DERIVED $A$-INFINITY ALGEBRAS AND THEIR HOMOTOPIES

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Abstract. The notion of a derived $A$-infinity algebra, introduced by Sagave, is a generalization of the classical $A$-infinity algebra, relevant to the case where one works over a commutative ring rather than a field. We initiate a study of the homotopy theory of these algebras, by introducing a hierarchy of notions of homotopy between the morphisms of such algebras. We define $r$-homotopy, for non-negative integers $r$, in such a way that $r$-homotopy equivalences underlie $E_r$-quasi-isomorphisms, defined via an associated spectral sequence. We study the special case of twisted complexes (also known as multicomples) first since it is of independent interest and this simpler case clearly exemplifies the structure we study. We also give two new interpretations of derived $A$-infinity algebras as $A$-infinity algebras in twisted complexes and as $A$-infinity algebras in split filtered cochain complexes.

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1. Introduction

The homotopy invariant version of an associative algebra, known as an $A_\infty$-algebra, has become an important idea in many areas of mathematics, including algebra, geometry and mathematical physics. An introduction to these structures and a discussion of various applications can be found in Keller’s survey [Kel01]. Kadeishvili’s work on minimal models used the existence of $A_\infty$-structures in order to classify differential graded algebras over a field up to quasi-isomorphism [Kad80].

The homotopy theory of these algebras has been studied by several authors, including Prouté, Grandis and Lefèvre-Hasegawa [Pro11, Gra99, LH03]. Grandis gave a notion of homotopy between morphisms of $A_\infty$-algebras via a functorial path construction. Working over a field, Lefèvre-Hasegawa establishes the structure of a “model category without limits” on the category of $A_\infty$-algebras.

In order to formulate a generalization of Kadeishvili’s work over a general commutative ground ring, Sagave developed the notion of derived $A_\infty$-algebra [Sag10]. Sagave establishes the existence of minimal models for differential graded algebras (dgas) by showing that the structure of a derived $A_\infty$-algebra arises on some projective resolution of the homology of a differential graded algebra. Furthermore, the dga can be recovered up to quasi-isomorphism from this data.

In [LRW13] an operadic description of derived $A_\infty$-algebras was developed, working with non-symmetric operads in the category $\mathrm{vbc}_{CR}$ of bicomplexes with zero horizontal differential. There is an operad $dA_\infty$ in this category encoding bidgas, which are simply monoids in bicomplexes. Derived $A_\infty$-algebras are precisely algebras over the operad

$$d A_\infty = (d A_\infty)^{\infty} = \Omega((d A_\infty)^{\infty}).$$

Here $(d A_\infty)^{\infty}$ is the Koszul dual cooperad of the operad $d A_\infty$, and $\Omega$ denotes the cobar construction. Further development of the operadic theory of these algebras was carried out in [ALR15]. The recent PhD thesis of Maes [Mac16] studies derived $P_\infty$-algebras, replacing the associative operad $A_\infty$ with a suitable operad $P$.

A derived $A_\infty$-algebra has an underlying twisted complex, also known as a multicomplex or $D_\infty$-algebra. Twisted complexes arise as a natural generalization of the notion of double complex by considering a family of “differentials” indexed over the non-negative integers. These objects were first considered by Wall [Wal61] in his work on resolutions for extensions of groups and subsequently they have arisen in the work of many authors. They were studied by Gugenheim and May [GM74] in their approach to differential homological algebra. Meyer [Mey78] introduced homotopies between morphisms of twisted complexes and proved an acyclic models theorem for these objects. More recently, twisted complexes have proven to be an important tool in homological perturbation theory (see for example [Hue14]). Saneblidze [San07] introduced projective twisted complexes and showed that every (possibly unbounded) chain complex over an abelian category $\mathcal{A}$ is weakly equivalent to a projective multicompact, provided that $\mathcal{A}$ has enough projectives and countable coprodtopics. This result provides a good description of the derived category of $\mathcal{A}$. The work of Sagave on minimal models for differential graded algebras can be thought of as a multiplicative enhancement of Saneblidze’s result [San07].

For twisted complexes over a field, Lapin [Lap03] gave a homotopy transfer theorem (see also [DSV15]).

In this paper, we initiate a study of the homotopy theory of derived $A_\infty$-algebras, by introducing a hierarchy of notions of homotopy between the morphisms of such algebras. We define $r$-homotopy for $r \geq 0$ and consider a related notion of $E_r$-quasi-isomorphism. Denoting the set of $r$-homotopy equivalences by $S_r$ and the set of $E_r$-quasi-isomorphisms by $\mathcal{E}_r$, we have the following inclusions

$$S_0 \subset S_1 \subset \ldots \subset S_r \subset S_{r+1} \subset \ldots$$
$$\mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_r \subset \mathcal{E}_{r+1} \subset \ldots$$

We treat the special case of twisted complexes first, since it is of independent interest and the theory is simpler in this case. Every twisted complex has an associated spectral sequence, defined via the column filtration of its total complex. In fact, the totalization functor gives rise to an isomorphism of categories between the category of twisted complexes and the full subcategory of filtered complexes whose objects have split filtrations (Theorem 3.3). The class of $E_r$-quasi-isomorphisms is given by those
morphisms of twisted complexes inducing a quasi-isomorphism on the $r$-th stage of their associated spectral sequence. The notion of $r$-homotopy for twisted complexes that we consider corresponds to the notion of homotopy of order $r$ introduced in [CE56] and further developed in [CG16] in the context of filtered complexes. We study the localized category of twisted complexes with respect to $r$-homotopies (Theorem 3.26). We present several equivalent formulations of $r$-homotopy: via a functorial path, via explicit formulas and an operadic approach (Theorem 3.37).

A substantial part of the paper is devoted to developing new interpretations of derived $A_{\infty}$-algebras. As well as being interesting in themselves, these new viewpoints are used in establishing properties of homotopy and they provide a key idea for the proof of the equivalence of the various formulations of $r$-homotopy in the derived $A_{\infty}$ case. Firstly, we show that derived $A_{\infty}$-algebras can be interpreted as $A_{\infty}$-algebras in twisted complexes (Theorem 1.47). This formulation has the potential to be a very useful tool for the future development of different aspects of the theory of derived $A_{\infty}$-algebras. Secondly, under suitable boundedness conditions, we show that derived $A_{\infty}$-algebras can be viewed as split filtered $A_{\infty}$-algebras (Theorem 1.53). This result allows one to transfer known constructions in the category of $A_{\infty}$-algebras to the category of derived $A_{\infty}$-algebras, by checking compatibility with filtrations.

The context for these new interpretations is the theory of operadic algebras for monoidal categories over a base, as developed by Fresse [Fre09]. We endow the categories of twisted complexes and filtered complexes with a monoidal structure over the category of vertical bicomplexes and use this to enrich them over vertical bicomplexes. This allows us to formulate these algebra structures by means of an enriched endomorphism operad. The totalization functor extends to the enriched setting and gives an isomorphism between the vertical bicomplexes-enriched categories of twisted complexes and split filtered complexes (Theorem 1.37). We use this isomorphism to show that the different interpretations of derived $A_{\infty}$-algebras are equivalent.

We then turn to $r$-homotopy for derived $A_{\infty}$-algebras. Here, the class of $E_r$-quasi-isomorphisms is defined by lifting $E_r$-quasi-isomorphisms of the underlying twisted complexes. The notion of $r$-homotopy that we consider arises as a combination of the notion of $r$-homotopy for twisted complexes and the classical notion of homotopy between morphisms of $A_{\infty}$-algebras. We again present different approaches: via a functorial path construction, via explicit formulas and in operadic terms. For the operadic description we formulate the general notion of a $(g, f)$-coderivation for morphisms $g, f$ of cofree coalgebras over a (non-symmetric) operad. In Theorem 5.31 we show that the different approaches are equivalent.

The results of this paper set up the foundations for a homotopy theory of derived $A_{\infty}$-algebras, contextualizing the ad-hoc notion of homotopy introduced by Sagave in a very particular case and generalizing the work of Grandis on functorial paths for $A_{\infty}$-algebras. We expect that both the new descriptions of derived $A_{\infty}$-algebras and the properties of homotopies developed here will allow us to endow the category of derived $A_{\infty}$-algebras with the structure of a model category without limits in the future, with weak equivalences being $E_r$-quasi-isomorphisms.

The paper is organized as follows. Section 2 covers background material introducing some of the categories we work with. The notion of $r$-homotopy for twisted complexes is covered in Section 3. Our new interpretations of derived $A_{\infty}$-algebras are presented in Section 4. Finally, Section 5 studies $r$-homotopy for derived $A_{\infty}$-algebras.

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2. Preliminaries

We first set up some notation which we will use throughout this paper.

Notation 2.1. Throughout this paper $R$ will denote a commutative ring with unit. Unless stated otherwise, all tensor products will be taken over $R$. Let $C$ be a category and $A, B$ be arbitrary objects
in $C$. We denote by $\text{Hom}_C(A, B)$ the set of morphisms from $A$ to $B$ in $C$. If $(C, \otimes, 1)$ is symmetric monoidal closed, then we denote its internal hom-object by $[A, B] \in C$ in which case we have by definition an isomorphism of sets

$$\text{Hom}_C(A \otimes B, C) \cong \text{Hom}_C(A, [B, C])$$

which is natural in $A, B$ and $C$.

2.1. Filtered modules and filtered cochain complexes. We collect some algebraic preliminaries about filtered $R$-modules and filtered complexes of $R$-modules. Our complexes are cochain complexes and our filtrations are increasing.

**Definition 2.2.** A filtered $R$-module $(A, F)$ is given by a family of $R$-modules $\{F_p A\}_{p \in \mathbb{Z}}$ indexed by the integers such that $F_{p+1} A \subseteq F_p A$ for all $p \in \mathbb{Z}$ and $A = \cup F_p A$. A morphism $f : A \to B$ of filtered modules is a morphism $f : A \to B$ of $R$-modules which is compatible with filtrations: $f(F_p A) \subseteq F_p B$ for all $p \in \mathbb{Z}$.

**Definition 2.3.** The tensor product of two filtered $R$-modules $(A, F)$ and $(B, G)$ is a filtered $R$-module, with

$$F_p (A \otimes B) := \sum_{i+j=p} \text{Im}(F_i A \otimes F_j B \to A \otimes B).$$

This makes the category of filtered $R$-modules into a symmetric monoidal category, where the unit is given by $R$ with the trivial filtration $0 = F_{-1} R \subseteq F_0 R = R$.

Denote by $C_R$ the category of cochain complexes of $R$-modules. The standard tensor product endows this with a symmetric monoidal structure, with unit $R$ concentrated in degree zero.

**Definition 2.4.** A filtered complex $(K, d, F)$ is a complex $(K, d) \in C_R$ together with a filtration $F$ of each $R$-module $K^n$ such that $d(F_p K^n) \subseteq F_p K^{n+1}$ for all $p, n \in \mathbb{Z}$.

We denote by $fC_R$ the category of filtered complexes of $R$-modules. Its morphisms are given by morphisms of complexes $f : K \to L$ compatible with filtrations: $f(F_p K) \subseteq F_p L$ for all $p \in \mathbb{Z}$. It is a symmetric monoidal category, with the tensor on the tensor product defined as above.

We next recall the notion of homotopy of order $r$ due to Cartan and Eilenberg.

**Definition 2.5.** [CE56, p321] Let $f, g : (K, F) \to (L, F)$ be two morphisms of filtered complexes and let $r \geq 0$ be an integer. A homotopy of order $r$ from $f$ to $g$ is a map of graded $R$-modules $H : K^* \to L^{*+1}$ such that $dH + Hd = g - f$ and $H(F_p K^n) \subseteq F_{p+r} L^{n+1}$ for all $p, n \in \mathbb{Z}$.

**Remark 2.6.** Every filtered complex $(K, d, F)$ has an associated spectral sequence $\{E_r(K), \delta_r\}$ which is functorial for morphisms of filtered complexes. Given a homotopy of order $r$ between morphisms of filtered complexes $f, g : (K, F) \to (L, F)$, the induced morphisms at the $k > r$ stages of the spectral sequences coincide: $f^r_k = g^r_k : E_k(K) \to E_k(L)$ for all $k > r$ (see [CE56 Proposition XV.3.1]). This result indicates how the notion of homotopy of order $r$ is the suitable notion to study the $r$-derived category defined by inverting those morphisms of filtered complexes which induce an isomorphism at the $E_{r+1}$-stage of the associated spectral sequences (see [Par96, CG16]).

2.2. Bigraded modules, vertical bicomplexes, sign conventions. We consider $(\mathbb{Z}, \mathbb{Z})$-bigraded $R$-modules $A = \{A^i_j\}$, where elements of $A^i_j$ are said to have bidegree $(i, j)$. We sometimes refer to $i$ as the horizontal degree and $j$ the vertical degree. The total degree of an element $a \in A^i_j$ is $|a| = j - i$. A morphism of bidegree $(p, q)$ maps $A^i_j$ to $A^{i+q}_{j+p}$. The tensor product of two bigraded $R$-modules $A$ and $B$ is the bigraded $R$-module $A \otimes B$ given by

$$(A \otimes B)^i_j := \bigoplus_{p,q} A^i_p \otimes B^j_{-q-p}.$$

We introduce the following scalar product notation for bidegrees: for $x, y$ of bidegree $(x_1, x_2), (y_1, y_2)$ respectively, we let $(x, y) = x_1 y_1 + x_2 y_2$. We follow the Koszul sign rule: if $f : A \to B$ and $g : C \to D$ are bigraded morphisms, then the morphism $f \otimes g : A \otimes C \to B \otimes D$ is defined by

$$(f \otimes g)(a \otimes c) := (-1)^{(g, a)} f(a) \otimes g(c).$$
We denote by $\text{bgMod}_R$ the category of bigraded $R$-modules. It is symmetric monoidal with the above tensor product.

We will use a standard (vertical) shift $S$ of bigraded modules, following Sagave’s conventions, as in the first part of [Sag10, Section 4]. So $S$ is the shift of bidegree $(0, 1)$; it is an endofunctor on bigraded $R$-modules, where $S(A)^k_j = A^{j+1}_{i-k}$ and on morphisms $Sf = (-1)^{u}f$, if $f$ has bidegree $(u, v)$. We write $\sigma$ for the corresponding canonical isomorphism of bidegree $(0, 1)$ from $S$ to the identity; this means that
\[
  f\sigma_A = \sigma_BS(f) = (-1)^{u}\sigma_Bf.
\]
Then $\Psi_k$ is the induced isomorphism from homomorphisms on a $k$-fold tensor power:
\[
  \Psi_k : \text{Hom}(A^{\otimes k}, B) \to \text{Hom}(SA^{\otimes k}, SB),
\]
where $\sigma_B\Psi_k(f) = (-1)^{\langle\Psi_k(f), \sigma\rangle} f\sigma_A^{\otimes k}$. If the bidegree of $f$ is $(u, v)$, then the bidegree of $\Psi_k(f)$ is $(u, v + k - 1)$, and then since that of $\sigma$ is $(0, 1)$, we have $\langle\Psi_k(f), \sigma\rangle = v + k - 1$.

**Definition 2.7.** A vertical bicomplex is a bigraded $R$-module $A$ equipped with a vertical differential $d_A : A \to A$ of bidegree $(0, 1)$. A morphism of vertical bicomplexes is a morphism of bigraded modules commuting with the vertical differential.

We denote by $\text{vbC}_R$ the category of vertical bicomplexes. The tensor product of two vertical bicomplexes $A$ and $B$ is given by endowing the tensor product of underlying bigraded modules with vertical differential $d_{A \otimes B} := d_A \otimes 1 + 1 \otimes d_B : (A \otimes B)^u_v \to (A \otimes B)^{u+1}_v$. This makes $\text{vbC}_R$ into a symmetric monoidal category.

The symmetric monoidal categories $(C_R, \otimes, R)$, $(\text{bgMod}_R, \otimes, R)$ and $(\text{vbC}_R, \otimes, R)$ are related by embeddings $C_R \to \text{vbC}_R$ and $\text{bgMod}_R \to \text{vbC}_R$ which are monoidal and full.

**Definition 2.8.** Let $A, B$ be bigraded modules. We define $[A, B]^*_v : \text{Hom}(A, B)^u_v$ to be the bigraded module of homomorphisms $A \to B$. Furthermore, if $A, B$ are vertical bicomplexes, and $f \in [A, B]^u_v$, we define
\[
  \delta(f) := d_B f - (-1)^{u}f d_A.
\]

**Lemma 2.9.** If $A, B$ are vertical bicomplexes, then $([A, B]^*_v, \delta)$ is a vertical bicomplex.

**Proof.** A direct computation gives that $\delta^2 = 0$. \qed

This gives an internal hom on $\text{vbC}_R$, making it symmetric monoidal closed. It restricts to give the standard internal hom on the categories $\text{bgMod}_R$ and $C_R$.

We denote by $\text{vbC}_R$, $C_R$, and $\text{bgMod}_R$, the categories of vertical bicomplexes, complexes, and bigraded modules respectively, enriched over themselves via their symmetric monoidal closed structure.

### 3. Twisted complexes and $r$-homotopy

In this section, we recall some key properties of the category of twisted complexes. We define the totalization functor from twisted complexes to filtered cochain complexes and show that it induces an isomorphism of categories onto its image. We then introduce $r$-homotopies between morphisms of twisted complexes, consider their interplay with spectral sequences and study the localized category $tC_R[S_r^{-1}]$.

#### 3.1. The category of twisted complexes.

**Definition 3.1.** A twisted complex $(A, d_m)$ is a bigraded $R$-module $A = \{A^i_j\}$ together with a family of morphisms $\{d_m : A \to A\}_{m \geq 0}$ of bidegree $(-m, -m + 1)$ such that for all $m \geq 0$,
\[
  \sum_{i+j=m} (-1)^i d_id_j d_{i+j} = 0. \quad (A_{m1})
\]

**Definition 3.2.** A morphism of twisted complexes $f : (A, d^A_m) \to (B, d^B_m)$ is given by a family of morphisms $\{f_m : A \to B\}_{m \geq 0}$ of bidegree $(-m, -m)$ such that for all $m \geq 0$,
\[
  \sum_{i+j=m} d^B_m f_j = \sum_{i+j=m} (-1)^i f_id^A_j. \quad (B_{m1})
\]
The composition of morphisms is given by \((g \circ f)_m := \sum_{i+j=m} g_if_j\). A morphism \(f = \{f_m\}_{m \geq 0}\) is said to be\(\textit{strict}\) if \(f_i = 0\) for all \(i > 0\). The identity morphism \(1_A : A \to A\) is the strict morphism given by \(1_{A_0}(x) = x\). A morphism \(f = \{f_i\}\) is an isomorphism if and only if \(f_0\) is an isomorphism of bigraded \(R\)-modules. Indeed, an inverse of \(f\) is obtained from an inverse of \(f_0\) by solving a triangular system.

Denote by \(tC_R\) the category of twisted complexes of \(R\)-modules. Also, denote by \(\text{bgMod}_{\infty}^R\) the full subcategory of \(tC_R\) on twisted complexes with trivial structure i.e., \(d_m = 0\) for all \(m \geq 0\).

The following construction endows \(tC_R\) with a symmetric monoidal structure.

**Lemma 3.3.** The category \((tC_R, \otimes, R)\) is symmetric monoidal, where the monoidal structure is given by the bifunctor
\[
\otimes : tC_R \times tC_R \to tC_R
\]
which on objects is given by \(\langle (A, d_{\text{md}}^A), (B, d_{\text{md}}^B) \rangle \mapsto (A \otimes B, d_{\text{md}}^A \otimes 1 + 1 \otimes d_{\text{md}}^B)\) and on morphisms is given by \(f \otimes g\), where \((f \otimes g)_m := \sum_{i+j=m} f_i \otimes g_j\). In particular, by the Koszul sign rule we have that \((f_i \otimes g_j)(a \otimes b) = (-1)^{ij} f_i(a) \otimes g_j(b)\). This functor describes a symmetric monoidal structure on \(\text{bgMod}_{\infty}^R\) by restriction.

**Proof.** We check that \((A \otimes B, d_{\text{md}} \otimes 1 + 1 \otimes d_{\text{md}})\) is a twisted complex: for all \(m \geq 0\) we have
\[
\sum_{i+j=m} (-1)^i \partial_i \partial_j = \sum_{i+j=m} (-1)^i(d_{i,j}^A \otimes 1 + 1 \otimes d_{i,j}^B) + d_i^A \otimes d_j^B + (-1)^{ij}(-1)^{ij-1} d_i^A \otimes d_j^B = 0.
\]
Similarly, one checks that \(f \otimes g\) is a morphism of twisted complexes. It only remains to see that this construction is functorial. A direct computation shows that
\[
(f \otimes g) \circ (f' \otimes g')_m = (f \circ f' \otimes g \circ g')_m.
\]

We extend the internal hom on bigraded modules to twisted complexes.

**Lemma 3.4.** Let \(A, B\) be twisted complexes, let \([A, B]^*_u\) be the bigraded module of homomorphisms \(A \to B\) and \(f \in [A, B]^*_u\). Setting
\[
(d_i f) := (-1)^{ij+1} d_{i+1} f - (-1)^v f d_{i+1}^A,
\]
for \(i \geq 0\), endows \([A, B]^*_u\) with the structure of a twisted complex.

**Proof.** It is a matter of calculation that \(\sum_i (-1)^i (d_i d_{m-i} f) = 0\), for all \(m \geq 0\). Thus the maps \(d_i : [A, B]^*_u \to [A, B]^*_u-i+1\) make \([A, B]^*_u\) into a twisted complex.

This construction gives an internal hom and we denote by \(tC_R\) the category of twisted complexes enriched over itself via this symmetric monoidal closed structure.

**Definition 3.5.** Let \(r \geq 0\) be a non-negative integer. An \(r\)-bigraded complex is a twisted complex \((A, d_m)\) such that \(d_m = 0\) for all \(m \neq r\).

Note that in this case, condition \(\{A_{m,1}\}\) implies that \(d_r d_r = 0\). For \(r = 0\), this coincides with the notion of vertical bicomplex. Denote by \(r-tC_R\) the full subcategory of \(tC_R\) of \(r\)-bigraded complexes. The prototypical example of such an object is given by the \(r\)-th term of the spectral sequence associated with a twisted complex, as we will see in Section 4.5.3.

### 3.2. Total cochain complex of a twisted complex.

**Definition 3.6.** The total graded \(R\)-module \(\text{Tot}(A)\) of a bigraded \(R\)-module \(A = \{A^i\}\) is given by
\[
\text{Tot}(A)^n := \prod_{i \geq 0} A^{n+i}_i \oplus \bigoplus_{i > 0} A^{n+i}_i.
\]
The \textit{column filtration} of \(\text{Tot}(A)\) is the filtration given by \(F_p \text{Tot}(A)^n := \prod_{i \geq p} A^{n+i}_i\) for all \(p, n \in \mathbb{Z}\).

We will show that, via the totalization functor, the category of twisted complexes is isomorphic to a full subcategory of that of filtered complexes. We use the following.
Definition 3.7. A filtered complex \((K, d, F)\) is said to be split if \(K = \text{Tot}(A)\) is the total graded module of a bigraded \(R\)-module \(A = \{A^n_i\}\) and \(F\) is the column filtration of \(\text{Tot}(A)\). We denote by \(\text{sfC}_R\) the full subcategory of \(\text{IC}_R\) whose objects are split filtered complexes.

Given a twisted complex \((A, d_m)\), define a map \(d : \text{Tot}(A) \rightarrow \text{Tot}(A)\) of degree 1 by letting

\[
d(a) := \sum_{m \geq 0} (-1)^mn d_m(a_{j+m}), \quad \text{for } a = (a_i)_{i \in \mathbb{Z}} \in \text{Tot}(A)^n,
\]

where \(a_i \in A^{n+i}_i\) denotes the \(i\)-th component of \(a\), and \(d(a)_j\) denotes the \(j\)-th component of \(d(a)\). Note that, for a given \(j \in \mathbb{Z}\) there is a sufficiently large \(m \geq 0\) such that \(a_{j+m'} = 0\) for all \(m' \geq m\). Hence \(d(a)_j\) is given by a finite sum. Also, for \(j\) sufficiently large, one has \(a_{j+m} = 0\) for all \(m \geq 0\), which implies \(d(a)_j = 0\).

Given a morphism \(f : (A, d_m) \rightarrow (B, d_m)\) of twisted complexes, let \(\text{Tot}(f) : \text{Tot}(A) \rightarrow \text{Tot}(B)\) be the map of degree 0 defined by

\[
(\text{Tot}(f)(a))_j := \sum_{m \geq 0} (-1)^mn f_m(a_{j+m}), \quad \text{for } a = (a_i)_{i \in \mathbb{Z}} \in \text{Tot}(A)^n.
\]

Theorem 3.8. The assignments \((A, d_m) \mapsto (\text{Tot}(A), d, F)\), where \(F\) is the column filtration of \(\text{Tot}(A)\), and \(f \mapsto \text{Tot}(f)\) define a functor \(\text{Tot} : \text{tC}_R \rightarrow \text{IC}_R\) which is an isomorphism of categories when restricted to its image \(\text{sfC}_R\).

Proof. Let \((A, d_m)\) be a twisted complex and let \(a = (a_i)_{i \in \mathbb{Z}} \in \text{Tot}(A)^n\). To see that \((\text{Tot}(A), d)\) is a complex it suffices to note that:

\[
(dd)(a)_j = \sum_{p \geq 0} \sum_{m \geq 0} (-1)^{p(n+1)+mn} d_p(d_m(a_{j+m+p})) = \sum_{l \geq 0} (-1)^l \sum_{m, p \geq 0, m + p = l} (-1)^p d_p d_m(a_{j+l}) = 0.
\]

One easily verifies that \(F_{p-1} \text{Tot}(A)^n \subset F_p \text{Tot}(A)^n\) and that \((d F_p \text{Tot}(A)^n) \subset F_p \text{Tot}(A)^{n+1}\).

Let \(f : (A, d_A^n m) \rightarrow (B, d_B^n m)\) be a morphism of twisted complexes. If \(a = (a_i) \in \text{Tot}(A)^n\) then

\[
(\text{Tot}(f) \circ d)(a)_j = \sum_{m \geq 0} \sum_{p+q = m} (-1)^{p(n+1)} (-1)^q f_p d_q(a_{j+m}) = \sum_{m \geq 0} (-1)^{mn} \sum_{p+q = m} d_q f_p(a_{j+m}) = \sum_{m \geq 0} (-1)^{q+pn} f_q d_p(a_{j+m}) = (d \circ \text{Tot}(f))(a)_j.
\]

Note that \(\text{Tot}(f)\) is compatible with the filtration \(F\) and that \(\text{Tot}(fg) = \text{Tot}(f) \text{Tot}(g)\). This proves that \(\text{Tot}\) is a functor with values in the category of split filtered complexes.

We next define a functor \(\text{Tot}^{-1} : \text{sfC}_R \rightarrow \text{tC}_R\) inverse to the restriction of \(\text{Tot}\) onto its image. Let \((\text{Tot}(A), d, F)\) be a split filtered complex, where \(A = \{A^m_i\}\) is a bigraded \(R\)-module. For all \(m \geq 0\), let \(d_m : A \rightarrow A\) be the morphism of bidegree \((-m, -m+1)\) defined by \(d_m(a) = (-1)^m d(a)_{i-m}\), where \(a \in A^{n+i}_i\) and \(d(a)_k\) denotes the \(k\)-th component of \(d(a)\), which lies in \(A^{n+1+k}_k\). Since \(d\) is compatible with the filtration \(F\), we have \(d_i = 0\) for \(i < 0\). Then \((A, d_m)\) is a twisted complex and its filtered total complex is \((\text{Tot}(A), d, F)\). Lastly, let \(f : (\text{Tot}(A), d, F) \rightarrow (\text{Tot}(B), d, F)\) be a morphism of split filtered complexes. For all \(m \geq 0\), let \(f_m : A \rightarrow B\) be the morphism of bidegree \((-m, -m)\) defined by \(f_m(a) = (-1)^m f(a)_{i-m}\), where \(a \in A^{n+i}_i\) and \(f(a)_k\) denotes the \(k\)-th component of \(f(a)\), which lies in \(B^{n+k}_k\). Since \(f\) is compatible with the filtration \(F\), we have that \(f_i = 0\) for \(i < 0\). Then the family \(\{f_m\}_{m \geq 0}\) is a morphism of twisted complexes whose total morphism is \(f\). It is straightforward to see that the above constructions define an inverse functor to the restriction of \(\text{Tot}\).

Remark 3.9. Strict morphisms of twisted complexes correspond, via the above isomorphism of categories, to strict morphisms of split filtered complexes, that is, morphisms preserving the splittings.

We will also consider the following bounded versions of our categories, since the totalization functor has better properties when restricted to these.
Definition 3.10. We let $tC_R^b$, $vbC_R^b$, $bgMod_R^b$ be the full subcategories of $(\mathbb{N}, \mathbb{Z})$-graded twisted complexes, vertical bicomplexes and bigraded modules respectively. We let $fMod_R^b$, $sfMod_R^b$, $fC_R^b$, $sfC_R^b$ be the full subcategories of (split) non-negatively filtered modules, respectively complexes, i.e. the full subcategories of objects $(K, F)$ such that $F_p K^n = 0$ for all $p < 0$. We refer to all of these as the bounded subcategories of $tC_R$, $vbC_R$, $bgMod_R$, $fMod_R$, $sfMod_R$, $fC_R$ and $sfC_R$ respectively.

In the following proposition, we show that the monoidal structures of twisted complexes and filtered complexes are compatible under the totalization functor.

Proposition 3.11. The functors $\text{Tot} : bgMod_R \to fMod_R$ and $\text{Tot} : tC_R \to fC_R$ are lax symmetric monoidal, with structure maps

$$\epsilon : R \to \text{Tot}(R) \quad \text{and} \quad \mu_{A,B} : \text{Tot}(A) \otimes \text{Tot}(B) \to \text{Tot}(A \otimes B),$$

given by $\epsilon = 1_R$ and for $a = (a_i)_i \in \text{Tot}(A)^{n_1}$ and $b = (b_j)_j \in \text{Tot}(B)^{n_2}$,

$$(\mu_{A,B}(a \otimes b))_k := \sum_{k_1 + k_2 = k} (-1)^{k_1 n_2} a_{k_1} \otimes b_{k_2}.$$

When restricted to the bounded case, the functors $\text{Tot} : bgMod_R^b \to fMod_R^b$ and $\text{Tot} : tC_R^b \to fC_R^b$ are strong symmetric monoidal functors.

Proof. Clearly $\epsilon$ is a map of filtered complexes and a direct computation shows that the same is true for $\mu_{A,B}$. We now show that $\mu_{A,B}$ respects the symmetric structure i.e., that Diagram (1) commutes.

$$\begin{array}{ccc}
\text{Tot}(A) \otimes \text{Tot}(B) & \xrightarrow{\mu_{A,B}} & \text{Tot}(A \otimes B) \\
\tau_{A,B}^{tC_R} & \xrightarrow{\text{Tot}(\tau_{A,B}^{tC_R})} & \text{Tot}(f \otimes g)
\end{array}$$

$$\begin{array}{ccc}
\text{Tot}(B) \otimes \text{Tot}(A) & \xrightarrow{\mu_{B,A}} & \text{Tot}(B \otimes A) \\
\tau_{A,B}^{tC_R} & \xrightarrow{\text{Tot}(\tau_{A,B}^{tC_R})} & \text{Tot}(f \otimes g)
\end{array}$$

$$\begin{array}{ccc}
\text{Tot}(A') \otimes \text{Tot}(B') & \xrightarrow{\mu_{A',B'}} & \text{Tot}(A' \otimes B') \\
\tau_{A,B}^{tC_R} & \xrightarrow{\text{Tot}(\tau_{A,B}^{tC_R})} & \text{Tot}(f \otimes g)
\end{array}$$

Let $a \otimes b \in \text{Tot}(A)^{n_1} \otimes \text{Tot}(B)^{n_2}$, with $n_1 + n_2 = n$. Then

$$\text{Tot}(\tau_{A,B}^{tC_R}) \mu_{A,B}(a \otimes b) = \sum_{k_1 + k_2 = k} (-1)^{k_1 n_2 + k_1 k_2 + (k_1 + n_1)(k_2 + n_2)} b_{k_2} \otimes a_{k_1}$$

$$= \sum_{k_1 + k_2 = k} (-1)^{n_1 n_2 + k_2 n_1} b_{k_2} \otimes a_{k_1}$$

$$= \mu_{B,A'} \tau_{A,B}^{tC_R}(a \otimes b).$$

The fact that $\mu_{A,B}$ defines a natural transformation, i.e. that Diagram (2) commutes, follows directly by restriction from the proof of Proposition 4.38. The coherence axioms are left to the reader. In the bounded case $\text{Tot}$ is strong symmetric monoidal since $\otimes$ distributes over $\oplus$, therefore the natural transformation $\mu$ is a natural isomorphism.

We summarize the categories we study and their relations in the following commutative diagram.
3.3. Spectral sequence associated to a twisted complex. Every twisted complex \((A,d_m)\) has an associated spectral sequence

\[ E_r^{*,*}(A,d_m) := E_r^{*,*}(\text{Tot}(A,d_m)), \]

which is functorial for morphisms of twisted complexes. Denote by \(\delta_r\) the differential of the \(r\)-th term. We choose the bigrading in such a way that for all \(r \geq 0\), the pair \((E_r(A,d_m), \delta_r)\) is an \(r\)-bigraded complex, so we have a functor \(E_r : \text{tC}_R \to r\text{-tC}_R\). With this choice we have \(E_r^{0,0}(A,d_m) = A_0^0\) and \(\delta_0 = d_0\). For \(r \geq 1\), we have \(E_r^{p,q}(A,d_m) = H^q(E_{r-1}^{p,q}(A,d_m),\delta_{r-1})\) and the map \(\delta_r\) depends on the maps \(d_m\) for \(m \leq r\).

The map \(\delta_r\) is induced by \(d_r\) only on those classes that have a representative \(a \in A_i^j\) for which \(d_k(a) = 0\) for all \(k < r\). In particular, for \(r = 1\) we have

\[ E_1^{p,q}(A,d_m) = H^q(A_p^0, d_0) = \frac{A_0^p \cap \text{Ker}(d_0)}{d_0(A_0^{p-1})} \]

and \(\delta_1 = H_{d_0}(d_1)\).

The morphism of spectral sequences

\[ E_r(f) := E_r(\text{Tot}(f)) : E_r^{*,*}(A,d_m^A) \to E_r^{*,*}(B,d_m^B) \]

associated with a morphism of twisted complexes \(f : (A,d_m^A) \to (B,d_m^B)\) is given by \(E_0(f) = f_0\) and \(E_r(f) = H(E_{r-1}(f),\delta_{r-1})\) for \(r \geq 1\). In particular, for \(r = 1\) we have \(E_1(f) = H_{d_0}(f_0)\). We refer to [Boa99] and [Hur10] for further properties of the spectral sequence associated to a twisted complex.

For the rest of this section, let \(r \geq 0\) be an integer. We shall consider the following notion of weak equivalence in the category of twisted complexes.

**Definition 3.12.** A morphism of twisted complexes \(f : A \to B\) is called an \(E_r\)-quasi-isomorphism if the morphism \(E_r^{*,*}(f) : E_r^{*,*}(A) \to E_r^{*,*}(B)\) at the \(r\)-stage of the associated spectral sequence is a quasi-isomorphism of \(r\)-bigraded complexes (that is, \(E_{r+1}^{*,*}(f)\) is an isomorphism).

Denote by \(\mathcal{E}_r\) the class of \(E_r\)-quasi-isomorphisms of \(\text{tC}_R\). This class is closed under composition and contains all isomorphisms of \(\text{tC}_R\). Denote by

\[ \text{Ho}_r(\text{tC}_R) := \text{tC}_R[\mathcal{E}_r^{-1}] \]

the \(r\)-homotopy category of twisted complexes defined by inverting \(E_r\)-quasi-isomorphisms. Since \(\mathcal{E}_r \subset \mathcal{E}_{r+1}\) for all \(r \geq 0\), we have a chain of functors

\[ \text{Ho}_0(\text{tC}_R) \to \text{Ho}_1(\text{tC}_R) \to \cdots \to \text{Ho}_r(\text{tC}_R) \to \cdots \]
Remark 3.13. The class $\mathcal{E}_1$ of $E_1$-quasi-isomorphisms corresponds to the class of weak multi-equivalences defined by Huebschmann \cite{Hue04} and the class of $E_2$-equivalences considered by Sagave \cite{Sag10}.

3.4. $r$-homotopies and $r$-homotopy equivalences. We next define a collection of functorial paths indexed by an integer $r \geq 0$ on the category of twisted complexes, giving rise to the corresponding notions of $r$-homotopy.

**Definition 3.15.** The $r$-path of a twisted complex $(A, d_m)$ is the twisted complex given by

$$P_r(A)_i := A_i^r \oplus A_i^{i+r-1} \oplus A_i^r,$$

with the maps $D_m : P_r(A) \to P_r(A)$ of bidegree $(-m, -m+1)$ given by

$$D_r := \begin{pmatrix}
   d_r & 0 & 0 \\
   -1 & -d_r & 1 \\
   0 & 0 & d_r
\end{pmatrix} \quad \text{and} \quad D_m := \begin{pmatrix}
   d_m & 0 & 0 \\
   0 & (1)^m_r m+1 d_m & 0 \\
   0 & 0 & d_m
\end{pmatrix} \quad \text{for } m \neq r.$$

For all $m \geq 0$ we have $\sum_{i+j=m} (-1)^i D_i D_j = 0$. Hence $(P_r(A), D_m)$ is indeed a twisted complex.

We have strict morphisms of twisted complexes

$$A \xrightarrow{\iota_A} P_r(A) \xrightarrow{\partial_A^+} \quad \text{and} \quad \partial_A^+ \circ \iota_A = 1_A,$$

given by $\partial_A^+(x, y, z) = x$, $\partial_A^+(x, y, z) = z$ and $\iota_A(x) = (x, 0, x)$. We will denote by $\partial_A^+ : P_r(A) \to A$ the map of bidegree $(r, r - 1)$ given by $(x, y, z) \mapsto y$. We will often omit the subscripts of these maps when there is no danger of confusion.

**Definition 3.16.** The $r$-path of a morphism $f : (A, d_A^m) \to (B, d_B^m)$ of twisted complexes is the morphism of twisted complexes $P_r(f) : (P_r(A), D_A^m) \to (P_r(B), D_B^m)$ given by

$$P_r(f)_m := (f_m, (-1)^m \delta f_m, f_m).$$

The above definitions give rise to a functorial path $P_r : tC_R \to tC_R$ in the category of twisted complexes. This gives a natural notion of homotopy.

**Definition 3.17.** Let $f, g : A \to B$ be two morphisms of twisted complexes. An $r$-homotopy from $f$ to $g$ is given by a morphism of twisted complexes $h : A \to P_r(B)$ such that $\partial_B^+ \circ h = f$ and $\partial_B^+ \circ h = g$. We use the notation $h : f \sim^r g$.

**Remark 3.18.** Let $\Lambda_r$ be the $r$-bigraded complex generated by $e_-$, $e_+$ in bidegree $(0, 0)$ and $u$ in bidegree $(-r, 1 - r)$, with the differential $\delta_i(e_-) = -u$, $\delta_i(e_+) = u$ and $\delta_i(e_\pm) = 0$ for all $i \neq r$. Then the assignment $(x, y, z) \mapsto e_- \otimes x + u \otimes y + e_+ \otimes z$ defines a strict isomorphism of twisted complexes from the $r$-path $(P_r(A), D_m)$ of a twisted complex $(A, d_m)$ to the twisted complex $(\Lambda_r \otimes A, \partial_m)$ where $\partial_m = \delta_m \otimes 1 + 1 \otimes d_m$.

**Proposition 3.19.** Let $f, g : (A, d_A^m) \to (B, d_B^m)$ be two morphisms of twisted complexes. Giving an $r$-homotopy $h : f \sim^r g$ is equivalent to giving a collection of morphisms $\hat{h}_m : A \to B$ of bidegree $(-m+r, -m+r-1)$ such that for all $m \geq 0$,

$$\sum_{i+j=m} (-1)^i d_i^r \hat{h}_j + (-1)^i d_i^r \hat{h}_d^j = \begin{cases} 
   0 & \text{if } m < r, \\
   g_{m-r} - f_{m-r} & \text{if } m \geq r.
\end{cases} \quad (H_m)$$

**Proof.** Let $h : f \sim^r g$. For every $m \geq 0$ we may write $h_m(x) = (f_m(x), \hat{h}_m(x), g_m(x))$, where $\hat{h}_m = \delta_B h_m$. It is a matter of verification to see that the family $\{\hat{h}_m\}_{m \geq 0}$ satisfies $(H_m)$ for all $m \geq 0$. Conversely, one may check that given a family $\{\hat{h}_m\}_{m \geq 0}$ satisfying $(H_m)$, then the family $h_m(x) := (f_m(x), \hat{h}_m(x), g_m(x))$ satisfies

$$\sum_{i+j=m} (-1)^i d_i^r \hat{h}_j = \sum_{i+j=m} d_i^r \hat{h}_j. \quad \square$$
Lemma 3.20. Let $f, g : (A, d^A_m) \to (B, d^B_n)$ be morphisms of twisted complexes. Giving an $r$-homotopy $h : f \not\simeq g$ is equivalent to giving a homotopy of order $r$ $\bar{H} : \mathsf{Tot}(A)^r \to \mathsf{Tot}(B)^{r-1}$ from $\mathsf{Tot}(f)$ to $\mathsf{Tot}(g)$, that is, $\bar{H}(\mathsf{F}_p\mathsf{Tot}(A)) \subset \mathsf{F}_{p+r}(\mathsf{Tot}(B))$, where $F$ is the column filtration.

Proof. Given an $r$-homotopy $h : A \to P_r(B)$ from $f$ to $g$ we obtain a morphism of filtered complexes $\mathsf{Tot}(h) : \mathsf{Tot}(A) \to \mathsf{Tot}(P_r(B))$. Since $\mathsf{F}_p(\mathsf{Tot}(P_r(B)))^n = \mathsf{F}_p\mathsf{Tot}(B)^n \oplus \mathsf{F}_{p+r}\mathsf{Tot}(B)^{n-1} \oplus \mathsf{F}_p\mathsf{Tot}(B)^n$, we may write $\mathsf{Tot}(h)(a) = (\mathsf{Tot}(f)(a), \bar{H}(a), \mathsf{Tot}(g)(a))$, where $\bar{H} : \mathsf{Tot}(A)^r \to \mathsf{Tot}(B)^{r-1}$ satisfies the desired conditions.

Conversely, given $\bar{H} : \mathsf{Tot}(A)^r \to \mathsf{Tot}(B)^{r-1}$ such that $\bar{d}\bar{H} + \bar{H}d = \mathsf{Tot}(g) - \mathsf{Tot}(f)$ and $\bar{H}(\mathsf{F}_pA) \subset \mathsf{F}_{p+r}B$ we define a morphism of filtered complexes $H : \mathsf{Tot}(A) \to \mathsf{Tot}(P_r(B))$ by letting $H(a) := (\mathsf{Tot}(f)(a), \bar{H}(a), \mathsf{Tot}(g)(a))$. By Theorem 3.18 there is a morphism $h : A \to P_r(B)$ of twisted complexes such that $\mathsf{Tot}(h) = H$. By construction, $h$ is an $r$-homotopy from $f$ to $g$.

Proposition 3.21. The notion of $r$-homotopy defines an equivalence relation on the set of morphisms between two given twisted complexes, which is compatible with the composition.

Proof. The homotopy relation defined by a functorial path is reflexive and compatible with the composition (see for example [KP97, Lemma I.2.3]. Symmetry is clear. We prove transitivity. Let $h : f \not\simeq f'$ and $h' : f' \not\simeq f''$. Using the equivalent notion of $r$-homotopy of Proposition 3.18 we get an $r$-homotopy $h''$ by letting $\hat{h}'' = \hat{h} + \hat{h}'$.

Definition 3.22. A morphism of twisted complexes $f : A \to B$ is called an $r$-homotopy equivalence if there exists a morphism $g : B \to A$ satisfying $f \circ g \not\simeq 1_B$ and $g \circ f \not\simeq 1_A$.

Denote by $S_r$ the class of $r$-homotopy equivalences of $\mathsf{tC}_R$. This class is closed under composition and contains all isomorphisms.

Proposition 3.23. For all $r \geq 0$, we have $S_r \subset S_{r+1}$.

Proof. Using the equivalent notion of homotopy of Proposition 3.18 it is straightforward to see that given an $r$-homotopy $h$ from $f$ to $g$, we obtain an $(r+1)$-homotopy $h'$ from $f$ to $g$ by letting $\hat{h}'_m = 0$ and $\hat{h}'_m = \hat{h}'_{m-1}$ for $m > 0$.

Proposition 3.24. For all $r \geq 0$, we have $S_r \subset E_r$.

Proof. By Lemma 3.20 an $r$-homotopy from $f$ to $g$ in $\mathsf{tC}_R$ gives a chain homotopy $H$ from $\mathsf{Tot}(f)$ to $\mathsf{Tot}(g)$ satisfying $H(\mathsf{F}_p) \subset \mathsf{F}_{p+r}$. By [CE56, Proposition XV.3.1], we have $E_{r+1}(f) = E_{r+1}(g)$.

Lemma 3.25. Let $(A, d_m)$ be a twisted complex. The strict morphism $\iota_A : (A, d_m) \to (P_r(A), D_m)$ given by $\iota_A(x) = (x, 0, x)$ is an $r$-homotopy equivalence.

Proof. Since $\partial_A^r \iota_A = 1_A$, it suffices to define an $r$-homotopy from $1_{P_r(A)}$ to $\iota_A \partial_A^r$. Consider the morphisms $\hat{h}_m : P_r(A) \to P_r(A)$ of bidegree $(-m+r, -m+r-1)$ defined by $\hat{h}_0(x, y, z) = (0, 0, y)$ and $\hat{h}_i = 0$ for all $i > 0$. It only remains to verify condition $(H_{r+1})$ of Proposition 3.18. We have

\[
\begin{align*}
(D_r\hat{h}_0 + \hat{h}_0 D_r)(x, y, z) &= D_r(0, 0, y) + \hat{h}_0(d_r x, -x - d_r y + z, d_r z) \\
&= (0, y, d_r y) + (0, 0, -x - d_r y + z) = (0, y, -x + z) \\
&= (1_{P_r(A)} - \iota_A \partial_A^r)(x, y, z).
\end{align*}
\]

\]

\[
\begin{align*}
\text{Theorem 3.26.} \quad \text{The localized category $\mathsf{tC}_R[S_r^{-1}]$ is canonically isomorphic to the quotient category $\pi_r(\mathsf{tC}_R) := \mathsf{tC}_R/\simeq$.}
\]
Proof. Denote by $\gamma_r : tC_R \to tC_R[S_r^{-1}]$ the localization functor. It suffices to show that if $h : f \sim g$ then $\gamma_r(f) = \gamma_r(g)$ (see [GNPR10, Proposition 1.3.3]). Consider the following diagram of morphisms of twisted complexes.

\[
\begin{array}{ccc}
A & \xrightarrow{h} & P_r(B) \\
B & \xrightarrow{g} & B
\end{array}
\]

By Lemma 3.25 the morphism $\iota_B$ is an $r$-homotopy equivalence. Hence the above diagram is a hammock between the $S_r$-zigzags $f$ and $g$ in the sense of [DHKS04]. This gives $f = g$ in $tC_R[S_r^{-1}]$. \qed

3.5. The $r$-translation and the $r$-cone. The $r$-path construction is related to a translation functor depending on $r$ (see [CG16] and [CG14] for similar constructions in the categories of filtered complexes and filtered commutative dgas respectively). Furthermore, the cone obtained via this translation allows one to detect $E_r$-quasi-isomorphisms, as we shall see next.

Definition 3.27. The $r$-translation of a twisted complex $(A, d_m)$ is the twisted complex $(T_r(A), T_r(d_m))$ given by $T_r(A)_i := A_i^{r-i+1}$ and $T_r(d_m) := (-1)^{m+r+1}d_m$.

Definition 3.28. The $r$-cone of a morphism $f : (A, d^A_m) \to (B, d^B_m)$ of twisted complexes is the twisted complex $(C_r(f), D_m)$ given by $C_r(f)_i := A_i^{r-i+1} \oplus B_i$ with the maps $D_m : C_r(f) \to C_r(f)$ of bidegree $(-m, -m + 1)$ given by

$$D_m(a, b) := ((-1)^{m+r+1}d_m(a), (-1)^{m+r+1}f_m(a) + d_m(b)),$$

where we adopt the convention that $f_m = 0$.

We have strict morphisms $(B, d^B_m) \to (C_r(f), D_m)$ and $(C_r(f), D_m) \to (T_r(A), T_r(d^A_m))$ given by $b \mapsto (0, b)$ and $(a, b) \mapsto a$ respectively. These fit into a short exact sequence

$$0 \to (B, d^B_m) \to (C_r(f), D_m) \to (T_r(A), T_r(d^A_m)) \to 0.$$

The following is a matter of verification.

Lemma 3.29. Let $w : A \to B$ be a morphism of twisted complexes and $X$ a twisted complex. Giving a morphism $\tau : C_r(w) \to X$ of twisted complexes is equivalent to giving a pair $(f, h)$ where $f : B \to X$ is a morphism of twisted complexes and $h : 0 \sim fw$ is an $r$-homotopy from $0$ to $fw$.

Proof. Let $\tau : C_r(w) \to X$ be a morphism of twisted complexes. Define a morphism of twisted complexes $f : B \to X$ by letting $f_m(b) := \tau_m(0, b)$. Let $\hat{h}_m : A \to X$ be defined by $\hat{h}_m(a) := (-1)^m\tau_m(a, 0)$. By Proposition 3.18 this gives an $r$-homotopy $h$ from $0$ to $fw$. Conversely, given $(f, h)$, we let $\tau_m(a, b) := (-1)^m\hat{h}_m(a) + f_m(b)$. \qed

Proposition 3.30. Let $r \geq 0$ and let $f : (A, d^A_m) \to (B, d^B_m)$ be a morphism of twisted complexes. We have a long exact sequence

$$\ldots \longrightarrow E_{r+1}^{p, q}(A) \longrightarrow E_{r+1}^{p, q}(B) \longrightarrow E_{r+1}^{p, q}(C_r(f)) \longrightarrow E_{r+1}^{p-r, q-r+1}(A) \longrightarrow \ldots$$

In particular, the morphism $f$ is an $E_r$-quasi-isomorphism if and only if the $r$-cone of $f$ is $E_r$-acyclic, that is, $E_{r+1}^{p, q}(C_r(f)) = 0$.

Proof. For every $p$ we have a short exact sequence of complexes

$$0 \to (E_0^{p, q}(B), d_0^B) \to E_0^{p, q}(C_0(f), D_0) \to (E_0^{p, q}(A), d_0^A)[1] \to 0,$$

which induces a long exact sequence in cohomology. This proves the result for $r = 0$. Assume that $r > 0$. For $m < r$ we have $D_m(a, b) = ((-1)^{m+r+1}d_m(a), d_m(b))$, so the contribution of $f$ to the differential vanishes. This gives a direct sum decomposition $E_r^{p, q}(C_r(f)) \cong E_r^{p-r, q-r+1}(A) \oplus E_r^{p, q}(B)$ inducing a long exact sequence in cohomology. \qed
3.6. Operadic approach. In this section we recall how to view twisted complexes as algebras over the operad $D_\infty$. We then study $r$-homotopy from this point of view.

Let $D$ be the operad of dual numbers in vertical bicomplexes. Here $D = R[e]/(e^2)$, where the bidegree of $e$ is $(-1, 0)$. This has trivial vertical differential and contains only arity one operations, so it can be thought of as simply a bigraded $R$-algebra.

The category of twisted complexes $tC_R$ is isomorphic to the category of $D_\infty$-algebras in vertical bicomplexes (see [LRW13] Section 3.1 or [LV12] 10.3.17 for the singly-graded analogue). Using the so-called Rosetta Stone [LV12], this means that twisted complexes can be studied via structure on cofree coalgebras over the Koszul dual cooperad $D^i$. We first recall some details from [ALR+15] Section 3.4 about $D^i$-coalgebras. We then make explicit how twisted complexes and their morphisms may be encoded via cofree coalgebras, before putting $r$-homotopies into this context.

The Koszul dual $D^i$ of $D$ is again concentrated in arity one and can be thought of as just an $R$-coalgebra. We have $D^i = R[x]$, where $x = S^{-1}e$, $x$ has bidegree $(-1, -1)$ and the comultiplication is determined by $\Delta(x^n) = \sum_{i+j=n} x^i \otimes x^j$.

A $D^i$-coalgebra is a (left)-comodule $C$ over this coalgebra and this turns out to just be a pair $(C, f)$, where $C$ is an $R$-module and $f$ is a linear map $f: C \to C$ of bidegree $(1, 1)$. (Given a coaction $\rho: C \to D^i \otimes C = R[x] \otimes C$, write $f_I$ for the projection onto $R x^I \otimes C$; then coassociativity gives $f_{m+n} = f_m f_n$, so the coaction is determined by $f_1$.) A coderivation is a linear map $d: C \to C$ of bidegree $(s, t)$ such that $df = (-1)^{d(f)} fd$, that is $df = (-1)^{s+t}fd$. In particular, if $d$ has bidegree $(0, 1)$ then it anti-commutes with $f$.

Remark 3.31. As an example, the cofree conilpotent $D^i$-coalgebra generated by a bigraded module $A$ is given by $D^i(A) = R[x] \otimes A$ with linear map $d^A: R[x] \otimes A \to R[x] \otimes A$ determined by $d^A(x^i \otimes a) = x^{i-1} \otimes a$. A map of $D^i$-coalgebras $h: (C, f_C) \to (D, f_D)$ of bidegree $(u, v)$ is a map of bigraded modules satisfying $f_D h = (-1)^{u+v}h f_C$.

It will be useful to introduce the following basic object.

Definition 3.32. Let $A, B, C$ be bigraded modules. We denote by $\underline{\text{bgMod}}_R(A, B)$ the bigraded module given by

$$\underline{\text{bgMod}}_R(A, B)_u := \prod_u [A, B]_{u-j}$$

where $[A, B]$ is the inner hom-object of bigraded modules. More precisely, $g \in \underline{\text{bgMod}}_R(A, B)_u$ is given by $g := (g_0, g_1, g_2, \ldots)$, where $g_j: A \to B$ is a map of bigraded modules of bidegree $(u - j, v - j)$. Moreover, we define a composition morphism

$$c: \underline{\text{bgMod}}_R(B, C) \otimes \underline{\text{bgMod}}_R(A, B) \to \underline{\text{bgMod}}_R(A, C)$$

by

$$c(f, g)_m := \sum_{i+j=m} (-1)^{|i|} f_i g_j.$$
Here a map of bidegree \((u, v)\) of bigraded modules \(F : UD^i(A) \to B\) uniquely lifts as a morphism of \(D^i\)-coalgebras \(\tilde{F} : D^i(A) \to D^i(B)\), of bidegree \((u, v)\) with formula

\[
\tilde{F}(x^n \otimes a) = \sum_{i \geq 0} (-1)^i (u + v) x^i \otimes F(x^{n-i} \otimes a).
\]

Furthermore if \(B = A\) then \(\tilde{F}\) is also a coderivation of \(D^i\)-coalgebras. We associate to such a map \(F\) the collection of maps \(f_n : A \to B\) given by \(f_n(a) = F(x^n \otimes a)\).

(2) If \(d^A : D^i(A) \to D^i(A)\) is a square-zero coderivation of \(D^i\)-coalgebras of bidegree \((0, 1)\), then the corresponding collection of maps \(d^A_n\) makes \(A\) into a twisted complex.

(3) If \(d^A_0\) and \(d^B_0\) are square-zero coderivations of bidegree \((0, 1)\) on \(D^i(A)\) and \(D^i(B)\) respectively, and \(\tilde{F} : D^i(A) \to D^i(B)\) is a morphism of \(D^i\)-coalgebras of bidegree \((0, 0)\) with \(d^B \tilde{F} = \tilde{F} d^A\) then \(f = (f_n)\) is a morphism of twisted complexes from \((A, (d^A_0))\) to \((B, (d^B_0))\).

(4) Composition of coalgebra morphisms of bidegree \((0, 0)\), \(\tilde{G} : D^i(A) \to D^i(B)\) and \(\tilde{F} : D^i(B) \to D^i(C)\), corresponds to composition of morphisms of twisted complexes.

**Proof.**

(1) As above, let \(\Delta : D^i \to D^i \circ D^i\) be the co-composition in the cooperad \(D^i\) (which sends \(x^n\) to \(\sum x^i \otimes x^{n-i}\)). Then one checks easily that \(\Delta \tilde{F} = D^i(\tilde{F}) \Delta\), for \(\tilde{F}\) to be a morphism or a coderivation.

One obtains the map \(f_m : A \to B\) by setting \(f_m(a) = \pi_B \tilde{F}(x^m \otimes a)\) and similarly for any map from \(D^i(A) \to D^i(B)\).

(2) Considering \((d^A)^2\) \((x^n \otimes a) = 0\), we read off \(\sum_{i+j=m} (-1)^i d^A_j d^A_i = 0\).

(3) Considering \(d^B \tilde{F}(x^n \otimes a) = \tilde{F} d^A(x^n \otimes a)\), we read off

\[
\sum_{i+j=m} d^B_i f_j = \sum_{i+j=m} (-1)^i f_i d^A_j.
\]

(4) One checks the statement about composition similarly. \(\square\)

**Remark 3.34.** The fact that the condition is the same to be a morphism of coalgebras as to be a coderivation arises because the cooperad \(D^i\) has only unary operations.

In order to formulate \(r\)-homotopy in this context we use a certain kind of shift operation on morphisms.

**Definition 3.35.** Let \(S : \text{bgMod}_{\text{R}}(A, B)_u^v \to \text{bgMod}_{\text{R}}(A, B)_{u+1}^{v+1}\) be the following map of \(R\)-modules.

For \(f = (f_0, f_1, f_2, \ldots) \in \text{bgMod}_{\text{R}}(A, B)_u^v\), we define \(Sf = (\text{sg} f)_{n := f_{n-1}}\), for \(n \geq 1\) and \((\text{sg} f)_0 := 0\). That is, \((\text{sg} f)_{0, 0, f_1, f_2, \ldots} := (0, f_0, f_1, f_2, \ldots)\).

We write \(S^r\) for the \(r\)-th iterate of this operation.

**Proposition 3.36.** If \(f \in \text{bgMod}_{\text{R}}(A, B)_u^v\) corresponds to \(\tilde{F} \in \text{Hom}_{D^i-\text{coalg}}(D^i(A), D^i(B))_u^v\) under the bijection of Proposition 3.35, then \(Sf\) corresponds to \(\tilde{F} d^A_{x^r}\).

**Proof.** Let \(\tilde{G}\) be the map corresponding to \(Sf\). Then, for all \(n \geq 0\) and all \(a \in A\),

\[
\tilde{G}(x^n \otimes a) = \sum_{i=0}^n (-1)^i f_i x^i \otimes (\text{sg} f)_{n-i} = \sum_{i=0}^{n-1} (-1)^i f_i x^i \otimes f_{n-1-i}(a) = \tilde{F}(x^{n-1} \otimes a) = \tilde{F} d^A_x(x^n \otimes a).
\]

\(\square\)

Now we see what \(r\)-homotopy looks like in this context.

**Theorem 3.37.** Let \(A, B \in tC_R\), with \(d^A, d^B\) the square-zero coderivations of bidegree \((0, 1)\) on \(D^i(A)\) and \(D^i(B)\) respectively encoding the twisted complex structures of \(A\) and \(B\). Let \(f, g \in \text{Hom}_{C_R}(A, B)\), with corresponding \(D^i\)-coalgebra maps \(\tilde{F}, \tilde{G} : D^i(A) \to D^i(B)\) of bidegree \((0, 0)\). Then having an \(r\)-homotopy \(h\) between \(f\) and \(g\) is equivalent to having a \(D^i\)-coalgebra map \(\tilde{H} : D^i(A) \to D^i(B)\) of bidegree \((r, r-1)\) such that

\[
(-1)^r d^B \tilde{H} + \tilde{H} d^A = S^r \tilde{G} - S^r \tilde{F}.
\]
Proof. Considering \(((−1)^r \delta^B H + \delta^A H)(x^n \otimes a) = (\Sigma^r G - \Sigma^r F)(x^n \otimes a)\), we read off

\[ \sum_{i+j=m} (-1)^r i d_i^B h_j(a) + (-1)^i h_i d_j^A(a) = \begin{cases} g_{m-r}(a) - f_{m-r}(a), & \text{if } m \geq r \\ 0, & \text{if } m < r \end{cases} \quad (H_m) \]

which is equivalent to the \(r\)-homotopy condition, by Proposition 3.18.

\[ \square \]

4. NEW INTERPRETATIONS OF DERIVED \(A_\infty\)-ALGEBRAS

In this section we reinterpret derived \(A_\infty\)-algebras both as \(A_\infty\)-algebras in twisted chain complexes and as split filtered \(A_\infty\)-algebras. First we recall the basic notions regarding derived \(A_\infty\)-algebras. Next we endow the categories of twisted complexes and filtered complexes with a monoidal structure over a base, in the sense of Fresse [Fre09], and explicitly describe the enrichments that these structures induce. Then we show that the totalization functor and its properties extend to this enriched setting and use this to prove our main results.

4.1. The category of derived \(A_\infty\)-algebras

We begin by recalling the basic definitions for derived \(A_\infty\)-algebras, also known as \(dA_\infty\)-algebras.

**Definition 4.1.** A (non-unital) \(dA_\infty\)-algebra \((A, m_{ij})\) is a \((\mathbb{Z}, \mathbb{Z})\)-bigraded \(R\)-module \(A = \{A^i_j\}\) equipped with morphisms \(\{m_{ij} : A^{\otimes j} \to A\}_{i,j \geq 1}\) of bidegree \((-i, 2-i-j)\) such that for all \(u \geq 0\) and all \(v \geq 1\),

\[ \sum_{u=i+p, v=j+q-1} (-1)^{rq+t+p} m_{ij} (1^\otimes r \otimes m_{pq} \otimes 1^\otimes t) = 0. \quad (A_{uv}) \]

**Definition 4.2.** A morphism \(f : (A, m_{ij}^A) \to (B, m_{ij}^B)\) of \(dA_\infty\)-algebras is given by a family of morphisms \(\{f_{ij} : A^{\otimes j} \to B\}_{i \geq 0, j \geq 1}\) of bidegree \((-i, 1-i-j)\) such that for all \(u \geq 0\) and all \(v \geq 1\),

\[ \sum_{u=i+p, v=j+q-1} (-1)^{rq+t+p} f_{ij} (1^\otimes r \otimes m_{pq}^A \otimes 1^\otimes t) = \sum_{u=i+p_1+\cdots+p_j, v=q_1+\cdots+q_j} (-1)^\sigma m_{ij}^B (f_{p_1q_1} \otimes \cdots \otimes f_{p_jq_j}). \quad (B_{uv}) \]

where \(\sigma = u + \sum_{l=1}^j (p_l + q_l)(j + t) + qt \sum_{w=1}^j (p_w + q_w).\)

**Proposition 4.3.** Let \(g = (g_{ij}) : A \to B\) and \(f = (f_{ij}) : B \to C\) be two morphisms of \(dA_\infty\)-algebras. Then the composite morphism \(fg : A \to C\) of \(dA_\infty\)-algebras has components

\[ (fg)_{uk} = \sum_{i=p-u} \sum_{r} \sum_{p_1+\cdots+p_r=p} (-1)^\sigma f_{ir} (g_{p_1q_1} \otimes \cdots \otimes g_{p_rq_r}), \]

where \(\sigma = \sum_{l=1}^r (p_l + q_l)(r + t) + qt \sum_{w=1}^r (p_w + q_w).\)

**Proof.** This is a direct consequence of Equations (4) and (5) in [LRW13 Theorem 2.8].

A morphism \(f = \{f_{ij}\}\) is said to be strict if \(f_{ij} = 0\) for all \(i > 0\) and all \(j > 1\). The identity morphism \(1_A : A \to A\) is the strict morphism given by \(1_A 1(x) = x\).

**Lemma 4.4.** A morphism \(f = \{f_{ij}\}\) of \(dA_\infty\)-algebras is an isomorphism if and only if \(f_{01}\) is an isomorphism of bigraded \(R\)-modules.

**Proof.** If \(fg = 1\), then \(f_{01}g_{01} = 1\), so \(f_{01}\) is an isomorphism and \(g_{01} = f_{01}^{-1}\). Then the equation giving the \((uk)\) component of the composite has a “top term” \(f_{01}g_{uk}\), with all other summands involving components \(g_{ij}\) with \(i < u\) or \(j < k\). Thus we can successively solve for each \(g_{uk}\) if and only if \(f_{01}\) is an isomorphism. This gives a right inverse \(g\) for \(f\) if and only if \(f_{01}\) is an isomorphism. The same argument shows that \(g\) also has a right inverse, and this must be \(f\), so \(f\) and \(g\) are two-sided inverses.

Denote by \(dA_\infty(R)\) the category of \(dA_\infty\)-algebras over \(R\).
Example 4.5 ($A_\infty$-algebras). The category $A_\infty(R)$ of $A_\infty$-algebras is a full subcategory of $dA_\infty(R)$. Indeed, if a $dA_\infty$-algebra $(A, m_{ij})$ is concentrated in horizontal degree 0, that is, $A^0_i = 0$ and $m_{ij} = 0$ for all $i > 0$, then $(A, m_{ij})$ is an $A_\infty$-algebra.

Example 4.6 (Underlying twisted complex). There is a forgetful functor $U : dA_\infty(R) \rightarrow tC_R$ defined by sending a $dA_\infty$-algebra $(A, m_{ij})$ to the twisted complex $(A, m_{ij})$ and a morphism $f = \{f_{ij}\}$ of $dA_\infty$-algebras to the morphism of twisted complexes given by $U(f) = \{f_{ij}\}$.

We define $E_r$-quasi-isomorphism for $dA_\infty$-algebras via their underlying twisted complexes.

Definition 4.7. Let $r \geq 0$. A morphism $f = \{f_{ij}\}$ of $dA_\infty$-algebras is said to be an $E_r$-quasi-isomorphism if the corresponding map $U(f) := \{f_{ij}\}$ of twisted complexes is an $E_r$-quasi-isomorphism.

Denote by $E_r$ the class of $E_r$-quasi-isomorphisms of $dA_\infty(R)$ and by $Ho_r(dA_\infty(R)) := dA_\infty(R)[E_r^{-1}]$ the $r$-homotopy category defined by inverting $E_r$-quasi-isomorphisms. Note that $E_r = U^{-1}(E_r^{C_R})$. The forgetful functor induces a functor $U : Ho_r(dA_\infty(R)) \rightarrow Ho_r(tC_R)$.

4.2. Monoidal categories over a base. In the following sections, all our notions of enriched category theory follow [Bor91]. Our new descriptions of $dA_\infty$-algebras will use certain enriched categories coming from monoidal categories over a base as defined in [Fre09]. We recall this notion first.

Definition 4.8. Let $(\mathcal{V}, \otimes, 1)$ be a symmetric monoidal category and $(\mathcal{C}, \otimes, 1)$ be a monoidal category. We say that $\mathcal{C}$ is a monoidal category over $\mathcal{V}$ if we have an external tensor product $\ast : \mathcal{V} \times \mathcal{C} \rightarrow \mathcal{C}$ such that we have natural isomorphisms:

- $1 \ast X \cong X$ for all $X \in \mathcal{C}$,
- $(C \otimes D) \ast X \cong C \ast (D \ast X)$ for all $C, D \in \mathcal{V}$ and $X \in \mathcal{C}$,
- $C \ast (X \otimes Y) \cong (C \ast X) \otimes Y = X \otimes (C \ast Y)$ for all $C \in \mathcal{V}$ and $X, Y \in \mathcal{C}$.

Remark 4.9. If we have, in addition, a bifunctor $\underline{\mathcal{C}}(-, -) : \mathcal{C}^{op} \times \mathcal{C} \rightarrow \mathcal{V}$ such that we have natural isomorphisms

$\text{Hom}_\mathcal{C}(C \ast X, Y) \cong \text{Hom}_\mathcal{V}(C, \underline{\mathcal{C}}(X, Y))$

(for example, if $\ast$ preserves colimits on the left and certain smallness conditions hold) we get a $\mathcal{V}$-enriched category $\underline{\mathcal{C}}$ with the same objects as $\mathcal{C}$ and with hom-objects given by $\underline{\mathcal{C}}(-, -)$. The unit morphism $u_A : 1 \rightarrow \underline{\mathcal{C}}(A, A)$ corresponds to the identity map in $\mathcal{C}$ under the adjunction and the composition morphism is given by the adjoint of the composite

$$(\underline{\mathcal{C}}(B, C) \otimes \underline{\mathcal{C}}(A, B)) \ast A \xrightarrow{\cong} \underline{\mathcal{C}}(B, C) \ast (\underline{\mathcal{C}}(A, B) \ast A) \xrightarrow{id_{ev_{AB}}} \underline{\mathcal{C}}(B, C) \ast B \xrightarrow{ev_{BC}} C,$$

where $ev_{AB}$ is the adjoint of the identity $\underline{\mathcal{C}}(A, B) \rightarrow \underline{\mathcal{C}}(A, B)$. Note that by construction, the underlying category of $\underline{\mathcal{C}}$ is $\mathcal{C}$. Furthermore, $\underline{\mathcal{C}}$ is a monoidal $\mathcal{V}$-enriched category, namely we have an enriched functor

$$\otimes : \underline{\mathcal{C}} \times \underline{\mathcal{C}} \rightarrow \underline{\mathcal{C}}$$

where $\underline{\mathcal{C}} \times \underline{\mathcal{C}}$ is the enriched category with objects $\text{Ob}(\underline{\mathcal{C}}) \times \text{Ob}(\underline{\mathcal{C}})$ and hom-objects

$$\underline{\mathcal{C}}(\underline{\mathcal{C}}((X, Y), (W, Z)) := \underline{\mathcal{C}}(X, W) \otimes \underline{\mathcal{C}}(Y, Z).$$

In particular we get maps in $\mathcal{V}$

$$\underline{\mathcal{C}}(X, W) \otimes \underline{\mathcal{C}}(Y, Z) \rightarrow \underline{\mathcal{C}}(X \otimes Y, W \otimes Z),$$

given by the adjoint of the composite

$$(\underline{\mathcal{C}}(X, W) \otimes \underline{\mathcal{C}}(Y, Z)) \ast (X \otimes Y) \xrightarrow{\cong} (\underline{\mathcal{C}}(X, W) \ast X) \otimes (\underline{\mathcal{C}}(Y, Z) \ast Y) \xrightarrow{ev_{XY} \otimes ev_{YZ}} W \otimes Z.$$

The monoidal structures of vertical bicomplexes and twisted complexes are compatible. More precisely, we can use the tensor product in twisted complexes to define monoidal structures over a base by restriction.
Lemma 4.10. The category $t\text{C}_R$ is a monoidal category over $v\text{bC}_R$ and the category $\text{bgMod}^\infty_R$ is a monoidal category over $\text{bgMod}_R$. In both cases, the external tensor product is given by restricting the tensor product in twisted complexes. We use the notation $\otimes$ instead of $*$ since the external tensor product coincides with the internal tensor product in $t\text{C}_R$.

Proof. The axioms of Definition 4.8 hold because $\otimes$ is a symmetric monoidal structure on $t\text{C}_R$ and the vertical arrows in the following diagram are monoidal embeddings.

We can also endow filtered complexes with a monoidal structure over $v\text{bC}_R$. We use the following.

Definition 4.11. The totalization with compact support of a vertical bicomplex $A$ is the filtered complex given by

$$\text{Tot}_c(A)^n := \bigoplus_{i \in \mathbb{Z}} A_{n+i}$$

with the column filtration and with differential as for the totalization functor. Given a morphism of vertical bicomplexes $f : A \to B$ we get a morphism of filtered complexes $\text{Tot}_c(f) : \text{Tot}_c(A) \to \text{Tot}_c(B)$ constructed analogously to $\text{Tot}(f)$.

Remark 4.12. Note $\text{Tot}_c$ is well-defined since vertical bicomplexes have only one differential and the category $v\text{bC}_R$ has only strict morphisms. Moreover, for any $A$ we have a natural map $\text{Tot}_c(A) \to \text{Tot}(A)$ which is the identity if $A$ is bounded.

Lemma 4.13. The category $f\text{C}_R$ is monoidal over $v\text{bC}_R$ with external tensor product given by

$$* : v\text{bC}_R \times f\text{C}_R \to f\text{C}_R$$

$$(A, K) \mapsto A * K := \text{Tot}_c(A) \otimes K.$$  

On morphisms it is given by the assignment $(f, g) \mapsto \text{Tot}_c(f) \otimes g$. This induces by restriction a monoidal structure on $f\text{Mod}_R$ over $\text{bgMod}_R$.

Proof. The assignments define a bifunctor since $\text{Tot}_c : v\text{bC}_R \to f\text{Mod}_R$ is a functor. Furthermore, the axioms of Definition 4.8 hold since $\otimes$ is a symmetric monoidal product in $f\text{C}_R$ and $\text{Tot}_c$ is strong symmetric monoidal since $\otimes$ distributes over $\oplus$. Finally, since $f\text{Mod}_R \hookrightarrow f\text{C}_R$ and $\text{bgMod}_R \hookrightarrow v\text{bC}_R$ are full embeddings, this construction induces by restriction a bifunctor $* : \text{bgMod}_R \times f\text{Mod}_R \to f\text{Mod}_R$ which gives a monoidal structure on $f\text{Mod}_R$ over $\text{bgMod}_R$.

4.3. Enrichments from monoidal structures over a base. We now explain how to give enrichments to the categories of bigraded modules, filtered modules, twisted complexes and filtered complexes, using their monoidal structure over a base. This will give the $v\text{bC}_R$-enriched categories $t\text{C}_R$ and $f\text{C}_R$ and $\text{bgMod}_R$-enriched categories $\text{bgMod}^R_R$ and $f\text{Mod}_R$. We emphasize that $\text{bgMod}_R$ is different from its standard enrichment coming from its symmetric monoidal closed structure.

Recall from Definition 3.32 that for bigraded modules $A, B, C$, we have already defined a bigraded module $\text{bgMod}_R(A, B)$ and a composition

$$c : \text{bgMod}_R(B, C) \otimes \text{bgMod}_R(A, B) \to \text{bgMod}_R(A, C).$$

Lemma 4.14. The composition morphism respects the identity and is associative.
Proof. For $f, g, h$,

$$c(c(f, g), h)_m = \sum_{i+j+k=m} (-1)^{(i|g|+|h|)+j|h|} f_igjh_k = c(f, c(g, h)).$$

□

Definition 4.15. Let $(A, d^A_i), (B, d^B_i)$ be twisted complexes, $f \in \underline{bgMod}_R(A, B)_u$ and consider $d^A := (d^A_i) \in \underline{bgMod}_R(A, A)_i$ and $d^B := (d^B_i) \in \underline{bgMod}_R(B, B)_i$. We define

$$\delta(f) := c(d^B, f) - (-1)^{<f, d^A>} c(f, d^A) \in \underline{bgMod}_R(A, B)_{u+1},$$

where $<f, d^A>$ is the scalar product for the bidegrees and $c$ is the composition morphism described in Definition 3.3. More precisely,

$$(\delta(f))_m := \sum_{i+j=m} (-1)^{|f|} d^B_i f_j - (-1)^{v+i} d^A_i f^2.$$ 

Lemma 4.16. The following equations hold

$$c(d^A, d^A) = 0,$$

$$\delta^2 = 0,$$

where the bidegree of $f$ is $(u, v)$. Furthermore, $f \in \underline{bgMod}_R(A, B)$ is a map of twisted complexes if and only if $\delta(f) = 0$. In particular, $f$ is a morphism in $\underline{tC}_R$ if and only if the bidegree of $f$ is $(0, 0)$ and $\delta(f) = 0$. Moreover, for $f, g$ morphisms in $\underline{tC}_R$, we have that $c(f, g) = f \circ g$, where the latter denotes composition in $\underline{tC}_R$.

Proof. One has

$$c(d^A, d^A)_m = \sum_{i+j=m} (-1)^i d^A_i d^A_j = 0,$$

so that

$$\delta^2(f) = c(d^B, \delta(f)) - (-1)^{<\delta(f), d^A>} c(\delta(f), d^A)$$

$$= c(d^B, c(d^B, f)) - (-1)^{<f, d^A>} c(d^B, c(f, d^A)) + (-1)^{<f, d^A>} c(c(d^B, f), d^A) - c(c(f, d^A), d^A) = 0.$$ 

The last equation follows from the associativity of $c$. □

Since $\delta$ is of bidegree $(0, 1)$, Lemma 4.16 allows us to make the following definition.

Definition 4.17. For $A, B$ twisted complexes, we define $\underline{tC}_R(A, B)$ to be the vertical bicomplex $\underline{tC}_R(A, B) := (\underline{bgMod}_R(A, B), \delta)$.

Proposition 4.18. If $B, C$ are twisted complexes, then the construction of the vertical bicomplex $\underline{tC}_R(B, C) := (\underline{bgMod}_R(B, C), \delta)$ extends to a bifunctor

$$\underline{tC}_R(-, -) : \underline{tC}_R^{op} \times \underline{tC}_R \to \underline{vbC}_R,$$

where for $f : C \to C'$ in $\underline{tC}_R$ we set

$$\underline{tC}_R(B, f) : \underline{tC}_R(B, C) \to \underline{tC}_R(B, C') \quad \text{and} \quad \underline{tC}_R(f, B) : \underline{tC}_R(C, B) \to \underline{tC}_R(C, B).$$

Moreover, the functor $- \otimes B : \underline{vbC}_R \to \underline{tC}_R$ is left adjoint to the functor $\underline{tC}_R(B, -) : \underline{tC}_R \to \underline{vbC}_R$, i.e., for all $A \in \underline{vbC}_R, B, C \in \underline{tC}_R$ we have natural isomorphisms

$$\text{Hom}_{\underline{tC}_R}(A \otimes B, C) \cong \text{Hom}_{\underline{vbC}_R}(A, \underline{tC}_R(B, C)),$$

given by $f \mapsto \tilde{f}$ where for $a \in A, \tilde{f}(a)_m$ is given by $b \mapsto (-1)^m[a]f_m(a \otimes b)$. (3)
Proof. If \( f \) is a map in \( tC_R \) then it is of bidegree \((0, 0)\) and \( \delta(f) = 0 \) and thus by (2)
\[
\delta(c(f, g)) = c(\delta(f), g) + (1)^{\nu}c(f, \delta(g)) = c(f, \delta(g)),
\]
showing that \( tC_R(B, f) \) is a map of vertical bicomplexes. A similar argument shows that \( tC_R(f, B) \) is a map of vertical bicomplexes. Finally, the fact that \( tC_R(-, -) \) is a bifunctor follows directly from Lemmas 3.13 and 4.10. Now, to see the adjointness property we describe a map \( \text{Hom}_{C_R}(A \otimes B, C) \to \text{Hom}_{vbC_R}(A, tC_R(B, C)) \), which sends a map of twisted complexes \( f = (f_m) : A \otimes B \to C \), to the map
\[
\tilde{f} : A \to tC_R(B, C)
\]
\[
a \mapsto \{\tilde{f}(a)_m : b \mapsto (-1)^{m|a|}f_m(a \otimes b)\}_{m \geq 0}.
\]
It is clear that \( \tilde{f} \) is a bidegree \((0, 0)\) map of bimodules. To show that it is a map of vertical bicomplexes we will show that \( \delta \tilde{f} = f d^A \). Let \( a \in A_u \), then \( \tilde{f}(a) \) has bidegree \((u, v)\) so that
\[
\delta(\tilde{f}(a)) = \sum_{i+j=m} (-1)^{i|a|+j|a|}d_i^C(\tilde{f}(a))_j - (-1)^{v+i}(\tilde{f}(a))_j d_j^B.
\]
Applying this to \( b \in B_u \), one gets
\[
(\delta(\tilde{f}(a)))_m(b) = \sum_{i+j=m} (-1)^{i|a|+j|a|}d_i^C f_j(a \otimes b) - (-1)^{v+i+|a|}f_i(a \otimes d_j^B(b))
\]
\[
= \sum_{i+j=m} (-1)^{m|a|}f_j(d_i^A(a) \otimes b) + \sum_{i+j=m} (-1)^{m|a|+j|a|+i}f_j(a \otimes d_i^B(b))
\]
\[
- \sum_{i+j=m} (-1)^{v+j+|a|}f_j(a \otimes d_i^B(b)).
\]
Since \( A \) is a vertical bicomplex we have that \( d_i^A a = 0 \) for \( i > 0 \). Thus,
\[
(\delta(\tilde{f}(a)))_m = (-1)^{m|a|+m}f_m(d^A(a) \otimes b) = \tilde{f}(d^A(a))_m(b).
\]
The inverse map is constructed in a similar fashion. \(\square\)

**Definition 4.19.** We define \( bgMod_{R}(-, -) \) to be the restriction of the bifunctor \( tC_R(-, -) \) to the full subcategories \( bgMod^{\infty}_{R} \times bgMod^{\infty}_{R} \).

**Proposition 4.20.** The image of \( bgMod_{R}(-, -) \) factors through \( bgMod_{R} \). Therefore it defines a bifunctor
\[
bgMod_{R}(-, -) : bgMod^{\infty}_{R} \times bgMod^{\infty}_{R} \to bgMod_{R}
\]
and \( - \otimes B : bgMod_{R} \to bgMod^{\infty}_{R} \) is left adjoint to the functor \( bgMod_{R}(-, B) : bgMod^{\infty}_{R} \to bgMod_{R} \), i.e., for all \( A \in bgMod_{R} \), \( B, C \in bgMod^{\infty}_{R} \) we have natural isomorphisms
\[
\text{Hom}_{bgMod_{R}^{\infty}} (A \otimes B, C) \cong \text{Hom}_{bgMod_{R}} (A, tC_R(B, C)).
\]

**Proof.** This follows directly from the facts that if \( B, C \in bgMod^{\infty}_{R} \) then \( tC_R(B, C) \) has trivial differential and that the functors \( bgMod_{R} \to vbC_R \) and \( bgMod^{\infty}_{R} \to tC_R \) are full embeddings. \(\square\)

This construction gives us our different enrichments of twisted complexes and bigraded modules, which we describe now.

**Definition 4.21.** The \( vbC_{R} \)-enriched category of twisted complexes \( tC_R \) is the enriched category given by the following data.

1. The objects of \( tC_R \) are twisted complexes.
2. For \( A, B \) twisted complexes the hom-object is the vertical bicomplex \( tC_R(A, B) \)
3. The composition morphism \( c : tC_R(B, C) \otimes tC_R(A, B) \to tC_R(A, C) \) is given by Definition 3.32
4. The unit morphism \( R \to tC_R(A, A) \) is given by the morphism of vertical bicomplexes sending \( 1 \in R \) to \( 1_A : A \to A \), the strict morphism of twisted complexes given by the identity of \( A \).

**Definition 4.22.** We denote by \( bgMod_{R} \) the \( bgMod_{R} \)-enriched category of bigraded modules given by the following data.
(1) The objects of $\text{bgMod}_R$ are bigraded modules.
(2) For $A, B$ bigraded modules the hom-object is the bigraded module $\text{bgMod}_R(A, B)$.
(3) The composition morphism $c : \text{bgMod}_R(B, C) \otimes \text{bgMod}_R(A, B) \to \text{bgMod}_R(A, C)$ is given by Definition 3.32.
(4) The unit morphism $R \to \text{bgMod}_R(A, A)$ is given by the morphism of bigraded modules that sends $1 \in R$ to $1_A : A \to A$, the strict morphism given by the identity of $A$.

**Lemma 4.23.** $\text{tC}_R$ and $\text{bgMod}_R$ are well-defined and their enrichments are the ones induced by the external tensor products $\otimes : \text{vbC}_R \times \text{tC}_R \to \text{tC}_R$ and $\otimes : \text{bgMod}_R \times \text{bgMod}_R^\infty \to \text{bgMod}_R^\infty$. Therefore, these are also monoidal enriched categories and their underlying categories are $\text{tC}_R$ and $\text{bgMod}_R^\infty$ respectively.

**Proof.** This follows directly from Propositions 4.18 and 4.20. Notice in particular that in the case of $\text{tC}_R$, the fact that the composition morphism $c$ is a map of vertical bicomplexes is equivalent to equation (2) in Lemma 4.16. □

**Remark 4.24.** There is an interpretation of $\text{bgMod}_R^\infty$ and $\text{bgMod}_R$ via a standard categorical construction, the co-Kleisli category for a comonad. Recall that $\mathcal{D}$ denotes the cofree $\mathcal{D}$-coalgebra functor from $\text{bgMod}_R$ to the category of $\mathcal{D}$-coalgebras, with left adjoint the forgetful functor $U$. Then $\text{bgMod}_R^\infty$ is the co-Kleisli category of $\text{bgMod}_R$ for the comonad $U\mathcal{D}$. And this construction enriches. Namely, recall that $\text{bgMod}_R$ denotes the category of bigraded modules enriched over itself via its symmetric monoidal closed structure. Then $\text{bgMod}_R^\infty$ is the enriched co-Kleisli category of $\text{bgMod}_R$ for the enriched comonad $U\mathcal{D}$. To see this, note that the objects are the same, Proposition 3.33 gives the isomorphism on morphisms or hom-objects and this isomorphism respects the composition.

We can describe the monoidal structures of $\text{tC}_R$ and $\text{bgMod}_R$ explicitly.

**Lemma 4.25.** The monoidal structure of $\text{tC}_R$ is given by the following map of vertical bicomplexes.

$$\hat{\otimes} : \text{tC}_R(A, B) \otimes \text{tC}_R(A', B') \to \text{tC}_R(A \otimes A', B \otimes B')$$

$$(f, g) \mapsto (f \hat{\otimes} g)_m := \sum_{i+j=m} (-1)^{ij} f_i \otimes g_j$$

The monoidal structure of $\text{bgMod}_R$ is given by the restriction of this map.

**Proof.** The same argument as in the proof of Proposition 4.18 shows that this is indeed a map of vertical bicomplexes. To see that this is indeed the enriched monoidal structure induced by the monoidal structure of $\text{tC}_R$ over $\text{vbC}_R$, one can follow closely Remark 4.9. □

Next we introduce enriched structures on filtered modules and filtered complexes. We enrich filtered modules over bigraded modules in the following way.

**Definition 4.26.** The $\text{bgMod}_R$-enriched category of filtered modules $\text{fMod}_R$ is the enriched category given by the following data.

1. The objects of $\text{fMod}_R$ are filtered modules.
2. For filtered modules $(K, F)$ and $(L, F)$, the bigraded module $\text{fMod}_R(K, L)$ is given by

$$\text{fMod}_R(K, L)_u := \{ f : K \to L \mid f(F_qK^m) \subset F_{q+u}L^{m+u-v}, \forall m, q \in \mathbb{Z} \}.$$  

3. The composition morphism is given by $c(f, g) = (-1)^{uv}fg$, where $f$ has bidegree $(u, v)$.
4. The unit morphism is given by the map $R \to \text{fMod}_R(K, K)$ given by $1 \to 1_K$.

We denote by $\text{sfMod}_R$ the full subcategory of $\text{fMod}_R$ on objects split filtered modules.

**Lemma 4.27.** The above definition gives a well-defined $\text{bgMod}_R$-enriched category, $\text{fMod}_R$.

**Proof.** Let $f \in \text{fMod}_R(L, M)_u$ and $g \in \text{fMod}_R(K, L)_{u'}$, where $(K, F)$, $(L, F)$ and $(M, F)$ are filtered modules. For all $q \in \mathbb{Z}$ one has

$$c(f, g)(F_qK^m) \subset f(F_{q+u'}L^{m+u'-v'}) \subset F_{q+u+u'}M^{m+v-u+v'-u'},$$

where $F_qK^m$ and $F_{q+u'}L^{m+u'-v'}$ are filtered modules.

From the above, we have that $c(f, g)$ is well-defined. □
so \( c(f, g) \in \mathcal{F}_{\mathcal{B} \mathcal{M} \mathcal{D}}(K, M)_{v+u'} \) showing that the composition morphism is a map in \( \mathcal{B} \mathcal{M} \mathcal{D}_R \) and \( 1_K \in \mathcal{F}_{\mathcal{B} \mathcal{M} \mathcal{D}}(K, K)_{0} \). It is a short computation to show that the associativity and unit axiom hold. \( \square \)

**Remark 4.28.** Notice that morphisms \( f \in \mathcal{F}_{\mathcal{B} \mathcal{M} \mathcal{D}}(K, L)_{0} \) correspond precisely to degree \( v \) morphisms of filtered modules which respect the filtration.

We will enrich filtered complexes over vertical bicomplexes analogously.

**Definition 4.29.** Let \((K, d^K, F)\) and \((L, d^L, F)\) be filtered complexes. We define \( \mathcal{F}_{\mathcal{C} \mathcal{R}}(K, L) \) to be the vertical bicomplex whose underlying bigraded module is \( \mathcal{F}_{\mathcal{B} \mathcal{M} \mathcal{D}}(K, L) \) with vertical differential

\[
\delta(f) := c(d^K, f) - (-1)^{v} c(f, d^K) = d^K f - (-1)^{v+u} f d^K = d^K f - (-1)^{|f|} f d^K
\]

for \( f \in \mathcal{F}_{\mathcal{B} \mathcal{M} \mathcal{D}}(K, L)_{u} \).

Note that \( d^K \in \mathcal{F}_{\mathcal{B} \mathcal{M} \mathcal{D}}(K, K)_{0} \) and \( d^L \in \mathcal{F}_{\mathcal{B} \mathcal{M} \mathcal{D}}(L, L)_{0} \). So \( \delta(f) \) has bidegree \((u, v+1)\). Also \( \delta^2 = 0 \) and thus \( \mathcal{F}_{\mathcal{C} \mathcal{R}}(K, L) \) is indeed a vertical bicomplex.

**Remark 4.30.** Notice that \( f \in \mathcal{F}_{\mathcal{C} \mathcal{R}}(K, L)_{u} \) is a map of complexes if and only if \( \delta(f) = 0 \). In particular, \( f \) is a morphism in \( \mathcal{C} \mathcal{R} \) if and only if \( f \in \mathcal{F}_{\mathcal{C} \mathcal{R}}(K, L)_{0} \) and \( \delta(f) = 0 \).

**Definition 4.31.** The \( \mathcal{V} \mathcal{B} \mathcal{C} \mathcal{R} \)-enriched category of filtered complexes \( \mathcal{F}_{\mathcal{C} \mathcal{R}} \) is the enriched category given by the following data.

1. The objects of \( \mathcal{F}_{\mathcal{C} \mathcal{R}} \) are filtered complexes.
2. For \( K, L \) filtered complexes the hom-object is the vertical bicomplex \( \mathcal{F}_{\mathcal{C} \mathcal{R}}(K, L) \).
3. The composition morphism is given as in \( \mathcal{F}_{\mathcal{B} \mathcal{M} \mathcal{D}} \) in Definition 4.26.
4. The unit morphism is given by the map \( K \to \mathcal{F}_{\mathcal{C} \mathcal{R}}(K, K) \) given by \( 1 \to 1_K \).

We denote by \( \mathcal{S}_{\mathcal{C} \mathcal{R}} \) the full subcategory of \( \mathcal{F}_{\mathcal{C} \mathcal{R}} \) on objects split filtered complexes.

**Lemma 4.32.** The above definition gives a well-defined \( \mathcal{V} \mathcal{B} \mathcal{C} \mathcal{R} \)-enriched category, \( \mathcal{F}_{\mathcal{C} \mathcal{R}} \).

**Proof.** To see that the composition and unit maps are maps of vertical bicomplexes note that for \((K, d^K, F)\) and \((L, d^L, F)\) filtered complexes, \( f \in \mathcal{F}_{\mathcal{B} \mathcal{M} \mathcal{D}}(K, L)_{u} \) and \( g \in \mathcal{F}_{\mathcal{B} \mathcal{M} \mathcal{D}}(L, M)_{v'} \)

\[
\delta(c(f, g)) = c(\delta(f), g) + (-1)^{|f|} c(f, \delta(g)).
\]

The associativity and unit axiom hold, because they hold in \( \mathcal{F}_{\mathcal{B} \mathcal{M} \mathcal{D}} \). \( \square \)

**Lemma 4.33.** The enrichment of filtered complexes and filtered modules is the one induced by the external tensor products \( \ast : \mathcal{V} \mathcal{B} \mathcal{C} \mathcal{R} \times \mathcal{C} \mathcal{R} \to \mathcal{C} \mathcal{R} \) and \( \ast : \mathcal{B} \mathcal{M} \mathcal{D}_R \times \mathcal{B} \mathcal{M} \mathcal{D}_R \to \mathcal{B} \mathcal{M} \mathcal{D}_R \). Therefore, the enriched categories \( \mathcal{F}_{\mathcal{C} \mathcal{R}} \) and \( \mathcal{F}_{\mathcal{B} \mathcal{M} \mathcal{D}} \) are also monoidal enriched categories and their underlying categories are \( \mathcal{C} \mathcal{R} \) and \( \mathcal{B} \mathcal{M} \mathcal{D}_R \) respectively.

**Proof.** Since \( \mathcal{F}_{\mathcal{C} \mathcal{R}} \) is a well-defined \( \mathcal{V} \mathcal{B} \mathcal{C} \mathcal{R} \)-enriched category we have a bifunctor

\[
\mathcal{F}_{\mathcal{C} \mathcal{R}}(-, -) : \mathcal{C} \mathcal{R}^{op} \times \mathcal{C} \mathcal{R} \to \mathcal{V} \mathcal{B} \mathcal{C} \mathcal{R}.
\]

It is left to show that we have natural isomorphisms \( \text{Hom}_{\mathcal{C} \mathcal{R}}(A \ast K, L) \cong \text{Hom}_{\mathcal{F}_{\mathcal{C} \mathcal{R}}}(A, \mathcal{F}_{\mathcal{C} \mathcal{R}}(K, L)) \). In one direction we have a map

\[
\text{Hom}_{\mathcal{C} \mathcal{R}}(\text{Tot}_c(A) \otimes K, L) \to \text{Hom}_{\mathcal{F}_{\mathcal{C} \mathcal{R}}}(A, \mathcal{F}_{\mathcal{C} \mathcal{R}}(K, L)),
\]

\[
f \mapsto \tilde{f} : a \mapsto (k \mapsto f(a \otimes k)).
\]

In the inverse direction we have a map

\[
\text{Hom}_{\mathcal{F}_{\mathcal{C} \mathcal{R}}}(A, \mathcal{F}_{\mathcal{C} \mathcal{R}}(K, L)) \to \text{Hom}_{\mathcal{C} \mathcal{R}}(\text{Tot}_c(A) \otimes K, L)
\]

\[
\tilde{g} : a \mapsto g_a \mapsto g : (a_i) \otimes k \mapsto \sum g_a(k).
\]

These constructions are inverse to each other and natural. \( \square \)
We can define the monoidal structure of \( \mathcal{C}_R \) explicitly.

**Lemma 4.34.** The monoidal structure of \( \mathcal{C}_R \) is given by the following map of vertical bicomplexes.

\[
\otimes : \mathcal{C}_R(K, L) \otimes \mathcal{C}_R(K', L') \to \mathcal{C}_R(K \otimes K', L \otimes L'),
\]

\[
(f, g) \mapsto f \otimes g := (-1)^{u|g|} f \otimes g,
\]

where \( f \) has bidegree \((u, v)\).

**Proof.** A direct computation using the Koszul rule gives

\[
c(f, f') \otimes c(g, g') = (-1)^{u_1|f'| + u_2|g'| + (u_1 + u_2)(|g| + |g'|)} f f' \otimes gg',
\]

which shows that the construction is functorial. To see that this is indeed the enriched monoidal structure induced by the monoidal structure of \( \mathcal{C}_R \) over \( \mathcal{C}_R \), one can follow closely Remark 4.9. \(\square\)

### 4.4. Enriched totalization

With our sign conventions the totalization functor extends to the enriched setting. The formal setup is the following. For \( A \in \mathcal{C}_R \) and \( B \in \mathcal{C}_R \) we have a map of filtered complexes \( \nu_{A,B} : A \ast \text{Tot}(B) \to \text{Tot}(A \otimes B) \) which defines a natural transformation \( \nu : - \ast \text{Tot}(-) \Rightarrow \text{Tot}(- \otimes -) \). Thus, one could say that \( \text{Tot} \) is “lax monoidal over \( \mathcal{C}_R \)." Fresse’s setup extends to this context and it implies in particular that the enrichments of twisted complexes and filtered complexes are compatible under the totalization functor, i.e. \( \text{Tot} \) extends to an enriched functor \( \mathfrak{Tot} : \mathcal{C}_R \to \mathcal{C}_R \), given by the assignment \( \mathfrak{Tot}_{A,B} : \mathcal{C}_R(A, B) \to \mathcal{C}_R(\text{Tot}(A), \text{Tot}(B)) \), which is the map of vertical bicomplexes adjoint to the composite

\[
\mathfrak{C}_R(A, B) \ast \text{Tot}(A) \xrightarrow{\nu} \text{Tot}(\mathfrak{C}_R(A, B) \otimes A) \xrightarrow{\text{Tot}(ev)} \text{Tot}(B).
\]

This justifies our sign conventions. We now describe this structure explicitly and show that the properties of the totalization functor also hold in the enriched sense. In order to do this, we first extend the definition of \( \text{Tot} \) to morphisms of any bidegree.

**Definition 4.35.** Let \( A, B \) be bigraded modules and \( f \in \text{bgMod}_R(A, B)_u^v \) we define

\[
\text{Tot}(f) \in \text{bgMod}_R(\text{Tot}(A), \text{Tot}(B))_u^v
\]

to be given on any \( a \in \text{Tot}(A)^m \) by

\[
(\text{Tot}(f)(a))_{j+u} := \sum_{m \geq 0} (-1)^{(m+u)n} f_m(a_{j+m}) \in B_{j+u}^{j+n+u} \subset \text{Tot}(B)^{n+v-u}.
\]

Let \( K = \text{Tot}(A), L = \text{Tot}(B) \) and \( g \in \text{bgMod}_R(K, L)_u^v \) we define

\[
f := \text{Tot}^{-1}(g) \in \text{bgMod}_R(A, B)_u^v
\]

to be \( f := (f_0, f_1, \ldots) \) where \( f_i \) is given on each \( A_j^{m+j} \) by the composite

\[
f_i : A_j^{m+j} \hookrightarrow \prod_{k \leq j} A_k^{m+k} = F_j(\text{Tot}(A)^m) \xrightarrow{g} F_j(\text{Tot}(B)^{m+v-u}) \]

\[
\xrightarrow{\prod (\text{projection})} \prod_{l \leq j+u} B_l^{m+v-u+l} \xrightarrow{\text{multiplication}} B_{j+u}^{m+j+v-u},
\]

where the last map is a projection and multiplication with the indicated sign.

**Lemma 4.36.** If \( f \in \text{bgMod}_R(A, B)_u^v \), then \( \text{Tot}(f) \in \text{bgMod}_R(\text{Tot}(A), \text{Tot}(B))_u^v \). Moreover

\[
c(\text{Tot}(f), \text{Tot}(g)) = \text{Tot}(c(f, g)).
\]
Proof. One has to prove that, for all $m, q \in \mathbb{Z}$, $\text{Tot}(f)(F_q(\text{Tot}(A)^m)) \subset F_{q+u}(\text{Tot}(B)^{m+v-u})$. Let $a \in F_q(\text{Tot}(A)^m)$, that is to say $a_k = 0$ for $k > q$. Hence for $j > q$ one has $\text{Tot}(f)(a)_{j+u} = 0$. Hence $\text{Tot}(f)(a) \in F_{q+u}(\text{Tot}(B)^{m+v-u})$. Recall that in $\mathsf{bgMod}_R$ and in $\mathsf{fMod}_R$, the compositions $c(f, g)$ and $c(\text{Tot}(f), \text{Tot}(g))$, with $f$ and $g$ of bidegree $(u, v)$ and $(u', v')$ respectively, are defined by

$$c(f, g)_m = \sum_{i+j=m} (-1)^{|i||j|} f_i g_j, \quad c(\text{Tot}(f), \text{Tot}(g)) = (-1)^{|u||g|} \text{Tot}(f) \text{Tot}(g).$$

Let $a \in \text{Tot}(A)^n$.

$$(-1)^{|u||g|} ((\text{Tot}(f) \text{Tot}(g))(a))_{j+u+u'} = \sum_{m \geq 0} (-1)^{|u||g|+(m+u)(|g|+n)} f_m (\text{Tot}(g)(a))_{j+u'+m}$$

$$= \sum_{m \geq 0, p \geq 0} (-1)^{|u||g|+(m+u)(|g|+n)+(p+u')n} f_m (g_p(a_{j+m+p}))$$

$$= \sum_{l \geq 0} \sum_{m+p = l} (-1)^{m+p+u+u'}(c(f, g))_l (a_{j+l})$$

$$= (\text{Tot}(c(f, g))(a))_{j+u+u'}.$$

\[\square\]

**Theorem 4.37.** Let $A, B$ be bigraded modules. The assignments $\mathfrak{T}ot(A) := \text{Tot}(A)$ and

$$\mathfrak{T}ot_{A, B} : \mathsf{bgMod}_R(A, B) \to \mathsf{fMod}_R(\text{Tot}(A), \text{Tot}(B))$$

define a $\mathsf{bgMod}_R$-enriched functor $\mathfrak{T}ot : \mathsf{bgMod}_R \to \mathsf{fMod}_R$ which restricts to an isomorphism onto its image $\mathsf{vbC}_R$. Furthermore, this functor extends to a $\mathsf{vbC}_R$-enriched functor

$$\mathfrak{T}ot : \mathsf{C}_R \to \mathsf{C}_R$$

which restricts to an isomorphism onto its image $\mathsf{C}_R$.

Proof. By construction, $\mathfrak{T}ot_{A, B}$ is a morphism in $\mathsf{bgMod}_R$. To show that $\mathfrak{T}ot$ defines an enriched functor it is only left to check that this assignment is associative and unital. This holds by Lemma 4.36. To see it restricts to an isomorphism onto its image we construct a $\mathsf{bgMod}_R$-enriched functor which is inverse to the restriction of $\mathfrak{T}ot$ onto its image. Let $K, L$ be split filtered modules, then we define $\mathfrak{T}ot^{-1}$ to be given by the assignments $\mathfrak{T}ot^{-1}(K) := \text{Tot}^{-1}(K)$ and

$$\mathfrak{T}ot^{-1}_{K, L} : \mathsf{fMod}_R(K, M) \to \mathsf{bgMod}_R(\text{Tot}^{-1}(K), \text{Tot}^{-1}(L))$$

$$f \mapsto \text{Tot}^{-1}(f).$$

We have that, $\text{Tot}(\text{Tot}^{-1}(K)) = K$ and $\text{Tot}^{-1}(\text{Tot}(A)) = A$. Now we check that $\text{Tot}^{-1}(\text{Tot}(f)) = f$. Write $\pi_l$ for the projection $\prod_{l \leq j+u} P_l^{m+v-u+l} \to B_l^{m+v-u+l}$ (without a sign). For $f$ of bidegree $(u, v)$ and $a_j \in A_j^{n+j}$ one has that

$$((\text{Tot}^{-1} \circ \text{Tot}(f))_l(a_j) = (-1)^{(i+u)\pi} (-1)^{|k+u|m} f_{k}(a_{j-i+k})) = f_i(a_j),$$

showing that $\text{Tot}^{-1}(\text{Tot}(f)) = f$. To see that $\text{Tot}(\text{Tot}^{-1}(g)) = g$, let $g : (\text{Tot}(A))^n \to (\text{Tot}(B))^{n+v-u}$ be a map of bidegree $(u, v)$. For $(a) \in (\text{Tot}(A))^n$ we write $g(a) = (g_k(a))_{k \in \mathbb{Z}}$ where $g_k(a) \in B_k^{n+v-u+k}$. Recall that for $q$, we have that $g(F_q(\text{Tot}(A)^n)) \subset F_{q+u}(\text{Tot}(B))^{n+v-u}$. Fix $q \in \mathbb{Z}$. If $(a) \in (\text{Tot}(A))^n$
then write $a = \alpha_{q-1} + \sum_{m \geq 0} a_{q+m}$ where $\alpha_{q-1} \in \prod_{r \leq q-1} A_r^{n+r} = F_{q-1}(\text{Tot}(A))^n$ and $\sum_{m \geq 0} a_{q+m}$ is finite. As a consequence
\[ g_{q+u}(a) = \sum_{m \geq 0} g_{q+u}(a_{q+m}), \]
\[ \text{Tot}^{-1}(g)_m(a_{j+m}) = (-1)^{(m+u)n} g_{j+u}(a_{j+m}). \]
Hence
\[ (\text{Tot} \circ \text{Tot}^{-1}(g))(a_{j+u}) = \sum_{m \geq 0} (-1)^{(m+u)n} \text{Tot}^{-1}(g)_m(a_{j+m}) = \sum_{m \geq 0} g_{j+u}(a_{j+m}) = g_{j+u}(a), \]
showing that $\text{Tot}^{-1}(g)$ is a vbC$_R$-enriched functor. Hence $\text{Tot}(\text{Tot}^{-1}(g)) = g$. Finally, it follows that $\mathfrak{Tot}$ is associative and unital since it is the inverse to $\mathfrak{Tot}$ and thus it defines a bgMod$_R$-enriched functor $\mathfrak{Tot}^{-1} : \mathcal{fMod}_R \rightarrow \mathcal{gMod}_R$ which is inverse to the restriction of $\mathfrak{Tot}$ onto its image.

Now to see that both $\mathfrak{Tot}$ and $\mathfrak{Tot}^{-1}$ extend to vbC$_R$-enriched functors notice first that for any twisted complex $(A, d^A)$ it holds that $\text{Tot}(d^A) = d^{\text{Tot}(A)}$. Now, for $f \in \mathcal{tC}_R(A, B)$ of bidegree $(u, v)$, $\delta \text{Tot}(f) = d^{\text{Tot}(B)} \text{Tot}(f) - (-1)^v \text{Tot}(f) d^{\text{Tot}(A)} = \text{Tot}(\delta(f))$. Thus, $\delta \mathfrak{Tot} = \mathfrak{Tot} \delta$. Therefore, the maps defining $\mathfrak{Tot}$ are maps in vbC$_R$. The same argument shows that $\mathfrak{Tot}^{-1}$ is a vbC$_R$-enriched functor. 

**Proposition 4.38.** The enriched functors
\[ \mathfrak{Tot} : \mathcal{bgMod}_R \rightarrow \mathcal{fMod}_R, \quad \mathfrak{Tot} : \mathcal{tC}_R \rightarrow \mathcal{fC}_R \]
are lax symmetric monoidal in the enriched sense and when restricted to the bounded case they are strong symmetric monoidal in the enriched sense. More precisely, there are enriched natural transformations
\[ \mu : \mathfrak{Tot} \circ \mathfrak{Tot} \Rightarrow \mathfrak{Tot}(- \otimes -), \quad \epsilon : R \Rightarrow \mathfrak{Tot}(R), \]
where $R$ and $\mathfrak{Tot}(R)$ are the constant (enriched) functors on $R$ and $\mathfrak{Tot}(R)$ respectively, which restrict to natural isomorphisms in the bounded case.

**Proof.** We show that $\mathfrak{Tot} : \mathcal{tC}_R \rightarrow \mathcal{fC}_R$ is lax symmetric monoidal in the enriched sense. The same argument will show by restriction that $\mathfrak{Tot} : \mathcal{bgMod}_R \rightarrow \mathcal{fMod}_R$ is also enriched lax symmetric monoidal.

Let $(A, B)$ be an object in $\mathcal{tC}_R \times \mathcal{tC}_R$. The vbC$_R$-enriched natural transformations $\mu$ and $\epsilon$ are given by the following assignments:
\[ \mu_{(A,B)} : \begin{array}{ll}
R & \rightarrow \mathcal{fC}_R(\text{Tot}(A) \otimes \text{Tot}(B), \text{Tot}(A \otimes B)), \\
1 & \mapsto \mu_{A,B} \\
\end{array} \]
\[ \epsilon : \begin{array}{ll}
R & \rightarrow \mathcal{fC}_R(R, \text{Tot}(R)), \\
1 & \mapsto \epsilon = 1_R \\
\end{array} \]
where $\mu_{A,B}$ is given as in Proposition 3.11. That is, for $a = (a_i)_i \in \text{Tot}(A)^{n_1}$ and $b = (b_j)_j \in \text{Tot}(B)^{n_2}$,
\[ ((\mu_{A,B}(a \otimes b))_k) = \sum_{k_1+k_2=k} (-1)^{k_1n_2} a_{k_1} \otimes b_{k_2}. \]
Note first that since $\mu_{A,B}$ is a map of complexes of degree 0 that respects the filtration, $\mu_{A,B}$ is indeed a map in vbC$_R$, and similarly for $\epsilon$. To show that $\mu$ indeed defines an enriched natural transformation amounts to showing that for $f \in \mathcal{tC}_R(A, A')$ and $g \in \mathcal{tC}_R(B, B')$ the following holds
\[ c(\text{Tot}(f \otimes g), \mu_{AB}) = c(\mu_{A'B'}, \text{Tot}(f) \otimes \text{Tot}(g)). \]
Let $a \otimes b \in \text{Tot}(A)^{n_1} \otimes \text{Tot}(B)^{n_2}$, with $n_1 + n_2 = n$. Calculating the left-hand side we get
\[ (c(\text{Tot}(f \otimes g), \mu_{AB})(a \otimes b))_{j+a+u'_{\beta}} = \sum_{m \geq 0} \pm (f \otimes g)_m((\mu_{AB}(a \otimes b))_{j+m}) \]
\[ = \sum_{m_1, m_2 \geq 0, k_1+k_2=j} (-1)^{\epsilon_{\beta}} f_{m_1}(a_{k_1+m_1}) \otimes g_{m_2}(b_{k_2+m_2}), \]
where
\[ \epsilon_{\beta} = k_1n_2 + m_1n_1 + m_2n_2 + un + (k_1 + n_1)|\beta| + u'n_2. \]
On the other hand, evaluating the right-hand side we get
\[
(\epsilon_{\mu A'B'}, \text{Tot}(f) \hat{\otimes} \text{Tot}(g))(a \otimes b)_{j+u+u'} = \sum_{k_1+k_2=j} \pm \text{Tot}(f)(a)_{k_1+u} \otimes \text{Tot}(g)(b)_{k_2+u'}
\]
\[
= \sum_{m_1,n_2 \geq 0} (-1)^{\epsilon_l} f_{m_1}(a_{k_1+m_1}) \otimes g_{m_2}(a_{k_2+m_2}),
\]
where
\[
\epsilon_l = k_1n_2 + m_1n_1 + m_2n_2 + un + (k_1 + n_1)\lvert g \rvert + u'n_2,
\]
showing that the equality holds. That this construction respects the symmetric structure was already shown in the proof of Proposition 3.11. The coherence axioms are left to the reader. □

4.5. Derived $A_\infty$-algebras as $A_\infty$-algebras in twisted complexes. We reinterpret the category of derived $A_\infty$-algebras as the category of $A_\infty$-algebras in twisted complexes. Generally, given an operad $P$ on a symmetric monoidal category one studies $P$-algebra structures on objects of the same category. We can extend this to the case of monoidal categories over a base via the following definition due to [Fre09].

Definition 4.39. Let $C$ be a monoidal category over $V$ and let $P$ be an operad in $V$. A $P$-algebra in $C$ consists of an object $A \in C$, together with maps
\[
P(n) \otimes A^\otimes n \to A
\]
for which the unit and associativity axioms hold.

We can give an equivalent definition by means of an enriched endomorphism operad.

Definition 4.40. Let $C$ be a monoidal $V$-enriched category and $A$ an object of $C$. We define $\text{End}_A$ to be the collection in $V$ given by
\[
\text{End}_A(n) := C(A^\otimes n, A) \quad \text{for } n \geq 1.
\]

Lemma 4.41. For any $A \in C$, the collection $\text{End}_A$ defines an operad in $V$ with unit
\[
1 \xrightarrow{u_A} C(A, A) = \text{End}_A(1)
\]
and composition
\[
\text{End}_A(r) \otimes \text{End}_A(n_1) \otimes \text{End}_A(n_2) \otimes \cdots \otimes \text{End}_A(n_r) \to \text{End}_A(r) \otimes C(A^\otimes n, A^\otimes r) \to \text{End}_A(n),
\]
where $n = n_1 + \cdots + n_r$. The first morphism is given by the monoidal structure of $C$ and the second is the composition of the symmetry morphism of $V$ with the composition morphism of $C$.

Proof. The appropriate diagrams commute by associativity of composition in an enriched category. We also refer the reader to [Fre09, Definition 3.4.1]. □

Example 4.42. For any twisted complex $A$ and any filtered complex $K$ we have operads in vertical bicomplexes $\text{End}_A$ and $\text{End}_K$.

The following result gives an equivalent interpretation of $P$-algebras in monoidal categories over a base.

Proposition 4.43. [Fre09, Proposition 3.4.3] Let $C$ be a monoidal category over $V$, let $P$ be an operad in $V$ and $A$ an object in $C$. Then there is a one-to-one correspondence between $P$-algebra structures on $A$ and morphisms of operads $P \to \text{End}_A$. □

We reinterpret $dA_\infty$-algebras as $A_\infty$-algebras in twisted complexes using the structure of $tC_R$ as a monoidal category over $vbC_R$. 

DERIVED $A$-INFINITY ALGEBRAS AND THEIR HOMOTopies
Proposition 4.44. Let $(A,d^A)$ be a twisted complex, $A$ its underlying bigraded module and consider $A_\infty$ as an operad in $vbC_R$ sitting in horizontal degree zero. There is a one-to-one correspondence between $A_\infty$-algebra structures on $(A,d^A)$ and $dA_\infty$-algebra structures on $A$ which respect the twisted complex structure of $A$. More precisely, let $\text{End}_A$ be the operad in $vbC_R$ corresponding to the bigraded module $A$. We have a natural isomorphism of sets

$$\text{Hom}_{vb\text{Op}}(A_\infty, \mathcal{E}_{\text{End}}(A)) \cong \text{Hom}_{vb\text{Op},d^A}(dA_\infty, \text{End}_A),$$

where $vb\text{Op}$ denotes the category of operads in vertical bicomplexes and $\text{Hom}_{vb\text{Op},d^A}$ denotes the subset of morphisms which send $\mu_{i1}$ to $d^A_i$, $i \geq 1$.

Proof. Let $f : A_\infty \rightarrow \mathcal{E}_{\text{End}}(A)$ be a map of operads in $vbC_R$. Since $A_\infty$ is quasi-free, this is equivalent to maps in $vbC_R$

$$f : (A_\infty(v), \partial_\infty) \rightarrow (\mathcal{E}_{\text{End}}(A)(v), \delta)$$

for each $v \geq 1$, which are determined by elements $M_v := f(\mu_v) \in \mathcal{E}_{\text{End}}(A)(v)$ for $v \geq 2$ of bidegree $(0,2-v)$ such that

$$\delta(M_v) = f(\partial_\infty(\mu_v)). \quad (4)$$

Moreover, $M_v := (m_{0v}, m_{1v}, \ldots)$ where $m_{uv} := (M_v)_u : A^{\otimes v} \rightarrow A$ is a map of bidegree $(-u,2-u-v)$. We first compute the left-hand side of (4). Since $\delta(M_v) = c(d^A, M_v) - (-1)^v c(M_v, dA^{\otimes v})$, we have

$$(\delta(M_v))_u = \sum_{u=i+p} (-1)^{r+1} d^A_q(M_v)_p - (-1)^v \sum_{u=i+p} (-1)^t (M_v)_t(\hat{1}^{\otimes r} \otimes d^A_p \otimes \hat{1}^{\otimes t})$$

and

$$f(\partial_\infty(\mu_v))_u = - \sum_{v=r+q+t} (-1)^{r+q+t} c(M_j, (\hat{1}^{\otimes r} \otimes M_q \otimes \hat{1}^{\otimes t}))_u$$

$$= - \sum_{i+p=u} (-1)^{r+q+t+i} (M_j)_i(\hat{1}^{\otimes r} \otimes (M_q)_p \otimes 1^{\otimes t})$$

$$= - \sum_{i+p=u} (-1)^{r+q+t+i} \tilde{m}_{ij} (1^{\otimes r} \otimes \tilde{m}_{pq} \otimes 1^{\otimes t}).$$

Then by setting the notation $\tilde{m}_{i1} = d^A_i$, the relation

$$\delta(M_n) - f(\partial_\infty(\mu_n)) = 0$$

gives us the relations

$$\sum_{u=i+p} (-1)^{r+q+t+i} \tilde{m}_{ij} (1^{\otimes r} \otimes \tilde{m}_{pq} \otimes 1^{\otimes t}) = 0.$$

By setting $m_{ij} = (-1)^{ij} \tilde{m}_{ij}$ one obtains the relation

$$\sum_{u=i+p} (-1)^{r+t+p} m_{ij} (1^{\otimes r} \otimes m_{pq} \otimes 1^{\otimes t}) = 0 \quad (A_{uv})$$

which by [LRW13] is equivalent to giving a map $dA_\infty \rightarrow \text{End}_A$ of operads in $vbC_R$. \qed

Remark 4.45. Note that as in the case of $A_\infty$-algebras in $C_R$ we have two equivalent descriptions of $A_\infty$-algebras in $tC_R$. 


(1) A twisted complex \((A, d^A)\) together with a morphism \(A_\infty \to \text{End}_A\) of operads in \(\text{vbCR}\), which is determined by a family of elements \(M_i^A \in \mathcal{C}_R(A^{\otimes i}, A)_0^{i-1}\) for \(i \geq 2\) for which the \((A_0^i)\) relations hold, where the composition is the one prescribed by the composition morphisms of \(\mathcal{C}_R\).

(2) A bigraded module \(A\) together with a family of elements \(M_i \in \text{bgMod}_R(A^{\otimes i}, A)_0^{i-1}\) for \(i \geq 1\) for which the \((A_0^i)\) relations hold, where the composition is the one prescribed by the composition morphisms of \(\text{bgMod}_R\).

Here \((A_0^i)\) and \((A_0^i)\) are the \(A_\infty\) relations for \(i \geq 2\) or \(i \geq 1\) respectively. Since the composition morphism in \(\text{bgMod}_R\) is induced from the one in \(\mathcal{C}_R\) by forgetting the differential, these two presentations are equivalent.

We now consider infinity morphisms and composition.

**Definition 4.46.** Let \(A, B, C\) be twisted complexes and \((A, M_i^A), (B, M_i^B)\) and \((C, M_i^C)\) be \(A_\infty\)-algebra structures on them. An \(A_\infty\)-morphism in twisted complexes \(F : (B, M_i^B) \to (C, M_i^C)\) is a family of elements \(F : \{F_j \in \mathcal{C}_R(B^{\otimes j}, C)_0^{j-1}\}_{j \geq 1}\) for which the \(A_\infty\) relations hold i.e.,

\[
\sum_{v=1+q+t}^{v+r+q+t} (-1)^{r+q+t} c(F_j, (1^{\otimes q} \otimes M_q^{1,1} \otimes 1^{\otimes t})) = \sum_{v=q_1+\cdots+q_j} (-1)^{q} c(M_j^{C}, (F_{q_1} \otimes \cdots \otimes F_{q_j})), \quad (B_{0v})
\]

where \(c = \sum_{k=1}^{j-1} q_k(j+k) + q_j \left( \sum_{s=k+1}^{j} q_s \right)\) and \(c\) is described in Definition 3.32.

Let \(F : (B, M_i^B) \to (C, M_i^C)\) and \(G : (A, M_i^A) \to (B, M_i^B)\) be \(A_\infty\)-morphisms in twisted complexes. Their composite is the \(A_\infty\)-morphism in twisted complexes \(F \circ G : (A, M_i^A) \to (C, M_i^C)\) given by

\[
(F \circ G)_v := \sum_{v=q_1+\cdots+q_j} (-1)^{q} c(F_j, (G_{q_1} \otimes \cdots \otimes G_{q_j})), \quad (C_{0v})
\]

The category of \(A_\infty\)-algebras in twisted complexes, denoted \(A_{\infty}^{IC}(R)\), is the category with objects \(A_\infty\)-algebras in twisted complexes and whose morphisms are \(A_\infty\)-morphisms in twisted complexes.

**Theorem 4.47.** The construction above extends to a functor \(\Psi : A_{\infty}^{IC}(R) \to dA_{\infty}(R)\) which is an isomorphism of categories.

**Proof.** On objects \(\Psi(A, M_i^A) = (A, m_i^A)\) takes an \(A_\infty\)-algebra in twisted complexes and associates to it its corresponding \(dA_{\infty}\)-algebra as described in Proposition 3.44.

On morphisms, consider \(F : (A, M_i^A) \to (B, M_i^B)\) a morphism in \(A_\infty\)-algs in \(\text{tCR}\) which is given by \(F := \{F_j \in \mathcal{C}_R(A^{\otimes j}, B)_0^{j-1}\}_{j \geq 1}\), where each \(F_j := (f_{\tilde{f}_{ij}1}, f_{\tilde{f}_{ij}2}, \ldots)\). The relations \((B_{0v})\) translate to

\[
\sum_{u=1+p}^{u+i+p} \sum_{v=q+r+q+t}^{v+i+q+t} (-1)^{i+r+q+t} m_{pq}^A \tilde{f}_{ij}(1^{\otimes r} \otimes m_{pq}^A \otimes 1^{\otimes t}) = \sum_{u=1+i+p}^{u+i+p} \sum_{v=q_1+\cdots+q_j} (-1)^{r} m_{ij}^B \tilde{f}_{pq1} \otimes \cdots \otimes \tilde{f}_{pqj}). \quad (\tilde{B}_{uv})
\]

Multiplying the equation by \((-1)^{i+q+p}\) one gets the equation

\[
\sum_{u=1+i+p}^{u+i+p} \sum_{v=q+r+q+t}^{v+i+q+t} (-1)^{i+r+q+t} m_{pq}^A \tilde{f}_{ij}(1^{\otimes r} \otimes m_{pq}^A \otimes 1^{\otimes t}) = \sum_{u=1+i+p}^{u+i+p} \sum_{v=q_1+\cdots+q_j} (-1)^{r} m_{ij}^B \tilde{f}_{pq1} \otimes \cdots \otimes \tilde{f}_{pqj}).
\]

Let us compute the signs modulo 2

\[
\tilde{\sigma} = rq + t + (i + p)(j + q) + ij + pq + iq = rq + t + pj,
\]
\[ \tilde{\sigma}_r = \sum_{k=1}^{j-1} q_k (j+k) + q_k (\sum_{s=k+1}^{j} q_s) + u + uv + iv + j \sum_{k=1}^{j} p_k q_k + i (\sum_{k=1}^{j} (q_k + 1)) + \sum_{k=1}^{j} (1 + q_k) (\sum_{s=1}^{k-1} p_s) \]
\[ = \sum_{k=1}^{j-1} q_k (j+k) + q_k (\sum_{s=k+1}^{j} q_s) + u + uv + iv + \sum_{k=1}^{j} q_k (\sum_{s=1}^{k} p_s) + \sum_{s=1}^{j} p_s (\sum_{k=s+1}^{j} 1) \]
\[ = u + \sum_{k=1}^{j-1} (p_k + q_k) (j+k) + \sum_{k=1}^{j} q_k (\sum_{s=k+1}^{j} p_s + q_s). \]

This gives exactly the relation defining morphisms of \(dA_\infty\)-algebras:
\[ \sum_{u=i+p, v=r+q+t, j=1+r+t} (-1)^{r+q+t+p} f_{ij} (1^\otimes r \otimes m_{pq}^A \otimes 1^\otimes t) = \sum_{u=i+p_1+\cdots+p_j, v=q_1+\cdots+q_j} (-1)^{\sigma} m_{ij}^B (f_{p_1 q_1} \otimes \cdots \otimes f_{p_j q_j}), \]

\( (B_{uv}) \)

where \( \sigma = u + \sum_{k=1}^{j-1} (p_k + q_k) (j+k) + \sum_{k=1}^{j} q_k (\sum_{s=k+1}^{j} p_s + q_s) \). Moreover, any morphism between \(dA_\infty\)-algebras can be constructed in this way. Therefore this construction is a bijection on morphisms. Finally, this construction is functorial. The relations \((C_{uv})\) translate to
\[ (F \circ G)_{uv} = \sum_{u=i+p_1+\cdots+p_j, v=q_1+\cdots+q_j} (-1)^{\sigma'} f_{ij} (g_{p_1 q_1} \otimes \cdots \otimes g_{p_j q_j}). \]

Setting \( f_{ij} = (-1)^{ij} \tilde{f}_{ij} \) and \( g_{pq} = (-1)^{pq} \tilde{g}_{pq} \), \( (F \circ G)_{uv} = (-1)^{uv} (F \circ G)_{uv} \) one gets the equation
\[ (F \circ G)_{uv} = \sum_{u=i+p_1+\cdots+p_j, v=q_1+\cdots+q_j} (-1)^{\sigma'_r} f_{ij} (g_{p_1 q_1} \otimes \cdots \otimes g_{p_j q_j}), \]

with \( \sigma'_r = \sigma + u + uv + uv = j-1 \sum_{k=1}^{j} (p_k + q_k) (j+k) + j-1 \sum_{k=1}^{j} q_k (\sum_{s=k+1}^{j} p_s + q_s) \), which is the composition of morphisms of \(dA_\infty\)-algebras. \( \square \)

4.6. Derived \( A_\infty \)-algebras as filtered \( A_\infty \)-algebras. From the fact that \( U_{R}^b \) and \( sfMod_{R}^b \) are isomorphic vbC_{p}-enriched monoidal categories, we now reinterpret \( dA_\infty \)-algebras, in the bounded case, in terms of split filtered \( A_\infty \)-algebras. First we define filtered \( A_\infty \)-algebras and their morphisms.

**Definition 4.48.** A **filtered \( A_\infty \)-algebra** is an \( A_\infty \)-algebra \((A, m_i)\) together with a filtration \( \{F_p A^i\}_{p \in \mathbb{Z}} \) on each \( R \)-module \( A^i \) such that for all \( i \geq 1 \) and all \( p_1, \ldots, p_i \in \mathbb{Z} \) and \( n_1, \ldots, n_i \geq 0 \),
\[ m_i(F_{p_1 A^{n_1}} \otimes \cdots \otimes F_{p_i A^{n_i}}) \subseteq F_{p_1 + \cdots + p_i A^{n_1 + \cdots + n_i + 2^{-i}}} \]

Such a filtered \( A_\infty \)-algebra is said to be **split** if \( A = \text{Tot}(B) \) is the total graded module of a bigraded \( R \)-module \( B = \{B_i^j\} \) and \( F \) is the column filtration of \( \text{Tot}(B) \).

**Remark 4.49.** Consider \( A_\infty \) as an operad in filtered complexes with the trivial filtration and let \( K \) be a filtered complex. There is a one-to-one correspondence between filtered \( A_\infty \)-structures on \( K \) and morphisms of operads in filtered complexes \( A_\infty \rightarrow \text{End}_K \). To see this, notice that if one forgets the filtrations such a map of operads gives an \( A_\infty \)-structure on \( K \). The fact that this is a map of operads in filtered complexes implies that all the \( m_i \)s respect the filtrations.

**Definition 4.50.** A **morphism of filtered \( A_\infty \)-algebras** from \((A, m_i, F)\) to \((B, m_i, F)\) is a morphism \( f : (A, m_i) \rightarrow (B, m_i) \) of \( A_\infty \)-algebras such that each map \( f_j : A^{\otimes j} \rightarrow A \) is compatible with filtrations:
\[ f_j(F_{p_1 A^{n_1}} \otimes \cdots \otimes F_{p_j A^{n_j}}) \subseteq F_{p_1 + \cdots + p_j B^{n_1 + \cdots + n_j + 1-j}}, \]

for all \( j \geq 1, p_1, \ldots, p_j \in \mathbb{Z} \) and \( n_1, \ldots, n_j \geq 0 \).
Denote by $fA_\infty(R)$ the category of filtered $A_\infty$-algebras. Composition is given as in the unfiltered case (this respects the filtration). We consider the following full subcategories of $fA_\infty(R)$.

- $sfA_\infty(R)$: the subcategory on split filtered $A_\infty$-algebras.
- $fA^b_\infty(R)$: the subcategory on non-negatively filtered $A_\infty$-algebras.
- $sfA^b_\infty(R)$: the subcategory on split non-negatively filtered $A_\infty$-algebras.

That is, we have full embeddings

$$sfA^b_\infty(R) \rightarrow fA_\infty(R) \rightarrow fA^b_\infty(R)$$

(5)

**Lemma 4.51.** For any twisted complex $A$ there is a morphism of operads

$$\mathcal{End}_A \rightarrow \mathcal{End}_{\text{Tot}(A)},$$

which is an isomorphism of operads if $A$ is bounded.

**Proof.** In each arity $n$ this map is given by the composition of the following morphisms in vertical bicomplexes

$$\mathcal{End}_A(n) := \mathcal{T}C_R(A^{\otimes n}, A) \xrightarrow{\mathcal{T}t_{A^{\otimes n},A}} fC_R(\text{Tot}(A^{\otimes n}), \text{Tot}(A)) \cong fC_R(\text{Tot}(A^{\otimes n}), \text{Tot}(A)) \otimes R$$

$$\xrightarrow{1 \otimes \mu_n} fC_R(\text{Tot}(A^{\otimes n}), \text{Tot}(A)) \otimes fC_R(\text{Tot}(A)^{\otimes n}, \text{Tot}(A^{\otimes n}))$$

$$\cong fC_R(\text{Tot}(A)^{\otimes n}, \text{Tot}(A)) = \mathcal{End}_{\text{Tot}(A)}(n),$$

where $\mu_n$ is the map of vertical bicomplexes constructed from $\mu(A,A)$ recursively as the following composite

$$\mu_n : R \cong R \otimes R \xrightarrow{\mu_{n-1} \otimes 1_{\text{Tot}(A)}} fC_R(\text{Tot}(A)^{\otimes n-1} \otimes \text{Tot}(A), \text{Tot}(A^{\otimes n-1}) \otimes \text{Tot}(A))$$

$$\cong R \otimes fC_R(\text{Tot}(A)^{\otimes n}, \text{Tot}(A^{\otimes n}) \otimes \text{Tot}(A))$$

$$\xrightarrow{\mu(A^{\otimes n-1},A)^{\otimes 1}} fC_R(\text{Tot}(A^{\otimes n-1}) \otimes \text{Tot}(A), \text{Tot}(A^{\otimes n}) \otimes fC_R(\text{Tot}(A)^{\otimes n}, \text{Tot}(A^{\otimes n}) \otimes \text{Tot}(A))$$

$$\cong fC_R(\text{Tot}(A)^{\otimes n}, \text{Tot}(A)).$$

These assemble to a morphism of operads since $\mathcal{T}t$ is a vb$C_R$-enriched functor and $\mu$ is a vb$C_R$-enriched natural transformation. Furthermore, if $A$ is bounded all maps are isomorphisms and thus the map gives an isomorphism of operads.

** Proposition 4.52.** Let $(A, d^A) \in tC^b_R$ be an $(\mathbb{N}, \mathbb{Z})$-graded twisted complex and $A$ its underlying bigraded module. There is a one-to-one correspondence between filtered $A_\infty$-algebra structures on $\text{Tot}(A)$ and $dA_\infty$-algebra structures on $A$ which respect the twisted complex structure of $A$. This bijection is induced by a one-to-one correspondence between filtered $A_\infty$-algebra structures on $\text{Tot}(A)$ and $A_\infty$-algebra structures on $(A, d^A)$. More precisely we have natural isomorphisms of sets

$$\text{Hom}_{\text{vbOp}, d^A}(dA_\infty, \text{End}_A) \cong \text{Hom}_{\text{vbOp}}(A_\infty, \mathcal{End}_A)$$

$$\cong \text{Hom}_{\text{vbOp}}(A_\infty, \mathcal{End}_{\text{Tot}(A)})$$

$$\cong \text{Hom}_{\text{ICOp}}(A_\infty, \text{End}_{\text{Tot}(A)}),$$

where vbOp and fICOp denote the categories of operads in vb$C_R$ and f$C_R$ respectively, and $\text{Hom}_{\text{vbOp}, d^A}$ denotes the subset of morphisms which send $\mu_{i1}$ to $d^A_i$. We view $A_\infty$ as an operad in vb$C_R$ sitting in horizontal degree zero or as an operad in filtered complexes with trivial filtration.
Lemma 4.51. Finally, to see the third isomorphism let $f : A_\infty \to \mathcal{E}_{\text{Tot}(A)}$ be a map of operads in $\text{vbC}_R^\infty$. Again, since $A_\infty$ is quasi-free, this is equivalent to maps in $\text{vbC}_R^\infty$:

$$(A_\infty(n), \partial_\infty) \to (\mathcal{E}_{\text{Tot}(A)}(n), \delta)$$

which are determined by elements $M_n := f(\mu_n) \in \mathcal{E}_{\text{Tot}(A)}(n)$ of bidegree $(0, 2 - n)$ such that

$$\delta(M_n) = f(\partial_\infty(\mu_n)).$$

Since $A$ is $(\mathbb{N}, \mathbb{Z})$-graded, $\text{Tot}$ is symmetric monoidal, and thus we have that

$$\delta M_n = c(d^{\text{Tot}(A)}(M_n)) + (-1)^{2-n} c(M_n, d^{\text{Tot}(A)^{\otimes n}}).$$

So these maps give the complex $\text{Tot}(A)$ the structure of an $A_\infty$-algebra. Moreover, all of the $M_n$s respect the filtration since they have horizontal degree zero. Therefore, the map $f$ gives $\text{Tot}(A)$ the structure of a split filtered $A_\infty$-algebra and it is clear that any filtered $A_\infty$-algebra structure on $\text{Tot}(A)$ can be described by such $f$. \(\square\)

This construction extends to infinity morphisms.

**Theorem 4.53.** The totalization functor extends to a functor

$$\Phi : A_{\infty}^C(R) \to f A_{\infty}(R)$$

which in the bounded case restricts to an isomorphism between the categories of bounded $A_\infty$-algebras in twisted complexes and split non-negatively filtered $A_\infty$-algebras.

**Proof.** The functor on objects is given as in Proposition 4.51. Here we describe this explicitly on elements. Let $(A, M_i)$ be an $A_\infty$-algebra in $\mathcal{C}_R$, that is $A \in \mathcal{C}_R$ and $M_i \in \mathcal{C}_R^{0-1}$ satisfying the $A_\infty$-relations

$$\sum_{v=v+q+t} (-1)^{r+1} c(M_i, 1^{\otimes r} \otimes M_q \otimes 1^{\otimes t}) = 0. \quad (A_{0v})$$

Following the notation of Proposition 4.38 let

$$\mu_i := \mu_{A, \ldots, A} : \text{Tot}(A)^{\otimes i} \to \text{Tot}(A^{\otimes i})$$

$$\mu_{r,q,t} := \mu_{A, \ldots, A, A^{\otimes q}, A, \ldots, A} : \text{Tot}(A)^{\otimes r} \otimes \text{Tot}(A^{\otimes q}) \otimes \text{Tot}(A)^{\otimes t} \to \text{Tot}(A^{\otimes r+q+t})$$

and define

$$m_i := c(\text{Tot}(M_i), \mu_i) : \text{Tot}(A)^{\otimes i} \to \text{Tot}(A).$$

Note first that for any $i$ the map $m_i$ has horizontal degree 0 thus it respects the filtrations. Now, we compute

$$\sum_{v=v+q+t \atop j=1+r+t} (-1)^{r+1} m_i(1^{\otimes r} \otimes m_q \otimes 1^{\otimes t})$$

$$= \sum_{v=v+q+t \atop j=1+r+t} (-1)^{r+1} c(\text{Tot}(M_i), \mu_i)(1^{\otimes r} \otimes c(\text{Tot}(M_q), \mu_q) \otimes 1^{\otimes t})$$

$$= \sum_{v=v+q+t \atop j=1+r+t} (-1)^{r+1} c(\text{Tot}(M_i)(1^{\otimes r} \otimes M_q \otimes 1^{\otimes t}))(\mu_{r,q,t}(1^{\otimes r} \otimes M_q \otimes 1^{\otimes t})) = 0.$$

Here the first equality holds by definition, the second by naturality of $\mu$ and the third because the $M_i$s satisfy the relations $(A_{0v})$. Thus $(\text{Tot}(A), m_i)$ is a filtered $A_\infty$-algebra. The same computation gives the result on morphisms and this is stable under the composition of morphisms giving the functor $\Phi$. Furthermore, when $A \in \mathcal{C}_R$ this functor restricts to an isomorphism by Proposition 4.52. \(\square\)
**Corollary 4.54.** Let Tot denote the composite
\[ \text{Tot} : da^b(\mathbb{R}) \xrightarrow{\Psi^{-1}} A^{\text{tw}}_{\text{b}}(\mathbb{R}) \xrightarrow{\Phi} fA^b(\mathbb{R}). \]

The functor Tot restricts to an isomorphism between the category of \((\mathbb{N}, \mathbb{Z})\)-graded \(dA_{\infty}\)-algebras and the category of split non-negatively filtered \(A_{\infty}\)-algebras. Furthermore, this functor fits into a commutative diagram of categories
\[
\begin{array}{ccc}
dA^b_{\infty}(\mathbb{R}) & \xrightarrow{U} & tC_R^b \\
\text{Tot} & & \downarrow \text{Tot} \\
fA^b_{\infty}(\mathbb{R}) & \xrightarrow{U} & fC_R^b
\end{array}
\]
where the horizontal arrows are forgetful functors and the vertical arrows are full embeddings.

**Proof.** This follows directly from Theorems 4.47 and 4.53. \(\square\)

5. **Derived \(A_{\infty}\)-algebras and \(r\)-homotopy**

The main goal of this section is to study different but equivalent interpretations of the notion of \(r\)-homotopy for derived \(A_{\infty}\)-algebras. We first define \(r\)-homotopy by constructing a functorial \(r\)-path object. Then, we study some of the properties of \(r\)-homotopy. Most notably, we show that 0-homotopy defines an equivalence relation and we study the localized category \(dA_{\infty}(\mathbb{R})[S_r^{-1}]\). Finally, we give an operadic interpretation of \(r\)-homotopy and show that both notions are equivalent.

5.1. **Twisted dgas and tensor product.** In general, the tensor product of two \(dA_{\infty}\)-algebras does not inherit a natural \(dA_{\infty}\)-algebra structure giving rise to a monoidal structure on \(dA_{\infty}(\mathbb{R})\). The construction works if one of the components is a twisted differential graded algebra, as we next show.

**Definition 5.1.** A twisted dga is a \(dA_{\infty}\)-algebra \((A, \mu_{ij})\) whose only non-zero structure morphisms are \(\mu_{i1}\) for \(i \geq 0\) and \(\mu_{02}\).

**Lemma 5.2.** For \((A, \mu_{ij})\) a twisted dga, the following hold.

1. \(\mu_{02}\) is associative.
2. \(\mu_{i1}(\mu_{02}) = \mu_{02}(1 \otimes \mu_{i1}) + \mu_{02}(\mu_{i1} \otimes 1)\) for all \(i \geq 0\).
3. Let \(\mu_n : A^{\otimes n} \rightarrow A\) be defined iteratively by \(\mu_n = \mu_{02}(\mu_{n-1} \otimes 1)\), with \(\mu_2 = \mu_{02}\). Then
   \[\mu_{i1}(\mu_n) = \sum_{r+t+1=n} \mu_n(1^{\otimes r} \otimes \mu_{i1} \otimes 1^{\otimes t})\text{ for all } i \geq 0.\]

**Proof.** It suffices to check (2). Relation \((A_{i2})\) reads \(-\mu_{02}(1 \otimes \mu_{i1}) + \mu_{02}(\mu_{i1} \otimes 1)) = 0. \(\square\)

**Proposition 5.3.** Let \((\Lambda, \mu_{ij})\) be a twisted dga and let \((\Lambda, m_{ij})\) be a \(dA_{\infty}\)-algebra. The bigraded module \(\Lambda \otimes A\) is endowed with a \(dA_{\infty}\)-algebra structure given by
   \[\widehat{m}_{i1} = \mu_{02}(\mu_{i1} \otimes 1_{A} + 1_{\Lambda} \otimes m_{i1}) \text{ and } \widehat{m}_{ij} = (\mu_{j} \otimes m_{ij})\tau_j \text{ for all } j \geq 2.\]

Here \(\tau_j : (\Lambda \otimes A)^{\otimes j} \rightarrow \Lambda^{\otimes j} \otimes A^{\otimes j}\) denotes the standard isomorphism given by the symmetric monoidal structure and \(\mu_{ij}\) is defined in Lemma 5.2.

**Proof.** For all \(n \geq 0\), we have
\[
\sum_{i+j=n} (-1)^j \widehat{m}_{i1}\widehat{m}_{j1} = \sum_{i+j=n} (-1)^j (\mu_{i1}\mu_{j1} \otimes 1 + 1 \otimes m_{i1}m_{j1} + \mu_{i1} \otimes m_{j1} + (-1)^{i+j+1}(1-i)(1-j)\mu_{j1} \otimes m_{i1}) = 0.
\]

Note that, for \(j, q \geq 2\), we have
\[
\widehat{m}_{ij}(1^{\otimes r} \otimes \widehat{m}_{pq} \otimes 1^{\otimes t}) = (\mu_{r+q+t} \otimes m_{ij}(1^{\otimes r} \otimes m_{pq} \otimes 1^{\otimes t}))\tau_{r+q+t}.
\]
Using this, for all \( u \geq 0 \) and all \( v \geq 2 \), we have

\[
\sum_{u=i+p, v=j+q-1} (-1)^{rq+t+pq} \tilde{m}_{ij}(1^{\otimes x} A \otimes \tilde{m}_{pq} \otimes 1^{\otimes y} A) = \\
\sum_{u=i+p, e=j+q-1} (-1)^{rq+t+pq} (\mu_v \otimes m_{ij})(1^{\otimes x} \otimes m_{pq} \otimes 1^{\otimes y} A) \tau_v + \\
\sum_{u=i+p} (-1)^p(\mu_{ij} \otimes 1_A)(\mu_v \otimes m_{pv}) \tau_v \\
+ \sum_{u=i+p} (-1)^{v+1+pv}(\mu_v \otimes m_{iv})(1^{\otimes x} A \otimes (\mu_p \otimes 1_A) \otimes 1^{\otimes y} A) \\
= \sum_{u=i+p} (-1)^p(\mu_{ij} \mu_v \otimes m_{pv}) \tau_v + \\
\sum_{v=r+t+1} (-1)^{p+1}(\mu_v(1^{\otimes x} \otimes \mu_p \otimes 1^{\otimes y} \otimes m_{iv}) \tau_v \\
= \sum_{i+p} (-1)^p(\mu_{ij} \mu_v \otimes m_{pv} + \\
\sum_{v=r+t+1} (-1)^{p+1}(\mu_v(1^{\otimes x} \otimes \mu_{ij} \otimes 1^{\otimes y} \otimes m_{pv}) \tau_v = 0. \quad \Box
\]

**Proposition 5.4.** Let \( \Lambda, \mu_{ij} \) be a twisted dga. The above construction gives rise to a functor \( \Lambda \otimes - : dA_{\infty}(R) \to dA_{\infty}(R) \), sending a morphism \( f : A \to B \) of \( dA_{\infty} \)-algebras to the morphism \( f : \Lambda \otimes A \to \Lambda \otimes B \) given by \( f_{i,j} = 1_\Lambda \otimes f_{ij} \) and \( f_{i,j} = (\mu_{ij} \otimes f_{ij}) \), for all \( j \geq 2 \). Furthermore, for a \( dA_{\infty} \)-algebra \( (A, m_{ij}) \), the construction above gives rise to a functor \( - \otimes A \) sending a strict morphism \( f : \Lambda \to \Lambda' \) to the strict morphism \( f \otimes 1_A : \Lambda \otimes A \to \Lambda' \otimes A \).

**Proof.** The proof follows exactly the same lines of computation as the preceding proof.

\[5.2. r\)-homotopies and \( r\)-homotopy equivalences.\] We next define a collection of functorial paths indexed by an integer \( r \geq 0 \) on the category of \( dA_{\infty} \)-algebras, giving rise to the corresponding notions of \( r \)-homotopy.

We will use specific twisted dgas defined in the following proposition whose proof is left to the reader.

**Proposition 5.5.** Let \( r \geq 0 \) be an integer. Let \( \Lambda_r \) be the bigraded module generated by \( e_- \) and \( e_+ \) in bidegree \((0,0)\) and \( u \) in bidegree \((-r, 1-r)\). The only non-trivial operations given by

\[
\mu_r(1_{-}) = -u, \quad \mu_r(1_{+}) = u, \quad \mu_0(1_{-}, 1_{-}) = e_-, \quad \mu_0(1_{+}, 1_{-}) = e_+, \quad \mu_0(1_{-}, u) = \mu_0(u, 1_{+}) = u,
\]

make \( \Lambda_r \) into a twisted dga. The morphisms

\[
\Lambda_r \xrightarrow{\partial^+} R \xrightarrow{\partial^{-}} \Lambda_r \xrightarrow{\iota} R
\]

given by \( \partial^-(e_-) = 1_R \), \( \partial^+(e_+) = 1_R \) and \( \iota(1) = e_- + e_+ \) and 0 elsewhere are strict morphisms of twisted dgas.

**Definition 5.6.** Let \( r \geq 0 \) be an integer. The functorial \( r \)-path \( P_r : dA_{\infty}(R) \to dA_{\infty}(R) \) is defined as \( P_r := \Lambda_r \otimes - \).

Note that for a \( dA_{\infty} \)-algebra \( A \), one has \( P_r(A) = (R e_- \otimes A_1^r) \otimes (R u \otimes A_1^{i+r}) \otimes (R e_+ \otimes A_1^r) \).

Hence we may identify \( P_r(A) \) with the bigraded \( R \)-module given by \( P_r(A) = A_1^r \otimes A_1^{i+r-1} \otimes A_1^r \) as in Definition 3.14. More precisely, the triple \((x, y, z)\) is identified with \( e_- \otimes x + u \otimes y + e_+ \otimes z \).

Given bigraded \( R \)-modules \( A \) and \( B \), let

\[
t_2 : P_r(A) \otimes P_r(B) \to P_r(A \otimes B)
\]

be the map given by

\[
t_2((x, y, z) \otimes (x', y', z')) = (x \otimes x' \otimes y' + y \otimes z', z \otimes z'),
\]

where \( \overline{x} := (-1)^{r_1 + (1-r)z_2} x \) and \((x_1, x_2)\) denotes the bidegree of \( x \). Likewise, for \( n \geq 2 \) we let

\[
t_n : P_r(A_1) \otimes \cdots \otimes P_r(A_n) \to P_r(A_1 \otimes \cdots \otimes A_n)
\]
be the map given by
\[ t_n((x_1, y_1, z_1) \otimes \cdots \otimes (x_n, y_n, z_n)) = (x_1 \otimes \cdots \otimes x_n, \sum_{1 \leq j \leq n} \prod_{i=1}^{j-1} z_i \otimes \prod_{i=j}^{n} y_i \otimes z_{j+1} \otimes \cdots \otimes z_n, z_1 \otimes \cdots \otimes z_n). \]

Note that under the identification above, the map \( t_n \) is obtained as the composite
\[(\mu_n \otimes 1) \circ \tau_n : (\Lambda_r \otimes A_1) \otimes \cdots \otimes (\Lambda_r \otimes A_n) \to (\Lambda_r)^{\otimes n} \otimes A_1 \otimes \cdots \otimes A_n \to \Lambda_r \otimes A_1 \otimes \cdots \otimes A_n.\]

As a consequence, combining Propositions 5.3 and 5.4 one obtains the following.

**Proposition 5.7.** The \( r \)-path \((P_r(A), M_{ij})\) of a \( dA_\infty \)-algebra \((A, m_{ij})\) is given by the bigraded module \( P_r(A) \otimes P_r(A) \) together with the morphisms \( M_{ij} : P_r(A)^{\otimes j} \to P_r(A) \) of bidegree \((-i, 2-i-j)\) given by
\[ M_{r}: \begin{pmatrix} m_{r+1} & 0 & 0 \\ -1 & -m_{r+1} & 1 \\ 0 & 0 & m_{r+1} \end{pmatrix} \quad \text{and} \quad M_{ij} := \begin{pmatrix} m_{ij} & 0 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & m_{ij} \end{pmatrix} \circ t_{ij}, \quad \text{for } i \geq 0 \text{ and } j \geq 2. \]

The \( r \)-path of a morphism \( f : (A, m_{ij}) \to (B, m'_{ij}) \) of \( dA_\infty \)-algebras is the morphism of \( dA_\infty \)-algebras \( P_r(f) : (P_r(A), M_{ij}) \to (P_r(B), M'_{ij}) \) given by \( P_r(f)_{ij} = (f_{ij}, (-1)^{(r+1)(j-1)}f_{ij}, f_{ij}) \).

The structure morphisms of the \( r \)-path
\[ A \xrightarrow{\iota_A} P_r(A) \xrightarrow{\partial_A} A ; \quad \partial_A^\dagger \circ \iota_A = 1_A \]
are given by \( \partial_A^\dagger(x, y, z) = x, \partial_A^\dagger(x, y, z) = z \) and \( \iota_A(x) = (x, 0, x) \). \( \square \)

It follows directly from the above proposition that the \( r \)-path is compatible with the forgetful functor \( U : dA_\infty(R) \to \text{tC}_R \). Also, if \((A, m_{ij})\) is a \( dA_\infty \)-algebra concentrated in horizontal degree 0, then its 0-path \( P_0(A) \) coincides with its path object as an \( A_\infty \)-algebra as defined by Grandis in [Gra99]. Hence the 0-path is compatible with the inclusion \( A_\infty(R) \to dA_\infty(R) \).

**Definition 5.8.** Let \( f, g : A \to B \) be two morphisms of \( dA_\infty \)-algebras. An \( r \)-homotopy from \( f \) to \( g \) is given by a morphism of \( dA_\infty \)-algebras \( h : A \to P_r(B) \) such that \( \partial_R^\dagger \circ h = f \) and \( \partial_B^\dagger \circ h = g \). We use the notation \( h : f \cong g \).

We postpone until later giving an explicit version of \( r \)-homotopy, in terms of a collection of morphisms \( \hat{h}_{ij} : A^{\otimes j} \to B; \) see Proposition 5.32.

In the category of \( A_\infty \)-algebras, the notion of homotopy defines an equivalence relation on the sets of \( A_\infty \)-morphisms (see [Pro11], see also [Gra99]). We next prove an analogous result in the context of \( dA_\infty \)-algebras, for \( 0 \)-homotopies. Our proof is an adaptation of the proof given by Grandis for \( A_\infty \)-algebras.

**Proposition 5.9.** The notion of \( r \)-homotopy is reflexive and compatible with the composition. Furthermore, for \( r = 0 \) it defines an equivalence relation on the set of morphisms of \( dA_\infty \)-algebras from \( A \) to \( B \), provided \( A \) and \( B \) are \((\mathbb{N}, \mathbb{Z})\)-graded.

**Proof.** Since the notion of \( r \)-homotopy is defined via a functorial path, it is reflexive and compatible with the composition. To show that \( 0 \)-homotopy is symmetric we define a natural reversion morphism of the \( r \)-path \( \zeta : P_0(A) \to P_0(A) \) of a \( dA_\infty \)-algebra \((A, m_{ij})\) such that \( \partial_R^\dagger \circ \zeta = \partial_B^\dagger \). Then, given a \( 0 \)-homotopy \( h : f \cong g \) we will have a \( 0 \)-homotopy \( \zeta \circ h : g \cong f \).

Consider the filtered \( A_\infty \)-algebra defined by applying \( \text{Tot} \) to \( P_0(A) \). This is given by:
\[ F_p \text{Tot}(P_0(A))^n = F_p \text{Tot}(A)^n \oplus F_p \text{Tot}(A)^{n-1} \oplus F_p \text{Tot}(A)^n, \]
with structure morphisms

\[ M_1 = \begin{pmatrix} \text{Tot}(m_{i1}) & 0 & 0 \\ -1 & -\text{Tot}(m_{i1}) & 1 \\ 0 & 0 & \text{Tot}(m_{i1}) \end{pmatrix} \quad \text{and} \quad M_j = \begin{pmatrix} \text{Tot}(m_{ij}) & 0 & 0 \\ 0 & (-1)^j\text{Tot}(m_{ij}) & 0 \\ 0 & 0 & \text{Tot}(m_{ij}) \end{pmatrix} \circ t_j, \]

for \( j \geq 2 \). We will next define a morphism of filtered \( A_\infty \)-algebras \( \zeta : \text{Tot}(P_0(A)) \to \text{Tot}(P_0(A)) \). Note that such a map is determined by its composition with the three maps \( \partial^-, \partial^0 \) and \( \partial^+ \) defined by projection to each of the direct summands of \( \text{Tot}(P_0(A)) \). We let \( \partial^- \xi_1 = \partial^+, \partial^0 \xi_1 = -\partial^0 \) and \( \partial^+ \xi_1 = \partial^+ \). For \( j > 1 \), we let \( \partial^0 \xi_j = 0 \) and define \( \partial^j \xi_j \) inductively by

\[ \partial^j \xi_j = \sum_{\substack{p+q=n+1 \\ x+y+z=n+1}} S_{nqxp} \text{Tot}(m_{ij}) ((\partial^+)^{x-1} \otimes \xi_q \otimes (\partial^-)^{y-1} \otimes \xi_1 \otimes (\partial^0)^{z-1}), \]

where all indices in the sum are integers \( \geq 1 \) and \( S_{nqxp} \) is a sign coefficient (see [Gra99, p56]). By [Gra99, Theorem 7.1], the family \( \{\xi_j\}_{j \geq 1} \) is a morphism of \( A_\infty \)-algebras. Since for all \( p \in \mathbb{Z} \), \( \partial^p(F_p \text{Tot}(P_0(A)) \subseteq F_p P_0(A) \) for \( \epsilon \in \{-, 0, +\} \), the morphism \( \zeta \) is compatible with filtrations. Therefore by Corollary [1.54] it gives the desired reversion of \( dA_\infty \)-algebras.

We next prove transitivity. Consider the pull-back of \( dA_\infty \)-algebras

\[ Q(A) \xrightarrow{\pi^-} P_0(A) \]

\[ \pi^- \downarrow \quad \partial^+ \downarrow \]

\[ P_0(A) \xrightarrow{\partial^-} A. \]

To prove transitivity it suffices to define a morphism \( \xi : Q(A) \to P_0(A) \) of \( dA_\infty \)-algebras such that

\[ \partial^\pm \xi = \partial^\pm \pi^\pm \] (see for example [KP97, Proposition 1.4.5(b)]).

Consider the filtered \( A_\infty \)-algebra given by applying \( \text{Tot} \) to \( Q(A) \). We will denote by \( \partial^\epsilon \eta := \partial^0 \pi^\epsilon \), with \( \epsilon = \pm \) and \( \eta \in \{-, 0, +\} \), the five projections \( \text{Tot}(Q(A)) \to \text{Tot}(A) \), noting that \( \partial^+- = \partial^+ \).

We next define \( \xi : \text{Tot}(Q(A)) \to \text{Tot}(P_0(A)) \). Let \( \xi_1 \) be defined by \( \partial^- \xi_1 := \partial^- \), \( \partial^0 \xi_1 := \partial^0 + \partial^+ \) and \( \partial^+ \xi_1 := \partial^+ + \). For \( j > 1 \), we let \( \partial^\pm \xi_j = 0 \) and define the central components by letting

\[ \partial^0 \xi_j = (-1)^j \text{Tot}(m_{ij}) \sum_{\substack{x+y+z=j+1 \\ x, y, z \geq 1}} (\partial^-)^{x-1} \otimes \partial^0 \otimes (\partial^0)^{y-1} \otimes \partial^0 \otimes (\partial^+)^{z-1}. \]

By [Gra99, Theorem 6.3], the family \( \{\xi_j\}_{j \geq 1} \) is a morphism of \( A_\infty \)-algebras. By construction, it is compatible with filtrations. Therefore by Corollary [1.54] it gives the desired morphism of \( dA_{\infty'} \) algebras.

**Remark 5.10.** The proof of symmetry and transitivity of \( 0 \)-homotopies given above does not extend to \( r \)-homotopies, due to the fact that for \( r > 0 \), the projection \( \partial^0 : \text{Tot}(P_r(A)) \to \text{Tot}(A) \) is not necessarily compatible with filtrations. Note that we have \( \partial^0(F_p \text{Tot}(P_r(A)) \subseteq F_{p+r} \text{Tot}(A)) \).

Denote by \( \sim_r \) the congruence of \( dA_\infty(R) \) generated by \( r \)-homotopies: \( f \sim_r g \) if and only if there is a chain of \( r \)-homotopies \( f \sim_r \cdots \sim_r g \) from \( f \) to \( g \) or a chain \( g \sim \cdots \sim_r f \) from \( g \) to \( f \).

**Definition 5.11.** A morphism of \( dA_\infty \)-algebras \( f : A \to B \) is called an \( r \)-homotopy equivalence if there exists a morphism \( g : B \to A \) satisfying \( f \circ g \sim_r 1_B \) and \( g \circ f \sim_r 1_A \).

Denote by \( S_r \) the class of \( r \)-homotopy equivalences of \( dA_\infty(R) \). This class is closed under composition and contains all isomorphisms. Since the \( r \)-path commutes with the forgetful functor \( U : dA_\infty(R) \to tC_R \), we have \( S_r = U^{-1}(S^{tC_R}_r) \). Note as well, that \( S_r \subset S_{r+1} \) and \( S_r \subset E_r \) for all \( r \geq 0 \).

**Lemma 5.12.** Let \((A, m_{ij})\) be a \( dA_\infty \)-algebra. The strict morphism \( \iota_A : (A, m_{ij}) \to (P_r(A), M_{ij}) \) given by \( \iota_A(x) = (x, 0, x) \) is an \( r \)-homotopy equivalence.
Proof. Note first that the category of twisted dgas together with strict morphisms is a monoidal category, hence \( A_r \otimes A_r \) is a twisted dga. Let \( \Delta : A_r \to A_r \otimes A_r \) be the map given by
\[
\Delta(e_-) = e_- \otimes (e_- + e_+) + e_+ \otimes e_-, \quad \Delta(e_+) = e_+ \otimes e_+, \quad \text{and} \quad \Delta(u) = u \otimes e_+ + e_+ \otimes u.
\]
That \( \Delta \) is a strict morphism of dgas is a matter of computation. Furthermore one has
\[
(\partial^\perp \otimes 1) \Delta = \text{id} \quad \text{and} \quad (\partial^\perp \otimes 1) \Delta = \iota \circ \partial^\perp.
\]
Consequently \( \Delta \otimes 1_A : P_r(A) \to P_r(P_r(A)) \) is an \( r \)-homotopy from \( \iota_A \circ \partial_A \) to the identity. \( \square \)

**Theorem 5.13.** The localized category \( dA_\infty(R)[S_r^{-1}] \) is canonically isomorphic to the quotient category \( \pi_r(dA_\infty(R)) := dA_\infty(R) / \sim_r \).

**Proof.** The proof is analogous to that of Proposition 5.2.6 using Lemma 5.12. \( \square \)

### 5.3. Operadic approach.

Since \( dA_\infty \)-algebras are algebras for the operad \( (dA_s)_\infty \), one expects to be able to describe homotopies in terms of structure on cofree \( (dA_s)^i \)-coalgebras, where \((dA_s)^i\) is the Koszul dual cooperad. We carry this out in this section, giving several equivalent formulations and showing that they agree with the definition via a path object presented above.

#### 5.3.1. \((dA_s)^i\)-coalgebras and coderivations.

In this setting, a homotopy between morphisms \( g \) and \( f \) should be a *coderivation homotopy*, that is, it satisfies two conditions, a usual homotopy relation together with a condition of compatibility with the comultiplication, called a \((g,f)\)-coderivation condition. (See [ALR+15] for the coderivation notion in an operadic context and [LH03] for \((g,f)\)-coderivations in the setting of \( A_\infty \)-algebras.)

First we recall the \((g,f)\)-coderivation condition for coassociative coalgebras.

**Definition 5.14.** Let \((A,\Delta^A), (B,\Delta^B)\) be coassociative \( R \)-coalgebras and let \( f, g : A \to B \) be coalgebra morphisms. A \((g,f)\)-coderivation is an \( R \)-linear map \( h : A \to B \) such that
\[
(g \otimes h + h \otimes f) \Delta^A = \Delta^B h.
\]

Next we define \((g,f)\)-coderivations in a suitable operadic setting.

**Definition 5.15.** Let \( C \) be a non-symmetric cooperad in vertical bicomplexes. For \( X,Y,Z \) vertical bicomplexes, the vertical bicomplex \( C(X; Y; Z) \) is given by
\[
C(X; Y; Z) := \bigoplus_{n \geq 1} C(n) \otimes \left( \bigoplus_{a+b+1 = n} X^{\otimes a} \otimes Y \otimes Z^{\otimes b} \right).
\]
If \( f : X \to X', h : Y \to Y' \) and \( g : Z \to Z' \) are maps of vertical bicomplexes, the map
\[
C(f; h; g) : C(X; Y; Z) \to C(X'; Y'; Z')
\]
is defined as the direct sum of the maps \( 1 \otimes f^{\otimes a} \otimes h \otimes g^{\otimes b} \).

**Definition 5.16.** Let \( C \) be a non-symmetric cooperad in vertical bicomplexes and let \( A \) and \( B \) be vertical bicomplexes. Let \( g \) and \( f \) be maps of \( C \)-coalgebras \( C(A) \to C(B) \). For \( r \geq 0 \), an \( r-(g,f) \)-coderivation is a map of vertical bicomplexes \( h : C(A) \to C(B) \) of bidegree \((r, r-1)\) such that the following diagram commutes.

\[
\begin{array}{ccc}
C(A) = C(A; A; A) & \xrightarrow{\Delta^C} & C \circ C(A; A; A) \\
\downarrow h & & \downarrow \text{C}(\text{C}(A; A; A) \circ C(A; A; A) = C(C(A); C(A; A; A); C(A)))} \\
C(B) = C(B; B; B) & \xrightarrow{\Delta^C} & C \circ C(B; B; B) \\
\end{array}
\]

In order to understand very explicitly what such a thing looks like in the \((dA_s)^i\) case, we first recall Proposition 3.2 of [ALR+15] and then extend it to \((g,f)\)-coderivations. As there, it is slightly simpler to use cooperadic suspension and work with the suspended cooperad \( \Lambda (dA_s)^i \).

Consider triples \((C,\Delta, f)\) where \((C,\Delta)\) is a conilpotent coassociative coalgebra and \( f : C \to C \) is a linear map of bidegree \((1,1)\) satisfying \((f \otimes 1)\Delta = (1 \otimes f)\Delta = \Delta f \). A morphism between two such triples is a morphism of coalgebras commuting with the given linear maps.
Proposition 5.17. \cite{ALR+15} Proposition 3.2] Cooperadic suspension gives rise to an isomorphism of categories between the category of conilpotent coalgebras over the cooperad \((d\mathcal{A}s)^i\) and the category of triples \((C, \Delta, f)\) as above.

An operadic coderivation of bidegree \((0, 1)\) of a \((d\mathcal{A}s)^i\)-coalgebra \(S^{-1}C\) corresponds on \((C, \Delta, f)\) to a coderivation of bidegree \((0, 1)\) of the coalgebra \(C\), anti-commuting with the linear map \(f\). □

Example 5.18. \cite{ALR+15} Example 3.3] As an example we give the structure corresponding to the cofree \(\Lambda(d\mathcal{A}s)^i\)-coalgebra cogenerated by \(C\). We have \(\Lambda(d\mathcal{A}s)^i(C) \cong R[x] \otimes \mathcal{T}C\), where \(\mathcal{T}C\) denotes the reduced tensor coalgebra on \(C\).

The coalgebra structure is given by

\[
\Delta(x^i \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{k=1}^{n-1} \sum_{r+s=i} (-1)^{\epsilon}(x^r \otimes a_1 \otimes \cdots \otimes a_k \otimes (x^s \otimes a_{k+1} \otimes \cdots \otimes a_n),
\]

where \(\epsilon = rn + ik + (s, s)(|a_1| + \cdots + |a_k|)\).

Let \(\pi_0\) denote the projection of \(R[x] \otimes \mathcal{T}C\) onto \(kx^0 \otimes \mathcal{T}C \cong \mathcal{T}C\). Then

\[
\Delta \pi_0 = (\pi_0 \otimes \pi_0) \Delta
\]

where the first \(\Delta\) is the usual deconcatenation product defined on \(\mathcal{T}C\).

The linear map \(f\) will be denoted \(d_x\) in this cofree case and

\[
d_x : R[x] \otimes \mathcal{T}C \rightarrow R[x] \otimes \mathcal{T}C
\]
is determined by \(d_x(x^n \otimes a) = (-1)^{j+1}x^{n-1} \otimes a\), for \(a \in C^{\otimes j}\).

Proposition 5.19. Let \(g\) and \(f\) be maps of \((d\mathcal{A}s)^i\)-coalgebras \((d\mathcal{A}s)^i(A) \rightarrow (d\mathcal{A}s)^i(B)\) and let \(h : (d\mathcal{A}s)^i(A) \rightarrow (d\mathcal{A}s)^i(B)\) be an \(r\)-(\(g, f\))-coderivation. If \(f, g\) correspond under the above isomorphism of categories to coalgebra morphisms \(\tilde{f}, \tilde{g} : R[x] \otimes \mathcal{T}SA \rightarrow R[x] \otimes \mathcal{T}SB\), (commuting with \(d_x\)), then \(h\) corresponds to a \((\tilde{g}, f)\)-coderivation of coalgebras \(h : R[x] \otimes \mathcal{T}SA \rightarrow R[x] \otimes \mathcal{T}SB\) of bidegree \((r, r-1)\), graded commuting with \(d_x\).

Proof. After applying cooperadic suspension, the \((g, f)\)-coderivation condition gives the commutative diagram

\[
\begin{array}{ccc}
R[x] \otimes \mathcal{T}SA & \xrightarrow{\rho_{\mathcal{A}}} & R[x] \otimes \mathcal{T}(R[x] \otimes \mathcal{T}SA) \\
\downarrow{\tilde{h}} & & \downarrow{(d\mathcal{A}s)^i(\tilde{g};\tilde{h};\tilde{f})} \\
R[x] \otimes \mathcal{T}SB & \xrightarrow{\rho_{\mathcal{B}}} & R[x] \otimes \mathcal{T}(R[x] \otimes \mathcal{T}SB)
\end{array}
\]

Let \(C = SA\) and \(D = SB\).

As in \cite{ALR+15} Example 3.3], the structure map \(\rho_C\) of the cofree coalgebra on \(C\) is completely determined by \(\Delta = \rho_{0,2}\) and \(d_x = \rho_{1,1}\), where \(\rho_{i,j} : C \rightarrow C^{\otimes j}\) is the following composite

\[
\rho_{i,j} : C \xrightarrow{\rho} R[x] \otimes \mathcal{T}C \xrightarrow{\pi_{0,2}} R^{i}x^{0} \otimes C^{\otimes j} \cong C^{\otimes j}.
\]

Post-composing the horizontal maps with \(\pi_{0,2}\) in the commutative diagram above, we find that we obtain a commutative diagram

\[
\begin{array}{ccc}
R[x] \otimes \mathcal{T}C & \xrightarrow{\rho_{\mathcal{A}}} & R[x] \otimes \mathcal{T}(R[x] \otimes \mathcal{T}C) \\
\downarrow{\tilde{h}} & & \downarrow{(d\mathcal{A}s)^i(\tilde{g};\tilde{h};\tilde{f})} \\
R[x] \otimes \mathcal{T}D & \xrightarrow{\rho_{\mathcal{B}}} & R[x] \otimes \mathcal{T}(R[x] \otimes \mathcal{T}D)
\end{array}
\]

and the outer commuting square gives the \(r\)-(\(\tilde{g}, \tilde{f}\))-coderivation condition for \(\tilde{h}\).

Similarly, one may post-compose the horizontal maps with \(\pi_{1,1}\) and check that the resulting condition is

\[
d_x^{D} \tilde{h} = (-1)^|\tilde{h}| \tilde{h}d_x^{C}.
\]

□
Next we want to pass from morphisms on $R[x] \otimes \mathcal{T}C$, commuting with $d_x$, to families of morphisms on $\mathcal{T}C$. As in Example 3.3 of [ALR+15], given a morphism of bigraded modules $f : R[x] \otimes \mathcal{T}C \to R[x] \otimes \mathcal{T}C$, write

$$f(x^n \otimes a) = \sum_i x^i \otimes f^{n,i}(a),$$

where $f^{n,i} : \mathcal{T}(C) \to \mathcal{T}(D)$ and $a \in C^{\otimes j}$. Then commuting with the map $d_x$ means that $f$ is completely determined by the family of maps $f^{n,0}$.

Define $f_n : \mathcal{T}C \to \mathcal{T}C$ by $f_n(a) = (-1)^{nj}f^{n,0}(a) = (-1)^{nj}\pi_0f(x^n \otimes a)$, where $a \in C^{\otimes j}$.

The correspondence gives a bijection between the subset of morphisms in $\text{bgMod}_R(R[x] \otimes \mathcal{T}C, R[x] \otimes \mathcal{T}D)$ which commute with $d_x$ and $\text{bgMod}_R^\infty(\mathcal{T}C, \mathcal{T}D)$.

Recall from [ALR+15] that under this assignment, a square-zero coderivation $\delta$ on $R[x] \otimes \mathcal{T}C$ corresponds to a twisted complex structure on $\mathcal{T}C$.

**Proposition 5.20.** Let $\tilde{\gamma}, \tilde{f}$ be coalgebra morphisms $R[x] \otimes \mathcal{T}C \to R[x] \otimes \mathcal{T}D$, commuting with $d_x$. If $\tilde{h}$ is an $r$-$(\tilde{\gamma}, \tilde{f})$-coderivation of coalgebras $\tilde{h} : R[x] \otimes \mathcal{T}C \to R[x] \otimes \mathcal{T}D$, graded commuting with $d_x$, the corresponding family of morphisms $\tilde{h}_n : \mathcal{T}C \to \mathcal{T}D$ satisfies

$$\left(\sum_j (-1)^j \tilde{\gamma}_j \otimes \tilde{h}_{i-j} + \sum \tilde{h}_j \otimes \tilde{f}_{i-j}\right) \Delta = \Delta \tilde{h}_i \quad \text{for all } i \geq 0.$$

**Proof.** First note that for $\alpha, \beta : R[x] \otimes \mathcal{T}C \to R[x] \otimes \mathcal{T}D$, $a \in C^{\otimes k}$ and $b \in C^{\otimes l}$, we have

$$(\pi_0 \otimes \pi_0)(\alpha \otimes \beta)(x^i \otimes a \otimes x^j \otimes b) = (-1)^{i(l(a'+\nu)+j(a_1+a_2)+i+k+j)(\alpha \otimes \beta)}(a \otimes b),$$

Then we calculate $\Delta \tilde{h}_i(a)$ for $a = a_1 \otimes \cdots \otimes a_n \in C^{\otimes n}$. The $\epsilon$ appearing in the sign in the following calculation is given as in Example 5.18 above. We also use that $|\tilde{h}| = -1$ and $|\tilde{f}| = 0$.

$$\begin{align*}
\Delta \tilde{h}_i(a) &= (-1)^{in} \Delta \pi_0 \tilde{h}(x^i \otimes a) \\
&= (-1)^{in} (\pi_0 \otimes \pi_0) \Delta \tilde{h}(x^i \otimes a) \\
&= (-1)^{in} (\pi_0 \otimes \pi_0)(\tilde{\gamma} \otimes \tilde{h} + \tilde{h} \otimes \tilde{f}) \Delta(x^i \otimes a) \\
&= (-1)^{in} (\pi_0 \otimes \pi_0)(\tilde{\gamma} \otimes \tilde{h} + \tilde{h} \otimes \tilde{f}) \sum_{k=1}^{n-1} (-1)^{\epsilon}(x^s \otimes a_1 \otimes \cdots \otimes a_k) \otimes (x^i \otimes a_{k+1} \otimes \cdots \otimes a_n) \\
&= (-1)^{in} \sum_{k=1}^{n-1} (-1)^{in+k+s+n-k}((\tilde{\gamma}_k \otimes \tilde{h}_i)((a_1 \otimes \cdots \otimes a_k) \otimes (a_{k+1} \otimes \cdots \otimes a_n)) \\
&\quad + (-1)^{in} \sum_{k=1}^{n-1} (-1)^{in+k+s+n-k}((\tilde{h}_k \otimes \tilde{f}_i)(a_1 \otimes \cdots \otimes a_k) \otimes (a_{k+1} \otimes \cdots \otimes a_n)) \\
&= \sum_{s+t=i} \left((-1)^s \tilde{\gamma}_s \otimes \tilde{h}_t + \tilde{h}_s \otimes \tilde{f}_t\right) \Delta(a).
\end{align*}$$

We now consider an $r$-shifted version of the usual homotopy relation and explain how $r$-homotopy of twisted complexes appears in this context.

**Definition 5.21.** For $F : \Lambda(dAs)^{(C)} \to \Lambda(dAs)^{(D)}$ a morphism of bigraded modules, let $SF : \Lambda(dAs)^{(C)} \to \Lambda(dAs)^{(D)}$ be given by $SF := -Fd_x^C$.

**Proposition 5.22.** The corresponding sequence of maps $\mathcal{T}C \to \mathcal{T}D$ is given by $(SF)_i = F_{i-1}$ if $i \geq 1$ and $(SF)_0 = 0$. 
Proof. For \( a \in C^{\otimes n} \), and setting \( F_{-1} = 0 \),
\[
(SF)_i(a) = (-1)^{in} \pi_0(SF)(x^i \otimes a) = (-1)^{in+1} \pi_0 F d_x^C(x^i \otimes a)
\]
\[
= (-1)^{in+r} \pi_0 F(x^{i-1} \otimes a) = (-1)^{in+(i-1)n} F_{i-1}(a)
\]
\[
= F_{i-1}(a).
\]

We defined a shift \( S : bgMod_R(A, B)^0 \rightarrow bgMod_R(A, B)^{r+1} \) in Definition 3.35. Proposition 5.22 shows that the one defined here corresponds to that one, hence we use the same notation.

**Proposition 5.23.** Let \( f, g : \Lambda(dA)_r(C) \rightarrow \Lambda(dA)_r(D) \) be morphisms of \( \Lambda(dA)_r \)-coalgebras and let \( h : \Lambda(dA)_r(C) \rightarrow \Lambda(dA)_r(D) \) be a morphism of bigraded modules of bidegree \((r, r-1)\) satisfying
\[
(1)^{r} \delta^D h + h \delta^C = S^r (g-f).
\]
Then the corresponding family of morphisms \( \tilde{h}_n : T C \rightarrow T D \) gives an r-homotopy of twisted complexes.

**Proof.** We extract the \( i \)-th map in the families corresponding to the two sides of the given equation. On the right-hand side, by Proposition 5.22 we obtain \( g_{i-r} - f_{i-r} \) for \( i \geq r \) and 0s for \( i < r \).

Recall from [ALR+15], that one can write \( \delta^{i-j}(a) = \sum \delta^{i-j}(a) \), with \( \delta^{i-j}(a) \in D^{\otimes r}, \) and similarly for \( h^{i-j}(a) \). Anti-commuting with the map \( d_x \) implies that \( \delta^{i-j}(a) = (-1)^{(n+l+1)} \delta^{i-j,0,l}(a) \) for \( a \in C^{\otimes n} \), and similarly for \( h \).

On the left-hand side we calculate to check the signs. For \( a \in C^{\otimes n} \),
\[
((-1)^{r} \delta^D h + h \delta^C)_i(a) = (-1)^{in} \pi_0 ((-1)^{r} \delta^D h + h \delta^C)(x^i \otimes a)
\]
\[
= (-1)^{in+r} \pi_0 \delta(\sum_j x^j \otimes h^{i,j}(a)) + (-1)^{in} \pi_0 h(\sum_j x^j \otimes \delta^{i,j}(a))
\]
\[
= (-1)^{in+r} \pi_0 (\sum_{j,k} x^j \otimes \delta^{i,0} h^{i,j}(a)) + (-1)^{in} \pi_0 (\sum_{j,k} x^k \otimes h^{i,0} \delta^{i,j}(a))
\]
\[
= (-1)^{in+r} \sum_j \delta^{i,0} h^{i,j}(a) + (-1)^{in} \sum_j h^{i,0} \delta^{i,j}(a)
\]
\[
+ (-1)^{in} (\sum_j (-1)^{(n+l+1)} h^{i,0} \delta^{i-j,0,l}(a)),
\]
where \( \delta^{i,j}(a) = \sum_j \delta^{i,j,l}(a) \), with \( \delta^{i,j,l}(a) \in D^{\otimes r} \), and similarly for \( h^{i,j}(a) \). Continuing the above calculation, we obtain
\[
(-1)^{in+r} (\sum_{j,l} (-1)^{(n+l+1)} \delta^{i-j,0,l}(a)) + (-1)^{in} (\sum_{j,l} (-1)^{(n+l+1)} h^{i,0} \delta^{i-j,0,l}(a))
\]
\[
= (-1)^{in+r} (\sum_{j} (-1)^{(n+1)+i-j} n \delta^{i-j,0,l}(a)) + (-1)^{in} (\sum_{j} (-1)^{(n+1)+i-j} n h^{i,j} \delta^{i-j,0,l}(a))
\]
\[
= \sum_j (-1)^{i+j} \delta^{i,j}(a) + (-1)^{i} h^{i,j} \delta^{i-j,0,l}(a),
\]
as required. \( \square \)

Thus an \( r \)-shifted version of the operadic notion of coderivation homotopy corresponds to a sequence of maps of bigraded \( R \)-modules \( h_n : T C \rightarrow T D \), where \( h_n \) has bidegree \((r-n, r-n-1)\), satisfying both the condition in Proposition 5.20 and the condition in Proposition 5.23. We are going to show that this is equivalent to an \( r \)-homotopy of \( dA_{\infty} \)-algebras, as defined via the path construction.

As an intermediate step, we reformulate the conditions using the composition in \( bgMod_R \).
5.3.2. Tensor coalgebra viewed in $\text{bgMod}_R$. First we make two definitions about families of maps on reduced tensor coalgebras. The first one is a coalgebra-morphism type condition for a family of maps; see also [Sag10, Section 4].

**Definition 5.24.** Let $(\tilde{f}_p) \in \text{bgMod}_R(\mathcal{T}SA, \mathcal{T}SB)^0_0$. Write $\tilde{f}^i_{pq}$ for the map $(SA)^{\otimes i} \to (SB)^{\otimes j}$ coming from $\tilde{f}_p$. We say that $(\tilde{f}_p)$ is a coalgebra-family of morphisms if for all $j, p, q$, we have

$$\tilde{f}^i_{pq} = \sum_{p_i + \cdots + p_j = p \atop q_1 + \cdots + q_j = q} f^1_{p_1 q_1} \otimes f^2_{p_2 q_2} \otimes \cdots \otimes f^i_{p_i q_i}.$$  

Now we consider coderivations between such families.

**Definition 5.25.** Let $(\tilde{f}_p, (\tilde{g}_p)) \in \text{bgMod}_R(\mathcal{T}SA, \mathcal{T}SB)^0_0$ be two coalgebra-families of morphisms. Let $(\tilde{h}_p) \in \text{bgMod}_R(\mathcal{T}SA, \mathcal{T}SB)^{\otimes r-1}_r$. Write $\tilde{h}^i_{pq}$ for the map $(SA)^{\otimes i} \to (SB)^{\otimes j}$ coming from $\tilde{h}_p$. We say that $(\tilde{h}_p)$ is an $r$-family of coderivations of morphisms if for all $i, k, l$, we have

$$\tilde{h}^i_{pk} = \sum_{0 \leq s \leq l-1} (-1)^{p_1 + \cdots + p_s} g^1_{p_1 q_1} \otimes \cdots \otimes g^1_{p_s q_s} \otimes \tilde{h}^i_{p_{s+1} q_{s+1}} \otimes f^1_{p_{s+2} q_{s+2}} \otimes \cdots \otimes f^i_{p_l q_l}.$$  

**Notation 5.26.** For $A \in \text{bgMod}_R$, we let $\mathcal{T}SA$ denote the object of $\text{bgMod}_R$ given by the underlying bigraded module of the reduced tensor coalgebra on $SA$. We let $\Delta \in \text{bgMod}_R(\mathcal{T}SA, \mathcal{T}SA \otimes \mathcal{T}SA)^0_0$ be given by $\Delta = (\Delta_0, \Delta_1, \ldots)$ with $\Delta_0 := \Delta$, the usual deconcatenation comultiplication and $\Delta_i := 0$ for $i > 0$.

Similarly, for $f \in \text{hom}_{\text{bgMod}_R}(A, B)$, write $\mathcal{T}sf$ for $\mathcal{T}sf := (\mathcal{T}SA, 0, 0, \ldots) \in \text{bgMod}_R(\mathcal{T}SA, \mathcal{T}SB)$.

In the light of the results of the previous section, and noting Remark 4.43 one expects to be able to describe $A_{\infty}$-algebras, their morphisms and so on, in terms of structure in $\text{bgMod}_R$ on the pairs $(\mathcal{T}SA, \Delta)$. We give the details next.

In the following proposition, we use the natural notions of coderivations, coalgebra morphisms and $(g, f)$-coderivations in $\text{bgMod}_R$. For example, a coalgebra morphism $\mathcal{T}SA \to \mathcal{T}SB$ in $\text{bgMod}_R$ means a morphism in $\text{bgMod}_R(\mathcal{T}SA, \mathcal{T}SB)^0_0$ satisfying

$$c(f \otimes f, \Delta) = c(\Delta, f).$$  

**Proposition 5.27.**

1. Let $\delta \in \text{bgMod}_R(\mathcal{T}SA, \mathcal{T}SA)^0_0$. If $\delta$ is a coderivation such that $c(\delta, \delta) = 0$ then it corresponds to a collection of coderivations $\delta_i : \mathcal{T}SA \to \mathcal{T}SA$ in $\text{bgMod}_R$, $i \geq 0$, together making $\mathcal{T}SA$ into a twisted complex. Thus, a $A_{\infty}$-algebra structure on a bigraded module $A$ is equivalent to specifying such a square-zero coderivation $\delta$.

2. Let $f \in \text{bgMod}_R(\mathcal{T}SA, \mathcal{T}SB)^0_0$. If $f$ is a coalgebra morphism commuting with given square-zero coderivations $d^1, d^2$, it corresponds to a coalgebra-family of morphisms $f_i : \mathcal{T}SA \to \mathcal{T}SB$, $i \geq 0$, together making a morphism of twisted complexes. Thus, a morphism of $A_{\infty}$-algebras from $A$ to $B$ is equivalent to specifying such a coalgebra morphism $f$.

3. Let $f, g \in \text{bgMod}_R(\mathcal{T}SA, \mathcal{T}SB)^0_0$ be coalgebra morphisms.

A $(g, f)$-coderivation $h \in \text{bgMod}_R(\mathcal{T}SA, \mathcal{T}SB)^{\otimes r-1}_r$ corresponds to an $r$-family of morphisms $h_i : \mathcal{T}SA \to \mathcal{T}SB$.

4. Let $f, g \in \text{bgMod}_R(\mathcal{T}SA, \mathcal{T}SB)^0_0$ be coalgebra morphisms.

A morphism $h \in \text{bgMod}_R(\mathcal{T}SA, \mathcal{T}SB)^{\otimes r-1}_r$ satisfying

$$(-1)^r c(d^B, h) + c(h, d^A) = S^r(g - f)$$

corresponds to an $r$-homotopy of morphisms of twisted complexes $h_i : \mathcal{T}SA \to \mathcal{T}SB$, $i \geq 0$. 

Proof. (1) We have that $c(\delta, \delta) = 0$ if and only if the $\delta_i$ satisfy the twisted complex relations. And the coderivation condition on $\delta$ translates into the coderivation condition on the individual $\delta_i$. In more detail,
\[
\begin{align*}
c(\delta \otimes 1 + 1 \otimes \delta, \Delta) &= c(\Delta, \delta) \\
&\iff (c(\delta \otimes 1 + 1 \otimes \delta, \Delta))_i = (c(\Delta, \delta))_i \quad \text{for all } i \geq 0 \\
&\iff (\delta \otimes 1 + 1 \otimes \delta_i)\Delta = \Delta \delta_i \quad \text{for all } i \geq 0, \text{ since } \Delta = (\Delta, 0, 0, \ldots ) \\
&\iff (\delta_i \otimes 1 + 1 \otimes \delta_i)\Delta = \Delta \delta_i \quad \text{for all } i \geq 0, \text{ since } 1 = (1_A, 0, 0, \ldots ).
\end{align*}
\]
The final part follows from \cite[4.1]{Sag10}.

(2) We have that $c(f, d^A) = c(d^B, f)$ if and only if the $f_i$ satisfy the relations to be a morphism of twisted complexes $(TSA, d^A_i) \to (TSB, d^B_i)$. Then $f$ is a coalgebra morphism means
\[
\begin{align*}
c(f \otimes f, \Delta) &= c(\Delta, f) \\
&\iff (c(f \otimes f, \Delta))_i = (c(\Delta, f))_i \quad \text{for all } i \geq 0 \\
&\iff (f \otimes f)_i\Delta = \Delta f_i \quad \text{for all } i \geq 0, \text{ since } \Delta = (\Delta, 0, 0, \ldots ) \\
&\iff \sum_j (f_j \otimes f_{i-j})\Delta = \Delta f_i \quad \text{for all } i \geq 0.
\end{align*}
\]
The last condition can be recursively reduced to the coalgebra-family of morphisms condition. The final part follows from \cite[4.3]{Sag10}.

(3) $c(g \otimes h + h \otimes f, \Delta) = c(\Delta, h)$
\[
\begin{align*}
&\iff (c(g \otimes h + h \otimes f, \Delta))_i = (c(\Delta, h))_i \quad \text{for all } i \geq 0 \\
&\iff (g \otimes h + h \otimes f)_i\Delta = \Delta h_i \quad \text{for all } i \geq 0, \text{ since } \Delta = (\Delta, 0, 0, \ldots ) \\
&\iff \left( \sum_j (-1)^j g_j \otimes h_{i-j} + \sum_j h_j \otimes f_{i-j} \right) \Delta = \Delta h_i \quad \text{for all } i \geq 0.
\end{align*}
\]
The last condition recursively reduces to the coderivation-family one.

(4) We calculate:
\[
\begin{align*}
(-1)^r c(d^B, h) + c(h, d^A) &= \mathbb{S}^r(g - f) \\
&\iff ((-1)^r c(d^B, h) + c(h, d^A))_i = (\mathbb{S}^r(g - f))_i \quad \text{for all } i \geq 0 \\
&\iff \sum_j (-1)^j (-1)^{i+r} d^B_j h_{i-j} + (-1)^j h_j d^A_{i-j} = \begin{cases} g_{i-r} - f_{i-r} & \text{if } i \geq r, \\ 0 & \text{if } i < r \end{cases} \quad (H_{11}).
\end{align*}
\]

\]

5.3.3. Comparison with path definition. We will show that the path definition of $r$-homotopy (Definition \ref{defn:homotopy}) corresponds to imposing conditions (3) and (4) in Proposition \ref{prop:homotopy}. In section \ref{subsection:comparison} we have checked that these match up with the corresponding $(dA^*)$-coalgebra notions.

Let $f, g : A \to B$ be two morphisms of $dA_{\Sigma}$-algebras and let $h : A \to P_r(B)$ be an $r$-homotopy from $f$ to $g$. Recall that this is a morphism of $dA_{\Sigma}$-algebras satisfying $\partial^-_B \circ h = f$ and $\partial^+_B \circ h = g$.

Let $F, G : \overline{T}SA \to \overline{T}SB$ be the coalgebra morphisms in $\overline{bgMod}_R$ corresponding to $f, g$ and let $H : \overline{T}SB \to \overline{T}SP_r(B)$ be the coalgebra morphism in $\overline{bgMod}_R$ corresponding to $h$.

We have three projection maps from $P_r(B)$ to $B$, on to the left, middle and right copies of $B$, denoted $\partial^-_B, \partial^+_B$ and $\partial^+_B$ respectively. We denote the corresponding maps $SP_r(B)$ to $SB$ by $\pi_L, \pi_M$ and $\pi_R$ respectively. Then let $\pi \in \overline{bgMod}_R(\overline{T}SP_r(B), \overline{T}SB)$ be the strict map
\[
\sum_s \pi^{\partial^-} \otimes \pi^{\partial^+} \otimes \pi^{\partial^+}.
\]

Proposition 5.28. In the situation above, $c(\pi, H) : \overline{T}SA \to \overline{T}SB$ is a $(G, F)$-coderivation in $\overline{bgMod}_R$. 

**Proof.** Consider the diagram

\[
\text{TSA} \xrightarrow{\Delta} \text{TSA} \otimes \text{TSA}
\]

\[
\text{HSA} \xrightarrow{\Delta} \text{HSA} \otimes \text{HSA}
\]

\[
\text{TP}(B) \xrightarrow{\Delta} \text{TP}(B) \otimes \text{TP}(B)
\]

\[
\pi \xrightarrow{\Delta} \text{TP}(\pi) \otimes \pi \otimes \text{TP}(\pi)
\]

We claim this is a commutative diagram in the category $\text{bgMod}_R$. The top square is such a commutative diagram since $H$ is a coalgebra morphism in $\text{bgMod}_R$. In the bottom square all the morphisms are strict, so composition and tensor agree with the usual ones and we just need to see that this commutes in the usual sense, which can be easily checked.

Finally, one can check that $c(\text{T}S(\partial_B^+), H) = G$ and $c(\text{T}S(\partial_B^-), H) = F$, so the commuting of the outer square reads $c(G \circ c(\pi, H) + c(\pi, H) \otimes F), \Delta) = c(\Delta, c(\pi, H))$, which is the $(G, F)$-coderivation condition for $c(\pi, H)$.

□

**Lemma 5.29.**

\[
(c(\pi, d^P B) + (-1)^r c(d^B, \pi))_i = \begin{cases} (\text{T}(\pi_R) - \text{T}(\pi_L)) & \text{if } i = r, \\ 0 & \text{otherwise.} \end{cases}
\]

**Proof.** Since $\pi$ is a strict map,

\[
(c(\pi, d^P B) + (-1)^r c(d^B, \pi))_i = \pi d_i^P B + (-1)^{r+i} d_i^B \pi.
\]

Now

\[
d_i^B \pi = \sum_{u,v} (1^s_B \otimes \tilde{m}_{ij} \otimes 1^t_B) (\sum_{u,v} \pi^u_L \otimes \pi^M \otimes \pi^v_R),
\]

where, following Sagave’s conventions, $\tilde{m}_{ij} = \Psi_j(m_{ij}) : SB^\otimes j \rightarrow SB$.

Thus we have

\[
\pi_L = S \partial_B^{-1} S^{-1}, \quad \pi_R = S \partial_B^{-1} S^{-1}, \quad \pi_M = S \partial_B^{-1} S^{-1}(-1)^{r-1}
\]

with $\pi_L$ and $\pi_R$ of bidegree $(0, 0)$ and $\pi_M$ of bidegree $(r, 1 - r)$ and $\tilde{m}_{ij} = Sm_{ij}(S^{-1})^\otimes (1)^{1+i}$, $\tilde{M}_{ij} = SM_{ij}(S^{-1})^\otimes (1)^{1+i}$, both of bidegree $(-i, 1 - i)$.

When we expand out the composition, there are three types of terms appearing, according to whether the input to $\tilde{m}_{ij}$ is of the form $\pi_L^a$, $\pi^a_L \otimes \pi^M \otimes \pi^v_R$ or $\pi^v_R$.

And

\[
\pi d_i^P B = \left( \sum_{u,v} \pi^u_L \otimes \pi^v_R \otimes \pi^M \right) \left( \sum_{u,v} (1^s_B \otimes \tilde{M}_{ij} \otimes 1^t_B) \right),
\]

where $\tilde{M}_{ij} : SP_r(B)^\otimes j \rightarrow SP_r(B)$. Again there are three sorts of terms, according to whether they involve $\pi_L \tilde{M}_{ij}$, $\pi_M M_{ij}$ or $\pi_R \tilde{M}_{ij}$.

From the definition of the $M_{ij}$ for $P_r(B)$, we have, for all $(i, j)$,

\[
\partial_B^R M_{ij} = \tilde{m}_{ij}(\partial_B^R)^{\otimes j}, \quad \partial_B^L M_{ij} = \tilde{m}_{ij}(\partial_B^L)^{\otimes j}.
\]

And

\[
(-1)^{r+j+i} \partial_B^0 M_{ij} = \sum_{a+b+1=1} m_{ij}((\partial_B^R)^{\otimes a} \otimes \partial_B^0 \otimes (\partial_B^L)^{\otimes b}), \quad \text{for } (i, j) \neq (r, 1),
\]

\[
\partial_B^0 M_{r1} = -m_{r1} \partial_B^+ \partial_B^- - \partial_B^0.
\]
Sign calculations with the suspension show that these convert to
\[
\pi_L \tilde{M}_{ij} = \tilde{m}_{ij} \pi_L^{\otimes j}, \quad \pi_R \tilde{M}_{ij} = \tilde{m}_{ij} \pi_R^{\otimes j}, \quad \text{for all } (i,j)
\]
and
\[
(-1)^{r+i+1} \pi_M \tilde{M}_{ij} = \sum_{a+b+1=1} \tilde{m}_{ij} (\pi_L^{\otimes a} \otimes \pi_M \otimes \pi_R^{\otimes b}),
\]
\[
\pi_M \tilde{M}_{1} = -\tilde{m}_{1} \pi_M + \pi_R - \pi_L.
\]

Using the above, one may now check that in \((c(\pi, dP_r B) + (-1)^r c(dB, \pi))\), almost all terms cancel pairwise, with the exception, when \(i = r\), of the extra terms in \(c(\pi, dP_r B)\) coming from the special form of \(M_{1}\). These contribute
\[
\sum_{s,t} 1^\otimes_s \otimes (\pi_R - \pi_L) \otimes 1^\otimes_t = \sum_j \pi_R^{\otimes j} - \pi_L^{\otimes j} = (\overline{T}(\pi_R) - \overline{T}(\pi_L)) \triangleleft
\]

**Proposition 5.30.** The map \(c(\pi, H) : \overline{T} SA \rightarrow \overline{T} SB\) satisfies the r-homotopy condition in \(bg\text{Mod}_R\) of part (4) of Proposition 5.29.

**Proof.** Using associativity of composition (Lemma 4.14), Lemma 5.29 and the relation \(c(H, dA) = c(dP_r(B), H)\), we have
\[
(c(c(\pi, H), dA) + (-1)^r c(dB, c(\pi, H))) = (c(c(\pi, dP_r B) + (-1)^r c(dB, \pi), H)) = (\overline{T}(\pi_R) - \overline{T}(\pi_L)) H_{i-r} = G_{i-r} - F_{i-r}.
\]

So
\[
c(c(\pi, H), dA) + (-1)^r c(dB, c(\pi, H)) = \mathcal{S'}(G - F),
\]
as required. \(\triangleleft\)

We now collect everything together.

**Theorem 5.31.** Let \(f, g : A \rightarrow B\) be two morphisms of \(dA_\infty\)-algebras. Then \(h : A \rightarrow P_r(B)\) being an r-homotopy from \(f\) to \(g\) in the sense of Definition 5.8 is equivalent to conditions (3) and (4) of Proposition 5.22 on \(c(\pi, H) : \overline{T} SA \rightarrow \overline{T} SB\), where \(H : \overline{T} SA \rightarrow \overline{T} SP_r(B)\) is the coalgebra morphism in \(bg\text{Mod}_R\) corresponding to \(h\).

**Proof.** We have already seen that \(c(\pi, H)\) does satisfy conditions (3) and (4) of Proposition 5.22. It is also straightforward to see that we can recover \(h\) from \(c(\pi, H)\). Indeed, from \(c(\pi, H)\) we can extract \(G, F\) and \(c(\pi_M, H^1)\), where \(H^1 : \overline{T} SA \rightarrow SP_r(B)\) is the composite of \(H\) with the strict projection \(\overline{T} SP_r(B) \rightarrow SP_r(B)\) and \(c(\pi, H)\) is uniquely determined by this data. Now \(H^1\) is also uniquely determined by \(G, F\) and \(c(\pi_M, H^1)\) and we have \(\tilde{h}_{ij} = H^1_{ij}\). \(\triangleleft\)

Thus the notion of r-homotopy defined via the path construction coincides with the operadic one.

5.3.4. **Explicit r-homotopy.**

**Proposition 5.32.** Giving an r-homotopy \(h : A \rightarrow P_r(B)\) between morphisms of \(dA_\infty\)-algebras \(f, g : A \rightarrow B\) is equivalent to giving a collection of morphisms \(h_{ik} : A^{\otimes k} \rightarrow B\) of bidegree \((r-i, r-i-k)\),
satisfying, for all \( m \) and \( k \),
\[
(-1)^{m-r} \sum_{i+p=m} \left( \sum_{l} m_{il}^{B} \sum_{0 \leq s \leq t-l-1} (-1)^{p+\sum_{i=1}^{s} p_{i} q_{1}} \otimes \cdots \otimes g_{p_{i} q_{s}} \otimes h_{p_{s+1} q_{s+1}} \otimes f_{p_{s+2} q_{s+2}} \otimes \cdots \otimes f_{p q}
\right. \\
+ \sum_{l} h_{il} \sum_{q+i+l=k} (-1)^{\beta} 1^{\otimes s} \otimes m_{pq}^{A} \otimes 1^{\otimes t}\right) = \begin{cases} 
0 & \text{if } m < r, \\
g_{m-r,k} - f_{m-r} & \text{if } m \geq r.
\end{cases} \quad (H_{mk})
\]

Here the signs are given by
\[
\alpha = \sum_{u=1}^{l} (p_{u} + q_{u}) (l + u) + \sum_{u=1}^{l} q_{u} (\sum_{v=u+1}^{l} p_{v} + q_{v}) + (r-1)(l+1+s + \sum_{u=1}^{s} q_{u}), \\
\beta = sq + t + pl + r.
\]

Proof. We have seen that the path definition of \( r \)-homotopy for \( dA_{\infty} \)-algebras agrees with the \( r \)-coderivation-family and \( r \)-homotopy of twisted complex conditions on the corresponding families of maps \( \mathcal{T}SA \to \mathcal{T}SB \).

Suppose that \( dA_{\infty} \)-algebra structures of \( A \) and \( B \) are encoded in families of coderivations \( d^{i}_{A} \) and \( d^{i}_{B} \) making \( \mathcal{T}SA \) and \( \mathcal{T}SB \) respectively into twisted chain complexes. Suppose also that \( f \) corresponds to the coalgebra-family of maps \( \tilde{f}_{i} : \mathcal{T}SA \to \mathcal{T}SB \), giving a morphism of twisted complexes; similarly \( g \) corresponds to a coalgebra-family of maps \( \tilde{g}_{i} : \mathcal{T}SA \to \mathcal{T}SB \).

The \( r \)-homotopy condition on \( \tilde{h}_{i} \) between maps of twisted complexes is equivalent to, for all \( m \geq 0 \),
\[
\sum_{i+j=m} (-1)^{i+r} d^{B}_{i} \tilde{h}_{j} + (-1)^{i} \tilde{h}_{i} \delta^{A}_{j} = \begin{cases} 
0 & \text{if } m < r, \\
\tilde{g}_{m-r} - \tilde{f}_{m-r} & \text{if } m \geq r.
\end{cases} \quad (H_{m1})
\]

This is an equality of maps \( \mathcal{T}SA \to \mathcal{T}SB \). Here the bidegree of \( \tilde{h}_{i} \) is \((r - i, r - i - 1)\), that of \( d_{i}^{A} \) and \( d_{i}^{B} \) is \((-i, i + 1)\) and that of \( \tilde{f}_{i} \) and \( \tilde{g}_{i} \) is \((-i, -i)\).

For each \( m \geq 0 \) and for \( k \geq 1 \), we consider the “component of equation \((H_{m1})\)” from \((SA)^{\otimes k} \to SB\). That is, we pre-compose with the inclusion \((SA)^{\otimes k} \to \mathcal{T}SA\) and post-compose with the projection to \( SB \). We will show that this gives, after shifting, the required statement.

The various conditions on the morphisms mean that everything is determined by components.

Firstly, the coderivation condition on each \( d_{i}^{A} \) means that they are determined by components \((SA)^{\otimes k} \to SB\), corresponding to \( m_{ik}^{A} : A^{\otimes k} \to A \) and similarly for \( d_{i}^{B} \). Secondly, since \( \tilde{f}_{i} : \mathcal{T}SA \to \mathcal{T}SB \) form a coalgebra-family of morphisms, it is determined by components \( \tilde{f}_{ik} : (SA)^{\otimes k} \to SB \) and similarly for \( \tilde{g}_{i} \). And finally, recall that the \( r \)-\((g_{i}, f_{i})\)-coderivation-family condition means that \( \tilde{h}_{i} \) is determined by components \( \tilde{h}_{ik} = \tilde{h}_{ik}^{1} : (SA)^{\otimes k} \to SB \), where \( \tilde{h}_{ik}^{1} : (SA)^{\otimes k} \to (SB)^{\otimes s} \) is given by
\[
\tilde{h}_{ik} = \sum_{i_{1}+\cdots+i_{s}+l_{1}+\cdots+l_{t}=k} (-1)^{i_{1}+\cdots+i_{s}+l_{1}+\cdots+l_{t}} \tilde{g}_{i_{1}k_{1}} \otimes \cdots \otimes \tilde{g}_{i_{s}k_{s}} \otimes h_{pq} \otimes \tilde{f}_{l_{1}j_{1}} \otimes \cdots \otimes \tilde{f}_{l_{t}j_{t}}.
\]

Let \( f_{ik} = \Psi_{k}^{-1}(\tilde{f}_{ik}) : A^{\otimes k} \to B \), \( g_{ik} = \Psi_{k}^{-1}(\tilde{g}_{ik}) : A^{\otimes k} \to B \) and \( h_{ik} = \Psi_{k}^{-1}(\tilde{h}_{ik}) : A^{\otimes k} \to B \) and note that the bidegree of \( h_{ik} \) is \((i-r, i-r-k)\). We have \( \tilde{m}_{ik} = \Psi_{k}(m_{ik}) = (-1)^{i+1} \sigma^{-1} m_{ik} \sigma^{\otimes k}, \)
\( f_{ik} = \Psi_{k}(f_{ik}) = (-1)^{i} \sigma^{-1} f_{ik} \sigma^{\otimes k} \) and similarly for \( g_{ik} \) and \( h_{ik} = \Psi_{k}(h_{ik}) = (-1)^{i+1} \sigma^{-1} h_{ik} \sigma^{\otimes k}. \)

Then it is a matter of direct calculation that the extraction of the relevant component maps from equation \((H_{m1})\), together with removing the shifts using the isomorphism \( \Psi_{k} \), gives the required result. 

\[ \Box \]

Remark 5.33. Sagave [Sag10] Definition 4.9] defined homotopy of morphisms from \( A \) to \( B \) of \( dA_{\infty} \)-algebras only in the special case where \( A \) is minimal and \( B \) is a bida. One may check that his definition is equivalent to our 0-homotopy in that case.
Appendix A. List of notation for categories

For easy reference, we provide a list of the notation for the main categories appearing in this paper.

- \((C_R, \otimes, R)\) is the category with objects cochain complexes and morphisms degree 0 chain maps. The unit \(R\) is the cochain complex concentrated in degree zero.
- \((\text{bgMod}_R, \otimes, R)\) is the category with objects bigraded modules and morphisms degree \((0,0)\) maps of bigraded modules. The unit \(R\) is as above.
- \((\text{vbC}_R, \otimes, R)\) is the category with objects vertical bicomplexes and morphisms degree \((0,0)\) maps of vertical bicomplexes. The unit \(R\) is as above. See Definition 2.7.
- \((tC_R, \otimes, R)\) is the category of twisted complexes with morphisms degree \((0,0)\) maps of twisted complexes i.e., infinity morphisms. The unit \(R\) is the twisted complex concentrated in degree \((0,0)\). See Definitions 3.1 and 3.2.
- \((\text{bgMod}^{f}_R, \otimes, R)\) is the full subcategory of \(tC_R\) on twisted complexes with trivial structure i.e., zero differentials.
- \((\text{fMod}_R, \otimes, R)\) is the symmetric monoidal category with objects filtered graded modules and morphisms degree 0 morphisms which respect the filtration. The unit is the base ring \(R\) sitting in degree 0 with trivial filtration. See Definition 2.2.
- \(\text{sfMod}_R\) is the full subcategory of \((\text{fMod}_R, \otimes, R)\) on split filtered modules. See Definition 3.7.
- \((\text{fC}_R, \otimes, R)\) is the category with objects filtered complexes and morphisms degree 0 chain maps that respect the filtration, the unit as above. See Definition 2.4.
- \(\text{sfC}_R\) is the full subcategory of \((\text{fC}_R, \otimes, R)\) on split filtered complexes. See Definition 3.7.
- \(tC^{f}_R, \text{vbC}^{f}_R, \text{bgMod}^{f}_R\) are the full subcategories of \((\mathbb{N}, \mathbb{Z})\)-graded twisted complexes, vertical bicomplexes and bigraded modules respectively. See Definition 3.10.
- \(\text{fMod}^{b}_R, \text{sfMod}^{b}_R, \text{fC}^{b}_R, \text{sfC}^{b}_R\) are the full subcategories of \((\text{split})\) non-negatively filtered modules respectively complexes, i.e. the full subcategories of objects \((K,F)\) such that \(F_pK^n = 0\) for all \(p < 0\). See Definition 3.10.
- \(A_{\infty}(R)\) is the category of \(A_{\infty}\)-algebras over \(R\).
- \(dA_{\infty}(R)\) is the category of derived \(A_{\infty}\)-algebras over \(R\). See Definitions 4.1 and 4.2.
- \(A^{C}_{\infty}(R)\) is the category of derived \(A_{\infty}\)-algebras over \(R\) in the category of twisted chain complexes. See Definition 4.16.
- \(fA_{\infty}\) is the category of filtered \(A_{\infty}\)-algebras. See Definitions 4.48 and 4.50.
- \(\text{sfA}_{\infty}, fA^{b}_{\infty}\) and \(\text{sfA}^{b}_{\infty}\) are the subcategories on split filtered \(A_{\infty}\)-algebras, on non-negatively filtered \(A_{\infty}\)-algebras and of split non-negatively filtered \(A_{\infty}\)-algebras respectively. See Diagram 1.
- \(\text{bgMod}^{b}_R\) is the \(\text{bgMod}_R\)-enriched category of bigraded modules. See Definition 4.22.
- \(tC^{b}_R\) is the \(\text{vbC}_R\)-enriched category of twisted complexes. See Definition 4.21.
- \(\text{sfMod}^{b}_R\) is the full subcategory of \(tC^{b}_R\) on \((\mathbb{N}, \mathbb{Z})\)-graded twisted complexes.
- \(\text{fMod}^{b}_R\) is the \(\text{bgMod}_R\)-enriched category of filtered graded modules. See Definition 4.26.
- \(\text{fC}^{b}_R\) is the \(\text{vbC}_R\)-enriched category of filtered complexes. See Definition 4.31.
- \(\text{sfMod}^{b}_R\) and \(\text{sfMod}^{f}_R\) are the full subcategories of \(\text{fMod}^{b}_R\) on split filtered modules and on split non-negatively filtered modules respectively.
- \(\text{sfC}^{b}_R\) and \(\text{sfC}^{f}_R\) are the full subcategories of \(\text{fC}^{b}_R\) on split filtered complexes and on split non-negatively filtered complexes respectively.
- \(\otimes\) denotes the enriched monoidal structure on \(tC^{b}_R, \text{bgMod}^{b}_R, \text{fMod}^{b}_R\) and \(\text{fC}^{b}_R\). See Lemmas 4.25 and 4.34.

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