Quotient Equations and Integrals of Motion for Vector Massless Field

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Abstract

In this article a group-theoretical aspect of the method of dimensional reduction is presented. Then, on the base of symmetry analysis of an anisotropic space geometrical description of dimensional reduction of equations for vector massless field is given. Formula for calculating components of the energy-momentum tensor from the variables of the field factor-equations is derived.
The covariant equations for massless vector field in the absence of sources are of the form

\[ \nabla_\mu F^{\mu\nu} = 0, \quad \nabla_\mu (\ast F)^{\mu\nu} = 0, \]  

(1)

where \( F^{\mu\nu} \) is tensor of vector field, \( (\ast F)^{\mu\nu} \) is conjugate magnitude defined by the equation

\[ (\ast F)^{\alpha\beta} = \frac{1}{2\sqrt{-g}} [\alpha\beta\gamma\eta] F_{\gamma\eta}; \]

\( [\alpha\beta\gamma\eta] \) is completely antisymmetric tensor with \([0123]=1\).

Because metric tensor \( (g_{\mu\nu}) \) does not depend on space coordinates, the theory under consideration is invariant under space translations, and consequently, the space transformation of Fourier-components of tensor is rendered possible:

\[ F^{\mu\nu}(t, x) = \int d^3 k \exp(i k, x) f^{\mu\nu}(t, k), \]
\[ (\ast F)^{\mu\nu}(t, x) = \int d^3 k \exp(ikx)(\ast f)^{\mu\nu}(t, k). \]

(2)

Substituting (2) in (1), one comes to the equation for \( f^{\mu\nu}(t, k) \):

\[ -i k_j f^{0j} = 0, \]
\[ - \left[ \frac{1}{\sqrt{-g}} \partial_0 \left( \sqrt{-g} f^{0j} \right) + i k_j f^{ji} \right] = 0. \]

(3)

Because \( f^{ji} = -[jr i]A_i^2(t, f)^{m0}/\sqrt{-g} \), the previous equation takes the form

\[ \frac{d}{dt} \left( \sqrt{-g} f^{0j} \right) = -i[jml]k_i A_i^2(t)(\ast f)^{m0}. \]

(4)

Similarly, for the conjugate magnitude we have

\[ k_j (\ast f)^{j0} = 0, \]
\[ \frac{d}{dt} \left( \sqrt{-g} (\ast f)^{j0} \right) = i[jml]k_i A_i^2(t) f^{0m}. \]

(5)
To simplify the set of equations (3)–(5), we introduce the new variables
\[ S_j^+ = \sqrt{-g} [f^0 j \pm i(f^j)^0]. \] (6)
This enables us to separate the equations (3)–(5) for \( S_j^+ \) and \( S_j^- \):
\[ k_j S_j^+ = 0, \quad \frac{d}{dt} S_j^+ = \mp \sqrt{-g} [j ml] k_i A_m^2 S_m^+. \] (7)
The further simplification consists of the introduction spherical coordinates in momentum space according to the relation
\[ (k_1, k_2, k_3) = k (\sin(\delta) \cos(\xi), \sin(\delta) \sin(\xi), \cos(\delta)). \] (8)
The equation (7) can be automatically satisfied going from (6) to the magnitudes
\[ S_\delta^+ = \cos(\delta) \cos(\xi) S_1^+ + \cos(\delta) \sin(\xi) S_2^+ - \sin(\delta) S_3^+, \]
\[ S_\xi^+ = -\sin(\xi) S_1^+ + \cos(\xi) S_2^+. \] (9)
\( S_\delta^+ \) is defined from the equation (7), rewritten as
\[ \sin(\delta) \left( \cos(\xi) S_1^+ + \sin(\xi) S_2^+ \right) + \cos(\delta) S_3^+ = 0, \]
and definition of \( S_\delta^+ \) it is possible to derive the relations
\[ S_\delta^+ = \frac{1}{\cos(\delta)} \left( \cos(\xi) S_1^+ + \sin(\xi) S_2^+ \right), \quad S_3^+ = -S_\delta^+ \sin(\delta). \]
Considering the previous relations and the definition \( S_\xi^+ \) as a set of equations, we obtain for \( S_1^\pm \) and \( S_2^\pm \) the following coupling relations
\[ S_1^\pm = S_\delta^+ \cos(\xi) \cos(\delta) - S_\xi^+ \sin(\xi), \]
\[ S_2^\pm = S_\delta^+ \sin(\xi) \cos(\delta) + S_\xi^+ \cos(\xi), \]
\[ S_3^\pm = -S_\delta^+ \sin(\delta). \] (10)
To derive the equations for \( S_\delta^+ \) and \( S_\xi^+ \), let us differentiate the magnitude (9) by \( t \):
\[ \dot{S}_\pm^\pm = \cos(\delta) \cos(\xi) \dot{S}_1^\pm + \cos(\delta) \sin(\xi) \dot{S}_2^\pm - \sin(\delta) \dot{S}_3^\pm, \]
\[ \dot{S}_\pm^\mp = -\sin(\xi) \dot{S}_1^\mp + \cos(\xi) \dot{S}_2^\mp. \] (11)

Further we use the equations (7)
\[ \pm \dot{S}_1^\pm = -\frac{k}{\sqrt{-g}} \left( \cos(\delta) A_2^2 S_2^\pm - \sin(\delta) \sin(\xi) A_3^2 S_3^\pm \right), \]
\[ \pm \dot{S}_2^\pm = +\frac{k}{\sqrt{-g}} \left( \cos(\delta) A_1^2 S_1^\pm - \sin(\delta) \cos(\xi) A_3^2 S_3^\pm \right), \]
\[ \pm \dot{S}_3^\pm = \frac{k}{\sqrt{-g}} \left( \sin(\delta) \sin(\xi) A_1^2 S_1^\pm - \sin(\delta) \cos(\xi) A_2^2 S_2^\pm \right), \]
and then (10) with the result that for \( S_\delta^\pm \) and \( S_\xi^\pm \) we obtain the equations
\[ \pm \dot{S}_\delta^\pm = -ka S_\delta^\pm - kb S_\xi^\pm, \]
\[ \pm \dot{S}_\xi^\pm = +kc S_\delta^\pm + ka S_\xi^\pm. \] (12)

The parameters \( a, b \) and \( c \) can be presented as follows
\[ a = \frac{\cos(\delta) \cos(\xi) \sin(\xi)}{\sqrt{-g}} \left( A_2^2(t) - A_1^2(t) \right), \]
\[ b = \frac{1}{\sqrt{-g}} \left( A_1^2(t) \cos^2(\xi) + A_2^2(t) \sin^2(\xi) \right), \]
\[ c = \frac{1}{\sqrt{-g}} \left( A_1^2(t) \cos^2(\delta) \cos^2(\xi) + A_2^2(t) \cos^2(\delta) \sin^2(\xi) + A_3^2(t) \sin^2(\delta) \right). \] (13)

To continue the analysis, it is necessary to discuss the dependence of basis vectors of space on time. The metric \( g_{\mu\nu} \) defines the natural covariant "unit" vector \( e_\mu \) with covariant components \( e_\mu^\alpha = \delta_\mu^\alpha \) and the natural contravariant vector \( e^\mu \) with contravariant components \( e^\mu_\alpha = \delta^\mu_\alpha \) with \( (e_\mu e_\nu) = g_{\mu\nu} \). The absence of the off-diagonal terms from metric means that the basis consisting of the vectors \( e_\mu \) is orthogonal, but the length of space basis vectors varies with time according to the relation
\[ (e_i e_i) = A_i^2(t). \]
Hence it follows that the components of various tensors calculated in this basis do not carry complete information on the corresponding fields. The difficulty mentioned above can be easily avoided by introduction of tetrad basis vectors according to the relations \( e_{(0)} = e_0, \ e_{(i)} = e_i / A_i(t) \). It turns out that, doing so, we obtain \( (e_{(\mu)} e_{(\nu)}) = \eta_{\mu\nu} \).

In the case under consideration, because \( g_{\mu\nu} \) does not depend on space coordinates, the tetrad basis can be introduced for all points of space at any given instant of time.

The vectors of electric and magnetic field strength can be written in tetrad basis as follows

\[
E(t, x) = \int d^3 k \exp(i k x) \sum_{i=1}^{3} A_i(t) f^0i(t, k) e_{(i)},
\]

\[
H(t, x) = \int d^3 k \exp(i k x) \sum_{i=1}^{3} A_i(t)(\ast f)^0i(t, k) e_{(i)}. \tag{14}
\]

The spectral components \( E \) and \( H \), calculated in tetrad basis, are orthogonal to the direction, varying with time, which is determined by vector \( k_i = k_i / A_i(t) \). In this connection it is convenient to introduce the unit vectors varying with time

\[
e_k = \sin(\theta) \cos(\varphi)e_{(1)} + \sin(\theta) \sin(\varphi)e_{(2)} + \cos(\theta)e_{(3)},
\]

\[
e_\theta = \cos(\theta) \cos(\varphi)e_{(1)} + \cos(\theta) \sin(\varphi)e_{(2)} - \sin(\theta)e_{(3)},
\]

\[
e_\varphi = -\sin(\varphi)e_{(1)} + \cos(\varphi)e_{(2)}, \tag{15}
\]

where the angles \( \theta \) and \( \varphi \) are defined by the relation

\[
\left( \sin(\theta) \cos(\varphi), \sin(\theta) \sin(\varphi), \cos(\theta) \right) = \mu^{-1} \left( \frac{\sin(\delta) \cos(\xi)}{A_1}, \frac{\sin(\delta) \sin(\xi)}{A_2}, \frac{\cos(\delta)}{A_3} \right), \tag{16}
\]

\[
\mu = \left( \frac{\sin^2(\delta) \cos^2(\xi)}{A_1^2} + \frac{\sin^2(\delta) \sin^2(\xi)}{A_2^2} + \frac{\cos^2(\delta)}{A_3^2} \right)^{1/2}.
\]

Let us transform the expression (14), using the functions \( S_\delta^\pm, S_\xi^\pm \), introduced earlier. We shall do this for magnetic field \( H \). Using (16), let us present \( H \) in the form
\[ H(t, x) = \int d^3k \exp(i k x) \sum_{l=1}^{3} \frac{1}{2i \sqrt{-g}} \left( S^+_l - S^-_l \right) A_l(t) e_{(l)} = \]
\[ = \frac{1}{2i \sqrt{-g}} \sum_{r=\pm 1} \int d^3k \exp(i k x) r \sum_{l=1}^{3} S^r_l A_l(t) e_{(l)}. \]

In the last relation we went from the notation \( S^\pm_l \) to the notation \( S^r_l \), where \( r = \pm 1 \). Let us consider the internal sum. The change \( S^\pm_l \to S^r \delta \) and \( S^r \xi \) according to (10) brings it to the form

\[ \sum_{l=1}^{3} S^r_l A_l(t) e_{(l)} = S^r_\delta \left[ -\sin(\xi) A_1 e_{(1)} + \cos(\xi) A_2 e_{(2)} \right] + \]
\[ + S^r_\xi \left[ \cos(\xi) \cos(\delta) A_1 e_{(1)} + \sin(\xi) \cos(\delta) A_2 e_{(2)} - \sin(\delta) A_3 e_{(3)} \right]. \]

Let us go from the angles \( \xi, \delta \) to the angles \( \varphi, \theta \). To this end we shall use the following transition formulas, which can be derived from (16):

\[ \sin(\varphi) = \frac{A_1}{(-g)^{1/4} b^{1/2}} \sin(\xi), \quad \cos(\varphi) = \frac{A_2}{(-g)^{1/4} b^{1/2}} \cos(\xi), \]
\[ \sin(\theta) = \frac{A_3 b^{1/2}}{\mu(-g)^{1/4}} \sin(\delta), \quad \cos(\theta) = \frac{1}{\mu^{-1} A_3} \cos(\delta). \]

(17)

As a result, taking into account the definition \( e_\varphi \), we obtain

\[ \sum_{l=1}^{3} S^r_\delta A_l(t) e_{(l)} = \left(\frac{(-g)^{1/4}}{b^{1/2}} \right) \left[ S^r_\delta \mu \left( \frac{A_1 A_3}{A_2} b \cos(\varphi) \cos(\theta) e_{(1)} + \right. \right. \]
\[ + \left. \left. \frac{A_2 A_3}{A_1} b \sin(\varphi) \cos(\theta) e_{(2)} - \sin(\theta) e_{(3)} \right) + S^r_\xi b e_{(\varphi)} \right]. \]

It is easy to verify that the following equation is satisfied

\[ \left( \frac{A_1 A_3}{A_2} b - 1 \right) \cos(\varphi) \cos(\theta) e_{(1)} + \]
\[ + \left( \frac{A_2 A_3}{A_1} b - 1 \right) \sin(\varphi) \cos(\theta) e_{(2)} = \frac{a}{\mu} e_\varphi. \]
This can be done, using the relation
\[
\left( \frac{\sqrt{-g} b}{A_1^2} - 1 \right) = \frac{A_1^2 - A_2^2}{A_2^2} \sin^2(\xi), \quad \left( \frac{\sqrt{-g} b}{A_2^2} - 1 \right) = \frac{A_2^2 - A_1^2}{A_1^2} \cos^2(\xi)
\]
and the connection formulas (17).

Then the next result is
\[
\sum_{l=1}^{3} S_r^l \mathcal{A}_l(t)e_l = \frac{(-g)^{1/4}}{b^{1/2}} \left[ S_r^\mu (\mu e_\theta + a e_\varphi) + S_r^\xi b e_\varphi \right] = \frac{(-g)^{1/4}}{b^{1/2}} \left[ S_r^\mu \mu e_\theta - \frac{r}{k} S_r^\xi e_\varphi \right].
\]

The magnetic field strength can be written now as follows:
\[
\mathbf{H} = \frac{1}{2i(-g)^{1/4}} \sum_{r=\pm1} \int d^3 k \exp(ikx) r b^{-1/2} \left[ S_r^\mu \mu e_\theta - \frac{r}{k} S_r^\xi e_\varphi \right]. \tag{18}
\]

For the vector of electric field strength the calculations are carried out in analogous way and give the following result:
\[
\mathbf{E} = \frac{4}{2i(-g)^{1/4}} \sum_{r=\pm1} \int d^3 k \exp(ikx) r b^{-1/2} \left[ S_r^\mu \mu e_\theta - \frac{r}{k} S_r^\xi e_\varphi \right]. \tag{19}
\]

The formulas (18) and (19) bring out that the electric and magnetic fields are completely described by the function \( S_r^\delta \). That is why it is convenient to go from (12) to the equation of the second order for \( S_r^\delta \). To this end let us differentiate the first equation (12) by \( t \), substitute the second equation and eliminate the rest terms, using again the first equation. As a result, we obtain
\[
\frac{d}{dt} \dot{S}_\delta^r = \frac{\dot{b}}{b} S_\delta^r + S_\delta^r \left( -kr \dot{a} + k^2 (a^2 - bc) + \frac{\dot{b}}{b} ka \right).
\]

Using the fact that \( a^2 - bc = -\mu^2 \), we come to
\[
\ddot{S}_\delta^r - \frac{\dot{b}}{b} S_\delta^r + \left( k^2 \mu^2 + k \Lambda^r \right) S_\delta^r = 0, \tag{20}
\]
with $\Lambda^r = r(\dot{a} - \frac{\dot{t}}{b}a)$.

Let us find out the restrictions on the functions $S^r_\delta$, following from the real valuedness of the fields $E$ and $H$. It stems from the conditions

$$F^{\alpha\beta}(t, x) = \ast F^{\alpha\beta}(t, x), \quad (\ast F)^{\alpha\beta}(t, x) = (\ast \ast F)^{\alpha\beta}(t, x)$$

immediately the requirement that $f^{\alpha\beta}(t, k) = \ast f^{\alpha\beta}(t, k), \quad (\ast f)^{\alpha\beta}(t, k) = (\ast \ast f)^{\alpha\beta}(t, -k)$. Then it follows from definition (6) straightforwardly that

$$S^+_{\gamma}(t, k) = \ast S^-_{\gamma}(t, -k), \quad S^-_{\gamma}(t, k) = \ast S^+_{\gamma}(t, -k). \quad (21)$$

Because the reflection in the momentum space corresponds to the change

$$\delta' = \pi - \delta, \quad \xi' = \pi + \xi, \quad \cos(\delta') = -\cos(\delta), \quad \sin(\delta') = \sin(\delta),$$

$$\cos(\xi') = -\cos(\xi), \quad \sin(\xi') = -\sin(\xi),$$

we obtain that

$$S^r_{\delta}(t, k) = \ast S^r_{-\delta}(t, -k)$$

(vectors $e_\theta$ and $e_\varphi$ under the change $k$ for $-k$ transform themselves in the same way).

Let us present $S^r_{\delta}$ as

$$S^r_{\delta} = C^r_1(k) Y^r(t, k) + C^r_2(k) \ast Y^r(t, k). \quad (22)$$

This representation corresponds to the fact that $S^r_{\delta}$ satisfies the equation of the second order (20).

The magnitudes $C^r_{1,2}(k)$ are coefficients which depend on momentum $k$, and the functions $Y^r(t, k)$ carry the information about the dependence on time $t$.

Let us find out the properties of $C^r_{1,2}(k)$ and $Y^r(t, k)$. First of all it is necessary to establish the connection between $C^r_1$ and $C^r_2$. To this end we shall consider the isotropic case $A_1 = A_2 = A_3 = R$. It is easy to verify that $a = 0, b = 1/R, \mu = 1/R$. Substituting these values in (20), we bring this equation to the form

$$\ddot{S}^r_{\delta} + \frac{\dot{R}}{R} \dot{S}^r_{\delta} + \frac{k^2}{R^2} S^r_{\delta} = 0.$$
It is easy to check up that the general solution of the latter is of the form

\[ S_\delta^r = C_1^r(k) \exp(-i \eta k) + C_2^r(k) \exp(i \eta k) \] with \( \eta = \int dt/R(t) \).

Then the equation (21) can be rewritten as

\[ \star S_\delta^+ (-k) = C_1^+(k) \exp(i \eta k) + C_2^+(k) \exp(i \eta k), \]

\[ \star S_\delta^- (k) = C_1^-(k) \exp(-i \eta k) + C_2^-(k) \exp(i \eta k), \]

hence, \( C_1^+ (-k) = C_2^- (k) \), \( C_2^+ (-k) = C_1^- (k) \).

In general case of anisotropic metric the equation (21) is of the form

\[ \star S_\delta^+ (-k) \mathcal{Y}^+ (-k) + \star C_1^+(k) \mathcal{Y}^+ \mathcal{Y}^-(k) = \]

\[ = C_1^-(k) \mathcal{Y}^-(k) + C_2^- (k) \mathcal{Y}^- \]

or

\[ C_2^- (k) \mathcal{Y}^+ (-k) + C_1^- (k) \mathcal{Y}^+ \mathcal{Y}^- (k) = C_1^- (k) \mathcal{Y}^- (k) + C_2^- (k) \mathcal{Y}^- \]

The last equation can be easily satisfied by putting \( \mathcal{Y}^+(–k) = \mathcal{Y}^-(k) \). In the isotropic case \( \mathcal{Y}^\pm (k) = C_0 \exp(i k \eta) \), i.e. the last equation is satisfied trivially.

Let us substitute (22) in (19). It is easy to verify that

\[ \sum_{r=\pm 1} \left[ S_\delta^r \mu e_\theta - \frac{r}{k} \dot{S}_\delta^r e_\varphi \right] = \sum_{r=\pm 1} \left[ C_1^r(k) U_0^r (t, k) + C_2^r(k) \mathcal{U}_0^r (t, k) \right] \]

with

\[ U_0^r (t, k) = \mu \mathcal{Y}^r e_\theta - \frac{r}{k} \dot{\mathcal{Y}}^r e_\varphi. \]

Then \( E \) equires the form

\[ E = \frac{1}{2(-g)^{1/4}} \sum_{r=\pm 1} \int d^3 k \exp(i k x) b^{-1/2} [C_1^r(k) U_0^r (t, k) + \]

\[ + C_2^r(k) \mathcal{U}_0^r (t, k) \].
The expression for $E$ contains two integrals. Let us change in the second integral the integration variable $k$ for $-k$ and consider the relation

$$\sum_{r=\pm 1} C^r_2(-k) U^r_0(t, -k) =$$

$$= \sum_{r=\pm 1} C^r_2(-k) \left( \mu(-k) - \mathbf{Y}^r(t, -k) e_\theta(-k) - \frac{r}{k} \hat{\mathbf{y}}^r(t, -k) e_\varphi(-k) \right) =$$

$$= \sum_{r=\pm 1} C^r_1(-k) \left( \mu(k) \hat{\mathbf{y}}^r(t, k) e_\theta(k) + \frac{r}{k} \hat{\mathbf{y}}^r(t, k) e_\varphi(k) \right) =$$

$$= \sum_{r=\pm 1} \hat{C}^r_1(k) \hat{U}^r_0(t, k).$$

After doing so, the expression for $E$ acquires the form

$$E = \frac{1}{2(-g)^{1/4}} \sum_{r=\pm 1} \int d^3 k b^{1/2} [C^r_1(k) U^r_0(t, k) \exp(ikx) +$$

$$+ \hat{C}^r_1(k) \hat{U}^r_0(t, k) \exp(-ikx)]. \quad (23)$$

Let us present $E$ as expansion in respect to complete system of eigenfunctions of the corresponding equation with amplitudes $C^r_1(k)$ (the index 1 shall be further omitted).

The vectors $U^r_0(t, k)$, involved in the expansion (23), in the case of anisotropic space are analogues of the vectors determining a certain state of circular polarization (in quantum language – determining the states of particles with a certain projection of spin on the direction of movement).

The expansion for magnetic field $H$ similar to (23) is of the form

$$H = \frac{1}{2i(-g)^{1/4}} \sum_{r=\pm 1} \int d^3 k b^{1/2} [C^r_1(k) U^r_0(t, k) \exp(ikx) +$$

$$+ \hat{C}^r_1(k) \hat{U}^r_0(t, k) \exp(-ikx)]. \quad (24)$$

Further we shall modify (23) and (24), introducing in the definition of the vectors $U^r_0(t, k)$ all functions of time:

$$U^r(t, k) = \frac{1}{(-g)^{1/4} b^{1/2}} \left( \mu \mathbf{Y}^r e_\theta - \frac{r}{k} \hat{\mathbf{y}}^r e_\varphi \right).$$
The ultimate expansion for electric and magnetic fields (EMF) is as follows:

\[
E = \frac{1}{2} \sum_{r=\pm 1} \int d^3k \left[ C^r(k) U^r(t, k) \exp(ikx) + C^+(r) \right] \left( k \right) U^r(t, k) \exp(-ikx) \],
\]

\[
H = \frac{1}{2} \sum_{r=\pm 1} \int d^3k \left[ C^r(k) U^r(t, k) \exp(ikx) + C^+(r) \right] \left( k \right) U^r(t, k) \exp(-ikx) \].
\] (25)

The main characteristic of quantum effects of EMF in anisotropic space are vacuum averages of the operators of symmetrized energy-momentum tensor (EMT) of EMF

\[
T_{\alpha\mu}(t) = \langle 0 | N_t \hat{T}_{\alpha\mu}(t, x) | 0 \rangle,
\]

\[
N_t \hat{T}_{\alpha\mu} = \hat{T}_{\alpha\mu} - \langle 0 | \hat{T}_{\alpha\mu} | 0 \rangle,
\]

\[
\hat{T}_{\alpha\mu} = -\frac{g^\alpha{}^\mu}{2} \left\{ \hat{F}_{\mu\rho}, \hat{F}^\rho_{\alpha\nu} \right\} + \frac{1}{8} g_{\alpha\mu} \left\{ \hat{F}^\beta{}^\nu, \hat{F}^\nu_{\beta\nu} \right\},
\]

\[
\left\{ \hat{A}, \hat{B} \right\} = \hat{A} \hat{B} - \hat{B} \hat{A}.
\]

In particular, \(T_0^0(t) = \frac{1}{V} \langle 0 | N_t \hat{H}(t) | 0 \rangle\). Similarly to the fact that the solution of Maxwell equations can be presented as superposition of plane waves with wave vector \(k\), EMT can be expanded as follows:

\[
T_{\mu}^\nu = \int d\xi d\delta \sin(\delta) \int dK_0(t, k, \delta, \xi) \tilde{T}_{\mu}^\nu(t, k, \delta, \xi),
\] (26)

where \(K_0(t, k, \delta, \xi) = k\mu(t, \delta, \xi)\) is photon physical frequency.

The nonzero spectral components of (26) are

\[
\tilde{T}_0^0 = \frac{k^3}{V} \sum_{r=1}^{2} 2S^r,
\]

\[
\tilde{T}_1^1 = \frac{k^3}{V} \sum_{r=1}^{2} \left[ -\cos(2\varphi)X^r + \sin(2\varphi)Y^r - \frac{\sin^2(\theta)}{2} (2S^r + U^r) \right],
\]
\[ \tilde{T}_2^2 = \frac{k^3}{V} \sum_{r=1}^{2} \left[ \cos(2\varphi)X^r - \sin(2\varphi)Y^r - \frac{\sin^2(\theta)}{2}(2S^r + U^r) \right], \]
\[ \tilde{T}_3^3 = \frac{k^3}{V} \sum_{r=1}^{2} \left[ -\cos(\theta)2S^r + \sin^2(\theta)U^r \right], \]
\[ \tilde{T}_{12}^{12} = \frac{k^3}{A_1A_2V} \sum_{r=1}^{2} \left[ \sin(2\varphi)X^r + \cos(2\varphi)Y^r \right], \]
\[ \tilde{T}_{13}^{13} = \frac{k^3}{A_1A_3V} \sum_{r=1}^{2} \left[ \cos(\varphi)\frac{\sin(2\theta)}{2}(2S^r + U^r) + \sin(\varphi)\text{tg}(\theta)Y^r \right], \]
\[ \tilde{T}_{23}^{23} = \frac{k^3}{A_2A_3V} \sum_{r=1}^{2} \left[ \sin(\varphi)\frac{\sin(2\theta)}{2}(2S^r + U^r) - \cos(\varphi)\text{tg}(\theta)Y^r \right]. \] (27)

The components \( T^{0i} \) vanish due to spaceous homogeneity of metric. The trace \( \tilde{T}_\mu^\mu \) of (27) vanishes, because the field is massless. The condition of conservativeness \( \nabla_\mu T^{\mu0} = 0 \) is satisfied, which in the chosen metric is substantial only for diagonal components of EMT.

In contrast to the case of isotropic space, where EMT is diagonal for all types of fields, EMT in anisotropic space is nondiagonal in respect to space indices. This is related with nonorthogonality of polarization vectors \( U^{(1)}(t, k) \) and \( U^{(-1)}(t, k) \):

\[
\left( U^{(+1)}(t, k), U^{*(-1)} (t, k) \right) = b^{-1}(t) \left( -g(t) \right)^{-1/2} \left( \mu^2(t, k) \right) Y^{(+1)}(t, k) \times \nonumber \\
\times Y^{(-1)} (t, k) - \frac{1}{k^2} \dot{Y}^{(+1)}(t, k) \dot{Y}^{(-1)} (t, k). \nonumber
\]

(At \( t = t_0 \), according to initial conditions, \( (U^{(+1)}, U^{*(-1)}) = 0 \).) For this reason it is impossible to rotate the orthogonal axes, in which a given components of EMT is evaluated under fixed \( k \) to make them coinciding with the direction of vectors \( U^r(t, k) \).

In formulas (27) for spectral components of EMT the following notation is introduced:

\[ X^r = 2S^r - (2S^r + U^r) \cos^2(\theta) + \frac{1}{2}, \quad Y^r = -r \cos(\theta)V^r. \]

The introducing of the functions \( S^r, U^r \) and \( V^r \) is carried out in two steps. First we introduce the functions \( \Phi^r \) and \( \Psi^r \) according to formulas...
\[ Y^r = \left( \frac{kb}{\mu} \right)^{1/2} (\Phi^r e_+ + \Psi^r e_-), \quad \dot{Y}^r = iK_0 \left( \frac{kb}{\mu} \right)^{1/2} (\Phi^r e_+ - \Psi^r e_-). \]

The functions \( \Phi^r \) and \( \Psi^r \) satisfy the set of equations

\[
\begin{align*}
\dot{\Psi}^r &= \Phi^r e^2_+ \left( \frac{W}{2} + ir \frac{\overline{W}}{2} \right) - ir \frac{\overline{W}}{2} \Psi^r, \\
\dot{\Phi}^r &= \Psi^r e^2_- \left( \frac{W}{2} + ir \frac{\overline{W}}{2} \right) + ir \frac{\overline{W}}{2} \Phi^r,
\end{align*}
\]

(28)

where \( W \) and \( \overline{W} \) are defined by the relations

\[ W = \dot{\mu}/\mu - \dot{b}/b, \quad \overline{W} = \overline{\Delta H}/\mu, \quad H_i \equiv A_i^{-1} \dot{A}_1, \quad \overline{\Delta H} = H_3 - H_1. \]

Let us define the functions \( S^r, U^r \) and \( V^r \) as

\[ S^r = |\Psi^r|^2, \quad U^r = 2\text{Re} \left( \Psi^r \Phi^r_+ e^2_+ \right), \quad V^r = 2\text{Im} \left( \Psi^r \Phi^r_+ e^2_+ \right). \]

Differentiating these expressions by \( t \) and taking into account (28), we obtain a set of equations, satisfied by the functions \( S^r, U^r, V^r \):

\[
\begin{align*}
\dot{S}^r &= \frac{W}{2} U^r + r \frac{\overline{W}}{2} V^r, \\
\dot{U}^r &= W (2S^r + 1) - (r \overline{W} + 2K_0) V^r, \\
\dot{V}^r &= r \overline{W} (2S^r + 1) + (r \overline{W} + 2K_0) U^r
\end{align*}
\]

(29)

with initial conditions \( S^r = U^r = V^r = 0 \) at \( t = t_0 \). Moreover, there is connection formula

\[ (U^r)^2 + (V^r)^2 = 4S^r (S^r + 1). \]

Comparing the results for all types of fields, we get the conclusion that the set of equations for functions \( S^r, U^r, V^r \) can be written in general form as
\[ \frac{d}{dt} \begin{pmatrix} S^r \\ U^r \\ V^r \end{pmatrix} = \begin{pmatrix} 0 & \frac{W}{2} & -\frac{rW}{2} \\ 2W & 0 & -(rW + 2K_0) \\ 2rW & rW + 2K_0 & 0 \end{pmatrix} \begin{pmatrix} S^r \\ U^r \\ V^r \end{pmatrix} + \begin{pmatrix} 0 \\ W \\ rW \end{pmatrix}. \]  

(30)

As it can be seen, the structure of the set of equations (30) is similar to that of the set of equations derived for scalar field and massive spinor field [1,2].

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