On Weak and Strong Convergence of the Projected Gradient Method for Convex Optimization in Hilbert Spaces

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Abstract

This work focuses on convergence analysis of the projected gradient method for solving constrained convex minimization problem in Hilbert spaces. We show that the sequence of points generated by the method employing the Armijo linesearch converges weakly to a solution of the considered convex optimization problem. Weak convergence is established by assuming convexity and Gâteaux differentiability of the objective function, whose Gâteaux derivative is supposed to be uniformly continuous on bounded sets. Furthermore, we propose some modifications in the classical projected gradient method in order to obtain strong convergence. The new variant has the following desirable properties: the sequence of generated points is entirely contained in a ball with diameter equal to the distance between the initial point and the solution set; and the whole sequence converges strongly to the solution of the problem that lies closest to the initial iterate. Convergence analysis of both methods is presented without Lipschitz continuity assumption.

Keywords: Armijo linesearch; Convex minimization; Projection method, Strong and weak convergence.

Mathematical Subject Classification (2010): 90C25, 90C30.

1 Introduction

In this work we are interested in weak and strong convergence of projected gradient methods for the following optimization problem

\[ \min f(x) \text{ s.t. } x \in C \subset \mathcal{H}, \]  

under the following assumptions: \( C \) is a nonempty, closed and convex subset of a real Hilbert space \( \mathcal{H} \); \( f : \mathcal{H} \rightarrow \mathbb{R} \) is a convex and Gâteaux differentiable function, and \( S^* \) is the solution set of problem (1), which is assumed to be nonempty. Moreover, we assume the Gâteaux derivative \( \nabla f \) of \( f \) to be uniformly continuous on bounded sets. The latter is a weaker assumption than the commonly adopted one, which requires \( \nabla f \) to be Lipschitz continuous in the whole space \( \mathcal{H} \).

Due to its simplicity, the projected gradient method has been widely used in practical applications. The method has several useful advantages. Primarily, it is easy to implement (especially, for optimization problems with relatively simple constraints). The method uses little storage and readily exploits any sparsity or separable structure of \( \nabla f \) or \( C \). Furthermore, it is able to drop or add active constraints during the iterations. Some important references on the projected gradient method in finite dimensional spaces are, for instance, [12, 28].

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A general description of the classical projected gradient method can be stated as follows, where we denote the projection of a given point $x$ onto $C$ by $P_C(x)$.

**Projected Gradient Method**

**Initialization step:** Take $x^0 \in C$ and set $k = 0$.

**Iterative step:** Given $x^k$, compute

$$z^k = x^k - \beta_k \nabla f(x^k)$$

$$x^{k+1} = \alpha_k P_C(z^k) + (1 - \alpha_k)x^k,$$

where $\alpha_k \in (0, 1]$ and $\beta_k$ is positive for all $k$.

Several choices are possible for the stepsizes $\beta_k$ and $\alpha_k$. We focus our attention on the description of the following four strategies:

(a) **Constant stepsize:** $\beta_k = \beta$ for all $k$ where $\beta > 0$ is a fixed number and $\alpha_k = 1$ for all $k$.

(b) **Armijo linesearch along the boundary of $C$:** $\alpha_k = 1$ for all $k$ and $\beta_k$ is determined by $\beta_k = \bar{\beta}\theta^k$, for some $\bar{\beta} > 0$, $\theta, \delta \in (0, 1)$ where

$$\ell(k) = \min \{ \ell \in \mathbb{N} \mid f(P_C(x^{k,\ell})) \leq f(x^k) - \delta(\nabla f(x^k), x^k - P_C(x^{k,\ell})) \}$$

and $x^{k,\ell} = x^k - \bar{\beta}\theta^\ell \nabla f(x^k)$.

(c) **Armijo linesearch along the feasible direction:** $(\beta_k)_{k \in \mathbb{N}} \subset [\hat{\beta}, \bar{\beta}]$ for some $0 < \hat{\beta} \leq \bar{\beta} < \infty$ and $\alpha_k$ determined by the following Armijo rule $\alpha_k = \hat{\theta}^k$, for some $\theta, \delta \in (0, 1)$ where

$$j(k) = \min \{ j \in \mathbb{N} \mid f(x^{k,j}) \leq f(x^k) - \delta\theta^j(\nabla f(x^k), x^k - P_C(z^k)) \}$$

and $x^{k,j} = \theta^j P_C(z^k) + (1 - \theta^j)x^k$.

(d) **Exogenous stepsize before projecting:** $\alpha_k = 1$ for all $k$ and $\beta_k$ given by

$$\beta_k = \frac{\delta_k}{\|\nabla f(x^k)\|}, \quad \text{with} \quad \sum_{k=0}^\infty \delta_k = \infty \quad \text{and} \quad \sum_{k=0}^\infty \delta_k^2 < \infty.$$

Strategy (a) was analyzed in [13] and its weak convergence was proved under Lipschitz continuity of $\nabla f$. The main difficulty is the necessity of taking $\beta \in (0, 2/L)$, where $L$ is the Lipschitz constant $\nabla f$; see also [12].

Note that Strategy (b) requires one projection onto $C$ for each step of the inner loop resulting from the Armijo linesearch. Therefore, many projections might be performed for each iteration $k$, making Strategy (b) inefficient when the projection onto $C$ is not explicitly computed. On the other hand, Strategy (c) demands only one projection for each outer step, i.e., for each iteration $k$. Strategies (b) and (c) are the constrained versions of the linesearch proposed in [6] for solving unconstrained optimization problems. Under existence of minimizers and convexity assumptions for problem (11), it is possible to prove, for Strategies (b) and (c), convergence of the whole sequence to a minimizer of $f$ in finite dimensional spaces; see [15]. No additional assumption on boundedness of level sets is required, as shown in [23].

Strategy (d), as its counterpart in the unconstrained case, fails to be a descent method. Furthermore, it is easy to show that this approach satisfies $\|x^{k+1} - x^k\| \leq \delta_k$ for all $k$, with $\delta_k$ exogenous and satisfying (4). This reveals that convergence of the sequence of points generated by this approach can be very slow: in view of (4), stepsizes are small (notice that Strategies (b) and (c) allow for occasionally long stepsizes because both strategies employ all information available at each iteration). Moreover, Strategy (d) does not take into account functional values for determining the stepsizes. These characteristics, in general, entail poor computational performance. The strategy’s redeeming feature is that its convergence properties also hold in the nonsmooth case, in which the two Armijo line searches given by (b) and (c) may be unsuccessful; see [28]. By assuming existence of solutions of problem (11), defined in an arbitrary Hilbert space $H$, and replacing, at each iteration, $\nabla f(x^k)$ by any subgradient $s_k$ of $f$ at $x^k$, the work [2] establishes that the sequence $(x^k)_{k \in \mathbb{N}}$ generated by Strategy (d) converges weakly to a solution of problem (11), providing that the subdifferential of $f$ is bounded on bounded sets.

In finite dimensional spaces and without assuming convexity of the function $f$, convergence results for the above strategies closely mirror the ones for the steepest descent method in the unconstrained case: cluster
points may fail to exist, even when (1) has solutions. However, if cluster points exist, they are stationary and feasible; see for instance [12, Section 2.3.2]. The work [17] proves convergence of the sequence of points generated by Strategy (b) to a stationary point of problem (1) by assuming that the starting iterate $x^0$ belongs to a bounded level set of $f$.

As already mentioned, in this paper we are interested in weak and strong convergence of projected gradient methods applied to convex programs like (1). To the best of our knowledge, weak convergence of the projected gradient method has only been shown under the assumption of Lipschitz continuity of $\nabla f$ or using exogenous stepsize, like Strategy (d) above. In the present work we prove, without Lipschitz continuity assumption, weak convergence of the projected gradient method employing Strategy (c). Moreover, we propose a few modifications of Strategy (c) in order to ensure that the resulting method is strongly convergent.

The paper is organized as follows. The next section provides some definitions and preliminary results that will be used in the remainder of this work. Weak convergence of the projected gradient methods is presented in Section 3. In Section 4 we propose and study strong convergence of a modified projected gradient method. Finally, some comments and remarks are presented in Section 5.

Our notation is standard: the inner product in $H$ is denoted by $\langle \cdot, \cdot \rangle$ and the norm induced by the inner product is $\| \cdot \|$. For an element $x \in H$, we define the orthogonal projection of $x$ onto $C$, denoted by $P_C(x)$, as the unique point in $C$ such that $\|P_C(x) - y\| \leq \|x - y\|$ for all $y \in C$. The indicator function of $C$, written as $I_C$, is given by $I_C(x) = 0$ if $x \in C$, and $I_C(x) = \infty$ otherwise, and the normal cone to $C$ is $N_C = \partial I_C$.

Furthermore, if we define the function $\hat{f} := f + I_C$, then the problem (1) is equivalent to find $x \in H$ such that $0 \in \partial \hat{f}(x) = \partial f(x) + N_C(x)$, which will be used in the proof of Theorem 3.1.

2 Preliminaries

We begin by stating the directional derivative of $f$ at $x \in \text{dom}(f)$ in the direction $d$, that is

$$f'(x; d) := \lim_{t \to 0+} \frac{f(x + td) - f(x)}{t},$$

when the limit exists. If the directional derivative $f'(x; d)$ exists for all directions $d$ and the functional $\nabla f(x) : H \to \mathbb{R}$ defined by $\langle \nabla f(x), \cdot \rangle := f'(x; \cdot)$ is linear and bounded, then we say that $f$ is Gâteaux differentiable at $x$, and $\nabla f(x)$ is called the Gâteaux derivative. Every convex and lower semicontinuous function $f : H \to \mathbb{R}$ that is Gâteaux differentiable at $x$ is also continuous at $x$.

We now remark some necessary and sufficient optimality conditions for problem (1), whose proof can be found in [7, Prop. 17.4].

**Proposition 2.1.** Let $f : H \to \mathbb{R}$ be a proper convex and Gâteaux differentiable function. Then the point $x_* \in C$ is a minimizer of problem (1) if and only if $0 \in \nabla f(x_*) + N_C(x_*)$ if and only if $\langle \nabla f(x_*), y - x_* \rangle \geq 0$ for all $y \in C$ if and only if $x_* = P_C(x_* - \beta \nabla f(x_*))$ with $\beta > 0$.

In the following a well known fact on orthogonal projections; see for instance Lemmas 1.1 and 1.2 in [32].

**Proposition 2.2.** Let $K$ be a nonempty, closed and convex set in $H$. For all $x, y \in H$ and all $z \in K$, the following properties hold:

(i) $\langle x - P_K(x), z - P_K(x) \rangle \leq 0$.

(ii) $\langle z - y, z - P_K(y) \rangle \geq \|z - P_K(y)\|^2$.

Next we deal with the so called quasi-Fejér convergence and its properties.

**Definition 2.1.** Let $S$ be a nonempty subset of $H$. A sequence $(x^k)_{k \in \mathbb{N}}$ in $H$ is said to be quasi-Fejér convergent to $S$ if and only if there exists a summable sequence $(\epsilon_k)_{k \in \mathbb{N}}$ such that for all $x \in S$, $\|x^{k+1} - x\|^2 \leq \|x^k - x\|^2 + \epsilon_k$ for all $k$.

This definition originates in [14] and has been further elaborated in [16,24]. A useful result on quasi-Fejér sequences is the following, which is proven in [14, Lemma 6].
Lemma 2.1. If \((x^k)_{k \in \mathbb{N}}\) is quasi-Fejér convergent to \(S\), then: (i) \((x^k)_{k \in \mathbb{N}}\) is bounded; (ii) if all weak cluster point of \((x^k)_{k \in \mathbb{N}}\) belong to \(S\), then the sequence \((x^k)_{k \in \mathbb{N}}\) is weakly convergent.

We finalize this section by showing that uniform continuity of the derivative \(\nabla f\) is a weaker assumption than Lipschitz continuity of \(\nabla f\) in \(\mathcal{H}\).

Example 2.1. Take in problem (11) \(f(x) = (1/p)||x||^p\), with \(p > 1\). It is not difficult to show that \(f\) is a convex function and that \(\nabla f\) is uniformly continuous for all \(p > 1\). However, \(\nabla f\) is globally Lipschitz continuous only for \(p = 2\).

3 Weak Convergence of the Projected Gradient Method

In this section we state the classical projected gradient method, with the linesearch along the feasible direction (i.e., Strategy (c)). Provided that the underlying problem has a solution, we show that the sequence of point generated by the project gradient method converges weakly to a solution of the problem. We will not assume Lipschitz continuity of the mapping \(\nabla f\).

We now remind the formal definition of the projected gradient method. Let \((\beta_k)_{k \in \mathbb{N}}\) be a sequence such that \((\beta_k)_{k \in \mathbb{N}} \subset [\hat{\beta}, \tilde{\beta}]\) with \(0 < \hat{\beta} \leq \tilde{\beta} < \infty\), and be \(\theta, \delta \in (0, 1)\). The algorithm is stated as follows:

Algorithm A1

Initialization step: Take \(x^0 \in C\) and set \(k = 0\).

Iterative step 1: Given \(x^k\), compute
\[
z^k := x^k - \beta_k \nabla f(x^k).
\] (5)

Stop Criteria: If \(x^k = P_C(z^k)\) stop.

Inner Loop: Otherwise, set \(\alpha_k = \theta^j(k)\), where
\[
j(k) := \min \{ j \in \mathbb{N} \mid f(x^{k-j}) \leq f(x^k) - \delta \theta^j \langle \nabla f(x^k), x^k - P_C(z^k) \rangle \} \quad \text{and} \quad x^{k-j} = \theta^j P_C(z^k) + (1 - \theta^j)x^k. \] (6)

Iterative step 2: Compute
\[
x^{k+1} = \alpha_k P_C(z^k) + (1 - \alpha_k)x^k. \] (7)

Set \(k = k + 1\) and go back to Step 1.

It follows from Proposition 2.2(ii) that the iterates of Algorithm A1 satisfy
\[
\langle \nabla f(x^k), x^k - P_C(z^k) \rangle \geq \frac{1}{\beta_k} \|x^k - P_C(z^k)\|^2 \quad \text{for all} \quad k.
\] (8)

Moreover, if Algorithm A1 stops then \(x^k = P_C(z^k) = P_C(x^k - \beta_k \nabla f(x^k))\). Since \(\beta_k \geq \hat{\beta} > 0\), it follows from Proposition 2.1 that \(x^k\) is a solution to problem (11). Moreover, from (6) and (8) we have
\[
f(x^{k+1}) \leq f(x^k) - \delta \alpha_k \langle \nabla f(x^k), x^k - P_C(z^k) \rangle \leq f(x^k) - \delta \frac{\alpha_k}{\beta_k} \|x^k - P_C(z^k)\|^2 \quad \text{for all} \quad k.
\]

Therefore, if the algorithm does not stop we obtain the inequality
\[
\delta \frac{\alpha_k}{\beta} \|x^k - P_C(z^k)\|^2 \leq f(x^k) - f(x^{k+1}),
\]

showing that \((f(x^k))_{k \in \mathbb{N}}\) is a monotone decreasing sequence. Since such sequence is bounded from below by the optimal value of problem (11), we conclude that \(\lim_{k \to \infty} (f(x^k) - f(x^{k+1})) = 0\). It thus follows from the above inequality that
\[
\lim_{k \to \infty} \alpha_k \|x^k - P_C(z^k)\|^2 = 0,
\] (9)
a crucial result to be considered in Theorem 3.1 below. In the following we show that the inner loop in Algorithm A1 is well-defined.
Moreover, we have that $\alpha x_k$. We thus have shown that $\alpha_k = \theta^{i(k)}$ satisfying (6).

Proof. The proof is by contradiction: suppose that (6) does not hold for all $j \geq 0$, i.e.,
\[
\frac{f(x^k + \theta^j(P_C(z_k) - x^k)) - f(x^k)}{\theta^j} < -\delta \langle \nabla f(x^k), x^k - P_C(z^k) \rangle \quad \text{for all } j \geq 0.
\]
Passing to the limit when $j$ goes to infinity and using the Gâteaux differentiability of $f$ and (8), we conclude that
\[
0 \geq (1 - \delta) \langle \nabla f(x^k), x^k - P_C(z^k) \rangle \geq \frac{(1 - \delta)}{\beta_k} \|x^k - P_C(z^k)\|^2 \geq \frac{(1 - \delta)}{\beta} \|x^k - P_C(z^k)\|^2,
\]
which contradicts, by Proposition 2.1, the assumption that $x^k$ is not a solution to the problem.

To the best of our knowledge, from now on all the presented results are new in Hilbert spaces.

Lemma 3.1. The sequence generated by Algorithm A1 is quasi-Fejér convergent to $S_*$.

Proof. Take any $x_* \in S_*$. Note that $\|x^{k+1} - x_*\|^2 + \|x^k - x_*\|^2 - \|x^{k+1} - x_*\|^2 = 2(x^k - x^{k+1}, x^k - x_*)$. Moreover,
\[
2(x^k - x^{k+1}, x^k - x_*) = 2 \alpha_k \langle P_C(z^k) - x^k, x^k - x_* \rangle
\]
\[
= 2 \alpha_k \beta_k \langle \nabla f(x^k), x^k - x_* \rangle - 2 \alpha_k \langle P_C(z^k) - x^k + \beta_k \nabla f(x^k), x^k - x_* \rangle
\]
\[
= 2 \alpha_k \beta_k \langle \nabla f(x^k), x^k - x_* \rangle - 2 \alpha_k \langle P_C(z^k) - x^k + \beta_k \nabla f(x^k), x^k - P_C(z^k) \rangle
\]
\[
- 2 \alpha_k \langle P_C(z^k) - (x^k - \beta_k \nabla f(x^k)), P_C(z^k) - x_* \rangle
\]
\[
\geq 2 \alpha_k \beta_k \langle \nabla f(x^k), x^k - x_* \rangle - 2 \alpha_k \langle P_C(z^k) - x^k + \beta_k \nabla f(x^k), x^k - P_C(z^k) \rangle
\]
\[
\geq 2 \alpha_k \beta_k \langle f(x^k) - f(x_*), x^k - P_C(z^k) \rangle - 2 \alpha_k \langle P_C(z^k) - x^k + \beta_k \nabla f(x^k), x^k - P_C(z^k) \rangle
\]
\[
\geq -2 \alpha_k \langle P_C(z^k) - x^k + \beta_k \nabla f(x^k), x^k - P_C(z^k) \rangle
\]
\[
= 2 \alpha_k \|x^k - P_C(z^k)\|^2 - 2 \alpha_k \beta_k \langle \nabla f(x^k), x^k - P_C(z^k) \rangle,
\]
where the first equality follows from (7), the second one by adding and subtracting $\beta_k \nabla f(x^k)$, and the third equality follows from adding and subtracting $P_C(z^k)$. The first inequality above is due to Proposition 2.2(i), the second one follows from convexity, and the third inequality holds because $x_*$ is a solution to problem (1).

We thus have shown that
\[
\|x^{k+1} - x_*\|^2 \leq \|x^k - x_*\|^2 + \|x^k - x^{k+1}\|^2 - 2 \alpha_k \|x^k - P_C(z^k)\|^2 + 2 \alpha_k \beta_k \langle \nabla f(x^k), x^k - P_C(z^k) \rangle.
\]
Since $x^{k+1} - x^k = \alpha_k (P_C(z^k) - x^k)$ by (7), we conclude that
\[
\|x^{k+1} - x_*\|^2 \leq \|x^k - x_*\|^2 + \alpha_k^2 \|x^k - P_C(z^k)\|^2 - 2 \alpha_k \|x^k - P_C(z^k)\|^2 + 2 \alpha_k \beta_k \langle \nabla f(x^k), x^k - P_C(z^k) \rangle.
\]
Moreover, we have that $\alpha_k^2 - 2 \alpha_k \leq -\alpha_k$, because $0 \leq \alpha_k \leq 1$ in the inner loop of Algorithm A1. This gives
\[
\|x^{k+1} - x_*\|^2 \leq \|x^k - x_*\|^2 - \alpha_k \|x^k - P_C(z^k)\|^2 + 2 \frac{\beta}{\delta} (f(x^k) - f(x^{k+1})),
\]
where we have used (8). In order to show that $(x^k)_{k \in \mathbb{N}}$ is quasi-Fejér convergent to $S_*$, it remains to prove that $\epsilon_k = -\alpha_k \|x^k - P_C(z^k)\|^2 + 2 \frac{\beta}{\delta} (f(x^k) - f(x^{k+1}))$ forms a convergent series. Indeed, this is true by the following development
\[
\sum_{k=0}^{\infty} \epsilon_k \leq 2 \frac{\beta}{\delta} \sum_{k=0}^{\infty} (f(x^k) - f(x^{k+1})) = 2 \frac{\beta}{\delta} \left( f(x^0) - \lim_{k \to \infty} f(x^{k+1}) \right) \leq 2 \frac{\beta}{\delta} (f(x^0) - f(x_*)) < \infty.
\]
We are now ready to prove the main result of this section.

**Theorem 3.1.** Suppose that $\nabla f$ is uniformly continuous on bounded sets. Then the sequence $(x^k)_{k \in \mathbb{N}}$ generated by Algorithm A1 is bounded and each of its weak cluster points belongs to $S_*$.

**Proof.** Since the sequence $(x^k)_{k \in \mathbb{N}}$ is quasi-Fejér convergent to $S_*$, it is bounded. Therefore, there exists a subsequence $(x^{i_k})_{k \in \mathbb{N}}$ of $(x^k)_{k \in \mathbb{N}}$ that converges weakly to some $x_*$. Moreover, since $\nabla f$ is uniformly continuous on bounded sets we conclude that $(\nabla f(x^k))_{k \in \mathbb{N}}$ is also a bounded sequence. Thus, it follows from (5) and (6) that $(P_C(z^i))_{k \in \mathbb{N}}$ is a bounded sequence as well. We now split our analysis into two distinct cases.

**Case 1.** Suppose that the sequence $(\alpha_k)_{k \in \mathbb{N}}$ does not converge to 0, i.e. there exists a subsequence $(\alpha_{i_k})_{k \in \mathbb{N}}$ of $(\alpha_k)_{k \in \mathbb{N}}$ and some $\alpha > 0$ such that $\alpha_{i_k} \geq \alpha$ for all $k$. Let $w^k := P_C(z^k)$; it follows from (9) and our assumption on $\nabla f$ that
\[
\lim_{k \to \infty} \|x^{i_k} - w^k\| = 0 \quad \text{and} \quad \lim_{k \to \infty} \|\nabla f(x^{i_k}) - \nabla f(w^k)\| = 0. \tag{10}
\]

Let $x_*$ be a weak cluster point of the subsequence $(x^{i_k})_{k \in \mathbb{N}}$. By (10), it is also a weak cluster point of $(w^k)_{k \in \mathbb{N}}$. Without loss of generality, we assume that $(x^{i_k})_{k \in \mathbb{N}}$ and $(w^k)_{k \in \mathbb{N}}$ converges weakly to $x_*$. In order to prove that $x_* \in S_*$, we define the function $\hat{f} := f + I_C$. It is well known that $\partial \hat{f}(x) := \nabla f(x) + N_C(x)$, for all $x \in C$, is a maximal monotone, and that $0 \in \partial \hat{f}(x)$ if and only if $x \in S_*$; see for instance [29]. Therefore, we need to show that $0 \in \partial \hat{f}(x_*)$. In order to do that, we take $(x, u) \in G(\partial \hat{f})$ with $x \in C$. Thus, $u \in \partial \hat{f}(x) = \nabla f(x) + N_C(x)$, implying that $u - \nabla f(x) \in N_C(x)$. So, we have $\langle x - y, u - \nabla f(x) \rangle \geq 0$ for all $y \in C$. In particular,
\[
\langle x - w^k, u \rangle \geq \langle x - w^k, \nabla f(x) \rangle. \tag{11}
\]

On the other hand, since $w^k = P_C(x^{i_k} - \beta_{i_k} \nabla f(x^{i_k}))$ and $x^{i_k} \in C$, it follows from Proposition 2.2(i), with $K = C$ and $x = x^{i_k} - \beta_{i_k} \nabla f(x^{i_k})$, that $(x - w^k, x^{i_k} - \beta_{i_k} \nabla f(x^{i_k}) - w^k) \leq 0$ for all $x \in C$ and $i_k \geq 0$. Rearranging terms and taking into account that $\beta_{i_k} > 0$, we get
\[
\left\langle x - w^k, x^{i_k} - w^k - \nabla f(x^{i_k}) \right\rangle \leq 0 \quad \forall x \in C \text{ and } i_k \geq 0.
\]

Together with (11), we conclude that
\[
\langle x - w^k, u \rangle \geq \langle x - w^k, \nabla f(x) \rangle \geq \langle x - w^k, \nabla f(x) \rangle + \left\langle x - w^k, \frac{x^{i_k} - w^k}{\beta_{i_k}} - \nabla f(x^{i_k}) \right\rangle
\]
\[
= \langle x - w^k, \nabla f(x) - \nabla f(w^k) \rangle + \langle x - w^k, \nabla f(w^k) - \nabla f(x^{i_k}) \rangle + \left\langle x - w^k, \frac{x^{i_k} - w^k}{\beta_{i_k}} \right\rangle.
\]

Monotonicity of $\nabla f$ gives $\langle x - w^k, \nabla f(x) - \nabla f(w^k) \rangle \geq 0$. Thus,
\[
\langle x - w^k, u \rangle \geq \langle x - w^k, \nabla f(w^k) - \nabla f(x^{i_k}) \rangle + \left\langle x - w^k, \frac{x^{i_k} - w^k}{\beta_{i_k}} \right\rangle
\]
\[
\geq -\|x - w^k\| \left(\|\nabla f(w^k) - \nabla f(x^{i_k})\| + \frac{1}{\beta_{i_k}} \|w^k - x^{i_k}\| \right)
\]
\[
\geq -\|x - w^k\| \left(\|\nabla f(x^{i_k}) - \nabla f(x^{i_k})\| + \frac{1}{\beta} \|w^k - x^{i_k}\| \right),
\]

where we have used Cauchy-Schwartz inequality in the third inequality and the fact that $\beta_k \geq \hat{\beta} > 0$ for all $k$ in the last one. Remember that $(w^k)_{k \in \mathbb{N}}$ is bounded and converges weakly to $x_*$. Thus, passing to the limit in the above relations and recalling to (10), we obtain that
\[
\langle x - x_*, u \rangle \geq 0 \quad \forall (x, u) \in G(\partial \hat{f}).
\]
Since $\partial f$ is maximal monotone, it follows from the above inequality that $(x_*, 0) \in G(\partial f)$, implying that $0 \in \partial f(x_*) = \nabla f(x_*) + N_C(x_*)$ and hence $x_* \in S_*$.

**Case 2.** Suppose now that $\lim_{k \to \infty} \alpha_k = 0$. Take, with $\alpha_k > 0$,
\[
y^k = \frac{\alpha_k}{\theta} P_C(z^k) + \left(1 - \frac{\alpha_k}{\theta}\right) x^k = x^k - \frac{\alpha_k}{\theta} (x^k - w^k). \tag{12}
\]
It follows from the definition of $j(k)$ in (6) that $f(y^k) - f(x_k) > -\delta \frac{\alpha_k}{\theta} \langle \nabla f(x_k), x^k - w^k \rangle$. Thus,
\[
\delta \frac{\alpha_k}{\theta} \langle \nabla f(x_k), x^k - w^k \rangle > f(x_k) - f(y^k) \geq \langle \nabla f(y^k), x^k - y^k \rangle = \frac{\alpha_k}{\theta} \langle \nabla f(y^k), x^k - w^k \rangle
\]
\[
= \frac{\alpha_k}{\theta} (\nabla f(y^k) - \nabla f(x_k), x^k - w^k) + \frac{\alpha_k}{\theta} \langle \nabla f(x_k), x^k - w^k \rangle
\]
\[
\geq -\frac{\alpha_k}{\theta} \| \nabla f(y^k) - \nabla f(x_k) \| \| x^k - w^k \| + \frac{\alpha_k}{\theta} \langle \nabla f(x_k), x^k - w^k \rangle,
\]
where we have used convexity of $f$ in the second inequality and Cauchy-Schwartz inequality in the last one. Rearrangement terms and using (8) yields
\[
\| \nabla f(y^k) - \nabla f(x_k) \| \| x^k - w^k \| \geq (1 - \delta) \langle \nabla f(x_k), x^k - w^k \rangle \geq \frac{(1 - \delta)}{\beta_k} \| x^k - w^k \|^2,
\]
which implies
\[
\| \nabla f(y^k) - \nabla f(x_k) \| \geq \frac{(1 - \delta)}{\beta} \| x^k - w^k \|. \tag{13}
\]

Since both sequences $(x_k)_{k \in \mathbb{N}}$ and $(w_k)_{k \in \mathbb{N}}$ are bounded and $\lim_{k \to \infty} \alpha_k = 0$, it follows from (12) that $\lim_{k \to \infty} \| y^k - x^k \| = 0$. As $\nabla f$ is uniformly continuous on bounded sets, we get $\lim_{k \to \infty} \| \nabla f(y^k) - \nabla f(x_k) \| = 0$. It thus follows from (13) that $\lim_{k \to \infty} \| x^k - w^k \| = 0$. We have show that **Case 2** also satisfies the key relations in (10) of **Case 1**. Hence, the remain of the proof can be done similarly to **Case 1**, mutatis mutandis.

**Theorem 3.2.** Suppose that $\nabla f$ is uniformly continuous on bounded sets. Then the sequence $(x^k)_{k \in \mathbb{N}}$ generated by Algorithm A1 converges weakly to a solution of problem (1).

**Proof.** By Lemma 3.1 $(x^k)_{k \in \mathbb{N}}$ is quasi-Fejér convergent to $S_*$ and by Theorem 3.1 all weak cluster points of $(x^k)_{k \in \mathbb{N}}$ belong to $S_*$. The result thus follows from Lemma 2.1(ii). \[ \square \]

### 4 Strongly Convergent Projected Gradient Method

In this section we consider a modification of the projected gradient method forcing strong convergence in Hilbert spaces. The modified projected method, employing linesearch (c), was inspired by Polyak’s method [9, 11, 27] for nondifferentiable optimization, and it uses an idea similar to that exposed in [10, 30], with the same goal, upgrading weak to strong convergence.

Additionally, our algorithm has the distinctive feature that the limit of the generated sequence is the solution of the problem closest to the initial iterate $x^0$. This property is useful in many specific applications, e.g., in image reconstruction [19, 22, 26] and in minimal norm solution problems, as discussed in [1]. We emphasize that this feature is of interest also in finite-dimensional spaces, differently from the strong versus weak convergence issue.

#### 4.1 Some Comments on Strong Convergence

Clearly weak and strong convergence are only distinguishable in the infinite-dimensional setting. Naturally, even when we have to solve infinite-dimensional problems, numerical implementations of algorithms are performed in finite-dimensional approximations of these problems. Nevertheless, it is interesting to have good
convergence theory for the infinite-dimensional setting in order to guarantee robustness and stability of the finite-dimensional approximations. This issue is closely related to the so-called Mesh Independence Principle presented in [3][1][25]. This principle relies on infinite-dimensional convergence to predict the convergence properties of a discretized finite-dimensional method. Moreover, the Mesh Independence Principle provides theoretical justification for the design of refinement strategies, which are crucial for having appropriate approximation to the true solution of the infinite-dimensional problem being solved. We suggest the reader to see [21], where a variety of applications are described. A strong convergence principle in Hilbert spaces is extensively analyzed in [5][8].

The importance of strong convergence is also underlined in [20], where it is shown, for the proximal-point algorithm, that the rate of convergence of the value sequence \((f(x^k))_{k\in\mathbb{N}}\) is better, when \((x^k)_{k\in\mathbb{N}}\) converges strongly. It is important to say that only weak convergence has been established for the projected gradient method in Hilbert spaces; see [18, 31]. In these cases the weak convergence has been established by assuming Lipschitz continuity of \(\nabla f\) or by employing exogenous stepsizes that may lead to small-length steps, as discussed in the Introduction. In our scheme we use the classical Armijo linesearch along the feasible direction establishing the strong convergence.

4.2 Algorithm and Convergence Analysis

Let \((\beta_k)_{k\in\mathbb{N}}\) be a sequence such that \((\beta_k)_{k\in\mathbb{N}} \subset [\hat{\beta}, \hat{\beta}]\) with \(0 < \hat{\beta} \leq \hat{\beta} < \infty\), and be \(\theta, \delta \in (0, 1)\). The algorithm of the proposed strongly convergence projected gradient is stated as follows.

Algorithm A2
Initialization step. Take \(x^0 \in C\) and set \(f^\text{lev}_0 = \infty\).
Iterative step 1. Given \(x^k\), compute \(x^{k+1} = x^k - \beta_k \nabla f(x^k)\).
Stop Criteria 1. If \(x^k = P_C(x^k)\) stop.
Inner Loop: Find \(j(k)\) as in (16) and set \(\alpha_k = \theta j(k)\) and \(f^\text{lev}_k = \min\{f^\text{lev}_{k-1}, f(x^k, j(k))\}\).
Iterative step 2. Define
\[
H_k := \{x \in \mathcal{H} \mid \langle \nabla f(x^k), x - x^k \rangle + f(x^k) - f^\text{lev}_k \leq 0\}, \quad \text{and} \quad W_k := \{x \in \mathcal{H} \mid \langle x - x^k, x^0 - x^k \rangle \leq 0\}.
\]

Compute
\[
x^{k+1} := P_{C \cap W_k \cap H_k}(x^0).
\]

Stop Criteria 2. If \(x^{k+1} = x^k\) then stop. Otherwise, set \(k = k + 1\) and go back to Step 1.

Algorithm A2 is a particular case of Algorithm 2 in [11], for nonsmooth convex optimization. In contrast to the algorithm in [11], Algorithm A2 gives a straightforward way to define the level sequence \((f^\text{lev}_k)_{k\in\mathbb{N}}\).

Suppose that \(x^k \notin S_*\). Since \(\langle \nabla f(x^k), x^k - P_C(z^k) \rangle \geq \frac{1}{\beta_k} \|x^k - P_C(z^k)\|\), the definition of \(f^\text{lev}_k\) does satisfies the inequalities given in [11], Eq. (4):
\[
f(x^k) > f^\text{lev}_k \geq f_* \quad \text{for all } k.
\]

We thus conclude that if \(x^k \notin S_*\) then \(f(x^k) > f^\text{lev}_k\), yielding that \(x^k \notin H_k\). In order to analyze convergence of Algorithm A2 we present below some key inequalities.

Lemma 4.1. For all \(k \geq 0\) it holds that
\[
\|x^{k+1} - x^0\|^2 \geq \|x^k - x^0\|^2 + \|x^k - x^{k+1}\|^2, \quad \text{with}
\]
\[
\|x^k - x^{k+1}\| \geq \frac{f(x^k) - f^\text{lev}_k}{\|\nabla f(x^k)\|} \geq \delta \frac{\alpha_k}{\beta_k} \|x^k - P_C(z^k)\|^2 \geq 0.
\]

Proof. Since \(x^{k+1} \in W_k\), then \(0 \geq \langle x^{k+1} - x^k, x^0 - x^k \rangle = \frac{1}{2} \left(\|x^{k+1} - x^k\|^2 - \|x^{k+1} - x^0\|^2 + \|x^k - x^0\|^2\right)\), which implies (16). Moreover, \(x^{k+1}\) belongs to \(H_k\) and thus
\[
\langle \nabla f(x^k), x^k - x^{k+1} \rangle \geq f(x^k) - f^\text{lev}_k, \quad \text{i.e.,} \quad \|x^k - x^{k+1}\| \geq \frac{f(x^k) - f^\text{lev}_k}{\|\nabla f(x^k)\|}.
\]
Using (10) and (8), \( f(x^k) - f^\nu_k \geq f(x^k) - f(x^{k,j(k)}) \geq \delta \alpha_k \langle \nabla f(x^k), x^k - P_C(z^k) \rangle \geq \delta \frac{\alpha_k}{\beta_k} \| x^k - P_C(z^k) \|^2 \geq 0 \), and the result follows because \( \beta_k \leq \hat{\beta} \).

We now put together some results from [11], yielded by (14) and (15).

**Lemma 4.2.** Assume that \( \nabla f \) is uniformly bounded on bounded sets, and that the solution set \( S^* \) of problem (11) is nonempty. Let \((x^k)_{k \in \mathbb{N}} \) be the sequence of points generated by Algorithm A2, and let \( x^* \) be the projection of \( x^0 \) onto \( S^* \), i.e., \( x^* = P_{S^*}(x^0) \). Then,

(i) \( S_* \subseteq H_k \cap W_k \cap C \) for all \( k \);

(ii) \((x^k)_{k \in \mathbb{N}} \) is contained in the closed ball centered in \((x^0 + x^*)/2\) and with radius \( \| x^* - x^0 \|/2 \);

(iii) all weak cluster points of \((x^k)_{k \in \mathbb{N}} \) belongs to \( S^* \).

**Proof.** It follows from our assumptions on problem (11) that \( x^* \) is well-defined. Since (15) holds, equation (11) in [11] holds for the choice \( i = 0 \) and \( j = \infty \) in the definition of \( I^j_0 \) therein. Therefore, both items (i) and (ii) follows from taking \( \hat{x} = x^0 \) in [11] Prop. 2.2.

As a result of (ii), \((x^k)_{k \in \mathbb{N}} \) is bounded and has at least a weak cluster point. It follows from (10) that the sequence \((\| x^k - x^0 \|)_{k \in \mathbb{N}} \) is nondecreasing and bounded, hence convergent. Again, by (10):

\[
0 \leq \| x^{k+1} - x^k \|^2 \leq \| x^{k+1} - x^0 \|^2 - \| x^k - x^0 \|^2,
\]

and we conclude that

\[
\lim_{k \to \infty} \| x^{k+1} - x^k \|^2 = 0. \tag{18}
\]

Boundedness of \((\nabla f(x^k))_{k \in \mathbb{N}} \) follows from the boundedness of \((x^k)_{k \in \mathbb{N}} \). Thus, (17) and (18) yields

\[
\lim_{k \to \infty} \alpha_k \| x^k - P_C(z^k) \|^2 = 0.
\]

The proof of item (iii) follows now from repeating the proof of Theorem 3.1 cases 1 and 2.

Since items (i)-(iii) above hold, convergence of Algorithm A2 follows from [11] Thm. 3.4. We repeat the result here for completeness.

**Theorem 4.1.** Assume that \( \nabla f \) is uniformly bounded on bounded sets. Then, the sequence \((x^k)_{k \in \mathbb{N}} \) generated by Algorithm A2 converges strongly to \( x_* = P_{S_*}(x^0) \).

**Proof.** It follows from the definition of \( x^{k+1} \) that \( \| x^{k+1} - x^0 \| \leq \| x - x^0 \| \) for all \( x \in H_k \cap W_k \cap C \). In particular, \( x_* \in H_k \cap W_k \cap C \) by Lemma 4.2(i). Thus,

\[
\| x^k - x^0 \| \leq \| x_* - x^0 \| \text{ for all } k. \tag{19}
\]

By items (ii) and (iii) of Lemma 4.2, \((x^k)_{k \in \mathbb{N}} \) is bounded and each of its weak cluster points belongs to \( S_* \). Let \( \{x^{i_k}\} \) be any weakly convergent subsequence of \((x^k)_{k \in \mathbb{N}} \), and let \( \hat{x} \in S_* \) be its weak limit. Observe that

\[
\| x^{i_k} - x_* \|^2 = \| x^{i_k} - x^0 - (x_* - x^0) \|^2 \\
= \| x^{i_k} - x^0 \|^2 + \| x_* - x^0 \|^2 - 2 \langle x^{i_k} - x^0, x_* - x^0 \rangle \\
\leq 2 \| x_* - x^0 \|^2 - 2 \langle x^{i_k} - x^0, x_* - x^0 \rangle,
\]

where the inequality follows from (19). By the weak convergence of \( \{x^{i_k}\} \) to \( \hat{x} \), we obtain

\[
\lim_{k \to \infty} \sup_k \| x^{i_k} - x_* \|^2 \leq 2(\| x_* - x^0 \|^2 - \langle \hat{x} - x^0, x_* - x^0 \rangle). \tag{20}
\]

Applying Proposition 2.2(i), with \( K = S_* \), \( x = x^0 \) and \( z = \hat{x} \in S_* \), and taking into account that \( x_* \) is the projection of \( x^0 \) onto \( S_* \), we have that \( \langle x^0 - x_* , \hat{x} - x_* \rangle \leq 0 \). This inequality yields

\[
0 \geq - \langle x_* - x^0 , x_* - x^0 \rangle = - \langle \hat{x} - x^0 , x_* - x^0 \rangle - \langle x^0 - x_* , x_* - x^0 \rangle \\
\geq - \langle \hat{x} - x^0 , x_* - x^0 \rangle + \| x_* - x^0 \|^2.
\]
It follows that \( \langle \hat{x} - x^0, x^*_0 - x^0 \rangle \geq \| x^*_0 - x^0 \|^2 \). By combining this last inequality with (20) we conclude that \( (x^k)_{k \in \mathbb{N}} \) converges strongly to \( x^*_0 \). Thus, we have shown that every weakly convergent subsequence of \( (x^k)_{k \in \mathbb{N}} \) converges strongly to \( x^*_0 \). Hence, the whole sequence \( (x^k)_{k \in \mathbb{N}} \) converges strongly to \( x^*_0 \in S^* \).

5 Final Remarks

It is well-known in Hilbert spaces that global Lipschitz continuity of the derivative \( \nabla f \) is sufficient for providing convergence of the sequence generated by the projected gradient method, since stepsizes are sufficiently small with respect to the Lipschitz constant. Naturally, small steps may lead to slow convergence, not mentioning that having gradients globally Lipschitz is a very restricted assumption.

In this work we dealt with weak and strong convergence of projected gradient methods for convex (Gâteaux) differentiable optimization problems. We focused on the classical Armijo linesearch along the feasible direction, eliminating thus the undesired small stepsizes. Moreover, we relaxed the Lipschitz assumption by supposing only uniform continuity of the derivatives, a much weaker assumption as illustrated in Example 2.1. Furthermore, we proposed a strongly convergent variant of the projected gradient method, which has advantages over the classical projected gradient method.

We hope that this study will serve as basis for future research on others more efficient variants, as well as including sophisticated line searches on the gradient methods in Hilbert spaces.

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