New $q$-supercongruences arising from a summation of basic hypergeometric series

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Abstract. With the help of a summation of basic hypergeometric series, the creative microscoping method recently introduced by Guo and Zudilin, and the Chinese remainder theorem for coprime polynomials, we find some new $q$-supercongruences. Especially, we give a $q$-analogue of a formula due to Liu [J. Math. Anal. Appl. 497 (2021), Art. 124915].

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1 Introduction

For any complex variable $x$, define the shifted-factorial to be

$$(x)_0 = 1 \quad \text{and} \quad (x)_n = x(x+1) \cdots (x+n-1) \quad \text{when} \quad n \in \mathbb{Z}^+.$$ 

Let $p$ be an odd prime and let $\mathbb{Z}_p$ denote the ring of all $p$-adic integers. Define Morita’s $p$-adic Gamma function (cf. [15, Chapter 7]) by

$$
\Gamma_p(0) = 1 \quad \text{and} \quad \Gamma_p(n) = (-1)^n \prod_{1 \leq k < n \atop p \nmid k} k, \quad \text{when} \quad n \in \mathbb{Z}^+.
$$

Noting $\mathbb{N}$ is a dense subset of $\mathbb{Z}_p$ related to the $p$-adic norm $| \cdot |_p$, for each $x \in \mathbb{Z}_p$, the definition of $p$-adic Gamma function can be extended as

$$\Gamma_p(x) = \lim_{n \in \mathbb{N}, |x-n|_p \to 0} \Gamma_p(n).$$

Two properties of the $p$-adic Gamma function in common use can be stated as follows:

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\[
\frac{\Gamma_p(x + 1)}{\Gamma_p(x)} = \begin{cases} 
-x, & \text{if } p \nmid x, \\
-1, & \text{if } p \mid x,
\end{cases}
\]

\[
\Gamma_p(x)\Gamma_p(1 - x) = (-1)^{\langle -x \rangle_p - 1},
\]

where \(\langle x \rangle_p\) indicates the least nonnegative residue of \(x\) modulo \(p\), i.e., \(\langle x \rangle_p \equiv x \pmod{p}\) and \(\langle x \rangle_p \in \{0, 1, \ldots, p - 1\}\). In 2016, Long and Ramakrishna \cite{LR16}, Proposition 25] showed that, for any prime \(p\),

\[
\sum_{k=0}^{p-1} \frac{(1/3)_k^3}{k!^3} \equiv \begin{cases} 
\frac{\Gamma_p(1/3)}{p^3} \pmod{p^3}, & \text{if } p \equiv 1 \pmod{6}, \\
-\frac{p^2}{3} \frac{\Gamma_p(1/3)}{p^3} \pmod{p^3}, & \text{if } p \equiv 5 \pmod{6}.
\end{cases}
\]

Similarly, Liu \cite{Liu13} Theorem 1.1] proved that, for any prime \(p\),

\[
\sum_{k=0}^{p-1} \frac{(-1/3)_k^3}{k!^3} \equiv \begin{cases} 
-18p^2 \frac{\Gamma_p(2/3)}{p^3} \pmod{p^3}, & \text{if } p \equiv 1 \pmod{6}, \\
54 \frac{\Gamma_p(2/3)}{p^3} \pmod{p^3}, & \text{if } p \equiv 5 \pmod{6}.
\end{cases}
\]

For any complex numbers \(x\) and \(q\), define the \(q\)-shifted factorial to be

\[(x; q)_0 = 1 \quad \text{and} \quad (x; q)_n = (1 - x)(1 - xq) \cdots (1 - xq^{n-1}) \quad \text{when} \quad n \in \mathbb{Z}^+.
\]

For simplicity, we also adopt the compact notation

\[(x_1, x_2, \ldots, x_m; q)_n = (x_1; q)_n(x_2; q)_n \cdots (x_m; q)_n.
\]

Following Gasper and Rahman \cite{GR90}, define the basic hypergeometric series \(r+1\varphi_r\) by

\[r+1\varphi_r \left[a_1, a_2, \ldots, a_{r+1}; b_1, b_2, \ldots, b_r; q, z \right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_k}{(q, b_1, b_2, \ldots, b_r; q)_k} z^k.
\]

Recently, Guo \cite{Guo07} established three \(q\)-supercongruences via the creative microscoping method (introduced by Guo and Zudilin \cite{GuoZudilin06}), and the Chinese remainder theorem for polynomials. Similarly, Wei, Liu, and Wang \cite{WLW13} Theorems 1.1 and 1.2] provided a \(q\)-analogue of \((1.1)\). For more \(q\)-analogues of supercongruences, we refer the reader to \cite{Guo07,GuoZudilin06,LiuWang13,LiuWangZhang13,WeiLiuWang13,WeiZhang13,WeiZhangWang13,WeiZhangWang14}.

Let \([n] = (1 - q^n)/(1 - q)\) be the \(q\)-integer and \(\Phi_n(q)\) the \(n\)-th cyclotomic polynomial in \(q\):

\[\Phi_n(q) = \prod_{1 \leq k \leq n \atop \gcd(k, n) = 1} (q - \zeta^k),
\]

where \(\zeta\) is an \(n\)-th primitive root of unity. Motivated by the work just mentioned, we shall establish the following two theorems.

\[\text{(1.1)}\]

\[\text{(1.2)}\]
Corollary 1.3. Letting \( p \equiv 1 \) (mod 6) be an prime such that \( \Phi_p(q) \equiv 1 \) (mod 6), then, modulo \( \Phi_p(q)^3 \),

\[
\sum_{k=0}^{(2n+1)/3} \frac{(q^{-1}; q^3)^3_k}{(q^3; q^3)^3_k} q^{9k} \equiv q^{(2-2n)/3} (1 + q) \frac{(q; q^3)^2_{(2n+1)/3}}{(q^3; q^3)^2_{(2n+1)/3}}
\times \left\{ 3 - [2n]^2 \left( \sum_{i=1}^{(2n+1)/3} \frac{3q^{3i-2}}{[3i - 2]^2} - \frac{1 + 5q + 3q^2}{1 + q} \right) \right\}.
\]

Theorem 1.2. Let \( n \) be a positive integer with \( n \equiv 5 \) (mod 6). Then, modulo \( \Phi_n(q)^3 \),

\[
\sum_{k=0}^{(n+1)/3} \frac{(q^{-1}; q^3)^3_k}{(q^3; q^3)^3_k} q^{9k} \equiv q^{(2-n)/3} (1 + q) \frac{(q; q^3)^2_{(n+1)/3}}{(q^3; q^3)^2_{(n+1)/3}}
\times \left\{ \theta_n(q) + [n]^2 \left( \sum_{i=1}^{(n+1)/3} \frac{3q^{3i-2}}{[3i - 2]^2} - \frac{1 + 5q + 3q^2}{1 + q} \right) \right\},
\]

where

\[
\theta_n(q) = (1 - q - 3q^2)(1 - 2q^n) + (4 - 4q - 6q^2 + 3q^3)q^{2n}.
\]

It is not difficult to understand that Theorems 1.1 and 1.2 give a \( q \)-analog of (1.2).

Letting \( n = p \) be an prime and taking \( q \rightarrow 1 \) in the above two theorems, we obtain the following conclusions.

Corollary 1.3. Let \( p \) be an prime such that \( p \equiv 1 \) (mod 6). Then

\[
\sum_{k=0}^{(p+1)/3} \frac{(-1/3)^3_k}{k!^3} \equiv \frac{6(1/3)^2_{(2p+1)/3}}{(1)^2_{(2p+1)/3}} \left\{ 1 + 6p^2 - \sum_{i=1}^{(2p+1)/3} \frac{4p^2}{(3i - 2)^2} \right\} \pmod{p^3}.
\]

Corollary 1.4. Let \( p \) be an prime such that \( p \equiv 5 \) (mod 6). Then

\[
\sum_{k=0}^{(p+1)/3} \frac{(-1/3)^3_k}{k!^3} \equiv \frac{54(1/3)^2_{(p+1)/3}}{(1)^2_{(p-2)/3}} \left\{ 1 + \frac{p^2}{(p + 1)^2} \sum_{i=1}^{(p+1)/3} \frac{1}{(3i - 2)^2} \right\} \pmod{p^3}.
\]

In order to explain the equivalence of (1.2) and Corollaries 1.3 and 1.4, we need to verify the following relations.

Proposition 1.5. Let \( p \) be a prime such that \( p \equiv 1 \) (mod 6). Then

\[
\frac{(1/3)^2_{(2p+1)/3}}{1} \left\{ 1 + 6p^2 - \sum_{i=1}^{(2p+1)/3} \frac{4p^2}{(3i - 2)^2} \right\} \equiv -3p^2\Gamma_p(2/3)^6 \pmod{p^3}.
\]

Proposition 1.6. Let \( p \) be a prime such that \( p \equiv 5 \) (mod 6). Then

\[
\frac{(1/3)^2_{(p+1)/3}}{1} \left\{ 1 + \frac{p^2}{(p + 1)^2} \sum_{i=1}^{(p+1)/3} \frac{1}{(3i - 2)^2} \right\} \equiv \Gamma_p(2/3)^6 \pmod{p^3}.
\]
The rest of the paper is arranged as follows. The proof of Theorems 1.1 and 1.2 will be given in Section 2. To this end, we first derive a $q$-supercongruence modulo $(1-aq^n)(a-q^n)(b-q^n)$, where $t \in \{1, 2\}$, by using a summation of basic hypergeometric series, the creative microscoping method, and the Chinese remainder theorem for coprime polynomials. Finally, the proof of Propositions 1.5 and 1.6 will be displayed in Section 3.

2 Proof of Theorems 1.1 and 1.2

In order to prove Theorems 1.1 and 1.2 we require the following lemma.

Lemma 2.1.

$$3\phi_2 \left[ \frac{a, b, q^{-m}}{q, abq^{2-m} \cdot q, q^3} \right] = \frac{(1/a, 1/b; q)_m}{(q, 1/ab, q)_m} \times \left\{ \frac{q^m(1-q^m)(q-abq^2-(1+q-aq-bq)q^m)}{(1-abq)(aq-q^m)(bq-q^m)} - \frac{1-ab-(2-a-b)q^m}{(1-a)(1-b)} \right\}. $$

Proof. By comparing the $k$-th summands in the summations, it is easy to see that

$$4\phi_3 \left[ \frac{a, b, xq, q^{-m}}{cq, x, abq^{1-m}/c \cdot q, q} \right] = \frac{(1-c)(ab-cxq^m)}{(1-x)(ab-c^2q^m)} \times \left\{ \frac{a, b, q^{-m}}{c, abq^{1-m}/c \cdot q, q} \right\} + \frac{(c-x)(ab-cq^m)}{(1-x)(ab-c^2q^m)} \times \left\{ \frac{a, b, q^{-m}}{c, abq^{1-m}/c \cdot q, q} \right\}. $$

Evaluating the two series on the right-hand side by $q$-Saalschütz identity (cf. [1, Appendix (II.12)]):

$$3\phi_2 \left[ \frac{a, b, q^{-m}}{c, abq^{1-m}/c \cdot q, q} \right] = \frac{(c/a, c/b; q)_m}{(c, c/ab; q)_m}, $$

we get

$$4\phi_3 \left[ \frac{a, b, xq, q^{-m}}{cq, x, abq^{1-m}/c \cdot q, q} \right] = \Omega_m(q; a, b, c, x), \quad (2.1) $$

where

$$\Omega_m(q; a, b, c, x) = \frac{(c/a, c/b; q)_m}{(qc, c/ab; q)_m} \times \left\{ \frac{1-cq^m(ab-cxq^m)}{(1-x)(ab-c^2q^m)} + \frac{(c-x)(ab-c)(a-cq^m)(b-cq^m)}{(1-x)(a-c)(b-c)(ab-c^2q^m)} \right\}. $$
Similarly, it is also routine to confirm the relation

\[
\phi_4 \left[ \frac{a, b, xq, yq, q^{-m}}{cq^2, x, y, abq^{1-m/c}; q, q} \right] = \frac{(1 - cq)(ab - cyq^m)}{(1 - y)(ab - c^2q^{m+1})} \phi_3 \left[ \frac{a, b, xq, q^{-m}}{cq, x, abq^{1-m/c}; q, q} \right] + \frac{(cq - y)(ab - c^2q^m)}{(1 - y)(ab - c^2q^{m+1})} \phi_3 \left[ \frac{a, b, xq, q^{-m}}{cq^2, x, abq^{-m/c}; q, q} \right].
\]

Calculating the two series on the right-hand side via (2.1), we arrive at

\[
\phi_4 \left[ \frac{a, b, xq, yq, q^{-m}}{cq^2, x, y, abq^{1-m/c}; q, q} \right] = \frac{(1 - cq)(ab - cyq^m)}{(1 - y)(ab - c^2q^{m+1})} \Omega_m(q; a, b, c, x) + \frac{(cq - y)(ab - c^2q^m)}{(1 - y)(ab - c^2q^{m+1})} \Omega_m(q; a, b, cq, x).
\]

Letting \( c \to q^{-1} \), \( x \to \infty \), \( y \to \infty \) in the last equation, we are led to Lemma 2.1.

Subsequently, we shall deduce the following united parametric extension of Theorems 1.1 and 1.2.

**Theorem 2.2.** Let \( n \) be a positive integer with \( n \equiv 3 - t \pmod{3} \) and \( t \in \{1, 2\} \). Then, modulo \((1 - aq^{tn})(a - q^m)(b - q^n)\),

\[
\sum_{k=0}^{(tn+1)/3} \frac{(aq^{-1}, q^{-1}/a, q^{-1}/b; q^3)_k}{(q^2; q^3)_k^2(q^2/b; q^3)_k} q^k \equiv \frac{(b - q^m(ab - 1 - a^2 + aq^{tn}))}{(a - b)(1 - ab)} A_n(q; b, t) + \frac{(aq - q^m)(a - q^m)}{(a - b)(1 - ab)} B(q; a, b),
\]

(2.2)

where

\[
A_n(q; b, t) = \frac{b(1 - q^{tn+1})(q^{tn+2}/b - q + q^{tn-1}(1 + q^3 - q^{tn+2} - q^2/b))}{(1 - q)(1 - bq^{tn-1})(1 - q^{tn+1}/b)} - \frac{1 - q^{tn-2}/b - q^{tn+1}(2 - q^{tn-1} - q^{-1}/b)}{(1 - q^{tn-1})(1 - q^{-1}/b)},
\]

\[
B(q; a, b) = \frac{(1 - bq)(1 - q - b(q^{-2} + q - a - 1/a))}{q(1 - q)(1 - ab/q)(1 - b/aq)} - \frac{1 - q^{-2} - b(2q - a - 1/a)}{bq(1 - aq^{-1})(1 - q^{-1}/a)}.
\]
Proof. When \( a = q^{-tn} \) or \( a = q^{tn} \), the left-hand side of (2.2) is equal to

\[
\sum_{k=0}^{(tn+1)/3} \frac{(q^{-1-tn}; q^{-1+tn}, q^{-1}/b; q^3)_k q^{9k}}{(q^3; q^3)_k(q^3/b; q^3)_k} = 3\phi_2 \left[ \frac{q^{-1-tn}; q^{-1+tn}, q^{-1}/b}{q^3, q^3/b}; q^3, q^9 \right]. \tag{2.3}
\]

According to Lemma 2.1, the right-hand side of (2.3) can be written as

\[
\frac{(bq, q^3)(tn+1)/3}{(bq)^{(tn+1)/3}(1/b, q^3; q^3)(tn+1)/3} A_n(q; b, t).
\]

Since \((1 - aq^{tn})\) and \((a - q^{tn})\) are relatively prime polynomials, we have the following result: modulo \((1 - aq^{tn})(a - q^{tn})\),

\[
\sum_{k=0}^{(tn+1)/3} \frac{(aq^{-1}, q^{-1}/a, q^{-1}/b; q^3)_k q^{9k}}{(q^3; q^3)_k(q^3/b; q^3)_k} = \frac{(bq, q^3)(tn+1)/3}{(bq)^{(tn+1)/3}(1/b, q^3; q^3)(tn+1)/3} A_n(q; b, t). \tag{2.4}
\]

When \( b = q^{tn} \), the left-hand side of (2.2) is equal to

\[
\sum_{k=0}^{(tn+1)/3} \frac{(aq^{-1}, q^{-1}/a, q^{-1-tn}; q^3)_k q^{9k}}{(q^3; q^3)_k(q^3/b; q^3)_k} = 3\phi_2 \left[ \frac{aq^{-1}, q^{-1}/a, q^{-1-tn}}{q^3, q^3-tn}; q^3, q^9 \right]. \tag{2.5}
\]

By Lemma 2.1, the right-hand side of (2.5) can be expressed as

\[
\frac{(aq, q/a; q^3)(tn+1)/3}{(q^2, q^3; q^3)(tn+1)/3} \times \left\{ \frac{q^{tn}(1 - q^{tn+1})\{1 - q - q^{tn}(q^{-2} + q - a - 1/a)\}}{(1 - q)(1 - aq^{tn-1})(1 - q^{tn-1}/a)} - \frac{1 - q^{-2} - q^{tn}(2q - a - 1/a)}{(1 - aq^{-1})(1 - q^{-1}/a)} \right\}.
\]

Then we obtain the conclusion: modulo \((b - q^{tn})\),

\[
\sum_{k=0}^{(tn+1)/3} \frac{(aq^{-1}, q^{-1}/a, q^{-1}/b; q^3)_k q^{9k}}{(q^3; q^3)_k(q^3/b; q^3)_k} \equiv \frac{(aq, q/a; q^3)(tn+1)/3}{b^{(tn+1)/3}(1/b, bq; q^3)(tn+1)/3} B(q; a, b). \tag{2.6}
\]

It is clear that the polynomials \((1 - aq^{tn})(a - q^{tn})\) and \((b - q^{tn})\) are relatively prime. Noting the \(q\)-congruences

\[
\frac{(b - q^{tn})(ab - 1 - a^2 + aq^{tn})}{(a - b)(1 - ab)} \equiv 1 \pmod{(1 - aq^{tn})(a - q^{tn})},
\]

\[
\frac{(1 - aq^{tn})(a - q^{tn})}{(a - b)(1 - ab)} \equiv 1 \pmod{(b - q^{tn})}
\]

and employing the Chinese remainder theorem for coprime polynomials, we get Theorem 2.2 from (2.4) and (2.6). \(\square\)
Proof of Theorem 1.1. Letting $b \to 1, t = 2$ in Theorem 2.2 we arrive at the formula: modulo $\Phi_n(q)(1 - aq^{2n})(a - q^{2n})$,
\[
\sum_{k=0}^{(2n+1)/3} \frac{(aq^{-1}, q^{-1}/a, q^{-1}; q^3)_k}{(q^3; q^3)_k^3} q^{nk} \equiv \frac{(1 - a)^2 + (1 - aq^{2n})(a - q^{2n})}{(1 - a)^2} \left( \frac{(q; q^3)^2}{q^{(2n+1)/3}(q^3; q^3)^{(2n+1)/3}} C_n(q) \right.
\]
\[+ \frac{(1 - aq^{2n})(a - q^{2n})}{(1 - a)^2} \frac{(aq, q/a; q^3)_{(2n+1)/3}}{(q^2, q^3; q^3)_{(2n-2)/3}} D(q; a) \equiv \frac{(q; q^3)^2_{(2n+1)/3}}{q^{(2n+1)/3}(q^3; q^3)^{(2n+1)/3}} C_n(q) + \frac{(1 - aq^{2n})(a - q^{2n})}{q^{(2n+1)/3}(1 - a)^2} \left( \frac{(q; q^3)^2_{(2n+1)/3}}{q^3; q^3)^{(2n+1)/3}} (3q + 3q^2) + \frac{(aq, q/a; q^3)_{(2n+1)/3}}{(q^3; q^3)^{(2n+1)/3}} (1 - q)^2 D(q; a) \right),\]
The expression above, \[\left(3q + 3q^2\right) + \frac{(aq, q/a; q^3)_{(2n+1)/3}}{(q^3; q^3)^{(2n+1)/3}} (1 - q)^2 D(q; a)\] \[= \frac{(aq^{-1}, q^{-1}/a, q^{-1}; q^3)_k}{(q^3; q^3)_k^3} q^{nk} \]
where \[C_n(q) = \frac{q^3 + q^{2n}(1 + q^{4n})(1 - 3q + q^3 - 3q^4)}{q(1 - q)^2(1 - q^{2n-1})^2} + \frac{q^{2n}(1 - 3q + 6q^2 + 2q^3 - 3q^4 + 3q^5) + q^{8n+3}}{q(1 - q)^2(1 - q^{2n-1})^2},\]
\[D(q; a) = \frac{(1 + a + a^2)(a - 3aq + q^3 + a^2q^3 - 3aq^4) + 3a^2q^2(2 + q^3)}{q(1 - q)^2(1 - aq)^2(a - a/q)^2}.\]

By the L'Hôpital rule, we have
\[
\lim_{a \to 1} \frac{(1 - aq^{2n})(a - q^{2n})}{(1 - a)^2} \left\{ - (q; q^3)^2_{(2n+1)/3} (3q + 3q^2) + (aq, q/a; q^3)_{(2n+1)/3} (1 - q)^2 D(q; a) \right\}
\]
\[= -q(1 + q)[2n]^2(q; q^3)^2_{(2n+1)/3} \left\{ \sum_{i=1}^{(2n+1)/3} \frac{3q^{3i-2}}{[3i - 2]^2} - \frac{1 + 5q + 3q^2}{1 + q} \right\}.\]

Letting $a \to 1$ in (2.7) and utilizing the above limit, we are led to the $q$-supercongruence:
modulo $\Phi_n(q)^3$,

$$
\sum_{k=0}^{(2n+1)/3} \frac{(q^{-1}; q^3)_k}{(q^3; q^3)_k} q^{9k} 
\equiv \frac{(q; q^3)^2}{q^{(2n+1)/3}(q^3; q^3)^2} C_n(q)
\equiv q^{2-2n/3}(1 + q) \frac{(q; q^3)^2}{(q^3; q^3)^2} (2n+1)/3
\equiv q^{2-2n/3}(1 + q) \frac{(q; q^3)^2}{(q^3; q^3)^2} (2n+1)/3
\times \left\{ 3 - \left[ 2n \right]^2 \left( \sum_{i=1}^{(2n+1)/3} \frac{3q^{3i-2}}{[3i-2]^2} - \frac{1 + 5q + 3q^2}{1 + q} \right) \right\}.
$$

This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Letting $b \to 1, t = 1$ in Theorem 2.2 we obtain the result: modulo $\Phi_n(q)(1 - aq^n)(a - q^n)$,

$$
\sum_{k=0}^{(n+1)/3} \frac{(aq^{-1}; q^{3/2})_k}{(q; q^3)_k} q^{9k} 
\equiv \frac{1 - a^2 + (1 - aq^n)(a - q^n)}{(1 - a)^2} \frac{(q; q^3)^2}{q^{(n+1)/3}(q^3; q^3)^2} C_{n/2}(q)
\equiv \frac{(1 - aq^n)(a - q^n)(aq, q/a; q^3)^{(n+1)/3}}{(1 - a)^2} \frac{(q; q^3)^2}{q^{(n+1)/3}(q^3; q^3)^2} C_{n/2}(q)
\equiv \frac{(q; q^3)^2}{q^{(n+1)/3}(q^3; q^3)^2} C_{n/2}(q) + \frac{1 - aq^n(a - q^n)}{q^{(n+1)/3}(1 - a)^2} \left\{ (q; q^3)^{(n+1)/3}(3q + 3q^2) - \frac{(aq, q/a; q^3)^{(n+1)/3}}{q^3; q^3} (1 - q)^2 D(q; a) \right\}.
$$

By the L'Hôpital rule, we have

$$
\lim_{a \to 1} \frac{1 - aq^n(a - q^n)}{(1 - a)^2} \left\{ (q; q^3)^{(n+1)/3}(3q + 3q^2) - \frac{(aq, q/a; q^3)^{(n+1)/3}}{q^3; q^3} (1 - q)^2 D(q; a) \right\}
= q(1 + q)[n]q^3(2n+1)/3 \left\{ \sum_{i=1}^{(n+1)/3} \frac{3q^{3i-2}}{[3i-2]^2} - \frac{1 + 5q + 3q^2}{1 + q} \right\}.
$$
Letting $a \to 1$ in (2.8) and employing the upper limit, we get the $q$-supercongruence:

modulo $\Phi_n(q)^3$,

\[
\sum_{k=0}^{(n+1)/3} \frac{(q^{-1}; q^3)^3_k}{(q^3; q^3)^3_k} q^k \equiv \frac{(q; q^3)^2}{q^{(n+1)/3}(q^3; q^3)^2(\phi^2)} C_{n/2}(q)
\]

\[
+ q(1 + q)[n]^2 \frac{(q; q^3)^2}{q^{(n+1)/3}(q^3; q^3)^2(\phi^2)} \left\{ \sum_{i=1}^{(n+1)/3} \frac{3q^{3i-2}}{[3i - 2]^2} - \frac{1 + 5q + 3q^2}{1 + q} \right\}
\]

\[
\equiv q^{(2-n)/3}(1 + q) \frac{(q; q^3)^2}{q^{(n+1)/3}(q^3; q^3)^2(\phi^2)}
\]

\[
\times \left\{ \theta_n(q) + [n]^2 \left( \sum_{i=1}^{(n+1)/3} \frac{3q^{3i-2}}{[3i - 2]^2} - \frac{1 + 5q + 3q^2}{1 + q} \right) \right\}.
\]

Thus we finish the proof of Theorem 1.2. \hfill \Box

3 \quad Proof of Propositions 1.5 and 1.6

Let $\Gamma_p(x)$ and $\Gamma''_p(x)$ be the first derivative and second derivative of $\Gamma_p(x)$ respectively. Now we begin to provide the Proof of Propositions 1.5 and 1.6.

Proof of Proposition 1.5 \quad By means of the properties of the $p$-adic Gamma function, we arrive at

\[
\frac{(1/3)^2}{(2p+1)/3} = \frac{p^2}{(2p+1)^2} \left\{ \frac{\Gamma_p((2 + 2p)/3)\Gamma_p(1)}{\Gamma_p(1/3)\Gamma_p((1+2p)/3)} \right\}^2
\]

\[
= \frac{p^2}{(2p+1)^2} \left\{ \frac{\Gamma_p(2/3)\Gamma_p((2 + 2p)/3)\Gamma_p((2 - 2p)/3)}{\Gamma_p(1/3)\Gamma_p((1+2p)/3)} \right\}^2.
\]

Moreover, it is not difficult to understand that

\[
1 + 6p^2 - \sum_{i=1}^{(p+1)/3} \frac{4p^2}{(3i - 2)^2} = -3 + 6p^2 - \sum_{i=1}^{(p-1)/3} \frac{4p^2}{(3i - 2)^2} - \sum_{i=(p+5)/3}^{(2p+1)/3} \frac{4p^2}{(3i - 2)^2}.
\]

Then we can proceed as follows:
\[
\frac{(1/3)^2_{(2p+1)/3}}{(1)^2_{(2p+1)/3}} \left\{ 1 + 6p^2 - \sum_{i=1}^{(2p+1)/3} \frac{4p^2}{(3i-2)^2} \right\} \\
= \frac{p^2}{(2p + 1)^2} \left\{ \Gamma_p((2/3))\Gamma_p((2 + 2p)/3)\Gamma_p((2 - 2p)/3) \right\}^2 \\
\times \left\{ -3 + 6p^2 - \sum_{i=1}^{(p-1)/3} \frac{4p^2}{(3i-2)^2} - \sum_{i=(p+5)/3}^{(2p+1)/3} \frac{4p^2}{(3i-2)^2} \right\} \\
\equiv \frac{-3p^2}{(2p + 1)^2} \Gamma_p(2/3)^6 \\
\equiv -3p^2 \Gamma_p(2/3)^6 \pmod{p^3}.
\]

This verifies the correctness of Proposition 1.5. \(\square\)

**Proof of Proposition 1.6.** Through the properties of the \(p\)-adic Gamma function, we have

\[
\frac{(1/3)^2_{(p+1)/3}}{(1)^2_{(p-2)/3}} = \left\{ \frac{\Gamma_p((2 + p)/3)\Gamma_p(1)}{\Gamma_p(1/3)\Gamma_p((1 + p)/3)} \right\}^2 \\
= \left\{ \Gamma_p(2/3)\Gamma_p((2 + p)/3)\Gamma_p((2 - p)/3) \right\}^2 \\
\equiv \Gamma_p(2/3)^2 \left\{ \Gamma_p(2/3) + \Gamma_p'(2/3)\frac{p}{3} + \Gamma_p''(2/3)\frac{p^2}{18} \right\}^2 \\
\times \left\{ \Gamma_p(2/3) - \Gamma_p'(2/3)\frac{p}{3} + \Gamma_p''(2/3)\frac{p^2}{18} \right\}^2 \\
\equiv \Gamma_p(2/3)^6 \left\{ 1 - \frac{2p^2}{9}G_1(2/3)^2 + \frac{2p^2}{9}G_2(2/3) \right\} \pmod{p^3}, \quad (3.1)
\]

where \(G_1(x) = \Gamma_p'(x)/\Gamma_p(x)\) and \(G_2(x) = \Gamma_p''(x)/\Gamma_p(x)\).

Let

\[
H_m = \sum_{k=1}^{m} \frac{1}{k}, \quad H_m^{(2)} = \sum_{k=1}^{m} \frac{1}{k^2}.
\]

In light of the three relations from Wang and Pan [17] Lemmas 2.3 and 2.4:

\[
G_2(0) = G_1(0)^2 \\
G_1(2/3) \equiv G_1(0) + H_{(2p-1)/3} \pmod{p}, \\
G_2(2/3) \equiv G_2(0) + 2G_1(0)H_{(2p-1)/3} + H_{(2p-1)/3}^2 - H_{(2p-1)/3}^{(2)} \pmod{p},
\]

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we get
\[ G_2(2/3) - G_1(2/3)^2 \equiv -H_{(2p-1)/3}^{(2)} (\text{mod } p). \] (3.2)

In view of (3.1) and (3.2), we are led to
\[ \frac{(1/3)^2}{(1/3)^2} \left\{ 1 + \frac{p^2}{(p + 1)^2} \sum_{i=1}^{(p+1)/3} \frac{1}{(3i - 2)^2} \right\} \]
\[ \equiv \Gamma_{p/3}^6 \left\{ 1 - \frac{2p^2}{9} H_{(2p-1)/3}^{(2)} \right\} \left\{ 1 + \frac{p^2}{(p + 1)^2} \sum_{i=1}^{(p+1)/3} \frac{1}{(3i - 2)^2} \right\} \]
\[ \equiv \Gamma_{p/3}^6 \left\{ 1 - \frac{2p^2}{9} H_{(2p-1)/3}^{(2)} + \frac{p^2}{(p + 1)^2} \sum_{i=1}^{(p+1)/3} \frac{1}{(3i - 2)^2} \right\} (\text{mod } p^3). \] (3.3)

It is easy to see that
\[ \sum_{i=1}^{(p+1)/3} \frac{1}{(3i - 2)^2} = H_{(p-1)/3}^{(2)} - \frac{1}{9} H_{(p-2)/3}^{(2)} - \sum_{i=1}^{(p-2)/3} \frac{1}{(3i - 1)^2} \]
\[ \equiv -\frac{1}{9} H_{(p-2)/3}^{(2)} - \sum_{i=1}^{(p-2)/3} \frac{1}{(3i - 1)^2} \]
\[ = -\frac{1}{9} H_{(p-2)/3}^{(2)} - \sum_{i=1}^{(p-2)/3} \frac{1}{(p - 3i)^2} \]
\[ = -\frac{2}{9} H_{(p-2)/3}^{(2)} \]
\[ = -\frac{2}{9} \sum_{i=(2p-2)/3}^{p-1} \frac{1}{(p - i)^2} \]
\[ \equiv -\frac{2}{9} \sum_{i=(2p-2)/3}^{p-1} \frac{1}{i^2} \]
\[ \equiv \frac{2}{9} H_{(2p-1)/3}^{(2)} (\text{mod } p^3). \] (3.5)

Substituting (3.5) into (3.4), we confirm the validity of Proposition 1.6. \(\square\)

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References

[1] G. Gasper, M. Rahman, Basic Hypergeometric Series (2nd edition), Cambridge University Press, Cambridge, 2004.

[2] V.J.W. Guo, q-Supercongruences modulo the fourth power of a cyclotomic polynomial via creative microscoping, Adv. Appl. Math. 120 (2020), Art. 102078.

[3] V.J.W. Guo, Some variations of a “divergent” Ramanujan-type q-supercongruence, J. Difference Equ. Appl. 27 (2021), 376–388.

[4] V.J.W. Guo, A further q-analogue of Van Hamme’s (H.2) supercongruence for primes $p \equiv 3 \pmod{4}$, Int. J. Number Theory, in press; https://doi.org/10.1142/S1793042121500329.

[5] V.J.W. Guo, M.J. Schlosser, A new family of q-supercongruences modulo the fourth power of a cyclotomic polynomial, Results Math. 75 (2020), Art. 155.

[6] V.J.W. Guo, M.J. Schlosser, A family of q-hypergeometric congruences modulo the fourth power of a cyclotomic polynomial, Israel J. Math. 240 (2020), 821–835.

[7] V.J.W. Guo, M.J. Schlosser, Some q-supercongruences from transformation formulas for basic hypergeometric series, Constr. Approx. 53 (2021), 155–200.

[8] V.J.W. Guo, W. Zudilin, A q-microscope for supercongruences, Adv. Math. 346 (2019), 329–358.

[9] V.J.W. Guo, W. Zudilin, Dwork-type supercongruences through a creative $q$-microscope, J. Combin. Theory, Ser. A 178 (2021), Art. 105362.

[10] L. Li, Some q-supercongruences for truncated forms of squares of basic hypergeometric series, J. Difference Equ. Appl. 27 (2021), 16–25.

[11] L. Li, S.-D. Wang, Proof of a q-supercongruence conjectured by Guo and Schlosser, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. 114 (2020), Art. 190.

[12] J.-C. Liu, F. Petrov, Congruences on sums of q-binomial coefficients, Adv. Appl. Math. 116 (2020), Art. 102003.

[13] J.-C. Liu, Supercongruences arising from transformations of hypergeometric series, J. Math. Anal. Appl. 497 (2021), Art. 124915.

[14] L. Long, R. Ramakrishna, Some supercongruences occurring in truncated hypergeometric series, Adv. Math. 290 (2016), 773–808.

[15] A.M. Robert, A Course in p-Adic Analysis, Graduate Texts in Mathematics, Springer-Verlag, New York, 2000.

[16] R. Tauraso, q-Analogs of some congruences involving Catalan numbers, Adv. Appl. Math. 48 (2009), 603–614.

[17] C. Wang, H. Pan, Supercongruences concerning truncated hypergeometric series, Math. Z., accepted.

[18] X. Wang, M. Yue, A $q$-analogue of the (A.2) supercongruence of Van Hamme for any prime $p \equiv 3 \pmod{4}$, Int. J. Number Theory 16 (2020), 1325–1335.

[19] X. Wang, M. Yue, Some $q$-supercongruences from Watson’s $s\phi_7$ transformation formula, Results Math. 75 (2020), Art. 71.

[20] X. Wang, M. Yu, Some new $q$-congruences on double sums, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. 115 (2020), Art. 9.
[21] X. Wang, M. Yue, A $q$-analogue of a Dwork-type supercongruence, B. Aust. Math. Soc. 103 (2021), 303–310.

[22] C. Wei, Some $q$-supercongruences modulo the fourth power of a cyclotomic polynomial, J. Combin. Theory, Ser. A 182 (2021), Art. 105469.

[23] C. Wei, Y. Liu, X. Wang, $q$-Supercongruences form the $q$-Saalschütz identity, Proc. Amer. Math. Soc., accepted.

[24] W. Zudilin, Congruences for $q$-binomial coefficients, Ann. Combin. 23 (2019), 1123–1135.