Neumann-Rosochatius system for (m,n)
string in $AdS_3 \times S^3$ with mixed flux

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Abstract: $(m,n)$ string in $AdS_3 \times S^3 \times T^4$ with mixed three-form fluxes can be
described by an integrable deformation of an one-dimensional Neumann-Rosochatius
(NR) system. We study general class of rotating and pulsating solutions in La-
grangian and Hamiltonian formulation. For the rotating string, the explicit solutions
can be expressed in terms of elliptic functions. We compute integrals of motion and
find out the scaling relation among conserved charges for the particular case of con-
stant radii solutions. Then we study the closed $(m,n)$ string pulsating in $R \times S^3$.
We find the string profile and calculate the total energy of such pulsating sting in
terms of oscillation number $(N)$ and angular momentum $(J)$.

Keywords: AdS/CFT correspondence, Semiclassical string
1 Introduction

Planar integrability on both sides of the famous AdS/CFT correspondence [1–3] has been proved to be very promising technique to unravel a deeper understanding in the study of string spectrum in different semisymmetric superspaces [4]. Minahan and Zarembo in [5] first established the matching between the one-loop dilatation operators of the $SU(2)$ sector of $\mathcal{N} = 4$ SYM theory with the Hamiltonian of integrable $SO(6)$ spin chain model [6, 7]. Over the last few years, $AdS_3 \times S^3 \times M^4$ geometry supported by mixed three form fluxes (both NS-NS and R-R) has been studied as a nice tool in the context of classical integrability in $AdS_3/CFT_2$ correspondence [8–14]. The appearance of integrability as a symmetry in Green-Schwarz action of type IIB string in compactified $AdS_3 \times S^3 \times M^4$ background with mixed R-R and NS-NS three form flux has put forth a renewed interest to analyse the $AdS_3/CFT_2$ duality in the presence of mixed fluxes by means of well known classical integrable models.

In proving the AdS/CFT duality in large charge limit, varieties of rigidly rotating strings have been studied in different backgrounds to enhance our understanding of the dual states in the gravity side. Among them special cases on sphere consist
of giant magnon\cite{15} and spiky string\cite{16} solutions which are dual to elementary excitation with large momentum $p$ and higher twist operators respectively in the field theory side of the duality. Also in\cite{17} an exact correspondence was proposed between string states and some dual field theory operators by considering spinning as well as pulsating string solutions, the later first developed in\cite{18}. Despite being more stable solutions these types of string state has not been explored enough as compared to the rigidly rotating strings.

A general description of the finite gap solutions of classically integrable string sigma model has been demonstrated in\cite{19, 20} in terms of solutions of certain integrable models. In this regard, it is worth emphasizing that a generalised ansatz for string coordinates \cite{21–23} provides a method to reformulate a large class of string configurations in terms of solutions of very well-known one dimensional integrable Neumann or Neumann-Rosochatius (NR) system. The Neumann integrable model describes harmonic oscillator restricted to move on a sphere whereas NR system is an integrable extension of the former with an additional centrifugal potential barrier of the form $\frac{1}{r^2}$. These mechanical models have been proved to be very effective to deal with the problem of geodesics on ellipsoid or equivariant harmonic maps into sphere\cite{24–27}. Earlier, in the context of classical integrability, classical giant magnon \cite{21–23} and spiky string\cite{28–30} solutions on $AdS_5 \times S^5$ was obtained by using NR model approach. Moreover, there are several recent works \cite{31–37} which reveals that general solutions corresponding to closed rotating and pulsating string in various 10D AdS/S compact spaces with pure or mixed flux may be constructed from systematic analysis of one dimensional Lagrangian of NR system with integrable deformation. Closed circular type solutions for rotating string with constant radii\cite{9} and their finite gap effects\cite{32} has been confirmed to be possible to achieve with this method in hand. Also there are very recent observations\cite{37} on energy-angular momenta relations for pulsating string in terms of oscillation number, which is credibly supported by NR model.

In a seminal paper, J. H. Schwarz \cite{38} had constructed an $SL(2, \mathbb{Z})$ multiplet of string-like solutions in type IIB supergravity starting from the fundamental string solution. It is also known that the equations of motion of type IIB theory are invariant under an $SL(2, \mathbb{R})$ rotation group. This enables one to generate new supergravity solutions in type IIB theory by applying this rotation to known solutions in supergravity. A discrete subgroup $SL(2, \mathbb{Z})$ of this $SL(2, \mathbb{R})$ group has been conjectured later to be the exact symmetry group of the type IIB string theory. Till date, a lot of work has been devoted for constructing various string-like as well as five-brane solutions of type IIB supergravity equations using $SL(2, \mathbb{Z})$ invariance of low energy effective action of this theory\cite{38–44}. In the near horizon limit, an $SL(2, \mathbb{Z})$ transformed bound state solution of $Q_5$ NS5-branes and $Q_1$ F-strings, gives rise to
AdS$_3 \times S^3$ background with mixed three form fluxes with integer charges. It has also been recently shown that the $SL(2, \mathbb{Z})$-transformation and the near horizon limit commute. This allows us to map the $(m, n)$ string action in $AdS_3 \times S^3$ background with mixed three form fluxes to $(m', n')$ string action in $AdS_3 \times S^3$ background with NS-NS two form flux using the relation

$$m' = \begin{pmatrix} m' \\ n' \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix},$$

(1.1)

where $a, b, c$ and $d$ are the integer entries of $SL(2, \mathbb{Z})$ matrix with $ad - bc = 1$. Interesting analysis of this fact has so far been put forth with $(m, n)$ string action as natural probe in various backgrounds\cite{43–47}. In \cite{46}, some fascinating observations on giant magnon and single spike solutions for rigidly rotating string along with the analysis of pulsating string as probe $(m, n)$ string in mixed flux background have been performed by mapping it into $(m', n')$ string in $AdS_3 \times S^3$ background with NSNS two form flux.

Motivated by such analysis, we cast our focus on the manifest $SL(2, \mathbb{Z})$ covariant action of probe $(m, n)$ string in $AdS_3 \times S^3 \times T^4$ background considering the presence of both the three form NSNS and RR fluxes using general rotating and pulsating string ansatz in the light of classical integrability of both sides of the correspondence. In this article we wish to construct integrable deformed NR system and utilize the solutions of integrable equations of motion of that system to obtain the relations between conserved charges for both rotating and pulsating $(m, n)$ string. The rest of the paper is organized as follows. In section 2, we present the construction of deformed Lagrangian and Hamiltonian of integrable NR model for closed $(m, n)$-string rotating in $R_t \times S^3$ background in the presence of mixed three form fluxes. Also we discuss for corresponding Uhlenbeck integrals of motion for such system. We derive elliptic solutions for rotating $(m, n)$ string which resembles the circular-type string solutions. We further study of conserved charges and energy-angular momenta dispersion relation for constant radii case for the closed rotating string. Section 4 is devoted to the study of $(m, n)$ type pulsating string in mixed flux background from the solutions of the modified NR model. We study the string profile and compute small energy correction to the scaling relation in terms of oscillation number in short string limit. Finally, we conclude our results in section 4.
2 Modified NR model for rotating \((m, n)\) string in \(AdS_3 \times S^3\) with mixed flux

We start by writing down the background metric and associated NS-NS and R-R fluxes for \(AdS_3 \times S^3\) background

\[
\begin{align*}
    ds^2 &= L^2 \left[ ds^2_{\tilde{AdS}_3} + ds^2_{S^3} \right] \\
    H &= 2L^2 (\tilde{\epsilon}_{AdS_3} + \epsilon_{S^3}), \quad L^2 = r_5^2 \\
    e^{-2\phi} &= \frac{r_1^2}{g_s^2 r_5^2}, \quad r_1^2 = \frac{16\pi^4 g_s^2 \alpha'^3 Q_1}{V_4}, \quad r_5^2 = Q_5 \alpha', \quad V_4 = (2\pi)^4 \alpha'^2 v
\end{align*}
\]  

(2.1a) (2.1b) (2.1c)

Here \(ds^2_{\tilde{AdS}_3}\) is the line element of the corresponding background in terms of dimensionless variables. In this section, we wish to study the dynamics of \((m, n)\) string in general background with mixed three form flux. On the other hand, the action of \((m, n)\) string in general background with metric \(g_{MN}\) and flux \(B_{MN}\) is given by

\[
S_{m,n} = -T_{D1} \int d\tau d\sigma \left[ \sqrt{\mathbf{M}} \mathcal{M}^{-1} \mathbf{m} \sqrt{-\det g_{MN} \partial_\alpha x^M \partial_\beta x^N - e^{\alpha\beta} \mathbf{m}^T \mathbf{B}_{MN} \partial_\alpha x^M \partial_\beta x^N} \right],
\]

(2.2)

with \(g_{MN}\) to be the Einstein frame metric which is invariant under \(SL(2, \mathbb{Z})\) transformations. This action was derived in [44] by applying \(SL(2, \mathbb{Z})\) covariance of the action of \(n\) coincident D1-branes in general background. The vector \(\mathbf{B}\) is defined as

\[
\mathbf{B} = \begin{pmatrix} B^{(2)} \\ C^{(2)} \end{pmatrix}
\]

(2.3)

where \(B^{(2)}\) and \(C^{(2)}\) are NS-NS and R-R two form fluxes respectively. It transforms under \(SL(2, \mathbb{Z})\) transformation as

\[
\hat{\mathbf{B}} = (\Lambda^T)^{-1} \mathbf{B}
\]

(2.4)

where \(\Lambda\) is given by

\[
\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{with} \quad ad - bc = 1
\]

(2.5)

It is convenient to introduce a complex field \(\tau = \chi + ie^{-\phi}\) containing both the axion scalar \(\chi\) and dilaton \(\phi\) corresponding to NS-NS and R-R sectors respectively. With this complex field, we introduce a matrix in the following form:

\[
\mathcal{M} = e^\phi \begin{pmatrix} \tau^* & \chi \\ \chi & 1 \end{pmatrix} = e^\phi \begin{pmatrix} \chi^2 + e^{-2\phi} \chi \\ \chi \end{pmatrix}, \quad \det \mathcal{M} = 1
\]

(2.6)

which transforms under \(SL(2, \mathbb{Z})\) transformation as,

\[
\hat{\mathcal{M}} = \Lambda \mathcal{M} \Lambda^T.
\]

(2.7)
On the other hand, the matrix $m$ in the action \((2.2)\) is expressed as

\[
m = \begin{pmatrix} m \\ n \end{pmatrix}
\]  \hspace{1cm} (2.8)

where $m$ and $n$ are the integers counting for the number of fundamental string and D1-brane respectively. It transforms under $SL(2, \mathbb{Z})$ as $\hat{m} = \Lambda m$ whereas the $SL(2, \mathbb{Z})$ transformation for $B$ and $\mathcal{M}$ reads as equations \((2.4)\) and \((2.7)\) respectively. Hence this action becomes manifestly $SL(2, \mathbb{Z})$ covariant. Using \((2.3)\) and \((2.6)\) we have

\[
m^T \mathcal{M}^{-1} m = m_i^T (\mathcal{M}^{-1})^{ij} m_j = e^\phi \left[ (m - n\chi)^2 + n^2 e^{-2\phi} \right],
\]

\[
m^T B = m_i B^i = m B_{MN}^{(2)} + n C_{MN}^{(2)}
\]  \hspace{1cm} (2.9)

Here $m, n$ are the numbers of fundamental string coupled to the NS-NS flux and D1 branes coupled to the R-R flux respectively. Therefore substitution of equation \((2.9)\) reduces the action as

\[
S_{m,n} = - T_{D1} \sqrt{\left( m - n\chi \right)^2 + n^2 e^{-2\phi}} \int d\tau d\sigma \sqrt{-\det G_{\alpha\beta} \partial_{\alpha} x^M \partial_{\beta} x^N}
\]

\[
+ \frac{T_{D1}}{2} n \epsilon^{\alpha\beta} \int d\tau d\sigma B_{MN}^{(2)} \partial_{\alpha} x^M \partial_{\beta} x^N + \frac{T_{D1}}{2} m \epsilon^{\alpha\beta} \int d\tau d\sigma C_{MN}^{(2)} \partial_{\alpha} x^M \partial_{\beta} x^N
\]  \hspace{1cm} (2.10)

It is to be noted that this action is taken in string frame with metric $G_{MN}^{\text{string}}$ given by $G_{MN}^{\text{string}} = e^{\frac{\phi}{2}} g_{MN}^{E}$. The nonlinearity of this action due to the presence of the square root of the determinant results in the equations of motion which are not desirable. To avoid such discrepancy it may be modified by introducing an auxiliary metric $h_{\alpha\beta}$. With such consideration, the action becomes

\[
S_{m,n} = - \tau_{(m,n)} \int d\tau d\sigma \sqrt{-h} h^{\alpha\beta} G_{\alpha\beta} + q_{(m,n)} \epsilon^{\alpha\beta} \int d\tau d\sigma B_{MN}^{(2)} \partial_{\alpha} x^M \partial_{\beta} x^N
\]

\[
+ \tilde{q}_{(m,n)} \epsilon^{\alpha\beta} \int d\tau d\sigma C_{MN}^{(2)} \partial_{\alpha} x^M \partial_{\beta} x^N
\]  \hspace{1cm} (2.11)

where,

\[
\tau_{(m,n)} = T_{D1} \sqrt{\left( m - n\chi \right)^2 + n^2 e^{-2\phi}}, \quad q_{(m,n)} = m \frac{T_F}{g_s}, \quad \tilde{q}_{(m,n)} = n T_{D1}
\]  \hspace{1cm} (2.12)

where tensions of D1-string and fundamental string, respectively, are

\[
T_{D1} = \frac{1}{2\pi \alpha'} g_s, \quad T_F = \frac{1}{2\pi \alpha'}
\]  \hspace{1cm} (2.13)

with $g_s$ being the string coupling. From this action equation of motion for $h_{\alpha\beta}$ may be obtained as

\[
T_{\alpha\beta} = - \frac{2}{\sqrt{-h}} \partial S_{m,n} \partial h_{\alpha\beta} = \frac{1}{2} h_{\alpha\beta} h^{\rho\xi} G_{\rho\xi} - G_{\alpha\beta} = 0
\]  \hspace{1cm} (2.14)
which gives nothing but tracelessness of the stress-energy tensor and can reproduce the action (2.10) from (2.11) by inserting $h_{\alpha\beta} = G_{\alpha\beta}$. Equation of motion for $x^M$ can be easily found as

$$
-2\partial_\alpha \left[ \sqrt{-h} h^{\alpha\beta} G_{MN} \partial_\beta x^N \right] + \sqrt{-h} h^{\alpha\beta} \partial_\alpha x^K \partial_\beta x^L \partial_M G_{KL} 
+ 2q_{(m,n)} H_{MKL} \partial_\tau x^K \partial_\sigma x^L + 2\tilde{q}_{(m,n)} F_{MKL} \partial_\tau x^K \partial_\sigma x^L = 0 \tag{2.15}
$$

where,

$$
H_{MNK} = \partial_M B^{(2)}_{NK} + \partial_N B^{(2)}_{MK} + \partial_K B^{(2)}_{MN}, \quad F_{MKL} = \partial_M C^{(2)}_{NK} + \partial_N C^{(2)}_{KM} + \partial_K C^{(2)}_{MN} \tag{2.16}
$$

In what follows, we will use the metric for $AdS_3 \times S^3$ background in terms of the global background coordinates,

$$
\text{ds}^2 = L^2 \left[ -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\phi^2 + d\theta^2 + \sin^2 \theta d\phi_1^2 + \cos^2 \theta d\phi_2^2 \right]. \tag{2.17}
$$

The accompanying flux components are given by

$$
B^{(2)}_{t\phi} = L^2 q^2 \sinh^2 \rho, \quad B^{(2)}_{\phi_1\phi_2} = -L^2 q^2 \cos^2 \theta \\
C^{(2)}_{t\phi} = L^2 \sqrt{1 - q^2} \sinh^2 \rho, \quad C^{(2)}_{\phi_1\phi_2} = -L^2 \sqrt{1 - q^2} \cos^2 \theta \tag{2.18}
$$

where $q$ and $\sqrt{1 - q^2} = \tilde{q}$, say, are the parameters associated to field strengths of NS-NS and R-R fluxes respectively. It satisfies, $0 \leq q \leq 1$ and $q^2 + \sqrt{1 - q^2} = 1$. For $q = 0$, it is a case of pure RR flux and the worldsheet theory can be described in terms of a Green-Schwarz coset [48–50]. On the other hand, for $q = 1$ it corresponds to pure NS-NS background, where the theory can be described in terms of a class of supersymmetric WZW model. For intermediate value of $q$ the theory has not been completely understood till date.

### 2.1 Lagrangian and Hamiltonian formulation

To specify the geometry of $AdS_3$ and $S_3$ it is convenient to consider the embedding coordinates to be $Y_i$’s and $W_i$’s respectively. These are related to the global coordinates as

$$
Y_1 + iY_2 = \sinh \rho e^{i\phi}, \quad Y_3 + iY_0 = \cosh \rho e^{it} \tag{2.19} \\
W_1 + iW_2 = \sin \theta e^{i\phi_1}, \quad W_3 + iW_4 = \cos \theta e^{i\phi_2} \tag{2.20}
$$

The relations (2.19) and (2.20) satisfy the following constraints[51]

$$
-Y_0^2 + Y_1^2 + Y_2^2 - Y_3^2 = -1 \tag{2.21} \\
W_1^2 + W_2^2 + W_3^2 + W_4^2 = 1 \tag{2.22}
$$

which precisely define the geometry of $AdS_3$ and $S_3$ respectively. As we are interested in the dynamics in $R_t \times S_3$, we take $Y_1 = Y_2 = 0$ so that $Y_3 + iY_0 = e^{it}$. Taking
\( r_1(\xi) = \sin \theta, r_2(\xi) = \cos \theta, \Phi_1(\xi) = \phi_1 \) and \( \Phi_2(\xi) = \phi_2 \) with \( \xi = \alpha \sigma + \beta \tau \), the Lagrangian of \((m, n)\) string in \( R_t \times S^3 \) geometry with mixed three form fluxes becomes

\[
\mathcal{L} = -\frac{\tau_{(m,n)} L^2}{2} \left[ (\partial_{\tau} r_1)^2 - (\partial_{\tau} r_1)^2 + (\partial_{\sigma} r_1)^2 - (\partial_{\sigma} r_1)^2 - (\partial_{\tau} r_2)^2 - (\partial_{\tau} r_2)^2 \right]
\]

\[
- \frac{\tau_{(m,n)} L^2}{2} \left[ r_1^2 \left\{ (\partial_{\tau} \Phi_1)^2 - (\partial_{\tau} \Phi_2)^2 \right\} + r_2^2 \left\{ (\partial_{\sigma} \Phi_2)^2 - (\partial_{\sigma} \Phi_2)^2 \right\} \right] + \frac{\Lambda}{2} \left( r_1^2 + r_2^2 - 1 \right)
\]

\[
- q_{(m,n)} q L^2 \cos^2 \theta \left[ (\partial_{\tau} \Phi_1)(\partial_{\tau} \Phi_2) - (\partial_{\tau} \Phi_1)(\partial_{\tau} \Phi_2) \right]
\]

where \( \Lambda \) being the Lagrange multipliers. Let us consider the following parametrization

\[
t = \kappa \tau, \quad r_a = r_a(\xi), \quad \Phi_a = \Phi_a(\xi) = \omega_a \tau + f_a(\xi), \quad \xi = \alpha \sigma + \beta \tau.
\]

where \( \kappa, \omega_a, \alpha \) and \( \beta \) are constants. For closed strings \( r_a \) and \( f_a \) should follow periodic conditions

\[
r_a(\xi + 2\pi \alpha) = r_a(\xi), \quad f_a(\xi + 2\pi \alpha) = f_a(\xi) + 2\pi \tilde{m}_a,
\]

\( \tilde{m}_a \) being integer winding numbers. Using such parametrization for the string rotating in \( R_t \times S^3 \), the Lagrangian becomes

\[
\mathcal{L} = -\frac{\tau_{(m,n)} L^2}{2} \left[ \kappa^2 + (\alpha^2 - \beta^2) \sum_{a=1}^{2} \left( r_a'^2 + r_a^2 \left( f_a' - \frac{\beta \omega_a}{\alpha^2 - \beta^2} \right)^2 - \frac{\alpha^2 \omega_a^2 r_a^2}{(\alpha^2 - \beta^2)^2} \right) \right]
\]

\[
- \left( q_{(m,n)} q + \tilde{q}_{(m,n)} \sqrt{1 - q^2} \right) L^2 \alpha^2 r_2^2 \left( \omega_2 f_1' - \omega_1 f_2' \right) + \frac{\Lambda}{2} \left( r_1^2 + r_2^2 - 1 \right),
\]

where derivatives with respect to \( \xi \) is denoted by prime. Equations of motion for \( f_a \)'s can be calculated as

\[
f_a' = \frac{1}{\alpha^2 - \beta^2} \left[ \frac{C_a}{r_a^2} + \beta \omega_a + \frac{Q \alpha^2 r_a^2 \omega_b}{r_a^2} \epsilon_{ba} \right]
\]

where \( \epsilon_{ba} = 1 \) and \( C_a \)'s are suitable integration constants. Also

\[
Q = \frac{q_{(m,n)} q + \tilde{q}_{(m,n)} \sqrt{1 - q^2}}{\tau_{(m,n)}} = \frac{mq + n \sqrt{1 - q^2}}{\sqrt{(m - n \chi)^2 + n^2 e^{-2\phi}}}
\]

where we have used (2.12) and (2.13). Putting this expression of \( f_a' \) in the Lagrangian (2.26) we achieve,

\[
\mathcal{L} = -\frac{\tau_{(m,n)} L^2}{2} \left[ \kappa^2 + (\alpha^2 - \beta^2) \sum_{a=1}^{2} r_a'^2 + \frac{1}{(\alpha^2 - \beta^2)^2} \sum_{a=1}^{2} \left( \frac{C_a^2 + Q \alpha^4 r_a^2 \omega_b^2}{r_a^2} \right) \right]
\]

\[
+ \frac{\tau_{(m,n)} L^2}{2} \left[ \frac{\alpha^2}{\alpha^2 - \beta^2} \sum_{a=1}^{2} \left( \omega_2^2 r_a^2 + 2 C_a Q \omega_b r_a^2 \epsilon_{ba} \right) \right] + \frac{\Lambda}{2} \left( r_1^2 + r_2^2 - 1 \right) - Q \tau_{(m,n)} \frac{\alpha^2 L^2 r_2^2}{\alpha^2 - \beta^2} \left[ \omega_2 C_1 - \frac{C_2 \omega_1}{r_2^2} + \frac{Q \alpha^2 (\omega_2^2 r_1^2 + \omega_2^2 r_2^2)}{r_1^2} \right].
\]
The Lagrangian contains both $r_1^2$ and $\frac{1}{r_2^4}$-terms representing harmonic oscillator type potential and centrifugal potential barrier respectively. Hence it can be identified as the Lagrangian of one dimensional integrable Neumann-Rosochatius system with extra terms due to presence of mixed flux. We can get the equations of motion for $r_1$ and $r_2$ as
\[ r''_1 = r_1 \left[ \left( f'_1 - \frac{\beta \omega_1}{\alpha^2 - \beta^2} \right)^2 - \frac{\alpha^2 \omega_1^2}{(\alpha^2 - \beta^2)^2} \right] + Ar_1, \tag{2.30} \]
and
\[ r''_2 = r_2 \left[ \left( f'_2 - \frac{\beta \omega_2}{\alpha^2 - \beta^2} \right)^2 - \frac{\alpha^2 \omega_2^2}{(\alpha^2 - \beta^2)^2} \right] + Ar_2 + \frac{2Q\alpha^2 r_2}{(\alpha^2 - \beta^2)^2} \left( \omega_2 f'_1 - \omega_1 f'_2 \right), \tag{2.31} \]
where $A = \frac{\Lambda}{\tau(m,n)^2}$ is a constant. Putting the expressions of $f_a$’s from (2.27) in equations (2.30) and (2.31) we achieve,
\[
(\alpha^2 - \beta^2) r''_1 + \frac{1}{(\alpha^2 - \beta^2)^2} \left[ \frac{C_1^2}{r_1^3} + \frac{Q^2 \alpha^4 r_2^2 \omega_2^2}{r_1^3} \right] + \frac{\alpha^2}{(\alpha^2 - \beta^2)^2} \omega_1 r_1 - Ar_1
+ \frac{2Q\alpha^2 r_2}{(\alpha^2 - \beta^2)^2} \left[ \frac{\omega_1 r_1}{r_1^3} + \frac{Q\alpha^2 r_2^2 \omega_2^2}{r_1^3} \right] = 0, \tag{2.32}
\]
and
\[
(\alpha^2 - \beta^2) r''_2 - \frac{2Q^2 \alpha^4 r_2^3 \omega_2^2}{r_1^2 (\alpha^2 - \beta^2)} - \frac{Q^2 \alpha^4 r_2 \omega_1^2}{(\alpha^2 - \beta^2)^2} + \frac{1}{(\alpha^2 - \beta^2)^2} \frac{C_2^2}{r_2^3} + \frac{\alpha^2}{(\alpha^2 - \beta^2)^2} \omega_2 r_2
+ \frac{2Q\alpha^2}{(\alpha^2 - \beta^2)} (\omega_2 C_1 - \omega_1 C_2) r_2 - Ar_2 - \frac{2Q\alpha^2}{(\alpha^2 - \beta^2)^2} \left[ \frac{\omega_1 r_1}{r_1^3} + \frac{2Q\alpha^2 r_2^3 \omega_2^2}{r_1^3} + \frac{Q\alpha^2 \omega_2^2 r_2^2}{r_1^3} \right] = 0. \tag{2.33}
\]

Above two equations can also be obtained from the Euler-Lagrange equations of the following equivalent Lagrangian of an integrable NR system
\[
\mathcal{L} = \frac{(\alpha^2 - \beta^2)}{2} \sum_{a=1}^{2} r'_a - \frac{1}{2(\alpha^2 - \beta^2)} \sum_{a=1}^{2} \left( \frac{C_a^2 + Q^2 \alpha^4 r_2^2 \omega_2^2}{r_a^2} \right)
+ \left[ \frac{\alpha^2}{2(\alpha^2 - \beta^2)} \sum_{a=1}^{2} (\omega_a^2 r_a^2 + 2C_a Q\omega_2 r_2^2 \epsilon_{ba}) \right] + \frac{A}{2} (r_1^2 + r_2^2) \tag{2.34}
- \frac{Q\alpha^2 r_2^2}{2(\alpha^2 - \beta^2)} \left[ \frac{\omega_2 C_1}{r_1^2} - \frac{C_2 \omega_1}{r_2^2} + \frac{Q\alpha^2 (\omega_1^2 r_1^2 + \omega_2^2 r_2^2)}{r_1^2} \right]
\]
for general coordinates $r_1$ and $r_2$. The Virasoro constraints may be expressed as
\[ T_{r\tau} + T_{\tau\sigma} = 0, \quad T_{\tau\tau} = T_{\tau\sigma} = 0. \tag{2.35a} \]
With the embedding (2.21) and (2.22) and using the ansatz (2.24), the constraints reduce to
\[ (x^2 + \beta^2)(r_1^2 + r_2^2) + r_1^2 \left[ (\omega_1 + \beta f_1)^2 \right] + r_2^2 \left[ (\omega_2 + \beta f_2)^2 \right] = \kappa^2 \] (2.36)
and
\[ \alpha \beta (r_1^2 + r_2^2) + \alpha \left[ \omega_1 r_1 f_1 + \omega_2 r_2 f_2 \right] + \alpha \beta \left[ f_1^2 r_1^2 + f_2^2 r_2^2 \right] = 0 \] (2.37)
respectively. The Hamiltonian of such a system can be written as
\[ \mathcal{H} = \frac{(\alpha^2 - \beta^2)}{2} \sum_{a=1}^{2} r_a^2 = \frac{1}{2} \sum_{a=1}^{2} \left( \frac{C_a^2 + Q^2 \alpha^4 r_a^4 \omega_a^2}{r_a^2} \right) \]
\[ - \frac{\alpha^2}{2 (\alpha^2 - \beta^2)} \sum_{a=1}^{2} \left( \omega_a^2 r_a^2 + 2C_a Q \omega_b r_b^2 \epsilon_{ba} \right) \]
\[ \frac{Q \alpha^2 r_a^2}{2 (\alpha^2 - \beta^2)} \left[ \frac{\omega_2 C_1}{r_1^2} - \frac{C_2 \omega_1}{r_2^2} + \frac{Q \alpha^2 (\omega_1^2 r_1^2 + \omega_2^2 r_2^2)}{r_1^2} \right] \] (2.38)
This again takes the form of the Hamiltonian of integrable one dimensional NR model. Hence it is obvious that classically integrable NR model may represent a nice tool to study for probe \((m, n)\) string in the desired background, even in the presence of flux.

### 2.2 Integrals of Motion

Any classically integrable system contains infinite number of conserved quantities, also known as integrals of motion, in involution, i.e., they must Poisson commute each other. The integrability of Neumann-Rosochatius system demands the existence of a set of integrals of motion, named as Uhlenbeck constants, in involution. These were first introduced by K. Uhlenbeck [24]. In the case of closed rotating string in \(S^3\) there are two integrals of motion \(I_1\) and \(I_2\) constrained by the relation \(I_1 + I_2 = 1\). A general form of Uhlenbeck integrals of motion for the case of closed rotating string can be written as
\[ I_a = \alpha^2 x_a x_a' + \sum_{b \neq a} \left| \frac{\tilde{x}_b p_a - x_a \tilde{p}_b}{\omega_a^2 - \omega_b^2} \right|^2, \] (2.39)
which eventually gives
\[ I_a = \alpha^2 \frac{r_a^2}{\omega_a^2} + (\alpha^2 - \beta^2) \sum_{b \neq a} \frac{(r_a' r_b' - r_a r_b')^2}{\omega_a^2 - \omega_b^2} + \sum_{b \neq a} \left[ \frac{1}{\omega_a^2 - \omega_b^2} \left( \frac{C_a^2 r_b^2}{r_a^2} + \frac{C_b^2 r_a^2}{r_b^2} \right) \right], \] (2.40)
for some arbitrary values of constants \(C_a\)’s. Here we have used the embeddings (2.21), (2.22) and ansatz (2.24) for closed string rotating on \(S^3\) with \(x_a(\xi) = r_a e^{i f_a(\xi)}\) [21]. To find out the integrals of motion in the deformed background, let us proceed with the methodology described in [32, 33] where a term \(g = g(r_1, r_2, Q)\) representing the
deformation due to fluxes is added without affecting the integrable properties of \( I_a \) such that,

\[
\bar{I}_a = \alpha^2 r_a^2 + (\alpha^2 - \beta^2) \sum_{b \neq a} \left( \frac{r'_a r'_b - r_a r'_b}{\omega_a^2 - \omega_b^2} \right)^2 + \omega_a^2 - \omega_b^2 \left( \frac{C_a^2 r_b^2}{r_a^2} + \frac{C_b^2 r_a^2}{r_b^2} \right) + \sum_{b \neq a} \frac{2g}{\omega_a^2 - \omega_b^2} .
\]

(2.41)

The function \( g \) may be derived by setting \( \bar{I}_a = 0 \). This will yield,

\[
g = (\omega_1^2 - \omega_2^2) r_1^a \alpha^2 - (\alpha^2 - \beta^2)(r_1 r_2^2 + r_2^2) - \left( \frac{C_1^2}{r_1^4} + \frac{C_2^2 r_2^2}{r_2^2} \right)
\]

(2.42)

Now, by using the constraints

\[
r_1^2 + r_2^2 = 1, \quad r_1 r_1' + r_2 r_2' = 0, \quad r_1 r_1'' + r_1^2 + r_2^2 + r_2 r_2'' = 0
\]

(2.43)

and the equations (2.30), (2.31) and (2.27), we get the integral of motion in the following form:

\[
\bar{I}_1 = \frac{\alpha^2 - \beta^2}{\omega_1^2 - \omega_2^2} \left( r_1 r_1' - r_1 r_2 \right)^2 + \frac{2}{\omega_1^2 - \omega_2^2} \left( \frac{C_1 - Q \alpha^2 r_2^2 \omega_2}{r_1} \right)^2 + \left( \frac{C_2 + Q \alpha^2 r_2^2 \omega_1}{r_2} \right)^2
\]

\[
- \frac{2\alpha^2}{\omega_1^2 - \omega_2^2} \left[ \left( 1 + \frac{2Q^2 \alpha^2 r_2^2}{r_1^2} \right) \left( \omega_1^2 r_1^2 + \omega_2^2 r_2^2 \right) + 2 Q r_2^2 \left( \frac{C_1 \omega_2}{r_1^2} - \frac{C_2 \omega_1}{r_2^2} \right) \right]
\]

\[
+ \frac{1}{\omega_1^2 - \omega_2^2} \left( \frac{C_1^2}{r_1^2} + \frac{C_2^2 r_2^4}{r_2^2} \right) .
\]

(2.44)

It is henceforth clear that in the absence of flux, the deformed constants satisfy the relation \( \sum_{a=1}^2 \bar{I}_a = 1 \).

### 2.3 Elliptic solutions for \( r_1 \) and \( r_2 \)

In this section we present the solutions for such system for \((m, n)\) string rotating in \( S^3 \) with two different angular momenta. In this context we can introduce three parameters on a sphere like \( \zeta_1, \zeta_2 \) and \( \zeta_3 \) to express the roots of the equation in ellipsoidal coordinate \( \zeta \)

\[
\frac{r_1^2}{\zeta - \omega_1^2} + \frac{r_2^2}{\zeta - \omega_2^2} = 0 .
\]

(2.45)

With angular frequencies \( \omega_1 < \omega_2 \), this equation is defined on a sphere while being invariant under \( r_\alpha \to \lambda r_\alpha \) [52]. The range of ellipsoidal coordinate is \( \omega_1^2 \leq \zeta \leq \omega_2^2 \) for which \( \zeta \) covers \( 1 \text{th} \) of a sphere corresponding to \( r_i \geq 0 \) and the whole sphere may be thought of as a covering of the domain of \( \zeta \) having branches along its boundary. If
we enter into the equations of motion we may get a set of second order differential equations for \( \zeta \). But for the sake of simplicity, we inject this ellipsoidal coordinates \( \zeta \) into the Uhlenbeck integrals of motion \([22, 23]\) and can have a first order differential equation for \( \zeta \). Differentiating equation (2.45) we find

\[
\frac{r_1^2}{\omega_1^2 - \omega_2^2} = \frac{\zeta - \omega_2^2}{\omega_1^2 - \omega_2^2}, \quad \frac{r_2^2}{\omega_1^2 - \omega_3^2} = \frac{\zeta - \omega_3^2}{\omega_1^2 - \omega_3^2}, \quad \left( r_1 r_2' - r_1' r_2 \right)^2 = \frac{\zeta'^2}{4(\omega_1^2 - \omega_2^2) (\omega_1^2 - \omega_3^2)}.
\]  

(2.46)

The integral of motion (2.44) in terms of the ellipsoidal coordinate takes the following form

\[
\bar{I}_1 = \frac{\alpha^2 - \beta^2}{\omega_1^2 - \omega_2^2} \frac{\zeta'^2}{4(\omega_1^2 - \zeta)(\zeta - \omega_2^2)} + \frac{2Q^2 \alpha^4 (\zeta - \omega_2^2)(\zeta + \omega_1^2 + \omega_2^2)}{(\omega_1^2 - \omega_2^2)(\zeta - \omega_2^2)} + \frac{2\alpha^2(\omega_1^2 - \zeta)}{(\omega_1^2 - \omega_2^2)} + \frac{2\alpha^2 \omega_3^2 Q}{(\omega_1^2 - \zeta)} + \frac{2\alpha^2}{\omega_1^2 - \omega_2^2} \left[ Q \omega_1^2 + \omega_2^2 (1 - Q) \right].
\]  

(2.47)

Solving for \( \zeta'^2 \) from this expression of deformed Uhlenbeck constant we explore that

\[
\zeta'^2 = -4P_3(\zeta), \quad P_3(\zeta) = \frac{2\alpha^2(1 - Q)}{\alpha^2 - \beta^2} \prod_{i=1}^{3} (\zeta - \zeta_i).
\]  

(2.48)

Here \( P_3(\zeta) \) is evidently a third order polynomial defining an elliptic curve \( s^2 + P_3(\zeta) = 0 \). Now changing the variables as

\[
\zeta = \zeta_2 + (\zeta_3 - \zeta_2) \eta^2
\]  

(2.49)

and substituting for \( \eta \) in equation (2.48) we get

\[
\eta(\xi) = cn \left( \alpha \xi \sqrt{\frac{2 (1 - Q) (\zeta_3 - \zeta_1)}{\alpha^2 - \beta^2}} + \xi_0, k \right),
\]  

(2.50)

where \( k = \frac{\zeta_3 - \zeta_2}{\zeta_3 - \zeta_1} \) is the elliptic modulus \( \xi_0 \) is the integration constant which may be set to zero by a rotation. This gives the expression for \( r_1 \) from equation (2.45) as

\[
r_1^2(\xi) = \frac{\zeta_3 - \omega_2^2}{\omega_3^2 - \omega_1^2} + \frac{\zeta_2 - \zeta_3}{\omega_2^2 - \omega_1^2} \operatorname{sn}^2 \left( \alpha \xi \sqrt{\frac{2 (1 - Q) (\zeta_3 - \zeta_1)}{\alpha^2 - \beta^2}}, k \right)
\]  

(2.51)

and similarly the expression for \( r_2 \) is

\[
r_2^2(\xi) = \frac{\omega_2^2 - \zeta_3}{\omega_3^2 - \omega_1^2} + \frac{\zeta_2 - \zeta_3}{\omega_2^2 - \omega_1^2} \operatorname{sn}^2 \left( \alpha \xi \sqrt{\frac{2 (1 - Q) (\zeta_3 - \zeta_1)}{\alpha^2 - \beta^2}}, k \right)
\]  

(2.52)

The range of elliptic modulus \( k \) should be \( 0 < k < 1 \) which eventually needs \( \zeta_1 < \zeta_2 < \zeta_3 \). Also we can achieve circular type solution for \( \omega_1^2 \leq \zeta_2, \omega_3 \leq \omega_2^2 \) in the codomain of equations (2.51) and (2.52) between 0 and 1 provided that no such restriction is needed for \( \zeta_1 \).
2.4 Conserved charges and dispersion relation

In this section we are interested in finding out the scaling relation among various conserved charges. The energy and angular momenta for this system are expressed as

\[ E = -\int d\sigma \frac{\partial L}{\partial (\partial_t \sigma)}, \quad J_a = \int d\sigma \frac{\partial L}{\partial (\partial_\tau \Phi_a)}, \]  

(2.53)

with \(a = 1, 2\). Therefore using the Lagrangian (2.23), conserved quantities may be expressed as

\[ E = 2\pi \tau (m, n) L^2 \kappa, \]

(2.54a)

\[ J_1 = \frac{1}{\alpha} \int_0^{2\pi} d\xi \tau (m, n) L^2 [r_1^2 (\omega_1^2 + \beta f_1') + Q\alpha r_2^2 f_1'], \]

(2.54b)

\[ J_2 = \frac{1}{\alpha} \int_0^{2\pi} d\xi \tau (m, n) L^2 [r_2^2 (\omega_2^2 + \beta f_2') + Q\alpha r_2^2 f_1']. \]

(2.54c)

2.5 Constant radii solutions

We are interested in deriving the constant radii solutions for the conserved energy and momenta which may be obtained by taking the limit \(\zeta_2 \to \zeta_3\) in the equations (2.30) and (2.31). These type of solutions have been constructed by using Neumann-Rosochatius integrable system in [32] for circular strings. With this limit, the radial coordinates become constant such as

\[ r_1 = \sqrt{\zeta_3 - \omega_1^2} = a_1, \quad \text{say and} \quad r_2 = \sqrt{\omega_2^2 - \zeta_3} = a_2, \quad \text{say.} \]

Then the derivatives of the angles become constant, thereby giving

\[ f_a = \tilde{m}_a \xi + f_{0a}, \]

(2.55)

where the integration constants \(f_{0a}\) can be chosen to be zero through a rotation and \(\tilde{m}_a = \frac{1}{\alpha^2 - \beta^2} \left[ \frac{C_a}{\alpha^2} + \beta \omega_a + \frac{Q\alpha^2 \omega_a \omega_b}{\alpha^2} \epsilon_{ba} \right]\) are assumed to be the constant integer windings of the string satisfying the closed string periodicity conditions. Also the Virasoro constraints (2.36), (2.37) and currents (2.54b), (2.54c) yield

\[ a_1^2 = \frac{\bar{m}_2 (\omega_2 + \beta \bar{m}_2)}{\bar{m}_2 (\omega_2 + \beta \bar{m}_2) - \bar{m}_1 (\omega_1 + \beta \bar{m}_1)}, \quad a_2^2 = \frac{\bar{m}_1 (\omega_1 + \beta \bar{m}_1)}{\bar{m}_1 (\omega_1 + \beta \bar{m}_1) - \bar{m}_2 (\omega_2 + \beta \bar{m}_2)}, \]

(2.56)

and

\[ \bar{m}_1 J_1 + \bar{m}_2 J_2 = 0. \]

(2.57)

These above relations along with the conserved quantities yield from (2.36),

\[ \frac{E^2}{\alpha^2} = (J_1 + J_2)^2 + \frac{(1 - w)^2}{w} J_1 J_2 - 2Q T (m, n) \bar{m}_1 (J_2 + J_1 w) \]

\[ + T (m, n) \left( \bar{m}_1 \bar{m}_2 - Q^2 \bar{m}_1^2 w \right) \left( \bar{m}_1 - \bar{m}_2 w \right) \left( \bar{m}_2 - \bar{m}_1 w \right), \]

(2.58)
where we have used \( T_{(m,n)} = 2\pi r_{(m,n)} L^2 \) and \( w = \frac{\omega_1 + \beta \bar{m}_1}{\omega_2 + \beta \bar{m}_2} \). To derive the dispersion relation it would be convenient to find out \( w \) in terms of windings \( \bar{m}_a \) and angular momenta \( J_a \). To do this let us write the reduced equations of motion for constant radii solutions as

\[
\bar{m}_1^2 - \frac{2\beta \omega_1 \bar{m}_1}{\alpha^2 - \beta^2} - \frac{\omega_1^2}{\alpha^2 - \beta^2} + A = 0 ,
\]

\[
\bar{m}_2^2 - \frac{2\beta \omega_2 \bar{m}_2}{\alpha^2 - \beta^2} - \frac{\omega_2^2}{\alpha^2 - \beta^2} + A + \frac{2Q\alpha^2}{(\alpha^2 - \beta^2)^2} (\omega_2 \bar{m}_1 - \omega_1 \bar{m}_2) = 0.
\]

Now adding equations (2.54b) and (2.54c), subtracting equation (2.60) from (2.59) and solving the resulting system of equations we get the relation

\[
\bar{m}_1^2 - \bar{m}_2^2 + \frac{2\beta}{\alpha^2 - \beta^2} \left[ \frac{\bar{m}_2}{w} (\omega_1 + \beta \bar{m}_1) - \beta \bar{m}_2 - \omega_1 \bar{m}_1 \right]
\]

\[
+ \left[ \frac{1}{w} (\omega_1 + \beta \bar{m}_1) - \beta \bar{m}_2 \right]^2 - \frac{\omega_1^2}{\alpha^2 - \beta^2} - \frac{2Q\alpha^2}{(\alpha^2 - \beta^2)^2} \left[ \frac{\bar{m}_1}{w} (\omega_1 + \beta \bar{m}_1) - \beta \bar{m}_2 \bar{m}_1 - \omega_1 \bar{m}_2 \right] = 0 ,
\]

where \( \omega_1 \) is expressed as

\[
\omega_1 = \frac{\bar{m}_1 w \left[ J - T_{(m,n)} Q \alpha (\bar{m}_1 - \bar{m}_2) \right] - \bar{m}_2 J}{T_{(m,n)} (\bar{m}_1 - \bar{m}_2)} - \beta \bar{m}_1 ,
\]

where \( J = J_1 + J_2 \) as the total angular momentum. Eliminating \( \omega_1 \) from these two relations we are left with a quartic equation of \( w \) given by

\[
(1 - w^2) (\bar{m}_1 w - \bar{m}_2)^2 J^2 - 4Q T_{(m,n)} \bar{m}_1 w (\bar{m}_1 w - \bar{m}_2) (\bar{m}_1 - \bar{m}_2) (1 - w^2)
\]

\[
+ \frac{w^2 Q^2 T_{(m,n)} ^2}{J^2} (\bar{m}_1 - \bar{m}_2)^2 \left[ 4 \bar{m}_1^2 (1 - w^2) (\bar{m}_1 - \bar{m}_2) + \bar{m}_2 (\bar{m}_2 - w) \right]
\]

\[
+ \frac{w^2 T_{(m,n)} ^2}{J^2} (\bar{m}_1 - \bar{m}_2)^3 (\bar{m}_1 + \bar{m}_2) (1 - Q^2) = 0 .
\]

Instead of solving this equation explicitly, it is convenient to get an approximate solution as a power series expansion in large \( \frac{J}{T_{(m,n)}} \) which is given by

\[
w = 1 + \frac{QT_{(m,n)} \bar{m}_2}{\bar{m}_1} (\bar{m}_1 - \bar{m}_2) - \frac{Q^2 T_{(m,n)} ^2}{J^2} \frac{\bar{m}_1^2}{2\bar{m}_2^2} (\bar{m}_1 - \bar{m}_2)^2 .
\]

where we have neglected the other higher order terms in the series expansion. Substituting equation (2.64) in equation (2.58) we get a relation between energy and angular momenta as follows

\[
E^2 = J^2 - 2QT_{(m,n)} \bar{m}_1 J + T_{(m,n)} \frac{\bar{m}_2^2}{\bar{m}_1} (\bar{m}_1 - \bar{m}_2) (1 - Q^2) - \frac{2Q^2 T_{(m,n)} ^2}{J} J_1 \bar{m}_1 \times
\]

\[
x (\bar{m}_1 - \bar{m}_2) \left[ \frac{\bar{m}_2}{\bar{m}_1} - \frac{QT_{(m,n)} \bar{m}_1}{J} \frac{\bar{m}_1^2}{2\bar{m}_2^2} (\bar{m}_1 - \bar{m}_2) \right] ,
\]

\[
(2.65)
\]
where we have considered $\alpha = 1$. For two equal angular momentum $J_1 = J_2$ (which sets $\tilde{m}_1 = -\tilde{m}_2 = \tilde{m}$), one gets

$$E^2 = J^2 - 4\pi T_{(m,n)} L^2 Q\tilde{m}J + 2\pi^2 T_{(m,n)}^2 L^4 Q^2 \tilde{m}^2$$  \hspace{1cm} (2.66)

where $T_{(m,n)} = 2\pi \tau_{(m,n)} L^2$. For large $J$, the above expression reduces to $E \approx J$. Such linear dependence of $E$ on $J$ stems for the identification of the string state with the gauge theory operator corresponding to the maximal-spin state of $XXX$ spin chain.

### 3 NR system for pulsating $(m,n)$ strings in $AdS_3 \times S^3$ with mixed flux

In this section, we study the circular closed string pulsating in $R_t \times S^3$ by using a deformed integrable one dimensional Neumann-Rosochatius model. We use the following embedding [36]

$$Y_3 + iY_0 = \cosh \rho e^{i\tau} = z_0(\tau) e^{i\theta_0(\tau)}, \hspace{1cm} (3.1a)$$

$$W_1 + iW_2 = \sin \theta e^{i\phi_1} = r_1(\tau) e^{i\Phi_1(\tau)} + \bar{m}_1 \sigma, \hspace{1cm} (3.1b)$$

$$W_3 + iW_4 = \cos \theta e^{i\phi_2} = r_2(\tau) e^{i\Phi_2(\tau)} + \bar{m}_2 \sigma. \hspace{1cm} (3.1c)$$

These will follow the relations between global and local coordinates as

$$\cosh \rho = z_0(\tau), \hspace{0.5cm} t = h_0(\tau),$$

$$\sin \theta = r_1(\tau), \hspace{0.5cm} \phi_1 = \Phi_1(\tau, \sigma) = f_1(\tau) + \bar{m}_1 \sigma;$$

$$\cos \theta = r_2(\tau), \hspace{0.5cm} \phi_2 = \Phi_2(\tau, \sigma) = f_2(\tau) + \bar{m}_2 \sigma. \hspace{1cm} (3.2)$$

Winding numbers $\bar{m}_a$’s are assumed along the $\sigma$ direction only to make the time-direction single-valued. With this embedding, the Lagrangian can now be expressed as,

$$\mathcal{L} = \frac{\tau_{(m,n)} L^2}{2} \left[ - \left( \dot{z}_0^2 + \dot{z}_0 \dot{h}_0 \right) + \sum_{a=1}^{2} \left( \dot{r}_a^2 + \dot{r}_a \dot{f}_a^2 - \dot{r}_a \bar{m}_a^2 \right) \right]$$

$$+ \left( q_{(m,n)} + \tilde{q}_{(m,n)} \sqrt{1 - q^2} \right) L^2 r_2^2 \left( \tilde{m}_2 \dot{f}_1 - \tilde{m}_1 \dot{f}_2 \right) - \frac{\Lambda}{2} \left( \sum_{a=1}^{2} r_a^2 - 1 \right) - \frac{\tilde{\Lambda}}{2} \left( z_0^2 + 1 \right), \hspace{1cm} (3.3)$$

where the derivative with respect to $\tau$ is denoted by dots. $\Lambda, \tilde{\Lambda}$ are suitable Lagrange multipliers.
3.1 Lagrangian and Hamiltonian formulation

Euler-Lagrange equations of motion for $z_0$ and $f_i$’s can be derived from the Lagrangian (3.3) as

\[ \ddot{z}_0 - \frac{C_0^2}{4z_0^3} + \ddot{A}z_0 = 0 \]  
\[ \dot{f}_1 = \frac{C_1}{r_1^2} - \frac{Qr_2^2\bar{m}_2}{r_1^2} \]  
\[ \dot{f}_2 = \frac{C_2}{r_2^2} + Q\bar{m}_1 , \]

where $\dot{h}_0 = \frac{C_0}{2z_0^2}$ and $\ddot{A} = \frac{\ddot{\Lambda}}{r_{(m,n)}^2}$. Putting the expressions for $\dot{f}_1$ and $\dot{f}_2$ in terms of $r_1$ and $r_2$ in the Lagrangian (3.3) we achieve,

\[
\mathcal{L} = \frac{\tau_{(m,n)}L^2}{2} \left[ -\left( \dot{z}_0^2 + \dot{z}_0^2 \dot{r}_0^2 \right) + \sum_{a=1}^{2} \left( \dot{r}_a^2 + \frac{(C_a + Q\bar{m}_a r_2^2)\dot{r}_a^2}{r_a^2} - \dot{r}_a^2 \bar{m}_a^2 \right) \right] - \left( q_{(m,n)} + \dot{q}_{(m,n)} \sqrt{1 - q^2} \right) L^2 r_2^2 \left( \frac{\ddot{m}_1 C_2}{r_2^2} - \frac{\ddot{m}_2 C_1}{r_1^2} + \frac{Q\ddot{m}_2^2 r_2^2}{r_1^2} \right) \right] - \frac{\Lambda}{2} \sum_{a=1}^{2} r_a^2 - 1 - \frac{\ddot{\Lambda}}{2} (z_0^2 + 1) .
\]

We can get the equations of motion for $r_1$ and $r_2$ as

\[ \ddot{r}_1 + \frac{(C_1 - Q\bar{m}_2 r_2^2)^2}{r_1^3} + \ddot{r}_1^2 + \frac{2Q(\ddot{m}_2 C_1 - Q\bar{m}_2 r_2^2)\ddot{r}_2}{r_1^3} + Ar_1 = 0 , \]  
\[ \ddot{r}_2 + \frac{Q\ddot{m}_2 r_2 (C_1 - Q\bar{m}_2 r_2^2)}{r_1^3} - \frac{Q\ddot{m}_1 r_2 (C_2 + Q\bar{m}_1 r_2^2)}{r_1^3} + \left( \frac{C_2 + Q\ddot{m}_1 r_2^2}{r_1^3} \right) + \ddot{m}_2^2 r_2 \]  
\[ + 2Q \left( \frac{\ddot{m}_1 C_2}{r_2^2} - \frac{\ddot{m}_2 C_1}{r_2^2} + \frac{2Q\ddot{m}_1^2 r_2^2}{r_1^3} \right) + Ar_2 = 0 . \]

These three equations of motion for $z_0$, $r_1$ and $r_2$ can also be obtained from the following Lagrangian

\[
L_{NR} = \frac{1}{2} (\ddot{z}_0^2 + \ddot{r}_1^2 + \ddot{r}_2^2) - \frac{1}{2} \frac{(C_1 - Q\bar{m}_1^2 r_2^2)^2}{r_1^2} - \frac{1}{2} \frac{(C_2 + Q\ddot{m}_1 r_2^2)^2}{r_1^2} - \frac{C_0^2}{8z_0^2} + \frac{1}{2} \left( \frac{\ddot{m}_1^2 C_2}{r_1^2} - \frac{\ddot{m}_2 C_1}{r_1^2} + \frac{Q\ddot{m}_1^2 r_2^2}{r_1^3} + \frac{Q\ddot{m}_2^2 r_2^2}{r_1^3} \right) \right) .
\]
Therefore the Hamiltonian of the system is
\[
H_{NR} = \frac{1}{2} \left( z_0^2 + v_1^2 + v_2^2 \right) + \frac{1}{2} \left( C_1 - Q\tilde{m}_2 r_2 \right)^2 \frac{1}{r_1^2} + \frac{1}{2} \left( C_2 + Q\tilde{m}_1 r_2 \right)^2 \frac{1}{r_2^2} + \frac{C_0^2}{8 z_0^2} - \frac{1}{2} \left( m_1 r_1^2 + \tilde{m}_2 r_2^2 \right)
- \frac{A}{2} \left( r_1^2 + r_2^2 - 1 \right) - \frac{\tilde{A}}{2} z_0 - \frac{1}{2} Q r_2^2 \left( \frac{\tilde{m}_1 C_2}{r_2^2} - \frac{\tilde{m}_2 C_1}{r_1^2} + \frac{Q \tilde{m}_1^2 r_2^2}{r_2^2} + \frac{Q \tilde{m}_2^2 r_2^2}{r_1^2} \right).
\]

(3.9)

From these two expressions it is obvious that the forms of both the Lagrangian and Hamiltonian are in agreement with those of the one-dimensional Neumann-Rosochatius system, only with extra terms due to the presence of mixed flux in the background.

The Uhlenbeck integrals of motion may be obtained in a similar way as used in the case of the spinning string ansatz. These yield the expressions as
\[
\tilde{I}_a = \frac{1}{m_1^2 - \tilde{m}_2^2} \left( r_1 \dot{r}_2 - \dot{r}_1 r_2 \right)^2 + \frac{2}{m_1^2 - \tilde{m}_2^2} \left[ \left( \frac{C_1 - Q r_2^2 \tilde{m}_2}{r_1} \right)^2 + \left( \frac{C_2 + Q r_2^2 \tilde{m}_1}{r_2} \right)^2 \right]
- \frac{2}{m_1^2 - \tilde{m}_2^2} \left[ \left( 1 + \frac{2Q r_2^2}{r_1} \right) \left( \tilde{m}_1^2 r_1^2 + \tilde{m}_2^2 r_2^2 \right) + 2Q r_2^2 \left( \frac{C_1 \tilde{m}_2}{r_1^2} - \frac{C_2 \tilde{m}_1}{r_2^2} \right) \right]
+ \frac{1}{m_1^2 - \tilde{m}_2^2} \left( \frac{C_1^2}{r_1^2} + \frac{C_2^2 r_1^2}{r_2^2} \right)
\]

(3.10)

The two Virasoro constraints \( G_{\tau\tau} + G_{\sigma\sigma} = 0 \) and \( G_{\tau\sigma} = G_{\sigma\tau} = 0 \) can be calculated as
\[
\begin{align*}
\dot{r}_1^2 + \dot{r}_2^2 + \dot{j}_1^2 r_1^2 + \dot{j}_2^2 r_2^2 + \tilde{m}_1^2 r_1^2 + \tilde{m}_2^2 r_2^2 &= \dot{z}_0^2 + \dot{\zeta}_0^2, \quad (3.11a) \\
\tilde{m}_1 \dot{r}_1 \dot{j}_1 + \tilde{m}_2 \dot{r}_2 \dot{j}_2 &= 0 \quad (3.11b)
\end{align*}
\]

respectively.

### 3.2 Solutions for \( r_1 \) and \( r_2 \) and string profile

Here we use the same procedure for finding the pulsating solutions to this integrable system by choosing ellipsoidal coordinates keeping the conditions as mentioned in section 3.3 intact. In case of pulsating string the ellipsoidal coordinates are taken to be the functions of \( \tau \) only and the roots of the equation
\[
\frac{\dot{r}_1^2}{\zeta - \tilde{m}_1^2} + \frac{r_2^2}{\zeta - \tilde{m}_2^2} = 0,
\]

(3.12)

may be found in terms of \( \zeta \). The range of \( \zeta \) extends from \( \tilde{m}_1^2 \) to \( \tilde{m}_2^2 \). The time derivative of this equation gives
\[
\left( r_1 \dot{r}_2 - \dot{r}_1 r_2 \right)^2 = \frac{\dot{\zeta}^2}{4(\tilde{m}_1^2 - \zeta)(\zeta - \tilde{m}_2^2)}.
\]

(3.13)
The conserved quantities, in the case of pulsating string may be found from the

\[ r_1^2 = \frac{\bar{m}_1^2 - \zeta}{\bar{m}_1^2 - \bar{m}_2^2}, \quad r_2^2 = \frac{\zeta - \bar{m}_2^2}{\bar{m}_1^2 - \bar{m}_2^2}. \]  

(3.14)

Solving equation (3.10) for \( \dot{\zeta}^2 \) we can get a similar solution containing 3rd order polynomial exactly like the case for rotating string and it is given as

\[ \dot{\zeta}^2 = -4P_3(\zeta), \quad P_3(\zeta) = 2(1 - Q) \prod_{i=1}^{3} (\zeta - \zeta_i). \]  

(3.15)

Using the same change of variables as given in equation (2.49) and with \( \eta \) taken to be a function of \( \tau \) only we can get

\[ \eta(\tau) = cn \left( \tau \sqrt{2(1 - Q)} (\zeta_3 - \zeta_1) + \tau_0, k \right) \]  

(3.16)

We may choose integration constant \( \tau_0 \) and the elliptic modulus \( k = \frac{\zeta_3 - \zeta_1}{\zeta_3 - \zeta_2} \). From equation (3.12) we get the expressions for \( r_1^2 \) and \( r_2^2 \) as

\[ r_1^2(\tau) = \frac{\zeta_2 - \bar{m}_1^2}{\bar{m}_2^2 - \bar{m}_1^2} + \frac{\zeta_2 - \zeta_3}{\bar{m}_2^2 - \bar{m}_1^2} sn^2 \left( \tau \sqrt{2(1 - Q)} (\zeta_3 - \zeta_1), k \right), \]  

(3.17)

and

\[ r_2^2(\tau) = \frac{\bar{m}_1^2 - \zeta_2}{\bar{m}_2^2 - \bar{m}_1^2} + \frac{\zeta_2 - \zeta_3}{\bar{m}_2^2 - \bar{m}_1^2} sn^2 \left( \tau \sqrt{2(1 - Q)} (\zeta_3 - \zeta_1), k \right). \]  

(3.18)

To restrict the elliptic modulus \( k \) within the fundamental domain, i.e, \( 0 < k < 1 \), we should choose the order of the \( \zeta_i \)'s as \( \zeta_1 < \zeta_2 < \zeta_3 \). Similar to the case for rotating string, here also \( \zeta \) is assumed to cover \( \frac{1}{3} \)th of the sphere with \( r_1^2 \geq 0 \). This consequently binds the range of \( \zeta \) as \( \bar{m}_1^2 \leq \zeta \leq \bar{m}_2^2 \).

### 3.3 Conserved Charges and Dispersion Relation

The conserved quantities, in the case of pulsating string may be found from the target space Lagrangian in the given background, as follows

\[ E = \frac{\partial L}{\partial (\partial_\tau t)} = -\tau(m,n) L^2 z_0^2 \dot{h}_0, \quad J_a = \frac{\partial L}{\partial (\partial_\tau \phi_a)} = \tau(m,n) L^2 \left( r_a^2 \dot{f}_a + Q r_a^2 \bar{m}_b \epsilon_{ab} \right) \]  

(3.19)

Here, by taking \( z_0 = 1 \) we get \( \dot{z}_0^2 + \bar{z}_0^2 \dot{h}_0^2 = \frac{E}{\tau(m,n)L^2} \) with \( \dot{h}_0^2 = \frac{\zeta^2}{\bar{z}_0^2} \). We may express the Virasoro constraint (3.11a) as a quartic equation of \( r_1 \) by substituting for \( \dot{h}_0 \) and \( \dot{f}_a \) in terms of conserved energy \( E = \frac{E}{\tau(m,n)L^2} \) and angular momenta \( J_a = \frac{J_a}{\tau(m,n)L^2} \). To make it simple, let us consider the case \( J_1 = J_2 \) and \( \bar{m}_1 = -\bar{m}_2 \). With the above, we get

\[ r_1 \dot{r}_1 = \sqrt{[\bar{m}_1^2(1 + Q) - E^2] r_1^4 + [E^2 - \bar{m}_1^2(1 + 2Q) + 2Q \bar{m}_1 J_1] r_1^2 - (J_1^2 - \bar{m}_1^2 + 2Q \bar{m}_1 J_1)} \]  

(3.20)
It is obvious from the above equation that the expression \( r_2^2 \) assumes infinite values for both \( r_1 \to 0 \) and \( r_1 \to \infty \) and oscillates in between. Therefore it must have some minimum value which can be derived to be [\( E^2 - \bar{m}_1^2 (1 + 2Q) + 2Q \bar{m}_1 J_1 \)] at \( r_1^2 = \pm \sqrt{\frac{J_2^2 - \bar{m}_1^2 + 2Q \bar{m}_1 J_1}{\bar{m}_1^2 (1 + Q) - E^2}} \). This yields the quantum oscillation number \([53]\) as

\[
N = \frac{N}{\tau_{(m,n) \parallel L^2}} = \int_0^{\sqrt{a_+}} r_1 \dot{r}_1 dr_1 \\
= \int_0^{\sqrt{a_+}} dr_1 \sqrt{[\bar{m}_1^2 (1 + Q) - E^2] r_1^4 + [E^2 - \bar{m}_1^2 (1 + 2Q) + 2Q \bar{m}_1 J_1] r_1^2 - (J_1^2 - \bar{m}_1^2 + 2Q \bar{m}_1 J_1)} \\
= \int_0^{\sqrt{a_+}} dr_1 \sqrt{(r_1^2 - a_-) (a_+ - r_1^2)},
\]

(3.21)

where \( a_{\pm} \) are the roots of the biquadratic function under the square root. Taking partial derivative of the above equation with respect to \( m_1 \) we get

\[
\frac{\partial N}{\partial \bar{m}_1} = \int_0^{\sqrt{a_+}} dr_1 \left[ \frac{\bar{m}_1 (1 + Q) r_1^4 - [\bar{m}_1 (1 + 2Q) - Q J_1] r_1^2 + (\bar{m}_1 - Q J_1)}{\sqrt{(r_1^2 - a_-) (a_+ - r_1^2)}} \right].
\]

(3.22)

Expressing the integrals in terms of standard elliptic integrals we are left with

\[
\frac{\partial N}{\partial \bar{m}_1} = (\bar{m}_1 - Q J_1) \frac{1}{\sqrt{a_+}} K(\epsilon) - [\bar{m}_1 (1 + 2Q) - Q J_1] \sqrt{a_+} [K(\epsilon) - E(\epsilon)] \\
- \frac{\bar{m}_1 (1 + Q)}{3} \sqrt{a_+} [a_- + 2 (a_+ + a_-) E(\epsilon) - (a_- + 2a_+) K(\epsilon)],
\]

(3.23)

where we have used \( \epsilon = \frac{a_-}{a_+} \). The standard expansions of elliptic integrals of first and second kinds are respectively

\[
K(\epsilon) = \frac{\pi}{2} + \frac{\pi \epsilon}{8} + \frac{9\pi \epsilon^2}{128} + \frac{25\pi \epsilon^3}{512} + \frac{1225\pi \epsilon^4}{32768} + \mathcal{O}[\epsilon]^5,
\]

\[
E(\epsilon) = \frac{\pi}{2} - \frac{\pi \epsilon}{8} - \frac{3\pi \epsilon^2}{128} - \frac{5\pi \epsilon^3}{512} - \frac{175\pi \epsilon^4}{32768} + \mathcal{O}[\epsilon]^5.
\]

(3.24a)

(3.24b)
Substituting for these expansions in equation (3.22) and taking the short string limit, we achieve

\[
\frac{\partial N}{\partial \bar{m}_1} = \sqrt{2} (\bar{\mathcal{m}}_1 - Q J_1) \left[ \frac{89\pi}{128} + \frac{123\pi}{128} \frac{Q J_1}{\bar{m}_1(1 + Q)} + O \left( \frac{J_1^2}{\bar{m}_1^2} \right) \right] + \nonumber
\]

\[
\frac{1}{\sqrt{2}} \{ \bar{m}_1(1 + 2Q) - Q J_1 \left[ \frac{11\pi}{32} + \frac{11\pi}{32} \frac{Q J_1}{\bar{m}_1(1 + Q)} + O \left( \frac{J_1^2}{\bar{m}_1^2} \right) \right] \nonumber
\]

\[
+ \sqrt{2} (\bar{\mathcal{m}}_1 - Q J_1) \left[ \frac{89\pi}{256 \bar{m}_1^2(1 + Q)^2} \frac{Q}{128 \bar{m}_1^3(1 + Q)^3} + O \left( \frac{J_1^2}{\bar{m}_1^4} \right) \right] \nonumber
\]

\[
+ \frac{1}{\sqrt{2}} \{ \bar{m}_1(1 + 2Q) - Q J_1 \left[ \frac{11\pi}{64} \frac{Q}{\bar{m}_1^2(1 + Q)} + \frac{37\pi}{32} \frac{Q^2 J_1}{\bar{m}_1^3(1 + Q)^3} + O \left( \frac{J_1^2}{\bar{m}_1^4} \right) \right] \nonumber
\]

\[
+ \sqrt{2} (\bar{\mathcal{m}}_1 - Q J_1) \left[ -\frac{15\pi}{32} \frac{Q^2}{\bar{m}_1^4(1 + Q)^4} + \frac{45\pi}{128} \frac{Q^3 J_1}{\bar{m}_1^5(1 + Q)^5} + O \left( \frac{J_1^2}{\bar{m}_1^6} \right) \right] \nonumber
\]

\[
+ \frac{1}{\sqrt{2}} \{ \bar{m}_1(1 + 2Q) - Q J_1 \left[ \frac{5\pi}{16} \frac{Q^2}{\bar{m}_1^4(1 + Q)^4} + \frac{9\pi}{32} \frac{Q^3 J_1}{\bar{m}_1^5(1 + Q)^5} + O \left( \frac{J_1^2}{\bar{m}_1^6} \right) \right] \nonumber
\]

\[
+ \bar{m}_1(1 + Q) \left[ \frac{\pi}{12} \frac{Q^2}{\bar{m}_1^4(1 + Q)^4} - \frac{11\pi}{48} \frac{Q^3 J_1}{\bar{m}_1^5(1 + Q)^5} + O \left( \frac{J_1^2}{\bar{m}_1^6} \right) \right] \nonumber
\]

\[
\right]
onumber
\]

\[
(3.25)
\]

Integrating the equation (3.25) with respect to \( \bar{m}_1 \) and then expanding \( \mathcal{E} \) as a function of oscillation number \( N \) and angular momentum \( J_1 \) for small values of \( J_1 \) we achieve

\[
\mathcal{E}^2 = (25.6704 + 1.8333 N) + (24.7749 + 1.2526 N) J_1 + (6.0259 + 0.0082 N) J_1^2 - (0.311 + 0.0082 N) J_1^3 + O(\mathcal{J}_1^4),
\]

\[
(3.26)
\]

where we have taken \( Q = 1 \) and considered the limit for \( \bar{m}_1 \to 1 \). With such limit in hand, small angular momenta, i.e., \( J_1 \to 0 \) yields from equation (3.26)

\[
\mathcal{E} \approx 1.354 \sqrt{N}.
\]

This result resembles the energy-oscillation number relation with small energy limit for fundamental string pulsating in background with pure NSNS flux [37] which can be achieved by assuming \( m = 1, n = 0 \) and \( q = 1 \) in equation (2.28). Also it matches with the energy and oscillation number expansion up to its leading order term in short string limit in [54] for the string pulsating in one plane.

4 Conclusion

We have investigated, in this paper, the integrable features of \((m, n)\) string in \(AdS_3 \times S^3\) background in the presence of mixed flux. We have computed the La-
The Lagrangian, Hamiltonian and integrals of motion and shown them similar to those of integrable deformed NR models. First, we have derived scaling relations among various conserved charges for closed circular strings of constant radii. We have also elucidated the periodic circular-type string profile for both the cases of closed strings rotating and pulsating in $\mathbb{R}_t \times S^3$ as solutions of this deformed NR integrable model in the presence of mixed three-form flux. For the pulsating string, we have computed the small energy correction to the scaling relation in terms of oscillation number in short string limit. It will be interesting to construct an NR model for rigidly rotating and pulsating string in $AdS_3 \times S^3 \times S^3 \times S^1$ background with various mixed fluxes. In connection with the integrable model, it will be quite interesting to establish relation between the hyperbolic Calogero-Sutherland integrable model to some string sigma model in AdS background with flux and check whether the integrability of $AdS_3/CFT_2$ duality remains intact with the solutions of those integrable models. We wish to return to these problems in future.

References

[1] J. M. Maldacena, “The Large N limit of superconformal field theories and supergravity,” Int. J. Theor. Phys. 38, 1113-1133 (1999), [arXiv:hep-th/9711200 [hep-th]].

[2] S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from noncritical string theory,” Phys. Lett. B 428, 105-114 (1998) [arXiv:hep-th/9802109 [hep-th]].

[3] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2, 253-291 (1998) [arXiv:hep-th/9802150 [hep-th]].

[4] K. Zarembo, “Strings on Semisymmetric Superspaces,” JHEP 05, 002 (2010) [arXiv:1003.0465 [hep-th]].

[5] J. Minahan and K. Zarembo, “The Bethe ansatz for N=4 superYang-Mills,” JHEP 03, 013 (2003) [arXiv:hep-th/0212208 [hep-th]].

[6] N. Beisert, “The complete one loop dilatation operator of N=4 superYang-Mills theory,” Nucl. Phys. B 676, 3-42 (2004) [arXiv:hep-th/0307015 [hep-th]].

[7] N. Beisert and M. Staudacher, “The N=4 SYM integrable super spin chain,” Nucl. Phys. B 670, 439-463 (2003) [arXiv:hep-th/0307042 [hep-th]].

[8] A. Cagnazzo and K. Zarembo, “B-field in AdS(3)/CFT(2) Correspondence and Integrability,” JHEP 11, 133 (2012) [arXiv:1209.4049 [hep-th]].

[9] B. Hoare, A. Stepanchuk and A. Tseytlin, “Giant magnon solution and dispersion relation in string theory in $AdS_3 \times S^3 \times T^4$ with mixed flux,” Nucl. Phys. B 879, 318-347 (2014) [arXiv:1311.1794 [hep-th]].
[10] B. Hoare and A. Tseytlin, “Massive S-matrix of $AdS_3 \times S^3 \times T^4$ superstring theory with mixed 3-form flux,” Nucl. Phys. B 873, 395-418 (2013) [arXiv:1304.4099 [hep-th]].

[11] B. Hoare and A. Tseytlin, “On string theory on $AdS_3 \times S^3 \times T^4$ with mixed 3-form flux: tree-level S-matrix,” Nucl. Phys. B 873, 682-727 (2013) [arXiv:1303.1037 [hep-th]].

[12] M. Baggio and A. Sfondrini, “Strings on NS-NS Backgrounds as Integrable Deformations,” Phys. Rev. D 98, no.2, 021902 (2018) [arXiv:1804.01998 [hep-th]].

[13] A. Pittelli, A. Torrielli and M. Wolf, “Secret symmetries of type IIB superstring theory on $AdS_3 \times S^3 \times M^4$,” J. Phys. A 47, no.45, 455402 (2014) [arXiv:1406.2840 [hep-th]].

[14] A. Pittelli, “Yangian Symmetry of String Theory on $AdS_3 \times S^3 \times S^3 \times S^1$ with Mixed 3-form Flux,” Nucl. Phys. B 935, 271-289 (2018) [arXiv:1711.02468 [hep-th]].

[15] D. M. Hofman and J. M. Maldacena, “Giant Magnons,” J. Phys. A 39, 13095-13118 (2006) [arXiv:hep-th/0604135 [hep-th]].

[16] M. Kruczenski, “Spiky strings and single trace operators in gauge theories,” JHEP 08, 014 (2005) [arXiv:hep-th/0410226 [hep-th]].

[17] J. A. Minahan, “Circular semiclassical string solutions on $AdS(5) \times S(5)$,” Nucl. Phys. B 648, 203-214 (2002) [arXiv:hep-th/0209047 [hep-th]].

[18] S. Gubser, I. Klebanov and A. M. Polyakov, “A Semiclassical limit of the gauge / string correspondence,” Nucl. Phys. B 636, 99-114 (2002) [arXiv:hep-th/0204051 [hep-th]].

[19] V. Kazakov, A. Marshakov, J. Minahan and K. Zarembo, “Classical/quantum integrability in AdS/CFT,” JHEP 05, 024 (2004) [arXiv:hep-th/0402207 [hep-th]].

[20] N. Beisert, V. Kazakov and K. Sakai, “Algebraic curve for the SO(6) sector of AdS/CFT,” Commun. Math. Phys. 263, 611-657 (2006) [arXiv:hep-th/0410253 [hep-th]].

[21] M. Kruczenski, J. Russo and A. A. Tseytlin, “Spiky strings and giant magnons on S**5,” JHEP 10, 002 (2006) [arXiv:hep-th/0607044 [hep-th]].

[22] G. Arutyunov, S. Frolov, J. Russo and A. A. Tseytlin, “Spinning strings in $AdS(5) \times S**5$ and integrable systems,” Nucl. Phys. B 671, 3-50 (2003) [arXiv:hep-th/0307191 [hep-th]].

[23] G. Arutyunov, J. Russo and A. A. Tseytlin, “Spinning strings in $AdS(5) \times S**5$: New integrable system relations,” Phys. Rev. D 69, 086009 (2004) [arXiv:hep-th/0311004 [hep-th]].

[24] K. Uhlenbeck, “Equivariant harmonic maps into spheres”, Lect. Notes Math. 949 39, (1982).
[25] O. Babelon, D. Bernard and M. Talon, “Introduction to Classical Integrable Systems,” doi:10.1017/CBO9780511535024

[26] J. Avan and M. Talon, “Alternative lax structures for the classical and quantum Neumann model,” Phys. Lett. B 268, 209-216 (1991)

[27] J. Avan and M. Talon, “Poisson Structure and Integrability of the Neumann-moser-uhlenbeck Model,” Int. J. Mod. Phys. A 5, 4477-4488 (1990)

[28] R. Ishizeki and M. Kruczenski, “Single spike solutions for strings on $S^2$ and $S^3$,” Phys. Rev. D 76, 126006 (2007) [arXiv:0705.2429 [hep-th]].

[29] A. Mosaffa and B. Safarzadeh, “Dual spikes: New spiky string solutions,” JHEP 08, 017 (2007) [arXiv:0705.3131 [hep-th]].

[30] H. Hayashi, K. Okamura, R. Suzuki and B. Vicedo, “Large Winding Sector of AdS/CFT,” JHEP 11, 033 (2007) [arXiv:0709.4033 [hep-th]].

[31] C. Ahn, P. Bozhilov and R. Rashkov, “Neumann-Rosochatius integrable system for strings on AdS(4) x CP**3,” JHEP 09, 017 (2008) [arXiv:0807.3134 [hep-th]].

[32] R. Hernández and J. M. Nieto, “Spinning strings in $AdS_3 \times S^3$ with NSâ„“NS flux,” Nucl. Phys. B 888, 236-247 (2014) [arXiv:1407.7475 [hep-th]].

[33] R. Hernandez and J. M. Nieto, “Elliptic solutions in the Neumannâ„“Rosochatius system with mixed flux,” Phys. Rev. D 91, no.12, 126006 (2015) [arXiv:1502.05203 [hep-th]].

[34] G. Arutyunov, M. Heinze and D. Medina-Rincon, “Integrability of the $n$-deformed Neumannâ„“Rosochatius model,” J. Phys. A 50, no.3, 035401 (2017) [arXiv:1607.05190 [hep-th]].

[35] R. Hernandez and J. M. Nieto, “Spinning strings in the $n$-deformed Neumann-Rosochatius system,” Phys. Rev. D 96, no.8, 086010 (2017) [arXiv:1707.08032 [hep-th]].

[36] R. Hernández, J. M. Nieto and R. Ruiz, “Pulsating strings with mixed three-form flux,” JHEP 04, 078 (2018) [arXiv:1803.03078 [hep-th]].

[37] A. Chakraborty and K. L. Panigrahi, “Neumann-Rosochatius system for strings in ABJ Model,” JHEP 12, 024 (2019) [arXiv:1909.12632 [hep-th]].

[38] J. H. Schwarz, “An SL(2,Z) multiplet of type IIB superstrings,” Phys. Lett. B 360, 13-18 (1995) [arXiv:hep-th/9508143 [hep-th]].

[39] A. A. Tseytlin, “Selfduality of Born-Infeld action and Dirichlet three-brane of type IIB superstring theory,” Nucl. Phys. B 469, 51-67 (1996) [arXiv:hep-th/9602064 [hep-th]].

[40] J. Lu and S. Roy, “An SL(2,Z) multiplet of type IIB super five-branes,” Phys. Lett. B 428, 289-296 (1998) [arXiv:hep-th/9802080 [hep-th]].

[41] Y. Lozano, “D-brane dualities as canonical transformations,” Phys. Lett. B 399, 233-242 (1997) [arXiv:hep-th/9701186 [hep-th]].
[42] M. Cederwall and P. Townsend, “The Manifestly Sl(2,Z) covariant superstring,” JHEP 09, 003 (1997) [arXiv:hep-th/9709002 [hep-th]].

[43] J. KlusoÅĹ, “(p, q)-five brane and (p, q)-string solutions, their bound state and its near horizon limit,” JHEP 06, 002 (2016) [arXiv:1603.05196 [hep-th]].

[44] J. Kluson, “(m, n)-String in (p, q)-string and (p, q)-five-brane background,” Eur. Phys. J. C 76, no.11, 582 (2016) [arXiv:1602.08275 [hep-th]].

[45] A. Banerjee, S. Biswas and R. R. Nayak, “D1 string dynamics in curved backgrounds with fluxes,” JHEP 04, 172 (2016) [arXiv:1601.06360 [hep-th]].

[46] S. P. Barik, M. Khouchen, J. KlusoÅĹ and K. L. Panigrahi, “SL(2, Z) invariant rotating (m, n) strings in AdS_3 x S^3 with mixed flux,” Eur. Phys. J. C 77, no.5, 298 (2017) [arXiv:1610.03402 [hep-th]].

[47] J. KlusoÅĹ, “(m,n)-String and D1-Brane in Stringy Newton-Cartan Background,” JHEP 04, 163 (2019) [arXiv:1901.11292 [hep-th]].

[48] R. Metsaev and A. A. Tseytlin, “Type IIB superstring action in AdS(5) x S**5 background,” Nucl. Phys. B 533, 109-126 (1998) [arXiv:hep-th/9805028 [hep-th]].

[49] B. Stefanski, Jr, “Green-Schwarz action for Type IIA strings on AdS(4) x CP**3,” Nucl. Phys. B 808, 80-87 (2009) [arXiv:0806.4948 [hep-th]].

[50] A. Babichenko, B. Stefanski, Jr. and K. Zarembo, “Integrability and the AdS(3)/CFT(2) correspondence,” JHEP 03, 058 (2010) [arXiv:0912.1723 [hep-th]].

[51] G. Grignani, T. Harmark, M. Orselli and G. W. Semenoff, “Finite size Giant Magnons in the string dual of N=6 superconformal Chern-Simons theory,” JHEP 12, 008 (2008) [arXiv:0807.0205 [hep-th]].

[52] O. Babelon and M. Talon, “Separation of variables for the classical and quantum Neumann model,” Nucl. Phys. B 379, 321-339 (1992) [arXiv:hep-th/9201035 [hep-th]].

[53] I. Y. Park, A. Tirziu and A. A. Tseytlin, “Semiclassical circular strings in AdS(5) and 'long' gauge field strength operators,” Phys. Rev. D 71, 126008 (2005) [arXiv:hep-th/0505130 [hep-th]].

[54] P. M. Pradhan and K. L. Panigrahi, “Pulsating Strings With Angular Momenta,” Phys. Rev. D 88, no.8, 086005 (2013) [arXiv:1306.0457 [hep-th]].