Five-Dimensional Gauge Theories and Local Mirror Symmetry

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Abstract

We study the dynamics of 5-dimensional gauge theory on \( M_4 \times S^1 \) by compactifying type II/M theory on degenerate Calabi-Yau manifolds. We use the local mirror symmetry and shall show that the prepotential of the 5-dimensional \( SU(2) \) gauge theory without matter is given exactly by that of the type II string theory compactified on the local \( F_2 \), i.e. Hirzebruch surface \( F_2 \) lying inside a non-compact Calabi-Yau manifold. It is shown that our result reproduces the Seiberg-Witten theory at the 4-dimensional limit \( R \to 0 \) (\( R \) denotes the radius of \( S^1 \)) and also the result of the uncompactified 5-dimensional theory at \( R \to \infty \).

We also discuss \( SU(2) \) gauge theory with \( 1 \leq N_f \leq 4 \) matter in vector representations and show that they are described by the geometry of the local \( F_2 \) blown up at \( N_f \) points.
1 Introduction

Recent developments in non-perturbative string theory and M theory have led to new insights into the relation between low energy field theory and string theory: it has been argued in particular that non-perturbative dynamics takes place in low energy field theory and higher gauge symmetries emerge when compactifying Calabi-Yau and $K_3$ manifolds degenerate and some of their homology cycles vanish. For instance, when $K_3$ surface develops an $A-D-E$ singularity, there appears an enhanced $A-D-E$ gauge symmetry in 6-dimensions. Similarly, when a family of $P^1$’s shrinks to zero size along a rational curve in Calabi-Yau threefold, we obtain a 4-dimensional $SU(2)$ gauge theory with $N = 2$ supersymmetry.

In this article we would like to study the dynamics of SUSY gauge theories in 5-dimensions by compactifying the M theory on degenerate Calabi-Yau manifolds. We construct an effective action of an $SU(2)$ gauge theory on a 5-dimensional space $M_4 \times S^1$ where $M_4$ is the Minkowki space and $S^1$ is a circle of radius $R$. In the limit of $R \to 0$ this theory reproduces the standard $\mathcal{N} = 2$ SUSY gauge theory in 4-dimensions, i.e. Seiberg-Witten theory [1]. Thus our 5-dimensional model gives an M-theoretic generalization of Seiberg-Witten theory by incorporating Kaluza-Klein excitations. In the opposite limit $R \to \infty$ our model reduces to the gauge theory in uncompactified 5-dimensions $M_5$. Characteristic features of this theory have been studied using the brane-probe picture [2] and also from the point of view of classical geometry of collapsing del Pezzo surfaces [3, 4, 5] and the behavior of the low-energy effective gauge coupling has been determined exactly.

In this paper we would like to propose an exact solution of the 5-dimensional theory on $M_4 \times S^1$ which reproduces the known results at both limits $R \to 0$ and $R \to \infty$. As it turns out, our prepotential follows directly from that of the type II string theory compactified on singular Calabi-Yau manifolds using the method of local mirror symmetry [6, 7, 8, 9]. In the case of pure $SU(2)$ gauge theory without matter our result is obtained from the type II theory compactified on the local $F_2$, i.e. Hirzebruch surface $F_2$ lying inside a Calabi-Yau threefold which is the canonical bundle over $F_2$. Similarly, $SU(2)$ gauge theories with $N_f$ matter in vector representations are also obtained from the type II theory compactified on the local $F_2$ blown up at $N_f$ points ($0 \leq N_f \leq 4$). We will find that our model at $R = \infty$ has an infinite bare coupling constant and yields a non-trivial interacting field theory in the infra-red limit.

The local mirror symmetry is a method of mirror symmetry adapted in the case of non-compact Calabi-Yau manifolds. Suppose, for instance, we are given a compact Calabi-Yau threefold which is an elliptic fibration over $F_n$. One considers the limit of the size of the fiber $t_E$ going to $\infty$. Then the resulting non-compact manifold is modeled by the local $F_n$, i.e. $F_n$ inside a Calabi-Yau with the normal bundle being given by the canonical bundle of $F_n$. The limit of $t_E \to \infty$ may also be considered
as the limit of shrinking $F_n$ with the size of the fiber kept fixed. $F_n$ ($n = 0, 1, 2$) and its various blow ups are the del Pezzo surfaces and these are in fact the type of manifolds which featured in the geometrical interpretation of 5-dimensional gauge theory \[3, 4, 5\].

2 \(SU(2)\) Gauge Theory without Matter

Let us start from the case of 5-dimensional gauge theory without matter. We consider the local $F_2$ model which is described by the toric data given in the Appendix. Following the standard procedure \[6, 7, 8, 9\] one obtains a curve in the B-model given by

\[ P = a_0 x + a_1 x^2 + a_2 \zeta + a_3 + a_4 \frac{1}{\zeta} = 0. \] (2.1)

Introducing a new variable $y = a_2 \zeta - a_4 / \zeta$ the curve is rewritten as

\[ y^2 = (a_1 x^2 + a_0 x + a_3)^2 - 4 a_2 a_4. \] (2.2)

Complex moduli of the B-model are defined by

\[ z_F = \frac{a_1 a_3}{a_0^2}, \quad z_B = \frac{a_2 a_4}{a_3^2}. \] (2.3)

If we choose $a_1 = a_3 = 1, a_0 = s, 4a_2a_4 = K^4$, we find

\[ y^2 = (x^2 + sx + 1)^2 - K^4, \quad z_F = \frac{1}{s^2}, \quad z_B = \frac{K^4}{4}. \] (2.4)

(2.4) is in fact the curve proposed by Nekrasov \[10\] for the description of 5-dimensional gauge theory and its properties have been studied in \[11\] in detail.

If we introduce a variable $U$ which is the analogue of $u$ of the Seiberg-Witten solution, the parameter $s$ is written as

\[ s = 2R^2 U \] (2.5)

where $R$ is the radius of $S^1$. In terms of $U$ and $R$ the curve reads as

\[ y^2 = (x^2 - R^4 (U^2 - \frac{1}{R^4}))^2 - K^4. \] (2.6)

By comparing (2.6) with the Seiberg-Witten curve $y^2 = (x^2 - u)^2 - \Lambda^4$, we find the correspondence between the parameters $U$ and $u$

\[ u \leftrightarrow R^2 (U^2 - \frac{1}{R^4}). \] (2.7)
As we see from (2.7), \( U \) variable describes two copies of the \( u \)-plane. Strong coupling region \( u \approx 0 \) maps to \( U \approx \pm 1/R^2 \) and thus the two copies are separated by a distance of order \( 1/R^2 \).

In the brane-probe interpretation of Seiberg-Witten solution [12, 13], \( u \)-plane is identified as a local region around one of the four fixed points (O-7 planes) in type I' theory compactification to 8-dimensions on \( T^2 \) \((u = 0 \) is identified as the location of the O-7 plane\). Then the curve (2.6) describes a theory which contains two of these O-7 planes. Since the fixed plane acts like a reflecting mirror, D3-brane probe will possess an infinite number of mirror images when inserted into a background of two orientifold planes. These mirror images are separated by distances \( n/R^2 \), \( n \in \mathbb{Z} \) and open strings connecting them generate Kaluza-Klein modes of supersymmetric gauge fields. Thus the curve (2.6) effectively describes a theory on a 5-dimensional manifold \( M_4 \times S^1 \) with \( R \) being the radius of \( S^1 \). In the limit of \( R \to 0 \) one of the \( u \)-planes moves off to \( \infty \) and the model (2.6) is expected to reduce to the Seiberg-Witten theory.

Periods of the B-model of local \( F_2 \) (2.4) is determined by solving differential equations (Gelfand-Kapranov-Zelevinskij (GKZ) system) associated with the toric data (see Appendix). Differential operators are given by

\[
\begin{align*}
\mathcal{L}_1 &= z_F(4\theta_{z_F}^2 + 2\theta_{z_F}) + \theta_{z_F}(2\theta_{z_B} - \theta_{z_F}), & (2.8) \\
\mathcal{L}_2 &= z_B(2\theta_{z_B} - \theta_{z_F} + 1)(2\theta_{z_B} - \theta_{z_F}) - \theta_{z_B}^2, & (2.9)
\end{align*}
\]

where

\[
\theta_{z_F} \equiv z_F \frac{\partial}{\partial z_F}, \quad \theta_{z_B} \equiv z_B \frac{\partial}{\partial z_B}. \tag{2.10}
\]

These operators have a regular singular point at \( z_F = z_B = 0 \): they possess “single-log” solutions \( \omega_F, \omega_B \) behaving as \( \omega_F = \log z_F + \cdots \) and \( \omega_B = \log z_B + \cdots \) at \( z_F, z_B \approx 0 \). There also exists a “double-log” solution \( \Omega \) behaving as \( \Omega = (\log z_F)^2 + (\log z_F)(\log z_B) + \cdots \). We identify the two single-log solutions as the Kähler parameters \( t_F, t_B \) of the A-model: \( t_F \) represents the size of the \( \mathbb{P}^1 \) fiber of \( F_2 \) and \( t_B \) the size of its base \( \mathbb{P}^1 \). \( t_F \) is given by

\[
-t_F \equiv \omega_F = \log z_F + \left[ \sum_{n \geq 1, m \geq 0} \frac{2(2n - 1)!}{(n - 2m)!n!m!} z_F^n z_B^m - \sum_{m \geq 1} \frac{(2m - 1)!}{m!^2} z_B^m \right],
\]

\[
= -2 \log \left( \frac{1}{\sqrt{4z_F}} + \sqrt{\frac{1}{4z_F} - 1} \right)
\]

\[
+ \left[ \sum_{n \geq 1, m \geq 1} \frac{2(2n - 1)!}{(n - 2m)!n!m!} z_F^n z_B^m - \sum_{m \geq 1} \frac{(2m - 1)!}{m!^2} z_B^m \right]. \tag{2.11}
\]
Similarly, \( t_B \) is given by
\[
-t_B \equiv \omega_B = -2 \log \left( \frac{1}{\sqrt{4z_B}} + \sqrt{\frac{1}{4z_B} - 1} \right). \tag{2.12}
\]

In our interpretation as the 5-dimensional gauge theory, the size of the fiber \( t_F \) is identified as the vacuum expectation value \( A \) of the scalar field in the vector-multiplet
\[
t_F = 4RA. \tag{2.13}
\]
On the other hand, the size of the base \( t_B \) is related to the dynamical mass parameter \( \Lambda \) as
\[
e^{-t_B} = 4R^4\Lambda^4. \tag{2.14}
\]
Then the mirror transformation (2.12) becomes
\[
K = \frac{2RA}{\sqrt{1 + 4R^4\Lambda^4}}. \tag{2.15}
\]
Note that (2.13) and (2.14) are in fact the identification of variables suggested in [6, 14].

One can invert the relations (2.11), (2.12) perturbatively and express the B-model parameters \( z_F, z_B \) in terms of \( t_F, t_B \). In the case of local mirror symmetry the holomorphic solution of GKZ system is a constant and the mirror transformation is simpler than in the compact Calabi-Yau case. One may then represent the double-log solution in terms of the Kähler parameters
\[
\Omega = t_F^2 + t_F t_B + 4 \sum_{n=1} \frac{1}{n^2} q_F^n + 4q_B \left( \sum_{n=1} n^2 q_F^n \right) + q_B^2(q_F^2 + 36q_F^3 + 260q_F^4 + 1100q_F^5 + \cdots) + \mathcal{O}(q_F^3), \tag{2.16}
\]
where
\[
q_F = e^{-t_F}, \quad q_B = e^{-t_B}. \tag{2.17}
\]
We identify the double-log solution as \( A_D \), the dual of the variable \( A \)
\[
\Omega = -8\pi i RA_D. \tag{2.18}
\]

First two terms of (2.16) represent the classical intersection numbers of the Calabi-Yau manifold and the remaining terms represent the contribution of worldsheet instantons. According to ref [9] the double-log solution has a generic form
\[
\Omega = \sum_{i,j=1}^2 t_i t_j \langle j^i J^j \rangle + \sum_{k=1} \sum_{n,m \geq 0} \left( \sum_{i} x^i \partial_i \right) d_{n,m} q_{1,k}^{kn} q_{2,m}^{km} \frac{1}{k^3}, \tag{2.19}
\]
for a local model of a surface $S$ (with two Kähler parameters). $J_i$ denotes the Kähler classes of $S$ and $\langle J_i J_j \rangle$ their intersection numbers. Numerical coefficients $x^i$ are defined by

$$c_1(S) = \sum_i x^i J_i,$$

(2.20)

where $c_1(S)$ is the first Chern class of $S$. $d_{n,m}$ gives the number of rational holomorphic curves in the homology class $n J_1 + m J_2$. Sum over $k$ in (2.19) represents the multiple-cover factor.

By comparing (2.16) and (2.19) (set $q_1 = q_F, q_2 = q_B$) we find \cite{6, 9}

$$d_{1,0} = -2, \quad d_{n,0} = 0, \quad n > 1, \quad d_{n,1} = -2n,$$

$$d_{1,2} = d_{2,2} = 0, \quad d_{3,2} = -6, \quad d_{4,2} = -32, \cdots$$

(2.21)

By integrating $A_D$ over $A$ we have the prepotential $F$ for the local $F_2$ model

$$F = \frac{1}{32 \pi i R^2} \left[ -\frac{t_B^3}{3} - \frac{t_F^2 t_B}{2} + 4 \sum_{n=1} 1 \frac{q^n}{n} + 4 q_B \left( \sum_{n=1} n q^n \right) + q_B^2 \left( \frac{1}{2} q^2_F + 12 q^3_F + 65 q^4_F + 220 q^5_F + \cdots \right) + O(q^3_B) \right].$$

(2.22)

### 3 Small and Large Radius Limits

Let us next examine the small and large radius limits $R \to 0, \infty$ of (2.22). We first consider the 4-dimensional limit $R \to 0$. Due to the relations (2.13), (2.14) small $R$ corresponds to the base of $F_2$ becoming large ($t_B \to \infty$) and the fiber becoming small ($t_F \to 0$) while the ratio $e^{-t_B} / t^4_F$ is kept fixed. At small $R$, we have $K \approx 2 R A$ and $U \approx \cosh(2 R A)$. As explained by Katz, Klemm and Vafa \cite{9}, quantum parts of $F$ are suppressed because of the powers of $q_B \approx R^4$ and the surviving contributions come from the divergent parts of the series $\sum d_{n,m} q^n_F$ over $q_F$ as $q_F = e^{-4 R A} \to 1$.

At small $R$ the gauge coupling $\tau = \partial^2 F / \partial A^2$ behaves as

$$\tau = \frac{4i}{\pi} R A - \frac{i}{2 \pi} \log(4 R^4 A^4) - \frac{2i}{\pi} \sum_{n=1} \frac{q^n}{n} - \frac{8i}{\pi} R^4 A^4 \sum_{n=1} n^3 q^n_F + \cdots$$

$$\approx \frac{2i}{\pi} \log(2 \sqrt{2} A / \Lambda) - \frac{3i}{16 \pi} A^4 + \cdots$$

(3.1)

(3.1) reproduces the one-loop beta function and the one-instanton contribution to the Seiberg-Witten solution \cite{1}. We can check the agreement with Seiberg-Witten theory for higher instanton terms.

In (3.1) the world-sheet instanton expansion of type II theory is converted into the space-time instanton expansion of gauge theory in the $R \to 0$ limit. Coefficients
of the $m$-instanton amplitudes of gauge theory are determined by the asymptotic behavior of the number of holomorphic curves $d_{n,m}$ as $n \to \infty$ with fixed $m$.

Let us next examine the $R \to \infty$ limit of the uncompactified 5-dimensional gauge theory. It is known \cite{15, 16} that the gauge theory on $M_5$ has no quantum corrections and its gauge coupling is simply expressed in terms of classical intersection numbers of Calabi-Yau manifold. In fact by taking $R \to \infty$ world-sheet instanton terms disappear and we have

$$\lim_{R \to \infty} \frac{\tau}{2\pi i R} \equiv \tau_5 = \frac{2}{\pi^2} A,$$

(3.2)

(we have rescaled $\tau$ so that $\tau_5$ corresponds to the gauge coupling of 5-dimensional theory, $\tau_5 = 1/g_5^2$). In the next section we will discuss local $F_2$ model blown up at $N_f$ points. We then find that the above formula is generalized as

$$\tau_5 = \frac{2}{\pi^2} \left( A - \sum_{i=1}^{N_f} \frac{1}{16} |A - M_i| - \sum_{i=1}^{N_f} \frac{1}{16} |A + M_i| \right).$$

(3.3)

(3.3) is exactly the behavior of gauge coupling of $SU(2)$ theory with $N_f$ matter in vector representations (at infinite bare coupling) \cite{3, 3, 4}. Thus we reproduce correct results also in the uncompactified limit $R \to \infty$.

We should note that the local models of $F_0$ and $F_1$ also reproduce Seiberg-Witten solution in the $R \to 0$ limit \cite{8} since the asymptotic behavior of the number of holomorphic curves $d_{n,m}$ are the same for all models $F_i$, $i = 0, 1, 2$. $F_0, F_1$, however, have a different classical topology from $F_2$ and do not reproduce (3.3) at $R = \infty$. Thus the $F_2$ model is singled out as the unique candidate for the description of 5-dimensional gauge theory on $M_4 \times S^1$.

4 $SU(2)$ theory with matter

Let us next consider the case of $SU(2)$ gauge theory coupled to $N_f$ matter ($1 \leq N_f \leq 4$) in vector representations. We first discuss the $N_f = 1$ case. Relevant geometry is given by the local $F_2$ with one-point blown up. Corresponding curve is given by (see Appendix)

$$y^2 = (x^2 + sx + 1)^2 - K^3(xw + \frac{1}{w}).$$

(4.1)

Comparison with the $N_f = 1$ Seiberg-Witten curve suggests the identification

$$w = e^{RM},$$

(4.2)

where $M$ denotes the bare mass of the matter multiplet. Complex moduli of the B-model are given by (we denote $z_{F-E}$ as $z_F$ and $z_{F+E}$ as $z_{F'}$ for notational simplicity)

$$z_F = \frac{w^2}{s}, \quad z_{F'} = \frac{1}{sw^2}, \quad z_B = \frac{K^3}{4w}. $$

(4.3)

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In this case the GKZ system is given by five partial differential equations in three variables (see Appendix) and is of considerable complexity. Here we content ourselves with the analysis of prepotential at the tree and one-loop level ignoring the space-time instantons. This suffices for our purpose of extracting the $R \to \infty$ behavior of the theory. The small $R$ behavior has already been studied in \cite{7} and argued to reproduce Seiberg-Witten theory (we have also verified that the 1-instanton term is correctly reproduced).

When we ignore instantons, single log solutions are given by

$$-t_F \equiv \omega_F = \log(z_F) + \sum_{n=1}^{\infty} \frac{(2n-1)!}{n!^2} (z_F z_{F'})^n, \quad (4.4)$$

$$-t_{F'} \equiv \omega_{F'} = \log(z_{F'}) + \sum_{n=1}^{\infty} \frac{(2n-1)!}{n!^2} (z_F z_{F'})^n, \quad (4.5)$$

$$-t_B \equiv \omega_B = \log(z_B). \quad (4.6)$$

Identification with the variables of the gauge theory is given by

$$t_F = 2R(A - M), \quad t_{F'} = 2R(A + M), \quad e^{-t_B} = \frac{2R^3 \Lambda^3}{w}. \quad (4.7)$$

Then by inverting relations (4.4),(4.5) we find

$$z_F = \frac{e^{-2R(A-M)}}{1 + e^{-4RA}}, \quad z_{F'} = \frac{e^{-2R(A+M)}}{1 + e^{-4RA}}. \quad (4.8)$$

The double log solution

$$\Omega = t_F^2 + \frac{1}{2} t_{F'}^2 + 2t_F t_{F'} + t_B(t_F + t_{F'}) + \sum_{n=1}^{\infty} \frac{4(2n-1)!}{n!^2} \sum_{j=1}^{n} \frac{1}{j} (z_F z_{F'})^n + \left( - \sum_{n>\ell} + \sum_{n<\ell} \right) \frac{(n+\ell-1)!}{n! \ell!} \frac{1}{n-\ell} z_F^n z_{F'}^\ell, \quad (4.9)$$

can then be re-expressed as

$$\Omega = t_F^2 + \frac{1}{2} t_{F'}^2 + 2t_F t_{F'} + t_B(t_F + t_{F'}) + 4 \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-4nRA} - \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-2nR(A+M)} - \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-2nR(A-M)}, \quad (4.10)$$

using (4.8).

When we take the derivative in $A$, (4.10) gives

$$\tau = \frac{i}{\pi} \left[ 2 \log \sinh(2RA) - \frac{1}{4} \log \sinh(R(A+M)) \sinh(R(A-M)) \right]. \quad (4.11)$$
In the 4-dimensional limit (4.11) becomes
\[
\tau \approx \frac{i}{\pi} (2 - \frac{1}{2}) \log A
\]
which gives the 1-loop beta function of Seiberg-Witten theory. On the other hand in the 5-dimensional limit we find
\[
\tau_5 = \lim_{R \to \infty} \frac{\tau}{2\pi i R} = \frac{2}{\pi^2} \left[ A - \frac{1}{16} |A - M| - \frac{1}{16} |A + M| \right]
\]
as we have claimed before. We note that in formulas (3.2) and (4.13) \(\tau_5\) does not contain an additive constant which is identified as the bare coupling constant \(\tau_{5,B} = 1/g_{5,B}^2\). Thus the theory sits at the infinite bare coupling constant limit \(g_{5,B}^2 = \infty\) which yields non-trivial 5-dimensional theory in the infra-red regime [2].

We can similarly study the system with more matter up to \(N_f = 4\) using the local mirror symmetry. Curves of the B-model are given by (see Appendix)
\[
y^2 = (x^2 + sx + 1)^2 - K^{4-N_f} \prod_{i=1}^{N_f} (w_i x + \frac{1}{w_i}) , \tag{4.14}
\]
where parameters \(w_i\) correspond to the bare masses of the matter multiplets
\[
w_i = e^{RM_i}, \quad i = 1, \cdots, 4 \tag{4.15}
\]
(4.14) agrees with the curve suggested by [10] and [17] (at \(N_f = 4\) the factor \(K^{4-N_f}\) should be replaced by a dimensionless parameter \(q\)).

In the cases \(N_f \geq 2\) the analysis of GKZ system becomes further involved: we will instead use a simpler method based on the Picard-Fuchs (PF) equation derived for the elliptic curves (4.14). It turns out that the PF equation of the elliptic curve is an ordinary differential equation of third order in the variable \(A\) and can be studied relatively easily. This is the method used in ref [13]. This system, however, is not complete unlike that of GKZ case. The periods are determined only up to integration constants and one can not precisely fix the mirror transformation. This ambiguity, however, affects only the quantum part of the computation and one still obtains precise results for the prepotential at the tree and one-loop level.

First we note that the quadratic curve \(y^2 = ax^4 + 4bx^3 + 6cx^2 + 4dx + e\) is transformed into the Weierstrass form
\[
y^2 = 4x^3 - g_2 x - g_3 , \tag{4.16}
\]
by the relation
\[
g_2 = ae - 4bd + 3c^2, \quad g_3 = ace + 2bcd - ad^2 - b^2 e - c^3. \tag{4.17}
\]
We regard \( g_2, g_3 \) as functions of the parameter \( s \). Periods \( \omega \) of the elliptic curve \((4.16)\) obey the PF equation \([18, 19]\)

\[
\frac{d^2 \omega}{ds^2} + c_1(s) \frac{d\omega}{ds} + c_0(s) \omega = 0 ,
\]

where

\[
c_1 = - \frac{d}{ds} \log \left( \frac{3}{2\Delta} (2g_2 \frac{dg_3}{ds} - 3 \frac{dg_2}{ds} g_3) \right) , \tag{4.19}
\]

\[
c_0 = \frac{1}{12} c_1 \frac{d}{ds} \log \Delta + \frac{1}{12} \frac{d^2}{ds^2} \log \Delta - \frac{1}{16\Delta} \left( g_2 \left( \frac{dg_2}{ds} \right)^2 - 12 \left( \frac{dg_3}{ds} \right)^2 \right) \tag{4.20}
\]

and \( \Delta = (g_2)^3 - 27(g_3)^2 \) denotes the discriminant of the curve. Since \( dA/ds \) is one of the periods of the curve, it satisfies the PF equation. Regarding \( s \) as a function of \( A \), we obtain

\[
\frac{ds}{dA} d^3 s = 3(\frac{d^2 s}{dA^2})^2 + c_1(\frac{ds}{dA})^2 \frac{d^2 s}{dA^2} - c_0(\frac{ds}{dA})^4 = 0 . \tag{4.21}
\]

This determines \( s \) in terms of \( A \).

Similarly \( dA_D/ds = dA/ds \cdot d^2 F/dA^2 \) satisfies the PF equation and we obtain

\[
\left( \frac{ds}{dA} \right)^{-1} \frac{d^4 F}{dA^4} = 3(\frac{ds}{dA})^{-2} \frac{d^2 s}{dA^2} \frac{d^3 F}{dA^3} + c_1 \frac{d^3 F}{dA^3} = 0 . \tag{4.22}
\]

This equation can be integrated once and we find

\[
\frac{d^3 F}{dA^3} = \frac{\text{const.}}{\Delta} \left( 2g_2 \frac{dg_3}{ds} - 3 \frac{dg_2}{ds} g_3 \right) \left( \frac{ds}{dA} \right)^3 . \tag{4.23}
\]

Solving (4.23) determines the prepotential (const. in the right-hand-side is fixed by a suitable normalization of \( F \)).

In the case \( N_f = 2 \), we find the gauge coupling at the classical and one-loop level as

\[
\tau = \frac{i}{\pi} \left[ 2 \log \sinh(2RA) - \frac{1}{4} \prod_{i=1}^{2} \log \sinh(R(A + M_i)) \sinh(R(A - M_i)) \right] . \tag{4.24}
\]

In the 4-dimensional limit (4.24) reads as

\[
\tau \approx \frac{2i}{\pi} \left( 2 - \frac{1}{2} \cdot 2 \right) \log A , \tag{4.25}
\]

which gives the beta function of \( N_f = 2 \) Seiberg-Witten theory. On the other hand, in the five-dimensional limit we have

\[
\tau_5 = \frac{2}{\pi^2} \left( A - \sum_{i=1}^{2} \frac{1}{16} |A - M_i| - \sum_{i=1}^{2} \frac{1}{16} |A + M_i| \right) \tag{4.26}
\]
in agreement with (3.3).

We have checked that the local model of $F_2$ blown up at 3 and 4 points (see Appendix) also reproduce (3.3). Thus we have obtained a model which appears to describe correctly the physics of 5-dimensional theory up to the number of flavors $N_f = 4$ making use of the mirror symmetry.

5 Discussions

Since $F_2$ is obtained from $F_1$ which is a one-point blow up of $P^2$, we effectively have up to 5-point blow ups of $P^2$ describing the gauge theory. Unfortunately, beyond the 5-point blow up mirror symmetry of del Pezzo surfaces can not be described by toric geometry and we can not apply our analysis for these cases. In fact in the range $N_f \geq 5$ something drastic must happen, since in this asymptotically non-free region Seiberg-Witten solution does not exist and we should not have a smooth 4-dimensional limit. On the other hand, in this range the 5-dimensional gauge theories are expected to possess $E_n$, $n = 6, 7, 8$ global symmetries and are of particular interests. It is a challenging problem to clarify the physics of gauge theory with $E_n$ symmetry. It may shed some light on the nature of asymptotically non-free field theories in 4-dimensions.

Our result shows that from the point of view of the type II/M theory compactified on Calabi-Yau manifold, low energy 5-dimensional theory on $M_4 \times S^1$ emerges most naturally with its prepotential being exactly the same as that of the string theory. On the other hand, 4-dimensional Seiberg-Witten theory appears only in the fine-tuned limit of the Kähler parameters. It seems possible that our 5-dimensional model is an example of an “M-theoretic” lift of various 4-dimensional quantum field theories. We may imagine most of the 4-dimensional SUSY field theories in fact have a lift to 5-dimensions where the quantum effects of the loops and instantons are replaced by purely geometrical effects of world-sheet instantons. It will be also quite interesting to see if there is a further lift of quantum field theory to 6-dimensions as suggested by the duality between F- and M-theory.

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Appendix
A.1 $N_f = 0$

For pure Yang-Mills case we take the Hirzebruch surface $F_2$ lying inside a Calabi-Yau manifold. Its toric diagram has five vertices;

$$\nu_0 = (0, 0) \ , \ \nu_1 = (1, 0) \ , \ \nu_2 = (0, 1) \ , \ \nu_3 = (-1, 0) \ , \ \nu_4 = (-2, -1). \quad (A.1)$$

This is a two dimensional reflexive polyhedra no.4 in Fig.1 of [9]. The charge vectors satisfying the linear relation

$$\sum_i \ell_i^{(k)} \nu_i = 0 \quad (A.2)$$

are given by

$$\ell^{(F)} = (-2; 1, 0, 1, 0) \ , \ \ell^{(B)} = (0; 0, 1, -2, 1). \quad (A.3)$$

The constraint among B-model variables $Y_i, i = 0, 1, \cdots, 4$

$$\prod_{\ell_i^{(k)}>0} Y_i^{\ell_i^{(k)}} = \prod_{\ell_i^{(k)}<0} Y_i^{-\ell_i^{(k)}} \quad (A.4)$$

gives

$$Y_0^2 = Y_1 Y_3, \quad Y_2 Y_4 = Y_3^2, \quad (A.5)$$

and we have a solution

$$Y = (sx, x^2, \zeta, s^2, \frac{s^4}{\zeta}). \quad (A.6)$$

Set $s = 1$. Then we obtain the following curve in the B model side

$$P_{N_f=0} = \sum_i a_i Y_i, \quad = a_0 x + a_1 x^2 + a_2 \zeta + a_3 + a_4 \frac{1}{\zeta} = 0. \quad (A.7)$$

Introducing a new variable $y = a_2 \zeta - (a_4/\zeta)$, we can rewrite the curve as

$$y^2 = (a_1 x^2 + a_0 x + a_3)^2 - 4a_2 a_4. \quad (A.8)$$

For each charge vector $\ell^{(k)}$, a corresponding complex structure modulus is given by

$$z_k = \prod_i a_i^{\ell_i^{(k)}}. \quad (A.9)$$

In the present case we have

$$z_F = \frac{a_1 a_3}{a_0^2}, \quad z_B = \frac{a_2 a_4}{a_3^2}. \quad (A.10)$$
By setting 

\[ a_1 = a_3 = 1, \quad a_0 = s, \quad 4a_2a_4 = K^4, \quad (A.11) \]

we have

\[ z_F = \frac{1}{s^2}, \quad z_B = \frac{K^4}{4}. \quad (A.12) \]

GKZ system is defined by a system of differential operators

\[ \prod_{\ell_i^{(k)} > 0} \left( \frac{\partial}{\partial a_i} \right)^{\ell_i^{(k)}} = \prod_{\ell_i^{(k)} < 0} \left( \frac{\partial}{\partial a_i} \right)^{-\ell_i^{(k)}}, \quad (A.13) \]

In the \( F_2 \) case it is given by

\[ L_1 = z_F(4\theta z_F + 2\theta z_F) + \theta z_F(2\theta z_B - \theta z_F), \quad (A.14) \]
\[ L_2 = z_B(2\theta z_B - \theta z_F + 1)(2\theta z_B - \theta z_F) - \theta z_B^2, \quad (A.15) \]
\[ \theta z_i \equiv z_i \frac{\partial}{\partial z_i}, \quad i = F, B. \]

### A.2 \( N_f = 1 \)

It is known that matters in the vector representation are generated by blowing up the manifold (see, for instance [20]). We expect that a blow up of the local \( F_2 \) provides the description of gauge theory with \( N_f = 1 \) matter. The corresponding reflexive polyhedra is no.6 of [19] and given by the vertices

\[ \nu_0 = (0, 0), \quad \nu_1 = (1, 0), \quad \nu_2 = (0, 1), \]
\[ \nu_3 = (-1, 1), \quad \nu_4 = (-1, 0), \quad \nu_5 = (-1, -1). \quad (A.16) \]

We see that the charge vectors are

\[ \ell^{(B)} = (0; 0, 0, 1, -2, 1), \]
\[ \ell^{(F-E)} = (-1; 0, 1, -1, 1, 0), \]
\[ \ell^{(E)} = (-1; 1, -1, 1, 0, 0). \quad (A.17) \]

The constraint

\[ Y_3Y_5 = Y_4^2, \quad Y_0Y_3 = Y_2Y_4, \quad Y_0Y_2 = Y_1Y_3, \quad (A.18) \]

is solved by

\[ Y = (sx, sx^2, \frac{tx}{\zeta}, \frac{s}{\zeta}, s, \zeta s). \quad (A.19) \]

Setting \( s = 1 \) gives the curve

\[ P_{N_f=1} = a_0x + a_1x^2 + a_2\frac{x}{\zeta} + a_3\frac{1}{\zeta} + a_4 + a_5\zeta, \quad (A.20) \]
or
\[ y^2 = (a_1 x^2 + a_0 x + a_4)^2 - 4a_5(a_2 x + a_3) \, . \tag{A.21} \]

If we set
\[ a_1 = a_4 = 1 \, , \quad a_0 = s \, , \quad 4a_5 = K^3 \, , \quad a_2 = w \, , \quad a_3 = w^{-1} \, , \tag{A.22} \]

we obtain the curve [4]
\[ y^2 = (x^2 + sx + 1)^2 - K^3(wx + \frac{1}{w}) \tag{A.23} \]

Complex moduli are given by
\[ z_B = \frac{a_3 a_5}{a_4^2} = \frac{K^3}{4w} \, , \quad z_{F-E} = \frac{a_2 a_4}{a_0 a_3} = \frac{w^2}{s} \, , \quad z_{F+E} = \frac{a_1 a_3}{a_0 a_2} = \frac{1}{sw^2} \, . \tag{A.24} \]

Complete set of differential equations is given by [3]
\[ \mathcal{L}_1 = \theta_B(\theta_{F-E} + \theta_{F+E}) - z_B(2\theta_B - \theta_{F-E})(2\theta_B - \theta_{F-E} + 1) \, , \tag{A.25} \]
\[ \mathcal{L}_2 = (\theta_{F-E} - \theta_{F+E})(\theta_{F-E} - 2\theta_B) - z_{F-E}(\theta_{F-E} + \theta_{F+E})(\theta_{F-E} - \theta_B - \theta_{F+E}) \, , \tag{A.26} \]
\[ \mathcal{L}_3 = \theta_{F+E}(\theta_{F+E} + \theta_{F-E}) - z_{F+E}(\theta_{F+E} + \theta_{F-E})(\theta_{F+E} - \theta_{F-E}) \, , \tag{A.27} \]
\[ \mathcal{L}_4 = (\theta_{F-E} - \theta_{F+E})\theta_B - z_B z_{F-E}(\theta_{F-E} + \theta_{F+E})(2\theta_B - \theta_{F-E}) \, , \tag{A.28} \]
\[ \mathcal{L}_5 = \theta_{F+E}(\theta_{F-E} - 2\theta_B) - z_{F-E} z_{F+E}(\theta_{F-E} + \theta_{F+E})(\theta_{F-E} + \theta_{F+E} + 1) \, . \tag{A.29} \]

We note that the last two operators are necessary to obtain a unique double log solution \( \Omega \).

### A.3 \( N_f = 2 \)

We choose the following vertices
\[ \nu_0 = (0, 0) \, , \quad \nu_1 = (1, 0) \, , \quad \nu_2 = (0, 1) \, , \quad \nu_3 = (-1, 1) \, , \quad \nu_4 = (-1, 0) \, , \quad \nu_5 = (-1, -1) \, , \quad \nu_6 = (1, 1) \, . \tag{A.30} \]

This is a reflexive polyhedra no.8 of [3]. The curve is given by
\[ P_{N_f=2} = a_0 x + a_1 x^2 + a_2 \frac{x}{\zeta} + a_3 \frac{1}{\zeta} + a_4 + a_5 \zeta + a_6 \frac{x^2}{\zeta} = 0 \, , \tag{A.31} \]

or
\[ y^2 = (a_1 x^2 + a_0 x + a_4)^2 - 4a_5(a_6 x^2 + a_2 x + a_3) \, . \tag{A.32} \]

Substituting the relation
\[ a_1 = a_4 = 1 \, , \quad a_0 = s \, , \quad 4a_5 = K^2 \, , \quad a_6 = w_1 w_2 \, , \quad a_2 = \left( \frac{w_2}{w_1} + \frac{w_1}{w_2} \right) \, , \quad a_3 = (w_1 w_2)^{-1} \, . \tag{A.33} \]
we obtain the complex structure moduli

$z_B = \frac{a_3a_5}{a_4^2} = \frac{K^2}{4w_1w_2}, \quad z_{F_1} = \frac{a_2a_4}{a_0a_3} = \frac{w_1^2 + w_2^2}{s},$

$z_{F_2} = \frac{a_1a_3}{a_0a_2} = \frac{1}{s(w_1^2 + w_2^2)}, \quad z_{F_3} = \frac{a_1a_4}{a_0a_6} = \frac{w_1^2 + w_2^2}{sw_1w_2}.$  \hfill (A.34)

\textbf{A.4} \quad N_f = 3

Polyhedron for the 3-point blow up is obtained by adding a vertex $\nu_7 = (0, -1)$ to (A.30) (no. 12 of [9]). Elliptic curve is given by

$$P_{N_f=3} = a_0x + a_1x^2 + a_2x + a_3 + a_4x + a_5\zeta + a_6x^2 + a_7x\zeta = 0.$$  \hfill (A.35)

or

$$y^2 = (a_1x^2 + a_0x + a_4)^2 - 4(a_6x^2 + a_2x + a_3)(a_7x + a_5).$$  \hfill (A.36)

By choosing the variables as

$$a_0 = s, \quad a_1 = 1, \quad a_2 = K \frac{w_1}{4w_2} + w_2, \quad a_3 = K \frac{w_1}{4w_2}, \quad a_4 = 1, \quad a_5 = \frac{1}{w_3},$$

$$a_6 = \frac{Kw_1w_2}{4}, \quad a_7 = w_3,$$  \hfill (A.37)

we find the complex moduli

$$z_B = \frac{a_3a_5}{a_4^2} = \frac{K}{4w_1w_2w_3}, \quad z_{F_1} = \frac{a_2a_4}{a_0a_3} = \frac{1}{s(w_1^2 + w_2^2)}, \quad z_{F_2} = \frac{a_1a_3}{a_0a_2} = \frac{1}{s w_1^2 + w_2^2},$$

$$z_{F_3} = \frac{a_1a_4}{a_0a_6} = \frac{1}{s(w_1^2 + w_2^2)}, \quad z_{F_4} = \frac{a_1a_7}{a_0a_5} = \frac{w_2^2}{s}.$$  \hfill (A.38)

\textbf{A.5} \quad N_f = 4

Polyhedron for the 4-point blow up is obtained by further adding a vertex $\nu_8 = (1, -1)$ to the polyhedron of $N_f = 3$ (no. 15 of [9]). Elliptic curve is given by

$$P_{N_f=4} = a_0x + a_1x^2 + a_2x + a_3 + a_4 + a_5\zeta + a_6x^2 + a_7x\zeta + a_8x^2\zeta = 0.$$  \hfill (A.39)

By eliminating $\zeta$ it becomes

$$y^2 = (a_1x^2 + a_0x + a_4)^2 - 4(a_6x^2 + a_2x + a_3)(a_8x^2 + a_7x + a_5).$$  \hfill (A.40)
By choosing the variables as

\[ a_0 = s, \quad a_1 = 1, \quad a_2 = (w_1 + w_2)q, \quad a_3 = \frac{1}{w_1w_2} q, \quad a_4 = 1, \quad a_5 = \frac{1}{w_3w_4}, \]
\[ a_6 = w_1w_2q, \quad a_7 = \frac{w_3}{w_4} + \frac{w_4}{w_3}, \quad a_8 = w_3w_4 \quad (A.41) \]

we find the complex moduli

\[ z_B = \frac{a_3a_5}{a_4^2} = \frac{q}{w_1w_2w_3w_4}, \quad z_{F_1} = \frac{a_2a_4}{a_6a_3} = \frac{1}{s} (w_1^2 + w_2^2), \quad z_{F_2} = \frac{a_1a_3}{a_0a_2} = \frac{1}{s w_1^2 + w_2^2}, \]
\[ z_{F_3} = \frac{a_1a_2}{a_0a_6} = \frac{1}{s} \left( \frac{1}{w_1^2} + \frac{1}{w_2^2} \right), \quad z_{F_4} = \frac{a_4a_7}{a_0a_5} = \frac{w_3^2 + w_4^2}{s}, \quad z_{F_5} = \frac{a_1a_7}{a_0a_8} = \frac{1}{s} \left( \frac{1}{w_3^2} + \frac{1}{w_4^2} \right), \quad (A.42) \]
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