Bumpless Pipedreams, Reduced Word Tableaux and Stanley Symmetric Functions

Neil J.Y. Fan¹, Peter L. Guo², Sophie C.C. Sun³

¹Department of Mathematics
Sichuan University, Chengdu, Sichuan 610064, P.R. China

²,³Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300071, P.R. China

¹fan@scu.edu.cn, ²lguo@nankai.edu.cn, ³suncongcong@mail.nankai.edu.cn

Abstract

Lam, Lee and Shimozono introduced the structure of bumpless pipedreams in their study of back stable Schubert calculus. They found that a specific family of bumpless pipedreams, called EG-pipedreams, can be used to interpret the Edelman-Greene coefficients appearing in the expansion of a Stanley symmetric function in the basis of Schur functions. It is well known that the Edelman-Greene coefficients can also be interpreted in terms of reduced word tableaux for permutations. Lam, Lee and Shimozono proposed the problem of finding a shape preserving bijection between reduced word tableaux for a permutation $w$ and EG-pipedreams of $w$. In this paper, we construct such a bijection. The key ingredients are two new developed isomorphic tree structures associated to $w$: the modified Lascoux-Schützenberger tree of $w$ and the Edelman-Greene tree of $w$. Using the Little map, we show that the leaves in the modified Lascoux-Schützenberger of $w$ are in bijection with the reduced word tableaux for $w$. On the other hand, applying the droop operation on bumpless pipedreams also introduced by Lam, Lee and Shimozono, we show that the leaves in the Edelman-Greene tree of $w$ are in bijection with the EG-pipedreams of $w$. This allows us to establish a shape preserving one-to-one correspondence between reduced word tableaux for $w$ and EG-pipedreams of $w$.

1 Introduction

The structure of bumpless pipedreams was recently introduced by Lam, Lee and Shimozono [19] in their study of backward stable Schubert calculus. They proved that bumpless pipedreams can generate back stable Schubert polynomials, which are polynomial representatives of the Schubert classes of the infinite flag varieties. Restricting to finite flag varieties, bumpless pipedreams serve as a new combinatorial model for the Schubert polynomials. Moreover, they found that for a permutation $w$, the coefficients $c^w_\lambda$ in the expansion of the Stanley symmetric function

$$F_w(x) = \sum_\lambda c^w_\lambda s_\lambda(x)$$

(1.1)
in the basis of Schur functions $s_\lambda(x)$ can be interpreted by a specific family of bumpless pipedreams, namely, the EG-pipedreams of $w$.

Stanley symmetric functions were invented by Stanley in his seminal paper [28] in order to enumerate the reduced decompositions of permutations. The coefficients $c^w_\lambda$, now called the Edelman-Greene coefficients or the Stanley coefficients, were first conjectured by Stanley [28] and then proved by Edelman and Greene [9] to be nonnegative integers. More precisely, by developing the Edelman-Greene algorithm (also called the Coxeter-Knuth algorithm), Edelman and Greene [9] showed that the coefficient $c^w_\lambda$ is equal to the number of reduced word tableaux for $w$ with shape $\lambda$, see also Fomin and Greene [8], Lam [18] or Stanley [29].

Lam, Lee and Shimozono [19] proposed the following problem.

**Problem 1.1** (Lam-Lee-Shimozono [19, Problem 5.19]). Find a direct shape preserving bijection between EG-pipedreams of $w$ and reduced word tableaux for $w$.

In this paper, we provide a desired bijection as asked in Problem 1.1. The construction of our bijection relies on the Edelman-Greene insertion algorithm [9], the Little map [23] as well as two new structures developed in this paper: the modified Lascoux-Schützenberger tree and the Edelman-Greene tree.

Before outlining our strategy, let us recall the classical Lascoux-Schützenberger tree (LS-tree), which is an alternative approach to explain the Edelman-Greene coefficients. For a permutation $w$, the LS-tree of $w$ was introduced by Lascoux and Schützenberger [21] by utilizing the maximal transition formula for Stanley symmetric functions. In the LS-tree of $w$, the children of a node $u$ are the results of applying maximal transitions to $u$. A combinatorial proof of such transition relations was found by Little [23]. In the LS-tree of $w$, each leaf is labeled by a Grassmannian permutation, whose Stanley symmetric function is a Schur function. Thus the coefficient $c^w_\lambda$ is equal to the number of leaves in the LS-tree of $w$ whose Stanley symmetric function is $s_\lambda(x)$.

We introduce the modified Lascoux-Schützenberger tree (modified LS-tree) of a permutation $w$, and show that each leaf in the modified LS-tree of $w$ corresponds to a reduced word tableau for $w$. Unlike the classical LS-tree, the construction of a modified LS-tree is based on general transition relations satisfied by Stanley symmetric functions. More precisely, in the modified LS-tree of $w$, the children of a node $u$ are the results of applying some general (not necessarily maximal) transitions to $u$. Each leaf in the modified LS-tree of $w$ is a dominant permutation. It is known that for a dominant permutation $u$, the corresponding Stanley symmetric function $F_u(x)$ equals a Schur function $s_{\lambda(u)}(x)$, where $\lambda(u)$ is the Lehmer code of $u$. Therefore, the modified LS-tree of $w$ also allows us to expand $F_w(x)$ in terms of Schur functions, that is, the Edelman-Greene coefficient $c^w_\lambda$ is equal to the number of leaves in the modified LS-tree of $w$ whose Lehmer codes are $\lambda$. Moreover, employing the Little map [23], we can associate each leaf in the modified LS-tree of $w$ to a reduced word tableau for $w$.

It is worth mentioning another difference between the modified LS-tree and the classical LS-tree. Let $w$ be a permutation on $\{1, 2, \ldots, n\}$. Since maximal transitions may increase the size of permutations, there may exist nodes in the LS-tree of $w$ which are labeled with permutations on $\{1, 2, \ldots, m\}$ with $m > n$. However, the general transi-
tions used in this paper do not increase the size of permutations, that is, each node in the modified LS-tree of \( w \) is also labeled with a permutation on \( \{1, 2, \ldots, n\} \). Therefore, in some sense, the modified LS-tree seems to be more controllable than the LS-tree in the process of expanding a Stanley symmetric function into Schur functions.

On the other hand, we show that the EG-pipedreams of \( w \) can also be generated as the leaves of a tree associated to \( w \), which is isomorphic to the modified LS-tree of \( w \). Lam, Lee and Shimozono [19] introduced an operation, called droops, on bumpless pipedreams. They showed that any bumpless pipedream of \( w \) can be obtained by applying a sequence of droops to the Rothe pipedream of \( w \). By applying the droop operations, we construct a tree of bumpless pipedreams of \( w \), called the Edelman-Greene tree (EG-tree) of \( w \). In the EG-tree of \( w \), each node is a bumpless pipedream of \( w \), and the children of the node \( u \) are obtained by applying some specific droops to \( u \), which correspond to the general transitions in the process of constructing the modified LS-tree of \( w \). Thus the EG-tree of \( w \) and the modified LS-tree of \( w \) are isomorphic. In particular, the leaves in the EG-tree of \( w \) are exactly labeled with the EG-pipedreams of \( w \). Since the leaves in the modified LS-tree of \( w \) are in one-to-one correspondence with the reduced word tableaux for \( w \), we obtain a bijection between reduced word tableaux for \( w \) and EG-pipedreams of \( w \).

This paper is organized as follows. In Section 2, we give overviews of the Stanley symmetric function, the Edelman-Greene insertion algorithm, the Lascoux-Schützenberger tree and the Little map. In Section 3, we describe the structure of bumpless pipedreams as well as the droop operation introduced by Lam, Lee and Shimozono [19]. In Section 4 we introduce the structure of a modified LS-tree. Based on Section 4, we construct the structure of an EG-tree from a modified LS-tree in Section 5. In Section 6 using the structures developed in Sections 4 and 5 together with the Edelman-Greene algorithm and the Little map, we establish a shape preserving bijection between reduced word tableaux and EG-pipedreams.

## 2 Preliminaries

In this section, we collect some notions and structures that we are concerned with in this paper, including the Stanley symmetric function, the Edelman-Greene insertion algorithm, the Lascoux-Schützenberger tree and the Little map. The reader familiar with these structures could skip this section.

### 2.1 Stanley symmetric functions

Let \( w = w_1w_2\cdots w_n \in S_n \) be a permutation on \( \{1, 2, \ldots, n\} \). As usual, let \( s_i \) denote the simple transposition interchanging the elements \( i \) and \( i+1 \). Note that \( ws_i \) is the permutation obtained from \( w \) by swapping \( w_i \) and \( w_{i+1} \). A decomposition of \( w \) as a product of simple transpositions is called reduced if it consists of a minimum number of simple transpositions. A sequence \((a_1, a_2, \ldots, a_\ell)\) is called a reduced word of \( w \) if \( s_{a_1}s_{a_2}\cdots s_{a_\ell} \) is a reduced decomposition of \( w \). The length of \( w \), denoted \( \ell(w) \), is the number of simple transpositions in a reduced decomposition of \( w \).
Stanley symmetric functions were introduced by Stanley [28] to enumerate the reduced decompositions of a permutation. For a permutation $w$, the Stanley symmetric function $F_w(x)$ is defined as

$$F_w(x) = \sum_{(a_1, a_2, \ldots, a_\ell)} \sum_{1 \leq b_1 \leq \cdots \leq b_\ell} x_{b_1} x_{b_2} \cdots x_{b_\ell},$$  \hspace{1cm} (2.1)$$

where $(a_1, a_2, \ldots, a_\ell)$ ranges over the reduced words of $w$. Note that the definition of $F_w(x)$ in (2.1) is $F_w^{-1}(x)$ in Stanley’s original definition [28].

It is well known that $F_w(x)$ can be regarded as the stable limit of the Schubert polynomial $S_w(x)$, see for example [3, 4, 24, 26]. Let us recall the definition of a double Schubert polynomial $S_w(x; y)$, which reduces to $S_w(x)$ by setting $y_i = 0$ and will also be used in Section 3. Double Schubert polynomials were introduced by Lascoux and Schützenberger [20] as polynomial representatives of the $T$-equivariant classes for Schubert varieties in the flag manifold, which can be defined recursively as follows. If $w$ is the longest permutation $w_0 = n(n-1) \cdots 1$, then set

$$S_{w_0}(x; y) = \prod_{i+j \leq n} (x_i - y_j).$$

If $w \neq w_0$, then choose a simple transposition $s_i$ such that $\ell(ws_i) = \ell(w) + 1$, and let

$$S_w(x; y) = \partial_i S_{ws_i}(x; y).$$

Here, $\partial_i$ is the divided difference operator applies only to the $x$ variables. That is, $\partial_i f = (f - s_i f)/(x_i - x_{i+1})$, where $f$ is a polynomial in $x$ and $s_i f$ is obtained by interchanging $x_i$ and $x_{i+1}$ in $f$. Setting $y_i = 0$, $S_w(x; y)$ reduces to the (single) Schubert polynomial $S_w(x)$.

There have been several combinatorial rules to generate $S_w(x)$ and $S_w(x; y)$, see, for example [1, 3, 10, 11, 14, 15, 32, 33]. To explain the relation between $F_w(x)$ and $S_w(x)$, let us recall the combinatorial construction of $S_w(x)$ due to Billey, Jockusch and Stanley [3]:

$$S_w(x) = \sum_{(a_1, a_2, \ldots, a_\ell)} \sum_{1 \leq b_1 \leq \cdots \leq b_\ell} x_{b_1} x_{b_2} \cdots x_{b_\ell},$$  \hspace{1cm} (2.2)$$

where $(a_1, a_2, \ldots, a_\ell)$ ranges over the reduced words of $w$. In view of (2.1) and (2.2), it is clear that

$$F_w(x) = \lim_{m \to \infty} S_{1^m \times w}(x),$$  \hspace{1cm} (2.3)$$

where $1^m \times w$ is the permutation $12 \cdots m(w_1 + m) \cdots (w_n + m)$.

### 2.2 Edelman-Greene insertion algorithm

The Edelman-Greene algorithm [9] inserts a positive integer into an increasing tableau to yield a new increasing tableau. An increasing tableau of shape $\lambda$ is a filling of positive integers into the boxes of $\lambda$ such that the entries in each row and each column are strictly
increasing, see for example [5, 6, 30]. It is worth mentioning that the Hecke insertion algorithm developed by Buch, Kresch, Shimozono, Tamvakis and Yong [5] specializes to the Edelman-Greene algorithm when applying to the reduced words of permutations.

Given a reduced word \( a = (a_1, a_2, \ldots, a_\ell) \) of a permutation \( w \), the Edelman-Greene algorithm transforms \( a \) into a pair \( (P(a), Q(a)) \) of tableaux of the same shape. The tableaux \( P(a) \) and \( Q(a) \) are called the insertion tableau and the recording tableau, respectively. The tableau \( P(a) \) is an increasing tableau which can be obtained by constructing a sequence of increasing tableaux \( P_1, \ldots, P_\ell \) as follows. Let \( P_1 \) be the tableau with one box filled with \( a_1 \). Suppose that \( P_i \) \((1 \leq i \leq \ell - 1)\) has been constructed. Let us generate \( P_{i+1} \) by inserting the integer \( a_{i+1} \) into \( P_i \). Set \( x = a_{i+1} \). Let \( R \) be the first row of \( P_i \). Roughly speaking, an element in \( R \) may be bumped out and then inserted into the next row. The process is repeated until no element is bumped out. There are two cases.

Case 1: The integer \( x \) is strictly larger than all the entries in \( R \). Let \( P_{i+1} \) be the tableau by adding \( x \) as a new box to the end of \( R \), and the process terminates.

Case 2: The integer \( x \) is strictly smaller than some element in \( R \). Let \( y \) be the leftmost entry in \( R \) that is strictly larger than \( x \). If replacing \( y \) by \( x \) results in an increasing tableau, then \( y \) is bumped out by \( x \) and \( y \) will be inserted into the next row. If replacing \( y \) by \( x \) does not result in an increasing tableau, then keep the row \( R \) unchanged and the element \( y \) will be inserted into the next row.

Iterating the above procedure, we finally get the tableau \( P_{i+1} \). Set \( P(a) = P_\ell \). It should be noted that in the process of inserting an integer into a row of \( P_i \), we will not encounter the situation that this integer is equal to the largest element in that row. The tableau \( Q(a) \) is the standard Young tableau which records the changes of the shapes of \( P_1, P_2, \ldots, P_\ell \) as the insertion is performed.

For example, let \( a = (2, 3, 1, 6, 4, 3, 2) \) be a reduced word of the permutation \( w = 3514276 \). Then \( P(a) \) and \( Q(a) \) can be constructed as in Figure 2.1

\[
P_1 : \quad \begin{array}{cccc}
2 & 1 & 3 & 2 \\
2 & 3 & 4 & 2 \\
1 & 3 & 6 & 2 \\
1 & 3 & 4 & 6 \\
1 & 3 & 4 & 2 \\
2 & 3 & 6 & 4 \\
2 & 3 & 4 & 6 \\
1 & 2 & 4 & 6
\end{array}
\]

\[
Q_1 : \quad \begin{array}{cccc}
1 & 1 & 2 & 1 \\
3 & 2 & 4 & 3 \\
3 & 2 & 4 & 3 \\
3 & 5 & 6 & 3 \\
3 & 5 & 6 & 3 \\
3 & 5 & 6 & 3 \\
3 & 5 & 6 & 3 \\
6 & 7 & 8 & 9
\end{array}
\]

Figure 2.1: Construction of \( P(a) \) and \( Q(a) \) for \( a = (2, 3, 1, 6, 4, 3, 2) \).

To characterize the insertion tableau \( P(a) \), we need the notion of the row reading word of a tableau \( T \), that is, the word obtained by reading the entries of \( T \) along the rows from left to right, bottom to top. For example, the tableau \( P(a) \) in Figure 2.1 has row reading word \((6, 4, 2, 3, 1, 2, 4)\).
**Theorem 2.1** (Edelman-Greene [9]). The Edelman-Greene correspondence is a bijection between the set of reduced words of $w$ and the set of pairs $(P, Q)$ of tableaux of the same shape, where $P$ is an increasing tableau whose row reading word is a reduced word of $w$, and $Q$ is a standard Young tableau.

The following theorem is attributed to Edelman and Greene [9], see also Fomin and Greene [8], Lam [18] or Stanley [29].

**Theorem 2.2** (Edelman-Greene [9]). The coefficient $c^w_\lambda$ equals the number of increasing tableaux of shape $\lambda$ whose row reading words are reduced words of $w^{-1}$.

Theorem 2.2 has an equivalent statement in terms of column reading words. The column reading word of $T$, denoted column$(T)$, is obtained by reading the entries of $T$ along the columns from top to bottom, right to left.

The following theorem follows from [5, Theorem 1] by restricting a Hecke word to a reduced word. Here we give a self-contained proof based on properties of the Edelman-Greene algorithm.

**Theorem 2.3** (Buch-Kresch-Shimozono-Tamvakis-Yong [5]). The coefficient $c^w_\lambda$ equals the number of increasing tableaux of shape $\lambda$ whose column reading words are reduced words of $w$.

**Proof.** Let $T$ be an increasing tableau such that the row reading word of $T$ is a reduced word of $w^{-1}$. We claim that column$(T)$ is a reduced word of $w$. Fix a standard Young tableau $Q$ which has the same shape as $T$. Let $a = (a_1, a_2, \ldots, a_\ell)$ be the reduced word of $w^{-1}$ corresponding to the pair $(T, Q)$. Denote $T^t$ by the transpose of $T$, and write

$$a^{rev} = (a_\ell, a_{\ell-1}, \ldots, a_1)$$

for the reverse of $a$. Note that $a^{rev}$ is a reduced word of $w$. By Edelman and Greene [9] (see also Felsner [7]), the insertion tableau of $a^{rev}$ is $T^t$. By Theorem 2.1, the row reading word of $T^t$ is a reduced word of $w$, or equivalently, the column reading word of $T$ is a reduced word of $w$. This verifies the claim.

Conversely, we can show that if $T$ is an increasing tableau whose column reading word is a reduced word of $w$, then its row reading word is a reduced word of $w^{-1}$. This completes the proof.

Throughout this paper, an increasing tableau $T$ is called a reduced word tableau for $w$ if column$(T)$ is a reduced word of $w$. Equivalently, by the proof of Theorem 2.3, an increasing tableau $T$ is a reduced word tableau for $w$ if its row reading word is a reduced word of $w^{-1}$.

### 2.3 Lascoux-Schützenberger tree

The LS-tree is built based on maximal transitions on Stanley symmetric functions. For a permutation $w = w_1 w_2 \cdots w_n$, let

$$r = \max\{i \mid w_i > w_{i+1}\},$$

Throughout this paper, an increasing tableau $T$ is called a reduced word tableau for $w$ if column$(T)$ is a reduced word of $w$. Equivalently, by the proof of Theorem 2.3, an increasing tableau $T$ is a reduced word tableau for $w$ if its row reading word is a reduced word of $w^{-1}$.
\[ s = \max\{i > r \mid w_i < w_r\}, \]

\[ I(w) = \{i < r \mid w_i < w_s \text{ and } \forall j \in (i, r), w_j \not\in (w_i, w_s)\}, \]

where for two integers \( a \) and \( b \) with \( a < b \), we use \((a, b)\) to denote the interval \( \{a+1, a+2, \ldots, b-1\} \). In other words, \( r \) is the last decent of \( w \), \( s \) is the unique position after \( r \) such that \( \ell(w_{t,s}) = \ell(w) - 1 \), where \( w_{t,s} \) is the permutation obtained by swapping \( w_r \) and \( w_s \). And \( I(w) \) is the set of positions \( i \) before \( r \) such that \( \ell(w_{t,s,t_i,r}) = \ell(w) \). With the above notation, set

\[
\Phi(w) = \begin{cases} 
\{w_{t_s,t_i,r} \mid i \in I(w)\}, & \text{if } I(w) \neq \emptyset; \\
\Phi(1 \times w), & \text{if } I(w) = \emptyset,
\end{cases}
\]

where \( 1 \times w = 1(w_1 + 1) \cdots (w_n + 1) \) as defined in Section 2.1.

Each permutation in the set \( \Phi(w) \) is called a child of \( w \). The LS-tree of \( w \) is obtained by recursively applying the above operation until every leaf is a Grassmanian permutation. A Grassmanian permutation is a permutation with at most one descent. In Figure 2.2, we illustrate the LS-tree of \( w = 231654 \), where, as used in [23], the entries \( w_r, w_s \) and \( w_i \) with \( i \in I(w) \) are boxed, barred and underlined, respectively.

![Figure 2.2: The LS-tree for \( w = 231654 \).](image)

According to the transition formula for Schubert polynomials due to Lascoux and Schützenberger [21], one has the following transition relation for Stanley symmetric functions:

\[
F_w(x) = \sum_{w' \in \Phi(w)} F_{w'}(x),
\]

see also [23, Theorem 1] or Garsia [12]. Since each leaf of a LS-tree is labeled with a Grassmannian permutation, \( F_w(x) \) can be eventually written as a sum of Stanley symmetric functions indexed by Grassmannian permutations. For a Grassmannian permutation, the Stanley symmetric function becomes a single Schur function. More generally,
as proved by Stanley [28], the Stanley symmetric function for a vexillary permutation is a single Schur function. Since Grassmannian permutations are vexillary, the LS-tree implies the Schur positivity of Stanley symmetric functions.

Recall that a permutation is called a vexillary (or 2143-avoiding) permutation if it does not contain a subsequence order-isomorphic to 2143, that is, there are no indices \( i_1 < i_2 < i_3 < i_4 \) such that \( w_{i_2} < w_{i_1} < w_{i_4} < w_{i_3} \). For a permutation \( w \in S_n \), let 

\[
    c(w) = (c_1(w), c_2(w), \ldots, c_n(w))
\]

be the Lehmer code of \( w \), where \( c_i(w) = |\{ j \mid j > i, w_j < w_i \}| \). Define a partition \( \lambda(w) \) by rearranging the Lehmer code of \( w \) in weakly decreasing order. For example, the Lehmer code of \( w = 35412 \) is \( (2, 3, 2, 0, 0) \), and so we have \( \lambda(w) = (3, 2, 2) \). For a vexillary permutation \( w \), Stanley [28] proved that 

\[
    F_w(x) = s_{\lambda(w)}(x),
\]

which also follows from (2.3) together with a tableau formula for Schubert polynomials of vexillary permutations due to Wachs [31] or Knutson, Miller and Yong [16].

In particular, a permutation \( w \) is called a dominant permutation if \( w \) is 132-avoiding. It is clear that a dominant permutation is also a vexillary permutation. The Lehmer code of a dominant permutation \( w \) is a weakly decreasing sequence which form a partition shape \( \lambda(w) \), see Stanley [27, Chapter 1]. Hence, for a dominant permutation \( w \), we have 

\[
    F_w(x) = s_{\lambda(w)}(x),
\]

where \( \lambda(w) = c(w) \) is the partition equal to the Lehmer code of \( w \). Therefore, there is only one reduced word tableau for \( w \). In fact, the only reduced word tableau \( T(w) \) for \( w \) can be obtained as follows: fill the entry in the box \((i, j)\) of \( \lambda(w) \) with \( i + j - 1 \). The tableau \( T(w) \) is also called a frozen tableau, see [22].

There is a more general relation satisfied by Stanley symmetric functions, including (2.5) as a special case. For a permutation \( u = u_1u_2 \cdots u_n \) and \( 1 \leq k \leq n \), let 

\[
    I(u, k) = \{ i < k \mid \ell(ut_{i,k}) = \ell(u) + 1 \},
\]

and

\[
    S(u, k) = \{ j > k \mid \ell(ut_{k,j}) = \ell(u) + 1 \}.
\]

Define two sets of permutations by

\[
    \Phi(u, k) = \begin{cases} 
    \{ ut_{i,k} \mid i \in I(u, k) \}, & \text{if } I(u, k) \neq \emptyset; \\
    \Phi(1 \times u, k + 1), & \text{otherwise},
    \end{cases}
\]

and

\[
    \Psi(u, k) = \begin{cases} 
    \{ ut_{k,j} \mid j \in S(u, k) \}, & \text{if } S(u, k) \neq \emptyset; \\
    \Psi(u \times 1, k), & \text{otherwise},
    \end{cases}
\]

where \( u \times 1 = u_1u_2 \cdots u_n(n + 1) \).

By the Monk’s rule for Schubert polynomials (see for example [1,26]) together with (2.3), one can easily establish the following relation:

\[
    \sum_{w \in \Psi(u, k)} F_w(x) = \sum_{w \in \Phi(u, k)} F_w(x).
\]

Indeed, (2.10) contains (2.5) as a special case, since

\[
    \Phi(wt_{r,s}, r) = \Phi(w) \quad \text{and} \quad \Psi(wt_{r,s}, r) = \{w\}.
\]
2.4 Little map

Little [23] developed a bijection, known as the Little map, to give a combinatorial proof of (2.10). Let $\text{Red}(w)$ denote the set of reduced words of a permutation $w$. The Little map is a descent preserving bijection:

$$\theta_k : \bigcup_{w \in \Psi(u,k)} \text{Red}(w) \longrightarrow \bigcup_{w' \in \Phi(u,k)} \text{Red}(w').$$

(2.11)

Indeed, in view of (2.1), the Little map yields a combinatorial proof of (2.10).

The Little map is defined based on a bumping algorithm acting on the line diagrams for permutations. Assume that $a = (a_1, a_2, \ldots, a_\ell)$ is a word (not necessarily reduced) of a permutation $w$. The line diagram of $a$ is the array $\{1, 2, \ldots, \ell\} \times \{1, 2, \ldots, n\}$ in the Cartesian coordinates, which describes the trajectories of the numbers $1, 2, \ldots, n$ as they are arranged into the permutation $w$ by successive simple transpositions. Note that $a$ is reduced if and only if no two lines cross more than once. For example, Figure 2.3 is the line diagram of the reduced word $a = (5, 4, 1, 2, 5)$ of $w = 231654$.

![Figure 2.3: The line diagram of $a = (5, 4, 1, 2, 5)$.](image)

Given a word $a = (a_1, a_2, \ldots, a_m)$ and an integer $1 \leq t \leq m$, let

$$a^{(t)} = (a_1, a_2, \ldots, a_{t-1}, a_{t+1}, \ldots, a_m)$$

and define

$$a \uparrow_t = \begin{cases} (a_1, a_2, \ldots, a_{t-1}, a_t - 1, a_{t+1}, \ldots, a_m), & \text{if } a_t > 1; \\ (a_1 + 1, a_2 + 1, \ldots, a_{t-1} + 1, a_t, a_{t+1} + 1, \ldots, a_m + 1), & \text{if } a_t = 1, \end{cases}$$

where $a \uparrow_t$ is called the word obtained from $a$ by bumping at time $t$. Denote the word obtained after a sequence of bumps by

$$a \uparrow_{t_1, t_2, \ldots, t_i} = ((a \uparrow_{t_1}) \uparrow_{t_2}) \cdots \uparrow_{t_i}.$$ 

The bumping algorithm, called the Little bump, transforms a reduced word into another reduced word, which can be sketched as follows. For more detailed information,
see Little [23] or Hamaker and Young [25]. Let \( a = (a_1, a_2, \ldots, a_m) \) be a reduced word. Assume that \( t_1 \) is an index such that \( a(t_1) \) is also reduced. Consider the word \( a \uparrow t_1 \). If \( a \uparrow t_1 \) is a reduced word, then the algorithm terminates. Otherwise, it can be shown that there is a unique index, say \( t_2 \), such that \( (a \uparrow t_1)\{t_2\} \) is reduced. Now consider the word \( a \uparrow t_1, t_2 \). Repeating the above procedure, one is eventually left with a reduced word \( a \uparrow t_1, t_2, \ldots, t_i \). The above process is referred to as the Little bump.

Using the above bumping algorithm, we can define the Little map \( \theta_k \). Given a permutation \( w \in \Psi(u, k) \), let \( a \) be a reduced word of \( w \). By the definition of the set \( \Psi(u, k) \), there is an index \( j > k \) in \( S(u, k) \) such that \( w = ut_{k,j} \) and \( \ell(w) = \ell(u) + 1 \). Let us first define a map \( \theta_{k,w_j} \) as follows. Since \( k < j \), \( w_k > w_j \) and \( a \) is a reduced word, there is exactly one letter, say \( a_{t_1} \), that interchanges \( w_k \) and \( w_j \). Because \( \ell(w) = \ell(u) + 1 \), the word \( a(t_1) \) is reduced. Thus we can invoke the Little bump beginning at the position \( t_1 \). Let \( a' \) be the reduced word after applying the Little bump. Define

\[
\theta_k(a) = \theta_{k,w_j}(a) = a'.
\]

Little [23] showed that

\[
a' \in \text{Red}(w') \text{ for some } w' \in \Phi(u, k).
\]

We remark that although the subscript \( w_j \) in \( \theta_{k,w_j} \) is determined once a reduce word of \( w \in \Psi(u, k) \) is given, we would like to use the two parameters \( k \) and \( w_j \) since this would be more convenient to describe the inverse of the Little map.

For example, let \( u = 241536 \) and \( k = 5 \). Then \( w = ut_{5,6} = 241563 \) is the unique permutation in \( \Psi(u, k) \). Let \( a = (3, 1, 4, 5, 2) \) be a reduced word of \( w \). Then \( w_6 = 6, w_7 = 3 \) and \( t_1 = 4 \). Applying the Little map \( \theta_5 \) to \( a \), the resulting word is \((2, 1, 3, 4, 2)\), which is a reduced word of the permutation \( w' = 341526 \) in \( \Phi(u, k) \). The bumping process is illustrated in Figure 2.4, where the dotted crossings indicate the bumped positions.

![Figure 2.4](image_url)

**Figure 2.4:** Applying the Little map \( \theta_5 \) to \( a = (3, 1, 4, 5, 2) \).

**Theorem 2.4** (Little [23]). The map \( \theta_k \) is a descent preserving bijection.

As described by Little [23], the inverse of \( \theta_k \) can be stated as follows. Assume that \( w' \) is a permutation in \( \Phi(u, k) \) and that \( i < k \) is the position in \( I(u, k) \) such that \( w' = wt_{i,k} \). Let \( a' = (a'_1, a'_2, \ldots, a'_\ell) \) be a reduced word of \( w' \) and denote \( (a')^e = (n - a'_1, \ldots, n - a'_\ell) \). Note that \( (a')^e \) is a reduced word of the permutation \( v = v_1v_2 \cdots v_n \) where \( v_i = n + 1 - w'_{n+1-i} \). Then one has

\[
\theta_k^{-1}(a') = (\theta_{n+1-k,n+1-w'_i}((a')^e))^c.
\]
We conclude with two properties of the Little bump due to Hamaker and Young [25], which will be used in Section 6. First, the Little bump preserves the recording tableaux of reduced words when applying the Edelman-Greene algorithm, which was conjectured by Lam [18] and proved by Hamaker and Young [25].

**Theorem 2.5** (Hamaker-Young [25, Theorem 2]). Let \( a \) and \( a' \) be two reduced words such that there exists a sequence of Little bumps changing \( a \) to \( a' \). Then \( Q(a) = Q(a') \).

The second property was essentially implied in the proof of Lemma 6 in [25].

**Theorem 2.6** (Hamaker-Young [25, Lemma 6]). Let \( T \) be a reduced word tableau and \( a = \text{column}(T) \). Assume that \( b \) is obtained from \( a \) by applying a Little bump, and let \( T' \) be the insertion tableau of the reverse \( b^{rev} \) of \( b \) under the Edelman-Greene algorithm. Then \( \text{column}(T') = b \).

### 3 Bumpless pipedreams

In this section, we give an overview of the structure of bumpless pipedreams as well as the droop operation on bumpless pipedreams. The droop operation has a close connection to the modified LS-tree, as will be seen in Section 4.

Lam, Lee and Shimozono [19] introduced several versions of bumpless pipedreams for a permutation \( w \in S_n \). For the purpose of this paper, we shall only be concerned with the bumpless pipedreams in the region of a square grid, which are called \( w \)-square bumpless pipedreams in [19]. Given an \( n \times n \) square grid, we use the matrix coordinates for unit squares, that is, the row coordinates increase from top to bottom and the column coordinates increase from left to right. Let us use \((i, j)\) to denote the box in row \( i \) and column \( j \). A bumpless pipedream for \( w \) consists of \( n \) pipes labeled \( 1, 2, \ldots, n \), flowing from the south boundary of the \( n \times n \) square grid to the east boundary, such that

1. the pipe labeled \( i \) enters from the south boundary in column \( i \) and exits from the east boundary in row \( w^{-1}(i) \);
2. pipes can only go north or east;
3. no two pipes overlap any step or cross more than once;
4. no two pipes change their directions in a box simultaneously. In other words, each box looks like one of the first six tiles as shown in Figure 3.5: an empty box, an NW elbow, an SE elbow, a horizontal line, a vertical line, and a crossing.

![Figure 3.5: Boxes in a bumpless pipedream.](image-url)
For ease of drawing pictures, an NW elbow and an SE elbow are replaced by the tiles given in Figure 3.6. Since two pipes cannot bump at the same box, there is no ambiguity for this simplification. We illustrate four bumpless pipedreams of $w = 2761453$ in Figure 3.7.

![Figure 3.6: NW elbow and SE elbow.](image)

![Figure 3.7: Four Bumpless pipedreams of $w = 2761453$.](image)

Lam, Lee and Shimozono [19] showed that the bumpless pipedreams of $w$ can generate the Schubert polynomial $\mathcal{S}_w(x)$, or more generally the double Schubert polynomial $\mathcal{S}_w(x; y)$. For a bumpless pipedream $P$, define the weight $\text{wt}(P)$ of $P$ to be the product of $x_i - y_j$ over all empty boxes of $P$ in row $i$ and column $j$.

**Theorem 3.1** (Lam-Lee-Shimozono [19, Theorem 5.13]). For any permutation $w$,

$$
\mathcal{S}_w(x; y) = \sum_P \text{wt}(P),
$$

(3.1)

where the sum is over the bumpless pipedreams of $w$.

Lam, Lee and Shimozono [19] also discovered that a specific family of bumpless pipedreams, called EG-pipedreams, can be used to interpret the Edelman-Greene coefficients. A bumpless pipedream $P$ is called an EG-pipedream if all the empty boxes of $P$ are at the northwest corner, where they form a Young diagram $\lambda = \lambda(P)$, called the shape of $P$. For example, Figure 3.7(d) is an EG-pipedream with shape $\lambda = (5, 4, 2, 1)$.

**Theorem 3.2** (Lam-Lee-Shimozono [19, Theorem 5.14]). The Edelman-Greene coefficient $c^w_\lambda$ is equal to the number of EG-pipedreams of $w$ with shape $\lambda$.

For example, there are three EG-pipedreams for $w = 321654$, as illustrated in Figure 3.8. By Theorem 3.2 we see that $F_w(x) = s_{(4,2)}(x) + 2 s_{(3,2,1)}(x)$.

In the remaining of this section, we recall the droop operation on bumpless pipedreams introduced by Lam, Lee and Shimozono [19]. As will be seen in Corollary 4.7, the droop operation is closely related to the modified LS-tree.
Let $P$ be a bumpless pipedream of $w$. The pipes in $P$ are determined by the locations of the NW elbows and the SE elbows. Generally speaking, a droop is a local move which swaps an SE elbow $e$ with an empty box $t$, when the SE elbow $e$ lies strictly to the northwest of the empty box $t$. To be more specific, let $R$ be the rectangle with northwest corner $e$ and southeast corner $t$, and let $L$ be the pipe passing through the SE elbow $e$. A droop is allowed only if

1. the pipe $L$ passes through the westmost column and northmost row of $R$;
2. the rectangle $R$ contains only one elbow: the SE elbow which is at $e$;
3. after the droop we obtain another bumpless pipedream.

After a droop, the pipe $L$ travels along the southmost row and eastmost column of $R$, and an NW elbow occupies the box that used to be empty while the box that contained an SE elbow becomes an empty box. Pipes in $P$ except the pipe $L$ do not change after the droop. Figure 3.9 is an illustration of the local move of the droop operation. For more examples, Figure 3.7(b) is obtained from Figure 3.7(a) by a droop, and Figure 3.7(c) is obtained from Figure 3.7(b) by a droop.

Lam, Lee and Shimozono [19] proved that each bumpless pipedream of $w$ can be generated from the Rothe pipedream of $w$ by applying a sequence of droops. The Rothe pipedream of $w$, denoted $D(w)$, is the unique bumpless pipedream of $w$ that does not contain any NW elbows, see Figure 3.7(a) for an example. Clearly, for $1 \leq i \leq n$, pipe $i$ passes through exactly the SE elbow $(i, w_i)$ in row $i$ and column $w_i$.

The Rothe pipedream $D(w)$ can also be constructed as follows. From the center of each box $(i, w_i)$, draw a horizontal line to the right and a vertical line to the bottom. This forms $n$ hooks with turning points at the center of the boxes $(i, w_i)$. Thinking of each
hook as a pipe, the pipes together with the \( n \times n \) grid form the Rothe pipedream \( D(w) \). The empty boxes of \( D(w) \) are known as the Rothe diagram of \( w \), denoted \( \text{Rothe}(w) \), which encode the positions of inversions of \( w \). That is, there is an empty box of \( \text{Rothe}(w) \) at \((i, j)\) if and only if \( w_i > j \) and the number \( j \) appears in \( w \) after the position \( i \).

**Proposition 3.3** (Lam-Lee-Shimozono [19, Proposition 5.3]). For a permutation \( w \), every bumpless pipedream of \( w \) can be obtained from the Rothe pipedream \( D(w) \) by a sequence of droops.

### 4 Modified Lascoux-Schützenberger tree

In this section, we introduce the structure of the modified LS-tree of a permutation. We also explain the relation between the modified LS-trees and the droop operations.

For a permutation \( w = w_1w_2 \cdots w_n \), let \( p \) be the largest index in \( w \) such that there are indices \( i \) and \( j \) satisfying

\[
i < p < j \quad \text{and} \quad w_i < w_j < w_p,
\]

namely,

\[
p = \max\{t \mid 1 \leq t \leq n-1, \exists \ i < t < j, \ \text{s.t.} \ w_i < w_j < w_t\}.
\]

In other words, \( p \) is the largest position such that there is a subsequence \( w_iw_pw_j \) which is order-isomorphic to 132. For example, for the permutation \( w = 2431 \) we have \( p = 2 \), whereas for the permutation \( w = 3421 \) we cannot find a position satisfying the condition in (4.1).

By definition, there does not exist an index \( p \) satisfying (4.1) if and only if \( w \) is 132-avoiding. If a permutation \( w \) contains a 132 pattern, we also call \( w \) a non-dominant permutation.

Assume that \( w \) is a non-dominant permutation. Let \( q \) be the largest index after \( p \) such that \( w_p > w_q \) and there exists an index \( i < p \) such that \( w_i < w_q \), namely,

\[
q = \max\{j \mid j > p, \ w_j < w_p, \ \exists \ i < p, \ \text{s.t.} \ w_i < w_j\}.
\]

For example, for \( w = 645978321 \), we have \( p = 4 \) and \( q = 6 \). We have the following equivalent description of the index \( q \).

**Lemma 4.1.** Let \( w \) be a non-dominant permutation. Then the index \( q \) defined in (4.3) is the index of the largest number following \( w_p \) which is less than \( w_p \).

**Proof.** Suppose otherwise that \( q' \neq q \) is the index of the largest number following \( w_p \) which is less than \( w_p \). By the definition of \( q \) in (4.3), we see that \( p < q' < q \). On the other hand, by the definition of \( p \), there is an index \( i < p \) such that \( w_i < w_q < w_p \). Thus we have \( w_i < w_q < w_{q'} \), implying that \( q' \) satisfies (4.1). This contradicts the choice of the index \( p \). So the proof is complete.

The following proposition implies that the indices \( p \) and \( q \) play similar roles to the indices \( r \) and \( s \) as defined in Section 2.3.
Proposition 4.2. Let \( w \) be a non-dominant permutation. Then we have
\[
\Psi(w_{t_{p,q}}, p) = \{ w \}. \tag{4.4}
\]

Proof. Write \( u = w_{t_{p,q}} = u_1 u_2 \cdots u_n \), where \( u_p = w_q \), \( u_q = w_p \), and \( u_i = w_i \) for \( i \neq p, q \).
By Lemma 4.1, we see that \( \ell(w_{t_{p,q}}) = \ell(w) - 1 \), which implies \( \ell(u_{t_{p,q}}) = \ell(w) = \ell(u) + 1 \).
So we have \( w \in \Psi(u, p) \).

Suppose that there is another permutation \( w' \in \Psi(u, p) \) which is not equal to \( w \). By the definition of \( \Psi(u, p) \) in (2.9), there exists an index \( q' \neq q \) such that \( q' > p \), \( w' = u_{t_{p,q'}} \) and \( \ell(w') = \ell(u) + 1 \). Then we have \( u_{q'} > u_p \), that is,
\[
w_{q'} > w_q = u_p. \tag{4.5}
\]

There are two cases.

Case 1: \( p < q' < q \). By the definition of \( q' \), there is an index \( i < p \) such that \( w_i < w_q < w_p \). In view of (1.5), we see that \( w_i < w_q < w_{q'} \), which implies that the index \( q' \) satisfies the condition in (4.1). This contradicts the choice of the index \( p \).

Case 2: \( q' > q \). In this case, since \( \ell(w') = \ell(u) + 1 \), we must have \( w_{q'} < w_p \), which, together with (4.5), implies that \( w_q < w_{q'} < w_p \). This is contrary to Lemma 4.1. So the proof is complete.

Combining (2.10) and Proposition 4.2, we arrive at the following transition relation satisfied by Stanley symmetric functions.

Theorem 4.3. Let \( w \in \mathcal{S}_n \) be a non-dominant permutation. Then we have
\[
F_w(x) = \sum_{w' \in \Phi(w_{t_{p,q}}, p)} F_{w'}(x). \tag{4.6}
\]
Moreover, by the choices of the indices \( p \) and \( q \), the set \( I(w_{t_{p,q}}, p) \) is not empty, and hence each permutation in \( \Phi(w_{t_{p,q}}, p) \) is still a permutation on \( \{1, 2, \ldots, n\} \).

For example, for \( w = 645978321 \), we see that \( p = 4 \), \( q = 6 \) and so we have \( w_{t_{p,q}} = 645879321 \). Moreover, one can check that \( I(w_{t_{p,q}}, p) = \{1, 3\} \). Therefore,
\[
\Phi(w_{t_{p,q}}, p) = \{845679321, 648579321\}.
\]

The construction of the modified LS-tree of \( w \) relies on Theorem 4.3. To be specific, for a permutation \( w \in \mathcal{S}_n \), iterate the relation in (4.6) until each leaf is a dominant permutation on \( \{1, 2, \ldots, n\} \). The resulting tree is called the modified LS-tree of \( w \). Figure 4.10 illustrates the modified LS-tree of \( w = 231654 \).

Remark 4.4. The construction of a modified LS-tree is feasible because the process eventually terminates. This can be seen as follows. For two sequences \( c = (c_1, c_2, \ldots, c_n) \) and \( c' = (c'_1, c'_2, \ldots, c'_n) \) of nonnegative integers of length \( n \), write \( c \geq c' \) if for \( 1 \leq i \leq n \),
\[
c_1 + \cdots + c_i \geq c'_1 + \cdots + c'_i.
\]
This defines a partial order on the sequences of nonnegative integers of length $n$. Assume that $w \in S_n$ is a non-dominant permutation. It is easy to check that for $w' \in \Phi(wt_{p,q}, p)$, the Lehmer code of $w$ is smaller than the Lehmer code of $w'$ under the above order. Hence, the nodes in any path from the root to a leaf in the modified LS-tree are labeled by distinct permutations on $\{1, 2, \ldots, n\}$. So the process terminates.

The differences between a modified LS-tree and an ordinary LS-tree are obvious. First, each node in a modified LS-tree is labeled with a permutation on $\{1, 2, \ldots, n\}$, whereas in an ordinary LS-tree, there may exist nodes which are labeled with permutations on $\{1, 2, \ldots, m\}$ with $m > n$. Second, each leaf in a modified LS-tree is labeled with a dominant permutation, whereas each leaf in an ordinary LS-tree is labeled with a Grassmannian permutation. To make a comparison, see Figure 2.2 and Figure 4.10.

Since each leaf in the modified LS-tree of $w$ is a dominant permutation, according to Theorem 4.3, we obtain the following interpretation for the Edelman-Greene coefficients.

**Theorem 4.5.** The Edelman-Greene coefficient $c^\lambda_w$ equals the number of leaves in the modified LS-tree of $w$ whose labels are the dominant permutations with Lehmer code $\lambda$.

In the modified LS-tree of $w$, if replacing the label of each node with its corresponding Rothe pipedream, then we obtain a tree labeled with Rothe pipedreams. Figure 4.11 is the Rothe pipedream version of Figure 4.10. By using Rothe pipedreams to label the nodes, we can directly read off the Schur functions in the expansion of a Stanley symmetric function. Moreover, it will be more convenient to use the Rothe pipedream version of a modified LS-tree in the construction of an EG-tree in Section 5.

The remaining of this section are devoted to several propositions that will be used in Section 5 to construct the EG-tree of $w$ from the modified LS-tree of $w$. 

Figure 4.10: The modified LS-tree of $w = 231654$. 

![Figure 4.10: The modified LS-tree of $w = 231654$.](image-url)
Propositions 4.6 and 4.7 tell us how to use the droop operation to generate the children of a node in a modified LS-tree. To describe these two propositions, we define the notion of a pivot of an empty box in the Rothe pipedream of a permutation $w$. Note that for the specific empty box $(r, w_s)$, where $r$ and $s$ are defined in Section 2.3, the notion of pivots has been defined by Knutson and Yong [17]. Let $(i, j)$ be an empty box in the Rothe pipedream $D(w)$ of $w$. We say that an SE elbow $e$ of $D(w)$ is a pivot of $(i, j)$ if $e$ is northwest of $(i, j)$ and there are no other elbows contained in the rectangle with northwest corner $e$ and southeast corner $(i, j)$. In other words, among the SE elbows that are northwest of $(i, j)$, the pivots are maximally southeast. For example, the Rothe pipedream of $w = 2761453$ is displayed in Figure 3.7(a). The empty box $(3, 1)$ has no pivots, whereas the empty box $(6, 3)$ has two pivots: $(1, 2)$ and $(4, 1)$.

For a non-dominant permutation $w$, it is easily seen that $(p, w_q)$ is an empty box of $D(w)$. The following proposition gives a characterization of the pivots of this specific empty box.

**Proposition 4.6.** Let $w$ be a non-dominant permutation. Then the set of pivots of the empty box $(p, w_q)$ is

$$\{(i, w_i) \mid i \in I(wt_{p,q}, p)\},$$

(4.7)

where $I(wt_{p,q}, p)$ is the index set as defined in (2.6). 

**Proof.** The assertion follows from the definition of pivots as well as the definition of the index set $I(wt_{p,q}, p)$.

Clearly, we can apply a droop to the Rothe pipedream of $w$ with respect to the empty box $(p, w_q)$ and a pivot of $(p, w_q)$. 

![Diagram](image-url)
Proposition 4.7. Assume that \( w \) is a non-dominant permutation. Let \( P \) be the bumpless pipedream of \( w \) obtained from the Rothe pipedream \( D(w) \) by applying a droop swapping the empty box \((p, w_q)\) and a pivot \((i, w_i)\), where \( i \in I(w_{p,q}, p) \). Then the Rothe diagram of \( w' = wt_{p,q}t_{i,p} \in \Phi(w_{p,q}, p) \) equals the set of empty boxes of \( P \).

Proof. Let us first explain that the Rothe diagram \( \text{Rothe}(w') \) of \( w' \) can be constructed from the Rothe diagram \( \text{Rothe}(w) \) of \( w \) as follows. First, note that \( \text{Rothe}(wt_{p,q}) \) can be obtained from \( \text{Rothe}(w) \) by deleting the box \((p, w_q)\), see Figure 4.12 for an illustration. Let us proceed to determine \( \text{Rothe}(w') \) from \( \text{Rothe}(wt_{p,q}) \). Locate the rectangle \( R \) in the \( n \times n \) square grid, such that the northwest corner is \((i, w_i)\) and the southeast corner is \((p, w_q)\). Then \( \text{Rothe}(w') \) can be obtained from \( \text{Rothe}(wt_{p,q}) \) as follows:

1. Move each box of \( \text{Rothe}(wt_{p,q}) \) in the bottom row of \( R \) to the top row of \( R \);
2. Move each box of \( \text{Rothe}(wt_{p,q}) \) in the rightmost column of \( R \) to the leftmost column of \( R \);
3. Add a new box to the position \((i, w_i)\).

Figure 4.13 illustrates the above construction of \( \text{Rothe}(w') \) from \( \text{Rothe}(wt_{p,q}) \).

One the other hand, one can apply a droop on \( D(w) \) swapping \((i, w_i)\) and \((p, w_q)\) to obtain a bumpless pipedream \( P \) of \( w \), see Figure 4.14 for an illustration.

Evidently, the droop has the same effect on the empty boxes of \( D(w) \) (that is, the Rothe diagram of \( w \)) as the operation illustrated in Figures 4.12 and 4.13. This competes the proof.
Remark 4.8. When $p$ is the last descent $r$ of $w$, Knutson and Yong [17] introduced the marching operation to generate the Rothe diagram of a child of $w$ from the Rothe diagram $\text{Rothe}(w)$ of $w$. In this specific case, the marching operation has the same effect on $\text{Rothe}(w)$ as the droop operation.

Propositions 4.6 and 4.7 give an explicit way to generate the children of a non-dominant permutation $w$ in the modified LS-tree of $w$. Assume that $e_1,\ldots,e_m$ are the pivots of $(p,w_q)$. For $1 \leq i \leq m$, let $P_i$ be the bumpless pipedream of $w$ obtained from $D(w)$ by applying a droop with respect to $(p,w_q)$ and $e_i$. Denote by $\text{Box}(P_i)$ the empty boxes of $P_i$. Then $\text{Box}(P_1),\ldots,\text{Box}(P_m)$ are the Rothe diagrams of the children of $w$. Equivalently, $\text{Box}(P_1),\ldots,\text{Box}(P_m)$ are the empty boxes of Rothe pipedreams of the children of $w$.

The next two propositions investigate the properties concerning the specific empty box $(p,w_q)$. Define a total order on the boxes of the $n \times n$ grid by letting $(i,j) < (k,\ell)$ if either $i < k$, or $i = k$ and $j < \ell$. (4.8)

Let $\text{pivot}(w)$ denote the set of empty boxes in the Rothe pipedream of $w$ which has at least one pivot.

**Proposition 4.9.** Let $w$ be a non-dominant permutation. Then $(p,w_q)$ is the largest box in the set $\text{pivot}(w)$ under the total order defined in (4.8).

**Proof.** Suppose otherwise that $(p',j) \neq (p,w_q)$ is the largest box in $\text{pivot}(w)$. Then there is an index $q' > p'$ such that $w_{q'} = j$. Since $(p,w_q)$ is the rightmost empty box in the $p$-th row of $D(w)$, we must have $p' > p$. Let $(i',w_{i'})$ be a pivot of $(p',w_{q'})$, where $i' < p'$. By the definition of a pivot, we have $w_{i'} < j = w_{q'}$. Moreover, notice that $w_{q'} = j < w_{q'}$. So, for the indices $i' < p' < q'$, we find that $w_{i'} < w_{q'} < w_{q'}$. This implies that $p'$ satisfies the condition (4.1), contrary to the choice of $p$. This concludes the proof.

Using Proposition 4.9 we can prove the following assertion.

**Proposition 4.10.** Let $w$ be a non-dominant permutation, and let $w' = \text{wt}_{p,q}t_{i,p} \in \Phi(\text{wt}_{p,q},p)$ be a child of $w$. Suppose that $w'$ is also a non-dominant permutation. Write $p'$ and $q'$ for the indices as defined in (4.2) and (4.3) for $w'$, respectively. Then

$$(p',w_{q'}) < (p,w_q) \quad (4.9)$$

under the total order defined in (4.8).
Proof. The proof is best understood by means of the pictures of the Rothe pipedreams of \( w \) and \( w' \), as illustrated in Figure 4.15. The dashed rectangle in Figure 4.15 signifies the rectangle where the droop operation on \( D(w) \), as described in Proposition 4.7, took place. The shaded area of \( D(w') \) in Figure 4.15 contains the boxes of \( D(w') \) smaller than the SE elbow \((p, w_i)\) under the total order (4.8). Since \( w'_p = w_i \), there are no empty boxes in the \( p \)-th row of \( D(w') \) after the box \((p, w_i)\). So (4.9) is equivalent to saying that the empty box \((p', w'_q)\) of \( D(w') \) lies in the shaded area.

Suppose otherwise that \((p', w'_q) > (p, w_q)\). Then \((p', w'_q)\) lies strictly below row \( p \). Note that \((p', w'_q)\) is also an empty box of \( D(w) \). There are two cases to discuss. 

Case 1: \( w_i < w'_q \). In this case, in the Rothe pipedream of \( w \), \((i, w_i)\) is an SE elbow that is northwest of \((p', w'_q)\). Hence \((p', w'_q)\) has a pivot in the Rothe pipedream of \( w \), which is contrary to Proposition 4.9.

Case 2: \( w_i > w'_q \). Let \( e \) be an SE elbow in the Rothe pipedream of \( w' \) which is a pivot of \((p', w'_q)\). Still, from Figure 4.15, we see that \( e \) is also an SE elbow in the Rothe pipedream of \( w \). This implies that \((p', w'_q)\) has a pivot in the Rothe pipedream of \( w \), which again contradicts Proposition 4.9.

5 Edelman-Greene tree

In this section, we introduce the structure of the Edelman-Greene tree (EG-tree) of a permutation \( w \). Each node in the EG-tree of \( w \) is labeled with a bumpless pipedream of \( w \). In particular, we show that the leaves in the EG-tree of \( w \) are exactly labeled with the EG-pipedreams of \( w \).

Let us proceed with the construction of the EG-tree of \( w \). The idea behind is that for each path in the modified LS-tree of \( w \), we construct a corresponding path of bumpless pipedreams of \( w \). Assume that

\[
w = w^{(0)} \rightarrow w^{(1)} \rightarrow \cdots \rightarrow w^{(m)}
\]

is a path in the modified LS-tree of \( w \) from the root \( w^{(0)} = w \) to a leaf \( w^{(m)} \). We aim to construct a sequence

\[
P_0 \rightarrow P_1 \rightarrow \cdots \rightarrow P_m
\]

(5.1)
of bumpless pipedreams of \( w \) such that
Let us first consider the case

**Proof.** Let \( P_0 \) be the Rothe pipedream of \( w \) and \( P_m \) be an EG-pipedream of \( w \);

(2) For \( 0 \leq j \leq m \), the empty boxes of \( P_j \) are the same as the empty boxes of the Rothe pipedream \( D(w^{(j+1)}) \) of \( w^{(j)} \).

We now describe the construction of the pipedreams in (5.1). For \( 0 \leq j \leq m - 1 \), let \( p_j \) and \( q_j \) be the indices of the permutation \( w^{(j)} \) as defined in (4.2) and (4.3). By the construction of the modified LS-tree of \( w \), there exists an index \( i_j \in I(w^{(j)})t_{p_j,q_j} \) such that \( w^{(j+1)} = w^{(j)}t_{p_j,q_j} \). By Proposition 4.7 we can generate the empty boxes of \( D(w^{(j+1)}) \) by applying a droop to \( D(w^{(j)}) \) with respect to the empty box \((p_j, w^{(j)}_{q_j}) \) and the pivot \((i_j, w^{(j)}_{i_j}) \). As will be seen in the proof of Proposition 5.1 we can also apply a droop to \( P_j \) with respect to \((p_j, w^{(j)}_{q_j}) \) and \((i_j, w^{(j)}_{i_j}) \). Let \( P_{j+1} \) be the bumpless pipedream of \( w \) obtained by applying a droop to \( P_j \) with respect to \((p_j, w^{(j)}_{q_j}) \) and \((i_j, w^{(j)}_{i_j}) \).

**Proposition 5.1.** The above construction from \( P_j \) to \( P_{j+1} \) is feasible.

**Proof.** Let us first consider the case \( j = 0 \). Since \( P_0 = D(w) \), by Proposition 4.7 we can apply a droop to \( P_0 \) by swapping the empty box \((p_0, w^{(0)}_{q_0}) \) and its pivot \((i_0, w^{(0)}_{i_0}) \), resulting in the bumpless pipedream \( P_1 \).

We proceed to consider the case \( j = 1 \). By Proposition 4.7 the empty boxes of \( D(w^{(2)}) \) can be obtained from \( D(w^{(1)}) \) by applying a droop with respect to the empty box \((p_1, w^{(1)}_{q_1}) \) and its pivot \((i_1, w^{(1)}_{i_1}) \). By Proposition 4.10 there holds that \((p_1, w^{(1)}_{i_1}) < (p_0, w^{(0)}_{q_0}) \). By the construction of a droop operation, it is easy to check that \( P_1 \) and \( D(w^{(1)}) \) have the same tile at each position smaller than \((p_0, w^{(0)}_{q_0}) \). Thus we can apply a droop to \( P_1 \) with respect to \((p_1, w^{(1)}_{q_1}) \) and \((i_1, w^{(1)}_{i_1}) \), yielding the bumpless pipedream \( P_2 \) of \( w \).

For the same reason as above, in the general case for \( j \geq 2 \), we can check that \( P_j \) and \( D(w^{(j)}) \) have the same tile at each position smaller than \((p_{j-1}, w^{(j-1)}_{q_{j-1}}) \), and so that we can apply a droop to \( P_j \) with respect to \((p_j, w^{(j)}_{q_j}) \) and \((i_j, w^{(j)}_{i_j}) \). This generates the bumpless pipedream \( P_{j+1} \) of \( w \).

By the proof of Proposition 5.1 the empty boxes of \( P_j \) are the same as the empty boxes of the Rothe pipedream of \( w^{(j)} \). Notice that the Rothe diagram of a dominant permutation is a partition shape at the northwest corner of the \( n \times n \) grid, see Fulton [13] or Stanley [27, Chapter 1]. Thus the bumpless pipedream \( P_m \) is an EG-pipedream of \( w \).

We apply the above procedure to each path from the root to a leaf in the modified LS-tree of \( w \). The resulting tree is called the Edelman-Greene tree (EG-tree) of \( w \). Figure 5.10 is the EG-tree of \( w = 231654 \).

By the construction of an EG-tree of \( w \), we summarize its relation to a modified LS-tree of \( w \) in the following theorem.

**Theorem 5.2.** For permutation \( w \), the modified LS-tree of \( w \) and the EG-tree of \( w \) are isomorphic. For any given node labeled with a permutation \( u \) in the modified LS-tree of \( w \), let \( P \) be the corresponding bumpless pipedream of \( w \) in the EG-tree of \( w \). Then the empty boxes of the Rothe pipedream of \( u \) are the same as the empty boxes of \( P \).
We have obtained a map from the leaves of an EG-tree of $w$ to the EG-pipedreams of $w$. In the following theorem, we shall give the reverse procedure.

**Theorem 5.3.** Let $w \in S_n$ be a permutation and $P$ be an EG-pipedream of $w$. Then there is a leaf in the EG-tree of $w$ whose label is $P$.

**Proof.** Assume that $P$ has $m$ NW elbows, say, 
\[(i_m, j_m) < (i_{m-1}, j_{m-1}) < \cdots < (i_1, j_1),\]
which are listed in the total order as defined in (4.8). We construct a sequence
\[P = P_m \rightarrow P_{m-1} \rightarrow \cdots \rightarrow P_0\]  
(5.2)
of bumpless pipedreams of $w$ such that $P_m = P$ and $P_0 = D(w)$.

First, we construct the pipedream $P_{m-1}$ from $P_m = P$. The construction is the same as that in the proof of [19, Proposition 5.3] and is sketched below. Let $L$ be the pipe in $P_m$ passing through the NW elbow $(i_m, j_m)$. Then $L$ passes through an SE elbow $(i_m, y)$ (respectively, $(x, j_m)$) in the same row (respectively, column). Let $R$ be the rectangle with corners $(i_m, y)$ and $(x, j_m)$. It is easy to check that the northwest corner of $R$ is an empty box, and that there are no any other elbows in $R$. Let $P_{m-1}$ be the pipedream obtained from $P_m$ by a “reverse droop”, that is, change the pipe $L$ to travel along the westmost column and northmost row of $R$, see Figure 5.17 for an illustration.

Using the same procedure as above, we can construct $P_{k-1}$ from $P_k$ ($1 \leq k \leq m$) by applying a “reverse droop” corresponding to the NW elbow $(i_k, j_k)$. Figure 5.18 gives an example to illustrate the generation of the chain from an EG-pipedream.
We show that the sequence in (5.2) is a path from a leaf to the root in the EG-tree of $w$. Since $P_{k-1}$ has one fewer NW elbows than $P_k$, the bumpless pipedream $P_0$ has no NW elbows and hence is the Rothe pipedream of $w$. Set $u^{(0)} = w$. By the construction of the sequence (5.2) together with the fact that the empty boxes of $P_m$ form a partition at the northwest corner of the $n \times n$ grid, it is not hard to check that $(i_1, j_1)$ is the largest box in the set $\text{pivot}(w)$ under the total order defined in (4.8). In view of Proposition 4.7, the empty boxes of $P_1$ are also the empty boxes of the Rothe pipedream of $u^{(1)}$ for some $u^{(1)} \in \Phi(w_{p',q}, p)$. Along the same line, we can deduce that for $2 \leq k \leq m$, the empty boxes of $P_k$ form the empty boxes of the Rothe pipedream of some $u^{(k)}$. In particular, for $1 \leq k \leq m$, $u^{(k)}$ is a child of $u^{(k-1)}$ in the modified LS-tree of $w$. Hence, the sequence

$$w = u^{(0)} \rightarrow u^{(1)} \rightarrow \cdots \rightarrow u^{(m)}$$

forms a path in the modified LS-tree of $w$ from the root to a leaf $u^{(m)}$. This shows that (5.2) is a path from a leaf to the root in the EG-tree of $w$. Now we see that $P$ is the label of a leaf in the EG-tree of $w$. This completes the proof.

It is easy to verify that the construction of (5.2) is the reverse process of the construction of (5.1). Hence we arrive at the following conclusion.

**Corollary 5.4.** For a permutation $w$, the labels of leaves in the EG-tree of $w$ are in bijection with the EG-pipedreams of $w$.

By Theorem 4.5 and Corollary 5.4, we obtain an alternative proof of Theorem 3.2.

# 6 The bijection

In this section, we establish the promised bijection between reduced word tableaux and EG-pipedreams. For a permutation $w$, let $\text{RT}(w)$ denote the set of reduced word tableaux
for $w$, namely, the set of increasing tableaux whose column reading words are reduced words of $w$. Let $\text{EG}(w)$ denote the set of EG-pipedreams of $w$.

**Theorem 6.1.** There is a shape preserving bijection between $\text{RT}(w)$ and $\text{EG}(w)$.

By Theorem 5.2 and Corollary 5.4, we need only to establish a bijection between the set $\text{RT}(w)$ and the set of leaves in the modified LS-tree of $w$. However, we shall directly construct a shape preserving bijection between $\text{RT}(w)$ and $\text{EG}(w)$. Of course, such a construction implies a bijection between the set $\text{RT}(w)$ and the set of leaves in the modified LS-tree of $w$.

We first define two maps

$$\Gamma : \text{RT}(w) \rightarrow \text{EG}(w).$$

and

$$\tilde{\Gamma} : \text{EG}(w) \rightarrow \text{RT}(w).$$

Then we show that they are the inverses of each other.

We need to employ the Little map for the transition in Theorem 4.3. In this case, the Little map can be written as $\theta_{p,w}(q)$, since $q$ is the unique element of $S(wt_{p,q},p)$ defined in (2.7). Note that $(p, w(q))$ is the maximum empty box in the Rothe pipedream of $w$ which has a pivot. Moreover, for a reduced word $a$ of a permutation in $\Phi(wt_{p,q},p)$, we have

$$\theta_{p,w(q)}^{-1}(a) = (\theta_{n+1-p,n+1-w(q)}(a^c))^c,$$

which is a reduced word of $w$.

Let $T \in \text{RT}(w)$ be a reduced word tableau for $w$ with shape $\lambda$. We first construct a path of permutations

$$w = w^{(0)} \rightarrow w^{(1)} \rightarrow \cdots \rightarrow w^{(m)}$$

in the modified LS-tree of $w$ from the root $w^{(0)} = w$ to a leaf $w^{(m)}$. Then there is a path in the EG-tree of $w$ which corresponds to the path in (6.1). Let $P$ be the EG-pipedream corresponding to $w^{(m)}$ in the EG-tree of $w$. Define $\Gamma(T) = P$.

To construct (6.1), we need a path of reduced words

$$\tau = \tau^{(0)} \rightarrow \tau^{(1)} \rightarrow \cdots \rightarrow \tau^{(m)}$$

such that $\tau^{(i)}$ is a reduced word of $w^{(i)}$ for $0 \leq i \leq m$. Let $\tau^{(0)} = \text{column}(T)$. By definition, $\text{column}(T)$ is a reduced word of $w$. Note that $T$ is the insertion tableau of $\text{column}(T)^{rev}$ under the Edelman-Greene algorithm. For $0 \leq i \leq m - 1$, let $\tau^{(i+1)}$ be obtained from $\tau^{(i)}$ by applying the Little map $\theta_{p_i,w(q_i)}^{(i)}$, where $(p_i, w^{(i)}(q_i))$ is the maximum empty box in the Rothe pipedream of $w^{(i)}$ which has a pivot.

Conversely, let $P$ be an EG-pipedream of $w$ with $m$ NW elbows, say

$$(i_m, j_m) < \cdots < (i_1, j_1)$$

in the order (1.8). Assume that $w'$ is the permutation in the modified LS-tree of $w$ which corresponds to $P$. Then $w'$ is a dominant permutation with a unique reduced
word tableau, say $T'$, namely, the frozen tableau of $w'$ as mentioned in Section 2.3. Take the column reading word column($T'$) of $w'$. Let $w(P)$ be the reduced word of $w$ defined by

$$w(P) = \theta^{-1}_{i_{1,j_1}} \circ \cdots \circ \theta^{-1}_{i_{m,j_m}} \text{column}(T')).$$  \hspace{1cm} (6.3)

Let $\tilde{T}$ be the insertion tableau of $w(P)^{\text{rev}}$ by the Edelman-Greene algorithm. Define $\tilde{\Gamma}(P) = \tilde{T}$. By Theorem 2.1 the row reading word of $\tilde{T}$ is a reduced word of $w^{-1}$. By the proof of Theorem 2.3, the column reading word of $\tilde{T}$ is a reduced word of $w$, and thus $\tilde{T}$ is a reduced word tableau for $w$.

**Proof of Theorem 6.1.** We show that $\tilde{\Gamma}$ is the inverse of $\Gamma$. Let $P = \Gamma(T)$ where $T \in \text{RT}(w)$. We need to show that $\tilde{T} = \tilde{\Gamma}(P) = T$. For $0 \leq i \leq m$, let $T_i$ be the insertion tableau of $(\tau^{(i)})^{\text{rev}}$ for the reduced words in (6.2). Note that $T_0 = T$. By Theorem 2.5 the tableaux $T_i$ have the same shape. Moreover, by Theorem 2.6 the column reading word of $T_i$ is $\tau^{(i)}$. By the construction of the EG-tree of $w$ and in view of the construction of $\tilde{T}$, it is easy to check that $\tilde{T} = T$. In the same manner, it is also easy to verify that $\Gamma$ is the inverse of $\tilde{\Gamma}$. So the proof is complete. 

For example, let $w = 231654$ and let $T$ be the following reduced word tableau of $w$:

$$T = \begin{array}{ccc}
1 & 4 & 5 \\
2 \\
5
\end{array}$$

We see that column($T$) = (5, 4, 1, 2, 5). Let us construct the path of permutations in (6.1). Since the maximum box in pivot($w$) is (5, 4), we should apply $\theta_{5,4}$ to column($T$) = (5, 4, 1, 2, 5). That is,

$$\theta_{5,4}(5, 4, 1, 2, 5) = (5, 3, 1, 2, 4),$$

which is a reduced word for $w^{(1)} = 241635$. The maximum box in pivot($w^{(1)}$) is (4, 5). Then we should apply $\theta_{4,5}$ to 53124 to yield

$$\theta_{4,5}(5, 3, 1, 2, 4) = (4, 3, 1, 2, 4),$$

which is a reduced word of $w^{(2)} = 251436$. The maximum box in pivot($w^{(2)}$) is (4, 3). Thus we apply $\theta_{4,3}$ to 43124 to obtain

$$\theta_{4,3}(4, 3, 1, 2, 4) = (4, 3, 1, 2, 3),$$

which is a reduced word of $w^{(3)} = 253146$. The maximum box in pivot($w^{(3)}$) is (2, 4). Then we apply $\theta_{2,4}$ to (4, 3, 1, 2, 3) to yield

$$\theta_{2,4}(4, 3, 1, 2, 3) = (3, 2, 1, 2, 3),$$

which is a reduced word of $w^{(4)} = 423156$. Since $w^{(4)}$ is a dominant permutation, the Rothe diagram Rothe($w^{(4)}$) is a partition, and we stop. Therefore, the path of permutations in the modified LS-tree of $w$ is

$$231654 \rightarrow 241635 \rightarrow 251436 \rightarrow 253146 \rightarrow 423156.$$
By the EG-tree of $w$ displayed in Figure 5.16, the EG-pipedream corresponding to the leaf 423156 is

Conversely, let $P$ be the second EG-pipedream in the bottom row of Figure 5.16. The set of NW elbows of $P$ is $\{ (2, 4), (4, 3), (4, 5), (5, 4) \} <$. The leaf in the modified LS-tree of $w$ corresponding to $P$ is $w' = 423156$, which has a unique reduced word tableau

$$ T' = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} $$

Then column($T'$) = $(3, 2, 1, 2, 3)$ and

$$ w(P) = \theta_{5,4}^{-1} \circ \theta_{4,5}^{-1} \circ \theta_{4,3}^{-1} \circ \theta_{2,4}^{-1}(3, 2, 1, 2, 3). $$

We have the following calculations:

$$ \theta_{2,4}^{-1}(3, 2, 1, 2, 3) = (\theta_{5,3}^{-1}(3, 4, 5, 4, 3))^c = (2, 3, 5, 4, 3)^c = (4, 3, 1, 2, 3); $$

$$ \theta_{4,3}^{-1}(4, 3, 1, 2, 3) = (\theta_{3,4}^{-1}(2, 3, 5, 4, 3))^c = (2, 3, 5, 4, 2)^c = (4, 3, 1, 2, 4); $$

$$ \theta_{4,5}^{-1}(4, 3, 1, 2, 4) = (\theta_{5,2}^{-1}(2, 3, 5, 4, 2))^c = (1, 3, 5, 4, 2)^c = (5, 3, 1, 2, 4); $$

$$ \theta_{5,4}^{-1}(5, 3, 1, 2, 4) = (\theta_{2,3}^{-1}(1, 3, 5, 4, 2))^c = (1, 2, 5, 4, 1)^c = (5, 4, 1, 2, 5). $$

Therefore, $w(P) = (5, 4, 1, 2, 5)$. Finally, insert $w(P)^{rev} = (5, 2, 1, 4, 5)$ by the Edelman-Greene algorithm to obtain

$$ T = \begin{array}{c} 1 \\ 2 \\ 4 \\ 5 \end{array} $$

which is a reduced word tableau for $w = 231654$.

**Acknowledgments.** Part of this work was completed during Neil Fan was visiting the Department of Mathematics at the University of Illinois at Urbana-Champaign, he wishes to thank the department for its hospitality and thank Alexander Yong for helpful conversations. This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, the National Science Foundation of China.

**References**

[1] N. Bergeron and S. Billey, RC-graphs and Schubert polynomials, Experiment. Math. 2 (1993), 257–269.
[2] N. Bergeron and F. Sottile, Schubert polynomials, the Bruhat order, and the geometry of flag manifolds, Duke Math. J. 95 (1998), 373–423.

[3] S. Billey, W. Jockusch and R.P. Stanley, Some combinatorial properties of Schubert polynomials, J. Algebraic Combin. 2 (1993), 345–374.

[4] S. Billey and B. Pawlowski, Permutation patterns, Stanley symmetric functions, and generalized Specht modules, J. Combin. Theory Ser. A 127 (2014), 85–120.

[5] A.S. Buch, A. Kresch, M. Shimozono, H. Tamvakis and A. Yong, Stable Grothendieck polynomials and K-theoretic factor sequences, Math. Ann. 340 (2008), 359–382.

[6] W.Y.C. Chen, P.L. Guo and S.X.M. Pang, Vacillating Hecke tableaux and linked partitions, Math. Z. 281 (2015), 661–672.

[7] S. Felsner, The Skeleton of a reduced word and a correspondence of Edelman and Greene, Electron. J. Combin. 8 (2001), #R10.

[8] S. Fomin and C. Greene, Noncommutative Schur functions and their applications, Discrete Math. 193 (1998), 179–200.

[9] P. Edelman and C. Greene, Balanced tableaux, Adv. Math., 63 (1987), 42–99.

[10] S. Fomin and A. N. Kirillov, The Yang-Baxter equation, symmetric functions, and Schubert polynomials, Proceedings of the 5th Conference on Formal Power Series and Algebraic Combinatorics (Florence 1993), Discrete Math. 153 (1996), 123–143.

[11] S. Fomin and R.P. Stanley, Schubert polynomials and the NilCoxeter algebra, Adv. Math. 103 (1994), 196–207.

[12] A.M. Garsia, The Saga of Reduced Factorizations of Elements of the Symmetric Group, Publications du LaCIM, Université du Québec à Montréal, Canada, Vol. 29, 2002.

[13] W. Fulton, Flags, Schubert polynomials, degeneracy loci, and determinantal formulae, Duke Math. J. 65 (1992), 381–420.

[14] A. Knutson and E. Miller, Gröbner geometry of Schubert polynomials, Ann. Math. 161 (2005), 1245–1318.

[15] A. Knutson and E. Miller, Subword complexes in Coxeter groups, Adv. Math. 184 (2004), 161–176.

[16] A. Knutson, E. Miller and A. Yong, Gröbner geometry of vertex decompositions and of flagged tableaux, J. Reine Angew. Math. 63 (2009), 1–31.

[17] A. Knutson and A. Yong, A formula for K-theory truncation Schubert calculus, Int. Math. Res. Not. 70 (2004), 3741–3756.
[18] T. Lam, Stanley symmetric functions and Peterson algebras. In: Lam, T., Lapointe, L., Morse, J., Schilling, A., Shimozono, M., Zabrocki, M. (eds) k-Schur Functions and Affine Schubert Calculus, Fields Institute Monographs, Vol. 33, pp. 133–168. Springer, New York, 2014.

[19] T. Lam, S. Lee and M. Shimozono, Back stable Schubert calculus, arXiv:1806.11233v1.

[20] A. Lascoux and M. Schützenberger, Polynômes de Schubert, C.R. Acad. Sci. Paris 294 (1982), 447–450.

[21] A. Lascoux and M. Schützenberger, Schubert polynomials and the Littlewood-Richardson rule, Lett. Math. Phys. 10 (2-3) (1985), 111–124.

[22] S. Linusson and S. Potka, New properties of the Edelman-Greene bijection, arXiv:1804.10034v1.

[23] D. P. Little, Combinatorial aspects of the Lascoux-Schrödinger tree, Adv. Math. 174 (2003), 236–253.

[24] Z. Hamaker, E. Marberg and B. Pawlowski, Schur P-positivity and involution Stanley symmetric functions, Int. Math. Res. Not. (2017), rnx274.

[25] Z. Hamaker and B. Young, Relating Edelman-Greene insertion to the Little map, J. Algebr. Comb., 40 (2014), 693–710.

[26] I.G. Macdonald, Notes on Schubert Polynomials, Montréal: Dép. de mathématique et d’informatique, Université du Québec à Montréal, 1991.

[27] R.P. Stanley, Enumerative Combinatorics, Vol. 1, Second edition, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2012.

[28] R. Stanley, On the number of reduced decompositions of elements of Coxeter groups, European J. Combin. 5 (1984), 359–372.

[29] R.P. Stanley, Reduced decompositions, http://www-math.mit.edu/~rstan/transparencies/redec-ams.pdf

[30] H. Thomas and A. Yong, A jeu de taquin theory for increasing tableaux, with applications to K-theoretic Schubert calculus, Algebra Number Theory 3 (2009), 121–148.

[31] M. Wachs, Flagged Schur functions, Schubert polynomials, and symmetrizing operators, J. Combin. Theory Ser. A. 40 (1985), 276–289.

[32] A. Weigandt and A. Yong, The prism tableau model for Schubert polynomials, J. Combin. Theory Ser. A 154 (2018), 551–582.

[33] R. Winkel, Diagram rules for the generation of Schubert polynomials, J. Combin. Theory Ser. A. 86 (1999), 14–48.