Covariant Linearization of elasticity

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Abstract

In this paper we derive a general linearized theory for first-order continuum dynamics on manifolds with particular application to incompatible elasticity. We adopt a global approach viewing the equations of motion as a 1-form on the configuration space which is the Banach manifold of $C^1$ time-dependent embeddings of a body manifold $\mathcal{B}$ into a space manifold $\mathcal{S}$. The linearization is done by differentiating the equations 1-form with respect to an affine connection which we construct and study extensively. We provide detailed coordinate computations for the linearized equations of a large class of problems in continuum dynamics on manifolds.

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1 introduction

The derivation of a linear theory from a nonlinear theorem is a central theme in mathematics, with innumerable applications in the various sciences. In the context of continuum mechanics, and notably in the theory of elasticity, the linear theories actually preceded the nonlinear theories (see Mau- 
gin [Mau16]). In fact, the equations of linear elasticity are commonly de-

d-derived directly from the balance laws (assuming small defor-

mations) (Gurtin [Gur73]), rather than as approximation to the nonlinear theory.

Linear theories of elasticity play several key roles in the analysis of non-

linear theories: (i) they serve as an intermediate step for proving the exis-

tence and the uniqueness of solutions for nonlinear theories, (ii) solutions 
of nonlinear problems can sometimes be obtained as limits of sequences 
of solutions of linearized problems, and (iii) they serve as a central tool in 
stability analysis [MH83].

The linearization of nonlinear continuum theories is nowadays a stan-


dard, however, its current scope does not fully cover the wealth of systems 
of current interest. To a large extent, existing linear theories address sys-

tems that are geometrically Euclidean. From a mathematical perspective, 
the state-space in continuum mechanics can be described as the embeddings 
of a body into a space, both viewed as differentiable manifolds.

For example, in a class of elastic systems dealing with residually-stressed 

bodies, the body manifold is viewed as a smooth manifold endowed with a 
Riemannian metric; the metric represents local equilibrium distances and 

angles between neighboring material elements. A configuration is an em-

bedding of the body manifold into the ambient space, which is usually as-

sumed Euclidean, although, non-Euclidean ambient spaces are of relevance 
even without recurring to relativistic theories [KOS17a]. When the geomet-

ries of the body of the ambient space are incompatible, there is no notion of 
stress-free reference configuration, hence the very notion of small defor-
mations is not naturally defined as it is when both body and space are assumed 
Euclidean. Incompatible elasticity is just one example in which complex 

geometries interact in a non-trivial way with mechanical laws and material
properties.

Physical theories in which non-Euclidean geometry plays a central role are best formulated in a covariant manner, i.e., in a way that does not rely on a particular system of coordinates. The classical reference for the covariant linearization of elasticity theories is the book of Marsden and Hughes [MH83]. Their starting point is a general notion of linearization, which we hereby define:

**Definition 1.1** Let \( \pi : E \rightarrow M \) be a smooth (possibly infinite-dimensional) vector bundle endowed with a connection \( \nabla \). Let \( s \in \Gamma(E) \) be a \( C^\infty \)-section of \( E \). The linearization of \( s \) at \( x \in M \) is an affine mapping \( L_x(s) : T_xM \rightarrow E_{s|_x} \) given by

\[
L_x(s)(w) = s_x + (\nabla_w s)_x, \quad \forall w \in T_xM.
\]

Marsden and Hughes formulate the equations of nonlinear elasticity as a section of an infinite-dimensional vector bundle over the manifold of configurations and compute their linearization for a general class of constitutive relations. In their calculation, however, it is implicitly assumed that the ambient space is Euclidian, hence that the manifold of configurations is a vector space. This assumption is reflected in the linearization of the acceleration vector field and more subtly, in the linearization of the stress tensor. Accounting for a non-Euclidean ambient space is not just a matter of technicalities, which might be overcome, for example, by adopting a local coordinate system. A curved space affects the basic notion of inertia, and may destroy the symmetries that are at the heart of the classical derivation of continuum theories; this lack of symmetries reflects, for example, in the presence of so-called self-forces, which arise from interactions of the body with inhomogeneous geometric incompatibilities.

Other approaches to covariant linearization can be found in Yavari and Ozakin [AA08], where the authors linearize the energy and momentum balance laws, and in [GLM13] where linearization is computed around a normal state.

In this paper, we derive a general linearized theory for first-order continuum dynamics on manifolds, with a particular application to incompatible elasticity. We adopt a global approach, where the space of configurations is the Banach manifold of \( C^1 \) time-dependent embeddings of a body manifold into a space manifold. In this setting, the equations of motion are a 1-form on the configuration space. The linearization of those equation is in the sense of Definition 1.1, where the connection on the cotangent bundle of the configuration space is induced in a natural way from a given connection on the tangent bundle of the space manifold.
In the global approach to continuum dynamics, the equations of motion can be viewed as a natural generalization to Newton’s laws. Velocity is the time derivative of the configuration; the acceleration is the covariant time-derivative of the velocity field with respect to the connection $\nabla^Q$; the force field, which is a 1-form $F \in \Omega^1(\mathcal{Q})$, is composed of external loadings and internal forces, where the latter are determined by the material properties through a constitutive relation. The equations of motion is obtained by pairing the acceleration to the force via a Riemannian metric $\mathcal{G}$ on the configuration space $\mathcal{Q}$.

Generally, elements of $T^*\mathcal{Q}$ are represented by vector-valued measures. Hence, the linearized equations of motion may be as singular as measures and in particular, assume no local differential form. However, in the case where the loadings and the constituting relations satisfy certain regularity properties, the equations of motion as well as their linearization have local forms, which we derive as well.

The structure of the paper is as follows: In Section 2 we discuss the geometric structure of the space $C^1(M,N)$ where $M$ is a compact smooth manifold and $N$ is a smooth manifold without boundary. We first introduce the Banach manifold structure of $C^1(M,N)$ and its tangent bundle $TC^1(M,N)$. Next, we construct a metric and connection on $TC^1(M,N)$. To this end, we assume that a Riemannian metric $G$ is given on the target space $N$ and that a volume form $\theta$ is prescribed for the source manifold $M$. The connection $\nabla^Q$ is induced by a connection $\nabla^N$ for $TN$. We discuss the construction of $\nabla^Q$ in detail, and show that if $\nabla^N$ is metric with respect to $G$ then so is $\nabla^Q$ with respect to $\mathcal{G}$.

In Section 3 we use the results of Section 2 to formulate Newton’s equations for continuum dynamics. We identify the configuration space $\mathcal{Q}$ of time-dependent $C^1$-embeddings as an open subset of the manifold $C^1(I \times \mathcal{B}, \mathcal{S})$. The connection for $T\mathcal{Q}$ gives a notion of covariant derivative that defines the acceleration, whereas the metric for $T\mathcal{Q}$ pairs the acceleration with force. The force part of the equation is induced by a constitutive relation (which is assumed time-independent) and a loading; the whole equation is viewed as a section of the cotangent bundle of the configuration space.

In Section 4 we derive the linearized form of the nonlinear equations of motion derived in Section 3. We first obtain a general expression for general, time-independent constitutive relations. We then derive a local differential representation for the case of a smooth constitutive relation; the linearized equations are formulated both in a covariant manner and in local coordinates.
2 Geometric preliminaries

In this section we present the geometric foundations for continuum dynamics on manifolds. We start by briefly recalling the notion of jets, which are the covariant constructs for encoding functions along with their derivatives.

2.1 Jet bundles

Definition 2.1 Let \( M \) and \( N \) be smooth manifolds of dimensions \( m \) and \( n \). A 1-jet from \( M \) to \( N \) is an equivalence class of triples \((f, U, p)\), where \( p \in M \), \( U \subset M \) is a neighborhood of \( p \) and \( f \in C^1(U, N) \). Two triples \((f, U, p)\) and \((g, V, q)\) are equivalent if

1. \( p = q \).
2. \( f(p) = g(q) \).
3. There exists local charts in \( M \) and \( N \), with respect to which the local representatives of \( f \) and \( g \) have the same values and first derivatives at \( p \).

Equivalently, \((f, U, p)\) and \((g, V, q)\) are equivalent if

\[(Tf)_p = (Tg)_q\]

where \((Tf)_p : T_pM \to T_{f(p)}N\) is the tangent map of \( f \) at \( p \). We denote the 1-jet of \( f \) at \( p \) by

\[[(f, U, p)] = j^1_p f.\]

Remark: The third condition in the definition of a 1-jet implies that \( f \) and \( g \) have the same values at \( p \) and the same first derivatives at \( p \) with respect to any local coordinate charts.

We denote by \( J^1(M, N) \) the set of all 1-jets from \( M \) to \( N \). The set \( J^1(M, N) \) can be given the structure of a smooth manifold of dimension \( m + n + mn \); it is also a fiber bundle over \( M \) with respect to the (source) projection map

\[\pi^1 : J^1(M, N) \to M, \quad j^1_p f \mapsto p.\]

Let \( \pi : E \to M \) be a smooth vector bundle over \( M \). Define

\[J^1(E) = \{j^1_p s : p \in M \text{ and } s \text{ is a local } C^1\text{-section of } E \text{ at } p\}.

Then \( \pi^1 : J^1(E) \to M \) is a vector bundle over \( M \). The first jet extension,

\[j^1 : C^1(E) \to C^0(J^1(E)), \quad s \mapsto j^1 s,

is a linear immersion.
2.2 The manifold $C^1(M,N)$

Let $M$ be a smooth, compact, orientable $d$-dimensional manifold, and let $N$ be a smooth orientable $m$-dimensional manifold without boundary endowed with a Riemannian metric $G$. Let $C^1(M,N)$ be the space of $C^1$ mappings $M \to N$. Endow $C^1(M,N)$ with the Whitney $C^1$-topology [Mic80], a subbase of which consists of sets of the form

$$\{ f \in C^1(M,N) : j^1 f(M) \subset U \}, \quad U \subset J^1(M,N) \text{ is open.}$$

Loosely speaking, the Whitney $C^1$-topology is the topology of uniform convergence of the function and its first derivative.

The space $C^1(M,N)$ is not a vector space, since $N$ is not a linear space. However, $C^1(M,N)$ can be given a structure of an infinite-dimensional Banach manifold: a topological space locally homeomorphic to a Banach space and equipped with a smooth structure (see Lang [Lan99]).

Given a mapping $\kappa \in C^1(M,N)$, a coordinate chart for $C^1(M,N)$ at $\kappa$ is constructed as follows: Let $\nabla^N$ be the Levi-Civita connection of $G$ and let $\exp^N : D \to S$ be the corresponding exponential map, where $D \subset TN$ is a neighborhood of the zero section of $TN$, such that $(\pi^N, \exp^N) : D \to N \times N$ is an embedding (i.e., a diffeomorphism onto its image). Let $(\kappa^* \pi^N, \kappa^* \exp) : \kappa^* D \to M \times N$

be the embedding induced by the pullback with $\kappa$, and denote its image by $V_\kappa$. Then, the canonical chart at $\kappa$

$$\phi_\kappa : C^1(\kappa^* D) \to U_\kappa, \quad U_\kappa = \{ f \in C^1(M,N) : \text{Graph}(f) \in V_\kappa \}$$

is given by

$$\phi_\kappa(v)(p) = \exp(v_p). \quad (2.1)$$

It’s inverse $\phi_\kappa^{-1} : U_\kappa \to C^1(\kappa^* D)$ is given by

$$\phi_\kappa^{-1}(f) = (\kappa^* \pi^N, \kappa^* \exp)^{-1}(Id, f).$$

The differentiable structure obtained by the atlas

$$\{(\phi_\kappa, U_\kappa) : \kappa \in C^1(M,N)\}$$

is independent of the choice of connection on $N$. For more detailed constructions see [Eli67, Pal68, Mic80] and for alternative approaches see also [PT01].
Since $D \subset TN$ is open, $C^1(\kappa^* D) \subset C^1(\kappa^* TN)$ is open. Since $C^1(M, N)$ is locally identified with $C^1(\kappa^* D)$, it follows that the tangent space $T_\kappa C^1(M, N)$ is isomorphic to the Banachable space of vector fields along $\kappa$,

$$C^1(\kappa^* TN) \simeq \{ v \in C^1(M, TN) : \pi_N \circ v = \kappa \}.$$ 

The Banach space structure for $C^1(\kappa^* TN)$ may be constructed as follows: Let $|| \cdot || : J^1(\kappa^* TN) \to \mathbb{R}$ be a Finsler structure on $J^1(\kappa^* TN)$, that is, for every $p \in M$, $|| \cdot ||_p : J^1(\kappa^* TN)_p \to \mathbb{R}$ is a norm and $|| \cdot ||$ varies smoothly between the fibers of $J^1(\kappa^* TN)$. Since $M$ is compact, a Finsler structure exists, and moreover, any two Finsler structures on $J^1(\kappa^* TS)$ are equivalent. We define a complete norm on $C^1(\kappa^* TS)$ by

$$||w||_{C^1} = \sup_{p \in M} ||j^1_p w||.$$ 

One may verify that the topology induced by the norm $|| \cdot ||_{C^1}$ on $C^1(\kappa^* TS)$ coincides with its Whitney $C^1$ topology. Thus, the canonical chart $\phi_\kappa$ is indeed a homeomorphisms onto its image.

The tangent bundle $TC^1(M, N)$ may be identified with the bundle

$$C^1(\pi_N) : C^1(M, TN) \to C^1(M, N),$$

where

$$C^1(\pi_N)(w) = \pi_N \circ w.$$ 

Moreover, for every $\kappa \in C^1(M, N)$ the mapping

$$\Phi_\kappa : C^1(\kappa^* D) \times C^1(\kappa^* TN) \to C^1(M, TN),$$

given by

$$\Phi_\kappa(u, w) = T(\exp_{\pi_N(u)})u(w),$$

is a trivialisation for $C^1(M, TN)$ along the canonical chart $\phi_\kappa$ corresponding to the trivialisation $T(\phi_\kappa)$ for $TC^1(M, N)$ under the bundle equivalence. For details see Eliasson [Eli67]. Note that for $(u, w) \in \kappa^* D \times_M \kappa^* TN$

$$T(\exp_{\pi_N(u)})u(w) = J_{\kappa, w}(1)$$

where $J_{\kappa, w} : [0, 1] \to TN$ is the unique Jacobi field along along the geodesic $\exp(tu)$ satisfying $J_{\kappa, w}(0) = 0$ and $D_{\kappa, w} \frac{dt}{dt}(0) = w$. 

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2.3 Connection and metric for $C^1(M, N)$

Following Eliasson [Eli67], we construct a connection for $TC^1(M, N)$. Let $\pi : E \to M$ be a (possibly infinite dimensional) fiber bundle over a smooth manifold $M$ and let $VE \subset TE \to E$ be the vertical bundle defined by $VE = \ker(d\pi)$. An Ehresmann connection is a splitting $\tilde{K} : TE \to VE$ of the short exact sequence

$$
0 \to VE \xrightarrow{t_V} TE \xrightarrow{d\pi} \pi^*TM \xrightarrow{\pi} 0
$$

satisfying $\tilde{K} \circ t_V = \text{id}_{VE}$ where $t_V : VE \to TE$ is the inclusion. $K$ is often referred to as the connection form of the Ehresmann connection. The horizontal bundle $HE$ is then identified with $\ker(\tilde{K})$. In case that $\pi : E \to M$ is a vector bundle we have a canonical identification $VE \simeq \pi^*E$. Thus, $\tilde{K}$ induces a unique mapping $K : TE \to E$ which we call a connection map for $E$.

A linear connection should also satisfy, the following condition: for every $\lambda \in \mathbb{R}$ denote by $S_\lambda : E \to E$ scalar multiplication by $\lambda$, then for every $e \in E$,

$$
K \circ (dS_\lambda)_e = \lambda K_e.
$$

Suppose that $M$ and $E$ are modelled over the Banach spaces $\tilde{M}$ and $\hat{E}$ respectively. Then $K$ has the local form

$$
\tilde{K} : \tilde{M} \times \hat{E} \times \tilde{M} \times \hat{E} \to \tilde{M} \times \hat{E}
$$

$$
\tilde{K}(x, \xi, \tilde{x}, \tilde{\xi}) = (x, S(x, \xi)(\tilde{x}, \tilde{\xi}))
$$

where $S(x, v)(\tilde{x}, \tilde{\xi})$ is linear in $\tilde{x}$ and $\tilde{\xi}$. The condition $K \circ t_V = \text{id}$ implies that $S$ is of the form $S(x, v)(\tilde{x}, \tilde{\xi}) = \tilde{\xi} - \Gamma(x, \xi)(\tilde{x})$ and the linearity condition implies that $\Gamma$ is linear in $\xi$. Thus, a linear connection map has the local form

$$
\tilde{K}(x, \xi, \tilde{x}, \tilde{\xi}) = (x, \tilde{\xi} - \Gamma(x)(\xi, \tilde{x})),
$$
where $\Gamma(x) : \tilde{M} \times \tilde{E} \to \tilde{E}$ is a bilinear transformation called the local connector of $K$ at $x$.

In the particular case where $M$ is finite-dimensional and $E = TM$, the local connector $\Gamma$ is given by the Christoffel symbols,

$$\Gamma(x)(v_i, w_j) = \Gamma^k_{ij}(x)v_iw_j e_k.$$

Given a connection map $K$ for $E$, one can define a covariant derivative $\nabla$ on $E$ in the following way: For a section $\xi \in \Gamma(E)$, set its covariant derivative as $\nabla_\xi = K \circ T_\xi \in \Gamma(\text{Hom}(TM, E))$. That is, for $p \in M$ and $w \in T_pM$

$$(\nabla_w \xi)_p = K(T_\xi(w)) \in E_p.$$ If a section $\xi$ is represented by $\tilde{\xi} : \tilde{M} \to \tilde{E}$, that is, locally $\xi(\cdot) = (\cdot, \tilde{\xi}(\cdot))$, and $w \in T_pM$ has a local representation $(x, \tilde{w})$, then a simple computation gives that the coordinate representation of $(\nabla_w \xi)_p$ is

$$(x, D_{\tilde{\xi}}(x)(\tilde{w}) + \Gamma(x)(\tilde{w}, \tilde{\xi}(x))),$$

where $x$ is the coordinate corresponding to $p$.

Turning back to the problem at hand, let $E = TN$ and let $K^N : T^2N \to TN$ be the connection map corresponding to the Levi Civita connection $\nabla^N$ on $TN$. One can then show (see [Eli67] for details) that $K^N$ induces a connection map

$$C^1(K^N) : T^2C^1(M, N) \simeq C^1(M, T^2N) \to TC^1(M, N) \simeq C^1(M, TN)$$

defined by composition,

$$C^1(K^N)(A) = K^N \circ A, \quad A \in C^1(M, T^2N). \quad (2.4)$$

Denote the corresponding connection by $\nabla^{C^1}$. By definition, for $\xi \in \Gamma(TC^1(M, N))$, $\kappa \in C^1(M, N)$ and $w \in T_\kappa C^1(M, N)$,

$$(\nabla^{C^1}_w \xi)_\kappa = (C^1(K^N) \circ (T_\kappa \xi)(w) = K^N \circ ((T_\kappa \xi)(w)) \quad (2.5)$$

Note that on the right-hand side, $(T_\xi)(w) : M \to T^2N$ and $K^N : T^2N \to TN$, hence, we obtain indeed a map $M \to TN$, i.e., an element of $TC^1(M, N)$. The exponential map for $C^1(M, N)$ with respect to $\nabla^{C^1}$ is given by composition with exp, thus, the canonical coordinate charts $\phi_\kappa$ are normal coordinates in the following sense: for every $\xi \in T_\kappa C^1(M, N)$, $t \mapsto \phi_\kappa(t\xi)$ is a $\nabla^{C^1}$-geodesic.
In particular, the local connector of $\nabla^{C^1}$ in the canonical coordinate chart $\phi_\kappa$ vanishes at the zero section $0 \in C^1(\kappa^*TN)$ (corresponding to $\kappa$).

We next turn to construct a Riemannian metric for $C^1(M,N)$. Assume that a mass form, which is a positive $d$-form $\theta$ on $M$ is given. Using the isomorphism $TC^1(M,N) \simeq C^1(M,TN)$, define a metric $G$ for $C^1(M,N)$ by

$$G_\kappa(u,w) = \int_M \kappa^* G(u,w) \theta, \quad u,w \in T_\kappa C^1(M,N) \simeq C^1(\kappa^*TN).$$

(2.6)

The mass density of $M$ is incorporated in the mass form $\theta$. Locally,

$$\theta = \rho \, dx^1 \wedge \cdots \wedge dx^d,$$

where $\rho : M \to \mathbb{R}_+$ is a mass density function. In cases where $M$ is endowed with a Riemannian metric $g$, it is often natural to take for mass form the Riemannian volume form $\theta = Vol_g$, corresponding to the mass destiny $\rho = \sqrt{\det(g_{ij})}$.

Remark: As always, the metric $G$ induces an isometric immersion $b^G : TC^1(M,N) \to T^* C^1(M,N)$ given by

$$b^G(w) = G(w,\cdot), \quad \forall w \in TC^1(M,N).$$

However, since the manifold is not a Hilbert manifold, $b^G$ is not an isomorphism. For this reason, $G$ is often called a weak Riemannian structure (as opposed to a strong Riemannian structure).

2.4 Metricity of the connection

We next show that the connection $\nabla^{C^1}$ and the metric $G$ for $C^1(M,N)$ are compatible, namely, for $u,v,w \in \Gamma(TC^1(M,N))$,

$$u(G(v,w)) = G(\nabla^C u v, w) + G(v, \nabla^C u w).$$

The metricity of the connection will be used in several instances in the mechanical context.

Lemma 2.1 Let $y \in N$, $v,w \in T_yN$ and $t > 0$ sufficiently small. Then, for every $s \in [0,1]$,

$$J_{tv,w}(s) = J_{t^2v,w}(ts),$$

where $J_{v,w}$ is the Jacobi field as defined in Section 2.2.
Proof: Define $J_1, J_2 : I \to TN$ by $J_1(s) = J_{vw}(s)$ and $J_2(s) = J_{vw/t}(ts)$. We need to prove that $J_1 = J_2$. $J_1$ is a Jacobi field along the geodesic $\gamma(t) = \exp(stv)$. Since $\dot{\gamma}(t) = t\dot{\gamma}(ts)$ and $J_{vw}$ satisfies the Jacobi equation, we get that

$$\frac{D^2}{ds^2} J_2 \bigg|_{s} = t^2 \frac{D^2}{ds^2} J_{vw/t} \bigg|_{ts} = -t^2 R(J_{vw/t}(ts), \dot{\gamma}(ts)), \dot{\gamma}(ts) = -R(J_2(s), \dot{\gamma}(s)), \dot{\gamma}(s).$$

In other words, $J_2$ is also a Jacobi field along $\gamma$. Moreover, $J_1(0) = J_2(0) = 0$ and $D\frac{d}{ds} J_2(0) = t \frac{d}{ds} J_{vw/t}(0) = t \cdot \frac{w}{t} = w = D\frac{d}{ds} J_1(0)$. The result follows from the existence and uniqueness of solutions to ordinary differential equations. 

The following lemma is a standard result in the theory of Jacobi fields (see e.g. [DOC92]).

**Lemma 2.2** Let $y \in N$, and $(y^i)$ be normal coordinates centered at $y$, and let $\gamma(t) = \exp(tv) (v \in T_yN)$ be a radial geodesic emanating from $y$. Then for any $w \in T_yN$ given locally by $w = w^i \partial_{y^i}$, the Jacobi field along $\gamma$ with initial conditions $J(0) = 0, (D/dt) J(0) = w$ is given locally by

$$J(t) = tw^i (\gamma^* \partial_{\gamma^i}).$$

**Theorem 2.1** The connection $\nabla^{c^1}$ is metric with respect to $\mathcal{G}$. In other words, for every $u, v, w \in \Gamma(TC^1(M, N))$

$$(\nabla^{c^1} \mathcal{G})(v, w) := u(\mathcal{G}(v, w)) - \mathcal{G}(\nabla^u v, w) - \mathcal{G}(v, \nabla^u w) = 0.$$

Proof: Let $\kappa \in C^1(M, N)$. It suffices to show that $\nabla^{c^1} \mathcal{G}$ vanishes at some coordinate chart at $\kappa$. Let

$$\phi_\kappa : C^1(\kappa^*D) \to C^1(M, N).$$

be the canonical chart around $\kappa$ and let

$$\Phi_\kappa : C^1(\kappa^*D) \times C^1(\kappa^*TN) \to C^1(M, TN) \simeq TC^1(M, N)$$

be the corresponding trivialization of $TC^1(M, N)$ along $\phi_\kappa$ given by (2.1) and (2.2). Since $\phi_\kappa$ is a normal coordinate chart, the Christoffel symbols (i.e., the local connector) of $\nabla^{c^1}$ vanish at $\kappa$. Therefore, it suffices to prove
that the derivative of the local representative of $\mathcal{G}$ vanishes at the zero section $\vec{0} \in C^1(\kappa^* TN)$ (corresponding to $\kappa$).

The local representative of $\mathcal{G}$,

$$\tilde{\mathcal{G}} : C^1(\kappa^* TN)^* \otimes C^1(\kappa^* TN)^*,$$

is given by

$$\tilde{\mathcal{G}}(\xi_0)(\xi_1, \xi_2) = \mathcal{G}_{\Phi_p(\xi_0)}(\Phi_{\kappa}(\xi_0, \xi_1), \Phi_{\kappa}(\xi_0, \xi_2)),$$

where $\xi_0 \in C^1(\kappa^* D)$ and $\xi_1, \xi_2 \in C^1(\kappa^* TN)$. More explicitly, using (2.3),

$$\tilde{\mathcal{G}}(\xi_0)(\xi_1, \xi_2) = \int_M G_{\exp(\xi_0)}(T(\exp_{\pi}(\xi_0))\xi_1, T(\exp_{\pi}(\xi_0))\xi_2)\theta$$

$$= \int_M G_{\exp(\xi_0)}(J_{\xi_0, \xi_1}(1), J_{\xi_0, \xi_2}(1))\theta.$$

Note that the vector field $J_{\xi_0, \xi_1}(1)$ evaluated at $p \in M$ is given by $J_{\xi_0(p), \xi_1(p)}(1)$.

Now, let $\xi_0 = \vec{0} \in C^1(\kappa^* D)$ (so that $\Phi_{\kappa}(\xi_0) = \kappa$), $\eta \in C^1(\kappa^* TN)$ and $\xi_1, \xi_2$ as before. Then

$$D\tilde{\mathcal{G}}_0(\eta)(\xi_1, \xi_2) = \frac{d}{dt} \bigg|_{t=0} \tilde{\mathcal{G}}(t\eta)(\xi_1, \xi_2)$$

$$= \frac{d}{dt} \bigg|_{t=0} \int_M G_{\exp(t\eta)}(J_{t\eta, \xi_1}(1), J_{t\eta, \xi_2})\theta$$

$$= \int_M \frac{d}{dt} \bigg|_{t=0} G_{\exp(t\eta)}(J_{t\eta, \xi_1}(1), J_{t\eta, \xi_2})\theta$$

$$= \int_M \frac{d}{dt} \bigg|_{t=0} G_{\exp(t\eta)}(J_{t\eta, \xi_1}(t), J_{t\eta, \xi_2}(t))\theta,$$

where in the passage to the third line we interchange integration over $M$ and differentiation with respect to time, and the last equality follows from lemma 2.1.

It suffices to show the the integrand vanishes at every $p \in M$. Let $p \in M$ and $v, u, w \in T_{\kappa(p)} N$ we need to prove that

$$\lim_{t \to 0} \frac{d}{dt} G_{\exp(tv)}(J_{v,u/t}(t), J_{v,w/t}(t)) = 0. \quad (2.7)$$

Since $\nabla^N$ is metric with respect to $G$,

$$\frac{d}{dt} G_{\exp(tv)}(J_{v,u/t}(t), J_{v,w/t}(t)) = G_{\exp(tv)} \left( \frac{D}{dt} J_{v,u/t}(t), J_{v,w/t}(t) \right)$$

$$+ G_{\exp(tv)} \left( J_{v,u/t}(t), \frac{D}{dt} J_{v,w/t}(t) \right).$$
Let \((y^i)\) be normal coordinates centred at \(\kappa(p)\), and let \(\gamma(s) = \exp(s v)\). Then by lemma 2.2, \(J\) is given locally by \(J = sw^i / t(\gamma^* \partial_{y^i})\) hence, \(J(\gamma^*(t)) = w^i \partial_{y^i}|_{\gamma^*(t)}\) is a constant vector field along \(\gamma\) and

\[
\lim_{t \to 0} D_{t=0} J(\gamma(t)) = 0,
\]

which completes the proof.

\[\blacksquare\]

**Remark:** The proof of theorem 2.1 shows in fact, that \(\nabla^C\) is metric with respect to \(\mathcal{G}\) whenever \(\nabla^N\) is metric with respect to \(G\). Note that metricity does not depend on the choice of mass form \(\theta\) for \(B\).

## 3 Elastodynamics

In this section we give a brief review of the geometric setting of elastodynamics. The exposition, which builds upon the geometric construction in Section 2, follows the lines of [KOS17a].

### 3.1 The manifold of configurations

**Definition 3.1** A body manifold \(B\) is a smooth compact and orientable \(d\)-dimensional manifold. A space manifold \(S\) is a smooth orientable \(m\)-dimensional manifold without boundary.

We assume that \(S\) is equipped with a Riemannian metric \(G\) and that \(B\) is equipped with a mass form \(\theta \in \Omega^d(B)\). The canonical charts for \(C^1(B, S)\) are constructed as in Section 2 using the exponential map induced by the Levi-Civita connection \(\nabla^S\) of \(S\).

**Definition 3.2** Denote by

\[
\mathcal{Q} = \text{Emb}^1(B, S)
\]

the space of \(C^1\)-embeddings of \(B\) in \(S\). Let \(I \subset \mathbb{R}\) be a closed time interval. The configuration space,

\[
\mathcal{Q} = C^1(I, \mathcal{Q}),
\]

is the space of \(C^1\)-paths of embeddings of \(B\) in \(S\).
Since $Q$ is an open subset of $C^1(B,S)$ with respect to the Whitney $C^1$-topology (see [Mic80]), it inherits the Banach manifold structure of $C^1(B,S)$. Moreover, as (see [Eli67])

$$C^1(I \times B,S) \simeq C^1(I, C^1(B,S)),$$

we may view $Q$ as an open subset of $C^1(I \times B,S)$. $Q$ therefore inherits the Banach manifold structure of $C^1(I \times B,S)$.

Note that there is a natural inclusion $t_0 : Q \hookrightarrow \mathfrak{Q}$, given by

$$(t_0 \kappa)(t,p) = \kappa(p). \quad (3.1)$$

We refer to $Q$ as the space of stationary configurations.

The tangent bundle $TQ$ is called the bundle of virtual displacements, or generalised velocities. For $\kappa \in \mathfrak{Q}$, an element $v \in T_\kappa Q$ is called a virtual displacement at $\kappa$. As in the general case, we have the isomorphisms,

$$T_\kappa \mathfrak{Q} \simeq C^1(\kappa^*TS) \simeq \{ v \in C^1(I \times B,TS) : \pi_S \circ v = \kappa \},$$

and

$$T\mathfrak{Q} \simeq \{ v \in C^1(I \times B,TS) : \pi_S \circ v \in \mathfrak{Q} \} \subset C^1(I \times B,TS).$$

where the above inclusion is open; in other words, we view $T\mathfrak{Q}$ as an open submanifold of $C^1(I \times B,TS)$.

Denote the restriction of the connection map $C^1(K^S)$ (see (2.4)) to $\mathfrak{Q}$ by $K^\mathfrak{Q}$, that is,

$$K^\mathfrak{Q} : T^2\mathfrak{Q} \subset C^1(I \times B,T^2S) \to T\mathfrak{Q} \subset C^1(I \times B,TS).$$

Denote the corresponding connection by $\nabla^\mathfrak{Q}$, namely,

$$\nabla^\mathfrak{Q} : \Gamma(T\mathfrak{Q}) \times \Gamma(T\mathfrak{Q}) \to \Gamma(T\mathfrak{Q}).$$

The metric $g^\mathfrak{Q}$ for $\mathfrak{Q}$ is given by

$$g^\mathfrak{Q}_{\kappa}(v,w) = \int_{I \times B} \kappa^*G(v,w) \theta \wedge dt, \quad v,w \in T_\kappa \mathfrak{Q} \simeq C^1(\kappa^*TS). \quad (3.2)$$

By Theorem 2.1 $\nabla^\mathfrak{Q}$ is metrically consistent with $g^\mathfrak{Q}$.

Throughout this paper, points in $I \times B$ and $S$ are denoted by $(t,x)$ and $y$ respectively. The indices of coordinates in $I \times B$ will be denoted by Greek letters, whereas indices of coordinates in $S$ will be denoted by Roman letters. A point $(t,x) \in I \times B$ is represented by $(x^\alpha)_{\alpha=0}^d = (t,x^1,\ldots,x^d)$. 

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3.2 Forces and stresses

**Definition 3.3** Let $\kappa \in \mathcal{Q}$. A force at $\kappa$ is an element $f \in T^*_\kappa \mathcal{Q} \simeq (C^1(\kappa^* T S))^*$. The action $f(w)$ of a force $f \in T^*_\kappa \mathcal{Q}$ on a virtual displacement $w \in T_\kappa \mathcal{Q}$ is called a virtual power.

For simplicity, we will focus our attention on forces that are independent of time derivatives; that is, forces $f \in T^*_\kappa \mathcal{Q}$ of the form

$$f(w) = \int f_t(w_t) dt, \quad \forall w \in T_\kappa \mathcal{Q} \simeq C^1(\kappa^* T S), \quad (3.3)$$

where $\{f_t\}_{t \in I}$ is a smooth family of elements $f_t \in T^*_\kappa \mathcal{Q}, \kappa_t = \kappa(t, \cdot) \in \mathcal{Q}$ and $w_t := w(t, \cdot) \in T_{\kappa_t} \mathcal{Q} \simeq C^1(\kappa^*_t T S)$. With a slight abuse of terminology, we refer to elements of $T^*_\mathcal{Q}$ as forces as well.

We therefore turn to present the structure of $T^*_\mathcal{Q}$, the space of forces over stationary configurations. First, note that unlike in finite dimensions, the tangent and cotangent bundles $T \mathcal{Q}$ and $T^* \mathcal{Q}$ are not isomorphic. In particular, given a stationary configuration $\varphi \in \mathcal{Q}$, the dual space $T^*_\varphi \mathcal{Q} \simeq (C^1(\varphi^* T S))^*$ depends on the topology of $C^1(\varphi^* T S)$. Since the topology of $C^1(\varphi^* T S)$ takes into account first derivatives, so do the elements of $(C^1(\varphi^* T S))^*$.

More formally, let $\varphi \in \mathcal{Q}$, and consider the first jet extension

$$j^1 : C^1(\varphi^* T S) \to C^0(J^1(\varphi^* T S)), \quad v \mapsto j^1 v,$$

which is a continuous linear embedding. By the Hahn-Banach theorem, its dual map,

$$(j^1)^* : (C^0(J^1(\varphi^* T S)))^* \to (C^1(\varphi^* T S))^*,$$

is onto. We conclude that to every force $f$ at $\varphi$ corresponds a (non-unique) $\sigma \in (C^0(J^1(\varphi^* T S)))^*$, satisfying

$$f(w) = (j^1)^* \sigma(w) := \sigma(j^1 w), \quad \forall w \in C^1(\varphi^* T S). \quad (3.4)$$

We call $\sigma$ a stress at $\varphi$. We say that a stress $\sigma$ at $\varphi$ represents the force $f$ if the relation $(3.4)$ holds. Note however, that for a given force $f$, there may be more than one stress representing it. This reflects the well-known static indeterminacy of continuum mechanics.

In fact, stresses may also be viewed as cotangent vectors of some other manifold; Let $\mathcal{E} = C^0(J^1(\mathcal{B}, S))$ be the manifold of $C^0$-sections $\mathcal{B} \to J^1(\mathcal{B}, S)$. Then for every $\varphi \in \mathcal{Q}$ one has a canonical isomorphism

$$(C^0(J^1(\varphi^* T S)))^* \simeq T^*_\varphi \mathcal{E}. \quad (3.5)$$
For more details see [KOS17a].

In general, stresses and forces, which are continuous linear functionals on differentiable sections, may be singular. Locally, and in particular, if $\mathcal{B}$ can be covered by a single chart, every stress $\sigma$ is represented by a collection of measures on $\mathcal{B}$,

$$\{\mu_i, \mu^\alpha_i : 1 \leq \alpha \leq d, 1 \leq i \leq m\}$$

by the formula

$$\sigma(j^1w) = \int_{\mathcal{B}} w^i d\mu_i + \int_{\mathcal{B}} w^i_{,\alpha} d\mu^\alpha_i.$$

If the measures $\{\mu_i, \mu^\alpha_i\}$ are absolutely continuous, we may write

$$\mu_i = R_i \text{Vol} \quad \text{and} \quad \mu^\alpha_i = S^\alpha_i \text{Vol},$$

where $R_i, S^\alpha_i \in C^1(\mathcal{B})$ and $\text{Vol} := dx^1 \wedge \cdots \wedge dx^d$. This suggests the following definition (see [Seg86]):

**Definition 3.4** Let $\phi \in Q$. A variational stress density $S$ at $\phi$ is a smooth $d$-form valued in the vector bundle $(J^1(\phi^*T\mathcal{S}))^*$. In other words

$$S \in \Gamma(\text{Hom}(J^1(\phi^*T\mathcal{S}), \Lambda^d T^*\mathcal{B})).$$

We say that a stress $\sigma$ at $\phi$ is smooth, if there exists a variational stress density $S \in \Gamma(\text{Hom}(J^1(\phi^*T\mathcal{S}), \Lambda^d T^*\mathcal{B}))$, such that

$$\sigma(j^1v) = \int_{\mathcal{B}} S(j^1v)$$

for every $v \in C^1(\phi^*T\mathcal{S})$.

Let $S$ be a variational stress density at $\phi$. As shown in [Seg02, Seg13], we may decompose $S$ into body and surface terms as follows,

$$\int_{\mathcal{B}} S(j^1w) = -\int_{\mathcal{B}} \text{div} S(w) + \int_{\partial\mathcal{B}} p_\sigma S(w). \quad (3.6)$$

Here, $\text{div} S$ and $p_\sigma S$ are vector-valued forms,

$$\text{div} S \in \Gamma(\text{Hom}(\phi^*T\mathcal{S}, \Lambda^d T^*\mathcal{B}))$$

$$p_\sigma S \in \Gamma(\text{Hom}(\phi^*T\mathcal{S}, \Lambda^{d-1} T^*\mathcal{B})).$$

In coordinates, the action of a variational stress on the jet extension of a virtual velocity is of form

$$S(j^1w) = (R_i w^i + S^\alpha_i w^i_{,\alpha}) \text{Vol},$$
where \( R_i, S_i^\alpha \in C^1(\mathcal{B}) \). The vector-valued forms \( \text{div} S \) and \( p_\sigma S \) are then given by

\[
\text{div} S(w) = (\text{div} S)_i w^i \text{Vol} \\
p_\sigma S(w) = (p_\sigma S)^\alpha_i w^i \partial_\alpha \text{Vol},
\]

where

\[
(\text{div} S)_i = S_i^\alpha - R_i \quad \text{and} \quad (p_\sigma S)^\alpha_i = S_i^\alpha. \quad (3.7)
\]

Let \( \varphi \in \Omega \). Suppose that a force \( f \in T^*_\varphi \Omega \) is given by body and surface force densities \( b \in \Gamma(\text{Hom}(\varphi^*TS, \Lambda^d T^*\mathcal{B})) \) and \( \mathcal{T} \in \Gamma(\text{Hom}(\varphi^*TS|_{\partial \mathcal{B}}, \Lambda^{d-1} T^*\partial \mathcal{B})) \). That is, for every \( w \in C^1(\varphi^*TS) \)

\[
f(w) = \int_\mathcal{B} b(w) + \int_{\partial \mathcal{B}} \mathcal{T}(w), \quad (3.8)
\]

Then, it follows from (3.6) that \( f \) is represented by a smooth stress \( \sigma \) at \( \varphi \) with variational stress density \( S \),

\[
f(w) = \int_\mathcal{B} S(j^1 w),
\]

if and only if, for every virtual displacement \( w \in C^1(\varphi^*TS) \),

\[
\int_\mathcal{B} (\text{div} S(w) + b(w)) = 0
\]

and

\[
\int_{\partial \mathcal{B}} (p_\sigma S|_{\partial \mathcal{B}}(w) - \mathcal{T}(w)) = 0.
\]

We conclude that \( f \) is represented by a variational stress density \( S \), if and only if

\[
\text{div} S + b = 0 \quad \text{and} \quad p_\sigma S|_{\partial \mathcal{B}} = \mathcal{T}. \quad (3.9)
\]

Equation (3.9) is only a representation theorem. In other words, for every fixed \( \varphi \in \Omega \) and force \( f \in T^*_\varphi \Omega \) of the form (3.8), a smooth stress \( \sigma \), given by a variational stress density \( S \) at \( \varphi \), represents \( f \) if and only if \( S \) satisfies the boundary value problem (3.9).

Note also that equation (3.9) is underdetermined: in local charts it constitutes \( d \) equations for the \( (d \times m + m) \) components \( (R_i, S_i^\alpha) \) of \( S \). In order to obtain a well-posed system, one must specify the dependence of stress and force on the configuration \( \varphi \).

Back to the time-dependent context, of the force \( f \in T^*_\kappa \Omega \) is of the form (3.3), where \( f_t \in T^*_\kappa \Omega \), then there exists a family \( \sigma_t \) of stresses at \( \kappa_t \), such that

\[
f(w) = \int_T \sigma_t(j^1 w_t) dt.
\]
If, furthermore, every $\sigma_t$ is smooth with a family of variational stress densities $S_t$, then

$$f(w) = \int_I \left( -\int_B \text{div} S_t(w_t) + \int_{\partial B} p_\sigma S_t(w_t) \right) dt.$$ 

The representation theorem states then that a force $f$, given by time-dependent body and surface force densities,

$$f(w) = \int_I \left( \int_B b_t(w_t) + \int_{\partial B} T_t(w_t) \right) dt,$$

is represented by a family of smooth stresses with densities $S_t$, then

$$\text{div} S_t + b_t = 0 \quad \text{and} \quad p_\sigma S_t|_{\partial B} = T_t.$$

### 3.3 Loadings and constitutive relations

A mechanical system, whether finite- or infinite-dimensional, is specified by its configuration space, and by a force field, assigning a force to every configuration. It is customary in mechanics to partition the total force $F_T$ into external and internal components; in continuum mechanics external forces are due to loadings, and internal forces result from a constitutive relation.

In our setting, a force field is a 1-form on the configuration space, $F_T \in \Gamma(T^*\mathcal{Q})$. We will focus our interest on time-independent force fields, i.e., force fields induced by section of $T^*\mathcal{Q}$. To this end, define the extension map $E : \Gamma(T^*\mathcal{Q}) \to \Gamma(T^*\mathcal{Q})$,

$$E(F)(\kappa)(w) = \frac{1}{|I|} \int_I F_{\kappa}(w_t) dt, \quad \kappa \in \mathcal{Q}, w \in T_{\kappa}\mathcal{Q}, \quad (3.10)$$

where for every $t \in I$, $\kappa_t$ and $w_t$ were defined above. This extension is natural for the following reason: The inclusion $i_{\Omega}$, defined by (3.1), induces a pullback of sections,

$$i^*_{\Omega} : \Gamma(T^*\mathcal{Q}) \to \Gamma(T^*\mathcal{Q}),$$

defined by

$$(i^*_{\Omega}F)(\varphi)(u) = F_{i_{\Omega}(\varphi)}(T_{i_{\Omega}}(u)), \quad \varphi \in \mathcal{Q}, u \in T_{\varphi}\mathcal{Q}.$$ 

A straightforward calculation shows that $E$ is a right-inverse for $i^*_{\Omega}$,

$$i^*_{\Omega} \circ E = \text{Id}_{\Gamma(T^*\Omega)}, \quad (3.11)$$

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**Definition 3.5** We say that a force field $F \in \Gamma(T^*Q)$ is time-independent if $F = \mathbb{E}(\Phi)$ for some $\Phi \in \Gamma(T^*Q)$.

Thus, a time-independent force field depends on time only through the time dependence of the configuration; moreover, by definition, its action on a virtual displacement $w \in T_\kappa Q$ does not involve explicitly the time derivative of $w$.

With the aid of the extension operator, we now show how the total force is composed from a loading and a constitutive relation:

**Definition 3.6** A loading is a 1-form $\Phi : \Omega \to T^*\Omega$, assigning to every $\phi \in \Omega$ a force $\Phi_\phi \in T^*_\phi \Omega$. A constitutive relation is a section $\Psi : C^0(J^1(B,S)) \to T^*C^0(J^1(B,S))$; its induced force field $(j^1)^*\Psi : \Omega \to T^*\Omega$ is given by

$$( (j^1)^*\Psi)_\phi(w) = \Psi_{j^1\phi}(j^1w).$$

The total force at a given configuration $\kappa \in \Omega$ is given by

$$F_T = \mathbb{E}(\Phi - (j^1)^*\Psi) \in \Gamma(T^*\Omega).$$

That is, for $\kappa \in \Omega$ and $w \in T_\kappa Q \simeq C^1(\kappa^*TS)$

$$(F_T)_\kappa(w) = \frac{1}{|T|} \int_I (\Phi_\kappa(w_t) - \Psi_{j^1\kappa}(j^1w_t)) dt.$$  

Note that by the isomorphism (3.5),

$$( (j^1)^*\Psi)_\phi \in T^*_\phi C^0(J^1(B,S)) \simeq (C^0(J^1(\phi^*TS)))^*.$$  

We next restrict our attention to smooth loading and smooth constitutive relations, which are induced by densities in the form of sections of vector bundles over $B \times S$.

**Definition 3.7** A loading $\Phi \in \Gamma(T^*\Omega)$ is called smooth if there exists a body loading density $b \in \Gamma(\text{Hom}(B \times TS, \Lambda^d T^*B \times S)),$

and a surface loading density

$$\mathcal{T} \in \Gamma(\text{Hom}(\partial B \times TS, \Lambda^{d-1} T^*\partial B \times S)),$$

such that for every $\phi \in \Omega$ and $v \in T_\kappa Q \simeq C^1(\phi^*TS)$

$$\Phi_\phi(v) = \int_B \phi^*b(v) + \int_{\partial B} \phi^*\mathcal{T}(v).$$
Note that
\[ \varphi^*b \in \Gamma(\text{Hom}(\varphi^*T S, \wedge^d T^* B)), \]
so that \( \varphi^*b(\nu) \) is a \( d \)-form on \( B \), as required.

**Definition 3.8** A constitutive relation \( \Psi \) is called smooth if there exists a constitutive density,

\[ \psi \in \Gamma(\text{Hom}(VJ^1(B, S), (\pi^1)^* \wedge^d T^* B)), \]
such that for every \( \varphi \in \Omega \) and \( v \in T\varphi \Omega \)
\[ \Psi_{j^1\varphi}(j^1 v) = \int_B ((j^1 \varphi)^* \psi)(j^1 v). \tag{3.12} \]

Note that in (3.12) we used the canonical isomorphism [KOS17b],
\[ (j^1 \varphi)^* VJ^1(M, N) \simeq J^1(\varphi^* T N). \]

### 3.4 The equations of motion

In this section we establish the equations of motion as a generalization of Newton’s second law of classical mechanics. We view the equations as a section of \( T^* \Omega \), thus, velocity, momentum and acceleration are defined as sections of \( T \Omega \) or \( T^* \Omega \).

The velocity \( V \in \Gamma(T \Omega) \) is defined by
\[ V_\kappa = \frac{\partial \kappa}{\partial t} : T \kappa(\partial_t) \in T\kappa \Omega. \]

The tangent map of \( V \),
\[ TV : T \Omega \to T^2 \Omega \]
can be computed explicitly. Let \( \kappa \in \Omega \) and let \( w \in T\kappa \Omega \) be represented by a path \( \gamma : (-\varepsilon, \varepsilon) \to \Omega \) satisfying \( \gamma(0) = \kappa \) and \( \dot{\gamma}(s) = w \). Then,
\[ (TV)_\kappa(w) = \left. \frac{d}{ds} V_{\gamma(s)} \right|_{s=0} \left. \frac{d}{ds} \left( \frac{d\gamma}{dt} \right) \right|_{s=0} = \left. \frac{dw}{dt} \right|_{s=0} = TV(\partial_t). \]

Note that we view \( w \) as an element of \( C^2(I \times B, T S) \) (with \( \pi_S(w) = \kappa \)), hence, \( dw/dt : I \times B \to T^2 S \); moreover, \( \pi_{T S} \circ dw/dt = V_\kappa \). In other words, \( dw/dt \) is an element of \( C^1(V^*_\kappa T^2 S) \hookrightarrow C^1(I \times B, T^2 S) \), consistent with the isomorphism
\[ T^2 \kappa \Omega \simeq C^1(V^*_\kappa T^2 S). \]
Next, define the acceleration \( A \in \Gamma(T\mathcal{Q}) \) by
\[
A = \nabla^\mathcal{Q}_V V = K^\mathcal{Q} \circ TV(V).
\]
Let \( \kappa \in \mathcal{Q} \). Then, \( A_\kappa \in C^1(\kappa^*T\mathcal{S}) \) is given by
\[
A_\kappa = K^\mathcal{Q}(T(V)(V))_\kappa = K^S \circ ((TV)_\kappa(V_\kappa)) \\
= (K^S \circ TV_\kappa(\partial_t)) = (\kappa^*\nabla^S)_{\kappa}V_\kappa := \frac{DV_\kappa}{dt},
\]
where the second equality follows from the definition of \( K^\mathcal{Q} \), the third equality follows from the expression for \( TV \), and the fourth equality follows from the definition of the pullback connection.

The momentum \( P \in \Gamma(T^*\mathcal{Q}) \) is the dual pairing of the velocity \( V \in \Gamma(T\mathcal{Q}) \) with respect to the metric \( \mathcal{G} \) defined in (3.2),
\[
P = \flat^\mathcal{G}(V) := \mathcal{G}(V, \cdot).
\]
For \( \kappa \in \mathcal{Q} \) and \( w \in T_\kappa\mathcal{Q} \),
\[
P(w) = \mathcal{G}(V, w) = \int_{I \times B} \kappa^*G(V_\kappa, w) \theta \wedge dt.
\]
The inertial force \( DP/\!\!/dt \in \Gamma(T^*\mathcal{Q}) \) is defined by
\[
\frac{DP}{dt} := \nabla^\mathcal{Q} V P,
\]
where \( \nabla^\mathcal{Q} \) is the dual connection of \( \nabla^\mathcal{Q} \) for \( T^*\mathcal{Q} \): given \( \xi \in \Gamma(T\mathcal{Q}) \),
\[
\frac{DP}{dt}(\xi) = \left( \nabla^\mathcal{Q} V P \right)(\xi) = V \cdot P(\xi) - P(\nabla^\mathcal{Q} V \xi).
\]

**Proposition 3.1** The inertial force is dual to the acceleration
\[
\frac{DP}{dt} = \flat^\mathcal{G}(A).
\]
**Proof:** Let \( \xi \in \Gamma(T\mathcal{Q}) \). By Theorem 2.11, \( \nabla^\mathcal{Q} \) is metric with respect to \( \mathcal{G} \). Hence,
\[
V \cdot P(\xi) = V \cdot \mathcal{G}(V, \xi) = \mathcal{G}(\nabla^\mathcal{Q}_V V, \xi) + \mathcal{G}(V, \nabla^\mathcal{Q}_V \xi),
\]
and
\[
\frac{DP}{dt}(\xi) = V \cdot P(\xi) - P(\nabla^\mathcal{Q}_V \xi) = \mathcal{G}(A, \xi) = \flat^\mathcal{G}(A)(\xi).
\]
\[\blacksquare\]
The equations of motion equate the inertial force with the forces induced by loadings and constitutive relations,

\[
\frac{DP}{dt} = \mathcal{E}(\Phi - (j^1)^*\Psi).
\]  

(3.14)

It is an equation taking values in \( T^*\mathcal{Q} \); its solutions are configurations \( \kappa \in \mathcal{Q} \). Generally, (3.14) has to be augmented by initial conditions; boundary conditions are already incorporated in the loadings and the constitutive relations.

Loadings and the constitutive relations may be singular, in which case (3.14) may not have a local differential form. If the loading and the constitutive relation are smooth, then (3.14) at \( \kappa \) transforms into

\[
\int_{I \times \mathcal{B}} \kappa^* G(A_{\kappa}, w) \theta \wedge dt = \int_{I} \int_{\mathcal{B}} \left( \kappa^* b(w_t) + \text{div}(j^1 \kappa)^* \Psi(w_t) \right) dt \\
+ \int_{I} \int_{\partial \mathcal{B}} \left( \kappa^* T(w_t) - p_{\sigma}((j^1 \kappa)^* \Psi)(w_t) \right) dt, \quad \forall w \in T_{\kappa}\mathcal{Q}.
\]  

(3.15)

Since equation (3.15) holds for every vector field \( w \), we obtain the following differential system:

\[
\kappa^* G(A_{\kappa}, \cdot) \theta = \kappa^* b(w) + \text{div}(j^1 \kappa)^* \Psi,
\]  

(3.16)

which is an identity of vector valued forms in \( I \times \mathcal{B} \) together with boundary conditions

\[
T_{\kappa} = p_{\sigma}((j^1 \kappa)^* \Psi)|_{I \times \partial \mathcal{B}}.
\]

A stationary configuration \( \varphi \in \mathcal{Q} \) is called an equilibrium configuration if

\[
\Phi_{\varphi} - ((j^1)^*\Psi)_{\varphi} = 0
\]

or equivalently, if the constant motion \( t_{\mathcal{Q}}(\varphi) \in \mathcal{Q} \) is a solution of (3.14). In the smooth case, the equilibrium condition yields the boundary value problem

\[
\text{div}(j^1 \varphi)^* \Psi + \varphi^* b = 0 \quad \text{in } \mathcal{B} \quad T_{\varphi} = p_{\sigma}((j^1 \varphi)^* \Psi_{\varphi}) \quad \text{on } \partial \mathcal{B}.
\]

Remark: The solution of the force free equation

\[
\frac{DP}{dt} = \mathcal{S}(A, \cdot) = 0
\]

is a geodesic flow of \( \mathcal{B} \) in \( S \). This is a covariant version of Newton’s law of inertia for non-Euclidian continuum dynamics.
3.5 The hyperelastic case

Definition 3.9 A constitutive relation $\Psi$ is called conservative if there exists a differentiable function $U : C^0(j^1(\mathcal{B}, \mathcal{S})) \to \mathbb{R}$ such that for every $\phi \in \Omega$

$$\Psi_{j^1 \phi} = (dU)_{j^1 \phi} \in T^*_j \Psi C^0(J^1(\mathcal{B}, \mathcal{S})) \simeq C^0(J^1(\mathcal{B}, \mathcal{S}))^*.$$

A constitutive relation $\Psi$ is called hyperelastic if $\Psi$ is conservative and $U$ is of the form

$$U(j^1 \phi) = \int_{\mathcal{B}} L(j^1 \phi) \theta,$$

where $L \in C^\infty(J^1(\mathcal{B}, \mathcal{S}))$ is a Lagrangian density function.

Proposition 3.2 Let $\Psi$ be a hyperelastic constitutive relation with Lagrangian destiny $L$. Then $\Psi$ is smooth and the constitutive density $\psi$ is given by

$$\psi = \delta L \otimes \theta$$

where $\delta L$ is the fiber derivative of $L$, i.e., the restriction of $dL$ to $VJ^1(\mathcal{B}, \mathcal{S})$. Thus, for every $\phi \in \Omega$, $(j^1 \phi)^* \psi = \delta_{j^1 \phi} L \otimes \theta$ where $\delta_{j^1 \phi} L := (j^1 \phi)^* \delta L \in \Gamma(J^1(\mathcal{B}, \mathcal{S}))^*$.

For a proof see [KOS17b].

Locally, $L$ is represented by a function $\mathbb{R}^m \times \mathbb{R}^{d \times m} \to \mathbb{R}$, and for every $w \in T_{\phi} Q$

$$(j^1 \phi)^* \psi(w_i, w_{i,\alpha}) = (R_i w^j + \psi^{i,\alpha}_j) \text{Vol},$$

where

$$R_i = \rho \frac{\partial L}{\partial y^i} (j^1 \phi) \quad \text{and} \quad \psi^{i,\alpha}_j = \rho \frac{\partial L}{\partial y^i} (j^1 \phi),$$

(3.17)

and $\rho$ is the mass density. In the absence of external loadings the equation of motion (3.16) take the form

$$G_{ij} \left( \frac{\partial^2 \kappa^i}{\partial t^2} + \Gamma^i_{lk} \frac{\partial \kappa^j}{\partial t} \frac{\partial \kappa^k}{\partial t} \right) = \frac{1}{\rho} \partial_{\alpha} \left( \rho \frac{\partial L}{\partial y^i} (j^1 \kappa) \right) - \frac{\partial L}{\partial y^i} (j^1 \kappa),$$

(3.18)

with boundary conditions

$$\frac{\partial L}{\partial y^i} (j^1 \kappa) (\partial_{\alpha} \cdot \text{Vol}) = 0 \quad \text{on} \; I \times \partial \mathcal{B}.$$

Eq. (3.18) is the equation of motion for the configuration $\kappa$ of a hyperelastic body in the absence of external loadings. It should supplemented by initial conditions $\kappa_0 \in \Omega$ and $V_0 \in T_{\kappa_0} \Omega$. 

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4 Linearization

We begin by defining the notion of linearization in a general context:

**Definition 4.1** Let \( \pi : E \to M \) be a (possibly infinite dimensional) vector bundle, \( \nabla \) a connection on \( E \) and \( s \in \Gamma(E) \). The linearization of \( p \) at a point \( p \in M \), denoted by \( L_p s \in \text{Aff}(T_pM, E_p) \), is defined by

\[
L_p s(v) := s_p + (\nabla_v s)_p, \quad v \in T_pM.
\]

Linearizations are used, in particular, in the following context: one seeks a solution \( p \in M \) to the (generally nonlinear) equation

\[
s_p = 0.
\]

Instead, one takes an approximate solution \( p_0 \), and then solves the linear equation

\[
L_p s(v) = 0.
\]

Then, \( p_1 = \exp_{p_0}(v) \) can be viewed as a next order iterate for the solution.

In our setting, \( E = T^*Q, M = Q \) and the section \( s \in \Gamma(T^*Q) \) is given by the equations of motion (3.14)

\[
s = \frac{DP}{dt} - \mathcal{E}(\Phi - (j^1)^\Psi) = 0.
\]

In this case, \( L_\kappa s \in \text{Aff}(T_\kappa Q, T_\kappa^*Q) \). The linearized equation of motion at \( \kappa \in Q \) for \( w \in T_\kappa Q \) is

\[
L_\kappa \left( \frac{DP}{dt} \right)(w) = L_\kappa(\mathcal{E}(\Phi - (j^1)^\Psi))(w). \tag{4.1}
\]

A solution \( w \in T_\kappa Q \) for (4.1) induces an approximate solution \( \kappa_1 = \phi_\kappa(w) \in Q \) to (3.14), where \( \phi_\kappa \) is a canonical chart at \( \kappa \).

Note that a solution \( w \) of (4.1) satisfies

\[
L_\kappa \left( \frac{DP}{dt} \right)(\xi)(\xi) = L_\kappa(\mathcal{E}(\Phi - (j^1)^\Psi))(\xi)(\xi), \tag{4.2}
\]

for every \( \xi \in T_\kappa Q \). In order to compute (4.2) explicitly, one needs to consider a local extension of \( \xi \), that is a local section \( \xi_0 \in \Gamma(TQ) \) satisfying \( \xi_0 = \xi \) and the same value is obtained regardless of how \( \xi \) is extended in a vicinity of \( \kappa \). Noting that \( \xi \in C^1(\kappa^*TS) \) is a vector field along \( \kappa \), we may extend \( \xi \) to a vector field on \( S \), \( \xi_0 \in \Gamma(TS) \). In particular, \( \xi = \kappa^*\xi_0 \). Thus, it suffices to impose that (4.2) be satisfied for \( \xi \) of the form

\[
\xi : \kappa \mapsto \kappa^*\xi_0,
\]

where \( \xi_0 \in \Gamma(TS) \).
4.1 Linearization of acceleration term

Let $\kappa \in \mathcal{Q}$ and $w \in T_\kappa \mathcal{Q}$, then

$$L_\kappa \left( \frac{DP}{dt} \right) (w) = L_\kappa (b^G(A))(w) = b^G(A_\kappa) + \nabla_\kappa^G \left( b^G(A) \right).$$

We therefore turn to compute $\nabla_\kappa^G \left( b^G(A) \right)$. Let $\xi \in \Gamma(T\mathcal{Q})$, then by the metricity of $G$,

$$\nabla_\kappa^G \left( b^G(A) \right)(\xi) = \left( w \cdot b^G(A)(\xi) \right)_\kappa - b^G(A)(\nabla_\kappa^G \xi) = G(\nabla_\kappa^G A, \xi)_\kappa.$$

In other words, metricity implies that

$$\nabla_\kappa^G \left( b^G(A) \right) = \nabla^G \left( \nabla_\kappa^G A \right).$$

It remains to compute $\nabla^G \nabla_\kappa^G A$. Let $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{Q}$ be a curve representing $w \in T_\kappa \mathcal{Q}$, that is $\gamma(0) = \kappa$ and $\dot{\gamma}(0) = w$. Then,

$$\nabla^G \nabla_\kappa^G A = K^G \circ T A_\kappa(w) = \left. \frac{d}{ds} A_{\gamma(s)} \right|_{s=0}.$$

Hence by the definition of the pullback connection $\gamma^* \nabla^S$,

$$\nabla^G \nabla_\kappa^G A = \frac{d}{ds} \left|_{s=0} \frac{D}{dt} \frac{d\gamma}{dt} \right. = \frac{D^2}{dt^2} \left. \left( \frac{d\gamma}{ds} \right|_{s=0} \right) - R^S \left( \left. \frac{d\gamma}{dt} \right|_{s=0}, \left. \frac{d\gamma}{ds} \right|_{s=0}, \left. \frac{d\gamma}{dt} \right|_{s=0} \right) = \frac{D^2 w}{dt^2} + R^S(w, V_{\kappa}, V_{\kappa}),$$

where $R^S$ is the curvature tensor corresponding to $\nabla^S$,

$$R^S(X, Y, Z) = \nabla^S_X \nabla^S_Y Z - \nabla^S_Y \nabla^S_X Z - \nabla^S_{[X, Y]} Z, \quad X, Y, Z \in \Gamma(TS).$$

To conclude, for every $\kappa \in \mathcal{Q}$ and $w \in T_\kappa \mathcal{Q} \simeq C^1(\kappa^*TS)$,

$$L_\kappa \left( \frac{DP}{dt} \right) (w) = b^G(A_\kappa) + b^G \left( \frac{D^2 w}{dt^2} + R^S(w, V_{\kappa}, V_{\kappa}) \right). \tag{4.3}$$

In other words, the linearization of acceleration term is the Jacobi equation.
4.2 Linearization of force

We now turn to linearize the right-hand side of the equations of motion (3.14). Without loss of generality, we may assume that there are no external loadings, \( \Phi = 0 \), as the loading may be incorporated into the constitutive relation \( \Psi \). Thus the total force is given by,

\[
F_{\kappa}(u) = \mathbb{E}((j^1)^*\Psi)_{\kappa}(u) = \frac{1}{|I|} \int_I \Psi_{j^1\kappa}(j^1u_t) \, dt, \quad \kappa \in \mathcal{Q}, u \in T_{\kappa}\mathcal{Q}. \tag{4.4}
\]

Then,

\[
L_{\kappa}(F)(w) = F_{\kappa} + (\nabla_{\kappa}^{\mathcal{Q}} F)_{\kappa},
\]

where by definition, for \( \xi \in \Gamma(T\mathcal{Q}) \),

\[
(\nabla_{\kappa}^{\mathcal{Q}} F)(\xi) = w(F(\xi)) - F(\nabla_{\kappa}^{\mathcal{Q}} \xi).
\]

**Lemma 4.1** Let \( F \in \Gamma(T^*\mathcal{Q}) \) be given by (4.4). Then, for every vector field \( \xi \in \Gamma(T\mathcal{Q}) \),

\[
L_{\kappa}(F)(w)(\xi) = \frac{1}{|I|} \int_I L_{\kappa}^{\mathcal{Q}}((j^1)^*\Psi)(w_t)((\xi_{\kappa})_t) \, dt, \tag{4.5}
\]

where the linearization on the right-hand side takes place in the space of stationary configurations \( \mathcal{Q} \).

**Proof**: The constant part of the identity is immediate since \( F \) is the extension of \( (j^1)^*\Psi \). To proceed as noted above, it suffices to consider vector field \( \xi \) of the form \( \kappa \mapsto \kappa^*\xi^S \), where \( \xi^S \in \Gamma(T\mathcal{S}) \). Note also that the mapping \( \varphi \mapsto \varphi^*\xi^S \) for \( \varphi \in \mathcal{Q} \) is a section of \( T\mathcal{Q} \), which we denote by \( \xi^Q \). Then,

\[
(\xi_{\kappa})_t = \xi_{\kappa^t}^Q.
\]

It remains to show the identity of the linear parts: that for every \( \xi^S \in \Gamma(T\mathcal{S}) \),

\[
w(F(\xi)) = \frac{1}{|I|} \int_I w_t \left((j^1)^*\Psi(\xi^Q_t)\right) \, dt, \tag{4.6}
\]

and

\[
F(\nabla_{\kappa}^{\mathcal{Q}} \xi) = \frac{1}{|I|} \int_I (j^1)^*\Psi(\nabla_{\kappa}^{\mathcal{Q}} \xi^Q) \, dt. \tag{4.7}
\]

To show (4.6), let \( \gamma : (-\varepsilon, \varepsilon) \to \mathcal{Q} \) satisfy \( \gamma(0) = \kappa \) and \( \dot{\gamma}(0) = w \), and let \( \gamma_t : (-\varepsilon, \varepsilon) \to \mathcal{Q} \) be the evaluation of \( \gamma \) at time \( t \), so that \( \dot{\gamma}_t(0) = w_t \in T_{\kappa}\mathcal{Q} \).
Then,
\[
\frac{d}{ds}w(F(\xi)) = \frac{d}{ds}\left. \left( \frac{1}{|I|} \int_I j^*\gamma(s)(j^1\xi^\Omega) dt \right) \right|_{s=0} = \frac{1}{|I|} \int_I j^*\gamma(s)(j^1\xi^\Omega) dt = \frac{1}{|I|} \int_I w_1((j^1)^*\gamma(\xi^\Omega)) dt.
\]

To show (4.7), we first simplify the term \(\nabla_{\xi w}^\xi \). By the chain rule,
\[
(T^\xi)_{\kappa}(w) = \frac{d}{ds}\left. \xi_{\gamma(s)} \right|_{s=0} = \frac{d}{ds}\left. \gamma(s)^*\xi^\Omega = \kappa^*T^\xi \circ w, \right.
\]
which is an identity in \(T^2_{\xi_w} \mathcal{Q} \simeq C^1(\xi_wT^2S)\). Hence,
\[
\nabla_{\xi w}^\xi = K^C((T^\xi)_{\kappa}(w)) = K^\xi \circ \kappa^*T^\xi \circ w = \kappa^*(\nabla^\xi \xi^\Omega)(w),
\]
where \(\kappa^*(\nabla^\xi \xi^\Omega) \in \Gamma(\text{Hom}(\kappa^*TS, \kappa^*TS))\) is the pullback of \(\nabla^\xi \xi^\Omega \in \Gamma(\text{Hom}(T\xi S, T\xi S))\).

For \( t \in I \),
\[
(\nabla_{\xi w}^\xi)_t = \kappa^*(\nabla^\xi \xi^\Omega)(w_t) = \nabla_{\xi_t w}^\xi \xi^\Omega,
\]
where the last equality follows from the calculation yielding (4.8) over \( \mathcal{B} \), rather than \( I \times \mathcal{B} \). Then,
\[
F(\nabla_{\xi w}^\xi) = \frac{1}{|I|} \int_I (j^1)^*\gamma((\nabla_{\xi w}^\xi)_t) dt = \frac{1}{|I|} \int_I (j^1)^*\gamma(\nabla_{\xi_t w}^\xi \xi^\Omega) dt,
\]
which concludes the proof. \( \square \)

For every \( \xi^\Omega \in \Gamma(T\xi S) \) and \( \varepsilon > 0 \) sufficiently small, consider the vector field \( \xi_\varepsilon \in \Gamma(T\xi \mathcal{Q}) \) given by \( \kappa \to \chi_\varepsilon(\kappa^*\xi^\Omega) \), where \( \chi_\varepsilon : I \times \mathcal{B} \to \mathbb{R} \) is a smooth cutoff function supported on \( (-\varepsilon, \varepsilon) \times \mathcal{B} \). By evaluating (4.5) and (4.3) at \( \xi_\varepsilon \) and letting \( \varepsilon \to 0 \) the linearized equations of motion (4.2) can be localized in time:

**Corollary 4.1** For a time-independent force induced by a constitutive relation \( \Psi \) and zero loading, the linearized equations of motion (4.2) is local in time; \( w \in T_{\kappa}\mathcal{Q} \) solves (4.2) if and only if, for every \( t \in I \),
\[
\int_{\mathcal{B}} G \left( (A_{\kappa})_t + \frac{D^2w}{dt^2} + R^\xi(w_t, (V_{\kappa})_t, (V_k)_t), \right) \theta = L^\xi_{\kappa}(j^1)^*\gamma(\xi^\Omega) \]
which is an equality of co-vectors in \( T_{\kappa}\mathcal{Q} \).
In view of (4.5), we need to calculate linearizations of the form

\[ L_\phi((j^1)^*\Psi)(v), \]

where \( \phi \in Q \) and \( v \in T_\phi Q \). We focus on the case where \( \Psi \) is smooth, given by the constitutive density \( \psi \in \Gamma(\text{Hom}(VJ^1(B,S), (\pi_1)^*\Lambda^d\overline{T}^*B)) \).

For every vector field \( \xi Q \in \Gamma(TQ) \),

\[
L_\phi((j^1)^*\Psi)(v)(\xi Q) = \int_B \psi j^1 \phi (j^1 \xi Q) + v \left( \int_B \psi j^1 \phi (j^1 \xi Q) - \int_B \psi j^1 \phi (j^1 \nabla^Q \xi Q) \right) - \int_{\partial B} \psi \sigma (\xi Q)
\]

where we substituted the decomposition (3.6) of \( \psi \) into a divergence term and a boundary term.

To further simplify the last equation, we note that for \( \xi^Q \) of the form \( \phi \mapsto \phi^* \xi^S \), in which case

\[
(\nabla^Q v \xi^Q) = (\phi^* \nabla^S \xi^S)(v).
\]

Moreover, the equation is tensorial in \( \xi^Q \), so that it can be represented as

\[
L^Q_\phi((j^1)^*\Psi)(v)(\xi^Q) = ((j^1)^*\Psi)\phi(\xi^Q) + \int_B A(\phi,v)(\xi^Q) + \int_{\partial B} B(\phi,v)(\xi^Q),
\]

where \( A(\phi,v) \) is a \( d \)-form valued in \( (\kappa^* T^*S)^* \) and \( B(\phi,v) \) is a \( (d-1) \)-form valued in the same vector bundle. At this stage and generality, \( A(\phi,v) \) and \( B(\phi,v) \) cannot be significantly simplified. We therefore turn to calculate their local representatives in a given coordinate chart.

4.3 Local form of the linearized equations of motion

Substituting (4.9) into Corollary 4.1, we obtain linearized equations of the motion in local form,

\[
\gamma^G \left( A_\kappa + \frac{D^2 W}{dt^2} + R^S (w, V_\kappa) \right) \otimes \theta = - \text{div}(\psi j^1 \kappa) + A(\kappa, w) \tag{4.10}
\]
\[ p_\sigma(\psi_{j_1^\kappa}) + B(\kappa, w) = 0 \]
in \( I \times B \) and
\[ p_\sigma(\psi_{j_1^\kappa}) + B(\kappa, w) = 0 \]
in \( I \times \partial B \).

If \( \kappa \) is a solution of (3.14), then only the terms that are linear in \( w \) remain.
In the particular case where \( \kappa = t_2(\varphi) \) is a stationary solution of (3.14), \( V_\kappa = 0 \), hence
\[ \mathcal{J}_\kappa \left( \frac{D_w^2}{dt^2} \right) \otimes \theta = A(\kappa, w) \quad \text{in} \ I \times B, \]
and
\[ B(\kappa, w) = 0 \quad \text{in} \ I \times \partial B. \]

Moreover, since \( \text{div}(\psi_{j_1^\kappa}) = 0 \) and \( p_\sigma(\psi_{j_1^\kappa}) = 0 \), the implicit expressions for \( A \) and \( B \) reduce to
\[ \int_B A(\varphi, w)(\xi) = -v \left( \int_B \text{div}(\psi_{j_1^\kappa})(\xi) \right) \]
\[ \int_B B(\varphi, w)(\xi) = v \left( \int_{\partial B} p_\sigma(\psi_{j_1^\kappa})(\xi) \right). \]

### 4.4 Coordinate representation

We hereby give a local expression for the terms \( A(\kappa, w) \) and \( B(\kappa, w) \) in (4.10) for the general case. For \( (t, p) \in I \times B \) let \( x_\alpha : U_p \subset B \rightarrow \mathbb{R} \) (1 \( \leq \alpha \leq d \)) and \( y_i : V_{\kappa(t, p)} \subset S \rightarrow \mathbb{R} \) (1 \( \leq i \leq m \)) be coordinate charts for \( B \) and \( S \) at \( p \) and \( \kappa(t, p) \) respectively such that \( \kappa(I \times U_p) \subset V_{\kappa(t, p)} \). Then \( \kappa |_{I \times U_p} \) is represented by
\[ \kappa^i = y^i \circ \kappa \circ (\text{id}, x^{-1}) : I \times \mathbb{R}^d \rightarrow \mathbb{R}. \]

\( w \in C^1(\kappa^*TS) \) then has the local form
\[ w = w^i \kappa^i \partial_i \]
where \( w^i : I \times B \rightarrow \mathbb{R} \) and \( \{ \partial_i \}_{i=1}^m \) is the local frame for \( TS \) induces by the charts \( y^i \). With a slight abuse of notation, let
\[ (x^\alpha, y^i, A_\alpha) : J^1(U_p, V_{\varphi(p)}) \subset J^1(B, S) \rightarrow \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{d \times m} \]
be the induced coordinate chart for \( J^1(B, S) \). That is, for \( j^i_q f \in J^1(U_p, V_{\varphi(p)}) \),
\[ x^\alpha(j^i_q f) = x^\alpha(q), \quad y^i(j^i_q f) = y^i(f(q)) \quad \text{and} \quad A_\alpha(j^i_q f) = \frac{\partial f^i}{\partial x^\alpha}(x(q)). \]
The variational stress density $\psi \in \Gamma(L(VJ^1(\mathcal{B}, S), \Lambda^d T^* \mathcal{B}))$ has the local form

$$\psi = (\psi^\alpha_i dA^i_\alpha + R_j dy^j) \text{ Vol}$$

where $\text{Vol} = dx^1 \wedge \cdots \wedge dx^d$ and $\psi^\alpha_i, R_j : J^1(U, V_{\psi(p)}) \to \mathbb{R}$, hence,

$$\psi^i_j \kappa (j^i w_j) = (\psi^\alpha_i (j^i \kappa) w_j^i + R_j (j^i \kappa) w_j^i) \text{ Vol}.$$ 

Finally, denote by $\Gamma^{ij} : V_{\psi(p)} \to \mathbb{R}$ the Christoffel symbols of $\nabla^g$. A straightforward calculation then gives:

Let $\Psi$ be a smooth constitutive relation represented by a constitutive density $\psi$ and let $\kappa \in \mathcal{G}$. Then the linearization of the equation of motion $\frac{DP}{dt} = E((j^i)^* \Psi)$, at $\kappa \in \mathcal{G}$,

$$L_\kappa \left( \frac{DP}{dt} \right)(w) = L_\kappa (E((j^i)^* \Psi))(w), \quad w \in T_\kappa \mathcal{G}$$

has the local form,

$$\rho G_{ij} \left( \frac{D^2 w^i}{dt^2} + R_{kli} \frac{\partial \kappa^h}{\partial t} \frac{\partial \kappa^k}{\partial t} \frac{\partial w^l}{\partial t} \right) = A^1 (\kappa^i_{ij}) w^i + A^2 (\kappa^i_{ij}) w^i_{\delta} + A^3 (\kappa^i_{ij}) w^i_{\alpha \beta}$$

$$+ (\text{div} \psi_{\kappa})_k (w^i \Gamma^k_{ij} - (\text{div} \psi_{\kappa})_j),$$

(4.11)

in $I \times \mathcal{B}$, and

$$\left( \psi^\alpha_i (j^i \left( 1 - w^j \Gamma^j_{ij} \right) \frac{\partial \psi^\alpha_{ij}}{\partial x^i} w^j + \frac{\partial \psi^\alpha_{ij}}{\partial A_{ij}^\beta} \frac{\partial w^j}{\partial A_{ij}^\beta} \right) (t_{\alpha u} \text{ Vol}) \mid_{\partial \mathcal{B}}$$

on $I \times \partial \mathcal{B}$. The function $A_1, A_2$ and $A_3$ are given by

$$A^1 (\varphi)_{ij} = \frac{\partial \psi^\alpha_{ij}}{\partial x^i} \frac{\partial x^i}{\partial \varphi^\alpha_{ij}} + \frac{\partial \psi^\alpha_{ij}}{\partial y^i} \frac{\partial y^i}{\partial \varphi^\alpha_{ij}} + \frac{\partial \psi^\alpha_{ij}}{\partial A_{ij}^\alpha} \frac{\partial A_{ij}^\alpha}{\partial \varphi^\alpha_{ij}} - \frac{\partial R_{ij}}{\partial \varphi^\alpha_{ij}},$$

$$A^2 (\varphi)_{ij} = \frac{\partial \psi^\alpha_{ij}}{\partial A_{ij}^\delta} \frac{\partial x^i}{\partial \varphi^\alpha_{ij}} + \frac{\partial \psi^\alpha_{ij}}{\partial A_{ij}^\delta} \frac{\partial y^i}{\partial \varphi^\alpha_{ij}} + \frac{\partial \psi^\alpha_{ij}}{\partial A_{ij}^\alpha} \frac{\partial A_{ij}^\alpha}{\partial \varphi^\alpha_{ij}} - \frac{\partial R_{ij}}{\partial A_{ij}^\alpha},$$

and

$$A^3 (\varphi)_{ij} = \frac{\partial \psi^\alpha_{ij}}{\partial A_{ij}^\beta},$$

In these equations, the entries $G_{ij}, R^i_{kli}$ and $\Gamma^k_{ij}$ of the metric, the curvature and the connection are evaluated at $\kappa$; the entries $\psi^\alpha_{ij}, R_j$ of the constitutive density and their derivatives are evaluated at $j^i \kappa$. 

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