Schrödinger equations with smooth measure potential and general measure data

Tomasz Klimesiak
Faculty of Mathematics and Computer Science, Nicolaus Copernicus University
Chopina 12/18, 87–100 Toruń, Poland
E-mail address: tomas@mat.umk.pl

Abstract

We study equations driven by Schrödinger operators consisting of a self-adjoint Dirichlet operator and a singular potential, which belongs to a class of positive Borel measures absolutely continuous with respect to a capacity generated by the operator. In particular, we cover positive potentials exploding on a set of capacity zero. The right-hand side of equations is allowed to be a general bounded Borel measure. The class of self-adjoint Dirichlet operators is quite large. Examples include integro-differential operators with the local part of divergence form. We give a necessary and sufficient condition for the existence of a solution, and prove some regularity and stability results.

1 Introduction

Let $E$ be a locally compact separable metric space and $m$ be a Radon measure on $E$ with full support. Let $A$ be a non-positive self-adjoint operator on $L^2(E;m)$ generating a Markov semigroup of contractions $(T_t)_{t \geq 0}$ on $L^2(E;m)$ (so-called Dirichlet operators). We also assume that there exists the Green function $G$ for $-A$ (see Section 2.2). In the paper, we give a necessary and sufficient condition for the existence of a solution to the following Schrödinger equation

$$-Au + u \cdot \nu = \mu. \quad (1.1)$$

Here $\nu$ belongs to the set $S_A$ consisting of positive smooth measures: Borel measures absolutely continuous with respect to a capacity $Cap_A$ generated by $A$ for which there exists a strictly positive quasi-continuous function $\eta$ such that $\int_E \eta \, d\nu < \infty$. The class of Dirichlet operators is quite wide. In the important case of $E = \mathbb{R}^d$, it includes densely defined operators $A$ of the form

$$Au(x) = \sum_{i,j=1}^d (a_{i,j}(x)u_{x_i})_{x_j} + P.V. \int_{\mathbb{R}^d \setminus \{0\}} (u(x) - u(y))N(x,dy) + u(x)c(x), \quad (1.2)$$

with an elliptic matrix $a$, symmetric kernel $N$, and $c \geq 0$ (see [20, Theorem 3.2.3]).

Mathematics Subject Classification: Primary 35J10, 60J45; Secondary 35B25, 35J08, 31C25, 47G20.

Keywords: Schrödinger operator, smooth measure, singular potential, Dirichlet form, Markov process, Green function, additive functional.
Since the seventies the Schrödinger operators with singular potentials have attracted growing interest in the literature (see e.g. [1, 2, 3, 19] and references therein). The very important, among others, class of singular potentials appearing in applications, including, as particular case, Coulomb potentials and harmonic potentials, is the family of repulsive potentials of the form

\[ \nu_1(dx) = \sum_{j=1}^{N} \frac{c_j}{|x - x_j|^\beta_j} \, dx, \quad \nu_2(dx) = \frac{c_1}{\delta_D^\beta_1(x)} \, dx \] (1.3)

with any \( c_j \geq 0, \beta_j \in \mathbb{R}, x_j \in \mathbb{R}^d, j = 1, \ldots, N \). Here and in what follows \( \delta_D(x) = \text{dist}(x, \partial D) \). Another wide class, disjoint from the one above, of singular potentials is the class of generalized potentials, i.e. Borel measures \( \nu \) concentrated on some \( m \)-measure zero set \( N \subset E \), e.g. in case \( A \) is the fractional Laplacian \( \Delta^{\alpha/2} \), i.e. the operator of the form (1.2) with \( a_{i,j} \equiv c \equiv 0 \) and \( N(dx, dy) = |x - y|^{-d-\alpha} \) for some \( \alpha \in (0, 2) \), any \( \sigma \)-finite positive Borel measure \( \nu \) satisfying

\[ \nu(dx) \ll \mathcal{H}^\lambda \]

for some \( \lambda \in (d - \alpha, d) \), where \( \mathcal{H}^\lambda \) is the Hausdorff measure of order \( \lambda \), falls within the class of generalized potentials. These types of potentials have been considered in variety of models in nuclear physics, solid-state physics and quantum field theory (see e.g. [3]). The class \( S_A \) of smooth measures includes, as particular cases, the above mentioned types of potentials. Although the class \( S_A \) depends on \( A \), we always have the inclusion

\[ L^{1,+}_{\text{loc}}(E; m) \subset S_A. \]

Note that smooth measure need not be a Radon measure. In fact, it can be a nowhere Radon measure. As an example of such a measure can serve \( \nu_1 \) defined by (1.3) with \( N = \infty \) and suitably chosen \( \{c_j\} \), \( \{\beta_j\} \) and \( \{x_j\} \) (see [5]).

One can look at (1.1) from two different perspectives. In the first one, we regard (1.1) as equation of the form

\[ -A_\nu v = \mu, \] (1.4)

where \( -A_\nu \) is a non-negative self-adjoint operator on \( L^2(E; m) \) being the perturbation of \( -A \) by the smooth measure potential \( \nu \), that is \( -A_\nu = -A + \nu \). In the second one, we regard (1.1) as the equation

\[ -Au = -u \cdot \nu + \mu \] (1.5)

with absorption term on the right-hand side. The difference between (1.4) and (1.5) is very subtle and appears only in the case where the concentrated part \( \mu_c \) of the measure \( \mu \), i.e. the singular part of \( \mu \) with respect to the capacity associated with \( A \) is non-trivial. The main goal of the paper is to study (1.1) from the above two perspectives. First, we provide definitions of solutions to (1.4) and (1.5). The problem of proper definitions of solutions is rather delicate and requires using some deep results from the potential theory. We then give some necessary and sufficient condition for the existence of solutions to (1.4) and to (1.5), and we compare the two approaches to (1.1). Finally, we give some results on regularity and stability of solutions.

In the paper, a solution \( v \) to (1.4) will be called a \textit{duality solution} to (1.1), and a solution \( u \) to (1.5) will be called a \textit{strong duality solution} to (1.1). Heuristically,

\[ v = R^\nu \mu, \quad u = R(-u \cdot \nu + \mu), \] (1.6)
where 
\[ R_{\nu} = (-A_{\nu})^{-1}, \quad R = (-A)^{-1}. \]

Note that both operators \( R_{\nu} \) and \( R \) are well-defined on \( L^{\infty,+}(E;m) \cap L^2(E;m) \) by
\[
R_{\nu} \eta = \text{ess sup}_{\alpha > 0} R_{\alpha} \eta, \quad R \eta = \text{ess sup}_{\alpha > 0} R_{\alpha} \eta, \quad \eta \in L^{\infty,+}(E;m) \cap L^2(E;m),
\]

where \((R_{\alpha})_{\alpha > 0}, (R_{\alpha})_{\alpha > 0}\) are the resolvents of \( A_{\nu} \) and \( A \), respectively (see, e.g., [4, 47]).

Since the operators \( R_{\nu} \) and \( R \) are linear and positive definite, we may extend them to \( L^{\infty,+}(E;m) \cap L^2(E;m) \) by
\[
R_{\nu} \eta = \text{ess sup}_{\alpha > 0} R_{\alpha} \eta, \quad \eta \in L^{\infty,+}(E;m) \cap L^2(E;m),
\]

where \((R_{\alpha})_{\alpha > 0}, (R_{\alpha})_{\alpha > 0}\) are the resolvents of \( A_{\nu} \) and \( A \), respectively (see, e.g., [4, 47]).

Since the operators \( R_{\nu} \) and \( R \) are linear and positive definite, we may extend them to \( L^{\infty,+}(E;m) \) (with possibly infinite values). For these extensions, we have \( R_{\nu} \eta \leq R \eta, m\text{-a.e.} \) \( \eta \in L^{\infty,+}(E;m) \). Since the operators \( R_{\nu}, R \) are not defined on the space of measures, the idea is to understand (1.6) in the duality sense, i.e., we require that
\[
\int_E v \eta \, dm = \int_E R_{\nu} \eta \, d\mu, \quad \int_E u \eta = - \int_E u R \eta \, d\nu + \int_E R \eta \, d\mu (1.7)
\]
for every \( \eta \in L^{\infty}(E;m) \) such that \( R|\eta| \) is bounded \( m\text{-a.e.} \). In the second equation, we additionally require that \( u \in L^1(E;\nu) \). Although this idea is simple and natural, its implementation is complicated by the fact that \( \mu, \nu \) are measures, and what is more, \( \mu \) is an arbitrary bounded Borel measure. For this reason (1.7) is meaningful only if the operators \( R_{\nu} \) and \( R \) are defined pointwise, i.e., the functions \( R_{\nu} \eta \) and \( R \eta \) are well defined at every point of \( E \) for every positive \( \eta \in B(E) \). We can define \( R \) pointwise by using the Green function \( G \) for \( -A \). Namely, we put
\[
R \eta(x) = \int_E G(x,y) \eta(y) \, dy, \quad x \in E. (1.8)
\]

Unfortunately, in general, there is no Green function for \( -A_{\nu} \). One of the results of the paper consists in finding a natural pointwise meaning for \( R_{\nu} \). We propose such a version and denote by \( \hat{R}_{\nu} \). In the case of uniformly elliptic divergence form operator
\[
A = \sum_{i,j=1}^d \frac{\partial}{\partial x_j} \left( a_{ij} \frac{\partial}{\partial x_i} \right) (1.9)
\]
Malusa and Orsina [37] used the notion of Lebesgue’s points to define the following version of the resolvent \( R_{\nu} \):
\[
\hat{R}_{\nu} \eta(x) = \lim_{r \to 0^+} \frac{1}{|B(x,r)|} \int_{B(x,r)} R_{\nu} \eta(y) \, m(dy). (1.10)
\]

Unfortunately, this recipe for pointwise version of \( R_{\nu} \) can be used only for a subclass of operators whose harmonic functions are characterized by the mean value property (or are comparable, via Green function, with operators for which the mean value characterization of harmonic functions holds; see [37, Lemma 4.10]). We propose a completely new approach based on the probabilistic potential theory. Our approach considers that the key role is played by the set
\[
E_{\nu} = \{ x \in E : \int_{V_x} G(x,y) \nu(dy) < \infty \text{ for some finely open neighborhood } V_x \text{ of } x \}
\]
considered in the case of equations with operator (1.9) in [6, 48]. Write \( N_{\nu} := E \setminus E_{\nu} \) and recall that the fine topology is the smallest topology under which all excessive
functions are continuous. Let $X$ be a Hunt process with life time $\zeta$ associated with the operator $A$. The first result of the paper (see Section 3), although rather technical, plays a pivotal rule in our approach. It states that for every positive smooth measure $\nu$ there exists a unique positive continuous additive functional $A^\nu$ (PCAF) of $X$ with exceptional set $N = N_\nu$ such that for all $x \in E_\nu$ and $\eta \in B^+(E)$,

$$E_x \int_0^\zeta \eta(X_r) \, dA^\nu_r = \int_E G(x, y) \eta(y) \nu(dy), \quad (1.11)$$

for every $x \in N_\nu$,

$$P_x(A^\nu_t = \infty, t > 0) = 1, \quad (1.12)$$

and $\psi_{A^\nu}$ defined as

$$\psi_{A^\nu}(x) = E_x \int_0^\zeta e^{-t} e^{-A^\nu_t} \, dt, \quad x \in E,$$

is finely continuous on $E$. Moreover, we prove that $N_\nu$ is the minimal exceptional set, in the sense that, if there exists a PCAF $A$ of $X$ with exceptional set $N \subset N_\nu$ such that (1.11) holds and $\psi_A(x) = E_x \int_0^\zeta e^{-t} e^{-A_t} \, dt$ is finely continuous on $E$, then $N = N_\nu$ and $P_x(A_t = A^\nu_t, t > 0) = 1, x \in E$. The above result was proved by Baxter, Dal Maso and Mosco [4] in the case of Brownian motion (see also [48]). Although this result is purely probabilistic in nature, it plays key role in defining duality solutions to (1.1). We put

$$R^\nu \eta(x) = E_x \int_0^\zeta e^{-A^\nu_t} \eta(X_t) \, dt, \quad x \in E, \quad (1.13)$$

and show that this formula agrees with (1.10) in the case where $A$ is defined by (1.9). With this notion in hand, in Section 4.1 we introduce the definition of a duality solution to (1.1) by using the first formula in (1.7) with $R^\nu \eta$ replaced by $R^\nu \eta$. We then show that for every bounded Borel measure $\mu$ on $E$ there exists a unique duality solution to (1.1).

It is worth noting here, that the above-mentioned approach to (1.11) goes back to Stampacchia [46], where equations with measure data and operator (1.9) defined on a bounded regular domain $D \subset \mathbb{R}^d$ are considered. In [46], the potential measure $\nu$ is of the form $\nu(dx) = V(x) \, dx$, where $V \in L^p(D; m)$ with $p > d/2$. Under this assumption, there exists the Green function for $-A_\nu$, so $R^\nu$ can be defined through its Green function.

As we mentioned earlier formula (1.13) gives a natural pointwise meaning for the resolvent $R^\nu$. In Section 1.2 we spend some time to explain why we use here the term “natural”. First, we show that if $\nu$ is a positive smooth measure such that there exists the Green function $G^\nu$ for the operator $-A_\nu$ ($\nu$ is then called a strictly smooth measure), then

$$R^\nu \eta(x) = \int_E G^\nu(x, y) \eta(y) \, m(dy), \quad x \in E.$$  

Moreover, for every sequence $\{\nu_n\}$ of positive strictly smooth measures such that $\nu_n \rightharpoonup \nu$,

$$R^\nu \eta(x) = \lim_{n \to \infty} \int_E G^{\nu_n}(x, y) \eta(y) \, m(dy), \quad x \in E.$$  

Then we show that if $\mu$ is additionally continuous functional on the extended domain $D_\nu(\mathcal{E})$, i.e. $\mu \in D'_\nu(\mathcal{E})$ (with the inner product $\mathcal{E}(\cdot, \cdot)$), then the duality solution $\nu$ to
(1.1) is the unique minimizer of the energy
\[ E(\eta) = \frac{1}{2} \mathcal{E}(\eta, \eta) + \frac{1}{2} \int_E |\tilde{\eta}|^2 \, d\nu - \langle \mu, \eta \rangle_{D'_e(\mathcal{E}), D_e(\mathcal{E})}, \quad \eta \in D_e(\mathcal{E}) \cap L^2(E; \nu). \]

Here \( \tilde{\eta} \) stands for the quasi-continuous \( m \)-version of \( \eta \). In other words, \( v \) is a variational solution to (1.1). Moreover, we show that for every bounded Borel measure \( \mu \), there exists a sequence \( \{\mu_n\} \) of bounded Borel measures in \( D'_e(\mathcal{E}) \) such that \( \mu_n \to \mu \) narrowly and \( v_n \to v \), where \( v_n \) is a variational solution to (1.1) with \( \mu \) replaced by \( \mu_n \). This stability property of \( v \) is sometimes used in the literature as the definition of the so-called SOLA solution (see e.g. [25]).

Let us note here that variational approach to (1.1) with the operator (1.9) on a bounded regular domain \( D \subset \mathbb{R}^d \) and \( \mu \in H^{-1}(D) \) was applied in Dal Maso and Mosco [14, 15] in the context of the so called relaxed Dirichlet problem. In [14, Example 3.10] it is observed that, in general, a variational solution to (1.1) is not a distributional solution to (1.1). It is also worth mentioning that in [14, 15] the authors considered even more general class of perturbations \( \nu \) which do not satisfy quasi-finiteness condition (see condition (b) in Section 2.1).

In Section 4.3, we prove basic regularity properties of a duality solution \( v \) to (1.1). We show that \( v \) possesses an \( m \)-version \( \tilde{v} \) which is quasi-continuous and \( \tilde{v} \in L^1(E; \nu) \). Moreover, we show that for every \( k \geq 0 \), \( T_k(v) := \min\{k, \max\{v, -k\}\} \in D_e(\mathcal{E}) \) and
\[ E(T_k(v), T_k(v)) \leq k\|\mu\|_{TV}. \]

Section 5 is devoted to strong duality solutions to (1.1). By a solution we mean a quasi-continuous function \( u \) on \( E \) such that \( u \in L^1(E; \nu) \) and the second equation in (1.7) holds with \( R \eta \) defined pointwise by (1.8). To understand the subtle difference between the notion of duality and strong duality solution to (1.1), we have to take a closer look at the formulations of both definitions (see (1.7)). Observe that in the case of duality solutions, we consider some class of test functions included in the range of the operator \( \tilde{R}' \), and in the case of strong duality solution, we consider possibly wider class of test functions included in the range of operator \( \tilde{R} \) (see Proposition 4.4). In the first case, by (1.12) and (1.13), each test function equals zero on the set \( N_\nu \). Hence, for every \( x \in N_\nu \), the function \( v \equiv 0 \) is a duality solution to
\[ -Au + v \cdot \nu = \delta_{\{x\}}. \]

However, this function is not a strong duality solution to the above equation as in such a case, by the definition, we would have
\[ 0 = \int_E R\eta \, d\delta_{\{x\}} = R\eta(x) = \int_E G(x, y) \eta(y) \, m(dy). \]

This implies that \( G(x, \cdot) = 0 \), which contradicts the definition of the Green function. We prove that there exists a strong duality solution \( u \) to (1.1) if and only if
\[ |\mu_c|(N_\nu) = 0, \]
and in this case \( u \) is also a duality solution to (1.1). Recently, this result was proved by Orsina and Ponce [40] in the case where \( A = \Delta |D \) and \( \nu(dx) = V(x) \, dx \), and by Gómez-Castro and Vazquez [25] in case \( A = \Delta^{\alpha/2} \), \( \nu(dx) = V(x) \, dx \) with \( V \) belonging to the class \( (1.15) \) defined below.
An existence result for strong duality solutions to (1.1) is a direct consequence of the following, interesting in its own right, result which we prove in Section 5. It states that if \( v \) is a duality solution to (1.1), then it is a strong duality solution to
\[-Av + v \cdot \nu = \mu|_{E_v}.\]
We have already mentioned that \( v \) is a limit of variational solutions to the Schrödinger equations
\[-Av_n + v_n \nu = \mu_n\]
with more regular than \( \mu \) measures \( \mu_n \) approximating \( \mu \) in the narrow topology. This means that when passing to the limit in (1.14) some reduction of the measure \( \mu \) occurs.

This phenomenon is somewhat reminiscent of the phenomena occurring in the theory of reduced measures introduced by Brezis, Marcus and Ponce \[11, 12\] for the Dirichlet Laplacian and next generalized by Klimsiak \[29\] to a wide class of Dirichlet operators. In our context, the measure \( \mu|_{E_v} \) may be considered as a reduced measure for \( \mu \). The same reduction takes place when we approximate monotonically the measure \( \nu \) (see Proposition 4.7). This result is a far-reaching generalization of \[25, Theorem 8.2\].

In Section 6 we briefly comment on the easy extension of our existence results to the weighted measure spaces \( \mathcal{M}_\rho \) consisting of Borel measures \( \mu \) with
\[\int_E \rho d|\mu| < \infty,\]
where \( \rho \) is a strictly positive excessive function, e.g. the principle eigenfunction for \(-A\).

In particular, we cover the class \( \mathcal{M}_{\delta_D^{\alpha/2}} \) considered in \[18\] for \( C^{1,1} \) open domain \( D \subset \mathbb{R}^d \) and \( A = (\Delta^{\alpha/2})|_D \), i.e. Dirichlet fractional Laplacian with zero exterior condition. It is well known that for \( C^{1,1} \) domains \( \delta_D^{\alpha/2} \) is comparable to the principle eigenfunction of \((\Delta^{\alpha/2})|_D \). We close the section with comments on the notion of renormalized solutions and its relation to the notion of duality solutions.

Self-adjoint Schrödinger operators with smooth measure potentials (also called generalized Schrödinger operators) and their applications to quantum theory were intensively studied in the late ’70s and ’80s of the last century by using methods of Dirichlet forms, probabilistic potential theory and harmonic spaces, see the papers by Albeverio, Ma and Röckner \[3, 4, 5\] and the paper by Boukricha, Hansen and Hueber \[10\] for a nice account of results in this direction. At the same time, Baxter, Dal Maso and Mosco \[6, 14, 15\] studied equations of the form (1.1) with the classical Laplacian and \( \mu \in H^{-1}(D) \) in the context of the so-called relaxed Dirichlet problem with even more general class of potentials, which do not satisfy the quasi-finiteness condition which is required in the definition of a smooth measure. In ’90s and 2000s, equation (1.1) with smooth both \( \nu \) and \( \mu \) was studied by Getoor \[22, 23, 24\] and Beznea and Boboc \[7\] with more general class of operators \( A \) generated by right Markov semigroups.

Recently, Orsina and Ponce \[40\] and Ponce and Wilmet \[42\] considered Schrödinger equations of the form (1.1) with \( A \) being the classical Laplacian, \( \nu(dx) = V(x) \, dx \) for some positive Borel measurable \( V \), and \( \mu \) being a general bounded Borel measure. As to the non-local Schrödinger operators, Diaz, Gómez-Castro and Vazquez proved in \[18\] existence results for (1.1) with \( A = \Delta^{\alpha/2}, \nu(dx) = V(x) \, dx, V \in L^{1,+}_{loc}(D; m), \) and \( \mu \in L^1(D; \delta_D^{\alpha/2} \cdot m) \). Next, Gómez-Castro and Vazquez in \[25\] generalized results of \[18\] by considering more relaxed class of potentials:
\[L^{1,+}_{loc}(D) + \{ g \in B^+(D) : \exists_{\text{finite } Y \subset D} \forall_{y \in Y} g \in L^\infty(D \backslash \bigcup_{x \in Y} B(x, r)) \text{ for any } r > 0 \} \cdot (1.15)\]
They also considered bounded Borel measures \( \mu \) in (1.1). It is an elementary calculation that for a potential \( V \) belonging to (1.15), we have \( V(x) \, dx \in S_A \). However, the general case, which we are interested in the paper, i.e. equations of the form (1.1) with smooth measure \( \nu \) and general (not necessarily smooth) measure \( \mu \) were until now considered only by Malusa and Orsina [37] in the case where \( A \) is a uniformly elliptic divergence form operator, and under additional condition that \( \mu \) has compact support.

The main goal of the present paper is twofold. First, we give a necessary and sufficient condition for the existence of a solution to the Schrödinger equation (1.1) in our general framework, thus we recover and give a far-reaching generalization of the existence results of [18, 25, 37, 40, 42]. Secondly, we provide a unified method, based on the probabilistic potential theory, to investigate Schrödinger equations of type (1.1) with general measure data. Of course, probabilistic methods based on Feynman-Kac formula have been applied successfully to Schrödinger equations in the past (see e.g. [2]), however in vast majority of the papers \( \mu \) in (1.1) was assumed to be a smooth measure. This is due to the fact that only smooth measures are in Revuz duality with positive additive functionals, and this duality is crucial for the Feynman-Kac representation formulas.

2 Preliminary results

In this section, we make standing assumptions on the Dirichlet operator and the associated Dirichlet form considered in the paper. For the convenience of the reader, we also recall some basic facts from the potential theory and the probabilistic potential theory.

2.1 Dirichlet forms and potential theory

In the whole paper, we assume that \((A, D(A))\) is a non-positive self-adjoint operator on \(L^2(E; m)\) generating a strongly continuous Markov semigroup of contractions \((T_t)_{t \geq 0}\) on \(L^2(E; m)\). It is well known (see [20, Section 1]) that there exists a unique symmetric Dirichlet form \((E, D(E))\) on \(L^2(E; m)\) such that \(D(A) \subset D(E)\) and

\[
E(u, v) = (-Au, v), \quad u \in D(A), v \in D(E).
\]

We denote by \((J_\alpha)_{\alpha > 0}\) the resolvent generated by \(A\). We assume that \((E, D(E))\) is transient and regular, i.e. there exists a strictly positive bounded function \(g\) on \(E\) such that

\[
\int_E |u| g \, dm \leq \sqrt{E(u, u)}, \quad u \in D(E),
\]

and \(D(E) \cap C_c(E)\) is dense in \(C_c(E)\) and in \(D(E)\) with natural topologies on these spaces. Since \(E\) is transient, there exists an extension \(D_e(E) \subset L^1(E; g \cdot m)\) of the domain \(D(E)\) such that the pair \((E, D_e(E))\) is a Hilbert space. For an open \(U \subset E\) we set

\[
\text{Cap}(U) = \inf \{E(u, u) : u \geq 1_U \text{ m-a.e., } u \in D(E)\}.
\]

and then, for arbitrary \(B \subset E\), we set \(\text{Cap}(B) = \inf \text{Cap}(U)\), where the infimum is taken over all open subsets of \(E\) such that \(B \subset U\). We say that a property \(P\) holds \((E)\)-q.e. if it holds except a set of capacity Cap zero.
We say that a function $u$ on $E$ is $(\mathcal{E})$-quasi-continuous if for every $\varepsilon > 0$ there exists a closed set $F_\varepsilon \subset E$ such that $\text{Cap}(E \setminus F_\varepsilon) \leq \varepsilon$ and $u|_{F_\varepsilon}$ is continuous. By [20] Theorem 2.1.3, each function $u \in D_0(\mathcal{E})$ admits a quasi-continuous $m$-version. In the sequel, for $u \in D_0(\mathcal{E})$, we denote by $\tilde{u}$ its quasi-continuous $m$-version. By [20] Lemma 2.1.4 if for given $(\mathcal{E})$-quasi-continuous functions $u, v$ we have $u = v$, $m$-a.e., then $u = v$ (\mathcal{E})$-q.e.

We say that a positive Borel measure $\nu$ on $E$ is $(\mathcal{E})$-smooth if

(a) $\nu$ is absolutely continuous with respect to the capacity $\text{Cap}$ generated by the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$,

(b) there exists an increasing sequence $\{F_n\}$ of closed subsets of $E$ such that for any compact $K \subset E$,

$$\nu(F_n) < \infty, \quad n \geq 1, \quad \lim_{n \to \infty} \text{Cap}(K \setminus F_n) = 0.$$ 

A signed Borel measure $\nu$ on $E$ is smooth if its variation $|\nu|$ is smooth. A sequence $\{F_n\}$ satisfying condition (b) of the above definition is called a generalized nest.

Let $\mathcal{B}(E)$ (resp. $\mathcal{B}^+(E)$) denote the set of all Borel (resp. non-negative Borel) measurable functions on $E$. We adopt the following notation: for a positive Borel measure $\mu$ on $E$ and $f \in \mathcal{B}^+(E)$ we set

$$\langle \mu, f \rangle = \int_E f \, d\mu,$$

and we denote by $f \cdot \mu$ the Borel measure on $E$ such that

$$\langle f \cdot \mu, \eta \rangle = \langle \mu, f \eta \rangle, \quad \eta \in \mathcal{B}^+(E).$$

If $\mu = m$, we write $\langle f, \eta \rangle = \langle f \cdot m, \eta \rangle$. By $\mathcal{M}_1$ we denote the set of Borel measures $\mu$ on $E$ for which $\langle |\mu|, 1 \rangle < \infty$. We say that $\mu$ is a bounded Borel measure if $\mu \in \mathcal{M}_1$.

Let $\nu$ be a positive smooth measure. We set

$$D(\mathcal{E}_\nu) = D(\mathcal{E}) \cap L^2(E; \nu), \quad \mathcal{E}_\nu(u, v) = \mathcal{E}(u, v) + \langle \tilde{u} \cdot \nu, \tilde{v} \rangle, \quad u, v \in D(\mathcal{E}_\nu).$$

By [38], Theorem 4.6, $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ is a quasi-regular symmetric Dirichlet form on $L^2(E; \nu)$. By [38] Corollary 2.10, there exists a unique non-positive self-adjoint operator $(A_\nu, D(A_\nu))$ such that $D(A_\nu) \subset D(\mathcal{E}_\nu)$ and

$$\mathcal{E}_\nu(u, v) = (-A_\nu u, v), \quad u \in D(A_\nu), v \in D(\mathcal{E}_\nu).$$

We denote by $(J_\nu^{(t)})_{t \geq 0}$ the resolvent generated by $-A_\nu$, and by $(T_\nu^{(t)})_{t \geq 0}$ the strongly continuous Markov semigroup of contractions generated by $-A_\nu$.

2.2 Probabilistic potential theory

Let $\Delta$ be a one-point compactification of $E$, in case $E$ is not compact, or an isolated point, in case $E$ is compact. Let $\mathcal{D}$ denote the set of all functions $\omega : [0, \infty) \to E \cup \{\Delta\}$, that are right continuous and possess the left limits for all $t \geq 0$ (càdlàg), and have the property that if $\omega(t) = \Delta$, then $\omega(s) = \Delta$, $s \geq t$. We equip $\mathcal{D}$ with the Skorokhod topology, see e.g. Section 12 of [38]. Define the canonical process: $X : \mathcal{D} \to \mathcal{D}$, $X_t(\omega) := \omega(t)$, $\omega \in \mathcal{D}$, shift operator: $\theta_s : \mathcal{D} \to \mathcal{D}$, $\theta_s(\omega)(t) = \omega(t + s)$, and life time:
Observe that 

\[ \zeta : \mathcal{D} \to [0, \infty], \zeta(\omega) := \inf \{ t > 0 : X_t(\omega) = \Delta \}. \]

By [20] Theorem 7.2.1, there exists a Hunt process

\[ X = ((P_x)_{x \in E \cup \{\Delta\}}, \mathcal{F} = (\mathcal{F}_t)_{t \geq 0}) \]

- here \( P_x \) is a probability measure on \( \mathcal{D} \) for fixed \( x \in E \) (\( P_{\Delta} = \delta(\Delta) \)), \( \mathcal{F} \) is a filtration (non-decreasing sequence of \( \sigma \)-fields) - such that for all \( t \geq 0 \) and \( f \in \mathcal{B}(E) \cap L^2(E; m) \),

\[ T_t f(x) = E_x f(X_t) := \int_\mathcal{D} f(X_t(\omega)) \, dP_x(\omega) \quad \text{m.a.e. } x \in E, \]

with the convention that \( f(\Delta) = 0 \). For \( f \in \mathcal{B}^+(E) \), we put

\[ P_t f(x) = E_x f(X_t), \quad t \geq 0, \quad R_\alpha f(x) = E_x \int_0^\infty e^{-\alpha t} f(X_t) \, dt, \quad \alpha \geq 0, \]

and \( R = R_0 \). Recall that \( f \in \mathcal{B}^+(E) \) is an excessive function (with respect to \( X \)) if

\[ \sup_{\alpha > 0} \alpha R_\alpha f(x) = f(x), \quad x \in E. \]

If in the above condition \( R_\alpha \) is replaced by \( R_{\alpha+\beta} \) for some \( \beta > 0 \), then \( f \) is called \( \beta \)-excessive (with respect to \( X \)). We assume that \( X \) satisfies the absolute continuity condition, i.e. for any \( f \in \mathcal{B}_B^+(E) \),

\[ P_t f(x) = 0, \quad t > 0, \quad x \in E \quad \text{whenever} \quad \int_E f \, dm = 0. \]

By [20] Lemma 4.2.4] (by transiency of \( X \), [20] Lemma 4.2.4] holds for \( \alpha = 0 \), too) for any \( \alpha \geq 0 \) there exists \( r_\alpha \in \mathcal{B}^+(E \times E) \) such that \( r_\alpha(x,\cdot), r_\alpha(\cdot, y) \) is \( \alpha \)-excessive for fixed \( x, y \in E \), and

\[ R_\alpha f(x) = \int_E f(y) r_\alpha(x, y) \, m(dy), \quad x \in E, \ f \in \mathcal{B}^+(E). \]

We set \( G_\alpha = r_\alpha \), and \( G = G_0 \). We call \( G \) the Green function for \(-A\) (or \( X \)). For a given positive Borel measure \( \mu \) on \( E \), we set

\[ R_\alpha \mu(x) = \int_E G_\alpha(x, y) \, \mu(dy), \quad x \in E. \quad (2.1) \]

Observe that \( P_t(R_\alpha \mu) = \int_E P_t G_\alpha(\cdot, y) \, \mu(dy) \). Therefore, since \( G_\alpha(\cdot, y) \) is \( \alpha \)-excessive, for any positive \( \mu \) we have that \( R_\alpha \mu \) is \( \alpha \)-excessive as well. A Borel measure \( \mu \) on \( E \) is called strictly smooth (with respect to \( X \)) if it is \((\mathcal{E})\)-smooth and there exists an increasing sequence \( \{B_n\} \) of Borel subsets of \( E \) such that \( \bigcup_{n \geq 1} B_n = E \) and \( R(1_{B_n} \cdot |\mu|) \) is bounded for every \( n \geq 1 \).

From the definition of an excessive function it follows directly that under the absolute continuity condition for \( X \), if \( f \leq g \) m-a.e. for some excessive functions \( f, g \), then \( f \leq g \) on \( E \). We will use this property frequently in the paper without special mentioning.

We say that \( A \subset E \) is nearly Borel (with respect to \( X \)) if there exist \( B_1, B_2 \in \mathcal{B}(E) \) such that \( B_1 \subset A \subset B_2 \) and for every finite positive Borel measure \( \mu \) on \( E \),

\[ P_\mu(\exists t \geq 0 \ X_t \in B_2 \setminus B_1) = 0, \]
where \( P_\mu(d\omega) = \int_E P_x(d\omega) \mu(dx) \). By \( \mathcal{B}^n(E) \), we denote the class of all nearly Borel sets. It is clear that \( \mathcal{B}(E) \subset \mathcal{B}^n(E) \). For \( A \in \mathcal{B}^n(E) \), we set

\[
\sigma_A = \inf\{t > 0 : X_t \in A\}, \quad \tau_A = \sigma_{E \setminus A} \land \zeta.
\]

We say that a set \( A \subset E \) is polar (with respect to \( \mathcal{X} \)) if there exists \( B \in \mathcal{B}^n(E) \) such that \( A \subset B \) and

\[
P_x(\sigma_B < \infty) = 0, \quad x \in E.
\]

By [20, Theorem 4.1.2, Theorem 4.2.1], \( \text{Cap}(A) = 0 \) if and only if \( A \) is polar. A nearly Borel set \( D \) is an absorbing set (with respect to \( \mathcal{X} \)) if \( P_x(\tau_D = \zeta) = 1 \), \( x \in D \). Observe that if \( N \) is a polar set, then \( E \setminus N \) is an absorbing set.

Let \( T \) be the topology generated by the metric on \( E \). By \( T_f \) we denote the smallest topology on \( E \) for which all excessive functions (with respect to \( \mathcal{X} \)) are continuous. This topology is called in the literature the fine topology (with respect to \( \mathcal{X} \)). By [9, Section II.4], \( T \subset T_f \) and \( A \) is a finely open set if and only if for every \( x \in A \) there exists \( D \in \mathcal{B}^n(E) \) such that \( D \subset A \) and

\[
P_x(\tau_D > 0) = 1.
\]

In other words, starting from \( x \in A \), the process \( X \) spends some nonzero time in \( A \) until it exits \( A \). Observe that each polar set is finely closed. By [9, Theorem II.4.8], \( f \in \mathcal{B}^n(E) \) is finely continuous if and only if the process \( f(X) \) is right-continuous under the measure \( P_x \) for every \( x \in E \).

Let \( D \subset E \) be finely open. We denote by

\[
\mathcal{X}^D = ((P^D_x)_{x \in D \cup \{\Delta\}}, R^D = (\mathcal{F}^D_t)_{t \geq 0})
\]

the part of the process \( \mathcal{X} \) on \( D \) (see [20, Section A.2]). Its life time satisfies \( \zeta = \tau_D \). By [20, Theorem A.2.8], \( \mathcal{X}^D \) is again a Hunt process and

\[
P^D_t f(x) := E^D_x f(X_t) = E_x f(X_t) 1_{\{t < \tau_D\}}, \quad x \in D,
\]

where \( E^D_x \) denotes the expectation with respect to the measure \( P^D_x \). We also set

\[
R^D f(x) = E_x \int_0^{\tau_D} f(X_t) dt = \int_0^{\infty} P^D_t f(x) dt, \quad x \in D. \tag{2.2}
\]

From this formula, we deduce that if \( D \) is an absorbing set (with respect to \( \mathcal{X} \)), i.e. \( P_x(\tau_D = \zeta) = 1 \), \( x \in D \), then

\[
R^D f(x) = Rf(x), \quad x \in D. \tag{2.3}
\]

It is clear that \( \mathcal{X}^D \) satisfies the absolute continuity condition. Therefore there exists a Green function \( G^D \) for \( \mathcal{X}^D \). From (2.3) it follows that if \( D \) is an absorbing set, then

\[
G^D(x, y) = G(x, y), \quad x, y \in D. \tag{2.4}
\]

Observe that most of the notions introduced in Section 2.1, 2.2 depend on the form \( \mathcal{E} \) or the process \( \mathcal{X} \). To underline this in all the foregoing definitions, where this dependence occurs, we added \( \mathcal{E} \) or \( \mathcal{X} \) in parenthesis. In what follows all the notions depending on a form or a process are by default understood as the notions with respect to \( \mathcal{E} \) or \( \mathcal{X} \) (we do not indicate them in the sequel) unless it is stated otherwise.
3 PCAFs of $\mathbb{X}$ with minimal exceptional set

The notion of a positive continuous additive functional (PCAF) of a general Markov process was introduced by Revuz in [43]. This notion allowed the author to show in [43] a duality between a subclass of smooth measures and a class of increasing processes with additivity property. Unfortunately, the subclass of smooth measures considered by Revuz was too restrictive in many applications. It did not even cover the class of bounded smooth measures. To get the general duality, covering the class of all smooth measures, Fukushima (see [20]) and Silverstein (see [45]) extended the notion of PCAF (see Definition 3.1 below). The crucial ingredient of the extended definition is the notion of so-called exceptional set which depends on the particular PCAF. In the extended definition the desirable properties of the functional - additivity, continuity etc. - are satisfied under the measure $P_x$ for $x \notin N$. Thanks to the more general notion of PCAF one can get a one-to-one correspondence between the class of all smooth measures and PCAFs. Nowadays, PCAFs in the sense of Revuz are called strict PCAFs to distinguish them from PCAFs introduced by Fukushima and Silverstein. Smooth measures associated with strict PCAFs are called strictly smooth measures.

In the present section, we show that the exceptional set $N$ for a given PCAF can be chosen in a canonical way and that this choice is in some sense minimal. In the special case when $\mathbb{X}$ is a Brownian motion, our result follows from the paper by Baxter, Dal Maso and Mosco [6] (see also [48]).

In what follows, we say that some property holds a.s. if it holds $P_x$-a.s. for every $x \in E$.

Definition 3.1. We say that an $\mathcal{F}$-adapted process $A = (A_t)_{t \geq 0}$ is a positive continuous additive functional of $\mathbb{X}$ (PCAF) if there exists a polar set $N$ and $\Lambda \in \mathcal{F}_\infty$ such that

(a) $P_x(\Lambda) = 1$, $x \in E \setminus N$,
(b) $P_x(A_0 = 0)$, $x \in E \setminus N$, and $A_t(\omega) \geq 0$, $t \geq 0$, $\omega \in \Lambda$,
(c) $\theta_t(\Lambda) \subset \Lambda$, $t > 0$, and for every $\omega \in \Lambda$, $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_s\omega)$, $s,t \geq 0$,
(d) $A_t(\omega) < \infty$, $t < \zeta(\omega)$, $\omega \in \Lambda$, and $A_t(\omega) = A_{\zeta(\omega)}(\omega)$, $t \geq \zeta(\omega)$, $\omega \in \Lambda$,
(e) $[0,\infty) \ni t \mapsto A_t(\omega)$ is continuous for every $\omega \in \Lambda$.

The set $N$ is called an exceptional set for $A$ and $\Lambda$ is called a defining set for $A$. If $N = \emptyset$, then $A$ is called a strict PCAF of $\mathbb{X}$. Notice that $A$ is a PCAF of $\mathbb{X}$ if and only if $A$ is a strict PCAF of $\mathbb{X}^{E \setminus N}$.

The one-to-one correspondence between PCAFs $A$ of $\mathbb{X}$ and positive smooth measures $\nu$ is characterized by the relation

$$\lim_{t \to 0^+} \frac{1}{t} E_m \int_0^t f(X_t) \, dA_t = \langle \nu, f \rangle, \quad f \in \mathcal{B}_+^+(E).$$

In the literature it is often called the Revuz duality. If $A$ is a strict PCAF, then under our assumption of absolute continuity of $\mathbb{X}$, the above relation may be expressed equivalently as

$$E_x \int_0^\zeta f(X_t) \, dA_t = \int_E G(x,y) f(y) \nu(dy), \quad x \in E,$$
for all $f \in \mathcal{B}^+(E)$.

By [20, Theorem 5.1.7], for every positive smooth measure $\nu$ such that $R\nu$ is bounded, there exists a strict PCAF $A$ of $\mathbb{X}$ in the Revuz duality with $\nu$. Let $\nu$ be a positive smooth measure. By [20, Theorem 2.2.4], there exists a generalized nest $\{F_n\}$ of closed subsets of $E$ such that $R\nu_n \leq n$, $n \geq 1$, and $\nu_n \to \nu$, where $\nu_n = 1_{F_n} \cdot \nu$ (the convergence follows from the fact that $\{F_n\}$ is a generalized nest).

In what follows we adopt the following notation. For given $\alpha \geq 0$, $f \in \mathcal{B}^+(E)$, and non-negative $\mathbb{F}$-adapted càdlàg process $Y$, we let

$$\phi_{Y}^{\alpha,f}(x) = E_x \int_0^\zeta e^{-at}f(X_t)e^{-Y_t} \, dt, \quad x \in E. \quad (3.1)$$

We also set $\phi_{Y}^{\alpha} := \phi_{Y}^{\alpha,1}$.

**Proposition 3.2.** Let $\nu$ be a positive smooth measure on $E$ and let

$$E_{\nu} = \{x \in E : \exists V_x \text{-finely open neighborhood of } x \text{ such that } \int_{V_x} G(x,y) \nu(dy) < \infty\}. \quad (3.2)$$

Let $\{\nu_n\}$ be a sequence of positive strictly smooth measures such that $\nu_n \not\to \nu$, and for $n \geq 1$ let $A^n$ be a strict PCAF of $\mathbb{X}$ in the Revuz correspondence with $\nu_n$. Set $A_t := \sup_{n \geq 1} A^n_t$, $t \geq 0$. Set $N_{\nu} := E \setminus E_{\nu}$. Then

(i) $\phi_{A}^{\alpha,f}$ is finely continuous for any $\alpha > 0$ and $f \in \mathcal{B}^+_b(E)$,

(ii) If $f \in \mathcal{B}^+(E)$ and $Rf$ is finite, then $\phi_{A}^{0,f}$ is finely continuous.

(iii) The process $A$ is a PCAF of $\mathbb{X}$ with the exceptional set $N_{\nu}$.

(iv) For all $x \in E_{\nu}$ and $\eta \in \mathcal{B}^+(E)$,

$$E_x \int_0^\zeta \eta(X_t) \, dA_t = \int_E G(x,y) \eta(y) \nu(dy). \quad (3.3)$$

(v) For any $\alpha \geq 0$, $E_{\nu} = \{\phi_{A}^{\alpha} > 0\}$.

(vi) For every $x \in N_{\nu}$, $P_x(A_t = \infty, t > 0) = 1$.

**Proof.** (i) Set

$$u_{\alpha}^n(x) = E_x \int_0^\zeta e^{-at}f(X_t)(1 - e^{-A^n_t}) \, dt, \quad x \in E. \quad (3.4)$$

Clearly, $(u_{\alpha}^n)_{n \geq 1}$ is nondecreasing. Set $u_{\alpha} = \sup_{n \geq 1} u_{\alpha}^n$. By the strong Markov property and additivity of $A^n$,

$$e^{-at}P_t u_{\alpha}^n(x) = E_x \int_t^\zeta e^{-as}f(X_s)(1 - e^{-(A^n_s - A^n_t)}) \, ds, \quad x \in E. \quad (3.5)$$

From this we easily deduce that $u_{\alpha}^n$ is $\alpha$-excessive for any $\alpha > 0, n \geq 1$. Hence, by [17, 4.d), page 222], $u_{\alpha}$ is $\alpha$-excessive. Observe that

$$\phi_{A}^{\alpha,f}(x) = u_{\alpha}^n(x) - R_\alpha f(x), \quad x \in E.$$
Letting \( n \to \infty \), we get

\[
\phi_A^{\alpha,f}(x) = u_\alpha(x) - R_\alpha f(x), \quad x \in E.
\]

Since \( u_\alpha, R_\alpha f \) are finely continuous (as excessive functions) and finite on \( E \), we conclude from the above equation that \( \phi_A^{\alpha,f} \) is finely continuous for any \( \alpha > 0 \). For (ii), we let \( \alpha \to 0^+ \) in the above equation. We thus get

\[
\phi_A^{0,f}(x) = u_0(x) - Rf(x), \quad x \in E.
\]

Clearly, \( u_0, Rf \) are excessive functions, and hence finely-continuous. Since \( Rf \) is finite, \( u_0 - Rf \) is finely continuous. For (iii), observe first that \( \phi_A^{0}(x) > 0, \ x \in E_\nu \). Indeed,

\[
\int_{V_x} G(x,y) \nu(dy) = \lim_{n \to \infty} \int_E 1_{V_x}(y) G(x,y) \nu^n(dy)
\]

\[
= \lim_{n \to \infty} E_x \int_0^\zeta 1_{V_x}(X_t) dA_t^n \geq \lim_{n \to \infty} E_x A_t^n \varpi \zeta = E_x A_t \varpi \zeta = E_x A_{\tau_{V_x}}.
\]

(3.4)

Since \( V_x \) is finely open, \( P_x(\tau_{V_x} > 0) = 1 \). From this and the definitions of \( E_\nu \) and \( \phi_A^{0} \) we deduce that \( \phi_A^{0} \) is strictly positive on \( E_\nu \). By [20, Lemma 5.1.5(ii)], we have\( E_x \int_0^\zeta e^{-t} \phi_A^k(X_t) dA_t^k \leq 1, \ x \in E, \) so \( E_x \int_0^\zeta e^{-t} \phi_A^k(X_t) dA_t^k \leq 1, \ x \in E \) for \( k \geq 1 \).

Letting \( k \to \infty \), we obtain

\[
E_x \int_0^\zeta e^{-t} \phi_A^k(X_t) dA_t^k \leq 1, \quad x \in E, \quad l \geq 1.
\]

(3.5)

Since \( \phi_A^k \) is finely continuous and strictly positive on \( E_\nu \), and \( N_\nu \) is polar, we conclude from \([35]\) that \( A_t < \infty, t < \zeta, \ P_x\text{-a.s. for } x \in E_\nu \). Hence, by \([38] \) Lemma 1, page 182) [see also \([33] \) Lemma 1.1] applied to the sequence \( \{A^n\} \) regarded as a sequence of strict PCAFs of \( X^{E_\nu} \), we get that \( A \) is a strict PCAF of \( X^{E_\nu} \), which is equivalent to the statement that \( A \) is a PCAF of \( X \) with the exceptional set \( N_\nu \). This proves (iii).

For \( x \in E_\nu \), and \( \eta \in B^+(E) \), we have

\[
E_x \int_0^\zeta \eta(X_t) dA_t = \lim_{n \to \infty} E_x \int_0^\zeta \eta(X_t) dA_t^n
\]

\[
= \lim_{n \to \infty} \int_E \eta(y) G(x,y) \nu^n(dy) = \int_E \eta(y) G(x,y) \nu(dy).
\]

This completes the proof of (iv). By \([34]\), \( E_\nu \subset \{ \phi_A^\alpha > 0 \} \) for any \( \alpha \geq 0 \). Suppose that \( \phi_A^\alpha(x_0) > 0 \) for some \( x_0 \in E \). Then there exists \( c > 0 \) such that \( \phi_A^\alpha(x_0) > c \). By \([41] \) Corollary 1.3.6], there exists a strictly positive bounded Borel measurable function \( g \) such that \( Rg \) is bounded. Let \( c_1 = \sup_{x \in E} Rg(x) \). By the Lebesgue monotone convergence theorem

\[
\phi_A^{1/n, g_n}(x) \uparrow \phi_A^0(x), \ x \in E,
\]

where \( g_n = ng/(1+ng), \ n \geq 1 \). Thus, there exists \( n_0 \) such that \( \phi_A^{1/n_0, g_{n_0}}(x_0) > c \). Write \( V_{x_0} = \{ \phi_A^{1/n_0, g_{n_0}}(x_0) > c \} \). Of course \( x_0 \in V_{x_0} \), and since \( \phi_A^{1/n_0, g_{n_0}} \) is finely continuous (by (i)), \( V_{x_0} \) is finely open. By \([20] \) Lemma 5.1.5(ii)],

\[
R_{1/n_0 g_{n_0}}(x) \geq E_x \int_0^\zeta \phi_A^{1/n_0, g_{n_0}}(X_t) dA_t^l, \quad x \in E, \quad l \geq 0.
\]

(3.6)
So, as in the case of (3.1), we get

\[ n_0c_1 \geq R_{1/n_0}g_{n_0}(x) \geq E_x \int_0^c \phi_A^{1/n_0} g_{n_0}(X_t) \, dA_t^l, \quad x \in E, \, l \geq 1. \]

Hence

\[ \frac{n_0c_1}{c} \geq E_x \int_0^c 1_{V_{n_0}}(X_t) \, dA_t^l = \int_{V_{n_0}} G(x_0, y) \nu^l(dy), \quad l \geq 1. \]

From this we conclude that \( x_0 \in E_\nu \). Thus, \( \{ \phi_A^\alpha > 0 \} \subset E_\nu \), which combined with the reverse inequality proved above implies \((v)\). \((vi)\) is a direct consequence of \((v)\). \qed

**Corollary 3.3.** Let \( \nu \) be a positive smooth measure on \( E \) and \( B \) a PCAF of \( X \) with exceptional set \( N \subset N_\nu \) such that \( B \) is in the Revuz correspondence with \( \nu \) and \( \phi_B^1 \) (cf. (3.3)) is finely continuous. Then \( N = N_\nu \), and for every \( x \in E \), \( P_x(A_t = B_t, t > 0) = 1 \), where \( A \) is the PCAF constructed in Proposition 3.2.

**Proof.** Since \( A \) and \( B \) satisfy (3.3) for every \( x \in E_\nu \), applying [3] Proposition IV.2.12 yields \( P_x(A_t = B_t, t > 0) = 1 \), \( x \in E_\nu \). This implies that \( \phi_A^1 = \phi_B^1 \) on \( E_\nu \). Since \( \phi_A^1 \) and \( \phi_B^1 \) are finely continuous, in fact \( \phi_A^1 = \phi_B^1 \). Thus \( P_x(B_t = \infty, t > 0) = 1 \), \( x \in N_\nu \), which implies that \( N = N_\nu \) and \( P_x(A_t = B_t, t > 0) = 1 \), \( x \in E \). \qed

**Corollary 3.4.** Define \( E_\nu^\alpha \) by (3.2) but with \( G \) replaced by \( G_{\alpha} \). Then for every \( \alpha > 0 \), \( E_\nu^\alpha = E_\nu \).

From now on, for a given smooth measure \( \nu \), we denote by \( A^\nu \) the PCAF of \( X \) with exceptional set \( N_\nu \) constructed in Proposition 3.2.

We close this section with a remark concerning the case when \( X \) is a Brownian motion. In that case Baxter, Dal Maso and Mosco [6] and Sturm [45] have shown a duality between positive Borel measures absolutely continuous with respect to Newtonian capacity (i.e. satisfying only condition (a) of the definition of a smooth measure) and so called positive additive functionals (PAFs) of \( X \), i.e. \( \mathcal{F} \)-adapted right continuous processes \( A : \Omega \times [0, \infty) \rightarrow [0, \infty] \) such that

(a) \( \forall s,t \geq 0 \quad A_{t+s} = A_t + A_s \circ \theta_t \) a.s.,

(b) \( \forall s \geq 0 \quad t \mapsto A_{s-t} \circ \theta_t \) is a.s. right-continuous on \([0, s]\).

Observe that for given smooth measure \( \nu \), we have that \( \hat{A}_{t}^\nu := A_{t+}^\nu, \, t \geq 0 \) is a PAF of \( X \) in Revuz duality with \( \nu \) considered in [6] and [45]. So, in other words, \( \hat{A}^\nu \) is the unique PAF of \( X \) associated with \( \nu \).

## 4 Duality solutions to Schrödinger equations

Let \( \nu \) be a positive smooth measure on \( E \) and \((\mathcal{E}^\nu, D(\mathcal{E}^\nu))\) be a quasi-regular Dirichlet form on \( L^2(E;m) \) being the perturbation of a regular Dirichlet form \((\mathcal{E}, D(\mathcal{E}))\) by a measure \( \nu \). We denote by \(-A + \nu\) the self-adjoint operator on \( L^2(E;m) \) generated by \((\mathcal{E}^\nu, D(\mathcal{E}^\nu))\), and by \((T_t^\nu)_{t \geq 0}\) the Markov semigroup of contractions on \( L^2(E;m) \) generated by \(-A + \nu\). The resolvent determined by \((T_t^\nu)_{t \geq 0}\) shall be denoted by \((J_t^\nu)_{t > 0}\).

In [37] the Schrödinger equation (1.1) with \( A \) defined by (1.9), and \( \mu \) having compact support is considered. Let \( a \) be a symmetric matrix-valued bounded Borel measurable
function on a bounded domain $D$ such that $\lambda I \leq a$ for some $\lambda > 0$. In [37, Definition 5.4] the following definition of a solution is adopted: $u \in L^1(D;m)$ is a duality solution to (1.1) in the sense of [37] if

$$\langle u, \eta \rangle = \langle \mu, \hat{\zeta}_\eta \rangle, \quad \eta \in L^\infty(D;m),$$

where $\zeta_\eta$ is the unique minimizer of the energy functional

$$E(u) = \frac{1}{2} \int_D |\sigma \nabla u|^2 \, dm + \frac{1}{2} \int_D \bar{u}^2 \, d\nu - \int_D u \eta \, dm,$$

$\sigma \cdot \sigma^* = a$, and $\hat{\zeta}_\eta$ is defined as

$$\hat{\zeta}_\eta(x) = \lim_{r \to 0^+} \frac{1}{|B(x,r)|} \int_{B(x,r)} \zeta_\eta(y) \, dy, \quad x \in D.$$

The limit above is well defined in each point $x \in D$ because $\zeta_\eta$ is a difference of bounded superharmonic functions, so each point in $D$ is a Lebesgue point for $\zeta_\eta$. This definition is a generalization of the notion of solution considered in [14, 15] in case $A$ is defined by (1.9) and the measure $\mu$ belongs to $H^{-1}(D)$. Under this additional assumption on $\mu$, $u$ is a duality solution to (1.1) if and only if $u \in H^1_0(D) \cap L^2(D;\nu)$, see [37, Remark 5.5].

The goal of this section is to extend the notion of duality solutions to Schrödinger equation (1.1) with general Dirichlet operator $A$. Before giving the rigorous definition, some remarks are in order. Observe that in fact, $\zeta_\eta = J^\nu_\eta$, $m$-a.e., so the key in the definition of duality solution is to give a proper pointwise meaning to the function $J^\nu_\eta$. For general $A$, we can no longer apply the notion of Lebesgue’s points. Instead, we use the notion of Green’s function. The problem is that without additional assumptions on the measure $\nu$, in general there is no Green function $G^\nu$ for the operator $-A + \nu$ on the whole $E$. To overcome this difficulty, we use the fact that there always exists a Green function $G^{E_\nu,\nu}$ on $E_\nu$ (cf. (3.2)) for the operator $(-A + \nu)|_{E_\nu}$ being the restriction of $-A + \nu$ to $E_\nu$, and then we show how to extend $G^{E_\nu,\nu}$ in a canonical way to the whole $E$.

### 4.1 Existence and uniqueness of duality solutions

By [20, Theorem A.2.11], there exists a Hunt process

$$\mathfrak{X}^\nu = ((P^\nu_x)_{x \in E \cup \{\Delta\}}, \mathbb{F}^\nu := \{\mathcal{F}^\nu_t, t \geq 0\})$$

associated with the form $(\mathcal{E}^\nu, D(\mathcal{E}^\nu))$ in the sense that for every $f \in \mathcal{B}(E) \cap L^2(E;m)$,

$$T^\nu_t f(x) = E^\nu_x f(X_t) \quad m\text{-a.e. } x \in E. \quad (4.2)$$

We set

$$P^\nu_t f(x) = E^\nu_x f(X_t), \quad R^\nu_t f(x) = E^\nu_x \int_0^t f(X_s) \, ds, \quad x \in E,$$
where $E^\nu_x$ stands for the expectation with respect to $P^\nu_x$. By the construction of the process $X^\nu$,

$$R^\nu f(x) = E^\nu_x \int_0^\zeta e^{-A^\nu_t} f(X_t) \, dt, \quad x \in E^\nu. \quad (4.3)$$

Since $A^\nu$ is a strict PCAF of $X^{E^\nu}$, by [20, Exercise 6.1.1] the process $X^{E^\nu,\nu}$ satisfies the absolute continuity condition. Therefore there exists a Green function $G^{E^\nu,\nu}$ on $E^\nu \times E^\nu$, i.e. for every $\eta \in \mathcal{B}^+(E)$,

$$E^\nu_x \int_0^\zeta \eta(X_t) \, dt = \int_{E^\nu} G^{E^\nu,\nu}(x,y) \eta(y) \, m(dy), \quad x \in E^\nu. \quad (4.4)$$

Moreover, by [20, Exercise 6.1.1] again,

$$G^{E^\nu,\nu}(x,y) + \int_{E^\nu} G^{E^\nu}(x,z) G^{E^\nu,\nu}(z,y) \nu(dz) = G^{E^\nu}(x,y), \quad x, y \in E^\nu. \quad (4.5)$$

Observe that by (2.2) and (4.3),

$$E^\nu_x \int_{\tau_{E^\nu}}^\zeta \eta(X_t) \, dt = E^\nu_x \int_0^\zeta \eta(X_t) \, dt, \quad x \in E^\nu. \quad (4.6)$$

From this and (4.4), (4.6), we conclude that for $f \in \mathcal{B}^+(E)$,

$$R^\nu f(x) = \int_{E^\nu} G^{E^\nu,\nu}(x,y) f(y) \, m(dy) = R^{E^\nu,\nu} f(x), \quad x \in E^\nu. \quad (4.8)$$

Observe also, that by (4.5) and the symmetry of $G^{E^\nu}, G^{E^\nu,\nu}$, we have that for any positive smooth measures $\nu, \beta$,

$$R^\nu(R^\nu \beta \cdot \nu) = R(R^\nu \beta \cdot \nu), \quad \text{on } E^\nu. \quad (4.9)$$

A careful look at the construction of the process $X^\nu$ (see the comments before [20, Theorem 6.1.1]) reveals that

$$R^\nu f(x) = \infty \cdot f(x), \quad x \in N^\nu.$$

We set

$$\hat{R}^\nu f(x) = 1_{E^\nu} R^\nu f(x), \quad x \in E.$$

By (4.3) and Proposition 3.2

$$\hat{R}^\nu f(x) = E^\nu_x \int_0^\zeta e^{-A^\nu_t} f(X_t) \, dt, \quad x \in E. \quad (4.10)$$

Define

$$\hat{G}^\nu(x,y) = \begin{cases} G^{E^\nu,\nu}(x,y), & (x,y) \in E^\nu \times E^\nu, \\ 0, & (x,y) \in (N^\nu \times E) \cup (E \times N^\nu). \end{cases} \quad (4.11)$$
With the above notation, we have
\[
\tilde{R}^\nu f(x) = \int_E \tilde{G}^\nu(x, y) f(y) \, m(dy), \quad x \in E.
\]
We can now extend \(\tilde{R}^\nu\) to an arbitrary positive Borel measure \(\mu\) on \(E\) by putting
\[
\tilde{R}^\nu \mu(x) = \int_E \tilde{G}^\nu(x, y) \mu(dy), \quad x \in E. \quad (4.12)
\]
Of course, if \(\nu\) is a strictly smooth measure, then \(R^\nu = R^\nu\) and \(G^\nu = G^\nu\) since \(N_\nu = \emptyset\). Observe also that
\[
\tilde{R}^\nu \leq R^\nu \leq R. \quad (4.13)
\]
By analogous arguing to that following (2.1), we get that for any positive \(\mu\), \(\tilde{R} \mu\) is excessive with respect to \(\mathbb{R}^{E_\nu, \nu}\).

**Definition 4.1.** We say that \(u \in \mathcal{B}(E)\) is a duality solution to (1.1) if for every \(\eta \in \mathcal{B}(E)\) such that \(R|\eta|\) is bounded we have
\[
\langle u, \eta \rangle = \langle \mu, \tilde{R}^\nu \eta \rangle. \quad (4.14)
\]

**Remark 4.2.** Observe that both integrals in (4.14) are well defined. Indeed, we have
\[
\langle |\mu|, |\tilde{R}^\nu \eta| \rangle \leq \langle |\mu|, R|\eta| \rangle \leq ||\mu||_{TV} ||R|\eta||_\infty.
\]
Moreover,
\[
\langle |u|, |\eta| \rangle = \langle u, \text{sgn}(u)|\eta| \rangle = \langle \mu, \tilde{R}^\nu (\text{sgn}(u)|\eta|) \rangle \leq ||\mu||_{TV} ||R|\eta||_\infty.
\]
Note also that thanks to the assumption that \((\mathcal{E}, D(\mathcal{E}))\) is transient there exists a strictly positive Borel function \(\eta\) on \(E\) such that \(R\eta\) is bounded (see [41, Corollary 1.3.6]).

**Theorem 4.3.** Assume that \(\nu\) is a positive smooth measure on \(E\) and \(\mu\) is a bounded Borel measure on \(E\). Then there exists a unique duality solution \(u\) to (1.1). Moreover, there exists a quasi-continuous function \(\tilde{u}\) which is an \(m\)-version of \(u\) and
\[
\tilde{u}(x) = \int_E \tilde{G}^\nu(x, y) \mu(dy) \quad \text{q.e.} \quad (4.15)
\]

**Proof.** By [29, Proposition 3.2], for every bounded Borel measure \(\mu\), \(R|\mu|\) is finite q.e. Since \(\tilde{R}^\nu|\mu| \leq R|\mu|\), we see that \(\tilde{R}^\nu|\mu|\) is finite q.e. Let \(\tilde{u}(x) = \tilde{R}^\nu \mu(x)\) for \(x \in E\) such that \(R|\mu|(x) < \infty\) and \(\tilde{u}(x) = 0\) otherwise. \(\tilde{R}^\nu \mu^+\) and \(\tilde{R}^\nu \mu^-\) are excessive functions with respect to \(\mathbb{X}^{E_\nu, \nu}\) (see the comment following (4.13)), so they are finely continuous with respect to \(\mathbb{X}^{E_\nu, \nu}\) (see Section 2.2). In particular, by (2.2) processes \(e^{-A^\nu} \tilde{R}^\nu \mu^+(X)\), \(e^{-A^\nu} \tilde{R}^\nu \mu^-(X)\) are right-continuous under measure \(P_x\) for every \(x \in \mathcal{E}_\nu\). Thus, by continuity of \(A^\nu\) and the definition of \(E_\nu\), \(\tilde{R}^\nu \mu^+(X)\), \(\tilde{R}^\nu \mu^-(X)\) are right-continuous under measure \(P_x\) for every \(x \in \mathcal{E}_\nu\). By [29, Theorem 4.6.1], \(\tilde{R}^\nu \mu^+, \tilde{R}^\nu \mu^-\) are quasi-continuous. Since \(\tilde{u} = \tilde{R}^\nu \mu^+ - \tilde{R}^\nu \mu^-\) q.e., we get that \(\tilde{u}\) is quasi-continuous too. Let \(\eta \in \mathcal{B}(E)\) be a function such that \(R|\eta|\) is bounded. Then
\[
\langle \tilde{u}, \eta \rangle = \langle \tilde{R}^\nu \mu^+, \eta \rangle - \langle \tilde{R}^\nu \mu^-, \eta \rangle = \langle \mu^+, \tilde{R}^\nu \eta \rangle - \langle \mu^-, \tilde{R}^\nu \eta \rangle = \langle \mu, \tilde{R}^\nu \eta \rangle.
\]
Therefore \(\tilde{u}\) is a duality solution to (1.1) and of course (4.15) is satisfied. Now, suppose that \(u, w\) are duality solutions to (1.1). Let \(\eta\) be a strictly positive Borel function on \(E\) such that \(R\eta\) is bounded. Then, by (4.14), \(\langle u - w, \text{sgn}(u - w) \eta \rangle = 0\). This implies that \(\langle |u - w|, \eta \rangle = 0\), hence that \(u = w\) m-a.e. \(\square\)
To show that for $A$ defined by (1.9) duality solutions coincide with solutions defined by (4.1) we will need the following proposition which is a generalization of [37, Theorem 5.2].

**Proposition 4.4.** For every $\eta \in \mathcal{B}^+(E)$ such that $R\eta$ is bounded there exists a unique positive smooth measure $\gamma_\eta$ such that

$$\check{R}^\nu \eta + R\gamma_\eta = R\eta.$$

Moreover, $\gamma_\eta = (R^\nu \eta) \cdot \nu$.

**Proof.** Multiplying both sides of (4.5) by $\eta$ and integrating with respect to $y$ over $E_\nu$ yields

$$R^{E_\nu \nu} \eta(x) = R^{E_\nu \eta}(x) - R^{E_\nu}((R^{E_\nu \nu} \eta) \cdot \nu)(x), \quad x \in E_\nu.$$

By (4.12), $R^{E_\nu \nu} \eta(x) = \check{R}^\nu \eta(x)$, $x \in E_\nu$. By (2.3), $R^{E_\nu} \eta(x) = R\eta(x)$, $x \in E_\nu$. By (2.4) and (4.8),

$$R^{E_\nu}((R^{E_\nu \nu} \eta) \cdot \nu)(x) = \int_{E_\nu} R^{E_\nu} \eta(y) G(x, y) \nu(dy) = R((R^\nu \eta) \cdot \nu)(x), \quad x \in E_\nu.$$

In the last equation we have used the fact that $\nu$ is smooth, which implies that $\nu(N_\nu) = 0$. By the above equations,

$$\check{R}^\nu \eta(x) + R\gamma_\eta(x) = R\eta(x), \quad x \in E_\nu,$$

with $\gamma_\eta = R^\nu \eta \cdot \nu$. Since $\check{R}^\nu \eta, R\gamma_\eta, R\eta$ are finely continuous (see Proposition 3.2(ii)) and $N_\nu$ is polar, we get the desired result.

**Corollary 4.5.** Let $a$ be a symmetric matrix-valued bounded Borel measurable function on a bounded domain $D$ such that $\lambda I \leq a$ for some $\lambda > 0$, and let $A$ be defined by (1.9).

Then $u$ is a duality solution to (1.1) if and only if $u$ satisfies (1.1).

**Proof.** Follows from Proposition 4.4 and [37, Theorem 5.2].

### 4.2 Duality solutions vs. variational solutions and stability results

We start with some stability results for duality solutions.

**Proposition 4.6.** Let $\nu$ be a positive smooth measure on $E$ and $\{\nu_n\}$ be a sequence of positive strictly smooth measures on $E$ such that $\nu_n \not\to \nu$. Then for every $\eta \in \mathcal{B}^+(E)$ such that $R\eta$ is bounded,

$$R^{\nu_n} \eta(x) \to R^\nu \eta(x), \quad x \in E.$$

**Proof.** By (1.3), for any $\eta$ as in the proposition, we have

$$R^{\nu_n} \eta(x) = E_x \int_0^\zeta e^{-A^{\nu_n} t} \eta(X_t) dt, \quad x \in E.
By our assumptions and Proposition 3.2, \( A_t^\nu \not
rightarrow A_t^\nu \), \( t \geq 0 \), \( P_x \cdot \text{a.s.} \) for every \( x \in E \). Hence
\[
E_x \int_0^\zeta e^{-A^\nu_t} \eta(X_t) \, dt \rightarrow E_x \int_0^\zeta e^{-A^\nu_t} \eta(X_t) \, dt, \quad x \in E.
\]
Therefore, by (4.10), we get the result. \( \square \)

**Proposition 4.7.** Let \( \nu \) be a positive smooth measure on \( E \) and \( \{ \nu_n \} \) be a sequence of positive strictly smooth measures on \( E \) such that \( \nu_n \not
rightarrow \nu \). Let \( u \) be a duality solution to (1.1) and \( u_n \) be a duality solution to (1.1) with \( \nu \) replaced by \( \nu_n \). Then, for every \( \rho \in B^+(E) \) such that \( R \rho \) is bounded, \( u_n \rightarrow u \) in \( L^1(E; \rho \cdot m) \).

**Proof.** Set \( u_n^\oplus = R^\nu u_n^+ \), \( u_n^\ominus = R^\nu u_n^- \). Then \( u_n^\oplus \), \( u_n^\ominus \) are excessive functions with respect to \( X^\nu \). Therefore, by (4.3),
\[
\alpha R^\nu u_n^\oplus(x) \leq \alpha R^\nu u_n^\ominus(x) \leq u_n^\ominus(x), \quad x \in E^\nu.
\]
This implies that \( \{ u_n^\ominus \} \) is a sequence of excessive functions with respect to \( X^\nu \). Analogously, we get that \( \{ u_n^\oplus \} \) is also a sequence of excessive functions with respect to \( X^\nu \). Since there exists the Green function for \( X^E X^\nu \), by [17, Lemma 94, page 306], there exists a subsequence (still denoted by \( (n) \)) such that \( \{ u_n^\ominus \}, \{ u_n^\oplus \} \) are convergent \( m \cdot \text{a.e.} \) Observe that
\[
|u_n| \leq R|\mu|, \quad m \cdot \text{a.e.}
\]
Hence, by the Lebesgue dominated convergence theorem, there exists \( u \in L^1(E; R \cdot m) \) such that \( \{ u_n \} \) converges to \( u \) in \( L^1(E; R \cdot m) \). As a consequence, for every \( \eta \in B(E) \) such that \( R|\eta| \) is bounded,
\[
\langle u_n, \eta \rangle \rightarrow \langle u, \eta \rangle.
\]
By Proposition 4.6
\[
\langle \mu, R^\nu \eta \rangle \rightarrow \langle \mu, R^\nu \eta \rangle.
\]
From these two convergences, we conclude that \( u \) is a duality solution to (1.1). Applying a uniqueness argument (see Theorem 4.3) shows that in fact the whole sequence \( \{ u_n \} \) converges to \( u \) in \( L^1(E; R \cdot m) \). \( \square \)

Since in the paper we assume that \( (E, D(E)) \) is transient, \( (D_c(E), E) \) is a Hilbert space. In what follows we denote by \( D_c^\ell(E) \) the dual space of \( D_c(E) \). Let \( \mu \) be a bounded Borel measure on \( E \). We write \( \mu \in D_c^\ell(E) \) if
\[
\langle \mu, \eta \rangle \leq c||\eta||_{D_c(E)}, \quad \eta \in C_b(E) \cap D_c(E).
\]
By [33, Proposition 3.1], each bounded Borel measure \( \mu \in D_c^\ell(E) \) is a smooth measure. Moreover, for every bounded \( \eta \in D_c(E) \),
\[
\langle \mu, \bar{\eta} \rangle \leq c||\eta||_{D_c(E)}.
\]
By the Hahn-Banach theorem, \( \mu \) can be uniquely extended to \( D_c(E) \) as a continuous linear functional. We denote this extension again by \( \mu \). With this notation, for every bounded \( \eta \in D_c(E) \),
\[
\langle \mu, \eta \rangle_{D_c^\ell(E), D_c(E)} = \int_E \bar{\eta} \, d\mu.
\] (4.16)
Definition 4.8. Let $\nu$ be a positive smooth measure and $\mu$ be a bounded Borel measure in $D'_e(\mathcal{E})$. We say that $u \in D'_e(\mathcal{E}) \cap L^2(E; \nu)$ is a variational solution to (1.1) if

$$\mathcal{E}(u, \eta) + \langle \nu, \bar{u}\eta \rangle = \langle \mu, \bar{\eta} \rangle_{D'_e(\mathcal{E}), D_e(\mathcal{E})}, \quad \eta \in D_e(\mathcal{E}) \cap L^2(E; \nu).$$

Remark 4.9. Since $D'_e(\mathcal{E}^\nu) \subset D'_e(\mathcal{E})$, the existence and uniqueness of a variational solution to (1.1) follows easily from the Lax-Milgram theorem.

Theorem 4.10. Let $\nu$ be a positive smooth measure on $E$ and $\mu$ be a bounded Borel measure on $E$ such that $\mu \in D'_e(\mathcal{E})$. Then $u$ is a variational solution to (1.1) if and only if it is a duality solution to (1.1).

Proof. By the existence and uniqueness results for variational and duality solutions to (1.1) it is enough to prove that if $u$ is a variational solution to (1.1), then $u$ is a duality solution to (1.1). Suppose that $u \in D'_e(\mathcal{E}) \cap L^2(E; \nu)$ is a variational solution to (1.1). Then, by the definition,

$$\mathcal{E}'(u, v) = \langle \mu, \bar{v} \rangle_{D'_e(\mathcal{E}), D_e(\mathcal{E})}, \quad v \in D_e(\mathcal{E}) \cap L^2(E; \nu). \quad (4.17)$$

Let $\eta \in B^+(E)$ be such that $R\eta$ is bounded. By [33, Lemma 2.1], there exists a generalized nest $\{F_n\}$ such that $\eta_n := 1_{F_n} \eta \in D'_e(\mathcal{E}^\nu)$. We have $v_n := \tilde{R}^\nu \eta_n \in D'_e(\mathcal{E}^\nu) = D'_e(\mathcal{E}) \cap L^2(E; \nu)$. Taking $v_n$ as a test function in (4.17) and using (4.16) we get

$$\langle u, \eta_n \rangle = \langle \mu, \tilde{R}^\nu \eta_n \rangle.$$ 

By (4.10) and Proposition 3.2(ii), $\tilde{R}^\nu \eta_n$ is finely continuous, so by [20, Theorem 4.2.2] it is quasi-continuous. Thus $\tilde{R}^\nu \eta_n = \tilde{R}^\nu \eta_n$ q.e. Since $\mu$ is smooth, we conclude that

$$\langle u, \eta_n \rangle = \langle \mu, \tilde{R}^\nu \eta_n \rangle.$$ 

Letting $n \to \infty$ in the above equation (see Remark 4.2) yields the desired result. \qed

Now we are going to show that in some sense the notion of duality solution to Schrödinger equation (1.1) is natural. To be precise, we will show that each duality solution to (1.1) (with measure $\mu$ on the right-hand side) is a limit of variational solutions to (1.1) with suitable chosen $\mu_n \in D'_e(\mathcal{E})$ approximating the measure $\mu$ in the narrow topology, i.e.

$$\int_E \eta \, d\mu_n \to \int_E \eta \, d\mu, \quad \eta \in C_b(E).$$

Let $\mu$ be a Borel measure on $E$. In the sequel, $\| \mu \|_{TV}$ stands for its total variation norm.

Proposition 4.11. Let $\nu$ be a positive smooth measure on $E$, $\mu$ be a bounded Borel measure on $E$, and let $u$ be a duality solution to (1.1) and $u_n$ be a duality solution to (1.1) with $\mu$ replaced by $\mu_n := nR\eta \mu$. Then $u_n \to u$ in $L^1(E; \rho \cdot m)$ for every strictly positive Borel function $\rho$ on $E$ such that $R\rho$ is bounded.

Proof. Let $\eta \in B^+(E)$ be such that $R\eta$ is bounded. Since $\tilde{R}^\nu \eta \leq R\eta$, $\tilde{R}^\nu \eta$ is bounded. This, when combined with the fact that $\tilde{R}^\nu \eta$ is finely continuous, implies that $nR\eta_n(\tilde{R}^\nu \eta) \to \tilde{R}^\nu \eta$. We have $nR\eta_n(\tilde{R}^\nu \eta) \leq \|R\eta\|_\infty$. Hence

$$\langle \mu, nR\eta_n(\tilde{R}^\nu \eta) \rangle \to \langle \mu, \tilde{R}^\nu \eta \rangle.$$ 


Therefore
\[ \langle \mu_n, \bar{R}^n \eta \rangle = \langle \mu, nR_n(\bar{R}^n \eta) \rangle \rightarrow \langle \mu, \bar{R}^n \eta \rangle. \]

By the definition of a duality solution,
\[ \langle u_n, \eta \rangle = \langle \mu_n, \bar{R}^n \eta \rangle. \quad (4.18) \]

Since duality solutions are unique, to complete the proof it is enough to show that, up to a subsequence, \( \{u_n\} \) is convergent in \( L^1(E; \rho \cdot m) \). To show this, we first observe that \( u_n \) is an excessive function with respect to \( X_{E^\circ, \nu} \) since \( u_n = \bar{R}^n(\mu_n) \). Since \( X_{E^\circ, \nu} \) satisfies the absolute continuity condition, by [17, Lemma 9.4, p. 306], there exists a subsequence (still denoted by \( n \)) such that \( \{u_n\} \) is convergent m.a.e. Moreover,
\[ |u_n| = |\bar{R}^n \mu_n| = |\bar{R}^n(nR_n \mu)| \leq R(nR_n |\mu|) = nR_n(R|\mu|) \leq R|\mu|. \]

In the last inequality, we used the fact that \( R|\mu| \) is an excessive function. Observe that
\[ \langle R|\mu|, \rho \rangle = \langle |\mu|, R\rho \rangle \leq \|\mu\|_{TV} \|R\rho\|_\infty. \]

Therefore, applying the Lebesgue dominated convergence to (4.18) yields the desired result. \( \square \)

**Corollary 4.12.** Let \( \rho \) be a strictly positive Borel function on \( E \) such that \( R\rho \) is bounded. There exists a sequence \( \{\mu_n\} \subset D_0'(E) \cap L^2(E; m) \) such that \( \mu_n \rightarrow \mu \) in the narrow topology and \( u_n \rightarrow u \) in \( L^1(E; \rho \cdot m) \), where \( u_n \) is the variational solution to (1.1) with \( \mu \) replaced by \( \mu_n \) and \( u \) is the duality solution to (1.1).

**Proof.** Let \( u \) be the duality solution to (1.1). Set \( \mu_n = nR_n \mu \) and let \( \{F_{n,k}\}_{k \geq 1} \) be a generalized nest such that \( \mu_{n,k} := 1_{F_{n,k}} \cdot \mu_n \in D_0'(E) \), \( k \geq 1 \). It is clear that \( \|\mu_{n,k} - \mu_n\|_{TV} \rightarrow 0 \) as \( k \rightarrow \infty \). Let \( u_{n,k} \) be a duality solution to (1.1) with \( \mu \) replaced by \( \mu_{n,k} \) and \( u_n \) be a duality solution to (1.1) with \( \mu \) replaced by \( \mu_n \). Observe that
\[ |u_{n,k} - u_n| \leq R|\mu_{n,k} - \mu_n|, \quad \text{m.a.e.} \]

Let \( k_n \in \mathbb{N} \) be such that \( \|\mu_{n,k_n} - \mu_n\|_{TV} \leq 1/n. \) Then
\[ \|u_{n,k_n} - u_n\|_{L^1(E; \rho \cdot m)} \leq \|R\rho\|_\infty \|\mu_{n,k_n} - \mu_n\|_{TV} \leq \|R\rho\|_\infty /n. \]

By Proposition 4.11, \( u_n \rightarrow u \) in \( L^1(E; \rho \cdot m) \) as \( n \rightarrow \infty \). Consequently, \( \|u_{n,k_n} - u\|_{L^1(E; \rho \cdot m)} \rightarrow 0. \) Now, we shall prove that \( \mu_{n,k_n} \rightarrow \mu \) in the narrow topology. Let \( \eta \) be a bounded continuous function on \( E \). Then
\[ |\langle \mu_{n,k_n} - \mu, \eta \rangle| \leq |\langle \mu_{n,k_n} - \mu_n, \eta \rangle| + |\langle \mu_n - \mu, \eta \rangle| \]
\[ \leq \|\eta\|_\infty \|\mu_{n,k_n} - \mu_n\|_{TV} + |\langle \mu_n - \mu, \eta \rangle| \]
\[ \leq \|\eta\|_\infty \|R\rho\|_\infty /n + |\langle \mu_n - \mu, \eta \rangle|. \]

Since \( |\langle \mu_n - \mu, \eta \rangle| = |\langle \mu, nR_n \eta - \eta \rangle| \) converges to zero as \( n \rightarrow \infty \), this shows that the sequence \( \{\mu_{n,k_n}\} \) has the desired properties. \( \square \)
4.3 Regularity results for duality solutions

For \( k \geq 0 \), we denote
\[
T_k(u) = \min\{k, \max\{u, -k\}\}.
\]

**Lemma 4.13.** For any excessive function \( \rho \) and positive smooth measure \( \nu \),
\[
\bar{R}^\nu (\rho \cdot \nu) \leq \rho, \quad \text{on } E.
\]

**Proof.** Let \( \rho \) and \( \nu \) be as in the assertion of the lemma. By [9, Proposition II.2.6] (see also [9, Exercise II.2.19]), there exists a sequence \( \{\eta_n\} \subset B^+_b(E) \) such that \( \rho_n := R\eta_n \not\nearrow \rho \). By (4.9) and [20, Lemma 5.1.5(ii)],
\[
R^\nu (\rho_n \cdot \nu) = R(R^\nu \eta_n \cdot \nu) \leq \rho_n, \quad \text{on } E
\]
Letting \( n \to \infty \) and using (4.11), (4.13), we get the result.

**Theorem 4.14.** Let \( u \) be a duality solution to (1.1). Then

(i) \( u \) has a quasi-continuous \( m \)-version \( \tilde{u} \) such that \( \tilde{u}(x) = \int_E G^\nu(x,y) \mu(dy) \) q.e.

(ii) \( \tilde{u} \in L^1(E; \nu) \) and \( \int_E |\tilde{u}| \, d\nu \leq \|\mu\|_{TV} \).

(iii) \( T_k(u) \in D_c(\mathcal{E}) \), \( k \geq 0 \), and \( \mathcal{E}(T_k(u), T_k(u)) \leq k\|\mu\|_{TV}, \ k \geq 0 \).

(iv) \( |\tilde{u}| \leq R|\mu| \).

(v) If \( R(\mathcal{B}_b(E)) \subset \mathcal{B}_b(E) \), then \( u \in L^1(E; m) \).

**Proof.** Assertion (i) is a consequence of Theorem 4.3. Let \( \rho \) be an excessive function. By Lemma 4.13,
\[
\langle \rho \cdot \nu, |\tilde{u}| \rangle \leq \langle \rho \cdot \nu, \bar{R}^\nu |\mu| \rangle = \langle \bar{R}^\nu (\rho \cdot \nu), |\mu| \rangle \leq \langle \rho, |\mu| \rangle,
\]
so we get (ii) by taking \( \rho \equiv 1 \). Let \( \nu_n \) be a sequence of bounded strictly smooth measures such that \( \nu_n \not\nearrow \nu \). Let \( u_n \) be a duality solution to (1.1) with \( \nu \) replaced by \( \nu_n \). We have
\[
\tilde{u}_n(x) = \int_E G^{\nu_n}(x,y) \mu(dy), \quad \text{q.e.}
\]
Since \( \nu_n \) is bounded, \( (\mathcal{E}^{\nu_n}, D(\mathcal{E}^{\nu_n})) \) is a regular symmetric Dirichlet form. Hence, by [31, Proposition 5.9], \( T_k(u_n) \in D_c(\mathcal{E}^{\nu_n}) \) and
\[
\mathcal{E}(T_k(u_n), T_k(u_n)) \leq \mathcal{E}^{\nu_n}(T_k(u_n), T_k(u_n)) \leq k\|\mu\|_{TV}, \quad k \geq 0.
\]
This when combined with Proposition 4.7 gives (iii). Assertion (iv) follows from (i) and 4.13. By (iv), we have \( |u| \leq R|\mu| \) \( m \)-a.e. Hence
\[
\|u\|_{L^1(E; m)} = \langle |u|, 1 \rangle \leq \langle R|\mu|, 1 \rangle = \langle |\mu|, R1 \rangle \leq \|R1\|_{\infty} \|\mu\|_{TV}.
\]
From this and the inclusion \( R(\mathcal{B}_b(E)) \subset \mathcal{B}_b(E) \) we get (v).
5 Strong duality solutions to Schrödinger equations

In this section, we compare the notion of duality solutions to (1.1) with the notion of strong duality solutions to (1.1), i.e. solutions to (1.5). We next provide a necessary and sufficient condition for the existence of a strong duality solution to (1.1). We also give some remarks concerning the concept of renormalized solutions.

It is well known (see [21]) that each bounded Borel measure $\mu$ admits a unique decomposition

$$\mu = \mu_d + \mu_c$$

into an absolutely continuous with respect to Cap part $\mu_d$ (called the diffuse part of $\mu$) and an orthogonal to Cap part $\mu_c$ (called the concentrated part of $\mu$).

**Definition 5.1.** Let $\nu$ be a positive smooth measure on $E$ and $\mu$ be a bounded measure on $E$. We say that a Borel measurable quasi-continuous function $u$ on $E$ is a strong duality solution to (1.1) if $u \in L^1(E; \nu)$ and for $m$-a.e. $x \in E$,

$$u(x) + \int_E u(y) G(x, y) \nu(dy) = \int_E G(x, y) \mu(dy).$$

(5.1)

**Remark 5.2.** By [29] Proposition 3.2, both integrals in (5.1) are well defined for q.e. $x \in E$. Since in Definition 5.1 we required from $u$ to be quasi-continuous, we have that in fact (5.1) holds q.e., see Section 2.1.

**Remark 5.3.** Let $u$ be a strong duality solution to (1.1). Integrating both sides of (5.1) with respect to a smooth measure $\beta$ such that $R|\beta|$ is bounded yields

$$\langle u, \beta \rangle + \langle u \cdot \nu, R\beta \rangle = \langle \mu, R\beta \rangle.$$  

(5.2)

Clearly, the above formula gives an equivalent definition of a strong duality solution to (1.1). In fact this is true if (5.2) is satisfied merely for any positive Borel function $\beta$ on $E$ such that $R\beta$ is bounded (see [31]).

For a measure $\mu$, we denote by $\mu |_{E_\nu}$ its restriction to the set $E_\nu$, where $E_\nu$ is defined by (3.2).

**Theorem 5.4.** Let $\nu$ be a positive smooth measure on $E$ and $\mu$ be a bounded Borel measure on $E$.

(i) If $u$ is a duality solution to (1.1), then its quasi-continuous $m$-version $\tilde{u}$ is a strong duality solution to (1.1) with $\mu$ replaced by $\mu |_{E_\nu}$.

(ii) If $u$ is a strong duality solution to (1.1), then $u$ is a duality solution to (1.1).

**Proof.** Let $u$ be a duality solution to (1.1). By Proposition 3.2 there exists a PCAF $A^\nu$ of $X$ in the Revuz duality with $\nu$ with the exceptional set $N_\nu$. Since $N_\nu$ is polar for $X$, the process $X^{E_\nu}$ satisfies the absolute continuity condition and its Green function $G^{E_\nu}$ satisfies

$$G^{E_\nu}(x, y) = G(x, y), \quad x, y \in E_\nu$$

(5.3)

(see (2.4) and the comment following it). Let $X^{E_\nu,\nu}$ be a Hunt process perturbed by the strict PCAF $A^\nu$ of $X^{E_\nu}$. By Theorem 4.1(i)-(ii), $\tilde{u} \in L^1(E; \nu)$, and

$$\tilde{u}(x) = \int_E \tilde{G}^{\nu}(x, y) \mu(dy), \quad \text{q.e.}$$

23
By the definition of $\tilde{G}^\nu(x,y)$,

$$\tilde{u}(x) = \int_{E_\nu} G_{E_\nu,\nu}(x,y) \mu(dy), \quad \text{q.e.} \quad (5.4)$$

Integrating both sides of (4.5) with respect to $\mu(dy)$ over $E_\nu$ yields

$$\int_{E_\nu} G_{E_\nu,\nu}(x,y) \mu(dy) + \int_{E_\nu} \left(G_{E_\nu}(x,z) \int_{E_\nu} G_{E_\nu,\nu}(z,y) \mu(dy)\right) \nu(dz) = \int_{E_\nu} G_{E_\nu}(x,y) \mu(dy) \quad (5.5)$$

for $x \in E_\nu$. From (5.3), (5.4), (5.5) and smoothness of $\nu$ we conclude that

$$\tilde{u}(x) + \int_E G(x,y) \tilde{u}(y) \nu(dy) = \int_E G(x,y) \mu\lfloor_{E_\nu}(dy), \quad \text{q.e.,}$$

which implies that $\tilde{u}$ is a strong duality solution to (1.1) with $\mu$ replaced by $\mu\lfloor_{E_\nu}$.

Now suppose that $u$ is a strong duality solution to (1.1). Then, by Remark 5.3 for every smooth measure $\beta$ such that $R|\beta|$ is bounded, we have

$$\langle u, \beta \rangle + \langle u \cdot \nu, R\beta \rangle = \langle \mu, R\beta \rangle. \quad (5.6)$$

By Proposition 4.4 $R\eta = R(\tilde{R}^\nu \cdot \nu) + \tilde{R}^\nu \eta$, so for every $\eta \in B^+(E)$ such that $R\eta$ is bounded,

$$R(\eta - \tilde{R}^\nu \cdot \nu) = \tilde{R}^\nu \eta.$$

From the above equation and (5.6) with $\beta = \eta - \tilde{R}^\nu \cdot \nu$ we get $\langle u, \eta \rangle = \langle \mu, \tilde{R}^\nu \eta \rangle$, which shows that $u$ is a duality solution to (1.1).

\textbf{Corollary 5.5.} There exists at most one strong duality solution to (1.1).  
\textbf{Proof.} Follows from Theorem 5.4(ii) and Theorem 4.3. \hfill \Box

\textbf{Theorem 5.6.} Let $\nu$ be a positive smooth measure on $E$ and $\mu$ be a bounded Borel measure on $E$. Then there exists a strong duality solution to (1.1) if and only if $|\mu_c|(N_\nu) = 0$.

\textbf{Proof.} Assume that $|\mu_c|(N_\nu) = 0$. Then, since $N_\nu$ is polar, $\mu = \mu\lfloor_{E_\nu}$. Therefore, by Theorem 4.3 and Theorem 5.4(i), there exists a solution to (1.1).

Now assume that that there exists a strong duality solution $u$ to (1.1). Then, by Theorem 5.4(ii), $u$ is a duality solution to (1.1). Consequently, by Theorem 5.4(i), $u$ is a strong duality solution to

$$-Au + u \cdot \nu = \mu\lfloor_{E_\nu}. \quad (5.7)$$

Therefore $u$ is a strong duality solution to (1.1), and at the same time, a strong duality solution to (5.7). By the definition of a strong duality solution and Remark 5.3 we have $\langle \mu\lfloor_{E_\nu}, R\beta \rangle = \langle \mu, R\beta \rangle$ for every smooth measure $\beta$ such that $R|\beta|$ is bounded. This implies that $\mu\lfloor_{E_\nu} = \mu$, so $|\mu_c|(N_\nu) = 0$. \hfill \Box
6 Extension of the class $\mathcal{M}_1$ and renormalized solutions

In this section we show that the methods of proofs of existence results for (1.1) achieved in the previous sections in fact apply to the broader class of measures $\mu$ on the right-hand side of (1.1). Denote by $\mathcal{W}$ the set of strictly positive, m.a.e. finite excessive functions on $E$. For $\rho \in \mathcal{W}$ we let $\mathcal{M}_\rho$ denote the set of Borel measures $\mu$ on $E$ such that

$$\int_E \rho(x) |\mu|(dx) < \infty,$$

Taking $\rho \equiv 1$ we get $\mathcal{M}_1$. Recall that if the Green function $G$ for $-A$ is strictly positive, then any function of the form $\rho = R\beta$ for non-trivial positive measure $\beta$ belongs to $\mathcal{W}$. In particular, if $\phi$ is a strictly positive principle eigenfunction for $-A$, then $\phi = \lambda^{-1}_1 R\phi$ belongs to $\mathcal{W}$. Another very important in applications class of functions included in $\mathcal{W}$ is the class of strictly positive harmonic functions.

Lemma 6.1. Let $\rho \in \mathcal{W}$. Then there exists a strictly positive function $g$ such that $Rg \leq \rho$.

Proof. Clearly, without loss of generality we may assume that $\rho$ is bounded. Let $\{U_n\}$ be an increasing sequence of relatively compact open subsets of $E$ such that $\bigcup_{n \geq 1} U_n = E$. By [13, Theorem 1.2.5, Lemma 2.3.5] for any $n \geq 1$ there exists a positive smooth measure $\nu_n$ such that $R\nu_n \leq \rho$, and $R\nu_n = \rho$ q.e. on $U_n$. Set $g_n := R_1 \nu_n$. Since $R\nu_n = \rho$ q.e. on $U_n$, we have that $R\nu_n > 0$ q.e. on $U_n$, and hence $R_1 \nu_n > 0$ q.e. on $U_n$. Thus, for any $n \geq 1$, $g_n > 0$ q.e. on $U_n$. Moreover,

$$Rg_n = RR_1 \nu_n = R_1 R\nu_n \leq R_1 \rho \leq \rho.$$

Set $g = \sum_{n=1}^{\infty} 2^{-n} g_n$. Then $g$ is the desired function.

Remark 6.2. For $\mu \in \mathcal{M}_\rho$, we define duality solutions to (1.1) as in Definition 4.14 but with the class of test functions consisting of $\eta \in \mathcal{B}(E)$ such that $R|\eta| \leq c \rho$,

$$\langle g, R|\mu| \rangle \leq \langle \rho, |\mu| \rangle < \infty.$$

The only change in the definition of strong duality solution (Definition 5.1) to (1.1) with $\mu \in \mathcal{M}_\rho$ is that we require from $u$ to be in $L^1(E; \rho \cdot \nu)$. The integrals in (5.1) are well defined and finite for q.e. $x \in E$. Indeed, by Lemma 6.1 there exists a strictly positive function $g$ such that $Rg \leq \rho$. Thus,

$$\langle g, R|\mu| \rangle \leq \langle \rho, |\mu| \rangle < \infty.$$

Therefore, since $g$ is strictly positive, $R|\mu| < \infty$, m.a.e. Since $R|\mu|$ is an excessive function, $R|\mu| < \infty$ q.e. The same reasoning applies to $u \cdot \nu$. Clearly, for $\mu \in \mathcal{M}_\rho$, (5.2) holds for smooth $\beta$ such that

$$R|\beta| \leq c \rho, \text{ for some } c > 0.$$

Repeating step by step the proof of Theorem 4.3, Theorem 4.14(i)-(ii), Theorem 5.4, and Theorem 5.6 but using test functions/measures satisfying (6.1), (6.2) we get assertions of these theorems for $\mu \in \mathcal{M}_\rho$ (clearly, in Theorem 4.14(ii) with $L^1(E; \nu)$ replaced by $L^1(E; \rho \cdot \nu)$).
Example 6.3. To show an application of the Remark 6.2 consider for a bounded open
\( D \subset \mathbb{R}^d, d \geq 2, \) and \( \alpha \in (0, 2) \)
\[ A = (\Delta^{\alpha/2})|_D, \]  
(6.3)
i.e. Dirichlet fractional Laplacian with zero exterior condition. Taking \( \rho = \varphi_1^D \), where \( \varphi_1^D \) is the principal eigenfunction of \( (\Delta^{\alpha/2})|_D \), we get the existence result for (1.1) with \( \mu \) in the class
\[ \mathcal{M}_{\varphi_1^D} := \{ \mu \text{ is a Borel measure and } \int_D \varphi_1^D \, d|\mu| < \infty \}. \]
Furthermore, if \( D \) is a \( C^{1,1} \) domain, then it is well known (see e.g. [28]) that
\[ c^{-1} \delta^{\alpha/2} \leq \varphi_1^D \leq c \delta^{\alpha/2} \]  
(6.4)
for some \( c > 0 \). Thus, we cover the class of measures considered in [18]. It is worth mentioning that (6.4) does not hold for arbitrary open domain \( D \). On the other hand, since \( (P_t)_{t \geq 0} \) is intrinsically ultracontractive (see e.g. [26]), we have that
\[ G(x, y) \geq c \varphi_1^D(x) \varphi_1^D(y), \quad x, y \in D. \]  
(6.5)
Therefore, we see that if \( u \) is a solution to (1.1) with \( \nu \equiv 0, \) and positive \( \mu \), then
\[ u(x) = R\mu(x) \geq c \varphi_1^D(x) \int_D \varphi_1^D(y) \mu(dy), \quad \text{q.e.} \]
So, the class \( \mathcal{M}_{\varphi_1^D} \) is optimal for problems of type (1.1) with \( A \) given by (6.3) and \( \nu \equiv 0 \) (cf. [18, Proposition 3.10]). In fact, by Theorem 4.3, the class \( \mathcal{M}_{\varphi_1^D} \) is optimal for (1.1) with \( A \) given by (6.3), and \( \nu \) satisfying
\[ c^{-1} G^\nu \leq G \leq c G^\nu, \]
see [27] for the sufficient conditions on \( \nu \) guaranteeing the above comparability of \( G \) and \( G^\nu \).

We close this section with some comments on the notion of renormalized solutions.

For semilinear equations with Dirichlet operators and general measure data this notion was introduced in [34]. However, the concept of renormalized solutions goes back to the paper by Dal Maso, Murat, Orsina and Prignet [16], where equations with local nonlinear Leray-Lions type operators are considered. In [32] we observed that one of the equivalent formulation of a renormalized solution to local equation with measure data considered in [16] is also suitable for equations with non-local operators and smooth measure data in the sense that it ensures uniqueness. In [34] we generalized this result to non-local equations with general measure data.

The definition adopted in [34] reads as follows.

Definition 6.4. We say that a quasi-continuous function \( u \) is a renormalized solution to (1.1) if
(a) \( u \in L^1(E; \nu) \), and \( T_k(u) \in D_e(\mathcal{E}) \), \( k \geq 0 \),
(b) There exists a sequence \( \{ \beta_k \} \) of bounded smooth measures such that for any bounded \( \eta \in D_e(\mathcal{E}) \) and any \( k \geq 0 \),
\[ \mathcal{E}(T_k(u), \eta) + \langle u \cdot \nu, \eta \rangle = \langle \mu_d, \eta \rangle + \langle \beta_k, \eta \rangle, \]
where \( \mathcal{E}(\cdot, \eta) = \langle \cdot \cdot \eta \rangle + \langle \cdot \cdot \eta \rangle \).
(c) \( R\beta_k \rightarrow R\mu_e \) q.e. as \( k \rightarrow \infty \).

By [30], under an additional assumption on the operator \( A \), e.g. \( R_1(B^+_b(E)) \subset C_b(E) \), the above definition is equivalent to the following one: \( u \) is a quasi-continuous function satisfying (a), (b), and

\[ \beta_k \rightarrow \mu_e \text{ narrowly as } k \rightarrow \infty. \]

**Proposition 6.5.** A quasi-continuous function \( u \) on \( E \) is a strong duality solution to (1.1) if and only if it is a renormalized solution to (1.1).

**Proof.** Follows from [34, Theorem 4.4] applied to (1.5), i.e. to the linear equation with bounded measure \(-u \cdot \nu + \mu\) on the right-hand side. \( \square \)

**Acknowledgements**

This work was supported by Polish National Science Centre (Grant No. 2017/25/B/ST1/00878).

**References**

[1] Aguilera-Navarro, V. C.; Estevez, G. A.; Guardiola, R. J.: Singular potentials in Quantum Mechanics. *Math Phys* 31 (1990).

[2] Aizenman, M., Simon, B.: Brownian motion and Harnack inequality for Schrödinger Operators, *Commun. Pure and Applied Math.* 35 (1982) 209–273.

[3] Albeverio, S., Brasche, J., Röckner, M.: Dirichlet forms and generalized Schrödinger operators. *Lecture Notes in Physics* 345, Springer, Berlin (1989) pp. 1–42.

[4] Albeverio, S., Ma, Z.: Perturbation of Dirichlet forms - Lower semiboundedness, closability, and form cores. *J. Funct. Anal.* 99 (1991) 332–356.

[5] Albeverio, S., Ma, Z.: Additive functionals, nowhere Radon and Kato class smooth measures associated with Dirichlet forms. *Osaka J. Math.* 29 (1992) 247–265.

[6] Baxter, J., Dal Maso, G., Mosco, U.: Stopping times and \( \Gamma \)-convergence. *Trans. Amer. Math. Soc.* 303 (1987) 1–38.

[7] Beznea, L., Boboc, N.: Measures not charging polar sets and Schrödinger equations in \( L^p \). *Acta Math. Sin. (Engl. Ser.)* 26 (2010) 249–264.

[8] Billingsley, P., Convergence of probability measures. Second edition. Wiley Series in Probability and Statistics: Probability and Statistics. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1999.

[9] Blumenthal, M.R., Getoor, R.K.: *Markov Processes and Potential Theory.* Academic Press, New York and London (1968).

[10] Boukricha, A., Hansen, W., Hueber, H.: Continuous solutions of the generalized Schrödinger equation and perturbation of harmonic spaces. *Explo. Math.* 5 (1987) 97–135.
[11] Brezis, H., Marcus, M., Ponce, A.C.: A new concept of reduced measure for nonlinear elliptic equations. C. R. Math. Acad. Sci. 339 (2004) 169–174.

[12] Brezis, H., Marcus, M., Ponce, A.C.: Nonlinear elliptic equations with measures revisited. In: Mathematical Aspects of Nonlinear Dispersive Equations (J. Bourgain, C. Kenig, S. Klainerman, eds.), Annals of Mathematics Studies, 163, Princeton University Press, Princeton, NJ, 55–110 (2007).

[13] Chen, Z.-Q., Fukushima, M.: Symmetric Markov Processes, Time Change, and Boundary Theory. Princeton University Press, Princeton (2012).

[14] Dal Maso, G., Mosco, U.: Wiener criteria and energy decay for relaxed Dirichlet problems. Arch. Rational Mech. Anal. 95 (1986) 345–387.

[15] Dal Maso, G., Mosco, U.: Wiener’s Criterion and Γ-Convergence. Appl. Math. Optim. 15 (1987) 15–63.

[16] Dal Maso, G., Murat, F., Orsina, L., Prignet, A.: Renormalized solutions of elliptic equations with general measure data. Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) 28 (1999) 741–808.

[17] Dellacherie, C., Meyer, P.A.: Probabilities and Potential C. North-Holland, Amsterdam (1988).

[18] Diaz, J.I., Gómez-Castro, D., Vazquez, J.L.: The fractional Schrödinger equation with general non-negative potentials. The weighted space approach. Nonlinear Anal. 177 (2018) 325–360.

[19] Esposito, G.: Scattering from singular potentials in quantum mechanics. J. Phys. A 31 (1998) No 47.

[20] Fukushima, M., Oshima, Y., Takeda, M.: Dirichlet forms and symmetric Markov processes. Second revised and extended edition. Walter de Gruyter, Berlin (2011).

[21] Fukushima, M., Sato, K., Taniguchi, S.: On the closable parts of pre-Dirichlet forms and the fine supports of underlying measures. Osaka J. Math. 28 (1991) 517–535.

[22] Getoor, R.K.: Measures not charging semipolars and equations of Schrödinger type. Potential Anal. 4 (1995) 79–100.

[23] Getoor, R.K.: Measure perturbations of Markovian semigroups. Potential Anal. 11 (1999) 101–133.

[24] Getoor, R.K.: An extended generator and Schrödinger equations. Electron. J. Probab. 4 (1999), no. 19, pp. 23.

[25] Gómez-Castro, D., Vázquez, J. L.: The fractional Schrödinger equation with singular potential and measure data. Discrete and Continuous Dynamical Systems, 39 (2019) 7113–7139

[26] Grzywny, T.: Intrinsic ultracontractivity for Lévy processes. Probab. Math. Statist. 28 (2008) 91–106.
[27] Hansen, W.: Global comparison of perturbed Green functions. *Math. Ann.* **334** (2006) 643–678.

[28] Kulczycki, T.: Properties of Green function of symmetric stable processes. *Probab. Math. Statist.* **17** 339–364 (1997).

[29] Klimsiak, T.: Reduced measures for semilinear elliptic equations involving Dirichlet operators. *Calc. Var. Partial Differential Equations* **55** (2016) Art. 78, 27 pp.

[30] Klimsiak, T.: On uniqueness and structure of renormalized solutions to integro-differential equations with general measure data. To appear in *NoDEA Nonlinear Differential Equations Appl.*

[31] Klimsiak, T., Rozkosz, A.: Dirichlet forms and semilinear elliptic equations with measure data. *J. Funct. Anal.* **265** (2013) 890–925.

[32] Klimsiak, T., Rozkosz, A.: Renormalized solutions of semilinear equations involving measure data and operator corresponding to Dirichlet form. *NoDEA Nonlinear Differential Equations Appl.* **22** (2015) 1911–1934.

[33] Klimsiak, T., Rozkosz, A.: On the structure of bounded smooth measures associated with a quasi-regular Dirichlet form. *Bull. Pol. Acad. Sci. Math.* **65** (2017) 45–56.

[34] Klimsiak, T., Rozkosz, A.: Renormalized solutions of semilinear elliptic equations with general measure data. *Monatsh. Math.* **188** (2019) 689–702.

[35] Lions, J.-L.: *Quelques méthodes de résolutions des problèmes aux limites non linéaires*. Dunod, Gauthier Villars, Paris. (1969).

[36] Ma, Z.-M., Röckner, M.: *Introduction to the Theory of (Non-Symmetric) Dirichlet Forms*. Springer, Berlin (1992).

[37] Malusa, A., Orsina, L.: Existence and regularity results for relaxed Dirichlet problems with measure data. *Ann. Mat. Pura Appl.* (4) **170** (1996) 57–87.

[38] Meyer, P. A.: Fonctionnelles multiplicatives et additives de Markov. *Ann. Inst. Fourier* **12** (1962) 125–230.

[39] Motoo, M., Watanabe, S.: On a class of additive functionals of Markov processes. *J. Math. Kyoto Univ.* **4** (1965) 429–469.

[40] Orsina, L., Ponce, A.C.: On the nonexistence of Green’s function and failure of the strong maximum principle. *J. Math. Pures Appl.* DOI: 10.1016/j.matpur.2019.06.001.

[41] Oshima, Y.: *Semi-Dirichlet forms and Markov processes*. Walter de Gruyter, Berlin (2013).

[42] Ponce, A.C., Wilmet, N.: Schrödinger operators involving singular potentials and measure data. *J. Differential Equations* **263** (2017) 3581–3610.
[43] Revuz, D.: Mésures associées aux fonctionelles additives de Markov I. *Trans. Amer. Math. Soc.* **148** (1970) 501–531.

[44] Sharpe, M.: *General Theory of Markov Processes*, Academic Press, New York (1988).

[45] Silverstein, M.L.: Symmetric Markov processes. *Lecture Notes in Math.* **426**, Springer (1974).

[46] Stampacchia, G.: Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. *Ann. Inst. Fourier* (Grenoble) **15** (1965) 189–258.

[47] Stollmann, P.: Smooth perturbations of regular Dirichlet forms. *Proc. Amer. Math. Soc.* **116** (1992) 747–752.

[48] Sturm, K-T.: Measures charging no polar sets and additive functionals of Brownian motion. *Forum Math.* **4** (1992) 257–297.