Superintegrable non-autonomous Hamiltonian systems

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The Mishenko–Fomenko theorem on action-angle coordinates for superintegrable autonomous Hamiltonian systems is extended to the non-autonomous ones.

I. INTRODUCTION

The Liouville – Arnold theorem for completely integrable systems\(^1\)\(^–\)\(^3\) and the Mishchenko – Fomenko theorem for the superintegrable ones\(^4\)\(^–\)\(^6\) state the existence of action-angle coordinates around a compact invariant submanifold. These theorems were generalized to the case of non-compact invariant submanifolds\(^7\)\(^–\)\(^10\). This generalization enable us to analyze completely integrable non-autonomous Hamiltonian systems whose invariant submanifolds are necessarily non-compact\(^11\),\(^12\). Here we aim to extend this analysis to superintegrable non-autonomous Hamiltonian systems.

We consider a non-autonomous mechanical system whose configuration space is a fibre bundle \(Q \to \mathbb{R}\) over the time axis \(\mathbb{R}\) endowed with the Cartesian coordinate \(t\) possessing transition functions \(t' = t + \text{const}\). Its phase space is the vertical cotangent bundle \(V^*Q \to Q\) of \(Q \to \mathbb{R}\) endowed with the Poisson structure \(\{\cdot, \cdot\}_V\)\(^13\),\(^14\). A Hamiltonian of a non-autonomous mechanical system is a section \(H\)\(^8\) of the one-dimensional fibre bundle

\[
\zeta : T^*Q \to V^*Q,
\]

where \(T^*Q\) is the cotangent bundle of \(Q\) endowed with the canonical symplectic form \((7)\).

**Definition 1:** A non-autonomous Hamiltonian system of \(m = \dim Q - 1\) degrees of freedom is called superintegrable if admits \(m \leq n < 2m\) integrals of motion \(\Phi_1, \ldots, \Phi_n\) obeying the following conditions.

(i) All the functions \(\Phi_\alpha\) are independent, i.e. the \(n\)-form \(d\Phi_1 \wedge \cdots \wedge d\Phi_n\) nowhere vanishes on \(V^* Q\). It follows that the map

\[
\Phi : V^* Q \to N = (\Phi_1(V^* Q), \ldots, \Phi_n(V^* Q)) \subset \mathbb{R}^n
\]

is a fibred manifold over a connected open subset \(N \subset \mathbb{R}^n\).

(ii) There exist smooth real functions \(s_{ij}\) on \(N\) such that

\[
\{\Phi_\alpha, \Phi_\beta\}_V = s_{\alpha\beta} \circ \Phi, \quad \alpha, \beta = 1, \ldots, n.
\]
(iii) The matrix function with the entries $s_{\alpha\beta}$ (3) is of constant corank $k = 2m - n$ at all points of $N$.

To describe this non-autonomous Hamiltonian system, we use the fact that there exists an equivalent autonomous Hamiltonian system on the cotangent bundle $T^*Q$ (Theorem 3) which is superintegrable (Theorem 7). Our goal is the following.

**Theorem 2:** Let Hamiltonian vector fields of the functions $\Phi_\alpha$ be complete, and let fibres of the fibred manifold $\Phi$ (2) be connected and mutually diffeomorphic. Then there exists an open neighborhood $U_M$ of a fibre $M$ of $\Phi$ (2) which is a trivial principal bundle with the structure group

$$\mathbb{R}^{1+k-r} \times T^r$$

whose bundle coordinates are the generalized action-angle coordinates

$$(p_A, q^A, I_\lambda, t, y^\lambda), \quad A = 1, \ldots, n - m, \quad \lambda = 1, \ldots, k,$$

such that:

(i) $(t, y^\lambda)$ are coordinates on the toroidal cylinder (4),

(ii) the Poisson bracket $\{,\}_V$ on $U_M$ reads

$$\{f, g\}_V = \partial^A f \partial_A g - \partial^A g \partial_A f + \partial^\lambda f \partial_\lambda g - \partial^\lambda g \partial_\lambda f,$$

(iii) the Hamiltonian $H$ depends only on the action coordinates $I_\lambda$,

(iv) the integrals of motion $\Phi_1, \ldots, \Phi_n$ are independent of coordinates $(t, y^\lambda)$.

If $n = m$, we are in the case of a completely integrable non-autonomous Hamiltonian system (Theorem 6).

**II. NON-AUTONOMOUS HAMILTONIAN MECHANICS**

The configuration space $Q \to \mathbb{R}$ of a non-autonomous Hamiltonian system is equipped with bundle coordinates $(t, q^i), \ i = 1, \ldots, m$. Its phase space $V^*Q$ is provided with holonomic coordinates $(t, q^i, p_i = \dot{q}_i)$ with respect to fibre bases $\{dq^i\}$ for $V^*Q$. The cotangent bundle $T^*Q$ of $Q$ plays a role of the homogeneous phase space endowed with holonomic coordinates $(t, q^i, p_0, p_i)$ possessing transition functions

$$p'_i = \frac{\partial q^j}{\partial q^i} p_j, \quad p'_0 = p_0 + \frac{\partial q^j}{\partial t^j} p_j.$$  \hspace{1cm} (6)

The cotangent bundle $T^*Q$ admits the canonical symplectic form

$$\Omega = dp_0 \wedge dt + dp_i \wedge dq^i,$$  \hspace{1cm} (7)
and the corresponding Poisson bracket

\[ \{ f, g \} = \partial^0 f \partial t g - \partial^0 g \partial t f + \partial^i f \partial_i g - \partial^i g \partial_i f, \quad f, g \in C^\infty(T^*Q). \]

A glance at the transformation law (6) shows that the one-dimensional fibre bundle \( \zeta \) (1) is a trivial affine bundle. Given its global section

\[ H : V^*Q \rightarrow T^*Q, \quad \rho_0 \circ H = -\mathcal{H}(t, q^i, p_j), \quad (8) \]

the cotangent bundle \( T^*Q \) is equipped with the fibre coordinate

\[ I_0 = p_0 + \mathcal{H}, \quad I_0 \circ H = 0, \quad (9) \]

possessing the identity transition functions. With respect to the coordinates

\[ (t, q^i, I_0, p_i), \quad i = 1, \ldots, m, \quad (10) \]

the fibration (1) reads

\[ \zeta : \mathbb{R} \times V^*Q \ni (t, q^i, I_0, p_i) \rightarrow (t, q^i, p_i) \in V^*Q. \quad (11) \]

The fibre bundle (1) provides the vertical cotangent bundle \( V^*Q \) with the canonical Poisson structure \( \{ , \}_V \) such that

\[ \zeta^* \{ f, g \}_V = \{ \zeta^* f, \zeta^* g \}, \quad \{ f, g \}_V = \partial^i f \partial_i g - \partial^i g \partial_i f, \quad f, g \in C^\infty(V^*Q). \quad (12, 13) \]

The Hamiltonian vector fields of functions on \( V^*Q \) with respect to the Poisson bracket (13) are vertical vector fields

\[ \vartheta f = \partial^i f \partial_i - \partial_i f \partial^i, \quad f \in C^\infty(V^*Q), \quad (14) \]

\[ [\vartheta f, \vartheta f'] = \vartheta \{ f, f' \}_V, \quad (15) \]

on \( V^*Q \rightarrow \mathbb{R} \). Accordingly, the corresponding symplectic foliation on the phase space \( V^*Q \) coincides with the fibration \( V^*Q \rightarrow \mathbb{R} \).

A Hamiltonian of non-autonomous mechanics on the phase space \( V^*Q \) is defined as a global section (8) of the affine bundle \( \zeta \) (1). Then there exists a unique vector field \( \gamma_H \) on \( V^*Q \) such that

\[ \gamma_H \} dt = 1, \quad \gamma_H \} H^* \Omega = 0, \]

\[ \gamma_H = \partial_t + \partial^i \mathcal{H} \partial_i - \partial_i \mathcal{H} \partial^i. \quad (16) \]
This vector field, called the Hamilton vector field, defines the first order Hamilton equations

\[ \dot{q}_i = \partial^i \mathcal{H}, \quad \dot{p}_i = -\partial_i \mathcal{H} \quad (17) \]

on a fibre bundle \( V^*Q \rightarrow \mathbb{R} \) with respect to the adapted coordinates \((t, q^i, p_i, q^i_t, p^i_t)\) on the first order jet manifold \( J^1V^*Q \) of \( V^*Q \rightarrow \mathbb{R} \). Due to the canonical imbedding \( J^1V^*Q \rightarrow TV^*Q \), the Hamilton equations (17) are equivalent to the first order differential equations

\[ \dot{i} = 1, \quad \dot{q}^i = \partial^i \mathcal{H}, \quad \dot{p}_i = -\partial_i \mathcal{H} \quad (18) \]

on a manifold \( V^*Q \).

In order to describe evolution of a mechanical system, the Hamilton vector field \( \gamma_H (16) \) is assumed to be complete. It defines a trivialization

\[ V^*Q \cong \mathbb{R} \times P \]

which is a canonical automorphism of the Poisson manifold \( V^*Q \) such that the corresponding coordinates \((t, \overline{q}^i, \overline{p}_i)\) are the initial date coordinates.\(^{14}\) With respect to these coordinates, the Hamiltonian (8) reads \( \mathcal{H} = 0 \), and the Hamilton equations (17) take the form

\[ \overline{q}^i_t = 0, \quad \overline{p}_{ti} = 0. \]

We agree to call \((V^*Q, H)\) the non-autonomous Hamiltonian system of \( m \) degrees of freedom.

**Theorem 3:** A non-autonomous Hamiltonian system \((V^*Q, H)\) is equivalent to an autonomous Hamiltonian system \((T^*Q, \mathcal{H}^*)\) of \( m + 1 \) degrees of freedom on a symplectic manifold \((T^*Q, \Omega)\) whose Hamiltonian is the function\(^{15,16}\)

\[ \mathcal{H}^* = I_0 = p_0 + \mathcal{H}. \quad (19) \]

The Hamiltonian vector field \( u_{\mathcal{H}^*} \) of \( \mathcal{H}^* \) (19) on \( T^*Q \) is

\[ u_{\mathcal{H}^*} = \partial_t - \partial_i \mathcal{H} \partial^i + \partial^i \mathcal{H} \partial_i - \partial_i \mathcal{H} \partial^i. \quad (20) \]

Written relative to the coordinates (10), this vector field reads

\[ u_{\mathcal{H}^*} = \partial_t + \partial^i \mathcal{H} \partial_i - \partial_i \mathcal{H} \partial^i. \quad (21) \]

It is projected onto the Hamilton vector field \( \gamma_H (16) \) on \( V^*Q \) such that

\[ \zeta^* (L_{\gamma_H} f) = \{ \mathcal{H}^*, \zeta^* f \}, \quad f \in C^\infty(V^*Q). \quad (22) \]
The corresponding autonomous Hamilton equations on $T^*Q$ take the form

$$
\begin{align*}
\dot{t} &= 1, \\
\dot{p}_0 &= -\partial_t \mathcal{H}, \\
\dot{q}^i &= \partial^i \mathcal{H}, \\
\dot{p}_i &= -\partial_i \mathcal{H}.
\end{align*}
$$

They are equivalent to the Hamilton equations (18).

Obviously, the vector field $u_{\mathcal{H}}$ (21) is complete if the Hamilton vector field $\gamma_{\mathcal{H}}$ (16) is complete.

### III. COMPLETELY INTEGRABLE NON-AUTONOMOUS HAMILTONIAN SYSTEMS

An integral of motion of a non-autonomous Hamiltonian system $(V^*Q, H)$ is defined as a smooth real function $F$ on $V^*Q$ whose Lie derivative

$$
L_{\gamma_{\mathcal{H}}} F = \gamma_{\mathcal{H}} \partial_t F + \{\mathcal{H}, F\}_V
$$

along the Hamilton vector field $\gamma_{\mathcal{H}}$ (16) vanishes. Given the Hamiltonian vector field $\vartheta_F$ (14) of $F$ with respect to the Poisson bracket (13), it is easily justified that

$$
[\gamma_{\mathcal{H}}, \vartheta_F] = \vartheta_{L_{\gamma_{\mathcal{H}}} F}.
$$

**Definition 4:** A non-autonomous Hamiltonian system $(V^*Q, H)$ of $m$ degrees of freedom is said to be completely integrable if it admits $m$ integrals of motion $F_1, \ldots, F_m$ which are in involution with respect to the Poisson bracket $\{,\}_V$ (13) and whose differentials $dF_\alpha$ are linearly independent, i.e.

$$
dF_1 \wedge \cdots \wedge dF_m \neq 0.
$$

By virtue of the relations (15) and (24), the vector fields

$$
(\gamma_{\mathcal{H}}, \vartheta_{F_1}, \ldots, \vartheta_{F_m}),
$$

mutually commute and, therefore, they span an $(m+1)$-dimensional involutive distribution $\mathcal{V}$ on $V^*Q$. Let $G$ be the group of local diffeomorphisms of $V^*Q$ generated by the flows of vector fields (25). Maximal integral manifolds of $\mathcal{V}$ are the orbits of $G$ and invariant submanifolds of vector fields (25). They yield a foliation $\mathcal{F}$ of $V^*Q$.

Let $(V^*Q, H)$ be a non-autonomous Hamiltonian system and $(T^*Q, \mathcal{H}^*)$ an equivalent autonomous Hamiltonian system on $T^*Q$. An immediate consequence of the relations (12) and (22) is the following. \(^9\)
Theorem 5: Given a completely integrable non-autonomous Hamiltonian system
\[ (\gamma_H, F_1, \ldots, F_m) \] (26)
of \( m \) degrees of freedom on \( V^*Q \), the autonomous Hamiltonian system
\[ (\mathcal{H}^*, \zeta^* F_1, \ldots, \zeta^* F_m) \] (27)
of \( m + 1 \) degrees of freedom on \( T^*Q \) is completely integrable.

The Hamiltonian vector fields
\[ \left( u_{\mathcal{H}^*}, u_{\zeta^* F_1}, \ldots, u_{\zeta^* F_m} \right), \] (28)
of the autonomous integrals of motion (27) span an \((m + 1)\)-dimensional involutive distribution \( \mathcal{V}_T \) on \( T^*Q \) such that
\[ T\zeta(\mathcal{V}_T) = \mathcal{V}, \quad TH(\mathcal{V}) = \mathcal{V}_{T|H(V^*Q)=I_0=0}, \] (29)
where
\[ TH : TV^*Q \ni (t, q^i, p_i, \dot{t}, \dot{q}^i, \dot{p}_i) \rightarrow (t, q^i, p_i, I_0 = 0, \dot{t}, \dot{q}^i, \dot{p}_i, \dot{I}_0 = 0) \in TT^*Q. \]

It follows that, if \( M \) is an invariant submanifold of the completely integrable non-autonomous Hamiltonian system (26), then \( H(M) \) is an invariant submanifold of the completely integrable autonomous Hamiltonian system (27).

In order to introduce generalized action-angle coordinates around an invariant submanifold \( M \) of the completely integrable non-autonomous Hamiltonian system (26), let us suppose that the vector fields (25) on \( M \) are complete. It follows that \( M \) is a locally affine manifold diffeomorphic to a toroidal cylinder
\[ (\mathbb{R}^{1+m-r} \times T^r). \] (30)
Moreover, let assume that there exists an open neighbourhood \( U \) of \( M \) such that the foliation \( \mathcal{F} \) of \( U \) is a fibred manifold \( \phi : U \rightarrow N \) over a domain \( N \subset \mathbb{R}^m \) whose fibres are mutually diffeomorphic.\(^{11}\)

Because the morphism \( TH \) (29) is a bundle isomorphism, the Hamiltonian vector fields (28) on the invariant submanifold \( H(M) \) of the completely integrable autonomous Hamiltonian system are complete. Since the affine bundle \( \zeta \) (11) is trivial, the open neighbourhood \( \zeta^{-1}(U) \) of the invariant submanifold \( H(M) \) is a fibred manifold
\[ \bar{\phi} : \zeta^{-1}(U) = \mathbb{R} \times U \xrightarrow{(\text{Id}_\mathbb{R}, \phi)} \mathbb{R} \times N = N'. \]
over a domain \( N' \subset \mathbb{R}^{m+1} \) whose fibres are diffeomorphic to the toroidal cylinder (30). In accordance with the Liouville – Arnold theorem extended to the case of non-compact invariant submanifolds, the open neighbourhood \( \zeta^{-1}(U) \) of \( H(M) \) is a trivial principal bundle
\[
\zeta^{-1}(U) = N' \times (\mathbb{R}^{1+m-r} \times T^r) \rightarrow N'
\]
with the structure group (30) whose bundle coordinates are the generalized action-angle coordinates
\[
(I_0, I_1, \ldots, I_m, t, z^1, \ldots, z^m)
\]
such that:
(i) \((t, z^a)\) are coordinates on the toroidal cylinder (30),
(ii) the symplectic form (7) on \( \zeta^{-1}(U) \) reads
\[
\Omega = dI_0 \wedge dt + dI_a \wedge dz^a,
\]
(iii) \( \mathcal{H}^* = I_0 \),
(iv) the integrals of motion \( \zeta^* F_1, \ldots, \zeta^* F_m \) depend only on the action coordinates \( I_1, \ldots, I_m \).
Provided with the coordinates (32), \( \zeta^{-1}(U) = U \times \mathbb{R} \) is a trivial bundle possessing the fibre coordinate \( I_0 \) (9). Consequently, the open neighbourhood \( U \) of an invariant submanifold \( M \) of the completely integrable non-autonomous Hamiltonian system (25) is diffeomorphic to the Poisson annulus
\[
U = N \times (\mathbb{R}^{1+m-r} \times T^r)
\]
endowed with the generalized action-angle coordinates
\[
(I_1, \ldots, I_m, t, z^1, \ldots, z^m)
\]
such that:
(i) the Poisson structure (13) on \( U \) takes the form
\[
\{f, g\}_V = \partial^a f \partial_a g - \partial^a g \partial_a f,
\]
(ii) the Hamiltonian (8) reads \( \mathcal{H} = 0 \),
(iii) the integrals of motion \( F_1, \ldots, F_m \) depend only on the action coordinates \( I_1, \ldots, I_m \).
The Hamilton equations (17) relative to the generalized action-angle coordinates (34) take the form
\[
z^a_t = 0, \quad I_{ta} = 0.
\]
It follows that the generalized action-angle coordinates (34) are the initial date coordinates.
Note that the generalized action-angle coordinates \((34)\) by no means are unique. Given a smooth function \(H'\) on \(\mathbb{R}^m\), one can provide \(\zeta^{-1}(U)\) with the generalized action-angle coordinates

\[
t, \quad z'^a = z^a - t\partial^a H', \quad I'_0 = I_0 + \mathcal{H}'(I_b), \quad I'_a = I_a.
\]

With respect to these coordinates, a Hamiltonian of the autonomous Hamiltonian system on \(\zeta^{-1}(U)\) reads \(H'^* = I'_0 - H'\). A Hamiltonian of the non-autonomous Hamiltonian system on \(U\) endowed with the generalized action-angle coordinates \((t, z'^a, I_a)\) is \(\mathcal{H}'\).

Thus, the following has been proved.

**Theorem 6:** Let \((\gamma_H, F_1, \ldots, F_m)\) be a completely integrable non-autonomous Hamiltonian system. Let \(M\) be its invariant submanifold such that the vector fields \((25)\) on \(M\) are complete and there exists an open neighbourhood \(U\) of \(M\) which is a fibred manifold in mutually diffeomorphic invariant submanifolds. Then \(U\) is diffeomorphic to the Poisson annulus \((33)\), and it can be provided with the generalized action-angle coordinates \((34)\) such that the integrals of motion \((F_1, \ldots, F_m)\) and the Hamiltonian \(H\) depend only on the action coordinates \(I_1, \ldots, I_m\).

### IV. SUPERINTEGRABLE NON-AUTONOMOUS HAMILTONIAN SYSTEMS

Let \((\gamma_H, \Phi_1, \ldots, \Phi_n)\) be a superintegrable non-autonomous Hamiltonian system in accordance with Definition 1. The associated autonomous Hamiltonian system on \(T^*Q\) possesses \(n + 1\) integrals of motion

\[
(\mathcal{H}^*, \zeta^*\Phi_1, \ldots, \zeta^*\Phi_n)
\]

with the following properties.

(i) The functions \((35)\) are mutually independent, and the map

\[
\Phi: T^*Q \to (\mathcal{H}^*(T^*Q), \zeta^*\Phi_1(T^*Q), \ldots, \zeta^*\Phi_n(T^*Q)) = (I_0, \Phi_1(V^*Q), \ldots, \Phi_n(V^*Q)) = \mathbb{R} \times N = N'
\]

is a fibred manifold.

(ii) The functions \((35)\) obey the relations

\[
\{\zeta^*\Phi_\alpha, \zeta^*\Phi_\beta\} = s_{\alpha\beta} \circ \zeta^*\Phi, \quad \{\mathcal{H}^*, \zeta^*\Phi_\alpha\} = s_{0\alpha} = 0
\]

so that the matrix function with the entries \((s_{0\alpha}, s_{\alpha\beta})\) on \(N'\) is of constant corank \(2m + 1 - n\).

Refereing to the definition of an autonomous superintegrable system,\(^4-6\) we come to the following.
**Theorem 7:** Given a superintegrable non-autonomous Hamiltonian system \((\gamma_H, \Phi_\alpha)\) on \(V^*Q\), the associated autonomous Hamiltonian system (35) on \(T^*Q\) is superintegrable.

There is the commutative diagram

\[
\begin{array}{ccc}
T^*Q & \xrightarrow{\zeta} & V^*Q \\
\Phi \downarrow & & \downarrow \Phi \\
N' & \xrightarrow{\xi} & N
\end{array}
\]

where \(\zeta\) (11) and

\[
\xi : N' = \mathbb{R} \times N \to N
\]

are trivial bundles. It follows that the fibred manifold (36) is the pull-back \(\tilde{\Phi} = \xi^*\Phi\) of the fibred manifold \(\Phi\) (2) onto \(N'\).

Let the conditions of Theorem 2 hold. If the Hamiltonian vector fields

\[
(\gamma_H, \partial_{\Phi_1}, \ldots, \partial_{\Phi_n}), \quad \partial_{\Phi_\alpha} = \partial^i \Phi_\alpha \partial_i - \partial_i \Phi_\alpha \partial^i,
\]

of integrals of motion \(\Phi_\alpha\) on \(V^*Q\) are complete, the Hamiltonian vector fields

\[
(u_{\gamma H^*}, u_{\zeta^*\Phi_1}, \ldots, u_{\zeta^*\Phi_n}), \quad u_{\zeta^*\Phi_\alpha} = \partial^i \Phi_\alpha \partial_i - \partial_i \Phi_\alpha \partial^i,
\]

on \(T^*Q\) are complete. If fibres of the fibred manifold \(\Phi\) (2) are connected and mutually diffeomorphic, the fibres of the fibred manifold \(\tilde{\Phi}\) (36) also are well.

Let \(M\) be a fibre of \(\Phi\) (2) and \(H(M)\) the corresponding fibre of \(\tilde{\Phi}\) (36). In accordance with the Mishchenko – Fomenko theorem extended to the case of non-compact invariant submanifolds,\(^{9,10}\) there exists an open neighbourhood \(U'\) of \(H(M)\) which is a trivial principal bundle with the structure group (4) whose bundle coordinates are the generalized action-angle coordinates

\[
(p_A, q^A, I_0, I_\lambda, t, y^\lambda), \quad A = 1, \ldots, n - m, \quad \lambda = 1, \ldots, k, \quad (37)
\]

such that:

(i) \((t, y^\lambda)\) are coordinates on the toroidal cylinder (4),

(ii) the symplectic form \(\Omega\) (7) on \(U'\) reads

\[
\Omega = dp_A \wedge dq^A + dI_0 \wedge dt + dI_\alpha \wedge dy^\alpha,
\]

(iii) the action coordinates \((I_0, I_\alpha)\) are expressed into the values of the Casimir functions \(C_0 = I_0, C_\alpha\) of the coinduced Poisson structure

\[
w = \partial^A \wedge \partial_A
\]
on $N'$,

(iv) the Hamiltonian $\mathcal{H}^*$ depends on the action coordinates, namely, $\mathcal{H}^* = I_0$,

(iv) the integrals of motion $\zeta^* \Phi_1, \ldots, \zeta^* \Phi_n$ are independent of the coordinates $(t, y^\lambda)$.

Provided with the generalized action-angle coordinates (37), the above mentioned neighbourhood $U'$ is a trivial bundle $U' = \mathbb{R} \times U_M$ where $U_M = \zeta(U')$ is an open neighbourhood of the fibre $M$ of the fibre bundle $\Phi$ (2). As a result, we come to Theorem 2.

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