ALTERNATING FORMULAS FOR $K$-THEORETIC QUIVER POLYNOMIALS

EZRA MILLER

Abstract. The main theorem here is the $K$-theoretic analogue of the cohomological ‘stable double component formula’ for quiver functions in [KMS03]. This $K$-theoretic version is still in terms of lacing diagrams, but nonminimal diagrams contribute terms of higher degree. The motivating consequence is a conjecture of Buch on the sign-alternation of the coefficients appearing in his expansion of quiver $K$-polynomials in terms of stable Grothendieck polynomials for partitions [Buc02a].

Introduction

The study of combinatorial formulas for the degeneracy loci of quivers of vector bundles with arbitrary ranks was initiated by Buch and Fulton [BF99]. In that paper they proved that the cohomology classes of such degeneracy loci can be expressed as integer sums of products of Schur polynomials evaluated on the Chern classes of the bundles in the quiver. After giving an explicit algorithmic (but nonpositive) expression for the quiver coefficients appearing therein, they also conjectured a positive combinatorial formula for them. This conjecture was proved in [KMS03], by way of three other positive combinatorial formulas for the quiver polynomials.

The cohomological ideas of [BF99] were extended to $K$-theory in [Buc02a], where the classes of the structure sheaves of the aforementioned degeneracy loci are expressed as integer sums of products of stable double Grothendieck polynomials for grassmannian permutations. Buch proved a formula for the coefficients in this expansion [Buc02a, Theorem 4.1], and conjectured that the signs of these coefficients alternate in a simple manner [Buc02a, Conjecture 4.2].

The main result here is Theorem 12, which extends the stable double component formula in terms of minimal lacing diagrams [KMS03, Theorem 6.20], as well as some combinatorial methods surrounding it, from cohomology to $K$-theory. Theorem 12 is still in terms of lacing diagrams, but nonminimal diagrams contribute terms of higher degree. The purpose is to prove Buch’s conjecture as a consequence (Theorem 17), using a sign-alternation theorem of Lascoux [Las01, Theorem 4]. The $K$-theory analogue of a formula [KMS03, Theorem 5.5] for quiver polynomials in terms of the pipe dreams of Fomin and Kirillov [FK96] enters along the way (Theorem 3).

The proof of Theorem 12 generalizes a procedure suggested by [KMS03] (see Remark 6.21 there), and carried out in [Yon03], for constructing pipe dreams associated to given lacing diagrams. This technique is combined with those developed in [KM03b] for dealing with nonreduced subwords of reduced expressions for permutations.

Date: 27 November 2003.

The author was partly supported by the National Science Foundation.
Buch \[\text{Buc03}\] independently arrived at the main results and definitions here (and more) by applying general techniques of Fomin and Kirillov \[\text{FK96, FK94}\]. A special case of the sign conjecture and \(K\)-component formula already appeared in \[\text{BKT03}\].

**Organization.** A notion of double quiver \(K\)-polynomial is identified via a ratio formula in Section 11 in analogy with the way (cohomological) double quiver polynomials arise in \[\text{KMS03}\]. The ‘pipe formula’ for quiver \(K\)-polynomials is proved in Section 2 after background on nonreduced pipe dreams and Demazure products. The condition on nonminimal lacing diagrams that turns out to make them occur with sign \(\pm1\) in the \(K\)-component formula is defined in Section 3. Rank stability of these nonminimal lacing diagrams, proved in Section 4, plays the same role here as it did for the cohomological component formula in \[\text{KMS03}\]. The stable \(K\)-component formula is derived in Section 5 after reviewing basics regarding Grothendieck polynomials and their stable limits. Finally, Buch’s sign alternation conjecture is proved in Section 6.

### 1. Double quiver \(K\)-polynomials

A \(k \times \ell\) partial permutation is a \(k \times \ell\) matrix \(w\) whose entries are either 0 or 1, with at most one nonzero entry in each row or column. Each such matrix \(w\) can be completed to a permutation matrix—that is, with exactly one 1 in each row and column—having \(w\) as its upper-left \(k \times \ell\) corner. Viewing permutations as lying in the union \(S_{\infty} = \bigcup_k S_k\) of all symmetric groups \(S_k\), there is a unique completion \(\tilde{w}\) of \(w\) that has minimal length \(l(\tilde{w})\). For any partial permutation \(w\), we write \(q = w(p)\) if the entry of \(w\) in row \(p\) and column \(q\) equals 1. If \(v\) is a permutation matrix, then the assignment \(p \mapsto v(p)\) defines a permutation in \(S_{\infty}\).

Let \(z = z_1, z_2, \ldots\) and \(\hat{z} = \hat{z}_1, \hat{z}_2, \ldots\) be alphabets. Writing a given polynomial \(f\) in these two alphabets over the integers \(\mathbb{Z}\) as a polynomial in \(z_i\) and \(z_{i+1}\) with coefficients that are polynomials in the other variables, the \(i\)th Demazure operator \(\overline{\partial}_i\) sends \(f\) to

\[
\overline{\partial}_i f = \frac{z_{i+1}f(z_{i+1}, z_{i+1}) - z_if(z_i, z_i)}{z_{i+1} - z_i}.
\]

Let \(w_0^k\) be the permutation of maximal length in \(S_k\), and write \(s_i \in S_{\infty}\) for the transposition switching \(i\) and \(i + 1\). Following \[\text{LS82}\], the double Grothendieck polynomial for a permutation \(v \in S_k\) is defined from the “top” double Grothendieck polynomial \(G_{w_0^k}(z/\hat{z}) = \prod_{i+j \leq k} (1 - z_i/\hat{z}_j)\) by the recursion

\[
G_{vs_i}(z/\hat{z}) = \overline{\partial}_i G_v(z/\hat{z}),
\]

whenever \(vs_i\) is lower in Bruhat order than \(v\). This definition is independent of the choice of \(k\). \[\text{LS82}\] If \(w\) is a partial permutation, then set \(G_w(z/\hat{z}) = G_{\tilde{w}}(z/\hat{z})\).

The permutation matrices \(v\) of central importance here are those associated to the ‘Zelevinsky permutations’ of \[\text{KMS03}\], which are defined as follows. Fix a positive integer \(d\) and an expression \(d = \sum_{j=0}^n r_j\) of \(d\) as a sum of \(n + 1\) “ranks” \(r_j\). Endow each \(d \times d\) permutation matrix \(v\) with a block decomposition in which the \(j\)th block row from the top has height \(r_j\), and the \(i\)th block column from the right has width \(r_i\). Thus each \(d \times d\) permutation matrix \(v\) is composed of \((n + 1)^2\) blocks \(B_{ji}\), each of size \(r_j \times r_i\). The matrix \(v\) is a Zelevinsky permutation as in \[\text{KMS03}\] Definition 1.7 if
\[ B_{ji} \text{ has all zero entries whenever } i \geq j + 2, \text{ and the nonzero entries of } v \text{ proceed from northwest to southeast within every block row or block column (so } v \text{ has no 1 entry that is northeast of another within the same block row or block column). Pictures and examples can be found in [KMS03, Section 1.2].} \]

If a Zelevinsky permutation \( v \) is given, define \( r_{ij} \) to be the number of nonzero entries of \( v \) in the union of all blocks \( B_{qp} \) for which \( q \geq j \) and \( p \leq i \) (that is, blocks \( B_{qp} \) weakly southeast of \( B_{ji} \)). This results in a rank array \( r = (r_{ij})_{i \geq j} \). Since \( r \) uniquely determines \( v \) by [KMS03 Proposition 1.6], the notation \( v = v(r) \) makes sense.

Double Grothendieck polynomials for Zelevinsky permutations \( v(r) \) are naturally written as \( G_v(r)(x/\hat{y}) \), using two alphabets \( z = x \) and \( z = \hat{y} \) each of which is an ordered sequence of \( n + 1 \) alphabets of sizes \( r_0, \ldots, r_n \) and \( r_n, \ldots, r_0 \), respectively:

\[
\begin{align*}
x &= x^0, \ldots, x^n \quad \text{and} \quad \hat{y} &= y^n, \ldots, y^0, \\
\text{where} \quad &x^i = x^i_1, \ldots, x^i_{r_j} \quad \text{and} \quad y^j = y^j_1, \ldots, y^j_{r_j}.
\end{align*}
\]

It is convenient to think of the \( x \) variables as labeling the rows of the \( d \times d \) grid, while the \( y \) variables label its columns (see [KMS03 Section 2.2] for pictures and examples). Most partial permutations \( w \) that occur in the sequel will have size \( r_{j-1} \times r_j \) for some \( j \in \{1, \ldots, n\} \); in that case we consider \( G_w(x^{j-1}/y^j) \).

Among all \( d \times d \) Zelevinsky permutations with block decompositions determined by \( d = \sum_{j=0}^n r_j \), there is a unique one \( v(\text{Hom}) \) whose rank array \( r(\text{Hom}) \) is maximal, in the sense that \( r_{ij}(\text{Hom}) \geq r_{ij} \) for all other \( d \times d \) Zelevinsky permutations \( v(r) \).

**Definition 1.** The **double quiver \( K \)-polynomial** is the ratio

\[
KQ_v(x/\hat{y}) = \frac{G_v(r)(x/\hat{y})}{G_v(\text{Hom})(x/\hat{y})}
\]

of double Grothendieck polynomials for \( v(r) \) and \( v(\text{Hom}) \).

The “ordinary” specialization of the polynomial \( KQ_v(x/\hat{y}) \) appears in the \( K \)-theoretic ratio formula [KMS03 Theorem 2.7]. It will follow from Theorem \( \Box \) below, that \( G_{v(\text{Hom})}(x/\hat{y}) \) divides \( G_v(r)(x/\hat{y}) \), so the right hand side of Definition 1 is actually a (Laurent) polynomial rather than simply a rational function.

**2. Nonreduced pipe dreams**

A \( k \times \ell \) pipe dream is a subset of the \( k \times \ell \) grid, identified as the set of crosses in a tiling of the \( k \times \ell \) grid by crosses \( \bigcirc \) and elbow joints \( \bigtriangledown \), as in the following diagrams:

\[
\begin{array}{c}
+ + + \\
+ + + \\
+ + + \\
+ + + \\
\end{array}
= 
\begin{array}{c}
\bigcirc \bigcirc \bigcirc \\
\bigcirc \bigcirc \bigcirc \\
\bigcirc \bigcirc \bigcirc \\
\bigcirc \bigcirc \bigcirc \\
\end{array}
\]

The square tile boundaries are omitted from the tilings forming the newtworks of pipes on right sides of these equalities. Pipe dreams are special cases of diagrams introduced by Fomin and Kirillov [FK96]; for more background, see [KM03a Section 1.4].

A pipe dream \( P \) yields a word in the Coxeter generators \( s_1, s_2, s_3, \ldots \) of \( S_{\infty} \) by reading the antidiagonal indices of the crosses in \( P \) along rows, right to left, starting
The Demazure product $\delta(P)$ is obtained (as in [KM03a, Definition 3.1]) by omitting adjacent transpositions that decrease length. More precisely, $\delta(P)$ is obtained by multiplying the word of $P$ using the idempotence relation $s_i^2 = s_i$ along with the usual braid relations $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ and $s_i s_j = s_j s_i$ for $|i - j| \geq 2$. Up to signs, this amounts to taking the product of the word of $P$ in the degenerate Hecke algebra [FK96]. Let

$$P(w) = \{\text{pipe dreams } P \mid \delta(P) = \tilde{w}\}$$

for a $k \times \ell$ partial permutation $w$ be the set of pipe dreams whose Demazure product is the minimal completion of $w$ to a permutation $\tilde{w} \in S_{\infty}$. Every pipe dream in $P(w)$ fits inside the $k \times \ell$ rectangle, and is to be considered as a pipe dream of size $k \times \ell$.

The subset of $P(w)$ consisting of reduced pipe dreams (or rc-graphs [BB93]), where no pair of pipes crosses more than once, is denoted by $RP(w)$.

**Lemma 2.** Suppose that $P \in P(w)$. Then the crossing tiles in $P$ lie in the union of all reduced pipe dreams for $w$.

**Proof.** The statement is obvious if $P$ is reduced, so suppose otherwise. Then some pipe dream $P' \in P(w)$ be can be obtained by deleting a single crossing tile from $P$. By induction, every crossing tile in $P'$ lies in some reduced pipe dream for $w$. On the other hand, [KM03a, Theorem 3.7] implies that a second pipe dream $P'' \in P(w)$ can be obtained from $P$ by deleting a different crossing tile. Induction shows that every crossing tile in $P''$, including the tile $P \setminus P'$, lies in a reduced pipe dream for $w$. □

Lemma 2 implies that [KMS03] Corollary 6.10 holds as well for every pipe dream with Demazure product $v(r)$. This claim will be made precise in Proposition 4 below.

The exponential reverse monomial associated to a $d \times d$ pipe dream $P$ is

$$(1 - \bar{x}/\bar{y})^P = \prod_{+ \in P} (1 - \bar{x}_+/\bar{y}_+),$$

where the variable $\bar{x}_+$ sits at the left end of the row containing $\bar{+}$ after reversing each of the $x$ alphabets before Definition 11 and the variable $\bar{y}_+$ sits atop the column containing $\bar{+}$ after reversing each of the $y$ alphabets there. (The row and column labeling in [KMS03] Section 2.2 is the one meant on the unreversed alphabets here.)

As in [KMS03] Definition 1.10, let $D_{\text{Hom}}$ be the Ferrers shape of all locations strictly above the block superantidiagonal. To make the meaning of $\delta(P)$ clear, it is necessary to consider all crosses in $P$, including those in $D_{\text{Hom}}$ (unlike the convention of [KMS03]). Here is the $K$-theoretic analogue of [KMS03] Proposition 6.9.

**Theorem 3** (Pipe formula). The double quiver $K$-polynomial is the alternating sum

$$KQ_r(x/\bar{y}) = \sum_{\delta(P) = v(r)} (-1)^{|P| - l(v(r))} (1 - \bar{x}/\bar{y})^{P \setminus D_{\text{Hom}}}$$

of exponential reverse monomials associated to pipe dreams $P \setminus D_{\text{Hom}}$ for $P \in P(v(r))$. The exponent on $-1$ is the number crosses in $P$ minus the length $l(v(r))$ of $v(r)$. 
Proof. Use Definition 1 and the symmetry of the double Grothendieck polynomial for $v(r)$ in each of its $2n + 2$ alphabets, along with the formula of Fomin and Kirillov [FK94, Theorem 2.3 and p. 190] (or see [KM03b, Theorem 4.1 and Corollary 5.4]). □

As in [KMS03] Section 4.4], let $m + r$ be the rank array obtained from $r$ by adding the nonnegative integer $m$ to each entry of $r$. Let $x_{m+r}$ be a list of finite alphabets of sizes $m + r_0, \ldots, m + r_n$, and let the alphabets in $\hat{y}_{m+r}$ have sizes $m + r_n, \ldots, m + r_0$. Denote by $D_{Hom}(m)$ the unique reduced pipe dream for the Zelevinsky permutation $v(m + r(Hom))$ in $S_{d+m(n+1)}$ associated to the maximal irreducible rank array.

**Proposition 4.** There is a fixed integer $\ell$, independent of $m$, such that for every pipe dream $P \in \mathcal{P}(v(m + r))$ with at least one cross $\bigcirc$ in an antidiagonal block, setting the last $\ell$ variables to 1 in every finite alphabet from the lists $x_{m+r}$ and $\hat{y}_{m+r}$ kills the exponential reverse monomial $(1 - \hat{x}/\hat{y})^{P-D_{Hom}(m)}$.

**Proof.** This follows immediately from [KMS03] Proposition 6.10] and Lemma 2. □

Observe that any pipe dream $P \in \mathcal{P}(v(r))$ with no crossing tiles in its antidiagonal blocks has its “interesting” crosses confined to the block superantidiagonal. All other blocks above the antidiagonal are filled completely with crossing tiles (in [KMS03] these are the * entries), while blocks below the block antidiagonal are empty. These kinds of pipe dreams $P \in \mathcal{P}(v(r))$ are central to the next section.

### 3. Nonminimal lacing diagrams

Suppose that $w = (w_1, \ldots, w_n)$ is a list of partial permutations in which $w_j$ has size $r_{j-1} \times r_j$. The list $w$ can be identified with the (nonembedded) graph in the plane called its *lacing diagram* in [KMS03 Section 3.1], based on diagrams of Abeasis and Del Fra [AD80]. The vertex set of the graph consists of $r_j$ bottom-justified dots in column $j$ for $j = 0, \ldots, n$, with an edge connecting the dot at height $\alpha$ (from the bottom) in column $j - 1$ with the dot at height $\beta$ in column $j$ if and only if the entry of $w_j$ at $(\alpha, \beta)$ is 1. A *lace* is a connected component of a lacing diagram. For example, here is the lacing diagram associated to a partial permutation list:

\[
\begin{array}{c}
\bullet \\
\bigcirc
\end{array} \quad \leftrightarrow \quad \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

The goal of this section is to define what it means for a rank array to equal the Demazure product $\delta(w)$ of a lacing diagram $w$. That $\delta(w)$ is a rank array rather than a minimal lacing diagram is in analogy with Demazure products of lists of simple reflections, which are permutations rather than reduced decompositions. Usually $\delta(w)$ will not equal the rank array of $w$ itself. In analogy with Demazure products of *reduced* words, however, the Demazure product of a *minimal* lacing diagram will equal its own rank array.

Given a pipe dream $P$, as in [KM03b Theorem 4.4] say that $P$ *simplifies* to $D \subseteq P$ if $D$ is the lexicographically first subword of $P$ with Demazure product $\delta(P)$. Equivalently, denoting by $P_{\leq i}$ the length $i$ initial string of simple reflections in $P$, the
simplification $D$ is obtained from $P$ by omitting the $i$th reflection from $P$ for all $i$ such that $\delta(P_{\leq i-1}) = \delta(P_{\leq i})$.

**Lemma 5.** Suppose that $P$ is a $k \times \ell$ pipe dream and let $\alpha|P$ be the $k \times (\ell + \alpha)$ pipe dream obtained by adding $\alpha$ columns of $k \begin{array}{c} \mathcal{R} \end{array}$ tiles to the left side of $P$. The pipe dream $P$ simplifies to $D$ if and only if $\alpha|P$ simplifies to $\alpha|D$.

**Proof.** Since $P$ is reduced if and only if $\alpha|P$ is reduced, we may assume that $P$ is not reduced. Moreover, by adding $\begin{array}{c} \mathcal{R} \end{array}$ tiles to $P$ one by one (from right to left in each row and top to bottom, as usual), it is enough to prove the lemma when $|P| = 1 + |D|$. In this case, a single pair of pipes in $P$ crosses twice, as does the corresponding pair of pipes (shifted to the right by $\alpha$) in $\alpha|P$. The simplifications of $P$ and $\alpha|P$ are obtained by deleting the southwestern crossings of the corresponding pairs of pipes.

**Definition 6.** Suppose $P_1, \ldots, P_n$ are pipe dreams of sizes $r_0 \times r_1, \ldots, r_{n-1} \times r_n$, and set $d = r_0 + \cdots + r_n$. Denote by $P(P_1, \ldots, P_n)$ the $d \times d$ pipe dream in which every block strictly above the block superantidiagonal is filled with crossing tiles, and the superantidiagonal $r_{j-1} \times r_j$ block in block row $j-1$ is the pipe dream $P_j$.

Given a $k \times \ell$ pipe dream $P$, let $\hat{P}$ be the $k \times \ell$ pipe dream that results after rotating $P$ through $180^\circ$. Also, recall from [BB93] Theorem 3.7 the notion of top pipe dream for a partial permutation $w$, which is the unique reduced pipe dream in $\mathcal{R}\mathcal{P}(w)$ that has no elbow tile due north of a crossing tile.

**Proposition 7.** Fix a lacing diagram $w = (w_1, \ldots, w_n)$. The Demazure product of $P(P_1, \ldots, P_n)$ is independent of $P_1, \ldots, P_n$, as long as $\hat{P}_j \in \mathcal{P}(w_j)$ for all $j = 1, \ldots, n$.

**Proof.** Associativity of Demazure products implies that we can take Demazure products first in each block row. By Lemma 4 these stripwise Demazure products don’t change when each $P_j$ is replaced by its simplification. Neither do the Demazure products of the pipe dreams $\hat{P}_j$. Therefore we can assume that each $\hat{P}_j$—and hence each block row of $P(P_1, \ldots, P_n)$—is reduced.

The Demazure product of each block row is unchanged by chute and inverse chute moves [BB93] that remain within block rows, because the Demazure product equals the usual product on reduced expressions. In addition, under these operations the reduced pipe dreams $\hat{P}_j$ remain inside of $\mathcal{R}\mathcal{P}(w_j)$ for all $j$. Therefore $\delta(P(P_1, \ldots, P_n))$ equals the Demazure product of the pipe dream $P(D_1, \ldots, D_n)$ in which $D_j$ is the unique “top” reduced pipe dream for $w_j$ by [BB93] Theorem 3.7.

**Definition 8.** Fix a lacing diagram $w$. If, for some (and hence, by Proposition 7 every) sequence $P_1, \ldots, P_n$ of pipe dreams satisfying $\hat{P}_j \in \mathcal{P}(w_j)$ for all $j$, the Demazure product of $P(P_1, \ldots, P_n)$ is a Zelevinsky permutation $v(r)$, then we write $\delta(w) = r$ and call the rank array $r$ the **Demazure product** of the lacing diagram $w$.

4. **Rank stability of lacing diagrams**

Next we show that lacing diagrams with Demazure product $r$ are stable, in the appropriate sense, under uniformly increasing ranks obtained by replacing $r$ with $m + r$. To ease the language, we use ‘horizontal strip $j$’ as a synonym for ‘block row $j$’.
Lemma 9. If \( P(P_1, \ldots, P_n) \in \mathcal{P}(v(1 + r)) \) and each \( \hat{P}_j \) is the top pipe dream for a \((1 + r_{j-1}) \times (1 + r_j)\) partial permutation \( w_j \), then all crossing tiles of \( P_j \) lie in the southwest \( r_{j-1} \times r_j \) rectangle of the antidiagonal block in horizontal strip \( j - 1 \).

Thus the antidiagonal block in the Lemma is supposed to have one blank row on top and one blank column to the right of the southwest \( r_{j-1} \times r_j \) rectangle in question.

Proof. No reduced pipe dream for \( v(1 + r) \) has a crossing tile on the main superantidiagonal, by \cite[Proposition 5.15]{KMS03}. Lemma 2 implies that the same is true of \( P \). It follows that \( w_j = 1 + w'_j \) for some \( r_{j-1} \times r_j \) partial permutation \( w'_j \). Consequently, the left column of \( \hat{P}_j \) has no crossing tiles, and shifting all crossing tiles in \( \hat{P}_j \) one unit to the left results in the top pipe dream for \( w'_j \). This top pipe dream fits inside the rectangle of size \( r_{j-1} \times r_j \).

Suppose \( P = P(P_1, \ldots, P_n) \) is a pipe dream in which

\[
\text{(SW)} \quad P_j \text{ has size } (1 + r_{j-1}) \times (1 + r_j), \text{ but every } \square \text{ in } P_j \text{ lies in the southwest } r_{j-1} \times r_j \text{ rectangle.}
\]

Write \( P'_j \) for the \( r_{j-1} \times r_j \) pipe dream consisting of the southwest rectangle of \( P_j \), and then write \( P' = P(P'_1, \ldots, P'_n) \). Thus \( P \) has block sizes consistent with ranks \( 1 + r \), while \( P' \) has block sizes consistent with ranks \( r \). The construction can also be reversed to create \( P \) having been given the pipe dream called \( P' \).

Given a reduced pipe dream \( D \), an elbow tile is absorbable \cite[Section 4]{KM03b} if the two pipes passing through it intersect in a crossing tile to its northeast. It follows from the definitions that a pipe dream \( P \) simplifies to \( D \) if and only if \( P \) is obtained from \( D \) by changing (at will) some of its absorbable elbow tiles into crossing tiles.

Lemma 10. Suppose \( D = (D_1, \ldots, D_n) \) satisfies the (SW) condition. Then \( D \) is a reduced pipe dream for \( v(1 + r) \) if and only if \( D' = (D'_1, \ldots, D'_n) \) is a reduced pipe dream for \( v(r) \). In this case, the absorbable elbow tiles in horizontal strip \( j - 1 \) of \( D' \) are in bijection with the absorbable elbow tiles in the southwest \( r_{j-1} \times r_j \) rectangle of the antidiagonal block in horizontal strip \( j - 1 \) of \( D \).

Proof. The first claim is a straightforward consequence of \cite[Proposition 5.15]{KMS03}. The second claim follows because the corresponding pairs of pipes in \( D \) and \( D' \) pass through corresponding elbow tiles. The rest of the proof makes this statement precise.

Given a nonzero entry of the Zelevinsky permutation \( v(1 + r) \), exactly one of the following three conditions must hold: (i) the entry lies in the northwest corner of some superantidiagonal block; (ii) the entry lies in the southeast corner of the whole matrix; or (iii) there is a corresponding nonzero entry in \( v(r) \). This means that the pipes in \( D' \) are in bijection with those pipes in \( D \) corresponding to nonzero entries of \( v(1 + r) \) that do not satisfy (i) or (ii). Furthermore, it is easily checked that the pipes in \( D \) of type (i) or (ii) can only intersect a superantidiagonal block in its top row or rightmost column. Hence to say

the two pipes passing through an elbow tile in the southwest \( r_{j-1} \times r_j \) rectangle of the antidiagonal block in horizontal strip \( j - 1 \) of \( D \) correspond to the pipes passing through the corresponding elbow tile in \( D' \)
Lemma 9 there is a corresponding pipe dream \( RP(w_j) \) after Lemma 9. On the other hand, the pipe dream results by changing back into crossing tiles those elbow tiles in the obvious manner instead of on \( \mathbb{Z}_{>0} = \{1, 2, \ldots \} \). For a list \( w = (w_1, \ldots, w_n) \) of partial permutations, set \( m + w = (m+w_1, \ldots, m+w_n) \).

**Proposition 11.** For each array \( r \), let \( L(r) = \{w \mid \delta(w) = r\} \) be the set of lacing diagrams \( w \) with Demazure product \( r \). Then \( L(r) \) and \( L(m+r) \) are in canonical bijection:

\[
L(m+r) = \{m+w \mid w \in L(r)\}.
\]

**Proof.** It suffices to prove the case \( m = 1 \), so suppose \( w \in L(1+r) \). Let \( P = P(P_1, \ldots, P_n) \) be the pipe dream in \( \mathcal{P}(v(1+r)) \) for which each \( P_j \) is the top pipe dream in \( \mathcal{RP}(w_j) \). Then \( P \) simplifies to a reduced pipe dream \( D \in \mathcal{RP}(v(1+r)) \). By Lemma 9 there is a corresponding pipe dream \( D' \in \mathcal{RP}(v(r)) \), constructed via the procedure after Lemma 9. On the other hand, the pipe dream \( P' \) constructed from \( P \) results by changing back into crossing tiles those elbow tiles in \( D' \) that correspond to the tiles deleted from \( P \) to get \( D \). Lemma 10 says that \( P' \) has Demazure product \( v(r) \). Defining \( w' \) by the equality \( 1 + w' = w \), which can be done by Lemma 9, it follows that \( w' \in L(r) \).

In summary, we have constructed \( P' \) from \( P \) via the intermediate steps

\[
P \in \mathcal{P}(v(1+r)) \leadsto D \in \mathcal{RP}(v(1+r)) \leadsto D' \in \mathcal{RP}(v(r)) \leadsto P' \in \mathcal{P}(v(r)),
\]

where the first and third steps are simplification and “unsimplification”. Consequently, \( L(1+r) \subseteq \{1+w' \mid w' \in L(r)\} \). But the arguments justifying these steps are all reversible, so the reverse containment holds, as well. \( \square \)

5. **Stable double component formula**

The main result in this paper, namely Theorem 12, involves stable double Grothendieck polynomials \( \hat{G}_w(z/z) \) for \( k \times \ell \) partial permutations \( w \) [FK94], which we recall presently. Suppose that the argument of a Laurent polynomial \( G \) is naturally a pair of alphabets \( z \) and \( \hat{z} \) of sizes \( k \) and \( \ell \), respectively. In this section and the next, the convention is that if \( G(z/\hat{z}) \) is written, but \( z \) or \( \hat{z} \) has fewer than the required number of letters, then the rest of the letters are assumed to equal 1. For example, the notation \( KQ_{m+r}(x_r/y_{r'}) \) indicates that all variables in \( x_{m+r} \setminus x_r \) and \( y_{m+r} \setminus y_{r'} \) (see the paragraph preceding Proposition 11) are to be set equal to 1.

Under this convention, let \( w \) be a \( k \times \ell \) partial permutation, and write \( G_{m+w}(z_k/\hat{z}_\ell) \) for each \( m \geq 0 \) to mean the Laurent polynomial \( G_{m+w} \) applied to alphabets \( z_k \) and \( \hat{z}_\ell \) of fixed sizes \( k \) and \( \ell \). As \( m \) gets large, these Laurent polynomials eventually stabilize, allowing the notation \( \hat{G}_w(z/\hat{z}) = \lim_{m \to \infty} G_{m+w}(z_k/\hat{z}_\ell) \) for the stable double Grothendieck polynomial.

Given a lacing diagram \( w \) with \( r_j \) dots in column \( j \), for \( j = 0, \ldots, n \) denote by

\[
G_w(x/y) = G_{w_1}(x^0/y^1) \cdots G_{w_n}(x^{n-1}/y^n)
\]
the product of double Grothendieck polynomials taken over partial permutations in the list \( w = (w_1, \ldots, w_n) \). Add hats over every \( G \) for the stable Grothendieck case.

Here now is the main result, the \( K \)-theoretic analogue of the (cohomological) component formula for stable double quiver polynomials [KAMS03 Theorem 6.20].

**Theorem 12.** The limit of double quiver \( K \)-polynomials \( KQ_{m+r}(x_r/\hat{y}_r) \) for \( m \) approaching \( \infty \) exists and equals the alternating sum

\[
G_r(x/\hat{y}) := \lim_{m \to \infty} KQ_{m+r}(x_r/\hat{y}_r) = \sum_{w \in L(r)} (-1)^{|l(w) - d(r)|} \hat{G}_w(x/\hat{y})
\]

of products of stable double Grothendieck polynomials, where \( L(r) = \{ w | \delta(w) = r \} \), \( l(w) = \sum_{i=1}^n l(\hat{\delta}(i)) \), and \( d(r) = l(v(r)) - l(v(Hom)) \). The limit polynomial \( G_r(x/\hat{y}) \) is symmetric separately in each of the \( 2n + 2 \) finite alphabets \( x^0, \ldots, x^n, y^n, \ldots, y^0 \).

**Definition 13.** \( G_r(x/\hat{y}) \) is called the stable double quiver \( K \)-polynomial.

As we shall see in Corollary 10 and the comments after it, the Laurent polynomial \( G_r(x/\hat{y}) \) is not a new object: it is obtained from Buch’s power series \( \bar{P} \) [Buc02a Section 4] by substituting \( 1 - x_i \) for \( x_i \) and \( 1 - y_j^{-1} \) for \( y_j \) in each polynomial \( G_{\mu_k} \) there.

**Proof.** Define \( KQ_{m+r}(x/\hat{y})_\ell \) by setting the last \( \ell \) variables to 1 in every finite alphabet from the lists \( x_{m+r} \) and \( \hat{y}_{m+r} \). Similarly, for each lacing diagram \( w \), define \( \hat{G}_{m+w}(x/\hat{y})_\ell \) by setting the same variables to 1 in \( \hat{G}_{m+w}(x/\hat{y}) \). Because of the nature of the limit in question, and the defining properties of stable Grothendieck polynomials, it suffices to prove that for all \( m \geq 0 \) and some fixed \( \ell \) independent of \( m \),

\[
KQ_{m+r}(x/\hat{y})_\ell = \sum_{w \in L(r)} (-1)^{|l(w) - d(r)|} \hat{G}_{m+w}(x/\hat{y})_\ell.
\]

Fix \( \ell \) as in Proposition 4 and apply Theorem 3 to \( m + r \) instead of \( r \). Setting the last \( \ell \) variables in each alphabet to 1 on the right hand side there kills all summands corresponding to pipe dreams \( P \) that are not expressible as \( P(P_1, \ldots, P_n) \) for some list of pipe dreams \( P_j \) of sizes \( (m + r_j) \times (m + r_j) \); this is the content of Proposition 4. What remains on the right side of Theorem 3 is a sum of terms having the form 

\[
(-1)^{|P| - l(v(m+r))}(1 - \hat{x}/\hat{y})_\ell^{P-\Hom(m)}
\]

for pipe dreams \( P = P(P_1, \ldots, P_n) \) in \( P(v(m+r)) \). If \( P_j \in P(m + w_j) \) for each \( j \), then this term equals the product

\[
(-1)^{|l(w) - d(r)|} \prod_{i=1}^n (-1)^{|P_j| - l(\hat{\delta}(i))}(1 - \hat{x}_{i}^{-1}/\hat{y}_{i})_\ell^{P_j}
\]

for \( w = (w_1, \ldots, w_n) \). The signs in \( \ast \) are correct because \( |P| - l(v(m+r)) = \sum_j |P_j| - d(m + r_j) \), and \( d(m + r_j) = d(r_j) \). To make sense of \( (1 - \hat{x}_{i}^{-1}/\hat{y}_{i})_\ell^{P_j} \), identify \( P_j \) with the \( d \times d \) pipe dream consisting of just \( P_j \) on the \( j \)th superantidiagonal block.

For each lacing diagram \( w \in L(r) \), let \( P_w(m + r) \) be the set of pipe dreams \( P(P_1, \ldots, P_n) \in P(v(m+r)) \) such that \( P_j \in P(m + w_j) \) for all \( j \). Summing the products in \( \ast \) over pipe dreams \( P \in P_w(m+r) \) yields \((-1)^{|l(w) - d(r)|} \hat{G}_{m+w}(x/\hat{y})_\ell \) by [FK94 Theorem 2.3 and p. 190] (see also [KM03b Section 5]). Summing over \( w \in L(r) \) completes the proof, by Proposition 11.  \( \square \)
Remark 14. Theorem 12 implies that $G_{m+r}(x_r/y_r) = G_r(x/y)$, in analogy with the (defining) stability properties of stable double Grothendieck polynomials.

Remark 15. Theorem 12 gives an explicit combinatorial formula, but the characterization of the Demazure product $\delta(w)$ of a lacing diagram via Zelevinsky permutations would be more satisfying if it were intrinsic. That is, it would be better to identify those partial permutation lists that fit stripwise into a pipe dream with Demazure product $\nu(r)$ using the language of lacing diagrams, without referring to Zelevinsky permutations or pipe dreams. Such an intrinsic method appears in [BFR03].

6. Sign alternation

A permutation $\mu \in S_\infty$ is grassmannian if it has at most one descent—that is, at most one index $p$ such that $\mu(p) > \mu(p+1)$. A crucial property of arbitrary stable double Grothendieck polynomials, proved in [Buc02b, Theorem 6.13], is that every such polynomial $\hat{G}_w(z/z)$ has a unique expression

$$\hat{G}_w(z/z) = \sum_{\mu \text{ grassmannian}} \alpha_{\mu}^w \hat{G}_\mu(z/z)$$

as a sum of stable Grothendieck polynomials $\hat{G}_\mu$ for grassmannian permutations. If $\underline{\mu} = (\mu_1, \ldots, \mu_n)$ is a sequence of partial permutations such that the minimal completions $\tilde{\mu}_1, \ldots, \tilde{\mu}_n$ are grassmannian, then let us call $\underline{\mu}$ a grassmannian lacing diagram.

Corollary 16. If $\alpha_{\underline{\mu}}^w = \prod_{i=1}^n \alpha_{\mu_i}^w$ for each lacing diagram $w$ and grassmannian $\underline{\mu}$, then

$$G_r(x/y) = \sum_{\underline{\mu}} c_{\underline{\mu}}(r) \hat{G}_\underline{\mu}(x/y)$$

for the constants

$$c_{\underline{\mu}}(r) = \sum_{w \in L(r)} (-1)^{l(w) - d(r)} \alpha_{\underline{\mu}}^w,$$

where the first sum above is over all grassmannian lacing diagrams $\underline{\mu}$.

Proof. Expand the right hand side of Theorem 12 using $\hat{G}_w = \sum_{\mu} \alpha_{\mu}^w \hat{G}_\mu$. □

Let $G_r(x/\hat{x})$ be the specialization of the stable double quiver $K$-polynomial obtained by setting $x_j = x^j$ for $j = 0, \ldots, n$. Independently from Corollary 16, it follows from [Buc02a, Theorem 4.1] that the (ordinary) stable quiver $K$-polynomial

$$G_r(x/\hat{x}) = \sum_{\underline{\mu}} c_{\underline{\mu}}(r) \hat{G}_\underline{\mu}(x/\hat{x})$$

is a sum of products of stable double Grothendieck polynomials $\hat{G}_{\mu_i}(x^{i-1}/x^i)$ for grassmannian permutations $\tilde{\mu}_i$, with uniquely determined integer coefficients $c_{\underline{\mu}}(r)$. That these coefficients are the same as in Corollary 16 follows from the fact that the right side above determines the same element in the $n$th tensor power of Buch’s bialgebra $\Gamma$ from [Buc02b, Buc02a] as does the right side of the top formula in Corollary 16.

In addition to proving the expansion of $\hat{G}_w$ as a sum of terms $\alpha_{w}^\mu \hat{G}_\mu$, Buch showed in [Buc02b, Theorem 6.13] that the coefficients $\alpha_{w}^\mu$ can only be nonzero if $l(\mu) \geq l(w)$, and he conjectured that the sign of $\alpha_{w}^\mu$ equals $(-1)^{l(\mu)-l(w)}$. This was proved by...
Lascoux [Las01, Theorem 4] as part of his extension of “transition” from Schubert polynomials to Grothendieck polynomials. Since, as shown in [Buc02a, Section 5], the coefficients $\alpha_w^\mu$ are special cases of the coefficients $c_w^\mu(\mathbf{r})$, Lascoux’s result is evidence for the following more general statement that was surmised by Buch (prior to [Las01]).

**Theorem 17** ([Buc02a, Conjecture 4.2]). The coefficients $c_w^\mu(\mathbf{r})$ alternate in sign; that is, $(-1)^{l(\mu)-d(\mathbf{r})}c_w^\mu(\mathbf{r}) \geq 0$ is a nonnegative integer.

**Proof.** By [Las01, Theorem 4] the sign of $\alpha_w^\mu$ is $(-1)^{l(\mu)-l(w)}$. Thus the sign of $c_w^\mu(\mathbf{r})$ is $(-1)^{l(w)-d(\mathbf{r})}(-1)^{l(\mu)-l(w)} = (-1)^{l(\mu)-d(\mathbf{r})}$, by the second formula in Corollary 16. □

**References**

[AD80] S. Abeasis and A. Del Fra, *Degenerations for the representations of a quiver of type $A_n$*, Boll. Un. Mat. Ital. Suppl. (1980) no. 2, 157–171.

[BB93] Nantel Bergeron and Sara Billey, *RC-graphs and Schubert polynomials*, Experimental Math. 2 (1993), no. 4, 257–269.

[BF99] Anders Skovsted Buch and William Fulton, *Chern class formulas for quiver varieties*, Invent. Math. 135 (1999), no. 3, 665–687.

[BFR03] Anders S. Buch, László M. Fehér, and Richárd Rimányi, *Positivity of quiver coefficients through Thom polynomials*, preprint, 2003. arXiv:math.AG/0311203

[BKTY03] Anders S. Buch, Andrew Kresch, Harry Tamvakis, and Alexander Yong, *Grothendieck polynomials and quiver formulas*, 2003.

[Buc02a] Anders S. Buch, *Grothendieck classes of quiver varieties*, Duke Math. J. 115 (2002), no. 1, 75–103.

[Buc02b] Anders S. Buch, *A Littlewood–Richardson rule for the $K$-theory of Grassmannians*, Acta Math. 189 (2002), 37–78.

[Buc03] Anders S. Buch, *Alternating signs of quiver coefficients*, preprint, 2003.

[FK94] Sergey Fomin and Anatol N. Kirillov, *Grothendieck polynomials and the Yang–Baxter equation*, 1994, Proceedings of the Sixth Conference in Formal Power Series and Algebraic Combinatorics, DIMACS, pp. 183–190.

[FK96] Sergey Fomin and Anatol N. Kirillov, *The Yang-Baxter equation, symmetric functions, and Schubert polynomials*, Discrete Math. 153 (1996), no. 1–3, 123–143, Proceedings of the 5th Conference on Formal Power Series and Algebraic Combinatorics (Florence, 1993).

[KM03a] Allen Knutson and Ezra Miller, *Gröbner geometry of Schubert polynomials*, to appear in Ann. of Math (2), 2003.

[KM03b] Allen Knutson and Ezra Miller, *Subword complexes in Coxeter groups*, to appear in Adv. in Math., 2003. arXiv:math.AG/0309259

[KMS03] Allen Knutson, Ezra Miller, and Mark Shimozono, *Four positive formulae for type $A$ quiver polynomials*, preprint, 2003. arXiv:math.AG/0308142

[Las01] A. Lascoux, *Transition on Grothendieck polynomials*, Physics and combinatorics, 2000 (Nagoya), 164–179, World Sci. Publishing, River Edge, NJ, 2001.

[LS82] Alain Lascoux and Marcel-Paul Schützenberger, *Structure de Hopf de l’anneau de cohomologie et de l’anneau de Grothendieck d’une variété de drapeaux*, C. R. Acad. Sci. Paris Sér. I Math. 295 (1982), no. 11, 629–633.

[Yon03] Alexander Yong, *On combinatorics of quiver component formulas*, preprint, 2003.

University of Minnesota, Minneapolis, Minnesota

E-mail address: ezra@math.umn.edu