ON THE COST OF DECIDING CONSENSUS

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Abstract. We study the computational complexity of a general consensus problem for switched systems. A set of \( n \times n \) stochastic matrices \( \{P_1, \ldots, P_k\} \) is a consensus set if for every switching map \( \tau : \mathbb{N} \to \{1, \ldots, k\} \) and for every initial state \( x(0) \), the sequence of states defined by \( x(t + 1) = P_{\tau(t)} x(t) \) converges to a state whose entries are all identical. We show in this paper that, unless \( P = NP \), the problem of determining if a set of matrices is a consensus set cannot be decided in polynomial-time. As a consequence, unless \( P = NP \), it is not possible to give efficiently checkable necessary and sufficient conditions for consensus. This provides a possible explanation for the absence of such conditions in the current literature on consensus. On the positive side, we provide a simple algorithm which checks whether \( \{P_1, \ldots, P_k\} \) is a consensus set in \( O\left(Bk^{n^2} + k^2n^3\right) \) operations where \( B \) is the number of bits needed to specify each entry of \( P_1, \ldots, P_k \).

Key words. computational complexity, consensus, switched systems.

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1. Introduction. There has been much recent interest in the design of control policies for distributed systems of mobile agents. This has been motivated by the emergence of large scale networks without centralized control and with uncertain or time-varying connectivity. Controllers deployed in such systems ought to be completely decentralized, relying only on local information, and resilient to unexpected node and link failures.

Consensus algorithms are a class of iterative update schemes which are commonly used as building blocks for the design of such controllers. They have recently been used in a variety of contexts, such as coverage control [4], formation control [11,12], distributed estimation [17,18], distributed task assignment [3], distributed optimization [13] and [9]. These algorithms provide a means by which distributed systems with limited or uncertain connectivity can reach a common state. They are often used as subroutines in various application areas.

The problem of finding convenient conditions for the convergence of consensus algorithms is an active area of current research. Much of the present literature is focused on sufficient conditions for consensus algorithms to converge. Such conditions have become progressively more general with time; we refer the reader to [14, 5, 8, 1, 6, 16, 7] for a sampling of the literature. However, despite the significant attention attracted by this problem, there is no known efficiently verifiable condition that is both necessary and sufficient for consensus.

In this paper, we investigate the computational complexity of consensus algorithms of the following form: We are given a set of row-stochastic matrices \( P_1, \ldots, P_k \), an initial state \( x(0) \) and we consider the sequence of states \( (x(0), x(1), \ldots) \) resulting from

\[
x(t + 1) = P_{\tau(t)} x(t) \quad t = 0, 1, \ldots
\]

for some switching map \( \tau : \mathbb{N} \to \{1, \ldots, k\} \). A set of matrices \( P_1, \ldots, P_k \) is said to be a consensus set if such sequences always converge to a state whose entries are all identical. In other words, consensus requires

\[
\lim_{t \to \infty} x(t) = \alpha \mathbf{1}
\]

for all possible switching maps and initial state \( x(0) \) (here \( \mathbf{1} \) denotes the vector of all ones and \( \alpha \) is some scalar which depends on \( x(0) \) and on the switching map). As we will show in the next section, the consensus property is known to be equivalent to the condition that for every switching map \( \tau \) the limit

\[
\lim_{t \to \infty} P_{\tau(t)} \cdots P_{\tau(2)} P_{\tau(1)}
\]

exists and has rank one.

The purpose of this paper is to investigate the computational complexity of deciding consensus. Before we state our main results, we illustrate with a small example that deciding consensus may be a non-trivial
Consider the two graphs of Figure 1.1. To a graph \( G \) we associate a stochastic matrix \( A \) by defining \( a_{ij} = 1/d(i) \) if \((i,j)\) is an edge in \( G \), and \( a_{ij} = 0 \) otherwise. Here \( d(i) \) is the number of neighbors of node \( i \), where we count node \( i \) as its own neighbor if it has a self-loop. To the graphs of Figure 1.1 we associate to the two stochastic matrices \( A_1, A_2 \) and we claim that these do not form a consensus set. Indeed, it is easy to verify that the product \( A_1A_2 \) has the property that the rows corresponding to nodes \( A \) and \( E \) have only a single nonzero entry, namely a 1 on the diagonal. It therefore follows that the limit \( \lim_{k \to \infty} (A_1A_2)^k \) does not approach a rank-1 matrix and the matrices \( A_1 \) and \( A_2 \) do not form a consensus set.

Consider now instead the stochastic transition matrices associated to the two graphs \( G'_1 \) and \( G'_2 \) of Figure 1.2. Note that the only difference with the graphs of Figure 1.1 is the additional self-loop at node \( E \). This turns out to make a difference for consensus: a simple computation reveals that every product of length 4 of these two matrices has a column whose entries are strictly positive. It is well-known (and, in any case, follows out of the more general results we will show in Section 2) that this implies these two matrices form a consensus set.

After this small motivating example, let us now briefly describe our results.

In Section 2, we show that a set of matrices is a consensus set if and only if every product of length \( 2^n \) of matrices from the set has no two rows with disjoint support (\( n \) is the dimension of the matrices). This is a stronger version of a result of Wolfowitz [19]. Verifying if a matrix has rows with disjoint support is an easy computational task and so it immediately follows that consensus is algorithmically decidable.

In Section 3, we turn our attention to the computational complexity of consensus and we show that, although decidable, the problem of deciding consensus is computationally hard. We first prove in Theorem 3.1 that, unless \( P = NP \), the problem of determining if a given set of two stochastic matrices is a consensus set cannot be solved in polynomial-time. We then prove a similar result for undirected matrices. A matrix
$P$ is undirected (or sign pattern symmetric) if $p_{ij} \neq 0$ implies $p_{ji} \neq 0$. Undirected matrices are important for consensus problems because they often appear in applications where information exchange goes in both directions: whenever $i$ communicates with $j$, $j$ communicates with $i$ (with possibly different strengths). For example, suppose the iteration

$$x(t + 1) = P(t)x(t)$$

is implemented by a collection of vehicles seeking to perform some distributed control task. If the variables $x_i(t)$ represent the positions of the vehicles, and if each vehicle will measure and take into account the positions of all neighbors with which it has line-of-sight communication, the resulting matrices $P(t)$ will be undirected. More generally, undirected matrices are likely to appear whenever a consensus process is implemented in a system with symmetry in the sensing or information exchange process. In Theorem 3.2 we prove that the problem of determining if a set is a consensus set cannot be decided in polynomial time (unless $P = NP$) even if we assume all matrices to be undirected. Our proof is for the case of three (or more) matrices. We believe that the same result holds true if there are only two undirected matrices but we haven’t been able to derive the corresponding proof.

As a consequence of these two results there cannot be a necessary and sufficient condition for consensus which is verifiable in polynomial-time (unless $P = NP$). This probably provides an explanation for the state of the current literature in which one can find several sufficient conditions as well as several necessary conditions for consensus, but no tractable conditions that are both necessary and sufficient.

In Section 3 we make a final observation that if the matrices are symmetric (rather than sign pattern symmetric), then the problem of consensus becomes decidable in polynomial time.

The sections 2 and 3 are not independent, as the proofs in Section 3 depend on certain lemmas in Section 2. We conclude our contribution with some remarks and open problems in Section 4.

2. Decidability of the consensus problem. In this section we provide an algorithm for checking whether a set of matrices is a consensus set, thus proving that the consensus problem is decidable. The algorithm is based on a certain finitary condition for consensus. While consensus was initially defined in terms of infinite product of matrices, Theorem 2.2 below shows that a set of matrices is a consensus sets if and only if all products of matrices from this set of length $2^n$ satisfy a certain easily-checkable property.

We begin with a formal statement of our results.

**Definition 2.1.** A (row) stochastic matrix is called scrambling if no two of its rows have disjoint supports.

**Theorem 2.2.** The set of stochastic matrices $\mathcal{P} = \{P_1, \ldots, P_k\}$ is a consensus set if and only if every product of matrices from $\mathcal{P}$ of length $2^n$ is scrambling.

This theorem is a stronger version of a result of Wolfowitz [19]. Arguments based on scrambling have often been used in the literature to show that sets of matrices satisfying certain conditions are consensus sets [13]. Theorem 2.2 may be viewed as a partial converse to some of these results. Since it is easy to check whether or not a matrix is scrambling, it also follows from the theorem that consensus is algorithmically decidable.

**Corollary 2.3.** Given a set of nonnegative, stochastic matrices $\mathcal{P} = \{P_1, \ldots, P_k\}$ with rational entries whose bit-size is at most $B$, there is an algorithm which checks whether $\mathcal{P}$ is a consensus set in

$$O\left( Bk n^2 + k^{2n} n^3 \right)$$

The rest of this section will be dedicated to the proof of Theorem 2.2 and Corollary 2.3.

We now give a brief outline of the proof. Our first step in Lemma 2.5 is to show that $\mathcal{P}$ is a consensus set if and only if all infinite right-products of matrices from $\mathcal{P}$ have row differences which asymptotically vanish. In the subsequent Lemma 2.8 we argue that this in turn is equivalent to the property that the supports of the rows of any product of matrices from $\mathcal{P}$ eventually overlap; moreover, the auxiliary Lemma 2.7 shows that if this overlap occurs, it must occur in products of length at most $2^n$. Putting these pieces together leads to the proof of Theorem 2.2. Corollary 2.3 follows as a straightforward consequence after some simple manipulations are used to obtain a neater expression for the total number of operations.
We now proceed to the details of the proof.

**Definition 2.4.** For a nonnegative stochastic matrix $P$ whose $i$'th row we will denote by $p_i$, we define

$$\delta(P) = \max_{i,j=1,\ldots,n}||p_i - p_j||_1.$$  

It is not hard to see that for stochastic matrices $P_1, P_2$,

(2.1) \hspace{1cm} \delta(P_1P_2) \leq \delta(P_2).

Indeed, this follows from the observation that the rows of $P_1P_2$ are convex combinations of the rows of $P_2$.

Our first lemma characterizes consensus sets in terms of the asymptotic properties of the rows of right-products of matrices; this is item 4 of the lemma.

**Lemma 2.5.** The following are equivalent:

1. $\mathcal{P} = \{P_1, \ldots, P_k\}$ is a consensus set.
2. For every map $\tau : \mathbb{N} \to \{1, \ldots, k\}$, the limit

$$\lim_{t \to \infty} P_{\tau(t)} \cdots P_{\tau(2)} P_{\tau(1)}$$

exists and has identical rows.
3. For every $\epsilon > 0$ there exists an integer $t(\epsilon)$ such that if $P$ is the product of $t(\epsilon)$ matrices from $\mathcal{P}$, then $\delta(P) < \epsilon$.
4. For all $i, j$, and all $\tau : \mathbb{N} \to \{1, \ldots, k\}$,

$$\lim_{t \to \infty} (e_i - e_j) P_{\tau(1)} P_{\tau(2)} P_{\tau(3)} \cdots P_{\tau(t)} = 0.$$ 

5. For all $\tau : \mathbb{Z} \to \{1, \ldots, k\}$ and all nonnegative vectors $p_1, p_2$ whose entries sum to 1,

$$\lim_{t \to \infty} (p_1 - p_2) P_{\tau(1)} P_{\tau(2)} P_{\tau(3)} \cdots P_{\tau(t)} = 0.$$ 

**Proof.**

• **Equivalence of (1) and (2)** is an elementary consequence of the definition of a consensus set.

• **(2) implies (3):** Suppose item (3) did not hold. That would mean there exists some $\epsilon > 0$ such that for every $i = 1, 2, \ldots$ there is a product of $i$ matrices from $\mathcal{P}$ whose $\delta$ is at least $\epsilon$.

Consider this infinite sequence of products. Observe that there is some $i_1 \in \{1, \ldots, k\}$ such that $P_{i_1}$ appears infinitely many times as the rightmost matrix in these products; out of the products with $P_{i_1}$ as the rightmost matrix, there is some $i_2 \in \{1, \ldots, k\}$ such that $P_{i_2}$ appears infinitely as the second matrix from the right; and proceeding in this way, we can find $i_3, i_4, \ldots$ and so forth.

Observe that the infinite product

$$\cdots P_{i_3} P_{i_2} P_{i_1}$$

has the property that each truncation

$$P_{i_k} \cdots P_{i_1}$$

appears as the rightmost part of a product which equals some matrix $P$ with $\delta(P) \geq \epsilon$. But applying Eq. (2.1), this implies that

$$\epsilon \leq \delta(P) \leq \delta(P_{i_k} \cdots P_{i_1})$$

for all $i_k$. In turn, this implies that the infinite left-product

$$\cdots P_{i_3} P_{i_2} P_{i_1}$$

cannot approach a matrix with identical rows, which contradicts item (2).
• (3) implies (2): Observe that item (3) implies that the sequence \( \Pi_t \) of products \( \Pi_t = P_{\tau(t)} \cdots P_{\tau(1)} \) is a Cauchy sequence. Indeed, for any \( t_1, t_2 > t(\varepsilon) \) the difference between a row of \( \Pi_{t_1} \) and a row of \( \Pi_{t_2} \) cannot be larger than \( \varepsilon \) in the 1-norm because they are both convex combinations of the rows of the matrix \( \Pi_{t(c)} \). Thus \( \Pi_t \) has a limit. Now applying item (3) we immediately get that this limit is a matrix with identical rows.

• Equivalence of (3) and (4): That item (3) implies (4) is trivial; conversely, a sequence of counterexamples to (3) easily yields a sequence of a counterexamples to (4).

• Equivalence of (4) and (5): Item (4) is a special case of item (5). To argue that item (4) implies item (5) observe that every vector whose entries sum to zero can be written as a linear combination of the vectors \( e_i - e_j \). Thus for any \( p_1, p_2 \) satisfying the conditions of the lemma, there exist \( a_{ij} \) such that

\[
p_1 - p_2 = \sum_{i<j} a_{ij} (e_i - e_j),
\]

and so

\[
\lim_{t \to \infty} (p_1 - p_2) P_{\tau(1)} \cdots P_{\tau(t)} = \sum_{i<j} a_{ij} \lim_{t \to \infty} (e_i - e_j) P_{\tau(1)} \cdots P_{\tau(t)} = 0.
\]

\[\square\]

**Remark 2.6.** We wish to remark on the following equivalence which was stated without proof in the introduction: \( \{P_1, \ldots, P_k\} \) is a consensus set if and only if for every map \( \tau : \mathbb{N} \to \{1, \ldots, k\} \), the limit

\[
\lim_{t \to \infty} P_{\tau(t)} \cdots P_{\tau(2)} P_{\tau(1)}
\]

exists and has rank one. Indeed, if \( \{P_1, \ldots, P_k\} \) is a consensus set then this statement immediately follows from item (2) of the preceding lemma. Conversely, if \( \lim_{t} P_{\tau(t)} \cdots P_{\tau(1)} = uv^T \) for some vectors \( u, v \in \mathbb{R}^n \), then because the set of stochastic matrices is a closed set which is also closed under multiplication, \( uv^T \) must be stochastic as well; this forces \( u = 1 \) which implies item (2) of the preceding lemma.

In the next lemma, we take a slight detour to consider matrix products with rows whose supports do not overlap. This result will be useful in what follows.

**Lemma 2.7.** Suppose that \( \Delta_1, \Delta_2 \) are nonnegative, nonzero row vectors with disjoint supports and \( P_1, \ldots, P_l \) is a sequence of matrices from \( \mathcal{P} \) of length \( l \geq 2^n \). Moreover, suppose that the supports of

\[
\Delta_1 P_1 \cdots P_l \quad \text{and} \quad \Delta_2 P_1 \cdots P_l
\]

do not overlap for all \( t = 1, \ldots, l \). Then, there exist nonnegative vectors \( \Delta_1', \Delta_2' \) with disjoint supports and an infinite sequence \( P_1', P_2', \ldots \) of matrices from \( \mathcal{P} \) such that the supports of

\[
\Delta_1' P_1' P_2' \cdots P_l' \quad \text{and} \quad \Delta_2' P_1' P_2' \cdots P_l'
\]

do not overlap for any \( t \).

**Proof.** Define \( S_1^t \) be the support of the row vector \( \Delta_1 P_1 \cdots P_t \), and \( S_2^t \) to be the support of \( \Delta_2 P_1 \cdots P_t \) (for consistency, we will use \( S_0^t \) to mean the support of \( \Delta_1 \) and \( S_0^t \) to mean the support of \( \Delta_2 \)). By assumption, \( S_1^t \cap S_2^t = \emptyset \) for all \( t = 0, 1, \ldots, l \). Note that the pair \( (S_1^t, S_2^t) \) can take on at most \( 2^{2n} \) possible distinct values. Since \( l + 1 > 2^{2n} \) there exist \( t_A < t_B \) in \( 0, 1, \ldots, l \) such that \( (S_1^{t_A}, S_2^{t_A}) = (S_1^{t_B}, S_2^{t_B}) \).

Next, define

\[
\Delta_1' = \Delta_1 P_1 \cdots P_{t_A} \quad \Delta_2' = \Delta_2 P_1 \cdots P_{t_A},
\]

and let the sequence \( P_1', P_2', \ldots \) to be simply the finite sequence \( P_{t_A+1}, \ldots, P_{t_B} \) repeated infinitely many times. By construction, the supports of

\[
\Delta_1' P_1' P_2' \cdots P_l' \quad \text{and} \quad \Delta_2' P_1' P_2' \cdots P_l'
\]
that for a nonnegative stochastic row vector $v^T$ and nonnegative stochastic matrices $Q_1, \ldots, Q_k$, the support of
$$v^T Q_1 \cdots Q_k$$
depends only on the support of $v$ (and not on the values of its positive entries). We can then therefore conclude that the supports of
$$\Delta_1' P_1' P_2' \cdots P_t'$$
and
$$\Delta_1' P_1' P_2' \cdots P_{t \mod t_B - t_A}'$$
are identical, and the same conclusion holds with $\Delta_1'$ replaced with $\Delta_2'$. This now implies that for any $t$, the supports of
$$\Delta_1' P_1' P_2' \cdots P_t'$$
and
$$\Delta_2' P_1' P_2' \cdots P_t'$$
do not overlap. \[\blacksquare\]

Our next lemma shows the equivalence between a characterization of consensus obtained in Lemma 2.3 and a scrambling property of matrix products.

**Lemma 2.8.** The following are equivalent:

1. Item (6) of Lemma 2.3.
2. For any sequence of matrices $P_1, P_2, \ldots$ from $\mathcal{P}$, and for all nonnegative, nonzero vectors $\Delta, \Delta'$ with disjoint support, there is some $t \leq 2^{2^n}$ such that the vectors
$$\Delta P_1 P_2 \cdots P_t$$
and
$$\Delta' P_1 P_2 \cdots P_t$$
have supports which overlap.
3. Every product of $2^{2^n}$ matrices from $\mathcal{P}$ of length $2^{2^n}$ is scrambling.

**Proof.**

- (1) implies (2): Suppose that (2) is false. By the previous lemma, this means there exists an infinite sequence $P_1, P_2, \ldots$, and nonnegative, nonzero vectors $\Delta_1, \Delta_2$ such that the supports of $\Delta_1 P_1 \cdots P_t, \Delta_2 P_1 \cdots P_t$ don’t overlap for any $t$. It’s clear that this remains true even after we normalize the entries of $\Delta_1, \Delta_2$ to sum up to 1. Thus $(\Delta_1 - \Delta_2) P_1 P_2 \cdots P_t$ is a vector with positive entries which add up to 1 and negative entries which add up to $-1$. This means that $\lim_{t \to \infty} (\Delta_1 - \Delta_2) P_1 P_2 \cdots P_t \neq 0$, contradicting item (1).
- (2) implies (1): Suppose item (2) is true. Given any vectors $p_1, p_2$ which are nonnegative with entries summing to 1, define
$$p_1^t = p_1 P_1 \cdots P_t, \quad p_2^t = p_2 P_1 \cdots P_t.$$  
Next, define $p_{\min}^t$ to be the nonnegative vector whose $i$th entry is the smallest of the $i$’th entries of $p_1^t, p_2^t$. Then, we define $\Delta_1^t, \Delta_2^t$ through the relations
$$p_1^t = p_{\min}^t + \Delta_1^t, \quad p_2^t = p_{\min}^t + \Delta_2^t.$$  
Observe that for any $t$, $\Delta_1^t, \Delta_2^t$ are nonnegative, have disjoint support, and their entries must sum to the same value, which we will call $\Delta_t$. Moreover,

$$\begin{align*}
(p_1 - p_2) P_1 \cdots P_t &= p_1^t - p_2^t = \Delta_1^t - \Delta_2^t.
\end{align*}$$
We claim that there exists some \( c > 0 \) with the following property: for any \( t \) there exists a later time \( \hat{t} \) such that
\[
\|P_{t+\hat{t}}^1 - P_{t+\hat{t}}^2\|_1 \leq (1-c)\|P_t^1 - P_t^2\|_1.
\]
This claim immediately implies that item (1) must be true.

We now prove the claim. Now for any \( t \), if one of \( \Delta_t^1, \Delta_t^2 \) is the zero vector, then the conclusion immediately follows; else, they both must be nonzero (because their entries have the same sum), and we may assume without loss of generality that both of them belong to the simplex. Observe that for any sequence \( P_{t+1}, \ldots, P_{t+i} \),
\[
p_{t+i}^1 = p_{t+i}^{\text{min}} P_{t+1} \cdots P_{t+i} + \Delta_t^1 P_{t+1} \cdots P_{t+i}, \quad p_{t+i}^2 = p_{t+i}^{\text{min}} P_{t+1} \cdots P_{t+i} + \Delta_t^2 P_{t+1} \cdots P_{t+i},
\]
so that it suffices to show that there is some \( \hat{t} \) and some \( c > 0 \) such that for any \( P_{t+1}, \ldots, P_{t+i} \), there is some index \( i \) so that the \( i \)th entries of \( \Delta_t^1 P_{t+1} \cdots P_{t+i} \) and \( \Delta_t^2 P_{t+1} \cdots P_{t+i} \) are both at least \( c \).

Now item (2) says that there exists some \( \hat{t} \leq 2^n \) such that for any choice of \( P_{t+1}, \ldots, P_{t+i} \) from \( \mathcal{P} \), the supports of
\[
\Delta_t^1 P_{t+1} \cdots P_{t+i}, \quad \Delta_t^2 P_{t+1} \cdots P_{t+i}
\]
intersect. Because there are only finitely many possible products \( P_{t+1} \cdots P_{t+i} \), it follows that for any \( \Delta_t^1, \Delta_t^2 \) there exists a \( c(\Delta_t^1, \Delta_t^2) > 0 \) such that for any \( P_{t+1}, \ldots, P_{t+i} \) there is some index \( i \) such that both \( \Delta_{t+i}^1 \) and \( \Delta_{t+i}^2 \) have \( i \)'th entry which is at least \( c(\Delta_t^1, \Delta_t^2) \). Now a compactness argument can be used to establish that there exists some \( c > 0 \) such that \( c(\Delta_t^1, \Delta_t^2) > c \) for all \( \Delta_t^1, \Delta_t^2 \) in the simplex - indeed, if this were not so, then we would be able to find some \( \Delta_t^1, \Delta_t^2 \) with disjoint support in the simplex such that the supports of \( \Delta_{t+i}^1 \) and \( \Delta_{t+i}^2 \) do not intersect.

- (2) and (3) are equivalent: Observe that a nonnegative stochastic matrix \( Q \) is scrambling if and only if for all \( i, j \), \( e_i^T Q \) and \( e_j^T Q \) have supports which intersect, or, equivalently, if for all nonnegative, nonzero vectors \( v_1, v_2 \), the vectors \( v_1^T Q \) and \( v_2^T Q \) have supports which intersect.

\[\Box\]

**Proof of Theorem 2.2.** Putting Lemmas 2.5 and 2.8 together, we see that the first item of the former is equivalent to the last item of the latter. This is exactly the statement of the theorem. \[\Box\]

**Proof of Corollary 2.3.** To check whether a set of matrices is a consensus set, we check whether all products of length \( 2^n \) are scrambling and output "yes" if they are and "no" if they are not; by Theorem 2.2, this produces the correct answer. We next argue that this may be completed in \( O(Bk n^2 + k^2 n^3) \) operations.

We first observe that to check whether a given matrix is scrambling, we do not need to know its entries exactly; we only need to know whether they are positive. To make use of this observation, we will go through all the entries of the matrices \( P_1, \ldots, P_k \) and replace each positive number by the symbol \( 1 \) and each zero by the symbol \( 0 \). In algebraic manipulation of these symbols we apply the usual rules with the added convention that \( 1 + 1 = 1 \). Its easy to see that a product of matrices is scrambling if and only if the product of the matrices with symbolic entries as above has rows with overlapping supports. Note that producing the matrices with symbolic entries takes \( O(Bkn^2) \) operations; we next turn to the complexity of checking that all products of length \( 2^n \) of these matrices have supports which overlap.

To compute all products of length \( l \), we first compute all products of size \( l/2 \), and then multiply all possible pairs from the latter; this takes \( O(k^4 n^3) \) operations. Thus to compute all products of length \( 2^n \) takes
\[
O(k^2 n^3) + O(k^2 n^3) + \cdots = O(k^2 n^3).
\]
Finally, to check that each of the resulting matrices has rows with overlapping supports takes \( O(k^2 n^2) \) operations. Thus the final count is \( O(Bkn^2 + k^2 n^3) \). \[\Box\]
3. Complexity of the consensus problem. In this section, we show that unless $P = NP$ there does not exist a polynomial-time algorithm for the consensus problem even in the case of two matrices. We prove a similar result for three undirected matrices. We begin by stating our main results.

**Theorem 3.1.** Unless $P = NP$, the problem of determining if a given set of two stochastic matrices is a consensus set cannot be decided in polynomial-time.

**Theorem 3.2.** Unless $P = NP$, the problem of determining if a given set of three undirected stochastic matrices is a consensus set cannot be decided in polynomial-time.

The assumptions of Theorem 3.2 cannot be relaxed from requiring each of the matrices to be undirected to requiring the matrices to be symmetric. In fact, one can check whether a set of stochastic and symmetric matrices $\{A_1, \ldots, A_k\}$ is a consensus set in polynomial time. Indeed, defining $G(A)$ to be the (undirected) graph with edges $(i, j)$ whenever $a_{ij} > 0$, it is not hard to see that $\{A_1, \ldots, A_k\}$ is a consensus set if and only if each of the graphs $G(A_1), \ldots, G(A_k)$ is connected and non-bipartite.

The remainder of this section will be devoted to the proof of these theorems, which we now briefly outline.

Our strategy will be to show both consensus and the non-satisfiability of a 3-SAT problem are equivalent to the non-existence of a path between two nodes in every possible sequence of two particularly chosen graphs $G_1, G_2$ (our notion of a path in a graph sequence is the straightforward one; the interested reader may refer to Definition 3.4 below for a precise statement). One side of this equivalence is easy: the equivalence of consensus and the non-existence of a path is easily obtained by taking scalings of the adjacency matrices of the graphs and applying the results of the previous section on supports which eventually overlap. On the other hand, to show that the insatisfiability of 3-SAT and the non-existence of a path are equivalent we use a variation of a previous construction from [15]. Finally, to prove the result for undirected matrices we will construct a gadget for simulating directed paths on sequences of two graphs with undirected paths on sequences of three graphs.

We next proceed to the details of the proofs.

**Definition 3.3.** Given a 3-SAT formula $f$, we will define two graphs $G_0(f), G_1(f)$ as follows (the reader may wish to refer to Figure 3.1 for an example).

We will create a node for each clause/variable pair; the node corresponding to clause $i$ and variable $j$ will be called $C_{i,j}$. There will be a node $F$ (intuitively thought of as a “failure node”) and nodes $S_2, \ldots, S_{n+1}$ (intuitively thought of as “intermediate success nodes.”).

For convenience, we will also adopt the following notation. For any $i$, $C_{i,n+1}$ will refer to the node $F$. Moreover, $S_{n+1}$ will sometimes be labeled simply as $S$.

If setting variable $i$ to 0 satisfies clause $j$, then we will put a directed link from $C_{i,j}$ to $S_{i+1}$ in $G_0(f)$; else, we will put a directed link from $C_{i,j}$ to $C_{i,j+1}$ in $G_0(f)$ (note that when $j = n$, this link leads to $F$).

If setting variable $i$ to 1 satisfies clause $j$, then we will put a directed link from $C_{i,j}$ to $S_{i+1}$ in $G_1(f)$; else, we will put a directed link from $C_{i,j}$ to $C_{i,j+1}$ in $G_1(f)$ (note that when $j = n$ this link leads to $F$).

Finally, both $G_1, G_2$ will have self-loops at $F$, links from each $S_i$ to $S_{i+1}$ and links from $S = S_{n+1}$ to all $C_{i,1}$.

**Definition 3.4.** Consider a sequence of directed graphs $G_1, G_2, G_3, \ldots$ on the same vertex set $V$. We will say that $e_1, \ldots, e_{t_2}$ is a path from node $u$ to node $v$ if: (i) $e_i$ is an edge in $G_t$, (ii) the source of $e_i$ is $u$ and the destination of $e_i$ is $v$ (iii) the source of $e_{t_2}$ is $u$ and the same as the destination of $e_{t_2}$ for $i = t_1 + 1, \ldots, t_2$.

**Lemma 3.5.** A 3-SAT formula $f$ is satisfiable if and only there exists a sequence of $G_0(f), G_1(f)$ of length at least $n + 1$ without a path from $S$ to $F$.

**Proof.** Suppose that the 3-SAT problem is satisfiable, i.e., there exists an assignment of $\{0, 1\}$ to the variables satisfying every clause. We need to find a sequence of the graphs $G_0(f), G_1(f)$ of length $n + 1$
without a path from $S$ to $F$. Because the links going out of node $S$ are the same in both $G_0(f), G_1(f)$ (they lead to all of the nodes $C_{i,1}$), equivalently we need to find a sequence of the graphs $G_0(f), G_1(f)$ of length $n$ such that there is no path from any $C_{i,1}$ to $F$.

We will construct such a sequence as follows: we will set the $i$'th graph to be $G_0(f)$ if the $i$'th variable in the satisfying assignment is 0 and $G_1(f)$ if the $i$'th variable in the satisfying assignment is 1.

We argue there is no path from any $C_{i,1}$ to $F$ in this sequence. Indeed, the links from any $C_{i,j}$ either lead “to the right” to $C_{i,j+1}$ or “down” to $S_{j+1}$. Because there is no path of length $n$ or less from any $S_j$ to $F$, it suffices to argue that at some time step, only the “down” edge to some $S_{j+1}$ will be available. But clause $i$ is satisfied by at least one variable, say by variable $k$; and so by construction the only edge from $C_{i,k}$ at time $k$ will lead to $S_{k+1}$. Thus there is no path from $C_{i,1}$ to $C_{i,k+1}$ of length $k$, which means that every path beginning at $C_{i,1}$ of length $k$ arrives at $S_{k+1}$. This proves the “only if” part.

To prove the “if” part, assume that there is a sequence of length $n + 1$ with no path from $S$ to $F$; as we remarked above, this is equivalent to the assumption that there is a sequence of length $n$ with no path from all the $C_{i,1}$ to $F$. Let $x_i$ be 0 if the $i$'th element of the sequence is $G_0(f)$ and 1 if the $i$'th element of the sequence is $G_1(f)$. We argue that this is a satisfying assignment.

Indeed, consider some clause $i$. We know that there is no path of length $n$ from $C_{i,1}$ to $F$. Note that if at every time step $k$ there was a link from $C_{i,k}$ to $C_{i,k+1}$ then a path from $C_{i,1}$ to $F$ would exist. Thus there is some time step $k$ at which there is only a link from $C_{i,k}$ to $S_{k+1}$.

**Definition 3.6.** Define $d_0(f,i)$ to mean the out-degree of node $i$ in $G_0(f)$; $d_1(f,i)$ is defined similarly. We will define the matrix $A_0(f)$ as $[A_0(f)]_{ij} = 1/d_0(f,i)$ if $(i,j)$ is an edge in $G_0(f)$ and 0 otherwise; the matrix $A_1(f)$ is defined similarly. Note that $A_0(f)$ and $A_1(f)$ are nonnegative stochastic matrices.

**Lemma 3.7.** \{A_0(f), A_1(f)\} is a consensus set if and only if every sequence of graphs $G_0(f), G_1(f)$ of length $n + 1$ has a path from $S$ to $F$.

**Proof.** By item 8 of Lemma 2.8, \{A_0(f), A_1(f)\} is a consensus set if and only if every product of $A_0(f), A_1(f)$ of length $2^{2n}$ is scrambling. Note, however, that the support of row $i$ of any such product is exactly the set of nodes reachable from node $i$ via a path in the corresponding sequence of graphs. Thus \{A_0(f), A_1(f)\} is a consensus set if and only if for every sequence of $G_0(f), G_1(f)$ of length $2^{2n}$ and every pair of indices $i, j$, there is a third index $k$ reachable from both $i$ and $j$ through a path in that graph sequence.
This, however, is equivalent to the assertion that every graph sequence of length \(n+1\) has a path from \(S\) to \(F\). Indeed, if every graph sequence of length \(n+1\) had such a path, then from every node \(k\) (and for every infinite graph sequence) there is a path from \(k\) to \(F\) of length \(2(n+1)\) (by going from \(k\) to one of \(\{S, F\}\) in at most \(n+1\) steps and then either taking the self-loop at \(F\) or going from \(S\) to \(F\) in \(n+1\) more). This means that there is a path from \(k\) to \(F\) of any length larger than \(2(n+1)\) by taking the self-loop at \(F\), and in particular of length \(2^{2n}\).

Conversely, if there is a sequence of length \(n+1\) with no path from \(S\) to \(F\), then by concatenating it with itself we would obtain an infinite graph sequence with the property that the set of nodes reachable from \(F\) (which is just \(F\)) and the set of nodes reachable from \(S\) never intersect. In particular, they do not intersect in the first \(2^{2n}\) steps.

Remark 3.8. We may restate Lemma 3.7 as follows: \(\{A_0(f), A_1(f)\}\) is a consensus set if and only if there does not exist an infinite sequence of graphs \(G_0(f), G_1(f)\) without a path from \(S\) to \(F\). The proof of Lemma 3.7 proves the equivalence of these two formulations as a byproduct.

Proof of Theorem 3.1 By putting together Lemma 3.5 and Lemma 3.7 we observe that, starting from an instance of 3-SAT with variables \(x_1, \ldots, x_n\) and \(m\) clauses, we can construct in polynomial time nonnegative, stochastic matrices \(P_1, P_2 \in \mathbb{R}^{(m+1)n+1 \times (m+1)n+1}\) such that the 3-SAT instance is unsatisfiable if and only if \(\{P_1, P_2\}\) is a consensus set. The theorem immediately follows.

Remark 3.9. Observe that for any formula \(f\), an infinite left-product of the matrices \(\{A_0(f), A_1(f)\}\) either does not converge to a rank-1 matrix or converges to the matrix \(e_F 1^T\). It may be argued that this represents a "pathological" case of convergence to consensus as in many applications it is desired that the final matrix equal \(v 1^T\) where \(v\) is a strictly positive vector; intuitively, one wants every node to "contribute" to the final result.

We give a brief informal sketch here of a construction that remedies this issue. We will create two new matrices \(A_0(f), A_1(f)\) with the property that they form a consensus set if and only if old \(A_0(f), A_1(f)\) formed a consensus set; however, any convergent infinite left-product of the new matrices must converge to a strictly positive matrix (which, consequently, can only be written as \(v 1^T\) with strictly positive \(v\)). These matrices are constructed exactly as before in Definition 3.6 from, however, new graphs \(F_0, F_1\) with strictly positive entries, adding links going from each of the \(F\) nodes to all the nodes in its copy, and adding links going both ways between the two \(F\) nodes. The new \(G_1(f)\) is produced in the same way. We recommend the reader consult Figure 3.2 for an illustration.

Observe that if the old \(\{A_0(f), A_1(f)\}\) was not a consensus set, the new matrices do not form a consensus set either. Indeed, by Remark 3.8 the old set was a consensus set if and only if there existed a sequence of the old \(G_0(f), G_1(f)\) such that starting from \(S\) one never reached \(F\); this implies that in the same sequence of the new \(G_0(f), G_1(f)\), paths starting from the two \(S\) nodes never reach their corresponding \(F\) nodes and so never intersect. This implies the new matrices \(A_0(f), A_1(f)\) do not form a consensus set.

Conversely, if the old \(\{A_0(f), A_1(f)\}\) was a consensus set, the new matrices form a consensus set as well. By Lemma 3.7 we know that the old matrices formed a consensus set if and only if in any sequence of graphs there is a path of length \(2(n+1)\) from each node to node \(F\) (by going first to one of \(\{S, F\}\) in at most \(n+1\) steps and then either taking the self-loop at \(F\) \(n+1\) times or going from \(S\) to \(F\)). It immediately follows that in any sequence of the new graphs, there is a path from any node to any other node of length \(2(n+1)+2\).

By Lemma 2.8 this implies that \(\{A_0(f), A_1(f)\}\) is a consensus set; moreover, it implies that any product of the matrices \(A_0(f), A_1(f)\) of length \(2(n+1)+2\) has entries bounded below by \((1/2(m+1)(n+1)))^{2(n+1)+2}\), which means that any limiting matrix of an infinite left-product must be positive.

Proof of Theorem 3.2 Given a 3-SAT formula \(f\), we will construct three undirected matrices \(B_0(f), B_1(f), B_2(f) \in \mathbb{R}^{4(m+1)n+4 \times 4(m+1)n+4}\), and we will prove that \(\{B_0(f), B_1(f), B_2(f)\}\) is a consensus set if and only if \(\{A_0(f), A_1(f)\}\) is a consensus set. Together with Theorem 3.1 this proves the current theorem.

We will construct these matrices as in Definition 3.6 from undirected graphs \(g_0(f), g_1(f), g_2(f)\) defined next. Note that the undirectedness of the graphs will imply the undirectedness of the corresponding matrices.
Fig. 3.2: The graphs corresponding to the formula \((x_1 \lor x_2 \lor x_3) \land (\overline{x_1} \lor \overline{x_2} \lor \overline{x_3})\). Edges which are present in both \(G_0(f), G_1(f)\) are shown in brown; edges which are only in \(G_0(f)\) are shown in blue and edges which are only in \(G_1(f)\) are in red.

Fig. 3.3: The graphs corresponding to the formula \((x_1 \lor x_2 \lor x_3) \land (\overline{x_1} \lor \overline{x_2} \lor \overline{x_3})\). The common edges of \(s_0(f), s_1(f)\) are in brown; the edges which are only in \(s_0(f)\) are in blue, while the edges which are only in \(s_1(f)\) are in red. The graph \(s_2(f)\) is given in green.
We will first construct undirected graphs $s_0(f), s_1(f), s_2(f)$ by taking every node of $G_0(f), G_1(f)$ (recall that these graphs share a common set of nodes) except $F$ and splitting them into two nodes - a “top” node and “bottom” node. We will use the subscripts $T, B$ to refer to the new nodes, e.g., node $a \neq F$ splits into a top node $a_T$ and a bottom node $a_B$. For convenience of notation, both $F_T$ and $F_B$ will refer to the (single) node $F$. We will put edges in the new (undirected) graphs $s_0(f), s_1(f)$ based on old edges in the (directed) graphs $G_0(f), G_1(f)$ as follows: if $(k, l)$ was an edge in $G_i(f)$ then we will put the edge $(k_T, l_B)$ in $s_i(f)$.

The graph $s_0(f)$ will contain all edges from the top node of a split pair to the bottom node of the same split pair (which we will understand to include a self-loop at node $F$). We refer the reader to Figure 3.3 for a concrete example of these graphs.

We construct $g_0(f), g_1(f)$ as follows. We take $s_0(f), s_1(f)$ and add edges from every bottom node to $F$; then we take two copies of the resulting graph and connect their $F$ nodes. The graph $g_3(f)$ is constructed in the same way, except that the new edges in each copy go from every top node to $F$ instead.

Our construction has the following property: if there is a path from node $a$ to node $b$ in some sequence of $G_0(f), G_1(f)$ (say in the sequence $G_0(f)G_0(f)G_0(f)G_0(f)$) then in each of the two copies, there is a path from $a_T$ to $b_T$ in the graph sequence obtained by replacing $G_0(f)$ by $g_0(f)$, $G_1(f)$ by $g_1(f)$, and inserting $g_2(f)$ into every even time step (for the above sequence this translates to $g_0(f)g_2(f)g_0(f)g_2(f)g_1(f)g_2(f)g_2(f)g_0(f)g_2(f)$).

Thus paths in graph sequences of $G_0(f), G_1(f)$ can be “ported” into sequences of the graphs $g_0(f), g_1(f), g_2(f)$ wherein $g_2(f)$ appears at, and only at, the even time steps. We will make use of this property below.

We now proceed to the proof of the current theorem. Suppose $\{A_0(f), A_1(f)\}$ is not a consensus set. By Remark 3.8 this means that there exists an infinite sequence of $G_0(f), G_1(f)$ such that there is no path from $S$ to $F$. Consider replacing every $G_0(f)$ by $g_0(f)$, every $G_1(f)$ by $g_1(f)$ and inserting $g_2(f)$ into every even time slot. By construction, we have that paths beginning at the two $S_T$ nodes never reach the $F$ nodes in their halves of the graph. Since the only edges connecting the two halves of the graphs connect the $F$-nodes, we have that the set of reachable nodes from the two $S$ nodes never intersect. By item 4 of Lemma 2.9 this implies $\{B_0(f), B_1(f), B_2(f)\}$ is not a consensus set.

Conversely, suppose that $\{A_0(f), A_1(f)\}$ is a consensus set. We next show that for every pair of nodes $i, j$ and every sequence of $g_0(f), g_1(f), g_2(f)$ of length $4(n + 1) + 3$, there is some third node $k$ reachable by a path starting from both $i$ and $j$.

Indeed: our assumption that $\{A_0(f), A_1(f)\}$ is a consensus set along with Lemma 3.7 assures us that every sequence of $G_1(f), G_2(f)$ has a path from $S$ to $F$ of length at most $n + 1$. Now consider any sequence of $g_0(f), g_1(f), g_2(f)$. These sequences will either have or not have the property that $g_2(f)$ appears at, and only at, the even time steps.

If the sequence does have this property, then there is a path from any node to the $F$ node in its copy in $4(n + 1)$ steps (indeed, at most $2(n + 1)$ steps to reach either $F$ or the $S_T$ in its copy, and then $2(n + 1)$ additional steps to go from $S_T$ to $F$ and necessary, and take self-loops at $F$ the remainder of the time). Thus the sets of reachable nodes starting at any disjoint pair of vertices intersect after $4(n + 1) + 1$ steps.

If not, then either $g_2(f)$ appears at an odd time step or one of $g_0(f), g_1(f)$ appears at an even time step. It is not hard to prove by induction that there is a path from any node to the $F$-node of its copy the first time this happens, and naturally at the following time step the sets of nodes reachable from any two distinct vertices must intersect.

Putting these two facts together, we complete the proof of the assertion that for all sequences of $g_0(f), g_1(f), g_2(f)$, there is a common reachable node from any pair of vertices in at most $4(n + 1) + 3$ steps. But now this implies that every matrix product of $B_0(f), B_1(f)$ of length $4(n + 1) + 3$ is scrambling and Lemma 2.8 implies that $\{B_0(f), B_1(f), B_2(f)\}$ is a consensus set.

4. Conclusions. Our goal has been to investigate the complexity of the consensus problem, and we have obtained some partial results to that end. We have shown that the consensus problem is decidable but that it is NP-hard for two matrices and for three undirected matrices.

Our work points to some intriguing open questions. We have left open the complexity of deciding whether a set of two undirected matrices is a consensus set. Moreover, our algorithm for deciding whether a set of matrices is a consensus set is doubly-exponential in the matrix dimension $n$, and it is interesting to ask whether a singly-exponential algorithm exists.

More broadly, even though stability determination for switched systems often turns out to be undecidable
it may turn out to be decidable (and even tractable) for many important subclasses of switched systems. It would be very interesting to relate the complexity of stability questions such as the one we consider here to the structural properties of the systems.

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