Monodromy of hypergeometric functions arising from arrangements of hyperplanes

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Abstract

Given an arrangement of hyperplanes in $\mathbb{P}^n$, possibly with non-normal crossings, we give a vanishing lemma for the cohomology of the sheaf of $q$-forms with logarithmic poles along our arrangement. We give a basis for the ideal $\mathcal{J}$ of relations for the Orlik-Solomon’s algebra. Under certain genericity conditions it was shown by H. Esnault, V. Schechtman and E. Viehweg that the cohomology of a local system is given by the Aomoto complex. We generalize this result to a deformation of local systems obtained via a deformation of our arrangement. We calculate the Gauß-Manin connection for this case. We give a basis for the Gauß-Manin bundle for which, with help of the basis for $\mathcal{J}$, we give then a method to calculate a representation of this connection. From here, with the results of K-T. Chen or P. Deligne, one can calculate the monodromy representation. This gives a generalization of the hypergeometric functions.

Introduction

Let $\{H_i\}_{i \in I}$ be a collection of different hyperplanes in $\mathbb{P}^n$, let $U = \mathbb{P}^n \setminus \{H_i\}_{i \in I}$ and let $X$ be a smooth compactification of $U$ in such a way that the divisor $D = X \setminus U$ has normal crossings. Let $\omega$ be a holomorphic $gl(n, U)$-valued 1-form with logarithmic poles along $D$. One can define a holomorphic connection $\nabla$ on $\mathcal{O}_U \otimes \mathbb{C}^n$ as

$$\nabla = df + f\omega.$$  

On $\Omega_U^p \otimes \mathbb{C}^r$ one defines $\nabla$ by $\nabla(\alpha \otimes v) = d\alpha \otimes v + (-1)^p \alpha \nabla(v)$. If $d\omega - \omega \wedge \omega = 0$ then $\nabla^2 = 0$ and one says that the connection is integrable. Let $V$ be the rank $r$ system of flat sections of $(\mathcal{O}_U, \nabla)$, i.e. $V$ is the kernel of $\nabla$ which locally analytically is isomorphic to $\mathbb{C}^r$.

In this paper we attempt to describe the monodromy of hypergeometric functions arising as solutions of ordinary differential equations with regular singularities along an arrangement of hyperplanes in the projective space. As it is well known, from Riemann’s integral representation formula one can express the hypergeometric system of differential equations as a direct image (as a variation of cohomology) of a rank one system, see [S, Theorem 2] and [M, Proposition in p. 373]. We consider a
deformation of rank one systems on the complement of a configuration in $\mathbb{P}^n$ and obtain this generalization of the hypergeometric function. We use the following results on rank one systems.

Consider a collection of different hyperplanes in $\mathbb{P}^n$ as before. Let $\omega$ be a global holomorphic differential form over $U = \mathbb{P}^n \setminus \{H_i\}_{i \in I}$ such that it has logarithmic poles along a divisor $D$. As before, this form induces an integral connection $\nabla$ on $\mathcal{O}_X$ which as flat section gives a rank 1 local system. Deligne proved in [D1] that if one takes $\omega$ in such a way that it has no positive integers as residues then the cohomology of the local system of flat sections of $\nabla$ is given by the cohomology of the de Rham complex induced by $\nabla$. Moreover, if we suppose that $\omega$ has no integers as residues, Esnault, Schechtman and Viehweg showed in [ESV] that the cohomology of the local system is given by the Aomoto complex which is the subcomplex of global sections of the de Rham complex. Under the same genericity conditions, Esnault and Viehweg proved in [EV] that the cohomology of $V$ is concentrated in the $n$-th term. Actually all these results are not only given for higher rank systems.

To describe the monodromy we use the previous results and consider deformations of local systems. We take a connection over a rank one bundle with logarithmic poles along $\mathcal{A}$, as before, and make deformation of this connection in such a way that all the residues remain constant. For this we consider a topologically trivial deformation of the arrangement $\mathcal{A} = \sum_{i \in I} H_i \subset \mathbb{P}^n$. This deformation is given by moving one hyperplane and leaving the other ones fixed in such a way that we don’t get new non-normal crossings. Let $H_{i_0}$ with $i_0 \in I$ be such hyperplane. We then have a family of arrangements parametrized by the complement of the discriminant of the arrangement $\mathcal{A}' = \cup_{i \in I \setminus i_0} H_i$ which is given by our original arrangement without the hyperplane that we want to move. We obtain a relative connection on our family of arrangements. We show that one can extend this connection to an absolute connection $\overline{\nabla}$. In this way, and it is the deciding point, one can construct the Gauß-Manin connection. This connection is defined on the Gauß-Manin bundle which is defined as the relative de Rham cohomology sheaves and has as flat section the direct images of the absolute local system on the family. Theorem 4.2 is a generalization of the results obtained by Esnault, Schechtman, Viehweg concerning the cohomology of the family. They imply that the Gauß-Manin bundle is generated by global sections. One can then, in an standard way, calculate a representation of the Gauß-Manin connection. In the general case, to calculate the monodromy, one can use the ideas of K-T. Chen of iterated integrals to calculate the solutions of our system of differential equations but in Example 1 we rather prefer to use other results on differential equations, see [D1, II.5.6].

In the first section we give a vanishing lemma for the cohomology of differential forms with logarithmic poles along our arrangement. We will use this lemma to show that the cohomology of the local system is given by the Aomoto complex. The second section is devoted to give a description of the Gauß-Manin connection. We consider a family of arrangements parametrized by the complement of the discriminant of an arrangement in $\mathbb{P}^n$. Given a relative connection we construct the Gauß-Manin connection with logarithmic poles along the divisor obtained from a desingularization of the divisor $\text{Discr}(\mathcal{A}) \subset \mathbb{P}^{n\vee}$.
In the third section contains several results concerning some aspects of the combinatorics for the theory of hyperplane arrangements. We present two important results. The first one due to Björner, see [Bj], where he gives a base for the cohomology with constant coefficients on the complement of an arrangement. By a result of Deligne [D2] this cohomology is equal to the cohomology of the sheaf of \( q \)-differential forms with poles along an arrangement. The second result is a basis for the ideal of relations \( \mathcal{J} \) for the Orlik-Solomon’s Algebra.

In section 4 we have the principal results. We show in Theorem 4.2 that for non normal crossing case the cohomology of the relative local system is also given by the Aomoto relative complex. From here one can give a representation of the Gauß-Manin connection. In section 5 and 6 we give some examples. We construct the Gauß-Manin connection for the following arrangement:

This example is of particular interest because as discriminant one obtains the “Ceva” configuration which has been deeply studied in [BHH] for the construction of Ball quotient surfaces. The second example is an example in which the arrangement chosen has non-normal crossings.

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1 Some Vanishing Theorems

Let \( \{H_i\}_{i \in I} \) be a family of distinct hyperplanes in \( \mathbb{P}^n \), \( H = \sum_{i \in I} H_i \) the associated effective divisor and \( U = \mathbb{P}^n \setminus H \) be the complementary affine open set. We have the following definition given in [ESV, Definition (Bad)].

\[ I_L = \{ i \in I \mid L \subset H_i \} . \]
b) We define the set
\[ \mathcal{L}_j(H) = \{ L \subseteq \mathbb{P}^n \text{ linear} \mid \dim L = j \text{ and} \]
\[ L = \cap_{i \in I_L \setminus \{i_0\}} H_i \text{ for every } i_0 \in I_L \} \]
for \( 0 \leq j \leq n - 2 \). Let
\[ \mathcal{L}(H) = \bigcup_{j=0}^{n-2} \mathcal{L}_j(H). \]
The loci where \( H \) has non-normal crossings are exactly the linear subspaces contained in \( \mathcal{L}(H) \). When there is no possible confusion about the divisors we will write only \( \mathcal{L} \). Let \( X \) be the variety obtained by considering successive blow ups along the elements of \( \mathcal{L} \) in the following way.

Let \( \pi^{(r)} = \tau_1 \circ \ldots \circ \tau_r \)

\[ X \xrightarrow{\tau_r} X_{r-1} \xrightarrow{\tau_{r-1}} \ldots \xrightarrow{\tau_1} \mathbb{P}^n \]

where \( \tau_i \) is the blow up of \( X_{i-1} \) along the proper transform \( T_{i-1} \) under \( \pi^{(i-1)} \) of the elements of \( \mathcal{L}_{i-1} \). As shown in [ESV, Claim], \( T_{i-1} \) is the disjoint union of closed nonsingular subschemes. Writing \( X_0 = \mathbb{P}^n \) and \( X = X_{n-1} \) let us set for \( r \leq n - 1 \)

\[ \pi^{(r)} = \tau_{r+1} \circ \ldots \circ \tau_{n-1} : X = X_{n-1} \longrightarrow X_r \]

and

\[ \pi = \pi^{(0)} = \pi^{(n-1)}. \]

The variety \( X \) is nonsingular and \( \pi \) will be called a standard resolution of \( H \).

**Lemma 1.2** Let \( I' \subset I \) and let us consider the divisor \( H' = \sum_{i \in I'} H_i \). Let \( \pi' : X' \longrightarrow \mathbb{P}^n \) be the standard resolution of \( H' \). Then there exists a morphism \( \gamma : X \longrightarrow X' \) such that

\[ X \xrightarrow{\gamma} X' \]

\[ \mathbb{P}^n \xleftarrow{\pi'} \]

commutes.

**Proof:** Let \( \mathcal{L}'_j \) be the bad strata of dimension \( j \) of \( H' \) and let \( \mathcal{L}' = \bigcup_{j=0}^{n-2} \mathcal{L}'_j \).

Note that \( \mathcal{L}'_j \subset \mathcal{L}_j \). Let \( \tau'_j : X'_j \longrightarrow X'_{j-1} \) be the \( j \)-th blow up of \( H' \) and let \( \pi'^{(j)} = \tau'_1 \circ \ldots \circ \tau'_j \). The proof follows by induction over the dimension of the bad loci. \( \square \)

We have two special cases in which we would like to apply the previous lemma.

a) Let \( L \in \mathcal{L}_j \) and \( I' = I_L \), i.e.,

\[ I' = I_L = \{ i \in I \mid L \subset H_i \} \]
so we have the commutative diagram

\[ \begin{array}{ccc} 
X & \xrightarrow{\gamma} & X_L \\
\pi \downarrow & & \downarrow \pi_L \\
\mathbb{P}^n & & 
\end{array} \]

where we write \( X_L = X' \) and \( \pi_L = \pi' \). In this case the exceptional divisor \( E_L = \pi^*(L) \) is a \( \mathbb{P}^{n-j-1} \) bundle over \( L \) for which on every fiber, the proper transform of \( H_L = \cup_{i \in I_L} H_i \) gives a configuration \( \Gamma = \sum_{i \in I_L} \Gamma_i \subset \mathbb{P}^{n-j-1} \).

On the other hand, for \( L \in \mathcal{L}_j \) we have a configuration in \( L \sim \mathbb{P}^j \) given by

\[ \sum_{i \in I \setminus I_L} H_i |_L. \]

b) For any \( i_0 \in I \) let \( I' = I \setminus \{i_0\} \) then we have

\[ \begin{array}{ccc} 
X & \xrightarrow{\gamma} & X' \\
\pi \downarrow & & \downarrow \pi' \\
\mathbb{P}^n & & 
\end{array} \]

Claim 1.3 Let \( H = \sum_{i \in I} H_i \) as before, \( \pi : X \rightarrow \mathbb{P}^n \) the standard resolution and \( D = \pi^*(H) \). Let \( \varpi : Z \rightarrow \mathbb{P}^n \) be any other resolution for which there exists a morphism \( \gamma : Z \rightarrow X \) with \( \pi \circ \gamma = \varpi \). If \( B = \gamma^*(D) \), then for \( \nu \geq 0 \) and for all \( p, q \geq 0 \) we have

\[ H^p(Z, \Omega^q_Z(\log B) \otimes \gamma^* \pi^*(\mathcal{O}(\nu))) = H^p(X, \gamma^* \pi^* \mathcal{O}(\nu)) = H^p(X, \Omega^q_X(\log D) \otimes \pi^*(\mathcal{O}(\nu))). \]

Proof: By [EV, Lemma 3.22] we know that for \( p > 0 \) \( R^p \gamma_* \Omega_X^q(\log D) = 0 \) and that \( \gamma_* \Omega^q_Z(\log B) = \Omega^q_X(\log D) \). Applying the projection formula we have

\[ R^p \gamma_* (\Omega^q_Z(\log B) \otimes \gamma^* \pi^* \mathcal{O}(\nu)) = R^p \gamma_* \Omega^q_X(\log B) \otimes \pi^* \mathcal{O}(\nu) = 0 \quad \text{for} \quad p > 0. \]

We can now apply the Leray spectral sequence to obtain the first equality. The second equality is clear, since both sheaves are isomorphic. \( \square \)

Considering \( H = \sum_{i \in I} H_i \) as before and \( \pi : X \rightarrow \mathbb{P}^n \) the standard resolution along \( H \), we have the following lemma.

Lemma 1.4 Let \( H = \sum_{i \in I} H_i \) be a non trivial configuration of hyperplanes in \( \mathbb{P}^n \), \( \pi : X \rightarrow \mathbb{P}^n \) a standard resolution and \( D = \pi^*(H) \) be the reduced pull back divisor of \( H \) then, for \( p > 0, \nu \geq 0 \), we have

\[ H^p(X, \Omega^q_X(\log D) \otimes \pi^* \mathcal{O}_{\mathbb{P}^n}(\nu)) = 0. \]
Proof: The proof will be by double induction over the number of hyperplanes and over $\nu$. Let $H = H_1$. We will consider first the case when $\nu = 0$.

As for $k > 0$ we know that

$$H^k(\mathbb{A}^n, \mathbb{C}) = 0,$$

by the degeneration of the Hodge to de Rham spectral sequence, see [D2, Corollary 3.2.13], we have that for $p + q > 0$

$$H^p(\mathbb{P}^n, \Omega^q(\log H_1)) = 0.$$

In this case one can argue by considering the long exact sequence of cohomology associated to the short exact sequence

$$0 \longrightarrow \Omega^q_{\mathbb{P}^n} \longrightarrow \Omega^q_{\mathbb{P}^n}(\log H_1) \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow \Omega^q_{\mathbb{P}^{n-1}} \longrightarrow 0. \quad (2)$$

The connecting morphism for the long exact sequence of cohomology obtained from (2) is an isomorphism. As $H^p(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}) = \begin{cases} 0 & \text{for } q \neq p \\ \mathbb{C} & \text{for } q = p \end{cases}$

we have the result.

For the case where $\nu > 0$ we tensor the sequence (2) by $\mathcal{O}_{\mathbb{P}^n}(\nu)$ to obtain the sequence

$$0 \longrightarrow \Omega^q_{\mathbb{P}^n}(\nu) \longrightarrow \Omega^q_{\mathbb{P}^n}(\log H_1) \otimes \mathcal{O}_{\mathbb{P}^n}(\nu) \longrightarrow \Omega^q_{\mathbb{P}^{n-1}}(\nu) \longrightarrow 0. \quad (3)$$

By Bott’s formula, see [OSS, p. 8], we have that for $\nu > 0$ and $p > 0$

$$H^p(\mathbb{P}^r, \Omega^q_{\mathbb{P}^r}(\nu)) = 0.$$

Thus from (3) we have

$$H^p(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(\log H_1) \otimes \mathcal{O}_{\mathbb{P}^n}(\nu)) = 0 \quad \text{for } p > 0$$

which proves the lemma for the case when $|I| = 1$.

Let $I' \subset I$ be a proper subset. As induction hypothesis we can assume that for $H' = \sum_{i \in I'} H_i$

$$H^p(X', \Omega^q_{X'}(\log D') \otimes \pi'^* \mathcal{O}_{\mathbb{P}^n}(\nu)) = 0$$

where $\pi': X' \longrightarrow \mathbb{P}^n$ is the standard resolution of $H' = \sum_{i \in I', \nu} H_i$ and $D' = \pi'^*(H')$. Let $i_0 \in I$ fixed and $I' = I \setminus \{i_0\}$, by Lemma 1.2.(a) we have the morphism

$$\gamma: X \longrightarrow X'$$

such that, for $\pi: X \longrightarrow \mathbb{P}^n$ the standard resolution of $H$, the following diagram

$$\begin{array}{ccc}
X & \xrightarrow{\gamma} & X' \\
\pi \downarrow & & \downarrow \pi' \\
\mathbb{P}^n & & \\
\end{array}$$
is commutative. Let $D = \pi^*(H)$ be the pull back of $H$ under $\pi$, $D' = \pi''^*(H')$ the pull back of $H'$ under $\pi'$ and $D'' = \gamma^*D'$ the pull back of $D'$ under $\gamma$ all taken to be reduced. Let $D_{i_0} = D - D''$ which is equal to the proper transform of $H_{i_0}$ under $\pi$. We have from [EV, 2.3.b] the following exact sequence

$$0 \longrightarrow \Omega^q_X(\log \gamma^*(D')) \longrightarrow \Omega^q_X(\log D) \longrightarrow \Omega^q_{D_{i_0}}(\log \gamma^*(D')|_{D_{i_0}}) \longrightarrow 0.$$  (4)

Tensoring with $\pi^*O(\nu)$ gives

$$0 \longrightarrow \Omega_X^q(\log D'') \otimes \pi^*O(\nu) \longrightarrow \Omega_X^q(\log D) \otimes \pi^*O(\nu) \longrightarrow \Omega_{D_{i_0}}^q(\log D''|_{D_{i_0}}) \otimes \pi^*O(\nu) \longrightarrow 0.$$  (5)

Applying Claim 1.3, by induction on the dimension

$$H^p(D_{i_0}, \Omega^q_{D_{i_0}}(\log D'') \otimes (\pi|_{D_{i_0}})^*O(\nu)) = 0.$$  (6)

In fact, $D_{i_0}$ is a resolution of a configuration in $\mathbb{P}^{n-1} \cong H_{i_0}$. It is easy to see, that $D_{i_0}$ is the standard resolution for this configuration, but by Claim 1.3, this is not necessary for the vanishing.

On the other hand, again by projection formula and Claim 1.3 we have

$$H^p(X, \Omega^q_X(\log D'') \otimes \pi^*O_{\mathbb{P}^n}(\nu)) = H^p(X', \Omega^q_{X'}(\log D') \otimes \pi''^*O_{\mathbb{P}^n}(\nu)).$$  (6)

By our induction hypothesis on the number of hyperplanes both groups are zero, for $p > 0$. From the long exact sequence of cohomology obtained from (5) we have

$$H^p(X, \Omega^q_X(\log D) \otimes O_{\mathbb{P}^n}(\nu)) = 0 \quad \text{for} \quad p > 0.$$  (9)

As an interesting application of this lemma we have:

**Corollary 1.5** Let $H = \sum_{i \in I} H_i$ and $\pi : X \longrightarrow \mathbb{P}^n$ as in Lemma 1.4, for $p > 0$ we have

$$R^p\pi_*(\Omega^q_X(\log D)) = 0$$  (7)

**Proof:** As $\Omega^q_X(\log D)$ is coherent, by being locally free, and as $\pi$ is proper, we have that $R^p\pi_*(\Omega^q_X(\log D) \otimes \pi^*O(\nu))$ is coherent, see [Ha, Theorem III 8.8]. From Serre’s Vanishing Theorem, there exist $\nu_0$ such that for every $\nu \geq \nu_0$

$$H^p(\mathbb{P}^n, R^j\pi_*(\Omega^q_X(\log D) \otimes \pi^*O(\nu))) = 0$$  (8)

for $p > 0$. By the Leray spectral sequence we then have

$$H^0(\mathbb{P}^n, R^j\pi_*(\Omega^q_X(\log D) \otimes \pi^*O(\nu))) = H^j(X, \Omega^q_X(\log D) \otimes \pi^*O(\nu)).$$  (9)
On the other hand, as the sheaf $\mathcal{O}(\nu)$ is ample, there exists $\nu_1$ such that for every $\nu \geq \max\{\nu_0, \nu_1\}$ we have that $R^p\pi_* (\Omega^q_X (\log D)) \otimes \mathcal{O}(\nu)$ is generated by global sections. By Lemma [1.4], $H^p(X, \Omega^q_X (\log D) \otimes \mathcal{O}(\nu)) = 0$. This implies, via [2], that $R^p\pi_* (\Omega^q_X (\log D)) \otimes \pi^* \mathcal{O}(\nu)$ has no global sections, which means that $R^p\pi_* (\Omega^q_X (\log D))$ must be zero.

Let $\nu = 0$. Let $z_i$ be the projective defining equation for $H_i$. We fix $H_{i_{\infty}}$ with $i_{\infty} \in I$ as the hyperplane at infinity and we will denote it by $H_{\infty}$. Let $x_i = z_i/z_{i_{\infty}}$ and let $w_i = d \log x_i$ be the differential form with a logarithmic pole along $H_i$ with residue 1 and a logarithmic pole along $H_{\infty}$ with residue $-1$. Let $\omega \in H^0(U, \Omega^1_U)$ be given as

$$\omega = \sum_{i \in I \setminus \{i_{\infty}\}} \alpha_i \omega_i$$

with $\alpha_i \in \mathbb{C}$. Then $\omega$ has a logarithmic pole along $H_{\infty}$ with residue $a_{i_{\infty}} = -\sum_{i \in I \setminus \{i_{\infty}\}} \alpha_i$. Let $\tilde{\omega} = \pi^* \omega$, where $\pi : X \rightarrow \mathbb{P}^n$ is the standard resolution of $H$ and let $D = \pi^* (H)$. As $H^0(U, \Omega^1_U)$ injects into $H^0(X, \Omega_X (\log \pi^{-1}(A)))$, we still denote $\pi^* \omega_i$ again by $\omega_i$. The form $\omega$ defines a connection $d + \omega$ on the rank 1 bundle $\mathcal{O}$ which, as $d \omega = 0$, is integrable, i.e. it has zero curvature. Let $U = X \setminus D$ and $j : U \rightarrow X$ the inclusion. Let $\Omega^\bullet_X$ be the de Rham complex with the differential $\nabla = d + \omega$. We have a local constant system $V$ over $U$ given as $V = \ker (\nabla)$. If for every $i \in I$ and for every $l \in \mathcal{L}$, the residues $\alpha_i$ and $\sum_{i \in I} \alpha_i$ are not positive integers, then the cohomology of the local system $V$ is then given by

$$H^p(U, V) = \mathbb{H}^p(X, Rj_* V) = \mathbb{H}^p(X, \Omega^\bullet_X (\log (D)), \tilde{\nabla})$$

where $\tilde{\nabla} = \pi^* \nabla$, see [D1, II.6]. Moreover, by Lemma [1.4], see [D1, II.6] too, we have that

$$\mathbb{H}^p(X, \Omega^\bullet_X (\log (D)), \tilde{\nabla}) = H^p(H^0(X, \Omega^\bullet_X (\log (D)), \wedge \omega))$$

where $H^p(H^0(X, \Omega^\bullet_X (\log (D)), \wedge \omega))$ is the $p$-homology of the complex of vector spaces $H^0(X, \Omega^\bullet_X (\log (D)), \wedge \omega)$.

Let $A^p = H^0(X, \Omega^p_X (\log (D)))$. The exterior product by $\omega$ induces the complex of vector spaces

$$0 \rightarrow A^0 \xrightarrow{\omega} A^1 \xrightarrow{\omega} A^2 \xrightarrow{\omega} \ldots \xrightarrow{\omega} A^n \rightarrow 0.$$  

(11)

This complex appeared for the first time in [A] and will play a central role in the following sections.

**Theorem 1.6 (EV1, 1.5 and 1.7)** Under the hypothesis of Lemma [1.4], let $\omega = \sum_{i \in I \setminus \{i_{\infty}\}} \alpha_i \omega_i \in H^0(U, \Omega^1_U)$ and $V = \ker (d + \omega)$ be a local system. If $\alpha_i \notin \mathbb{Z}$ for every $i \in I$ and if for every $L \in \mathcal{L}$ $\sum_{i \in I_L} \alpha_i \notin \mathbb{Z}$ then, for $U$ affine, we have that

$$\mathbb{H}^p(X, Rj_* V) = \mathbb{H}^p(X, j_* V) = 0$$

for $p \neq n$.  

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Corollary 1.7  Under the hypothesis of Theorem 1.6 one has:

1) \( H^p(A^*, \wedge \omega) = H^p(U, V) \)

2) The complex \((A^*, \wedge \omega)\) is exact, except in degree \(n\).

Let \( \Gamma^p \subset A^p \) be given as

\[ \Gamma^p = \{ \bigwedge_{j=1}^{p} \omega_{i_j} \mid i_j \in I \setminus \{i_\infty\} \} \]

where, as above, \( \omega_i \) is the pull back of the logarithmic differential form \( \omega_i \).

We have the following result due to Brieskorn, see [B, Lemma 5]. As we later, in Section 4, apply a similar method we give a proof here.

Claim 1.8  The set \( \Gamma^p \) generates \( A^p \) as a \( \mathbb{C} \) vector space.

Proof: As in Lemma 1.4 the proof will be by induction on \( |I| \). For \( |I| = 1 \) we only have one hyperplane, namely the one at infinity so \( \Gamma^p = \emptyset \). On the other hand, from (2) we have that

\[ H^0(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}(\log(H_{\infty}))) = 0 \]

for \( p > 0 \).

Let \( |I| > 1 \). For \( I' \subset I \) a proper not empty subset, we can assume that \( i_\infty \in I' \) otherwise we can choose another hyperplane as the one at infinity. Let

\[ A'^p = H^0(X, \Omega^p_X(\log \gamma^*(D'))(\wedge \omega') \]

where \( \pi' : X' \to \mathbb{P}^n \) is the standard resolution of \( H' = \sum_{i \in I'} H_i, D' = \pi'^*(H') \) and \( \gamma : X \to X' \) is the morphism given by Lemma 1.2(a). Let \( \Gamma'^p = \{ \wedge_{i_j} \omega_{i_j} \mid i_j \in I' \setminus \{i_\infty\} \} \). As inductions hypothesis we assume that the claim holds true for any proper subset \( I' \subset I \). We fix \( i_0 \in I \) with \( i_0 \neq i_\infty \) and let \( I' = I \setminus \{i_0\} \). From (4) we have the following exact sequence

\[ 0 \to H^0(X, \Omega^p_X(\log \gamma^*(D'))) \to H^0(X, \Omega^p_X(\log(D))) \to H^0(D_{i_0}, \Omega^{p-1}_{D_{i_0}}(\log(D')|_{D_{i_0}})) \to 0. \]  \hspace{1cm} (12)

The left map in (12) is given by the natural inclusion and the right one is given as

\[
\omega_{i_1} \wedge \ldots \wedge \omega_{i_q} \mapsto \begin{cases} 
0 & \text{if } i_j \neq i_0 \text{ for } 1 \leq j \leq p \\
\omega_{i_1} \wedge \ldots \wedge \omega_{i_j} \wedge \ldots \wedge \omega_{i_q}|_{D_{i_0}} & \text{if } i_j = i_0 \text{ for } 1 \leq j \leq p
\end{cases}.
\]  \hspace{1cm} (13)

By induction on the dimension the restriction of this map to \( A'^{p-1} \wedge \omega_{i_0} \) is surjective and one obtains

\[ A^p = A'^p + A'^{p-1} \wedge \omega_{i_0}. \]  \hspace{1cm} (14)

Induction on \( |I| \) implies the claim.  \hspace{1cm} \square
Remark 1.9 The sum (14) is a direct sum for the case when \( l \cap H_{i_0} \neq l \) for every \( l \in \mathcal{L} \), i.e. when \( H_{i_0} \) does not contain "bad loci".

Proof: The result follows directly from 3.13. \( \square \)

2 The Gauß-Manin Connection

Let \( \mathcal{A} \) be an arrangement of \( m = n + r + 1 \) hyperplanes in \( \mathbb{P}^n \) which does not necessarily have normal crossings. We fix once and for all an order "<" on the set of hyperplanes such that \( \{H_0, ..., H_n\} \), the first \( n + 1 \) hyperplanes, are linearly independent. Let \( (z_0, ..., z_n) \) be homogeneous coordinates for \( \mathbb{P}^n \). We choose the coordinates in such a way that \( z_i \) is a homogeneous defining equation of \( H_i \) for \( i = 0, ..., n \). We have that

\[
H_j := \sum_{i=0}^{n} \lambda_{j,i} z_i = 0 \quad (j = n + 1, ..., m).
\]

We denote by \( \mathbb{P}^{n^\vee} \) the projective space dual to \( \mathbb{P}^n \). As every point \( p \in \mathbb{P}^{n^\vee} \) represents a hyperplane \( H_p \subset \mathbb{P}^n \) we can now consider the locus in \( \mathbb{P}^{n^\vee} \) defined as the set \( p \in \mathbb{P}^n \) such that the configuration \( \mathcal{A} \cup H_p \subset \mathbb{P}^n \) has more non-normal crossings than \( \mathcal{A} \). This set is known as the discriminant of \( \mathcal{A} \), it forms a divisor in \( \mathbb{P}^{n^\vee} \) and will be denoted by \( \text{Discr}(\mathcal{A}) \). The discriminant is not necessarily a normal crossings divisor even though if \( \mathcal{A} \) was one.

Let \( \{h_0, ..., h_n\} \) be homogeneous coordinates of \( \mathbb{P}^{n^\vee} \) dual to \( (z_0, ..., z_n) \) and let \( S^\vee \) be the homogeneous coordinate ring of \( \mathbb{P}^{n^\vee} \). We can identify \( S^\vee \) with the set of homogeneous polynomials in the \( h_i \)'s. Let \( J \in M_{m+1,n+1}(S^\vee) \) be given by

\[
J = \begin{pmatrix}
1 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & 1 \\
\lambda_{0,0} & \ldots & \lambda_{0,n} \\
\vdots & \ddots & \vdots \\
\lambda_{r,0} & \ldots & \lambda_{r,n} \\
h_0 & \ldots & h_n
\end{pmatrix}
\]  \hspace{1cm} (15)

From \( J \) one can write the discriminant of \( \mathcal{A} \) as the union of the zero set of all non trivial \( (n + 1) \)-minors that contain the row \( \{h_0, ..., h_n\} \). The fact that the arrangement can have non-normal crossings means that some of the \( n + 1 \) minors of the matrix \( J' \) can be zero.

Let us consider now a family of arrangements in \( \mathbb{P}^n \) given by the projection

\[
\pi : \mathbb{P}^{n^\vee} \times \mathbb{P}^n \longrightarrow \mathbb{P}^{n^\vee}.
\]
The fiber of $\pi$ on $p = (h_0, ..., h_n) \in \mathbb{P}^n_\Lambda$ represents again the arrangement $\mathcal{A}$ plus a hyperplane that moves smoothly when we move smoothly on $\mathbb{P}^n_\Lambda$. The extra hyperplane is defined by the fibers of $\Delta = \{(h_0, ..., h_n) \times (z_0, ..., z_n) \in \mathbb{P}^n_\Lambda \times \mathbb{P}^n \mid z_h := \sum_{i=0}^n h_i z_i = 0\}$. We have a divisor in $\mathbb{P}^n_\Lambda \times \mathbb{P}^n$ given as $[\mathbb{P}^n_\Lambda \times \mathcal{A}] \cup \Delta \cup [\text{Discr}(\mathcal{A}) \times \mathbb{P}^n]$. This divisor does not have normal crossings. We will denote this divisor by $D$.

Let

$$\rho: \tilde{S} \longrightarrow \mathbb{P}^n_\Lambda$$

be a blow up along the elements of $\mathcal{L}(\text{Discr}(\mathcal{A}))$ as shown in section [1]. Let $X = \tilde{S} \times \mathbb{P}^n$ and $D' = (\rho \times \text{id}_{\mathbb{P}^n})^{-1}D$. Let $\rho: \tilde{X} \longrightarrow X$ be a blow up, again as shown in section [1], along the elements of $\mathcal{L}(D')$ and let $\tilde{D} = \rho^*D'$, such that $\tilde{D}$ has normal crossings. We have the following diagram

$$\begin{array}{c}
\tilde{X} \\
\rho \downarrow \\
X \longrightarrow \mathbb{P}^n_\Lambda \times \mathbb{P}^n \xrightarrow{\pi} \mathbb{P}^n \\
\downarrow \\
\tilde{S} \xrightarrow{\sigma} \mathbb{P}^n_\Lambda.
\end{array} \tag{16}$$

We denote the morphism $\tilde{X} \xrightarrow{\rho} X \longrightarrow \tilde{S}$ again by $\pi$. Let $\Omega_{\tilde{X}/\tilde{S}}^i(\log \tilde{D})$ be the coherent sheaf of $\mathcal{O}_{\tilde{X}}$-modules of relative $i$-forms of $\tilde{X}$ over $\tilde{S}$ with logarithmic poles along $\tilde{D}$.

We fix once and for all the hyperplane $H_0$ as the hyperplane at infinity. For $1 \leq i \leq m$ let $\omega_i = d_{\text{rel}} x_i / x_i$, with $x_i = z_i / z_0$, be the differential form holomorphic on $\mathbb{U} = \mathbb{P}^n \setminus \mathcal{A}$ with a logarithmic pole along $H_i$ with 1 as residue, and a logarithmic pole along $H_0$ with residue $-1$. Let $\tilde{\omega}_i = \sigma^* \pi^* \omega_i$. Let $\tilde{\mathbb{U}} = \mathbb{P}^n_\Lambda \times \mathbb{P}^n \setminus D$ and let $\omega_s = d_{\text{rel}} x_s / x_s$ be the differential form holomorphic on $\tilde{\mathbb{U}}$ where $x_s := 1 + \sum_{i=1}^n l_i x_i$ with $l_i = h_i / h_0$, $x_i = z_i / z_0$ and where the differential $d_{\text{rel}}$ is the relative differential, i.e. $d_{\text{rel}}|_{\pi^{-1} \mathcal{O}_\mathbb{S}} \equiv 0$. The differential form $\omega_s$ has then a logarithmic pole along $\Delta$ with residue 1, and a logarithmic pole with residue $-1$ along $z_0 = 0$.

**Remark 2.1** The absolute differential form $\omega_s = dx_s / x_s$ has also a logarithmic pole along $h_0 = 0$ with residue $-1$.

**Notation 2.2** Let $H_0$ and $H'$ two hyperplanes in $\mathbb{P}^n$ with $z_0$ and $z'$ as homogeneous defining equations. We take $H_0$ as the hyperplane at infinity. Let $x' = z'/z_0$ be the affine defining equation of the hyperplane induced by $H'$ on the affine space $\mathbb{P}^n \setminus H_0$. We denote by $[dx']$ or by $[dx' - d\frac{x'}{z'}]$ the global differential form, holomorphic on $\mathbb{U} = \mathbb{P}^n \setminus H' \cap H_0$, with a logarithmic pole along $H'$ with residue 1 and a logarithmic pole along $H_0$ with residue $-1$.

Let $\omega \in H^0(\tilde{\mathbb{U}}, \Omega_{\tilde{\mathbb{U}}/\tilde{S}}^1)$ be given by

$$\omega = \sum_{i=1}^m a_i \omega_i + a_0 \omega_s.$$
The pull back $\sigma^*\omega$ induces the differential form $\tilde{\omega} \in H^0(\tilde{X}, \Omega^1_{\tilde{X}/\tilde{S}}(\log \tilde{D}))$. This form has residue $a_i$ along the pole $H_i$, $a_h$ as residue along $\Delta$ and $\sum_{i \in I_L} a_i$ as residue along the exceptional divisor $e_L = \sigma^{-1}(L)$ for $L \in \mathcal{L}(D)$.

We consider the operator $\nabla = d_{rel} + \tilde{\omega}$. As $d_{rel}\tilde{\omega} = 0$ we have a logarithmic de Rham complex

$$0 \to \mathcal{O}_{\tilde{X}} \to \Omega^1_{\tilde{X}/\tilde{S}}(\log \tilde{D}) \to \cdots \to \Omega^m_{\tilde{X}/\tilde{S}}(\log \tilde{D}) \to 0.$$ (17)

The $\ker(\nabla : \mathcal{O}_{\tilde{X}} \to \Omega^1_{\tilde{X}/\tilde{S}}(\log \tilde{D}))$ defines a relative local constant system $V_{rel}$ over the complement of $\tilde{D}$ in $\tilde{X}$ relative to $\tilde{S}$.

Let $H^i_{DR}(\tilde{X}/\tilde{S}, \tilde{D}, \nabla)$ be the $i$-th de Rham cohomology of $\tilde{X}$ relative to $\tilde{S}$ with respect to $\nabla$. This is the sheaf of $\mathcal{O}_{\tilde{S}}$ modules

$$H^i_{DR}(\tilde{X}/\tilde{S}, \tilde{D}, \nabla) = \mathbb{R}^i\pi_*(\Omega^*_{\tilde{X}/\tilde{S}}(\log \tilde{D}), \nabla)$$

where $\mathbb{R}^i\pi_*(\Omega^*_{\tilde{X}/\tilde{S}}(\log \tilde{D})$ are the hyperderived functors of the functor $\mathbb{R}^0\pi_*$. Under the assumptions that all the residues of $\omega$ are not positive integers, we have, see [D1, II.6], that these are the sheaves of cohomology groups of the relative local system $V_{rel}$.

As $\pi : \tilde{X} \to \tilde{S}$ we have the following exact sequence

$$0 \to \pi^*\Omega^1_{\tilde{S}}(\log(\tilde{\omega}^{*}\text{Discr}(\mathcal{A}))) \to \Omega^1_{\tilde{X}}(\log \tilde{D}) \to \Omega^1_{\tilde{X}/\tilde{S}}(\log \tilde{D}) \to 0.$$ (18)

One can extend the differential on the relative complex by taking $\tilde{\Omega} \in H^0(\tilde{X}, \Omega^1_{\tilde{X}}(\log \tilde{D}))$ as $\sigma^*(\sum_{i=1}^m a_i \frac{dx_i}{x_i} + a_h \frac{dx_s}{x_s})$ where again $x_i$ and $x_s$ are as before and where $\tilde{d}$ is the absolute differential. We define $\nabla = d + \tilde{\Omega}$. As $d\tilde{\Omega} = 0$ we have $\nabla^2 = 0$. This defines the complex $(\Omega^*_{\tilde{X}}(\log \tilde{D}), \nabla)$. Let $V = \ker(\nabla : \mathcal{O}_{\tilde{X}} \to \Omega^1_{\tilde{X}/\tilde{S}}(\log \tilde{D}))$.

Filtering the complex $(\Omega^*_{\tilde{X}}(\log \tilde{D}), \nabla)$ by

$$\cdots \mathcal{F}^{i+1} \subset \mathcal{F}^i \subset \cdots \subset \mathcal{F}^0 = \Omega^*_{\tilde{X}}(\log \tilde{D})$$

where

$$\mathcal{F}^i = \pi^*\Omega^i_{\tilde{S}}(\log(\tilde{\omega}^{*}\text{Discr}(\mathcal{A}))) \wedge \Omega^{* -i}_{\tilde{X}}(\log \tilde{D})$$

we can construct a spectral sequence abutting to $\mathbb{R}^*\pi_*(\Omega^*_{\tilde{X}}(\log \tilde{D}))$. The $E_1^{a,b}$ terms are equal to $\Omega^a_{\tilde{S}}(\log(\tilde{\omega}^{*}(\text{Discr}(\mathcal{A}))) \otimes_{\mathcal{O}_{\tilde{S}}} \mathbb{R}^b\pi_*(\Omega^{*}_{\tilde{X}/\tilde{S}}(\log \tilde{D}))$ and the differential

$$d_1 : E_1^{a,b} \to E_1^{a+1,b}$$ (19)

has bidegree $(1, 0)$.

From the filtration we have

$$0 \to \mathcal{F}^1 \to \mathcal{F}^0 \to \mathcal{F}^0 \to 0.$$
This is just the exact sequence of complexes
\[
0 \rightarrow \pi^* \Omega^1_S(\log \rho^*(\text{Discr}(A))) \otimes \Omega^{-1}_{\tilde{X}/\tilde{S}}(\log \tilde{D})
\]
\[
\rightarrow \pi^* \Omega^2_S(\log \rho^*(\text{Discr}(A))) \wedge \Omega^{-1}_{\tilde{X}/\tilde{S}}(\log \tilde{D}) \rightarrow \Omega^0_{\tilde{X}/\tilde{S}}(\log \tilde{D}) \rightarrow 0.
\] (20)

The differential (19), for the case when \(a = 0\), is the connecting morphism for the long exact sequence of cohomology obtained from (20). Using projection formula, with local calculation one can show that it has the Leibniz properties of a connection. It is called the Gauß-Manin connection, see [K,4.6]. We will still denote it by \(\bar{\nabla}\).

For the integrability of the Gauß-Manin connection we have the following diagram.

\[
\begin{array}{cccc}
0 & 0 & 0 \\
\uparrow & \uparrow & \uparrow \\
0 & F^1/F^2 & F^0/F^2 & F^0/F^1 & 0 \\
\uparrow & \uparrow & || \\
0 & F^1/F^3 & F^0/F^3 & F^0/F^1 & 0 \\
\uparrow & \uparrow \\
F^2/F^3 &= F^2/F^3 \\
\uparrow & \uparrow \\
0 & 0 \\
\end{array}
\] (21)

The curvature is then given by the map
\[
\nabla^2 : R^a\pi_*(F^0/F^1) \rightarrow R^{a+2}\pi_*(F^2/F^3).
\]

For an element \(\alpha \in R^a\pi_*(F^0/F^1)\), the connecting morphism of the middle horizontal exact sequence in (21) gives us an element in \(R^a\pi_*(F^1/F^3)\). From the left vertical exact sequence in (21), one has the integrability of the Gauß-Manin connection.

**Proposition 2.3** For \(\pi\) and \(\omega\) as above we have that
(i) \(H_{DR}(\tilde{X}/\tilde{S}, D, \nabla)\) has an integrable connection.
(ii) Under the assumptions that the residues of \(\omega\) are not integers then \(R^i\pi_*(U, V)\) is a local system equal to \(R^i\pi_*(\Omega^\bullet_{\tilde{X}/\tilde{S}}(\log \tilde{D}), \nabla)\) where \(V\) is the local system of flat sections of \(\nabla\).

## 3 Some Combinatorics

Let \(\mathcal{A}\) be an arrangement of \(m = n + r\) hyperplanes in the affine space \(\mathbb{C}^n\).

**Definition 3.1** Let \(L = L(\mathcal{A})\) be the set of all non-empty intersections \(\cap_{i \in I} H_i \neq \emptyset\) of elements of \(\mathcal{A}\). We assume \(\mathbb{C}^n \in L(\mathcal{A})\) as the intersection over the empty set, i.e. \(I = \emptyset\).
One has a partial order “≤” on the elements of $L$ given by reverse inclusion: $X \preceq Y$, for $X, Y \in L$ if and only if $Y \subseteq X$. Let $X, Y \in L$ be such that $X \prec Y$. A chain from $X$ to $Y$ is a set $\{Z_0, \ldots, Z_n\} \subseteq L$ such that $X = Z_0 \prec Z_1 \prec \ldots \prec Z_n = Y$. We say then, that this chain has length $n$. A chain is maximal if for every $i \in \{0, \ldots, n-1\}$ there does not exist $W \in L$ such that $Z_i \prec W \prec Z_{i+1}$.

We have two binary operations on the elements of $L$, namely the meet defined as $X \land Y = \cap \{Z \in L \mid X \subseteq Z \subseteq Y\}$ and the join defined as $X \lor Y = X \cap Y$ when $X \cap Y \neq \emptyset$. We have a rank function given as $\text{rank}(X) = \text{codim}X$ for $X \in L$. For the case when $X \lor Y$ exists, this function satisfies

$$\text{rank}(X \land Y) + \text{rank}(X \lor Y) \leq \text{rank}(X) + \text{rank}(Y). \quad (22)$$

**Lemma 3.2** If $X, Y \in L$ and $X \prec Y$, then any maximal chain from $X$ to $Y$ has the same length and one says that $L$ satisfies the Jordan-Dedekind chain condition.

*Proof:* The lemma follows from seeing that the length of any maximal chain is equal to $\text{rank}(Y) - \text{rank}(X)$. \qed

**Definition 3.3** Given $X, Y \in L$ such that $X \preceq Y$ we define the closed interval as $[X, Y] = \{Z \in L \mid X \preceq Z \preceq Y\}$. One can define the open interval by taking strict inequalities. For $X \in L$ we say it is maximal (resp. minimal) if there does not exist $Y \in L$ such that $X \prec Y$ (resp. $Y \prec X$).

A partially ordered set $L$ has the structure of a lattice if the two operations “$\land$” and “$\lor$” exist for every pair of elements in $L$. A lattice is called geometric if there is a rank function $\text{rank} : L \rightarrow \mathbb{N}$ satisfying (22). If $L$ is a partially ordered set for which only $X \land Y$ exists for all $X, Y \in L$ then $L$ is called semi-lattice.

One can see that $L(A)$ has the structure of a geometric semi-lattice and is called the intersection semi-lattice of the arrangement $A$.

If $A$ is central, (i.e. $\cap_{H \in A} H \neq \emptyset$) then the “join” of any two elements exists and $L(A)$ has a unique maximal element. We have then that $L(A)$ has the structure of a geometric lattice.

From a $n$-dimensional non-central arrangement one can construct a $(n+1)$-dimensional central arrangement. This process is know as the coning process and is given as follows: Let $f_i \in \mathbb{C}[x_1, \ldots, x_n]$ the defining equation for the hyperplane $H_i \in A$ with $A$ non-central. Let $F_i \in \mathbb{C}[z_0, \ldots, z_0]$ be the homogenized polynomial of $f_i$ obtained by substituting $x_j = z_j / z_0$ in $f_i$ and multiplying by $z_0$. Let $F_0 = z_0$. We define the cone over $A$ as the $(n+1)$-dimensional arrangement $cA = \cup_{i=0}^n H_i$ where $H_i = \ker(F_i)$.

As the arrangement $cA$ is central, the inequality (22) is always satisfied. Elements $l \in L(A)$ of codimension $n - r$ (resp. dimension $r$) give rise to elements of codimension $(n + 1) - r$ (resp. dimension $r$) of $L(cA)$. We have that $|cA| = |A| + 1$.

The deconing process is then given as the inverse process of the coning process. Let $A = \cup_{i \in I} H_i$ be a central arrangement. We fix a hyperplane $H_0$ as the hyperplane.
at infinity. Let \( F_i \in \mathbb{C}[z_0, \ldots, z_n] \) be the homogeneous defining equations for \( \bar{H}_i \).

We can choose the set of coordinates such that \( H_0 \) is given by \( z_0 = 0 \). Let then \( \mathcal{A}' \) be the affine arrangement given by the hyperplanes \( H_i \) with defining equation \( f_i = F_i/z_0 \) and where \( x_i = z_i/z_0 \).

With this method one can translate the definitions and results from central arrangement to non-central arrangements and back.

**Definition 3.4** Let \( L_{n-r} = L^r = \{ Z \in L \mid \text{rank}(Z) = r \} \). We have that \( L^0 = \mathbb{C}^n \), \( L_{n-1} = L^1 \) is the set of hyperplanes of our arrangement and \( L_0 \) is a set of points in \( \{ \mathbb{C}^n \} \).

**Lemma 3.5** Maximal elements of \( L(\mathcal{A}) \) have the same rank.

*Proof:* See [OT, lemma 2.4]

We have the following definition.

**Definition 3.6** We define rank of \( L = L(\mathcal{A}) \) as the rank of any maximal element.

Let “\( \prec \)” be an arbitrary but fixed linear order for the elements of \( \mathcal{A} \), i.e. we fix a linear order for the elements of \( L^1 \). Let \( \mathcal{E}_1 \) be the complex vector space freely generated by elements \( \{ e_H : H \in \mathcal{A} \} \) and let \( \mathcal{E} \) be its exterior algebra. For \( S \subset \mathcal{A} \) we denote the element \( \wedge_{H \in S} e_H \) by \( e_S \) respecting the order chosen. A subset \( S \subset \mathcal{A} \) is said to be dependent if there exists \( H_0 \in \mathcal{A} \) such that \( \cap_{H \in S \setminus \{ H_0 \}} H = \cap_{H \in S} H \). We have the morphism

\[
\partial : \mathcal{E} \longrightarrow \mathcal{E}
\]

where for \( e_{(H_1, \ldots, H_p)} \in \mathcal{E}_p \) is given as \( \partial e_{(H_1, \ldots, H_p)} = \sum_{i=1}^{p} (-1)^i e_1 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_p \). It is a morphism of algebras and it is easy to see that \( \partial^2 = 0 \). Let \( \mathcal{J} \) be the graded ideal of \( \mathcal{E} \) generated by \( \partial e_S \) with \( S \subset \mathcal{A} \) dependent. The quotient \( \mathcal{E}/\mathcal{J} \) is a graded algebra known as the Orlik-Solomon algebra and appeared for the first time in [OS], see [OT] too. From [OS, Theorem 5.2] we have the isomorphism

\[
\mathcal{E}/\mathcal{J} \longrightarrow H^\ast(U, \mathbb{C})
\]

with \( U = \mathbb{C}^n \setminus \mathcal{A} \).

For the intersection semi-lattice \( L(\mathcal{A}) \) we have the following definition of a neat base-family which we will denote as Nbf. As before we denote by 0 the minimal element of our semi-lattice \( L \).

**Definition 3.7** (i) If \( rk(L) = 1 \), then \( \{ H \in L \mid H \text{ maximal} \} \) is a Nbf.

(ii) We assume the existence of Nbf for lattices of rank \( \leq (n - 1) \) and suppose that \( rk(L) = n \). For every \( p \in L^n \), an upper bound of \( L \), chose a fixed \( H_p \in L^1 \). For every \( l \in L^{n-1} \) such that \( H \not\preceq l \prec p \) let \( \mathcal{B}_l \) be a Nbf for the lattice \( [0, l] \). We define

\[
\mathcal{B} = \bigcup_{p \in L^n} \{ B \cup \{ H_p \} / B \in \bigcup_{H \not\preceq l} \mathcal{B}_l \}
\]

is a Nbf for \( L \).
This definition was first made by A. Björner for the case when $L$ is a lattice. With respect to his definition the neat base family defined above is the union of the neat base families of the lattices $[0, l]$ with $l \in L(A)$ maximal, see [Bj, Section 3].

Given a subset $S \subseteq L^1$ of points in a semi-lattice we have that $\text{rank}(\lor_{H \in S} H) \leq |S|$. We have the following definition, see [OT, definition 3.2].

**Definition 3.8** For $S \subseteq L^1$, with $L^1$ as in definition 3.4 we say that $S$ is independent if when $\lor_{H \in S} H$ exists $\text{rank}(\lor_{H \in S} H) = |S|$ otherwise we say that $S$ is dependent.

This notion of dependence is compatible with the one above.

**Definition 3.9** Maximal independent subsets will be called bases and the minimal dependent sets will be called circuits.

We will denote the set of independent elements of $L$ by $I(A)$ and the set of circuits by $C(A)$. In our case, one can easily see that, for $L$ as in Definition 1.1, we have that $l \in L$ if and only if the set $\{H \in I(l)\}$ contains a circuit.

**Definition 3.10** Let $C \subseteq L^1$ be a circuit and $p \in L^1$ the least element of $C$. Then we say that $C - \{p\}$ is a broken circuit. For a broken circuit $C \subseteq L^1$ let $\text{princ}(C) \in L^1$ be such that $C - \{\text{princ}(C)\}$ is a circuit and is the smallest element with this property. The family of sets which do not contain a broken circuit will be called non broken circuits; we will call them nbc-elements. A maximal nbc-element will be called an nbc-base.

We have the next proposition due to A. Björner, see [Bj, Proposition 3.8]

**Proposition 3.11** Let $L$ be a geometric semi-lattice of rank $r$ and “$\preceq$” a linear order of the elements of $L^1$. Then the collection nbc-bases are an Nbf for $L$.

Let $x_i$ be the defining equation for the hyperplane $H_i \in A$ and let

$$A^p = \{\wedge_{j=1}^p \frac{dx_{i_j}}{x_{i_j}} | i_j \in 1, \ldots, m\}.$$ 

One can obtain a basis for $A^n$ and $A^{n-1}$ by following the construction defined in [SV, 1.6]. To each element $X \in L$ we associate a hyperplane $H_X$ such that $H_X \preceq X$ and such that for every $H < H_X$ we have that $H \neq X$. Taking complete flags over $L$ of the form $X_{l-1} \prec \ldots \prec X_1 \prec X_0$ where $X_i \in L_i$ and is such that $H_X \npreceq X_i$ when $X$ is an element of the flag and $X_i \prec X$. The set of such flags of length $l$ will be denoted by $FL_l$. To every such flag $X_{n-1} \prec \ldots \prec X_1 \prec X_0 \in FL_n$ one can associate the $n$-form $\omega_{\alpha_0} \wedge \ldots \wedge \omega_{\alpha_{n-1}} \in A^n$, where $\alpha_i$ is a defining equation for $H_{X_i}$ and $\omega_{\alpha_i} = dx_i/\alpha_i$. From construction is clear that our $n$-form is not trivial. We denote by $B^n$ the set of all $n$-forms obtained in this way. Following the same construction but now taking flags starting over elements $X \in L_{n-r}$. In this case, let us denote the complete flags of length $l \leq r$ by $FL^*_l$ and by $B^r$ the set of all the
forms obtained from elements in $FL^r_r$, that is, the forms that come from maximal complete flag constructed over elements of our intersection lattice of codimension $r$. One can see that $FL^r_r$ forms a neat base family for the lattice $L(A)$. We have the following theorem due to A. Björner, see [Bj, Theorem 4.2] and [SV].

**Theorem 3.12 (Bj)** *The set of $B^r$ forms a basis for $A^n$.***

We would like to calculate a basis for $\ker(\phi : A^n \rightarrow H^n(U, \mathbb{C}))$. By theorem 3.12 we know that the set of $nbc$-bases is a basis for $A^n$. Let $A = \bigcup_{i=1}^{m=n+r} H_i \subset \mathbb{C}^n$ be an arrangement, $E_1$ be the complex vector space generated by elements $\{e_H | H \in A\}$ and $E/J$ be the Orlik-Solomon algebra. Let $B \subset A$ a base but not an $nbc$, then $B$ contains at least one broken circuit. Take $\hat{H} \in A$ to be the least element in $A$ with the property that there exists $C \subset B$ such that $C$ is a broken circuit and $\hat{H} = \text{princ}(C)$, i.e. $\hat{H}$ is the least element in $A$ such that $\{H\} \cup C$ is dependent. Let $\hat{C}$ denote the circuit $C \cup \hat{H}$ and $\hat{B} = B \cup \hat{H}$.

Let $J_n = J \cap E_n$ where $J$ and $E_n$ are as before. From the isomorphism (23), to give a basis for $\ker(\phi : A^n \rightarrow H^n(U, \mathbb{C}))$ is equivalent to give a basis for $J_n$. We have the following proposition obtained together with V. Welker.

**Proposition 3.13** *Let $A = \bigcup_{i=0}^{m=n+r} H_i \subset \mathbb{C}^n$ be an arrangement, $E$ the Orlik-Solomon algebra, then a basis for $J_n$ is given by elements of $E_n$ of the form:

- i) $e_B$ for $B \subset A$ dependent and $|B| = n$.
- ii) $\partial e_{\hat{B}}$ for $B$ a base but not an $nbc$.***

**Proof:** We have that

$$|A^n| = |nbc\text{-bases}| + \{|\text{circuits}| + |\text{broken circuits}|\}.$$ 

As the number of elements in (i) together with (ii) equal $|A^n| - |nbc|$, to show that they form a basis we only have to show that they generate $\ker(\phi)$. We will prove this by induction by showing that, with help of the elements in (i) and (ii) one can write any element of $A^n$ as a linear combination of non broken circuits. Taking the lexicographic order on the elements of $A^n$, induced by the order chosen for the set of hyperplanes, the induction will be taken on this order.

As the first element of $A^n$ is already a non broken circuit so the statement is true for the base of induction. Let $C = (H_1, \ldots, H_n)$ be a base but not $nbc$. By inductions hypothesis we can assume that the statement is true for any base $B \leq C$. Let $\hat{C} = C \cup \text{princ}(C)$ as before, then

$$\partial e_{\hat{C}} = \sum_{i=1}^{\text{princ}(C)} e_{\hat{C} - \{H_i\}} + \sum_{i=\text{princ}(C)}^{n} e_{\hat{C} - \{H_i\}}.$$  

Every summand in the first sum of (24) contains a circuit so they all are elements of $i$. For the second sum, the first element is $C$ and the rest contain $\text{princ}(C)$ so they have lexicographic order smaller than $C$. Applying our induction hypothesis we can write $C$ as linear combination of circuits and $nbc$-bases. \qed

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**Remark 3.14** One can obtain a basis for the graded ideal \( J \) of relations for the Orlik-Solomon Algebra by defining a basis for every level \( J_r \) in a similar way as in proposition 3.13.

**Example**

Let \( \mathcal{A} = \bigcup_{i=0}^{5} H_i \) be an arrangement in \( \mathbb{P}^2 \) given by:

\[
\begin{align*}
H_0 &:= z_0 = 0 \\
H_1 &:= z_1 = 0 \\
H_2 &:= z_1 = 0 \\
H_3 &:= z_3 := z_0 - z_1 = 0 \\
H_4 &:= z_4 := z_0 - z_2 = 0 \\
H_5 &:= z_5 := z_1 - z_2 = 0
\end{align*}
\]  

(25)

Take \( H_0 \) to be the hyperplane at infinity. On the affine complement of \( H_0 \) we have the following arrangement.

\[
\begin{align*}
L_1 &:= x_1 = 0 \\
L_2 &:= x_2 = 0 \\
L_3 &:= x_3 := x_1 - 1 = 0 \\
L_4 &:= x_4 := x_2 - 1 = 0 \\
L_5 &:= x_5 := x_1 - x_2 = 0
\end{align*}
\]  

(26)

We have that the set of circuits is

\[
C(\mathcal{A}) = \{(L_1, L_3), (L_2, L_4), (L_1, L_2, L_5), (L_3, L_4, L_5)\}
\]
and are the only dependent subsets of $A$. The $nbc$’s are

$$nbc(A) = \{(L_1, L_2), (L_1, L_4), (L_1, L_5), (L_2, L_3), (L_3, L_4), (L_3, L_5)\}.$$ 

Clearly we have that the broken circuits are only $\{(L_2, L_5), (L_4, L_5)\}$ for which $(L_1) = \text{princ}(L_2, L_5)$ and $(L_3) = \text{princ}(L_4, L_5)$.

Let $\mathcal{E}_1$ be freely generated by $\{e_i | L_i \in A\}$ and let $\mathcal{E}$ be its exterior algebra. Let $J$ the ideal of $\mathcal{E}$ generated by $\partial e_S$ for $S \subset A$ dependent. From proposition 3.13 we have that $J_n = J \cap \mathcal{E}_n$ is generated by

$$J_n = \langle e_{13}; e_{24}; \partial e_{125}; \partial e_{345} \rangle = \langle e_{13}; e_{24}; e_{12} - e_{15} + e_{25}; e_{34} - e_{35} + e_{45} \rangle.$$

Under the natural identification of $\mathcal{E}_1$ with $\Omega_U^1$, where $U = \mathbb{P}^2 \setminus A$, given by $e_i \mapsto \frac{dx_i}{x_i}$, the these relations gives place to the following relations of 2-forms

$$\frac{dx_1 dx_4}{x_1 x_3} = 0$$

$$\frac{dx_2 dx_4}{x_2 x_4} = 0$$

$$\frac{dx_1 dx_2}{x_1 x_2} - \frac{dx_1 dx_5}{x_1 x_5} + \frac{dx_2 dx_5}{x_2 x_5} = 0$$

$$\frac{dx_3 dx_4}{x_3 x_4} - \frac{dx_3 dx_5}{x_3 x_5} + \frac{dx_4 dx_5}{x_4 x_5} = 0. \quad (27)$$

By proposition 3.13 these relations are linearly independent.

### 4 The Gauß-Manin Matrix

Let $A$ be an arrangement of $m = n + r + 1$ hyperplanes in $\mathbb{P}^n$ as taken in Section 2. We consider the same situation as in Section 2 but to make things easier we don’t compactify the space of parameters. We have a family of arrangements in $\mathbb{P}^n$ given by the projection

$$\pi : S \times \mathbb{P}^n \longrightarrow S$$

where $S = \mathbb{P}^n \setminus \text{Discr}(A)$. Let $D, U = S \times \mathbb{P}^n \setminus D$ and $\omega \in H^0(U, \Omega_U^1)$ be given as in Section 2 but restricted to $S \times \mathbb{P}^n$.

Let $\rho : \tilde{X} \longrightarrow S \times \mathbb{P}^n$ be the blow up, as in (13) along $L(D) = \mathcal{L}(S \times A + \Delta \cap [S \times \mathbb{P}^n])$ taken in the same way as in Section 1. As our space of parameters is taken as the non compactified space, we have that $\mathcal{L}(S \times A + \Delta \cap [S \times \mathbb{P}^n]) = \mathcal{L}(S \times A)$. We have the following remark.

**Remark 4.1** We have that $\mathcal{L}(D) = S \times \mathcal{L}(A) \subset D$, i.e., the bad loci can only have at most codimension $n$. Letting $\pi' : \tilde{Y} \longrightarrow \mathbb{P}^n$ be the standard resolution along elements of $\mathcal{L}(A)$ as described in Section 1, one has $\tilde{X} = S \times \tilde{Y}$. 

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We denote $\rho^*(D)$ by $\tilde{D}$. Let $\Omega^i_{\tilde{X}/S}(\log \tilde{D})$ be the coherent sheaf of $\mathcal{O}_\tilde{X}$-modules of relative $i$-forms of $\tilde{X}$ relative to $S$ with logarithmic poles along $\tilde{D}$. Let $\tilde{\omega} = \rho^*\omega \in H^0(S \times \mathbb{P}^n, \Omega^1_{\tilde{X}/S}(\log \tilde{D}))$. Then $\tilde{\omega}$ is the differential form with residues $a_h$ along $\sum_{i=0}^n h_iz_i = 0$, $a_i$ along $H_i$ with $a_0 = \sum_{i=1}^m a_i - a_h$ and where for every $L \in (\mathcal{A})$ the form $\tilde{\omega}$ has residue $\sum_{i \in I_L} a_i$ along the exceptional divisor $e_L = \rho^{-1}(S \times L)$.

We consider lifting of $\nabla$ to $\tilde{X}$ given as the operator

$$\nabla = d_{rel} + \tilde{\omega}.$$  

Again, as $d_{rel}\tilde{\omega} = 0$, it gives a logarithmic de Rham complex

$$0 \longrightarrow \mathcal{O}_{\tilde{X}} \xrightarrow{\tilde{\nabla}} \Omega^1_{\tilde{X}/S}(\log \tilde{D}) \xrightarrow{\tilde{\nabla}} \ldots \xrightarrow{\tilde{\nabla}} \Omega^n_{\tilde{X}/S}(\log \tilde{D}) \longrightarrow 0. \quad (28)$$

The $\ker(\nabla : \rho^*\mathcal{O}_{\tilde{X}} \longrightarrow \Omega^1_{\tilde{X}/S}(\log \tilde{D}))$ defines a relative local system $V_{rel}$ over the complement of $\tilde{D}$ in $\tilde{X}$ relative to $S$.

From Proposition 2.3 we have that $H^1_{\text{DR}}(\tilde{X}/S, \tilde{D}, \nabla)$ are the cohomology sheaves of groups of the relative local system $V_{rel}$ when the residues of $\tilde{\omega}_s$ along the components of $\rho^*(D)$ are not positive integers, see [D1, II.6].

Let $A^p \subset \pi_*\mathcal{O}_{U/S}^p$ be generated over $\mathcal{O}_S$ by

$$\left\{ \wedge_{j=1}^p \frac{dx_{ij}}{x_{ij}} ; \wedge_{j=1}^{p-1} \frac{dx_{ij}}{x_{ij}} \wedge \frac{dx_{rel}x_s}{x_s} \mid i_j \in \{1, \ldots, m\} \right\}.$$  

We have the subcomplex $A^\bullet \subset \pi_*\mathcal{O}_{U/S}^\bullet$ given by

$$0 \longrightarrow \mathcal{O}_S \xrightarrow{\nabla} A^1 \xrightarrow{\nabla} \ldots \xrightarrow{\nabla} A^n \longrightarrow 0. \quad (29)$$

For $s \in S$, we consider the restriction $\nabla_s$ of the connection to the fiber $\pi^{-1}(s) = s \times \mathbb{P}^n \simeq \mathbb{P}^n$. From Remark [3] the restriction $\rho|_s$ of the blow up $\rho$ to the fiber $\pi^{-1}(s)$ is a resolution of the configuration $A \cup H_s \subset \mathbb{P}^n$. We will denote $\rho|_s^*(\pi^{-1}(s))$ and $\rho|_s^*(A \cup H_s)$ as $\tilde{X}_s|_s$ and $\tilde{D}_s$ respectively. Let $A^p_s$ be the finite dimensional subspace of $H^0(\mathbb{P}^n \setminus \{A \cup H_s\}, \Omega^p_{\mathbb{P}^n \setminus \{A \cup H_s\}})$ generated by $\left\{ \wedge_{j=1}^p \frac{dx_{ij}}{x_{ij}} ; \wedge_{j=1}^{p-1} \frac{dx_{ij}}{x_{ij}} \wedge \frac{dx_{rel}x_s}{x_s} \mid i_j \in \{1, \ldots, m\} \text{ and } x_{ij} < x_{ik} \text{ if } j < k \right\}$. We have the following complex:

$$0 \longrightarrow \mathbb{C} \xrightarrow{\nabla} A^1_s \xrightarrow{\nabla} A^2_s \xrightarrow{\nabla} \ldots \xrightarrow{\nabla} A^n_s \longrightarrow 0. \quad (30)$$

By Lemma [4] we have that $H^p(\tilde{X}_s|_s, \Omega^i_{\tilde{X}_s|_s}(\log \tilde{D}_s|_s)) = 0$ for $p > 0$. This implies that the cohomology of the de Rham complex $(\Omega^i_{\tilde{X}_s|_s}(\log \tilde{D}_s|_s), \nabla_s)$ is given by the complex $A^i_s$. Moreover, if we assume that $a_i, a_h$ and $\sum_{i \in I_L} a_i$ for all $L \in (S \times A)$ are not in $\mathbb{Z}$ one has that this cohomology is equal to the cohomology of the local system defined by $\ker(\nabla_s)$, see [D1, II.6]. We know, from [ESV], that the cohomology of this local system is concentrated in degree $n$, see Theorem [5]. We have the following theorem:
《Theorem 4.2》Let $S \times \mathbb{P}^n, S$ and $\pi : S \times \mathbb{P}^n \rightarrow S$ as before. Then, if $a_i \notin \mathbb{Z}$ for $i \in \left\{0, \ldots, m, h\right\}$ with $a_0 = -\sum_{i=1}^{m} a_i - a_h$ and if $\sum_{i \in I_L} a_i \notin \mathbb{Z}$ for every $L \in \mathcal{L}$, then we have that $R^p(\pi \circ \rho)_* \Omega^q_{X/S}(\log \tilde{D}) = 0$ for $p \neq n$ and $A^p = (\pi \circ \rho)_* \Omega^p_{X/S}(\log \tilde{D})$.

**Proof:** For $\Omega^q_{X}(\log \tilde{D})$ we define the function $h^p$ on $S$ as

$$h^p(s, \Omega^q_{X/S}(\log \tilde{D})) = \dim H^p(\tilde{X}|_s, \Omega^q_{X/S}(\log \tilde{D})|_s)$$

where $\Omega^q_{X/S}(\log \tilde{D})|_s$ is just the restriction to the fiber $\pi^{-1}(s) = s \times \mathbb{P}^n$. As from Remark 1.3 we have that $\Omega^q_{X/S}(\log \tilde{D})|_s = \Omega^q_{X|s}(\log \tilde{D}|_s)$, from Lemma 1.4 we have that $H^p(X, \Omega^q_{X|s}(\log \tilde{D}|_s)) = 0$, for $p > 0$. This implies that for $p > 0$ $h^p$ is the constant function with value zero. As $S$ is the complement of the discriminant, from Theorem 3.12 we have that $h^0$ is constant equal to $|nbc(A)|$. We conclude that $h^p$ is constant for every $p$.

As $\pi \circ \rho$ is projective and $h^p$ is constant, from [Ha, III, 12.9], we have that for every $s \in S$ the natural map

$$R^p(\pi \circ \rho)_* \Omega^q_{X/S}(\log \tilde{D}) \otimes k(s) \longrightarrow H^p(\tilde{X}|_s, \Omega^q_{X/S}(\log \tilde{D})|_s)$$

(31)

where $k(s)$ is the residue field over $s$, is an isomorphism for every $p$. By Lemma 1.4 we have $H^p(\tilde{X}|_s, \Omega^q_{X/S}(\log \tilde{D})|_s) = 0$ for $p > 0$ which from (31) implies that

$$R^p(\pi \circ \rho)_* \Omega^q_{X/S}(\log \tilde{D}) = 0$$

(32)

for $p > 0$.

For $p = 0$ we have that $H^0(\tilde{X}|_s, \Omega^q_{X/S}(\log \tilde{D})|_s)$ has constant dimension equal to $\dim A^p_s$ for every $s \in S$. Thus by base change and [Ha, III, 12.9] the inclusion

$$A^p \hookrightarrow \pi_* \Omega^p_S \times \mathbb{P}^n/S (\log D)$$

is an isomorphism.

From the spectral sequence

$$E^{p,q}_1 = R^p(\pi \circ \rho)_* \Omega^q_{X/S}(\log \tilde{D}) \Longrightarrow R^{p+q}(\pi \circ \rho)_* \Omega^*_{X/S}(\log \tilde{D})$$

and (32) we have that

$$R^p(\pi \circ \rho)_* \Omega^*_{X/S}(\log \tilde{D}) =$$

$$\ker((\pi \circ \rho)_* \Omega^p_{X/S}(\log \tilde{D}) \xrightarrow{d+\omega} (\pi \circ \rho)_* \Omega^{p+1}_{X/S}(\log \tilde{D}))$$

im$$((\pi \circ \rho)_* \Omega^{p-1}_{X/S}(\log \tilde{D}) \xrightarrow{d+\omega} (\pi \circ \rho)_* \Omega^{p}_{X/S}(\log \tilde{D})).$$

(33)

As $d_{rel} = 0$ on $(\pi \circ \rho)_* \Omega^*_{X/S}(\log \tilde{D})$, the differential $d_{rel} + \omega = \omega$ and $R^p(\pi \circ \rho)_* \Omega^*_{X/S}(\log \tilde{D})$ is coherent.
If we denote the maximal ideal of $O_{S,s}$ as $M_s$, from (33), one can show that one has the base change formula
\[
R^p(\pi \circ \rho)_*((\pi \circ \rho)^* M_s \otimes \Omega^\bullet_{\tilde{X}/S}(\log \tilde{D})) = M_s \otimes R^p(\pi \circ \rho)_* \Omega^\bullet_{\tilde{X}/S}(\log \tilde{D}). \tag{34}
\]
We have the following exact sequence
\[
0 \rightarrow (\pi \circ \rho)^* M_s \otimes \Omega^\bullet_{\tilde{X}/S}(\log \tilde{D}) \rightarrow \Omega^\bullet_{\tilde{X}/S}(\log \tilde{D}) \rightarrow \Omega^\bullet_{\tilde{X}/S}(\log \tilde{D})|_s \rightarrow 0. \tag{35}
\]
For the long exact sequence of cohomology, from Theorem 1.6, we have that for $p \neq n$
\[
R^p(\pi \circ \rho)_*(\Omega^\bullet_{\tilde{X}/S}(\log \tilde{D})|_s) = 0.
\]
Applying projection formula (34) gives the following surjection
\[
M_s \otimes R^p(\pi \circ \rho)_* \Omega^\bullet_{\tilde{X}/S}(\log \tilde{D}) \rightarrow R^p(\pi \circ \rho)_* \Omega^\bullet_{\tilde{X}/S}(\log \tilde{D}) \rightarrow 0
\]
for $p \neq n$. By Nakayama’s lemma gives
\[
R^p(\pi \circ \rho)_* \Omega^\bullet_{\tilde{X}/S}(\log \tilde{D}) = 0
\]
for $p \neq n$.

For $p = n$, by (33) we have that $R^n(\pi \circ \rho)_* \Omega^\bullet_{\tilde{X}/S}(\log \tilde{D}) = A^n/\omega A^{n-1}$ is a coherent $O_S$-module.

**Remark 4.3** The sheaf of $O_S$-modules $R^n(\pi \circ \rho)_* \Omega^\bullet_{\tilde{X}/S}(\log \tilde{D})$ is free over $O_S$.

Let $A^p \subset A^p$ be the subalgebra generated by $\{\bigwedge_{j=1}^n dx_{ij} | i_j \in \{1, ..., m\} \text{ and } x_{ij} < x_{ik} \text{ if } j < k\}$. On every fiber we have the arrangement $A \cup H_s$, where $H_s$ is defined by $x_s := 1 + \sum_{i=1}^n l_i x_i = 0$ with $s = (1, l_1, ..., l_n) \in S \subset \mathbb{P}^n$. As for every $l \in \mathcal{L}(A \cup H_s)$, where $\mathcal{L}$ is as in definition 1.1, we have that $l \not\subset H_s$ which from Proposition 3.13 implies that
\[
A^p_s = A^p_s \oplus \frac{dx_s}{x_s} \wedge A^{p-1}_s. \tag{36}
\]
From claim 1.8 we have that $A^p_s$ generates $H^p_s(\tilde{X} |_s, \Omega^p_{\tilde{X}/S}(\log \tilde{D})|_s)$. As a consequence of Theorem 4.2 from (31) we can extend the decomposition (36) to global sections as
\[
A^p = A^p \oplus \frac{dx_s}{x_s} \wedge A^{p-1}. \tag{37}
\]
Applying the Euler characteristic to the sequence
\[
0 \rightarrow A^0 \rightarrow A^1 \rightarrow \ldots \rightarrow A^n \rightarrow R^n(\pi \circ \rho)_* \Omega^\bullet_{\tilde{X}/S}(\log \tilde{D}) \rightarrow 0
\]
we have proved the following corollary.
Corollary 4.4 Under the same assumptions as in Theorem 4.2 we have that \( A' \) generates \( R_n^\pi (\log \bar{D}) \) where \( A' \subset A \) is generated by \( \{ \wedge_{j=1}^m dx_{ij} | i_j \in \{1, \ldots, m\} \} \) for which from Theorem 3.12 the set \( \text{nbc}(A) \) forms a basis.

Remark 4.5 When the arrangement \( A \) has normal crossings the sheaf of \( \mathcal{O}_S \)-modules \( R_n^\pi \Omega_{\bar{X}/S} (\log \bar{D}) \) is free of rank \( \binom{m}{n} \) over \( \mathcal{O}_S \) with basis \( \{ \wedge_{k=1}^n dx_{ik} x_{ik} | i_k \in \{1, \ldots, n+r\} \} \) and \( i_j < i_k \) when \( j < k \).

As under the hypothesis of Theorem 4.2, the cohomology of the complex \( \Omega_{\bar{X}/S} (\log \bar{D}) \) is concentrated in degree \( n \), the Gauß-Manin connection is given as

\[
\nabla : R_n^\pi \Omega_{\bar{X}/S} (\log \bar{D}) \longrightarrow \Omega_S^1 \otimes R_n^\pi \Omega_{\bar{X}/S} (\log \bar{D}).
\]

Remark 4.6 The order on the set of hyperplanes induces an order “\(<\)” on the elements of \( A^1 \). This order induces an order on the basis of Corollary 4.4 for \( R_n^\pi \Omega_{\bar{X}/S} (\log \bar{D}) \), where we say that \( \wedge_{k=1}^n dx_{ik} x_{ik} \leq \wedge_{k=1}^n dx_{jk} x_{jk} \) when there is \( k \in \{1, \ldots, n\} \) such that \( \frac{dx_{ik}}{x_{ik}} \leq \frac{dx_{jk}}{x_{jk}} \) and \( \frac{dx_{il}}{x_{il}} = \frac{dx_{jl}}{x_{jl}} \) for \( l < k \).

The procedure to write matrix of the Gauß-Manin connection with respect to the basis given in corollary 4.4 is as follows: We take, as before, affine coordinates for the complement of \( z_0 = 0 \) in \( \mathbb{P}^n \) as \( x_i = z_i / z_0 \). We do the same for the complement of \( h_0 = 0 \) in \( \mathbb{P}^n \) by taking \( l_i = h_i / h_0 \). We extend the relative differential form \( \omega \) to a global form \( \Omega \), as in Section 2. We have that in affine coordinates \( \Omega = \sum_{i=1}^n a_i \frac{dx_i}{x_i} + a_{h} \frac{dx_{h}}{x_{h}} \) where \( x_i = 1 + l_1 x_1 + \ldots + l_n x_n \) and where the differential is the absolute one. The procedure is the standard one, we take an element of the basis given in 4.4, we apply to it the connection and write its image again in terms of this basis. With help of the basis of relations given in Proposition 3.13 one can write the image canonically back in terms of the basis. We have basically to cases. The first one is when, by taking an element of the basis 4.4, the hyperplanes involved with this elements are given by the set of affine coordinates chosen. Under the order induced to the basis this is the first element of our basis. For the rest of the elements we suggest to make a change of basis for the affine coordinates.

As the basis 4.4, for the Gauß-Manin bundle, depends on the combinatorics of our arrangement we cannot give and explicit form for the matrix. Nevertheless, the basis \( \text{nbc}(A) \) and the basis of relations in Proposition 3.13 are given in a so precise way that, for any explicit example, with the method above, one can compute the matrix for the Gauß-Manin connection.

5 Example I

In this section we give an example for the method given in the previous section. We take an arrangement of six lines in \( \mathbb{P}^2 \) in general position. The discriminant in this case is Ceva’s arrangement. This configuration has been deeply studied in [BHH].
Let $A = \cup_{i=0}^{3} H_i$ be the arrangement in $\mathbb{P}^2$ given as:

\[
\begin{align*}
H_0 &:= z_0 = 0 \\
H_1 &:= z_1 = 0 \\
H_2 &:= z_2 = 0 \\
H_3 &:= z_3 := z_0 + z_1 + z_2 = 0
\end{align*}
\]  

(39)

where we can take $z_0, z_1, z_2$ as a local frame for $\mathbb{P}^2$. In this case, the discriminant is given as $\text{Discr}(A) = \{ h_0 = 0, h_1 = 0, h_2 = 0, h_0 - h_1 = 0, h_0 - h_2 = 0, h_2 - h_1 = 0 \}$.

Let $X = S \times \mathbb{P}^2 \setminus \{ \Delta := h_0 z_0 + h_1 z_1 + h_2 z_2 = 0 \} \cup \{ S \times A \}$ and $\pi : X \to S$ be a family of arrangements parametrized by $S = \mathbb{P}^{2\nu} \setminus \text{Discr}(A)$. We denote the divisor $S \times A \cup \{ \Delta \cap S \times \mathbb{P}^2 \}$ by $D$.

We fix $H_0$ as the hyperplane at infinity of the arrangement (39). Let $\omega \in H^0(S \times \mathbb{P}^2, \Omega^1_{S \times \mathbb{P}^2}(\log D))$ be given as $\omega = \sum_{i=0}^{3} a_i \, d_{rel}(x_i) + a_h \, d_{rel}(x_l)$ where $x_i = z_i / z_0$ and $l_i = h_i / h_0$ and $x_l = l_1 x_1 + l_2 x_2 + 1$, and where the differential is taken as the relative differential along $S$, and where $d_{rel}$ is taken as in Remark 2.4. We assume that $a_i \notin \mathbb{Z}$ for $i \in \{0, \ldots, 3, h\}$, and that $\sum_{i=0}^{3} a_i + a_h = 0$.

The operator $\nabla = d_{rel} + \omega$ defines the complex

\[
0 \to \mathcal{O}_X \to \Omega^1_{S \times \mathbb{P}^2/S}(\log D) \to \Omega^2_{S \times \mathbb{P}^2/S}(\log D) \to 0.
\]

Let $V$ be the relative local system defined as the sheaf of flat sections of $\nabla$. Under the above assumptions on $\omega$, from 4.2 we have that the cohomology

\[
H^i(X/S, V) = R^i\pi_*\Omega^*_{S \times \mathbb{P}^2/S}(\log D) = 0
\]  

(41)
for \(i = 0, 1\).

For \(i = 2\) we have

\[
H^2(X/S, V) = \mathbb{R}^2 \pi_* \Omega^*_S \times \mathbb{P}^2 / S (\log D) =
\]

\[
O_S \frac{dx_1 \wedge dx_2}{x_1 x_2} \oplus O_S \frac{dx_1 \wedge dx_3}{x_1 x_3} \oplus O_S \frac{dx_2 \wedge dx_3}{x_2 x_3}
\]

(42)

We can now extend \(\omega\) to

\[
\Omega = \sum_{i=0}^3 a_i \frac{dz_i}{z_i} + a_h \frac{dz_h}{z_h}
\]

where the differential is no longer the relative differential. We have the operator

\[
\nabla = d + \Omega
\]

which, when using affine coordinates namely on the complement of \(z_0 = 0\) and \(h_0 = 0\), takes the form

\[
\nabla = d + \sum_{i=1}^3 a_i \frac{dx_i}{x_i} + a_h \frac{dx_h}{x_h}.
\]

The Gauß-Manin connection is then obtained with help of the canonical filtration applied to the de Rham complex \((\Omega^*_S \times \mathbb{P}^n (\log D), \nabla)\) and is given as

\[
\nabla : H^2(X/S, V) \rightarrow H^2(X/S, V) \otimes \Omega^1_S (\log (\text{Discr}(A))).
\]

With respect to the basis \((12)\) one can represent this connection by the matrix

\[
\begin{pmatrix}
-a_1 \left[ \frac{dh_1}{h_1} - \frac{dh_0}{h_0} \right] & -a_2 \left[ \frac{dh_2}{h_2} - \frac{d(h_0 - h_2)}{h_0 - h_2} \right] & a_1 \left[ \frac{dh_1}{h_1} - \frac{d(h_0 - h_1)}{h_0 - h_1} \right] \\
-a_3 \left[ \frac{dh_2}{h_2} - \frac{dh_0}{h_0} \right] & -a_1 \left[ \frac{d(h_1 - h_2)}{h_1 - h_2} - \frac{d(h_0 - h_2)}{h_0 - h_2} \right] & a_1 \left[ \frac{d(h_1 - h_2)}{h_1 - h_2} - \frac{d(h_0 - h_1)}{h_0 - h_1} \right] \\
a_3 \left[ \frac{dh_1}{h_1} - \frac{dh_0}{h_0} \right] & -a_2 \left[ \frac{d(h_1 - h_2)}{h_1 - h_2} - \frac{d(h_0 - h_2)}{h_0 - h_2} \right] & -a_2 \left[ \frac{d(h_1 - h_2)}{h_1 - h_2} - \frac{d(h_0 - h_1)}{h_0 - h_1} \right]
\end{pmatrix}
\]

(43)

We would now like to calculate the monodromy of the Gauß-Manin connection along different elements of the fundamental group of \(S\).

For \(H_i \in \text{Discr}(A)\) we have the residue map along \(H_i\)

\[
\text{Res}_{H_i}(\nabla) : H^2(X/S, V) \rightarrow H^2(X/S, V) \otimes \Omega^1_S (\log \text{Discr}(A)) \rightarrow H^2(X/S, V) \otimes O_{H_i}
\]

and defined in the usual way, see [D1, II.3.7].

Fix a base point \(p \in S\) and let \(\gamma_i \in \pi(S, p)\) be a loop around \(H_i \in \text{Discr}(A)\) with base point \(p\).

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Let
\[ T_i = \exp(-2\pi i \cdot Res_{H_i}(\nabla)). \] (44)

If we suppose that \( a_i + a_j \not\in \mathbb{Z} \setminus \{0\} \) for \( 1 \leq i < j \leq 3 \) then, by [D1, II.5.6], the monodromy transformation \( M_p \) when we go around \( H_i \) along \( \gamma_i \) is given by a conjugacy class of \( T_i \).

The monodromy around \( H_1 : h_1 = 0 \) is given as follows.

Let \( A_{H_1} \) be the residue matrix of the connection along \( H_1 \). From (43) we have that
\[
A_{H_1} = \text{Res}_{H_1}(\nabla) = \begin{pmatrix}
-a_1 & 0 & a_1 \\
0 & 0 & 0 \\
a_3 & 0 & -a_3
\end{pmatrix}.
\]

We have that for \( n \geq 1 \)
\[
A_{H_1}^n = (-a_1 - a_3)^{n-1}A_{H_1}.
\]

One can see that \((-a_1 - a_3)\) is the trace of the matrix \( A_{H_1} \) which is an eigenvalue.

We have that \( A_{H_1}^n = tr(A_{H_1})^{n-1}A_{H_1} \) where \( tr \) is the trace of the matrix. If \( a_1 + a_3 \not\in \mathbb{Z} \setminus \{0\} \) from (43) we have then that the monodromy transform is given by a conjugacy class of

\[
T_1 = I + (\exp(-2\pi i \cdot (-a_1 - a_3)) - 1) \cdot \begin{pmatrix}
-a_1 & 0 & a_1 \\
0 & 0 & 0 \\
a_3 & 0 & -a_3
\end{pmatrix}.
\]

This is
\[
M_p(\gamma_1) = I + (\exp(-2\pi i \cdot tr(A_{H_1})) - 1)tr(A_{H_1})^{-1} \cdot A_{H_1} \alpha^{-1}.
\]

For \( H_2 : h_2 = 0 \) we have the following.

Let \( A_{H_2} \) be the residue matrix of the connection along \( H_2 \). From (43) we have that
\[
A_{H_2} = \text{Res}_{H_2}(\nabla) = \begin{pmatrix}
-a_2 & -a_2 & 0 \\
a_3 & -a_3 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

We have that for \( n \geq 1 \)
\[
A_{H_2}^n = (-a_2 - a_3)^{n-1}A_{H_2}.
\]

Again we have that \( A_{H_2}^n = tr(A_{H_2})^{n-1}A_{H_2} \) where \( tr \) is the trace and an eigenvalue of the matrix \( A_{H_2} \). If \( a_2 + a_3 \not\in \mathbb{Z} \setminus \{0\} \), from (44) we have then that the monodromy transform is given by a conjugacy class of

\[
T_2 = I + (\exp(-2\pi i \cdot tr(A_{H_2})) - 1)tr(A_{H_2})^{-1} \cdot A_{H_2}.
\]

We calculate now the monodromy around \( H_0 : h_0 = 0 \).
As before let $A_{H_0}$ be the residue matrix of the connection along $H_0$. From (43) we have that
\[
A_{H_0} = Res_{H_0}(\nabla) = \begin{pmatrix} a_1 + a_2 & 0 & 0 \\ a_3 & 0 & 0 \\ -a_3 & 0 & 0 \end{pmatrix}.
\]

We have that for $n \geq 1$
\[
A^n_{H_0} = (a_1 + a_2)^{n-1}A_{H_0}.
\]

Again we see that $A^n_{H_0} = tr(A_{H_0})^{n-1}A_{H_0}$ where $tr$ is the trace and an eigenvalue of the matrix $A_{H_0}$. If $a_1 + a_2 \not\in \mathbb{Z} \setminus \{0\}$, from (44) we have then that the monodromy transform is given as a conjugacy class of
\[
T_0 = I + (\exp(-2\pi i \cdot tr(A_{H_0})) - 1)tr(A_{H_0})^{-1} \cdot A_{H_0}.
\]

The monodromy around $H_3 : h_0 - h_1 = 0$ is given as follows.
Let $A_{H_3}$ be the residue matrix of the connection along $H_3$. From (43) we have that
\[
A_{H_3} = Res_{H_3}(\nabla) = \begin{pmatrix} 0 & 0 & -a_1 \\ 0 & 0 & a_1 \\ 0 & a_2 + a_3 & 0 \end{pmatrix}.
\]

For $n \geq 1$ we have that
\[
A^n_{H_3} = (a_2 + a_3)^{n-1}A_{H_3}.
\]

One can see that $(a_2 + a_3)$ is the trace and an eigenvalue of the matrix $A_{H_3}$. We have again that $A^n_{H_3} = tr(A_{H_3})^{n-1}A_{H_3}$ where $tr$ is the trace of the matrix. If $a_1 + a_3 \not\in \mathbb{Z} \setminus \{0\}$, from (44) we have then that the monodromy transform is given as a conjugacy class of
\[
T_3 = I + (\exp(-2\pi i \cdot tr(A_{H_3})) - 1)tr(A_{H_3})^{-1} \cdot A_{H_3}.
\]

Around $H_4 : h_0 - h_2 = 0$ the monodromy is given as follows.
Let $A_{H_4}$ be the residue matrix of the connection along $H_4$. From (43) we have that
\[
A_{H_4} = Res_{H_4}(\nabla) = \begin{pmatrix} 0 & a_2 & 0 \\ 0 & a_1 + a_3 & 0 \\ 0 & a_2 & 0 \end{pmatrix}.
\]

We have that for $n \geq 1$
\[
A^n_{H_4} = (a_1 + a_3)^{n-1}A_{H_4}.
\]

One can see again that $(a_1 + a_3)$ is the trace and an eigenvalue of the matrix $A_{H_4}$. We have that $A^n_{H_4} = tr(A_{H_4})^{n-1}A_{H_4}$ where $tr$ is the trace of the matrix. If $a_1 + a_3 \not\in \mathbb{Z} \setminus \{0\}$, from (44) we have then that the monodromy transform is given as a conjugacy class of
\[
T_4 = I + (\exp(-2\pi i \cdot tr(A_{H_4})) - 1)tr(A_{H_4})^{-1} \cdot A_{H_4}.
\]
The monodromy around \( H_5 : h_2 - h_1 = 0 \) is given as follows.

Let \( A_{H_5} \) be the residue matrix of the connection along \( H_5 \). From [13] we have that
\[
A_{H_5} = Res_{H_5}(\nabla) = \begin{pmatrix}
0 & 0 & 0 \\
0 & -a_1 & -a_1 \\
0 & -a_2 & -a_2
\end{pmatrix}.
\]

We have that for \( n \geq 1 \)
\[
A_{H_5}^n = (-a_1 - a_2)^{n-1} A_{H_5}.
\]

One can see that \((-a_1 - a_2)\) is the trace and an eigenvalue of the matrix \( A_{H_5} \). We have that
\[
A_{H_5}^n = tr(A_{H_5})^{n-1} A_{H_5}.
\]

If \( a_1 + a_2 \notin \mathbb{Z} \setminus \{0\} \), then we have then that the monodromy transform is given as a conjugacy class of
\[
T_5 = I + (\exp(-2\pi i \cdot tr(A_{H_5})) - 1) tr(A_{H_5})^{-1} \cdot A_{H_5}.
\]

6 Ceva’s Configuration

Let \( \mathcal{A} = \bigcup_{i=0}^{5} H_i \) be an arrangement in \( \mathbb{P}^2 \) given as:

\[
\begin{align*}
H_0 &:= z_0 = 0 \\
H_1 &:= z_1 = 0 \\
H_2 &:= z_1 = 0 \\
H_3 &:= z_3 := z_0 - z_1 = 0 \\
H_4 &:= z_4 := z_0 - z_2 = 0 \\
H_5 &:= z_5 := z_1 - z_2 = 0
\end{align*}
\]

The discriminant is \( Discr(\mathcal{A}) = \{h_0 = 0, h_1 = 0, h_2 = 0, h_0 + h_1 = 0, h_1 + h_2 = 0, h_0 + h_2 = 0, h_0 + h_1 + h_2 = 0\} \).

Let \( X = S \times \mathbb{P}^2 \setminus \{\Delta := h_0 z_0 + h_1 z_1 + h_2 z_2 = 0\} \cup \{S \times \mathcal{A}\} \) and \( \pi : X \to S \) be a family of arrangements parametrized by \( S = \mathbb{P}^2 \setminus Discr(\mathcal{A}) \). We denote the divisor \((S \times \mathcal{A}) \cup \{\Delta \cap (S \times \mathbb{P}^2)\}\) as \( D \).

Let \( \rho : \tilde{X} \to S \times \mathbb{P}^2 \) be the blow up along the elements of \( \mathcal{L}(D) \) as in [16], see remark 4.1. Let \( \tilde{D} = \rho^*(D) \).
Let $H_0$ be the hyperplane at infinity of the projective arrangement (15). The affine arrangement on the complement of $H_0$ is given then by the equations $\{x_1 = 0, x_2 = 0, x_3 := x_1 - 1 = 0, x_4 := x_2 - 1 = 0, x_5 := x_1 - x_2 = 0\}$. Let $U = S \times \mathbb{P}^2 \setminus D$.

Let $\omega \in H^0(U, \Omega^1_U)$ be given as $\omega = \sum_{i=1}^5 a_i \frac{dx_i}{x_i} + a_h \frac{dx_i}{x_i}$ where $x_i = z_i/z_0$ and $l_i = h_i/h_0$ and $x_i = l_1 x_1 + l_2 x_2 + 1$ and where $\frac{dx_i}{x_i}$ is taken as in remark 2.2 and where the differential is taken as the relative differential along $S$. We assume that $\sum_{i=0}^5 a_i + a_h = 0, a_i \notin \mathbb{Z}$ for $i \in \{0, \ldots, 5, h\}$ and $\sum_{i \in I_L} a_i \not\in \mathbb{Z}$ for $L \in \mathcal{L}(A)$. Let $\tilde{\omega} \in H^0(\tilde{X}, \Omega^1_X(\log \tilde{D}))$ be given as $\tilde{\omega} = \rho^* \omega$.

Again as in section 3 we have that the operator $\nabla = d + \tilde{\omega}$ defines the complex $0 \rightarrow \mathcal{O}_{\tilde{X}} \rightarrow \Omega^1_X(\log \tilde{D}) \rightarrow \Omega^2_{\tilde{X}/S}(\log \tilde{D}) \rightarrow 0$.

From Theorem 4.2 and [D1, II.6] we have that the cohomology of the local system $V$ obtained as the flat sections of $\nabla$ is

$$H^i(\tilde{X}/S, V) = R^i \pi_* \Omega^*_{\tilde{X}/S}(\log \tilde{D}) = 0$$

for $i = 0, 1$.

For $i = 2$ we have from corollary 4.4

$$H^2(\tilde{X}/S, V) = R^2 \pi_* \Omega^*_{\tilde{X}/S}(\log \tilde{D}) =$$

$$\mathcal{O}_S \frac{dx_1 \wedge dx_2}{x_1 x_2} \oplus \mathcal{O}_S \frac{dx_1 \wedge dx_4}{x_1 x_4} \oplus \mathcal{O}_S \frac{dx_1 \wedge dx_5}{x_1 x_5} \oplus \mathcal{O}_S \frac{dx_2 \wedge dx_3}{x_2 x_3} \oplus \mathcal{O}_S \frac{dx_3 \wedge dx_4}{x_3 x_4} \oplus \mathcal{O}_S \frac{dx_3 \wedge dx_5}{x_3 x_5}$$

(47)

We have the basis of relations of elements of $A^2$ given in (27).

Let $\Omega \in H^0(U, \Omega^1_U)$ be given by

$$\Omega = \sum_{i=1}^5 a_i \frac{dx_i}{x_i} + a_h \frac{dx_i}{x_i}$$

where the differential is no longer the relative differential and where $x_i = z_i/z_0$, $l_i = h_i/h_0$, $x_i = 1 + l_1 x_1 + l_2 x_2$ and $dx_i/x_i$ is taken as in remark 2.2. We extend $\tilde{\omega}$ to $\tilde{X}$ to an element $\tilde{\Omega} \in H^0(\tilde{X}, \Omega^1_X(\log \tilde{D}))$ as $\tilde{\Omega} = \rho^* \Omega$. We have the operator $\nabla = d + \tilde{\Omega}$. We have the Gauß-Manin connection

$$\nabla : H^2(\tilde{X}/S, V) \rightarrow H^2(\tilde{X}/S, V) \otimes \mathcal{O}_S.$$
The first column is

\[
\begin{pmatrix}
-a_1 - a_5 \left[ \frac{dh_3}{h_1} - \frac{dh_0}{h_0} \right] - a_2 \left[ \frac{dh_2}{h_2} - \frac{dh_0}{h_0} \right] \\
-a_4 \left[ \frac{dh_2}{h_2} - \frac{dh_0}{h_0} \right] \\
a_5 \left[ \frac{dh_1}{h_1} - \frac{dh_2}{h_2} \right] \\
a_3 \left[ \frac{dh_1}{h_1} - \frac{dh_0}{h_0} \right] \\
0 \\
0
\end{pmatrix}
\]

(48)

The second column is

\[
\begin{pmatrix}
-a_2 \left[ \frac{dh_2}{h_2} - \frac{dh_0 + h_2}{(h_0 + h_2)^2} \right] \\
-a_1 \left[ \frac{dh_1}{h_1} - \frac{dh_0 + h_2}{(h_0 + h_2)^2} \right] - a_4 \left[ \frac{dh_2}{h_2} - \frac{dh_0 + h_2}{(h_0 + h_2)^2} \right] \\
-a_5 \left[ \frac{dh_2}{h_2} - \frac{dh_0 + h_2}{(h_0 + h_2)^2} \right] \\
0 \\
(-a_3 - a_5) \left[ \frac{dh_1}{h_1} - \frac{dh_0 + h_2}{(h_0 + h_2)^2} \right] \\
a_5 \left[ \frac{dh_1}{h_1} - \frac{dh_0 + h_2}{(h_0 + h_2)^2} \right]
\end{pmatrix}
\]

The third column is

\[
\begin{pmatrix}
a_2 \left[ \frac{dh_1 + h_2}{h_1 + h_2} - \frac{dh_0}{h_2} \right] \\
-a_4 \left[ \frac{dh_2}{h_2} - \frac{dh_0}{h_0} \right] \\
(-a_1 - a_2) \left[ \frac{dh_1 + h_2}{h_1 + h_2} - \frac{dh_0}{h_0} \right] - a_5 \left[ \frac{dh_2}{h_2} - \frac{dh_0}{h_0} \right] \\
0 \\
a_4 \left[ \frac{dh_1 + h_2}{h_1 + h_2} - \frac{dh_0}{h_0} \right] \\
(-a_3 - a_4) \left[ \frac{dh_1 + h_2}{h_1 + h_2} - \frac{dh_0}{h_0} \right]
\end{pmatrix}
\]

The fourth column is

\[
\begin{pmatrix}
30
\end{pmatrix}
\]
\[
\begin{pmatrix}
(a_1 + a_5) \left[ \frac{dh_1}{h_1} - \frac{d(h_0 + h_1)}{h_0 + h_1} \right] \\
\left[ -a_5 \left( \frac{dh_1}{h_1} - \frac{d(h_0 + h_1)}{h_0 + h_1} \right) \right] \\
-a_3 \left( \frac{dh_1}{h_1} - \frac{d(h_0 + h_1)}{h_0 + h_1} \right) - a_2 \left( \frac{dh_2}{h_2} - \frac{d(h_0 + h_1)}{h_0 + h_1} \right) \\
a_4 \left( \frac{dh_2}{h_2} - \frac{d(h_0 + h_1)}{h_0 + h_1} \right) \\
a_5 \left( \frac{dh_2}{h_2} - \frac{d(h_0 + h_1)}{h_0 + h_1} \right)
\end{pmatrix}.
\]
The fifth column is

\[
\begin{pmatrix}
0 \\
-a_1 \left[ \frac{dh_1}{h_1} - \frac{d(h_0+h_1+h_2)}{h_0+h_1+h_2} \right] \\
0 \\
a_2 \left[ \frac{dh_2}{h_2} - \frac{d(h_0+h_1+h_2)}{h_0+h_1+h_2} \right] \\
(-a_3 - a_5) \left[ \frac{dh_1}{h_1} - \frac{d(h_0+h_1+h_2)}{h_0+h_1+h_2} \right] - a_4 \left[ \frac{dh_2}{h_2} - \frac{d(h_0+h_1+h_2)}{h_0+h_1+h_2} \right] \\
a_5 \left[ \frac{dh_1}{h_1} - \frac{dh_2}{h_2} \right]
\end{pmatrix}
\]

The sixth column is

\[
\begin{pmatrix}
a_2 \left[ \frac{d(h_1+h_2)}{h_1+h_2} - \frac{d(h_0+h_1+h_2)}{h_0+h_1+h_2} \right] \\
0 \\
(-a_1 - a_2) \left[ \frac{d(h_1+h_2)}{h_1+h_2} - \frac{d(h_0+h_1+h_2)}{h_0+h_1+h_2} \right] \\
a_2 \left[ \frac{dh_2}{h_2} - \frac{d(h_0+h_1+h_2)}{h_0+h_1+h_2} \right] \\
a_4 \left[ \frac{d(h_1+h_2)}{h_1+h_2} - \frac{dh_2}{h_2} \right] \\
(-a_3 - a_4) \left[ \frac{d(h_1+h_2)}{h_1+h_2} - \frac{d(h_0+h_1+h_2)}{h_0+h_1+h_2} \right] - a_5 \left[ \frac{dh_2}{h_2} - \frac{d(h_0+h_1+h_2)}{h_0+h_1+h_2} \right]
\end{pmatrix}
\]

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