On the Exact Operator Formalism of Two-Dimensional Liouville Quantum Gravity in Minkowski Spacetime

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Abstract

A detailed reexamination is made of the exact operator formalism of two-dimensional Liouville quantum gravity in Minkowski spacetime with the cosmological term fully taken into account. Making use of the canonical mapping from the interacting Liouville field into a free field, we focus on the problem of how the Liouville exponential operator should be properly defined. In particular, the condition of mutual locality among the exponential operators is carefully analyzed, and a new solution, which is neither smoothly connected nor relatively local to the existing solution, is found. Our analysis indicates that, in Minkowski spacetime, coupling gravity to matter with central charge $d < 1$ is problematical. For $d = 1$, our new solution appears to be the appropriate one; for this value of $d$, we demonstrate that the operator equation of motion is satisfied to all orders in the cosmological constant with a certain regularization. As an application of the formalism, an attempt is made to study how the basic generators of the ground ring get modified due to the inclusion of the cosmological term. Our investigation, although incomplete, suggests that in terms of the canonically mapped free field the ground ring is not modified.

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1 Introduction

Beginning with the discovery [1] that the matrix model techniques together with the idea of the double scaling limit can be used to define and solve models of two-dimensional Euclidean gravity (or random surfaces), our understanding of the subject has been advanced considerably. (For review see for example [2] and references cited therein.) Despite these successes, however, there still remain a number of important issues yet to be clarified. Notable among them is the problem of space-time interpretation. Due to its formulation without the explicit appearance of the metric, the matrix model offers only a modest insight into how the gravitational and the matter degrees of freedom interact.

One possible way to clarify this problem is to study how one can make contact with the continuum formulation, for example in the conformal gauge, where the metric degree of freedom appears explicitly as the Liouville field. Here the problem is how to treat the notoriously tricky dynamics generated by the cosmological term $\mu^2 e^\phi$.

Recently there have been attempts to understand Liouville dynamics via path-integral formalism with certain amount of success [3]–[9]. In this method one first performs the integration over the Liouville zero mode and observes that if the total “momentum”, which is in general fractional, were a positive integer the remaining integral becomes effectively that for a free field with vertex operator insertions (although with a formally divergent factor). Upon performing this free-field integral one “analytically continues” back to the original fractional momentum. The method appears to be non-rigorous (see however [9] for justification) but it reproduces the matrix model results [10] for up to 3 point functions on a sphere and a torus.

Although it has captured an important aspect of the Liouville theory, the formalism above which integrates out the Liouville zero mode in the beginning appears to be awkward for unravelling the deep symmetry structure of the theory. For this purpose, the operator formalism would appear to be more suitable. Indeed, the existence of discrete physical states [11]–[17] and the associated $W_\infty$ algebra [18]–[23] first found in the matrix model approach have been analyzed in the operator formalism and such intriguing structures as the “ground ring” [24]–[26] have been discovered. These developments have clearly demonstrated the utility of the operator formalism but there is a caveat: in these works the Liouville field is treated as a free field and hence the role of the Liouville dynamics remains unclear. Also it can be suspected that the naive perturbative treatment of the exponential interaction may not be justified. In view of this situation, it is an appropriate time to focus on the exact operator formalism for the Liouville theory and examine how it can be applied to the clarification of the aforementioned problems.
Exact operator quantization for the Liouville theory has been developed over the past ten years [27]–[35] (for a review see [37].) In [35], applications of the operator formalism to two-dimensional quantum gravity coupled to matter are discussed from a modern point of view, where special emphasis is placed on the quantum group structure of Liouville theory, and various results previously obtained by matrix model techniques are rederived (see also [36] for a recent review). Despite the fact that many intriguing results have been obtained in these works, we feel that some subtle issues have been left unsettled (or overlooked). A major problem, and one that has still not been satisfactorily resolved in our opinion, concerns the rigorous construction of exponential Liouville operators of arbitrary weight having the right conformal properties and satisfying locality. The purpose of this article is to try to point out these and other subtleties as clearly as possible and propose solutions to some of these problems. Because of the technicalities involved we shall not explain the nature of the subtleties in detail here, but nevertheless let us mention one novel result of our investigation which may have serious consequences. As in the traditional treatment, we shall perform a canonical operator quantization in Minkowski space by a non-linear and non-local canonical transformation which maps the interacting Liouville field into a free field. In this approach, the requirement of mutual locality among exponential operators plays a crucial role. The locality condition was recently reanalyzed by Otto and Weigt in [32]. They proposed a new definition of the exponential operator that is obtained from the usual exponential series by a kind of quantum deformation, and checked the locality of these operators in a perturbation expansion in terms of the cosmological constant $\mu^2$ up to third order. We have carefully reexamined this locality equation and found an additional solution (valid to all orders in $\mu^2$) which is not smoothly connected to the solution of [32]. Moreover, our analysis casts some doubt (at least in Minkowski space) on the validity of the conventional treatment of the theory coupled with $d \leq 1$ matter. If we postulate that the operator $e^\varphi$, which appears in the Liouville equation of motion, should belong to the set of admissible (i.e. mutually local) operators, we find that (i) $d < 1$ is not allowed and (ii) for $d = 1$ our solution, not the one constructed in [32], is relevant. Since the notion of locality (or micro-causality) does not enter in the Euclidean treatment, our result is not necessarily in direct conflict with the matrix model, but it certainly calls for further investigation.

With the formalism developed for the $d = 1$ case, we then made an attempt to study how the structure of the ground ring gets modified when the exponential interaction is turned on. The results we have obtained so far are unfortunately difficult to interpret.\footnote{J. Schnittger has informed us that he has now verified the locality condition for the deformed quantum Liouville operator proposed in [32] to much higher orders.}
To be sure, it is straightforward to write down candidates for the basic generators of the ground ring in terms of the interacting fields with the expected conformal transformation properties. However, the BRST transformation properties are not completely dictated by the conformal properties alone, and these candidates, unlike their free field counterparts, turn out not to be BRST invariants. This seems to indicate that we have exactly the same ground ring generators as for vanishing cosmological constant. Nonetheless, there is a large difference with the the $\mu^2 = 0$ case. The free field in our case is not the Liouville field itself, but the canonically mapped field which is difficult to interpret physically. To put it differently, a simple generator in terms of the free field is actually a very complicated one in terms of the Liouville field. It is not entirely clear to us how our results are related to those of [38],[39], where the ground ring with non-vanishing cosmological constant is also discussed. In [39], the basic idea is to determine the deformations of the ground ring by “fusion”, i.e. by evaluating operator products of the free field ground ring operators in a perturbation expansion in $\mu^2$; in this way one arrives at the conclusion that there appear extra operators besides the ones already present in the free field case. Since we have not computed products of (interacting) ground ring operators, we cannot ascertain the compatibility of the two approaches at this point. We note the fact that the expansion in $\mu^2$ of the exact ground ring operators, on which our analysis is based, breaks off after finitely many terms precisely for the special weights which appear in the associated Liouville exponentials. It would be interesting to see whether this observation is related to the fact that only finitely many terms remain in the perturbative expansion of [39] due to charge screening. Obviously, more work is needed to clarify these issues.

Despite a long history, pedagogical expositions of the operator approach to Liouville theory seem to be scarce. For this reason, we have decided to organize our paper in such a way that it can (hopefully) also serve as a relatively self-contained review. We start in section 2 with a review of the classical properties of the Liouville theory. In particular, we provide a rather detailed exposition of the non-linear and non-local canonical transformation which maps the interacting Liouville field into a free field. With the use of this canonical transformation, we proceed to the operator quantization of the theory in section 3. The central issue will be the question of the proper definition of exponential operators and their locality properties. We will also demonstrate that the operator Liouville equation of motion is satisfied to all orders in $\mu^2$ for certain set of values for the matter central charge including $d = 1$. In section 4 we shall describe an attempt to construct the basic generators of the ground ring including the cosmological term. Finally in section 5 we will give discussions on the issues which are left unsolved in this work. An appendix is provided to discuss the conversion between Minkowski and Euclidean formulations.
2 Classical Properties of Liouville Theory

2.1 General Remarks

In this section we describe some classical properties of Liouville theory [27]–[34]; as for our conventions and notation we will mostly follow [32]. Let us briefly recall how the Liouville action arises in string theory, and why it plays an important role there. Any conformal field theory can be formulated as a two-dimensional field theory of matter fields coupled to a gravitational background, which is described by a metric tensor $g_{\mu\nu}$. In two dimensions, $g_{\mu\nu}$ carries no propagating but only topological degrees of freedom. More specifically, it can always be brought to the form

$$g_{\mu\nu} = e^\varphi \hat{g}_{\mu\nu}$$

by means of two-dimensional diffeomorphisms. Here, the background metric $\hat{g}_{\mu\nu}$ locally can be chosen equal to the flat Minkowski metric (i.e. in each coordinate patch, we can set $\hat{g}_{\mu\nu} = \eta_{\mu\nu}$). Globally, it will depend on the moduli characterizing the conformal equivalence class of the world sheet; we will, however, ignore these topological degrees of freedom in most of the remainder. The factor multiplying the background metric $\hat{g}_{\mu\nu}$ will be referred to as the “conformal factor” in the sequel, and $\varphi$ is the Liouville field. In a conformally invariant theory, the classical Lagrangian is not only invariant under two-dimensional diffeomorphisms, but in addition invariant under Weyl rescalings of the metric. This invariance allows us to gauge the conformal factor to unity, and the Liouville field therefore decouples from the matter sector. However, as is well known [40], this decoupling in general cannot be maintained at the quantum level, because integration over the matter fields induces the quantum action

$$S = \frac{1}{\gamma^2} \int d\tau d\sigma \sqrt{\hat{g}} \left( \frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \mu^2 e^\varphi - Q \varphi \hat{R} \right)$$

or, with $\hat{g}_{\mu\nu} = \eta_{\mu\nu}$,

$$S = \frac{1}{\gamma^2} \int d\tau d\sigma \left( \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \mu^2 e^\varphi \right) ,$$

where $\mu^2$ is the cosmological constant, $Q$ the background charge. Furthermore, we have

$$\frac{1}{\gamma^2} \equiv \frac{26 - d}{48\pi} .$$

$d$ is the number of scalar matter fields, or, more generally, the total central charge of the matter sector (which need not even be described by a Lagrangian). Consequently,

\footnotetext{For definiteness, we consider only bosonic theories.}
at the quantum level, the Liouville field decouples only for \( d = 26 \), which is the critical dimension for a bosonic target space. For \( d < 26 \), on the other hand, the Liouville field \( \varphi \) must be taken into account as an extra degree of freedom, whose dynamics is governed by the non-polynomial action (2.1.2) \[ \square \]. In order to make sense out of such a “subcritical” theory, one is thus forced to study the Liouville action in the context of two-dimensional quantum field theory. The hope, then, is that through its non-linear dynamics, the Liouville field adjusts its contribution to the conformal anomaly in such a way that conformal invariance is restored in the full theory, and thereby a consistent theory can be formulated also outside the critical dimension. We remark that, in many approaches, the (renormalized) cosmological constant is set to zero, so the complicated dynamics of the Liouville field is masked by an effective free theory. However, we will not adopt this point of view and assume \( \mu^2 \neq 0 \) in this paper.

One of the central problems of Liouville theory is the construction of quantum operators corresponding to exponentials of the Liouville field, i.e. \( e^{\lambda \varphi} \) for arbitrary weights \( \lambda \). We will discuss this problem in much detail in section 3.2, but let us emphasize already at this point that these operators are indispensable for the construction of physical vertex operators. The main reason is that ordinary string vertex operators built out of matter fields only will usually fail to have the correct conformal dimension, and must be “dressed up” by suitable powers of the worldsheet metric and its determinant before they can be consistently integrated over the world sheet. Since the conformal dimension of \( e^{\lambda \varphi} \) depends upon \( d \), the conformal dimensions of the matter field vertex operators will also have to be modified away from the critical dimension.

Before moving on to describe the classical aspects of Liouville theory, we note another important point. Due to quantum effects, we will have to replace the coupling constant multiplying the action (2.1.2) by the renormalized value \[ 27 \]
\[
\left( \frac{1}{\gamma^2} \right)_{\text{ren}} \equiv \frac{25 - d}{48\pi} = \frac{1}{16\pi \tilde{h}}.
\] (2.1.5)

The shift from 26 to 25 is a quantum effect and will be explained below. In the above equation, we have also introduced “Planck’s constant” \( \tilde{h} \equiv \frac{\hbar}{25 - d} \). This terminology has the advantage that we can think of the limit \( d \to -\infty \) as some kind of semi-classical limit in which the theory simplifies considerably.

\[ ^3 \text{The case } d > 26 \text{ is not of interest as there are always negative norm states in the physical Hilbert space.} \]
2.2 General Solution

The classical equation of motion following from the action (2.1.3) reads

\[ 4 \partial_+ \partial_- \varphi + \mu^2 e^\varphi = 0, \]  

(2.2.1)

where, for later convenience, we have introduced the light-cone coordinates

\[ \zeta^\pm \equiv \tau \pm \sigma \quad \partial^\pm \equiv \frac{1}{2}(\partial_\tau \pm \partial_\sigma) \Rightarrow \partial^2_\tau - \partial^2_\sigma = 4\partial_+ \partial_- . \]  

(2.2.2)

(2.2.1) is a non-linear partial differential equation, whose general solution has been known for a long time (and was, in fact, known to Liouville [42]). It is most conveniently expressed in terms of left and right moving waves, i.e. two arbitrary functions \( A = A(\xi^+) \) and \( B = B(\xi^-) \):

\[ \varphi(\tau, \sigma) = \log \left( \frac{8}{\mu^2} \frac{\partial_+ A(\xi^+) \partial_- B(\xi^-)}{1 + A(\xi^+) B(\xi^-)^2} \right) . \]  

(2.2.3)

Observe that this representation is not unique and differs from the one used in earlier papers [27]. The above parametrization was chosen by [31] [32], and will turn out to be the most convenient for our subsequent considerations. The fact that the above equation can be solved exactly may be regarded as a consequence of the integrability of Liouville theory, which follows from the existence of a Lax pair [41]. Periodicity in the space coordinate requires the boundary condition \( \varphi(\tau, \sigma + 2\pi) = \varphi(\tau, \sigma) \). This does not imply that the “constituent functions” \( A, B \) are themselves periodic; rather, the periodicity constraint on the Liouville field is compatible with \( SL(2, \mathbb{R}) \) (i.e. Möbius) transformations on \( A \) and \( B \), viz.

\[ A(\xi^+ + 2\pi) = \frac{\alpha A(\xi^+) + \beta}{\gamma A(\xi^+) + \delta}, \quad B(\xi^- - 2\pi) = \frac{\delta B(\xi^-) - \gamma}{-\beta B(\xi^-) + \alpha}, \]  

(2.2.4)

where \( \alpha \delta - \beta \gamma = 1 \). Note that \( B \) transforms “contragrediently” to \( A \); this is the reason why some authors prefer to work with the field \(-B(\xi^-)^{-1}\) instead, which transforms like \( A \). In choosing boundary conditions below, we will not make use of the full Möbius group, but rather put \( \beta = \gamma = 0 \). The solution (2.2.3) can be equivalently expressed as

\[ \varphi(\tau, \sigma) = \log \partial_+ A(\xi^+) + \log \left( \frac{8}{\mu^2} \partial_- B(\xi^-) \right) - 2 \log (1 + Y(\tau, \sigma)) , \]  

(2.2.5)

where

\[ Y(\tau, \sigma) \equiv A(\xi^+) B(\xi^-) \]  

(2.2.6)

is the only part of \( \varphi \) mixing left and right movers.
Having exhibited the explicit solution in terms of left and right movers, the next task is to construct a free field $\psi$ out of them. In view of the different parametrizations alluded to before, the identification of the free field is not unambiguous. We will here follow [31] and [32] and define $\psi$ by

$$\psi(\tau, \sigma) = \psi^+(\xi^+) + \psi^-(\xi^-) \equiv \log \partial_+ A(\xi^+) + \log \left( \frac{8}{\mu^2} \partial_- B(\xi^-) \right)$$

(2.2.7)

so that

$$\varphi = \psi - 2 \log(1 + Y).$$

(2.2.8)

These equations will serve as our basis for all subsequent considerations, in particular those concerning the quantum theory.

Integration of the differential equations $\partial_+ A = e^{\psi^+}$ and $\partial_- B = (\mu^2/8)e^{\psi^-}$ is straightforward. Assuming the simplified boundary conditions $A(\xi^+ + 2\pi) = \alpha A(\xi^+)$ and $B(\xi^- - 2\pi) = \alpha^{-1} B(\xi^-)$ we obtain [32]

$$A(\xi^+) = c(\alpha) \int_0^{2\pi} d\xi' E_\alpha(\xi^+ - \xi') e^{\psi^+(\xi')}$$

(2.2.9)

$$B(\xi^-) = \frac{\mu^2}{8} c(\alpha) \int_0^{2\pi} d\xi'' E_\alpha(\xi^- - \xi'') e^{\psi^-(\xi'')}$$

(2.2.10)

where

$$c(\alpha) \equiv \left( \sqrt{\alpha} - \frac{1}{\sqrt{\alpha}} \right)^{-1},$$

(2.2.11)

and

$$E_\alpha(\xi) \equiv \exp \left( \frac{1}{2} \log \alpha \epsilon(\xi) \right) = E_{\alpha}^{\frac{1}{2}}(-\xi),$$

(2.2.12)

with $\epsilon(\xi)$ the usual staircase function, i.e. $\epsilon(\xi) = 2n + 1$ for $2n\pi < \xi < 2(n + 1)\pi$. When written out, (2.2.9) reads

$$A(\xi^+) = c(\alpha) \left( \int_0^{\xi^+} d\xi' e^{\frac{1}{2} \log \alpha} e^{\psi^+(\xi')} + \int_{\xi^+}^{2\pi} d\xi' e^{-\frac{1}{2} \log \alpha} e^{\psi^+(\xi')} \right).$$

(2.2.13)

The function $E_\alpha$ (which is actually a distribution) obeys

$$\frac{\partial}{\partial \xi} E_\alpha(\xi) = \frac{1}{c(\alpha)} \delta(\xi),$$

(2.2.14)

where $\delta(\xi)$ is the 2$\pi$-periodic delta-function. Note that we cannot choose the functions $A, B$ to be simply periodic because $c(\alpha)$ diverges at $\alpha = 1$. Solving for $Y(\tau, \sigma)$, we obtain

$$Y(\tau, \sigma) = \frac{\mu^2}{8} c(\alpha)^2 \int_0^{2\pi} d\sigma' \int_0^{2\pi} d\sigma'' E_\alpha(\sigma - \sigma') E_{\alpha}^{\frac{1}{2}}(\sigma - \sigma'') e^{\psi^+(\tau + \sigma')} e^{\psi^-(\tau - \sigma'')}.$$  

(2.2.15)
It is now easy to check that $Y$ is a periodic function of $\sigma$, although $A$ and $B$ are not periodic separately (note that with the more general boundary condition (2.2.4) $Y$ would not have been periodic). Since, for the canonical formalism we will be mostly concerned with equal time expressions, we have explicitly displayed the dependence on the time and space variables in (2.2.15); this will permit us to put $\tau = 0$ wherever appropriate. Although we will not make use of this fact, we note that $Y(\tau, \sigma)$ can also be represented directly in terms of the free field $\psi$ and its time derivative $\dot{\psi}$ at time $\tau$. The resulting expression is local in $\tau$, but non-local in the space variable $\sigma$.

Given the result for the Liouville field as a non-local function of the free field $\psi$, we can now define exponentials with arbitrary weights $\lambda$ through the expansion

$$e^{\lambda \psi} = e^{\lambda \psi} (1 + Y)^{-2\lambda} = e^{\lambda \psi} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(2\lambda + n)}{n! \Gamma(2\lambda)} Y^n. \quad (2.2.16)$$

(We have displayed the expression appropriate for positive $\lambda$) Observe that this is actually an expansion in powers of the cosmological constant, as $Y$ is proportional to $\mu^2$ by (2.2.15). Note also that when $-2\lambda$ is a non-negative integer the expansion contains only a finite number of terms.

In the remainder, we will also make use of the Fourier expansion of the free field $\psi$ in terms of oscillators. For the left and right moving fields $\psi^\pm$ this expansion takes the form

$$\psi^\pm(\xi^\pm) = \frac{1}{2} \gamma Q + \frac{\gamma}{4\pi} P \xi^\pm + \frac{i\gamma}{\sqrt{4\pi}} \sum_{n \neq 0} \frac{1}{n} a_n^{(\pm)} e^{-i n \xi^\pm}, \quad (2.2.17)$$

where the extra factor of $\gamma$ has been inserted so as to obtain the canonically normalized Poisson bracket relations for $Q, P$ and the oscillators $a_n^{(\pm)}$. Comparison with (2.2.5) leads to the identification

$$\log \alpha = \frac{1}{2} \gamma P. \quad (2.2.18)$$

We have already remarked above that we cannot put $\alpha = 1$ in the expressions for the interacting Liouville field. Now we can understand this result in a more physical way: $\alpha \neq 1$ is equivalent to $P \neq 0$, i.e. Liouville theory does not possess a translationally invariant groundstate; of course, this problem becomes apparent only for $\mu^2 \neq 0$. There is another, and related, difficulty here. Any shift of the Liouville field by a constant can be absorbed into the cosmological constant, which is therefore an essentially undetermined parameter. This, in turn, means that there is no way in which (2.2.16) can be thought of as an expansion in a small parameter. In the absence of a better method, we will proceed in spite of this difficulty.
2.3 Canonical Transformation

The Poisson brackets are determined from the action (2.1.3). The equal time brackets are given by

\[ \{ \varphi(\tau, \sigma) , \varphi(\tau, \sigma') \} = \{ \dot{\varphi}(\tau, \sigma) , \dot{\varphi}(\tau, \sigma') \} = 0 \quad \text{for} \quad \sigma \neq \sigma' , \quad (2.3.1) \]

\[ \{ \varphi(\tau, \sigma) , \frac{1}{\gamma^2} \dot{\varphi}(\tau, \sigma') \} = \delta(\sigma - \sigma') , \quad (2.3.2) \]

where the dot denotes the derivative with respect to \( \tau \). The factor in front of \( \dot{\varphi} \) is just the coupling constant multiplying the action (2.1.3).

It is quite remarkable that the non-local transformation connecting the Liouville field \( \varphi \) and the free field \( \psi \) is canonical \([27],[28]\). In the present formulation, this means that the above brackets are equivalent to

\[ \{ \psi(\tau, \sigma) , \psi(\tau, \sigma') \} = \{ \dot{\psi}(\tau, \sigma) , \dot{\psi}(\tau, \sigma') \} = 0 \quad \text{for} \quad \sigma \neq \sigma' , \quad (2.3.3) \]

\[ \{ \psi(\tau, \sigma) , \frac{1}{\gamma^2} \dot{\psi}(\tau, \sigma') \} = \delta(\sigma - \sigma') . \quad (2.3.4) \]

The proof of this assertion requires a rather tedious computation. Since our setup, and in particular, our definition of the free field in (2.2.7), is different from the one in \([27],[28]\), we present the details of this proof here. We have found that the simplest method is to proceed “backwards” by showing that the free field Poisson brackets imply the ones in terms of \( \varphi \) above. We now give some useful intermediate relations. Since all of these are equal time commutators, we will omit the time coordinate in the formulas listed below; however, we alert the reader that in all relations involving time derivatives, we must keep \( \tau \) as a variable at the intermediate stages of the calculation and can put \( \tau = 0 \) only at the very end. Also one must not forget that \( \alpha \) contains the zero mode \( P \) and hence has non-trivial bracket with \( \psi \), which contains \( Q \). A prime \( ' \) denotes the derivative with respect to the spatial coordinate \( \sigma \).

Let us first list the brackets between the free field \( \psi \) and the quantities \( A, B \); to make the formulas less cumbersome, we put \( \gamma^2 = 0 \) in eqs. (2.3.5–21). So we obtain

\[ \{ \psi(\sigma_1) , A(\sigma_2) \} = \frac{\sigma_1}{4\pi} A(\sigma_2) - \frac{c(\alpha)}{2} A(\sigma_1) E_\frac{1}{\pi} (\sigma_1 - \sigma_2) \]

\[ -\frac{1}{4} A(\sigma_2) \epsilon(\sigma_1 - \sigma_2) , \quad (2.3.5) \]

\[ \{ \psi(\sigma_1) , A'(\sigma_2) \} = \frac{\sigma_1}{4\pi} A'(\sigma_2) - \frac{1}{4} A'(\sigma_2) \epsilon(\sigma_1 - \sigma_2) , \quad (2.3.6) \]
\[ \{ \dot{\psi}(\sigma_1), A(\sigma_2) \} = -\frac{c(\alpha)}{2} A'(\sigma_1) E_{\alpha}^{\perp} (\sigma_1 - \sigma_2), \] (2.3.7)

\[ \{ \dot{\psi}(\sigma_1), A'(\sigma_2) \} = -\frac{1}{2} A'(\sigma_2) \delta(\sigma_1 - \sigma_2), \] (2.3.8)

\[ \{ \psi(\sigma_1), B(\sigma_2) \} = -\frac{\sigma_1}{4\pi} B(\sigma_2) - \frac{c(\alpha)}{2} E_{\alpha}^{\perp} (\sigma_2 - \sigma_1) B(\sigma_1) \]
\[ + \frac{1}{4} \epsilon(\sigma_1 - \sigma_2) B(\sigma_2), \] (2.3.9)

\[ \{ \psi(\sigma_1), B'(\sigma_2) \} = -\frac{\sigma_1}{4\pi} B'(\sigma_2) - \frac{1}{4} \epsilon(\sigma_1 - \sigma_2) B'(\sigma_2), \] (2.3.10)

\[ \{ \dot{\psi}(\sigma_1), B(\sigma_2) \} = +\frac{c(\alpha)}{2} E_{\alpha}^{\perp} (\sigma_2 - \sigma_1) B'(\sigma_1), \] (2.3.11)

\[ \{ \dot{\psi}(\sigma_1), B'(\sigma_2) \} = -\frac{1}{2} B'(\sigma_1) \delta(\sigma_1 - \sigma_2). \] (2.3.12)

From these brackets, we derive

\[ \{ \psi(\sigma_1), Y(\sigma_2) \} = -\frac{1}{2} c(\alpha) \left( A(\sigma_1) E_{\alpha}^{\perp} (\sigma_1 - \sigma_2) B(\sigma_2) + A(\sigma_2) E_{\alpha}^{\perp} (\sigma_2 - \sigma_1) B(\sigma_1) \right) \]
\[ = \{ \dot{\psi}(\sigma_2), Y(\sigma_1) \}, \] (2.3.13)

\[ \{ \psi(\sigma_1), \dot{Y}(\sigma_2) \} = -\frac{1}{2} c(\alpha) \left( A'(\sigma_1) E_{\alpha}^{\perp} (\sigma_1 - \sigma_2) B(\sigma_2) - A(\sigma_2) E_{\alpha}^{\perp} (\sigma_2 - \sigma_1) B'(\sigma_1) \right) \]
\[ = \{ \dot{\psi}(\sigma_2), \dot{Y}(\sigma_1) \}, \] (2.3.14)

\[ \{ \psi(\sigma_1), \ddot{Y}(\sigma_2) \} = \frac{1}{2} c(\alpha) \left( A(\sigma_1) E_{\alpha}^{\perp} (\sigma_1 - \sigma_2) B'(\sigma_2) - A'(\sigma_2) E_{\alpha}^{\perp} (\sigma_2 - \sigma_1) B(\sigma_1) \right) \]
\[ = \{ \dot{\psi}(\sigma_2), Y(\sigma_1) \}, \] (2.3.15)

\[ \{ \dot{\psi}(\sigma_1), \dddot{Y}(\sigma_2) \} = -\frac{1}{2} \delta(\sigma_1 - \sigma_2) \dddot{Y}(\sigma_2) \]
\[ + \frac{1}{2} c(\alpha) \left( A'(\sigma_1) E_{\alpha}^{\perp} (\sigma_1 - \sigma_2) B'(\sigma_2) + A'(\sigma_2) E_{\alpha}^{\perp} (\sigma_2 - \sigma_1) B'(\sigma_1) \right). \] (2.3.16)

The equal time relations involving commutators of \( Y \) and \( \dot{Y} \) read

\[ \{ Y(\sigma_1), Y(\sigma_2) \} = \frac{1}{2} (Y(\sigma_1) - Y(\sigma_2)) \{ \psi(\sigma_1), Y(\sigma_2) \} \]
\[ = \frac{1}{2} (Y(\sigma_1) - Y(\sigma_2)) \{ \dot{\psi}(\sigma_2), Y(\sigma_1) \}, \] (2.3.17)
\[
\{Y(\sigma_1), \dot{Y}(\sigma_2)\} = -\frac{1}{2}\{\psi(\sigma_1), Y(\sigma_2)\}\dot{Y}(\sigma_2)
+ \frac{1}{2}(Y(\sigma_1) - Y(\sigma_2))\{\psi(\sigma_1), \dot{Y}(\sigma_2)\},
\]
(2.3.18)

\[
\{\dot{Y}(\sigma_1), \dot{Y}(\sigma_2)\} = -\frac{1}{2}\{\psi(\sigma_2), \dot{Y}(\sigma_1)\}\dot{Y}(\sigma_2)
+ \frac{1}{2}(Y(\sigma_1) - Y(\sigma_2))\{\dot{\psi}(\sigma_1), \dot{Y}(\sigma_2)\}.
\]
(2.3.19)

These results can now be inserted to rewrite (2.3.1) and (2.3.2) in terms of free field commutators. For instance, we get

\[
\{\varphi(\sigma_1), \varphi(\sigma_2)\} = -\frac{2}{1 + Y(\sigma_2)}\{\psi(\sigma_1), Y(\sigma_2)\} + \frac{2}{1 + Y(\sigma_1)}\{\psi(\sigma_2), Y(\sigma_1)\}
+ \frac{4}{(1 + Y(\sigma_1))(1 + Y(\sigma_2))}\{Y(\sigma_1), Y(\sigma_2)\}.
\]
(2.3.20)

The last term can be rewritten by use of (2.3.17)

\[
\frac{Y(\sigma_1) - Y(\sigma_2)}{(1 + Y(\sigma_1))(1 + Y(\sigma_2))}\left(\{\psi(\sigma_1), Y(\sigma_2)\} + \{\psi(\sigma_2), Y(\sigma_1)\}\right)
= \frac{2}{1 + Y(\sigma_2)}\{\psi(\sigma_1), Y(\sigma_2)\} - \frac{2}{1 + Y(\sigma_1)}\{\psi(\sigma_2), Y(\sigma_1)\}.
\]
(2.3.21)

and thus precisely cancels the first two terms. The proof of the remaining Poisson brackets along these lines is now completely analogous.

By standard arguments it can be shown that the free field Poisson brackets are equivalent to the following brackets for the oscillators

\[
\{Q, P\} = 1, \quad i\{a_m^{(+), a_n^{(+)}}\} = i\{a_m^{(-), a_n^{(-)}}\} = m\delta_{m+n,0}.
\]
(2.3.22)

(The brackets that have not been listed vanish).

Although the proof of the canonical nature of the transformation is already complete, it is instructive to display explicitly how the Hamiltonian gets transformed. The Hamiltonian following from the action (2.1.3) is given by

\[
H = \int d\sigma \left(\frac{1}{2}\gamma^2 \Pi^2 + \frac{1}{2\gamma^2} \varphi'^2 + \frac{\mu^2}{\gamma^2} e^\varphi\right)
= \frac{1}{\gamma^2} \int d\sigma \left(\frac{1}{2} \dot{\varphi}^2 + \frac{1}{2} \varphi'^2 + \mu^2 e^\varphi\right),
\]
(2.3.23)
where $\Pi_\varphi = \frac{\dot{\varphi}}{\gamma}$ is the canonical momentum. We now substitute the expression (2.2.8) and make use of identities such as

\[
\partial_+ \psi \partial_+ Y = \partial_+^2 Y, \\
\partial_- \psi \partial_- Y = \partial_-^2 Y, \\
\partial_+ \partial_- Y = \mu^2 \frac{e^\psi}{8}, \\
\partial_+ Y \partial_- Y - Y \partial_+ \partial_- Y = 0,
\]

which easily follow from the definitions of $Y$ and $\psi$. After some calculations we obtain

\[
H = \frac{1}{\gamma^2} \int_0^{2\pi} d\sigma \left[ \frac{1}{2} \left( \dot{\psi}^2 + \psi'^2 \right) - 4\dot{\sigma}^2 \ln(1 + Y) \right] = \frac{1}{\gamma^2} \int_0^{2\pi} d\sigma \frac{1}{2} \left( \dot{\psi}^2 + \psi'^2 \right).
\]

This is nothing but a Hamiltonian for a free field $\psi$. So there are two ways to look at the Liouville dynamics: one can say that the Liouville field evolves in a complicated way because of the exponential interaction. Alternatively one may say that the Liouville dynamics is non-trivial because the Liouville field is a complicated combination of a free field which evolves trivially.

### 2.4 Energy-Momentum Tensor and Conformal Transformations

The energy momentum tensor is defined by

\[
T_{\mu\nu} = 4\pi \frac{\delta S}{\delta \hat{g}^{\mu\nu}}.
\]

It is not difficult to see that the trace of $T_{\mu\nu}$ is proportional to the Liouville equation of motion and therefore vanishes on-shell. It is in this sense that we can regard Liouville theory as a conformal field theory, although the basic Liouville field does not decompose into a sum of left and right movers. Switching to light-cone notation, we find for the remaining traceless components

\[
T_{\pm\pm} = 4\pi \frac{\delta S}{\delta \hat{g}^{\pm\pm}} = \frac{2\pi}{\gamma^2} \left( \partial_\pm \varphi \partial_\pm \varphi - 2\partial_\pm^2 \varphi \right).
\]

Note that we must keep the fiducial metric $\hat{g}_{\mu\nu}$ in order to get the improvement term, and can equate it with the flat Minkowski metric only at the very end of the calculation. From conservation of the energy momentum tensor (i.e. $\partial_- T_{++} + \partial_+ T_{--} = 0$) and its tracelessness (i.e. $T_{++} = 0$), we expect $T_{++}$ ($T_{--}$) to depend only on $\xi^+$ ($\xi^-$). This is borne out by an explicit computation, which yields

\[
T_{++} = \frac{2\pi}{\gamma^2} \left( \partial_+ \psi \partial_+ \psi - 2\partial_+^2 \psi \right).
\]
Therefore, despite the complicated dependence of the Liouville field on $\tau$ and $\sigma$, the $++$ and $--$ components of the energy momentum tensor take a rather simple form in terms of free fields, being equivalent to those of a free field theory, apart from the improvement term containing second derivatives. In other words, as far as its energy momentum tensor is concerned, Liouville theory behaves very much like a free field theory (with a background charge).

The Virasoro generators $L_m^{(\pm)}$, defined from the expansion

$$T_{\pm\pm}(\xi^{\pm}) = \sum_{m} L_m^{(\pm)} e^{-im\xi^{\pm}}$$

are given by (cf. (2.2.17))

$$L_m^{(\pm)} = \frac{1}{2} \sum_{n} a_n^{(\pm)} a_{m-n}^{(\pm)} + \frac{\sqrt{4\pi}}{\gamma} ima_m^{(\pm)}.$$  \hfill (2.4.5)

In the sequel, we will omit the superscript $(\pm)$ if there is no danger of confusion. By use of the canonical brackets given in the preceding section, one recovers the Virasoro algebra

$$i\{L_m, L_n\} = (m - n) L_{m+n} + \frac{4\pi}{\gamma^2} m^3 \delta_{m+n,0}. \hfill (2.4.6)$$

Consequently, there is a central term already at the classical level. Note the the anomaly term proportional to $m^3$ can be converted into the more usual form $m(m^2 - 1)$ through the replacement of $ma_m^{(+)}$ by $(m + 1)a_m^{(+)}$ in (2.4.5).

At this point, we can explain the shift from 26 to 25 mentioned in section 2.1. As is well known, in the quantum theory, the expression for the Virasoro generators must be normal ordered if it is to be remain well defined. This results in an extra contribution of $\frac{1}{12}$ to the central charge. Since we want the total central charge to be given by 26, we have to readjust its classical value in such a way that

$$\frac{c}{12} = \frac{1}{12} + \left(\frac{4\pi}{\gamma^2}\right)_{\text{ren}} = \frac{26 - d}{12},$$

which leads to the renormalized value given in (2.1.5). We will henceforth drop the subscript “ren”, and always assume the renormalized value for the coupling constant.

3 Quantization of Liouville Theory

3.1 Free Field Quantization

The main result of the foregoing chapter was the demonstration that the interacting Liouville field can be canonically reexpressed in terms of the free field $\psi$ defined in (2.2.7).
In quantizing Liouville theory, one tries to exploit this equivalence by performing the quantization in terms of this free field. In this subsection, we set up our conventions and notation and collect some basic (and well known) formulas for free fields.

First replace Poisson brackets by commutators in the usual fashion so that
\[
[Q, P] = i, \quad [a_m^{(+)}, a_n^{(+)}] = [a_m^{(-)}, a_n^{(-)}] = m\delta_{m+n,0} \tag{3.1.1}
\]
(again, commutators that have not been listed vanish). This can be inserted into (2.2.17) for the free field \(\psi\) to derive the quantum commutators
\[
[\psi(\tau, \sigma), \psi(\tau, \sigma')] = [\dot{\psi}(\tau, \sigma), \dot{\psi}(\tau, \sigma')] = 0 \quad \text{for} \quad \sigma \neq \sigma', \tag{3.1.2}
\]
\[
[\psi(\tau, \sigma), \frac{1}{\gamma^2} \dot{\psi}(\tau, \sigma')] = i\delta(\sigma - \sigma'). \tag{3.1.3}
\]
For the derivation of short distance expansions and the computation of vacuum expectation values it will be convenient to split the field \(\psi\) into creation and annihilation parts by defining
\[
\psi^+_a(\xi^\pm) \equiv \frac{i\gamma}{\sqrt{4\pi}} \sum_{n=1}^{\infty} \frac{1}{n} a_n^{(\pm)} e^{-in\xi^\pm}, \tag{3.1.4}
\]
and
\[
\psi^-_a(\xi^\pm) \equiv -\frac{i\gamma}{\sqrt{4\pi}} \sum_{n=1}^{\infty} \frac{1}{n} a_n^{(\pm)} e^{+in\xi^\pm}, \tag{3.1.5}
\]
where zero mode terms involving \(Q\) and \(P\) have been left out.

All relevant formulas can now be derived from the reordering relation (for the non-zero modes)
\[
e^{a\psi^+_a(\xi_1^+)} e^{b\psi^+_b(\xi_2^+)} = e^{b\psi^+_b(\xi_2^+)} e^{a\psi^+_a(\xi_1^+)} \cdot |1 - e^{i(\xi_1^+ - \xi_2^+)\frac{ab}{2\pi}}| \cdot e^{-i\pi\frac{a^2b}{2\pi}\epsilon(\xi_1^+ - \xi_2^+)} e^{i\pi\frac{a^2b^2}{2\pi}\epsilon(\xi_1^+ - \xi_2^+)} \tag{3.1.6}
\]
and the corresponding formula for exponentials of \(\psi^-(\xi^-)\), which reads exactly the same. The relation for the zero modes is
\[
e^{aQ} f(P) = f(P + ia)e^{aQ}. \tag{3.1.7}
\]
To establish these relations, we need the formula
\[
\sum_{n \geq 1} \frac{1}{n} e^{-in(\theta - i\epsilon)} = -\ln \left(1 - e^{-i(\theta - i\epsilon)}\right) = -\ln |1 - e^{-i\theta}| - i\frac{\pi}{2}\epsilon(\theta) + i\frac{\theta}{2}. \tag{3.1.8}
\]
which is obtained by carefully looking at the phase of the logarithm in the region where $\theta$ is small. In deriving the reordering relations above, one encounters infinite sums which fail to converge without regularization. The sums are regularized by adding a small imaginary part to the argument $\theta$; of course, $\epsilon$ must be taken to zero at the end of the calculation (this shift is very similar to the well known $i\epsilon$-prescription needed to make Feynman integrals well defined). The phase factors will be especially important in our analysis of the locality condition in section 3.3.

Normal ordering is now defined in the usual fashion by putting all annihilation operators to the right, except that in the zero mode sector we use the symmetric normal ordering given by

$$
:e^aQf(P) : \equiv e^{\frac{1}{2}aQ}f(P)e^{\frac{1}{2}aQ}.
$$

(3.1.9)

In accordance with these remarks, the fully normal ordered exponential is now given by

$$
: \exp (a\psi(\tau, \sigma)) : = \exp \left( \frac{1}{4} a \gamma Q \right) \exp \left( \frac{a \gamma}{4 \pi} P \tau \right) \exp \left( \frac{1}{2} a \gamma Q \right) \exp \left( a \psi^+(\tau + \sigma) \right)
\cdot \exp \left( a \psi^-(\tau - \sigma) \right) \exp \left( a \psi^-(-\tau + \sigma) \right).
$$

(3.1.10)

By means of the reordering relation (3.1.6), it is straightforward to prove that

$$
: e^{a\psi^+(\xi^+)} : \cdot e^{b\psi^+(\xi^+)} : = : e^{(a\psi^+(\xi^+)+b\psi^+(\xi^+))} : \\
\cdot \left| 1 - e^{i(\xi^+ \xi^2)} \right|^{-a b \sqrt{2} \pi} e^{-\frac{i \gamma}{8} a b (\xi^+ - \xi^2)} e^{ia b \sqrt{2} \pi (\xi^+ - \xi^2)}
$$

(3.1.11)

and

$$
: e^{a\psi^-(\xi^+)} : \cdot e^{b\psi^-(\xi^+)} : = : e^{(a\psi^-(\xi^+)+b\psi^-(\xi^+))} : \\
\cdot \left| 1 - e^{i(\xi^+ \xi^2)} \right|^{-a b \sqrt{2} \pi} e^{-\frac{i \gamma}{8} a b (\xi^+ - \xi^2)} e^{ia b \sqrt{2} \pi (\xi^+ - \xi^2)}
$$

(3.1.12)

so the product of two normal ordered exponentials equals the fully normal ordered product up to a “short-distance factor”. Multiplying these two expressions and noticing that, at equal times, $\xi^1 - \xi^2 = -(\xi^- - \xi^-) = \sigma_1 - \sigma_2$, we see that the phase factor cancels between the left and the right moving sectors. The final result is therefore symmetrical, so that for arbitrary weights $a$ and $b$, we deduce

$$
: e^{a\psi(\tau, \sigma_1)} : \cdot e^{b\psi(\tau, \sigma_2)} : = : e^{a\psi(\tau, \sigma_2)} : \cdot e^{b\psi(\tau, \sigma_1)} : \quad \text{for } \sigma_1 \neq \sigma_2.
$$

(3.1.13)
This shows that two free-field exponentials are indeed mutually local, although this property does not hold for the left- and right-moving sectors separately because of the extra phase factors.

It is also well known that the normal ordered free field exponentials behave properly under conformal transformations. With the normal ordered Virasoro generators

\[ L_m^{(\pm)} = \frac{1}{2} \sum_n : a_n^{(\pm)} a_{m-n}^{(\pm)} : + \frac{\sqrt{4\pi}}{\gamma} m a_m^{(\pm)}, \]  

(3.1.14)

one obtains

\[ [L_m, L_n] = (m - n) L_{m+n} + \frac{c}{12} m^3 \delta_{m+n,0}, \]  

(3.1.15)

where the central charge is now given by (2.4.7); as we explained there, the contribution from the renormalized coupling constant, which appears already in the classical algebra (2.4.6), is augmented by \( \frac{1}{12} \) from the normal ordering of the oscillators above. A standard calculation shows that

\[ [L_m^{\pm}, : e^{a\psi(\tau,\sigma)} : ] = e^{im\xi^{\pm}} \left( -i \partial_{\pm} + m(a - \frac{\gamma^2}{8\pi} a^2) \right) : e^{a\psi(\tau,\sigma)} :, \]  

(3.1.16)

The customary Euclidean formulation of conformal field theory makes use of operator product expansions. These are most conveniently derived by the use of Wick’s theorem. For convenience, we define the variables

\[ z^{\pm} = e^{i(\tau^{\pm}\sigma)} \]  

(3.1.17)

(or just \( z, w, \ldots \) if there is no danger of confusion). As is well known, these variables can be analytically continued to imaginary times by the replacement \( \tau \to -i\tau \) (in the appendix, we will provide further details of the transcription between Minkowski and Euclidean conventions). It is now elementary to show that

\[ \partial_{\pm} \psi(z) \partial_{\pm} \psi(w) =: \partial_{\pm} \psi(z) \partial_{\pm} \psi(w) : + \frac{\gamma^2}{4\pi} \frac{wz}{(z - w)^2}. \]  

(3.1.18)

Here, the \( i\epsilon \)-prescription introduced above is understood; in the Euclidean formulation, it becomes unnecessary as the singularity at coincident points is avoided by radial ordering. Now using Wick’s theorem to evaluate such products, we can derive the important operator product expansion

\[ T_{++}(z)T_{++}(w) = \frac{c}{2} \frac{z^2 w^2}{(z - w)^4} + \frac{2zw}{(z - w)^2} T_{++}(w) + \frac{zw}{(z - w)} \partial T_{++}(w), \]  

(3.1.19)

where \( \partial \) now denotes the partial derivative with respect \( w^{\pm} \) (and not \( \xi^{\pm} \)); as is customary in conformal field theory, we have dropped the non-singular terms on the right hand side. From the formulas given in the appendix, one can easily verify that the somewhat unusual factors of \( zw \) disappear upon conversion of these formulas into Euclidean language.
3.2 Quantum Definition of $\exp(\lambda \varphi)$

Our aim in this section is to define the quantized exponential of the (interacting) Liouville field $\varphi$, which will provide the necessary “gravitational dressing” of string vertex operators in a non-trivial gravitational background. For lack of a better notation, we will continue to designate the renormalized exponential operators by $: e^{\lambda \varphi} :$, although the semicolons now mean something different from and should not be confused with ordinary free field normal ordering. The obvious point of departure for this construction is the expansion (2.2.16) of the Liouville exponential in terms of the free field $\psi$. Since the quantization of free fields is well understood (see foregoing section), one hopes that the interacting theory can also be exactly quantized by exploiting this relation. However, matters are not so easy, and although considerable progress can be made, there remain some unresolved problems, most notably the question of whether the expansion in $\mu^2$ makes sense at all.

The main difficulty is the complicated non-local and non-linear character of the relation between the Liouville field and $\psi$. So, for instance, it will become apparent that free field normal ordering by itself is not sufficient to achieve consistency with the two main requirements, which we expect the exponential operators to satisfy. These are:

(i) The exponential operator $: e^{\lambda \varphi} :$ should transform properly under the conformal group, i.e.

$$\left[ L^\pm_m, : e^{\lambda \varphi(\tau,\sigma)} : \right] = e^{im\xi^\pm} (-i\partial_\pm + m \Delta^\pm(\lambda)) : e^{\lambda \varphi(\tau,\sigma)} :.$$  \hspace{1cm} (3.2.1)

Here $\Delta^\pm(\lambda)$ denotes the renormalized (“dressed”) conformal dimension of the exponential operator which, in general, will be different from the classical dimension as we already pointed out.

(ii) Two exponential operators at spacelike distances should commute, i.e. they should be mutually local:

$$\left[ : e^{\lambda \varphi(\tau,\sigma)} :, : e^{\nu \varphi(\tau,\sigma')} : \right] = 0 \quad \text{for} \quad \sigma \neq \sigma'.$$  \hspace{1cm} (3.2.2)

To evaluate condition (i), it is crucial that we use the free field energy momentum tensor to define $L^\pm_0$ since there is no way to make sense out of $T^\pm\pm(\varphi)$ before the Liouville field $\varphi$ itself has been defined as a quantum operator. It is an open question, whether $T^\pm\pm(\varphi)$ can be defined as a quantum operator and whether this operator coincides with the free field energy momentum tensor.

As we have shown in the preceding section, both requirements are met for normal ordered exponentials of free fields, but they are far from trivial to satisfy for the full
Liouville operator and will lead to stringent and unexpected constraints. For instance, the obvious idea of defining \( e^{\lambda \phi} \) by simply putting (free field) normal ordering symbols around the full sum which defines \( e^{\lambda \phi} \) in terms of the free field at the classical level, fails because it clashes with the first condition (as will become clear in a moment). The locality condition is not considered in the customary Euclidean formulation. This omission could be justified if one had an Osterwalder-Schrader type reconstruction theorem for Liouville theory, which is, however, not available at this point. This is one of our main reasons for sticking with a Minkowskian world-sheet: the locality condition will lead to further restrictions on the theory, which are “invisible” in the Euclidean formulation. These will be discussed in the following section.

One of the additional elements needed in the construction of the full theory is a finite multiplicative (i.e. wave function) renormalization of the free field, whose necessity was first pointed in [27]. We thus rescale \( \psi \) according to

\[
\psi \rightarrow \eta \psi ,
\]

where the constant \( \eta \) will be determined shortly. The rescaling of \( \psi \) must be performed wherever \( \psi \) appears, i.e. in particular inside the expression for \( Y \). The definition of \( c(\alpha) \) (cf. (2.2.18) must be modified accordingly, viz.

\[
\log \alpha = \frac{1}{2} \eta \gamma P.
\]

As another example, consider formula (3.1.18), which is replaced by

\[
\partial_+ \psi(z) \partial_+ \psi(w) =: \partial_+ \psi(z) \partial_+ \psi(w) : + 2g \frac{wz}{(z - w)^2} ,
\]

where, for later convenience (and in order to remain in unison with the literature), we have introduced the constant \( g \)

\[
g \equiv \frac{\eta^2 \gamma^2}{8\pi} = 2h\eta^2 .
\]

The renormalization implied by the rescaling (3.2.3) will entail a shift in the canonical conformal dimension of vertex operators and is therefore at the origin of “gravitational dressing”.

The necessity of introducing \( \eta \) can be seen as follows. The expansion (2.2.16) contains arbitrary powers of \( Y \). Clearly, a definite conformal dimension can be assigned to the infinite sum if and only if \( Y \) has conformal dimension zero, and the full conformal weight is carried by the prefactor, which is local in the free field \( \psi \). So, we first make \( Y \) well defined, replacing it by \( : Y : \), where semicolons denote symmetric free field normal ordering. As a consequence, the conformal weight of the full expression is determined by the
prefactor alone, which we now replace by $e^{\lambda \eta \psi}$, taking into account the multiplicative renormalization introduced above. To compute the conformal dimension of $Y$, we first rewrite (3.1.16) in the form

$$[L_m^{(\pm)}, e^{\lambda \eta \psi(\tau, \sigma)}] = e^{im\xi^\pm} \left(-i\partial_{\pm} + m(\lambda \eta - \frac{\gamma^2 \eta^2}{8\pi} \lambda^2)\right): e^{\lambda \psi(\tau, \sigma)}:. \quad (3.2.7)$$

In order to arrange for $Y$ to have conformal dimension zero, we demand that

$$\eta - \frac{\gamma^2 \eta^2}{8\pi} = \eta - 2\hbar \eta^2 = \eta - g = 1. \quad (3.2.8)$$

Putting $\lambda = 1$ in (3.2.7) and assuming (3.2.8) to hold, we get

$$[L_m^{(+)} e^{\eta \psi(\tau, \sigma)}] = -i\partial_+ \left(e^{im\xi^+} e^{\eta \psi(\tau, \sigma)}\right) \quad (3.2.9)$$

(and the analogous result for $L_m^{(-)}$). Although the exponentials of $\psi^+$ and $\psi^-$ appear with different spatial arguments under the integral, this is already enough to compute the commutator of $L_m^{(+)}$ with $Y$ because $L_m^{(+)}$ commutes with the non-zero mode part of $\psi^-$. As for the zero modes, the momentum dependence of the integrand can be ignored because $L_m^{\pm}$ is independent of $Q$. Furthermore, the dependence of the integrand on the center of mass coordinate $Q$ is the same as in $e^{\eta \psi(\tau, \sigma)}$. After a little algebra, we obtain

$$[L_m^{(+)} : Y(\tau, \sigma) :] = \mu^2 \left[c(\alpha)^2 \int_0^{2\pi} d\sigma' \int_0^{2\pi} d\sigma'' E_\alpha(\sigma - \sigma') E_\alpha(\sigma - \sigma'') \left(-i\frac{\partial}{\partial \sigma'}\right) (e^{im(\tau + \sigma')} e^{\eta \psi^+(\tau + \sigma')} e^{\eta \psi^-(\tau - \sigma'')} \right]:. \quad (3.2.10)$$

This can now be integrated by parts. The boundary terms cancel, and since the derivative on $E_\alpha$ is proportional to the $\delta$-function by (2.2.14), we can replace $e^{im(\tau + \sigma')}$ in the integrand by $e^{im(\tau + \sigma)}$, which can be pulled out of the integral. In this way, we arrive at the desired result

$$[L_m^{\pm}, Y(\tau, \sigma)] = -ie^{im\xi^+} \frac{\partial}{\partial \xi^+} : Y(\tau, \sigma):. \quad (3.2.11)$$

Let us now return to equation (3.2.8); it is solved by

$$\eta = \frac{1}{12} \left(25 - d \pm \sqrt{(25 - d)(1 - d)}\right), \quad (3.2.12)$$

where we have inserted the (renormalized) value for the coupling constant in terms of $d$. This result shows not only that $\eta$ depends on $d$ (the central charge of the matter system), but, more significantly, that the solution imposes a rather stringent constraint on $d$ itself, and thus on the dimension of the target space in which the string moves. Since $\eta$ cannot
be complex, we must restrict \( d \) to the ranges \( d \leq 1 \) or \( d \geq 25 \). Especially, for \( d = 1 \) and \( d = 25 \), one obtains \( \eta = 2 \) \((g = 1)\) and \( \eta = 0 \) \((g = -1)\), respectively; in the latter case, however, the renormalized coupling constant diverges in such a way that the product \( \eta \gamma = 2\sqrt{2\pi i} \) stays finite. In the remainder, we will be mostly interested in the borderline case \( d = 1 \).

The renormalized dimension of the exponential, which now comes entirely from the factor :\( e^{\lambda \eta \psi} :\), is given by

\[
\Delta(\lambda) = \eta \lambda - \frac{\eta^2 \gamma^2}{8\pi} \lambda^2 = \eta \lambda - g \lambda^2. \tag{3.2.13}
\]

When coupling the exponential of the Liouville field to a string vertex operator \( \Phi \) of dimension \( \Delta_0 \), conformal invariance requires that \( \Delta(\lambda) + \Delta_0 = 1 \), so that the total vertex operator can be integrated over the world sheet. Classically, \( \Delta(\lambda) = \lambda \), so \( \Delta_0 = 1 - \lambda \). When \( \Delta(\lambda) \) is deformed to the expression above, \( \Delta_0 \) must be modified accordingly. Denoting the renormalized (“gravitationally dressed”) dimension of \( \Phi \) by \( \Delta \), a little algebra shows that, with the above result, we must demand

\[
\Delta - \Delta_0 = 2h \eta^2 \Delta_0 (1 - \Delta_0) \tag{3.2.14}
\]

in order to maintain total conformal dimension one. Formula \((3.2.14)\) is nothing but the famous KPZ condition \([13]\).

In summary, requiring correct behavior of the exponential operator with respect to conformal transformations, we have been led to the result

\[
: e^{\lambda \varphi} :=: e^{\lambda \eta \psi} : = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(2\lambda + n)}{n! \Gamma(2\lambda)} (\;Y\;)^n \tag{3.2.15}
\]

with \( \eta \) given by \((3.2.12)\). The renormalized (“dressed”) dimension of this operator is given by \((3.2.13)\). Let us stress once more that this formula defines what we mean by the semicolons on the left hand side of this equation. If instead, we had defined the renormalized operator by free field normal ordering, the result would have differed by certain short distance factors (see also the discussion below), which would have led to additional and anomalous contributions in the commutator of \( L_m^\pm \) with the exponential operator, and thus would have destroyed the nice behavior of the operator under conformal transformations.

It must be emphasized at this point that the solution as written down in \((3.2.15)\) is not yet unique. Since the Virasoro generators do not depend on \( Q \), and the desired

\[g_+ = \frac{m}{m+1} \quad \text{and} \quad g_- = \frac{m+1}{m} = g_+^{-1}\]

\(^4\)

We mention that for the unitary minimal models with \( d = 1 - \frac{6}{m(m+1)} \), one obtains two solutions \( g_+ = \frac{m}{m+1} \) and \( g_- = \frac{m+1}{m} = g_+^{-1} \).
commutation relation with $L_m^{(\pm)}$ works order by order, we are free to replace (3.2.15) by the modified expansion

$$e^{\lambda \phi} := e^{\lambda \eta \psi} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(2\lambda + n)}{n! \Gamma(2\lambda)} f_n(P) (\psi)$$

where $f_n(P)$ are arbitrary functions of the center of mass momentum $P$; the modified exponential operator behaves properly under the conformal group. One can exploit (and eliminate) this remaining freedom by imposing locality.

However, there is a price to pay. Even ignoring questions concerning the convergence of the expansion, one may wonder whether the individual terms in this expansion are actually well defined in view of the appearance of arbitrary powers of $\psi$ in it. Naively and from one’s experience with ordinary conformal field theory, one would not anticipate difficulties, as $\psi$ has dimension zero by construction, and therefore the product of two such operators should be non-singular at coincident points if there are no negative dimension operators in the theory. However, trouble is caused by the non-local dependence of $\psi$ on $\psi$, which will lead to non-integrable singularities inside the integrals defining $\psi$. To make these explicit, consider the product $\psi(\tau, \sigma_1) \psi(\tau, \sigma_2)$. Making use of (3.1.11) and (3.1.12) to rewrite this as a fully normal ordered expression, we get additional short distance factors inside the integral. Concentrating on the “critical region”, where the integration variables coincide, we can approximate the integral by

$$\int d\xi^1 \int d\xi^2 \int d\xi^3 \int d\xi^4 \left( \frac{1}{(\xi^1 - \xi^2 - i\epsilon)^2g (\xi^3 - \xi^4 - i\epsilon)^2g} \right)$$

where we have only exhibited the singular factor, and the dots stand for the harmless (and normal ordered) part of the integrand. We repeat that we could have avoided this problem by defining the exponential operator in terms of fully normal ordered products right away, but at the expense of spoiling the conformal properties, since $\psi(\tau, \sigma)$ is not a conformal field in the sense of (3.1.16). Remarkably, the singularities thus occur only in the integration variables, whereas the dependence of the integrand on the “external” variables $\xi_1^1 = \tau + \sigma_1$ and $\xi_2^1 = \tau + \sigma_2$ is completely regular. Performing two of the (indefinite) integrals, we end up with

$$\int d\xi^1 (\xi^1 - \xi^2 - i\epsilon)^{1-2g} \int d\xi^3 (\xi^3 - \xi^4 - i\epsilon)^{1-2g}$$

These integrals exist for $g < 1$; at $g = 1$, they are still well defined because of the $i\epsilon$ prescription. This means that at low orders in the expansion, the singularities are still integrable for a suitable range of values of $g$; yet, at higher orders, the number of poles
at coinciding arguments will increase faster (namely as $n(n-1)/2$ for $(\lambda Y)^n$) than the number of integrations (of which there are only $n$). Consequently, the singularities will eventually become non-integrable for sufficiently large $n$, no matter how we choose $g$. The problem disappears only at special values of the parameter $\lambda$ (namely when $2\lambda$ is a negative integer), for which the series terminates after a finite number of terms.

3.3 Locality Condition

Having constructed the quantum Liouville operator, we now proceed to exploit the consequences of the locality requirement stated in (3.2.2). We will find two (mutually incompatible) ways to satisfy it. The first severely restricts the allowed values of $\lambda$, but works to all orders. The second solution, proposed in [32], is based on a modification of the exponential operator (3.2.15), which exploits (and eliminates) the remaining freedom of choosing the $P$ dependence of the expansion coefficients, cf. (3.2.14). This proposal has the advantage that it salvages locality for arbitrary values of $\lambda$, but has so far only been shown to work to cubic order in $\mu^2$ [32]; in fact, we will present the proof to lowest non-trivial order only. To simplify the notation, we will drop the normal ordering symbols, and set $\tau = 0$ for convenience.

For ease of comparison with [32], we shall trade the field $Y$ for another one, called $S$, which differs in that the zero mode dependence has been pulled out. So we write

$$e^{\lambda\varphi} = e^{\lambda \eta \psi} \sum_m C_m(\bar{P}, \lambda) S^m,$$

(3.3.1)

where we only indicate the dependence on the center of mass momentum and adopt the normalization

$$C_0(\bar{P}, \lambda) = 1.$$

(3.3.2)

For convenience, we have also defined

$$\bar{Q} \equiv \gamma \eta Q, \quad \bar{P} \equiv \frac{1}{4} \gamma \eta P,$$

(3.3.3)

in terms of which the canonical commutator reads

$$[\bar{Q}, \bar{P}] = 2\pi i g.$$

(3.3.4)

The quantity $S$ is related to $Y$ and defined by

$$S(0, \sigma) = \int d\sigma' d\sigma'' e^{Q/2} e^{P(\sigma)} e^{\bar{P}(\sigma' - \sigma'')} e^{Q/2} e^{\eta \psi(\sigma')} e^{\eta \psi(\sigma' - \sigma'')} e^{\eta \psi(-\sigma'')},$$

(3.3.5)

5See, however, the footnote in the introduction.
where
\[ \theta(\sigma) \equiv \theta(\sigma; \sigma', \sigma'') \equiv \epsilon(\sigma - \sigma') - \epsilon(\sigma - \sigma'') . \quad (3.3.6) \]

It should be clear that this is essentially the same expression as the one that we derived for \( Y \) in section 2.1. The integrand has been written in a slightly different fashion by means of the above definition (3.3.3) and the identity
\[ E_\alpha(\sigma - \sigma')E_1/\alpha(\sigma - \sigma'') = \exp\left(\bar{P}\left(\epsilon(\sigma - \sigma') - \epsilon(\sigma - \sigma'')\right)\right) . \quad (3.3.7) \]

Evidently, the operators \( Y \) and \( S \) differ only in their dependence on the momentum \( \bar{P} \). The precise relation is
\[ Y = \frac{8}{\mu^2} \left[ C(\bar{P} + i\pi g) \right]^2 S, \]
\[ C(\bar{P}) = \frac{1}{2\sinh(\bar{P})}, \quad (3.3.8) \]
where the shift in the argument of \( C(\bar{P}) \) comes from pulling the \( \bar{P} \)-dependence out of the integral and the normal ordering symbols. Expanding both sides and comparing term by term, one arrives at the result
\[ C_m(\bar{P}, \lambda) = \frac{(-1)^m \Gamma(2\lambda + m)}{m! \Gamma(2\lambda)} \frac{\mu^2}{8} \left[C(\bar{P} + i\pi g)\right]^{2m} . \quad (3.3.9) \]

It will be important to note that this expression is invariant under the shift
\[ \bar{P} \rightarrow \bar{P} + i\pi n \quad n \in \mathbb{Z} . \quad (3.3.10) \]

For convenience, we here repeat the exchange relation (3.1.6) in a form suitable for the analysis of the locality condition. Not forgetting to take into account the rescaling (3.2.3), we have
\[ e^{a\eta \bar{\psi}_a^+(\sigma_1)}e^{b\eta \bar{\psi}_a^+(\sigma_2)} = \left| 1 - e^{i|\sigma_1 - \sigma_2|} \right|^{-2abg} \]
\[ \cdot e^{-i\pi abge(\sigma_1 - \sigma_2)}e^{iabg(\sigma_1 - \sigma_2)} \]
\[ \cdot e^{b\eta \bar{\psi}_a^+(\sigma_2)}e^{a\eta \bar{\psi}_a^+(\sigma_1)} , \]
\[ e^{a\eta \bar{\psi}_a^-(\sigma_1)}e^{b\eta \bar{\psi}_a^-(\sigma_2)} = \left| 1 - e^{i|\sigma_1 - \sigma_2|} \right|^{-2abg} \]
\[ \cdot e^{i\pi abge(\sigma_1 - \sigma_2)}e^{-iabg(\sigma_1 - \sigma_2)} \]
\[ \cdot e^{b\eta \bar{\psi}_a^- (\sigma_2)}e^{a\eta \bar{\psi}_a^- (\sigma_1)} , \quad (3.3.11) \]
where \( \bar{\psi}_a^\pm \) and \( \bar{\psi}_c^\pm \) were already defined in (3.1.4) and (3.1.3). The corresponding formulas for the zero modes read
\[ e^{aQ} f(\bar{P}) = f(\bar{P} + ia2\pi g)e^{aQ}, \]
\[ f(\bar{P})e^{aQ} = e^{aQ} f(\bar{P} - ia2\pi g) . \quad (3.3.12) \]
This means
\[ e^{a\bar{Q}} f_a(\bar{P}) e^{b\bar{Q}} f_b(\bar{P}) e^{b\bar{Q}} = e^{b\bar{Q}} f_b(\bar{P} + ia4\pi g) e^{b\bar{Q}} e^{a\bar{Q}} f_a(\bar{P} - ib4\pi g) e^{a\bar{Q}}. \] (3.3.13)

We are now ready to analyze the consequences of the locality condition. For this purpose, we expand
\[ \left[ : e^{\lambda\varphi(\tau,\sigma)} : , : e^{\nu\varphi(\tau,\rho)} : \right] = 0 \quad \text{for} \quad \sigma \neq \rho, \] (3.3.14)
and require the result to vanish order by order. At lowest non-trivial order, we obtain the following condition (the \(O(1)\) contribution is trivial), which must hold for all values of the arguments \(\sigma\) and \(\rho\)
\[ 0 = e^{\lambda\eta(\sigma)} e^{\nu\eta(\rho)} C_1(\bar{P}, \nu) S(\rho) - e^{\nu\eta(\rho)} C_1(\bar{P}, \nu) S(\rho) e^{\lambda\eta(\sigma)} + e^{\lambda\eta(\sigma)} C_1(\bar{P}, \lambda) S(\sigma) e^{\nu\eta(\rho)} - e^{\nu\eta(\rho)} e^{\lambda\eta(\sigma)} C_1(\bar{P}, \lambda) S(\sigma). \] (3.3.15)

Since the non-zero mode contributions coincide in all four terms, we must only look at the zero mode contributions. It is now straightforward to show that this leads to the following conditions on the coefficients:
\[ 0 = C_1(\bar{P}, \nu) e^{(P+i\pi g)\theta(\rho)} - C_1(\bar{P} - i\pi 2\lambda g, \nu) e^{(P+i\pi g(1-2\lambda))\theta(\rho) + i\pi 2\lambda g\theta(\sigma)} + C_1(\bar{P} - i\pi 2\nu g, \lambda) e^{(P+i\pi g(1-2\nu))\theta(\sigma) + i\pi 2\nu g\theta(\rho)} - C_1(\bar{P}, \lambda) e^{(P+i\pi g)\theta(\sigma)}. \] (3.3.16)

This equation must be satisfied in all the regions of \((\sigma, \rho)\) space. From its definition, \(\theta(\sigma)\) is easily seen to take the values
\[ \theta(\sigma) = \begin{cases} 0 & \text{if} \quad \sigma > \sigma', \sigma'' \quad \text{or} \quad \sigma', \sigma'' > \sigma \\ 2 & \text{if} \quad \sigma'' > \sigma > \sigma' \\ -2 & \text{if} \quad \sigma' > \sigma > \sigma'' \end{cases} \]

Therefore, \(\theta(\sigma)\) and \(\theta(\rho)\) can take separately the values 0, \(\pm 2\), except for the combination \(\theta(\sigma) = \pm 2, \theta(\rho) = \mp 2\). It is not difficult to see that many of the regions give identical conditions and there are only three independent equations (plus the corresponding ones in which \(\lambda\) and \(\nu\) are interchanged):
\[ 0 = C_1(\bar{P}, \nu) - C_1(\bar{P} - i\pi 2g\lambda, \nu). \]
\[ +C_1(P - i\pi 2g\nu, \lambda) - C_1(P, \lambda), \]  
\[ 0 = C_1(\bar{P}, \nu)e^{2(P+4\pi g)} - C_1(\bar{P}, \nu)e^{2(P+4\pi g(1-2\lambda))} + C_1(\bar{P} - i\pi 2g\nu, \lambda)e^{4\pi i\nu g} - C_1(\bar{P}, \lambda), \]  
\[ 0 = C_1(\bar{P}, \nu)e^{2(P+4\pi g)} - C_1(\bar{P}, \nu)e^{2(P+4\pi g(1-2\lambda))} + C_1(\bar{P} - i\pi 2g\nu, \lambda)e^{2\pi i2\nu g} - C_1(\bar{P}, \lambda) \]  

By subtracting (3.3.18) and (3.3.19) from (3.3.17), we get

\[ C_1(\bar{P}, \nu) \left( 1 - e^{2(P+4\pi g)} \right) - C_1(\bar{P} - i\pi 2g\lambda, \nu) \left( 1 - e^{2(P+4\pi g(1-2\lambda))} \right) + C_1(\bar{P} - i\pi 2g\nu, \lambda) \left( 1 - e^{4\pi i\nu g} \right) = 0, \]  
\[ C_1(\bar{P}, \nu) \left( 1 - e^{-2(P+4\pi g)} \right) - C_1(\bar{P} - i\pi 2g\lambda, \nu) \left( 1 - e^{-2(P+4\pi g(1-2\lambda))} \right) + C_1(\bar{P} - i\pi 2g\nu, \lambda) \left( 1 - e^{-4\pi i\nu g} \right) = 0. \]  

Notice that in both of these equations a factor of the form \(1 - e^{\pm 4\pi i\nu g}\) appears in the last term. Although the notations are somewhat different, effectively this factor was implicitly assumed to be non-vanishing in the existing analysis [32]. As we shall show, the case where this factor vanishes will yield a new solution which is neither smoothly connected nor relatively local to the solution obtained in [32]. Thus we now solve (3.3.20) and (3.3.21) for these two cases separately. Since our analysis will involve the consideration of special discrete values of \(\lambda\), we emphasize that, except for \(d = 1, 25\), the equation for \(g\) admits two solutions.

**Case 1:** Begin with the case where the factor above vanishes. We then have

\[ 2\nu g = \text{integer}, \]  
and from (3.3.20) and (3.3.21) we easily obtain

\[ C_1(\bar{P} - i\pi 2g\lambda, \nu) = \frac{1 - e^{2(P+4\pi g)}}{1 - e^{2(P+4\pi g(1-2\lambda))}} C_1(\bar{P}, \nu), \]  
\[ C_1(\bar{P} - i\pi 2g\nu, \lambda) = \frac{1 - e^{-2(P+4\pi g)}}{1 - e^{-2(P+4\pi g(1-2\lambda))}} C_1(\bar{P}, \nu). \]  

From the compatibility of these equations, we get

\[ 2\lambda g = \text{integer}, \]  
\[ C_1(P - i\pi 2g\lambda, \nu) = C_1(P, \nu). \]
It is easy to see that the classical form of the coefficient (3.3.9) satisfies this requirement.

Case 2: Now consider the case where $2\nu g \neq \text{integer}$. After some calculation using (3.3.20) and (3.3.21), we obtain

$$C_1(\bar{P}, \nu) = \frac{a(\nu)}{\sinh(P + i\pi g) \sinh(P + i\pi g(1 - 2\nu))}, \quad (3.3.26)$$

where $a(\nu)$ is an unknown function only of $\nu$. It is determined by substituting (3.3.26) into (3.3.20). The result is

$$a(\nu) = a \sin(2\pi \nu g) \quad (a = \text{const.}). \quad (3.3.27)$$

Thus we finally obtain

$$C_1(\bar{P}, \nu) = \frac{a \sin(2\pi \nu g)}{\sinh(P + i\pi g) \sinh(P + i\pi g(1 - 2\nu))}. \quad (3.3.28)$$

It can be checked by a rather tedious calculation that this form of the coefficient does satisfy the locality equations in all the regions without any additional conditions on the values of $\lambda$ and $\nu$. Apart from an inessential normalization, this is the form obtained in [32]. Clearly, this formula is very suggestive of a hidden quantum group structure of quantum Liouville theory, and this has led the authors of [32] to conjecture a formula for the quantum deformed Liouville operator to all orders.

Let us summarize the implications of the $\mathcal{O}(\mu^2)$ analysis just performed: First, if $2\nu g$ and $2\lambda g$ are both not integers, then $\exp(\nu \varphi)$ and $\exp(\lambda \varphi)$ are mutually local with the choice of the coefficients given by Otto and Weigt. On the other hand, if $2\nu g \in \mathbb{Z}$, then $\exp(\lambda \varphi)$ is mutually local with respect to $\exp(\nu \varphi)$ only if $2\lambda g \in \mathbb{Z}$. This latter case is precisely the one which is relevant for the Liouville theory since the operator $e^\varphi$ appearing in the equation of motion must certainly exist. One might wonder if Case 1 is but a special case of Case 2. This is not so: In order for the solution for Case 2 to correctly reproduce the classical limit (i.e. $g \to 0$), the constant $a$ in (3.3.28) must be proportional to $1/\sin(\pi g)$, namely

$$C_1(\bar{P}, \nu) \propto \frac{\sin(2\pi \nu g) / \sin(\pi g)}{\sinh(P + i\pi g) \sinh(P + i\pi g(1 - 2\nu))}. \quad (3.3.29)$$

(Strictly speaking, the factor $\sin(\pi g)$ can be replaced by any function going linearly to zero as $g \to 0$.) However, if we now take the limit $\nu \to n/(2g)$ where $n$ is a non-zero integer, then the coefficient $C_1(\bar{P}, \nu)$ above vanishes. Therefore the cases 1 and 2 above are not smoothly connected.

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6 The quantum group structure of Liouville theory has also been extensively discussed in [35], although from somewhat different point of view.
Having seen that we must take \( 2g \in \mathbb{Z} \) for the Liouville theory to make sense, we will now show that the locality condition is satisfied to all orders in \( \mu^2 \) by taking the classical form of the coefficients \( C_m(\bar{P}, \lambda) \).

Let us first consider the effect of commuting the free-field exponential \( \exp(\lambda \eta \psi) \) through various operators. When commuted through \( \bar{P} \), it produces the shift \( \bar{P} \rightarrow \bar{P} + i\pi \) where \( n \) is an integer. Since the coefficient \( C_m(\bar{P}, \lambda) \) has the periodicity

\[
C_m(\bar{P} + i\pi n, \lambda) = C_m(\bar{P}, \lambda),
\]

it does not affect \( C_m(\bar{P}) \). Also the operator

\[
E_\alpha(\sigma - \sigma')E_{1/\alpha}(\sigma - \sigma'') = \exp \left( \bar{P}(\epsilon(\sigma - \sigma') - \epsilon(\sigma - \sigma'')) \right)
\]

is not affected, since \( \epsilon(\sigma - \sigma') - \epsilon(\sigma - \sigma'') \) is always an even integer.

One more change \( \exp(\lambda \eta \psi) \) produces when commuted with the operator \( S(\rho) \) is a phase of the type \( \exp(\rho' - \rho'') a \). From the exchange formula, we can check that while the non-zero mode exchange produces the phase \( \exp(-i(\rho' - \rho'')2\lambda g) \), it is precisely compensated by the opposite phase coming from the zero-mode exchange. Thus we conclude that, for the case under consideration, \( \exp(\lambda \eta \psi) \) has no effect in the locality equation and hence can be ignored. It is therefore sufficient to prove

\[
\sum_{m,n} \left\{ C_m(\bar{P}, \lambda)S^m(\sigma)C_n(\bar{P}, \nu)S^n(\rho) - C_n(\bar{P}, \nu)S^n(\rho)C_m(\bar{P}, \lambda)S^m(\sigma) \right\} = 0.
\]

(3.3.32)

Since \( S^m(\sigma) \) contains the zero mode part \( \exp(m\bar{Q}) \), the left hand side of the above equation becomes

\[
\sum_{m,n} \left\{ C_m(\bar{P}, \lambda)C_n(\bar{P} + 2i\pi gm, \nu)S^m(\sigma)S^n(\rho) - C_n(\bar{P}, \nu)C_m(\bar{P} + 2i\pi gn, \lambda)S^n(\rho)S^m(\sigma) \right\}.
\]

(3.3.33)

For each term in the sum, the exchange of \( S^n(\rho) \) and \( S^m(\sigma) \) produces the phase

\[
S^n(\rho)S^m(\sigma) = S^m(\sigma)S^n(\rho)e^{i(\theta(\sigma, \rho) - \theta(\rho, \sigma)) + \sum_{\rho'j} \theta(\rho', j) - \sum_{\sigma'i} \theta(\sigma', i)}.
\]

(3.3.34)

But since each sum in the exponent is an even integer, the phase factor is actually 1 for \( 2g \in \mathbb{Z} \). Also, in this case, we have \( C_n(\bar{P} + 2i\pi gm, \nu) = C_n(\bar{P}) \) etc.. This means that all the operators appearing in the locality equations commute and therefore the locality requirement is satisfied to all orders in \( \mu^2 \) with the classical expression for \( C_m(\bar{P}, \lambda) \).
3.4 Operator Equation of Motion

The procedure by which we constructed the exponential operator (3.2.16) relied heavily on the existence of the associated free field $\psi$, which itself was built out of the solution of the classical Liouville equation in terms of the functions $A, B$. We can now ask under what circumstances this equation can remain valid in the quantized theory. The main question we have to address here is how to define the quantum Liouville field itself (and not its exponential). This is a rather subtle issue in view of the results of the preceding section. If the proposal of [32] could be shown to work to all orders, we could define the quantum field $\varphi$ by taking the derivative of $: e^{\lambda \psi} :$ with respect to $\lambda$ and putting $\lambda = 0$ afterwards. If, on the other hand, only the discrete and mutually local set of operators with $2g \lambda \in \mathbb{Z}$ is available, this idea does not work. In this case, we can define the Liouville field by its normal ordered expansion, i.e.

$$\varphi := \eta \psi + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} (: Y :)^n.$$  

(3.4.1)

Of course, we must keep in mind that this definition is afflicted with the same problems that we encountered in defining the exponential operator, especially since the expansion (3.4.1) contains infinitely many terms (cf. the discussion at the end of section 3.2); furthermore, with this definition the relation between $: \varphi :$ and the exponential operator defined before is somewhat obscure. With this caveat, let us proceed nonetheless and analyze the quantum Liouville equation. To do so, we need some relations involving derivatives of $: Y :$. From (2.2.14) and (2.2.15), we easily derive

$$: \partial_{+} Y(\tau, \sigma) : =: \left( c(\alpha) \int_0^{2\pi} d\sigma'' E_{\alpha}(\sigma - \sigma'') e^{\psi^+(\tau + \sigma) e^{\psi^-(\tau - \sigma'')}} \right) :,$$  

(3.4.2)

and

$$: \partial_- Y(\tau, \sigma) : =: \left( c(\alpha) \int_0^{2\pi} d\sigma' E_{\alpha}(\sigma - \sigma') e^{\psi^+(\tau + \sigma') e^{\psi^-(\tau - \sigma)}} \right) : .$$  

(3.4.3)

(We put $\mu^2 = 8$ for simplicity.) For the double derivative, we obtain

$$: \partial_{+} \partial_{-} Y(\tau, \sigma) : =: e^{\eta \psi^+(\xi^+) e^{\eta \psi^-(\xi^-)}} := e^{\eta \psi(\tau, \sigma)} : .$$  

(3.4.4)

A separate calculation shows that

$$[ : \partial_{\pm} Y(\tau, \sigma) :, Y(\tau, \rho) :] = 0 .$$  

(3.4.5)

We can therefore differentiate (3.4.1) term by term and move the derivatives to the right; the calculation is then almost identical with the classical case. Using the above relations,
we thus get
\[
\partial_+ \partial_- \varphi = 2 \sum_{n \geq 1} (-1)^n \left( : Y : \right)^{n-1} \cdot e^{\eta \psi} + \\
+ 2 \sum_{n \geq 1} (-1)^n (n-1) \left( : Y : \right)^{n-2} \cdot \partial_+ Y \cdot \partial_- Y. \tag{3.4.6}
\]

Now, it is a little exercise in re-normal ordering to show that
\[
\partial_+ Y(\tau, \sigma) \cdot : \partial_- Y(\tau, \rho) : = \left( (-1)^g \int_0^{2\pi} d\sigma' \int_0^{2\pi} d\sigma'' \right) E_\alpha(\rho - \sigma') e^{\psi^+ (\tau + \sigma')} e^{\psi^- (\tau - \sigma'')} \tag{3.4.7}
\]
\[
= (-1)^g \left( c(\alpha)^2 \int_0^{2\pi} d\sigma' \int_0^{2\pi} d\sigma'' \right) E_\alpha(\rho - \sigma') e^{\psi^+ (\tau + \sigma')} e^{\psi^- (\tau - \sigma'')} \cdot e^{\eta \psi(\sigma)} e^{\eta \psi(\sigma')} :. \tag{3.4.8}
\]

Putting \(\sigma = \rho\), we get
\[
\partial_+ Y(\tau, \sigma) \cdot : \partial_- Y(\tau, \sigma) := (-1)^g \cdot : Y(\tau, \sigma) \cdot e^{\eta \psi(\tau, \sigma)} :. \tag{3.4.9}
\]

Inserting this into \((3.4.6)\), we see that for \(g \in \mathbb{Z}\), the right hand side of \((3.4.6)\) can be simplified to
\[
2 \sum_{n \geq 1} (-1)^n n \left( : Y : \right)^{n-1} \cdot e^{\eta \psi} := - : e^{\varphi} :, \tag{3.4.10}
\]

which is the desired result. We conclude that under the assumptions made above the quantum Liouville equation is satisfied only for integer values of \(g\), and, in particular, for \(d = 1\). For the \(d < 1\) models, \(g\) is not an integer, as is obvious from \((3.2.8)\). In the foregoing section, we demonstrated that locality holds to all orders if \(2g\lambda\) is an integer. For non-integer \(g\), \(\lambda\) is not integer either; but then, : \(e^{\varphi} :\) cannot be a local operator. This seems to indicate that the quantum Liouville equation is not consistent with locality for \(d < 1\)! However, we repeat that this conclusion is subject to the caveats mentioned above, and therefore does not necessarily imply any inconsistency of the \(d < 1\) models.

## 4 Ground Ring

As an application of the formalism developed in the preceding sections, we shall make an attempt to construct the generators of the so called “ground ring” \([24]\), which was found to play an important role in characterizing the symmetry structure of the \(d = 1\) string theory. Our prime concern here is to see how its structure changes when the dependence on the cosmological constant is fully taken into account in the operator formalism.
4.1 Euclidean Case with $\mu^2 = 0$

To begin with, let us briefly recall the Euclidean case with $\mu^2$ set to zero, the case first discussed by Witten [24]. When the Liouville field is regarded as a free field, one can consider the ring of BRST invariant operators with vanishing ghost number for the holomorphic and the anti-holomorphic sectors separately. Each of them is hence termed “a chiral ground ring”. In the holomorphic sector, the chiral ring was shown to be generated by the following two operators, called $x$ and $y$:

$$x = \left( cb + \frac{i}{\sqrt{2}} (\partial X - i \partial \phi) \right) \cdot e^{i(X+i\phi)/\sqrt{2}}, \quad (4.1.1)$$

$$y = \left( cb - \frac{i}{\sqrt{2}} (\partial X + i \partial \phi) \right) \cdot e^{-i(X-i\phi)/\sqrt{2}}, \quad (4.1.2)$$

where $b(z)$ and $c(z)$ are the ghost fields, while $\phi(z)$ and $X(z)$ are respectively the Liouville and the matter field with the free-field operator products

$$\phi(z)\phi(w) \sim X(z)X(w) \sim -\ln(z - w). \quad (4.1.3)$$

These operators correspond to special discrete physical states and are invariant with respect to the BRST operator $Q_E$ given by

$$Q_E = \sum c_{-n} L_n^E - \frac{1}{2} \sum (m - n) : c_{-m} c_{-n} b_{m+n} :_{\text{inv}}, \quad (4.1.4)$$

where the subscript $\text{inv}$ signifies $sl(2)$ invariant normal ordering. The Virasoro generator $L_n^E$ is of the form

$$L_n^E = L_n^X + L_n^\phi, \quad (4.1.5)$$

$$L_n^X = \frac{1}{2} \sum : \alpha_{n-m}^X \alpha_m^X :,$$  

$$L_n^\phi = \frac{1}{2} \sum : \alpha_{n-m}^\phi \alpha_m^\phi : + iq(n+1) \alpha_n^\phi, \quad (4.1.7)$$

where the background charge $q$ takes the value $\sqrt{2}$. The anti-holomorphic chiral ground ring is generated by similar operators called $\bar{x}$ and $\bar{y}$.

In the following, we shall concentrate on the generator $x$ and for ease of notation express it as

$$x = j \mathcal{X} \Phi_0 + \partial \mathcal{X} \Phi_0 - \mathcal{X} \partial \Phi_0, \quad (4.1.8)$$

$$j \equiv cb, \quad (4.1.9)$$

$$\mathcal{X} \equiv e^{\frac{i}{\sqrt{2}} X}, \quad (4.1.10)$$

$$\Phi_0 \equiv e^{-\frac{i}{\sqrt{2}} \phi}. \quad (4.1.11)$$

The subscript “0” emphasizes that the Liouville field is treated as a free field and the usual normal ordering for composite operators is understood.
4.2 Minkowski Case with $\mu^2 = 0$

Before tackling the fully interacting case, we need to clarify in some detail how the free field case should be treated in Minkowski formulation. Many of the calculations to be performed in this subsection will be utilized in the interacting case.

To facilitate the comparison with the Euclidean case, we shall use the canonically normalized Liouville field $\phi(x^+)$ which, in terms of the field $\psi(x^+)$, is given by

$$\phi(x^+) = \frac{1}{\sqrt{2}} \psi(x^+). \quad (4.2.1)$$

(To be precise, $\phi(x^+)$ is defined to be that part of $\phi(x^+, x^-)$ which is independent of $x^-$. That is, it includes the full $Q$-zero mode of $\phi(x^+, x^-)$.) Then the energy-momentum tensor for the Liouville sector is given by

$$T_M^\phi(x^+)^{\mu} = \frac{1}{2} (\partial_\mu \phi)^2 - q \partial_\mu^2 \phi, \quad (4.2.2)$$

and the corresponding Virasoro generator is of the form (still with $q = \sqrt{2}$)

$$L_n^{M,\phi} = \frac{1}{2} \sum c_{-n} c_m : \alpha_n \alpha_m : + i q n \alpha_n. \quad (4.2.3)$$

Notice that the term containing the background charge takes a slightly different form compared with the Euclidean case. It is straightforward to show that the exponential operator $\Phi_0$ is primary with respect to this Virasoro generator provided that it is defined with symmetric normal ordering (see the appendix.) Specifically,

$$\left[ L_n^{M,\phi}, \Phi_0(x^+) \right] = z^n \left( \frac{1}{i} \partial_+ + \left( \frac{5}{4} \right) n \right) \Phi_0(x^+). \quad (4.2.4)$$

It is easy to check that $L_n^{M,\phi}$'s satisfy the Virasoro algebra of the form

$$\left[ L_m^{M,\phi}, L_n^{M,\phi} \right] = (m - n) L_{m+n}^{M,\phi} + \delta_{m+n,0} \left( \frac{1 + 12q^2}{12} (m^3 - m) + q^2 m \right). \quad (4.2.5)$$

In order to construct the correct nilpotent BRST operator, we need Virasoro generators which satisfy the above algebra with the central term proportional to $m^3 - m$. As is well-known, such a standard form can be obtained by a shift

$$L_n^{M,\phi} \rightarrow L_n^{M,\phi} + \frac{1}{2} q^2 \delta_{n,0} = L_n^{M,\phi} + \delta_{n,0}. \quad (4.2.6)$$

If we denote the total Virasoro operator (including the matter part $L_n^{M,X}$) by $L_n^M$, the BRST operator takes the form

$$Q_M = \sum c_{-n} \left( L_n^M + \delta_{n,0} \right) - \frac{1}{2} \sum (m - n) : c_m c_n b_{m+n} :_{\text{inv}}, \quad (4.2.7)$$
We are now ready to display the generator $x_M$, corresponding to the operator $x$, for the Minkowski case. It takes the form

$$x_M = \left( j - \frac{3}{2} \right) \mathcal{X} \Phi_0 + \frac{1}{i} \partial_+ \mathcal{X} \Phi_0 - \frac{1}{i} \mathcal{X} \partial_+ \Phi_0,$$  \hspace{1cm} \text{(4.2.8)}

where the ghost current $j$ is normal-ordered with respect to the $sl(2)$ invariant vacuum as in the Euclidean case, and the operators $\Phi_0$ and $\mathcal{X}$ are defined with the symmetric normal ordering. The shift of $j$ by the amount $3/2$ compared with (4.1.8) can be understood as follows: The BRST operator $Q_M$ given in (4.2.7) actually takes a better form if we make a change of normal-ordering from the $sl(2)$ invariant one to the “physical” one defined by the rule

$$b_n, c_n \quad \text{annihilation for } n \geq 1,$$  \hspace{1cm} \text{(4.2.9)}

$$\frac{1}{2} (c_0 b_0 - b_0 c_0) \quad \text{for zero mode.}$$  \hspace{1cm} \text{(4.2.10)}

$Q_M$ then becomes

$$Q_M = \sum c_{-n} L_n^M - \frac{1}{2} \sum (m - n) : c_{-m} c_{-n} b_{m+n} :_{\text{phys}}.$$  \hspace{1cm} \text{(4.2.11)}

It is easily checked that under this normal ordering $j$ becomes anti-hermitian and is related to the previous definition by

$$j^{\text{inv}} - \frac{3}{2} = j^{\text{phys}}.$$  \hspace{1cm} \text{(4.2.12)}

BRST invariance of $x_M$ given above can be demonstrated as follows. First we write the commutator $[Q_M, x_M]$ as

$$[Q_M, x_M] = A_1 + A_2 + A_3 - \frac{3}{2} A_4,$$  \hspace{1cm} \text{(4.2.13)}

where

$$A_1 = \lim_{y^+ \to x^+} \left[ Q_M, j(x^+) \mathcal{X} \Phi_0(y^+) \right],$$  \hspace{1cm} \text{(4.2.14)}

$$A_2 = \left[ Q_M, -\mathcal{X} \frac{1}{i} \partial_+ \Phi_0 \right],$$  \hspace{1cm} \text{(4.2.15)}

$$A_3 = \left[ Q_M, \frac{1}{i} \partial_+ \mathcal{X} \Phi_0 \right],$$  \hspace{1cm} \text{(4.2.16)}

$$A_4 = \left[ Q_M, \mathcal{X} \Phi_0 \right].$$  \hspace{1cm} \text{(4.2.17)}

$A_2, A_3$ and $A_4$ are easy to evaluate using the fact that $\mathcal{X}$ and $\Phi_0$ are primary with respect to the Virasoro operators inside $Q_M$. The result is

$$A_2 = c \partial_+ (\mathcal{X} \partial_+ \Phi_0) - \frac{5}{4} \partial_+^2 \mathcal{X} \Phi_0,$$  \hspace{1cm} \text{(4.2.18)}

$$A_3 = -c \partial_+ (\partial_+ \mathcal{X} \Phi_0) - \frac{1}{4} \partial_+^2 \mathcal{X} \Phi_0,$$  \hspace{1cm} \text{(4.2.19)}

$$-\frac{3}{2} A_4 = -\frac{3}{2} c \frac{1}{i} \partial_+ (\mathcal{X} \Phi_0) + \frac{3}{2} \frac{1}{i} \partial_+ c \mathcal{X} \Phi_0.$$  \hspace{1cm} \text{(4.2.20)}
Therefore,
\[
A_2 + A_3 - \frac{3}{2} A_4 = -c \left( \partial_+^2 \mathcal{X} \Phi_0 - \mathcal{X} \partial_+^2 \Phi_0 \right) - \frac{3}{2} \partial_+^2 c \mathcal{X} \Phi_0 \\
- \frac{3}{2} c \frac{1}{i} \partial_+ (\mathcal{X} \Phi_0) + \frac{3}{2} \frac{1}{i} \partial_+ c \mathcal{X} \Phi_0.
\] (4.2.21)

To compute \( A_1 \), we will need the following formulae which can be obtained by straightforward calculations:

\[
[Q_M, j(x^+)] = -c \left( T_M(x^+) + 1 \right) - \frac{1}{i} c \partial_+ c b + \frac{3}{2} \partial_+^2 c - \frac{3}{2} \frac{1}{i} \partial_+ c,
\] (4.2.22)

\[
[Q_M, \mathcal{X} \Phi_0(y^+)] = \frac{1}{i} \partial_+ (\mathcal{X} \Phi_0) - \frac{1}{i} \partial_+ c \mathcal{X} \Phi_0,
\] (4.2.23)

\[
T_M^x(x^+) \mathcal{X}(y^+) = \frac{1}{2} \left( \frac{1}{\sqrt{2}} \frac{w}{z-w} + \frac{1}{2} \frac{1}{i} \partial_+ \mathcal{X} \right) + \left( \frac{1}{i} \frac{1}{\sqrt{2}} \partial_+^2 \phi - \frac{1}{16} \right) \Phi_0 + : T_M^x(x^+) \mathcal{X} :,
\] (4.2.24)

\[
T_M^\phi(x^+) \Phi_0(y^+) = -\frac{5}{4} \frac{zw}{(z-w)^2} \Phi_0 + \left( \frac{w}{z-w} + \frac{1}{2} \right) \frac{1}{i} \partial_+ \Phi_0 + \frac{1}{\sqrt{2}} \partial_+^2 \phi \Phi_0 + : T_M^\phi(x^+) \Phi_0 :,
\] (4.2.25)

\[
j(x^+) c(y^+) = c(x^+) \left( \frac{w}{z-w} + 2 \right) + : j(x^+) c(y^+) :,
\] (4.2.26)

\[
j(x^+) \frac{1}{i} \partial_+ c(y^+) = c(x^+) \left( \frac{w}{z-w} \right)^2 + \frac{w}{z-w} - 1 - \frac{1}{i} c \partial_+ c b.
\] (4.2.27)

Using these formulae, it is easy to verify that all singular terms cancel, and we are left with

\[
A_1 = -c \left( \frac{1}{2} \left( \partial_+ \mathcal{X} \right)^2 - \frac{i}{\sqrt{2}} \partial_+^2 \mathcal{X} + \frac{1}{2} \left( \partial_+ \phi \right)^2 - \frac{1}{\sqrt{2}} \partial_+^2 \phi \right) \mathcal{X} \Phi_0 \\
+ \frac{3}{2} c \frac{1}{i} \partial_+ \mathcal{X} \Phi_0 + \frac{3}{2} \frac{1}{i} \partial_+ c \mathcal{X} \Phi_0 - \frac{3}{2} \frac{1}{i} \partial_+ c \mathcal{X} \Phi_0.
\] (4.2.29)

From (4.2.21) and (4.2.29) it is evident that the sum \( A_1 + A_2 + A_3 - \frac{3}{2} A_4 \) indeed vanishes.

### 4.3 Minkowski Case with \( \mu^2 \neq 0 \)

We now describe an attempt to construct the generators of the ground ring for the interacting case. As we switch on the cosmological term, left and right moving modes are coupled and we must necessarily consider the full ground ring. In the free field case its
generators are obtained as products of chiral generators, such as \( x_M \bar{x}_M, x_M \bar{y}_M \), etc. Since the interacting generators must reduce to these forms in the limit \( \mu^2 \to 0 \), it is reasonable to take our candidates to be obtained from these products by appropriate replacements of free fields by corresponding interacting fields. More specifically, we shall consider in the following the operator \( a_1 \) which is obtained from \( x_M \bar{x}_M \) by the replacements

\[
\mathcal{X}(x^+) \mathcal{X}(x^-) \rightarrow \mathcal{X}(x^+, x^-), \quad (4.3.1)
\]
\[
\partial_+ \mathcal{X}(x^+) \mathcal{X}(x^-) \rightarrow \partial_+ \mathcal{X}(x^+, x^-), \quad (4.3.2)
\]
\[
\partial_+ \Phi_0(x^+) \partial_- \Phi_0(x^-) \rightarrow \partial_+ \partial_- \Phi(x^+, x^-), \quad \text{etc.} \quad (4.3.3)
\]

Since the commutation relations with the right and left BRST charges can be considered separately, we shall concentrate on the action of the left charge. Then we can effectively work with only the \( x_M \) factor of \( a_1 \) (with interacting fields) and at the end of the calculation we can multiply by \( \bar{x}_M \) and make the replacements above.

We note the remarkable fact that the weight in the exponent of \( \Phi \) is precisely such that the expansion in powers of \( \mu^2 \) breaks off after the first non-trivial order; namely, we have exactly \( \Phi = \Phi_0(1 + Y) \). This allows us to perform the calculations without any approximation.

Let us now calculate the commutator \([Q_M, x_M]\). All the calculations which make use of the conformal properties of the fields \( \mathcal{X} \) and \( \Phi \) proceed exactly as in the free case. The only place we must not use the free-field expressions is the computation of \( T^\phi(x^+)\Phi(y^+) \) and \( \partial^2 \Phi \). The result using the conformal properties is

\[
[Q_M, x_M] = -c \mathcal{X}(x) \left( T^\phi_M(x^+)\Phi(y) + \left( \frac{5}{4} \left( \frac{w}{z-w} \right)^2 + \frac{5}{4} \frac{w}{z-w} + \frac{1}{16} \right) \Phi \right.
\]
\[
- \partial_+^2 \Phi - \left( \frac{w}{z-w} + \frac{1}{2} \right) \frac{1}{4} \partial_+ \Phi \Bigg\} \quad (4.3.4)
\]

This is precisely the combination which vanished in the free field calculation.

To simplify this expression it is useful to factorize \( \Phi \) as

\[
\Phi = \Phi_0(1 + Y), \quad (4.3.5)
\]

where \( \Phi_0 \) is as defined for free theory and symmetric normal ordering for \( \Phi_0 \) and \( Y \) is implicit. Now we shall prove the simple yet non-trivial result

\[
\partial_+ \Phi = \partial_+ \Phi_0(1 + Y). \quad (4.3.6)
\]

To prove this, we must show that \( \Phi_0 \partial_+ Y \) vanishes. First from the definition of \( Y \) we easily get

\[
\partial_+ Y(x) = \frac{\mu^2}{8} \int d^2 \sigma'' e^{\bar{Q}/2} C(P) e^{-\bar{P} e^{-\bar{Q}/2}} e^{\frac{1}{2} \ln z} e^{\frac{1}{2} \ln z''} e^{\bar{Q}/2}
\]

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\[ e^{\sqrt{2} \phi^+(z)} e^{\sqrt{2} \phi^-(z')} e^{\sqrt{2} \phi^+(z')} e^{\sqrt{2} \phi^-}(z'). \]  

(4.3.7)

A better representation is obtained if we move the second \( e^{\bar{Q}/2} \) factor through the parts involving \( \sigma'' \). Then we can express \( \partial_+ Y(x) \) in the form

\[ \partial_+ Y(x) = e^{\sqrt{2} \phi^+(x^+)} : Y^-(y) , \]  

(4.3.8)

(4.3.9)

where

\[ Y^-(y) = \mu^2 8 \int d\sigma'' C(\bar{P} - i\pi) e^{-(\bar{P} - i\pi)\epsilon(\sigma - \sigma'')} e^{(\frac{\bar{P} - i\pi}{2}) \ln z''} e^{\sqrt{2} \phi^-}(y) \]  

(4.3.10)

With this expression, we can compute the operator product between \( \Phi_0 \) and \( \partial_+ Y \) at different arguments. We get

\[ \Phi_0(x^+) \partial_+ Y(y) = e^{-\frac{\phi^+(x^+)}{\sqrt{2}} - \frac{\phi^-(x^-)}{\sqrt{2}}} : e^{\sqrt{2} \phi^+(y^+)} : Y^-(y) \]

\[ \cdot : e^{\sqrt{2} \phi^+(y^+)} : Y^-(y) \]

\[ = e^{-\frac{1}{2} \ln(w/z)} (1 - \frac{w}{z}) \]

\[ \cdot : e^{-\frac{1}{\sqrt{2}} \phi^+(x^+) + \sqrt{2} \phi^+(y^+) - \frac{1}{\sqrt{2}} \phi^-(x^-)} : Y^-(y) \]  

(4.3.11)

This is seen to vanish in the limit \( y^+ \to x^+ \) and we have the announced result (4.3.6).

With the use of various formulae developed for the free field case, we then get, after some calculations

\[ \partial^2_+ \Phi = : T^\phi \Phi_0 : (1 + Y) + \frac{1}{\sqrt{2}} \partial^2_+ \phi \Phi_0 : (1 + Y) + \partial_+ \Phi_0 \partial_+ Y . \]  

(4.3.12)

Now look at \( T^\phi_M(x^+) \Phi(y) \) in \([Q_M, x_M]\). We first re-normal order \( T^\phi_M \) and \( \Phi_0 \) and leave the factor \( 1 + Y \) untouched. The result is

\[ T^\phi_M(x^+) \Phi(y) = - \left( \frac{5}{4} \left( \frac{w}{z - w} \right)^2 + \frac{5}{4} \frac{w}{z - w} + \frac{1}{16} \right) \Phi(y) \]

\[ + \left( \frac{w}{z - w} + \frac{1}{2} \right) \frac{1}{i} \partial_+ \Phi_0 (1 + Y) \]

\[ + : \frac{1}{\sqrt{2}} \partial^2_+ \phi(y) \Phi_0(y) : (1 + Y) + : T^\phi(x^+) \Phi_0(y) : (1 + Y) . \]  

(4.3.13)

(4.3.14)

Substituting (4.3.12) and (4.3.13) into (4.3.14) we get a non-vanishing result of order \( \mu^2 \):

\[ [Q_M, x_M] = \alpha \chi \partial_+ \Phi_0 \partial_+ Y . \]  

(4.3.14)
Since this result is not as expected, let us make a few checks before contemplating upon its implications. One check of the correctness of the calculation above is to show that the right hand side of \((4.3.14)\)

\[ A \equiv c \mathcal{X} \partial_+ \Phi_0 \partial_+ Y \]  

is \(Q_M\) closed. We need the following formulae:

\[
\begin{align*}
\{Q_M, c(x^+)\} &= \frac{1}{i} \partial_+ c(x^+) \\
\{Q_M, \Phi_0\} &= \frac{1}{i} \partial_+ \Phi_0 - \frac{5}{4} \partial_+ c \Phi_0 \\
\{Q_M, \partial_+ \Phi_0\} &= \partial_+ \left(\frac{1}{i} \partial_+ \Phi_0 \right) + \frac{1}{i} \partial^2_+ \Phi_0 \\
&\quad - \frac{5}{4} \partial^2_+ c \Phi_0 - \frac{5}{4} \partial_+ c \partial_+ \Phi_0 \\
\{Q_M, Y\} &= \frac{1}{i} \partial_+ Y \\
\{Q_M, \partial_+ Y\} &= \partial_+ \left(\frac{1}{i} \partial_+ Y \right) + \frac{1}{i} \partial^2_+ Y \\
\{Q_M, \mathcal{X}\} &= \frac{1}{i} \partial_+ \mathcal{X} + \frac{1}{4} \partial_+ c \mathcal{X}
\end{align*}
\]  

(4.3.16) – (4.3.21)

Using these expressions and \(\Phi_0 \partial_+ Y = 0\), we get

\[
\begin{align*}
\{Q_M, A\} &= \frac{1}{i} \partial_+ c \mathcal{X} \partial_+ \Phi_0 \partial_+ Y \\
&\quad - \left\{ \frac{1}{i} \partial_+ c \mathcal{X} \partial_+ \Phi_0 \partial_+ Y \\
&\quad - \frac{1}{4} \partial_+ c \mathcal{X} \partial_+ \Phi_0 \partial_+ Y \\
&\quad + \frac{1}{4} \partial_+ c \mathcal{X} \partial_+ Y \right\}
\end{align*}
\]

= 0.  

(4.3.22)

Thus indeed \((4.3.13)\) is BRST closed.

We now provide a further argument of more general nature which supports the result above. This argument will lead to the conclusion that as long as we do not change the ghost structure of the operator \(x_M\) it is not possible to construct a BRST invariant.

It is well-known that the total zero-level Virasoro generator \(L_0^{\text{tot}}\), including the ghost part, can be written as \(L_0^{\text{tot}} = \{b_0, Q_M\}\). On the other hand we have, for any operator \(\mathcal{O}\) with a definite global dimension, \([L_0^{\text{tot}}, \mathcal{O}] = \frac{1}{i} \partial_+ \mathcal{O}\). Combining these relations, we get

\[
\begin{align*}
[L_0^{\text{tot}}, \mathcal{O}] &= \frac{1}{i} \partial_+ \mathcal{O} \\
&= \{b_0, Q\} \mathcal{O} \\
&= \{b_0, [Q, \mathcal{O}]\} + \{Q, [b_0, \mathcal{O}]\}.
\end{align*}
\]  

(4.3.23)
Thus if $\mathcal{O}$ is $Q_M$-closed, then
\[
\frac{1}{i} \partial_+ \mathcal{O} = \{Q,\left[ b_0, \mathcal{O} \right] \} . \tag{4.3.24}
\]
This equation states that $\partial_+ \mathcal{O}$ is necessarily BRST exact and is determined solely by the part of $\mathcal{O}$ which contains the ghost $c$.

Let us apply this logic to see if we can construct a BRST invariant operator of the form
\[
\mathcal{O} = cb\mathcal{X}\Phi + \text{terms not containing } c . \tag{4.3.25}
\]
For this class of operators, we have
\[
\left[ b_0, \mathcal{O} \right] = \left[ b_0, cb\mathcal{X}\Phi \right] = b\mathcal{X}\Phi . \tag{4.3.26}
\]
Therefore
\[
\frac{1}{i} \partial_+ \mathcal{O} = \{Q, b\mathcal{X}\Phi \} = \{Q, b\} \mathcal{X}\Phi - b\{Q, \mathcal{X}\} \Phi - b\mathcal{X}\{Q, \Phi \} . \tag{4.3.27}
\]
Using the various formulae developed previously for the calculation of $x_M$, we can compute the right hand side. The result is
\[
\partial_+ \mathcal{O} = \partial_+ \left( j\mathcal{X}\Phi + \frac{1}{i} \partial_+ \mathcal{X}\Phi - \frac{1}{i} \mathcal{X}\partial_+ \Phi - \frac{3}{2} \mathcal{X}\Phi \right) + \frac{1}{i} \mathcal{X}\partial_+ \Phi_0 \partial_+ Y . \tag{4.3.28}
\]
(One can check explicitly that $\{Q, \partial_+ \mathcal{O} \} = 0$.) As expected, the expression in the parenthesis is precisely our candidate $x_M$, but the additional term cannot be written as a total derivative. The closest we can get is
\[
\frac{1}{i} \mathcal{X}\partial_+ \Phi_0 \partial_+ Y = \partial_+ \left( \frac{1}{i} \int_a^{x^+} dy^+ \mathcal{X}\partial_+ \Phi_0 \partial_+ Y(y) \right) . \tag{4.3.29}
\]
But the operator in parenthesis is not BRST invariant due to the lower limit of the integration.

Thus we have shown that as long as we keep intact the structure of the term involving the $c$-ghost, we cannot construct a BRST invariant in terms of the the fully interacting field in analogy to the free field case. The situation does not improve even when we take into account the right-moving sector.

This result would mean that the structure of the ground ring remains identical to that of the theory without interactions. Of course, one might question our use of the free
field form of the BRST operator in reaching this conclusion. However, the only part of this operator that could be plausibly affected and altered by the interactions is the term with the Liouville energy momentum tensor. We are hesitant about this possibility as it is difficult to see how the construction can be modified without upsetting all our results up to this point. After all, it is the free field form of the Liouville energy momentum tensor that we have been using in our construction of the exponential operator in section 3.2 (since we did not even know how to define the energy momentum tensor otherwise!). Despite the fact that we have been using the free field form of the BRST operator, our conclusion is not as trivial as it may seem. To underline this point, we note that, as far as the conformal properties are concerned, we do have operators such as $e^{\varphi}$ which exhibit the same conformal properties as the corresponding free-field operator. It should also be noted that, if we express the generators of the ring in terms of the fully interacting fields, their forms certainly change into complicated expressions involving $\mu^2$. Since the result of the matrix model, which incorporates the full interaction, indicates the existence of the ground ring of the free-field structure, this interpretation seems to be the correct one. The point is that the free-field is not the Liouville field, but it is a complicated combination of the Liouville field.

5 Discussion

Clearly, difficult problems remain. The rigorous construction of the exponential Liouville operator still has not been accomplished. Apart from the technical difficulties discussed at the end of section 3.2, it is far from clear whether a perturbative expansion in the cosmological constant $\mu^2$ makes sense at all. As long as these problems are not solved, there is little point in addressing other issues, such as the question of whether the Liouville energy momentum tensor can be consistently defined in terms of the interacting Liouville field and shown to coincide with the free field energy momentum tensor, or whether the transformation between the Liouville field and the free field $\psi$ is canonical also at the quantum level. The hidden quantum group structure revealed in the construction of [32] is certainly intriguing, but the existence of two mutually exclusive solutions to the locality condition remains an unsatisfactory feature. Perhaps the fact that there exist two solutions for the constant $g$ (except for $d = 1$ and $d = 25$) plays a role in resolving this issue. In any case, we find it remarkable that the weights which appear in the ground ring operators are precisely in agreement with our discretization condition (3.3.25) and such that the expansion (3.2.15) has only finitely many terms.

A possible way out may be to shelve these questions for the time being and to pursue
the study of the operator formalism for Liouville theory along the lines advocated by Gervais and collaborators [35],[36]. Rather than insisting on a rigorous construction of the Liouville exponential operator, these authors emphasize the importance of the hidden quantum group structure. This point of view is supported by the fact that many results can be deduced by requiring covariance with respect to the hidden $SL(2)_q$, i.e. without invoking the explicit form of the Liouville operator, and are found to agree with the results obtained by other methods. A detailed comparison between this approach and the results obtained in this paper certainly merits further investigation.

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Appendix Euclidean-Minkowski Conversion

In this appendix, we summarize how Euclidean and Minkowski formulations of conformal field theories are related. When there is a background charge, transcription is somewhat non-trivial. We shall mostly deal with the chiral sector.

We begin with the Euclidean case. With $z$ the “plane” coordinate, the energy-momentum tensor is expanded as

$$T(z) = \sum_n L_n z^{-n-2},$$  \hspace{1cm} (A.1)

where $L_n$’s are assumed to satisfy the “standard form” of the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \hspace{1cm} (A.2)$$

A primary field $\phi(z)$ of dimension $\Delta$ is characterized by

$$\phi(z) = \sum_n \phi_n z^{-n-\Delta}, \hspace{1cm} (A.3)$$

and

$$[L_n, \phi(z)] = z^n \left( z \frac{d}{dz} + (n+1)\Delta \right) \phi(z). \hspace{1cm} (A.4)$$
The “cylinder” coordinates \( \tilde{z}, \bar{\tilde{z}} \), which will be directly related the Minkowski light-cone coordinates, are defined by the conformal transformation

\[
\begin{align*}
  z &= e^{\tilde{z}}, & \tilde{z} &= \tau + i\sigma, \\
  \bar{z} &= e^{\bar{\tilde{z}}}, & \bar{\tilde{z}} &= \tau - i\sigma.
\end{align*}
\]

Then the energy-momentum tensor is transformed into

\[
T(\tilde{z}) = \left( \frac{dz}{d\tilde{z}} \right)^2 T(z) + \frac{c}{12} \{ z, \tilde{z} \},
\]

where \( \{ z, \tilde{z} \} \) is the Schwarzian derivative

\[
\{ z, \tilde{z} \} = \frac{z'''}{z'-\frac{3}{2} \left( \frac{z''}{z'} \right)^2}.
\]

For the transformation (A.5) above, the Schwarzian derivative is \(-\frac{1}{2}\). Thus we get

\[
T(\tilde{z}) = z^2 T(z) - \frac{c}{24},
\]

\[
L_n \tilde{z} = L_n - \frac{c}{24} \delta_{n,0}.
\]

The generators \( L_n \tilde{z} \) satisfy the algebra

\[
\left[ L_m \tilde{z}, L_n \tilde{z} \right] = (m - n) L_{m+n} \tilde{z} + \frac{c}{12} m^3 \delta_{m+n,0}
\]

that is, without a central term linear in \( m \).

The primary field \( \phi(z) \) is transformed into

\[
\phi(\tilde{z}) = \left( \frac{dz}{d\tilde{z}} \right)^\Delta \phi(z) = \sum \phi_n z^n = \sum \phi_n e^{-n\tilde{z}}.
\]

A simple calculation leads to

\[
\left[ L_n, \phi(\tilde{z}) \right] = e^{n\tilde{z}} \left( \frac{d}{d\tilde{z}} + n\Delta \right) \phi(\tilde{z}).
\]

Note that the factor in front of \( \Delta \) is changed from \( n + 1 \) to \( n \).

Minkowski formulation is obtained from Euclidean cylinder formulation by the replacement \( \tau \rightarrow i\tau \), which converts the cylinder coordinates into the light-cone coordinates:

\[
\begin{align*}
  \tilde{z} &= \tau_E + i\sigma & \rightarrow & & i(\tau_M + \sigma) = ix^+, \\
  \bar{\tilde{z}} &= \tau_E - i\sigma & \rightarrow & & i(\tau_M - \sigma) = ix^-, \\
  iz\partial & \rightarrow & \partial_+ & \equiv \frac{\partial}{\partial x^+}, \\
  i\bar{\tilde{z}}\bar{\partial} & \rightarrow & \partial_- & \equiv \frac{\partial}{\partial x^-}.
\end{align*}
\]
Here and hereafter, $\partial$ ($\bar{\partial}$) means $\partial/\partial z$ ($\partial/\partial \bar{z}$). Using these formulae, it is easy to obtain operator product expansions, such as (3.1.19), from the corresponding ones in the Euclidean plane coordinate formulation.

Before discussing the important case of a free boson with a background charge, let us briefly describe the notion of marginal deformation of Virasoro generators. Let $L_m$’s satisfy the standard form of the Virasoro algebra (A.2). We also suppose that there exists a $U(1)$ type current with the mode oscillators $\alpha_m$, which satisfy

$$\begin{align*}
[\alpha_m, \alpha_n] &= m\delta_{m+n,0}, \\
[L_m, \alpha_n] &= -n\alpha_{m+n} + (A_1 m + A_2 m^2)\delta_{m+n,0}.
\end{align*}$$

(A.17) (A.18)

Note that we allow for a term proportional to $\delta_{m+n,0}$ in the second equation. This will occur later when we deal with a system with a background charge.

With this setting one can deform the Virasoro generator in the following manner:

$$\tilde{L}_m \equiv L_m + (B_0 + B_1 m)\alpha_m + C\delta_{m,0}. \quad \text{(A.19)}$$

Then a straightforward calculation shows

$$\begin{align*}
[\tilde{L}_m, \tilde{L}_n] &= (m-n)\tilde{L}_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \\
&\quad -m^3B_1(2A_2 + B_1)\delta_{m+n,0} + m(B_0^2 + 2A_1 B_0 - 2C)\delta_{m+n,0}.
\end{align*}$$

(A.20)

Thus if we take

$$\begin{align*}
A_2 &= \begin{cases} -\frac{B_1^2}{4} & \text{if } B_1 \neq 0 \\ \text{arbitrary} & \text{if } B_1 = 0 \end{cases} \quad \text{(A.21)} \\
C &= \frac{B_0^2}{2} + A_1 B_0, \quad \text{(A.22)}
\end{align*}$$

$\tilde{L}_m$ satisfy the standard form of the Virasoro algebra with the same central charge.

Now let us consider a system of a free boson $\phi$ with a background charge $q$. In Euclidean plane coordinate formulation, $\phi(z, \bar{z})$ is expanded as

$$\phi(z, \bar{z}) = \phi_0 - i\alpha_0 \ln(z\bar{z}) + \frac{1}{i} \sum_{n \neq 0} \frac{1}{n} (\alpha_{-n} z^n + \bar{\alpha}_{-n} \bar{z}^n), \quad \text{(A.23)}$$

where the mode operators satisfy the usual commutation relations. The holomorphic energy-momentum tensor is given by

$$\begin{align*}
T(z) &= -\frac{1}{2}(\partial\phi)^2 + q\partial^2 \phi \equiv \sum L_n^E z^{-n-2}, \\
L_n^E &= L_n^{(0)} + iq(n + 1)\alpha_n, \quad \text{(A.24)} \\
L^{(0)} &= \frac{1}{2} \sum : \alpha_{n-m} \alpha_m :.
\end{align*}$$

(A.25) (A.26)
$L^E_n$ satisfy the standard form of the Virasoro algebra with $c = 1 + 12q^2$.

It is important to note that $\phi$ transforms under conformal transformation with an additional inhomogeneous term proportional to the background charge $q$:

$$[L^E_n, \phi(z)] = z^n (z \partial \phi(z) + q(n + 1)) \cdot \quad (A.27)$$

Now we make a conversion to the Minkowski formulation. Due to the the non-trivial transformation property just mentioned, $\phi$ undergoes the replacement

$$\phi(z, \bar{z}) \rightarrow \phi(x^+, x^-) - iq(x^+ + x^-) - i\pi q. \quad (A.28)$$

In terms of modes, this is equivalent to the shifts

$$\begin{align*}
\phi_0 &\rightarrow \phi_0^M - iq\pi, \\
\alpha_0 &\rightarrow \alpha_0 - iq, \\
\alpha_n &\rightarrow \alpha_n.
\end{align*} \quad (A.29)$$

To convert the energy-momentum tensor, one must take into account this replacement in addition to the transformation (A.7). After a simple calculation we obtain

$$T(x^+) = \frac{1}{2} (\partial_+ \phi)^2 - q\partial^2_+ \phi \quad \equiv \sum L^M_n e^{-inx^+}, \quad (A.30)$$

$$L^M_n = L^{(0)} + iqn\alpha_n = L^E_n - iq\alpha_n, \quad (A.32)$$

where the inessential overall additive constant $-1/24$ has been dropped. $L^M_n$ satisfies the Virasoro algebra of the form

$$[L^M_m, L^M_n] = (m - n)L^M_{m+n} + \delta_{m+n,0} \left( \frac{c}{12} (m^3 - m) + q^2 m \right), \quad (A.33)$$

which is not of the standard form. But since the difference is linear in $m$, we can shift $L^M_0$ to cancel this term. It is easy to see that

$$\tilde{L}^M_n = L^M_n + \frac{q^2}{2} \delta_{n,0}$$

$$= L^E_n - iq\alpha_n + \frac{q^2}{2} \delta_{n,0} \quad (A.35)$$

satisfies the standard form. Notice that

$$[L^E_m, \alpha_n] = -n\alpha_{m+n} + iq(m + 1)\delta_{mn}. \quad (A.36)$$
This is of the form treated in the discussion of marginal deformation of the Virasoro algebra. Indeed we can identify

\[ A_1 = iq, \quad A_2 = iq, \]
\[ B_0 = -iq, \quad B_1 = 0, \quad C = \frac{q^2}{2}, \]

and easily check that these coefficients precisely satisfy the conditions for the marginal deformation.

It is extremely useful to regard the shift (A.29) as a similarity transformation \( \mathcal{U} \) defined by

\[ \mathcal{U} = e^{-q\phi_0} e^{q\pi\alpha_0}. \] (A.38)

Then the conversion of \( \phi \) can be succinctly expressed as

\[ \phi \rightarrow \mathcal{U} \phi \mathcal{U}^{-1}. \] (A.39)

Moreover, it is not difficult to check explicitly that this operation correctly converts the Virasoro generators. Namely,

\[ \mathcal{U} L_n^E \mathcal{U}^{-1} = L_n^M, \] (A.40)

where \( L_n^E \) and \( L_n^M \) are as defined in (A.25) and (A.32). This makes obvious the previously mentioned fact that both \( L_n^E \) and \( L_n^M \) satisfy the standard form of the Virasoro algebra.

As an application of the similarity transformation, let us give the conversion formula for the exponential operator \( e^{\lambda \phi} \) carrying the conformal weight \( \Delta = -(\lambda^2/2) + \lambda q \). If we denote by : : \( E \) and : : \( S \) the Euclidean and the symmetric normal orderings respectively, a simple calculation gives

\[ \mathcal{U} : e^{\lambda \phi(z,\bar{z})} (\frac{dz}{dx^+} \frac{d\bar{z}}{dx^-})^\Delta \mathcal{U}^{-1} = : e^{\lambda \phi} : S e^{-i\pi\lambda^2/2}. \] (A.41)

Apart from a coordinate independent phase factor, the usual Euclidean normal ordering is precisely converted to the symmetric normal ordering This clearly shows the necessity of symmetric normal ordering adopted in the text.

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