A LOCAL MATHEMATICAL MODEL FOR EPR-EXPERIMENTS

W. Philipp
Beckman Institute
Department of Statistics and Department of Mathematics
University of Illinois, Urbana, IL 61801
E-mail: wphilipp@uiuc.edu

K. Hess
Beckman Institute
Department of Electrical Engineering and Department of Physics
University of Illinois, Urbana, IL 61801
E-mail: k-hess@uiuc.edu

In this paper we give a detailed and simplified version of our original mathematical model published first in the Proceedings of the National Academy of Science. We hope that this will clarify some misinterpretations of our original paper.

1 Introduction

In \cite{1,2,3,4} we presented a local mathematical model for EPR-type experiments. Our model is in agreement with the results predicted by Quantum Mechanics which in turn were confirmed by large scale experiments, first by Aspect, Roger and Dalibard\cite{5} and later by several other teams. Due to space limitations the presentation of our model was rather terse in places. The purpose of this paper is to present our model in much greater detail and at the same time mathematically simplified because concerns and questions have been raised about non-locality and parameter dependence in a few recent publications \cite{6,7,8,9,10}. Although we have answered these concerns at various occasions \cite{11,12,13}, a comprehensive detailed exposition might be a better way to address these concerns.

At first we give a brief summary of our model. In EPR type experiments two particles having their spin in a singlet state are emitted from a source and are sent to spin analyzers at two spatially separated stations $S_1$ and $S_2$. We assume with Bell that the particles emitted from the source are permitted to carry information in form of arbitrary hidden parameter random variables $\Lambda$ that can assume values in some abstract space.

In the original experiment by Aspect \cite{5}, pairs of photons were emitted from the source once every few microseconds over a hour period governed by
a random process. We model this process mathematically in the following way. Imagine the time axis wrapped around a circle of circumference that corresponds to a time interval related to a simple measurement and normalized to 1. We suppose that for a fixed \( N \) each interval \( [(m - 1)/N, m/N] \), \( m = 1, 2, \ldots, N \) of arc length \( 1/N \) on the circle gets about its proper share of time measurement points over the measurement period. This assumption, in turn, induces a random variable \( R \), that we call the labelling variable, which assumes the values \( m = 1, 2, \ldots, N \) with equal probability, i.e.,

\[
P(R = m) = 1/N \quad m = 1, 2, \ldots, N.
\] (1)

In Section 4 below, we present better motivation for the generation of the labels \( m \) and the random variable \( R \) by means of the Poisson process, commonly used to model spontaneous emissions.

In our papers\(^1,2,3,4\), using notation standard in Bell type proofs, the source parameter now denoted by \( \Lambda \) was not assigned a separate letter. The letter \( \lambda \), which in the standard notation generically symbolizes randomness, was used instead. As a consequence the label \( m \) being a function of the random emission times was misinterpreted by various authors as a function of \( \lambda \). In the present paper we have therefore decided to use the standard probability notation instead: Random variables are given separate names, they will be denoted by capital letters, they are measurable functions of \( \omega \) or \( \lambda \), attached to some experiments, and there will be a clear distinction between the random variables, the values they can assume, and the set of measured data.

After a pair of particles has been emitted from the source, the time of emission and thus of measurement are known, and so is the interval \( [(m - 1)/N, m/N] \) on the unit circle into which the time of measurement falls. This determines the label \( m \). While the pair of particles travels to their designated analyzer stations the experimenters (or a random number generator) can exercise their free will and choose in their respective stations the directional settings, say \( a \) in \( S_1 \) and \( b \) in \( S_2 \). Our model calls for hidden parameter random variables\(^\ref{hidden-variables}\) \( \Lambda^*_{at} \) in \( S_1 \) and \( \Lambda^*_{bt} \) in \( S_2 \) which depend on the respective settings and on the time of measurement. The time of measurement is known (either the same or connected by a linear relation) in both stations. Consequently, the label \( m \) is known at both stations. This provides for the time correlation we elaborated in papers\(^1,2,3\) and \(^4\).

Thus we have four random variables in operation, \( R, \Lambda, \Lambda^*_{at}, \) and \( \Lambda^*_{bt} \). The joint density of all these variables \( \rho_{ab} \), is permitted\(^\ref{joint-density} \) to depend on the settings \( a \) and \( b \), and is given in Eq. (29). As a consequence of our construction we obtain certain properties of stochastic dependence relations which we state as a preview. In the general case where the distribution of \( \Lambda \) may depend on
time we have the following stochastic dependence relations between these four random variables signifying their time correlations.

(i) The random variables $\Lambda_{a t}^*$ and $\Lambda_{b t}^{**}$ are stochastically independent.

(ii) Given the random variable $R$ the pair $(\Lambda_{a t}^*, \Lambda_{b t}^{**})$ is conditionally independent of $\Lambda = \Lambda_t$.

As to the probability distributions our construction yields the following properties.

(iii) The probability distribution of $\Lambda_t$ can be chosen arbitrary.

(iv) The probability distributions of $\Lambda_{a t}^*$ and $\Lambda_{b t}^{**}$ do not depend on $a$, $b$, nor $t$.

In the special case where the distribution of $\Lambda$ does not depend on time we have in addition to the above properties

(ii)* The random variables $\Lambda, \Lambda_{a t}^*$, and $\Lambda_{b t}^{**}$ are stochastically independent.

(vi)* The random variables $R$ and $\Lambda$ are stochastically independent.

The random variables $A_a = \pm 1$ and $B_b = \pm 1$ symbolize the possible spin values and are functions only of $a$, $\Lambda$, $R$, and $\Lambda_{a t}^*$, and of $b$, $\Lambda$, $R$, and $\Lambda_{b t}^{**}$, respectively, thus obeying Einstein locality; that is

$$A_a = A_a(\Lambda_{a t}^*, \Lambda; R)$$

and

$$B_b = B_b(\Lambda_{b t}^{**}, \Lambda; R).$$

Moreover, we have

$$E\{A_a B_b\} = -a \cdot b = -\cos \langle a, b \rangle$$

and thus we have with probability 1

$$B_a = -A_a.$$  \hspace{1cm} (3)

In addition, we have with probability 1

$$E\{A_a \mid \Lambda, \Lambda_{a t}^*\} = E\{B_b \mid \Lambda, \Lambda_{b t}^{**}\} = 0.$$  \hspace{1cm} (4)
Note that the integration is, in essence, only performed with respect to $R$. If, in addition we integrate Eq. (4) with respect to $\Lambda^t_a$ and $\Lambda^{**}t_b$, we obtain with probability 1

$$E\{A_a \mid \Lambda\} = E\{B_b \mid \Lambda\} = 0. \quad (5)$$

Since $A_a = \pm 1$ and $B_b = \pm 1$, Eq. (5) implies parameter independence. However, our variables $A_a$ and $B_b$ depend on time through their functional dependence on $R$, a feature that is not considered in the original Bell definition of $A_a$ and $B_b$.

Thus our model is distinguished from standard Bell-type models by the introduction of the time related labelling variable $R$. If in Eq. (5) we also condition on $R$, besides conditioning on $\Lambda$, these equations no longer will hold. Moreover, we note that in the general case we have:

(v) Given the random variable $R$ the random variables $\Lambda^t_a$ and $\Lambda^{**}t_b$ are stochastically dependent.

(vi) The random variables $R$ and $\Lambda_t$ are stochastically dependent.

(vii) The conditional probability distributions of $\Lambda^t_a$ given $R$ and of $\Lambda^{**}t_b$ given $R$ depend on both settings $a$ and $b$.

On a more basic level probabilities, conditional probabilities or even conditional expectations, such as the one in Eq. (5), can be interpreted as long term averages of outcomes of certain experiments. These long term averages can be thought of being taken over certain points on the time axis. We separate this averaging process into two parts by introducing the random variable $R$. We first average over the concatenated time intervals associated with a fixed label $m = 1, 2, \ldots, N$. Subsequently, we average the first averages over the values $m$ that $R$ can assume to obtain the overall averages.

As a consequence we do not view the conditional stochastic dependence in (v), nor the dependence on the settings of the conditional probability distributions in (vii) as a violation of Einstein locality. These dependencies only signify the time correlations between the events in stations $S_1$ and $S_2$. To express this in physical terms we point the reader to the following facts. The label $m$ represents a concatenation of short time segments and not a given time. $m$ therefore does not relate to or permit any instantaneous signalling. It can not be influenced by the experimenter in any significant way since it depends on the random spontaneous emission times and the largely arbitrary way of concatenating these short time intervals in a specific interval $[(m-1)/N, m/N]$. Therefore $m$ is not an element of reality as opposed to, for example, the source parameter $\Lambda$. 

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Any argument for instantaneous action at a distance involving probabilities conditional on \( \{R = m\} \) must therefore be counterfactual. As an important example, one could argue that instead of a setting pair \( a, b \) the experimenters might have chosen \( a, c \). Then, since the joint probability conditional on \( \{R = m\} \) depends on both settings, the marginal distribution of \( \Lambda_{at}^* \) for setting \( a \) conditional on \( \{R = m\} \) may be different. How can this be without instantaneous action at a distance? The answer is that if \( c \) would have been chosen, then over a whole sequence of measuring times all the settings would be different. In order to have setting \( b \) with equal probability to setting \( c \), the experimenters would have had to decide to choose \( c \) instead of \( b \) at other occasions. In other words, the whole history of settings would have to be different. Because all the involved parameters, as well as the possible outcomes for the spin pair values may depend on the history, the probability distribution of \( \Lambda_{at}^* \) conditional on \( \{R = m\} \) may depend on the history and can therefore be different for the setting pairs \( a, b \) and \( a, c \). The EPR argument postulates a physical reality of the source parameter \( \Lambda \); in our papers we postulate also physical reality for the station parameters \( \Lambda_{at}^* \) and \( \Lambda_{bt}^{**} \). However, we do not attach a physical reality in the same sense to the labelling random variable \( R \). Fulfillment and violation of Einstein locality with respect to random variables such as the labelling variable \( R \) becomes a highly complex problem. Let us note, in passing, that the exclusion of setting dependence conditional to any concatenation of time segments such as represented by \( m \) will automatically also exclude the result of the actual experiments which can be regarded as performed by concatenating the results obtained in certain time segments.

We would like to emphasize that the joint probability measure given by Eq. (29) below is not canonical, i.e., not unique. This makes the model highly flexible to accommodate other possible set-ups of experiments. In fact, we hope to show with our work that the choice of the particular form of variables is mathematically highly flexible and can go far beyond simple ideas of elements of physical reality. We do not claim that the particular model actually exists in nature. All we want to show is that Bell type proofs actually can not do justice to the complexities involved in EPR experiments and therefore can not be used to draw conclusions about nonlocal effects as epitomized by instantaneous action at a distance.

For clarity of presentation we develop our construction in several steps. The first one, almost identical with the presentation in \(^1\), \(^3\), will be given in Section 2. In Section 3 we define the probability distribution of the time and setting dependent station parameters \( \Lambda_{at}^* \) and \( \Lambda_{bt}^{**} \). The measure we construct in these sections is not quite a probability measure (see Eq. (8), below). However, it is a routine exercise to derive from it a probability measure by applying
some basic facts from the theory of weak convergence of probability measures. We may present the details in a paper to be submitted to a mathematics journal.

We hope that this introduction provides enough of a guiding line through the mathematical intricacies that will follow.

2 The First Step in Establishing the Model

Before we start with the mathematics, let us recall that a pair of particles has been emitted from the source. The emission time and thus the measuring time is known. As a consequence the value of the labelling variable is determined. The experimenters have subsequently chosen their vectors in and respectively. In effect, we assume that the measuring time, considered as a random variable and the labelling variable are independent of the choice of vectors and.

Let and be unit vectors. Our goal is to show that under our generalized conditions, it is possible to obtain the quantum result, the scalar product for the spin pair expectation value . Here we formulate a theorem which provides the first stepping stone for this procedure.

We define functions and and choose the underlying measure space , i.e., the Euclidean plane with Borel measurability, symbolized by . We set

\[
A_a(u) = \begin{cases}
\text{sign}(a_k) & \text{if } -k \leq u < -k + 1, \ k = 1, 2, 3 \\
-1 & \text{if } j \leq u < j + \frac{1}{2}, \ j = 0, 1, \ldots \\
+1 & \text{if } j + \frac{1}{2} \leq u < j + 1, \ j = 0, 1, \ldots \\
+1 & \text{elsewhere.}
\end{cases}
\]

Thus depends here on and only. We will return below to the complete list of dependencies which only here would complicate the notation and not add to the present purpose. Here and throughout, we set \(\text{sign}(0) = 1\). Similarly, we define

\[
B_b(v) = \begin{cases}
-\text{sign}(b_k) & \text{if } -k \leq v < -k + 1, \ k = 1, 2, 3 \\
+1 & \text{if } j \leq v < j + \frac{1}{2}, \ j = 0, 1, \ldots \\
-1 & \text{if } j + \frac{1}{2} \leq v < j + 1, \ j = 0, 1, \ldots \\
-1 & \text{elsewhere.}
\end{cases}
\]

As in the case for above depends for the moment on and only. We now formulate the first step as a theorem.
Theorem 2.1 Let \( n \geq 4 \) be an integer. Then there exists a finite measure \( \mu = \mu^{(n)}_{ab} \) with the following properties: \( \mu \) depends only on \( n, a \) and \( b \), has compact support \( \Omega \), satisfies

\[
1 \leq \mu(R^2) < 1 + 1/n^2
\]

and has a density \( \rho = \rho^{(n)}_{ab} \) with respect to Lebesgue measure. Further

\[
\int_{\Omega} A_a(u)B_b(v)\rho^{(n)}_{ab}(u,v)dudv = -a \cdot b
\]

and for each vector \( a \) the following equation holds for all \( x \):

\[
B_a(x) = -A_a(x).
\]

The proof of the theorem requires the following fact which follows from a basic theorem on B-splines. We state the fact here in form of a lemma.

Lemma 2.2 Let \( n \geq 4 \) be an integer. Then there exist real-valued functions \( N_i(x), \psi_i(y) \) with \( 1 \leq i \leq n \) depending only on real variables \( x \) and \( y \), respectively, such that

\[
0 \leq N_i(x) \leq 1, \quad 0 \leq \psi_i(y) \leq 2 \quad \text{for } 0 \leq x, y \leq 1
\]

and

\[
0 \leq \sum_{i=1}^{n} \psi_i(y)N_i(x) - (y-x)^2 \leq \frac{1}{4}n^{-2} \quad \text{for } 0 \leq x, y \leq 1.
\]

The proof of this lemma is given in Appendix 1. We now proceed to prove Theorem 2.1.

Proof of Theorem 2.1: We first observe that Eq. (10) follows from the above definitions of \( A_a \) and \( B_b \). Let \( \Omega = [-3,3]^2 \) and let \( \kappa \) be the indicator function of the union of the unit squares \( \bigcup_{i=1}^{3n} (i-1,i)^2 \), lined up along the main diagonal of \( \Omega \), in symbols

\[
\kappa(u,v) = \sum_{i,j=-2,-1,\ldots,3n} \delta_{ij} \cdot 1\{i-1 \leq u < i\} \cdot 1\{j-1 \leq v < j\}.
\]

Here \( 1\{\cdot\} \) denotes the indicator function of the set in curly brackets and

\[
\delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}
\]
denotes the Kronecker symbol. On each of these \(3n + 3\) unit squares we place uniform mass, that comes from a product measure on each of the squares, where the first factor only depends on the setting \(a\) and the second factor only depends on the setting \(b\). Although this will make the mathematics quite a bit more complicated, we can envision further experiments where this feature of our construction may be of importance. The details are as follows. We define

\[
\sigma_a(u) = |a_k| \cdot 1\{−k \leq u < −k + 1\} \quad k \in I_1
\]

\[
N_k(|a_1|) \cdot 1\{k - 1 \leq u < k\} \quad k \in I_2
\]

\[
\tau_b(v) = |b_k| \cdot 1\{−k \leq v < −k + 1\} \quad k \in I_1
\]

\[
N_k(|b_1|) \cdot 1\{k - 1 \leq v < k\} \quad k \in I_2
\]

\[
|a_k| \cdot 1\{−k \leq u < −k + 1\} \quad k \in I_3
\]

\[
N_k(|a_2|) \cdot 1\{k - 1 \leq u < k\} \quad k \in I_4
\]

\[
0 \quad \text{elsewhere}
\]

\[
0 \quad \text{elsewhere}
\]

The symbols \(I_1, \ldots, I_4\) stand for \(I_1 = +3, +2, +1; I_2 = 1, \ldots, n; I_3 = n + 1, \ldots, 2n; I_4 = 2n + 1, \ldots, 3n\). We finally define the density \(\rho^{(n)}_{ab}\) by

\[
\rho^{(n)}_{ab}(u, v) = \sigma_a(u)\tau_b(v)\kappa(u, v)
\]

and the measure \(\mu\) by having density \(\rho^{(n)}_{ab}\) with respect to Lebesgue measure.

Hence we obtain from the above definitions the following integrals needed for the calculation of the spin pair correlation function:

\[
\int_{[-3,0]^2} A_a(u)B_b(v)\rho_{ab}(u,v)dudv = \sum_{k=1}^{3} |a_k| |b_k| \text{sign}(a_k)\text{sign}(b_k) = -a \cdot b.
\]

Furthermore, the integral over the complement of the square \([-3,0]^2\) vanishes, i.e.,

\[
\int_{\Omega \backslash [-3,0]^2} A_a(u)B_b(v)\rho_{ab}(u,v)dudv = 0
\]
which proves Eq. (9).

It remains to be shown that \( \rho_{ab} \) defines a measure \( \mu \) that is close to a probability measure, i.e., fulfills Eq. (8). For this, we consider the mass distribution between the square \([-3, 0)^2\) and its complement. The amount of mass \( M_1 \) distributed over \([-3, 0)^2\) is

\[
M_1 = \sum_{k=1}^{3} |a_k\|b_k|.
\]

(20)

The mass \( M_2 \) of \( \Omega \backslash [-3, 0)^2 \) equals

\[
M_2 = \frac{1}{2} \sum_{k=1}^{3} \sum_{i=1}^{n} N_i(|a_k|)\psi_i(|b_k|).
\]

(21)

Thus the total mass distributed equals in view of Eq. (12)

\[
M_1 + M_2 = \sum_{k=1}^{3} |a_k\|b_k| + \frac{1}{2} \sum_{k=1}^{3} \sum_{i=1}^{n} N_i(|a_k|)\psi_i(|b_k|)
\]

\[
= \sum_{k=1}^{3} |a_k\|b_k| + \frac{1}{2} \sum_{k=1}^{3} (|a_k| - |b_k|)^2 + \theta \cdot n^{-2}
\]

\[
M_1 + M_2 = 1 + \theta \cdot n^{-2}
\]

(22)

where \( 0 \leq \theta < 1/4 \).

This completes the proof of the theorem which is the first stepping stone of our construction of a suitable probability measure.

Obviously, if instead of Eq. (21), we would define

\[
M_2 := \sum_{K=1}^{3} (|a_k| - |b_k|)^2,
\]

\( \Omega = [-3, 3]^2 \) and would place the mass represented by these three summands on any of the nine unit squares of \([0, 3)^2\), we would produce a genuine probability measure, satisfying all conclusions of Theorem 2.1.

Finally, let \( L \geq 1 \), and \( p_\ell \geq 0 \) with \( \sum_{\ell=1}^{L} p_\ell = 1 \). For \( 0 \leq w < 1 \), we define

\[
s(w) = (-1)^\ell, \quad \frac{\ell-1}{L} \leq w < \frac{\ell}{L}, \quad \ell = 1, \ldots, L
\]

\[
q(w) = p_\ell, \quad \frac{\ell-1}{L} \leq w < \frac{\ell}{L}, \quad \ell = 1, \ldots, L.
\]

(23)
For $-3 \leq u, v < 3n$ and $0 \leq w < 1$ we define $\tilde{\Omega} = \Omega \times [0,1)$,

\[
\begin{align*}
\tilde{A}_a(u, w) &= A_a(u)s(w) \\
\tilde{B}_b(v, w) &= B_b(v)s(w) \\
\tilde{\rho}_{ab}(u, v, w) &= \rho_{ab}(u,v)q(w).
\end{align*}
\]

Then $\tilde{A}_a$ and $\tilde{B}_b$ only depend on $a, u, w$, and $b, v, w$, respectively. Moreover, they satisfy the properly modified conclusion of Theorem 2.1. This procedure extends $\Omega$ to $\tilde{\Omega}$ by adding as a factor the unit interval $0 \leq w < 1$ with a given mass distribution.

3 Definition of the Layers

We call the construction including the unit interval as factor, given in Section 2 the first layer. To simplify the notation we shall omit the $\sim$ sign from the $\Omega$, $A$, $B$, and $\rho$. As we noted in 1 and 3 the first layer does not yet provide a model that guarantees absence of action at a distance. To achieve this goal we will now define a system of layers. These layers will be obtained by permuting all the unit squares contained in $\Omega$, including the mass distribution and the corresponding strips on which $A$ and $B$ are defined. In addition we shall duplicate the mass distribution of each layer labeled $m$, labelling the duplicate layer $m'$. On the layer labeled $m$ the functions $A_a^{(m)}$ and $B_b^{(m)}$ will remain unchanged. However, on the companion layer labeled $m'$ we shall switch the signs of $A_a$ and $B_b$, by setting $A_a^{(m')} = -A_a^{(m)}$ and $B_b^{(m')} = -B_b^{(m)}$. As we observed in a recent paper 13, this simple modification of our original construction encompasses all the desired features to achieve so called parameter independence. We now present this program in detail.

Think of each of the unit cubes $[i-1,i) \times [j-1,j) \times [0,1)$, $i, j = -2, -1, \ldots, 3n$ together with their respective mass distribution and the values of $A_a$ and $B_b$ defined on them as a unit ensemble. We permute these unit ensembles in the following way. Choose three vertical strips $[i-1,i) \times [-3,3n) \times [0,1)$, $i = -2, -1, \ldots, 3n$ and three horizontal strips $[-3,3n) \times [j-1,j) \times [0,1)$, $j = -2, -1, \ldots, 3n$. These intersect in nine unit cubes. Place the three unit ensembles $[i-1,i) \times [j-1,j) \times [0,1)$, $i = -2, -1, 0$ of the first layer onto three of these nine unit cubes, such that each vertical and each horizontal strip contains exactly one of these three unit ensembles of the first layer, and move with them the vertical and horizontal strips of the first layer. This can be done in $36 \left(\frac{3n+3}{3}\right)^2$ different ways. There are still $9n^2$ unit cubes left to be assigned their unit ensembles. Choose $3n$ of them and place on them these unit ensembles of the first layer where the density was defined by $\frac{1}{2}N_i(|a_k|)\psi_i(|b_k|)$.
\[i = 1, 2, \ldots, n; \ k = 1, 2, 3. \] This can be done in \(\binom{9n^2}{3n}(3n)!\) different ways. Place the remaining \(9n^2 - 3n\) unit ensembles to fill up the empty spaces. They have total mass 0. This yields a grand total of

\[
36 \left( \frac{3n + 3}{3} \right)^2 \left( \frac{9n^2}{3n} \right) (3n)!
\]

arrangements, which we call “layers”. We call this number \(\frac{1}{2}N\).

At this point we exercise our option to let \(p_\ell = p_{m\ell}\) in Eq. (23), \(m = 1, 2, \ldots, \frac{1}{2}N\) depend on the label of the layer.

In summary, on each layer the functions \(A_a(u, w; m)\) and \(B_b(v, w; m)\) only depend on \((u, w)\) and \((v, w)\), respectively. Each layer supports a measure \(\mu_m = \mu_{abm}\) satisfying

\[1 \leq \mu_m(\mathbb{R}^3) < 1 + 1/n^2.\]

Each measure \(\mu_m\) has a density \(\rho(u, v, w; m)\) with respect to Lebesgue measure that can be written in the form

\[\rho(u, v, w; m) = \sigma_a(u; m)\tau_b(v; m)\kappa(u, v; m)q(w; m)\]

for \(-3 \leq u, v < 3n, 0 \leq w < 1\), with the obvious interpretation of \(\sigma_a, \tau_b, \kappa\) and \(q\). Moreover, by Eq. (9), we have for each \(m = 1, \ldots, \frac{1}{2}N\),

\[
\int_{\Omega^{(m)}} A_a(u, w; m)B_b(v, w; m)\rho(u, v, w; m)dudvdw = -a \cdot b. \tag{25}
\]

As indicated at the beginning of this section, we shall duplicate each layer so that at the end we will have a total of \(N\) layers. We renumber the original layers \(m\) by the odd positive integers, \(2m - 1\), say, \(m = 1, 2, \ldots, \frac{1}{2}N\). The companion layer to the layer \(2m - 1\) will be assigned label \(2m, m = 1, 2, \ldots, \frac{1}{2}N\). Each layer \(2m - 1\) and companion layer \(2m\) will be assigned density previously denoted \(\rho(u, v, w; m)\). Each layer \(2m - 1\) will carry the functions, originally denoted \(A_a(u, w; m)\) and \(B_b(v, w; m)\), whereas the companion layer \(2m\) will carry \(-A_a(u, w; m)\) and \(-B_b(v, w; m)\), instead. Thus after renumbering the functions \(A_a\) and \(B_b\) and the densities \(\rho\) accordingly we have for all \(u, v, w,\) and all \(m = 1, 2, \ldots, \frac{1}{2}N\)

\[
A_a(u, w; 2m - 1) + A_a(u, w; 2m) = 0
\]

\[
B_b(v, w; 2m - 1) + B_b(v, w; 2m) = 0 \tag{26}
\]

and

\[
\rho(u, v, w; 2m - 1) = \rho(u, v, w; 2m). \tag{27}
\]
Moreover, Eq. (25) continues to hold for all \( m = 1, 2, \ldots, N \). Of course, the equivalent effect had been achieved by adding a fourth dimension \( t \) and multiplying the original functions \( A_a \) and \( B_b \) by a Rademacher function \( r(t) \). This was done in Section 5.3 of our paper \(^1\).

With all the mathematical objects properly in place we now finalize the second step of the construction of our model. The emission time of the \( i \)-th particle determines the measurement time and thus the label \( m \) where \( m = 1, 2, \ldots, N \). Recall that the labelling variable \( R \) has uniform distribution over the integers \( m = 1, 2, \ldots, N \), given by Eq. (1).

Apart from the random variable \( R \) the construction so far is plain calculus in \( \mathbb{R}^3 \). Only now we do define a realization of the random variables \( \Lambda^*_a \), \( \Lambda^*_b \), and \( \Lambda_t \) by defining the conditional density of \( \Lambda^*_a \), \( \Lambda^*_b \), and \( \Lambda_t \) given the random variable \( R \) by

\[
\text{Prob}(\Lambda^*_a \in [u, u + \Delta u], \Lambda^*_b \in [v, v + \Delta v], \Lambda_t \in [w, w + \Delta w] \mid R = m) = \int_1^0 q(w; m) \, dw.
\]

This is the same as saying that the joint density of the four random variables \( \Lambda^*_a \), \( \Lambda^*_b \), \( \Lambda_t \), and \( R \) is given by

\[
\rho(u, v, w; m) \cdot \frac{1}{N}, \quad -\infty < u, v < \infty, 0 \leq w < 1; m = 1, 2, \ldots, N.
\] (29)

A few remarks are in order. First, in previous write-ups we have included mappings \( f \) and \( g \) to accommodate more general random variables \( \Lambda^*_a \) and \( \Lambda^*_b \). Obviously, this can be done here, too. Second, we changed the model by defining \( \rho_{ab} \) to be the joint conditional density of \( \Lambda^*_a \) and \( \Lambda^*_b \) given \( R \), rather than by defining \( \rho_{ab} \) given by Eq. (17) to be the joint conditional density of the mixed parameters \( \Lambda^1_{at} \) and \( \Lambda^2_{bt} \), given \( R \), as was done in 1,3. This makes for a more streamlined presentation when the source parameter \( \Lambda_t \) is taken into account since obviously \( \Lambda^1_{at} \) and \( \Lambda^2_{bt} \), are functions of \( \Lambda_t \) and the station parameters \( \Lambda^*_a \) and \( \Lambda^*_b \), respectively, and thus cannot be independent of \( \Lambda_t \) (compare to condition (ii*) in Section 1). Hence, the expression for the joint density corresponding to Eq. (29) would be more complicated.

We now discuss the stochastic dependence relations between the four random variables \( \Lambda^*_a \), \( \Lambda^*_b \), \( \Lambda_t \), and \( R \) that are direct consequences of Eq. (29). First, the joint density of \( \Lambda^*_a \) and \( \Lambda^*_b \) is given by

\[
\frac{1}{N} \sum_{m=1}^{N} \sigma_a(u; m) \tau_b(v; m) \kappa(u, v; m) \int_0^1 q(w; m) \, dw.
\]
Since the last integral equals 1, this reduces to, in view of Eq. (22),
\[
\frac{1}{(3n + 3)^2} \left( \sum_{k=1}^{3} |a_k||b_k| + \frac{1}{2} \sum_{k=1}^{3} \sum_{i=1}^{3} N_i(|a_k|) \psi_i(|b_k|) \right) = \frac{M_1 + M_2}{(3n + 3)^2} = 1 + \theta \cdot n^{-2}
\]
with \(0 \leq \theta < \frac{1}{4}\). We conclude that the joint density of \(\Lambda^*_a\) and \(\Lambda^*_b\) is approximately uniform over the square \([-3, 3n)\) and, as a consequence, equals the product of its two marginal densities which are themselves approximately uniform over the interval \([-3, 3n)\).

We conclude that approximately:

(i) \(\Lambda^*_a\) and \(\Lambda^*_b\) are independent random variables, and

(iv) the distributions of \(\Lambda^*_a\) and \(\Lambda^*_b\) do not depend on \(a\), \(b\), and \(t\).

Moreover, summation over \(\ell = 1, 2, \ldots, L\) yields

\[
\text{Prob}(\Lambda^*_a \in [u, u + \Delta u], (\Lambda^*_b \in [v, v + \Delta v]) | R = m) = \sigma_a(u; m) \tau_b(v; m) \kappa(u, v; m) \Delta u \Delta v, \quad m = 1, \ldots, N, -\infty < u, v < \infty
\]

and integration over \(-\infty < u, v < \infty\) yields

\[
\text{Prob}(\Lambda_t \in [w, w + \Delta w] | R = m) \sim q(w; m) \Delta w \quad 0 \leq w < 1, m = 1, \ldots, N.
\]

Thus by Eq. (28), we have approximately

(ii) Given \(R\) the pair \((\Lambda^*_a, \Lambda^*_b)\) is conditionally independent of \(\Lambda_t\).

Also, approximately,

\[
\text{Prob}(\Lambda^*_a \in [u, u + \Delta u], (\Lambda^*_b \in [v, v + \Delta v] | \Lambda_t \in [w, w + \Delta w], R = m) = \frac{\sigma_a(u; m) \tau_b(v; m) \kappa(u, v; m) \Delta u \Delta v \Delta w}{\text{Prob}(\Lambda_t \in [w, w + \Delta w] | R = m) P(R = m)}
\]

\[
= \text{Prob}(\Lambda^*_a \in [u, u + \Delta u], (\Lambda^*_b \in [v, v + \Delta v] | R = m).
\]

(ii*) Further if \(p_{m,t} = p_t\) independent of \(m\), i.e., if \(\Lambda_t\) and \(R\) are independent, then \(\Lambda^*_a\), \(\Lambda^*_b\), and \(\Lambda_t\) are independent random variables.
Moreover, we obtain for the pair correlation integral

\[ E\{A_a B_b\} := E\{A_a(\Lambda^*_a, \Lambda_t, R) B_b(\Lambda^*_b, \Lambda_t, R)\} \]

\[ = \frac{1}{N} \sum_{m=1}^{N} E\{A^{(m)}_a B^{(m)}_b\} = -a \cdot b \]

by Eq. (25) and Eq. (29). Since by construction (see Eq. (28))

\[ A^{(2m-1)}_a + A^{(2m)}_a = 0, \quad B^{(2m-1)}_b + B^{(2m)}_b = 0, \quad m = 1, 2, \ldots, \frac{N}{2}, \]

we obtain parameter independence first summing over \( m \) to obtain Eq. (4) and then by keeping the desired variables fixed and by integrating over the remaining ones.

4 A Model Based on the Poisson Process

The original experiment of Aspect et al.\(^5\) took hours to complete. Currently, improvements of the technique have been accomplished by various teams of experimenters\(^5\) and the length of time it takes to perform these experiments has been reduced substantially. The time between subsequent measurements is still limited by the recovery (essentially a random process) of the detectors between two measurements.

From the logistical angle the present section is designed to replace the third paragraph of Section 1 and the parts of Sections 2 and 3 corresponding to it. Thus, overall, the present section is a variant of that part of the model dealing with generating the labels \( m \). This will be done by considering the waiting times between consecutive “jumps” of a Poisson process. Since we are entering more advanced mathematical territory we present some of the relevant definitions and theorems in a basic form rather than to send the reader searching through the literature.

We first recall a few definitions from the theory of uniform distribution mod1. For more details see\(^19\) and\(^20\). For a real number \( x \), denote by \( [x] \) the integer part and by \( \{x\} = x - [x] \) the fractional part of \( x \). Let \((x_i)_{i=1}^{\infty}\) be a sequence of real numbers. For \( k \geq 1 \) and \( 0 \leq \alpha < \beta \leq 1 \) denote by

\[ A_k(\alpha, \beta) := \sum_{i \leq k} 1(\alpha \leq \{x_i\} < \beta) \]

the number of elements \( x_i, i \leq k \) such that their fractional part \( \{x_i\} \) is contained in a given interval \( [\alpha, \beta) \subset [0,1) \). The sequence \((x_i)\) is called uniformly
distributed mod1 if its discrepancy

\[ D_k := \sup_{\alpha, \beta} \left| \frac{1}{k} A_k(\alpha, \beta) - (\beta - \alpha) \right| \to 0, \]

as \( k \to \infty \), that is, if in the long run, each interval \([\alpha, \beta]\) uniformly contains its proper share of points \( \{x_i\} \). Equivalently, we could define \( A_k \) by wrapping the real axis around a circle of circumference 1 and count the number of hits a given interval \([\alpha, \beta]\), now located on the circle, receives from the sequence \((x_i), i \leq k\), itself.

The standard mathematical model for spontaneous emissions of particles, such as photons or electrons, is a Poisson process with intensity \( 1/\Theta \), say. The waiting times \( T_i \) between successive emissions are independent identically distributed random variables having exponential distribution with parameter \( \Theta \). The following theorem is a special case of Theorem 2 of H. Robbins.

**Theorem 4.1** Let \((T_i)\) be the sequence of waiting times between consecutive jumps of a Poisson process. Then with probability 1 the sequence \((T_1 + \cdots + T_i)_{i=1}^\infty\) is uniformly distributed mod1.

**Remark** In fact, it follows easily from Robbins’ proof and the Erdős-Turán inequality that with probability 1 the discrepancy \( D_k \) tends to zero at least with speed \( k^{-\frac{1}{2}}(\log k)^{\frac{3}{2}} \). There are more than a dozen other papers extending Robbins’ theorem.

In terms of weak convergence of probability measures Theorem 4.1 can be reformulated in the following way. (See e.g., Billingsley 22, pp. 15–25.) Let \( \omega \) be an element of the set \( \Omega^* \) of probability 1 as in Theorem 4.1. Set \( x_i = T_1(\omega) + \cdots + T_i(\omega) \). Let \( P_k \) be the probability measure that assigns point mass \( \frac{1}{k} \) to each \( \{x_i\}, 1 \leq i \leq k \). If several \( \{x_i\} \) coincide, let the mass add. Then \( P_k \Rightarrow P \) in the sense of weak convergence. Here \( P \) denotes Lebesgue measure on \([0, 1)\). For ease of presentation let us reformulate Theorem 4.1 in terms of random variables. Let \( Y_k \) be a random variable defined on some probability space such that

\[ \text{Prob}(Y_k = \{x_i\}) = \frac{1}{k} \quad i = 1, 2, \ldots, k \]

and let \( U \) be a random variable having uniform distribution on \([0, 1)\), i.e., \( \text{Prob}(U \leq x) = x, 0 \leq x \leq 1 \). Then Theorem 4.1 can be restated as follows. For each \( \omega \in \Omega^* \), we have as \( k \to \infty \)

\[ Y_k \Rightarrow U \]
in the sense of weak convergence. Let \( N \) be defined in Eq. (24). For \( m = 1, 2, \ldots, N \) define intervals \( I_m \), of length \( 1/N \) by

\[
I_m := \left[ \frac{m-1}{N}, \frac{m}{N} \right).
\]

Define a new random variable \( R \) by setting

\[
R = m \quad \text{if} \quad U \in I_m, m = 1, 2, \ldots, N.
\]

Then uniformly over all intervals \( I_m, m = 1, 2, \ldots, N \), we have

\[
\text{Prob}(R = m) = \text{Prob}(U \in I_m) = \lim_{k \to \infty} \text{Prob}(Y_k \in I_m) = \frac{1}{N}.
\]

(30)

As in Section 3, suppose that the \( i \)-th pair of particles has been emitted. Fix \( \omega \in \Omega^* \). The time of emission of the \( i \)-th pair equals \( x_i = T_1(w) + \cdots + T_i(\omega) \). When reduced mod1 the fractional part \( \{x_i\} \) determines a label \( m \) with \( 1 \leq m \leq N \). The labelling variable \( R \) has uniform distribution over the integers \( m = 1, 2, \ldots, N \) given by Eq. (30) or Eq (1). However, at the time the pair of particles arrive at their respective measuring stations, the devices may not yet be ready to provide a measurement, because of recovery problems, etc. For \( r = 1, 2 \) we define the random variable \( D_r \) by setting

\[
D_r = +1 \quad \text{if device is ready at station } S_r
\]

\[
= 0 \quad \text{if not.}
\]

Obviously, by stochastic independence,

\[
P(R = m|D_1, D_2) = P(R = m) = \frac{1}{N} \quad m = 1, 2, \ldots, N.
\]

Hence, given that the devices at both stations are ready for measurement, the labelling random variable \( R \) still has uniform distribution.

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Appendix 1

The lemma is an immediate consequence of Theorem 4.1 of Schumaker\(^{17}\) for the special values of \( m = 3, \ell = 0, r = n \), and the knots chosen to be \( y_{ni} = \frac{n}{n} \)
with $\nu = 0, \pm 1, \pm 2, \ldots$. Then by Schumaker’s equation (4.33) we have (dropping the fixed superscript 3 of $N_i^3$):

$$(y - x)^2 = \sum_{i=-2}^{n} \phi_{i,3}(y)N_i(x) \text{ for all } 0 \leq x \leq 1 \text{ and } y \in R.$$ 

Here

$$\phi_{i,3}(y) = (y - y_{i+1})(y - y_{i+2})$$

and

$$0 \leq N_i(x) \leq 1 \text{ for all } x.$$ 

We now restrict $y$ to $0 \leq y \leq 1$. Then for $-2 \leq i \leq n$, we have $0 \leq \phi_{i,3}(y) < 2$ unless $y \in [y_{i+1}, y_{i+2}]$. Since we must avoid negative $\phi$, we set $\phi = 0$ in this interval by defining new functions $\psi$:

$$\psi_i(y) = 0 \text{ if } y \in [y_{i+1}, y_{i+2}]$$

$$\psi_i(y) = \phi_{i,3}(y) \text{ otherwise.}$$

Since for $y \in [y_{i+1}, y_{i+2}]$ we have

$$|(y - y_{i+1})(y - y_{i+2})| \leq \frac{1}{4n^2}$$

we have

$$0 \leq \sum_{i=-2}^{n} (\psi_i(y)N_i(x) - \phi_{i,3}(y)N_i(x)) \leq \frac{1}{4n^2}$$

because for any given $y$ and for all $x$, only one term in the sum can be off by at most $\frac{1}{4n^2}$. This proves the lemma.

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14. In 1, 2, 3, and 4, we used lower case letters for these random variables.
15. Our model here is not canonical. We have shown in quant-ph/0211117 that Bell type proofs come to a halt already under more general conditions that do not require a joint probability distribution as outlined in this paper.
16. In our previous papers, the source parameter $\Lambda$ is thought to interact with the station parameters $\Lambda_{at}$ in $S_1$ and $\Lambda_{bt}$ in $S_2$, to produce mixed parameters $\Lambda_{at}^1$ in $S_1$ and $\Lambda_{bt}^1$ in $S_2$. This transition from $\Lambda$ and $\Lambda_{at}$ to $\Lambda_{at}^1$ and from $\Lambda$ and $\Lambda_{bt}$ to $\Lambda_{bt}^1$ is thought to be defined by certain rules that can be represented by station specific operators $O_{at}^1$ and $O_{bt}^1$, respectively. These time dependent operators are thought of depending on the globally known time that is the same at the stations, as well as at the source. In the present paper, we will make no further use of the mixed parameters, nor the time dependent operators. Here we use only instrument parameters, source parameters and time intervals, as well as functions of them. From a mathematical viewpoint this makes the introduction of the mixed parameters and the time operators unnecessary. In other words, $m$ was previously labelling a concatenation of time and setting dependent operators. This concatenation is here replaced by a concatenation of short time segments. These time segments are thought to have the duration of the interaction time of the particles with the instruments. In essence this is what we mean when we talk about measuring times. We believe this can be done without much loss of generality and with the gain of mathematical clarity.
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