The semiflow of a reaction diffusion equation with a singular potential

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Abstract

We study the semiflow \( S(t) \) defined by a semilinear parabolic equation with a singular square potential \( V(x) = \frac{\mu}{|x|^2} \). It is known that the Hardy-Poincaré inequality and its improved versions, have a prominent role on the definition of the natural phase space. Our study concerns the case \( 0 < \mu \leq \mu^* \), where \( \mu^* \) is the optimal constant for the Hardy-Poincaré inequality. On a bounded domain of \( \mathbb{R}^N \), we justify the global bifurcation of nontrivial equilibrium solutions for a reaction term \( f(s) = \lambda s - |s|^{2\gamma} \), with \( \lambda \) as a bifurcation parameter. The global bifurcation result is used to show that any solution \( \phi(t) = S(t)\phi_0 \), initiating form initial data \( \phi_0 \geq 0 \) (\( \phi_0 \leq 0 \)), \( \phi_0 \not\equiv 0 \), tends to the unique nonnegative (nonpositive) equilibrium.

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1 Introduction

Fundamental issues of the linear heat equation with a singular potential

\[
\begin{align*}
    \partial_t \phi - \Delta \phi - \frac{\mu}{|x|^2} \phi &= 0, \quad x \in \Omega, \quad t > 0, \\
    \phi(x, 0) &= \phi_0(x), \quad x \in \Omega, \\
    \phi|_{\partial \Omega} &= 0, \quad t > 0,
\end{align*}
\]

(1.1)

where \( \Omega \) is in general an open set of \( \mathbb{R}^N \), have been analyzed in the works \[8,10,24]. The behavior of the solutions depends heavily on the critical value of the parameter \( \mu \) (denoted by \( \mu^* \)), which is the best constant of the Hardy’s inequality. The first fundamental result was that of \[3\] for the Cauchy-Dirichlet problem in an open set of \( \mathbb{R}^N \): If \( \phi_0(x) \geq 0, \phi_0(x) \not\equiv 0 \), there exists a global solution if \( 0 < \mu \leq \mu_* \). Not even a local solution exists if \( \mu > \mu_* \) (complete instantaneous blowup). The importance of Hardy’s inequality for the result of \[3\] was shown in \[10\]. Further fundamental results ranging from the removal of the sign condition on the initial data, the uniqueness of solutions in an appropriate functional space, the possible decay of solutions and its rate when \( \mu < \mu^* \), to the description of the behavior of solutions at the critical value \( \mu^* \) as well as analysis of the Cauchy problem, have been addressed in \[24\]. The legitimate analysis at the transition \( \mu = \mu^* \) and beyond, for the aforementioned questions is related to the Hardy inequality and to its improved versions \[24\], in bounded as well in unbounded domains.

Concerning the bounded domain case, the situation regarding the behavior of solutions of \( (1.1) \), can be described in summary as follows: When \( 0 < \mu < \mu^* \) the sign-condition on the initial data can be removed
and for any $\phi_0 \in L^2(\Omega)$ there exist a unique, global in time solution $\phi \in C([0, \infty); L^2(\Omega)) \cap L^2([0, \infty); H^1_0(\Omega))$ which decays at exponential rate. In the critical transition value $\mu = \mu^*$ solutions which still exist globally in $L^2(\Omega)$, blow up instantaneously in $H^1_0(\Omega)$ but exist globally in the generalized Sobolev space $H^1_0(\Omega)$. The Hilbert space $H^1_0(\Omega)$ is defined for any fixed $0 < \mu \leq \mu^*$, as the completion of the $C^\infty(\Omega)$ functions under the norm $||\phi||^2_\mu = \int_\Omega |\nabla \phi|^2 \, dx - \mu \int_\Omega \frac{\phi^2}{|\lambda|^2} \, dx$. The $H^1_0(\Omega)$-solution, $\phi(x,t) \sim O(e^{-\lambda t})$ with the rate $\lambda > 0$ explicitly given. This is only a basic framework, since there are important and deep phenomena in the range $0 < \mu \leq \mu^*$ (e.g. singular behavior at the origin with prescribed rate, even for the solutions $\phi \geq 0$ associated with good initial data and even non uniqueness of nonnegative distributional solutions). When $\mu > \mu^*$, there exist initial data of oscillating type for which the solution exists globally in time. Extensions of the results of the bounded domain when $0 < \mu \leq \mu^*$ (but with major differences e.g. on the rate of decay) have been made on appropriate weighted spaces based on weighted improvements of the Hardy’s inequality.

Strongly motivated by the results of [24], for (1.4) on the bounded domain case, we shall discuss the dynamics of a semilinear analogue of (1.1).

$$
\begin{align*}
\partial_t \phi - \Delta \phi - \frac{\mu}{|x|^2} \phi &= \lambda \phi - |\phi|^{2\gamma} \phi, \quad x \in \Omega, \ t > 0, \\
\phi(x,0) &= \phi_0(x), \quad x \in \Omega, \\
\phi|_{\partial \Omega} &= 0, \quad t > 0.
\end{align*}
$$

(1.2)

with our attention restricted in this work, up to the critical case $\mu = \mu^*$.

We start with the analysis of the set of equilibrium solutions of (1.2). The equilibrium solutions in this case satisfy the semilinear elliptic equation

$$
\begin{align*}
- \Delta u - \frac{\mu}{|x|^2} u &= \lambda u - |u|^{2\gamma} u, \\
u|_{\partial \Omega} &= 0,
\end{align*}
$$

(1.3)

The results of Section 2, concern the bifurcation of equilibrium solutions with respect to the parameter $\lambda \in \mathbb{R}$. Considering this type of nonlinear term with $\lambda$ as a varying parameter, is of importance, having in mind the Ginzburg-Landau nonlinearity. Hardy’s inequality implies for the subcritical case $0 < \mu < \mu^*$, the equivalence $H^1_0(\Omega) \equiv H^1_0(\Omega)$. In this case, the operator $L = -\Delta - \frac{\mu}{|x|^2}$ defines an unbounded self-adjoint operator in $L^2(\Omega)$ with compact inverse. Thus, a global branch of nonnegative solutions of (1.2) bifurcating from the trivial solution at $(\lambda_1,0)$, where $\lambda_1$ is the positive principal eigenvalue of the linear eigenvalue problem

$$
\begin{align*}
- \Delta u - \frac{\mu}{|x|^2} u &= \lambda u, \\
u|_{\partial \Omega} &= 0,
\end{align*}
$$

(1.4)

is naturally expected.

The analysis carried out in [24] for the critical case $\mu = \mu^*$, suggests that we cannot expect $H^1_0$-solutions for the eigenvalue problem (1.3). Instead, the main result of Section 2, is stated in the following

**THEOREM 1.1** Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded domain. Assume that $0 < \mu \leq \mu^*$, and that

$$
0 < \gamma \leq \frac{Nq - 2N + 2q}{2(N-q)} := \gamma_* \quad \text{for any} \quad \frac{2N}{N+2} < q < 2.
$$

(1.5)

Then, the principal eigenvalue $\lambda_{1,\mu}$ of (1.4) considered in $H^1_0(\Omega)$, is a bifurcating point of the problem (1.3) (in the sense of Rabinowitz) and $C_{\lambda_{1,\mu}}$ is a global branch of nonnegative $H^1_0(\Omega)$-solutions of (1.3).

For comparison results and properties of the linear eigenvalue problem (1.3), we will refer to [13].

The global bifurcation results of Section 2 are of a twofold meaning. On the one hand, they establish the existence of a global branch $C_{\lambda_{1,\mu}}$, of nonnegative solutions in the critical value $\mu = \mu^*$. The global branch has the properties proved in

**PROPOSITION 1.2** Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded domain. Assume that $0 < \mu \leq \mu^*$ and that (1.3) holds. Then (i) The global branch $C_{\lambda_{1,\mu}}$ bends to the right of $\lambda_{1,\mu}$ (supercritical bifurcation) and it is bounded for $\lambda$ bounded.

(ii) Every solution $u \in C_{\lambda_{1,\mu}}$ is the unique nonnegative solution for the problem (1.3).
Theorem 1.3 Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded domain. Assume that $0 < \mu \leq \mu^*$ and that (1.5) holds. Then $C_{\lambda_n, \mu} \to C_{\lambda, \mu}$ in $H^1(\Omega)$, as $\mu \uparrow \mu^*$.

It seems even more interesting to discuss how the branches $C_{\lambda_n, \mu}$ for $\mu < \mu^*$ behave as $\mu \uparrow \mu^*$. Regarding the behavior of the global branch $C_{\lambda_1, \mu}$ as the parameter $\mu$ varies to the transition value $\mu^*$, the answer is given in the following theorem, showing that the situation in $H^1_0(\Omega)$ and in $H^1(\Omega)$ is qualitatively totally different (figure 1 demonstrates a possible configuration).

Theorem 1.4 Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. We assume that

$$0 < \gamma \leq \frac{5q - 6}{2(3 - q)} := \gamma_*, \text{ for any } \frac{3}{2} < q < 2, \text{ if } N = 3,$$

and when $N \geq 4$, we assume condition (1.5).

A. Let $\mu_n \uparrow \mu^*$, as $n \to \infty$. Assume that $(\lambda_n, u_n) \in C_{\lambda_n, \mu_n}$, be such that $\lambda_n$ is bounded, i.e. $|\lambda_n| < L$. Then, $u_n$ must be bounded too, in $H^s(\Omega)$. Moreover, $(\lambda_n, u_n) \to (\lambda_*, u_*)$ in $\mathbb{R} \times H_{\mu^*}(\Omega)$, with $(\lambda_*, u_*) \in C_{\lambda_1, \mu^*}$.

B. Let $\mu_n \uparrow \mu^*$, as $n \to \infty$. Assume that $(\lambda_n, u_n) \in C_{\lambda_n, \mu_n}$, be such that $\lambda_n \to \lambda_1 \mu^*$. Then, $u_n$ must be unbounded in $H^1_0(\Omega)$.

Observe that the condition (1.5) is slightly modified, distinguishing between the cases $N = 3$ and $N \geq 4$.

In Section 3, and in the spirit of our recent work [15], we shall use Theorem 1.4 to discuss the stability properties of equilibria and the asymptotic behavior of solutions of (1.2). We discuss first stability by linearization: Using the improved Hardy inequality of [24] and its consequences, we consider appropriate Garding forms to prove the asymptotic stability of the trivial equilibrium when $\lambda \leq \lambda_{1, \mu}$ and the asymptotic stability of the unique nonnegative equilibrium when $\lambda > \lambda_{1, \mu}$, for $0 < \mu \leq \mu^*$. However the setting of [24], enables for a stronger result: Following closely the semiflow theory [2, 17, 23], we define a gradient semiflow in $H^1_0(\Omega)$, for any $0 < \mu \leq \mu^*$. This is one of the basic results proved in Section 3, stated in

Proposition 1.5 Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded domain, $0 < \mu \leq \mu^*$ and condition (1.5) be fulfilled. The semiflow $\varphi(t)$, possesses a global attractor $A$ in $H^s(\Omega)$. Let $E$ denote the bounded set of equilibrium points of $S(t)$. For each complete orbit $\phi$ lying in $A$, the limit sets $\alpha(\phi)$ and $\omega(\phi)$ are connected subsets of $E$ on which the Lyapunov functional $\mathcal{J}$ associated to $S(t)$, is constant. If $E$ is totally disconnected (in particular if $E$ is countable), the limit

$$z_- = \lim_{t \to -\infty} \phi(t), \quad z_+ = \lim_{t \to +\infty} \phi(t).$$
exist and are equilibrium points. Furthermore, any solution of \((1.2)\), tends to an equilibrium point as \(t \to \infty\).

Armed with the fact, that the limit set \(\omega(\phi)\) for each positive orbit \(\phi\) lying in the global attractor \(A\), is a connected subset of the bounded set \(E\) of the equilibrium solutions, the global bifurcation result of Theorem 1.1 will be crucial: It actually shows that \(E = \{0\}\) when \(\lambda \leq \lambda_{1,\mu}\), \(\mu\), and is totally disconnected when \(\lambda > \lambda_{1,\mu}\), \(\mu\), \(\mu^*\).

THEOREM 1.6 Let \(\Omega \subset \mathbb{R}^N, N \geq 3\), be a bounded domain. Assume that \(0 < \mu \leq \mu^*\) and that \((1.3)\) is fulfilled. Let \(\phi_0 \in H^1_0(\Omega), \phi_0 \neq 0\). If \(\lambda \leq \lambda_{1,\mu}\), then \(A = \{0\}\). If \(\lambda > \lambda_{1,\mu}\), then \(\omega(\phi_0) = \{u\}\) when \(\phi_0 \geq 0\) and \(\omega(\phi_0) = \{u_\cdot\}\) when \(\phi_0 \leq 0\).

The above result is a rigorous verification that \((1.2)\) which undergoes a pitchfork bifurcation of supercritical type for any \(\mu < \mu^*\) in \(H^1_0(\Omega)\) preserves this behavior up to the transition \(\mu = \mu^*\) in the \(H^1_{\mu^*}(\Omega)\)-phase space (see figure 2). We remark that in the case \(\lambda > \lambda_{1,\mu}\) Proposition 1.5 clearly implies that for any \(\phi_0 \neq 0\), any solution \(\phi(t) = S(t)\phi_0\) converges to one of the equilibrium solutions \(u_0\) or \(u_\cdot\), possibly through an heteroclinic orbit connecting them.

However, Theorem 1.4B. combined with Theorem 1.6 indicate for the “explosive” behavior of the attractor \(A\) in \(H^1_0(\Omega)\) when \(\mu \to \mu^*\). Theorem 1.6 could also be viewed as the analogue of \([24, \text{Theorem 4.1, pg. 123}]\) for \((1.2)\), with the exponential decay, replaced by the convergence to the unique nonnegative or the unique nonpositive equilibrium, for any \(\lambda > \lambda_{1,\mu}\), according to the sign of the initial data \(\phi_0\).

Figure 2: Supercritical pitchfork bifurcation for the semiflow defined by \((1.2)\) in \(H^{1,*}_\mu(\Omega)\).

At this point, we also remark \([12]\) for bifurcation results on \(H^1_0(\Omega)\) with \(\mu\) as a bifurcation parameter, regarding the semilinear elliptic problem

\[- \Delta u - \frac{\mu}{|x|^2}u = u^q, \quad u > 0, \quad u|_{\partial\Omega} = 0.\]

For bifurcation results on the degenerate elliptic problem

\[-|x|^2 \Delta u = \lambda f(u), \quad u > 0, \quad u|_{\partial\Omega} = 0,\]

related to the Hardy inequality, we refer to \([14]\). We also point out \([7]\), on recent bifurcation results for the elliptic problem

\[- \Delta u = \lambda m(x)u + b(x)u^\gamma, \quad \frac{\partial u}{\partial n}|_{\partial\Omega} = 0,\]

where the functions \(m, b : \bar{\Omega} \to \mathbb{R}\) are this time, smooth functions but of changing sign. For a brief reference to existing results on the issue of convergence of solutions of global solutions of evolution equations to steady states, we refer to \([9]\) (see also \([18, \text{pg. 366}]\)). For improvements related to second order Hardy-type inequalities, we refer to the recent work \([22]\).
2 Global bifurcation of equilibrium solutions

This section is devoted to the proof of the existence of bifurcation branches for the equilibrium solutions of (1.2) given by the semilinear elliptic equation (1.3). Here $\Omega$ will be an open bounded and connected subset of $\mathbb{R}^N$, $N \geq 3$ including the origin. We shall assume that $0 < \mu \leq \mu^*$, where

$$\mu^* := \left(\frac{N-2}{2}\right)^2,$$

is the best constant of Hardy’s inequality

$$\int_\Omega |\nabla u|^2 \, dx > \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} \, dx. \tag{2.1}$$

In subsection 2.1 we recall the basic properties of the delicate functional framework developed in [24, Section 4, pg. 121-123], and we present some auxiliary results regarding the nonlinear maps defined in this setting. Subsection 2.3 refers to the proof of Theorem 1.1, while subsection 2.3 is devoted to the approximation of the global branch by the associated branches of systems considered in domains not containing the origin. In subsection 2.4 we discuss the proof of Theorem 1.4.

2.1 Basic properties of the phase space.

It well known that the constant $\mu^*$ is optimal and it is not attained in $H_0^1(\Omega)$. In [6] it was given the following improved version of (2.1)

$$\int_\Omega |\nabla u|^2 \, dx \geq \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} \, dx + \lambda_\Omega \int_\Omega u^2 \, dx, \tag{2.2}$$

where $\lambda_\Omega = z_0^2 \omega_N^\frac{2}{N} |\Omega|^{-\frac{2}{N}}$, where $\omega_N$ and $|\Omega|$ denote the volume of the unit ball and $\Omega$ respectively, and $z_0 = 2.4048 \ldots$ denotes the first zero of the Bessel function $J_0(z)$. This constant is optimal when $\Omega$ is a ball, but it is also not achieved in $H_0^1(\Omega)$. In [15] was proved that inequality (2.1) admits an infinite series of correction terms.

The analysis of [24], recovered that the natural phase space for the study of linear equation (1.1) system (1.2) is the Hilbert space $H_\mu(\Omega)$, defined for any fixed $0 < \mu \leq \mu^*$, as the completion of the $C_0^\infty(\Omega)$ functions under the norm

$$||\phi||_\mu^2 = \int_\Omega |\nabla \phi|^2 \, dx - \mu \int_\Omega \frac{\phi^2}{|x|^2} \, dx, \tag{2.3}$$

and endowed with the scalar product

$$(\phi, \psi)_\mu = \int_\Omega \nabla \phi \nabla \psi \, dx - \mu \int_\Omega \frac{\phi \psi}{|x|^2} \, dx.$$

Consequently, this is also the case for the semilinear analogue (1.2). Friedrich’s extension theory is applicable due to the inequality (2.2), is the main ingredient which can be used to consider the operator $\mathcal{L} = -\Delta - V(x)$ as a positive and self adjoint operator with domain of definition

$$D(\mathcal{L}) = \{ \phi \in H_\mu(\Omega) : \mathcal{L}\phi \in L^2(\Omega) \}. \tag{2.4}$$

The improved Hardy-Poincaré inequalities

$$\int_\Omega \left|\nabla \phi - \mu^* \frac{\phi^2}{|x|^2}\right|^2 \, dx \geq C(q, \Omega)||\phi||_{W^{1,q}(\Omega)}^2, \quad 1 \leq q < 2, \tag{2.5}$$

$$\int_\Omega \left|\nabla \phi - \mu^* \frac{\phi^2}{|x|^2}\right|^2 \, dx \geq C(s, r, \Omega)||\phi||_{W^{s,r}(\Omega)}^2, \quad 0 \leq s < 1, \quad 1 \leq r < r_* = \frac{2N}{N-2(1-s)}. \tag{2.6}$$

for all $\phi \in C_0^\infty(\Omega)$, imply the continuous embeddings,

$$H_\mu(\Omega) \hookrightarrow W^{1,q}_0(\Omega), \quad H_\mu(\Omega) \hookrightarrow H_0^s(\Omega), \quad 1 \leq q < 2, \quad 0 \leq s < 1. \tag{2.7}$$
if $1 \leq q < 2$ and $0 \leq s < 1$. Furthermore, since $W^{1,q}_0(\Omega)$ is compactly embedded in $H^s_0$ for suitable $q = q(s)$ close enough to 2, and $H^s_0(\Omega)$ is compactly embedded in $L^2(\Omega)$, we infer the compact embeddings

$$H^s_0(\Omega) \hookrightarrow L^2(\Omega), \quad H^s_0(\Omega) \hookrightarrow H^s_0(\Omega), \quad 0 \leq s < 1. \quad (2.8)$$

In the subcritical case $0 < \mu < \mu^*$ we have the following property of $H^s_0(\Omega)$.

**Lemma 2.1** ([24]) Let $0 < \mu < \mu^*$. Then $H^s_0(\Omega) \equiv H^s_0(\Omega)$.

**Proof:** Clearly from (2.3),

$$||u||^2_{H^s_0(\Omega)} \leq \int_{\Omega} |\nabla u|^2 \, dx = ||u||^2_{H^s_0(\Omega)}. \quad (2.9)$$

On the other hand, Hardy’s inequality (2.1), implies that

$$||u||^2_{H^s_0(\Omega)} \geq \left[1 - \left(\frac{N - 2}{2}\right)^2 \mu\right] ||u||^2_{H^s_0(\Omega)}. \quad (2.10)$$

Thus, from inequalities (2.9) and (2.10) we conclude that

$$c ||u||^2_{H^s_0(\Omega)} \leq ||u||^2_{H^s_0(\Omega)} \leq C ||u||^2_{H^s_0(\Omega)},$$

for $c = 1 - \left(\frac{N - 2}{2}\right)^2 \mu > 0$ if $0 < \mu < \mu^*$, and $C = 1$. ■

A remarkable property was shown in [24] concerning the critical case $\mu = \mu^*$: $H^s_0(\Omega)$ is larger than $H^s_0(\Omega)$, since it contains singularities of the form $f \sim |x|^{(N-2)/2}$, and it is smaller than $C_{q<2} W^{1,q}(\Omega)$.

With the continuous embeddings (2.7) at hand, we can handle the nonlinearity of (1.2).

**Lemma 2.2** Let condition (1.3) be satisfied and assume that $\mu \leq \mu^*$. The function $g(s) = |s|^{2\gamma} s, s \in \mathbb{R}$, defines a sequentially weakly continuous map $g : H^s_0(\Omega) \rightarrow L^2(\Omega)$. Let $G(\phi) := \int_0^\phi g(s) ds$. The functional $G : H^s_0(\Omega) \rightarrow \mathbb{R}$ defined by $G(\phi) = \int_\Omega G(\phi) \, dx$, is $C^1$ and sequentially weakly continuous.

**Proof:** Starting by the standard Sobolev embeddings, we recall that

$$W^{1,q}(\Omega) \hookrightarrow L^p(\Omega) \quad \text{for any} \quad 1 \leq p \leq \frac{qN}{N - q}, \quad q < N. \quad (2.11)$$

We consider the critical exponent

$$p^* := \frac{qN}{N - q} \quad \text{for any} \quad 1 \leq q < 2. \quad (2.12)$$

Thus, as an immediate consequence of the embedding (2.7) we infer that

$$H^s_0(\Omega) \hookrightarrow L^p(\Omega), \quad \text{for any} \quad 1 \leq p \leq p^*. \quad (2.13)$$

Using (2.13) it can be easily deduced that the functional $g$ is well defined, under the restriction (1.3). Furthermore, it follows from (2.13), that $G$ is well defined if

$$0 < \gamma \leq \frac{Nq - 2N + 2q}{(N - q)} := \gamma^*, \quad \text{for any} \quad \frac{2N}{N + 2} < q < 2.$$  

noting that $\gamma^* < \gamma^*$.

That both functional are sequentially weakly continuous, can be verified by using the compact embeddings (2.8) and repeating the arguments of [2] Lemma 3.3, pg. 38 & Theorem 3.6, pg. 40. We will check that $G$ is a $C^1$-functional, and its derivative is

$$G'(\phi)(z) = \langle g(\phi), z \rangle, \quad \text{for every} \quad \phi \in H^s_0(\Omega), \quad z \in H^{-1}_0(\Omega). \quad (2.14)$$
We consider for \( \phi, \psi \in H_\mu(\Omega) \), the quantity
\[
\frac{G(\phi + s\psi) - G(\phi)}{s} = \frac{1}{s} \int_\Omega \int_0^1 \frac{d}{d\theta} G(\phi + \theta s\psi)d\theta dx = \int_\Omega \int_0^1 g(\phi + s\theta s\psi)d\theta dx.
\] (2.15)

We set \( \sigma = \frac{qN}{N(q-1)q+1} \), \( \sigma^{-1} + p^* - 1 = 1 \), and we get
\[
\left| \int_\Omega g(\phi + \theta s\psi)d\theta dx \right| \leq c \left( \int_\Omega (|\phi|^{2\gamma+1} + |\psi|^{2\gamma+1})^\sigma dx \right)^{1/\sigma} \left( \int_\Omega |\psi|^{p^*} dx \right)^{1/p^*}. \] (2.16)

To apply the continuous embedding (2.13) we need the requirement
\[
(2\gamma + 1)\sigma \leq p^*. \]

This requirement produces the restriction (1.5). Letting \( s \to 0 \), and using the dominated convergence theorem, we infer that \( G \) is differentiable with the derivative (2.14).

For the continuity, we consider a sequence \( \{\phi_n\} \in H_\mu(\Omega) \) such that \( \phi_n \to \phi \) in \( H_\mu(\Omega) \) as \( n \to \infty \). We note first, that
\[
\langle G'(\phi_n) - G'(\phi), z \rangle \leq ||g(\phi_n) - g(\phi)||_{L^\sigma} ||z||_{L^{p^*}}. \] (2.17)

Setting then \( p_1 = \frac{p^*}{\sigma} \), the requirement for \( p_1 > 1 \), produces again the restriction for \( \frac{2N}{N+2} < q < 2 \). Now for
\[
p_2 = \frac{N(q-1) + q}{N(q-1) - N + 2q}, \quad p_2^{-1} + p_1^{-1} = 1,
\]
we get the inequality
\[
||g(\phi_n) - g(\phi)||_{L^\sigma} \leq c \left( \int_\Omega (|\phi_n|^{2\gamma} + |\phi|^{2\gamma})^{\sigma} dx \right)^{1/\sigma} \left( \int_\Omega |\phi_n - \phi|^{p^*} dx \right)^{1/p^*}.
\]
The embedding (2.13) is applicable if \( 2\gamma\sigma p_2 < p^* \), giving (1.5). Under this condition and as
\[
\lim_{n \to \infty} \int_\Omega |\phi_n - \phi|^{p^*} dx = 0,
\]
we conclude from (2.17), the continuity of \( G' \). \( \blacksquare \)

### 2.2 Existence of a global branch of positive solutions for any \( 0 < \mu \leq \mu^* \)

The existence of a global branch of nonnegative solutions will be proved via the classical Rabinowitz’s theorem:

**THEOREM 2.3** Assume that \( X \) is a Banach space with norm \( || \cdot || \) and consider \( G(\lambda, \cdot) = \lambda L \cdot + H(\lambda, \cdot) \), where \( L \) is a compact linear map on \( X \) and \( H(\lambda, \cdot) \) is compact and satisfies
\[
\lim_{||u|| \to 0} \frac{||H(\lambda, u)||}{||u||} = 0. \] (2.18)

If \( \lambda \) is a simple eigenvalue of \( L \) then the closure of the set
\[
C = \{ (\lambda, u) \in \mathbb{R} \times X : (\lambda, u) \text{ solves } u = G(\lambda, u), \ u \neq 0 \},
\]
possesses a maximal continuum (i.e. connected branch) of solutions, \( C_\lambda \), such that \( (\lambda, 0) \in C_\lambda \) and \( C_\lambda \) either:

(i) meets infinity in \( \mathbb{R} \times X \) or,

(ii) meets \( (\lambda^*, 0) \), where \( \lambda^* \neq \lambda \) is also an eigenvalue of \( L \).
We will prove that there exists a global branch (i.e. the second alternative of Theorem 2.3 cannot happen) of solutions bifurcating from the principal eigenvalue $\lambda_{1,\mu}$ of the problem (1.4), for any $\mu \leq \mu^*$.

**Lemma 2.4** Assume that $0 < \mu \leq \mu^*$. Problem (1.4), admits a positive principal eigenvalue $\lambda_{1,\mu}$, given by

$$
\lambda_{1,\mu} = \inf_{\phi \in H_\mu(\Omega), \phi \neq 0} \frac{\int_\Omega |\nabla \phi|^2 \, dx - \mu \int_\Omega \frac{\phi^2}{|x|^2} \, dx}{\int_\Omega |\phi|^2 \, dx}.
$$

(2.19)

with the following properties:

(i) $\lambda_{1,\mu}$ is simple with a positive associated eigenfunction $u_{1,\mu}$, which belongs at least to $C^1_{\text{loc}}(\Omega \setminus \{0\})$, for some $\zeta \in (0, 1)$,

(ii) $\lambda_{1,\mu}$ is the only eigenvalue of (1.4) with nonnegative associated eigenfunction.

Proof: The existence and the variational characterization (2.19) of the principal eigenvalue follows from the compactness of the embeddings (2.3) implying that $\mathcal{L} = -\Delta - \frac{\mu}{|x|^2}$ for $0 < \mu \leq \mu^*$, has an orthonormal basis of eigenfunctions in $H_\mu(\Omega)$ with an eigenvalue sequence $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \to \infty$, (2.20)

(cf. [24, pg. 122]) The regularity results (cf. [16, Theorem 8.22]) imply that if $u$ is a weak solution of the problem (1.4), then $u \in C^2_{\text{loc}}(\Omega \setminus \{0\})$, for some $\zeta \in (0, 1)$. The positivity of $u_{1,\mu}$ follows from [13, Lemma 2.2]-we also refer to the weak maximum principle of [4]. The simplicity and the uniqueness up to positive eigenfunctions of $\lambda_{1,\mu}$ can be verified, by using Picone's identity [18].

For some further properties of the principal eigenvalue and the corresponding eigenfunction, we refer to [13].

We remark [15], where the weighted space Hilbert space $W^{1,2}_0(\Omega; |x|^{-(N-2)})$ was used, defined as the completion of $C^\infty_0$-functions under the norm

$$
||u||_{W^{1,2}_0(\Omega; |x|^{-(N-2)})} = \int_\Omega |x|^{-(N-2)} |\nabla u|^2 \, dx + \int_\Omega |x|^{-(N-2)} u^2 \, dx
$$

and endowed with the inner product

$$
<u, v>_{W^{1,2}_0(\Omega; |x|^{-(N-2)})} = \int_\Omega |x|^{-(N-2)} \nabla f \nabla g \, dx + \int_\Omega |x|^{-(N-2)} f g \, dx.
$$

In [13], the space $W^{1,2}_0(\Omega; |x|^{-(N-2)})$ was considered for the proof of the existence of principal eigenvalues for the eigenvalue problem

$$
-\Delta u - \frac{\mu}{|x|^2} u = \lambda V(x) u
$$

(2.21)

and

$$
u|_{\partial \Omega} = 0,$$

Furthermore, it was assumed that $V(x) \geq 0$, $V(x) \in L^p(\Omega)$, $p = N/2$. A comparison of the spaces $H_{\mu^*}(\Omega)$ and $W^{1,2}_0(\Omega; |x|^{-(N-2)})$ implies that $u \in H_{\mu^*}(\Omega)$ if and only if $|x|^{(N-2)/2} u \in W^{1,2}_0(\Omega; |x|^{-(N-2)})$.

Proceeding to the proof of the global bifurcation result, we discuss first the behavior of $\lambda_{1,\mu}$, $0 < \mu < \mu^*$ as $\mu \uparrow \mu^*$. Next lemma demonstrates the qualitative differences in $H^1_0(\Omega)$ for the solutions of the linear eigenvalue problem (1.4) as $\mu$ converges to the transition value $\mu^*$.

**Proposition 2.5** Let $\mu \uparrow \mu^*$. Then,

(i) $\lambda_{1,\mu}$ is a decreasing sequence, and there exists $\lambda_\ast > 0$ such that $\lambda_{1,\mu} \downarrow \lambda_\ast$.

(ii) The corresponding normalized eigenfunctions $u_{1,\mu}$ are converging weakly to 0, in $H^1_0(\Omega)$.

Proof: (i) Let $\mu_1 < \mu_2$. Then the variational characterization of the principal eigenvalue $\lambda_{1,\mu}$ (2.19) implies that $\lambda_{1,\mu_1} > \lambda_{1,\mu_2}$. Thus $\lambda_{1,\mu}$ is decreasing. Applying next the improved Hardy’s inequality (2.2) we infer that $\lambda_{1,\mu}$ is bounded from below by $\lambda_{\Omega}$. Thus, there exists $\lambda_\ast > 0$, such that $\lambda_{1,\mu} \downarrow \lambda_\ast$. 


(ii) The eigenfunctions $u_{1,\mu}$ should satisfy the weak formula
\[\int \nabla u_{1,\mu} \nabla \phi dx - \mu \int \frac{u_{1,\mu} \phi}{|x|^2} dx = \lambda_{1,\mu} \int u_{1,\mu} \phi dx,\] (2.22)
for any $\phi \in C^\infty_0(\Omega)$. We still denote by $u_{1,\mu}$ the sequence of normalized eigenfunctions, forming a bounded sequence in $H^1_0(\Omega)$. We deduce that there exists some $u \in H^1_0(\Omega)$ such that up to a subsequence (not relabelled), $u_{1,\mu} \rightharpoonup u$ in $H^1_0(\Omega)$ and $u_{1,\mu} \to u$ in $L^q(\Omega)$, for any $1 < q < \frac{2N}{N-2}$. For some fixed $\varepsilon > 0$, small enough and any $\phi \in C^\infty_0(\Omega)$, we have that
\[\int \frac{(u_{1,\mu} - u) \phi}{|x|^2} dx \leq \|\phi\|_{L^\infty(\Omega)} \left( \int \frac{u_{1,\mu}}{|x|^2} \right)^{\frac{N+2+\varepsilon}{2(N-\varepsilon)}} \left( \int |x|^{-N+\varepsilon} \right)^{2/(N-\varepsilon)} \to 0,
\]
thus
\[\int \frac{u_{1,\mu} \phi}{|x|^2} dx \to \int \frac{u \phi}{|x|^2} dx, \quad \text{as} \quad \mu \uparrow \mu^*.\]
Let us now assume by contradiction that $u \not\equiv 0$. Passing to the limit in (2.22), we get that $u$ must satisfy
\[\int \nabla u \nabla \phi dx - \mu^* \int \frac{u \phi}{|x|^2} dx = \lambda_* \int u \phi dx,
\]
for any $\phi \in C^\infty_0(\Omega)$, or equivalently that $u$ must be a nontrivial solution of the problem
\[-\Delta u - \mu^* \frac{u}{|x|^2} = \lambda_* u, \quad u \in H^1_0(\Omega).\] (2.23)
However, since $\mu^*$ is the optimal constant of (2.2) which is not achieved in $H^1_0(\Omega)$, (2.24) implies that $u \equiv 0$. ■

Proof of Theorem 1.1. For the justification of Theorem 2.3, the improved Hardy’s inequality (2.2), will allow us to employ the method developed in [8]: On the account of (2.19), we define a bilinear form in $C^\infty_0(\Omega)$ by
\[\langle u, v \rangle_X = \int \nabla u \nabla v dx - \mu \int \frac{u v}{|x|^2} dx - \frac{c}{2} \int u v dx, \quad \text{for all} \quad u, v \in C^\infty_0(\Omega), \quad c = \lambda_{1,\mu}.\] (2.24)
We define the next space $X$, as the completion of $C^\infty_0(\Omega)$ with respect to the norm induced by (2.24), $\|u\|_X^2 = \langle u, u \rangle_X$. Then due to the improved Hardy’s inequality (2.2), we deduce the equivalence of norms
\[\frac{1}{2} \|u\|^2_{H^2_\mu(\Omega)} \leq \|u\|^2_X \leq \frac{3}{2} \|u\|^2_{H^2_\mu(\Omega)}, \quad \text{for all} \quad u, v \in C^\infty_0(\Omega),\]
Since $C^\infty(\Omega)$ is dense both in $X$ and $H^2_\mu(\Omega)$, it follows that $X = H^2_\mu(\Omega)$. Henceforth we may suppose that the norm in $X$ coincides with the norm in $H^2_\mu(\Omega)$ and that the inner product in $X$ is given by $\langle u, v \rangle_X = \langle u, v \rangle_{H^2_\mu(\Omega)}$. Let us note that the identification principle [25] (Identification Principle 21.18, pg. 254) implies that if $\langle \cdot, \cdot \rangle_{X^*,X}$ denotes the duality pairing on $X$, then $\langle \cdot, \cdot \rangle_{X^*,X} = \langle \cdot, \cdot \rangle_X$. To proceed further we note that the bilinear form
\[a(u, v) = \int u v dx, \quad \text{for all} \quad u, v \in X,\]
is clearly continuous in $X$. The Riesz representation theorem implies that we can define a bounded linear operator $L$ such that
\[a(u, v) = \langle Lu, v \rangle, \quad \text{for all} \quad u, v \in X.\] (2.25)
The operator $L$ is self adjoint and compact and its largest eigenvalue $\nu_1$ is characterized by
\[\nu_1 = \sup_{u \in X} \frac{\langle Lu, u \rangle}{\langle u, u \rangle} = \sup_{u \in X} \frac{\int u^2 dx}{\int |\nabla u|^2 dx - \mu \int \frac{u^2}{|x|^2} dx}.
\]
Then, by Lemma 2.4 it readily follows that the positive eigenfunction $u_1$ of (1.4) corresponding to $\lambda_{1,\mu}$ is a positive eigenfunction of $L$ corresponding to $\nu_1 = 1/\lambda_{1,\mu}$. 
After these preparations, we may define the nonlinear operator \( N(\lambda, \cdot) : \mathbb{R} \times X \to X^* \) as
\[
< N(\lambda, u), v > = \int_{\Omega} \nabla u \nabla v \, dx - \mu \int_{\Omega} \frac{u v}{|x|^2} \, dx - \lambda \int_{\Omega} u v \, dx + \int_{\Omega} |u|^{2\gamma} u \, v \, dx,
\]
for all \( v \in X \). Since the functional \( S : X \to \mathbb{R} \) defined by
\[
S(v) = \int_{\Omega} \nabla u \nabla v \, dx - \mu \int_{\Omega} \frac{u v}{|x|^2} \, dx - \lambda \int_{\Omega} u v \, dx + \int_{\Omega} |u|^{2\gamma} u \, v \, dx, \quad v \in X,
\]
is a bounded linear functional we have that \( N(\lambda, u) \) is well defined from (2.26). Moreover by using the fact that \( X = H^{1}_0(\Omega) \) and relation (2.27), we can rewrite \( N(\lambda, u) \) in the form \( N(\lambda, u) = u - G(\lambda, u) \) where \( G(\lambda, u) := \lambda \mathbf{L} u - \mathbf{H}(u), \)
\[
< H(u), v > = \int_{\Omega} |u|^{2\gamma} u \, v \, dx \quad \text{for all} \quad v \in X.
\]
Under condition (1.5), the embedding \( H^{1}_0(\Omega) \to L^{2\gamma+2}(\Omega) \) is compact, implying that the map \( H \) is compact. To check condition (2.10) of Theorem (2.9), we derive first the inequality
\[
\frac{1}{|u||x|} < H(u), v > \leq \frac{1}{|u||x|} |u|^{2\gamma+2} |v|_{L^{2\gamma+2}} \leq c_1 |u|^{2\gamma} |v|_{X}. \tag{2.27}
\]
Then, we get from (2.27) that
\[
\lim_{|u||x| \to 0} \frac{|H(u)||x*|}{|u||x|} = \lim_{|u||x| \to 0} \sup_{|v|_{L^{2\gamma+2}} \leq 1} \frac{1}{|u||x|} |< H(u), v > | = 0.
\]
It remains to prove that \( C_{\lambda_1, \mu} \) is global. We proceed in two steps, adapting the arguments of [18].

(a) We shall prove first that all solutions \( (\lambda, u) \in C_{\lambda_1, \mu} \) close to \( (\lambda_1, \mu, 0) \) are positive for all \( x \in \Omega \). More precisely we shall prove that there exists \( \epsilon_0 > 0 \), such that any \( (\lambda, u(x)) \in C_{\lambda_1, \mu} \cap B_{\epsilon_0}((\lambda_1, \mu, 0)) \), satisfies \( u(x) > 0 \), for any \( x \in \Omega \). Here \( B_{\epsilon_0}((\lambda_1, \mu, 0)) \), stands for the open ball of \( C_{\lambda_1, \mu} \) of center \( (\lambda_1, \mu, 0) \) and radius \( \epsilon_0 \).

We argue by contradiction, assuming that \( (\lambda_n, u_n) \) is a sequence of solutions of (1.3), such that \( (\lambda_n, u_n) \to (\lambda_1, \mu, 0) \) and that \( u_n \) are changing sign in \( \Omega \). Let \( u^-_n := \min\{0, u_n\} \) and \( U^-_n = \{ x \in \Omega : u_n(x) < 0 \} \). Since \( u_n = u^+_n - u^-_n \) is a solution of the problem (1.3) it can easily seen that \( u_n \), satisfies (in the weak sense) the equation
\[
- \Delta u^-_n - \mu \frac{u^-_n}{|x|^2} = \lambda_n u^-_n + |u_n|^{2\gamma} u^-_n = 0,
\]
\[
u^-_n|_{\partial \Omega} = 0. \tag{2.28}
\]
Then, multiplying (2.28) with \( u^-_n \) and integrating over \( \Omega \) we have that
\[
\int_{U^-_n} |\nabla u^-_n|^2 \, dx - \mu \int_{U^-_n} \frac{|u^-_n|^2}{|x|^2} \, dx - \lambda_n \int_{U^-_n} |u^-_n|^2 \, dx + \int_{U^-_n} |u^-_n|^{2\gamma} |u^-_n|^2 \, dx = 0. \tag{2.29}
\]
Since \( \lambda_n \) is a bounded sequence, it follows from (2.27) and Hölder’s inequality that
\[
||u^-_n||_{H^{1}_0(U^-_n)}^2 \leq \lambda_n \int_{U^-_n} |u^-_n|^2 \, dx \leq C |U^-_n|^\frac{2N-2\gamma+2}{\gamma} \left( \int_{U^-_n} |u^-_n|^\gamma \right)^\frac{2}{\gamma} \leq C |U^-_n|^\frac{2N-2\gamma+2}{\gamma} ||u^-_n||_{H^{1}_0(U^-_n)}. \tag{2.30}
\]
where $p^*$ is the critical exponent defined in (2.12), for any $q \in [1, 2)$. Then, from (2.30) we get that
\[
M \leq \|u_n\|, \quad \text{for all } n, \tag{2.31}
\]
with the constant $M$ being independent of $n$. We denote now by $\tilde{u}_n = u_n/\|u_n\|$ the normalization of $u_n$. Then there exists a subsequence of $\tilde{u}_n$ (not relabelled) converging weakly in $H_\mu(\Omega)$ to some function $\tilde{u}_0$. It can be seen that $\tilde{u}_0 = u_{1,1}$. Moreover, $\tilde{u}_n \rightharpoonup u_{1,1} > 0$ in $L^2(\Omega)$. Passing to a further subsequence if necessary, by Egorov’s Theorem, $\tilde{u}_n \rightharpoonup u_{1,1}$ uniformly on $\Omega$ with the exception of a set of arbitrary small measure. This contradicts (2.31) and we conclude the functions $u_n$ cannot change sign.

(b) We shall exclude next that for some solution $(\lambda, u) \in C_{\lambda,1,\mu}$, there exists a point $\xi \in \Omega$, such that $u(\xi) < 0$: Using (a), the fact that the continuum $C_{\lambda,1,\mu}$ is connected (see Theorem 1.3 and the $C^{1,\xi}_c(\Omega \setminus \{0\})$-regularity of solutions, we deduce that there exists $(\lambda_0, u_0) \in C_{\lambda,1,\mu}$, such that $u_0(\xi) \geq 0$, for all $\xi \in \Omega$, except possibly some point $x_0 \in \Omega$, such that $u_0(x_0) = 0$. Then, the maximum principle (see [3, 13]) and the fact that the solutions are singular at the origin imply that $u_0 \equiv 0$ on $\Omega$. Thus, we may construct a sequence $\{(\lambda_n, u_n)\} \subseteq C_{\lambda,1,\mu}$, such that $u_n(x) > 0$, for all $n$ and $x \in \Omega$, $u_n \rightarrow 0$ in $H_\mu(\Omega)$, and $\lambda_n \rightarrow \lambda_0$. However, this is true only for $\lambda_0 = \lambda_{1,1}$. As a consequence, we have that $C_{\lambda,1,\mu}$ cannot cross $(\lambda, 0)$ for some $\lambda \neq \lambda_1$, and every function which belongs to $C_{\lambda,1,\mu}$ is strictly positive. ■

### 2.3 Approximation by bounded domains not containing the origin

In this subsection we prove Theorem 1.3. The proof is also an alternative approach, to show Theorem 1.1 approximating (1.3) by the family of problems, 

\[ (A)_r \begin{cases} -\Delta u - \mu \frac{u}{|x|^2} = \lambda u - |u|^{2^*} u, \quad \text{in } \Omega_r = \Omega \setminus B_r(0), \\ u|_{\partial \Omega_r} = 0, \end{cases} \]

for some $r > 0$ sufficiently small. Standard regularity results imply that if $u$ is a weak solution of the problem $((A)_r)$, for some $r > 0$ small enough, then $u$ belongs at least in $C^{1,\xi}_c(\Omega_r)$, for some $\xi \in (0, 1)$.

The corresponding approximating linear eigenvalue problems

\[ (AL)_r \begin{cases} -\Delta u - \mu \frac{u}{|x|^2} = \lambda u, \quad \text{in } \Omega_r, \\ u|_{\partial \Omega_r} = 0, \end{cases} \]

admit for any $r > 0$, a positive principal eigenvalue $\lambda_{1,1,\mu,\mu}$, characterized by

\[
\lambda_{1,1,\mu,\mu} = \inf_{\phi \in H^1_0(\Omega_r), \phi \neq 0} \frac{\int_{\Omega_r} |\nabla \phi|^2 \, dx - \mu \int_{\Omega_r} \frac{\phi^2}{|x|^2} \, dx}{\int_{\Omega_r} \phi^2 \, dx}. \tag{3.3}
\]

with the following properties: $\lambda_{1,1,\mu,\mu}$ is simple with a positive associated eigenfunction $u_{1,1,\mu,\mu}$ and $\lambda_{1,1,\mu,\mu}$ is the only eigenvalue of $(PL)_r$, with positive associated eigenfunction. Furthermore, we have the following

**LEMMA 2.6** Let $0 < \mu \leq \mu^*$, and $\lambda_{1,1}$ and $\lambda_{1,1,\mu,\mu}$, be the positive principal eigenvalues of the problems $(1.4)$ and $(AL)_r$, respectively. Then

(i) $u_{1,1,\mu,\mu}(x) \leq u_{1,1}(x)$, for any $x \in \Omega_r$, and any $r > 0$.

(ii) $u_{1,1,\mu,\mu} \rightharpoonup u_{1,1}$ in $H_\mu(\Omega) \cap L^{2^*_\mu}(\Omega \setminus \{0\})$, and $\lambda_{1,1,\mu,\mu} \uparrow \lambda_{1,1}$, as $r \downarrow 0$.

**Proof:** (i) Having in mind, that both $u_{\lambda,\mu}$ and $u_\lambda$ are sufficiently smooth and positive functions on $\hat{\Omega}_r$, the assertion follows from the comparison principle (cf. [20, Theorem 10.5]).

(ii) We extend $u_{1,1,\mu,\mu}$ on $\Omega$ as

\[
\hat{u}_{1,1,\mu,\mu}(x) = \begin{cases} u_{1,1,\mu,\mu}(x), & x \in \Omega_r, \\ 0, & x \in B_r, \end{cases}
\]
for any sufficiently small $r > 0$, using in the sequel for convenience, the same notation $u_{1,\mu,r} \equiv \hat{u}_{1,\mu,r}$. We note first that
\[
\lambda_{1,\mu} = \frac{\int_\Omega |\nabla u_{1,\mu,r}|^2 \, dx - \mu \int_{\Omega_r} \frac{|u_{1,\mu,r}|^2}{|x|^2}}{\int_\Omega |u_{1,\mu,r}|^2 \, dx} \geq \lambda_1.
\]
Since $\Omega_r \subset \Omega_{r_2}$, for any $r_1 > r_2$, we deduce that $\lambda_{1,\mu,r}$ is an decreasing sequence as $r \to 0$. Moreover, $u_{1,\mu,r}$ forms a bounded sequence in $H_\mu(\Omega)$, thus $u_{1,\mu,r} \to u^*$ in $H_\mu(\Omega)$ (up to a subsequence), and $\lambda_{1,\mu,r} \to \lambda^*$ in $\mathbb{R}$. Then, by the compact embedding $H_\mu(\Omega) \hookrightarrow L^2(\Omega)$ we get that
\[
\lambda_{1,r} \int_\Omega |u_{1,r}|^2 \, dx \to \lambda^* \int_\Omega |u^*|^2 \, dx,
\]
as $r \to 0$. Therefore,
\[
\|u_{1,\mu,r}\|_{H_\mu(\Omega)} \to \|u^*\|_{H_\mu(\Omega)}.
\]
Hence $(\lambda^*, u^*)$ must be an eigenpair of (1.4) and from Lemma 2.6 (ii), we infer that $(\lambda^*, u^*) \equiv (\lambda_1, u_1)$. Finally, we consider the difference $\psi = u - u_{\lambda,r}$. Standard regularity results imply that
\[
\|\psi\|_{W^{2,2}(\Omega \setminus \{0\})} \leq C \|\psi\|_{W^{1,2}(\Omega)} + O(r), \quad \text{as } r \to 0,
\]
for some positive constant $C$ independent from $r$. By a bootstrap argument, we conclude that $u_{1,\mu,r} \to u_{1,\mu}$ in $L^\infty(\Omega \setminus \{0\})$. \■

Rabinowitz’s Theorem 2.3 is applicable for the approximating problems $(A)_r$, by following closely the arguments used in proof of Theorem 1.1.

**Lemma 2.7** Assume that $0 < \mu \leq \mu^*$, The principal eigenvalue $\lambda_{1,\mu,r}$ of $(PL)_r$ is a bifurcating point of the problem $(P)_r$ (in the sense of Rabinowitz) and $C_{1,\mu,r}$ is a global branch of nonnegative solutions, which "bends" to the right of $\lambda_{1,\mu}$. For any fixed $\lambda > \lambda_{1,\mu}$ these solutions are unique.

The properties of the global branch $C_{1,\mu,r}$ can be proved as in Proposition 1.2 (see Subsection 2.4). The nonlinear analogue of Lemma 2.6 is stated in

**Proposition 2.8** Assume that $0 < \mu \leq \mu^*$, and let $\lambda$ be a fixed number, such that $(\lambda, u_{\lambda,r}) \in C_{1,\mu,r}$. Then,

(i) $u_{\lambda,r} \to u_\lambda$ in $H_\mu(\Omega)$, with $(\lambda, u_\lambda) \in C_{1,\mu}$,

(ii) $u_{\lambda,r}(x) \leq u_\lambda(x)$, for any $x \in \Omega_r$, and any $r \downarrow 0$,

(iii) $u_{\lambda,r} \to u_\lambda$ in $L^\infty(\Omega \setminus \{0\})$, as $r \downarrow 0$.

**Proof:** (i) We shall prove first that $u_{\lambda,r}$ is a bounded sequence in $H_\mu(\Omega)$. We argue by contradiction, assuming that
\[
\|u_{\lambda,r}\|_{H_\mu(\Omega)} \to \infty \quad \text{as } r \downarrow 0. \tag{2.32}
\]
From the weak formulation of the problems $(A)_r$ we get that $u_{\lambda,r}$ satisfies the equation
\[
\int_{\Omega_r} |\nabla u_{\lambda,r}|^2 \, dx - \mu \int_{\Omega_r} \frac{|u_{\lambda,r}|^2}{|x|^2} = \lambda \int_{\Omega_r} |u_{\lambda,r}|^2 \, dx - \int_{\Omega_r} |u_{\lambda,r}|^{2\gamma+2} \, dx, \tag{2.33}
\]
which implies that
\[
\|u_{\lambda,r}\|_{H_\mu(\Omega)} \leq \lambda \|u_{\lambda,r}\|_{L^2(\Omega)}, \tag{2.34}
\]
for any $r$ small enough. Setting
\[
\tilde{u}_{\lambda,r} = \frac{u_{\lambda,r}}{\|u_{\lambda,r}\|_{H_\mu(\Omega)}},
\]
we get that $\|\tilde{u}_{\lambda,r}\|_{H_\mu(\Omega)} = 1$, for any $r > 0$ small enough. Consequently (up to a subsequence) $\tilde{u}_{\lambda,r}$ converges weakly to some $\tilde{u}_*$ in $H_\mu(\Omega)$, as $r \downarrow 0$, and so $u_{\lambda,r} \to u_*$ in $L^2(\Omega)$ as well as in $L^{2\gamma+2}(\Omega)$, as $r \downarrow 0$. In addition, it follows from (2.33) that
\[
\|\tilde{u}_{\lambda,r}\|_{H_\mu(\Omega)} \leq \lambda \|\tilde{u}_{\lambda,r}\|_{L^2(\Omega)}, \quad \text{for any } r > 0,
\]
We conclude in this section, with the discussion on the properties of the global branches 2.4 Behavior of the branch $C$

Proof of Proposition 1.2: We start with the proof of Proposition 1.2, which actually shows that the global bifurcation is of supercritical type.

Proof of Theorem 1.3: We are making use of Whyburn’s Theorem (see [14] and the references therein).

Proof of Proposition 1.2: (i) Assume by contradiction that $C_{\lambda, \mu}$ bends to the left of $\lambda_{1, \mu}$. Then there exists a pair $(\lambda, u) \in \mathbb{R} \times H_\mu(\Omega)$ with $0 < \lambda < \lambda_{1, \mu}$, such that

$$
\int_{\Omega} |\nabla u|^2 \, dx - \mu \int_{\Omega} \frac{u^2}{|x|^2} \, dx = \lambda \int_{\Omega} |u|^2 \, dx - \int_{\Omega} |u|^{2r+2} \, dx,
$$

Last equation implies that

$$
||u||_{H_\mu(\Omega)}^2 \leq \lambda ||u||_{L^2(\Omega)}^2, \quad \text{with} \ \lambda < \lambda_{1, \mu},
$$

contradicting the variational characterization of $\lambda_{1, \mu}$. Thus, $C_{\lambda, \mu}$ must bend to the right of $\lambda_{1, \mu}$. To show that $C_{\lambda, \mu}$ is bounded for $\lambda$ bounded, we consider the weak formula satisfied by any $u \in C_{\lambda, \mu}$,

$$
\int_{\Omega} \nabla u \nabla \psi \, dx - \int_{\Omega} \frac{u\psi}{|x|^2} \, dx - \lambda \int_{\Omega} u\psi \, dx + \int_{\Omega} |u|^{2r} u\psi \, dx = 0, \quad \text{for all} \ \psi \in H_\mu(\Omega).
$$

2.4 Behavior of the branch $C_{\lambda, \mu}$ as $\mu \to \mu^*$

We conclude in this section, with the discussion on the properties of the global branches $C_{\lambda, \mu}$ when $0 < \mu \leq \mu^*$. We start with the proof of Proposition 1.2 which actually shows that the global bifurcation is of supercritical type.

Proof of Proposition 1.2: (i) Assume by contradiction that $C_{\lambda, \mu}$ bends to the left of $\lambda_{1, \mu}$. Then there exists a pair $(\lambda, u) \in \mathbb{R} \times H_\mu(\Omega)$ with $0 < \lambda < \lambda_{1, \mu}$, such that

$$
\int_{\Omega} |\nabla u|^2 \, dx - \mu \int_{\Omega} \frac{u^2}{|x|^2} \, dx = \lambda \int_{\Omega} |u|^2 \, dx - \int_{\Omega} |u|^{2r+2} \, dx,
$$

Last equation implies that

$$
||u||_{H_\mu(\Omega)}^2 \leq \lambda ||u||_{L^2(\Omega)}^2, \quad \text{with} \ \lambda < \lambda_{1, \mu},
$$

contradicting the variational characterization of $\lambda_{1, \mu}$. Thus, $C_{\lambda, \mu}$ must bend to the right of $\lambda_{1, \mu}$. To show that $C_{\lambda, \mu}$ is bounded for $\lambda$ bounded, we consider the weak formula satisfied by any $u \in C_{\lambda, \mu}$,

$$
\int_{\Omega} \nabla u \nabla \psi \, dx - \int_{\Omega} \frac{u\psi}{|x|^2} \, dx - \lambda \int_{\Omega} u\psi \, dx + \int_{\Omega} |u|^{2r} u\psi \, dx = 0, \quad \text{for all} \ \psi \in H_\mu(\Omega).
$$
Setting \( \psi = u \) in (2.37) and using the inequality
\[
2\lambda \int_\Omega |u|^2 \, dx \leq 2\lambda |\Omega|^{\frac{2}{2\gamma + 2}} ||u||^2_{L^{2\gamma + 2}} \leq \frac{1}{2} ||u||^2_{L^{2\gamma + 2}} + R_0, \tag{2.38}
\]
we get that any \( u \in C_{\lambda_n, \mu} \), satisfies the bound
\[
||u||^2_{H_{\mu_n}(\Omega)} \leq R_0. \tag{2.39}
\]
The bound (2.39), shows that any \( u \in C_{\lambda_n, \mu} \), is bounded for each fixed \( \lambda \).

(ii) Let \( u \in C_{\lambda_n, \mu} \), and suppose that \( v \) is a nonnegative solution of (1.3) with \( u \neq v \). Considering the approximating solutions \( u_{\lambda, \gamma} \) of the problems \((A)_r\), we get from Proposition 2.8 (ii) (comparison principle) that
\[
u_{\lambda, \gamma}(x) \leq \min_{x \in \Omega} \{u(x), v(x)\}. \tag{2.40}
\]
Then, by the \( L^{2\gamma}_0 \)-convergence of \( u_{\lambda, \gamma} \) to \( u \) of Lemma 2.8 (iii) and (2.40), we infer that
\[
u(x) \leq v(x). \tag{2.41}
\]
We apply next the weak formula (2.37) for the solutions \( u \) and \( v \), setting \( \psi = v \) and \( \psi = u \) respectively. Subtracting the resulting equations, we get that
\[
\int_\Omega (|u_\gamma^2 v - |v_\gamma^2 u|) \, dx = 0,
\]
contradicting (2.41), unless \( u \equiv v \), \( \blacksquare \)

Finally, we discuss the behavior of the branches \( C_{\lambda_n, \mu} \), as \( \mu \uparrow \mu^* \). The eigenfunction \( u_{1, \mu^*} \) does not belong in \( H^2_0(\Omega) \), although the eigenfunctions \( u_{1, \mu}, 0 < \mu < \mu^* \), belong in \( H^1_0(\Omega) \). Therefore, the behavior of the branches \( C_{\lambda_n, \mu} \) as \( \mu \uparrow \mu^* \) should be completely different if considered in \( H^{\mu^*}_0(\Omega) \) and in \( H^1_0(\Omega) \) respectively.

**Proof of Theorem 1.4** A. By assumption, the pair \( (\lambda_n, u_n) \), satisfies
\[
\int_\Omega |\nabla u_n|^2 \, dx - \mu_n \int_\Omega \frac{|u_n|^2}{|x|^2} \, dx = \lambda_n \int_\Omega |u_n|^2 \, dx - \int_\Omega |u_n|^{2\gamma + 2} \, dx, \tag{2.42}
\]
which implies that
\[
\int_\Omega |\nabla u_n|^2 \, dx - \mu_n \int_\Omega \frac{|u_n|^2}{|x|^2} \, dx \leq \lambda_n \int_\Omega |u_n|^2 \, dx. \tag{2.43}
\]
On the other hand, by the definition of the \( H^{\mu^*}_0(\Omega) \)-norm and the hypothesis \( \mu_n \uparrow \mu^* \), it follows that
\[
||u_n||_{H^{\mu^*}_0(\Omega)} \leq \int_\Omega |\nabla u_n|^2 \, dx - \mu_n \int_\Omega \frac{|u_n|^2}{|x|^2} \, dx. \tag{2.44}
\]
Combining (2.43) and (2.44) with the assumption that \( |\lambda_n| \leq L \), we get the estimate
\[
||u_n||_{H^{\mu^*}_0(\Omega)} \leq \lambda_n ||u_n||_{L^2(\Omega)} + L ||u_n||_{L^2(\Omega)} \tag{2.45}
\]
We employ an argument similar to the one used in the proof of Proposition 2.8, assuming by contradiction that \( ||u_n||_{H^{\mu^*}_0(\Omega)} \to \infty \) as \( n \to \infty \). We consider the normalization \( \hat{u}_n \) of \( u_n \) in \( H^{\mu^*}_0(\Omega) \),
\[
\hat{u}_n = \frac{u_n}{||u_n||_{H^{\mu^*}_0(\Omega)}},
\]
which is a bounded sequence in \( H^{\mu^*_n}(\Omega) \). Hence, we may extract a subsequence (not relabelled), converging weakly to some \( \hat{u}_* \) in \( H^{\mu^*_n}(\Omega) \). The compact embedding \( H^{\mu^*_n}(\Omega) \to L^2(\Omega) \) and inequality (2.45) imply that \( \hat{u}_* \neq 0 \). Dividing (2.43) by \( ||u_n||_{H^{\mu^*}_0(\Omega)} \), we get the inequality
\[
\int_\Omega |\nabla \hat{u}_n|^2 \, dx - \mu_n \int_\Omega \frac{|\hat{u}_n|^2}{|x|^2} \, dx \leq \lambda_n \int_\Omega |\hat{u}_n|^2 \, dx \to \infty. \tag{2.46}
\]
Moreover, dividing (2.42) by $||u_n||_{H^{\mu^*}\ast}(\Omega)$, we get the equation
\[
\int_{\Omega} |\nabla u_n|^{2\gamma + 2} dx = \frac{\lambda_n}{||u_n||_{H^{\mu^*}\ast}(\Omega)} \int_{\Omega} |\nabla u_n|^2 dx - \frac{1}{||u_n||_{H^{\mu^*}\ast}(\Omega)} \left( \int_{\Omega} |\nabla u_n|^2 dx - \mu_n \int_{\Omega} |u_n|^2 dx \right). \tag{2.47}
\]
Passing to the limit in (2.47) as $n \to \infty$, we deduce that $u_\ast \equiv 0$, which is the contradiction. Thus $u_n$ must be bounded in $H^{\mu^*}\ast$, and (up to some subsequence) converges weakly to some $u_\ast$ in $H^{\mu^\ast}_{\ast}(\Omega)$.

The strong convergence $(\lambda_n, u_n) \to (\lambda_\ast, u_\ast)$ in $\mathbb{R} \times H^{\mu^*}\ast(\Omega)$, follows from the compactness of the embedding $H^{\mu^*}\ast(\Omega) \hookrightarrow L^\infty(\Omega)$ and (2.45). Let us remark that if $u_\ast \equiv 0$, the same argument implies that $u_n \to 0$ in $H^{\mu^\ast}_{\ast}(\Omega)$. In this case, division of (2.42) by $||u_n||_{H^{\mu^*}\ast}(\Omega)$ and passage to the limit, shows that $\lambda_n \to \lambda_{\ast, \mu^\ast}$.

It remains to prove that the limit $(\lambda_\ast, u_\ast) \in C_{\lambda, \mu^\ast}$. Note that for any $\phi \in C_0^\infty(\Omega)$,
\[
\int_{\Omega} \nabla u_n \nabla \phi dx - \mu^\ast \int_{\Omega} \frac{u_n \phi}{|x|^2} dx - (\mu_n - \mu^\ast) \int_{\Omega} \frac{u_n \phi}{|x|^2} dx = \lambda_n \int_{\Omega} u_n \phi dx - \int_{\Omega} |u_n|^{2\gamma} u_n \phi dx.
\]
Passing to the limit as $n \to \infty$, we need to show that the integral
\[
\int_{\Omega} \frac{u_n \phi}{|x|^2} dx,
\]
remains bounded for any $\phi \in C_0^\infty(\Omega)$ and any $n \in \mathbb{N}$. This claim follows by Hölder’s inequality and the continuous embedding $H^{\mu^*}_{\ast}(\Omega) \hookrightarrow L^p(\Omega)$, since
\[
\left| \int_{\Omega} \frac{u_n \phi}{|x|^2} dx \right| \leq ||\phi||_{L^\infty(\Omega)} ||u_n||_{L^p(\Omega)} \int_{\Omega} |x|^{-\frac{2N\gamma}{N-1-\gamma}} dx. \tag{2.48}
\]
The integral in the right hand side of (2.48) converges if $q > \frac{N}{N-1-\gamma}$. Combining this requirement with (1.5), the condition (1.6) follows for the case $N = 3$. When $N \geq 4$, the claim is valid under the condition (1.5).

B. Let $(\lambda_n, u_n) \in C_{\lambda, \mu^\ast}$, and assume that $\mu_n \uparrow \mu^\ast$ and $\lambda_n \to \lambda_{\ast, \mu^\ast}$ as $n \to \infty$. Assuming further that $u_n$ remains bounded in $H^1_0(\Omega)$, we may extract a subsequence still denoted by $u_n$, which converges weakly to some $u_\ast$ in $H^1_0(\Omega)$. Passing to the limit in the weak formula as $n \to \infty$, it follows that $u_\ast, \mu^\ast, \lambda_{\ast, \mu^\ast}$ satisfy
\[
\int_{\Omega} \nabla u_\ast \nabla \phi dx - \mu^\ast \int_{\Omega} \frac{u_\ast \phi}{|x|^2} dx = \lambda_{\ast, \mu^\ast} \int_{\Omega} u_\ast \phi dx - \int_{\Omega} |u_\ast|^{2\gamma} u_\ast \phi dx,
\]
for any $\phi \in C_0^\infty(\Omega)$. However, the variational characterization of $\lambda_{\ast, \mu^\ast}$ implies that this is true only if $u_\ast \equiv 0$. Therefore $u_n \to 0$, in $H^1_0(\Omega)$. On the other hand, arguing as in part A., it can be seen from (2.42) that the normalization $u_n = u_n / ||u_n||_{H^1_0(\Omega)}$ converges (up to a subsequence) weakly to $u_{\ast, \mu^\ast}$ in $H^1_0(\Omega)$ which is impossible. Thus, $u_n$ must be unbounded in $H^1_0(\Omega)$. \[\blacksquare\]

### 3 Definition of a gradient semiflow

In this section we shall define a gradient semiflow associated to the semilinear parabolic equation (1.2),
\[
S(t) : H^{\mu}(\Omega) \to H^{\mu}(\Omega), \quad 0 < \mu \leq \mu^*, \tag{3.49}
\]
with
\[
\mathcal{J}(\phi) := \frac{1}{2} \int_{\Omega} |\nabla \phi|^2 dx - \frac{\mu}{2} \int_{\Omega} |\phi|^2 dx - \frac{\lambda}{2} \int_{\Omega} |\phi|^2 dx + \frac{1}{2\gamma + 2} \int_{\Omega} |\phi|^{2\gamma + 2} dx, \quad 0 < \mu \leq \mu^*, \tag{3.50}
\]
as a Lyapunov functional. In subsection 3.1 we discuss the stability properties of the equilibrium solutions by linearization. In subsection 3.2 and by following closely the general semiflow theory \cite{2, 17, 28}, we present the proof of Theorem 1.5 as well as the description of the limit set $\omega(\phi_0)$ for nonnegative (nonpositive) initial data $\phi_0$, $\phi_0 \neq 0$ given in Corollary 1.6.
3.1 Stability of equilibrium solutions by linearization

Seeking for nonpositive stationary solutions \( u = -u_\ast \) with \( u_\ast \geq 0, u_\ast \neq 0 \), it is clear that \( u_\ast \) satisfies (1.3). Therefore, Theorem 1.1 can be restated as

**COROLLARY 3.1** Let \( \Omega \subset \mathbb{R}^N, N \geq 3 \), be a bounded domain. Assume that \( 0 < \mu \leq \mu^* \), and that condition (1.4) is satisfied. Then, the principal eigenvalue \( \lambda_{1,\mu} \) of (1.4) considered in \( H^1_\mu(\Omega) \), is a bifurcating point of the problem (1.3) (in the sense of Rabinowitz) and \( C_{\lambda_{1,\mu}} \) and \( C_{-\lambda_{1,\mu}} \) are global branches of nonnegative and nonpositive \( H^1_\mu(\Omega) \)-solutions respectively, which bend to the right of \( \lambda_{1,\mu} \). For any fixed \( \lambda > \lambda_{1,\mu} \), every solution \( u \in C_{\lambda_{1,\mu}} \) and \( u_\ast \in C_{-\lambda_{1,\mu}} \) is the unique nonnegative and unique nonpositive solutions for the problem (1.3) and \( u_\ast = -u \).

We first verify that solutions of (1.2) initiating from nonnegative (nonpositive) initial data remain nonnegative (nonpositive) for all times. Then, we will proceed with the asymptotic stability of the nonnegative equilibrium by linearization. For the latter, Hardy’s inequalities and their improvements, allow for the definition of appropriate Garding forms, helping us to verify that zero is not an eigenvalue for the linearized flow around the nonnegative (nonpositive) equilibrium.

**LEMMA 3.2** Assume that \( \mu \leq \mu^* \). The set

\[
D_{+(-)} := \{ \phi \in H^1_\mu(\Omega) : \phi(x) \geq (\leq) 0 \text{ on } \Omega \},
\]

is a positively invariant set for the semiflow \( S(t) \).

**Proof:** The argument of [11] Proposition 5.3.1 for the linear heat equation, can be repeated here (see also [18]). We assume that \( \phi_0 \in H^1_\mu(\Omega), \phi_0 \geq 0 \text{ a.e in } \Omega, \) and \( \phi(t) = S(t)\phi_0 \), the global in time solution of (1.2), initiating from \( \phi_0 \). We consider \( \phi^+ := \max\{\phi, 0\}, \phi^- := -\min\{\phi, 0\} \). Both \( \phi^+ \) and \( \phi^- \) are nonnegative, and \( \phi = \phi^+ - \phi^- \).

It can be seen from (1.2) that \( \phi^- \) satisfies the equation

\[
\partial_t \phi^- - \Delta \phi^- - \mu \frac{\phi^-}{|x|^2} - \lambda \phi^- + |\phi|^2 \gamma \phi^- = 0.
\]  

Moreover, \( \phi^- \) satisfies the energy equation (see Proposition 1.5),

\[
\frac{1}{2} \frac{d}{dt} \|\phi^-\|_{L^2}^2 + \int_{\Omega} |\nabla \phi^-|^2 dx - \mu \int_{\Omega} \frac{|\phi^-|^2}{|x|^2} dx - \lambda \|\phi^-\|_{L^2}^2 + \int_{\Omega} |\phi|^2 \gamma |\phi^-|^2 dx = 0.
\]

From (3.2) and (2.19), we get that

\[
\frac{1}{2} \frac{d}{dt} \|\phi^-\|_{L^2}^2 \leq c \|\phi^-\|_{L^2}^2.
\]

where \( c = \lambda_{1,\mu} - \lambda \). Thus \( \phi^- \) satisfies

\[
\|\phi^- - t\|_{L^2}^2 \leq e^{ct} \|\phi_0^-\|_{L^2}^2 = 0, \text{ for every } t \in [0, +\infty),
\]

implying that \( \phi \geq 0 \) for all \( t \in (0, +\infty) \), a.e. in \( \Omega \).

**PROPOSITION 3.3** Let \( \mu \leq \mu^* \). The unique nonnegative (nonpositive) equilibrium point which exists for \( \lambda > \lambda_{1,\mu} \) is uniformly asymptotically stable.

On the account of Corollary 3.1 we consider only the nonnegative equilibrium \( u \geq 0, u \neq 0 \). First, we observe that the linearized semiflow around the zero solution, is defined by the Cauchy-Dirichlet problem

\[
\partial_t \psi - \Delta \psi - \mu \frac{u}{|x|^2} \psi - \lambda \psi = 0, \quad x \in \Omega,
\]

\[
\psi|_{\partial \Omega} = 0.
\]

We have that \( \psi = 0 \) is asymptotically stable in \( H^1_\mu(\Omega) \) if \( \lambda \leq \lambda_{1,\mu} \), and unstable in \( H^1_\mu(\Omega) \) if \( \lambda > \lambda_{1,\mu} \). The linearized semiflow around the nonnegative equilibrium point \( u \) of (1.2), is defined by the Cauchy-Dirichlet problem

\[
- \Delta \psi - \mu \frac{\psi}{|x|^2} - \lambda \psi + (2\gamma + 1)|u|^{2\gamma} \psi = 0,
\]

\[
\psi|_{\partial \Omega} = 0,
\]
To confirm the asymptotic stability of $u$, we will prove that $\tilde{\mu} = 0$, is not an eigenvalue for the eigenvalue problem

$$- \Delta \psi - \mu \frac{\psi}{|x|^2} - \lambda \psi + (2\gamma + 1)|u|^{2\gamma} \psi = \tilde{\mu} \psi, \quad \psi|_{\partial \Omega} = 0. \quad (3.5)$$

The weak formulation of (3.5) is

$$A_\mu(\psi, \omega) := \int_\Omega \nabla \psi \nabla \omega \, dx - \mu \int_\Omega \frac{\psi \omega}{|x|^2} \, dx - \lambda \int_\Omega \psi \omega \, dx + (2\gamma + 1) \int_\Omega |u|^{2\gamma} \psi \omega \, dx = \tilde{\mu} \int_\Omega \psi \omega \, dx, \quad (3.6)$$

for every $\omega \in H_\mu(\Omega)$. Using the improved Hardy’s inequality and the properties of the $H_\mu(\Omega)$-space for any $0 < \mu \leq \mu^*$, we may consider a symmetric bilinear form $A_\mu : H_\mu(\Omega) \times H_\mu(\Omega) \to \mathbb{R}$, which in turns, defines a Garding form [25, pg. 366]: Since

$$A_\mu(\psi, \psi) \geq ||\psi||^2_{H_\mu(\Omega)} - \lambda ||\psi||^2_{L^2(\Omega)},$$

Garding’s inequality is satisfied. Then, it follows from [25, Theorem 22.G pg. 369-370] and (2.8), that the problem (3.5) has infinitely many eigenvalues of finite multiplicity. Counting the eigenvalues according to their multiplicity, we derive the sequence

$$-\lambda < \tilde{\mu}_1 \leq \tilde{\mu}_2 \leq \cdots, \quad \text{and} \quad \tilde{\mu}_j \to \infty \quad \text{as} \quad j \to \infty. \quad (3.7)$$

The smallest eigenvalue can be characterized by the minimization problem

$$\tilde{\mu}_1 = \min A_\mu(\psi, \psi), \quad \psi \in H_\mu(\Omega), \quad ||\psi||_{L^2} = 1. \quad (3.8)$$

The $j$-th eigenvalue, can be characterized by the minimum-maximum principle

$$\tilde{\mu}_j = \min_{M \in \mathcal{L}_j} \max_{\psi \in M} A_\mu(\psi, \psi), \quad (3.9)$$

where $M = \{\psi \in H_\mu(\Omega) : ||\psi||_{L^2} = 1\}$ and $\mathcal{L}_j$ denotes the class of all sets $M \cap L$ with $L$ an arbitrary $j$-dimensional linear subspace of $H_\mu(\Omega)$.

By using similar arguments as for the proof of Lemma 2.4 (see also Lemma 2.6), we may see that for (3.5), the (nontrivial) eigenfunction corresponding to the principal eigenvalue $\tilde{\mu}_1$ is nonnegative, i.e $\psi_1 \geq 0$ a.e. on $\Omega$. Since $\tilde{\mu}_1, \psi_1$ satisfy (3.6) we get by setting $\omega = u$ that

$$\int_\Omega \nabla \psi_1 \nabla u \, dx - \mu \int_\Omega \frac{\psi_1 u}{|x|^2} \, dx - \lambda \int_\Omega \psi_1 u \, dx + (2\gamma + 1) \int_\Omega |u|^{2\gamma} \psi_1 u \, dx = \tilde{\mu}_1 \int_\Omega \psi_1 u \, dx. \quad (3.10)$$

On the other hand, by setting $\psi = \psi_1$ to the weak formula (2.37) we get

$$\int_\Omega \nabla \psi_1 \nabla u \, dx - \mu \int_\Omega \frac{\psi_1 u}{|x|^2} \, dx - \lambda \int_\Omega \psi_1 u \, dx + \int_\Omega |u|^{2\gamma} \psi_1 u \, dx = 0. \quad \text{Subtracting these equations, we obtain that} \quad 2\gamma \int_\Omega |u|^{2\gamma} u \psi_1 \, dx = \tilde{\mu}_1 \int_\Omega u \psi_1 \, dx.$$
3.2 Global attractor in $H_\mu(\Omega)$ for any $0 < \mu \leq \mu^*$

The proof of Proposition 1.5 is based on the analogue of [2] Theorem 3.6, pg. 40, this time for the parabolic equation (1.2).

**Proof of Proposition 1.5.** It follows from [24], that the operator $L = -\Delta - \mu \frac{\partial}{|x|^2}$ with domain of definition (2.4) is a generator of a strongly continuous semigroup $T(t)$, for any $0 < \mu < \mu^*$, while the function $f(s) = |s|^{2\gamma} s - \lambda s$, defines a locally Lipschitz map $f : H_\mu(\Omega) \to L^2(\Omega)$ as it can be easily deduced by Lemma 2.2. Thus for any $0 < \mu \leq \mu^*$ and any $\phi_0 \in H_\mu(\Omega)$, there exists a unique solution $\phi(t)$ of (1.2), defined on a maximal interval $[0, T_{max})$ and in the class $C([0,T]; H_\mu(\Omega) \cap C^1([0,T]; L^2(\Omega))$. The solution satisfies the variation of constants formula

$$\phi(t) = T(t)\phi_0 + \int_0^t T(t-s)f(\phi(s))ds.$$  \hspace{1cm} (3.11)

Lemma 2.2 implies also that the functional $J \in C^1(\mathbb{R}, H_\mu(\Omega))$, for any $0 < \mu \leq \mu^*$. Moreover, for all $\phi \in D(L)$ and any $t \in [0,T]$, $T < T_{max}$,

$$\left< \Delta \phi + \mu \frac{\phi}{|x|^2} + f(\phi), J'(\phi) \right> = - \int_\Omega \left| \Delta \phi + \mu \frac{\phi}{|x|^2} + f(\phi) \right|^2 dx = - \int_\Omega |\partial_t \phi|^2 dx \leq 0.$$ \hspace{1cm} (3.12)

Setting $h(t) = f(\phi(t))$, we consider the sequence $h_n(t) \in C^1([0,T]; H_\mu(\Omega))$ and $\phi_{0n} \in D(L)$ such that

$$h_n \to h, \text{ in } C^1([0,T]; H_\mu(\Omega)), \phi_{0n} \to \phi_0, \text{ in } H_\mu(\Omega).$$

We define $\phi_n(t) = T(t)\phi_{0n} + \int_0^t T(t-s)h_n(s)ds$, and it follows from [19] Corollary 2.5, p107 that $\phi_n(t) \in D(L)$, $\phi_n \in C^1([0,T]; H_\mu(\Omega))$ and that they satisfy

$$\partial_t \phi_n - \Delta \phi_n - \mu \frac{\phi_n}{|x|^2} + f(\phi_n) = 0.$$ \hspace{1cm} (3.13)

Moreover, from [1] Lemma 5.5, pg. 246-247 or [2] Theorem 3.6, pg. 41, we deduce that

$$\phi_n \to \phi, \text{ in } H_\mu(\Omega).$$

Finally, by using the continuity of $J$ and (3.12), and passing to the limit to the equation

$$J(\phi_n(t)) - J(\phi_{0n}) = \int_0^t \left< J'(\phi_n(s)), \Delta \phi_n(s) + \mu \frac{\phi_n(s)}{|x|^2} + h_n(s) \right> ds$$

$$= - \int_\Omega |\partial_t \phi_n(s)|^2 dx + \int_0^t \left< J'(\phi_n(s)), h_n(s) - f(\phi_n(s)) \right> ds,$$

we derive

$$\frac{d}{dt} J(\phi(t)) = - \int_\Omega |\partial_t \phi|^2 dx, \text{ for all } 0 < \mu \leq \mu^* \text{ and } t \in [0,T], T < T_{max}.$$ \hspace{1cm} (3.14)

From (3.14) we infer that the unique solution $\phi$, satisfies the energy equation

$$\frac{1}{2} \frac{d}{dt} ||\phi||_{L^2}^2 + \int_\Omega |\nabla \phi|^2 dx - \mu \int_\Omega \frac{||\phi||_{L^2}^2}{|x|^2} - \lambda ||\phi||_{L^2}^2 + \int_\Omega |\phi|^{2\gamma+2} dx = 0, \text{ for all } 0 < \mu \leq \mu^*.$$ \hspace{1cm} (3.15)

When $\lambda \leq \lambda_{1,\mu}$, we observe by using (2.19), that $\lim_{t \to \infty} ||\phi(t)||_{L^2}^2 = 0$. For the case $\lambda > \lambda_{1,\mu}$, we insert (2.38) to (3.15), to get the estimate

$$\frac{1}{2} \frac{d}{dt} ||\phi||_{L^2}^2 + \frac{1}{2} ||\phi||_{H^s_\mu(\Omega)}^2 + \lambda ||\phi||_{L^2}^2 + \frac{1}{2} ||\phi||_{L^{2\gamma+2}}^{2\gamma+2} dx \leq R_0.$$
Then by Gronwall’s Lemma

\[ ||φ(t)||_{L^2}^2 \leq ||φ(0)||_{L^2}^2 \exp(-2λt) + \frac{R_0}{λ}(1 - \exp(-2λt)). \]  (3.16)

Letting \( t \to \infty \), from (3.16) we obtain that

\[ \limsup_{t \to \infty} ||φ(t)||_{L^2}^2 \leq ρ^2, \quad ρ^2 = \frac{R_0}{λ}. \]  (3.17)

Now assume that \( φ_0 \) is in a bounded set \( B \) of \( H_μ(Ω) \). Then (3.17) implies that for any \( ρ_1 > ρ \), there exists \( t_0(B, ρ_1) \), such that

\[ ||φ(t)||_{L^2}^2 \leq ρ_1^2, \quad \text{for any } t \geq t_0(B, ρ_1). \]  (3.18)

By the definition of the Lyapunov functional \( J \) and (3.18), we have the inequality

\[ J(φ(t)) \geq \frac{1}{2} \int_Ω |∇φ|^2 \, dx - \frac{μ}{2} \int_Ω \frac{|φ|^2}{|x|^2} \, dx - \frac{λ}{2} \int_Ω |φ|^2 \, dx \]

\[ \geq \frac{1}{2} \int_Ω |∇φ|^2 \, dx - \frac{μ}{2} \int_Ω \frac{|φ|^2}{|x|^2} \, dx - \frac{λ}{2} R_1^2, \quad t \geq t_0. \]  (3.19)

Since \( J \) is nonincreasing in \( t \), we conclude with the bound

\[ ||φ(t)||_{H_μ,Ω}^2 \leq 2J(φ_0) + λρ_1^2, \quad t \geq t_0. \]  (3.20)

establishing that solutions are globally defined in \( H_μ(Ω) \), for any \( 0 < μ \leq μ^* \) and \( λ > λ_{1,μ} \). In addition, (3.20) implies that the semiflow \( S(t) \) is eventually bounded and since the operator \( L \) has compact resolvent, \( S(t) \) is asymptotically compact (cf. [2, Proposition 2.3, pg. 36], [17, 23]). Thus, the positive orbit \( γ^+(φ_0) \) for any \( φ_0 \in H_μ(Ω) \) is precompact and has a nonempty compact and connected invariant \( ω \)-limit set \( ω(φ_0) \). Moreover (3.14) implies that \( ω(φ_0) \in E \). Equilibria of \( S(t) \) are extreme points of \( J \), satisfying the weak formula (2.37). From (2.39), we have that \( E \) is bounded for any fixed \( λ \). Hence \( S(t) \) is point dissipative. ■

Proof of Theorem 1.6. Lemma 3.2 and Proposition 1.5 imply that the solution \( φ(t) = S(t)φ_0 \), initiating from initial data \( φ_0 ≥ 0 \) (\( φ_0 ≤ 0 \), \( φ_0 \neq 0 \) converge towards the set of nonnegative (nonpositive) solutions of (1.3) as \( t \to \infty \), in \( H_μ(Ω) \), for any \( 0 < μ \leq μ^* \). In fact, it follows from Theorem 1.1 that the set of equilibrium solutions \( E = \{ u_-, 0, u \} \), when \( λ > λ_{1,μ} \), the trivial solution being unstable by Proposition 3.3. Thus for any nonnegative (nonpositive) initial condition \( φ_0, \omega(φ_0) = \{ u \} \) \( ω(φ_0) = \{ u_− \} \). While in the case \( λ < λ_{1,μ} \), Theorem 1.5 combined with Propositions 3.3 and 1.5 imply that \( \text{dist}(S(t)B, \{ 0 \}) \to 0 \) as \( t \to ∞ \), for every bounded set \( B \subset H_μ(Ω) \). Thus, when \( λ < λ_{1,μ} \) the global attractor \( A = \{ 0 \} \). ■

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