'Universality' of the Ablowitz-Ladik hierarchy.

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Abstract

The aim of this paper is to summarize some recently obtained relations between the Ablowitz-Ladik hierarchy (ALH) and other integrable equations. It has been shown that solutions of finite subsystems of the ALH can be used to derive a wide range of solutions for, e.g., the 2D Toda lattice, nonlinear Schrödinger, Davey-Stewartson, Kadomtsev-Petviashvili (KP) and some other equations. Similar approach has been used to construct new integrable models: O(3,1) and multi-field sigma models. Such 'universality' of the ALH becomes more transparent in the framework of the Hirota’s bilinear method. The ALH, which is usually considered as an infinite set of differential-difference equations, has been presented as a finite system of functional-difference equations, which can be viewed as a generalization of the famous bilinear identities for the KP tau-functions.

1 Introduction

One of the characteristic features of the theory of integrable systems is the fact that various models, sometimes rather different apparently, turn out to be, in such or other way, closely interrelated. For example, the nonlinear Schrödinger equation (NLSE) is gauge equivalent to the Landau-Lifshitz equation, sine-Gordon to Thirring model, Ablowitz-Ladik model – to the classical Heisenberg chain, etc. A large number of the integrable equations can be obtained as reductions of, e.g., the KP or the self-dual Yang-Mills equations. Poles of the rational solutions of the KdV evolve according to the Calogero model and so on.

Another kind of interrelations has been discovered by Levi, Benguria [1, 2], Flaschka [3], and also Shabat, Yamilov [4]. They considered sequences of Backlund transformations (BT) for some integrable nonlinear problems

\[ N[\psi] = 0, \quad \ldots \rightarrow \psi \rightarrow \psi_1 \rightarrow \psi_2 \rightarrow \ldots \]

and demonstrated that these sequences can be described by differential-difference equations (DDE) which are also integrable. For example, sequences of BT’s for the NLSE provide solutions for the Toda chain.

Similar ideas, in a somewhat transformed form, have been used in the works [4, 5, 6]. It turns out that in some situations solutions of the DDE’s can be used to generate families of solutions for some related partial differential equations (PDE). Namely such cases have been discussed in [4, 5, 6]. There the approach of [4, 5, 6] has been extended by taking into account several DDE’s simultaneously, which enabled to deal with multidimensional PDE’s, such as the 2D Toda lattice (2DTL), and the Davey-Stewartson (DS) equation.

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(2+1-dimensional systems) and others. The method of the works [5, 6, 7] can be explained by discussing the following very simple examples.

**Toy example 1.**
Consider the system of differential-difference equations

\[
\begin{align*}
    i\partial_x q_n &= q_{n+1} + q_{n-1} \\
    \partial_y q_n &= q_{n+1} - q_{n-1}
\end{align*}
\]

(1.1)

By trivial algebra one can show that, first, this system is compatible and, second, that for every \(n\) the quantity \(q = q_n\) satisfies the Helmholtz equation.

\[
\Delta q + 4q = 0, \quad \Delta = \partial_{xx} + \partial_{yy}
\]

(1.2)

This example is aimed to demonstrate that the system of DDE’s can be converted to PDE. In the linear case this is not very interesting, but in the nonlinear one, as will be shown below, such transformations may be fruitful.

**Toy example 2.**
As the second example consider the system (1.1) enlarged with one equation more:

\[
\begin{align*}
    i\partial_x q_n &= q_{n+1} + q_{n-1} \\
    \partial_y q_n &= q_{n+1} - q_{n-1} \\
    i\partial_t q_n &= q_{n+2} + q_{n-2}
\end{align*}
\]

(1.3)

Again, one can straightforwardly verify that this system is compatible and obtain that for every \(n\) the quantity \(q = q_n\) satisfies the equation

\[
\partial_t q + \frac{1}{2} \Box q = 0, \quad \Box = \partial_{xx} - \partial_{yy}
\]

(1.4)

As in the first example we have converted the system of DDE’s into a PDE. But it should be noted that we started with three 1+1, i.e. 2 dimensional equations, while the resulting one is 2+1, i.e. 3 dimensional. This is not crucial in the linear case, but is very important in the nonlinear one. The methods elaborated for multidimensional \((d \geq 3)\) nonlinear integrable systems (2DTL, DS, KP, etc.) such as, e.g., the multidimensional inverse scattering transform (MIST), are much more complicated than the traditional one (IST) developed for the 2- and (1+1)-dimensional systems (KdV, NLSE, Ablowitz-Ladik model, etc.). It will be shown below that in the nonlinear case such ‘decomposing’ of a multidimensional PDE in a system of low-dimensional DDE’s can be rather useful, at least from the practical viewpoint.

Transformations similar to the ones described above, their nonlinear variant, are the core of the works [5, 6, 7]. But before proceeding further I would like to discuss the following very important question, namely the question of compatibility. It is rather easy to write down compatible linear systems as the ones above, but in the nonlinear case the problem is a little bit more difficult. Nevertheless it is surely possible to construct systems, say, ones similar to (3.1), (3.3). But there is no need to do that, because such nonlinear and compatible systems are already known – they are the integrable hierarchies. Every integrable equation does not appear alone, it is always a member of some infinite set of related equations. And commutativity of corresponding flows, or involutivity of corresponding Hamiltonians, or, in simpler words, compatibility of corresponding equations, is one of the ingredients of integrability. Thus, in what follows I will ‘nonlinearize’ the toy examples using an integrable hierarchy, namely the Ablowitz-Ladik hierarchy.

The plan of the present paper is as follows. After outlining some basic facts related to the ALH (section 2) I will establish relations between this hierarchy and some other equations, such as 2DTL, DS, NLSE, KP (section 3) as well as few vector models (section 4). To make this ‘universality’ of the ALH more transparent I will discuss it in the framework of the Hirota’s bilinear method and present in section 5 the recently obtained functional representation of the ALH which can be viewed as a generalization of the famous bilinear identities for the KP tau-functions.
2 Ablowitz-Ladik hierarchy.

The ALH is an infinite set of ordinal differential-difference equations, that has been introduced by Ablowitz and Ladik in 1975 [1]. The most well known of these equations is the discrete nonlinear Schrödinger equation (DNLSE)

\[ i\dot{q}_n = q_{n+1} - 2q_n + q_{n-1} - q_n r_n (q_{n+1} + q_{n-1}) \]  

(2.1)

and the discrete modified KdV equation (DMKdV) (see, e.g., [11]),

\[ \dot{q}_n = p_n (q_{n+1} - q_{n-1}) \]  

(2.2)

where

\[ p_n = 1 - q_n r_n, \quad r_n = -\kappa q_n, \quad \kappa = \pm 1 \]  

(2.3)

All equations of the ALH can be presented as the compatibility condition for the linear system

\[ \Psi_{n+1} = U_n \Psi_n \]  

(2.4)

\[ \partial_t \Psi_n = V_n \Psi_n \]  

(2.5)

where \( \partial_t \) stands for \( \partial/\partial t \), which leads to their zero-curvature representation:

\[ \partial_t U_n = V_{n+1} U_n - U_n V_n \]  

(2.6)

In the standard IST approach developed in [9] the matrix \( U_n \) for the ALH is given by

\[ U_n = \begin{pmatrix} \lambda & r_n \\ q_n & \lambda^{-1} \end{pmatrix} \]  

(2.7)

where \( \lambda \) is the auxiliary (spectral) constant parameter.

According to [9], elements of the matrix \( V_n \), can be chosen as Laurent polynomials in \( \lambda \) in such a way that (2.4) holds automatically for all \( \lambda \)'s provided \( q_n \)'s and \( r_n \)'s satisfy some differential relations. It should be noted that one can obtain an infinite number of the matrices \( V_n \) (which are Laurent polynomials of different order) which leads to the infinite number of differential equations \( \partial q_n/\partial t = F_n \), \( (l = 1, 2, ...) \). Using the widely accepted viewpoint one can consider \( q_n \)'s and \( r_n \)'s as depending on the infinite number of 'times', \( q_n = q_n(t_1, t_2, ...) \) and consider the \( l \)th equation of the ALH as describing the flow with respect to the \( l \)th variable, \( \partial q_n/\partial t_l = F_n \). Traditionally it is implied that all 'times' \( t_l \) are real, which is grounded from the standpoint of physical applications, and also is convenient in the framework of the inverse scattering technique. However, in some cases it is more convenient to use instead of real 'times' \( t_l \) some complex variables \( z_j, \tilde{z}_j, \) \( j = 1, 2, ... \) (as in, say, 2D Toda theory), which, as will be shown below, exhibit in a more explicit way some intrinsic properties of the ALH. A simple analysis yields that the family of possible solutions of (2.9) (and hence the equations of the hierarchy) can be divided in two subsystems. One of them consists of \( V \)-matrices which are polynomials in \( \lambda^{-1} \) (I will term the corresponding equations as a 'positive' part of hierarchy), and the other consists of matrices which are polynomials in \( \lambda \) ('negative' subhierarchy), while in the standard, 'real-time', approach all \( V \)-matrices contain terms proportional to \( \lambda^m \) together with the terms proportional to \( \lambda^{-m} \) \( (m \geq 0) \). Let us consider first the 'positive' case. An infinite number of polynomial in \( 1/\lambda \) solutions \( V_n^j \) \( (j = 1, 2, ...) \) possesses the following structure:

\[ V_n^j = \lambda^{-2} V_{n+1}^j - \left( \frac{\lambda^{-2} \alpha_n^j \beta_n^j - \lambda^{-1} \delta_n^j}{\alpha_n^j \beta_n^j} \right) \]  

(2.8)

where the elements \( \alpha_n^j, ..., \delta_n^j \) satisfy the equations

\[ \alpha_{n+1}^j - \alpha_n^j = -q_n \beta_{n+1}^j + r_n \gamma_n^j \]  

(2.9)

\[ \delta_{n+1}^j - \delta_n^j = q_n \beta_n^j - r_n \gamma_{n+1}^j \]  

(2.10)

\[ \partial_j q_n = q_n \delta_{n+1}^j + \alpha_{n+1}^j = q_n \alpha_{n+1}^j + \gamma_{n+1}^j \]  

(2.11)

\[ \partial_j r_n = -r_n \delta_n^j - \beta_n^j = -r_n \alpha_{n+1}^j - \beta_{n+1}^j \]  

(2.12)
with \( \partial_j = \partial / \partial z_j \). Choosing
\[
\alpha_n^0 = \beta_n^0 = \gamma_n^0 = 0, \quad \delta_n^0 = -i
\]
we can obtain consequently
\[
\begin{align*}
\alpha_n^1 &= 0 \\
\beta_n^1 &= -ir_{n-1} q_n \\
\gamma_n^1 &= -ir_n q_n \\
\delta_n^1 &= i r_{n-1} q_n
\end{align*}
\]
and, in principle, all other matrices \( V_n^j \). This leads to the infinite system of equations for \( q_n, r_n \), some first of which are
\[
\begin{align*}
\partial_2 q_n &= -i p_n q_{n+1} \\
\partial_1 r_n &= i r_{n-1} p_n \\
\partial_2 q_n &= -i p_n p_{n+1} q_{n+2} + i r_{n-1} p_n q_n q_{n+1} + i p_n r_n q_{n+1}^2 \\
\partial_2 r_n &= i r_{n-2} p_{n-1} p_n - i r_{n-1} p_n r_{n+1} - i r_{n-1} p_n q_n
\end{align*}
\]
Analogously, looking for the \( V \)-matrices of the form
\[
V_n^{-j} = \lambda^2 V_n^{-j+1} + \left( \begin{array}{cc}
\alpha_n^{-j} & \lambda \beta_n^{-j} \\
\lambda \gamma_n^{-j} & \lambda^2 \delta_n^{-j}
\end{array} \right)
\]
one can obtain the 'negative' part of the ALH. Some first of its equations are
\[
\begin{align*}
\tilde{\partial}_1 q_n &= -i q_{n-1} p_n \\
\tilde{\partial}_1 r_n &= i p_n r_{n+1} \\
\tilde{\partial}_2 q_n &= -i q_{n-2} p_{n-1} p_n + i q_{n-1} p_n q_n r_{n+1} + i q_{n-1} p_n r_n \\
\tilde{\partial}_2 r_n &= i p_n p_{n+1} r_{n+2} - i q_{n-1} p_n r_{n+1} - i p_n q_n r_{n+1}^2
\end{align*}
\]
where \( \tilde{\partial}_j = \partial_j - \lambda / \partial \bar{z}_j \) and \( \bar{z}_j \) is the complex conjugated of \( z_j \).

I will not discuss here this hierarchy in detail, because now we have all the necessary to derive relations between the ALH and other integrable models and namely this is the main theme of the present paper.

3 "Embedding" into the ALH.

3.1 The Ablowitz-Ladik hierarchy and the O(3,1) \( \sigma \)-model.

The first, and the simplest, implementation of the 'embedding' into the ALH method was carried out in [5] and can be viewed as a nonlinearized version of the first toy example.

Consider the following system of two equations of the ALH:
\[
\begin{cases}
\partial_x q_n = p_n (q_{n+1} + q_{n-1}) \\
\partial_y q_n = p_n (q_{n+1} - q_{n-1})
\end{cases}
\]
The first of them is the DNLSE (2.1) transformed by means of the substitution \( q_n \rightarrow q_n \exp(2ix) \), while the second is the DMKdV (2.2). One can also view these equations as \( \partial \gamma_n^1 = i r_{n-1} p_n, \) \( \partial \gamma_n^2 = i r_{n-1} p_n r_{n+1} \) rewritten in terms of the variables \( x = \text{Re} \ z_j \) and \( y = \text{Im} \ z_j \).

This system is compatible because it is a system of equations belonging to one hierarchy (this fact is not hard to verify directly in this case). Differentiating the first equation with respect to \( x \) and the second one with respect to \( y \) one can get the identity
\[
\text{div} \left( \frac{1}{p_n} \nabla q_n + 2 (p_{n-1} + p_{n+1}) q_n \right) = 0
\]
from which, using again (3.1), one can derive the following one:

\[
\text{div} \left( \frac{1}{p_n} \nabla q_n \right) + \left[ 4 - \frac{1}{p_n} |\nabla q_n|^2 \right] q_n = 0
\] (3.3)

Thus, we have obtained that for all \( n \)'s the quantities \( q = q_n \) solve the PDE

\[
\Delta q - \frac{\kappa \bar{q}}{p} (\nabla q, \nabla q) + 4pq = 0
\] (3.4)

where

\[
p = 1 + \kappa |q|^2, \quad \Delta = \partial_{xx} + \partial_{yy}
\] (3.5)

which is a nonlinear analog of the Helmholtz equation (1.2). It turns out that this equation is not merely some 'nonlinearization' of the Helmholtz equation. It has a rather interesting physical origin. This is the field equation of some \( \sigma \)-model which has been proposed and discussed in \( \Box \) and which should be called, using the terminology by Pohlmeyer \( \Box \), the O(3,1) \( \sigma \)-model. This model arises from the problem

\[
\Delta \phi^\mu + \lambda \phi^\mu = 0
\] (3.6)

under the restriction

\[
\phi_\mu \phi^\mu = -1
\] (3.7)

Here \( \Delta \) is the two-dimensional Laplacian, \( \phi^\mu \) is a space-like vector from the Minkowski space,

\[
\phi^\mu = (\phi^0, \phi^1, \phi^2, \phi^3)
\] (3.8)

with the scalar product

\[
\phi^\mu \psi^\nu = \phi^0 \psi^0 - \phi^1 \psi^1 - \phi^2 \psi^2 - \phi^3 \psi^3
\] (3.9)

To satisfy the condition (3.7) the Lagrange multiplier \( \lambda \) is to be set to \( \lambda = -(\nabla \phi_\mu, \nabla \phi^\mu) \). The vectors \( \phi^\mu, \partial_x \phi^\mu, \partial_y \phi^\mu \) together with the time-like unit vector \( \chi^\mu \) which is normal to the surface \( \phi^\mu(x,y) \), form a local basis in the Minkowski space which satisfies the Gauss-Weingarten system,

\[
\partial_x \mathcal{F} = U \mathcal{F}, \quad \partial_y \mathcal{F} = V \mathcal{F}, \quad \mathcal{F} = (\phi^\mu, \partial_x \phi^\mu, \partial_y \phi^\mu, \chi^\mu)^T
\] (3.10)

where \( U \) and \( V \) are some \( 4 \times 4 \)-matrices (not written here). The integrability conditions for the system (3.10), the so-called Gauss-Codazzi equations, can be presented as follows (see \( \Box \) for details):

\[
\Delta \alpha - 4 \sin \alpha \cos \alpha + \frac{\cos \alpha}{\sin^3 \alpha} (\nabla \beta, \nabla \beta) = 0
\] (3.11)

and

\[
\text{div} (\cot^2 \alpha \nabla \beta) = 0
\] (3.12)

(note that the particular case of the above equations, \( \beta = 0 \), is the well-known elliptic sine-Gordon equation). These equations are the Euler-Lagrange equations for the action

\[
S = \iint dx dy \mathcal{L}
\] (3.13)

with the Lagrangian

\[
\mathcal{L} = (\nabla \alpha, \nabla \alpha) + \cot^2 \alpha (\nabla \beta, \nabla \beta) - 4 \cos^2 \alpha
\] (3.14)

which can be rewritten in terms of the function

\[
q = \cos \alpha \cdot \exp(i\beta)
\] (3.15)

as
\[ \mathcal{L} = \frac{(\nabla q, \nabla \bar{q})}{1 - |q|^2} - 4|q|^2 \]  

(3.16)

This Lagrangian can also be obtained as the reduction of the Euclidean version of the principal chiral field model, as is outlined in the Appendix of [3].

The field equation corresponding to the Lagrangian (3.16),

\[ \Delta q + \frac{\bar{q}}{1 - |q|^2} (\nabla q, \nabla q) + 4 \left( 1 - |q|^2 \right) q = 0 \]  

(3.17)

is nothing other than equation (3.4) for \( \kappa = -1 \).

The field equation (3.17) is integrable and it is possible to develop the inverse scattering scheme applicable to this equation (it has been done in the paper [13]). However, it turns out that the derived relationship between this model and the ALH provides almost all results that are usually obtained in the framework of the ISM. It can be used to demonstrate that the \( O(3, 1) \) \( \sigma \)-model possesses an infinite number of symmetries and conserved quantities and to obtain soliton and some other solutions for the field equation.

3.2 The Ablowitz-Ladik hierarchy and the 2DTL.

In this section I want to present some results that have no nontrivial linear analogues. Let us consider again the DNLSE-DMKdV system, now written in terms of the complex variables \( z_1 \) and \( \bar{z}_1 \), i.e. the system (2.15), (2.20):

\[ \begin{align*}
\partial q_n &= -ip_n q_{n+1} \\
\partial \bar{q}_n &= -ip_n q_{n-1}
\end{align*} \]  

(3.18)

(here \( \partial \) stands for \( \partial / \partial z_1 \) and \( \bar{\partial} \) for \( \partial / \partial \bar{z}_1 \)) and turn our attention to the quantity \( p_n \). One can derive from (3.18) the corresponding equations for \( p_n \)

\[ \begin{align*}
\partial p_n &= i\kappa p_n (\bar{q}_n q_{n-1} - \bar{q}_n q_{n+1}) \\
\bar{\partial} p_n &= i\kappa p_n (q_n \bar{q}_{n+1} - q_n \bar{q}_{n-1})
\end{align*} \]  

(3.19)

and obtain from this system, together with (3.18), that the quantities \( p_n \) satisfy the following equation:

\[ \partial \bar{\partial} \ln p_n = p_{n-1} - 2p_n + p_{n+1} \]  

(3.20)

This equation is the famous 2DTL equation which can be rewritten in terms of the functions \( u_n \) defined by \( p_n = \exp (u_n - u_{n+1}) \) and the real variables \( x \) and \( y \) (\( z_1 = x + iy \)) as follows:

\[ \frac{1}{4} \Delta u_n = \exp (u_{n-1} - u_n) - \exp (u_n - u_{n+1}) \]  

(3.21)

Thus, we have shown that the 2D Toda lattice turns out to be hidden in the simplest equations of the ALH or, in other terms, the 2DTL can be 'embedded' into the ALH.

The main, so to say, 'practical' value of the ALH–2DTL correspondence is in the fact that we can obtain a wide number of results for the 2DTL, which is 2+1 dimensional system, by means of the traditional version of the inverse scattering transform which has been developed for the 1+1 dimensional systems such as DNLSE, DMKdV, without invoking the MIST (that has been elaborated for the 2DTL by Lipovsky V.D. and Shirokov A.V. in [14]). All known solutions for the ALH (read for the DNLSE or DMKdV) can be converted to solutions for the 2DTL.

Another interesting problem is the question of the conservation laws [13]. One can now obtain an infinite number of the conservation laws in terms of the ALH and then 'convert' them to the 2DTL case (this idea has been described in [3]). Some new (comparing with the paper [13]) conservation laws have been obtained in [6] using the following simple procedure. Take one of them

\[ \text{div} \ J = 0 \]  

(3.22)

and apply operator \( \partial / \partial t_j \) where \( t_j \) is one of the hierarchy's times. This will yield an infinite number of conservation laws

\[ \text{div} \ J_j = 0 \]  

(3.23)
An interesting manifestation of the ALH-2DTL relation is the fact that the so-called conserved quantities 'of discrete variable direction' which are of the form

$$\partial I = 0$$  \hspace{1cm} (3.24)

derived by Kajiwara and Satsuma [15] when considered in the framework of the approach of [3] are nothing other than constants of motion of the equations of the ALH, i.e. Hamiltonians of the ALH-flows.

3.3 The Ablowitz-Ladik hierarchy and the DS equation.

Now let us discuss the bigger subsystems of the ALH, and start with the nonlinearization of the second toy example. Consider the system (2.15) - (2.22)

$$\partial_1 q_n = -ip_n q_{n+1} \hspace{1cm} (3.25)$$
$$\bar{\partial}_1 q_n = -iq_{n-1} p_n \hspace{1cm} (3.26)$$
$$\partial_2 q_n = -ip_n p_{n+1} q_{n+2} + ir_{n-1} p_n q_n q_{n+1} + iq_n r_n q_{n+1}^2 \hspace{1cm} (3.27)$$
$$\bar{\partial}_2 q_n = -iq_{n-2} p_{n-1} p_n + iq_{n-1} p_n r_{n+1} + iq_{n-1} p_n r_n \hspace{1cm} (3.28)$$

Here I do not write the corresponding equations for $r_n$'s, which can be restored from (3.25) - (3.28) using the involution $r_n = -\kappa q_n.$

Differentiating (3.25) with respect to $z_1$ one can straightforwardly obtain, using (3.27), that

$$i\partial_2 q_n + \bar{\partial}_1^2 q_n = -2C_n q_n \hspace{1cm} (3.29)$$

where $C_n = r_{n-1} p_n q_{n+1}.$ On the other hand, it follows from (3.26) that

$$\bar{\partial}_1 C_n = ip_n (r_n q_{n+1} - r_{n-1} q_n) = \partial_1 p_n \hspace{1cm} (3.30)$$

which leads to

$$\Delta C_n = 4\partial_1^2 p_n \hspace{1cm} (3.31)$$

where $\Delta = 4\partial_1 \bar{\partial}_1.$ Performing analogous computations starting from the equations (3.26) and (3.28) one can obtain similar relations for $r_n$ and $\mathcal{C}_n = q_{n-1} p_n r_{n+1}.$ Using the real variables $x, y$ and $t$

$$x = \text{Re} \ z_1, \quad y = \text{Im} \ z_1, \quad t = \text{Re} \ z_2 \hspace{1cm} (3.32)$$

and introducing the real quantities $A_n$ given by $A_n = \text{Re} \ C_n,$

$$A_n = \frac{1}{2} p_n (r_{n-1} q_{n+1} + q_{n-1} r_{n+1}) \hspace{1cm} (3.33)$$

one can show that $q_n$ and $A_n$ satisfy the following system

$$i\partial_t q_n + \frac{1}{2} \Box q_n + 4A_n q_n = 0 \hspace{1cm} (3.34)$$
$$\Delta A_n = \kappa \Box |q_n|^2 \hspace{1cm} (3.35)$$

where $\Box = \partial_{xx} - \partial_{yy}.$ This system is nothing other than the DS system [16] or, to be more precise, the DS-II equation, according to the generally accepted classification.

In such a way we have obtained the following result: solutions of the equations (3.27) – (3.28), belonging to the ALH, can be used to obtain solutions for the DS equation, i.e. the DS equation can be 'embedded' into the ALH. Moreover, when solving (3.27) – (3.28), we obtain simultaneously an infinite (for an infinite ALH chain) number of solutions for the DS: the pairs $(q_n, A_n)$ for all $n$'s solve the DS system. Thus, equations (3.27) – (3.28) can be viewed as sequences of the Backlund transformations. And again, as in the case of the 2DTL, we can obtain a wide number of solutions for this (2+1)-dimensional system by, so to say, little efforts: without invoking the MIST. But here we come to an important for this approach question. Each
satisfies the KP equation: the KP equation can be 'embedded' into the ALH. Thus, it follows from equations (2.15) – (2.18) that the quantities that we obtain by 'embedding' into the ALH are only some subclass of all possible solutions of the DS. But it turns out that this subclass is rather rich: it contains solitons, 'Wronskian' solutions, finite-gap quasiperiodic ones (these solutions are discussed in [3] and many other. It seems to be interesting that the pioneering papers by Ablowitz and Ladik \[9, 10\], which have been written twenty years ago, possess everything necessary to construct, say, the N-soliton solutions for the DS equation.

3.4 The Ablowitz-Ladik hierarchy and the KP equation.

The result I want present in this section is, in some sense, the most important example of 'embedding' into the ALH approach. To derive it one has to consider the system of three first equations of the 'positive' subhierarchy, that is the system consisting of the equations (2.15), (2.17) and the equation determining the third flow, ∂/∂z3:

\[
\begin{align*}
\partial_1 q_n &= -ip_n q_{n+1} + 2i q_n r_{n+1} q_{n+2} + ir_{n-1} q_n q_{n+1} + ip_n r_n q_{n+1} + i q_n r_{n+1} q_{n+2} + 2i p_n r_n q_{n+1} q_{n+2} + 2i r_{n-1} q_n q_{n+1} + & 2i p_n r_n q_{n+1} q_{n+2} + 2i r_{n-1} q_n q_{n+1} + \text{terms involving } q_n \text{'s the quantity } q_n \text{ of the O(3,1) } \sigma \text{-model. So, } q_n \text{ satisfies the KP equation:}
\end{align*}
\]

\[
\partial_1 (4\partial_3 u + \partial_{111} u + 12u \partial_1 u) = 3\partial_{22} u
\]

By very lengthy but straightforward calculations one can verify that for all n's the quantity

\[
u = -\kappa \bar{q}_{n-1} p_n q_{n+1}
\]
satisfies the KP equation:

Thus, the KP equation can be 'embedded' into the ALH.

3.5 The Ablowitz-Ladik hierarchy and the AKNS hierarchy.

It follows from equations (2.15) – (2.18) that the quantities q_n and r_{n-1} satisfy the closed system:

\[
\begin{align*}
i\partial_2 q_n + \partial_1^2 q_n + 2i q_n r_{n-1} \partial_1 q_n &= 0 \\
i\partial_2 r_{n-1} - \partial_1^2 r_{n-1} + 2i q_n r_{n-1} \partial_1 r_{n-1} &= 0
\end{align*}
\]

This is one of the forms of the derivative NLSE. To rewrite it in the standard way one can use the gauge transform which leads to the following result: the quantities

\[
Q = q_n \exp(-i\phi), \quad R = r_{n-1} \exp(i\phi)
\]

where

\[
\phi = \int_{z_1}^{z_2} dz q_n(z, z_2) r_{n-1}(z, z_2)
\]
satisfy the system

\[
\begin{align*}
i\partial_2 Q + \partial_1^2 Q + 2i \partial_1 Q^2 R &= 0 \\
i\partial_2 R - \partial_1^2 R + 2i \partial_1 Q R^2 &= 0
\end{align*}
\]

which is the complexified derivative NLSE (I use the term 'complexified' to indicate that Q and R are not related by the involution Q = ±R which is typical for physical applications). Thus, the derivative NLSE can be 'embedded' into the ALH.
Having obtained this result, one can expect that similar relations exist between the ALH and AKNS hierarchy because, using the paraphrase of Kipling’s words given by Newell, “The AKNS thing and the DNLS string are sisters under the skin.” However this cannot be done using only local in $q_n$’s and $r_n$’s functions, but is possible in terms of the $\tau$-functions of the ALH,

$$p_n = \frac{\tau_{n-1} \tau_{n+1}}{\tau_n^2}$$  \hspace{1cm} (3.47)

(see (5.4) below). It follows from (2.13) and (2.18) that the quantities $Q$ and $R$ given by

$$Q = \frac{\tau_{n+1}}{\tau_n} q_{n+1}, \quad Q = \frac{\tau_{n-1}}{\tau_n} r_{n-1}$$  \hspace{1cm} (3.48)

solve the complexified NLSE

$$i \partial_2 Q + \partial_{11} Q + 2Q^2 R = 0$$  \hspace{1cm} (3.49)

$$-i \partial_2 R + \partial_{11} R + 2QR^2 = 0$$  \hspace{1cm} (3.50)

Moreover, considering also the third flow one can derive that $Q$ and $R$ solve the first of the higher DNLSE’s as well:

$$\partial_3 Q + \partial_{111} Q + 6QR \partial_1 Q = 0$$  \hspace{1cm} (3.51)

$$\partial_3 R + \partial_{111} R + 6QR \partial_1 R = 0$$  \hspace{1cm} (3.52)

This is a demonstration of the fact that the AKNS hierarchy can be ’embedded’ into the ALH.

4 ’Embedding’ into the ’vectorized’ ALH.

The relations between the O(3,1) $\sigma$-model, 2DTL, DS, KP equations and the ALH, discussed in the previous sections were formulated in terms of $q_n$’s and $r_n$’s: solutions of the equations of the ALH yield solutions to other problems. Now I want to present some results on the ’embedding’ not into the ALH itself but into a set of related equations, which are equations for some combinations of the solutions of the linear problems (2.4), (2.5) associated with the ALH. Such relation is called in literature ’gauge equivalence’: NLSE is gauge equivalent to the Landau-Lifshitz equation, sine-Gordon — to the Thirring model, DS equation — to the Ishimori spin model.

Consider the matrices $\sigma_a^a, a = 1, 2, 3$ defined by

$$\sigma_a^a = \Psi_n^{-1} \sigma^a \Psi_n$$  \hspace{1cm} (4.1)

where $\sigma^a (a = 1, 2, 3)$ are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$  \hspace{1cm} (4.2)

and $\Psi_n$ is a $2 \times 2$ invertible matrix solution of the system (2.4), (2.5)

$$\Psi_{n+1} = U_n \Psi_n$$  \hspace{1cm} (4.3)

$$\partial_j \Psi_n = V^j_n \Psi_n$$  \hspace{1cm} (4.4)

It follows from (4.1) and (4.4) that

$$\partial_j \sigma_a^a = \Psi_n^{-1} \left[ \sigma^a, V^j_n \right] \Psi_n$$  \hspace{1cm} (4.5)

The right-hand sides of these equations can be written in terms of the matrices $\sigma_a^a$ for different $a$’s and $n$’s. Moreover, we can obtain the closed system of equations for the matrices $\sigma_a^a$ with the index $a$ being fixed (in what follows we will deal with the matrices $\sigma_3^a$) by means of the ’scattering’ problem (4.3) which can be used to relate $\sigma_a^a$’s and $\sigma_3^{a \pm 1}$
\[ p_n \sigma^3_{n+1} = (1 + q_n r_n) \sigma^3_n + 2 \lambda^{-1} r_n \sigma^+_n - 2 \lambda q_n \sigma^-_n \quad (4.6) \]
\[ p_{n-1} \sigma^3_{n-1} = (1 + q_{n-1} r_{n-1}) \sigma^3_n - 2 \lambda r_{n-1} \sigma^+_n + 2 \lambda^{-1} q_{n-1} \sigma^-_n \quad (4.7) \]

where the matrices \( \sigma^\pm_n \) are given by
\[ \sigma^\pm_n = \frac{1}{2} (\sigma^1_n \pm i \sigma^2_n) \quad (4.8) \]

These identities also enable to express some combinations of the "ALH quantities" \( q_n, r_n \) and \( p_n \) in terms of the "matrix" ones. For example, it follows from (4.6) that
\[ p_n = \frac{2}{1 + \frac{1}{2} \text{tr} \sigma_{n+1} \sigma_n} \quad (4.9) \]

For our further purposes it is convenient to use the matrix-vector correspondence
\[ S = \sum_{a=1}^3 S_a \sigma^a \rightarrow \vec{S} = (S_1, S_2, S_3) \quad (4.10) \]

where \( \sigma^a \) are the Pauli matrices (4.2).

The equations (4.5) together with (4.6), (4.7) can be viewed then as a system of DDE’s for the matrices \( \sigma^3_n \) or the vectors \( \vec{\sigma}_n \) which correspond to \( \sigma^3_n \)'s by (4.10),
\[ \sigma^3_n \rightarrow \vec{\sigma}_n \quad (4.11) \]

Namely these equations will play the role of the starting system of DDE’s, and from them I will derive some interesting difference and differential consequences. Our first example is the 'gauge' analog of the O(3,1) \( \sigma \)-model.

### 4.1 The Ablowitz-Ladik hierarchy and the Landau-Lifshitz equation.

Using the expressions for \( V^\pm_n \),
\[ V^1_n = -i \begin{pmatrix} 0 & \lambda^{-1} q_n & - \lambda^{-2} r_{n-1} \\ \lambda^{-1} q_n & \lambda^{-2} - r_{n-1} q_n & 0 \\ - \lambda^{-2} r_{n-1} q_n & 0 & \lambda r_n \end{pmatrix}, \quad V^{-1}_n = i \begin{pmatrix} \lambda^2 - q_{n-1} r_n & \lambda r_n & 0 \\ \lambda r_n & \lambda^{-2} - q_n r_n & 0 \\ 0 & 0 & \lambda^{-1} q_n \end{pmatrix} \quad (4.12) \]

one can express the derivatives of the matrices \( \sigma^3_n \) in terms of the matrices \( \sigma^\pm_n \) given by (4.8) as follows:
\[ (i \lambda / 2) \partial \sigma^3_n = r_{n-1} \sigma^+_n - q_n \sigma^-_n, \quad (4.13) \]
\[ (i / 2 \lambda) \bar{\partial} \sigma^3_n = -r_n \sigma^+_n + q_{n-1} \sigma^-_n. \quad (4.14) \]

(the symbols \( \partial \) and \( \bar{\partial} \) stand, remind, for \( \partial_1 \) and \( \partial_5 \)). From these relations, using analogous expressions for the derivatives \( \partial \sigma^\pm_n \), \( \bar{\partial} \sigma^\pm_n \) and formulae (2.15), (2.20), one can obtain, after straightforward calculations, omitted here, that for every \( n \) the matrix \( S = \sigma^3_n \) solves the equation
\[ \partial \bar{\partial} S + \frac{1}{2} \left( \text{tr} \; \partial S \; \bar{\partial} S \right) S + \frac{1}{2} \left[ \lambda^2 \partial S + \lambda^{-2} \bar{\partial} S, \; S \right] = 0 \quad (4.15) \]

In the vector form equation (4.15) becomes
\[ \vec{S} \times \left( \vec{S} \cdot \vec{S} \right) \vec{S} + \left[ \left( \lambda^2 \vec{S} + \lambda^{-2} \bar{\vec{S}} \right) \times \vec{S} \right] = 0 \quad (4.16) \]

which turns out to be the (0+2)-dimensional version of the Landau-Lifshitz equation for the isotropic two-dimensional classical Heisenberg ferromagnets,
\[ \partial_t \vec{S} = g \left[ \vec{S} \times \Delta \vec{S} \right], \quad \vec{S}^2 = 1 \quad (4.17) \]
where $\Delta$ is the two-dimensional Laplacian. Indeed, if we restrict ourselves with the stationary structures of the form $\vec{S} = \vec{S}(x - v_xt, y - v_yt)$, then equation (4.17) becomes

$$g \left[ \vec{S} \times \Delta \vec{S} \right] + \left( \vec{\sigma}, \nabla \vec{S} \right) = 0.$$  \hfill (4.18)

Introducing the variables

$$z = \frac{v}{4g} \left[ x - v_x t + i(y - v_y t) \right], \quad \bar{z} = \frac{v}{4g} \left[ x - v_x t - i(y - v_y t) \right]$$  \hfill (4.19)

where $v = |\vec{v}|$, the latter can be rewritten as

$$\left[ \vec{S} \times \vec{S} \right] + \lambda^2 \vec{S} + \lambda^{-2} \vec{S} = 0$$  \hfill (4.20)

with $\lambda = \exp(i\gamma/2)$, where the angle $\gamma$ is defined by $v_x = v \cos \gamma$, $v_y = v \sin \gamma$ which is equivalent to (4.16).

Thus, the $(0+2)$-dimensional Landau-Lifshitz equation can be 'embedded' into the ALH. The equation (4.16) is known to be integrable (its zero-curvature representation one can find in the paper [18]), and one can tackle it by elaborating the corresponding inverse scattering transform, but, as in the cases discussed above, we can now obtain a wide range of physically interesting solutions using the already known solutions for the ALH. This has been done in the paper [19].

4.2 The Ablowitz-Ladik hierarchy and a multi-$\sigma$-field model.

In this section we will fix the quantity $\lambda$,

$$\lambda = 1$$  \hfill (4.21)

noting that the case of the arbitrary $\lambda$'s ($|\lambda| = 1$) can be restored by the rotation of the coordinates $x$ and $y$ ($z \equiv z_1 = x + iy$).

Differentiating (4.13) with respect to $\bar{z}$ (or (4.14) with respect to $z$, which leads to the same result) one can obtain

$$\partial \bar{\partial} \sigma_n = 2 (2 - p_{n-1} - p_n) \sigma_n + 2 (r_n - r_{n-1}) \sigma_n^+ + 2 (q_{n-1} - q_n) \sigma_n^-$$  \hfill (4.22)

Using the identities

$$2 (r_n - r_{n-1}) \sigma_n^+ + 2 (q_{n-1} - q_n) \sigma_n^- = p_n \sigma_{n+1} + (p_{n-1} + p_n - 4) \sigma_n + p_{n-1} \sigma_{n-1}$$  \hfill (4.23)

which follow from (4.6), (4.7) and (4.9) one can rewrite (4.22) as

$$\partial \bar{\partial} \sigma_n = p_{n-1} \sigma_{n-1} - (p_{n-1} + p_n) \sigma_n + p_n \sigma_{n+1}$$  \hfill (4.24)

or, using again (4.13) - (4.14), as

$$\partial \bar{\partial} \sigma_n^3 + \frac{1}{2} (\text{tr} \partial \sigma_n^3 \bar{\partial} \sigma_n^3) \sigma_n = p_{n-1} \sigma_{n-1} - (p_{n-1} + p_n) \sigma_n + p_n \sigma_{n+1}$$  \hfill (4.25)

The last equation turns out to be the field equation of some model which to author’s knowledge is new and which can be viewed as a multi-field generalization of the well-known O(3) $\sigma$-model.

The O(3) $\sigma$ model is described by the Lagrangian

$$\mathcal{L} = \mathcal{T} + \mathcal{L}'$$  \hfill (4.26)

where $\mathcal{T}$ is the 'kinetic-energy' term in two dimensions

$$\mathcal{T} = \frac{1}{2} \left( \nabla \vec{S}, \nabla \vec{S} \right), \quad \nabla = (\partial_x, \partial_y)$$  \hfill (4.27)

and $\mathcal{L}'$ takes into account the restriction

$$\vec{S}^2 = 1$$  \hfill (4.28)

Our generalization is, first, in taking instead of one field $\vec{S}$ an infinite number of fields.
\[ \vec{S}_n, \quad n = 0, \pm 1, ..., \quad \vec{S}_n^2 = 1 \]  
(4.29)
i.e., in replacing \( T \) (4.27) with
\[ T = \frac{1}{2} \sum_n \left( \nabla \vec{S}_n, \nabla \vec{S}_n \right) \]  
(4.30)
It is easily understood that in absence of interactions between different fields (spins) our generalization would be trivial and of little interest both from physical and mathematical standpoints. So, we consider also the interaction, the nearest-neighbour one,
\[ \mathcal{U} = \sum_n u \left( \vec{S}_n, \vec{S}_{n+1} \right) \]  
(4.31)
In what follows our attention will be restricted to the particular pair potential \( u \) which has no, so to say, physical origin, but is remarkable by the fact that it preserves integrability of the corresponding equations. Namely, we will study the potential of the classical Heisenberg ferromagnets model (which is known to be integrable, see, e.g., [20, 21] and to be in close relations with the ALH [22]):
\[ \mathcal{U} = -J \sum_n \ln \left[ 1 + \left( \vec{S}_n, \vec{S}_{n+1} \right) \right] \]  
(4.32)
Summarizing, the model considered is given by
\[ \mathcal{L} = \frac{1}{2} \sum_n \left( \nabla \vec{S}_n, \nabla \vec{S}_n \right) + J \sum_n \ln \left[ 1 + \left( \vec{S}_n, \vec{S}_{n+1} \right) \right] + \mathcal{L}' \]  
(4.33)
where
\[ \mathcal{L}' = \sum_n \Lambda_n \left( \vec{S}_n^2 - 1 \right) \]  
(4.34)
and \( \Lambda_n \)'s are the Lagrange multipliers, which should be chosen to satisfy the conditions \( \vec{S}_n^2 = 1 \).
The field equations corresponding to (4.33) are of the form
\[ \Delta \vec{S}_n = \vec{F}_n + \Lambda_n \vec{S}_n \]  
(4.35)
where
\[ \vec{F}_n = -\frac{\partial}{\partial \vec{S}_n} \mathcal{U} = \frac{1}{2} J \left( f_{n-1} \vec{S}_{n-1} + f_n \vec{S}_n \right) \]  
(4.36)
with
\[ f_n = \frac{2}{1 + \left( \vec{S}_n, \vec{S}_{n+1} \right)} \]  
(4.37)
and after calculating \( \Lambda_n \)'s they become
\[ \Delta \vec{S}_n + \left( \nabla \vec{S}_n, \nabla \vec{S}_n \right) \vec{S}_n = \frac{1}{2} J \left\{ f_{n-1} \vec{S}_{n-1} + (f_{n-1} + f_n - 4) \vec{S}_n + f_n \vec{S}_{n+1} \right\} \]  
(4.38)
These equations after identifying \( f_n \) and \( p_n \) (compare (4.37) and (4.39)) and rescaling the coordinates become the vector form of the matrix equations (4.29). Thus, the multi-\( \sigma \)-field model described above, which is a (2+1)-dimensional system gauge equivalent to the 2DTL, can be 'embedded' into the ALH.
4.3 The Ablowitz-Ladik hierarchy and the Ishimori equation.

The following example is a manifestation of the already known fact that the Ishimori model is gauge equivalent to the DS equation [23].

Using the expressions for \(V_n^{(\pm)}\), one can express the derivatives of the matrices \(\sigma_3^n\) as follows:

\[
(i\lambda/2) \frac{\partial_1}{\partial t} \sigma_3^n = r_{n-1}^2 - q_n \sigma_3^n, \tag{4.39}
\]

\[
(i/2\lambda) \frac{\partial_1}{\partial t} \sigma_3^n = -r_n \sigma_3^n + q_{n-1} \sigma_3^n, \tag{4.40}
\]

\[
(i\lambda/2) \frac{\partial_2}{\partial t} \sigma_3^n = \left(\frac{\lambda^2}{2} q_{n-1} + q_{n-1}^2 r_n - q_{n-1}^2 \right) \sigma_3^n + \left(\lambda q_{n-1} - q_{n-1}^2 - 2q_{n-1}^2 r_n \right) \sigma_3^n, \tag{4.41}
\]

\[
(i/2\lambda) \frac{\partial_2}{\partial t} \sigma_3^n = -\left(\frac{\lambda^2}{2} q_{n-1} + q_{n-1}^2 r_n - q_{n-1}^2 \right) \sigma_3^n + \left(\lambda q_{n-1} - q_{n-1}^2 - 2q_{n-1}^2 r_n \right) \sigma_3^n. \tag{4.42}
\]

From these relations, using analogous expressions for the derivatives \(\partial_j \sigma_3^\pm\), one can obtain that for every \(n\) the matrix \(S = \sigma_3^n\) (the index \(n\) will be fixed and omitted hereafter) with \(|\lambda| = 1\) satisfies the following equations:

\[
-2i\partial_2 S + [S, \partial_2 S] = 4iw\partial_1 S, \tag{4.43}
\]

\[
-2i\partial_2 S + [S, \partial_2 S] = 4i\bar{w}\bar{\partial}_1 S, \tag{4.44}
\]

where

\[
w \equiv r_{n-1} q_n - \lambda^2, \quad \bar{w} \equiv q_{n-1} r_n - \lambda^2. \tag{4.45}
\]

From the other hand, it follows from (2.15)-(2.16) that

\[
i\bar{\partial}_1 w = -i\partial_1 \bar{w} = p_n - p_{n-1}. \tag{4.46}
\]

The r.h.s. of (4.46) can be expressed in terms of the matrix \(S\)

\[
p_n - p_{n-1} = -\frac{1}{8} \text{tr} \{ S \left[ \partial_1 S, \bar{\partial}_1 S \right] \}. \tag{4.47}
\]

Finally, using the real variables \(x, y, t\) (3.32) and the quantity \(f\) defined by

\[
w = -i\partial f \tag{4.48}
\]

which satisfy the equation \(\Delta f = 4(p_n - p_{n-1})\), equations (4.43), (4.44) and (4.46) can be presented as

\[
iS_t = \frac{1}{4} \{ S, \square S \} + i(f_y S_x + f_x S_y), \tag{4.49}
\]

\[
\Delta f = \frac{1}{4t} \text{tr} \{ S [S_x, S_y] \}. \tag{4.50}
\]

This system, when rewritten in terms of the vector \(\vec{S}\) related to the matrix \(S\) by (4.10) becomes the Ishimori equation [24]

\[
\vec{S}_t = \frac{1}{2} \{ \vec{S} \times \square \vec{S} \} + f_y \vec{S}_x + f_x \vec{S}_y, \tag{4.51}
\]

\[
\Delta f = \left( \vec{S} \cdot \{ \vec{S}_x \times \vec{S}_y \} \right). \tag{4.52}
\]

This is the main result of this section: the Ishimori equation can be 'embedded' into the ALH.
5 Functional representation of the Ablowitz-Ladik hierarchy.

In the above sections one can find many examples of the relations between the ALH and other integrable models. These examples show that the ALH possesses some kind of ‘universality’. A hypothesis arises that all known integrable models can be 'embedded' into the ALH. So, the main question one should answer now is the following: is the ALH in some sense a distinguished hierarchy? It seems to me that there is some sense in these words. I cannot give any rigorous proof of such a statement (and even formulate it in strict terms). Nevertheless, I think this question is worth studying. The results discussed above are in some sense 'empirical' facts: one can easily verify them by simple calculations, but one can hardly find there an answer to the question why do such apparently different models turn out to be interconnected. To do this one probably has to consider the problem, and hence the ALH, in some more general terms. Namely this was the motivation of the work [8].

The main result of the paper [8] is the representation of the ALH, which has been originally introduced as an infinite system of differential-difference equations, as a finite system of difference-functional equations. I will not repeat here the calculations outlined there and only write down the main result. The ‘positive’ Ablowitz-Ladik subhierarchy is shown to be equivalent to following equations:

\[
\sigma_n \left( z + \frac{i}{2} [\delta] \right) \tau_n \left( z - \frac{i}{2} [\delta] \right) - \tau_n \left( z + \frac{i}{2} [\delta] \right) \sigma_n \left( z - \frac{i}{2} [\delta] \right) = (5.1)
\]

\[
= \delta \sigma_{n+1} \left( z + \frac{i}{2} [\delta] \right) \tau_{n-1} \left( z - \frac{i}{2} [\delta] \right)
\]

\[
\tau_n \left( z + \frac{i}{2} [\delta] \right) \rho_n \left( z - \frac{i}{2} [\delta] \right) - \rho_n \left( z + \frac{i}{2} [\delta] \right) \tau_n \left( z - \frac{i}{2} [\delta] \right) = (5.2)
\]

\[
= \delta \tau_{n+1} \left( z + \frac{i}{2} [\delta] \right) \rho_{n-1} \left( z - \frac{i}{2} [\delta] \right)
\]

\[
\tau_n \left( z + \frac{i}{2} [\delta] \right) \tau_n \left( z - \frac{i}{2} [\delta] \right) - \sigma_n \left( z + \frac{i}{2} [\delta] \right) \rho_n \left( z - \frac{i}{2} [\delta] \right) = (5.3)
\]

\[
= \tau_{n+1} \left( z + \frac{i}{2} [\delta] \right) \tau_{n-1} \left( z - \frac{i}{2} [\delta] \right)
\]

Here \( z \) stands for \((z_1, z_2, ..., z_k, ...)\), \([\delta]\) is the traditional now designation for \((\delta, \delta^2/2, ..., \delta^k/k, ...)\), and \( \tau_n, \sigma_n \) and \( \rho_n \) are \( \tau \)-functions defined by

\[
ge_n = \frac{\sigma_n}{\tau_n}, \quad r_n = \frac{\rho_n}{\tau_n}, \quad p_n = \frac{\tau_{n-1} \tau_{n+1}}{\tau_n^2}
\]

Expanding \((5.1)-(5.3)\) in the power series in \( \delta \) one can obtain all equations of the 'positive' subhierarchy. As is seen, in the above formulae only the 'positive' coordinates \( z_j \) are used. An analogous representation, involving \( \bar{z}_j \)'s, can be obtained for the 'negative' subhierarchy.

Using Hirota’s bilinear approach one can rewrite \((5.1)-(5.3)\) as

\[
\exp \left[ \frac{i}{2} D(\delta) \right] \begin{pmatrix} \sigma_n \cdot \tau_n - \tau_n \cdot \sigma_n - \delta_{n+1} \cdot \tau_{n-1} \\ \rho_n \cdot \tau_n - \tau_n \cdot \rho_n + \delta \tau_{n+1} \cdot \rho_{n-1} \\ \sigma_n \cdot \rho_n - \tau_n \cdot \sigma_n + \delta \tau_{n+1} \cdot \rho_{n-1} \end{pmatrix} = 0 (5.5)
\]

where

\[
D(\delta) = \sum_{k=1}^{\infty} \frac{\delta^k}{k} D_k, \quad D_j = D_{z_j} (5.6)
\]

and the operators \( D_{z_j} \) are defined by

\[
D_j^a ... D_j^b u \cdot v = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x} \right)^a ... \left( \frac{\partial}{\partial y} - \frac{\partial}{\partial y} \right)^b u(x, y, ...) v(x', y', ...) \bigg|_{x'=x, y'=y,...} (5.7)
\]

Thus, we have a system of difference-functional equations for an infinite number of triplets of \( \tau \)-functions. It has been shown in [8] that from this system one can derive a closed system for only one triplet \( (\sigma, \rho, \tau) : \)
\[
\hat{G}(\delta) \begin{pmatrix} \sigma \cdot \tau \\ \tau \cdot \rho \\ \sigma \cdot \rho + \tau \cdot \tau \end{pmatrix} = 0 \quad (5.8)
\]

where \(\sigma, \rho\) and \(\tau\) stand for \(\sigma_n, \rho_n\) and \(\tau_n\) with \(n\) being fixed and the operator \(\hat{G}(\delta)\) is defined by
\[
\hat{G}(\delta) = 2 \sin \left[ \frac{1}{2} D(\delta) \right] - \delta D_1 \exp \left[ \frac{i}{2} D(\delta) \right] \quad (5.9)
\]

These equations, when expanded in power series in \(\delta\) yield a hierarchy of partial differential equations of the NLSE type.

This procedure of 'shortening' can be continued by excluding \(\sigma\) and \(\rho\) and obtaining the 'scalar' equations for only one function, \(\tau\),
\[
\hat{H}(\delta) \tau \cdot \tau = 0 \quad (5.10)
\]

where the operator \(\hat{H}(\delta)\) is given by
\[
\hat{H}(\delta) = [2D_1 - \delta (D_2 + iD_{11})] \exp \left[ \frac{i}{2} D(\delta) \right] \quad (5.11)
\]

Expanding the operator \(\hat{H}(\delta)\) in powers of \(\delta\) one can obtain an infinite number of operators. The first of them is
\[
4D_{31} - 3D_{22} + D_{1111} \quad (5.12)
\]
i.e. the operator which appears in the bilinear representation of the KP equation.

Thus, we have a chain: the infinite system \((5.1)-(5.3)\) \(\rightarrow\) the system of three equations \((5.8)\) \(\rightarrow\) one equation \((5.10)\). This chain leads from the ALH, through the NLSE-like hierarchies, to the KP-like one and may be useful to understand the place of the ALH among other integrable hierarchies.

6 Conclusion.

The results presented in this paper are aimed to demonstrate that the ALH indeed possesses some kind of 'universality'. One more indication of, in some sense, distinguished character of the ALH, which has not been mentioned above and which will be discussed in a separate paper, comes from the theory of the \(\theta\)-functions. It is well known that the \(\theta\)-functions of the finite genus Riemann surfaces satisfy some functional relations, I mean the Fay’s trisecant identity \([25]\). There is also a well-elaborated procedure how to derive from the latter a number of differential relations satisfied by the \(\theta\)-functions, and one can easily find in literature a lot of identities involving some combinations of some differential operators \(\partial_j\) (usually ones that appear in the KdV, KP, 2DTL and other equations). It is surprising, but to author’s knowledge the following, rather natural, question has not been given due attention. Consider the problem of how to express the action of the operators \(\partial_j\) taken separately on the \(\theta\)-functions (or some ratios of the \(\theta\)-functions, \(\theta(\zeta + \alpha)/\theta(\zeta + \beta)\)) without invoking other operators \(\partial_k, (k \neq j)\), but only in terms of \(\theta\)-functions, maybe of some other arguments. This problem is not very difficult: it can be done using the already known (and widely used) algorithm, by expanding the Fay’s identity. It turns out that these relations can be written in the form of the equations of the ALH. This fact becomes more transparent if we consider this question in the framework of the functional representation of the ALH: equations \((5.1) - (5.3)\) in the quasiperiodical case become equivalent to the Fay’s identity.

However the word 'universality' often used in this paper should not be understood literally (that is why I used it in the quotation marks) because of the following. The relations between the ALH and other equations are not, in general, ones of equivalence. The correspondence between the ALH subsystem \(\(5.3)\) and the field equations of the O(3,1) \(\sigma\)-model \(\(3.4)\) is one-to-one: one can find in \(\[3\] the inverse transform (from the field equations to the DNLSE-DMKdV chain). At the same time in most of the other examples the situation is different. Not all solutions of, say, (2+1)-dimensional models can be obtained by the 'embedding' into the ALH method. And a logical continuation of the works \(\[3, 4, 5, 6, 7, 8\) is to study the question of what kind of reduction do we have when 'split' some system (2DTL, DS, KP) into few equations from the ALH. In the case of, e.g., the DS equation this is easy to do: we can start from the fact that the quantities \(q_n\) satisfy not
only the (2+1)-dimensional DS system (3.34), (3.35) but also the 2-dimensional O(3,1) σ-model equations, and to formulate the ansatz we have implicitly used. However in other cases, the 2DTL for example, this problem is somewhat more difficult: I cannot at present write down some, so to say, more simple equations which are satisfied by the quantities \( p_n \), i.e. I cannot formulate, using the language of equations, what restrictions do we impose on solutions of the 2DTL when substituting the latter by the ALH equations (3.4). I think some progress can be made in the framework of a more general approach, e.g., as one discussed in [8]. Maybe it will be easier to understand the ALH-(other equations) reductions studying the chain from the ALH to the KP equation mentioned at the end of the previous section. To conclude, I want to stress once more that this question is not answered yet, while it seems to be one of the most important problems to be solved before (if ever) to call the ALH universal (without the quotation marks).
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