Progressively Enlargement of Filtrations and Control Problems for Step Processes

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Abstract

In the present paper we address stochastic optimal control problems for a step process \((X, \mathcal{F})\) under a progressive enlargement of the filtration. The global information is obtained adding to the reference filtration \(\mathcal{F}\) the point process \(H = 1_{[\tau, +\infty)}\). Here \(\tau\) is a random time that can be regarded as the occurrence time of an external shock event. We study two classes of control problems, over \([0, T]\) and over the random horizon \([0, T \wedge \tau]\).

We solve these control problems following a dynamical approach based on a class of BSDEs driven by the jump measure \(\mu^Z\) of the semimartingale \(Z = (X, H)\), which is a step process with respect to the enlarged filtration \(\mathcal{G}\). The BSDEs that we consider can be solved in \(\mathcal{G}\) thanks to a martingale representation theorem which we also establish here. To solve the BSDEs and the control problems we need to ensure that \(Z\) is quasi-left continuous in the enlarged filtration \(\mathcal{G}\). Therefore, in addition to the \(\mathcal{F}\)-quasi left continuity of \(X\), we assume some further conditions on \(\tau\): the avoidance of \(\mathcal{F}\)-stopping times and the immersion property, or alternatively Jacod’s absolutely continuity hypothesis.

Keywords: Progressive enlargement of filtration, stochastic optimal control, marked point processes, Backward Stochastic Differential Equations (BSDEs).

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1 Introduction

The enlargement of filtrations is an important subject in probability theory. Starting from the first works published in the 80s (see e.g. Jeulin [28] and Jacod [24]), this topic has received growing interest due to its application in credit risk research to model default events. The reference filtration, which describes the default-free information structure, is enlarged by the knowledge of a default time when it occurs, leading to a global filtration, called progressive enlargement of filtration. Usually, it assumes that the credit event should arrive by surprise, i.e. it is a totally inaccessible random time for the reference filtration, see e.g. Bielecki and Rutkowski [7], Duffie and Singleton [8], and

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Schönbucher [36]. In this framework various stochastic optimal control problems can naturally arise. Here the notion of information plays a crucial role. The decision-maker studies a dynamical systems subject to random perturbations and can act on it in order to optimize some performance criterion. In line with intuition, the more information the controller has available, the better his choice will be.

In the present work we build a bridge between the theory of enlargement of filtrations and control problems for marked point processes. More precisely, let $X$ be a step process, that is, a semimartingale $X$ of the form $X = \sum_{0 \leq s < \cdot} \Delta X_s$ with finitely many jumps over compact time intervals. We denote by $\mathbb{F}$ the filtration generated by $X$: This filtration represents the information available in a market in which an agent (i.e., the controller) handles. We progressively enlarge $\mathbb{F}$ to $\mathbb{G}$ by a random time $\tau$, that can be regarded as the occurrence time of an external shock event, as the death of the agent (e.g., life insurance) or the default of part of the market (e.g., credit risk). We then study two related classes of control problems. The first one consists in optimization problems over $[0, T]$. They can be regarded as control problems of an insider trader who has private information on $\tau$, that is, she can use $\mathbb{G}$-predictable controls. Moreover, the control problem over $[0, T]$ allows to consider terminal costs which may depend on the default time $\tau$, i.e., defualtable costs, for example of the form $g = g_1 1_{\{\tau < T\}} + g_2 1_{\{\tau \leq T\}}$, see also Remark 5.10. It is evident that the associated control problem cannot be solved in the reference filtration $\mathbb{F}$ because the random variable $g$ is $\mathcal{G}_T$-measurable but not $\mathcal{F}_T$-measurable, in general. The second class of control problems we look at is over the random horizon $[0, T \wedge \tau]$. These can be understood as control problems of an agent who only disposes of the information available in the market, that is, she only uses $\mathbb{F}$-predictable controls but, for some reasons, she has exclusively access to the market up to $\tau$, see also Remark 5.19. In this case the current costs are assumed to be $\mathbb{F}$-predictable, while the terminal condition will be $\mathcal{G}_{T\wedge \tau}$-measurable.

Stochastic optimal control problems have been intensively studied over these last years, inspired especially by economics and finance, where they arise in risk management, portfolio selection or optimal investment. Several approaches have been developed ranging from the dynamic programming method, Hamilton- Jacobi-Bellman Partial Differential Equations and Backward Stochastic Differential Equations (BSDEs) to convex martingale duality methods. We refer to the monographs of Fleming and Soner [20] and Yong and Zhou [37]. In particular, the theory of BSDEs has known important developments also in the progressive enlarged filtration setting. For instance in Pham [35] an optimal investment problem for an agent delivering the defaultable claim at maturity $T$ is solved with the BSDE approach. Similar problems have been addressed in Lim and Quenez [33], Ankirchner, Blanchet-Scalliet, and Eyraud-Loisel [5], and Jiao and Pham [29].

Other results concern expected utility optimization problems with random terminal time, where the progressively enlarged filtration is used to handle the time $\tau$ of a shock that affects the market or the agent. These problems can be solved by introducing a suitable BSDE over $[0, T \wedge \tau]$. In Kharroubi and Lim [31] and Kharroubi, Lim and Ngoupeyou [30], the authors consider optimization problems on $[0, T \wedge \tau]$ in a progressively enlarged Brownian filtration $\mathbb{G}$ reducing the study of the BSDEs on $[0, T \wedge \tau]$ to the one of an associated BSDEs with deterministic horizon $[0, T]$ in the reference Brownian filtration $\mathbb{F}$. This method is often called in the literature reduction method. Following the approach of [30], Jeanblanc et al. study in [26] an exponential utility maximization problem over $[0, T \wedge \tau]$ in a progressively enlarged Brownian filtration. In both [31] and [26] the random time $\tau$ avoids stopping times and satisfies the immersion property. More general BSDEs over $[0, T \wedge \tau]$ have been considered by Aksamit, Lim and Rutkowski in the recent preprint [4] where $\tau$ is a random time satisfying some mild conditions. If from one side the reduction approach, as shown in [4], allows very general random times $\tau$, it only allows to solve BSDEs in $\mathbb{G}$ on $[0, T \wedge \tau]$ and not over $[0, T]$.

In the present paper, we show that the problem over $[0, T \wedge \tau]$ in $\mathbb{G}$ can be solved as a problem in $\mathbb{G}$ over the deterministic horizon $[0, T]$: We prove that the two value functions coincide, and there
is an explicit relationship between the two optimal control processes, see Theorem 5.26. This can be done thanks to a martingale representation theorem in $G$ holding for all martingales and not only for martingales stopped at $\tau$, so that the BSDEs on $[0,T]$ can be globally solved. To our opinion this is for step processes the most natural approach since, as shown below, a martingale representation theorem always holds in the progressively enlargement $G$. We remark that this approach was initiated by Di Tella in [15], where an expected utility maximization problem for a continuous price processes was considered. Thanks to our method, we are also able to consider defaultable costs and we can interpret the two control problems over $[0,T]$ and $[0, T \land \tau]$ as explained above. We stress that in the present paper we are able to consider a more general random time $\tau$ in comparison to [15]: The intensity of $\tau$ need not be bounded here and we do not necessarily require the immersion property. Furthermore, we can also allow the presence of running costs, which were not present in [15].

We now give more details on the control problems we deal with. They consist in minimizing a cost functional $J$ defined as the expectation of a cumulated running cost plus a terminal cost. They are formulated in a weak way, namely the expectation appearing in $J$ is taken with respect to a probability measure which depends on the control process itself. We solve these control problems following a dynamical approach based on a class of BSDEs with Lipschitz generators of sub linear growth and driven by a particular random measure. More precisely, since the filtration $G$ is obtained adding to the reference filtration $\mathbb{F}$ the point process $H = 1_{[\tau, +\infty)}$, we consider a class of BSDEs driven by the jump measure $\mu^Z$ of the semimartingale $Z = (X, H)$, which is a $G$-marked point process. Existence and uniqueness of the solution of the involved BSDEs relay on the theory developed by Confortola and Fuhrman in [10]. In [10] the authors assume that the driving marked point process is quasi-left continuous, i.e., its compensator is continuous. This assumption is crucial: Indeed, Confortola, Fuhrman and Jacod have shown in [11, Remark 10.1] that if the compensator is not continuous, one cannot expect that the corresponding BSDE admits a solution, in general. On the other side, Di Tella and Jeanblanc gave in [17] a counterexample showing that the $\mathbb{F}$-quasi-left continuity of the marked point process $\mu^X$ is not preserved in $G$, that is, $\mu^Z$ need not be $G$-quasi-left continuous, see [17, Counterexample 4.7]. Furthermore, in order to deal with a treatable class of control problems we need the explicit form for the $G$-compensator $\nu^{G,Z}$ of $\mu^{Z}$, that in general is a challenging open problem.

For these reasons, in Section 3 we assume sufficient conditions on $\tau$, namely the avoidance of $\mathbb{F}$-stopping times, see Definition 4.5 and the immersion property, see Definition 4.6 or alternatively, Jacod’s absolutely continuity hypotheses, see formulae (4.4)-(4.5). Under these assumptions, we are able to explicitly compute the $G$-compensator $\nu^{G,Z}$ and show the $G$-quasi-left continuity of $\mu^Z$, see Theorems 4.7 and 4.8. To the best of our knowledge, the explicit form of $\nu^{G,Z}$ under Jacod’s absolutely continuity hypothesis is new and of independent interest.

Finally, to apply the theory for BSDEs developed in [10], we also need a martingale representation theorem in the enlarged filtration $G$. In Theorem 5.1 we provide a martingale representation theorem when the filtration $\mathbb{F}$ generated by a step process $X$ is enlarged by the filtration $\mathbb{H}$ generated by a step process $H$. Taking $H = 1_{[\tau, +\infty)}$, we obtain the special setting of the control problem. The results of Section 3 generalize the paper [17] by Di Tella and Jeanblanc, in which $X$ and $H$ are simple point processes, i.e. $\Delta X, \Delta H \in \{0,1\}$. In the recent preprint [9] by Calzolari and Torti a martingale representation theorem for progressively enlarged filtrations generated by marked point processes is also obtained. The results of [9] are very general and go beyond the semimartingales context, while here we give a concise independent proof of the martingale representation theorem in the special case of step processes.

The present paper has the following structure: In Section 2 we recall same basic notions. In Section 3 we obtain a martingale representation theorem in the enlarged filtration of a step process. Section 5 is devoted to the study of the $G$-quasi-left continuity of the step process $Z = (X, H)$, if
\[ H = 1_{[\tau, +\infty)} \], and to the computations for getting the explicit form of the compensator \( v^{G,Z} \). The applications to different kinds of control problems both on \([0,T]\) and \([0,T \wedge \tau]\) are presented in Section 5 while the proofs of technical results of Sections 4 and 5 are postponed respectively to Appendices A and B.

2 Basic Notions

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. We denote by \(\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}\) a right-continuous filtration of subsets of \(\mathcal{F}\) and by \(\mathcal{G}(\mathcal{F})\) (resp. \(\mathcal{P}(\mathcal{F})\)) the \(\mathcal{F}\)-optional (resp. \(\mathcal{F}\)-predictable) \(\sigma\)-algebra on \(\Omega \times \mathbb{R}_+\).

We define \(\mathcal{F}_\infty := \bigvee_{t \geq 0} \mathcal{F}_t\).

Let \(X\) be a stochastic process. We sometimes use the notation \((X, \mathcal{F})\) to mean that \(X\) is \(\mathcal{F}\)-adapted. By \(\mathbb{F}^X\) we denote the smallest right-continuous filtration such that \(X\) is adapted. If \(X\) is càdlàg, we denote by \(\Delta X\) the jump process and use the convention \(\Delta X_0 = 0\).

We say that an \(\mathcal{F}\)-adapted càdlàg process \(X\) is \(\mathbb{F}\)\(-\text{quasi-left continuous if } \Delta X_\sigma = 0 \text{ a.s. for every finite-valued } \mathbb{F}\)-predictable stopping time \(\sigma\).

**Random measures.** For a Borel subset \(E\) of \(\mathbb{R}^d\), we introduce \(\tilde{\Omega} := \Omega \times \mathbb{R}_+ \times E\) and the product \(\sigma\)-algebras \(\mathcal{G}(\mathcal{F}) := \mathcal{G}(\mathcal{F}) \otimes \mathcal{B}(E)\) and \(\mathcal{P}(\mathcal{F}) := \mathcal{P}(\mathcal{F}) \otimes \mathcal{B}(E)\). If \(W\) is an \(\mathcal{G}(\mathcal{F})\)-measurable (resp. \(\mathcal{P}(\mathcal{F})\)-measurable) mapping from \(\tilde{\Omega}\) into \(\mathbb{R}\), it is called an \(\mathcal{F}\)-optional (resp. \(\mathcal{F}\)-predictable) function.

Let \(\mu\) be a random measure on \(\mathbb{R}_+ \times E\) (see [25, Definition II.1.3]). For a nonnegative \(\mathcal{F}\)-optional function \(W\), we write \(W * \mu = (W * \mu_t)_{t \geq 0}\), where \(W * \mu_t(\omega) := \int_{[0,t] \times E} W(\omega, s, x) \mu(\omega, ds, dx)\) is the process defined by the (Lebesgue–Stieltjes) integral of \(W\) with respect to \(\mu\) (see [25, II.1.5] for details). If \(W * \mu\) is \(\mathbb{F}\)-optional (resp. \(\mathbb{F}\)-predictable), for every optional (resp. \(\mathbb{F}\)-predictable) function \(W\), then \(\mu\) is called \(\mathbb{F}\)\(-optional (resp. \(\mathbb{F}\)-predictable).

**Semimartingales.** When we say that \(X\) is a semimartingale, we always assume that it is càdlàg.

For an \(\mathbb{R}^d\)-valued \(\mathbb{F}\)-semimartingale \(X\), we denote by \(\mu^X\) the jump measure of \(X\), that is, \(\mu^X(\omega, dt, dx) = \sum_{\Delta X(\omega) \neq 0} \delta_{\Delta X(\omega)}(dt, dx)\), where, here and in the whole paper, \(\delta_\omega\) denotes the Dirac measure at point \(\omega\). From [25, Theorem II.1.16], \(\mu^X\) is an integer-valued random measure on \(\mathbb{R}_+ \times \mathbb{R}^d\) with respect to \(\mathcal{F}\) (see [25, Definition II.1.13]). Thus, \(\mu^X\) is, in particular, an \(\mathcal{F}\)-optional random measure. According to [25, Definition III.1.23], \(\mu^X\) is called an \(\mathbb{R}^d\)-valued marked point process (with respect to \(\mathcal{F}\)) if \(\mu^X(\omega; [0,t] \times \mathbb{R}^d) < +\infty\), for every \(\omega \in \Omega\) and \(t \in \mathbb{R}_+\). By \(v^X\) we denote the \(\mathbb{F}\)-predictable compensator of \(\mu^X\) (see [25, Definition II.2.6]). We recall that \(v^X\) is a predictable random measure characterized by the following properties: For any \(\mathbb{F}\)-predictable function \(W\) such that \(|W| * \mu^X \in \mathcal{A}^+_\text{loc}^\perp\), we have \(|W| * v^X \in \mathcal{A}^+_\text{loc}\) and \(W * \mu^X - W * v^X \in \mathcal{H}^1_{\text{loc}}(\mathcal{F}), \mathcal{H}^1_{\text{loc}}(\mathcal{F})\) denoting the space of \(\mathbb{F}\)-local martingales and \(\mathcal{A}^+_\text{loc}^\perp(\mathcal{F})\) the space of \(\mathbb{F}\)-adapted locally integrable càdlàg increasing processes starting at zero.

If \(X = Y - Z\) with \(Y, Z \in \mathcal{A}^+_\text{loc}^\perp(\mathcal{F})\), we then write \(X \in \mathcal{A}^+_\text{loc}(\mathcal{F})\). For \(X \in \mathcal{A}^\perp_{\text{loc}}(\mathcal{F})\) we denote by \(X^{p,\mathbb{F}}\) the \(\mathbb{F}\)-dual predictable projection of \(X\), that is the unique \(\mathbb{F}\)-predictable process in \(\mathcal{A}^\perp_{\text{loc}}(\mathcal{F})\) such that \(X - X^{p,\mathbb{F}} \in \mathcal{H}^1_{\text{loc}}(\mathcal{F})\).

An \(\mathbb{R}^d\)-valued semimartingale \(X\) is a step process with respect to \(\mathbb{F}\) if it can be represented in the form \(X = \sum_{n=1}^{\infty} \xi_n 1_{[\tau_n, +\infty)}\), where \((\tau_n)\) is a sequence of \(\mathbb{F}\)-stopping times such that \(\tau_n \uparrow +\infty\), \(\tau_n < \tau_{n+1}\) on \(\{\tau_n < +\infty\}\) and \((\xi_n)_{n \geq 1}\) is a sequence of \(\mathbb{R}^d\)-valued random variables such that \(\xi_n\) is \(\mathcal{F}_{\tau_n}\)-measurable and \(\xi_0 \neq 0\) if and only if \(\tau_0 < +\infty\) (see [21, Definition 11.48]). The process \(N^X = \sum_{n=1}^{\infty} 1_{[\tau_n, +\infty)}\) is called the point process associated to \(X\). If \(X\) is a step process with respect to
If \( \mathcal{F} \), we then obviously have \( \tau_n = \inf \{ t > \tau_{n-1} : X_t \neq X_{\tau_{n-1}} \} \) (\( \tau_0 := 0 \)), \( \xi_n = \Delta X_{\tau_n} 1_{\{ \tau_n < +\infty \}} \) and

\[
\mu^X(dt, dx) = \sum_{n=1}^{\infty} 1_{\{ \tau_n < +\infty \}} \delta_{(\tau_n, \xi_n)}(dt, dx).
\] (2.1)

We say that a semimartingale \( X \) is a sum of jumps with respect to \( \mathcal{F} \) if \( X \) is \( \mathcal{F} \)-adapted, of finite variation and \( X = \sum_{0 \leq s < \infty} \Delta X_s \). If \( X \) is a sum of jumps, then \( X = \text{Id} + \mu^X \) holds, where \( \text{Id}(x) := x \). Furthermore, it is evident, that \( \mu^X \) is a marked step process if and only if \( X \) is a step process (see III.1.21 and Proposition II.1.14), that is, if \( X \) has finitely many jumps over compact time intervals.

## 3 Martingale Representation

For an \( \mathbb{R}^d \)-valued step process \( (X, \mathbb{F}^X) \) and a \( \sigma \)-field \( \mathcal{B}^X \), called the initial \( \sigma \)-field, we denote by \( \mathbb{F} = (\mathcal{F}_t)_{t \geq 0} \) the filtration \( \mathbb{F}^X \) initially enlarged by \( \mathcal{B}^X \), thus \( \mathcal{F}_t := \mathcal{B}^X \vee \mathcal{F}^X_t \). It is well-known that \( \mathbb{F} \) is right-continuous and clearly, \( (X, \mathbb{F}) \) is a step process. We stress that non-trivial initial \( \sigma \)-field \( \mathcal{B}^X \) allows to include in the theory developed in the present paper, without any additional effort, also step processes with \( \mathcal{F}_0 \)-measurable semimartingale characteristics, that is, step-processes with conditionally independent increments with respect to \( \mathbb{F} \) given \( \mathcal{F}_0 \).

We now consider an \( \mathbb{R}^\ell \)-valued step process \( H \) and introduce \( \mathcal{H} = (\mathcal{H}_t)_{t \geq 0} \) by \( \mathcal{H}_t := \mathcal{B}^H \vee \mathcal{F}^H_t \), \( t \geq 0 \), where \( \mathcal{B}^H \) denotes a \( \sigma \)-field.

The progressive enlargement of \( \mathbb{F} \) by \( \mathcal{H} \) we denote by \( \mathbb{G} = (\mathcal{G}_t)_{t \geq 0} \), where

\[
\mathcal{G}_t := \bigcap_{s \geq t} \mathcal{F}_s \vee \mathcal{H}_s \quad t \geq 0.
\]

It is evident that \( \mathbb{G} \) is the smallest right-continuous filtration containing \( \mathbb{F}^X, \mathcal{B}^H, \mathcal{B}^X \) and \( \mathcal{B}^H \) (i.e., \( \mathbb{F} \) and \( \mathcal{H} \)).

As a special example of \( H \), one can take the default process associated with a random time \( \tau \), i.e., \( H_t(\omega) := 1_{[\tau_{n-1}+1]}(\omega, t) \), where \( \tau \) is a \((0, +\infty]\)-valued random variable. In this case \( (H, \mathcal{H}) \) is a point process. If \( \mathcal{B}^H \) is trivial, \( \mathbb{G} \) is called the progressive enlargement of \( \mathbb{F} \) by \( \tau \) and it is the smallest right-continuous filtration containing \( \mathbb{F} \) and such that \( \tau \) is a \( \mathbb{G} \)-stopping time.

We now introduce the \( \mathbb{R}^d \times \mathbb{R}^\ell \)-valued \( \mathbb{G} \)-semimartingale \( Z = (X, H)^\uparrow \). Clearly, \( Z \) is a sum of jumps with respect to \( \mathbb{G} \), hence it is a \( \mathbb{G} \)-semimartingale. The jump measure \( \mu^Z \) of \( Z \) is an integer-valued random measure on \( \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^\ell \) and satisfies

\[
\mu^Z(\omega, dt, dx_1, dx_2) = \sum_{s < 0} 1_{\{ \Delta Z_s(\omega) \neq 0 \}} \delta_{(s, \Delta Z_s(\omega))}(ds, dx_1, dx_2).
\]

**Theorem 3.1.** Let \( (X, \mathbb{F}^X) \) and \( (H, \mathbb{F}^H) \) be step processes taking values in \( \mathbb{R}^d \) and \( \mathbb{R}^\ell \) respectively and consider two initial \( \sigma \)-fields \( \mathcal{B}^X \) and \( \mathcal{B}^H \). We define the filtrations \( \mathbb{F}, \mathcal{H} \) and \( \mathbb{G} \) as above and set \( Z := (X, H)^\uparrow \). We then have:

(i) \( \mu^Z \) is an \( \mathbb{R}^d \times \mathbb{R}^\ell \)-valued marked point process, that is \( (Z, \mathbb{G}) \) is a step process.

(ii) \( \mathbb{G} \) is the smallest right-continuous filtration containing \( \mathcal{B} := \mathcal{B}^X \vee \mathcal{B}^H \) and such that \( \mu^Z \) is optional.

If furthermore \( \mathcal{F} = \mathcal{G}_\infty \), then every \( Y \in \mathcal{H}^{1}\text{loc}(\mathbb{G}) \) can be represented as

\[
Y = Y_0 + W * \mu^Z - W * v^Z
\] (3.1)

where \( (\omega, t, x_1, x_2) \mapsto W(\omega, t, x_1, x_2) \) is a \( \mathcal{P}(\mathcal{G}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^\ell) \)-measurable function such that \( |W| * \mu^Z \in \mathcal{H}^{1}\text{loc}(\mathbb{G}) \) and \( v^Z \) denotes the \( \mathbb{G} \)-dual predictable projection of the jump measure \( \mu^Z \) of \( Z \).
We now come to (3.1). If we assume \( F \) and an meaning that (a) the filtration \( R \) and an \( \ell \)-valued step processes, respectively. Therefore, we have

\[
\mu^Z([0,t] \times \mathbb{R}^d \times \mathbb{R}^\ell) = \sum_{0 \leq s \leq t} 1_{\{\Delta s \neq 0\}} = \sum_{0 \leq s \leq t} 1_{\{\Delta X_s \neq 0\} \cup \{\Delta H_s \neq 0\}} \\
\leq \sum_{0 \leq s \leq t} (1_{\{\Delta X_s \neq 0\}} + 1_{\{\Delta H_s \neq 0\}}) \\
= \mu^X([0,t] \times \mathbb{R}^d) + \mu^H([0,t] \times \mathbb{R}^\ell) < +\infty,
\]

meaning that \( \mu^Z \) is an \( \mathbb{R}^d \times \mathbb{R}^\ell \)-valued marked point process with respect to \( G \). This concludes the proof of (i). We now come to (ii). Let us denote by \( G' \) the smallest right continuous filtration such that \( \mu^Z \) is optional. We first show the identity \( G' = \mathbb{F}^Z \). Since \( Z \) is an \( \mathbb{F}^Z \)-semimartingale, \( \mu^Z \) is an \( \mathbb{F}^Z \)-optional integer-valued random measure. So, \( G' \subseteq \mathbb{F}^Z \) holds. We now show the converse inclusion \( G' \supseteq \mathbb{F} \). We denote \( g_1(x_1,x_2) = x_1 \) and \( g_2(x_1,x_2) = x_2 \). By definition of \( \mu^Z \) we have

\[
|g| \ast \mu^Z = \sum_{s \leq t} |\Delta X_s| 1_{\{\Delta Z \neq 0\}} \\
= \sum_{s \leq t} |\Delta X_s| (1_{\{\Delta X_s \neq 0, \Delta H_s = 0\}} + 1_{\{\Delta X_s = 0, \Delta H_s \neq 0\}} + 1_{\{\Delta X_s \neq 0, \Delta H_s \neq 0\}}) \\
= \sum_{s \leq t} |\Delta X_s| 1_{\{\Delta X_s \neq 0\}} \leq \text{Var}(X), < +\infty,
\]

where \( \text{Var}(X) \) denotes the total variation of \( s \mapsto X_s(\omega) \) on \([0,t]\). Hence, the integral \( g_1 \ast \mu^Z \) is well defined and satisfies \( X = g_1 \ast \mu^Z \). Analogously, \( H = g_2 \ast \mu^Z \) holds. This yields that \( X \) and \( H \) are \( G' \)-optional processes. Since \( G' \) is right-continuous, we get \( G' \supseteq \mathbb{F}^Z \). From [22, Proposition 3.39 (a)] the filtration \( \mathcal{F} \cup G' \) is right-continuous. Therefore, \( \mathcal{F} \cup G' \) and \( G \) coincide: They are both the smallest right continuous filtrations containing \( \mathbb{F}^X, \mathbb{F}^H, \mathcal{F}^X \) and \( \mathcal{F}^H \). The proof of (ii) is complete. We now come to (3.1). If we assume \( \mathcal{F} = \mathcal{G}_\alpha \), this is an immediate consequence of (i), (ii) and [25, Theorem III.4.37]. The proof is complete.

As an application of Theorem 3.1 we can easily show by induction the following result.

**Corollary 3.2.** We consider the \( \mathbb{R}^d \)-valued step processes \((X^i, \mathbb{F}^X)\) and the initial \( \sigma \)-fields \( \mathcal{F}^X \), \( i = 1, \ldots, n \). We set \( \mathcal{F} := \mathbb{F}^X \vee \mathcal{F}^X \) and denote by \( G \) the smallest right-continuous filtration containing \( \{\mathcal{F}^i, i = 1, \ldots, n\} \). Then the \( E := \mathbb{R}^d \times \cdots \times \mathbb{R}^d \)-valued semimartingale \( Z = (X^1, \ldots, X^n) \) satisfies:

(i) \( \mu^Z \) is an \( E \)-valued marked point process with respect to \( G \), that is \( Z \) is an \( E \)-valued step process with respect to \( G \).

(ii) \( G \) is the smallest right continuous filtration containing \( \mathcal{F} := \bigvee_{i=1}^n \mathcal{F}^X \) and such that \( \mu^Z \) is an optional random measure.

If furthermore \( \mathcal{F} = \mathcal{G}_\alpha \), then every \( Y \in \mathcal{G}^\mathcal{F}_\mathcal{G}_\alpha \) can be represented as

\[
Y = Y_0 + W \ast \mu^Z - W \ast \nu^Z,
\]

where \( (\omega, t, x_1, \ldots, x_n) \mapsto W(\omega, t, x_1, \ldots, x_n) \) is a \( \mathcal{P}(G) \otimes \mathcal{B}(E) \)-measurable function such that \( |W| \ast \mu^Z \in \mathcal{G}^\mathcal{F}_\mathcal{G}_\alpha \) and \( \nu^Z \) denotes the \( G \)-dual predictable projection of the jump measure \( \mu^Z \) of \( Z \).
4 The $\mathcal{G}$-dual predictable projection

Consider an $\mathbb{R}^l$-valued step process $H$ and introduce $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$ by $\mathcal{H}_t := \mathcal{R}^H \vee \mathcal{F}_t^H$, $t \geq 0$, where $\mathcal{R}^H$ denotes a $\sigma$-field. We denote by $\mathcal{G} = (\mathcal{G}_t)_{t \geq 0}$ the progressive enlargement of $\mathbb{F}$ by $\mathbb{H}$.

In this section we shall concentrate on the special case $H = 1_{[\tau, +\infty)}$, where $\tau$ denotes a random time: In particular, we aim to establish sufficient conditions to ensure the $\mathcal{G}$-quasi-left continuity of $Z = (X, H)$. To this end we need to determine the form of the $\mathcal{G}$-dual predictable projection $\nu^{G,Z}$ of $\mu^{Z}$. The next result, which holds for general step processes, gives the structure of $\nu^{G,Z}$ if $H$ and $X$ have no common jumps. Its proof is postponed to the Appendix A.

**Theorem 4.1.** Let $(X, \mathbb{F})$ be an $\mathbb{R}^d$-valued step-process and let $(H, \mathbb{H})$ be an $\mathbb{R}^l$-valued step-process. If $\Delta X \Delta H = 0$, then the following identities hold for the $\mathbb{R}^d \times \mathbb{R}^l$-valued $\mathcal{G}$-step process $Z = (X, H)$:

(i) $\mu^{Z}(\omega, dt, dx_1, dx_2) = \mu^X(\omega, dt, dx_1) \delta_0(dx_2) + \mu^H(\omega, dt, dx_2) \delta_0(dx_1)$.

(ii) $\nu^{G,Z}(\omega, dt, dx_1, dx_2) = \nu^{G,X}(\omega, dt, dx_1) \delta_0(dx_2) + \nu^{G,H}(\omega, dt, dx_2) \delta_0(dx_1)$.

4.1 Progressive enlargement by a random time

We now denote by $\mathcal{G}$ the progressive enlargement of $\mathbb{F}$ by a random time $\tau : \Omega \longrightarrow (0, +\infty)$, that is, $\mathcal{G}$ is the smallest right-continuous filtration containing $\mathbb{F}$ and such that $\tau$ is a $\mathcal{G}$-stopping time.

We denote by $H = 1_{[\tau, +\infty)}$ the default process of $\tau$ and by $A = \mathcal{G}(1 - H) = \mathcal{G}1_{[0, \tau]}$ the $\mathcal{F}$-optional projection of $(1 - H) = 1_{[0, \tau]}^\mathcal{G}$ (see [14, Theorem V.14 and V.15]). The process $A$ is a càdlàg $\mathbb{F}$-supermartingale, called Azéma supermartingale, satisfying $A_t = \mathbb{P}[\tau > t | \mathcal{F}_t]$ a.s., for every $t \geq 0$. Let $m$ be the martingale defined by $m_t = \mathbb{E}[H^0 + 1_{[\tau, +\infty)} | \mathcal{F}_t]$ a.s., $t \geq 0$, where $H^0$ denotes the $\mathcal{F}$-dual optional projection of $H$. The martingale $m$ belongs to the class BMO with respect to $\mathbb{F}$ (see [1, Proposition 1.49]) and $A$ has the following $\mathbb{F}$-optional decomposition: $A = m - H^0$. It is well-known that $\{A_\cdot > 0\} \subseteq [0, \tau]$ holds (see, e.g., [1, Lemma 2.14]), so the process $\frac{1}{\mathbb{E}} 1_{[0, \tau]}$ is well defined. For every $\mathbb{F}$-local martingale $Y$, the process $[Y, m]$ belongs to $\mathcal{A}_{loc}(\mathbb{F})$. Therefore, the $\mathbb{F}$-dual predictable projection $[Y, m]^{p,\mathbb{F}}$ of $[Y, m]$ is well defined and we set $([Y, m]^{p,\mathbb{F}}) := [Y, m]^{p,\mathbb{F}}$.

The $\mathcal{G}$-dual predictable projection of $H$ is denoted by $\Lambda^G$ and, by [1, Proposition 2.15] it satisfies

$$\Lambda^G = \int_0^{\tau^\land} \frac{1}{A_s} \, dH^p_{s, \mathbb{F}},$$

$H^p_{s, \mathbb{F}}$ denoting the $\mathcal{F}$-dual predictable projection of $H$.

Because of the special structure of the enlarged filtration, the following result holds, whose proof is given in Appendix A.

**Lemma 4.2.** Let $(\omega, t, x) \mapsto W(\omega, t, x)$ be a $\mathcal{G}$-predictable function. Then, there exists an $\mathbb{F}$-predictable function $(\omega, t, x) \mapsto \overline{W}(\omega, t, x)$ such that $W(\omega, t, x) 1_{[0, \tau]}(\omega, t) = \overline{W}(\omega, t, x) 1_{[0, \tau]}(\omega, t)$. If furthermore $W$ is bounded, then $\overline{W}$ is bounded too.

4.2 Quasi-left continuity in the enlarged filtration

Let $X \in \mathcal{A}_{loc}(\mathbb{F})$ be an $\mathbb{F}$-quasi left continuous step process. We are interested in the following question: Is $X$ $\mathcal{G}$-quasi left continuous? The reason to consider this problem is that to solve the BSDEs associated to the control problems studied in Section 5 we want to apply the theory developed in [10] and, to this aim, it is important that the $\mathcal{G}$-step process $Z = (X, H)$ is $\mathcal{G}$-quasi left continuous. However, in general, this is not true: Intuitively, the larger filtration $\mathcal{G}$ supports more predictable stopping times than $\mathbb{F}$. To see this we recall [13, Counterexample 4.8]:


Counterexample 4.3. Let $X$ be a homogeneous Poisson process with respect to $\mathbb{P}^X$ and let $(\tau_n)_{n \geq 1}$ be the sequence of the jump-times of $\mathbb{P}^X$. The process $X$ is not quasi-left continuous in the filtration $\mathbb{G}$ obtained enlarging $\mathbb{F}^X$ progressively by the random time $\tau = \frac{1}{n}(\tau_1 + \tau_2)$. Indeed, the jump-time $\tau_2$ of $X$ is announced in $\mathbb{G}$ by $(\vartheta_n)_{n \geq 1}$, $\vartheta_n := \frac{1}{n}\tau + (1 - \frac{1}{n})\tau_2$, and $\vartheta_n > \tau$ is a $\mathbb{G}$-stopping time for every $n \geq 1$ by [14, Theorem III.16]. Hence, $\tau_2$ is a $\mathbb{G}$-predictable jump-time of $X$.

Therefore, we need to state sufficient conditions of $\tau$ to ensure the $\mathbb{G}$-quasi-left continuity of $Z$. Notice that the quasi-left continuity of $X$ can get lost only over $(\tau, +\infty)$, as the following result shows. The proof is given in Appendix A.

Proposition 4.4. Let $X \in \mathcal{A}_{loc}(\mathbb{F})$ be an $\mathbb{F}$-quasi-left continuous step process. Then, the $\mathbb{G}$-adapted stopped process $X^\tau$ (defined by $X^\tau_t := X_{t\wedge \tau}$, $t \geq 0$) is $\mathbb{G}$-quasi-left continuous.

In the remaining part of this section we state sufficient conditions for the $\mathbb{G}$-quasi-left continuity of $Z$.

Avoidance of $\mathbb{F}$-stopping times. The first property we are going to recall is the avoidance of $\mathbb{F}$-stopping times, from now on referred as assumption ($\mathcal{A}$).

Definition 4.5. The random time $\tau$ satisfies assumption ($\mathcal{A}$) if $\mathbb{P}[\tau = \sigma < +\infty] = 0$ for every $\mathbb{F}$-stopping time $\sigma$.

The interpretation of assumption ($\mathcal{A}$) is the following: The random time $\tau$ carries an information which is completely exogenous: Nothing about $\tau$ can be inferred from the information contained in the reference filtration $\mathbb{F}$. In particular, $\tau$ satisfies ($\mathcal{A}$) if and only if $H^\varphi$ is continuous (see, e.g., [1, Lemma 1.48(a)] or [16, Lemma 3.4]). This implies that $H^\varphi = H^{b, R}$ and, according to (4.1), we get that $\Lambda^G$ is a continuous process. Therefore, $H$ is a quasi-left continuous process and $\tau$ is a totally inaccessible stopping time.

If $\tau$ satisfies assumption ($\mathcal{A}$), then $\Delta X \Delta H = 0$. Indeed,

$$[X, H]_t = \sum_{s \leq t} \Delta X_s \Delta H_s = \sum_{s \leq t} \Delta X_s \Delta H_s 1_{\{\Delta H \neq 0\} \cap \{\Delta X \neq 0\}}.$$  (4.2)

Denoting by $(\sigma_n)_{n \geq 1}$ a sequence exhausting the thin set $\{\Delta X \neq 0\}$ we obviously have $\{\Delta X \neq 0\} \cap \{\Delta H \neq 0\} = \bigcup_{n=1}^{\infty} [\sigma_n] \cap [\tau]$, where for a stopping time $\eta$ we denote by $[\eta]$ the graph of $\eta$. Because of assumption ($\mathcal{A}$), the random set $[\sigma_n] \cap [\tau]$ is evanescent, for every $n \geq 1$. Hence, (4.2) yields $[X, H] = 0$ and therefore $\Delta X \Delta H = \Delta [X, H] = 0$. Thus, according to Theorem 4.1 if assumption ($\mathcal{A}$) is satisfied, by the special form of $H$, we have

$$\mu^Z(\omega, dt, x_1, dx_2) = \mu^X(\omega, dt, dx_1) \delta_0(dx_2) + dH_t(\omega) \delta_1(dx_2) \delta_0(dx_1),$$

$$v^{G, Z}(\omega, dt, x_1, dx_2) = v^{G, X}(\omega, dt, dx_1) \delta_0(dx_2) + d\Lambda^G_t(\omega) \delta_1(dx_2) \delta_0(dx_1).$$  (4.3)

Immersion property. As we have seen above, if $\tau$ satisfies assumption ($\mathcal{A}$), then the process $H$ is $\mathbb{G}$-quasi-left continuous. This however, does not imply that the joint process $Z = (X, H)$ is $\mathbb{G}$-quasi-left continuous. Indeed, the random time $\tau$ from Counterexample 4.3 avoids $\mathbb{F}$-stopping times. However, $X$ is not $\mathbb{G}$-quasi-left continuous. So, we need further assumptions to ensure the $\mathbb{G}$-quasi-left continuity of $Z$. To this aim we state the following definition

Definition 4.6. We say that the random time $\tau$ satisfies the immersion property, from now on assumption ($\mathcal{H}$), if $\mathbb{F}$-martingales remain $\mathbb{G}$-martingales.
If $X$ is an $\mathbb{F}$-step process and $\tau$ is a random time satisfying assumption ($\mathcal{H}$), then [25, Theorem 2.21] yields $\nu^{G,X} = \nu^{F,X}$. Therefore, if $X$ is $\mathbb{F}$-quasi-left continuous, i.e., $\nu^{F,X}(\omega, \{t\} \times \mathbb{R}^d) = 0$, $t \geq 0$, the same holds in $\mathbb{G}$. We therefore have the following result, whose proof follows from the above discussion and is therefore omitted.

**Theorem 4.7.** Let $X$ be a step process and let $\tau$ be a random time satisfying both assumption ($\mathcal{A}$) and assumption ($\mathcal{H}$). Then, the $\mathbb{G}$-dual predictable projection of $\mu^X$, where $Z = (X, H)$, is given by

$$
\nu^{G,Z}(\omega, d\tau, dx_1, dx_2) = \nu^{F,X}(\omega, d\tau, dx_1)\delta_0(dx_2) + \delta_1(dx_2)\delta_0(dx_1)d\Lambda^G_\tau(\omega).
$$

In particular, if $X$ is $\mathbb{F}$-quasi-left continuous, then $Z$ is $\mathbb{G}$-quasi-left continuous as well.

We stress that, although assumption ($\mathcal{H}$) is of technical nature, it is equivalent to require that the $\sigma$-fields $\mathcal{F}_\infty$ and $\mathcal{G}_t$ are conditionally independent given $\mathcal{F}_t$ (see [1, Theorem 3.2]). Furthermore, because of the Cox construction (see [1, §2.3.1]), it is easy to construct random times $\tau$ satisfying the assumptions ($\mathcal{A}$) and ($\mathcal{H}$) (see [26, Remark 3.8] for details).

**Jacod’s absolute continuity condition.** Let $\eta$ denote the law of the random time $\tau$, that is, $\eta(A) := \mathbb{P}(\tau^{-1}(A))$, for every $A \in \mathcal{B}(\mathbb{R})$. We denote by $P\mathbb{P}(\omega, A)$ a regular version of the conditional distribution $\mathbb{P}[\tau \in A|\mathcal{F}_t], A \in \mathcal{B}(\mathbb{R})$. We make the following assumptions.

1. $\eta$ is a diffused probability measure. \hspace{1cm} (4.4)
2. $P\mathbb{P}(\omega, du)$ is absolutely continuous with respect to $\eta(du)$. \hspace{1cm} (4.5)

We stress that random times of this type can be constructed following the approach presented by Jeanblanc and Le Cam in [27, §5].

We say that a random time $\tau$ with (4.4) and (4.5) satisfies Jacod’s absolute continuity condition. According to [19, Corollary 2.2], (4.4) ensures property ($\mathcal{A}$). Thanks to [1, Proposition 4.17], (4.5) implies that there exists a nonnegative and $\mathcal{G}(\mathbb{F}) \otimes \mathcal{B}([0, +\infty])$-measurable function $(\omega, t, u) \mapsto p_t(\omega, u)$ such that $(p_t(u))_{t \geq 0}$ is an $\mathbb{F}$-martingale for every $u \in [0, +\infty]$ and

$$
\mathbb{E}[f(\tau)|\mathcal{F}_t] = \int_{\mathbb{R}} f(u)p_t(\omega, u)\eta(du), \hspace{1cm} t \geq 0
$$

for every bounded Borel function $f$.

We stress that Jacod’s absolute continuity does not imply, in general, that $\tau$ satisfies assumption ($\mathcal{H}$). This is only true if and only if $p_t(u) = p_u(u)$ \(\eta\)-a.s., if $u < t$ (see [1, Proposition 5.28]).

In the next result we give the form of the $\mathbb{G}$-dual predictable projection of $\mu^X$. To the best of our knowledge, this result is new and of independent interest.

**Theorem 4.8.** Let $(X, \mathbb{F})$ be an $\mathbb{F}$-quasi left continuous step process, where $\mathbb{F} := \mathbb{F}^X \vee \mathcal{G}$, and let $\tau$ be a random time satisfying (4.4) and (4.5). Then the $\mathbb{G}$-dual predictable projection $\nu^{G,X}$ of $\mu^X$ is given by

$$
\nu^{G,X}(\omega, d\tau, dx) = \left(1_{[0, \tau]}(\omega, t)\left(1 + \frac{W'(\omega, t, x)}{A_\tau}\right) + 1_{(\tau, +\infty]}(\omega, t)(1 + U(\omega, t, x))\right)\nu^{F,X}(\omega, d\tau, dx)
$$

(4.7)

where $W'$ is an $\mathbb{F}$-predictable function such that $A_\tau + W' \geq 0$ and $U$ is a $\mathbb{G}$-predictable function such that $1 + U \geq 0$ identically. In particular, $X$ is a $\mathbb{G}$-quasi left continuous step process.
The proof of Theorem 4.8 is technical and we give it in Appendix A. At this point we only observe that the function \( U \) in Theorem 4.8 can be constructed in the following way, as in the proof of [24, Proposition 3.14 and Theorem 4.1]. Let \( p_t(u) \) be the process appearing in (4.6). Since \( p_t(u) \) is an \( \mathbb{F} \)-martingale for every \( u \), we can represent it as \( p_t(u) = W^u + \mu^X - W^u + \nu^X \). The function \( (\omega, t, x, u) \mapsto W^u(\omega, t, x) \) can be chosen \( \mathcal{P}(\mathbb{F}) \otimes \mathcal{B}([0, +\infty]) \)-measurable and such that \( W^u(\omega, t, x) + p_{t-}(\omega, u) \geq 0 \) and \( p_{t-}(u) = 0 \Longrightarrow W^u(\omega, t, x) = 0 \). So using the convention \( \theta := 0 \), one can define \( V(\omega, t, x) := W^u(\omega, t, x) \) which satisfies \( 1 + V^u \geq 1 \) identically. The \( \mathbb{G} \)-predictable function \( U \) is then given by

\[
U(\omega, t) = V^T(\omega)(\omega, t, x) 1_{(\tau, +\infty)}(\omega, t) = \frac{W^T(\omega)(\omega, t, x)}{p_{t-}(\omega, \tau(\omega))} 1_{(\tau, +\infty)}(\omega, t).
\]

The following result is a direct application of (4.3), Theorem 4.8 and (4.1).

**Corollary 4.9.** Let \( \tau \) be a random time satisfying conditions (4.4) and (4.5). Then

\[
v^G, Z(\omega, dt, dx_1, dx_2) = \left( 1_{[0, \tau]}(\omega, t) \left( 1 + \frac{W(\omega, t, x_1)}{A_{t-}(\omega)} \right) + 1_{(\tau, +\infty)}(\omega, t)(1 + U(\omega, t, x_1)) \right) v^F, X(\omega, dt, dx_1) \delta_0(dx_2)
+ 1_{[0, \tau]}(\omega, t) \frac{1}{A_{t-}(\omega)} dH^{\mathbb{F}}(\omega) \delta_0(dx_1) \delta_1(dx_2),
\]

where \( W' \) is an \( \mathbb{F} \)-predictable function such that \( A_- + W' \geq 0 \) and \( U \) is a \( \mathbb{G} \)-predictable function such that \( 1 + U \geq 0 \) identically.

We remark that if \( \tau \) is a random time satisfying conditions (4.4) and (4.5), then Corollary 4.9 yields the \( \mathbb{G} \)-quasi left continuity of the step process \( Z \), whenever \( X \) is \( \mathbb{F} \)-quasi left continuous.

## 5 Applications to stochastic control theory

In this section we consider an optimization problem for marked point processes, in presence of an additional exogenous risk source that cannot be inferred from the information available in the market, represented by the filtration \( \mathbb{F} \). The additional risk source can be a shock event, as the death of the investor or the default of part of the market. Its occurrence time is modelled by a jump measure \( \mu^X \). The fundamental tool in order to apply the results in [10] to the present context is Theorem 3.1.

Let \( T > 0 \) be a fixed finite time horizon. Let \( X \) be an \( \mathbb{R}^d \)-valued step process, and set \( \mathbb{F} = (\mathcal{F}_t)_{t \geq 0} \) where \( \mathcal{F}_t := \mathcal{F}^X_t \). In particular, \( \mathcal{F}_0 \) is trivial, since \( X_0 = 0, X \) being a step process. Let \( \tau : \Omega \to (0, +\infty] \) be a random time and \( H \) be the default process associated with \( \tau \). We denote by \( \mathbb{G} \) the progressive enlargement of \( \mathbb{F} \) by \( \tau \), and by \( \mathbb{G}_T = (\mathcal{G}_t)_{t \in [0, T]} \) the restriction of \( \mathbb{G} \) to \([0, T]\). We also introduce the jump measure \( \mu^X \) of \( X \) and the corresponding \( \mathbb{F} \)-dual predictable projection \( v^F, X \).

Introduce the step process \( Z = (X, H) \) with jump measure \( \mu^Z \) and corresponding \( \mathbb{G} \)-dual predictable projection \( v^G, Z \). By Theorem 3.1 we know that \( Z \) satisfies the WRT with respect to \( \mathbb{G}_T \).

To ensure that the theory developed in [10] can be applied to the enlarged filtration \( \mathbb{G} \) we now state the following assumptions.

**Assumption 5.1.** The process \( X \) is \( \mathbb{F} \)-quasi-left continuous.

**Assumption 5.2.** \( \tau \) satisfies (\( \mathcal{A} \)) and (\( \mathcal{F} \)).
Assumption 5.3. \( \tau \) satisfies (4.4),(4.5) (hence \((\mathcal{A})\)).

In the following we will always assume Assumption 5.1 together with Assumption 5.2 or alternatively with Assumption 5.3.

Remark 5.4. (i) The \( \mathbb{F} \)-compensator \( v^{F,X} \) of \( \mu^X \) admits the decomposition
\[
v^{F,X}(dt, dx_1) = \phi^F_X(dx_1) dC^{F,X} t,
\]
where \( \phi^F_X \) is a transition probability from \( (\Omega \times [0,T], \mathcal{P}(\mathbb{F})) \) into \( (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \), and, by Assumption 5.1, \( C^{F,X} \in \mathcal{A}^+_{loc}(\mathbb{F}) \) is a continuous process. Analogously, the \( \mathbb{G} \)-compensator \( v^{G,X} \) of \( \mu^X \) admits the decomposition
\[
v^{G,X}(dt, dx_1) = \phi^G_X(dx_1) dC^{G,X} t,
\]
where \( \phi^G_X \) is a transition probability from \( (\Omega \times [0,T], \mathcal{P}(\mathbb{G})) \) into \( (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \), with \( C^{G,X} \in \mathcal{A}^+_{loc}(\mathbb{G}) \). If furthermore Assumption 5.1 together with Assumption 5.2 or with Assumption 5.3 hold, by Theorem 4.7 or by Theorem 4.8, \( C^{G,X} \) is also a continuous process.

(ii) From Assumption 5.1 and condition \((\mathcal{A})\) it follows that the \( \sigma \)-algebra \( \mathcal{G}_0 \) is trivial.

Remark 5.5. Because of \((\mathcal{A})\), by Theorem 4.1 we get that
\[
\mu^Z(\omega, dt, dx_1, dx_2) = \mu^X(\omega, dt, dx_1) \delta_0(dx_2) + dH_t(\omega) \delta_t(dx_2) \delta_0(dx_1)
\]
and the corresponding \( \mathbb{G} \)-dual predictable projection is given by
\[
v^{Z}(\omega, dt, dx_1, dx_2) = \delta_0(dx_2) \phi^G_X(\omega, dx_1) dC^{G,X} _t(\omega) + \delta_0(dx_1) \delta_t(dx_2) d\Lambda^G_t(\omega).
\]
By Theorem 4.7 (if Assumption 5.2 holds) or Theorem 4.8 (if Assumption 5.3 holds) we have that \( Z \) is a \( \mathbb{G} \)-quasi left continuous step process.

We notice that the random measure \( v^Z \) in (5.1) can be rewritten as
\[
v^{Z}(\omega, dt, dx_1, dx_2) = \phi_t(\omega, dx_1, dx_2) dC_t(\omega),
\]
where \( C \) is defined by
\[
C_t(\omega) := C^{G,X}_t(\omega) + \Lambda^G_t(\omega)
\]
and \( \phi \) is a transition probability from \( (\Omega \times [0,T], \mathcal{P}(\mathbb{G})) \) into \( (\mathbb{R}^{d+1}, \mathcal{B}(\mathbb{R}^{d+1})) \).

We also observe that, by (5.1), the following identities hold:
\[
\begin{align*}
1_{\{x_2=0\}} v^Z(\omega, dt, dx_1, dx_2) & = \delta_0(dx_2) \phi^G_X(\omega, dx_1) dC^{G,X} _t(\omega), \\
1_{\{x_2\neq 0\}} v^Z(\omega, dt, dx_1, dx_2) & = \delta_t(dx_2) \delta_0(dx_1) d\Lambda^G_t(\omega).
\end{align*}
\]
In particular, integrating previous expressions on \( \mathbb{R}^{d+1} \), we get
\[
\begin{align*}
d_1(\omega, t) dC_t(\omega) & = dC^{G,X}_t(\omega),
\end{align*}
\]
\[
\begin{align*}
d_2(\omega, t) dC_t(\omega) & = d\Lambda^G_t,
\end{align*}
\]
with
\[
\begin{align*}
d_1(\omega, t) & := \int_{\mathbb{R}^{d+1}} 1_{\{x_2=0\}} \phi_t(\omega, dx_1, dx_2), \\
d_2(\omega, t) & := \int_{\mathbb{R}^{d+1}} 1_{\{x_2\neq 0\}} \phi_t(\omega, dx_1, dx_2).
\end{align*}
\]
Remark 5.6. Under Assumptions 5.1-5.2 we have \( \phi^G_{t}(\omega, dx_1) = \phi^F_{t}(\omega, dx_1) \) while under Assumptions 5.1, 5.3 one gets
\[
\phi^G_{t}(\omega, dx_1) = \frac{\kappa(\omega, t, x_1)}{\int_{\mathbb{R}^d} \kappa(\omega, t, x_1) \phi^F_{t}(\omega, dx_1)} \phi^F_{t}(\omega, dx_1),
\]
\[
dC^G_{t} = \int_{\mathbb{R}^d} \kappa(\omega, t, x_1) \phi^F_{t}(\omega, dx_1) dC^F_{t}(\omega),
\]
where
\[
\kappa(\omega, t, x_1) := 1_{[0, \tau]}(\omega, t) \left(1 + \frac{W'(\omega, t, x_1)}{A_{t, \tau}(\omega)}\right) + 1_{(\tau, +\infty)}(\omega, t) \left(1 + U(\omega, t, x_1)\right)
\]
is the density appearing in Theorem 4.8.

5.1 The control problem on \([0, T]\)

The data specifying the optimal control problem are an action space \( U \), a running cost function \( l \), a terminal cost function \( g \), and another function \( r \) specifying the effect of the control process. They are assumed to satisfy the following conditions.

Assumption 5.7. \((U, \mathcal{U})\) is a measurable space.

Assumption 5.8. The functions \( r, l : \Omega \times [0, T] \times \mathbb{R}^d \times U \to \mathbb{R} \) are \( \mathcal{P}(\mathcal{G}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{U} \)-measurable and there exist constants \( M_r > 1, M_l > 0 \) such that, \( \mathbb{P} \)-a.s.,
\[
0 \leq r_t(x_1, u) \leq M_r, \quad |l_t(x_1, u)| \leq M_l, \quad t \in [0, T], x_1 \in \mathbb{R}^d, u \in U.
\]

Assumption 5.9. The function \( g : \Omega \times \mathbb{R}^d \to \mathbb{R} \) is \( \mathcal{G}_T \otimes \mathcal{B}(\mathbb{R}^d) \)-measurable, and there exists a constant \( \beta \) such that \( \beta > \sup |r - 1|^2 \), and
\[
\mathbb{E}[e^{\beta C_T}] < +\infty, \quad (5.9)
\]
\[
\mathbb{E}[|g(X_T)|^2 e^{\beta C_T}] < +\infty. \quad (5.10)
\]

To every admissible control process \( u \in \mathcal{U} \) we will associate the cost functional
\[
J(u) = \mathbb{E}_u \left[ \int_0^T l_t(X_t, u_t) dC^G_t + g(X_T) \right], \quad (5.11)
\]
where \( \mathbb{E}_u \) denotes the expectation under a probability measure \( \mathbb{P}_u \), absolutely continuous with respect to \( \mathbb{P} \), that will be specified below. The control problem will consists in minimizing \( J \) over all the admissible controls. Because of the structure of the control problem, it is evident that in general it cannot be solved in the filtration \( \mathcal{F} \). Therefore, we have to allow \( \mathcal{G} \)-predictable strategies: The set of admissible control processes, denoted \( \mathcal{U} \), consists of all \( U \)-valued and \( \mathcal{G} \)-predictable processes \( u(\cdot) = (u_t)_{t \in [0, T]} \).

Remark 5.10 (Interpretation). We now give an interpretation of the control problem associated with 5.11. One could consider for instance a defaultable terminal cost \( g : \Omega \times \mathbb{R}^d \to \mathbb{R} \) of the form
\[
g(\omega, x_1) = g_1(x_1) 1_{(\tau < \tau(\omega))} + g_2(x_1) 1_{(\tau \geq \tau(\omega))}.
\]
In this case, \( g_1 \) is the terminal cost if the default does not occur before the maturity \( T \) while \( g_2 \) is the cost to pay in case of default up to maturity. Similarly, one can allow a defaultable running cost \( l \). So, the agent acting in the enlarged market can minimize the cost functional also over defaultable terminal costs and running costs. This is not possible when acting in the reference market represented by \( \mathcal{F} \). One could also regard the minimization problem associated to 5.11 as the problem of an insider who disposes of private information about \( \tau \) and whose strategies are \( U \)-valued and \( \mathcal{G} \)-predictable.
Related to every control \( u \in \mathcal{C} \), we introduce the predictable random measure
\[
v^{Z,u}(\omega, dt, dx_1, dx_2) = r_t(\omega, X_t(\omega), u_t(\omega)) \phi_t^{G,X}(\omega, dx_1) dC_t^{G,X}(\omega) + \delta_t(dx_2) \delta_0(dx_1) d\Lambda_t^G.
\] (5.12)

Let us now set
\[
R_t(x_1, x_2, u) := r_t(x_1, u) 1_{\{x_2=0\}} + 1_{\{x_2\neq 0\}}, \quad x_1 \in \mathbb{R}^d, \ x_2 \in \{0, 1\}, \ u \in U.
\] (5.13)

By (5.2)- (5.6), we have \( v^{Z,u} = R_t(X_t, H_t, u_t) v^Z \).

We denote by \((T_n)_{n \geq 1}\) the sequence of jump times of \( Z \) and, for any \( u \in \mathcal{C} \), we consider the process
\[
L_t^u = \exp \left( \int_0^t \int_{\mathbb{R}^d} (1 - R_s(x_1, x_2, u_s)) v^Z(ds, dx_1, dx_2) \right) \prod_{n \geq 1: T_n \leq t} R_{T_n}(X_{T_n}, H_{T_n}, u_{T_n}),
\] (5.14)

with the convention that the last product equals 1 if there are no indices \( n \geq 1 \) satisfying \( T_n \leq t \). We notice that \( L_t^u \) is a Doléans-Dade stochastic exponential, solution to the equation
\[
L_t^u = 1 + \int_0^t \int_{\mathbb{R}^d} (R_s(x_1, x_2, u_s) - 1)(\mu^Z - v^Z)(ds, dx_1, dx_2).
\]

Hence, \( L_t^u \) is a \( \mathcal{G} \)-local martingale, for every \( u \in \mathcal{C} \). Furthermore, \( L_t^u \) is nonnegative (see [23, Proposition 4.3] for details), thus it is a nonnegative supermartingale.

Taking into account (5.13), we remark that
\[
\int_0^t \int_{\mathbb{R}^d} (1 - R_s(x_1, x_2, u_s)) v^Z(ds, dx_1, dx_2) = \int_0^t \int_{\mathbb{R}^d} (1 - r_s(x_1, u)) \delta_0(dx_2) \phi_s^{G,X}(dx_1) dC_s^{G,X}
\]
so that (5.14) reads
\[
L_t^u = \exp \left( \int_0^t \int_{\mathbb{R}^d} (1 - r_s(x_1, u_s)) \phi_s^{G,X}(dx_1) dC_s^{G,X} \right) \prod_{n \geq 1: T_n \leq t} (r_{T_n}(X_{T_n}, u_{T_n}) 1_{\{H_{T_n} = 0\}} + 1_{\{H_{T_n} \neq 0\}}).
\] (5.15)

The result below follows from [10, Lemma 4.2] with \( \gamma = 2 \).

**Lemma 5.11.** Assume that
\[
\mathbb{E}[e^{(3+M_t)\mathcal{C}_T}] < +\infty.
\] (5.16)

Then, for every \( u \in \mathcal{C} \), \( \sup_{t \in [0,T]} \mathbb{E}[|L_t^u|^2] < \infty \) and \( \mathbb{E}[L_T^u] = 1 \). In particular, \( L_t^u \) is a square integrable \( \mathcal{G} \)-martingale for every \( u \in \mathcal{C} \).

By Lemma 5.11 we can define an absolutely continuous probability measure \( \mathbb{P}_u \) by setting
\[
\mathbb{P}_u(d\omega) = L_T^u(\omega) \mathbb{P}(d\omega).
\]

It can then be proven (see e.g. [23, Theorem 4.5]) that the \( \mathcal{G} \)-compensator \( v^{Z,u} \) of \( \mu^Z \) under \( \mathbb{P}_u \) is given by (5.12). To every \( u \in \mathcal{C} \) we can then associate the cost functional
\[
\inf_{u \in \mathcal{C}} J(u) = \inf_{u \in \mathcal{C}} \mathbb{E}_u \left[ \int_0^T \ell(X_t, u_t) dC_t^{G,X} + g(X_T) \right].
\] (5.17)
Notice that \( J \) in (5.11) is finite for every admissible control. Moreover, \( g(X_T) \) is integrable under \( \mathbb{P}_u \), since
\[
\mathbb{E}_u[|g(X_T)|] = \mathbb{E}[|L_T^u g(X_T)|] \leq (\mathbb{E}[|L_T^u|^2])^{1/2} (\mathbb{E}[|g(X_T)|^2])^{1/2} < \infty \tag{5.18}
\]
where the latter inequality follows from (5.10) in Assumptions 5.9. Moreover, under Assumption 5.8 and recalling (5.9) and (5.4), we get
\[
\mathbb{E}_u \left[ \int_0^T l_i(x_t, u_t) dC_t^{G,X} \right] = \mathbb{E}_u \left[ \int_0^T l_i(x_t, u_t) d1(t) dC_t \right] \leq M_I \mathbb{E}_u[C_T] < \infty.
\]

**Remark 5.12** (The action of the insider). Because of (5.12), in the optimal control problem (5.17) the insider acts under \( \mathbb{P}^u \) by changing the \( G \)-compensator of \( X \), while the one of \( H \) (and hence of \( \tau \)) remains untouched.

**The associated BSDE.** We next proceed to the solution of the optimal control problem formulated above. A fundamental role is played by the following BSDE: \( \mathbb{P} \)-a.s., for all \( t \in [0, T] \),
\[
Y_t + \int_t^T \int_{\mathbb{R}^{d+1}} \Theta_s(x_s, x_s) (\mu^Z - v^Z)(ds, dx_1, dx_2) = g(X_T) + \int_t^T f(s, X_s, \Theta_s(\cdot)) dC_s^{G,X}. \tag{5.19}
\]
The generator of BSDE (5.19) is defined by means of the Hamiltonian function
\[
f(\omega, t, y_1, \theta(\cdot)) := \inf_{u \in U} \left\{ l_t(\omega, y_1, u) + \int_{\mathbb{R}^d} \theta(x_1, 0) \left( r_t(\omega, x_1, u) - 1 \right) \phi_1^{G,X}(\omega, dx_1) \right\}, \tag{5.20}
\]
for every \( \omega \in \Omega \), \( t \in [0, T] \), \( y_1 \in \mathbb{R}^d \), \( y_2 \in \mathbb{R}^1 \) and \( \theta \in \mathcal{L}^1(\mathbb{R}^{d+1}, \mathcal{B}(\mathbb{R}^{d+1}), \phi_1(\omega, dx_1, dx_2)) \).

For \( \beta > 0 \), we look for a solution \((Y, \Theta(\cdot))\) to (5.19) in the space \( L^2_{\text{Prog}}(\Omega \times [0, T], G) \times L^2(\mu^Z, G) \), where \( L^2_{\text{Prog}}(\Omega \times [0, T], G) \) denotes the set of real-valued \( G \)-progressively measurable processes \( Y \) such that
\[
\mathbb{E} \left[ \int_0^T e^{\beta C_t} |Y_t|^2 dC_t \right] < \infty,
\]
and \( L^2(\mu^Z, G) \) denotes the set of \( \mathcal{P}(G) \otimes \mathcal{B}(\mathbb{R}^{d+1}) \)-measurable functions \( \Theta \) such that
\[
\mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^{d+1}} e^{\beta C_t} |\Theta_t(x_1, x_2)|^2 \phi_1(dx_1, dx_2) dC_t \right] = E \left[ \int_0^T \int_{\mathbb{R}^d} e^{\beta G} |\Theta_t(x_1, 0)|^2 \phi^{G,X}(dx_1) dC_t^{G,X} \right] + \mathbb{E} \left[ \int_0^T e^{\beta G} |\Theta_t(0, 1)|^2 d\Lambda_t^G \right] < \infty.
\]
By \( L^{1,0}(\mu^Z, G) \) we denote the set of \( \mathcal{P}(G) \otimes \mathcal{B}(\mathbb{R}^{d+1}) \)-measurable functions \( \Theta \) such that
\[
\mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^{d+1}} |\Theta_t(x_1, x_2)| \phi_1(dx_1, dx_2) dC_t \right] = \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} |\Theta_t(x_1, 0)| \phi^{G,X}(dx_1) dC_t^{G,X} \right] + \mathbb{E} \left[ \int_0^T |\Theta_t(0, 1)| d\Lambda_t^G \right] < \infty.
\]
We note the inclusion \( L^2(\mu^Z) \subseteq L^{1,0}(\mu^Z) \) for all \( \beta > 0 \) holds (see [10, Remark 3.2-2.]). We will consider the following additional assumption:
Assumption 5.13. For every $\Theta \in L^{1,0}(M^Z, \mathbb{G})$ there exists a $\mathbb{G}$-predictable process (i.e., an admissible control) $\underline{u}^\Theta: \Omega \times [0, T] \to U$, such that, for $d_1(\omega, t) \, dc_1(\omega) \, P(d\omega)$-almost all $(\omega, t)$, we have

$$f(\omega, t, X_t(\omega), \Theta_t(\omega, \cdot)) = l_t(\omega, X_t(\omega), \underline{u}^\Theta(\omega, t))$$

$$+ \int_{\mathbb{R}^{d+1}} \Theta_t(\omega, x_1, 0) (r_t(\omega, x_1, \underline{u}^\Theta(\omega, t)) - 1) \phi_t^{G,X}(\omega, dx_1).$$

(5.21)

Remark 5.14. Assumption 5.13 can be verified in specific situations when it is possible to compute explicitly the function $\underline{u}^\Theta$. General conditions for its validity can also be formulated using appropriate measurable selection theorems, as the case $U$ compact metric space, and $l_t(\omega, x, \cdot), r_t(\omega, x, \cdot): U \to \mathbb{R}$ continuous functions, see [10, Proposition 4.8].

Thanks to the WRT for $Z$ with respect to $\mathbb{G}_T$ provided in Theorem 5.1 one can show the existence and the uniqueness of the solution of BSDE (5.19). The proof of the proposition below is postponed to Appendix B.

Proposition 5.15. Let Assumptions 5.1, 5.7, 5.8 and 5.13 hold true. Assume that Assumption 5.2 or 5.3 holds true. Set

$$L := \text{ess sup}_{\omega} \left( \sup \{ |r_t(x, u)| - 1 : t \in [0, T], x \in \mathbb{R}^d, u \in U \} \right)$$

(5.22)

and let Assumption 5.9 hold true with $\beta > L^2$. Then BSDE (5.19) admits a unique solution $(Y, \Theta(\cdot)) \in L^2_{prog}(\Omega \times [0, T], \mathbb{G}) \times L^2_{\mathbb{P}}(\mu^Z, \mathbb{G})$.

Solution to the optimal control problem. At this point we can give the main result of the section.

Theorem 5.16. Let Assumptions 5.1, 5.7, 5.8 and 5.13 hold true. Assume also that Assumption 5.9 holds true with $\beta > L^2$, with $L$ in (5.22), and that condition (5.16) holds true. Let Assumption 5.2 or 5.3 holds true, and let $(Y, \Theta) \in L^2_{prog}(\Omega \times [0, T], \mathbb{G}) \times L^2_{\mathbb{P}}(\mu^Z, \mathbb{G})$ denote the unique solution to BSDE (5.19), with corresponding admissible control $\underline{u}^\Theta \in \mathcal{C}$ satisfying (5.21). Then $\underline{u}^\Theta$ is optimal and $Y_0$ is the optimal cost, i.e.

$$Y_0 = J(\underline{u}^\Theta) = \inf_{u \in \mathcal{C}} J(u).$$

Proof. The proof consists in proving the so-called fundamental relation. We first recall that, by Lemma 5.11 for every $u \in \mathcal{C}$, we have $\sup_{t \in [0, T]} \mathbb{E}[|L^u_t|^{2}] < \infty$. Moreover, by (5.18), $\mathbb{E}_u[|g(X_T)|] < +\infty$. Let $u \in \mathcal{C}$ be fixed. Then, Hölder inequality and Assumption 5.8 yield $\Theta(\cdot) \in L^{1,0}(\mu^Z, \mathbb{G})$ under $\mathbb{P}_u$. Setting $t = 0$ and taking the expectation $\mathbb{E}_u[\cdot]$ in BSDE (5.19), we get

$$Y_0 + \mathbb{E}_u \left[ \int_0^T \int_{\mathbb{R}^d} \Theta(x_1, 0) (r_s(x_1, u) - 1) \phi_s^{G,X}(dx_1) \, dc_s^{G,X} \right] = \mathbb{E}_u[|g(X_T)|] + \mathbb{E}_u \left[ \int_0^T f(s, X_s, \Theta_s(\cdot)) \, dc_s^{G,X} \right].$$

Then, adding and subtracting $\mathbb{E}_u \left[ \int_0^T l_s(X_s, u) \, dc_s^{X,G} \right]$, we obtain

$$Y_0 = J(u)$$

$$+ \mathbb{E}_u \left[ \int_0^T \left[ f(s, X_s, \Theta_s(\cdot)) - l(s, X_s, u) - \int_{\mathbb{R}^d} \Theta_s(x_1, 0) (r_s(x_1, u) - 1) \phi_s^{G,X}(dx_1) \right] \, dc_s^{X,G} \right]$$

where we have also used the continuity of $C$. The conclusion follows from the definition of $f$ in (5.20), noticing that the term in the square brackets is non positive, and it equals 0 if $u(\cdot) = \underline{u}^\Theta(\cdot)$. □
5.2 The control problem on $[0, T \wedge \tau]$

We now consider the problem of an agent for whom the available information is exclusively given by $\mathbb{F}$ (that is, she pursues $\mathbb{F}$-predictable strategies) but, for some reasons, she has only access to the market up to the occurrence of the exogenous shock event, whose occurrence time is modelled by $\tau$. For example, the problem over $[0, T \wedge \tau]$ can be regarded as the optimization problem of an agent who minimizes running costs not up to the maturity $T > 0$, but only up to $T \wedge \tau$.

For simplicity we consider in this section only the case where Assumptions 5.1 - 5.2 are satisfied, so that, according to Remark 5.6, $d\mathbb{F}X = d\mathbb{G}X$ and $\phi^{\mathbb{F}X}(dx_1) = \phi^{\mathbb{G}X}(dx_1)$. In this context, this seems to be a natural assumption. Indeed, let Jacod’s assumption hold for $\tau$. Then, by [27, Corollary 3.1], $\tau$ satisfies ($\mathcal{H}$) if and only if $p(u)$ is constant after $u$, that is, $p_t(u) = p_t(0), t \geq u$, a.s. As observed in [19, p.1016], this is substantially equivalent to say that the “information contained in the reference filtration after the default time gives no new information on the conditional distribution of the default”. But, since we restrict our attention to $[0, T \wedge \tau]$, that is, before the default, we are neglecting all information after default.

We still consider a measurable space $(U, \mathcal{U})$ satisfying Assumption 5.7. The other data specifying the optimal control problem are a running cost function $l$, a terminal cost function $\bar{g}$, and a function $\bar{f}$, that are assumed to satisfy the following conditions.

Assumption 5.17. The functions $\bar{f}, l : \Omega \times [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}$ are $\mathcal{F}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{U}$-measurable and there exist constants $M_f > 1, M_l > 0$ such that, $\mathbb{P}$-a.s.,

$$0 \leq \bar{f}(x_1,u) \leq M_f, \quad |l(x_1,u)| \leq M_l, \quad t \in [0,T], x_1 \in \mathbb{R}^d, u \in U. \quad (5.23)$$

Assumption 5.18. The function $\bar{g} : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is $\mathcal{F}_{T \wedge \tau} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable, and there exists a constant $\beta$ such that $\beta > \sup |\bar{f} - 1|^2$, and

$$\mathbb{E}[e^{\beta C_T}] < +\infty, \quad (5.24)$$
$$\mathbb{E}[(\bar{g}(X_{T \wedge \tau}))^2 e^{\beta C_T}] < +\infty. \quad (5.25)$$

Let $\mathcal{C}$ be the set of admissible strategies for the optimization problem introduced in Section 5.1. For any $u \in \mathcal{C}$, we define $\hat{u} := 1_{[0,T \wedge \tau]}u$. Clearly $\hat{u} \in \mathcal{C}$ holds true. We define now the new set of admissible strategies as

$$\hat{\mathcal{C}} := \{u \in \mathcal{C} : 1_{[T \wedge \tau]}u = 0\} \subseteq \mathcal{C}. \quad (5.26)$$

Since $\mathbb{F}$-predictable and $\mathbb{G}$-predictable processes coincide on $[0, \tau]$ (see [28, Lemma 4.4, b])), the set $\hat{\mathcal{C}}$ given in (5.26) consists of strategies which are morally $\mathbb{F}$-predictable.

To every control $\hat{u} \in \hat{\mathcal{C}}$, we associate the predictable random measure $\nu^Z, \hat{u}(\omega, dt, dx_1, dx_2)$ of the same form as (5.12). We have $\nu^Z, \hat{u} = (\bar{f}(X_t, \hat{u}_t)d_1 + d_2)\nu^Z$, where $d_1$ and $d_2$ are the densities in (5.6) - (5.7).

For any $\hat{u} \in \hat{\mathcal{C}}$, under condition (5.16) with $M_f$ replaced by $M_f$, we can consider then the Doléans-Dade exponential martingale $L^\hat{u}$ in (5.15), and we can introduce the absolutely continuous probability measure $\mathbb{P}_{\hat{u}}$ defined as $\mathbb{P}_{\hat{u}}(d\omega) = L^\hat{u}_T(d\omega)\mathbb{P}(d\omega)$. We then consider a cost functional of the form

$$J(\hat{u}) = \mathbb{E}_{\hat{u}} \left[ \int_0^{T \wedge \tau} \bar{f}(X_t, \hat{u}_t)d\mathbb{F}X_t + \bar{g}(X_{T \wedge \tau}) \right], \quad \hat{u} \in \hat{\mathcal{C}}, \quad (5.27)$$

where $\mathbb{E}_{\hat{u}}$ denotes the expectation under $\mathbb{P}_{\hat{u}}$. The control problem is now

$$\inf_{\hat{u} \in \hat{\mathcal{C}}} J(\hat{u}) = \inf_{\hat{u} \in \hat{\mathcal{C}}} \mathbb{E}_{\hat{u}} \left[ \int_0^{T \wedge \tau} \bar{f}(X_t, \hat{u}_t)d\mathbb{F}X_t + \bar{g}(X_{T \wedge \tau}) \right]. \quad (5.28)$$
Remark 5.19. The control problem in (5.28) can be interpreted as the one of an agent who only controls $X$ using $\mathbb{F}$-predictable strategies but only up to the occurrence $\tau$ of an external risky event. Hence, because of the exogenous risk source, this control problem cannot be solved in $\mathbb{F}$.

The associated BSDE. The optimal control problem in (5.28) can be solved by means of the following BSDE: $\mathbb{P}$-a.s., for all $t \in [0, T]$,

$$
R_t + \int_{t \wedge \tau}^{T \wedge \tau} \sum_{s \in [t+1]} (\mu^Z(ds, dx_1, dx_2) - v^Z(ds, dx_1, dx_2)) = \bar{g}(X_{T \wedge \tau}) + \int_{t \wedge \tau}^{T \wedge \tau} \bar{f}(s, X_s, \Sigma_s(\cdot)) \, dC_s^X. 
$$

(5.29)

with

$$
\bar{f}(\omega, t, y_1, \theta(\cdot)) = \inf_{u \in U} \left\{ I_t(\omega, y_1, u) + \int_{\mathbb{R}^d} \theta(x_1, 0)(\bar{r}_t(\omega, x_1, u) - 1) \phi_t^X(\omega, dx_1) \right\}
$$

(5.30)

for every $\omega \in \Omega$, $t \in [0, T]$, $y_1 \in \mathbb{R}^d$, and $\theta \in \mathcal{L}^1(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \phi_\tau(\omega, dx_1, dx_2))$.

Assumption 5.20. For every $\Theta \in L^1(\mathbb{R}^d)$ there exists $\bar{u}_t^\Theta \in \mathcal{C}$ such that, for almost all $(\omega, t)$ with respect to the measure $d_1(\omega, t)\, dC_t(\omega) \mathbb{P}(d\omega)$,

$$
\bar{f}(\omega, t, X_{t-}(\omega), \Theta_t(\omega, \cdot)) = I_t(\omega, X_{t-}(\omega), \bar{u}_t^\Theta(\omega, t)) + \int_{\mathbb{R}^d} \Theta_t(\omega, x_1, 0)(\bar{r}_t(\omega, x_1, \bar{u}_t^\Theta(\omega, t)) - 1) \phi_t^X(\omega, dx_1).
$$

(5.31)

In order to prove existence and uniqueness for BSDE (5.29) we will use the following auxiliary equation: $\mathbb{P}$-a.s., for all $t \in [0, T]$,

$$
\bar{Y}_t + \int_t^T \bar{\Theta}_s(x_1, x_2) (\mu^Z - v^Z)(ds, dx_1, dx_2)
= \bar{g}(X_{T \wedge \tau}) + \int_t^T \bar{f}(s, X_s, \bar{\Theta}_s(\cdot)) \mathbb{1}_{[0, T \wedge \tau]}(s) \, dC_s^X. 
$$

(5.32)

The proofs of the two following results are postponed to Appendix B.

Proposition 5.21. Let Assumptions 5.1, 5.2, 5.7, 5.17 and 5.20 hold. Set

$$
\bar{L} := \text{ess sup}_{\omega} \left( \sup \{ |\bar{r}_t(x_1, u) - 1| : t \in [0, T], x_1 \in \mathbb{R}^d, u \in U \} \right),
$$

(5.33)

and let Assumption 5.18 hold true with $\beta > L^2$. Then BSDE (5.32) admits a unique solution $(\bar{Y}, \bar{\Theta}(\cdot)) \in L^2(\mathbb{P}; \mathbb{F}; [0, T], \mathcal{G}) \times L^2(\mathbb{P}; \mathbb{F}; [0, T], \mathcal{G})$.

Theorem 5.22. Let Assumptions 5.1, 5.2, 5.7, 5.17 and 5.20 hold true. Assume also that Assumptions 5.18 hold true with $\beta > L^2$, with $\bar{L}$ in (5.33), and let $(\bar{Y}, \bar{\Theta}(\cdot)) \in L^2(\mathbb{P}; \mathbb{F}; [0, T], \mathcal{G}) \times L^2(\mathbb{P}; \mathbb{F}; [0, T], \mathcal{G})$ denote the unique solution to BSDE (5.32). Then the BSDE (5.29) admits a unique solution given by $(R, \Sigma) = (\bar{Y}_{\wedge \tau}, \Theta_{\wedge \tau}) \mathbb{1}_{[0, T \wedge \tau]}$. In particular, $\bar{Y} = \bar{Y}_{\wedge \tau}, \mathbb{P}(d\omega)$-a.e. and $\Theta = \Theta_{\wedge \tau}, \mathbb{P}(d\omega)$-a.e.

Solution to the optimal control problem. We can then give following result, that is the analogous of Theorem 5.16 in the present framework.
Theorem 5.23. Let Assumptions 5.7, 5.2, 5.7, 5.7, and 5.20 hold true. Let also Assumption 5.18 hold true with $\beta > L^2$, with $\bar{L}$ in (5.33), and condition (5.16) hold true with $M_T$ in place of $M_r$. Let $(\bar{Y}, \bar{\Theta}) = (\bar{Y}_{\wedge \tau}, \bar{\Theta}1_{[0,T \wedge \tau]} \in L^2\bar{\beta}(\Omega \times [0, T], \mathbb{G}) \times L^2\bar{\beta}(\mu^Z, \mathbb{G})$ denote the unique solution to BSDE (5.29), with corresponding admissible control $\mathbf{u}^{\Theta} \in \mathbf{C}$ satisfying (5.31). Then $\mathbf{u}^{\Theta}$ is optimal and $\bar{Y}_0$ is the optimal cost, i.e.

$$\bar{Y}_0 = \bar{J}(\mathbf{u}^{\Theta}) = \inf_{\tilde{u} \in \mathbf{C}} \bar{J}(\tilde{u}).$$

Proof. The proof consists once again in proving the fundamental relation. By the analogous result of Lemma 5.11 for the present framework, for every $\tilde{u} \in \mathcal{G}$, we have $\sup_{t \in [0, T]} \mathbb{E}[|L_t^{\tilde{u}}|^2] < \infty$ and $\mathbb{E}[L_T^{\tilde{u}}] = 1$. In particular, $L^{\tilde{u}}$ is a square integrable $\mathbb{G}$-martingale for every $\tilde{u} \in \mathcal{G}$. Let $\tilde{u} \in \mathcal{G}$ be fixed. Proceeding as in (5.18), we see that $\mathbb{E}_{\tilde{u}}[|\tilde{g}(X_T)|] < +\infty$, while Hölder inequality and Assumption 5.8 yield $\Theta(\cdot) \in L^{1,0}(\mu^Z, \mathbb{G})$ under $\mathbb{P}_{\tilde{u}}$. Setting $t = 0$ and taking the expectation $\mathbb{E}_{\tilde{u}}[\cdot]$ in BSDE (5.29), we get

$$\bar{Y}_0 + \mathbb{E}_{\tilde{u}}\left[ \int_0^{T \wedge \tau} \tilde{f}(s, X_s, \bar{\Theta}_s(\cdot)) dC_s^{\tilde{u}X} \right] = \mathbb{E}_{\tilde{u}}[\tilde{g}(X_{T \wedge \tau})] + \mathbb{E}_{\tilde{u}}\left[ \int_0^{T \wedge \tau} \tilde{f}(s, X_s, \bar{\Theta}_s(\cdot)) dC_s^{\tilde{u}X} \right].$$

Then, adding and subtracting $\mathbb{E}_{\tilde{u}}\left[ \int_0^{T \wedge \tau} \tilde{I}(s, X_s, \tilde{u}) dC_s^{\tilde{u}X} \right]$ we obtain

$$\bar{Y}_0 = \bar{J}(\tilde{u}) + \mathbb{E}_{\tilde{u}}\left[ \int_0^{T \wedge \tau} \tilde{f}(s, X_s, \bar{\Theta}_s(\cdot)) - \tilde{I}(s, X_s, \tilde{u}) - \int_{\mathbb{R}^d} \bar{\Theta}_s(x_1, 0) (\tilde{r}_s(x_1, \tilde{u}) - 1) \phi_s^{\tilde{u}X}(dx_1) \right] dC_s^{\tilde{u}X} X \right]$$

(5.34)

where we have also used the continuity of $C$. The conclusion follows from the definition of $\tilde{f}$ in (5.31), noticing that the term in the square brackets in (5.34) is non positive, and it equals 0 if $\tilde{u}(\cdot) = \tilde{u}^{\Theta}(\cdot)$. \hfill \Box

5.3 Relationship between the two control problems

We end this section with some additional considerations on the two optimal control problems (5.17) and (5.28). We first notice that the functional cost in optimal control problem in (5.28) can be equivalently rewritten in the form

$$\bar{J}(\tilde{u}) = \mathbb{E}_{\tilde{u}}\left[ \int_0^T \tilde{I}(X_t, \tilde{u}_t) 1_{[0,T \wedge \tau])}(t) dC_t^{\tilde{u}X} + \tilde{g}(X_{T \wedge \tau}) \right], \quad \tilde{u} \in \mathcal{C}.$$

Clearly $\tilde{I}1_{[0,T \wedge \tau]}$ is $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{U}$-measurable and $\tilde{g}(X_{T \wedge \tau})$ is $\mathcal{G}_T$-measurable. Recalling Remark 5.6 we see that the control problem in (5.28) can be seen as one of the type studied in Section 5.1, where however a subclass of admissible controls is considered.

Let us now consider the enlarged optimal control problem obtained from (5.28) by taking the infimum over all the $\mathbb{G}$-predictable processes $u(\cdot)$:

$$\inf_{u \in \mathcal{C}} \bar{J}(u) = \mathbb{E}_u\left[ \int_0^T \tilde{I}(X_t, u_t) 1_{[0,T \wedge \tau])}(t) dC_t^{uX} + \tilde{g}(X_{T \wedge \tau}) \right].$$

(5.35)
According to Section 5.1, one can solve optimal control problem (5.35) by considering the following BSDE: P-a.s., for all \( t \in [0, T] \),
\[
Y_t + \int_t^T \int_{\mathbb{R}^{d+1}} \Theta_s(x_1, x_2) (\mu^Z - \nu^Z)(dx, dx_1, dx_2) = \bar{g}(X_{T \wedge \tau}) + \int_t^T f(s, X_s, \Theta_s(\cdot)) \, dC^F_s, \tag{5.36}
\]
where
\[
f(\omega, t, y, \theta(\cdot)) := \inf_{u \in U} \left\{ l_t(\omega, y, u) 1_{[0, T \wedge \tau]}(t) + \int_{\mathbb{R}^d} \theta(x_1, 0) (\bar{r}_t(\omega, x_1, u) - 1) \phi_t^{F,X}(\omega, dx_1) \right\}
\]
for every \( \omega \in \Omega, t \in [0, T], y \in \mathbb{R}^d \), and \( \theta \in \mathcal{L}^1(\mathbb{R}^{d+1}) \).

**Assumption 5.24.** For every \( \Theta \in L_2(\mathbb{R}^Z) \) there exists a \( \mathbb{G} \)-predictable process \( \bar{u}^{\Theta} : \Omega \times [0, T] \to U \), such that, for almost all \( \omega \) with respect to the measure \( d \mu_t(\omega) \int_{\mathbb{R}^d} \phi_t^{F,X}(\omega, dx_1) \),
\[
f(\omega, t, X_t(\cdot)(\omega), \Theta_t(\omega, \cdot)) = \bar{l}_t(\omega, X_t(\cdot)(\omega), \bar{u}^{\Theta}(\omega, t)) 1_{[0, T \wedge \tau]}(t)
+ \int_{\mathbb{R}^d} \Theta_t(\omega, x_1, 0) (\bar{r}_t(\omega, x_1, \bar{u}^{\Theta}(\omega, t)) - 1) \phi_t^{F,X}(\omega, dx_1). \tag{5.37}
\]

Under Assumption 5.24, the well-posedness of BSDE (5.36) directly follows from Proposition 5.15 noticing that \( l_t(\omega, x_1, u) := l_t(\omega, x_1, u) 1_{[0, T \wedge \tau]}(t) \) and \( g(\omega, x_1) := g(\omega) = \bar{g}(X_{T \wedge \tau}(\omega)) \) satisfy respectively Assumption 5.8 with \( M_t = M_t^f \) and Assumption 5.9 with \( \beta > L^2 \). Then Theorem 5.16 reads in the present framework as follows.

**Theorem 5.25.** Let Assumptions 5.1 5.2, 5.7, and 5.24 hold true. Assume also that Assumption 5.18 holds true with \( \beta > L^2 \), with \( L \) in 5.33, and that condition 5.16 holds true with \( M \) in place of \( M_t \). Let \( (Y, \Theta(\cdot)) \in L_2^\beta_{\mathbb{G}^q}((\Omega \times [0, T], \mathcal{F}), L_2^\beta(\mathbb{R}^Z, \mathcal{G})) \) denote the unique solution to BSDE (5.36), with corresponding admissible control \( \bar{u}^{\Theta} \in \mathcal{C} \) satisfying 5.37. Set
\[
\bar{J}(u) := \mathbb{E}_u \left[ \int_0^T \bar{l}_t(X_t, \bar{u}_t) 1_{[0, T \wedge \tau]}(t) \, dC^F_t + \bar{g}(X_{T \wedge \tau}) \right], \quad u \in \mathcal{C}.
\]

Then
\[
Y_0 = \bar{J}(\bar{u}^{\Theta}) = \inf_{u \in \mathcal{C}} \bar{J}(u).
\]

Let us now go back to optimal control problem (5.28). In order to solve it, we aim at finding an admissible process \( \tilde{u} \in \mathcal{C} \) such that
\[
\bar{J}(\tilde{u}) = \inf_{u \in \mathcal{C}} \bar{J}(u).
\]

We prove below that such an optimal control process exists and is provided by \( \tilde{u}^{\Theta} := \bar{u}^{\Theta} 1_{[0, T \wedge \tau]} \) with \( \tilde{u}^{\Theta} \in \mathcal{C} \) the process given in Theorem 5.25. Moreover, the value functions of the optimal control problems (5.28) and (5.35) coincide.

**Theorem 5.26.** Let Assumptions 5.1 5.2, 5.7, and 5.24 hold true. Assume also that Assumption 5.18 holds true with \( \beta > L^2 \), with \( L \) in 5.33, and that condition 5.16 holds true with \( M \) in place of \( M_t \). Let \( (Y, \Theta(\cdot)) \in L_2^\beta_{\mathbb{G}^q}((\Omega \times [0, T], \mathcal{F}), L_2^\beta(\mathbb{R}^Z, \mathcal{G})) \) denote the unique solution to BSDE (5.36), with corresponding admissible control \( \tilde{u}^{\Theta} \in \mathcal{C} \) satisfying 5.37. Set
\[
\tilde{u}^{\Theta} := \bar{u}^{\Theta} 1_{[0, T \wedge \tau]} \in \mathcal{C}.
\]

Then \( \tilde{u}^{\Theta} \) is an optimal control process for the control problem (5.28) and the value function is represented by the initial solution to BSDE (5.36), namely
\[
Y_0 = \bar{J}(\tilde{u}^{\Theta}) = \inf_{u \in \mathcal{C}} \bar{J}(\tilde{u}) = \inf_{u \in \mathcal{C}} \bar{J}(u). \tag{5.38}
\]
Proof. We divise the proof into two steps.

Step 1. The unique solution \((Y, \Theta(\cdot))\) to BSDE (5.36) coincides with the one to BSDE (5.32). In particular, \(Y = Y_{\wedge \tau}, \mathbb{P}(\omega)\)-a.e. and \(\Theta = \Theta_{1,[0,T\wedge\tau]} \cdot \phi_1(\omega, dx_1, dx_2) dC_t(\omega) \mathbb{P}(d\omega)\)-a.e.

Step 2. Identity (5.38) holds true.

Proof of Step 1. Let \((\tilde{Y}, \tilde{\Theta})\) be the unique solution to BSDE (5.32). It is enough to show the identity

\[
\tilde{f}(\omega, s, X_s(\omega), \tilde{\Theta}_s(\omega, \cdot)) \mathbb{1}_{[0,T\wedge\tau(\omega)]}(s) = f(\omega, s, X_s(\omega), \Theta_s(\omega, \cdot)), d_1(\omega, s) dC_s(\omega) \mathbb{P}(d\omega)\text{-a.e.} \quad (5.39)
\]

As a matter of fact, plugging (5.39) in BSDE (5.32), we would get BSDE (5.36). Then, by the uniqueness of the solution, \((Y, \Theta(\cdot))\) would coincide with the solution to BSDE (5.32) and, by Theorem 5.22, \(Y = Y_{\wedge \tau}, \mathbb{P}(d\omega)\)-a.e. and \(\Theta = \Theta_{1,[0,T\wedge\tau]} \cdot \phi_1(\omega, dx_1, dx_2) dC_t(\omega) \mathbb{P}(d\omega)\)-a.e.

Let us thus prove (5.39). By Theorem 5.22, \(\Theta = \Theta_{1,[0,T\wedge\tau]} \cdot \phi_1(\omega, dx_1, dx_2) dC_t(\omega) \mathbb{P}(d\omega)\)-a.e., so that

\[
f(\omega, s, X_s(\omega), \Theta_s(\omega, \cdot)) \mathbb{1}_{[0,T\wedge\tau(\omega)]}(s) = f(\omega, s, X_s(\omega), \Theta_s(\omega, \cdot)). \quad (5.40)
\]

On the other hand, recalling (5.30), we have that, \(dC_s(\omega) \mathbb{P}(d\omega)\)-almost surely on \(\Omega \times [0, T]\),

\[
\begin{align*}
&\tilde{f}(\omega, s, X_s(\omega), \Theta_s(\omega, \cdot)) \mathbb{1}_{[0,T\wedge\tau(\omega)]}(s) \\
&= \inf_{u \in U} \left\{ \tilde{l}_s(\omega, X_s(\omega), u) \mathbb{1}_{[0,T\wedge\tau(\omega)]}(s) + \int_{\mathbb{R}^d} \tilde{\Theta}_s(x, 0) \mathbb{1}_{[0,T\wedge\tau(\omega)]}(s) (\tilde{r}_s(\omega, x, u) - 1) \phi_{s, X}^{\tau, \omega}(\omega, dx_1) \right\} \\
&= \mathbb{1}_{[0,T\wedge\tau(\omega)]}(t) \inf_{u \in U} \left\{ \tilde{l}_s(\omega, X_s(\omega), u) + \int_{\mathbb{R}^d} \tilde{\Theta}_s(x, 0) (\tilde{r}_s(\omega, x, u) - 1) \phi_{s}^{\omega, X}(\omega, dx_1) \right\} \\
&= \tilde{f}(\omega, s, X_s(\omega), \tilde{\Theta}_s(\omega, \cdot)) \mathbb{1}_{[0,T\wedge\tau(\omega)]}(s). \quad (5.41)
\end{align*}
\]

Collecting (5.40) and (5.41), we get (5.39).

Proof of Step 2. Let \((Y, \Theta(\cdot)) \in L_{\text{Prog}}^2(\Omega \times [0, T], \mathbb{G}) \times L_{\text{ad}}^2(\mu, \mathbb{G})\) denote the unique solution to BSDE (5.36), with corresponding admissible control \(\mu^\Theta \in \mathcal{G}\) satisfying (5.37). Recalling the proof of Theorem 5.23, it is enough to show that, for almost all \((\omega, t)\) such that \(t \leq T \cap \tau(\omega)\) with respect to the measure \(d_1(\omega, t) dC_t(\omega) \mathbb{P}(d\omega)\),

\[
\tilde{f}(\omega, t, X_t(\omega), \Theta_t(\omega, \cdot)) = \tilde{l}_t(\omega, X_t(\omega), \mu^\Theta(\omega, t)) \\
+ \int_{\mathbb{R}^d} \Theta_t(\omega, x, 0) (\tilde{r}_t(\omega, x, \mu^\Theta(\omega, t)) - 1) \phi_{t, X}^{\omega, X}(\omega, dx_1). \quad (5.42)
\]

Identities (5.39) and (5.37), together with Step 1, yield

\[
\begin{align*}
&\tilde{f}(\omega, t, X_t(\omega), \Theta_t(\omega, \cdot)) \mathbb{1}_{[0,T\wedge\tau(\omega)]}(t) \\
&= f(\omega, t, X_t(\omega), \Theta_t(\omega, \cdot)) \\
&= \mathbb{1}_{[0,T\wedge\tau(\omega)]}(t) \tilde{l}_t(\omega, X_t(\omega), \mu^\Theta(\omega, t)) \\
&+ \int_{\mathbb{R}^d} \Theta_t(\omega, x, 0) (\tilde{r}_t(\omega, x, \mu^\Theta(\omega, t)) - 1) \phi_{t, X}^{\omega, X}(\omega, dx_1) \\
&= \mathbb{1}_{[0,T\wedge\tau(\omega)]}(t) \tilde{l}_t(\omega, X_t(\omega), \mu^\Theta(\omega, t)) \\
&+ \int_{\mathbb{R}^d} \Theta_t(\omega, x, 0) \mathbb{1}_{[0,T\wedge\tau(\omega)]}(t) (\tilde{r}_t(\omega, x, \mu^\Theta(\omega, t)) - 1) \phi_{t, X}^{\omega, X}(\omega, dx_1) \\
&= \mathbb{1}_{[0,T\wedge\tau(\omega)]}(t) \left\{ \tilde{l}_t(\omega, X_t(\omega), \mu^\Theta(\omega, t)) + \int_{\mathbb{R}^d} \Theta_t(\omega, x, 0) (\tilde{r}_t(\omega, x, \mu^\Theta(\omega, t)) - 1) \phi_{t, X}^{\omega, X}(\omega, dx_1) \right\}
\end{align*}
\]

that provides (5.42). \(\square\)
Appendix

A Proofs of technical results of Section 4

Proof of Theorem 4.7 Let us start by proving (i). We have

$$\mu^Z(\omega, dt, dx_1, dx_2) = \sum_{s>0} 1\{\Delta Z_s(\omega) \neq 0\} \delta(s, \Delta Z_s(\omega))(dt, dx_1, dx_2)$$

$$= \sum_{s>0} 1\{\Delta X_s(\omega) \neq 0, \Delta H_s(\omega) \neq 0\} \delta(s, (\Delta X_s(\omega), \Delta H_s(\omega)))(dt, dx_1, dx_2)$$

$$+ \sum_{s>0} 1\{\Delta X_s(\omega) \neq 0, \Delta H_s(\omega) = 0\} \delta(s, (\Delta X_s(\omega), 0)))(dt, dx_1, dx_2)$$

$$+ \sum_{s>0} 1\{\Delta X_s(\omega) = 0, \Delta H_s(\omega) \neq 0\} \delta(s, (0, \Delta H_s(\omega)))(dt, dx_1, dx_2).$$

Now we notice that, since by assumption $\Delta X \Delta H = 0$, we have $\{\Delta X \neq 0\} \cap \{\Delta H \neq 0\} = \emptyset$. Therefore, noticing that moreover $\{\Delta X \neq 0\} \subseteq \{\Delta H = 0\}$ and $\{\Delta H \neq 0\} \subseteq \{\Delta X = 0\}$, previous expression reads

$$\mu^Z(\omega, dt, dx_1, dx_2) = \sum_{s>0} 1\{\Delta X_s(\omega) \neq 0\} \delta(s, \Delta X_s(\omega))(dt, dx_1) \delta_0(dx_2)$$

$$+ \sum_{s>0} 1\{\Delta H_s(\omega) \neq 0\} \delta(s, \Delta H_s(\omega))(dt, dx_2) \delta_0(dx_1)$$

$$= \mu^X(\omega, dt, dx_1) \delta_0(dx_2) + \mu^H(\omega, dt, dx_2) \delta_0(dx_1).$$

Let us now prove (ii). Set

$$v^{G Z}(\omega, dt, dx_1, dx_2) := v^{G X}(\omega, dt, dx_1) \delta_0(dx_2) + v^{G H}(\omega, dt, dx_2) \delta_0(dx_1).$$

We have to prove that $v^G$ is the $G$-dual predictable projection of $\mu^Z$. To this end it is sufficient to show that, for every $G$-predictable function $W$ satisfying $W \geq 0$ and $W \ast \mu^Z \in \mathcal{A}_{\text{loc}}^+(G)$, the process $W \ast \mu^Z - W \ast v^G \in \mathcal{H}_{\text{loc}}^1(G)$. So, let us consider such a $G$-predictable function $W$. We then have

$$W \ast \mu^Z = \int_0^t \int_{R^d} W(\omega, s, x_1, 0) \mu^X(\omega, ds, dx_1) + \int_0^t \int_{R^d} W(\omega, s, 0, x_2) \mu^H(\omega, ds, dx_2)$$

and

$$W \ast v^G = \int_0^t \int_{R^d} W(\omega, s, x_1, 0) v^{G X}(\omega, ds, dx_1) + \int_0^t \int_{R^d} W(\omega, s, 0, x_2) v^{G H}(\omega, ds, dx_2).$$

Since $W \ast \mu^Z \in \mathcal{A}_{\text{loc}}^+(G)$ and $W \geq 0$, we get

$$\int_0^t \int_{R^d} W(s, x_1, 0) \mu^X(ds, dx_1), \int_0^t \int_{R^d} W(s, 0, x_2) \mu^H(ds, dx_2) \in \mathcal{A}_{\text{loc}}^+(G)$$

and therefore

$$\int_0^t \int_{R^d} W(s, x_1, 0) v^{G X}(ds, dx_1), \int_0^t \int_{R^d} W(s, 0, x_2) v^{G H}(ds, dx_2) \in \mathcal{A}_{\text{loc}}^+(G)$$

that yields $W \ast v^G \in \mathcal{A}_{\text{loc}}^+(G)$. It remains to show that $W \ast \mu^Z - W \ast v^G \in \mathcal{H}_{\text{loc}}^1(G)$. By definition of $v^G$, we have

$$W \ast \mu^Z - W \ast v^G = \int_0^t \int_{R^d} W(\omega, s, x_1, 0) \mu^X(\omega, ds, dx_1) - \int_0^t \int_{R^d} W(\omega, s, x_1, 0) v^{G X}(\omega, ds, dx_1)$$

$$+ \int_0^t \int_{R^d} W(\omega, s, 0, x_2) \mu^H(\omega, ds, dx_2) - \int_0^t \int_{R^d} W(\omega, s, 0, x_2) v^{G H}(\omega, ds, dx_2).$$
By linearity, it follows that $W \ast \mu^Z - W \ast v^G \in \mathcal{H}^1_{\text{loc}}(\mathbb{G})$, $v^{G,X}$ and $v^{G,H}$ being the $\mathbb{G}$-compensator of $\mu^X$ and $\mu^H$, respectively. Let now $W$ be an arbitrary $\mathbb{G}$-predictable function such that $|W| \ast \mu^Z \in \mathcal{A}^+_{\text{loc}}(\mathbb{G})$. Applying the previous step to $W^+$ and $W^-$ we get that $|W| \ast v^G \in \mathcal{A}^+_{\text{loc}}(\mathbb{G})$ and $W \ast \mu^Z - W \ast v^G \in \mathcal{H}^1_{\text{loc}}(\mathbb{G})$. The proof is complete. 

**Proof of Lemma 4.2** Let $W$ be a $\mathbb{G}$-predictable bounded function of the form $W(\omega, t, x) = f(x)\mathcal{I}_t(\omega)$, where $f$ is a bounded $\mathcal{B}(\mathbb{R}^d)$-measurable function and $\mathcal{I}$ is a bounded $\mathbb{G}$-predictable process. Then, by [28, Lemma 4.4 (b)] we have $W(\omega, t, x)1_{[0,t]}(\omega, t) = f(x)\mathcal{I}_t(\omega)1_{[0,\tau]}(\omega, t)$, where $\mathcal{I}$ is an $\mathbb{F}$-predictable bounded process. By a monotone class argument we get the statement for arbitrary bounded $\mathbb{G}$-predictable functions. Then, by approximation, we get the statement for arbitrary $\mathbb{G}$-predictable functions. The proof is complete. 

**Proof of Proposition 4.4** Because of the $\mathbb{F}$-quasi-left continuity of $X$, $X^{p,F}$ is an $\mathbb{F}$-adapted continuous process (see [21, Corollary 5.28 (3)]). Furthermore, by [1, Theorem 5.1], the process

$$X^\tau - (X^{p,F})^\tau - \int_0^{\tau \wedge t} \frac{1}{A_s} d(X - X^{p,F}, m)_s^{F}$$

is a $\mathbb{G}$-local martingale. This means that $(X^{p,F})^\tau + \int_0^{\tau \wedge t} \frac{1}{A_s} d(X - X^{p,F}, m)_s^{F}$ is the $\mathbb{G}$-dual predictable projection of $X$. We denote this process by $(X^{p,G})^\tau$. Since $X$ is $\mathbb{F}$-quasi-left continuous, $(X - X^{p,F}, m)_F$ is a continuous process. Indeed, be the property of the dual predictable projection, for every $\mathbb{F}$-predictable finite valued stopping time $\sigma$ we have

$$\Delta(X - X^{p,F}, m)_\sigma = \mathbb{E}[\Delta[X - X^{p,F}, m]_\sigma | \mathcal{F}_\sigma] = \mathbb{E}[\Delta X_{\sigma} \Delta m_\sigma | \mathcal{F}_\sigma] = 0.$$ 

Hence, by the predictable section theorem, $(X - X^{p,F}, m)_F$ being $\mathbb{F}$-predictable, $\Delta(X - X^{p,F}, m)_F = 0$ up to an evanescent set. Therefore, $(X^{p,F})^\tau$ being continuous, we deduce that $(X^{p,G})^\tau$ is continuous as well. Hence, by [21, Corollary 5.28 (3)], $X^\tau$ is $\mathbb{G}$-quasi-left continuous. The proof is complete. 

**Proof of Theorem 4.8** Let $(\omega, t, x) \mapsto W(\omega, t, x)$ be a $\mathbb{G}$-predictable bounded and nonnegative function. We then have that that $W \ast \mu^X$ is locally bounded, and hence belongs to $\mathcal{A}^+_{\text{loc}}(\mathbb{G})$, because of the estimate $W \ast \mu^X \leq cN^X$, where $c > 0$ is such that $W(\omega, t, x) \geq c$ and $N^X$ is the point process associated to $X$. Because of Lemma 4.2 there exists a bounded $\mathbb{F}$-predictable function $\overline{W}$ such that

$$W = \overline{W}1_{[0,\tau]} + W1_{(\tau,+\infty)}.$$ 

We now analyse separately the two integrals $\overline{W}1_{[0,\tau]} \ast \mu^X$ and $W1_{(\tau,+\infty)} \ast \mu^X$.

We start with $\overline{W}1_{[0,\tau]} \ast \mu^X$. We observe that $\overline{W} \ast \mu^X$ is locally bounded and hence it belongs to $\mathcal{A}^+_{\text{loc}}(\mathbb{F})$, $\overline{W}$ being an $\mathbb{F}$-predictable bounded function. Hence, $(\overline{W} \ast \mu^X)^{p,F}$ exists and is equal to $\overline{W} \ast v^{F,X}$. So, the process $\overline{W} \ast \mu^X - \overline{W} \ast v^{F,X}$ is an $\mathbb{F}$-local martingale and, by [1, Theorem 5.1],

$$\overline{W}1_{[0,\tau]} \ast \mu^X - \overline{W}1_{[0,\tau]} \ast v^{F,X} - \int_0^{\tau \wedge t} \frac{1}{A_s} d(\overline{W} \ast \mu^X - \overline{W} \ast v^{F,X}, m)_s^{F}$$

is a $\mathbb{G}$-local martingale. Since $m$ is an $\mathbb{F}$-martingale, we find an $\mathbb{F}$-predictable function $W^m$ such that $m = m_0 + W^m \ast \mu^X - \overline{W} \ast v^{F,X}$. Furthermore, $m$ is in the class $\text{BMO}$. So, the $\mathbb{F}$-predictable covariation $(\overline{W} \ast \mu^X - \overline{W} \ast v^{F,X}, m)_F$ is well defined and satisfies $(\overline{W} \ast \mu^X - \overline{W} \ast v^{F,X}, m)_F = \overline{W}W^m \ast v^{F,X}$, where we used [25, Theorem II.1.13] and that $v^{F,X}$ is non-atomic in $t$, $X$ being $\mathbb{F}$-quasi-left continuous (see [25, Proposition II.2.9]). So, by linearity we deduce that

$$\overline{W}1_{[0,\tau]} \ast \mu^X - \overline{W}\left(1 + \frac{W^m}{A_{-\tau}}\right)1_{[0,\tau]} \ast v^{F,X}.$$
is a $G$-local martingale. We also have that $\Delta m = W^m(\cdot, \cdot, \Delta X) 1_{\{\Delta X \neq 0\}} = \Delta A$. Therefore, we obtain

$$A_- + W^m(\cdot, \cdot, \Delta X) 1_{\{\Delta X \neq 0\}} = A_- + \Delta A = A \geq 0.$$ 

We now introduce the $\mathcal{B}(\mathbb{R})$-measurable set $D := \{(\omega, t, x) \in \Omega : A_{t\gamma}(\omega) + W^m(\omega, t, x) < 0\}$. We denote by $M^X_\mu$ the Doléans measure induced by $\mu^X$, that is, $M^X_\mu(B) = \mathbb{E}[1_B \mu^X]$, for every $B \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$. We then have $M^X_\mu(D) = 0$. Therefore, we can define the $\mathbb{F}$-predictable function $W'(\omega, t, x) := W^m(\omega, t, x) 1_{D^\tau}(\omega, t, x)$ which again satisfies $m = W' \ast \mu^X - W' \ast \mathcal{F}X$ and moreover $A_{t\gamma}(\omega) + W'(\omega, t, x) \geq 0$ identically. We now define the $G$-predictable measure

$$v^{G, \leq \tau}(\omega, dt, dx) = 1_{[0, \tau]} \left(1 + \frac{W'(\omega, t, x)}{A_{t\gamma}(\omega)}\right) v^{\mathcal{F}X}(\omega, dt, dx).$$

We then clearly have that $W[1_{[0, \tau]} \ast \mu^X - W \ast v^{G, \leq \tau}$ is a $G$-local martingale.

We now come to the integral $W_1(\tau, +\infty) \ast \mu^X$. To begin with, we introduce the filtration $G^\tau = (\mathcal{F}_t^\tau)_{t \geq 0}$ by $\mathcal{F}_t^\tau := \bigcap_{r > 0} \mathcal{F}_{r + t} \vee \sigma(\tau)$, that is, $G^\tau$ is the initial enlargement of $\mathcal{F}$ by $\tau$. It is clear that $G \subseteq G^\tau$ and that $G = G^\tau$ over the stochastic interval $(\tau, +\infty)$. Following the proof of [24, Proposition 3.14 and Theorem 4.1] we can show that there exists a $G^\tau$-predictable function $U$ such that $1 + U \geq 0$ and

$$\bar{v}(\omega, dt, dx) = (1 + U(\omega, t, x)) v^{\mathcal{F}X}(\omega, dt, dx)$$

is the $G^\tau$-dual predictable projection of $\mu^X$. In particular, $W_1(\tau, +\infty) \ast \mu^X$ being $G^\tau$-adapted and locally bounded, we deduce that $W_1(\tau, +\infty) \ast \mu^X - W_1(\tau, +\infty) \ast \bar{v}$ is a $G^\tau$-local martingale. We now observe that the function $1(\tau, +\infty)(1 + U)$ is indeed $G$-predictable, since $G$ and $G^\tau$ coincides over $(\tau, +\infty)$. This implies that the $G^\tau$-local martingale $W_1(\tau, +\infty) \ast \mu^X - W_1(\tau, +\infty) \ast \bar{v}$ is actually $G$-adapted. Furthermore, this is a martingale with bounded jumps, the process $W_1(\tau, +\infty) \ast \bar{v}$ being continuous and $W_1(\tau, +\infty)$ being bounded. We can therefore apply [25, Proposition 9.18 (iii) and the subsequent comment] to obtain that $W_1(\tau, +\infty) \ast \mu^X - W_1(\tau, +\infty) \ast \bar{v}$ is indeed a $G$-local martingale. We now define the $G$-predictable random measures $v^{G, > \tau}(\omega, dt, dx) := 1_{(\tau, +\infty)}(\omega, t) (1 + U(\omega, t, x)) v^{\mathcal{F}X}(\omega, dt, dx)$ and

$$v^{G}(\omega, dt, dx) = v^{G, \leq \tau}(\omega, dt, dx) + v^{G, > \tau}(\omega, dt, dx)$$

Putting together the two previous steps, we get that the process $W \ast \mu^X - W \ast v^G$ is a $G$-local martingale, for every bounded nonnegative $G$-predictable function $W$.

Let now $W$ be a nonnegative $G$-predictable function and define $W^n(\omega, t, x) := W(\omega, t, x) \wedge n$. Because of the previous step, the process $W^n \ast \mu^X - W^n \ast v^G$ is a $G$-local martingale. Let $(\sigma_n)_n$ be a localizing sequence. For every $n \geq 0$ we get $\mathbb{E}[W^n 1_{[0, \sigma_n]} \ast \mu^X] = \mathbb{E}[W^n 1_{[0, \sigma_n]} \ast v^G]$. Since $W^n 1_{[0, \sigma_n]}$ converges monotonically to $W$, by monotone convergence we obtain the identity $\mathbb{E}[W \ast \mu^X] = \mathbb{E}[W \ast v^G]$, for every nonnegative $G$-predictable function $W$. By [25, Theorem II.1.18 (i)] we deduce the identity $v^G = v^G X$. In particular, since $v^{\mathcal{F}X}(\{t\} \times \mathbb{R}^d) = 0$ for every $t$, $X$ being $\mathbb{F}$-quasi-left continuous, we deduce that $v^{G, X}(\{t\} \times \mathbb{R}^d) = 0$ for every $t$, meaning that $X$ is also $G$-quasi left continuous. The proof of the theorem is complete.

\[\Box\]

**B  Proofs of technical results of Section 5**

**Proof of Proposition 5.75** To show the result we apply [10, Theorem 3.4] to the present framework. Let us then check that all the hypotheses of [10, Theorem 3.4] are satisfied. More precisely, setting

$$F(\omega, t, X_\tau(\omega), \Theta_t(\omega)) = d_1(\omega, t) f(\omega, t, X_\tau(\omega), \Theta_t(\omega)),$$

B  Proofs of technical results of Section 5

**Proof of Proposition 5.75** To show the result we apply [10, Theorem 3.4] to the present framework. Let us then check that all the hypotheses of [10, Theorem 3.4] are satisfied. More precisely, setting

$$F(\omega, t, X_\tau(\omega), \Theta_t(\omega)) = d_1(\omega, t) f(\omega, t, X_\tau(\omega), \Theta_t(\omega)),$$
with $d_1$ given in (5.6), we have to verify that:

1. The terminal cost $g(X_T)$ is $\mathcal{G}_T$-measurable and there exists $\beta > 0$ such that $\mathbb{E}[e^{\beta C}] < \infty$ and $\mathbb{E}\left[\int_0^T e^{\beta C} |F(t,X_t,0)|^2 \,dC_t\right] < \infty$.

2. For every $\omega \in \Omega$, $t \in [0,T]$, $\Theta(\cdot) \in L^{2,\beta}(\mu^Z,\mathbb{G})$, the mapping $(t,\omega) \mapsto F(t,\omega,X_t(\omega),\Theta_t(\omega,\cdot))$ is $\mathbb{G}$-progressively measurable.

3. For every $\omega \in \Omega$, $t \in [0,T]$, and $\zeta, \zeta' \in \mathcal{L}^2(\mathbb{R}^{d+1},\mathcal{B}(\mathbb{R}^{d+1}),\phi(\omega,\omega_1,\omega_2))$,

$$|F(t,\omega,X_t(\omega),\zeta) - F(t,\omega,X_t(\omega),\zeta')| \leq L_F \left(\int_{\mathbb{R}^{d+1}} |\zeta(x_1,x_2) - \zeta'(x_1,x_2)| \phi(\omega,\omega_1,\omega_2)\right)^{1/2}, \quad (B.1)$$

where $L_F > 0$ is a constant.

Point (1) follows from Assumption 5.9 with $\beta > L^2$, being

$$\mathbb{E}\left[\int_0^T e^{\beta C} |F(t,X_t,0)|^2 \,dC_t\right] < \mathbb{E}\left[\int_0^T e^{\beta C} \inf_{u \in \mathcal{U}} l_t(X_t,\omega)|^2 \,dC_t\right] < M_1^2 \beta^{-1} \mathbb{E}[e^{\beta C}] < +\infty.$$ 

Concerning (3), by the boundedness conditions (5.8) in Assumption 5.8 it is easy to check that estimate (B.1) holds with $L_F = L$. Finally, the measurability requirements in (2) for the Hamiltonian $f$ hold thanks to Assumption 5.12. By (5.21), for every $\Theta(\cdot) \in L^{2,\beta}(\mu^Z,\mathbb{G})$ (recalling the inclusion $L^{2,\beta}(\mu^Z) \subseteq L^{1,0}(\mu^Z)$ for all $\beta > 0$), the map $(t,\omega) \mapsto f(\omega,t,X_t(\omega),\Theta_t(\omega,\cdot))$ is $\mathbb{G}$-progressively measurable; since by Remark 5.8 the process $C$ is continuous and $X$ has piecewise constant paths, the same holds (after modification on a set of measure zero) for $(\omega,t) \mapsto F(\omega,t,X_t(\omega),\Theta_t(\omega,\cdot))$.

**Proof of Proposition 5.21** It suffices to verify that all the hypotheses of [10, Theorem 3.4] are satisfied. By Assumption 5.18 with $\beta > L^2$, we have that $\bar{g}(X_{T\wedge \tau})$ is $\mathcal{G}_T$-measurable and $\mathbb{E}[e^{\beta C}] < \infty$. Moreover, by Assumption 5.20 for every $\omega \in \Omega$, $t \in [0,T]$, $\Theta(\cdot) \in L^{2,\beta}(\mu^Z)$, the mapping

$$\bar{F}(\omega,t,X_t(\omega),\Theta_t(\omega)) = d_1(\omega,t) \bar{f}(\omega,t,X_t(\omega),\Theta_t(\omega)) 1_{[0,T\wedge \tau]}(t)$$

is $\mathbb{G}$-progressively measurable. Finally, by Assumptions 5.17 and 5.18 $\mathbb{E}\left[\int_0^T e^{\beta C} |\bar{F}(t,X_t,0)|^2 \,dC_t\right]$ is finite for every $\beta > L^2$, and

$$|\bar{F}(t,\omega,X_t(\omega),\zeta) - \bar{F}(t,\omega,X_t(\omega),\zeta')| \leq L \left(\int_{\mathbb{R}^{d+1}} |\zeta(x_1,x_2) - \zeta'(x_1,x_2)| \phi(\omega,\omega_1,\omega_2)\right)^{1/2}.$$ 

**Proof of Theorem 5.22** Let $(\bar{Y},\bar{\Theta}(\cdot))$ be the unique solution to BSDE (5.32). We know that the process $\bar{Y}$ is defined as

$$\bar{Y}_t = \mathbb{E}\left[\bar{g}(X_{T\wedge \tau}) + \int_0^t \bar{f}(s,X_s,\bar{\Theta}_s(\cdot)) 1_{[0,T\wedge \tau]}(s) \,dC^\mathbb{G}_s \right] |\mathcal{G}_t] - \int_0^t \bar{f}(s,X_s,\bar{\Theta}_s(\cdot)) 1_{[0,T\wedge \tau]}(s) \,dC^\mathbb{G}_s$$

and moreover the following representation holds:

$$\bar{g}(X_{T\wedge \tau}) + \int_0^T \bar{f}(s,X_s,\bar{\Theta}_s(\cdot)) 1_{[0,T\wedge \tau]}(s) \,dC^\mathbb{G}_s$$

$$= \mathbb{E}\left[\bar{g}(X_{T\wedge \tau}) + \int_0^T \bar{f}(s,X_s,\bar{\Theta}_s(\cdot)) 1_{[0,T\wedge \tau]}(s) \,dC^\mathbb{G}_s\right] + \int_0^T \bar{\Theta}_s(x_1,x_2)(\mu^Z - \nu^Z)(ds,\omega_1,\omega_2).$$
where the last integral is a martingale. Consequently, for all $t \in [0, T]$,

$$\bar{Y}_t = \mathbb{E}\left[\bar{g}(X_{T \wedge \tau}) + \int_0^T \bar{f}(s, X_s, \bar{\Theta}_s(\cdot)) \, 1_{[0, T \wedge \tau]}(s) \, dC^X_s \right]$$

$$+ \int_0^t \int_{\mathbb{R}^{d+1}} \bar{\Theta}_s(x_1, x_2) (\mu^Z - v^Z) (ds, dx_1, dx_2) - \int_0^t \bar{f}(s, X_s, \bar{\Theta}_s(\cdot)) \, 1_{[0, T \wedge \tau]}(s) \, dC^X_s.$$

By Doob’s stopping theorem and (5.32), previous expression gives

$$\bar{Y}_{t \wedge \tau} = \mathbb{E}\left[\bar{g}(X_{T \wedge \tau}) + \int_0^{T \wedge \tau} \bar{f}(s, X_s, \bar{\Theta}_s(\cdot)) \, dC^X_s \right]$$

$$+ \int_0^{t \wedge \tau} \int_{\mathbb{R}^{d+1}} \bar{\Theta}_s(x_1, x_2) (\mu^Z - v^Z) (ds, dx_1, dx_2) - \int_0^{t \wedge \tau} \bar{f}(s, X_s, \bar{\Theta}_s(\cdot)) \, dC^X_s.$$

Now we notice that for $t = 0$ and $t = T \wedge \tau$ in (B.2) we obtain respectively

$$\bar{Y}_0 = \mathbb{E}\left[\bar{g}(X_{T \wedge \tau}) + \int_0^{T \wedge \tau} \bar{f}(s, X_s, \bar{\Theta}_s(\cdot)) \, dC^X_s \right], \quad \bar{Y}_{T \wedge \tau} = \bar{g}(X_{T \wedge \tau}).$$

Then,

$$\bar{g}(X_{T \wedge \tau}) + \int_0^{T \wedge \tau} \bar{f}(s, X_s, \bar{\Theta}_s(\cdot)) \, dC^X_s$$

$$= \bar{Y}_{T \wedge \tau} + \int_{t \wedge \tau}^{T \wedge \tau} \bar{\Theta}_s(x_1, x_2) (\mu^Z - v^Z) (ds, dx_1, dx_2)$$

or, equivalently,

$$\bar{g}(X_{T \wedge \tau}) + \int_{t \wedge \tau}^{T \wedge \tau} \bar{f}(s, X_s, \bar{\Theta}_s(\cdot)) \, 1_{[0, T \wedge \tau]}(s) \, dC^X_s$$

$$= \bar{Y}_{T \wedge \tau} + \int_{t \wedge \tau}^{T \wedge \tau} \int_{\mathbb{R}^{d+1}} 1_{[0, T \wedge \tau]}(s) \bar{\Theta}_s(x_1, x_2) (\mu^Z - v^Z) (ds, dx_1, dx_2).$$

that proves that $(\bar{Y}_{T \wedge \tau}, \bar{\Theta}(\cdot) \, 1_{[0, T \wedge \tau]})$ satisfies (5.29).

At this point, we notice that

$$\bar{Y}_{t \wedge \tau} + \int_{t \wedge \tau}^{T \wedge \tau} \int_{\mathbb{R}^{d+1}} \bar{\Theta}_s(x_1, x_2) \, 1_{[0, T \wedge \tau]}(s) (\mu^Z - v^Z) (ds, dx_1, dx_2)$$

$$= \bar{g}(X_{T \wedge \tau}) + \int_{t \wedge \tau}^{T \wedge \tau} \bar{f}(s, X_s, \bar{\Theta}_s(\cdot)) \, 1_{[0, T \wedge \tau]}(s) \, dC^X_s, \quad t \in [0, T],$$

Therefore,

$$\bar{Y}_{t \wedge \tau} + \int_{t}^{T} \int_{\mathbb{R}^{d+1}} \bar{\Theta}_s(x_1, x_2) \, 1_{[0, T \wedge \tau]}(s) (\mu^Z - v^Z) (ds, dx_1, dx_2)$$

$$= \bar{g}(X_{T \wedge \tau}) + \int_{t}^{T} \bar{f}(s, X_s, \bar{\Theta}_s(\cdot)) \, 1_{[0, T \wedge \tau]}(s) \, dC^X_s, \quad t \in [0, T],$$

namely the pair $(\bar{Y}_{T \wedge \tau}, \bar{\Theta} \, 1_{[0, T \wedge \tau]})$ solves BSDE (5.32) as well. The conclusion follows from the uniqueness of the solution to (5.32) in $L^2_{\text{Prog}}(\Omega \times [0, T], \mathcal{G}) \times L^2(\mu^Z, \mathcal{G}).$ \qed
References

[1] A. Aksamit and M. Jeanblanc. *Enlargement of filtration with finance in view*. SpringerBriefs in Quantitative Finance, Springer, 2017.

[2] A. Aksamit, T. Choulli and M. Jeanblanc. Thin times and random times’ decomposition. *Electron. J. Probab.*, 26 (2021), Paper No. 31, 22 pp, DOI 10.1214/20-EJP569.

[3] A. Aksamit, M. Jeanblanc and M. Rutkowski. Predictable representation property for progressive enlargements of a poisson filtration. *Stoch. Proc. Appl.*, 129(4):1229–1258, 2019.

[4] A. Aksamit, L. Li and M. Rutkowski. Generalized BSDEs with random time horizon in a progressively enlarged filtration, *Preprint 2021*, Arxiv version: http://arXiv.org/abs/2105.06654

[5] S. Ankirchner, C. Blanchet-Scalliet, and A. Eyraud-Loisel. Credit risk premia and quadratic BSDEs with a single jump, *Int. J. Theor. Appl. Finance*, 13(07):1103-1129, 2010.

[6] M. T. Barlow. Study of filtration expanded to include an honest time. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, 44:307–323, 1978.

[7] T. Bielecki and M. Rutkowski. *Credit risk: modelling, valuation and hedging*. Springer Finance, 2002.

[8] D. Duffie and K. Singleton. *Credit risk: Pricing, measurement and management*. Princeton University Press, 2003.

[9] A. Calzolari and B. Torti. Martingale representations in progressive enlargement by marked point processes. *Preprint 2021*. Arxiv version: https://arxiv.org/abs/2107.04087

[10] F. Confortola and M. Fuhrman. Backward stochastic differential equations and optimal control of marked point processes. *SIAM J. Control Optim.*, 51(5):3592–3623, 2013.

[11] F. Confortola, M. Fuhrman and J. Jacod. Backward stochastic differential equation driven by a marked point process: an elementary approach with an application to optimal control. *Ann. Appl. Probab.*, 26(3):1743–1773, 2016.

[12] S. Alsheyab and T. Choulli. Reflected backward stochastic differential equations under stopping with an arbitrary random time. *Preprint 2021*. Arxiv version: https://arXiv.org/abs/2107.11896

[13] M. H. A. Davis and P. Varaiya. On the multiplicity of an increasing family of sigma-fields. *Ann. Probab.*, 2:958–963, 1974.

[14] C. Dellacherie. *Capacités et processus stochastiques*. Springer, Berlin, 1972.

[15] P. Di Tella. On the weak representation property in progressively enlarged filtrations with an application in exponential utility maximization. *Stoch. Proc. Appl.*, 130, 760–784, 2020.

[16] P. Di Tella and H.-J. Engelbert. Martingale Representation in Progressively Enlarged Lévy Filtrations. *Stochastics*, published online June 2021.
[17] P. Di Tella and M. Jeanblanc. Martingale representation in the enlargement of the filtration generated by a point process, *Stoch. Proc. Appl.*, 131:103–121, 2021.

[18] D. Duffie. Stochastic equilibria: Existence, spanning number, and the no expected financial gain from ‘Trade’hypothesis. *Econometrica*, 1161–1183, 1986.

[19] N. El Karoui, M. Jeanblanc and Y. Jiao. What happens after a default: The conditional density approach, *Stoch. Proc. Appl.*, 120(7):1011–1032, 2010.

[20] W. Fleming and M. Soner. *Controlled Markov Processes and Viscosity Solutions*. Springer Verlag, 2006.

[21] S. He, J. Wang and J. Yan. *Semimartingale theory and stochastic calculus*. CRC, 1992.

[22] J. Jacod. *Calcul stochastique et problèmes de martingales*. Springer, 1979.

[23] J. Jacod. Multivariate point processes: predictable projection, Radon-Nikodym derivatives, representation of martingales. *Z. Wahrs. Verw. Gebiete*, 31:235–253, 1974/75.

[24] J. Jacod. Grossissement initial, hypothèse \( (\mathcal{H}) \) et théorème de Girsanov, p. 15-35 in *Grossissement de filtrations: examples et applications*, Lecture Notes in Mathematics, vol. 118, Sém. de Calcul Stochastique 1982/83, Université Paris VI, Springer, 1985. Springer, 1979.

[25] J. Jacod and A. Shiryaev. *Limit theorems for stochastic processes*. Springer, 2003.

[26] M. Jeanblanc, T. Mastrolia, D. Possamaï and A. Réveillac. Utility maximization with random horizon: A BSDE approach. *International Journal of Theoretical and Applied Finance*, 18(07):1550045, 2015.

[27] M. Jeanblanc and Y. Le Cam. Progressive enlargement of filtrations with initial times. *Stoch. Proc. Appl.*, 119:2523–2543, 2009.

[28] T. Jeulin. *Semi-martingales et grossissement d’une filtration*. Springer, 1980.

[29] Y. Jiao and H. Pham. Optimal investment under counterparty risk: a default-density approach, *Finance Stoch.*, 2009.

[30] I. Kharroubi, T. Lim and A. Nguyen. Mean-variance hedging on uncertain time horizon in a market with a jump, *Appl. Math. Optim.*, 68(3):413–444, 2013.

[31] I. Kharroubi and T. Lim. Mean-variance hedging on uncertain time horizon in a market with a jump, *J. Theoret. Probab.*, 68(3):683–724, 2012.

[32] S. Kusuoka. A remark on default risk models. *Adv. Math. Econ.*, 69–82. Springer, 1999.

[33] T. Lim and M.C. Quenez. Exponential utility maximization in incomplete markets with defaults, *Electron. J. Probab.*, 16(53):143-1464, 2011.

[34] R. Mansuy and M. Yor. *Random Times and Enlargements of Filtrations in Brownian Setting.* Lect. Notes in Mathematics, Springer 2006.

[35] H. Pham. Stochastic control under progressive enlargement of filtrations and applications to multiple defaults risk management. *Stochastic Processes and their Applications* 120, 1795-1820, 2010.
[36] P. Schönbucher. *Credit derivatives pricing models*. Wiley Finance, 2003.

[37] J. Yong and X.Y. Zhou. *Stochastic Controls: Hamiltonian Systems and HJB Equations*. Springer Verlag, 1999.