Spherically symmetric black holes in minimally modified self-dual gravity

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Abstract

We discuss spherically symmetric black holes in the modified self-dual theory of gravity recently studied by Krasnov, obtained by adding a Weyl curvature-dependent ‘cosmological term’ to the Plebanski lagrangian for general relativity. This type of modified gravity admits two different types of singularities: one is a true singularity for the theory where the fundamental fields of the theory, as well as the (auxiliary) spacetime metric, become singular, and the other one is a milder ‘non-metric singularity’ where the metric description of the spacetime breaks down but the fundamental fields themselves are regular. We first generalize this modified self-dual gravity to include Maxwell’s field and then study the basic features of spherically symmetric, charged black holes, with particular focus on whether these two types of singularities are hidden or naked. We restrict our attention to minimal forms of the modification, and find that the theory exhibits ‘screening’ effects of the electric charge (or ‘anti-screening’, depending upon the sign of the modification term), in the sense that it leads to the possibility of charging the black hole more (or less) than it would be possible in general relativity without exposing a naked singularity. We also find that for any (even arbitrarily large) value of charge, true singularities of the theory appear to be either achronal (non-timelike) covered by the hypersurface of a harmless non-metric singularity or simply hidden inside at least one Killing horizon.

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(Some figures in this article are in colour only in the electronic version)
1. Introduction

Ever since general relativity was formulated, various different types of modifications of the theory have been proposed, aiming at, e.g., obtaining better control of gravity in quantum regime or attempting to account for astrophysical/cosmological observations without invoking unknown, exotic matter fields in the cosmic inventory. Recently, Krasnov has introduced yet another modification of general relativity [1–6] where the standard cosmological constant term is made into a function $\Phi$ of the Weyl curvature, in such a fashion so as not to add extra degrees of freedom. The absence of extra degrees of freedom is a remarkable property which distinguishes the case at hand from usual modifications of gravity. The key to this result is to give up the metric as the fundamental field of the theory (hence the definition ‘non-metric theory of gravity’ was used for instance in [1–3]), and to use the formulation of general relativity in terms of a self-dual 2-form as the fundamental field. This idea was introduced long ago by Plebanski [8], and later developed by Capovilla, Dell, Jacobson and Mason [9, 10]. A metric can be derived from this 2-form through the imposition of suitable constraints. Krasnov’s main result is that these constraints can be generalized to a class of theories labelled by the choice of $\Phi$, all of which give dynamics for a metric with only two degrees of freedom. Such generalization had already been considered in the Hamiltonian framework by Bengtsson and Peldan [11, 12]. See also [13–15].

It is tempting to speculate that such a modification of general relativity gives an effective description of quantum gravity models based on the Plebanski lagrangian, e.g. the spin foam formalism for loop quantum gravity [18]. In this paper, however, we consider the theory only from a classical viewpoint, with the goal of investigating the physical consequences of the modification, and its viability as a theory of classical gravity.

The nature of the modification makes gravity behave differently in regions of different Weyl curvature. As the Weyl curvature vanishes on conformally flat metrics, such as Minkowski or FLRW cosmological spacetimes, these are also exact solutions in the modified theory. The effect of the modification can be studied perturbing around these solutions, but one needs to go to second order in perturbation theory. It is then easier to investigate the classical effects of the modification for spherical symmetric solutions with non-zero Weyl curvature, where departures from general relativity are expected right away. This case has been considered by Krasnov and Shtanov [3]. They looked at specific choices of $\Phi$ without matter, and found solution analogues of the Schwarzschild black hole4. A characteristic feature discovered in [3] is the existence of a new class of singularities, where the derived metric is singular but the fundamental fields of the theory remain finite.

In this paper, we extend the analysis of the theory to the coupling to electromagnetism. In spite of the derived nature of the metric field, a natural coupling in terms of the fundamental 2-form was found by Capovilla, Dell, Jacobson and Mason [10], and as anticipated by Krasnov [2], such coupling can be extended straightforwardly to the modified theory. This allows us to study spherically symmetric electrovacuum black hole solutions, for which we prove staticity for general form of $\Phi$. We then restrict our attention to the two simplest profiles of $\Phi$, and give explicit solutions. We describe how the asymptotic structure of the solutions depends on $\Phi$, and for one of the two chosen profiles we find the standard Reissner–Nordstøm–de Sitter asymptotic structure. On such a solution, we investigate its geometry near the horizon and the way this is affected by the modification.

4 A central motivation of Krasnov and Shtanov [3, 6] is to investigate whether the modification to gravity can be suitable to explaining the anomalous rotational curve of spiral galaxies without appealing to dark matter, and they have considered profiles of $\Phi$ tailored to that end.
We find that the non-metric singularities are still present and their location depends on the value of the charge. The number and locations of the horizons are also affected by the modification. With respect to general relativity, we find that more/less charge can be poured into the hole before the event horizon disappears, depending upon the sign of the modifying term: more charge when the sign of the modification is coherent with a negative cosmological constant, and less charge when it is coherent with a positive cosmological constant. In a sense, one could say that the electric charge is screened in one case, and anti-screened in the other case. In particular, in the first case, a sector of the theory exists where arbitrarily charged solutions admit a (Killing) horizon which is located inside a non-metric singularity (for this reason, it cannot be identified with the event horizon) and covers the central singularity at \( r = 0 \). This striking departure from general relativity shows that useful bounds on the physically allowed types of \( \Phi \) can be obtained by looking at classical solutions coupled with matter.

The plan of the paper is as follows. In order to keep the paper as self-contained as possible, and also to establish our notation and conventions, the next two sections are devoted to reviews of self-dual gravity theories: Plebański’s self-dual formulation of general relativity in the next section and Krasnov’s modified self-dual gravity in section 3. We also refer the reader to the literature provided in the bibliography for more details on both the standard Plebański formulation of gravity and the modification proposed by Krasnov. Section 4 describes the coupling to electromagnetism. In section 5, we restrict our attention to the spherical symmetric case, and show how to obtain the field equations. We roughly follow [3], although the notation and presentation differ here and there. We also reprove Birkhoff’s theorem in a slightly more conventional way. In section 6, we discuss the rationale for \( \Phi \) and possible links to the existence of asymptotically flat solutions. In section 7, we discuss the field equations for the minimal modification which allows asymptotic flatness, and the existence of Reissner–Nordstrøm and Nariai–Bertotti–Robinson solutions. We then focus on this minimal modification; in section 8 we review the analytic solution for the vacuum black hole found by Krasnov and Shtanov in [3], and in section 9 we present the solutions for the charged case. In the final section 10, we give an overview of the properties of the solutions. Throughout the paper, we set \( c \equiv 4\pi \epsilon_0 = 1 \).

2. Self-duality and the Plebański action

The notion of self-duality plays a key role in the Plebański construction of general relativity. Consider a 2-form \( F = \frac{1}{2} F_{\mu\nu} \, dx^\mu \wedge dx^\nu \) on a pseudo-Riemannian manifold with the metric \( g_{\mu\nu} \) and non-zero determinant \( g < 0 \). Introduce the identity \( \delta_{\rho\sigma} \equiv \delta_\rho \sigma \equiv \delta_{\rho\sigma} \delta_{\eta\zeta} \) (here and in the following a square bracket on the indices means normalized anti-symmetrization) and the Hodge star \( \star \equiv \frac{1}{\sqrt{-g}} \epsilon^{\mu\rho\sigma} \) (here \( \epsilon \) is the completely anti-symmetric tensor density with \( \epsilon_{0123} = 1 \)). The latter satisfies \( \star^2 = -1 \), as can be seen immediately using

\[
\epsilon^{\rho\sigma} \epsilon_{\mu\nu} \delta^\mu_{\lambda\tau} = 4g \delta^\rho_{\lambda\tau}.
\]

Then, the complex orthogonal projectors \( \mathcal{P}^\pm = \frac{1}{2} (1 \pm i \star) \) (i.e. \( \mathcal{P}^\pm \mathcal{P}^\mp = 0 \)) define respectively the self-dual and anti-self-dual parts of the 2-form \( F \),

\[
F^{(\pm)}_{\mu\nu} = (\mathcal{P}^\pm F)_{\mu\nu} = \frac{1}{2} \left( F_{\mu\nu} \pm \frac{i}{\sqrt{-g}} \epsilon^{\rho\sigma} \epsilon_{\mu\nu} F_{\rho\sigma} \right)
\]

These are respectively the positive and negative eigenvectors of \( i \star \),

\[
i \star F^{(\pm)} = \pm F^{(\pm)}.
\]
and we have \( F = F^{(+)} + F^{(-)}. \) An important property which will play a role in the model of the \( \mu \) under conformal transformations of the metric \( g_{\mu\nu}. \)

In the following, we will work with forms valued in the Lorentz algebra \( so(3, 1) \). A typical example is the tetrad \( e_i^\mu(x) \), defined by \( g_{\mu\nu} = e_i^\mu e_j^\nu \eta_{ij} \). The tetrad is a 1-form with indices \( I = 0, \ldots, 3 \) in the fundamental representation of the Lorentz algebra. Another well-known example is the spin connection \( \omega_{\mu}^I(e) \), a 1-form with values in the adjoint representation of \( so(3, 1) \). For an object in the adjoint representation, we can define the algebraic self-dual and anti-self-dual components, as we did in (2). This time we have to use the Hodge operator on the Lorentz bundle, which reads \( \frac{1}{2} \epsilon^{IJK} \). These two projectors realize explicitly the familiar homomorphism \( so(3, 1; C) = su_2(2) \oplus su_1(2) \) of the Lorentz algebra into two \( su(2) \) algebras, which rather than self-dual and anti-self-dual are more commonly dubbed right handed and left handed. To make the mapping more explicit, it is convenient to pick out the time direction \( I = 0 \), and define \( \omega_{\mu}^{\pm i} \equiv \omega_{\mu}^{0i}, \) with \( i = 1, 2, 3 \) an \( SU(2) \) index. (Anti)self-duality then means \( \omega_{\mu}^{\pm i} \equiv \pm \frac{1}{2} e^i_j \omega_{\mu}^{i[j]}. \)

Consider now a 2-form with values in the adjoint representation, \( B_{ij} \). For this object, we can consider the self-dual projections in both spacetime and algebra indices. Plebański [8, 10] found that the two notions of self-duality coincide if the 2-form is constrained to satisfy the following quadratic condition:

\[
B^i \wedge B^j = \frac{1}{2} \delta^{ij} B^k \wedge B_k = 0. \tag{4}
\]

This equation is called metricity constraint, and it is solved by \( B^i \propto \Sigma^i(e) \), where

\[
\Sigma^i(e) = e^0 \wedge e^i + \frac{i}{2} e^j e^i \wedge e^k \tag{5}
\]

is (twice) the self-dual part of the two form \( e \wedge e \). It satisfies

\[
\Sigma^i \wedge \Sigma^j = 2i e^i \delta^{ij} d^4 x, \tag{6}
\]

with \( e = e^{\mu \nu}, e^0, e^1, e^2, e^3 \), the determinant of the tetrad, related to the determinant of the metric by \( \det g_{\mu\nu} = -e^2 \). The proportionality coefficient between \( B \) and \( \Sigma \) can thus be fixed to 1 requiring

\[
B^i \wedge B_i = 6i e d^4 x. \tag{7}
\]

Therefore, a \( B \) field solving (4) encodes a metric via the tetrad field in (5). In order for these to give a real Lorentzian metric, reality conditions on \( B \) have also to be imposed. These are [10]

\[
B^i \wedge B^{*i} = 0, \quad \text{Im}(iB^i \wedge B_i) = 0, \tag{8}
\]

which imply respectively that \( B^{*i} \) is proportional to the anti-self-dual form, and that the determinant (7) is real. The \( B \) field itself can well be complex. These additional reality conditions are the trade-off for using the simpler self-dual variables instead of the full Lorentz group.

In the following, we will refer to a \( B \) field satisfying (4), (7) and (8) and the reality conditions as ‘metric’. To come to the dynamics, let us also introduce an \( SU(2) \) connection \( \omega_{\mu}^I \)—as an independent field, not the spin connection—and its curvature 2-form \( F_{\mu\nu}^i(\omega) = d\omega_{\mu}^i + \frac{1}{2} e^j_{\mu} \omega_{\nu}^i \wedge \omega_{\mu}^j. \)

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5 We put the label \( \pm \) of self- and anti-self-dual forms within a bracket, to avoid confusion with \( SU(2) \) indices later on. In the literature, one also finds the convention \( P^\pm = \frac{1}{2} (1 \mp i\sigma) \), so that self and anti-self are the \( \pm i \) eigenvectors of \( \star \).

6 Note that also the anti-self-dual part of \( e \wedge e \) solves (4). The two solutions are distinguished by the sign of the right-hand side of (6).
Including the constraint (4) with a symmetric and traceless Lagrange multiplier $\psi_{ij}$, the action

$$S_{G}[B, \omega, \psi] = \frac{1}{8\pi G_N} \int \left[ B^i \wedge F_i(\omega) + \frac{1}{2} \left( \psi_{ij} - \frac{1}{3} \lambda \delta_{ij} \right) B^i \wedge B^i \right]$$  \hspace{1cm} (9)

is equivalent, once the reality conditions for the fields are imposed, to $(i$ times) the Einstein–Hilbert action for general relativity with the cosmological constant $\lambda$, plus a total divergence [8, 9]. It is instructive to see the equivalence at the level of the equations of motion. Next to (4), the other field equations are

$$\frac{dB^i}{dt} + i \epsilon^{ijk} \omega_j \wedge B_k = 0, \hspace{1cm} (10)$$

$$F^i(\omega) + \left( \psi_{ij} - \frac{1}{3} \lambda \delta_{ij} \right) B_j = 0. \hspace{1cm} (11)$$

When $B$ is metric, (10) becomes the first Cartan structure equation, which—upon requiring invertibility of $e^i_{\mu}$—identifies $\omega$ as the (self-dual part of the) spin connection $\omega(e)$. This in turns implies that $F^i(\omega(e))$ is the (self-dual part of the) Riemann tensor (via the second Cartan structure equation). Finally, the 18 equations (11) have the following double role: ten of them give the Einstein equations for the tetrad, while five identify the Lagrange multiplier $\psi_{ij}$ with the (self-dual) Weyl part of the Riemann tensor. The remaining three equations are gauge. See [2, 10, 16] and references therein for more details.

Before moving on, let us add a brief comment on Plebański’s theory. The metric solution (5) to the constraints is unique if $B^i \wedge B_i \neq 0$ is required, a condition that translates into the invertibility of the tetrad. Degenerate solutions with $B^i \wedge B_i = 0$ also exist, and they have no analogue in general relativity. The existence of this other sector makes Plebański’s gravity more general than Einstein’s theory. Let us also recall that the canonical analysis of (9) gives the Ashtekar formalism [17], which is at the roots of loop quantum gravity [18].

### 3. Modified Plebański gravity and non-metricity

The modification introduced by Krasnov is to promote the cosmological constant term in (9) to a scalar function of the Lagrange multiplier itself, say $\Phi(\Psi)$. The action then reads

$$S[B, \omega, \Psi] = \frac{1}{8\pi G_N} \int \left[ B^i \wedge F_i(\omega) + \frac{1}{2} \left( \psi_{ij} + \delta_{ij} \Phi(\Psi) \right) B^i \wedge B^i \right]. \hspace{1cm} (12)$$

As before, $i = 1, 2, 3$ is an $SU(2)$ index, $\psi_{ij}$ is symmetric and traceless and the fields are complex. The reality conditions are the same as before. For the moment, we leave $\Phi$ arbitrary.

When $\Phi = \Phi_0$ is constant, we obtain Plebański’s gravity with $\Lambda = -3\Phi_0$; a non-constant $\Phi$ is responsible for deviations from general relativity. To understand the nature of the deviations, let us look at the field equations. The $\omega$ and $B$ variations yield

$$\frac{dB^i}{dt} + i \epsilon^{ijk} \omega_j \wedge B_k = 0, \hspace{1cm} (13)$$

$$F^i(\omega) + \left( \psi_{ij} + \Phi(\Psi) \delta_{ij} \right) B_j = 0. \hspace{1cm} (14)$$

The key difference with Plebański’s gravity comes from the variation with respect to the field $\Psi$. In order to take this variation, note that only two algebra scalars can be built out of a symmetric and traceless $\psi_{ij}$, $z_1 \equiv \text{Tr} \Psi^2 = \psi_{ij} \psi_{ij}$ and $z_2 \equiv \text{Tr} \Psi^3 = \psi_{ij} \psi_{ij} \psi_{ij}$. Therefore, $\Phi$ is truly a function of $z_1$ and $z_2$. Taking also into account the tracelessness of $\Psi$, the variation gives

$$\left( \frac{\delta \Phi(\Psi)}{\delta \psi} \right)^{ij} = 2 \psi^{ij} \frac{\delta \Phi}{\delta z_1} + 3 \left( \left[ \delta^{ij} \psi_{km} - \frac{1}{3} \delta^{ij} \text{Tr} \Psi^2 \right] \frac{\delta \Phi}{\delta z_2} \right). \hspace{1cm} (15)$$
Thus instead of the metricity constraints (4), we now get the following equation:

\[ B^i \wedge B^j - \frac{1}{3} \delta^{ij} B^k \wedge B_k = - \left( \frac{\delta \Phi(\Psi)}{\delta \Psi} \right)^{ij} B^k \wedge B_k. \]  

(16)

When the right-hand side of the equation is non-zero, (5) is no longer a solution, and the theory is inequivalent to general relativity. Hence, the quantity \( \delta \Phi/\delta \Psi \) controls the departures from general relativity. In the previous section, we recalled that when the right-hand side of (16) vanishes, the solution \( B^i = \Sigma^i(e) \) encodes a metric \( e^I_\mu \) through (5). We are now going to show how a metric is encoded in \( B^i \) also when the right-hand side does not vanish.

### 3.1. Extracting the metric from the modified constraints

Equations (16) are the key to understand the nature of the modification to gravity. They are manifestly not anymore constraints for \( B \), but rather five equations relating \( B \) and \( \Psi \). For instance, they can be used to determine the five \( \Psi_{ij} \)s as functions of the \( B^i \)'s, once a specific choice of \( \Phi \) is made. The \( \Psi \) so obtained can be then plugged in equations (14) to obtain the dynamics of \( B \).

There is however an alternative procedure, as emphasized by Krasnov [2], that allows us to describe the theory in more familiar metric terms. This is based on the fact that the notion of Hodge self-duality naturally defines a metric (up to a conformal factor), even before imposing (4). Hence, simply declaring \( B \) to be self-dual suffices to endow the theory with a metric, and one can derive its field equations from the action (12).

To be more explicit, consider the following two symmetric tensors, \( g^{\mu \nu} \) and \( h^{ij} \), that can be constructed out of \( B^i \) [7, 15, 19],

\[ \sqrt{|g^{\mu \nu}|} g^{\mu \nu} = \frac{1}{12} \epsilon^{\alpha \beta \gamma \delta} \epsilon_{ijk} B^i_{\mu \alpha} B^j_{\nu \beta} B^k_{\delta \gamma}, \]  

(17)

\[ B^i \wedge B^j = \frac{1}{3} B^k \wedge B_k h^{ij}, \]  

(18)

Here \( g^{\mu \nu} \) is the determinant of the Urbantke metric \( g^{\mu \nu} \) introduced in [19, 10]. The crucial fact [7, 15] is that whenever \( h^{ij} \) is invertible, \( B^i \) is self-dual with respect to the Urbantke metric (17)—or to any metric related to it by a conformal transformation, because of the invariance of \( \star \). Hence the notion of self-duality together with \( B^i \wedge B_i \neq 0 \) suffices to have a unique metric, up to conformal transformations; we do not need to impose any constraints.

But if a metric is already present, or rather a conformal class of metrics, what is the role of the metricity constraints (4) in the standard theory? It is instructive at this point to count the variables. A generic triple of 2-forms \( B^i \) has \( 3 \times 6 = 18 \) components, of which 3 can be gauged away using the \( SU(2) \) symmetry. The remaining 15, thanks to the above result and when \( B^i \wedge B^j \) is non-degenerate, can be parametrized in terms of the two tensors (17) and (18): \( B^i = B^i (g^{\mu \nu}, h^{ij}) \). Then, the role of the five equations (16) is to fix the five components of \( h^{ij} \): to constants in the standard case, to functions of \( \Psi_{ij} \) in the modified case.

To see how the fixing goes, consider first the standard case. Plugging (18) into the metricity constraints (4), we get \( h^{ij} = \delta^{ij} \). This kills the extra five components in the \( (g^{\mu \nu}, h^{ij}) \) parametrization of \( B^i \), and we are left with only the metric \( g^{\mu \nu} \) as the variable of the theory. An explicit computation then shows that \( g^{\mu \nu} \) coincides with the metric \( e^I_\mu e^J_\nu \eta_{IJ} \) encoded in \( \Sigma^i(e) \) via the Plebański solution (5). The only conformal ambiguity left is the rescaling by a constant, but this can be fixed as in (7), or reabsorbed in the definition of \( G_N \) and \( \Lambda \). So the standard constraints guarantee the existence of a preferred metric in the

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7 The trace of (16) vanishes identically.
conformal class. In the modified case when the right-hand side of (16) is non-vanishing, we can still use these equations to eliminate $h^i_j$, but rather than being fixed proportional to the identity, this time its components satisfy algebraic relations with (the derivatives of) the $\Psi^{ij}$'s. Thus after solving (16) we have a new parametrization $B^i(g^U_{\mu\nu}, \Psi^{ij})$. Plugging the latter in the torsionless condition (13) and solving this equation we obtain a ($\Psi$-dependent) spin connection $\omega(g^U_{\mu\nu}, \Psi^{ij})$. Finally, inserting the solutions $B(g^U_{\mu\nu}, \Psi^{ij})$ and $\omega(g^U_{\mu\nu}, \Psi^{ij})$ into (14) gives us the dynamics for $g^U_{\mu\nu}$ and $\Psi^{ij}$.

However, the conformal ambiguity is not trivial anymore in the modified case, for we can also consider metrics which differ from (17) by a $\Psi$-dependent conformal factor. One such metric arises if we solve the modified metricity constraints in terms of the metric encoded in $\Sigma^i(e)$, instead of (17). As remarked in [2], this alternative construction is more convenient for practical reasons, and it is the one that we use in this paper. First of all, we restrict to the $B^i \wedge B_i \neq 0$ sector, and assume $h^i_j$ to be invertible, and thus diagonalizable since symmetric. Consider the matrix $D^i_j$ of normalized eigenvectors, which is such that $h^i_j = D^i_k \delta^k_j (D^{-1})^l_j$. $h^i_j$ has five independent components, hence also the diagonalizing matrix $D^i_j$. Call them $c_n, n = 1, \ldots, 5$. Using $D(c_n)$ we can bring (18) in a form reminiscent of the metricity constraint (4). Defining $\hat{B}^i = (D^{-1})^i_j B_j$, equation (18) becomes

$$\hat{B}^i \wedge \hat{B}^j = \frac{1}{3} \hat{B}^k \delta^{ij}.$$  

(19)

This equation is solved by $\hat{B}^i = \Sigma^i(e)$, and therefore

$$B^i = D^i_j(c_n) \Sigma^j(e).$$

(20)

Inserting this ansatz for $B^i$ into (16) gives us algebraic relations between $c_n$ and (the derivatives of) $\Psi^{ij}$, from which we obtain $D^i_j(c_n(\Psi)) \equiv D^i_j(\Psi)$. This gives a parametrization $B^i = B^i(e^i_\mu, \Psi^{ij})$ which solves (16).

This way of constructing a solution to equations (16) has the advantage of a simple link to the usual Plebański form (5). Using this $B$ to evaluate the Urbantke metric (17) one obtains $g^U_{\mu\nu} = f(\Psi) e^i_\mu e^j_\nu \eta^{ij}$, where $f$ is a scalar function obtained from contractions of three matrices $D^i_j(\Psi)$. As anticipated above, the metric $e^i_\mu e^j_\nu \eta^{ij}$ contained in (5) and the Urbantke metric (17) now differ by a non-trivial conformal factor which depends on $\Psi$.

Which of these two should be taken as the physical metric? As it turns out, there is no physically distinguished metric in the modified theory, and only the conformal class of the metric is specified. Thus both $e^i_\mu e^j_\nu \eta^{ij}$ and $g^U_{\mu\nu}$, as well as any other metric in the same class, are treated on equal footing. This conformal ambiguity distinguishes the modified theory from Plebański's.

The ambiguity is expected to be resolved by matter couplings, and indeed in [26] a mechanism to distinguish a physical metric was shown, based on the requirement that test particles follow geodesics. In this paper on the other hand we consider only the vacuum theory, or its coupling to electromagnetism, which being conformally invariant is insensitive to the ambiguity. Thus the ambiguity is fully present, and it becomes relevant when discussing the singularities of the theory, because a metric has to be singled out in order to do so. To that end, we will consider a specific metric in the sections on black holes below. We anticipate here that this is the metric $e^i_\mu e^j_\nu \eta^{ij}$ which arises naturally in our construction of the spherically symmetric solution.

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8 That this is possible in general is shown e.g. in [7, 15, 20–22].
9 For a discussion on this, see the literature by Krasnov and in particular [26].
3.2. Reality conditions and degrees of freedom

One would expect the procedure just described to expose the presence of new propagating degrees of freedom with respect to general relativity. Remarkably, the Hamiltonian analysis performed in [5] (see also [11]) shows that this is not the case: also the modified theory has only two propagating degrees of freedom. The key for the absence of new physical degrees of freedom is in the fact that the field equations for $\Psi_{ij}$ obtained from (14) are purely algebraic [7, 15]. The property is related to the peculiar complex nature of the action, and to the fact that reality conditions have to be additionally imposed, by hand. It is ultimately this imposition that is responsible for the absence of extra degrees of freedom, both in the Plebański theory (9) and in the modified theory (12).

In this perspective, let us add a remark on the reality constraints before continuing. After the revival of Plebański’s complex formalism by Capovilla, Dell, Jacobson and Mason, it was later realized by Reisenberger [23] (see also [24]) that one can use real fields. In order to do so, one gives up the self-duality of the fields and takes $B^{kJ}$, $\omega^{ij}$ and $\Psi^{IJK}L$ in the full Lorentz group. The action is identical to (9) with Lorentz indices, and it is often called $SO(3,1)$ (or non-chiral) Plebański action. The absence of reality constraints is a manifest advantage of this formulation.

The same modification proposed by Krasnov in (12) can be applied to the non-chiral $SO(3,1)$ action. In its simplest form $\Phi \propto Tr \Psi^2$, this was introduced by Smolin [14]. However, although the chiral and non-chiral Plebański actions lead to the same classical physics, this might not be true for the modified actions. Above we mentioned how crucial the reality conditions are to ensure that the modified theory (12) only has two propagating degrees of freedom. The non-chiral action on the other hand does not have the reality conditions, thus the modification might include new degrees of freedom. This was recently pointed out by Alexandrov and Krasnov in [25], where a counting of the constraints in the Hamiltonian framework showed the presence of six additional degrees of freedom. The modified non-chiral action thus appears to be a rather different theory, and its relevance for gravity is still to be explored.

4. Maxwell action

The coupling of matter to the standard Plebański action (9) was studied in [10], where actions for scalar, spinor and vector fields were proposed starting from a reformulation of the conventional curved-space matter lagrangians in terms of the fundamental fields $B^i$ and $\omega^i$. This conservative approach guarantees that when the $B$ field is metric, the standard minimal coupling of matter fields to the metric is recovered.

Such a reformulation is particularly natural for gauge theories, whose action can be written directly in terms of the spacetime Hodge star $\star = (1/2)e\epsilon^{\mu\nu\rho\sigma}$. Thanks to this fact, the gauge theory lagrangian proposed in [10] can be immediately coupled to the modified action (12), and we will restrict our attention to this case. In units $4\pi \epsilon_0 = 1$, the Maxwell action reads

$$S(g_{\mu\nu}, A_\mu) = \frac{1}{8\pi} \int F \wedge \star F = -\frac{1}{16\pi} \int d^4x \sqrt{-g}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}(A)F_{\rho\sigma}(A). \quad (21)$$

\footnote{The actions for scalar and fermion fields proposed in [10] cannot be coupled to (12) the way they are, because they carry additional constraints, which are compatible only with the standard Plebański ones (4) and not with (16) [4]. Hence suitable modifications of these actions have to be studied before scalars or fermions can be coupled to the modified theory.}
To express this action in terms of the $B$ field, a simple choice is\(^{11}\) \[ S_{\text{M}}[\varphi, A, B] = \frac{1}{2\pi} \int \left[ \varphi_i B^i \wedge F(A) - \frac{1}{2} \varphi_i \varphi_j B^i \wedge B^j \right]. \tag{22} \]

This is a first-order formulation of electromagnetism, where the vector potential $A$ is flanked by an auxiliary field $\varphi$. This action, which can be straightforwardly generalized to the Yang–Mills non-Abelian case, is reminiscent of the first order formulation of Yang–Mills theory in flat space, sometimes called field strength formulation, or BFYM theory \[\tag{21}\text{[21, 28]}\].

When $B$ is metric, (22) is equivalent to ($i$ times) the Maxwell action (21). To see this, note that the vanishing of the $\varphi$-variation gives

\[ B^i \wedge [F(A) - \varphi_j B^j] = 0. \tag{23} \]

Recall that self- and anti-self-dual forms make a basis in the space of 2-forms. When the reality conditions (8) hold, a self-dual $B$ implies that its complex conjugate $B^\ast$ is anti-self-dual. Then, a generic Maxwell field can be decomposed as

\[ F = F^{(+)} + F^{(-)} = F^{(+)}B^i + F^{(-)}B^i, \tag{24} \]

and thus $B \wedge F \equiv B \wedge F^{(+)}$. Hence (23) implies

\[ F^{(+)}_{\rho\sigma}(A) = \varphi_j B^{j\rho\sigma}. \tag{25} \]

Substituting this result back into (22) we obtain

\[ S_{\text{M}}[\varphi, A, B] = \frac{1}{16\pi} \int d^4x \epsilon_{\mu\nu} F^{(+)}_{\mu\nu}(A) F^{(+)}_{\rho\sigma}(A) - \frac{1}{8\pi} \int F \wedge (1 - i \ast) F. \tag{26} \]

Finally, if $B^i$ is metric we can use (1) and get explicitly

\[ S_{\text{M}}[\varphi, A, B] = -\frac{i}{16\pi} \int d^4x \left[ \sqrt{-g} g^{\mu\nu} g^{\rho\sigma} F_{\mu\nu}(A) F_{\rho\sigma}(A) - \frac{i}{2} \epsilon_{\mu\nu} F_{\mu\nu}(A) F_{\rho\sigma}(A) \right]. \tag{27} \]

As promised, the first term of (27) is ($i$ times) the Maxwell lagrangian (21). The second term is a total divergence, and as such does not contribute to the equations of motion\(^{12}\). Neglecting this total divergence (and the similar one arising in the gravity sector), (9) plus (22) is equivalent to

\[ S_{\text{tot}} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R - \frac{1}{16\pi} \int d^4x \sqrt{-g g^{\mu\nu} g^{\rho\sigma} F_{\mu\nu} F_{\rho\sigma}} \tag{28} \]

in agreement with the standard conventions \[\tag{29}\text{[29]}\].

This proves the validity of the action (22) for coupling Maxwell theory to Plebański’s gravity. The constraints (23) are compatible with the metricity constraints (4) as well as with the modified constraints (16); thus the action (22) can also be coupled to the modified theory. This is what we do in this paper. That is, we consider (22) plus (12) as the total action for modified gravity coupled to electromagnetism.

Since the Maxwell action (22) does not depend on the spin connection $\omega$ nor on the field $\Psi$, the coupled action leads to the field equations (13) and (16) unchanged. The only equations to acquire a source are (14), which now read

\[ F^{i}(\omega) + (\Psi^{i} + \Phi(\Psi)\delta^{i}) B^{j} = 8\pi G \tau^{i}, \tag{29} \]

\(^{11}\) Alternatives for the coupling of Yang–Mills theory to the non-chiral $SO(3,1)$ Plebański action have been considered in \[\tag{14, 27}\].

\(^{12}\) The second term is the standard CP-violating $\theta_{\text{QCD}}$ term, but with an $i$ factor that makes it purely imaginary. As explained in \[\text{[10]}\], this complex action for Maxwell theory can be obtained from the usual one in the Hamiltonian framework through a complex canonical transformation, where the momentum conjugated to the connection is $E + iB$. This is analogous to what happens in the gravitational sector with Ashtekar’s self-dual variables, where the momentum conjugated to the densitized triad is $\Gamma + iK$ \[\text{[17]}\].
where we conveniently defined
\[
\frac{\delta S_{\text{matter}}}{\delta B^\mu_{\rho \nu}} = -\frac{1}{4} \epsilon^\mu_{\rho \nu} \epsilon^i_{\rho \sigma}.
\]  
(30)

The algebra-valued 2-form \( \tau^i \) = \( \frac{1}{2} \epsilon^i_{\rho \sigma} \) \( \text{d}x^\rho \wedge \text{d}x^\sigma \) plays the role of the energy–momentum tensor in the standard metric formalism, and we shall refer to it as such. Note however that unlike the standard energy–momentum tensor, \( \tau^i \) is antisymmetric in the spacetime indices. For \( S_{\text{matter}} = S_M \) given by (22), we get
\[
\tau^i_{\rho \sigma} = -\frac{1}{2\pi} \varphi^i F^{(-)}(A).
\]  
(31)

This satisfies \( B^i \wedge \tau_i = 0 \), the analogue of the standard tracelessness condition.

5. Spherically symmetric spacetime

In the rest of the paper, we consider the spherically symmetric reduced sector of the theory. This sector has been investigated by Krasnov and Shtanov [3, 6], who obtained the reduced field equations in vacuum. In this section, we review their results and extend them to the coupling to electromagnetism.

5.1. Preliminaries: Changing SU(2) basis

Let us look back at the solution (5) of the standard Plebański theory, which encodes a metric \( ds^2 = \epsilon^i_{\rho \sigma} \otimes \epsilon^i_{\rho \sigma} - e^i_{\rho} \otimes e^i_{\sigma} - e^i \otimes e^i - e^i \otimes e^j \) via the tetrad \( e^j_{\mu} \). With the prospect of spherical symmetry in mind, it is convenient to work with a Newman–Penrose tetrad \( \ell^I_{\mu} = (l^\mu, n^\mu, m^\mu, \bar{m}^\mu) \), which defines the metric in the form
\[
ds^2 = l \otimes n + n \otimes l - m \otimes \bar{m} - \bar{m} \otimes m, \quad l \wedge n \wedge m \wedge \bar{m} = -i e^4 \text{d}^4 x.
\]  
(32)

The transformation between tetrad basis is
\[
l^\mu = (e^0 - e^1) / \sqrt{2}, \quad n^\mu = (e^0 + e^1) / \sqrt{2}, \quad m^\mu = (e^2 + ie^3) / \sqrt{2} \quad \text{and} \quad \bar{m}^\mu = m^\mu^*.
\]

With the Newman–Penrose tetrad, the Plebański solution (5) reads
\[
\Sigma^{\pm} = i(n \wedge l - m \wedge \bar{m}) \quad \text{and} \quad i(n \wedge m - l \wedge \bar{m}) \quad \text{for} \quad \Sigma^0 = l \wedge n \wedge m \wedge \bar{m}, \quad \Sigma^+ = n \wedge \bar{m} + l \wedge m, \quad \text{and} \quad \Sigma^- = m \wedge l.
\]  
(33)

A straightforward computation checks that this triple is still self-dual with respect to (32), and that the reality conditions (8) are satisfied provided \( l \) and \( n \) are real and \( \bar{m} = m^* \). Just as (33), the triple (34) provides a basis in the space of self-dual 2-forms.

In the spherical basis \( \bar{t}_i \), the Killing–Cartan metric is not diagonal any longer, instead
\[
\text{Tr}(\bar{t}_i \bar{t}_j) = \frac{1}{2} \delta_{ij}, \quad \delta_{ij} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & -2 \\
0 & -2 & 0
\end{pmatrix}.
\]  
(35)

Accordingly, we have
\[
\text{Tr}(\Sigma \wedge \Sigma) = \frac{1}{2} \bar{\Sigma}^0 \wedge \bar{\Sigma}^0 - 2 \bar{\Sigma}^+ \wedge \bar{\Sigma}^-.
\]  
(36)
Consistency with (6) can be straightforwardly verified,
\[ \delta_{ij} \Sigma^i \wedge \Sigma^j = \tilde{\delta}_{ij} \overline{\Sigma}^i \wedge \overline{\Sigma}^j = -6l \wedge n \wedge m \wedge \tilde{m} = 6ie d^4x. \quad (37) \]

The non-trivial scalar product given by \( \tilde{\delta}_{ij} \) is the trade-off for the simpler form of (34). However, the advantage of the spherical basis \( \tilde{\tau}_i \) is not only to simplify the triple of self-dual 2-forms from (33) to (34), but also to allow us to deal with tensor product of irreps using the familiar Clebsch–Gordan decomposition. This will be needed to describe \( \Psi \) and the modified constraint equation (16). Before doing so, note that the \( \tilde{\tau}_i \) are a basis in the adjoint representation \( 1 \), but they are not normalized with respect to the scalar product (35). The transformation to the standard normalized basis \( |j, m \rangle \) is characterized in the spherically symmetric sector by the functional \( \Phi(\beta) \). To write equation (16), we also need the identity in the spherical basis. The latter is the identity in the spherical basis. The identity naturally belongs to the singlet \( 0 \) representation of (35), which is a symmetric and traceless tensor product of two irreps, \( 1 \otimes 0 \) and \( 0 \otimes 1 \), and corresponds to the standard \( SU(2) \) scalar product \( g^{mn} = (-1)^m \delta_{m,-n} \), once the different normalization of the \( \tilde{\tau}_k \) is taken into account.

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We now have all the ingredients to write the metricity equations (16). Using the metric \( \tilde{\delta}_{ij} \) in (35) to take the scalar products, we compute
\[ \Psi^i_k \Psi^k_j \tilde{\tau}_i \otimes \tilde{\tau}_j = \beta^2 (2\tilde{\tau}_+ \otimes \tilde{\tau}_- - 2\tilde{\tau}_- \otimes \tilde{\tau}_+ + 16\tilde{\tau}_0 \otimes \tilde{\tau}_0), \quad (41) \]
\[ \Psi^i_k \Psi^k_j \tilde{\tau}_i \otimes \tilde{\tau}_j = -\beta^3 (4\tilde{\tau}_+ \otimes \tilde{\tau}_+ + 4\tilde{\tau}_- \otimes \tilde{\tau}_- + 64\tilde{\tau}_0 \otimes \tilde{\tau}_0), \quad (42) \]
from which \( z_1 = \Psi^i_j \Psi^j_i = 24\beta^2 \), \( z_2 = \Psi^i_k \Psi^k_j \Psi^j_l = -48\beta^3 \).

Being both \( z_1 \) and \( z_2 \) functions of \( \beta \), it follows that the modification to general relativity is characterized in the spherically symmetric sector by the functional \( \Phi(\beta) \) of a single

\[ \begin{align*}
13 & \text{Alternatively, one can work with the normalized } SU(2) \text{ basis. This, however, introduces factors of } \sqrt{2} \text{ in (34), complicating the field equations.}
\end{align*} \]
scalar function. As a consequence we also have that the partial derivatives are simply
d_2 \Phi = \Phi_\beta / 48 \beta, d_2 \Phi = -\Phi_\beta / 144 \beta^2, where we have introduced the shorthand notation
\Phi_\beta = \delta \Phi / \delta \beta. Note that \Phi and \Psi have the same dimensions, and thus \Phi_\beta is dimensionless.

Putting all this together, equation (16) gives
\[ B^i \wedge B^j \bar{\tau}_i \otimes \bar{\tau}_j = \frac{1}{3} B^k \wedge B_k \left[ \left( \frac{1}{4} \Phi_\beta - \frac{1}{2} \right) (\bar{\tau}_+ \otimes \bar{\tau}_- + \bar{\tau}_- \otimes \bar{\tau}_+) + (\Phi_\beta + 1) \bar{\tau}_0 \otimes \bar{\tau}_0 \right]. \] (43)

From this we can obtain the following five independent equations:
\[ B^+ \wedge B^+ = B^- \wedge B^- = 0, \] (44a)
\[ B^+ \wedge B^0 = B^- \wedge B^0 = 0, \] (44b)
\[ 2 B^+ \wedge B^- + B^0 \wedge B^0 = \frac{1}{2} \Phi_\beta B^k \wedge B_k, \] (44c)

Using the scalar product (35), the last equation can be written as
\[ \frac{B^+ \wedge B^-}{B^0 \wedge B^0} = \frac{\Phi_\beta - 2}{4(1 + \Phi_\beta)}. \] (44d)

Proceeding as discussed above in section (3.1), we now look for a solution in the form
\[ B^i = D^i_j (c_0, e) \bar{\Sigma}^j \bar{e}, \] (45)
with \( \bar{\Sigma}^j (e) \) given by (34), and \( c_0 \) five parameters to be related to \( \Psi^{ij} \) through (16). In the
spherical basis, a generic linear combination that preserves self-duality is given by (see e.g. the appendix of [2])
\[ D(c_0) = \begin{pmatrix} c_0 & 0 & 0 \\ 0 & c_+ & c_- \\ 0 & c_- & c_+ \end{pmatrix}. \] (46)

Plugging this into (44a), we have
\[ c_+ c_- = 0, \quad c_- c_+ = 0, \quad \frac{c_+ c_- + c_+ c_+}{-2(c_0)^2} = \frac{\Phi_\beta - 2}{4(1 + \Phi_\beta)}. \] (47)

A simple parametrization of the solution is given by \( c_+ = c_- = 0, c_0 = 1 \) and \( c+ = c- = c \) with
\[ c^2 = \frac{2 - \Phi_\beta}{2(1 + \Phi_\beta)}. \] (48)

In conclusion, the \( B^i \) field given by the triple of 2-forms
\[ B^+ = cn \wedge n, \quad B^- = cm \wedge m, \quad B^0 = l \wedge n - m \wedge \bar{m}, \] (49)
solves the metricity equations (16) and the reality conditions, and it is self-dual with respect to the metric (32). Note that although \( c^2 \) might be negative, hence \( B \) complex, (32) is always real and Lorentzian (up to singularities). Note also that the relation between \( B^i \) and the metric (32) breaks down when \( \Phi_\beta = 2 \) or \( \Phi_\beta = -1 \). These situations can indeed arise, and are the
non-metric singularities which we will discuss below.

Needless to say, having a metric at disposal is instrumental for the physical interpretation
of the theory. The strategy is to use (49) to write the remaining field equations as differential
equations for \( f, g \) and \( \beta \). Once the theory has been reformulated in such more familiar terms,
the effect of the modification also becomes more transparent.

Thus far, the only place where we used spherical symmetry was the fact that \( \Psi \) has the
single component (39). (In the general case equations (44a) have non-vanishing right-hand
sides and depend on the whole five components of \( \Psi \), which get related to the coefficients \( c_n \).

We now specialize the Newman–Penrose tetrad, and thus the solution (49) for the \( B \) field, to the case of spherical symmetry. We choose spacetime coordinates as follows:

\[
l = 1 \sqrt{2} (f \, dt - g \, dr), \quad n = 1 \sqrt{2} (f \, dt + g \, dr), \quad m = R \sqrt{2} (d\theta + i \sin \theta \, d\phi),
\]

and \( \bar{m} = m^* \). Through equation (32), they define the metric

\[
d s^2 = f^2 \, dt^2 - g^2 \, dr^2 - R^2 (d\theta^2 + \sin^2 \theta \, d\phi^2), \quad e = fgR^2 \sin \theta,
\]

which can be recognized as the standard ansatz for a spherically symmetric spacetime. We initially take \( f(t, r), g(t, r) \) and \( R(t, r) \) functions of time as well as the radius.

Using the tetrad (50), we can write the components \( B_{\mu\nu} \) in (49)

\[
B^0_{0\nu} = \begin{pmatrix} 0 & fg & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & R^2 \sin^2 \theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B^=_{\mu\nu} = \frac{c}{2} \begin{pmatrix} 0 & 0 & \pm RF & -iRF \sin \theta \\ 0 & Rg & \mp iRg \sin \theta & 0 \\ Rg & \mp iRg \sin \theta & 0 & 0 \\ \pm RF & iRF \sin \theta & 0 & 0 \end{pmatrix}.
\]

### 5.3. Maxwell fields

With the explicit form of \( B^i \) at hand, we can immediately compute the Maxwell stress-energy tensor (31). It is straightforward to show that when a Maxwell field respects the spherical symmetry, the field strength takes the form

\[
F(A) = E(t, r)l \wedge n + B(t, r)m \wedge \bar{m}
\]

with two scalars, \( E(t, r) \) and \( B(t, r) \), corresponding respectively to the electric and magnetic field. Then, it follows from the source-free Maxwell equations \( dF = 0 \) and \( d*F = 0 \) (which are still valid in the modified theory in terms of the derived metric) that \( B(t, r) = \text{const} \times R(t, r)^{-2} \), \( E(t, r) = \text{const} \times R(t, r)^{-2} \). For simplicity, hereafter we restrict our attention to the case \( B(t, r) = 0 \), and thus in units \( 4\pi \epsilon_0 = 1 \),

\[
F(A) = \frac{Q}{R(t, r)^2} l \wedge n,
\]

where the constant \( Q \) is chosen to be the conserved electric charge defined by

\[
Q = \frac{1}{4\pi} \int_{S^2} *F,
\]

on a 2-sphere \( S^2 \).

To evaluate equation (31) we need the self and anti-self-dual parts of (54), which can be computed from (24) and the explicit form (52) of \( B \). The only non-vanishing components turn out to be

\[
F^{(\pm)}_{tr} = \frac{Q fg}{2R^2}, \quad F^{(\pm)}_{\theta\phi} = \pm i \frac{Q}{2} \sin \theta.
\]

From the equation of motion (25) we have

\[
F^{(*)}_{\mu\nu} = \phi_i B^i_{\mu\nu} = \psi^0 B^0_{\mu\nu} = 2\psi^+ B^+_{\mu\nu} = 2\psi^- B^-_{\mu\nu},
\]

which gives

\[
\psi^0 = \frac{Q}{2fgR^2}, \quad \psi^+ = \psi^- = 0.
\]

Finally, plugging (56) and (58) into (31), the only non-zero components of the energy–momentum 2-form compute to

\[
\tau^0_{tr} = -\frac{Q^2 fg}{8\pi R^4}, \quad \tau^0_{t\phi} = i \frac{Q^2}{8\pi R^2} \sin \theta.
\]
5.4. Cartan equations

The next step is to solve the Cartan equation (13), using the explicit form of $B^i$ in (52), to compute the connection $\omega$ in terms of the metric and $\beta$. Since we are using the spherical basis, we have to adapt the covariant derivative. Recalling its general definition $d_\omega B = d\omega + [\omega, B]$ and using the commutators in the $\tilde{r}_i$ basis, we get

$$dB^0 + 2(\omega^- \wedge B^+ - \omega^+ \wedge B^-) = 0, \quad dB^\pm \pm (\omega^0 \wedge B^4 - \omega^\pm \wedge B^0) = 0.$$  \hspace{1cm} (60)

To solve the equations, we proceed as follows. First, we use (52) and write down the non-zero components of equations (60) with all four basis vectors $l$, $..$, $m$, and read off the components of $\omega$ in this basis, as functions of the metric and $\beta$. The procedure is straightforward but space consuming, so we report here only the final result (more details can be found in [2, 3])

$$\omega^\pm = -\frac{1}{2} P_\pm (d\theta \mp i \sin \theta \, d\phi), \quad \omega^0 = F \, dt + G \, dr - i \cos \theta \, d\phi,$$  \hspace{1cm} (61)

where we have introduced the following shorthand notation:

$$P_\pm = \frac{R_1'}{g^*} \pm \frac{\dot{R}}{f_0}, \quad F = (Rf_0)' \mp \frac{R'}{Rg_*} - \frac{c^2 Rg_*}{f_0}, \quad G = \frac{(Rg_*)'}{Rf_0} - \frac{\dot{R}g_*}{c^2 Rf_0}.$$  \hspace{1cm} (62)

with $f_0 = cf$ and $g_* = cg$. Here and in the following, $'$ and $\dot{}$ denote, respectively, the derivative by $t$ and $r$.

The curvature in the spherical basis is

$$F^0 = d\omega^0 + 2\omega^- \wedge \omega^+, \quad F^\pm = d\omega^\pm \pm \omega^0 \wedge \omega^\pm,$$  \hspace{1cm} (63)

where from (61) we have

$$\partial_\mu \omega^0_\mu = \begin{pmatrix} \dot{F} & \dot{G} & 0 & 0 \\ F' & G' & 0 & 0 \\ 0 & 0 & 0 & i s_\theta \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \partial_\mu \omega^\pm_\mu = \frac{1}{2} \begin{pmatrix} 0 & 0 & -\dot{P}_\pm & \pm is_\theta \dot{P}_\pm \\ 0 & 0 & -\dot{P}'_\pm & \pm ic_\theta \dot{P}'_\pm \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$  \hspace{1cm} (64a)

$$\omega^0_\mu \omega^+_\mu = \frac{P_\pm P_\mp}{4} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -is_\theta \\ 0 & 0 & is_\theta & s_\theta^2 \end{pmatrix}, \quad \omega^0_\mu \omega^-_\mu = \frac{P_\pm}{2} \begin{pmatrix} 0 & 0 & -F & \pm is_\theta F \\ 0 & 0 & -G & \pm is_\theta G \\ 0 & 0 & 0 & 0 \\ 0 & 0 & ic_\theta & \pm c_\theta s_\theta \end{pmatrix}. \hspace{1cm} (64b)$$

Here $s_\theta = \sin \theta$ and $c_\theta = \cos \theta$.

5.5. Field equations

Using (39) and (63), the field equations (14) in the spherical basis read

$$d\omega^\pm \pm \omega^0 \wedge \omega^\pm + (\beta + \Phi) B^\pm = 8\pi G_N \tau^\pm,$$  \hspace{1cm} (65a)

$$d\omega^0 + 2\omega^- \wedge \omega^+ + (\Phi - 2\beta) B^0 = 8\pi G_N \tau^0.$$  \hspace{1cm} (65b)

Here $\omega = \omega(B, \beta)$ through (61) and $\Phi = \Phi(\beta)$ as seen above. For a spherically symmetric configuration of Maxwell’s field, we found that the only non-zero components of the energy–momentum tensor are $\tau^0_0$ and $\tau^{\pm}_0$. Using also the explicit expressions (52) and (64), the non-vanishing components of equations (65a) are

$$P_\pm \pm P_\pm F = (\beta + \Phi) Rf_0 = 0.$$  \hspace{1cm} (66a)
\[ P' \pm P \pm G - (\beta + \Phi)Rg_\ast = 0, \] (66b)

and those of (65b)

\[ G - F' + (\Phi - 2\beta) \frac{f_\ast g_\ast}{c^2} = 8\pi G N^0_{\tau \tau}, \] (66c)

\[ 1 - P_\ast P_- + R^2(\Phi - 2\beta) = 8\pi G N^0_{\tau \phi \phi}. \] (66d)

Among the eight equations (66), only three are independent because of the Bianchi identities [3]. In section (5.7), we write the three independent equations in the form suggested in [3]. Before doing so, we show the validity of part of the Birkhoff theorem that any spherically symmetric vacuum solutions are static.

### 5.6. Birkhoff’s theorem

In [3], Krasnov and Shatanov proved Birkhoff’s theorem that the spherically symmetric solutions of this theory are necessarily static for the vacuum case. Their proof is slightly unconventional, so we find it useful to reproduce the same results for our case, following a more standard procedure [30]. We assume that \(\overline{R}(t, r)\) be not constant, \(d\overline{R} \neq 0\). This, in particular, implies that we can choose the gauge so that \(\overline{R}(r, t) = r\). Then, the field equations above reduce to

\[ P_\pm = \frac{1}{g_\ast}, \quad F = \frac{f'}{g_\ast} + \frac{f_\ast}{rg_\ast} \left(1 - \frac{1}{c^2}\right), \quad G = \frac{g'}{f_\ast}. \] (67)

The first two of equations (66) simplify to

\[ -\frac{1}{g_\ast} g'_\ast \pm \frac{1}{g_\ast} F = (\beta + \Phi)r f_\ast = 0, \] (68a)

\[ -\frac{1}{g_\ast} g'_\ast \pm \frac{g'}{f_\ast g_\ast} G - (\beta + \Phi)rg_\ast = 0. \] (68b)

We now show that the solution is necessarily static, i.e. that \(\dot{f} = \dot{g} = \dot{c} = 0\). First of all, summing the two equations in (68a) gives \(g'_\ast = 0\), thus \(G = 0\) from the third of (67). It then follows from (68b) that \(\beta + \Phi\) is a function of only \(r\) and thus \((1 + \Phi_\beta)\beta = 0\). As we briefly mentioned above, \(\Phi_\beta = -1\) corresponds to the singular case, which we are not going to consider here. We therefore conclude that \(\beta\) itself is \(t\)-independent and so are \(\Phi\) and \(c^2\).

Subtracting the two equations (68a) gives \(F/f_\ast = rg_\ast(\beta + \Phi)\), implying that \((F/f_\ast)'' = 0\). Then, it follows from (67) and \((F/f_\ast)'' = 0\) that \((f''/f_\ast)'' = 0\), namely \(f''/f_\ast = a(r)\) with \(a(r)\) being some function of \(r\); the latter equation allows for a time-dependent integration constant, \(f_\ast(t, r) = b(t) \int a(r) \, dr\), but this is of the type that can always be absorbed by a change of coordinates \(t \mapsto \tau\) with \(d\tau = b(t) \, dt\).

Note that besides this staticity property, Birkhoff’s theorem also asserts that such a spherically symmetric static metric, if satisfying the vacuum Einstein equations, is necessarily locally isometric to the Schwarzschild metric [30]. For our case, in order to have the corresponding assertion, we have to integrate equations (66c) and (66d) with the stress tensor for non-vanishing Maxwell fields. We perform this integration only numerically and do not have analytic solutions. For this reason, we do not have the corresponding assertion in the same decisive level as the original Birkhoff’s theorem for the vacuum Einstein gravity.

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5.7. Equations of motion for static fields

Using the staticity and the choice of gauge $R(r) = r$, the equations reduce to

$$\beta + \Phi = \frac{1}{rf_s g_s} F = - \frac{g_s}{r g_s}$$

$$-F' + (\Phi - 2\beta) \frac{f_s g_s}{c^2} = 8\pi G_N \tau_r^0$$

$$1 - \frac{1}{g_s} + r^2(\Phi - 2\beta) = \frac{1}{8\pi G_N \tau_{\theta\phi}^0}$$

(69a)

(69b)

(69c)

Here (69c) is an algebraic relation, and (69a), (69b) are the differential equations for $f_s$, $g_s$ and $\beta$, of which only two are independent thanks to the Bianchi identities. Following [3], we now reexpress these equations in a more convenient form. First of all, (69a) and the second of (67) immediately imply that

$$\frac{(f_s g_s)'}{f_s g_s} = -\frac{1}{r} \left(1 - \frac{1}{c^2}\right).$$

(70)

Secondly, taking the derivative of (69b) and using the above equation, we obtain

$$\beta' (2 - \Phi) = -\frac{3}{r} \left(2\beta - \frac{2G_N Q^2}{r^5}\right).$$

(71)

The three equations (69c), (70) and (71) are the field equations of the theory in the spherically symmetric sector. They give dynamics for $f_s$, $g_s$, and $\beta$, and thus for the metric (51). For $Q = 0$, these equations reduce to equation (35) of [3]. Finally, using the explicit expressions of the energy–momentum tensor (59) and of $c^3$ (equation (48)), and defining $x(r) = f_s(r)g_s(r)$, we have the independent equations

$$\frac{x'}{x} = \frac{3\Phi}{r(2 - \Phi)},$$

$$\frac{(2 - \Phi)\beta'}{r} = -\frac{6\beta}{r} + \frac{2G_N Q^2}{r^5},$$

$$\frac{1}{g_s} = 1 - (2\beta - \Phi) r^2 - \frac{G_N Q^2}{r^2}.$$  

(72a)

(72b)

(72c)

Before proceeding, let us check that for the constant $\Phi = -\Lambda/3$ we recover the standard results of the general relativity case.

5.8. Recovering the standard Reissner–Nordstrøm metric

When $\Phi = 0$, we have $c^3 = 1$ and, from (72a), that $f_s g_s = fg$ is a constant. Taking the latter to be 1, $f = 1/g$. We are left with a single differential equation, from (72b),

$$\beta' = -\frac{3\beta}{r} + \frac{G_N Q^2}{r^5}.$$  

(73)

This equation is solved by

$$\beta(r) = \frac{G_N M}{r^3} - \frac{G_N Q^2}{r^4}.$$  

(74)
where \( M \) is the integration constant. Finally from (72c), we read
\[
\frac{1}{g^2} = 1 - \frac{2G_N M}{r} + \frac{G_N Q^2}{r^2} - \frac{1}{3} \Lambda r^2,
\]
which can be immediately recognized to give the \( g_{00} = g^{-2} \) component of the Reissner–Nordstrøm metric with the cosmological constant \( \Lambda \).

5.9. Metric singularities

The field equations (72) give the dynamics for \( f^*, g^* \), and \( \beta \), in terms of which the metric (51) reads
\[
d s^2 = 2 \left( 1 + \Phi \beta \right)^2 [f^*_2 \, dt^2 - g^*_2 \, dr^2] - r^2 \, d^2 \Omega.
\]
Note the possibility of singularities if values of \( r \) exist such that \( \Phi = 2, -1 \). These are the non-metric singularities found and discussed in [3]. As shown in section 5.2, these are the points where the relation between \( B \) and the metric breaks down. To understand the theory, it is useful to distinguish the following three types of apparent singularities which can occur.

- Points where \( g^2 (r) \) diverges and \( x (r) \) is finite are Killing horizons, the outermost of which may be viewed as the black hole (event) horizon, depending upon the asymptotic structure.
- Points where \( \Phi = 2 \). We call this non-metric singularities of type (1).
- Points where \( \Phi = -1 \). We call this non-metric singularities of type (2).

The nature of these non-metric singularities will be discussed below, with explicit solutions at hand.

It is at this point important to comment on the conformal ambiguity present in the theory. As discussed earlier in section (3.1), the procedure to extract a metric is defined up to a conformal transformation, and in the electrovacuum theory there is no principle which would select a physical metric in the conformal class. In particular, multiplying (76) by a conformal factor generically depending on \( \beta \) gives an equally valid line element in this modified theory of gravity. A possible mechanism to distinguish a preferred metric has been studied in [26], by introducing test particles and demanding that they follow geodesics.

Since we restrict our analysis to the coupling to electromagnetism and Maxwell’s equations are conformally invariant, the ambiguity is present in our case, and we have to arbitrarily select a metric in order to investigate its singularities. Thus, in the absence of a physical criterium to distinguish a preferred metric, we choose to work with the line element (76), which emerged naturally in our construction\(^{14}\).

6. Choices of \( \Phi \)

Thus far we have kept \( \Phi \) arbitrary, without worrying about the origin of its non-constancy. Different choices of \( \Phi \) have an impact on the classical behaviour of gravity in different ways and at different scales, allowing one to entertain the possibility of explaining in this way current puzzles, such as an anomalous rotational curve in spiral galaxies, without appealing to dark matter scenarios. To that end, tailored profiles of \( \Phi \) have been investigated in [3, 6].

In introducing this modification of gravity, Krasnov pointed out that it could be seen as an effective action coming from a model of quantum gravity where the fundamental field is not

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\(^{14}\) Another natural choice considered in the literature [3] is to work with the metric such that the volume form coincides with \( B^4 \wedge B_6 / 3 \). This amounts to multiplying (76) by \( (1 + \Phi \beta)^{1/2} \).
the metric, but rather the 2-form $B$. Recall that $\Phi$ has length dimensions $[\Phi] = [\Psi] = -2$, and that it is a function of $z_1 = \text{Tr} \Psi^2$ and $z_2 = \text{Tr} \Psi^3$ with dimensions $[z_1] = -4$ and $[z_2] = -6$; we see that a non-trivial dependence of $\Phi$ on $\Psi$ introduces coupling constants with non-zero length dimensions. A natural coupling constant with such dimensions is the Planck length, which leads us to postulate a quantum mechanical origin of a non-constant $\Phi$. At present, it is not clear what quantum gravity model could lead to such an effective action, but the use of $B$ instead of the metric as the fundamental field suggests to look at spin foam models as possible candidates.

In this paper we do not attempt to quantize the theory; we are only interested in classical consequences of different choices of $\Phi$. Consider for instance the case in which $\Phi$ is ‘analytic’ near the origin in the sense that it can be Taylor expanded as

$$\Phi(\Psi) = \Phi_0 + a_1 \text{Tr} \Psi^2 + a_2 \text{Tr} \Psi^3 + \cdots. \quad (77)$$

As $[\Psi] = -2$, it follows that $[a_n] = 2n$, e.g. $a_n = b_n (\hbar G_N)^n$ with $b_n$ dimensionless. Lack of control on the possible origin and on the quantum properties of the theory makes it hard to restrict the admissible forms of $\Phi$, and a priori we have no reasons to demand analyticity in the sense of (77). On the other hand, by studying the implications of a non-constant $\Phi$ at the classical level we can hopefully obtain restrictions on the admissible forms of $\Phi$.

### 6.1. On asymptotic flatness

A useful requirement is that in the modified theory, there exist such solutions that approach at large distances an asymptotically flat solution of general relativity. This can put restrictions on the admissible choices of $\Phi$. Taking $\Lambda = 0$ for simplicity, we demand that

$$c^2 \to 1, \quad (78)$$
$$f^2, g^2 \to 1 + O(1/r), \quad (79)$$
$$\beta \to 0, \quad (80)$$

in the limit of large area radius $r \to \infty$. Since $c^2 = (2 - \Phi_\beta)/2(1 + \Phi_\beta)$, the first of the asymptotic flatness conditions, equation (78), implies that $\Phi_\beta$ cannot approach a non-vanishing constant, and hence that with equation (80), $\Phi$ cannot be linear to $\beta$. Therefore, the asymptotic flatness conditions above require that if Taylor expanded in $\beta$, $\Phi$ must start from at least the quadratic order of $\beta$: $\Phi \propto \hbar G_N \beta^2 + O(\beta^3)$. This is also a justification of our discussion about the form, equation (77), from a physical view point. The simplest modification to general relativity is thus to truncate the expansion (77) to the first non-trivial term, $\Phi = \Phi_0 + a_1 \text{Tr} \Psi^2 = \Phi_0 + 24 a_1 \beta^2$. (When $\Phi_0 \neq 0$, the solution would be asymptotically (anti-)de Sitter.) This quadratic case will be discussed in detail in the subsequent sections.

Since our present metric is static, admitting the time-like Killing vector field $\xi^\mu = (\partial/\partial t)^\mu$, the ADM mass may be given by the Komar integral on 2-sphere, $S^2_\infty$, at spatial infinity

$$M = -\frac{1}{8\pi G_N} \int_{S^2_\infty} * (\nabla \xi). \quad (81)$$

(Note that the conserved electric charge, $Q$, is always given by (55), irrespective of the conditions, equation (79)) Since

$$f^2 = \frac{x^2}{c^2 g^2} = x^2 \left[ 1 - \left( \frac{2\beta - \frac{A}{2} \beta^2}{r^2} \right)^2 + O(1/r^2) \right], \quad (82)$$

18
the second of equation (79) implies that
\[ x^2 \to 1 + O(1/r), \quad \beta \to C r^3 + O(1/r^4), \] (83)
with some constant \( C \). (In fact, integrating equation (72a) at the asymptotic region implies
\[ x^2 \sim \exp\left[-(AC)/r^3\right] \sim 1 + O(1/r^3). \]) Then, the ADM mass—which is now written
\[ M = r^4 c^2 f'(f^2)'/2x \]—is well defined and becomes, \( C = G_N M \), in accordance with the
standard GR results, equation (74).

6.2. Linear modification

When \( \Phi \) is linear in \( \beta \), then \( x \) would either diverge or vanish at \( r \to \infty \). Therefore, solutions
in a modified theory linear in \( \beta \) would not be asymptotically flat. Since linearity in \( \beta \) can only
happen if \( \Phi(\Psi) \) is non-analytic at the origin in the sense of the expansion equation (77), one
may ask whether the lack of asymptotically flat static solutions characterizes all ‘non-analytic’
choices of \( \Phi \). This would be a rather useful restriction on the admissible choices of \( \Phi \). We
leave this question open for future investigations.

Nevertheless, the linear modified case can be solved easily, and thus it is instructive to
calculate this case first. Let us take \( \Phi_{\beta} = \Phi_0 + a\beta \), with \( a \) being some dimensionless parameter.
In this case \( \Phi_{\beta} \) is constant and equation (48) reads
\[ c^2 = \frac{2 - a}{2(1 + a)}. \] (84)
The constancy of \( c^2 \) trivializes some of the nonlinearities of the field equations, thus
significantly simplifying the study of the spacetime structure. Assuming \( a \neq 2, -1 \), the
field equations are
\[ \frac{x'}{x} = \frac{3a}{(2 - a) r}, \] (85a)
\[ (2 - a)\beta' = -\frac{6\beta}{r} + \frac{2G_N Q^2}{r^3}. \] (85b)
\[ \frac{1}{g_*^2} = 1 + \Phi_0 r^2 - (2 - a)\beta r^2 - \frac{G_N Q^2}{r^2}. \] (85c)
Equation (85a) can be easily solved to give
\[ x(r) = c_1 r^{3a/(2 - a)} \] (86)
with \( c_1 \) being an integration constant. From (85b), we obtain
\[ \beta(r) = c_2 r^{-3a/(2 - a)} + \frac{G_N Q^2}{(2a - 1)} r^{-4}, \] (87)
where \( c_2 \) is another integration constant. The integration constants can be fixed to reproduce
the standard GR result (74) for \( a = 0 \), specifically \( c_1 = 1, c_2 = G_N M \). Then we have
\[ \frac{1}{g_*^2} = 1 + \Phi_0 r^2 - \frac{(2 - a)G_N M}{r^{1/c^2}} + \frac{3(1 + a) G_N Q^2}{2a - 1} \frac{1}{r^2}. \] (88)
Note the different asymptotic behaviour from the standard Reissner–Nordstrøm solution, and
in particular the absence of asymptotic flatness. This means, in particular, that \( M \) cannot be
interpreted as the ADM mass. Note how the location of the zeroes of \( 1/g_*^2 \) depends also on
\( a \). In the next sections, when we treat the quadratic modification, we will see that a similar
modification of the horizon structure happens within solutions which are asymptotically the
Reissner–Nordstrøm metric.
7. Modified self-dual gravity with Maxwell field: quadratic case

In the rest of the paper, we focus on the quadratic modification $\Phi_1(\beta) = \Phi_0 + 24a_1\beta^2$. For commodity, we redefine $a_1 = \hbar G_N A / 48$, and from now on we work in units $\hbar = G_N = 1$. We then have $\Phi_1(\beta) = \Phi_0 + A\beta^2/2, \Phi_\beta = A\beta$ and

$$c^2 = \frac{2 - A\beta}{2(1 + A\beta)}, \quad (89)$$

which is positive for $A\beta \in (-1, 2)$ and negative otherwise.

As a consequence of this choice of $\Phi_1$, we expect larger departures from general relativity in regions of larger curvature.

7.1. Nariai–Bertotti–Robinson solutions

Our general formulation of the field equations (66) allows us to look for solutions with $R(t, r) = R_0$ constant. Such solutions exist in general relativity: they are not asymptotically flat and have a Cartesian product structure. The simplest case is the neutral Nariai metric [31], a solution to the Einstein equation with a positive cosmological constant given by a direct product $dS_2 \times S^2$. Setting $R(r, t) = R_0$ in (66), we have

$$P_\perp = 0, \quad F = \frac{f r}{g}, \quad G = \frac{g_r}{f_r}. \quad (90)$$

Equations (66a) and (66b) reduce to

$$(\beta + \Phi(\beta)) f = (\beta + \Phi(\beta)) g = 0, \quad (91)$$

whereas (66d) reduces to

$$1 + R_0^2(\Phi(\beta) - 2\beta) = 0. \quad (92)$$

Avoiding the unphysical case of $f_\perp$ and $g_\perp$ everywhere vanishing, equations (91) give an algebraic constraint for $\beta$. Thus, the solutions to the constraint $\beta + \Phi(\beta) = 0$ will have constant Weyl curvature.

In general relativity, i.e. $\Phi = \Phi_0 = -\Lambda/3$, the constraint is $\beta = \Lambda/3$, which plugged into equation (92) gives

$$R_0^2 = \frac{1}{\Lambda}. \quad (93)$$

This fixes the value of $R_0$ dynamically, and the Nariai solution follows from (66c).

In the quadratically modified case, the constraint is

$$\frac{1}{2} A\beta^2 + \beta - \frac{1}{\Lambda} = 0, \quad (94)$$

from which $\beta = (-1 \pm \sqrt{1 + (2/3)A\Lambda})/A$. Substituting this value into (92), we have

$$R_0^2 = \frac{A/3}{\pm\sqrt{1 + (2/3)(A\Lambda) - 1}}. \quad (95)$$

Therefore, the Nariai solution still exists, but with a different value of its constant radius. This new value now depends on the modification parameter $A$, as well as the cosmological constant $\Lambda$. For $A > 0$, we take the solution of $\beta$ with a positive sign, and $R_0^2$ in (95) is positive provided $\Lambda$ is positive, as in (93) above. For $A < 0$, the solution with a positive sign again requires $\Lambda > 0$, but the solution with a negative sign allows any $\Lambda < 3/2|A|$. This, in particular, implies that the Nariai metric of $dS_2 \times S^2$ can be a solution even when $\Lambda < 0$ if $A < 0$ with a negative sign of equation (95). It is however only the solutions with a positive
sign and $\Lambda > 0$ that reduces to equation (93) in the limit $A \mapsto 0$, i.e. the limit to the standard general relativity case.

Another case is the Bertotti–Robinson metric [32, 33], a solution to the Einstein–Maxwell equations with vanishing cosmological constant, given by the direct product of $AdS_2 \times S^2$ with each two-dimensional manifold having the same constant curvature radius. This metric has vanishing Weyl curvature and therefore is still a solution to the modified theory. Indeed, taking $\Lambda_1 = 0$ in (94) we see that $\beta = 0$ is a solution also for $A \neq 0$.

7.2. Perturbative analysis of charged black holes

Let us now look for asymptotically flat solutions. Fixing $R(r) = r$, equations (72) take the form

\begin{align}
\frac{x'}{x} &= \frac{1}{r} \frac{3A\beta}{2 - A\beta}, \quad (96a) \\
\beta'(2 - A\beta) &= -\frac{6\beta}{r} + \frac{2Q^2}{r^3}, \quad (96b) \\
\frac{1}{g^2} &= 1 - r^2 \left(2\beta - \frac{1}{2}A\beta^2 - \Phi_0\right) - \frac{Q^2}{r^2}. \quad (96c)
\end{align}

With respect to general relativity, two immediate effects of the modification are visible.

- From (96a), $x = \text{const.}$ is not anymore a solution, so we do not have the familiar form, $f^2 = 1/g^2$, of the spherically symmetric Einstein–Maxwell system.
- In equation (96b), the term proportional to $A$ makes the differential equation nonlinear.

The nonlinearity of the system is still manageable for $Q = 0$, the case that was solved exactly in [3], and which we review below. However, for $Q \neq 0$ the situation is more involved, and unfortunately the system eluded our attempts to find exact analytic solutions. A solution can be found perturbatively in the regime $|A\beta| \ll 1$, giving

\begin{align}
x(r) &= 1 + A \left(\frac{M}{2r^3} + \frac{3Q^2}{8r^4}\right) + o(A^2), \quad (97) \\
\beta(r) &= \frac{M}{r^3} = \frac{Q^2}{r^4} + \frac{A}{4r^6} \left(M - \frac{Q^2}{r}\right)^2 + o(A^2), \quad (98)
\end{align}

where we fixed the integration constants to match general relativity at zeroth order in $A$. 

\textbf{A posteriori} we see that the approximation is valid for $r \gg (AM)^{1/3}$. The perturbative solution is useful to fix the integration constants, because it turns out that, unlike the linear modification (cf (87)), the quadratic one maintains the right asymptotic behaviour. This important property was obtained for the vacuum theory in [3], and here we extend it to the coupling to electromagnetism by a perturbative analysis. In section (9) we show it also using the non-perturbative numerical solution. Before studying numerically the system, let us discuss the vacuum case $Q = 0$, that can be solved analytically as found in [3].

8. Vacuum black holes revisited

In [3], Krasnov and Shtanov solved the system (96) for $Q = 0$ and found an analogue of the Schwarzschild black hole. Here we review their results and extend them to include the
locations of the event horizon. The knowledge of the analytic solution allows us to test our numerical integration code, which we use for the plots of this section and the next section on the charged case.

The Krasnov–Shtanov solution is given by

$$x(r) = e^{-\frac{1}{2}A\beta},$$

(99)

with \(\beta(r)\) implicitly defined by

$$r(\beta) = \left(\frac{M}{\beta}\right)^{\frac{1}{3}} e^{\frac{1}{2}A\beta},$$

(100)

\(M\) being an integration constant, corresponding to \(M\) in equation (98). From (96) with \(Q = 0\), we can immediately extract the non-trivial components of the metric

$$g_{00}(r) = \frac{1}{c^2} f^2 = \frac{2(1 + A\beta)}{2 - A\beta} e^{-A\beta} \left[1 - \left(2\beta - \frac{1}{2}A\beta^2 - \Phi_0\right)r^2\right].$$

(101)

$$g_{11}(r) = -\frac{1}{c^2} g^2 = -\frac{2(1 + A\beta)}{2 - A\beta} \left[1 - \left(2\beta - \frac{1}{2}A\beta^2 - \Phi_0\right)r^2\right]^{-1}.$$  

(102)

One can easily check that setting \(A = 0\) gives back the standard Schwarzschild–de Sitter metric representing a black hole of mass \(M\).\(^{16}\)

At values of \(r\) where \(1 - \left(2\beta(r) - \frac{1}{2}A\beta^2(r) - \Phi_0\right)r^2 = 0\), we have a coordinate singularity that can be removed via a change of coordinates. If \(c^2 > 0\) outside, then the coordinate singularity may be interpreted as the event horizon. The metric becomes degenerate also at the points such that \(\Phi_\beta = A\beta = 2, -1\). These are non-metric singularities where \(c^2 \to 0\) (type (1)) or \(c^2 \to \infty\) (type (2)), and where the relation between the fundamental field \(B\) and the metric breaks down. To see whether and when these singularities occur, note first that the solution for \(\beta(r)\) obtained inverting equation (100) is always positive for \(r > 0\). This means that \(A\beta = 2\) can only occur if \(A > 0\), and the one at \(A\beta = -1\) if \(A < 0\). Recalling that \(\Lambda = -3\Phi_0\), a positive (resp. negative) \(A\) has a coherent sign with a negative (resp. positive) cosmological constant.

Note that \(\Phi_0\) enters the equations in the same unmodified way that the cosmological constant does in general relativity. In the following, we will restrict our attention to \(\Phi_0 = 0\) and focus on the effect of the modification introduced by a non-constant \(\Phi_\beta\).

What is the nature of the non-metric singularity? At these locations, the metric becomes singular or divergent, see equations (101) and (102). However, it was shown in [3] that these are not physical singularities for the theory: a change of coordinates exists such that all the fundamental fields \(B\) and \(\omega\) are finite. To see supporting evidence that the singularity is not physical, we can also look at scalar invariants. One simple scalar invariant can be constructed out of the field equations (14) as follows: first, contract the equations with the wedge product of \(B^i\). This gives a scalar density. Second, divide by the volume element provided by \(V = \frac{1}{2} B^i \land B_i\). In the standard general relativity this procedure would give the scalar density

$$\Rightarrow \frac{1}{2} B^i \land B_i.$$

We thank Robert Mann for pointing this out to us.

\(^{15}\)This solution can be expressed in terms of the Lambert \(W\) function (see e.g. [34]),

$$\beta(r) = -\frac{2}{A} W\left(-\frac{AM}{2r^3}\right).$$

We thank Robert Mann for pointing this out to us.

\(^{16}\)Note that here and in the following, by \(g_{00}\) and \(g_{11}\) we mean the \((t, t)\)- and \((r, r)\)-coordinate components of the metric; they are not the components with respect to frames, \(\{e^t\}\), or \(\{l^t\}\), introduced in section (5).
Figure 1. Paradigmatic plots of $r(\beta)$ from (100), for $A > 0$ (red) and $A < 0$ (blue). In the first case a minimal possible radius arises. In the second case the function is always invertible.

Ricci scalar. In the modified theory, it defines a scalar quantity $S \equiv B^i \wedge F_i / V$. Taking $\Phi_0 = 0$ for simplicity, equations (14) give after some algebra

$$S = \frac{9}{2} A \beta^2. \tag{103}$$

This shows explicitly the finiteness of this scalar quantity at the non-metric singularities of both type (1) $A \beta = 2$ and (2) $A \beta = -1$, and justifies, in part, the claim that these non-metric singularities are not physical singularities of the theory. What is happening is not that the theory is breaking down, but rather at the locations of these non-metric singularities, there does not exist any metric with respect to which the 2-forms, equation (49), is self-dual.

The two cases of positive and negative $A$ are qualitatively different (see figure 1), and it is convenient to treat them separately.

8.1. Positive $A$ branch

When $A$ is positive, we see from figure 1 that $r(\beta)$ in equation (100) has a minimum. Consequently, there is a minimal value of $r$ that can be reached. Imposing the derivative of equation (100) to vanish, this minimum is found at $r_{NM} = (e A M / 2)^{1/3}$. We call this point 'r$_{NM}$' because it is precisely where $A \beta = 2$, thus at this point there is a non-metric singularity of type (1). $\beta$ can be inverted in either the domain $r \in (r_{NM}, \infty)$ or $r \in (0, r_{NM})$. The first one has the standard asymptotic behaviour $\beta \sim r^{-3}$ ($r \to \infty$) of general relativity, and we look at this case first. In this branch we have $A \beta \in (0, 2)$, thus $c^2$ is always positive; consequently, the outmost zero of $f_2^*$ corresponds to the event horizon, and changes in the signature of the metric come from changes in the sign of $f_2^*$ only. It is natural to ask whether the singularity at $r_{NM}$ lies within the event horizon, or it is naked. The horizon is located at $f_2^*(r_H) = 0$, but this equation cannot be solved analytically. To visualize the situation, we find numerically the zeroes $r_H(A)$ of $f_2^*$ and compared them with $r_{NM}(A)$. See figure 2. To further illustrate the structure of the spacetime metric, we plot $\beta$ and $f_2^*$ for various values of $A$ in figure 3. The complete behaviour of the metric can be easily inferred from these plots and equations (101) and (102).

We see from figure 2 that both $r_H(A)$ and $r_{NM}(A)$ increase with $A$. The radius of the non-metric singularity grows in $A$ faster than the horizon radius, and there is a value of $A$ for which the radius of the singularity reaches the horizon radius. This value can be computed by requiring $f_2^*(A \beta = 2)$ to be non-positive (see figure 3), and one finds $A \leq 2 e^2 M^2$. This means that for fixed $A$ there is a minimal mass, $M_0 = \sqrt{A / (2e^2)}$ for which the black hole exists: for smaller values of the mass, the non-metric singularity is naked.

At the non-metric singularity both $g_{00}$ and $g_{11}$ are diverging, but this is not a physical singularity as we have discussed above. Hence, the theory itself is not breaking down and the
Figure 2. Location of the horizons and non-metric singularities as functions of $A$, for $M = 1$. The red continuous curve is the non-metric singularity $r_{\text{NM}}$ for $A > 0$ and the green continuous curve for $A < 0$. The blue interpolated dots are numerically found zeros of $f_2^*$ representing the radius $r_H$ of the event horizon. For negative $A$, we have two horizons. The values of $A$ which expose the singularity compute analytically to $A = 2e^2 \approx 14.78$ and $A = -125/(8e) \approx -5.75$. Analytically, we can compute that the two zeros of $f_2^*$ present for negative $A$ merge at $A = -54e^{-2} \approx -7.31$. Yet note that the zeros numerically found do not match exactly at this value: this shows explicitly the limitation of the numerical integration. The error can be estimated to be $\sim 5\%$.

Figure 3. Plots of $f_2^*$ (continuous) and $2 - A\beta$ (dotted) for different values of $A > 0$, with $M = 1$. From left to right, $A = 0.2$ (brown), $A = 1$ (blue), the limiting case $A = 2e^2$ (green) and $A = 50$ (red). The dashed black line is the standard general relativity result, drawn here for the comparison of the asymptotic behaviour. The zeros of the dashed lines are the locations of the non-metric singularities $A\beta = 2$, where $c^2 = 0$ and this branch of the solution interrupts.

solution can be continued to describe a ‘spacetime’ beyond this point. From the point of view of the metric, continuing the solution means using the other branch of the inverse of $r(\beta)$ in equation (100). This is the branch where $A\beta \in (2, \infty)$ and we move from $r = r_{\text{NM}}$ to $r = \infty$ as we recede from the non-metric singularity. Since $c^2$ is always negative there, it follows from equations (101) and (102) that the behaviour of the metric inside the non-metric singularity is similar to the case of Kantowski–Sachs cosmological spacetime, with the singularity at $r \to \infty$ (and thus $\beta \to \infty$, $g_{00} \to 0^-$, $g_{11} \to 0^+$) being a true curvature singularity. We do not report here plots of the solution in this branch. The overall picture can be summarized in the Penrose diagram already present in [3] (e.g. for the region of Kantowski–Sachs spacetime, see VI and VIII in figure 1 of [3]), where further discussion of the global structure is also given.
8.2. Negative $A$ branch

When $A$ is negative, the function $r(\beta)$ in equation (100) is invertible for any $r > 0$. Since $A\beta \in (-\infty, 0)$, the $A\beta = 2$ singularity cannot occur, but rather the $A\beta = -1$ one. At large radii, $\beta$ approaches zero with the standard $r^{-3}$ behaviour. As we decrease $r$, we reach the value $r_{NM}$ at which $A\beta = -1$ and the type (2) non-metric singularity occurs. The location is easily determined plugging $A\beta = -1$ in (100): we find $r_{NM} = (M|A|)^{1/3}e^{-1/6}$. There, $e^2 = 0$, and both $g_{00}$ and $g_{11}$ vanish. As already seen, this is not a physical singularity of the theory. The solution continues after this singularity (recall $\beta'$ never vanishes, so equation (100) can be inverted for all $r > 0$), with $e^2 < 0$ and the metric signature is switched to $-\text{+}\text{-}\text{-}$ as $\beta$ becomes large.

Again, we can numerically find the zeroes of $f_2^2$ to see whether the type (2) non-metric singularity is hidden or not by the event horizon, see figure 2 for $A < 0$. In figure 4, we plot $f_2^2$ and $\beta$. The plots show that $f_2^2$ approaches the Schwarzschild solution at large distances, and hence we can naturally expect that all solutions are asymptotically flat.

The plots also show the near-horizon structure, the relative positions of the horizon and the non-metric singularity, as zeroes of $f_2^2$ and $1 + A\beta$ (see also figure 2). Note the presence of two zeroes of $f_2^2$. For $A$ small enough, the non-metric singularity lies inside the event horizon given by the outmost zero of $f_2^2$, i.e. at the outmost zero of $f_2^2$, $1 + A\beta > 0$. We find that the event horizon is at a smaller radius than the Schwarzschild one. As we increase $A$, the radius of the event horizon shrinks, whereas that of the non-metric singularity grows. The critical value of $A$ at which the singularity becomes naked can be computed by requiring $f_2^2(r_{NM}) \leq 0$, and one gets $|A| \leq \left(\frac{2}{5}\right)^3\frac{M^2}{r}$. Therefore, we have a situation similar to the positive $A$ branch with the minimal black hole, having this time the mass $M_0 = \left(\frac{2}{5}\right)^3/2\sqrt{e}|A|$. For larger values of $|A|$, the non-metric singularity $A\beta = -1$ is exposed, and the two zeroes of $f_2^2$ keep moving towards each other. The two zeroes merge in a similar manner that two Killing horizons of the Reissner–Nordstrom metric merge and form a single degenerate horizon when the extremal limit $|Q| \to M$ is taken. However, for the present case, both the zeroes of $f_2^2$ occur in the region where $1 + A\beta < 0$: both the ‘horizons’ defined as the zeroes of $f_2^2$ are located inside the non-metric singularity. The value at which the two zeroes of $f_2^2$ merge and disappear can be computed by requiring the minimum of $f_2^2$ to be non-negative. We find $A \leq -54M^2/e^2 \approx -7.31M^2$ (green line). In [3] this relation was interpreted to give, for
fixed $A$, the minimal value of the mass for the black hole to exist. However we have seen above that well before (at $|A| = (\frac{3}{2})^3 \frac{M^2}{e} \simeq -5.7 M^2$), the black hole has already ceased to exist because the non-metric singularity at $\Phi_\beta = -1$ is naked. After the zeroes of $f_2^*$ disappear, even the true singularity at $r = 0$ is exposed.

Summarizing with the help of figure 2, we see that in both branches we have at fixed $M$ a maximal value of $|A|$ above which the black hole ceases to exist. This means that for a given $A$ there is always a minimal mass that a black hole can have, $M_0 = \sqrt{A}/(2\pi)$ for positive $A$ and $M_0 = (\frac{3}{2})^{3/2} \sqrt{\pi |A|}$ for negative $A$. In the range of $A$ for which the horizon exists, its location at fixed $M$ is an increasing function of $A$. It is smaller than the Schwarzschild value $r_H = 2M$ for $A < 0$ and larger for $A > 0$. The spacetime structure in the case of a naked non-metric singularity with no horizons present is shown in the conformal diagram in figure 5.

9. Charged black holes

We now consider the full system of equations (96) in the presence of an electric charge. The numerical studies that we present below show that the metric components approach those corresponding to the Reissner–Nordstrøm metric at large distances, and that $\beta$ goes to zero, thus $c^2 > 0$ and the asymptotic signature is $++--$. This confirms the asymptotic flatness of the perturbative solution presented earlier (see equation (98)). Thanks to this asymptotic behaviour, which conforms to the conditions (78)–(80), the parameters $M$ and $Q$ can be interpreted as the ADM mass and the electric charge in the standard sense discussed in section 6.1.\footnote{The analytic perturbative solution (98) allows us to identify $M$ and $Q$ as the ADM total mass and charge, respectively. However, for practical purposes in the numerical code one has to assign the boundary conditions at a finite value of the radius, say $r_0$. The larger is $r_0$, the closer the boundary values $M$ and $Q$ are to the ADM charges; but also the less precise are the numerical solutions near the horizon. Conversely, a small $r_0$ increases the precision of the solution near the horizon, but limits the interpretation of $M$ and $Q$. After investigating the stability of our code (by comparing it with the analytical cases known at $Q = 0$, see the left panel of figure 4), we took $r_0 = 50$}
Figure 6. The numerical solutions for $M = 1$ and $A = 1$ of the modified theory (solid lines) plotted against the Reissner–Nordstrøm solution of general relativity (dashed lines) for $Q = 0.5$ (blue), $Q = 0.81$ (green), $Q = 1$ (orange) and $Q = 1.5$ (red). Note the matching asymptotic behaviour. $g_{00}$ diverges at minus infinity at the radius where the type (1) $A\beta = 2$ singularity occurs (blue line), whereas diverges at plus infinity at $r = 0$, as the Reissner–Nordstrøm solution, when this type (1) singularity is avoided. In the latter case, extra zeroes may appear in $g_{00}$ due to crossings of the type (2) $A\beta = -1$ singularity. The value $Q = 0.81$ estimates the turning point from one regime to the other.

Decreasing $r$ from the asymptotic regime, the curvature increases and we start seeing departures from the Reissner–Nordstrøm solution in general relativity. In particular,

- the number and the location of the horizons are changed;
- there appear again non-metric singularities at $A\beta = 2$ (of type (1)) or $A\beta = -1$ (of type (2)).

Concerning the non-metric singularities, they now occur at locations $r_{NM}(A, M, Q)$ depending also on $Q$. At these points the metric given by (76) is degenerate or diverging, but as seen in the uncharged case, this does not mean that the singularities are harmful to the theory itself. In this perspective, we point out that the scalar $S$ constructed in (103) is unchanged by the presence of an electromagnetic field, thanks to the property $B^i \wedge \tau_i = 0$. This is analogous to what happens in general relativity, where the presence of an electric charge does not affect the Ricci scalar. Therefore, $S$ is again finite at the non-metric singularities. In view of this, although we have not fully studied the behaviour of other curvature components at the singularities, we anticipate that a similar change of coordinates as the one made in [3] would show finiteness of the fundamental fields of the theory.

As in the uncharged case, the solutions in the radial coordinate $r$ interrupt at $A\beta = 2$, where $\beta'$ diverges (see equation (96b)), whereas they continue after $A\beta = -1$. So again the range of our plots is $r \in (0, \infty)$ if no type (1) singularity occurs, whereas it is $(r_{NM}(A\beta = 2), \infty)$ if it does. However, unlike the uncharged case, both types of non-metric singularities can occur regardless of the sign of $A$, since now the sign of $\beta$ is not fixed (see e.g. the perturbative solution (98)).

To get a first feeling of departures from the general relativity case, in figure 6, we plot the numerical solutions of $g_{00}$ for $A = M = 1$ and various values of the charge, and compare them with the Reissner–Nordstrom solution in general relativity. These plots show the matching asymptotic behaviour anticipated above. The behaviour of $g_{00}$ near the horizon, on the other hand, becomes rather different in the modified theory. Note that zeroes of $g_{00}$ are not necessarily horizons, as they can also correspond to a non-metric singularity of type (2).

As we vary $A$, the departures from general relativity change significantly, and we have to face rich fauna of solutions. To give a useful overview of the situation, we plot in figure 7...
the locations of horizons and non-metric singularities as functions of $Q$, for a few significant values of $A$. We also illustrate the global spacetime structures for a few specific cases in figures 9, 10 and 11.

The initial parameter space in which the numerical integrations are performed is four dimensional: $(\Phi_0, A, M, Q)$. However, $\Phi_0$ and $M$ affect the solutions in a trivial way: a non-vanishing cosmological constant $\Phi_0$ can be straightforwardly added using (96c), and when changing the boundary condition $M$, we simply reproduce the same features at appropriately rescaled $A$ and $Q$. Therefore, we can without loss of generality restrict the parameter space to the two dimensional $(A, Q)$.

Given the plethora of possibilities, it is useful again to discuss separately the two branches with $A$ positive or negative.

9.1. Positive branch

Looking at figure 6 and the top-left panel of figure 7, we can divide the branch with $A > 0$ in two regions:

- region (i), where the non-metric singularity of type (1) is present, and all plots interrupt at its location $r_{NM}(A\beta = 2)$;
- region (ii), where the type (1) singularity is absent, and all plots continue to $r = 0$.

For given $A$, the demarcation between the two regions is given by a certain critical value of the charge, which we call $Q_0(A)$: when below this value we are in region (i), and when above we are in region (ii). In region (i), the location $r_{NM}$ of the non-metric singularity is a decreasing function of $Q$. This decreasing behaviour continues until the critical charge,
e.g. $Q_0 \sim 0.8$ for $A = 1$, where a minimum value of $r_{NM}$ is reached and we pass into region (ii). In region (ii), the type (1) singularity is replaced by the type (2), and its location is now an increasing function of $Q$. Numerical analysis shows that the value $Q_0(A)$ which marks the turning point from region (i) to region (ii) is a slowly growing function of $A$. For instance, $Q_0(0.1) \sim 0.55, Q_0(1) \sim 0.8$ (see the top-left panel of figure 7) and $Q_0(12) \sim 1.2$ (the top-right panel).

Next, let us move our attention to the horizons, drawn in blue. These are zeroes of $f_2^*$ which are simultaneously infinities of $g_2^*$. At small $Q$, the location of the event horizon is a decreasing function of $Q$ similar to the one of general relativity, $r_+ = M + \sqrt{M^2 - Q^2}$, which is also shown in the plot for comparison. The difference is that at each $Q$ the location of the event horizon is at a larger radius than in the case of general relativity. This generalizes to the charged case what already seen for $Q = 0$ in the previous section. A characteristic feature of the Reissner–Nordstrøm solution is the presence of an inner horizon. The top-left panel of figure 7 shows that in the modified theory there is only a small range of parameters in which the inner horizon is present, and this happens within region (ii). Furthermore, note that the value of the charge at which the two horizons merge is slightly larger than the $Q = M$ value of the Reissner–Nordstrøm solution. This result suggests the possibility of screening effects of the electric charge in this region of parameter space.

A stronger departure from general relativity is the presence of a third horizon: a close look at the top-left panel of figure 7 tells that for $Q$ in a narrow range around $Q = 1$, we have three horizons, the event-, inner- and the third horizon (see also the right panel of figures 6 and 11 for the global structure of the spacetime). Then, when $Q$ increases, the first two (event and inner) horizons merge and cease to exist (as in the Reissner–Nordstrom case), and only the third one continues to exist for larger value of $Q$—passed the critical value at which the third horizon first appears, and is always behind the type (2) non-metric singularity. The behaviour of horizons around $Q = 1$ and the appearance of the third horizon can be much clearly seen for smaller values of $A$ than $A = 1$. In figure 8, we show plots similar to figure 7 but with $A = 0.1$. The locations of both the horizon and non-metric singularity grow with $Q$, but the singularity grows faster. This suggests that there is a region of parameter space in the theory where we can arbitrarily overcharge a black hole without ever exposing the $r = 0$ singularity, i.e. the central singularities are always behind non-metric singularities (or otherwise covered by an event horizon).

When we increase $A$, the value $Q_0(A)$ at which we pass from region (i) to region (ii) slowly grows. Also, the location itself of the non-metric singularities grows with $A$. Specifically,
Figure 9. Conformal diagram for the solution with $A = 1$ and $M = 1, Q \gtrsim 1.1$ or $A = 12, M = 1, Q \gtrsim 1.2$. See the top two panels of figure 7. The double line, $\mathcal{I}$, denotes null infinity. There is a naked non-metric singularity of type (2) (green line), and inside it, a horizon (blue line) that covers a true singularity at the centre (thick dashed line). Here, the motion of time is taken to be vertical. As a consequence, in this figure, the two green lines—one vertical and the other horizontal—should be identified, as they denote the same non-metric singularity. Covered by a horizon (blue line) inside the non-metric singularity, the central singularity at $r = 0$ is non-spacelike. The full extension of the spacetime is obtained by gluing together copies of the fundamental block depicted here, at the blue solid line.

This grows faster than the location of the event horizon, a phenomenon already seen in the uncharged case that led to the existence of a notion of ‘minimal’ black hole. The same thing happens in the charged case and, because the presence of a charge shrinks the radius of the event horizon, we can expect that at fixed $M$ the non-metric singularity of type (1) is exposed at smaller values of $A$ as we increase the charge. This situation is depicted in the top-right panel of figure 7, where we consider the value $A = 12$. This value is smaller than $A \simeq 14.78$ required—for $M = 1$—to expose the non-metric singularity in the uncharged case (cf figure 2), but can make the type (1) singularity naked if enough charge is present (e.g. $Q \sim 0.8$ for $A = 12$, see figure 7).

For such a large $A$ (still the top-right panel), we also note the complete disappearance of the event horizon, swallowed up by the type (1) non-metric singularity. This does not exclude the possibility that a horizon is still present beyond the singularity, but we do not have access to that region. If we continue to increase $A$, the event horizon disappears for smaller and smaller values of $Q$, until the value $A = 2e^2M^2$ found in the previous section. This is the value for which the event horizon disappears also in the uncharged case. Note however that a horizon is still present in region (ii), behind the non-metric singularity of type (2). Therefore, even at such large $A$, we can still have a horizon that covers the true singularity at the centre, if the charge is high enough.

In figures 9–11, we report the Penrose diagrams of different configurations for which the solution is complete, i.e. $r \in (0, \infty)$. 

30
9.2. Negative branch

In the negative branch the standard two horizons of the Reissner–Nordstrøm solution are more easily distinguishable (see the bottom panels of figure 7). The effect of a negative $A$ to shrink the location of the event horizon seen in the uncharged case generalizes to the charged case in a way that qualitatively maintains the standard dependence on $Q$. At the same time, the inner horizon is pushed to a larger radius. As a consequence, the charge at which the two horizons merge and disappear is smaller than the $Q = M$ value of the general relativity case. This hints at possible effects of anti-screening of the electric charge in this theory.

Concerning the appearance of non-metric singularities, we see from the bottom panels of figure 7 that

- the solutions always possess the type (1) singularity;
- the type (2) singularities can also occur in the solutions, for $Q$ smaller than a certain value, $Q_1(A)$.

Comparing the two bottom panels of figure 7, we see that the value $Q_1(A)$ at which the two singularities of type (2) merge and disappear grows with $|A|$. The simultaneous presence of the two singularities makes the near-horizon behaviour differ significantly from the case of general relativity: see the blue plots in figure 12 which have $A = -1$, $M = 1$ and $Q = 0.5$.

Let us look at the bottom-right panel of figure 7. We see that when increasing $|A|$, the two horizons approach each other, and the extremal case happens at a smaller and smaller
Figure 11. Conformal diagram for the charged black hole solution with $A = 0.1$ and $M = 1, Q = 0.9$. See figure 8. The double line denotes null infinity. There is a type (2) non-metric singularity (green line) inside the inner horizon, and the third horizon is located inside the non-metric singularity. The flow of time is vertical, and the two green lines—one vertical and the other horizontal—denoting the same non-metric singularity are identified. The central singularity (thick dashed line) inside the third-horizon is non-spacelike. The full extension of the spacetime is obtained by gluing together copies of the fundamental block depicted here, at the blue solid line.

Figure 12. The numerical solutions for $M = 1$ and $A = -1$ of the modified theory (solid lines) plotted against the Reissner–Nordstrom solution of general relativity (dashed lines) for $Q = 0.5$ (blue), $Q = 1$ (green) and $Q = 1.5$ (red). Note the matching asymptotic behaviour. $g_{00}$ always diverges at the radius where the $A\beta = 2$ singularity occurs. For $Q = 1$ (green line) there is no horizon, unlike in the Reissner–Nordstrom solution. For $Q = 0.5$ (blue lines) the singularity at $A\beta = -1$ is crossed twice, and consequently the plot for $g_{00}$ shows two additional zeros before the divergence, which is now at minus infinity.
charge. The location of the outmost horizon still maintains the qualitative dependence on $Q$ of the Reissner–Nordstrom solution, but the location of the inner one has now a very deformed dependence. The two horizons disappear completely at the value $A = -54M^2/e^2$ computed in the uncharged case, and as already seen then, the type (2) singularity becomes exposed before the two horizons disappear: the bottom-right panel shows that at $A = -7$ the horizons are still present but inside the singularity of type (2).

10. Summary

We have studied spherically symmetric black holes in the framework of Krasnov’s modified self-dual gravity [2], in which a Weyl curvature-dependent functional is added to the Plebansky action of general relativity. The remarkable novelty of this modification is that there appear no extra physical degrees of freedom for gravity. This is a significant difference from other modified theories of gravity proposed in the literature, and it requires the use of a self-dual 2-form $B$ as the fundamental field: the metric describing the geometry of spacetime is only a derived quantity. The derived metric can be obtained using the self-duality of $B$, and the field equations it satisfies are still second order as in general relativity. At least for the case of spherical symmetry, the field equations can be casted in a way which is significantly similar to general relativity, with the effect of the modification clearly distinguishable. The modification increases the nonlinearity of the equations, due to the extra feedback effect coming from the additional Weyl curvature dependent—otherwise ‘cosmological constant’—term.

The first result of the paper is to show how to couple electromagnetism to the theory. This uses the coupling to Plebański’s theory developed in [10], which is shown to apply straightforwardly also to the modified theory, as anticipated by Krasnov [2]. We have then proved the staticity of the electrovacuum spherically symmetric solutions for any form of $\Phi$ (Birkhoff’s theorem, section 5.6). Under the assumption of spherical symmetry and accordingly with the staticity result, we have considered spherically symmetric, static black hole spacetimes for the two simplest profiles of $\Phi$, i.e. the linear modification $\Phi = -\Lambda/3 + a\beta(r)$ and the quadratic modification $\Phi = -\Lambda/3 + A\beta^2(r)/2$, with $a$ and $A$ parameters and $\beta(r)$ the ‘magnitude’ of Weyl curvature of our spherically symmetric spacetimes. In both cases, the structures near the horizon and near the central singularity significantly differ from general relativity, and the notion of extremal black hole changes with the parameters. We summarize our main results below.

In the linearly modified case, the solutions do not approach asymptotically flat geometries at large distances even when the cosmological constant $\Phi_0 = -\Lambda/3$ is set to zero, as we have briefly discussed in section 6.1, and hence the solutions have no direct physical relevance. Since the linear modification fails to satisfy a certain notion of analyticity, the question arises if there is a relation between the latter and the possibility of asymptotic flatness. Such a relation would be useful to restrict the class of $\Phi$s with physically relevant solutions. The linear case is nonetheless worth examining since it can easily be solved analytically. The solutions obtained in section 6.2 explicitly show how the $Q$-dependence of the locations of the horizons is affected by the parameter $a$.

In the quadratically modified case, the nonlinearity of the equations of motion is more involved, and we have not been able to find analytic solutions for $Q \neq 0$. We resorted to numerical computations to study the charged black holes. For this purpose, we first considered a perturbative approach to the field equations. The approximate solutions found in this way are valid at large distances, and allowed us to show that the solutions have the standard Reissner–Nordstrøm asymptotic behaviour, and thus the parameters $(M, Q)$ appeared there.
can be interpreted in the standard sense, the ADM mass and the total charge, respectively, in accordance with the arguments of section 6.1.

The numerical methods allowed us also to further investigate the quadratic modification case with \( Q = 0 = \Lambda \), whose vacuum black hole solutions had already been analytically found in [3]. We studied the location of the event horizon, and showed that a black hole of mass \( M \) can only exist for \( A \in (\frac{-1}{2} e^{-1} M^2, 2 e^2 M^2) \), where the lower bound differs from the one proposed in [3]. For values of \( A \) outside this interval, a non-metric singularity is naked outside the horizon: it is of type (1) if \( A > 0 \), and of type (2) if \( A < 0 \). Concerning the nature of the singularity, it was shown in [3] that a change of coordinates can be found in which the fundamental fields are finite. Therefore, this is not a physical singularity for the theory, and the fundamental fields can be continued beyond it. Here we have introduced also a scalar quantity, analogue of the Ricci scalar, and showed explicitly that it is finite at the non-metric singularities. The vacuum \( Q = 0 = \Lambda \) case results are summarized as follows.

(i) The location of the horizon, \( r_H(A, M) \), at fixed \( M \) is pushed at larger/smaller values than in GR according to the sign of \( A \).

(ii) The internal structure becomes more intricate: the true singularity is surrounded by the non-metric singularity and an internal horizon. The non-metric singularity is of type (1), \( A\beta = 2 \), for \( A > 0 \) and type (2), \( A\beta = -1 \), for \( A < 0 \). The new internal horizon can be seen explicitly for \( A < 0 \), but for \( A > 0 \) one has to change the branch of the solution after the singularity to see it, or go to \( \beta \) as radial coordinate, as it is done in [3].

(iii) For \(|A|\) too large, the non-metric singularity is exposed outside the event horizon.

Therefore, in a given modified action with fixed \( A \), there is a minimal mass that a black hole can have. Beyond this limit, distant observers would not see the event horizon, but a non-metric singularity first.

We next summarize the charged case. Our main focus has been on which type of (e.g. (1) \( A\beta = 2 \) or (2) \( A\beta = -1 \)) non-metric singularities appear and whether those singularities are covered by the event horizon. Given a fixed \( M \), the answer depends upon the values of the parameters \( A \) and \( Q \).

(i) As in GR, one can find an event horizon as well as an inner horizon. The location \( r_H(A, Q, M) \) of the event horizon for fixed \( M \) and \( Q \) is pushed at larger or smaller values than in GR according to the sign of \( A \). For the inner horizon, we have the opposite behaviour. As a consequence, the extremal case is now at a larger or smaller value than \( Q = M \) as in GR. This suggests that in the theory, electromagnetic charges might experience the screening or anti-screening effects according to the sign of \( A \), as we have discussed in section (9). The physical implication of the screening effects, however, remains an open issue.

(ii) The true singularity at \( r = 0 \) is surrounded by non-metric singularities, and (if exists) inner horizons. The outmost non-metric singularity is

- type (1) at small \( Q \), then (2) at larger \( Q \) for \( A > 0 \)
- type (2) at small \( Q \), then (1) at larger \( Q \) for \( A < 0 \)

(iii) Increasing \(|A|\) too much, the non-metric singularity is exposed outside the event horizon.

This happens earlier and earlier, as we pour a larger charge into the hole. Therefore, in a given modified action with fixed \( A \), there is a minimal mass that a black hole of charge \( Q \) can have, and this value increases as we increase \( Q \).

(iv) A region of parameter space exists, with \( A > 0 \), where the type (2) singularity is naked, but inside it there always exists a horizon which hides the singularity at \( r = 0 \) for arbitrarily large values of the charge.
Note that there could be a further structure behind all singularities of type (1), which we do not attempt to access numerically. This would require a change of coordinates, or the knowledge of the analytic solution as it was done for the vacuum solution in [3]. (Compare with section (8) for the positive branch of the uncharged case.) It would certainly be interesting to do so, changing the radial coordinate as in the vacuum case. However, given the limitations of our understanding of analytic solutions, we prefer to restrain from doing so, and instead focus the attention of the reader on the near-horizon structure.

Our results also bring some insight into how it is possible to restrict the class of physically relevant types of $\Phi$. We showed that spherically symmetric black holes (charged or not) are not asymptotically flat in some type of $\Phi$ non-analytic at the origin, and one is tempted to conjecture that this generalizes to any non-analytic choice. Although charged black holes are not relevant in astrophysical context, the results obtained show that a deeper understanding of the modified theory, as well as further useful restrictions on $\Phi$, can be found by looking at matter coupling.

Finally, we would like to recall that given $M$, there are parameter regions of $(A, Q)$ in which both types of non-metric singularities can be naked. In this case, distant observers would see the breakdown of the standard spacetime picture based on the metric description outside the horizon. Nevertheless, from the view point of this modified self-dual gravity, there is nothing to be afraid of, since the non-metric singularities are harmless for the fundamental fields, even when naked. Therefore, in this modified self-dual gravity, the notion of the cosmic censorship itself may also need be modified. In this regard, we should also recall that a non-metric singularity is specified as a $(r = \text{const})$ hypersurface, across which the time and spatial (radial)-coordinates change their spacetime role. This, in particular, implies that, as illustrated by the conformal diagram in figure 5 for the simplest vacuum case, a true singularity at the centre $(r = 0)$ covered by a naked non-metric singularity is achronal (non-timelike), unless covered by any other horizons. We also note from figure 5 that the hypersurface of a naked non-metric singularity plays the role of the (partial) Cauchy surface for the interior region, and therefore once a causal curve goes across a non-metric singularity and enters the interior region, then it has to fall into the central singularity and never come back to the region outside the non-metric singularity. Therefore, in this sense the hypersurface of a non-metric singularity may be viewed as a one-way membrane, or a type of ‘horizon’. This view point may appear to be in favour of the cosmic censorship in the sense that the true singularities of the theory appear to always be covered by either the event horizon or a non-metric singular horizon. Note that in the present context, one cannot tell which horizon, future or past, a non-metric singularity corresponds to, due to the time-symmetric nature of static spacetimes. However, the situation would be clear when dynamical (non-stationary) cases are considered. It would be interesting to consider a sensible definition of a Cauchy problem for the evolution in such a spacetime and the notion of cosmic censorship, within the framework of the modified self-dual gravity theory.

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