ON THE PHASES OF A COMPLEX MATRIX *
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Abstract. In this paper, we define the phases of a complex cramped matrix to be the phases of the eigenvalues of the unitary part of an associated sectoral decomposition. While a cramped matrix admits distinct sectoral decompositions, its phases defined in this manner are unique. In particular, we show that specific forms of sectoral decompositions are indeed unique, namely the symmetric polar decomposition and generalized Cholesky factorization. Various properties of matrix phases are studied, including those of compressions, Schur complements, matrix products, and sums. Importantly, by exploiting a notion known as the compound numerical range, we establish a majorization relation between the phases of the eigenvalues of a matrix product and the phases of the individual matrices. This is then applied to investigate the rank of the sum of a matrix product and the identity matrix under phasic perturbations. The phases of the Kronecker and Hadamard products are also discussed.

Key words. phases, arguments, cramped matrices, interlacing, majorization, rank robustness

AMS subject classifications. 15A18, 15A23, 15A42, 15A60, 15B48, 15B57

1. Introduction. A nonzero complex scalar $c$ can be represented in the polar form as
$$c = \sigma e^{i\phi},$$
where $\sigma > 0$ is called the modulus or the magnitude and $\phi$ the argument or the phase. That is,
$$\sigma = |c| \text{ and } \phi = \angle c.$$
To be more specific, the phase $\phi$ takes values in a half open $2\pi$-interval, typically $[0, 2\pi)$ or $(-\pi, \pi]$. If $c = 0$, the magnitude $|c|$ is defined as 0, but the phase $\angle c$ is undefined.

The magnitude and phase are invariant under certain operations. Specifically, the magnitude of $c$ is “unitarily invariant” in the sense that $|c| = |ucv|$ for all $u, v \in \mathbb{C}$ satisfying $|u| = |v| = 1$. On the other hand, the phase is “congruence invariant” in the sense that $\angle c = \angle w^*cw$ for all nonzero $w \in \mathbb{C}$, where the superscript $*$ denotes the complex conjugate transpose.

The magnitude and phase have the following fundamental properties:

\[(1.1) \quad |ab| = |a||b| \quad \text{(multiplicativity)}\]
\[(1.2) \quad |a + b| \leq |a| + |b| \quad \text{(subadditivity)}\]
\[(1.3) \quad \angle(ab) = \angle a + \angle b \mod 2\pi\]
\[(1.4) \quad \angle(a + b) \in \text{Co}\{\angle a, \angle b\} \quad \text{if } |\angle a - \angle b| < \pi,
\]
where Co denotes the convex hull. The observation that (1.1) and (1.3) are simple equalities whereas (1.2) and (1.4) are not enhances our understanding that the

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multiplication operation is easier to perform using the polar form while the addition
operation the rectangular form of complex numbers.

It is well understood that an \( n \times n \) complex matrix \( C \) has \( n \) magnitudes, served
by the \( n \) singular values

\[ \sigma(C) = [\sigma_1(C) \, \sigma_2(C) \, \ldots \, \sigma_n(C)] \]

arranged in such a way that

\[ \sigma(C) = \sigma_1(C) \geq \sigma_2(C) \geq \cdots \geq \sigma_n(C) = \sigma(C). \]

The singular values can be obtained from a singular value decomposition \( C = U S V^* \),
where \( U, V \) are unitary, and \( S = \text{diag} \{ \sigma_1(C), \sigma_2(C), \ldots, \sigma_n(C) \} \) [12, Theorem 2.6.3].
They can also be derived from the following maximin and minimax expressions [3, Corollary III.1.2][12, Theorem 4.2.6]:

\[
\sigma_i(C) = \max_{M: \dim M = i} \min_{x \in M, \|x\| = 1} \|Cx\| = \min_{N: \dim N = n-i+1} \max_{x \in N, \|x\| = 1} \|Cx\|.
\]

The singular values are unitarily invariant in the sense that

\[ \sigma(C) = \sigma(U^*CV) \]

for all unitary matrices \( U \) and \( V \). In particular, permuting the rows or columns of \( C \)
does not change its singular values.

The singular values provide a bound on the magnitudes of the eigenvalues of \( C \)
in the majorization order. Given two vectors \( x, y \in \mathbb{R}^n \), denote by \( x^k \) and \( y^k \) the
rearranged versions of \( x \) and \( y \), respectively, in which their elements are sorted in a
non-increasing order. Then, \( x \) is said to be majorized by \( y \), denoted by \( x \prec y \), if

\[
\sum_{i=1}^{k} x_i^k \leq \sum_{i=1}^{k} y_i^k, \quad k = 1, \ldots, n-1, \quad \text{and} \quad \sum_{i=1}^{n} x_i^k = \sum_{i=1}^{n} y_i^k.
\]

Moreover, \( x \) is said to be weakly majorized by \( y \) from below, denoted by \( x \prec_w y \), if
the last equality sign is changed to \( \leq \). For two nonnegative vectors \( x, y \in \mathbb{R}^n \), \( x \)
is said to be log-majorized by \( y \), denoted by \( x \prec_{\log} y \), if

\[
\prod_{i=1}^{k} x_i^k \leq \prod_{i=1}^{k} y_i^k, \quad k = 1, \ldots, n-1, \quad \text{and} \quad \prod_{i=1}^{n} x_i^k = \prod_{i=1}^{n} y_i^k.
\]

Note that log-majorization is stronger than weak-majorization, i.e., \( x \prec_{\log} y \) implies
\( x \prec_w y \) [16, Chapter 5, A.2.b]. For more details on the theory of majorization, we
refer to [16] and the references therein. Denote the vector of eigenvalues of \( C \) by

\[ \lambda(C) = [\lambda_1(C) \, \lambda_2(C) \, \ldots \, \lambda_n(C)]. \]

It is well known that the magnitudes of the eigenvalues are bounded by the singular
values in the following manner [16, Theorem 9.E.1]:

\[ |\lambda(C)| \prec_{\log} \sigma(C), \]
which implies, among other things, that

\[ |\lambda(C)| \prec_w \sigma(C). \]

It is also of interest to relate the singular values of \( C \) with those of any \((n-k) \times (n-k)\) submatrix of \( C \). In this regard, we have the following interlacing property [3, Corollary III.1.5]:

\[ \sigma_i(C) \geq \sigma_i(U^*CV) \geq \sigma_{i+k}(C), \quad i = 1, \ldots, n-k, \]

for all \( n \times (n-k) \) isometries \( U \) and \( V \), which are matrices with orthonormal columns. In particular, when \( k = 1 \), we have

\[ \sigma_1(C) \geq \sigma_1(U^*CV) \geq \sigma_2(C) \geq \cdots \geq \sigma_{n-1}(U^*CV) \geq \sigma_n(C). \]

Let \( C \in \mathbb{C}^{n \times n} \) be partitioned as \( C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \), where \( C_{11} \in \mathbb{C}^{k \times k} \). If \( C_{11} \) is nonsingular, then the Schur complement of \( C_{11} \) in \( C \), denoted by \( C/_{11} \), exists and is given by \( C/_{11} = C_{22} - C_{21}C_{11}^{-1}C_{12} \). It can be inferred from [22, Corollary 2.3] that \( \sigma(C/_{11}) \) and \( \sigma(C) \) satisfy the following interlacing relation:

\[ \sigma_i(C) \geq \sigma_i(C/_{11}) \geq \sigma_{i+k}(C), \quad i = 1, \ldots, n-k. \]

The singular values of \( AB \) and \( A + B \) are related to those of \( A \) and \( B \) through the following majorization type inequalities resembling (1.1) and (1.2), respectively, [16, Theorem 9.H.1, 9.G.1.d]

\[ \sigma(AB) \prec_{\log} \sigma(A) \odot \sigma(B), \]
\[ \sigma(A + B) \prec_w \sigma(A) + \sigma(B), \]

where \( \odot \) denotes the Hadamard product, i.e., the elementwise product.

In contrast to the magnitudes of a complex matrix \( C \), there does not exist a universally accepted definition of phases of \( C \). What properties should the phases satisfy? In this paper, we advocate a definition of matrix phases and derive some of their properties as counterparts to those of the singular values described above.

An early attempt [18] defined the phases of \( C \) as the phases of the eigenvalues of the unitary part of its polar decomposition, as motivated by the seeming generalization of the polar form of a scalar to the polar decomposition of a matrix. As in the scalar case, ambiguity arises when defining phases for a singular \( C \). Hence, only nonsingular matrices are relevant. More precisely, let the left and right polar decompositions of a nonsingular matrix \( C \) be given by \( C = PU = UQ \), where \( P \) and \( Q \) are positive definite and \( U \) is unitary [12, Theorem 7.3.1]. The authors of [18] proposed to define the phases of \( C \) as

\[ \psi(C) = \angle \lambda(U). \]

The \( \angle \) function may take values in a fixed interval of length \( 2\pi \), such as \( (-\pi, \pi] \) or \( [0, 2\pi) \). Phases defined through the polar decomposition have the advantage that they are applicable to any square nonsingular matrices. However, they do not possess certain desired properties, which will be discussed later.

In this paper, we shall revisit the canonical angles introduced in [9] and propose to adopt them as the phases of a complex matrix. At present, we limit the definition to the class of cramped matrices, i.e., matrices whose numerical ranges do not contain the origin.
2. Definition of phases. First, we review the concepts of numerical range, angular numerical range, and canonical angles of a matrix [13, Chapter 1]. The numerical range, also called the field of values, of a matrix $C \in \mathbb{C}^{n \times n}$ is defined as
\[
W(C) = \{ x^* C x : x \in \mathbb{C}^n \text{ with } ||x|| = 1 \},
\]
which, as a subset of $\mathbb{C}$, is compact and contains the spectrum of $C$. In addition, according to the Toeplitz-Hausdorff theorem, $W(C)$ is convex. The angular numerical range, also called the angular field of values, of $C$ is defined as
\[
W'(C) = \{ x^* C x : x \in \mathbb{C}^n, x \neq 0 \}.
\]
If $0 \not\in W(C)$, then $W(C)$ is completely contained in an open half plane by its convexity. In this case, $C$ is said to be a cramped matrix. We note that [2] defined a cramped matrix to mean $0 \not\in W(C)$, but only when $C$ is unitary. Here we say a matrix $C$ is cramped if $0 \not\in W(C)$, regardless of whether it is unitary or not. Early studies of cramped unitary matrices can also be found in [20, 15], among others.

Let $C$ be a cramped matrix. It is well known [11, 5, 7] that $C$ is congruent to a unitary matrix. In particular, the unitary matrix can be chosen to be diagonal, i.e., there exist a nonsingular matrix $T$ and a diagonal unitary matrix $D$ such that
\[
C = T^* D T. \tag{2.1}
\]
The factorization (2.1) is called sectoral decomposition [23]. It is clear that $W'(D) = W'(C)$ and thus $D$ is also cramped. Let $\delta(C)$ be the field angle of $C$, i.e., the angle subtended by the two supporting rays of $W'(C)$ at the origin. We define the phases of $C$, denoted by $\phi_1(C), \phi_2(C), \ldots, \phi_n(C)$, to be the phases of the eigenvalues (i.e., diagonal elements) of $D$, taking values in an interval $(\theta, \theta + \pi)$ for some $\theta \in [-\pi, \delta(C))$.

The phases defined in this way coincide with the canonical angles of $C$ introduced in [9].

A sectoral decomposition for a cramped matrix is not unique. Nevertheless, the diagonal unitary matrix $D$ is unique up to a permutation, as pointed out in [23]. As such, the phases of a cramped matrix are unique. We adopt the convention of labeling the phases in such a way that
\[
\bar{\phi}(C) = \phi_1(C) \geq \phi_2(C) \geq \cdots \geq \phi_n(C) = \phi(C).
\]
Define $\phi(C) = [\phi_1(C) \phi_2(C) \ldots \phi_n(C)]$. The uniqueness of the phases also follow directly from the maximin and minimax expressions [11, Lemma 8]:
\[
\phi_i(C) = \max_{M: \dim M = i} \min_{x \in M, ||x|| = 1} \angle x^* C x
\]
\[
= \min_{N: \dim N = n - i + 1} \max_{x \in N, ||x|| = 1} \angle x^* C x. \tag{2.2}
\]
In particular,
\[
\bar{\phi}(C) = \max_{x \in \mathbb{C}^n, ||x|| = 1} \angle x^* C x,
\]
\[
\phi(C) = \min_{x \in \mathbb{C}^n, ||x|| = 1} \angle x^* C x,
\]
from which we can observe that $\bar{\phi}(C)$ and $\phi(C)$ enjoy good geometric interpretations.
rays can be determined uniquely. Figure 1 illustrates an example of $W(C)$ and its supporting rays. The two angles from the positive real axis to the supporting rays are $\phi(C)$ and $\phi(C)$ respectively. The other phases of $C$ lie in between.

It is worth noting that the class of unitoids was introduced in [14] to consist of matrices that are congruent to unitary matrices. Clearly, cramped matrices constitute a special type of unitoid matrices. A noncramped unitoid matrix admits a factorization of the form (2.1). However, in this case, the eigenvalues of $D$ lie on an arc of the unit circle of length no less than $\pi$. In this paper, we do not define the phases of such matrices.

**Example 2.1.** We have

$$
\phi\left(\begin{bmatrix}
1 & 0 & 0 \\
0 & e^{i\pi/4} & 0 \\
0 & 0 & e^{-i\pi/4}
\end{bmatrix}\right) = \left[\frac{\pi}{4} \ 0 \ -\frac{\pi}{4}\right]
$$

and

$$
\phi\left(\begin{bmatrix}
-1 & 0 & 0 \\
0 & e^{i3\pi/4} & 0 \\
0 & 0 & e^{-i3\pi/4}
\end{bmatrix}\right) = \left[\frac{5\pi}{4} \ \pi \ 3\pi/4\right].
$$

The matrix

$$
\begin{bmatrix}
1 & 0 & 0 \\
0 & e^{i2\pi/3} & 0 \\
0 & 0 & e^{-i2\pi/3}
\end{bmatrix}
$$

is not cramped. We do not define the phases of this matrix, since in this endeavor there would clearly be an ambiguity in deciding whether the phases should be taken as $2\pi/3, 0, -2\pi/3$ or $4\pi/3, 2\pi/3, 0$.

It is easy to see that the phases have the following simple properties:

1. The phases of a cramped normal matrix are the phases of its eigenvalues.
2. The phases are invariant under congruence transformation, i.e., $\phi(C) = \phi(P^*CP)$ for every nonsingular $P$.
3. The phases of $C$ belong to $(-\pi/2, \pi/2)$ if and only if $C$ has a positive definite Hermitian part, i.e., $(C + C^*)/2 > 0$. Such matrices are said to be strongly positive real in the engineering literature [1, p. 51].

If we were to use $\psi(C)$ in (1.6) as the definition of phases, then property 1 would be satisfied. However, properties 2-3 would not hold as illustrated in the following example.
Example 2.2. We have

$$\psi\left( \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \right) = [\theta \ -\theta].$$

But

$$\psi\left( \begin{bmatrix} \cos \alpha & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \right) \begin{bmatrix} \cos \alpha & 0 \\ 0 & \sin \alpha \end{bmatrix} = \begin{bmatrix} \arccos \frac{\cos \theta}{\sqrt{1 - \sin^2 \theta \cos^2 2\alpha}} & -\arccos \frac{\cos \theta}{\sqrt{1 - \sin^2 \theta \cos^2 2\alpha}} \\ \arccos \frac{\cos \theta}{\sqrt{1 - \sin^2 \theta \cos^2 2\alpha}} & -\arccos \frac{\cos \theta}{\sqrt{1 - \sin^2 \theta \cos^2 2\alpha}} \end{bmatrix},$$

which differs from $[\theta \ -\theta]$.

We also have

$$\psi\left( \begin{bmatrix} \cos^2 \alpha & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \right) = [\theta \ -\theta],$$

which is invariant of $\alpha \in (0, \pi/2)$. However, the determinant of the Hermitian part is given by

$$\sin^2 2\alpha \cos^2 \theta - \cos^2 2\alpha \sin^2 \theta,$$

which is negative for some $\theta$ close to $\pi/2$ and $\alpha$ away from $\pi/4$, such as $\theta = \pi/4$ and $\alpha = \pi/12$. That means the matrix is not positive real for these values of $\theta$ and $\alpha$.

The following lemma shows that the phases of a cramped matrix provide a bound on the phases of its eigenvalues, just as the gains of a matrix provide a bound on the magnitudes of its eigenvalues. It has been proved implicitly in [11, Lemma 9] and also shown in [9, Theorem 1].

**Lemma 2.3.** Let $C$ be cramped with phases in $(\theta, \theta + \pi)$, where $\theta \in [-\pi, \delta(C)]$. Then the phases of the eigenvalues of $C$ can be chosen so that

$$\angle \lambda(C) \prec \phi(C).$$

Consider the matrix

$$C = \begin{bmatrix} \cos \alpha & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \alpha & 0 \\ 0 & \sin \alpha \end{bmatrix},$$

in Example 2.2, where $\alpha, \theta \in (0, \frac{\pi}{2})$. One can see from the discussion therein that $\psi(C) \prec \phi(C)$. In fact, this is generally true for any cramped matrix $C$ with phases in $(\theta, \theta + \pi)$. If $\psi(C)$ takes values in the same interval, which can be inferred from [11, Lemma 9].

3. Cramped matrix decompositions. As discussed in the previous section, a cramped matrix $C$ admits a sectoral decomposition as in (2.1), which is not unique. If the unitary matrix in (2.1) is not restricted to be diagonal, one gains more flexibility. In fact, letting $T = VP$ be the unique right polar decomposition and $U = V^*DV$ leads to a new decomposition

$$C = PUP,$$

where $P$ is positive definite and $U$ is unitary. This is called the symmetric polar decomposition (SPD). Meanwhile, letting $T = QR$ be the unique QR factorization
[12, Theorem 2.1.14] with \( Q \) being unitary and \( R \) being upper triangular with positive diagonal elements and letting \( W = Q^*DQ \) leads to another decomposition

\[ C = R^*WR. \]

This is called the generalized Cholesky factorization (GCF).

**Theorem 3.1.** The SPD and GCF of a cramped matrix \( C \) are unique.

**Proof.** We first show the uniqueness of the SPD. Suppose \( C \) admits two SPDs:

\[ C = P_1U_1P_1 = P_2U_2P_2. \]

Since the phases of a cramped matrix are unique, we can find unitary matrices \( W_1 \) and \( W_2 \) such that

\[ U_1 = W_1^*DW_1 \quad \text{and} \quad U_2 = W_2^*DW_2, \]

where \( D = \text{diag}\{e^{i\phi_1(C)}, e^{i\phi_2(C)}, \ldots, e^{i\phi_n(C)}\} \).

It follows that

\[ P_1W_1^*DW_1P_1 = P_2W_2^*DW_2P_2, \]

yielding

\[ D = T^*DT, \tag{3.1} \]

where \( T = W_2P_2P_1^{-1}W_1^* \). Suppose within the phases of \( C \) there are in total \( m \) distinct values \( \phi_1, \phi_2, \ldots, \phi_m \), arranged in a decreasing order with respective multiplicities given by \( n_1, n_2, \ldots, n_m \). Then, we can rewrite \( D \) as

\[ D = \text{diag}\{e^{i\phi_1}I_{n_1}, e^{i\phi_2}I_{n_2}, \ldots, e^{i\phi_m}I_{n_m}\} \]

and partition \( T \) with compatible dimensions into

\[ T = \begin{bmatrix}
T_{11} & T_{12} & \cdots & T_{1m} \\
T_{21} & T_{22} & \cdots & T_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
T_{m1} & T_{m2} & \cdots & T_{mm}
\end{bmatrix}, \]

where \( T_{ij} \in \mathbb{C}^{n_i \times n_j} \).

Simple computations from (3.1) yield

\[ e^{i\phi_l}I_{n_k} = \sum_{l=1}^{m} e^{i\phi_l}T_{lk}^*T_{lk}, \quad k = 1, 2, \ldots, m \]

and

\[ 0 = \sum_{l=1}^{m} e^{i\phi_l}T_{jk}^*T_{lk}, \quad j, k = 1, 2, \ldots, m, \quad j \neq k. \tag{3.2} \]

From the first equation in (3.2) and the fact that all phases lie in an interval \((\theta, \theta + \pi)\), it follows that

\[ T_{11}^*T_{11} = I_{n_1} \quad \text{and} \quad T_{11} = 0, l \neq 1. \]

Consequently, by the second equation in (3.2), we have

\[ T_{1k} = 0, k \neq 1. \]
Repeated applications of the arguments above then yield
\[ T^*_k T_{kk} = I_{n_k}, \quad k = 1, 2, \ldots, m, \quad \text{and} \quad T_{lk} = 0, \quad l \neq k, \]
which means \( T \) is a block diagonal unitary matrix. Therefore, \( W_2^* T W_1 \) is unitary and \( P_2 = W_2^* T W_1 P_1 \) gives a polar decomposition of \( P_2 \). By the uniqueness of polar decomposition, we have \( P_2 = P_1 \) and thus \( U_1 = U_2 \).

The uniqueness of the GCF can be shown using the same lines of arguments and the uniqueness of the QR decomposition in lieu of that of the polar decomposition. \( \square \)

For a cramped matrix \( C \) with the polar decomposition \( C = \tilde{P} \tilde{U} \) and the SPD \( C = P U P^* \), where \( \tilde{P}, P \) are positive definite, and \( \tilde{U}, U \) are unitary, we have introduced the majorization relationship between \( \angle \lambda(C) \), \( \psi(C) \) and \( \phi(C) \) in the previous section. There is an analogous relationship between \( |\lambda(C)|, \sigma(\tilde{P}) \) and \( \sigma(P) \). From [16, Theorem 9.E.1], we know
\[ |\lambda(C)| \prec \log \sigma(\tilde{P}). \]
Since \( \tilde{P} = P U P^* \), in view of inequality (1.5), we have
\[ \sigma(\tilde{P}) \prec \log \sigma^2(P), \]
where \( \sigma^2(P) \) is the elementwise square of \( \sigma(P) \).

For a noncramped unitoid matrix, one can also write its SPD and GCF as above. However, neither SPD nor GCF is unique in this case.

**Example 3.2.** Consider the unitoid matrix \( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \). For any \( a > 0 \),
\[ \begin{bmatrix} a & 0 \\ 0 & \frac{1}{a} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & \frac{1}{a} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]
is both an SPD and a GCF.

In the case where \( C \) is a real cramped matrix, sectoral decomposition, SPD, and GCF have their respective real counterparts. To see this, note that the numerical range of a real cramped matrix is symmetric about the real axis. Let
\[ H = \frac{1}{2}(C + C^*) \quad \text{and} \quad S = \frac{1}{2}(C - C^*). \]
Without loss of generality, suppose that \( H \) is positive definite (otherwise we could work with \(-C\)). Let \( H^{-\frac{1}{2}} \) be the inverse of the unique positive definite square root of \( H \). Then
\[ H^{-\frac{1}{2}} C H^{-\frac{1}{2}} = H^{-\frac{1}{2}} (H + S) H^{-\frac{1}{2}} = I + H^{-\frac{1}{2}} S H^{-\frac{1}{2}}, \]
which is a real normal matrix. In other words, \( C \) is congruent to a real normal matrix via a real congruence. Since a real normal matrix \( A \) can be decomposed [12, Theorem 2.5.8] into
\[ A = Q^* \text{diag}\{A_1, \ldots, A_k\} Q, \quad 1 \leq k \leq n, \]
where \( Q \) is a real orthogonal matrix and \( A_k \) is either a real scalar or a real 2-by-2 matrix of the form
\[ A_k = \begin{bmatrix} \alpha_k & \beta_k \\ -\beta_k & \alpha_k \end{bmatrix}, \]
it follows that a real cramped matrix $C$ can be factorized as
\[ C = T^* D T, \]
where $T$ is a nonsingular real matrix and $D$ is a real block orthogonal matrix with each block being either a scalar or a 2-by-2 matrix. We call this a real sectoral decomposition, which is nonunique in general. By performing the real polar decomposition and real QR decomposition of $T$, we arrive, respectively, at the real SPD and GCF, whose uniqueness follows from Theorem 3.1. To be specific, there exist unique real positive definite $P$, real upper triangular $R$ with positive diagonal elements, and real orthogonal $U$ and $W$ such that
\[ C = P U P = R^* W R. \]

4. Phases of compressions and Schur complements. As introduced in Section 1, there is an interlacing relation between the magnitudes of a matrix and those of its submatrices. In this section, we establish its counterpart between the phases of a cramped matrix and those of its submatrices. In this section, we establish its counterpart between the phases of a cramped matrix and those of its compressions and Schur complements.

Let $U$ be an $n \times (n-k)$ isometry, then $\tilde{C} = U^* C U$ is said to be a compression of $C$ [3, p. 59]. The phases of $\tilde{C}$ and those of $C$ have the following interlacing property.

**Theorem 4.1.** Let $C \in \mathbb{C}^{n \times n}$ be cramped with phases in $(\theta, \theta + \pi)$, $\theta \in [-\pi, \delta(C))$, and $\tilde{C} \in \mathbb{C}^{(n-k) \times (n-k)}$ be a compression of $C$. Then $\tilde{C}$ is also cramped and
\[ \phi_j(C) \geq \phi_j(\tilde{C}) \geq \phi_{j+k}(C), \text{ for } j = 1, \ldots, n-k. \]

**Proof.** Since $W(\tilde{C}) \subset W(C)$ and $W(C)$ is in an open half plane, it follows that $W(\tilde{C})$ is in the same half plane and thus $\tilde{C}$ is cramped.

Now, from the minimax characterization of phases in (2.2) and the fact $W(\tilde{C}) \subset W(C)$, it follows that
\[ \phi_j(\tilde{C}) \leq \phi_j(C), \]
which is the first inequality in (4.1). Analogously, the second inequality in (4.1) follows from the maximin characterization of phases in (2.2). This completes the proof. \[ \square \]

Note that in the special case where $k = 1$, i.e., $\tilde{C} \in \mathbb{C}^{(n-1) \times (n-1)}$, the inequality (4.1) becomes
\[ \phi_1(C) \geq \phi_1(\tilde{C}) \geq \phi_2(C) \geq \cdots \geq \phi_{n-1}(\tilde{C}) \geq \phi_n(C). \]

**Remark 4.2.** Since any full-rank $X \in \mathbb{C}^{n \times (n-k)}$ can be QR-decomposed as $X = QR$, where $Q \in \mathbb{C}^{n \times (n-k)}$ is an isometry and $R \in \mathbb{C}^{(n-k) \times (n-k)}$ is a nonsingular matrix, it follows from Theorem 4.1 that for any cramped $C \in \mathbb{C}^{n \times n}$ and $j = 1, \ldots, n-k,$
\[ \phi_j(C) \geq \phi_j(X^* C X) = \phi_j(Q^* C Q) \geq \phi_{j+k}(C). \]

By exploiting the interlacing property of the phases of compressions of a cramped matrix, we can deduce an interlacing property of the phases of the Schur complements of a cramped matrix. Let $C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \in \mathbb{C}^{n \times n}$ be cramped, where $C_{11} \in \mathbb{C}^{k \times k}$. In light of Theorem 4.1, $C_{11}$ is cramped and hence nonsingular.

**Theorem 4.3.** Let $C \in \mathbb{C}^{n \times n}$ be cramped with phases in $(\theta, \theta + \pi)$, $\theta \in [-\pi, \delta(C))$. Then $C_{11}$ is also cramped and
\[ \phi_j(C) \geq \phi_j(C_{11}) \geq \phi_{j+k}(C), \text{ for } j = 1, \ldots, n-k. \]
A perspective of characterizing matrix phases.

We elucidate below that the former is more informative from the matrices. While the compound numerical range is related to the numerical range of crucial to deriving the majorization result in the succeeding section on products of The section explores a useful notion known as the compound numerical range. It is of (4.1), we have

\[
C^{-1} = P^{-1}U^{-1}P^{-1}
\]

Hence, \(C^{-1}\) is cramped. Moreover, if \(\phi(C)\) take values in the interval \((\theta, \theta + \pi)\), then \(\phi(C^{-1}) = [-\phi_n(C), \ldots, -\phi_1(C)]\), taking values in the interval \((-\theta - \pi, -\theta)\).

We partition \(C^{-1}\) into \(C^{-1} = \begin{bmatrix} (C^{-1})_{11} & (C^{-1})_{12} \\ (C^{-1})_{21} & (C^{-1})_{22} \end{bmatrix}\). Then \(C^{-1}_{11} = (C^{-1})_{22}\). By Theorem 4.1, \((C^{-1})_{22}\) is cramped and thus, \(C^{-1}_{11}\) is cramped and so is \(C_{11}\). In view of (4.1), we have

\[
\phi_j(C^{-1}) \geq \phi_j((C^{-1})_{22}) = \phi_j(C_{11}^{-1}) \geq \phi_{j+k}(C^{-1}), \quad j = 1, \ldots, n - k.
\]

Since

\[
\phi_j(C^{-1}) = -\phi_{n-j+1}(C),
\]

\[
\phi_j(C_{11}^{-1}) = -\phi_{n-k-j+1}(C_{11}),
\]

it follows that

\[
\phi_j(C) \geq \phi_j(C_{11}) \geq \phi_{j+k}(C), \quad j = 1, \ldots, n - k.
\]

Theorem 4.1 can also be used to construct a simple proof to the following result, which is a special case of [6, Theorem 3.11].

**Lemma 4.4.** Let \(C \in \mathbb{C}^{n \times n}\) be cramped with phases in \((\theta, \theta + \pi)\), where \(\theta \in [-\pi, \delta(C)]\). Then

\[
\begin{align*}
\max_{X \in \mathbb{C}^{n \times k} \text{ is full rank}} & \sum_{i=1}^{k} \phi_i(X^*CX) = \sum_{i=1}^{k} \phi_i(C), \\
\min_{X \in \mathbb{C}^{n \times k} \text{ is full rank}} & \sum_{i=1}^{k} \phi_i(X^*CX) = \sum_{i=n-k+1}^{n} \phi_i(C).
\end{align*}
\]

**Proof.** We prove (4.2) below. The validity of (4.3) can be shown similarly.

Application of Theorem 4.1 and Remark 4.2 yields

\[
\sum_{i=1}^{k} \phi_i(X^*CX) \leq \sum_{i=1}^{k} \phi_i(C)
\]

for any full-rank matrix \(X \in \mathbb{C}^{n \times k}\). Let \(C = T^*DT\) be a sectoral decomposition. Then defining the full-rank \(X = T^{-1} [I_k \ 0]^*\) gives \(\sum_{i=1}^{k} \phi_i(X^*CX) = \sum_{i=1}^{k} \phi_i(C)\). \(\square\)

5. Compound numerical ranges and numerical ranges of compounds.

This section explores a useful notion known as the compound numerical range. It is crucial to deriving the majorization result in the succeeding section on products of matrices. While the compound numerical range is related to the numerical range of a compound matrix, we elucidate below that the former is more informative from the perspective of characterizing matrix phases.

Let \(A \in \mathbb{C}^{n \times m}\) and \(1 \leq k \leq \min \{n, m\}\). The \(k\)th compound of \(A\), denoted by \(A^{(k)}\), is the \(\binom{n}{k} \times \binom{m}{k}\) matrix whose elements are

\[
\det \left( A \begin{bmatrix} i_1, & \ldots, & i_k \\ j_1, & \ldots, & j_k \end{bmatrix} \right)
\]
arranged lexicographically, where

\[
A \begin{bmatrix} i_1, \ldots, i_k \\ j_1, \ldots, j_k \end{bmatrix}
\]

represents the submatrix of \( A \) consisting of rows \( i_1, \ldots, i_k \) and columns \( j_1, \ldots, j_k \). It is known that the eigenvalues of the \( k \)th compound of a square \( A \) are products (\( k \) at a time) of the eigenvalues of \( A \). In particular, \( A_{(1)} = A \) and when \( A \in \mathbb{C}^{n \times n} \), \( A_{(n)} = \det A \). Moreover, if \( A \) is Hermitian (resp. unitary), \( A_{(k)} \) is also Hermitian (resp. unitary). It also holds that \((AB)_{(k)} = A_{(k)}B_{(k)}\) by the Binet-Cauchy theorem.

For more details on compound matrices, we refer to [16, Chapter 19] and [8, Chapter 6].

Given \( A \in \mathbb{C}^{n \times n} \) and \( k = 1, \ldots, n \), we define the \( k \)th compound spectrum as

\[
\Lambda_{(k)}(A) = \left\{ \prod_{m=1}^{k} \lambda_m(A) : 1 \leq i_1 < \cdots < i_k \leq n \right\}.
\]

It follows that \( \Lambda_{(k)}(A) \subset W(A_{(k)}) \), the numerical range of the compound matrix \( A_{(k)} \). Note that even if \( A \) is cramped, its compound \( A_{(k)} \) is not necessarily cramped, which means we can have the origin lying in the interior of \( W(A_{(k)}) \) for a cramped \( A \). This shows that the numerical range of a compound matrix is not generally suitable for characterizing compound spectra from an angular point of view.

For \( k = 1, \ldots, n \), define the \( k \)th compound numerical range of \( A \in \mathbb{C}^{n \times n} \) as

\[
W_{(k)}(A) = \left\{ \prod_{m=1}^{k} \lambda_m(X^*AX) : X \in \mathbb{C}^{n \times k} \text{ is isometric} \right\},
\]

and the \( k \)th compound angular numerical range of \( A \) as

\[
W'_{(k)}(A) = \left\{ \prod_{m=1}^{k} \lambda_m(X^*AX) : X \in \mathbb{C}^{n \times k} \text{ is full rank} \right\}.
\]

When \( k = 1 \), the above definitions specialize to the familiar notions of numerical range and angular numerical range, respectively. It is straightforward to see that if \( 0 \notin W(A) \), then \( 0 \notin W_{(k)}(A) \). Furthermore, \( W_{(k)}(A) \) is always compact, but is not convex in general.

The following lemma generalizes [13, Theorem 1.7.6], which covers the special case of \( k = 1 \).

**Lemma 5.1.** Let \( A, B \in \mathbb{C}^{n \times n} \) for which \( B \) is cramped. Then

\[
\Lambda_{(k)}(AB^{-1}) \subset W_{(k)}(A)/W_{(k)}(B).
\]

**Proof.** First consider the case where \( AB^{-1} \) is diagonalizable. By the definition of eigenvectors and eigenvalues, there exists a full-rank \( X = [x_{i_1} \ x_{i_2} \ \ldots \ x_{i_k}] \in \mathbb{C}^{n \times k} \) such that

\[
X^*AB^{-1} = \text{diag}\{\lambda_{i_1}(AB^{-1}), \lambda_{i_2}(AB^{-1}), \ldots, \lambda_{i_k}(AB^{-1})\}X^*
\]

for \( 1 \leq i_1 < \cdots < i_k \leq n \). Factorize \( X \) as \( X = UP \) by applying the polar decomposition, where \( U \in \mathbb{C}^{n \times k} \) is isometric and \( P \in \mathbb{C}^{k \times k} \) is positive definite. Consequently,

\[
U^*AB^{-1} = P^{-1}\text{diag}\{\lambda_{i_1}(AB^{-1}), \lambda_{i_2}(AB^{-1}), \ldots, \lambda_{i_k}(AB^{-1})\}PU^*.
\]
Post-multiplying both sides by $BU$ and taking the determinants yields

$$
\prod_{i=1}^{k} \lambda_i(U^*AU) = \prod_{m=1}^{k} \lambda_m(AB^{-1}) \prod_{i=1}^{k} \lambda_i(U^*BU).
$$

The claim then follows by dividing both sides by $\prod_{i=1}^{k} \lambda_i(U^*BU)$.

When $AB^{-1}$ is not diagonalizable, choose a sequence $\{A_i\}$ with limit $A$ such that $A_iB^{-1}$ is diagonalizable for all $i = 1, 2, \ldots$. It follows from the arguments above that

$$
\Lambda_i(A_iB^{-1}) \subset W(k)(A_i)/W(k)(B) \text{ for all } i = 1, 2, \ldots.
$$

Given an isometric $X \in \mathbb{C}^{n \times k}$, we have $\prod_{m=1}^{k} \lambda_m(X^*A_iX) \to \prod_{m=1}^{k} \lambda_m(X^*AX)$ by the continuity of eigenvalues. Correspondingly, $W(k)(A_i) \to W(k)(A)$ in the Hausdorff metric. Similarly, we have $\Lambda_i(A_iB^{-1}) \to \Lambda_i(AB^{-1})$. Thus the desired claim follows.

The next result demonstrates that the compound numerical range provides a tighter characterization of the compound spectrum than the numerical range of a compound matrix.

**Corollary 5.2.** $\Lambda(k)(A) \subset W(k)(A) \subset W(A(k))$.

**Proof.** Letting $B = I$ in Lemma 5.1 and noting that $W(k)(I) = \{1\}$ yields the first inclusion. The second inclusion follows from the fact that for any isometric $X \in \mathbb{C}^{n \times k}$,

$$
\det(X^*AX) = (X^*AX)(k) = X^*_kA(k)X_k
$$

and $X(k)$ is a normalized column vector, since

$$
X^*X = I \implies X^*_kX(k) = I(k) = 1.
$$

The following lemma generalizes [13, Theorem 1.7.8], which covers the special case of $k = 1$.

**Lemma 5.3.** Let $A, B \in \mathbb{C}^{n \times n}$ for which $B$ is cramped. Then $\Lambda(k)(AB) \subset W'(k)(A)W'(k)(B)$.

**Proof.** By Lemma 5.1, we have

$$
\Lambda(k)(AB) \subset W(k)(A)/W(k)(B^{-1}.
$$

We show below that

$$
1/W(k)(B^{-1}) \subset W'(k)(B),
$$

from which the claimed result follows. To this end, let $c \in 1/W(k)(B^{-1})$. This means for some isometric $X \in \mathbb{C}^{n \times k}$, we have

$$
c = \prod_{m=1}^{k} \frac{1}{\lambda_m(X^*B^{-1}X)} = \prod_{m=1}^{k} \frac{1}{\lambda_m(X^*B^{-1}X)} = \prod_{m=1}^{k} \frac{1}{\lambda_m(Y^*BY)},
$$

where $Y = B^{-1}X$ is full rank. Noting that $c = |c|^2/\bar{c}$, we have

$$
c = |c|^2 \prod_{m=1}^{k} \lambda_m(Y^*B^*Y) = |c|^2 \prod_{m=1}^{k} \lambda_m(Y^*BY) \in W'(k)(B),
$$

as desired. \(\square\)
6. Phases of matrix product. The phases of an individual complex matrix and their properties have been the focus in Sections 2 to 4. Here, we study the product of two cramped matrices $A$ and $B$.

In view of the majorization result on the magnitudes of the product of two matrices as in (1.5), it would be desirable to have a phase counterpart in the form of

$$
\phi(AB) < \phi(A) + \phi(B)
$$

for cramped matrices $A, B$. We know that this is true when $A, B$ are unitary [17, 21]. Unfortunately, this fails to hold in general, as shown in the following example.

Example 6.1. Let $A, B$ be $n \times n$ positive definite matrices, then $\phi(A) = \phi(B) = 0$. However, $AB$ is in general not a positive definite matrix, and hence $\phi(AB) \neq 0$.

Notwithstanding, the following weaker but very useful result can be derived.

Theorem 6.2. Let $A, B \in \mathbb{C}^{n \times n}$ be cramped matrices with phases in $(\theta_1, \theta_1 + \pi)$ and $(\theta_2, \theta_2 + \pi)$, respectively, where $\theta_1 \in [-\pi, \delta(A))$ and $\theta_2 \in [-\pi, \delta(B))$. Let $\angle \lambda(AB)$ take values in $(\theta_1 + \theta_2, \theta_1 + \theta_2 + 2\pi)$. Then

(6.1) \[ \angle \lambda(AB) < \phi(A) + \phi(B). \]

Proof. Let $\hat{A} = e^{i(-\frac{\pi}{2} - \theta_1)}A$ and $\hat{B} = e^{i(-\frac{\pi}{2} - \theta_2)}B$. Then both $\hat{A}$ and $\hat{B}$ are cramped matrices with phases in $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $\angle \lambda(\hat{A}\hat{B})$ take values in $(-\pi, \pi)$. Label the eigenvalues of $\hat{A}\hat{B}$ so that $\angle \lambda_1(\hat{A}\hat{B}) \geq \angle \lambda_2(\hat{A}\hat{B}) \geq \cdots \geq \angle \lambda_n(\hat{A}\hat{B})$. Since $\phi_i(\hat{A}) = \phi_i(A) - \frac{\pi}{2} - \theta_1$, $\phi_i(\hat{B}) = \phi_i(B) - \frac{\pi}{2} - \theta_2$ and $\angle \lambda_i(\hat{A}\hat{B}) = \angle \lambda_i(AB) - \pi - \theta_1 - \theta_2$ for $i = 1, \ldots, n$, the inequality (6.1) holds if and only if

$$
\angle \lambda(\hat{A}\hat{B}) < \phi(\hat{A}) + \phi(\hat{B}).
$$

To show the above inequality, we note that $\prod_{i=1}^k \lambda_i(\hat{A}\hat{B}) \in \Lambda_{(k)}(\hat{A}\hat{B}), k = 1, \ldots, n$. In view of Lemma 5.3, it follows that $\prod_{i=1}^k \lambda_i(\hat{A}\hat{B}) \in W'(k)(\hat{A})W'(k)(\hat{B})$. This means there exist full-rank matrices $X, Y \in \mathbb{C}^{n \times k}$ such that

$$
\prod_{i=1}^k \lambda_i(\hat{A}\hat{B}) = \prod_{i=1}^k \lambda_i(X^*\hat{A}X) \prod_{i=1}^k \lambda_i(Y^*\hat{B}Y).
$$

Since phases of $\hat{A}$ and $\hat{B}$ are in $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $\angle \lambda(\hat{A}\hat{B})$ take values in $(-\pi, \pi)$, we have

$$
\sum_{i=1}^k \angle \lambda_i(\hat{A}\hat{B}) = \sum_{i=1}^k \angle \lambda_i(X^*\hat{A}X) + \sum_{i=1}^k \angle \lambda_i(Y^*\hat{B}Y)
$$

$$
= \sum_{i=1}^k \phi_i(X^*\hat{A}X) + \sum_{i=1}^k \phi_i(Y^*\hat{B}Y)
$$

$$
\leq \sum_{i=1}^k \phi_i(\hat{A}) + \sum_{i=1}^k \phi_i(\hat{B}),
$$

(6.2)

where the inequality (6.2) is due to Lemma 4.4. Note that when $k = n$, the inequality (6.2) becomes equality, as matrix phases are invariant under congruence transformation. This completes the proof. \[\square\]
For any $A, B \in \mathbb{C}^{n \times n}$, it is known that

$$\sigma_i(AB) \leq \sigma_i(A)\sigma_1(B) \quad \text{for } i = 1, \ldots, n.$$  

This is a special form of the more general Lidskii-Wielandt inequality, which can be found in [4]. While it would be desirable to have a counterpart to this inequality for cramped $A$ and $B$ in the form of

$$\angle \lambda_i(AB) \leq \phi_i(A) + \phi_1(B) \quad \text{for } i = 1, \ldots, n,$$

this is generally not true. For a simple counterexample, let $B = I$ and note that

$$\angle \lambda_i(A) \leq \phi_i(A) \quad \text{for } i = 1, \ldots, n$$  

does not hold in general.

7. Phases of matrix sum. Given $\alpha, \beta$ for which $0 \leq \beta - \alpha < 2\pi$, define

$$\mathcal{C}[\alpha, \beta] = \{ C \in \mathbb{C}^{n \times n} : C \text{ is cramped and } \phi_1(C) \leq \beta, \phi_n(C) \geq \alpha \}.$$  

The following theorem can be inferred from the discussions on [23, p. 2]. A proof is presented here for completeness.

**Theorem 7.1.** Let $A, B \in \mathcal{C}[\alpha, \beta]$ with $\beta - \alpha < \pi$. Then $A + B \in \mathcal{C}[\alpha, \beta]$.

**Proof.** Since $A, B \in \mathcal{C}[\alpha, \beta]$, $\beta - \alpha < \pi$, there exists an open half plane containing both $W(A)$ and $W(B)$. In view of the subadditivity of numerical range, i.e., $W(A + B) \subset W(A) + W(B)$ [13, Property 1.2.7], $W(A + B)$ is contained in the same open half plane. Thus, $A + B$ is cramped. Moreover,

$$\phi_1(A + B) = \max_{x \in \mathbb{C}^n, \|x\| = 1} \angle (x^*Ax + x^*Bx)$$

(7.1)

$$\leq \max_{x \in \mathbb{C}^n, \|x\| = 1} \max(\angle x^*Ax, \angle x^*Bx)$$

$$= \max(\phi_1(A), \phi_1(B))$$

$$\leq \beta,$$

$$\phi_n(A + B) = \min_{x \in \mathbb{C}^n, \|x\| = 1} \angle (x^*Ax + x^*Bx)$$

(7.2)

$$\geq \min_{x \in \mathbb{C}^n, \|x\| = 1} \min(\angle x^*Ax, \angle x^*Bx)$$

$$= \min(\phi_n(A), \phi_n(B))$$

$$\geq \alpha,$$

where the inequalities (7.1) and (7.2) follow from property (1.4) of complex scalars. The proof is complete. \qed

This theorem says that when $\beta - \alpha < \pi$, $\mathcal{C}[\alpha, \beta]$ is a convex cone. A special case is given by $\mathcal{C}[0, 0]$, i.e., the cone of positive definite matrices. In general, when $\pi < \beta - \alpha < 2\pi$, $\mathcal{C}[\alpha, \beta]$ is a cone, but not a convex cone.

Theorem 7.1 provides an upper and a lower bound on the phases of $A + B$. It would be interesting to explore for more results regarding $\phi_i(A + B)$, such as a majorization property similar to Theorem 6.2 or involving other geometric descriptions of $\phi_i(A + B)$.  

8. Rank robustness against perturbations. The rank of $I + AB$ plays an important role in the field of systems and control [24]. It is straightforward to see that if $\sigma(A)$ and $\sigma(B)$ are sufficiently small, then $I + AB$ has full rank. If the magnitudes of $A$ are known in advance, a problem well studied [24, 10] in the field of robust control is how large the magnitudes of $B$ need to be before $I + AB$ loses rank by 1, by 2, ..., or by $k$. It can be inferred from the Schmidt-Mirsky theorem [19, Theorem 4.18] that for a given matrix $A \in \mathbb{C}^{n \times n},$

$$\inf \{ \sigma(B) : \text{rank}(I + AB) \leq n - k \} = \sigma_k(A)^{-1}. $$

(8.1)

Define $B[\gamma] = \{ C \in \mathbb{C}^{n \times n} : \pi(C) \leq \gamma \}$ to be the set of matrices whose largest singular values are bounded above by $\gamma > 0$. Then (8.1) is equivalent to that $\pi(I + AB) > n - k$ for all $B \in B[\gamma]$ if and only if $\gamma < \sigma_k(A)^{-1}$.

On the other hand, intuitively we can see that if $\phi(A)$ and $\phi(B)$ are sufficiently small in magnitudes, then $I + AB$ has full rank. This motivates us to establish a phase counterpart to the analysis of rank robustness.

For $\alpha \in [0, k\pi)$, define

$$C_k[\alpha] = \left\{ C \in \mathbb{C}^{n \times n} : C \text{ is cramped and } \sum_{i=1}^{k} \phi_i(C) \leq \alpha, \sum_{i=n-k+1}^{n} \phi_i(C) \geq -\alpha \right\}. $$

This set is a cone but possibly non-convex unless $k = 1$ and $\alpha < \pi/2$. Clearly, $C_1[\alpha]$ is simply $C[\alpha]$. The next theorem is concerned with the robustness of the rank of $I + AB$ under phaseal perturbations on $B$.

**Theorem 8.1.** Let $A \in \mathbb{C}^{n \times n}$ be cramped with phases in $(-\pi, \pi)$ and $k = 1, \ldots, n$. Then $\text{rank}(I + AB) > n - k$ for all $B \in C_k[\alpha]$, $\alpha \in [0, k\pi)$, if and only if

$$\alpha < \min \left\{ k\pi - \sum_{i=1}^{k} \phi_i(A) + \frac{\pi}{k}, \sum_{i=n-k+1}^{n} \phi_i(A) \right\}. $$

**Proof.** First, label the eigenvalues of $AB$ so that $\angle \lambda_1(AB) \geq \angle \lambda_2(AB) \geq \cdots \geq \angle \lambda_n(AB)$. Since $A$ and $B$ are cramped, $I + AB$ loses rank by $k$ only if

$$\angle \lambda_1(AB) = \cdots = \angle \lambda_k(AB) = \pi \quad \text{or} \quad \angle \lambda_{n-k+1}(AB) = \cdots = \angle \lambda_n(AB) = -\pi. $$

Sufficiency thus follows from Theorem 6.2, whereby

$$\sum_{i=1}^{k} \angle \lambda_i(AB) \leq \sum_{i=1}^{k} \phi_i(A) + \phi_i(B) < k\pi$$

and

$$\sum_{i=n-k+1}^{n} \angle \lambda_i(AB) \geq \sum_{i=n-k+1}^{n} \phi_i(A) + \phi_i(B) > -k\pi. $$

For necessity, suppose to the contraposition that $\sum_{i=1}^{k} \phi_i(A) + \alpha \geq k\pi$. We construct below a matrix $B \in C_k[\alpha]$ such that the rank of $I + AB$ is $n - k$. Let $A = T^*DT$ be a sectoral decomposition. Define $B = T^{-1}ET^{-*}$, where $E = \text{diag}\{e_1, e_2, \ldots, e_n\}$ satisfies

$$|e_i| = 1 \quad \text{for } i = 1, \ldots, k,$$

$$\phi_i(A) + \angle e_i = \pi \quad \text{for } i = 1, \ldots, k,$$

and

$$e_i = 1 \quad \text{for } i = k + 1, \ldots, n.$$
Clearly, $B$ is cramped. Moreover, 
\[ \sum_{i=1}^{k} \phi_i(B) = \sum_{i=1}^{k} \angle e_i \leq \alpha, \quad \sum_{i=n-k+1}^{n} \phi_i(B) \geq -\alpha \]
and
\[ AB = T^* D T T^{-1} E T^{-*} = T^* D E T^{-*} \]
has $k$ eigenvalues at $-1$, whereby the rank of $I + AB$ is $n - k$.

The necessity of \( \sum_{i=n-k+1}^{n} \phi_i(A) - \alpha > -k\pi \) can be shown in a similar manner.\[\]

The specialization of Theorem 8.1 to the case where $k = 1$ is of particular importance in the study of robust control. Specifically, the theorem says that for a cramped matrix $A \in \mathbb{C}^{n \times n}$ with phases in $(-\pi, \pi)$ and $\alpha \in [0, \pi)$, there holds $\text{rank}(I + AB) = n$ for all $B \in \mathcal{C}_1[\alpha]$ if and only if $\alpha < \min\{\pi - \phi_1(A), \pi + \phi_n(A)\}$.

Combining this understanding and (8.1), we have the following result. The proof is straightforward and is thus omitted for brevity.

**Theorem 8.2.** Let $A \in \mathbb{C}^{n \times n}$ be cramped with phases in $(-\pi, \pi)$. Then $\text{rank}(I + AB) = n$ for all $B \in \mathbb{B}[\gamma] \cup \mathcal{C}_1[\alpha]$ with $\gamma > 0$, $\alpha \in [0, \pi)$, if and only if $\gamma < \sigma_i(A)^{-1}$ and $\alpha < \min\{\pi - \phi_1(A), \pi + \phi_n(A)\}$.

**9. Kronecker and Hadamard products.** The Kronecker product of $A \in \mathbb{C}^{n \times n}$ and $B \in \mathbb{C}^{m \times m}$, denoted by $A \otimes B$, is given by
\[
A \otimes B = \begin{bmatrix}
a_{11}B & \cdots & a_{1n}B \\
\vdots & \ddots & \vdots \\
a_{n1}B & \cdots & a_{nn}B
\end{bmatrix} \in \mathbb{C}^{nm \times nm}.
\]

It is known that the singular values of $A \otimes B$ are given by $\sigma_i(A)\sigma_j(B), 1 \leq i \leq n, 1 \leq j \leq m$ [13, Theorem 4.2.15]. Regarding the phases of $A \otimes B$, we have the following result.

**Theorem 9.1.** Let $A \in \mathbb{C}^{n \times n}$ and $B \in \mathbb{C}^{m \times m}$ be cramped matrices with phases in $(\theta_1, \theta_1 + \pi)$ and $(\theta_2, \theta_2 + \pi)$, where $\theta_1 \in [-\pi, \delta(A))$ and $\theta_2 \in [-\pi, \delta(B))$, respectively. If $\phi_1(A) + \phi_1(B) - \phi_n(A) - \phi_n(B) < \pi$, then $A \otimes B$ is cramped and its phases are given by $\phi_i(A) + \phi_j(B), 1 \leq i \leq n, 1 \leq j \leq m$.

**Proof.** Let $A = T^* DT$ and $B = R^* ER$ be sectoral decompositions of $A$ and $B$, respectively. Then,
\[
(9.1) \quad A \otimes B = (T^* DT) \otimes (R^* ER) = (T \otimes R)^*(D \otimes E)(T \otimes R),
\]
where $T \otimes R$ is nonsingular and $D \otimes E$ is diagonal unitary. Note that the eigenvalues of $D \otimes E$ are given by $\lambda_i(D)\lambda_j(E), 1 \leq i \leq n, 1 \leq j \leq m$, where
\[
\angle(\lambda_i(D)\lambda_j(E)) = \angle \lambda_i(D) + \angle \lambda_j(E) = \phi_i(A) + \phi_j(B).
\]

Since $\phi_1(A) + \phi_1(B) - \phi_n(A) - \phi_n(B) < \pi$, it follows that $A \otimes B$ is cramped and (9.1) is a sectoral decomposition of $A \otimes B$. Furthermore, the phases of $A \otimes B$ are given by $\angle(\lambda_i(D)\lambda_j(E)), \text{i.e., } \phi_i(A) + \phi_j(B), 1 \leq i \leq n, 1 \leq j \leq m$.\[\]

As for the Hadamard product $A \odot B$, a notably elegant result on its singular values is that [13, Theorem 5.5.4]
\[
\sigma(A \odot B) \precsim \sigma(A) \odot \sigma(B).
\]

One may expect a phase counterpart in the form of $\phi(A \odot B) \precsim \phi(A) + \phi(B)$. The following example demonstrates that this is not true in general.
Example 9.2. Let

\[
A = \begin{bmatrix}
3 - 2i & 1 - 2i & 1 & 1 + i \\
1 - 2i & 2 & -i & -i \\
1 & -i & 1 + 3i & 3i \\
1 + i & -i & 3i & 1 + 4i
\end{bmatrix}
\text{ and } B = I.
\]

Then

\[
\phi(A \odot B) = [1.3258 \ 1.249 \ 0 \ -0.588],
\]

\[
\phi(A) + \phi(B) = [1.5303 \ 0.7684 \ 0.3561 \ -0.7926].
\]

It can be seen that no majorization type relation holds between \(\phi(A \odot B)\) and \(\phi(A) + \phi(B)\). This example also invalidates \(\angle \lambda(\phi(A \odot B)) < \angle \lambda(\phi(A) + \phi(B))\).

Notwithstanding, we have the following weaker result.

**Theorem 9.3.** Let \(A, B \in \mathbb{C}^{n \times n}\) be cramped matrices with phases in \((\theta_1, \theta_1 + \pi)\) and \((\theta_2, \theta_2 + \pi)\), where \(\theta_1 \in [-\pi, \delta(A)]\) and \(\theta_2 \in [-\pi, \delta(B)]\), respectively. If \(\phi_1(A) + \phi_1(B) - \phi_n(A) - \phi_n(B) < \pi\), then \(A \odot B\) is cramped,

\[
\phi_1(A \odot B) \leq \phi_1(A) + \phi_1(B) \quad \text{and} \quad \phi_n(A \odot B) \geq \phi_n(A) + \phi_n(B).
\]

**Proof.** The claim follows from Theorems 9.1 and 4.1, since \(A \odot B\) can be expressed as a compression of \(A \odot B\). We offer an alternative proof below.

Let \(x \in \mathbb{C}^n, x \neq 0\). By [13, Lemma 5.1.5],

\[
x^*(A \odot B)x = \text{trace}(D_x^*AD_xB^T) = \sum_{i=1}^{n} \lambda_i(D_x^*AD_xB^T),
\]

where \(D_x\) is a diagonal matrix with diagonal entries given by \(x_i\). Since

\[
\lambda_i(D_x^*AD_xB^T) \in W'(D_x^*AD_x)W'(B^T) \subset W'(A)W'(B),
\]

we have \(x^*(A \odot B)x \neq 0\) and \(\angle x^*(A \odot B)x \in [\phi_n(A) + \phi_n(B), \phi_1(A) + \phi_1(B)]\), as required.

**10. Conclusions.** In this paper, we define the phases of a cramped matrix and study their properties. We introduce the symmetric polar decomposition and generalized Cholesky factorization of a cramped matrix and establish their uniqueness. The symmetric polar decomposition seems to have advantages over the usual polar decomposition, at least in defining the matrix phases. In the scalar case, the symmetric polar decomposition takes the form \(c = \sqrt{\sigma}e^{i\phi}\sqrt{\sigma}\). A number of useful properties of the matrix phases have been studied, including those of compressions, Schur complements, matrix products, and sums. The rank robustness of a matrix against magnitude/phase perturbations, motivated from applications in robust control, has also been examined.

The definition of phases can be generalized to matrices whose numerical ranges contain the origin on their boundaries. We call such matrices semicramped. A number of results in this paper have potential extensions to semicramped matrices, including those on sectoral and symmetric polar decompositions, matrix products, sums, and rank robustness.

The definition of phases can be further extended to some matrices which are not cramped, such as block diagonal matrices with cramped diagonal blocks, the
Kronecker products of cramped matrices, compound cramped matrices, etc. These matrices are constructed from cramped matrices and their phases can be defined by exploiting the phases of the original cramped matrices.

How to define phases for general noncramped unitoid matrices, which are congruent to unitary matrices, remains open. A critical issue in this problem involves determining how the phases take values and deriving their corresponding properties. We expect that the notion of Riemann surface would play an important role in studying the phases of unitoid matrices.

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