Remarks* on Kim’s
Strong Subadditivity Matrix Inequality:
Extensions and Equality Conditions

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Abstract

We describe recent work of Kim in [arXiv:1210.5190] to show that operator
convex functions associated with quasi-entropies can be used to prove a large
class of new matrix inequalities in the tri-partite and bi-partite setting by
taking a judiciously chosen partial trace over all but one of the spaces. We give
some additional examples in both settings. Furthermore, we observe that the
equality conditions for all the new inequalities are essentially the same as those
for strong subadditivity.

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*It is noteworthy that Kim’s note [arXiv:1210.5190] was posted very close to the 40th anniversary
of the proof of SSA which was completed in October, 1972.
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1 Background

In a recent paper Kim [16] showed that for operators on a tensor product space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ one can obtain an interesting new matrix inequality on one space $\mathcal{H}_C$ by taking the partial trace $\text{Tr}_{AB}$ over a quantity for which the full $\text{Tr}_{ABC}$ would yield strong subadditivity (SSA) of von Neumann entropy. A key ingredient is the inclusion of an additional operator $K$ used in the Wigner-Yanase-Dyson (WYD) skew information [30, 18], and then choosing a special form for $K$. As observed at the end of [16], Kim’s protocol can be applied to a large class of operator convex functions, including those associated with the WYD skew information, to produce additional matrix inequalities. Although Kim used the recent elegant approach of Effros [7], earlier work, going back to Petz [22], based on Araki’s relative modular operator [3] will suffice. We remark at the end on the different approaches which lead to well-known properties used below.

Kim’s result seems remarkable in view of certain well known facts. The concavity of the von Neumann entropy $S(\rho) = -\text{Tr} \rho \log \rho$ is an easy consequence of the much stronger operator convexity of $g(x) = x \log x$. (See, e.g., [4] and references therein.) However, it is also known that the operator $h(\rho, \gamma) = \sqrt{\rho} (\log \rho - \log \gamma) \sqrt{\rho}$ is not jointly operator convex, although one has separate operator convexity in the sense that the first term is operator convex in $\rho$ and the second in $\gamma$. A trace is needed to obtain the joint convexity of relative entropy $H(\rho, \gamma) = \text{Tr} \rho (\log \rho - \log \gamma)$. Thus, it is surprising that one can prove an operator inequality whose trace would yield something which is an immediate corollary of the joint convexity of relative entropy. It should perhaps be emphasized that this does not provide a new proof of SSA; rather, the new inequality emerges from the proof of a mild strengthening of SSA by the inclusion of an additional operator $K$ as in the WYD setting (from which the usual relative entropy can be obtained as a limit) and then making a judicious special choice for $K$ in the tripartite setting.

As he observed, Kim’s approach can be used with other convex operator functions; some examples are worked out below. In addition, we show that in almost all cases, the equality conditions are identical to those in [12] for strong subadditivity.
2 General theory

2.1 Some basics

Let $\mathcal{G}$ denote the class of operator convex functions $g(x)$ on $(0, \infty)$ with $g(1) = 1$. For any $g \in \mathcal{G}$, one can define a generalized relative entropy \cite{17}, also known as an $f$-divergence \cite{2, 11} or “quasi-entropy” \cite{21, 22} as

$$H_g(K, P, Q) \equiv \text{Tr} K^* g(L_{P} R_{Q}^{-1}) R_{Q}(K) = \text{Tr} K^* g(L_{P} R_{Q}^{-1}) K Q$$

(1)

where $P, Q > 0$ are positive definite matrices and $L_{P}(X) = PX$ and $R_{Q}(X) = XR$ denote left and right multiplication respectively. It is by now well-known that the map $(P, Q) \mapsto H_{g}(K, P, Q)$ is jointly convex in $P, Q$ for any pair of positive definite $P, Q$ and any fixed $K$. For $g(x) = -\log x$,

$$H_{-\log x}(K, P, Q) = \text{Tr} \left( K^* K Q \log Q - K Q K^* \log P \right)$$

(2)

which reduces to the usual relative entropy when $K = I$.

Whenever $g \in \mathcal{G}$, then $\tilde{g} \equiv xg(x^{-1})$ is also in $\mathcal{G}$ and

$$H_{\tilde{g}}(K^*, P, Q) = H_{g}(K, Q, P).$$

(3)

In the case $g(x) = -\log x$ above, $\tilde{g} = x \log x$ and

$$H_{x \log x}(K, P, Q) = \text{Tr} \left( K K^* P \log P - K^* P K \log Q \right)$$

(4)

Because $K \neq I$ is important in what follows, we observe that for any function $f$

$$\text{Tr} K^* f(L_{P}) K Q = \text{Tr} K^* f(P) K Q$$

but

$$\text{Tr} K^* f(R_{Q}) K Q = \text{Tr} K^* K f(Q) Q$$

which leads to the “sandwiched” expressions $K Q K^*$ in (2) and $K^* P K$ in (4). This is inevitable unless $K$ happens to commute with $Q$ and/or $P$.

It is also well-known that joint convexity of $H_{g}(P, Q)$ for fixed $K$ implies monotonicity under partial traces in the following sense

$$H_{g}(I_{A} \otimes K_{BC}, P_{ABC}, Q_{ABC}) \geq H_{g}(K_{BC}, P_{BC}, Q_{BC})$$

(5)

and that one can not replace $I_{A} \otimes K_{BC}$ by a general $K_{ABC}$. The best one can do is the minor generalization $V_{A} \otimes K_{BC}$ where $V_{A}$ is unitary \cite{14}. 


2.2 Main result

Using the definition (1), we can rewrite (5) as

\[ \text{Tr}_{ABC} I_A \otimes K_{BC}^* g(L_{P_{ABC}} R_{Q_{ABC}}^{-1}) K_{BC} Q_{ABC} \]
\[
\geq \text{Tr}_{BC} K_{BC}^* g(L_{P_{BC}} R_{Q_{BC}}^{-1}) K_{BC} Q_{BC} \] (6)
\[
= \text{Tr}_{ABC} K_{BC}^* g(L_{P_{BC}} R_{Q_{BC}}^{-1}) K_{BC} Q_{ABC} \]

where the equality follows from \( \text{Tr}_A Q_{ABC} = Q_{BC} \) since there is no explicit dependence on \( \mathcal{H}_A \) in the rest of the expression. We also suppress tensor products with the identity so that, e.g., it is understood that \( P_{AB} \) means \( P_{AB} \otimes I_C \), etc. Next, consider the special case \( P_{ABC} = P_{AB} \otimes I_C \) to get

\[ \text{Tr}_{ABC} I_A \otimes K_{BC}^* g(L_{P_{AB}} R_{Q_{ABC}}^{-1}) K_{BC} Q_{ABC} \]
\[
\geq \text{Tr}_{ABC} K_{BC}^* g(L_{P_{BC}} R_{Q_{BC}}^{-1}) K_{BC} Q_{ABC} \] (7)

and choose \( K_{BC} = I_B \otimes K_C \). Then \( K \) commutes with \( P_{ABC} = P_{AB} \otimes I_C \) so that (7) can be rewritten as

\[ \text{Tr}_{ABC} I_{AB} \otimes K_{BC}^* K_C g(L_{P_{AB}} R_{Q_{ABC}}^{-1}) K_{BC} Q_{ABC} \]
\[
\geq \text{Tr}_{ABC} I_{AB} \otimes K_{BC}^* K_C g(L_{P_{BC}} R_{Q_{BC}}^{-1}) K_{BC} Q_{ABC} \] (8)

Furthermore, we can choose \( K_C = |\phi_C\rangle \langle \phi_C| \) to be a rank one projection so that (8) becomes

\[ \langle \phi_C, \text{Tr}_{AB} \left[ g(L_{P_{AB}} R_{Q_{ABC}}^{-1}) - g(L_{P_{BC}} R_{Q_{BC}}^{-1}) \right] Q_{ABC} \phi_C \rangle \geq 0 \] (9)

where \( |\phi_C\rangle \) is an arbitrary vector in \( \mathcal{H}_C \). This implies that the operator

\[ \text{Tr}_{AB} \left[ g(L_{P_{AB}} R_{Q_{ABC}}^{-1}) - g(L_{P_{BC}} R_{Q_{BC}}^{-1}) \right] Q_{ABC} \]
\[
= \text{Tr}_{AB} g(L_{P_{AB}} R_{Q_{ABC}}^{-1}) Q_{ABC} - \text{Tr}_{B} g(L_{P_{BC}} R_{Q_{BC}}^{-1}) Q_{BC} \geq 0 \] (10)

is positive semi-definite on \( \mathcal{H}_C \).

We can choose \( \mathcal{H}_B \) to be one dimensional to obtain bipartite formulas, e.g.,

\[ \text{Tr}_A \left[ g(L_{P_{AC}} R_{Q_A}) - g(L_{P_{C}}^{-1}) \right] P_{AC} \geq 0 \] (11)

which is also useful. However, we presented the development in the tripartite situation because the most important application is to SSA.
2.3 Adjoint form

If one replaces \( g(x) \) by \( \tilde{g} = xg(x^{-1}) \) and interchanges \( P \leftrightarrow Q \), one obtains

\[
\text{Tr}_{AB} P_{ABC} \left[ g(L_{P_{AB}}^{-1} R_{Q_{AB}}) - g(L_{P_{BC}}^{-1} R_{Q_{B}}) \right] \geq 0 \tag{12}
\]

in which the LHS is formally the adjoint of that in (10); however, since any positive semi-definite operator is self-adjoint, (12) is equivalent to (10). For additional insight into why this is so, recall the basic property that \( (WX)^* = X^*W^* \) reverses the order and hence, reverses left and right multiplication by self-adjoint operators. Thus

\[
[g(L_P R_Q^{-1}) R_Q(X)]^* = g(R_P L_Q^{-1}) L_Q(X^*)
\]

\[
= L_Q R_P^{-1} g\left[ (L_Q R_P^{-1})^{-1} \right] R_P(X^*)
\]

\[
= \tilde{g}(L_Q R_P^{-1}) R_P(X^*)
\]

For \( X = I_{AB} \otimes |\phi_C\rangle\langle \phi_C| = X^* \) the equivalence of (10) and (12) is then clear.

3 Specific inequalities

3.1 Subadditive type

The choice \( Q_{ABC} = \rho_{ABC}, P_{AB} = \rho_{AB} \) and \( g(x) = -\log x \) in (10) gives a result reminiscent of SSA, i.e.,

\[
\text{Tr}_{AB} \left[ \log \rho_{ABC} - \log \rho_{AB} - \log \rho_{BC} + \log \rho_B \right] \rho_{ABC} \geq 0 \tag{13a}
\]

as an operator inequality on \( \mathcal{H}_C \). Using, instead, \( \tilde{g}(x) = x \log x = xg(x^{-1}) \) and (12) with the choices \( P_{ABC} = \rho_{ABC}, Q_{AB} = \rho_{AB} \) gives the result in the form written by Kim in [16], i.e.,

\[
\text{Tr}_{AB} \rho_{ABC} \left[ \log \rho_{ABC} - \log \rho_{AB} + \log \rho_B - \log \rho_{BC} \right] \geq 0 \tag{13b}
\]

which is formally the adjoint of (13a). However, as remarked above, these are equivalent since a positive semi-definite operator on \( \mathcal{H}_C \) is necessarily self-adjoint. If one uses \( \tilde{g} = x \log x \) without the exchange \( P \leftrightarrow Q \), i.e, with the choice \( Q_{ABC} = \rho_{ABC}, P_{AB} = \rho_{AB} \), one gets

\[
\text{Tr}_{AB} \rho_{AB} \left[ \log \rho_{AB} - \log \rho_{ABC} - \log \rho_B + \log \rho_{BC} \right] \geq 0 \tag{14}
\]

in which the simple replacement of \( \rho_{ABC} \) by \( \rho_{AB} \) on the left appears to reverse the usual form of SSA. Note, however, that taking \( \text{Tr}_C \) in (14) does not yield SSA!

When \( \mathcal{H}_B \) is one dimensional, (13b) becomes

\[
\text{Tr}_A \rho_{AC} \left[ \log \rho_{AC} - \log \rho_A - \log \rho_C \right] \geq 0 \tag{15}
\]

which is an operator version of ordinary subadditivity.
3.2 Relative entropy

The more general choice \( Q_{AB} = \gamma_{AB} \) changes (14) to

\[
\text{Tr}_B \rho_{ABC} \left[ \log \rho_{ABC} - \log \gamma_{AB} - \log \rho_{BC} + \log \gamma_B \right] \geq 0
\]

which is naturally associated with the monotonicity of relative entropy under partial traces. When \( H_B \) is one dimensional, this becomes

\[
\text{Tr}_A \rho_{AC} \left[ \log \rho_{AC} - \log \gamma_A - \log \rho_C + \log \gamma_C \right] \geq 0
\]

and for \( \gamma_A = \frac{1}{d_A} I_A \)

\[
- \text{Tr}_A \rho_{AC} \log \rho_{AC} + \rho_C \log \rho_C \leq (\log d_A) \rho_C
\]

which gives an upper bound on an operator version of conditional information, although this can be negative in the quantum setting.

Choosing \( Q_{ABC} = Q_{AB} \otimes I_C \) in (12) is essential to ensure that it commutes with \( I_{AB} \otimes \mathcal{K}_C \). This precludes a proof of the full-fledged operator analogue of monotonicity of relative entropy by this method since

\[
\text{Tr}_A \rho_{AC} \left[ \log \rho_{AC} - \log \gamma_{AC} - \log \rho_C + \log \gamma_C \right]
\]

is not even Hermitian and, hence, can not be positive semi-definite.

3.3 WYD inequalities

The functions \( g(x) = \frac{1}{t(1-t)}(1 - x^t) \) and \( \tilde{g}(x) = \frac{1}{t(1-t)}(x - x^{1-t}) \) generate the WYD skew information, for which \( H_g(K, P, P) \geq 0 \) and \( H_g(K, P, Q) \) is jointly operator convex in \( P, Q \) in the maximal range \([-1, 2]\) as observed implicitly \footnote{Although first Lieb \cite{18} and then Ando \cite{4} obtained the key convexity result for the WYD entropy with \( t \in (0, 1) \), the seemingly innocuous omission of the obvious linear term precludes writing their results in the general framework used here. Ando also showed that the concavity of Lieb’s expression changes to convexity for \( t \in (1, 2) \).} in \cite{4} and explicitly by Hasegawa \cite{8}. (See also \cite{14}.)

Since, as is well known, \( \lim_{p \to 1} g(x) = -\log x \) and \( \lim_{p \to 0} \tilde{g}(x) = x \log x \), one can also recover the results of Section 3.1.

Using \( g(x) \) in (9) with \( Q_{ABC} = \rho_{ABC} \), and \( P_{AB} = \gamma_{AB} \) gives the inequalities

\[
\frac{1}{t(1-t)} \left[ \text{Tr}_A \rho_{ABC} - \text{Tr}_B \rho_{BC} \gamma_B^t \right] \geq 0
\]

for any \( t \in [-1, 2] \). Since \( \text{Tr}_A \rho_{ABC} = \text{Tr}_B \rho_{BC} = \rho_C \), this becomes

\[
\frac{1}{t(1-t)} \left[ \text{Tr}_A \rho_{ABC} \gamma_B^t \right] \geq 0
\]
where it is important to retain the factor \( \frac{1}{t(1-t)} \) which changes sign at \( t = 0, 1 \). When \( \mathcal{H}_B \) is one-dimensional \(^{[21]}\) implies

\[
\frac{1}{t(1-t)} \text{Tr}_A \rho_A^{1-t} \gamma_A \geq \frac{1}{t(1-t)} \rho_C^{1-t} \tag{22}
\]

### 3.4 But Cauchy-Schwarz matrix inequalities are not new

Using \( g(x) = (x - 1)^2 \) is equivalent to using \( x^2 \) since the linear terms cancel. This gives

\[
\text{Tr}_{AB} P_{AB}^2 Q_{ABC}^{-1} - \text{Tr}_B P_B^2 Q_{BC}^{-1} \geq 0 \tag{23}
\]

Since \( P_{AB} \) does not depend upon \( \mathcal{H}_C \) one can use the cyclicity of the trace to rewrite this in a more symmetric form as

\[
\text{Tr}_{AB} P_{AB} Q_{AC}^{-1} P_{AB} \geq \text{Tr}_B P_B Q_{BC}^{-1} P_B \tag{24}
\]

However, the inequality \(^{[24]}\) is not new; indeed when \( \mathcal{H}_B \) is one-dimensional it reduces to something slightly less general than

\[
\text{Tr}_A X_{AC}^* Q_{AC}^{-1} X_{AC} \geq X_{AC}^* Q_{AC}^{-1} X_{AC} \tag{25}
\]

which was proved\(^2\) in \(^{[20]}\) with \( X - AC \) arbitrary and \( Q_{AC} \) positive semi-definite with \( \ker Q_{AC} \subseteq \ker X_{AC}^* \). Moreover, this is equivalent to the joint operator convexity of the map \( (X, P) \mapsto X^* P^{-1} X \) also proved in \(^{[20]}\) by Lieb and Ruskai, who were unaware until 2010 that the latter had been proved much earlier by Kiefer \(^{[15]}\) in 1957.

However, the slightly modified joint convexity

\[
(X, P, Q) \mapsto \text{Tr} X^* \frac{1}{L_P + tR_Q} X \quad \forall \ t \in (0, \infty) \tag{26}
\]

does not hold as an operator inequality (even for \( P = Q \)) without the trace. Both \(^{[25]}\) and \(^{[20]}\) can be proved by very elementary and similar arguments, as shown in \(^{[20]}\) for the former and for the latter in \(^{[25]}\) and the Appendix of \(^{[14]}\). One can use \(^{[25]}\) to prove subadditivity and related inequalities, but the operator inequality \(^{[25]}\) does not suffice. This subtle difference makes it even more surprising that Kim’s method allows one to essentially extract operator inequalities from \(^{[26]}\) in the bi-partite and tri-partite settings.

\(^2\)In \(^{[20]}\) Lieb and Ruskai proved the slightly more general result that \( [\Phi(X)]^* \Phi(A)^{-1} \Phi(X) \leq \Phi(X^* A^{-1} X) \) for a completely positive map \( \Phi \), of which the partial trace is a special case. Later, Choi \(^{[5]}\) showed that the hypothesis could be weakened to 2-positivity. The special case \( \Phi(X)^* \Phi(X) \leq \Phi(X^* X) \) was shown earlier for unital maps by Kadison and played an important role in Petz’s work \(^{[22, 21, 11]}\).
3.5 More examples of new inequalities

Although the functions mentioned above are the most commonly considered, there are many more. As shown in [17], any operator convex function \( k(x) : (0, \infty) \mapsto (0, \infty) \) satisfying the symmetry condition \( k(x) = k(x^{-1}) \) defines an operator convex function \( g(x) = (1 - x)^2 k(x) \) with the symmetry property \( g(x) = x g(x^{-1}) = g(x) \) which can be used to define an \( H_g(K, P, Q) \) as above. The symmetrization

\[
g(x) + \tilde{g}(x) = -\log x + x \log x = (x - 1) \log x
\]
yields \( k(x) = (\log x) / (x - 1) \). However, the symmetrized version yields a less transparent inequality since one would have the sum of (13a) and (14).

Several families of functions \( k \) have been studied by Petz [23] (who uses \( f = 1/k \) operator monotone) in the context of monotone Riemannian metrics and, more recently, by Hiai and Kosaki [9, 10] who developed a theory of operator means. A fairly comprehensive list is given in [13, Section 4].

The symmetrized version of the primitive example in Section 3.4 is

\[
\frac{1}{2} [g(x) + \tilde{g}(x)] = (1 - x)^2 \frac{1 + x}{2x}
\]
which yields \( k(x) = (1 + x) / 2x \). It is well-known [23, 17, 13] that the functions \( k(x) \) satisfy a partial order with

\[
\frac{2}{1 + x} \leq k(x) \leq \frac{1 + x}{2x}
\]  \hspace{1cm} (27)

The smallest element \( k(x) = 2/(1 + x) \) is associated with the Bures metric but \( g(L_P R_Q^{-1}) R_Q = (L_P - R_Q)^2 / (L_P + R_Q) \) does not seem to yield particularly transparent inequalities when inserted in (10).

The function \( k(x) = x^{-1/2} \) also plays a special role in some situations [9, 10, 13, 26]. In this case, we can “unsymmetrize” to \( g(x) = x^{-1/2} - x^{1/2} \) and \( \tilde{g}(x) = x(x^{1/2} - x^{-1/2}) \) to obtain the inequality

\[
\text{Tr}_{AB} \gamma_{AB}^{-1/2} [P_{ABC} - \gamma_{AB}] \rho_{ABC}^{1/2} - \text{Tr}_B \gamma_B^{-1/2} [\rho_{BC} - \gamma_B] \rho_{BC}^{1/2} \geq 0 \]  \hspace{1cm} (28)

where we used \( g \) with \( P_{AB} = \gamma_{AB}, Q_{ABC} = \rho_{ABC} \).

4 Equality conditions

It is natural to ask under what conditions equality holds in these inequalities. In the case of those related to SSA, i.e., (13) and (14), it is easy to see that the equality conditions given in [12] suffice. In the simplest case, \( \mathcal{H}_B = \mathcal{H}_{B'} \otimes \mathcal{H}_{B''} \)
and \( \rho_{ABC} = \rho_{AB'} \otimes \rho_{B'C} \). The general case is a direct sum of this situation, i.e., \( \mathcal{H}_B = \bigoplus_k \mathcal{H}^k_B \otimes \mathcal{H}^k_{B'} \), and

\[
\rho_{ABC} = \bigoplus_k \rho^k_{AB'} \otimes \rho^k_{B'C} \tag{29}
\]

Since a positive semi-definite matrix \( A \geq 0 \) is equal to zero if and only if \( \text{Tr} A = 0 \), it is immediate that (29) is necessary and sufficient for equality. This also gives conditions for equality in (10) for the other examples with \( Q_{ABC} = \rho_{ABC} \) and \( P_{AB} = \rho_{AB} \) and are essentially independent of the function \( g \). This is because Nevanlinna’s theorem [1, Section 59, Theorem 2] implies that any operator convex function \( g \) on \((0, \infty)\) with \( g(1) = 0 \) has an integral representation of the general form

\[
g(x) = ax + bx^2 + \int_0^\infty \frac{f(x, t)}{x + t} d\mu_g(t) \tag{30}
\]

(The precise representations are written in equivalent, but slightly differently forms in several references, including eq. (8.2) in [11] or eq. (17) in [14] or eq. (13) in [17]. The details are not relevant here.) Whenever the corresponding measure \( \mu_g(t) \) is supported on \((0, \infty)\), the conditions (29) are necessary and sufficient for equality. In the approach of [14] the equality conditions arise as the condition for equality in the joint convexity in (26) for all \( t \in (0, \infty) \), which makes the somewhat surprising lack of dependence on \( g \) transparent.

For inequalities with more general choices of \( P_{AB} \) in (10), as in Sections 3.2, 3.3 and 3.5, \( \gamma_{AB} \) must also have a similar block representation with \( \rho^k_{AB'} = \gamma^k_{AB'} \) for equality.

5 Historical remarks

The proofs of joint convexity in \( P, Q \) of functions of the type defined in (11) for operator convex functions \( g \in \mathcal{G} \) are based on the relative modular operator \( \Delta_{PQ}^\dag \) introduced by Araki in a much more general context. The operator \( L/R \) in Effros’s perspective is essentially \( \Delta_{PQ} \). Using an integral representation of the form (30) one can reduce the joint convexity of \( (P, Q) \mapsto H_g(K, P, Q) \) to the joint convexity of the map in (26) which, as mentioned above, can be proved by a very elementary argument, as shown in [25] and the Appendix of [14].

Uhlmann [28] seems to have been the first to realize that one could take the partial trace by integrating unitary conjugations over Haar measure. A pedestrian equivalent is to use the discrete Weyl-Heisenberg group (as in, e.g., [14]), a process sometimes called “twirling”, although other orthogonal unitary bases can also be used [27]. Remarkably, Uhlmann [28] also realized that the concavity of the map \( A \mapsto \text{Tr} e^{K + \log A} \) could then be used to prove SSA. Without knowing about
Uhlmann’s work, Lieb found and proved this concavity \cite[Theorem 6]{18} which was the key ingredient in the two original proofs of SSA presented in \cite{19}, both of which are different from Uhlmann’s.

Instead of a two-step argument using joint convexity and unitary conjugation, one can go directly to the monotonicity under quantum channels, i.e., completely positive trace-preserving maps of which the partial trace is a special case. This was first done by Petz under the slightly weaker condition of 2-positivity and a form of Jensen’s inequality \cite{11, 21, 22}. Another argument was given in \cite{17} based on the integral representation of convex operator functions and an elementary Schwarz argument similar to the one in \cite{25} and \cite{14} Appendix.

One advantage of this approach is that one does not need to add the $\text{Tr}_{A} \frac{1}{\sigma_{A}}$ to the RHS of (5) as Kim did in \cite{16} to perform twirling. (See also \cite{14}.) One can prove (5) directly and then use $\text{Tr}_{AB} (\ _{A}^{B} Q_{AB} = \text{Tr}_{ABC} (\ _{A}^{B} Q_{ABC}$ as in the last line of (6).

It is also well-known that one can use a block matrix representation to show that monotonicity under partial traces implies convexity, as noted in \cite{18, 19, 24, 29}. See \cite{24, 29} for additional background.

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\footnote{To be more precise, Petz uses the weak Kadison form of the operator Schwarz inequality in Section 6.3 which follows from 2-positivity. See \cite{11} for details.}
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