Spectral asymptotics induced by approaching and diverging planar circles

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Dedicated to Pavel Exner on the occasion of his 70th birthday.

Abstract

We consider two-dimensional system governed by the Hamiltonian with delta interaction supported by two concentric circles separated by distance $d$. We analyze the asymptotics of the discrete eigenvalues for $d \to 0$ as well as for $d \to \infty$.

1 Introduction

The paper belongs to research often called Schrödinger operators with delta potentials. We study a special model: two-dimensional quantum system with delta potential supported by two concentric circles: $C_R$ and $C_{R+d}$, where $C_R := \{(x, y) \in \mathbb{R}^2 : (x^2 + y^2)^{1/2} = R\}$ and $C_{R+d}$ is defined analogously for $R_d := R + d$, $d > 0$. The Hamiltonian of such system can be symbolically written as

$$-\Delta - \beta \delta_{C_R} - \alpha \delta_{C_{R+d}}, \quad \text{where} \quad \alpha, \beta \in \mathbb{R}, \quad (1.1)$$

where $\delta_{C_r}$ stands for the Dirac delta supported on $C_r$. To define a self-adjoint operator $H_{\alpha,\beta,d}$ corresponding formally to $(1.1)$ we employ the form sum method.

The main results. We investigate the behaviour of the discrete eigenvalues for $d \to 0$ and $d \to \infty$. In fact, in both asymptotics one can observe certain "spectral memory" on single circle system. Therefore, it is convenient to introduce a special notation $H_{\gamma,R}$ for the Hamiltonian corresponding to the formal expression

$$-\Delta - \gamma \delta_{C_R}, \quad \gamma \in \mathbb{R}. \quad (1.2)$$

In the following $\gamma$ will be expressed by means of the coupling constants $\alpha$ and $\beta$, however, this dependence will be different in two considered cases. If $\gamma > 0$ then operator $H_{\gamma,R}$ has $2M_{\gamma,R} + 1$ eigenvalues (counting multiplicity), where $M_{\gamma,R} := \max\{m \in \mathbb{Z} : 2|m| < R\gamma\}$.

The first result concerns the eigenvalue asymptotics in the approaching circles system and the statement can be formulated as follows.
• Let $E_m$ denote an eigenvalue of $H_{\alpha+\beta, R}$. Then the eigenvalues of $H_{\alpha, \beta, d}$ admit the following asymptotics

$$E_m + t_m d + o(d),$$

for $d \to 0$. The explicit form for the first correction term $t_m$ is derived in Theorem [3.1]. The analysed system enables separation of variables and, consequently, relying on the implicit function theorem we can reproduce $t_m$ in the terms of the Bessel functions and their derivatives. In Section [3.1] we also study certain properties of $t_m$; for example, we show that the sign of $t_m$ is not defined generally.

The second result is addressed to the system with circles separated by a large distance.

• Assume that $d \to \infty$. Then the system has a "tendency for the decoupling". This is manifested as the localization of eigenvalues of $H_{\alpha, \beta, d}$ near the eigenvalues of $H_{\beta, R}$ as well as $H_{\alpha, R}$. Precisely, the eigenvalues of $H_{\alpha, \beta, d}$ behave as

$$\begin{align*}
E_m,_{\beta} + o(|m|), & |m| \leq M_{\alpha, R}, \\
E_m,_{\beta} + w_m \varepsilon + o(\varepsilon), & |m| \leq M_{\beta, R},
\end{align*}$$

(1.3)

where $\varepsilon := \exp(-2d\kappa_{m, \beta})$ and $E_m,_{\beta}$ stand for the eigenvalues of $H_{\beta, R}$. Note that the expression in the first line of (1.3) reflects the asymptotics of eigenvalues of $H_{\alpha, R}$.

The models of delta interactions supported by circles or spheres has been already studied in the various contexts and dimensions, see for example [3], [4], [6], [7], [8], [9], [12], [17].

2 Preliminaries and the main result

Single ring: spectral properties of the system. Spectral properties of the single circle Hamiltonian will be essential for the both asymptotics considered in this paper. Therefore, we start our analysis from recalling some useful known facts, cf [9]. Consider the Hamiltonian $H_{\gamma, R}$ associated to the sesquilinear form

$$h_{\gamma, R}(f, g) = (\nabla f, \nabla g)_{L^2(\mathbb{R}^2)} - \gamma \int_{C_R} fg ds, \quad f, g \in W^{1,2}(\mathbb{R}^2), \quad \gamma \in \mathbb{R};$$

where the functions in the second component are understood in the sense of the trace embedding $W^{1,2}(\mathbb{R}^2) \hookrightarrow L^2(C_R)$ and the arc length parameter $s$ ranges $s \in [0, 2\pi R]$. In fact, $L^2(C_R)$ can be identify with $L^2((0, 2\pi R))$. Applying the results of [5] we conclude that the operator $H_{\gamma, R}$ associated to $h_{\gamma, R}$ via the first representation theorem is self-adjoint. $H_{\gamma, R}$ gives a mathematical meaning to the formal expression [1].

To be specific we introduce the polar system of coordinates $(r, \phi)$ where $r > 0, \phi \in [0, 2\pi]$. The delta potential support $C_R$ decomposes $\mathbb{R}^2$ onto two disjoint

\footnote{In view of the above trace embedding operator it seems natural to consider $s$ parameter instead of $\phi$; however, since we going to implement the second circle it is more convenient to stay with standard polar coordinates.}
open sets $\Omega^i$, $\Omega^e$; denote by $\bar{\Omega}^i$, $\bar{\Omega}^e$ their closures and $\mathcal{C}^1_R := C^1(\bar{\Omega}^i) \cup C^1(\bar{\Omega}^e)$. Assume that $f \in \mathcal{C}^1_R$ satisfies
\begin{align*}
\lim_{r \to R^+} f(r, \phi) &= \lim_{r \to R^-} f(r, \phi) =: f_R(\phi), \\
\lim_{r \to R^+} \partial_r f(r, \phi) - \lim_{r \to R^-} \partial_r f(r, \phi) &= -\gamma f_R(\phi).
\end{align*}
(2.1)
Then the operator which acts as
\[
\hat{H}_{\gamma,R} f = -\Delta f \quad \text{a.e. in } \mathbb{R}^2,
\]
(2.2)
on the domain
\[
D(\hat{H}_{\gamma,R}) = \{ f \in \mathcal{C}^1_R \cap W^{2,2}(\mathbb{R}^2 \setminus C_R) : f \text{ satisfies (2.1)} \}
\]
(2.3)
is essentially self-adjoint and its closure coincides with $H_{\gamma,R}$, cf [5].

Since the delta potential is compactly supported the essential spectrum of $H_{\gamma,R}$ is stable under such “perturbation,” i.e.
\[
\sigma_{\text{ess}}(H_{\gamma,R}) = [0, \infty),
\]
cf [5].

Henceforth we will be interested in negative eigenvalues. In view of the rotational symmetry we postulate that the eigenfunctions of $H_{\gamma,R}$ take the form
\[
\sqrt{2} \pi \varphi_m(r)e^{im\phi}, \quad m \in \mathbb{Z}.
\]
Let $\kappa > 0$. The behaviour of eigenfunctions at the infinity and origin imposes
\begin{align*}
\varphi_m(r) &= c_1 K_m(\kappa r), \quad \text{for } r > R, \\
\varphi_m(r) &= c_2 I_m(\kappa r), \quad \text{for } r < R,
\end{align*}
(2.4)
where $K_m(\cdot)$ and $I_m(\cdot)$ denote the modified Bessel functions, cf [1]. Using the boundary conditions (2.1) we get the following spectral condition
\[
K_m(\kappa R)I_m(\kappa R) = \frac{1}{\gamma R}, \quad m \in \mathbb{Z},
\]
(2.5)
cf [9].

It follows from (2.4) that $\kappa$ determines the spectral parameter and the solutions of (2.5) reproduce negative eigenvalues $E$ of $H_{\gamma,R}$ by means of the relation
\[
E = -\kappa^2.
\]

Remarks 2.1. A. Relying on asymptotics formulae (5.1)-(5.4) and using the fact that $(K_m I_m)(\cdot)$ is monotonously decreasing we state that the equation (2.5) has exactly one solution for $\kappa > 0$ provided $2|m| < R\gamma$ or equivalently $|m| \leq M_{\gamma,R}$ and no solution otherwise; recall that the notation $M_{\gamma,R}$ was introduced in introduction.
B. It is also useful to recall that for $\gamma R$ large the solution of (2.5) behaves as
\[
-m^2 = -\frac{\gamma^2}{4} + \frac{m^2}{R^2} + O(\gamma^{-2} R^{-4}).
\]
Hamiltonian with the delta potential supported by two concentric rings. Let 
$s \in [0, 2\pi R)$ and $s_4 \in [0, 2\pi R_d)$ stand for the arc length parameters 
associated to $C_R$ and $C_{R_d}$ respectively. For $\alpha, \beta \in \mathbb{R}$ let us define the 
following sesquilinear form

$$h_{\alpha, \beta, d}(f, g) = (\nabla f, \nabla g)_{L^2(\mathbb{R}^2)} - \beta \int_{C_R} fg \, ds - \alpha \int_{C_{R_d}} fg \, ds_d,$$

where we employ the trace embedding of $W^{1,2}(\mathbb{R}^2)$ to $L^2(0, 2\pi r)$, 
$r = R, R_d$. Similarly as for the single circle case we define the operator $H_{\alpha, \beta, d}$ 
associated to $h_{\alpha, \beta, d}$ via the first representation theorem.

Analogously for the single circle we can characterize $H_{\alpha, \beta, d}$ by means of 
boundary conditions. Note that circles $C_R$ and $C_{R_d}$ decompose $\mathbb{R}^2$ onto three 
open sets $\Omega_1, \Omega_2$ and $\Omega^*$. Denote $C^1_{R, R_d} := C^1(\Omega_1) \cup C^1(\Omega_2) \cup C^1(\Omega^*)$ and 
assume that $f \in C^1_{R, R_d}$ satisfies

$$\lim_{r \to R^+} f(r, \phi) = \lim_{r \to R^-} f(r, \phi) =: f_R(\phi),$$

$$\lim_{r \to R_d^+} f(r, \phi) - \lim_{r \to R_d^-} f(r, \phi) = -\beta f_R(\phi),$$

$$\lim_{r \to R_d} f(r, \phi) = \lim_{r \to R_d} f(r, \phi) =: f_{R_d}(\phi),$$

(2.6)

$$\lim_{r \to R_d} \partial_r f(r, \phi) - \lim_{r \to R_d} \partial_r f(r, \phi) = -\alpha f_{R_d}(\phi).$$

In fact, $H_{\alpha, \beta, d}$ stands for the closure

$$\hat{H}_{\alpha, \beta, d} f = -\Delta f \quad \text{a.e. in } \mathbb{R}^2,$$

$$D(\hat{H}_{\alpha, \beta, d}) = \left\{ f \in C^1_{R, R_d} \cap W^{2,2}(\mathbb{R}^2 \setminus (C_R \cup C_{R_d})) : f \text{ satisfies (2.6)} \right\}. \quad (2.7)$$

2.1 Spectral equation for the double ring system

To derive the spectral equation for the double ring system we proceed analogously as in the previous case. The system again admits separation variables. Consequently, the eigenfunctions of $H_{\alpha, \beta, d}$ can be written as $
\rho_{m}(r)e^{im\phi}$, \nwhere $m \in \mathbb{Z}$ and

$$\rho_{m}(r) = C_1 K_m(\kappa r), \quad \text{for } r > R_d,$$

$$\rho_{m}(r) = C_2 K_m(\kappa r) + C_3 I_m(\kappa r), \quad \text{for } R < r < R_d$$

and

$$\rho_{m}(r) = C_4 I_m(\kappa r), \quad \text{for } r < R.$$

Inserting the above formulae to (2.6) we obtain four equations

$$C_1 K_m(\kappa R_d) - C_2 K_m(\kappa R_d) - C_3 I_m(\kappa R_d) = 0,$$

$$C_1 (\kappa K_m'(\kappa R_d) + \alpha K_m(\kappa R_d)) - C_2 \kappa K_m'(\kappa R_d) - C_3 \kappa I_m'(\kappa R_d) = 0,$$

$$C_2 \kappa K_m'(\kappa R) + C_3 I_m(\kappa R) - C_4 I_m(\kappa R) = 0,$$

$$C_2 \kappa K_m'(\kappa R) + C_3 \kappa I_m(\kappa R) + C_2 (\beta I_m(\kappa R) - \kappa I_m'(\kappa R)) = 0.$$
Spectral equation. The above system of equations admits a solution if and only if the determinant of the corresponding matrix vanishes. This condition can be written by means of the following equation

\[ \eta_m(\kappa, d) = 0, \quad m \in \mathbb{Z}, \kappa > 0, \quad d \geq 0, \]  

for \( m \in \mathbb{Z}, \kappa > 0, d \geq 0 \), \( \eta_m(\kappa, d) = \nu_m(\kappa, d) - \xi_{m,\alpha}(\kappa)\xi_{m,\beta}(\kappa), \) \( \eta_m(\kappa, d) \equiv \xi_{m,\alpha}(\kappa) := \alpha R_d(K_m I_m)(\kappa R_d) - 1, \quad \xi_{m,\beta}(\kappa) := \beta R(K_m I_m)(\kappa R) - 1, \) \( \nu_m(\kappa, d) := \alpha\beta R_d R K_m^2(\kappa R_d) I_m^2(\kappa R). \)

Formulae (2.8) constitute the spectral equations for \( H_{\alpha,R,d}. \)

Remark 2.2. Note that the functions \( \xi_{m,\tau}(\kappa), \) where \( \tau = \alpha, \beta \) are related to the single circle systems. More precisely, the relations \( \xi_{m,\tau}(\kappa) = 0, \) determine spectral equations of \( H_{\alpha,R} \) and \( H_{\beta,R}. \)

3 Approaching rings

In this section we consider the eigenvalue asymptotics for \( d \to 0 \) and \( \alpha + \beta > 0. \) Note that for \( d = 0 \) equation (2.3) reads

\[ K_m(\kappa R)I_m(\kappa R) = \frac{1}{(\alpha + \beta)R}; \]  

the latter corresponds to the single ring Hamiltonian \( H_{\gamma,R} \) with coupling constant \( \gamma = \alpha + \beta, \) cf. (2.5). As follows from the previous discussion, see Remark 2.1, equation (3.1) has exactly one solution \( \kappa_m \) provided \( |m| \leq M_{\alpha+\beta,R}. \)

The following theorem provides the spectral asymptotics for approaching rings.

Theorem 3.1. Assume \( \gamma = \alpha + \beta > 0. \) Let \( E_m, \) where \( |m| \leq M_{\gamma,R}, \) stand for an eigenvalue of \( H_{\gamma,R}. \) Then the eigenvalues of \( H_{\alpha\beta,d} \) admit the following asymptotics for \( d \to 0: \)

\[ E_m(d) = E_m + t_m d + o(d), \]

where \( t_m \) is given by

\[ t_m := \frac{2\kappa_m I_m K_m (-\alpha\beta R I_m K_m + \alpha\kappa_m R (I_m K_m)' + \alpha I_m K_m)}{R(I_m K_m)'}; \]

moreover, functions \( K_m(\cdot) \) and \( I_m(\cdot) \) as well as their derivatives contributing to (3.3) are defined for the value \( R\kappa_m. \)
Proof. Suppose $|m| \leq M_{\gamma,R}$. Eigenvalues of $H_{\alpha,\beta,d}$ are determined by the solutions of (2.8). Note that
\[ \eta_m(\kappa,0) = 0. \]
Using the regularity of $K_m$ and $I_m$ we state for $d \in \mathbb{R}$ and $\kappa > 0$ the functions $\frac{\partial \eta_m}{\partial \kappa}$ and $\frac{\partial \eta_m}{\partial d}$ are $C^\infty$. Furthermore, using (3.1) we get
\[ \frac{\partial \eta_m}{\partial \kappa} = (\alpha + \beta)R^2(I_m K_m)' = R(I_m K_m)', \]
where the derivative at the left hand side is defined at $(\kappa_m,0)$. Moreover, $Z_m = Z_m(R\kappa_m)$, $Z_m = K_m, I_m$ and the analogous notation is applied for the derivatives contributing to right hand side of the above expression. Since the function $(I_m K_m)(\cdot)$ is monotonously decreasing we have $\frac{\partial \eta_m}{\partial \kappa} < 0$. Consequently, we can employ the implicit function theorem which states that there exists a neighbourhood $U \in \mathbb{R}$ of 0 and the unique function $U \ni d \mapsto \kappa_m(d) \in \mathbb{R}$ such that
\[ \eta_m(\kappa_m(d),d) = 0 \]
and
\[ \kappa_m(d) = \kappa_m - \left( \frac{\partial \eta_m}{\partial d} \right) \left( \frac{\partial \eta_m}{\partial \kappa} \right)^{-1} d + o(d) , \]
where all derivatives in the second component are determined for $d = 0$, $\kappa = \kappa_m$. Using (2.9) and the Wroński equation
\[ (I_m' K_m)(z) - (K_m' I_m)(z) = \frac{1}{z} \]
we get by a straightforward calculation
\[ \frac{\partial \eta_m}{\partial d} = -\alpha \beta R(K_m I_m) + \alpha \kappa_m R(I_m K_m)' + \alpha (I_m K_m). \]
Combining the above derivatives together with (3.5) we arrive at
\[ E_m(d) = -\kappa_m(d)^2 = -\kappa_m^2 + 2\kappa_m \frac{\partial \eta_m}{\partial d} \left( \frac{\partial \eta_m}{\partial \kappa} \right)^{-1} d + o(d) = E_m + t_m d + o(d), \]
with $t_m$ given by (3.3) \qed

3.1 Discussion on the first order correction

In this section we discuss some properties of the first order correction for converging rings.

The first order correction in the terms of unperturbed eigenfunctions. The following analysis will be conducted for $m = 0$. Let $f_0$ stand for the ground state of $H_{\alpha,\beta,R}$. Using (2.1) and (2.4) we conclude that $c_1 = \frac{1}{c_2}c_2$; recall $K_0 = K_0(R\kappa_0)$, $I_0 = I_0(R\kappa_0)$ and the analogous notation is employed for derivatives. Applying the relation
\[ \int x Z_0^2(x)dx = \frac{1}{2}(Z_0^2(x) - (Z_0'(x))^2), \quad Z_0 = I_0, K_0, \]
one can show that the norm of eigenfunction $f_0$ is given by
\[ \| f_0 \|_{L_2(R^2)}^2 = |c_2|^2 \frac{R^2}{2K_0^2}((IK')^2 - (K'I)^2). \]

Using again the Wronskian equation \((3.6)\) one gets
\[ \| f_0 \|_{L_2(R^2)}^2 = -|c_2|^2 \frac{R}{2K_0 K_0'} (I_0 K_0)'. \]

Applying the above formula together with boundary conditions \((2.1)\) and comparing this with \((3.3)\) we obtain
\[ t_0 = -\alpha (\int_{C(R)} \partial_r \| f_0 \|^2 ds + \alpha \int_{C(R)} |f_0|^2 ds) - \frac{\alpha}{R} \int_{C(R)} |f_0|^2 ds \]
where we abbreviate \( \partial_r f(r, \phi) = \lim_{r \to R} \partial_r f(r, \phi) \); recall that the equation \( s = R \phi \) states the relation the relation between \( \phi \in [0, 2\pi) \) and \( s \in [0, 2\pi R) \).

Let us mention that the above formula describes a very particular case of the class considered in the forthcoming paper \[13\]. In this paper the spectral asymptotics for approaching hypersurfaces in \( \mathbb{R}^d \) is analyzed. The method developed in \[13\] allows to reconstruct asymptotics of eigenvalues by means of the "unperturbed" eigenfunctions. The technics enables generalization for complex coupling constants.

The last component of \((3.9)\) reflects contribution of the curvature to the first correction term. More general situation shows a presence of the first mean curvature in eigenvalue asymptotics, cf \[13\]. Furthermore, let us note that a contribution of the first mean curvature in spectral asymptotics has been recently showed in related problems, see \[14\], \[15\] and \[16\].

The second component of \((3.9)\) is a consequence of singular character of delta potential. Suppose \( f_0 \) and \( f_d \) denote the normalized ground states of \( H_{\alpha^+ \beta, R} \) and \( H_{\alpha, \beta, d} \), respectively. The second component of \((3.9)\) comes directly from the fact that \( \partial_r (f_0 - f_d)(r, \phi) \) do not tend to 0 if \( d \to 0 \) and \( r \in (R, R_d) \).

The sign of \( t_m \). For the one dimensional system with two converging points of interaction the first order correction is always positive, cf \[2\]. This means that the splitting of the singular potential from one point to two points leads to pushing up the eigenvalue. The situation is slightly different in the case of converging circles. As formula \((3.3)\) shows the sign of \( t_0 \) depends on
\[ \varsigma := \alpha (1 - \beta R) I_0 K_0 + \alpha \kappa_0 R (I_0 K_0)', \]
i.e. \( \text{sign} \varsigma = -\text{sign} t_0 \).

First, let us consider the situation when \( R \to 0 \); then \( \kappa_0 \to 0 \) as well, cf. \[9\]. Employing asymptotics formulae for \( Z_0 \), where \( Z_0 = I_0, K_0 \), see \[15.3\] and \[15.4\] together with \[15.5\] and \[15.7\] one gets
\[ \varsigma \sim -\alpha (1 - \beta R) \ln(\kappa_0 R), \]
which implies \( \varsigma > 0 \) for \( R \) small enough and, consequently, leads to \( t_0 < 0 \).
• Now we assume that \( R \to \infty \). Then \( \kappa_0 \sim \alpha/2 \) and using again formulae (5.3) and (5.4) we obtain:

\[
\varsigma = -\alpha \beta R \left( \frac{1}{2\kappa_0 R} + O \left( (\kappa_0 R)^{-3} \right) \right).
\]

(3.10)

This shows that for \( R \) large enough we have \( \varsigma < 0 \) and \( t_0 > 0 \).

The above discussion establishes that the sign of the first order correction term is generally undefined.

4 Diverging rings

In this section we consider the asymptotics for circles separated by a large distance, i.e. for \( d \to \infty \).

Following the convention introduced in the previous discussion we denote by \( H_{\alpha,R_d} \) and \( H_{\beta,R} \) the corresponding single circle Hamiltonians. Operator \( H_{\alpha,R_d} \) has \( 2M_{\alpha,R_d} + 1 \) (counting multiplicities) eigenvalues \( \{ E_m,\alpha \}_{|m| \leq M_{\alpha,R_d}} \) and \( H_{\beta,R} \) has \( 2M_{\beta,R} + 1 \) eigenvalues \( \{ E_m,\beta \}_{|m| \leq M_{\beta,R}} \). Suppose \( \tau = \alpha, \beta \). In fact, \( E_m,\tau \) can be recovered from the spectral equations, i.e. \( E_m,\tau = -\kappa_m^{2,\tau} \) where \( \kappa_m,\tau \) stand for the solutions of

\[
\xi_m,\tau(\kappa) = 0, \quad \text{for} \quad \kappa > 0.
\]

Recall that \( \xi_m,\tau \) are defined by (2.10). Moreover, using the statement of Remark 2.1 we conclude that

\[
E_m,\alpha = -\frac{\alpha^2}{4} + \frac{m^2 - \frac{1}{4}}{R^2} + O(d^{-4}).
\]

(4.1)

**Theorem 4.1.** Assume that \( \alpha \) and \( \beta \) are positive and \( E_m,\beta \neq -\frac{\alpha^2}{4} \) for all \( |m| \leq M_{\beta,R} \). Then the eigenvalues of \( H_{\alpha,\beta,d} \) admit the following asymptotics for \( d \to \infty \):

\[
\epsilon_d = \begin{cases} 
-\frac{\alpha^2}{4} + \frac{m^2 - \frac{1}{4}}{R^2} + o(d^{-2}), & |m| \leq M_{\alpha,R_d}, \\
E_m,\beta + w_m \epsilon + o(\epsilon), & |m| \leq M_{\beta,R},
\end{cases}
\]

(4.2)

where

\[
\epsilon := \exp(-2d\kappa_{m,\beta}), \quad w_m := \frac{\pi \alpha \beta R e^{-2\kappa_{m,\beta} R} I_m(R\kappa_{m,\beta})^2}{(1 - \alpha/(2\kappa_{m,\beta})) I_m'(\kappa_{m,\beta})}. \quad (4.3)
\]

**Remark 4.2.** In fact, \( \epsilon_d \) reflects the asymptotics of \( 2(M_{\alpha} + M_{\beta}) + 2 \) eigenvalues of \( H_d \). However, since \( \epsilon_d \) converge to \( E_m,\alpha \) and \( E_m,\beta \) we leave the labelling inherited from the discrete eigenvalues of the single circle Hamiltonians.

**Proof.** The analysis is based on investigating spectral equation (2.8) which reads

\[
\eta_m(\kappa, d) = \nu_m(\kappa, d) - \xi_m(\kappa, d) = 0,
\]

(4.4)

where

\[
\xi_m(\kappa, d) := \xi_m,\alpha,d(\kappa) \xi_m,\beta(\kappa).
\]
First, assume that $|m| \leq M_{\beta,R}$. Then for $d$ large enough we have $|m| \leq M_{\alpha,R}$. Combining equations (2.10) and (2.11) with the formulae (5.1) and (5.2) we get the following asymptotics for $\kappa \to \infty$ and any $m \in \mathbb{Z}$:

\[
\begin{cases}
\xi_m(\kappa, d) = 1 - \frac{\alpha + \beta}{2m} + O(\kappa^{-2}), \\
\nu_m(\kappa, d) = \frac{\alpha \beta}{4\kappa} e^{-2d\kappa} (1 + o_{\kappa}(1)),
\end{cases}
\]  

(4.5)

the error terms in the above expressions are uniform with respect to $d > C$ where $C$ is a positive number\(^2\). The symbol $o_{\kappa}(1)$ denotes that the asymptotics understood with respect to $\kappa$. On the other hand, for $\kappa \to 0$ we have

\[
\xi_m(\kappa, d) = \left\{ \begin{array}{ll}
\frac{\alpha R^d}{2m} - 1 \left( \frac{\beta R^d}{2m} - 1 \right) (1 + o_{\kappa}(1)), & m \neq 0 \\
\alpha \beta \log(\kappa R) \log(\kappa R_d) R R_d (1 + o_{\kappa}(1)), & m = 0
\end{array} \right.
\]

(4.6)

and

\[
\nu_m(\kappa, d) = \left\{ \begin{array}{ll}
\frac{\alpha \beta R^{2m+1}}{4m^2 R_d^{m+1}} (1 + o_{\kappa}(1)), & m \neq 0 \\
\alpha \beta \log^2 (\kappa R_d) R R_d (1 + o_{\kappa}(1)), & m = 0
\end{array} \right.
\]

(4.7)

where all error terms are uniform with respect to $d > C$. We have $\nu_m(\kappa, d) > 0$.

Moreover, $\nu_m(\kappa, d) \to 0$ as $d \to \infty$ and the limit is uniform with respect to $\kappa > C$. It follows from (4.6) and (4.7) that if $m \neq 0$ then $\xi_m(0, d) > \nu_m(0, d)$ for $d$ large enough. If $m = 0$ then the corresponding limits for $\kappa \to 0$ do not exist, however, $\xi_m(\kappa, d) > \nu_m(\kappa, d)$ holds for $\kappa$ from a neighbourhood of $0$ and $d$ large enough. The function $\xi_m(\kappa, d)$ has two roots: $\kappa_{m,\alpha}$ and $\kappa_{m,\beta}$. Moreover $\xi_m(\kappa, d) \to 1$ for $\kappa \to \infty$ and the limit is uniform with respect to $d > C$. It follows from the discussed properties of $\xi_m$ and $\nu_m$ and their continuity that equation (4.4) has at least two solutions. Furthermore, for $d \to \infty$ we have $\xi_m(\kappa, d) = (\alpha/(2\kappa) + 1 + o_d(1)) \xi_m(\kappa, d)$ and the error is uniform with respect to $\kappa > C$. This implies that (4.4) has exactly two solutions for $\kappa > C$ which we denote as $\kappa_{m,\alpha}(d)$ and $\kappa_{m,\beta}(d)$ and they approach to $\kappa_{m,\alpha}$ and $\kappa_{m,\beta}$ as $d \to \infty$.

Note that, since $C$ can be chosen arbitrary small we can conclude, in view of the behaviour of $\nu_m$ and $\xi_m$ in a neighbourhood of $0$, that (4.4) has exactly two solutions for $\kappa \in (0, \infty)$. Let us consider $\kappa_{m,\beta}(d)$. We have

\[
\kappa_{m,\beta}(d) = \kappa_{m,\beta} + \delta_{m,\beta}(d),
\]

where $\delta_{m,\beta}(d)$ converges to $0$ as $d \to \infty$. Inserting $\kappa_{m,\beta}(d)$ to $\eta(\kappa, d)$ one gets

\[
\eta(\kappa_{m,\beta}(d), d) = \nu_m(\kappa_{m,\beta}(d), d) - \xi_m(\kappa_{m,\beta}) \xi_m(\kappa_{m,\beta}) \delta_{m,\beta} + o(\delta_{m,\beta}).
\]

(4.8)

The equation

\[
\eta(\kappa_{m,\beta}(d), d) = 0
\]

(4.9)

and the behaviour of $\nu_m(\kappa_{m,\beta}(d), d)$ imposes $d \delta_{m,\beta}(d) \to 0$ as $d \to \infty$. Therefore, we get

\[
\nu_m(\kappa_{m,\beta}(d), d) = \frac{\pi \alpha \beta R}{2 \kappa_{m,\beta}} e^{-2 \kappa_{m,\beta} R I_m(R \kappa_{m,\beta})^2 \varepsilon + o(\varepsilon)}.
\]

Implementing the above expression and (4.5) to (4.9) and comparing appropriated terms leads to (4.11). To derive the second eigenvalue we employ (4.11)

---

\(2\)Note that in this proof $C$ denotes a positive constant which can change from line to line.
together with the asymptotics of $\xi_{m,\alpha}$ which depends on $d$ as well. The analogous analysis as above establishes the asymptotics of eigenvalues localized near $-\alpha^2/4$, see the first line of (4.2).

If $|m| > M_{\beta,R}$ and $|m| \leq M_{\alpha,R}$ then $\xi_m(\kappa, d) = 0$ has only one solution $\kappa_{m,\alpha}$. Then repeating the above steps one shows the existence of one solution of the spectral equation; this solution admits the asymptotics specified in the first line of (4.2).

The result of the above theorem corresponds to the phenomena known for regular potentials. It was shown in [10] that the introducing a second well to the single well system leads to the splitting of original eigenvalues and the corresponding spectral gaps can be expressed by the current. Consequently, the asymptotics of the gaps, if the wells are separated by a large distance, is determined by the exponential decay of eigenvectors. Theorem 4.1 shows that the introducing interaction supported by circle $C_R$ to the system governed by $H_{\beta,R}$ leads to the shifting of original energies $E_{m,\beta}$ and this spectral shifting is determined by exponential decay of corresponding eigenfunctions.

One the other hand, the system governed by $H_{\alpha,R}$ admits eigenvalues $E_{m,\alpha}$ which depend on $d$, see (4.1). Formula (4.2) shows that the leading terms of this eigenvalues asymptotics are preserved if we introduce also interaction supported by $C_R$.

5 Appendix

We complete here the asymptotics of functions contributing to the spectral equations, see [1]. Namely, for $z \to \infty$ and $m \in \mathbb{Z}$ we have

$$I_m(z) = \frac{e^z}{\sqrt{2\pi}} \left( 1 - \frac{4m^2 - 1}{8z} + O(z^{-2}) \right), \quad (5.1)$$

and

$$K_m(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \left( 1 - \frac{4m^2 - 1}{8z} + O(z^{-2}) \right). \quad (5.2)$$

Furthermore, for $z \to 0$ we have

$$I_m(z) \sim \frac{1}{\Gamma(m+1)} \left( \frac{z}{2} \right)^m, \quad m \in \mathbb{Z} \quad (5.3)$$

and

$$\begin{cases} K_m(z) \sim \frac{\Gamma(m)}{2} \left( \frac{z}{2} \right)^m, & m \neq 0 \\ K_0(z) \sim -\ln(z/2). \end{cases} \quad (5.4)$$

Recall $\Gamma(m) = (m - 1)!$.

For the asymptotics of derivatives the following formulae will be useful

$$I_m'(z) = \frac{I_{m-1}(z) + I_{m+1}(z)}{2} \quad (5.5)$$

and

$$K_m'(z) = -\frac{K_{m-1}(z) + K_{m+1}(z)}{2}. \quad (5.6)$$

Furthermore, since $Z_m = Z_{-m}$, where $Z_m = I_m, K_m$ we, for example, have

$$I_0'(z) = I_1(z), \quad K_0'(z) = -K_1(z). \quad (5.7)$$
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