Two-tone distortion products in hardware-efficient cochlea model based on asynchronous cellular automaton oscillator

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Abstract In this paper, we present a cochlea model whose nonlinear dynamics are described by an asynchronous cellular automaton. Our proposed model is demonstrated to enable the reproduction of two-tone distortion products. The proposed model is implemented on a field-programmable gate array (FPGA). It is subsequently demonstrated that our proposed model can be implemented using fewer hardware resources than a conventional cochlea model, which is a Hopf-type cochlea model, implemented on a digital signal processor performing a numerical integration.

Keywords: cochlea, field-programmable gate array, Hopf oscillator, asynchronous cellular automaton

Classification: Integrated circuits

1. Introduction

A laser Doppler velocimeter performed on live mammalian cochleas has observed that they have an active amplifier and essential nonlinearities [1, 2, 3]. For example, Fig. 1 shows typical nonlinear response characteristics of a mammalian cochlea measured in an anesthetized chinchilla [3]. In this figure, the basilar membrane (BM) is stimulated by two-tone frequencies \( f_1 \) and \( f_2 \) \((f_2 > f_1)\), and it perceives tones with a combination of two frequencies, namely \( n_1 f_1 \pm n_2 f_2 \) \((n_1, n_2 \in \mathbb{Z})\), where the characteristic frequency (CF) in the stimulated site of the BM corresponds to \( 2f_1 - f_2 \). Such responses to tones with frequencies not included in two-tone stimuli are called two-tone distortion products. Eguiluz et al. were the first to introduce a cochlea model based on Hopf oscillators to understand the nonlinearities of hearing [4]. Subsequently, Stoop et al. succeeded in demonstrating that the model enables the reproduction of a number of nonlinear hearing phenomena such as nonlinear compression, two-tone suppression, combination tone generation, and first (second) pitch shift [5, 6, 7, 8, 9, 10]. Since then, there has been an increasing interest in implementing a cochlea model utilizing nonlinear oscillators on digital and analog electronic circuits [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30]. Such an electronic circuit can be applied to hearing aids and cochlear implants that incorporate nonlinear signal processing.

Traditional methods applicable to modeling and im-

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Fig. 1 Spectrum of cochlear basilar membrane (BM) responses to two-tone stimuli with frequencies \( f_1 \) and \( f_2 \) \((f_2 > f_1)\) measured in chinchilla adapted from [3]. The two-tone distortion products are perceived at frequencies not included in the stimulus \( (e.g., 2f_1 - f_2, 3f_1 - 2f_2, 2f_2 - f_1, \text{and } 3f_2 - 2f_1)\), where the characteristic frequency (CF) in the stimulated site of the BM corresponds to \( 2f_1 - f_2 \).

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Implementing nonlinear biological oscillators, including the cochlear amplifier, can be classified into the following three ways based on the continuousness of time and state.

(i) The first method to model a nonlinear biological oscillator is by using an ordinary differential equation, which has a continuous time and state. Such a model is implemented by a nonlinear electronic analog circuit [11, 12, 13, 14].

(ii) The second method to model a nonlinear biological oscillator is by using a difference equation, which has discrete time and continuous states. Such a model is implemented by a switched-capacitor circuit [19, 20, 21, 22].

(iii) The third method to model a nonlinear biological oscillator is by using a discrete difference equation or a cellular automaton (CA), which has discrete times and states. Such models are implemented using a digital signal processor (DSP) that performs numerical integration or sequential logic [15, 16, 17, 18].

Most nonlinear biological oscillators that have been studied belong to one of the above-mentioned three categories [11, 12, 13, 14, 15, 16, 17, 18]. Recently, our group and a few other groups have designed a nonlinear biological oscillator using the below-mentioned fourth method.

(iv) The fourth method to model a nonlinear biological oscillator is by using asynchronous CA, which has a continuous state transition time and discrete states. Such a model is implemented by an asynchronous sequential logic [23, 24, 25, 26, 27, 28, 29, 30].

Inspired by the Hopf-type cochlea [4, 5, 6, 7, 8, 9, 10], our group has proposed a cochlea model designed by the fourth method and has shown that the model can reproduce the frequency-threshold tuning curves of a mammalian cochlea [27]. However, the reproduction of nonlinear response char-
acteristics to two-tone stimuli remains to be demonstrated.

In this paper, a cochlea model with nonlinear dynamics described by an asynchronous CA was studied. We demonstrated that the proposed cochlea model enables the reproduction of two-tone distortion products [3]. The proposed model was implemented on a field-programmable gate array (FPGA). Subsequently, we demonstrated that the proposed cochlea model can be implemented using fewer hardware resources than a Hopf cochlea-type cochlea model implemented on a DSP performing a numerical integration.

2. Cochlea model based on asynchronous CA oscillator

In this section, a hardware-efficient cochlea model based on an asynchronous CA oscillator is proposed. Subsequently, we demonstrated that the proposed cochlea model enables the reproduction of two-tone distortion products. Fig. 2(a) shows a hardware configuration of the proposed cochlea model. As shown in the figure, the proposed cochlea model comprises a sigma-delta analog-to-digital converter (ADC) and an asynchronous CA oscillator.

2.1 Asynchronous CA oscillator

Fig. 2(b) shows a schematic of the asynchronous CA oscillator. As shown in the figure, the oscillator accepts the following clock.

\[ C_{int}(t) \equiv \sum_{k=0}^{\infty} \delta_R(t - kT_{int}), \quad C_{int} \in B \equiv \{0, 1\}, \quad (1) \]

where \( t \in \mathbb{R} \) and \( T_{int} \in \mathbb{R}^+ \equiv \{ x \in \mathbb{R} \mid x > 0 \} \) represent the continuous time and a period of the clock, respectively, and the function \( \delta_R : \mathbb{R} \to B \) represents the unit impulse.

\[ \delta_R(x) \equiv \begin{cases} 1 & (x = 0), \\ 0 & (x \neq 0). \end{cases} \]

Furthermore, as shown in Fig. 2(b), the oscillator has the following two discrete state variables,

\[ X \in \mathbb{Z}_N \equiv \{0, \ldots, N - 1\}, \quad Y \in \mathbb{Z}_N, \]

and the following two discrete auxiliary variables

\[ P_X \in \mathbb{Z}_M \equiv \{0, \ldots, M - 1\}, \quad P_Y \in \mathbb{Z}_M, \]

where \( N \geq 2 \) and \( M \geq 2 \) are integers determining the resolution of a state space \( Z \equiv \{(X,Y,P_X,P_Y) \mid X \in \mathbb{Z}_N, Y \in \mathbb{Z}_N, P_X \in \mathbb{Z}_M, P_Y \in \mathbb{Z}_M\} \) of the oscillator. To design a nonlinear vector field, the following two functions \( f_X : \mathbb{Z}_N \times \mathbb{Z}_N \to \mathbb{Z} \) and \( f_Y : \mathbb{Z}_N \times \mathbb{Z}_N \to \mathbb{Z} \) are introduced.

\[ f_X(x,y) \equiv \frac{1}{T_{int}(\mu(x - \frac{N}{2}) - \omega(y - \frac{N}{2})) - l^2((x - \frac{N}{2})^2 + (y - \frac{N}{2})^2)}, \]

\[ f_Y(x,y) \equiv \frac{1}{T_{int}(\omega(x - \frac{N}{2}) + \mu(y - \frac{N}{2})) - l^2((x - \frac{N}{2})^2 + (y - \frac{N}{2})^2)}, \quad (2) \]

where \( l \in \mathbb{R}^+ \), \( \mu \in \mathbb{R} \), and \( \omega \in \mathbb{R}^+ \) are the parameters. The derivation of the functions \( f_X \) and \( f_Y \) is presented in Appendix. Furthermore, \( \lfloor \cdot \rfloor \) denotes the following floor function.

\[ \lfloor x \rfloor \equiv \max\{l \in \mathbb{Z} \mid l \leq x\}, \quad x \in \mathbb{R}. \]

The functions \( f_X \) and \( f_Y \) are implemented in the lookup tables (LUTs). Fig. 3 shows the timing chart of the oscillator.

As shown in Fig. 3, the clock at \( C_{int}(t) = 1 \), then

\[ X(t) := \text{sat}_N(Y(t) + \delta_Z(P_X(t)) \text{sgn}(\text{sat}_M(f_X(X(t),Y(t)))), \]

\[ Y(t) := \text{sat}_N(Y(t) + \delta_Z(P_Y(t)) \text{sgn}(\text{sat}_M(f_Y(X(t),Y(t)))), \quad (3) \]

where the symbol “\( \lim_{l \to +0} + \epsilon \)” represents an “instantaneous state transition,” and \( \text{sat}_N : \mathbb{Z} \to \mathbb{Z}_N \) and \( \text{sat}_M : \mathbb{Z} \to \mathbb{Z}_M \equiv \{- (M - 1), \ldots, 0, \ldots, M - 1\} \) denote the following saturation functions.

Fig. 2 (a) Hardware configuration of the proposed cochlea model. (b) Schematic of asynchronous cellular automaton (CA) oscillator.
Fig. 4 (a) Typical responses of asynchronous CA oscillator without input stimulus. The parameter values are the same as those chosen in Fig. 3. (b) Typical responses of asynchronous CA oscillator with two-tone input stimulus. The parameter values are the same as those chosen in Fig. 3.

2.2 Input stimulus modulated by sigma-delta ADC
In this study, an input stimulus is assumed to be

\[ u(t) \equiv A_1 e^{j\omega_1 t} + A_2 e^{j\omega_2 t}, \quad u \in \mathbb{C} \]

where \( A_1, A_2 \in \mathbb{R}^+ \) and \( \omega_1, \omega_2 \in \mathbb{R}^+ \) represent the amplitude and angular frequency of the input stimulus, respectively. The input stimulus \( u(t) \) is transformed into a Cartesian coordinate representation for applying to the asynchronous CA oscillator as follows.

\[
\begin{align*}
  u_X(t) & \equiv A_1 \cos(\omega_1 t) + A_2 \cos(\omega_2 t), \quad u_X \in \mathbb{R}, \\
  u_Y(t) & \equiv A_1 \sin(\omega_1 t) + A_2 \sin(\omega_2 t), \quad u_Y \in \mathbb{R}.
\end{align*}
\]

The input stimuli \( u_X \) and \( u_Y \) are converted by using a sigma-delta ADC, and a schematic of the ADC is shown in Fig. 5. As shown in this figure, for sampling an analog signal, the following clock is introduced.

\[
C_{ext}(t) \equiv \sum_{k=0}^{\infty} \delta(t-kT_{ext}), \quad C_{ext} \in \mathbb{B},
\]

where \( T_{ext} \in \mathbb{R}^+ \) represents a period of the clock. Furthermore, the input stimuli \( u_X \) and \( u_Y \) are sampled by clock \( C_{ext} \) and integrated by the following signals \( W_X, W_Y \in \mathbb{R}^+ \).

If \( C_{ext}(t) = 1 \), then

\[
\begin{align*}
  W_X(t_{int}) & \equiv u_X(t) + W_X(t) - B_0 S_X(t), \\
  W_Y(t_{int}) & \equiv u_Y(t) + W_Y(t) - B_0 S_Y(t),
\end{align*}
\]

where \( S_X(t) = q(W_X(t)), \quad S_Y(t) = q(W_Y(t)) \) and \( B_0 \in \mathbb{R}^+ \), and the function \( Q : \mathbb{R}^+ \to \mathbb{Q} \) denotes the following quantizer.

\[
q(x) \equiv \begin{cases} 1 & (x \geq 0), \\
-1 & (x < 0).
\end{cases}
\]

Subsequently, the clock \( C_{ext}(t) \) triggers the following transitions of the discrete state variables \( X \) and \( Y \).

If \( C_{ext}(t) = 1 \), then

\[
\begin{align*}
  X(t_{int}) & \equiv X(t) + S_X(t), \\
  Y(t_{int}) & \equiv Y(t) + S_Y(t).
\end{align*}
\]

Fig. 4(b) shows the typical time waveforms of the asynchronous CA oscillator with a two-tone input stimulus. The transitions of the discrete state variables \( X \) and \( Y \) are triggered by the uncoupled clocks \( C_{int} \) and \( C_{ext} \), as shown in Fig. 3. The oscillator can be regarded as an asynchronous CA.

2.3 Reproductions of two-tone distortion products
Fig. 6 shows a frequency spectrum of a time waveform of the proposed cochlea model stimulated by two tones, where CF corresponds to \( 2f_1 - f_2 \). The frequency spectrum herein was obtained via fast Fourier transform (FFT) using a Hanning...

\[
\begin{align*}
  \text{Fig. 4 (a) Typical responses of asynchronous CA oscillator without input stimulus. The parameter values are } N = 2^7, M = 2^5, \mu = -10, \omega = 2\pi \times 10^3, \quad & l = 0.25, \quad \text{and } T_{int} = 2^{-7}. \text{ (b) Typical responses of asynchronous CA oscillator with two-tone input stimulus. The parameter values are the same as those chosen in Fig. 3.}
\end{align*}
\]

\[
\begin{align*}
  \text{sat}_N(x) & \equiv \begin{cases} N-1 & (x > N-1), \\
0 & (0 \leq x \leq N-1), \\
0 & (x < 0),
\end{cases}
\end{align*}
\]

Furthermore, \( \delta_Z : \mathbb{Z} \to \mathbb{B} \) denotes the following unit impulse function.

\[
\delta_Z(x) \equiv \begin{cases} 1 & (x = 0), \\
0 & (x \neq 0),
\end{cases}
\]

where \( \text{sgn} : \mathbb{Z} \to \mathbb{Q} \equiv \{-1, 1\} \) denotes the following signum function.

\[
\text{sgn}(x) \equiv \begin{cases} 1 & (x \geq 0), \\
-1 & (x < 0),
\end{cases}
\]

As shown in Fig. 3, the clock \( C_{int} \) triggers the following transitions of the discrete auxiliary variables \( P_X \) and \( P_Y \).

If \( C_{int}(t) = 1 \), then

\[
\begin{align*}
  P_X(t_{int}) & \equiv P_X(t) + F_D(P_X(t), |\text{sat}_M(f_X(X(t), Y(t)))|), \\
  P_Y(t_{int}) & \equiv P_Y(t) + F_D(P_Y(t), |\text{sat}_M(f_Y(X(t), Y(t)))|),
\end{align*}
\]

where \( F : \mathbb{Z}_2^M \to \mathbb{Z}_2^M \equiv \{-M-1, \cdots, 0, 1\} \) denotes the following function.

\[
F_D(x, a) \equiv \begin{cases} 1 & (x < a), \\
-x & (x \geq a).
\end{cases}
\]

Fig. 4(a) shows a typical time waveform of the discrete state variables \( X \) and \( Y \) without an input stimulus.

\[
\text{Fig. 5 Schematic of sigma-delta ADC.}
\]
window with an applied amplitude factor. The proposed cochlea model perceives tones with frequencies not included in the two-tone stimulus, for example, $2f_1 - f_2$. Hence, the FFT analysis results shown in Fig. 6 verified that our proposed cochlea model can generate two-tone distortion products (see also Fig. 1).

3. FPGA implementation and comparison

3.1 FPGA implementation

The proposed cochlea model is implemented on a field-programmable gate array (FPGA) in a register transfer level (RTL) code using VHDL as follows: The discrete state variables $X$ and $Y$ are implemented by registers as $n$-bit unsigned integers, where $n = \lceil \log_2 N \rceil$. The discrete auxiliary variables $P_X$ and $P_Y$ are implemented by using registers as $m$-bit unsigned integers, where $m = \lceil \log_2 M \rceil$. The functions $s_a M (f_X (x, y))$ and $s_a M (f_Y (x, y))$ are implemented in lookup tables (LUTs) with an $n$-bit unsigned integer input and an $m + 1$-bit signed integer output in the two’s complement format. The dynamic equations in Eqs. (3), (4), and (7) are written by sequential statements triggered by the clocks $C_{\text{int}}$ and $C_{\text{ext}}$ in Eqs. (1) and (5), respectively. The RTL code was synthesized by Xilinx Vivado Design Suite v2020.1, and a generated bitstream file was downloaded into Xilinx FPGA Artix-7 XC7A100T-1CSG324C. Fig. 7(a) shows the resulting RTL schematic of our proposed cochlea model. The sigma-delta ADC is implemented by a switched-capacitor technique on a field-programmable analog array (FPAA) (Anadigm Single Apex 3.3 V Development Kit). The dynamic equation in Eq. (6) is written using Anadigm Designer 2, where the resulting schematic of the sigma-delta ADC is shown in Fig. 7(b). Fig. 8 shows the frequency spectrum of the proposed cochlea model stimulated by two-tone stimuli implemented on the FPGA. The two-tone distortion product can be confirmed from the frequency spectrum (see also Figs. 1 and 6).

3.2 Comparison with Hopf-type cochlea model

For comparison, we implemented a Hopf-type cochlea model\(^1\) on the same FPGA, where the hardware configuration is shown in Fig. 9. The model comprises a sigma-delta ADC\(^2\) and the following Hopf oscillator.

$$
\dot{z} = (b + j)\omega_{ch} z - |z|^2 z + F(t), \quad z \in \mathbb{C},
$$

where $b \in \mathbb{R}$ and $\omega_{ch} \in \mathbb{R}$ are parameters, and $F(t) \equiv E_1 e^{j\omega_{ch} t} + E_2 e^{j\omega_{ch} t}$ is an external input representing a two-tone stimulus. To implement an electronic circuit, the model in Eq. (8) is transformed into a Cartesian coordinate representation as follows.

$$
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2) + F_1(t), \quad x_1 \in \mathbb{R}, \\
\dot{x}_2 &= f_2(x_1, x_2) + F_2(t), \quad x_2 \in \mathbb{R},
\end{align*}
$$

where

\(^1\) The sigma-delta modulator is not necessarily chosen for the ADC in the Hopf-type cochlea model.

\(^2\)
were not used. 6-input lookup tables (LUTs) and eight flip-flops (FFs). DSP slices and block RAM. IIR LPF. Artix-7 XC7A100T-1CSG324C has 15,850 slices; each slice contains four

Note: The hardware resources of the Hopf oscillator do not include those of the IIR LPF. Artix-7 XC7A100T-1CSG324C has 15,850 slices; each slice contains four 6-input lookup tables (LUTs) and eight flip-flops (FFs). DSP slices and block RAM were not used.

\[
\begin{align*}
  f_1(x_1, x_2) &\equiv bx_1 - \omega_{c_b} x_2 - x_1(x_1^2 + x_2^2), \\
  f_2(x_1, x_2) &\equiv \omega_{c_b} x_1 + bx_2 - x_2(x_1^2 + x_2^2), \\
  F_1(t) &\equiv E_1 \cos(\omega_1 t) + E_2 \cos(\omega_2 t), \\
  F_2(t) &\equiv E_1 \sin(\omega_1 t) + E_2 \sin(\omega_2 t).
\end{align*}
\]

The Hopf oscillator in Eq. (9) is discretized using the forward Euler method, which is one of the simplest numerical integration methods as follows.

\[
\begin{align*}
  x_1(t + \Delta t) &= x_1(t) + \Delta t(f_1(x_1(t), x_2(t)) + F_1(t)), \\
  x_2(t + \Delta t) &= x_2(t) + \Delta t(f_2(x_1(t), x_2(t)) + F_2(t)),
\end{align*}
\]

where \( \Delta t \in \mathbb{R} \) denotes a discretized time step. Furthermore, \( F_1(t) \) and \( F_2(t) \) are modulated by the sigma-delta ADC, which is the same as that shown in Fig. 7(b). As with Section 3.1, the discretized Hopf oscillator in Eq. (10) is implemented on the FPGA in an RTL code using VHDL, where the state variables and parameters are represented by fixed-point numbers in the two’s complement format. Furthermore, as shown in Fig. 9, an infinite impulse response (IIR) low-pass filter (LPF) is implemented on the FPGA for demodulation of a sigma-delta ADC signal. Note that this digital filter is not required in the proposed cochlea model.

The bit-lengths of all variables are reduced to be as short as possible under the condition that the model can generate two-tone distortion products. The RTL code is synthesized using the same development environment as that used in Section 3.1. Table I summarizes the comparison results of hardware resources. As shown in this table, the proposed cochlea model can be implemented using fewer hardware resources than the Hopf oscillator.

### 4. Conclusions

In this paper, a cochlea model based on an asynchronous CA oscillator is proposed. Our proposed cochlea model enables the reproduction of two-tone distortion products. The proposed cochlea and Hopf-type cochlea models were implemented on the same FPGA. The comparison revealed the following advantages of the proposed cochlea model: (i) the asynchronous CA oscillator can be implemented using fewer hardware resources than the Hopf oscillator and (ii) the proposed cochlea model does not require a digital filter for the demodulation of the sigma-delta ADC signal. Hence, this study contributes to the development of hearing aids and cochlear implants implemented in small-scale circuits. Our future work will include (i) extensive analyses of the two-tone distortion products in the proposed cochlea model and (ii) reproduction of other types of nonlinear response characteristics in a biological cochlea such as two-tone suppression and first (second) pitch shift.

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### References

[1] M.A. Ruggero: “Responses to sound of the basilar membrane of the mammalian cochlea,” Curr. Opin. Neurobiol. 2 (1992) 449 (DOI: 10.1016/0959-4388(92)90179-0)

[2] M.A. Ruggero, et al.: “Basilar-membrane responses to tones at the base of the chinchilla cochlea,” J. Acoust. Soc. Am. 101 (1997) 2151 (DOI: 10.1121/1.418265)

[3] L. Robles, et al.: “Two-Tone Distortion on the Basilar Membrane of the Chinchilla Cochlea,” J. Neurophysiol. 77 (1997) 2385 (DOI: 10.1152/jn.1997.77.5.2385)

[4] V.M. Eguiluz, et al.: “Essential nonlinearities in hearing,” Phys. Rev. Lett. 84, (2000) 5232 (DOI: 10.1103/PhysRevLett.84.5232)

[5] A. Kern and R. Stoop: “Essential Role of Couplings between Hearing Nonlinearities,” Phys. Rev. Lett. 91 (2003) 128101 (DOI: 10.1103/PhysRevLett.91.128101)

[6] R. Stoop and A. Kern: “Two-Tone Suppression and Combination Tone Generation as Computations Performed by the Hopf Cochlea,” Phys. Rev. Lett. 93 (2004) 268103 (DOI: 10.1103/PhysRevLett.93.268103)

[7] R. Stoop, et al.: “Auditory two-tone suppression from a subcritical Hopf cochlea,” Physica A: Statistical Mechanics and its Applications, 351 (2005) 175 (DOI: 10.1016/j.physa.2004.12.019)

[8] S. Martignoli and R. Stoop: “Local Cochlear Correlations of Perceived Pitch,” Phys. Rev. Lett. 105 (2010) 048101 (DOI: 10.1103/PhysRevLett.105.048101)

[9] S. Martignoli, et al.: “Pitch sensation involves stochastic resonance,” Sci. Rep. 3 (2013) 2676 (DOI: 10.1038/srep02676)

[10] F. Gomez and R. Stoop: “Mammalian pitch sensation shaped by the cochlear fluid,” Nature Phys. 10 (2014) 530 (DOI: 10.1038/nphys2975)

[11] S. Martignoli, et al.: “Analog electronic cochlea with mammalian hearing characteristics,” Appl. Phys. Lett. 91 (2007) 064108 (DOI: 10.1063/1.2768204)

[12] R. Stoop: “From Hearing to Listening: Design and Properties of an Actively Tunable Electronic Hearing Sensor,” Sensors 7 (2007) 3287 (DOI: 10.3390/s7123287)

[13] T.J. Hamilton, et al.: “A 2-D Cochlea with Hopf Oscillators,” in Proc. IEEE Biomedical Circuits and Systems Conference (2007) 91 (DOI: 10.1109/BIOCAS.2007.440414)
\[ x_1(t) + \frac{\Delta x}{f_1(x_1(t), x_2(t))} = x_1(t) + \Delta x \text{sgn}(f_1(x_1(t), x_2(t))), \]
\[ x_2(t) + \frac{\Delta x}{f_2(x_1(t), x_2(t))} = x_2(t) + \Delta x \text{sgn}(f_2(x_1(t), x_2(t))), \]

(11)

where \( \Delta x \in \mathbb{R} \) denotes a discretized state step and the function \( \text{sgn} : \mathbb{R} \rightarrow \mathbb{Q} \) denotes \( \text{sgn}(x) = 1 \) for \( x \geq 0 \) and \( \text{sgn}(x) = -1 \) for \( x < 0 \). In the left-hand side of Eq. (11), \( |\Delta x/f_1(x_1(t), x_2(t))| \) and \( |\Delta x/f_2(x_1(t), x_2(t))| \) represent the amounts of time advance per unit distance \( \Delta x \). Furthermore, in the right-hand side of Eq. (11), the amounts of state transitions of \( x_1 \) and \( x_2 \) are restricted to \( \Delta x \) or \(-\Delta x\). Fig. 10 shows the relationship between the forward Euler method and our method. In the proposed cochlea model, the discrete state variables \( X, Y \) and the functions \( \{f_X, f_Y\} \) correspond to \( \{x_1, x_2\} \) and \( \{\Delta x/f_1(x_1(t), x_2(t)), \Delta x/f_2(x_1(t), x_2(t))\} \) in Eq. (11), where \( \Delta x = 1 \). Also, the discrete auxiliary variables \( \{P_X, P_Y\} \) whose dynamic equations are defined in Eq. (4) work as state-dependent frequency dividers using the functions \( \{fx, fy\} \) (see also the bent black arrows in Fig. 3).

Appendix: Derivation of functions \( fx \) and \( fy \)

The proposed cochlea model is based on the Hopf-type cochlea model in Eq. (8). The forward Euler formula for the model in Eq. (10) without the input term is rewritten as follows.