Holomorphic simplicity constraints for 4D spinfoam models

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Abstract
Within the framework of spinfoam models, we revisit the simplicity constraints reducing topological BF theory to 4D Riemannian gravity. We use the reformulation of SU(2) intertwiners and spin networks in terms of spinors, which has come out from both the recently developed U(N) framework for SU(2) intertwiners and the twisted geometry approach to spin networks and spinfoam boundary states. Using these tools, we are able to perform a holomorphic/anti-holomorphic splitting of the simplicity constraints and define a new set of holomorphic simplicity constraints, which are equivalent to the standard ones at the classical level and which can be imposed strongly on intertwiners at the quantum level. We then show how to solve these new holomorphic simplicity constraints using coherent intertwiner states. We further define the corresponding coherent spin network functionals and introduce a new spinfoam model for 4D Riemannian gravity based on these holomorphic simplicity constraints and whose amplitudes are defined from the evaluation of the new coherent spin networks.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The spinfoam framework for quantum gravity is a formalism for a regularized path integral for general relativity. It is based on a reformulation of gravity as a quasi-topological field theory, or more precisely as a topological BF theory with extra constraints which break the topological invariance and (re-)introduce local degrees of freedom in the field theory\(^{20}\). These are called the simplicity constraints and are at the heart of the spinfoam quantization program. They correspond to the second class constraints appearing in the canonical treatment of the first order (Holst-)Palatini action for general relativity as a gauge field theory\(^{3}\).

\(^{3}\) They also appear as the reality conditions in the self-dual Ashtekar formulation of general relativity as an SU(2) gauge theory.
The standard procedure to construct a spinfoam path integral for quantum gravity is to start from the discretized path integral for the topological BF theory, either under its state-sum formulation or derived from a discrete BF action. Then, one discretizes the simplicity constraints, investigates their geometrical and physical meaning at the spinfoam level and imposes them. Finally, the goal is to check whether this leads to the correct degrees of freedom both at the fundamental discrete level and in the semi-classical continuum level at large scales.

The first explicit spinfoam model for 4D quantum gravity is the Barrett–Crane model, in both its Riemannian version [1] and Lorentzian version [2]. It relies on a strong imposition of the discrete simplicity constraints. Since then, it has been argued that this is a too strong requirement. Indeed the strong imposition seems to kill too many degrees of freedom (e.g. [3]). Moreover, it turns out that the Hilbert space of boundary states of the Barrett–Crane model does not seem to fit with the space of canonical states of loop quantum gravity, which is another inconvenience (see e.g. [4, 5]). It was thus later argued that a weaker imposition of the (discrete) simplicity constraints would improve the spinfoam procedure and the semi-classical behavior of the model [6, 7]. This is related to the fact that the simplicity constraints correspond to second class constraints in the canonical analysis of the classical theory. It was then proposed to solve them only weakly in the spinfoam path integral through coherent state techniques [6, 8, 9] or through a Gupta–Bleuer-like procedure [7, 10]. This leads to the definition of the EPRL–FK spinfoam models [8, 10] which exist for both Riemannian and Lorentzian spacetime signature (also see [11] for a more thorough definition of the Lorentzian spinfoam model).

This is the current state-of-the-art spinfoam proposal for 4D quantum gravity.

Nevertheless, the construction of the EPRL–FK models relies on imposing the simplicity constraints on the expectation values with small uncertainty. This ensures a nice behavior of the states in the semi-classical regime. But a drawback with this construction is that the states are not properly defined as actual (strong) solutions to a set of constraints. In particular, they do not come from an actual Gupta–Bleuer procedure with a holomorphic/anti-holomorphic factorization of the constraints in terms of creation and annihilation operators. This means that we cannot define the EPRL–FK states through a simple algebraic equation.

Recently in [12], we have proposed a detailed analysis of the algebraic properties of the discrete simplicity constraints for 4D Riemannian gravity using the $\text{U}(N)$ framework for $\text{SU}(2)$ intertwiners [13–15]. This leads to the identification of a holomorphic/anti-holomorphic factorization of the simplicity constraints, which can now be used to perform a true Gupta–Bleuer procedure in order to take into account the simplicity constraints. We can indeed use these new F-simplicity constraints and impose them strongly. In [12], it was shown that this solves the simplicity constraints weakly and that we can construct solutions as coherent states using tools from the $\text{U}(N)$ framework. In this paper, we propose to investigate further the definition and properties of these F-simplicity constraints, both at the classical and quantum levels. Furthermore, we build new spinfoam amplitudes based on imposing these F-simplicity constraints and we compare them to the standard EPRL–FK models.

In section 2, we start by reviewing the classical phase space formulation for $\text{SU}(2)$ intertwiners in terms of spinors. This provides a clear geometrical framework to translate the simplicity constraints. We introduce our new F-simplicity constraints, or holomorphic simplicity constraints, and show their equivalence with the standard ways to formulate

4 Actually, the requirement is slightly stronger and we require the vanishing of the matrix elements of the constraints between solution states. Calling $H_s$ the Hilbert space of solution states and $C$ the simplicity constraints, we ask [7, 9]

$$\forall \psi, \phi \in H_s, \quad \langle \psi | C | \phi \rangle = 0.$$ 

This is stronger than simply requiring the vanishing of the expectation values $\langle \psi | C | \psi \rangle = 0$, but in practice it amounts simply to vanishing of the expectation values with small (almost-minimal) uncertainty.
the simplicity constraints. We discuss the geometrical meaning of the various simplicity constraints. In section 3, we move to the quantum level and review the U(N) framework for SU(2) intertwiners, which is obtained as a quantization of the classical spinor phase space. We review how to define coherent intertwiner states in this framework. Then, we show how to solve the holomorphic simplicity constraints at the quantum level using these tools. We propose a classical action principle to define the classical dynamics of discrete geometry states solving the holomorphic simplicity constraints. In section 4, we define a new spinfoam model solving strongly the holomorphic simplicity constraints and weakly the standard simplicity constraints, and we show its link to the EPRL–FK spinfoam amplitudes.

We work in this whole paper with 4D Riemannian quantum gravity. The Lorentzian case will be investigated elsewhere [16].

2. Classical geometry of the simplicity constraints

SU(2) intertwiners are the basic building blocks of the spin network states for quantum geometry in loop quantum gravity. Recently, the U(N) framework for the Hilbert space of intertwiners was developed [12–15, 17] and it was realized that it leads to a re-formulation of intertwiner states as the quantization of a classical phase space parameterized by spinors [15, 17]. When gluing the intertwiners back together along graphs to form spin network states, this actually led back to the twisted geometries, introduced independently to describe the discrete phase space of loop quantum gravity on a fixed graph [18, 19].

In this section, we will review this classical spinor phase space formalism for SU(2)-invariant states and generalize it to Spin(4) ∼ SU_L(2) × SU_R(2). We will then discuss the simplicity constraints in this framework: we will define our new holomorphic simplicity constraints and show their equivalence with the standard expressions of the simplicity constraints at the classical level.

2.1. The classical spinor framework for SU(2)

Following the previous ideas on the U(N) formalism for intertwiners [12, 14, 15] and on twisted geometries for loop gravity and spin foams [18, 19], it was realized that loop quantum gravity’s spin network states are the quantization of some classical spinor networks [17]. We review this formalism below.

Spin networks, and thus spinor networks, are constructed on a given graph. Let us thus choose a closed oriented graph Γ with E edges and V vertices. We will label its vertices as v and its edges as e, calling s(e) and t(e), respectively, the source and target vertices of each edge e. A spin network state (or exactly a gauge-invariant cylindrical function of the SU(2) connection on the graph Γ) is a function ϕ of SU(2) group elements ge living on each edge e and satisfying a gauge invariance at each vertex:

ϕ(ge) = ϕ(h−1 s(e)geh t(e)), \quad \forall g_e \in SU(2)^E, \quad \forall h_v \in SU(2)^V. \quad (1)

The classical data on a graph are thus given by (the equivalence classes under SU(2) gauge transformations at the vertices of) the group elements g_e living on each edge e and satisfying a gauge invariance at each vertex:

ϕ((ge)) = ϕ((h−1 s(e)geh t(e))), \quad \forall g_e \in SU(2)^E, \quad \forall h_v \in SU(2)^V. \quad (1)

The classical data on a graph are thus given by (the equivalence classes under SU(2) gauge transformations at the vertices of) the group elements g_e living on each edge e. It was shown that this setting can be replaced by spinors z_v^e living at the vertices and labeled by the edges attached to that vertex, which allow a more direct geometrical interpretation of the classical data as defined on a discrete 3D space geometry.

Let us start by focusing on a single intertwiner or vertex. We assume that it has N edges attached to it. We attach one spinor z_v^e to each leg of the vertex, with e running from 1 to N. A
spinor \( z_e \) determines a 3-vector \( \vec{V}(z) \) through its projection on Pauli matrices:
\[
|z\rangle \langle z| = \frac{1}{4}((|z\rangle \langle z|) + \vec{V}(z) \cdot \vec{\sigma}),
\]
with \( \vec{V}(z) = \langle z|z\rangle \). Conversely the original spinor \( z \) is entirely determined by the 3-vector \( \vec{V}(z) \) up to a phase. We give more details on this in appendix A, where we also define the dual spinor \( |z\rangle \) such that
\[
[z|z] = 0, \quad [z|z] = \langle z|z\rangle, \quad \vec{V}(|z\rangle) = -\vec{V}(|z\rangle).
\]

The phase space is defined by simply postulating that the spinor \( z_e \) is dual to its complex conjugate \( \bar{z}_e \):
\[
\{ z_e^\mu, \bar{z}_e^{\bar{\nu}} \} = -i\delta_{\mu\bar{\nu}}.
\]
The (components of the) 3-vector \( \vec{V}(z_e) \) can be seen to generate SU(2) transformations on the spinor \( z_e \) and we can check that their Poisson bracket truly forms the \( su(2) \) algebra:
\[
[V_i(z_e), V_j(z_e)] = 2\epsilon_{ijk}V_k(z_e).
\]
Then, we further impose the closure constraints \( \sum_e \vec{V}(z_e) = 0 \), which generates global SU(2) transformations on the spinors \( z_e \). This constraint is easily translated in terms of the spinors \( \bar{z}_e \):
\[
\sum_e |z_e\rangle \langle z_e| = \frac{1}{2} \sum_e \langle z_e|z_e\rangle I_2.
\]
The reader can find more details in the appendix or in references [12, 15, 17]. Thus, we can write a classical action principle which defines this constrained phase space as
\[
S_c[z_e] = \int dt \sum_e \left(-i\langle z_e|\dot{\vec{z}}_e z_e\rangle + \langle z_e|\Lambda|z_e\rangle\right),
\]
where \( \Lambda \) is the \( 2 \times 2 \) matrix Lagrange multiplier, with \( \text{Tr} \Lambda = 0 \), which imposes the closure constraint. Let us stress that this action principle only defines the kinematics on this phase space and we have not yet added dynamics to it.

The work done in [17] was to identify suitable SU(2) observables:
\[
E_{ef} = \langle z_e|z_f\rangle, \quad F_{ef} = |z_e|z_f\rangle, \quad \vec{F}_{ef} = \langle z_f|z_e\rangle.
\]
It is clear that these quadratic combinations are invariant under global SU(2) transformations on the spinors \( z_e \rightarrow g z_e \) for \( g \in \text{SU}(2) \). We can also check that their Poisson bracket with the closure constraints vanishes. They thus turn out to be a nice complete set of observables on the constrained phase space. The U(N) formalism for intertwiners is actually based on promoting these observables to operators acting on the Hilbert space of intertwiner states [14, 15, 17].

Now, having a spinorial description of each vertex, we can glue these structures together along the edges and form a spinor network. We now have spinors \( z_e^\nu \) around each vertex \( \nu \), which satisfy the closure constraints around each vertex. The gluing will induce a further constraint on each edge \( e \):
\[
\{ z_e^{\nu()}|z_e^{\nu()}\} = \{ z_e^{\nu()}|z_e^{\nu()}\}.
\]
This ensures that the two 3-vectors living on the same edge \( e \), but attached to the source and target vertices, have the same norm \( |\vec{V}(z_e^{\nu()})| = |\vec{V}(z_e^{\nu()})| \) and are thus related by an SU(2) rotation, which is exactly the group element \( g_e \) of the standard formulation.

The corresponding action principle, which defines the classical kinematical structure, of the spinor network is [17]
\[
S_{\mathcal{F}}[z_e^\nu] = \int dt \sum_{e,\nu} -i\langle z_e^{\nu}|\dot{\vec{z}}_e^{\nu} z_e^{\nu}\rangle + \langle z_e^{\nu}|\Lambda^{\nu}|z_e^{\nu}\rangle + \sum_e \lambda^{\nu}\langle z_e^{\nu}|z_e^{\nu}\rangle - \langle z_e^{\nu}|z_e^{\nu}\rangle),
\]
where the \( \lambda^{\nu} \)'s are Lagrangian multipliers imposing the new gluing constraints.
2.2. The classical spinor framework for Spin(4)

We now adapt this spinorial framework for SU(2) to Spin(4), which is the relevant gauge group for Riemannian 4D quantum gravity. Since, Spin(4) \(\sim SU_L(2) \times SU_R(2)\) exactly factorizes in its left and right SU(2)-subgroups, we take exactly two independent copies of the spinors introduced above.

Starting around a single vertex as before, we now have two spinors for edge \(e\) attached to that vertex, \(z^L_e\) and \(z^R_e\). The corresponding 3-vectors \(\tilde{V}(z^L_e)\) and \(\tilde{V}(z^R_e)\) respectively generate respectively SU(L) and SU(R) transformations. We can combine them to reconstruct the standard \(su(2)\) generators and boost generators:

\[
\begin{align*}
J'_e &= \frac{1}{2}(\tilde{V}(z^L_e) + \tilde{V}(z^R_e)), \\
K'_e &= \frac{1}{2}(\tilde{V}(z^L_e) - \tilde{V}(z^R_e)),
\end{align*}
\]

which satisfy the expected Poisson brackets:

\[
\{J'_e, J'_f\} = \epsilon^{ijk} J'_e \rho^i_k, \quad \{J'_e, K'_f\} = \epsilon^{ijk} K'_e \rho^i_k, \quad \{K'_e, K'_f\} = \epsilon^{ijk} J'_e \rho^i_k.
\]

Therefore, an SU(2)-rotation will act on the two spinors \(z^L_e\) and \(z^R_e\) with the space SU(2) group element, \((z^L_e, z^R_e) \rightarrow (gz^L_e, gz^R_e)\), while boosts will act on the two spinors with inverse group elements, \((z^L_e, z^R_e) \rightarrow (gz^L_e, g^{-1}z^R_e)\).

Then, we impose the invariance under the global Spin(4)-action on the spinors around the vertex, which is generated by both left and right closure constraints \(\sum_e \tilde{V}(z^L_e) = \sum_e \tilde{V}(z^R_e) = 0\), which are trivially equivalent to \(\sum_e J'_e = \sum_e K'_e = 0\).

Now moving back to the full graph and spinor network, we have two copies of the spinors \(z^L_e\) and \(z^R_e\) and the action principle defining the kinematical phase space structure is just the sum of the two previously defined actions (10):

\[
S_T[z^L_e, z^R_e] = S_T[z^L_e] + S_T[z^R_e].
\]

2.3. The simplicity constraints: differences and equivalence

Now we would like to discuss the simplicity constraints. They are constructed through their action on each intertwiner independently, so we will focus on one vertex \(v\) and drop the index \(v\) in this section. We introduce our holomorphic simplicity constraints:

\[
\forall e, f, \quad F^L_{ef} = \rho^2 F^R_{ef}, \quad \text{i.e.} \quad \forall e, f, \quad [z^L_{ef}, z^L_{ef}] = \rho^2 [z^R_{ef}, z^R_{ef}],
\]

where \(\rho\) is a fixed parameter, related to the Immirzi parameter. We will also refer to these as the \(F\)-constraints or \(F\)-simplicity.

A first remark is that \(F_{ef}\) is anti-symmetric in \(e \leftrightarrow f\), and in particular \(F_{ee}\) vanishes for \(e = f\). Therefore, the \(F\)-constraints are trivial for \(e = f\) and symmetric under the exchange \(e \leftrightarrow f\).

A second remark is that the \(F_{ef}\) are holomorphic in the spinors, thus the name ‘holomorphic simplicity constraints’. In particular, their Poisson brackets with each other vanish:

\[
\{F^L_{ef} - \rho^2 F^R_{ef}, F^L_{ef} - \rho^2 F^R_{ef}\} = 0.
\]

First, we show that \(F\)-simplicity implies the standard simplicity constraints.

**Proposition 2.1.** Assuming the holomorphic simplicity constraints, \(\rho^2 [z^L_{ef}, z^L_{ef}] = \rho^2 [z^R_{ef}, z^R_{ef}]\) for all couple of edges \(e, f\), and assuming the closure constraints \(\sum_e z^L_{ef}(z^L_{ef}) = \sum_e z^R_{ef}(z^R_{ef}) = 1\), the following amongst the spinors and 3-vectors are implied:

\[
\begin{align*}
\tilde{V}^L_{ef} &\equiv |\rho^2|^2 (\tilde{V}^R_{ef}, \tilde{V}^L_{ef}), \\
\langle z^L_{ef}, z^L_{ef} \rangle &\equiv |\rho^2| |z^R_{ef}, z^R_{ef}|.
\end{align*}
\]
These are the standard simplicity constraints. In particular, for \( e = f \), we get the diagonal simplicity constraints, which we can express in terms of the spin(4) generators \( \bar{F} \) and \( \tilde{K}_e \):

\[
(1 - |\rho|^2)(\bar{F}^2 + \tilde{K}^2) + (1 + |\rho|^2)(2\bar{F} \cdot \tilde{K}) = 0.
\]

**Proof.** We are going to take the norm squared of the \( F \)-observables. First, one can check that

\[
|F_{ef}|^2 = \langle z_f | z_e \rangle \langle z_e | z_f \rangle = \text{Tr}(z_e \otimes z_f) = \frac{1}{2} \text{Tr}(\bar{F}^e_0 \cdot \sigma_1 + \bar{F}^e_1 \cdot \sigma_2)
\]

Thus, taking the norm squared of the \( F \)-simplicity constraint gives

\[
|\bar{V}^e_0|^2 |\bar{V}^e_1|^2 - |\bar{V}^e_0|^2 |\bar{V}^e_1|^2 = |\rho|^2 \left( |\bar{V}^R_0|^2 |\bar{V}^R_1|^2 - |\bar{V}^R_0|^2 |\bar{V}^R_1|^2 \right).
\]

Summing this relation over both edges \( e \) and \( f \) and taking into account that the vectors satisfy the closure condition \( \sum_e \bar{V}^e_0 = \sum_e \bar{V}^e_1 = 0 \), we easily get

\[
\sum_e |\bar{V}^e_0|^2 = |\rho|^2 \sum_e |\bar{V}^R_0|^2 = 0.
\]

Then, coming back to the previous relation and only summing over the index \( f \) while keeping \( e \) fixed, we get

\[
\forall e, \quad |\bar{V}^e_0| = |\rho| |\bar{V}^R_0|.
\]

Plugging this back in the expression of \( |F_{ef}|^2 \), we finally get

\[
\forall e, f, \quad \bar{V}^e_0 \cdot \bar{V}^f_0 = |\rho|^2 (\bar{V}^R_e \cdot \bar{V}^R_f),
\]

which are the standard quadratic simplicity conditions.

We also check that the \( F \)-simplicity constraints imply the \( E \)-simplicity constraints, i.e

\[
\forall e, f, \quad |z^e_f| = 2 \frac{\sum_f |z^e_f|^2 |z^e_f|^2}{\sum_f |z^e_f|^2} - 2 \frac{|\rho|^2 \sum_f |z^e_f|^2 |z^e_f|^2}{|\rho|^2 \sum_f |z^e_f|^2} = |\rho|^2 |\bar{V}^R_0|^2,
\]

where we use the fact that we already know that \( |z^e_f|^2 = |\bar{V}^e_1|^2 = |\rho|^2 |\bar{V}^R_0|^2 \).

Now, we can make the link between these new holomorphic simplicity constraints and the ‘linear’ simplicity constraints involving the time normal. As already explained in previous work [8, 9, 12], the time normal gets encoded as an SU(2) transformation between the left spinors and the right spinors.

**Proposition 2.2.** Assuming the closure constraints, the holomorphic simplicity constraints are equivalent to the linear simplicity constraints, i.e. there exists a group element \( g \in \text{SU}(2) \) such that

\[
\exists g \in \text{SU}(2), \quad \forall e, \quad g|z^e_e| = \rho |z^e_e|.
\]

Let us translate this linear simplicity constraint into a constraint on the 3-vectors, which is easier to understand geometrically. Dropping the index \( e \), the condition \( g|z^e| = \rho |z^e| \) implies that \( g|z^e| = \rho |z^e| g^{-1} = |\rho|^2 |z^e| g |z^e| \), i.e. that \( g \) rotates the 3-vector \( \bar{V}^0 \) onto its right counterpart \( \bar{V}^R \):

\[
g|z^e| = \rho |z^e| \Rightarrow g \cdot \bar{V}^0 = |\rho|^2 \bar{V}^R.
\]

Now, let us think in terms of a bivector, or Lie algebra element in \( \text{spin}(4) \). We can write a bivector \( B \) either in terms of its left/right components \( \langle \bar{V}^L, \bar{V}^R \rangle \) or in terms of its rotation/boost
components \( (\vec{J}, \vec{K}) \), with the correspondence given by \( \vec{V}_L = \vec{J} + \vec{K} \) and \( \vec{V}_R = \vec{J} - \vec{K} \). The Hodge dual of the bivector \( \bullet B \) is obtained by switching the rotation and boost components \( (\vec{K}, \vec{J}) \) or by simply switching the sign of its right part \( (\vec{V}_L, -\vec{V}_R) \).

We consider the combination \( B + \gamma \bullet B = (1 + \gamma) \vec{V}_L, (1 - \gamma) \vec{V}_R \). The parameter \( \gamma \) is the Immirzi parameter. We act on \( B + \gamma \bullet B \) with the Spin(4) transformation \( G = g_{(L)} \otimes g_{(R)} \):

\[
G \triangleright (B + \gamma \bullet B) = (G \triangleright B) + \gamma \bullet (G \triangleright B) = ((1 + \gamma) g \triangleright \vec{V}_L, (1 - \gamma) \vec{V}_R).
\]

We distinguish two cases:

- \(|\gamma| < 1\): then we take \(|\rho|^2 = (1 - \gamma)/(1 + \gamma) > 0\). Then, the boost part of \( G \triangleright (B + \gamma \bullet B) \) vanishes (i.e. its left component is equal to its right component). Thus, the linear simplicity constraint \( g |z^L_e\rangle = \rho |z^R_e\rangle \) for all \( e \) means that there exists a common time normal to all bivectors:

\[
\forall e, \quad \mathcal{N}^\ell (B_e + \gamma \bullet B_e)_{\mu\nu} = 0, \quad \mathcal{N}^\ell G^{-1} \triangleright (1, 0, 0, 0). \tag{26}
\]

- \(|\gamma| > 1\): because of the sign switch, we take \(|\rho|^2 = (\gamma - 1)/(1 + \gamma) > 0\). Then, the rotation part of \( G \triangleright (B + \gamma \bullet B) \) vanishes (i.e. its left component is equal to minus its right component), or equivalently the boost part of \( G \triangleright (\bullet B + \gamma B) \) vanishes. Thus, the linear simplicity constraint \( g |z^L_e\rangle = \rho |z^R_e\rangle \) for all \( e \) means once again that there exists a common time normal to all bivectors:

\[
\forall e, \quad \mathcal{N}^\ell (B_e + 1 \gamma \bullet B_e)_{\mu\nu} = 0, \quad \mathcal{N}^\ell G^{-1} \triangleright (1, 0, 0, 0). \tag{27}
\]

Let us now get back to proving the previous proposition.

**Proof.** To start with, it is easy to see that the existence of such a group element implies \( F \)-simplicity since the \( F_{\mu\nu} \)'s are SU(2)-invariant observables:

\[
[g_{\mu
u} |z^L_e\rangle |z^R_e\rangle] = \rho^2 [g_{\mu\nu}^{-1} |z^L_e\rangle |z^R_e\rangle] = \rho^2 [z^L_e |z^R_e\rangle]. \tag{28}
\]

It is actually straightforward to show the converse statement. It means that the \( F \)-observables are a complete set of SU(2)-invariant observables.

Let us start by assuming \( F \)-simplicity. Let us choose one index \( e \) and consider the two spinors \( z^L_e, z^R_e \). The \( F \)-simplicity implies that the ratio of the norms of these two spinors is given by \(|\rho|^2 \vert z^L_e \vert^2 |z^R_e\rangle = \rho^2 \vert z^R_e \vert^2 |z^L_e\rangle \). Then, there exists a (unique) SU(2) group element which maps one onto the other (the interested reader can find more details in the appendix):

\[
g_e = \frac{|z^R_e\rangle \langle z^L_e| + |z^L_e\rangle \langle z^R_e|}{\sqrt{|z^R_e\rangle \langle z^R_e| |z^L_e\rangle \langle z^L_e|}}, \quad g_e |z^L_e\rangle = |\rho| |z^R_e\rangle. \tag{29}
\]

For the sake of simplicity, we will now assume that \( \rho \in \mathbb{R}^+ \) is real and (strictly) positive. Then, we can check that this group element \( g_e \) actually maps any spinor \( z^L_e \) to its right counterpart

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5 We can similarly treat the generic case of a complex parameter \( \rho \) by being careful with phase factors. Writing \( \rho = r e^{i\phi} \), \( F \)-simplicity implies that \( |z^L_e \rangle \langle z^R_e| = r^2 |z^R_e \rangle \langle z^L_e| \). Then, we define the group element \( g_e \) such that \( g_e |z^L_e\rangle = |\rho| |z^R_e\rangle \):

\[
g_e = \frac{r e^{i\phi} |z^R_e\rangle \langle z^L_e| + e^{-i\phi} |z^L_e\rangle \langle z^R_e|}{r |z^R_e\rangle \langle z^R_e| + |z^L_e\rangle \langle z^L_e|}.
\]

Then, we can check its action on all other left spinors:

\[
g_e |z^L_f\rangle = \frac{r e^{i\phi} |z^R_f\rangle \langle z^L_f| + e^{-i\phi} |z^L_f\rangle \langle z^R_f|}{r |z^R_f\rangle \langle z^R_f| + |z^L_f\rangle \langle z^L_f|} = r^2 e^{i\phi} |z^R_f\rangle \langle z^L_f| + e^{-i\phi} r^2 e^{i\phi} |z^R_f\rangle \langle z^L_f| = |\rho| |z^R_f\rangle.
\]
The dynamics of the theory will reflect the Spin $\mathbb{S}^2$. Should compute transition amplitudes for the SU(2) spin networks from the point of view that spinfoam models based on simple Spin(2) spin networks be in one-to-one correspondence with SU(2) spin network states. This gives us an action which depends on the left spinors with the gluing condition. We start by solving for the right spinors in terms of the left spinors.

This construction also allows us to see how to go between the two sectors that we distinguished above with $|\gamma| < 1$ and $|\gamma| > 1$. Indeed, if we now assume that we have a group element such that $g |z^L_e| = \rho |z^R_e|$ for all $e$’s, then it is equivalent to requiring the conjugate F-simplicity $F^L_f = \rho^2 F^R_f$. This condition implies that $g \triangleright V^L = -|\rho|^2 V^R$. This sign switch allows us to swap the two cases $|\gamma| < 1$ and $|\gamma| > 1$.

2.4. Classical phase space for simple intertwiners

Let us look at the classical Spin(4)-invariant phase space for a single intertwiner and impose the holomorphic simplicity constraints. Thus, we start with the (free) action:

$$S^\text{simple}_v = \int \mathcal{D}v \sum \left| z^L_e \right| \left| \partial_e z^L_e \right| + \left| z^R_e \right| \left| \Lambda^L \right| \left| z^L_e \right| + \sum_{e,f} \Phi_{ef} \left( \left| z^L_e \right| \left| z^L_f \right| - \rho^2 \left| z^R_e \right| \left| z^R_f \right| \right),$$

where $\Phi_{ef}$ is the Lagrange multiplier imposing the simplicity constraints. Then, as was shown above, we can solve these constraints exactly and write the right spinors in terms of the left spinors:

$$\exists g \in SU(2), \quad \forall e, \quad |z^R_e| = \rho^{-2} g |z^L_e|.$$ 

This group element $g$ is then re-absorbed in the Lagrange multiplier $\Lambda^R$ and we are left with solely the left sector:

$$S^\text{simple}_v = \int \mathcal{D}v \sum \left| z^L_e \right| \left| \partial_e z^L_e \right| + \left| z^L_e \right| \left| \Lambda^L \right| \left| z^L_e \right|.$$

This reduces to the phase space for a single SU(2) intertwiner. This shows that imposing the holomorphic simplicity constraints on a single intertwiner reduces effectively the degrees of freedom down back to the SU(2) theory.

We can further move to a full spinor network on a graph $\Gamma$. A priori, we have to be careful with the gluing condition. We start by solving for the right spinors in terms of the left spinors. This gives us an action which depends on the left spinors $z^L_e$ and on group elements $g_e$ at each vertex. Then, we realize that the gluing conditions involve only norms of the spinors and thus do not see the group elements $g_e$ at all. And finally, we arrive back at the action for a pure SU(2) spinor network.

This shows that, at the kinematical level, the holomorphic simplicity constraints allow us to reduce effectively the Spin(4) phase space down to the SU(2) phase space. This means that, after quantization, at the kinematical level, the simple Spin(4) spin network states should be in one-to-one correspondence with SU(2) spin network states. This is actually a desired feature from the point of view that spinfoam models based on simple Spin(4) spin networks should compute transition amplitudes for the SU(2) spin networks of loop quantum gravity [7, 10].

We would like to underline that this is at the kinematical level and that we expect that the dynamics of the theory will reflect the Spin(4) structure and invariance of the theory.
3. Coherent states and simplicity at the quantum level

In this section, we will quantize all the classical spinorial structures defined in the previous section. This will lead us to SU(2) and Spin(4) intertwiner states and to the quantum simplicity constraints. We will then show how to solve these holomorphic simplicity constraints using coherent intertwiner states, which we will dub simple intertwiners.

Our starting point is the work on the U(N) formalism for SU(2) intertwiners \([14, 15]\) and on solving the quantum simplicity constraint using U(N) coherent states \([12]\). We will review these previous results in a concise and consistent fashion. We will also describe how to glue coherent intertwiners into coherent SU(3) spin network states and how to similarly glue simple intertwiners into simple spin networks. This will set the proper foundations in order to build the corresponding spin foam amplitudes, as we will do in the next section.

3.1. SU(2)-intertwiner spaces and U(N) representations

Let us start with the quantization of the classical phase space for SU(2). We will focus on a single vertex \(v\) of valence \(N\) and we will drop the index \(v\). This is a review of the U(N) formalism developed in \([13–15]\).

Quantizing the spinor phase space can be done in a straightforward way. Considering a spinor \(z\) and its canonically conjugated \(\bar{z}\), we quantize its two components as creation/annihilation operators of two harmonic oscillators:

\[
\begin{align*}
\hat{z}_0 &\to a, \\
\hat{z}_1 &\to b, \\
\hat{\bar{z}}_0 &\to a^\dagger, \\
\hat{\bar{z}}_1 &\to b^\dagger,
\end{align*}
\]

\([a, a^\dagger] = [b, b^\dagger] = 1, \quad [a, b] = 0. \quad (33)

So, on each leg \(e\) around the vertex \(v\), we have a couple of harmonic oscillators \(a_e, b_e\), and we use the standard basis \(|n_a^e, n_b^e\rangle\) labeled with the number of quanta for both oscillators.

Using this quantization procedure, we directly quantize the vectors \(\hat{V}(z_e)\) and the observables \(E_{e f}, F_{e f}\). The components of the 3-vectors are given by the Schwinger representation of the \(su(2)\) algebra in terms of a couple of harmonic oscillators:

\[
\begin{align*}
\frac{i}{2} V^3 &= \frac{i}{2} (|z_0|^2 - |z_1|^2) \quad \to \quad \frac{i}{2} \hat{V}^3 = \frac{i}{2} (a^* a - b^* b) \\
\frac{i}{2} V^+ &= \frac{i}{2} z_0^{*} z_1 \quad \to \quad \frac{i}{2} \hat{V}^+ = a^* b \\
\frac{i}{2} V^- &= \frac{i}{2} z_0 z_1^{*} \quad \to \quad \frac{i}{2} \hat{V}^- = a b^* \\
|V| &= \langle z | z \rangle = (|z_0|^2 + |z_1|^2) \quad \to \quad |\hat{V}| = (a^* a + b^* b).
\end{align*}
\]

(34)

The components of \(\hat{V}(z_e)\) form an \(su(2)\) Lie algebra as expected from the Poisson brackets. So we now have that each leg \(e\) carries an SU(2)-representation.

The norm operator \(|\hat{V}(z_e)|\) is the total energy of the oscillators and is SU(2)-invariant. Fixing its value projects us onto an irreducible SU(2)-representation and its value gives twice the spin \(j_e\) of that representation. More precisely, we go from the standard oscillator basis \(|n_a^e, n_b^e\rangle\) to the usual magnetic momentum basis \(|j_e, m_e\rangle\) for spin systems by diagonalizing the operators \(\hat{V}_e^3\) and \(|\hat{V}_e|\), which gives the simple correspondence

\[
j_e = \frac{n_a^e + n_b^e}{2}, \quad m_e = \frac{n_a^e - n_b^e}{2}. \quad (35)
\]

So fixing the total energy of the two harmonic oscillators, we fix the spin \(j_e\) of the SU(2)-representation attached to the leg \(e\). Calling \(\mathcal{H}^{HO} = \bigoplus_n \mathbb{C}|n\rangle\) the Hilbert space of a single harmonic oscillator, we can write more generally:

\[
\mathcal{H}^{HO} \otimes \mathcal{H}^{HO} = \bigoplus_{j \in \mathbb{N}/2} \mathcal{V}^j, \quad (36)
\]

where we write \(\mathcal{V}^j\) for the SU(2)-representation of spin \(j\).
The next step is to impose the closure constraints $\sum_e \hat{\chi}_e = 0$. This means imposing the invariance under the global SU(2)-action, which implies considering SU(2)-invariant states in the tensor product of the SU(2)-representations living on the legs $e$ around the vertex, i.e. intertwiners between the spins $j_e$. This leads to the whole Hilbert space of $N$-valent intertwiners:

$$\mathcal{H}_N = \text{InvSU}(2) \bigotimes_e (\mathcal{H}_e^{\text{hol}} \otimes \mathcal{H}_e^{\text{hol}}) = \bigoplus_{\{j_e\} \in \mathbb{N}/2} \mathcal{V}^{j_e}.$$  

Then, we quantize the observables $E_{ef}, F_{ef}$ following the same quantization procedure:

$$E_{ef} = (z_e | z_f) \quad \Rightarrow \quad \hat{E}_{ef} = a_e^\dagger a_f + b_f^\dagger b_e$$
$$F_{ef} = | z_e z_f \rangle \quad \Rightarrow \quad \hat{F}_{ef} = a_e b_f - b_e a_f$$

It is straightforward to compute the commutators between these observables and to check that their algebra closes [15]. Moreover, as was shown in [17], these commutators provide the expected quantization of their classical Poisson brackets. The interested reader can find the relevant Poisson brackets and commutators in appendix B.

It is also straightforward to check that these operators $\hat{E}_{ef}, \hat{F}_{ef}, \hat{F}_{ef}^\dagger$ commute with the SU(2) generators $\sum_e \hat{\chi}_e$, so that they are still SU(2)-invariant observables at the quantum level.

The important fact is that the $\hat{E}$-operators form a $u(N)$ Lie algebra. This comes from the fact that the $E_{ef}$ generate U(N)-transformations on the spinors through the Poisson bracket at the classical level (as was checked in $[12, 15, 17]$):

$$\{E_{ef}, z_l\} = i\delta_{el} z_f, \quad \sum_{ef, \alpha, \beta} U_{\alpha\beta} (E_{ef})_{\alpha\beta} z_l = \sum_f (e_i^{\alpha^\dagger} z_f = (e_i^{\alpha^\dagger} z_l,$$  

where $\alpha = \alpha^\dagger$ is an $N \times N$ Hermitian matrix and $U = e^{i\alpha}$ the corresponding unitary transformation in U(N). Similarly, at the quantum level, the operators $\hat{E}_{ef}$ generate a U(N)-action on the space of $N$-valent intertwiners [13, 14]. Without going into the details (which the interested reader will find in [14]), the final result is that each space of intertwiners at fixed total area $J = \sum_{j_e}$ carries an irreducible representation of U(N) (whose highest weight vector is a bivalent intertwiner). This is summarized by the following decomposition of the intertwiner space:

$$\mathcal{H}_N = \bigoplus_{\{j_e\} \in \mathbb{N}/2} \text{InvSU}(2) \bigotimes_e \mathcal{V}^{j_e} = \bigoplus_{J} \mathcal{R}^J, \quad \mathcal{R}^J = \bigoplus_{J = \sum_{j_e}} \text{InvSU}(2) \bigotimes_e \mathcal{V}^{j_e}.$$  

The $u(N)$-generators $\hat{E}_{ef}$ act within each U(N)-representation $\mathcal{R}^J$ at fixed $J$, while the operators $\hat{F}_{ef}$ (resp. $\hat{F}_{ef}^\dagger$) act as annihilation operators (resp. creation operators) and allow transitions from a subspace $\mathcal{R}^J$ to $\mathcal{R}^{J-1}$ (resp. $\mathcal{R}^{J+1}$).

3.2. Coherent intertwiner states

Following the logic of the series of papers [12, 14, 15, 17] on the U(N) formalism for SU(2) intertwiners, we introduce coherent intertwiner states which are peaked on each point of the spinor phase space. We start with SU(2) coherent states and review the various definitions of coherent intertwiners until the most advanced one which will allow us to solve the holomorphic simplicity constraints.
We start by introducing SU(2) coherent states living in each SU(2)-representation at fixed spin $j$. We define them by acting with the relevant creation operators on the vacuum of the harmonic oscillators:

$$|j, z\rangle = \frac{(z^0 a^\dagger + z^1 b^\dagger)^{2j}}{\sqrt{(2j)!}}|0\rangle = \sum_{m=-j}^{+j} \frac{(2j)!}{(j+m)!(j-m)!} (z^0)^{j+m} (z^1)^{j-m}|j, m\rangle. \quad (41)$$

It is pretty easy to compute the norm of these vectors:

$$\langle j, z| j, z\rangle = \langle z|z\rangle^{2j}. \quad (42)$$

These are coherent states under the SU(2)-action, i.e. they transform covariantly under SU(2)-transformations (see e.g. [12, 15]):

$$\forall g \in \text{SU}(2), \quad g|j, z\rangle = |j, gz\rangle, \quad (43)$$

where the action of $g \in \text{SU}(2)$ on the spinor $z$ is simply the standard action of the $2 \times 2$ matrix (in the fundamental representation). From this, we can simply deduce another fundamental property of these SU(2) coherent states: they can all be obtained from the highest weight spinor $\Omega$ (in the fundamental representation). From this, we can simply deduce another fundamental property of these SU(2) coherent states: they can all be obtained from the highest weight spinor $\Omega$ (in the fundamental representation). From this, we can simply deduce another fundamental property of these SU(2) coherent states: they can all be obtained from the highest weight spinor $\Omega = (1, 0)$ by a (unique) SU(2) transformation:

$$g(z)|\Omega\rangle = \frac{|z\rangle}{\sqrt{|z|z\rangle}}, \quad g(z) = \frac{1}{\sqrt{|z|z\rangle}} \begin{pmatrix} z^0 & -z^1 \\ z^1 & z^0 \end{pmatrix} = \frac{1}{\sqrt{|z|z\rangle}}(|z\rangle, |z\rangle). \quad (44)$$

This implies a similar relation on the coherent states:

$$\frac{1}{\sqrt{|z|z\rangle^2}}|j, z\rangle = g(z)|j, \Omega\rangle = g(z)|j, z\rangle. \quad (45)$$

From here, taking into account that $|j, j\rangle = |1_j, 1_j\rangle^{\otimes j}$, we deduce the tensoring properties of the SU(2) coherent states:

$$|j, z\rangle = |1_j, z\rangle^{\otimes j}, \quad |1_j, z\rangle = |z\rangle. \quad (46)$$

Finally, these states are semi-classical and the expectation value of the $su(2)$-generators $\vec{J}$ on them is as expected:

$$\frac{\langle j, z| \vec{J} |j, z\rangle}{\langle j, z| j, z\rangle} = 2j \frac{|z|^2}{|z|} = j \frac{V(z)}{|V(z)|}. \quad (47)$$

The first notion of the coherent intertwiner was then introduced in [6] from tensoring together SU(2) coherent states and group-averaging in order to get SU(2)-invariant states. This was re-cast in terms of spinors in [12, 15]. Thus, such an $N$-valent coherent intertwiner is labeled by a list of $N$ spins $j_e$ and $N$ spinors $z_e$ attached to each leg $e$ and we define

$$|\{j_e, z_e\}\rangle = \bigotimes_e |j_e, z_e\rangle, \quad ||\{j_e, z_e\}\rangle = \int_{\text{SU}(2)} dg g \bigotimes_e |j_e, z_e\rangle. \quad (48)$$

As shown in [6], these states have nice semi-classical properties. They are used to construct semi-classical spin network states and to define the EPRL–FK spinfoam models [8, 9]. Nevertheless, most of their peakedness properties are only known approximatively in the large spin asymptotic regime $j_e \gg 1$ (through saddle point approximations [6]).

Then, reference [15] introduced a new class of coherent intertwinsers whose properties are under much better control than the previous coherent intertwiners constructed through group-averaging. These coherent intertwiners are covariant under the $U(N)$-action and are constructed from the vacuum state using the $F^\dagger$ as creation operators. These $U(N)$ coherent
intertwiner states are labeled by their total area $J \in \mathbb{N}$ and $N$ spinors $z_e$ living on each leg $e$ of the intertwiners. They are defined as

$$|J, \{z_e\} \rangle = \frac{1}{\sqrt{J!(J+1)!}} \left( \frac{1}{2} \sum_{e,f} |z_e| |z_f\rangle \hat{F}_{ef}^\dagger \right)^J |0\rangle. \quad (49)$$

Then, it is possible to show that they are given by a superposition of the previous coherent intertwiners defined by Livine–Speziale (LS) as proved in [15]:

$$\frac{1}{\sqrt{J!(J+1)!}} |J, \{z_e\} \rangle = \sum_{J=\sum_e j_e} \frac{1}{\sqrt{\prod_e (2j_e)!}} |\{j_e, z_e\}\rangle. \quad (50)$$

Due to their definition in terms of the creation operators $\hat{F}_{ef}^\dagger$, it is straightforward to prove that these states are covariant under the $U(N)$-action [15]:

$$\hat{U} |J, \{z_e\} \rangle = |J, \{(Uz_e)\} \rangle, \quad U = e^{iu}, \quad \hat{U} = e^{i \sum_e \alpha_j \hat{F}_{ef}^\dagger}, \quad (51)$$

where $\alpha$ is an arbitrary $N \times N$ Hermitian matrix. The behavior of these states under global rescaling of the spinors is also very simple:

$$|J, \{\beta z_e\} \rangle = \beta^2 |J, \{z_e\} \rangle, \quad \forall \beta \in \mathbb{C}. \quad (52)$$

We can also compute explicitly their scalar products and norms:

$$\langle J, \{w_e\}|J, \{z_e\} \rangle = \det \left( \sum_e |z_e\rangle \langle w_e| \right)^J = \left( \frac{1}{2} \sum_{e,f} \langle z_f|z_e\rangle \langle w_e|w_f\rangle \right)^J, \quad (53)$$

where we have introduced the new notation $A(z)$ for the sake of shortening the equations:

$$A(z) = \frac{1}{2} \sum_e (z_e|z_e\rangle = \frac{1}{2} \sum_e |\tilde{V}(z_e)|. \quad$$

Finally, the property which will be most interesting to us in the following is that the action of the annihilation operators $\hat{F}_{ef}^\dagger$ on those states can be simply computed [12, 15, 17] (using either of two definitions of the coherent states in terms of the creation operators $\hat{F}_{ef}^\dagger$ or in terms of the LS coherent intertwiners):

$$\hat{F}_{ef}^\dagger |J, \{z_e\} \rangle = \sqrt{J(J+1)} |z_e| |z_f\rangle |(J-1), \{z_e\}\rangle. \quad (54)$$

All these properties allow the exact calculation of the expectation values of the $\hat{E}$ operators on these new coherent states [15]:

$$\frac{\langle J, \{z_e\}|\hat{E}_{ef}|J, \{z_e\} \rangle}{\langle J, \{z_e\}|J, \{z_e\} \rangle} = \frac{J \langle z_e|z_f\rangle}{A(z)}. \quad (55)$$

Let us stress that this expectation value is exact while the expectation values of the SU(2)-observables on the LS coherent intertwiners are only known approximatively in the large spin asymptotic limit.

The ultimate notions of coherent intertwiners that we would like to introduce are the ones that were found in [12] to be useful when investigating the simplicity constraints at the quantum level. They are simply defined as the eigenstates of the annihilation operators $\hat{F}_{ef}$, just as in the case of the harmonic oscillator. This is possible since the operators $\hat{F}_{ef}$ all commute
with each other. More precisely, we define following [12] coherent states labeled by a complex number \( \lambda \in \mathbb{C} \) and the set of \( N \) spinors \( \{ z_i \} \):

\[
|\lambda, \{ z_i \} \rangle = \sum_j \frac{\lambda^{2j}}{\sqrt{j!(j+1)!}} |j, \{ z_i \} \rangle
\]

\[
= \sum_{\{e\}} \prod_i \frac{\lambda^{2e_i}}{\sqrt{(2e_i)!}} |\lambda, \{ z_i \} \rangle
\]

\[
= \int \text{d}gg \ e^{i \sum a_i^1 + z_i^1 b_i^1} |0\rangle,
\]

where we have used the explicit expression of the LS coherent intertwiners as the group averaging of the tensor product of SU(2) coherent states. The last equality shows the clear relation between our coherent states \( |\lambda, \{ z_i \} \rangle \) and the standard (unnormalized) coherent states for the harmonic oscillators.

Using the previous action of the annihilation operators on the \( |j, \{ z_i \} \rangle \) states\(^6\), we can easily check that these new coherent intertwiners are eigenstates of the annihilation operators:

\[
\hat{F}_{ef} |\lambda, \{ z_i \} \rangle = \lambda^2 |z_i z_f \rangle |\lambda, \{ z_i \} \rangle = \lambda^2 |z_i z_f \rangle |\lambda, \{ z_i \} \rangle.
\]

At this point, we note from the definition above that we have \( |\beta \lambda, \{ z_i \} \rangle = |\lambda, \{ \beta z_i \} \rangle \) for an arbitrary complex factor \( \beta \in \mathbb{C} \). We can thus re-absorb the factor \( \lambda \) directly in the spinor labels and choose to set \( \lambda = 1 \) in the definition of \( |\lambda, \{ z_i \} \rangle \) without loss of generality. We will therefore work in the following with the states

\[
||\{ z_i \} \rangle = |\lambda = 1, \{ z_i \} \rangle,
\]

which only refer to the \( N \) spinors. This defines coherent intertwiners simply labeled by the classical spinor variables.

We can compute the norm and scalar product of these states\(^7\) in terms of the modified Bessel functions of the first kind \( I_k \) [12]:

\[
\langle \{ z_i \} | |\{ z_i \} \rangle = \sum_j \frac{A(z)^{2j}}{j!(j+1)!} = \frac{I_1(2A(z))}{A(z)}
\]

\[
\langle \{ w_i \} | |\{ z_i \} \rangle = \sum_j \frac{1}{j!(j+1)!} \langle j, \{ w_i \} | j, \{ z_i \} \rangle = \sum_j \frac{1}{j!(j+1)!} \left( \det \sum_i |z_i \rangle \langle w_i| \right)^j
\]

and also the expectation value of the \( \hat{E} \)-operators:

\[
\langle \{ z_i \} | \hat{E}_{ef} | |\{ z_i \} \rangle = \frac{\langle z_i z_f \rangle}{A(z)} \prod_{j=1}^\infty \frac{(A(z))^{2j}}{(j-1)!(j+1)!} = \frac{\langle z_i z_f \rangle}{A(z)} \frac{I_1(2A(z))}{I_1(2A(z))},
\]

\[
\langle \{ z_i \} | \hat{E}_{ef} | |\{ z_i \} \rangle = \frac{\langle z_i z_f \rangle}{A(z)} \prod_{j=1}^\infty \frac{(A(z))^{2j}}{(j-1)!(j+1)!} = \frac{\langle z_i z_f \rangle}{A(z)} \frac{I_1(2A(z))}{I_1(2A(z))}.
\]

---

\(^6\) Since the operator \( \hat{F}_{ef} \) is SU(2)-invariant and thus commutes with the SU(2)-action, we could more simply compute its commutator with the usual operator \( e^{i \sum a_i^1 + z_i^1 b_i^1} \). Since we have

\[
[\hat{F}_{ef}, \sum z_i^1 a_i^1 + z_i^1 b_i^1] = (z_i^0 b_f + z_i^1 a_e - z_i^0 b_e - z_i^1 a_f), \quad \left( z_i^0 b_f + z_i^1 a_e - z_i^0 b_e - z_i^1 a_f \right), \sum z_i^0 a_i^1 + z_i^1 b_i^1 = 2F_{ef},
\]

this leads back to the same result.

\(^7\) An interesting particular case is when the two sets of spinors only differ through a constant phase \( e^{i \phi} \):

\[
\langle \lambda, \{ z_i \} | \lambda, \{ e^{i \phi} z_i \} \rangle = \sum_j \frac{(e^{i \phi})^j (A(z))^{2j}}{j!(j+1)!} = e^{-i \phi} \frac{I_1(2e^{i \phi}|z|^2 A(z))}{|z|^2 A(z)}.
\]
Figure 1. From left to right, plots of the modified Bessel function $I_1(x)$, of its logarithm $\ln I_1(x)$ which illustrates its asymptotic behavior for large $x \gg 1$ and the ratio $I_2(x)/I_1(x)$ which quickly converges to 1 as $x \to +\infty$.

Once again, we would like to underline the fact that these expectation values are computed exactly, while the expectation values of the LS coherent intertwiners (used as a basis to build the EPRL–FK spinfoam models) are only computed (up to now) approximatively at leading order in the large spin asymptotic limit.

In our case, the asymptotic behavior of the $\langle |z_J| \rangle$ coherent states is given by

$$\langle |z_J| \rangle = A(z) \frac{I_2(2A(z))}{I_1(2A(z))},$$  

$$\langle |z_J| \rangle E_{Jf} \langle |z_J| \rangle = \sum_J J^J A(z)^{2J} J!(J+1)!,$$

where the asymptotics are taken for large area $A(z) \gg 1$ (see figure 1 for plots of the Bessel functions).

To better understand these expectation values, let us have a look at the probability distributions on the area $J$ and spin labels $j_f$ induced by these coherent intertwiners $\langle |z_J| \rangle$. We start with the area $J$, which is given as an operator by $\hat{J} = \frac{1}{2} \sum f E_{ff}$. Copying the formulas above, we have

$$\langle |z_J| \rangle \hat{J} \langle |z_J| \rangle = \sum_J J^J A(z)^{2J} J!(J+1)!.$$

The (un-normalized) probability distribution for the area observable $J$ is thus

$$P_J = \frac{A(z)^{2J}}{J!(J+1)!}.$$  

We can approximate it at large $J$ by using Stirling’s formula for the factorials:

$$P_J \sim \frac{1}{2\pi J^2} \left( \frac{eA(z)}{J} \right)^{2J} \sim \frac{1}{2\pi} e^{2J(\ln A(z)+1)-(2J+2)\ln J}.$$

Calling $S[J] = 2J(\ln A(z)+1)-(2J+2)\ln J$ the exponent, we can investigate its behavior and check whether it has any extremum:

$$\partial_J S = 2 \left( \ln A(z) - \ln J - \frac{1}{J} \right), \quad \partial_J^2 S = -\frac{2}{J} \left( 1 - \frac{1}{J} \right).$$

8 The asymptotics for the modified Bessel functions $I_n(x)$ for $x \in \mathbb{R}$ do not depend on the label $n$ at leading order:

$$I_n(x) \sim \frac{e^x}{\sqrt{2\pi x}} \left( 1 + \mathcal{O} \left( \frac{1}{x} \right) \right).$$
Thus, $S$ has a unique extremum, which is a maximum, for the value $J_0$, which is approximately $J_0 \sim A(z)$ when $A(z) \gg 1$. Thus, for a large classical area $A(z)$, we do recover that the probability distribution for $J$ is peaked on this classical value $J_0 \sim |\lambda|^2$. Moreover, we have approximately a Gaussian around this value:

$$P_A[J] \sim e^{S[J]}e^{-\frac{1}{2N}(J-A(z))^2}.$$  

We can directly see this very simple behavior on numerical simulations of this probability distribution in figure 2.

We can do a similar analysis for the individual spin labels $j_f$ associated with each leg $f$ of the intertwiners. The corresponding operator is $\hat{j}_f \equiv \frac{1}{2}E_{j_f}$ and we can compute

$$\langle \{zf\} || \hat{j}_f || \{zf\} \rangle = \frac{1}{2} \langle zf|zf\rangle \frac{I_2(2A(z))}{I_1(2A(z))}.$$  

Going step by step, we can derive the probability distribution on the spin label $j_e$ by using equation (56):

$$\langle \{ze\} || \hat{j}_e || \{ze\} \rangle = \sum_{\{j_f\}} j_f \prod_{e} \frac{1}{(2j_e)!} \langle \{je, ze\} || \{je, ze\} \rangle.$$  

Actually, we can generalize this formula to any observable $O(j_e)$ depending on the spin labels instead of the single observable $j_f$. To study the behavior of the probability distribution in the $j_e$’s, we need the norm of the LS coherent intertwiners $\langle \{je, ze\} || \{je, ze\} \rangle$. These norms do not have a closed formula (up to now) despite detailed analyses [21], but we do have their asymptotic behavior obtained through a saddle point approximation [6]. Nevertheless, even without knowing the full exact behavior of this probability distribution, we can still extract some (minimal) information.

Indeed, the LS coherent intertwiner $\langle \{je, ze\} \rangle$ is the group-averaged state of the tensor product of the SU(2) coherent states $|je, ze\rangle$ which are not normalized. The norm of the SU(2) coherent states is simple $\langle je, ze|je, ze\rangle = \langle ze|ze\rangle^{2j_e}$. We can extract this norm for all the states. Then, assuming that the LS coherent intertwiner defined as the group averaging of the tensor product of the normalized coherent states contains information about the coupling between the various legs of the intertwiner, the other factors can be considered as describing approximatively the probability distribution for each decoupled single spin label $j_f$:

$$P[j_f] = \frac{1}{(2j_f)!} (zf|zf)^{2j_f}.$$  

One directly recognizes a Poisson distribution. It is peaked about $2j_f = \langle zf|zf\rangle$ as expected. One can use the Stirling approximation for the factorial as before and check that (as is well...
Figure 3. Left: plot of the Poisson distribution \( P[j_f] \) describing the probability distribution of the spin label \( j_f \) for a value \( x = \langle z_f | z_f \rangle = 50 \). The x-axis gives the value of \( 2 j_f \). Right: its superposition with its Gaussian approximation around its maximum \( 2 j_f \sim x = 50 \).

\[ P[j_f] = \frac{\lambda^{2j_f}}{(2j_f)!} \frac{1}{\sqrt{4\pi j_f}} e^{2j_f(1+\ln x-\ln 2j_f)} \sim \frac{\lambda^{2j_f}}{2j_f} \frac{e^{-\lambda(2j_f-x)^2}}{\sqrt{4\pi j_f}} \]

where the second approximation \( \sim \) gives the Gaussian behavior of the Poisson distribution around its maximum, as shown in figure 3.

3.3. Decomposing the identity on the intertwiner space

To conclude our summary of coherent intertwiner states, we would like to review the property that these coherent intertwiners provide us with an over-complete basis of the intertwiner space \( \mathcal{H}_N \). More exactly, the SU(2) coherent states \( |j, z\rangle \) form an over-complete basis of the space \( \mathcal{V}_j \) at fixed spin \( j \), the LS coherent intertwiners \( ||\{j_e, z_e\}\rangle \) form an over-complete basis of the space \( \text{Inv}_{SU(2)} \otimes \mathcal{V}_j \) for fixed spins on all the legs, the U(\(N\)) coherent states \( |J, \{z_e\}\rangle \) form an over-complete basis of the space \( \mathcal{R}_J \) for the fixed total area \( J = \sum_j J_e \) (or equivalently fixed \( \text{U}(N) \) representation) and finally our coherent states \( ||\{z_e\}\rangle \) span the whole space of \( N \)-valent intertwiners \( \mathcal{H}_N \).

Furthermore, we can write a decomposition of the identity on \( \mathcal{H}_N \) using our new coherent intertwiners. It is directly inherited from the decomposition of the identity on the Hilbert space \( \mathcal{H}^{HO} \) of a harmonic oscillator using the standard coherent states\(^9\):

\[ \mathbb{I}_{\mathcal{H}^{HO}} = \frac{1}{\pi} \int d^2 z \ e^{-|z|^2} |z\rangle_{\text{HO}} \langle z|, \quad \text{with} \quad |z\rangle_{\text{HO}} \equiv e^{z a^\dagger} |0\rangle = \sum_n z^n \sqrt{n!} |n\rangle_{\text{HO}}. \]  

\(^9\) This identity is rather straightforward to be shown explicitly:

\[ \int d^2 z e^{-|z|^2} \sum_{n,m} \frac{z^n \sqrt{n!} |n\rangle_{\text{HO}} |m\rangle_{\text{HO}}}{\sqrt{m!} n!} = \sum_n \int d^2 z e^{-|z|^2} \frac{(|z|^2)^n}{n!} |n\rangle_{\text{HO}} |n\rangle_{\text{HO}} = \pi \sum_n |n\rangle_{\text{HO}} \langle n|. \]
Starting by applying this decomposition of the identity to the space $\mathcal{H}^{HO} \otimes \mathcal{H}^{HO}$ and projecting down to $\mathcal{V}^J$ by fixing the total energy and thus the spin, we obtain\(^\text{10}\)

$$\mathbb{I}_{\mathcal{V}^J} = \frac{1}{\pi^2} \int d^4 z e^{-\|z\|^2} |j, z\rangle \langle j, z|.$$  

\hspace{1cm} (72)

Next, applying the decomposition of $\mathbb{I}_{\mathcal{H}^{HO}}$ to the tensor product $\bigotimes_e \mathcal{H}^{HO}$, we can decompose the identity using $2N$ copies of coherent states. Then, we simply go down to the intertwiner space by group averaging $\mathcal{H}_N = \text{Inv}_{\text{SU}(2)} \bigotimes_e \mathcal{H}^{(2)}_N$ as explained in section 3.1 (equation (37)). This provides an easy decomposition of the identity on $\mathcal{H}_N$:

$$\mathbb{I}_{\mathcal{H}_N} = \frac{1}{\pi^{2N}} \int [d^4 z] e^{-\sum_e \langle z_e | z_e \rangle} \int_{\text{SU}(2)} \text{d}\mathcal{g} \mathcal{g}^* \langle e \sum_e \phi_i^a \phi_i^b|0\rangle \langle 0 | (e \sum_e \phi_i^a \phi_i^b)^*$

where $\text{SU}(2)$ group elements act by conjugation. Taking into account the definition of the coherent intertwiners in (56), this simply means that our coherent intertwiners provide a decomposition of the identity

$$\mathbb{I}_{\mathcal{H}_N} = \frac{1}{\pi^{2N}} \int [d^4 z] e^{-\sum_e \langle z_e | z_e \rangle} \langle |z_e\rangle |\langle z_e|\rangle \rangle.$$

(73)

From this decomposition of the identity, one can project it on spaces at fixed $J$ or further on spaces at fixed $\{J_e\}$ and derive the formulas in terms of the other coherent intertwiner states.

Here, we need to point out that we are integrating over all sets of spinors $\{z_e\}$ and not restricting ourselves to spinors satisfying the closure constraint $\sum_e |z_e\rangle \langle z_e| \propto \mathbb{1}$. Actually, all the definitions of LS coherent intertwiners or $U(N)$ coherent states or our new coherent intertwiners that we gave in the previous section do not depend on the closure constraints and work for generic spinors. The only things that change are the explicit expressions for the scalar products and norms. More precisely, for generic spinors, we have

$$\langle J, \{w_e\}|J, \{z_e\}\rangle = \left(\frac{1}{2} \sum_{e,f} |z_f| |w_f| \right)^J,$$

$$\langle J, \{z_e\}|J, \{z_e\}\rangle = \left(\frac{1}{2} \sum_{e,f} |z_f| |z_e|\right)^J = \left(\frac{1}{4} \sum_e |\tilde{V}(z_e)|\right)^2 - \frac{1}{4} \sum_e |\tilde{V}(z_e)|^2.$$  

When the closure constraint is satisfied, $\sum_e \tilde{V}(z_e) = 0$, the norm of the state reduces to $A(z)^{2J}$ as before. We see that the more the closure constraint is violated, the more suppressed is the norm of the coherent intertwiner. This is similar to what happens with the LS coherent intertwiners [6].

The important point about relaxing the closure constraint is that we can go in and out of it by a straightforward global SL(2, C) action on the spinors $[15, 21]$. SL(2, C) transformations act as $2 \times 2$ matrices on the spinors:

$$|z_e\rangle \rightarrow A |z_e\rangle = |A z_e\rangle,$$

$$\sum_e |z_e\rangle \langle z_e| \rightarrow A \left(\sum_e |z_e\rangle \langle z_e|\right) A^\dagger,$$

$$|z_e| \rightarrow |z_e| A^{-1} A |z_f\rangle = |z_e| z_f\rangle.$$

\(^{10}\) It is possible to check this formula directly by decomposing the states $|j, z\rangle$ on the basis $|j, m\rangle$. We can also use the covariance property of the coherent states $|j, z\rangle$ under the SU(2)-action, which implies that the integral over $d^4 z$ is proportional to the identity on $\mathcal{V}^J$ by Schur’s lemma. Then, we just need to check the trace of the operator:

$$\int d^4 z e^{-\|z\|^2} (j, z|j, z\rangle = \int d^4 z e^{-\|z\|^2} (z|z\rangle = \pi^2 (2J + 1)!$$
On the one hand, starting with an arbitrary set of spinors $z_e$, one can always choose a suitable
$\Lambda \in \text{SL}(2, \mathbb{C})$ so that $\Lambda \left( \sum_e |z_e \rangle \langle z_e| \right) \Lambda^\dagger \propto I$. On the other hand, the SU(2)-invariant
observables $[z_e | z_f \rangle \langle z_f | z_e]$ are furthermore invariant under the SL(2, $\mathbb{C}$)-action, so that the coherent
intertwiners $| J, \{ z_e \} \rangle$ and $|| \{ z_e \} \rangle$ are themselves invariant under the change $z_e \rightarrow \Lambda z_e$ (for
more details, the interested reader can refer to [12, 15]).

By gauging out this SL(2, $\mathbb{C}$)-invariance, one can restrict the full integral over all spinors
$z_e$ to an integral over sets of spinors satisfying the closure constraint. In order to do this
consistently, one has to compute the corresponding Fadeev–Popov determinant. For the LS
coherent intertwiners, this was investigated in [21]. For the U(N) coherent states $| J, \{ z_e \} \rangle$ and
our holomorphic coherent intertwiners $|| \{ z_e \} \rangle$, this determinant is trivial due to the invariance
of those states under SL(2, $\mathbb{C}$) [12, 15].

Now that we have consistently defined coherent intertwiner states and describe how they
provide a decomposition of the identity on the intertwiner space, we will first explain how to
 glue them together to build coherent spin network states, and then we will show how to use
them in order to solve the holomorphic simplicity constraints at the quantum level.

### 3.4. Coherent spin network states

Let us generalize the previous construction of coherent intertwiners to full coherent spin
network states on an arbitrary graph $\Gamma$. We start with the classical spinor networks on
the graph $\Gamma$ and consider the corresponding phase space parameterized by the spinors $z^e_v$
satisfying the closure constraints $\sum_{e \in \partial v} V (z^e_v) = 0$ at every vertex and the gluing constraints
$V(\{ z^e_v \}) = \overline{V}(\{ z^e_v \})$ on every edge, and invariant under the symmetries generated by those
constraints, thus SU(2)-transformations at every vertex and U(1)-transformations on every
dge, in short, our classical phase space on the graph $\Gamma$

$$(\mathbb{C}^4)^E / (\text{SU}(2)^V \times \text{U}(1)^E),$$

where $V$ and $E$ are, respectively, the number of vertices and edges of $\Gamma$. The symbol //
denotes the symplectic reduction. It amounts to both imposing the constraints and considering
the orbits under the corresponding group action.

The standard spin network states provide an orthonormal basis of quantum state on this
phase space. And we now would like to define coherent spin network states, which are peaked
around each point of this phase space. Starting with a set of spinors on $\Gamma$, $\{ z^e_v \}$ satisfying the
closure and gluing constraints, we will use the previous definition of coherent intertwiners to
define a coherent state on the whole $\Gamma$ graph labeled by this set of spinors.

To this purpose, we follow a straightforward logic and associate the coherent intertwiner $|| \{ z^e_v \} \rangle$ with each vertex $v \in \Gamma$. Then we simply define our coherent quantum state on $\Gamma$ as the
tensor product of these intertwiner states:

$$\psi(\{ z^e_v \}) = \bigotimes_v || \{ z^e_v \} \rangle \rangle_{\forall v}. \quad (74)$$

These states are usually defined through their evaluation on group elements $\{ g_e \} \in \text{SU}(2)^E$:

$$\psi(\{ z^e_v \})(g_e) = \text{Tr}_e \bigotimes_e g_e \otimes \bigotimes_v || \{ z^e_v \} \rangle \rangle_{\forall v}. \quad (75)$$

To better understand the meaning of this evaluation on SU(2) group elements, we first expand
our coherent intertwiners onto the LS coherent intertwiners. This re-introduces explicitly the
SU(2) representation labels $j_e$. A subtlety is that this a priori leads to two spin labels $j^{(\nu)}_e$ and
$j^{(\nu)}_e$ per edge for each of the two intertwiners living at the source and target vertices of the
dge. However, since we evaluate this expression on SU(2) group elements $g_e$, these two
SU(2) representation labels must necessarily match \( j^\text{left}_e = j^\text{right}_e \) and the evaluation is given in terms of a single spin \( j_e \) per edge of the graph\(^{11}\).

Thus, expanding the previous expression, we get

\[
\psi_{\{z\}_1}(g_e) = \sum_{\{j\}_0} \frac{1}{(2j_e)!} \text{Tr}_e \bigotimes_{e} D^j_e(g_e) \otimes \bigotimes_{v} \{j_v, z_v\}_{e \in \partial v} 
\]

\[
= \int [dh_e] \sum_{\{j\}_0} \prod_{e} \frac{1}{(2j_e)!} [j_e z^e(e)| h^{-1}_{e(e)} g_e h_{e(e)} | j_e z^e(e)] 
\]

where we have as before \(|j, z\rangle = |j, z^j\rangle = |z|^{2j}\). In this form, it is clear that this coherent state functional \(\psi_{\{z\}_1}(g_e)\) is SU(2)-invariant at every vertex \(v\) and is fully holomorphic in the spinor labels \(z^e\).

Moreover, this expression is clearly invariant under U(1)-transformations on the spinors on each edge:

\[
z^e(e) \rightarrow e^{i\theta} z^e(e), \quad z^j(e) \rightarrow e^{-i\theta} z^j(e),
\]

and under SU(2)-transformations on the spinors at each vertex since the coherent intertwiner states were themselves invariant. Thus, our coherent spin network states \(\psi_{\{z\}_1}\) are truly labeled by points (sets of spinors or spinor networks) in our constrained phase space \((C^4)^E / (SU(2)^V \times U(1)^E)\).

Obviously, we have a sum over SU(2) spins \(j_e\) and those are not fixed. We would like to point out that their distribution is not only fixed by the factor \(1/(2j_e)!\) but also by the norm factors coming from the fact that the spinors are not normalized.

Here, we have expanded our coherent spin network functional as a sum over spin labels \(j_e\) because this is the usual way to discuss spin network states. However, in our case, the sum over the labels \(j_e\) is straightforward due to the specific form of the coherent intertwiners and we have

\[
\psi_{\{z\}_1}(g_e) = \int [dh_e] e^{\sum_{e} z^e(e) h^{-1}_{e(e)} g_e h_{e(e)} z^e(e)} 
\]

where the matrix elements are all simply taken in the fundamental \(2 \times 2\) representation.

### 3.5. Solving the holomorphic simplicity constraints

Now that we have reviewed the quantization of the classical spinor phase space for SU(2) intertwiners and spin networks, we can apply the construction of coherent intertwiners to the Spin(4) case and define simple coherent intertwiners satisfying the holomorphic simplicity constraints.

As we have already seen earlier in section 2.2, since \(\text{Spin}(4) = SU_L(2) \times SU_R(2)\) factorizes exactly as the product of its two SU(2) subgroups, the classical phase space for Spin(4) intertwiners (and more generally spin networks) is just the tensor product of two copies of the classical phase space for SU(2) intertwiners (and spin networks). As a direct consequence, the quantization is straightforward: we define Spin(4) intertwiners as tensor products of two SU(2) intertwiners (one for the left SU(2) subgroup and the other for the

\(^{11}\) A natural extension of the evaluation on SU(2) group elements is the evaluation on SL(2, \(\mathbb{C}\)) group elements. Indeed, as is well known (e.g. see the appendix of \([13]\)), the Schwinger representation of SU(2) in terms of two harmonic oscillators also carries a unitary SL(2, \(\mathbb{C}\)) representation. Then, we can evaluate our coherent spin network states on SL(2, \(\mathbb{C}\)) group elements, in which case the matching of the spins \(j^\text{left}_e\) and \(j^\text{right}_e\) on each edge will be relaxed and we will have to describe the SL(2, \(\mathbb{C}\))-evaluation keeping these two spin labels per edge.

This natural extension to the evaluation on SL(2, \(\mathbb{C}\)) suggests a link between our coherent spin network states and the complexifier coherent states introduced earlier by Thiemann et al \([23]\). This should be investigated later.
right copy) and we obtain coherent Spin(4) intertwiners as tensor products of two coherent intertwiners:

\[ ||\{J^L e, z^L\}|| \otimes ||\{J^R e, z^R\}|| \quad \text{and} \quad ||\{J^L e, z^L\}|| \otimes ||\{J^R e, z^R\}|| \quad \text{and} \quad ||\{J^L e, z^L\}|| \otimes ||\{J^R e, z^R\}||.\]

depending on the type of coherent intertwiners which we consider.

We are interested in writing the simplicity constraints at the quantum level and solving them using coherent intertwiner states. As we have seen in section 2.3, the holomorphic simplicity constraints read at the classical level as

\[ F^L_{ef} - \rho^2 F^R_{ef} = 0, \quad \forall e \neq f. \]

We define the quantum simplicity constraints as the direct quantization of these classical constraints:

\[ \hat{F}^L_{ef} - \rho^2 \hat{F}^R_{ef} = 0, \quad \forall e \neq f. \]  \hspace{1cm} (79)

It is easy to check that these constraints all commute with each other since the \( \hat{F} \)-operators only involve annihilation operators (which comes from the fact that the \( \hat{F} \)-observables are all holomorphic at the classical level). We call \( \mathcal{H}^{\text{simple}} \) the Hilbert space of intertwiners solving these constraints. Since our coherent intertwiners diagonalize the \( \hat{F}_{ef} \) operators, it is direct to give an overcomplete basis of simple coherent intertwiners for \( \mathcal{H}^{\text{simple}} \):

\[ ||\{z_L\}| \equiv ||\{\rho z_L\}_L \otimes ||\{z_R\}|_R, \quad (\hat{F}^L_{ef} - \rho^2 \hat{F}^R_{ef})||\{z_L\}|_\rho = 0, \]  \hspace{1cm} (80)

where the simplicity constraints are solved since \( \hat{z}^L = \rho \hat{z}^R \) and thus \( \hat{z}^L \hat{z}^L = \rho^2 \hat{z}^R \hat{z}^R \).

We have left the indices \( L \) and \( R \) on the quantum states to keep note of which intertwiner corresponds to \( \text{SU}_L(2) \) or to \( \text{SU}_R(2) \).

We can give the expectation values of the \( \hat{E}_{ef} \) operators, which measures the scalar product between the spinors on the legs \( e \) and \( f \) of the intertwiners, using the formula (61) for \( \lambda = 1 \):

\[
\frac{\rho ||\{z_L\}| \hat{E}^L_{ef} ||\{z_L\}|_\rho}{\rho ||\{z_L\}|_\rho} = \rho^2 \langle z_L | z_f \rangle I_3 (2 \rho^2 A(z)) I_1 (2 \rho^2 A(z)) \]
\[
= \rho^2 \frac{\rho ||\{z_L\}| \hat{E}^R_{ef} ||\{z_L\}|_\rho}{\rho ||\{z_L\}|_\rho} I_3 (2 \rho^2 A(z)) I_1 (2 \rho^2 A(z)) I_2 (2 A(z)) , \]
\hspace{1cm} (81)

where the last factor quickly converges to 1 when the area \( A(z) \) grows large. For the expectation values of further operators such as the scalar product operators, the interested reader can find more expressions in [12].

In the following, we are going to use these new simple intertwiners, which solve the holomorphic simplicity constraints, to build simple coherent spin network states for the Spin(4) gauge group and to construct a new spinfoam model for (Riemannian) quantum gravity.

3.6. \( U(N) \)-action on simple intertwiners

A very interesting property of these simple coherent intertwiners is that they are covariant under a \( U(N) \) action, which can thus be used to deform them from one to another. This comes directly from the property of the SU(2) coherent intertwiners themselves (51):

\[
\hat{U} ||\{z_L\}|_\rho = ||\{U z_L\}|_\rho, \quad U = e^{i \omega},
\]
\[
\hat{U} = e^{i \sum_{\alpha} \alpha_i (\hat{E}^L_{ef} + \hat{E}^R_{ef})} = e^{i \sum_{\alpha} \alpha_i \hat{E}^L_{fL} + \sum_{\alpha} \alpha_i \hat{E}^R_{fL}} .
\]  \hspace{1cm} (82)

The generators of these \( U(N) \)-transformations are the operators \( \hat{E}^L_{ef} + \hat{E}^R_{ef} \).
It is actually interesting to check that the whole Hilbert space of simple intertwiners $\mathcal{H}_{\text{simple}}$ is invariant under this $U(N)$-action. Indeed, we compute the commutator of the $u(N)$-generators with the simplicity constraints

$$[F_{e\ell}^L + \rho^L \overline{F}_{e\ell}^L, \overline{F}_{gh}^R - \rho^R \overline{F}_{gh}^R] = \delta_{gh}(F_{f\ell}^L - \rho^L \overline{F}_{f\ell}^L) - \delta_{\ell f}(F_{f\ell}^L - \rho^L \overline{F}_{f\ell}^L).$$

Thus, if we start with a state $\psi$ satisfying the holomorphic simplicity constraints $(F_{gh}^L - \rho^L \overline{F}_{gh}^L)\psi = 0$ for all pairs of legs $g, h$, and act on it with an operator $(F_{e\ell}^L + \rho^L \overline{F}_{e\ell}^L)$ for an arbitrary pair of legs $e, f$, then the resulting state also satisfies the simplicity constraints for all pairs of legs. This shows that the space of simple intertwiners $\mathcal{H}_{\text{simple}}$ carries a $U(N)$-representation.

We hope that these $U(N)$ transformations will later become useful to deform spin network states and study discrete diffeomorphisms of spinfoam amplitudes.

### 3.7. Simple spin network states

Similarly to how we build an overcomplete basis of coherent spin network states for $SU(2)$ by assigning a coherent intertwiner state at each vertex of the graph, we can construct a basis of coherent spin network states for $Spin(4)$. Here, we would like to focus on the construction of a basis of simple spin network states made of simple intertwiners which satisfy the holomorphic simplicity constraints at every vertex of the graph:

$$\rho \psi_{\{\ell\}} = \bigotimes_v \{z_v^e\}_{e\ell|v}. \quad (84)$$

These states are defined through their evaluation on group elements $\{G_e = (g_e^L, g_e^R)\} \in Spin(4)$:

$$\rho \psi_{\{\ell\}}(G_e) = \psi_{\{\ell\}}(g_e^L, g_e^R) = \text{Tr}_e \bigotimes_v (g_e^L \otimes g_e^R) \otimes \bigotimes_v \{\rho z_v^e\}_{e\ell|v} \otimes \{z_v^e\}_{e\ell|v}. \quad (85)$$

$$= \sum_{\{f\}} \prod_{e} \left(2j_e^L\right)! \left(2j_e^R\right) ! \text{Tr}_e D_{e}^L(g_e^L) D_{e}^R(g_e^R) \bigotimes \bigotimes_v \{\rho z_v^e\}_{e\ell|v} \otimes \{z_v^e\}_{e\ell|v}$$

$$= \int [d\rho^L_R] \sum_{\{f\}} \prod_{e} \left(2j_e^L\right)! \left(2j_e^R\right) ! \left[\rho e^{\ell(e)(1)} h_{\ell(e)}^L h_{\ell(e)}^L \right]^{j_e^L} \rho z_v^e \bigotimes_v \{\rho z_v^e\}_{e\ell|v} \otimes \{z_v^e\}_{e\ell|v}$$

$$= \int [d\rho^L_R] \sum_{\{f\}} \prod_{e} \rho e^{\ell(e)(1)} h_{\ell(e)}^L h_{\ell(e)}^L \left[\rho e^{\ell(e)(1)} h_{\ell(e)}^L h_{\ell(e)}^L \right]^{j_e^L} \rho z_v^e \bigotimes_v \{\rho z_v^e\}_{e\ell|v} \otimes \{z_v^e\}_{e\ell|v}$$

$$= \int [d\rho^L_R] e^{\sum_{e} \rho e^{\ell(e)(1)} h_{\ell(e)}^L h_{\ell(e)}^L \left[\rho e^{\ell(e)(1)} h_{\ell(e)}^L h_{\ell(e)}^L \right]^{j_e^L} \rho z_v^e \bigotimes_v \{\rho z_v^e\}_{e\ell|v} \otimes \{z_v^e\}_{e\ell|v}}. \quad (86)$$

The big difference of the present proposal with the coherent intertwiner approach to the EPRL–FK spinfoam models [8–10] based on the LS coherent intertwiner states [6] is that the EPRL–FK ansatz imposes strongly the diagonal simplicity constraints, i.e. $j_e^L = \rho j_e^R$. This reduces our double sum over both $j_e^L$ and $j_e^R$ to a single sum over only the right (or left) spin labels. However, in our framework, we do not need to enforce the diagonal simplicity constraints by hand since, first, the holomorphic simplicity constraint implies the diagonal simplicity at the classical level, and second, the spin labels are actually still peaked on the relation $j_e^L = \rho j_e^R$ but simply have a non-trivial spread around it.
4. A new spinfoam model

Since we have introduced a new formulation for the simplicity constraints both at the classical and quantum levels and defined new coherent intertwiners and coherent spin networks that solved these holomorphic simplicity constraints, we would like to propose a new spinfoam model for 4D Riemannian quantum gravity constructed from these new coherent intertwiners. This can be considered as an improved version of the EPRL–FK spinfoam models [7, 8, 10] based on a rigorous Gupta–Bleuler resolution of the simplicity constraints. As we will see, the definition of the model will be rather simple and we hope that the asymptotical analysis at large scale will similarly be simple.

As a first step, we will start by re-writing the spinfoam path integral for 4D BF theory with SU(2) as the gauge group in terms of spinors and holomorphic coherent intertwiners. This is similar to the procedure which was started in [6] to write the discretized BF path integral in terms of the LS coherent intertwiners before imposing the simplicity constraints on the coherent intertwiners to derive the EPRL–FK spinfoam models [8, 9]. Then, our second step will be to impose our holomorphic simplicity constraints and define our new proposal for a spinfoam model.

4.1. BF theory in terms of spinors?

Let us start by trying to define the topological spinfoam model for BF theory with the gauge group SU(2). We will not review the spinfoam program here and we will assume that the reader is familiar with the spinfoam framework and structures. For reviews, the interested reader can refer to [23].

We will restrict ourselves for simplicity’s sake to simplicial triangulations of the 4D spacetime manifold with 4-simplices glued together along tetrahedra. Nevertheless, it is straightforward and obvious to generalize it to arbitrary cellular decomposition by considering the coherent intertwiners and the decomposition of the identity on the space of \( N \)-valent intertwiners for arbitrary \( N \), and by defining the spinfoam amplitudes from the evaluation of the boundary spin networks for each 4-cell.

Thus, let us consider a 4D triangulation made of 4-simplices. Spinfoams are made of two ingredients: a vertex amplitude which defines the local amplitude for the geometry of each 4-cell and an edge amplitude which defines how to glue the 4-cells together. The natural ans¨atze are as follows.

- A vertex amplitude attached to each 4-simplex.

It defines the probability amplitude of the geometry of the 4-simplex. It is given by the evaluation of boundary spin network of the 4-simplex on the identity group element. The 4-simplex boundary graph is made of five nodes fully connected to each other, as shown in figure 4. Each of those nodes corresponds to a tetrahedron of the 4-simplex. To each of the nodes or intertwiners, we attach a coherent intertwiner and build the resulting coherent spin network living on the 4-simplex boundary. Finally evaluating this spin network functional at the identity provides us with a amplitude depending on \( 2 \times 10 \) spinors:

\[
\mathcal{A}_\sigma (z^\tau_\Delta, t^\tau_\Delta) = \psi_{\{z^\tau_\Delta\}}(I) = \text{Tr} \bigotimes_{\tau} \langle |z^\tau_\Delta| \rangle = \int [dh_\tau]^5 e^{\sum_{\Delta} \iota_{\Delta} b_{\Delta}^* \pi_{\Delta} (t^\tau_\Delta) b_{\Delta} (s^\tau_\Delta) |z^\tau_\Delta\rangle}, \tag{87}
\]

where \( \sigma \) denotes the 4-simplex, \( \Delta \) and \( \tau \) respectively label the ten triangles and five tetrahedra of \( \sigma \), \( s(\Delta) \) and \( t(\Delta) \), respectively, denote the source and target tetrahedra sharing the triangle \( \Delta \) as depicted in figure 4. This vertex amplitude is completely holomorphic in the spinor variables \( z^\tau_\Delta \).
An edge amplitude describing the gluing of the 4-simplices.

Each tetrahedron $\tau$ of the 4D triangulation glues two 4-simplices together. In each of these 4-simplices, an intertwiner is associated with this tetrahedron. For BF theory, the standard ansatz for the gluing is to insert the identity on the intertwiner space between the two 4-simplices. Using the decomposition of the identity in terms of coherent intertwiners, we have

$$I^{\sigma}_{\tau} = \frac{1}{\pi^2} \int [d^4 z_\Delta] e^{-\sum_{\Delta_1 \in \tau} \langle z_\Delta | z_\Delta \rangle \prod_{\sigma_1} \int d h_{\sigma_1} e^{\sum_{\Delta_1 = \sigma_1 \Delta} \epsilon_{\sigma_1} (| z^{\sigma_1}_{\Delta_1} | z^{\sigma_1}_{\Delta_1} )' - 1} | z^{\sigma_1}_{\Delta_1} | z^{\sigma_1}_{\Delta_1} \rangle \langle z^{\sigma_1}_{\Delta_1} | z^{\sigma_1}_{\Delta_1} \rangle}, \quad (88)$$

where the sign $\epsilon_{\sigma_1}$ is equal to 1 if the 4-simplex $\sigma$ is the source for the tetrahedron $\tau$ and is 0 if $\sigma = T(\tau)$.

This partition function is simply derived by associating with each tetrahedron a coherent intertwiner labeled by the appropriate spinors and gluing these intertwiners within each 4-simplex. A very interesting property of this spinfoam model is that it is directly defined by a discrete action principle. It could be interesting to compare it with other discrete action principles proposed for spinfoam models. Here, we will focus on showing that this partition function correctly defines the discrete path integral for topological BF theory.

We focus on a plaquette, dual to a given triangle $\Delta$. For simplicity’s sake, we assume that it is consistently oriented all around the plaquette, as shown in figure 4:

$$S(\tau_i) = \sigma_i, \quad T(\tau_i) = \sigma_{i+1},$$

and in the 4-simplex $\sigma_i$:

$$s(\Delta) = \tau_{i-1}, \quad t(\Delta) = \tau_i,$$

with the obvious identification $(n + 1) \equiv 1$. Dropping the subscript $\Delta$, the terms of the action involving the spinors all around the plaquettes are (taking care with the signs and relative...
orientations)
\[
\sum_{\tau=1}^{n} \left| z^{\tau-1} (h_{r_{\tau}}^{\dagger})^{-1} h_{r_{\tau}}^i z^{\tau} \right| = \sum_{\tau=1}^{n} \left( z^{\tau} (h_{r_{\tau}}^{\dagger})^{-1} h_{r_{\tau}}^i \right) z^{\tau-1}.
\]  
(89)

Then, we can perform the integration over the spinor variables keeping in mind that each spinor \( z \) enters the action twice, once as \( |z| \) and once as \( \langle z \rangle \). We only need the following Gaussian integral\(^{12}\):

\[
\frac{1}{\pi^2} \int d^4 z e^{-\langle z | z \rangle} = \pi^2 |w| \langle w | w \rangle.
\]  
(90)

Applying this to the integrals around the plaquette, we get

\[
\frac{1}{\pi^2} \int d^4 z_1 e^{-\langle z_1 | z_1 \rangle} = e^{\langle w | w \rangle}.
\]

Calling \( g_r = h_{r_{\tau}}^{\dagger} (h_{r_{\tau}}^{\dagger})^{-1} \) or more generally \( g_r = h_{r_{\tau}}^{\dagger} (h_{r_{\tau}}^{\dagger})^{-1} \), we can integrate this and finally obtain

\[
\prod_{i} \frac{e^{-\langle z_i | z_i \rangle}}{\pi^2} d^4 z = \frac{1}{\pi^2} \int d^4 z e^{-\langle z | z \rangle} = \int d^4 G e^{\langle G | z \rangle} \quad \text{with} \quad G = g_n \ldots g_1.
\]  
(91)

We can evaluate this last integral by expanding it into irreps of SU(2)\(^{13}\):

\[
\frac{1}{\pi^2} \int d^4 z e^{-\langle z | z \rangle} e^{\langle G | z \rangle} = \sum_j \frac{1}{\pi^2 (2j)!} \int d^4 z e^{-\langle z | z \rangle} (z | G | z)^{2j} = \sum_j \chi_j(G) = \tilde{\delta}(G),
\]  
where we used the decomposition (72) of the identity on \( V^j \) in terms of coherent states. The character \( \chi_j(G) \) is by definition the trace of the matrix representing the group element \( G \) in

\(^{12}\) We can either compute the Gaussian integral explicitly or use the decomposition (72) of the identity on \( V^j \) in terms of coherent states:

\[
\frac{1}{\pi^2} \int d^4 z e^{-\langle z | z \rangle} e^{\langle w | z \rangle} = \frac{1}{\pi^2} \int d^4 z e^{-\langle z | z \rangle} \sum_j \frac{(w | z)^{2j}}{(2j)!} \left( \frac{z | w}{2j} \right)^{2j} \int d^4 z e^{-\langle z | z \rangle} (j, w | z) = \chi_j(G).
\]

\(^{13}\) We can change variables from the spinor \( z \) to the 3-vector \( \vec{V} \), which would allow us to see the relation between the modes \( e^{i\theta V} \) and the usual more functional \( e^{i\theta \vec{V}} \) usually used in spinfoam constructions [27, 28]. Thus, using the change of integration measure derived in (A6), we get

\[
\frac{1}{\pi^4} \int d^4 V e^{-\langle \vec{V} | \vec{V} \rangle} e^{i\theta V} = \frac{1}{4\pi} \int \frac{d^4 \vec{V}}{|\vec{V}|} e^{-\langle \vec{V} | \vec{V} \rangle} e^{i\theta \vec{V}} \tilde{\delta}(G) \quad \text{Tr}G.
\]

Introducing the parametrization of the group element \( G \) as \( G = \cos \theta \hat{1} + i \sin \theta \hat{a} \cdot \hat{a} \) in terms of the class angle \( \theta \in [0, 2\pi] \) and the rotation axis \( \hat{a} \in S^2 \), we can separate the integration over the radius \( |V| \) and the integration over the angular part \( d^\theta \vec{V} \) and we get

\[
\frac{1}{\pi^4} \int d^3 \vec{V} e^{-\langle \vec{V} | \vec{V} \rangle} e^{i\theta V} = 0 \int d^\theta \sin \theta \sin \theta \sin \theta \sin \theta = \frac{1}{2(1 - \cos \theta)}.
\]

where the last equality is a standard integral. This can be compared to the formula for the \( \tilde{\delta} \)-distribution as a sum over spin labels, if we compute the sum as a mere geometrical series:

\[
\sum_j \chi_j(G) = \sum_{n \in \mathbb{N}} \sin(n + 1) \frac{\theta}{\sin \theta} = \frac{1}{2i \sin \theta} \left( \sum_{n \geq 1} e^{in\theta} - \sum_{n \geq 1} e^{-in\theta} \right) = \frac{\cos \frac{\theta}{2}}{2 \sin \theta \sin \theta} = \frac{1}{2(1 - \cos \theta)}.
\]
the representation of spin $j$. This distribution $\widetilde{\delta}(G)$ should be compared to the $\delta$-distribution on $SU(2)$:

$$\delta(G) = \sum_j (2j + 1)\chi_j(G) = \frac{1}{\pi^2} \int d^4z e^{-\langle z|z \rangle} \sum_j (2j + 1) (\langle z|G|z \rangle)^{2j}$$

$$= \frac{1}{\pi^2} \int d^4z e^{-\langle z|z \rangle} (1 + \langle z|G|z \rangle) e^{\langle z|G|z \rangle}.$$  (93)

It is the factor $d_j = (2j + 1) = \dim \mathcal{V}^j$ that messes up the relation with the topological BF theory.

Indeed, performing all the integrations over the spinor variables and doing a change of variables from the $h^*_r$ to the $g_r$, our spinfoam partition function in terms of coherent intertwiners is very similar to the standard discretized path integral for BF theory:

$$Z[\mathcal{M}] = \int \prod_r [d_{\mathcal{H}_3}] \prod_\Delta \widetilde{\delta} \left( \prod_j \tau_3 \Delta r \right),$$  (94)

but the difference resides in the fact that $\widetilde{\delta}$ is not the $\delta$-distribution on $SU(2)$. For instance, $\widetilde{\delta}$ is not stable under convolution and, as a consequence, the partition function $Z[\mathcal{M}]$ is not topological. Assuming that we have not made any mistake in the normalization of the coherent states or in the measures of integration over the spinors or in the decomposition of the identity on the intertwiner space, we see a few possibilities to fix the issue of the $(2j + 1)$-factor and recover BF theory.

- For each triangle, we can insert by hand the observable $(2j + 1)$ on the link of a 4-simplex around the plaquette. This is done by inserting the operator $(E_{ee} + 1)$ in the path integral where $e$ stands for the link corresponding to the triangle $\Delta$ within the chosen 4-simplex. The operator $E_{ee}$ is a differential operator in the relevant spinor $z$, which is simply $\langle z|\partial_z \rangle$. This method is straightforward to implement and gives the desired result. Nevertheless, it does not help us to understand where the $(2j + 1)$-factor comes from.

- It seems that the $(2j + 1)$-factor is the factor that enters the orthonormality of the matrix elements $D^j(h)$ of the Peter–Weyl theorem for functions in $L^2(SU(2))$. This would mean that we have to modify our edge amplitude and that we should not insert directly the identity on the intertwiner space $\mathbb{1}_{\mathcal{H}_3}$, but maybe insert a decomposition of the identity on $L^2(SU(2)^{\times 4})$ instead. We have not yet investigated how this can be implemented in terms of the spinor variables and there does not seem to be a natural alternative to the insertion of $\mathbb{1}_{\mathcal{H}_3}$ between 4-cells.

- Putting aside the coherent states and intertwiners and focusing on the discrete path integral defined in terms of spinors, another possibility is to modify the terms in the action $e^{\langle \overline{g}|g \rangle}$ to $e^{\langle \overline{g}|g \rangle + 1}$ as suggested by the decomposition of $\delta(g)$ as an integral over spinors. This actually simply amounts to the insertion of a $(2j + 1)$-factor or equivalently of the operator $(E_{ee} + 1)$ on the corresponding wedge:

$$e^{\langle \overline{g}|g \rangle} = \sum_j \frac{(\langle \overline{g}|g \rangle)^{2j}}{(2j)!}$$

$$\rightarrow \quad (\langle \overline{g}|g \rangle + 1) e^{\langle \overline{g}|g \rangle} = \sum_j \frac{(\langle \overline{g}|g \rangle + 1)^{2j}}{(2j)!}.$$  

The problem is that the ‘convolution’ property of these modes is not nice. Indeed the equivalent of (90) is now

$$\frac{1}{\pi^2} \int d^4z e^{-\langle z|z \rangle} (\langle w|z \rangle + 1) e^{\langle w|z \rangle} (\langle z|w \rangle + 1) e^{\langle z|\overline{w} \rangle} = (\langle w|\overline{w} \rangle)^2 + 3 (\langle w|\overline{w} \rangle + 1) e^{\langle w|\overline{w} \rangle},$$

and the power of the factor in front of the exponential will increase as we integrate over the spinor variables around the plaquette. This simply means that we pick up extra $(2j + 1)$-factors as we perform the integrations around the plaquette.
A way out is to insert the factor \((|w|z) + 1\) only once around the plaquette. Indeed we do have
\[
\frac{1}{\pi^2} \int d^4 z e^{-[(|w|z) + 1]} e^{(|w|z) + 1} e^{(|w|\tilde{w})} = (|w|\tilde{w}) + 1\).
\]
This means choosing one ‘origin’ 4-simplex for the plaquette and inserting that factor there. This is exactly equivalent to the insertion of a \((2j + 1)\)-factor (or of the operator \((E_{ee} + 1)\)) on the corresponding wedge of the plaquette, which was our first proposed solution!

Another way out would be to introduce a \(\star\)-product, which would deform the multiplication between modes so that they remain stable under convolution:
\[
\frac{1}{\pi^2} \int d^4 z e^{-[(|w|z) + 1]} e^{(|w|z) + 1} \star \left[ (|z|\tilde{w}) + 1 \right] e^{(|z|\tilde{w})} = (|w|\tilde{w}) + 1\).
\]
This is very similar to what happens when writing the discrete action principle for BF theory in terms of local terms \([24–26]\). Indeed, it turns out to be useful and more convenient to define the discretized path integral using the \(\star\)-product on \(\mathbb{R}^3\) dual to the convolution product on \(SU(2)\) [28]. It would actually be interesting to compare that \(\star\)-product previously introduced to the new \(\star\)-product between functions over spinors that we need here.

Finally, instead of introducing a \(\star\)-product, maybe a suitable change of integration measure over the spinors could allow us to realize the same procedure.

Finally, it seems that the most straightforward method to truly write the spinfoam amplitudes for BF theory is to insert by hand a factor \((|\tilde{z}|g|z) + 1\) on one wedge (i.e. 4-simplex) of each plaquette. We can choose the ‘origin’ ‘for the plaquette for instance at \(i = 1\). And this insertion simply amounts to the insertion of the operator \((E_{ee} + 1)\), which produces the required factor \(d_j = (2j + 1)\) to turn the distribution \(\delta(G)\) into \(\delta(G)\). That way, we do recover an exact discretization of the path integral for the topological BF theory.

We will investigate the other possibility of using a \(\star\)-product in the future and see if there is a way to write the exact discretized BF path integral in terms of the holomorphic coherent intertwiners without the insertions discussed above.

We compare our new coherent state approach with the more standard method of expanding spinfoam amplitudes as sums over discrete spin labels. Besides the obvious disadvantage that the most natural spinfoam ansatz presented here does not exactly reproduce the topological partition function for \(SU(2)\) BF theory, it still has some promising aspects.

- Even if the most natural ansatz in our framework does not lead to the spinfoam amplitudes for BF theory, a slight modification with suitable (simple) observable insertions does allow us to recover the proper partition function.
- The path integral is directly expressed in terms of coherent states and coherent intertwiners, which should simplify the study of the semi-classical limit.
- We have exchanged the sum over spin labels, with integrals over complex variables. The path integral defined through integrals over coherent intertwiners can be directly written as a discretized action principle. This should simplify the study of the large scale asymptotics and the (semi-)classical regime of the amplitudes.
- It is possible to expand explicitly the coherent intertwiners as sums over spin labels through the exact formulas given in the earlier sections. These sums are more intricate than usual because spin labels are not \(a priori\) forced to be the same around a plaquette: given a triangle \(\Delta\), we would have one spin \(\rho_\sigma\) for each 4-simplex \(\sigma \ni \Delta\), i.e. for each wedge of the plaquette. It is the integrals over the spinors which allow us to identify (or not) the wedge spins around the same plaquette.
4.2. A new holomorphic vertex amplitude

Now that we have reformulated the spinfoam partition function for topological BF theory in terms of coherent intertwiners and spinors, we can propose our spinfoam model for 4D gravity with Riemannian signature.

To start with, we focus on the vertex amplitude associated with 4-cells of the triangulated manifold. In general, we will put a simple spin network on the boundary graph of the 4-cell, with coherent intertwiners solving the (holomorphic) simplicity constraints on the nodes and we will define the vertex amplitude as the evaluation of the boundary spin network. For the sake of notational simplicity, we will focus on a simplicial triangulation made out of 4-simplices.

Then, the boundary spin network of the 4-simplex is labeled by $2 \times 10$ spinors living on each triangle in each tetrahedron, just as in the case of the pure SU(2) BF theory. The vertex amplitude for a 4-simplex $\sigma$ is then

$$\rho A_\sigma(z_{A/\Delta 1}) = \rho \psi^{\{z_{\tau/\Delta 1}/\Delta 1\}}(I) = \psi^{\{z_{\tau/\Delta 1}/\Delta 1\}}(I) \rho = \int [d h_{\tau}]^5 e^{\sum_{A=1}^{5} \rho \{f^{\{z_{\tau/\Delta 1}/\Delta 1\}}(I) - f^{\{z_{\tau/\Delta 1}/\Delta 1\}}(I)\}}. \quad (95)$$

Gluing these vertex amplitudes with the decomposition of the identity on the intertwiner space, we obtain the full spinfoam amplitude for a 4D triangulation, being careful of the relative orientations of tetrahedra and 4-simplices as in the previous section. This automatically provides us with a spinfoam amplitude given by the integrals over spinors $z_{\tau/\Delta 1}$ and auxiliary group elements $h_{\tau}^{L,R}$ of a discrete Lagrangian. It will be very interesting in the future to compare this action principle with the other proposed discretized action for general relativity as a constrained BF theory [24, 25].

The arbitrariness in our construction is the gluing of the vertex amplitude into a full spinfoam associated with the whole triangulation. Here, we chose the natural ansatz from the perspective of our spinorial construction, which is given by the insertion of the identity on the intertwiner space. However, as we have seen earlier, this is not the choice of edge amplitude that allows the recovery of the spinfoam amplitudes for topological BF theory. Nevertheless, for spinfoam models which are not topological invariant, the edge amplitude is a priori not fixed and should be kept as an ambiguity in the definition of the model. It would be fixed a posteriori by the identification of a symmetry of the discrete partition function (such as discrete diffeomorphisms) and could change under coarse-graining (renormalization flow).

5. Conclusion and outlook

Using the recently developed formulation of SU(2) group elements and functionals over SU(2) in terms of spinors, we have discussed the simplicity constraints (with the Immirzi parameter) for discretized Riemannian 4D quantum gravity. Following the approach started in [12], we have introduced a new set of holomorphic simplicity constraints. We have shown their equivalence at the classical level with the standard simplicity constraints. Then, we have shown how to solve them using new coherent intertwiners, which diagonalize the annihilation operators of the U(N) formalism for SU(2) intertwiners [12, 14, 15]. This truly realizes a quantization in the manner of Gupta–Bleuler. Finally, we have explained how to glue these coherent intertwiners into coherent spin network states and defined a new spinfoam model for discretized Riemannian 4D quantum gravity whose boundary states solve the holomorphic simplicity constraints and whose amplitudes are given by the evaluation of the new coherent spin network states.
This new spinfoam model is formulated without reference to spin labels but directly through a discrete action principle and integrals over spinor variables. The diagonal simplicity constraints are not strongly enforced and we are no more restricted to simple irreducible representations of Spin(4). A possible side effect is that this might allow a more detailed discussion of the possible renormalization and running of the Immirzi parameter in this spinfoam model.

Since there exist other proposals for coherent intertwiner states as we have reviewed in section 3.2 and that surely one could always come up with another definition of semi-classical states for spin networks, we would like to insist on the reasons leading to the coherent intertwiners $$|\{|z_e\} \rangle$$ which we introduced. First, they are labeled exactly by points in the classical phase space, i.e. spinor variables $$z_e$$ satisfying the closure constraints and up to SU(2) gauge transformations. They are moreover semi-classical states peaked on these classical phase space points. Second, they are eigenstates of the annihilation operators $$F_{ef}$$ acting on the space of $$N$$-legged intertwiners. We did not know of any coherent intertwiner state satisfying one of these criteria before this work. Third, the new coherent intertwiners transform covariantly under the action of $$\text{U}(N)$$, which describes deformations of the polyhedron/intertwiner at fixed total boundary area, which could be an advantage when investigating the action of discrete diffeomorphisms on intertwiners and spin network states.

Our new coherent intertwiners and the resulting new spinfoam model naturally open the door to various questions.

- We should compare our new discrete Lagrangian to the other proposals for discretized Riemannian 4D quantum gravity.
- We could study the asymptotics at large scale (large area) of the vertex amplitude of our new model and compare it to the asymptotics formula of the EPRL–FK models [29]. It would provide us with a first check that the semi-classical behavior of our model is correct.
- It would be interesting to see if the $$\text{U}(N)$$ covariance of the coherent intertwiners can be turned into a $$\text{U}(N)$$ symmetry for the spinfoam amplitude, in the hope of potentially understanding an action of discrete diffeomorphisms on our new spinfoam model.
- It would also be interesting to investigate if our new vertex amplitude satisfies recursion relations, which would be written as differential equations in terms of the spinors. As is understood that recursion relations are deeply linked to the topological/diffeomorphism invariance and dynamics of the spinfoam model [30], the hope is that such differential equations would reflect the dynamics and Hamiltonian constraints as was recently shown for BF theory [31].
- We should see if we can generate our new spinfoam amplitudes from a suitable group field theory.
- Since we are discussing the implementation of the simplicity constraints at the discrete level in spinfoams and we are relaxing them, it would be interesting to look at our new spinfoam models from the point of view of modified gravity theories defined from topological BF theory with relaxed simplicity constraints, such as bi-metric gravity theories as defined in [32]. Indeed, such modified gravity theories could arise at large scales from the renormalization of spinfoams.
- It is necessary to generalize our approach to the Lorentzian case and build a spinfoam model for Lorentzian 4D quantum gravity. We need to investigate if we can have similar holomorphic simplicity constraints and coherent intertwiners. This is currently under investigation [16].
- Finally, we can also investigate the application of our spinorial framework and new spinfoam model to the recently introduced spinfoam cosmology [33]. It turns out that it simplifies
both the formulation of the boundary data and the transition amplitudes, and allows us to see that the new spinfoam amplitudes satisfy a Hamiltonian constraint in this symmetry-reduced setting [34].

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Appendix A Spinors and notations

In this preliminary section, we introduce spinors and the related useful notations, following the previous works [12, 17, 19].

A.1. Spinors

Considering a spinor $z$, $|z\rangle = (z^0 \ z^1)$, $\langle z| = (\bar{z}^0 \ \bar{z}^1)$, we define its dual spinor through the duality map $\varsigma$ acting

$$\varsigma \left( \begin{array}{c} z^0 \\ z^1 \end{array} \right) = \left( \begin{array}{c} -\bar{z}^1 \\ \bar{z}^0 \end{array} \right), \quad \varsigma^2 = -1. \tag{A.1}$$

This is an anti-unitary map, $\langle \varsigma z | \varsigma w \rangle = \langle w | z \rangle = \langle z | w \rangle$, and we will write the dual spinor as $|z\rangle \equiv \varsigma |z\rangle$, $[z|w\rangle = \langle z|w\rangle$.

We associate with the spinor $z \in \mathbb{C}^2$ a 3-vector $\vec{V}(z) \in \mathbb{R}^3$ defined from the projection of the $2 \times 2$ matrix $|z\rangle\langle z|$ onto Pauli matrices $\sigma_a$ (taken Hermitian and normalized so that $(\sigma_a)^2 = \mathbb{I}$):

$$|z\rangle\langle z| = \frac{1}{2} (\langle z|z\rangle \mathbb{I} + \vec{V}(z) \cdot \vec{\sigma}). \tag{A.2}$$

The norm of this vector is $|\vec{V}(z)| = \langle z|z \rangle = |z^0|^2 + |z^1|^2$ and its components are given explicitly as

$$V^z = |z^0|^2 - |z^1|^2, \quad V^x = 2\Re(z^0 z^1), \quad V^y = 2\Im(z^0 z^1). \tag{A.3}$$

The spinor $z$ is entirely determined by the corresponding 3-vector $\vec{V}(z)$ up to a global phase. We can give the inverse map

$$z^0 = e^{i\phi} \sqrt{\frac{|\vec{V}| + V^z}{2}}, \quad z^1 = e^{i(\phi-\theta)} \sqrt{\frac{|\vec{V}|-V^z}{2}}, \quad \tan \theta = \frac{V^y}{V^x}, \quad \text{where } e^{i\phi} \text{ is an arbitrary phase.} \tag{A.4}$$

Then, the map $\varsigma$ sends the 3-vector $\vec{V}(z)$ onto its opposite:

$$|z||z\rangle = \frac{1}{2} (|z|z\rangle - \vec{V}(z) \cdot \vec{\sigma}) \tag{A.5}$$
A.1.2 Change of integration variables

Since the spinor $z$ is entirely determined by the 3-vector $\vec{V}$ and a phase $\phi$, we can compute the change of integration variable from $d^4z$ to a measure $d^3\mu(\vec{V}, \phi)$. Actually it is more interesting to consider functions of the spinor $z$ which do not depend on the phase $\phi$, for instance, functions of the matrix $|z\rangle\langle z|$. In this case, we can show that

$$\frac{1}{4\pi^2} \int d^dz e^{-i\langle z|\hat{g}|z\rangle} = \frac{1}{4\pi} \int |\vec{V}|^{-1} e^{-i\langle \vec{V}|\hat{g}|\vec{V}\rangle}.$$  

(A.6)

It is straightforward to prove by assuming that the measure on $\vec{V}$ should be invariant under 3D rotations and then evaluating it over the basis of functions $|\vec{V}|^n = (z|z)^n$ of function invariant 3D rotations.

A.1.3 Closure of $N$ spinors

Considering the setting necessary to describe intertwiners with $N$ legs, we consider $N$ spinors $z_e$ and their corresponding 3-vectors $\vec{V}(z_e)$. We require that the $N$ spinors satisfy a closure condition, i.e. that the sum of the corresponding 3-vectors vanishes, $\sum_e \vec{V}(z_e) = 0$. Coming back to the definition of the 3-vectors $\vec{V}(z_e)$, the closure condition is easily translated in terms of 2 × 2 matrices as the condition $\sum_e |z_e\rangle\langle z_e| \propto 1$:

$$\sum_e |z_e\rangle\langle z_e| = A(z)I,$$

with

$$A(z) \equiv \frac{1}{2} \sum_e \langle z_e|z_e\rangle = \frac{1}{2} \sum_e |\vec{V}(z_e)|.$$  

(A.7)

This further translates into quadratic constraints on the spinors:

$$\sum_e z_e^0 z_e^0 = 0, \quad \sum_e |z_e^0|^2 = \sum_e |z_e^1|^2 = A(z).$$

(A.8)

In simple terms, it means that the two components of the spinors, $z_e^0$ and $z_e^1$, are orthogonal $N$-vectors of equal norm.

A.1.4 Spinors and SU(2) group elements

Given two spinors, $|z\rangle$ and $|w\rangle$, there exists a unique group element $g \in U(2)$ which maps one onto the other, i.e. such that

$$g = \frac{|z\rangle}{\sqrt{\langle z|z\rangle}} = \frac{|w\rangle}{\sqrt{\langle w|w\rangle}}, \quad g^\dagger g = I.$$  

(A.9)

Its explicit expression in terms of the spinors is

$$g = \frac{|w\rangle\langle z| + |z\rangle|w\rangle}{\sqrt{\langle z|z\rangle\langle w|w\rangle}}.$$  

(A.10)

It is direct to realize that this defines an SU(2) group element by checking that $g^\dagger g = I$ and $\text{Tr}g \in \mathbb{R}$.

Appendix B Observables $E$ and $F$: Poisson brackets and commutation relations

The Poisson brackets of the SU(2) observables are

\begin{align}
\{E_{ef}, E_{gh}\} &= -i(\delta_{fh}E_{eg} - \delta_{eh}E_{gf}), \\
\{E_{ef}, F_{gh}\} &= -i(\delta_{eh}F_{fg} - \delta_{fg}F_{eh}), \\
\{E_{ef}, \bar{F}_{gh}\} &= -i(\delta_{fh}\bar{F}_{eg} - \delta_{eh}\bar{F}_{fg}), \\
\{\bar{F}_{ef}, F_{gh}\} &= -i(\delta_{eg}E_{fh} - \delta_{fg}E_{eh} - \delta_{fh}E_{eg} + \delta_{eh}E_{fg}), \\
\{\bar{F}_{ef}, \bar{F}_{gh}\} &= 0, \\
\{E_{ef}, \bar{F}_{gh}\} &= 0.
\end{align}

(B.1)
At the quantum level, these observables become $SU(2)$-invariant operators:
\[
\hat{E}_{ef} = a^+_e a_f + b^+_f b_e,
\hat{F}_{ef} = a_e b_f - b_e a_f,
\hat{F}^+_{ef} = a^+_e b^+_f - b^+_e a^+_f.
\]
They form a closed algebra, which mirrors the Poisson algebra given above:
\[
\begin{align*}
[\hat{E}_{ef}, \hat{E}_{gh}] &= \delta_{fg} \hat{E}_{eh} - \delta_{eh} \hat{E}_{gf}, \\
[\hat{E}_{ef}, \hat{F}_{gh}] &= \delta_{eg} \hat{F}_{fh} - \delta_{fh} \hat{F}_{eg}, \\
[\hat{E}_{ef}, \hat{F}^+_{gh}] &= \delta_{fg} \hat{F}^+_{eh} - \delta_{eh} \hat{F}^+_{gf}, \\
[\hat{F}_{ef}, \hat{F}^+_{gh}] &= \delta_{eg} \hat{F}^+_{hf} - \delta_{fh} \hat{F}^+_{eg} + 2(\delta_{eg} \delta_{fh} - \delta_{eh} \delta_{fg}), \\
[\hat{F}_{ef}, \hat{F}_{gh}] &= 0, \\
[\hat{F}^+_{ef}, \hat{F}^+_{gh}] &= 0.
\end{align*}
\]
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