Gribov’s Picture of Confinement and Chiral Symmetry Breaking

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Abstract

Gribov’s scenario of confinement caused by supercritical charges is described and the present status of Gribov’s equation for the Green function of light quarks is discussed.
1 Introduction

One of the most challenging problems in quantum field theory is to understand the confinement mechanism in QCD. V. N. Gribov attacked this problem in a long lasting effort in which he developed both a physical picture and a quantitative approach to describing the confinement mechanism. His main concern was not just to find a confinement mechanism but rather the confinement mechanism relevant to the real world in which we live. It was a key observation for him [1] that the existence of very light quarks can drastically change the vacuum structure of QCD. Therefore a solution of the confinement problem appeared impossible to him without understanding the effects of light quarks. From this point of view the question of confinement in pure gluodynamics is interesting but probably irrelevant for our real world.

From the early 90’s onwards the problem of light quarks therefore became the main focus of Gribov’s research [2]. In this context he addressed several key questions. One of them is the way in which the Pauli principle can provide a binding mechanism even for almost massless quarks due to the occurrence of supercritical charges. Another one is the relation between the infrared and ultraviolet regimes of the theory which appear to be linked much more closely than one might naïvely expect. He also addressed the question whether and how the perturbative expansion has to be reconsidered in a theory in which the fundamental fields of the Lagrangian do not exist as asymptotic states. This is in fact a very deep problem as one recognizes by recalling that the usual treatment of propagators with the Feynman $i\epsilon$-prescription requires that the perturbative vacuum is stable – which is not true in QCD where quarks and gluons cannot propagate freely.

In 1997 V. N. Gribov was convinced that he had finally found answers to these questions and that he understood the physics of light quarks sufficiently well. He then collected his results in two papers [3, 4] which were supposed to conclude his study of confinement in QCD. Unfortunately, the second paper had to remain unfinished when he passed away.

In the present contribution we describe the idea of supercritical charges in QCD, give a review of the Gribov equation for the propagator of light quarks, and summarize the present status of that equation. For a more detailed exposition of the motivations and physical ideas behind Gribov’s approach we refer the reader to the recent review [5]. All of Gribov’s original papers on confinement (as well as many of his papers and lectures on high energy scattering) are collected in the book [6].

2 Light Quarks and Supercritical Charges

The phenomenon of supercritical charges is well known in QED. Consider the energy levels that an electron would have in the static external field of a heavy nucleus $A_Z$ with charge $Z$. When $Z$ is increased the energy levels are lowered. At $Z > 137$ the lowest $(1s)$ level sinks below $-me^2$ and hence dives into the Dirac sea.\(^1\) In this situation an electron from the Dirac sea fills this level and forms a strongly bound state with the nucleus while a positron is emitted. This is also reflected in the quantum mechanical

\(^1\)The critical value $Z_c = 137$ holds for a pointlike nucleus, in the more realistic situation of an extended nucleus it would be somewhat higher.
energy of the electron as it results from the Dirac equation which becomes complex at $Z > 137$, signaling an instability that corresponds to the so-called ‘falling onto the center’. In summary, the heavy nucleus decays into a supercritically bound state $A_{Z-1}$ with a smaller charge $Z - 1$ and a positron,

$$A_Z \rightarrow A_{Z-1} + e^+.$$  

Starting from a highly charged nucleus that process continues for higher energy levels until the resulting bound state is subcritically charged.

Gribov’s idea is that a similar mechanism might apply to the color charges of QCD if light quarks exist with a Compton wave length that is larger than the typical color charge radius or confinement radius. He proposed the intriguing possibility that in QCD already the color charge of a single quark could be supercritical, in contrast to QED where a very large charge is required. Note that in QED the relevant parameter for the phenomenon of supercritical charges is in fact $\alpha_{em}Z$ and that also a large electromagnetic coupling $\alpha_{em}$ could trigger the instability. In QCD the role of the supercritical charge is in fact played by the strong coupling constant $\alpha_s$ which becomes large at low momentum scales. As we will see in section 4 below supercritical behavior is expected if the strong coupling constant exceeds a critical value $\alpha_c$ given by

$$\frac{\alpha_c}{\pi} = C_F^{-1} \left(1 - \sqrt{\frac{2}{3}}\right) = 0.137,$$

with the Casimir operator $C_F = (N_c^2 - 1)/(2N_c) = 4/3$. Note that this critical value is remarkably small.

Let us now briefly discuss how supercritical color charges change the vacuum structure of light quarks. (A more detailed account of the confinement mechanism caused by light quarks can be found in Refs. [7, 8].) The emerging picture is again most conveniently discussed in the language of the Dirac sea. For subcritical coupling we have the usual situation with empty positive energy quark states and a filled Dirac sea of negative energy states. If the coupling is larger than the critical value $\alpha_c$ additional states appear. Supercritical bound states can form consisting of a quark and an antiquark which both have positive kinetic energy. But due to the supercritical binding, the bound state has a negative total energy. In order to avoid having infinitely many mesonic negative energy states the corresponding quark and antiquark states have to be filled in the vacuum. In particular, the vacuum then contains occupied quark states of positive kinetic energy, in addition to the negative energy states in the Dirac sea. In such a vacuum state the Pauli exclusion principle prevents single quarks from propagating freely since the states required for that are already occupied. In that sense the Pauli principle can bind even massless quarks which usually tend to move at the speed of light. A single quark put into the QCD vacuum would undergo a decay into a supercritically bound meson and another quark, $q \rightarrow M + q$. Since the resulting quark is again supercritically charged that process would continue and eventually shield the color charge completely, such that a single quark would only exist as a resonance but not as an asymptotic state.

Obviously, this confinement mechanism based on the Pauli principle works only for quarks, that is for fermions in general, but does not apply to gluons. It remains to
be investigated how gluons are affected by the light quarks in such a picture. It is conceivable that the confinement of gluons is, at least to some extent, caused by their coupling to confining light quarks, and would then be an indirect or second-order effect.

From a technical point of view the situation in QCD is considerably more complicated than the example of a heavy nucleus in QED. While a heavy quark can be treated in a similar way as the heavy nucleus, the supercritical charge of a light quark can clearly not be considered as an external field. In view of this problem Gribov invented a new approach to the Dyson-Schwinger equation for light quarks that we discuss in the next section.

Interestingly, the dispersive approach \[9\] to power corrections to event shape variables in $e^+e^-$ collisions and in deep inelastic scattering suggests that the integrated strong coupling constant in the infrared region (assuming of course that such a concept exists) satisfies

$$\alpha_0 = \frac{1}{2 \text{GeV}} \int_0^{2 \text{GeV}} \alpha_s(k) \, dk \simeq 0.5,$$  \hspace{1cm} (3)

see Ref. \[10\] for a recent review. From (3) one can deduce that the strong coupling becomes in fact larger than the critical value (2) at least in some range of momenta, but remains comparatively small in most if not all of the small-momentum region. This observation raises some hopes that – unexpectedly – some aspects of low-momentum phenomena like for example hadronization might be accessible at least qualitatively in a semi-perturbative way.

3 Gribov’s Equation for the Green Function of Light Quarks in Feynman Gauge

Let us now turn to the Gribov equation (or Gribov-Dyson-Schwinger equation) for the Green function of light quarks in Feynman gauge \[3\]. For Gribov this equation was mainly a technical step on the way to a quantitative support of the physical picture discussed in the previous section. But it can also be viewed as a new approach to the quark’s Dyson-Schwinger equation which is very interesting already by itself, independently of the supercritical confinement scenario.

The Green functions of QCD satisfy a tower of coupled Dyson-Schwinger equations. Since it is virtually impossible to solve this system of integral equations as a whole one usually has to resort to simplifications. Most current approaches to the Dyson-Schwinger equations are based on truncation schemes (based for example on certain topological classes of diagrams), but their accuracy is often very difficult to control. Gribov discovered that it is possible to attack the Dyson-Schwinger equation for the quark’s Green function in Feynman gauge based on an approximation rather than a mere truncation scheme. The guiding principle in this approximation scheme is to collect the most singular contributions from the infrared momentum region, that is those contributions that are expected to cause chiral symmetry breaking and confinement.

In Feynman gauge the gluon propagator has the general form

$$D_{\mu\nu}(k) = -\frac{g_{\mu\nu}}{k^2} \alpha_s(k^2),$$ \hspace{1cm} (4)
which can be interpreted as a nonperturbative definition of the strong coupling constant $\alpha_s$ at low momentum scales. We now make the (rather mild) assumption that $\alpha_s(k^2)$ is not divergent as $k^2 \to 0$ and does not vary too strongly. For spacelike momenta a typical example for such a coupling is the model

$$\alpha_s(k^2) = \frac{4\pi}{\left(11 - \frac{2}{3}n_f\right) \ln\left(-k^2/\Lambda^2_{\text{QCD}} + a\right)}, \quad (5)$$

which for $a = 6$ satisfies $[3]$ and agrees with the perturbative one-loop running at large momenta. It has been shown $[5]$ that the qualitative results discussed below are independent of the particular model used for $\alpha_s$ at low momentum scales.

One now starts with the Dyson-Schwinger equation for the quark’s inverse Green function $G^{-1}(q)$ and applies the d’Alembert operator $\partial^2 = \partial^\mu \partial_\mu$ with

$$\partial_\mu = \frac{\partial}{\partial q^\mu}. \quad (6)$$

Let us illustrate this procedure for the one-loop self-energy diagram

![Diagram](image)

which is given by

$$\Sigma = -i \, C_F \, \frac{\alpha_s}{\pi} \int \frac{d^4k}{4\pi^2} \gamma^\mu G(k) \gamma_\mu \frac{1}{(q-k)^2}. \quad (7)$$

Firstly, differentiating twice with respect to the external momentum $q_\mu$ makes this integral convergent. Secondly, since $1/q^2$ is the Green function of the four-dimensional d’Alembert operator $\partial^2$,

$$\partial^2 \frac{1}{(q-q')^2 + i\epsilon} = -4\pi^2 i \delta^{(4)}(q-q'), \quad (8)$$

we can eliminate the integral in $[7]$ and obtain to first order in $\alpha_s$

$$\partial^2 G^{-1}(q) = C_F \frac{\alpha_s}{\pi} \gamma_\mu G(k) \gamma_\mu. \quad (9)$$

One can now systematically proceed to higher order diagrams. In each diagram the most singular contribution from the infrared is obtained when both derivatives are applied to the same gluon line, since only those terms give rise to a delta function according to $[5]$. All other terms are clearly less singular and still involve integrals over loop momenta.

Including higher orders one builds up two full quark-gluon vertices $\Gamma_\mu$ which due to the delta functions have to be taken at vanishing gluon momentum. Note that as a consequence of the differentiations we obtain two full vertices in spite of the fact that the original Dyson-Schwinger equation involves one bare and one full vertex. The two zero-momentum vertex functions $\Gamma_\mu(q,q,0)$ can be related to the inverse Green function via the Ward identity

$$\Gamma_\mu(q,q,0) = \partial_\mu G^{-1}(q), \quad (10)$$
and the corrections to the Ward identity due to the nonabelian structure of QCD can be argued to be subleading in the Dyson-Schwinger equation in the approximation scheme used here. We hence obtain the Gribov equation for the light quark’s Green function in Feynman gauge,

$$\partial^2 G^{-1} = g (\partial^\mu G^{-1}) G (\partial_\mu G^{-1}) + \cdots,$$

(11)

where

$$g = C_F \frac{\alpha_s(q)}{\pi}.$$  

(12)

and the ellipsis in (11) stands for less IR-singular terms of order $g^2$ which can – at least in principle – be systematically computed. Note that via (12) we have inserted the running strong coupling constant $\alpha_s(q)$ in the Gribov equation. Following the derivation outlined above one would first obtain the coupling at zero momentum due to the delta function (8), but the difference between the two is subleading in the sense of our approximation.

With the running coupling the Gribov equation has the remarkable property that it can reproduce the usual renormalization group behavior for the quark propagator in the perturbative region of large spacelike momenta. Therefore the equation collects the most important terms both in the infrared as well as in the ultraviolet, and hence has a good chance of giving a valid description of the quark propagator for all momenta.

The derivation of the equation strongly relies on the mathematical identity (8) and hence on the Feynman gauge. Most other approaches to the Dyson-Schwinger equations use Landau gauge such that a direct comparison with those approaches is rather difficult. If one succeeds in deriving a genuinely gauge-independent phenomenon like chiral symmetry breaking or confinement in one gauge, however, it will clearly hold in any gauge.

We finally point out that the same equation (11) applies also for electrons in QED after replacing $C_F$ by 1 and $\alpha_s$ by $\alpha_{em}$. However, the behavior of the coupling constants and the boundary conditions are very different in the two theories.

### 4 Chiral Symmetry Breaking

On general grounds the quark’s Green function has the form

$$G^{-1}(q^2) = Z^{-1}(q - M),$$

(13)

where $M(q^2)$ is the dynamical mass function of the quark and $Z(q^2)$ stands for the wave function renormalization.

In QCD the infrared and ultraviolet limits of Green functions are very closely related [3]. In order to see this let us consider the limit of vanishing coupling $\alpha_s \to 0$, corresponding to large spacelike momenta, $-q^2 \to \infty$, for which we obtain the free equation $\partial^2 G^{-1} = 0$. One readily verifies that this equation has the solution

$$G^{-1}(q) = Z^{-1} \left( q - m - \frac{\nu_1^3}{q^2} - \frac{\nu_2^4}{q^4} q \right).$$

(14)

Note that in addition to the usual two constants, namely the wave function renormalization $Z$ and the bare mass $m$, we find two further integration constants $\nu_1^3$ and $\nu_2^4$. 
since the Gribov equation is a second order differential equation. According to their mass dimension one expects that they can be related (up to factors) to the quark and gluon condensates. The additional terms involving these two constants are singular at low $q^2$. In QED one can simply drop these terms since the behavior of the Green function is perturbative at low momentum scales and the perturbative degrees of freedom in the Lagrangian coincide with the observable fields. In QCD the situation is drastically different since there are no freely propagating quarks at low momenta. Gribov’s differential equation (11) hence relates the asymptotic solution (14) at high momenta to a complicated and qualitatively different structure at low momenta. In order to obtain the full and correct solution at low momenta one must therefore in general keep all four terms of (14) even in the regime of asymptotic freedom. The potential consequences of this result for applications of perturbative QCD still need to be explored.

We now turn to the behavior of the dynamical mass function of the quark in the whole region of spacelike momenta in order to see how chiral symmetry breaking occurs [3]. One finds from Gribov’s equation (11) that the dynamical mass function can have oscillations and change its sign in an interval of momenta if the coupling constant exceeds its critical value (2) in that interval [2] [3]. That critical value can be obtained analytically from (11). The full mass function can be calculated numerically for spacelike momenta [8]. Choosing the model (5) with $a = 6$ for the running coupling one obtains\(^2\) the mass function $M(q^2)$ shown in Figure 1 for three different initial values at $q^2 = 0$. In the figure $q$ denotes the spacelike momentum, $q = \sqrt{-q^2}$. Depending on the initial value the mass function can stay positive (upper curve) or can have oscillations resulting in the spikes in the logarithmic plot in the two lower lines in Figure 1.

We now define the renormalized mass $m_R$ as the value of the dynamical mass \(^2\)This figure has in fact been obtained using the modified equation (18) that includes pion corrections instead of the original equation (14). For spacelike momenta the mass functions resulting from the two equations have qualitatively the same behavior and their difference is rather small, see section 6 below.

Figure 1: The mass function for three different values of the renormalized mass

| $|M(q^2)|$ [GeV] |
|------------------|
| 10^-5 | 1 | 0.1 | 0.01 | 0.001 | 0.0001 |
| $q$ [GeV]       |
| 0    | 0.2 | 0.4 | 0.6 | 0.8 | 1 | 1.2 | 1.4 | 1.6 |

Dependent on the
function at vanishing quark momentum,

\[ m_R = \lim_{q \to 0} M(q^2), \quad (15) \]

and further define a perturbative mass \( m_P \) as the value of the mass function at a large (perturbative) momentum scale \( \lambda \),

\[ m_P = M(\lambda^2). \quad (16) \]

The precise value chosen for the scale \( \lambda \) is not relevant for the present discussion as long as the coupling is no longer supercritical at that scale. In practice \( \lambda \) can therefore be chosen as small as 2 GeV, for example.

Recall that by solving the Gribov equation (11) one obtains a certain value for \( m_P \) for each particular choice of \( m_R \). We can hence consider the dependence of the renormalized mass \( m_R \) on the perturbative mass \( m_P \) (or vice versa). If the coupling constant is subcritical in the whole range of momenta (for example if \( a \) in (5) is chosen sufficiently large) this dependence is one-to-one. If on the other hand the coupling is supercritical, the mass function can oscillate and different \( m_R \) can correspond to the same \( m_P \) at the scale \( \lambda \). In this situation the dependence of \( m_R \) on \( m_P \) is no longer one-to-one and instead becomes multivalued, as is shown in Figure 2.

![Figure 2](image_url)

**Figure 2:** The dependence of the renormalized mass \( m_R \) on the perturbative mass \( m_P \) reflecting chiral symmetry breaking.

It can be easily seen that even for vanishing perturbative quark mass \( m_P \) a large renormalized mass \( m_R \) is generated. Hence chiral symmetry breaking occurs dynamically if the coupling is supercritical in the infrared.

One can then proceed and compute the solutions to the Gribov equation (11) in the whole complex \( q^2 \)-plane in order to study their analytic structure. In general this requires further assumptions about the behavior of \( \alpha_s \) and hence of the gluon propagator \( D_{\mu \nu} \), see (4), for complex \( q^2 \). If one assumes that \( D_{\mu \nu} \) does not have dramatic singularities off the real \( q^2 \)-axis one finds that even for supercritical coupling...
the analytic structure of the quark’s Green function does not correspond to a confined quark \[12\]. Instead, the Green function exhibits a pole and a cut on the positive real \(q^2\)-axis corresponding to the free propagation of the quark.

5 Pion Effects on the Quark’s Green Function

In the process of chiral symmetry breaking pions are generated as Goldstone bosons and appear in the physical spectrum as massless states. Their Bethe-Salpeter amplitude \(\phi_\pi\) is described by an equation \[3\] that is derived in the same approximation as the equation \(\text{(11)}\) for the quark’s Green function described above. For massless pions the solution of that equation can be found analytically in terms of the quark’s Green function,

\[ \phi_\pi \sim \{\gamma_5, G^{-1}\}. \] (17)

Once pions appear as massless states in the physical spectrum due to chiral symmetry breaking they might have a considerable effect on the quark’s Green function. This backreaction is not properly taken into account in the original equation \(\text{(11)}\). In order to obtain an improved equation for the Green function of light quarks one should therefore add their contribution separately \[4\]. Incidentally, the pion propagator is proportional to \(1/k^2\) such that one can treat pion loops on the quark using the same mathematical identity \[8\] as for the gluon propagator in Feynman gauge, hence again extracting the most IR-singular contributions. The coupling of the pion to the quark can be related to the pion decay constant \(f_\pi\) via a Goldberger-Treiman relation, and taking into account the proper isospin factor for two light quark flavors one obtains the improved Gribov equation

\[ \partial^2 G^{-1} = g (\partial^\mu G^{-1}) G (\partial_\mu G^{-1}) - \frac{3}{16\pi^2 f_\pi^2} \{i\gamma_5, G^{-1}\} G \{i\gamma_5, G^{-1}\}. \] (18)

The pion decay constant \(f_\pi\) can further be shown to fulfill in a selfconsistent way the equation

\[ f_\pi^2 = \frac{1}{8} \int \frac{d^4 q}{(2\pi)^4 i} \text{Tr} \left[ \{i\gamma_5, G^{-1}\} G \{i\gamma_5, G^{-1}\} G (\partial_\mu G^{-1}G)^2 \right] \]

\[ + \frac{1}{64\pi^2 f_\pi^2} \int \frac{d^4 q}{(2\pi)^4 i} \text{Tr} \left[ \{i\gamma_5, G^{-1}\} G \right]^4. \] (19)

Note that also the modified Gribov equation \[18\] is a differential equation rather than an integral equation and that it again involves only the quark’s Green function.

Choosing for \(f_\pi\) in \[18\] its phenomenological value one can repeat the numerical analysis of the quark’s Green function based on the improved Gribov equation \[11\] \[12\]. One finds that for spacelike momenta the mass function \(M(q^2)\) is affected only very little by the pion correction, as is illustrated in Figure \[4\] for the choice \(m_R = 0.1\text{ GeV}\). As a consequence the general behavior of the Green function for spacelike momenta as it was found already with the original equation \[11\] remains the same when pion corrections are taken into account. In particular, the oscillations of the mass functions for supercritical coupling again lead to chiral symmetry breaking in exactly the same way as discussed above.
Preliminary numerical results indicate that the situation is quite different when one moves away from spacelike momenta into the complex $q^2$-plane. Here the pion corrections become important and appear to induce considerable changes in the analytic structure of the Green function of light quarks. Gribov argued that the modified equation (18) will lead to an analytic structure of the Green function corresponding to confined quarks and hence to the general confinement of color charges in QCD [4]. A full numerical study of this conjecture remains to be carried out.

6 Summary and Outlook

Gribov has developed an attractive physical picture of QCD in which chiral symmetry breaking and confinement are caused by the existence of light quarks which can trigger the mechanism of supercritical color charges. He has derived an equation for the Green function of light quarks that collects the most important contributions both in the infrared and in the ultraviolet momentum region. The equation describes chiral symmetry breaking and gives rise to pions as Goldstone bosons. There are some indications that the backreaction of the pions on the quark leads to confinement.

The next step in a systematic study of Gribov’s approach should be a full investigation of the equation (18) which includes pion corrections, in particular a study of the resulting analytic structure of the quark’s Green function. A more difficult but extremely important future step should be to develop equations that describe the gluon sector of the theory in a similar approximation. Another crucial goal is to find observables that are sensitive to the particular confinement mechanism due to supercritical charges and to devise phenomenological tests of Gribov’s scenario.

In the course of his study of confinement Gribov has raised a number of questions concerning the general treatment of gauge theories in which the elementary fields of Lagrangian are not the ones that can propagate freely. In Gribov’s picture a satisfactory solution of the confinement problem can only be obtained if these questions are
addressed at the same time. That makes it a very challenging task to fully understand his ideas and to develop his program further in the future, but it is worth the effort.

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