A new action for nonrelativistic bosonic string in flat space time

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Abstract

A new action for nonrelativistic bosonic string in flat space time is shown to emerge from a holistic Hamiltonian analysis of the minimal action for the string. The proposed action under appropriate limits interpolates between the minimal action (Nambu Goto type) where the string metric is taken to be that induced by the embedding and the Polyakov type of action where the world sheet metric components are independent fields. The equivalence among different actions is established by a detailed study of symmetries based on constraint analysis. The interpolating action mooted here is shown to reveal the geometry of the string and may be useful in analyzing the nonrelativistic string coupled with curved background.

1 Introduction

String theory has been developed as an approach towards quantum gravity [1]. Though many interesting results have been deduced (including the theory of Einsteins gravity itself), there are many difficulties also [2]. It is not our purpose to go in to the details of the issues. We only mention that string moving in a nonrelativistic background is an interesting subject on its own that has many welcome features and low energy stringy phenomena can be investigated with more confidence. For instance, such field theories have been proved to be unitary and ultra violet free [3]. Nonrelativistic string theories (NRST) are useful in the study of nonrelativistic holpgraphy which have found applications in the strongly corelated systems in condensed matter physics [4]. The literature of NRST is quite rich and expanding [3],[5],[7],[8]. There are also many surprising outcomes in the geometry of spacetime when the string moves under gravity [3], [4]; the likes of which are not seen in relativistic theories. Of course there are many intricate aspects of non relativistic curved space time geometry, which are not shared by Riemannian...
geometry. Thus NRSTs are quite important for theoretical studies of the Newton Cartan (NC) geometry, the geometry of Newtonian gravity. In the present paper our goal will be to investigate the basic actions of the non relativistic string, taking a bosonic string for simplicity. In this process we introduce a completely new action. In the NRST it is difficult to establish the equivalence of the minimal action of Nambu - Goto (NG) type with the action of Polyakov type. The introduction of the new action removes these difficulties, as we will see.

The action of the relativistic string can be most economically written by an extension of the action for the relativistic particle model. For string the area swept out by the string constitutes the action in the Nambu - Goto form. Though this is the primary form of the string action containing minimum number of fields, it is not very convenient, because of the occurrence of the square root operator in the measure of the area. Polyakov highlighted a new action where the world sheet metric defines the independent variables. It is quite straightforward to demonstrate the equivalence of the two descriptions in relativistic string theories. The story changes dramatically when we go to the literature on nonrelativistic string theories. Here the Polyakov type action contains two additional fields which are not related to the geometry of the problem a priori. So it is quite a long and arduous task to pass from the Polyakov action to the Nambu Goto by eliminating the independent metric fields.

In this situation it may be noted that for relativistic strings a new action of the interpolating type was prescribed that was shown to be equivalent to the Nambu Goto action under certain conditions and passes to the Polyakov form in some other limit. It will certainly be much more welcome in the nonrelativistic case. In this paper such an action will be mooted for NRST. This action reveals the connection between dynamics and geometry. As we have mentioned there are many interesting sides of the geometry of the string world sheet and that of the bulk in which the string is embedded for NRSTs. It appears that our proposed new action may be useful in these contexts. All these possibilities (and many others) will be treated in our analysis.

One aspect of our analysis should be showcased before we proceed. It is the multifaceted use of the Hamiltonian analysis of nonrelativistic strings provided here. The new action that we propose rests directly on this canonical procedure. Nonrelativistic generally covariant models are very interesting singular systems. In the present work Hamiltonian analysis plays a pivotal role in the emergence of the new interpolating action, as we will see. Also a comprehensive account of the Lagrangian and Hamiltonian structure of the new action are provided. Almost all the results concerning the Hamiltonian analysis of different nonrelativistic string models discussed here are new.

It is appropriate to give an account of the organization of the paper. In section 2 we have derived the non relativistic action of the string in Nambu - Goto (N - G) form from its relativistic counterpart. Several authors discussed this model. We took a simple limit and tried to explain what really non relativistic string implies during building of the action. In the next section detailed canonical analysis of the model is provided. Symmetries in the canonical level are retained by working in the gauge independent approach. Particularly

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1 The geometrical theory of Newtonian gravity was discovered by Cartan.
remarkable is the identification of the gauge generator with the diffeomorphisms of the string world sheet. The dynamics in phase space is then analysed by fixing the gauge. In section 4, The new action for the non relativistic string is obtained. We have demonstrated that this action may be reduced to the previous (Nambu Goto) form. More important is the evolution of the Polyakov type action from the interpolating action including the two extra fields. This section also provides a canonical analysis of the interpolating action. The connection between the hamiltonian analysis of different actions have been analysed to show the equivalence of the new action with the Nambu Goto action from the canonical point of view. In the fifth section the geometrical connection was pushed further. Finally, we conclude in section 6.

2 The Nambu - Goto action for the bosonic string

Unlike the particle which is represented by a point, the string is an one dimensional object which is described by a parameter $\sigma$. So during its evolution it traces a two dimensional world sheet. This surface is mapped by two coordinates, $\tau$ and $\sigma$, where $\tau$ is time-like and $\sigma$ is space-like. The world sheet is embedded in a background space time. If the background is Poincare symmetric we obtain a flat relativistic string. On the other hand, a flat nonrelativistic (NR) string is defined if the background symmetry is Galilean.

The relativistic Nambu Goto action of the bosonic string is given by,

$$ S_{NG} = -N \int d\sigma d\tau \sqrt{-\det h_{\alpha\beta}} $$

where $N$ is a dimension-full constant. The metric $h_{\alpha\beta}$ is induced by the target space and given by,

$$ h_{\alpha\beta} = \eta_{\mu\nu} \partial_{\alpha} X^\mu \partial_{\beta} X^\nu $$

Here $X^\mu = X^\mu(\tau,\sigma)$, corresponds to the coordinates of a point on the string with $\tau$ labelling time like and $\sigma$ labelling a space like direction on the world sheet while $\mu$ represents the coordinates of the background space time in which the world sheet traced by the string is embedded. So, in this formalism the metric are not independent fields. Note that $\eta_{\mu\nu} = \text{diag} -1, +1, +1...$ is the Lorentzian metric in the target space. Now, expanding the determinant we get,

$$ S_{NG} = -N \int d\sigma d\tau \left[ \left( \frac{\partial X^\mu}{\partial \tau} \frac{\partial X_\mu}{\partial \sigma} \right)^2 - \left( \frac{\partial X^\mu}{\partial \tau} \frac{\partial X_\mu}{\partial \sigma} \right) \left( \frac{\partial X^\mu}{\partial \sigma} \frac{\partial X_\mu}{\partial \sigma} \right) \right]^{\frac{1}{2}} $$

If we consider the low energy phenomenology of the relativistic string, then the effects in the target space are nonrelativistic. Let the dimension of the embedding space be $(D+2)$. At a given time the string intersects the embedding space along a line. We take this line as the $X^1$ coordinate line. Then $X^0$ and $X^1$ are longitudinal to the string and the rest is transverse.

Collectively, they will be represented by $\sigma^\alpha, \alpha = 1, 2$. 

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Note that we could take any of $X^2, \ldots, X^D$ in place of $X^1$. So no particular gauge choice is associated with this prescription.

There are different methods of obtaining the NR approximation of (3) which are variants of the Inonu-Wigner contraction method. Here, one or more of the coordinates $X^\mu$ are scaled and finally a limit of the scaling parameter is taken. While this may be algebraically tenable, it is not physically motivated. Indeed the most natural prescription would be to reintroduce $'c'$ in the time like variable as $X^0 = ct$ and then take the $c \to \infty$ limit. This is adopted here.

Simplification of (3) using $X^0 = ct$ gives

$$S_{NG} = -N \int d\sigma d\tau \left[ c^2 \left( i X'^1 - \dot{X}^1 t' \right)^2 + \sum_k c^2 \left( i X'^k - \dot{X}^k t' \right)^2 - \sum_k \left( \dot{X}^1 X'^k - \dot{X}^k X'^1 \right)^2 + \sum_{k,l} \left( \dot{X}^k X'^l \dot{X}^l - \dot{X}^k \dot{X}^l X'^1 X'^1 \right) \right]^{1/2}$$

(4)

where a dot over a symbol denotes derivative with respect to $\tau$ while a prime as a superscript implies differentiation with respect to $\sigma$. Also $c$ is made explicit.

So far our result is relativistic. Now, remember that the $X^1$ axis is longitudinal to the string and the rest ($k=2,3,\ldots D$) are in the transverse section. So $\dot{X}^1 \ll c$ and $dX^k \ll dX^1$ since in the low energy scenario the slope of the transverse vibration of the string is very small. Thus the non relativistic limit of the action (4) is

$$S_{NG} = -N \int d\sigma d\tau \left( c \left( i X'^1 - \dot{X}^1 t' \right) \right) \left[ 1 + \frac{\sum_k c^2 \left( i X'^k - \dot{X}^k t' \right)^2}{c^2 \left( i X'^1 - \dot{X}^1 t' \right)^2} \right]^{1/2} - \frac{\sum_k \left( \dot{X}^1 X'^k - \dot{X}^k X'^1 \right)^2}{c^2 \left( i X'^1 - \dot{X}^1 t' \right)^2} \right]$$

(5)

where, we have neglected the last term of (4), as it is of higher order of smallness. Now $ct = X^0$ and taking the leading term of the small quantities (also making sum over $k$ implicit), we get

$$\mathcal{L}_{NG} = -N \left[ \frac{\left( \dot{X}^0 X'^k - \dot{X}^k X'^0 \right)^2}{2 \left( \dot{X}^0 X'^1 - \dot{X}^1 X'^0 \right)} - \frac{\left( \dot{X}^1 X'^k - \dot{X}^k X'^1 \right)^2}{2 \left( \dot{X}^0 X'^1 - \dot{X}^1 X'^0 \right)} \right]$$

(6)

Note that we have dropped the first term within the square bracket as it is a total boundary,

$$\left( \dot{X}^0 X'^1 - \dot{X}^1 X'^0 \right) = \frac{\partial}{\partial \tau} \left( X^0 X'^1 \right) - \frac{\partial}{\partial \sigma} \left( X^0 \dot{X}^1 \right)$$

(7)
Equation (6) is the NR Nambu-Goto form of the Lagrangian for a bosonic string. The derivation is based on the usual $c \to \infty$ limit along with certain physical inputs. This result was obtained in [5], using a variant of the Inonu-Wigner contraction referred earlier which somewhat obscures the physical origin inherent in our derivation. Another aspect of the construction is the impact of the relativistic nature of the string which enforces that the metric in the $0 - 1$ plane is Lorentzian, even in our example of non relativistic phenomena. We denote this metric by $\eta_{\mu\nu}; \mu, \nu = 0, 1 = \text{diag}(1,-1)$. We can now rewrite the Lagrangian (6) in a less clumsy form,

$$L_{\text{NG}} = -N \left( 2\epsilon_{\mu\nu} \dot{X}^\mu X'^\nu \right)^{-1} \left( \dot{X}^\mu X'^\mu - \dot{X}^k X'^k \right)^2 \tag{8}$$

by using the covariant notation.

One can wonder in what sense the action

$$S_{\text{NG}} = \int d\sigma d\tau L_{\text{NG}} \tag{9}$$

with $L_{\text{NG}}$ given by (8), is Galilean invariant. That the string is essentially relativistic makes the question non trivial. Let us now discuss the issue. We have already mentioned that we are considering the string to be in motion in a non relativistic background. So one would expect that physics in the background remains unaltered under the Galilean transformations. But not all elements of the group can be included here. The values of $\tau$ and $\sigma$, which specify a point on the string world sheet should not change. Consider the Galilean transformations in the usual way,

$$\begin{align*}
\delta X^0 &= -\epsilon \\
\delta X^1 &= \epsilon^1 + \omega^1_j X^j - v^1 X^0 \\
\delta X^k &= \epsilon^k + \omega^k_l X^l - v^k X^0
\end{align*} \tag{10}$$

We now calculate the change of the Lagrangian (8) under (10). The result is,

$$\delta L_{\text{NG}} = N \omega^{1i}(\dot{X}^0 X'^i - X^0 \dot{X}^i) \left[ (\epsilon_{\mu\nu} \dot{X}^\mu X'^\nu)^{-2} (\dot{X}^\mu X'^\mu - \dot{X}^k X'^k)^2 \right] \tag{11}$$

Now the $X^1$ axis is assumed to be lying along the $\tau = \text{constant}$ direction. So a non zero $\omega^{1i}$ would mean the change of the world sheet parameters, which is contrary to the concept of the global coordinate transformations in the target space. Hence $\omega^{1i} = 0$. So the Galilean transformations under which the theory (8) is symmetric are (10), supplemented by this condition. For ready reference we write the modified symmetry transformations as,

$$\begin{align*}
\delta X^0 &= -\epsilon \\
\delta X^1 &= \epsilon^1 - v^1 X^0 \\
\delta X^k &= \epsilon^k + \omega^k_l X^l - v^k X^0; \ k, l > 1
\end{align*} \tag{12}$$

\[3\text{Just as the value of proper time locating a particle on its world line is insensitive to the Galilean transformations of the background}\]
Another demand we would like to place on (8) is, it should reduce to the Lagrangian of the classical vibrating string in the appropriate limit. Let us take a string stretched between $x = 0$ to $x = a$ along the $x$ - axis, vibrating transversely along the $y$ - axis. From elementary analysis we get the Lagrangian as

$$L = \left[ \frac{1}{2} \mu^0 \left( \frac{\partial y}{\partial t} \right)^2 - \frac{1}{2} T^0 \left( \frac{\partial y}{\partial x} \right)^2 \right]$$

(13)

where $\mu^0$ is the linear mass density and $T^0$ is the tension of the string. Now let us see in what way we may reproduce this result from (8). Putting $X^0 = c\tau = ct$ and $X^1 = \sigma$ we get

$$L_{NG} = \left[ \frac{1}{2} Nc \left( \frac{\partial X^k}{\partial \tau} \right)^2 - \frac{1}{2} Nc \left( \frac{\partial X^k}{\partial \sigma} \right)^2 \right]$$

(14)

We get $k$-copies of the transversally vibrating classical string if we identify, $\frac{N}{c} = \mu^0$ and $Nc = T^0$. Thus the constant $N$ in (8) is related to the tension of the string. Expectedly, the velocity is given by $c^2 = \frac{T^0}{\mu^0}$, as happens classically.

3 Canonical analysis of the model

We have now established our Lagrangian. The next step is a Hamiltonian analysis of the model. Now, being a reparametrization invariant theory, it is already covariant [12]. The model is thus an interesting example of a constrained system, and because of its NR nature, more involved than its relativistic counterpart. As we have already mentioned, these studies are entirely new as the hamiltonian analysis available in the literature [14] does not provide a faithful Dirac treatment of the model.

A constrained system with first class constraints necessarily possesses gauge degrees of freedom [18]. General symmetries of such systems can be derived without fixing the gauges. In fact we will see that this gauge independent approach will lead to the derivation of a new action. The importance of canonical analysis can thus be hardly overestimated. On the contrary, this aspect has been less emphasized in the literature. We will therefore try to give a holistic account of the topic. For clarity of presentation we divide our results in a number of subsections.

3.1 Phase space structure

Here the fields are $X^0(\tau, \sigma)$ , $X^1(\tau, \sigma)$ , $X^k(\tau, \sigma)$ (where k=2,3,...,D ).

The canonical momenta corresponding to $X^0$ is $\frac{4}{5}$

\footnote{That such a choice is possible is discussed in section 3.4}

\footnote{We rename the NG lagrangian as $L_{NG} \rightarrow L$ and set $N = 1$.}
\[ \Pi_0 = \frac{\partial L}{\partial \dot{X}^0} = \left[ X^{\mu \nu} (\epsilon_{\mu \nu} \dot{X}^k X^\nu)^{-1} (\dot{X}^0 X^k - \dot{X}^k X^0) - \frac{X^{\mu \nu}}{2} (\epsilon_{\mu \nu} \dot{X}^k X^\nu)^{-2} \left( \dot{X}^k X^\mu - \dot{X}^k X^\mu \right) \right] \] (15)

Similarly that for \( X^1 \) is

\[ \Pi_1 = \frac{\partial L}{\partial \dot{X}^1} = \left[ -X^{\mu \nu} (\epsilon_{\mu \nu} \dot{X}^k X^\nu)^{-1} (\dot{X}^1 X^k - \dot{X}^k X^1) + \frac{X^{\mu \nu}}{2} (\epsilon_{\mu \nu} \dot{X}^k X^\nu)^{-2} \left( \dot{X}^k X^\mu - \dot{X}^k X^\mu \right) \right] \] (16)

Also for \( X_k \),

\[ \Pi_k = \frac{\partial L}{\partial \dot{X}^k} = \left( -\epsilon_{\mu \nu} \dot{X}^\nu X^\mu \right)^{-1} \left[ X^\mu (\dot{X}^\mu X^k - \dot{X}^k X^\mu) \right] \] (17)

Using these in the definition of the canonical Hamiltonian we get,

\[ H_c(\tau) = \int d\sigma \left( \Pi^\mu \dot{X}_\mu - L \right) \]

A straightforward calculation gives , \( H_c(\tau) = 0 \). This is a characteristic of the already co-

variant theories. From an inspection of the expressions of the momenta, the following primary constraints emerge,

\[ \Omega_1 = \Pi^\mu X^\mu \approx 0 \]
\[ 2\Omega_2 = \Pi^2 + X^{\mu \nu} - 2\sigma_{\alpha \beta} \Pi^\alpha X^\beta \approx 0 \] (18)

where \( \sigma_{\alpha \beta} \) is second Pauli matrix. The fundamental Poisson’s brackets of the theory are given by

\[ \{ X^\mu (\tau, \sigma), \Pi_\nu (\tau, \sigma') \} = \eta^\mu_\nu \delta (\sigma - \sigma') \] (19)

while the others vanish. Using these Poisson brackets it is easy to work out the algebra of the constraints,

\[ \{ \Omega_1 (\sigma), \Omega_2 (\sigma') \} = (\Omega_2 (\sigma) + \Omega_2 (\sigma')) \partial_\sigma \delta (\sigma - \sigma') \]
\[ \{ \Omega_1 (\sigma), \Omega_1 (\sigma') \} = (\Omega_1 (\sigma) + \Omega_1 (\sigma')) \partial_\sigma \delta (\sigma - \sigma') \]
\[ \{ \Omega_2 (\sigma), \Omega_2 (\sigma') \} = (\Omega_2 (\sigma) + \Omega_2 (\sigma')) \partial_\sigma \delta (\sigma - \sigma') \] (20)

Clearly, the Poisson brackets between the constraints (18) are weakly involutive. So the set (18) is first class.
The total Hamiltonian is

\[ H_T = \int d\sigma (\rho \Omega_1 + \lambda \Omega_2) \]  

(21)

where \( \rho \) and \( \lambda \) are Lagrange multipliers and \( \Omega_1 \) and \( \Omega_2 \) are shown in [18].

Conserving the primary constraints no new secondary constraints emerge since the constraint algebra simply closes. The total set of constraints of the N - G theory is then given by the first class system (18).

The Nambu Goto string is a constrained system. So its description is redundant. If it is embedded in a \( D + 1 \) dimensional space time, the number of fields in the configuration space is \( D + 1 \). The corresponding number of variables in the phase space is \( 2(D + 1) \). Then the number of degrees of freedom in the configuration space is given by,

\[ n = \frac{1}{2} (2(D + 1) - 4) = D - 1 \]  

(22)

This result is consistent with our understanding about the non relativistic excitation taking place in the transverse direction and we identify the \((D - 1)\) variables \( X^k \) as the physical set.

Also, we understand from another angle, why \( \omega^{1i} = 0 \) should hold (Recall the second equation of (12)).

3.2 Studies of local symmetries

The string world sheet is a two dimensional manifold which is charted by the parameters \( \tau \) and \( \sigma \). The physical theory should not depend on any particular parametrization. In other words we should have invariance under reparametrization (a mapping of the manifold on itself i.e. a diffeomorphism)

\[ \tau' = \tau'(\tau, \sigma) \]
\[ \sigma' = \sigma'(\tau, \sigma) \]  

(23)

which becomes for infinitesimal diffeomorphism

\[ \tau' = \tau + \delta \tau \]
\[ \sigma' = \sigma + \delta \sigma \]  

(24)

The increments \( \delta \sigma \) and \( \delta \tau \) both are functions of \( \sigma \) and \( \tau \). In the Lagrangian level the diffeomorphism invariance is conceptually clear. The Lagrangian is a world sheet scalar. Its form variation under (24) is given by \( \delta \mathcal{L} = \delta \sigma_a \frac{\partial \mathcal{L}}{\partial \sigma_a} \) where, \( \sigma^0 = \sigma \) and \( \sigma^1 = \tau \).

The Jacobian of the transformation is \((1 + \partial_a \delta \sigma_a)\). Direct substitution in the action gives

\[ \delta S = \int d\sigma d\tau \frac{\partial}{\partial \sigma_a} (\mathcal{L} \delta \sigma_a) = 0 \]  

(25)
since the variations vanish at the boundary. So the theory (8) is invariant under (24).

Looking from the Hamiltonian point of view such action level symmetries should correspond to the gauge symmetries of the model. The gauge redundancy account for the diffeomorphism invariances and vice versa. So one should be able to establish an exact mapping between gauge and reparametrization parameters. We will derive the explicit form of the mapping now.

3.3 Mapping between gauge and reparametrization symmetries

According to the Dirac conjecture all the first class constraints generate gauge transformations. But the gauge parameters associated with these transformations are not independent. It is known that the number of independent primary first class constraints equals the number of independent gauge parameters [16], [17]. Since in the present example two primary first class constraints form the set of constraints, the gauge generator can be written down immediately,

\[ G(\tau) = \int d\sigma (\omega_1(\sigma)\Omega_1 + \omega_2(\sigma)\Omega_2) \]  
(26)

The corresponding gauge variations are,
\[ \delta_G X^0 = [X^0, G]_{PB} = \omega_1(\sigma)X^0\nu - \omega_2X^1\nu \]
\[ \delta_G X^1 = [X^1, G]_{PB} = \omega_1(\sigma)X^1\nu - \omega_2(\sigma)X^0\nu \]
\[ \delta_G X^k = [X^k, G]_{PB} = \omega_1(\sigma)X^{k\nu} + \omega_2(\sigma)\Pi^{k\nu} \]  
(27)

with the exact number of independent parameters.

Now, the variation due to diffeomorphism (24) are,
\[ \delta_D X^\mu = \xi_1 X^{\nu} + \xi_2 \dot{X}^\mu \]  
(28)

where \( \xi_1 = -\delta\sigma \) and \( \xi_2 = -\delta\tau \) defined in (24). We see "velocities" appearing in the expression of variations (28). To exhibit the one to one correspondence we have to substitute \( \dot{X}^\mu \) in (28) from its equation of motion, \( \dot{X}^\mu = [X^\mu, H_T] \), where \( H_T \) is the total Hamiltonian given by (21). This calculation gives us for \( \mu = 0 \)

\[ \dot{X}^0 = \rho X^0 - \lambda X^1 \]  
(29)

Substitution of this in (28), we get,
\[ \delta_D X^0 = (\xi_1 + \rho\xi_2) X^0 - \xi_2\lambda X^1 \]  
(30)

Comparison with \( \delta_G X^0 \) gives us the desired mapping,
\[ \omega_1 = (\xi_1 + \xi_2\rho) \]
\[ \omega_2 = \xi_2\lambda \]  
(31)

The same results will be obtained for any component of \( \mu \). Thus the mapping (31) exhibits the complete equivalence of the gauge symmetries with the diffeomorphism invariances of the model.
3.4 Gauge fixed analysis

We have seen that the analysis of the symmetries is best done in the gauge independent approach. However, for the study of dynamics in the phase space one must eradicate the gauge redundancy by a gauge choice. A gauge is a condition in phase space which makes a first class constraint second class, thereby reducing two degrees of freedom. While this is a necessary condition it is not sufficient. In order to specify the physical set, one has to properly obtain the canonical variables. These variables can always be obtained (this is the content of the Maskawa-Nakajima theorem [15]) such that the Dirac brackets among these variables is the same as the Poisson brackets. Then one can proceed with the quantisation by replacing these brackets by suitable commutators.

We assume the standard gauges,

\[ \Omega_3 = X^1 - \sigma \approx 0 \quad \Omega_4 = X^0 + c\tau \approx 0 \] (32)

Note that if we include the gauge conditions as constraints with the existing set \{\Omega_1 \approx 0, \Omega_1 \approx 0\}, then all the constraints become second class. The phase space can now be reduced by implementing these constraints strongly. The canonical structure is now optimum, described by the phase space variables \{X^k, \Pi_k\} and the symplectic structure is given by the Dirac brackets [18, 12]. The structure of the Dirac brackets between the coordinates are rich with physical significance and worthy to be studied carefully. The Dirac bracket between two phase space variables is defined by [18],

\[
[A(\sigma), B(\sigma')]_{DB} = [A(\sigma), B(\sigma')]_{PB} - \int d\sigma_1 d\sigma_2 [A(\sigma), \Omega_i(\sigma_1)]_{PB} \Delta^{ij}(\sigma_1, \sigma_2) [\Omega_j(\sigma_2), B(\sigma')]_{PB}
\] (33)

where, the matrix \(\Delta^{ij}\) is the inverse of the matrix \(\Delta_{ij}\) formed by the constraint algebra,

\[ \Delta_{ij} = [\Omega_i, \Omega_j]_{PB} ; \quad i = 1, 2, 3, 4 \] (34)

which is necessarily non singular and admits an inverse. Using these definitions and the Poisson brackets between the second class constraints [18] and (32), we get,

\[
\Delta_{ij} = \begin{bmatrix}
0 & 0 & -\delta(\sigma - \sigma') & 0 \\
0 & 0 & 0 & \delta(\sigma - \sigma') \\
\delta(\sigma - \sigma') & 0 & 0 & 0 \\
0 & -\delta(\sigma - \sigma') & 0 & 0
\end{bmatrix}
\]

The inverse is easily found,

\[
\Delta^{ij} = \begin{bmatrix}
0 & 0 & \delta(\sigma - \sigma') & 0 \\
0 & 0 & 0 & -\delta(\sigma - \sigma') \\
-\delta(\sigma - \sigma') & 0 & 0 & 0 \\
0 & \delta(\sigma - \sigma') & 0 & 0
\end{bmatrix}
\]
It is then straightforward to calculate the Dirac brackets between the previous canonically
conjugate variables in the gauge independent analysis,

\[
[X^0(\sigma, \tau), \Pi_0(\sigma', \tau)]_{DB} = 0 \\
[X^1(\sigma, \tau), \Pi_1(\sigma', \tau)]_{DB} = 0 \\
[X^i(\sigma, \tau), \Pi_j(\sigma', \tau)]_{DB} = \delta^i_j \delta(\sigma - \sigma') \quad (35)
\]

We will provide a detailed derivation of the first equation. Starting from the definition of
the Dirac bracket we get,

\[
[X^0(\sigma, \tau), \Pi_0(\sigma', \tau)]_{DB} = [X^0(\sigma, \tau), \Pi_0(\sigma', \tau)]_{PB} \\
- \int d\sigma_1 d\sigma_2 [X^0(\sigma), \Omega_2(\sigma_1)] \Delta^{24}(\sigma_1, \sigma_2) [\Omega_4(\sigma_2), \Pi_0] \\
= \delta(\sigma - \sigma') - \int d\sigma_1 d\sigma_2 \delta(\sigma - \sigma_1) X^1(\sigma_1 - \sigma_2) \delta(\sigma_2 - \sigma') \\
= 0 \quad (36)
\]

where we have imposed the constraint \(\Omega_4\) strongly\(^6\) so that \(X^1' = 1\). Similarly we can derive
the other two equations of the set (35). Incidentally the last relation of (35) is the only non-zero
bracket.

Thus \(X^0, X^1\) (and their conjugate momenta) are out of the dynamics and \((X^i, \Pi_i), i=2,3,...\)
are the canonical pairs. Also note that the total Hamiltonian vanishes when the constraints
are strongly implemented. In this situation we have to identify a Hamiltonian in the reduced
phase space which will generate the equations of motion with respect to the Dirac brackets.

Remember that canonically the Hamiltonian may be considered as the conjugate to the
time parameter. So we identify the new Hamiltonian as

\[
H = c \int d\sigma \left[\Pi^0\right]_g \quad (37)
\]

where the subscript \(g\) denotes that gauge fixed value Using the gauge conditions (32) which
now allow us to put, \(X^0 = -c\) and \(X^1 = \sigma\) we find by substitution from (15),

\[
H = \frac{c}{2} \int d\sigma \left[\left(\partial_\sigma X^k\right)^2 + \frac{1}{c^2} (\partial_\tau X^k)^2\right] \quad (38)
\]

We see that the proposed Hamiltonian is positive definite. Remarkably, this Hamiltonian is a
sum of harmonic oscillator terms. Use the definition of \(\Pi^k\) and the gauge fixing conditions to
get,

\[
\Pi^k = \frac{1}{c} \partial_\tau X^k \quad (39)
\]

\(^6\)This is permitted as all the Poisson brackets have already been evaluated
Then from (38) we can write,

\[ H = \frac{c}{2} \int \left[ (\partial_{\sigma} X^k)^2 + (\Pi^k)^2 \right] \]  

This is however the transversely vibrating string Hamiltonian. The corresponding lagrangian, obtained by an inverse Legendre transformation reproduces (14) with \( N = 1 \).

4 The interpolating Lagrangian

In this section the canonical analysis of the previous section will be used from an inverse approach to develop a new Lagrangian, which will be shown to have the remarkable property of interpolating between NG and Polyakov Lagrangians. In the case of relativistic string one can easily deduce the NG string from the Polyakov string on shell, by substituting the metric from its equation of motion in the original Lagrangian under definite conditions. But for the non relativistic string, the Polyakov action requires to be supplemented by two world sheet fields, the origin of which is hard to trace [3]. Thus the equivalence of the two actions becomes problematic. In the following discussion we will see that the Hamiltonian analysis can be used to identify the source of the additional fields in the interpolating action which eventually permeates to the Polyakov form. Also, this action is remarkable due to its connection with the geometry and may be useful in coupling the string with curved background.

The Lagrangian corresponding to the Hamiltonian \( H_T \), see equation (21), see is

\[ \mathcal{L}_I = \Pi_\mu \dot{X}^\mu - \rho \Omega_1 - \lambda \Omega_2 \]  

where the multipliers \( \rho \) and \( \lambda \) are given the status of independent fields. The Lagrange equations corresponding to \( \Pi^0 \), \( \Pi^1 \) and \( \Pi^k \) are now respectively,

\[ \begin{align*}
\dot{X}^0 - \rho X^0 + \lambda X^1 &= 0 \\
\dot{X}^1 - \rho X^1 + \lambda X^0 &= 0 \\
\dot{X}^k - \rho X^k - \lambda \Pi^k &= 0
\end{align*} \]  

(42)

Solving \( \rho \) and \( \lambda \) from the set (42), we get,

\[ \begin{align*}
\rho &= \frac{\dot{X}^1 X^1 - \dot{X}^0 X^0}{(X^1)^2 - (X^0)^2} \\
\lambda &= \frac{-\dot{X}^0 X^1 + \dot{X}^1 X^0}{(X^1)^2 - (X^0)^2}
\end{align*} \]  

(43)

It is possible to eliminate all the momenta from (41) using (42). Now, reinterpret \( \rho \) and \( \lambda \) as independent fields. Also, equations (43) will now be promoted to Lagrangian constraints.
Doing all these steps the following lagrangian is obtained,

\[
\mathcal{L}_I = \frac{1}{2\lambda} \left[ (\dot{X}^k)^2 - 2\rho \dot{X}^k X^k + (\rho^2 - \lambda^2) (X^k)^2 \right] + \beta \left( \rho - \frac{\dot{X}^1 X^0}{X^{12} - X^{02}} \right) + \alpha \left( \lambda + \frac{\dot{X}^0 X^1}{X^{12} - X^{02}} \right)
\]

This is the appearance of the new Lagrangian mooted in this paper. We will presently show that we can derive both the Nambu Goto and Polyakov forms of the nonrelativistic string action from (44). This is the reason for dubbing (44) as the interpolating Lagrangian.

4.1 Passage to the Nambu Goto type

The derivation of the Nambu Goto action (4) from (44) is trivial. The multipliers simply enforce the solutions (43)). Putting it back in (44) reproduces the expected result.

4.2 Passage to the Polyakov form

This sounds really interesting for the Polyakov form brings in an independent metric on the world sheet but in the interpolating Lagrangian there is no explicit reference to such a metric. Also, the world sheet fields that are invoked in the Lagrangian look like a hurdle. The startling observation in the following analysis is the solution at one stroke, where aspects of Riemannian geometry of the world sheet converges with the canonical structure of the non relativistic string. From this confluence the action emerges including the extra fields.

Let us first observe that with the help of the fields \(\rho\) and \(\lambda\) we can construct a 2 by 2 matrix,

\[
h^{ij} = (-h)^{-\frac{1}{2}} \begin{pmatrix} \frac{1}{\lambda} & \frac{-\rho}{\lambda^2} \\ \frac{-\rho}{\lambda} & \frac{\rho^2 - \lambda^2}{\lambda} \end{pmatrix}
\]

where \(h\) is the determinant of the inverse matrix \(h_{ij}\). The consistency of the construction can be verified by the computation of \(\det h^{ij}\), which yields

\[
\det h^{ij} = \frac{1}{h}
\]

showing that the matrix \(h_{ij}\) is a 2 by 2 real symmetric matrix. Also the inverse to \(h_{ij}\) is \(h^{ij}\). Note that obeying the constraints, \(h\) may assume any non zero value. We propose \(h_{ij}\) as the metric on the world sheet and write the interpolating Lagrangian (44), in terms of the elements of \(h^{ij}\), as,

\[
\mathcal{L}_I = \frac{1}{2} \sqrt{-h} h^{ij} \partial_i X^k \partial_j X^k + \mathcal{L}_c
\]
The first part is formally the same as the relativistic bosonic string, though here only transverse degrees of freedom are dynamical. The second part is due to the constraints \((43)\), enforced in the Lagrangian, again by multipliers,

\[
\mathcal{L}_e = \beta \left( -\frac{h^{01}}{h^{00}} - \frac{\dot{X}^1 X^0 - \dot{X}^0 X^1}{(X^1)^2 - X^{02}} \right) + \alpha \left( \frac{1}{\sqrt{-h}h^{00}} + \frac{\dot{X}^0 X^1 - \dot{X}^1 X^0}{X^1 - X^{02}} \right)
\]

Note that from the identification of the metric we can show that,

\[
\lambda = \frac{1}{\sqrt{-h}h^{00}}; \quad \rho = -\frac{h^{01}}{h^{00}}
\]

which has been used in writing \((48)\). The Lagrangian \((47)\), with \(\mathcal{L}_e\) given by \((48)\) is the string action given in the Polyakov form. We see that the matrix \(h^{ij}\) now represents independent fields. Indeed, \(h^{ij}\) resembles the metric on the world sheet. We can compare the action with the corresponding relativistic action. The first term is of the same form but only transverse degrees of freedom are involved. This is consistent with the previous analysis given here. The fields \(\alpha\) and \(\beta\) are the two extra fields included in the Polyakov form. For nonrelativistic string action considered in previous studies [3] such fields are included. However, their appearance was not explained. We have seen they follow from the canonical analysis. The issue will be considered in more detail in the following section.

### 4.3 The interpolating Lagrangian – canonical analysis

The last section has established that the interpolating action \((44)\) is a versatile tool to study nonrelativistic strings. We have shown that under appropriate conditions the model interpolates between the Nambu Goto and Polyakov forms of the string action. This equivalence will further be elucidated by the canonical analysis.

We have already shown the action level reduction of the interpolating action to the Nambu-Goto (NG) string. But that does not necessarily imply that the two theories have the same physical content. The canonical structure of the interpolating theory determines its energy spectrum. Also, the first class constraints determine the gauge symmetry of the models. In the following discussion a canonical analysis of the new action is provided which further elucidates its equivalence with the NG form.

The interpolating Lagrangian will be the starting point of the comparison. The momenta corresponding to the basic fields in \((44)\) lead to one genuine momentum \(\Pi^k\),

\[
\Pi_k = \frac{\dot{X}^k}{\lambda} - \frac{\rho}{\lambda} X^k
\]

while the rest are constraints, two of which are first class defined as,

\[
\pi_\rho \approx 0; \quad \pi_\lambda \approx 0
\]
while the remaining are second class,

\[ \Sigma_1 = \pi_\alpha \approx 0; \quad \Sigma_2 = \pi_\beta \approx 0 \]
\[ \Sigma_3 = \Pi_0 - \frac{\alpha X'^1 + \beta X'^0}{(X'^1)^2 - (X'^2)^2} \approx 0 \]
\[ \Sigma_4 = \Pi_1 + \frac{\alpha X'^0 + \beta X'^1}{(X'^1)^2 - (X'^2)^2} \approx 0 \]  

(52)

The canonical Hamiltonian is

\[ H_c = \int d\sigma \left[ \lambda \left( \frac{\Pi_k^2 + X'^k}{2} - \alpha \right) + \rho \left( \Pi^k X'^k - \beta \right) \right] \]  

(53)

The second class constraints are strongly implemented by using Dirac brackets instead of the Poisson. The total Hamiltonian is then defined as,

\[ H_T = H_c + \int d\sigma (\nu_1 \pi_\lambda + \nu_2 \pi_\rho) \]  

(54)

From conserving the first class constraints we get two more constraints. Note that now we have to use, instead of the Poisson brackets, the relevant Dirac brackets. However it is easy to see that the Dirac brackets and the Poisson brackets are identical for the variables that are involved in the iterative computation of constraints. The new constraints are,

\[ [\pi_\rho, H_T] \approx 0 \rightarrow \Pi^k X'^k - \beta = \Phi_1 \approx 0 \]
\[ [\pi_\lambda, H_T] \approx 0 \rightarrow \frac{1}{2} (\Pi^k X'^k) - \alpha = \Phi_2 \approx 0 \]  

(55)

The constraints are now all first class. Their algebra is strongly involutive, except for the pair \((\Phi_1, \Phi_2)\) which satisfies an algebra identical to \((20)\),

\[ \{ \Phi_1 (\sigma), \Phi_2 (\sigma') \} = (\Phi_2 (\sigma) + \Phi_2 (\sigma')) \partial_\sigma \delta (\sigma - \sigma') \]
\[ \{ \Phi_1 (\sigma), \Phi_1 (\sigma') \} = (\Phi_1 (\sigma) + \Phi_1 (\sigma')) \partial_\sigma \delta (\sigma - \sigma') \]
\[ \{ \Phi_2 (\sigma), \Phi_2 (\sigma') \} = (\Phi_1 (\sigma) + \Phi_1 (\sigma')) \partial_\sigma \delta (\sigma - \sigma') \]  

(56)

Let us next perform a degree of freedom count. The total number of phase space degrees of freedom are \(2(6 + k)\). There are 4 second class constraints and 4 first class constraints. Hence the number of independent phase space degrees of freedom are,

\[ n = 2(6 + k) - 4 - 2 \times 4 = 2k \]  

(57)

Hence the independent number of configuration space degrees of freedom is \(k\), which we take to be the \(X^k\) variables. This precisely matches with our earlier counting and identification.

Since the constraints in \((52)\) are strongly implemented, it is possible to solve for \(\alpha\) and \(\beta\). We get,

\[ \alpha = \Pi^0 X'^1 + \Pi_1 X'^0 \]
\[ \beta = -\Pi^0 X'^0 - \pi_1 X'^1 \]  

(58)
Substituting these in (55) we obtain,
\[ \Phi_1 = \Omega_1 ; \Phi_2 = \Omega_2 \] (59)
which are identical to the two first class constraints (18) of the original Nambu - Goto model.

5 Connection with geometry

In the above we have introduced a new action for the nonrelativistic bosonic string which has the merit of interpolating between the Nambu Goto form on one side and the Polyakov form of the action on the other. Thus both types of actions can be related in one go. In the nonrelativistic variety this task is not simple, as evidenced in the literature [5, 3]. First of all, metric components in the transverse directions only appear in the Polyakov type action and the target space there is a clear compartmentalization of the directions horizontal to the string and transverse to the string. The low energy excitations are entirely transverse. The horizontal components of the metric are not dynamical. Probably due to this proviso two new fields are included in the action which are devoid of any dynamics [5]. How these fields are related with the geometry (i.e. their connection with the metric) is not known. Clearly the action level correspondence between the different forms of the action, so transparent in the relativistic formulations appear to be missing. The equivalence could be established by an arduous path [4]. The new action found here not only was shown to bridge the different forms, it also generated the additional fields. Moreover it has connected the geometric elements with the multipliers in the Hamiltonian. In the following we will further investigate the connection and one would appreciate that this connection is not accidental.

So far, the metric induced on the world sheet was not discussed because the discussion could proceed without reference to the metric. A metric on the world sheet was derived as the metric induced by the embedding (see equation (1)). But the Polyakov form gives an independent metric. So geometry of the world sheet is now interwoven with the dynamics. In deriving the Polyakov type action from the Nambu Goto through the interpolating action, we have connected the form of the metric (45) composed with the Hamiltonian constructs \( \rho \) and \( \lambda \), with the world sheet geometry. This evolution of the metric as dynamical fields is certainly a new input in the existing literature.

Our construction (45) is reminiscent of the Arnowitt–Deser–Misner (ADM) decomposition of general relativity [11]. In the ADM representation the metric of the four dimensional Riemannian space time \( (4)\gamma^{\mu\nu} \) is split as\(^7\)

\[
\begin{pmatrix}
(4)\gamma_{00} & (4)\gamma_{0m} \\
(4)\gamma_{0k} & (4)\gamma_{km}
\end{pmatrix}
= \begin{pmatrix}
-\frac{1}{(N)^2} & \frac{(N^m)}{(N)^2} \\
\frac{(N^k)}{(N)^2} & \gamma_{km} - \frac{(N^k)(N^m)}{(N)^2}
\end{pmatrix}
\] (60)

Here, \( k, m \) take the values 1, 2, 3. \( \gamma^{km} \) is the metric on a three dimensional hypersurface embedded in the four dimensional space time. \( N_l \) and \( N^k \) are the arbitrary lapse and shift

\(^7\)For the metric of the total space time the dimension is mentioned as a (pre)superscript
variables which are nothing but the Lagrange multipliers of the theory. From (60), a similar structure for \( d = 2 \) assumes the following form

\[
\begin{pmatrix}
(2)\gamma_{00} & (2)\gamma_{01} \\
(2)\gamma_{01} & (2)\gamma_{11}
\end{pmatrix}
= 
\begin{pmatrix}
-\frac{1}{(N)^2} & \frac{(N^1)}{(N)^2} \\
\frac{(N^1)}{(N)^2} & \left(\gamma^{11} - \frac{(N^1)^2}{(N)^2}\right)
\end{pmatrix}
\] (61)

Comparing this with (45) we can easily establish

\[ N^1 \mapsto \rho \quad \text{and} \quad (N)^2 \mapsto -\lambda \sqrt{-g} \quad \text{and} \quad \gamma \mapsto g \] (62)

Thus the identification (45) is the same as the ADM foliation of the world-sheet. The fields \( \lambda \) and \( \rho \) are manifestations of the arbitrariness along the string (the shift \( N \)) and an arbitrariness transverse to the string i.e. the relative time direction (the lapse \( N^1 \)), both on the world sheet.

For further insight and comparison with existing results [3], we rewrite the interpolating Lagrangian in the light-cone coordinates,

\[
X = X^0 + X^1 \quad \bar{X} = X^0 - X^1
\] (63)

The extra piece can now be written as,

\[
\mathcal{L}_e = \alpha \left( \frac{1}{\sqrt{-h}h^{00}} + \frac{\epsilon^{\alpha\beta}\partial_\alpha X \partial_\beta \bar{X}}{2X'^{i}X^{i'}} \right) + \beta \left( -\frac{h^{01}}{h^{00}} - \frac{\sigma^{\alpha\beta}\partial_\alpha X \partial_\beta \bar{X}}{2X'^{i}X^{i'}} \right)
\] (64)

where,

\[
\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\] (65)

and \( \sigma^{\alpha\beta} \) is

\[
\sigma^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\] (66)

Combining (47) and (64) we get the interpolating action in light cone coordinates

\[
\mathcal{L}_I = \frac{1}{2} \sqrt{-h}h^{\alpha\beta}\partial_\alpha X^k \partial_\beta X^k + \alpha \left( \frac{1}{\sqrt{-h}h^{00}} + \frac{\epsilon^{\alpha\beta}\partial_\alpha X \partial_\beta \bar{X}}{2X'^{i}X^{i'}} \right) + \beta \left( -\frac{h^{01}}{h^{00}} - \frac{\sigma^{\alpha\beta}\partial_\alpha X \partial_\beta \bar{X}}{2X'^{i}X^{i'}} \right)
\] (67)

Now note that the coordinates \( X^a \) are defined in a Lorentz plane with metric \( \eta_{ab} = (1, -1) \). The string world sheet, in its ground state is parallel with the Lorentz plane. When the world sheet metric is an independent field, one has to introduce the tangent space at every point on the world sheet. The tangent space is locally Lorentzian. The coordinates, \( X^0 \) and \( X^1 \) are referred to these coordinates. Let \( e_\alpha \) and \( e_a \) be the bases at a point on the world sheet and the tangent space at that point, respectively. The vierbein \( \Lambda^\alpha_a \) and its inverse connect the two bases,

\[
e_\alpha = \Lambda^\alpha_a e_a.
\] (68)
The inverse of $\Lambda_a^\alpha$ will be denoted by $\Lambda_a^{\alpha}$.

The vierbeins may be used to factorize the metric,

$$h^{\alpha\beta} = \Lambda_\alpha^a \Lambda_\beta^b \delta^{ab}$$ (69)

We now give special attention to the part of the Lagrangian where the vierbeins explicitly appear in the theory. It is that part of the Lagrangian which is special to the non relativistic theory having no relativistic analog. Now, both $X$ and $\bar{X}$ are world sheet scalars. So we can easily derive the following relations

$$\partial_\alpha X = \Lambda_\alpha^a \partial_a X = \Lambda_\alpha^a \left( \delta_0^a + \delta_1^a \right) = e_\alpha$$

$$\partial_\alpha \bar{X} = \Lambda_\alpha^a \partial_a \bar{X} = \Lambda_\alpha^a \left( \delta_0^a - \delta_1^a \right) = \bar{e}_\alpha$$ (70)

where

$$e_\alpha = \Lambda_0^\alpha + \Lambda_1^\alpha$$

$$\bar{e}_\alpha = \Lambda_0^\alpha - \Lambda_1^\alpha$$ (71)

Now we have all the intermediate quantities. We can then write the expression in terms of the basis vectors,

$$\mathcal{L}_I = \frac{1}{2} \sqrt{-h} h^{\alpha\beta} \partial_\alpha X^k \partial_\beta X^k + \alpha \left( \frac{1}{\sqrt{-h} h^{00}} + \epsilon^{\alpha\beta} e_\alpha \bar{e}_\beta \right) + \beta \left( -\frac{h^{01}}{h^{00}} - \frac{\sigma^{\alpha\beta} e_\alpha \bar{e}_\beta}{2X'X'} \right)$$ (72)

The strig action in the form [72] clearly reveals the connection with the world sheet geometry. The deduction of this from the canonical analysis of the model [8] is indeed remarkable.

6 Conclusion

Nonrelativistic string theories (NRST) have recently come to prominence in the literature [7, 5, 4, 3]. Just as recent studies of nonrelativistic field theories have emphasised the role of geometry to tackle the issue of coupling nonrelativistic field theories with gravity, studies in string theories have raised new questions about geometry. But nay, the NRSTs are interesting even in flat space. Similar to their relativistic counterpart, different actions have been proposed. Broadly we can divide these in two classes – minimal action which contains the world sheet area swept by the string as the Lagrangian [5] and more redundant form of action where the metric elements on the world sheet [3] are considered as independent fields. The former is comparable with the Nambu - Goto type and the latter with the Polyakov type in relativistic strings. But there is one significant difference. In NRSTs in the second type there are two extra fields on the world sheet. In relativistic field theories of Polyakov type one can easily substitute the metric components from their equations of motion to get the Nambu - Goto string. This action level
correspondence is not apparent in case of the NRST theories. We have derived a completely new action in this paper which can be identified with the Polyakov type. Thereby, we regain the action level correspondence in NRSTs.

Another remarkable aspect is the role of Hamiltonian analysis in the formulation of the new action. We have started from the minimal action. To facilitate the introduction of geometry on the world sheet of the string we have carried out a comprehensive canonical analysis of the model. The action is then enriched by the introduction of the constraints in the Lagrangian by the Lagrange multiplier technique and lifting the status of the multipliers to independent fields. Eliminating the reference to the phase space variables by the inverse Legendre procedure, the desired action is obtained. A surprising connection of the new fields with the Arnowit - Deser - Misner construction in general relativity emerged, whereby geometry was introduced in the theory. It was then an easy journey towards the Polyakov type action. Remarkably, the two extra fields appeared spontaneously in the process. We have provided a detailed canonical analysis of the new action. Its phase space structure has been studied. Throughout the paper symmetries of the different actions have been investigated from the canonical point of view and the interpolating Lagrangian is no exception. This analysis has been used to deduce further geometrical connections.

The interplay of canonical analysis and geometry, as evidenced here, brings out a clear picture of the connection of canonical analysis with the metric of the world sheet. Thus the possible use of the action obtained here in case of coupling with gravity appears to be feasible. One has to find out the ways to relate the background curvature These and other issues may open up further research.

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