CAUCHY PROBLEM FOR THE GENERALIZED DAVEY-STEWARTSON SYSTEMS IN BESOV SPACES AND SOME COUNTEREXAMPLES

Wenjing Song\textsuperscript{1} and Ganshan Yang\textsuperscript{2,†}

\textbf{Abstract} In this paper, the Cauchy problem of the generalized ellipse-ellipse type Davey-Stewartson systems is discussed. When the dimension of space is greater than or equal to two, we get a unique global solution in Besov spaces by contraction mapping argument. Moreover, by using the F-expansion method, the exact periodic wave solutions for the generalized ellipse-ellipse type Davey-Stewartson systems are discussed, some counter examples are given.

\textbf{Keywords} Davey-Stewartson systems, F-expansion method, multi-order exact solutions, Lam function, Cauchy problem.

\textbf{MSC(2010)} 31A30.

1. Introduction

A large amount of work are devoted to the study of Davey-Stewartson systems. The classical Davey-Stewartson systems

\[
\begin{aligned}
    iu_t + D_1 u &= r |u|^2 u + \mu uv, \\
    D_2 v &= D_3(|u|^2),
\end{aligned}
\]

(1.1)

are originally derived by Davey and Stewartson in [4] to describe quasi-monochromatic wave pockets on the surface of a shallow liquid. Here $D_1$, $D_2$ and $D_3$ are partial differential operators of the second order in $x, y (x, y) \in \mathbb{R}^2$. $D_1 = \delta \partial_x^2 + \partial_y^2$ is either elliptic or hyperbolic, and $D_2 = \partial_x^2 + m \partial_y^2 (m > 0)$ is elliptic. Later, the case that $D_3$ is hyperbolic, i.e. $m < 0$, is derived in [5] by taking account of the effect of surface tension (or capillary). Generally, $D_3 = \partial_x^2 u$ is a complex-valued function of $(t; x, y) \in R_+ \times \mathbb{R}^2$, and $v$ is a real-valued function of $(t; x, y) \in R_+ \times \mathbb{R}^2$. And $u$ and $v$ are related to the amplitude and the mean velocity potential of the water wave, respectively.

As in [7], the Davey-Stewartson equations are usually classified as elliptic-elliptic, elliptic-hyperbolic, hyperbolic-elliptic and hyperbolic-hyperbolic types according to the respective types of $D_1$ and $D_2$. In recent years, Davey-Stewartson equation-s(1.1) have drawn much attention from many physicists and mathematicians due...
to the abundant physical and mathematical properties. In [3], by using the generalized Kudryashov method, Demiray and Bulut found the dark soliton solutions of DSE systems. Later, by employing the Bäcklund transformation, the Hamiltonian approach and the G-expansion method to the DSEs, new traveling solitary and kink wave solutions are obtained, see [23]. The well-posedness, decay of solutions, soliton solutions, solitary and standing waves, etc., have been quite extensively studied by many authors (see [7,9,14,15,17,25,26]).

In [10], Wang and Guo study the generalized Davey-Stewartson systems

\[
\begin{align*}
    iu_t + Au &= \lambda_1 |u|^{p_1} u + \lambda_2 |u|^{p_2} u + \mu uv_{x_1}, \\
    Bv &= (|u|^2)_{x_1},
\end{align*}
\]

(1.2)

where \( A := \sum_{1 \leq i,j \leq n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}, \quad B := \sum_{1 \leq i,j \leq n} b_{ij} \frac{\partial^2}{\partial x_i \partial x_j}, \) \((a_{ij})\) and \((b_{ij})\) are real invertible matrix. They discuss the initial value problem of the Davey-Stewartson systems for the elliptic-elliptic and hyperbolic-elliptic cases. The local and global existence, as well as uniqueness of solutions in \( H^s \) are shown. Moreover, they prove that the scattering operator carries a band in \( H^s \) into \( H^s \). By using the Hirota’s bilinear method, the homoclinic orbits of the Davey-Stewartson Equations are obtained through the dependent variable transformation, see [30].

In [27], Wang shows that there exists \( s_c > 0 \) such that the cubic (quartic) non-elliptic derivative Schrödinger equations with small data in modulation spaces \( M^s_{2,1}(\mathbb{R}^n) \) for \( n \geq 3 (n = 2) \) are globally well-posed if \( s \geq s_c \), and ill-posed if \( s < s_c \).

It should be pointed that there is little result about the existence of solutions if the second equation of (1.2) is replaced by the following equations

\[
    Bv = (|u|^q u)_{x_1}, \quad q \neq 2.
\]

Furthermore, if \( \lambda_1 |u|^{p_1} u + \lambda_2 |u|^{p_2} u \) in the first equation of (1.2) is replaced by a more general nonlinear function \( f(u) \), the existence of solutions is also little known. And some physical models, such as Landau-Lifshitz equation, Navier Stokes equation, can be transformed into the equation with more general nonlinear function \( f(u) \). So it is more meaningful and more difficult to study the Davey-Stewartson systems with general nonlinear term \( f(u) \). System (1.1) and (1.2) are two important cases of general nonlinear Davey-Stewartson equation. Although our study (2.1) is a special case of general nonlinear Davey-Stewartson equation, it includes (1.1) and (1.2).

The purpose of this paper is to investigate the Cauchy problem of the two cases mentioned above, i.e., the generalized ellipse-ellipse and hyperbolic-ellipse type. When the dimension of space is greater than or equal to two, we get a unique global solution in Besov spaces by contraction mapping argument (see Section 2).

Naturally, the reader will curious about whether these results are valid for bounded domain? To answer those interesting questions, we construct some exact periodic wave solutions for the generalized ellipse-ellipse type Davey-Stewartson systems by using the F-expansion method, and some counter examples are given (see Section 3). Furthermore, we construct some exact periodic wave solutions to show the existence of solutions for generalized ellipse-hyperbolic and hyperbolic-hyperbolic type Davey-Stewartson systems which is still open (see Section 4).

This paper is based on many kinds of methods which are raised, such as homogeneous balance method [28], hyperbolic function expansion method [6], nonlinear
transformation method [11, 13], trial function method [19], sine-cosine method [29],
Jacobi elliptic function expansion method, etc. The solution gotten by these methods
are mainly solitary wave solutions, shock solutions, see [6, 11, 13, 19, 28, 29] and
elliptic function, see [20–22]. To search the stability of the solutions and inspired
by [18], we add perturbation in our research and discuss the evolution of the pertur-

bation. Essentially, it is to expand the solution of nonlinear evolution equation to
$\varepsilon$- power series and try to get the multi-order exact solutions of it. The symmetry
group properties of the variable coefficient Davey-Stewartson (vCDs) systems are
studied in [8]. The dromion of the Davey-Stewartson-1 equation is studied under
perturbation on the large time [12]. Through the Hirota bilinear method, Ma for-
mulate a combined fourth-order nonlinear equation while guaranteeing the existence
of lump solutions of new (2+1)-dimensional nonlinear equations [16].

2. Cauchy problem

In this section we study the Cauchy problem for the generalized Davey-Stewartson
systems

\[
\begin{cases}
iu_t + Au = f(u) + \mu uv_x, \\
Bv = (|u|^q u)_x, \\
u(0, x) = u_0(x),
\end{cases}
\]

(2.1)

where \( A := \sum_{1 \leq i, j \leq n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}, \) \( B := \sum_{1 \leq i, j \leq n} b_{ij} \frac{\partial^2}{\partial x_i \partial x_j}, \) \((a_{ij})\) and \((b_{ij})\) are real invertible matrices. \( \lambda, \mu \in \mathbb{C}. \) Let exist \( C > 0, \) satisfied

\[\sum b_{ij} \xi_i \xi_j \geq C |\xi|^2, \forall \xi \in \mathbb{R}^n.\] (2.2)

Denote \( E(\psi) = F^{-1} \left[ \frac{\xi}{\sum_{1 \leq i, j \leq n} b_{ij} \xi_i \xi_j} \right] F\psi, \) where \( F \) is Fourier transformation. Then (2.1) is equivalent to the following form

\[
\begin{cases}
iu_t + Au = f(u) + \mu E(|u|^q u)u, \\
u(0, x) = u_0(x).
\end{cases}
\] (2.3)

Here \( f(u) \in C^k(k \in \mathbb{Z}), \) is a nonlinear function and for any given \( p, \)

\[|f^{(k)}(u)| \leq C|u|^{p+1-k}, \quad k = 0, 1, \ldots.\] (2.4)

For brevity, in the following, we denote

\[s(p) = \frac{n}{2} - \frac{2}{p}, \quad \frac{2}{\gamma(r)} = n\left(\frac{1}{2} - \frac{1}{r}\right), \quad r(p) = \frac{2n(2+p)}{n(2+p) - 4},\]

(2.5)

where \( \frac{4}{n} \leq p < \infty, \quad 1 \leq r < \infty, \)

\[\alpha(n) = \begin{cases}
\infty, & n = 2, \\
\frac{2n}{n-2}, & n > 2.
\end{cases}\] (2.6)
Let \( n \geq 2, \frac{4}{n} \leq p, q < \infty, \max(s(q), s(p)) \leq s < \infty, [s] \leq p. \) If \( u_0 \in H^s \) and there exist a \( \delta > 0 \) such that \( \|u_0\|_{H^s} < \delta \), then there exists a unique solution \( u \) of the Cauchy problem (2.3) satisfying

\[
u \in C(0, \infty; H^s) \cap L^{p+2}(0, \infty; B_{r(p),2}^s) \cap L^{q+3}(0, \infty; B_{r(q),2}^s).
\]

Throughout this paper, we use a variety of function spaces, Lebesgue spaces \( L^r \), Bessel potential spaces \( H^{s,r} \), Besov spaces \( B_{r,2}^s \). The definition of \( L^r \) and \( H^{s,r} \) is as usual, and an equivalent definition of the norm on \( B_{r,2}^s \) is that

\[
\|u\|_{B_{r,2}^s} = \left( \int_0^\infty t^{2\langle s-|s| \rangle} \sum_{|\alpha|=|s|} \|\Delta_{\alpha} D^\alpha u\|_{L^r}^2 \frac{dt}{t} \right)^\frac{1}{2},
\]

where \( |s| \) denotes the largest integer less than or equal to \( s \), \( \Delta_{\alpha} u(x) = u(x + h) - u(x) = u_h - u \). For some additional basic results on Besov spaces, one can refer to [1, 24].

In the following, \( C \) stands for a constant that may be different numbers in different places. For any \( r \in [1, \infty] \), \( r' \) denotes the duality number of \( r \), i.e. \( \frac{1}{r} + \frac{1}{r'} = 1 \).

### 2.1. Main lemmas

The main tools used here are time spaces \( L^p - L^{p'} \) estimates for solutions of linear Schrödinger equations in Lebesgue space. These estimates are usually named generalized Strichartz inequalities. The method of the proof of the main result is a contraction mapping argument. Let us recall that some estimates for linear Schrödinger equations in Lebesgue-Besov spaces which are established by Cazenave and Weissler in [2].

**Lemma 2.1.** For all \( s \in R, r, q \in [2, \alpha(n)) \). \( S(t) \) is semi-group of operator and \( i\partial / \partial t + A \) is generating operator of \( S(t) \), then we have

(i) If \( u_0 \in \dot{H}^s \), then \( S(t)u_0 \in L^{\gamma'(r)}(0, \infty; \dot{B}_{r',2}^*) \), and there exists a constant \( C > 0 \), such that

\[
\|S(t)u_0\|_{L^{\gamma'(r)}(0, \infty; \dot{B}_{r',2}^*)} \leq C \|u_0\|_{\dot{H}^s};
\]

(ii) If \( f \in L^{\gamma(r)'}(0, \infty; \dot{B}_{r',2}^*) \), then \( \int_0^t S(t-\tau)f(\tau)d\tau \in L^{\gamma(q)}(0, \infty; \dot{B}_{r',2}^*) \), and there exists \( C > 0 \), such that

\[
\left\| \int_0^t S(t-\tau) f(\tau)d\tau \right\|_{L^{\gamma(q)}(0, \infty; \dot{B}_{r',2}^*)} \leq C \|f\|_{L^{\gamma(r)'}(0, \infty; \dot{B}_{r',2}^*)},
\]

for all \( f \in L^{\gamma(r)'}(0, \infty; \dot{B}_{r',2}^*) \).

The proof of this lemma can be found in [10].

**Lemma 2.2 ([10]).** Let \( 0 \leq s < \infty, 0 \leq r' < \infty, l_k, m_k, p_k, q_k > 0 \) and

\[
\frac{1}{r'} = \frac{1}{l_k} + \frac{1}{m_k} = \frac{1}{p_k} + \frac{1}{q_k}, k = 0, \cdots, [s].
\]
Let $\|uv\|_{H^s} \leq C \sum_{k=0}^{s} (\|u\|_{H^s} \|v\|_{H^s} + \|u\|_{H^s} \|v\|_{H^s} \|v\|_{H^s} \|v\|_{H^s})$. \hfill (2.11)

Lemma 2.3 ([10]). Let $-\infty < \sigma < \infty$, $1 < r$, $\mu < \infty$. Then there exists a constant $C > 0$ such that, for all $u \in \dot{H}^{\sigma}_{r,\mu}$,

$$\|E(u)\|_{\dot{H}^{\sigma}_{r,\mu}} \leq C \|u\|_{\dot{H}^{\sigma}_{r,\mu}}.$$ \hfill (2.12)

By a similar method in [9], one can have the following result.

Lemma 2.4. Let $s(p) \leq s < \frac{n}{2}$ and $\rho = \frac{2n(p + 2)}{n(p + 2) - 2}$. If $f \in C^{[s]+1}(R, R)$ satisfying one of the following conditions:

(i) $|f^{(k)}(u)| \leq C|u|^{p+1-k}$, where $k = 0, 1, \ldots, [s] + 1, [s] < p + 1$;

(ii) $|f^{(k)}(u)| \leq C|u|^{p+1-k}$, when $k < p + 1$; $f^{(k)}(u) = 0$, when $k < p + 1$.

Then

$$\|f(u)\|_{\dot{B}^{r}_{\rho,\mu}} \leq C \|u\|_{\dot{B}^{r}_{\rho,\mu}}.$$ \hfill (2.13)

Proof. The proof can be divided into the following steps.

Step 1 First, consider the case $[s - s(p)] \geq 1$, one has

$$\|f(u)\|_{\dot{B}^{r}_{\rho,\mu}} = \left(\int_{0}^{\infty} t^{-2(s-[s])} \sup_{|\alpha| \leq t} \sum_{|\alpha| = [s]} \|\triangle_{\alpha} D^{\alpha} u\|_{L^\rho}^2 \frac{dt}{t}\right)^{\frac{1}{2}},$$ \hfill (2.14)

and recalling that $|f^{(q)}(u)| \leq C|u|^{p+1-q}$ to obtain

$$|f^{(q)}(u) - f^{(q)}(v)| \leq (|u|^{p-q} + |v|^{p-q})|u - v|.$$ 

Notice that $[s] \geq 1$ and (2.11) to get

$$\sum_{|\alpha| = [s]} \|\triangle_{\alpha} D^{\alpha} f(u)\|_{L^\rho} \leq C \sum_{q=1}^{[s]} \sum_{\Lambda_{\sigma}^q} \|(u_{\alpha})_{\sigma} - (u_{\alpha})_{\sigma} - u_{\alpha} \|_{L^\rho} \|D^{\alpha} u\|_{L^\rho}$$

$$+ \sum_{q=1}^{[s]} \sum_{\Lambda_{\sigma}^q} \sum_{i=1}^{q} \|u_{\alpha} - u_{\alpha} \|_{L^\rho} \|D^{\alpha} u\|_{L^\rho}.$$ 

Let

$$\Gamma_1 := \|(u_{\alpha})_{\sigma} - (u_{\alpha})_{\sigma} - u_{\alpha} \|_{L^\rho},$$ \hfill (2.15)

$$\Gamma_2 := \sum_{i=1}^{q} \|u_{\alpha} - u_{\alpha} \|_{L^\rho},$$ \hfill (2.16)

$$\Gamma_3 := \sum_{j=1}^{i-1} \sum_{j=i+1}^{q} \|D^{\alpha} u\|_{L^\rho}.$$
thus
\[
\sum_{[\alpha]=[s]} \|\triangle h D^{\alpha} f(u)\|_{L_{D}} \leq C \sum_{q=1}^{[s]} \sum_{\Lambda_{[s]}^{q}} (\Gamma_{1} + \Gamma_{2}). \tag{2.17}
\]

Next, we estimate \(\Gamma_{1}\) and \(\Gamma_{2}\), respectively. Without lose of generality, it can be considered \(\Lambda_{q}^{[s]} \geq |\alpha_{q}| \geq |\alpha_{q-1}| \geq \cdots \geq |\alpha_{2}| \geq |\alpha_{1}|\). Firstly, when \(q = 1\), let
\[
\begin{align*}
a_{0} &= (p-1)(\frac{1}{\rho} - \frac{s - s(p)}{n}), \\
a_{1} &= \frac{1}{\rho} - \frac{s - s(p)}{n}, \\
a_{2} &= \frac{1}{\rho}.
\end{align*}
\]

It is easy to see \(a_{0}, a_{1}, a_{2} > 0\), and
\[
a_{0} + a_{1} + a_{2} = p(\frac{1}{\rho} - \frac{s - s(p)}{n}) + 1 - \frac{1}{\rho} = 1 - \rho'. \tag{2.18}
\]

By using \(\dot{B}_{\rho}^{s-s(p)} \hookrightarrow \dot{H}_{\rho}^{s-s(p)}\), (2.15), and Hölder inequality, one gets
\[
\begin{align*}
\Gamma_{1} &\leq C\|u\|_{\dot{B}_{\rho}^{s-s(p)}}^{-1} \|u_{h} - u\|_{L_{\rho}^{s}}^{-1} \|u\|_{L_{\rho}^{s}} \\
&\leq C\|u\|_{\dot{B}_{\rho}^{s-s(p)}}^{-1} \|u_{h} - u\|_{L_{\rho}^{s}}^{-1} \|u\|_{L_{\rho}^{s}}. \tag{2.19}
\end{align*}
\]

Since \([s - s(p)] \leq [s]\), then
\[
\begin{align*}
\Gamma_{1} &\leq C\|u\|_{\dot{B}_{\rho}^{s-s(p)}}^{-1} \|u\|_{\dot{B}_{\rho}^{[s]}} \|u\|_{\dot{B}_{\rho}^{s}} \\
&\leq C\|u\|_{\dot{B}_{\rho}^{s-s(p)}}^{-1} \|u\|_{\dot{B}_{\rho}^{s-s(p)}} \|u\|_{\dot{B}_{\rho}^{[s]}} \\
&\leq C\|u\|_{\dot{B}_{\rho}^{s-s(p)}}^{-1} \|u\|_{\dot{B}_{\rho}^{s}}. \tag{2.20}
\end{align*}
\]

Second, consider the case \(q \geq 2\), by (2.15), and let
\[
\begin{align*}
a_{0} &= (p - q)(\frac{1}{\rho} - \frac{s - s(p)}{n}), \\
a'_{0} &= \frac{1}{\rho} - \frac{s - s(p)}{n}, \\
a_{i} &= \frac{1}{\rho} - \frac{s - s(p) - |a_{i}|}{n}, \quad i = 1, 2, \cdots, q - 1, \\
a_{q} &= \frac{1}{\rho} - \frac{[s] - |a_{q}|}{n}.
\end{align*}
\]

Clearly \(a_{0}, a'_{0}, a_{i} > 0\), \((i = 1, 2, \cdots, q)\) and
\[
a_{0} + a'_{0} + \sum_{i=1}^{q} a_{i} = p(\frac{1}{\rho} - \frac{s - s(p)}{n}) + 1 - \frac{1}{\rho'} \tag{2.21}
\]
By H"older inequality, (2.15) yields
\[
\Gamma_1 \leq C \|u\|^{p-q}_{\dot{B}^s_{\rho^{-s(p)}}} \|u\| \prod_{i=1}^{q-1} \|D^{\alpha_i} u\|_{\dot{H}^{s-s(p)}_{\rho^{s}}} \|u\|_{\dot{H}^{[s]}_{\rho}}. \tag{2.22}
\]

Since \(\dot{B}^s_{\rho^{-s(p)}} \hookrightarrow \dot{H}^{s-s(p)}_{\rho}\), then (2.22) implies
\[
\Gamma_1 \leq C \|u\|^p_{\dot{B}^s_{\rho^{-s(p)}}} \|u\|_{\dot{B}^{[s]}_{\rho}}. \tag{2.23}
\]

Let’s consider \(\Gamma_2 := \sum_{i=1}^{q} \prod_{j=1}^{q-i} D^{\alpha_i} u_h \prod_{j=i+1}^{q} D^{\alpha_i} (u_h - u) \|u_h\|^{p-q+1}_{L^{p'}}\). There are also two scenarios to consider. First of all, when \(q = 1\),
\[
\Gamma_2 = \|D^{\alpha_1} (u_h - u)\|_{L^{p'}}. \tag{2.24}
\]

Let
\[
a_0 = p\left(\frac{1}{\rho} - \frac{s-s(p)}{n}\right),
\]
\[
a_1 = \frac{1}{\rho} - \frac{|\alpha_1| - |\alpha_1|}{n},
\]

then \(a_0, a_1 > 0\), and \(a_0 + a_1 = \frac{1}{\rho'}\) and by H"older inequality, one gets
\[
\Gamma_2 \leq C \|u\|^p_{\dot{B}^s_{\rho^{-s(p)}}} \|u\|_{\dot{B}^{[s]}_{\rho}}. \tag{2.25}
\]

Notice that \(q = 1, |\alpha_1| = [s]\), thus
\[
\Gamma_2 \leq C \|u\|^p_{\dot{B}^s_{\rho^{-s(p)}}} \|u\|_{\dot{B}^{[s]}_{\rho}}. \tag{2.26}
\]

Second, let’s consider the case \(q \geq 2\),
\[
a_0 = (p-q+1)\left(\frac{1}{\rho} - \frac{s-s(p)}{n}\right),
\]
\[
ai = \frac{1}{\rho} - \frac{s-s(p) - |\alpha_i|}{n}, \quad i = 1, 2, \ldots, q-1,
\]
\[
aq = \frac{1}{\rho} - \frac{|s| - |aq|}{n},
\]

thus, \(a_0, a_i > 0, (i = 1, 2, \ldots, q)\)
\[
a_0 + \sum_{i=1}^{q} a_i = p\left(\frac{1}{\rho} - \frac{s-s(p)}{n}\right) + 1 = \frac{1}{\rho'}.
\tag{2.27}
\]

It follows from H"older inequality,
\[
\Gamma_2 \leq C \|u\|^{p-q+1}_{\dot{B}^s_{\rho^{-s(p)}}} \|u\|^{q-1}_{\dot{B}^s_{\rho^{-s(p)}}} \|u\|_{\dot{B}^{[s]}_{\rho}} \leq C \|u\|^{p}_{\dot{B}^s_{\rho^{-s(p)}}} \|u\|_{\dot{B}^{[s]}_{\rho}}, \tag{2.28}
\]
which combination (2.20), (2.23), (2.26) and (2.28) yields
\[
\|f(u)\|_{\dot{B}_{p}^{s-r(p)}} \leq C\|u\|_{\dot{B}_{p}^{s-r(p)}}^{p}\|u\|_{\dot{B}_{p}^{s-r(p)}},
\]  
(2.29)

**Step 2** Let’s consider the case \([s-s(p)] = 0,\)
\[
\|f(u)\|_{\dot{B}_{p}^{s-r(p)}} = \left( \int_{0}^{\infty} t^{-2(s-s(p))} \sup_{|h|\leq t} \|\Delta_{h} f(u)\|_{L^{p'}}^{2} \frac{dt}{t} \right)^{\frac{1}{2}}.
\]  
(2.30)

Since \(\dot{B}_{p}^{s-r(p)} \hookrightarrow \dot{H}_{r}^{s}\) and \(|f(u)| \leq |u|^{p+1},\) one has
\[
\frac{1}{\beta} = \frac{1}{p_{r}} - \frac{s-s(p)}{n},
\]
and utilizing Hőlder inequality to get
\[
\|f(u)\|_{\dot{B}_{p}^{s-r(p)}} = \|\|u\|_{p}^{p+1}\|_{\dot{B}_{p}^{s-r(p)}} \leq C\|u\|_{\dot{B}_{p}^{s-r(p)}}^{p}\|u\|_{\dot{B}_{p}^{s-r(p)}} \leq C\|u\|_{\dot{B}_{p}^{s-r(p)}}^{p}\|u\|_{\dot{B}_{p}^{s-r(p)}}.
\]  
(2.32)

From (2.29) and (2.32), it can be seen (2.13) holds which is
\[
\|f(u)\|_{\dot{B}_{p}^{s-r(p)}} \leq C\|u\|_{\dot{B}_{p}^{s-r(p)}}^{p}\|u\|_{\dot{B}_{p}^{s-r(p)}}.
\]

Thus the proof of this lemma is completed. \(\square\)

**Lemma 2.5.** Let \(n \geq 2, \frac{4}{n} \leq q < \infty, \lfloor s \rfloor \leq q, \ 0 \leq s < \infty, \ r = r(q),\) then we have
\[
\|E(|u|^q u)\|_{\dot{B}_{r}^{s-[r]}} \leq C\|u\|^q_{\dot{B}_{r}^{s-[r]}} \|u\|^2_{\dot{B}_{r}^{s-[r]}},
\]  
(2.33)

**Proof.** From Lemma 2.2, it is easy to see that
\[
\|E(|u|^q u)\|_{\dot{B}_{r}^{s-[r]}} \leq C\sum_{k=0}^{[s]} \left[ \|u\|_{\dot{H}^{k,p_{k}}} \|E(|u|^q u)\|_{\dot{B}_{r}^{s-k}} + \|u\|_{\dot{H}^{s-k}} \|E(|u|^q u)\|_{\dot{H}_{r}^{k,m_k}} \right],
\]  
(2.34)

where \(\frac{1}{p_{r}} = \frac{1}{k} + \frac{1}{m_k}, \ \frac{1}{p_{r}} = \frac{1}{p_k} + \frac{1}{q_k}, \ k = 0, 1, \ldots, \lfloor s \rfloor.\)

In the following, we estimate \(I\) and \(II,\) firstly,
\[
\dot{H}^{k,p_{k}} \supseteq \dot{B}_{r}^{s},
\]
and
\[
\|u\|_{\dot{H}^{k,p_{k}}} \leq C\|u\|_{\dot{B}_{r}^{s-[r]}}.
\]  
(2.35)
Setting $\frac{1}{p_u} = 1 - \frac{1}{r} - \frac{k}{n}, \frac{1}{q_u} = \frac{k}{n}$, then

$$
\|E(|u|^q u)\|_{\dot{B}^{r_q}_{t,q,2}} \leq C \|E(|u|^{q_u} u)\|_{\dot{B}^{r_q}_{t,q,2}}. \quad (2.36)
$$

From Lemma 2.3 and Lemma 2.4, one has

$$
\|E(|u|^q u)\|_{\dot{B}^{r_q}_{t,q,2}} \leq C \|u|^{q_u} u\|_{\dot{B}^{r_q}_{t,q,2}} \leq C \|u|^{q_u} u\|_{\dot{B}^{r_q}_{t,q,2}} \leq C \|u\|_{\dot{B}^{r_q}_{t,q,2}}. \quad (2.37)
$$

Thus,

$$
I \leq C \|u\|_{\dot{B}^{r_q}_{t,q,2}}^q \|u\|_{\dot{B}^{r_q}_{t,q,2}}^2. \quad (2.38)
$$

Similarly,

$$
II \leq C \|u\|_{\dot{B}^{r_q}_{t,q,2}}^q \|u\|_{\dot{B}^{r_q}_{t,q,2}}^2. \quad (2.39)
$$

Then, one gets

$$
\|E(|u|^q u)u\|_{\dot{B}^{r_q}_{t,q,2}} \leq C \|u\|_{\dot{B}^{r_q}_{t,q,2}}^q \|u\|_{\dot{B}^{r_q}_{t,q,2}}^2, \quad (2.40)
$$

and the proof of this lemma is completed.

\square

2.2. The proof of Theorem 2.1

The Cauchy problem of (2.1) is essentially equivalent to the following integral equation

$$
u(t) = S(t)u_0 - i \int_0^t S(t - \tau) F(u(\tau)) d\tau, \quad (2.41)$$

where $F(u) = f(u) + \mu E(|u|^q u).

For all $\delta > 0$, define

$$
D = \{ u \in L^{p+2}(0, \infty; \dot{B}^r_{t,q,2}) \cap L^{q+3}(0, \infty; \dot{B}^r_{t,q,2}) : \|u\|_{L^{p+2}(0, \infty; \dot{B}^r_{t,q,2})} \cap L^{q+3}(0, \infty; \dot{B}^r_{t,q,2}) \leq \delta \}. \quad (2.42)
$$

And for any $u, v \in D$, the metric $d(u, v)$ is

$$
d(u, v) = \|u - v\|_{L^{p+2}(0, \infty; L^r(t)) \cap L^{q+3}(0, \infty; L^r(t))}. \quad (2.43)
$$

Considering the mapping

$$
J : u(t) \rightarrow S(t)u_0 - i \int_0^t S(t - \tau) F(u(\tau)) d\tau, \quad (2.44)
$$

and we claim that $J : (D, d) \rightarrow (D, d)$ is a contraction mapping. To show this claim, in view of Lemma 2.4 and Lemma 2.5 to get

$$
\|f(u)\|_{\dot{B}^{r_{t(p,q)+}}_{t,q,2}} \leq C \|u\|_{\dot{B}^{r_{t(p,q)+}}_{t,q,2}}^{p+1}, \quad (2.45)
$$

and

$$
\|E(|u|^q u)u\|_{\dot{B}^{r_q}_{t,q,2}} \leq C \|u\|_{\dot{B}^{r_q}_{t,q,2}}^{q+2}. \quad (2.46)
$$
From \( \frac{1}{(p+2)} = \frac{p+1}{p+2} \) and \( \frac{1}{(q+3)} = \frac{q+2}{q+3} \), one gets

\[
\|f(u)\|_{L^{(p+2)'}(0, \infty; B^s_{r(p), 2})} \leq C \|u\|_{L^{(p+2)}(0, \infty; B^s_{r(p), 2})}^{p+1},
\]

and

\[
\|E(|u|^q) u\|_{L^{(q+3)'}(0, \infty; B^s_{r(q), 2})} \leq C \|u\|_{L^{(q+3)}(0, \infty; B^s_{r(q), 2})}^{q+2}.
\]

So, for any \( u \in D \),

\[
\|Ju\| \leq \|S(t)u_0\| + \left\| \int_0^t S(t-\tau)F(u(\tau))d\tau \right\|_L
\]

\[
\leq C \|u_0\|_{H^s} + C \left( \|f(u)\|_{L^{(p+2)'}(0, \infty; B^s_{r(p), 2})} + \|E(|u|^q) u\|_{L^{(q+3)'}(0, \infty; B^s_{r(q), 2})} \right)
\]

\[
\leq C \|u_0\|_{H^s} + 2C(p+1 + q+2)
\]

\[
\leq C \|u_0\|_{H^s} + 2 \delta\]

where

\[
L := L^{p+2}(0, \infty; B^s_{r(p), 2}) \cap L^{q+3}(0, \infty; B^s_{r(q), 2}).
\]

Now, one can get \( J : D \to D \). Further, for any \( u, v \in D \),

\[
d(Ju, Jv)
\]

\[
= \|Ju - Jv\|_{L^{p+2}(0, \infty; L^{r(p)}) \cap L^{q+3}(0, \infty; L^{r(q)})}
\]

\[
= \left\| \int_0^t S(t-\tau)(F(u(\tau)) - F(v(\tau)))d\tau \right\|_{L^{p+2}(0, \infty; L^{r(p)}) \cap L^{q+3}(0, \infty; L^{r(q)})}
\]

\[
\leq C \|f(u) - f(v)\|_{L^{(p+2)'}(0, \infty; L^{r(p)})} + C \|E(|u|^q) u - E(|v|^q) v\|_{L^{(q+3)'}(0, \infty; L^{r(q)})}
\]

\[
\leq C \|u - v\| \left( \|u\|^p + \|v\|^p \right)_{L^{(p+2)'}(0, \infty; L^{r(p)})}
\]

\[
+ C \|u - v\| \left( E(|u|^q) u - E(|v|^q) v \right)_{L^{(q+3)'}(0, \infty; L^{r(q)})}
\]

\[
\leq C \|u - v\|_{L^{p+2}(0, \infty; L^{r(p)}) \cap L^{q+3}(0, \infty; L^{r(q)})} \left( \|u\|_{L^{(p+2)}(0, \infty; L^{r(p)})} + \|v\|_{L^{(p+2)}(0, \infty; L^{r(p)})}
\]

\[
+ \|u\|_{L^{q+3}(0, \infty; L^{r(q)})} + \|v\|_{L^{q+3}(0, \infty; L^{r(q)})} \right)
\]

\[
\leq C \left( \|u\|_{L^{p+2}(0, \infty; L^{r(p)})} + \|v\|_{L^{q+3}(0, \infty; L^{r(q)})} \right) (\delta^p + \delta^{q+1})
\]

\[
\leq \frac{1}{2} d(u,v).
\]

Thus, \( J \) is a contraction mapping on \( (D, d) \), and has a unique fixed point \( u \in D \). From Lemma 1, we deduce that there exists a unique solution \( u \) of the Cauchy problem (2.1) satisfying

\[
u \in C(0, \infty; H^s) \cap L^{p+2}(0, \infty; B^s_{r(p), 2}) \cap L^{q+3}(0, \infty; B^s_{r(q), 2}).
\]

This finishes the proof of the Theorem.
3. Explicit periodic wave solutions and some counter examples

The F-expansion method is the generalization of Jacobi elliptic function expansion method. In this section we mainly consider the general Davey-Stewartson systems

\[
\begin{cases}
  iu_t + \Delta u + r |u|^2 u - \mu uv_{x_1} = 0, \\
  \Delta v + b(|u|^2)_{x_1} = 0,
\end{cases}
\]

(3.1)

where \( u \) is a complex-valued function, \( r, \mu, b \) are real constants.

Let

\[ u = \exp(i\eta)w(x,t), \quad \eta = \sum_{i=1}^{n} \alpha_i x_i + \lambda t + \eta_0, \]

(3.2)

where \( w(x,z) \) is real, \( \alpha_i (i = 1, 2, \cdots, n) \), \( \lambda \) are undetermined coefficients, \( \eta_0 \) is an arbitrary \( n \)-dimensional constant vector.

From (3.2), one gets

\[ u_t = i\lambda \exp(i\eta)w + \exp(i\eta)w_t, \]

(3.3)

and

\[ u_{x_i x_i} = -\alpha_i^2 \exp(i\eta)w + 2\alpha w_{x_i} \exp(i\eta)i + \exp(i\eta)w_{x_i x_i}. \]

(3.4)

Combining (3.2) and (3.3)-(3.4), it follows that

\[
\begin{cases}
  w_t + 2\sum_{i=1}^{n} \alpha_i w_{x_i} = 0, \\
  \Delta w + r w^3 - \mu w v_{x_1} - (\lambda + \sum_{i=1}^{n} \alpha_i^2)w = 0, \\
  \Delta v + b(w^2)_{x_1} = 0.
\end{cases}
\]

(3.5)

Supposing the problem (3.5) has wave solution as follows

\[ w = w(\xi) = w(\sum_{i=1}^{n} k_i x_i + nt + \xi_0), \]

(3.6)

and

\[ v = v(\xi) = v(\sum_{i=1}^{n} k_i x_i + nt + \xi_0), \]

(3.7)

where \( k_i (i = 1, 2, \cdots, n) \), are undetermined constants, \( \xi_0 \) is an arbitrary constant.

Combining (3.5) and (3.6)-(3.7), one can gets simultaneous differential equations of \( w(\xi), v(\xi) \),

\[
\begin{align*}
  n + 2\sum_{i=1}^{n} \alpha_i k_i &= 0, \\
  (\sum_{i=1}^{n} k_i^2) w'' + r w^3 - \mu k_1 w - (\lambda + \sum_{i=1}^{n} \alpha_i^2)w &= 0, \\
  (\sum_{i=1}^{n} k_i^2) v'' + 2b k_1 w w' &= 0.
\end{align*}
\]

(3.8)

(3.9)

(3.10)

In view of the F-expansion, the homogeneous balance of \( (\sum_{i=1}^{n} k_i^2) w'' \) and \( r w^3 - \mu k_1 w v' \) in (3.9), \( (\sum_{i=1}^{n} k_i^2) v'' \) and \( 2b k_1 w w' \) in (3.10) should be considered. So, let

\[
\begin{align*}
  w &= a_1 F + a_0, \\
  v &= b_1 F + b_0,
\end{align*}
\]

(3.11)

(3.12)
where $a_0, a_1, b_0, b_1$ are undetermined constants, $F(\xi)$ satisfies

$$F' = PF^4 + QF^2 + R, \quad (3.13)$$

where $P, Q, R$ are real constants.

Combining (3.8)-(3.12), one gets the polynomials of $F(\xi)$,

$$\begin{align*}
[a_1(S_0^n k_0^2)(2PF^3 + QF) + r(a_1 F + a_0)^3 - (\lambda + \Sigma_{i=1}^n \alpha_i^2)(a_1 F + a_0)]^2 \\
- [\mu k_1 b_1(a_1 F + a_0)]^2 (PF^4 + QF^2 + R) = 0.
\end{align*} \quad (3.14)$$

$$\begin{align*}
[b_1(S_0^n k_0^2)(2PF^3 + QF)^2 - [2a_1 b k_1(a_1 F + a_0)]^2 (PF^4 + QF^2 + R) = 0. \quad (3.15)
\end{align*}$$

Setting the coefficients of the polynomials to zeros, one can get the functions of the undetermined parameters as follows,

$$\begin{align*}
F^6 : & \quad 4(S_0^n k_0^2)^2 P^2 + r^2 a_4^2 + 4r(S_0^n k_0^2)P a_1^2 = \mu^2 k_1^2 b_1^2 P, \quad (3.16) \\
F^5 : & \quad 3a_3^2 a_0 r^2 + 6(S_0^n k_0^2) a_1 a_0 P = \mu^2 k_1^2 b_1^2 P a_0, \quad (3.17) \\
F^4 : & \quad 4a_2^2 (S_0^n k_0^2)^2 PQ + 9r^2 a_1^2 a_3^2 + 6r^2 a_1^2 a_2^2 + 2a_1^2 r (S_0^n k_0^2) Q \\
& \quad + 12a_1^2 a_2^2 r (S_0^n k_0^2) P - 4a_1^2 (\lambda + S_0^n \alpha_i^2)(S_0^n k_0^2) P \\
& \quad - 2r a_1^2 (\lambda + S_0^n \alpha_i^2) = \mu^2 k_1^2 b_1^2 (a_1^2 Q + a_2^2 P), \quad (3.18) \\
F^3 : & \quad a_1^2 a_0^3 a_2^2 + 9a_1 a_0^2 r^2 + 3a_1 a_0 r (S_0^n k_0^2) - 2P a_0 (S_0^n k_0^2) (\lambda + S_0^n \alpha_i^2) \\
& \quad + 2P a_1 a_0^3 r (S_0^n k_0^2) - r (\lambda + S_0^n \alpha_i^2)(3a_1 a_0 + a_1^3 a_0) = \mu^2 k_1^2 b_1^2 Q a_0, \quad (3.19) \\
F^2 : & \quad a_1^2 (S_0^n k_0^2)^2 Q^2 + 9r^2 a_1^2 a_3^2 + 6r^2 a_1^2 a_2^2 + a_1^2 (\lambda + S_0^n \alpha_i^2)^2 \\
& \quad - 2a_1^2 Q (\lambda + S_0^n \alpha_i^2)(S_0^n k_0^2) - 6r (\lambda + S_0^n \alpha_i^2)(a_1^2 a_2^2 + a_1 a_0^2) \\
& \quad + 6a_1^2 a_2^2 Q r (S_0^n k_0^2) = \mu^2 k_1^2 b_1^2 (a_1^2 Q + a_2^2 Q), \quad (3.20) \\
F^1 : & \quad 3r^2 a_1^2 (\lambda + S_0^n \alpha_i^2)^2 a_0^3 - Q(\lambda + S_0^n \alpha_i^2)(S_0^n k_0^2) a_0 \\
& \quad + r Q (S_0^n k_0^2) a_0 - 4r (\lambda + S_0^n \alpha_i^2) a_0^3 = \mu^2 k_1^2 b_1^2 R a_0, \quad (3.21) \\
F^0 : & \quad r^2 a_0^6 + (\lambda + S_0^n \alpha_i^2)^2 a_0^2 - 2r (\lambda + S_0^n \alpha_i^2) a_0^4 = \mu^2 k_1^2 b_1^2 R a_0^2, \quad (3.22) \\
F^6 : & \quad P^2 (S_0^n k_0^2)^2 b_1^2 = b_1^2 k_1^2 P a_1^4, \quad (3.23) \\
F^5 : & \quad b_1^2 k_1^2 P a_1^3 a_0 = 0, \quad (3.24) \\
F^4 : & \quad P Q (S_0^n k_0^2)^2 b_1^2 = b_1^2 k_1^2 a_1^2 (a_1^2 Q + a_2^2 P), \quad (3.25) \\
F^3 : & \quad b_1^2 k_1^2 Q a_1^3 a_0 = 0, \quad (3.26) \\
F^2 : & \quad Q^2 (S_0^n k_0^2)^2 b_1^2 = 4b_1^2 k_1^2 a_1^2 (a_1^2 R + a_2^2 Q), \quad (3.27) \\
F^1 : & \quad b_1^2 k_1^2 R a_1^3 a_0 = 0, \quad (3.28) \\
F^0 : & \quad b_1^2 k_1^2 R a_0^3 = 0. \quad (3.29)
\end{align*}$$
Solving the algebraic equations (3.16)-(3.29) to get

\[
\begin{align*}
  a_0 &= 0, a_1 = \pm (\Sigma_{i=1}^{n} k_i^2) \sqrt{\frac{-2P}{r(\Sigma_{i=1}^{n} k_i^2) + \mu b k_1^2}}, \\
  b_0 &= \text{const}, b_1 = \pm \frac{2bk_1 (\Sigma_{i=1}^{n} k_i^2)}{r(\Sigma_{i=1}^{n} k_i^2) + \mu b k_1^2} \sqrt{P}, \\
  \lambda &= (\Sigma_{i=1}^{n} k_i^2)Q - (\Sigma_{i=1}^{n} \alpha_i^2), \\
  Q^2 &= 4PR, 
\end{align*}
\]

(3.30)

where \(k_i, \alpha_i(i = 1, 2, \cdots, n)\) are constants, and \(r(\Sigma_{i=1}^{n} k_i^2) + \mu b k_1^2 < 0\).

Since \(Q^2 = 4PR\), in view of (3.13), one has

\[
F = \frac{R}{P}(\tan[\sqrt{PR}(\xi + c)])^4, 
\]

(3.31)

where \(c\) is a constant.

Combining (3.30)-(3.31) and (3.11)-(3.12), and in view of (3.3), one can get solutions of (3.1), which are as follows

\[
\begin{align*}
  u &= \pm (\Sigma_{i=1}^{n} k_i^2) \sqrt{\frac{-2P}{r(\Sigma_{i=1}^{n} k_i^2) + \mu b k_1^2}} \exp(i\eta) \frac{R}{P}(\tan[\sqrt{PR}(\xi + c)])^4, \\
  v &= b_0 \pm \frac{2bk_1 (\Sigma_{i=1}^{n} k_i^2)}{r(\Sigma_{i=1}^{n} k_i^2) + \mu b k_1^2} \frac{R}{\sqrt{P}}(\tan[\sqrt{PR}(\xi + c)])^4, 
\end{align*}
\]

(3.32) (3.33)

where

\[
\begin{align*}
  \eta &= \Sigma_{i=1}^{n} \alpha_i x_i + [(\Sigma_{i=1}^{n} k_i^2)Q - (\Sigma_{i=1}^{n} \alpha_i^2)]t + \eta_0, \\
  \xi &= \Sigma_{i=1}^{n} k_i x_i - 2(\Sigma_{i=1}^{n} \alpha_i k_i) t + \xi_0, 
\end{align*}
\]

(3.34)

\(k_i, \alpha_i\) are constants, \(r\Sigma_{i=1}^{n} k_i^2 + \mu b k_1^2 < 0, \eta_0\) is an arbitrary constant.

**Remark 3.1.** (3.32) and (3.33) show that if \(R^n\) is replaced by bounded domain, then there are some counter examples for nonhomogeneous initial values problems to elliptic-elliptic Davey-Stewartson systems.

### 4. Multi-order exact solutions

#### 4.1. Lam equation and Lam function

In this chapter, we aim to construct Multi-order exact solutions for DSI (Davey-Stewartson systems of elliptic-hyperbolic types). First, we recall the Lam equation and Lam function. Usually, the Lam equation of \(y(x)\) can be written as

\[
\frac{d^2y}{dx^2} + [\lambda - n(n + 1)m^2 sn^2 x]y = 0, 
\]

(4.1)

where \(\lambda\) is eigenvalue, \(n\) is positive integer, \(snx\) is Jacobi elliptic sine function, \(m\) is the modulus and \(0 < m < 1, x \in R^1\) in this subsection.

Making a change of independent variable

\[
z = sn^2 x, 
\]

(4.2)
then, (4.1) is rewritten as
\[
\frac{d^2 y}{dz^2} + \frac{1}{2} \frac{1}{z} \left( \frac{1}{z - 1} + \frac{1}{z - h} \right) \frac{dy}{dz} - \frac{\mu n(n + 1) z}{4z(z - 1)(z - h)} y = 0, \tag{4.3}
\]
where \( h = m^{-2} > 1, \mu = -h \lambda \). The equation (4.3) is a Fuch-type equation which has four singular points, i.e. \( z = 0, 1, h, \infty \). The solution of (4.3) is called Lam function.

Especially,

(i) When \( n = 2, \lambda = 1 + m^2, \mu = -(1 + m^2) \), the Lam equation is
\[
\frac{d^2 y}{dx^2} + [(1 + m^2) - 6m^2 \text{sn}^2 x] y = 0. \tag{4.4}
\]
The corresponding Lam function is defined by
\( L^c_2(x) \equiv \text{snxdnx} \).

(ii) When \( n = 2, \lambda = (1 + 4m^2) \), the Lam equation is
\[
\frac{d^2 y}{dx^2} + [(1 + 4m^2) - 6m^2 \text{sn}^2 x] y = 0. \tag{4.5}
\]
The corresponding Lam function is defined by
\( L^c_2(x) \equiv \text{snxdnx} \).

(iii) When \( n = 2, \lambda = 4 + m^2 \), the Lam equation is
\[
\frac{d^2 y}{dx^2} + [(4 + m^2) - 6m^2 \text{sn}^2 x] y = 0. \tag{4.7}
\]
The corresponding Lam function is defined by
\( L^d_2(x) \equiv \text{snxcnx} \).

(iv) When \( n = 3, \lambda = 4(1 + m^2), [\mu = -4(1 + m^{-2})] \), the Lam equation is
\[
\frac{d^2 y}{dx^2} + [4(1 + m^2) - 12m^2 \text{sn}^2 x] y = 0. \tag{4.9}
\]
The corresponding lam function is defined by
\( L_3(x) \equiv z^{1/2}(1 - z)^{1/2}(1 - h^{-1}z)^{1/2} = \text{snxcnxdxn} \).

4.2. Multi-order exact solutions of DSI

In this section, we shall consider the following elliptic-hyperbolic types systems (DSI)
\[
\begin{align*}
iu_t + \Delta u + r |u|^2 u - 2uv & = 0 \\
\sum_{j=1}^l v_{x_j x_j} - \sum_{j=l+1}^n v_{x_j x_j} - \sum_{j=1}^k r_j \sum_{j=l+1}^n (|u|^2)_{x_j x_j} & = 0, \quad 1 \leq l, k < n.
\end{align*}
\tag{4.11}
\]
Setting
\[ u = \exp(in)w(x,t), \quad x \in R^n, \quad \eta = \sum_{i=1}^{n} \alpha_i x_i + \lambda t + \eta_0, \] (4.12)
\[ w = w(\xi) = w(\sum_{i=1}^{n} k_i x_i + nt + \xi_0), \] (4.13)
\[ v = v(\xi) = v(\sum_{i=1}^{n} k_i x_i + nt + \xi_0). \] (4.14)

Thus, (4.11) can be rewrite as
\[ n + 2\sum_{j=1}^{n} \alpha_j k_j = 0, \] (4.15)
\[ \left( \sum_{j=1}^{n} k_j^2 \right) w'' + rw^3 - 2wv - (\lambda + \sum_{j=1}^{n} \alpha_j^2)w = 0. \] (4.16)
\[ \left( \sum_{j=1}^{l} k_j^2 - \sum_{j=l+1}^{n} k_j^2 \right) v'' - 2\sum_{j=1}^{k} r_j k_j^2 (w'' + w w'') = 0, \] (4.17)

Let
\[ w = w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \cdots, \] (4.18)
\[ v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \cdots, \] (4.19)

where \( 0 < \varepsilon << 1 \), \( w_0, w_1, w_2 \cdots v_1, v_2 \cdots \) are the exact solutions of the zeroth-order equation, the first-order equation and the second-order equation and so on, respectively.

Combining (4.16)-(4.19), one gets equation of each order. The equation of \( \varepsilon^0 \)-order is
\[ \begin{cases}
\left( \sum_{j=1}^{n} k_j^2 \right) w_0'' + rw_0^3 - 2w_0v_0 - (\lambda + \sum_{j=1}^{n} \alpha_j^2)w_0 = 0, \\
\left( \sum_{j=1}^{l} k_j^2 - \sum_{j=l+1}^{n} k_j^2 \right) v_0'' - 2\sum_{j=1}^{k} r_j k_j^2 (w_0'' + w_0 w_0'') = 0.
\end{cases} \] (4.20)

The equation of \( \varepsilon^1 \)-order is
\[ \begin{cases}
\left( \sum_{j=1}^{n} k_j^2 \right) w_1'' + 3rw_0^2 w_1 - 2(w_0 v_1 + w_1 v_0) - (\lambda + \sum_{j=1}^{n} \alpha_j^2)w_0 = 0, \\
\left( \sum_{j=1}^{l} k_j^2 - \sum_{j=l+1}^{n} k_j^2 \right) v_1'' - 2\sum_{j=1}^{k} r_j k_j^2 (w_0'' w_1' + w_0 w_0'' + w_0'' w_1) = 0.
\end{cases} \] (4.21)

The equation of \( \varepsilon^2 \)-order is
\[ \begin{cases}
\left( \sum_{j=1}^{n} k_j^2 \right) w_2'' + 3rw_0^2 w_2 + w_0 w_2^2 - 2(w_0 v_2 + w_1 v_1 + w_2 v_0) = (\lambda + \sum_{j=1}^{n} \alpha_j^2)w_2, \\
\left( \sum_{j=1}^{l} k_j^2 - \sum_{j=l+1}^{n} k_j^2 \right) v_2'' - 2\sum_{j=1}^{k} r_j k_j^2 (2w_0'' w_2' + w_0 w_2'' + w_0 w_0'' + w_1'' v_1') = 0.
\end{cases} \] (4.22)

For (4.20), one can apply the Jacobi elliptic function expansion method. Firstly, setting
\[ w_0 = a_0 + a_1 \sin \xi, \quad v_0 = b_0 + b_1 \sin \xi + b_2 \sin^2 \xi. \] (4.23)

Combining (4.20) and (4.23), it can easily be obtained
\[ \begin{cases}
a_0 = 0, a_1 = \pm \sqrt{\frac{2m^2(\sum_{j=1}^{l} k_j^2 - \sum_{j=l+1}^{n} k_j^2)}{r - \sum_{j=1}^{n} \alpha_j^2}}, \\
b_0 = \text{const}, b_1 = 0, b_2 = 2\sum_{j=1}^{l} k_j^2 m^2, \\
\lambda = -(\sum_{j=1}^{n} k_j^2)(1 + m^2) - \sum_{j=1}^{n} \alpha_j^2 - 2c.
\end{cases} \] (4.24)
Thus, the zeroth-order solution of (4.18) can be get
\[ w_0 = \pm m \sqrt{2(\Sigma_{j=1}^{l} k_j^2 - \Sigma_{j=l+1}^{n} k_j^2)} \; sn \xi, \]
\[ v_0 = c + 2\Sigma_{j=1}^{l} k_j^2 m^2 sn^2 \xi, \]  
\[ \xi = \Sigma_{j=1}^{n} k_j x_j - (2\Sigma_{j=1}^{n} \alpha_j k_j) t + \xi_0, \]
where \( k_j, \alpha_j \) are constants, \( \xi_0 \) is arbitrary constant, and \( \frac{(\Sigma_{j=1}^{l} k_j^2 - \Sigma_{j=l+1}^{n} k_j^2)}{r} > 0. \)
Notice that one can deduce that \( v_0 = \frac{\Sigma_{j=1}^{l} k_j^2 rw_0}{\Sigma_{j=1}^{l} k_j^2 - \Sigma_{j=l+1}^{n} k_j^2} + c \) from (4.25) and
\[ v_1 = \frac{2\Sigma_{j=1}^{l} k_j^2 r}{\Sigma_{j=1}^{l} k_j^2 - \Sigma_{j=l+1}^{n} k_j^2} w_0 w_1 \] from the second equation of (4.21).
Then one gets the transformation of (4.21)
\[ (\Sigma_{j=1}^{n} k_j^2)w''_1 + 6(r - \frac{2\Sigma_{j=1}^{l} k_j^2 r}{\Sigma_{j=1}^{l} k_j^2 - \Sigma_{j=l+1}^{n} k_j^2}) \frac{m^2(\Sigma_{j=1}^{l} k_j^2 - \Sigma_{j=l+1}^{n} k_j^2)}{r} \; sn^2 \xi w_1 \]
\[ + (\Sigma_{j=1}^{n} k_j^2)(1 + m^2)w_1 = 0. \]  
Simplifying this equation to have
\[ w'' + [(1 + m^2) - 6m^2 sn^2 \xi]w_1 = 0. \]  
From (4.26), the first-order term of (4.18) is
\[ w_1(\xi) = AL_2 = Acn \xi dn \xi, \]
\[ v_1(\xi) = \pm 2A\Sigma_{j=1}^{l} k_j^2 m \sqrt{2r} \frac{2r}{(\Sigma_{j=1}^{l} k_j^2 - \Sigma_{j=l+1}^{n} k_j^2) sn \xi cn \xi dn \xi}. \]  
For the second-order equation of (4.22), combining (4.27), (4.25) and \( v_2 = \frac{\Sigma_{j=1}^{l} k_j^2 r}{\Sigma_{j=1}^{l} k_j^2 - \Sigma_{j=l+1}^{n} k_j^2}(2w_0 w_2 + w_1^2) \) from the second equation of (4.22), one has
\[ w''_2 + [(1 + m^2) - 6m^2 sn^2 \xi]w_2 = \pm 3 \sqrt{2r} \frac{2r}{(\Sigma_{j=1}^{l} k_j^2 - \Sigma_{j=l+1}^{n} k_j^2) m^2 \xi cn \xi dn \xi}, \]
i.e.
\[ w''_2 + [(1 + m^2) - 6m^2 sn^2 \xi]w_2 = \pm 3 \sqrt{2r} \frac{2r}{(\Sigma_{j=1}^{l} k_j^2 - \Sigma_{j=l+1}^{n} k_j^2) m^2 \xi cn \xi dn \xi}, \]
by using \( cn^2 \xi = 1 - s\xi, dn^2 \xi = 1 - m^2 s^2 \xi. \)
Noticing that (4.28) is an inhomogeneous Lam equation and the key step is to find a particular solution of the inhomogeneous term of (4.28).
Letting
\[ w_2 = c_1 sn \xi + c_3 sn^3 \xi, \]
Considering the (4.28), one gets
\[ c_1 = \pm \frac{1 + m^2}{4m} \sqrt{2r} \frac{2r}{(\Sigma_{j=1}^{l} k_j^2 - \Sigma_{j=l+1}^{n} k_j^2) A^2}, \]
Then the second-order solution of (4.18) is

\[ w_2(\xi) = \pm \frac{1 + m^2}{4m} \sqrt{\frac{2r}{\sum_{j=1}^{l} k_j^2 - \sum_{j=l+1}^{n} k_j^2}} A^2 sn\xi \]

\[ \pm \frac{1}{2} \sqrt{\frac{2r}{\sum_{j=1}^{l} k_j^2 - \sum_{j=l+1}^{n} k_j^2}} mA^2 sn^3\xi, \]

where \( \xi = \sum_{j=1}^{n} k_j x_j - (2\sum_{j=1}^{n} \alpha_j k_j)t + \xi_0, \eta = \sum_{j=1}^{n} \alpha_j k_j - [\sum_{j=1}^{n} k_j^2](1 + m^2) + \sum_{j=1}^{n} \alpha_j^2 + 2c|t + \eta_0, k_j, \alpha_j \) are constants, \( \xi_0, \eta_0 \) are arbitrary constants, and \( \sum_{j=1}^{n} k_j^2 - \sum_{j=l+1}^{n} k_j^2 > 0 \).

Thus one has the multi-order solution of DSI

\[ u_0(\xi) = \pm m \sqrt{\frac{2r}{\sum_{j=1}^{l} k_j^2 - \sum_{j=l+1}^{n} k_j^2}} \exp(i\eta) \]

\[ v_0(\xi) = c + 2\sum_{j=1}^{n} k_j^2 m^2 sn^2\xi, \quad (4.34) \]

\[ u_1(\xi) = AL_2 = Acn\xi dn\xi \exp(i\eta) \]

\[ v_1 = \pm 2A\sum_{j=1}^{n} k_j^2 m \sqrt{\frac{2r}{\sum_{j=1}^{l} k_j^2 - \sum_{j=l+1}^{n} k_j^2}} \exp(i\eta) \]

\[ v_2(\xi) = \pm \frac{1 + m^2}{4m} \sqrt{\frac{2r}{\sum_{j=1}^{l} k_j^2 - \sum_{j=l+1}^{n} k_j^2}} A^2 sn\xi \exp(1 - \frac{2m^2}{1 + m^2} sn^2\xi) \exp(i\eta) \]

\[ v_2(\xi) = A^2 \sqrt{\frac{2r}{\sum_{j=1}^{l} k_j^2 - \sum_{j=l+1}^{n} k_j^2}} [cn^2\xi dn^2\xi \exp(1 + m^2) sn^2\xi + 2m^2 sn^4\xi], \quad (4.36) \]

4.3 Degenerate solution

When \( m \to 1, sn\xi \to tanh\xi \), the zeroth-order solution of DSI degenerates into

\[ u_0 = \pm \sqrt{\frac{2r}{\sum_{j=1}^{l} k_j^2 - \sum_{j=l+1}^{n} k_j^2}} \exp(i\eta) \]

\[ v_0 = c + 2\sum_{j=1}^{n} k_j^2 tanh^2\xi, \quad (4.37) \]

where

\[ \xi = \sum_{j=1}^{n} k_j x_j - (2\sum_{j=1}^{n} \alpha_j k_j)t + \xi_0, \]

\[ \eta = \sum_{j=1}^{n} \alpha_j k_j - [2(\sum_{j=1}^{n} k_j^2) + \sum_{j=1}^{n} \alpha_j^2 + 2c]|t + \eta_0, \]

and \( k_j, \alpha_j \) are constants, \( \xi_0, \eta_0 \) are arbitrary constants, and \( \sum_{j=1}^{n} k_j^2 - \sum_{j=l+1}^{n} k_j^2 > 0 \).

This is solitary wave solution that we frequently see, and we call it shock wave solution.
Similarly by $cn\xi \to \sec h$, $dn\xi \to \sec h\xi$, when $m \to 1$, the first-order solution of DSI is to degenerate into

$$u_1(\xi) = A \sec h^2\xi \exp(i\eta),$$

$$v_1(\xi) = \pm A \Sigma_{j=1}^{l} k_j^2 \sqrt{\frac{2r}{(\Sigma_{j=1}^{l} k_j^2 - \Sigma_{j=l+1}^{n} k_j^2)^2} \tanh \xi \sec h^2\xi}. \quad (4.39)$$

This is a bell shaped solitary wave solution, pulse shock wave solution.

The second-order solution of DSI is to degenerate into

$$u_2(\xi) = \pm \frac{1}{2} \sqrt{\sum_{j=1}^{l} k_j^2} \frac{2r}{(\sum_{j=1}^{l} k_j^2 - \sum_{j=l+1}^{n} k_j^2)^2} A^2 \tanh \xi(1 - \tanh^2\xi) \exp(i\eta),$$

$$v_2(\xi) = A^2 \sum_{j=1}^{l} k_j^2 r \left[ \sec h^4\xi \mp 2(\tanh^2\xi - \tanh^4\xi) \right]. \quad (4.40)$$

It is a new solitary wave solution.

### 4.4. The more exact solution of DSI

One can get more solutions of Davey-Stewartson equations:

(i) If $w_0 = a_0 + a_1 cn\xi$, $v_0 = b_0 + b_1 cn\xi + b_2 cn^2\xi$ in (4.23), one can get the zeroth-order solution of DSI which is

$$u_0 = \pm m \sqrt{-2[(\Sigma_{j=1}^{l} k_j^2 - \Sigma_{j=l+1}^{n} k_j^2)/r] cn\xi \exp(i\eta)}, \quad -2[(\Sigma_{j=1}^{l} k_j^2 - \Sigma_{j=l+1}^{n} k_j^2)/r] > 0,$$

$$v_0 = c - 2\Sigma_{j=1}^{l} k_j^2 m^2 cn^2\xi,$$

$$\eta = \Sigma_{j=1}^{n} \alpha_j k_j + [(\Sigma_{j=1}^{n} k_j^2)(2m^2 - 1) - \Sigma_{j=1}^{n} \alpha_j^2 - 2c]t + \eta_0. \quad (4.43)$$

The first-order solution is

$$u_1(\xi) = AL^2 \exp(i\eta) = A sn\xi dn\xi \exp(i\eta),$$

$$v_1(\xi) = \pm 2A \sum_{j=1}^{l} k_j^2 m \sqrt{-2[(\Sigma_{j=1}^{l} k_j^2 - \Sigma_{j=l+1}^{n} k_j^2)/r] sn\xi cn\xi dn\xi}. \quad (4.44)$$

The second-order solution is

$$u_2(\xi) = \mp \sqrt{\sum_{j=1}^{l} k_j^2} \frac{-2r}{(\sum_{j=1}^{l} k_j^2 - \sum_{j=l+1}^{n} k_j^2)^2} A^2(2m^2 - 1) \frac{cn\xi(1 - \frac{2m^2}{2m^2 - 1} cn^2\xi)}{4m} \exp(i\eta),$$

$$v_2(\xi) = A^2 \sum_{j=1}^{l} k_j^2 r \left[ sn^2\xi dn^2\xi \mp (2m^2 - 1) cn^2\xi \pm 2m^2 cn^4\xi \right]. \quad (4.45)$$

(ii) If $w_0 = a_0 + a_1 dn\xi$, $v_0 = b_0 + b_1 dn\xi + b_2 dn^2\xi$ in (4.23), one can get the zeroth-order solution of DSI which is

$$u_0 = \pm \sqrt{-2[(\Sigma_{j=1}^{l} k_j^2 - \Sigma_{j=l+1}^{n} k_j^2)/r] dn\xi \exp(i\eta)}, \quad -2[(\Sigma_{j=1}^{l} k_j^2 - \Sigma_{j=l+1}^{n} k_j^2)/r] > 0.$$
\[ v_0 = c - 2\text{d}n^2 \xi, \]
\[ \eta = \Sigma_{j=1}^{n} \alpha_j k_j + [(\Sigma_{j=1}^{n} k_j^2) (2 - m^2) - \Sigma_{j=1}^{n} \alpha_j^2 - 2c] t + \eta_0. \]  

(4.46)

The first-order solution is
\[ u_1(\xi) = AL_j^d \exp(i\eta) = A\sin\xi\cos\xi \exp(i\eta), \]
\[ v_1(\xi) = \pm 2A\Sigma_{j=1}^l k_j^2 \sqrt{-\frac{2r}{\left(\Sigma_{j=1}^l k_j^2 - \Sigma_{j=l+1}^n k_j^2\right)^2}} \sin \xi \cos \xi \text{d}n\xi. \]  

(4.47)

The second-order solution is
\[ u_2(\xi) = \mp \sqrt{-\frac{2r}{\Sigma_{j=1}^l k_j^2 - \Sigma_{j=l+1}^n k_j^2}} \frac{A^2 (2 - m^2)}{4m^4} \sin \xi \text{d}n(1 - \frac{2}{2 - m^2} \sin^2 \xi) \exp(i\eta), \]
\[ v_2(\xi) = A^2 \frac{\Sigma_{j=1}^l k_j^2 r}{\Sigma_{j=1}^l k_j^2 - \Sigma_{j=l+1}^n k_j^2} [\sin^2 \xi \cos^2 \xi + \frac{2 - m^2}{m^4} \sin^2 \xi (1 - \frac{2}{2 - m^2} \sin^2 \xi)]. \]  

(4.48)

These are periodic wave solutions of DSI expressed by Jacobi elliptic functions.

When \( m \to 1 \), one can get the degenerate solutions,
\[ u_0(\xi) = \pm \sqrt{-\frac{2(\Sigma_{j=1}^l k_j^2 - \Sigma_{j=l+1}^n k_j^2)}{r}} \sec h\xi \exp(i\eta), \]
\[ v_0(\xi) = c - 2\Sigma_{j=1}^l k_j^2 \sec h^2 \xi, \]  

(4.49)

\[ \eta = \Sigma_{j=1}^n \alpha_j k_j + [(\Sigma_{j=1}^n k_j^2) - \Sigma_{j=1}^n \alpha_j^2 - 2c] t + \eta_0, \]
\[ u_1(\xi) = A \tanh \xi \sec h\xi \exp(i\eta), \]
\[ v_1(\xi) = \pm 2A\Sigma_{j=1}^l k_j^2 \sqrt{-\frac{2r}{\left(\Sigma_{j=1}^l k_j^2 - \Sigma_{j=l+1}^n k_j^2\right)^2}} \tanh \xi \sec h^2 \xi, \]  

(4.50)

\[ u_2(\xi) = \mp \sqrt{-\frac{2r}{\Sigma_{j=1}^l k_j^2 - \Sigma_{j=l+1}^n k_j^2}} \frac{A^2}{4} \sec h(1 - 2 \sec h^2 \xi) \exp(i\eta), \]
\[ v_2(\xi) = A^2 \frac{\Sigma_{j=1}^l k_j^2 r}{\Sigma_{j=1}^l k_j^2 - \Sigma_{j=l+1}^n k_j^2} [\tanh^2 \xi \sec h^2 \xi \mp 2 \sec h^2 \xi]. \]  

(4.51)

**Remark 4.1.** It is open on the existence of global smooth solutions for Cauchy problems to elliptic-hyperbolic types Davey-Stewartson systems. (4.27) indicates that there are some examples of global smooth solutions.

### 5. Conclusion

In this paper, we prove that the Cauchy problem of generalized Davey-Stewartson systems has a unique solution in \( C(0, \infty; H^s) \cap L^{p(d)}(0, \infty; B^{s(p)}_{r(d)}) \cap L^{q(d)}(0, \infty; B^{q(p)}_{r(d)}) \).

What’s more interesting, we construct some explicit period wave solution of the generalized Davey-Stewartson by F-expansion method, as well as some multi-order exact solutions.

From the discussion above, it can be seen that
(i) One can get many zeroth-order solutions of nonlinear evolution equations by using F-expansion or Jacobi elliptic function expansion, which only related to the correlation chart of \( P, Q, R \) and the solution of \( PF^4 + QF^2 + R \).

(ii) The form of the first-order equation is the same as that of the Lam equation. So one can get the first-order solution by solving the Lam equation. The form of the second-order equation is the same as the inhomogeneous Lam equation, and one can obtain the second-order solution by the particular solution of the inhomogeneous term.

(iii) One can obtain the degenerate solution by discussing the limit cases of the multi-order exact solutions. The method is valid to get the multi-order exact solutions of some other nonlinear evolution equations. At the same time, one can get many kinds of solitary wave solutions.

(iv) By the contraction mapping theorem, one can deduce that there exists a unique solution of the Cauchy problem of generalized Davey-Stewartson systems.

References

[1] J. Bergh and J. Löfström, Interpolation space. An introduction, grundlehren der mathemati-schen wissenschafen, Springer-Verlag, Berlin-New york, 1976.

[2] T. Cazenave and F. B. Weissler, The cauchy problem for the nonlinear schrödinger equation in \( H^s \), Manuscripta Math., 1988, 61(4), 477–494.

[3] S. T. Demiray and H. Bulut, New soliton solutions of Davey-Stewartson equation with power-law nonlinearity, Opt. and Quant. Electron., 2017, 3, 1–8.

[4] A. Davey and K. Stewartson, On three dimensional packets of surface waves, Proc. Roy. Soc. London Ser. A, 1974, 338(1613), 101–110.

[5] V. D. Djordjevic and L. G. Redekopp, On two-dimensional packets of capillary-gravity waves, J. Fluid Mech., 1977, 79(04), 703–714.

[6] E. Fan, Extended tanh-function method and its applications to nonlinear equations, Phys. Lett. A, 2000, 277(4), 212–218.

[7] J. M. Ghidaglia and J. C. Saut, On the initial value problem for the Davey-Stewartson systems, Nonlinearity, 1990, 3(2), 475–506.

[8] F. Güngör and C. Özemir, Variable coefficient Davey-Stewartson system with a Kac-Moody-Virasoro symmetry algebra, J. Math. Phys, 2016, 57(6), 101–110.

[9] B. Guo and B. Wang, The cauchy problem for Davey-Stewartson systems Comm. Pure Appl. Math, 1999, 52(12), 1477–1490.

[10] B. Guo and B. Wang, On the initial value problem and scattering of solutions for the generalized Davey-Stewartson systems, Science of China (Series A), 2001, 44(8), 994–1002.

[11] R. Hirota, Exact N-soliton solutions of the wave equation of long waves in shallow-water and in nonlinear lattices, J. Math. Phys, 1973, 14(7), 810–814.

[12] O. M. Kiselev, Asymptotics of soliton solution for the perturbed Davey-Stewartson-1 equations, Physics, 1999.

[13] N. A. Kudryashov, Exact solutions of the generalized Kuramoto-Sivashinsky equation, Phys. Lett. A, 1990, 147(5), 287–291.

[14] C. Klein and N. Stolov, A numerical study of blow-up mechanisms for Davey-Stewartson II systems, Stud. Appl. Math., 2018, DOI:10.1111/sapm.1221.
[15] F. Linares and G. Ponce, *On the D-Stewartson systems*, Ann Inst Henri Poincaré, Anal nonlinéaire, 1993, 10(5), 523–548.

[16] W. Ma, *A search for lump solutions to a combined fourth-order nonlinear PDE in (2+1)-dimensions*, J. Appl. Anal. Comput., 2019, 9(4), 1319–1332.

[17] Muslu and M. Gulcin, *Numerical study of blow-up to the purely elliptic generalized DaveyCStertson system*, J. Comput. Appl. Math., 2017, 317, 331–342.

[18] A. H. Nayfeh, *Perturbation Methods*, New York, John Wiley and Sons Inc., 1973.

[19] M. Otwinowski, R. Paul, and W. G. Laidlaw, *Exact travelling wave solutions of a class of nonlinear diffusion equations by reduction to a quadrature*, Phys. Lett. A, 1989, 128(9), 483–487.

[20] A. V. Porubov, *Periodical solution to the nonlinear dissipative equation for surface waves in a convecting liquid layer*, Phys. Lett. A, 1996, 221(6), 391–394.

[21] A. V. Porubov and D. F. Parker, *Some general periodic solutions to coupled nonlinear schrödinger equations*, Wave Motion, 1999, 29(2), 97–109.

[22] A. V. Porubov and M. G. Velarde, *Exact periodic solutions of the complex Ginzburg-Landau equation*, J. Math. Phys., 1999, 40(2), 884–896.

[23] E. S. Selima and Y. Mao and et al, *Applicable symbolic computations on dynamics of small-amplitude long waves and Davey-Stewartson equations in finite water depth*, Appl. Math. Model., 2018, 57, 376–390.

[24] H. Triebel, *Theory of function spaces*. Monographs in Mathematics, 1983. DOI,10.1007/3–7643–7582–5.

[25] M. Tsutsumi, *Decay of weak solutions to the Davey-Stewartson systems*, J. Math. Anal. Appl., 1994, 182(3), 680–704.

[26] T. Tsuchida, *Integrable semi-discretizations of the Davey-Stewartson system and a (2 + 1)-dimensional Yajima-Oikawa system. I*, 2019. arXiv,1904.07924v1 [nlin.SI].

[27] B. Wang, *Globally well and ill posedness for non-elliptic derivative Schrödinger equations with small rough data*, J. Funct. Anal., 2013, 265(12), 3009–3052.

[28] M. Wang, *Solitary wave solutions for variant Boussinesq equations*, Phys. Lett. A, 1995, 199(3–4), 169–172.

[29] C. Yan, *A simple transformation for nonlinear waves*, Phys. Lett. A, 1996, 224(1–2), 77–84.

[30] J. Zhang, B. Guo, and S. Shen, *Homoclinic orbits of the Davey-Stewartson equations*, Appl. Math. Mech, 2005, 26(2), 139–141.