A NOTE ON SOME SYSTEMS OF GENERALIZED SYLVESTER EQUATIONS *

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Abstract. In this paper, we study two systems of generalized Sylvester operator equations. We derive necessary and sufficient conditions for the existence of a solution and provide the general form of a solution. We extend some recent results to more general settings.

Key words: Sylvester equations, generalized inverses, Matrix equations and identities

1. Introduction

Let $H, K, F, G, L, M, N$ be complex Hilbert spaces and let $B(H, K)$ denote the set of all bounded linear operators from $H$ to $K$. For a given $A \in B(H, K)$, the symbols $N(A)$ and $R(A)$ denote the null space and the range of operator $A$, respectively. The identity operator is always denoted by $I$. If $A \in B(H, K)$ has a closed range, then there exists unique operator $X \in B(K, H)$ satisfying the following equations

\begin{align*}
(1) \quad AXA &= A \\
(2) \quad XAX &= X \\
(3) \quad (AX)^* &= AX \\
(4) \quad (XA)^* &= XA.
\end{align*}

Such operator is called the Moore-Penrose inverse of an operator $A \in B(H, K)$ which is denoted by $A^\dagger$. If $X \in B(K, H)$ satisfies the equation (1), i.e. $AXA = A$, then $X$ is an inner generalized inverse of $A$, and is usually denoted by $A^\perp$. For $A \in B(H, K)$ there exists a Moore-Penrose inverse, if and only if there exists its

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inner generalized inverse if and only if $\mathcal{R}(A)$ is closed. In this case, we say that $A$ is regular. Furthermore, $L_A$ and $R_A$ stand for two projections $L_A = I - A^\dagger A$ and $R_A = I - AA^\dagger$, induced by $A$, respectively.

In this paper, we study two systems of generalized Sylvester operator equations

\begin{align}
(1.1) \quad & A_1 X_1 - X_2 B_1 = C_1, \quad A_2 X_3 - X_2 B_2 = C_2, \\
(1.2) \quad & A_1 X_1 - X_2 B_1 = C_1, \quad A_2 X_2 - X_3 B_2 = C_2,
\end{align}

where $A_1 \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $B_1 \in \mathcal{B}(\mathcal{F}, \mathcal{G})$, $C_1 \in \mathcal{B}(\mathcal{F}, \mathcal{K})$, $A_2 \in \mathcal{B}(\mathcal{M}, \mathcal{K})$, $B_2 \in \mathcal{B}(\mathcal{L}, \mathcal{G})$, $C_2 \in \mathcal{B}(\mathcal{L}, \mathcal{K})$, and $C_2 \in \mathcal{B}(\mathcal{G}, \mathcal{M})$.

Systems of such type of matrix equations have been considered by many authors [3, 4, 5, 6, 7]. In this paper, we extended recent results [7] on systems of quaternion matrix equations to infinite dimensional settings and provide much simpler proofs to existing conditions.

2. Main results

The following two lemmas play a key role in this paper:

**Lemma 2.1.** [1] Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $B \in \mathcal{B}(\mathcal{F}, \mathcal{G})$ and $C \in \mathcal{B}(\mathcal{F}, \mathcal{K})$ be such that $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed. Then the operator equation

$$AXB = C$$

is consistent if and only if

$$AA^{-} CB^{-} B = C,$$

for some $A^{-}$ and $B^{-}$, in which case the general solution is given by

$$X = A^{-} CB^{-} + Y - A^{-} AY BB^{-},$$

for arbitrary $Y \in \mathcal{B}(\mathcal{G}, \mathcal{H})$.

**Lemma 2.2.** [2] Let $E, F, G, D, N, M$ be Banach spaces. Let $A_1 \in \mathcal{B}(F, E)$, $A_2 \in \mathcal{B}(F, N)$, $B_1 \in \mathcal{B}(D, G)$, $B_2 \in \mathcal{B}(M, G)$ and

$$T := (I_G - B_1 B_1^{-}) B_2 \quad \text{and} \quad S := A_2 (I_F - A_1^{-} A_1)$$

be all regular. Moreover, let $A_1 A_1^{-} C_1 B_1^{-} B_1 = C_1$ and $A_2 A_2^{-} C_2 B_2^{-} B_2 = C_2$. Then the equations

$$A_1 XB_1 = C_1 \quad \text{and} \quad A_2 XB_2 = C_2$$

have a common solution if and only if

$$(I_N - SS^{-}) C_2 (I_M - T^{-} T) = (I_N - SS^{-}) A_2 A_1^{-} C_1 B_1^{-} B_2 (I_M - T^{-} T).$$
In this case, the general common solution is given by
\[ X = (A_1^+ C_1 - (I_F - A_1^+ A_1)S^-(A_2A_1^+ C_1 - W))B_1^-(I_G - B_2T^-(I_G - B_1B_1^-)) + ((I_F - (I_F - A_1^+ A_1)S^- A_2)A_1^+ V + (I_F - A_1^+ A_1)S^- C_2)T^-(I_G - B_1B_1^-) + Z - (A_1^- A_1 + (I_F - A_1^+ A_1)S^+ S)Z(B_1B_1^- + TT^-(I_G - B_1B_1^-)), \]

where
\[ V = C_1B_1^- B_2(I_M - T^{-T}) + A_1A_1^- (I_N - SS^-)C_2T^{-T} + A_1A_1^- QT^{-T} - A_1A_2^- (I_N - SS^-)A_2A_1^- QT^{-T}, \]
\[ W = (I_N - SS^-)A_2A_1^- C_1 + SS^- C_2(I_M - T^{-T})B_2 B_1 + SS^- PB_1^- B_1 - SS^- PB_1^- B_2(I_M - T^{-T}) B_2 B_1, \]
in which \( P, Q, Z \) are arbitrary elements of \( \mathcal{B}(D, N), \mathcal{B}(M, E) \) and \( \mathcal{B}(G, F) \), respectively.

Note that in the preceding lemmas, in the solvability conditions and formulas for general solutions, arbitrary inner generalized inverses can be replaced by the Moore-Penrose inverse. For example, in Lemma 2.1, if
\[ AA^- CB^- B = C \]
holds for some \( A^- \) and \( B^- \), then
\[ AA^\dagger CB^\dagger B = AA^\dagger (AA^- CB^- B)B^\dagger B = AA^- CB^- B = C. \]

Conversely, if
\[ AA^\dagger CB^\dagger B = C \]
holds, then for arbitrary \( A^- \) and \( B^- \) it follows
\[ AA^- CB^- B = AA^- (AA^\dagger CB^\dagger B)B^- B = AA^\dagger CB^\dagger B = C. \]

So, for \( A^- \) and \( B^- \) in the solvability conditions and formulas for general solutions, we can choose exactly \( A^\dagger \) and \( B^\dagger \), respectively.

**Theorem 2.1.** Let \( A_1 \in \mathcal{B}(H, K), B_1 \in \mathcal{B}(F, G), C_1 \in \mathcal{B}(F, K), A_2 \in \mathcal{B}(M, K), B_2 \in \mathcal{B}(L, G), C_2 \in \mathcal{B}(L, K) \) be such that \( A_1, A_2, B_1, B_2, S \) and \( T \) are all regular. Put
\[ T = (I - B_1B_1^\dagger)B_2, \quad S = (I - A_2A_2^\dagger)A_1A_1^\dagger, \]
\[ C = (I - A_2A_2^\dagger)(C_2 - (I - A_1A_1^\dagger)C_1B_1^\dagger B_2)(I - T^\dagger T). \]

The following statements are equivalent:
(i) The system (1.1) is consistent;
where

\[ \text{Proof.} \ (i) \Rightarrow (ii): \text{Since the system (1.1) is consistent, there exists } X_2 \in \mathcal{B}(G,K) \text{ such that equations} \]

\[ A_1X_1 - X_2B_1 = C_1 \]
\[ A_2X_2 - X_2B_2 = C_2 \]

are solvable for \( X_1 \) and \( X_3 \), respectively. According to Lemma 2.1 equation

\[ A_1X_1 - X_2B_1 = C_1 \]

is solvable for \( X_1 \) if and only if

\[ (I - A_1A_1^\dagger)(C_1 + X_2B_2) = 0, \]

and equation

\[ A_2X_3 - X_2B_2 = C_2 \]
is solvable for $X_2$ if and only if

$$(I - A_2A_1^T)(C_2 + X_2B_2) = 0.$$  

(2.2)

So, from (2.1) and (2.2) it follows that equations

$$(I - A_1A_1^T)X_2B_1 = -(I - A_1A_1^T)C_1,$$  

(2.3)

$$(I - A_2A_2^T)X_2B_2 = -(I - A_2A_2^T)C_2$$

have a common solution. From Lemma 2.1 and Lemma 2.2 system (2.3) is consistent if and only if

$$(I - A_1A_1^T)C_1(I - B_1^TB_1) = 0,$$  

$$(I - A_2A_2^T)C_2(I - B_2^TB_2) = 0,$$  

$$(I - SS^\dagger)C = 0.$$  

(ii) $\Rightarrow$ (i): If (ii) holds, then by Lemma 2.2 it follows that system (2.3) is consistent. Let $X_2 \in \mathcal{B}(G, K)$ be the solution to the system (2.3) and let $X_1 = A_1^T(X_2B_1 + C_1)$ and $X_3 = A_2^T(X_2B_2 + C_2)$. Then it is easy to see that such $X_1, X_2$ and $X_3$ satisfy (1.1).

(ii) $\Rightarrow$ (iii): Suppose that

$$(I - A_1A_1^T)C_1(I - B_1^TB_1) = 0,$$  

(2.4)

$$(I - A_2A_2^T)C_2(I - B_2^TB_2) = 0$$  

(2.5)

and

$$(I - SS^\dagger)C = 0$$  

(2.6)

hold. From (2.6) we get

$$C(I - (B_2L_T)^\dagger (B_2L_T)) = C(I - (B_2(I - T^\dagger T))^\dagger (B_2(I - T^\dagger T)))$$

$$= (I - A_2A_2^T)C_2(I - T^\dagger T)(I - (B_2(I - T^\dagger T))^\dagger (B_2(I - T^\dagger T)))$$

$$-(I - A_2A_2^T)(I - A_1A_1^T)C_2B_2(I - T^\dagger T)(I - (B_2(I - T^\dagger T))^\dagger (B_2(I - T^\dagger T)))$$

$$= (I - A_2A_2^T)C_2(I - T^\dagger T)(I - (B_2(I - T^\dagger T))^\dagger (B_2(I - T^\dagger T)))$$

$$= (I - A_2A_2^T)C_2B_2(I - T^\dagger T)(I - (B_2(I - T^\dagger T))^\dagger (B_2(I - T^\dagger T)))$$

$$= 0.$$  

(iii) $\Rightarrow$ (ii): Suppose that

$$(I - A_1A_1^T)C_1(I - B_1^TB_1) = 0,$$  

(2.7)
\[(2.8) \quad C(I - (B_2(I - T^T))\diff(B_2(I - T^T))) = 0 \]

and

\[(2.9) \quad (I - SS^\dagger)C = 0 \]

hold. From (2.8) we get

\[
\begin{align*}
R_A^2C_2(I - T\dagger T)(I - (B_2(I - T^T))\diff(B_2(I - T^T))) &= R_A^2R_A^1C_1B_1^\dagger B_2(I - T^T)L_{B_2}(I - T\dagger T) \\
&= 0.
\end{align*}
\]

Note that

\[
\begin{align*}
(I - T\dagger T)B_2 &= (I - (I - B_1B_1^\dagger)B_2)\diffB_2(I - B_2^\dagger B_2) \\
&= I - B_2^\dagger B_2 \\
&= L_{B_2},
\end{align*}
\]

so from (2.11) and (2.10) we get

\[
\begin{align*}
R_A^2C_2L_{B_2} &= R_A^2C_2(I - T\dagger T)L_{B_2} \\
&= R_A^2C_2(I - T\dagger T)(B_2(I - T^T))\diffB_2(I - T\dagger T)L_{B_2} \\
&= R_A^2C_2(I - T\dagger T)(B_2(I - T^T))\diff(I - T\dagger R_{B_1})B_2L_{B_2} \\
&= 0.
\end{align*}
\]

Suppose that system (1.1) is consistent.

Since $X_2 \in \mathcal{B}(G, K)$ is a solution to (1.1) if and only if it satisfies (2.3), its general form, according to Lemma 2.2, is given by

\[
X_2 = (-R_{A_1}C_1 + S\dagger(R_{A_1}C_1 + W))B_1^\dagger(I - B_2T) + ((I - S\dagger)R_{A_2}V - S\dagger C_2)T \dagger + Z - (I - A_1A_1^\dagger + S\dagger S)Z(B_1B_1^\dagger + TT^\dagger),
\]

where $Z$ is an arbitrary element of $\mathcal{B}(G, K)$, and

\[
V = -R_{A_1}C_1B_1^\dagger B_2L_T - R_{A_1}R_{A_2}R_SC_2T^\dagger T \\
+ R_{A_1}Q^\dagger T - R_{A_1}R_{A_2}R_SC_2R_{A_1}Q^\dagger T
\]

and

\[
W = -R_SR_{A_2}R_{A_1}C_1 - SS^\dagger C_2L_TB_2^\dagger B_1 \\
+ SS^\dagger PB_1^\dagger B_1 - SS^\dagger PB_2^\dagger B_2L_TB_1^\dagger B_1,
\]
where $P$ and $Q$ are arbitrary elements of $B(F, K)$ and $B(G, K)$.

From the first equation in (1.1) we have
\[ A_1X_1 = X_2B_1 + C_1, \]
so, by Lemma 2.1 we get
\[ X_1 = A_1^T(X_2B_1 + C_1) + L_{A_1}R, \]
\[ = A_1^T\bigl( (R_{A_1}C_1 + W)B_1^\dagger B_1 + A_1^T S^T S Z B_1 - A_1^T S^T S Z B_1 + A_1^T C_1 + L_{A_1} R, \]
where $R$ is an arbitrary element of $B(F, H)$.

From the second equation in (1.1) we have
\[ A_2X_3 = X_2B_2 + C_2, \]
so, by Lemma 2.1 we get
\[ X_3 = A_2^T(X_2B_2 + C_2) + L_{A_2}Y, \]
\[ = A_2^T(-R_{A_2}C_2 - S^T(R_{A_2}C_1 + W))B_2^\dagger B_2 L_T \]
\[ + A_2^T((I - S^\dagger)R_{A_2}V + S^T C_2)T^\dagger B_2 \]
\[ + A_2^T Z B_2 - A_2^T(I - A_2^\dagger) + S^T S Z (B_1 B_1^\dagger B_2 + T) + A_2^T C_2 + L_{A_2} Y, \]
where $Y$ is an arbitrary element of $B(L, K)$. □

**Theorem 2.2.** Let $A_1 \in B(H, K)$, $B_1 \in B(M, L)$, $C_1 \in B(M, K)$, $A_2 \in B(K, N)$, $B_2 \in B(L, G)$, $C_2 \in B(L, N)$ be such that $A_1$, $A_2$, $B_1$, $B_2$, $S$ and $T$ are all regular.

Put
\[ T = (I - B_1 B_2^\dagger)(I - B_1^\dagger B_2), \quad S = A_2 A_1 A_1^\dagger, \]
\[ C = (I - (A_2 A_1)(A_2 A_1)^\dagger)C_2 + A_2(I - A_2 A_1^\dagger)C_1 B_1^\dagger (I - B_1^\dagger B_2). \]

The following statements are equivalent:

(i) The system (1.2) is consistent;

(ii) $R_{A_1} C_1 L B_1 = 0$, $R_{A_2} C_2 L B_2 = 0$, $C L T = 0$;

(iii) $R_{A_1} C_1 L B_1 = 0$, $(I - R_{A_2} A_1 A_2(R_{A_2} A_1 A_2)\dagger)C = 0$, $C L T = 0$.

In this case, the general solution to the system (1.2) is given by
\[ X_1 = A_1^T S^T A_2 R_{A_1} C_1 + A_1^T S^T W B_1^\dagger B_1 + A_1^T(I - S^\dagger) V B_1 \]
\[ + A_1^T Z B_1 - A_1^T S^T S Z B_1 + A_1^T C_1 + R_{A_1} R, \]
\[ X_2 = (-R_{A_1} C_1 + S^T(A_2 R_{A_1} C_1 + W)) B_2^\dagger(I - T^\dagger) \]
\[ + ((I - S^\dagger A_2) R_{A_2} V + S^T C_2 L B_2) T^\dagger \]
\[ + Z - (R_{A_1} + S^T S Z (B_1 B_1^\dagger B_2 + T T^\dagger), \]
\[ X_3 = A_2 \left(-R_A C_1 + S^\dagger (A_2 R_A C_1 + W)\right) B_1^\dagger (I - T^\dagger) B_2^\dagger + A_2 \left( (I - S^\dagger A_2) R_A V + S^\dagger C_2 L_{B_2} \right) T^\dagger B_2^\dagger + A_2 Z B_2^\dagger - A_2 (R_A + S^\dagger S) Z (B_1 B_2^\dagger + T T^\dagger) B_2^\dagger - C_2 B_2^\dagger + Y R_{B_2}, \]

where
\[ V = -R_A C_1 B_1^\dagger L_{B_2} L_T + R_A QT_T - R_A A_1^\dagger R_S A_2 R_A QT_T \]
and
\[ W = -R_S A_2 R_A C_1 + S S^\dagger C_2 L_{B_2} B_1 + S S^\dagger P B_1^\dagger B_1 - S S^\dagger P B_1^\dagger L_{B_2} B_1 \]

with \( P, Q, Z \) and \( Y \) arbitrary elements of \( B(F, K), B(N, K), B(G, K), \) and \( B(N, M), \) respectively.

**Proof.** \((i) \Rightarrow (ii): \) Since the system (1.1) is consistent, there exists \( X_2 \in B(G, K) \) such that equations
\[ A_1 X_1 - X_2 B_1 = C_1 \]
\[ A_2 X_2 - X_3 B_2 = C_2 \]
are solvable for \( X_1 \) and \( X_3, \) respectively. According to Lemma 2.1 equation
\[ (2.12) \]
\[ A_1 X_1 - X_2 B_1 = C_1 \]
is solvable for \( X_1 \) if and only if
\[ (2.13) \]
\[ (I - A_1 A_1^\dagger) (C_1 + X_2 B_2) = 0 \]
and equation
\[ (2.14) \]
\[ A_2 X_2 - X_3 B_2 = C_2 \]
is solvable for \( X_3 \) if and only if
\[ (2.15) \]
\[ (A_2 X_2 - C_2) (I - B_2^\dagger B_2) = 0. \]
So, from (2.13) and (2.15) it follows that equations
\[ (2.16) \]
\[ (I - A_1 A_1^\dagger) X_2 B_1 = -(I - A_1 A_1^\dagger) C_1, \]
\[ A_2 X_2 (I - B_2^\dagger B_2) = C_2 (I - B_2^\dagger B_2) \]
have a common solution. From Lemma 2.1 and Lemma 2.2 system (2.16) is consistent if and only if
\[ (I - A_1 A_1^\dagger) C_1 (I - B_1^\dagger B_1) = 0, \]
\[ (I - A_2 A_2^\dagger) C_2 (I - B_2^\dagger B_2) = 0, \]
\[ C' (I - T T^\dagger) = 0, \]
where
\[ C' = (I - SS^T)(C_2 + A_2(I - A_1A_1^T)C_1B_1^T)(I - B_2^TB_2). \]
Note that condition
\begin{equation}
C'(I - T^\dagger T) = 0
\end{equation}
is equivalent to
\begin{equation}
C(I - T^\dagger T) = 0,
\end{equation}
since (2.17) implies
\begin{align*}
C(I - T^\dagger T) &= R_A A_1 A_2 (C_2 + A_2(I - A_1A_1^T)C_1B_1^T)L_{B_1}L_T \\
&= R_A A_1 SS^\dagger (C_2 + A_2(I - A_1A_1^T)C_1B_1^T)L_{B_1}L_T \\
&= R_A A_1 A_2 A_1 A_2^S S^\dagger (C_2 + A_2(I - A_1A_1^T)C_1B_1^T)L_{B_1}L_T \\
&= 0,
\end{align*}
and (2.18) implies
\begin{align*}
C'(I - T^\dagger T) &= R_S (C_2 + A_2(I - A_1A_1^T)C_1B_1^T)L_{B_1}L_T \\
&= R_S (A_2 A_1)(A_2 A_1)^\dagger (C_2 + A_2(I - A_1A_1^T)C_1B_1^T)L_{B_1}L_T \\
&= (I - (A_2 A_1 A_1^\dagger) (A_2 A_1 A_1^\dagger)^\dagger (A_2 A_1)(A_2 A_1)^\dagger (C_2 + A_2(I - A_1A_1^T)C_1B_1^T)L_{B_1}L_T \\
&= 0.
\end{align*}
I follows that
\begin{align*}
(I - A_1A_1^T)C_1(I - B_1^TB_1) &= 0, \\
(I - A_2A_2^T)C_2(I - B_2^TB_2) &= 0, \\
C(I - T^\dagger T) &= 0.
\end{align*}

(ii) ⇒ (i): If (ii) holds, then by Lemma 2.2 it follows that system (2.16) is consistent. Let \( X_2 \in B(G, K) \) be the solution to the system (2.16) and let \( X_1 = A_1^T(X_2B_1 + C_1) \) and \( X_3 = (A_2X_2 - C_2)B_2^T \). Then it is easy to see that such \( X_1, X_2 \) and \( X_3 \) satisfy (1.2).

(ii) ⇒ (iii): Suppose that
\begin{align*}
(I - A_1A_1^T)C_1(I - B_1^TB_1) &= 0, \\
(I - A_2A_2^T)C_2(I - B_2^TB_2) &= 0
\end{align*}
and

\[(2.21) \quad C(I - T^\dagger T) = 0.\]

From (2.20) we obtain

\[
(I - R_{A_2}A_1A_2(R_{A_2}A_1A_2)^\dagger)C \\
= (I - R_{A_2}A_2(R_{A_2}A_1A_2)^\dagger)R_{A_2}A_1(C_2 + A_2(I - A_1A_1^\dagger)C_1B_1)B_{B_2} \\
= (I - R_{A_2}A_2(R_{A_2}A_1A_2)^\dagger)R_{A_2}A_1C_2B_{B_2} \\
+ (I - R_{A_2}A_2(R_{A_2}A_1A_2)^\dagger)R_{A_2}A_1A_2(I - A_1A_1^\dagger)C_1B_1^\dagger L_{B_2} \\
= (I - R_{A_2}A_2(R_{A_2}A_1A_2)^\dagger)R_{A_2}A_1A_2A_2^\dagger C_2B_{B_2} \\
= 0.
\]

\((ii) \Rightarrow (iii):\) Suppose that

\[(2.22) \quad (I - A_1A_1^\dagger)C_1(I - B_1^\dagger B_1) = 0,\]

\[(2.23) \quad (I - R_{A_2}A_2A_2(R_{A_2}A_1A_2)^\dagger)C = 0\]

and

\[(2.24) \quad C(I - T^\dagger T) = 0.\]

From (2.23) we get

\[
(I - A_2A_2^\dagger)C_2(I - B_2^\dagger B_2) \\
= (I - A_2A_2^\dagger)C \\
= (I - A_2A_2^\dagger)R_{A_2}A_2A_2(R_{A_2}A_1A_2)^\dagger C \\
= 0.
\]

Suppose that system (1.2) is consistent. Since \(X_2 \in \mathcal{B}(G, K)\) is a solution to (1.2) if and only if it is solution to (2.16), its general form, according to Lemma 2.2, is given by

\[
X_2 = (-R_{A_2}C_1 + S^\dagger(A_2R_{A_1}C_1 + W))B_1^\dagger(I - T^\dagger) \\
+ (S^\dagger A_2R_{A_1}V + S^\dagger C_2B_{B_2})T^\dagger \\
+ Z - (R_{A_1} + S^\dagger S)Z(B_1B_1^\dagger + TT^\dagger),
\]

where

\[
V = -R_{A_2}C_1B_1^\dagger B_{B_2}L_T + R_{A_2}QT^\dagger T - R_{A_2}A_2^\dagger R_{S_2}A_2R_{A_2}QT^\dagger T
\]

and

\[
W = -R_{S_2}A_2R_{A_2}C_1 + SS^\dagger C_2B_{B_2}B_1 + SS^\dagger P_{B_1}B_1 - SS^\dagger P_{B_1}B_{B_2}B_1
\]
with \( P, Q, Z \) arbitrary elements of \( \mathcal{B}(F, \mathcal{M}), \mathcal{B}(G, \mathcal{K}) \) and \( \mathcal{B}(G, \mathcal{K}) \), respectively.

From the first equation in (1.2) we have

\[
A_1X_1 = X_2B_1 + C_1,
\]

so, by Lemma 2.1 we get

\[
X_1 = A_1^\dagger(X_2B_1 + C_1) + L_{A_1}R,
\]

where \( R \) is an arbitrary element of \( \mathcal{B}(F, \mathcal{H}) \).

From the second equation in (1.2) we have

\[
X_3B_2 = A_2X_2 - C_2,
\]

so, by Lemma 2.1 we get

\[
X_3 = (A_2X_2 - C_2)B_2^\dagger + YR_{B_2}
\]

\[
= A_2 (-R_{A_1}C_1 + S^\dagger(A_2R_{A_1}C_1 + W)) B_1^\dagger(I - T^\dagger)B_2^\dagger
\]

\[
+ A_2 ((I - S^\dagger A_2)R_{A_1}V + S^\dagger C_2L_{B_2}) T^\dagger B_2^\dagger
\]

\[
+ A_2ZB_2^\dagger - A_2(R_{A_1} + S^\dagger S)Z(B_1B_1^\dagger + TT^\dagger)B_2^\dagger - C_2B_2^\dagger + YR_{B_2},
\]

where \( Y \) is an arbitrary element of \( \mathcal{B}(N, \mathcal{M}) \).

### REFERENCES

1. A. Ben-Israel, T. N. E. Greville, *Generalized Inverse: Theory and Applications*, 2nd Edition, Springer, New York, 2003.
2. A. Dajic, *Common solutions of linear equations in ring, with applications*, Electron. J. Linear Algebra, 30 (2015), 66–79.
3. S.G. Lee, Q.P. Vu, *Simultaneous solutions of matrix equations and simultaneous equivalence of matrices*, Linear Algebra Appl., 437 (2012), 2325–2339.
4. Y. H. Liu, *Ranks of solutions of the linear matrix equation AX + YB = C*. Comput. Math. Appl., 52 (2006), 861–872.
5. Q.W. Wang, Z.H. He, *Solvability conditions and general solution for the mixed Sylvester equations*, Automatica, 49 (2013), 2713–2719.
6. Z.H. He, Q.W. Wang, *A pair of mixed generalized Sylvester matrix equations*, Journal of Shanghai University (Natural Science), 20 (2014), 138–156.
7. Z.-H. He, Q.-W. Wang, Y. Zhang, *A system of quaternary coupled Sylvester-type real quaternion matrix equations*, Automatica, 87 (2018), 25–31.