Majorana Corner Modes in a High-Temperature Platform

Zhongbo Yan,¹ Fei Song,¹ and Zhong Wang¹,²,⊥

¹Institute for Advanced Study, Tsinghua University, Beijing, 100084, China
²Collaborative Innovation Center of Quantum Matter, Beijing, 100871, China

We introduce two-dimensional topological insulators in proximity to high-temperature cuprate or iron-based superconductors as high-temperature platforms of Majorana Kramers pairs of zero modes. The proximity-induced pairing at the helical edge states of topological insulator serves as a Dirac mass, whose sign changes at the sample corner because of the pairing symmetry of high-TC superconductors. This sign changing naturally creates at each corner a pair of Majorana zero modes protected by time-reversal symmetry. Conceptually, this is a topologically-trivial-superconductor-based approach for Majorana zero modes. We provide quantitative criteria and suggest candidate materials for this proposal.

Majorana zero modes (MZMs)¹–³ have been actively pursued in recent years as building blocks of topological quantum computations.⁴–¹¹ These emergent excitations can generate robust ground-state degeneracy, supporting storage of nonlocal qubits robust to local decoherence.¹² Moreover, quantum gates can be implemented by their braiding operations.¹³–¹⁷ As platforms of MZMs, a variety of realizations, including topological insulators in proximity to conventional superconductors,¹⁸–²² semiconductor heterostructures,²³–²⁵ cold-atom systems,²⁶–³¹ quantum wires,³²–³⁵, to name a few; meanwhile, remarkable experimental progress has been witnessed.³⁶–³⁸

A single MZM entails breaking the time-reversal symmetry (TRS); in contrast, time-reversal-invariant (TRI) topological superconductors host Majorana Kramers pairs (MKPs) of zero modes, which are robust in the presence of TRS, and have interesting consequences such as TRS-protected non-Abelian statistics,⁶³–⁶⁵ TRS as local supersymmetry,⁵⁵, TRS as local supersymmetry,⁵⁵, and novel Kondo effects⁶⁶ and Josephson effects,⁶⁷–⁷¹ indicating their potentials in qubit storage or manipulation and other applications. In addition, MKPs can be used as tunable generators of MZMs by breaking the TRS.⁷²–⁷⁴ There have been a few interesting proposals for realizing TRI topological superconductors and MKPs,⁷⁵–⁷⁸,⁷²–⁸⁴, though experimental realizations are yet to come.

In this Letter, we show that simple structures of two-dimensional topological insulators (2D TIs) (also known as quantum spin Hall insulators) in proximity to high-temperature superconductors naturally generate MKPs (Fig.1). Since 2D TIs have been experimentally realized at temperatures as high as 100 Kelvin,⁸⁵–⁸⁶, this setup can be a high-temperature platform of MKPs. The physical picture can be readily described as follows. The helical edge states of TI, described as 1D massless Dirac fermions, are gapped out by the induced superconducting gap, which introduces a Dirac mass. Due to the nature of pairing symmetry (say d-wave), the induced Dirac mass changes sign at the corner, which generates a MKP as domain-wall excitations.

It is interesting to note that we do not propose here any realization of TRI topological superconductor. In fact, the helical Majorana edge states of Z₂-nontrivial superconductors cannot be gapped out without breaking TRS. In our setup, the 2D TI with a proximity-induced pairing has gapped edges, therefore it is a Z₂-trivial superconductor. If the sign of the pairing mass changes at the corner, it creates at each corner a pair of Majorana zero modes protected by time-reversal symmetry. Conceptually, this is a topologically-trivial-superconductor-based approach for Majorana zero modes. We provide quantitative criteria and suggest candidate materials for this proposal.

FIG. 1. Schematic illustration. A 2D TI is grown on a d-wave or sₓ−sᵧ-wave high-Tc superconductor. Majorana Kramers pairs (MKPs) of zero modes emerge at the corners of TI.
the pairing is removed, the Hamiltonian becomes the paradigmatic BHZ model of 2D TIs\cite{85, 86, 90}. The Hamiltonian has TRS $TH(k)T^{-1} = H(-k)$ with $T = i\sigma_z K$ (where $K$ is the complex conjugation), and particle-hole symmetry $CH(k)C^{-1} = -H(-k)$ with $C = \tau_y K$.

We first consider the $d$-wave pairing that is relevant to cuprate superconductors, which is

$$\Delta_0 = 0, \quad \Delta_x = -\Delta_y \equiv \Delta_d,$$  \hspace{1cm} (3)

The spectra on a cylinder geometry are shown in Fig.2(a), indicating that the helical edge states of TI are gapped out by $d$-wave pairing. From the numerical results for the square geometry [Fig.2(b)], it is clear that each corner hosts a MKP, whose energy is pinned to zero.

It is interesting to note that, unlike the more familiar vortex or end modes, the Majorana modes here are corner modes. As such, they may be viewed in the framework of recently proposed higher-order topological insulators\cite{91, 100} and superconductors\cite{102, 106}, for which crystal symmetries have been highlighted; the present scheme does not rely sensitively on the crystal symmetries.

We also mention that the bulk of $d$-wave superconductor is gapless and the MKPs may hybridize with these gapless modes. Nevertheless, the MKPs remain observable in scanning tunneling microscopy (STM). Near a MKP, the tunneling conductance displays a zero-bias peak (though broadened by hybridization), which is absent in the usual $c$-axis tunneling conductance\cite{107} (In other directions there is zero bias peak\cite{108, 114}, which is irrelevant in our setup). Later, we also study the $s_z$-wave case with an entirely gapped setup, in which case MKPs are the only low-energy modes.

**Edge theory.**--To gain intuitive understandings, we study the edge theory. To simplify the picture, we take $\mu = 0$ and focus on the continuum model by expanding the lattice Hamiltonian in Eq. (1) to second order around $k = (0, 0)$:

$$H(k) = \left(m + \frac{k_x^2}{2}\right)\sigma_x \tau_z + A_0 k_x \sigma_z s_z + A_0 k_y \sigma_y \tau_z - \frac{1}{2}(\Delta_x k_x^2 + \Delta_y k_y^2) s_y \tau_y,$$  \hspace{1cm} (4)

where $\Delta_x + \Delta_y = 0$ has been used for the $d$ wave, and $m = m_0 - t_x - t_y < 0$ is assumed to ensure that the 2D insulator without pairing is in the topologically nontrivial regime. We label the four edges of a square as (I), (II), (III), and (IV) [Fig.2(b)], and we focus on the edge (I) first. We can replace $k_x \rightarrow -i\delta_x$ and decompose the Hamiltonian as $H = H_0 + H_p$, in which

$$H_0(-i\delta_x, k_y) = (m - t_x \delta_x^2/2)\sigma_x \tau_z - iA_0 \sigma_z s_z \delta_x,$$

$$H_p(-i\delta_x, k_y) = A_0 k_y \sigma_y \tau_z + (\Delta_x/2) s_y \delta_x^2,$$  \hspace{1cm} (5)

where the insignificant $k_y^2$ term has been omitted. The purpose of this decomposition is to solve $H_0$ first, and then treat $H_p$ as a perturbation, which is justified when the pairing is relatively small (This is the case in real samples).

Solving the eigenvalue equation $H_0 \psi_{\alpha}(x) = E_{\alpha} \psi_{\alpha}(x)$ under the boundary condition $\psi_{\alpha}(-\infty) = 0$, we find four zero-energy solutions, whose forms are

$$\psi_{\alpha}(x) = N_\alpha \sin(k_1 x) e^{-ik_2 x} e^{iK x} \chi_\alpha,$$  \hspace{1cm} (6)

with normalization given by $|N_\alpha|^2 = 4|\kappa_2^2(\kappa_1^2 + \kappa_y^2)/k_1^2|$ (Here, $\kappa_1 = \sqrt{2m_0 - \frac{\Delta_x}{\Delta_y}}$, $\kappa_2 = \frac{\Delta_x}{\Delta_y}$. The result remains valid even when $k_1$ is imaginary). The eigenvectors $\chi_\alpha$ satisfy $\sigma_z s_\tau \chi_\alpha = -\chi_\alpha$. We can explicitly choose them as

$$\chi_1 = |\sigma_y = -1 \rangle \otimes | \uparrow \rangle \otimes | \tau_z = +1 \rangle,$$

$$\chi_2 = |\sigma_y = +1 \rangle \otimes | \downarrow \rangle \otimes | \tau_z = +1 \rangle,$$

$$\chi_3 = |\sigma_y = +1 \rangle \otimes | \uparrow \rangle \otimes | \tau_z = -1 \rangle,$$

$$\chi_4 = |\sigma_y = -1 \rangle \otimes | \downarrow \rangle \otimes | \tau_z = -1 \rangle,$$  \hspace{1cm} (7)

then the matrix elements of the perturbation $H_p$ in this basis are

$$H_{k,\alpha\beta}(k_y) = \int_{-1}^{+1} dx \psi^*_\alpha(x) H_p(-i\delta_x, k_y) \psi_\beta(x),$$  \hspace{1cm} (8)

therefore, the final form of the effective Hamiltonian is

$$H_1(k_y) = -A_0 k_x s_z + M_1 s_y \tau_y,$$  \hspace{1cm} (9)

where

$$M_1 = (\Delta_c/2) \int_{-1}^{+1} dx \psi^*_\alpha(x) \partial_x^2 \psi_\alpha(x) = \Delta_c m/t_c.$$  \hspace{1cm} (10)

Similarly, the low-energy effective Hamiltonians for the other three edges are

$$H_{II}(k_y) = A_0 k_x s_z + M_{II} s_y \tau_y,$$

$$H_{III}(k_y) = A_0 k_y s_z + M_{III} s_y \tau_y,$$

$$H_{IV}(k_y) = A_0 k_y s_z + M_{IV} s_y \tau_y$$  \hspace{1cm} (11)

with $M_{II} = M_{IV} = \Delta_c m/t_y$, and $M_{III} = M_1$. To be more transparent, let us take an “edge coordinate” $l$, which grows
MKPs in triangle samples. (a) The existence or absence of MKPs depends on the edge directions at the corner, which can be explained in the edge theory. The lower corner has a sign change in the edge Dirac mass, while the right corner does not. (b) For a $\pi/4$ angle, the edge Dirac mass vanishes, and the edge states display a gapless feature (see the inset, and compare it to that of (a)). $m_0 = 1.5$, $t_s = t_t = 2.0$, $A_t = A_s = 2.0$, $\Delta_A = -\Delta_t = 1.0$, $\mu = 0$.

in the anticlockwise direction (apparently, $l$ is defined mod $2(L_x + L_y)$), then the low-energy edge theory becomes

$$H_{\text{edge}} = -iA(l)s_y\partial_t + M(l)s_y\tau_y.$$  

The kinetic-energy coefficient $A(l)$ and the Dirac mass $M(l)$ are step functions: $A(l) = A_x, A_x, A_y, A_t$ and $M(l) = \Delta m/t_x, -\Delta m/t_x, \Delta m/t_y, -\Delta m/t_t$ for (I), (II), (III), and (IV), respectively. At each corner, the $A_{x,y}$ coefficient does not change sign, while the Dirac mass does, which is due to the sign changing in the $d$-wave pairing: $\Delta_d = -\Delta_d$. Consequently, there is a MKP at each corner (analogous to the Jackiw-Rebbi zero modes \[15,16\]). For example, at the corner between (I) and (II), we have

$$|\psi_{\text{MKP}}(l)\rangle \propto e^{-\frac{i}{2}d(l,M(l))}d(l,M(l))\sigma_s = \tau_y = \pm 1.$$  

TRS ensures that these two modes cannot be coupled to generate an energy gap. In essence, the edge theory above can be regarded as two copies of that of Ref.\[88\], with TRS as the key additional input.

By a similar calculation, one can find that the sign changing in $M(l)$ occurs at a corner when one of the edges has a polar angle within $[-\pi/4, \pi/4]$ and the other within $[\pi/4, 3\pi/4]$ (the gap-maximum direction is taken as the zero polar angle). In Fig 3(a), the lower corner has a sign changing while the right corner does not, and the existence or absence of MKP is consistent with the edge-theory prediction. If one of the edges lies in the $\pi/4$ direction, the edge states become gapless, which also manifests in the numerical spectrum in Fig 3(b).

Finally, we mention that cuprate superconductors in proximity to 3D topological insulators have been experimentally studied for the purpose of creating vertex (instead of corner) MZMs\[117–120\]. In these setups, the 2D topological surface states (instead of the 1D edge states) are the key ingredients.

$s_x$-wave pairing.-Now we consider fully gapped $s_x$-wave superconductors with sign changing in the pairing. A host of candidates can be found in high $T_c$ iron-based superconductor\[121\]–\[122\], whose pairing at the Fermi surfaces near the Brillouin zone center and the Brillouin zone boundary have both $s$-wave nature but with opposite signs. The Fermi surfaces do not cross the pairing nodal rings, therefore, the superconductor is fully gapped. A simplest form of $s_x$-wave pairing is:

$$\Delta(k) = \Delta_0 - \Delta_1(\cos k_x + \cos k_y),$$  

with $0 < \Delta_0 < 2\Delta_1$. The pairing node is $\cos k_x + \cos k_y = \Delta_0/\Delta_1$.

Let us first study the edge theory of TI. Expanding the Hamiltonian near $k = (0, 0)$ and taking $\mu = 0$, we have

$$H(k) = (m + \frac{t_s^2}{2} + \frac{t_t^2}{2})\sigma_x \tau_z + A_x\sigma_x \tau_z + A_y\sigma_y \tau_z + \frac{\Delta_0 - 2\Delta_1}{2}(k_x^2 + k_y^2)\tau_y\tau_z.$$  

Following a similar approach as the previous section, for the edge (I), we decompose the Hamiltonian as $H = H_0 + H_p$, where

$$H_0(-i\partial_y, k_y) = (m - \frac{t_y^2}{2})\sigma_y \tau_z - iA_x\sigma_x \tau_z \partial_x,$$

$$H_p(-i\partial_y, k_y) = A_x\kappa_x \sigma_y \tau_z + [\Delta_0 - 2\Delta_1 - (\Delta_1/2)^2]s_y\tau_y.$$  

Similar to the previous section, four zero-energy solutions of $H_0$ can be found, and $H_p$ takes the following form within this four-dimensional low-energy subspace:

$$H_1(k_y) = -A_x\kappa_x \sigma_y + M_1s_y \tau_y,$$  

with $M_1 = \int_0^{\infty} dx \psi_0(x)[\Delta_0 - 2\Delta_1 - (\Delta_1/2)^2]\psi_0(x) = \Delta_0 - 2\Delta_1 - \Delta_1m/t_x$. The low-energy effective Hamiltonians for the other three edges take the same forms as in Eq.\[11\], with Dirac masses $M_{III} = \Delta_0 - 2\Delta_1 - \Delta_1m/t_x = M_1$, and $M_{II} = M_{IV} = \int_0^{\infty} dy \psi_0(y)[\Delta_0 - 2\Delta_1 - (\Delta_1/2)^2]\psi_0(y) = \Delta_0 - 2\Delta_1 - \Delta_1m/t_x$. Using the edge coordinate $l$, the effective edge Hamiltonian is the same as Eq.\[12\] with the same $A(l)$ but different $M(l)$, namely, $M(l) = -\Delta_0 - \Delta_1m/t_x, -\Delta_0 - \Delta_1m/t_y, -\Delta_0 - \Delta_1m/t_t, -\Delta_0 - \Delta_1m/t_3$ for (I), (II), (III), and (IV), respectively, where we have defined $\Delta_0 = 2\Delta_1 - \Delta_0$.

To have MKP at each corner, the sign of Dirac mass $M(l)$ must change from an edge to its adjacent, which leads to the following criterion:

$$(\Delta_0 + \Delta_1m/t_x)(\Delta_0 + \Delta_1m/t_t) < 0.$$  


Let us define \( R_x \equiv \sqrt{2\Delta_0/\Delta_1} \), whose physical meaning is the radius of the ring of the pairing node, across which the pairing changes sign, and \( R_y \equiv \sqrt{2m/T_c} \), whose meanings are the two semi-axes of the ellipse determined by \( m + \frac{x^2}{a^2} + \frac{y^2}{b^2} = 0 \) (i.e., the “band-inversion ring” of TI, where the sign of the \( \sigma_z \) term changes). The mode existence criterion in Eq. (19) (for \( \mu = 0 \)) becomes
\[
(R_x - R_z)(R_x - R_y) < 0,
\]
which means that the band-inversion ring has to cross the pairing nodal ring [Fig. 4a]. Although derived from the continuum model, Eq. (19) is quite accurate according to our lattice-model numerical results [123]. Intuitively, the low-energy edge modes come mainly from states near the band inversion (where the bulk states have the lowest energies), and inherit the sign of pairing there. When the two rings cross, the pairing sign at band inversion is opposite at two adjacent edges, which supports MKPs. We emphasize that the TI has to be anisotropic in the \( x, y \) directions to satisfy Eq. (19) \((R_x \neq R_y)\), which is the case for the high-transition-temperature TI WTe\(_2\) [85]. In Fig. 4b, one finds the existence of MKP when Eq. (19) is satisfied. Including a modest chemical potential with a Fermi surface is innocuous [Fig. 4c], as the Fermi surface can be gapped out by the induced pairing as long as it does not cross the pairing node. A \((m, \Delta_0/\Delta_1, \mu)\) phase diagram is shown in Fig. 4d.

So far, we have not discussed disorders. We have numerically confirmed that usual disorders such as on-site random potential does not destroy the MKPs [123]. In addition, our proposal does not require atomically precise edges. Modest edge imperfections do not affect the MKPs because they are pinned to zero energy by particle-hole symmetry; “big” edge imperfections just create new corners that host their own MKPs, which offer more opportunity to observe MKPs [123].

Finally, it is useful to mention that, in both the \( d \)-wave and \( s_x \)-wave cases, a single MZM can be created from the MKP at the corner by killing one mode in the pair. Apparently, TRS must be broken. For example, it can be achieved by adding an in-plane magnetic field with an appropriate magnitude. The physical picture is most transparent in the edge theory (see the Supplemental Material for details [123]).

Experimental estimations.—For concreteness, let us focus on the high-temperature \( s_x \)-wave iron-based superconductors. As emphasized above, in the \( s_x \)-wave case the TI band structure is required to be anisotropic in the \( x \) and \( y \) directions [due to Eq. (19)]. Notably, the monolayer WTe\(_2\), which has recently been confirmed as a high-temperature TI in experiments [85] (up to 100 Kelvin), has the desired band structure [86]. According to the \( \mathbf{k} \cdot \mathbf{p} \) model in Ref. [86], we fit the parameters to be \( R_x = 0.41 \AA^{-1}, R_y = 0.15 \AA^{-1} \) (details are given in the Supplemental Material [123]). The reciprocal lattice vectors of WTe\(_2\) along the \( x \) and \( y \) directions are \( G_x \approx 1.0 \AA^{-1} \) and \( G_y \approx 1.8 \AA^{-1} \). Thus, the band-inversion ring reaches close to the Brillouin zone boundary in the \( x \) direction, while it stays close to the zone center in the \( y \) direction, resembling the advantageous shape of the band-inversion ring in Fig. 4a. Although an accurate estimation of the magnitude of the induced pairing gap is not available, we note that cuprate superconductors can induce a gap of tens of meV at the surface states of topological insulators [117, 118]; presumably similar order of magnitude can be expected in the present setup. Therefore, among other options, a setup composed of a WTe\(_2\) monolayer in proximity to a high-\( T_c \) iron-based superconductor is promising for the present proposal. A WTe\(_2\) monolayer in proximity to cuprate superconductors is also promising.

Conclusions.—We have shown that a 2D TI with proximity-induced \( d \)-wave or \( s_x \)-wave pairing, though being topologically trivial as a TRI superconductor, is a promising candidate of high-temperature platform for realizing robust Majorana corner modes. We provide quantitative criteria for this proposal. This Letter may also stimulate further studies of topologically-trivial-superconductor-based Majorana modes.

Acknowledgements.—We would like to thank Wei Li for helpful discussions. This work is supported by NSFC (No. 11674189). Z. Y. is supported in part by the China Postdoctoral Science Foundation (2016M590082).

Note added: Recently, there appeared a related preprint [124] that focuses on the \( s_x \)-wave case.
Nature Physics 8, 795–799 (2012).

[39] MT Deng, CL Yu, GY Huang, Marcus Larsson, Philippe Caroff, and HQ Xu, “Anomalous zero-bias conductance peak in a nb-insb nanowire-nb hybrid device,” Nano letters 12, 6414–6419 (2012).

[40] Anindya Das, Yuval Ronen, Yonatan Most, Yuval Oreg, Moty Heiblum, and Hadas Shtrikman, “Zero-bias peaks and splitting in an al-mnas nanowire topological semiconductor as a signature of majorana fermions,” Nature Physics 8, 887–895 (2012).

[41] A. D. K. Finck, D. J. Van Harlingen, P. K. Mohseni, K. Jung, and X. Li, “Anomalous modulation of a zero-bias peak in a hybrid nanowire-supersolid device,” Phys. Rev. Lett. 110, 126406 (2013)

[42] HOH Churchill, V Fatemi, Kasper Grove-Rasmussen, MT Deng, Philippe Caroff, HQ Xu, and Charles M Marcus, “Superconductor-nanowire devices from tunneling to the multichannel regime: Zero-bias oscillations and magetoconductance crossover,” Physical Review B 87, 241401 (2013).

[43] Y.-F. Lv, W.-L. Wang, Y.-M. Zhang, H. Ding, W. Li, L. Wang, K. He, C.-L. Song, X.-C. Ma, and Q.-K. Xue, “Experimental Observation of Topological Superconductivity and Majorana Zero Modes on beta-Bi2Pd Thin Films,” ArXiv e-prints (2016), arXiv:1607.07551 [cond-mat.supr-con]

[44] Wei-Miao Feng, Xiaochuan Jiang, X. J. Zhou, Xucun Ma, Q. K. Xue, and Feng Nie, “Topological edge states in a high-temperature superconductor,” Phys. Rev. B 92, 081406 (2015).

[45] ZF Wang, Huimin Zhang, Delfi Liu, Chong Liu, Chenjia Tang, Canli Song, Yong Zhong, Junping Peng, Fansen Li, Cina Nie, LiLi Wang, Y. J. Zhou, Xucun Ma, Q. K. Xue, and Feng Nie. “Topological edge states in a high-temperature superconductor fese/srto3 (001) film,” Nature Materials 15, 968–973 (2016).

[46] SM Albrecht, AP Higginbotham, M Madsen, F Kueemth, TS Jespersen, Jesper Nygård, P Kroghstrup, and CM Marcus, “Exponential protection of zero modes in majorana islands,” Nature 531, 206–209 (2016).

[47] MT Deng, S Vaitiekunas, EB Hansen, J Danon, M Leijnse, K Flensberg, J Nygård, P Krogstrup, and CM Marcus, “Majorana bound state in a coupled quantum-dot hybrid-nanowire system,” Science 354, 1557–1562 (2016).

[48] R. Pawlak, M. Kisiel, J. Klinovaja, T. Meier, S. Kawai, T. Glatzel, D. Loss, and E. Meyer, “Probing atomic structure and Majorana wavefunctions in mono-atomic Fe chains on superconducting Pb surface,” npj Quantum Mechanics 2, 16035 (2016), arXiv:1505.06078 [physics.atom-clust]

[49] Jin-Peng Xu, Mei-Xiao Wang, Zhi Long Liu, Jian-Feng Ge, Xiaojun Yang, Canhua Liu, Zhu An Xu, Dandan Guan, Chun Lei Gao, Dong Qian, Ying Liu, Qiang-Hua Wang, Fu-Chun Zhong, Qi-Kun Xu, and Jin-Feng Jia, “Experimental detection of a majorana mode in the core of a magnetic vortex inside a topological insulator-supersolid bismuth/niobium heterostructure,” Phys. Rev. Lett. 114, 017001 (2015).

[50] Hao-Hua Sun, Kai-Wei Zhang, Lun-Hui Hu, Chuang Li, Guang-Yong Wang, Hai-Yang Ma, Zhu-An Xu, Chun-Lei Gao, Dan-Dan Guan, Yao-Yi Li, Canhua Liu, Dong Qian, Yi Zhou, Liang Fu, Shao-Chun Li, Fu-Chun Zhong, and Jin-Feng Jia, “Majorana zero mode detected with spin selective andrewe reflection in the vortex of a topological superconductor,” Phys. Rev. Lett. 116, 257003 (2016).

[51] Q. L. He, L. Fan, A. L. Stern, E. Burks, X. Che, G. Yin, J. Wang, B. Lian, Q. Zhou, E. S. Choi, K. Murata, X. Kou, T. Nie, Q. Shao, Y. Fan, S.-C. Zhang, K. Liu, J. Xia, and K. L. Wang, “Chiral Majorana edge state in a quantum anomalous Hall insulator-supersolid structure,” ArXiv e-prints (2016), arXiv:1606.05712 [cond-mat.supr-con]

[52] Hao Zhang, Chun-Xiao Liu, Sasa Gazibegovic, Di Xu, John A Logan, Guanzhong Wang, Nick van Loo, Jouri DS Bommer, Michiel WA de Moor, Diana Car, et al., “Quantized majorana conductance,” arXiv preprint arXiv:1710.10701 (2017).

[53] Peng Zhang, Koichiro Yaji, Takahiro Hashimoto, Yuichi Ota, Takeshi Kondo, Kozo Okazaki, Zhijun Wang, Jinseng Wen, GD Gu, Hong Ding, et al., “Observation of topological superconductivity on the surface of an iron-based superconductor,” Science , eaan4596 (2018).

[54] Dongfei Wang, Lingyun Kong, Peng Fan, Hui Chen, Yujie Sun, Shixuan Du, John Schneeloch, RD Zhong, GD Li, Guan-Yong Wang, Hai-Yang Ma, Zhu-An Xu, Chun-Lei Gao, Dong Qian, Yi Zhou, et al., “Observation of pristine majorana bound state in iron-based superconductor,” arXiv preprint arXiv:1706.06074 (2017).

[55] Xiao-Liang Qi, Taylor L. Hughes, S. Raghuv, and Shou-Cheng Zhang, “Time-reversal-invariant topological superconductors and superfluids in two and three dimensions,” Phys. Rev. Lett. 102, 187001 (2009).

[56] Xiao-Liang Qi, Taylor L. Hughes, and Shou-Cheng Zhang, “Topological invariants for the fermi surface of a time-reversal-invariant superconductor,” Phys. Rev. B 81, 134508 (2010).

[57] Fan Zhang, C. L. Kane, and E. J. Mele, “Time-reversal-invariant topological superconductivity and majorana knmers pairs,” Phys. Rev. Lett. 111, 056402 (2013).

[58] Chris L. M. Wong and K. T. Law, “Majorana kramers doublets in d2−2 wave superconductors with rashba spin-orbit coupling,” Phys. Rev. B 86, 184516 (2012).

[59] Masatoshi Sato, “Topological odd-parity superconductors,” Phys. Rev. B 81, 220504 (2010).

[60] Fan Zhang, C. L. Kane, and E. J. Mele, “Topological mirror superconductivity,” Phys. Rev. Lett. 111, 056403 (2013).

[61] Liang Fu and Erez Berg, “Odd-parity topological superconductors: Theory and application to cu,bis3,,” Phys. Rev. Lett. 105, 097001 (2010).

[62] Arbel Haiim, Erez Berg, Karsten Flensberg, and Yuval Oreg, “No-go theorem for a time-reversal invariant topological phase in noninteracting systems coupled to conventional superconductors,” Phys. Rev. B 94, 161110 (2016).

[63] Xiong-Jun Liu, Chris L. M. Wong, and K. T. Law, “Non-abelian majorana doublets in time-reversal-invariant topological superconductors,” Phys. Rev. X 4, 021018 (2014).

[64] Fan Zhang and C. L. Kane, “Anomalous topological pumps and fractional josephson effects,” Phys. Rev. B 90, 020501 (2014).

[65] Pin Gao, Ying-Ping He, and Xiong-Jun Liu, “Symmetry-protected non-abelian braiding of majorana knmers pairs,” Phys. Rev. B 94, 224509 (2016).

[66] Zhi-qiag Bao and Fan Zhang, “Topological majorana two-channel interface,” Phys. Rev. Lett. 111, 056403 (2013).

[67] Peng Zhang, C. L. Kane, and E. J. Mele, “Topological mirror superconductivity,” Phys. Rev. Lett. 111, 056403 (2013).

[68] Christoph P. Orth, Rakesh P. Tiwari, Tobias Meng, and Thomas L. Schmidt, “Non-abelian parafermions in time-reversal-invariant topological insulators and kramers pairs,” Phys. Rev. B 91, 081406 (2015).

[69] Constantin Schrade, A. A. Zyuzin, Jelena Klinovaja, and Thomas L. Schmidt, “Non-abelian parafermions in time-reversal-invariant interacting helical systems,” Phys. Rev. B 91, 081406 (2015).

[70] Christof Schrade, A. A. Zyuzin, Jelena Klinovaja, and Daniel Loss, “Proximity-induced π josephson junctions in topological insulators and kramers pairs of majorana
fermions,” Phys. Rev. Lett. 115, 237001 (2015).

[70] Constantin Schrade and Liang Fu, “Parity-controlled 2π Josephson effect mediated by majorana kramers pairs,” Phys. Rev. Lett. 120, 267002 (2018).

[71] Alberto Camjayi, Liliana Arrachea, Armando Aligia, and Felix von Oppen, “Fractional spin and josephson effect in time-reversal-invariant topological superconductors,” Phys. Rev. Lett. 119, 046801 (2017).

[72] Anna Keselman, Liang Fu, Ady Stern, and Erez Berg, “Inducing time-reversal-invariant topological superconductivity and fermion parity pumping in quantum wires,” Phys. Rev. Lett. 111, 116402 (2013).

[73] S. Nakosai, J. C. Budich, Y. Tanaka, B. Trauzettel, and N. Nagaosa, “Majorana Bound States and Nonlocal Spin Correlations in a Quantum Wire on an Unconventional Superconductor,” Physical Review Letters 110, 117002 (2013). arXiv:1211.2307 [cond-mat.supr-con].

[74] Sho Nakosai, Yukio Tanaka, and Naoto Nagaosa, “Topological superconductivity in bilayer rashba system,” Phys. Rev. Lett. 108, 147003 (2012).

[75] Jing Wang, Yong Xu, and Shou-Cheng Zhang, “Two-dimensional time-reversal-invariant topological superconductivity in a doped quantum spin-hall insulator,” Phys. Rev. B 90, 054503 (2014).

[76] Arbel Haim, Anna Keselman, Erez Berg, and Yuval Oreg, “Time-reversal-invariant topological superconductivity induced by repulsive interactions in quantum wires,” Phys. Rev. B 89, 220504 (2014).

[77] Jelena Klinovaja, Amir Yacoby, and Daniel Loss, “Kramers pairs of majorana fermions and parafermions in fractional topological insulators,” Phys. Rev. B 90, 155447 (2014).

[78] Fan Yang, Cheng-Cheng Liu, Yu-Zhong Zhang, Yuqiu Yao, and Dung-Hai Lee, “Time-reversal-invariant topological superconductivity in n-doped bii,” Physical Review B 91, 134514 (2015).

[79] F. Trani, G. Campagnano, A. Tagliazucchi, and P. Lucignano, “High critical temperature nodal superconductors as building block for time-reversal invariant topological superconductivity,” Phys. Rev. B 94, 134518 (2016).

[80] Jian Li, Wei Pan, B. Andrei Bernevig, and Roman M. Lutchyn, “Detection of majorana kramers pairs using a quantum point contact,” Phys. Rev. Lett. 117, 046806 (2016).

[81] Fengcheng Wu and Ivar Martin, “Majorana kramers pair in a nematic vortex,” Physical Review B 95, 224503 (2017).

[82] Christopher Reeg, Constantin Schrade, Jelena Klinovaja, and Daniel Loss, “Dit topology superconductivity with emergent time-reversal symmetry,” Phys. Rev. B 96, 161407 (2017).

[83] J. Wang, “Electrically tunable topological superconductivity and Majorana fermions in two dimensions,” Phys. Rev. B 94, 214502 (2016). arXiv:1608.04870 [cond-mat.supr-con].

[84] H. Hu, F. Zhang, and C. Zhang, “Majorana Doublets, Flat Bands, and Dirac Nodes in s-Wave Superfluids,” ArXiv e-prints (2017). arXiv:1710.06388 [cond-mat.quant-gas].

[85] Sanfeng Wu, Valla Fatemi, Quinn D. Gibson, Kenji Watanabe, Takashi Taniguchi, Robert J. Cava, and Pablo Jarillo-Herrero, “Observation of the quantum spin hall effect up to 100 kelvin in a monolayer crystal,” Science 359, 76–79 (2018).

[86] Xiaoqiang Qian, Junwei Liu, Liang Fu, and Ju Li, “Quantum spin hall effect in two-dimensional transition metal dichalcogenides,” Science (2014), 10.1126/science.1256815.

[87] J. C. Y. Teo and C. L. Kane, “Topological defects and gapless modes in insulators and superconductors,” Phys. Rev. B 82, 115120 (2010).

[88] Zhongbo Yan, Ren Bi, and Zhong Wang, “Majorana zero modes protected by a hopf invariant in topologically trivial superconductors,” Phys. Rev. Lett. 118, 147003 (2017).

[89] Cheung Chan, Lin Zhang, Ting Fung Jeffrey Poon, Ying-Ping He, Yan-Qi Wang, and Xiong-Jun Liu, “Generic theory for majorana zero modes in 2d superconductors,” Physical review letters 119, 047001 (2017).

[90] B. A. Bernevig, T. L. Hughes, and S. C. Zhang, “Quantum spin Hall effect and topological phase transition in HgTe quantum wells,” Science 314, 1757 (2006).

[91] Wladimir A Benalcazar, B Andrei Bernevig, and Taylor L Hughes, “Quantized electric multipole insulators,” Science 357, 61–66 (2017).

[92] F. Schindler, A. M. Cook, M. G. Vergniory, Z. Wang, S. S. Parkin, B. A. Bernevig, and T. Neupert, “Higher-Order Topological Insulators,” ArXiv e-prints (2017), arXiv:1708.03636 [cond-mat.mes-hall].

[93] Fan Zhang, C. L. Kane, and E. J. Mele, “Surface state magnetization and chiral edge states on topological insulators,” Phys. Rev. Lett. 110, 046404 (2013).

[94] Wladimir A. Benalcazar, B. Andrei Bernevig, and Taylor L. Hughes, “Electric multipole moments, topological multipole moment pumping, and chiral hinge states in crystalline insulators,” Phys. Rev. B 96, 245115 (2017).

[95] Zhida Song, Zhong Fang, and Chen Fang, “(d – 2)-dimensional edge states of rotation symmetry protected topological states,” Phys. Rev. Lett. 119, 246402 (2017).

[96] J. Langbehn, Yang Peng, L. Trifunovic, Felix von Oppen, and Piet W. Brouwer, “Reflection-symmetric second-order topological insulators and superconductors,” Phys. Rev. Lett. 119, 246401 (2017).

[97] Yang Peng, Yimu Bao, and Felix von Oppen, “Boundary green functions of topological insulators and superconductors,” Phys. Rev. B 95, 235143 (2017).

[98] S. Imhof, C. Berger, F. Bayer, J. Brehm, L. Molenkamp, T. Kiessling, F. Schindler, C. H. Lee, M. Greiter, T. Neupert, and R. Thomale, “Topolectrical circuit realization of topological corner modes,” ArXiv e-prints (2017), arXiv:1708.03647 [cond-mat.mes-hall].

[99] M. Serra-Garcia, V. Peri, R. Susstrunk, O. R. Bilal, T. Larsen, L. G. Villanueva, and S. D. Huber, “Observation of a phononic quadrupole topological insulator,” ArXiv e-prints (2017), arXiv:1708.05015 [cond-mat.mtrl-sci].

[100] F. Schindler, Z. Wang, M. G. Vergniory, A. M. Cook, A. Murani, S. Sengupta, A. Y. Kasumov, R. Deblock, S. Jeon, I. Dzoldz, H. Bouchiat, S. Guérard, A. Yazdiani, B. A. Bernevig, and T. Neupert, “Higher-Order Topology in Bismuth,” ArXiv e-prints (2018), arXiv:1802.02585 [cond-mat.mtrl-sci].

[101] Motohiko Ezawa, “Higher-order topological insulators and semimetals on the breathing kagome and pyrochlore lattices,” Phys. Rev. Lett. 120, 026801 (2018).

[102] C. W. Peterson, W. A. Benalcazar, T. L. Hughes, and G. Bahl, “Demonstration of a quantized microwave quadrupole insulator with topologically protected corner states,” ArXiv e-prints (2017), arXiv:1710.03231 [cond-mat.mes-hall].

[103] M. Geier, L. Trifunovic, M. Hoskam, and P. W. Brouwer, “Second-order topological insulators and superconductors with an order-two crystalline symmetry,” ArXiv e-prints (2018), arXiv:1802.02585 [cond-mat.mtrl-sci].

[104] X. Zhu, “Tunable Majorana corner states in a two-dimensional second-order topological superconductor induced by magnetic fields,” ArXiv e-prints (2018), arXiv:1801.10053 [cond-mat.mes-hall].
Supplemental Material

This supplemental material contains eight parts: (i) The derivation of the edge theory for the II, III, IV edges. (ii) The derivation of the edge theory for the $s_\pm$-wave pairing via the lattice model. (iii) Demonstrating the absence of Majorana Kramers pair when the pairing nodal ring does not cross the band-inversion ring. (iv) Experimental estimations. (v) Realizing single Majorana zero mode at the corner. (vi) Stability of the Majorana corner modes against disorders. (vii) Effects of edge imperfections. (viii) Effects of chemical potential and phase diagram.

I. EDGE THEORY OF THE II, III, IV EDGES FOR THE $d$-WAVE PAIRING

We start from the low-energy bulk Hamiltonian around $k = (0, 0)$. Having $d$-wave pairing in mind, we take $\Delta_0 = 0$. The low-energy Hamiltonian at $\mu = 0$ reads (not imposing any constraint on $\Delta_y$ at this stage):

$$H(k) = (m + \frac{t_x}{2}k_x^2 + \frac{t_y}{2}k_y^2)\sigma_x\tau_z + A_x k_x \sigma_x \sigma_z + A_y k_y \sigma_y \tau_z$$

$$+ [\Delta_x + \Delta_y - \frac{1}{2}(\Delta_x k_x^2 + \Delta_y k_y^2)] s_y \tau_y. \quad (20)$$

Note that $m < 0$. For the edge III, the $k_y^2$ terms can be neglected and the Hamiltonian is decomposed as $H = H_0 + H_p$, with

$$H_0(-i\partial_x, k_y) = (m - t_x \partial_x^2/2)\sigma_x\tau_z - iA_x \partial_x \sigma_z s_z,$$

$$H_p(-i\partial_x, k_y) = A_y k_y \sigma_y \tau_z + (\Delta_x + \Delta_y + \frac{2}{\Delta_x} \partial_x^2) s_y \tau_y. \quad (21)$$

When solving the eigenvalue equation $H_0\psi_a(x) = E_0\psi_a(x)$, the boundary condition is $\psi_a(0) = \psi_a(-\infty) = 0$. A straightforward calculation gives four solutions with $E_0 = 0$, whose forms are

$$\psi_a(x) = N_a \sin(k_1 x)e^{k_1 x} e^{i\gamma_1 y} \tilde{\chi}_a$$

with the normalization constant $N_a = 2 \sqrt{k_1(k_1^2 + k_2^2)}$ and the two parameters $k_1$ and $k_2$ given by

$$k_1 = \sqrt{-\frac{2m}{t_x} - \frac{A_x^2}{t_x^2}}, \quad k_2 = \frac{A_y}{t_x}. \quad (22)$$

$\tilde{\chi}_a$ are eigenvectors satisfying $\sigma_x \tilde{\chi}_a = \chi_a$. Here we choose

$$\tilde{\chi}_1 = |\sigma_y = +1\rangle \otimes |\uparrow\rangle \otimes |\tau_z = +1\rangle,$$

$$\tilde{\chi}_2 = |\sigma_y = -1\rangle \otimes |\downarrow\rangle \otimes |\tau_z = +1\rangle,$$

$$\tilde{\chi}_3 = |\sigma_y = +1\rangle \otimes |\uparrow\rangle \otimes |\tau_z = -1\rangle,$$

$$\tilde{\chi}_4 = |\sigma_y = +1\rangle \otimes |\downarrow\rangle \otimes |\tau_z = -1\rangle. \quad (23)$$

Then the matrix elements of the perturbation $H_p$ in this basis are

$$H_{III,p}(k_x) = \int_{-\infty}^{0} dx \psi_a^*(x)H_p(-i\partial_x, k_y)\psi_a(x). \quad (24)$$

In terms of the Pauli matrices, the final form of the effective Hamiltonian is

$$H_{III}(k_x) = A_x k_x s_z + M_{III} s_y \tau_y, \quad (25)$$

where

$$M_{III} = \int_{-\infty}^{0} dx \psi_a^*(x)(\Delta_x + \Delta_y + \frac{2}{\Delta_x} \partial_x^2)\psi_a(x)$$

$$= \Delta_x + \Delta_y + \Delta_x m/t_x. \quad (26)$$

Similarly, for the edge II, we also decompose the Hamiltonian into two parts, discarding terms of the order of $k_x^2$:

$$H_0(k_x, -i\partial_y) = (m - t_x \partial_x^2/2)\sigma_x\tau_z - iA_x \partial_x \sigma_z \tau_z,$$

$$H_p(k_x, -i\partial_y) = A_x k_x \sigma_x \sigma_z + (\Delta_x + \Delta_y + \frac{2}{\Delta_x} \partial_x^2)s_y \tau_y. \quad (27)$$

By solving the eigenvalue equation $H_0\psi_a(y) = E_p\psi_a(y)$ with the boundary condition $\psi_a(0) = \psi_a(-\infty) = 0$, we find that there are four solutions with $E_a = 0$, whose forms are

$$\psi_a(y) = N_a \sin(\gamma_1 y) e^{-y\gamma_1} e^{ik_1 x} \xi_a$$

with the normalization constant $N_a = 2 \sqrt{\gamma_1^2 + \gamma_2^2}/\gamma_1^2$ and the two parameters $\gamma_1$ and $\gamma_2$ given by

$$\gamma_1 = \sqrt{-\frac{2m}{t_y} - \frac{A_y^2}{t_y^2}}, \quad \gamma_2 = \frac{A_x}{t_y}. \quad (28)$$

$\xi_a$ are the eigenvectors satisfying $\sigma_x \xi_a = \xi_a$. Here we choose

$$\xi_1 = |\sigma_x = +1\rangle \otimes |\uparrow\rangle \otimes |\tau_z = +1\rangle,$$

$$\xi_2 = |\sigma_x = +1\rangle \otimes |\downarrow\rangle \otimes |\tau_z = +1\rangle,$$

$$\xi_3 = |\sigma_x = +1\rangle \otimes |\uparrow\rangle \otimes |\tau_z = -1\rangle,$$

$$\xi_4 = |\sigma_x = +1\rangle \otimes |\downarrow\rangle \otimes |\tau_z = -1\rangle. \quad (29)$$

In this basis, the matrix elements of the perturbation $H_p$ are

$$H_{III,\partial}(k_x) = \int_{0}^{\infty} dy \psi_a^*(y)H_p(k_x, -i\partial_y)\psi_a(y). \quad (30)$$

In terms of the Pauli matrices, the final form of the effective Hamiltonian is

$$H_{II}(k_x) = A_x k_x s_z + M_{II} s_y \tau_y, \quad (31)$$

where

$$M_{II} = \int_{0}^{\infty} dy \psi_a^*(y)(\Delta_x + \Delta_y + \frac{2}{\Delta_x} \partial_x^2)\psi_a(y)$$

$$= \Delta_x + \Delta_y + \Delta_x m/t_y. \quad (32)$$

Similarly, for the edge IV, the effective Hamiltonian is

$$H_{IV}(k_x) = -A_x k_x s_z + M_{IV} s_y \tau_y, \quad (33)$$

and $M_{IV} = M_{II}$.

For the $d$-wave pairing with amplitude satisfying $\Delta_x = -\Delta_y$, the $\Delta_x + \Delta_y$ term appearing in the mass term vanishes. Let
\[ \Delta_s = -\Delta_t = \Delta_d, \text{ then the effective Hamiltonian of the four } \]
\[ \text{edges are} \]
\begin{align*}
    H_1(k_y) &= -A_y k_y s_z + M_1 s_y \tau_y, \\
    H_0(k_y) &= A_y k_y s_z + M_0 s_y \tau_y, \\
    H_{III}(k_y) &= A_y k_y s_z + M_{III} s_y \tau_y, \\
    H_{IV}(k_y) &= -A_y k_y s_z + M_{IV} s_y \tau_y, \\
\end{align*}
\[ \text{(36)} \]
where \( M_1 = M_{III} = \Delta_s m/t_s, M_0 = M_{IV} = -\Delta_s m/t_s. \] It is immediately clear that on the mass terms on two edges always have opposite signs.

where \( x \) is the integer-valued coordinate (the lattice constant \( a = 1 \)) taking values from 1 to \( L \). In the basis \( \{ \psi_{1,x}, \psi_{2,x}, \ldots \} \), the Hamiltonian can be expressed in a matrix form \( H(k_y) = H_0(k_y) + H_1(k_y) + H_2(k_y) \) with

\[
H_0(k_y) = \begin{pmatrix}
    D_0 & T_0 & 0 & \cdots \\
    T_0^\dagger & D_0 & T_0 & \cdots \\
    0 & T_0^\dagger & D_0 & \cdots \\
    \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]
\[ \text{where } D_0 = (m_0 - t_s \cos k_y), T_0 = (-t_s \Gamma_1 + iA_s \Gamma_2)/2, \]
\[
H_1(k_y) = \begin{pmatrix}
    D_1 & T_1 & 0 & \cdots \\
    T_1^\dagger & D_1 & T_1 & \cdots \\
    0 & T_1^\dagger & D_1 & \cdots \\
    \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]
\[ \text{where } D_1 = (\Delta_0 - \Delta_1 \cos k_y) \Gamma_4, T_1 = -\Delta_1 \Gamma_4/2, \]
\[
H_2(k_y) = \begin{pmatrix}
    A_y \sin k_y \Gamma_3 & 0 & 0 & \cdots \\
    0 & A_y \sin k_y \Gamma_3 & 0 & \cdots \\
    0 & 0 & A_y \sin k_y \Gamma_3 & \cdots \\
    \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]
\[ \text{(41)} \]
To simply the calculation, we take \( k_y \) to be close to 0 and the pairing amplitude to be small, so that \( H_1(k_y) \) and \( H_2(k_y) \) can be treated as perturbations. We first solve for the eigenstates of \( H_0(k_y = 0) \). Their forms can be written as \( \psi = (\psi_1, \psi_2 \cdots)^T \), which satisfies the following iteration relation,

\[ \frac{1}{2}(-\Gamma_1 + iA_s \Gamma_2)\psi_{n+1} + m\Gamma_1 \psi_n + \frac{1}{2}(-\Gamma_1 - iA_s \Gamma_2)\psi_{n-1} = 0, \]
\[ \text{(42)} \]
for convenience we have renamed \( m_0 = t_s, t_s, A_s \) as \( m, t, A, \) respectively (Do not confuse it with the \( m \) in the main text). Because of the anticommutation relation between \( \Gamma_1 \) and \( \Gamma_2 \), the eigenvalues of \( \Gamma_1 \Gamma_2 \) are \( \pm1 \). To solve this equation, we choose a trial solution \( \psi_{n,s} = A^s \phi_n \), where \( \phi_n \) satisfies \(-\Gamma_1 \phi_n = s \phi_n \) (\( s = \pm1 \)). Note that \(-\Gamma_1 \Gamma_2 \) is an \( 8 \times 8 \) matrix, therefore, there are four \( \phi_n \)'s satisfying this equation. Substituting this trial solution into Eq. (42), it is readily found that

\[ \lambda_s = \frac{m \pm \sqrt{m^2 - (t^2 - A^2)}}{t + A}, \]
\[ \text{(43)} \]
It can be shown that \( |\lambda_s| < 1 \) when \( s = \text{sgn}(t/A) \) and \( m^2 < t^2 \). If we consider a semi-infinite geometry with \( L \rightarrow \infty \), the wave function should satisfy the boundary conditions: \( \psi_{n,0} = 0 \) and \( \psi_{n,\infty} = 0 \) (For convenience, we have added an artificial site \( x = 0, \) on which the wavefunction is zero). The solution is of the form \( \psi_{n,s} = N(A^s - A^\dagger)^s \phi_n \), where \( N \) is the normalization constant:

\[ |N|^2 = \left[ \frac{18 \sqrt{m^2 - (t^2 - A^2)^2}}{A^2} \right]^{-1}. \]
\[ \text{(44)} \]
Now the effective Hamiltonian for the edge states is obtained from perturbation theory,

\[ H_{\text{eff}}(k_y) = \psi_{A}^\dagger [H_1(k_y) + H_2(k_y)] \psi_B. \]  
\[ \text{(45)} \]
The mass term of the effective Hamiltonian comes solely from $H_1(k_\nu)$. Ignoring all terms of orders higher than $k_\nu$, we get

$$M_1 = \Delta_0 - \Delta_1 - \frac{\Delta_1}{2}\left(\sum_{n=1}^4 \psi_{n+1}^\dagger \psi_n + \sum_{n=2}^4 \psi_{n-1}^\dagger \psi_n\right).$$

After straightforward calculations, we find

$$M_1 = \Delta_0 - \Delta_1 - \Delta_1|V|^2 \frac{m_s}{A} \frac{m^2 - (r^2 - A^2)}{r^2 - m^2} = \Delta_0 - \Delta_1 - \Delta_1 \frac{m_0 - t_x}{t_x}.$$  \hspace{1cm} (46)

Similar calculations lead to the mass terms for the other three edges, II, III, IV, which are $M_{II} = M_1$ and

$$M_{II} = M_{IV} = \Delta_0 - \Delta_1 - \Delta_1 \frac{m_0 - t_x}{t_y}.$$ \hspace{1cm} (47)

To create corner Majorana Kramers pairs, the mass term must change sign at the corner, which requires

$$(\Delta_0 - \Delta_1 - \Delta_1 \frac{m_0 - t_x}{t_y}) (\Delta_0 - \Delta_1 - \Delta_1 \frac{m_0 - t_y}{t_y}) < 0. \hspace{1cm} (48)$$

This criterion is the same as the one obtained from continuum model in the main text.

\begin{equation}
H(k) = \begin{pmatrix}
-\delta + \frac{\hbar^2 k_x^2}{2m_0} - \frac{\hbar^2 k_y^2}{2m_0} & 0 & -i\hbar v_1 k_x & h v_2 k_y \\
0 & -\delta + \frac{\hbar^2 k_x^2}{2m_0} + \frac{\hbar^2 k_y^2}{2m_0} & h v_2 k_y & -i\hbar v_1 k_x \\
-i\hbar v_1 k_x & h v_2 k_y & \delta + \frac{\hbar^2 k_x^2}{2m_0} + \frac{\hbar^2 k_y^2}{2m_0} & 0 \\
0 & -i\hbar v_1 k_x & 0 & \delta + \frac{\hbar^2 k_x^2}{2m_0} + \frac{\hbar^2 k_y^2}{2m_0}
\end{pmatrix},
\end{equation}

where $v_1 = 3.87 \times 10^5$ m/s, $v_2 = 0.46 \times 10^5$ m/s, $\delta = 0.33$ eV, $m_0 = 0.50 m_e$, $m_0^l = 0.16 m_e$, $m_0^d = 2.48 m_e$, $m_e$ is the free electron mass. The lattice constant of WTee is $a = 6.25\text{Å}, b = 3.48\text{Å}$. To simply the calculation, we make an approximation that $m_x = \sqrt{m^l_e m^d_e} = 1.11 m_e$ and $m_y = \sqrt{m^l_e m^d_y} = 0.24 m_e$, and then transform the $k \cdot p$ model to the lattice form, which is

$$H(k) = [m_0 + t_x \cos (k_x a) + t_y \cos (k_y b)] \sigma_z + A_x \sin(k_x a) \sigma_y + A_y \sin(k_y b) \sigma_x,$$ \hspace{1cm} (50)

where $t_x = \frac{\hbar^2}{m_x a^2} = 0.18$ eV, $t_y = \frac{\hbar^2}{m_y b^2} = 2.64$ eV, $m_0 = -\delta - t_x - t_y = -2.49$ eV, $A_x = h v_1 / a = 0.41$ eV, $A_y = h v_2 / b = 0.09$ eV. The band-inversion ring intersects the $k_x$ axis at $R_x = \arccos(-m_0 - t_y) / t_x / a = 0.41\text{Å}^{-1}$, and the $y$ axis at $R_y = \arccos(-m_0 + t_x) / t_y / b = 0.15\text{Å}^{-1}$. The energy gap is 0.087 eV (at $(k_x, k_y) = (0, \pm R_y)$). It is notable that $R_y$ and the energy gap thus obtained agree excellently with the results (0.146Å$^{-1}$, about 0.08 eV) based on the DFT calculation[88], indicating this lattice model gives an accurate description of the relevant band structure. In addition, the reciprocal lattice vectors are $G_x = 2\pi / a = 1.00\text{Å}^{-1}$, $G_y = 2\pi / b = 1.80\text{Å}^{-1}$, and it is straightforward to find that $R_x / (G_x / 2) = 0.82, R_y / (G_y / 2) = 0.16$, indicating that the band-inversion ring reaches close to the Brillouin-zone boundary in the $x$ direction, while stays close to the zone center in the $y$ direction.

In the presence of pairing, the Hamiltonian becomes

$$H(k) = [m_0 + t_x \cos(k_x a) + t_y \cos(k_y b)] \sigma_z \tau_z + A_x \sin(k_x a) \sigma_y \tau_z + A_y \sin(k_y b) \sigma_x \tau_y + \Delta(k) \sigma_y \tau_y,$$ \hspace{1cm} (51)

where $\Delta(k) = \Delta_0 + \Delta_x \cos(k_x a) + \Delta_y \cos(k_y b)$. The only difference between this Hamiltonian and the one in Eq.(1) of the main text is in the basis choices.

In high-temperature cuprate superconductor, the reduced gap $2\Delta/k_B T$ can be much larger than the expected BCS value 4.3 for $d$-wave pairing[125], indicating that the pairing amplitude can be quite large. e.g., it was found that $\Delta$ can be as high

III. MAJORANA KRAMERS PAIR IS ABSENT WHEN THE PAIRING NODAL RING DOES NOT CROSS THE BAND-INVERSION RING

In the main text, we have shown when the pairing nodal ring crosses the band-inversion ring, Majorana Kramers pairs are created at the corner of TI. To display the opposite situation, we tune the pairing nodal ring of Fig.4(a) in the main text to be around the $(\pi, \pi)$ point while keeping the band-inversion ring unchanged, so that the two rings no longer cross each other, as shown in Fig 5(a). In this regime, the numerical results demonstrate that there is no zero energy state (see Fig 5(b)), indicating the absence of Majorana Kramers pair.

IV. DETAILS OF EXPERIMENTAL ESTIMATIONS

In this section, we give an experimental estimation based on the band structures of WTee, which has recently been found as a 2D TI at temperature as high as 100 Kelvin[85]. In Ref.[86], a $k \cdot p$ model has been obtained to fit the band structure near the $\Gamma$ point (the band-inverted region), which is
are m concreteness, suppose that the magnetic field is added in the corner, time-reversal symmetry must be broken. We find that it can be achieved by adding an amplitude can also be higher than 10 meV, e.g., $\Delta = 15$ meV in superconductor Ba$_{1-x}$K$_x$Fe$_2$As$_2$ ($T_c = 37$ K)\cite{127}.

In Fig\textsuperscript{5} we use the model parameters of WTe$_2$ extracted from the $k \cdot p$ model, and take the $s_\pm$-wave pairing with an amplitude $\sim 100$ meV (which has been exaggerated, yet the result is qualitatively unchanged; similarly, the value of $A_y$ has also been increased from 0.09 eV to 0.2 eV). Fig\textsuperscript{5} demonstrates that Majorana Kramers pairs are created at the corners when the pairing nodal ring crosses the band-inversion ring.

V. REALIZING SINGLE MAJORANA ZERO MODE AT THE CORNER

To realize a single Majorana zero mode instead of a Majorana Kramers pair per corner, time-reversal symmetry must be broken. We find that it can be achieved by adding an appropriate in-plane magnetic field in either the $d$-wave or the $s_\pm$-wave case, provided that the system is anisotropic. For concreteness, suppose that the magnetic field is added in the $x$ direction, then it introduces a Zeeman energy $V_x s_x \tau_z$. The low-energy effective theory becomes

$$H_{\text{edge}} = -iA(l)s_x \partial_t + M(l)s_y \tau_3 + V_x s_x \tau_z.$$

(52)

Now we can split this $4 \times 4$ Hamiltonian into two decoupled blocks. In fact, in the two-dimensional $\tau_z = \pm s_z$ subspace, we have $V_x s_x \tau_z = -i V_x s_z \tau_3 = \mp i V_x s_y \tau_y$, therefore, the two decoupled blocks have Dirac mass $M(l) - V_x$ and $M(l) + V_x$, respectively, and single MZM appears when one of them has a sign changing, while the other does not. This is satisfied, for example, when $|\Delta m/t_x| < V_x < |\Delta m/t_z|$ for the $d$-wave case.

VI. STABILITY OF THE MAJORANA CORNER MODES AGAINST DISORDERS

In this section, we investigate the stability of the Majorana Kramers pairs against random disorders. For concreteness, we consider random on-site potentials. The real-space Hamiltonian reads
The last term represents the random on-site potential, which is taken to obey a Gaussian distribution, namely the unit matrix. The last term represents the random on-site potential, which is taken to obey a Gaussian distribution, namely the unit matrix. The last term represents the random on-site potential, which is taken to obey a Gaussian distribution, namely the unit matrix.

$$H = \sum_{i,j,\alpha,\beta} m_0 c_{i,j,\alpha}^\dagger (\sigma_z)_{\alpha\alpha'} (s_0)_{\beta \beta'} c_{i,j,\alpha'} \beta + \left( \frac{t_v}{2} c_{i,j,\alpha}^\dagger (\sigma_x)_{\alpha\alpha'} (s_0)_{\beta \beta'} c_{i,j+1,\alpha} \beta + \frac{t_v}{2} c_{i,j,\alpha}^\dagger (\sigma_z)_{\alpha\alpha'} (s_0)_{\beta \beta'} c_{i,j+1,\alpha'} \beta + h.c. \right)$$

where $s_{x,y,z}$ and $\sigma_{x,y,z}$ are Pauli matrices acting on spin and orbital degree of freedom, respectively, $s_0$ and $\sigma_0$ are the 2 $\times$ 2 unit matrix. The last term represents the random on-site potential, which is taken to obey a Gaussian distribution, namely $\langle V_{ij} \rangle = 0, \langle V_{ij} V_{km} \rangle = D_0^2 \delta_{ij} \delta_{jk}$ with $D_0$ characterizing the strength of the randomness.

Fig 7 shows several profiles of Gaussian disorders and the spectra close to zero energy. It is apparent that the Majorana Kramers pairs are robust against disorders for a quite broad range of disorder strength.

VII. EFFECTS OF EDGE IMPERFECTIONS

In real experiments, the edges are usually not atomically precise, and the edge orientations can vary in space. It is useful to study the stability of Majorana Kramers pairs in the presence of these imperfections. Thanks to the protection of particle-hole symmetry, a Majorana Kramers pair of zero modes remains robust in the presence of small or modest imperfections, because shifting the energy of a spatially isolated Majorana Kramers pair away from zero is inconsistent with the inherent particle-hole symmetry of BdG equation.

As a representative example of edge imperfections, we calculate the energies and eigenstates on a non-ideal square lattice with a small square removed at a corner (see Fig 8). In Fig 8(b), additional in-gap modes are created with energies far away from zero. Such nonzero-energy modes do not affect detection of the original Majorana Kramers pairs, which remain at zero energy. When the size of the removed square is increased [Fig 8(c)], the energies of in-gap modes are lowered, and finally, their energies come to zero, namely, additional Majorana Kramers pairs are created [Fig 8(d)(e)]. From this example, we can see that lattice imperfections generally do not destroy the Majorana Kramers pairs; instead, they provide more opportunity to detect them: The imperfections provide additional corners built in the samples, hosting additional Majorana Kramers pairs.
VIII. EFFECTS OF CHEMICAL POTENTIAL

In this section, we provide more details about the effects of a nonzero chemical potential. We will see that including the chemical potential does not change the physics qualitatively compared to the case of vanishing chemical potential.

First, we study the effect of chemical in the low-energy effective edge theory. At the edge 1, nonzero \( \mu \) simply generates the following term in the effective Hamiltonian:

\[
(\Delta H)_{\text{eff}} = - \int_0^{+\infty} dx \psi_0^*(x) \mu \tau_z \psi_0(x) = -\mu(\tau_z)_{\text{eff}}.
\]

Now the edge effective Hamiltonian reads

\[
H_{\text{edge}} = -iA(l)s_z \partial_l - \mu \tau_z + M(l)s_y \tau_y.
\]

This Hamiltonian can be decomposed as the sum of two independent \( 2 \times 2 \) Hamiltonians

\[
H_{\text{edge},-} = -iA(l)s_z \partial_l - \mu \tau_z - M(l)s_x,
\]

\[
H_{\text{edge},+} = -iA(l)s_z \partial_l + \mu \tau_z + M(l)s_x,
\]

where \( H_{\text{edge},-} \) acts in the \( \tau_z = s_z \) subspace spanned by \(|s_z = 1, \tau_z = 1\rangle, |s_z = -1, \tau_z = -1\rangle \), and \( H_{\text{edge},+} \) acts in the \( \tau_z = -s_z \) subspace spanned by \(|s_z = 1, \tau_z = -1\rangle, |s_z = -1, \tau_z = 1\rangle \).

Now the zero modes can be found as

\[
|\psi^\pm(l)\rangle \propto e^{-\int^l dp \langle \text{M}(p) \mp \mu \text{M}(p) \rangle}.
\]

Here, \( \mu \) only adds trivial phase factors to the wavefunctions without modifying their profiles. It indicates that a small or modest \( \mu \) (to which the low-energy theory is applicable) does not qualitatively change the physics.

Now we go beyond the low-energy continuum Hamiltonian to the lattice Hamiltonian. As we increase the chemical potential \( \mu \) away from 0, there are two possible scenarios that can kill the Majorana Kramers pairs: (i) The bulk gap closes; (ii) the edge gap closes. First, let us consider the first scenario. The BdG Hamiltonian of the bulk is

\[
H(k) = M(k) \sigma_z \tau_z + A_x \sin k_x \sigma_x \tau_x + A_y \sin k_y \sigma_y \tau_y + \Delta(k) s_y \tau_y - \mu \tau_z,
\]

whose energy spectra are

\[
E(k) = \pm \sqrt{(M^2(k) + A_x^2 \sin^2 k_x + A_y^2 \sin^2 k_y \pm \mu)^2 + \Delta^2(k)}.
\]

Without the pairing, the gap closing condition is

\[
\sqrt{M^2(k) + A_x^2 \sin^2 k_x + A_y^2 \sin^2 k_y} = |\mu|,
\]

which determines the bulk Fermi surface of the doped TI. Without losing generality.
we only focus on the $\mu > 0$ case. The Fermi surface appears when $\mu > \min[\sqrt{M^2(k) + A^2_1 \sin^2 k_x} + A^2_2 \sin^2 k_y]$. When the Fermi surface determined by

$$\sqrt{M^2(k) + A^2_1 \sin^2 k_x} + A^2_2 \sin^2 k_y = \mu$$

(59)

and the pairing nodal ring determined by $\Delta(k) = 0$ cross each other, the energy gap closes. For the pairing we considered in the main text, $\Delta(k) = \Delta_0 - \Delta_1 (\cos k_x + \cos k_y)$ with $0 < \Delta_0 < 2\Delta_1$, we find the gapless region is

$$\mu_1 < \mu < \mu_2,$$

(60)

where

$$\mu_1 = \min\left\{\frac{\sqrt{(m_0 - t_x)\Delta_0 \Delta_1}}{\Delta_1}, \frac{(m_0 - t_x)\Delta_0 \Delta_1}{\Delta_1}\right\}$$

and

$$\mu_2 = \max\left\{\frac{\sqrt{(m_0 - t_x)\Delta_0 \Delta_1}}{\Delta_1}, \frac{(m_0 - t_x)\Delta_0 \Delta_1}{\Delta_1}\right\}.$$

The scenario (ii) mentioned above, namely closing the edge gap to kill the Majorana Kramers pairs, is not as convenient to quantify. The reason is that the edge states of TI do not exist as a 1D system by itself, and the edge cannot be separately studied beyond the low-energy theory. Nevertheless, we can numerically find the critical chemical potential $\mu_c$ above which the Majorana Kramers pairs disappear. When the Cooper pairing is small, we find that $\mu_c < \mu_1$ in general, which indicates that the gap closing appear at the edge, i.e., the scenario (ii).

A 3D phase diagram with varied $(m_0, \Delta_0/\Delta_1, \mu)$ is numerically calculated and shown in Fig.4(d) in the main article. In this supplementary material, let us focus on the the $\Delta_0/\Delta_1 = 1$ cross section of this 3D phase diagram, which is shown in Fig.9(a). For the chosen parameters, the paring nodal ring and the band-inversion ring cross each other when $0.4 < m_0 < 1.3$. From Fig.9(b)(c), it is clear that zero modes are indeed found when $0.4 < m_0 < 1.3$, in accordance with the criterion Eq.(19) of the main article. Along the path $\gamma$, Majorana corner modes exist for $\mu \leq 0.38$ (Fig.9(d)). Note that the bulk gap closes at $\mu_1 = \sqrt{(m_0 - 1.3)^2 + 0.4^2} = 0.5$, therefore, the disappearance of Majorana corner modes is due to the gap closing at the edges.

FIG. 9. (a) A cross section of the 3D phase diagram in Fig.4(d) of the main text. The cross section corresponds to $\Delta_1 = \Delta_1$. In the shadow region the Majorana corner modes are found to exist. (b)(c)(d) show the energy spectrum when the parameters change along the path $\alpha, \beta, \gamma$ shown in (a), respectively. The system size is (b) $L_x \times L_y = 80 \times 20$, (c) $L_x \times L_y = 20 \times 90$, (d) $L_x \times L_y = 40 \times 40$. Common parameters are $t_x = A_x = 0.4, t_y = A_y = 1.3, \Delta_0 = \Delta_1 = 0.4$. 

\[\text{Image}\]