Nonautonomous mixed mKdV–sinh–Gordon hierarchy

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Abstract

The construction of a nonautonomous mixed mKdV/sine–Gordon model is proposed by employing an infinite-dimensional affine Lie algebraic structure within the zero-curvature representation. A systematic construction of soliton solutions is provided by an adaptation of the dressing method which takes into account arbitrary time-dependent functions. A particular choice of those arbitrary functions provides an interesting solution describing the transition of a pure mKdV system into a pure sine–Gordon soliton.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Sometime ago, the study of nonlinear effects in lattice dynamics under the influence of a weak dislocation potential has lead to a mixed mKdV/sine–Gordon equation [1]. The system was shown to admit multisoliton solutions and an infinite set of conservation laws [1]. More recently the two-breather solution was discussed in [2] in connection with the propagation of few cycle pulses (FCP) in nonlinear optical media. According to [2], the general mKdV/sine–Gordon equation, in fact, describes the propagation of a ultrashort optical pulses in the Kerr media. Moreover, it was shown in [3] that when the ressonance frequency of atoms in the physical system is well above or well below the characteristic duration of the pulse, the propagation is described by the mKdV or sine–Gordon equations, respectively. The main object of this paper is to provide a systematic construction of soliton solutions that describe the transition between the two regimes, i.e. governed by the mKdV and sine–Gordon equations. This is accomplished by considering the mixed integrable model proposed in [1] with two arbitrary time-dependent coefficients. In this paper we show the integrability of the mixed model with time-dependent coefficients and that by a suitable choice of these coefficients as a smooth step-type functions (as shown in figures 1 and 3) we obtain exact solutions for the mKdV–SG transition and hence a more realistic description of such phenomena.
2. Algebraic formalism

In [4] the algebraic structure of the mixed mKdV/sine–Gordon equation was formulated within the zero-curvature representation and a graded infinite-dimensional Lie algebraic structure as we now briefly review. Consider the associated \( \mathfrak{g} = sl(2) \) Lie algebra with generators satisfying

\[
[h, E_{\pm\alpha}] = \pm 2E_{\pm\alpha}, \quad [E_\alpha, E_{-\alpha}] = h
\]

and the grading operator \( Q = 2\lambda d\lambda + \frac{1}{2}h \).

The grade-by-grade decomposing of equation (2) leads to

\[
\begin{align*}
\partial_t x &= 0, \quad c_2 = a_3 v, \quad b_1 = \frac{1}{2}\partial_t c_2, \\
\partial_t a_1 + 2vb_1 &= 0, \quad \partial_t b_1 + 2va_1 - 2c_0 = 0, \\
\partial_t a_{-1} + 2vb_{-1} &= 0, \quad \partial_t b_{-1} + 2va_{-1} = 0,
\end{align*}
\]

together with the equation of motion

\[
\partial_t c_0 - \partial_t v - 2b_{-1} = 0.
\]

In solving equations (4), we find

\[
a_3 = a_3(t), \quad b_1 = \frac{a_3(t)}{2}v_x, \quad c_2 = a_3(t)v,
\]
where \( a_3(t) \) is an arbitrary function of \( t \). Introducing (8) into the first equation of (5), we obtain
\[ \partial_x \left( a_1 + a_3(t) \frac{v^2}{2} \right) = 0, \]
which implies that
\[ a_1 + a_3(t) \frac{v^2}{2} = f_1(t), \]
where \( f_1(t) \) is another arbitrary function of \( t \). It therefore follows that
\[ a_1 = f_1(t) - a_3(t) \frac{v^2}{2}. \]
(9)
Substituting (9) into the second equation of (5), we get
\[ c_0 = a_3(t) \frac{v^2}{4} (v_{xx} - 2v^3) + f_1(t)v. \]
(10)
Adding and subtracting equations (6), we obtain
\[ \partial_x a_\pm = \mp 2v a_\pm, \]
(11)
where we have denoted
\[ a_\pm = a_{-1} \pm b_{-1}. \]
Without loss of generality, we may solve (11) by changing the variable
\[ \nu h = -\partial_x BB^{-1} = \phi_x h, \quad B = e^{-\phi h}, \]
(12)
which leads us to \( a_\pm = f_{-1}(t)e^{\mp 2\phi} \), where \( f_{-1}(t) \) is another arbitrary function of \( t \). Writing
\[ a_{-1} = \frac{a_+ + a_-}{2}, \quad b_{-1} = \frac{a_+ - a_-}{2}, \]
we find
\[ a_{-1} = f_{-1}(t) \cosh(2\phi), \quad b_{-1} = -f_{-1}(t) \sinh(2\phi). \]
(13)
Substituting (10), (12) and (13) into (7), we finally obtain
\[ \frac{a_3(t)}{4} \left( \phi_{xxxx} - 6\phi_x^2 \phi_{xx} \right) + f_1(t) \phi_{xx} - \phi_{tt} + 2f_{-1}(t) \sinh(2\phi) = 0. \]
(14)
Considering \( f_1(t) = 0 \), \( a_3(t) = \) constant and \( f_{-1}(t) = \) constant we find the usual mixed mKdV/sine–Gordon equation. For \( f_1(t) = 0 \), \( a_3(t) \) a given numerical constant \( \neq 0 \), we recover equation (10) of [5]. Moreover for \( f_{-1}(t) = 0 \), we recover equation considered in [6] with a choice of coefficients that makes the model integrable.

We point out that by the change of coordinates (see for instance [7]) \((x, t) \rightarrow (\tilde{x}, \tilde{t}) = (x + V(t), t)\), where \( V_t = f_1(t) \) followed by a subsequently change \( \tilde{t} \rightarrow T = \int a_3(t) \, dt \) and re-scaling \( f_{-1} \rightarrow \tilde{\eta} \) leads to
\[ \frac{1}{4} \left( \phi_{xxxx} - 6\phi_x^2 \phi_{xx} \right) - \phi_{tt} + 2\tilde{\eta}(t) \sinh(2\phi) = 0. \]
(15)
Although equation (15) corresponds to the equation discussed in [5], the object of this paper is to consider a class of solutions that interpolates between the mKdV and the sine–Gordon equations. This is more conveniently accomplished by employing equation (14) where the two arbitrary functions \( a_3(t) \) and \( f_{-1}(t) \) (with \( f_1(t) = 0 \)) can be chosen as step-like limiting functions (figures 1 and 3) as we will see.
3. Construction of soliton solutions

In order to construct, in a systematic manner, the soliton solutions of the mixed model, let us now recall some basic aspects of the dressing method (see for instance [8]). The zero-curvature representation (2) implies in a pure gauge configuration, i.e.,

$$\partial_t + E + A_0 = \partial_t TT^{-1}, \quad \partial_t + D^{(1)} + \cdots + D^{(-1)} = \partial_t TT^{-1}. \quad (16)$$

In particular, the vacuum is obtained by setting $\phi_{\text{vac}} = 0$ which implies

$$\partial_x T_0 T_0^{-1} = -E_1^{(1)}, \quad \partial_x T_0 T_0^{-1} = -a_3(t) E_1^{(3)} - f_{-1}(t) E_{-1}^{(-1)} - f_1(t) E_1^{(1)},$$

which after integration yields

$$T_0 = \exp \left( -\int dt' a_3(t') E_1^{(3)} - \int dt' f_{-1}(t') E_{-1}^{(-1)} - \int dt' f_1(t') E_1^{(1)} \right) \exp(-x E_1^{(1)}). \quad (17)$$

Following the dressing method explained in [8] and employed in [4] we define the tau-functions

$$\tau_n \equiv \langle \lambda_n | B | \lambda_n \rangle = \langle \lambda_n | T_0 g T_0^{-1} | \lambda_n \rangle,$$

where $\lambda_n, n = 0, 1$ are the fundamental weights of the full affine Kac–Moody algebra $\hat{sl}(2)$, $g$ is a constant group element which classifies the soliton solutions and $B$ is a zero-grade group element containing the physical fields. In order to ensure the highest weight representations we now introduce central extensions within the affine Lie algebra, characterized by $\hat{c}$, i.e.

$$[h(n), E_{\pm \alpha}] = \pm 2 E_{\pm (n+\alpha)},$$

$$[E_{\alpha}, E_{-\alpha}] = h(n) + n \delta_{n,0} \hat{c},$$

$$[h(n), h(m)] = 2n \delta_{n+m,0} \hat{c},$$

and define the highest weight representations, i.e.

$$h |\lambda_n\rangle = \delta_{n,1} |\lambda_n\rangle, \quad \hat{c} |\lambda_n\rangle = |\lambda_n\rangle, \quad g_i |\lambda_n\rangle = 0, \quad i > 0,$$

where $n = 0, 1$. Under this affine picture the group element $B$ acquires a central term contribution:

$$B = e^{-\phi \hat{c}} e^{-\frac{1}{2} \hat{c} \delta_{n,0}}. \quad (21)$$

In order to obtain explicit spacetime dependence from the rhs of (19) we consider the vertex operators,

$$V(\gamma) = \sum_{n=-\infty}^{\infty} \left( \lambda^n h - \frac{1}{2} \hat{c} \delta_{n,0} \right) \gamma^{-2n} + E_{-1}^{(2n+1)} \gamma^{-2n-1}, \quad (22)$$

satisfying

$$[E_{-1}^{(2n+1)}, V(\gamma)] = -2 \gamma^{2n+1} V(\gamma). \quad (23)$$

For a general $M$-soliton solution, the group element $g$ in (19) is written as

$$g = \prod_{j=1}^{M} e^{a_j V(\gamma_j)}, \quad (24)$$

1 For a general member of the hierarchy evolving according to $t = t_{2n+1}$, the vacuum configuration implies

$$\partial_x T_0 T_0^{-1} = -E_1^{(1)}, \quad \partial_{2n+1} T_0 T_0^{-1} = -a_{2n+1}(t) E_{-1}^{(2n+1)} - f_{-1}(t) E_{-1}^{(-1)} - \sum_{k=1}^{n} f_{2k-1}(t) E_{-1}^{(2k-1)}.$$
where $\alpha_j$ are the arbitrary constants. We therefore obtain

$$\tau_0 = e^{-\nu} = \langle \lambda_0 | \Pi_{j=1}^M e^{\alpha_j \rho_j(x,t)V(\gamma_j)} | \lambda_0 \rangle,$$

$$\tau_1 = e^{-\phi - \nu} = \langle \lambda_1 | \Pi_{j=1}^M e^{\alpha_j \rho_j(x,t)V(\gamma_j)} | \lambda_1 \rangle,$$

where

$$\rho_j(x, t) = e^{2\gamma_j x} + 2\gamma_j A_2 n+1(t) + 2\gamma_j F_{2n-1}(t),$$

$$A_2 n+1(t) = \frac{1}{2} \frac{d}{dt} a_{2n+1}(t),$$

$$F_{2n-1}(t) = \frac{1}{2} \frac{d}{dt} f_{2n-1}(t).$$

As an illustrative example, we consider the one- and two-soliton cases, $M = 1, 2$, where

$$\tau_0^{1\text{-sol}} = e^{-\nu} = 1 - \frac{\alpha_1}{2} \rho_1,$$

$$\tau_1^{1\text{-sol}} = e^{-\phi - \nu} = 1 + \frac{\alpha_1}{2} \rho_1,$$

and

$$\tau_0^{2\text{-sol}} = e^{-\nu} = 1 - \frac{\alpha_1}{2} \rho_1 - \frac{\alpha_2}{2} \rho_2 + \alpha_1 \alpha_2 A_{1,2} \rho_1 \rho_2,$$

$$\tau_1^{2\text{-sol}} = e^{-\phi - \nu} = 1 + \frac{\alpha_1}{2} \rho_1 + \frac{\alpha_2}{2} \rho_2 + \alpha_1 \alpha_2 A_{1,2} \rho_1 \rho_2,$$

respectively. In order to obtain (26) and (27) where we have used the fact that

$$\langle \lambda_n | V(\gamma) | \lambda_n \rangle = \delta_n, \quad n = 0, 1,$$

$$\langle \lambda_n | V(\gamma_1) V(\gamma_2) | \lambda_n \rangle = A_{1,2} \equiv \frac{1}{4} \left( \frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2} \right)^2.$$

The general two-soliton solution can then be written as

$$\phi = \ln \left( \frac{\tau_0}{\tau_1} \right) = \ln \left( \frac{1 - \frac{\alpha_1}{2} \rho_1 - \frac{\alpha_2}{2} \rho_2 + \alpha_1 \alpha_2 A_{1,2} \rho_1 \rho_2}{1 + \frac{\alpha_1}{2} \rho_1 + \frac{\alpha_2}{2} \rho_2 + \alpha_1 \alpha_2 A_{1,2} \rho_1 \rho_2} \right),$$

while the one soliton is obtained from (28) by setting $\alpha_2 = 0$.

### 4. Applications

According to [2] the propagation of a FCP with frequency $\omega$ on a dielectric media with characteristic frequency $\Omega$, $\omega \ll \Omega$, can be described by the mKdV equation

$$\phi_{zt} + a \left( \frac{1}{2} \phi_z^2 \phi_{zz} + \phi_{tt} \right) = 0,$$

where the coordinates $z$ and $t$ correspond respectively to the propagation distance and the retarded time, while the electric field $E = \phi_z$. For the case, where $\omega \gg \Omega$, the system is described by the sine–Gordon equation

$$\phi_{zt} - b \sin \phi = 0.$$
If we now consider a dielectric media with two characteristic frequencies $\Omega_1$ and $\Omega_2$, the case in the regime $\Omega_1 \ll \omega \ll \Omega_2$ is described by the mixed mKdV–SG equation

$$\phi_{\tau \tau} + a \left( \frac{3}{2} \phi_x^2 \phi_{\tau \tau} + \phi_{\tau \tau \tau \tau} \right) - b \sin \phi = 0,$$

where the two constants $a$ and $b$ are related to the nonlinear and dispersion properties of the media.

In order to adapt model (14) to such a situation, define

$$a_3(t) = -4a_1(t), \quad f_1(t) = 0, \quad f_{-1}(t) = \frac{b}{4} \theta_2(t),$$

rescaling $\phi \rightarrow \phi, t \rightarrow z, x \rightarrow \tau$, equation (14) becomes

$$a_1 \theta_1(z) \left( \phi_{\tau \tau \tau \tau} + \frac{3}{2} \phi_x^2 \phi_{\tau \tau} \right) + \phi_{\tau \tau} - b \theta_2(z) \sin \phi = 0. \quad (29)$$

If we now substitute $\alpha_k \rightarrow -2\alpha_k$ and make use of the identity

$$\arctan X = \frac{1}{2i} \ln \left( \frac{1 + iX}{1 - iX} \right),$$

we find that the two-soliton solution (28) may be written as

$$\phi = 4 \arctan \left( \frac{\alpha_1 \rho_1 + \alpha_2 \rho_2}{1 - 4\alpha_1 \alpha_2 A_{1,2} \rho_1 \rho_2} \right), \quad (30)$$

where

$$\rho_j = \exp \left( 2\gamma_j \tau + 2\gamma_j^3 A_3(z) + 2\gamma_j^{-1} A_{-1}(z) \right),$$

$$A_3(z) = -4a \int z \frac{dz'}{\phi_{z'}}, \quad A_{-1}(z) = \frac{b}{4} \int z \frac{dz'}{\phi_{z'}}. \quad (31)$$

### 4.1. Transition mKdV–SG

Consider now a dielectric media in which $\Omega_1 \gg 2\gamma_j$ in the region $z < z_1$ and $\Omega_2 \ll 2\gamma_j$ in the region $z > z_2$ with $z_1 > z_2$ such that there exist an overlap region in which the media admits the two characteristic frequencies, $\Omega_2 \ll 2\gamma_j \ll \Omega_1$. As an example to describe such a realistic situation, we take both $\theta_1(z)$ and $\theta_2(z)$ as step-like functions (see figure 1) with

$$\theta_1(z) = \frac{1}{2} - \frac{1}{\pi} \arctan \left[ \beta_1(z - z_1) \right], \quad \theta_2(z) = \frac{1}{2} + \frac{1}{\pi} \arctan \left[ \beta_2(z - z_2) \right],$$

where $\beta_1$ and $\beta_2$ are the phenomenological parameters describing the transition between the two medias (see figure 1).

They can therefore be integrated from (32) to yield

$$-\frac{1}{4a} A_3(z) = \frac{z}{2} - \frac{z - z_1}{\pi} \arctan \left[ \beta_1(z - z_1) \right] + \frac{1}{2\pi \beta_1} \ln \left[ 1 + \beta_1^2 (z - z_1)^2 \right],$$

$$\frac{4}{b} A_{-1}(z) = \frac{z}{2} + \frac{z - z_2}{\pi} \arctan \left[ \beta_2(z - z_2) \right] - \frac{1}{2\pi \beta_2} \ln \left[ 1 + \beta_2^2 (z - z_2)^2 \right].$$

The $\beta_1, \beta_2 \rightarrow +\infty$ limit for $z_1 = z_2 = 0$ corresponds to the system governed by the pure mKdV in the region $z < 0$ and by the pure sine–Gordon equation for $z > 0$. In such a limit, we have

$$\theta_1(z) = \frac{1}{2} \left( 1 - \frac{|z|}{z} \right), \quad \theta_2(z) = \frac{1}{2} \left( 1 + \frac{|z|}{z} \right) \quad (32)$$

and

$$-\frac{1}{4a} A_3(z) = \frac{1}{2} (z - |z|), \quad \frac{4}{b} A_{-1}(z) = \frac{1}{2} (z + |z|). \quad (33)$$

Figure 2 shows the transition mKdV–SG for the one soliton solution, $\alpha_1 = 1, \alpha_2 = 0$. The plot on the right shows the soliton solution viewed from the above and displays the transition (different velocities) from the mKdV to the SG solitons.
4.2. Transition mKdV–SG–mKdV

Another example consists in two equal media separated by a second one describing, for instance, the mKdV–SG–mKdV transition. Mathematically this situation may be described by combining theta-type functions (see figure 3), i.e.

\[
\theta_1(z) = 1 - \frac{1}{\pi} \arctan[\tilde{\beta}_1(z - \tilde{z}_1)] + \frac{1}{\pi} \arctan[\tilde{\beta}_2(z - \tilde{z}_2)], \quad \tilde{z}_1 < \tilde{z}_2,
\]

\[
\theta_2(z) = \frac{1}{\pi} \arctan [\beta_1(z - z_1)] - \frac{1}{\pi} \arctan [\beta_2(z - z_2)], \quad z_1 < z_2,
\]

with \(z_1 \leq \tilde{z}_1\) and \(\tilde{z}_2 \leq z_2\) to guarantee the existence of an overlap region between the two medias.
After integration (32), we find

\[- \frac{1}{4a} A_3(z) = z - \frac{(z - \bar{z}_1)}{\pi} \arctan[\beta_1(z - \bar{z}_1)] + \frac{(z - \bar{z}_2)}{\pi} \arctan[\beta_2(z - \bar{z}_2)] + \frac{1}{2\pi\beta_1} \ln \left[ 1 + \beta_1^2 (z - \bar{z}_1)^2 \right] - \frac{1}{2\pi\beta_2} \ln \left[ 1 + \beta_2^2 (z - \bar{z}_2)^2 \right],\]

\[\frac{4}{b} A_{-1}(z) = \frac{(z - z_1)}{\pi} \arctan[\beta_1(z - z_1)] - \frac{(z - z_2)}{\pi} \arctan[\beta_2(z - z_2)] - \frac{1}{2\pi\beta_1} \ln \left[ 1 + \beta_1^2 (z - z_1)^2 \right] + \frac{1}{2\pi\beta_2} \ln \left[ 1 + \beta_2^2 (z - z_2)^2 \right].\]
The limit $\beta_1, \beta_2, \bar{\beta}_1, \bar{\beta}_2 \to +\infty$, when $z_1 = \bar{z}_1$ and $z_2 = \bar{z}_2$ with $z_1 < z_2$, corresponds to the pure mKdV case in the region $z < z_1$ and $z > z_2$, and pure sine–Gordon in the region $z_1 < z < z_2$. Under such a limiting case, we have

$$\theta_1(z) = 1 - \frac{1}{2} \left( \frac{|z - z_1| - |z - z_2|}{(z - z_1) - (z - z_2)} \right), \quad \theta_2(z) = \frac{1}{2} \left( \frac{|z - z_1| - |z - z_2|}{(z - z_1) - (z - z_2)} \right),$$

such that

$$-\frac{1}{4\rho} A_3(z) = z - \frac{1}{2} (|z - z_1| - |z - z_2|), \quad \frac{4}{\rho} A_{-1}(z) = \frac{1}{2} (|z - z_1| - |z - z_2|).$$

Figure 4 represents the mKdV–SG–mKdV transition for the one-soliton solution.

4.3. Two-soliton solution

The two-soliton solution is shown in figure 5.

As a conclusion, we have adopted the general dressing construction of soliton solutions to the mixed mKdV–SG hierarchy with arbitrary ‘time’-dependent functions. The choice of such arbitrary functions as step-type functions allowed exact solutions describing smooth transitions from the mKdV to the sine–Gordon regime and therefore more realistic models.

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