EIGENVALUES OF THE LAPLACIAN ON RIEMANNIAN MANIFOLDS*

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Abstract. For a bounded domain \( \Omega \) with a piecewise smooth boundary in a complete Riemannian manifold \( M \), we study eigenvalues of the Dirichlet eigenvalue problem of the Laplacian. By making use of a fact that eigenfunctions form an orthonormal basis of \( L^2(\Omega) \) in place of the Rayleigh-Ritz formula, we obtain inequalities for eigenvalues of the Laplacian. In particular, for lower order eigenvalues, our results extend the results of Chen and Cheng [7].

1. Introduction

Let \( \Omega \subset M \) be a bounded domain with a piecewise smooth boundary \( \partial \Omega \) in an \( n \)-dimensional complete Riemannian manifold \( M \). We consider the following Dirichlet eigenvalue problem of the Laplacian:

\[
\begin{aligned}
\Delta u &= -\lambda u \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

It is well known that the spectrum of this problem is real and discrete:

\[
0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \nearrow \infty,
\]

where each \( \lambda_i \) has finite multiplicity which is repeated according to its multiplicity.

When \( M \) is an \( n \)-dimensional Euclidean space \( \mathbb{R}^n \), Payne, Pólya and Weinberger [19] proved

\[
\lambda_{k+1} - \lambda_k \leq \frac{4}{kn} \sum_{i=1}^{k} \lambda_i.
\]

Hile and Protter [17] generalized the above result to

\[
\sum_{i=1}^{k} \frac{\lambda_i}{\lambda_{k+1} - \lambda_i} \geq \frac{kn}{4}.
\]

In 1991, a much sharper inequality was obtained by Yang [21] (cf. [12]):

\[
\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \lambda_i.
\]

When \( M \) is an \( n \)-dimensional unit sphere \( S^n(1) \), Cheng and Yang [10] have proved an optimal universal inequality:

\[
\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) (\lambda_i + \frac{n^2}{4}).
\]
For the Dirichlet eigenvalue problem of the Laplacian on a bounded domain in an $n$-dimensional complete Riemannian manifold $M$, Chen and Cheng [7] and El Soufi, Harrell and Ilias [16] have proved, independently,

\begin{equation}
\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)(\lambda_i + \frac{n^2}{4} H_0^2),
\end{equation}

where $H_0^2$ is a nonnegative constant which only depends on $M$ and $\Omega$. When $M$ is the unit sphere, $H_0^2 = 1$, the above inequality is best possible, which becomes the result of Cheng and Yang [10]. For the Dirichlet eigenvalue problem of the Laplacian on a bounded domain in a hyperbolic space, universal inequalities for eigenvalues have been obtained by Cheng and Yang [13]. For complex projective spaces and so on, see [6], [10] and [11].

For lower order eigenvalues of the eigenvalue problem (1.1), when $M$ is the Euclidean space $\mathbb{R}^n$, the following conjecture of Payne, Pólya and Weinberger is well known:

**Conjecture of PPW.** For a bounded domain $\Omega$ in $\mathbb{R}^n$, eigenvalues of the eigenvalue problem (1.1) satisfy

\begin{align*}
(1) & \quad \frac{\lambda_2}{\lambda_1} \leq \frac{\lambda_2}{\lambda_1} \bigg|_{B^n} = \frac{j_{2,1}^2}{j_{n-1,1}^2}, \\
(2) & \quad \frac{\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}}{\lambda_1} \leq \frac{n}{j_{n-1,1}^2},
\end{align*}

where $B^n$ is the $n$-dimensional unit ball in $\mathbb{R}^n$, $j_{p,k}$ denotes the $k$-th positive zero of the standard Bessel function $J_p(x)$ of the first kind of order $p$.

For the conjecture (1) of Payne, Pólya and Weinberger, many mathematicians studied it. For examples, Payne, Pólya and Weinberger [19], Brands [5], de Vries [15], Chiti [14], Hile and Protter [17], Marcellini [18] and so on. Finally, Ashbaugh and Benguria [2] (cf. [1] and [3]) solved this conjecture.

For the conjecture (2) of Payne, Pólya and Weinberger, when $n = 2$, Brands [5] improved the bound $\frac{\lambda_2 + \lambda_3}{\lambda_1} \leq 6$ of Payne, Pólya and Weinberger [19], he proved $\frac{\lambda_2 + \lambda_3}{\lambda_1} \leq 3 + \sqrt{7}$. Furthermore, Hile and Protter [17] obtained $\frac{\lambda_2 + \lambda_3}{\lambda_1} \leq 5.622$. In [18], Marcellini proved $\frac{\lambda_2 + \lambda_3}{\lambda_1} \leq (15 + \sqrt{345})/6$. Recently, Chen and Zheng [8] have proved $\frac{\lambda_2 + \lambda_3}{\lambda_1} \leq 5.3507$. For a general dimension $n \geq 2$, Ashbaugh and Benguria [4] proved

\begin{equation}
\frac{\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}}{\lambda_1} \leq n + 4.
\end{equation}

Furthermore, Ashbaugh and Benguria [4] (cf. Hile and Protter [17] ) improved the above result to

\begin{equation}
\frac{\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}}{\lambda_1} \leq n + 3 + \frac{\lambda_1}{\lambda_2}.
\end{equation}

Very recently, Cheng and Qi [9] have proved that, for any $1 \leq j \leq n + 2$, eigenvalues satisfy at least one of the following:

\[ \frac{\lambda_2}{\lambda_1} < 2 - \frac{\lambda_1}{\lambda_j}, \]
\[
\frac{\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}}{\lambda_1} \leq n + 3 + \frac{\lambda_1}{\lambda_j}.
\]

When \( M \) is the \( n \)-dimensional unit sphere \( S^n(1) \), that is, for a bounded domain \( \Omega \) in \( S^n(1) \), Cheng, Sun and Yang \[20\] have proved

\[(1.9) \quad \frac{\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}}{\lambda_1} \leq n + 4 + \frac{n^2}{\lambda_1}.\]

For a general complete Riemannian manifold \( M \), Chen and Cheng \[7\] have proved that there exists a non-negative constant \( H_0 \) such that

\[(1.10) \quad \frac{\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}}{\lambda_1} \leq n + 4 + \frac{n^2 H_0^2}{\lambda_1}.
\]

In this paper, by making use of the fact that eigenfunctions form an orthonormal basis of \( L^2(\Omega) \) in place of the Rayleigh-Ritz formula, we obtain inequalities for eigenvalues of the Laplacian. In particular, we improve the above result.

2. Estimates for lower order eigenvalues

In this section, first of all, we will mainly focus our mind on the investigation for lower order eigenvalues of the Dirichlet eigenvalue problem of the Laplacian by making use of the fact that eigenfunctions form an orthonormal basis of \( L^2(\Omega) \) in place of the Rayleigh-Ritz formula. We prove the following:

**Theorem 2.1.** Let \( M \) be an \( n \)-dimensional complete Riemannian manifold, \( \Omega \subset M \) a bounded domain with a piecewise smooth boundary \( \partial \Omega \). Then, the lower order eigenvalues of the Dirichlet eigenvalue problem of the Laplacian satisfy

\[
\frac{\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}}{\lambda_1} \leq n + \sqrt{\left(\frac{n^2 H_0^2}{\lambda_1} + 4\right) (2 - \frac{\lambda_1}{\lambda_2}) \frac{n^2 H_0^2}{\lambda_1} + 3 + \frac{\lambda_1}{\lambda_2} + \sqrt{\left(3 + \frac{\lambda_1}{\lambda_2} + \frac{n^2 H_0^2}{\lambda_2}\right)^2 + 4(1 - \lambda_1) \frac{n^2 H_0^2}{\lambda_2}}},
\]

where \( H_0 \) is a non-negative constant depending on \( M \) and \( \Omega \) only.

**Remark 2.1.** It is not hard to prove, from \( \frac{\lambda_1}{\lambda_2} < 1 \),

\[
(2 - \frac{\lambda_1}{\lambda_2}) \frac{n^2 H_0^2}{\lambda_1} + 3 + \frac{\lambda_1}{\lambda_2} + \sqrt{\left(3 + \frac{\lambda_1}{\lambda_2} + \frac{n^2 H_0^2}{\lambda_2}\right)^2 + 4(1 - \lambda_1) \frac{n^2 H_0^2}{\lambda_2}} < \frac{n^2 H_0^2}{\lambda_1} + 4.
\]

In particular, when \( M \) is an \( n \)-dimensional complete minimal submanifold in the Euclidean space \( \mathbb{R}^N \), we have

**Corollary 2.1.** Let \( \Omega \) be a bounded domain in an \( n \)-dimensional complete minimal submanifold \( M \) in \( \mathbb{R}^N \). Then, we have

\[
\frac{\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}}{\lambda_1} \leq n + \sqrt{3 + \frac{\lambda_1}{\lambda_2}}.
\]
Since $M$ is a complete Riemannian manifold, from a theorem of Nash, there exists an isometric immersion $\varphi : M \to \mathbb{R}^N$ from $M$ into a Euclidean space $\mathbb{R}^N$. Let $(x^1, \cdots, x^n)$ denote an arbitrary local coordinate system of $M$. For any point $p \in \Omega$, we can write $\varphi(p) = (y_1, y_2, \cdots, y_N)$ with

$$y_\alpha = y_\alpha(x^1, \cdots, x^n), \quad 1 \leq \alpha \leq N,$$

which is the position vector of $p$ in $\mathbb{R}^N$. Thus, we have

$$g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \sum_{\alpha=1}^{N} \frac{\partial y_\alpha}{\partial x^i} \sum_{\beta=1}^{N} \frac{\partial y_\beta}{\partial x^j} = \sum_{\alpha=1}^{N} \frac{\partial y_\alpha}{\partial x^i} \frac{\partial y_\alpha}{\partial x^j},$$

where $g$ denotes the induced metric of $M$ from $\mathbb{R}^N$, $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^N$.

We denote the gradient of a function $f$ by $\nabla f$. Then, the following lemma holds, which is proved by Chen and Cheng [7].

**Lemma 2.1.**

\[ \sum_{\alpha=1}^{N} g(\nabla y_\alpha, \nabla y_\alpha) = \sum_{\alpha=1}^{N} |\nabla y_\alpha|^2 = n, \]
\[ \sum_{\alpha=1}^{N} (\Delta y_\alpha)^2 = n^2 |H|^2, \]
\[ \sum_{\alpha=1}^{N} \Delta y_\alpha \nabla y_\alpha = 0, \]

and for any function $u \in C^\infty(M)$,

\[ \sum_{\alpha=1}^{N} \left( g(\nabla y_\alpha, \nabla u) \right)^2 = \sum_{\alpha=1}^{N} \left( \nabla y_\alpha \cdot \nabla u \right)^2 = |\nabla u|^2, \]

where $|H|$ is the mean curvature of $M$.

**Proof of Theorem 2.1.** Let $u_j$ be the eigenfunction corresponding to the eigenvalue $\lambda_j$ such that \( \{u_j\}_{j=1}^\infty \) becomes an orthonormal basis of $L^2(\Omega)$. Hence, $\int_\Omega u_i u_j = \delta_{ij}$ for $i, j = 1, 2, \cdots$.

Defining

$$a_{\alpha j} = \int_\Omega y_\alpha u_1 u_{j+1},$$

since $u_1$ does not change sign in $\Omega$, we can assume $u_1 > 0$ in $\Omega$. We consider the $N \times N$-matrix $A = (a_{\alpha j})$. From the orthogonalization of Gram and Schmidt, there exist an upper triangle matrix $R = (R_{\alpha j})$ and an orthogonal matrix $Q = (q_{\alpha \beta})$ such that $R = QA$. Thus,

$$R_{\alpha j} = \sum_{\beta=1}^{N} q_{\alpha \beta} a_{\beta j} = \int_\Omega \sum_{\beta=1}^{N} q_{\alpha \beta} y_\beta u_1 u_{j+1} = 0, \quad 1 \leq j < \alpha \leq N.$$  

Defining $\bar{y}_\alpha = \sum_{\gamma=1}^{N} q_{\alpha \gamma} y_\gamma$, we have

$$\int_\Omega \bar{y}_\alpha u_1 u_{j+1} = \int_\Omega \sum_{\gamma=1}^{N} q_{\alpha \gamma} y_\gamma u_1 u_{j+1} = 0, \quad 1 \leq j < \alpha \leq N.$$  

Putting

$$z_\alpha = \bar{y}_\alpha - b_\alpha, \quad b_\alpha = \int_\Omega \bar{y}_\alpha u_1^2, \quad 1 \leq \alpha \leq N$$
and

\[ A_{\alpha j} = \int_{\Omega} z_\alpha u_1 u_j, \]

we have

(2.1) \quad A_{\alpha j} = 0, \quad \text{for } 1 \leq j \leq \alpha \leq N.

Defining

\[ B_{\alpha j} = \int_{\Omega} u_j \nabla z_\alpha \cdot \nabla u_1 \]

and

\[ C_{\alpha j} = \int_{\Omega} u_j u_1 \Delta z_\alpha, \]

from the Stokes theorem, we obtain

\[ -\lambda_j A_{\alpha j} = \int_{\Omega} z_\alpha u_1 \Delta u_j = \int_{\Omega} \Delta(z_\alpha u_1) u_j = \int_{\Omega} \left(2\nabla z_\alpha \cdot \nabla u_1 - \lambda_1 z_\alpha u_1 + u_1 \Delta z_\alpha \right) u_j = -\lambda_1 A_{\alpha j} + 2B_{\alpha j} + C_{\alpha j}, \]

namely,

(2.2) \quad 2B_{\alpha j} = (\lambda_1 - \lambda_j) A_{\alpha j} - C_{\alpha j}.

Since \( \{u_j\}_{j=1}^{\infty} \) is an orthonormal basis in \( L^2(\Omega) \) and \( A_{\alpha j} = 0, \) for \( 1 \leq j \leq \alpha \leq N, \) we have

(2.3) \quad z_\alpha u_1 = \sum_{j=\alpha+1}^{\infty} A_{\alpha j} u_j \quad \text{and} \quad \|z_\alpha u_1\|^2 = \sum_{j=\alpha+1}^{\infty} A_{\alpha j}^2.

Furthermore,

(2.4) \quad \int_{\Omega} u_j^2 z_\alpha \Delta z_\alpha = \sum_{j=\alpha+1}^{\infty} A_{\alpha j} C_{\alpha j},

(2.5) \quad 2 \int_{\Omega} z_\alpha u_1 \nabla z_\alpha \cdot \nabla u_1 = 2 \sum_{j=\alpha+1}^{\infty} A_{\alpha j} B_{\alpha j} = \sum_{j=\alpha+1}^{\infty} (\lambda_1 - \lambda_j) A_{\alpha j}^2 - \sum_{j=\alpha+1}^{\infty} A_{\alpha j} C_{\alpha j}.

Since for any function \( f \in C^2(\Omega) \cap C(\bar{\Omega}) \),

(2.6) \quad -2 \int_{\Omega} f u_1 \nabla f \cdot \nabla u_1 = \int_{\Omega} u_1^2 f \Delta f + \int_{\Omega} |\nabla f|^2 u_1^2,

we have

(2.7) \quad \int_{\Omega} |\nabla z_\alpha|^2 u_1^2 = -\int_{\Omega} z_\alpha u_1 \left(2\nabla z_\alpha \cdot \nabla u_1 + u_1 \Delta z_\alpha \right).

We obtain

(2.8) \quad \sum_{j=\alpha+1}^{\infty} (\lambda_j - \lambda_1) A_{\alpha j}^2 = \int_{\Omega} |\nabla z_\alpha|^2 u_1^2.
For any positive integer \( k \), we have

\[
\sum_{j=\alpha+1}^{\infty} (\lambda_j - \lambda_1) A_{\alpha j}^2 = \sum_{j=\alpha+1}^{k} (\lambda_j - \lambda_1) A_{\alpha j}^2 + \sum_{j=k+1}^{\infty} (\lambda_j - \lambda_1) A_{\alpha j}^2 \geq \sum_{j=\alpha+1}^{k} (\lambda_j - \lambda_1) A_{\alpha j}^2 + (\lambda_{k+1} - \lambda_1) \sum_{j=\alpha+1}^{\infty} A_{\alpha j}^2 - (\lambda_{k+1} - \lambda_1) \sum_{j=\alpha+1}^{k} A_{\alpha j}^2 = \sum_{j=\alpha+1}^{k} (\lambda_j - \lambda_{k+1}) A_{\alpha j}^2 + (\lambda_{k+1} - \lambda_1) \sum_{j=\alpha+1}^{\infty} A_{\alpha j}^2.
\]

Thus, we infer

\[(2.9)\] 

\[(\lambda_{k+1} - \lambda_1)\|z_\alpha u_1\|^2 \leq \sum_{j=\alpha+1}^{k} (\lambda_{k+1} - \lambda_j) A_{\alpha j}^2 + \int_{\Omega} |\nabla z_\alpha|^2 u_1^2,
\]

and, in particular,

\[(2.10)\] 

\[(\lambda_{\alpha+1} - \lambda_1)\|z_\alpha u_1\|^2 \leq \int_{\Omega} |\nabla z_\alpha|^2 u_1^2.
\]

For any \( \alpha \), we have

\[(2.11)\] 

\[|\nabla z_\alpha|^2 \leq 1.
\]

In fact, for any fixed point \( p_0 \in \Omega \), we can choose a new coordinate system \( \tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_N) \) of \( \mathbb{R}^N \) given by \( \varphi(p) - \varphi(p_0) = \tilde{y}(p)B \) such that \( \frac{\partial}{\partial \tilde{y}_i}|_{p_0}, \ldots, \frac{\partial}{\partial \tilde{y}_N}|_{p_0} \) span \( T_{p_0} M \) and at \( p_0 \), \( g\left( \frac{\partial}{\partial \tilde{y}_i}, \frac{\partial}{\partial \tilde{y}_j} \right) = \delta_{ij} \), where \( B = (b_{\alpha\beta}) \in O(N) \) is an \( N \times N \) orthogonal matrix.

\[
|\nabla z_\alpha|^2(p_0) = g(\nabla z_\alpha, \nabla z_\alpha) = \sum_{\beta,\gamma=1}^{N} q_{\alpha\gamma} q_{\alpha\beta} g(\nabla y_\gamma, \nabla y_\beta)
\]

\[
= \sum_{\beta,\gamma=1}^{N} q_{\alpha\gamma} q_{\alpha\beta} g(\sum_{\mu=1}^{N} b_{\gamma\mu} \nabla \tilde{y}_\mu, \sum_{\nu=1}^{N} b_{\beta\nu} \nabla \tilde{y}_\nu)
\]

\[(2.12)\] 

\[= \sum_{\beta,\gamma,\mu,\nu=1}^{N} q_{\alpha\gamma} b_{\gamma\mu} q_{\alpha\beta} b_{\beta\nu} g(\nabla \tilde{y}_\mu, \nabla \tilde{y}_\nu)
\]

\[= \sum_{j=1}^{n} \left( \sum_{\beta=1}^{N} q_{\alpha\beta} b_{\beta j} \right)^2 \leq 1,
\]

since \( QB \) is an orthogonal matrix when \( B \) and \( Q \) are orthogonal matrices. Therefore, (2.11) holds because \( p_0 \) is an arbitrary point. Since Lemma 2.1 also holds for \( z_\alpha \) from the definition of
z_\alpha$, for any positive constant $t > \frac{1}{2}$, we have, from Lemma 2.1 and (2.11),

$$
\sum_{\alpha=1}^{N} (\lambda_{\alpha+1} - \lambda_1) \int_{\Omega} |\nabla z_\alpha|^2 u_1^{t+1} \\
geq \sum_{j=1}^{n} (\lambda_{j+1} - \lambda_1) \int_{\Omega} |\nabla z_j|^2 u_1^{t+1} + (\lambda_{n+1} - \lambda_1) \sum_{A=n+1}^{N} \int_{\Omega} |\nabla z_A|^2 u_1^{t+1} \\
= \sum_{j=1}^{n} (\lambda_{j+1} - \lambda_1) \int_{\Omega} |\nabla z_j|^2 u_1^{t+1} + (\lambda_{n+1} - \lambda_1) \int_{\Omega} (n - \sum_{j=1}^{n} |\nabla z_j|^2) u_1^{t+1} \\
= \sum_{j=1}^{n} (\lambda_{j+1} - \lambda_1) \int_{\Omega} |\nabla z_j|^2 u_1^{t+1} + \int_{\Omega} (\sum_{j=1}^{n} (\lambda_{j+1} - \lambda_1)(1 - |\nabla z_j|^2) u_1^{t+1} \\
= \sum_{j=1}^{n} (\lambda_{j+1} - \lambda_1) \int_{\Omega} u_1^{t+1}.
$$

(2.13)

On the other hand, from the Stokes theorem and the Cauchy-Schwarz inequality, we obtain

$$
\int_{\Omega} |\nabla z_\alpha|^2 u_1^{t+1} = - \int_{\Omega} z_\alpha u_1 \left( u_1^t \Delta z_\alpha + (1 + t)u_1^{t-1} \nabla z_\alpha \cdot \nabla u_1 \right) \\
\leq \|z_\alpha u_1\| \cdot \|u_1^t \Delta z_\alpha + (1 + t)u_1^{t-1} \nabla z_\alpha \cdot \nabla u_1\|,
$$

(2.14)

and

$$
\int_{\Omega} |\nabla z_\alpha|^2 u_1^2 = - \int_{\Omega} z_\alpha u_1 \left( u_1 \Delta z_\alpha + 2\nabla z_\alpha \cdot \nabla u_1 \right) \\
\leq \|z_\alpha u_1\| \cdot \|u_1 \Delta z_\alpha + 2\nabla z_\alpha \cdot \nabla u_1\|.
$$

(2.15)

From (2.10), (2.13), (2.14) and (2.15), we derive

$$
\sum_{j=1}^{n} (\lambda_{j+1} - \lambda_1) \int_{\Omega} u_1^{t+1} \\
\leq \sum_{\alpha=1}^{N} (\lambda_{\alpha+1} - \lambda_1) \int_{\Omega} |\nabla z_\alpha|^2 u_1^{t+1} \\
\leq \sum_{\alpha=1}^{N} \int_{\Omega} |\nabla z_\alpha|^2 u_1^2 \int_{\Omega} |\nabla z_\alpha|^2 u_1^{t+1} \\
\leq \sum_{\alpha=1}^{N} \|u_1 \Delta z_\alpha + 2\nabla z_\alpha \cdot \nabla u_1\| \cdot \|u_1^t \Delta z_\alpha + (1 + t)u_1^{t-1} \nabla z_\alpha \cdot \nabla u_1\| \\
\leq \sqrt{\sum_{\alpha=1}^{N} \|u_1 \Delta z_\alpha + 2\nabla z_\alpha \cdot \nabla u_1\|^2 \cdot \sum_{\alpha=1}^{N} \|u_1^t \Delta z_\alpha + (1 + t)u_1^{t-1} \nabla z_\alpha \cdot \nabla u_1\|^2}.
$$

(2.16)
Since Lemma 2.1 also holds for \( z_\alpha \) from the definition of \( z_\alpha \), we have

\[
\sum_{\alpha=1}^{N} \| u_1 \Delta z_\alpha + 2 \nabla z_\alpha \cdot \nabla u_1 \|^2 = \int_{\Omega} (n^2 |H|^2 u_1^2 + 4 |\nabla u_1|^2)
\]

\[
\leq n^2 \sup_{\Omega} |H|^2 + 4 \lambda_1
\]

and

\[
\sum_{\alpha=1}^{N} \| u_1' \Delta z_\alpha + (1 + t) u_1' \nabla z_\alpha \cdot \nabla u_1' \|^2 = \int_{\Omega} (n^2 |H|^2 u_1'^2 + \frac{(1 + t)^2}{t^2} |\nabla u_1'|^2)
\]

\[
\leq \left( n^2 \sup_{\Omega} |H|^2 + \frac{(1 + t)^2}{2t - 1} \lambda_1 \right) \int_{\Omega} u_1'^2.
\]

Putting (2.17) and (2.18) into (2.16), we obtain

\[
\sum_{j=1}^{n} (\lambda_{j+1} - \lambda_1) \leq B(t) \sqrt{\left( n^2 \sup_{\Omega} |H|^2 + 4 \lambda_1 \right) \left( n^2 \sup_{\Omega} |H|^2 + \frac{(1 + t)^2}{2t - 1} \lambda_1 \right)},
\]

where

\[
B(t) = \frac{\int_{\Omega} u_1'^2}{\int_{\Omega} u_1'^{t+1}}.
\]

Since the spectrum of the Dirichlet eigenvalue problem of the Laplacian is an invariant of isometries, we know that the above inequality holds for any isometric immersion from \( M \) into a Euclidean space. Now we define \( \Phi \) by

\[
\Phi = \{ \varphi; \varphi \text{ is an isometric immersion from } M \text{ into a Euclidean space} \}.
\]

Defining \( H_0^2 = \inf_{\varphi \in \Phi} \sup_{\Omega} |H|^2 \), we have

\[
\sum_{j=1}^{n} (\lambda_{j+1} - \lambda_1) \leq B(t) \sqrt{\left( n^2 H_0^2 + 4 \lambda_1 \right) \left( n^2 H_0^2 + \frac{(1 + t)^2}{2t - 1} \lambda_1 \right)}.
\]

Next, we need to estimate \( B(t) \) as a function of \( t \) by making use of the same method as Brands [5]. Let \( u = u_1' - u_1 \int_{\Omega} u_1'^{t+1} \). We know that \( u \) is a trial function for \( \lambda_2 \). Hence, we have

\[
\lambda_2 \leq \frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} u^2}.
\]

According to a direct calculation, we obtain

\[
\frac{\lambda_2}{\lambda_1} \leq \frac{\frac{t^2}{2t - 1} B(t)^2 - 1}{B(t)^2 - 1}
\]

since \( B(t)^2 - 1 = \frac{\int_{\Omega} u^2}{\left( \int_{\Omega} u_1'^{t+1} \right)^2} > 0 \) for \( t > 1 \). Let \( a = \frac{\lambda_2}{\lambda_1} > 1 \). We have

\[
\left( a - \frac{t^2}{2t - 1} \right) B(t)^2 \leq a - 1.
\]
When \(1 < t < a + \sqrt{a^2 - a}\), we can infer

\[
B(t) \leq \sqrt{\frac{(a - 1)(2t - 1)}{a(2t - 1) - t^2}}.
\]

Therefore, we obtain

\[
\sum_{j=1}^{n} (\lambda_{j+1} - \lambda_1) \leq \sqrt{\left(n^2H^2_0 + 4\lambda_1\right)\left(n^2H^2_0 + \frac{(1 + t)^2}{2t - 1} \lambda_1\right)\frac{(a - 1)(2t - 1)}{a(2t - 1) - t^2}}.
\]

Letting \(b = \frac{n^2H^2_0}{\lambda_1}\) and defining a function

\[
f(t) = \frac{b(2t - 1) + (1 + t)^2}{a(2t - 1) - t^2},
\]

we have

\[
\sum_{j=1}^{n} (\lambda_{j+1} - \lambda_1) \leq \sqrt{\lambda_1(a - 1)\left(n^2H^2_0 + 4\lambda_1\right) f(t)}.
\]

If we take \(t = 1\), \(f(1) = \frac{b + 4}{a - 1}\). Thus, we obtain the result of Chen and Cheng [7]. Furthermore, we try to get the minimum of \(f(t)\) under \(1 \leq t \leq a + \sqrt{a^2 - a}\). It is not difficult to prove that the minimum of \(f(t)\) is attained at

\[
t_0 = \frac{a + b - 1 + \sqrt{(a + b - 1)^2 + 8a(a + b + 1)}}{2(a + b + 1)}.
\]

Since \(g(s) = t_0\) as a function of \(s = a + b\), is a decreasing function of \(s\) in the interval \([a, \infty)\), we have

\[
1 = g(\infty) < t_0 \leq g(a) < a + \sqrt{a^2 - a}.
\]

By a direct computation, we have

\[
a(2t_0 - 1) - t_0^2 = \frac{2\sqrt{(a + b - 1)^2 + 8a(a + b + 1)}}{4(a + b + 1)^2}
\]

\[
\times \left(2a(a + b + 1) - (a + b - 1) - \sqrt{(a + b - 1)^2 + 8a(a + b + 1)}\right).
\]

From \(\left(3(a + b) + 1\right)^2 - 8b(a + b + 1) = (a + b - 1)^2 + 8a(a + b + 1)\), we get

\[
b(2t_0 - 1) + (1 + t_0)^2 = \frac{2\sqrt{(a + b - 1)^2 + 8a(a + b + 1)}}{4(a + b + 1)^2}
\]

\[
\times \left(2b(a + b + 1) + 3(a + b) + 1 + \sqrt{(a + b - 1)^2 + 8a(a + b + 1)}\right).
\]
Thus, we have
\[ f(t_0) = \frac{2b(a + b + 1) + 3(a + b) + 1 + \sqrt{(a + b - 1)^2 + 8a(a + b + 1)}}{2a(a + b + 1) - (a + b - 1) - \sqrt{(a + b - 1)^2 + 8a(a + b + 1)}}, \]
\[ = \frac{2(a + b + 1)^2 \{(2a - 1)b + 3a + 1 + \sqrt{(a + b - 1)^2 + 8a(a + b + 1)}\}}{4a(a + b + 1)^2(a - 1)} \]
\[ = \frac{(2a - 1)b + 3a + 1 + \sqrt{(a + b - 1)^2 + 8a(a + b + 1)}}{2a(a - 1)}. \]

From (2.23) and (2.24), we obtain
\[ \sum_{j=1}^{n} (\lambda_{j+1} - \lambda_1) \leq \sqrt{\lambda_1 \left( n^2 H_0^2 + 4\lambda_1 \right) \frac{(2a - 1)b + 3a + 1 + \sqrt{(a + b - 1)^2 + 8a(a + b + 1)}}{2a}}, \]
\[ = \lambda_1 \left( \frac{n^2 H_0^2}{\lambda_1} + 4 \right) \left( 2 - \frac{\lambda_1}{\lambda_2} \right) \frac{n^2 H_0^2}{\lambda_1} + 3 + \frac{\lambda_1}{\lambda_2} + \sqrt{\left( 1 + \frac{n^2 H_0^2}{\lambda_1} - \frac{\lambda_1}{\lambda_2} \right)^2 + 8(1 + \frac{n^2 H_0^2}{\lambda_2} + \frac{\lambda_1}{\lambda_2})} \right), \]
\[ = \lambda_1 \left( \frac{n^2 H_0^2}{\lambda_1} + 4 \right) \left( 2 - \frac{\lambda_1}{\lambda_2} \right) \frac{n^2 H_0^2}{\lambda_1} + 3 + \frac{\lambda_1}{\lambda_2} + \sqrt{\left( 3 + \frac{\lambda_1}{\lambda_2} + \frac{n^2 H_0^2}{\lambda_2} \right)^2 + 4(1 - \frac{\lambda_1}{\lambda_2}) \frac{n^2 H_0^2}{\lambda_2}} \right), \]
because \( a = \frac{\lambda_2}{\lambda_1} \) and \( b = \frac{n^2 H_0^2}{\lambda_1} \). This finishes the proof of Theorem 2.1. \( \square \)

For any positive integer \( k \), we have from (2.9)
\[ \left( \lambda_{k+1} - \lambda_1 \right) \sum_{j=\alpha+1}^{k} (\lambda_{k+1} - \lambda_j) A_{\alpha j}^2 + \int_{\Omega} |\nabla z_\alpha|^2 u_1^2 \leq \frac{1}{\|z_\alpha u_1\|^2}. \]

From (2.14) and (2.15), we obtain
\[ \sum_{j=\alpha+1}^{k} (\lambda_{k+1} - \lambda_j) A_{\alpha j}^2 + \int_{\Omega} |\nabla z_\alpha|^2 u_1^2 \]
\[ = \left( \lambda_{k+1} - \lambda_1 \right) \int_{\Omega} |\nabla z_\alpha|^2 u_1^{t+1} \int_{\Omega} |\nabla z_\alpha|^2 u_1^2 \]
\[ \leq \|u_1^t \Delta z_\alpha + (1 + t) u_1^{t-1} \nabla z_\alpha \cdot \nabla u_1\| \cdot \|u_1 \Delta z_\alpha + 2 \nabla z_\alpha \cdot \nabla u_1\|. \]

For any positive integer \( k \), we can find some \( \alpha_0 \) such that
\[ \sum_{j=\alpha_0+1}^{k} \left( \lambda_{k+1} - \lambda_j \right) A_{\alpha_0 j}^2 \leq \max_{1 \leq \alpha \leq N} \sum_{j=\alpha+1}^{k} \left( \lambda_{k+1} - \lambda_j \right) A_{\alpha j}^2. \]
Hence, from Lemma 2.1, we obtain

\[
\frac{n(\lambda_{k+1} - \lambda_1) \int_\Omega u_1^{t+1}}{1 + \sum_{j=\alpha_0+1}^k (\lambda_{k+1} - \lambda_j) \frac{A^2_{\alpha_0 j}}{\int_\Omega |\nabla z_{\alpha_0}|^2 u_1^2}} 
\leq \sum_{\alpha=1}^N \|u_1^1 \Delta z_{\alpha} + (1 + t)u_1^{t-1} \nabla z_{\alpha} \cdot \nabla u_1 \| \cdot \|u_1 \Delta z_{\alpha} + 2 \nabla z_{\alpha} \cdot \nabla u_1\|
\]

\[
\leq \sqrt{\left( n^2 \sup_{\Omega} |H|^2 + 4 \lambda_1 \right) \left( n^2 \sup_{\Omega} |H|^2 + \frac{(1 + t)^2}{2t - 1} \lambda_1 \right) \int_\Omega u_1^{2t}}
\]

that is, we have

\[
\frac{n(\lambda_{k+1} - \lambda_1)}{1 + \sum_{j=\alpha_0+1}^k (\lambda_{k+1} - \lambda_j) \frac{A^2_{\alpha_0 j}}{\int_\Omega |\nabla z_{\alpha_0}|^2 u_1^2}} \leq B(t) \sqrt{\left( n^2 \sup_{\Omega} |H|^2 + 4 \lambda_1 \right) \left( n^2 \sup_{\Omega} |H|^2 + \frac{(1 + t)^2}{2t - 1} \lambda_1 \right)}.
\] (2.27)

On the other hand, we have

\[
\int_\Omega |\nabla u_1^{t-1}|^2 u_1^2 = \frac{(t - 1)^2}{2t - 1} \int_\Omega \nabla u_1 \cdot \nabla u_1^{2t-1} = \frac{(t - 1)^2}{2t - 1} \lambda_1 \int_\Omega u_1^{2t}.
\] (2.28)

Letting

\[
D_j = \int_\Omega u_1^t u_j,
\]

we know

\[
u_1^t = \sum_{j=1}^{\infty} D_j u_j, \quad \int_\Omega u_1^{2t} = \sum_{j=1}^{\infty} D_j^2.
\] (2.29)

Taking \(f = u_1^{t-1}\) in (2.6), we get

\[
\int_\Omega |\nabla u_1^{t-1}|^2 u_1^2 = -2 \int_\Omega u_1^t \nabla u_1^{t-1} \cdot \nabla u_1 - \int_\Omega u_1^{t+1} \Delta u_1^{t-1}
\]

\[
= -\sum_{j=1}^{\infty} D_j \left( 2 \int_\Omega u_j \nabla u_1^{t-1} \cdot \nabla u_1 + \int_\Omega u_j u_1 \Delta u_1^{t-1} \right)
\]

\[
= -\sum_{j=1}^{\infty} D_j \left( \int_\Omega u_j \Delta u_1^t - \int_\Omega u_j u_1^{t-1} \Delta u_1 \right)
\]

\[
= -\sum_{j=1}^{\infty} D_j \left( \int_\Omega u_1^t \Delta u_j - \int_\Omega u_j u_1^{t-1} \Delta u_1 \right)
\]

\[
= \sum_{j=1}^{\infty} D_j \left( \lambda_j \int_\Omega u_1^t u_j - \lambda_1 \int_\Omega u_1^t u_j \right)
\]

\[
= \sum_{j=2}^{\infty} (\lambda_j - \lambda_1) D_j^2.
\]
Thus, we infer
\[(2.30) \sum_{j=2}^{\infty} (\lambda_j - \lambda_1) D_j^2 = \frac{(t-1)^2}{2t-1} \lambda_1 \int_{\Omega} u_1^{2t}.\]

Defining
\[(2.31) \beta_j = \frac{D_j}{\sqrt{\frac{(t-1)^2}{2t-1} \lambda_1 \int_{\Omega} u_1^{2t}}},\]

we have
\[(2.32) \sum_{j=2}^{\infty} (\lambda_j - \lambda_1) \beta_j^2 = 1.\]

For any positive integer \(l\),
\[
1 = \sum_{j=2}^{\infty} (\lambda_j - \lambda_1) \beta_j^2 \\
= \sum_{j=2}^{l} (\lambda_j - \lambda_1) \beta_j^2 + \sum_{j=l+1}^{\infty} (\lambda_j - \lambda_1) \beta_j^2 \\
\geq \sum_{j=2}^{l} (\lambda_j - \lambda_1) \beta_j^2 + (\lambda_{t+1} - \lambda_1) \sum_{j=l+1}^{\infty} \beta_j^2 \\
= \sum_{j=2}^{l} (\lambda_j - \lambda_1) \beta_j^2 + (\lambda_{t+1} - \lambda_1) \sum_{j=2}^{\infty} \beta_j^2 - (\lambda_{t+1} - \lambda_1) \sum_{j=2}^{l} \beta_j^2 \\
= \sum_{j=2}^{l} (\lambda_j - \lambda_{t+1}) \beta_j^2 + (\lambda_{t+1} - \lambda_1) \sum_{j=2}^{\infty} \beta_j^2,
\]

namely,
\[(2.33) (\lambda_{t+1} - \lambda_1) \sum_{j=2}^{\infty} \beta_j^2 \leq 1 + \sum_{j=2}^{l} (\lambda_{t+1} - \lambda_j) \beta_j^2.\]

From (2.29) and (2.30), we infer
\[(2.34) \frac{(t-1)^2}{2t-1} \lambda_1 \sum_{j=1}^{\infty} \beta_j^2 = 1.\]

Since
\[
\left( \int_{\Omega} u_1^{t+1} \right)^2 = D_1^2 = \frac{(t-1)^2}{2t-1} \lambda_1 \beta_1^2 \int_{\Omega} u_1^{2t},
\]

according to the definition of \(B(t)\), we have
\[(2.35) B(t)^2 = \frac{\int_{\Omega} u_1^{2t}}{\left( \int_{\Omega} u_1^{t+1} \right)^2} = \frac{1}{\frac{(t-1)^2}{2t-1} \lambda_1 \beta_1^2} = \frac{1}{1 - \frac{(t-1)^2}{2t-1} \lambda_1 \sum_{j=2}^{\infty} \beta_j^2}.\]

From (2.27), (2.33) and (2.35), we have
**Proposition 2.1.** Let $M$ be an $n$-dimensional complete submanifold in $\mathbb{R}^N$, $\Omega \subset M$ a bounded domain with a piecewise smooth boundary $\partial \Omega$. Then, for any positive integer $k$, there exists an integer $\alpha_0$ with $1 \leq \alpha_0 \leq N$ such that eigenvalues of the Dirichlet eigenvalue problem of the Laplacian satisfy, for any positive integer $l$ and $t > \frac{1}{2}$,

\[
\frac{n(\lambda_{k+1} - \lambda_1)}{1 + \sum_{j=\alpha_0+1}^{k} (\lambda_{k+1} - \lambda_j) \frac{A_{\alpha_0}^2}{\int_{\Omega} |\nabla z_{\alpha_0}|^2 u_1^2}} \leq \left( \frac{n^2 \sup_{\Omega} |H|^2 + 4\lambda_1}{2t - 1 + \lambda_1} \right)^2 \left( \frac{n^2 \sup_{\Omega} |H|^2 + (1 + t)^2}{2t - 1 + \lambda_1} \right).
\]

3. Estimates for eigenvalues on Minimal submanifolds

In this section, we will deal with eigenvalues of the Laplacian on bounded domains in complete minimal submanifolds of Euclidean spaces. Thus, let $\Omega \subset M$ be a bounded domain with a piecewise smooth boundary $\partial \Omega$ in an $n$-dimensional complete minimal submanifold $M$ of the Euclidean space $\mathbb{R}^N$. We consider the following Dirichlet eigenvalue problem of the Laplacian:

\[
\begin{cases}
\Delta u = -\lambda u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Since $M$ is an $n$-dimensional complete minimal submanifold in $\mathbb{R}^N$, we have from Lemma 2.1 and the definition of $C_{\alpha_j}$,

\[C_{\alpha_j} = 0\]

for any $\alpha$ and $j$. Hence, we have from (2.2),

\[2B_{\alpha_j} = (\lambda_1 - \lambda_j)A_{\alpha_j}.\]

For any $\alpha$, we have

\[
0 = -\frac{2}{t+1} \int_{\Omega} u_1^{t+1} \Delta z_\alpha
= 2 \int_{\Omega} u_1^{t} \nabla z_\alpha \cdot \nabla u_1
= 2 \sum_{i=1}^{\infty} D_i B_{\alpha_i} = \sum_{i=1}^{\infty} (\lambda_1 - \lambda_i) D_i A_{\alpha_i}.
\]

Thus, from (2.31) we obtain

\[\sum_{i=2}^{\infty} (\lambda_i - \lambda_1) \beta_i A_{\alpha_i} = 0.\]

For any positive integer $j \geq 2$, since

\[\left( (\lambda_j - \lambda_1) \beta_j A_{\alpha_j} \right)^2 \leq \left( \sum_{i=2,i\neq j}^{\infty} (\lambda_i - \lambda_1) \beta_i^2 \right) \left( \sum_{i=2,i\neq j}^{\infty} (\lambda_i - \lambda_1) A_{\alpha_i}^2 \right),\]

according to (2.8) and (2.32), we derive

\[\left( \lambda_j - \lambda_1 \right)^2 \beta_j^2 A_{\alpha_j}^2 \leq \left( 1 - (\lambda_j - \lambda_1) \beta_j^2 \right) \left( \int_{\Omega} |\nabla z_\alpha|^2 u_1^2 - (\lambda_j - \lambda_1) A_{\alpha_j}^2 \right).
\]

Hence, we have

\[\left( \lambda_j - \lambda_1 \right)^2 \beta_j^2 A_{\alpha_j}^2 + (\lambda_j - \lambda_1) \frac{A_{\alpha_j}^2}{\int_{\Omega} |\nabla z_\alpha|^2 u_1^2} \leq 1.\]
From Lemma 2.1 and the Cauchy-Schwarz inequality, we get

\[(\lambda_{k+1} - \lambda_1) \sum_{\alpha=1}^{N} \|z_\alpha u_1\|^2 \leq n + \sum_{\alpha=1}^{N} \sum_{j=\alpha+1}^{k} (\lambda_{k+1} - \lambda_j) A_{\alpha j}^2\]

On the other hand, from the Stokes theorem, we have

\[\int_{\Omega} u_1^{t+1} |\nabla z_\alpha|^2 = \frac{1}{2} \int_{\Omega} u_1^{t+1} \Delta z_\alpha^2 = -(t + 1) \int_{\Omega} z_\alpha u_1^{t} \nabla z_\alpha \cdot \nabla u_1.\]

From Lemma 2.1 and the Cauchy-Schwarz inequality, we get

\[n \int_{\Omega} u_1^{t+1} = -(t + 1) \sum_{\alpha=1}^{N} \int_{\Omega} z_\alpha u_1^{t} \nabla z_\alpha \cdot \nabla u_1\]

\[\leq (t + 1) \left( \sum_{\alpha=1}^{N} \|z_\alpha u_1\|^2 \right)^{\frac{1}{2}} \left( \sum_{\alpha=1}^{N} \int_{\Omega} u_1^{2t-2} (\nabla z_\alpha \cdot \nabla u_1)^2 \right)^{\frac{1}{2}}\]

\[= (t + 1) \left( \sum_{\alpha=1}^{N} \|z_\alpha u_1\|^2 \right)^{\frac{1}{2}} \left( \int_{\Omega} u_1^{2t-2} |\nabla u_1|^2 \right)^{\frac{1}{2}}\]

\[= \frac{1}{2t-1} \int_{\Omega} \nabla u_1^{2t-1} \cdot \nabla u_1\]

namely,

\[(\lambda_{k+1} - \lambda_1) \sum_{\alpha=1}^{N} \|z_\alpha u_1\|^2 \geq \frac{n^2}{2t-1} \frac{f_\Omega u_1^{t+1}}{\lambda_1} \int_{\Omega} u_1^{2t} = \frac{n^2}{2t-1} \frac{(t + 1)^2}{\lambda_1 B(t)^2}\]

From (3.3)-(3.5), (2.35) and (2.33), we have

\[
\frac{n^2(\lambda_{k+1} - \lambda_1)}{n + \sum_{\alpha=1}^{N} \sum_{j=\alpha+1}^{k} (\lambda_{k+1} - \lambda_j) A_{\alpha j}^2} \leq \frac{(t + 1)^2 \lambda_1 B(t)^2}{2t-1} = \frac{\lambda_1}{2t-1} \left(1 + \sum_{j=2}^{\infty} \beta_j^2\right)\]

\[\leq \frac{(t + 1)^2}{2t-1} \frac{\lambda_1}{1 - \frac{(t - 1)^2}{2t-1} \frac{\lambda_1}{\lambda_{t+1} - \lambda_1} \left(1 + \sum_{j=2}^{t} (\lambda_{t+1} - \lambda_j) \beta_j^2\right)}\]

\[\leq \frac{(t + 1)^2}{2t-1} \frac{\lambda_1}{1 - \frac{(t - 1)^2}{2t-1} \frac{\lambda_1}{\lambda_{t+1} - \lambda_1} \left(1 + \sum_{j=2}^{t} \frac{\lambda_{t+1} - \lambda_j}{\lambda_j - \lambda_1} \left(1 - (\lambda_j - \lambda_1) \frac{A_{\alpha j}^2}{\int_{\Omega} |\nabla z_\alpha|^2 u_1^{2t}}\right)\right)}.\]
EIGENVALUES OF THE LAPLACIAN

Defining

\[ \sigma_{al} = \lambda_1 + \frac{\lambda_{l+1} - \lambda_1}{1 + \sum_{j=2}^{l} \frac{\lambda_{l+1} - \lambda_j}{\lambda_j - \lambda_1} \left[ 1 - (\lambda_j - \lambda_1) \frac{A_{\alpha j}^2}{\int_{\Omega} |\nabla z_{\alpha}|^2 u_1^2} \right]} \]

and taking

\[ t = \frac{2\sigma_{al}}{\sigma_{al} + \lambda_1}, \]

we obtain the following:

**Theorem 3.1.** Let \( M \) be an \( n \)-dimensional complete minimal submanifold in \( \mathbb{R}^N \), \( \Omega \subset M \) a bounded domain with a piecewise smooth boundary \( \partial \Omega \). Then, for positive integers \( k, l \), eigenvalues of the Dirichlet eigenvalue problem of the Laplacian satisfy, for \( 1 \leq \alpha \leq N \),

\[ \frac{n^2(\lambda_{k+1} - \lambda_1)}{n + \sum_{\alpha=1}^{N} \sum_{j=\alpha+1}^{k} (\lambda_{k+1} - \lambda_j)A_{\alpha j}^2} \leq 3\lambda_1 + \frac{\lambda_1^2}{\sigma_{al}}. \]  

(3.6)

**Corollary 3.1.** Let \( M \) be an \( n \)-dimensional complete minimal submanifold in \( \mathbb{R}^N \), \( \Omega \subset M \) a bounded domain with a piecewise smooth boundary \( \partial \Omega \). Then, for the Dirichlet eigenvalue problem of the Laplacian, we have

\[ \frac{\lambda_2}{\lambda_1} \leq \frac{n + 3 + \sqrt{n^2 + 10n + 9}}{2n}. \]  

(3.7)

**Proof.** Taking \( k = l = 1 \) in (3.6), we have

\[ n(\lambda_2 - \lambda_1) \leq 3\lambda_1 + \frac{\lambda_1^2}{\lambda_2}. \]

The above inequality can be written by the following quadratic inequality:

\[ n \left( \frac{\lambda_2}{\lambda_1} \right)^2 - (n + 3) \frac{\lambda_2}{\lambda_1} - 1 \leq 0. \]

Therefore, we can obtain (3.7). \( \square \)

**Remark 3.1.** When \( n = 2 \), the inequality (3.7) becomes the following form:

\[ \frac{\lambda_2}{\lambda_1} \leq \frac{5 + \sqrt{33}}{4}. \]

Thus, the result of Brands \[5\] for a bounded domain in the Euclidean space is also included here.

For any positive integer \( k \), we can find some \( \alpha_0 \) such that

\[ \sum_{j=\alpha_0+1}^{k} \frac{(\lambda_{k+1} - \lambda_j)A_{\alpha j}^2}{\int_{\Omega} |\nabla z_{\alpha}|^2 u_1^2} = \max_{1 \leq \alpha \leq N} \sum_{j=\alpha+1}^{k} \frac{(\lambda_{k+1} - \lambda_j)A_{\alpha j}^2}{\int_{\Omega} |\nabla z_{\alpha}|^2 u_1^2}. \]
Then, from Lemma 2.1, we get

\[ n + \sum_{\alpha=1}^{N} \sum_{j=\alpha+1}^{k} (\lambda_{k+1} - \lambda_j) A_{\alpha j}^2 \]

\[ \leq n + \sum_{j=\alpha_0+1}^{k} (\lambda_{k+1} - \lambda_j) \frac{A_{\alpha_0 j}^2}{\int_{\Omega} |\nabla z_{\alpha_0}|^2 u_1^2} \sum_{\alpha=1}^{N} \int_{\Omega} |\nabla z_\alpha|^2 u_1^2 \]

\[ = n \left( 1 + \sum_{j=\alpha_0+1}^{k} (\lambda_{k+1} - \lambda_j) \frac{A_{\alpha_0 j}^2}{\int_{\Omega} |\nabla z_{\alpha_0}|^2 u_1^2} \right). \]

Therefore, we have the following

Corollary 3.2. Let \( M \) be an \( n \)-dimensional complete minimal submanifold in \( \mathbb{R}^N \), \( \Omega \subset M \) a bounded domain with a piecewise smooth boundary \( \partial \Omega \). Then, for any positive integer \( k \), there exists an integer \( \alpha_0 \) with \( 1 \leq \alpha_0 \leq N \) such that eigenvalues of the Dirichlet eigenvalue problem of the Laplacian satisfy, for any positive integer \( l \),

\[ \frac{n(\lambda_{k+1} - \lambda_1)}{1 + \sum_{j=2}^{k} (\lambda_{k+1} - \lambda_j)} \leq 3\lambda_1 + \frac{\lambda_1^2}{\sigma_{\alpha_0 l}}. \]

Since (3.3) holds for any \( j \) and any \( \alpha \), from Corollary 3.2, we have

Corollary 3.3. Let \( M \) be an \( n \)-dimensional complete minimal submanifold in \( \mathbb{R}^N \), \( \Omega \subset M \) a bounded domain with a piecewise smooth boundary \( \partial \Omega \). Then, for positive integers \( k, l \), eigenvalues of the Dirichlet eigenvalue problem of the Laplacian satisfy, for \( 1 \leq \alpha \leq N \),

\[ \frac{n(\lambda_{k+1} - \lambda_1)}{1 + \sum_{j=2}^{k} \frac{\lambda_{k+1} - \lambda_j}{\lambda_j - \lambda_1}} \leq 3\lambda_1 + \frac{\lambda_1^2}{\sigma_{\alpha l}}. \]

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