Some results concerning the constant astigmatism equation

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Abstract. In this paper we continue investigation of the constant astigmatism equation $z_{yy} + (1/z)_{xx} + 2 = 0$. We newly interpret its solutions as describing spherical orthogonal equiareal patterns, with relevance to two-dimensional plasticity. We show how the classical Bianchi superposition principle for the sine-Gordon equation can be extended to generate an arbitrary number of solutions of the constant astigmatism equation by algebraic manipulations. As a by-product, we show that sine-Gordon solutions give slip line fields on the sphere. Finally, we compute the solutions corresponding to classical Lipschitz surfaces of constant astigmatism via the corresponding equiareal patterns.

1. Introduction

It is well known that the classical Bäcklund transformation [4] for the sine-Gordon equation $u_{\xi\eta} = \sin u$ as well as the Bianchi permutability property [7] have been discovered in the context of pseudospherical surfaces, i.e., surfaces of constant negative Gaussian curvature. It is perhaps less known that historical roots of these developments lie in another class of surfaces, characterised by the constancy of the difference $\rho_2 - \rho_1$ between the principal radii of curvature $\rho_1, \rho_2$; see [26] for the historical account. Lying covered with dust and oblivion for almost a century, the surfaces satisfying $\rho_2 - \rho_1 = \text{const}$ reemerged recently from the systematic search for integrable classes of Weingarten surfaces conducted by Baran and one of us [5]. Although nameless in the nineteenth century, in [5] they have been named the surfaces of constant astigmatism in connotation with the astigmatic interval [31] of the geometric optics, albeit without suggesting any specific application.

Undoubtedly, the most important results about constant astigmatism surfaces are due to Bianchi. In [6] (see also [8, §130]), Bianchi observed that evolutes (i.e., focal surfaces) of surfaces satisfying $\rho_2 - \rho_1 = \text{const}$ are pseudospherical. In the same paper he also constructed surfaces satisfying $\rho_2 - \rho_1 = \text{const}$ as involutes corresponding to parabolic geodesic systems on pseudospherical surfaces. Apparently, Bianchi was the first to obtain surfaces of constant astigmatism explicitly, namely, surfaces [6, eq. (30)] corresponding to Dini’s pseudospherical helicoids (see, e.g., [27, §1.4.2] or [32, p. 183]).
Lipschitz [22] obtained another class of surfaces of constant astigmatism; within the full class given in terms of elliptic integrals he pointed out a subclass of surfaces of revolution, further investigated by von Lilienthal.

Let us stress that the aforementioned constructions of Lipschitz and Bianchi refer to ad hoc parameterisations. Bianchi used the rotation angle and a parameterisation of the generating tractrix of the helicoid, while Lipschitz employed spherical coordinates on the Gaussian sphere. In [5] we observed that under an adapted parameterisation by lines of curvature the constant astigmatism surfaces correspond to solutions of the constant astigmatism equation

\[ z_{yy} + \left( \frac{1}{z} \right)_{xx} + 2 = 0 \]  

(1)

(\(x, y\) are natural parameters in the sense of Ganchev and Mihova [16]). The geometric link to pseudospherical surfaces induces a nonlocal transformation to the sine-Gordon equation and vice versa. Since curvature coordinates on constant astigmatism surfaces correspond to parabolic geodesic coordinates on the pseudospherical surfaces, and these are not the coordinates the sine-Gordon equation is referred to, the transformations change both the dependent and independent variables. Explicit formulas can be found in [5], ready to be applied to the sine-Gordon solutions, which are known in abundance, see [1, 14, 24, 25] and references therein. However, the only explicit instance of such a relationship we were able to find in the literature was that of the Bianchi surfaces [6, eq. (30)] to the Dini helicoid; Fig. 1 presents a plot of them as unparameterised surfaces.

In this paper we continue the investigation of the constant astigmatism equation (1) and its solutions. Firstly, we show that equation (1) describes orthogonal equiareal patterns (Sadowsky [29, 30]) on the sphere, i.e., a system of local coordinates \(x^1, x^2\) such that the metric coefficients satisfy \(g_{12} = 0, \det g = 1\). Hence, the area element is simply \(dx^1 \wedge dx^2\) and the area of the curvilinear rectangle \(a^i \leq x^i \leq b^i, i = 1, 2\), is equal to \((b^1 - a^1)(b^2 - a^2)\). It follows that the curvilinear rectangles formed by “uniformly spaced” coordinate lines are of equal area, which explains the terminology.
The equiareal property in plasticity theory can be traced back to Boussinesq [11]. Seventy years later, Sadowsky [29, 30], rediscovered the “equiareal patterns” as configurations of the principal stress lines under the Tresca yield condition (see [19]) and gave them their name. Later Hill [18] gave a kinematic interpretation of these patterns. Coburn [13, Thm. 1] established the same equiareal property, this time for slip lines under a different yield condition. Ament [2] discovered a relation to the class of Weingarten surfaces, determined by the constancy of the difference between the principal curvatures (as opposed to the difference between the principal radii of curvature). Finally, Fialkow [15, Th. 4.1] observed relevance of orthogonal equiareal patterns to conformal geometry.

The contents of this paper are as follows. Section 2 contains the necessary background. In Section 3, we observe that adapted curvature coordinates on constant astigmatism surfaces correspond to orthogonal equiareal patterns on the Gaussian sphere. Inspired by the aforementioned relation to plasticity, we construct a two-dimensional stress tensor formally satisfying both the Tresca yield condition and the equilibrium equations. Physical relevance of our purely mathematical construction is not a primary concern, yet the flow of a thin plastic layer around a sphere seems to be a realistic picture. Guided by this picture we investigate the maximum shear stress directions, positioned at the angle of $\pi/4$ to the principal stress directions. The corresponding trajectories are known as slip lines; we show them to be related to solutions of the sine-Gordon equation later in section 4.

Section 4 is devoted to a simplified reconstruction of constant astigmatism surfaces from a pair of complementary pseudospherical surfaces [6] or [8, §136], i.e., under the frequently occurring condition that both evolutes are known. Complementary pseudospherical surfaces are easy to find among those resulting from the famous and powerful Bianchi permutability theorem [7]. It turns out that given a pair of complementary pseudospherical surfaces, the corresponding (unparameterised) constant astigmatism surface can be obtained by pure algebraic manipulations and differentiation. This is also true for geodesics (as proved by Bianchi himself) and, hence, for one of the curvature coordinates, while obtaining the other requires one integration. However, owing to a suitable extension of the Bianchi superposition principle this integration needs to be done only once. We also observe (Proposition 4) that the coordinates $\xi, \eta$ the sine-Gordon equation is referred to correspond to slip line fields on the spherical image of the constant astigmatism surface.

Finally, in section 5 we pay another longstanding debt and find the function $z(x, y)$ corresponding to the Lipschitz surfaces. As we already mentioned, Lipschitz computed a class of constant astigmatism surfaces in terms of spherical coordinates on the Gaussian image. The result being not easily transformable to curvature coordinates, we compute the associated orthogonal equiareal pattern directly from the definitions to observe that solutions of the Lipschitz class are invariant solutions with respect to Lie symmetries.
2. Preliminaries

In this section we recall previous results about the constant astigmatism surfaces; see [5] for details. We consider surfaces immersed in Euclidean space under parameterisation by the lines of curvature (also known as curvature coordinates). Hence, the fundamental forms can be written as

\[ I = u^2 \, dx^2 + v^2 \, dy^2, \quad II = \frac{u^2}{\rho_1} \, dx^2 + \frac{v^2}{\rho_2} \, dy^2, \quad III = \frac{u^2}{\rho_1^2} \, dx^2 + \frac{v^2}{\rho_2^2} \, dy^2, \]

where \( \rho_1, \rho_2 \) are the principal radii of curvature. The first two forms determine the surface up to the rigid motions (Bonnet theorem).

A surface is said to be of constant astigmatism if the difference \( \rho_2 - \rho_1 \) between the principal radii of curvature is a nonzero constant (if zero, then the surface is a part of the sphere). We assume the ambient space to be scaled so that \( \rho_2 - \rho_1 = \pm 1 \).

**Definition 1.** A parameterisation by lines of curvature is said to be adapted if

\[ uv \left( \frac{1}{\rho_1} - \frac{1}{\rho_2} \right) = \pm 1 \]

(2)

holds.

This is the natural parameterisation recently introduced by Ganchev and Mihova [16, Prop. 5.6] with the arbitrary constant being normalised to \( \pm 1 \). Every constant astigmatism (more generally, Weingarten) surface can be equipped with an adapted parameterisation by lines of curvature, see [16, Prop. 5.6] or [5]. Henceforth we assume that \( x, y \) are adapted coordinates. Then, according to [5], the nonzero coefficients of the three fundamental forms of a surface of constant astigmatism can be expressed through a single variable \( z(x, y) \):

\[ u = \frac{z^{3/2}(\ln z - 2)}{2}, \quad v = \frac{\ln z}{2z}, \quad \rho_1 = \frac{\ln z - 2}{2}, \quad \rho_2 = \frac{\ln z}{2}. \]

Obviously, condition (2) is satisfied.

Let \( r(x, y) \) be the surface of constant astigmatism corresponding to \( z(x, y) \), let \( n(x, y) \) denote the unit normal vector. Then \( r, n \) satisfy the Gauss–Weingarten system

\[ \begin{align*}
    r_{xx} &= \frac{(\ln z)z_x}{2(\ln z - 2)z} r_x - \frac{(\ln z - 2)z y_y}{2 \ln z} r_y + \frac{1}{2} (\ln z - 2) zn, \\
    r_{xy} &= \frac{(\ln z) z_y}{2(\ln z - 2)z} r_x - \frac{(\ln z - 2)z x_x}{2 \ln z} r_y, \\
    r_{yy} &= \frac{(\ln z) z_x}{2(\ln z - 2)z} r_x - \frac{(\ln z - 2)z y_y}{2 \ln z} r_y + \frac{\ln z}{2z} n, \\
    n_x &= -\frac{2}{\ln z - 2} r_x, \quad n_y = -\frac{2}{\ln z} r_y.
\end{align*} \]

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Note that \( e_1 = \mathbf{r}_x / u \), \( e_2 = \mathbf{r}_y / v \), and \( \mathbf{n} = e_1 \times e_2 \) constitute an orthonormal frame.

Compatibility conditions of the Gauss–Weingarten system constitute the Gauss–Mainardi–Codazzi system, which in our case amounts to the Gauss equation alone, and coincides with the constant astigmatism equation (1).

According to Bianchi [6] (see also [9, §136]), if \( \mathbf{r} \) is a surface of constant astigmatism and \( \mathbf{n} \) is its normal, then the two evolutes

\[
\mathbf{r} + \rho_1 \mathbf{n}, \quad \mathbf{r} + \rho_2 \mathbf{n}
\]

are pseudospherical surfaces. These are said to be complementary.

For further reference, we also recall a list of symmetries of equation (1). Lie symmetries are completely known, see [5]. They are the \( x \)-translation \( T_x c = (x+c, y, z) \), the \( y \)-translation \( T_y c = (x, y+c, z) \), and the scaling \( S_c(x, y, z) = (e^{-c}, e^c y, e^{2c} z) \), where \( c \) is a real parameter.

We shall also refer to a discrete symmetry \( I(x, y, z) = (y, x, 1/z) \), called the involution. Obviously,

\[
I \circ I = \text{Id},
\]

\[
I \circ T_x a = T_y a \circ I,
\]

\[
I \circ T_y b = T_x b \circ I,
\]

\[
S_c \circ T_x a/c = T_x a/c \circ S_c,
\]

\[
S_c \circ T_y b/c = T_y b/c \circ S_c,
\]

\[
S_c \circ I = I \circ S_1/c.
\]

Translations are mere reparameterisations of the corresponding constant astigmatism surface. The scaling symmetry corresponds to an offsetting, i.e., takes a surface to a parallel surface (moves every point a unit distance along the normal). The involution interchanges \( x \) and \( y \) (swaps the orientation), followed by a unit offsetting.

3. Orthogonal equiareal patterns and slip line fields

The geometric meaning of the variable \( z \) can be seen from the third fundamental form, which turns out to be simply

\[
\text{III} = z \, dx^2 + \frac{1}{z} \, dy^2.
\]

Since \( \text{III} = d\mathbf{n} \cdot d\mathbf{n} \) coincides with the first fundamental form of the Gaussian sphere \( \mathbf{n}(x, y) \), it follows that one obtains a rather special parameterisation of the latter.

**Definition 2.** By an orthogonal equiareal pattern on a surface \( S \) we shall mean a parameterization \( x, y \) such that the corresponding first fundamental form is

\[
\text{I}_S = z \, dx^2 + \frac{1}{z} \, dy^2,
\]

(5)

\( z \) being an arbitrary function of \( x, y \).
Let $\mathbf{R}$ denote the position vector of a point on the surface $S$. Since $\det \mathbf{I}_S = 1$, the local parameterisation $\mathbf{R}(x, y)$ is an area preserving map from the plane to the surface $S$. Moreover, the coordinate lines are, obviously, orthogonal. These two properties imply that evenly distributed coordinate lines cover the surface with curvilinear rectangles of equal area (see the Introduction).

**Example 1.** *The Archimedean projection.* A simple example of an orthogonal equiareal pattern on the sphere that can be seen on Fig. 2 is delivered by the well-known Archimedean projection of the cylinder $(\cos y, \sin y, x)$ onto an inscribed sphere. In this case, $(x, y)$ is sent to $(\sqrt{1-x^2} \cos y, \sqrt{1-x^2} \sin y, x)$ and we have

$$I_{\text{Arch}} = \frac{d x^2}{1-x^2} + (1-x^2) d y^2,$$

i.e., $z = 1/(1-x^2)$. According to [5], this solution of the constant astigmatism equation corresponds to von Lilienthal surfaces [21].

![Figure 2. The Archimedean equiareal parameterisation of the sphere](image)

Not only every constant astigmatism surface generates an orthogonal equiareal parameterization of the unit sphere; a converse statement is also available.

**Proposition 1.** Let $\mathbf{n}(x, y), \| \mathbf{n} \| = 1$, be an orthogonal equiareal pattern on the unit sphere $S$. Then $z$ defined by formula (5) is a solution of the constant astigmatism equation (1).

**Proof.** Using the well-known Brioschi formula to compute the Gaussian curvature of the sphere, we obtain

$$1 = -\frac{1}{2} \frac{z_{yy}}{z^3} - \frac{1}{2} \left( \frac{1}{z} \right)_{xx}. $$

The constant astigmatism equation (1) easily follows. \qed
The corresponding constant astigmatism surface can be reconstructed from the last two equations of the Gauss–Weingarten system (3).

Let us stress that all the point symmetries given in Sect. 2 can be understood as reparameterisations of the corresponding orthogonal equiareal pattern on the Gaussian sphere. In particular, scaling \( \mathcal{E}_c \) means shrinking the pattern along one family of lines, compensated by stretching it along the orthogonal family of lines.

In the case of \( S \) being a plane, the notion of an orthogonal equiareal pattern was introduced by Sadowski [29, 30] in the context of two-dimensional plasticity. Choosing the vectors \( \partial_x, \partial_y \) along the principal stress directions (i.e., eigenvectors of the stress tensor \( \sigma^j_i \)), Sadowski derived the equiareal property from the equilibrium condition \( \text{div}\sigma = 0 \) and the Tresca yield condition \( \sigma^1_1 - \sigma^2_2 = \text{const.} \)

Let us reverse the line of reasoning and reconstruct a two-dimensional stress tensor from a given orthogonal equiareal pattern \( g = I_S \). In what follows, all components are taken with respect to the basis \( \partial_x, \partial_y \) of the tangent space and indices are raised and lowered with the metric.

**Proposition 2.** Consider an orthogonal equiareal pattern \( g = g_{ij} \, dx^i \, dx^j \) such that

\[
g_{11} = z, \quad g_{12} = g_{21} = 0, \quad g_{22} = 1/z.
\]

Then the tensor \( \sigma \) given by the components

\[
\sigma^1_1 = \frac{1}{2} \ln z, \quad \sigma^1_2 = \sigma^2_1 = 0, \quad \sigma^2_2 = \frac{1}{2} (\ln z - 2).
\]

satisfies \( \sigma^{ij}_j = 0 \) (the equilibrium equation) and \( \sigma^1_1 - \sigma^2_2 = 1 \) (the Tresca yield condition).

**Proof.** From the metric coefficients we produce the Christoffel symbols

\[
\Gamma^1_{11} = -\Gamma^2_{12} = \frac{\dot{z}_x}{2z}, \quad \Gamma^1_{12} = -\Gamma^2_{22} = \frac{\dot{z}_y}{2z}, \quad \Gamma^1_{22} = \frac{\dot{z}_x}{2z^3}, \quad \Gamma^2_{11} = -\frac{\dot{z}_y}{2}
\]

as well as the twice contravariant tensor

\[
\sigma^{11} = \frac{\ln z}{2z}, \quad \sigma^{12} = \sigma^{21} = 0, \quad \sigma^{22} = \frac{\ln z - 2}{2z}.
\]

The yield condition \( \sigma^1_1 - \sigma^2_2 = 1 \) is obvious, while checking the equilibrium equation \( \sigma^{ij}_j = 0 \) is a matter of routine.

This proposition holds for any surface \( S \) equipped with an orthogonal equiareal pattern \( g \). Note that if \( S \) is a unit sphere, then the Tresca yield condition follows from the constant astigmatism property, since \( \sigma^1_1 = \rho_2 \) and \( \sigma^2_2 = \rho_1 \). Conversely, if \( \sigma^1_1 \) and \( \sigma^2_2 \) are arbitrary functions of \( z \), then the equilibrium equation and the yield condition imply the same \( \sigma^1_1 \) and \( \sigma^2_2 \) as in (6) up to an additive and a multiplicative constant.

In the rest of this section we recall the derivation of the Mohr circle and slip lines in two-dimensional plasticity (see, e.g., [19]). We consider a symmetric stress tensor \( \sigma \) diagonalised along the principal stress directions \( \partial_x, \partial_y \), i.e.,

\[
\sigma^1_1 = p, \quad \sigma^1_2 = \sigma^2_1 = 0, \quad \sigma^2_2 = q, \quad g_{12} = 0.
\]
Let \( \mathbf{w} \) be a unit vector with components \((\cos \phi / \sqrt{g_{11}}, \sin \phi / \sqrt{g_{22}})\) with respect to the basis \( \partial_x, \partial_y \). The stress \( \sigma(\mathbf{w}) = p \cos \phi / \sqrt{g_{11}} + q \sin \phi / \sqrt{g_{22}} \) can be decomposed into a sum of the normal stress \( \sigma_N(\mathbf{w}) \) and the shear stress \( \sigma_T(\mathbf{w}) \), where \( \sigma_N(\mathbf{w}) \perp \sigma_T(\mathbf{w}) \) and, by definition, \( \sigma_N(\mathbf{w}) = \alpha \mathbf{w} \) is a multiple of \( \mathbf{w} \). Hence,

\[
\sigma_T(\mathbf{w}) = \sigma(\mathbf{w}) - \alpha \mathbf{w} = ((p - \alpha) \cos \phi / \sqrt{g_{11}}, (q - \alpha) \sin \phi / \sqrt{g_{22}}).
\]

Obviously, \( ||\sigma_N(\mathbf{w})|| = |\alpha| \); likewise, we introduce \( \beta = ||\sigma_T(\mathbf{w})|| \). By orthogonality,

\[
0 = \sigma_N(\mathbf{w}) \cdot \sigma_T(\mathbf{w}) = \alpha (p - \alpha) \cos^2 \phi + \alpha (q - \alpha) \sin^2 \phi.
\]

Excluding \( \cos \phi \) and \( \sin \phi \) from the last equation and the condition

\[
0 = \beta^2 - \sigma_T(\mathbf{w}) \cdot \sigma_T(\mathbf{w}) = (p - \alpha)^2 \cos^2 \phi + (q - \alpha)^2 \sin^2 \phi,
\]

we conclude that all admissible values of \( \alpha, \beta \) belong to the Mohr circle [23]

\[
\left( \alpha - \frac{p + q}{2} \right)^2 + \beta^2 = \left( \frac{p - q}{2} \right)^2.
\]

It follows that the extremal values \( \alpha = p, q \) of the normal stress magnitude \( \alpha \) are achieved when \( \beta = 0 \), i.e., when \( \mathbf{w} \) lies in one of the principal stress directions, as it should be. The extremal values \( |\frac{1}{2}(p - q)| \) of the shear stress magnitude \( \beta \) are achieved when \( \alpha = \frac{1}{2}(p + q) \). To find the corresponding vectors \( \mathbf{w} \), we determine the acute angle \( \phi \) between \( \mathbf{w} \) and \( \pm \partial_x = (\pm 1, 0) \). Substituting \( \alpha = \frac{1}{2}(p + q) \) into \( \cos^2 \phi = (\alpha - q)/(p - q) \), we obtain \( \cos^2 \phi = \frac{1}{2} \), meaning that \( \phi = \frac{1}{4} \pi \).

Now, the Tresca criterion (see, e.g., [19]) says that yielding occurs whenever the maximal shear stress magnitude \( \beta \) achieves a threshold depending on the material. It follows that the stress tensor satisfies \( p - q = \text{const} \), which is called the Tresca yield condition. The lines along the maximal shear stress direction are called slip lines and, as we have already seen, have a constant deviation of \( \frac{1}{4} \pi \) from the principal stress directions.

**Definition 3.** By a *slip line field* associated with the orthogonal equiareal pattern (5) on a surface \( S \) we shall mean a parameterization \( \xi, \eta \) such that the angle between \( \partial_x \) and \( \partial_\xi \) as well as the angle between \( \partial_y \) and \( \partial_\eta \) is equal to \( \frac{1}{4} \pi \).

It follows that slip lines form an orthogonal net. Note the available freedom of reparameterisation of each \( \xi \) or \( \eta \) separately.

**Remark 1.** In plane plasticity slip line lines form what is called a *Hencky net* [17]. These have been fully described by Carathéodory and Schmidt [12]; a full description of orthogonal equiareal patterns in the plane follows. However, the famous Hencky conditions satisfied by slip lines in the plane fail on surfaces of non-vanishing curvature.

**Example 2.** Continuing Example 1, we easily see that the corresponding orthogonal net of slip lines is, by definition, formed by the \( \pm 45^\circ \) loxodromes (lines of constant bearing); see Fig. 3 or model No. 249 in the Göttingen collection [28]. Also compare with Zelin’s superplastic sheet stretched with a spherical punch [33, Fig. 5b].

In the next section (Prop. 4) we shall see that solutions of the sine–Gordon equation produce slip line fields on the Gaussian sphere of the associated constant astigmatism surface.
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4. Bäcklund transformation and the superposition principle

Available already at the beginning of the nineteenth century, the classical Bäcklund transformation [4] in combination with the Bianchi permutability theorem [7] is a powerful way to generate pseudospherical surfaces and solutions of the sine–Gordon equation. In this section we extend these methods to generate constant astigmatism surfaces, solutions of the constant astigmatism equation as well as equiareal patterns and slip line fields on the sphere.

To start with we briefly recall the Bäcklund transformation [4], see, e.g., [9, 27, 32]. Let us consider a pseudospherical surface \( r(\xi, \eta) \), where the parameters \( \xi, \eta \) are both Chebyshev and asymptotic (which is always possible), i.e.,

\[
I = d\xi^2 + 2 \cos(2\omega) d\xi d\eta + d\eta^2, \quad II = 2 \sin(2\omega) d\xi d\eta.
\]

The integrability conditions of the above system reduce to the sine-Gordon equation

\[
\omega_{\xi\eta} = \frac{1}{2} \sin 2\omega.
\]

The position vector \( r(\xi, \eta) \) and the unit normal \( n(\xi, \eta) \) satisfy the Gauss–Weingarten system

\[
\begin{align*}
\mathbf{r}_{\xi\xi} &= \frac{\sin 4\omega}{\sin^2 2\omega} \omega_{\eta} \mathbf{r}_\xi - \frac{2}{\sin 2w} \omega_x \mathbf{r}_\eta, \\
\mathbf{r}_{\xi\eta} &= (\sin 2\omega)n, \\
\mathbf{r}_{\eta\eta} &= \frac{\sin 4\omega}{\sin^2 2\omega} \omega_{\xi} \mathbf{r}_\eta - \frac{2}{\sin 2w} \omega_y \mathbf{r}_\xi, \\
\mathbf{n}_\xi &= \frac{\sin 4\omega}{2 \sin^2 2\omega} \mathbf{r}_\eta - \frac{1}{\sin 2w} \omega_x \mathbf{r}_\eta, \\
\mathbf{n}_\eta &= \frac{\sin 4\omega}{2 \sin^2 2\omega} \mathbf{r}_\xi - \frac{1}{\sin 2w} \omega_y \mathbf{r}_\xi.
\end{align*}
\]
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The Bäcklund transform of our surface is

$$ r^{(\lambda)} = r + \frac{2\lambda}{1 + \lambda^2} \left( \frac{\sin(\omega - \omega^{(\lambda)})}{\sin(2\omega)} r_\xi + \frac{\sin(\omega + \omega^{(\lambda)})}{\sin(2\omega)} r_\eta \right), $$

(9)

where $\omega^{(\lambda)}$ is another sine-Gordon solution, obtained from the pair of compatible first-order equations

$$ \omega^{(\lambda)}_\xi = \omega_\xi + \lambda \sin(\omega^{(\lambda)} + \omega), \quad \omega^{(\lambda)}_\eta = -\omega_\eta + \frac{1}{\lambda} \sin(\omega^{(\lambda)} - \omega). $$

(10)

Here $\lambda$ is a constant called the Bäcklund parameter.

The Bäcklund transformation is particularly useful in combination with Bianchi’s permutability theorem [7]; see also, e.g., [9, 27]. To simplify exposition, we shall write $B^{(\lambda)}_c \omega$ to denote a solution $\omega^{(\lambda)}_c$ of system (10) for a specified value of the integration constant $c$. The Bianchi permutability theorem says that given a pair of Bäcklund parameters $\lambda_1 \neq \lambda_2$, then for every choice of integration constants $c_1, c_2$ there is a unique choice of integration constants $c_1', c_2'$ such that

$$ B^{(\lambda_2)}_{c_2} B^{(\lambda_1)}_{c_1} \omega = B^{(\lambda_1)}_{c_1} B^{(\lambda_2)}_{c_2} \omega $$

(11)

and, moreover, denoting by $\omega^{(\lambda_1, \lambda_2)}$ the common value in (11), then $\omega^{(\lambda_1, \lambda_2)}$ can be obtained from the superposition principle

$$ \tan \frac{\omega^{(\lambda_1, \lambda_2)} - \omega}{2} = \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} \tan \frac{\omega^{(\lambda_1)} - \omega^{(\lambda_2)}}{2} $$

(12)

without further integration, by purely algebraic manipulations. The corresponding pseudospherical surfaces $r^{(\lambda_1, \lambda_2)}$ can be easily obtained by iterating formula (9).

Assume now that a general solution of system (10) is known for every value of the Bäcklund parameter $\lambda$. Substituting $\omega = \omega^{(\lambda_1)}$ into formula (12), one can also compute the Bäcklund transform $\omega^{(\lambda_1, \lambda_2, \lambda_3)} = B^{(\lambda_3)} \omega^{(\lambda_1, \lambda_2)} = B^{(\lambda_2)} \omega^{(\lambda_1, \lambda_3)}$, by purely algebraic manipulations. In principle, this process may be repeated indefinitely, leading to solutions $\omega^{(\lambda_1, \lambda_2, \ldots, \lambda_s)}$ depending on any finite number of Bäcklund parameters and integration constants, which are thereby obtained by purely algebraic manipulations. Needless to say, the corresponding pseudospherical surfaces $r^{(\lambda_1, \lambda_2, \ldots, \lambda_s)}$ can be computed by iterating the formula (9). Having summarized the Bäcklund transformation and the Bianchi superposition principle, we proceed to generation of surfaces of constant astigmatism by purely algebraic means.

To start with, we remind the reader that in the particular case of $\lambda = 1$ the Bäcklund transformation coincides with Bianchi’s [6] complementarity relation, cf. (4) (actually, the Bäcklund transformation is a combination of complementarity and Lie’s transformation, and the latter is identity if $\lambda = 1$). Otherwise said, the complementary pseudospherical surfaces result from the particular case $B^{(1)}$ of the Bäcklund transformation. Consequently, the superposition formula (12) yields a method to obtain abundant pairs of complementary sine-Gordon solutions $\omega^{(\lambda_1, \lambda_2, \ldots, \lambda_s)}$.
and $\omega^{(\lambda_1 \lambda_2 \cdots \lambda_s)}$. Likewise, one can also obtain abundant pairs of complementary pseudospherical surfaces $r^{(\lambda_1 \lambda_2 \cdots \lambda_s)}$ and $r^{(\lambda_1 \lambda_2 \cdots \lambda_s)}$ by using formula (9).

Let $r$ be a pseudospherical surface, corresponding to a sine-Gordon solution $\omega$. Substituting $\lambda = 1$ into formulas (9) and (10), we immediately see that the complementary surface is

$$r^{(1)} = r + \frac{\sin(\omega - \omega^{(1)})}{\sin(2\omega)} r_\xi + \frac{\sin(\omega + \omega^{(1)})}{\sin(2\omega)} r_\eta, \tag{13}$$

where $\omega^{(1)}$ is the complementary solution of the sine-Gordon equation, satisfying the compatible first-order equations

$$\omega^{(1)}_\xi = \omega_\xi + \sin(\omega^{(1)} + \omega), \quad \omega^{(1)}_\eta = -\omega_\eta + \sin(\omega^{(1)} - \omega). \tag{14}$$

Before proceeding further, we recall two important observations due to Bianchi [6] (see also [9, §386]). Considering the dependence of $\omega^{(1)}$ on the integration constant $c$ and denoting $f = \ln(d\omega^{(1)}/dc)$, differentiation of (14) gives

$$f_\xi = \cos(\omega^{(1)} + \omega), \quad f_\eta = \cos(\omega^{(1)} - \omega). \tag{15}$$

Similarly, taking one more derivative $f' = df/dc$, we get

$$f'_\xi = -e^f \sin(\omega^{(1)} + \omega), \quad f'_\eta = -e^f \sin(\omega^{(1)} - \omega). \tag{16}$$

It follows that, knowing solutions of system (14), we can also obtain solutions of systems (15) and (16) by purely algebraic manipulations and differentiation.

All this is important because surfaces of constant astigmatism are easier to obtain from a pair of complementary pseudospherical surfaces $r$ and $r^{(1)}$ than from a single pseudospherical surface (as considered in [5]). Denote

$$\tilde{n} = r^{(1)} - r = \frac{\sin(\omega - \bar{\omega})}{\sin(2\omega)} r_\xi + \frac{\sin(\omega + \bar{\omega})}{\sin(2\omega)} r_\eta, \tag{17}$$

Then $\tilde{n}$ is a unit vector tangent to both surfaces $r$ and $r^{(1)}$ and determines what is called a pseudospherical congruence. Normal surfaces of this congruence are the constant astigmatism surfaces sought.

**Proposition 3.** Let $\omega^{(1)}(\xi, \eta, c)$ be a general solution of system (14), where $c$ is an integration constant. Then $\tilde{r} = r - f\tilde{n}$, where $f = \ln(d\omega^{(1)}/dc)$ and $\tilde{n}$ is the unit vector given by formula (17), is a surface of constant astigmatism having surfaces $r$ and $r^{(1)}$ as evolutes.

**Proof.** The surface $\tilde{r} = r - f\tilde{n}$ is normal to the congruence determined by the surface $r$ and vectors $\tilde{n}$, if and only if $\tilde{r}_\xi \cdot \tilde{n} = \tilde{r}_\eta \cdot \tilde{n} = 0$. By virtue of the Gauss–Weingarten
system (8) above, the derivatives of \( \mathbf{r} = \mathbf{r} - f \hat{n} \) can be written as

\[
\begin{align*}
\mathbf{r}_\xi &= \left(1 + \frac{\sin 2\omega^{(1)} + \sin \omega}{2 \sin 2\omega} f + \frac{\sin(\omega^{(1)} - \omega)}{\sin 2\omega} f_\xi\right) \mathbf{r}_\xi \\
&\quad - \left(\frac{\sin 2(\omega^{(1)} + \omega)}{2 \sin 2\omega} f + \frac{\sin(\omega^{(1)} + \omega)}{\sin 2\omega} f_\xi\right) \mathbf{r}_\eta - \sin(\omega^{(1)} + \omega)f \hat{n}, \\
\mathbf{r}_\eta &= \left(1 - \frac{\sin 2\omega^{(1)} - \sin \omega}{2 \sin 2\omega} f - \frac{\sin(\omega^{(1)} + \omega)}{\sin 2\omega} f_\eta\right) \mathbf{r}_\eta \\
&\quad + \left(\frac{\sin 2(\omega^{(1)} - \omega)}{2 \sin 2\omega} f - \frac{\sin(\omega^{(1)} - \omega)}{\sin 2\omega} f_\eta\right) \mathbf{r}_\xi + \sin(\omega^{(1)} - \omega)f \hat{n},
\end{align*}
\]

Now it is straightforward to check that the conditions \( \mathbf{r}_\xi \cdot \hat{n} = 0 \) and \( \mathbf{r}_\eta \cdot \hat{n} = 0 \) reduce to equations (15). Moreover, it is a routine to verify that \( \mathbf{r} \) is of constant astigmatism equal to 1. Actually an equivalent computation will be done in the proof of the next proposition under the same assumptions.

Based on Bianchi’s observation above, Proposition 3 shows that the constant astigmatism surfaces \( \mathbf{r} = \mathbf{r} - f \hat{n} \) can be found by purely algebraic manipulations and differentiation once a one-parameter family of pseudopotentials \( \omega^{(1)} \) is known.

However, since \( \xi, \eta \) need not be curvature coordinates on the constant astigmatism surfaces \( \mathbf{r} = \mathbf{r} - f \hat{n} \), Proposition 3, as it stands, yields neither a solution of the constant astigmatism equation nor an orthogonal equiareal pattern on the sphere \( \hat{n} \). Yet the coordinates \( \xi, \eta \) have a geometric meaning of a slip line field according to Definition 3.

**Proposition 4.** Let \( \omega^{(1)}(\xi, \eta, c) \) be a general solution of system (14), where \( c \) is an integration constant, let \( f = \ln(d\omega^{(1)}/dc) \) and \( x = df/dc \). Let \( y(\xi, \eta) \) be a solution of the system

\[
y_\xi = e^{-f} \sin(\omega + \omega^{(1)}), \quad y_\eta = e^{-f} \sin(\omega - \omega^{(1)}).
\]

Then \( x, y \) are adapted curvature coordinates on the surface \( \mathbf{r} \). Moreover, if \( z = e^{-2f} \), then \( z(x, y) \) is a solution of the constant astigmatism equation (1). Finally, \( z \, dx^2 + dy^2/z \) is an orthogonal equiareal pattern on the unit sphere \( \hat{n} \), while \( \xi, \eta \) is the associated slip line field according to Definition 3.

**Proof.** Continuing the routine computations started in the proof of Proposition 3 we obtain

\[
\begin{align*}
\mathbf{I} &= \frac{1}{2} (1 + 2f + 2f^2) (1 - \cos 2(\omega + \omega^{(1)})) \, d\xi^2 \\
\quad &\quad + (1 + 2f)(\cos 2\omega - \cos 2\omega^{(1)}) \, d\xi \, d\eta \\
\quad &\quad + \frac{1}{2} (1 + 2f + 2f^2) (1 - \cos 2(\omega - \omega^{(1)})) \, d\xi^2, \\
\mathbf{II} &= \frac{1}{2} (1 + 2f)(1 - \cos 2(\omega + \omega^{(1)})) \, d\xi^2 \\
\quad &\quad + (\cos 2\omega - \cos 2\omega^{(1)}) \, d\xi \, d\eta \\
\quad &\quad + \frac{1}{2} (1 + 2f)(1 - \cos 2(\omega - \omega^{(1)})) \, d\xi^2.
\end{align*}
\]
Some results concerning the constant astigmatism equation

The corresponding shape operator is

\[
\frac{1}{2f(f+1)} \begin{pmatrix}
2f + 1 & 1 - \cos 2(\omega^{(1)} - \omega) \\
1 - \cos 2(\omega^{(1)} + \omega) & \cos 2\omega^{(1)} - \cos 2\omega \\
\cos 2\omega^{(1)} - \cos 2\omega & 2f + 1
\end{pmatrix}.
\]

Its eigenvalues are the curvatures, namely \(1/f\) and \(1/(f+1)\). We choose \(\rho_1 = f + 1\), \(\rho_2 = f\) to have \(\rho_1 - \rho_2 = 1\). The eigenvectors yield two principal directions

\[\Pi_\pm = (\cos 2\omega^{(1)} - \cos 2\omega) \frac{\partial}{\partial \xi} \pm (1 - \cos 2(\omega + \omega^{(1)})) \frac{\partial}{\partial \eta}.\]

It is easy to check that \(x\) and \(y\) are integrals of the fields \(\Pi_+\) and \(\Pi_-\), respectively. This implies that \(x, y\) are curvature coordinates. Now, equations

\[
\tilde{I} = u^2 dx^2 + v^2 dy^2, \quad \tilde{II} = (u^2/\rho_1) dx^2 + (v^2/\rho_2) dy^2.
\]

yield

\[u = (f + 1)/e^f, \quad v = fe^f.\]  \hfill (19)

Substituting into condition (2), we see that the curvature coordinates \(x, y\) are adapted. Moreover, it is straightforward to verify that \(z = e^{-2f}\) satisfies the equation of constant astigmatism (1) with respect to independent variables \(x, y\).

To prove that \(\xi, \eta\) is the associated slip line field on the sphere \(\tilde{n}\), it suffices to check that

\[
\tilde{n}_\xi = \frac{\cos 3\omega - \cos \omega - \cos(2\omega^{(1)} - \omega) + \cos(2\omega^{(1)} + \omega)}{8 \sin^2 \omega \cos \omega} r_\xi
\]

\[+ \frac{\cos(2\omega^{(1)} + \omega) - \cos(2\omega^{(1)} + 3\omega)}{8 \sin^2 \omega \cos \omega} r_\eta + \sin(\omega^{(1)} + \omega)n\]

\[
\tilde{n}_\eta = \frac{\cos 3\omega - \cos \omega + \cos(2\omega^{(1)} - \omega) - \cos(2\omega^{(1)} + \omega)}{8 \sin^2 \omega \cos \omega} r_\eta
\]

\[+ \frac{\cos(2\omega^{(1)} - \omega) - \cos(2\omega^{(1)} - 3\omega)}{8 \sin^2 \omega \cos \omega} r_\xi - \sin(\omega^{(1)} - \omega)n\]

bisect the right angle between

\[
\tilde{n}_x = -\frac{\cos(\omega^{(1)} - \omega)}{\sin 2\omega e^f} r_\xi + \frac{\cos(\omega^{(1)} + \omega)}{\sin 2\omega e^f} r_\eta \quad \text{and} \quad \tilde{n}_y = e^f n,
\]

according to Definition 3. This is straightforward.

Corollary 1. If \(S\) is a constant astigmatism surface, then the asymptotic coordinates on the focal surfaces of \(S\) correspond to slip line fields on the Gaussian image of \(S\).

Proposition 4 allows us to construct one of the adapted curvature coordinates by purely algebraic manipulations and differentiation, while the other curvature coordinate has to be obtained by integration. It is therefore natural to ask whether one could obtain superposition formulas for \(f, x, y\) similar to formula (12). The answer is positive.
Definition 4. Given two sine-Gordon solutions $\omega$ and $\omega^{(\lambda)}$ related by the Bäcklund transformation $B^{(\lambda)}$, let $f^{(\lambda)}$, $x^{(\lambda)}$, $y^{(\lambda)}$ denote functions satisfying the compatible equations

$$
\begin{align*}
 f^{(\lambda)}_\xi &= \lambda \cos(\omega^{(\lambda)} + \omega), & f^{(\lambda)}_\eta &= \frac{1}{\lambda} \cos(\omega^{(\lambda)} - \omega), \\
 x^{(\lambda)}_\xi &= \lambda e^{f^{(\lambda)}} \sin(\omega^{(\lambda)} + \omega), & x^{(\lambda)}_\eta &= \frac{1}{\lambda} e^{f^{(\lambda)}} \sin(\omega^{(\lambda)} - \omega), \\
 y^{(\lambda)}_\xi &= \lambda e^{-f^{(\lambda)}} \sin(\omega^{(\lambda)} + \omega), & y^{(\lambda)}_\eta &= -\frac{1}{\lambda} e^{-f^{(\lambda)}} \sin(\omega^{(\lambda)} - \omega).
\end{align*}
$$

(20)

The quantities $f^{(\lambda)}$, $x^{(\lambda)}$, $y^{(\lambda)}$ will be called associated potentials corresponding to the pair $\omega, \omega^{(\lambda)}$.

Proposition 5. Let $\omega, \omega^{(\lambda_1)}, \omega^{(\lambda_2)}, \omega^{(\lambda_1\lambda_2)}$ be four sine-Gordon solutions related by the Bianchi superposition principle (12). Then the associated potentials $f^{(\lambda_1\lambda_2)}$, $x^{(\lambda_1\lambda_2)}$, $y^{(\lambda_1\lambda_2)}$ corresponding to the pair $\omega^{(\lambda_1)}, \omega^{(\lambda_1\lambda_2)}$ are related to the associated potentials $f^{(\lambda_2)}$, $x^{(\lambda_2)}$, $y^{(\lambda_2)}$ corresponding to the pair $\omega, \omega^{(\lambda_2)}$ by formulas

$$
\begin{align*}
 f^{(\lambda_1\lambda_2)} &= f^{(\lambda_2)} - \ln\left(2 \cos(\omega^{(\lambda_1)} - \omega^{(\lambda_2)}) - \frac{\lambda_1}{\lambda_2} - \frac{\lambda_2}{\lambda_1}\right), \\
x^{(\lambda_1\lambda_2)} &= \frac{\lambda_1 \lambda_2}{\lambda_1^2 - \lambda_2^2} \left(x^{(\lambda_2)} - \frac{2 \lambda_1 \lambda_2 \sin(\omega^{(\lambda_1)} - \omega^{(\lambda_2)})}{\lambda_1^2 - 2 \lambda_1 \lambda_2 \cos(\omega^{(\lambda_1)} - \omega^{(\lambda_2)}) + \lambda_2^2 e^{f^{(\lambda_2)}}}\right), \\
y^{(\lambda_1\lambda_2)} &= \left(\frac{\lambda_1}{\lambda_2} - \frac{\lambda_2}{\lambda_1}\right) y^{(\lambda_2)} - 2 e^{-f^{(\lambda_2)}} \sin(\omega^{(\lambda_1)} - \omega^{(\lambda_2)}),
\end{align*}
$$

(21)

up to an additive constant.

Proof. It is straightforward to check that $f^{(\lambda_1\lambda_2)}$, $x^{(\lambda_1\lambda_2)}$, $y^{(\lambda_1\lambda_2)}$ given by formulas (21) satisfy

$$
\begin{align*}
 f^{(\lambda_1\lambda_2)}_\xi &= \lambda_2 \cos(\omega^{(\lambda_1\lambda_2)} + \omega^{(\lambda_1)}), & f^{(\lambda_1\lambda_2)}_\eta &= \frac{1}{\lambda_2} \cos(\omega^{(\lambda_1\lambda_2)} - \omega^{(\lambda_1)}), \\
x^{(\lambda_1\lambda_2)}_\xi &= \lambda_2 e^{f^{(\lambda_2)}} \sin(\omega^{(\lambda_1\lambda_2)} + \omega^{(\lambda_1)}), & x^{(\lambda_1\lambda_2)}_\eta &= \frac{1}{\lambda_2} e^{f^{(\lambda_1\lambda_2)}} \sin(\omega^{(\lambda_1\lambda_2)} - \omega^{(\lambda_1)}), \\
y^{(\lambda_1\lambda_2)}_\xi &= \lambda_2 e^{-f^{(\lambda_1\lambda_2)}} \sin(\omega^{(\lambda_1\lambda_2)} + \omega^{(\lambda_1)}), & y^{(\lambda_1\lambda_2)}_\eta &= -\frac{1}{\lambda_2} e^{-f^{(\lambda_1\lambda_2)}} \sin(\omega^{(\lambda_1\lambda_2)} - \omega^{(\lambda_1)}).
\end{align*}
$$

whenever $f^{(\lambda_2)}$, $x^{(\lambda_2)}$, $y^{(\lambda_2)}$ satisfy (20) with $\lambda = \lambda_2$. \hfill \qed

Example 3. (One-soliton solutions) Let us apply the procedure outlined above to the one-soliton solutions

$$
\omega^{(\lambda)} = B^{(\lambda)}_0(0) = 2 \arctan \exp p_\lambda,
$$

of the sine-Gordon equation. Here and in what follows we denote

$$
p_\lambda = \lambda \xi + \frac{\eta}{\lambda}, \quad q = \xi - \eta.
$$
As is well known, these one-soliton solutions correspond to the Dini surfaces (helicoids of the tractrix)

\[ r^{(\lambda)} = \frac{2\lambda}{1 + \lambda^2} \begin{pmatrix} \text{sech } p_{\lambda} \sin q \\ \text{sech } p_{\lambda} \cos q \\ p_{\lambda} - \tanh p_{\lambda} \end{pmatrix} + \frac{1 - \lambda^2}{1 + \lambda^2} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \]

We now proceed to the complementary surfaces of the Dini surfaces, which correspond to the nonlinear superposition of \( \omega^{(\lambda)} \) and \( \omega^{(1)} \), i.e., the two-soliton solutions

\[ \omega^{(\lambda_1)} = B_{c_1}^{(1)}(\omega^{(\lambda)}) = 2 \arctan \frac{(\lambda + 1)(e^{p_{\lambda}} - e^{p_{1}+c})}{(\lambda - 1)(1 + e^{p_{\lambda}+p_{1}+c})}, \]

where, obviously, \( p_1 = \xi + \eta \). The particular case of \( \lambda = 1 \) (the Beltrami pseudosphere) is excluded from consideration.

After tedious computations, one obtains the resulting quantities \( x = x^{(\lambda_1)}, y = y^{(\lambda_1)}, z = e^{-2f^{(\lambda_1)}} \). They are

\[ x = \frac{\lambda}{\lambda^2 - 1} \times \frac{(\lambda - 1)^2(c_2 A^2 B^2 - c_1) - (\lambda + 1)^2(c_1 B^2 + c_2 A^2) + 4(c_1 - c_2)\lambda AB}{(\lambda - 1)^2(A^2 B^2 + 1) + (\lambda + 1)^2(B^2 + A^2) - 8\lambda AB}, \]

\[ y = \frac{4 \ln B}{c_1 + c_2} - 2 \frac{2(\lambda^2 + 1) \ln A}{(c_1 + c_2)\lambda} + \frac{4\lambda(AB + 1)(A - B) + c_3(c_1 + c_2)(\lambda^2 - 1)A(1 + B^2)}{(c_1 + c_2)\lambda e^{2(1 + e^{2p_{\lambda}})}} \]

\[ z = \left( \frac{(\lambda - 1)^2(A^2 B^2 + 1) + (\lambda + 1)^2(B^2 + A^2) - 8\lambda AB}{(c_1 + c_2)\lambda A(1 + B^2)} \right)^2, \]

where \( A = e^{p_{\lambda}} = e^{\xi + \eta} \) and \( B = e^{p_{\lambda}} = e^{\lambda \xi + \eta / \lambda} \), while \( c_1, c_2, c_3 \) are arbitrary constants. By eliminating \( \xi, \eta \) one obtains

\[ y = \frac{1}{c_1 + c_2} \left( \frac{4(AB + 1)(A - B)}{(B^2 + 1)A} - 2 \frac{\lambda^2 + 1}{\lambda} \ln A + 4 \ln B \right) + \frac{(\lambda^2 - 1)c_3}{\lambda}, \quad (22) \]

where

\[ A = \frac{\lambda(\lambda^2 + 1)(c_1 + c_2)\sqrt{z} - \sqrt{k}}{(\lambda^2 - 1)^2 + (\lambda^2 x - \lambda c_2 - x)^2z}, \]

\[ B = \frac{2\lambda^2(c_1 + c_2)\sqrt{z} + \sqrt{k}}{(\lambda^2 - 1)^2 + (\lambda^2 x - \lambda c_2 - x)(\lambda^2 x + \lambda c_1 - x)z}, \]

\[ k = -[(\lambda^2 - 1)^2 + 2(c_1 + c_2)\lambda^2 \sqrt{z} + (\lambda^2 x - \lambda c_2 - x)(\lambda^2 x + \lambda c_1 - x)z] \]

\[ \times [(\lambda^2 - 1)^2 - 2(c_1 + c_2)\lambda^2 \sqrt{z} + (\lambda^2 x - \lambda c_2 - x)(\lambda^2 x + \lambda c_1 - x)z]. \]

Eq. (22) is an implicit formula for a solution \( z(x, y) \) of the constant astigmatism equation. Using Proposition 3 it is now easy to construct the surface of constant astigmatism from its two evolutes \( r^{(\lambda)} \) and \( r^{(\lambda_1)} \) as well as the orthogonal slip line net on the Gaussian sphere \( \hat{n} = r^{(\lambda_1)} - r^{(\lambda)} \), part of which can be seen on Fig. 4 (actually, the sphere is multiply covered). This example also demonstrates that sphere’s slip line fields are prone to developing singularities.

Construction of constant astigmatism surfaces related to the sine-Gordon \( n \)-soliton solutions is postponed to a separate paper.
Some results concerning the constant astigmatism equation

Figure 4. Sphere’s slip line field with features

5. Lipschitz surfaces in principal coordinates

In 1887 Lipschitz [22] presented a class of surfaces of constant astigmatism in terms of spherical coordinates related to the Gaussian image. To find the corresponding solutions of the constant astigmatism equation, one has to obtain the surfaces in terms of the principal coordinates. Redoing the computation is easier than transforming the Lipschitz result.

Consider the unit sphere $n = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)$ parameterised by the latitude $\theta$ and longitude $\phi$. To specify an orthogonal equiareal pattern we let $\theta, \phi$ denote yet unknown functions of parameters $x, y$. Lipschitz defines a *Stellungswinkel* to be the angle $\omega$ between $n_\theta$ and $n_x = \phi_x n_\phi + \phi_\theta n_\theta$. The Lipschitz class is specified by allowing the Stellungswinkel to depend solely on the latitude $\theta$.

**Theorem 1.** The general Lipschitz solution of the constant astigmatism equation (1) depends on four constants $h_{11}, h_{10}, h_{01}, h_{00}$ and consists of functions

$$z = \frac{1 - h^2 + \sqrt{(1 - h^2)^2 - 4(H_1 h - H_2)^2}}{2(h_{11} x + h_{01})^2},$$

(23)

where

$$h = h_{11} xy + h_{10} x + h_{01} y + h_{00}, \quad H_1 = h_{11}, \quad H_2 = h_{11} h_{00} - h_{10} h_{01}.$$

The constants $H_1, H_2$ are invariant with respect to the translations $\Sigma^x, \Sigma^y$, the scaling $\mathcal{S}$ and the involution $\mathfrak{I}$. Formula (23) covers all Lipshitz solutions except a particular solution

$$z = \frac{1}{c_1 - (x - c_0)^2},$$

$c_1, c_0$ being arbitrary constants.
Some results concerning the constant astigmatism equation

Proof. Computing the first fundamental form in two ways we have
\[ d\theta^2 + \sin^2 \theta \, d\phi^2 = z \, dx^2 + (1/z) \, dy^2, \]
i.e.,
\[ \theta_x^2 + \phi_x^2 \sin^2 \theta = z, \quad \theta_x \theta_y + \phi_x \phi_y \sin^2 \theta = 0, \quad \theta_y^2 + \phi_y^2 \sin^2 \theta = 1/z. \]
Eliminating \( z \), we have
\[ (\theta_x^2 + \phi_x^2 \sin^2 \theta)(\theta_y^2 + \phi_y^2 \sin^2 \theta) = 1, \quad \theta_x \theta_y + \phi_x \phi_y \sin^2 \theta = 0. \quad (24) \]
Since
\[ \mathbf{n}_\theta = (\cos \phi \cos \theta, \sin \phi \cos \theta, -\sin \theta), \quad \mathbf{n}_\phi = (-\sin \phi \sin \theta, \cos \phi \sin \theta, 0), \]
\[ \mathbf{n}_x = (\theta_x \cos \phi \cos \theta - \phi_x \sin \phi \sin \theta, \theta_x \sin \phi \cos \theta + \phi_x \cos \phi \sin \theta, -\theta_x \sin \theta), \]
we easily compute
\[ \cos \omega = \frac{\mathbf{n}_\theta \cdot \mathbf{n}_x}{|\mathbf{n}_\theta| |\mathbf{n}_x|} = \frac{\theta_x}{\sqrt{\theta_x^2 + \phi_x^2 \sin^2 \theta}}. \quad (25) \]
Hence, the Lipschitz’ condition amounts to \( \phi_x / \theta_x \) being a function of \( \theta \) alone. This can be conveniently written in the form
\[ \phi_x = \frac{\Theta(\theta)}{\sin \theta} \theta_x \]
(intentionally leaving \( \sin \theta \) unabsorbed). Conditions (24) and (26) combine into the system
\[ \phi_x = \frac{\Theta}{\sin \theta} \theta_x, \quad \phi_y = -\frac{1}{(\sin \theta)(\Theta^2 + 1)\theta_x}, \quad \theta_y = \frac{\Theta}{(\Theta^2 + 1)\theta_x}. \quad (27) \]
Computing the compatibility conditions for \( \phi \) we get \((\sin \theta)\theta_{xx} + (\cos \theta)\theta_x^2 = 0\) with the general solution
\[ \theta = \arccos h, \quad h = h_1 x + h_0, \]
\( h_1, h_0 \) being arbitrary functions of \( y \). Now the third equation of (27) implies
\[ \frac{\Theta}{\Theta^2 + 1} = \theta_x \theta_y = \frac{h_1}{1 + (h_1 x + h_0)^2} \left( \frac{\partial h_1}{\partial y} x + \frac{\partial h_0}{\partial y} \right). \quad (28) \]
Since the left-hand side is a function of \( \theta \) alone, the same is true for the right-hand side. Checking the Jacobian, we get
\[ 0 = \frac{h_1^2}{(1 + (h_1 x + h_0)^2)^{3/2}} \left( \frac{\partial h_1^2}{\partial y^2} x + \frac{\partial h_0^2}{\partial y^2} \right). \quad (29) \]
If \( h_1 = 0 \), then \( h = h_0 \) is a function of \( y \) alone and from system (27) we easily obtain \( h_0 = h_{01} y + h_{00} \) and \( z = 1/h_{01}^2 - (y + h_{00}/h_{01})^2 \), which is the particular solution.
Some results concerning the constant astigmatism equation

Otherwise $h_1 \neq 0$; it follows from (29) that $h_1, h_0$ are linear in $y$ and we can write

$$h = h_{11}xy + h_{10}x + h_{01}y + h_{00},$$

with $h_{ij}$ being arbitrary constants. Now, eq. (28) gives

$$\Theta = \frac{1 - h^2 + \sqrt{(1 - h^2)^2 - 4(H_1 h - H_2)^2}}{2(H_1 h - H_2)}.$$ 

Since $H_1, H_2$ are constants, $\Theta$ is a function of $h$ and, consequently, of $\theta$ alone. Inserting

$$z = \theta^2 + \phi^2 \sin^2 \theta,$$

we obtain the general solution (23).

\[\square\]

It is now easy to obtain the corresponding orthogonal equiareal pattern.

**Theorem 2.** The orthogonal equiareal pattern corresponding to the general Lipschitz solution is $\mathbf{n} = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)$, where

$$\theta = \arccos h,$$

$$\phi = -\frac{\ln(h_{11}x + h_{01})}{h_{11}} + \int \frac{1 - h^2 + \sqrt{(1 - h^2)^2 - 4(H_1 h - H_2)^2}}{2(H_1 h - H_2)(1 - h^2)} dh,$$

$$h = h_{11}xy + h_{10}x + h_{01}y + h_{00}, \quad H_1 = h_{11}, \quad H_2 = h_{11}h_{00} - h_{10}h_{01}.$$ 

**Proof.** We need to know $\phi$, i.e., we have to integrate the first two equations (27). It is easily observed that $\phi_y h_y - \phi_x h_x + 1 = 0$. Solving this PDE for $\phi$, we obtain

$$\phi = -\frac{\ln(h_{11}x + h_{01})}{h_{11}} + \Phi(h),$$

while for $\Phi(h)$ we get

$$\frac{d\Phi}{dh} = \frac{1 - h^2 + \sqrt{(1 - h^2)^2 - 4(H_1 h - H_2)^2}}{2(H_1 h - H_2)(1 - h^2)}.$$ 

\[\square\]

The Stellungswinkel $\omega$ is a function of the lattitude $\theta$ as required; namely

$$\cos^2 \omega = \frac{1}{\Theta^2 + 1} = \frac{1 - h^2 + \sqrt{(1 - h^2)^2 - 4(H_1 h - H_2)^2}}{2(1 - h^2)}$$

$$= \frac{\sin^2 \theta + \sqrt{\sin^4 \theta - 4(H_1 \cos \theta - H_2)^2}}{2 \sin^2 \theta}.$$
Remark 2. It is easy to check that the general Lipschitz solution (23) satisfies

\[ h_{11}s + h_{01}t^x - h_{10}t^y = 0, \]

where \( t^x = z_x, \ t^y = z_y, \ s = xz_x - yz_y + 2z \) are generators (see, e.g., [10]) of the Lie symmetries \( \mathfrak{T}^x, \mathfrak{T}^y, \mathfrak{S} \), respectively. This means that (23) is a symmetry-invariant solution of the constant astigmatism equation.

Example 4. When the integral in (31) can be expressed in terms of elementary functions? Assuming that \( h_{11} \) is nonzero, \( h_{10} \) and \( h_{01} \) can be removed by shifts, so we set \( h_{10} = h_{01} = 0 \). Consider the expression under the square root in (31). Its discriminant with respect to \( h \) is proportional to

\[
(1 + H_1^2 + 2H_2)(1 + H_1^2 - 2H_2)(H_1 - H_2)^2(H_1 + H_2)^2
= h_{11}^4(1 + h_{11}^2 + 2h_{11}h_{00})(1 + h_{11}^2 - 2h_{11}h_{00})(1 - h_{00})^2(1 + h_{00})^2,
\]

which is zero if and only if

\[ h_{00} = \pm 1 \quad \text{or} \quad h_{00} = \pm \frac{1 + h_{11}^2}{2h_{11}}. \]

In these cases, \( \phi \) can be expressed in terms of elementary functions. For \( h_{00} = \pm 1 \) we have

\[
\phi = \mp \frac{\sqrt{1 - C^2}}{2C} \ln \left( \frac{4(1 - C^2 \pm h + \sqrt{1 - C^2 (h \mp 1)^2 - 4(C^2 \mp h)})}{h \mp 1} \right) - \frac{\ln(Cx)}{C} + \frac{\ln[h^2 - 1 + (h \mp 1) \sqrt{(h \mp 1)^2 - 4(C^2 \mp h)}]}{2C}
\pm \frac{1}{2} \arctan \left( \frac{\sqrt{(h \mp 1)^2 - 4C^2}}{2C} \right),
\]

where \( C = H_1 = h_{11} \) is a constant and \( h = Cxy + 1 \). The orthogonal equiareal pattern corresponding to \( h_{00} = 1 \) and \( h_{11} = 1/4 \) can be seen on Fig. 5.

In the second case, when \( h_{00} = \pm (1 + h_{11}^2)/2h_{11} \), we obtain

\[
\phi = -\frac{\ln(Cx)}{C} \mp \frac{\ln(h \pm C + \sqrt{h^2 \pm 2Ch - C^2 - 2})}{2C}
- \frac{1}{2C} \ln \left( \frac{-C^3 - 3C \pm h(3C^2 + 1) - (1 - C^2) \sqrt{h^2 \pm 2Ch - C^2 - 2}}{(C^2 \mp 2Ch + 1)^2} \right)
\pm \frac{1}{2} \arctan \left( \frac{2(Ch \pm 1) \sqrt{h^2 \pm 2Ch - C^2 - 2}}{h^2 - C^2h^2 - C^2 \pm 4Ch - 3} \right).
\]

Here \( h, C \) have the same meaning as above. No figure is provided in this case, since \( \phi \) and \( \theta \) cannot be simultaneously real.
6. Conclusions

Summarizing, we identified the constant astigmatism equation and the sine-Gordon equation as integrable models in two-dimensional plasticity on the sphere with respect to parameterisation by principal stress lines and slip lines, respectively. We remark in this context that the majority of exact solutions in plasticity that can be found in the literature are either due to linearisable systems or come from symmetry methods; see, e.g., [3, 20] and references therein.

We have also extended the classical Bianchi superposition principle so as to be able to generate solutions of the constant astigmatism equation by algebraic manipulations. Finally, revisiting the classical Lipschitz surfaces of constant astigmatism, we have identified them as corresponding to invariant solutions.

In conclusion, we are able to say that obtaining large families of exact solutions of the constant astigmatism equation as well as interpreting them as plastic flows on a sphere is merely a matter of routine. Since computations quickly leave the realm of elementary functions, the examples and illustrations scattered throughout this paper are only the simplest ones. More are to follow in a subsequent paper.

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