ON THE ISOMETRY GROUP AND THE GEOMETRIC STRUCTURE OF COMPACT STATIONARY LORENTZIAN MANIFOLDS

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ABSTRACT. We study the geometry of compact Lorentzian manifolds that admit a somewhere timelike Killing vector field, and whose isometry group has infinitely many connected components. Up to a finite cover, such manifolds are products (or amalgamated products) of a flat Lorentzian torus and a compact Riemannian (resp., lightlike) manifold.

1. INTRODUCTION

Paradigmatic example. We will deal with dynamics and geometry of the following flavor. Let $q$ be a Lorentz form on $\mathbb{R}^n$; this induces a (flat) Lorentz metric on the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. The linear isometry group of $\mathbb{T}^n$ is $O(q, \mathbb{Z}) = GL(n, \mathbb{Z}) \cap O(q)$, and its full isometry group is the semi-direct product $O(q, \mathbb{Z}) \rtimes \mathbb{T}^n$.

The global and individual structure of $O(q, \mathbb{Z})$ involves interesting geometric, arithmetic and dynamical interactions. For generic $q$, $O(q, \mathbb{Z})$ is trivial. Nonetheless, if $q$ is rational, i.e., if $q(x) = \sum a_{ij}x_i x_j$, where $a_{ij}$ are rational numbers, then $O(q, \mathbb{Z})$ is big in $O(q)$; more precisely, by Harich-Chandra-Borel theorem, it is a lattice in $O(q)$. When $q$ is not rational, many intermediate situations are possible. It is a finite volume non co-compact lattice in the case of the standard form $q_0 = -x_1^2 + x_2^2 + \ldots + x_n^2$, but can be co-compact for other forms. On the other hand, a given element $A \in O(q_0, \mathbb{Z})$, could have complicated dynamics. For instance, if $A$ is hyperbolic, i.e., its spectrum is not contained in $S^1$, then the remainder of eigenvalues are roots of unity or Salem numbers. Conversely, any Salem number is the eigenvalue of such a hyperbolic $A \in O(q_0, \mathbb{Z})$, for some dimension $n$.

Lorentz geometry and dynamics. The global geometry of compact manifolds endowed with a non positive definite metric (pseudo-Riemannian manifolds) can be quite different from the geometry of Riemannian manifolds. For instance, compact pseudo-Riemannian manifolds may fail to be geodesically complete or geodesically connected; moreover, the isometry group of a compact pseudo-Riemannian manifold fails to be compact in general. The main goal of this paper is to investigate the geometric structure of Lorentz manifolds essentially non Riemannian, i.e., with non compact isometry group.

Lorentzian manifolds, i.e., manifolds endowed with metric tensors of index 1, play a special role in pseudo-Riemannian geometry, due to their relations with General Relativity. The lack of compactness of the isometry group is due to the

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fact that, unlike the Riemannian case, Lorentzian isometries need not be equicontinuous, and may generate chaotic dynamics on the manifold. For instance, the dynamics of Lorentz isometries can be of Anosov type, evocative of the fact that in General Relativity one can have contractions of time and expansion in space. A celebrated result of D’Ambra (see [3]) states that the isometry group of a real analytic simply-connected compact Lorentzian manifold is compact. It is not known whether this results holds in the $C^\infty$ case. In the last decade several authors have studied isometric group actions on Lorentz manifolds. Most notably, a complete classification of (connected) Lie groups that act locally faithfully and isometrically on compact Lorentzian manifolds has been obtained independently by Adams and Stuck (see [1]) and the second author (see [17]). Roughly speaking, the identity component $G_0$ of the isometry group of a compact Lorentz manifold is the direct product of an abelian group, a compact semi-simple group, and, possibly, a third factor which is locally isomorphic to either $\text{SL}(2, \mathbb{R})$ or to an oscillator group, or else to a Heisenberg group. The geometric structure of a compact Lorentz manifold that admits a faithful isometric action of a group $G$ isomorphic to $\text{SL}(2, \mathbb{R})$ or to an oscillator group is well understood; such manifolds can be described using right quotients $G/\Gamma$, where $\Gamma$ is a co-compact lattice of $G$, and warped products, see Section 3 for more details. Observe that such constructions produce Lorentz manifolds on which the $G_0$-action has some timelike orbit. Recall that a Lorentz manifold is said to be stationary if it admits an everywhere timelike Killing vector field. Our first result (Theorem 3.1) is that when the identity component of the isometry group is non compact and it has some timelike orbit, then it must contain a non trivial factor locally isomorphic to $\text{SL}(2, \mathbb{R})$ or to an oscillator group.

Thus, next natural question is to study the geometry of manifolds whose isometry group is non compact for having an infinite number of connected components.

Results. We will show in this paper that compact Lorentz manifolds with a large isometry group are essentially constructed in the same way. In order to define the appropriate notion of the lack of compactness of the isometry group of a Lorentzian manifold, let us give the following:

Definition. Let $\rho : \Gamma \to \text{GL}(E)$ be a representation of the group $G$ on the vector space $E$. Then, $\rho$ is said to be of Riemannian type if it preserves some positive definite inner product on $E$. We say that $\rho$ is of post-Riemannian type if it preserves a positive semi-definite inner product on $E$ having kernel of dimension equal to 1.

Observe that $\rho$ is of Riemannian type if and only if it is precompact, i.e., $\rho(\Gamma)$ is precompact in $\text{GL}(E)$.

Given a Lorentzian manifold $(M, g)$, we will denote by $\text{Iso}(M, g)$ its isometry group, and by $\text{Iso}_0(M, g)$ the identity connected component of $\text{Iso}(M, g)$. The Lie algebra of $\text{Iso}(M, g)$ will be denoted by $\mathfrak{iso}(M, g)$. By a large isometry group, we mean that the adjoint action of the discrete part $\Gamma = \text{Iso}(M, g)/\text{Iso}_0(M, g)$ of $\text{Iso}(M, g)$ is not of post-Riemannian type. This implies, in particular, that $\Gamma$ is not finite.

The main result of the paper is that compact Lorentz manifolds with large isometry group are essentially built up by tori. More precisely, we prove the following structure theorem.
1. **Theorem.** Let $(M, g)$ be a compact Lorentz manifold, and assume that the conjugacy action of $\Gamma$ on $\text{Iso}_0(M, g)$ is not post-Riemannian. Then, $\text{Iso}_0(M, g)$ contains a torus $T = T^d$, endowed with a Lorentz form $q$, such that $\Gamma$ is a subgroup of $O(q, \mathbb{Z})$.

Up to a finite cover, there is a new Lorentz metric $g^{\text{new}}$ on $M$ having a larger isometry group than $g$, such that $\Gamma = O(q, \mathbb{Z})$. Geometrically, $M$ is metric direct product $T \times N$, where $N$ is a compact Riemannian manifold, or $M$ is an amalgamated metric product $T \times_{S^1} L$, where $L$ is a lightlike manifold with an isometric $S^1$-action. The last possibility holds when $\Gamma$ is a parabolic subgroup of $O(q)$.

A more precise description of the original metric $g$ is given in Section 8 for the hyperbolic case and in Section 9 for the parabolic case. We will in fact prove Theorem 1 under an assumption weaker than the non post-Riemannian hypothesis for the conjugacy action of $\Gamma$. The more general statement proved here is the following:

2. **Theorem.** Assume that $\Gamma$ is infinite and that $\text{Iso}_0(M, g)$ has a somewhere timelike orbit. Then the conclusion of Theorem 1 holds.

Theorem 1 will follow from Theorem 2 once we show that, under the assumption that the conjugacy action of $\Gamma$ is not of post-Riemannian type, then the connected component of the identity of the isometry group must have some timelike orbits, see Subsection 2.3.

A first consequence of our main result is the following:

3. **Corollary.** Assume that $(M, g)$ is a compact Lorentzian manifold with infinite discrete part $\Gamma$. If $(M, g)$ has a somewhere timelike Killing vector field, then $(M, g)$ has an everywhere timelike Killing vector field.

We will also prove (Proposition 4.1) that, when $(M, g)$ has a Killing vector field which is timelike somewhere, then the two situations (a) and (b) below are mutually exclusive:

(a) the connected component of the identity $\text{Iso}_0(M, g)$ of $\text{Iso}(M, g)$ is non compact;
(b) $\text{Iso}(M, g)$ has infinitely many connected components, as in the case of the flat Lorentzian torus.

The point here is that, in a compact Lorentzian manifold, the flow of a Killing vector field which is timelike somewhere generates a non trivial precompact group in the (connected component of the identity of the) isometry group. Thus, by continuity, the Lie algebra of the isometry group of such manifolds must contain a non empty open cone of vectors generating precompact 1-parameter subgroups in the isometry group. The proof of Proposition 4.1 is obtained by ruling out the existence of a non compact abelian or nilpotent factor in the connected component of the isometry group. The argument is based on an algebraic precompactness criterion for 1-parameter subgroups of Lie groups proved in Proposition 2.11.

Moreover, using Theorem 2 and previous classification results by the second author, we prove the following partial extension of D’Ambra’s result to the $C^\infty$-realm:

4. **Theorem.** The isometry group of a simply connected compact Lorentzian manifold that admits a Killing vector field which is somewhere timelike is compact.
Lemma 2.2. Every Lorentz quadratic form relating the dynamics of an isometry whose identity component \(G\) is compact abelian, it has some timelike orbit, and the quotient \(\Gamma = G/G_0\) is torsion free. Moreover, the conjugacy action of \(G\) on \(G_0 \cong T^k\), denoted by \(\rho : G \to \text{Out}(T^k) \cong \text{GL}(k, \mathbb{Z})\), preserves some Lorentz form of \(\mathbb{R}^k\); thus, the image \(\rho(G)\) is contained in \(\text{GL}(k, \mathbb{Z}) \cap \text{SO}(q)\) for some Lorentz quadratic form \(q\). The proof of our structure result is then obtained by relating the dynamics of an isometry \(f \in G, f \not\in G_0\), on \(M\), and the dynamics of the conjugacy action of \(f\) on the toral factor \(G_0\). In this step the proof splits into two cases, according to whether \(\rho(f)\) is a parabolic or a hyperbolic element of \(\text{SO}(q)\). We will make full use of the techniques developed by the second author in [18], in particular we will employ the notions of approximate stable foliation and of strongly approximate stable foliation of an isometry of a compact Lorentz manifold. This analysis will show the existence of a possibly smaller torus \(T^d \subset T^k, d \geq 2\), whose action on \(M\) is everywhere free and timelike. In the hyperbolic case, the orthogonal distribution to the \(T^d\)-orbits is integrable; a general covering Lemma (Section 7) will yield the product structure of (a finite covering of) \(M\). The amalgamated product structure in parabolic case is somewhat more involved, and it is discussed in Section 9.

2. Preliminaries

In this section we will collect several auxiliary results needed in the rest of the paper.

2.1. Toral subgroups. Our aim here is to determine the freeness of isometric toral actions on manifolds. The key fact is that the set \(S(T^d)\) of all closed subgroups of the \(d\)-torus \(T^d\) is countable, and it satisfies a uniform discreteness property.

Lemma 2.1. Let \(X\) be a locally compact metric space, and let \(\phi : X \to S(T^d)\) be a semi-continuous map, that is, if \(x_n \to x\), then any limit of \(\phi(x_n)\) is contained in \(\phi(x)\). Then, there exists \(A \in S(T^d)\) such that \(\phi^{-1}(A)\) has non empty interior.

Proof. For \(A \in S(T), \) set \(F_A = \{x \in X : \text{such that } A \subset \phi(x)\}\). By the semi-continuity, the closure \(\phi^{-1}(A)\) \(\subset F_A\) for all \(A \in S(T^d)\). Clearly, \(X = \bigcup_{A \in S(T^d)} \phi^{-1}(A)\). By Baire’s theorem, the interior \(\text{int}(\phi^{-1}(A))\) of some \(\phi^{-1}(A)\) must be non empty. Thus, the intersection \(\text{int}(\phi^{-1}(A)) \cap \phi^{-1}(A)\) is non empty.

Let \(x\) be a point of such intersection, so that \(A = \phi(x)\), and there is a neighborhood \(V\) of \(x\) such that \(\phi(y) \supseteq \phi(x)\) for all \(y \in V\). By semi-continuity, we must have equality \(\phi(y) = \phi(x)\) for \(y\) in some neighborhood \(V' \subset V\). This follows from the fact that \(A\) is an isolated point of the set:

\[ S(T^d; A) = \{B \in S(T^d) : A \subset B\}, \]

see Lemma 2.2. Hence, \(\phi^{-1}(A)\) has non empty interior. \(\Box\)

Lemma 2.2. Every \(A \in S(T^d)\) is isolated in \(S(T^d; A)\).

Proof. Let us consider the case that \(A\) is the trivial subgroup. To prove that \(A = \{1\}\) is isolated in \(S(T^d)\) it suffices to observe that there exist two disjoint closed subsets \(C_1, C_2 \subset T^d\) such that \(C_1\) is a neighborhood of \(1\), \(C_1 \cap C_2 = \emptyset\), and with
the property that if $B \in S(T^d)$ is such that $B \cap C_1 \neq \{1\}$, then $B \cap C_2 \neq \emptyset$. For instance, one can take $C_1$ to be the closed ball around 1 of radius $r > 0$ small, and $C_2 = \{p \in T^d : 2r \leq \text{dist}(p, 1) \leq 3r\}$. Here we are considering the distance on $T^d = \mathbb{R}^d / \mathbb{Z}^d$ induced by the Euclidean metric of $\mathbb{R}^d$. The number $r$ can be chosen in such a way that $C_1 \cap C_2 = \emptyset$. Every non trivial element in $C_1$ has some power in $C_2$, which proves that $C_1$ and $C_2$ have the required properties. Thus, if $A_n \in S(T^d)$ is any sequence which is not eventually equal to $A$, then some limit point of $A_n$ must be contained in $C_2$, and therefore $\lim A_n \neq A$. If one replaces $T^d$ by a finite quotient of $T^d$, then one gets to the same conclusion by essentially the same proof. The general case is obtained by considering the quotient $T^d/A$, which is equal to a finite quotient of a torus. □

**Corollary 2.3.** If $X$ is a locally compact metric space and $\phi : X \to S(T^d)$ is semi-continuous, then, there is a dense open subset $U \subset X$, where $\phi$ is locally constant, i.e., any $x \in U$ has a neighborhood $V$ where $\phi$ is constant.

**Proof.** Let $U$ be the open subset of $X$ given by the union of the interiors of the sets $\phi^{-1}(A)$, with $A$ running in $S(T^d)$. This is the largest open subset of $X$ where $\phi$ is locally constant. If $U$ were not dense, then there would exist a non empty open subset $V \subset X$ with $V \cap U = \emptyset$. The restriction $\phi$ of $\phi$ to $V$ is a semi-continuous map, with the property that $\tilde{\phi}^{-1}(A)$ has empty interior for all $A \in S(T^d)$. By Lemma 2.1, this is impossible, hence $U$ is dense. □

**Corollary 2.4.** Any faithful isometric action of a torus $T^d$ on some pseudo-Riemannian manifold $(M, g)$ is free on a dense open subset of $M$.

**Proof.** Apply Corollary 2.3 to the map $\phi : M \to S(T^d)$ that associates to each $p \in M$ its stabilizer. Such map is obviously semi-continuous. Thus, on a dense open subset $U$ of $M$, the stabilizer of the isometric action is locally constant. No nontrivial isometry of a pseudo-Riemannian manifold fixes all points of a non empty open subset, and this implies that the stabilizer of each point of $U$ is trivial. □

2.2. **Linear dynamics.** Gauss maps (and variants) have the advantage to transform the dynamics on $M$ into a linear dynamics, i.e., an action of the group in question on a linear space or an associated projective space, via a linear representation. We will prove in the sequel a stability result: if a linear group “almost-preserves” a Lorentz form, then it (fully) preserves another one. We start with the individual case, i.e., with actions of the infinite cyclic group $\mathbb{Z}$, and then we will consider general groups.

2.2.1. **Individual dynamics.** Let $E$ be a vector space, and $A \in \text{GL}(E)$. It has a Jordan decomposition $A = EHU$, where $U$ is unipotent (i.e., $U - 1$ is nilpotent), $H$ hyperbolic (i.e., diagonalizable over $\mathbb{R}$), and $E$ is elliptic (i.e., diagonalizable over $\mathbb{C}$, and all its eigenvalues have norm equal to 1).

If $F$ is a space obtained from $E$ by functorial constructions, e.g., $F = \text{Sym}(E^*)$ the space of quadratic forms on $E$, or $F = \text{Gr}^d(E)$ the Grassmannian of $d$-dimensional subspaces of $E$, the associated $A$-action on $F$ will be denoted by $A^F$. Naturally, when $F$ is a vector space, we have $A^F = E^F H^F U^F$.

A point $p \in E$ is $A$-recurrent if there is $n_i \in \mathbb{N}, n_i \to \infty$, such that $A^{n_i}(p) \to p$ as $i \to \infty$. A point $p$ is $A$-escaping if for any compact subset $K \subset E$ there is $N$...
such that $A^n(p) \notin K$, for $n > N$. So, $p$ is non-escaping if there is $n_i \to \infty$, such that $A^{n_i}p$ stay in some compact set $K \subset \mathcal{E}$.

It is easy to prove the following

**Lemma 2.5.** Let $p \in F$ be a point recurrent under the $A^F$-action. Then $p$ is fixed by $H^F$ and $U^F$. If $F$ is a vector space, $p$ is $A^F$-non-escaping iff $p$ is fixed by $H^F$ and $U^F$.

Recall that an element $A$ of the orthogonal group of a Lorentz form $q$ is **hyperbolic** if it has one eigenvalue $\lambda$ with $|\lambda| \neq 1$, and $A$ is **parabolic** if it is not diagonalizable (over $\mathbb{C}$). In other words, $A$ is hyperbolic if it is conjugate in $\text{GL}(k,\mathbb{R})$ (in fact in $\text{SO}(q)$) to a matrix of the form:

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & R \end{pmatrix}$$

where $\lambda \in \mathbb{R}$, $\lambda > 1$, and $R \in \text{SO}(k-2)$. Similarly, $A$ is parabolic if it has the normal form:

$$\begin{pmatrix} 1 & t & -t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

with $t \in \mathbb{R}$ and $R \in \text{SO}(k-3)$.

**Lemma 2.6.** Set $F = \text{Sym}(\mathcal{E})$, and assume $A = \text{EHU}$ non-elliptic (i.e., either $H$ or $U$ is non-trivial). Suppose there is a Lorentz form $q_0$ which is $A$-recurrent, and let $K \subset \text{GL}(\mathcal{E})$ be the torus generated by the powers of $E$. Then, $\int_K B^F(q_0) \, d\mu(B)$ is an $A$-invariant Lorentz form, where $\mu$ is the Haar measure on $K$.

**Proof.** Assume $H \neq 1$. By Lemma 2.5, $H$ preserves $q_0$. Then, there exists $\lambda \in \mathbb{R} \setminus \{1\}$, such that $\lambda, \lambda^{-1} \in \sigma(A)$. The eigenspaces $V_\lambda$ and $V_{\lambda^{-1}}$ are isotropic, hence 1-dimensional. Namely, for $v \in V_\lambda$, $q_0(v,v) = q_0(Hv,Hv) = \lambda^2 q_0(v,v)$; similarly, for $v \in V_{\lambda^{-1}}$, $q_0(v,v) = \lambda^{-2} q_0(v,v)$. In both cases, one must have $q_0(v,v) = 0$. It follows that the direct sum $\mathcal{E}_\lambda = V_\lambda \oplus V_{\lambda^{-1}} \cong \mathbb{R}^2$ is timelike, and the Lorentzian form $q_0|_{\mathcal{E}_\lambda}$ is $H$-invariant.

Observe now that $\mathcal{E}_\lambda$ is the unique 2-space on which $H$ is non-elliptic (i.e., all its powers are uniformly unbounded). Thus, any endomorphism commuting with $H$ preserves $\mathcal{E}_\lambda$. Therefore, $E$ preserves $\mathcal{E}_\lambda$. A similar argument yields that $E$ preserves the orthogonal $\mathcal{E}_\lambda^\perp$ (with respect to $q_0$), since it is the maximal subspace on which $H$ acts elliptically.

On the other hand, up to multiples, the only quadratic form on $\mathbb{R}^2$ which is preserved by \( \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \) with $\lambda \neq 1$, is the form $(x_1,x_2) \mapsto x_1x_2$. As above, since $E$ commutes with $H$, it preserves this quadratic form on $\mathcal{E}_\lambda$.

From all this, it follows that $E^n$ preserves the fixed Lorentzian metric $q_0|_{\mathcal{E}_\lambda}$ for all $n$, while in the orthogonal space to $\mathcal{E}_\lambda$ the restriction of $q_0$ is positive definite. Hence, the average $\int_K B^F(q_0) \, d\mu(B)$, which is $A$-invariant, is equal to $q_0$ on $\mathcal{E}_\lambda$ and positive definite on the orthogonal of $\mathcal{E}_\lambda$. 


Now the case $H = 1$ but $U \neq 1$ can be treated in analogous way, using essentially that $E$ commutes with $U$, and that $U$ has the normal form of a parabolic element in $SO(q_0)$, see the proof of Lemma 6.2 for a similar argument. $\square$

2.2.2. Group dynamics. We consider now a group $\Gamma$ acting on $E$ via a representation $\rho : \Gamma \to GL(E)$. One can naturally develop a theory of recurrence leading in particular to a variant of the previous lemma for groups.

**Proposition 2.7.** Let $\rho : \Gamma \to GL(E)$ be such that $\rho(\alpha)$ is non-elliptic for any $\alpha \in \Gamma$. Let $F = Sym(E)$, and assume that the associated action $\rho^F$ preserves a compact set of $F$ contained in the (open) subset of Lorentz forms, and that $\rho^F$ leaves invariant a finite measure on such compact set. Then, $\rho(\Gamma)$ preserves some Lorentz form.

The proof of Proposition 2.7 occupies the remainder of this subsection.

For $a \in \Gamma$, let $H(a)$ and $U(a)$ be the hyperbolic and unipotent parts of $\rho(a)$, respectively. Let $q_0$ be a form recurrent under the $\Gamma$-action. By Lemma 2.5, $q_0$ is preserved by any $H(a)$ and any $U(a), a \in \Gamma$. Let $\Gamma^*$ be the group generated by $\{H(a), U(a) : a \in \Gamma\}$. Then $\Gamma^*$ is a subgroup of $SO(q_0)$.

A subgroup $L \subset SO(q_0)$ is non-elementary if it does not preserve a timelike or a lightlike direction in $E$. To simplify, let us assume that $\Gamma^*$ is not elementary; the elementary case can be treated separately. For instance, a subgroup of $SO(q)$ preserves a timelike direction if and only if it is pre-compact. More precisely, preserving a Lorentz form and a timelike direction is equivalent to preserving a positive definite inner product. The case of a lightlike direction looks like the previous case of a group generated by a single element.

**Lemma 2.8.** Let $L$ be a non-elementary subgroup of $SO(q_0)$. Then:

1. There is a unique decomposition $E = A \oplus B$, characterized by the fact that $H$ acts precompactly on $B$ and non precompactly on $A$. Furthermore, the decomposition is $q_0$-orthogonal, with $A$ timelike and $B$ spacelike.

2. Any Lorentz form $q$ invariant under $L$ decomposes as $\lambda q_0|_A + \delta$, where $\delta$ is a positive definite form on $B$.

It follows in particular that any $A \in GL(E)$ normalizing $L$ respects the decomposition, and the image $A^* q_0$ has the previous form.

**Proof.** For part (1), if $L$ does not act irreducibly, then it leaves invariant some subspace $R \subset E$. By the non-elementarity hypothesis, $R$ is spacelike, or timelike with dimension larger than 1. If $R$ is spacelike, then we consider its orthogonal $R^\perp$ which is timelike and with dimension greater than 1. Iterating the process, we get a timelike subspace $A$ of dimension greater than 1, having no proper $L$-invariant subspace, that is, $L$ acts irreducibly on $A$. One can show that the orbit $Lu$ is unbounded for any non zero vector $u \in A$. Namely, let $A_0$ be the subspace of $A$ consisting of vectors $u$ whose $L$-orbit is bounded. This is clearly $L$-invariant.

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1 If a subgroup $H$ of $SO(q)$ preserves a direction spanned by a timelike vector $e$, then it also preserves its spacelike orthogonal, and therefore it preserves the positive product obtained from $q$ by reversing the sign of $q(e)$. Conversely, if $H$ preserves both a Lorentz form $q$ and a positive definite inner product $q_0$, then it commutes with the $q_0$-symmetric operator $S$ defined by $q = q_0(S \cdot \cdot)$. Such operator $S$ has precisely one negative eigenvalue, and the corresponding timelike direction is preserved by $H$. Note that this implies in particular that no precompact subgroup of $SO(q)$ acts irreducibly on $E$. 


and by irreducibility it is either 0 or the whole \( \mathcal{E} \). But \( \mathcal{A}_0 \) cannot coincide with \( \mathcal{A} \), because otherwise the image of \( L \) in \( \text{SO}(q|_A) \) would be precompact, and no precompact subgroup of \( \text{SO}(q|_A) \) acts irreducibly. Thus, every non zero element of \( \mathcal{A} \) has unbounded \( L \)-orbit.

Clearly, the action of \( L \) on the orthogonal space \( \mathcal{B} = \mathcal{A}^\perp \) is precompact, as \( q_0 \) is positive definite on \( \mathcal{B} \).

Let \( (u, v) \in \mathcal{A} \oplus \mathcal{B} \), with \( u \neq 0 \). Then, there is a sequence \( l_i \in L \), such that the direction of \( l_i(u, v) \) converges to a direction in \( \mathcal{A} \), because \( l_i(u) \) is unbounded and \( l_i(v) \) is bounded. Hence, \( \mathcal{A} \) is an attractor, and such dynamical characterization implies the uniqueness of \( \mathcal{A} \) as an irreducible timelike \( L \)-invariant subspace.

For part (2), let \( q \) be another \( L \)-invariant Lorentz form. Using the dynamical properties of \( L \) on \( \mathcal{A} \) and \( \mathcal{B} \), one proves that \( \mathcal{A} \) and \( \mathcal{B} \) are \( q \)-orthogonal. Namely, consider the set \( \{ u \in \mathcal{A} : u \text{ is } q \text{-orthogonal to } \mathcal{B} \} \). This is an \( L \)-invariant subspace of \( \mathcal{A} \), and by irreducibility it is either \( \{0\} \) or equal to \( \mathcal{A} \). Let us show that it is non zero. Fix \( u \in \mathcal{A} \), \( u \neq 0 \), and let \( l_i \) be a sequence in \( L \) such that \( l_i u \) is unbounded. Choose an auxiliary norm \( \| \cdot \| \) in \( \mathcal{A} \) and set \( u_i = l_i u / \| l_i u \| \); up to subsequences, we can assume that \( \lim_i u_i = u_\infty \in \mathcal{A} \) and \( u_\infty \neq 0 \). We claim that \( u_\infty \) is \( q \)-orthogonal to \( \mathcal{B} \); namely, given \( v \in \mathcal{B} \), \( q(u_\infty, v) = \lim_i q(u_i, v) = \lim_i q(u, l_i^{-1} v) / \| l_i u \| = 0 \), because \( l_i u \) is bounded.

It remains to check the proportionality condition along \( \mathcal{A} \). For this, we can assume \( B = 0 \), and that \( L \) acts irreducibly on \( \mathcal{E} \). Then, \( q \) can be written by means of \( q_0 \) via an endomorphism \( S \) of \( \mathcal{E} \), i.e., as bilinear forms, \( q = q_0(S \cdot, \cdot) \). The fact that \( L \) preserves both \( q \) and \( q_0 \) implies that \( S \) commutes with the elements of \( L \). Let us show that \( S \) has at least one real eigenvalue. Consider the group \( R = \text{SO}(q) \cap \text{SO}(q_0) \); since \( R \) contains \( L \), then it acts irreducibly on \( \mathcal{E} \). Now, \( R \) must contain either a hyperbolic or a unipotent element. Namely, \( R \) is a non compact algebraic group, and it has finitely many connected components. The connected component of the identity of \( R \) is a connected non compact Lie group, so it must contain a non precompact 1-parameter subgroup. Hence, it contains at least one non elliptic element. Also, an algebraic group is closed under Jordan decomposition, i.e., if \( A \in R \), then its elliptic, its hyperbolic and its unipotent factors belong to \( R \). This proves that \( R \) contains either a hyperbolic or a unipotent element. Such element has at least one real eigenvalue of multiplicity 1, and the corresponding eigenspace is preserved by \( S \). Thus, \( S \) has at least one real eigenvalue. The corresponding \( S \)-eigenspace is \( L \)-invariant, and by irreducibility, such eigenspace must coincide with \( \mathcal{E} \). It follows that \( S \) is a homothety, and this concludes the proof. \( \square \)

Apply Lemma 2.8 to \( L = \Gamma^* \), with the key observation that \( \Gamma \) normalizes \( \Gamma^* \), that is, \( \rho(a)^T \rho(a)^{-1} = \Gamma^H \) for all \( a \), since \( H(aba^{-1}) = \rho(a)H(b)\rho(a)^{-1} \) and \( U(aba^{-1}) = \rho(a)U(b)\rho(a)^{-1} \). It follows that \( \Gamma \) preserves the decomposition, and \( \rho(a)^* q_0 = \lambda(a) q_0 \). We know that \( H(a) \) and \( U(a) \) preserve \( q_0 \), and thus the image by \( \rho(a) \) coincides with its image under the elliptic part \( E(a) \), that is, \( E(a) \) acts homothetically on \( (A, q_0) \). But, since \( E(a) \) is elliptic, we have, \( \lambda(a) = \pm 1 \). In fact, \( \lambda(a) = +1 \), since we know that \( \rho(a)^* q_0 \) is a Lorentz form.

Summarizing, the action of \( \Gamma \) is \( q_0 \)-isometric on \( A \), and each element \( a \) acts on \( B \) via its elliptic part \( E(a) \), in other words \( \rho(a)|_B = E(a)|_B \). In order to prove that \( \Gamma \) preserves a Lorentz form on \( \mathcal{E} \) it suffices to show that its action (via the \( E(a) \)) preserves a positive definite form on \( B \). More precisely, let \( \Gamma^E \) be the subgroup of \( \text{GL}(B) \) generated by the \( E(a) \)'s. It is true that any \( A \in \Gamma^E \) is elliptic, in particular
it preserves a positive definite form. However, there are exotic examples of groups with all elliptic elements that are not pre-compact (see [15])! This is why in the statement of Proposition 2.7 we make an assumption stronger than just recurrence, but rather the existence of a an invariant measure. The hypothesis can be here seen as equivalent to the existence of an invariant measure on the space $\text{Sym}(\mathcal{B})$ supported on the open subset of positive-definite quadratic forms. At this stage, one could use Furstenberg Lemma which says, roughly speaking, that linear groups preserving a volume act pre-compactly on its support, see [19]. The proof of this statement in the present situation is straightforward. If $\mu$ is the preserved measure, then the mean $\int_{\text{Sym}(\mathcal{B})} q \, d\mu(q)$ is a positive definite quadratic form, invariant under the $\Gamma$-action.

This completes the proof of Proposition 2.7. □

**Corollary 2.9.** Let $\Gamma$ be a subgroup of $\text{GL}(k, \mathbb{Z})$ which acts on $\text{Sym}(\mathbb{R}^k)$ by preserving a finite measure supported in a in the open set of Lorentz forms. Then, up to a finite index, $\Gamma$ preserves a Lorentz form.

**Proof.** Indeed, up to a finite index, we can assume that $\Gamma$ has no torsion elements. This follows from Selberg Lemma (which says that a finitely generated matrix group has a torsion free subgroup of finite index, see for instance [2]). But, in a discrete group an elliptic $A$ element has finite order, because the group $\{A_n, n \in \mathbb{Z}\}$ is bounded and thus finite. We deduce that, up to a finite index, $\Gamma$ has no elliptic element, and thus Proposition 2.7 applies. □

### 2.3. A Gauss map.

Let $(M, g)$ be a compact Lorentzian manifold, let $\text{Iso}(M, g)$ denote its isometry group, which is Lie group (see for instance [9]), and denote by $\text{Iso}_0(M, g)$ the connected component of the identity of $\text{Iso}(M, g)$. The Lie algebra of $\text{Iso}(M, g)$ will be denoted by $\mathfrak{Iso}(M, g)$; let us recall that there is a Lie algebra anti-isomorphism from $\mathfrak{Iso}(M, g)$ to the space of Killing vector field $\mathfrak{kill}(M, g)$ obtained by mapping a vector $\nu \in \mathfrak{Iso}(M, g)$ to the Killing field $K^{\nu}$ which is the infinitesimal generator of the one-parameter group of isometries $\mathbb{R} \ni t \mapsto \exp(t\nu) \in \text{Iso}(M, g)$. If $\Phi$ is a diffeomorphism of $M$ and $K$ is a vector field on $M$, we will denote by $\Phi_*(K)$ the push-forward of $K$ by $\Phi$, which is the vector field given by $\Phi_*(K)(p) = d\Phi(\Phi^{-1}(p))K(\Phi^{-1}(p))$ for all $p \in M$. If $\Phi$ is an isometry and $K$ is Killing, then $\Phi_*(K)$ is Killing.

If $\Phi \in \text{Iso}(M, g)$, then:

$$\Phi_*(K^{\nu}) = K^{\text{Ad}_\Phi(\nu)}, \quad \forall \nu \in \mathfrak{Iso}(M, g). \tag{2.1}$$

It will be useful to introduce the following map. Let $\text{Sym}(g)$ denote the vector space of symmetric bilinear forms on $g$. The **Gauss map** $\mathcal{G} : M \to \text{Sym}(g)$ is the map defined by

$$\mathcal{G}_p(\nu, \omega) = g_p(K^{\nu}(p), K^{\omega}(p)), \tag{2.2}$$

for $p \in M$ and $\nu, \omega \in g$. The following identity is immediate:

$$\mathcal{G}_{\Phi(p)} = \mathcal{G}_p(\text{Ad}_\Phi \cdot, \text{Ad}_\Phi \cdot), \tag{2.3}$$

for all $\Phi \in \text{Iso}(M, g)$. In this paper we will be interested in the case where $(M, g)$ admits a Killing vector field which is timelike somewhere. In this situation, the image of the Gauss map contains a Lorentzian (nondegenerate) symmetric bilinear form on $g$ (in fact, a non empty open subset consisting of Lorentzian forms, this will be used in Lemma 5.4).
We now have the necessary ingredients to show how the proof of Theorem 1 is obtained from Theorem 2.

Proof of Theorem 1 from Theorem 2. Let us assume that the action of the discrete part $\Gamma$ on $\text{Iso}_0(M, g)$ is not of post-Riemannian type; we will show by contradiction that $\text{Iso}_0(M, g)$ has a somewhere timelike orbit. Let $\kappa$ be the quadratic form on $\text{Iso}(M, g)$ defined by

$$\kappa(v, w) = \int_M G_p(v, w) \, dM(p),$$

the integral being taken relatively to the volume element of the Lorentzian metric $g$. By (2.1), $\kappa$ is invariant by the conjugacy action. If $\text{Iso}_0(M, g)$ has no timelike orbit, then $\kappa$ is positive semi-definite. The proof will be concluded if we show that the kernel $\mathcal{K} = \text{Ker}(\kappa)$ has dimension less than or equal to 1. Assume that $\mathcal{K}$ is not trivial, i.e., that $\kappa$ is positive semi-definite. If $v \in \mathcal{K}$, then for all $w \in \text{Iso}(M, g)$ and all $p \in M$, $g_p(K^\nu_p, K^\mu_p) = 0$; in particular, $g_p(K^\nu_p, K^\mu_p) = 0$, i.e., $K^\nu$ is an everywhere isotropic Killing vector field on $M$. Non trivial isotropic Killing vector fields are never vanishing, see for instance [4, Lemma 3.2]; this implies that the map $\mathcal{K} \ni v \mapsto K^\nu_p \in T_pM$ is an injective vector space homomorphism for all $p \in M$. On the other hand, its image has dimension 1, because an isotropic subspace of a Lorentz form has dimension at most 1, hence $\mathcal{K}$ has dimension 1. \hfill $\Box$

2.4. Precompactness of 1-parameter subgroups. It will be useful to recall that there is a natural smooth left action of $\text{Iso}(M, g)$ on the principal bundle $\mathcal{F}(M)$ of all linear frames of $TM$, defined as follows. If $b = (v_1, \ldots, v_n)$ is a linear basis of $T_pM$, then for $\Phi \in \text{Iso}(M, g)$ set $\Phi(b) = (d\Phi_p(v_1), \ldots, d\Phi_p(v_n))$, which is a basis of $T_{\Phi(p)}M$. The action of $\text{Iso}(M, g)$ on $\mathcal{F}(M)$ is defined by $\text{Iso}(M, g) \times \mathcal{F}(M) \ni (\Phi, b) \mapsto \Phi(b) \in \mathcal{F}(M)$. Given any frame $b \in \mathcal{F}(M)$, then the map $\text{Iso}(M, g) \ni \Phi \mapsto \Phi(b) \in \mathcal{F}(M)$ is an embedding of $\text{Iso}(M, g)$ onto a closed submanifold of $\mathcal{F}(M)$ (see [9, Theorem 1.2, Theorem 1.3]), and thus the topology and the differentiable structure of $\text{Iso}(M, g)$ can be studied by looking at one of its orbits in the frame bundle. In particular, the following will be used at several points:

Precompactness criterion. If $H \subset \text{Iso}(M, g)$ is a subgroup that has one orbit in $\mathcal{F}(M)$ which is contained in a compact subset of $\mathcal{F}(M)$, then $H$ is precompact. For instance, if $H$ preserves some Riemannian metric on $M$ and it leaves a non empty compact subset of $M$ invariant, then $H$ is precompact.

Lemma 2.10. Let $(M, g)$ be a compact Lorentzian manifold and $K$ be a Killing vector field on $M$. If $K$ is timelike at some point, then it generates a precompact 1-parameter subgroup of isometries in $\text{Iso}_0(M, g)$.

Proof. Let $p \in M$ be such that $g(K(p), K(p)) < 0$. Consider the compact subsets of $TM$ given by:

$$V = \left\{ K(q) : q \in M \text{ is such that } g(K(q), K(q)) = g(K(p), K(p)) \right\},$$

Here we use the following terminology: a vector $v \in TM$ is isotropic if $g(v, v) = 0$, and it is lightlike if it is isotropic and non zero.
2.5. An algebraic criterion for precompactness. Now observe that if a compact manifold \((M, g)\) admits a Killing vector field which is timelike somewhere, then, by continuity, sufficiently close Killing fields are also timelike somewhere. Thus, if one wants to study the (connected component of the) isometry group of a Lorentz manifold that has a Killing vector field which is timelike at some point, it is a natural question to ask is which (connected) Lie groups have open sets of precompact 1-parameter subgroups. The problem is better cast in terms of the Lie algebra; we will settle this question in our next:

**Proposition 2.11.** Let \(G\) be a connected Lie group, \(K \subset G\) be a maximal compact subgroup, and \(\mathfrak{k} \subset \mathfrak{g}\) be their Lie algebras. Let \(\mathfrak{m}\) be a \(\text{Ad}_K\)-invariant complement of \(\mathfrak{k}\) in \(\mathfrak{g}\). Then, \(\mathfrak{g}\) has a non empty open cone of vectors that generate precompact 1-parameter subgroups of \(G\) if and only if there exists \(v \in \mathfrak{k}\) such that the restriction of \(\text{ad}_v : \mathfrak{m} \to \mathfrak{m}\) is an isomorphism.

**Proof.** Let \(\mathfrak{c} \subset \mathfrak{g}\) be the cone of vectors that generate precompact 1-parameter subgroups of \(G\); we want to know when \(\mathfrak{c}\) has non empty interior. Clearly \(\mathfrak{c}\) contains \(\mathfrak{k}\), and every element of \(\mathfrak{c}\) is contained in the Lie algebra \(\mathfrak{t} \subset \mathfrak{k}\) of some maximal compact subgroup \(K'\) of \(G\). Since all maximal compact subgroups of \(G\) are conjugate (see for instance [8]), it follows that \(\mathfrak{c} = \text{Ad}_{K'}(\mathfrak{t})\), i.e., \(\mathfrak{c}\) is the image of the map \(F : G \times \mathfrak{t} \to \mathfrak{g}\) given by \(F(g, v) = \text{Ad}_g(v)\). We claim that \(\mathfrak{c}\) has non empty interior if and only if the differential \(dF\) has maximal rank at some point \((g, v) \in G \times \mathfrak{t}\). The condition is clearly sufficient, and by Sard’s theorem is also necessary; namely, if \(dF\) has never maximal rank then all the values of \(F\) are critical, and they must form a set with empty interior. The second claim is that it suffices to look at the rank of \(dF\) at the points \((e, v)\), where \(e\) is the identity of \(G\). This follows easily observing that the function is \(G\)-equivariant in the first variable. Now, the differential of \(F\) at \((e, v)\) is easily computed as:

\[
dF_{(e,v)}(g, \mathfrak{t}) = [g, v] + \mathfrak{t} = [\mathfrak{m}, v] + \mathfrak{t}.
\]

Thus, \(dF_{(e,v)}\) is surjective if and only if there exists \(v \in \mathfrak{k}\) such that \([\mathfrak{m}, v] = \mathfrak{m}\), which concludes the proof. \qed

3. The identity connected component of the isometry group

The geometric structure of compact Lorentz manifold whose isometry group contains a group which is locally isomorphic to an **oscillator group** or to \(\text{SL}(2, \mathbb{R})\) is well known. Let us recall (see [17, §1.6]) that a compact Lorentz manifold that admits a faithful isometric action of a group locally isomorphic to \(\text{SL}(2, \mathbb{R})\) has universal cover which is given by a warped product of the universal cover of \(\text{SL}(2, \mathbb{R})\), endowed with the bi-invariant Lorentz metric given by its Killing
form, and a Riemannian manifold. Every such manifold admits everywhere time-like Killing vector field, corresponding to the timelike vectors of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$.

Oscillator groups are characterized as the only simply-connected solvable and non commutative Lie groups that admit a bi-invariant Lorentz metric (see [11]); oscillator groups possess a lattice, i.e., a co-compact discrete subgroup. More precisely, an oscillator group $G$ is a semi-direct product $\mathbb{S}^1 \ltimes \text{Heis}$, where $\text{Heis}$ is a Heisenberg group (of some dimension $2d + 1$). There are positivity condition on the eigenvalues of the automorphic $\mathbb{S}^1$ action on the Lie algebra $\mathfrak{heis}$ (ensuring the existence of a bi-invariant Lorentz metric), and arithmetic conditions on them (ensuring existence of a lattice).

It is interesting and useful to consider oscillator groups as objects completely similar to $\text{SL}(2,\mathbb{R})$, from a Lorentz geometry viewpoint. In particular, regarding our arguments in the present paper, both cases are perfectly parallel. Let us notice however, some differences (but with no incidence on our investigation here). First, of course, the bi-invariant Lorentz metrics on an oscillator group do not correspond to its Killing form, since this latter is degenerate (because the group is solvable). Another fact, in non-uniqueness of these bi-invariant metrics, but surprisingly, their uniqueness up to automorphism. In the $\text{SL}(2,\mathbb{R})$-case, we have uniqueness up to a multiplicative constant. Also, we have essential uniqueness of lattices in an oscillator group, versus their abundance in $\text{SL}(2,\mathbb{R})$.

Let us describe briefly the construction of Lorentz manifolds endowed with a faithful isometric $G$-action, where $G$ is either $\text{SL}(2,\mathbb{R})$ or an oscillator group. The construction starts by considering right quotients $G/\Gamma$, where $\Gamma$ is a lattice of $G$. The $G$-left action is isometric exactly because the metric is bi-invariant. A slight generalization is obtained by considering a Riemannian manifold $(\tilde{N},\tilde{g})$ and quotients of the direct metric product $X = \tilde{N} \times G$ by a discrete subgroup $\Gamma$ of $\text{Iso}(\tilde{N},\tilde{g}) \times G$. Observe here that since the isometry group of the Lorentz manifold $X$ is $\text{Iso}(\tilde{N},\tilde{g}) \times (G \times G)$, it is possible to take a quotient by a subgroup $\Gamma$ contained in this full group. The point is that we assumed $G$ acting (on the left) on the quotient, and hence, $G$ normalizes $\Gamma$; but since $G$ is connected, it centralizes $\Gamma$. Therefore, only the right $G$ factor in the full group remains (since the centralizer of the left action is exactly the right factor). Observe however that $\Gamma$ does not necessarily split. Indeed, there are examples where $\Gamma$ is discrete co-compact in $\text{Iso}(\tilde{N},\tilde{g}) \times G$, but its projection on each factor is dense!

Next, warped products yield a more general construction. Rather than a direct product metric $\tilde{g} \oplus \kappa$, one endows $\tilde{N} \times G$ with a metric of the form $\tilde{g} \oplus w\kappa$, where $w$ is a positive function on $\tilde{N}$, and $\kappa$ is the bi-invariant Lorentz metric on $G$. Here, there is one difference between the case of $\text{SL}(2,\mathbb{R})$ and the oscillator case. For $\text{SL}(2,\mathbb{R})$ this is the more general construction, but in the oscillator case, some “mixing” between $G$ and $\tilde{N}$ and also a mixing of their metrics, is also possible, see [17, §1.2]. Finally, to be more accurate, it should be emphasized that it is the universal cover of $G$ that must be considered in these constructions. (It turns out however that, it is a finite cover of $G$ which acts faithfully and not the universal cover. This is because of the compactness assumption on the Lorentz manifold $M$, see [17, §2]).
Let us study now the situation when the isometry group does not contain any group which is locally isomorphic to \( \text{SL}(2, \mathbb{R}) \) or to an oscillatory group.

**Theorem 3.1.** Let \((M, g)\) be a compact Lorentz manifold that admits a Killing vector field which is timelike at some point. Then, the identity component of its isometry group is compact, unless it contains a group locally isomorphic to \( \text{SL}(2, \mathbb{R}) \) or to an oscillatory group.

**Proof.** By the classification result in [1, 17], if \( \text{ISO}_0(M, g) \) does not contain a group locally isomorphic to \( \text{SL}(2, \mathbb{R}) \) or to an oscillatory group, then \( \text{ISO}(M, g) \) can be written as a Lie algebra direct sum \( \mathfrak{h} + \mathfrak{a} + \mathfrak{c} \), where \( \mathfrak{h} \) is a Heisenberg algebra, \( \mathfrak{a} \) is abelian, and \( \mathfrak{c} \) is semi-simple and compact. Our aim is to show that the Heisenberg summand \( \mathfrak{h} \) in fact does not occur in the decomposition, and that the abelian group \( A \) corresponding to the summand \( \mathfrak{a} \) is compact. By the assumption that \((M, g)\) has a Killing vector field which is timelike at some point, \( \text{ISO}(M, g) \) must contain a non empty open cone of vectors that generate a precompact 1-parameter subgroup of \( \text{ISO}_0(M, g) \) (Lemma 2.10). The first observation is that, since \( \mathfrak{c} \) is compact, if \( \mathfrak{h} + \mathfrak{a} + \mathfrak{c} \) has an open cone of vectors that generate precompact 1-parameter subgroups, then so does the subalgebra \( \mathfrak{h} + \mathfrak{a} \). Moreover, by the same compactness argument, we can also assume that the abelian Lie subgroup \( A \) is simply connected. The proof of our result will be concluded once we show that given any Lie group \( G \) with Lie algebra \( \mathfrak{g} = \mathfrak{h} + \mathfrak{a} \), \( \mathfrak{h} \) and \( \mathfrak{a} \) as above, does not have an open set of precompact 1-parameter subgroups. To this aim, write \( \mathfrak{h} + \mathfrak{a} = \mathfrak{m} + \mathfrak{t} \), with \( \mathfrak{t} \) the Lie algebra of a maximal compact subgroup \( K \) of \( G \) and \( \mathfrak{m} \) a \( \mathfrak{t} \)-invariant complement of \( \mathfrak{t} \) in \( \mathfrak{g} \). If either \( \mathfrak{h} \) or \( \mathfrak{a} \) is not zero, then also \( \mathfrak{m} \) is non zero. Since \( \mathfrak{h} + \mathfrak{a} \) is nilpotent, for no \( x \in \mathfrak{t} \) the map \( \text{ad}_x : \mathfrak{m} \to \mathfrak{m} \) is injective, and by Proposition 2.11, \( G \) does not have an open set of precompact 1-parameter subgroups. \( \square \)

4. *When the isometry group has infinitely many connected components*

Let us now study the situation when the isometry group of a Lorentzian manifold with a timelike Killing vector field has infinitely many connected components.

**Example.** An important example is that of a flat torus, see for instance [5]. Consider a Lorentzian scalar product \( g_0 \) in \( \mathbb{R}^n \), and let the quotient \( M = T^n = \mathbb{R}^n / \mathbb{Z}^n \) endowed with the induced metric, still denoted by \( g_0 \). Then, \( \text{ISO}(M, g_0) \) is isomorphic to the semi-direct product \( T^n \rtimes O(g_0, \mathbb{Z}) \), where \( O(g_0, \mathbb{Z}) \) is the infinite discrete group \( O(g_0) \cap \text{GL}(n, \mathbb{Z}) \).

4.1. **Compactness of the identity connected component.** A first consequence of Theorem 3.1 is that when \( \text{ISO}(M, g) \) has infinitely many connected components, then the connected component of the identity has to be compact. Before proving this, let us recall a general fact on the isometry group of a covering. If \((M, g)\) is a pseudo-Riemannian manifold and \( \pi : \tilde{M} \to M \) is a covering, let \( \tilde{g} \) be the pseudo-Riemannian metric on \( \tilde{M} \) given by the pull-back of \( g \) by \( \pi \), so that \( \pi \) becomes a local isometry. Let \( H \subset \text{ISO}(\tilde{M}, \tilde{g}) \) be the closed subgroup consisting of all isometries \( \tilde{f} : \tilde{M} \to \tilde{M} \) that descend to the quotient \( M \), i.e., such that \( \pi(\tilde{f}(x)) = \pi(\tilde{f}(y)) \) whenever \( \pi(x) = \pi(y) \). There is a natural homomorphism \( \varphi : H \to \text{ISO}(M, g) \) that associates to each \( \tilde{f} \in H \) the unique isometry \( f \) of \( M \) such that \( f \circ \pi = \pi \circ \tilde{f} \). The kernel of \( \varphi \) is a discrete subgroup \( \Gamma \) of \( H \) given by the group
of covering automorphisms of \( \pi \). If \( \pi \) is the universal covering of \( M \), then the homomorphism \( \varphi \) is surjective, and \( H \) is precisely the normalizer \( \text{Nor}(\Gamma) \) of \( \Gamma \) in \( \text{Iso}(\tilde{M}, \tilde{g}) \), so that \( \text{Iso}(M, g) \) is isomorphic to the quotient \( \text{Nor}(\Gamma)/\Gamma \).

**Proposition 4.1.** Let \((M, g)\) be a compact Lorentzian manifold that admits a somewhere timelike Killing vector field. If \(\text{Iso}(M, g)\) has infinitely many connected components, then \(\text{Iso}_0(M, g)\) is compact.

**Proof.** By Theorem 3.1, if \(\text{Iso}_0(M, g)\) is not compact, then it contains a group which is locally isomorphic to \(\text{SL}(2, \mathbb{R})\) or to an oscillator group, in which case the geometric structure of \((M, g)\) is well understood (see [17, Theorem 1.13, Theorem 1.14]).

Assume that \(\text{Iso}_0(M, g)\) contains a group locally isomorphic to \(G = \text{SL}(2, \mathbb{R})\). Then, there exists a Riemannian manifold \((\tilde{N}, \tilde{g})\) such that, up to a finite cover, \(M\) is a quotient of \(X = \tilde{N} \times G\) by a subgroup \(\Gamma\) of \(\text{Iso}(\tilde{N}, \tilde{g}) \times G\). Here, \(G\) is endowed with the bi-invariant Lorentzian metric defined by the Killing form of \(\text{SL}(2, \mathbb{R})\) and the metric of \(\tilde{N} \times G\) may be warped rather than a direct product. Any isometry of \(X\) that normalizes the \(G\)-action on \(X\) is the product of an isometry \(a\) of the first factor \(\tilde{N}\), and an isometry of \(G\) of the form \((b, c) \in G \times G\), with \(b\) acting by right multiplication and \(c\) by left multiplication on \(G\). When the metric of \(X\) is warped, then \(a\) must preserve the warping function. We will denote by \((a, b, c)\) such an isometry of \(X\).

The fundamental group \(\Gamma\) commutes with the \(G\)-action, and therefore its elements have the form \((a, b, 1)\), for they must commute with all elements of the form \((1, 1, c)\). One can define a Riemanian metric on \(X\) by replacing the Lorentz metric of the factor \(G\) with a right invariant Riemannian metric on \(G\), that will be denoted by \(m\). By construction, such a new Riemannian metric on \(X\) is preserved by \(\Gamma\), thus it descends to a Riemannian metric \(h\) on \(M\). Now, the map \((a, b, c) \mapsto (a, b)\) defines a homomorphism

\[
\phi : \text{Iso}(M, g) \rightarrow \text{Iso}(M, h),
\]

whose kernel is exactly \(G\). Observe however that \(\text{Iso}(M, h)\) is compact, but \(\phi\) may not be onto. Denote by \(H_1\) the closed subgroup of \(\text{Iso}(M, h)\) consisting of elements that preserve the foliation of \(M\) determined by \(G\). This is a closed subgroup, and all its elements have the split form \((a, d)\), where \(d \in \text{Iso}(G, m)\).

Now, \(\text{Iso}(G, m)\) contains \(G\) (acting by right multiplication), but it may be strictly larger than \(G\) when the isotropy group of \((G, m)\) is non trivial, and a further reduction is needed. More precisely, let \(e_1, e_2, e_3\) be a basis of the Lie algebra of \(G\), and let \(E_1, E_2, E_3\) be the corresponding vector fields on \(M\). Denote by \(H_2\) the closed subgroup of \(H_1\) consisting of isometries \(f \in \text{Iso}(M, h)\) such that \(f_* E_i = E_i\), \(i = 1, 2, 3\). The lift to \(X\) of such an isometry \(f\) has the form \((a, b, 1)\), thus we have an exact sequence \(1 \rightarrow G \rightarrow \text{Iso}(M, g) \rightarrow H_2 \rightarrow 1\). Since \(H_2\) is compact, it has a finite number of connected components. Moreover, \(G\) is connected, it follows that \(\text{Iso}(M, g)\) has the same number of connected components as \(H_2\), and we are done.

The case where \(\text{Iso}_0(M, g)\) contains an oscillator group is treated following the same arguments. \(\square\)
4.2. A further compactness result.

**Lemma 4.2.** Let $G$ be a Lie group acting on a manifold $R$, and let $G_0$ be a compact normal subgroup of $G$ all of whose orbits in $R$ have the same dimension. Then:

- the distribution $\Delta$ tangent to the $G_0$-orbits is smooth, and it is preserved by $G$;
- there exists a Riemannian metric $h_0$ on $\Delta$ which is preserved by $G_0$ and by the centralizer $\text{Centr}(G_0)$ of $G_0$ in $G$. 

**Proof.** Introduce the following notation: for $x \in R$, let $\beta_x : G \to R$ be the map $\beta_x(g) = g \cdot x$, and let $L_x : \mathfrak{g} \to T_x M$ be its differential at the identity. Here $\mathfrak{g}$ is the Lie algebra of $G$. The map $M \times \mathfrak{g} \ni (x, v) \mapsto L_x(v) \in TM$ is a smooth vector bundle morphism from the trivial bundle $M \times \mathfrak{g}$ to $TM$. The distribution $\Delta$ is the image of the sub-bundle $M \times \mathfrak{g}_0$, where $\mathfrak{g}_0 \subset \mathfrak{g}$ is the Lie algebra of $G_0$. Since the orbits of $G_0$ have the same dimension, then the image of $M \times \mathfrak{g}_0$ is a smooth sub-bundle of $TM$ (recall that the image of a vector bundle morphism is a smooth sub-bundle if it has constant rank). The action of $G$ preserves $\Delta$ because $G_0$ is normal, which concludes the proof of the first assertion.

The construction of $h_0$ goes as follows. Choose a positive definite inner product $B$ on $\mathfrak{g}_0$ which is $\text{Ad}_{G_0}$-invariant; the existence of such $B$ follows from the compactness of $G_0$. For all $x \in R$, the restriction to $\mathfrak{g}_0$ of $L_x$ gives a surjection $L_x|_{\mathfrak{g}_0} : \mathfrak{g}_0 \to \Delta_x$; denote by $V_x$ the $B$-orthogonal complement of the kernel of this map, given by $\text{Ker}(L_x|_{\mathfrak{g}_0}) = \text{Ker}(L_x) \cap \mathfrak{g}_0$. The value of $h_0$ on $\Delta_x$ is defined to be the push-forward via the map $L_x$ of the restriction of $B$ to $V_x$. In order to see that such metric is invariant by the action of $G_0$ and of its centralizer, for $g \in G$ denote by $I_g : G_0 \to G_0$ the conjugation by $g$ (recall that $G_0$ is normal) and by $\gamma_g : M \to M$ the diffeomorphism $x \mapsto g \cdot x$; for fixed $x \in M$ we have a commutative diagram:

\[
\begin{array}{ccc}
G_0 & \xrightarrow{T_g} & G_0 \\
\downarrow{\beta_x} & & \downarrow{\beta_{g \cdot x}} \\
M & \xrightarrow{\gamma_g} & M
\end{array}
\]

Differentiating at the identity the diagram above we get:

\[
\begin{array}{ccc}
\mathfrak{g}_0 & \xrightarrow{\text{Ad}_g} & \mathfrak{g}_0 \\
\downarrow{L_x} & & \downarrow{L_{g \cdot x}} \\
\Delta_x & \xrightarrow{\text{d}\gamma_g(x)} & \Delta_{g \cdot x}
\end{array}
\]

Now assume that $g$ is such that $\text{Ad}_g$ preserves $B$; this holds by assumption when $g \in G_0$, and clearly also for $g$ in the centralizer of $G_0$ (in which case $\text{Ad}_g$ is the identity!). For such a $g$, since $\text{Ad}_g(\text{Ker}(L_x) \cap \mathfrak{g}_0) = \text{Ker}(L_{g \cdot x})$, then also $\text{Ad}_g(V_x) = V_{g \cdot x}$, and thus we have a commutative diagram:

\[
\begin{array}{ccc}
V_x & \xrightarrow{\text{Ad}_g} & V_{g \cdot x} \\
\downarrow{L_x} & & \downarrow{L_{g \cdot x}} \\
\Delta_x & \xrightarrow{\text{d}\gamma_g(x)} & \Delta_{g \cdot x}
\end{array}
\]
from which it follows that $d\gamma_g(x)$ preserves the metric $h_0$, proving the last statement in the thesis.

Let us now assume throughout that $(M, g)$ is a compact Lorentzian manifold that admits a Killing vector field $K$ which is timelike somewhere, and with $\text{Iso}_0(M, g)$ compact. To simplify notations, let us set $G = \text{Iso}(M, g)$, $G_0 = \text{Iso}_0(M, g)$, $g = \mathfrak{iso}(M, g)$, $\text{Aut}(G_0)$ the group of automorphisms of $G_0$, $\text{Inn}(G_0)$ the normal subgroup of inner automorphisms of $G_0$, and $\text{Out}(G_0) = \text{Aut}(G_0)/\text{Inn}(G_0)$. Since $G_0$ is normal in $G$, then $G$ acts by conjugation on $G_0$, and we have a homomorphism $G \to \text{Out}(G_0)$, and composing with the quotient map we have a homomorphism:

$$\rho : G \to \text{Out}(G_0),$$

whose kernel $G_1 = \text{Ker}(\rho)$ is the product $G_0 \cdot \text{Centr}(G_0)$.

**Proposition 4.3.** $G_1$ is compact.

**Proof.** Let $R$ be the (non empty) open subset of $M$ consisting of all points $x$ whose $G_0$-orbit $\mathcal{O}(x)$ has maximal dimension (among all $G_0$-orbits), and such that $\mathcal{O}(x)$ is timelike, i.e., the restriction of $g$ to $\mathcal{O}(x)$ is Lorentzian. Recall that the set of points whose $G_0$-orbit has maximal dimension is open and dense, and $R$ is the intersection of this dense open subset with the open subset of $M$ where $K$ is timelike. We claim that there exists a Riemannian metric $h$ on $R$ which is preserved by $G_1$. Such a metric $h$ is constructed as follows: on the distribution $\Delta$ tangent to the $G_0$-orbits is given by the metric $h_0$ as in Lemma 4.2, on the $g$-orthogonal distribution $\Delta'$ it is the (positive definite) restriction of $g$, furthermore $\Delta$ and $\Delta'$ are declared to be $h$-orthogonal. Note that the distributions $\Delta$ and $\Delta'$ are preserved by $G$, the metric $g$ on $\Delta'$ is preserved by all elements of $g$, and the metric $h_0$ on $\Delta$ is preserved by all elements of $G_1$, which proves our claim.

Observe that $G_1$ is closed in $G$. Using the precompactness criterion, in order to prove the compactness of $G_1$ it suffices to show that $G_1$ leaves some compact subset of $R$ invariant. If $d$ is the dimension of the principal $G_0$-orbits, let $\text{vol}_d$ denote $d$-dimensional volume induced by the Lorentzian restriction of $g$ on the $G_0$-orbits in $R$. Set $a = \sup_{x \in M} \text{vol}_d(\mathcal{O}(x))$, and let $T$ the compact subset of $M$ consisting of all points $x$ such that $\text{vol}_d(\mathcal{O}(x)) = a$. Note that the volume function is lower semi-continuous, see Lemma 4.4 below, and it admits maximum in the compact set of all $G_0$-orbits, so that $a$ is well defined and $T$ is not empty. Moreover, since the condition $\text{vol}_d(\mathcal{O}(x)) \geq a$ is closed, then $T$ is compact. Clearly, $T$ is contained in $R$, and it is preserved by $G_1$. This concludes the proof.

We have used the following result, which has some interest in its own:

**Lemma 4.4.** Let $G$ be a compact Lie group that acts smoothly by isometries on a Lorentzian manifold $(L, g_L)$ in such a way that:

(a) all the $G$-orbits have the same dimension;

(b) every $G$-orbit is timelike, i.e., the restriction of the metric $g_L$ to each orbit is Lorentzian.

Then, the function $L \ni x \mapsto \text{vol}_d \in \mathbb{R}$ is lower semi-continuous, where $\text{vol}_d$ denotes the volume of the orbit $\mathcal{O}(x)$ relative to the volume induced by the restriction to $\mathcal{O}(x)$ of the Lorentzian metric $g_L$. 

Proof. By standard result on group actions (see for instance [6]), through every $x \in L$ there is a slice for the action of $G$, i.e., a smooth submanifold $S_x \subset L$ containing $x$ that has (among others) the following properties:

1. the map $G \times S_x \ni (g, y) \mapsto g \cdot y \in L$ is open;
2. denoting by $G_x$ the isotropy of $x$, if $g \notin G_x$ then $g \cdot S_x \cap S_x = \emptyset$.

By (1), if $V \subset S_x$ is a neighborhood of $x$ in $S_x$, then $G \cdot V$ is a neighborhood of $x$ in $L$; thus, since the function volume is constant on each orbit, it suffices to show that the map $S_x \ni y \mapsto \text{vol}_y \in \mathbb{R}$ is lower semi-continuous.

By (2), if $y \in S_x$, then $G y \subset G_x$. Let $H \subset G_x$ be the identity connected component of $G_x$. By assumption (a), $\dim(\mathcal{O}(y)) = \dim(\mathcal{O}(x))$, from which it follows that $H$ is contained in $G_y$. We therefore have a well defined map $\overline{\beta} : G/H \times S_x \to L$, given by $\overline{\beta}(gH, y) = g \cdot y$. This map is smooth, because $q : G \times S_x \to (G/H) \times S_x$ is a smooth surjective submersion, and $\overline{\beta} \circ q : G \times S_x \to L$ is smooth.

For $y \in S_x$, consider the smooth map $\overline{\beta}_y : G/H \to L$ defined by $\overline{\beta}_y(gH) = \overline{\beta}(gH, y)$; its image is given by the embedded submanifold $\mathcal{O}(y)$, thus we have a smooth map $\overline{\beta}_y : G/H \to \mathcal{O}(y)$. Such map is a finite covering, with folding (cardinality of the fiber) equal to $|G_y/H|$. Namely, $\overline{\beta}_y$ can be written as the composition of the diffeomorphism $G/G_y \ni gG_y \mapsto g \cdot y \in \mathcal{O}(y)$ with the covering map $G/H \to G/G_y$. The latter is a smooth fibration, with discrete fiber, hence a covering map, with folding equal to $|G_y/H|$ (see for instance [12, Proposition 2.1.14]).

It follows that the map $\overline{\beta}_y : G/H \to L$ is an immersion (composition of an immersion and a local diffeomorphism), and we obtain a symmetric $(0,2)$-tensor on $G/H$ given by the pull-back $g_y = \overline{\beta}_y^* (g_H)$. In fact, we have a smoothly varying family $S_x \ni y \mapsto g_y$ of symmetric $(0,2)$-tensors on $G/H$. Such tensors are non-degenerate and Lorentzian, in fact $g_y$ is also given as the pull-back by the covering map $\overline{\beta}_y : G/H \to \mathcal{O}(y)$ of the restriction to $\mathcal{O}(y)$ of $g_L$. It follows that the volume of $G/H$ relatively to the metric $g_y$, which is a continuous function of $y$ (volume relative to a continuous family of measures), is equal to $|G_y/H| \text{vol}_y$.

The desired semi-continuity property of the function $y \mapsto \text{vol}_y$ follows now easily observing that for $y \in S_x$, $G_y \subset G_x$, thus $|G_y/H| \text{vol}_y \leq |G_x/H| \text{vol}_y$. □

5. The toral factor

We will now pursue the study of the action of subgroups of the isometry group of $M$. The idea is to consider a suitable reduction of $\text{Iso}(M, \mathbf{g})$, i.e., a closed subgroup $G \subset \text{Iso}(M, \mathbf{g})$ such that, denoting by $G_0$ its identity connected component, the following properties are satisfied:

$$G \text{ is non compact},$$

$$G_0 \text{ is compact},$$

$$G_0 \text{ has some timelike orbit}.$$ 

We observe that (5.1) is preserved by passing to finite index subgroups of $G$. Namely, if $G' \subset G$ has finite index, then it is closed, non compact, and it has the same connected component of the identity as $G$. 
5.1. Reduction of $G^0$. Recall the homomorphism $\rho : G \to \text{Out}(G_0)$ in (4.1); by Proposition 4.3, $G_1 = \text{Ker}(\rho)$ is compact, and in particular $\rho(G)$ is non compact.

If $G_0$ has an almost decomposition $T^k \times K$, where $K$ is semisimple, then the image of $\rho$ is contained in $\text{Out}(T^k) = \text{GL}(k, \mathbb{Z})$.

Since $K$ is normal, we have a representation $r : G \to \text{Aut}(K)$; let $G'$ be its kernel, which is the centralizer of $K$ in $G$. One can see that $G/G' = K/\text{Centr}(K)$. Indeed, since $\text{Aut}(K) = \text{Int}(K)$, for any $f \in G$, there exists $k \in K$ such that $r(f) = r(k)$, that is, $fk^{-1} \in G'$. Therefore, $G/G'$ is a quotient of $K$. This quotient is easily identified with $K/\text{Centr}(K)$.

In some sense, going from $G$ to $G'$ allows one to kill the semi-simple factor, that is to assume that the identity component is a torus, and that the discrete part $G/G^0$ has not changed. More precisely, let us now describe how to “forget” the semi-simple factor $K$ keeping the identity component with somewhere timelike orbits. Let $X$ be a somewhere timelike Killing field. The closure of its flow is a product (possibly trivial) of two tori, $K_1 \times K_2$, where $K_1$ (resp., $K_2$) is a subgroup of $T^k$ (resp., of $K$). Since $G'$ centralizes $K_2$, we have a direct product group $G' \times K_2$.

Summarizing, we have proven the following:

Lemma 5.1. There is a subgroup $G$ of $\text{Iso}(M, g)$ satisfying (5.1) and having an abelian identity component $G_0 = T^k$.

With such reduction of the group $G$, we can now consider the action of $G$ on $G_0 \cong T^k$ given by the representation $\rho : G \to \text{Out}(T^k) = \text{GL}(k, \mathbb{Z})$; in order to distinguish the action of $G$ on $M$ and on $T^k$, we will call the latter the $\rho$-action.

Corollary 5.2. Up to a finite index reduction, the quotient group $\Gamma = G/G_0$ is torsion free, i.e., all its non trivial elements have infinite order.

Proof. Choose any torsion free finite index subgroup $H$ of $\text{GL}(k, \mathbb{Z})$ (it exists by Selberg lemma [2]), and set $G' = \rho^{-1}(H)$. This is a finite index torsion-free subgroup of $G$. \qed

5.2. Preliminary properties of the $T^k$-action.

Lemma 5.3. The $T^k$-action on $M$ is locally free on a dense open set.

Proof. It follows immediately from Corollary 2.4. \qed

Lemma 5.4. After replacing $G$ by a finite index subgroup, the $\rho$-action on $T^k$ preserves some Lorentz metric. In particular, one can see $\rho(G)$ as lying in $\text{GL}(k, \mathbb{Z}) \cap \text{SO}(q)$, where $q$ is a Lorentz form on $\mathbb{R}^k$.

Proof. The $\rho$-action of $G$ on $G_0 \cong T^k$ by conjugation induces an action of $G$ on the space $\text{sym}(\mathbb{R}^k)$ of symmetric bilinear forms on $\mathbb{R}^k$. By (2.3), the compact subset given by the image of the Gauss map $G$ is invariant by this action; moreover, the image of the Lebesgue measure of $M$ by the Gauss map is also invariant by $\rho(G)$. Such compact subset contains a non empty open subset consisting of Lorentz forms, because $G_0$ has timelike orbits in $M$. By Corollary 2.9, there exists a finite index subgroup of $H'$ of $\rho(G)$ that preserves some Lorentz form on $\mathbb{R}^k$. The desired finite index subgroup of $G$ is $\rho^{-1}(H')$. \qed
6. Actions of Almost Cyclic Groups

Choose \( f \in G, f \not\in G_0 \); then, \( \rho(f) \in \text{GL}(k, \mathbb{Z}) \) has infinite order by Corollary 5.2. Consider the group \( G = G_f \) generated by \( f \) and \( T^k \). Up to a compact normal subgroup, \( G_f \) is cyclic, which justifies the name almost cyclic. One can prove:

Lemma 6.1. \( G = G_f \) is a closed subgroup of \( \text{ISO}(M, g) \) satisfying (5.1). It is isomorphic to a semi-direct product \( \mathbb{Z} \times T^k \).

Lemma 6.2. If \( A \in \text{GL}(k, \mathbb{Z}) \) is parabolic, then some power of \( A \) is rationally equivalent to:

\[
\begin{pmatrix}
1 & t & -t^2/2 \\
0 & 1 & t \\
0 & 0 & 1
\end{pmatrix}
\]

This means that the subspaces \( \{e_1\}, \{e_1, e_2\}, \{e_1, e_2, e_3\} \) and \( \{e_4, \ldots, e_k\} \) are rational.

In particular, there is an \( A \)-invariant rational 3-space, on whose orthogonal, which is not necessarily rational, the \( A \)-action is trivial.

Proof. The proof is quite standard. Let \( A \) have the normal form above. Consider \( \mathcal{E} \) the kernel of \( (A - 1)^3 \), which is a rational subspace, and it contains the subspace \( \mathcal{E}_0 = \{e_1, e_2, e_3\} \). On \( \mathcal{E}/\mathcal{E}_0 \), \( A \) is elliptic, but since it satisfies \( (A - 1)^3 = 0 \), its eigenvalues are roots of 1. More precisely, replacing \( A \) by a \( A^3 \), we can assume that \( A \) is trivial on its elliptic subspace in \( \mathcal{E} \).

Since \( \mathcal{E} \) is rational, \( A \) determines an integer matrix in \( \text{GL}(R^k/\mathcal{E}) \). This is an elliptic matrix. So, all the eigenvalues of \( A \) are roots of unity, and therefore, after passing to a power, we can assume that 1 is the unique eigenvalue of \( A \).

Consider \( A - 1 \) and \( (A - 1)^2 \). Their images are, respectively, the 2-plane generated by \( \{e_1, e_3\} \) and the line \( Re_1 \). These two subspaces are thus rational.

The 1-eigenspace of \( A \) is generated by \( \{e_1, e_4, \ldots, e_k\} \). It is rational. We can choose \( e_4, \ldots, e_k \) rational. For \( e_3 \), one can take any rational vector which does not belong to the space generated by \( \{e_1, e_2, e_4, \ldots, e_k\} \).

\[ \square \]

6.1. Structure Theorem.

Theorem 6.3. Let \( f \in \text{ISO}(M, g) \) act non-periodically on \( \text{ISO}_0(M, g) \). Then, there is a minimal timelike \( \rho(f) \)-invariant torus \( T^d \subset \text{ISO}_0(M, g) \) of dimension \( d = 3 \) or \( d \geq 2 \) according to whether \( \rho(f) \) is parabolic or hyperbolic, respectively. The action of \( T^d \) on \( M \) is (everywhere) free and timelike.

We will present the proof of the theorem in the parabolic case; the hyperbolic case is analogous, in fact, easier. So, let \( f \) be such that \( \rho(f) \) is parabolic. The 3-torus \( T^d = T^3 \) in question is the one corresponding to the rational 3-space associated to \( A \) in Lemma 6.2. The normal form of \( \rho(f) \) on this rational 3-space is:

\[
(6.1) \quad \rho(f) \equiv \begin{pmatrix} 1 & t & -t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, \quad t \neq 0.
\]
We need to show that this torus acts freely with timelike orbits on \( M \), and the idea is to relate the dynamics of \( f \) on \( M \) and the dynamics of \( \rho(f) \) on the toral factor. Towards this goal, we will use the approximately stable foliation of a Lorentz isometry, introduced in reference [18].

6.2. Recalls on approximate stability. Let \( \phi \) be a diffeomorphism of a compact manifold \( M \). A vector \( v \in T_xM \) is called approximately stable if there is a sequence \( v_n \in T_xM \), \( v_n \to v \) such that the sequence \( D_x\phi^nv_n \) is bounded in \( TM \). The vector \( v \) is called strongly approximately stable if \( D_x\phi^n v_n \to 0 \). The set of approximately stable vectors in \( T_xM \) is denoted \( \text{AS}(x, \phi) \), or sometimes \( \text{AS}(x, \phi, M) \). Their union over \( M \) is denoted \( \text{AS}(\phi) \), or \( \text{AS}(\phi, M) \). Similarly, \( \text{SAS}(x, \phi) \) will denote that set of strongly approximately stable vectors in \( T_xM \), and \( \text{SAS}(\phi) = \bigcup_{x \in M} \text{SAS}(x, \phi) \).

The structure of \( \text{AS}(\phi) \) when \( \phi \) is a Lorentzian isometry has been studied in [18]:

**Theorem 6.4 (Zeghib [18]).** Let \( \phi \) be an isometry of a compact Lorentz manifold \( (M, g) \) such that the powers \( \{\phi^n\}_{n \in \mathbb{N}} \) of \( \phi \) form an unbounded set (i.e., non pre-compact in \( \text{IsO}(M, g) \)). Then:

- \( \text{AS}(\phi) \) is a Lipschitz condimension 1 vector subbundle of \( TM \) which is tangent to a codimension 1 foliation of \( M \) by geodesic lightlike hypersurfaces;
- \( \text{SAS}(\phi) \) is a Lipschitz 1-dimensional subbundle of \( TM \) contained in \( \text{AS}(\phi) \) and everywhere lightlike.

6.3. The action on \( M \) vs the toral action. Denote by \( \mathcal{T} \) the Lie algebra of \( T^3 \), and by \( \rho_0(f) \) the linear representation in associated to \( \rho(f) \). More explicitly, \( \rho_0(f) \) is the push-forward by \( f \) of Killing vector fields, see formula (2.1).

**Lemma 6.5.** Let \( X \in \mathcal{T} \) be a Killing field which is approximately stable for \( \rho(f) \) at 1 \( \in T^3 \). Then, for all \( x \in M \), \( X(x) \in T_xM \) is approximately stable. In other words, if \( X \in \text{AS}(0, \rho_0(f), \mathcal{T}) \), then \( X(x) \in \text{AS}(x, f, M) \) for any \( x \in M \).

A totally analogous statement holds for the strong approximate stability.

**Proof.** Let \( X_n \) be a sequence of Killing fields in \( \mathcal{T} \) such that \( X_n \to X \) and with \( Y_n = f_x^n X_n \) bounded. Clearly \( X_n(x) \to X(x) \) for all \( x \in M \); moreover, by assumption, the \( Y_n \) are bounded vector fields, and so \( D_x f_x^n X_n(x) = Y_n(f^n_x x) \) is bounded, that is \( X(x) \in \text{AS}(f) \). \( \square \)

**Lemma 6.6.** Assume \( \rho(f) \) parabolic. Then, there is \( Z \in \mathcal{T} \) a Killing field such that

- (a) \( Z \) defines a periodic flow \( \phi^t \);
- (b) \( f \) preserves \( Z \), i.e., \( f \) commutes with the 1-parameter group of isometries \( \phi^t \) generated by \( Z \);
- (c) \( Z \) generates the strong approximate stable 1-dimensional bundle of \( f \);
- (d) \( Z \) is everywhere isotropic;
- (e) \( Z \) is non-singular, hence \( Z \) is everywhere lightlike.

**Proof.** Let \( Z \) to be a 1-eigenvector of \( \rho(f) \); since \( \rho_0(f) Z = f_* Z = Z \), then \( f \) preserves \( Z \). In the normal form (6.1) of \( \rho(f) \), the vector \( Z \) corresponds to the first element of the basis.
The $Z$-direction is rational, since it is the unique 1-eigendirection of $\rho_0(f)$. Thus $Z$ defines a periodic flow.

One verifies that $Z$ is strongly approximately stable for the $\rho_0(f)$-action at $0 \in \mathcal{T}$. Therefore, at any $x$ where it does not vanish, $Z(x)$ determines the strongly stable 1-dimensional bundle of $f$. In particular, $Z(x)$ is isotropic for all $x \in M$. But, non trivial isotropic Killing field cannot have singularities.

6.4. Proof of Theorem 6.3 (parabolic case).

**Lemma 6.7.** $\mathbb{T}^3$ preserves the approximate stable foliation $\mathcal{F}$ of $f$.

**Proof.** The group $G_f$ generated by $\mathbb{T}^3$ and $f$ is amenable (it is an extension of the abelian $\mathbb{T}^3$ by the abelian $\mathbb{Z}$). The statement follows then from [18, Theorem 2.4, Theorem 2.6].

**Lemma 6.8.** The $\mathbb{T}^3$-action is locally free.

**Proof.** Let $\Sigma$ be the set of points $x$ having a stabilizer $S_x$ of positive dimension. We claim that if $\Sigma$ is non empty, then there must be some point of $\Sigma$ whose stabilizer contains the flow $\phi^t$ of the vector field $Z$ given in Lemma 6.6. This is clearly a contradiction, because such $Z$ has no singularity.

In order to prove the claim, consider the set $\Sigma^2 = \{ x \in M : \dim(S_x) = 2 \}$. This is a closed subset of $M$, because 2 is the highest possible dimension of the stabilizers of the $\mathbb{T}^3$-action. If $\Sigma^2$ is non empty, then there exists an $f$-invariant measure on $\Sigma^2$, and by Poincaré recurrence theorem there is at least one recurrent point $x_0 \in \Sigma^2$. The Lie algebra $\mathfrak{s}_{x_0}$ of $S_{x_0}$ is then $\rho(f)$-recurrent, and since $\rho(f)$ is parabolic, by Lemma 2.5 (applied to the $\rho(f)$-action on the Grassmannian of 2-planes in $\mathcal{T}$), then $\mathfrak{s}_{x_0}$ is fixed by $\rho(f)$. There is only one 2-plane fixed by $\rho(f)$ in $\mathcal{T}$ (the one spanned by the first two vector of the basis that puts $\rho(f)$ in the normal form), and such plane contains $Z$.

Similarly, if $\Sigma^2$ is empty, then $\Sigma^1 = \{ x \in M : \dim(S_x) = 1 \}$ is closed in $M$. As above, there must be a recurrent point $x_0$ in $\Sigma^1$, and $\mathfrak{s}_{x_0}$ is fixed by $\rho(f)$. This implies that $\mathfrak{s}_{x_0}$ contains $Z$.

The proof is concluded.

**Lemma 6.9.** The $\mathbb{T}^3$-action is everywhere timelike.

**Proof.** If not, there exists $x \in M$ such that the restriction $g_x$ of the metric $g$ to $T_x \mathbb{T}^3 \cong \mathcal{T}$ is lightlike, i.e., positive semi-definite (note that the $Z$ is a lightlike vector of such restriction, that cannot be positive definite). Consider the $f$-invariant compact subset $M_+ = \{ x \in M : g_x$ is positive semi-definite $\}$; it has an $f$-invariant measure, and by Poincaré recurrence theorem there is a recurrent point $x_0 \in M_+$ for $f$. Also the metric $g_{x_0}$ on $\mathcal{T}$ is $\rho(f)$-recurrent, and by Lemma 2.5 (applied to the $\rho(f)$-action on the space of quadratic forms on $\mathcal{T}$), $g_{x_0}$ is fixed by $\rho(f)$. But there exists no non zero $\rho(f)$-invariant quadratic form on $\mathcal{T}$ whose kernel is $Z$. This is proved with an elementary computation using the normal form (6.1) of $\rho(f)$.

7. A GENERAL COVERING LEMMA

Let us now go back to the general case where $\rho(f)$ is either parabolic or hyperbolic, and proceed with the study of the geometrical structure of $M$. The product structure of (a finite covering of) $M$ willbe established using a general covering result.
Proposition 7.1. Let $M$ be a compact manifold, and let $X$ be a non singular vector field on $M$ of the manifold $\mathbb{R} \times N_0$ generating an equicontinuous flow $\phi^t$ (i.e., $\phi^t$ preserves some Riemannian metric). Assume there exists a codimension 1 foliation $\mathcal{N}$ such that:
- $\mathcal{N}$ is everywhere transverse to $X$;
- $\mathcal{N}$ is preserved by $\phi^t$.

Then, $X$ and $\mathcal{N}$ defines a global product structure in the universal cover $\tilde{M}$. More precisely, let $x_0 \in M$ and let $N_0$ be its $\mathcal{N}$-leaf. Then, the map $p : \mathbb{R} \times N_0 \to M$ defined by $p(t, x) = \phi^t x$ is a covering.

A generalization is available for some group actions. Namely, consider an action of a compact Lie group $K$ on a compact manifold $M$ such that:
- the action is locally free (in particular, all orbits have the same dimension);
- $K$ preserves a foliation $\mathcal{N}$ transverse to its orbits (with a complementary dimension).

Then, for all $x_0 \in M$, denoting by by $N_0$ the leaf of $\mathcal{N}$ through $x_0$, the map $p : K \times N_0 \to M$ defined by $p(g, x) = gx$ is an equivariant covering.

Proof. In order to prove the first statement, consider the set of Riemannian metrics for which $X$ and $\mathcal{N}$ are orthogonal, and $X$ has norm equal to 1. The equicontinuity assumption implies that the $\phi_t$'s generate a precompact subgroup $\Phi$ of the diffeomorphisms group of $M$. By averaging over the compact group $\Phi$, one obtains a metric $g_*$ on $M$ which is preserved by $\phi^t$. Now, endow $\mathbb{R} \times N_0$ with the product metric, where $\mathbb{R}$ is endowed with the Euclidean metric $dt^2$ and $N_0$ has the induced metric from $g_*$. Observe that the induced metric on $N_0$ is complete (leaves of foliations in compact manifolds have bounded geometry, i.e., they are complete, have bounded curvature and injectivity radius bounded from below). One then observes that $p$ is a local isometry; namely, $\mathbb{R} \times N_0$ is complete, and therefore $p$ is a covering.

For the second statement, one can choose a left-invariant Riemannian metric $h$ on $K$, and taking an average on $K$ one obtains a $K$-invariant Riemannian metric $g_*$ on $M$ such that:

(a) the $K$-orbits and the leaves of $\mathcal{N}$ are everywhere orthogonal;
(b) the map $K \ni k \mapsto kx_0 \in Kx_0$ is a local isometry when the orbit $Kx_0$ is endowed with the Riemannian metric induced by $g_*$.  

As above, with such a choice the equivariant map $p : K \times N_0 \to M$ defined by $p(k, x) = kx$ is a local isometry, and since $K \times N_0$ is complete, $p$ is a covering map. \hfill \Box

8. On the Product Structure: The Hyperbolic Case

Let us now assume that $\rho(f)$ is hyperbolic; in this section we will denote by $T$ the torus $\mathbb{T}^d \subset G_0$ given in Theorem 6.3.

Lemma 8.1. The orthogonal distribution $\mathcal{N}$ to the $T$-foliation is integrable.

Proof. Let $N$ be the quotient of $M$ by the $T$-action, and $\pi : M \to N$ the projection. It is a compact Riemannian orbifold. The $f$-action induces an isometry $g$ of $N$. Consider the Levi form (i.e., the integrability tensor of the distribution $\mathcal{N}$) $l : \mathcal{N} \times \mathcal{N} \to \mathcal{N}^\perp$. Observe that $\mathcal{N}^\perp$ is the tangent bundle of the $T$-foliation.  

\footnote{The action of $K$ on $K \times N_0$ is the left multiplication on the first factor.}
Let $X$ and $Y$ be two vector fields on $N$. Suppose they are $g$-invariant: $g_{*}X = X$ and $g_{*}Y = Y$. Let $\bar{X}$ and $\bar{Y}$ their horizontal lifts on $M$. Then, $f_{*}\bar{X} = \bar{X}$ and $f_{*}\bar{Y} = \bar{Y}$. Hence, $l(\bar{X}, \bar{Y})$ is an $f$-invariant vector field tangent to the $\mathbb{T}$-foliation. However, by definition of the minimal torus $\mathbb{T}$, the $\rho(f)$-action on it has no invariant vector field. This means $l(\bar{X}, \bar{Y}) = 0$.

This proof will be finished thanks to the following fact. $\square$

**Proposition 8.2.** Let $g$ be an isometry of a Riemannian manifold $N$. There is an open dense set $U$ such that for any $x \in U$, any vector $u \in T_{x}U$ can be extended locally to a $g$-invariant vector field.

**Proof.** The closure of $\{g^{n}, n \in \mathbb{Z}\}$ in the isometry group of $N$ is a compact group with a torus as identity connected component. Apply Corollary 2.4 to conclude that, for the associated isometric action on $N$, the isotropy group is trivial on an open dense set $U$. Given $x \in U$ and $v \in T_{x}N$, extend first $u$ to an arbitrary smooth vector field on the slice through $x$ of the $S$-action, and then extend to $U$ using the $S$-action. $\square$

We will now prove the compactness of the leaves of $N$.

**Lemma 8.3.** Let $N_{0}$ be a leaf of $N$. Then, $N_{0}$ is compact.

**Proof.** The distribution $\mathcal{N}$ can be seen as a connection on the $\mathbb{T}$-principal bundle $M \to N$. We have just proved that this connection is flat, i.e., $\mathcal{N}$ is integrable, which is equivalent to the fact that its holonomy group is discrete. The leaves will be compact if we prove that the holonomy group is indeed finite; for $x \in N$, we will denote by $T_{x}$ the fiber at $x$ of the principal bundle $M \to N$. Recall that if $c$ is a loop at $x \in N$, then the holonomy map $H(c) : T_{x} \to T_{x}$ is obtained by means of horizontal lifts of $c$. It commutes with the $\mathbb{T}$-action and therefore it is a translation itself. In fact, $H(c)$ can be seen as an element of the acting torus $\mathbb{T}$ (and so, it is independent on the base point $x$). We have a holonomy map $H : \pi_{1}(N, x) \to \mathbb{T}$. In fact, since $\mathbb{T}$ is commutative, we have canonical identification of holonomy maps defined on different base points. In other words $H(c) = H(c')$, whence $c$ and $c'$ are freely homotopic curves.

Up to replacing $f$ by some power, we can assume that the basic Riemannian isometry $g : N \to N$ is in the identity component of $\text{Iso}(N)$ (since this group is compact). Therefore, any loop $c$ is freely homotopic to $g(c)$, and hence $H(c) = H(g(c))$.

Now, $f$ preserves all the structure, and thus if $\tilde{c}$ is a horizontal lift of $c$, then $f \circ \tilde{c}$ is a horizontal lift of $g(c)$. So, $fH(c)f^{-1} = H(g(c))$. If $g(c)$ is freely homotopic to $c$, then $H(c)$ is a fixed point of $\rho(f)$. But we know that that $\rho(f)$ has only finitely many fixed points (by the definition of $\mathbb{T}$). Therefore, the holonomy group is finite. $\square$

Apply now Proposition 7.1 to deduce that $M$ is covered by a product $\mathbb{T} \times N_{0} \to M$. The covering is finite because $N_{0}$ is compact.

Observe that we can assume the leaf $N_{0}$ is $f$-invariant. Indeed, the leaf $N_{0}$ meets all the fibers $T_{x}$, and, say, it contains $\tilde{x} \in T_{x}$. So after composing with a suitable translation $t \in \mathbb{T}$, i.e., replacing $f$ by $t \circ f$, we can assume that $f(\tilde{x}) \in N_{0}$. This implies that $f(N_{0}) = N_{0}$. Summarizing, all things (the $\mathbb{T}$-action and $f$) can be lifted to the finite cover $\mathbb{T} \times N_{0}$.
8.1. **The metric.** The Lorentz metric $g$ is not necessarily a product of the Riemannian metric on $N_0$ by that of $T^2$. It is true that $T^2$ and $N$ are $g$-orthogonal. Also, two leaves $\{t\} \times N$ and $\{t'\} \times N$ are isometric, via the $T^2$-action. However, the metric induced on each $T^2 \times \{n\}$ may vary with $n$. Observe however that one can choose a same metric for all these toral orbits, of course keeping the same initial group acting isometrically.

8.2. **The non-elementary case.** Let $\Gamma$ be a discrete subgroup of $SO(1, k - 1)$. Its limit set $L_\Gamma$ in the sphere (boundary at infinity of the hyperbolic space $H^{k-1}$). The group $\Gamma$ is **elementary parabolic** if $L_\Gamma$ has cardinality equal to 1, and **elementary hyperbolic** if $L_\Gamma$ has cardinality equal to 2. It is known that if $\Gamma$ is not elementary, then $L_\Gamma$ is infinite, and the action of $\Gamma$ on $L_\Gamma$ is minimal, i.e., every orbit is dense, see [14].

If $\Gamma$ elementary hyperbolic, then $\Gamma$ is virtually a cyclic group, i.e., up to a finite index, it consists of powers $A^n$ of the same hyperbolic element element $A$. If $\Gamma$ is elementary parabolic, then it is virtually a free abelian group of rank $d \leq k - 2$, i.e., it has a finite index subgroup isomorphic to $\mathbb{Z}^d$.

One fact on non-elementary groups is that they contain hyperbolic elements. More precisely, the set of fixed points of hyperbolic elements in $L_\Gamma$ is dense in $L_\Gamma$.

All the previous considerations in the case of a hyperbolic isometry $f$, extend to the case of a non-elementary group. We get:

**Theorem 8.4.** Let $(M, g)$ be a compact Lorentz manifold with $\text{Iso}(M, g)$ non compact, but $\text{Iso}_0(M, g)$ compact, and let $\Gamma$ be the discrete part $\text{Iso}(M, g)/\text{Iso}_0(M, g)$. Assume that $\text{Iso}_0(M, g)$ has some timelike orbit. Then, there is a torus $T^k$ contained in $\text{Iso}_0(M, g)$, invariant under the action by conjugacy of $\Gamma$, and such that the $T^k$-action is everywhere locally free and timelike.

The $\Gamma$-action on $T^k$ preserves some Lorentz metric on $T^k$, which allows one to identify $\Gamma$ with a discrete subgroup of $SO(1, k - 1)$, as well as a subgroup of $\text{GL}(k, \mathbb{R})$.

If $\Gamma$ is not elementary parabolic, then, up to a finite covering, $M$ splits as a topological product $T^k \times N$, where $N$ is a compact Riemannian manifold. One can modify the original metric $g$ along the $T^k$ orbits, and get a new metric $g_{\text{new}}$ with a larger isometry group, $\text{Iso}(M, g_{\text{new}}) \supset \text{Iso}(M, g)$, such that $(M, g_{\text{new}})$ is a pseudo-Riemannian direct product $T^k \times N$.

9. **On the Product Structure: the Parabolic Case**

As above, we have a $T^3$-principal fibration $M \to N$ over a Riemannian orbifold $N$, and $N$ is seen as a connection. Let $Z$ be the Killing field (as defined above), that is the unique vector field which commutes with $f$.

Consider the codimension 2 bundle $\mathcal{L} = \mathcal{N} \oplus \mathbb{R}Z$.

**Lemma 9.1.** $\mathcal{L}$ is integrable.

**Proof.** Let $X$ and $Y$ be two vector fields tangent to $\mathcal{N}$. As in the proof above in the hyperbolic case, we can choose $X$ and $Y$ to be $f$-invariant, see Proposition 8.2. Therefore, $l(X, Y)$ is also $f$-invariant, where $l$ is the Levi form of $\mathcal{N}$ (not of $\mathcal{L}$!)

But $Z$ is the unique $\rho(f)$-invariant vector. Thus, $l(X, Y)$ is tangent to $\mathbb{R}Z$.

Now, consider the Lie bracket $[X, Z]$. The $T^3$-action preserves $\mathcal{N}$, in particular, $[Z, X]$ is tangent to $\mathcal{N}$, for any $X$ tangent to $\mathcal{N}$. Consequently, $\mathcal{L}$ is integrable. $\square$
As in the hyperbolic case, one can prove:

**Lemma 9.2.** The leaves of \( \mathcal{L} \) are compact.

One can then apply Proposition 7.1 to \( \mathcal{L} \) and any 2-dimensional torus \( T^2 \) transverse to \( Z \) (\( \mathcal{L} \) is invariant by \( T^3 \) and so also by any such a \( T^3 \)). One obtains that \( M \) is finitely covered by \( T^2 \times L_0 \).

However, since \( T^2 \) is not \( f \)-invariant, this product is not compatible with \( f \).

As in the hyperbolic case, we can choose \( L_0 \) invariant by \( f \). A slightly deeper analysis shows that up to a finite cover \( M \) is an amalgamated product, i.e., a quotient \( (T^3 \times L_0)/S^1 \), see Subsection 9.1 for details.

The metric structure of \( L_0 \) is that of a lightlike manifold, that is \( L_0 \) is endowed with a positive semi-definite (degenerate) metric with a 1-dimensional null space. Here, the null space corresponds to the foliation defined by an \( S^1 \)-action (the flow of the vector field \( Z \)). The circle \( S^1 \) acts isometrically on the lightlike \( L_0 \) and the Lorentz \( T^3 \). A Lorentz metric can be defined on \( (L_0 \times T^3)/S^1 \).

We have proven the following:

**Theorem 9.3.** Let \( f \) be an isometry of a compact Lorentz manifold \((M, g)\) such that the action \( \rho(f) \) on the toral component of \( \text{Isom}(M, g) \) is parabolic. Then, there is a new metric on \( M \) having a larger isometry group such that \( M \) is the amalgamated product of a Lorentz torus \( T^3 \), and a lightlike manifold \( L_0 \). Both have an isometric \( S^1 \)-action. The isometry \( f \) is obtained by means of an isometry \( h \) of \( L_0 \) commuting with the \( S^1 \)-action, and a linear isometry on the Lorentz \( T^3 \).

The same statement is valid if instead of a single parabolic \( f \), we have an elementary parabolic group \( \Gamma \) of rank \( d \). In this case, the torus has dimension \( 2 + d \).

### 9.1. Amalgamated products

Given any two manifolds \( X \) and \( Y \) carrying free (left) actions of the circle \( S^1 \), then one can consider the diagonal action of \( S^1 \) on the product \( X \times Y \): \( g(x, y) = (gx, gy) \) for all \( g \in S^1 \), \( x \in X \) and \( y \in Y \). Let \( Z \) be the quotient \( (X \times Y)/S^1 \) of this diagonal action. Assume that \( X \) is Lorentzian, \( Y \) is Riemannian, and the action of \( S^1 \) in each manifold is isometric; one can define a natural Lorentzian structure on \( Z \) as follows. Let \( A \in \mathfrak{X}(X) \) and \( B \in \mathfrak{X}(Y) \) be smooth vector fields tangent to the fibers of the \( S^1 \)-action on \( X \) and on \( Y \) respectively; for \( (x_0, y_0) \in X \times Y \), denote by \( [(x_0, y_0)] \in Z \) the \( S^1 \)-orbit \( \{(gx_0, gy_0) : g \in S^1\} \). The subspace \( T_{x_0}X \oplus B_{y_0} \perp \) is complementary to the one-dimensional subspace spanned by \( (A_{x_0}, B_{y_0}) \) in \( T_{x_0}X \oplus T_{y_0}Y \). If we denote by \( q : X \times Y \rightarrow Z \) the projection, then the linear map \( dq_{(x_0,y_0)} : T_{x_0}X \oplus T_{y_0}Y \rightarrow T_{[(x_0,y_0)]}Z \cong (T_{x_0}X \oplus T_{y_0}Y)/R \cdot (A_{x_0}, B_{y_0}) \) restricts to an isomorphism:

\[
dq_{(x_0,y_0)} : T_{x_0}X \oplus B_{y_0}^\perp \cong T_{[(x_0,y_0)]}Z.
\]

A Lorentzian metric can be defined on \( Z \) by requiring that such isomorphism be isometric; in order to see that this is well defined, we need to show that this definition is independent on the choice of \( (x_0, y_0) \) in the orbit \( [(x_0, y_0)] \). For \( g \in S^1 \), denote by \( \phi_g : X \rightarrow X \) and \( \psi_g : Y \rightarrow Y \) the isometries given by the action of \( g \) on \( X \) and on \( Y \) respectively. By differentiating at \( (x_0, y_0) \) the commutative diagram:

\[
\begin{array}{ccc}
X \times Y & \overset{q}{\longrightarrow} & X \times Y \\
\downarrow{\phi_g} & \quad & \quad \downarrow{\psi_g} \\
Z & \overset{q}{\longrightarrow} & Z
\end{array}
\]
we get a commutative diagram:

\[ T_{x_0}X \oplus T_{y_0}Y \xrightarrow{\left( (d\phi)_x, (d\psi)_y \right)} T_{g_{x_0}y_0}X \oplus T_{g_{y_0}y_0}Y. \]

As \(( (d\phi)_x, (d\psi)_y \)) carries \( T_{x_0}X \oplus B_{y_0} \) onto \( T_{g_{x_0}y_0}X \oplus B_{g_{y_0}} \), we get the following commutative diagram of isomorphisms:

\[ T_{x_0}X \oplus B_{y_0} \xrightarrow{\left( (d\phi)_x, (d\psi)_y \right)} T_{g_{x_0}y_0}X \oplus B_{g_{y_0}}. \]

Since \(( (d\phi)_x, (d\psi)_y \)) is an isometry, the above diagram shows that the metric induced by \( d_{\mathfrak{q}}(x_0,y_0) \) coincides with the metric induced by \( d_{\mathfrak{q}}(g_{x_0},g_{y_0}) \). This shows that the Lorentzian metric tensor on \( Z \) is well defined.

As to the topology of \( Z \), we have the following:

**Lemma 9.4.** If \( X \) and \( Y \) are simply connected, then \( Z \) is simply connected. If the product of the fundamental groups \( \pi_1(X) \times \pi_1(Y) \) is not a cyclic group, then \( Z \) is not simply connected.

**Proof.** The diagonal \( S^1 \)-action on \( X \times Y \) is free (and proper), and therefore the quotient map \( q : X \times Y \rightarrow Z \) is a smooth fibration. The thesis follows from an immediate analysis of the long exact homotopy sequence of the fibration, that reads:

\[ Z \cong \pi_1(S^1) \rightarrow \pi_1(X) \times \pi_1(Y) \rightarrow \pi_1(Z) \rightarrow \pi_0(S^1) \cong \{1\}. \]

**10. Proof of Corollary 3 and Theorem 4**

**Proof of Corollary 3.** This is one of the steps of the proof of our structure result, see Lemma 6.9.

**Proof of Theorem 4.** By the structure result of [17], compact Lorentzian manifolds admitting an isometric action of (some covering of) \( \text{SL}(2,\mathbb{R}) \) or of an oscillator group are not simply connected. Thus, if \( M \) is simply connected, by Theorem 3.1 \( \text{Iso}_0(M,g) \) is compact. Now, if \( \text{Iso}(M,g) \) has infinitely many connected components, then (a finite covering of) \( M \) is not simply connected. When \( \Gamma = \text{Iso}(M,g)/\text{Iso}_0(M,g) \) is not elementary parabolic, this follows directly from Theorem 8.4. When \( \Gamma \) is elementary parabolic, this follows from Theorem 9.3 and the second statement of Lemma 9.4. Hence, \( \text{Iso}(M,g) \) is compact. \( \square \)

**References**

[1] S. Adams, G. Stuck, *The isometry group of a compact Lorentz manifold. I, II*, Invent. Math. 129 (1997), no. 2, 239–261, 263–287.
[2] R. Alperin, *An elementary account of Selberg’s lemma*, Enseign. Math. (2) 33 (1987), no. 3-4, 269–273.
[3] G. D’Ambra, *Isometry groups of Lorentz manifolds*, Invent. Math. 92 (1988), no. 3, 555–565.
[4] J. K. Beem, P. E. Ehrlich, S. Markvorsen, Timelike isometries and Killing fields, Geom. Dedicata 26 (1988), no. 3, 247–258.
[5] T. Barbot, A. Zeghib, Group actions on Lorentz spaces, mathematical aspects: a survey, The Einstein equations and the large scale behavior of gravitational fields, 401–439, Birkhäuser, Basel, 2004.
[6] G. E. Bredon, Introduction to compact transformation groups, Pure and Applied Mathematics, Vol. 46, Academic Press, New York–London, 1972.
[7] J. L. Flores, M. A. Javaloyes, P. Piccione, Periodic geodesics and geometry of compact Lorentzian manifolds with a Killing vector field, to appear in Mathematische Zeitschrift.
[8] G. Hochschild, The structure of Lie groups, Holden-Day, Inc., San Francisco–London-Amsterdam 1963.
[9] S. Kobayashi, Transformation groups in differential geometry, Reprint of the 1972 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995.
[10] S. Kobayashi, K. Nomizu, Foundations of Global Differential Geometry, vol. 1, Interscience, New York, 1963.
[11] A. Medina, P. Revy, Les groupes oscillateurs et leurs réseaux, Manuscripta Math. 52 (1985), no. 1-3, 81–95.
[12] P. Piccione, D. V. Tausk, A student’s guide to symplectic spaces, Grassmannians and Maslov index, Publicações Matemáticas do IMPA, Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2008. xiv+301 pp., ISBN: 978-85-244-0283-8.
[13] M. Sánchez, Structure of Lorentzian tori with a Killing vector field, Trans. AMS 349, no. 3 (1997), 1063–1080.
[14] W. Thurston, Three-dimensional geometry and topology, Princeton Mathematical Series, 35. Princeton University Press, Princeton, NJ, 1997
[15] P. L. Waterman, Purely elliptic Möbius groups. Holomorphic functions and moduli, Vol. II (Berkeley, CA, 1986), 173–178, Math. Sci. Res. Inst. Publ., 11, Springer, New York, 1988.
[16] H. Wu, On the de Rham decomposition theorem, Illinois J. Math. 8 (1964), 291–311
[17] A. Zeghib, Sur les espaces-temps homogènes, The Epstein birthday schrift, 551–576 (electronic), Geom. Topol. Monogr. 1, Geom. Topol. Publ., Coventry, 1998.
[18] A. Zeghib, Isometry groups and geodesic foliations of Lorentz manifolds, Part I: Foundations of Lorentz dynamics, Geom. Funct. Anal. Vol. 9 (1999), 775–822.
[19] R. J. Zimmer, Ergodic theory and semisimple groups, Monographs in Mathematics, 81. Birkhäuser Verlag, Basel, 1984.

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