SOME GENERAL INTEGRAL INEQUALITIES FOR LIPSCHITZIAN FUNCTIONS VIA CONFORMABLE FRACTIONAL INTEGRAL

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ABSTRACT. In this paper, the author establishes some Hadamard-type and Bullen-type inequalities for Lipschitzian functions via Riemann Liouville fractional integral.

1. INTRODUCTION

Hermite-Hadamard Inequality. Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a convex function defined on the interval \( I \) of real numbers and \( a, b \in I \) with \( a < b \). The following inequality

\[
\frac{f(a) + f(b)}{2} \leq \frac{1}{b - a} \int_{a}^{b} f(x) \, dx \leq \frac{f(a) + f(b)}{2}
\]

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions (see [7]). Note that some of the classical inequalities for means can be derived from (1) for appropriate particular selections of the mapping \( f \).

Ostrowski’s Inequality. Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a mapping differentiable in \( I^\circ \), the interior of \( I \), and let \( a, b \in I^\circ \) with \( a < b \). If \( |f'(x)| \leq M \), \( x \in [a, b] \), then we have the following inequality holds

\[
\left| f(x) - \frac{1}{b - a} \int_{a}^{b} f(t) \, dt \right| \leq \frac{M}{b - a} \left[ \frac{(x - a)^2 + (b - x)^2}{2} \right]
\]

for all \( x \in [a, b] \) (see [1]).
Simpson’s Inequality. Let $f : [a, b] \to \mathbb{R}$ be a four times continuously differentiable mapping on $(a, b)$ and $\|f^{(4)}\|_\infty = \sup_{x \in (a, b)} |f^{(4)}(x)| < \infty$. Then the following inequality holds:

$$\left| \frac{1}{3} \left[ \frac{f(a) + f(b)}{2} + 2f \left( \frac{a + b}{2} \right) \right] - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{1}{2880} \|f^{(4)}\|_\infty (b - a)^4$$

(see [3, 11] and therein).

Bullen’s inequality. Suppose that $f : [a, b] \to \mathbb{R}$ is a convex function on $[a, b]$. Then we have the inequalities:

$$\frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{1}{2} \left[ f \left( \frac{a + b}{2} \right) + \frac{f(a) + f(b)}{2} \right]$$

(see [9] and [16]). In what follows we recall the following definition.

**Definition 1.** A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is called an $M$-Lipschitzian function on the interval $I$ of real numbers with $M \geq 0$, if

$$|f(x) - f(y)| \leq M |x - y|$$

for all $x, y \in I$.

For some recent results are connected with Hermite-Hadamard type integral inequalities for Lipschitzian functions, see [4, 8, 9, 17, 18]. In [17], Tseng et al. established some Hadamard-type and Bullen-type inequalities for Lipschitzian functions as follows:

**Theorem 2.** Let $I$ be an interval in $\mathbb{R}$, $a \leq A \leq B \leq b$ in $I$, $V = (1 - \alpha)A + \alpha b$, $\alpha \in [0, 1]$ and let $f : I \to \mathbb{R}$ be an $L$-Lipschitzian function with $L \geq 0$. Then we have the inequality

$$\left| \alpha f(A) + (1 - \alpha) f(B) - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{LV_\alpha(A, B)}{2(b - a)}, \quad (2)$$

where

$$V_\alpha(A, B) = \begin{cases} (A - a)^2 - (A - V)^2 + (B - V)^2 + (b - B)^2, & a \leq V \leq A \leq B \leq b, \\ (A - a)^2 + (V - A)^2 + (B - V)^2 + (b - B)^2, & a \leq A \leq V \leq B \leq b, \\ (A - a)^2 + (V - A)^2 + (b - B)^2 - (V - B)^2, & a \leq A \leq B \leq V \leq b \end{cases}$$
Theorem 3. Let $I$ be an interval in $\mathbb{R}$, $a \leq A \leq B \leq C \leq b$ in $I$, $V_1 = (1-\alpha)a+\alpha b$, $V_2 = \gamma a + (\alpha + \beta) b$, $\alpha, \beta, \gamma \in [0, 1]$, $\alpha + \beta + \gamma = 1$, and let $f : I \to \mathbb{R}$ be an $L$-Lipschitzian function with $L \geq 0$. Then we have the inequality
\[
\left| \alpha f(A) + \beta f(B) + \gamma f(C) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{LV_{\alpha,\beta,\gamma}(A, B, C)}{2 (b-a)},
\]
where $V_{\alpha,\beta,\gamma}$ is defined as in [17, Section 3].

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 4. Let $f \in L[a,b]$. The Riemann-Liouville fractional integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ are defined by
\[
J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a
\]
and
\[
J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b
\]
respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$ and $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$ (see [12]).

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. For some recent results connected with fractional integral inequalities, see [2, 10, 14, 15, 19]. In [15], Sarıkaya et al. represented Hermite–Hadamard’s inequalities in fractional integral forms as follows:

Theorem 5. Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L[a,b]$. If $f$ is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold
\[
f \left( \frac{a + b}{2} \right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[ J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) \right] \leq \frac{f(a) + f(b)}{2}
\]
with $\alpha > 0$.

Definition 6. Let $\alpha \in (n, n+1)$, $n = 0, 1, 2, \ldots$ and set $\beta = \alpha - n$. Then the left conformable fractional integral of any order $\alpha > 0$ is defined by
\[
(I_{a+}^\alpha f)(x) = \frac{1}{n!} \int_a^x (x-t)^n (t-a)^{\beta-1} f(t)dt,
\]
and analogously, the right conformable fractional integral of any order $\alpha > 0$ is defined by

$$\left( I_{a}^{b} \right) f(x) = \frac{1}{\Gamma(a)} \int_{x}^{b} (b-t)^{n-1} f(t) dt.$$ 

Notice that, if $\alpha = n + 1$ then $\beta = \alpha - n = 1$ and hence $(I_{a}^{b} f)(x) = J_{a}^{b+1} f(x)$ and $(I_{a}^{b} f)(x) = J_{b+1}^{a} f(x)$. Also, if $n = 0$ and $\alpha = 1$ then $\beta = 1$ and hence $(I_{a}^{b} f)(b) = (I_{a}^{b} f)(a) = \int_{a}^{b} f(t) dt$.

The Beta function defined as follows:

$$B(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a + b)} = \int_{0}^{1} t^{a-1} (1-t)^{b-1} dt, \quad a, b > 0.$$ 

The Incomplete Beta function is defined by

$$B_{x}(a, b) = \int_{0}^{x} t^{a-1} (1-t)^{b-1} dt, \quad x \in [0, 1], a, b > 0,$$

for $x = 1$, the incomplete beta function coincides with the complete beta function. In [12], Set et al. represented Hermite–Hadamard’s inequalities for conformable fractional integrals as follows:

\textbf{Theorem 7.} Let $f : [a, b] \rightarrow \mathbb{R}$ be a function with $0 \leq a < b$ and $f \in L[a, b]$. If $f$ is a convex function on $[a, b]$, then the following inequalities for conformable fractional integrals hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha + 1)}{2(\alpha - n)!} \left[ (I_{a}^{b} f)(b) + (I_{a}^{b} f)(a) \right] \leq \frac{f(a) + f(b)}{2}. \quad (5)$$

The aim of this paper is to indicate generalizations of some integral inequalities for Lipschitzian functions via conformable fractional integral. The results are obtained in this study is a generalization of the results which are obtained in Theorem 2 and Theorem 3 by using conformable fractional integrals.

2. A GENERALIZATION OF HADAMARD AND OSTROWSKI TYPE INEQUALITIES FOR LIPSCHITZIAN FUNCTIONS VIA FRACTIONAL INTEGRALS

Throughout this section, let $I$ be an interval in $\mathbb{R}$, $a \leq x \leq y \leq b$ in $I$ and let $f : I \rightarrow \mathbb{R}$ be an $M$-Lipschitzian function. In the next theorem, let $\lambda \in [0, 1]$, $A = (1 - \lambda)a + \lambda b$, and $A_{\alpha, \beta, n}$, $\alpha > 0$, $n = 0, 1, 2$, $\beta = \alpha - n$, as follows:

(1) If $a \leq A \leq x \leq y \leq b$, then

$$A_{\alpha, \beta, n}(x, y, A) = K_{\alpha, \beta, n}(x, y, A) + L_{\alpha, \beta, n}(x, y, A).$$
(2) If \(a \leq x \leq A \leq y \leq b\), then
\[
A_{\alpha,\beta,n}(x, y, A) = K_{\alpha,\beta,n}^*(x, y, A) + L_{\alpha,\beta,n}^*(x, y, A).
\]

(3) If \(a \leq x \leq y \leq A \leq b\), then
\[
A_{\alpha,\beta,n}(x, y, A) = K_{\alpha,\beta,n}^*(x, y, A) + L_{\alpha,\beta,n}(x, y, A).
\]

where
\[
K_{\alpha,\beta,n}(x, y, A) = (A - a)^{\alpha} \left[ (x - a) B(\beta, n + 1) - (A - a) B(\beta + 1, n + 1) \right]
\]
\[
K_{\alpha,\beta,n}^*(x, y, A) = (A - a)^{\alpha} \left\{ (x - a) \left[ 2B_{\alpha-1} (\beta, n + 1) - B(\beta, n + 1) \right] \right. \\
+ (A - a) \left[ B(\beta + 1, n + 1) - 2B_{\alpha-1} (\beta + 1, n + 1) \right] \right\}, A \neq a,
\]
\[
L_{\alpha,\beta,n}(x, y, A) = 0,
\]
\[
L_{\alpha,\beta,n}^*(x, y, A) = (b - A)^{\alpha} \left[ (A - y) B(n + 1, \beta) + (b - A) B(n + 2, \beta) \right],
\]
\[
L_{\alpha,\beta,n}^*(x, y, A) = (b - A)^{\alpha} \left\{ (y - A) \left[ 2B_{\alpha-1} (n + 1, \beta) - B(n + 1, \beta) \right] \right. \\
+ (b - A) \left[ B(n + 2, \beta) - 2B_{\alpha-1} (n + 2, \beta) \right] \right\}, A \neq b,
\]
\[
L_{\alpha,\beta,n}^*(x, y, b) = 0.
\]

**Theorem 8.** Let \(x, y, \alpha, \lambda, A, A_{\alpha,\beta,n}\) and the function \(f\) be defined as above. Then we have the inequality for fractional integrals
\[
\left| \lambda^\alpha f(x) + (1 - \lambda)^\alpha f(y) - \frac{\Gamma(\alpha + 1)}{(b - a)^{\alpha} \Gamma(\alpha - n)} \left[ \int_{A}^{\beta} (t - \alpha)^{n} (t - \alpha)^{\beta-1} dt \right] \right|
\]
\[
\leq \frac{\Gamma(\alpha + 1)A_{\alpha,\beta,n}(x, y, A)}{n! (b - a)^{\alpha} \Gamma(\alpha - n)} M. \tag{6}
\]

**Proof.** Using the hypothesis of \(f\), we have the following inequality
\[
\left| \lambda^\alpha f(x) + (1 - \lambda)^\alpha f(y) - \frac{\Gamma(\alpha + 1)}{(b - a)^{\alpha} \Gamma(\alpha - n)} \left[ \int_{A}^{\beta} (t - \alpha)^{n} (t - \alpha)^{\beta-1} dt \right] \right|
\]
\[
= \frac{\Gamma(\alpha + 1)}{n! (b - a)^{\alpha} \Gamma(\alpha - n)} \int_{A}^{\beta} \left| f(x) - f(t) \right| (t - \alpha)^{n} (t - \alpha)^{\beta-1} dt \\
+ \int_{A}^{B} \left| f(y) - f(t) \right| (t - A)^{n} (b - t)^{\beta-1} dt
\]
\[
\leq \frac{\Gamma(\alpha + 1)}{n! (b - a)^{\alpha} \Gamma(\alpha - n)} \int_{A}^{\beta} \left| f(x) - f(t) \right| (t - \alpha)^{n} (t - \alpha)^{\beta-1} dt
\]
\[
\begin{align*}
\int_a^b |f(y) - f(t)| (t - A)^n (b - t)^{\beta - 1} dt \\
\leq \frac{\Gamma(\alpha + 1)M}{n! (b - a)^n} \left[ \frac{\Gamma(\alpha - n)}{\Gamma(\alpha + 1 - n)} \int_a^A |x - t| (A - t)^n (t - a)^{\beta - 1} dt \right. \\
+ \left. \int_a^b |y - t| (t - A)^n (b - t)^{\beta - 1} dt \right].
\end{align*}
\] (7)

Now using simple calculations, we obtain the following identities

\[
\begin{align*}
\int_a^A |x - t| (A - t)^n (t - a)^{\beta - 1} dt \quad \text{and} \quad \int_a^b |y - t| (t - A)^n (b - t)^{\beta - 1} dt.
\end{align*}
\]

1. If \( a \leq A \leq x \leq y \leq b \), then

\[
\int_a^A |x - t| (A - t)^n (t - a)^{\beta - 1} dt = (A - a)^n [(x - a) B(\beta, n + 1) - (A - a) B(\beta + 1, n + 1)]
\]

\[
= K_{\alpha, \beta, n}(x, y, A).
\]

2. If \( a \leq x \leq A \leq y \leq b \), then

\[
\int_a^b |y - t| (t - A)^n (b - t)^{\beta - 1} dt
\]

\[
= (b - A)^\alpha \left\{ (y - A) \left[ 2B_{\frac{n + 1}{n + \alpha}}(n + 1, \beta) - B(n + 1, \beta) \right] \right. \\
+ (b - A) \left[ B(n + 2, \beta) - 2B_{\frac{n + 1}{n + \alpha}}(n + 2, \beta) \right] \right\}
\]

\[
= L_{\alpha, \beta, n}(x, y, A).
\]

2. If \( a \leq x \leq A \leq y \leq b \), then

\[
\int_a^A |x - t| (A - t)^n (t - a)^{\beta - 1} dt
\]

\[
= (A - a)^\alpha \left\{ (x - a) \left[ 2B_{\frac{n + 1}{n + \alpha}}(\beta, n + 1) - B(\beta, n + 1) \right] \right. \\
+ (A - a) \left[ B(\beta + 1, n + 1) - 2B_{\frac{n + 1}{n + \alpha}}(\beta + 1, n + 1) \right] \right\}
\]

\[
= K_{\alpha, \beta, n}(x, y, A).
\]
and

\[
\int_{A}^{b} |y - t| (t - A)^{n} (b - t)^{\beta - 1} \, dt = (b - A)^{n} \left\{ (y - A) \left[ 2B_{\frac{\beta - 1}{\alpha - \beta}} (n + 1, \beta) - B(n + 1, \beta) \right] \right. \\
+ (b - A) \left[ B(n + 2, \beta) - 2B_{\frac{\beta - 1}{\alpha - \beta}} (n + 2, \beta) \right] \right\} = L_{\alpha, \beta, n}(x, y, A).
\]

3. If \(a \leq x \leq y \leq A \leq b\), then

\[
\int_{A}^{b} |x - t| (A - t)^{n} (t - a)^{\beta - 1} \, dt \\
= (A - a)^{n} \left\{ (x - a) \left[ 2B_{\frac{\beta - 1}{\alpha - \beta}} (\beta, n + 1) - B(\beta, n + 1) \right] \right. \\
+ (A - a) \left[ B(\beta + 1, n + 1) - 2B_{\frac{\beta - 1}{\alpha - \beta}} (\beta + 1, n + 1) \right] \right\} \\
= K_{\alpha, \beta, n}(x, y, A),
\]

and

\[
\int_{A}^{b} |y - t| (t - A)^{n} (b - t)^{\beta - 1} \, dt \\
= (b - A)^{n} [(A - y) B(n + 1, \beta) + (b - A) B(n + 2, \beta)] = L_{\alpha, \beta, n}(x, y, A).
\]

Using the inequality \(7\) and the above identities \(\int_{A}^{b} |x - t| (A - t)^{n} (t - a)^{\beta - 1} \, dt\) and \(\int_{A}^{b} |y - t| (t - A)^{n} (b - t)^{\beta - 1} \, dt\), we derive the inequality \(6\). This completes the proof. \(\square\)

Under the assumptions of Theorem 8, we have the following corollaries and remarks as follows:

**Remark 9.** In Theorem 8, if we take \(\alpha = \beta = 1\) and \(n = 0\), then the inequality \(6\) reduces the inequality \(2\) in Theorem 8 under the appropriate symbols.

**Corollary 10.** In Theorem 8, let \(\delta \in \left[\frac{1}{2}, 1\right], x = \delta a + (1 - \delta) b\) and \(y = (1 - \delta) a + \delta b\). Then, we have the inequality

\[
|\lambda^{\alpha} f(\delta a + (1 - \delta) b) + (1 - \lambda)^{\alpha} f((1 - \delta) a + \delta b) - \frac{\Gamma(\alpha + 1)}{(b-a)^{\alpha} \Gamma(\alpha - n)} \left( (I_{\alpha} f)(A) + (b I_{\alpha} f)(A) \right) | \\
\leq \frac{\Gamma(\alpha + 1) A_{\alpha, \beta, n}(\delta a + (1 - \delta) b, (1 - \delta) a + \delta b, A)}{n! (b-a)^{\alpha} \Gamma(\alpha - n)} M.
\]

(8)
Specially if we choose, if we take \( x = y = A \), then we have Ostrowski-type inequality as follows:

\[
\left| \lambda^\alpha + (1 - \lambda)^\alpha \right| f(x) - \frac{\Gamma(\alpha + 1)}{(b - a)^\alpha \Gamma(\alpha - n)} \left[ (I^a_{\alpha}) (A) + (b I^a_{\alpha}) (A) \right] \leq \frac{\Gamma(\alpha + 1)A_{\alpha,\beta,n}(x, y, A)}{n! (b - a)^\alpha \Gamma(\alpha - n)} M,
\]

where

\[
A_{\alpha,\beta,n}(x, y, A) = (x - a)^{\alpha + 1} (B(\beta, n + 1) - B(\beta + 1, n + 1)) + (b - x)^{\alpha + 1} B(n + 2, \beta).
\]

**Remark 11.** In the inequality (9), if we take \( \alpha = n + 1 \), then the inequality (9) reduces the inequality (2.4) obtained via Riemann-Liouville fractional integrals in [10, Corollary 2.1].

**Corollary 12.** We have the following weighted Hadamard-type inequalities for Lipschitzian functions via conformable fractional integrals as follows:

In the inequality (8), if we take \( \delta = 1 \), then we have

\[
\left| \lambda^\alpha f(a) + (1 - \lambda)^\alpha f(b) - \frac{\Gamma(\alpha + 1)}{(b - a)^\alpha \Gamma(\alpha - n)} \left[ (I^a_{\alpha}) (A) + (b I^a_{\alpha}) (A) \right] \right|
\leq \frac{\Gamma(\alpha + 1)A_{\alpha,\beta,n}(a, b, A)}{n! (b - a)^\alpha \Gamma(\alpha - n)} M,
\]

where

\[
A_{\alpha,\beta,n}(a, b, A) = (a - A)^{\alpha + 1} [B(\beta + 1, n + 1) - B(\beta, n + 1)]
+ (b - A)^{\alpha + 1} [B(n + 1, \beta) - B(n + 2, \beta)],
\]

in this inequality, specially if we choose \( \lambda = \frac{x - a}{b - a} \) for \( x \in [a, b] \), then

\[
\left| \frac{(x - a)^\alpha f(a) + (b - x)^\alpha f(b)}{(b - a)^\alpha} - \frac{\Gamma(\alpha + 1)}{(b - a)^\alpha \Gamma(\alpha - n)} \left[ (I^a_{\alpha}) (x) + (b I^a_{\alpha}) (x) \right] \right|
\leq \frac{\Gamma(\alpha + 1)A_{\alpha,\beta,n}(a, b, x)}{n! (b - a)^\alpha \Gamma(\alpha - n)} M,
\]

(i) if we choose \( \lambda = \frac{1}{2} \), then

\[
\left| \frac{f \left( \frac{a + b}{2} \right) - \frac{2^{\alpha - 1}\Gamma(\alpha + 1)}{(b - a)^\alpha \Gamma(\alpha - n)} \left[ (I^a_{\alpha}) \left( \frac{a + b}{2} \right) + (b I^a_{\alpha}) \left( \frac{a + b}{2} \right) \right] }{n! (b - a)^\alpha \Gamma(\alpha - n)} \right|
\leq \frac{2^{\alpha - 1}\Gamma(\alpha + 1)A_{\alpha,\beta,n}(\frac{a + b}{2}, \frac{a + b}{2}, \frac{a + b}{2})}{n! (b - a)^\alpha \Gamma(\alpha - n)} M,
\]

where

\[
A_{\alpha,\beta,n}(\frac{a + b}{2}, \frac{a + b}{2}, \frac{a + b}{2})
\]
\[
\left(\frac{b-a}{2}\right)^{\alpha+1} \left[B(\beta, n+1) - B(\beta + 1, n + 1) + B(n+2, \beta)\right].
\]

(ii) In the inequality (9), if we take \(\lambda = \frac{1}{2}\) and \(\delta = \frac{3}{4}\) then
\[
\frac{1}{2} \left[f \left(\frac{3a+b}{4} \right)+ f \left(\frac{a+3b}{4}\right)\right]
- 2^{\alpha-1} \Gamma(\alpha + 1) \left[\left(\int_0^\beta f (\frac{a+b}{2}) + \left(\frac{a+b}{2}\right)\right)\right]
\leq \frac{2^{\alpha-1} \Gamma(\alpha + 1) A_{\alpha, \beta, n}(\frac{3a+b}{4}, \frac{a+3b}{4}, \frac{a+b}{4})}{n! \Gamma(n+1)}\] 

where
\[
A_{\alpha, \beta, n}(\frac{3a+b}{4}, \frac{a+3b}{4}, \frac{a+b}{4}) = \left(\frac{b-a}{2}\right)^{\alpha+1} \left[B_{1/2}(\beta, n+1) + B_{1/2}(n + 1, \beta) - 2B_{1/2}(\beta + 1, n + 1)
-2B_{1/2}(n+2, \beta) + B(\beta + 1, n + 1) + B(n+2, \beta) - B(\beta, n+1)\right].
\]

3. A generalization of Bullen and Simpson type inequalities for Lipschitzian functions via fractional integrals

Throughout this section, let \(I\) be an interval in \(\mathbb{R}\), \(a \leq x \leq y \leq z \leq b\) in \(I\) and \(f : I \rightarrow \mathbb{R}\) be an \(M\)-lipschitzian function. In the next theorem, let \(\lambda + \eta + \mu = 1\), \(\lambda, \eta, \mu \in [0, 1]\), \(A = (1-\lambda)a + \lambda b\), \(C = \mu a + (\lambda + \eta)b\), and define \(I_{\alpha, \lambda, \eta, \mu}\), \(\alpha > 0\), as follows:

(1) If \(A \leq C \leq x \leq y \leq z\) or \(A \leq x \leq C \leq y \leq z\), then
\[
I_{\alpha, \lambda, \eta, \mu}(x, y, z) = M_{\alpha, \lambda, \eta, \mu}(x, y, z) + N_{\alpha, \lambda, \eta, \mu}(x, y, z) + O_{\alpha, \lambda, \eta, \mu}(x, y, z).
\]
(2) If \(A \leq x \leq y \leq C \leq z\), then
\[
I_{\alpha, \lambda, \eta, \mu}(x, y, z) = M_{\alpha, \lambda, \eta, \mu}(x, y, z) + N_{\alpha, \lambda, \eta, \mu}(x, y, z) + O_{\alpha, \lambda, \eta, \mu}(x, y, z).
\]
(3) If \(A \leq x \leq y \leq z \leq C\), then
\[
I_{\alpha, \lambda, \eta, \mu}(x, y, z) = M_{\alpha, \lambda, \eta, \mu}(x, y, z) + N_{\alpha, \lambda, \eta, \mu}(x, y, z) + O_{\alpha, \lambda, \eta, \mu}(x, y, z).
\]
(4) If \(x \leq A \leq C \leq y \leq z\), then
\[
I_{\alpha, \lambda, \eta, \mu}(x, y, z) = M_{\alpha, \lambda, \eta, \mu}(x, y, z) + N_{\alpha, \lambda, \eta, \mu}(x, y, z) + O_{\alpha, \lambda, \eta, \mu}(x, y, z).
\]
(5) If \(x \leq A \leq y \leq C \leq z\), then
\[
I_{\alpha, \lambda, \eta, \mu}(x, y, z) = M_{\alpha, \lambda, \eta, \mu}(x, y, z) + N_{\alpha, \lambda, \eta, \mu}(x, y, z) + O_{\alpha, \lambda, \eta, \mu}(x, y, z).
\]
(6) If \(x \leq A \leq y \leq z \leq C\), then
\[
I_{\alpha, \lambda, \eta, \mu}(x, y, z) = M_{\alpha, \lambda, \eta, \mu}(x, y, z) + N_{\alpha, \lambda, \eta, \mu}(x, y, z) + O_{\alpha, \lambda, \eta, \mu}(x, y, z).
\]
(7) If \( x \leq y \leq A \leq C \leq z \), then
\[
I_{\alpha,\eta,\mu}(x, y, z) = M^*_{\alpha,\lambda,\eta,\mu}(x, y, z) - N_{\alpha,\lambda,\eta,\mu}(x, y, z) + O^*_{\alpha,\lambda,\eta,\mu}(x, y, z).
\]

(8) If \( x \leq y \leq A \leq z \leq C \) or \( x \leq y \leq z \leq A \leq C \), then
\[
I_{\alpha,\eta,\mu}(x, y, z) = M^*_{\alpha,\lambda,\eta,\mu}(x, y, z) - N_{\alpha,\lambda,\eta,\mu}(x, y, z) + O_{\alpha,\lambda,\eta,\mu}(x, y, z).
\]

Where
\[
M_{\alpha,\lambda,\eta,\mu}(x, y, z) = (A - a)^\alpha [(x - a) B(\beta, n + 1) - (A - a) B(\beta + 1, n + 1)],
\]
\[
N_{\alpha,\lambda,\eta,\mu}(x, y, z) = (C - A)^\alpha [(y - A) B(n + 1, \beta) - (C - A) B(n + 2, \beta)],
\]
\[
O_{\alpha,\lambda,\eta,\mu}(x, y, z) = (b - C)^\alpha [(C - z) B(n + 1, \beta) + (b - C) B(n + 2, \beta)],
\]
\[
M^*_{\alpha,\lambda,\eta,\mu}(x, y, z) = (A - a)^\alpha \left\{ (x - a) \left[ 2B \frac{x - a}{x - a} (\beta, n + 1) - B(\beta, n + 1) \right] \right\}, \ A \neq a \ (\text{or } \lambda \neq 0),
\]
\[
M^*_{\alpha,\lambda,\eta,\mu}(x, y, z) = 0,
\]
\[
N^*_{\alpha,\lambda,\eta,\mu}(x, y, z) = (C - A)^\alpha \left\{ (y - A) \left[ 2B \frac{y - a}{y - a} (n + 1, \beta) - B(n + 1, \beta) \right] \right\}, \ A \neq C \ (\text{or } \eta \neq 0),
\]
\[
N^*_{\alpha,\lambda,\eta,\mu}(x, y, z) = 0,
\]
\[
O^*_{\alpha,\lambda,\eta,\mu}(x, y, z) = (b - C)^\alpha \left\{ (z - C) \left[ 2B \frac{z - c}{z - c} (n + 1, \beta) - B(n + 1, \beta) \right] \right\}, \ C \neq b \ (\text{or } \mu \neq 0),
\]
\[
O^*_{\alpha,\lambda,\eta,\mu}(x, y, z) = 0.
\]

**Theorem 14.** Let \( x, y, z, \lambda, \eta, \mu, A_1, A_2, A_{\alpha,\lambda,\eta,\mu} \) and the function \( f \) be defined as above. Then we have the inequality
\[
|\lambda^\alpha f(x) + \eta^\alpha f(y) + \mu^\alpha f(z) - \frac{\Gamma(\alpha + 1)}{(b - a)^\alpha \Gamma(\alpha - n)} [(I_0^\alpha f)(A) + (C I_0 f)(A) + (b I_0 f)(C)] | \\
\leq \frac{\Gamma(\alpha + 1) I_{\alpha,\lambda,\eta,\mu}(x, y, z)}{n! (b - a)^\alpha \Gamma(\alpha - n)} M.
\]

**Proof.** Using the hypothesis of \( f \), we have the inequality
\[
|\lambda^\alpha f(x) + \eta^\alpha f(y) + \mu^\alpha f(z) - \frac{\Gamma(\alpha + 1)}{(b - a)^\alpha \Gamma(\alpha - n)} [(I_0^\alpha f)(A) + (C I_0 f)(A) + (b I_0 f)(C)] | \\
= \frac{\Gamma(\alpha + 1)}{n! (b - a)^\alpha \Gamma(\alpha - n)} \int_a^A [f(x) - f(t)] (A - t)^\alpha (t - a)^{\beta - 1} dt
\]
Now, using simple calculations, we obtain the following identities

\begin{align*}
\int_a^A |x-t| (A-t)^{\beta-1} \, dt &= (A-a)^\alpha \left[ (x-a) B(\beta, n+1) - (A-a) B(\beta+1, n+1) \right] \\
&= M_{\alpha,\lambda,\eta,\mu}(x, y, z), \\
\int_a^C |y-t| (t-A)^{\beta-1} \, dt &= (C-A)^\alpha \left[ (y-A) B(n+1, \beta) - (C-A) B(n+2, \beta) \right] \\
&= N_{\alpha,\lambda,\eta,\mu}(x, y, z), \\
\int_c^b |z-t| (t-C)^{\beta-1} \, dt &= (b-C)^\alpha \left\{ (z-C) \left[ 2B_{\frac{C}{b-C}}(n+1, \beta) - B(n+1, \beta) \right] \right. \\
&\quad + (b-C) \left[ \left( \frac{B(n+2, \beta) - 2B_{\frac{C}{b-C}}(n+2, \beta)}{2} \right) \right] \\
&\quad \left. + \frac{\Gamma(\alpha+1)}{n! (b-a)^\alpha \Gamma(\alpha-n)} \right|_a^A |f(x) - f(t)| (A-t)^{\beta-1} \, dt \\
&\quad + \int_a^C |f(y) - f(t)| (t-A)^{\beta-1} \, dt + \int_C^b |f(z) - f(t)| (t-C)^{\beta-1} \, dt \\
&\quad \leq \frac{\Gamma(\alpha+1)}{n! (b-a)^\alpha \Gamma(\alpha-n)} \right|_a^A |f(x) - f(t)| (A-t)^{\beta-1} \, dt \\
&\quad + \int_a^C |f(y) - f(t)| (t-A)^{\beta-1} \, dt + \int_C^b |f(z) - f(t)| (t-C)^{\beta-1} \, dt \\
&\quad \leq \frac{\Gamma(\alpha+1)M}{n! (b-a)^\alpha \Gamma(\alpha-n)} \right|_a^A |x-t| (A-t)^{\beta-1} \, dt \\
&\quad + \int_a^C |y-t| (t-A)^{\beta-1} \, dt + \int_C^b |z-t| (t-C)^{\beta-1} \, dt.
\end{align*}

Now, using simple calculations, we obtain the following identities

\begin{align*}
\int_a^A |x-t| (A-t)^{\beta-1} \, dt &= (A-a)^\alpha \left[ (x-a) B(\beta, n+1) - (A-a) B(\beta+1, n+1) \right] \\
&= M_{\alpha,\lambda,\eta,\mu}(x, y, z), \\
\int_a^C |y-t| (t-A)^{\beta-1} \, dt &= (C-A)^\alpha \left[ (y-A) B(n+1, \beta) - (C-A) B(n+2, \beta) \right] \\
&= N_{\alpha,\lambda,\eta,\mu}(x, y, z), \\
\int_c^b |z-t| (t-C)^{\beta-1} \, dt &= (b-C)^\alpha \left\{ (z-C) \left[ 2B_{\frac{C}{b-C}}(n+1, \beta) - B(n+1, \beta) \right] \right. \\
&\quad + (b-C) \left[ \left( \frac{B(n+2, \beta) - 2B_{\frac{C}{b-C}}(n+2, \beta)}{2} \right) \right] \\
&\quad \left. + \frac{\Gamma(\alpha+1)}{n! (b-a)^\alpha \Gamma(\alpha-n)} \right|_a^A |f(x) - f(t)| (A-t)^{\beta-1} \, dt \\
&\quad + \int_a^C |f(y) - f(t)| (t-A)^{\beta-1} \, dt + \int_C^b |f(z) - f(t)| (t-C)^{\beta-1} \, dt \\
&\quad \leq \frac{\Gamma(\alpha+1)M}{n! (b-a)^\alpha \Gamma(\alpha-n)} \right|_a^A |f(x) - f(t)| (A-t)^{\beta-1} \, dt \\
&\quad + \int_a^C |f(y) - f(t)| (t-A)^{\beta-1} \, dt + \int_C^b |f(z) - f(t)| (t-C)^{\beta-1} \, dt \\
&\quad \leq \frac{\Gamma(\alpha+1)M}{n! (b-a)^\alpha \Gamma(\alpha-n)} \right|_a^A |x-t| (A-t)^{\beta-1} \, dt \\
&\quad + \int_a^C |y-t| (t-A)^{\beta-1} \, dt + \int_C^b |z-t| (t-C)^{\beta-1} \, dt.
\end{align*}
= O_{\alpha,\lambda,\eta,\mu}^*(x, y, z).

(2) If \( A \leq x \leq y \leq C \leq z \), then we have
\[
\int_{a}^{A} |x - t| (A - t)^{n} (t - a)^{\beta - 1} \, dt = M_{\alpha,\lambda,\eta,\mu}(x, y, z),
\]
\[
\int_{a}^{A} |y - t| (A - t)^{n} (C - t)^{\beta - 1} \, dt
\]
\[
= (C - A)^{n} \left\{ (y - A) \left[ 2B_{y-A} (n + 1, \beta) - B(n + 1, \beta) \right] \\
+ (C - A) \left[ B(n + 2, \beta) - 2B_{y-A} (n + 2, \beta) \right] \right\}
\]
\[
= N_{\alpha,\lambda,\eta,\mu}^*(x, y, z),
\]
and
\[
\int_{b}^{C} |z - t| (t - C)^{n} (b - t)^{\beta - 1} \, dt = O_{\alpha,\lambda,\eta,\mu}^*(x, y, z).
\]

(3) If \( A \leq x \leq y \leq z \leq C \), then we have
\[
\int_{a}^{A} |x - t| (A - t)^{n} (t - a)^{\beta - 1} \, dt = M_{\alpha,\lambda,\eta,\mu}(x, y, z),
\]
\[
\int_{a}^{A} |y - t| (A - t)^{n} (C - t)^{\beta - 1} \, dt = N_{\alpha,\lambda,\eta,\mu}^*(x, y, z),
\]
and
\[
\int_{b}^{C} |z - t| (t - C)^{n} (b - t)^{\beta - 1} \, dt
\]
\[
= (b - C)^{n} \left[ [(C - z) B(n + 1, \beta) + (b - C) B(n + 2, \beta) \right]
\]
\[
= O_{\alpha,\lambda,\eta,\mu}(x, y, z).
\]

(4) If \( x \leq A \leq C \leq y \leq z \), then we have
\[
\int_{a}^{A} |x - t| (A - t)^{n} (t - a)^{\beta - 1} \, dt = M_{\alpha,\lambda,\eta,\mu}^*(x, y, z),
\]
\[
\int_{a}^{A} |y - t| (A - t)^{n} (C - t)^{\beta - 1} \, dt = N_{\alpha,\lambda,\eta,\mu}(x, y, z),
\]
and
\[
\int_{b}^{a} |z - t| (t - C)^n (b - t)^{\beta - 1} dt = O_{\alpha, \lambda, \eta, \mu}^{*}(x, y, z).
\]

(5) If \( x \leq A \leq y \leq C \leq z \), then we have
\[
\int_{A}^{a} |x - t| (A - t)^n (t - a)^{\beta - 1} dt = M_{\alpha, \lambda, \eta, \mu}^{*}(x, y, z),
\]
\[
\int_{C}^{A} |y - t| (t - A)^n (C - t)^{\beta - 1} dt = N_{\alpha, \lambda, \eta, \mu}^{*}(x, y, z),
\]
and
\[
\int_{C}^{a} |z - t| (t - C)^n (b - t)^{\beta - 1} dt = O_{\alpha, \lambda, \eta, \mu}^{*}(x, y, z).
\]

(6) If \( x \leq A \leq y \leq z \leq C \), then we have
\[
\int_{A}^{a} |x - t| (A - t)^n (t - a)^{\beta - 1} dt = M_{\alpha, \lambda, \eta, \mu}^{*}(x, y, z),
\]
\[
\int_{C}^{A} |y - t| (t - A)^n (C - t)^{\beta - 1} dt = N_{\alpha, \lambda, \eta, \mu}^{*}(x, y, z),
\]
and
\[
\int_{C}^{a} |z - t| (t - C)^n (b - t)^{\beta - 1} dt = O_{\alpha, \lambda, \eta, \mu}^{*}(x, y, z).
\]

(7) If \( x \leq y \leq A \leq C \leq z \), then we have
\[
\int_{A}^{a} |x - t| (A - t)^n (t - a)^{\beta - 1} dt = M_{\alpha, \lambda, \eta, \mu}^{*}(x, y, z),
\]
\[
\int_{C}^{A} |y - t| (t - A)^n (C - t)^{\beta - 1} dt = -N_{\alpha, \lambda, \eta, \mu}(x, y, z)
\]
and
\[
\int_{C}^{a} |z - t| (t - C)^n (b - t)^{\beta - 1} dt = O_{\alpha, \lambda, \eta, \mu}^{*}(x, y, z).
\]
If \( x \leq y \leq A \leq z \leq C \) or \( x \leq y \leq z \leq A \leq C \), then we have
\[
\int_a^A |x - t| (A - t)^n (t - a)^{\beta - 1} \, dt = M_{\alpha, \lambda, \eta, \mu}^*(x, y, z),
\]
\[
\int_A^C |y - t| (t - A)^n (C - t)^{\beta - 1} \, dt = -N_{\alpha, \lambda, \eta, \mu}(x, y, z),
\]
and
\[
\int_C^b |z - t| (t - C)^n (b - t)^{\beta - 1} \, dt = O_{\alpha, \lambda, \eta, \mu}(x, y, z).
\]

Using the inequality (11) and the above identities, we derive the inequality (10). This completes the proof.

Under the assumptions of Theorem 14, we have the following corollaries and remarks as follows:

**Remark 15.** In Theorem 14, if we take \( \alpha = \beta = 1 \) and \( n = 0 \), then the inequality (10) reduces the inequality (3) in Theorem 3 under the appropriate symbols.

**Corollary 16.** In Theorem 14, let \( x = a + (1 - \delta) b \), \( y = \alpha b + \beta b \) and \( z = (1 - \delta) a + \beta b \). Then, we have the inequality
\[
\left| \lambda^\alpha f(\delta a + (1 - \delta) b) + \eta^\alpha f\left(\frac{a + b}{2}\right) + \mu^\alpha f((1 - \delta) a + \beta b) \right|
\leq \frac{\Gamma(\alpha + 1)}{(b - a)^n \Gamma(\alpha - n)} \left[ (I_a^n f)(A) + (C I_a f)(A) + (b I_a f)(C) \right]
\leq \frac{\Gamma(\alpha + 1) I_{\alpha, \lambda, \eta, \mu}(\delta a + (1 - \delta) b, \frac{a + b}{2}, (1 - \delta) a + \beta b)}{n! (b - a)^n \Gamma(\alpha - n)} M.
\]

**Corollary 17.** In Corollary 16, if we take \( \delta = 1 \), \( \lambda = \mu = \frac{\theta}{2} \) and \( \eta = 1 - \theta \) with \( \theta \in [0, 1] \), then we have the following weighted Bullen-type inequality for \( M \)-Lipschitzian functions via fractional integrals
\[
\left| \left(\frac{\theta}{2}\right)^\alpha \left( f(a) + f(b) \right) + (1 - \theta)^\alpha f\left(\frac{a + b}{2}\right) \right|
\leq \frac{\Gamma(\alpha + 1)}{(b - a)^n \Gamma(\alpha - n)} \left[ (I_a^n f)(A) + (C I_a f)(A) + (b I_a f)(C) \right]
\leq \frac{\Gamma(\alpha + 1) I_{\alpha, \frac{\theta}{2}, 1 - \theta, \frac{\theta}{2}}(a, \frac{a + b}{2}, b)}{n! (b - a)^n \Gamma(\alpha - n)} M,
\]

(12)
where
\[ I_{n, \alpha} \left( \frac{a + b}{2}, \frac{a + b}{2} \right) = (b - a)^{\alpha + 1} \left\{ \left( \frac{\alpha}{2} \right)^{\alpha + 1} \left[ B \left( \beta + 1, n + 1 \right) + B \left( n + 1, \beta \right) - B \left( n + 2, \beta \right) \right] + (1 - \theta)^{\alpha + 1} \left[ B \left( n + 2, \beta \right) - \frac{1}{2} B \left( n + 1, \beta \right) + B_{1/2} \left( n + 1, \beta \right) \right] \right\}. \]

Specially, in the inequality (12), if we take \( n = 0 \) and \( \alpha = \beta = 1 \), then the inequality (12) reduces to the following general inequality for \( M \)-Lipschitzian functions
\[
\left| \frac{\theta}{2} \left( f(a) + f(b) \right) + (1 - \theta) f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(t) dt \right| \leq \frac{M}{4} (b - a) \left[ 2b^2 + (1 - \theta)^2 \right].
\]

**Remark 18.** In the inequality (12), if we take \( n = 0 \) and \( \alpha = 1 \), then the inequality (12) reduces the inequality obtained via Riemann-Liouville fractional integrals in [10, Corollary 3.2].

**Remark 19.** In the inequality (13), if we take \( \theta = \frac{1}{3} \), then the inequality (13) reduces to the following Simpson-type inequality for \( M \)-Lipschitzian functions
\[
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a + b}{2} \right) + f(b) \right] - \frac{1}{b - a} \int_a^b f(t) dt \right| \leq \frac{M}{6} (b - a).
\]

**Remark 20.** In the inequality (13), if we take \( \theta = \frac{1}{2} \), then the inequality (13) reduces to the following Bullen type inequality for \( M \)-Lipschitzian functions
\[
\left| \frac{1}{2} \left[ f(a) + f(b) \right] + f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(t) dt \right| \leq \frac{3M}{16} (b - a).
\]

**Remark 21.** In the inequality (13), if we take \( \theta = 0 \), then the inequality (13) reduces to the following Midpoint type inequality for \( M \)-Lipschitzian functions
\[
\left| f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(t) dt \right| \leq \frac{M}{4} (b - a).
\]

**Remark 22.** In the inequality (13), if we take \( \theta = 1 \), then the inequality (13) reduces to the following Trapezoid type inequality for \( M \)-Lipschitzian functions
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(t) dt \right| \leq \frac{M}{2} (b - a).
\]
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