Representations and cohomologies of Kleinian 4-rings

by

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REPRESENTATIONS AND COHOMOLOGIES OF KLEINIAN 4-RINGS

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Abstract. We introduce a new class of algebras over discrete valuation rings, called Kleinian 4-rings, which generalize the group algebra of the Kleinian 4-group. For these algebras we describe the lattices and their cohomologies. In the case of regular lattices we obtain an explicit form of cocycles defining the cohomology classes.

Introduction

Integral representations of the Kleinian 4-group $G$ (or $G$-lattices) were described by Nazarova [9]. Another description was proposed by Plakosh [10]. In the papers [6] and [7] cohomologies of these lattices were calculated. In this paper we consider a class of rings that generalizes group rings of the Kleinian 4-group. We call them Kleinian 4-rings. We give a description of lattices over such rings and calculate cohomologies of these lattices. In a special case of regular lattices we obtain an explicit form of cocycles defining cohomology classes.

1. Lattices over Kleinian 4-rings

In what follows $R$ denotes a complete discrete valuation ring with a prime element $p$, the field of fractions $Q$ and the field of residues $k = R/pR$. We write $\otimes$ instead of $\otimes_R$. If $A$ is an $R$-algebra, we call an $A$-module $M$ an $A$-lattice if it is finitely generated and free as $R$-module. Then we identify $M$ with its image $1 \otimes M$ in the vector space $Q \otimes M$ and an element $v \in M$ with $1 \otimes v \in Q \otimes M$. We denote by $A$-lat the category of $A$-lattices.

Definition 1.1. The Kleinian 4-ring over $R$ is the $R$-algebra $K = R[x, y]/(x(x - p), y(y - p))$. 
Note that if \( p = 2 \) this is just the group algebra over \( R \) of the Kleinian 4-group \( G = \langle a, b \mid a^2 = b^2 = 1, ab = ba \rangle \). One has to set \( x = a + 1, y = b + 1 \).

One easily sees that \( Q \otimes K \) is isomorphic to \( Q^4 \): just map \( x \) to \( \bar{x} = (p, p, 0, 0) \) and \( y \) to \( \bar{y} = (p, 0, p, 0) \). We consider \( K \) as embedded into \( Q^4 \) identifying \( x \) with \( \bar{x} \) and \( y \) with \( \bar{y} \). We also set \( z = (p, 0, 0, 0) \in Q^4 \) (note that \( z \notin K \) and \( z^2 = xy \)).

The maximal ideal \( r \) of \( K \) is \( (p, x, y) \) and \( K/r \simeq k \). Let \( A = \{ a \in Q^4 \mid aK \subset K \} \). One easily verifies that \( A = K + Rz \) and \( A/K \simeq k \). Hence \( K \) is a Gorenstein ring \([3, \text{Proposition } 6]\), i.e. \( \text{inj.dim}_K K = 1 \). Therefore, \( A \) is its unique minimal over-ring and every \( K \)-lattice is isomorphic to a direct sum of a free \( K \)-module and an \( A \)-lattice (see \([5, \text{Lemma } 2.9]\) or \([4, \text{Lemma } 3.2]\)). Note that the ring \( A \) is also local with the maximal ideal \( m = (p, x, y, z) \) and \( A/m \simeq k \).

Moreover, as the submodule of \( Q^4 \), \( m = pA^2 = \text{rad } A^2 \), where \( A^2 = R^4 \) is hereditary. Thus \( A \) is a Backström order in the sense of \([11]\). Therefore, \( A \)-lattices can be described by the representations of the quiver

\[
\begin{array}{cc}
pp & \downarrow f_{pp} \\
p0 & \downarrow f_{p0} \\
0p & \downarrow f_{0p} \\
00 \\
\end{array}
\]

over the field \( k \). Namely, denote by \( R_{\alpha\beta} \), where \( \alpha, \beta \in \{0,p\} \) the \( A \)-lattice such that \( R_{\alpha\beta} = R \) as \( R \)-module, \( xv = \alpha v \) and \( yv = \beta v \) for all \( v \in R_{\alpha\beta} \). For any \( A \)-lattice \( M \) and \( \alpha, \beta \in \{0,p\} \) set \( M_{\alpha\beta} = \{ v \in M \mid xv = \alpha v, yv = \beta v \} \). If \( M \) is an \( A \)-lattice, \( M^2 = A^2 M \) is an \( A^2 \)-module, hence \( M^2 = \bigoplus_{\alpha,\beta} M^2_{\alpha\beta} \). Let \( V_\bullet = M/mM \) and \( V_{\alpha\beta} = M^2_{\alpha\beta}/pM^2_{\alpha\beta} \). Note that \( M^2 \supset M \supset mM = pM^2 \). So the natural maps \( f_{\alpha\beta} : V_\bullet \to V_{\alpha\beta} \) are defined and we obtain a representation \( V \) of the quiver \( \Gamma \):

\[
V = V_\bullet
\]
We denote this representation by \( \Phi(M) \). It gives a functor \( \Phi : A\text{-lat} \to \text{rep } \Gamma \). The next result follows from [11].

**Theorem 1.2.** Let \( \text{rep}_+\Gamma \) be the full subcategory of \( \text{rep } \Gamma \) consisting of such representations \( V \) that all maps \( f_{\alpha \beta} \) are surjective and the map \( f_+ : V_+ \to V_+ \) is injective. The image of the functor \( \Phi \) is in \( \text{rep}_+\Gamma \) and, considered as the functor \( A\text{-lat} \to \text{rep}_+\Gamma \), the functor \( \Phi \) is an equivalence.

Recall that the term equivalence means that \( \Phi \) is full, maps non-isomorphic objects to non-isomorphic and every representation \( V \in \text{rep}_+\Gamma \) is isomorphic to some \( \Phi(M) \) (then \( \Phi \) maps indecomposable objects to indecomposable). Actually, this \( M \) can be reconstructed as follows. Set \( d_{\alpha \beta} = \dim V_{\alpha \beta}, V_+ = \bigoplus_{\alpha \beta} V_{\alpha \beta} \) and \( \bar{V} \) be the image of the map \( V_+ \to V_+ \) with the components \( f_{\alpha \beta} \). Then \( V_+ \cong M^\sharp/pM^\sharp \), where \( M^\sharp = \bigoplus_{\alpha \beta} R_{d_{\alpha \beta}} \). Let \( \Psi(V) \) be the preimage of \( V_+ \) in \( M^\sharp \). It is an \( A\text{-lattice} \) and \( \Phi(M) \cong V \). Moreover, \( M^\sharp = A^\sharp M \). Note also that the kernel of the map \( \text{Hom}_A(M,N) \to \text{Hom}_F(\Phi(M),\Phi(N)) \) coincides with \( \text{Hom}_A(M,mN) \).

The quintuple \( (d_\bullet | d_{pp}, d_{p0}, d_{0p}, d_{00}) \), where \( d_\bullet = \dim V_\bullet \), is called the vector dimension of the representation \( V \). We also call it the vector rank of the lattice \( M = \Psi(V) \) and denote it by \( \text{Rk} M \). For instance, \( \text{Rk} R_{pp} = (1 | 1, 0, 0, 0) \) and \( \text{Rk} A = (1 | 1, 1, 1, 1) \). Note that the rank of \( M \) as of \( R \)-module equals \( \sum_{\alpha \beta} d_{\alpha \beta} \), while \( d_\bullet = \dim_k M/mM \).

**Remark 1.3.** Note that the only indecomposable representations of \( \Gamma \) that do not belong to \( \text{rep}_+\Gamma \) are “trivial representations” \( V^j \), where \( j \in \{\bullet, \alpha \beta | \alpha, \beta \in \{0,p\}\} \) such that \( V^j_\bullet = k \) and \( V^j_i = 0 \) if \( i \neq j \). Therefore, the \( A \)-lattices are indeed classified by the representations of the quiver \( \Gamma \).

Let \( \tau_K (\tau_A) \) denote the Auslander-Reiten translate in the category \( K\text{-lat} \) (respectively, \( A\text{-lat} \)). Recall that \( \tau_K M \) for a non-projective indecomposable \( K \)-lattice \( M \) is an indecomposable \( K \)-lattice \( N \) such that there is an almost split sequence \( 0 \to N \to E \to M \to 0 \) [1]. The next result follows from [4].

**Proposition 1.4.**

1. \( \tau_K M \cong \tau_A M \) for any indecomposable \( A \)-lattice \( M \not\cong A \).
2. \( \tau_K A \cong \tau \) and it is a unique indecomposable \( A \)-lattice \( N \) such that \( \text{inj.dim}_N = 1 \).
3. \( \tau_K M \cong \Omega M \) for any \( A \)-lattice \( M \), where \( \Omega M \) denote the syzygy of \( M \) as of \( K \)-module.
Following [12], we can also restore the Auslander-Reiten quiver $Q(A)$ of the category $A$-lat from the Auslander-Reiten quiver $Q(\Gamma)_r$ of the category rep$\Gamma$. Recall that the quiver $Q(\Gamma)$ consists of the preprojective, preinjective and regular components. The quiver $Q((A))$ is obtained from $Q(\Gamma)$ by gluing the preprojective and preinjective components omitting trivial representations. The resulting preprojective-preinjective component is the following:

\[
\begin{array}{ccccccc}
R^2_{pp} & R^1_{pp} & R_{pp} & R^{-1}_{pp} & R^{-2}_{pp} & R^{-3}_{pp} \\
R^2_{p0} & R^1_{p0} & R_{p0} & R^{-1}_{p0} & R^{-2}_{p0} & R^{-3}_{p0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
R^2_{0p} & R^1_{0p} & R_{0p} & R^{-1}_{0p} & R^{-2}_{0p} & R^{-3}_{0p} \\
R^2_{00} & R^1_{00} & R_{00} & R^{-1}_{00} & R^{-2}_{00} & R^{-3}_{00} \\
\end{array}
\]

Here $M^k$ denotes $\tau^K_k M$. Note that $A^1 \simeq r \simeq A^\vee$, where $M^\vee = \text{Hom}_K(M, K)$. The representations belonging to this component are uniquely determined by their vector-ranks. One can verify that

\[
\text{Rk } A^k = \begin{cases} (2k - 1 \mid k, k, k) & \text{if } k > 0, \\ (1 - 2k \mid 1 - k, 1 - k, 1 - k, 1 - k) & \text{if } k < 0; \end{cases}
\]

\[
\text{Rk } R^k_{pp} = \begin{cases} (k + 1 \mid \left[\frac{k}{2}\right] - (-1)^k, \left[\frac{k}{2}\right], \left[\frac{k}{2}\right]) & \text{if } k > 0, \\ (-k \mid \left[\frac{1-k}{2}\right] + (-1)^k, \left[\frac{1-k}{2}\right], \left[\frac{1-k}{2}\right]) & \text{if } k < 0. \end{cases}
\]

The remaining (regular) components are tubes, where $\tau_K$ acts periodically. They are parametrized by the set

\[\mathbb{P} = \{\text{irreducible unital polynomials } f(t) \in \mathbb{k}[t] \} \cup \{\infty\}.\]

Actually, it is the set of closed points of the projective line over the field $\mathbb{k}$, that is of the projective scheme $\text{Proj} \mathbb{k}[x, y]$. If $f(t) = t - \lambda (\lambda \in \mathbb{k})$, we write $\mathcal{T}^F$ instead of $\mathcal{T}$. If $f \in \mathbb{P} \setminus \{t, t - 1, \infty\}$, the corresponding tube $\mathcal{T}^f$ is homogeneous, which means that $\tau_K M \simeq M$ for all $M \in \mathcal{T}^f$. It has the form

\[
\begin{array}{ccccccc}
T_1^f & \longrightarrow & T_2^f & \longrightarrow & T_3^f & \longrightarrow & \cdots
\end{array}
\]
and \( \text{Rk} T_n^f = (2dn \mid dn, dn, dn, dn) \), where \( d = \deg f(t) \). In this diagram all maps \( T_n^f \to T_{n+1}^f \) are monomorphisms with the cokernels \( T_n^f \), while all maps \( T_{n+1}^f \to T_n^f \) are epimorphisms with the kernels \( T_n^f \).

The exceptional tubes \( \mathcal{T}^\lambda (\lambda \in \{0, 1, \infty\}) \) are of the form

\[
\begin{array}{cccccc}
T_{11}^\lambda & \longrightarrow & T_{21}^\lambda & \longrightarrow & T_{31}^\lambda & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
T_{12}^\lambda & \longrightarrow & T_{22}^\lambda & \longrightarrow & T_{32}^\lambda & \longrightarrow & \cdots
\end{array}
\]

(2)

Here \( \tau_K T_{n+1}^\lambda = T_n^\lambda \) and \( \tau_K T_n^\lambda = T_{n+1}^\lambda \). In this diagram all maps \( T_n^\lambda \to T_{n+1}^\lambda \) are monomorphisms with the cokernels \( T_1^\lambda \), where \( j = i \) if \( n \) is even and \( j \neq i \) if \( n \) is odd. All maps \( T_{n+1}^\lambda \to T_n^\lambda \) (\( j \neq i \)) are epimorphisms with the kernels \( T_1^\lambda \).

For \( \lambda = 1 \) we have

\[
\begin{align*}
\text{Rk} T_{2m}^{1j} &= (2m \mid m, m, m, m) \quad \text{for both } j = 1 \text{ and } j = 2, \\
\text{Rk} T_{2m-1}^{1j} &= (2m-1 \mid m, m, m-1, m-1), \\
\text{Rk} T_{2m-1}^{12} &= (2m-1 \mid m-1, m-1, mm).
\end{align*}
\]

The vector-ranks for the tubes \( \mathcal{T}^0 \) and \( \mathcal{T}^\infty \) are obtained from those for \( \mathcal{T}^1 \) by permutation of \( d_{p0} \), respectively, with \( d_{00} \) and with \( d_{0p} \).

2. Cohomologies

A Kleinian 4-ring is a supplemented \( R \)-algebra in the sense of [2, Ch. X] with respect to the surjection \( \pi : K \to K/(x - p, y - p) \cong R \). Therefore, for any \( K \)-module \( M \) the homologies \( H_n(K, M) = \text{Tor}_n^K(R, M) \) and cohomologies \( H^n(K, M) = \text{Ext}_n^K(R, M) \) are defined. Moreover, if we consider \( M \) as \( K \)-bimodule setting \( mx = my = pm \) for all \( m \in M \), they coincide with the Hochschild homologies and cohomologies:

\[ H_n(K, M) \cong HH_n(K, M) \quad \text{and} \quad H^n(K, M) \cong HH^n(K, M). \]

(see [2, Theorem X.2.1]).

Remark 2.1. We have chosen the augmentation \( K \to R \) such that if \( p = 2 \), hence \( K \cong RG \) for the Kleinian 4-group \( G \), it coincides with the usual augmentation \( RG \to R \) mapping all elements of the group to 1. Thus in this case \( H_n(K, M) = H_n(G, M) \).

Proposition 2.2. For every \( K \)-module \( M \) and \( n \neq 0 \)

\[ xyH_n(K, M) = xyH^n(K, M) = p^2H_n(K, M) = p^2H^n(K, M) = 0 \]
Proof. The map \( \mu : r \mapsto rxy \) is a homomorphism of \( K \)-modules \( R \to K \) such that \( \pi \mu : R \to R \) is the multiplication by \( xy \) or, the same, by \( p^2 \). Therefore, the multiplication by \( xy \) or by \( p^2 \) in \( \text{Ext}_n^K(R, M) \) or in \( \text{Tor}_n^K(R, M) \) factors, respectively, through \( \text{Ext}_n^K(K, M) = 0 \) or through \( \text{Tor}_n^K(K, M) = 0 \). \( \square \)

Note that \( K \simeq \bar{K} \otimes_R K \), where \( \bar{K} = R[x]/(x(x - p)) \) A projective resolution \( \bar{P} \) for \( R \) as of \( K \)-module, where \( xr = pr \) for all \( r \in R \), is obtained if we set \( \bar{P}_n = \bar{K}u^n \) and

\[
d u^n = C_n(x)u^{n-1}, \quad \text{where} \quad C_i(x) = \begin{cases} 1 & \text{if } n \text{ is even}, \\ x^{-p} & \text{if } n \text{ is odd}. \end{cases}
\]

Then \( \bar{P} = \bar{P} \otimes_R \bar{P} \) is a projective resolution of \( R \) as of \( K \)-module. Here \( P_n \) is the module of homogeneous polynomials of degree \( n \) from \( K[u, v] \) and

\[
d(x^i y^j) = C_i(x)u^{i-1}v^j + (-1)^i C_j(y)u^i v^{j-1}.
\]

Denote \( H_n(\bar{K}, M) = \text{Tor}_n^K(R, M) \). Then

\[
H_n(\bar{K}, M) = \begin{cases} M/(x-p)M & \text{if } n = 0, \\ \ker(x-p)M/xM & \text{if } n \text{ is odd}, \\ \ker xM/(x-p)M & \text{if } n \text{ is even}, \end{cases}
\]

where \( aM \) denotes the multiplication by \( a \) in the module \( M \). Let \( R_0 = \bar{K}/(x), R_p = \bar{K}/(x - p) \). Then \( R_{\alpha \beta} \simeq R_\alpha \otimes_R R_\beta \). As the ring \( R \) is hereditary, the K"unneth formula [2, Theorem VI.3.2] implies that

\[
H_n(K, R_{\alpha \beta}) \simeq \left( \bigoplus_{i+j=n} H_i(\bar{K}, R_\alpha) \otimes_R H_j(\bar{K}, R_\beta) \right) \oplus \left( \bigoplus_{i+j=n-1} \text{Tor}_1^R(H_i(\bar{K}, R_\alpha), H_j(\bar{K}, R_\beta)) \right).
\]

Since

\[
H_n(\bar{K}, R_0) = \begin{cases} 0 & \text{if } n \text{ is odd}, \\ k & \text{if } n \text{ is even}; \end{cases} \\
H_n(\bar{K}, R_p) = \begin{cases} k & \text{if } n \text{ is odd}, \\ 0 & \text{if } n \text{ is even}; \end{cases}
\]

we obtain

\[
H_n(K, R_{pp}) = \begin{cases} R & \text{if } n = 0, \\ (R/p)^{(n+3)/2} & \text{if } n \text{ is odd}, \\ (R/p)^{n/2} & \text{if } n \text{ is even}; \end{cases}
\]
and if \((\alpha, \beta) \neq (p, p)\)

\[
H_n(K, R_{\alpha\beta}) = (R/p)^{[n+2]/2}.
\]

On the other hand, the exact sequence \(0 \to K \to A \to \mathbb{k} \to 0\) implies that for \(n > 0\)

\[
H_n(K, A) \simeq H_n(K, \mathbb{k}) \simeq P_n \otimes_K \mathbb{k} \simeq \mathbb{k}^{n+1},
\]

since \(H_n(K, K) = 0\) and the differential in \(P \otimes_K \mathbb{k}\) is zero.

As \(K\) is Gorenstein, the functor \(M \mapsto M^\vee = \text{Hom}_K(M, K)\) is an exact duality in the category \(\mathcal{K}\)-lat, i.e. the natural map \(M \mapsto M^{\vee\vee}\) is an isomorphism. If \(P\) is projective, then \(P \otimes_K M \simeq \text{Hom}_K(P^\vee, M)\).

Therefore, homologies of a module \(M\) can be obtained as \(H_n(\text{Hom}_K(P^\vee, M))\).

Note that the embedding \(R \to P_0^\vee \simeq K\) maps 1 to \(xy\). Hence, just as for finite groups, we can consider a full resolution \(\hat{P}\) setting

\[
\hat{P}_n = \begin{cases} 
P_n & \text{if } n \geq 0, \\
P_{n-1} & \text{if } n < 0
\end{cases}
\]

and defining \(d_0 : K = \hat{P}_0 \to \hat{P}_1 \simeq K\) as multiplication by \(xy\). Thus the Tate cohomologies \(\hat{H}^n(K, M)\) are defined as \(H^n(\text{Hom}_K(\hat{P}, M))\) with the usual properties

\[
\hat{H}^n(K, M) = \begin{cases} 
H^n(K, M) & \text{if } n > 0, \\
H_{-n-1}(K, M) & \text{if } n < -1, \\
M_{pp}/xyM & \text{if } n = 0, \\
\{m \mid xym = 0\}/((x - p)M + (y - p)M) & \text{if } n = -1,
\end{cases}
\]

where \(M_{pp} = \{m \mid zm = ym = pm\}\). In particular, \(xy\hat{H}^n(K, M) = p^2\hat{H}^n(K, M) = 0\) for all \(M\). Note also that, if \(M\) is an \(A\)-lattice, \(M_{pp} = \{m \mid zm = pm\}\) and \(xyM = \mathbb{k}^2M\).

A basis of \(\hat{P}_{-n}\) \((n > 0)\) can be chosen as \(\{\hat{u}^i\hat{v}^j \mid i + j = n - 1\}\), where \((\hat{u}^i\hat{v}^j)(u^kv^l) = \delta_{ik}\delta_{jl}\). Then

\[
d(\hat{u}^i\hat{v}^j) = C_{i+1}\hat{u}^{i+1}\hat{v}^j + (-1)^iC_{j+1}\hat{u}^i\hat{v}^{j+1}.
\]

**Proposition 2.3.** If \(M\) is an \(A\)-lattice that has no direct summands isomorphic to \(R_{pp}\), then

\[
\hat{H}^0(K, M) = M_{pp}/pM_{pp} \simeq \mathbb{k}^{d_{pp}},
\]

where \((d_\bullet \mid d_{pp}, d_{p0}, d_{0p}, d_{00}) = \mathbb{R}kM\).

**Proof.** Set \(M^\sharp = A^\sharp M = \bigoplus_{\alpha\beta} M^\sharp_{\alpha\beta}\). Note that \(xyA = Rxy = xyA^\sharp\), hence \(xyM = xym^\sharp = p^2M^\sharp_{pp}\). On the other hand, \(M^\sharp_{pp} \simeq R_{pp}^d\). If \(pM_{pp} \neq M^\sharp_{pp}, M_{pp}\) contains a direct summand
Denote $T = Q/R$, $DM = \text{Hom}_R(M, T)$. It is the Matlis duality between noetherian and artinian $R$-modules, as well as $K$-modules [8]. We have the following dualities for cohomologies.

**Proposition 2.4.** Let $M$ be a $K$-module. Then

$\hat{H}^n(K, DM) \simeq D\hat{H}^{-n-1}(K, M), \quad (7)$

and if $M$ is a lattice

$\hat{H}^n(K, DM) \simeq \hat{H}^{n+1}(K, M^\vee), \quad (8)$

$\hat{H}^n(K, M^\vee) \simeq D\hat{H}^{-n}(K, M). \quad (9)$

**Proof.** Note first that, since $K$ is local and Gorenstein, $\text{Hom}_R(K, R) \simeq K$, whence $M^\vee \simeq \text{Hom}_R(M, R)$ and we identify these modules. As $T$ is an injective $R$-module,

$\text{Ext}^n_K(R, \text{Hom}_R(M, T)) \simeq \text{Hom}_R(\text{Tor}_n^K(R, M), T),$

(see [2, Proposition VI.5.1]), which is just (7).

The exact sequence $0 \to R \to Q \to T \to 0$ gives, for any lattice $M$, the exact sequence

$0 \to M^\vee \to \text{Hom}_R(M, Q) \to DM \to 0.$

As multiplication by $p^2$ is an automorphism of $\text{Hom}_R(M, Q)$, Proposition 2.2 implies that $\hat{H}^n(\text{Hom}_R(M, Q)) = 0$. Then the long exact sequence for cohomologies implies (8).

(9) is a combination of (7) and (8). \hfill \square

Note also that $\hat{H}^n(K, F) = 0$ for any projective (hence free) $K$-module $F$. Therefore, Proposition 1.4 implies that, for any indecomposable $A$-lattice $M$,

$\hat{H}^n(K, M) \simeq \hat{H}^{n+1}(K, \tau_K M) \simeq \hat{H}^{n-1}(K, \tau_K^{-1} M). \quad (10)$

Hence from the formulae (4)-(6) and the duality (9) we obtain a complete description of cohomologies of $K$-lattices belonging to the preprojective-preinjective component.
Theorem 2.5.

\[ \hat{H}^n(K, A^k) \simeq \begin{cases} 
k^{n-k+1} & \text{if } n \geq k, \\k^{n-k} & \text{if } n < k; \end{cases} \]

\[ \hat{H}^n(K, R^{k}_{pp}) \simeq \begin{cases} 
k^{(n-k)/2+1} & \text{if } n - k \neq 0 \text{ is even}, \\
k^{n-k}/2 & \text{if } n - k \text{ is odd}, \\
R/xyR & \text{if } n = k; \end{cases} \]

\[ \hat{H}^n(K, R^{k}_{\alpha\beta}) \simeq nk^{(n-k+1)/2} & \text{if } (\alpha, \beta) \neq (p, p). \]

The description of cohomologies of \( A \)-lattices belonging to tubes are obtained from Proposition 2.3, since \( \hat{H}^n(K, M) \simeq \hat{H}^0(K, \tau^{-n}_K M) \) and we know the action of \( \tau_K \) in tubes.

Theorem 2.6.

1. If \( f \notin \{t, t - 1\} \), then \( \hat{H}^n(K, T^f_m) \simeq k^{dm} \), where \( d = \deg f \).

2. If \( \lambda \in \{0, 1, \infty\} \), then

\[ \hat{H}^n(K, T^\lambda_m) \simeq \begin{cases} 
k^{m/2} & \text{if } m \text{ is even,} \\
k^{m-(-1)^{n+1}}/2 & \text{if } m \text{ is odd.} \end{cases} \]

3. Regular lattices

An \( A \)-lattice \( M \) is called regular if all its indecomposable direct summands belong to tubes. As neither regular lattice is projective, \( \tau_K M = \tau_A M = \Omega M \). Note that if \( M \) is regular, then

\[ 2d_\bullet(M) = \sum_{\alpha\beta} d_{\alpha\beta}(M). \]

Therefore,

\[ d_\bullet(\Omega M) = d_\bullet(M) \quad \text{and} \quad d_{\alpha\beta}(\Omega M) = d_\bullet(M) - d_{\alpha\beta}(M). \]

These formulae imply the following fact.

Lemma 3.1. Every exact sequence of regular \( A \)-lattices \( 0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0 \) induces exact sequences

\[ 0 \rightarrow \Omega M \rightarrow \Omega N \rightarrow \Omega L \rightarrow 0, \]

\[ 0 \rightarrow \Omega^{-1} M \rightarrow \Omega^{-1} N \rightarrow \Omega^{-1} L \rightarrow 0. \]

Proof. Obviously, there is an exact sequence \( 0 \rightarrow \Omega M \rightarrow \Omega N \oplus P \rightarrow \Omega L \rightarrow 0 \) for some projective module \( P \). On the other hand, as \( d_{\alpha\beta}(N) = d_{\alpha\beta}(M) + d_{\alpha\beta}(L) \), the formulae (11) and (12) imply that \( Rk \Omega N = Rk \Omega M + Rk L \). Hence \( P = 0 \) and we obtain (13). By duality, we also have (14). \( \square \)
Corollary 3.2. Every exact sequence of regular $A$-lattices $0 \to M \to N \to L \to 0$ induces exact sequences of cohomologies
\[ 0 \to \hat{H}^n(K, M) \to \hat{H}^n(K, N) \to \hat{H}^n(K, L) \to 0. \]

Proof. If $n = 0$, it follows from Proposition 2.3. For other $n$ it is obtained by an easy induction using Lemma 3.1.

For regular lattices we can give an explicit form of cocycles defining cohomology classes. Namely, for an indecomposable regular lattice $M$ and an integer $n$ we set

\[ M(n) = \begin{cases} M_{pp} & \text{if } n \text{ is even}, \\ M_{0p} & \text{if } n \text{ is odd and } M \notin \mathcal{T}^\infty, \\ M_{p0} & \text{if } n \text{ is odd and } M \in \mathcal{T}^\infty. \end{cases} \]

For $n > 0$ we define a homomorphism $M(n) \to \hat{H}^n(K, M)$ mapping an element $a \in M(n)$ to the class of the cocycle $\xi_a : P_n \to M$ defined as follows:

- If $M \notin \mathcal{T}^\infty$, then
  \[ \xi_a(u^k v^{n-k}) = \begin{cases} a & \text{if } k = n, \\ 0 & \text{otherwise}. \end{cases} \]

- If $M \in \mathcal{T}^\infty$, then
  \[ \xi_a(u^k v^{n-k}) = \begin{cases} a & \text{if } k = 0, \\ 0 & \text{otherwise}. \end{cases} \]

Theorem 3.3. The map $a \mapsto \xi_a$ induces an isomorphism
\[ \xi : M(n)/pM(n) \simeq \hat{H}^n(K, M) \]
for every $n > 0$ and every regular indecomposable $A$-lattice $M$.

Proof. One easily sees that $\xi_a$ is a cocycle. Theorem 2.6 and formulae (3) show that $\hat{H}^n(K, M) \simeq M(n)/pM(n)$. Hence we only have to prove that $\xi_a$ is not a coboundary if $a \notin pM(n)$. First we check it for the lattices $T_1^f$ and $T_1^{\infty}$.

We consider the case when $n$ is even and $M = T_1^f$, where $\deg f = d$ and $f \notin \{t, t - 1\}$. The other cases are quite similar or even easier.
The corresponding representation of the quiver $\Gamma$ is

$$
\begin{align*}
\text{k}^d & \xrightarrow{(I \, 0)} \text{k}^d \\
\text{k}^{2d} & \xrightarrow{(I \, I)} \text{k}^d \\
\text{k}^d & \xrightarrow{(I \, F)} \text{k}^d
\end{align*}
$$

where $I$ is $d \times d$ unit matrix and $F$ is the Frobenius matrix with the characteristic polynomial $f(t)$. Therefore, $M$ is the submodule of $\bigoplus_{\alpha \beta} M^d_{\alpha \beta}$, where $M^d_{\alpha \beta} = R^d_{\alpha \beta}$ and $M$ consists of the quadruples $a = (a_{pp}, a_{p0}, a_{0p}, a_{00}) \equiv (r, r', r + r', r + \bar{F}r') \pmod{p}$, where $r, r' \in R^d$ and $\bar{F}$ is a $d \times d$ matrix over $R$ such that $F = \bar{F} \pmod{p}$. Hence, $M_{\alpha \beta} = pM^d_{\alpha \beta}$. In particular, elements $a \in M(n)$ are of the form $(pr, 0, 0, 0)$. Let

$$
\xi_a = d\gamma, \text{ where } \gamma(x^{k-1}y^{n-k}) = \gamma_k \equiv (r_k, r'_k, r_k + r'_k, r_k + \bar{F}r'_k) \pmod{p}
$$

for $1 \leq k \leq n$. Then

$$
d\gamma(u^n) = 0 = y\gamma_1 \equiv (pr_1, 0, p(r_1 + r'_1), 0) \pmod{p^2},
$$

hence $\gamma_1 \equiv 0 \pmod{p}$. Suppose that $\gamma_{k-1} \equiv 0 \pmod{p}$ for $1 < k \leq n$. If $k$ is odd, then

$$
d\gamma(u^{k-1}v^{n-k+1}) = 0 = x\gamma_{k-1} + y\gamma_k \equiv
\equiv (pr_k, 0, p(r_k + r'_k), 0) \pmod{p^2},
$$

If $k$ is even, then

$$
d\gamma(u^{k-1}v^{n-k+1}) = 0 = (x - p)\gamma_{k-1} - (y - p)\gamma_k \equiv
\equiv (0, pr'_k, 0, p(r_k + \bar{F}r'_k)) \pmod{p^2}.
$$

In both cases $\gamma_k \equiv 0 \pmod{p}$. Therefore, $\gamma_k \equiv 0 \pmod{p}$ for all $1 \leq k \leq n$. Then

$$
d\gamma(u^n) = (a, 0, 0, 0) = x\gamma_n \equiv 0 \pmod{p^2},
$$

so $a \in pM(n)$.

Suppose now that the theorem is valid for all $T^f_{k-1}$ and for all $T^{\lambda_i}_{k-1}$. If $M = T^f_k$ or $M = T^{\lambda_i}_k$, there is an exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$, where, respectively, $M' = T^f_1$, $M'' = T^f_{k-1}$ or $M' = T^{\lambda_i}$, $M'' = T^{\lambda_i}_{k-1}$.
\( T^{'*}_{k-1} \) \((j \neq i)\). It gives a commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \longrightarrow & M'(n) & \longrightarrow & M(n) & \longrightarrow & M''(n) & \longrightarrow & 0 \\
\downarrow \xi & & \downarrow \xi & & \downarrow \xi & & \downarrow \xi & & \\
0 & \longrightarrow & \hat{H}^n(K, M') & \longrightarrow & \hat{H}^n(K, M) & \longrightarrow & \hat{H}^n(K, M'') & \longrightarrow & 0
\end{array}
\]

Using induction, we may suppose that the first and the third homomorphisms \( \xi \) satisfy the theorem. Therefore, so does the second, which accomplishes the proof. \( \square \)

Dualizing this construction, we obtain an explicit description of Tate cohomologies with negative indices. Namely, for \( n < 0 \) we define a homomorphism \( M(n) \rightarrow \hat{H}^n(K, M) \) mapping an element \( a \in M(n) \) to the class of the cocycle \( \hat{\xi}_a : P_n \rightarrow M \) defined as follows:

- If \( M \notin \mathcal{T}^\infty \), then
  \[
  \hat{\xi}_a(\hat{u}^k \hat{v}^{|n|-1-k}) = \begin{cases}
  a & \text{if } k = |n| - 1, \\
  0 & \text{otherwise}.
  \end{cases}
  \]

- If \( M \in \mathcal{T}^\infty \), then
  \[
  \hat{\xi}_a(\hat{u}^k \hat{v}^{|n|-1-k}) = \begin{cases}
  a & \text{if } k = 0, \\
  0 & \text{otherwise}.
  \end{cases}
  \]

**Theorem 3.4.** The map \( a \mapsto \hat{\xi}_a \) induces an isomorphism
\[
\hat{\xi} : M(n)/pM(n) \simeq \hat{H}^n(K, M)
\]
for every \( n < 0 \) and every regular indecomposable \( A \)-lattice \( M \).

The proof just repeats that of Theorem 3.3, so we omit it.

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