An upper bound for the Waring rank of a form

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Abstract

In this paper we introduce the open Waring rank of a form of degree $d$ in $n$ variables and prove the that this rank in bounded from above by

\[
\binom{n + d - 2}{d - 1} - \binom{n + d - 6}{d - 3}
\]

whenever $n, d \geq 3$. This proves the same upper bound for the classical Waring rank of a form, improving the result of [BBS] and giving, as far as we know, the best upper bound known.

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1 Introduction

In [BBS] the authors introduce a stronger version of the Waring rank of a homogeneous polynomial and give an upper bound for this rank. Denoting this rank by $S(n, d)$, the key point of the proof is the inequality

\[
S(n, d) \leq S(n, d - 1) + S(n - 1, d),
\]

(1)

for $n, d \geq 3$, which gives a recursive step in the proof of the bound $S(n, d) \leq \binom{n + d - 2}{d - 1}$. The base cases of the recursion are the equalities

\[
S(2, d) = d \quad \text{and} \quad S(n, 2) = n,
\]

(2)

so the smallest case where the obtained upper bound may be sharp is $S(3, 3)$; [BBS] gives the bound $S(3, 3) \leq 6$. In this article we introduce an even slightly stronger version of the rank, denoted $Ork(n, d)$. We prove the inequality (1) together with the base cases (2) for $Ork(n, d)$ thus obtaining a bound

\[
Ork(n, d) \leq \binom{n + d - 2}{d - 1}.
\]

Next we prove that $Ork(3, 3) = 5$ which improves the upper bound to

\[
Ork(n, d) \leq \binom{n + d - 2}{d - 1} - \binom{n + d - 6}{d - 3}.
\]

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In the proof we will both adopt (in Lemma [14]) and reference the ideas from Johannes Kleppe Master thesis [Kl].

**Notation.** Let \( k \) be an algebraically closed field of characteristic 0 and \( S = k[x_1, \ldots, x_n] \) be a polynomial ring. We will often think of \( S_1 \) as an affine space; in this spirit let \( V \subseteq S_1 \) be a Zariski closed subset. Let \( S^* = k \left[ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right] \) be the ring of differential operators with its usual action on \( S \), which is denoted by \( (-) \lrcorner (-) : S^* \otimes S \to S \). For \( F \in S_d \) by \( F^\perp \subseteq S^* \) we denote the annihilator of \( F \) with respect to this action.

**Definition 1.** A form \( F \in S_d \) essentially depends on \( n \) variables if it cannot be written using less than \( n \) variables after a linear change of coordinates.

**Definition 2.** For a form \( F \) of degree \( d \) in \( S \) and \( V \subseteq S_1 \) let \( m = \text{Ork}(F,V) \) be the minimal natural number such that there exists a presentation

\[
F = \sum_{i=1}^{m} l_i^d, \text{ where } l_i \notin V.
\]

or \( \text{Ork}(F,V) = \infty \) if such presentation does not exist. Define the open Waring rank of \( F \) by

\[
\text{Ork}(F) := \sup \{ \text{Ork}(F,V) \mid V \subseteq S_1 \text{ homogeneous and Zariski closed} \}.
\]

Finally take

\[
\text{Ork}(n,d) := \sup \{ \text{Ork}(F) \mid F \in S_d \text{ essentially depends on } n \text{ variables} \}.
\]

**Remark 3.** The classical Waring rank of a form \( F \) is equal to \( \text{Ork}(F,\emptyset) \). The rank defined in [BBS] is similar to the one defined above, the difference is that the authors consider only subsets \( V \subseteq S_1 \) which are finite sums of hyperplanes through the origin. Thus we have an inequality \( \text{Ork}(n,d) \geq S(n,d) \).

The paper is divided into two sections, preceded by a preliminary part. In the first section we prove:

**Theorem 4.** Let \( n, d \geq 2 \) be integers. We have equations \( \text{Ork}(2,d) = d \) and \( \text{Ork}(n,2) = n \). Moreover

\[
\text{Ork}(n,d) \leq \text{Ork}(n-1,d) + \text{Ork}(n,d-1)
\]

for every \( n, d \geq 3 \).

The proof is a copy of the proof of [1] and [2] from [BBS]. Unfortunately the proof given there is, formally, just a special case of the proof required and moreover Białynicki-Birula and Schinzel are also concerned with non-homogeneous polynomials which makes their proof more complicated.

In the second part we prove the following theorem, with an immediate corollary bounding the open Waring rank:

**Theorem 5.** \( \text{Ork}(3,3) = 5 \).

**Corollary 6.** Let \( n, d \geq 3 \) be integers, then

\[
\text{Ork}(n,d) \leq \binom{n+d-2}{d-1} - \binom{n+d-6}{d-3}.
\]
2 Preliminaries

Let us recall a well known lemma, whose proof can be found e.g. in [Kl]

Lemma 7. If $F \in S_d$ and $\bigcap_{i=1}^k m_i \subseteq F^\perp$, where $m_i$ are homogeneous ideals of distinct points $[l_i] \in \mathbb{P}S_1$, then

$$F \in \text{lin} \left( l_1^d, l_2^d, \ldots, l_k^d \right).$$

This lemma will be used together with a special case of unmixedness property of complete intersections in $\mathbb{P}^2$:

Lemma 8. If two homogeneous forms $G, H$ intersect transversally in points $\{a_1, \ldots, a_k\} \subseteq \mathbb{P}^2$, then $\bigcap_i m_{a_i} = (G, H)$.

Another lemma, whose proof can be found in [Kl], is concerned with linear systems obtained from the apolar ideal of a form:

Lemma 9. Let $F$ be a form in $S_d$. Choose $e \leq d$ and consider $\mathcal{L} = (F^\perp)_e$ as a linear system on $\mathbb{P}S_1$. A point $[l] \in \mathbb{P}S_1$ is a base point of $\mathcal{L}$ if and only if there exists a differential $\partial \in S_{d-e}^*$ such that $\partial F = l^e$.

Sketch of proof. Fix a point $[l] \in \mathbb{P}S_1$ with homogeneous ideal $m_l$. The point $[l]$ is a base point of $\mathcal{L}$ iff $(m_l)_e \supseteq (F^\perp)_e = \bigcap \{(\partial F)_e \mid \partial \in S_{d-e}^*\}$ iff there exists $\partial \in S_{d-e}^*$ such that $l^e = \partial F$. □

Corollary 10. Fix $d \geq 1$. Denote by $Ess_{d,e}$ the set of forms $F \in S_d$ such that no nonzero element of $S_e^*$ annihilates $F$. The set of $F \in Ess_{d,e}$ such that $(F^\perp)_{d-e}$ has a base point in $\mathbb{P}S_1$ is closed in $Ess_{d,e}$.

Proof. Note that $Ess_{d,e}$ is Zariski open in $S_d$. Denote by $W$ the subset of forms $F \in Ess_{d,e}$ such that $(F^\perp)_{d-e}$ has a base point in $\mathbb{P}S_1$. Consider the closed subvariety

$$\left\{ (F, [\partial], [l]) \in Ess_{d,e} \times \mathbb{P}S_e^* \times \mathbb{P}S_1 \mid l^{d-e} \text{ and } \partial F \text{ are linearly dependent} \right\}.$$

The projection to the first coordinate gives the set of forms $F \in Ess_{d,e}$ such that there exist $\partial \in S_e^*$, $l \in S_1$ and $\lambda, \lambda' \in k$, not both equal zero, satisfying $\lambda l^{d-e} = \lambda' \partial F$. As $l^{d-e} \neq 0$ and $\partial F \neq 0$ from the definition of $Ess_{d,e}$ we have $\lambda \lambda' \neq 0$, which is equivalent, by Lemma 9, to $F \in W$. □

3 Proof of Theorem 4

The proof will be divided into three independent lemmas.

Lemma 11. Let $d \geq 2$ be an integer, then $\text{Ork} (2, d) = d$.

Proof. Let $S = k[x_1, x_2], F \in S_d$ and $V \subseteq S_1$ be homogeneous and Zariski-closed. We would like to prove that $\text{Ork} (F, V) \leq d$. It is a classical result by Sylvester that $F^\perp$ is a complete intersection generated by elements of degrees $d_1$ and $d_2$ such that $d_1 + d_2 = d + 2$. If $\text{min}(d_1, d_2) = 1$ then $F$ is does not essentially depend on two variables. Thus $\text{min}(d_1, d_2) \geq 2$ and $\text{max}(d_1, d_2) \leq d$, in particular the linear system $F_{d_1}^\perp$ on $\mathbb{P}S_1$ is base point free. By Bertini Theorem [Har] Thm III.10.9 a general element $D$ of $F_{d_1}^\perp$ is smooth and does not intersect $V$. The zero set of $D$ is a sum of $d$ points, which, by Lemma 7, gives a required presentation of $F$. On the other hand the apolar ideal $(x_1^{d-1} x_2)^\perp$ is generated by a square of a linear form and a $d$-th power of another linear form, thus there are no smooth forms of degree less than $d$ in this ideal and so the Waring rank of $x_1^{d-1} x_2^2$ is $d$. □
Lemma 12. Let \( n \geq 1 \) be an integer, then \( \text{Ork}(n, 2) = n \).

Proof. The inequality \( \text{Ork}(n, 2) \geq n \) is trivial because the sum of less than \( n \) squares does not essentially depend on \( n \) variables. We prove the other inequality by induction on \( n \), the base being clear. Let \( n \geq 2 \). Take \( F \in S_2 \) which essentially depends on \( n \) variables and \( V \subseteq S_1 \) homogeneous and Zariski-closed.

Think about \( S^*_1 \) as an affine space. For \( \partial \in S^*_1 \) the condition \( \partial^2 F = 0 \) is Zariski-closed. Take any \( \alpha \in S^*_1 \) such that \( \alpha^2 \notin 0 \) and \( V(\alpha) \not\subset V \). Let

\[
F' = F - \frac{(\alpha \cdot F)^2}{2 \cdot \alpha^2 F},
\]

then \( \alpha \cdot F' = 0 \), thus \( F' \) may be written, after a linear change of coordinates, in \( n - 1 \) variables \( x'_1, \ldots, x'_{n-1} \) such that \( \alpha \cdot x'_1 = 0 \). From the definition of \( F' \) it follows that \( F \) may be written using one more variable than \( F' \), thus \( F' \) essentially depends on \( n - 1 \) variables. Furthermore \( V' = V \cap V(\alpha) \neq V(\alpha) \) is a homogeneous Zariski-closed set, so that \( \text{Ork}(F', V') \leq n - 1 \) by induction, and we obtain \( \text{Ork}(F, V) \leq \text{Ork}(F', V') + 1 \leq n \). \( \square \)

Lemma 13. Let \( n, d \geq 3 \) be integers, then

\[
\text{Ork}(n, d) \leq \text{Ork}(n - 1, d) + \text{Ork}(n, d - 1).
\]

Proof. Take \( F \in S_2 \) which essentially depends on \( n \) variables and \( V \subseteq S_1 \) homogeneous and Zariski-closed. Take \( \alpha \in S^*_1 \) such that \( V(\alpha) \not\subset V \) and \( F' = \alpha \cdot F \) essentially depends on \( n \) variables (these are open non-empty conditions). The form \( F' \) has a presentation

\[
F' = \sum_{i=1}^{m} l_i^{d-1},
\]

where \( m = \text{Ork}(F', V \cup V(\alpha)) \) and \( l_i \not\in V \cup V(\alpha) \). Note that \( l_i \not\in V(\alpha) \) is equivalent to \( \alpha \cdot l_i \neq 0 \). Take

\[
F_1 = \sum_{i=1}^{m} \alpha_i \cdot l_i^d
\]

(3)

where \( \alpha_i = (d \cdot \alpha \cdot l_i)^{-1} \), then \( \alpha \cdot (F - F_1) = 0 \). Let \( T \subseteq \{1, 2, \ldots, m\} \) be a minimal set of indexes such that there exists \( 0 \neq \beta = \beta_T \in S^*_1 \) such that

1. \( V(\beta) \not\subset V \),
2. \( F_2 := F - \sum_{i \in T} \alpha_i \cdot l_i^d \) is annihilated by \( \beta \),

(the set \( T = \{1, \ldots, m\} \) with \( \alpha = \beta_T \) satisfies the above hypotheses except, perhaps, minimality). We claim that the form \( F_2 \) obtained from a minimal \( T \) essentially depends on \( n - 1 \) variables. If this is not the case then we take \( i \in T \) such that \( F_2 + \alpha_i \cdot l_i^d \) essentially depends on more variables than \( F_2 \). The space \( (F_2)_{1}^+ \) is at least two-dimensional, thus its intersection with \( (l_i^d)^+ \) contains a non-zero element \( \beta' \). Since \( l_i \in V(\beta') \setminus V \), we have \( V(\beta') \not\subset V \) and the set \( T' := T \setminus \{i\} \) satisfies the above conditions. This contradicts the minimality of \( T \).

Since \( F_2 \in k[x_1, \ldots, x_n] \) essentially depends on \( n - 1 \) variables lying in \( V(\beta) \) and \( V \cap V(\beta) \neq V(\beta) \), the form \( F_2 \) may be written as \( m_2 \leq \text{Ork}(n - 1, d) \) powers of linear forms taken from outside \( V \). The field \( k \) is algebraically closed, thus \( \boxempty \) shows that \( F = (F - F_2) + F_2 \) may be written using at most \( m + m_2 \leq \text{Ork}(n, d - 1) + \text{Ork}(n - 1, d) \) powers of linear forms taken from outside \( V \). \( \square \)
4 Proof of Theorem 5

From now on \( n = 3 \), i.e. \( S := k[x_1, x_2, x_3] \). First we deal with the majority of forms, using the following lemma:

**Lemma 14.** Let \( F \in S_3 \) be such that \( V((F^\perp)_2) \subseteq \mathbb{P}S_1 \) is an empty set. Then \( \text{Ork}(F, V) \leq 4 \) for any homogeneous closed \( V \subseteq S_1 \).

**Proof.** Let \( V' \subseteq \mathbb{P}S_1 \) be the image of \( V \setminus \{0\} \), then \( V' \) is closed and not equal to \( \mathbb{P}S_1 \).

By Bertini theorem [Har, Thm III.10.9] applied to the base point free linear system \((F^\perp)_2\) on \( \mathbb{P}S_1 \) we see that the general element \( D \) of this system is smooth. At the same time a general element \( D \) intersects \( V' \) properly i.e. \( \dim V(D) \cap V' < \dim V' \). We choose \( D_0 \) satisfying both properties.

Restricting to \( V(D_0) \) and using Bertini theorem once more we obtain an element \( D_1 \in (F^\perp)_2 \) such that \( V(D_0) \cap V(D_1) \) is smooth of dimension zero and \( V(D_0) \cap V(D_1) \cap V' \) is empty. From Lemmas 7-8 it follows that

\[
F \in \lin (l_{a_1}^3, l_{a_2}^3, l_{a_3}^3, l_{a_4}^3)
\]

where \( \{a_1, a_2, a_3, a_4\} = V(D_0, D_1) \) so \( \{a_1, a_2, a_3, a_4\} \cap V' = \emptyset \).

Now we would like to show that the set of “bad forms”, i.e. those which do not satisfy the assumptions of Lemma 14 is closed in the (open) set of all forms which essentially depend on three variables.

**Corollary 15.** Denote by \( \text{Ess} \) the (open) set of forms which essentially depend on three variables and let \( W \subseteq \text{Ess} \) be the subset consisting of forms such that \( V((F^\perp)_2) \subseteq \mathbb{P}S_1 \) is not an empty set. Then \( W \) is closed in \( \text{Ess} \).

**Proof.** This follows from Corollary 10 applied to the case \( d = 3, e = 1 \).

Finally we need an explicit characterisation of the “bad forms” due to Kleppe:

**Proposition 16.** Consider the set of forms \( F \in S_3 \) essentially dependent on three variables and such that \( V((F^\perp)_2) \subseteq \mathbb{P}S_1 \) is not an empty set. Every element of this set is an image, under a linear change of basis in \( S_1 \), of one of the following forms

\[
x_0x_1^2 + x_1x_2^2 \quad \text{or} \quad x_0^3 + g \quad \text{where} \quad g \in k[x_1, x_2].
\]

Furthermore the classical Waring rank of \( x_0x_1^2 + x_1x_2^2 \) is five.

**Proof.** See [Kl] Theorem 2.3.

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**Proof of Theorem 5** By Proposition 16 it suffices to prove \( \text{Ork}(3, 3) \leq 5 \). Take a form \( F \in S_3 \) which essentially depends on three variables and a homogeneous closed subset \( V \subseteq A^3 \).

If \( F \) satisfies the assumptions of Lemma 14 then \( \text{Ork}(F, V) \leq 4 \) and we are done. Denote the set of the forms which essentially depend on three variables and satisfy the assumptions of Lemma 14 by \( U \).

If \( F \notin U \), then \( F \in W \), where \( W \) was defined in Corollary 15. In this case we would like to find a linear form \( l \) such that \( F + l^3 \in U \). After a linear change of coordinates we can assume \( F \) is of the form from Lemma 16. For \( x_0x_1^2 + x_1x_2^2 \) the form \( l^3 = (x_0 + x_1)^3 \) will do and in the second case we can write \( g = x_1x_2(a_1x_1 + a_2x_2) \) where \( a_1 \neq 0 \), then \( l^3 = (x_0 + x_2)^3 \) will do.

The set of forms which essentially depend on three variables is open in the set of all forms and the set \( U \) is open in this set by Corollary 15 so that \( U \) is open in the set of all forms. We have just seen that \( U \) has non-empty intersection with \( \{F + l^3\} \), so \( U \cap \{F + l^3\} \) is open in this set, choosing \( l \notin V \) such that \( F + l^3 \in U \) we get the required result. 

\( \square \)
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