Dissecting a Resonance Wedge on Heteroclinic Bifurcations

Alexandre A. P. Rodrigues

Abstract
This article studies routes to chaos occurring within a resonance wedge for a 3-parametric family of differential equations acting on a 3-sphere. Our starting point is an autonomous vector field whose flow exhibits an attracting heteroclinic network made by two 1-dimensional connections and a 2-dimensional separatrix between two equilibria with different Morse indices. After changing the parameters, while keeping the 1-dimensional connections unaltered, we concentrate our study in the case where the 2-dimensional invariant manifolds of the equilibria do not intersect. We derive the first return map near the network and we reduce the analysis of the system to a 2-dimensional map on the cylinder. Complex dynamical features arise from a discrete-time Bogdanov–Takens singularity, which may be seen as the organizing center by which one can obtain infinitely many attracting tori, strange attractors, infinitely many sinks and non-trivial contracting wandering domains. These dynamical phenomena occur within a structure that we call resonance wedge. As an application, we may see the “classical” Arnold tongue as a projection of a resonance wedge. The results are general, extend to other contexts and lead to a fine-tuning of the theory.

Keywords Heteroclinic bifurcations · Torus-breakdown · Resonance wedge · Arnold tongue · Strange attractors

Mathematics Subject Classification 34C28 · 34C37 · 37D05 · 37D45 · 37G35

1 Introduction
To date there has been very little systematic investigation of the effects of perturbations that break an invariant torus, despite being natural for the modelling of many biological and physi-
cal effects [2,32,33,38,47,52]. In this paper, we describe the transition from regular dynamics to chaos associated to the Torus-breakdown Theory partially described in Afraimovich and Shilnikov [1], applied to a specific heteroclinic configuration involving 2-dimensional connecting manifolds (continuum of connections in the terminology of [11]).

In studying bifurcations associated to Torus-breakdown, it is natural to examine the bifurcation diagram in terms of Arnold tongues [8]. In planar maps, a \( \frac{p}{q} \)-resonance tongue can be defined as the locus, in the parameter space, where periodic points with rotation number \( \frac{p}{q} \) exist, for \( p, q \in \mathbb{N} \); in the engineering literature, these solutions are called \( p:q \) phase-locked [22,27]. The way in which resonance tongues overlap and evolve indicates how the rotational dynamics is changing.

Although tongues corresponding to different rotation numbers can overlap, in general there is no path where a periodic orbit from a resonance tongue becomes a periodic orbit from another tongue [15,22]. A singular phenomenon has been described by Kirk [32], who studied resonance zones for a 3-dimensional system that corresponds to the normal form associated to a codimension-two Hopf-zero singularity. Such a normal form is perturbed with nonsymmetric terms breaking the axial symmetry and the phenomenon of “merging of resonance wedges” has been described: far from the torus bifurcation, a periodic orbit modifies its shape and collapses with another periodic orbit coming from another tongue.

Assuming a strong 1-dimensional contracting direction, the structure of the Arnold tongue has been given by Arnold [8], Boyland [15] and Herman [27] who reduced the study of Torus-breakdown effects to the “canonical family” on the circle:

\[
x \mapsto x + a + \frac{b}{2\pi} \sin(2\pi x) \mod 1, \quad a, b \in \mathbb{R}.
\]

When the radial contracting foliation on the torus is lost, the bifurcation structure of the circle maps family is inadequate to explain the diverse phenomena that accompany the loss of the attracting torus and it is here that the study of resonance wedges plays an important role. Resonance tongues are associated to a wide range of behaviours such as: the existence of quasi-periodic solutions, sinks, saddle-node bifurcations, homoclinic orbits, bistability, rotational horseshoes and strange attractors (either “large” or “small” according to Broer et al. [17]). Early papers in this context are [1,2,7,10]. We address the reader to [3,9,15,17,25,45] for more information on the subject. For a tutorial, see Shilnikov et al. [53]. New directions of the theory and applications to periodically-kicked differential equations can be found in [13,14,20,37,50,55].

The goal of this paper is to construct a 3-parameter bifurcation diagram for a concrete configuration associated to a weakly attracting heteroclinic network with two saddle-foci previously studied in [50], the Bykov attractor. Our study has been motivated by numerical results obtained in [6,20,21,37].

Our purpose in writing this paper is not only to point out the range of phenomena that can occur when an invariant torus is broken, but to bring to the foreground the techniques that have allowed us to reach these conclusions in a relatively straightforward manner. These mechanisms are not limited to the heteroclinic network considered here.

1.1 The Novelty

While some progress has been made, both numerically and analytically, the number of explicit configurations whose flows have an invariant torus and for which the Torus-Breakdown description is available, has remained small.
By studying unfoldings of a Bykov attractor, we are able to delineate the ways in which the first return map to a cross section can make the transition from a single rotation number to an interval of rotation numbers.\textsuperscript{1} As a continuation of the project started in [50], we will prove analytically that a sequence of discrete-time Bogdanov–Takens bifurcations organise the dynamics that appear in the unfolding. Besides, within the Arnold wedge, we numerically find surfaces corresponding to the following bifurcations: Hopf, period-doubling and the transitions node $\leftrightarrow$ focus.\textsuperscript{2} Our results agree well with the literature about heteroclinic bifurcations, Arnold tongues and the scenarios described in [9,10,45].

Our object of study is not “just another dynamical system”, but representative for the case of 3-dimensional dissipative flows admitting 2-dimensional connections that are pulled apart. In similar models, the corresponding phenomenology should contain no further secrets.

### 1.2 Physical Setting

Our study allows us to understand the bifurcations from an invariant torus to strange attractors that appear in Ruelle and Takens [51] and Langford [38] (see §6.2 of [20]). In the context of turbulent flows, the author of [38] studied a two-parameter unfolding a Hopf-zero singularity and proved that axisymmetric perturbations generate an invariant torus. By slightly breaking the symmetry, Langford prove that the flow becomes more and more turbulent with fractal basins of attraction as a consequence of the emergence strange attractors. The bifurcations described in our paper have similarities with those described in [52] in the context of a low-order atmospheric circulation model.

### 1.3 The Structure

In Sect. 2, we describe precisely our object of study and we review the literature related to it. In Sect. 3 we state the main results of the article. The coordinates and other notation used in the rest of the article are presented in Sect. 4 to prove the main results of the manuscript in Sect. 5.

We refine the structure of Arnold tongues which appear in the context of heteroclinic bifurcations on Sect. 6. In Sect. 7, we briefly illustrate our theoretical results with an example explored in [21]. Symmetry-breaking effects will be described.

We finish the article with a discussion in Sect. 8 about the consequences of our findings. Dynamics similar to what we described is expected to occur near periodically forced attracting heteroclinic cycles. For reader’s convenience, we have compiled at the end of the article (Appendix A) a list of basic definitions.

### 2 Setting and State of Art

In this section, we describe the main hypotheses about the weakly heteroclinic network we are considering. We postpone the technical definitions used in this section to Appendix A.

\textsuperscript{1} The existence of rotational horseshoes, described in Appendix A.8, is responsible for the existence of an interval of rotation number—see [43].

\textsuperscript{2} Also called Belyakov transitions.
2.1 Starting Point

For $\varepsilon > 0$ small, consider the 3-parameter family of $C^3$–smooth differential equations

$$\dot{x} = f(A,\lambda,\omega)(x) \quad x \in S^3 \subset \mathbb{R}^4 \quad A, \lambda \in [0, \varepsilon], \quad \omega \in \mathbb{R}^+$$

(2.1)

where $S^3$ represents the unit three-sphere, endowed with the usual topology. Let us denote by $\varphi(t, x)$, $t \in \mathbb{R}$, the flow associated to (2.1) satisfying the following properties for $A = \lambda = 0$ and $\omega \in \mathbb{R}^+$:

(P1) There are two different equilibria, say $O_1$ and $O_2$.

(P2) The eigenvalues of $Df_X$ are:

(P2a) $E_1$ and $-C_1 \pm \omega i$ where $C_1, E_1 > 0$, for $X = O_1$;

(P2b) $-C_2$ and $E_2 \pm \omega i$ where $C_2, E_2 > 0$, for $X = O_2$.

For $W \subseteq S^3$, denoting by $\overline{W}$ the closure of $W$, we also assume that:

(P3) The manifolds $\overline{W^{u}(O_2)}$ and $W^{s}(O_1)$ coincide and the set $\overline{W^{u}(O_2)} \cap W^{s}(O_1)$ consists of a 2-dimensional sphere ($\overline{W^{u}(O_2)} \cap W^{s}(O_1)$ is called the 2D-connection).

and

(P4) There are two trajectories, say $\gamma_1, \gamma_2$, contained in $\overline{W^{u}(O_1)} \cap W^{s}(O_2)$, each one in each connected component of $S^3 \setminus \overline{W^{u}(O_2)}$ ($\gamma_1, \gamma_2$ are called the 1D-connections).

The equilibria $O_1$ and $O_2$, the 2D-connection referred in (P3) and the two trajectories $\gamma_1, \gamma_2$ of (P4) build a heteroclinic network, that we denote by $\Gamma$. This network, illustrated in Fig. 1, has two cycles. Assuming that the set of eigenvalues of $Df_{O_1}$ and $Df_{O_2}$ satisfy

(P5) $\frac{C_1 C_2}{E_1 E_2} \gtrsim 1$

the network $\Gamma$ is asymptotically stable: with exception of the origin, all trajectories are forward attracted to $\Gamma$. As a consequence, we may find a neighborhood $\mathcal{U}$ of $\Gamma$ having its boundary transverse to the flow and such that every solution starting in $\mathcal{U}$ is asymptotic to $\Gamma$. The set $\Gamma$ is usually called Bykov attractor.$^3$

---

$^3$ The terminology Bykov is a homage to V. Bykov who dedicated his latest research projects to similar cycles [18,19].
2.2 Chirality: A Topological Assumption

There are two possibilities for the geometry of the flow around the saddle-foci of $\Gamma_1$, depending on the direction the trajectories turn around the 1D-connections. This is related to the topological concept of *chirality* introduced in [36].

Let $V_1$ and $V_2$ be small disjoint neighborhoods of $O_1$ and $O_2$ with boundaries $\partial V_1$ and $\partial V_2$, respectively. These neighborhoods will be precisely constructed in Sect. 4. Trajectories starting at $\partial V_1 \setminus W_s(O_1)$ near $W_s(O_1)$ go into the interior of $V_1$ in positive time, then follow one of the solutions in $[O_1 \rightarrow O_2]$, go inside $V_2$, come out at $\partial V_2$ and then return to $\partial V_1$ (see Fig. 2). This trajectory is not closed since $\Gamma_1$ is attracting.

Let $Q$ be a piece of trajectory like this from $\partial V_1$ to $\partial V_1$. Within $\partial V_1 \setminus W_s(O_1)$, join its starting point to its end point by a segment as in Fig. 2, forming a closed curve, which we call the *loop* of $Q$. By construction, the loop of $Q$ and $\Gamma_1$ are disjoint closed sets.

**Definition 1** We say that $O_1$ and $O_2$ in $\Gamma_1$ have the *same chirality* if the loop of every trajectory starting near $O_1$ is linked to $\Gamma_1$ (i.e. the trajectories cannot be disconnected by an isotopy). Otherwise, we say that $O_1$ and $O_2$ have *different chirality*.

The next hypothesis may be written as:

**(P6)** The saddle-foci $O_1$ and $O_2$ have the same chirality.

For $r \geq 3$, denote by $\mathcal{X}^r(S^3)$, the set of 3-parameter family of $C^3$–vector fields on $S^3$ satisfying Properties (P1)–(P6), endowed with the $C^r$–topology.

2.3 Perturbing Terms

Concerning the effect of $A$, $\lambda$ and $\omega$ on the dynamics of (2.1), we assume that:

**(P7)** For $A > \lambda \geq 0$ and $\omega \in \mathbb{R}^+$, the two trajectories within $W^u(O_1) \cap W^s(O_2)$ persist.

By the Kupka-Smale Theorem, generically the invariant 2-dimensional manifolds $W^u(O_2)$ and $W^s(O_1)$ are transverse (either intersecting or not):

**(P8a)** For $A, \lambda \geq 0$ and $\omega \in \mathbb{R}^+$, the manifolds $W^u(O_2)$ and $W^s(O_1)$ intersect transversely.

**(P8b)** For $A, \lambda \geq 0$ and $\omega \in \mathbb{R}^+$, the manifolds $W^u(O_2)$ and $W^s(O_1)$ do not intersect.

and

**(P9)** Up to high order terms in $x, y$, the transitions along the connections $[O_1 \rightarrow O_2]$ and $[O_2 \rightarrow O_1]$ are given, in the local coordinates that will be defined in Sect. 4, by the *Identity map* and by

$$(x, y) \mapsto (x, y + A + \lambda \Phi(x))$$

respectively, where $\Phi : S^1 \rightarrow S^1$ is a Morse function with at least two non-degenerate critical points ($S^1 = \mathbb{R} \ (\text{mod} \ 2\pi)$). This assumption will be clearer later.
2.4 Constants

Once for all, we define the following notation that will be used throughout the present manuscript:

\[
\begin{align*}
\delta_1 &= \frac{C_1}{E_1} > 0 \\
\delta_2 &= \frac{C_2}{E_2} > 0 \\
\delta &= \delta_1 \delta_2 \gtrsim 1 \\
\omega^* &= \frac{2\ell \pi (\delta - 1)}{K \ln \delta} > 0 \\
K &= \frac{E_2 + C_1}{E_1 E_2} > 0 \\
M &= \delta \frac{1}{\tau_x} - \delta \frac{1}{\tau_y} \\
\mu &= (A, \lambda, \omega) \\
\nu &= (A, \lambda_0, \omega), \quad \lambda_0 > 0 \text{ fixed}
\end{align*}
\]

(2.2)

2.5 Digestive Remarks About the Hypotheses

We discuss the Hypotheses (P1)–(P9), stressing that they are natural in several settings. An illustrative scheme has been summarized in Table 1 of [50].

Remark 2.1 Although the fully description of the bifurcations associated to the heteroclinic attractor \( \Gamma \) is a phenomenon of codimension three [35], the setting described by (P1)–(P9) is natural in symmetric contexts [4,21] and also in some unfoldings of the Hopf-zero singularity [12,23,52].

Remark 2.2 The first hit of \( W^u(O_2) \) and \( W^s(O_1) \) to a global cross section \( \Sigma \) are two closed curves (see Fig. 3); the distance between these two curves can be written as

\[
A + \lambda \Phi(x), \quad x \in S^1,
\]

which may be seen as an approximation of the Melnikov integral associated to the intersection of \( W^u(O_2) \) and \( W^s(O_1) \) [26, §4.5].

Remark 2.3 Variable \( \omega \) represents the speed of rotation of the saddle-foci. Using different imaginary parts on the complex eigenvalues of \( Df_{O_1} \) and \( Df_{O_2} \) would complicate the expression of \( K \omega \) without any qualitative benefits in the final result.

Remark 2.4 Derivations of the first return map using a more general form for the transition \([O_1 \rightarrow O_2]\) have been performed in Sect. 6 of [48]. The transition along \([O_2 \rightarrow O_1]\) corresponds to the expected unfolding from the coincidence of the 2-dimensional invariant manifolds at \( f(0,0,\omega), \omega \in \mathbb{R}^+ \).
Remark 2.5 The technical Hypothesis (P9) modulates the two generic possibilities given by (P8a) and (P8b):

\[
\begin{align*}
\lambda &> A \geq 0, \quad \omega \in \mathbb{R}^+ \quad \Leftrightarrow \quad (P8a) : W^u(O_2) \text{ and } W^s(O_1) \text{ intersect transversely;} \\
A &> \lambda \geq 0, \quad \omega \in \mathbb{R}^+ \quad \Leftrightarrow \quad (P8b) : W^u(O_2) \text{ and } W^s(O_1) \text{ do not intersect.}
\end{align*}
\]

Assumption (P9) governs the transition maps along the heteroclinic connections and is necessary to make precise computations. All results are valid for any \(2\pi\)-periodic non-constant Morse function \(\Phi\).

2.6 State of Art

In this section, we give an overview of results for the class of vector fields satisfying either (P1)–(P8a)–(P9) or (P1)–(P8b)–(P9).

2.6.1 Heteroclinic Tangle

If \(f(A, \lambda, \omega) \in \mathfrak{X}(S^3)\) and satisfies (P7)–(P8a)–(P9), then the 2-dimensional invariant manifolds \(W^u(O_2)\) and \(W^s(O_1)\) meet transversely, giving rise to a union of Bykov cycles [19,48]. The dynamics in the maximal invariant set inside \(\mathcal{U}\), contains the suspension of horseshoes accumulating on the network described in [5,35,48]. Near the original heteroclinic attractor \(\Gamma\), the flow contains infinitely many homoclinic tangencies and sinks with long periods, coexisting with sets with positive entropy, giving rise the so called quasi-stochastic attractors [24]; more details in Appendices A.2 and A.3.

2.6.2 Torus-Breakdown

If \(f(A, \lambda, \omega) \in \mathfrak{X}(S^3)\) and satisfies (P7)–(P8b)–(P9), then the 2-dimensional invariant manifolds of the saddle-foci do not intersect. According to [50], in the bifurcation diagram \((\omega, \lambda/A)\) of Fig. 4, there are two curves, the graphs of \(h_1\) and \(h_2\), such that, for all \(\omega \in \mathbb{R}^+\), we have \(h_1(\omega) < h_2(\omega)\) and:

1. the region below the graph of \(h_1\) corresponds to parameters whose flow has an attracting normally hyperbolic 2-dimensional torus with zero topological entropy – Theorem B of [50]. In the bifurcation parameter \((\omega, \lambda/A)\), there exists a set of positive Lebesgue measure for which the whole torus is the minimal attractor.
2. for a fixed \(\omega > 0\), in the transition from \(h_1(\omega)\) to \(h_2(\omega)\), the attracting torus breaks. It starts to disintegrate into a finite collection of periodic saddles and sinks, a phenomenon occurring within an Arnold Tongue [50,53,55]. Each time the Floquet multipliers of periodic orbits cross a root of unity, a pair of saddle-node bifurcation curves may be defined. These curves limit locally the corresponding resonance tongue. In addition, there are regions corresponding to homoclinic tangencies to dissipative periodic solutions, responsible for the persistence of Hénon-like strange attractors [50].
3. the region above the graph of \(h_2\) corresponds to vector fields whose flows exhibit rotational horseshoes [50] (see Appendix A.8). Theorem D of [50] may be seen as a criterion to obtain rotational horseshoes near \(\Gamma\); once they develop, they persist for small perturbations.
Fig. 4 Overview of the results of [50]: location of the attracting 2-dimensional torus and chaotic regions with respect to $\omega$ and $\lambda$ for $f(A,\lambda,\omega) \in X_{Byk}^3(S^3)$.

The graphs of $h_1$ and $h_2$ of Fig. 4 are not bifurcation lines; they define regions inside which the transitional dynamics occurs. From now on, without loss of generality, let us also assume that $\Phi(x) = \sin x$, $x \in S^1$, which has exactly two critical points. It simplifies the computations and allows comparison with previous works. For $r \geq 3$, we denote by $X_{Byk}^r(S^3) \subset X^r(S^3)$, the set of $C^r$–vector fields on $S^3$ satisfying conditions (P1)–(P8b) and (P9).

3 Main Results

Let $T$ be a neighborhood of the heteroclinic attractor $\Gamma$, which exists for $A = \lambda = 0$ and $\omega \in \mathbb{R}^+$. For $\varepsilon > 0$ small, define the set

$$\mathcal{V} = \{ \mu = (A, \lambda, \omega) : \quad 0 \leq \lambda < A \leq \varepsilon \quad M \geq A + \lambda \quad \text{and} \quad \omega \in \mathbb{R}^+ \} \quad (3.1)$$

and let $(f_\mu)_{\mu \in \mathcal{V}}$ be a 3-parameter family of vector fields in $X_{Byk}^3(S^3)$. According to [50], there is $\tilde{\varepsilon} > 0$ such that the first return map to a given global cross section $\Sigma$ to $\Gamma$ can be expressed, in local coordinates $(x, y) \in \Sigma$, by:

$$F_\mu(x, y) = [x - K \omega \ln(y + A + \lambda \sin x) \pmod{2\pi}, \ (y + A + \lambda \sin x)^{\delta}] + \ldots$$

$$\quad (3.2)$$

where

$$(x, y) \in D = \{ x \in \mathbb{R} \pmod{2\pi}, \ y/\tilde{\varepsilon} \in [-1, 1] \quad \text{and} \quad y + A + \lambda \sin x > 0 \}$$

and $\ldots$ stand for small terms depending on $x$ and $y$ converging to zero along their derivatives. The main steps to get the expression (3.2) are revived in Sect. 4. Since $\delta > 1$, for $A > 0$ sufficiently small, the second component of $F_\mu$ is contracting in $y$ (Remark 4.4).

Definition 2 Let $(x_0, y_0) \in D$ and $\ell \in \mathbb{N}$. We say that $(x_0, y_0)$ is a $(1, \ell)$–fixed point of $F_\mu$ if $F_\mu(x_0, y_0) = (x_0 + 2\ell\pi, y_0)$.

The main contribution of this article is the following result:

$\square$ Springer
Theorem A  Let $f_\mu \in \mathcal{X}_{\text{Byk}}^3(S^3)$, $\ell \in \mathbb{N}$ and $\delta \geq 1$ be a control parameter. Then, there are two curves in the 3-dimensional parameter space $\mu = (A, \lambda, \omega) \in \mathcal{V}$ where $(1, \ell)$–fixed points of $F_\mu$ undergo a discrete-time Bogdanov–Takens bifurcation. The curves occur for $\omega = \omega^*_\ell$.

The proof of Theorem A is performed in Sect. 5.3, after the statement of some preliminary technical results. Around the Bogdanov–Takens bifurcation, we found the following (secondary) codimension one bifurcations:

Corollary B  Under the hypotheses of Theorem A, there are surfaces in the 3-dimensional parameter space $\mu = (A, \lambda, \omega) \in \mathcal{V}$ where $(1, \ell)$–fixed points of $F_\mu$ undergo:

(a) a saddle-node bifurcation;
(b) a Hopf bifurcation;
(c) generic (quadratic) homoclinic tangencies associated to a dissipative saddle point.\(^4\)

In the region between the surfaces corresponding to homoclinic tangencies, there is a transverse intersection of the stable and the unstable manifolds of an invariant saddle. The surfaces corresponding to Hopf and saddle-node bifurcations meet tangentially along a curve.

Around the transverse intersection of the manifolds, horseshoe dynamics occurs. Corollary B refines the findings of Sects. 3.2 and 3.3 of [31], where the resonance wedges are projection of bifurcation surfaces. See also [46]. The ‘necessity’ of a Hopf bifurcation surface to explain the transition from an invariant torus to chaos has been raised in [44]. Strange attractors contribute to the richness and complexity of a dynamical system. Sinai-Bowen-Ruelle (SRB) measures represent visible statistical laws in non-uniformly hyperbolic systems. More details of these concepts may be found in Appendices A.5 and A.6. Chaos associated with them is both sustained in the space of parameters and observable. Next result ensures the existence of open regions in the parameter region (3.1) for which we observe strange attractors with SRB measures and historic behaviour (see Appendix A.4):

Corollary C  Under the hypotheses of Theorem A, there is an open region in the space of parameters, say $\tilde{\mathcal{V}} \subset \mathcal{V}$, such that if $\mu \in \tilde{\mathcal{V}}$, then $F_\mu$ exhibits:

1. strange attractors with an ergodic SRB measure occurring in a subset of $\tilde{\mathcal{V}}$ with positive Lebesgue measure;
2. an open set of initial conditions exhibiting historic behaviour occurring in a subset of $\tilde{\mathcal{V}}$ whose topological closure is $\tilde{\mathcal{V}}$.\(^5\)

Taking advantage of the existence of the Hopf bifurcation surfaces, we may use the reasoning of Denjoy [22] to conclude the existence of a map $H$, arbitrarily $C^1$-close to $F_\mu$, with a contracting non-trivial wandering domain. As defined in Appendix A.7, a wandering domain for $F_\mu$ may be seen as a non-empty connected open set whose forward orbit is a sequence of pairwise disjoint open sets.

Corollary D  Under the hypotheses of Theorem A, there is an open region in the space of parameters, say $\tilde{\mathcal{V}} \subset \mathcal{V}$, such that if $\mu \in \tilde{\mathcal{V}}$, then:

1. $\mathcal{F}_\mu$ exhibits an attracting 2-dimensional torus, which is contractible\(^6\);

\(4\) A $\mathcal{F}_\mu$–fixed point $O$ is dissipative if $O$ is hyperbolic and $|\det F_\mu(O)| < 1$.

\(5\) The open set is defined in the phase space; the set $\tilde{\mathcal{V}}$ is defined in the space of parameters.

\(6\) The invariant circles, in the first return map, do not envelop the phase cylinder.
there exists a diffeomorphism $H$ arbitrarily $C^1$-close to $F_\mu$, exhibiting a contracting non-trivial wandering domain $D$ for which the union of the $\omega$-limit set of points in $D$ is a nonhyperbolic transitive Cantor set without periodic points.

The proofs of Corollaries B, C and D are performed in Sects. 5.4, 5.5 and 5.6, respectively. In Sect. 6, we analyse the continuation of these bifurcations by studying the precise expressions of the eigenvalues of $DF_\mu$ at the $(1, \ell)$–fixed points of $F_\mu$. We derive an analytical expression for the Hopf and period-doubling bifurcations, as well as for the transitions node $\leftrightarrow$ focus. These plethora of bifurcations limited by two saddle-node bifurcations surfaces is what we call a resonance wedge. An Arnold tongue may be seen as the projection of one of these wedges.

**Important Remarks About Theorem A**

**Remark 3.1** The Bogdanov–Takens bifurcations of Theorem A will be computed under the assumption that $M = G_\ell(\omega_\kappa^\ast) = A \pm \lambda$ (details in Prop. 5.2). Since $A$ and $\lambda$ are small, $M$ must be also small. This is achieved by assuming that $\delta \gtrsim 1$ is a control parameter (cf. Fig. 5). The condition $\delta \gtrsim 1$ means that $\delta - 1$ can be as small as $A \pm \lambda$.

**Remark 3.2** The bifurcations in (3.2) might be studied for non-small, but arbitrary values of parameters $A$ and $\lambda$. This means that Theorem A can be regarded more generally and not only for unfolding of weakly attracting cycles (i.e. those that satisfy $\delta \gtrsim 1$).

### 4 The First Return Map

We analyze the dynamics near $\Gamma$ through local maps, after selecting appropriate coordinates in neighborhoods of $O_1$ and $O_2$.

#### 4.1 Local Coordinates

In order to describe the dynamics around the cycles of $\Gamma$, we use the local coordinates near the equilibria $O_1$ and $O_2$ introduced in [42] (cf. [50]). In these coordinates, we use cylindrical neighborhoods $V_1$ and $V_2$ in $\mathbb{R}^3$ of $O_1$ and $O_2$, respectively, of radius $\rho = \tilde{\varepsilon} > 0$ and height $z = 2\tilde{\varepsilon}$. After a linear rescaling, we assume $\tilde{\varepsilon} = 1$. 

![Graph of $\delta$](image)
The boundaries of $V_1$ and $V_2$ consist of three components: the cylinder wall parametrised by $x \in \mathbb{R} \mod 2\pi$ and $|y| \leq 1$ with the cover
\[(x, y) \mapsto (1, x, y) = (\rho, \theta, z)\]
and two discs, the top and bottom of the cylinder. We consider polar coverings of these disks
\[(r, \phi) \mapsto (r, \phi, \pm 1) = (\rho, \theta, z)\]
where $0 \leq r \leq 1$ and $\phi \in \mathbb{R} \mod 2\pi$. In $V_1$, we use the notation:

- **In ($O_1$)**, the cylinder wall of $V_1$, consists of points that go inside $V_1$ in positive time;
- **Out ($O_1$)**, the top and bottom of $V_1$, consists of points that go outside $V_1$ in positive time.

We denote by $\text{In}^+(O_1)$ the upper part of the cylinder, parametrised by $(x, y)$, $y \in [0, 1]$ and by $\text{In}^-(O_1)$ its lower part. The local stable manifold of $O_1$, $W^s_{\text{loc}}(O_1)$, corresponds to the circle parametrised by $y = 0$. The cross-sections around $O_2$ are dual of the previous sections. The set $W^u_{\text{loc}}(O_2)$ corresponds to the intersection of the $z$-axis with the top and bottom of $V_2$; these two intersection points will be the origin of its coordinates. The set $W^u_{\text{loc}}(O_2)$ is defined by $y = 0$ and:

- **In ($O_2$)**, the top and bottom of $V_2$, consists of points that go inside $V_2$ in positive time;
- **Out ($O_2$)**, the cylinder wall of $V_2$, consists of points that go outside $V_2$ in positive time, with $\text{Out}^+(O_2)$ denoting its upper part, parametrised by $(x, y)$, $y \in ]0, 1]$ and $\text{Out}^-(O_2)$ its lower part parametrised by $(x, y)$, $x \in \mathbb{R}$ and $y \in [-1, 0[$.

By construction, the flow is transverse to these cross-sections and the boundaries of $V_1$ and of $V_2$ may be written as the topological closure of $\text{In} (O_1) \cup \text{Out} (O_1)$ and $\text{In} (O_2) \cup \text{Out} (O_2)$, respectively.

**Remark 4.1** The orientation of the angular coordinate near $O_2$ is chosen to be compatible with the direction induced by the angular coordinate in $O_1$.

### 4.2 Local Maps

Adapting [42], the trajectory of a point $(x, y) \in \text{In}^+(O_1)$, leaves $V_1$ at $\text{Out} (O_1)$ at
\[
\Phi_1(x, y) = \left( y^{\delta_1} + S_1(x, y; A, \lambda, \omega), x - \frac{\omega \ln y}{E_1} + S_2(x, y; A, \lambda, \omega) \right) = (r, \phi)
\]
where $\delta_1 = \frac{C_1}{E_1} > 1$, $S_1, S_2$ are smooth functions which depend on the parameters $A, \lambda$ and $\omega$ and satisfy:
\[
\left| \frac{\partial^{k+l+m}}{\partial \lambda^k \partial y^l \partial A^m_1 \partial \lambda^m_2 \partial \omega^m_3} S_i(x, y; A, \lambda, \omega) \right| \leq C y^{\delta_1 + \sigma - l},
\]
where the numbers $C, \sigma$ are positive constants and $k, l, m_1, m_2, m_3$ are non-negative integers. In a similar way, a point $(r, \phi)$ in $\text{In} (O_2) \setminus W^u_{\text{loc}}(O_2)$ leaves $V_2$ at $\text{Out} (O_2)$ at
\[
\Phi_2(r, \phi) = \left( \phi - \frac{\omega \ln r}{E_2} + R_1(r, \phi; A, \lambda, \omega), r^{\delta_2} + R_2(r, \phi; A, \lambda, \omega) \right) = (x, y)
\]
where $\delta_2 = \frac{C_2}{E_2} > 1$ and $R_1, R_2$ satisfy a condition similar to (4.2). The expressions $S_1, S_2, R_1, R_2$ correspond to terms that vanish when $y$ and $r$ go to zero.
4.3 Global Maps

The coordinates on \( V_1 \) and \( V_2 \) are chosen so that \( [O_1 \rightarrow O_2] \) connects points with \( z > 0 \) (resp. \( z < 0 \)) in \( V_1 \) to points with \( z > 0 \) (resp. \( z < 0 \)) in \( V_2 \). Points in \( \text{Out}(O_1) \setminus W^u_{\text{loc}}(O_1) \) near \( W^u(O_1) \) are mapped into \( \text{In}(O_2) \) along a flow-box around each of the connections of \( [O_1 \rightarrow O_2] \). Assuming (P9), the transition

\[
\Psi_{1 \rightarrow 2}: \quad \text{Out}(O_1) \rightarrow \text{In}(O_2)
\]
does not depend neither on \( \lambda, A \) nor \( \omega \) and is the Identity map, a choice compatible with (P5) and (P7). Denote by \( \eta \) the map:

\[
\eta = \Phi_2 \circ \Psi_{1 \rightarrow 2} \circ \Phi_1: \quad \text{In}(O_1) \setminus W^s_{\text{loc}}(O_1) \rightarrow \text{Out}(O_2).
\]

Omitting the higher order terms that appear in (4.1) and (4.3), for \( y > 0 \) we may write:

\[
\eta(x, y) = \left( x - K \omega \ln y, \ (mod \ 2\pi), \ y^\delta \right) \quad (4.4)
\]

with

\[
\delta = \delta_1 \delta_2 \geq 1 \quad \text{and} \quad K = \frac{C_1 + E_2}{E_1 E_2} > 0. \quad (4.5)
\]

A similar expression is valid for \( y < 0 \), after suitable changes. Using (P8) and (P9), for all \( A > \lambda \geq 0 \) and \( \omega \in \mathbb{R}^+ \), we may define the map \( \Psi_{2 \rightarrow 1} : \text{Out}(O_2) \rightarrow \text{In}(O_1) \) that depends on the parameters \( \lambda \) and \( A \) (see Fig. 6):

\[
\Psi_{2 \rightarrow 1}(x, y) = (x, y + A + \lambda \sin x) \quad \text{where} \quad \Phi(x) = \sin x. \quad (4.6)
\]

Observe that \( \Psi_{2 \rightarrow 1} \) does not depend on \( \omega \). The expression of the first return map \( \mathcal{F}_\mu \) follows by composing the local and global maps constructed above. Let

\[
\mathcal{F}_\mu = \eta \circ \Psi_{2 \rightarrow 1} = \mathcal{D} \subset \text{Out}(O_2) \rightarrow \mathcal{D} \subset \text{Out}(O_2) \quad (4.7)
\]

be the first return map to \( \text{Out}(O_2) \), where \( \mathcal{D} \neq \emptyset \) is the set of initial conditions \((x, y) \in \text{Out}(O_2)\) whose solution returns to \( \text{Out}(O_2) \). Composing (4.4) with \( \Psi_{2 \rightarrow 1} \) (4.6), the expression of \( \mathcal{F}_\mu \) is given by

\[
\mathcal{F}_\mu(x, y) = \left[ x - K \omega \ln [y + A + \lambda \sin x] \ (mod \ 2\pi), \ (y + A + \lambda \sin x)^\delta \right] \\
=: \left( \mathcal{F}^\mu_1(x, y), \mathcal{F}^\mu_2(x, y) \right).
\]

The following remarks will be useful in the sequel.
Remark 4.2 The map $F_\mu$ is $C^3$ and is well defined in a compact subset of $\text{Out}(O_2)$. Thus, results on circular maps [43] may be applied to $F_\mu$.

Remark 4.3 If $A = \lambda = 0$ and $\omega \in \mathbb{R}^+$, then $F_1^{(0,0,\omega)}(x, y) = x - K \omega \ln y$ may be identified with a rigid rotation on $S^1 = \mathbb{R}/2\pi$ and $F_2^{(0,0,\omega)}(x, y) = y^\delta$ defines an invariant contracting foliation.

Remark 4.4 Since $\delta > 1$ and $1 > \varepsilon > A > y \geq 0$, we may write:

$$\frac{\partial F_2^\mu(x, y)}{\partial y} = \left| \delta(y + A + \lambda \sin x)^{\delta-1} \right| = O\\((A + \lambda)^{\delta-1}) < 1,$$

where $O\\((A + \lambda)^{\delta-1})$ represents the standard Landau notation. This means that, under Hypotheses (P1)–(P8b)–(P9), if $A > 0$ small enough, then $F_2^\mu$ is a contraction in the variable $y$.

5 Proof of Theorem A and Its Corollaries

The main goal of this section is to prove Theorem A and its consequences. We start this task by giving preparatory results.

5.1 Fixed Points of $F_\mu$ and Their Stability

The $(1, \ell)$–fixed points of $F_\mu$ in $D \cap \text{Out}(O_2)$, say $p_\ell = (x_\ell, y_\ell) \in \text{Out}(O_2)$, $\ell \in \mathbb{N}$, are solutions of:

$$\begin{cases} x - K \omega \log(y + A + \lambda \sin x) = x + 2\ell\pi \\ (y + A + \lambda \sin x)^\delta = y. \end{cases}$$

Therefore,

$$y_\ell + A + \lambda \sin x_\ell = \exp\left(-\frac{2\ell\pi}{K \omega}\right) \quad \text{and} \quad y_\ell = \exp\left(-\frac{2\ell\pi \delta}{K \omega}\right),$$

which implies that

$$A + \lambda \sin x_\ell = \exp\left(-\frac{2\ell\pi}{K \omega}\right) - \exp\left(-\frac{2\ell\delta\pi}{K \omega}\right).$$

For $\ell \in \mathbb{N}$, define the real-valued map

$$G_\ell(\omega) = \exp\left(-\frac{2\ell\pi}{K \omega}\right) - \exp\left(-\frac{2\ell\delta\pi}{K \omega}\right), \quad \omega \in \mathbb{R}^+.$$

whose graph is depicted in Fig. 7, for different values of $\ell \in \mathbb{N}$. Observe that:

$$A + \lambda \sin x_\ell = G_\ell(\omega).$$

The next result summarises some basic properties of $G_\ell$.

Lemma 5.1 For $\ell, \ell_1, \ell_2 \in \mathbb{N}$, the following assertions are true:

1. The map $G_\ell$ has a global maximum $M = \delta^{\frac{1}{\delta-1}} - \delta^{\frac{1}{\delta-1}}$ at $\omega^*_\ell = \frac{2\ell\pi(\delta-1)}{K \ln \delta} > 0$. 

$\square$ Springer
Fig. 7  The map $G_\ell$ has a global maximum $M = \delta^{1-\delta} - \delta^{1+\delta}$ at
$\omega^*_\ell = \frac{2\ell\pi(\delta - 1)}{K \ln \delta} > 0$

(2) The maximum $M$ of $G_\ell$ is independent of $\ell$.
(3) $\lim_{\omega \to 0^+} G_\ell(\omega) = \lim_{\omega \to +\infty} G_\ell(\omega) = 0$.
(4) If $\ell_1 < \ell_2$, then $\omega^*_{\ell_1} < \omega^*_{\ell_2}$.

**Proof**  Differentiating $G_\ell$ with respect to $\omega$ and multiplying by $K > 0$, we get:

$$K G'_\ell(\omega) = \frac{2\ell\pi}{\omega^2} \exp\left(-\frac{2\ell\pi}{K \omega}\right) - \frac{2\ell\delta\pi}{\omega^2} \exp\left(-\frac{2\ell\delta\pi}{K \omega}\right)$$

$$= \frac{2\ell\pi}{\omega^2} \left[ \exp\left(-\frac{2\ell\pi}{K \omega}\right) - \delta \exp\left(-\frac{2\ell\delta\pi}{K \omega}\right) \right]$$

$$= \frac{2\ell\pi}{\omega^2} \exp\left(-\frac{2\ell\pi}{K \omega}\right) \left[ 1 - \delta \exp\left(-\frac{2\ell(\delta - 1)\pi}{K \omega}\right) \right].$$

Since $\frac{2\ell\pi}{\omega^2} \exp\left(-\frac{2\ell\pi}{K \omega}\right) > 0$ for all $\omega \in \mathbb{R}^+$ and $\ell \in \mathbb{N}$, we may conclude that:

$$G'_\ell(\omega) = 0 \iff 1 - \delta \exp\left(-\frac{2\ell(\delta - 1)\pi}{K \omega}\right) = 0$$

$$\iff \exp\left(-\frac{2\ell(\delta - 1)\pi}{K \omega}\right) = 1/\delta$$

$$\iff \frac{2\ell(\delta - 1)\pi}{K \omega} = \ln \delta$$

$$\iff \omega = \frac{2\ell\pi(\delta - 1)}{K \ln \delta} =: \omega^*_\ell.$$  

As suggested in Fig. 7, it is easy to check that $G'_\ell(\omega) > 0$ if $\omega \in [0, \omega^*_\ell]$ and $G'_\ell(\omega) < 0$ otherwise. This implies that $G_\ell$ is increasing in $[0, \omega^*_\ell$ [ and decreasing in $] \omega^*_\ell, \infty[$. Furthermore,

$$G_\ell(\omega^*_\ell) = \exp\left(-\frac{2\ell\pi \ln \delta}{2\ell\pi(\delta - 1)}\right) - \exp\left(-\frac{2\ell\delta\pi \ln \delta}{2\ell(\delta - 1)\pi}\right)$$

$$= \exp\left(-\frac{\ln \delta}{1 - \delta}\right) - \exp\left(-\frac{\delta \ln \delta}{1 - \delta}\right)$$

$$= \delta^{1-\delta} - \delta^{1+\delta} =: M.$$
The following two limits are zero as a result of the analytic expression of $G_\ell$:

$$\lim_{\omega \to 0^+} \left[ \exp \left( \frac{2\ell \pi}{K \omega} \right) - \exp \left( \frac{2\ell \delta \pi}{K \omega} \right) \right] = 0 = \lim_{\omega \to +\infty} \left[ \exp \left( \frac{2\ell \pi}{K \omega} \right) - \exp \left( \frac{2\ell \delta \pi}{K \omega} \right) \right].$$

The last assertion follows straightforwardly from the expression of $\omega^*_\ell$. \(\square\)

In order to determine the Lyapunov stability of the fixed points of $\mathcal{F}_\mu$, we compute the derivative of $\mathcal{F}_\mu$, at a general $(1, \ell)$–fixed point $(x_\ell, y_\ell)$.

$$D\mathcal{F}_\mu(x_\ell, y_\ell) = \begin{pmatrix} 1 - \frac{K \omega \lambda \cos x_\ell}{y_\ell + A + \lambda \sin x_\ell} & -\frac{K \omega}{y_\ell + A + \lambda \sin x_\ell} \\ \lambda \delta (y_\ell + A + \lambda \sin x_\ell)^{\delta-1} \cos x_\ell & \delta (y_\ell + A + \lambda \sin x_\ell)^{\delta-1} \end{pmatrix}.$$ \((5.5)\)

In order to find the $(1, \ell)$–fixed points of $\mathcal{F}_\mu$, we need to solve the equation:

$$\varphi(x) = G_\ell(\omega) \quad \text{where} \quad \varphi(x) = A + \lambda \sin x.$$ \((5.6)\)

For $A > \lambda > 0$ fixed and $\omega \in \mathbb{R}^+$, the graph of the left hand side of \((5.4)\), say $\varphi(x)$, depends on $x \in [0, 2\pi]$ and does not depend on $\omega$. On the other hand, the graph of the right hand side of \((5.4)\) does not depend on $x$.

As illustrated in Figs. 8 and 9, finding $(1, \ell)$–fixed points of $\mathcal{F}_\mu$ amounts to intersect the graph of $\varphi(x)$ with a horizontal line. The line moves first up and then down, as $\omega$ increases. Since the range of $\varphi$ is the interval $[A - \lambda, A + \lambda]$, and the range of $G_\ell(\omega)$ is the interval $(0, M)$, the geometry of the solution set depends on the relative positions of these intervals. From now on, we use the inequality $M \geq A + \lambda$, $A \in [0, \epsilon]$—see (3.1). The analytic treatment of the other cases are similar to the approach of Sect. 5 of [37].

As $\omega$ increases from 0, there is a threshold value $\omega_1$ for which the horizontal line at height $G_\ell(\omega_1)$ touches the graph of $\varphi$ at $x = 3\pi/2$. At this point we have $\sin(x) = -1$. As $\omega$ increases further, each tangency unfolds as two intersection points of the graph with the horizontal line. There is a saddle-node at the points $(x^{(1)}_\ell, G_\ell(\omega_1)) = (3\pi/2, G_\ell(\omega_1))$, as we will see in Proposition 5.2. The surface $G_\ell(\omega) = A + \lambda$ defines the boundaries of the $(1, \ell)$–resonance wedge. The horizontal line may move further up and a pair of solutions come together at a second saddle-node at $(x^{(2)}_\ell, G_\ell(\omega_2)) = (\pi/2, G_\ell(\omega_2))$ and reappear at a saddle-node at $(x^{(3)}_\ell, G_\ell(\omega_3)) = (\pi/2, G_\ell(\omega_3))$ coming together finally at $(x^{(4)}_\ell, G_\ell(\omega_4)) = (3\pi/2, G_\ell(\omega_4))$. The evolution of the geometry of solutions of \((5.4)\), as $\omega$ varies, is illustrated on the right side of Fig. 9. We show below that, at these points, the map $D\mathcal{F}_\mu$ has an eigenvalue equal to 1.
Based on (5.1), for each $\ell \in \mathbb{N}$, define the map

$$y(\omega) = \exp\left(\frac{-2\ell\pi \delta}{K \omega}\right) \quad \text{with} \quad \omega \in \mathbb{R}^+.$$  \hfill (5.7)

### 5.2 Double Eigenvalue 1

A discrete-time Bogdanov–Takens bifurcation occurs when the maximum of $G_\ell$ coincides with either the minimum or the maximum of $\varphi$. In this subsection, we check the necessary linear conditions for this bifurcation.

**Proposition 5.2** For $G_\ell(\omega_*^\ell) = A \pm \lambda$, the derivative $D\mathcal{F}_\mu(x^{(N)}, y(\omega_*^N))$ at a solution of (5.4) has 1 as a double eigenvalue and is not the identity, for $N = 1, \ldots, 4$.

**Proof** Computing the derivative $D\mathcal{F}_\mu$ at the points $(x^{(N)}, y(\omega_N))$, $N = 1, \ldots, 4$, where $\sin(x_N) = \pm 1$, we get:

$$D\mathcal{F}_\mu\left(x^{(N)}, y(\omega_N)\right) = \begin{pmatrix} 1 & -\frac{K\omega}{y(\omega_N) + A \pm \lambda} \\ 0 & \delta (y(\omega_N) + A \pm \lambda)^{\delta - 1} \end{pmatrix}.$$  

At $(x^{(N)}, y(\omega_N))$ the Jacobian matrix is triangular and so the two eigenvalues are

$$\Delta_1 = 1 \quad \text{and} \quad \Delta_2 = \delta(y(\omega_N) + A \pm \lambda)^{\delta - 1} > 0.$$  

Since $\omega_*^\ell$ was defined to be the value of $\omega$ where the function $G_\ell$ defined in (5.3) has a global maximum, then $\frac{dG_\ell}{dt}(\omega_*^\ell) = 0$. In particular,

$$\Delta_2 = \delta(y(\omega_*^\ell) + A \pm \lambda)^{\delta - 1}$$

$$= \delta \left( \exp\left(\frac{-2\ell\pi}{K \omega_*^\ell}\right) \right)^{\delta - 1}$$

$$= \delta \left( \exp\left(\frac{-2\ell\pi (\delta - 1)}{K \omega_*^\ell}\right) \right)$$

$$= \delta \left( \exp\left(\frac{-2\ell\pi (\delta - 1) \ln \delta}{2\ell\pi (\delta - 1)}\right) \right).$$
\[ = \delta \exp(\ln \delta^{-1}) = 1. \]  \hfill (5.8)

Hence the derivative \( D\mathcal{F}_\mu \), at the solutions of (5.4) with \( G_\ell(\omega^*_\ell) = A \pm \lambda \), has a double eigenvalue equal to 1, and is not the identity as we may confirm in (5.5). \( \square \)

### 5.3 Proof of Theorem A

Proposition 5.2 indicates a (possible) bifurcation of codimension 2, corresponding to a curve in the 3-dimensional parameter space \( \mu = (A, \lambda, \omega) \), where we expect to find a discrete-time Bogdanov–Takens bifurcation. This bifurcation occurs at points where 1 is a double eigenvalue, the derivative is not the identity and the map \( \mathcal{F}_\mu \) satisfies a finite number of non-degeneracy conditions. In this section, we check these nonlinear conditions. We recall the main ideas of [16,56] adapted to our purposes.

Along the surface defined by \( \pm \lambda = A - G_\ell(\omega^*_\ell) \), the map \( \mathcal{F}_\mu \) has a fixed point \( p_\ell = (x^{(N)}, y(\omega^*_\ell)), N = 1, \ldots, 4 \) and \( \ell \in \mathbb{N} \), such that \( D\mathcal{F}_\mu(p_\ell) \) has a double unit eigenvalue but is not the identity.

For \( \lambda = \lambda_0 > 0 \) fixed, by composing the translation \( (x_\ell, y_\ell) \mapsto (0, 0) \) with the following (local) change of coordinates:

\[
(x, y) \iff (x, C y) \quad \text{where} \quad C = \frac{-2K\ell \pi (\delta - 1)}{\delta \frac{1}{\delta} \ln \delta} < 0,
\]

for \( \nu = (A, \lambda_0, \omega) \) near \( (G_\ell(\omega^*_\ell) \mp \lambda_0, \lambda_0, \omega^*_\ell) \in \mathcal{V} \), the map \( D\mathcal{F}_\nu(0, 0) \) has the form:

\[
D\mathcal{F}_\nu(x, y) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a(x, y; \nu) \\ b(x, y; \nu) \end{pmatrix} + \mathcal{O}(\|x, y\|^3), \]  \hfill (5.9)

where the polynomial expansions of order 2 of \( a \) and \( b \) may be written as:

\[
a(x, y; \nu) = a_{00}(\nu) + a_{10}(\nu)x + a_{01}(\nu)y + \frac{1}{2} a_{20}(\nu)x^2 + a_{11}(\nu)xy + \frac{1}{2} a_{02}(\nu)y^2
\]

and

\[
b(x, y; \nu) = b_{00}(\nu) + b_{10}(\nu)x + b_{01}(\nu)y + \frac{1}{2} b_{20}(\nu)x^2 + b_{11}(\nu)xy + \frac{1}{2} b_{02}(\nu)y^2
\]

with

\[
a_{00}(0, 0) = a_{10}(0, 0) = a_{01}(0, 0) = b_{00}(0, 0) = b_{10}(0, 0) = b_{01}(0, 0) = 0.
\]

The leading coefficients of \( a \) and \( b \) that will be used in the sequel are listed in Table 1. We concentrate our attention on the fixed point associated to \( x = 3\pi/2 \); the other is similar.

By Proposition 3.1 of Yagasaki [56], since

\[
b_{20}(\nu) = (A + \lambda)^{\delta-2} \lambda \delta (1 - \delta) < 0 \quad \text{for} \quad \delta > 1,
\]

and

\[
a_{20}(\nu) + b_{11}(\nu) - b_{20}(\nu) = -\frac{C K \omega \lambda}{(A + \lambda)^2} = a_{20} > 0,
\]

then there exists a bifurcation point of codimension 2 at \( (x_N, y(\omega^*_\ell)) \) with \( N = 1, \ldots, 4 \) and \( \ell \in \mathbb{N} \) such that, nearby (see Fig. 10):

1. there exist two curves associated to saddle-node bifurcation (\( \text{sn}_1 \) and \( \text{sn}_2 \));
Fig. 10 Dynamics for the discrete-time Bogdanov–Takens bifurcation in \((A, \omega)\), after a smooth change of coordinates. \(\text{sn}_1\) and \(\text{sn}_2\): Saddle-node bifurcations; \(\text{hopf}\): Hopf bifurcation; \(h_1\) and \(h_2\): homoclinic tangencies associated to a dissipative fixed point. 

| Coefficient | Condition | Condition |
|-------------|-----------|-----------|
| \(a_{20}(v)\) | \(\frac{CK\omega\lambda}{(A-\lambda)^2} < 0\) | \(-\frac{CK\omega\lambda}{(A+\lambda)^2} > 0\) |
| \(b_{11}(v)\) | \(-(A-\lambda)^2 - 2\lambda \delta (1-\delta) > 0\) | \((A+\lambda)^2 - 2\lambda \delta (1-\delta) < 0\) |
| \(b_{20}(v)\) | \(-(A-\lambda)^2 - 2\lambda \delta (1-\delta) > 0\) | \((A+\lambda)^2 - 2\lambda \delta (1-\delta) < 0\) |

(2) there exists one curve associated to a Hopf bifurcation at the stable focus born at the saddle-node bifurcation (\(\text{hopf}\));

(3) there exists a region with a Lyapunov stable invariant circle created at the Hopf bifurcation, since

\[
b_{20}(a_{20} + b_{11} - b_{20}) < 0
\]

(5.10)

(all coefficients are computed at \((G(\omega^*_2) \mp \lambda_0, \lambda_0, \omega^*_2)\));

(4) there exist two curves, \(h_1\) and \(h_2\), associated to a homoclinic bifurcation where the stable and unstable manifolds of the saddle point born at (1) touch tangentially. The distance between the two homoclinic bifurcation curves is exponentially small with respect to \(\sqrt{\|v - (G(\omega^*_2) \mp \lambda_0, \omega^*_2)\|}\).
the invariant manifolds of a dissipative saddle intersect transversely inside the parameter region between the curves \( h_1 \) and \( h_2 \) and do not intersect outside it.

**Remark 5.3** In Proposition 3.1 of [56], there exists an extra condition: \( \det D_\mu v(0) \neq 0. \) This inequality serves to describe (in some system of coordinates) the explicit expression for the bifurcating curves, which is used to conclude that the hopf bifurcation curve is tangent to the saddle-node bifurcation curves \( sn_1 \) and \( sn_2 \), at the bifurcation point.

For \( \ell \in \mathbb{N} \), denote by \( BT_1^\ell \) and \( BT_2^\ell \) the two discrete-time Bogdanov–Takens bifurcation in the parameter space \((A, \lambda)\) such that \( A > \lambda = \lambda_0 \) and \( \omega = \omega_\ell^* \):

\[
BT_1^\ell \mapsto A = G_\ell(\omega_\ell^*) - \lambda, \quad BT_2^\ell \mapsto A = G_\ell(\omega_\ell^*) + \lambda. \tag{5.11}
\]

### 5.4 Proof of Corollary B

This is a direct corollary of Theorem A. For \( \lambda = \lambda_0 > 0 \) and \( \ell \in \mathbb{N} \) fixed, there exist two points of Bogdanov–Takens bifurcation for the map \( F_\mu \) at \( p_\ell \): \( BT_1^\ell \) and \( BT_2^\ell \) (see (5.11)). As depicted in Fig. 12, varying smoothly \( \lambda \gtrsim 0 \) around each Bogdanov–Takens bifurcation:

(a) there exist two surfaces of saddle-node bifurcations \( SN_1^\ell \) and \( SN_2^\ell \);
(b) there exists a surface of Hopf bifurcations \( Hopf_\ell \);
(c) there exist two surfaces of homoclinic tangencies \( H_1^\ell \) and \( H_2^\ell \).

The two surfaces (c) correspond to bifurcations at which the stable and unstable manifolds of a dissipative saddle point are tangent. In the region between these surfaces there is a transverse intersection of the stable and the unstable manifolds of a saddle. This configuration implies that the dynamics of \( F_\mu \) is equivalent to Smale’s horseshoe. Last assertion of Corollary B follows from Remark 5.3.

**Remark 5.4** We cannot exclude the possibility that the two surfaces \( H_1^\ell \) and \( H_2^\ell \), \( \ell \in \mathbb{N} \), coincide, although it would be a highly non-generic behaviour.

### 5.5 Proof of Corollary C

The existence of \( H_1^\ell \) and \( H_2^\ell \), \( \ell \in \mathbb{N} \), shows that there are surfaces in the space of parameters for which the map \( F_\mu \) has a quadratic (generic) homoclinic tangency associated to a dissipative periodic point. Using [39], there exists a positive measure set \( \Delta \) of parameter values, so that for every \( \mu \in \Delta \subset \mathcal{V} \), \( F_\mu \) admits a strange attractor of Hénon-type with an ergodic SRB measure. The existence of historic behaviour is a combination of the latter tangencies and Theorem A of Kiriki and Soma [34].

### 5.6 Proof of Corollary D

The first part of the corollary is a straightforward consequence of the Hopf bifurcation surface of Corollary B, from which a stable torus emerge (see (5.10)). The proof for the second part is a simple inspection of Theorem B of [49] (see also references therein). For the sake of completeness, we list the main steps of the proof:

1. for each \( \lambda = \lambda_0 > 0 \), we write explicitly the normal form for the family of Hopf bifurcation, which creates an attracting invariant circle;
(2) perturb (if necessary) the truncated normal form in order to obtain an irrational rotation on the circle, say $H_1$;
(3) perturb $H_1$, using Denjoy procedure [22], in order to obtain contracting wandering domains. The resulting map is $C^1$-close to $\mathcal{F}_\mu$, $\mu \in \mathcal{V} \cap \text{Hopf}_\ell$, $\ell \in \mathbb{N}$.

6 Dissecting a Resonance Wedge: Putting All Together

Theorem A may be seen as a “local” theorem. A new problem arises: how the surfaces of bifurcations of Corollary B are globally organised? This section provides a partial answer to this question. We plot the graphs of the maps that defines the Hopf, the transitions node $\leftrightarrow$ focus and the period-doubling bifurcation, as function of the parameters $(A, \lambda, \omega)$, where $A, \lambda$ are not necessarily small. These bifurcations arise in a form consistent with Corollary B and with the bifurcation diagram of Fig. 20 of [9].

6.1 Necessary Conditions for Bifurcations

To simplify the notation, denote by $\det$ and $\text{trace}$ the determinant and the trace of $D\mathcal{F}_\mu(x, y)$ (see (5.5)), respectively.

\[
\det := \det D\mathcal{F}_\mu(x, y) = \delta(y + A + \lambda \sin x)^{\delta-1}
\]

\[
\text{trace} := \text{trace} D\mathcal{F}_\mu(x, y) = 1 - \frac{K\omega\lambda \cos x}{y + A + \lambda \sin x} + \delta(y + A + \lambda \sin x)^{\delta-1}.
\]

Note that for a $2 \times 2$–real matrix $D\mathcal{F}_\mu(x, y)$, its eigenvalues are the roots of the polynomial in $t$ given by

\[
P(t) = t^2 - \text{trace} t + \det, \quad \text{say} \quad t = \frac{\text{trace} \pm \sqrt{\text{trace}^2 - 4\det}}{2}.
\]

At a $(1, \ell)$–fixed point of $\mathcal{F}_\mu$, we know that $G_\ell(\omega) = \varphi(x)$ – see (5.4). This means that

\[
\exp\left(-\frac{2\ell\pi}{K\omega}\right) - \exp\left(-\frac{2\ell\delta\pi}{K\omega}\right) = A + \lambda \sin x
\]

$\Leftrightarrow G_\ell(\omega) - A = \lambda \sin x$.

Since $\lambda \cos x = \pm\sqrt{\lambda^2 - \lambda^2 \sin x} = \pm\sqrt{\lambda^2 - (G_\ell(\omega) - A)^2}$, we may write:

\[
\text{trace} = 1 - \frac{K\omega\lambda \cos x}{y + A + \lambda \sin x} + \delta(y + A + \lambda \sin x)^{\delta-1}
\]

\[
= 1 \mp \frac{K\omega\sqrt{\lambda^2 - (G_\ell(\omega) - A)^2}}{\exp\left(-\frac{2\ell\pi}{K\omega}\right)} + \delta \exp\left(-\frac{2\ell(\delta - 1)\pi}{K\omega}\right)
\]

\[
\det = \delta(y + A + \lambda \sin x)^{\delta-1} \overset{\text{(5.1)}}{=} \delta \exp\left(-\frac{2\ell(\delta - 1)\pi}{K\omega}\right).
\]

6.1.1 Saddle-Node Bifurcation (SN)

One root of $P(t)$ is 1. Using the explicit expressions for $\text{trace}$ and $\det$, we get
Fig. 11  Graphs of $\text{trace}^2 - 4 \det = 0$ (NF and FN), $G_\ell(\omega) = A \pm \lambda$ (SN) and one of the roots of $P(t)$ equals to $-1$ (PD), with $\ell = K = 1$, $\delta = 1.3$, numerically plotted using Maple, $A \in [0; 0.5]$, $\lambda \in [0; 0.1]$, $\omega \in [1; 3.9]$. (a) and (b) are different perspectives of the intersection of the previous surfaces with the plane defined by $\omega = 3.9$

1 - $\text{trace} + \det = 0$
\[ \iff \frac{K \omega \lambda \cos x}{y + A + \lambda \sin x} - \delta (y + A + \lambda \sin x)^{\delta-1} + \delta (y + A + \lambda \sin x)^{\delta-1} = 0 \]
\[ \iff \cos x = 0 \iff \sin x = \pm 1 \quad (\text{see Fig. 8}) \quad G_\ell(\omega) = A \pm \lambda. \]

These bifurcations have been represented numerically in Figs. 11 and 12. They correspond to the borders of of the resonance wedge $T_\ell$.

6.1.2 Period-Doubling Bifurcation (PD)

One root of $P(t)$ is $-1$. Therefore, the characteristic equation reads as:
\[ 1 + \text{trace} + \det = 0 \]
\[ \iff 2 - \frac{K \omega \lambda \cos x}{y + A + \lambda \sin x} + 2\delta (y + A + \lambda \sin x)^{\delta-1} = 0 \]
\[ \iff 2 - K \omega \lambda \cos x \exp \left( \frac{2\ell \pi}{K \omega} \right) + 2\delta \exp \left( \frac{-2\ell (\delta - 1)\pi}{K \omega} \right) = 0 \]
\[ \iff \lambda \cos x = \frac{2 + 2\delta \exp \left( \frac{-2\ell (\delta - 1)\pi}{K \omega} \right)}{K \omega} \exp \left( \frac{-2\ell \pi}{K \omega} \right). \]

At the left-hand side stands a small parameter multiplied by a bounded function. At the right-hand side stands a finite value bounded from zero. These assertions mean that the bifurcation PD is not observed for $\lambda$ small. The intersection of the bifurcation surface PD with $\omega = 3.9$ has been represented numerically in Fig. 11 and schematically at the right-hand side of Fig. 13.

6.1.3 Andronov-Hopf Bifurcation (Hopf$_\ell$)

Up to nonlinear conditions, Andronov-Hopf bifurcation (for a map) occurs when the norm of the complex conjugate eigenvalues cross the unit circle. This happens when $\det = 1$ and $\text{trace} \in [-2, 2]$. Indeed,
Fig. 12 Bifurcation diagram in the plane \((\omega, A)\), for \(\lambda = \lambda_0 > 0\), \(\ell = K = 1\) and \(\delta = 1.3\). a numerical scheme using Maple with \(\ell = K = 1\), \(\delta = 1.3\), \(A \in [0; 0.5]\), \(\lambda = 0.1\), \(\omega \in [2; 5]\). b theoretical scheme. The gray plane corresponds to \(\lambda = 0.1\). **Bifurcations:** BT: Bogdanov–Takens, SN: saddle-node, FN/NF: transitions focus ↔ node, Hopf: Hopf. The right-hand side of this figure is distorted and nonlinearly scaled to enable some of the regions to be distinguished.

![Bifurcation Diagram](image)

Fig. 13 Schematic representation of the bifurcations of fixed points of \((5.4)\), giving rise to a **resonance wedge** (left) and an **Arnold tongue** (right) for \(A > \lambda \gtrsim 0\). Bifurcations: BT: Bogdanov–Takens, SN: saddle-node, H: homoclinic tangencies, FN/NF: transitions focus ↔ node, Hopf: Hopf, PD: period-doubling. This figure is distorted and nonlinearly scaled to enable some of the regions to be distinguished. More details in Fig. 14.

\[
\det = 1 \\
\iff \delta \exp\left(\frac{-2\ell(\delta - 1)\pi}{K \omega}\right) = 1 \\
\iff \omega = \frac{2\ell\pi(\delta - 1)}{K \ln \delta} \\
\iff \omega = \omega^*_\ell,
\]

which implies that the surface \(\text{Hopf}_\ell\) should lie in the plane defined by \(\omega = \omega^*_\ell\), \(\ell \in \mathbb{N}\) (see the scheme on the right side of Fig. 12).

### 6.1.4 Transitions Node ↔ Focus (NF/FN)

The characteristic equation associated to this transition is \(\text{trace}^2 - 4 \det = 0\). These bifurcations have been represented numerically in Figs. 11, 12 and 13.
Remark 6.1 The theory of the previous section has been performed for $(1, \ell)$–fixed points of $F_\mu$. However, all these phenomena can also be observed for wedges associated to other rotation number; of course, the analytic expressions for the bifurcation surfaces are different.

6.2 Summarizing Movie

In this subsection, we give a heuristic discussion of what is going on within an Arnold wedge. We compare our results with those found in previous works by other authors; all results agree well with the theoretical information of [41,53].

For $\varepsilon > 0$ small, the choice of parameters in Sect. 2 allows us to build the bifurcation diagram of Fig. 13 in $V$ (see (3.1)). Dynamical bifurcation surfaces in resonance wedge may be projected into a generic plane, giving rise to what the literature calls an Arnold tongue. This projection is sketched on the right-hand side of Figs. 13 and 14. All resonance wedges have the origin as a common point.

For $A > \lambda \geq 0$ fixed, if $\omega$ is sufficiently small, the flow of (2.1) exhibits a 2-dimensional non-contractible torus which is globally attracting and normally hyperbolic. The dynamics of $F_\mu$ is governed by the dynamics of a circle map. There is a positive measure set $\Delta \subset V$ so that the rotation number of $F_\mu$ is irrational if and only if $\mu \in \Delta$ [27,28].

Within $V$, for each $\ell \in \mathbb{N}$, we may define a resonance wedge, denoted by $T_\ell$, limited by the surfaces $SN : A = G_\ell(\omega) \pm \lambda$ adjoining the graph of

$$A = G_\ell(\omega), \quad \lambda = 0.$$ 

Parameters within this wedge correspond to first return maps with at least a pair of fixed points: one of the fixed points is a saddle (say $Q_\ell$); the other point is a sink (say $P_\ell$). As
suggested in Fig. 14, we suppose the existence of just one pair of fixed points for the following analysis, both with the same rotation number.

The borders of $T_\ell$ are the bifurcation surfaces $SN := SN^1_\ell \cup SN^2_\ell$ at which the fixed points $Q_\ell$ and $P_\ell$ merge to a saddle-node. These surfaces might touch the corresponding surfaces of other wedge, meaning that there are parameter values for which periodic points of periods $m$ and $\ell$ coexist, $\ell, m \in \mathbb{N}$. On the one hand, the surface Hopf connects both Hopf surfaces that appear near $BT^1_\ell$ and $BT^2_\ell$ given by Corollary B, from where a stable 2-dimensional torus emerges. On the other hand, it connects the surface defined by $FN$ (black dots in Fig. 12) at $\omega = \omega_\star^\ell$, as predicted in § 6.1.3, where the focus becomes a node.

The latter torus is contractible because it does not envelope the cylinder $Out(O_2)$; new tori might coexist (for different values of $\ell \in \mathbb{N}$) and are not diffeomorphic to the original torus that exists for $\lambda = 0$ and $A > 0$.

In the bifurcation plane $(A, \lambda)$ of Fig. 14, the Hopf surface is above the set $NF$ where the eigenvalues of the sink $P_\ell$ become complex. At the period-doubling bifurcation surface PD, one multiplier becomes equal to $-1$. At this stage, the curve resulting from the intersection of the torus’ ghost with a global cross section, is no longer homeomorphic to a circle.

Furthermore, the torus is no longer smooth as the unstable manifold of the saddle $Q_\ell$ winds around the focus infinitely many times (see Fig. 14). Along the surface $FN$, the eigenvalues of $P_\ell$ become real again. These bifurcations have been discussed in [41,55] where the authors relate the dynamics of an Arnold tongue with maps on the circle.

Along the bifurcation surfaces $H^1_\ell$ and $H^2_\ell$ described by Corollary B, one observes a homoclinic contact of the components $W^s(Q_\ell)$ and $W^u(Q_\ell)$, where $Q_\ell$ is a dissipative saddle for $\mathcal{F}_\mu$. There are small regions (in terms of measure) inside the resonance wedges where chaotic trajectories are observable: they correspond to strange attractors of Hénon type [50] and are associated to the historic behaviour of Corollary C. Other stable points of large period exist in the region above the surfaces $H^1_\ell$ and $H^2_\ell$, as a consequence of Newhouse phenomena [40]. Numerics in [17] also suggest the existence of bistability for open regions of the parameter space: coexistence of a stable periodic solution and an attracting torus.

### 7 An Example

Our study was initially motivated by the following example introduced in [4] and explored in [21]. Some preliminaries about symmetries of a vector field may be found in Appendix A.1. For $\tau_1, \tau_2 \in [0, 1]$ and $\omega \in \mathbb{R}^+$, our object of study is the two-parameter family of vector fields on $\mathbb{R}^4$

$$x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mapsto f(\tau_1, \tau_2, \omega)(x)$$

defined for each $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ by

$$\begin{cases}
\dot{x}_1 = x_1(1 - r^2) - \omega x_2 - \alpha x_1 x_4 + \beta x_1 x_2^2 + \tau_2 x_1 x_3 x_4 \\
\dot{x}_2 = x_2(1 - r^2) + \omega x_1 - \alpha x_2 x_4 + \beta x_2 x_3^2 \\
\dot{x}_3 = x_3(1 - r^2) + \alpha x_3 x_4 + \beta x_3 x_2^2 + \tau_1 x_3^4 - \tau_2 x_3^2 x_4 \\
\dot{x}_4 = x_4(1 - r^2) - \alpha(x_3^2 - x_1^2 - x_2^2) - \beta x_4(x_1^2 + x_2^2 + x_3^2) - \tau_1 x_3 x_4^2
\end{cases} \quad (7.1)$$

where $\dot{x}_i = \frac{dx_i}{d\tau}$, $r^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$, and

$$\omega > 0, \quad \beta < 0 < \alpha, \quad \beta^2 < 8\alpha^2 \quad \text{and} \quad |\beta| < |\alpha|.$$
The vector field $f_{(0,0)}$ is equivariant under the action of the compact Lie group $\mathbb{S} \mathbb{O}(2)(\gamma \psi) \oplus \mathbb{Z}_2(\gamma_2)$, where $\mathbb{S} \mathbb{O}(2)(\gamma \psi)$ and $\mathbb{Z}_2(\gamma_2)$ act on $\mathbb{R}^4$ as

$$\gamma \psi (x_1, x_2, x_3, x_4) = (x_1 \cos \psi - x_2 \sin \psi, x_1 \sin \psi + x_2 \cos \psi, x_3, x_4), \quad \psi \in [0, 2\pi]$$
given by a phase shift $\theta \mapsto \theta + \psi$ in the first two coordinates, and

$$\gamma_2 (x_1, x_2, x_3, x_4) = (x_1, x_2, -x_3, x_4).$$

By construction, $\tau_1$ is the controlling parameter of the $\mathbb{Z}_2(\gamma_2)$–symmetry breaking and $\tau_2$ controls the $\mathbb{S} \mathbb{O}(2)(\gamma \psi)$–symmetry breaking but keeping the $\mathbb{S} \mathbb{O}(2)(\gamma_\pi)$–symmetry, where

$$\gamma_\pi (x_1, x_2, x_3, x_4) = (-x_1, -x_2, x_3, x_4).$$

When restricted to the sphere $S^3$, for every $\tau_1, \tau_2 \in [0, 1]$, the flow of $f_{(\tau_1, \tau_2)}$ has two equilibria

$$O_1 = (0, 0, 0, +1) \quad \text{and} \quad O_2 = (0, 0, 0, -1),$$

which are hyperbolic saddle-foci. The linearization of $f_{(0,0)}$ at $O_1$ and $O_2$ has eigenvalues

$$-(\alpha - \beta) \pm \omega i, \quad \alpha + \beta \quad \text{and} \quad (\alpha + \beta) \pm \omega i, \quad -(\alpha - \beta)$$

respectively. The 1-dimensional heteroclinic connections are given by:

$$W^u(O_1) \cap S^3 = W^s(O_2) \cap S^3 = \text{Fix}(\mathbb{S} \mathbb{O}(2)(\gamma \psi)) \cap S^3 = \{(x_1, x_2, x_3, x_4) : x_1 = x_2 = 0, x_3^2 + x_4^2 = 1\}$$

and the 2-dimensional connection is contained in

$$W^u(O_2) \cap S^3 = W^s(O_1) \cap S^3 = \text{Fix}(\mathbb{Z}_2(\gamma_2)) \cap S^3 = \{(x_1, x_2, x_3, x_4) : x_1^2 + x_2^2 + x_4^2 = 1, x_3 = 0\}.$$

The two-dimensional invariant manifolds of $O_1$ and $O_2$ are contained in the two-sphere $\text{Fix}(\mathbb{Z}_2(\gamma_2)) \cap S^3$. It is precisely the symmetry $\mathbb{Z}_2(\gamma_2)$ that forces the two-invariant manifolds $W^u(O_2)$ and $W^s(O_1)$ to coincide. We denote by $\Gamma$ the heteroclinic network formed by the two equilibria, the two connections [$O_1 \rightarrow O_2$] and the sphere [$O_2 \rightarrow O_1$]. Keeping $\tau_1 = \tau_2 = 0$, the equilibria $O_1$ and $O_2$ have the same chirality. Therefore:

**Lemma 7.1** If $\tau_1 = \tau_2 = 0$, the flow of (7.1) satisfies (P1)–(P6) described in Sect. 2.1.

As a consequence, for $\tau_1 = \tau_2 = 0$, the flow of (7.1) exhibits an asymptotically stable heteroclinic network $\Gamma$ associated to $O_1$ and $O_2$. The parameters $\tau_1$ and $\tau_2$ play the role of $A$ and $\lambda$, respectively, of (P7)–(P8), after possible rescaling.

**Corollary 7.2** [21] For $\tau_1 > 0$ and $\tau_2 = 0$, close to the “ghost” of the attracting network $\Gamma$, the flow of (7.1) has an attracting invariant two-torus, which is normally hyperbolic.

When $\tau_1 \gg \tau_2 > 0$, although we break the $\mathbb{S} \mathbb{O}(2)(\gamma \psi)$–equivariance, the $\mathbb{Z}_2(\gamma_\pi)$–symmetry is preserved. This is why the connections lying in $x_1 = x_2 = 0$ persist.

**Lemma 7.3** [21] For $\tau_1, \tau_2 > 0$ small enough such that $\tau_1 \gg \tau_2$, the flow of (7.1) satisfies (P7)–(P8b).
Fig. 15  Number of non-negative Lyapunov exponents along the orbit with initial condition $(0.1; 0.1; 0; -0.99)$ near $W^u(O_2)$ for equation (7.1) with $\alpha = 1$, $\beta = -0.1$ and $\omega = 1$, $t \in [0, 3750]$. Red for 0; Blue for 1; Yellow for 2. I – Homoclinic bifurcations; Hénon-like strange attractors; II – Sink; III – Resonant tongue (Arnold tongue); IV – Hopf bifurcation; V – Saddle-node bifurcation (border of the Arnold tongue); VI – Irrational torus (thin yellow region). Figure performed by L. Castro adapted from [21](Color figure online)

Numerical simulations of (7.1) for $\tau_1 \gg \tau_2 > 0$ suggest the existence of regular and chaotic behaviour in the region of transition from an attracting 2-dimensional torus to rotational horseshoes [21]. Chaotic attractors with one positive Lyapunov exponent seem to exist, as suggested by the yellow regions occurring in the upper part of the Arnold tongues in Fig. 15. The description of Sect. 6 agrees quite well with the bifurcation diagram.

Hopf surfaces found in Corollary B seem to be responsible for the lower bound of the blue “bananas” that one observes in Fig. 15. This bifurcation gives rise to a stable 2-dimensional torus (blue region) in the flow of (7.1). Numerically we lose control of this stable torus, although we guess that it persists in other location of the phase space.

Technicalities on Numerics of Fig. 15

The parameter plane $(\tau_1, \tau_2)$ of Fig. 15 is scanned with a sufficiently small step along each coordinate axes. The software evaluates at each parameter value how many Lyapunov exponents along the orbit with initial condition $(0.1; 0.1; 0; -0.99) \in W^u(O_2)$ are non-negative (considered “positive” when greater than $5 \times 10^{-4}$ to discard uncertain positive Lyapunov exponents due to numerical precision issues). The parameter is painted according to the following rules: red for 0, blue for 1, yellow for 2. To estimate the complete Lyapunov spectra, the authors of [21] used the algorithm for differential equations with a Taylor series integrator.

8 Discussion

In this article, we have shown the existence of discrete-time Bogdanov–Takens bifurcations in the bifurcation diagram associated to an unfolding of a weekly attracting heteroclinic
network with a 2-dimensional connecting manifold, a natural configuration in symmetric systems and in some unfoldings of the Hopf-zero singularity [12,38,52].

We have concentrated our attention in a family of vector fields $f_\mu \in \mathcal{X}(\mathbb{S}^3)$ satisfying $(P7)$–$(P8b)$–$(P9)$. The bifurcation diagram of each element of the family is governed by an Arnold wedge, a structure through which an Arnold tongue may be seen as a projection. This (new) heteroclinic bifurcation is different from that obtained in [7], in which an equilibrium produces a periodic solution which, in turn, generates a 2-dimensional torus.

The structure of the Arnold tongue strongly depends on $\omega$. This suggested us to extend the 2-parameter “classical” bifurcation diagram $(A, \lambda A)$ of [50] to a 3-dimensional case, where $\omega$ is the additional parameter and $\delta$ a controlling parameter. Doing that, Arnold tongues give rise to resonance wedges bounded by two surfaces that correspond to saddle-node bifurcations. The resonance wedge contains a sequence of curves corresponding to a discrete-time Bogdanov–Takens bifurcation, a possibility already anticipated in [6,31]. Parameters within the wedge corresponds to maps whose periodic orbits share the same rotation number.

The structure of a resonance wedge is consistent with the Torus-breakdown theory [1,41,44,45], an essential route to understand the nature of turbulence [51]. For $A > \lambda > 0$ fixed and $\omega > 0$ small, the flow of (2.1) exhibits an attracting torus consisting of either locked or quasiperiodic solutions. As $\omega$ increases, the attracting torus disintegrates into isolated periodic sinks and saddles. Increasing the magnitude of $\omega$ further, the phase space is stretched and folded, creating rotational horseshoes [43], homoclinic tangencies and strange attractors of Hénon-type with ergodic SRB measures. In between, Hopf bifurcations are present and play an important role to explain the existence of tori in the flow. Our description refines the diagrams proposed by [9,10,25].

For $f_\mu \in \mathcal{X}(\mathbb{S}^3)$, we may distinguish the dynamics between heteroclinic tangle and rank-one like attractors; this depends on the hypotheses $(P8a)$ and $(P8b)$. In both cases, there exists complicated dynamics in $\mathcal{U}$, but chaotic dynamics are created by two independent mechanisms. They are:

$$\lambda > A \geq 0, \; \omega \in \mathbb{R}^+ \iff (P8a): W^u(O_2) \text{ and } W^s(O_1) \text{ intersect transversely}$$

- the expansion induced by intersection of invariant manifolds
- Smale horseshoes (heteroclinic tangles).

$$A > \lambda \geq 0, \; \omega \in \mathbb{R}^+ \iff (P8b): W^u(O_2) \text{ and } W^s(O_1) \text{ do not intersect}$$

- the invariant manifolds of the saddle-foci are pulled apart
- the expansion is induced by large $\omega$

- Rotational horseshoes.

In the first case, we conjecture the existence of a non-uniform expansion for a set with positive Lebesgue measure. The difficulties to prove the conjecture are linked with the existence of infinitely many points within $W^s(O_1)$ where the first return map is not well defined. In the second scenario, the dynamics is governed by strange attractors with large basins of attraction (in terms of Lebesgue measure) [54,55]. A lot more needs to be done before these two types of chaos are well understood.

The analysis in this paper is not sensitive to the particular configuration given by the heteroclinic attractor $\Gamma$; the results are valid for more general attracting networks with 2-dimensional heteroclinic connections which unfold generically from the coincidence.

Finally, we would like to point out that all results also hold for periodically-forced differential equations, natural in the study of seasonally forced systems, where “our” parameters
A, λ, ω, δ may be interpreted as (see [25, 37]):

A → Average of the periodic-forcing;
λ → Effect (fluctuations) of the unstable manifold on a global cross section;
ω → Frequency of the forcing;
δ → Measures the strength of attraction of the cycle/network in the absence of perturbations.

By moving parameters, the invariant manifolds of invariant saddles cause destruction and fusion of attractors. The full description of these metamorphoses is deferred for future work.

Acknowledgements

The author is grateful to Isabel Labouriau for the fruitful discussions during the research work performed in [37]. Special thanks to Andrey Shilnikov for pointing out the paper [52] on bifurcations analysis of a low-order atmospheric circulation model. The author is indebted to the two reviewers for the constructive comments, corrections and suggestions which helped to improve the readability of this manuscript.

Appendix A. Glossary

We record a miscellaneous collection of terms and terminology that are used throughout the text. For ε > 0 small, consider the 3-parameter family of $C^3$–smooth autonomous differential equations

$$\dot{x} = f(A, \lambda, \omega)(x), \quad x \in S^3 \subset \mathbb{R}^4, \quad A, \lambda \in [0, \varepsilon], \quad \omega \in \mathbb{R}^+. \quad (A.1)$$

Since $S^3$ is a compact set without boundary, the local solutions of \( (A.1) \) could be extended to $\mathbb{R}$. Denote by $\varphi(A, \lambda, \omega)(t, x), \ t \in \mathbb{R}$, the associated flow.

A.1 Symmetry

Given a compact Lie group $G$ of endomorphisms of $\mathbb{R}^4$, we will consider 3-parameter families of vector fields ($f(A, \lambda, \omega)$) under the equivariance assumption

$$f(A, \lambda, \omega)(\gamma x) = \gamma f(A, \lambda, \omega)(x)$$

for all $x \in S^3$, $\gamma \in G$ and $(A, \lambda, \omega) \in [0, \varepsilon]^2 \times \mathbb{R}^+$. For an isotropy subgroup $\bar{G} < G$, we write $\text{Fix}(\bar{G})$ for the vector subspace of points that are fixed by the elements of $\bar{G}$. Note that, for $G$–equivariant differential equations, the subspace $\text{Fix}(\bar{G})$ is flow-invariant.

A.2 Attracting Set

A subset $\Omega$ of $S^3$ for which there exists a neighborhood $U \subset S^3$ satisfying $\varphi(A, \lambda, \omega)(t, U) \subset U$ for all $t \geq 0$ and

$$\bigcap_{t \in \mathbb{R}^+} \varphi(A, \lambda, \omega)(t, U) = \Omega$$

is called an attracting set by the flow of \( (A.1) \). This set is not necessarily connected. Its basin of attraction, denoted by $B(\Omega)$, is the set of points in $S^3$ whose orbits have $\omega$–limit in $\Omega$. We say that $\Omega$ is asymptotically stable (or $\Omega$ is a global attractor) if $B(\Omega) = S^3$. An attracting set is said to be quasi-stochastic if it encloses periodic solutions with different
Morse indices (dimension of the unstable manifold), structurally unstable cycles, sinks and saddle-type invariant sets (cf. [24]).

A.3 Heteroclinic Structures

Suppose that $O_1$ and $O_2$ are two hyperbolic equilibria of (A.1) with different Morse indices (dimension of the unstable manifold). There is a heteroclinic cycle associated to $O_1$ and $O_2$ if

$$W^u(O_1) \cap W^s(O_2) \neq \emptyset \quad \text{and} \quad W^u(O_2) \cap W^s(O_1) \neq \emptyset.$$  

For $i \neq j \in \{1, 2\}$, the non-empty intersection of $W^u(O_i)$ with $W^s(O_j)$ is called a heteroclinic connection between $O_i$ and $O_j$, and will be denoted by $[O_i \rightarrow O_j]$. Although heteroclinic cycles involving equilibria are not a generic property within differential equations, they may be structurally stable within families of vector fields which are equivariant under the action of a compact Lie group $G \subset O(4)$, due to the existence of flow-invariant subspaces [26].

A heteroclinic cycle between two hyperbolic saddle-foci of different Morse indices, where one of the connections is transverse while the other is structurally unstable, is called a Bykov cycle. We address the reader to [30] for an overview of heteroclinic bifurcations and substantial information on the dynamics near different types of structures.

A.4 Historic Behaviour

We say that the solution of (A.1), $\varphi(t, x)$, has historic behaviour if there is a continuous function $H : \mathbb{S}^3 \to \mathbb{R}$ such that the time average $\frac{1}{T} \int_0^T H(\varphi(t, x)) dt$ fails to converge.

A.5 Strange Attractor

A (Hénon-type) strange attractor of a two-dimensional dissipative diffeomorphism $R$ defined in a Riemannian manifold $\mathcal{M}$, is a compact invariant set $\Lambda$ with the following properties:

- $\Lambda$ equals the closure of the unstable manifold of a hyperbolic periodic point;
- the basin of attraction of $\Lambda$ contains an open set;
- there is a dense orbit in $\Lambda$ with a positive Lyapounov exponent (exponential growth of the derivative along its orbit).

A vector field possesses a strange attractor if the first return map to a cross section does.

A.6 SRB Measure

Given an attracting set $\Omega$ for a continuous map $R : \mathcal{M} \to \mathcal{M}$ where $\mathcal{M}$ is a compact smooth manifold, consider the Birkhoff average with respect to the continuous function $T : \mathcal{M} \to \mathbb{R}$ on the $R$-orbit starting at $x \in \mathcal{M}$:

$$L(T, x) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} T \circ R^i(x).$$  

(A.2)
Suppose that, for Lebesgue almost all points \( x \in B(\Omega) \), the limit (A.2) exists and is independent on \( x \). Then \( L \) is a continuous linear functional in the set of continuous maps from \( M \) to \( \mathbb{R} \) (denoted by \( C(M, \mathbb{R}) \)). By the Riesz Representation Theorem, it defines a unique probability measure \( \mu \) such that:

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} T \circ R^i(x) = \int_{\Omega} T \, d\mu
\]  

(A.3)

for all \( T \in C(M, \mathbb{R}) \) and for Lebesgue almost all points \( x \in B(\Omega) \). If there exists an ergodic measure \( \mu \) supported in \( \Omega \) such that (A.3) is satisfied for all continuous maps \( T \in C(M, \mathbb{R}) \) for Lebesgue almost all points \( x \in B(\Omega) \), where \( B(\Omega) \) has positive Lebesgue measure, then \( \mu \) is called a SRB (Sinai-Ruelle-Bowen) measure and \( \Omega \) is a SRB attractor. More details in [55].

A.7 Non-trivial Wandering Domains

A non-trivial wandering domain for a given map \( R \) on a Riemannian manifold \( M \) is a non-empty connected open set \( D \subset M \) which satisfies the following conditions:

- \( R^i(D) \cap R^j(D) = \emptyset \) for every \( i, j \geq 0 \) (\( i \neq j \))
- the union of the \( \omega \)-limit sets of points in \( D \) for \( R \), denoted by \( \Omega(D, R) \), is not equal to a single periodic orbit.

A wandering domain \( D \) is called contracting if the diameter of \( R^n(D) \) converges to zero as \( n \to +\infty \).

A.8 Rotational Horseshoe

Let \( \mathcal{H} \) stand for the infinite annulus \( \mathcal{H} = S^1 \times \mathbb{R} \) (endowed with the usual inner product from \( \mathbb{R}^2 \)). We denote by \( \text{Homeo}^+(\mathcal{H}) \) the set of homeomorphisms of the annulus which preserve orientation. Given a homeomorphism \( f : X \to X \) and a partition of \( m \in \mathbb{N} \backslash \{ 1 \} \) elements \( R_0, ..., R_{m-1} \) of \( X \subset \mathcal{H} \), the itinerary function \( \xi : X \to \{ 0, ..., m - 1 \}^\mathbb{Z} = \Sigma_m \) is defined by:

\[
\xi(x)(j) = k \iff f^j(x) \in R_k, \quad \text{for every } j \in \mathbb{Z}.
\]

Following [43], we say that a compact invariant set \( \Lambda \subset \mathcal{H} \) of \( f \in \text{Homeo}^+(\mathcal{H}) \) is a rotational horseshoe if it admits a finite partition \( P = \{ R_0, ..., R_{m-1} \} \) by sets \( R_i \) with non-empty interior in \( \Lambda \) so that:

- the itinerary \( \xi \) defines a semi-conjugacy between \( f|_\Lambda \) and the full-shift \( \sigma : \Sigma_m \to \Sigma_m \), that is \( \xi \circ f = \sigma \circ \xi \) with \( \xi \) continuous and onto;
- for any lift \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) of \( f \), there exist \( k > 0 \) and \( m \) vectors \( v_0, ..., v_{m-1} \in \mathbb{Z} \times \{ 0 \} \) so that:

\[
\left\| (F^n(\hat{x}) - \hat{x}) - \sum_{i=0}^{n} v_{\xi(x)(i)} \right\| < k \quad \text{for every } \hat{x} \in \pi^{-1}(\Lambda), \quad n \in \mathbb{N},
\]

where \( \| \cdot \| \) is the usual norm on \( \mathbb{R}^2 \), \( \pi : \mathbb{R}^2 \to \mathcal{H} \) denotes the usual projection map and \( \hat{x} \in \pi^{-1}(\Lambda) \) is the lift of \( x \); more details in the proof of Lemma 3.1 of [43]. The existence of a rotational horseshoe for a map implies positive topological entropy at least \( \log m \).
References

1. Afraimovich, V.S., Shilnikov, L.P.: On invariant two-dimensional tori, their breakdown and stochasticity. In: Methods of the Qualitative Theory of Differential Equations, pp. 3–26, Gor’kov. Gos. University (1983). Translated in: Amer. Math. Soc. Transl., (2), 149, 201–212 (1991)
2. Afraimovich, V.S., Hsu, S.-B., Lin, H.E.: Chaotic behavior of three competing species of May-Leonard model under small periodic perturbations. Int. J. Bifurc. Chaos 11(2), 435–447 (2001)
3. Afraimovich, V.S., Hsu, S.B.: Lectures on Chaotic Dynamical Systems. American Mathematical Society and International Press, Cambridge (2002)
4. Aguiar, M.: Vector fields with heteroclinic networks, Ph.D. thesis, Departamento de Matemática Aplicada, Faculdade de Ciências da Universidade do Porto (2003)
5. Aguiar, M.A.D., Castro, S.B.S.D., Labouriau, I.S.: Dynamics near a heteroclinic network. Nonlinearity 18, 391–414 (2005)
6. Algaba, A., Merino, M., Rodríguez-Luis, A.: Takens-Bogdanov bifurcations of periodic orbits and Arnold’s tongues in a three-dimensional electronic model. Int. J. Bifurc. Chaos 11(02), 513–531 (2001)
7. Anishchenko, V., Safonova, M., Chua, L.: Confirmation of the Afraimovich-Shilnikov torus-breakdown theorem via a torus circuit. IEEE Trans. Circ. Syst. I: Fundam. Theory Appl. 40(11), 792–800 (1993)
8. Arnold, V.: Small denominators. I. Mapping the circle onto itself, Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya 25(1), 21–86 (1961)
9. Arrowsmith, D., Cartwright, J., Lansbury, A., Place, C.: The Bogdanov map: bifurcations, mode locking, and chaos in a dissipative system. Int. J. Bifurc. Chaos 3(04), 803–842 (1993)
10. Aronson, D., Chory, M., Hall, G., McGehee, R.: Bifurcations from an invariant circle for two-parameter families of maps of the plane: a computer-assisted study. Commun. Math. Phys. 83(3), 303–354 (1982)
11. Ashwin, P., Chossat, P.: Attractors for robust heteroclinic cycles with continua of connections. J. Nonlinear Sci. 8(2), 103–129 (1998)
12. Baldomá, I., Ibáñez, S., Seara, T.: Hopf-Zero singularities truly unfold chaos. Commun. Nonlinear Sci. Numer. Simul. 84, 105162 (2020)
13. Bakri, T., Verhulst, F.: Bifurcations of quasi-periodic dynamics: torus breakdown. Zeitschrift für angewandte Mathematik und Physik 65(6), 1053–1076 (2014)
14. Bakri, T., Kuznetsov, Y., Verhulst, F.: Torus bifurcations in a mechanical system. J. Dyn. Differ. Equ. 27(3–4), 371–403 (2015)
15. Boyland, P.L.: Bifurcations of circle maps: Arnold tongues, bistability and rotation intervals. Commun. Math. Phys. 106, 353–381 (1986)
16. Broer, H., Roussarie, R., Simó, C.: Invariant circles in the Bogdanov-Takens bifurcation for diffeomorphisms. Ergod. Theory Dynam. Syst. 16, 1147–1172 (1996)
17. Broer, H., Simó, C., Tatjer, J.C.: Towards global models near homoclinic tangencies of dissipative diffeomorphisms. Nonlinearity 11, 667–770 (1998)
18. Bykov, V.V.: On systems with separatrix contour containing two saddle-foci. J. Math. Sci. 95, 2513–2522 (1999)
19. Bykov, V.V.: Orbit Structure in a neighborhood of a separatrix cycle containing two saddle-foci. Am. Math. Soc. Transl. 200, 87–97 (2000)
20. Capiński, M.J., Fleurantin, E., James, J.M.: Computer assisted proofs of two-dimensional attracting invariant tori for ODEs. Discret. Contin. Dyn. Syst. A 40(12), 6681–6707 (2020)
21. Castro, M.L., Rodrigues, A.A.P.: Torus-breakdown near a heteroclinic attractor: acase study. Int. J. Bifurc.Chaos 31(10). (2021). https://doi.org/10.1142/S0218127421300299
22. Denjoy, A.: Sur les courbes définies par les équations différentielles a la surface du tore. J. Math. Pures Appl. 11, 333–375 (1932)
23. Gaspard, P.: Local birth of homoclinic chaos. Phys. D: Nonlinear Phenom. 62(1–4), 94–122 (1993)
24. Gonchenko, S.V., Shilnikov, L.P., Turaev, D.V.: Quasiattractors and homoclinic tangencies. Comput. Math. Appl. 34(2–4), 195–227 (1997)
25. Greenspan, B., Holmes, P.: Repeated resonance and homoclinic bifurcation in a periodically forced flow of oscillators. SIAM J. Math. Anal. 15(1), 69–97 (1984)
26. Guckenheimer, J., Holmes, P.: Nonlinear oscillations, dynamical systems, and bifurcations of vector fields. Appl. Math. Sci. 42, Springer-Verlag (1983)
27. Herman, M.: Mesure de Lebesgue et Nombre de Rotation. Lecture Notes in Mathematics, vol. 597, pp. 271–293, Springer, Berlin (1977)
28. Herman, M.: Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations. Publications Mathématiques de l’IHÉS 49, 5–233 (1979)
29. Hirsch, M.W., Pugh, C., Shub, M.: Invariant manifolds. Bull. Am. Math. Soc. 76(5), 1015–1019 (1970)
30. Homburg, A.J., Sandstede, B.: Homoclinic and heteroclinic bifurcations in vector fields. In: Handbook of Dynamical Systems, vol. 3, pp. 379–524. North Holland, Amsterdam (2010)
31. Kim, S., MacKay, R., Guckenheimer, J.: Resonance regions for families of torus maps. Nonlinearity 2(3), 391–404 (1989)
32. Kirk, V.: Merging of resonance tongues. Phys. D: Nonlinear Phenom. 66(3–4), 267–281 (1993)
33. Kirk, V., Rucklidge, A.: The effect of symmetry breaking on the dynamics near a structurally stable heteroclinic cycle between equilibria and a periodic orbit. Dynam. Syst. 23(1), 43–74 (2008)
34. Kiriki, S., Soma, T.: Takens’ last problem and existence of non-trivial wandering domains. Adv. Math. 306, 524–588 (2017)
35. Knobloch, J., Lamb, J.S.W., Webster, K.N.: Using Lin’s method to solve Bykov’s problems. J. Differ. Equ. 257(8), 2984–3047 (2014)
36. Labouriau, I.S., Rodrigues, A.A.P.: Dense heteroclinic tangencies near a Bykov cycle. J. Differ. Equ. 259(12), 5875–5902 (2015)
37. Labouriau, I.S., Rodrigues, A.A.P.: Bifurcations from an attracting heteroclinic cycle under periodic forcing. J. Differ. Equ. 269(5), 4137–4174 (2020)
38. Langford, W.F.: Numerical studies of torus bifurcations. In: Numerical Methods for Bifurcation Problems, pp. 285–295. Birkhäuser, Basel (1984)
39. Mora, L., Viana, M.: Abundance of strange attractors. Acta Math. 171(1), 1–71 (1993)
40. Newhouse, S.E.: The abundance of wild hyperbolic sets and non-smooth stable sets for diffeomorphisms. Publ. Math. Inst. Hautes Études Sci. 50, 101–151 (1979)
41. Ostlund, S., Rand, D., Sethna, J., Siggia, E.: Universal properties of the transition from quasi-periodicity to chaos in dissipative systems. Phys. D: Nonlinear Phenom. 8(3), 303–342 (1983)
42. Ovsyannikov, I.M., Shilnikov, L.P.: On systems with a saddle-focus homoclinic curve. Math. USSR Sb. 58, 557–574 (1987)
43. Passeggi, A., Potrie, R., Sambarino, M.: Rotation intervals and entropy on attracting annular continua. Geom. Topol. 22(4), 2145–2186 (2018)
44. Peckham, B.: The necessity of the Hopf bifurcation for periodically forced oscillators. Nonlinearity 3(2), 261–280 (1990)
45. Peckham, B., Frouzakis, C., Kevrekidis, I.: Bananas and banana splits: a parametric degeneracy in the Hopf bifurcation for maps. SIAM J. Math. Anal. 26(1), 190–217 (1995)
46. Peckham, B., Kevrekidis, I.: Lighting Arnold flames: resonance in doubly forced periodic oscillators. Nonlinearity 15, 405–428 (2002)
47. Rodrigues, A.A.P.: Persistent switching near a heteroclinic model for the geodynamo problem. Chaos Solitons Fractals 47, 73–86 (2013)
48. Rodrigues, A.A.P.: Repelling dynamics near a Bykov cycle. J. Dyn. Differ. Equ. 25(3), 605–625 (2013)
49. Rodrigues, A.A.P.: Strange attractors and wandering domains near a homoclinic cycle to a bifocus. J. Differ. Equ. 269(4), 3221–3258 (2020)
50. Rodrigues, A.A.P.: Unfolding a Bykov attractor: from an attracting torus to strange attractors. J. Dyn. Differ. Equ. (2020). https://doi.org/10.1007/s10884-020-09858-z
51. Ruelle, D., Takens, F.: On the nature of turbulence, Les rencontres physiciens-mathématiciens de Strasbourg-RCP25 12, 1–44 (1971)
52. Shilnikov, A., Nicolis, G., Nicolis, C.: Bifurcation and predictability analysis of a low-order atmospheric circulation model. Int. J. Bifurc. Chaos Appl. Sci. Eng. 5(06), 1701–1711 (1995)
53. Shilnikov, A., Shilnikov, L.P., Turaev, D.: On some mathematical topics in classical synchronization. A tutorial. Int. J. Bifurc. Chaos Appl. Sci. Eng. 14, 2143–2160 (2004)
54. Wang, Q., Oksasoglu, A.: Dynamics of homoclinic tangles in periodically perturbed second-order equations. J. Differ. Equ. 250(2), 710–751 (2011)
55. Wang, Q., Young, L.S.: From invariant curves to strange attractors. Commun. Math. Phys. 225, 275–304 (2002)
56. Yagasaki, K.: Melnikov’s method and codimension-two bifurcations in forced oscillations. J. Differ. Equ. 185, 1–24 (2002)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.