CHARACTERIZATION OF DISTRIBUTIONS WITH THE LENGTH-BIAS SCALING PROPERTY

MARCOS LÓPEZ-GARCÍA
Instituto de Matemáticas
Universidad Nacional Autónoma de México
México, D.F. C.P. 04510.
email: flopez@matem.unam.mx

Submitted November 26, 2008, accepted in final form April 15, 2009

AMS 2000 Subject classification: Primary: 60E05, Secondary: 44A60.
Keywords: Length-bias scaling property, Indeterminate moment problem, theta function.

Abstract
For $q \in (0,1)$ fixed, we characterize the density functions $f$ of absolutely continuous random variables $X > 0$ with finite expectation whose respective distribution functions satisfy the so-called (LBS) length-bias scaling property $X \overset{d}{=} q \bar{X}$, where $\bar{X}$ is a random variable having the distribution function $\bar{F}(x) = (E_X)^{-1} \int_0^x y f(y) dy$.

For an absolutely continuous random variable $X > 0$ with probability density function (pdf) $f$ and finite expectation $E_X$, we denote by $\bar{X}$ an absolutely continuous random variable having the probability density function $(E_X)^{-1} xf(x)$. In this case, $\bar{X}$ is called the size- or length-biased version of $X$ and $\cal L(\bar{X})$ is the corresponding length-biased distribution. It is well known that $\bar{X}$ is the stationary total lifetime in a renewal process with generic lifetime $X$ (see [2, Chapter 5]).

The length-biased distributions have been applied in various fields, such as biometry, ecology, environmental sciences, reliability and survival analysis. A review of these distributions and their applications are included in [5, Section 3], [6, 8, 12, 13].

In [9], Pakes and Khattree ask whether it is possible to randomly rescale the total lifetime to recover the lifetime law. More specifically, let $V \geq 0$ be a random variable independent of $X$ with a fixed law satisfying $P(V > 0) > 0$. For which laws $\cal L(\bar{X})$ does the following “equality in law”

$$X \overset{d}{=} V \bar{X},$$

hold? For instance, when $V$ has the uniform law on $[0,1]$ the last equality holds if and only if $\cal L(\bar{X})$ is an exponential law (see [9]).

In this note we consider the case where $V$ is a constant function: The law of $X$ has the so-called length-bias scaling property (abbreviated to LBS-property) if

$$X \overset{d}{=} q \bar{X},$$

(1)
with \( q \in (0, 1) \). Several authors, including Chihara \([3]\), Pakes and Khattree \([9]\), Pakes \([10, 11]\), Vardi et al. \([14]\), have studied the LBS-property. In \([1]\), Bertoin et al. analyze a random variable \( X \) that arises in the study of exponential functionals of Poisson processes; they show that 
\[
q X = \frac{X}{1 - q},
\]
with 
\[
E X = q^{-1}.
\]

An easy computation shows that (1) can be written as
\[
\int_0^x f(y) dy = \frac{1}{EX} \int_0^x y f \left( \frac{y}{q} \right) \frac{dy}{q}, \ x > 0,
\]
which is equivalent to
\[
(\text{EX}) q f (qx) = x f (x), \ a.e. \ x > 0.
\] (2)

By induction we have that
\[
(\text{EX})^n q^n f (q^n x) = q^{n^2/2 - n/2} x^n f (x), \ x > 0, \ n \in \mathbb{Z},
\]
and therefore
\[
\int_0^\infty x^n f (x) dx = (\text{EX})^n q^{n^2/2 - n/2}, \forall \ n \in \mathbb{Z}.
\] (3)

When \( X \) is an absolutely continuous random variable with probability density function \( f \), we sometimes write \( X \sim f \).

**Proposition 1.** If \( X \sim f \) and \( f \) satisfies (2), then the pdf \( g(x) = e^{ax} f(e^x) \) of the random variable \( Y = \log X \) satisfies the functional equation
\[
g(x - b) = Ce^{a(x-b)} g(x), \ x \in \mathbb{R},
\] (4)

with \( a = 1, b = -\ln q, C = (\text{EX})^{-1} \).

So, the main result of this note characterizes the probability density functions fulfilling the last functional equation. First, we recall that the theta function given by
\[
\theta(x, t) = (4\pi t)^{-1/2} \sum_{n \in \mathbb{Z}} e^{-(x+n)^2/(4t)} = \sum_{n \in \mathbb{Z}} e^{-4\pi^2 n^2 t + 2\pi ini} > 0,
\] (5)

for all \( (x, t) \in \mathbb{R}_+^2 \), satisfies the heat equation on \( \mathbb{R}_+^2 \) and
\[
\int_0^1 \theta(x, t) dx = 1, \text{ for all } t > 0. \text{ (see [15 Chapter V])}
\] (6)

**Theorem 1.** Let \( a, b, C \) be real numbers with \( ab > 0, C > 0 \). Then the pdf \( g \) satisfies the functional equation (2) if and only if there exists a 1-periodic function \( \varphi, \varphi \geq -1 \), such that the restriction of \( \varphi \) to \( (0, 1) \) belongs to \( L^1(0, 1) \),
\[
g(x) = \frac{1}{\sqrt{2\pi a^{-1}b}} \exp \left( -\frac{(ax - \mu)^2}{2ab} \right) \left\{ 1 + \varphi \left( \frac{ax - \mu}{ab} \right) \right\},
\] (7)

and
\[
\int_0^1 \theta \left( x, \frac{1}{2ab} \right) \varphi(x) dx = 0,
\] (8)

where \( -\mu = \ln C + ab/2 \).
Proof. For $b > 0$ the probability density function

$$
h(x) = \frac{1}{\sqrt{2\pi b}} e^{-(x-\mu)^2/(2b)},$$

where $-\mu = \ln C + b/2$, satisfies the functional equation \( \psi \) with $a = 1$. If the density function $g$ so does, then $g(x)/h(x) = g(x)/h(x)$, $x \in \mathbb{R}$; therefore there exists a $1$-periodic function $\psi$ such that $g(x) = h(x) \psi(b^{-1}x)$. By making the change of variable $y = (x-\mu)/b$, we obtain

$$1 = \int_{\mathbb{R}} g(x)dx = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi b^{-1}}} e^{-by^2/2} \psi(y + b^{-1}\mu) dy = \sum_{n \in \mathbb{Z}} \int_{J_n} \frac{1}{\sqrt{2\pi b^{-1}}} e^{-by^2/2} \psi(y + b^{-1}\mu) dy = \int_{0}^{1} \theta(y, 2^{-1}b^{-1}) \psi(y + b^{-1}\mu) dy.$$

By using (6), the result follows with $\varphi(x) = -1 + \psi(x + b^{-1}\mu)$. The general case follows by setting $g(x) = a^{-1}g\left(a^{-1}x\right)$, $\tilde{b} = ab$, $-\tilde{\mu} = \ln C + \tilde{b}/2$.

From Proposition 1 we obtain the characterization of the probability density functions with the LBS-property.

**Corollary 1.** Let $q \in (0, 1)$ be fixed, and let $X > 0$ be a random variable with pdf $f$ and $EX < \infty$. The law of $X$ has the LBS property if and only if there exists a $1$-periodic function $\varphi$, $\varphi \geq -1$, with the restriction of $\varphi$ to $(0, 1)$ in $L^1((0, 1))$, such that $\varphi$ satisfies (9) with $a = 1$, $b = -\ln q$, and

$$f(x) = \frac{1}{x\sqrt{-2\pi \ln q}} \exp\left(\frac{(\ln x - \mu)^2}{2\ln q}\right) \left\{1 + \varphi\left(\frac{\ln x - \mu}{-\ln q}\right)\right\},$$

where $\mu = \ln \left(q^{1/2}EX\right)$.

In [10] Theorem 3.1, Pakes uses a different approach to characterize the probability distribution functions $F = \mathcal{L}(X)$ satisfying (1) with $EX = 1$.

By (3), it follows that the probability density functions having the LBS-property are solutions of an indeterminate moment problem.

Let $N(\mu, -\ln q)$ be the normal density with mean $\mu$ and variance $-\ln q$. If $Y \sim N(\mu, -\ln q)$, we note that $\exp(Y)$ has the log-normal density, i.e.

$$\exp(Y) \sim \frac{1}{x\sqrt{-2\pi \ln q}} \exp\left(\frac{(\ln x - \mu)^2}{2\ln q}\right).$$

**Remark 1.** If $X$ is a positive absolutely continuous random variable with pdf $f$, then

$$cX^{-1} \sim cx^{-2}f\left(cx^{-1}\right), \text{ for all } c > 0.$$
So, for \( \nu \in \mathbb{R} \) the distributional identity \( X \overset{\mathcal{L}}{=} e^{2\nu}X^{-1} \) is equivalent to the functional equation
\[
f(x^{-1}) = e^{2\nu}x^2 f(e^{2\nu}x), \quad x > 0. \tag{10}
\]

If \( \varphi \) is a measurable function on \( \mathbb{R} \) and \( f \) is a pdf function given as follows
\[
f(x) = \frac{1}{x \sqrt{-2\pi \ln q}} \exp \left( \frac{(\ln x - \nu)^2}{2 \ln q} \right) \left\{ 1 + \varphi \left( \frac{\ln x - \nu}{\ln q} \right) \right\},
\]
\( x > 0 \), then \( f \) satisfies the latter functional equation if and only if \( \varphi \) is an even function.

As a consequence of Corollary 1 and the last remark with \( \nu = \ln \left( q^{1/2}EX \right) \), we have that a positive random variable \( X \) with probability density function \( f \) satisfies
\[qX \overset{\mathcal{L}}{=} X \overset{\mathcal{L}}{=} q \left( EX \right)^2 X^{-1}\]
if and only if \( f \) can be written as in Corollary 1 with \( \varphi \) being an even function.

Finally, we provide some families of functions satisfying (8).

**Examples**

From bounded functions, the following observation allows to construct functions with values in the non-negative axis.

**Remark 2.** For \( \alpha, \beta \in \mathbb{R} \) with \( \alpha < \beta \), it is easy to see that there exists an interval \( I \subset \mathbb{R} \) such that \( e^{[\alpha, \beta]} + 1 \subset \mathbb{R}^+ \) for all \( \epsilon \in I \). In fact, when \( \alpha < 0 < \beta \) we have that \( I = \left[ -\beta^{-1}, -\alpha^{-1} \right] \). For \( \alpha \geq 0, I = \left[ -\beta^{-1}, \infty \right), \) and for \( \beta \leq 0, I = ( -\infty, -\alpha^{-1} ) \).

**Example 1.** Let \( t > 0 \) be fixed and let \( (c_n)_{n \in \mathbb{Z}} \) be a sequence of complex numbers such that \( \varphi(x) = \sum_{n \in \mathbb{Z}} c_n e^{2\pi i nx} \in L^2(0, 1) \). Then \( \varphi \) satisfies
\[
\int_0^1 \theta(x, t) \varphi(x) \, dx = 0, \tag{11}
\]
if and only if \( \varphi(x) \) is orthogonal to \( \theta(x, t) \) in \( L^2(0, 1) \). By (5) this is equivalent to the orthogonality between \( (c_n)_{n \in \mathbb{Z}} \) and \( \left(e^{-4\pi^2 n^2 t} \right)_{n \in \mathbb{Z}}, \) i.e.
\[
\sum_{n \in \mathbb{Z}} c_n e^{-4\pi^2 n^2 t} = 0.
\]

In [11, page 1278] Pakes says that the continuous solutions of (2) probably are exceptions. But for any trigonometric polynomial \( p(x) = \sum_{|n| \leq N} c_n e^{2\pi i nx} \) whose coefficients \( c_n \in \mathbb{C} \) satisfy the last equality with \( t = b^{-1}/2 \), there is an interval \( I \) such that
\[\epsilon x \geq -1, \text{ for all } x \in \left[ \min \text{Re } p, \max \text{Re } p \right], \epsilon \in I, \]
therefore \( \varphi = \epsilon \text{Re } p \geq -1 \) on \( \mathbb{R} \) and the corresponding density function given by Corollary 1 is an infinitely differentiable function on \( \mathbb{R}^+ \).
Example 2. Let $c_m = -c_{-m} = i/2$, and $c_n = 0$ for all $n \in \mathbb{Z} \setminus \{ -m , m \}$, $m \neq 0$. So, the corresponding trigonometric polynomial $\varphi(x) = -\sin(2\pi mx)$ is a function satisfying (11) for all $t > 0$.

Example 3. Let $c_{-1} = c_1 = -1/2$, $c_0 = e^{-4\pi \xi}$, and $c_n = 0$ for all $|n| \geq 2$. Thus, the corresponding trigonometric polynomial $\varphi(x) = e^{-4\pi \xi t} - \cos(2\pi x) \geq -1$ is an even function satisfying (11).

Example 4. By (6) we have that

$$\varphi_c(x) = -1 + \left( \int_0^1 \frac{\theta(x,t)}{\theta(x+c,t)} \frac{dx}{\theta(x+c,t)} \right)^{-1} \frac{1}{\theta(x+c,t)}$$

is a 1-periodic, continuous function satisfying (11) for all $c \in [0,1)$. Since $\theta(x,t)$ is an even function for all $t > 0$, the function $\varphi_c$ is even if and only if $c = 0, 1/2$. In (4) equality (2.15) it is shown that

$$\theta(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)} \left( q_t; q_t \right)_\infty \left( -q_t^{1/2-x}; q_t \right)_\infty \left( -q_t^{1/2+x}; q_t \right)_\infty,$$

where $q_t = e^{-t^{-1/2}}$, $(p; q)_\infty = \prod_{k=0}^{\infty} (1 - pq^k)$. For $c \in (0,1)$ we have that (see (4) equality (2.17))

$$\int_0^1 \frac{\theta(x,t)}{\theta(x+c,t)} \frac{dx}{\theta(x+c,t)} = 2t \frac{\pi q_t^{t(c-1)/2}}{\sin(\pi c)} \frac{\left( q_t^{1+c_x}; q_t \right)_\infty}{\left( q_t; q_t \right)_\infty^2}.$$ 

To get more examples of functions fulfilling (8) see [4]. The results in [7] can be used to construct positive random variables having not the LBS-property but with moment sequence (3).

Acknowledgement I thank the anonymous referees for useful suggestions to the original manuscript.

References

[1] Bertoin J., Biane P., Yor M., Poissonian exponential functionals, $q$-series, $q$-integrals, and the moment problem for log-normal distributions, Seminar on Stochastic Analysis, Random Fields and Applications IV, 45–56, Progr. Probab., 58, Birkhauser, Basel, 2004. [MR2096279]

[2] Cox D. R., Renewal theory, Methuen & Co. Ltd., London; John Wiley & Sons, Inc., New York 1962 ix+142 pp. [MR0153061]

[3] Chihara T. S., A characterization and a class of distributions functions for the Stieltjes-Wigert polynomials, Canadian Math. Bull. 13 (1970), 529–532. [MR0280761]

[4] Gómez R., López-García M., A family of heat functions as solutions of indeterminate moment problems, Int. J. Math. Math. Sci., vol. 2007, Article ID 41526, 11 pages, doi:10.1155/2007/41526.

[5] Gupta R. C., Kirmani S. N. U. A., The role of weighted distributions in stochastic modeling, Comm. Statist. Theory Methods 19 (1990), no. 9, 3147–3162. [MR1089242]

[6] Leyva V., Sanhueza A., Angulo J. M., A length-biased version of the Birnbaum-Saunders distribution with application in water quality, Stoch. Environ Res Risk Assess (2009) 23:299–307, doi:10.1007/s00477-008-0215-9.
[7] López-García M., Characterization of solutions to the log-normal moment problem, to appear in Theory of Probability and its Applications.

[8] Oncel S. Y., Ahsanullah M., Aliev F. A., Aygun F., Switching record and order statistics via random contractions, Statist. Probab. Lett. 73 (2005), no. 3, 207–217. MR2179280

[9] Pakes A. G., Khattree, R., Length-biasing, characterization of laws, and the moment problem, Austral. J. Statist. 34 (1992), 307–322. MR1193781

[10] Pakes A. G., Length biasing and laws equivalent to the log-normal, J. Math. Anal. Appl. 197 (1996), 825–854. MR1373083

[11] Pakes A. G., Structure of Stieltjes classes of moment-equivalent probability laws, J. Math. Anal. Appl. 326 (2007), 1268–1290. MR2280980

[12] Pakes A. G., Navarro J., Distributional characterizations through scaling relations, Aust. N. Z. J. Stat. 49 (2007), no. 2, 115–135. MR2392366

[13] Rao C. R., Shanbhag D. N., Choquet-Deny type functional equations with applications to stochastic models. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons, Ltd., Chichester (1994). xii+290 pp. MR1329995

[14] Vardi Y., Shepp L. A., Logan B. F., Distribution functions invariant under residual-lifetime and length-biased sampling, Wahrscheinlichkeitstheorie verw. Gebiete 56 (1981), 415–426. MR0621657

[15] Widder D. V., The Heat Equation, Pure and Applied Mathematics, Vol. 67. Academic Press, New York-London (1975). xiv+267 pp. MR0466967