MATRIX FUNCTIONS THAT PRESERVE THE STRONG
PERRON-FROBENIUS PROPERTY

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Abstract. Using the real Jordan canonical form of a real matrix, we characterize matrix
functions that preserve the strong Perron-Frobenius property.

Key words. Matrix function, Real Jordan Canonical form, Perron-Frobenius Theorem, Even-
tually Positive Matrix

AMS subject classifications. 15A16, 15B48, 15A21.

1. Introduction. A real matrix $A$ has the Perron-Frobenius property if its spec-
tral radius is a positive eigenvalue corresponding to an entrywise nonnegative eigenvector. The strong Perron-Frobenius property further requires that the spectral radius
is simple; that it dominates in modulus every other eigenvalue of $A$; and that it has
an entrywise positive eigenvector.

In [17], Micchelli and Willoughby characterized matrix functions that preserve
doubly nonnegative matrices. In [7], Guillot et al. used these results to solve the critical
exponent conjecture established in [12]. In [1], Bharali and Holtz characterized entire
functions that preserve nonnegative matrices of a fixed order and, in addition, they
characterized matrix functions that preserve nonnegative block triangular, circulant,
and symmetric matrices. In [3], Elhashash and Szyld characterized entire functions
that preserve sets of generalized nonnegative matrices.

In this work, using the characterization of a matrix function via the real Jordan
form established in [16], we characterize matrix functions that preserve the strong
Perron-Frobenius property. Although our results are equivalent to those presented in
[4], the assumption of entirety of a function is dropped in favor of analyticity in some
domain containing the spectrum of a matrix.

2. Notation. Denote by $M_n(\mathbb{C})$ (respectively, $M_n(\mathbb{R})$) the algebra of complex
(respectively, real) $n \times n$ matrices. Given $A \in M_n(\mathbb{C})$, the spectrum of $A$ is denoted
by $\sigma(A)$, the spectral radius of $A$ is denoted by $\rho(A)$, and the peripheral spectrum,
denoted by $\pi(A)$, is the multi-set given by

$$\pi(A) = \{\lambda \in \sigma(A) : |\lambda| = \rho(A)\}.$$  

The direct sum of the matrices $A_1, \ldots, A_k$, where $A_i \in M_{n_i}(\mathbb{C})$, denoted by $A_1 \oplus \cdots \oplus A_k$, or $\bigoplus_{i=1}^{k} A_i$, or $\text{diag} A_1, \ldots, A_k$, is the $n \times n$ matrix

$$\begin{bmatrix}
A_1 \\
\vdots \\
A_k
\end{bmatrix},$$

where $n = \sum_{i=1}^{k} n_i$.

For $\lambda \in \mathbb{C}$, $J_n(\lambda)$ denotes the $n \times n$ Jordan block with eigenvalue $\lambda$. For $A \in M_n(\mathbb{C})$, denote by $J = Z^{-1}AZ = \bigoplus_{i=1}^{s} J_{n_i}(\lambda_i) = \bigoplus_{i=1}^{s} J_{n_i}$, where $\sum_{i=1}^{s} n_i = n$, a Jordan canonical form of $A$. Denote by $\lambda_1, \ldots, \lambda_s$ the distinct eigenvalues of $A$, and, for $i = 1, \ldots, s$, let $m_i$ denote the index of $\lambda_i$, i.e., the size of the largest Jordan block associated with $\lambda_i$. Denote by $i$ the imaginary unit, i.e., $i := \sqrt{-1}$.

A domain $D$ is any open and connected subset of $\mathbb{C}$. We call a domain real-symmetric if $\overline{\lambda} \in D$ whenever $\lambda \in D$ (i.e., $D$ is symmetric with respect to the real-axis). Given that an open connected set is also path-connected, it follows that if $D$ is real-symmetric, then $\mathbb{R} \cap D \neq \emptyset$.

3. Background. Although there are multiple ways to define a matrix function (see, e.g., [9]), our preference is via the Jordan Canonical Form.

Definition 3.1. Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a function and denote by $f^{(j)}$ the $j$th derivative of $f$. The function $f$ is said to be defined on the spectrum of $A$ if the values $f^{(j)}(\lambda_i), \ k = 0, \ldots, m_i - 1, \ i = 1, \ldots, s$, called the values of the function $f$ on the spectrum of $A$, exist.

Definition 3.2 (Matrix function via Jordan canonical form). If $f$ is defined on the spectrum of $A \in M_n(\mathbb{C})$, then

$$f(A) := Zf(J)Z^{-1} = Z \left( \bigoplus_{i=1}^{s} f(J_{n_i}) \right) Z^{-1},$$

where

$$f(J_{n_i}) := \begin{bmatrix}
f(\lambda_i) & f'(\lambda_i) & \cdots & \frac{f^{(n_i-1)}(\lambda_i)}{(n_i-1)!} \\
f(\lambda_i) & \ddots & \vdots \\
\vdots & \ddots & f'(\lambda_i) \\
\vdots & \vdots & \ddots & f(\lambda_i)
\end{bmatrix}. \quad (3.1)$$
The following theorem is well-known (for details see, e.g., [11], [14]; for a complete proof, see, e.g., [6]).

**Theorem 3.3 (Real Jordan canonical form).** If $A \in M_n(\mathbb{R})$ has $r$ real eigenvalues (including multiplicities) and $c$ complex conjugate pairs of eigenvalues (including multiplicities), then there exists an invertible matrix $R \in M_n(\mathbb{R})$ such that

$$R^{-1}AR = \left[ \bigoplus_{k=1}^{r} J_{n_k}(\lambda_k) \bigoplus_{k=r+1}^{r+c} C_{n_k}(\lambda_k) \right],$$

where:

1. $C_k(\lambda) := \begin{bmatrix} C(\lambda) & I_2 & & \cdots & \cdots & \cdots & I_2 \\ C(\lambda) & \ddots & & & \ddots & \cdots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ C(\lambda) & & & & & & \end{bmatrix} \in M_{2k}(\mathbb{R});$ (3.3)

2. $C(\lambda) := \begin{bmatrix} \Re(\lambda) & \Im(\lambda) \\ -\Im(\lambda) & \Re(\lambda) \end{bmatrix} \in M_2(\mathbb{R});$ (3.4)

3. $\Im(\lambda_k) = 0$, $k = 1, \ldots, r$; and
4. $\Im(\lambda_k) \neq 0$, $k = r+1, \ldots, r+c$.

**Proposition 3.4 ([16 Corollary 2.13]).** Let $\lambda \in \mathbb{C}$, $\lambda \neq 0$, and let $f$ be a function defined on the spectrum of $J_k(\lambda) \oplus J_k(\bar{\lambda})$. For $j$ a nonnegative integer, let $f^{(j)}_\lambda$ denote $f^{(j)}(\lambda)$. If $C_k(\lambda)$ and $C(\lambda)$ are defined as in (3.3) and (3.4), respectively, then

$$f(C_k(\lambda)) = \begin{bmatrix} f(C_\lambda) & f'(C_\lambda) & \cdots & \frac{f^{(k-1)}(C_\lambda)}{(k-1)!} \\ f(C_\lambda) & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & f'(C_\lambda) \\ \frac{f^{(k-1)}(C_\lambda)}{(k-1)!} & \cdots & \cdots & f(C_\lambda) \end{bmatrix} \in M_{2k}(\mathbb{C}),$$

and, moreover,

$$f(C_k(\lambda)) = \begin{bmatrix} C(f_\lambda) & C(f'_\lambda) & \cdots & C \left( \frac{f^{(k-1)}(f_\lambda)}{(k-1)!} \right) \\ C(f_\lambda) & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & C(f'_\lambda) \\ C(f_\lambda) & \cdots & \cdots & C(f_\lambda) \end{bmatrix} \in M_{2k}(\mathbb{R})$$
if and only if \( f^{(j)}_\lambda = f^{(j)}_{\bar{\lambda}} \).

We recall the Perron-Frobenius theorem for positive matrices (see [11, Theorem 8.2.11]).

**Theorem 3.5.** If \( A \in M_n(\mathbb{R}) \) is positive, then

(i) \( \rho := \rho(A) > 0 \);
(ii) \( \rho \in \sigma(A) \);
(iii) there exists a positive vector \( x \) such that \( Ax = \rho x \);
(iv) \( \rho \) is a simple eigenvalue of \( A \).
(v) \( |\lambda| < \rho \) for every \( \lambda \in \sigma(A) \) such that \( \lambda \neq \rho \).

There are nonnegative matrices that are not strictly positive for which the conclusions of Theorem 3.5 apply; recall that a nonnegative matrix \( A \in M_n(\mathbb{R}) \) is said to be primitive if it is irreducible and has only one eigenvalue of maximum modulus. The conclusions to Theorem 3.5 apply to primitive matrices ([11, Theorem 8.5.1]), and the following theorem is a useful characterization of primitivity ([11, Theorem 8.5.2]).

**Theorem 3.6.** If \( A \in M_n(\mathbb{R}) \) is nonnegative, then \( A \) is primitive if and only if \( A^k > 0 \) for some \( k \geq 1 \).

One can verify that the irreducible matrix
\[
\begin{bmatrix}
2 & 1 \\
2 & -1
\end{bmatrix}
\]
satisfies properties (i) through (v) of Theorem 3.5 but obviously contains a negative entry. This motivates the following concept.

**Definition 3.7.** A matrix \( A \in M_n(\mathbb{R}) \) is said to possess the strong Perron-Frobenius property if \( A \) satisfies properties (i) through (v) of Theorem 3.5.

The following theorem relates the strong Perron-Frobenius property with eventually positive matrices (see [8, Lemma 2.1], [13, Theorem 1], or [18, Theorem 2.2]).

**Theorem 3.8.** A real matrix \( A \) is eventually positive if and only if \( A \) and \( A^T \) possess the strong Perron-Frobenius property.

### 4. Main Results

Before we state our main results, we begin with the following definition and state some preliminary results.

**Definition 4.1.** A function \( f : \mathbb{C} \longrightarrow \mathbb{C} \) defined on a real-symmetric domain \( \mathcal{D} \), \( \mathcal{D} \cap \mathbb{R}^+ \neq \emptyset \), is called Frobenius if

(i) \( \overline{f(\lambda)} = f(\bar{\lambda}) \), \( \lambda \in \mathcal{D} \);
(ii) \( |f(\lambda)| < f(\rho) \), whenever \( |\lambda| < \rho \), and \( \lambda, \rho \in \mathcal{D} \).
Remark 4.2. We note that (i) implies \( f(r) \in \mathbb{R} \), whenever \( r \in D \cap \mathbb{R} \), and (ii) implies \( f(r) \in \mathbb{R}^+ \), whenever \( r \in D \cap \mathbb{R}^+ \).

Following Friedland [5], for a multi-set \( \sigma = \{ \lambda_i \}_{i=1}^n \subseteq \mathbb{C} \), let \( \rho(\sigma) := \max_i \{|\lambda_i|\} \), and \( \bar{\sigma} := \{ \bar{\lambda}_i \}_{i=1}^n \subseteq \mathbb{C} \). If \( \sigma = \bar{\sigma} \), then \( \sigma \) is said to be self-conjugate.

A (multi-)set \( \sigma \) is called a Frobenius (multi)-set if, for some positive integer \( h \leq n \), the following properties hold:

\( \rho(\sigma) > 0; \)
\( \sigma \cap \{ z \in \mathbb{C} : |z| = \rho(\sigma) \} = \{ \rho(\sigma) \exp(2\pi ik/h) : k = 0, 1, \ldots, h-1 \}; \) and
\( \sigma = \omega \sigma, \) i.e., \( \sigma \) is invariant under rotation by the angle \( 2\pi/h \).

The importance of Frobenius multi-sets becomes clear in view of the following result, which was introduced and stated without proof in [5, §4, Lemma 1] and proven in [20, Theorem 3.1]. Thus, we use the term ‘Frobenius,’ given that a Frobenius function preserves Frobenius sets.

Lemma 4.3. Let \( A \) be an eventually nonnegative matrix. If \( A \) is not nilpotent, then the spectrum of \( A \) is a union of self-conjugate Frobenius sets.

Theorem 4.4. Let \( A \in M_n(\mathbb{R}) \) and suppose that \( A \) is diagonalizable and possesses the strong Perron-Frobenius property. If \( f : \mathbb{C} \rightarrow \mathbb{C} \) is a function defined on the spectrum of \( A \), then \( f(A) \) possesses the strong Perron-Frobenius property if and only if \( f \) is Frobenius.

Example 4.5. Table 4.1 lists examples of Frobenius functions for diagonalizable matrices that possess the strong Perron-Frobenius property.

\[
\begin{array}{c|c}
\hline
f & \mathcal{D} \\
\hline
f(z) = z^p, \ p \in \mathbb{N} & \mathbb{C} \\
f(z) = |z| & \mathbb{C} \\
f(z) = z^{1/p}, \ p \in \mathbb{N}, \ p \ even & \{ z \in \mathbb{C} : z \notin \mathbb{R}^+ \} \\
f(z) = z^{1/p}, \ p \in \mathbb{N}, \ p \ odd, \ p > 1 & \mathbb{C} \\
f(z) = \sum_{k=0}^n a_k p^k, \ a_k > 0 & \mathbb{C} \\
f(z) = \exp(z) & \mathbb{C} \\
\hline
\end{array}
\]

Table 4.1 Examples of Frobenius functions.

For matrices that are not diagonalizable, i.e., possessing Jordan blocks of size two or greater, given (3.1) it is reasonable to assume that \( f \) is complex-differentiable, i.e., analytic. We note the following result, which is well known (see, e.g., [2], [3], [10]...
Theorem 3.2, [15], or [19]).

**Theorem 4.6 (Reflection Principle).** Let $f$ be analytic in a real-symmetric domain $D$ and suppose that $I := D \cap \mathbb{R} \neq \emptyset$. Then $\overline{f(\lambda)} = f(\lambda)$ for every $\lambda \in D$ if and only if $f(r) \in \mathbb{R}$ for all $r \in I$.

The Reflection Principle leads immediately to the following result.

**Corollary 4.7.** An analytic function $f : \mathbb{C} \to \mathbb{C}$ defined on a real-symmetric domain $D$, $D \cap \mathbb{R}^+ \neq \emptyset$, is Frobenius if and only if

(i) $f(r) \in \mathbb{R}$, whenever $r \in D \cap \mathbb{R}$; and

(ii) $|f(\lambda)| < f(\rho)$, whenever $|\lambda| < \rho$ and $\lambda, \rho \in D$.

**Lemma 4.8.** Let $f$ be analytic in a domain $D$ and suppose that $I := D \cap \mathbb{R} \neq \emptyset$. If $f(r) \in \mathbb{R}$ for all $r \in I$, then $f^{(j)}(r) \in \mathbb{R}$ for all $r \in I$ and $j \in \mathbb{N}$.

**Proof.** Proceed by induction on $j$: when $j = 1$, note that, since $f$ is analytic on $D$, it is holomorphic (i.e., complex-differentiable) on $D$. Thus,

$$f'(r) := \lim_{z \to r} \frac{f(z) - f(r)}{z - r}$$

exists for all $z \in D$; in particular,

$$f'(r) = \lim_{x \to r} \frac{f(x) - f(r)}{x - r}, \quad x \in I,$$

and the conclusion that $f'(r) \in \mathbb{R}$ follows by the hypothesis that $f(x) \in \mathbb{R}$ for all $x \in I$.

Next, assume that the result holds when $j = k - 1 > 1$. As above, note that $f^{(k)}(r)$ exists and

$$f^{(k)}(r) = \lim_{x \to r} \frac{f^{(k-1)}(x) - f^{(k-1)}(r)}{x - r}, \quad x \in I$$

so that $f^{(k)}(r) \in \mathbb{R}$.

**Theorem 4.9.** Let $A \in M_n(\mathbb{R})$ and suppose that $A$ possesses the strong Perron-Frobenius property. If $f : \mathbb{C} \to \mathbb{C}$ is an analytic function defined in a real-symmetric domain $D$ containing $\sigma(A)$, then $f(A)$ possesses the strong Perron-Frobenius property if and only if $f$ is Frobenius.

**Proof.** Suppose that $f$ is Frobenius. Following Theorem 3.3 and Theorem 3.5 there exists an invertible matrix $R$ such that

$$R^{-1}AR = \begin{bmatrix} \rho(A) & \bigoplus_{k=2}^r J_{n_k}(\lambda_k) \\ \bigoplus_{k=r+1}^{r+c} C_{n_k}(\lambda_k) \end{bmatrix},$$
where
\[ R = [x \ R'], \ x > 0. \]

Because \( f \) is Frobenius, following Theorem 4.6, \( f(\lambda) = f(\bar{\lambda}) \) for all \( \lambda \in \mathcal{D} \). Since \( f \) is analytic, \( f(j) \) is analytic for all \( j \in \mathbb{N} \) and, following Lemma 4.8, \( f^{(j)}(r) \in \mathbb{R} \) for all \( r \in I \). Another application of Theorem 4.6 yields that \( f(j)(\lambda) = f(j)(\bar{\lambda}) \) for all \( \lambda \in \mathcal{D} \).

Hence, following Proposition 3.4, the matrix
\[
\begin{pmatrix}
f(\rho(A)) \\
\bigoplus_{k=2}^r f(J_n(\lambda_k)) \\
\bigoplus_{k=r+1}^{r+c} f(C_n(\lambda_k))
\end{pmatrix}
\]
is real and possesses the strong Perron-Frobenius property.

The converse is clear given that if \( f \) is not Frobenius, then either \( f(A) \) is not real (e.g., \( \exists \lambda \in \sigma(A), \lambda \in \mathbb{R} \) such that \( f(\lambda) \notin \mathbb{R} \)) or \( f(A) \) does not retain the strong Perron-Frobenius property (e.g., \( \exists \lambda \in \sigma(A) \) such that \( |f(\lambda)| \geq f(\rho(A)) \)).

**Corollary 4.10.** Let \( A \in M_n(\mathbb{R}) \) and suppose that \( A \) is eventually positive. If \( f : \mathbb{C} \to \mathbb{C} \) is an analytic function defined in a real-symmetric domain \( \mathcal{D} \) containing \( \sigma(A) \), then \( f(A) \) is eventually positive if and only if \( f \) is Frobenius.

**Proof.** Follows from Theorem 4.9 and the fact that \( f(A^\top) = (f(A))^\top \) (\cite[Theorem 1.13(c)]{Theorem 4.9}).

**Remark 4.11.** Aside from the function \( f(z) = |z| \), which is nowhere differentiable, every function listed in Table 4.1 is analytic and Frobenius.

5. Acknowledgements. I gratefully acknowledge Hyunchul Park for discussions arising from this research.

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