The Fredholm index of a pair of commuting operators

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Abstract

This paper concerns Fredholm theory in several variables, and its applications to Hilbert spaces of analytic functions. One feature is the introduction of ideas from commutative algebra to operator theory.

Specifically, we introduce a method to calculate the Fredholm index of a pair of commuting operators. To achieve this, we define and study the Hilbert space analogs of Samuel multiplicities in commutative algebra.

Then the theory is applied to the symmetric Fock space. In particular, our results imply a satisfactory answer to Arveson’s program on developing a Fredholm theory for pure $d$-contractions when $d = 2$, including both the Fredholmness problem and the calculation of indices. We also show that Arveson’s curvature invariant is in fact always equal to the Samuel multiplicity for an arbitrary pure $d$-contraction with finite defect rank. It follows that the curvature is a similarity invariant.

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0 Introduction

0.1 Background and motivation

The idea of developing Fredholm theory, beyond that of a single operator, has been around for many years. In particular, a theory based on J. Taylor’s Koszul complex approach to multivariable spectral theory [46], [47] seems particularly appealing, and can be formulated as the index theory of an abstract Dirac operator [4], which can be regarded as an abstraction of the local model of Dirac operators ([29], page 310) in Riemannian geometry. But currently there are essentially no effective tools available to calculate the Fredholm index in several variables. In this paper we will introduce a general method to calculate the index in the two variable case, which can be regarded as a Hilbert space version of a classical theorem of J.-P. Serre in local algebra ([44], page 57, Theorem 1). A novelty of our approach is the extensive use of ideas from commutative algebra in a Hilbert space setting.

Among the applications to operator theory we mention two results next. Theorem 11 implies a satisfactory answer to Arveson’s program [6] on developing a Fredholm theory for pure d-contractions when d = 2, including both the Fredholmness problem and the calculation of indices. It is noteworthy that the Fredholmness problem is usually quite subtle, and previous results are scarce [5], [6]. Theorem 18 shows that Arveson’s curvature invariant is in fact always equal to the Samuel multiplicity, a notion borrowed from algebra, for an arbitrary pure d-contraction with finite defect rank. This provides an answer to Arveson’s question [4], [5] on how to express the curvature invariant in terms of other invariants which are determined by the d-contraction directly, and can immediately imply that the curvature is an integer. From Theorem 18 it follows that the curvature is a similarity invariant, which is previously unknown.

Next we discuss some background information and motivations. In the past much of the effort in higher dimensional Fredholm theory was devoted to the study of general Fredholm complexes, especially their stability under various perturbations. See Ambrozie-Vasilescu [1], Curto [11], [14], Eschmeier-Putinar [18], Segal [43], Vasilescu [49], and the references therein (most of these contain extensive bibliographies). What is more relevant to this paper is the study of numerical invariants. Along this line, Carey-Pincus [39], Levy [35], [36], Putinar [41] and the last chapter of the book of Eschmeier-Putinar [18], establish many connections between multivariable Fredholm
theory and various areas in mathematics, such as K-theory, index theory, geometric measure theory, sheaf theory, and even cyclic cohomology in non-commutative geometry. Also, see Arveson’s expository paper [6] for a recent development along a different line.

However, as far as the calculation of multivariable Fredholm indices is concerned, still no effective machineries are available. Although there are many evidences showing that such calculations are valuable and should have close connections with index theory of Dirac operators in geometry [34], [43] and function theory over various domains in $\mathbb{C}^n$ [18].

What we want to emphasize in this paper is that Koszul complexes are not just general Fredholm complexes: there exist many commutative algebraic features, such as tensorial properties, which are particular to Koszul complexes, and not enjoyed by general Fredholm complexes. These features play significant roles in many areas of commutative algebra, but have not been exploited in operator theory. One of the contributing factors to this situation might be that Koszul complexes over Hilbert spaces are somehow difficult to handle from both algebraic and analytic aspects: Hilbertian Koszul complexes lack basic properties such as Noetherian conditions to be algebraically amenable, while their algebraic structure is not familiar to many analysts.

Our motivation is the following observation. J. Taylor’s work shows that in order to define fundamental concepts such as invertibility in multivariable operator theory, objects (i.e., Koszul complexes) with a rich algebraic theory necessarily enter the picture; it suggests that a well developed multivariable operator theory will be inextricably interwoven with homological and commutative algebraic methods. The Hilbert module program of Douglas-Paulsen [15] provides such a framework.

On the other hand, J. Taylor’s framework is well suited for Fredholm theory [11]. This leads us to think about:

(A) how to formulate Fredholm theory from an algebraic viewpoint; and

(B) how to connect existing parts of operator theory to this algebraic viewpoint.

What appears to be plausible to us is that, for (A) Fredholm theory can be studied as a Koszul homology, defined on topological modules (or, Hilbert modules [15]), and we seek to develop methods to calculate this (topological) homology theory; for (B), we can interpret natural operator theoretic invariants, such as the kernel $\ker(T)$ and cokernel $H/TH$ of an
operator $T \in B(H)$, in terms of homological invariants: they are just the two homology groups of the length one Koszul complex $0 \to H \xrightarrow{T} H \to 0$.

More generally, a systematic way to introduce homological and commutative algebraic methods into operator theory is described in [24]. For a single operator, this program turns out to be fruitful and leads to results in operator theory [20], [21]. This paper represents some progress in the more than one variable case.

Thus the purpose of this paper is twofold. Firstly, we show that many algebraic methods carry over to the study of Koszul complexes over Hilbert modules; in the meanwhile, these algebraic considerations lead to new phenomena, which are unseen for algebraic modules (see Subsection 1.1). Secondly, and perhaps more importantly, we show that these purely algebraic machineries interact well with function theory and operator theory, and indeed lead to operator theoretic results.

Since the paper treats both algebraic theory and function theory, sometimes we have provided more details than necessary, for the benefit of readers with different backgrounds. In particular, operator theorists might be interested in more references in algebra: Eisenbud’s book [17] is a standard resource for Koszul complexes, Hilbert polynomials and Samuel multiplicities; the expository paper [17] of Auslander-Buchbaum provides a nice treatment of Serre’s multiplicity theory; various ideas in Serre’s book [44], P. Roberts’ book [42], and the papers of C. Lech [32], [33] have been very helpful. Also Balcerzyk-Jozefiak’s book [8] contains another proof of Serre’s theorem on the Euler characteristic and Samuel multiplicity.

0.2 Organization

The body of the paper contains four sections.

Section 1 develops the necessary algebraic theory for Koszul complexes over Hilbert spaces, which is needed in subsequent sections. In particular, it introduces a set of refined invariants to calculate the Fredholm index of any Fredholm pair of commuting operators (1.1). Moreover, it contains induction arguments (1.3), estimates of Hilbert functions (1.4), and examples.

Section 2 contains the applications to the Fredholm theory on the $L$-valued symmetric Fock space in two variables (Theorem 11). Here $L$ is a separable Hilbert space, not necessarily finite dimensional. In particular, we
completely characterize the Fredholmness of the pair of multiplication by coordinate functions on invariant subspaces. Then the Fredholm index is calculated in terms of the Samuel multiplicity.

Section 3 establishes a version of the additivity of Hilbert polynomials (Theorem 15). It follows a monotonicity property of Samuel multiplicities (Corollary 16), which can be regarded as a higher dimensional generalization of the codimension-one property for invariant subspaces of the Hardy or Dirichlet space over the unit disc.

The study of this additivity property is intimately related to the study of the curvature invariant introduced by Arveson [3]. We show that the curvature invariant is always equal to the Samuel multiplicity for an arbitrary pure d-contraction with finite defect rank (Theorem 18). As a consequence, the curvature invariant is shown to be preserved under similarity relations (Corollary 19).

Section 4 provides one more proof of the index formula established in Theorem 3 Part (ii), and some comments concerning further studies.

0.3 The one variable case

The one variable case can serve as a good illustration of the basic ideas, which we shall explore. Moreover, this paper makes essential use of the one variable theory developed in [20]. Because we shall apply results in [20] (mainly Theorem 2 and its proof) in a non-Hilbert-space setting, which is more general than what is actually proved in that paper, not only shall we use the theorems there, but also the arguments to prove them. Since it does not seem worthwhile to give a separate treatment of this non-Hilbert-space case, we advise readers unfamiliar with [20] to go through the proof of Theorem 2 there.

Recall that, for a single Fredholm operator $T$, its index is defined to be

$$\text{indexs}(T) = \dim \ker(T) - \dim H/TH.$$ 

Then $\text{index}(T)$ has remarkable stability, while, individually, $\ker(T)$ and $H/TH$ are sensitive to perturbations. The starting point is the simple fact

$$\text{index}(T^k) = k \cdot \text{index}(T),$$

or, equivalently,

$$\text{index}(T) = \frac{\dim \ker(T^k)}{k} - \frac{\dim H/T^kH}{k}. $$
So, instead of $\text{ker}(T)$ and $H/TH$, we look at the asymptotic behavior of $\text{ker}(T^k)$ and $H/T^kH$ as $k \to \infty$. At this point, experts can immediately recognize that this will connect operator theory with Hilbert polynomials in algebraic geometry, and the asymptotic behavior of Grothendieck’s local cohomology modules \cite{9}, which is a currently active research area in commutative algebra. These connections seem promising and are not fully understood.

When $k >> 0$, one has

\[
\text{dim ker}(T^k) = ak + b \\
\text{dim } H/T^kH = ck + d
\]

for some integers $a$, $b$, $c$, and $d$. These equalities might not be obvious, but they are simple consequences of the existence of Hilbert polynomials, or they can be directly proved by showing that, the dimension of the difference $T^kH \ominus T^{k+1}H$ stabilizes, as $k \to \infty$. By Equation (2)

\[
\text{index}(T) = a - c.
\]

Formally, $a$ and $c$ are analogous to Samuel multiplicities in algebra. Hence one can ask a variety of questions from an algebraic viewpoint. But in the case of operator theory, we feel that the real significance of the integers $a$ and $c$ lies in the fact that, they are defined on a module with a topology, that is, the Hilbert space $H$ as a $\mathbb{C}[z]$-module, hence they might have some geometric meaning; when compared with $\text{dim ker}(T)$ and $\text{dim } H/TH$, $a$ and $c$ have more stability, and the stability might lead to some geometric interpretation of $a$ and $c$. This is indeed true, and has lead to results in operator theory \cite{20}, \cite{21}.

1 Algebraic theory: Hilbertian Koszul complexes

1.1 Fredholm index and Samuel multiplicity

Given a tuple of commuting operators $T = (T_1, \cdots , T_d)$, acting on a Hilbert space $H$, we can endow $H$ with an $A = \mathbb{C}[z_1, \cdots , z_d]$-module structure by \cite{15}

\[
(p(z_1, \cdots , z_d), \xi) \in A \times H \to p(T_1, \cdots , T_d)\xi \in H.
\]
For any $x \in A$, we denote by
\[ K(x) = 0 \rightarrow A \xrightarrow{x} A \rightarrow 0 \]
the Koszul complex associated with $x$. For $x_1, \cdots, x_r \in A$, we define the Koszul complex
\[ K(x_1, \cdots, x_r) = K(x_1) \otimes_A K(x_2) \otimes_A \cdots \otimes_A K(x_r). \]
See [17] for basic definitions about tensor products of complexes. For any $A$-module $M$, we define
\[ K(x_1, \cdots, x_r; M) = M \otimes_A K(x_1, \cdots, x_r). \]
In the case of a Hilbert space $H$, whose $A$-module structure is induced by a commuting operator tuple $T = (T_1, \cdots, T_d)$, we also write $K(T_1, \cdots, T_d; H) = K(z_1, \cdots, z_d; H)$. Let $H_iK(\cdots; M)$ denote the $i$th homology group of a Koszul complex $K(\cdots; M)$.

Then comes J. Taylor’s definition of invertibility: for a single operator $T$ we say that $T$ is invertible if and only if
\[ H \otimes_{C[z]} K(z) \]
is exact, equivalently, $0 \rightarrow H \xrightarrow{T} H \rightarrow 0$ is exact; for a commuting tuple $T = (T_1, \cdots, T_d)$, $T$ is invertible if and only if
\[ H \otimes_A K(z_1) \otimes_A \cdots \otimes_A K(z_d) \]
is exact. We say that $(T_1, \cdots, T_d)$ is Fredholm if each $H_iK(T_1, \cdots, T_d; H)$ is finite dimensional as a vector space; then the Fredholm index of the tuple $T$ is defined as
\[ \text{index}(T) = (-1)^d \chi(K(T; H)) = \sum_{i=0}^{d} (-1)^{d-i} \dim H_iK(T_1, \cdots, T_d; H). \]

Our definition of the Koszul complex and Taylor invertibility has a strong algebraic flavor, and looks different from some of the past references in multivariable operator theory, say [4], [14], [18], [49]. They are of course equivalent. Our definition will allow one to work with algebraic properties more naturally.

Because of the following Proposition, we shall be able to adopt a strategy similar to that used for the study of a single operator.
Proposition 1  Let \((T_1,T_1',T_2,\cdots,T_d)\) be a \((d+1)\) tuple of commuting operators, acting on a Hilbert space \(H\). If any two of the following three \(d\)-tuples 
\((T_1,T_2,\cdots,T_d)\), \((T_1',T_2,\cdots,T_d)\), and \((T_1T_1',T_2,\cdots,T_d)\) are Fredholm, then so is the third one. When all are Fredholm, we have

\[\text{index}(T_1T_1',T_2,\cdots,T_d) = \text{index}(T_1,T_2,\cdots,T_d) + \text{index}(T_1',T_2,\cdots,T_d).\]

Corollary 2  For any \(d\)-tuple \(T = (T_1,\cdots,T_d)\) of commuting operators, if \(T\) is Fredholm, then

\[\text{index}(T_1^{n_1}\cdots T_d^{n_d}) = n_1\cdots n_d \text{index}(T_1,\cdots,T_d)\]

for all \(n_1,\cdots,n_d \in \mathbb{N}\).

Proof of Proposition 1  The proof is purely algebraic, and we adopt the arguments in \[42\] page 97. We also call the readers attention to Putinar’s \[40\] where many of the basic algebraic properties of a Koszul complex were established.

Consider \(H\) as a Hilbert module over the polynomial ring \(A\) in \((d+1)\) variables, with module actions given by \((T_1,T_1',T_2,\cdots,T_d)\).

First we remark that, if we also consider \(H\) as a Hilbert module over the polynomial ring in \(d\) variables, with module actions given by \((T_1,T_2,\cdots,T_d)\), then there are two natural ways to associate a Koszul complex to the tuple \((T_1,T_2,\cdots,T_d)\). The resulting Koszul complexes \(K(T_1,\cdots,T_d;H)\) are in fact independent of the base ring and hence produce the same Fredholm index.

Let \(E = 0 \rightarrow E_1 \xrightarrow{d_1} E_0 \rightarrow 0\) be the complex given by \(E_1 = E_0 = A \oplus A\), and \(d_1 = \begin{pmatrix} z_1 & 1 \\ 0 & z_2 \end{pmatrix}\). Then we have an exact sequence

\[0 \rightarrow K(z_1;A) \rightarrow E \rightarrow K(z_2;A) \rightarrow 0\] \hspace{1cm} (3)

by embedding \(K(z_1;A)\) into the first copy of \(A\) of \(E_1\) and \(E_0\).

Next, we relate \(E\) to \(K(z_1z_2;A)\) by the following exact sequence

\[0 \rightarrow K(z_1z_2;A) \xrightarrow{f_1} E \xrightarrow{g_1} K(1,A) \rightarrow 0.\] \hspace{1cm} (4)

Here \(f_1 = (-1,z_1), f_0 = (0,1), g_1 = (z_1,1),\) and \(g_0 = (1,0)\).
Now apply $H \otimes_A \otimes_A K(z_3, \cdots , z_{d+1})$ to sequence (3) and (4). Recall that tensoring with a complex of free modules preserves exactness. Moreover, $\chi(K(1, z_3, \cdots , z_{d+1}; H)) = 0$, since the d-tuple $(1, T_3, \cdots , T_{d+1})$ is invertible. Now we can complete the proof by using the additivity of the Euler characteristics. □

In particular, we have, for any $k \in \mathbb{N}$,

\[
\text{index}(T_1^k, T_2^k) = k^2 \text{index}(T_1, T_2). \tag{5}
\]

So for a pair $T = (T_1, T_2)$, we define the homological Hilbert functions

\[
h_i(k) = \dim H_i K(T_1^k, T_2^k; H) \tag{6}
\]

for $i = 0, 1, 2$, $k \in \mathbb{N}$. See [9], [42] for similar definitions in algebra. Then

\[
\text{index}(T_1, T_2) = \frac{h_2(k)}{k^2} - \frac{h_1(k)}{k^2} + \frac{h_0(k)}{k^2}. \tag{7}
\]

Now by analogy with Hilbert polynomials, it is natural to expect that each $h_i(k)$ is a polynomial in $k$, when $k >> 0$, and the leading coefficients of $h_i$ are the two-variable-analogs of the numbers $a$ and $c$, as discussed in the introduction. Unfortunately, this is not the case, as illustrated by the following example.

Example $h_i(k)$ may not be a polynomial when $k >> 0$.

For a similar result in algebra, see [25]. The author thanks P. Roberts for bringing this paper to his attention. However, there exists a related polynomial in terms of the length of local cohomology modules in the graded case ([9], page 317, Theorem 17.1.9).

Let $H^2(\mathbb{D})$ be the Hardy space over the unit disc, and $M_z$ be the multiplication by the coordinate function $z$. Let

\[
H = H^2(\mathbb{D}) \oplus H^2(\mathbb{D}), \quad T_1 = \begin{pmatrix} M_z & 0 \\ 0 & M_z \end{pmatrix}, \quad \text{and} \quad T_2 = \begin{pmatrix} 0 & M_z^2 \\ 1 & 0 \end{pmatrix}.
\]

Then $T_1, T_2$ commute. Observe that

\[
T_2^{2t} = \begin{pmatrix} M_z^{2t} & 0 \\ 0 & M_z^{2t} \end{pmatrix}, \quad \text{and} \quad T_2^{2t+1} = \begin{pmatrix} 0 & M_z^{2t+2} \\ M_z^{2t} & 0 \end{pmatrix}.
\]
So

\[ h_0(k) = \begin{cases} 2k, & \text{k even;} \\ 2k - 1, & \text{k odd,} \end{cases} \]

which is not a polynomial when \( k > 0 \).

Because \( h_2(k) = 0 \) for all \( k \), it follows from Equation (5) that, \( h_1(k) \) is also not a polynomial, when \( k > 0 \).

The function \( h_2(k) \) may not be a polynomial for \( k > 0 \) follows from considering \( (T_1^*, T_2^*) \).

However, we have

**Theorem 3** Let \( T = (T_1, T_2) \) be a pair of commuting operators, acting on a Hilbert space \( H \). Assume that \( T \) is Fredholm. Then

(i) \( h_i(k) \) \((i = 0, 1, 2)\) may not be a polynomial for \( k > 0 \), but the limit

\[ e_i = e_i(T) = \lim_{k \to \infty} \frac{h_i(k)}{k^2}, \]

exists, and is an integer. In particular, each \( e_i \) can be strictly positive.

(ii) \( \text{index}(T_1, T_2) = e_2 - e_1 + e_0 \).

(iii) \( e_i(T_1^*, T_2^*) = t \cdot s \cdot e_i(T_1, T_2) \), for \( t, s \geq 0, \; i = 0, 1, 2 \).

**Proof.** Parts (i) and (ii) of the theorem can be quickly reduced to the existence of \( e_0 \) (Subsection 1.2), as follows: First we have \( H/(T_1H + T_2H) \cong \ker(T_1^*) \cap \ker(T_2^*) \) when \( T_1H + T_2H \) is closed. It follows that, \( e_2(T_1, T_2) = e_0(T_1^*, T_2^*) \) exists, if \( e_0 \) always exists. Then, by Equation (7), we conclude that \( e_1 \) exists, and the formula in Theorem 3 Part (ii) is true.

Part (iii) follows from Corollary 9 in Subsection 1.4. \( \square \)

**Remark** The additional property of \( e_i \) as displayed in Part (iii) is well known for the Euler characteristic in algebra (Proposition 1 or 12, page 98, Corollary 5.2.4), but the fact that, in operator theory, each \( e_i \) satisfies the formula in Part (iii) is somehow unexpected.

Also, the fact that, each \( e_i \) can be strictly positive, stands in strong contrast with algebraic results because the algebraic analogs of \( e_i \) \((I \neq 0)\) are always zero under mild conditions.
Example Each $e_i$ can be strictly positive.

The following example was communicated to the author by R. Curto [13], and satisfies $e_0 = e_2 = 0$, and $e_1 = 1$. It is trivial to see that $e_0 = e_1 = 0$, $e_2 = 1$, or $e_1 = e_2 = 0$, $e_0 = 1$ can happen.

Let $S = \{(s_1, s_2) \in \mathbb{Z}^2, s_1 \geq 0, \text{ or } s_2 \geq 0\}$, and $H$ be the Hilbert space with an orthonormal basis $\{e_{s_1,s_2}, (s_1, s_2) \in S\}$. Define $T_1$, $T_2$ by $T_1 e_{s_1,s_2} = e_{s_1+1,s_2}$, and $T_2 e_{s_1,s_2} = e_{s_1,s_2+1}$. Then $h_0(k) = h_2(k) = 0$, and $h_1(k) = k^2$.

Note that $e_i > 0$ for $i \neq 0$ is new to operator theory, hence there is no corresponding results in algebra. See [42] page 99 Proposition 5.2.6 for more details in the corresponding algebraic situation. It in fact represents an essential difference between algebraic modules and Hilbertian modules. In the algebraic case, $h_i(k)$ is usually bounded by a polynomial in $k$ with degree $d - i$ ([12], page 99). In our case, $d = 2$. Hence, $e_1 = e_2 = 0$, and they are essentially invisible in algebra. It follows that $\chi(K(T;H)) = e_0$, which is a classical theorem of Serre [44]. In this sense our theorem is an extension of Serre's theorem to Hilbert modules.

An alternative way to look at the asymptotic behavior of $h_i(k)$ in algebra is to consider $\lim_{k \to \infty} \frac{h_i(k)}{k^{d-1}}$. However, the problem of whether or not these limits exist seems still open [30].

1.2 Lech’s formula

Now we turn to the existence of $e_0$, which follows immediately from Lech’s formula (Theorem 4), together with some basic facts on Hilbert polynomials which we recall now.

Let $H$ be a Hilbert module over $A = \mathbb{C}[z_1, \ldots, z_d]$, such that $\dim H/\overline{IH} < \infty$. Here $I = (z_1, \ldots, z_d) \subset A$ is the maximal ideal at the origin, and the bar denotes Hilbertian closure. If we form the graded module

$$gr(H) = \bigoplus_{k \geq 0} \frac{I^kH}{I^{k+1}H}$$

over $A$, then $gr(H)$ is generated by its first component $H/\overline{IH}$, which is finite dimensional. (Note that $H$ is usually not generated by $H \oplus \overline{IH}$.) Hence, by basic results on Hilbert polynomials in algebra [15], [17], the function $\phi_H(k) = \dim H/I^kH$, which counts the dimension of the first $k$ components
of $gr(H)$, is actually a polynomial of degree at most $d$, when $k \gg 0$. Moreover, its leading term (or, the degree $d$ term) has the form $\frac{e}{d!} k^d$, where $e$ is an integer. The generating property of the first component implies

$$\dim H/\mathcal{J}_{k}H \geq e.$$  \hfill (8)

We call $e = e(I; H)$ the Samuel multiplicity of $H$ with respect to $I$. Also note that, in the above discussion, we can replace $I$ by a general ideal $J \subset A$.

**Theorem 4 (Lech’s formula)** Let $H$ be a Hilbert module over $\mathbb{C}[z_1, \cdots, z_d]$, such that $\dim H/\mathcal{J}_{k}H < \infty$. Let $J_k = (z_1^k, \cdots, z_d^k)$. Then

$$\lim_{k \to \infty} \frac{\dim H/J_kH}{k^d} = d! \lim_{k \to \infty} \frac{\dim H/I^kH}{k^d}.$$  

Theorem 4 is apparently a Hilbertian version of the following algebraic Lech’s formula (Lemma 5), which plays a key role in our proof of Theorem 4. Lemma 5 is excerpted from P. Robert’s book [112] (page 101 Theorem 5.2.8.).

**Lemma 5** Let $\alpha = (x_1, \cdots, x_d)$ be an ideal in a local ring $R$, and let $M$ be a finitely generated $R$-module, such that $M/(x_1, \cdots, x_d)M$ has finite length. Then

$$\lim_{k \to \infty} \frac{\text{length}(M/(x_1^k, \cdots, x_d^k)M)}{k^d} = d! \lim_{k \to \infty} \frac{\text{length}(M/\alpha^kM)}{k^d}.$$  

Note that when the module $M$ is a $\mathbb{C}$-vector space, the above length can be replaced by $\dim$. Also, it is a standard fact in algebra that the above statement in local rings implies a corresponding version for graded rings/modules.

**Proof of Theorem 4.** By the definition of $gr(H)$, one has $\dim H/I^kH = \dim gr(H)/I^kgr(H)$. Apply the algebraic formula of Lech (Lemma 5) to the graded module $gr(H)$, we have

$$d! \lim_{k \to \infty} \frac{\dim H/I^kH}{k^d} = d! \lim_{k \to \infty} \frac{\dim gr(H)/I^kgr(H)}{k^d} = \lim_{k \to \infty} \frac{\dim gr(H)/J_{k}gr(H)}{k^d}.$$
Now consider the natural surjective map, induced by inclusions
\[
\text{gr}(H) = H \bigoplus \frac{TH}{I^2H} \oplus \cdots \rightarrow H \bigoplus \frac{TH + J_kH}{I^2H + J_kH} \bigoplus \cdots,
\] (9)
whose kernel contains the graded submodule \( J_k \text{gr}(H) \).

Observe that the latter graded module in (9) stops after a finite number of steps, since \( I^t \subset J_k \) when \( t \) is large enough. Moreover, it gives rise to a grading of \( H/J_kH \). It follows that
\[
\dim \text{gr}(H)/J_k \text{gr}(H) \geq \dim H/J_kH,
\] (10)
which yields one direction of Theorem 4
\[
d! \lim_{k \to \infty} \frac{\dim H/I^kH}{k^d} \geq \lim_{k \to \infty} \frac{\dim H/J_kH}{k^d}.
\]

On the other hand, we have \( \dim H/J_kH \geq e(J_k, H) \), according to inequality (1). We claim that
\[
e(J_k, H) = e(I^k, H) = k^d \cdot e(I, H).
\]
This claim follows from the following fact
\[
(I^k)^t + d \subset J^t_k \subset (I^k)^t,
\]
and the definition of \( e(\cdot, H) \). Thus the other direction of Theorem 4 follows
\[
\lim_{k \to \infty} \frac{\dim H/J_kH}{k^d} \geq e(I, H). \quad \square
\]

Now the proof of Part (i) and (ii) of Theorem 3 is completed. It also follows that, when \( d = 2 \), the Samuel multiplicity \( e(I, H) \) coincides with \( e_0 \), defined in Theorem 3.

**Remark** Now we want to make some comparisons between the one and two variable cases. So far, the idea to calculate the Fredholm index of a pair is parallel to that of the one variable theory in [20], even though for a pair the situation is more complicated. But, realistically, the numbers \( e_i \) are still quite difficult to calculate, especially \( e_1 \). So we need to develop more techniques (see 1.3, 1.4), before we can effectively apply Theorem 3 to the symmetric Fock space. Also, it is not clear how to obtain a matrix decomposition for the pair, based on \( e_i \), as was done in [20], Theorem 4.
1.3 Induction arguments

In algebra an advantage of considering Koszul complexes is that, they are well suited for induction arguments. By this it often means to consider modules of the form $H/z_1H$, on which the $z_1$-module-action is zero. This consideration turns out to be helpful in Section 2 of this paper. Since our main interests lie in the Fredholm index and the Samuel multiplicity, next we study these two invariants on $H/T_1H$.

Unlike in Lech’s formula, we cannot take the closure of $T_1H$, which means that, we have to enlarge the Hilbert modules to a larger category to include non-complete spaces.

Let $H$ be a Hilbert module over $A = \mathbb{C}[z_1, \cdots, z_d]$, with module actions given by a d-tuple $(T_1, \cdots, T_d)$. Then $H/T_1H$ admits a natural module structures over both $A$ and $A' = \mathbb{C}[z_2, \cdots, z_d]$, where the module action over $A'$ is given by $(\tilde{T}_2, \cdots, \tilde{T}_d)$, which is induced by $(T_2, \cdots, T_d)$. We can similarly consider the Koszul complex $K(z_2, \cdots, z_d; H/T_1H) = K(\tilde{T}_2, \cdots, \tilde{T}_d; H/T_1H)$, and define the Fredholm index, when all involved homology groups are finite dimensional vector spaces.

**Proposition 6** If $\ker(T_1) = \{0\}$, then

$$\text{index}(T_1, \cdots, T_d; H) = -\text{index}(\tilde{T}_2, \cdots, \tilde{T}_d; H/T_1H).$$

**Proof.** Observe that

$$K(T_1, \cdots, T_d; H) = K(T_1; H) \otimes_A K(z_2, \cdots, z_d),$$

and

$$K(\tilde{T}_2, \cdots, \tilde{T}_d; H/T_1H) = H/T_1H \otimes_A K(z_2, \cdots, z_d).$$

So the key is to relate $K(T_1; H)$ to $H/T_1H$. We begin with the exact sequence of $A$-modules

$$0 \to H \xrightarrow{T_1} H \to H/T_1H \to 0.$$

Then extend it to an exact sequence of complexes

$$0 \to K(1; H) \xrightarrow{f_1} K(T_1; H) \to H/T_1H \to 0.$$ 

Here $f_1 = id$, $f_0 = T_1$, and $H/T_1H$ is regarded as a complex such that the 0th component is $H/T_1H$, and other components are zero.
Since $K(z_2, \cdots, z_d)$ is a complex of free modules, tensoring with it preserves exactness. Thus we obtain the following exact sequence

$$0 \to K(1, z_2, \cdots, z_d; H) \to K(T_1, \cdots, T_d; H) \to K(z_2, \cdots, z_d; H/T_1 H) \to 0.$$ 

But the tuple $(I, T_2, \cdots, T_d)$ is invertible, hence has index zero. Now the conclusion follows from the additivity of Euler characteristics. □

**Remark:** If the condition $\ker(T_1) = \{0\}$ is dropped, then Proposition 6 is not true which can be seen by considering $(M^*_z, M^*_w)$, the adjoints of the multiplication operators on the Hardy space over the bidisk.

Now we take up the Samuel multiplicity of $H/T_1 H$. First we recall some general facts on Hilbert polynomials ([28] page 49). While not strictly required, they provide the “the right way” to present the material. The reason largely lies in the identity

$$(z^d) - (z-1)^d = (z-1)^{d-1} + \cdots + c_1.$$ (11)

Here $\binom{z}{r} = \frac{1}{r!} z(z-1) \cdots (z-r+1)$ is the binomial coefficient function.

Note that Hilbert polynomials are numerical polynomials. By a numerical polynomial we mean a polynomial $P(z) \in \mathbb{Q}[z]$ such that $P(k) \in \mathbb{Z}$ for $k >> 0$. Then for any numerical polynomial $P(z)$ of degree $d$, there are integers $c_0, \cdots, c_d$, such that

$$P(z) = c_d \binom{z}{d} + c_{d-1} \binom{z}{d-1} + \cdots + c_0.$$ (12)

In particular, $P(k) \in \mathbb{Z}$ for all $k$. We call $c_d$ the reduced leading coefficient of $P(z)$; it is actually the leading coefficient multiplied by $d!$. If we take $P(k) = \phi_H(k)$ $(k >> 0)$ to be the Hilbert polynomial of a Hilbert module $H$, then $c_d$ is just the Samuel multiplicity of $H$.

For a function $f$, defined on $\mathbb{Z}$, we define an operation $\Delta f$ by taking the difference : $(\Delta f)(k) = f(k+1) - f(k)$. Because of Equations (11) and (12), Hilbert polynomials behave nicely under $\Delta$

$$(\Delta P)(z) = c_d \binom{z}{d-1} + c_{d-1} \binom{z}{d-2} + \cdots + c_1.$$
In particular, the reduced leading coefficient is invariant under $\Delta$.

Fix a Hilbert module $H$ over $A$, such that $\dim H/IH < \infty$, here $I = (z_1, \ldots, z_d) \subset A$. Let $J = (z_2, \ldots, z_d) \subset A$. Then $H/z_1 H$ can be regarded as a module over the polynomial ring in $d - 1$ variables, say $z_2, \ldots, z_d$, such that $\dim (H/z_1 H)/J(H/z_1 H) < \infty$. Similarly, we define its Hilbert function

$$\phi_{H/z_1 H}(k) = \dim \frac{H/z_1 H}{J^k(H/z_1 H)} = \dim \frac{H}{z_1 H + J^k H},$$

and the Samuel multiplicity as the reduced leading coefficient of $\phi_{H/z_1 H}(k)$ at the degree $(d - 1)$ term.

Because of $I^k + z_1 H = J^k + z_1 H$, the following is exact

$$H/I^k H \xrightarrow{\tilde{z}_1} H/I^{k+1} H \xrightarrow{q} H/(z_1 H + J^{k+1} H) \to 0.$$  

Here $\tilde{z}_1$ is induced by $z_1$, and $q$ is the natural quotient map. It follows

**Proposition 7** For any Hilbert module $H$ over the polynomial ring $\mathbb{C}[z_1, \ldots, z_d]$, such that $\dim H/IH < \infty$ here $I = (z_1, \ldots, z_d)$, one has

$$\phi_{H/z_1 H}(k) \geq (\Delta \phi_H)(k)$$

for all $k \in \mathbb{N}$. In particular,

$$\dim(H/IH) \geq e(J, H/z_1 H) \geq e(I, H).$$

### 1.4 More estimates of Hilbert functions

It may appear natural to consider the two variable homology Hilbert functions

$$h_i(s, t) = \dim H_i K(T_1^s, T_2^t; H),$$

for $i = 0, 1, 2$, which, of course, are not necessarily of polynomial type, when $s, t >> 0$. But even when $h_i(k)$, defined by Equation (13), is a polynomial, $h_i(s, t)$ may still fail to be so. It is largely open how to characterize when $h_i(k)$ and $h_i(s, t)$ will be of polynomial type, when $k, s, t$ are large enough. See [48] for algebraic results along this line.

**Example** Let $H^2(\mathbb{D})$ and $H^2(\mathbb{D}^2)$ be the Hardy space over the unit disc and bidisc, respectively. Let $M_z, M_w$ be multiplication by coordinate functions
on $H^2(\mathbb{D}^2)$, and $M_\xi$ on $H^2(\mathbb{D})$. Let

$$H = H^2(\mathbb{D}^2) \oplus H^2(\mathbb{D}), \quad T_1 = \begin{pmatrix} M_z & 0 \\ P_0 & M_\xi \end{pmatrix}, \quad \text{and} \quad T_2 = \begin{pmatrix} M_w & 0 \\ P_0 & M_\xi \end{pmatrix}.$$  

Here $P_0 : H^2(\mathbb{D}^2) \to H^2(\mathbb{D})$ is the projection onto the constants, that is $P_0(f) = f(0,0)$. Then

$$T_1 T_2 = T_2 T_1 = \begin{pmatrix} M_z M_w & 0 \\ M_\xi P_0 & M_\xi^2 \end{pmatrix}, \quad \text{and} \quad T_1^s = \begin{pmatrix} M_z^s & 0 \\ M_\xi^s P_0 & M_\xi^s \end{pmatrix}.$$  

So

$$\dim H/(T_1^s H + T_2^s H) = st + \min\{s,t\},$$

which is not a polynomial, but collapses to one when $s = t$. □

The main result of this subsection is Theorem 8, which gives some estimates of Hilbert functions in more than one variable. There exists an algebraic version of Theorem 8 due to C. Lech [32]. The proof relies on the so called “form ideal” technique, which goes back to Krull [31]. Northcott’s book [38] contains a detailed discussion of this technique. However, our proof is modelled after Lech [32]. It is worth mentioning that, this technique of associating a homogenous ideal to a module, has great computational power in calculating Hilbert functions, especially when combined with Gröbner bases ([17], chapter 15).

**Theorem 8** Let $H$ be a Hilbert module over the polynomial ring $\mathbb{C}[z_1, \cdots, z_d]$, such that $\dim H/\overline{I H} < \infty$, here $I = (z_1, \cdots, z_d)$, let $e = e(I, H)$ be the Samuel multiplicity of $H$, we have

$$n_1 \cdots n_d \cdot e \leq \dim H/(z_1^{n_1} H + \cdots + z_d^{n_d} H) \leq n_1 \cdots n_d (e + \frac{C}{\min_i n_i})$$

for some constant $C$ and any $n_1, \cdots, n_d \in \mathbb{N}$.

Thus we obtain the following stronger form of Lech’s formula

**Corollary 9** If $H$ is a Hilbert module over $\mathbb{C}[z_1, \cdots, z_d]$ such that $\dim H/\overline{I H} < \infty$, then

$$e(I, H) = \lim_{(\min n_i) \to \infty} \frac{\dim H/z_1^{n_1} H + \cdots + z_d^{n_d} H}{n_1 \cdots n_d}$$
If we replace \( n_i \) by \( n_i \cdot t \), and let \( t \to \infty \), then Part (iii) of Theorem 3 follows.

Now specialize to \( d = 2 \). By considering the adjoints \( (T_1^*, T_2^*) \), and Theorem 8 we know that \( h_i(s, t) \geq s \cdot t \cdot e_i \) for \( i = 0, 2 \). Now by Corollary 2 we have the following corollary, which will be used in the next section.

**Corollary 10** For any pair \( T = (T_1, T_2) \) of commuting operators, if \( T \) is Fredholm, and let \( h_1(s, t) = \dim H_1 K(T_1^*, T_2^*) \), for \( s, t \in \mathbb{N} \), then

\[
h_1(s, t) \geq s \cdot t \cdot e_1.
\]

Recall that \( e_1 \) is defined in Theorem 3.

**Proof of Theorem 8** Let \( l = \dim H/TH \) (assuming \( l > 0 \)), and choose a sequence of submodules

\[
H = H_0 \supset H_1 \supset \cdots \supset H_l = TH,
\]

such that \( \dim H_i/H_{i+1} = 1 \). This is always possible by considering Jordan decomposition in linear algebra. Pick any \( \zeta_i \in H_i \setminus H_{i+1} \), such that \( 1 \to \zeta_i + H_{i+1} \) induces an isomorphism \( \mathbb{C} \to H_i/H_{i+1} \).

Now multiply the above sequence by \( I^\mu, \mu = 0, 1, \ldots, k-1 \), take closures in the Hilbert space \( H \), and link them together

\[
H = H_0 \supset H_1 \supset \cdots \supset H_l \supset TH_1 \supset TH_2 \supset \cdots \supset TH_l \supset \cdots \supset IH_l (= I^k H).
\]

Then for any ideal \( \alpha \subset \mathbb{C}[z_1, \ldots, z_d] \), we can add \( \alpha H \) to the above sequence to obtain a chain from \( H = H + \alpha H \) to \( \alpha H + TH \). Note that \( \alpha H + TH \) is automatically closed. Hence

\[
\dim H/(\alpha H + TH) = \sum_{\mu=0}^{k-1} \sum_{\nu=0}^{l-1} \dim \frac{\alpha H + TH_{\nu+1}}{\alpha H + TH_{\nu+1}}.
\]

(15)

**Fact:** For any vector spaces \( A, M \supset N \), there is a natural isomorphism between vector spaces \( A + M \simeq M \cap M + N \).

**Proof of the Fact.** Note that \( A + M \simeq M \cap (A + N) \simeq M \cap A + N \). Then verify directly that \( M \cap (A + N) = M \cap A + N \).
Now we continue with the proof of Theorem 8. It follows that, as vector spaces
\[
\frac{\alpha H + \overline{I^k H_\nu}}{\alpha H + \overline{I^k H_{\nu+1}}} \cong \frac{I^\mu H_\nu}{\alpha H \cap I^\mu H_\nu + \overline{I^\mu H_{\nu+1}}}.
\]

Let \( F_\mu \) denote the collection of homogeneous polynomials of degree \( \mu \) in \( \mathbb{C}[z_1, \cdots, z_d] \). For any \( \mu \) we define a \( \mathbb{C} \)-linear map \( T_{\nu,\mu} : F_\mu \to \overline{I^\mu H_{\nu+1}} \) by
\[
T_{\nu,\mu}(f) = f \zeta_\nu + \overline{I^\mu H_{\nu+1}}.
\]

It is clear that \( \z_j \zeta_\nu \in IH \subset H_{\nu+1} \), and one can then directly verify that, \( T_{\nu,\mu} \) is surjective.

Moreover, it induces, through the quotient map, the following
\[
F_\mu \to \frac{I^\mu H_\nu}{\alpha H \cap I^\mu H_\nu + \overline{I^\mu H_{\nu+1}}},
\]
whose kernel is denoted by
\[
K_{\nu,\mu}(\alpha) = \{ f \in F_\mu, T_{\nu,\mu} f \subset (\alpha H \cap I^\mu H_\nu) + \overline{I^\mu H_{\nu+1}} \}.
\]

Observe that \( z_i K_{\nu,\mu}(\alpha) \subset K_{\nu,\mu+1}(\alpha) \), it follows that \( I_\nu(\alpha) = \cup_{\mu=0}^\infty K_{\nu,\mu}(\alpha) \) is a graded ideal in \( \mathbb{C}[z_1, \cdots, z_d] \), called a “form ideal” of \( \alpha \). Then we have
\[
\dim H/(\alpha H + \overline{I^k H}) = \sum_{\nu=0}^{l-1} \sum_{\mu=0}^{k-1} \dim \frac{F_\mu}{I_\nu(\alpha) \cap F_\mu}.
\]

Two special cases are particularly important:

1. Let \( \alpha = 0 \), then the number of \( \nu \) such that \( I_\nu(0) = 0 \) is exactly \( e(I, H) \), because, for a nonzero ideal \( I_\nu(0) \), the function \( \sum_{\mu=0}^{k-1} \dim \frac{F_\mu}{I_\nu(\alpha) \cap F_\mu} \) is a polynomial of degree at most \( d - 1 \) for large \( k \);

2. Assume that \( \alpha \) is \( I \)-primary, that is, \( I \supset \alpha \supset I^t \) for some \( t \). Then for \( k > t \), one has
\[
\alpha H + \overline{I^k H} = \alpha H + \overline{I^k H} = \alpha H.
\]

So
\[
\dim H/\alpha H = \sum_{\nu=0}^{l-1} \dim \frac{\mathbb{C}[z_1, \cdots, z_d]}{I_\nu(\alpha)}.
\]
Now apply formula (17) to $\alpha = (z_1^{n_1}, \cdots, z_d^{n_d})$, and observe that $\alpha \subset I_\nu(\alpha)$, one has

$$\dim \frac{\mathbb{C}[z_1, \cdots, z_d]}{I_\nu(\alpha)} \leq n_1 \cdots n_d.$$  

In addition, if $0 \neq f \in I_\nu(\alpha)$ for some $\nu$, then, order the polynomials lexicographically, and assume that the highest power in $f$ is $z_1^{\sigma_1} \cdots z_d^{\sigma_d}$. Then, representatives of elements in $\frac{\mathbb{C}[z_1, \cdots, z_d]}{I_\nu(\alpha)}$ can be chosen in such a way that, any power product $z_1^{r_1} \cdots z_d^{r_d}$ appearing in any representative is subject to $r_i < n_i$, $i = 1, 2, \cdots, d$, and in addition at least $r_i < \sigma_i$ for some $i$. Hence

$$\dim \frac{\mathbb{C}[z_1, \cdots, z_d]}{I_\nu(\alpha)} \leq n_1 \cdots n_d(\frac{\sigma_1}{n_1} + \cdots + \frac{\sigma_d}{n_d}).$$

Putting all $I_\nu(\alpha)$ together, we have an upper estimate

$$\dim H/(z_1^{n_1}H + \cdots + z_d^{n_d}H) \leq n_1 \cdots n_d(e(I, H) + \frac{A}{\min_i n_i}), \quad (18)$$

where $A$ is a constant independent of all $n_i$.

For the other direction of Theorem 8 observe that, by inequality (8)

$$\dim H/(z_1^{n_1}H + \cdots + z_d^{n_d}H) \geq e((z_1^{n_1}, \cdots, z_d^{n_d}), H).$$

So it suffices to show

$$e((z_1^{n_1}, \cdots, z_d^{n_d}), H) \geq n_1 \cdots n_d e(I, H).$$

Note that the upper estimate (18) implies

$$e(I, H) \geq \lim_{(\min n_i) \to \infty} \frac{\dim H/(z_1^{n_1}H + \cdots + z_d^{n_d}H)}{n_1 \cdots n_d}. \quad (19)$$

Choose $m_1, \cdots, m_d$ such that $n_1 m_1 = \cdots = n_d m_d$, then apply the above inequality (19) to the ideal $(z_1^{m_1}, \cdots, z_d^{m_d})$, instead of $I$ to obtain

$$e((z_1^{n_1}, \cdots, z_d^{n_d}), H) \geq \lim_{k \to \infty} \frac{\dim H/((z_1^{m_1})^{m_1k}H + \cdots + (z_d^{m_d})^{m_dk}H)}{(m_1 k) \cdots (m_d k)} \geq n_1 \cdots n_d \lim_{t \to \infty} \frac{\dim H/(z_1^t H + \cdots + z_d^t H)}{t^d}.$$ 

By Theorem 4 (Lech’s formula), the last expression is just $n_1 \cdots n_d e(I, H)$. Now our proof of Theorem 8 is complete. □
2 Fredholm theory on the symmetric Fock space in two variables

In this section we show that the algebraic theory developed in Section 1 provides useful tools in the study of operator theory and function theory on the symmetric Fock space over the ball in $\mathbb{C}^2$. In particular, our results imply that Arveson’s program [6] on Fredholm theory for pure $d$-contractions can have satisfactory answers for both the Fredholmness problem and the calculation of Fredholm indices when $d = 2$. It is noteworthy to mention that the Fredholmness problem is a subtle one in general.

In multivariable Fredholm theory the lack of amenable examples has seriously hampered the development of the subject. When trying to obtain meaningful examples, it is a folk problem to determine the Fredholmness and the index of the tuples of multiplication operators on holomorphic function spaces. For example, a complete characterization of the Fredholmness of submodules of the vector-valued Hardy module over the polydisc $H^2(\mathbb{D}^n) \otimes \mathbb{C}^N$, and to calculate the index, seem out of reach at this time. This might require the development of more sophisticated machineries from both the analytic side, that is, function theory on the polydisc, and the algebraic side, that is, theory of Koszul complexes over Hilbert modules. When $n = 2$, $N = 1$, Yang [50] showed that the index of the pair of multiplication by coordinate functions is one for a fairly large class. See Curto-Salinas [12] for a study of Bergman-type spaces. Also see [15] for more discussions on Bergman spaces.

Through Arveson’s work [2], it is now known that Fredholm theory on the symmetric Fock space is in fact equivalent to that of pure $d$-contractions. In particular, in a series of paper [1, 5, 6], Arveson initiated a study of Fredholm theory for $d$-contractions. In [26] Gleason-Richter-Sundberg showed that, under certain regularity condition on the invariant subspace of the vector valued symmetric Fock space, the associated Koszul complex (in possibly more than two variables) is acyclic, that is, all homology groups are zero except for the last stage, hence the index is equal to the dimension of the last homology group.

In general, the difficulty in calculating the several variable Fredholm index is, mainly due to the lack of an effective way to estimate the intermediate homology groups, that is, those other than the first and last. A common phenomenon, which has occurred in calculating the index in past research, is that the Koszul complex $K(\cdot)$ is shown to be acyclic, often by nontrivial
arguments, and hence the index is identified with $\dim H_0 K(\cdot)$.

Now recall that the symmetric Fock space over the ball $B_2 \subset \mathbb{C}^2$, denoted by $H^2$, is a Hilbert space of holomorphic functions, defined on the ball $B_2 \subset \mathbb{C}^2$. It is determined by the reproducing kernel $k((z,w),(\zeta,\eta)) = \frac{1}{1-z\overline{\zeta}-w\overline{\eta}}$; equivalently, it can be obtained by symmetrizing the full Fock space, as was done by Arveson [2]. In what follows, let $\mathcal{M} \subset H^2 \otimes L$ be a multiplication invariant subspace of the $L$-valued symmetric Fock space, here $L$ is a Hilbert space; and $(M_z,M_w)$ be the pair obtained by restricting multiplication by coordinate functions $z,w$ to $\mathcal{M}$. Note that $H_2 K(M_z,M_w;\mathcal{M}) = 0$, and $H_0 K(M_z,M_w;\mathcal{M}) = \mathcal{M}/(z\mathcal{M}+w\mathcal{M})$. So, in order to have a Fredholm pair, an obvious necessary condition is to have $\dim \mathcal{M}/(z\mathcal{M}+w\mathcal{M}) < \infty$, which we shall show is also sufficient.

The following is the main result of this section.

**Theorem 11** Let $\mathcal{M} \subset H^2 \otimes L$ be a multiplication invariant subspace of the $L$-valued symmetric Fock space in two variables. Here $L$ is a Hilbert space. Let $(M_z,M_w)$ be the pair of multiplication by coordinate functions on $\mathcal{M}$.

1. $(M_z,M_w)$ is Fredholm if and only if $\dim \mathcal{M}/(z\mathcal{M}+w\mathcal{M}) < \infty$.
2. When Fredholm, the index is $\text{index}(M_z,M_w) = e_0$ ($= \text{Samuel multiplicity of } \mathcal{M}$).

**Proof.** We shall try to separate the algebraic and analytic components in our proof as much as possible. This is done through step 1 - 7 below. Basically there are three main ingredients: 1. general algebraic properties of Koszul complexes; 2. Fredholm theory developed in Section 1 and [20]; and 3. function-theoretic operator theory, such as reproducing kernels, holomorphic multipliers, and the Hardy space over the unit disc $H^2(\mathbb{D})$.

We first set two notations. For any set $S$, consisting of functions in two variables, let $S(a,b) = \{f(a,b), f \in S\}$. For any holomorphic function $f$ in one variable, defined on a neighborhood of the origin, if $f(x) = c_rx^r + c_{r+1}x^{r+1} + \cdots$ is its Taylor series, and $c_r \neq 0$, then we define $\text{ord}_x \ f = r$.

**Step 1.** Let $T_w$ be the map acting on the vector space $\mathcal{M}/z\mathcal{M}$, induced by multiplication by $w$. Then the two variable index is reduced to the one variable index.
Lemma 12 For any invariant subspace $\mathcal{M}$, one has
\[
\text{index}(M_z, M_w; \mathcal{M}) = -\text{index}(T_w; \mathcal{M}/z\mathcal{M}).
\] (20)

In particular, $(M_z, M_w)$ is Fredholm (semi-Fredholm, not Fredholm, resp.) if and only if $T_w$ is Fredholm (semi-Fredholm, not Fredholm, resp.).

Here the latter index is defined in the following more general setting: Let $T$ be a linear map on a vector space $V$, then its index is $\text{index}(T; V) = \dim \ker(T) - \dim V/TV$, if finite.

**Proof.** This result follows from Proposition 6. Since we shall need more precise information between $(M_z, M_w)$ and $T_w$ in what follows, we give a different approach. For any pair $(T_1, T_2)$ acting on $H$, we consider the following exact sequence of complexes
\[
0 \to K(T_1; H) \to K(T_1, T_2; H) \to \bar{K}(T_1; H) \to 0,
\]
where $\bar{K}$ is the complex obtained by shifting $K$ to the right by one step. We have the following long exact sequence
\[
0 \to H_2K(T_1, T_2; H) \to H_1K(T_1; H) \xrightarrow{T_2} H_1K(T_1; H) \to H_1K(T_1, T_2; H)
\]
\[
\to H_0K(T_1; H) \xrightarrow{T_2} H_0K(T_1; H) \to H_0K(T_1, T_2; H) \to 0.
\]
Here $T_2$ is the map induced by $T_2$. When applied to $(M_z, M_w)$ on $\mathcal{M}$, the above sequence yields
\[
0 \to H_1K(M_z, M_w; \mathcal{M}) \to \mathcal{M}/z\mathcal{M} \xrightarrow{T_w} \mathcal{M}/z\mathcal{M} \to H_0K(M_z, M_w; \mathcal{M}) \to 0.
\] (21)

In particular, we know that $\ker(T_w) \cong H_1K(M_z, M_w; \mathcal{M})$. Similarly, it follows that $\ker(T_w^k) \cong H_1K(M_z, M_w^k; \mathcal{M})$, for any $k \in \mathbb{N}$, which will be used in step 7.

*Step 2.*

**Lemma 13** For any pair $(T_1, T_2)$ of commuting operators on a Hilbert space $H$, if $U = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is a unitary matrix, then for $i = 0, 1, 2$,
\[
H_iK(T_1, T_2; H) \cong H_iK(aT_1 + bT_2, cT_1 + dT_2; H)
\] (22)
are natural isomorphisms between vector spaces.
The proof is essentially linear algebra, and is best carried out in the “exterior power definition” of Koszul complexes. Readers can show that the matrix $U$ actually induces an isomorphism between the Koszul complexes of $(T_1, T_2)$ and $(aT_1 + bT_2, cT_1 + dT_2)$. The details are skipped.

So in order to consider the Fredholm theory of $(T_1, T_2)$, it is equivalent to look at $(aT_1 + bT_2, cT_1 + dT_2)$. This is particularly applicable to “ball-oriented operator theory”. Note that $(z, w) \rightarrow (z', w') = (az + bw, cz + dw)$ is a change of variables on the ball $B_2$, which preserves the metric. Moreover, it preserves the metric on $H^2$, that is, for any two variable polynomial $p(\cdot, \cdot)$, one has

$$||p(z, w)||_{H^2} = ||p(az + bw, cz + dw)||_{H^2},$$

which can be seen easily from the reproducing kernel: the change of variables $(z, w) \rightarrow (z', w') = (az + bw, cz + dw)$ does not change the form of the kernel $\frac{1}{1 - z\bar{z} - w\bar{w}}$, because $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is unitary, hence if $(\zeta', \eta') = (a\zeta + b\eta, c\zeta + d\eta)$, then $z\bar{\zeta} + w\bar{\eta} = z'\bar{\zeta}' + w'\bar{\eta}'$.

So the pair of multiplication operator $(M_{az+bw}, M_{cz+dw})$ on an invariant subspace $M \subset H^2 \otimes L$ is unitarily equivalent to $(M_z, M_w)$ on another invariant subspace $M' \subset H^2 \otimes L$, obtained from $M$ by a change of variables.

Step 3. Now we need the Fredholm theory developed in [20]. We rephrase Theorem 2 in [20] in the setting of a general linear map $T : V \rightarrow V$, acting on a vector space. Define the shift and backward shift Samuel multiplicity of $T$ by

$$s_{\text{mul}}(T) = \lim_{k \rightarrow \infty} \frac{\dim V/T^kV}{k}, \quad \text{and} \quad b.s._{\text{mul}}(T) = \lim_{k \rightarrow \infty} \frac{\dim \ker(T^k)}{k}.$$ 

If $T$ is semi-Fredholm in the general sense, that is, if at least one of $\dim \ker(T)$ and $\dim V/TV$ is finite, then

(i) $s_{\text{mul}}(T), b.s._{\text{mul}}(T) \in \{0, 1, 2, \ldots, \infty\}$;

(ii) $\text{index}(T) = b.s._{\text{mul}}(T) - s_{\text{mul}}(T)$;

(iii) when $k >> 0$,

$$b.s._{\text{mul}}(T) = \dim(\ker(T) \cap T^kH) = \dim(\ker(T) \cap T^\infty H),$$
and
\[ s_{mul}(T) = \dim \left( \frac{H}{TH + \ker(T^k)} \right) = \dim \left( \frac{H}{TH + \ker(T^\infty)} \right). \]

Here \( T^\infty H = \cap_{k \geq 0} T^k H \), and \( \ker(T^\infty) = \cup_{k \geq 0} \ker(T^k) \).

The proof in [20] is for Hilbert space operators, but a careful examination of the proof will reveal that it carries over to the more general setting of vector spaces. Only a small change is needed: In [20], \( \ker(T^\infty) \) is defined to be \( \cup_{k \geq 0} \ker(T^k) \). Here, we do not have a Hilbert space structure, and hence cannot form the closure. On the other hand, \( \ker(T^\infty) \) appears only in \( TH + \ker(T^\infty) \). By arguments in [20], it is easy to see that \( TH + \ker(T^k) = TH + \ker(T^\infty) \), for large \( k \), and is always closed for a semi-Fredholm map in the general sense. So, it does not matter if we do not form the Hilbertian closure of \( \ker(T^\infty) \). Recall that \( \ker(T) \cap T^\infty H \) is defined to be the stabilized kernel of \( T \) [20].

**Step 4.** Now let \( V = \mathcal{M}/z\mathcal{M} \), and we apply step 3 to the linear map \( T_w : V \to V \) to calculate its Fredholm index by looking at the stabilized kernel and cokernel.

According to McCullough-Trent [37], we can assume that, there is a partial isometric multiplier \( \Phi : H^2 \otimes E \to H^2 \otimes L \), where \( E \) is a Hilbert space, such that \( \Phi(H^2 \otimes E) = \mathcal{M} \). Let \( \mathcal{M}' = \ker(\Phi) \).

Observe that
\[ T_w^\infty V \cong \frac{\cap_{k \geq 0}(w^k \mathcal{M} + z\mathcal{M})}{z\mathcal{M}}. \]

**Lemma 14** For any \( x = \Phi \xi \in \mathcal{M} \), we have

(i) \( x \in \cap_{k \geq 0}(w^k \mathcal{M} + z\mathcal{M}) \) if and only if
\[ \xi(0, w) \in \cap_{k \geq 0}(w^k H^2(0, w) \otimes E + \mathcal{M}'(0, w)). \]

(ii) \( x \in z\mathcal{M} \) if and only if \( \xi(0, w) \in \mathcal{M}'(0, w) \).

**Proof.** Part (i): The following statements are all equivalent.

1. \( \Phi \xi \in \cap_{k \geq 0}(\Phi(w^k H^2 \otimes E + zH^2 \otimes E)) \);
2. for any \( k \), there exists \( u, v \in H^2 \otimes E \), such that \( \Phi(\xi - w^k u - zv) = 0 \);
3. \( \xi \in w^k H^2 \otimes E + zH^2 \otimes E + \mathcal{M}' \), for any \( k \);
4. $\xi(0, w) \in w^k H^2(0, w) \otimes E + \mathcal{M}'(0, w)$, for any $k$.

For Part (ii), $\Phi \xi \in z \mathcal{M} = z \Phi(H^2 \otimes E)$ means $\Phi(\xi - z \eta) = 0$, for some $\eta \in H^2 \otimes E$. That is, $\xi \in \mathcal{M}' + z H^2 \otimes E$, which is equivalent to $\xi(0, w) \in \mathcal{M}(0, w)$. □

Step 5. Next we deal with $\mathcal{M}'(0, w)$, which is a subspace of $H^2(0, w) \otimes E$, which in turn is equivalent to the direct sum of $\dim(E)$ many copies of the Hardy space $H^2(\mathbb{D})$ over the unit disc $\mathbb{D} \subset \mathbb{C}$. Now the difficulty is that, we do not know whether $\mathcal{M}'(0, w)$ is closed.

Observe that $\Phi(0, w) H^2(0, w) \otimes E = \mathcal{M}'(0, w)$. Thanks to the result of Greene-Richter-Sundberg [27], we know that there exists a (thin) subset $Z \subset S = \partial B_2 \subset \mathbb{C}^2$ of the sphere, which is contained in the zero set of a nonzero holomorphic function, such that the boundary values of the partial isometric holomorphic multiplier $\Phi$ are partial isometries, with a constant rank on $S \setminus Z$. Now by arguments in step 2, we can apply a change of variables, if necessary, such that $Z \cap \{(0, w), |w| = 1\}$ is a thin subset of the circle $\{(0, w), |w| = 1\}$. It follows that $\Phi(0, w)$ is a one-variable, bounded, holomorphic multiplier, with boundary values that are partial isometries with constant rank almost everywhere on the unit circle.

Now by the familiar theory of $H^2(\mathbb{D})$, we know that $\Phi(0, w)$, acting on the direct sum of Hardy spaces $H^2(0, w) \otimes E$, is a partial isometry, and hence has closed range. That is, we can assume that $\mathcal{M}'(0, w)$ is closed.

Step 6. In this step we show that the stabilized kernel of $T_w$ is 0. That is to show that

if $x \in \cap_{k \geq 0}(w^k \mathcal{M} + z \mathcal{M})$, and $wx \in z \mathcal{M}$, then $x \in z \mathcal{M}$.

Let $x = \Phi \xi$, where $\Phi$ is from step 4. Then by step 4 we need to show that

if $\xi(0, w) \in \cap_{k \geq 0}(w^k H^2(0, w) \otimes E + \mathcal{M}'(0, w))$, and $w \xi(0, w) \in \mathcal{M}'(0, w)$, then $\xi(0, w) \in \mathcal{M}'(0, w)$.

By the Beurling-Lax-Halmos Theorem on the Hardy space $H^2(\mathbb{D})$, there exists an isometric (not just partially isometric) holomorphic multiplier $\Theta$, in terms of $w$, such that $\Theta(H^2(\mathbb{D}) \otimes F) = \mathcal{M}'(0, w)$ for some Hilbert space $F$. For each $k \geq 0$ we can write $\xi(0, w) = w^k h_k + \Theta g_k$ for some $h_k \in H^2(\mathbb{D}) \otimes E$, and $g_k \in H^2(\mathbb{D}) \otimes F$. Moreover, we write $w \xi(0, w) = \Theta g$, where $g \in H^2(\mathbb{D}) \otimes F$. 

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We first show that the sequence $h_k$ can be chosen in such a way that $\sup_{k \geq 0} ||h_k|| < \infty$. To achieve this, we write $g = g(0) + wg'$, and let $T_k(g')$ denote the Taylor polynomial of $g'$ at the origin with degree $k$. Now, if we let $g_k = T_k(g')$, then by using the above expressions of $\xi(0, w)$ and $w\xi(0, w)$, we have

$$w^{k+1}h_k = \Theta g(0) + w\Theta(g' - T_k(g')).$$

(23)

It follows that for this choice of $h_k$ one has $\lim_{k \to \infty} ||h_k|| = ||\Theta g(0)||$.

Now we form the inner product between $\Theta g(0)$ and both sides of Equation (23), and observe that $\Theta g(0)$ and $w\Theta(g' - T_k(g'))$ are orthogonal, we obtain

$$||\Theta g(0)||^2 = \langle \Theta g(0), w^{k+1}h_k \rangle.$$

Since $||h_k||$ is uniformly bounded, the right hand side tends to zero as $k \to \infty$. If follows $\Theta g(0) = 0$. Recall that $\Theta$ is an isometry, so $g(0) = 0$. Now $\xi(0, w) = \Theta g' \in \mathcal{M}'$.

**Remark:** The above arguments fail for the Bergman space: Take an invariant subspace $\mathcal{M} \subset L^2_\mathbb{D}$ of the Bergman space over the unit disc, such that $\dim \mathcal{M}/w\mathcal{M} = 2$ and $\mathcal{M}(0) \neq \{0\}$. Then there exists a function, say $h$, in $\mathcal{M} \ominus w\mathcal{M}$ such that $h(0) = 0$. So if we write $h = wh' \in \mathcal{M}$, then it does not follow that $h' \in \mathcal{M}$.

**Step 7.** Now we are ready to complete the proof of Theorem 11. Recall from [20] that, the difference between the kernel $\ker(T_w)$ and the stabilized kernel $\ker(T_w) \cap T_w^\infty V$ is

$$\frac{\ker(T_w)}{\ker(T_w) \cap T_w^k V} \cong \frac{\ker(T_w^k) + T_w V}{T_w V}$$

for large $k$, which is finite dimensional for a semi-Fredholm map. This is where we use the condition $\dim \mathcal{M}/(z\mathcal{M} + w\mathcal{M}) < \infty$. Since the stabilized kernel of $T_w$ is 0, it follows that $\ker(T_w) \cap T_w^k V = 0$ for large $k$. So we know that $\ker(T_w) \cong \frac{\ker(T_w^k) + T_w V}{T_w V}$ is finite dimensional, and hence $T_w$ is Fredholm.

By results in [20] or step 3, the stabilized kernel of $T_w$ is 0 which means that

$$\sup_{k \geq 0} \dim \ker(T_w^k) < \infty.$$

By the remark at the end of the proof of Lemma 12, this means

$$\sup_{k \geq 0} h_1(1, k) < \infty.$$
By Corollary 10 we know that \( h_1(t,k) \geq t \cdot k \cdot e_1 \), for any \( t,k \geq 0 \). It follows that \( e_1 = 0 \).

Obviously, \( e_2 = 0 \). So by Theorem 3 we have \( \text{index}(M_z, M_w) = e_0 \), and the proof of Theorem 11 is complete.

3 Additivity of Hilbert polynomials

3.1 Additivity and monotonicity of Samuel multiplicities

For any invariant subspace \( M \subset H^2(\mathbb{D}) \) of the Hardy space one has the well known codimension-one property \( \dim M \ominus zM = 1 \). This result has been exploited in one variable, but a multivariable generalization proves to be resistent. At the end of [21] it is conjectured that the right approach is probably through considering certain stabilized dimensions. In this section, as a consequence of some algebraic considerations on Hilbert spaces, we prove what might be called the generalization of the codimension-one property for the symmetric Fock space in two variables (Corollary 16).

As we have seen, the study of multivariable Fredholm index is closely related to the study of Hilbert polynomials. Hilbert polynomials play essential roles in algebraic geometry. One of their key properties is their additivity over short exact sequences, which can in fact characterize the Hilbert polynomials in some sense (see [17] Exercise 19.18).

For a short exact sequence of finitely generated modules over a Noetherian ring \( R \)
\[
0 \to L \to M \to N \to 0,
\]
with \( \phi_L, \phi_M, \) and \( \phi_N \) the Hilbert polynomials of \( L, M, \) and \( N \) respectively with respect to an ideal \( I \subset R \), the additivity of Hilbert polynomials in the graded case refers to
\[
\phi_L + \phi_N = \phi_M;
\]

In [17] Exercise 19.18, it is shown that the Hilbert functions are the universal additive invariants of graded modules over the polynomial ring \( \mathbb{C}[z_0, \cdots, z_d] \), or equivalently, they are the universal additive invariants of coherent sheaves on the complex projective space \( \mathbb{P}^d \). In a recent paper [10], Chan showed that this additivity of Hilbert polynomials is equivalent to the multiplicativity of Chern numbers over projective spaces.
In the non-graded case, the additivity of the leading terms, that is, the additivity of Samuel multiplicities, is still true under mild conditions. In [21] such an additivity is shown to be true on the Dirichlet space over the unit disc, which has several consequences in operator theory.

The purpose of this subsection is to establish a version of the additivity of Hilbert polynomials for $\mathcal{H}^2$ (Theorem 15). Theorem 11 plays an important role in the proof of Theorem 15. We conjecture that the conclusion of Theorem 15 still holds for three or more variables. The problem is open on the Hardy space over the polydisc or the ball in $\mathbb{C}^d$ ($d \geq 2$).

**Theorem 15** Let $\mathcal{M} \subset \mathcal{H}^2 \otimes \mathbb{C}^N$ ($N \in \mathbb{N}$) be an invariant subspace of the $\mathbb{C}^N$-valued symmetric Fock space in two variables; let $\mathcal{M}^\perp = \mathcal{H}^2 \otimes \mathbb{C}^N \ominus \mathcal{M}$. Equip $\mathcal{M}$, $\mathcal{H}^2 \otimes \mathbb{C}^N$, and $\mathcal{M}^\perp$ with the natural module structures over $\mathbb{C}[z,w]$. Let $I = (z,w)$.

If $\dim \mathcal{M}/IM < \infty$, then

$$e(I, \mathcal{M}) + e(I, \mathcal{M}^\perp) = N.$$ 

Observe that, in order to formulate the above additivity of Hilbert polynomials, it is necessary to have the condition $\dim \mathcal{M}/IM < \infty$, so that the Samuel multiplicity $e(I, \mathcal{M}) = d! \lim_{k \to \infty} \frac{\dim \mathcal{M}/IM^k}{k^d}$ is finite, where $d = 2$. Also, we do not form the closure $\overline{IM}$, in order to match up with the algebraic formulation.

We have the following “monotonicity property” of $e(I, \mathcal{M})$.

**Corollary 16** If $\mathcal{M}_1 \subset \mathcal{M}_2 \subset \mathcal{H}^2 \otimes \mathbb{C}^N$ ($N \in \mathbb{N}$) are two invariant subspaces of the $\mathbb{C}^N$-valued symmetric Fock space in two variables, such that $\dim \mathcal{M}_1/IM_1 < \infty$, and $\dim \mathcal{M}_2/IM_2 < \infty$, then

$$e(I, \mathcal{M}_1) \leq e(I, \mathcal{M}_2).$$

In particular, for any invariant subspace $\mathcal{M} \subset \mathcal{H}^2 \otimes \mathbb{C}^N$ ($N \in \mathbb{N}$), such that $\dim \mathcal{M}/IM < \infty$, one has

$$e(I, \mathcal{M}) \leq N.$$ 

**Proof.** Let $F_n$ denote the polynomials in $\mathcal{H}^2 \otimes \mathbb{C}^N$ of degree at most $n$. Observe that

$$\dim \mathcal{M}^\perp/IM^\perp = \dim \cap_{\alpha_1 + \alpha_2 = n} \ker(M^*_z M^*_w)^{\alpha_1 \alpha_2} = \dim \mathcal{M}^\perp \cap F_n,$$
where } M \subset H^2 \otimes \mathbb{C}^N \text{ is a general invariant subspace. Since } M \perp M' \supset M \perp n \cap F_n, \text{ the definition of the Samuel multiplicity implies } e(I, M \perp 1) \geq e(I, M \perp 2) \cap F_n. \text{ An application of Theorem 15 completes the proof.} \quad \Box

Remark The above corollary is somewhat perplexing, in the sense that, for an invariant subspace } M \subset H^2 \otimes \mathbb{C}^N \text{ with } \dim M/I \text{ is finite, its sub invariant subspace } M' \subset M \text{ may still have } \dim M'/IM' = \infty. \text{ Hence } e(I, M') = d! \lim_{k \to \infty} \dim M'/I^kM' = \infty, \text{ where } d = 2. \text{ But when its Samuel multiplicity is finite, it is dominated by that of } M.

If we replace } H^2 \text{ by the one variable Hardy or Dirichlet space over the unit disc, then the second part of Corollary 16 is the well known codimension-N-property. The straightforward generalization of this property to higher dimensions, obtained by considering the codimension } \dim M/IM, \text{ fails immediately; for instance, if } M \text{ is the invariant subspace } [z, w] \text{ of } H^2, \text{ consisting of functions vanishing at the origin, then } \dim M/IM = 2 > 1. \text{ In Section 7 of [21] an explanation of this phenomenon was given, and it was suggested that a notion of “stabilized codimension” might be what is really relevant. It is worth mentioning that, in the one variable case, the codimension is always stabilized. Corollary 16 provides such a “stabilized codimension-N-property” in more than one variable.}

Proof of Theorem 15. We first show that } e(I, M') \text{ is always well defined, even when } M \subset H^2 \otimes \mathbb{C}^N \text{ is not Fredholm. Then we find the corresponding invariant on } M, \text{ denoted by } \sigma = \sigma_M, \text{ which is additive with respect to } e(I, M') \text{ on } M', \text{ i.e., } \sigma + e(I, M') = N.

To achieve this it is common in commutative or homological algebra to consider the exactness of the functor sending the module } H = M/IM \text{ to } H/I^kH, \text{ which is in fact right half-exact. Hence with respect to the exact sequence of Hilbert modules } 0 \to M \to H^2 \otimes \mathbb{C}^N \to H \to 0, \text{ the following sequence is exact}

\[ \frac{M}{I^kM} \to \frac{H^2 \otimes \mathbb{C}^N}{I^kH^2 \otimes \mathbb{C}^N} \to \frac{H}{I^kH} \to 0. \tag{24} \]

Of course, the exactness of the above sequence can be verified directly. Since the above middle term is finite dimensional, so is } H/I^kH. \text{ It follows that } I^kH \text{ is closed in } H, \text{ and } e(I, M') \text{ exists, and is finite.}
To complete the above sequence, we look at the image of the second arrow and it follows that the following sequence is exact

\[ 0 \to \frac{(\mathcal{M} + I^k H^2 \otimes \mathbb{C}^N)}{I^k H^2 \otimes \mathbb{C}^N} \to \frac{H^2 \otimes \mathbb{C}^N}{I^k H^2 \otimes \mathbb{C}^N} \to \frac{H}{I^k H} \to 0. \] (25)

Observe that,

\[ \dim \frac{(\mathcal{M} + I^k H^2 \otimes \mathbb{C}^N)}{I^k H^2 \otimes \mathbb{C}^N} = \dim P_{k-1} \mathcal{M}, \]

where \( P_k \) is the projection onto the space of polynomials of degree \( \leq k \). By the above exact sequence (25), the function

\[ \varphi_M(k) = \dim P_{k-1} \mathcal{M} \] (26)

will be a polynomial of degree at most \( d \) for \( k >> 0 \), and \( d! \lim_{k \to \infty} \varphi_M(k) \) is an integer. Here \( d = 2 \). For convenience, we define

\[ (\sigma =) \sigma_M = d! \lim_{k \to \infty} \frac{\dim P_{k-1} \mathcal{M}}{k^d}. \] (27)

Then \( \sigma + e(I, \mathcal{M}^\perp) = N \).

Next we take up \( e(I, \mathcal{M}) \). From Theorem 11, we know that \((M_z, M_w)\) is Fredholm, and \( e(I, \mathcal{M}) \) is equal to \( \text{index}(M_z, M_w) \). Now we need a result of Gleason-Richter-Sundberg [26]: if \((M_z, M_w)\) is assumed to be Fredholm, then its index is equal to the fibre dimension \( f.d.(\mathcal{M}) \). Here the fibre dimension is defined by

\[ f.d.(\mathcal{M}) = \sup \{ \dim \mathcal{M}(z, w), \ |z|^2 + |w|^2 < 1 \}. \] (28)

Note that the sup is achieved almost everywhere.

Now, what we need to show is that

\[ (\ast) \quad \text{if } (M_z, M_w) \text{ is Fredholm, then } f.d.(\mathcal{M}) = \sigma. \]

Here it is worthwhile to point out that, the definition of both \( f.d.(\mathcal{M}) \) and \( \sigma \) (see (27), and (28)), does not require the Fredholmness condition. Hence, statement \((\ast)\) might be true more generally. This leads us to Theorem 17, which is true for the symmetric Fock space \( H^2_d \) in \( d \) variables, \( d \geq 2 \).
Theorem 17 Let $\mathcal{M} \subset H_d^2 \otimes \mathbb{C}^N$ be an invariant subspace of the $\mathbb{C}^N$-valued symmetric Fock space in $d$ variables ($d \in \mathbb{N}$). Then

$$f.d.(\mathcal{M}) = d! \lim_{k \to \infty} \frac{\dim P_k \mathcal{M}}{k^d}.$$ 

The proof of Theorem 17 is in Subsection 3.3, which will complete the proof of Theorem 15. A generalization of Theorem 17 will be given in [22]. Before giving the proof we list several other consequences of Theorem 17 in Subsection 3.2, mostly having something to do with the curvature invariant introduced by Arveson [3].

3.2 The curvature of a pure $d$-contraction

This subsection contains applications to Arveson's curvature invariant. This is largely in response to Arveson's question on expressing the curvature invariant by invariants which are directly determined by the spatial actions of the $d$-contractions, and through the expression one can immediately tell that the curvature is an integer [4], [5]. The content of this and the next subsections, together with Theorem 17, has been circulated in a preprint under the title “Samuel multiplicity and Arveson's curvature invariant”.

Recall that [2] a $d$-contraction is a $d$-tuple of commuting operators $T = (T_1, \cdots, T_d)$ acting on a Hilbert space $H$ that defines a row contraction in the sense that

$$||T_1 \xi_1 + \cdots + T_d \xi_d||^2 \leq ||\xi_1||^2 + \cdots + ||\xi_d||^2$$

for all $\xi_1, \cdots, \xi_d \in H$. For every $d$-contraction we have $T_1 T_1^* + \cdots + T_d T_d^* \leq 1$. Define the defect rank of the $d$-contraction $T$ to be the rank of the defect operator $\Delta_T = \sqrt{1 - T_1 T_1^* - \cdots - T_d T_d^*}$. $T$ is said to be pure if the completely positive map defined by

$$\psi(X) = T_1 X T_1^* + \cdots + T_d X T_d^*, \quad X \in B(H)$$

satisfies $\psi^n(1) \to 0$ strongly, as $n \to \infty$. In the case of a single contraction, it reduces to the condition that $T$ belongs to the class $C_{0,0}$, see Sz.-Nagy and Foias [45].

For any $z \in \mathbb{C}^d$, define $T(z) = \bar{z}_1 T_1 + \cdots + \bar{z}_d T_d$, and $F(z) = \Delta_T(1 - T(z)^*)^{-1}(1 - T(z))^{-1} \Delta_T$. Then the curvature $K(T)$ of Arveson is defined by
\[ K(T) = \int_{S_d} \lim_{r \to 1^+} (1 - r^2) \text{tr}(F(rz)) \, dz, \quad (29) \]

where \( dz \) is the normalized Lebesgue measure on the unit sphere \( S_d \) in \( \mathbb{C}^d \).

Results of Greene-Richter-Sundberg \[27\] shows that the curvature is known to be an integer. But it is not clear how it can be computed in terms of the actions of the operators \( T_i \) (see Arveson’s \[5\] for more comments). Here we shall show that the Samuel multiplicity provides such a way to compute the curvature, and that it is obviously an integer.

On the other hand, given a pure d-contraction with finite defect rank, i.e., such that \( I - T_1 T_1^* - \cdots - T_d T_d^* \) is finite rank, it follows that \( H/(T_1 H + \cdots + T_d H) \) is finite dimensional. Hence, by considering \( H \) as a Hilbert module over \( \mathbb{C}[z_1, \ldots, z_d] \), the Samuel multiplicity with respect to \( I = (z_1, \ldots, z_d) \)

\[ e(I, H) = d! \lim_{k \to \infty} \frac{\dim H/I^k H}{k^d} \quad (30) \]

exists, and is an integer.

The main result of this subsection is

**Theorem 18** Let \( T = (T_1, \ldots, T_d) \) be a pure d-contraction with finite defect rank, acting on a Hilbert space \( H \). Regard \( H \) as a Hilbert module over \( \mathbb{C}[z_1, \ldots, z_d] \), and define its Samuel multiplicity \( e(I, H) \) with respect to \( I = (z_1, \ldots, z_d) \).

Then the curvature of \( T \) is always equal to the Samuel multiplicity, that is, \( K(T) = e(I, H) \).

**Proof.** In the theory of pure d-contractions with finite defect rank, a significant reduction, due to Arveson \[2\], is that all these tuples can be realized as the compressions of the tuple of multiplications by coordinate functions onto coinvariant subspaces of the vector-valued symmetric Fock space \( H_d^2 \) over the unit ball in \( \mathbb{C}^d \). Our proof relies on this reduction.

Fix an invariant subspace \( \mathcal{M} \subset H_d^2 \otimes \mathbb{C}^N \) \((N \in \mathbb{N})\) of the \( \mathbb{C}^N \)-valued symmetric Fock space in \( d \) variables. Let \( T = (T_1, \ldots, T_d) \) acting on \( H = \mathcal{M}^\perp \) be the compression of the multiplication tuple \( M_z = (M_{z_1}, \ldots, M_{z_d}) \) acting on \( H_d^2 \otimes \mathbb{C}^N \) onto \( \mathcal{M}^\perp \). Then results in \[27\] implies that \( K(T) + f.d.(\mathcal{M}) = N \). By the above discussions on \( d! \lim_{k \to \infty} \frac{\dim P_k \mathcal{M}}{k^d} \), and Theorem \[17\] we can complete the proof. \( \square \)
In [4] Arveson pointed out that “unlike the Fredholm index, the curvature invariant is not known to be invariant under similarity.” Here as a quick application of Theorem 18, we show that it is.

**Corollary 19** If $T = (T_1, \cdots, T_d)$ acting on $H$, and $S = (S_1, \cdots, S_d)$ acting on $K$ are two $d$-contractions with finite defect rank, and $T$ is similar to $S$; that is, there exists an invertible operator $X \in B(H, K)$ such that $S_iX = XT_i$ for all $i$, then $K(T) = K(S)$.

Proof. By arguments similar to the proof of Theorem 2, part (1) in [19], we know that the Hilbert polynomial $\phi_H(k)$, hence the Samuel multiplicity $e(I, T)$, is invariant under similarity. By Theorem 3 so is $K(T)$. □

Note that $\varphi_M(k + 1) = \dim P_kM = \text{rank } P_kP_M$, where $P_M$ is the projection onto $M$. Also it is known [19], [27] that $d! \lim_{k \to \infty} \frac{\text{tr}(P_kP_M)}{k^d}$ is equal to the fibre dimension $f.d.(M)$. Thus it follows that

**Corollary 20** For any invariant subspace $M \subset H_d^2 \otimes \mathbb{C}^N$, $d! \lim_{k \to \infty} \frac{\text{tr}(P_kP_M)}{k^d} = d! \lim_{k \to \infty} \frac{\text{rank}(P_kP_M)}{k^d}$.

Note that for two projections $P$ and $Q$, $\text{tr}(PQ) = \text{rank}(PQ)$ if and only if $PQ$ is a projection, which means that the range of $P$ naturally splits into a direct sum with respect to the range and the kernel of $Q$. It suggests that an invariant subspace $M \subset H_d^2 \otimes \mathbb{C}^N$ is asymptotically splitting with respect to polynomials of large degrees.

We do not know how to give a direct proof of the above corollary.

Finally, in this subsection, we give an interesting interpretation of the curvature invariant in terms of dilation theory. For any invariant subspace $M \subset H_d^2 \otimes \mathbb{C}^N$, let $H = M^\perp$, then we have

$$\phi_H(k) = \dim H/I^kH = \dim M^\perp \cap F_{k-1}.$$ 

So in terms of dilation theory, the curvature $K(T)$ measures, in some sense, how many polynomials $H = M^\perp$ contains.
3.3 Proof of Theorem 17

The key to the proof is to introduce an auxiliary invariant on $M$, denoted by $\varepsilon_M$, and show that it is equal to the two invariants appearing in Theorem 17 respectively.

**Definition 21** For an invariant subspace $M \subset H^2_d \otimes \mathbb{C}^N$, define $\varepsilon_M = \varepsilon$ to be the maximal dimension of a subspace $E$ of $\mathbb{C}^N$ with the following property: there exists an orthonormal basis $e_1, \ldots, e_\varepsilon$ of $E$ and $h^1, \ldots, h^\varepsilon \in M$ such that

$$P_{H^2_d \otimes E} h^i (\neq 0) \in H^2_d \otimes e_i, \quad i = 1, \ldots, \varepsilon.$$  

When $E$ has the above property we say that $M$ occupies $H^2_d \otimes E$ in $H^2_d \otimes \mathbb{C}^N$.

**Remark** It should be pointed out that $\varepsilon$ is different from the smallest dimension of a subspace $S \subset \mathbb{C}^N$, such that $M \subset H^2_d \otimes S$. A simple example can illustrate the difference: take $N = 2$; for any two functions $f, g \in H^2_d$, let $M$ be the invariant subspace generated by the $\mathbb{C}^2$-valued function $(f, g)$; then $\varepsilon = 1$, while the above $S$ can be chosen to be one-dimensional if and only if $f$ and $g$ differ by a scalar multiple.

Next we give an elementary lemma which shows that the above definition is independent of the basis $e_i$. It might be tempting to guess that this conclusion has nothing to do with Nevanlinna-Pick reproducing kernels, but just depends on the weighted shift structure. But that is not the case. In fact, one can show that, the conclusion is not true for the Bergman space over the unit disc, using the fact that, two invariant subspaces of the scalar valued Bergman space may have a positive angle.

**Lemma 22** If $M$ occupies $H^2_d \otimes E$ for some $E \subset \mathbb{C}^N$, then for any vector $e (\neq 0) \in E$, there exists an element $h \in M$ such that $P_{H^2_d \otimes E} h (\neq 0) \in H^2_d \otimes e$.

**Proof.** Fix an orthonormal basis $e_1, \ldots, e_N$ for $\mathbb{C}^N$. Then we write any element $f \in H^2_d \otimes \mathbb{C}^N$ as $f = (f_1, \ldots, f_N)$ with respect to the decomposition $\oplus_{i=1}^N H^2_d \otimes e_i$. We say that $f$ has multiplier entries if each $f_i$ is a multiplier on $H^2_d$. It is easy to see that the condition that $f$ has multiplier entries is independent of the choice of the basis $e_i$. Because $H^2_d$ admits a Nevanlinna-Pick reproducing kernel, a theorem of McCullough and Trent implies that any invariant subspace $M \subset H^2_d \otimes \mathbb{C}^N$ is generated by elements with multiplier entries. In particular, for any element $h (\neq 0) \in H^2_d \otimes \mathbb{C}^N$ the
invariant subspace generated by \( h = (h_1, \ldots, h_N) \) contains a nonzero \( f = (f_1, \ldots, f_N) \) which has multiplier entries. Since \( f = gh \) for some holomorphic \( g \), we know that \( f_i = 0 \) if and only if \( h_i = 0 \). So in Definition 21 we can assume that all \( h^i \) have multiplier entries. In particular, let \( P_{H^2_d \otimes \mathbb{C}} h^i = g_i \otimes e_i \), where \( g_i \) is a multiplier. Then for any \( e = c_1 e_1 + \cdots + c_\varepsilon e_\varepsilon \in \mathcal{E} \) (\( c_i \in \mathbb{C} \)), let

\[
h = c_1 g_2 \cdots g_\varepsilon h^1 + \cdots + c_\varepsilon g_1 \cdots g_{\varepsilon-1} h^\varepsilon,
\]
then \( P_{H^2_d \otimes \varepsilon} h = g_1 \cdots g_\varepsilon \otimes e \in H^d_2 \otimes e. \quad \square \)

For convenience, let \( \delta = f.d. \langle \mathcal{M} \rangle \). The next lemma gives the first, and easier, equality needed to prove Theorem 17.

**Lemma 23** For any \( \mathcal{M} \subset H^2_d \otimes \mathbb{C}^N \), we have \( \delta = \varepsilon \).

Proof. It is obvious that \( \delta \geq \varepsilon \). For the other direction, choose \( \delta \) elements \( h^1, \ldots, h^\delta \) from \( \mathcal{M} \), all with multiplier entries, such that at some point \( \lambda \in B_d \) the \( \delta \) vectors \( h^1(\lambda), \ldots, h^\delta(\lambda) \) are linearly independent in \( \mathbb{C}^N \). Now we write \( h^i = (h^i_1, \ldots, h^i_N) \) with respect to some basis \( e_1, \ldots, e_N \) of \( \mathbb{C}^N \), and assume that the determinant of the \( \delta \times \delta \) matrix \( \Theta = (h^i_j)_{i,j=1}^\delta \) is nonzero. Note that the determinant \( det(\Theta) \) is still a multiplier on \( H^2_d \). Recall that the inverse matrix of \( \Theta \) is given by \( (\Theta^{-1})_{ij} = (h_j^i)^{\delta}_{i,j=1} \), where \( A_{i,j} \) is the \((\delta-1) \times (\delta-1)\) minor of \( \Theta \) associated with \( h^j_i \). It follows that \( (h^i_j)(A_{i,j}) = det(\Theta) \cdot 1_\delta \) at the level of matrix multiplication. For \( j = 1, \ldots, \delta \), if we set \( g^j = \sum_{i=1}^\delta A_{i,j} h^i \), then \( P_{H^2_d \otimes F} g^j = det(\Theta) e_j \), where \( F = e_1 \mathbb{C} + \cdots + e_\delta \mathbb{C} \). So \( \mathcal{M} \) occupies \( H^2_d \otimes F. \quad \square \)

The other equality needed to prove Theorem 17 is given by Lemma 24.

For any element \( f \in H^2_d \otimes \mathbb{C}^N \), we expand \( f \) into homogeneous terms \( f = f_1 + f_{c+1} + \cdots \), where \( f_i \) is a homogeneous polynomial of degree \( i \), and \( f_c \neq 0 \). Then we call \( c \) the order of \( f \) at the origin, denoted by \( \text{ord}(f) = c \). Let \( \mathcal{P}_k \) denote the set of all (scalar-valued) polynomials with degrees at most \( k \). (Note that we have used a different notation \( \mathcal{F}_k \) to denote the vector-valued polynomials.) When \( N = 1, \mathcal{M} = H^2_d \), we have \( \varphi_{\mathcal{M}}(k) = \dim \mathcal{P}_{k-1} = \left( \begin{array}{c} d + k - 1 \\ d \end{array} \right) \) and \( \sigma = d! \lim_{k \to \infty} \frac{\varphi_{\mathcal{M}}(k)}{k^d} = 1 \).

**Lemma 24** For any \( \mathcal{M} \subset H^2_d \otimes \mathbb{C}^N \), we have \( \sigma = d! \lim_{k \to \infty} \frac{\varphi_{\mathcal{M}}(k)}{k^d} = \varepsilon. \)

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Proof. We first show that \( \sigma = d! \lim_{k \to \infty} \frac{\varphi_M(k)}{k^d} = 1 \) when \( \mathcal{M} = [f] \), the invariant subspace generated by a single element \( f \in H_d^2 \otimes \mathbb{C}^N \). Note that

\[
\dim P_{k+\text{ord}(f)}[f] \geq \dim P_{k+\text{ord}(f)}(\text{span}\{P_k f\}) = \dim P_k = \binom{d+k}{d}.
\]

The first equality holds because if \( P_k f + \text{span}\{P_k f\} = 0 \) for some polynomial with degree at most \( k \), then we must have \( p = 0 \). Now it is easy to see \( \sigma = 1 \).

Choose a basis \( e_1, \ldots, e_N \) for \( \mathbb{C}^N \), and write any \( f = (f_1, \ldots, f_N) \in H_d^2 \otimes \mathbb{C}^N \) relative to the basis. Assume that \( \mathcal{M} \) occupies \( H_d^2 \otimes \mathcal{E} \), where \( \mathcal{E} = e_1 \mathbb{C} + \cdots + e_N \mathbb{C} \). Choose \( h^i \in \mathcal{M} \) such that \( P_{H_d^2 \otimes \mathcal{E}} h^i = h_i^i \otimes e_i (\neq 0) \). Let \( \mathcal{M}' = [h_1^i \otimes e_1, \ldots, h_N^\varepsilon \otimes e_N] \), which naturally splits into the direct sum of \( \varepsilon \) many singly generated invariant subspaces. Note that \( \mathcal{M}' \subset P_{H_d^2 \otimes \mathcal{E}}(\mathcal{M}) \).

Now

\[
\dim P_k(\mathcal{M}) \geq \dim P_{H_d^2 \otimes \mathcal{E}} P_k(\mathcal{M}) = \dim P_k P_{H_d^2 \otimes \mathcal{E}}(\mathcal{M}) \geq \dim P_k(\mathcal{M}') = \sum_{i=1}^\varepsilon \dim P_k([h_i^i \otimes e_i]).
\]

It follows that \( \sigma \geq \varepsilon \).

For the other direction we first recall that for any \( \lambda \in B_d \), \( \dim(\mathcal{M}(\lambda)) \leq \delta \), which is equal to \( \varepsilon \) by Lemma 23. Hence for any \( f = (f_1, \ldots, f_N) \in \mathcal{M} \), the determinant of the \((\varepsilon + 1) \times (\varepsilon + 1)\) matrix

\[
\begin{pmatrix}
  h_1^1 & 0 & \cdots & 0 & f_1 \\
  0 & h_2^2 & \cdots & 0 & f_2 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & \cdots & h_\varepsilon^\varepsilon & f_\varepsilon \\
  h_1^i & h_2^i & \cdots & h_\varepsilon^i & f_i
\end{pmatrix}
\]

is identically zero for any fixed \( i = \varepsilon + 1, \ldots, N \). It follows that

\[
g_1 f_1 + g_2 f_2 + \cdots + g_\varepsilon f_\varepsilon + g_i f_i = 0, \tag{31}
\]

where \( g_j = h_1^1 \cdots h_{j-1}^i \cdot h_j^i \cdot h_{j+1}^i \cdots h_\varepsilon^i \) for \( j = 1, \ldots, \varepsilon \), and \( g_i = h_1^1 \cdots h_\varepsilon^i \). In particular, \( g_i \) is nonzero, and independent of \( i \).
Now for $k \in \mathbb{N}$, we consider the natural map $J_k : P_k(\mathcal{M}) \to P_{H^2_0}$. If $\xi = P_k(f) \in \ker(J_k)$, that is,

$$P_k(f_1 \otimes e_1) = \cdots = P_k(f_\varepsilon \otimes e_\varepsilon) = 0,$$

then by Eq. 31

$$P_k(g_1 f_1 + \cdots + g_\varepsilon f_\varepsilon) = 0.$$

Note that $\text{ord}(g_i)$ is independent of $i$. Now by looking at the lowest degree term in $g_i f_i$, we conclude that $P_{k-\text{ord}(g_i)} f_i = 0$ and $i = \varepsilon + 1, \cdots, N$. This implies that the kernel $\ker(J_k)$ is contained in the range of $P_{H^2_0} (P_k - P_{k-\text{ord}(g_i)})$ whose rank is

$$(N - \varepsilon) \left( d + \frac{k}{d} \right) - \left( d + \frac{k-\text{ord}(g_i)}{d} \right),$$

which is a polynomial of degree $d - 1$. Hence

$$\sigma = d! \lim_{k \to \infty} \frac{\dim P_k(\mathcal{M})}{k^d} = d! \lim_{k \to \infty} \frac{\dim P_{H^2_0}P_k(\mathcal{M})}{k^d} \leq \varepsilon. \quad \Box$$

4 Concluding remarks

Since part of our purpose is to develop a multivariable Fredholm theory, we give a different proof of the index formula $\text{index}(M_z, M_w) = e_0$, appearing in Theorem 11. But we are not able to find a different proof for the characterization of the Fredholmness of $(M_z, M_w)$.

We have to assume that $(M_z, M_w)$ is Fredholm. Then by step 3, and 6 in the proof of Theorem 11 we have

$$\text{index}(T_w) = - \lim_{k \to \infty} \frac{\dim \mathcal{M}/(z\mathcal{M} + w^k\mathcal{M})}{k}. \quad (32)$$

By Proposition 7

$$\lim_{k \to \infty} \frac{\dim \mathcal{M}/(z\mathcal{M} + w^k\mathcal{M})}{k} \geq e(I, \mathcal{M}) = e_0,$$

where $I = (z, w) \subset \mathbb{C}[z, w]$ is the maximal ideal at the origin.

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By the result of Gleason-Richter-Sundberg [26], we know that, if \( \mathcal{M} \) is Fredholm, then

\[
\text{index}(M_z, M_w) = f.d.(\mathcal{M}).
\]

By Theorem 17, \( f.d.(\mathcal{M}) = d! \lim_{k \to \infty} \frac{\dim P_k \mathcal{M}}{k^d} \).

Note that the kernel of the following natural surjective map

\[
\mathcal{M} \to P_k \mathcal{M}
\]

contains \( I^{k+1} \mathcal{M} \), and hence it factors through

\[
\mathcal{M}/I^{k+1} \mathcal{M} \to P_k \mathcal{M}.
\]

It follows that

\[
e(I, \mathcal{M}) = d! \lim_{k \to \infty} \frac{\dim \mathcal{M}/I^{k+1} \mathcal{M}}{k^d} \geq d! \lim_{k \to \infty} \frac{\dim P_k \mathcal{M}}{k^d} = f.d.(\mathcal{M}).
\]

Now we have another proof of the index formula in Theorem 11.

Before we conclude the paper, we make some comments on further studies. Hopefully, this will spur the interests of some readers.

So far, we have several numerical invariants defined on \( \mathcal{M} \subset H^2 \otimes \mathbb{C}^N \), appearing in [19], [20], [21], and this paper:

1. \( \text{index}(M_z, M_w) \);
2. \( e(I, \mathcal{M}) \);
3. \( \lim_{k \to \infty} \frac{\dim \mathcal{M}/(z \mathcal{M} + w^k \mathcal{M})}{k} \);
4. \( \dim \mathcal{M}/[(z - \lambda) \mathcal{M} + (w - \mu) \mathcal{M}] \) for almost all \((\lambda, \mu)\) in the ball;
5. \( d! \lim_{k \to \infty} \frac{\text{rank} P_k \mathcal{M}}{k^d} \), here \( P_\mathcal{M} \) is the orthogonal projection onto \( \mathcal{M} \);
6. \( d! \lim_{k \to \infty} \frac{\text{trace} P_k \mathcal{M}}{k^d} \);
7. \( f.d.(\mathcal{M}) \);
8. the \( \varepsilon \) invariant defined in 3.3.

Among them the last five are always well-defined, and equal; moreover, when \( \dim \mathcal{M}/\mathcal{I} \mathcal{M} < \infty \), the first three are defined, and are equal to the last five.

It seems that the above equalities, or the failure of these equalities, together with the additivity of Hilbert polynomials as in Theorem 15 and [21], can serve as test problems when trying to develop operator theory in several variables or over different spaces. It is interesting to observe that, when one
looks at the Bergman space over various domains, almost all the above equal-
ities can fail to be true. This might pose a host of problems. For instance, is
the invariant in the above (6) always well defined for any invariant subspace
of the Bergman space $L^2_a(\mathbb{D})$ over the unit disc? A positive answer would
provide a numerical invariant, lying between 0 and 1, measuring the size of
the invariant subspaces of $L^2_a(\mathbb{D})$; in fact, if we assume its existence, then
elements show that it takes all the values in the interval $(0, 1]$.

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