Charged Black Hole Solutions in Einstein-Born-Infeld gravity with a Cosmological Constant

Sharmanthie Fernando* 1 and Don Krug** 2

*Department of Physics & Geology
**Department of Mathematics & Computer Science
Northern Kentucky University
Highland Heights
Kentucky 41099
U.S.A.

Abstract

We construct black hole solutions to Einstein-Born-Infeld gravity with a cosmological constant. Since an elliptic function appears in the solutions for the metric, we construct horizons numerically. The causal structure of these solutions differ drastically from their counterparts in Einstein-Maxwell gravity with a cosmological constant. The charged de-Sitter black holes can have up to three horizons and the charged anti-de Sitter black hole can have one or two depending on the parameters chosen.

Key words: Born-Infeld, Cosmological constant, Black Holes

1 Introduction

Born-Infeld electrodynamics was first introduced in 1930’s to obtain a finite energy density model for the electron [1]. It has attracted considerable interest in recent times due to various reasons. One of the motivations being the observations that it arises naturally in open superstrings and in D-branes [2]. The low energy effective action for an open superstring in loop calculations lead to Born-Infeld type actions [3]. It has also being observed that the Born-Infeld action arises as an effective action governing the dynamics of vector-fields on D-branes [4]. For a review of aspects of Born-Infeld theory in string theory see Gibbons [5].

1 fernando@nku.edu
2 krugd@nku.edu
In recent times, the cosmological constant has been considered for several theoretical and observational reasons. One of the strongest supports have been the recent results from a supernova which suggests a positive cosmological constant \[6\]. From the theoretical point of view, recent advances in string theory have been subjected to studies in all dimensions with a negative cosmological constant \[7\]. Asymptotically anti-de Sitter black holes are more interesting than their counterparts since they allow more geometries for black hole horizons than for the \( \Lambda = 0 \) case \[8\]. Also they are thermodynamically stable which has been an inspiration for much recent work \[9\]. Therefore studying black holes in (anti)de Sitter spaces has an important place in current research.

In this paper we explore the existence of electrically charged black holes in Einstein-Born-Infeld gravity with a cosmological constant for a U(1) gauge field. Our motivation is to see how the singular nature of black holes gets modified when coupled to non-linear electrodynamics. We find that depending on the value of the cosmological constant \( \Lambda \), mass parameter \( M \), coupling constant \( \beta \) and the charge \( Q \), the black hole solutions can have up to three horizons.

The paper is organized as follows: In section 2 the basic equations are derived. In section 3 the general solutions to static spherically symmetric solutions are constructed. In section 4 the solutions are studied in detail for various values of the parameters of the theory. In section 5 the temperature is calculated and finally in section 6 the conclusion are given.

\section{Basic Equations}

In this section we will derive the equations of motion for non-linear electrodynamics. The most general action for such a theory coupled to gravity with a cosmological constant is as follows:

\[
S = \int d^4 x \sqrt{-g} \left[ \frac{(R - 2\Lambda)}{16\pi G} + L(F) \right] \tag{1}
\]

Here, \( L(F) \) is a function of the field strength \( F_{\mu\nu} \) only. For the weak field limit, \( L(F) \) has to be of the form

\[
L(F) = -F_{\mu\nu}F_{\mu\nu} + O(F^4) \tag{2}
\]

In this paper, we will study a particular non-linear electrodynamics called Born-Infeld theory which has attracted lot of attention due to its relation to string effective actions. The function \( L(F) \) for Born-Infeld may be expanded to

\[
L(F) = 4\beta^2 \left( 1 - \sqrt{1 + \frac{F_{\mu\nu}F_{\mu\nu}}{2\beta^2}} \right) \tag{3}
\]

Here, \( \beta \) has dimensions \textit{length}^{-2} and \( G \, \textit{length}^2 \). In the following sections we will take \( 16\pi G = 1 \).
By extremising the Lagrangian in eq.(1), with respect to the metric \( g_{\mu\nu} \) and the electrodynamics potential \( A_{\mu} \), one obtains the corresponding field equations as follows:

\[
\nabla_{\mu} \left( \frac{F^{\mu\nu}}{\sqrt{1 + F^2/2\beta^2}} \right) = 0 \quad (4)
\]

\[
R_{\mu\nu} - g_{\mu\nu} \Lambda = \left( \frac{\partial L(F)}{\partial g^{\mu\nu}} + \frac{g_{\mu\nu} A(F)}{2} \right) \quad (5)
\]

where

\[
A(F) = -L(F) + \frac{\partial L(F)}{\partial g^{ab}} g^{ab}; \quad \frac{\partial L(F)}{\partial g^{\mu\nu}} = \frac{-2F_{\beta\nu}F_{\mu}^{\beta}}{\sqrt{1 + F^2/2\beta^2}} \quad (6)
\]

3 Static Spherically Symmetric Solutions

We will study static spherically symmetric charged solutions to Einstein-Born-Infeld gravity with a cosmological constant. The most general metric for such a configuration can be written as

\[
ds^2 = -e^{2\nu}dt^2 + e^{2\nu}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (7)
\]

To simplify the equations, we use tetrad formalism and Cartan structure equations. The obvious orthonomal basis is the following:

\[
\theta^0 = e^\mu dt; \quad \theta^1 = e^\nu dr; \quad \theta^2 = rd\theta; \quad \theta^3 = r\sin(\theta)d\phi \quad (8)
\]

The indices \( a, b = 0, 1, 2, 3 \) are for the orthonomal basis and \( \mu, \nu = 0, 1, 2, 3 \), for the coordinate basis with \( x^0 = t, x^1 = r, x^2 = \theta, x^3 = \phi \). Now, one can compute the \( R_{\mu\nu} \) for the orthonomal basis given in eq.(8) as follows:

\[
R_{00} = e^{-2\nu}(\mu'' + \mu' + 2\mu') + \frac{2\mu'}{r} \quad (9)
\]

\[
R_{11} = e^{-2\nu}(\mu'' + \lambda' + \mu' + 2\nu') + \frac{2\nu'}{r} \quad (10)
\]

\[
R_{22} = R_{33} = \frac{1}{r^2}(1 - e^{-2\nu}) + \frac{1}{r}(-\mu' + \nu')e^{-2\nu} \quad (11)
\]

For static electrically charged solutions, the only non-zero field strength is \( F_{rt} \) or \( F_{01} \). This leads to,

\[
\frac{\partial L(F)}{\partial g^{11}} = -\frac{\partial L(F)}{\partial g^{00}}; \quad \frac{\partial L(F)}{\partial g^{22}} = \frac{\partial L(F)}{\partial g^{33}} = 0. \quad (12)
\]

Note that the conditions in equation(12) are satisfied by any function \( L \) of \( F^2 \) for static electrically charged metric. The condition in eq.(12) and the fact that \( \eta_{ab} = -, +, +, + \), combined with the above field equations leads to,

\[
R_{11} = -R_{00} \quad \Rightarrow \mu + \nu = 0 \quad (13)
\]
In order to solve for the function $\mu$ and $\nu$, we will use the electromagnetic equations (4). The non-vanishing components of the electrodynamic field tensor in the coordinate basis are given by $F_{tr} = E(r)$ and in the orthonormal basis $F_{01} = \hat{E}$. They are related to each other by, $E(r) = \hat{E}(r)e^{\mu+\nu}$. Hence, the non-linear Lagrangian reduces to,

$$L(F) = 4\beta^2 \left(1 - \sqrt{1 - \frac{E^2}{\beta^2}}\right)$$

(14)

Note that the eq.(14) imposes an upper bound for $|E|$ to be smaller than $\beta$. This is a crucial characteristic of non-linear electrodynamics which leads to finite self energy of the electron. Th electric field $E$ follows from eq.(4) as,

$$E(r) = \frac{Q}{\sqrt{r^4 + Q^2/\beta^2}}$$

(15)

From Einstein’s equations one can derive a relation for $\nu$ as follows,

$$(re^{-2\nu})' = 1 - \Lambda r^2 + 2\beta (r^2 \beta - \sqrt{Q^2 + r^4 \beta^2})$$

(16)

Hence,

$$e^{-2\nu} = 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3} + 2\beta \left(\frac{r^2 \beta}{3} - \frac{1}{r} \int_{r}^{\infty} \sqrt{Q^2 + r^4 \beta^2} \right)$$

(17)

In the limit $\beta \to \infty$, the elliptic integral can be expanded to give,

$$(re^{-2\nu})' = 1 - \Lambda r^2 - \frac{Q^2}{r^2}$$

(18)

resulting the function $\nu(r)$ for the Reissner-Nordstrom-(anti)de Sitter solutions,

$$e^{-2\nu} = 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3} + \frac{Q^2}{r^2}$$

(19)

Here $M$ is an integrating constant which may be interpreted as a quasi-local mass when $\Lambda = 0$ [11].

4 Electrically Charged Black Hole Solutions

In this section we will explore the existence of black holes solutions from the above general solutions. For regular horizon to exist the metric function $g_{tt} = e^{-2\nu}$ has to have zeros. First, let us redefine $e^{-2\nu}$ as $f(r)$ and define $g_0(r)$ as the function $d(rf(r))/dr$.

$$f(r) = 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3} + 2\beta \left(\frac{r^2 \beta}{3} - \frac{1}{r} \int_{r}^{\infty} \sqrt{Q^2 + r^4 \beta^2} \right)$$

(4)
\[ g_0(r) = 1 + (2\beta^2 - \Lambda)r^2 - 2\beta\sqrt{Q^2 + \beta^2 r^4} \]  \hspace{1cm} (20)

The number of positive roots of \( f(r) \) and of \( rf(r) \) are the same. Therefore, the number of roots of \( f(r) \) would be at most the number of roots of \( g_0(r) \) plus one. (which is a result of Rolle’s theorem in calculus). Since we are looking for roots of \( g_0(r) \), we will consider the function \( g(r) \) which is obtained by simplifying \( g_0(r) = 0 \).

\[ g(r) = r^4(\Lambda^2 - 4\beta^2\Lambda) + r^2(4\beta^2 - 2\Lambda) + (1 - 4\beta^2Q^2) \]  \hspace{1cm} (21)

To understand the behavior of the black hole metric for very small \( r \), one can do a series expansion of \( f(r) \) around \( r = 0 \) as follows:

\[ f(r) \approx 1 - \frac{(2M - A)}{r} - \frac{10}{3}\beta Q + \frac{2\beta^2}{3}r^2 - \frac{\beta^2}{5}r^4 \]  \hspace{1cm} (22)

Here,

\[ A = \frac{1}{3}\sqrt{\frac{\beta}{\pi}}Q^{3/2}\Gamma\left(\frac{1}{4}\right)^2 \]  \hspace{1cm} (23)

We will define \( 2M' = 2M - A \) to discuss the behavior of the geometry. Hence when \( r \to 0 \), the behavior of \( f(r) \) is dominated by \( 2M'/r \) term irrespective of the value of \( Q \) and \( \Lambda \). One may recall that in the Einstein-Maxwell black hole solutions, it is \( Q^2/r^2 \) that dominates the behavior of the space-time around \( r = 0 \). This is a major difference incorporating Born-Infeld electrodynamics to gravity compared to the Maxwell case.

We will classify the solutions according to the value of the integration constant \( M' \) in the following sections. We will consider \( \Lambda > 0 \) and \( \Lambda < 0 \) cases separately. Note that the \( \Lambda = 0 \) case has been studied by Rasheed [11].

4.1 De Sitter Black Holes

For \( \Lambda > 0 \), \( g(r) \) is a 4th degree even function and can have at the most two positive roots. Hence \( f(r) \) can have at the most three roots. However, if one takes a closer look at the behavior of the function for \( r \to \infty \) and \( r \to 0 \) one can predict the number of roots exactly. From the expansions in eq.(22), it is obvious that for small \( r \), \( f(r) \to -2M'/r \) and for large \( r \), \( f(r) \to -\infty \).

When \( M' > 0 \), for small \( r \), the function \( f(r) \to -\infty \). Hence \( f(r) \) does not have a root at all or it has two roots or it has a degenerate root as shown in the Fig.1. In all these cases, the curvature scalars \( R_{\alpha\gamma\beta\rho}R^{\alpha\gamma\beta\rho}, R_{\alpha\beta}R^{\alpha\beta} \) and \( R \) explode at \( r = 0 \) and are finite else where. Hence there is a curvature singularity at \( r = 0 \). Since for large \( r \), \( f(r) < 0 \), \( r \) becomes a time coordinate. Hence the future infinity is space-like and such space-times have cosmological event horizons. Such space-times are similar to Schwarzschild-de Sitter black hole space-time [12].

In the Fig.1, the graph in plain line represents a solution with two roots to the function \( f(r) \). The smaller of these values, which can be called \( r_+ \) can be regarded
as the position of the black hole event horizon. The larger value of the roots \( r_{++} \) represents the position of the cosmological event horizon for observers on the world-lines of constant \( r \) between \( r_+ \) and \( r_{++} \). The Killing vector \( \frac{\partial}{\partial t} \) is null on both these horizons. These horizons can be removed by appropriate coordinate patches similar to the Schwarzschild-de-Sitter black hole space-time \[12\].

As \( M' \) increases, the black hole event horizon increases and the cosmological event horizon decreases. For special values of the \( M' \), the two coincides leading to a degenerate horizon as given in the Fig.1 (dashed).

![Figure 1](image)

Figure 1. The figure shows the function \( f(r) \) for \( \Lambda = 1.5, \beta = 0.01 \) and \( Q = 1 \). The lighter graph shows \( f(r) \) for \( M' = 0.1 \) and the dashed one is for \( M' = 0.4 \).

### 4.2 Anti de-Sitter Black Holes

For \( \Lambda < 0 \), one can apply Decartes’s rule on the discriminant to predict the behavior of the roots of \( g(r) \) as follows; Since \((2\beta^2 - \Lambda) > 0\), \( g(r) \) can have only one root leading \( f(r) \) to have two roots.

When \( M' > 0 \) the function \( f(r) \to -\infty \) for small \( r \) and goes to \( \infty \) for large \( r \). From the Fig. 2, there is only one horizon. There is a curvature singularity at \( r = 0 \). Hence this is very similar to Schwarzschild black hole with a space-like singularity at the origin.

![Figure 2](image)
When $M' = 0$ the function $f(r) \to (1 - 10\beta Q/3)$ for small $r$ and goes to $\infty$ for large $r$. From the Fig. 3, the function $f(r)$ can have one root or none. However, there is no curvature singularity at $r = 0$ and the function $f(r)$ is bounded at $r = 0$. Hence this is a particle-like solution.

Figure 2. The figure shows the function $f(r)$ for $\Lambda = -0.1$, $\beta = 0.05$, $M' = 1$ and $Q = 1$

Figure 3. The figure shows the function $f(r)$ for $\Lambda = -1$, $M' = 0$ and $Q = 1$. The plain graph is for $\beta = 0.01$, the heavy is for $\beta = 0.5$ and the dashing is for $\beta = 5$

5 Extreme Black Holes

Here we will also discuss the extreme black hole in detail since it is a counterpart to the BPS state in Reissner-Nordstrom solutions. For degenerate roots of $f(r)$, both $f(r)$ and $df(r)/dr$ are zero. Hence from eq.(16), one arrives at the following equation for horizon radii $r_{ex}$:

$$r_{ex}^4(\Lambda^2 - 4\beta^2\Lambda) + r_{ex}^2(-2\Lambda + 4\beta^2) + (1 - 4Q^2\beta^2) = 0 \tag{24}$$

which gives the solution as,

$$(r_{ex}^2)_{\pm} = \frac{(2\beta^2 - \Lambda) \pm \sqrt{\delta}}{\Lambda(4\beta^2 - \Lambda)} \tag{25}$$

Here,

$$\delta = (2\beta^2 - \Lambda)^2 + (1 - 4\beta^2Q^2)(4\beta^2 - \Lambda)\Lambda \tag{26}$$

The existence of $r_{ex}$ varies for various values of the parameters of the theory as follows:

For $\Lambda < 0$, $(4\beta^2 - \Lambda) > 0$, the denominator of $r_{ex}$ is negative. Since $(2\beta^2 - \Lambda) > 0$, there is at most one positive root for $r_{ex}^2$. It occurs only if $(1 - 4\beta^2Q^2)(4\beta^2 - \Lambda)\Lambda > 0$ leading to $\beta Q < 1/2$. The corresponding value is given by $r_{ex+}^2$ of eq.(24).

For $2\beta^2 > \Lambda > 0$, the denominator of eq.(24) is positive and so is the first element of the numerator. Thus there is always one positive root. A second will occur if $(1 - 4\beta^2Q^2) < 0$ while $\delta$ remains positive.
For $4\beta^2 > \Lambda > 2\beta^2$, the denominator of eq.(24) is positive, while the first element of the numerator is negative. There is a positive root if $\delta > 0$ i.e. $\beta Q < 1/2$.

For $\Lambda > 4\beta^2$, both the denominator and the first element of the numerator are negative. So there is always a positive root and there will be two if $(\delta - (2\beta^2 - \Lambda)^2) < 0$ (or $\beta Q > 1/2$) and $\delta$ remains positive.

6 Temperature

The Hawking temperature of the black hole solutions discussed above can be calculated as follows: $T = \kappa/2\pi$. Here $\kappa$ is the surface gravity given by

$$\kappa = -\frac{1}{2} \left. \frac{dg_{tt}}{dr} \right|_{r=r_+}$$

Here, $r_+$ is the event horizon of the black hole. Since $f(r) = 0$ at $r = r_+$, the eq.(16) can be used to calculate the surface gravity exactly and the corresponding temperature as,

$$T = \frac{1}{4\pi} \left( \frac{1}{r_+} - \Lambda r_+ \right) + 2\beta \left( r_+ \beta - \sqrt{\frac{Q^2 + r^2\beta^2}{r_+}} \right)$$  (27)

7 Conclusions

In this paper we presented the local properties of the solutions arising in Einstein-Born-Infeld gravity with a cosmological constant. The non-singular nature of the electric field at the origin changes the structure of the space-times drastically. The singularity at the origin is dominated by the mass term $M/r$ rather than the $Q^2/r^2$. Hence the number of horizons changes when compared to the Einstein-Maxwell gravity. Currently we are studying the global properties of these space-times in detail.

In extending this work, we would like to study the stability of these black holes in detail. The electrically charged black holes in anti-de Sitter space have been shown to be unstable for large black holes by using linear perturbation techniques [13]. It would be interesting to study how the non-linear nature effects the instability of such solutions.

From a string theoretical point of view, it is vital to study black hole solutions arising in Einstein-Born-Infeld-dilaton gravity. It is well known that the presence of a dilaton changes the space-time drastically. In such case, the Lagrangian to be considered would be a more general one with a $SL(2, R)$ invariant Born-Infeld term with a dilaton-axion coupled to it. Solitons and black hole solutions were constructed in a recent paper by Clement and Gal’tsov for Einstein-Born-Infeld-dilaton gravity without a cosmological constant [14]. It would be interesting to study the effects of
both the Born-Infeld term and the dilaton with a cosmological constant which we hope to report elsewhere.

One may recall that the Reissner-Nordstrom AdS black hole is supersymmetric for $Q = 0$ and for $Q^2 = M^2$ which was shown by Romans [15]. It would be interesting to embed the black hole solutions obtained in this paper in a supergravity theory.

Note: The references [16] and [17] was added after the paper was published.

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