AN INTEGER-VALUED VERSION OF THE BIRMAN-KREIN FORMULA

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To the memory of M. Sh. Birman

Abstract. We discuss an identity in abstract scattering theory which can be interpreted as an integer-valued version of the Birman-Krein formula.

1. Introduction

In the classic paper [5] by M. Sh. Birman and M. G. Krein, the following identity (known today as the Birman-Krein formula) was proven:

\[ e^{-2\pi i \xi(\lambda)} = \det S(\lambda). \]  

Here \( S(\lambda) \) and \( \xi(\lambda) \) are the scattering matrix and the spectral shift function of a pair of self-adjoint operators in a Hilbert space and \( \lambda \in \mathbb{R} \) is a spectral parameter; precise definitions will be given in Sections 2 and 3. The identity (1.1) first appeared in [22, 9, 10] in some important particular cases; in [5] it was recognised as an abstract theorem of mathematical scattering theory.

The Birman-Krein formula is invariant with respect to adding an integer to \( \xi(\lambda) \). Thus, one can say that (1.1) involves only the fractional part of \( \xi(\lambda) \); the information about the integer part of \( \xi(\lambda) \) is “lost”. The purpose of this note is to state and discuss an identity (see (4.2) below) which can be interpreted as an integer-valued version of the Birman-Krein formula.

In Sections 2 and 3 we give the necessary definitions. The main result is stated in Section 4. Proofs will appear in [27].

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2. \( \xi(\lambda) \) and \( \Xi(\lambda) \)

2.1. Naive definition. Throughout the paper, \( A \) and \( B \) are self-adjoint operators in a Hilbert space \( \mathcal{H} \) such that the difference

\[ V = B - A \]

is a compact operator. For simplicity of exposition we will also assume that \( A \) and \( B \) are bounded. The assumption of compactness of \( V \) implies, in particular, that

\[ Mathemati...
the essential spectra of $A$ and $B$ coincide: $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(B)$. We denote by $E_A(\lambda)$ the spectral projection of $A$ corresponding to the interval $(-\infty, \lambda)$ and let

$$N_A(\lambda) := \text{rank } E_A(\lambda) = \text{Tr } E_A(\lambda) \leq \infty$$

be the eigenvalue counting function of $A$. For $\lambda < \text{inf } \sigma_{\text{ess}}(A)$ the difference

$$(2.1)\quad N_A(\lambda) - N_B(\lambda)$$

is well defined, since both $N_A(\lambda)$ and $N_B(\lambda)$ are finite. The difference $(2.1)$ measures the shifts of the eigenvalues of $B$ relatively to the eigenvalues of $A$. For $\lambda > \text{inf } \sigma_{\text{ess}}(A)$ the difference $(2.1)$ formally gives $\infty - \infty$. Below we discuss two regularisations of $(2.1)$: the spectral shift function $\xi(\lambda)$ and the index function $\Xi(\lambda)$.

2.2. The spectral shift function $\xi(\lambda)$. Assume that the difference $V = B - A$ is a trace class operator. Then [23, 21] the following Lifshits-Krein trace formula holds:

$$(2.2)\quad \text{Tr}(\varphi(B) - \varphi(A)) = \int_{-\infty}^{\infty} \varphi'(t)\xi(t)dt, \quad \forall \varphi \in C^\infty_0(\mathbb{R}).$$

Here $\xi(\cdot) = \xi(\cdot; B, A)$ is a uniquely defined function in $L^1(\mathbb{R})$ which is called the spectral shift function (SSF). See [6] or [30, Section 8] for a detailed exposition of the SSF theory.

Formally taking $\varphi = \chi_{(-\infty, \lambda)}$ (= the characteristic function of $(-\infty, \lambda)$), we obtain

$$(2.3)\quad \xi(\lambda; B, A) = \text{Tr}(E_A(\lambda) - E_B(\lambda)) = \text{Tr } E_A(\lambda) - \text{Tr } E_B(\lambda) = N_A(\lambda) - N_B(\lambda),$$

whenever the r.h.s. makes sense. In particular, this calculation is not difficult to justify for $\lambda < \text{inf } \sigma_{\text{ess}}(A)$. It shows that $\xi(\lambda)$ is the natural regularisation of the difference $N_A(\lambda) - N_B(\lambda)$.

2.3. The index of a pair of projections. Let $P, Q$ be orthogonal projections in a Hilbert space; consider the spectrum of the difference $P - Q$. It is obvious that $\sigma(P - Q) \subset [-1, 1]$. By using the commutation relation

$$W(P - Q) = -(P - Q)W, \quad W = I - P - Q,$$

it is not difficult to see [3, Theorem 4.2] that

$$(2.4)\quad \dim \text{Ker}(P - Q - \lambda I) = \dim \text{Ker}(P - Q + \lambda I), \quad \lambda \neq \pm 1.$$ 

A pair $P, Q$ is called Fredholm, if

$$(2.5)\quad \{1, -1\} \cap \sigma_{\text{ess}}(P - Q) = \emptyset.$$ 

In particular, if $P - Q$ is compact, then the pair $P, Q$ is Fredholm. The index of a Fredholm pair is defined by the formula

$$(2.6)\quad \text{index}(P, Q) = \dim \text{Ker}(P - Q - I) - \dim \text{Ker}(P - Q + I).$$ 

We note that $\text{index}(P, Q)$ coincides with the Fredholm index of the operator $QP$ viewed as a map from $\text{Ran } P$ to $\text{Ran } Q$, see [3, Proposition 3.1].

If $P - Q$ is a trace class operator, then

$$(2.7)\quad \text{index}(P, Q) = \text{Tr}(P - Q),$$

where $\text{Tr}$ denotes the trace.
since all the eigenvalues of $P - Q$ apart from $1$ and $-1$ in the series $\text{Tr}(P - Q) = \sum_k \lambda_k(P - Q)$ cancel out by (2.4). It follows that in the simplest case of finite rank projections $P, Q$ we have

$$\text{(2.8)} \quad \text{index}(P, Q) = \text{rank } P - \text{rank } Q.$$ 

2.4. The index function $\Xi(\lambda)$. Suppose that for some $\lambda \in \mathbb{R}$, the pair $E_A(\lambda), E_B(\lambda)$ is Fredholm. Then we will say that the index $\Xi(\lambda) = \Xi(\lambda; B, A)$ exists and define it by

$$\text{(2.9)} \quad \Xi(\lambda; B, A) = \text{index}(E_A(\lambda), E_B(\lambda)).$$

The function $\Xi(\lambda; B, A)$ has already appeared in the literature in various guises (see e.g. [2, 11, 15, 29, 28, 14, 13, 8, 4, 19, 16, 17]); its properties were reviewed and proven in a systematic fashion in [25].

For $\lambda < \inf \sigma_{\text{ess}}(A)$, both projections $E_A(\lambda), E_B(\lambda)$ have finite rank and so by (2.8) we have

$$\Xi(\lambda; B, A) = N_A(\lambda) - N_B(\lambda), \quad \lambda < \inf \sigma_{\text{ess}}(A).$$

Thus, $\Xi(\lambda)$, along with $\xi(\lambda)$, is a natural regularisation of the difference $N_A(\lambda) - N_B(\lambda)$. Using the Riesz integral representation for the spectral projections and the resolvent identity, it is not difficult to prove that for all $\lambda \in \mathbb{R} \setminus \sigma_{\text{ess}}(A)$, the difference $E_A(\lambda) - E_B(\lambda)$ is compact and therefore $\Xi(\lambda)$ exists. Below we will also give a criterion for the existence of $\Xi(\lambda)$ on the essential spectrum of $A$ and $B$, see Theorem 4.1.

2.5. Comparison of $\xi$ and $\Xi$.

1. If $V$ is a trace class operator, we have

$$\Xi(\lambda) = \xi(\lambda), \quad \forall \lambda \in \mathbb{R} \setminus \sigma_{\text{ess}}(A).$$

In particular, this holds true for all $\lambda \in \mathbb{R}$ if $\dim \mathcal{H} < \infty$.

2. $\Xi(\lambda)$ is always integer-valued, whereas $\xi(\lambda)$ is, in general, real-valued when $\lambda$ belongs to the essential spectrum of $A, B$.

3. If $V$ is trace class then the SSF $\xi(\lambda)$ is automatically well defined for a.e. $\lambda \in \mathbb{R}$.

The function $\Xi(\lambda)$ may not exist on a set of positive measure even if $V$ is a rank one operator; see [20]. On the other hand, the existence of $\Xi(\lambda)$ does not require trace class assumptions.

3. The scattering matrix

3.1. Strong smoothness assumptions. Below we make assumptions typical for smooth scattering theory, which goes back to [12] and [18]. We fix a compact interval $\Delta = [a, b] \subset \mathbb{R}$ and assume that the spectrum of $A$ in $\Delta$ is purely absolutely continuous with constant multiplicity $N \leq \infty$. In the terminology of [30], we assume that the operator $G = |V|^{1/2}$ is strongly $A$-smooth on $\Delta$ with some exponent $\gamma \in (0, 1]$. This means the following. Let $\mathcal{F}$ be a unitary operator from $\text{Ran } E_A(\Delta)$ to $L^2(\Delta, \mathcal{N})$, $\dim \mathcal{N} = N$, such that $\mathcal{F}$ diagonalizes $A$: if $f \in \text{Ran } E_A(\Delta)$ then

$$\text{(3.1)} \quad (\mathcal{F}Af)(\lambda) = \lambda(\mathcal{F}f)(\lambda), \quad \lambda \in \Delta.$$
The strong $A$-smoothness of $G$ on the interval $\Delta$ means that the operator

$$G_\Delta \overset{\text{def}}{=} GE_A(\Delta) : \text{Ran } E_0(\Delta) \rightarrow \mathcal{H}$$

satisfies the condition

$$(3.2) \quad (FG^*_\Delta \psi)(\lambda) = Z(\lambda) \psi, \quad \forall \psi \in \mathcal{H}, \quad \lambda \in \Delta,$$

where $Z = Z(\lambda) : \mathcal{H} \rightarrow \mathcal{N}$ is a family of compact operators obeying

$$(3.3) \quad \|Z(\lambda)\| \leq C, \quad \|Z(\lambda) - Z(\lambda')\| \leq C|\lambda - \lambda'|^{\gamma}, \quad \lambda, \lambda' \in \Delta.$$  

3.2. The scattering matrix. Under the above strong smoothness assumption, the local wave operators

$$W_\pm = W_\pm(A, B; \Delta) = \lim_{t \to \pm \infty} e^{itB} e^{-itA} E_A(\Delta)$$

exist and are complete, i.e. $\text{Ran } W_+ = \text{Ran } W_- = \text{Ran } E_B^{(\text{ac})}(\Delta)$; here $E_B^{(\text{ac})}(\cdot)$ is the absolutely continuous part of the spectral measure of the operator $B$. The local scattering operator $S = W_+ W_-$ is unitary in $\text{Ran } E_A(\Delta)$ and commutes with $A$. Thus, we have a representation

$$(FSF^* f)(\lambda) = S(\lambda) f(\lambda), \quad \text{a.e. } \lambda \in \Delta,$$

where the operator $S(\lambda) : \mathcal{N} \rightarrow \mathcal{N}$ is called the scattering matrix for the pair of operators $A, B$. The scattering matrix is a unitary operator in $\mathcal{N}$. Under the above assumptions, one can prove that $S(\lambda) - I$ is a compact operator for all $\lambda \in \Delta$. Thus, the spectrum of $S(\lambda)$ consists of eigenvalues on the unit circle $e^{i\theta}$ with the only possible point of accumulation being 1. Further, one can prove that $S(\lambda)$ depends continuously on $\lambda$ in the operator norm. For the details, see the original papers [12, 18] or the survey [7] or the book [30].

3.3. The spectral flow of $S(\lambda)$. Let us recall the definition of the spectral flow of the family $\{S(\lambda)\}_{\lambda \in [a, b]}$. The spectral flow is an integer-valued function $\mu$ on $\mathbb{T} \setminus \{1\}$. The naive definition of the spectral flow is

$$(3.4) \quad \text{sp. flow}(e^{i\theta}; \{S(\lambda)\}_{\lambda \in [a, b]}) =$$

$$(\text{the number of eigenvalues of } S(\lambda) \text{ which cross } e^{i\theta} \text{ in the anti-clockwise direction})$$

$$(\text{the number of eigenvalues of } S(\lambda) \text{ which cross } e^{i\theta} \text{ in the clockwise direction}),$$

as $\lambda$ increases monotonically from $a$ to $b$. Here $\theta \in (0, 2\pi)$ and the eigenvalues are counted with multiplicities taken into account. The eigenvalues of $S(\lambda)$ may cross $e^{i\theta}$ infinitely many times, and thus the above naive definition needs to be replaced by a more rigorous one. Below we describe one such possible definition; there are other approaches to this definition in the literature, see e.g. [2, 28].

Let us introduce some notation for the eigenvalue counting function of $S(\lambda)$. For $\theta_1, \theta_2 \in (0, 2\pi)$ we denote

$$N(e^{i\theta_1}, e^{i\theta_2}; S(\lambda)) = \sum_{\theta \in [\theta_1, \theta_2]} \dim \text{Ker}(S(\lambda) - e^{i\theta} I), \quad \text{if } \theta_1 < \theta_2,$$
and
\[ N(e^{i\theta_1}, e^{i\theta_2}; S(\lambda)) = -N(e^{i\theta_2}, e^{i\theta_1}; S(\lambda)) \quad \text{if} \quad \theta_1 > \theta_2. \]

Assume first that there exists \( \theta_0 \in (0, 2\pi) \) such that \( e^{i\theta_0} \notin \sigma(S(\lambda)) \) for all \( \lambda \in [a, b] \). Then one can define the spectral flow of the family \( \{S(\lambda)\}_{\lambda \in [a, b]} \) by
\[ \text{sp. flow}(e^{i\theta}; \{S(\lambda)\}_{\lambda \in [a, b]}) = N(e^{i\theta}, e^{i\theta_0}; S(b)) - N(e^{i\theta}, e^{i\theta_0}; S(a)). \]

It is evident that this definition is independent of the choice of \( \theta_0 \) and agrees with the naive definition (3.4) whenever the latter makes sense.

In general, \( \theta_0 \) as above may not exist. However, by a standard argument based on the compactness of \([a, b]\) one can always find the values \( a = \lambda_0 < \lambda_1 < \cdots < \lambda_n = b \) such that for each of the subintervals \( \Delta_i = [\lambda_{i-1}, \lambda_i] \), a point \( \theta_0 \) with the required properties can be found. Thus, the spectral flow of each of the corresponding families \( \{S(\lambda)\}_{\lambda \in \Delta_i} \) is well defined. Now one can set
\[ \text{sp. flow}(e^{i\theta}; \{S(\lambda)\}_{\lambda \in [a, b]}) = \sum_{i=1}^n \text{sp. flow}(e^{i\theta}; \{S(\lambda)\}_{\lambda \in \Delta_i}). \]

It is not difficult to see that the above definition is independent of the choice of the subintervals \( \Delta_i \) and agrees with the naive definition (3.4).

4. Main results

4.1. Statement of the results. The first preliminary result concerns the existence of \( \Xi(\lambda) \) on the continuous spectrum.

**Theorem 4.1.** [24] Assume that for some interval \( \Delta = [a, b] \) the spectrum of \( A \) in \( \Delta \) is purely absolutely continuous and let the strong smoothness assumption (3.2), (3.3) hold true. Then for all \( \lambda \in (a, b) \) one has
\[ \sigma_{\text{ess}}(E_B(\lambda) - E_A(\lambda)) = [-\alpha(\lambda), \alpha(\lambda)], \quad \alpha(\lambda) = \frac{1}{2}\|S(\lambda) - I\|. \]

In particular, \( \Xi(\lambda; B, A) \) exists if and only if \(-1 \notin \sigma(S(\lambda))\).

A description of the absolutely continuous spectrum of the difference \( E_B(\lambda) - E_A(\lambda) \) is also available in terms of the spectrum of \( S(\lambda) \), see [26].

As \( \lambda \) increases monotonically in the interval \( \Delta \), the eigenvalues of \( S(\lambda) \) rotate on the unit circle and the quantity \( \alpha(\lambda) \) changes continuously in \( \lambda \). According to Theorem 4.1, the index \( \Xi(\lambda) \) exists if and only if \( \alpha(\lambda) < 1 \), i.e. if and only if the spectrum of \( S(\lambda) \) does not contain the point \(-1\).

Further simple analysis based on the stability of Fredholm index shows that the function \( \Xi(\lambda) \) is constant on the intervals where \(-1 \notin \sigma(S(\lambda))\). Thus, the integer-valued function \( \Xi(\lambda) \) can jump only at the points \( \lambda \) where \(-1 \in \sigma(S(\lambda))\). This leads to the natural question: what is the size of the jump of \( \Xi(\lambda) \) when an eigenvalue of \( S(\lambda) \) crosses \(-1\)? Our main result below answers this question.

**Theorem 4.2.** Assume that for some interval \( \Delta = [a, b] \) the spectrum of \( A \) in \( \Delta \) is purely absolutely continuous and let the strong smoothness assumption (3.2),
hold true. Fix \( \lambda_1, \lambda_2 \in (a, b) \), \( \lambda_1 < \lambda_2 \) and assume that \(-1 \notin \sigma(S(\lambda_1)) \) and \(-1 \notin \sigma(S(\lambda_2)) \). Then

\begin{equation}
(4.1) \quad \Xi(\lambda_2; B, A) - \Xi(\lambda_1; B, A) = -\text{sp. flow}(\lambda; \{S(\lambda)\}_{\lambda \in [\lambda_1, \lambda_2]}).
\end{equation}

The proof will appear in [27]. The proof is based, roughly speaking, on a continuous deformation of the pair of operators \( A, B \), into a pair of operators which has an “infinitesimal spectral gap” at a point \( \lambda \in (a, b) \). This deformation, together with the Birman-Schwinger principle, enables us to calculate \( \Xi(\lambda; A, B) \) in terms of some auxiliary operators. This calculation makes it possible to relate \( \Xi(\lambda; A, B) \) to the eigenvalue counting function \( N(-1, e^{i\theta}; S(\lambda)) \). From this relation it is not difficult to derive (4.1).

Theorem 4.2 can be extended to a certain class of unbounded operators \( A, B \). In many concrete examples of interest, the absolutely continuous spectra of \( A \) and \( B \) contain the semi-axis \([0, \infty)\), and \( \|S(\lambda) - I\| \to 0 \) as \( \lambda \to \infty \). In this case one can take \( \lambda_2 \to \infty \) in (4.1), which yields

\begin{equation}
(4.2) \quad \Xi(\lambda; B, A) = \text{sp. flow}(\lambda; \{S(\lambda')\}_{\lambda ' \in [\lambda, \infty]}).
\end{equation}

This is discussed in [27] for the case \( A = -\Delta, B = -\Delta + V \) in \( H = L^2(\mathbb{R}^d) \), with a short range potential \( V \).

4.2. Comparison of (4.2) with the Birman-Krein formula. Let \( \{e^{i\theta_n(\lambda)}\} \) be the eigenvalues of \( S(\lambda) \) distinct from 1. To compare (4.2) with the Birman-Krein formula (1.1), let us rewrite the latter as

\begin{equation}
(4.3) \quad \xi(\lambda) = -\frac{1}{2\pi} \sum_n \theta_n(\lambda) \pmod{1}.
\end{equation}

As discussed in Section 2, the left hand sides of (4.2) and (4.3) are two different regularisations of \( N_A(\lambda) - N_B(\lambda) \). The right hand sides of (4.2) and (4.3) are two different quantities related to the spectrum of the scattering matrix.

The identity (4.2) relates two integers. The identity (4.3) relates two real numbers modulo 1. Thus, in some sense (perhaps yet to be understood) they present complementary pieces of information.

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