Four-Dimensional Chern-Simons and Gauged Sigma Models

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January 26, 2024

Abstract

In this paper we introduce a new method for generating gauged sigma models from 4d CS theory and give a unified action for a class of these models. We begin with a review of recent work by several authors on the classical generation of integrable sigma models from four dimensional Chern-Simons theory. This approach involves introducing classes of two-dimensional defects into the bulk on which the gauge field must satisfy certain boundary conditions. One finds integrable sigma models from 4d CS theory by substituting the solutions to the equations of motion back into the action. The integrability of these sigma models is guaranteed because the gauge field is gauge equivalent to the Lax connection of the sigma model. By considering a theory with two 4d CS fields coupled together on two-dimensional surfaces in the bulk we are able to introduce new classes of ‘gauged’ defects. By solving the bulk equations of motion we find a unified action for a set of genus zero integrable gauged sigma models. The integrability of these models is guaranteed as the gauge fields remain gauge equivalent to Lax connections. Finally, we consider two examples in which we derive the gauged Wess-Zumino-Witten model with BF term and the nilpotent gauged Wess-Zumino-Witten models. This latter model is of note as one can find the conformal Toda models from it.

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1 Introduction

Over the last two decades, several groups have turned their focus to the question of whether one can use gauge theories to identify properties of conformal field theories (CFTs), vertex operator algebras, and integrable models. We know of three such examples: the first, by Fuchs et al. in [1–5], uses topological field theories to analyse conformal field theories. The second, by Beem et al., has shown a deep relationship between $\mathcal{N} = 2$ superconformal field theories in four dimensions and vertex operator algebras [6, 7]. The final example began with the work of Costello in [8, 9] and has since been expanded upon by Costello, Witten, and Yamazaki in [10–12]. In this series of papers the authors introduced a new gauge theory, called four-dimensional Chern-Simons theory (4d CS), and used it to explain several properties of two dimensional integrable models. In [10, 11] the authors were able to find the $R$-matrix and Quantum group structure of lattice and particle scattering models from Wilson lines on four-dimensional Chern-Simons theory. A fourth paper in this series [13], has also shown 't-Hooft operators are related to $Q$-operators.

We are interested in the third paper [12] in which the authors proved that classical four-dimensional Chern-Simons theory, with suitably chosen two-dimensional defects, reduces to a two-dimensional integrable sigma model. One finds these sigma models by solving the 4d CS equations of motion in terms of a group element $\hat{g}$, which becomes the sigma model’s field. The dynamical degrees of freedom of the sigma model are the values of this group element on certain defects. Integrable sigma models are of particular interest given they exhibit many of the phenomena present in non-abelian gauge theories, such as confinement, instantons or anomalies [14–17] while their integrability ensures they are exactly solvable [18–21].

This result was extended by Bittleston and Skinner in [22] where it was shown that higher dimensional Chern-Simons models can be used to generate higher dimensional integrable sigma models.

These constructions are analogous to the construction of the Wess-Zumino-Witten (WZW) model as the boundary theory of three-dimensional Chern-Simons given in [23]. However, what makes these constructions different is that these models sit on defects in the bulk rather than on the boundary.

Alongside these developments, Vicedo, in [24], observed that the gauge field $A$ of four-dimensional Chern-Simons theory is gauge equivalent to the Lax connection $A$ of the two-dimensional integrable sigma model (with or without a spectral parameter). This result was expanded upon in [25] by Delduc, Lacroix, Magro and Vicedo (DLMV) where they construct a general action for genus zero integrable sigma models called the unified sigma model action. This result is remarkable for two reasons; the first is that the Lax connection of an integrable sigma model can be found geometrically by solving the equations of motion of four-dimensional Chern-Simons theory; and the second is that it gives a general action from which the actions in this class of sigma models can be found - provided their Lax connections are known. We will refer to this construction as the DLMV construction throughout this paper.

In all of this work, the inability to generate gauged sigma models whose target spaces are cosets (manifolds of the form $G/H$ where $G$ and $H \subseteq G$ are groups) has been mentioned several times. This is with the exception of symmetric space models [12] or $\lambda$- and $\eta$-deformations (which can realise coset models for certain subgroups of $H \subset G$) [25–27]. Gauged sigma models are of particular interest given they include the GKO constructions [28–30] from which one can possibly find all rational conformal field theories (RCFTs).

The main result of this paper is to prove that one can generate coset sigma models by coupling together two four-dimensional Chern-Simons theories on new classes of two dimensional defects which are collectively called ‘gauged defects’. We call this theory doubled four-dimensional Chern-Simons theory (doubled 4d CS). This theory consists of two four-dimensional Chern-Simons theories, one for a group $G$ with gauge field $A$, and a second for a group $H \subset G$ with gauge field $B$; the two models are coupled only on the gauged defects. This result is analogous to the work of Moore and Seiberg in [31] where it was shown the GKO constructions are the boundary theory of a doubled three-dimensional Chern-Simons model - see also [32].

As before, $A$ and $B$ are gauge equivalent to two-dimensional Lax connections $A$ and $B$ on the defects.

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1 In this paper the process of solving the equations of motion is referred to as solving along the fibre.
(with or without spectral parameters), and the equations of the motion of the model are the Lax equations for $A$ and $B$, together with “boundary conditions” on the defects. By following arguments similar to those made by Delduc et al. in [25] we find a unified gauged sigma model from which a large class of integrable gauged sigma models can be found.

It could have been expected that a gauged 2d WZW model can be found from doubled four-dimensional Chern-Simons theory. Firstly, the gauged WZW model can be found from the difference of two WZW models (see appendix C) each of which can be found from four-dimensional Chern-Simons theory. The second reason is that four-dimensional Chern-Simons theory is T-dual to three-dimensional Chern-Simons, as was shown by Yamazaki in [33]. Since the GKO constructions are the boundary theory of a doubled three-dimensional Chern-Simons, it is natural to expect that they can be found in four-dimensional Chern-Simons theory.

The structure of this paper is as follows: in section 2 we define four-dimensional Chern-Simons theory, deriving its equations of motion and boundary conditions amongst other properties. In section 3 we review the construction of integrable sigma models by Delduc et al. in four-dimensional Chern-Simons theory. In this construction the authors solve four-dimensional Chern-Simons theory’s equations of motion and substitute them back into the action; where they differ from Costello et al. is in the choice of gauge in which they do these calculations. In section 4 we define the doubled four-dimensional Chern-Simons theory, deriving the gauged defects and describing its gauge invariance.

In section 5 we use the DLMV approach to derive the unified gauged sigma model and construct the normal gauged WZW model with a BF term and a nilpotent gauged WZW model. These examples are notable for two reasons: the first is that the normal gauged WZW model gives an action for the GKO constructions as described in [34–38]; the second reason is that the Toda fields theories can be found from both of these action. In the former case this is as a quantum equivalence with the $G_k \times G_1/G_{k+1}$ GKO model, as shown in [39], while in the latter case this is proven via a Hamiltonian reduction as shown in [40]. It was also shown in [40] that one can find the $W$-algebras from the nilpotent gauged WZW model.

In section 6 we summarise our results and comment on a few potential directions of this research.

Acknowledgements

I would like to thank my supervisor Gérard Watts for proposing this problem and the support he has provided during our many discussions. I would also like to thank Ellie Harris and Rishi Mouland for our discussions; Nadav Drukker who kindly provided comments on a previous version of this manuscript; Benoît Vicedo for his comments; and finally the anonymous referees whose comments we feel have greatly improved the following work. This work was funded by the STFC grant ST/T506187/1.

2 The four-dimensional Chern-Simons Theory

In this section we define the four-dimensional Chern-Simons theory, its regularisation and derive the equations of motion. We conclude the section with a discussion of the various defect boundary conditions that we use and describe the gauge-invariance of the action.

2.1 The Action

4d CS theory is a theory of a gauge field $A = A_\mu dx^\mu$. It is defined on a four-dimensional manifold of the form $M = \Sigma \times \tilde{C}$ where $\Sigma$ and $\tilde{C}$ are each two-dimensional surfaces. Further, $\tilde{C}$ is a punctured complex manifold with coordinate $z$, it is equipped with a meromorphic 1-form $\omega = \varphi(z) dz$ where $\varphi(z)$ is a rational function. We denote the set of zeros of $\omega$ by $Z$ and the set of poles by $P$. The punctures of $\tilde{C}$ occur at the zeros of $\omega$, i.e. $\tilde{C} = C \setminus Z$, where $C$ is an un-punctured complex manifold.
The Chern-Simons three form is:

\[ \text{CS}(A) = \left< A, dA + \frac{2}{3} A \wedge A \right> , \] (2.1)

and the action of the 4d CS theory is the integral:

\[ S(A) = \frac{1}{2\pi \hbar} \int_{\Sigma \times \tilde{C}} \omega \wedge \left< A, dA + \frac{2}{3} A \wedge A \right> , \] (2.2)

The gauge field \( A \) is a connection on a principal bundle over the four-dimensional manifold \( M = \Sigma \times C \), with complex Lie group \( G \). We take \( \langle \cdot, \cdot \rangle \) to be a non-degenerate symmetric bilinear form proportional to the killing form of the complex Lie algebra \( g \). The gauge field \( A_\mu = A_\mu^a T^a \) is in the adjoint representation of \( g \), and basis elements \( T^a \) are normalised such that \( T^a, T^b = \delta^{ab} \). Note, when both entries in \( \langle \cdot, \cdot \rangle \) are differential forms there is an implicit wedge product between the two entries. We will discuss two classes of the 4d CS action in the following and refer to (2.2) as the standard action, or theory, for short.

By analogy with three-dimensional Chern-Simons, we call \( \hbar \) the ‘level’; although irrelevant to classical 4d CS, \( \hbar \) will be relevant in section 4 when we introduce a second 4d CS field \( B \).

While \( \Sigma \) can be more general, we will usually take it to be \( \mathbb{R}^2 \). Generically, the theory does not depend on a metric on \( \Sigma \), but we find it helpful to write the coordinates on \( \Sigma \) as \( x^\pm \) as the result of the construction is a two-dimensional Lorentzian theory in which \( x^\pm \) are “light-cone coordinates”. We will see that \( \Sigma \) is the space-time of our sigma models.

The integrable models one can generate using 4d CS depend not only upon the choice of \( G \), but also the choice of \( b \) and 1-form \( \omega \). If \( b \) is a Riemann surface of genus \( g \), the Riemann-Roch theorem states that the number \( N_Z \) of zeros and \( N_P \) of poles (counted with multiplicity) of \( \omega \) satisfy the equation:

\[ N_Z - N_P = 2g - 2 . \] (2.3)

In this paper we will only discuss genus zero integrable field theories and follow [12] by fixing \( \tilde{C} \) to be the punctured Riemann sphere, \( \tilde{CP}^1 \), with the coordinates \( z \) and \( \bar{z} \). We restrict \( \omega \) to have at most double poles. The presence of poles and zeros of \( \omega \) requires the consideration of the behaviour of the field \( A \) near these locations, which we call “boundary conditions”. A point in \( \tilde{CP}^1 \) is a two-dimensional surface in \( \Sigma \times \tilde{CP}^1 \), therefore the poles and zeros of \( \omega \) and their boundary conditions define two-dimensional defects in the theory. We call the defects associated to zeros type A defects and those associated to poles, type B defects. We discuss these in sections 2.3 and 2.4 respectively.

Let \( P^\text{fin} \) denote the set of poles of \( \omega \) not including infinity, then \( \omega \) on \( \tilde{CP}^1 \) is of the form:

\[ \omega = \eta_\infty dz + \sum_{q \in P^\text{fin}} \sum_{l=0}^{n_q-1} \frac{\eta_q^{(l)}}{(z - q)^{l+1}} dz , \text{ where } \eta_q^{(l)} = \text{res}_q ((z - q)^l \varphi(z)) , \] (2.4)

and \( \eta_\infty = \text{res}_0 (\varphi(1/z)/z) \) while \( n_q \) denotes the order of the pole \( q \). We have restricted ourselves to at most doubles poles, including at \( \infty \), which is why there are no terms of \( \omega \) with degree greater than zero.

Let \( m_\zeta \) be the multiplicity of the zero at \( \zeta \in \mathbb{Z} \). We clearly have

\[ N_P = \sum_{q \in P} n_q , \quad N_Z = \sum_{\zeta \in \mathbb{Z}} m_\zeta . \] (2.5)

\(^2\)One should note that this is only possible when the adjoint representation is non-trivial. If the adjoint representation is degenerate, such as for \( U(1) \), then one must use an alternative representation.
Furthermore, since \( g = 0 \) we have \( N_P = N_Z + 2 \) and it follows that the number of poles (counted with multiplicity) in \( P \) is always greater than \( N_Z \), which could be zero.

Finally, we will only consider \( \omega \) with a simple or double pole at infinity. Since \( N_P = N_Z + 2 \), \( \omega \) must have at least one pole. If there is no pole at infinity, we pick the location of a pole: \( z = q \), say. Noting then that the isometry group of the Riemann sphere consists of Möbius transformations, one can send the pole \( q \) to infinity by the transformation \( z \rightarrow (1/z) + q \) (since inversions and translations are Möbius transformations). From here on, we assume \( \omega \) has a pole at infinity and \( P = P_{\text{fin}} \cup \{ \infty \} \).

Before deriving the equations of motion we emphasise two important facts. The first is that because (2.2) is constructed from wedge products, and contains no metric, one might reasonably expect (2.2) to be invariant under all diffeomorphisms, or in the vernacular “topological” - this is not the case. This is because \( \varphi(z) \) does not transform as a vector, unlike \( A_\mu \), and thus \( \omega \) is not topological. Hence, the action is not topological, but rather “semi-topological”, as it is invariant under all diffeomorphisms of \( \Sigma \).

The second fact is that (2.2) has an additional gauge invariance. The \( z \) components of the gauge field and the exterior derivative \( d, A_z dz \) and \( dz \partial_z \), both fall out of (2.2) because \( \omega = \varphi(z) dz \) and \( dz \wedge dz = 0 \). This means the action has an unusual gauge invariance under the transformation:

\[
A_z \rightarrow A_z + \chi_z ,
\]

where \( \chi_z \) can be any \( g \) valued function. As a result of this gauge invariance, all field configurations of \( A_z \) are gauge equivalent, allowing us to set \( A_z = 0 \). Thus, in the following the gauge field \( A \) is given:

\[
A = A_\Sigma + \tilde{A},
\]

where \( \tilde{A} = A_z d\bar{z} \) and \( A_\Sigma = A_z dx^+ + A_\Sigma dx^- \) - the restriction of \( A \) to \( \Sigma \).

We will also drop the \( dz \) terms from the exterior derivative \( d \). In general, one has

\[
d = d_\Sigma + \bar{\partial} + \partial ,
\]

where \( d_\Sigma \) is the exterior derivative on \( \Sigma \) and \( \bar{\partial} \) and \( \partial \) are the Dolbeault operators on \( C \). However, we can drop the \( \partial \) term since the exterior derivative \( d \) appears either in the action containing \( \omega \) (and \( dz \wedge dz = 0 \)) or in gauge transformations of \( A \) (and we will always set \( A_z \) to zero). By an abuse of notation, we will also write this modified exterior derivative simply as \( d \),

\[
d = d_\Sigma + \bar{\partial} .
\]

We note that \( \partial \) and \( \bar{\partial} \) act on functions as \( dz \partial_z \) and \( d\bar{z} \partial_{\bar{z}} \) respectively.

### 2.1.1 Regularisation and Localisation

When performing our analysis we need to be careful to ensure the action is well defined. In particular, the action needs to be finite for the field configurations we study, otherwise we will not realise sigma models. This is not generically the case if \( \omega \) has more than simple poles, as we now review. The derivation of the results in this subsection can be found in [41].

Let \( \Lambda = \sum f(z, \bar{z}, \xi) dz \wedge dx^+ \wedge dx^- \) and assume that any poles of \( f \) are at the zeros of \( \omega \). Further still, take the degree of a pole of \( f \) to be less than or equal to the degree of the zero of \( \omega \). The top form \( \omega \wedge \Lambda \) is given by:

\[
\omega \wedge \Lambda = \eta_{\infty} z \partial \Lambda + \sum_{q \in P_{\text{fin}}} \sum_{l=1}^{n_q} \frac{\eta_l}{l!} \frac{dz}{z-q} \wedge \partial_z^l \Lambda + \partial \psi ,
\]

\[\text{We will make these assumptions of } CS(A) \text{ in the following, allowing us to use the results of this subsection.}\]
where:

$$\psi = -\eta_\infty z \Lambda + \sum_{q \in P^{\text{sing}}} \sum_{l=1}^{n_r} \sum_{r=0}^{l-1} \frac{(-1)^{l-r}}{r! (l-r)!} \partial^l z \Lambda^{l-1} \left( \frac{\eta_q^{(l)}}{z - q} \right).$$  \tag{2.11}$$

with $P^{\text{sing}} \subseteq P^{\text{fin}}$ the set of poles whose multiplicity is greater than one; $\partial \psi$ is singular at these poles. Clearly, we must regularise $\int_{\Sigma \times \mathbb{C}P^1} \omega \wedge \Lambda$ to ensure any integral of this form, such as the 4d CS action, is well defined.

There are two ways in which the singular part $\partial \psi$ can be removed from the action, ensuring it is regular. The first, implicitly used in [12], is to restrict the bundles we consider to those satisfying boundary conditions such that $\psi$ is regular. This ensures integrals of $\partial \psi$ vanish since $\mathbb{C}P^1$ is compact. The second method is to subtract $\partial \psi$ from the action, as in [41]; we use this approach.

From here, it is simple to show the second term on the right hand side of (2.10) is regular by changing coordinates to polar coordinates centred on $q$. Since $\Lambda$ contains $dz$ it follows that $dz \wedge \partial^l_z \Lambda$ is proportional to $dz \wedge d\bar{z} = 2i r dr \wedge d\theta$ thus cancelling the pole from $z - q = r e^{i \theta}$. The same argument applies to the first term if one works in the coordinates $z = r^{-1} e^{-i \theta}$.

Our chosen regularisation scheme requires the introduction of a regularised wedge product:

$$\omega \wedge_{\text{reg}} \Lambda = \eta_\infty z \partial \Lambda + \sum_{q \in P^{\text{fin}}} \sum_{l=0}^{n_r-1} \eta_q^{(l)} \frac{dz}{l!} (z - q) \wedge \partial^l_z \Lambda.$$  \tag{2.12}$$

Rather than working with the unregularised action (2.2) we work with the following regularised four-dimensional Chern-Simons action:

$$S_{4\text{dCS}}(A) = \frac{1}{2 \pi i} \int_{\Sigma \times \mathbb{C}P^1} \omega \wedge_{\text{reg}} \left( A, dA + \frac{2}{3} A \wedge A \right).$$  \tag{2.13}$$

This ensures our action, equations of motion, etc. are all well defined.

Finally, we conclude this subsection by stating an identity which allows us to localise our action to the defects at the poles of $\omega$, it is:

$$I = \int_{\Sigma \times \mathbb{C}P^1} \omega \wedge_{\text{reg}} \partial \bar{\xi} = -2\pi i \eta_\infty \int_{\Sigma} \partial_w \xi + 2\pi i \sum_{q \in P^{\text{fin}}} \sum_{l=0}^{n_r-1} \eta_q^{(l)} \frac{dz}{l!} \int_{\Sigma} \partial^l_z \xi,$$  \tag{2.14}$$

where $\Sigma_q = \Sigma \times (q, \bar{q})$, with $w = 1/z$ the local coordinate at infinity.

Note, we can write, and evaluate, (2.14) as a sum over residues. Let $V_q \subset \mathbb{C}P^1$ denote an open region which contains only the pole $q$, thus we have:

$$\int_{\Sigma_q} \text{res}_q (\omega \wedge \xi) = \int_{\Sigma \times V_q} dz \wedge \delta^2 (z - q) \partial^p \omega \wedge \xi = \sum_{l=0}^{n_r-1} \int_{\Sigma \times V_q} dz \wedge \delta^2 (z - q) \partial^p \omega \wedge \xi = \sum_{l=0}^{n_r-1} \eta_q^{(l)} \frac{dz}{l!} \int_{\Sigma_q} \partial^l_z \xi,$$  \tag{2.15}$$

where in the final equality we have cancelled $(n_q - 1)!$ with the same term in the binomial coefficient, used $\eta^{(l)} = \text{res}_q ((z - q)^l \omega)$ and evaluated an integral over $V_q$. 

7
2.2 The Equations of Motion

Our analysis is entirely classical and so the standard action’s equations of motion are easily found from the variation of $S$:

$$
\delta S_{\text{4dCS}}(A) = \frac{1}{2\pi\hbar} \int_{\Sigma \times \mathbb{C}P^1} \omega \wedge_{\text{reg}} \langle 2F(A), \delta A \rangle - \frac{1}{2\pi\hbar} \int_{\Sigma \times \mathbb{C}P^1} \omega \wedge_{\text{reg}} \mathcal{J} \langle A, \delta A \rangle.
$$

The equations of motion for $A$ follow from demanding the variation vanish. We treat the two terms on the right hand side separately, by demanding that they vanish independently. We do this because the second term simplifies to a sum over terms evaluated at the poles of $\omega$, as we demonstrate momentarily.

By applying (2.12) to the first term on the right hand side of equation (2.16) and demanding it vanishes we find:

$$
\int_{\Sigma \times \mathbb{C}P^1} \sum_{q \in P} \sum_{l=0}^{n_q-1} \frac{\eta_l^{(i)}}{l!} \frac{dz}{z-q} \wedge \partial^i_z \langle F(A), \delta A \rangle = 0.
$$

This equation is solved by field configurations which are flat on $\Sigma \times \mathbb{C}P^1$, i.e. $F(A) = 0$. Because the zeros of $\omega$ have been cut out of $\mathbb{C}P^1$, solutions to $F(A) = 0$ are allowed to be singular at those locations. These singularities are the type A defects defined in the next subsection. In fact, the solutions we are interested in are those of the following bulk equations of motion:

$$
\omega \wedge (\mathcal{J}A_\Sigma + d\mathcal{X} + \mathcal{X} \wedge A_\Sigma + A_\Sigma \wedge \mathcal{X}) = 0, \quad d\mathcal{X}A_\Sigma + A_\Sigma \wedge A_\Sigma = 0.
$$

These are satisfied everywhere on $\Sigma \times \mathbb{C}P^1$ - we do not remove the locations of zeros and poles when discussing the above equations.

The second term on the right hand side of (2.16) collapses to a sum over the poles of $\omega$ if we use (2.14) and so we call this term the “boundary variation”, and the resulting equations “boundary conditions”:

$$
I_{\text{boundary}}(A, \delta A) = \frac{1}{2\pi\hbar} \int_{\Sigma \times \mathbb{C}P^1} \omega \wedge_{\text{reg}} \mathcal{J} \langle A, \delta A \rangle = 0,
$$

where we’ve sent a total derivative in $d\mathcal{X}$ to zero. The solutions to (2.19) are a set of boundary conditions which $A$ satisfies at the poles of $\omega$, thus this equation plays a role similar to that of a boundary equation of motion. Upon imposing these boundary conditions on $A$ we introduce a set of two dimensional defects which sit at the poles of $\omega$ and span $\Sigma$; we refer to these defects as type B defects. For this reason we call equation (2.19) the defect equations of motion.

We can write down the defect equations of motion explicitly by substituting $\xi = \langle A, \delta A \rangle$ into (2.14) and requiring the contribution to (2.14) from each pole vanishes separately. Since the poles of $\omega$ are at most of degree two, the sum over $l$ in (2.14) truncates to $l \leq 1$. Combining these conditions, we end up with the following defect equations of motion:

$$
\left( \eta^{(0)}_q + \eta^{(1)}_q \partial_z \right) \int_{\Sigma_q} \langle A, \delta A \rangle = 0,
$$

(for a simple pole $\eta^{(1)}_q = 0$).

Type B defects are defined by boundary conditions which solve this equation.

2.3 Type A Defects: Singular Solutions to the Bulk Equations of Motion

In this subsection we solve the first of equations (2.18); identifying solutions for $A_{\pm}$ which are singular at the zeros of $\omega$. These singularities are two-dimensional defects which we call type A defects. A classification
of these defects was given in [12]; we rephrase this classification as the following regularity conditions on $A$ at the zeros of $\omega$. To finish we give a solution of the bulk equations of motion (2.18) which satisfies these conditions.

We define a type $A$ defect of type $(k_+^\zeta, k_-^\zeta)$ at a zero $\zeta \in \mathbb{Z}$ by the conditions:

- $(z - \zeta)^{k_+^\zeta} A_+$ is regular;
- $(z - \zeta)^{k_-^\zeta} A_-$ is regular.

These boundary conditions must be preserved by gauge transformations, otherwise we would change the defect type. If the gauge transformations are regular at the zeros of $\omega$ then we preserve the above boundary conditions. We restrict the pair $(k_+^\zeta, k_-^\zeta)$ to satisfy the inequality $k_+^\zeta + k_-^\zeta \leq m_\zeta$.

Let's consider a solution to the bulk equations of motion which satisfies a collection of type $A$ defects. To do this we work in the following gauge:

$$\mathcal{A} = 0, \quad A_\Sigma = A.$$  \hspace{1cm} (2.21)

which we call the Lax gauge - the reason for this choice of nomenclature will become clear in section 3.3. With this gauge choice the first of equations (2.18) reduces to

$$\omega \wedge \mathcal{D} A = 0.$$  \hspace{1cm} (2.23)

holds only if $\mathcal{D} A_\pm = 0$ meaning $A_\pm$ is not a function of $\bar{z}$. However, at the zeros of $\omega$ the equality in (2.23) holds even when $\mathcal{D} A_\pm \neq 0$. Thus, $\mathcal{D} A_\pm$ has finite support meaning $A_\pm$ is meromorphic in $z$ with poles the zeros of $\omega$, this is because $\mathcal{D}$ derivatives of poles are delta functions (by $\mathcal{D}(z - \zeta)^{-1} = -2\pi i \delta^2(z - \zeta)$). We note, the equality in (2.23) only holds at a zero of $\omega$ if the pole of $A_\pm$ is of the same order or less than the multiplicity of the zero. On $\mathbb{C}P^1$ meromorphic functions are ratios of two polynomials in $z$ leading to the partial fraction expansion (2.22). The polynomial terms of (2.22) follow from the assumption that $\omega$ has a zero at $z = \infty$, these terms are clearly poles by the inversion $z \rightarrow 1/z$.

When $\varphi(z)$ is non-vanishing at infinity, (2.22) reduces to:

$$A_\pm = A_\pm^c(x^\pm) + \sum_{\zeta \in \mathbb{Z}} \sum_{l=0}^{k_-^\pm - 1} \frac{A_\pm^{c,l}(x^\pm)}{(z - \zeta)^{l+1}},$$  \hspace{1cm} (2.24)

where $A_\pm^c = \lim_{z \rightarrow \infty} A_\pm$. This includes our constructions because $\varphi(z)$ always has a pole at infinity.
2.4 Type B Defects: Boundary Conditions

The 4d CS action is independent of any metric on $\Sigma$, but the two-dimensional action to which it is equivalent is either Euclidean or Lorentzian. The nature of the action is determined by the nature of the boundary conditions we impose. For simplicity, we will just discuss the Lorentzian case and will take real coordinates $x^{\pm}$ on $\Sigma$ which will eventually be light-cone coordinates for the Lorentzian sigma model. The Euclidean case is easily achieved by substituting complex coordinates $(w, \bar{w})$ for $(x^+, x^-)$.

In this section we introduce three classes of ‘Type B’ defects first given in [12]. Type B defects are solutions of (2.20) and are associated to poles in $\omega$. The first two of these classes (which we will call chiral and anti-chiral Dirichlet) are associated to first order poles in $\omega$, while the third class (which we simply call Dirichlet) is associated to a second order pole. We note that this list is not exhaustive, others are discussed in [10, 25].

Before stating the chiral, anti-chiral and Dirichlet boundary conditions we first emphasise that the gauge transformation:

$$A \rightarrow uA = u(d + A)u^{-1}, \quad (2.25)$$

must preserve the boundary conditions - we call this a physical gauge transformations. This requirement implies a set of conditions on the group element $u : \Sigma \times \mathbb{CP}^1 \to G$, which will be useful when discussing the gauge invariance of the standard action (2.2).

**Chiral Boundary Conditions:** At a simple pole $q$ where $\eta_q^{(1)} = 0$ in (2.20) one defines the chiral boundary condition by the solution:

$$A_-|_q = 0, \quad (2.26)$$

which also implies $\delta A_-|_q = 0$. This boundary condition is only preserved by gauge transformations which satisfy:

$$\partial_- u|_q = 0. \quad (2.27)$$

One uses the nomenclature ‘chiral’ because $A_-$ gives a chiral Kac-Moody current on the defect, as will be shown later. Note, simple poles do not contribute to the (2.11) meaning there is no issue of irregularity.

**Anti-Chiral Boundary Conditions:** Similarly, the anti-chiral boundary condition is also defined at a simple pole and is the solution of (2.20) (where again $\eta_q^{(1)} = 0$):

$$A_+|_q = 0, \quad (2.28)$$

where the requirement the boundary condition is preserved by gauge transformations implies

$$\partial_+ u|_q = 0. \quad (2.29)$$

As with the chiral case one finds $A_-$ gives anti-chiral Kac-Moody currents on the defect.

**Dirichlet Boundary Conditions:** The Dirichlet boundary conditions are defined at double poles of $\omega$ and is the following solution of (2.20):

$$A_{\pm}|_q = 0, \quad (2.30)$$

which implies $\delta A_{\pm}|_q = 0$ meaning $\partial_{\pm} \langle A, \delta A \rangle = 0$. Again the preservation of the boundary condition by gauge transformations implies:

$$\partial_{\pm} u|_q = 0. \quad (2.31)$$
2.5 Gauge Invariance

We have already discussed the unusual gauge invariance of the four-dimensional action; we are now in a position to discuss invariance of the action under physical gauge transformations. The transformations are given by (2.25), with \( u \in G \) satisfy the aforementioned boundary conditions at poles of \( \omega \). Under such gauge transformations, the action (2.2), transforms as:

\[
S_{4dCS}(A) \rightarrow S_{4dCS}(A) + \frac{1}{2\pi \hbar} \int_{\Sigma \times \mathbb{CP}^1} \omega \wedge_{\text{reg}} \bar{J}(u^{-1} du, A) + \frac{1}{6\pi \hbar} \int_{\Sigma \times \mathbb{CP}^1} \omega \wedge_{\text{reg}} (u^{-1} du, (u^{-1} du)^2).
\]  

(2.32)

Note, we have sent a \( d_{\Sigma} \) total derivative to zero by requiring \( A \) to die off to zero, and \( u \) to the identity, at infinity in \( \Sigma \). In the following we denote the second term on the right hand side by \( I_1 \) and the third by \( I_2 \). These two terms vanish separately and we consider \( I_1 \) first.

Using (2.14) with \( \xi = (u^{-1} du, A) \), and noting that each pole is at most second order, we find:

\[
I_1 = \sum_{q \in P} I_q, \quad I_q = \left( \eta_q^{(0)} + \eta_q^{(1)} \partial_z \right) \int_{\Sigma_q} (u^{-1} du, A).
\]  

(2.33)

For the action to be gauge invariant the above expression must vanish. This is indeed the case for the three boundary conditions defined in the previous section, with \( I_q \) vanishing due to the conditions on \( u \) at \( (q, \bar{q}) \).

**Chiral boundary conditions:** We take \( \omega \) to have a simple pole at \( z = q \) (where \( \eta_q^{(1)} = 0 \)), at which we impose the chiral boundary condition where \( A_+|_q = 0 \) reducing equation (2.33) to:

\[
I_q = \eta_q^{(0)} \int_{\Sigma_q} (u^{-1} \partial_- u, A_+) dx^- \wedge dx^+ = 0,
\]  

(2.34)

where the final equality holds upon imposing the constraint \( \partial_- u|_q = 0 \). Hence any contribution due to a first order pole in the second term of equation (2.32) can be made to vanish upon imposing chiral boundary conditions.

**Anti-chiral boundary conditions:** We take \( \omega \) to have a simple pole at \( z = q \), at which we impose the anti-chiral boundary condition where \( A_-|_q = 0 \). After imposing this, the term \( I_q \) in equation (2.33) vanishes upon imposing the constraint \( \partial_+ u|_q = 0 \). Hence, any contribution due to a first order pole in the second term of equation (2.32) can be made to vanish upon imposing anti-chiral boundary conditions.

**Dirichlet boundary conditions:** Finally, we take \( \omega \) to have a double pole at \( z = q \), at which we impose the Dirichlet boundary conditions, hence (2.33) is:

\[
I_q = \int_{\Sigma_q} (\eta_q^{(0)} + \eta_q^{(1)} \partial_z) (u^{-1} du, A) = 0.
\]  

(2.35)

The condition \( A_+|_q = 0 \) means the first term in equation (2.35) vanishes. This leaves us with:

\[
I_q = \int_{\Sigma_q} \eta_q^{(1)} \partial_z (u^{-1} \partial_j u, A_k) dx^j \wedge dx^k,
\]  

(2.36)

for \( j, k = \pm \). Upon imposing \( \partial_\pm u|_q = 0 \) along with the constraint \( A_\pm|_q = 0 \) we find this term also vanishes. Hence any contribution due to a second order pole vanishes when we impose a Dirichlet boundary condition.
The Wess-Zumino Term: The final step in proving gauge invariance is to show the Wess-Zumino term:

\[ I_2 = \int_{\Sigma \times \mathbb{C}P^1} \omega \wedge_{\text{reg}} \langle u^{-1} du, (u^{-1} du)^2 \rangle, \tag{2.37} \]

vanishes. If we take the exterior derivative of the Wess-Zumino three form we find it is closed:

\[ d \langle u^{-1} du, (u^{-1} du)^2 \rangle = -\langle (u^{-1} du)^2, (u^{-1} du)^2 \rangle = 0, \tag{2.38} \]

because of cyclic symmetry of the trace. Since the three form is closed, it is natural to ask whether it is exact. We can answer this by calculating \( H^3_{\text{dR}}(\Sigma \times \mathbb{C}P^1) \), the third de Rham cohomology of \( \Sigma \times \mathbb{C}P^1 \). In the following sections we fix \( \Sigma = \mathbb{R}^2 \). In appendix A, we find \( H^3_{\text{dR}}(\mathbb{R}^2 \times \mathbb{C}P^1) = 0 \), using the Künneth theorem and the cohomologies of \( \mathbb{R}^2 \) and \( \mathbb{C}P^1 \). As a consequence, the Wess-Zumino three form is exact on \( \mathbb{R}^2 \times \mathbb{C}P^1 \). If we take the three form to be the exterior derivative of \( E(u) \) and integrate by parts then equation (2.37) becomes:

\[ I_2 = \int_{\mathbb{R}^2 \times \mathbb{C}P^1} \omega \wedge_{\text{reg}} \tilde{E}(u), \tag{2.39} \]

where we have sent a total derivative to zero by requiring \( u \) die off at infinity in \( \mathbb{R}^2 \).

This integral is of the form (2.14) where \( \xi = \xi_{i-} dx^+ \wedge dx^- \) is the \( dx^+ \wedge dx^- \) component of \( E(u) \). This component must take the form \( \xi_{i+} = f(\zeta)_{ab} \partial_{\xi-} \zeta^a \partial_{\xi-} \zeta^b \) where \( \zeta \) are coordinates on the group \( G \). However, we required \( \partial_{\xi-} u = \partial_i \zeta^a \partial_{\xi-} u = 0 \) at a pole of \( \omega \) for either \( i = + \) or \( i = - \) or both. It thus follows that \( \xi_{i-} \) is zero at a simple pole of \( \omega \) and that \( \xi_{i+} = \partial_i \xi_{i+} = 0 \) for a double pole. From (2.14), the integral (2.39) is given by a sum of integrals evaluated at the poles of \( \omega \), and hence is zero and the 4d CS theory is gauge invariant on \( \mathbb{R}^2 \times \mathbb{C}P^1 \) for the boundary conditions we have discussed.

3 Integrable Sigma Models on Type B Defects

In this section we review [25] whose techniques will be used in the subsequent sections to derive the unified gauged sigma model. We will refer to the work of [25] as the DLMV construction. In [25], by solving the bulk and defect equations of motion and imposing suitable gauge choices, the authors reduce the 4d CS action (2.2) to the two dimensional unified sigma model action (3.28). They start by following Costello and Yamazaki by asserting the existence of a certain class of group elements \( \hat{\sigma} \) in which \( A_\pm = \hat{\sigma} \partial_\pm \hat{\sigma}^{-1} \).

The next step is to show that one can work in the so-called ‘archipelago gauge’, expressed as a set of conditions on \( \hat{\sigma} \) called the ‘archipelago’ conditions. Finally, one can show that \( A \) is gauge equivalent to a field configuration \( \mathcal{A} \) which satisfies the conditions required of a Lax connection. This step requires the gauge field satisfy \( \omega F_{\pm \pm} = 0 \), which follows from the bulk equations of motion. The two dimensional unified sigma model action (3.28) is expressed in term of \( \mathcal{A} \) and is completely determined (up to gauge symmetry) by the values of \( \hat{\sigma} \) (and its \( \partial_\pm \) derivatives) at the poles of \( \omega \).

3.1 The class \( \{\hat{g}\} \) in the DLMV Construction

In [12] Costello and Yamazaki proved the existence of a class of group elements, \( \hat{g} \)'s, such that:

\[ A_\pm(x^\pm, z, \bar{z}) = \hat{\sigma} \partial_\pm \hat{\sigma}^{-1}. \tag{3.1} \]

where \( \hat{\sigma} : \Sigma \times \mathbb{C}P^1 \to G \). This is analogous to the construction of the Wess-Zumino-Witten model in [23], but here \( \hat{\sigma} \) is not found explicitly as a path ordered exponential. The equation (3.1) has a right acting symmetry transformation, which we call the right redundancy,

\[ \hat{\sigma} \rightarrow \hat{\sigma}' = \hat{\sigma} k_g. \tag{3.2} \]
For $\hat{\sigma}$ and $\hat{\sigma}'$ to give the same field $A_z$, $k_g$ must be holomorphic. However, any holomorphic function on $\mathbb{C}P^1$ is constant since $\mathbb{C}P^1$ is compact, thus $k_g$ is a function of $x^\pm$ only.

We will fix this right-redundancy by choosing a canonical group element $\hat{g}$ in the class $\{\hat{\sigma}\}$ to be the element which satisfies $\hat{g}|_{\infty} = 1$. The canonical element $\hat{g}$ can be found from any element $\hat{\sigma} \in \{\hat{\sigma}\}$ via a right redundancy transformation (3.2) by $k_g = \hat{\sigma}^{-1}|_{\infty}$. Under this transformation we find:

$$\hat{g}(x^\pm, z, \bar{z}) = \hat{\sigma}(x^\pm, z, \bar{z}) \cdot \hat{\sigma}(x^\pm, \infty, \infty)^{-1},$$

(3.3)

and clearly $\hat{g}|_{\infty} = 1$. For the sake of brevity and clarity in the following we use $\hat{g}$ and assume it is the identity at infinity since one can always make this choice. Finally, we note that a gauge transformation $A \rightarrow u A = u(d + A) u^{-1}$ induces the following transformation on $\hat{g}$:

$$\hat{g} \rightarrow u \hat{g} u^{-1},$$

(3.4)

where the right action by $u_{\infty}^{-1}$ appears because we have fixed the right redundancy. We make extensive use of this law in the following.

### 3.2 The Strong Archipelago Conditions

In this section we correct a minor error in [25] and prove that there exists a gauge (called the ‘archipelago’ gauge) in which a group element $\hat{g}$ in the class $\{\hat{\sigma}\}$ satisfies the ‘archipelago’ conditions of [25]. This is expressed in terms of $\hat{g}$, but since $A_z$ is determined by $\hat{g}$, these are a partial gauge choice on $A$. These conditions are the following.

For each pole $q$ in $P$ we define a distance $R_q$. For each pole in $P^{\text{fin}}$, let $U_q$ be a disc of radius $R_q$ such that for a finite pole $|z - q| < R_q$; for the pole at infinity, let $U_{\infty}$ be the region $|1/z| < R_{\infty}$. We require that the radii $R_q$ be chosen to ensure the these discs are disjoint. Using these discs we define the strong archipelago conditions by:

(i) $\hat{g} = 1$ outside the disjoint union $\Sigma \times \sqcup_{q \in P} U_q$;

(ii) Within each $\Sigma \times U_q$ we require that $\hat{g}$ depends only upon $x^\pm$ and the radial coordinate $r_q$ of the disc $U_q$. We choose the notation $\tilde{g}_q$ to indicate that $\hat{g}$ is in the disc $U_q$, this condition means that $\tilde{g}_q$ is rotationally invariant;

(iii) For every $q \in P$ there is an open disc $V_q \subset U_q$ of radius $S_q < R_q$ which is centred on $q$. In the disc $V_q$ $\tilde{g}_q$ depends upon $x^+$ and $x^-$ only. We denote $\tilde{g}_q$ in this region by $g_q = \tilde{g}|_{\Sigma \times V_q}$.

The first condition sets $A_z = 0$ outside $\Sigma \times \sqcup_{q \in P} U_q$. The second condition ensures that $A_z$ in rotationally invariant in $U_q$. The third condition sets $A_z = 0$ inside $\Sigma \times \sqcup_{q \in P} V_q$.

We now prove the existence of the archipelago gauge which we do in two steps. The first is, given a group element $\hat{g}$, to construct a group element $\tilde{g}$ that satisfies the strong archipelago conditions; the second is to show that the gauge transformation $u = \tilde{g} \hat{g}^{-1}$, which puts $A$ into the archipelago gauge, preserves the boundary conditions on $A$.

In [25] the authors proposed a construction of $\tilde{g}$ which satisfies the strong archipelago conditions, but their construction is not quite right because it involves expressing $\tilde{g}$ as an exponential of a Lie algebra element. Although $\tilde{g}$ is in the identity component of $G$ by the first archipelago condition it is not the case that $\tilde{g}$ can be constructed as an exponential everywhere in the identity component. For example, if we take $G = SL(2, \mathbb{C})$ then the group element:

$$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix},$$

(3.5)
\[ \tilde{g} = g_0 \]

\[ \tilde{g} = g_1 \]

\[ \tilde{g} = g_2 \]

\[ \tilde{g} = g_3 \]

\[ \tilde{g} = g_4 \]

\[ \tilde{g} = g_5 \]

### Figure 1: An illustration of the strong archipelago conditions for an \( \omega \) with seven poles and five zeros. The diamonds represent the poles of \( \omega \) with the enclosing circles illustrating the discs \( U_q \). Each \( g_i = \tilde{g}\vert_q \) denotes the value of \( \tilde{g} \) at the associated pole of \( \omega \). The five black triangles represent the zeros of \( \omega \) at which \( A \) can have poles.

is in the identity component of \( SL(2, \mathbb{C}) \) but cannot be written as an exponential of an element of the Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \). It is for this reason that the treatment presented below is slightly different to that presented in [25].

This minor issue is easily solved by the following argument. Let \( \hat{g} \) be the original group element and \( \tilde{g} \) the group element in archipelago gauge. At each pole \( q \in P \) we choose \( g_q = \tilde{g}\vert_q \) to be equal to \( \hat{g}\vert_q \):

\[ g_q = \hat{g}\vert_q . \tag{3.6} \]

Since \( g_\infty = \hat{g}\vert_\infty = 1 \) and \( \hat{g} \) varies smoothly over \( M \), \( \hat{g} \) must be in the identity component of \( G \) everywhere on \( M \). This means, for each disc \( U_q \), we can choose a smooth path in \( G \) between 1 and \( g_q \), \( \gamma_q(t, x^\pm) : \Sigma_q \times [0, 1] \rightarrow G \) where \( \gamma_q(0, x^\pm) = 1 \) and \( \gamma_q(1, x^\pm) = g_q \).

To construct the smooth path in \( U_q \) we use a bump function, which by definition is smooth with compact support in the domain. For each \( U_q \) let \( W_q \supset U_q \) be a disc of radius \( R_q + \epsilon \) (where \( \epsilon > 0 \)) which, by an abuse of notation, has the radial coordinate \( r_q \) and centre \( U_q \) on \( r_q = 0 \). We define for each \( W_q \), a bump function \( f_q : W_q \rightarrow \mathbb{R} \) with the following properties:

\[ f_q(r_q) = \begin{cases} f_q(r_q) = 1 & \text{if } r_q < S_q , \\ f_q(r_q) = 0 & \text{if } r_q \geq R_q , \\ f_q(r_q) \text{ interpolates smoothly between 1 and 0 if } S_q \leq r_q < R_q . \end{cases} \tag{3.7} \]

Hence, in the disc \( W_q \) we can define \( \tilde{g} \) by:

\[ \tilde{g}(r_q, x^\pm) = \gamma_q(f_q(|z - q|), x^\pm) . \tag{3.8} \]

This satisfies conditions (ii) and (iii) of the strong archipelago conditions.
Globally, we define $\tilde{g}$ by:

$$
\tilde{g}(z, \bar{z}, x^\pm) = \begin{cases} 
1 & z \in \mathbb{CP}^1 \setminus \bigcup_{q \in P} W_q, \\
\gamma_q(I_{q(q)}(|z - q|), x^\pm) & z \in W_q,
\end{cases}
$$

(3.9)

which satisfies all three strong archipelago conditions and is smooth by construction.

We now show that the gauge transformation $u = \tilde{g}\hat{g}^{-1}$ from $\hat{g}$ to $\tilde{g} = u\hat{g}$ satisfies the boundary condition (2.27), (2.29) and (2.31), as appropriate. By equations (3.9) and (3.6) it is clear $u = 1$ at a pole $q \in P$, thus $\partial_{x^\pm} u = 0$. Hence the boundary conditions are satisfied and the archipelago gauge is accessible.

### 3.3 Lax connection

In this subsection we introduce the notion of a Lax connection for an integrable model. Following [25], the field configuration $A$ is formally gauge equivalent\(^4\) to a field configuration $\mathcal{A}$ in the Lax gauge:

$$
\mathcal{A} = \tilde{g}^{-1} d\tilde{g} + \tilde{g}^{-1} A\tilde{g},
$$

(3.10)

where $\mathcal{A}_{\bar{z}} = 0$ since $A_{\bar{z}} = \tilde{g}\partial_{\bar{z}}\tilde{g}^{-1}$.

One can show $\mathcal{A}$ satisfies the conditions required of a Lax connection. This is done using both the 4d CS equations of motion and Wilson lines. In subsequent sections we fix $\mathcal{A}$ in terms of the group elements $\{g_q\}$, leading to a sigma model in which $\{g_q\}$ are the fields. We conclude this section by discussing the gauge transformations of $\mathcal{A}$ induced by physical transformation of $\tilde{g}$ and right redundancy transformations of $\tilde{g}$. It follows that there exists an equivalence class of Lax connections. This is as one would expect since a sigma model should not have a preferred Lax connection.

A one-form $A$ on a two-dimensional space $\Sigma$ whose coordinates are $(x^1, x^2)$, with components given as functions of the degrees of freedom of an integrable model, is called a Lax connection if it satisfies the following properties: [42]:

1. The equation $d_{\Sigma}A + A \wedge A = 0$ gives the equations of motion for the integrable model.

It is called a Lax connection with spectral parameter if, in addition, it satisfies

2. $A$ has a meromorphic dependence upon a complex parameter $z$, called the spectral parameter,

3. One can obtain an infinite set of Poisson-commuting charges from the Taylor expansion in $z$ of the trace of the monodromy matrix of $A$. The monodromy matrix is the path ordered exponential of the line integral of $A$.

When working in the Lax gauge (2.21) the bulk equations of motion are:

$$
\omega \wedge \partial A = 0, \quad d_{\Sigma}A + A \wedge A = 0.
$$

(3.11)

This first equation implies $A$ is meromorphic, with a solution of the form (2.24); the second equation is the statement that $A$ is flat on $\Sigma$.

The final property of the Lax connection follows from the Wilson line operators in 4d CS theory. For illustrative purposes, we present here the case where $\Sigma = S^1 \times \mathbb{R}$ with coordinates $(\theta, t)$, this can be found in [42]. The generic case follows from [24] in which Vicedo found that the 4d CS Poisson algebra (in an appropriate gauge) is that of a Lax connection. The conservation and Poisson commutativity of the infinite stack of charges then follows from the standard argument found in [43, 44].

\(^4\)We say formally, because $\tilde{g}$ does not satisfy the boundary conditions required of a physical gauge transformation.
The monodromy matrix of $A$ is:

$$U(z, t) = P \exp \left( \int_0^{2\pi} A_\theta d\theta \right) = \tilde{g}^{-1} P \exp \left( \int_0^{2\pi} A_\theta d\theta \right) \tilde{g}, \quad (3.12)$$

Following the standard argument, we take the trace of both matrices and find:

$$W(z, t) = \text{Tr} \left( P \exp \left( \int_0^{2\pi} A_\theta d\theta \right) \right) = \text{Tr} \left( P \exp \left( \int_0^{2\pi} A_\theta d\theta \right) \right), \quad (3.13)$$

where the right-hand side is a gauge invariant observable in 4d CS, implying the trace of the monodromy matrix is an observable. By taking the time derivative of $W(z, t)$ we find:

$$\partial_t W(z, t) = \text{Tr} \left( \left[ U(z, t), A_t \right] \right) = 0, \quad (3.14)$$

and thus that $W(z)$ is independent of their position along the length of the cylinder. By Taylor expanding $W(z)$ in $z$ it follows from (3.14) that the coefficient of each power is conserved in time. This set of coefficients is the infinite stack of charges associated to $A$; they are observables since $W_A(z)$ is.

We now turn to a discussion of the gauge symmetry of $A$. At the beginning of this section we discussed the right redundancy (3.2) of the class of group elements $\{\hat{\sigma}\}$. The right redundancy amongst $\{\hat{\sigma}\}$ left $A_\bar{z}$ invariant, meaning every element must give the same field configuration which is completely determined in terms of $A_\bar{z}$ by the equations of motion. By performing a right redundancy transformation on $\tilde{g}$ in (3.10), and using the fact that $A$ is invariant, we find the induced gauge transformation of $A$:

$$A \rightarrow \hat{k} A = (\tilde{g} k_\bar{g})^{-1} A (\tilde{g} k_\bar{g})^{-1} d (\tilde{g} k_\bar{g})$$

$$= k_\bar{g}^{-1} (\tilde{g}^{-1} A \tilde{g} + \tilde{g}^{-1} d \tilde{g}) k_\bar{g} + k_\bar{g}^{-1} d k_\bar{g}$$

$$= k_\bar{g}^{-1} A k_\bar{g} + k_\bar{g}^{-1} d k_\bar{g}, \quad (3.15)$$

where we have used $\partial_z k_\bar{g} = 0$. Hence, $A$ is invariant under the combined transformations:

$$\tilde{g} \rightarrow \tilde{g} k_\bar{g}, \quad A \rightarrow \hat{k} A = k_\bar{g}^{-1} A k_\bar{g} + k_\bar{g}^{-1} d k_\bar{g}. \quad (3.16)$$

The invariance of $A$ under the right redundancy is significant as it means that a field configuration $A$ is associated to a class of gauge equivalent Lax connections (via the right redundancy). This means that there is no preferred Lax connection, as one would expect given an integrable sigma model. This fact can be concretely proven by noting that any sigma model found by substituting a field configuration into the action is left invariant by the right redundancy because $A$ is. This was shown explicitly in [25].

What about transformations of $A$ induced by gauge transformations of $A$? By performing a gauge transformation of (3.10) and using $\tilde{g}' = u^{-1} \tilde{g} u^{-1}$ we have:

$$A \rightarrow {}^u A = \tilde{g}'^{-1} d \tilde{g}' + \tilde{g}'^{-1} (u A) \tilde{g}' = u_\infty d \Sigma u_\infty^{-1} + u_\infty A u_\infty^{-1}. \quad (3.17)$$

### 3.4 The Unified Sigma Model Action

In this section we rewrite the 4d CS action in terms of $\tilde{g}$ and $A$ and use the strong archipelago conditions to reduce the four-dimensional action to the two-dimensional unified sigma model of [25] defined on the defects at the poles of $\omega$. To do this we substitute:

$$A = \tilde{g} d \tilde{g}^{-1} + \tilde{g} A \tilde{g}^{-1}, \quad (3.18)$$
Upon combining this with the first term, we find the unified sigma model having used the third archipelago condition to remove $g$ is holomorphic, i.e.

$$S_4 \propto \int \langle \bar{A}, g^{-1} \bar{d}g \rangle + \frac{i}{3} \int \langle \bar{A}, \bar{d}g \rangle$$.

(3.21)

In $CS(C)$ the second term $C \wedge C \wedge C$ vanishes as $A$ is a one form with non-zero $\Sigma$ components only. Hence, $CS(C)$ reduces to:

$$CS(C) = \langle C, dC \rangle = \langle A, dA \rangle + 2 \langle \bar{g}^{-1} \bar{d}g, A \wedge A \rangle$$.

(3.22)

The second term in this expression cancels with with $2 \langle B, C \rangle = -2 \langle \bar{g}^{-1} \bar{d}g, 1 \rangle$ of (3.19). Hence, upon simplifying the fourth term we find:

$$CS(B + C) = \langle A, dA \rangle + \frac{1}{3} \langle \bar{g}^{-1} \bar{d}g, \bar{g}^{-1} \bar{d}g \rangle$$.

(3.23)

where we have used $d \langle B, C \rangle = -d \langle \bar{g}^{-1} \bar{d}g, A \rangle$ in equation (3.19). This leaves us with the action:

$$S_{4dCS}(A) = \frac{1}{2 \pi h} \int \langle A, dA \rangle - \frac{1}{2 \pi h} \int \langle A, \bar{d}A \rangle + \frac{1}{6 \pi h} \int \langle \bar{A}, \bar{d}A \rangle + \frac{1}{6 \pi h} \int \langle \bar{g}^{-1} \bar{d}g, \bar{g}^{-1} \bar{d}g \rangle \wedge \bar{g}^{-1} \bar{d}g$$.

(3.24)

One recovers sigma models from the above action by substituting in solutions to the equations of motion, thus we take $A$ to be of the form (2.24). Because the zeros of $\Sigma$ have been cut out of the manifold $\mathbb{C}P^1$, $A$ is holomorphic, i.e. $\bar{\partial}A = 0$. Thus the first term on the right hand side of (3.24) vanishes. Therefore, (3.24) reduces to:

$$S_{4dCS}(A) = -\frac{1}{2 \pi h} \int \langle A, dA \rangle + \frac{1}{6 \pi h} \int \langle \bar{A}, \bar{d}A \rangle + \frac{1}{6 \pi h} \int \langle \bar{g}^{-1} \bar{d}g, \bar{g}^{-1} \bar{d}g \rangle$$.

(3.25)

This action can be reduced further using the strong archipelago conditions as shown in [25]. The first term is easily calculated using (2.14) where we find:

$$-\frac{1}{2 \pi h} \int \langle A, dA \rangle = -\frac{i}{h} \sum_{q \in P} \int_{\Sigma_q} \langle \text{res}_q (\omega \wedge A), g_q^{-1} dq \rangle$$.

(3.26)

having used the third archipelago condition to remove $g_q^{-1} dq$ from the residue.

A detailed reduction of the final term of (3.25) can be found in appendix B. In particular, one finds:

$$\frac{1}{6 \pi h} \int \omega \wedge \text{res}_q (\omega \wedge A), \bar{g}_q^{-1} \bar{d}g_q, (\bar{g}_q^{-1} \bar{d}g_q)^2$$.

(3.27)

Upon combining this with the first term, we find the unified sigma model action:

$$S_{USM}(A, \bar{g}) \equiv S_{4dCS}(A) = -\frac{i}{h} \sum_{q \in P} \int_{\Sigma_q} \langle \text{res}_q (\omega \wedge A), g_q^{-1} dq \rangle$$.

(3.28)

$$-\frac{i}{3h} \sum_{q \in P} \int_{\Sigma \times [0, R_{\Sigma}]} \langle \bar{g}_q^{-1} \bar{d}g_q, (\bar{g}_q^{-1} \bar{d}g_q)^2 \rangle$$.
3.5 Examples of Models

The choice of 1-form $\omega$ and types of defects does not necessarily lead to an interesting sigma model, or to a Lax connection with spectral parameter. To illustrate this, we consider the cases of (a) no zeros, one double pole with Dirichlet boundary conditions (b) no zeros, two simple poles, both with chiral boundary conditions (c) no zeros, two simple poles, with chiral and anti-chiral boundary conditions (d) two zeros, two double poles with Dirichlet boundary conditions. To derive these models we use the solution to the equations of motion in the gauge $A_\bar{z} = 0$, equation (2.24), with an appropriately chosen $\omega$ and boundary conditions on $A$, to find $A$. This is done using:

$A_\pm|_q = g_\pm^{-1}A_\pm|q g_q + g_\pm^{-1}\partial_\pm g_q$.

(3.29)

Having found $A$ we calculate the unified sigma model (3.28). For simplicity’s sake, we specialise to the case where $\Sigma = \mathbb{R}^2$, which we parametrise with light-cone coordinates $x^+$, and $x^-$. Note, the strong archipelago conditions are compatible with the boundary conditions we use in this section, thus we work in the archipelago gauge and with the group element $\tilde{g}$. Each of these examples has a pole at infinity at which we fix the right redundancy with $g_\infty = 1$.

3.5.1 $\omega$ with a Single Double Pole

Consider $\omega$ of the form:

$\omega = dz$,

(3.30)

which has a single double pole at infinity. We impose the Dirichlet boundary condition at this pole:

$A_\pm|_\infty = 0$.

(3.31)

Since $\omega$ has no zeros the solution (2.24) is of the form:

$A = A^c_+(x^+, x^-)dx^+ + A^c_-(x^+, x^-)dx^-.$

(3.32)

If we substitute this equation into (3.29) and use $g_\infty = 1$ to fix the right redundancy it follows that the Dirichlet boundary condition implies:

$A = 0$.

(3.33)

Similarly, the evaluation of $\text{res}_\infty(\omega)$ vanishes. Hence, when we calculate the unified sigma model action (3.28) we find it vanishes.

3.5.2 $\omega$ with Two Simple Poles and a Common Boundary Condition

Consider $\omega$ of the form:

$\omega = \frac{dz}{z}$,

(3.34)

with simple poles at both zero and infinity at which we impose chiral boundary conditions:

$A_-|_0 = 0$, \quad $A_-|_\infty = 0$.

(3.35)

As in the previous example, $\omega$ does not have any zeros meaning we again consider solutions of the form:

$A = A^c_+(x^+, x^-)dx^+ + A^c_-(x^+, x^-)dx^-.$

(3.36)

Using $g_\infty = 1$ in (3.29) it follows the boundary condition at infinity implies:

$A_\pm = B_+(x^+, x^-)$.

(3.37)
where \( B_+ = A_+|_\infty \). Similarly, at \( z = 0 \) we have \( g_0 = g \) and thus by (3.29):

\[
A_- = g^{-1} \partial_- g = 0, \quad A_+ = g^{-1} \partial_+ g + g^{-1} A_+|_0 g = B_+.
\]  

(3.38)

Hence, the boundary conditions (3.35) constrain \( \tilde{g} \) to be a function of \( x^+ \) only at \( z = 0 \) and fail to fix \( A_+ \) as it is expressed everywhere in terms of the undetermined field \( B_+ \). Further still, this solution leads to a vanishing sigma model action when substituted into (3.28). This is because the conditions \( g_\infty = 1 \) and \( g_0 = g(x^+) \) ensure both the kinetic and Wess-Zumino terms vanish. This shows one must be careful to choose boundary conditions which completely fix the field configuration.

3.5.3  \( \omega \) with Two Simple Poles and Different Boundary Conditions

As shown in [12] one can recover the Wess-Zumino-Witten model for \( \omega \) of the form:

\[
\omega = \frac{dz}{z},
\]  

(3.39)

if appropriate boundary conditions are chosen. To be precise, we choose a pair of chiral and anti-chiral boundary conditions. There are two ways to do this (which are equivalent via a coordinate inversion), one being

\[
A_-|_0 = 0, \quad A_+|_\infty = 0.
\]  

(3.40)

In addition, we fix \( \tilde{g} \) to be of the form:

\[
\tilde{g}|_0 = g_0 = g, \quad \tilde{g}|_\infty = g_\infty = 1.
\]  

(3.41)

Again, \( A \) is of the form:

\[
A = A^c_+(x^+, x^-)dx^+ + A^c_-(x^+, x^-)dx^-.
\]  

(3.42)

Inserting these conditions into (3.29) we find:

\[
A_+ = 0, \quad A_- = g^{-1} \partial_- g.
\]  

(3.43)

This is an example of a Lax connection without a spectral parameter.

If we substitute this solution into (3.28) one finds the only non-zero contribution to the kinetic term is:

\[
\text{res}_0(\omega \wedge A) = g^{-1} \partial_- g dx^-,
\]  

(3.44)

while the Wess-Zumino terms vanishes at infinity since \( \tilde{g} \) is the identity at \( r_\infty = 0 \) and \( r_\infty = R_\infty \). Hence we find the Wess-Zumino-Witten model:

\[
S_{WZW}(g) = \kappa \frac{k}{4\pi} \int_{\Sigma_0} d^2 x \left< g^{-1} \partial_+ g, g^{-1} \partial_- g \right> - \frac{k}{12\pi} \int_{\Sigma \times [0, R_0]} \left< \tilde{g}^{-1} \tilde{d} \tilde{g}, (\tilde{g}^{-1} \tilde{d} \tilde{g})^2 \right>,
\]  

(3.45)

where \( d^2 x = dx^+ \wedge dx^- \) and \(-i\hbar = 4\pi/k\).

3.5.4  The Principal Chiral Model with Wess-Zumino Term

For the purpose of illustrating the case where \( \omega \) has zeros, we repeat the derivation of the principal chiral model with Wess-Zumino term done in [12, 25]. This should aid the clarity of subsequent sections.

We consider the 4d CS theory where \( \omega \) is given by:

\[
\omega = \frac{(z - \zeta_+)(z - \zeta_-)}{z^2} dz.
\]  

(3.46)
At the zero $z = \zeta_+$ we insert a $(1,0)$-defect such that $(z - \zeta_+)A_+$ and $A_-$ are regular, while at $z = \zeta_-$ we insert a $(0,1)$-defect such that $A_+$ and $(z - \zeta_-)A_-$ are regular. This allows a first order pole in $A_+$ at $z = \zeta_+$ and a first order pole in $A_-$ at $z = \zeta_-$. Hence, upon using (2.24) (as (3.46) has a pole at infinity) it follows the Lax connection is of the form:

$$\mathcal{A} = \left(A_+^e + \frac{A_+^{\zeta,0}}{z - \zeta_+}\right) dx^+ + \left(A_-^c + \frac{A_-^{\zeta,0}}{z - \zeta_-}\right) dx^- . \tag{3.47}$$

Our chosen one-form $\omega$ (3.46) has a double pole at both $z = 0$ and $z = \infty$, at each of which we impose the Dirichlet boundary condition:

$$A_\pm|_0 = 0, \quad A_\pm|_\infty = 0 . \tag{3.48}$$

Further, we fix $\tilde{g}$ to be the identity at infinity (fixing the right redundancy, as described above) and at $z = 0$ we denote it by $g$, thus:

$$\tilde{g}|_0 = g_0 = g, \quad \tilde{g}|_\infty = g_\infty = 1 . \tag{3.49}$$

By inserting this into equation (3.29) we find:

$$A_\pm|_0 = g^{-1} \partial_\pm g + g^{-1} A_\pm|_0 g , \quad A_\pm|_\infty = A_\pm|_\infty , \tag{3.50}$$

which we use to fix $A_\pm^c$ and $A_\pm^{\zeta,0}$ in terms of $g$. By using the boundary condition on $A$ at $z = \infty$ the second of these two equations implies:

$$A_\pm^c = 0 , \tag{3.51}$$

while the first equation gives:

$$A_\pm^{\zeta,0} = -\zeta_\pm g^{-1} \partial_\pm g . \tag{3.52}$$

Hence, we find the Lax connection of the principal chiral model with Wess-Zumino term:

$$A = -\frac{\zeta_+}{z - \zeta_+} g^{-1} \partial_+ g dx^+ - \frac{\zeta_-}{z - \zeta_-} g^{-1} \partial_- g dx^- . \tag{3.53}$$

Since we work in the archipelago gauge, the action is of the form (3.28), from which we recover the sigma model action by substituting in (3.53). Both of the poles of (3.46) contribute to the kinetic term of (3.28), however the term at infinity vanishes because $g_\infty = 1$. Thus we need only evaluate $\text{res}(\omega \wedge A)$:

$$\text{res}(\omega \wedge A) = -\zeta_+ g^{-1} \partial_+ g dx^+ - \zeta_- g^{-1} \partial_- g dx^- . \tag{3.54}$$

Similarly, the coefficient of the Wess-Zumino term at $z = 0$ is $\text{res}(\omega) = -(\zeta_+ + \zeta_-)$ and we needn’t calculate $\text{res}_\infty(\omega)$ since the associated Wess-Zumino term vanishes as $\tilde{g}$ is the identity at $r_\infty = 0$ and $r_\infty = R_\infty$. Therefore, we find the principal chiral model with Wess-Zumino term:\footnote{Our metric is $\eta^{+-} = 2, \eta^{++} = \eta^{--} = 0$ and $d^2x = dx^+ \wedge dx^-$.}

$$S_{\text{PMC+WZ}}(g) = i \frac{\zeta_+ - \zeta_-}{\hbar} \int_{\mathbb{R}^2} d^2x \left( g^{-1} \partial_+ g \cdot g^{-1} \partial_- g \right) + i \frac{\zeta_+ + \zeta_-}{3\hbar} \int_{\mathbb{R}^2 \times [0, R_\infty]} \left( \tilde{g}^{-1} d\tilde{g} \cdot (\tilde{g}^{-1} d\tilde{g})^2 \right) , \tag{3.55}$$

where $\mathbb{R}^2 = \mathbb{R}^2 \times (0, 0)$.

As a final remark, it is interesting to consider two limits of this theory. The first limit of interest is $\zeta_+ \to 0$ in which we recover the Wess-Zumino-Witten model (3.45), where we have set $i\hbar = 4\pi$ and $\zeta_- = k$. The second interesting limit is $\zeta_+ \to \zeta_-$ in which the kinetic term vanishes leaving us with a topological sigma model.
4 Doubled Four-dimensional Chern-Simons

4d CS described integrable sigma models because the gauge field $A$ is gauge equivalent to a Lax connection. It therefore follows that a set of 4d CS theories will describe a collection of integrable models. In this section we ask whether multiple 4d CS theories can be coupled together and still describe an integrable model.

The simplest version of this theory contains two gauge fields: $A$ with the gauge group $G$ and $B$ with the gauge group $H \subseteq G$. We respectively denote the Lie algebras of $G$ and $H$ by $\mathfrak{g}$ and $\mathfrak{h} \subseteq \mathfrak{g}$. In this section we take $\mathfrak{g}$ and $\mathfrak{h}$ to be semisimple and $\pi : \mathfrak{h} \to \mathfrak{g}$ to be the embedding of $\mathfrak{h}$ into $\mathfrak{g}$.

We define the bilinear form $\langle \cdot , \cdot \rangle_\mathfrak{g}$ on $\mathfrak{g}$ as before. The embedding $\pi$ induces $\langle \cdot , \cdot \rangle_\mathfrak{h}$ on $\mathfrak{h}$ by:

$$\iota \langle a, b \rangle_\mathfrak{h} = \langle \pi(a), \pi(b) \rangle_\mathfrak{g},$$

where $\iota$ is called the index of embedding [45]. In the following we take $\mathfrak{g}$ to be in the adjoint representation $R_{ad}$, this induces a representation $R_{ad} \circ \pi$ of $\mathfrak{h}$. Finally, we restrict ourselves to subgroups $H$ for which the coset $G/H$ is a reductive homogenous space, that is we can take $\mathfrak{h}$ and $\mathfrak{f}$ in the decomposition $\mathfrak{g} = \mathfrak{f} \oplus \mathfrak{h}$ to satisfy

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{f}] \subset \mathfrak{f}, \quad \langle \mathfrak{h}, \mathfrak{f} \rangle_\mathfrak{g} = 0. \tag{4.2}$$

Since $\mathfrak{f}$ is orthogonal to $\mathfrak{h}$ by the final expression in (4.2) we call it the orthogonal complement.

Thus, given the two fields $A \in \mathfrak{g}$ and $B \in \mathfrak{h}$, we define the doubled 4d CS theory using the difference of two regularised 4d CS actions, one for each field, and a new boundary term which couples $A$ and $B$ together:

$$S_{\text{Dbl}}(A, B) = S_{\text{4dCS}}(A) - S_{\text{4dCS}}(B) + S_{\text{bdry}}(A, B)$$

$$= \frac{1}{2\pi \hbar} \int_{\Sigma \times C} \omega \wedge_{\text{reg}} \left\langle A, dA + \frac{2}{3} A \wedge A \right\rangle_\mathfrak{g} - \frac{1}{2\pi \hbar} \int_{\Sigma \times C} \omega \wedge_{\text{reg}} \left\langle B, dB + \frac{2}{3} B \wedge B \right\rangle_\mathfrak{h}$$

$$- \frac{1}{2\pi \hbar} \int_{\Sigma \times C} \omega \wedge_{\text{reg}} \mathfrak{g} \left\langle A^b, B \right\rangle_\mathfrak{h}, \tag{4.3}$$

we will often refer to this action as the doubled theory for short. The final term in this action is the boundary term mentioned above, its only non-zero contributions are at the poles of $\omega$ and thus only modifies the defect equations of motion. The superscript $\mathfrak{h}$ on $A$ denotes the projection of $A$ onto $\mathfrak{h}$, so that $A^b \in \mathfrak{h}$. More generally, if an expression is projected onto $\mathfrak{h}$ then we place $|^\mathfrak{h}$ to the right. Likewise, a superscript $\mathfrak{f}$ and $|^{\mathfrak{f}}$ denote the projection onto $\mathfrak{f}$. Later in this section we show the doubled action is gauge invariant if the two levels $h_\mathfrak{g}$ and $h_\mathfrak{h}$ satisfy:

$$h_\mathfrak{g} = \iota h_\mathfrak{h}. \tag{4.4}$$

For now, we use equations (4.1) and (4.4) and simplify our notation to: $\langle \cdot , \cdot \rangle_\mathfrak{g} = \langle \cdot , \cdot \rangle$ and $h_\mathfrak{g} = h$. Upon doing this, we treat $B$ as a gauge field valued in $\mathfrak{g}$, whose components in $\mathfrak{f}$ vanish and drop the projection of $A$ in the boundary term since $\langle A^f, B \rangle$ vanishes by (4.2). Later, when discussing gauge invariance, we reintroduce the two bilinear forms and levels, and show (4.4) is necessary for the action to be gauge invariant.

Once a set of coordinates $z$ and $\bar{z}$ are chosen for $C$, it is clear the fields $A_z$ and $B_z$ fall out of the doubled action due to the wedge product with $\omega = \varphi(z)dz$ (since $dz \wedge dz = 0$). Thus the action is invariant under the additional gauge transformations:

$$A_z \mapsto A_z + \chi_z, \quad B_z \mapsto B_z + \xi_z, \tag{4.5}$$

where $\chi_z$ and $\xi_z$ are arbitrary functions valued respectively in $\mathfrak{g}$ and $\mathfrak{h}$. Since $\chi_z$ and $\xi_z$ are arbitrary functions it follows that all field configurations of $A_z$ and $B_z$ are gauge equivalent, thus we work in the gauge $A_z = B_z = 0$ while $A$ and $B$ are:

$$A = A_+ dx^+ + A_- dx^- + A_\bar{z} d\bar{z}, \quad B = B_+ dx^+ + B_- dx^- + B_\bar{z} d\bar{z}. \tag{4.6}$$
As in 4d CS, the doubled action (4.3) is topological in \( \Sigma \), but is not in \( C \) since \( \varphi(z) \) does not transform as a vector. The diffeomorphisms of \( C \) which leave (4.3) invariant are those which leave \( \omega \) invariant: if \( z \to w(z) \) is a diffeomorphism then \( \omega \) is invariant if \( \varphi(w(z)) \partial w/\partial z = \varphi(z) \).

The equations of motion of the doubled action (4.3) are found from the variation

\[
\delta S_{\text{Dbl}}(A, B) = \frac{1}{2 \pi \hbar} \int_{\Sigma \times C} \omega \wedge \text{reg} \langle 2F(A), \delta A \rangle - \frac{1}{2 \pi \hbar} \int_{\Sigma \times C} \omega \wedge \text{reg} \langle 2F(B), \delta B \rangle - \frac{1}{2 \pi \hbar} \int_{\Sigma \times C} \omega \wedge \text{reg} \langle A - B, \delta A + \delta B \rangle .
\]

(4.7)

For the reason given in section 2.2, the doubled bulk equations of motion are:

\[
\omega \wedge F(A) = 0, \quad \omega \wedge F(B) = 0,
\]

(4.8)
on \( \Sigma \times CP^1 \). The defect equation of motion is:

\[
\frac{1}{2 \pi \hbar} \int_{\Sigma \times C} \omega \wedge \text{reg} \langle A - B, \delta A + \delta B \rangle = 0 .
\]

(4.9)

In the next section we discuss the solutions to this equation.

### 4.1 Boundary Conditions and Gauged Type B Defects

In standard 4d CS, the defect equations of motion require boundary conditions on \( A \) at the poles of \( \omega \), which insert type B defects. Similarly, in the doubled theory the defect equations of motion (4.9) require boundary conditions on \( A \) and \( B \) at the poles of \( \omega \), which introduce analogues of the type B defects which we call ‘gauged’ type B defects. On these defects we find the \( H \) symmetry of \( B \) is gauged out of the \( G \) symmetry of \( A \), introducing an \( H \) gauge symmetry in our sigma models. In the following we define the gauged type B defects for simple and double poles of \( \omega \).

To solve (4.9) we use the decomposition \( g = f \oplus h \) and the orthogonality of \( f \) with respect to \( h \) to separate (4.9) into a set of equations in \( f \) and a set in \( h \). After using (2.14) these equations are:

\[
\sum_{q \in P} \sum_{l=0}^{n_q-1} \frac{\eta_q^{(l)}}{l!} \epsilon^{ijk} \partial_z^l \left\langle A^f_j, \delta A^f_k \right\rangle |_q = 0 ,
\]

(4.10)

\[
\sum_{q \in P} \sum_{l=0}^{n_q-1} \frac{\eta_q^{(l)}}{l!} \epsilon^{ijk} \partial_z^l \left\langle A^h_j - B_j, \delta A^h_k + \delta B_k \right\rangle |_q = 0 ,
\]

(4.11)

where \( i, j = \pm \), \( P \) the set of all poles of \( \omega \) and \( \eta_q^{(l)} \) the residue defined in (2.4). Note, we have dropped the integral over \( \Sigma \) as the boundary conditions we construct ensure the integrand, and thus integral, vanish. We solve these equations individually at each pole of \( \omega \), hence our defect equations of motion for an \( \omega \) with at most double poles reduce to:

\[
\left( \eta_q^{(0)} + \eta_q^{(1)} \partial_z \right) \epsilon^{ijk} \left\langle A^f_j, \delta A^f_k \right\rangle |_q = 0 ,
\]

(4.12)

\[
\left( \eta_q^{(0)} + \eta_q^{(1)} \partial_z \right) \epsilon^{ijk} \left\langle A^h_j - B_j, \delta A^h_k + \delta B_k \right\rangle |_q = 0 ,
\]

(4.13)

where \( \eta_q^{(1)} = 0 \) for simple poles.

In our solutions to (4.12) and (4.13) there are two types of boundary condition which have slightly different interpretations: constraints and dynamical equations of motion. Constraints force us to reduce the
phase space of solutions we consider, while dynamical equations of motion are differential equations that our solutions satisfy.

For example, one could take $\delta A$ and $\delta B$ to be unconstrained and thus independent of each other, but we will not do this - instead we chose boundary conditions which constrain $A^h_{\pm}$ and $B_{\pm}$ such that $\delta A^h_{\pm}$ and $\delta B_{\pm}$ are not independent. This means the variation of the action can be made to vanish. To be explicit, all the gauged boundary conditions in the following satisfy the constraint $A^h_{\pm} - B_{\pm} = 0$. If we view $\delta$ as a derivative acting on the space of field configuration and assume $\delta A^h_{\pm} - \delta B_{\pm} = \delta(A^h_{\pm} - B_{\pm}) \neq 0$ it follows that somewhere on the defect the condition $A^h_{\pm} - B_{\pm} = 0$ would not hold. Hence, $\delta A^h_{\pm} - \delta B_{\pm} = 0$ is imposed for consistency, thus reducing the phase space.

An example of a boundary condition which is a dynamical equations of motion is $\partial_z (A^h_{\pm} - B_{\pm}) = 0$. This will ensure the defect equations of motion are satisfied for double poles once you impose the constraints $A^h_{\pm} - B_{\pm} = 0$ and $\delta A^h_{\pm} - \delta B_{\pm} = 0$.

Finally, we emphasise that boundary conditions must hold in all gauges, thus in the gauge transformations:

$$A \rightarrow u A = u(d + A)u^{-1}, \quad B \rightarrow v B = v(d + B)v^{-1}, \quad (4.14)$$

the group elements $u \in G$ and $v \in H$ are constrained to ensure this is the case. It is necessary to include these constraints in our boundary conditions as they will be used to prove the action is gauge invariant under large gauge transformations in the next subsection. When defining the group of gauge transformations it will be useful to introduce $C_G(h)$, the centraliser of $h$ in $G$, and $N_G(f)$, the normaliser of $f$ in $G$.

**Gauged Chiral Boundary Conditions**

The Gauged chiral boundary condition is a solution to (4.12) and (4.13) for simple poles of $\omega$, thus our defect equations of motion are:

$$e^{jk} \left\langle A^j_j, \delta A^k_k \right\rangle |_q = 0, \quad e^{jk} \left\langle A^h_j - B_j, \delta A^h_k + \delta B_k \right\rangle |_q = 0, \quad (4.15)$$

where we have dropped an $\eta^0_{(q)}$ as it is an arbitrary overall constant. The gauged chiral boundary condition at a pole $q$ is the solution:

$$A^f|_q = 0, \quad (A^h_{\pm} - B_{\pm})|_q = 0. \quad (4.16)$$

This solution solves (4.15) because the first condition implies the constraint $\delta A^f|_q = 0$.

In order to preserve (4.16) under the gauge transformation (4.14) we impose the constraints:

$$y|_q = (u^{-1}v)|_q \in K, \quad \partial_y y|_q = 0, \quad y^{-1} \partial_y y^b|_q = 0, \quad (4.17)$$

where:

$$K = C_G(h) \cap N_G(f). \quad (4.18)$$

We have found the conditions (4.17) by solving the following equations at the pole $q$:

$$(y^{-1}B_- y + y^{-1} \partial_y)|^f_q = 0, \quad (4.19)$$

$$(y^{-1}A_+ y + y^{-1} \partial_y)|^b_q = B_{\pm}|_q, \quad \text{where} \quad y = u^{-1}v. \quad (4.20)$$

The first of these equations is the statement that $A^f|_q = 0$ is preserved under the gauge transformation (4.14):

$$A^f|_q \rightarrow u A^f|_q = (uB_- u^{-1} + u \partial_- u^{-1})|^f_q = 0, \quad (4.21)$$
where in the first equality we have used (4.16). Let \( y = u^{-1}v \in G \) be the group element which \( u \) and \( v \) differ by and substitute in \( u = vy^{-1} \). If we decompose \( y^{-1}B_+ y + y^{-1}\partial_- y \) into its \( \mathfrak{f} \) and \( \mathfrak{h} \) parts, use the conditions \( vfv^{-1} = \mathfrak{f}, \mathfrak{fhv}^{-1} = \mathfrak{h} \) (as \( v \in H \) and \([\mathfrak{h}, \mathfrak{f}] \subset \mathfrak{f}\)) and project into \( \mathfrak{f} \) we find:

\[
(y^{-1}B_+ y + y^{-1}\partial_- y)|_q = 0. \tag{4.22}
\]

Similarly, the second equation in (4.16) transforms as:

\[
(A^b_\pm - B^\pm_\pm)|_q \rightarrow (uA^b_\pm - vB^\pm_\pm)|_q = (uA_\pm u^{-1} + u\partial_\pm u^{-1})|_q - (vB_\pm v^{-1} + v\partial_\pm v^{-1})|_q = 0. \tag{4.23}
\]

If we again substitute in \( u = vy^{-1} \) this equation becomes:

\[
v(y^{-1}A_\pm y + y^{-1}\partial_\pm y - B_\pm)v^{-1}|_q = 0, \tag{4.24}
\]

where we have used \( vfv^{-1} = \mathfrak{f} \) and \( \mathfrak{fhv}^{-1} = \mathfrak{h} \). Hence (4.24) is equivalent to

\[
(y^{-1}A_\pm y + y^{-1}\partial_\pm y)|_q = B_\pm|_q. \tag{4.25}
\]

When discussing the gauge invariance of an action one considers arbitrary field configurations satisfying the boundary conditions, not just those which are on-shell, for this reason the equations (4.21) and (4.25) must hold for every \( A \) and \( B \). Thus, (4.21) implies the two conditions: \( y|q \in C_G(\mathfrak{h}), \) and \( \partial_- y|q = 0 \). Using these conditions, equation (4.25) reduces to \((y^{-1}A_+|^f y + y^{-1}\partial_+ y)|_q^b = 0\). This implies \( y^{-1}\partial_+ y|_q^b = 0 \) and \( y|q \in N_G(\mathfrak{f}) \).

**Gauged Anti-Chiral Boundary Conditions**

We also define the gauged anti-chiral boundary condition at a simple pole of \( \omega \) to be:

\[
A^f_\pm|_q = 0, \quad (A^b_\pm - B_\pm)|_q = 0, \tag{4.26}
\]

which imply the following constraints on a gauge transformation:

\[
y|q = (u^{-1}v)|_q \in K, \quad \partial_+ y|_q = 0, \quad y^{-1}\partial_+ y|_q^b = 0, \tag{4.27}
\]

This set of constraints follow from the conditions:

\[
(yB_+ y^{-1} + y\partial_+ y^{-1})|_q^f = 0, \tag{4.28}
\]

\[
(y^{-1}A_\pm y + y^{-1}\partial_\pm y)|_q^b = B_\pm. \tag{4.29}
\]

These are found and solved by following an argument similar to that used in the gauged chiral boundary condition.

**Gauged Dirichlet Boundary Conditions**

At double poles of \( \omega \) the defect equations of motion (4.12) and (4.13) are:

\[
\left(\eta_0^0 + \eta_0^1 \partial_n\right)\epsilon^{jk} \left<A^j_0 + \delta A^j_k\right>|_q = 0, \tag{4.30}
\]

\[
\left(\eta_0^0 + \eta_1^1 \partial_n\right)\epsilon^{jk} \left<A^b_0 - B_j + \delta A^b_k + \delta B_k\right>|_q = 0. \tag{4.31}
\]
Therefore the set of gauge transformations as defined is indeed closed.

where the second equality follows from

Taken sequentially, two gauge transformations are given by

we have

One class of solutions to this equation are the gauged Dirichlet boundary conditions:

which imply the following constraints on gauge transformations:

Following the argument used for the second gauge chiral boundary condition, the first two conditions in (4.32) are preserved by gauge transformations satisfying:

By again demanding that \( y \) be independent of \( A \) and \( B \) when solving these equations we find the first two conditions in equation (4.33). The third condition is preserved if:

which has been derived using (4.35). Upon expanding the \( z \) derivative and using the conditions already imposed upon \( y \) this simplifies to:

By again demanding that \( y \) be independent of the field configurations you find the conditions

One might worry that the composition of two gauge transformations, each satisfying (4.33), might not itself satisfy these constraints. The only potential impediment to this would be the conditions \( y^{-1}\partial_y y|_q = 0 \).

Taken sequentially, two gauge transformations are given by \( (A, B) \to (u v^u A, y^v B) \) where we assume \( y = u^{-1}v \) and \( y' = u^{-1}v' \) satisfy (4.33). For consistency, the composition of the two gauge transformations, given by \( y'' = (u'v)|^u = u^{-1}v' = u^{-1}v' \), ought to also satisfy the above conditions. In particular, we have:

where we've used two arguments to reach the second equality. The first is to note that \( y \in C_G(h) \), which leads to the first term. While the second is \( v^{-1}y^{-1}\partial_y y' y'|_q = v^{-1}(y'^{-1}\partial_y y'|_q + y'^{-1}\partial_y y'|_q)\) since \( y'^{-1}\partial_y y'|_q \in f \) and thus \( v^{-1}(y'^{-1}\partial_y y'|_q)\) since \( v^{-1}y^{-1}\partial_y y'|_q = 0 \). Using \( u|_q = v^{-1}|_q \) we find:

where the second equality follows from \( y \in C_G(h) \). By the boundary conditions we have \( y^{-1}\partial_y y|_q \in f \), which together with \( y' \in N_G(f) \) gives \( y'^{-1}y^{-1}\partial_y y|_q \in f \). Hence:

Therefore the set of gauge transformations as defined is indeed closed.
4.2 Gauge Invariance

In this section we prove that the doubled action (4.3) is gauge invariant for field configurations which satisfy any of the boundary conditions defined in the previous section at the poles of $\omega$. We reintroduce the trace $\langle \cdot, \cdot \rangle_\theta = \langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_h$ into the action and show the action is gauge invariant if $h_\theta = \imath h_h$ after having used $\text{Tr}_h \equiv \imath \text{Tr}_\theta$, (4.1). As a reminder, the gauge transformations of the gauge fields $A$ and $B$ are:

$$A \rightarrow uA = u(d + A)u^{-1}, \quad B \rightarrow vB = v(d + B)v^{-1},$$

and we define $y = u^{-1}v \in G$.

Under the gauge transformations (4.44), the action transforms as:

$$S_{\text{Dbld}}(A, B) \rightarrow S_{\text{Dbld}}(uA, vB) = S_{4\text{dCS}}(uA) - S_{4\text{dCS}}(vB) + S_{\text{Bdry}}(uA, vB),$$

where:

$$S_{4\text{dCS}}(uA) = S_{4\text{dCS}}(A) + \frac{1}{2\pi h_\theta} \int_{\Sigma \times C} \omega_{\text{reg}} \bar{\theta} \langle u^{-1}du, A \rangle_\theta + \frac{1}{6\pi h_\theta} \int_{\Sigma \times C} \omega_{\text{reg}} \langle u^{-1}du, (u^{-1}du)^2 \rangle_\theta,$$

$$S_{\text{Bdry}}(uA, vB) = -\frac{1}{2\pi h_\theta} \int_{\Sigma \times C} \omega_{\text{reg}} \bar{\theta} \langle uAu^{-1} - duu^{-1}, vBv^{-1} - dvv^{-1} \rangle_b.$$  

(4.45)

(4.46)

In these expressions we have sent a total derivative in $du$ to zero by requiring $A$, $B$, $u$ and $v$ die off sufficiently fast enough at infinity. Upon using the Polyakov-Wiegmann identity [46]:

$$\langle u^{-1}du, (u^{-1}du)^2 \rangle - \langle v^{-1}dv, (v^{-1}dv)^2 \rangle = \langle ydy^{-1}, (ydy^{-1})^2 \rangle + 3d \langle vdv^{-1}, duu^{-1} \rangle,$$

(4.48)

along with (4.1) and (4.4), the two Wess-Zumino terms in (4.45) can be written as:

$$\frac{1}{6\pi h_\theta} \int_{\Sigma \times C} \omega_{\text{reg}} \langle u^{-1}du, (u^{-1}du)^2 \rangle_\theta - \frac{1}{6\pi h_\theta} \int_{\Sigma \times C} \omega_{\text{reg}} \langle v^{-1}dv, (v^{-1}dv)^2 \rangle_b = \frac{1}{6\pi h} \int_{\Sigma \times C} \omega_{\text{reg}} \bar{\theta} \langle ydy^{-1}, (ydy^{-1})^2 \rangle + \frac{1}{2\pi h} \int_{\Sigma \times C} \omega_{\text{reg}} \bar{\theta} \langle vdv^{-1}, duu^{-1} \rangle,$$

(4.49)

where we have used (4.1) and then set $h_\theta = h$.

Using $u = vy^{-1}$ and cancelling several terms, this reduces to:

$$S_{\text{Dbld}}(uA, vB) = S_{4\text{dCS}}(A) - S_{4\text{dCS}}(B) + \frac{1}{6\pi h} \int_{\Sigma \times C} \omega_{\text{reg}} \langle ydy^{-1}, ydy^{-1} \wedge ydy^{-1} \rangle$$

$$+ \frac{1}{2\pi h} \int_{\Sigma \times C} \omega_{\text{reg}} \bar{\theta} \left[ \langle ydy^{-1}, A \rangle - \langle y^{-1}A, y^{-1}dy \rangle \right].$$

(4.50)

In the previous subsection we found a set of conditions that $y$ must satisfy at the poles $\omega$ in order to preserve our boundary conditions. Using these, we can reduce the final term of (4.50) to $S_{\text{Bdry}}(A, B)$ and thus find:

$$S_{\text{Dbld}}(uA, vB) = S_{\text{Dbld}}(A, B) + I_2 + I_3,$$

$$I_2 = \frac{1}{2\pi h} \int_{\Sigma \times C} \omega_{\text{reg}} \bar{\theta} \langle ydy^{-1}, A \rangle,$$

$$I_3 = \frac{1}{6\pi h} \int_{\Sigma \times C} \omega_{\text{reg}} \langle ydy^{-1}, ydy^{-1} \wedge ydy^{-1} \rangle.$$

(4.51)

(4.52)

(4.53)
In the following we focus on sigma models which are recovered from doubled 4d CS on $\mathbb{R}^2 \times \mathbb{C}P^1$, thus as described in appendix A and section 2.5 the three-form $\langle ydy^{-1}, ydy^{-1} \wedge ydy^{-1} \rangle$ is exact. This allows us to write the Wess-Zumino term of (4.53) as:

$$I_3 = \frac{1}{6 \pi \hbar} \int_{\Sigma \times C} \omega \wedge_{\text{reg}} \overline{\partial} E(y).$$  

(4.54)

Since $d(ydy^{-1})^2 = 0$ identically it is clear that $E(y) \neq \langle ydy^{-1}, ydy^{-1} \rangle$ and thus that $I_2$ and $I_3$ must vanish independently, as we now show. As in section 2.5 we achieve this by asking that the terms in $I_2$ and $I_3$ both vanish at each pole $q$ separately.

4.2.1 $I_3$, The Wess-Zumino Term:

As in section 2.5, the component of $E(y)$ which contributes to the Wess-Zumino term is $\xi_+dx^+ \wedge dx^-$. Using the coordinates, $\zeta^a$, on the group $G$, we have the expressions $\xi_+ = f(\zeta)_{ab}\partial_+ \zeta^a \partial_- \zeta^b$ and $\partial_\pm y = \partial_\pm \zeta^a \partial_a y$. For simple poles, $E(y)$ must at least be linear in $z$ for the contribution to vanish.

- The gauged chiral boundary condition imposed upon $y$ the condition $\partial_- y = 0$. Since $\partial_- = \partial_- \zeta^a \partial_a y$ it follows that $\partial_- \zeta^a = 0$ and thus that the Wess-Zumino term vanishes.
- The gauged chiral boundary condition imposed upon $y$ the condition $\partial_+ y = 0$. Since $\partial_+ = \partial_+ \zeta^a \partial_a y$ it follows that $\partial_+ \zeta^a = 0$ and thus that the Wess-Zumino term vanishes.
- For gauged Dirichlet boundary conditions, the contribution from double poles vanishes if $E(y) = 0$ and $\partial_\pm E(y) = 0$. This is guaranteed because $E(y)$ and $\partial_\pm E(y)$ are both proportional to $\partial_\pm \zeta^a$ or $\partial_\pm \zeta^a$ which both vanish since $\partial_\pm y = 0$.

4.2.2 $I_2$

The contribution to $I_2$ from the pole at $q$ is

$$I_2^q = \frac{1}{2 \pi \hbar} \sum_{l=0}^{n_q-1} \int_{\Sigma_q} d^2x \frac{\eta^{(l)}_q}{l!} \epsilon^{jk} \partial_x^l \langle y\partial_j y^{-1}, A_k \rangle. $$

(4.55)

- Gauged Chiral/Anti-Chiral Boundary Conditions:

At a simple pole, (4.55) is:

$$I_2^q, \text{sing} = \frac{1}{2 \pi \hbar} \int_{\Sigma_q} d^2x \epsilon^{jk} \langle y\partial_j y^{-1}, A_k \rangle. $$

(4.56)

Upon imposing the gauged chiral boundary conditions (4.16) and (4.17) this expression vanishes. The same holds true if one imposes the gauged anti-chiral boundary conditions (4.26) and (4.27).

- Gauged Dirichlet Boundary Conditions:

At a double pole, $I_2^q$ is:

$$I_2^q, \text{dble} = \frac{1}{2 \pi \hbar} \left( \eta^{(0)}_q + \eta^{(1)}_q \partial_x \right) \int_{\Sigma_q} d^2x \epsilon^{jk} \langle y\partial_j y^{-1}, A_k \rangle. $$

(4.57)

By imposing (4.32) and (4.33) the above vanishes - this requires noting that $y \in C_G(h)$ and using $\partial_\pm(y\partial_\pm y^{-1}) = y\partial_\pm(y^{-1}\partial_\pm y)y^{-1}$.

It follows from the above analysis that the doubled 4d CS action is indeed gauge invariant.
5 The Unified Gauged Sigma Model

In this section we reduce the doubled action (4.3) to a unified gauged sigma model following arguments similar to those used in [25]. As in [25] one constructs field configurations $A$ and $B$ of the doubled 4d CS equations of motion (4.8) which are gauge equivalent to two Lax connection $\mathcal{A}$ and $\mathcal{B}$. Using the unified gauged model, a set of field configurations (and thus Lax connections) determines a gauged sigma model whose integrability is determined by the existence of the Lax connections.

We begin by introducing two classes of group elements, $\{\hat{g}\}$ and $\{\hat{h}\}$, using them to rewrite the doubled action. We prove that one can construct group elements which satisfy the strong archipelago conditions of [25] which are used to reduce the rewritten doubled action to a two-dimensional theory which sits on the defects at the poles of $\omega$. By varying this action we show $A$ and $B$ are gauge equivalent to Lax connections.

Finally, we construct several examples of gauged sigma models from the unified gauged sigma model. In this section we fix $C = \mathcal{CP}^1$ with the coordinates $z$ and $\bar{z}$.

5.1 Gauge fields and group elements

As in section 3.1, the fields $\mathcal{A} = A_z d\bar{z}$ and $\mathcal{B} = B_z d\bar{z}$ can be expressed in terms of group elements $\hat{g} : \Sigma \times \mathcal{CP}^1 \to G$ and $\hat{h} : \Sigma \times \mathcal{CP}^1 \to H$ by:

$$\mathcal{A} = \hat{g} \bar{\partial} \hat{g}^{-1} = -(\bar{\partial} \hat{g}) \hat{g}^{-1}, \quad \mathcal{B} = \hat{h} \partial \hat{h}^{-1} = -(\partial \hat{h}) \hat{h}^{-1}. \quad (5.1)$$

These define equivalence classes of elements, $\{\hat{g}\}$ and $\{\hat{h}\}$, related by right-multiplication by group elements that are independent of $\bar{z}$ (which we called right-redundancy). We again will only consider forms $\omega$ with a pole at infinity and choose canonical representatives of $\hat{g}$ and $\hat{h}$ which are the identity at $z = \infty$, and from now on the notation $\hat{g}$ and $\hat{h}$ will always imply this canonical representative. As in the section 3.3, gauge transformations on $A$ and $B$ by $u$ and $v$ manifest as the following transformations on $\hat{g}$ and $\hat{h}$:

$$A \longrightarrow A' = uAu^{-1} + udu^{-1}, \quad B \longrightarrow B' = vBv^{-1} + vdv^{-1} \quad (5.2)$$

$$\hat{g} \longrightarrow \hat{g}' = u\hat{g}u^{-1}, \quad \hat{h} \longrightarrow \hat{h}' = v\hat{h}v^{-1}, \quad (5.3)$$

where $u_\infty$ and $v_\infty$ are the values of $u$ and $v$ at $z = \infty$, and these terms are included to ensure the group elements satisfy $\hat{g}'|_\infty = \hat{h}'|_\infty = 1$.

5.2 More Lax Connections

A connection $\mathcal{A}$ is a Lax connection if it satisfies properties 1 - 3 in section 3.3. Using the group elements $\hat{g}$ and $\hat{h}$ one can construct the fields $\mathcal{A}$ and $\mathcal{B}$ from $A$ and $B$ which satisfy the conditions required of a Lax connection. By gauge transforming $A$ by $\hat{g}^{-1}$ and $B$ by $\hat{h}^{-1}$ one finds:

$$\mathcal{A} = \hat{g}^{-1} d\hat{g} + \hat{g}^{-1} A\hat{g}, \quad \mathcal{B} = \hat{h}^{-1} d\hat{h} + \hat{h}^{-1} B\hat{h}, \quad (5.4)$$

where $\mathcal{A} = \mathcal{B} = 0$ follows from equation (5.1).

From equations (5.2) and (5.3), gauge transformations on $A$ and $B$ by $u$ and $v$ are equivalent to the following changes to $\mathcal{A}$ and $\mathcal{B}$:

$$A \longrightarrow A' = u_\infty Au_\infty^{-1} + u_\infty du_\infty^{-1}, \quad (5.5)$$

$$B \longrightarrow B' = v_\infty Bv_\infty^{-1} + v_\infty dv_\infty^{-1}, \quad (5.6)$$

where $u_\infty = u|_\infty$ and $v_\infty = v|_\infty$.
The equations of motion for $A$ and $B$ (4.8) imply:

$$\omega \wedge \partial A = 0, \quad d_{\Sigma} A + A \wedge A = 0, \quad (5.7)$$

$$\omega \wedge \partial B = 0, \quad d_{\Sigma} B + B \wedge B = 0. \quad (5.8)$$

The first and third equations mean $A$ and $B$ have a meromorphic dependence upon $z$ and thus satisfy the second property of a Lax connection. The second and fourth equations mean $A$ and $B$ are flat in the plane $\Sigma$ and we will demonstrate in the following that these are the equations of motion of our sigma model, ensuring we satisfy the first property of a Lax connection.

Following exactly the same arguments as in section 3.3 for $I = A, B$, in the case where $\Sigma = S^1 \times \mathbb{R}$ we construct monodromy matrices $U_I$:

$$U_A(z, t) = P \exp \left( \int_0^{2\pi} A_\theta d\theta \right) = \hat{g}^{-1} P \exp \left( \int_0^{2\pi} A_\theta d\theta \right) \hat{g}, \quad (5.9)$$

$$U_B(z, t) = P \exp \left( \int_0^{2\pi} B_\theta d\theta \right) = \hat{h}^{-1} P \exp \left( \int_0^{2\pi} B_\theta d\theta \right) \hat{h} \quad (5.10)$$

to find conserved quantities $W_I = \text{Tr} U_I$. In particular, the conserved charges are the coefficients of $W_I$ when expanded into a power series in $z$.

### 5.3 The Weak Archipelago Conditions

In this section we introduce the weak archipelago gauge for the doubled 4d CS theory. This in essence is a straightforward repackaging of a gauge choice made in [41], although given the techniques used in that paper we are required to give some motivation. In [41], the fundamental objects used by the authors to define their field theories are elements of a jet group, which we denote by $J$. To define $J$ we let $\mathbb{C}[\epsilon]/(\epsilon^n)$ be a polynomial ring of degree $n$. Given $\mathbb{C}[\epsilon]/(\epsilon^n)$ the elements of the jet group $J$, $\tilde{j}_n : \Sigma \times \mathbb{C}P^1 \rightarrow J$, are simply truncated Taylor expansions of elements in $G$, $g : \Sigma \times \mathbb{C}P^1 \rightarrow G$:

$$\tilde{j}_n(z, x^\pm) = \sum_{l=0}^{n-1} \frac{1}{l!} \partial^l g(z, x^\pm) \otimes \epsilon^l, \quad (5.11)$$

where $\epsilon \in \mathbb{C}[\epsilon]/(\epsilon^n)$ and $n$ is the maximal degree of the poles of $\omega \{n_q\}$.

Let $\tilde{g} \in G$ be defined as above and take $\tilde{j}_n$ to be its associated jet group element. In the jet groups language the weak archipelago gauge is defined by an element $\tilde{j}_n \in J$ which satisfies the following three properties:

(i) $\tilde{j}_n = 1$ outside the disjoint union $\Sigma \times \sqcup_{q \in P} U_q$;

(ii) Within each $\Sigma \times U_q$, $\tilde{j}_n$ depends smoothly only upon $x^\pm$ and the radial coordinate $r_q$ of the disc $U_q$;

(iii) At every pole $q \in P$, $\tilde{j}_n$ satisfies $\tilde{j}_{n_q} = \hat{j}_{n_q}$, where we’ve truncated both jet group elements to be of degree $n_q$.

It was shown that there exists such a group element in lemma 3.5 and proposition 3.6 of [41].

We could use the jet group $J$ to perform the analysis which follows. This however, introduces an additional level of abstraction that obscures a few important details. For this reason we do not use $J$, but instead note that the existence of $\tilde{j}_n$ implies the existence of a group element $\tilde{g}$ which satisfies the following weak archipelago conditions:
\( (i) \) \( \tilde{g} = 1 \) outside the disjoint union \( \Sigma \times \sqcup_{q \in P} U_q; \)

\( (ii) \) Within each \( \Sigma \times U_q, \tilde{g} \) depends smoothly only upon \( x^\pm \) and the radial coordinate \( r_q \) of the disc \( U_q; \)

\( (iii) \) At every pole \( q \in P, \tilde{g} \) satisfies \( \tilde{g}|_q = \tilde{g}^h|_q = g_q \) and \( \partial^l_z \tilde{g}|_q = \partial^l_z g_q \) for \( l = 1, \ldots, n_q - 1 \) (where \( n_q \) is the degree of the pole \( q \)).

The constructions we consider in the rest of this paper will use \( \tilde{g} \).

It is simple to define the weak gauge in the doubled 4d CS theory, as it is simply that the group elements \( \hat{g} \) and \( \hat{h} \) defined in section 5.1 must satisfy:

(a) \( \hat{g} \) satisfies the weak archipelago conditions,

(b) \( \hat{h} \) is the identity.

These conditions are equivalent to a gauge choice on \( A \) and \( B \). They allow one to reduce the doubled action to a unified gauged sigma model on the defects at the poles of \( \omega \) - we leave this to the next section.

We show that the weak archipelago conditions are a valid gauge choice by exhibiting a gauge transformation which will put any gauge field configuration into this gauge. If \( A \) and \( B \) define group elements \( \hat{g} \) and \( \hat{h} \), then this is equivalent to finding gauge transformations \( U \in G \) and \( V \in H \) consistent with the defect boundary conditions such that \( \tilde{g} = U \hat{g} U^{-1} \) satisfies (a) and \( \tilde{h} = V \hat{h} V^{-1} = 1 \) satisfies (b).

To show that the gauge transformation of \((A, B)\) by \((U, V)\) to the archipelago gauge preserves the defect boundary conditions, we shall split it into two consecutive gauge transformations

\[
\begin{align*}
(\hat{g}, \hat{h}) \twoheadrightarrow (\hat{h}^{-1} \hat{g}^{-1}) \twoheadrightarrow (\hat{h}^{-1} \hat{g}, 1) \equiv (\tilde{g}^h, 1) \twoheadrightarrow (\tilde{u} \tilde{g}^h, 1) \equiv (\tilde{g}, 1),
\end{align*}
\]

where we have used the facts that \( \hat{h}^\infty = \hat{u}^\infty = 1 \).

For this gauge transformation to be consistent it must satisfy (4.17) for gauged chiral conditions, (4.27) for gauged anti-chiral, and (4.33) for gauged Dirichlet. These conditions are all satisfied because \( u = \hat{h}^{-1} \) implies \( \tilde{y} = \tilde{u}^{-1} \tilde{v} = 1 \) everywhere on \( \Sigma \times \mathbb{C}P^1 \).

We now give the second step for each boundary condition in turn.

**Gauged Chiral/Anti-Chiral Boundary Conditions**

A gauged chiral boundary condition requires a gauge transformation \((A, B) \rightarrow (uA, vB)\) satisfy:

\[
y|_q = uv^{-1}|_q \in K, \quad \partial_- y|_q = 0, \quad y^{-1} \partial_+ y|_q = 0.
\]

In the second step our gauge transformation is defined by \((u, v) = (\tilde{u}, 1)\) where \( \tilde{u} = \tilde{g}(\tilde{g}^h)^{-1} \). The weak archipelago condition \( iii \) implies that \( y = \tilde{u}^{-1} = 1 \) at a simple pole \( q \); thus the above conditions are satisfied.

The same argument works for anti-chiral boundary conditions if one swaps \(+\) and \(-\) in the equation above.

\footnote{Note that \( \tilde{g}^h \equiv \hat{h}^{-1} \tilde{g} \) and does not denote conjugation by \( h \), as is often meant by this notation}
Gauged Dirichlet Boundary Conditions

The gauged Dirichlet conditions require:

\[ y|_q = uv^{-1}|_q \in K, \quad \partial y|_q = 0, \quad y^{-1} \partial y|_q = 0. \]  

(5.15)

As above, the second gauge transformation is defined by \((u, v) = (\tilde{u}, 1)\) where \(\tilde{u} = \tilde{g}(\tilde{g}^h)^{-1}\). All of the above conditions are satisfied because of the weak archipelago condition \(iii\). The first two requirements hold since \(iii\) implies \(y = \tilde{u}^{-1} = 1\) at a pole. While, when calculating \(y^{-1} \partial y\) at a double pole \(q\) one finds:

\[ y^{-1} \partial y|_q = (\tilde{g}(\tilde{g}^h)^{-1} \partial \tilde{g}(\tilde{g}^h)^{-1} + \tilde{g} \partial \tilde{g}^{-1})(q) = \partial_z g_q g_q^{-1} + g_q \partial_z g_q^{-1} = 0. \]  

(5.16)

In the second equality we have evaluated the expression at \((z, \tilde{z}) = (q, \tilde{q})\) and used the weak condition \(iii\), while in the final equality we’ve used the identity \(\partial_z g g^{-1} = -g \partial_z g\). Hence, we can gauge transform into the weak archipelago gauge.

As a result of the above, we are free to set \(\tilde{h} = 1\), \(\tilde{B} = 0\), and work in the weak archipelago gauge. Hence, the gauge fields take the form

\[ A = \tilde{g} d\tilde{g}^{-1} + \tilde{g} A\tilde{g}^{-1}, \quad B = B. \]  

(5.17)

5.4 The Unified Gauged Sigma Model Action

In this subsection we use the weak archipelago conditions to localise the doubled action (4.3) onto the two-dimensional defects at the poles of \(\omega\). This yields a two-dimensional action, as found for the standard 4d CS action in [25]. We give the equations of motion for this two-dimensional action. We will conclude this subsection by constructing two types of gauged WZW (GWZW) model as examples.

We start by assuming the gauge fields are in the weak archipelago gauge, so that they take the form (5.17). Substituting these expressions into the doubled action, we find:

\[
S_{\text{DBid}}(A, B) = -\frac{1}{2\pi \hbar} \int_{\Sigma \times \mathbb{C}^2} \omega \wedge_{\text{reg}} \partial \left( (\langle A, \tilde{g}^{-1} d\Sigma \tilde{g} \rangle + \langle -d\Sigma \tilde{g}^{-1} + \tilde{g} A\tilde{g}^{-1}, B \rangle \right)
\]

\[
+ \frac{1}{2\pi \hbar} \int_{\Sigma \times \mathbb{C}^2} \omega \wedge_{\text{reg}} \langle A, \tilde{g} \tilde{A} \rangle - \frac{1}{2\pi \hbar} \int_{\Sigma \times \mathbb{C}^2} \omega \wedge_{\text{reg}} \langle B, \tilde{g} \tilde{B} \rangle
\]

\[+ \frac{1}{6\pi \hbar} \int_{\Sigma \times \mathbb{C}^2} \omega \wedge_{\text{reg}} \langle \tilde{g}^{-1} d\tilde{g}, (\tilde{g}^{-1} d\tilde{g})^2 \rangle
\]

(5.18)

Using equation (2.14), the first two terms are localised to the defects at the poles of \(\omega\). For double poles this requires expanding out a first derivative in \(z\) whose action on \(\tilde{g}\) introduces a new scalar field \(\Delta_q = \tilde{g}^{-1} \partial_z \tilde{g}|_q\). By applying the third weak archipelago condition and using (2.15) we find the first two terms are given by:

\[-\frac{i}{\hbar} \sum_{q \in P} \int_{\Sigma_q} \left( \langle \text{res}_q (\omega \wedge A), g_q^{-1} d\Sigma g_q \rangle + \text{res}_q (\omega \wedge \langle -d\Sigma g_q g_q^{-1} + g_q A g_q^{-1}, B \rangle) \right)
\]

\[+ \eta_q^{(1)} \langle A, g_q^{-1} d\Sigma (g_q \Delta_q g_q^{-1}) g_q \rangle + \eta_q^{(1)} \langle g_q^{-1} B g_q, d\Sigma \Delta_q + [A, \Delta_q] \rangle
\]

where \(g_q = g_q(x^+, x^-)\). By integrating by parts the terms \(\langle g_q A g_q^{-1}, d\Sigma (g_q \Delta_q g_q^{-1}) \rangle\) and \(\langle g_q^{-1} B g_q, d\Sigma \Delta_q \rangle\), sending to zero any total derivatives; and using the symmetry of \(\langle \cdot, \cdot \rangle\) one finds:

\[-\frac{i}{\hbar} \sum_{q \in P} \int_{\Sigma_q} \left( \langle \text{res}_q (\omega \wedge A), g_q^{-1} d\Sigma g_q \rangle + \text{res}_q (\omega \wedge \langle -d\Sigma g_q g_q^{-1} + g_q A g_q^{-1}, B \rangle) \right)
\]

\[+ \eta_q^{(1)} \langle g_q \Delta_q g_q^{-1}, d\Sigma (g_q A g_q^{-1}) \rangle + d\Sigma B + [g_q d\Sigma g_q^{-1} + g_q A g_q^{-1}, B] \right)\]

(5.19)
where \( \eta^{(1)}_q = 0 \) if \( q \) is a simple pole.

Using proposition 3.7 of [41], the Wess-Zumino term is found to be:

\[
-\frac{i}{\hbar} \sum_{q \in p} \left( \text{res}_q (\omega) \frac{1}{3} \int_{\Sigma_q \times [0,R_q]} \langle g_q^{-1} dg_q, (g_q^{-1} dg_q)^2 \rangle + \eta^{(1)}_q \int_{\Sigma_q} \langle g_q \Delta_q g_q^{-1}, d\Sigma (g_q d\Sigma g_q^{-1}) \rangle \right),
\]

whose derivation can be found in appendix B.

Thus, the doubled action reduces to:

\[
S_{\text{dbl}}(A,B) = \frac{1}{2\pi \hbar} \int_{\Sigma \times \mathbb{C}^2} \omega \wedge \langle A, \overline{\partial} A \rangle - \frac{1}{2\pi \hbar} \int_{\Sigma \times \mathbb{C}^2} \omega \wedge \langle B, \overline{\partial} B \rangle + S_{\text{USM}}(g,A)
\]

\[
+ \frac{i}{\hbar} \sum_{q \in p} \text{res}_q (\omega \wedge \langle -d\Sigma g_q g_q^{-1} + g_q A g_q^{-1}, B \rangle)
\]

\[
- \frac{i}{\hbar} \int \sum_{q \in \text{sing}} \eta^{(1)}_q \int_{\Sigma_q} \langle g_q \Delta_q g_q^{-1}, d\Sigma (g_q d\Sigma g_q^{-1} + g_q A g_q^{-1}) + d\Sigma B + [g_q d\Sigma g_q^{-1} + g_q A g_q^{-1}, B] \rangle.
\]

In this form, the interpretation of the scalar field \( \Delta_q \) is obvious, it is a Lagrange multiplier imposing that \( \mathcal{D}_\Sigma \nabla_\Sigma + \nabla_\Sigma \mathcal{D}_\Sigma = 0 \) on the defect, where \( \mathcal{D} = d + A \) and \( \nabla = d + B \). This is analogous to the scalar \( B \) field in two-dimensional BF theory [47].

When constructing integrable sigma models from this action we substitute in solutions for \( A \) and \( B \) which satisfy \( \overline{\partial} A = \overline{\partial} B = 0 \). Thus, the doubled 4d CS action reduces to the unified gauged sigma model action:

\[
S_{\text{UGSM}} \equiv S_{\text{USM}}(g,A) - \frac{i}{\hbar} \sum_{q \in p} \text{res}_q (\omega \wedge \langle -d\Sigma g_q g_q^{-1} + g_q A g_q^{-1}, B \rangle)
\]

\[
- \left( \frac{i}{\hbar} \int \sum_{q \in \text{sing}} \eta^{(1)}_q \int_{\Sigma_q} \langle g_q \Delta_q g_q^{-1}, d\Sigma (g_q d\Sigma g_q^{-1} + g_q A g_q^{-1}) + d\Sigma B + [g_q d\Sigma g_q^{-1} + g_q A g_q^{-1}, B] \rangle \right).
\]

### 5.4.1 Constraints and the Unified Gauged Model’s Equations of Motion

We now turn to the equations of motion of the unified gauged sigma model and a brief discussion of the scalar fields \( \Delta_q \).

The reader will recall from section 4.1 that we made a distinction between boundary condition which are dynamical, and those which are dynamical equations of motion. Our constraints were restrictions imposed upon the phase space. For the boundary conditions defined above these are:

\[
(\bar{g}_q \partial_\pm \bar{g}_q^{-1} + \bar{g}_q A_\pm | q \bar{g}_q^{-1} |^f) = 0, \quad (\bar{g}_q \partial_\pm \bar{g}_q^{-1} + \bar{g}_q A_\pm | q \bar{g}_q^{-1} |^b = B_\pm
\]

where in the first equation one chooses either \(-, +\) or both depending on whether the boundary condition is chiral, anti-chiral or Dirichlet. When deriving models from the unified gauged action we use the constraints to fix the coefficients of the Lax connections \( A \) and \( B \).

Boundary conditions which are dynamical equations of motion are not solved when deriving examples of models, but rather are equations of motion in the model. These supplement the flatness conditions on \( A \) and \( B \), which are also equations of motion because the unified model is found from the doubled action via a reduction to the defects. Hence, the unified gauged sigma model’s equations of motion are:

\[
d_\Sigma A + A \wedge A = 0, \quad d_\Sigma B + B \wedge B = 0,
\]

\[
\{g_q (\partial_q A) | q - d_\Sigma A_q - [A_q, \Delta_q] g_q^{-1} |^b = \partial_\Sigma B_q \}
\]

Where the last equation holds only at double poles with a gauged Dirichlet boundary condition, and follows from \( \partial_\Sigma (A^b_\Sigma - B_\Sigma = 0) \). The condition \( \mathcal{D}_\Sigma \nabla_\Sigma + \nabla_\Sigma \mathcal{D}_\Sigma = 0 \), imposed at double poles, holds trivially for gauged Dirichlet boundary conditions because \( A_\Sigma = B_\Sigma \) and \( \nabla_\Sigma \Sigma = 0 \).
5.5 Examples of Gauged Models

In this following section we use the boundary conditions of section 4.1 along with generic form of the Lax connection (2.24) and the unified gauged sigma model action (5.23) to derive various sigma models. The form of the Lax connections at a pole \( q \) of \( \omega \) is given by (5.27),

\[
A|_q = g_q^{-1} A|_q g_q + g_q^{-1} dq_q, \quad B|_q = B|_q
\]  

(5.27)

We always assume that \( \omega \) has a pole at infinity at which \( g_\infty = 1 \). For ease, in the following examples we fix \( \Sigma = \mathbb{R}^2 \) with Lorentzian signature and light-cone coordinates \( x^\pm \), and denote by \( w = 1/z \) the coordinate on \( \mathbb{C}P^1 \) at infinity.

In the following we consider two example. The first of these gives the GWZW model with a BF term. This example is illustrative of more general constructions where all boundary conditions of \( A \) are gauged. In particular, given a Lax connection in traditional 4d CS we expect the associated model to have a ‘gauged’ version found from the doubled theory. This is under the proviso that all boundary conditions of \( A \) can be replaced by gauged versions à la the gauged chiral boundary conditions etc. above. In the second example we construct a nilpotent version of the GWZW model. We give this example because it shows similar constructions are possible even if \( H \) does not satisfy the conditions assumed above.

5.5.1 The Gauged WZW Model + BF Term

We consider the 4d CS action where \( \omega \) is:

\[
\omega = \frac{z - k}{z} dz,
\]  

(5.28)

with a zero at \( z = k \), a simple pole at \( z = 0 \) and a double pole at \( z = \infty \). At \( z = 0 \) we impose the gauged chiral boundary condition:

\[
A_-|_0 = B_-|_0, \quad A^\pm_0|_0 = B^\pm_0,
\]  

(5.29)

while at \( z = \infty \) (\( w = 0 \)) we impose the gauged Dirichlet boundary condition:

\[
A^\pm_\infty = 0, \quad A^\pm_\infty = B^\pm_\infty.
\]  

(5.30)

with the final gauged Dirichlet condition:

\[
\partial_w A^\pm_w = 0 = \partial_w B^\pm_w,
\]  

(5.31)

an equation of motion of the model. We also choose \( \tilde{g} \) such that \( \tilde{g}|_\infty = g_\infty = 1 \) and denote \( \tilde{g}|_0 = g_0 = g \).

At the zero \( z = k \) we impose type A defects:

\[
A_+, \quad (z - k)A_-, \quad (z - k)B_+, \quad \text{and} \quad B_- \quad \text{are regular.}
\]  

(5.32)

It thus follows from the solution to the equations of motion in the Lax gauge, equation (2.24), that \( A \) and \( B \) are of the form:

\[
A = A^c_+ dx^+ + \left( A^c_- + \frac{A^{k,0}_-}{z - k} \right) dx^- , \quad B = \left( B^c_+ + \frac{B^{k,0}_+}{z - k} \right) dx^+ + B^- dx^-.
\]  

(5.33)

Using (5.27), the boundary condition at \( z = \infty \), equation (5.30), and \( g_\infty = 1 \) it follows that:

\[
A^c_\pm = B^c_\pm.
\]  

(5.34)
Similarly, the boundary condition at \( z = 0 \), (5.29), with (5.27) and \( g_0 = g \) implies:

\[
A_+^{k,0} = k \left( B_-^c - g^{-1} \partial_- g - g^{-1} B_-^c g \right), \quad B_-^c = g^{-1} \partial_+ g + g^{-1} \left( A_+ + B_+^c - \frac{B_+^{k,0}}{k} \right) g,
\]

(5.35)

Hence, the Lax connections are:

\[
A = B_+^c \, dx^+ + \left( B_-^c + \frac{k(B_-^c - g^{-1}\partial_- g - g^{-1} B_-^c g)}{z - k} \right) \, dx^-,
\]

\[
B = \left( B_+^c + \frac{k(B_+^c - g \partial_+ g^{-1} - g B_+^c g^{-1})}{z - k} \right) \, dx^+ + B_+^c \, dx^-.
\]

(5.36)

Having found the lax connections (5.36) we substitute them into the unified gauged sigma model action (5.23). Note, since \( g_\infty = 1 \) and thus \( d_{\Sigma} g_\infty = 0 \) we need only calculate \( \text{res}_0(\omega \wedge A) \):

\[
\text{res}_0(\omega \wedge A) = -k B_+ dx^+ - k(g^{-1}\partial_- g + g^{-1} B_- g) dx^-.
\]

(5.37)

Hence, the unified sigma model term of (5.23) is:

\[
S_{USM}(A, \tilde{g}) = -\frac{i k}{\hbar} \int_{\mathbb{R}^2} d^2 x \left( \langle g^{-1} \partial_+ g - B_+^c g^{-1} \partial_- g \rangle + \langle g g^{-1}, B_-^c \rangle \right) + \frac{i k}{\hbar} \int_{\mathbb{R}^2 \times [0, R_\infty)} \langle \tilde{g}^{-1} \tilde{d} \tilde{g}, (\tilde{g}^{-1} \tilde{d} \tilde{g})^2 \rangle,
\]

(5.38)

where \( d^2 x = dx^+ \wedge dx^- \) while the Wess-Zumino term at \( z = \infty \) vanishes since \( \tilde{g} = 1 \) at both \( r_\infty = 0 \) and \( r_\infty = R_\infty \). Similarly, the second term in (5.23) gives:

\[
-\frac{i}{\hbar} \sum_{q \in \ell^p \mathbb{R}^2} \int_{\mathbb{R}^2} \text{res}_q \left( \omega \wedge \langle -d_{\Sigma} q g^{-1} q^{-1} + g_q A q^{-1}, B \rangle \right)
\]

\[
= -\frac{i k}{\hbar} \int_{\mathbb{R}_\infty} d^2 x \left( B_+^c - g B_+^c g^{-1}, B_-^c \right) + \langle \partial_+ g g^{-1}, B_-^c \rangle - \langle B_+^c, g^{-1} \partial_- g \rangle.
\]

(5.39)

And the final term of (5.23) gives a BF term:

\[
+ \frac{2i}{\hbar} \int_{\mathbb{R}^2} \langle \Delta, d_{\Sigma} B^c + B^c \wedge B^c \rangle,
\]

(5.40)

where we’ve simplified notation by setting \( \Delta_{\infty} = \Delta \). Upon combining these four equations and setting \( i\hbar = 4\pi \), we find a gauged WZW model + BF term:

\[
S_{GWZ}(g, B_+, B_-) = S_{WZW}(g) + \frac{k}{2\pi} \int_{\mathbb{R}^2} dx^+ \wedge dx^- \left( \langle \partial_+ g g^{-1} - g B_+^c g^{-1} + B_+^c, B_+^c \rangle - \langle B_+^c, g^{-1} \partial_- g \rangle \right)
\]

\[
- \frac{1}{2\pi} \int_{\mathbb{R}^2} \langle \Delta, d_{\Sigma} B^c + B^c \wedge B^c \rangle,
\]

(5.41)

where \( S_{WZW}(g) \) is the Wess-Zumino-Witten model.

The gauged WZW model’s equations of motion follow naturally from the Lax connection and the boundary conditions we’ve imposed. Using \( g_\infty = 1 \), equation (5.31) simplifies to \( \partial_\omega A^b - d_{\Sigma} \Delta^b - [A, \Delta]^b = \partial_\omega B \) - evaluated at \( w = 0 \) - and thus gives:

\[
\partial_\omega \Delta^b + [B_+^c, \Delta]^b = k(B_-^c - g^{-1} \partial_- g - g^{-1} B_-^c g)^b,
\]

(5.42)

\[
\partial_\omega \Delta^b + [B_+^c, \Delta]^b = k(g \partial_+ g^{-1} + g B_+^c g^{-1} - B_+^c)^b.
\]

(5.43)
5.5.2 The Nilpotent Gauged WZW Model

In [40, 48], Balog et al. demonstrated the conformal Toda field theories and W-algebras can be found by constraining a version of the gauged WZW model; we call this version the nilpotent gauged WZW model. As we have discussed above, the Wess-Zumino-Witten model has the symmetry group, \(G_L \times G_R\) where the \(G_L\) acts from the left \(g \rightarrow ug\) and is a function of \(x^+, u(x^+)\), while the second acts on the right \(g \rightarrow g\bar{u}\) and depends on \(x^−\). What makes this version of the gauged WZW model unusual is that one gauges these two symmetries independently from each other, finding a model whose target space is \(G/(N− \times N+)\). By introducing a gauge field \(C\) we gauge the left symmetry by a maximal nilpotent subgroup of \(G\) associated to positive roots, denoted by \(N^+\); this field is valued in the Lie algebra \(n^+\) of \(N^+\). Similarly, we introduce the gauge field \(B\) to gauge the right symmetry by a maximal nilpotent subgroup of \(G\) associated to negative roots, denoted by \(N^−\); this field is valued in the Lie algebra \(n^−\) of \(N^−\). We note \(n^−, n^+ \subset g\). One recovers the Toda theories from the nilpotent gauged WZW model by fixing the gauge \(C = B = 0\) and performing a Gauss decomposition, as discussed in [40]. In this section we will assume \(G = SL(N, C)\) in which case we work in a basis where \(n^+\) is the set of strictly upper triangular matrices, and \(n^−\) is the set of strictly lower triangular matrices. The case of \(G\) is recovered by replacing \(n^+\) and \(n^−\) by a pair maximal nilpotent subalgebras associated to positive and negative roots.

Consider a tripled version of the 4d CS model with three gauge fields \(A \in \mathfrak{sl}(n), B \in n^−, C \in n^+\):

\[
S_{\text{triple}}(A, B, C) = S_{4\text{dCS}}(A) - S_{4\text{dCS}}(B) - S_{4\text{dCS}}(C) - i \int_{\mathbb{R}^4_3} \left( \langle A, C \rangle + 2 \langle A_- dx^−, \mu dx^+ \rangle \right) + i \int_{\mathbb{R}^4_3} \left( \langle A, B \rangle + 2 \langle \omega \wedge A_+ dx^+, \nu dx^− \rangle \right),
\]

where:

\[
\omega = \frac{dz}{z},
\]

while \(\mu \in n^−\) and \(\nu \in n^+\) are constants. We fix the manifold \(\Sigma \times C\) to be \(\mathbb{R}^2 \times \mathbb{C}P^1\) where \(\mathbb{R}^2\) has the light-cone coordinates \(x^±\) and metric \(\eta^− = 2, \eta^+ = \eta^− = 0\). We take \(A, B\) and \(C\) to be in the adjoint representation of \(g\).

For each of these algebras, as well as the Cartan subalgebra of \(\mathfrak{sl}(n)\), denoted \(t\), we define our basis in the following way. For \(n^+\) our basis is \(\{e_\alpha\}\), for \(n^−\) \(\{e_−\beta\}\), for \(t\) \(\{h_\gamma\}\), and for \(\mathfrak{sl}(n)\) \(\{h_\gamma, e_\alpha, e_−\beta\}\). The indices in each basis indicate that these elements are labelled by elements of root space of \(\mathfrak{sl}\), denoted \(\Phi\). The index \(\gamma\) is in the set simple roots \(\Delta\), while \(\alpha\) and \(\beta\) are positive roots in the space \(\Phi^+\). In this basis the trace of \(g\) is given by:

\[
\langle e_\alpha, e_\beta \rangle = \frac{2}{\alpha^2} \delta_{\alpha, −\beta}, \quad \langle h_\gamma, h_\tau \rangle = \gamma^\vee \cdot \tau^\vee, \quad \langle e_\alpha, h_\gamma \rangle = 0,
\]

where \(\gamma, \tau \in \Delta, \alpha, \beta \in \Phi\), and \(\alpha^\vee = 2\alpha/\alpha^2\) is the coroot [49, 50]. We have given the derivation of these traces in appendix D. If we expand the actions \(S_{4\text{dCS}}(B)\) and \(S_{4\text{dCS}}(C)\) into their Lie algebra components, it is clear that \(S_{4\text{dCS}}(B) = S_{4\text{dCS}}(C) = 0\) by the first of equation in (5.46) where \(\langle e_\alpha, e_\beta \rangle = 0\) since \(\beta \neq −\alpha\) as the elements of \(n^+\) are labelled by the positive roots \(\Phi^+\) while the elements of \(n^−\) are labelled by the negative roots \(\Phi^−\). Hence the action (5.44) reduces to:

\[
S_{\text{triple}}(A, B, C) = S_{4\text{dCS}}(A) - i \int_{\mathbb{R}^4_3} \left( \langle A, C \rangle + 2 \langle A_- dx^−, \mu dx^+ \rangle \right) + i \int_{\mathbb{R}^4_3} \left( \langle A, B \rangle + 2 \langle \omega \wedge A_+ dx^+, \nu dx^− \rangle \right),
\]

where the fields \(B\) and \(C\) behave as Lagrange multipliers.
Since $B$ and $C$ only appear in boundary terms we have one bulk equation of motion:

$$\omega \wedge F(A) = 0,$$

where $A$ is gauge equivalent to a Lax connection $A$ by $A = \hat{g}d\hat{g}^{-1} + \hat{g}A\hat{g}^{-1}$. We note that as above $\hat{g}$ is defined by $A_2 = \hat{g}\partial_2\hat{g}^{-1}$. Because $B$ and $C$ do not have any equations of motion in the bulk we assume $\partial_2 B = \partial_2 C = \partial_2 B = \partial_2 C = 0$.

If we vary $A$, $B$ and $C$ together while using (2.14) and (2.15) we find the defect equations of motion:

$$\int_{\mathbb{R}^2_0} \left( \langle A - C, \delta A \rangle + \langle A, \delta C \rangle + 2 \langle \delta A_- dx^-, \mu dx^+ \rangle \right) = 0,$$

$$\int_{\mathbb{R}^2_\infty} \left( \langle A - B, \delta A \rangle + \langle A, \delta B \rangle + 2 \langle \delta A_+ dx^+, \nu dx^- \rangle \right) = 0.$$  

We solve (5.50) by expanding the Lie algebra components into $t, n^+$, $n^-$ and introducing nilpotent versions of gauged chiral boundary conditions:

$$A_{n^+}|_0 = C_-, \quad A_{n^-+1}|_0 = 0, \quad A_{n^-}|_0 = \mu.$$  

Note, $A_{n^+}|_0 = \mu$ implies $\delta A_{n^+}|_0 = 0$. Similarly, (5.51) is solved by a nilpotent version of gauged anti-chiral boundary conditions:

$$A_{n^-}|_\infty = B_+, \quad A_{n^-+1}|_\infty = 0, \quad A_{n^+}|_\infty = \nu,$$

where we’ve used $\partial_w B = 0$. For the purposes of the following discussion it is enough to note that these boundary conditions are preserved by gauge transformations of $A$ which $u = 1$ at $z = 0$ and $z = \infty$.

Consider the gauge transformation generated by $u = \hat{g}\hat{g}^{-1}$, where $\hat{g}$ satisfies the strong archipelago conditions. By the third strong condition we have $u = 1$ at $z = 0$ and $z = \infty$. Thus, our boundary conditions are preserved.

We can therefore simplify the bulk action $S_{4dCS}(A)$ using the strong archipelago conditions, such that (5.48) becomes:

$$S_{Tripled}(A, B, C) = S_{USM}(\hat{g}, A) - \frac{i}{\hbar} \int_{\mathbb{R}^2_0} \left( \langle A, C \rangle + 2 \langle A_- dx^-, \mu dx^+ \rangle \right)$$

$$+ \frac{i}{\hbar} \int_{\mathbb{R}^2_\infty} \left( \langle \omega \wedge A, B \rangle + 2 \langle A_+ dx^+, \nu dx^- \rangle \right)$$

where $S_{USM}(\hat{g}, A)$ is the unified sigma model (3.28).

By demanding that $A$ be regular, our Lax connection is of the form:

$$A = A_+ dx^+ + A_- dx^-.$$  

We now use:

$$A_i|_q = g_q \partial_i g_q^{-1} + g_q A g_q^{-1},$$

where $i = \pm$ and the boundary conditions (5.52) and (5.53) to fix the coefficients of $A$. As above we take $\hat{g}$ to be of the form:

$$\hat{g}|_0 = g_0 = g, \quad \hat{g}|_\infty = g_\infty = 1.$$  

The boundary conditions at $z = \infty$ (5.53), together with $g_\infty = 1$, implies:

$$A_+^c = B_+, \quad A_-^c = \nu + A_{n^-+1}^-,$$  

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while the boundary conditions at \( z = 0 \) and \( g_0 = g \) give:

\[
A_\pm^0 = g^{-1} \partial_+ g + g^{-1} (A_\pm^+ + \mu) g, \quad A_\pm^- = g^{-1} \partial_- g + g^{-1} C_- g.
\] (5.60)

Thus, the Lax connection is:

\[
A = B_+ dx^+ + (g^{-1} \partial_- g + g^{-1} C_- g).
\] (5.61)

Note, the boundary conditions at \( z = 0 \) imply the condition:

\[
(g \partial_+ g^{-1} + g B_+ g^{-1})|^{n^-} = \mu,
\] (5.62)

while the conditions at \( z = \infty \) imply:

\[
(g^{-1} \partial_- g + g^{-1} C_- g)|^{n^+} = \nu.
\] (5.63)

We now show that substituting (5.61) into (5.55) gives the nilpotent gauged WZW model [40]. The unified sigma model term of (5.55) has residues at both \( z = \infty \) and \( \infty \), however we needn’t calculate \( \text{res}_\infty (\omega \wedge A) \) since \( d_2 g_\infty = 0 \) as \( g_\infty = 1 \) meaning there is no contribution from the pole at \( \infty \). Thus, we calculate \( \text{res}_0 (\omega \wedge A) \) where we find:

\[
\text{res}_0 (\omega \wedge A) = B_+ dx^+ + (g^{-1} \partial_- g + g^{-1} C_- g) dx^-,
\] (5.64)

thus the kinetic term of the unified sigma model is:

\[
\frac{-i}{\hbar} \sum_{q \in \{0, \infty\}} \int_{\Sigma_q} \langle \text{res}_q (\omega \wedge A), q^{-1} dg_q \rangle \tag{5.65}
\]

\[
= \frac{-i}{\hbar} \int_{\mathbb{R}^2_0} dx^+ \wedge dx^- \left( -\langle g^{-1} \partial_+ g, g^{-1} \partial_- g \rangle - \langle \partial_+ g g^{-1}, C_- \rangle + \langle B_+, g^{-1} \partial_- g \rangle \right),
\]

Similarly, the other two residues in (5.55) are:

\[
\frac{-i}{\hbar} \int_{\mathbb{R}^2_0} \left( \langle A, C \rangle + 2 \langle A_- dx^-, \mu dx^+ \rangle \right) \tag{5.66}
\]

\[
= \frac{i}{\hbar} \int_{\mathbb{R}^2_0} dx^+ \wedge dx^- \left( -\langle \partial_+ g g^{-1} + g B_+ g^{-1}, C_- \rangle - 2 \langle C_-, \mu \rangle \right),
\]

\[
\frac{i}{\hbar} \int_{\mathbb{R}^2_\infty} \left( \langle A, B \rangle + 2 \langle A_+ dx^+, \nu dx^- \rangle \right) \tag{5.67}
\]

\[
= \frac{-i}{\hbar} \int_{\mathbb{R}^2_\infty} dx^+ \wedge dx^- \left( -\langle g^{-1} \partial_- g + g^{-1} C_- g, B_+ \rangle + 2 \langle B_+, \nu \rangle \right),
\]

where we have used \( \langle C_+, C_- \rangle = \langle B_+, B_- \rangle = 0 \). Upon combining all of this together and setting \( \hbar = i4\pi/k \) we find the nilpotent gauged WZW model [40]:

\[
S_{\text{NGWZW}}(g, B_+, C_-) = S_{\text{WZW}}(g) + \frac{k}{2\pi} \int_{\mathbb{R}^2} d^2 x \left( \langle \partial_+ g g^{-1}, C_- \rangle - \langle B_+, g^{-1} \partial_- g g^{-1} C_- g \rangle + \langle \mu, C_- \rangle + \langle \nu, B_+ \rangle \right),
\] (5.68)

where \( S_{\text{WZW}}(g) \) is the WZW model and \( d^2 x = dx^+ \wedge dx^- \). When one varies the fields of this action one finds that the equations of motion are the requirement that the Lax connection (5.61) is flat and the constraints (5.62,5.63). It is known from [40] that one can classically find the Toda theories from this action. In this discussion we assumed \( G = SL(N, \mathbb{C}) \) – one easily recovers the case of an arbitrary \( G \) by replacing \( n^+ \) and \( n^- \) with the maximal nilpotent subalgebras associated to positive and negative roots.

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6 Conclusion

We have reviewed the recent work of Costello and Yamazaki [12], and Delduc et al. [25]. In these papers it was shown that one could solve the equations of motion of 4d CS theory (with two-dimensional defects inserted into the bulk) by defining a class of group elements \( \hat{g} \) in terms of \( A_\pm \). Given a solution to the equations of motion, one finds an integrable sigma model by substituting the solution back into the 4d CS action. These sigma models are classical field theories on the defects inserted in to the 4d CS theory. In [25] it was shown the equivalence class of Lax connections, \( \mathcal{A} \), of an integrable sigma model are the gauge invariant content of \( \mathcal{A} \), where \( \mathcal{A} \) is found from \( A \) by performing the Lax gauge transformation (3.10). That \( \mathcal{A} \) satisfies the conditions of a Lax connection was due to the Wilson lines and bulk equations of motion of \( A \).

In section 4 we introduced the doubled 4d CS theory, inspired by an analogous construction in three-dimensional Chern-Simons [31]. In this section we coupled together two 4d CS theory fields, where the second field was valued in a subgroup of the first, by introducing a boundary term. This boundary term had the effect of modifying the defect equations of motion enabling the introduction of new classes of gauged defects associated to the poles of \( \omega \). In the rest of this section it was shown that the properties of 4d CS theory, such as its semi-topological nature or the unusual gauge transformation, are also present in the doubled 4d CS theory, even with the introduction of the boundary term.

In section 5 we used the techniques of Delduc et al. in [25] to derive the unified gauged sigma model action (3.28). It was found that this model is associated to two Lax connections, one each for \( A \) and \( B \), and some boundary conditions associated to the defects inserted in the bulk of the doubled theory. The unified gauged sigma model’s equations of motion are the flatness of the Lax connections and the boundary conditions associated to the defects. We concluded in section 5.5.2 by deriving the Gauged WZW and Nilpotent Gauged WZW models, from which one finds the conformal Toda field theories [40].

Before we finish we wish to make some additional comments. The first of these is on the relation between the doubled four-dimensional action (4.3) and its equivalent in three-dimensions:

\[
S(A, B) = S_{CS}(A) - S_{CS}(B) - \frac{1}{2\pi} \int_M d \langle A, B \rangle.
\]

(6.1)

In [33] it was proven that the 4d CS action for \( \omega = dz/z \) is \( T \)-dual to the three-dimensional Chern-Simons action. By Yamazaki’s arguments it is clear that the boundary term of the doubled action (4.3) for \( \omega = dz/z \) is \( T \)-dual to the boundary term of (6.1), hence (4.3) and (6.1) are \( T \)-dual. As a result, we expect that arguments analogous to those used in section 5 can be used to derive the gauged WZW model from (6.1). It is important to note that this is different to the derivation of the gauged WZW model from Chern-Simons theory given in [31]. This is because the introduction of the boundary term leads to a modification of the defect equations of motion and therefore the boundary conditions. This contrasts with the construction given in [31] where a Lagrange multiplier was used to impose the relevant boundary conditions.

Our second comment concerns the boundary conditions defined above. The boundary conditions used in the doubled theory are not exhaustive; just as the two chiral, and Dirichlet conditions are not exhaustive in the standard theory. In [10, 12] the authors solve the defect equations of motion by requiring that \( A \) be valued in an isotropic subalgebra \( \mathfrak{l} \) at simple poles of \( \omega \) – here \textit{Isotropic} means that for \( a, b \in \mathfrak{l} \) we have \( \langle a, b \rangle = 0 \). Similarly, in the doubled theory one can introduce a gauged version of this isotropic boundary condition. Suppose there exists an isotropic subalgebra of \( \mathfrak{g} \) in \( \mathfrak{f} \), the gauged isotropic condition is then defined by the requirement that \( A^h_\pm = B_\pm \) and \( A^f_\pm \in \mathfrak{l} \). This opens up the possibility for a \( T \)-duality between certain gauged models, as we now argue.

In [25] a somewhat more restricted version of both 4d CS and the isotropic boundary conditions were considered. In particular, reality conditions were imposed such the unified sigma model action was real. This requirement meant that first order poles of \( \omega \) must be considered in pairs, \( (q_\pm) \), such that they are
either: (a) on the real line or (b) complex conjugates. In case (a) one introduces a Manin pair \((\mathfrak{d}, \mathfrak{l})\), where\[^7\] \(\mathfrak{d} = \mathfrak{g}^R \oplus \mathfrak{g}^R\) has a subalgebra \(\mathfrak{l}\) which is maximally isotropic (or lagrangian) with respect to the bilinear form 
\[ \langle (A|_{q_+}, A|_{q_-}), \delta(A|_{q_+}, A|_{q_-}) \rangle := \epsilon_{ij} (\eta_0 q_+ A_i|_{q_+}, \delta A_j|_{q_-}) + \eta_0^0 (\delta A_i|_{q_+}, A_j|_{q_-}) \] on \(\mathfrak{d}\). The restricted isotropic boundary condition of [25] is then defined by requiring that \((A_{\pm}|_{q_+}, A_{\pm}|_{q_-}) \in \mathfrak{l}\). In case (b) one does the same thing, but with \(\mathfrak{d}\) replaced by \(\mathfrak{g}\). We expect a similar, gauged, construction to exist in the doubled theory.

In [25] the Manin double construction is extended to a Manin triple \((\mathfrak{d}, \mathfrak{l}_1, \mathfrak{l}_2)\) - where \(\mathfrak{l}_1\) and \(\mathfrak{l}_2\) are both isotropic subalgebras of \(\mathfrak{d}\) such that\[^8\] \(\mathfrak{d} = \mathfrak{l}_1 \perp \mathfrak{l}_2\). Given the Manin triple, one solves the defect equations of motion by imposing that \(A\) is valued in \(\mathfrak{l}_1\) (or \(\mathfrak{l}_2\)) at both poles. It was suggested that for a fixed \(\omega\) the models found by imposing Manin triple boundary conditions in case (a) should be Poisson-Lie T-dual to those found from case (b), where one has also imposed Manin triple boundary conditions. We hope the same is true in a gauged Manin triple boundary condition.

Our third comment concerns the scalars \(\Delta_q\). In the example of GWZW + BF theory we saw that \(\Delta\) deforms the equations of motion of the GWZW model realised when \(\Delta = 0\). We expect this to be a general feature of such models, with \(\Delta_q\) inducing integrable deformations of a theory obtained when \(\Delta_q = 0\), à la the conformal Toda models found from deformations of Toda theories [51–53]. We intend to explore this elsewhere.

Finally, our hope is that one can find other new integrable gauged sigma models using the construction defined in section 5. This being said, there are several other problems which we have not discussed in this paper, but which we plan to cover in the future. These include \(\lambda\)–[54, 55], \(\eta\)–[56, 57], and \(\beta\)-deformations [58–60], this is expected to be similar to [61] and [62–64]; the generation of affine Toda models from 4d CS theory; the generation of gauged sigma models associated to a higher genus choice of \(C\) – we expect this to be analogous to the discussion near the end of [12]; how to find a set of Poisson commuting charges from \(A\) and \(B\) such that \(A\) and \(B\) are Lax connections; related to this is, the connection between our construction of gauged sigma models and that given by Gaudin models, this is likely similar to [24]; the quantum theory of the doubled action; and finally whether the results of [22] can be repeated for the doubled action, enabling us to find higher dimensional integrable gauged sigma models.

A The K"unneth Theorem and Cohomology

The K"unneth theorem gives one a relation between the cohomologies of a product space and the cohomologies of the manifolds which it is constructed from:

\[ H^k(X \times Y) = \bigoplus_{i+j=k} H^i(X) \otimes H^j(Y). \] (A.1)

The de Rham cohomology for \(\mathbb{R}^n\) is:

\[ H^k(\mathbb{R}^n) \cong \begin{cases} \mathbb{R}, & \text{if } k = 0, \\ 0, & \text{otherwise}. \end{cases} \] (A.2)

While for \(\mathbb{C}P^n\) this is:

\[ H^k(\mathbb{C}P^n) \cong \begin{cases} \mathbb{R}, & \text{for } k \text{ even and } 0 \leq k \leq 2n, \\ 0, & \text{otherwise}. \end{cases} \] (A.3)

\[^7\] \(\mathfrak{g}^R\) is a real form of \(\mathfrak{g}\).

\[^8\] Here \(\perp\) denotes the direct sum as a vector space.
B  Localisation of the Wess-Zumino term.

In this appendix we give the derivation of equation (5.21). To do this we use propositions 3.7 of [41], whose primary argument we repeat here. Consider the Wess-Zumino term constructed from a group element \( \tilde{g} \) satisfying the weak archipelago conditions. Using the first condition we expand the integral into a sum of integrals over each open disk \( U_q \):

\[
I = \int_{\Sigma \times \mathbb{P}^1} \omega \wedge \langle \tilde{g}^{-1}d\tilde{g}, (\tilde{g}^{-1}d\tilde{g})^2 \rangle = \sum_{q \in \partial \Sigma} \int_{\Sigma \times U_q} \omega \wedge \langle \tilde{g}_q^{-1}d\tilde{g}_q, (\tilde{g}_q^{-1}d\tilde{g}_q)^2 \rangle. 
\]

(B.1)

Let \( \gamma := [0, 1] \). Under the assumption that \( \Sigma \times U_q \) is contractible the form \( \langle \tilde{g}^{-1}d\tilde{g}, (\tilde{g}^{-1}d\tilde{g})^2 \rangle \) is closed, and therefore exact - thus we have:

\[
I = \sum_{q \in \partial \Sigma} \int_{\Sigma \times U_q} \omega \wedge d\left( -\int_{\gamma} \langle \tilde{g}_q^{-1}d\tilde{g}, (\tilde{g}_q^{-1}d\tilde{g})^2 \rangle \right). 
\]

(B.2)

Using equation (2.14) we find:

\[
I = \sum_{q \in \partial \Sigma} \sum_{l=0}^{n_q-1} \frac{\eta_q^{(l)}}{l!} \int_{\Sigma_q} \partial_z^l \left( -\int_{\gamma} \langle \tilde{g}_q^{-1}d\tilde{g}_q, (\tilde{g}_q^{-1}d\tilde{g}_q)^2 \rangle \right) = -\sum_{q \in \partial \Sigma} \sum_{l=0}^{n_q-1} \frac{\eta_q^{(l)}}{l!} \int_{\Sigma_q \times [0,1]} \partial_z^l \langle \tilde{g}_q^{-1}d\tilde{g}_q, (\tilde{g}_q^{-1}d\tilde{g}_q)^2 \rangle,
\]

(B.3)

where in the second equality we have identified \( \gamma \) with the radial interval of the disc. In all cases discussed in the paper \( n_q \) is at most 2, therefore we expand the \( z \) derivative to first order and find:

\[
I = -\sum_{q \in \partial \Sigma} \eta_q^{(0)} \int_{\Sigma_q \times I} \langle \tilde{g}_q^{-1}d\tilde{g}_q, (\tilde{g}_q^{-1}d\tilde{g}_q)^2 \rangle - \sum_{q \in \partial \Sigma} \int_{\Sigma_q} \langle g_q \Delta q g_q^{-1}, d_\Sigma (g_q d_\Sigma g_q^{-1}) \rangle ,
\]

(B.4)

where to find the final term one uses \( \partial_z (\tilde{g}^{-1}d_\Sigma \tilde{g}) |_q = g_q^{-1}d_\Sigma (g_q \Delta q g_q^{-1}) g_q \) and integrated by parts. Note, when working in the strong archipelago gauge \( \Delta_q = 0 \).

C  WZW and Gauged WZW Model Conventions

The WZW model is constructed from the field \( g : \mathbb{R}^2 \rightarrow G \), where \( G \) is a complex Lie group, and is defined by the action:

\[
S_{\text{WZW}}(g) = \frac{k}{8\pi} \int_{\mathbb{R}^2} d^2x \sqrt{-\eta_{\mu\nu}} \langle g^{-1}\partial_\mu g, g^{-1}\partial_\nu g \rangle + \frac{k}{12\pi} \int_B \langle g^{-1}dg, g^{-1}dg \wedge g^{-1}dg \rangle,
\]

(C.1)

where \( d^2x = dx^+ \wedge dx^- \), \( \eta_{\mu\nu} \) a metric on \( \mathbb{R}^2 \), \( \eta \) the determinant of \( \eta_{\mu\nu} \), and \( \tilde{g} \) the extension of \( g \) into the three-dimensional manifold \( B \), where \( \partial B = \mathbb{R}^2 \). In this paper we take \( B = \mathbb{R}^2 \times [0, R_0] \) with light-cone coordinates \( x^{\pm} \) on \( \mathbb{R}^2 \) and metric \( \eta^{+} = 2, \eta^{+} = \eta^{-} = 0 \). Our light-cone coordinates are connected to the Lorentzian coordinates \( x^0, x^1 \) by \( x^+ = x^0 + x^1 \) and \( x^- = x^0 - x^1 \) with the Minkowski metric \( \eta_{00} = -\eta_{11} = 1, \eta_{01} = 0 \).

The WZW action is invariant under transformations of the form \( g \rightarrow u(x^+)\tilde{g} \tilde{u}(x^-)^{-1} \) in \( G_L \times G_R \) where \( u \in G_L \) and \( \tilde{u} \in G_R \). To show this invariance one defines an extension of \( u \) and \( \tilde{u} \) into \( B \), denoted \( \bar{u} \), and uses the Polyakov-Wigmann identity:

\[
S_{\text{WZW}}(gh) = S(g) + S(h) + \frac{k}{2\pi} \int_{\mathbb{R}^2} dx^+ \wedge dx^- \langle g^{-1}\partial_- g, \partial_+ h h^{-1} \rangle,
\]

(C.2)
to expand $S_{WZW}(ug\bar{u})$ into a sum over WZW terms. Upon doing this one finds all terms other than $S_{WZW}(g)$ vanish. On $B = \mathbb{R}^2 \times [0, R_0]$ we parametrise $[0, R_0]$ by $z$ and define the extension $\bar{u}$ such that $\bar{u}|_{z=0} = \bar{u}$ and $\bar{u}|_{z=R_0} = u$, this ensures a cancellation of the Wess-Zumino terms associated to $u$ and $\bar{u}$. All other terms vanish due to $\partial_- u = \partial_+ \bar{u} = 0$.

From the variation $g \rightarrow g + \delta g$ in (C.1) one finds the variation of the action:

$$\delta S(g) = -\frac{k}{2\pi} \int_{\mathbb{R}^2} dx^+ \wedge dx^- \langle g^{-1}\delta g, \partial_+(g^{-1}\partial_- g) \rangle = -\frac{k}{2\pi} \int_{\mathbb{R}^2} dx^+ \wedge dx^- \langle \delta gg^{-1}, \partial_+ (\partial_+ gg^{-1}) \rangle,$$

(C.3)

which gives the equations of motion:

$$\partial_+(g^{-1}\partial_- g) = \partial_-(\partial_+ gg^{-1}) = 0,$$

(C.4)

where $J_+ = \partial_+ gg^{-1}$ and $J_- = g^{-1}\partial_- g$ are the currents of the model. These equations have the solution:

$$g(x^+, x^-) = g_l(x^+)g_r(x^-)^{-1},$$

(C.5)

where $g_l (g_r)$ is a generic holomorphic (anti-holomorphic) map into $G$.

One can define a version of the WZW model where the symmetry $g \rightarrow ugu^{-1}$ is gauged by a group $H \subseteq G$, this gives an action to the coset models [28–30] as shown in [34–38]. This gauged WZW model can be found from the normal WZW model by applying the Polyakov-Wigmann identity (C.2) to:

$$S_{\text{Gauged}}(g, h, \tilde{h}) = S_{WZW}(h\tilde{h}^{-1}) - S_{WZW}(h\tilde{h}^{-1}),$$

(C.6)

where $h(x^+, x^-), \tilde{h}(x^+, x^-) \in H$. It is clear that this equation is invariant under the transformation $g \rightarrow ugu^{-1}, h \rightarrow hu^{-1}, \tilde{h} \rightarrow hu^{-1}$ for $u(x^+, x^-) \in H$. After expanding (C.6) and setting $B_- = h^{-1}\partial_- h$ and $B_+ = \tilde{h}^{-1}\partial_+ \tilde{h}$ one finds gauged WZW model action:

$$S_{\text{Gauged}}(g, B_+, B_-) = S_{WZW}(g) + \frac{k}{2\pi} \int_{\mathbb{R}^2} dx^+ \wedge dx^- (\langle \partial_+ gg^{-1}, B_- \rangle - \langle B_+, g^{-1}\partial_- g \rangle - \langle gB_+ g^{-1}, B_- \rangle + \langle B_+, B_- \rangle),$$

(C.7)

where the symmetry $g \rightarrow ugu^{-1}, h \rightarrow hu^{-1}, \tilde{h} \rightarrow \tilde{h}u^{-1}$ corresponds to the gauge transformation:

$$g \longrightarrow ugu^{-1}, \quad B_\pm \longrightarrow u(\partial_\pm + B_\pm)u^{-1},$$

(C.8)

for $u(x^+, x^-) \in H$. This gauge symmetry means the orbits of $G$ which are mapped to each other by the action of $H$ are identified and therefore physical equivalent, hence the target space of the gauged WZW model is the coset $G/H$.

It is important to note that two conventions for the WZW model and Polyakov-Wigmann identity exist which are related by $g \rightarrow g^{-1}, h \rightarrow h^{-1}$. Further still, four conventions for the gauged WZW models exist found by taking $g \rightarrow g^{-1}$ and $B_+ \rightarrow -B_+$ in various combinations.

D The Cartan-Weyl Basis

Here we collect some facts about Lie algebras from [45]. A semi-simple Lie algebra $\mathfrak{g}$ can be decomposed into three subalgebras $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{t} \oplus \mathfrak{n}^-$. The first subalgebra is a Cartan subalgebra $\mathfrak{t}$ which is a maximal set of commuting semi-simple\(^9\) elements of $\mathfrak{g}$. We take $\{H_i\}$ to be a basis of $\mathfrak{t}$. We can choose a basis of $\mathfrak{n}^+$ to be

\(^9\)An element $x \in \mathfrak{g}$ is semi-simple if the matrix of eigenvalues formed by the adjoint action $\text{ad}_x$ is diagonalisable.
$\{e_\alpha\}$ where the set $\{\alpha\} = \Phi^+$ is called the set of positive roots. Similarly, $\{e_{-\alpha}\}$ span $\mathfrak{n}^-$ and $\{-\alpha\} = \Phi^-$ is the set of negative roots.

The Killing form on $\mathfrak{g}$ is $K(x,y) = \text{Tr}(\text{ad}_x \circ \text{ad}_y)$, where $\text{ad}_x$ denotes the adjoint action of $x$, $\circ$ the composition of maps, and $\text{Tr}$ over linear maps. Let $\langle \cdot, \cdot \rangle$ denote a symmetric invariant bilinear form proportional to the Killing form. We can always choose the basis elements $H_i$ to be orthonormal. With these choices, we can take the commutators to be

$$[H_i, H_j] = 0, \quad [H_i, e_\pm \alpha] = \pm \alpha^i e_{\pm \alpha}, \quad (D.1)$$

$$[e_\alpha, e_{-\alpha}] = \frac{2\alpha^i}{\alpha^2} H_i, \quad [e_{\pm \alpha}, e_{\pm \beta}] = \epsilon(\pm \alpha, \pm \beta) e_{\pm \alpha \pm \beta}. \quad (D.2)$$

where $\epsilon(\pm \alpha, \pm \beta)$ is a structure constant for any pair of $\pm$, $\alpha^i$ the $i$-th element of $\alpha \in \Phi^+$, and $\alpha^2 = \langle \alpha, \alpha \rangle$. If in the final equation $\pm \alpha \pm \beta \not\in \Phi^+$ then $\epsilon(\pm \alpha, \pm \beta) = 0$.

Let $\Delta$ denote the set of generators of $\Phi^+$, which are called simple roots. For each root $\alpha \in \Phi$ one can define an element of the Cartan Subalgebra given by $h_\alpha = \alpha^\vee H_i$, where $\alpha^\vee = 2\alpha_i/\alpha^2$ is the coroot. The set of elements $\{h_\alpha\}$ labelled by a simple root, $\alpha \in \Delta$, form a basis of the Cartan subalgebra. From this result the equations (D.1,D.2) can be rewritten as:

$$[h_\gamma, h_\beta] = 0, \quad [h_\gamma, e_{\pm \beta}] = \pm \gamma^\vee \cdot \beta e_{\pm \beta}, \quad (D.3)$$

$$[e_\alpha, e_{-\alpha}] = h_\alpha, \quad [e_{\pm \alpha}, e_{\pm \beta}] = \epsilon(\pm \alpha, \pm \beta) e_{\pm \alpha \pm \beta}, \quad (D.4)$$

where $\gamma, \tau \in \Delta$ and $\alpha, \beta \in \Phi^+$. Note, to each root $\alpha \in \Phi^+$ we can associate an $\mathfrak{sl}(2)$ given by $\mathfrak{g}_\alpha = \{e_\alpha, e_{-\alpha}, h_\alpha\}$.

The inner product $\langle h_\alpha, h_\beta \rangle$ is found by noting the basis elements $\{H_i\}$ are orthonormal, i.e. $\langle H_i, H_j \rangle = \delta_{ij}$, hence:

$$\langle h_\alpha, h_\beta \rangle = \frac{4\alpha_i \beta_j}{\alpha^2 \beta^2} \langle H_i, H_j \rangle = \langle \alpha^\vee, \beta^\vee \rangle, \quad (D.5)$$

where $\langle \alpha^\vee, \beta^\vee \rangle$ is the symmetrised Cartan matrix. The final bilinear form to be found is $\langle e_\alpha, e_{-\alpha} \rangle$. Using the identity $\langle X, [Y, Z] \rangle = ([X, Y], Z)$ it is clear that:

$$\langle \alpha^\vee, \alpha \rangle \langle e_\alpha, e_{-\alpha} \rangle = \langle h_\alpha, [e_\alpha, e_{-\alpha}] \rangle = \langle h_\alpha, h_\alpha \rangle = 4 \alpha^2, \quad (D.6)$$

hence our trace in the basis $\{h_\alpha, e_\alpha, e_{-\alpha}\}$ is:

$$\langle e_\alpha, e_\beta \rangle = \frac{2}{\alpha^2} \delta_{\alpha, -\beta}, \quad \langle h_\gamma, h_\tau \rangle = \gamma^\vee \cdot \tau^\vee, \quad \langle e_\alpha, h_\gamma \rangle = 0, \quad (D.7)$$

where $\gamma, \tau \in \Delta$ and $\alpha, \beta \in \Phi$.

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