Asymptotics of quantum weighted Hurwitz numbers

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Abstract
This work concerns both the semiclassical and zero temperature asymptotics of quantum weighted double Hurwitz numbers. The partition function for quantum weighted double Hurwitz numbers can be interpreted in terms of the energy distribution of a quantum Bose gas with vanishing fugacity. We compute the leading semiclassical term of the partition function for three versions of the quantum weighted Hurwitz numbers, as well as lower order semiclassical corrections. The classical limit $\hbar \to 0$ is shown to reproduce the simple single and double Hurwitz numbers studied by Okounkov and Pandharipande (2000 \textit{Math. Res. Lett.} \textbf{7} 447–53, 2000 \textit{Lett. Math. Phys.} \textbf{53} 59–74). The KP-Toda $\tau$-function that serves as generating function for the quantum Hurwitz numbers is shown to have the $\tau$-function of Okounkov and Pandharipande (2000 \textit{Math. Res. Lett.} \textbf{7} 447–53, 2000 \textit{Lett. Math. Phys.} \textbf{53} 59–74) as its leading term in the classical limit, and, with suitable scaling, the same holds for the partition function, the weights and expectations of Hurwitz numbers. We also compute the zero temperature limit $T \to 0$ of the partition function and quantum weighted Hurwitz numbers. The KP or Toda $\tau$-function serving as generating function for the quantum Hurwitz numbers are shown to give the one for Belyi curves in the zero temperature limit and, with suitable scaling, the same holds true for the partition function, the weights and the expectations of Hurwitz numbers.

Keywords: quantum Hurwitz number, $\tau$-function, zero temperature limit, asymptotics, generating function, semiclassical

(Some figures may appear in colour only in the online journal)
1. Introduction: weighted Hurwitz numbers and their generating functions

1.1. Hurwitz numbers

Multiparametric weighted Hurwitz numbers were introduced in [9–11, 13] as generalizations of simple Hurwitz numbers [14, 15, 20, 22] and other special cases [1–4, 8, 16, 24] previously studied. In general, parametric families of KP or 2D Toda $\tau$-functions of hypergeometric type [17, 21] serve as generating functions for the weighted Hurwitz numbers, which appear as coefficients in an expansion over the basis of power sum symmetric functions in an auxiliary set of variables. The weights are determined by a parametric family of weight generating functions $G(z, c)$, with parameters $c = (c_1, c_2, \ldots)$ that can either be expressed as a formal sum

$$G(z) = 1 + \sum_{i=1}^{\infty} g_iz^i$$ (1.1)

or an infinite product

$$G(z) = \prod_{i=1}^{\infty} (1 + zc_i),$$ (1.2)

or some limit thereof. Comparing the two formulae, $G(z)$ can be interpreted as the generating function for elementary symmetric functions in the variables $c = (c_1, c_2, \ldots)$.

$$g_i = e_i(c).$$ (1.3)

Another parametrization considered in [9–11, 13], consists of weight generating functions of the form

$$\tilde{G}(z) = \prod_{i=1}^{\infty} (1 - zc_i)^{-1}.$$ (1.4)

The corresponding power series expansions

$$\tilde{G}(z) = 1 + \sum_{i=1}^{\infty} \tilde{g}_iz^i$$ (1.5)

can similarly be interpreted as defining the complete symmetric functions

$$\tilde{g}_i = h_i(c).$$ (1.6)

Pure Hurwitz numbers $H(\mu^{(1)}, \ldots, \mu^{(k)})$ may be defined in one of two equivalent ways: geometrical or combinatorial. The geometrical definition is:

**Definition 1.1.** For a set of $k$ partitions $(\mu^{(1)}, \ldots, \mu^{(k)})$ of $n$, $H(\mu^{(1)}, \ldots, \mu^{(k)})$ is the number of distinct $n$-sheeted branched coverings $\Gamma \rightarrow \mathbb{P}^1$ of the Riemann sphere having $k$ branch points $(p_1, \ldots, p_k)$ with ramification profiles $\{\mu^{(i)}\}_{i=1,\ldots,k}$ divided by the order aut($\Gamma$) of the automorphism group of $\Gamma$.

The combinatorial definition is:

**Definition 1.2.** $H(\mu^{(1)}, \ldots, \mu^{(k)})$ is the number of distinct factorization of the identity element $I \in S_n$ of the symmetric group as an ordered product

$$I = h_1, \ldots, h_k, \quad h_i \in S_n, \quad i = 1, \ldots, k$$ (1.7)
where $h_i$ belongs to the conjugacy class with cycle lengths equal to the parts of $\mu^{(i)}$, divided by $n!$.

The fact that these coincide [10, 11] follows from the monodromy representation of the fundamental group of the sphere minus the branch points mapped into the symmetric group $S_n$. Let $P_n$ denote the set of integer partitions of $n$ and $p(n)$ its cardinality. The Frobenius–Schur formula [5, 6, 18, 23] expresses the Hurwitz numbers in terms of the irreducible characters of $S_n$

$$H(\mu^{(1)}, \ldots, \mu^{(k)}) = \sum_{\lambda \in P_n} h^{-1}_\lambda \prod_{i=1}^k z^{-1}_{\mu^{(i)}} \chi_\lambda(\mu^{(i)}),$$

(1.8)

where $\chi_\lambda(\mu)$ is the irreducible character of the representation with Young symmetry class $\lambda$ evaluated on the conjugacy class with cycle lengths equal to the parts of $\mu$; $h_\lambda$ is the product of hook lengths of the Young diagram of partition $\lambda$ and

$$z_{\mu} = \prod_{i=1}^{\ell(\mu)} m_i(\mu) p^{m(\mu)}$$

(1.9)

is the order of the stabilizer of any element of the conjugacy class $\mu$, with $m_i(\mu)$ equal to the number of times $i$ appears as a part of $\mu$. We denote the weight of a partition $|\mu|$, its length $\ell(\mu)$ and define its colength as

$$\ell^*(\mu) := |\mu| - \ell(\mu).$$

(1.10)

1.2. Weighted Hurwitz numbers

Following [9–11, 13] we define, for each positive integer $d$ and every pair of ramification profiles $(\mu, \nu)$ (i.e. partitions of $n$), the weighted double Hurwitz number

$$H^d_G(\mu, \nu) := \sum_{k=0}^\infty \sum_{\sum_{i=1}^k \ell^*(\mu^{(i)}) = d} \sum' m_{\lambda}(\mathbf{c}) H(\mu^{(1)}, \ldots, \mu^{(k)}, \mu, \nu),$$

(1.11)

where

$$m_{\lambda}(\mathbf{c}) := \frac{1}{|\text{aut}(\lambda)|} \sum_{\sigma \in S_k} \sum_{1 \leq i_1 < \cdots < i_k} c_{\sigma(1)}^{\lambda_1} \cdots c_{\sigma(k)}^{\lambda_k},$$

(1.12)

is the monomial sum symmetric function [19] corresponding to a partition $\lambda$ of weight

$$|\lambda| = d = \sum_{i=1}^\ell \ell^*(\mu^{(i)}),$$

(1.13)

whose parts $\{\lambda_i\}_{i=1, \ldots, \ell}$ are the colengths $\{\ell^*(\mu^{(i)})\}_{i=1, \ldots, \ell}$ in weakly descending order,

$$|\text{aut}(\lambda)| := \prod_{i=1}^{\ell(\lambda)} m(\lambda_i)!,$$

(1.14)
and \( \sum' \) denotes the sum over all \( k \)-tuples of partitions \((\mu^{(1)}, \ldots, \mu^{(k)})\) satisfying condition (1.13) other than the cycle type of the identity element. By the Riemann–Hurwitz formula, the Euler characteristic of the covering surface is

\[
\chi = 2 - 2g = \ell(\mu) + \ell(\nu) - d. \tag{1.15}
\]

For weight generating functions of the form (1.4), the weighted double Hurwitz number is defined as:

\[
H^d_G(\mu, \nu) := \sum_{k=0}^{\infty} \sum'_{\sum_{i=1}^{k} e^*(\mu^{(i)}):=d} f_\lambda(c) H^{(1)}(\ldots, \mu^{(k)}, \mu, \nu) \tag{1.16}
\]

where

\[
f_\lambda(c) := \frac{(-1)^{e^*(\lambda)}}{|\text{aut}(\lambda)|} \sum_{\sigma \in S_k} \sum_{1 \leq i_1 \leq \cdots \leq i_k} c_{\sigma_{i_1}(1)}^{\lambda_{i_1}} \cdots c_{\sigma_{i_k}(k)}^{\lambda_{i_k}}, \tag{1.17}
\]

is the ‘forgotten’ symmetric function [19].

The particular case where all the \( \mu_i \)’s represent simple branching (i.e. where they are all 2-cycles) was studied in [20, 22] and corresponds to the exponential weight generating function

\[
G = \exp, \quad G(z) = e^z = \lim_{k \to \infty} \left(1 + \frac{z}{k}\right)^k. \tag{1.18}
\]

The evaluation of the monomial sum symmetric function in this limit is

\[
\lim_{k \to \infty} m_{\lambda} \left(\frac{1}{m}, \ldots, \frac{1}{m}, 0, 0, \cdots\right) = \delta_{\lambda, (2, 1)^{n-2}}, \tag{1.19}
\]

so the weight is uniform on all \( k \)-tuples \((\mu^{(1)}, \ldots, \mu^{(k)})\) of partitions corresponding to simple branching

\[
\mu^{(i)} = (2, (1)^{n-2}) \tag{1.20}
\]

and vanishes on all others. The corresponding weighted Hurwitz numbers \( H^d_{\exp}(\mu, \nu) \) are what we refer to as the ‘classical’ simple (double) Hurwitz numbers.

1.3. The 2D-Toda \( \tau \)-function as generating function

Choosing a small parameter \( \beta \), the following double Schur function expansion defines a 2D-Toda \( \tau \)-function of hypergeometric type [21] (at the lattice point \( N = 0 \)).

\[
\tau^{(G, \beta)}(t, s) = \sum_{\lambda} r^{(G, \beta)}_{\lambda} s_{\lambda}(t)s_{\lambda}(s), \tag{1.21}
\]

where the coefficients \( r^{(G, \beta)}_{\lambda} \) are defined in terms of the weight generating function \( G \) by the following \textit{content product} formula

\[
r^{(G, \beta)}_{\lambda} := \prod_{(i,j) \in \lambda} G(\beta(j - i)). \tag{1.22}
\]
The same formulae apply *mutatis mutandis* with the replacement \( G \rightarrow \tilde{G} \) for the case of the second type of weight generating function \( \tilde{G} \) defined by (1.4).

Changing the expansion basis from diagonal products Schur functions to products \( p_\mu(t) p_\nu(s) \) of power sum symmetric functions, using the standard Frobenius character formula \([7, 19]\),

\[
s_\lambda = \sum_{\mu, |\mu|=|\lambda|} \chi_\lambda(\mu) p_\mu, \tag{1.23}
\]

it follows \([9–11, 13]\) that \( \tau^{(G, \beta)}(t, s) \) is interpretable as a generating function for the weighted double Hurwitz numbers \( H^d_G(\mu, \nu) \).

**Theorem 1.1** \([9–11, 13]\). The 2D Toda \( \tau \)-function \( \tau^{(G, \beta)}(t, s) \) can be expressed as

\[
\tau^{(G, \beta)}(t, s) = \sum_{d=0}^{\infty} \beta^d \sum_{|\mu|=|\nu|} H^d_G(\mu, \nu) p_\mu(t) p_\nu(s) \tag{1.24}
\]

and the same formula holds under the replacement \( G \rightarrow \tilde{G} \).

The case of the exponential weight generating function (1.18) gives the following content product coefficient in the \( \tau \)-function expansion (1.21)

\[
r^{(\text{exp}, \beta)}_\lambda = e^{\frac{\beta}{2} \sum_{i=1}^{l(\lambda)} \lambda_i (\lambda_i - 2i + 1)}, \tag{1.25}
\]

as in \([20]\), and the generating function expansion (1.24) becomes

\[
\tau^{(\text{exp}, \beta)}(t, s) = \sum_{k=0}^{\infty} \frac{\beta^k}{k!} \sum_{|\mu|=|\nu|} H^d_{\text{exp}}(\mu, \nu) p_\mu(t) p_\nu(s), \tag{1.26}
\]

where

\[
H^d_{\text{exp}}(\mu, \nu) := H((2, (1)^{n-2}), \ldots, (2, (1)^{n-2})), \tag{d times}
\]

1.4. Quantum Hurwitz numbers: relation to Bose gases

1.4.1. Quantum Hurwitz numbers. A special case of weighted Hurwitz numbers consists of *pure quantum Hurwitz numbers* \([10, 12]\), which are obtained by choosing the parameters \( c_i \) as

\[
c_i = q^i, \quad i = 1, 2, \ldots \tag{1.28}
\]

where \( q \) is a real parameter between 0 and 1. The justification for naming these *quantum Hurwitz numbers* is the relation, under suitable identification of parameters, of the energy distribution of the various branched configurations to that for a quantum Bose gas with linear energy spectrum, The classical limit will be seen to coincide with the case of the Dirac measure supported on the space of simple branchings of type (1.20), as studied by Okounkov and Pandharipande \([20, 22]\).

The corresponding weight generating function is

\[
G(z) = E^*(q, z) := \prod_{i=1}^{\infty} (1 + q^i z) = (-zq; q)_\infty := 1 + \sum_{i=0}^{\infty} E_i(q) z^i, \tag{1.29}
\]
\[ E_i'(q) := \frac{q^{i(i+1)}_1}{\prod_{j=1}^{i}(1-q^j)} = \frac{q^{i(i+1)}_1}{(q;q)_i}, \quad i \geq 1, \]  
(1.30)

where

\[ (z; q)_k := \prod_{j=0}^{k-1}(1 - zq^j), \quad (z; q)_\infty := \prod_{j=0}^{\infty}(1 - zq^j) \]  
(1.31)

is the quantum Pochhammer symbol. This is related to the quantum dilogarithm function by

\[
(1 + z)E'(q, z) = e^{-\sum_{k=0}^{\infty} \frac{z^k}{k(1-q^k)}}, \quad \text{Li}_2(q, z) := (1 - q) \sum_{k=1}^{\infty} \frac{z^k}{k(1-q^k)}. 
\]  
(1.32)

We thus have

\[ e_{\lambda}(e) \equiv E'(q, z) = E'(q) = \prod_{i=1}^{\ell(\lambda)} \frac{q^{1_{\lambda}(\lambda+1)}_1}{\prod_{j=1}^{\ell(\lambda)}(1 - q^j)} = \prod_{i=1}^{\ell(\lambda)} \frac{q^{1_{\lambda}(\lambda+1)}_1}{(q;q)_{\lambda(\lambda+1)}}. \]  
(1.33)

The content product coefficients entering in the \( \tau \)-function (1.21) for this case are

\[ r_j^{(E'(q), \beta, \lambda)}(z) = \prod_{k=1}^{\infty}(1 + q^j \beta^j) = (-q^j \beta^j; q)_\infty. \]  
(1.34)

\[ r_\lambda^{(E'(q), \beta, \lambda)} = \prod_{k=1}^{\infty}(1 + q^j \beta^j(z - i)) = \prod_{(i, j) \in \lambda} (-q^j \beta^j(z - i); q)_\infty. \]  
(1.35)

Making the substitutions (1.28), the weights entering in (1.11) evaluate to

\[ W_{E'}(\mu^{(1)}, \ldots, \mu^{(k)}) := m_\lambda(q, q^2, \ldots) \]  
\[ = \frac{1}{|\text{aut}(\lambda)|} \sum_{\sigma \in S_\lambda} (q^{E_{\mu^{(1)}}(\mu^{(1)})} - 1) \cdots (q^{E_{\mu^{(k)}}(\mu^{(k)})} - 1). \]  
(1.36)

The (unnormalized) weighted Hurwitz numbers therefore become

\[ H^d_{E'}(\mu, \nu) := \sum_{k=0}^{\infty} \sum_{\sum_{i=1}^{k} \mu^{(i)} = \mu} W_{E'}(\mu^{(1)}, \ldots, \mu^{(k)}) H(\mu^{(1)}, \ldots, \mu^{(k)}, \mu, \nu). \]  
(1.37)

Choosing \( G = E'(q) \) in equations (1.21) and (1.24), we obtain

\[ \tau^{(E'(q), \beta)}(t, s) = \sum_{\lambda} \mathcal{R}_{\lambda}^{(E'(q), \beta)}(z) s_\lambda(t) s_\lambda(s) \]  
\[ = \sum_{d=0}^{\infty} \sum_{|\mu| = |\nu|} H^d_{E'}(\mu, \nu) p_\mu(t) p_\nu(s) \]  
(1.38)

as generating function for simple quantum Hurwitz numbers.
1.4.2. Relation to Bose gas. The parameter $q$ may be interpreted as $q = e^{-\epsilon}$, for a small positive parameter $\epsilon$ identified as

$$\epsilon = \frac{\hbar \omega_0}{k_B T}, \quad T = \text{temperature}, \quad k_B = \text{Boltzman constant},$$

where $\hbar \omega_0$ is interpreted as a ground state energy (i.e. no branching), while the higher levels $\epsilon(\mu)$ are integer multiples proportional to the colength of the partition representing the ramification type of a branch point; i.e. the degree of degeneration of the sheets

$$\epsilon(\mu) = \ell^*(\mu) \epsilon.$$ (1.40)

The weight $W_{E'}(q)(\mu^{(1)}, \ldots, \mu^{(k)})$ for a branching configuration of type $(\mu^{(1)}, \ldots, \mu^{(k)}, \mu, \nu)$ is then

$$W_{E'}(q)(\mu^{(1)}, \ldots, \mu^{(k)}) = \frac{1}{|\text{aut}(\lambda)|} \sum_{\sigma \in S_k} \left( e^{\epsilon(\sigma^{(1)}(\mu^{(1)}))} - 1 \right) \cdots \left( e^{\sum_{i=1}^k \epsilon(\mu^{(i)}))} - 1 \right).$$ (1.41)

The weight $W_{E'}(q)(\mu^{(1)}, \ldots, \mu^{(k)})$ defined in (1.36) may thus be interpreted in terms of the distribution of a quantum boson gas with vanishing fugacity

$$n_{\epsilon(\mu)} = \frac{1}{e^{\epsilon(\mu)} - 1},$$ (1.42)

assuming that the energy for $k$ branch points $(\mu^{(1)}, \ldots, \mu^{(k)})$ is the sum of that for each

$$\epsilon(\mu^{(1)}, \ldots, \mu^{(k)}) = \sum_{i=1}^k \epsilon(\mu^{(i)}))$$ (1.43)

and that the total weight for configurations with up to $k$ weighted branch points plus two unweighted ones, with profiles $(\mu, \nu)$, is the sum over the products of those for $i = 1, 2, \ldots k$ branch points.

1.4.3. Other variants of quantum Hurwitz numbers. Another variant on the weight generating function for quantum Hurwitz numbers consists of choosing the parameters $c = (c_1, c_2, \ldots)$ in (1.2) to be

$$c_i := q^{i-1},$$ (1.44)

which gives

$$G(z) = E(q, z) := (-qz: q)_{\infty}.\quad$$ (1.45)

This is related to the quantum dilogarithm function by

$$E(q, z) = e^{-\frac{\ln(q-z)}{q}}.$$ (1.46)

The content product coefficients entering in the $\tau$-function (1.21) for this case are

$$r_j^{\tau}(q)(z) = \prod_{k=0}^\infty (1 + q^k z^j) = (-z^j; q)_{\infty},$$ (1.47)

$$r_\lambda^{\tau}(q)(z) = \prod_{k=0}^\infty \prod_{(i,j) \in \lambda} (1 + q^k z(j - i)) = \prod_{(i,j) \in \lambda} (-z(j - i); q)_{\infty}.\quad$$ (1.48)
The weights entering in (1.11) evaluate to
\[
W_{E(q)}(\mu^{(1)}, \ldots, \mu^{(k)}) := \frac{1}{|\text{aut}(\lambda)|} \sum_{\sigma \in S_\lambda} \sum_{0 \leq i_1 < \cdots < i_k} q^{i_1 \sigma^{(1)}} \cdots q^{i_k \sigma^{(k)}}
\]
and the weighted Hurwitz numbers therefore become
\[
H^d_{E(q)}(\mu, \nu) := \sum_{k=0}^{\infty} \sum_{\sum^{(k)}_{i=1} \mu^{(i)} = d} W_{E(q)}(\mu^{(1)}, \ldots, \mu^{(k)}) H(\mu^{(1)}, \ldots, \mu^{(k)}, \mu, \nu).
\]

A third variant on the weight generating function for quantum Hurwitz numbers consists of choosing it of the form (1.4) with parameters \(c = (c_1, c_2, \ldots)\) again chosen as in (1.44). This gives
\[
G(z) = H(q, z) := \prod_{k=0}^{\infty} (1 - q^k z)^{-1} = \frac{1}{(-z; q)_\infty} = q^{\text{Li}_2(q, z)} = \sum_{i=0}^{\infty} H_i(q) z^i.
\]

1.5. Asymptotics for quantum Hurwitz numbers

1.5.1. Classical asymptotics for quantum Hurwitz numbers. With the identification \(q = e^{-\epsilon}\) and \(\epsilon := \frac{h \omega}{4\pi}\) viewed as a small positive number, taking the limit \(\epsilon \to 0^+\) (or \(q \to 1^-\)) of the scaled quantum dilogarithm function \(\text{Li}_2(q, \epsilon z)\) gives
\[
\lim_{\epsilon \to 0^+} \frac{\text{Li}_2(e^{-\epsilon}, \epsilon z)}{1 - e^{-\epsilon}} = z.
\]
It follows that all three generating functions \( E(q, z), E'(q, z) \) and \( H(q, z) \) have as scaled limits the generating function for the Okounkov–Pandharipande simple (single and double) Hurwitz numbers
\[
\lim_{\epsilon \to 0^+} E(q, \epsilon z) = \lim_{\epsilon \to 0^+} E'(q, \epsilon z) = \lim_{\epsilon \to 0^+} H(q, \epsilon z) = e^\epsilon. \tag{1.59}
\]

The corresponding scaled limit of the generating \( \tau \)-functions for all three versions of quantum weighted Hurwitz numbers therefore coincides with the generating function for simple Hurwitz numbers considered in \([20, 22]\)
\[
\lim_{\epsilon \to 0^+} \tau_{E(q, \epsilon \beta)}(t, s) = \lim_{\epsilon \to 0^+} \tau_{E'(q, \epsilon \beta)}(t, s) = \lim_{\epsilon \to 0^+} \tau_{H(q, \epsilon \beta)}(t, s) = \tau_{\exp, \beta}(t, s). \tag{1.60}
\]

Equivalently, this implies the limit
\[
\lim_{\epsilon \to 0^+} e^\epsilon dH_{E'(q, \epsilon \beta)}^d = H_{E'(q=\epsilon \beta)}^d. \tag{1.61}
\]

(see theorem 3.5 and remark 3.2).

### 1.5.2. Zero temperature asymptotics for quantum Hurwitz numbers.

The zero temperature limit \( T \to 0 \), on the other hand, corresponds to \( q \to 0^+ \). The suitably scaled limit of the generating function \( E'(q, z) \) is
\[
\lim_{\epsilon \to \infty} E'(q = e^{-\epsilon}, ze^\epsilon) = 1 + z, \tag{1.62}
\]
and the corresponding limit of the \( \tau \)-function is
\[
\lim_{\epsilon \to \infty} \tau_{E'(q=e^{-\epsilon}, ze^\epsilon)}(t, s) = \tau_{E, \beta}(t, s) \tag{1.63}
\]
where the weight generating function \( E \) is
\[
E(z) = 1 + z. \tag{1.64}
\]

This is the generating function for uniformly weighted Hurwitz numbers supported on curves with just three branch points \((\mu(1), \mu, \nu)\) \([10, 11]\); i.e. those with \( k = 1 \), sometimes referred to as Belyi curves \([1, 16, 24]\). (see theorem 4.1 and remark 4.1).

### 2. Probabilistic approach to quantum Hurwitz numbers

Since \( W_{E'(q)}(\mu^{(1)}, \ldots, \mu^{(k)}) \) is real, positive and normalizable, we can interpret \( H_{E'(q)}^d \) in terms of an expectation value. For \( k \in \{1, \ldots, d\} \) consider the (finite) set of \( k \)-tuples
\[
\mathcal{M}_{d,k}^{(n)} = \left\{ (\mu^{(1)}, \ldots, \mu^{(k)}) \in (\mathbb{P}_d)^k : \sum_{j=1}^k \ell^\beta \left( \mu^{(j)} \right) = d \right\} \tag{2.1}
\]
and their disjoint union
\[
\mathcal{M}_d^{(n)} = \prod_{k=1}^d \mathcal{M}_{d,k}^{(n)} \tag{2.2}
\]

Define a measure \( \theta_{E'(q)}^{(n,d)} \) on \( \mathcal{M}_d^{(n)} \) by
\[ \theta^{(n,d)}_{E(q)} \left( \left( \mu^{(1)}, \ldots, \mu^{(k)} \right) \right) = \frac{1}{Z^{(n,d)}_{E(q)}} W^{(n,d)}_{E(q)} \left( \mu^{(1)}, \ldots, \mu^{(k)} \right), \]  

(2.3)

where the partition function \( \tilde{Z}_{E(q)}^{(n,d)} \) is defined so that \( \theta^{(n,d)}_{E(q)} \) is a probability measure; that is,

\[ \tilde{Z}_{E(q)}^{(n,d)} = \sum_{k=1}^{d} \sum_{\lambda \in \mathcal{M}^{(n)}_{d}} W_{E(q)}^{(n,d)} \left( \mu^{(1)}, \ldots, \mu^{(k)} \right). \]

(2.4)

We then have the expectation value

\[ \langle H (\cdot, \ldots, \cdot, \mu, \nu) \rangle_{\tilde{Z}^{(n,d)}_{E(q)}} = \frac{1}{Z^{(n,d)}_{E(q)}} \tilde{H}_{E(q)}^{d}(\mu, \nu), \]

(2.5)

where \( \langle \cdot \rangle_{\tilde{Z}^{(n,d)}_{E(q)}} \) denotes integration with respect to the measure \( \tilde{\theta}^{(n,d)}_{E(q)} \).

**Definition 2.1.** For \( n, d \in \mathbb{Z}_{>0} \) define the function \( \Lambda^{(n)}_{d} : \mathcal{M}^{(n)}_{d} \rightarrow \mathcal{P}_{d} \) as follows:

\[ \Lambda^{(n)}_{d} : \left( \mu^{(1)}, \ldots, \mu^{(k)} \right) \mapsto \lambda \]

(2.6)

where \( \lambda \) is the unique partition of \( d \) such that

\[ \{ \lambda_1, \ldots, \lambda_k \} = \left\{ \ell^{*} (\mu^{(1)}), \ldots, \ell^{*} (\mu^{(k)}) \right\}. \]

(2.7)

The weight of the partition \( \Lambda^{(n)}_{d} (\mu^{(1)}, \ldots, \mu^{(k)}) \) is thus the sum \( d \) of colengths of the partitions \( \{ \mu^{(1)}, \ldots, \mu^{(k)} \} \)

\[ d = \sum_{i=1}^{k} \ell^{*} (\mu^{(i)}). \]

(2.8)

Letting \( \mathcal{P}_{n,k} \) denote the set of partitions of \( n \) with \( k \) parts, the image of \( \mathcal{M}^{(n)}_{d} \) under \( \Lambda^{(n)}_{d} \) is thus \( \mathcal{P}_{n,k} \).

Since \( W_{E(q)}^{(n,d)} (\mu^{(1)}, \ldots, \mu^{(k)}) \) depends on the partitions \( \{ \mu^{(1)}, \ldots, \mu^{(k)} \} \) only through their colengths, it makes sense to consider the push-forward

\[ \tilde{\xi}^{(n,d)}_{E(q)} = \left( \Lambda^{(n)}_{d} \right)_{*} \theta^{(n,d)}_{E(q)} \]

(2.9)

of \( \tilde{\theta}^{(n,d)}_{E(q)} \) under \( \Lambda^{(n)}_{d} \) (as a measure on \( \mathcal{P}_{d} \)). Let \( p(n, k) := |\mathcal{P}_{n,k}| \) denote the cardinality of \( \mathcal{P}_{n,k} \) and observe that, for any \( \lambda \in \mathcal{P}_{d} \),

\[ \left( \Lambda^{(n)}_{d} \right)^{-1} (\lambda) = \prod_{j=1}^{\ell(\lambda)} p(n, n - \lambda_j). \]

(2.10)

Therefore

\[ \tilde{\xi}^{(n,d)}_{E(q)} (\lambda) = \frac{1}{Z^{(n,d)}_{E(q)}} \left( \prod_{j=1}^{\ell(\lambda)} p(n, n - \lambda_j) \right) w_{E(q)}^{(n,d)} (\lambda) \]

(2.11)

where \( w_{E(q)}^{(n,d)} \) is defined as the weight function \( w_{E(q)} : \mathcal{P}_{d} \rightarrow [0, \infty) \) satisfying
\[ w_{E(q)}^\prime(\lambda) = \frac{\Phi_{E(q)}(\lambda_1, \ldots, \lambda_{\ell(\lambda)})}{|\text{aut}(\lambda)|} \tag{2.12} \]

with \( \Phi_{E(q)} : \prod_{m \in \mathbb{N}} \mathbb{R}^m \to \mathbb{R} \) defined by
\[ \Phi_{E(q)}(x_1, \ldots, x_m) = \sum_{\sigma \in S_m} \prod_{j=1}^m (q^{-\sum_{i=1}^j x_{\sigma(i)} - 1})^{-1}. \tag{2.13} \]

**Lemma 2.1.** For any \( n, \ell \in \mathbb{N} \) with \( n \geq 2\ell \) we have
\[ p(n, n-\ell) = p(\ell). \tag{2.14} \]

The proof of this lemma is given in section 5. From now on we always assume that \( n \geq 2d \).

We also denote
\[ p(\lambda) = \prod_{j=1}^{\ell(\lambda)} p(\lambda_j). \tag{2.15} \]

From the above discussion and lemma 2.1 we have the following result. For \( d \in \mathbb{Z}_{>0} \) and \( q \in (0, 1) \) let
\[ Z_{E(q)}^{(d)} := \sum_{\lambda \in \mathcal{P}_d} p(\lambda) w_{E(q)}(\lambda) \tag{2.16} \]
and define a probability measure on \( \mathcal{P}_d \) by
\[ \xi_{E(q)}^{(d)}(\lambda) := \frac{1}{Z_{E(q)}^{(d)}} \, p(\lambda) w_{E(q)}(\lambda) \quad \forall \lambda \in \mathcal{P}_d. \tag{2.17} \]

**Proposition 2.2.** Let \( n, d \in \mathbb{Z}_{>0} \) with \( n \geq 2d \). Then

1. The partition function \( \tilde{Z}_{E(q)}^{(n,d)} \) does not depend on \( n \);
\[ \tilde{Z}_{E(q)}^{(n,d)} = Z_{E(q)}^{(d)} \tag{2.18} \]

2. The probability measure \( \tilde{\xi}_{E(q)}^{(n,d)} \) does not depend on \( n \): for any \( \lambda \in \mathcal{P}_d \)
\[ \tilde{\xi}_{E(q)}^{(n,d)}(\lambda) = \xi_{E(q)}^{(d)}(\lambda) \tag{2.19} \]

We conclude this section by explaining how this extends to the other two quantum weight generating functions \( E(q) \) and \( H(q) \).

**Definition 2.2.** Define probability measures \( \theta_{E(q)}^{(n,d)} \) and \( \theta_{H(q)}^{(n,d)} \) on \( \mathcal{P}_d \) as in (2.3) and (2.4), replacing \( W_{E(q)} \) by \( W_{E(q)} \) and \( W_{H(q)} \) respectively, whenever it occurs.

Equations (2.5), (2.9)–(2.13) and (2.16)–(2.19) apply *mutatis mutandis*, replacing \( E(q) \) by \( E(q) \) and \( H(q) \) respectively.
3. Semiclassical limits and asymptotic expansion

In this section we state our asymptotic results for the semiclassical limit \( q \rightarrow 1^- \). All proofs are given in the section 5.

3.1. Classical limit

We begin by stating the classical limits (i.e. the leading term).

**Definition 3.1.** The Dirac measure \( \delta_x \) at \( x \in S \) on a measurable space \((S, \Sigma)\) is defined by

\[
delta_x(A) = \begin{cases} 
1 & \text{if } x \in A \\
0 & \text{otherwise}
\end{cases}
\]

for all \( A \in \Sigma \).

Recall that \( m_i(\lambda) \) denotes the number of parts of \( \lambda \) equal to \( i \). We can then identify the partition alternatively as

\[
\lambda = \left( 1^{m_1(\lambda)}, 2^{m_2(\lambda)}, \ldots \right).
\]

We also use the following notation for partitions with at most two different part lengths \((\ell \in \mathbb{Z}^+, 1)\), such that

\[
m_\ell(\lambda) = m, \quad m_1(\lambda) = n - m.
\]

**Definition 3.2.** We denote such a partition as

\[
\ell^m_n := \left( 1^{n-m}, \ell^m \right).
\]

When the weight of the partition is clear from context we simply write \( \ell^m := \ell^m_n \). When \( m = 1 \) we write \( \ell^1_n := \ell^1 \), or simply \( \ell \) (see figure 1).

**Theorem 3.1.** Let \( d \in \mathbb{Z}_{>0} \). As \( q \rightarrow 1^- \), each of the sequence of measures \( \left( \xi^{(d)}_{E(q)} \right)_{q < 1} \) and \( \left( \xi^{(d)}_{H(q)} \right)_{q < 1} \) on \( \mathcal{P}_d \) converges weakly to the Dirac measure \( \delta_{\{1^d\}} \) at \( \{1^d\} \in \mathcal{P}_d \).
By the discussion in section 3 this translates to a convergence result on $\mathfrak{M}^{(n)}_d$:

**Corollary 3.2.** If $d \geq 2n$ then each of the sequence of measures $\theta^{(n,d)}_{E(q)}$, $\theta^{(n,d)}_{E(q)}$ and $\theta^{(n,d)}_{H(q)}$ on $\mathfrak{M}^{(n)}_d$ converges weakly, as $q \rightarrow 1^-$, to the Dirac measure at $(\underbrace{2, \ldots, 2}_d$ terms) (in the notation of (3.4))

**Remark 3.1.** Observe that, from (1.19), the limiting measure in corollary 3.2 corresponds to the Okounkov–Pandharipande measure [20, 22].

### 3.2. Semiclassical corrections

We now turn to the next order term in the semiclassical asymptotics. Throughout we set $q = e^{-\epsilon}$ and let $\epsilon \rightarrow 0^+$. We begin by giving the asymptotic expansion for each weight. For any $\lambda \in \mathcal{P}_d$ define

$$w_0(\lambda) = \sum_{\sigma \in S(\lambda)} \prod_{j=1}^{\ell(\lambda)} \frac{1}{\sum_{i=1}^{\lambda_j} \sigma(i)} \quad (3.5)$$

$$w_1(\lambda) = \frac{1}{2} \sum_{\sigma \in S(\lambda)} \sum_{r=1}^{\ell(\lambda)} \frac{\lambda_{\sigma(i)}}{\prod_{j=1}^{\ell(\lambda)} \sum_{i=1}^{\lambda_j} \sigma(i)} \quad (3.6)$$

**Theorem 3.3.** For any $\lambda \in \mathcal{P}_d$ we have

$$e^{-\ell(\lambda)}w_{E(q)}(e^{-\epsilon})(\lambda) = w_0(\lambda) + \epsilon w_1(\lambda) + O(\epsilon^2) \quad (3.7)$$

$$e^{-\ell(\lambda)}w_{E(q)}(e^{-\epsilon})(\lambda) = w_0(\lambda) + \epsilon (w_1(\lambda) - d w_0(\lambda)) + O(\epsilon^2) \quad (3.8)$$

$$e^{-\ell(\lambda)}w_{H(q)}(e^{-\epsilon})(\lambda) = w_0(\lambda) + \epsilon \left( w_1(\lambda) - \frac{\ell(\lambda)(\ell(\lambda) + 1)}{2} d w_0(\lambda) \right) + O(\epsilon^2). \quad (3.9)$$

From this result one can deduce the following semiclassical expansion for the partition function:

**Theorem 3.4.** For $d \in \mathbb{Z}_{>0}$ and $q = e^{-\epsilon}$ we have

$$e^d Z_{E(q)}(e^{-\epsilon}) = \frac{1}{d!} + \epsilon \frac{3 - d}{4(d - 1)!} + O(\epsilon^2), \quad (3.10)$$

$$e^d Z_{E(q)}(e^{-\epsilon}) = \frac{1}{d!} + \epsilon \frac{5 + d}{4(d - 1)!} + O(\epsilon^2), \quad (3.11)$$

$$e^d Z_{H(q)}(e^{-\epsilon}) = \frac{1}{d!} + \epsilon \frac{d + 1}{(d - 1)!} + O(\epsilon^2). \quad (3.12)$$

We also obtain a semiclassical convergence result for the weighted Hurwitz numbers. (Recall our notation for partitions from definition 3.2.)
Theorem 3.5. For any $\mu, \nu \in P_n$ and $G \in \{E', E, H\}$ we have

$$
\epsilon d \text{Hd}(G(q))_{\mu, \nu} = \frac{1}{d!} \text{H}(2, \ldots, 2, \mu, \nu) \left( 1 + \epsilon \left[ \gamma(1) \text{H}(2, \ldots, 2, 3, \mu, \nu) + \gamma(1) \text{H}(2, \ldots, 2, 2^2, \mu, \nu) \right] \right) + O(\epsilon^2) \tag{3.13}
$$

where $\gamma(1) = \frac{1}{(d-1)!}$ and

$$
\gamma_{E'(q)}(2) = -\frac{d+1}{4(d-1)!}, \quad \gamma_{E(q)}(2) = -\frac{3-d}{4(d-1)!}, \quad \gamma_{H(q)}(2) = \frac{d+1}{2(d-1)!}. \tag{3.14}
$$

Remark 3.2. In particular, for $G \in \{E', E, H\}$, we have

$$
\lim_{\epsilon \to 0} \text{H}(G(\epsilon^{-\gamma}))_{\mu, \nu} = \frac{1}{d!} \text{H}_{\text{exp}}(\mu, \nu), \tag{3.15}
$$

which includes (1.61) for $G = E'$.

4. Zero-temperature limit and asymptotic expansion

Recalling that the parameter $q$ is interpreted as

$$
q = e^{-\frac{\hbar \omega_0}{k_B T}} \tag{4.1}
$$

for some ground state energy $E_0 = \hbar \omega_0$, the zero temperature limit $T \to 0^+$ corresponds to $q \to 0^+$. In this section we state our asymptotic results in this limit. All proofs are given in the following section. We further assume throughout that $d \geq 2$ and $n \geq 2d$.

4.1. Zero-temperature limit: leading term

We begin by stating the leading order zero temperature limit.

Theorem 4.1. Let $d \in \mathbb{Z}_{>0}$. As $q \to 0^+$, the 1-parameter family of measures $\left( \xi^{(d)}_{E(q)} \right)$ on $P_d$ converges weakly to the Dirac measure $\delta_{(d)}$ at $(d) \in P_d$.

See figure 2. By the discussion in section 3 this translates to a convergence result on $\mathcal{M}^{(n)}_d$. 

Figure 2. The partition $(d)$ on which $\xi^{(d)}_{E(q)}$ concentrates asymptotically in the zero-temperature limit.
Corollary 4.2. If \( n \geq 2d \) then the measure \( \theta^{\langle n,d \rangle}_{E(q)} \) on \( \mathcal{M}^{\langle n \rangle}_{d,1} \) converges weakly, as \( q \to 0^+ \), to the uniform measure \( \nu \) on \( \mathcal{M}^{\langle n \rangle}_{d,1} \), the set of single partitions \( \mu^{(1)} \) of \( n \) with colength \( d \). That is,

\[
\nu(A) = \frac{|A \cap \mathcal{M}^{\langle n \rangle}_{d,1}|}{|\mathcal{M}^{\langle n \rangle}_{d,1}|}.
\] (4.2)

Remark 4.1. Note that this leading term contribution has support on the horizontal partition \( \lambda = (d) \). This corresponds to branched coverings with just \( k = 1 \) weighted branch point, with ramification profile \( \mu^{(1)} \) of colength \( \ell^*(\mu^{(1)}) = d \),

\[
\ell^*(\mu^{(1)}) = d,
\] (4.3)

plus two unweighted ones, with ramification profiles \( (\mu, \nu) \). This means three branch points in total, with profiles \( (\mu^{(1)}, \mu, \nu) \), where the first branched point, with profile \( \mu^{(1)} \), has uniform measure on the space of branch points with colength \( d \). The weighted enumeration of such branched coverings, sometimes known as Belyi curves, is known to be equivalent to the enumeration of Grothendieck’s Dessins d’Enfants [1, 16, 24]. They are also determined by a generating function \( \tau^{(E,\beta)}(t,s) \), that has been studied in [10, 11], in which the weight generating function \( G = E \) is chosen to be simply

\[
E(z) := 1 + z.
\] (4.4)

4.2. Higher-order corrections

We now consider higher order terms in the \( T \to 0 \) limit for the partition function and the quantum Hurwitz numbers. The following gives the two leading terms in weighted sums of functions on \( \mathcal{M}^{\langle n \rangle}_{d} \) with weights \( W_{E(q)} \) on \( \mathcal{P}_d \).

Theorem 4.3. For any function \( g: \mathcal{M}^{\langle n \rangle}_{d} \to \mathbb{R} \) we have

\[
\sum_{k=1}^{d} \sum_{(\mu^{(1)},\ldots,\mu^{(k)}) \in \mathcal{M}^{\langle n \rangle}_{d}} g \left( \mu^{(1)}, \ldots, \mu^{(k)} \right) W_{E(q)} \left( \mu^{(1)}, \ldots, \mu^{(k)} \right) = d^d \sum_{\ell^*(\mu^{(1)}) = d} \sum_{\ell^*(\mu^{(2)}) = d-1} \sum_{\ell^*(\mu^{(3)}) = d-1} g \left( \mu^{(1)}, \mu^{(2)} \right) + O \left( q^{d+2} \right).
\] (4.5)

In particular, we obtain the following leading terms in the \( T = 0 \) expansion of the partition function.

Corollary 4.4. As \( q \to 0^+ \),

\[
Z_{E(q)}^{(d)} = p(d)q^d + p(d-1)q^{d+1} + O \left( q^{d+2} \right).
\] (4.7)

(Recall that \( p(d) \) denotes the number of integer partitions of \( d \).)

For the zero temperature expansions of simple quantum Hurwitz numbers, we have the following leading terms
Corollary 4.5. For any $\mu, \nu \in \mathcal{P}_n$ we have, as $q \to 0^+$,
\begin{equation}
H_{E'(q)}^p(\mu, \nu) = q^d \sum_{\mu^{(1)} \in \mathcal{P}_{\nu}, \epsilon \in [0,1]} H(\mu^{(1)}, \mu, \nu) + q^{d+1} \sum_{\mu^{(1)} \in \mathcal{P}_{\nu}, \epsilon \in [0,1]} H(\mu^{(1)}, \mu, \mu, \nu) + O(q^{d+2}).
\end{equation}

5. Proofs

5.1. Classical and semiclassical asymptotics

Proof of lemma 2.1. Consider the function $f : \mathcal{P}_{n,n-\ell} \to \mathcal{P}_\ell$ defined as follows. Let $\lambda \in \mathcal{P}_{n,n-\ell}$, then the first column of the Young diagram of $\lambda$ has $n-\ell$ boxes. Remove these to obtain a partition $\nu := f(\lambda)$ of $\ell$. This function has an inverse: for $\nu \in \mathcal{P}_\ell$, simply add a new column with $n-\ell$ boxes to the left of the Young diagram of $\nu$. Since $n-\ell \geq \ell$ by assumption the result is the Young diagram of an integer partition $\lambda := f^{-1}(\nu)$. It is easy to see that $\lambda \in \mathcal{P}_n$ and that $\ell(\lambda) = n-\ell$.

We only detail the proofs of the results from sections 3 and 4 for the case $E'(q)$. The corresponding results for $E(q)$ and $H(q)$ follow analogously, using (1.50) and (1.56). The proofs all rely on the following asymptotic expansion of $\Phi_{E'(\epsilon^{-\cdot})}$ as $\epsilon \to 0$:

Lemma 5.1. Let $x_1, \ldots, x_m \in \mathbb{Z}_{>0}$. Then, as $\epsilon \to 0$
\begin{equation}
\Phi_{E'(\epsilon^{-\cdot})}(x_1, \ldots, x_m) = \epsilon^{-m} \sum_{\sigma \in \mathcal{S}_m} \left[ \frac{1}{\prod_{i=1}^m \sum_{j=1}^{x_i} x_{\sigma(i)}} - \frac{\epsilon}{2} \sum_{i=1}^m \frac{\sum_{j=1}^{x_{\sigma(i)}} x_{\sigma(i)}}{\prod_{i=1}^m \sum_{j=1}^{x_i} x_{\sigma(i)}} \right] + O(\epsilon^{2-m}).
\end{equation}

Proof. A direct computation yields
\begin{equation}
\Phi_{E'(\epsilon^{-\cdot})}(x_1, \ldots, x_m) = \sum_{\sigma \in \mathcal{S}_m} \prod_{j=1}^m \left( \epsilon \sum_{i=1}^{x_{\sigma(i)}} - 1 \right)^{-1}
\end{equation}
\begin{equation}
= \sum_{\sigma \in \mathcal{S}_m} \prod_{j=1}^m \left( \frac{\epsilon}{\sum_{i=1}^{x_{\sigma(i)}}} \right)^{-1} \left( 1 + \frac{\epsilon}{2} \sum_{i=1}^m x_{\sigma(i)} + O(\epsilon^2) \right)^{-1}
\end{equation}
\begin{equation}
= \epsilon^{-m} \sum_{\sigma \in \mathcal{S}_m} \prod_{j=1}^m \left( \frac{1}{\sum_{i=1}^{x_{\sigma(i)}}} - \frac{\epsilon}{2} + O(\epsilon^2) \right)
\end{equation}
\begin{equation}
= \epsilon^{-m} \sum_{\sigma \in \mathcal{S}_m} \left( \prod_{j=1}^m \frac{1}{\sum_{i=1}^{x_{\sigma(i)}}} - \frac{\epsilon}{2} \prod_{i=1}^m \sum_{j=1}^{x_{\sigma(i)}} + O(\epsilon^2) \right)
\end{equation}
as claimed.

By considering the highest order terms, it follows immediately that, letting
\begin{equation}
d = \sum_{r=1}^m x_r \geq m,
\end{equation}
we have
\[
\lim_{\epsilon \to 0} e^\epsilon \Phi_{E(\epsilon^{-r})} (x_1, \ldots, x_m) = \begin{cases} 
\frac{1}{d!} & \text{if } d = m \\
0 & \text{if } d > m. 
\end{cases} 
\] (5.7)

This completes the proof of theorem 3.1 and hence also corollary 3.2. Setting \( q = e^{-r} \) and considering additionally the terms of order \( \epsilon^{1-d} \) gives theorem 3.3.

Moreover we obtain the following intermediate result:

**Proposition 5.2.** For any function \( f: \mathcal{M}_d^{(n)} \to \mathbb{R} \),
\[
e^\epsilon \sum_{\mu^{(1)}, \ldots, \mu^{(k)} \in \mathcal{M}_d^{(n)}} f \left( \mu^{(1)}, \ldots, \mu^{(k)} \right) W_{E(q)} \left( \mu^{(1)}, \ldots, \mu^{(k)} \right) = \frac{p(1)}{\text{aut}(1)} \Phi_{e^{-r}, 1, \ldots, 1} \sum_{\lambda \in \mathcal{L}_d^{(n)}} f \left( \mu^{(1)}, \ldots, \mu^{(k)} \right) + O(\epsilon^2) 
\] (5.8)
\[
+ \frac{\epsilon}{(d-1)!} \left[ f \left( 2, \ldots, 2, 3 \right)_{d-1 \text{ times}} + f \left( 2, \ldots, 2^2 \right)_{d-1 \text{ times}} - \frac{d+1}{4} f \left( 2, \ldots, 2 \right)_{d \text{ times}} \right] + O(\epsilon^2) 
\] (5.9)
\[
+ O(\epsilon^2) 
\] (5.10)
where we recall that \( 2 = (1^{n-1}, 2) \) and \( 3 = (1^{n-3}, 3) \) and \( 2^2 = (1^{n-4}, 2^2) \).

**Proof.** From lemma 5.1 it follows that \( W_{E(q)} (\lambda) \) contributes terms of order \( e^{-\mu(\lambda)} \) and lower. Thus the only terms in (5.8) that are not \( o(\epsilon^{d+1}) \) correspond to elements \( (\mu^{(1)}, \ldots, \mu^{(k)}) \) of \( \mathcal{M}_d^{(n)} \) such that \( \lambda = \Lambda_{\mu^{(1)}, \ldots, \mu^{(k)}} \) has length \( d \) or \( d-1 \), i.e. \( \lambda \in \{1, 2\} \) (recalling once more the notation from definition (3.2)). Therefore,
\[
\sum_{\mu^{(1)}, \ldots, \mu^{(k)} \in \mathcal{M}_d^{(n)}} f \left( \mu^{(1)}, \ldots, \mu^{(k)} \right) W_{E(q)} \left( \mu^{(1)}, \ldots, \mu^{(k)} \right) = \frac{p(1)}{\text{aut}(1)} \Phi_{e^{-r}, 1, \ldots, 1} \sum_{\lambda \in \mathcal{L}_d^{(n)}} f \left( \mu^{(1)}, \ldots, \mu^{(k)} \right) + O(\epsilon^2). 
\] (5.11)
\[
+ \frac{p(2)}{\text{aut}(2)} \Phi_{e^{-r}, 2, 1, \ldots, 1} \sum_{\lambda \in \mathcal{L}_d^{(n)}} f \left( \mu^{(1)}, \ldots, \mu^{(k)} \right) + O(\epsilon^2). 
\] (5.12)

We first deal with the term in (5.11): \( p(1) = 1 \) and \( |\text{aut}(1)| = d! \). Further, by lemma 5.1,
\[
\Phi_{E(q)}(1, \ldots, 1) = e^{-d} \sum_{\sigma \in S_d} \left( \frac{1}{d!} - \frac{d+1}{2d!} + O(\epsilon^2) \right) = e^{-d} \left( 1 - \epsilon \frac{d+1}{4} \right) + O(\epsilon^{d+2}). 
\] (5.13)

For the terms in (5.12): \( p(2) = 2 \) and \( \text{aut}(2) = (d-1)! \). This time we only need the first order approximation of lemma 5.1, and obtain
\[
\Phi_{E(e^{-r})}(2, 1, \ldots, 1) = e^{-d+1} \sum_{\sigma \in S_{d-1}} \left( \prod_{j=1}^{d-1} \sum_{i=1}^{j} x_{\sigma(i)} \right)^{-1} + O(\epsilon^{d+2}). 
\] (5.14)

If \( x = (2, 1, \ldots, 1) \) then we have, for \( j \in \{1, \ldots, d-1\} \) and \( \sigma \in S_{d-1} \),
\[
\sum_{i=1}^{j} x_{\sigma(i)} = \begin{cases} 
\frac{j+1}{j} & \text{if } j < \sigma^{-1}(1) \\
0 & \text{otherwise}. 
\end{cases} 
\] (5.15)
and therefore
\[ \sum_{\sigma \in S_{d-1}} \left( \prod_{j=1}^{d-1} x_{\sigma(j)} \right)^{-1} \bigg|_{x(2,1,\ldots,1)} = \sum_{\sigma \in S_{d-1}} \left( \prod_{j=1}^{\sigma^{-1}(1)-1} j \right)^{-1} \left( \prod_{j=\sigma^{-1}(1)}^{d-1} (j+1) \right)^{-1} \] (5.16)
\[ = \sum_{\sigma \in S_{d-1}} \sigma^{-1}(1) \frac{1}{d!} \sum_{r=1}^{d-1} \sum_{\sigma^{-1}(1)=r} r \] (5.17)
\[ = \frac{(d-2)!}{d!} \cdot \frac{d(d-1)}{2} = \frac{1}{2}. \] (5.18)

It follows that
\[ \Phi_{E'(e^{-1})}(2, 1, \ldots, 1) = \frac{1}{2} e^{-d+1} + O(e^{-d+2}). \] (5.19)

Substituting (5.13) and (5.19) into (5.11) and (5.12) gives
\[ \sum_{\mathcal{W}_c^{(d)}} f(\mu(1), \ldots, \mu(k)) W_{E'(q)}(\mu(1), \ldots, \mu(k)) = \frac{e^{-d}}{d!} \left( 1 - \frac{d(d+1)}{4} \right) f \left( \frac{2, \ldots, 2}{d \text{ times}} \right) \] (5.20)
\[ + \frac{e^{-d+1}}{(d-1)!} \left( f(3, 2, \ldots, 2) + f(2, 2, 2, \ldots, 2) \right) \] (d-2 times) + O(e^{-d+2}) (5.21)
as required.

Choosing \( f(\mu(1), \ldots, \mu(k)) = H(\mu(1), \ldots, \mu(k), \mu, \nu) \) gives theorem 3.5. On the other hand by setting \( f(\mu(1), \ldots, \mu(k)) = 1 \) we obtain
\[ Z_{c^\infty}^{(d)} = \frac{e^{-d}}{d!} + e^{1-d} \left( \frac{1}{(d-1)!} - \frac{d(d+1)}{4d!} \right) \] (5.22)
\[ = \frac{e^{-d}}{d!} + e^{1-d} \frac{3-d}{4(d-1)!} + O(e^{2-d}) \] (5.23)
and so we have proved proposition 3.4.

5.2. Zero-temperature limit: leading order terms

**Proof of theorem 4.1.** For any \( \lambda \in \mathcal{P}_d \),
\[ w_{E'(q)}(\lambda) = \frac{1}{\text{aut}(\lambda)} \sum_{\sigma \in \mathcal{S}_{\ell}(\lambda)} \prod_{j=1}^{\ell(\lambda)} \frac{q_{\sum_{i=1}^{j} \lambda_{\sigma(i)}}}{1 - q_{\sum_{i=1}^{j} \lambda_{\sigma(i)}}} \] (5.24)
\[
\begin{align*}
&= \frac{1}{\text{aut}(\lambda)} \sum_{\sigma \in S(\lambda)} \prod_{j=1}^{\ell(\lambda)} q^{\sum_{i=1}^{j} \lambda_{\sigma(i)}} (1 + O(q)) \\
&= \frac{1}{\text{aut}(\lambda)} \sum_{\sigma \in S(\lambda)} q^{\sum_{i=1}^{\ell(\lambda)} (\ell(\lambda)-i+1) \lambda_{\sigma(i)}} (1 + O(q)) \\
&= \frac{1}{\text{aut}(\lambda)} \sum_{\sigma \in S(\lambda)} q^{\sum_{i=1}^{\ell(\lambda)} \lambda_{\sigma(i)}} (1 + O(q)) .
\end{align*}
\]

Since \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell(\lambda)} \) and \( q \) is small, the sum above is dominated by the contribution when \( \sigma \) is the identity permutation. In particular we obtain

\[
w_{E'}(q)(\lambda) = \frac{1}{\text{aut}(\lambda)} q^{\sum_{i=1}^{\ell(\lambda)} \lambda_i} (1 + O(q)).
\]

Thus, in the limit \( q \to 0^+ \) the dominant weight will be given by \( \lambda \) such that \( \sum_j \lambda_j \) is minimised; i.e. \( \lambda = (d) \). This completes the proof. \( \square \)

### 5.3. Zero-temperature limit: higher-order corrections

Having established the leading order result we turn to the next order corrections. It follows from (5.28) that

\[
\sum_{\lambda \neq (d), \lambda \neq (d-1, 1)} w_{E'}(q)(\lambda) = O(q^{d+2}) .
\]

Further,

\[
w_{E'}(q)((d)) = \frac{1}{\text{aut}((d))} \frac{q^d}{1-q^d} = q^d + O(q^{2d}) ,
\]

\[
w_{E'}(q)((d-1, 1)) = \frac{q^{d-1}}{1-q^d} \frac{q^d}{1-q^d} + \frac{q}{1-q} \frac{q^d}{1-q^d}
\]

\[
= \frac{q^{d+1}}{(1-q)(1-q^d)} + O(q^{2d-1})
\]

\[
= q^{d+1} \left( 1 + q + O(q^2) \right) \left( 1 + q^d + O(q^{2d}) \right)
\]

\[
= q^{d+1} + q^{d+2} + O(q^{d+3}) .
\]

For any \( f: \mathcal{P}_d \to \mathbb{R} \) we therefore have

\[
\sum_{\lambda \in \mathcal{P}_d} f(\lambda)w_{E'}(q)(\lambda) = f((d)) p(d)q^d + f(d-1, 1) p(d-1) q^{d+1} + O(q^{d+2}) .
\]

Theorem 4.3 now follows from (5.35) and the discussion in section 3.

By choosing \( g \) to be identically equal to 1 we obtain corollary 4.4, whereas choosing, for fixed \( (\mu, \nu) \),

\[
\sum_{\lambda \in \mathcal{P}_d} f(\lambda)w_{E'}(q)(\lambda) = f((d)) p(d)q^d + f(d-1, 1) p(d-1) q^{d+1} + O(q^{d+2}) .
\]
\[ g(\mu^{(1)}, \ldots, \mu^k) = H(\mu^{(1)}, \ldots, \mu^k, \mu, \nu). \] (5.36)

Corollary 4.5 is just theorem 4.3 for this particular case.

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