f_X(x) \) is the generalized Poincaré inequality in Lemma 1.

2. The Proof

We start with the following Lemma. It shows that to leading order all the interaction energy is concentrated in small balls. The proof we give here is a detailed version of the one sketched in \( [LS] \). Instead of referring to the methods of \( [LSY1] \), we follow a slightly different route and use the (partly simpler) methods of the review article \( [LSSoY] \).

Throughout the paper we suppress the dependence on \( N \) for simplicity of notation. For instance, in Eq. (2.2) both \( X \) and \( f_X(x) \) depend on \( N \).

LEMMA 1 (Localization of the Energy). For fixed \( X \) let

\[
\Omega_X = \left\{ x \in \mathbb{R}^3 \left| \min_{k \geq 2} |x - x_k| \geq N^{-7/17} \right. \right\} .
\] (2.1)

Then

\[
\lim_{N \to \infty} \int dX \int_{\Omega_X} d^3x |\phi_{\text{GP}}(x)|^2 |\nabla_x f_X(x)|^2 = 0 .
\] (2.2)

Note that (2.2) does not imply that the kinetic energy of \( f_X \) goes to zero. In fact, it can be shown \( [LS, CS] \) that if one replaces \( \Omega_X \) by \( \mathbb{R}^3 \) in (2.2), the result is

\[
\lim_{N \to \infty} \int dX \int_{\mathbb{R}^3} d^3x |\phi_{\text{GP}}(x)|^2 |\nabla_x f_X(x)|^2 = g s \int |\phi_{\text{GP}}(x)|^4 d^3x .
\] (2.3)

(compare with (1.9)). Here \( 0 < s \leq 1 \) is a parameter depending only on the interaction potential \( v \). (For example, \( s = 1 \) in the case of hard core Bosons; in general, \( s < 1 \).) The right side of (2.3) is \( O(1) \) in the limit considered, and is not necessarily small. Thus Lemma 1 shows that all the interaction energy is localized in small balls of radius \( N^{-7/17} \) surrounding each particle.

PROOF OF LEMMA 1. We shall show that

\[
\int dX \int_{\Omega^c_X} d^3x |\phi_{\text{GP}}(x)|^2 |\nabla_x f_X(x)|^2 
+ \int dX \int d^3x |\phi_{\text{GP}}(x)|^2 |f_X(x)|^2 \left[ \frac{1}{2} \sum_{k \geq 2} v(x - x_k) - 2g|\phi_{\text{GP}}(x)|^2 \right] 
geq -g \int |\phi_{\text{GP}}(x)|^4 d^3x - o(1)
\] (2.4)

as \( N \to \infty \). Here \( \Omega^c_X \) is the complement of \( \Omega_X \) in \( \mathbb{R}^3 \). We claim that this implies the assertion of the Lemma. To see this, note that the left side of (2.4) can be
written as
\[
\frac{1}{N} E^{QM} - \mu_{\text{GP}} - \int d\mathbf{X} \int_{\Omega_{\mathbf{X}}} d^3\mathbf{x} |\phi_{\text{GP}}(\mathbf{x})|^2 |\nabla_{\mathbf{x}} f_{\mathbf{X}}(\mathbf{x})|^2 ,
\] (2.5)

where we used partial integration and the GP equation (1.5), and also the symmetry of \(\Psi\). The convergence of the energies in Theorem 2 and the relation (1.6) now imply the desired result.

We are left with the proof of (2.4). This is actually just a detailed examination of the lower bounds to the energy derived in [LSY1] and [LY1], and we use the methods from there. The proof presented here follows closely [LSSoY] and differs slightly from [LSY1].

Writing \(f_{\mathbf{X}}(\mathbf{x}) = \Pi_{k \geq 2} \phi_{\text{GP}}(\mathbf{x}_k) F(\mathbf{x}, \mathbf{X})\) and using that \(F\) is symmetric in the particle coordinates, we see that (2.4) is equivalent to
\[
\frac{1}{N} Q(F) \geq -g \int |\phi_{\text{GP}}|^4 - o(1) ,
\] (2.6)

where \(Q\) is the quadratic form
\[
Q(F) = \sum_{i=1}^N \int_{\Omega_i^c} |\nabla_i F|^2 \prod_{k=1}^N |\phi_{\text{GP}}(\mathbf{x}_k)|^2 d^3\mathbf{x}_k,
\]
\[
+ \sum_{i=1}^N \left[ \sum_{j \neq i}^N v(\mathbf{x}_i - \mathbf{x}_j) - 2g |\phi_{\text{GP}}(\mathbf{x}_i)|^2 \right] |F|^2 \prod_{k=1}^N |\phi_{\text{GP}}(\mathbf{x}_k)|^2 d^3\mathbf{x}_k ,
\] (2.7)

with \(\Omega_i^c = \{ (\mathbf{x}_1, \mathbf{X}) \in \mathbb{R}^{3N} | \min_{k \neq i} |\mathbf{x}_i - \mathbf{x}_k| \leq N^{-7/17}\}\).

While (2.6) is not true for all conceivable \(F\)’s satisfying the normalization condition
\[
\int |F|^2 \prod_{k=1}^N |\phi_{\text{GP}}(\mathbf{x}_k)|^2 d^3\mathbf{x}_k = 1 ,
\] (2.8)

it is true for an \(F\), such as ours, that has bounded kinetic energy (2.3). In fact, we will show that
\[
\frac{1}{N} \left[ Q(F) + \varepsilon \sum_{i=1}^N \int_{\mathbb{R}^3N} |\nabla_i F|^2 \prod_{k=1}^N |\phi_{\text{GP}}(\mathbf{x}_k)|^2 d^3\mathbf{x}_k \right] \geq -g \int |\phi_{\text{GP}}|^4 - o(1) ,
\] (2.9)

for all \(F\) satisfying (2.8), with \(\varepsilon = o(1)\) as \(N \to \infty\).

To estimate the left side of (2.9) from below, we divide space into boxes of side length \(L\) labeled by \(\alpha\), and distribute the particles over the boxes. If we use Neumann boundary conditions in each box, this can only lower the energy, since we effectively allow discontinuous functions. Moreover, since \(v\) is positive, we can neglect interactions among particles in different boxes for the lower bound. To be precise, the left side of (2.9) is bounded below by
\[
\frac{1}{N} \inf_{\{n_{\alpha}\}} \sum_{\alpha} \inf_{F_{\alpha}} \frac{\tilde{Q}_{\alpha}(F_{\alpha})}{\|F_{\alpha}\|^2} ,
\] (2.10)
where the infimum is over all distributions of \( n_\alpha \) particles in the boxes \( \alpha \), under the constraint that \( \sum_\alpha n_\alpha = N \), and

\[
\tilde{Q}_\alpha(F_\alpha) = \sum_{i=1}^{n_\alpha} \left[ \int_{\alpha \cap \Omega_i^\epsilon} |\nabla_i F_\alpha|^2 + \varepsilon \int_\alpha |\nabla_i F_\alpha|^2 \right] \prod_{k=1}^{n_\alpha} |\phi^{GP}(x_k)|^2 d^3x_k
\]

\[
+ \sum_{i=1}^{n_\alpha} \int_\alpha \left( \sum_{j \neq i} v(x_i - x_j) - 2g|\phi^{GP}(x_i)|^2 \right) |F_\alpha|^2 \prod_{k=1}^{n_\alpha} |\phi^{GP}(x_k)|^2 d^3x_k . \tag{2.11}
\]

Here all the integrals are restricted to the box \( \alpha \), and \( F_\alpha \) is a function of \( n_\alpha \) variables.

We now fix some \( M > 0 \), that will eventually tend to \( \infty \), and restrict ourselves to boxes \( \alpha \) inside a cube \( \Lambda_M \) of side length \( M \). Since \( v \geq 0 \) the contribution to \( \ref{eq:2.11} \) of boxes outside this cube is easily estimated from below by \(-2gN\sup_{x \notin \Lambda_M} |\phi^{GP}(x)|^2\), which, divided by \( N \), is arbitrarily small for \( M \) large, since \( \phi^{GP} \) decreases faster than exponentially at infinity (\cite{LY1}, Lemma A.5).

For the boxes inside the cube \( \Lambda_M \) we want to use the results on the homogeneous Bose gas obtained in \cite{LY1}, and therefore we must approximate \( \phi^{GP} \) by constants in each box. Let \( \rho_{\max} \) and \( \rho_{\min} \), respectively, denote the maximal and minimal values of \( |\phi^{GP}|^2 \) in box \( \alpha \). Define

\[
\Psi_\alpha(x_1, \ldots, x_{n_\alpha}) = F_\alpha(x_1, \ldots, x_{n_\alpha}) \prod_{k=1}^{n_\alpha} \phi^{GP}(x_k) , \tag{2.12}
\]

and

\[
\Psi^{(i)}_\alpha(x_1, \ldots, x_{n_\alpha}) = F_\alpha(x_1, \ldots, x_{n_\alpha}) \prod_{k \neq i}^{n_\alpha} \phi^{GP}(x_k) . \tag{2.13}
\]

We have, for all \( 1 \leq i \leq n_\alpha \),

\[
\left[ \int_{\alpha \cap \Omega_i^\epsilon} |\nabla_i F_\alpha|^2 + \frac{1}{2} \sum_{j \neq i} \int_\alpha v(x_i - x_j)|F_\alpha|^2 \right] \prod_{k=1}^{n_\alpha} |\phi^{GP}(x_k)|^2 d^3x_k
\]

\[
\geq \rho_{\min} \left[ \int_{\alpha \cap \Omega_i^\epsilon} |\nabla_i \Psi^{(i)}_\alpha|^2 + \frac{1}{2} \sum_{j \neq i} \int_\alpha v(x_i - x_j)|\Psi^{(i)}_\alpha|^2 \right] \prod_{k=1}^{n_\alpha} d^3x_k . \tag{2.14}
\]

We now use the following Lemma, which was proved in \cite{LY1}. It allows us to replace \( v \) by a ‘soft’ potential, at the cost of sacrificing kinetic energy and increasing the effective range.

**Lemma 2.** Let \( v(x) \geq 0 \) with finite range \( R_0 \). Let \( U(r) \geq 0 \) be any function satisfying \( \int U(r)r^2dr \leq 1 \) and \( U(r) = 0 \) for \( r < R_0 \). Let \( B \subset \mathbb{R}^3 \) be star shaped with respect to 0 (e.g. convex with 0 \( \in B \)). Then for all functions \( \psi \)

\[
\int_B [ |\nabla \psi|^2 + \frac{1}{2} v|\psi|^2 ] \geq a \int_B |U \psi|^2 . \tag{2.15}
\]

By dividing \( \alpha \) for given points \( x_1, \ldots, x_{n_\alpha} \) into Voronoi cells \( B_i \) that contain all points closer to \( x_i \) than to \( x_j \) with \( j \neq i \) (these cells are star shaped w.r.t. \( x_i \), indeed convex), and choosing a \( U \) with radius \( R \leq N^{-7/17} \), we see that \( \ref{eq:2.14} \) is
bounded below by
\[ (2.14) \geq a \rho_{\min} \int_a U(t_i)|\Psi^{(i)}_\alpha|^2 \geq a \frac{\rho_{\min}}{\rho_{\max}} \int_a U(t_i)|\Psi_\alpha|^2 , \]
where \( t_i \) is the distance of \( x_i \) to its nearest neighbor among the other points \( x_j, j = 1, \ldots, n_\alpha \), i.e.,
\[ t_i(x_1, \ldots, x_{n_\alpha}) = \min_{j, j \neq i} |x_i - x_j| . \]
As in [LY1] we choose for \( U \) the potential
\[ U(r) = \begin{cases} 
3(R^3 - R_0^3)^{-1} & \text{for } R_0 < r < R \\
0 & \text{otherwise} \end{cases} , \]
with \( R \) determined by \( \Omega_X \) as \( R = N^{-7/17} \). Note that \( R \gg R_0 \) for \( N \) large enough, since \( R_0 = O(N^{-1}) \) in the limit considered.

Since \( \Psi_\alpha = \phi^{GP}(x_i)\Psi^{(i)}_\alpha \) we can estimate
\[ |\nabla_i \Psi_\alpha|^2 \leq 2 \rho_{\max}|\nabla_i \Psi^{(i)}_\alpha|^2 + 2|\Psi^{(i)}_\alpha|^2 C_M \]
with
\[ C_M = \sup_{x \in \Lambda_M} |\nabla \phi^{GP}(x)|^2 . \]
Inserting (2.19) into (2.11), summing over \( i \) and using \( \rho^{GP}(x_i) \leq \rho_{\max} \) in the last term of (2.11), we get
\[ \tilde{Q}(F_\alpha) \geq \frac{\rho_{\min}}{\rho_{\max}} E_{\varepsilon}(n_\alpha, L) - 2g\rho_{\max} n_\alpha - \varepsilon C_M n_\alpha , \]
where \( E_{\varepsilon}(n_\alpha, L) \) is the ground state energy of
\[ \sum_{i=1}^{n_\alpha} \left( -\frac{1}{2} \varepsilon \Delta_i + aU(t_i) \right) \]
in a box of side length \( L \). We want to minimize (2.21) with respect to \( n_\alpha \), and drop the subsidiary condition \( \sum_\alpha n_\alpha = N \) in (2.10). This can only lower the minimum. For the time being we also ignore the last term in (2.21). The total contribution of this term for all boxes is bounded by \( \varepsilon C_M N \) and will turn out to be negligible compared to the other terms.

It was shown in [LY1] (see also [LSSoY]) that
\[ E_{\varepsilon}(n_\alpha, L) \geq \frac{4\pi a^2 n_\alpha^2}{L^3}(1 - CY_\alpha^{1/17}) \]
with \( Y_\alpha = a^3 n_\alpha / L^3 \), provided \( \varepsilon \geq Y_\alpha^{1/17} \) and \( n_\alpha \geq (\text{const.})Y_\alpha^{-1/17} \). The condition on \( \varepsilon \) is certainly fulfilled if we choose \( \varepsilon = Y_\alpha^{1/17} \) with \( Y = a^3 N / L^3 \). We now want to show that the \( n_\alpha \) minimizing the right side of (2.21) is large enough for (2.23) to apply.

If the minimum of the right side of (2.21) (without the last term) is taken for some \( \bar{n}_\alpha \), we have
\[ \frac{\rho_{\min}}{\rho_{\max}} \left( E_{\varepsilon}(\bar{n}_\alpha + 1, L) - E_{\varepsilon}(\bar{n}_\alpha, L) \right) \geq 2g\rho_{\max} . \]
On the other hand, we claim that
Lemma 3. For any $n$
\[
E_n^U(n+1, L) - E_n^U(n, L) \leq 8 \pi a \frac{n}{L^3}.
\] (2.25)

Proof. Denote the operator (2.22) by $	ilde{H}$, with $n_\alpha = n$, and let $	ilde{\Psi}_n$ be its ground state. Let $t_i'$ be the distance to the nearest neighbor of $x_i$ among the $n+1$ points $x_1, \ldots, x_{n+1}$ (without $x_i$) and $t_i$ the corresponding distance excluding $x_{n+1}$. Obviously, for $1 \leq i \leq n$,
\[
U(t_i') \leq U(t_i) + U(|x_i - x_{n+1}|) \quad (2.26)
\]
and
\[
U(t_{n+1}') \leq n \sum_{i=1}^{n} U(|x_i - x_{n+1}|). \quad (2.27)
\]
Therefore
\[
\tilde{H}_{n+1} \leq \tilde{H}_n - \frac{1}{2} \epsilon \Delta_{n+1} + 2a \sum_{i=1}^{n} U(|x_i - x_{n+1}|). \quad (2.28)
\]
Using $	ilde{\Psi}_n/L^{3/2}$ as trial function for $\tilde{H}_{n+1}$ we arrive at (2.23). We shall choose $L \sim N^{-1/10}$, so the conditions needed for (2.23) are fulfilled for $N$ large enough, since $\rho_{\text{max}} = O(1)$ and hence $n_\alpha \sim N^{7/10}$ and $Y_\alpha \sim N^{-2}$.

In order to obtain a lower bound on (2.21), we can use $Y_\alpha \leq Y$ in the error term in (2.23). We therefore have to minimize
\[
4 \pi a \left( \frac{\rho_{\text{min}} n_\alpha^2}{L^3} \left( 1 - CY^{1/17} \right) - 2n_\alpha N \rho_{\text{max}} \right), \quad (2.29)
\]
and we can drop the requirement that $n_\alpha$ has to be an integer. The minimum of (2.29) is obtained for
\[
n_\alpha = \frac{\rho_{\text{max}}}{\rho_{\text{min}}} \frac{NL^3}{(1 - CY^{1/17})} \quad (2.30)
\]
Using (2.10), (2.21) and (2.23) this gives the following lower bound on the left side of (2.9), including now the last term in (2.21) as well as the contributions from the boxes outside $\Lambda_M$:
\[
-g \sum_{\alpha \subset \Lambda_M} \rho_{\text{min}}^2 L^3 \left( \frac{\rho_{\text{max}}}{\rho_{\text{min}}} \right)^3 \frac{1}{(1 - CY^{1/17})} - 2g \sup_{x \notin \Lambda_M} |\phi^{\text{GP}}(x)|^2 - \epsilon C_M. \quad (2.31)
\]
Now $\phi^{\text{GP}}$ is differentiable and strictly positive. Since all the boxes are in the fixed cube $\Lambda_M$ there are constants $C' < \infty$, $C'' > 0$, such that
\[
\rho_{\text{max}} - \rho_{\text{min}} \leq C' L, \quad \rho_{\text{min}} \geq C''. \quad (2.32)
\]
Since $L \sim N^{-1/10}$ and $Y \sim N^{-17/10}$ we therefore have, for large $N$,
\[
\frac{\rho_{\text{max}}^3}{\rho_{\text{min}}^3} \frac{1}{(1 - CY^{1/17})} \leq 1 + (\text{const}) N^{-1/10}. \quad (2.33)
\]
Also,
\[
g \sum_{\alpha \subset \Lambda_M} \rho_{\text{min}}^2 L^3 \leq g \int |\phi^{\text{GP}}|^4. \quad (2.34)
\]
Hence, for some constant depending only on \( g \) and \( M \),
\[
(2.31) \geq -g \int |\phi^{GP}|^4 - (\text{const.}) N^{-1/10} - 2g \sup_{x \notin \Lambda_M} |\phi^{GP}(x)|^2 .
\]
This proves the desired result, and finishes the proof of Lemma 1.

In the following, \( \mathcal{K} \subset \mathbb{R}^m \) denotes a bounded and connected set that is sufficiently nice so that the Poincaré-Sobolev inequality (see [LLo, Thm. 8.12]) holds on \( \mathcal{K} \). In particular, this is the case if \( \mathcal{K} \) satisfies the cone property [LLo] (e.g., if \( \mathcal{K} \) is a ball or a cube). The following Lemma is a generalization of the Poincaré inequality. It can be further generalized to the \( L^p \) case, and, with a different and more complicated proof, to the case of magnetic fields [LSY3].

**Lemma 4 (Generalized Poincaré Inequality).** For \( m \geq 2 \) let \( \mathcal{K} \subset \mathbb{R}^m \) be as explained above, and let \( h \) be a bounded function with \( \int_{\mathcal{K}} h = 1 \). There exists a constant \( C \) (depending only on \( \mathcal{K} \) and \( h \)) such that for all sets \( \Omega \subset \mathcal{K} \) and all \( f \in H^1(\mathcal{K}) \) with \( \int_{\mathcal{K}} fh \, d^m x = 0 \), the inequality
\[
\int_{\Omega} |\nabla f(x)|^2 d^m x + \left( \frac{|\Omega|}{|\mathcal{K}|} \right)^{2/m} \int_{\mathcal{K}} |\nabla f(x)|^2 d^m x \geq \frac{1}{C} \int_{\mathcal{K}} |f(x)|^2 d^m x
\]  
holds. Here \( |\cdot| \) is the volume of a set, and \( \Omega^c = \mathcal{K} \setminus \Omega \).

**Proof.** By the usual Poincaré-Sobolev inequality on \( \mathcal{K} \) (see [LLo, Thm. 8.12]),
\[
\|f\|_{L^2(\mathcal{K})}^2 \leq \tilde{C} \|\nabla f\|_{L^{2m/(m+2)}(\mathcal{K})}^2
\]
for some constant \( \tilde{C} \), if \( m \geq 2 \) and \( \int_{\mathcal{K}} fh = 0 \). Using the triangle inequality we can estimate
\[
\|f\|_{L^2(\mathcal{K})}^2 \leq 2\tilde{C} \left( \|\nabla f\|_{L^{2m/(m+2)}(\Omega)}^2 + \|\nabla f\|_{L^{2m/(m+2)}(\Omega^c)}^2 \right) .
\]
Applying Hölder’s inequality
\[
\|\nabla f\|_{L^{2m/(m+2)}(\Omega)} \leq \|\nabla f\|_{L^2(\Omega)} |\Omega|^{1/m}
\]
(and the analogue with \( \Omega \) replaced by \( \Omega^c \)), we see that (2.36) holds with \( C = 2|\mathcal{K}|^{2/m} \tilde{C} \).

The important point in Lemma 4 is that there is no restriction on \( \Omega \) concerning regularity or connectivity. Combining the results of Lemmas 1 and 4, we now are able to prove Theorem 1.

**Proof of Theorem 1.** For some \( M > 0 \) let \( \mathcal{K} = \{ x \in \mathbb{R}^3, |x| \leq M \} \), and define
\[
\langle f x \rangle_{\mathcal{K}} = \frac{1}{\int_{\mathcal{K}} |\phi^{GP}(x)|^2 d^3 x} \int_{\mathcal{K}} |\phi^{GP}(x)|^2 f x(x) d^3 x .
\]
We shall use Lemma 1 with \( m = 3 \), \( h(x) = |\phi^{GP}(x)|^2 / \int_{\mathcal{K}} |\phi^{GP}|^2 \), \( \Omega = \Omega x \cap \mathcal{K} \) and \( f(x) = f x(x) - \langle f x \rangle_{\mathcal{K}} \) (see (2.1) and (1.16)). Since \( \phi^{GP} \) is bounded on \( \mathcal{K} \) above and
below by some positive constants, this Lemma also holds (with a different constant \( C' \)) with \( d^3x \) replaced by \( |\phi^{GP}(x)|^2 d^3x \) in (2.30). Therefore,

\[
\int dX \int_{\mathcal{K}} d^3x |\phi^{GP}(x)|^2 [f(x) - \langle f(x) \rangle_{\mathcal{K}}]^2 \\
\leq C' \int dX \left[ \int_{\Omega \cap \mathcal{K}} |\phi^{GP}(x)|^2 |\nabla_x f(x)|^2 d^3x \\
+ \frac{N^{-8/51}}{M^2} \int_{\mathcal{K}} |\phi^{GP}(x)|^2 |\nabla_x f(x)|^2 d^3x \right],
\]

(2.41)

where we used that \(|\Omega \cap \mathcal{K}| \leq \frac{4\pi}{3} N^{-4/17} \). The first integral on the right side of (2.41) tends to zero as \( N \to \infty \) by Lemma 1, and the second is bounded by (2.3). We conclude that

\[
\lim_{N \to \infty} \int dX \int_{\mathcal{K}} d^3x |\phi^{GP}(x)|^2 [f(x) - \langle f(x) \rangle_{\mathcal{K}}]^2 = 0.
\]

(2.42)

Moreover, since

\[
\int_{\mathcal{K}} |\phi^{GP}(x)|^2 f(x) d^3x \leq \int_{\mathbb{R}^3} |\phi^{GP}(x)|^2 f(x) d^3x
\]

by the positivity of \( f \),

\[
\frac{1}{N} \langle \phi^{GP} | \gamma | \phi^{GP} \rangle \geq \left[ \int_{\mathcal{K}} |\phi^{GP}(x)|^2 d^3x \right]^2 \int dX \langle f(x) \rangle_{\mathcal{K}}^2.
\]

(2.44)

Hence, by (2.42),

\[
\liminf_{N \to \infty} \frac{1}{N} \langle \phi^{GP} | \gamma | \phi^{GP} \rangle \geq \int_{\mathcal{K}} |\phi^{GP}(x)|^2 d^3x \lim_{N \to \infty} \int dX \int_{\mathcal{K}} d^3x |\Psi(x, X)|^2.
\]

(2.45)

It follows from (1.11) that the right side of this inequality equals \( \left[ \int_{\mathcal{K}} |\phi^{GP}(x)|^2 d^3x \right]^2 \). Since the radius of \( \mathcal{K} \) was arbitrary, we conclude that

\[
\lim_{N \to \infty} \frac{1}{N} \langle \phi^{GP} | \gamma | \phi^{GP} \rangle = 1,
\]

(2.46)

implying convergence of \( \gamma/N \) to \( P^{GP} \) in Hilbert-Schmidt norm. Since the traces are equal, convergence even holds in trace class norm (cf. [13], Thm. 2.20), and Theorem 1 is proven.

\[\square\]

Throughout the paper we were dealing with Bosons in three-dimensional space. However, the method presented here also works in the case of a 2D Bose gas. The relevant parameter to be kept fixed in the GP limit is \( g = 4\pi N/|\ln(a^2 N)| \), all other considerations carry over without essential change, using the results in [LSY2, LY2]. We also point out that our method necessarily fails for the 1D Bose gas, where there is no BEC in the ground state [PiSt]. An analogue of Lemma 1 cannot hold in the 1D case since even a hard core potential with arbitrarily small range produces an interaction energy that is not localized on scales smaller than the mean particle spacing.
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