Two-Weighted Inequalities for Hausdorff Operators in Herz-Type Hardy Spaces*

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Abstract—In this paper, we prove the boundedness of matrix Hausdorff operators and rough Hausdorff operators in the two weighted Herz-type Hardy spaces associated with both power weights and Muckenhoupt weights. By applying the fact that the standard infinite atomic decomposition norm on two weighted Herz-type Hardy spaces is equivalent to the finite atomic norm on some dense subspaces of them, we generalize some previous known results due to Chen et al. [7] and Ruan, Fan [35].

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1. INTRODUCTION

The one-dimensional Hausdorff operator is defined by

$$H_{\Phi}(f)(x) = \int_0^\infty \frac{\Phi(y)}{y} f\left(\frac{x}{y}\right) dy,$$

where \( \Phi \) is an integrable function on the positive half-line. It is well known that the Hausdorff operator is one of the important operators in harmonic analysis, and it is used to solve certain classical problems in analysis, in particular, it is closely related to the summability of the classical Fourier series (see, for instance, [7], [17], [19], [25], and references therein). The Hausdorff operator was extended to high-dimensional spaces by Brown and Móricz [3] and independently by Lerner and Liflyand [21]. Let \( \Phi \) be a locally integrable function on \( \mathbb{R}^n \). The matrix Hausdorff operator \( H_{\Phi,A} \) associated to the kernel function \( \Phi \) is then defined by

$$H_{\Phi,A}(f)(x) = \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} f(A(y)x) dy, \quad x \in \mathbb{R}^n,$$

where \( A(y) \) is an \( n \times n \) invertible matrix for almost all \( y \) in the support of \( \Phi \) and \( x \) is assumed to be the column \( n \)-vector. It is worth pointing out that if the kernel function \( \Phi \) is chosen appropriately, then the Hausdorff operator reduces to many classical operators in analysis such as the Hardy operator, the Cesàro operator, the Riemann–Liouville fractional integral operator and the Hardy–Littlewood averaging operator (see, e.g., [1], [8], [9], [13], [15], [30], [32], [39] and references therein).

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In 2012, Chen, Fan and Li [7] introduced the rough Hausdorff operator on $\mathbb{R}^n$. More precisely, let $\Phi$ be a locally integrable and radial function on $\mathbb{R}^n$ and $\Omega : S_{n-1} \to \mathbb{C}$ be measurable functions such that $\Omega(y) \neq 0$ for almost everywhere $y$ in $S_{n-1}$. The rough Hausdorff operator $\mathcal{H}_{\Phi,\Omega}$ is then defined by

$$\mathcal{H}_{\Phi,\Omega}(f)(x) = \int_{\mathbb{R}^n} \frac{\Phi(x|y|^{-1})}{|y|^n} \Omega(y)f(y) \, dy, \quad x \in \mathbb{R}^n,$$

where $y' = y/|y|$. Note that, using polar coordinates, we can rewrite

$$\mathcal{H}_{\Phi,\Omega}(f)(x) = \int_0^{\infty} \int_{S_{n-1}} \frac{\Phi(t)}{t} \Omega(y)(f(t^{-1}|x|y'))y' \, dt \, dy', \quad x \in \mathbb{R}^n. \quad (1.3)$$

For simplicity, we denote $\mathcal{H}_{\Phi} := \mathcal{H}_{\Phi,\Omega}$ when $\Omega = 1$. It is useful to remark that if we choose $\Phi(t) = t^{-n}\chi_{(1,\infty)}(t)$ and $\Omega \equiv 1$, the rough Hausdorff operator $\mathcal{H}_{\Phi,\Omega}$ reduces to the famous Hardy operator

$$\mathcal{H}(f)(x) = \frac{1}{|x|^n} \int_{|y| \leq |x|} f(y) \, dy.$$

Also, if $\Omega \equiv 1$ and $\Phi(t) = \chi_{(0,1)}(t)$, then $\mathcal{H}_{\Phi,\Omega}$ reduces to the adjoint Hardy operator

$$\mathcal{H}^*(f)(x) = \int_{|y| > |x|} \frac{f(y)}{|y|^n} \, dy.$$

Recently, Chuong, Duong and Dung [10] have introduced a general class of multilinear Hausdorff operators defined by

$$H_{\Phi,A}(\vec{f})(x) = \int_{\mathbb{R}^n} \Phi(y) \prod_{i=1}^{m} f_i(A_i(y)x) \, dy, \quad x \in \mathbb{R}^n, \quad (1.4)$$

for $\vec{f} = (f_1, ..., f_m)$ and $A = (A_1, ..., A_m)$. The authors gave necessary and sufficient conditions for the boundedness of $H_{\Phi,A}$ on weighted Lebesgue, Herz, central Morrey, and Morrey–Herz type spaces with variable exponent. It is interesting to see that the weighted multilinear Hardy–Cesáro operator (see [11] for more details) is a special case of a multilinear Hausdorff operator.

It is well known that, in recent years, the theory of Hausdorff type operators has been significantly developed into different contexts (see [2], [7], [9], [12], [15], [30], [32], [38]). In particular, the problem establishing the boundedness of Hausdorff operators in Herz spaces has attracted many mathematicians (see, e.g., [21]–[25], [35]). It is useful to remark that Lïlyand and Miyachi [23] showed that, in the case $n = 1$, there exists a bounded function $\Phi$ whose support is contained in $[a, b] \subset (0, \infty)$ such that the one-dimensional Hausdorff operator $H_{\Phi}$ is not bounded on $H^p(\mathbb{R})$ for any $0 < p < 1$. Very recently, the authors of the papers [7], [35] showed that if Hardy spaces are replaced by Herz-type Hardy spaces, then the boundedness of Hausdorff operators is established.

The theory of Hardy spaces associated with Herz spaces has developed in the past few years and played important roles in harmonic analysis, partial differential equations (see [5], [6], [26], [27], [29] for more details). These new Hardy spaces can be regarded as the local version at the origin of the classical Hardy spaces $H^p(\mathbb{R}^n)$ and are good substitutes for $H^p(\mathbb{R}^n)$ when we study the boundedness of nontranslation invariant operators (see, for example, [28]). Using the results of Meyer [34], Bownik [4], Yang [40], [41], Meda [33], and Grafakos [16], the author of the paper [42] proved that the norms in two-weighted Herz-type Hardy spaces $HK^{\alpha,p}_{q,\omega_1,\omega_2}(\omega_1,\omega_2)$ can be achieved by finite central atomic decompositions in some dense subspaces of them. As an application, it was shown in [42] that if $T$ is a sublinear operator and maps all central $(\alpha, q, s, \omega_1, \omega_2)$-atoms into uniformly bounded elements of certain quasi–Banach space $B$ for certain nonnegative integer $s \geq [\alpha - n(1 - 1/q)]$, then $T$ uniquely extends to a bounded sublinear operator from $HK^{\alpha,p}_{q,\omega_1,\omega_2}(\omega_1,\omega_2)$ to $B$.

In this paper, by using the above-mentioned method, which is quite different from the previous method used in [35], we establish sufficient conditions for the boundedness of both matrix Hausdorff
operators $H_{\Phi,A}$ and rough Hausdorff operator $\mathcal{H}_{\Phi,\Omega}$ on two-weighted homogeneous Herz-type Hardy spaces $\mathcal{H}K_\alpha^{\alpha,p}(\omega_1,\omega_2)$.

Our paper is organized as follows. In Sec. 2, we present some notation and definitions of the homogeneous Herz spaces and the homogeneous Herz-type Hardy spaces associated with two weights. Our main theorems are given and proved in Sec. 3 and Sec. 4.

2. SOME NOTATION AND DEFINITIONS

Throughout the whole paper, we denote by $C$ a positive geometric constant that is independent of the main parameters, but can change from line to line. We also write $a \lesssim b$ to mean that there is a positive constant $C$, independent of the main parameters, such that $a \leq Cb$.

It is well known that the theory of $A_p$ weight was first introduced by Muckenhoupt [31] in the Euclidean spaces in order to characterize the weighted $L^p$ boundedness of Hardy–Littlewood maximal functions.

**Definition 1.** Let $1 < p < \infty$. It is said that a weight $\omega \in A_p(\mathbb{R}^n)$ if there exists a constant $C$ such that, for all balls $B \subset \mathbb{R}^n$,

$$ \left( \frac{1}{|B|} \int_B \omega(x) \, dx \right) \left( \frac{1}{|B|} \int_B \omega(x)^{-1/(p-1)} \, dx \right)^{p-1} \leq C. $$

It is said that a weight $\omega \in A_1(\mathbb{R}^n)$ if there is a constant $C$ such that, for all balls $B \subset \mathbb{R}^n$,

$$ \frac{1}{|B|} \int_B \omega(x) \, dx \leq C \operatorname{essinf}_{x \in B} \omega(x). $$

We denote $A_\infty(\mathbb{R}^n) = \bigcup_{1 \leq p < \infty} A_p(\mathbb{R}^n)$.

We note that the class $A_\infty(\mathbb{R}^n)$ is closely connected with the reverse Hölder condition. More precisely, if there exists an $r > 1$ and a fixed constant $C$ such that

$$ \left( \frac{1}{|B|} \int_B \omega(x)^r \, dx \right)^{1/r} \leq \frac{C}{|B|} \int_B \omega(x) \, dx, $$

for all balls $B \subset \mathbb{R}^n$, we then say that $\omega$ satisfies the reverse Hölder condition of order $r$ and write $\omega \in RH_r(\mathbb{R}^n)$. By Theorem 19 and Corollary 21 in [20], we have $\omega \in A_\infty(\mathbb{R}^n)$ if and only if there exists some $r > 1$ such that $\omega \in RH_r(\mathbb{R}^n)$. Moreover, if $\omega \in RH_r(\mathbb{R}^n)$, $r > 1$, then $\omega \in RH_{r+\varepsilon}(\mathbb{R}^n)$ for some $\varepsilon > 0$. We thus write $r_\omega \equiv \sup\{r > 1 : \omega \in RH_r(\mathbb{R}^n)\}$ to denote the critical index of $\omega$ for the reverse Hölder condition. For further properties of $A_p$ weights, one may see in the famous book [36].

**Proposition 1.** The following statements hold:

(i) $|x|^\alpha \in A_1(\mathbb{R}^n)$ if and only if $-n < \alpha \leq 0$;

(ii) $|x|^\alpha \in A_p(\mathbb{R}^n)$, $1 < p < \infty$, if and only if $-n < \alpha < n(p-1)$.

Let us give the following standard properties of $A_p$ weights that will be used in the sequel.

**Proposition 2.** Let $\omega \in A_p(\mathbb{R}^n) \cap RH_r(\mathbb{R}^n)$, $p \geq 1$ and $r > 1$. Then there exist constants $C_1, C_2 > 0$ such that

$$ C_1 \left( \frac{|E|}{|B|} \right)^p \leq \frac{\omega(E)}{\omega(B)} \leq C_2 \left( \frac{|E|}{|B|} \right)^{(r-1)/r}, $$

for any measurable subset $E$ of a ball $B$. 
Proposition 3. If \( \omega \in A_p(\mathbb{R}^n) \), \( 1 \leq p < \infty \), then, for any \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) and any ball \( B \subset \mathbb{R}^n \),

\[
\frac{1}{|B|} \int_B |f(x)| \, dx \leq C \left( \frac{1}{\omega(B)} \int_B |f(x)|^{\frac{q}{q-1}} \omega(x) \, dx \right)^{1/p}.
\]

As usual, the weighted function \( \omega \) is a nonnegative measurable function on \( \mathbb{R}^n \). Suppose that \( L^q(\omega) \), \( 0 < q < \infty \), is the space of all measurable functions \( f \) on \( \mathbb{R}^n \) such that

\[
\|f\|_{L^q(\omega)} = \left( \int_{\mathbb{R}^n} |f(x)|^q \omega(x) \, dx \right)^{1/q} < \infty.
\]

The space \( L^q_{\text{loc}}(\omega) \) is defined as the set of all measurable functions \( f \) on \( \mathbb{R}^n \) satisfying

\[
\int_K |f(x)|^q \omega(x) \, dx < \infty \quad \text{for any compact subset } K \text{ of } \mathbb{R}^n.
\]

The space \( L^q_{\text{loc}}(\omega, \mathbb{R}^n \setminus \{0\}) \) is also defined in a similar way to the space \( L^q_{\text{loc}}(\omega) \).

In what follows, denote \( \chi_k = \chi_{C_k}, C_k = B_k \setminus B_{k-1} \) for all \( k \in \mathbb{Z} \), where \( B_R = \{ x \in \mathbb{R}^n : |x| \leq 2^R \} \) and \( B_R^c = \{ x \in \mathbb{R}^n : |x| \leq R \} \) for all \( R \in \mathbb{R} \). Denote by \( \omega(K) \) the integral \( \int_K \omega(x) \, dx \) for all subsets \( K \) of \( \mathbb{R}^n \).

We are now ready to give some notation and definitions for homogeneous two-weighted Herz spaces and homogeneous two-weighted Herz-type Hardy spaces.

Definition 2. Let \( 0 < \alpha < \infty, \ 1 \leq q < \infty, \ 0 < p < \infty \), and let \( \omega_1 \) and \( \omega_2 \) be weighted functions. Then the homogeneous two-weighted Herz space \( \dot{K}^\alpha_p(\omega_1, \omega_2) \) is defined as the set of all measurable functions \( f \in L^q_{\text{loc}}(\omega_2, \mathbb{R}^n \setminus \{0\}) \) such that

\[
\|f\|_{\dot{K}^\alpha_p(\omega_1, \omega_2)} = \left( \sum_{k=-\infty}^{\infty} \omega_1(B_k)^{\alpha/p/n} \|f \chi_k\|_{L^q(\omega_2)}^p \right)^{1/p} < \infty.
\]

Denote \( \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \). Let \( S(\mathbb{R}^n) \) be the space of Schwartz functions, and denote by \( S'(\mathbb{R}^n) \) the dual space of \( S(\mathbb{R}^n) \). Given \( N \in \mathbb{N} \), we denote

\[
S_N(\mathbb{R}^n) = \{ \phi \in S(\mathbb{R}^n) : \|\phi\|_{m,\beta} \leq 1, m \leq n + N, |\beta| \leq N \},
\]

where \( \|\phi\|_{m,\beta} = \sup_{x \in \mathbb{R}^n} (1 + |x|^m)^{D^\beta(\phi)}(x) \), \( \beta = (\beta_1, \ldots, \beta_n) \), \( D^\beta \phi = (\frac{\partial}{\partial x_1})^{\beta_1} \cdots (\frac{\partial}{\partial x_n})^{\beta_n} \phi \). Next, the grand maximal function of \( f \in S'(\mathbb{R}^n) \) due to Fefferman–Stein [14] is defined by

\[
G_N(f)(x) = \sup_{\phi \in S_N} M_\phi(f)(x), \quad x \in \mathbb{R}^n,
\]

where \( M_\phi(f)(x) = \sup_{|y-x|<t} |\phi_t * f(y)| \) and \( \phi_t(x) = t^{-n} \phi(t^{-1} x) \) for all \( t > 0 \). Let us recall the definition of the Hardy spaces associated to the two-weighted Herz spaces due to Lu and Yang [27] as follows.

Definition 3. Let \( 0 < \alpha < \infty, \ 1 \leq q < \infty, \ 0 < p < \infty, \ N = \max\{[\alpha - n(1 - 1/q)] + 1, 1\} \), and let \( \omega_1, \omega_2 \in A_1 \). The homogeneous two-weighted Herz-type Hardy space \( H\dot{K}^\alpha_p(\omega_1, \omega_2) \) is defined as the set of all \( f \in S'(\mathbb{R}^n) \) such that

\[
\|f\|_{H\dot{K}^\alpha_p(\omega_1, \omega_2)} = \|G_N(f)\|_{\dot{K}^\alpha_p(\omega_1, \omega_2)} < \infty.
\]

Now we state the definition of central atom and dyadic central unit. Here we denote the integer part of real number \( x \) by \( [x] \).

Definition 4. Let \( 1 < q < \infty, \ \alpha \in \{n(1 - 1/q), \infty\}, \ s \geq [\alpha - n(1 - 1/q)] \) and \( \omega_1, \omega_2 \in A_1 \). A function \( a \) on \( \mathbb{R}^n \) is called a central \((\alpha, q, s; \omega_1, \omega_2)\)-atom if it satisfies the following four conditions:
(i) supp \(a\) ⊂ \(B(0, r)\) for some \(r > 0\);

(ii) \(\|a\|_{L^q(\omega_2)} \leq \omega_1(B(0, r))^{-\alpha/n}\);

(iii) \(\int_{\mathbb{R}^n} a(x)x^\beta\,dx = 0\) for all \(\beta \leq s\);

(iv) \(a(x) = 0\) in some neighborhood of 0.

A function \(a\) on \(\mathbb{R}^n\) is called a dyadic central \((\alpha, q; \omega_1, \omega_2)\)-unit if it satisfies (i) and (ii) associated to \(r = 2^k\) for some \(k \in \mathbb{Z}\).

**Theorem 1** (Theorem 1.1 in [26]). Let \(0 < \alpha < \infty\), \(0 < p < \infty\), \(1 \leq q < \infty\). Let \(\omega_1 \in A_1(\mathbb{R}^n)\) and \(\omega_2\) be a weighted function on \(\mathbb{R}^n\). We then have \(f \in K^\alpha_q(\omega_1, \omega_2)\), if and only if \(f = \sum_{k=-\infty}^{\infty} \lambda_kb_k\) point-wise, where \(\sum_{k=-\infty}^{\infty} |\lambda_k|^p < \infty\), and each \(b_k\) is a dyadic central \((\alpha, q, \omega_1, \omega_2)\)-unit with the support in \(B_k\). Moreover, \(\|f\|_{K^\alpha_q(\omega_1, \omega_2)} \simeq \inf\{\sum_{k=-\infty}^{\infty} |\lambda_k|^p\}^{1/p}\), where the infimum is taken over all decompositions of \(f\) as above.

Next, we present a useful result due to Zhou in [42], which states that the norms in \(HK^\alpha_q(\omega_1, \omega_2)\) can be achieved by finite central atomic decomposition in some dense subspaces of them.

Let \(0 < p < \infty\), \(1 < q < \infty\), \(\alpha \in [n(1 - 1/q), \infty)\), \(s \geq [\alpha - n(1 - 1/q)]\) and \(\omega_1, \omega_2 \in A_1\). Denote by \(\hat{\mathcal{F}}^\alpha_q(\omega_1, \omega_2)\) the collection of all finite linear combinations of central \((\alpha, q, s; \omega_1, \omega_2)\)-atoms. Then for \(f \in \hat{\mathcal{F}}^\alpha_q(\omega_1, \omega_2)\), we define

\[
\|f\|_{\hat{\mathcal{F}}^\alpha_q(\omega_1, \omega_2)} = \inf \left\{ \left( \frac{1}{m} \sum_{j=1}^{m} |\lambda_j|^p \right)^{1/p} : m \in \mathbb{N}, f = \sum_{j=1}^{m} \lambda_j a_j, \{a_j\}_{j=1}^{m} \text{ are central } (\alpha, q, s; \omega_1, \omega_2)\text{-atoms} \right\}.
\]

(2.1)

By \(\hat{\mathcal{F}}^\alpha_q(\omega_1, \omega_2)\) we denote the collection of all finite linear combinations of \(C^\infty(\mathbb{R}^n)\) central \((\alpha, q, s; \omega_1, \omega_2)\)-atoms. For \(f \in \hat{\mathcal{F}}^\alpha_q(\omega_1, \omega_2)\), we also define \(\|f\|_{C^\alpha_q(\omega_1, \omega_2)}\) in the same way as in (2.1) by replacing central \((\alpha, q, s; \omega_1, \omega_2)\)-atoms by \(C^\infty(\mathbb{R}^n)\) central \((\alpha, q, s; \omega_1, \omega_2)\)-atoms.

**Theorem 2** (Theorem 1 in [42]). Let \(p \in (0, \infty)\), \(q \in (1, \infty)\), \(\alpha \in [n(1 - 1/q), \infty)\) and nonnegative integer \(s \geq [\alpha - n(1 - 1/q)]\). Then, \(\| \cdot \|_{H^\alpha_q(\omega_1, \omega_2)}\) and \(\| \cdot \|_{\hat{\mathcal{F}}^\alpha_q(\omega_1, \omega_2)}\) are equivalent on \(\hat{\mathcal{F}}^\alpha_q(\omega_1, \omega_2)\) (resp., \(\| \cdot \|_{C^\alpha_q(\omega_1, \omega_2)}\) are equivalent on \(C^\alpha_q(\omega_1, \omega_2)\)).

To end this section, let us recall that a quasi-Banach space \(\mathcal{B}\) is a vector space endowed with a quasi-norm \(\| \cdot \|\) which is nonnegative, nondegenerate, homogeneous, and obeys the quasi-triangle inequality. Let \(p \in (0, 1)\). A quasi-Banach space \(\mathcal{B}_p\) with a quasi-norm \(\| \cdot \|_{\mathcal{B}_p}\) is said to be a \(p\)-quasi-Banach space if \(\|f + g\|_{\mathcal{B}_p} \leq \|f\|_{\mathcal{B}_p} + \|g\|_{\mathcal{B}_p}\), for any \(f, g \in \mathcal{B}_p\).

Recall that, for any given \(r\)-quasi-Banach space \(\mathcal{B}_r\) with \(r \in (0, 1]\) and linear space \(X\), an operator \(T\) from \(X\) to \(\mathcal{B}_r\) is called \(r\)-sublinear if for any \(f, g \in X\) and \(\lambda, \nu \in \mathbb{C}\), we have

\[
\|T(\lambda f + \nu g)\|_{\mathcal{B}_r} \leq (|\lambda|^r \|T(f)\|_{\mathcal{B}_r} + |\nu|^r \|T(g)\|_{\mathcal{B}_r})^{1/r}, \quad \|T(f) - T(g)\|_{\mathcal{B}_r} \leq \|T(f - g)\|_{\mathcal{B}_r}.
\]

Let us present the following helpful theorem that will be used in the sequel.
Theorem 3 (Theorem 2 in [42]). Let $0 < p \leq 1$, $p \leq r \leq 1$, $1 < q < \infty$, $\alpha \in [n(1 - 1/q), \infty)$ and nonnegative $s \geq [\alpha - n(1 - 1/q)]$. If $T$ is a $B_r$-sublinear operator defined on $F_p^{\alpha,q,s}(\omega_1,\omega_2)$ such that

$$S = \sup \{\|Ta\|_{B_r} : a \text{ is any central } (\alpha, q, s; \omega_1, \omega_2)_0 - \text{atom} \} < \infty$$

or defined on $C^0 F_p^{\alpha,q,s}(\omega_1,\omega_2)$ and satisfies

$$S = \sup \{||Ta||_{B_r} : a \text{ is any } C^\infty(\mathbb{R}^n) \text{ central } (\alpha, q, s; \omega_1, \omega_2)_0 - \text{atom} \} < \infty,$$

then $T$ uniquely extends to a bounded $B_r$-sublinear operator from $H^\alpha_k q^{\alpha,p}(\omega_1,\omega_2)$ to $B_r$.

3. THE MAIN RESULTS ABOUT THE BOUNDEDNESS OF $\mathcal{H}_{\Phi,\Omega}$

Our first main result is the following.

Theorem 4. Let $0 < p \leq 1$, $1 < q < \infty$, $n(1 - 1/q) \leq \alpha < 1 + n(1 - 1/q)$, and $\omega_1 = |x|^\beta_1$, $\omega_2 = |x|^\beta_2$ with $\beta_1, \beta_2 \in (n, 0]$. If $\Phi$ is a radial function and there exist $m, M \in \mathbb{Z}$ such that

$$\text{supp}(\Phi) \subset \{x \in \mathbb{R}^n : 2^m < |x| < 2^M\},$$

then $\mathcal{H}_\Phi$ is a bounded operator from $H^\alpha_k q^{\alpha,p}(\omega_1,\omega_2)$ into itself.

Proof. Let $a$ be any central $(\alpha, q, 0; \omega_1, \omega_2)_0$-atom. Thus, there exists $j_a \in \mathbb{Z}$ such that $\text{supp}(a) \subset B_{j_a}$ and $\|a\|_{L^q(\omega_2)} \lesssim \omega_1(B_{j_a})^{-\alpha/n}$. We will prove that

$$\|\mathcal{H}_\Phi(a)\|_{H^\alpha_k q^{\alpha,p}(\omega_1,\omega_2)} \lesssim \int_{(2^m, 2^M]} |\Phi(t)| \, dt. \quad (3.1)$$

Indeed, we can rewrite

$$\mathcal{H}_\Phi(a)(x) = \sum_{k=m+1}^M \int_{(2^{k-1}, 2^k]} \int_{B_{j_a+k}} \frac{\Phi(t)}{t} a(t^{-1}|x|y^t) \, dy \, dt := \sum_{k=m+1}^M b_k(x).$$

From the definition of $b_k$ and $\text{supp}(a) \subset B_{j_a}$, it is easy to see that

$$\text{supp}(b_k) \subset B_{j_a+k}. \quad (3.2)$$

By the Minkowski inequality and the H"{o}lder inequality, we have

$$\|b_k\|_{L^q(\omega_2)} \leq \int_{(2^{k-1}, 2^k]} \frac{|\Phi(t)|}{t} \int_{B_{j_a+k}} \|a(t^{-1}|x|y^t)\|_{L^q(\omega_2, B_{j_a+k})} \, dy \, dt \lesssim \int_{(2^{k-1}, 2^k]} \frac{|\Phi(t)|}{t} \left( \int_{B_{j_a+k}} |a(t^{-1}|x|y^t)|^q \omega_2(x) \, dx \, dy \right)^{1/q} \, dt.$$

By polar coordinates, we have

$$\int_{B_{j_a+k}} |a(t^{-1}|x|y^t)|^q \omega_2(x) \, dx \, dy \leq \int_{S_{n-1}} \int_0^{2^{j_a+k}} |a(r t^{-1}|y|y^r)|^q r^{n-1+\beta_2} \, dr \, dy \leq t^{n+\beta_2} \|a\|^q_{L^q(\omega_2)}.$$

Thus, by $\|a\|_{L^q(\omega_2)} \lesssim \omega_1(B_{j_a})^{-\alpha/n}$, it immediately follows that

$$\|b_k\|_{L^q(\omega_2)} \lesssim \left( \int_{(2^{k-1}, 2^k]} \frac{|\Phi(t)|}{t^{1-(n+\beta_2)/q}} \, dt \right) \omega_1(B_{j_a+k})^{-\alpha/n} \left( \frac{\omega_1(B_{j_a+k})}{\omega_1(B_{j_a})} \right)^{\alpha/n} \quad (3.3)$$

and

$$\|b_k\|_{L^q(\omega_2)} \lesssim \left( \int_{(2^{k-1}, 2^k]} \frac{|\Phi(t)|}{t^{1-(n+\beta_2)/q}} \, dt \right) \omega_1(B_{j_a+k})^{-\alpha/n} \left( \frac{\omega_1(B_{j_a+k})}{\omega_1(B_{j_a})} \right)^{\alpha/n} \quad (3.4)$$
In addition, because \( \omega_1 \) is a power weighted function, one has
\[
\left( \frac{\omega_1(B_{ja+k})}{\omega_1(B_{ja})} \right)^{\alpha/n} \simeq \left( \frac{q(ja+k)(\beta_1+n)}{2ja(\beta_1+n)} \right)^{\alpha/n} = 2^{k(\alpha+\beta_1\alpha/n)}. \tag{3.5}
\]
Consequently, by letting \( t \in (2^{k-1}, 2^k) \), we obtain
\[
\|b_k\|_{\mathcal{L}^q(\omega_2)} \lesssim \left( \int_{(2^{k-1},2^k)} \frac{|\Phi(t)|}{t^{1-(n+\beta_2)/q-\alpha-\beta_1\alpha/n}} dt \right) \omega_1(B_{ja+k})^{-\alpha/n} = \lambda_k \omega_1(B_{ja+k})^{-\alpha/n}, \tag{3.6}
\]
where
\[
\lambda_k = \int_{(2^{k-1},2^k)} \frac{|\Phi(t)|}{t^{1-(n+\beta_2)/q-\alpha-\beta_1\alpha/n}} dt.
\]
Next, by \( \omega_2 = |x|^{\beta_2} \in A_1 \) and \( \text{supp}(b_k) \subset B_{ja+k} \), we have \( b_k \in L^1(\mathbb{R}^n) \). Combining this inclusion with \( \int_{\mathbb{R}^n} a(x) \, dx = 0 \), using polar coordinates and the Fubini theorem, it is easy to check that
\[
\int_{\mathbb{R}^n} b_k(x) \, dx = |S_{n-1}| \int_{(2^{k-1},2^k)} \Phi(t)t^{n-1} dt \int_{\mathbb{R}^n} a(y) \, dy = 0. \tag{3.7}
\]
Since there exists \( r_n \in \mathbb{Z} \) such that \( a = 0 \) on \( B_{rn} \), we have
\[
b_k = 0 \quad \text{on} \quad B_{rn+k-1}. \tag{3.8}
\]
For convenience, we denote
\[
b_{ja,k} = \begin{cases} b_k \lambda_k^{-1} & \text{if } \lambda_k \neq 0, \\ 0 & \text{otherwise}. \end{cases}
\]
This shows that
\[
\mathcal{H}_f(a)(x) = \sum_{k=m+1}^M \lambda_k b_{ja,k}(x).
\]
By (3.2), (3.6), (3.7) and (3.8), we see that \( b_{ja,k} \) is a central \((\alpha, q, 0; \omega_1, \omega_2)\)-atom. This implies that
\[
\mathcal{H}_f(a) \in \dot{F}^\alpha_{p,q} (\omega_1, \omega_2).
\]
Consequently, by Theorem 2, we have
\[
\|\mathcal{H}_f(a)\|_{\dot{H}^\alpha_{q,p} (\omega_1, \omega_2)} \lesssim \|\mathcal{H}_f(a)\|_{\dot{F}^\alpha_{p,q} (\omega_1, \omega_2)} \lesssim \left( \sum_{k=m+1}^M |\lambda_k|^p \right)^{1/p} \lesssim \sum_{k=m+1}^M |\lambda_k|,
\]
which implies that inequality (3.1) holds. From this, by Theorem 3, we obtain
\[
\|\mathcal{H}_f(f)\|_{\dot{H}^\alpha_{q,p} (\omega_1, \omega_2)} \lesssim \left( \int_{(2^{m,2M})} |\Phi(t)| \, dt \right) \|f\|_{\dot{H}^\alpha_{q,p} (\omega_1, \omega_2)}
\]
for all \( f \in \dot{H}^\alpha_{q,p} (\omega_1, \omega_2) \). Therefore, the proof of this theorem is complete. \( \square \)

**Theorem 5.** Let \( 1 < q < \infty, \alpha \in [n(1 - 1/q), \infty), \omega_1 = |x|^{\beta_1}, \omega_2 = |x|^{\beta_2} \) with \( \beta_1, \beta_2 \in (-n, 0] \). Let \( \Omega \in L^q(S_{n-1}) \) and \( \Phi \) be a radial function.

(i) If \( p = 1 \) and
\[
C_1 = \int_0^\infty \frac{|\Phi(t)|}{t^{1-(n+\beta_2)/q-\alpha-\beta_1\alpha/n}} dt < \infty,
\]
then
\[
\|\mathcal{H}_f(\Omega(f))\|_{\dot{K}^\alpha_{q,1} (\omega_1, \omega_2)} \lesssim C_1 \|\Omega\|_{L^q(S_{n-1})} \|f\|_{\dot{H}^\alpha_{q,1} (\omega_1, \omega_2)} \quad \text{for all} \quad f \in \dot{H}^\alpha_{q,1} (\omega_1, \omega_2).}
\]
(ii) If \(0 < p < 1, \sigma > (1 - p)/p\), and
\[
C_2 = \int_0^\infty \frac{\|\Phi(t)\|}{t^{1-(n+\beta_2)/q-\alpha-\beta_1/\alpha}} (\log_2(t))^\sigma \chi_{(0,1]}(t) + (\log_2(t) + 1)^\sigma \chi_{(1,\infty)}(t) \, dt < \infty,
\]
then
\[
\|\mathcal{H}_{\Phi,\Omega}(f) \|_{\mathcal{K}^\alpha,p_{(\omega_1,\omega_2)}} \lesssim C_2 \|\Omega\|_{L^p(S_{n-1})} \|f\|_{H\mathcal{K}^\alpha,p_{(\omega_1,\omega_2)}} \quad \text{for all} \quad f \in H\mathcal{K}^\alpha,p_{(\omega_1,\omega_2)}.
\]

**Proof.** Let us fix a nonnegative integer \(s \geq [\alpha - n (1 - 1/q)]\), and let a function \(a\) be any central \((\alpha, q, s; \omega_1, \omega_2)\)-atom with supp \((a) \subset B_{ja}\) and \(\|a\|_{L^q(\omega_2)} \lesssim \omega_1(B_{ja})^{-\alpha/n}\). We must prove that
\[
\|\mathcal{H}_{\Phi,\Omega}(a) \|_{\mathcal{K}^\alpha,p_{(\omega_1,\omega_2)}} \lesssim \begin{cases} C_1 \|\Omega\|_{L^p(S_{n-1})}, & p = 1, \\ C_2 \|\Omega\|_{L^p(S_{n-1})}, & p \in (0, 1), \quad \sigma > (1 - p)/p. \end{cases} \quad (3.9)
\]
In fact, we have
\[
|\mathcal{H}_{\Phi,\Omega}(a)(x)| \leq \sum_{k \in \mathbb{Z}} \int_{(2k-1,2k]} \int_{S_{n-1}} \frac{\|\Phi(t)\|}{t} |\Omega'(t)| |\alpha(t^{-1}|x|) y'\| |\Omega'(t)| \, dy' \, dt := \sum_{k \in \mathbb{Z}} \tilde{b}_k(x). \quad (3.10)
\]
By estimating just as for (3.2) and (3.6) above, we see that
\[
\text{supp} (\tilde{b}_k) \subset B_{ja+k} \quad \text{and} \quad \|\tilde{b}_k\|_{L^q(\omega_2)} \lesssim \lambda_k \|\Omega\|_{L^q(S_{n-1})} \omega_1(B_{ja+k})^{-\alpha/n}.
\]
(3.11)
Denote
\[
\tilde{b}_{ja,k} = \begin{cases} \tilde{b}_k \|\Omega\|_{L^p(S_{n-1})}^{-1} \lambda_k^{-1}, & \text{if } \lambda_k \neq 0, \\ 0, & \text{otherwise}. \end{cases}
\]
Then
\[
\sum_{k \in \mathbb{Z}} \tilde{b}_{ja,k} = \sum_{k \in \mathbb{Z}} \|\Omega\|_{L^q(S_{n-1})} \lambda_k \tilde{b}_{ja,k}.
\]
By (3.11), we see that \(\tilde{b}_{ja,k}\) is a dyadic central \((\alpha, q, \omega_1, \omega_2)\)-unit. Therefore, by Theorem 1, we have
\[
\|\mathcal{H}_{\Phi,\Omega}(a) \|_{\mathcal{K}^\alpha,p_{(\omega_1,\omega_2)}} \lesssim \left( \sum_{k \in \mathbb{Z}} \lambda_k \right)^{1/p} \|\Omega\|_{L^q(S_{n-1})} \left( \sum_{k \in \mathbb{Z}} |\lambda_k|^p \right)^{1/p}. \quad (3.12)
\]
For \(p = 1\), we estimate
\[
\sum_{k \in \mathbb{Z}} |\lambda_k| = \sum_{k \in \mathbb{Z}} \int_{(2k-1,2k]} \frac{\|\Phi(t)\|}{t^{1-(n+\beta_2)/q-\alpha-\beta_1/\alpha}} \, dt = C_1. \quad (3.13)
\]
For \(p \in (0, 1)\) and \(\sigma > (1 - p)/p\), by the Hölder inequality, we have
\[
\left( \sum_{k \in \mathbb{Z}} |\lambda_k|^p \right)^{1/p} \lesssim \sum_{k \in \mathbb{Z}} |k|^p |\lambda_k| = \sum_{k = 1}^{\infty} \int_{(2k-1,2k]} \frac{|\Phi(t)|}{t^{1-(n+\beta_2)/q-\alpha-\beta_1/\alpha}} |k|^\sigma \, dt + \sum_{k = -\infty}^{0} \int_{(2k-1,2k]} \frac{|\Phi(t)|}{t^{1-(n+\beta_2)/q-\alpha-\beta_1/\alpha}} |k|^\sigma \, dt
\]
\[
\lesssim \sum_{k = 1}^{\infty} \int_{(2k-1,2k]} \frac{|\Phi(t)|}{t^{1-(n+\beta_2)/q-\alpha-\beta_1/\alpha}} (\log_2(t) + 1)^\sigma \, dt + \sum_{k = -\infty}^{0} \int_{(2k-1,2k]} \frac{|\Phi(t)|}{t^{1-(n+\beta_2)/q-\alpha-\beta_1/\alpha}} |\log_2(t)|^\sigma \, dt = C_2. \quad (3.14)
\]
From this, using (3.12) and (3.13), we obtain inequality (3.9). Combining Theorem 3 and inequality (3.9), we conclude the proof of the theorem. \(\Box\)
Theorem 6. Let \( 1 < q < \infty, \alpha \in [n(1-1/q), \infty), \omega_2 = |x|^{\beta_2} \) with \( \beta_2 \in (-n, 0) \). Let \( \Omega \in L^q(S_{n-1}) \), and let \( \Phi \) be a radial function. At the same time, let \( \omega_1 \in A_1 \) with the finite critical index \( r_{\omega_1} \) for the reverse H"older condition and \( \delta \in (1, r_{\omega_1}) \).

(i) If \( p = 1 \) and

\[
C_3 = \int_0^{\infty} \frac{\Phi(t)}{t^{1-(n+\beta_2)/2}} (t^{\alpha(\delta-1)/\delta} \chi_{(0,1)}(t) + t^{\alpha} \chi_{(1,\infty)}(t)) \, dt < \infty,
\]

then

\[
\| \mathcal{H}_q^{\alpha, 1}(\omega_1, \omega_2) \|_{K_q^{\alpha, 1}(\omega_1, \omega_2)} \leq C_3 \| \Omega \|_{L^q(S_{n-1})} \| f \|_{H^{K_q^{\alpha, 1}}(\omega_1, \omega_2)} \quad \text{for all} \quad f \in H^{K_q^{\alpha, 1}}(\omega_1, \omega_2).
\]

(ii) If \( 0 < p < 1, \sigma > (1-p)/p \) and

\[
C_4 = \int_0^{\infty} \frac{\Phi(t)}{t^{1-(n+\beta_2)/2}} (t^{\alpha(\delta-1)/\delta} |\log_2(t)|^\sigma \chi_{(0,1)}(t) + t^{\alpha} (\log_2(t) + 1)^\sigma \chi_{(1,\infty)}(t)) \, dt < \infty,
\]

then

\[
\| \mathcal{H}_q^{\alpha, p}(\omega_1, \omega_2) \|_{K_q^{\alpha, p}(\omega_1, \omega_2)} \leq C_4 \| \Omega \|_{L^q(S_{n-1})} \| f \|_{H^{K_q^{\alpha, p}}(\omega_1, \omega_2)} \quad \text{for all} \quad f \in H^{K_q^{\alpha, p}}(\omega_1, \omega_2).
\]

Proof. Just like Theorem 5, to prove the theorem, it suffices to show that

\[
\| \mathcal{H}_q^{\omega}(a) \|_{K_q^{\alpha, p}(\omega_1, \omega_2)} \leq \begin{cases} C_3 \| \Omega \|_{L^q(S_{n-1})}, & p = 1, \\ C_4 \| \Omega \|_{L^q(S_{n-1})}, & p \in (0, 1), \quad \sigma > (1-p)/p,
\end{cases}
\]

(3.15)

where the nonnegative integer \( s \) satisfies \( s \geq [\alpha - n(1-1/q)] \) and \( a \) is any central \((\alpha, q, s; \omega_1, \omega_2)\)-atom with \( \text{supp}(a) \subset B_{j_0} \). Let us recall that

\[
\tilde{b}_k(x) = \int_{(2^{k-1}, 2^k]} \int_{S_{n-1}} \frac{\Phi(t)}{t} |\Omega(y')| |a(t^{-1/2}|x|y')| \, dy' \, dt,
\]

with \( \text{supp}(\tilde{b}_k) \subset B_{j_0+k} \). Now, by estimating as (3.4) above, we also have

\[
\| \tilde{b}_k \|_{L^q(\omega_2)} \leq \| \Omega \|_{L^q(S_{n-1})} \left( \int_{(2^{k-1}, 2^k]} \frac{|\Phi(t)|}{t^{1-(n+\beta_2)/2}} \, dt \right) \omega_1(B_{j_0+k})^{-\alpha/n} \left( \frac{\omega_1(B_{j_0+k})}{\omega_1(B_{j_0})} \right)^{\alpha/n}.
\]

On the other hand, by \( \omega_1 \in A_1 \) and Proposition 2, we have

\[
\left( \frac{\omega_1(B_{j_0+k})}{\omega_1(B_{j_0})} \right)^{\alpha/n} \leq \begin{cases} 2^{k\alpha} & \text{if } k \geq 1, \\ 2^{k\alpha(\delta-1)/\delta} & \text{otherwise}.
\end{cases}
\]

(3.16)

Thus, by letting \( t \in (2^{k-1}, 2^k] \), we can write

\[
\| \tilde{b}_k \|_{L^q(\omega_2)} \leq \omega_1(B_{j_0+k})^{-\alpha/n} \| \Omega \|_{L^q(S_{n-1})} \begin{cases} \int_{(2^{k-1}, 2^k]} \frac{|\Phi(t)|}{t^{1-(n+\beta_2)/2}} \, dt & \text{if } k \geq 1, \\ \int_{(2^{k-1}, 2^k]} \frac{|\Phi(t)|}{t^{1-(n+\beta_2)/2}} \, dt & \text{otherwise},
\end{cases}
\]

:= \omega_1(B_{j_0+k})^{-\alpha/n} \| \Omega \|_{L^q(S_{n-1})} \cdot \mu_k
\]

Set

\[
\tilde{b}_{j_0,k} = \begin{cases} \tilde{b}_k \| \Omega \|_{L^q(S_{n-1})}^{-1} \mu_k^{-1} & \text{if } \mu_k \neq 0, \\ 0 & \text{otherwise}.
\end{cases}
\]

Thus, we have

\[
\sum_{k \in \mathbb{Z}} \tilde{b}_k = \sum_{k \in \mathbb{Z}} \| \Omega \|_{L^q(S_{n-1})} \mu_k \tilde{b}_{j_0,k}.
\]
where \( \tilde{b}_{j_a,k}^* \) is a dyadic central \((\alpha, q, \omega_1, \omega_2)\)-unit. From this, using inequality (3.10) and Theorem 1, we obtain

\[
\| \mathcal{H}_{\Phi,\Omega}(a) \|_{K_q^{\alpha,p}(\omega_1,\omega_2)} \lesssim \| \Omega \|_{L^q(S_{n-1})} \left( \sum_{k \in \mathbb{Z}} |\mu_k|^p \right)^{1/p}.
\]

For \( p = 1 \), it is evident to see that \( \sum_{k \in \mathbb{Z}} |\mu_k| = C_3 \). Next, we consider for the case \( p \in (0, 1) \) and \( \sigma > (1-p)/p \). By the arguments for (3.14) above, we have

\[
\left( \sum_{k \in \mathbb{Z}} |\mu_k|^p \right)^{1/p} \lesssim \sum_{k \in \mathbb{Z}} |k|^{\sigma} |\mu_k|
\]

\[
= \sum_{k=1}^{\infty} \int_{(2^{k-1},2^k]} \frac{|\Phi(t)|}{t^{1-(n+\beta_2)/q-\alpha}} |k|^\sigma \, dt + \sum_{k=-\infty}^{0} \int_{(2^k,2^{k-1}]} \frac{|\Phi(t)|}{t^{1-(n+\beta_2)/q-\alpha}} |k|^\sigma \, dt
\]

\[
\lesssim \sum_{k=1}^{\infty} \int_{(2^{k-1},2^k]} \frac{|\Phi(t)|}{t^{1-(n+\beta_2)/q-\alpha}} (\log_2(t) + 1)^\sigma \, dt + \sum_{k=-\infty}^{0} \int_{(2^k,2^{k-1}]} \frac{|\Phi(t)|}{t^{1-(n+\beta_2)/q-\alpha}} |\log_2(t)|^\sigma \, dt = C_4.
\]

Consequently, inequality (3.15) holds. Therefore, the proof of the theorem is complete. \( \square \)

**Theorem 7.** Let \( 1 \leq q^* < q < \infty, 0 < \alpha^* < \infty, \alpha \in [n(1-1/q), \infty) \). Let \( \Omega \in L^q(S_{n-1}) \), \( \Phi \) be a radial function, \( \omega_i \in A_1 \) with finite critical index \( \tau_{\omega_i} \) for the reverse Hölder condition and \( \delta_i \in (1, r_{\omega_i}), \) for all \( i = 1, 2 \). Assume that \( q > q^* r_{\omega_2} \) and the following conditions hold:

\[
1/q + \alpha/n = 1/q^* + \alpha^*/n,
\]

\[
\omega_2(B_k) \lesssim \omega_1(B_k) \quad \text{for all} \quad k \in \mathbb{Z}.
\]

Denote \( \gamma_1 = (\delta_2 - 1)\delta_2^{-1}(n/q + \alpha) - \alpha^*/\delta_1, \gamma_2 = n/q + \alpha + \alpha^*/\delta_2 \).

(i) If \( p = 1 \) and

\[
C_5 = \int_0^\infty \frac{|\Phi(t)|}{t} \left( t^{\gamma_1} \chi_{(0,1]}(t) + t^{\gamma_2} \chi_{(1,\infty]}(t) \right) \, dt < \infty,
\]

then

\[
\| \mathcal{H}_{\Phi,\Omega}(f) \|_{K_q^{\alpha^*,1}(\omega_1,\omega_2)} \lesssim C_5 \| \Omega \|_{L^q(S_{n-1})} \| f \|_{H \tilde{K}_q^{\alpha^*,1}(\omega_1,\omega_2)} \quad \text{for all} \quad f \in H \tilde{K}_q^{\alpha^*,1}(\omega_1,\omega_2).
\]

(ii) If \( 0 < p < 1, \sigma > (1-p)/p \) and

\[
C_6 = \int_0^\infty \frac{|\Phi(t)|}{t} \left( t^{\gamma_1} |\log_2(t)|^\sigma \chi_{(0,1]}(t) + t^{\gamma_2} |\log_2(t)| + 1 |\log_2(t)|^\sigma \chi_{(1,\infty]}(t) \right) \, dt < \infty,
\]

then

\[
\| \mathcal{H}_{\Phi,\Omega}(f) \|_{K_q^{\alpha^*,p}(\omega_1,\omega_2)} \lesssim C_6 \| \Omega \|_{L^q(S_{n-1})} \| f \|_{H \tilde{K}_q^{\alpha^*,p}(\omega_1,\omega_2)} \quad \text{for all} \quad f \in H \tilde{K}_q^{\alpha^*,p}(\omega_1,\omega_2).
\]

**Proof.** We fix a nonnegative integer \( s \geq [\alpha - n(1-1/q)] \). Then, let a function \( a \) be any central \((\alpha, q, s; \omega_1, \omega_2)\) atom with supp \((a) \subset B_{j_a}\). To prove the theorem, it suffices to show that

\[
\| \mathcal{H}_{\Phi,\Omega}(a) \|_{K_q^{\alpha^*,p}(\omega_1,\omega_2)} \lesssim \begin{cases} C_5 \| \Omega \|_{L^q(S_{n-1})}, & p = 1, \\ C_6 \| \Omega \|_{L^q(S_{n-1})}, & p \in (0, 1), \quad \sigma > (1-p)/p. \end{cases}
\]


As mentioned above, we have
\[ \tilde{b}_k(x) = \int_{(2^{k-1}, 2^k)} \int_{S_{n-1}} \frac{\Phi(t)}{t} |\Omega(y')| |a(t^{-1}|x,y)| dy' dt, \]
with \( \text{supp}(\tilde{b}_k) \subset B_{j_{a+k}} \). Using the Minkowski inequality, we obtain
\[ \|\tilde{b}_k\|_{L^{q}^{+}(\omega_2)} = \left( \int_{B_{j_{a+k}}} \left( \int_{(2^{k-1}, 2^k)} \int_{S_{n-1}} \frac{\Phi(t)}{t} \cdot |\Omega(y')| |a(t^{-1}|x,y)| dy' dt \right)^{q^*} \omega_2(x) dx \right)^{1/q^*} \leq \int_{(2^{k-1}, 2^k)} \frac{\Phi(t)}{t} \int_{S_{n-1}} |\Omega(y')| |a(t^{-1}|x,y)| \|L^{q}^{+}(\omega_2, B_{j_{a+k}})\| dy' dt. \]

By the assumption \( q > q^* r_{\omega_2} \), there exists an \( r \in (1, r_{\omega_2}) \) such that \( q = q^* r' \), where \( r' \) is the conjugate real number of \( r \). Thus, by the Hölder inequality and the reverse Hölder inequality, we have
\[ \|a(t^{-1}|y')\|_{L^{q}^{+}(\omega_2, B_{j_{a+k}})} \leq \|a(t^{-1}|y')\|_{L^{q}(B_{j_{a+k}})} \left( \int_{B_{j_{a+k}}} \omega_2(x) dx \right)^{1/(rq^*)} \leq \|a(t^{-1}|y')\|_{L^{q}(B_{j_{a+k}})} \|B_{j_{a+k}}\|^{-1/q} \omega_2(B_{j_{a+k}})^{1/q}. \]

As a consequence, using the Hölder inequality and the argument as for (3.3) above, we obtain
\[ \|\tilde{b}_k\|_{L^{q}^{+}(\omega_2)} \leq \|B_{j_{a+k}}\|^{-1/q} \omega_2(B_{j_{a+k}})^{1/q} \int_{(2^{k-1}, 2^k)} \frac{\Phi(t)}{t} \int_{S_{n-1}} \|\Omega(y')\|_{L^{q}(S_{n-1})} \|a(t^{-1}|y')\|_{L^{q}(B_{j_{a+k}})} dy' dt \leq \|B_{j_{a+k}}\|^{-1/q} \omega_2(B_{j_{a+k}})^{1/q} \|\Omega\|_{L^{q}(S_{n-1})} \int_{(2^{k-1}, 2^k)} \frac{\Phi(t)}{t} \int_{S_{n-1}} \|a(t^{-1}|y')\|_{L^{q}(B_{j_{a+k}})} dy' dt \leq \|B_{j_{a+k}}\|^{-1/q} \omega_2(B_{j_{a+k}})^{1/q} \|\Omega\|_{L^{q}(S_{n-1})} \int_{(2^{k-1}, 2^k)} \frac{\Phi(t)}{t} \int_{S_{n-1}} \|a(t^{-1}|y')\|_{L^{q}(B_{j_{a+k}})} dy' dt. \]

By applying Proposition 3 and the inequality \( \|a\|_{L^{q}(\omega_2)} \leq \omega_1(B_{j_{a}})^{-\alpha/n} \), we obtain
\[ \|a\|_{L^{q}(B_{t^{-1/2}j_{a+k}})} \leq \|B_{t^{-1/2}j_{a+k}}\|^{1/q} \omega_2(B_{t^{-1/2}j_{a+k}})^{-1/q} \|a\|_{L^{q}(\omega_2)} \]
\[ \leq \|B_{t^{-1/2}j_{a+k}}\|^{1/q} \omega_2(B_{t^{-1/2}j_{a+k}})^{-1/q} \omega_1(B_{j_{a}})^{-\alpha/n}. \]

Therefore, using the fact that
\[ \left( \frac{\|B_{t^{-1/2}j_{a+k}}\|}{\|B_{j_{a+k}}\|} \right)^{1/q} \approx t^{-\alpha/n}, \]
we can write
\[ \|\tilde{b}_k\|_{L^{q}^{+}(\omega_2)} \leq \|\Omega\|_{L^{q}(S_{n-1})} \int_{(2^{k-1}, 2^k)} \frac{\Phi(t)}{t} \omega_2(B_{j_{a+k}})^{1/q} \omega_1(B_{j_{a}})^{-\alpha/n} dt. \]

From \( 2^{k-1} < t \leq 2^k \), we have \( \omega_2(B_{t^{-1/2}j_{a+k}})^{-1/q} \leq \omega_2(B_{j_{a}})^{-1/q} \). This shows that
\[ \|\tilde{b}_k\|_{L^{q}^{+}(\omega_2)} \leq \omega_1(B_{j_{a+k}})^{-\alpha/n} \|\Omega\|_{L^{q}(S_{n-1})} \int_{(2^{k-1}, 2^k)} \frac{\Phi(t)}{t} U_{\omega_1, \omega_2, j_{a}, j_{a}} dt, \]
where \( U_{\omega_1, \omega_2, j_{a}, j_{a}} = \omega_1(B_{j_{a+k}})^{\alpha/n} \omega_2(B_{j_{a+k}})^{1/q} \omega_2(B_{j_{a}})^{-1/q} \omega_1(B_{j_{a}})^{-\alpha/n} \). Now, by (3.17) and (3.18), we have
\[ U_{\omega_1, \omega_2, j_{a}, j_{a}} = \frac{\omega_2(B_{j_{a+k}})^{1/q} \omega_2(B_{j_{a}})^{-1/q} \omega_1(B_{j_{a+k}})^{\alpha/n}}{\omega_2(B_{j_{a+k}})^{1/q} \omega_2(B_{j_{a}})^{1/q} \omega_1(B_{j_{a+k}})^{\alpha/n}}. \]
Using Proposition 2, we obtain

\[
\mathcal{U}_{\omega, k, j,a} = \begin{cases} 
\frac{|B_{j,a+k}^1|}{|B_{j,a}|} \frac{1}{1+\alpha/n} \left| \frac{|B_{j,a+k}^2|}{|B_{j,a}|} \right| \left( \alpha^*/\omega \right)^{\alpha/n} & \text{if } k \geq 1, \\
\frac{1}{1+\alpha/n} \left| \frac{|B_{j,a+k}^1|}{|B_{j,a}|} \right| \left( \alpha^*/\omega \right)^{\alpha/n} & \text{otherwise},
\end{cases}
\]

From (3.20) with \(2^{k-1} < t \leq 2^k\), we find

\[
\|\tilde{b}_l\|_{L^q(\omega_2)} \lesssim \omega_1(B_{j,a+k})^{-\alpha/n} \|\Omega\|_{L^{p'}(S_{n-1})} \cdot \begin{cases} 
\int_{(2^{k-1},2^k]} \frac{|\Phi(t)|}{t^{1/2}} \, dt & \text{if } k \geq 1, \\
\int_{(2^{k-1},2^k]} \frac{|\Phi(t)|}{t^{1/2}} \, dt & \text{otherwise},
\end{cases}
\]

Next, for simplicity, we denote

\[
\tilde{b}_{j,a,k}^{**} = \begin{cases} 
\tilde{b}_k \|\Omega\|_{L^{p'}(S_{n-1})}^{-1} \mu_k^{p-1} & \text{if } \mu_k^* \neq 0, \\
0 & \text{otherwise}.
\end{cases}
\]

Thus, we have

\[
\sum_{k \in \mathbb{Z}} \tilde{b}_k = \sum_{k \in \mathbb{Z}} \|\Omega\|_{L^{p'}(S_{n-1})} \mu_k^{p} \tilde{b}_{j,a,k}^{**}
\]

where \(\tilde{b}_{j,a,k}^{**}\) is a dyadic central \((\alpha^*, q^*, \omega_1, \omega_2)\)-unit. Hence, by inequality (3.10) and Theorem 1, it immediately follows that

\[
\|\mathcal{H}_{\Phi, \omega}(a)\|_{K_{q^*}^{\alpha^*, p}(\omega_1, \omega_2)} \lesssim \|\Omega\|_{L^{p'}(S_{n-1})} \left( \sum_{k \in \mathbb{Z}} |\mu_k^*|^{p} \right)^{1/p}.
\]

By estimating as in the final part of the proof of Theorem 6 above, we obtain

\[
\left( \sum_{k \in \mathbb{Z}} |\mu_k^*|^{p} \right)^{1/p} \lesssim \begin{cases} 
C_5, & p = 1, \\
C_6, & p \in (0, 1), \quad \sigma > (1-p)/p.
\end{cases}
\]

This implies that inequality (3.19) is valid. Hence the proof of this theorem is complete. \(\square\)

4. THE MAIN RESULTS ABOUT THE BOUNDEDNESS OF \(H_{\Phi, A}\)

For a matrix \(A = (a_{ij})_{n \times n}\), we define the norm of \(A\) as follows \(\|A\| = \left( \sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{1/2}\). It is known that \(|Ax| \leq \|A\| \|x\|\) for any vector \(x \in \mathbb{R}^n\). In particular, if \(A\) is invertible, then

\[
\|A\|^{-n} \leq \det(A^{-1}) \leq \|A^{-1}\|^n.
\]

Our first main result in this section is as follows.

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Theorem 8. Let $0 < p ≤ 1 < q < ∞$, $α ∈ [n(1−1/q), ∞)$, $ω_1 = |x|^{β_1}$, $ω_2 = |x|^{β_2}$ with $β_1, β_2 ∈ (-n, 0)$, and let there exist $m, M ∈ ℤ$ such that $2^m < \|A^{-1}(y)\| ≤ 2^M$, a.e $y ∈ supp(Φ)$. Suppose that the following conditions hold:

\[
ρ_A = \underset{y ∈ supp(Φ)}{ess \sup} \|A^{-1}(y)\| |A(y)| < ∞, \quad (4.1)
\]

\[
\int_{ℝ^n} \frac{|Φ(y)|}{|y|^n} dy < ∞. \quad (4.2)
\]

Then

\[
\|H_{Φ,A}(f)\|_{HK^{α,p}_q(ω_1,ω_2)} \lesssim \left( \int_{ℝ^n} \frac{|Φ(y)|}{|y|^n} dy \right) \|f\|_{HK^{α,p}_q(ω_1,ω_2)} \quad \text{for all } f ∈ HK^{α,p}_q(ω_1,ω_2).
\]

Proof. Just as in the proof of Theorem 4, it suffices to prove that

\[
\|H_{Φ,A}(a)\|_{HK^{α,p}_q(ω_1,ω_2)} \lesssim \int_{ℝ^n} \frac{|Φ(y)|}{|y|^n} dy,
\]

for a nonnegative integer $s ≥ [α − n(1−1/q)]$ and for any central $(α, q, s; ω_1, ω_2)$-atom $a$ with $supp(a) ⊂ B_{j_0}$ and $\|a\|_{L^q(ω_2)} \lesssim ω_1(B_{j_0})^{−α/n}$.

We can now write

\[
H_{Φ,A}(a)(x) = \sum_{k=m+1}^{M} \left( \int_{2^{k−1}<\|A^{-1}(y)\|≤2^k} \frac{Φ(y)}{|y|^n} a(A(y)x) dy := \sum_{k=m+1}^{M} c_k(x). \right.
\]

Note that $supp(a) ⊂ B_{j_0}$. Let $x ∈ ℝ^n$ satisfy $|x| > 2^{j_0+k}$. Since $\|A^{-1}(y)\| ∈ (2^{k−1}, 2^k]$, we have

\[
|A(y)x| ≥ \|A^{-1}(y)\|^{-1}|x| > 2^{-k}2^{j_0+k} = 2^{j_0}.
\]

This implies $c_k(x) = 0$. From this, we immediately deduce

\[
supp(c_k) ⊂ B_{j_0+k}. \quad (4.4)
\]

By the Minkowski inequality, we have

\[
\|c_k\|_{L^q(ω_2)} = \left( \int_{ℝ^n} \left| \int_{2^{k−1}<\|A^{-1}(y)\|≤2^k} \frac{Φ(y)}{|y|^n} a(A(y)x) dy \right|^q \omega_2(x) dx \right)^{1/q} ≤ \int_{2^{k−1}<\|A^{-1}(y)\|≤2^k} \frac{Φ(y)}{|y|^n} \left( \int_{ℝ^n} |a(A(y)x)|^q \omega_2(x) dx \right)^{1/q} dy.
\]

After a change of variables, using the relations $β_2 ≤ 0$ and $\|a\|_{L^q(ω_2)} ≤ ω_1(B_{j_0})^{−α/n}$, we obtain

\[
\left( \int_{ℝ^n} |a(A(y)x)|^q \omega_2(x) dx \right)^{1/q} ≤ |A(y)||−β_2/q|\det A^{-1}(y)|^{1/q} \left( \int_{ℝ^n} |a(z)|^q |z|^{β_2} dz \right)^{1/q} \lesssim |A(y)||−β_2/q|\det A^{-1}(y)|^{1/q}ω_1(B_{j_0})^{−α/n}.
\]

This implies that

\[
\|c_k\|_{L^q(ω_2)} \lesssim \left( \int_{2^{k−1}<\|A^{-1}(y)\|≤2^k} \frac{Φ(y)}{|y|^n} |A(y)||−β_2/q|\det A^{-1}(y)|^{1/q} dy \right) \times \omega_1(B_{j_0+k})^{−α/n} \left( \frac{ω_1(B_{j_0+k})}{ω_1(B_{j_0})} \right)^{α/n}.
\]

Consequently, by letting $\|A^{-1}(y)\| ∈ (2^{k−1}, 2^k]$ and using inequality (3.5) above, we obtain

\[
\|c_k\|_{L^q(ω_2)} ≤ θ_kω_1(B_{j_0+k})^{−α/n}, \quad (4.6)
\]
where
$$\theta_k = \int_{2^{k-1}<\|A^{-1}(y)\|\leq 2^k} \frac{|\Phi(y)|}{|y|^{\beta}|A(y)|^{-\beta_2/q}|\det A^{-1}(y)|^{1/q}|A^{-1}(y)|^{\alpha_2\alpha_3^{-1}}} dy.$$ 

For any $|\zeta| \leq s$, by $c_k x^\zeta \in L^1(\mathbb{R}^n)$, we have
$$\int_{\mathbb{R}^n} c_k(x)x^\zeta \, dx = \int_{\mathbb{R}^n} \left( \int_{2^{k-1}<\|A^{-1}(y)\|\leq 2^k} \frac{\Phi(y)}{|y|^{\beta}} a(A(y)x) \, dy \right) x^\zeta \, dx
= \int_{2^{k-1}<\|A^{-1}(y)\|\leq 2^k} \frac{\Phi(y)}{|y|^{\beta}} \left( \int_{\mathbb{R}^n} a(A(y)x)x^\zeta \, dx \right) \, dy.$$ 

Note that since $\int_{\mathbb{R}^n} a(x)x^\gamma \, dx = 0$ for all $|\gamma| \leq s$, we have
$$\int_{\mathbb{R}^n} a(A(y)x)x^\zeta \, dx = \int_{\mathbb{R}^n} a(z)(A^{-1}(y)z)^\zeta |\det A^{-1}(y)| \, dz
= |\det A^{-1}(y)| \int_{\mathbb{R}^n} a(z) \left( \sum_{|\gamma|\leq|\zeta|} d_{A^{-1}(y),\gamma} z^\gamma \right) \, dz = 0.$$ 

Hence
$$\int_{\mathbb{R}^n} a(A(y)x)x^\zeta \, dx = 0, \quad \text{for all } |\zeta| \leq s. \quad (4.7)$$

Also, note that there exists an $r_a \in \mathbb{Z}$ satisfying $a = 0$ on $B_{r_a}$. For $x \in \mathbb{R}^n$ such that $|x| \leq r_a 2^{k-1} \rho_A^{-1}$, using condition (4.1) and the inclusion $\|A^{-1}(y)\| \in (2^{k-1}, 2^k)$, we obtain
$$\|A(y)x\| \leq \|A(y)\| |x| \leq \rho_A \|A^{-1}(y)\|^{-1} |x| < \rho_A 2^{-k+1} r_a 2^{k-1} \rho_A^{-1} = r_a,$$
which implies $c_k(x) = 0$. Thus,
$$c_k = 0 \text{ on } B_{r_a 2^{k-1} (\rho_A)^{-1}}. \quad (4.8)$$

Denote
$$c_{j_{a,k}} = \left\{ \begin{array}{ll} c_k \theta_k^{-1} & \text{if } \theta_k \neq 0, \\
0 & \text{otherwise.} \end{array} \right.$$ 

We have
$$H_{\Phi, A}(a)(x) = \sum_{k=m+1}^{M} \theta_k c_{j_{a,k}}(x).$$

From (4.4), (4.6), (4.7), and (4.8), it is easy to see that $c_{j_{a,k}}$ is also a central $(\alpha, q, s; \omega_1, \omega_2)_0$-atom. Hence $H_{\Phi, A}(a) \in F_{p}^\alpha(q,s)(\omega_1, \omega_2)$. From this, using Theorem 2, (4.1), and the inequality
$$|\det A^{-1}(y)| \leq \|A^{-1}(y)\|^n,$$
we can write
$$\|H_{\Phi, A}(a)\|_{HK_{q}^\alpha(p,\omega_1, \omega_2)} \lesssim \|H_{\Phi, A}(a)\|_{F_{p}^\alpha(q,s)(\omega_1, \omega_2)} \leq \left( \sum_{k=m+1}^{M} |\theta_k|^p \right)^{1/p} \lesssim \sum_{k=m+1}^{M} |\theta_k|$$
$$\leq \sum_{k=m+1}^{M} \int_{2^{k-1}<\|A^{-1}(y)\|\leq 2^k} \frac{|\Phi(y)|}{|y|^{\beta}} |A(y)|^{-\beta_2/q}|\det A^{-1}(y)|^{1/q}|A^{-1}(y)|^{\alpha_2\alpha_3^{-1}} \, dy
\lesssim \sum_{k=m+1}^{M} \int_{2^{k-1}<\|A^{-1}(y)\|\leq 2^k} \frac{|\Phi(y)|}{|y|^{\beta}} |A(y)|^{-\beta_2/q}|\det A^{-1}(y)|^{1/q}|A^{-1}(y)|^{\alpha_2\alpha_3^{-1}} \, dy$$

This implies that inequality (4.3) is valid. Therefore, the proof of the theorem is complete. \qed
Theorem 9. Let $1 < q < \infty$, $\alpha \in [n(1-1/q), \infty)$, $\omega_1 = |x|^{\beta_1}$, $\omega_2 = |x|^{\beta_2}$ with $\beta_1, \beta_2 \in (-n, 0]$.

(i) If $p = 1$ and
\[
C_7 = \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} |A(y)|^{-\beta_2/q} |\det A^{-1}(y)|^{1/q} |A^{-1}(y)|^{\alpha + \beta_1 \alpha n^{-1}} \, dy < \infty,
\]
then
\[
\|H_{\Phi,A}(f)\|_{K_q^{\alpha-1}(\omega_1,\omega_2)} \leq C_7 \|f\|_{HK_q^{\alpha-1}(\omega_1,\omega_2)} \quad \text{for all } f \in HK_q^{\alpha-1}(\omega_1,\omega_2).
\]

(ii) If $0 < p < 1$, $\sigma > (1-p)/p$, and
\[
C_8 = \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} |A(y)|^{-\beta_2/q} |\det A^{-1}(y)|^{1/q} |A^{-1}(y)|^{\alpha + \beta_1 \alpha n^{-1}}
\times (\chi_{\{\|A^{-1}(y)\| \leq 1\}} \log_2 |A^{-1}(y)| |A^{-1}(y)| |A^{-1}(y)| + 1)^{\delta}) \, dy < \infty,
\]
then
\[
\|H_{\Phi,A}(f)\|_{K_q^{\alpha,p}(\omega_1,\omega_2)} \leq C_8 \|f\|_{HK_q^{\alpha,p}(\omega_1,\omega_2)} \quad \text{for all } f \in HK_q^{\alpha,p}(\omega_1,\omega_2).
\]

Proof. Let us fix a nonnegative integer $s \geq \lfloor \alpha - n(1-1/q) \rfloor$, and let $a$ be any central $(\alpha, q, s; \omega_1, \omega_2)0$-atom with $\text{supp} (a) \subset B_{j_a}$ and $\|a\|_{L^q(\omega_2)} \lesssim \omega_1(B_{j_a})^{-\alpha/n}$. To complete the proof of the theorem, it suffices to prove that the following inequality holds:
\[
\|H_{\Phi,A}(a)\|_{K_q^{\alpha,p}(\omega_1,\omega_2)} \leq \left\{ \begin{array}{ll}
C_7, & p = 1, \\
C_8, & p \in (0, 1), \quad \sigma > (1-p)/p.
\end{array} \right. \tag{4.9}
\]

We have
\[
|H_{\Phi,A}(a)(x)| \leq \sum_{k \in \mathbb{Z}} \int_{2^{k-1} < \|A^{-1}(y)\| \leq 2^k} \frac{|\Phi(y)|}{|y|^n} |a(A(y)x)| \, dy := \sum_{k \in \mathbb{Z}} \tilde{c}_k(x). \tag{4.10}
\]
From (4.4) and (4.6), we instantly have
\[
\text{supp} (\tilde{c}_k) \subset B_{j_a+k} \quad \text{and} \quad \|	ilde{c}_k\|_{L^q(\omega_2)} \lesssim \theta_k \omega_1(B_{j_a+k})^{-\alpha/n}. \tag{4.11}
\]
Set
\[
\tilde{c}_{j_a,k} = \left\{ \begin{array}{ll}
\tilde{c}_k \theta_k^{-1} & \text{if } \theta_k \neq 0, \\
0 & \text{otherwise}.
\end{array} \right.
\]
Thus, we obtain
\[
\sum_{k \in \mathbb{Z}} \tilde{c}_k = \sum_{k \in \mathbb{Z}} \theta_k \tilde{c}_{j_a,k},
\]
where
\[
\theta_k = \int_{2^{k-1} < \|A^{-1}(y)\| \leq 2^k} \frac{|\Phi(y)|}{|y|^n} |A(y)|^{-\beta_2/q} |\det A^{-1}(y)|^{1/q} |A^{-1}(y)|^{\alpha + \beta_1 \alpha n^{-1}} \, dy.
\]
From (4.11), it is easy to see that $\tilde{c}_{j_a,k}$ is a dyadic central $(\alpha, q, \omega_1, \omega_2)$-unit. As a consequence, by Theorem 1, we have
\[
\|H_{\Phi,A}(a)\|_{K_q^{\alpha,p}(\omega_1,\omega_2)} \leq \left( \sum_{k \in \mathbb{Z}} |\theta_k| \right)^{1/p}.
\]
Case 1: $p = 1$. It follows that
\[
C_7 = \sum_{k \in \mathbb{Z}} |\theta_k| = \sum_{k \in \mathbb{Z}} \int_{2^{k-1} < \|A^{-1}(y)\| \leq 2^k} \frac{|\Phi(y)|}{|y|^n} |A(y)|^{-\beta_2/q} |\det A^{-1}(y)|^{1/q} |A^{-1}(y)|^{\alpha + \beta_1 \alpha n^{-1}} \, dy.
\]
Case 2: $p \in (0,1)$ and $\sigma > (1-p)/p$. By putting
\[
G_{\Phi, A}(y) = \frac{|\Phi(y)|}{|y|^n} \|A(y)||^{-\beta_2/q}|\det A^{-1}(y)|^{1/q}\|A^{-1}(y)||^{n+\beta_1 n-1}
\]
and using the Hölder inequality again, we obtain
\[
\left(\sum_{k \in \mathbb{Z}} |\theta_k|^p\right)^{1/p} \lesssim \sum_{k \in \mathbb{Z}} |k|^\sigma |\theta_k|
\]
\[
= \sum_{k=1}^{\infty} \int_{2^{k-1} < \|A^{-1}(y)\| \leq 2^k} G_{\Phi, A}(y).|k|^{\sigma} dy + \sum_{k=\infty}^{0} \int_{2^{k-1} < \|A^{-1}(y)\| \leq 2^k} G_{\Phi, A}(y).|k|^{\sigma} dy
\]
\[
\lesssim \sum_{k=1}^{\infty} \int_{2^{k-1} < \|A^{-1}(y)\| \leq 2^k} G_{\Phi, A}(y). (\log_2 \|A^{-1}(y)\| + 1)^{\sigma} dy + \sum_{k=\infty}^{0} \int_{2^{k-1} < \|A^{-1}(y)\| \leq 2^k} G_{\Phi, A}(y). \log_2 \|A^{-1}(y)\|^{\sigma} dy = C_8.
\]
Thus, inequality (4.9) is established. Thus, the proof of the theorem is complete. \qed

By similar arguments as in the proofs of Theorem 6 and Theorem 7, we also obtain the following useful results. The details of the proofs are left to the reader.

**Theorem 10.** Suppose $1 < q < \infty$, $\alpha \in [n(1 - 1/q), \infty)$, $\omega_2 = |x|^{\beta_2}$ with $\beta_2 \in (-n,0]$ and $\omega_1 \in A_1$ with the finite critical index $r_{\omega_1}$ for the reverse Hölder condition and $\delta \in (1, r_{\omega_1})$.

(i) If $p = 1$ and
\[
C_9 = \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} \|A(y)||^{-\beta_2/q}|\det A^{-1}(y)|^{1/q}
\]
\[
\times (\chi_{\{\|A^{-1}(y)\| \leq 1\}} \|A^{-1}(y)\|^{(\delta - 1)\delta^{-1}} + \chi_{\{\|A^{-1}(y)\| > 1\}} \|A^{-1}(y)\|^{\alpha}) dy < \infty,
\]
then we have
\[
\|H_{\Phi, A}(f)\|_{K_q^{\alpha,1}(\omega_1, \omega_2)} \lesssim C_9 \|f\|_{HK_q^{\alpha,1}(\omega_1, \omega_2)} \quad \text{for all } f \in HK_q^{\alpha,1}(\omega_1, \omega_2).
\]

(ii) If $0 < p < 1$, $\sigma > (1-p)/p$ and
\[
C_{10} = \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} \|A(y)||^{-\beta_2/q}|\det A^{-1}(y)|^{1/q}(\chi_{\{\|A^{-1}(y)\| \leq 1\}} \|A^{-1}(y)\|^{(\delta - 1)\delta^{-1}} |\log_2 \|A^{-1}(y)\||^{\sigma}
\]
\[
+ \chi_{\{\|A^{-1}(y)\| > 1\}} \|A^{-1}(y)\|^{\alpha} |\log_2 \|A^{-1}(y)\|| + 1)^{\sigma}) dt < \infty,
\]
then
\[
\|H_{\Phi, A}(f)\|_{K_q^{\alpha,p}(\omega_1, \omega_2)} \lesssim C_{10} \|f\|_{HK_q^{\alpha,p}(\omega_1, \omega_2)} \quad \text{for all } f \in HK_q^{\alpha,p}(\omega_1, \omega_2).
\]

**Theorem 11.** Let $1 \leq q^* < q < \infty$, $0 < \alpha^* < 4$, $\alpha \in [n(1 - 1/q), \infty)$, $\omega_1 \in A_1$ with the finite critical index $r_{\omega_1}$ for the reverse Hölder condition and $\delta_i \in (1, r_{\omega_i})$, for all $i=1,2$. Suppose that $q > q^* r_{\omega_2}$ and that the assumptions (3.17) and (3.18) of Theorem 7 hold.

(i) If $p = 1$ and
\[
C_{11} = \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q} \|A(y)||^{\alpha/n/q}
\]
\[
\times (\chi_{\{\|A^{-1}(y)\| \leq 1\}} \|A^{-1}(y)\|^{\gamma_1} + \chi_{\{\|A^{-1}(y)\| > 1\}} \|A^{-1}(y)\|^{\gamma_2}) dy < \infty,
\]
then
\[
\|H_{\Phi, A}(f)\|_{K_q^{\alpha,1}(\omega_1, \omega_2)} \lesssim C_{11} \|f\|_{HK_q^{\alpha,1}(\omega_1, \omega_2)} \quad \text{for all } f \in HK_q^{\alpha,1}(\omega_1, \omega_2).
\]
(ii) If $0 < p < 1$, $\sigma > (1 - p)/p$ and
\[ C_{12} = \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q} \|A(y)\|^{n/q} \chi_{\{|A^{-1}(y)| \leq 1\}} \|A^{-1}(y)\|^{\gamma_1} \log_2 \|A^{-1}(y)\|^{\gamma_2} dy < \infty, \]
then
\[ \|H_{\Phi,A}(f)\|_{K_{q}^{\alpha,p}((\omega_1,\omega_2))} \lesssim C_{12} \|f\|_{H_{K_{q}^{\alpha,p}((\omega_1,\omega_2)} \text{ for all } f \in H_{K_{q}^{\alpha,p}((\omega_1,\omega_2)} \].

Here $\gamma_1, \gamma_2$ are defined in Theorem 7.

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REFERENCES

1. K. Andersen and E. Sawyer, “Weighted norm inequalities for the Riemann–Liouville and Weyl fractional integral operators,” Trans. Amer. Math. Soc. 308, 547–558 (1988).
2. K. F. Andersen, “Boundedness of Hausdorff operators on $L^p(\mathbb{R}^n)$, $H^1(\mathbb{R}^n)$, and $BMO(\mathbb{R}^n)$,” Acta Sci. Math. (Szeged). 69, 409–418 (2003).
3. G. Brown and F. Móricz, “Multivariate Hausdorff operators on the spaces $L^p(\mathbb{R}^n)$,” J. Math. Anal. Appl. 271, 443–454 (2002).
4. M. Bownik, “Boundedness of operators on Hardy spaces via atomic decompositions,” Proc. Amer. Math. Soc. 133, 3535–3542 (2005).
5. J. García-Cuerva, “Hardy spaces and Beurling algebras,” J. London Math. Soc. 39(2), 499–513 (1989).
6. J. García-Cuerva and M. L. Herrero, “A theory of Hardy spaces associated to the Herz spaces,” Proc. London Math. Soc. 69 (3), 605–628 (1994).
7. J. Chen, D. Fan, and J. Li, “Hausdorff operators on function spaces,” Chinese Annals of Mathematics, Series B. 33, 537–556 (2012).
8. N. M. Chuong, Pseudodifferential Operators and Wavelets over Real and p-Adic Fields (Springer-Verlag, Berlin, 2018).
9. N. M. Chuong, D. V. Duong and H. D. Hung, “Bounds for the weighted Hardy–Cesàro operator and its commutator on weighted Morrey–Herz type spaces,” Z. Anal. Anwend. 35, 489–504 (2016).
10. N. M. Chuong, D. V. Duong, and K. H. Dung, “Multilinear Hausdorff operators on some function spaces with variable exponent," arxiv.org/abs/1709.08185 (2017).
11. N. M. Chuong, N. T. Hong, and H. D. Hung, “Multilinear Hardy–Cesàro operator and commutator of the product of Morrey–Herz spaces,” Analysis Math. 43 (4), 547–565 (2017).
12. M. Dyachenko, E. Nursultanov, and S. Tikhonov, “Hardy–Littlewood and Pitt’s inequalities for Hausdorff operators,” Bull. Sci. Math. 147, 40–57 (2018).
13. M. M. Dzherbashyan, “A generalized Riemann–Liouville operator and some of its applications,” Izv. Akad. Nauk SSSR Ser. Mat. 32 (5), 1075–1111 (1968).
14. C. Fefferman and E. M. Stein, “$H^p$ spaces of several variables,” Acta Math. 129, 137–193 (1972).
15. Z. W. Fu, S. L. Gong, S. Z. Lu, and W. Yuan, “Weighted multilinear Hardy operators and commutators,” Forum Math. 27, 2825–2851 (2015).
16. L. Grafakos, L. Liu, and D. Yang, “Maximal function characterizations of Hardy spaces on RD-spaces and their applications,” Sci. China Ser. A. 51, 2253–2284 (2008).
17. F. Hausdorff, “Summation methoden und Momentenfolgen,” I. Math. Z. 9, 74–109 (1921).
18. T. Hytönen, C. Pérez, and E. Rela, “Sharp reverse Hölder property for $A_{\infty}$ weights on spaces of homogeneous type,” J. Funct. Anal. 263, 3883–3899 (2012).
19. W. A. Hurwitz and L. L. Silverman, “The consistency and equivalence of certain definitions of summabilities,” Trans. Amer. Math. Soc. 18, 1–20 (1917).
20. S. Indratno, D. Maldonado, and S. Silwal, “A visual formalism for weights satisfying reverse inequalities,” Expo. Math. 33, 1–29 (2015).
21. A. Lerner and E. Liflyand, “Multidimensional Hausdorff operators on the real Hardy space,” J. Austr. Math. Soc. 83, 79–86 (2007).
22. E. Liflyand and F. Móricz, “The Hausdorff operator is bounded on the real Hardy space $H^1(\mathbb{R})$,” Proc. Amer. Math. Soc. 128, 1391–1396 (2000).
23. E. Liflyand and A. Miyachi, “Boundedness of the Hausdorff operators in $H^p$ spaces, $0 < p < 1$,” Studia Math. 194, 279–292 (2009).
24. E. Liflyand and A. Miyachi, “Boundedness of multidimensional Hausdorff operators in $H^p$ spaces, $0 < p < 1$,” Trans. Amer. Math. Soc. 371, 1391–1396 (2019).
25. E. Liflyand, “Hausdorff operators on Hardy spaces,” Eurasian Math. J. 4, 101–141 (2013).
26. S. Lu and D. Yang, “The decomposition of weighted Herz space on $\mathbb{R}^n$ and its applications,” Sci. China Ser. A. 38, 147–158 (1995).
27. S. Lu and D. Yang, “The weighted Herz-type Hardy space and its applications,” Sci. China Ser. A. 38, 662–673 (1995).
28. S. Lu and D. Yang, “Oscillatory singular integrals on Hardy spaces associated with Herz spaces,” Proc. Amer. Math. Soc. 123, 1695–1701 (1995).
29. S. Lu and D. Yang, “Some characterizations of weighted Herz-type Hardy spaces and their applications,” Acta Math. Sinica (N. S.). 13, 45–58 (1997).
30. A. Miyachi, “Boundedness of the Cesàro operator in Hardy space,” J. Fourier Anal. Appl. 10, 83–92 (2004).
31. B. Muckenhoupt, “Weighted norm inequalities for the Hardy maximal function,” Trans. Amer. Math. Soc. 165, 207–226 (1972).
32. F. Móricz, “Multivariate Hausdorff operators on the spaces $H^1(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$.” Analysis Math. 31, 31–41 (2005).
33. S. Meda, P. Sjögren, and M. Vallarino, “On the $H^1 - L^1$ boundedness of operators,” Proc. Amer. Math. Soc. 136, 2921–2931 (2008).
34. Y. Meyer, M. Taibleson, and G. Weiss, “Some functional analytic properties of the spaces $B_q$ generated by blocks,” Indiana Univ. Math. J. 34, 493–515 (1985).
35. J. Ruan and D. Fan, “Hausdorff operators on the weighted Herz-type Hardy spaces,” Math. Inequal. Appl. 19, 565–587 (2016).
36. E. M. Stein, Harmonic Analysis, Real-Variable Methods, Orthogonality, and Oscillatory Integrals, (Princeton University Press, Princeton, NJ, 1993).
37. V. D. Stepanov and E. P. Ushakova, “Hardy–Steklov Operators and Duality Principle in Weighted Sobolev Spaces of the First Order,” Dokl. Math. 97(3), 232–235 (2018).
38. S. S. Volosivets, “Hausdorff operators on $p$-adic linear spaces and their properties in Hardy, BMO, and Hölder spaces,” Math. Notes 93, 382–391 (2013).
39. J. Xiao, “$L^p$ and BMO bounds of weighted Hardy–Littlewood averages,” J. Math. Anal. Appl. 262, 660–666 (2001).
40. D. Yang and Y. Zhou, “Boundedness of sublinear operators in Hardy spaces on RD spaces via atoms,” J. Math. Anal. Appl. 339, 622–635 (2008).
41. D. Yang and Y. Zhou, “A boundedness criterion via atoms for linear operators in Hardy spaces,” Constr. Approx. 29, 207–218 (2009).
42. Y. Zhou, “Boundedness of sublinear operators in Herz-type Hardy spaces,” Taiwanese J. Math. 13, 983–996 (2009).