LOCALISED SOLUTIONS OF THE DIRAC-MAXWELL EQUATIONS

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Abstract

The full classical Dirac-Maxwell equations are considered in a somewhat novel form and under various simplifying assumptions. A reduction of the equations is performed in the case when the Dirac field is static. A further reduction of the equations is made under the assumption of spherical symmetry. These static spherically symmetric equations are examined in some detail and a numerical solution presented. Some surprising results emerge from this investigation:

- Spherical symmetry necessitates the existence of a magnetic monopole.
- There exists a uniquely defined solution, determined only by the demand that the solution be analytic at infinity.
- The equations describe highly compact objects with an inner onion like shell structure.
INTRODUCTION

It is an interesting exercise to compare the current development of a quantum theory of gravitation, from the fully non-linear Einstein theory, to the development of QED from the linearized Dirac-Maxwell theory. The most startling difference is the large body of work on the classical, non-linear, theory of gravitation (General Relativity) – a theory which includes, in a self consistent manner, the interactions of the gravitational field itself. There is no comparable body of work on the full Dirac-Maxwell theory (Dirac equations with electromagnetic interaction, Maxwell equations with Dirac field source - the so-called “self interaction”). Of course this situation arose, historically, because of the rapid development and stunning success of QED.

Einstein’s equations provide a much better description of gravity than do the linear spin-2 equations. Indeed, one can “derive” classical general relativity from the linear, massless spin-2 theory by summing all the Feynman diagrams to tree level – see [1] and [2]. The full Dirac-Maxwell equations should provide a much better description of electronic matter than their linearized counterparts (in which self terms are ignored).

Solutions to the Dirac-Maxwell equations are rare, see [3]. There are, however, a number of solutions to the Yang-Mills-Dirac and Yang-Mills-Dirac-Higgs equations, see [3] and the comprehensive list of references contained therein.

The paper is organized as follows: in §1 we write the equations in 2-spinor form, this description then allows us to (covariantly) solve the Dirac equations for the electromagnetic potential and so write down a complete set of equations in terms of the Dirac field. In §2 we examine the static and spherically symmetric reductions of the equations. In §3 we deal with some general properties of the static spherically symmetric system and in §4 present a numerical solution to this system.

§1 The Dirac-Maxwell Equations

In standard notation the Dirac-Maxwell equations are

\[
\gamma^\alpha (\partial_\alpha - i e A_\alpha) \psi + im\psi = 0
\]

\[
F_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta}
\]

\[
\partial^\alpha F_{\alpha\beta} = -4\pi e j_\beta = -4\pi e \bar{\psi} \gamma_\beta \psi
\]

Employing the $\gamma_5$-diagonal or van der Waerden description, see for example [4], we have

\[
\gamma^\alpha = \sqrt{2} \begin{pmatrix}
0 & \sigma^\alpha_B \\
\sigma^{\alpha A} & 0
\end{pmatrix}
\]
with $\sigma_{\dot{A}\dot{A}}^\alpha$ the van der Waerden symbols, i.e.

\[
(\sigma_{\dot{A}\dot{A}}^0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad (\sigma_{\dot{A}\dot{A}}^j) = \frac{1}{\sqrt{2}} \times \text{Pauli Matrix}, \ j = 1, 2, 3.
\]

Where $A, B = 0, 1$ and $\dot{A}, \dot{B} = \dot{0}, \dot{1}$ are two-spinor indices (see [Pe]).

The Dirac bispinor, $\psi$ is

\[
\psi = \begin{pmatrix} u_A \\ \pi^\dot{B} \end{pmatrix} \quad \text{and} \quad \psi = (v^B, \pi^\dot{A})
\]

So that the Dirac equations become

\[
(\partial^{\dot{A}\dot{A}} - ieA^{\dot{A}\dot{A}})u_A + \frac{im}{\sqrt{2}}\pi^\dot{A} = 0
\]

\[
(\partial^{\dot{A}\dot{A}} + ieA^{\dot{A}\dot{A}})v_A + \frac{im}{\sqrt{2}}\pi^\dot{A} = 0
\]

(2)

where $\partial^{\dot{A}\dot{A}} \equiv \sigma^{\alpha\dot{A}\dot{A}}\partial_\alpha$, $A^{\dot{A}\dot{A}} = \sigma^{\alpha\dot{A}\dot{A}}A_\alpha$.

The Maxwell equations are

\[
\partial^\alpha F_{\alpha\beta} = -4\pi e j_\beta = -4\pi e \sigma^\dot{A}\dot{A}_\beta (u_A\pi^\dot{A} + v_A\pi^\dot{A})
\]

(3)

In the linearized theory the “self current” $j_\beta$ is ignored. We now eliminate the potential $A^{\dot{A}\dot{A}}$ using (2) – another approach is to eliminate “$A^{\text{self}}_\alpha$” using the formal Green’s function [6] – we will use purely algebraic methods. From the equations (2) we have

\[
v^A\partial^{\dot{A}\dot{A}}u_B + u^A\partial^{\dot{A}\dot{A}}v_B + \frac{im}{\sqrt{2}}(v^A\pi^\dot{A} + u^A\pi^\dot{A})
\]

\[
= ie[A^{\dot{B}\dot{A}}(v^A u_B - u^A v_B)]
\]

(4)

However, because of the 2-dimensionality of the 2-spinor space we have

\[
v_A u_B - u_A v_B = \epsilon_{AB}(u^C v_C).
\]

Here, $\epsilon_{01} = \epsilon_{01}^\text{01} = 1$, $\epsilon_{10} = \epsilon_{10}^\text{10} = -1$, $\epsilon_{00} = \epsilon_{00}^\text{00} = \epsilon_{11} = \epsilon_{11}^\text{11} = 0$; we define $\xi^A = \epsilon^{AB}\xi_B$ and $\xi^A = \epsilon_{BA}\xi^B$.

We assume that $u^C v_C \neq 0$ almost everywhere. Now, $j^\alpha j_\alpha = |u^A v_A|^2$, so $u^C v_C \equiv 0$ implies that the current vector, $j$, is null – a massive neutrino field.

2
We can now solve (4) for the electromagnetic potential \( A \),

\[
(5) \quad A^{A\dot{A}} = \frac{i}{e(u^c v_c)} \left\{ v^A \partial^{B\dot{A}} u_B + u^A \partial^{B\dot{A}} v_B + \frac{im}{\sqrt{2}} (u^A \bar{u}^{\dot{A}} + v^A \bar{v}^{\dot{A}}) \right\}
\]

Notice that, from (5), under the gauge transformation

\[
\begin{pmatrix} u_A \\ \bar{v}^B \end{pmatrix} \rightarrow e^{i\chi} \begin{pmatrix} u_A \\ \bar{v}^B \end{pmatrix}
\]

we have

\[
A_\alpha \rightarrow A_\alpha + \frac{1}{e} \partial_\alpha \chi
\]

as we should expect!

The four complex equations (2) actually over determine the four real quantities \( A_\alpha \). We must impose on (5) the condition that \( A_\alpha \) is real. These reality conditions can be written as

\[
\begin{align*}
(A^{A\dot{A}} u_A \bar{u}_{\dot{A}}) &= A^{A\dot{A}} u_A \bar{u}_{\dot{A}} \\
(A^{A\dot{A}} v_A \bar{v}_{\dot{A}}) &= A^{A\dot{A}} v_A \bar{v}_{\dot{A}} \\
(A^{A\dot{A}} u_A \bar{v}_{\dot{A}}) &= A^{A\dot{A}} v_A \bar{u}_{\dot{A}}
\end{align*}
\]

With the use of (5) these reality conditions become

\[
\begin{align*}
\partial^{A\dot{A}} (u_A \bar{u}_{\dot{A}}) &= -\frac{im}{\sqrt{2}} (u^C \bar{v}_C - \bar{u}^C v_C) \\
\partial^{A\dot{A}} (v_A \bar{v}_{\dot{A}}) &= \frac{im}{\sqrt{2}} (u^C \bar{v}_C - \bar{u}^C v_C) \\
u_A \partial^{A\dot{A}} v_{\dot{A}} - \bar{v}_{\dot{A}} \partial^{A\dot{A}} u_A &= 0
\end{align*}
\]

These equations constitute four real first order equations for the four complex quantities \( u^A \) and \( v^A \). A further four real third order equations for these quantities is obtained upon substitution of (5) into the Maxwell equations (3). Note that adding the first two equations of (6) leads to the equation of conservation for \( j^\alpha \).

**§2 Reduction of the System**

2.1 The Static Equations

Firstly, we impose the condition that the field is static. We assume that there exists a Cartesian Lorentz frame in which \( j^\alpha = \delta_0^\alpha j^0 \). Imposing this condition one quickly finds that

\[
v^A = e^{i\chi} \sqrt{2} \sigma^{A\dot{A}} u_A \bar{u}_{\dot{A}}, \quad \text{with } \chi \text{ a real function.}
\]

the current vector is now

\[
j^\alpha = \sqrt{2} (u^0 \bar{u}^1 + u^1 \bar{u}^0) \delta_0^\alpha
\]
Now write

\[ u^A = e^{-imt} \zeta^A. \]

The reality conditions are

\[ \partial_{A\dot{A}}(\zeta^A\dot{\zeta}^A) = \frac{-2m}{\sqrt{2}}(|\zeta^0|^2 + |\zeta^1|^2) \sin \chi \]

\[ (\partial_{\dot{0}0} + \partial_{11})(|\zeta^0|^2 + |\zeta^1|^2) = 0 \]

\[ \zeta^0(\partial_{\dot{0}0} + \partial_{11})\zeta^1 - \zeta^1(\partial_{\dot{0}0} + \partial_{11})\zeta^0 = i[\zeta^0\zeta^1(\partial_{\dot{0}0} - \partial_{11}) + (\zeta^1)^2\partial_{1\dot{0}} - (\zeta^0)^2\partial_{11}] \chi \]

the expressions for the potential \( A^{\dot{A}A} \) can now be written down, although we won’t do this at this stage.

Now under a gauge transformation we have

\[ \zeta^A \rightarrow e^{i\mu} \zeta^A \quad \text{and} \quad A_\alpha \rightarrow A_\alpha + \frac{1}{e} \partial_\alpha \mu. \]

We fix the gauge by defining real functions \( X, Y \) and \( \eta \) as follows

\[ \zeta^0 = X e^{i\frac{\chi + \eta}{2}} \]

\[ \zeta^1 = Y e^{i\frac{\chi - \eta}{2}} \]

Our equations can be given in a particularly suggestive \textit{three vector} form by writing (in our Cartesian coordinates)

\[ \ell = (\sigma_{A\dot{A}}^\alpha u^{A\alpha}) \]

\[ = (\ell^0, \frac{1}{\sqrt{2}} V) \]

with

\[ \ell^0 = \frac{1}{\sqrt{2}} (X^2 + Y^2) \]

and

\[ V = (2XY \cos \eta, 2XY \sin \eta, X^2 - Y^2) \]

The reality conditions become

\[ \frac{\partial}{\partial t} (X^2 + Y^2) = 0 \]

\[ \nabla \cdot V = -2m(X^2 + Y^2) \sin \chi \]

\[ \frac{\partial V}{\partial t} + (\nabla \chi) \times V = 0 \]
With electromagnetic potential

\[
A^0 = \frac{m}{e} (\cos \chi - 1) + \frac{(X^2 - Y^2)}{2e(X^2 + Y^2)} \frac{\partial \eta}{\partial t} + \frac{(\nabla \chi).V}{2e(X^2 + Y^2)}
\]

\[
A = \frac{1}{2e(X^2 + Y^2)} \left[ \frac{\partial \chi}{\partial t} V + (X^2 - Y^2) \nabla \eta - \nabla \times V \right]
\]

where \( A = (A^1, A^2, A^3) \)

The full system is given by the above two sets of equations and the Maxwell equations.

2.2 Spherical Symmetry

We now impose spherical symmetry upon our static system. A minimal requirement that the Dirac field be static and spherically symmetric (in any gauge) is that the vector \( \ell \), above, is spherically symmetric. We require

\[
[X_i, \ell] = 0
\]

where the \( X_i, i = 1, 2, 3 \), are the three (vector) generators of rotations.

These conditions imply that \( \ell \) has time and radial components only and that these components are functions of \( (t, r) \) only, \( r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} \).

Using the notation above we have

\[
X^2 + Y^2 = R = R(r) \text{, only}
\]

\[
V = |V| \hat{r} = R \hat{r}
\]

where \( \hat{r} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \) in terms of the polar coordinates \( r, \theta, \phi \). We have

\[
X = \sqrt{R} \cos(\theta/2)
\]

\[
Y = \sqrt{R} \sin(\theta/2)
\]

\[
\eta = \phi
\]

The Dirac bispinor is now

\[
\psi = e^{-imt} \sqrt{R} \begin{pmatrix}
-e^{i(\chi - \phi)} \sin(\frac{\theta}{2}) \\
e^{i(\chi + \phi)} \cos(\frac{\theta}{2}) \\
-e^{i(\chi + \phi)} \sin(\frac{\theta}{2}) \\
e^{i(\chi - \phi)} \cos(\frac{\theta}{2})
\end{pmatrix}
\]

The equations are now as follows

\[
\chi = \chi(r), \ R = R(r)
\]
\[
A = \frac{1}{2e} \frac{\cot \theta}{r} \hat{\phi}
\]

\[
A^0 = \frac{m}{e} (\cos \chi - 1) + \frac{1}{2e} \frac{d\chi}{dr}
\]

\[
\frac{d}{dr} (r^2 R) = -2mr^2 R \sin \chi
\]

\[
\frac{d}{dr} (r^2 \frac{dA^0}{dr}) = -4\sqrt{2} \pi e r^2 R
\]

(7)

The really surprising result here is the unavoidable appearance of the magnetic monopole term

\[
A = \frac{1}{2e} \frac{\cot \theta}{r} \hat{\phi}.
\]

Here \( \hat{\phi} \) is the usual azimuthal unit vector, in terms of a coordinate basis \( A = -\frac{1}{2e} \cos \theta d\phi \).

We should also impose a normalisation condition (or finite total charge condition) on any solution

\[
\int j^\alpha dS_\alpha < \infty
\]

on any space like hypersurface. This leads to the condition

\[
\int_{r=0}^{\infty} r^2 R dr < \infty
\]

To end this section we rewrite the determining radial equations in a more transparent form by introducing the following new (dimensionless) variables

\[
\rho = 2mr
\]

\[
a = \frac{e}{m} A^0
\]

\[
q = 4\pi \sqrt{2} e^2 mr^2 R.
\]

\[
\frac{d\chi}{d\rho} = a + 1 - \cos \chi
\]

\[
\frac{d}{d\rho} (\rho^2 \frac{da}{d\rho}) = -q
\]

\[
\frac{dq}{d\rho} = -q \sin \chi
\]

(8)

§3 Static Spherical Symmetry

3.1 Some General Properties

The system of equations (8) possesses the discrete symmetry

\[
\chi \rightarrow \pi - \chi
\]

\[
a + 1 \rightarrow -(a + 1)
\]

(9)

\[
q \rightarrow -q
\]
This is just the operation of charge conjugation; \( q \) needs to be reinterpreted (it was originally defined as non-negative) to account for the change in sign of the charge which manifests itself on the right hand side of the second equation of (8). We write \( q = \epsilon Q \), with \( \epsilon^2 = 1 \) and \( Q \geq 0 \), so \( Q = \sqrt{2m^2\rho^2}R \); then, under (9), we have \( \epsilon \to -\epsilon \) and \( Q \to Q \).

Our equations read,

\[
\begin{align*}
\frac{d\chi}{d\rho} &= a + 1 - \cos \chi \\
\frac{da}{d\rho} &= -\epsilon f / \rho^2 \\
\frac{df}{d\rho} &= Q \\
\frac{dQ}{d\rho} &= -Q \sin \chi
\end{align*}
\]

(10)

Where we have introduced the new variable \( f \) to give a set of four first order, ordinary differential equations. This new variable is directly related to the magnitude total (Dirac field) charge contained in a ball of radius \( r \), \( B(r) \),

\[
e \int_{B(r)} j^\alpha dS_\alpha = e \int_{B(r)} j^0 d^3 x = 4\pi e \sqrt{2} \int_{s=0}^{r} s^2 R(s) \, ds = \frac{2\pi}{e} \int_{\sigma=0}^{\rho} Q \, d\sigma = \frac{1}{2e} (f(\rho) - f(0))
\]

In view of this – and the fact that \( \frac{df}{d\rho} = Q = \epsilon q \) is proportional to the charge density on a shell of radius \( r \) – we will use the following condition on our system (10).

(C1) \[
\begin{cases}
\text{On } \rho > 0 \text{ , } f \text{ is a bounded } C^1 \text{ function, with bounded first derivative.} \\
\text{Both } f \text{ and } \frac{df}{d\rho} \text{ have well defined limits as } \rho \to \infty
\end{cases}
\]

We will now develop some qualitative results which indicate the types of solution which can exist under rather general (and physically reasonable) conditions.

**Lemma 1** Suppose \( (\chi, a, f, Q) \) is a solution of (10) on \( \rho > 0 \), then under C1 the function \( a \) is \( C^2 \) on \( \rho > 0 \) with \( a \) and \( \frac{da}{d\rho} \) bounded on intervals \( \rho \geq \rho_1 > 0 \) and \( \rho a \) bounded on \( 0 \leq \rho \leq \rho_1 < \infty \). If \( f(0) \neq 0 \) or \( \frac{df}{d\rho}(0) = Q(0) > 0 \) then \( a \) is unbounded as \( \rho \to 0 \).

**Proof.** We first establish that \( Q \) and \( f \) have well defined limits as \( \rho \) approaches 0. We are assuming that the solution \( (\chi, a, f, Q) \) exists on \( \rho > 0 \), so for \( \rho_2 > \rho_1 > 0 \) and using
equations (10), we have
\[ |Q(\rho_2) - Q(\rho_1)| = |\int_{\rho_1}^{\rho_2} Q(\sigma) \sin \chi(\sigma) d\sigma| \]
\[ \leq \int_{\rho_1}^{\rho_2} |Q| \sin |d\sigma| \]
\[ < M_1 (\rho_2 - \rho_1), \text{ here } M_1 = \sup_{\mathbb{R}_+} Q < \infty. \]

Letting \( \rho_1 \) and \( \rho_2 \) approach zero, we have \( |Q(\rho_2) - Q(\rho_1)| \rightarrow 0. \) Using Cauchy’s criterion we conclude that \( Q(0^+) \) exists. A similar argument can be given to demonstrate the existence of \( f(0^+) \).

The boundedness of \( \frac{da}{d\rho} \), on \( \rho \geq \rho_1 \) follows from
\[ \frac{da}{d\rho} = -f/\rho^2 \]

Integrate this expression to bound \( a \) on \( \rho \geq \rho_1 > 0. \)

Write \( \Omega = \epsilon \rho a \), then
\[ \rho \frac{d\Omega}{d\rho} - \Omega = -f, \text{ and } \]
\[ \rho \frac{d^2\Omega}{d\rho^2} = -Q \]

From the second of these equations we have, on \( (0, \rho_1] \),
\[ M_1 \ln \left( \frac{\rho_1}{\rho} \right) + \frac{d\Omega}{d\rho}(\rho_1) \geq \frac{d\Omega}{d\rho}(\rho) \geq \frac{d\Omega}{d\rho}(\rho_1) \]

So we have (note as \( a \) is \( C^2 \) away from \( \rho = 0 \) so is \( \Omega) \)
\[ \rho \frac{d\Omega}{d\rho} \rightarrow 0, \text{ as } \rho \rightarrow 0 \]

Hence from the first of the \( \Omega \) equations we see that \( \Omega \) has a well defined limit as \( \rho \rightarrow 0 \), in fact
\[ \lim_{\rho \rightarrow 0} \Omega(\rho) = f(0) \]

An immediate consequence is that if \( f(0) \neq 0 \) then \( a \) is unbounded as \( \rho \rightarrow 0. \)

Now from (10) we have \( Q \geq Q(0)e^{-\rho} \), so that \( f(\rho) - f(0) \geq \frac{df}{d\rho}(0)(1 - e^{-\rho}) - \text{recall } \frac{df}{d\rho} = \epsilon q = Q. \) Given the earlier result we may take \( f(0) = 0 - \text{otherwise } a \) is unbounded. We can now integrate our \( \frac{da}{d\rho} \) equation with this bound for \( f, \)
\[ \epsilon (a(\rho) - a(\rho_1)) \geq \frac{df}{d\rho}(0) \left[ \frac{(1 - e^{-\rho})}{\rho} - \frac{(1 - e^{-\rho_1})}{\rho_1} \right] + \frac{df}{d\rho}(0) \int_{\sigma=\rho_1}^{\rho_1} e^{-\sigma} d\sigma \]

As \( \rho \rightarrow 0 \) the integral on the right side of the inequality diverges to \( +\infty. \)
Lemma 2 Suppose \((\chi, a, f, Q)\) is a solution of (10) on \(\infty > \rho > 0\), then under \(C1\) the function \(\chi\) is \(C^1\) with \(\frac{d\chi}{d\rho}\) bounded on intervals \(\rho \geq \rho_1 > 0\). If \(f(0) \neq 0\) then \(\chi\) is unbounded as \(\rho \to 0\).

Proof. The regularity of \(\chi\) – on its presumed interval of existence – is established using standard theory (see, for example, [6] or [8]) after first noting that the right side of

\[
\frac{d\chi}{d\rho} = a + 1 - \cos \chi
\]

is \(C^2\) in \(\rho\) (treating \(a\) as a known function and using lemma 1) and \(C^\infty\) in \(\chi\).

From the above equation we also have

\[
\epsilon(a + 1) - 1 \leq \epsilon \frac{d\chi}{d\rho} \leq \epsilon(a + 1) + 1
\]

which gives the required bounds (using lemma 1). Working on \((0, \rho_1)\) we have (as in the proof of lemma 1)

\[
C_0 + \frac{f(0)}{\rho} \leq \epsilon \frac{d\chi}{d\rho} \leq C_1 + \frac{f(\rho_1)}{\rho}
\]

where \(C_0 = \epsilon(a(\rho_1) + 1) - 1 - \frac{f(0)}{\rho_1}\) and \(C_1 = \epsilon(a(\rho_1) + 1) + 1 - \frac{f(\rho_1)}{\rho_1}\).

Integrating,

\[
C_2 - C_1(\rho_1 - \rho) + f(\rho_1) \ln(\rho) \leq \epsilon \chi \leq C_3 - C_0(\rho_1 - \rho) + f(0) \ln(\rho)
\]

with \(C_2 = \epsilon \chi(\rho_1) - f(\rho_1) \ln(\rho_1)\) and \(C_3 = \epsilon \chi(\rho_1) - f(0) \ln(\rho_1)\). If \(f(0) < 0\) choose \(\rho_1\) near 0 so that \(f(\rho_1) < 0\) and we have from our last inequality that \(\epsilon \chi \to \infty\) as \(\rho \to 0\). If \(f(0) > 0\) our inequality yields \(\epsilon \chi \to -\infty\) as \(\rho \to 0\).

There is one other condition which makes sense “physically”: if we have an isolated system we expect the charge density should go to zero at infinity.

\[
(C2) \quad \frac{df}{d\rho} = Q \to 0, \text{ as } \rho \to \infty.
\]

Lemma 3 Suppose \((\chi, a, f, Q)\) is a solution of (10) on \(\rho > 0\) under conditions \(C1\) and \(C2\). Then \(-2 \leq a_\infty \leq 0\) where \(a \to a_\infty\) as \(\rho \to \infty\).

Proof. We first establish that \(a\) has a well defined limit, written as \(a_\infty\), as \(\rho \to \infty\); for \(\rho_2 > \rho_1 > 0\), we have

\[
|a(\rho_2) - a(\rho_1)| = \left| \int_{\rho_1}^{\rho_2} \frac{f(\sigma) d\sigma}{\sigma^2} \right| < M_2 \left( \frac{1}{\rho_1} - \frac{1}{\rho_2} \right), \text{ where } M_2 = \sup_{\mathbb{R}^+} |f|.
\]
Letting $\rho_1$ and $\rho_2$ approach $\infty$, we conclude that the limit $a \to a_\infty$ exists. From the first and second equations of (10) we have

$$\frac{d}{d\rho} \ln |a + 1 - \cos \chi| = \frac{\frac{da}{d\rho} + \sin \chi \frac{d\chi}{d\rho}}{a + 1 - \cos \chi} = \sin \chi - \frac{f}{\rho^2(a + 1 - \cos \chi)}$$

Consequently, on $\rho \geq \rho_1 > 0$, integrating the last equation of (10) we have

$$-\ln \left[ \frac{Q}{Q(\rho_1)} \right] = \int_{\rho_1}^{\rho} \sin \chi(\sigma) d\sigma$$

$$\leq \ln \left| \frac{a + 1 - \cos \chi}{a(\rho_1) + 1 - \cos \chi(\rho_1)} \right| + \int_{\rho_1}^{\rho} \frac{f(\sigma) d\sigma}{\sigma^2(a(\sigma) + 1 - \cos \chi(\sigma))}$$

Now assume $a_\infty > 0$. In fact, this also takes care of the case $a_\infty < -2$, since under the discrete conjugation symmetry, (9), $a \to -(a + 2)$ and in particular $a_\infty \to -(a_\infty + 2)$.

Working on $\rho \geq \rho_1 > 0$ we have, for $\rho_1$ large enough,

$$a(\rho) > a_0 > 0$$

for $\rho \geq \rho_1$ and some constant $a_0$.

Then, as $a \leq \frac{d\chi}{d\rho} \leq a + 2$, so

$$0 < a_0 < \frac{d\chi}{d\rho} < M$$

where $M$ is a finite constant – see lemma 1.

i.e., $a_0 < a + 1 - \cos \chi < M$

Clearly both terms on the right side of our equation for $-\ln [Q/Q(\rho_1)]$ are bounded. This contradicts our assumption C2 that $Q \to 0$ as $\rho \to \infty$.

The constant $a_\infty$ can be removed from the potential via a gauge transformation. Under $\psi \to e^{-ima_\infty t} \psi$, we have $a \to a - a_\infty$. After this transformation the Dirac field $\psi$ has time dependence $e^{-iEt}$, where $-m \leq E = (1 + a_\infty)m \leq m$.

The three lemmas give a basic characterisation of the solutions obeying C1 and C2. If $f(0) \neq 0$ or $Q(0) > 0$ then $a$ diverges at the origin, these are solutions which can be pictured as an Dirac field surrounding a central charged monopole - the numerical solution of §4 is of this type. There is also the possibility of solutions with $a$ and $R$ (recall: $Q = \sqrt{2} \frac{\varphi}{m} r^2 R$) everywhere bounded, such solutions were suggested by the work of Wakano [9], who examined numerical solutions for what could be called “half linearised” Dirac Maxwell equations – “half linearised”: if the electrostatic potential is “dominant” ignore the Maxwell equation involving the electromagnetic vector potential and vice versa. In fact, as the following theorem demonstrates, no such solutions exist.
**Theorem.** There does not exist a non-trivial solution of (10) on \( \rho \geq 0 \) under conditions C1 and C2 with \( a \) and \( P = Q/\rho^2 \) bounded on \( \rho \geq 0 \).

**Proof.** From lemma 1 we have, \( f(0) = Q(0) = 0 \). Next we establish that both \( \chi \) and \( a \) have well defined limits as \( \rho \to 0 \). Note that \( \frac{d\chi}{d\rho} = a + 1 - \cos \chi \) is bounded, under the hypothesis of the theorem, as \( \rho \to 0 \); hence, by an argument of the sort used previously, \( \chi \to \chi(0) = \chi_0 \), say, as \( \rho \to 0 \). We also have

\[
f = \int_{\sigma=0}^{\rho} Q(\sigma) \, d\sigma = \int_{\sigma=0}^{\rho} \sigma^2 P(\sigma) \, d\sigma \geq 0
\]

Using the mean value theorem we have, for some \( \rho_1, \rho > \rho_1 > 0 \)

\[
f(\rho) = \rho \rho_1^2 P(\rho_1) < \rho^3 M_3, \text{ where } M_3 = \sup_{\mathbb{R}^+} P < \infty
\]

Thus,

\[
|a(\rho_2) - a(\rho_1)| = \left| \int_{\rho_1}^{\rho_2} \frac{f(\rho) \, d\rho}{\rho^2} \right| < \int_{\rho_1}^{\rho_2} \rho M_3 \, d\rho = \frac{1}{2} M_3 (\rho_2^2 - \rho_1^2)
\]

Letting \( \rho_1, \rho_2 \to 0 \) establishes the existence of the limit for \( a \), we write

\[
\lim_{\rho \to 0} a = a_0
\]

Now we use an argument similar to that used in the proof of lemma 3 to show that \( \frac{d\chi}{d\rho} = a + 1 - \cos \chi \to 0 \), as \( \rho \to 0 \). On \( (0, \rho_1) \) we have

\[
(*) \quad -\ln[Q(\rho_1)/Q(\rho)] = \ln \left| \frac{a(\rho_1 + 1 - \cos \chi(\rho_1))}{a + 1 - \cos \chi} \right| + \int_{\sigma=\rho}^{\rho_1} \frac{f(\sigma) \, d\sigma}{\sigma^2 [a(\sigma) + 1 - \cos \chi(\sigma)]}
\]

Assume, \( \frac{d\chi}{d\rho}(0) = a_0 + 1 - \cos \chi_0 \neq 0 \). Then, choosing \( \rho_1 \) near 0 so that \( \frac{d\chi}{d\rho}(\rho) \frac{d\chi}{d\rho}(0) > 0 \), for \( \rho_1 > \rho > 0 \), we have

\[
\left| \int_{\rho}^{\rho_1} \frac{f(\sigma) \, d\sigma}{\sigma^2 [a(\sigma) + 1 - \cos \chi(\sigma)]} \right| < M_3 \int_{\rho}^{\rho_1} \frac{\sigma \, d\sigma}{|\frac{d\chi}{d\rho}(\sigma)|}
\]

The right side of this inequality is bounded as \( \rho \to 0 \). Consequently, the right side of (*) is bounded as \( \rho \to 0 \). But this contradicts the assumption of the theorem that \( Q = \rho^2 P \to 0 \) as \( \rho \to 0 \). Thus

\[
\frac{d\chi}{d\rho}(0) = a_0 + 1 - \cos \chi_0 = 0, \text{ or } a_0 = -1 + \cos \chi_0
\]
We now assume \( \epsilon = +1 \), the case \( \epsilon = -1 \) can (of course!) be obtained by conjugation. As \( \frac{\partial a}{\partial \rho} = -f/\rho^2 < 0 \) on \( \rho > 0 \), so
\[
a_0 = -1 + \cos \chi_0 > a > a_\infty \geq -2
\]
on \( \rho > 0 \).
Define new variables
\[
U = \sqrt{Q} \cos(\chi/2), \quad \text{and} \quad V = \sqrt{Q} \sin(\chi/2).
\]
We have
\[
\frac{dU}{d\rho} = -\frac{1}{2}(a + 2)V, \quad \text{and} \quad \frac{dV}{d\rho} = \frac{1}{2}aU.
\]
The pair \( U \) and \( V \) also satisfy the following linear, second order equations
\[
\frac{d^2U}{d\rho^2} + \frac{f}{\rho^2(a + 2)} \frac{dU}{d\rho} + \frac{1}{4}a(a + 2)U = 0, \quad \text{and} \quad \frac{d^2V}{d\rho^2} + \frac{f}{\rho^2a} \frac{dV}{d\rho} + \frac{1}{4}a(a + 2)V = 0
\]
(11)
We note that \( \frac{f}{\rho^2(a+2)} \) is bounded on intervals \([0, \rho_2]\), with \( 0 \leq \rho_2 < \infty \) and that \( \frac{f}{\rho^2a} \) is bounded on \((0, \infty)\) – in the first case we may have \( a_\infty = -2 \), whereas we may have \( a_0 = 0 \) in the second; we also have
\[
\frac{1}{4}a(a + 2) \leq 0, \quad \text{on, } \rho \geq 0.
\]
From the definitions of \( U \) and \( V \) we have \( U, V \to 0 \) as \( \rho \to 0 \) or \( \infty \). Thus, by the maximum principle for odes (see [10]), we conclude \( U \equiv V \equiv 0 \), so \( Q \equiv 0 \). There do not exist non-trivial solutions.

§4 Numerical Solutions

Numerical solutions to the system (10), with \( \epsilon = 1 \), were sought by first expanding in a power series from either \( s = 0 \) (\( \rho = \infty \), with \( s = 1/\rho \)) or \( \rho = 0 \) and then evolving the system in \( s \) or \( \rho \), respectively, using a MATLAB interface to the NAG library, [11].

4.1 Solutions Near \( \rho = 0 \)

In lemma 1 we found that \( \Omega = \rho a \) was bounded, near \( \rho = 0 \), with
\[
\lim_{\rho \to 0} \Omega(\rho) = f(0)
\]
It is natural then to seek solutions of the form \( a = \Omega(\rho)/\rho \), near \( \rho = 0 \), with \( \Omega \) analytic in \( \rho \). From (10) it can be seen that both \( f \) and \( Q \) must be analytic, with \( Q(0) = 0 \).

However, with \( Q \) analytic and \( Q(0) = 0 \), the last equation of (10) can only be satisfied with \( Q \equiv 0 \) which implies \( a = c_0 + c_1/\rho \), we will refer to such solutions as trivial. The behaviour of the system near \( \rho = 0 \) may be quite complex; from the proof of the theorem (see equations (11)), with \( a = \Omega/\rho \), we see that near \( \rho = 0 \) the second order equations for \( U \) and \( V \) have indicial equation (see [12])

\[
\lambda^2 + \frac{1}{4}\Omega(0)^2 = 0
\]

This implies that \( U \) and \( V \) have behaviour

\[
U \text{ or } V \sim \omega_1(\rho) \cos\left(\frac{\Omega(0)}{2} \ln \rho\right) + \omega_2(\rho) \sin\left(\frac{\Omega(0)}{2} \ln \rho\right), \text{ near } \rho = 0
\]

4.2 Solutions Near \( \rho = \infty \)

Near \( \rho = \infty \) we expect \( a \sim a_\infty + \frac{a_1}{\rho} + \frac{a_2}{\rho^2} + \cdots \). Assuming \( a \) is analytic in \( s = 1/\rho \), near \( s = 0 \), then implies that \( f \), \( Q \) and \( \sin \chi \) are also analytic in \( s \), in fact

\[
f = \frac{da}{ds}
\]

\[
Q = -s^2 \frac{d^2a}{ds^2}, \text{ and }
\]

\[
\sin \chi = 2s + s^2 \left(\frac{d^3a/ds^3}{d^2a/ds^2}\right)
\]

With the assumption that \( \chi \) is analytic near \( s = 0 \) a uniquely defined power series results if we demand that the solution be non-trivial (\( \chi \) does have the freedom to add integer multiples of \( 2\pi \)).

The resulting power series has no free parameters, it is uniquely determined.

The lower order portion of the power series solution is as follows

\[
\chi = 2s + \frac{1}{21}s^3 + \frac{3}{520}s^5 + O(s^7)
\]

\[
a = -4s^2 + \frac{3}{7}s^4 - \frac{341}{5096}s^6 + O(s^7)
\]

\[
f = -8s + \frac{12}{7}s^3 - \frac{1023}{2548}s^5 + O(s^7)
\]

\[
Q = 8s^2 - \frac{36}{7}s^4 + \frac{5115}{2548}s^6 + O(s^7)
\]

Using the power series to determine initial conditions it was found that the numerical results were very stable for a good range of initial values for \( s \) (\( s_0 = .000001 \) to \( s_0 = .01 \)), the results were somewhat unstable for \( s_0 < 0.0000001 \). The results presented in figures 1 to 4 were obtained by first shooting from near \( s = 0 \) towards \( \rho = 0 \) and then using the
final values of this run as initial conditions to shoot from near $\rho = 0$ towards $s = 0$, to verify the solution.

In figures 1 to 4 $\chi$, $a$, $f(r) - f(0)$ (proportional to the “electron” charge interior to a ball radius $r$) and $Q$ are plotted against the radial distance measured in units of the Compton wavelength (i.e. against $\frac{1}{2} \rho = mr$).

**Interpretation**: The solution represented in figures 1 to 4 can be thought of as a central, charged monopole (point source), surrounded by an oppositely charged Dirac field – near $\infty$ the electrostatic potential behaves as $A^0 = \frac{m_e a}{e} \sim \frac{-1/(me)}{r^2}$ and near $r = 0$ the potential behaves as $A^0 \sim \frac{-a/e}{r}$ (where $a \approx 5.7037$ is the magnitude of the slope of the line in figure 5, where $a$ is plotted against $1/(mr)$).

![Figure 1: The Angular Variable, $\chi$.](image1)

![Figure 2: The Potential, $a$.](image2)

At about one half the Compton wavelength from the center there is a shielding effect and the Coulomb nature of the central charge becomes apparent. At large distances from the center the electrostatic charges “cancel” each other. We can calculate the magnitude of the total charge due to the Dirac field (see §3.1), with $f(0)$ calculated numerically ($f(\infty) = 0$, in this case),

\[
e \int_{\mathbb{R}^3} j^\alpha dS_\alpha = \frac{1}{2e} (f(\infty) - f(0)) \approx \frac{1}{2e} \times 11.407391
\]

This calculation results in a charge of the same magnitude as the central charge.

The object is highly compact, with a radius of about a (reduced) Compton wavelength – see figures 4 and 6. It has an onion like structure consisting of an infinite series of spherical shells – the local maxima of $Q$ occur at points, $\rho = \rho_m$, where $\sin(\chi(\rho_m)) = 0$,
however from lemma 2 we see that $\chi$ must diverge as $\rho \to 0$ (in the present case) so there will be an infinite number of shells.

We note that, in the present case, $a_\infty = 0$ so the Dirac field has time dependence $e^{-imt}$ in the gauge for which $a \to 0$ as $\rho \to 0$. The Dirac field has mass

$$m \int_{\mathbb{R}^3} j^\alpha dS_\alpha = \frac{m}{2e^2} (f(\infty) - f(0))$$

$$\approx 11.407391 \frac{m}{2e^2}$$

Figure 3: The “Charge” Interior to $B(r)$, $f - f(0)$.

Figure 4: The “Charge” on a Shell, radius $mr$, $q$.

Figure 5: The Potential, $a$, from Infinity.

Figure 6: The Shell “Charge”, $q$, from Infinity.
To finish we calculate the energy-momentum density of the system. The symmetric energy-momentum tensor is

\[ T_{\alpha\beta} = T_{\alpha\beta}^D + T_{\alpha\beta}^{em}, \]

with

\[ T_{\alpha\beta}^D = \frac{i}{4} \left( \sigma_A^{\alpha A} (u_{A,\beta} + v_A \bar{v}_{A,\beta}) + \sigma^{A\tilde{A}} (u_{A,\alpha} + v_A \bar{v}_{A,\alpha}) \right) - \sigma^{\tilde{A}A} (u_{A,\alpha} \bar{v}_{A,\beta} - \sigma^{A\tilde{A}} (u_{A,\alpha} v_A + v_A \bar{v}_{A,\alpha}) \right) + e A_{(\alpha j \beta)} \]

\[ T_{\alpha\beta}^{em} = -\frac{1}{4\pi} \left( F_{\alpha\gamma} F_{\beta}^{\gamma} - \frac{1}{4} \eta_{\alpha\beta} F_{\mu\nu} F^{\mu\nu} \right) \]

These expressions are derived from the Lagrangian

\[ L = \frac{i}{2} \left( \bar{\psi}_A \partial^{AA} u_A - u_A \partial^{AA} \bar{\psi}_A - \bar{\psi}_A \partial^{\tilde{A}A} v_A + v_A \partial^{\tilde{A}A} \bar{\psi}_A \right) - \frac{m}{\sqrt{2}} \left( u_A v^A + \bar{v}_A \bar{v}^A \right) + e j_0 A^\alpha - \frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} \]

Notice that in the absence of the electromagnetic field, for a Dirac field with time dependence \( e^{-iEt} \), the energy density \( T_{00}^D \) is

\[ T_{00}^D = E j_0 \]

In the present case we have

\[ T_{00}^D = j_0 (m + e A_0) = \frac{m^2}{e^2} \frac{Q}{4\pi r^2} (1 + a) \]

\[ T_{00}^{em} = \frac{m^2}{4\pi r^2 e^2} \left( \left[ \frac{\rho^2}{2} \left( \frac{da}{d\rho} \right)^2 + \frac{1}{4\rho^2} \right] + \frac{a_0^2}{2} \left( \frac{da}{d\rho} \right)^2 + \frac{1}{4\rho^2} \right) \]

These expressions include terms due to the central Coloumb and magnetic monopole fields, they lead to an infinite total energy when integrated over \( \mathbb{R}^3 \), the energy can, however, be regularised by removing these singular terms. If we exclude the energy due to the interaction between the electromagnetic and Dirac fields, \( e A_{(\alpha j \beta)} \), then \( T_{00}^D \) gives the mass density \( mj_0 \) as above – however, this procedure is clearly not gauge invariant.

Finally, it is perhaps worth mentioning that the highly localised “multi-electron fields” described here may in fact have applications to objects described in recent experimental work \[13\] – “geonium” or “kilo-e” objects – consisting of highly localised (point like, from the experimental viewpoint) collections of electrons in atomic traps.
† We assume that the constants appearing in the Dirac equation, i.e. $e$ and $m$ have their usual meaning — $e$ the square root of the fine structure constant and $m$ the inverse of the (reduced) Compton wavelength.

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