Local-in-time Solvability and Space Analyticity for the Navier-Stokes Equations with BMO-type Initial Data

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Abstract

It is proved that there exists a local-in-time solution \( u \in C([0,T), bmo(\mathbb{R}^d)^d) \) of the Navier-Stokes equations such that every \( u(t) \) has an analytic extension on a complex domain whose size only depends on \( t \) (and increases with \( t \)) and the external force \( f \), assuming only that the initial velocity \( u_0 \) is a local \( BMO \) function. Our method for proving is a combination and refinement of the work by Grujić and Kukavica [13], Guberović [15] and Kozono et al. [18]. One challenging step is the estimation of the heat and Stokes semigroups from \( BMO \)-type spaces to \( L^\infty \); a result itself of independent interest. We also apply the idea to the analyticity of vorticity with the assistance of Calderón-Zygmund theory.

1 Introduction

We consider the space analyticity of solutions to the Navier-Stokes system in \( \mathbb{R}^d \) (\( d \geq 2 \))

\[
\begin{align*}
\partial_t u - \Delta u + u \cdot \nabla u + \nabla p &= f, \quad \text{in } \mathbb{R}^d \times (0,T) \\
\text{div} \ u &= 0, \quad \text{in } \mathbb{R}^d \times (0,T) \\
u(\cdot,0) &= u_0(\cdot), \quad \text{in } \mathbb{R}^d \times \{t = 0\}
\end{align*}
\]

where the force \( f(\cdot,t) \) is real-analytic in space with an uniform analyticity radius \( \delta_f \) for all \( t \in \mathbb{R}^+ \), which admits some analytic extension \( f + ig \), while \( u_0 \) is the given initial velocity vector field. We prove that there exists a solution \( u \in C([0,T^*), bmo(\mathbb{R}^d)^d) \) for the system (1.1)-(1.3) evolving from the initial value \( u_0 \in bmo(\mathbb{R}^d)^d \), which admits space analyticity in a domain \( \mathcal{D}_t \) of \( \mathbb{C}^d \) for every \( t \) such that both real and imaginary parts have bounded local mean oscillations, where the time bound \( T^* \) is characterized only by \( \|u_0\|_{bmo} \). With certain modification of the proof, the result also holds if \( bmo \) space is replaced by \( B^0_{\infty,\infty} \) space.

The study of analyticity radius of solutions to the NSE dates back to the seminal work by Foias and Temam [9] who applied Fourier techniques and Gevrey spaces in \( L^2 \). Related results for \( L^p \)
spaces can be found in Lemarié-Rieusset [20]. Similar types of approaches to other equations and function spaces had been developed by Levermore and Oliver [21], Paicu and Vicol [22], Biswas and Swanson [4], Biswas and Foias [3], Bae et al. [1] and Ignatova et al. [17]. A different method for analyticity of the NSE with \( L^p \) (\( 3 < p < \infty \)) initial value was first presented by Grujić and Kukavica [13] using the technique of complexified extension. The idea was adopted to the NSE with \( L^\infty \) initial value by Guberović [15]. This method was also applied to non-linear heat equations on bounded domains by Grujić and Kukavica [14] for the analyticity at interior points. Later, the technique was refined by Bradshaw et al. [6] for local analyticity of the NSE with locally analytic forcing term.

Among existing literatures of about NSE, very few had addressed the analyticity or even the strong solvability of the Navier-Stokes equations (\( d \geq 3 \)) with non-decaying or oscillatory type initial values. Giga et al. [11] proved strong solutions exist for \( u_0 \in BUC \) (bounded and uniformly continuous functions). Guberović [15] derived the space analyticity for \( u_0 \in L^\infty \), showing the time-local existence of mild solution for \( L^\infty \) initial value. Sawada [23] proved the time-local existence of solutions in Besov spaces with non-positive differential orders using various Hölder-type estimates in the Besov spaces. Later, Kozono et al. [18] showed the time-local existence in Besov spaces and in “time-logarithmically-weighted \( L^\infty \) spaces” for \( u_0 \in B_{\infty,\infty}^0 \) using a different converging algorithm. Until now, no result has been established for existence in \( C([0,T],X) \) where \( X \) is either a local or global BMO-type of space. This paper is the first attempt to deal with this challenge. We develop a new iteration scheme and converging argument from the approaches as in Grujić and Kukavica [13], Sawada [23] and Kozono et al. [18], and consider the strong solvability and spatial analyticity of (1.1)-(1.3) with \( u_0 \in bmo(\mathbb{R}^d)^d \). Having derived some results on the boundedness of the Stokes semigroups from BMO (or Besov) spaces to \( L^\infty \), with the chain of embeddings which connects \( L^p \), BMO and Besov spaces, we proved that the complexified solution of (1.1)-(1.3) exists locally in time with almost \( t_1^2 \) analyticity radius for each \( u(t) \), and the real solution, i.e. restriction on the real axis, is classical in the interior of the parabolic cylinder \( \mathbb{R}^d \times (0,T^*) \).

A natural follow-up question is whether we can extend the method to the usual BMO spaces (homogeneous type) and study the global oscillations of the solution to (1.1)-(1.3). A positive answer is given in a forthcoming paper. We also attempt to study the local analyticity and the bound of oscillations of the NSE on bounded domains, especially the ones with regular boundaries, i.e. straight and circular boundary lines. One of the motivations for studying the analyticity estimate and in-time bound of BMO-norms is their connection with the sparseness of the region of intense oscillations which can possibly provide a geometric type criterion weaker than the one presented in Grujić [12] which requires the sparseness of the level sets of velocity or vorticity truncated by the \( L^\infty \)-norm. Since the measures of the level sets of oscillation mean are naturally controlled by the BMO-norms (e.g. John-Nirenberg inequality), replacing the notion of intense velocity (or vorticity) by that of intense oscillation may potentially reduce the restriction on the geometry of fluid activity near the possible blow-up time. The fundamental step of this idea requires the in-time continuity of BMO norms of velocity and vorticity.

The paper is organized as follows. In Section 2, we exhibit some known theorems and develop some results about the heat and Stokes semigroups in Besov and BMO-type spaces tailored to our needs. In Section 3, we state the main theorem with the basic setup for complexified solutions and analyticity for the Navier-Stokes equations, followed by the proof of the theorem. Section 4 is an extension of the idea for proving a similar result about vorticity.
2 Preliminaries

In this section, we first take a brief survey of some results on the heat semigroup in $BMO$-type norm, Hölder type estimate in Besov spaces and equivalency and embedding among some oscillation function spaces. Then we prove some technical lemmas in preparation for the main theorem.

The following result shows the heat semigroup is bounded in $BMO(\mathbb{R}^d)$. The idea for proving is to define the norm of Hardy space $H^1$ through the convolution with the heat kernel and use the duality by Fefferman and Stein [8]. For the detailed definitions of $BMO$ and $H^1$, the reader may refer to Stein [24]

**Theorem 2.1** (Bolkart et al. [5]). Consider the equation $\partial_t u - \Delta u = 0$ in $\mathbb{R}^d \times [0, \infty)$ with $u(0) = u_0 \in BMO(\mathbb{R}^d)$. There is a solution $u(t)$ and constant $C$ satisfying the estimate

$$\sup_{t>0} \left( \|u(t)\|_{BMO} + t^{\frac{\alpha}{2}} \|\nabla u(t)\|_{L^\infty} + t \|\nabla^2 u(t)\|_{L^\infty} + t \|\partial_t u\|_{L^\infty} \right) \leq C\|u_0\|_{BMO}. \quad (2.1)$$

The following two lemmas list several Hölder-type estimates for the heat semigroup in $BMO$ and the Besov spaces. For the detailed definitions of Besov spaces, the reader may refer to Bahouri et al. [2], Triebel [25] and Kozono et al. [18].

**Lemma 2.2** (Giga et al. [11] and Kozono et al. [18]). For all $f \in BMO(\mathbb{R}^d)$ we have the estimate

$$\|(-\Delta)^{\alpha} e^{t\Delta} f\|_{L^\infty} \lesssim t^{-\alpha} \|f\|_{BMO}. \quad (2.2)$$

where $\alpha > 0$.

The proof by Giga et al. [11] is based on the estimate for the maximal function of $(-\Delta)^{\alpha} e^{t\Delta} f$. The idea by Kozono et al. [18] is to compute $\|(-\Delta)^{\alpha} G_t\|_{H^1}$ ($G_t$ is the heat kernel) and use $H^1$-$BMO$ duality.

**Lemma 2.3** (Kozono et al. [18]). If $s_0 \leq s_1$, $1 \leq p, q \leq \infty$, then there holds

$$\|e^{t\Delta} f\|_{B^s_{p,q}} \lesssim t^{-\frac{\alpha}{2}(s_1-s_0)} \|f\|_{B^{s_0}_{p,q}} \quad (2.3)$$

$$\|e^{t\Delta} f\|_{B^s_{p,q}} \lesssim \left(1 + t^{\frac{\alpha}{2}(s_1-s_0)}\right) \|f\|_{B^{s_0}_{p,q}} \quad (2.4)$$

$$\|e^{t\Delta} f\|_{B^s_{p,1}} \lesssim \left(1 + t^{\frac{\alpha}{2}(s_1-s_0)}\right) \ln(e + t^{-1}) \|f\|_{B^{s_0}_{p,\infty}} \quad (2.5)$$

for all $t > 0$. If $s_0 < s_1$, $1 \leq p \leq \infty$, then there holds

$$\|e^{t\Delta} f\|_{B^s_{p,1}} \lesssim t^{-\frac{\alpha}{2}(s_1-s_0)} \|f\|_{B^{s_0}_{p,\infty}}. \quad (2.6)$$

The proof of (2.3) and (2.4) is based on the $L^p$ estimate for each $\phi_j \ast f$ (mode of frequency) by using Young’s inequality. The idea for (2.5) and (2.6) is to truncate the Besov norms by high frequency and low frequency terms and apply some interpolation inequalities in the Besov spaces.

Before showing results on equivalency and embedding, we provide a short introduction of Triebel-Lizorkin-type spaces. For the detailed definitions of Triebel-Lizorkin spaces and their variations, see Triebel [25] and Yuan et al. [26].
Definition 2.4. Let \( s \in \mathbb{R} \), \( p \in (0, \infty) \) and \( q \in (0, \infty] \). Let \( \{\varphi_j\}_{j \in \mathbb{Z}} \) be Littlewood-Paley decomposition (homogeneous one) and \( \mathcal{P}(\mathbb{R}^d) \) be the set of all polynomials. The Triebel-Lizorkin space \( \dot{F}^s_{p,q}(\mathbb{R}^d) \) is the set of all \( f \in \mathcal{S}'(\mathbb{R}^d)/\mathcal{P}(\mathbb{R}^d) \) such that

\[
\|f\|_{\dot{F}^s_{p,q}} := \left\| \left( \sum_{j \in \mathbb{Z}} \left( 2^{js} |\varphi_j * f| \right)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^d)} < \infty
\]

where the \( \ell^q \)-norm is replaced by the supremum on \( j \) if \( p = \infty \). For \( p = \infty \), \( \dot{F}^s_{p,q}(\mathbb{R}^d) \) is defined to be the set of all \( f \in \mathcal{S}'(\mathbb{R}^d)/\mathcal{P}(\mathbb{R}^d) \) such that

\[
\|f\|_{\dot{F}^s_{\infty,q}} := \sup_{Q \in \mathcal{D}} \frac{1}{|Q|^{1/q}} \left\| \sum_{j=-\ln l(Q)}^\infty \left( 2^{js} |\varphi_j * f| \right)^q \right\|_{L^p(Q)}^{1/q} < \infty
\]

where \( \mathcal{D} \) denote the collection of all dyadic cubes and \( l(Q) \) is the side length of \( Q \).

Let \( s, \tau \in \mathbb{R} \), \( p \in (0, \infty) \) and \( q \in (0, \infty] \). Let \( \{\varphi_j\}_{j \in \mathbb{Z}} \) be Littlewood-Paley decomposition (homogeneous one). The Triebel-Lizorkin-Morrey space is defined to be the set of all \( f \in \mathcal{S}'(\mathbb{R}^d)/\mathcal{P}(\mathbb{R}^d) \) such that

\[
\|f\|_{\dot{F}^{s,\tau}_{p,q}} := \sup_{Q \in \mathcal{D}} \frac{1}{|Q|^\tau} \left\| \left( \sum_{j=-\ln l(Q)}^\infty \left( 2^{js} |\varphi_j * f| \right)^q \right)^{1/q} \right\|_{L^p(Q)} < \infty
\]

where \( \mathcal{D} \) and \( l(Q) \) are as above. For \( p = \infty \), modification of the norm should be made also.

The definitions of Triebel-Lizorkin spaces of non-homogeneous type are defined in a similar fashion (see Yuan et al. [26] for details).

The following two theorems connect \( BMO \)-type and Triebel-Lizorkin spaces. Both are proved by the equivalency \( b^p = F^0_{p,2} \) (resp. \( H^p = F^0_{p,2} \)) and the dualities \( F^0_{\infty,2} \approx (F^0_{1,2})^* \approx (b^1)^* \approx bmo \) (resp. \( \dot{F}^0_{\infty,2} \approx (\dot{F}^0_{1,2})^* \approx (H^1)^* \approx BMO \)). For the details of the proofs, see Triebel [25].

Theorem 2.5 (Triebel [25], Page 93, Theorem 2). The following equality holds

\[
bmo(\mathbb{R}^d) = F^0_{\infty,2}(\mathbb{R}^d)
\]

with norm equivalence.

Theorem 2.6 (Triebel [25], Theorem on Page 244, or Yuan et al. [26], Section 1.4.4). The following equality holds

\[
BMO(\mathbb{R}^d) = \dot{F}^0_{\infty,2}(\mathbb{R}^d)
\]

with equivalent quasi-norms.

Applying a result in Frazier and Jawerth [10] (representation of \( \dot{F}^s_{p,q} \) in the sequence space indexed by the dyadic system \( \mathcal{D} \)), Yuan et al. [26] proved that

\[
\text{(2.7)}
\]

\[
\text{(2.8)}
\]
Theorem 2.7 (Yuan et al. [26], Proposition 2.4, or Frazier and Jawerth [10], Corollary 5.7). Let $s \in \mathbb{R}$, $p \in (0, \infty)$ and $q \in (0, \infty)$. Then the followings hold with equivalent norms and equivalent quasi-norms respectively:

\[ F^s_{p,q}(\mathbb{R}^d) = F^s_{\infty,q}(\mathbb{R}^d) \]  
\[ \hat{F}^s_{p,q}(\mathbb{R}^d) = \hat{F}^s_{\infty,q}(\mathbb{R}^d) \]  

(2.9) \hspace{2cm} (2.10)

Now we are ready to prove some auxiliary results for the next two sections.

Lemma 2.8. The following chains of continuous embeddings hold:

\[ L^\infty \hookrightarrow bmo \hookrightarrow F^0_{\infty,\infty} = B^0_{\infty,\infty} \]  
\[ L^\infty \hookrightarrow bmo \hookrightarrow BMO \hookrightarrow \hat{F}^0_{\infty,\infty} = \hat{B}^0_{\infty,\infty} \]  

(2.11) \hspace{2cm} (2.12)

Proof. It suffices to show $bmo \hookrightarrow F^0_{\infty,\infty}$ and $BMO \hookrightarrow \hat{F}^0_{\infty,\infty}$; the rest easily follows by the definitions of $BMO$ and $L^\infty$. In the spirit of (2.7) and (2.9), to prove $bmo(\mathbb{R}^d) \hookrightarrow F^0_{\infty,\infty}(\mathbb{R}^d)$ is equivalent to showing

\[ F^0_{\infty,2}(\mathbb{R}^d) \hookrightarrow F^0_{p,\infty}(\mathbb{R}^d) \quad \text{for some } p \in (0, \infty). \]

Pick $p = 2$. By the definitions of Triebel-Lizorkin spaces and the variations, $F^0_{\infty,2}(\mathbb{R}^d) \hookrightarrow F^{0,1/2}_{2,\infty}(\mathbb{R}^d)$ is a consequence of the monotonicity of $\ell^p$-norms and Hölder’s inequalities. This proves $bmo(\mathbb{R}^d) \hookrightarrow F^0_{\infty,\infty}(\mathbb{R}^d)$. It follows similarly that $BMO \hookrightarrow F^0_{\infty,\infty}$. ∎

Lemma 2.9. Consider the equation $\partial_t u - \Delta u = 0$ in $\mathbb{R}^d \times [0, \infty)$ with $u(0) = u_0 \in bmo(\mathbb{R}^d)$. There is a solution $u(t)$ and constant $C$ satisfying the estimate

\[ \sup_{t > 0} \left( \| u(t) \|_{bmo} + t^{\frac{d}{2}} \| \nabla u(t) \|_{L^\infty} + t \| \nabla^2 u(t) \|_{L^\infty} + t \| \partial_t u \|_{L^\infty} \right) \leq C \| u_0 \|_{bmo}. \]  

(2.13)

Proof. The boundedness of $\| \nabla u(t) \|_{L^\infty}$, $\| \nabla^2 u(t) \|_{L^\infty}$ and $\| \partial_t u \|_{L^\infty}$ is a simple corollary of Lemma 2.1 since $bmo \hookrightarrow BMO$. It remains to show $\| u(t) \|_{bmo} \lesssim \| u_0 \|_{bmo}$. Similar to Bolkart et al. [5], we define the Hardy space $\mathcal{H}^1(\mathbb{R}^d)$ as

\[ \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^d) \mid \| f \|_{\mathcal{H}^1} := \sup_{0 < s < 1} \| G_s * f \|_{L^1} < \infty \right\}. \]

Let $\phi \in \mathcal{H}^1(\mathbb{R}^d)$. Then for all $t$

\[ |\langle u(t), \phi \rangle| \lesssim \| u_0 \|_{bmo} \| G_t * \phi \|_{\mathcal{H}^1} \lesssim \| u_0 \|_{bmo} \sup_{0 < s < 1} \| G_s * (G_t * \phi) \|_{L^1} \lesssim \| u_0 \|_{bmo} \| G_t \|_{\mathcal{H}^1} \sup_{0 < s < 1} \| G_s * \phi \|_{L^1} \lesssim \| u_0 \|_{bmo} \| \phi \|_{\mathcal{H}^1}. \]

Then, $\| u(t) \|_{bmo} \lesssim \| u_0 \|_{bmo}$ follows by $\mathcal{H}^1 - bmo$ duality. ∎
Lemma 2.10. Let $\partial_t u - \Delta u = \nabla \cdot f$ with $u(0) \in bmo(\mathbb{R}^d)$, where $f$ is analytic for every $t \in (0, T)$ and has bounded-in-time $BMO$-norms; more precisely

$$a_0(t)f \in C^\infty([\delta, T), C^\infty(\mathbb{R}^d)) \cap L^\infty([0, T), BMO(\mathbb{R}^d))$$

where $\delta > 0$ can be arbitrarily small. Then the function

$$u(t) = e^{t\Delta}u_0 + \int_0^t \nabla e^{(t-s)\Delta} f \, ds$$

solves the equations in $\mathbb{R}^d \times (0, T)$ and is real analytic for every $t \in (0, T)$. Moreover,

$$a_1(t)u \in C^\infty([\delta, T), C^\infty(\mathbb{R}^d)) \cap L^\infty([0, T), L^\infty(\mathbb{R}^d))$$

for any $\delta > 0$.

Proof. First of all, by Duhamel’s principle (for distributions) we know (2.15) solves the equation in $\mathcal{S}'(\mathbb{R}^d \times (0, T))$, and for any $t_0 \in (0, T)$ the function

$$\tilde{u}(t) = e^{(t-t_0)\Delta}u(t_0) + \int_{t_0}^t \nabla e^{(t-s)\Delta} f \, ds$$

solves the equation in $\mathcal{S}'(\mathbb{R}^d \times (t_0, T))$; moreover, $\tilde{u}$ and $u$ agree on $\mathbb{R}^d \times (t_0, T)$. The analyticity of $\tilde{u}$ is due to the following estimates: By Lemma 2.2 and the assumption (2.14), for any $t > t_0$ we have

$$\|\nabla u(t)\|_\infty \lesssim \|\nabla e^{(t-t_0)\Delta} u(t_0)\|_\infty + \left\| \nabla \int_{t_0}^t \nabla e^{(t-s)\Delta} f \, ds \right\|_\infty$$

$$\lesssim (t-t_0)^{-1/2}\|u(t_0)\|_{BMO} + \int_{t_0}^t \|e^{(t-s)\Delta} f\|_\infty ds$$

$$\lesssim (t-t_0)^{-1/2}\|u(t_0)\|_{BMO} + (t-t_0)\|f\|_\infty .$$

Similarly, one can show, for higher order derivatives,

$$\|\nabla^k u(t)\|_\infty \lesssim (t-t_0)^{-k/2}\|u(t_0)\|_{BMO} + (t-t_0)\|f\|_\infty .$$

This proves that for any $t > t_0$, $\tilde{u}(t)$ (hence $u(t)$) is analytic. Since $t_0$ is arbitrary, $u(t)$ is analytic for every $t \in (0, T)$.

Lemma 2.11 (Montel’s). Let $p \in [1, \infty]$ and let $\mathcal{F}$ be a set of analytic functions $f$ in an open set $\Omega \subset \mathbb{C}^d$ such that

$$\sup_{f \in \mathcal{F}} \|f\|_{L^p(\Omega)} < \infty .$$

Then $\mathcal{F}$ is a normal family.

Provided at the end of this section is a type of sharp $L^\infty$-estimates for the heat semigroup convolved with Calderón-Zygmund operators, as stated and proved in a more general setup, being prepared only for Section 4 (as well as for some follow-up questions in future).
Lemma 2.12. Let $T$ be a Calderón-Zygmund operator (see Stein [24] for the definition of C.Z.O.) with symmetric kernel $K(\cdot, \cdot)$ satisfying
\[
\int_{\mathbb{S}^d} K(x, z) d\sigma(z) = \int_{\mathbb{S}^d} K(z, y) d\sigma(z) = 0 \quad \text{for all } x, y
\]
where $\mathbb{S}^d$ denotes the unit sphere centered at $x$ or $y$. And let $\Phi \in L \log L(\mathbb{R}^d)$ be a non-negative, radially symmetric and radially decreasing function. Given $k > 0$, there exists a number $T^*$ (which only depends on $k$) such that, for any $t < T^*$ and for any function $g$ such that $|g| \leq \Phi$ and any $f \in L^\infty(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for some $1 \leq p < \infty$, we have
\[
\left\| g_1 * |Tf|^k \right\|_{L^\infty} \lesssim_{\Phi, p, k} \Psi_k(t) \left( \|f\|_{L^\infty}^k + \|f\|_{L^p}^k + \|f\|_{L^\infty}^k \|f\|_{L^p}^{k(1-\alpha)} \right)
\]
\[
(2.17)
\]
for some $\alpha$, where $g_1(x) := t^{-d} g(x/t)$ and $\Psi_k(t)$ grows logarithmically as $t \to 0^+$.

To make the proof shorter, we borrow a result from Calderón and Zygmund [7]:

Proposition 2.13. Suppose $T$ is a Calderón-Zygmund operator with $K$ satisfying (2.16). Let $f$ be a function such that
\[
\int_{\mathbb{R}^d} |f(x)| \left( 1 + \ln^+ |f(x)| \right) dx < \infty .
\]
\[
(2.18)
\]
Then $Tf(x)$ is integrable over any set $S$ of finite measure and
\[
\int_S |Tf(x)| d\mu(x) \lesssim \int_{\mathbb{R}^d} |f(x)| d\mu(x) + \int_{\mathbb{R}^d} |f(x)| \ln^+ \left( \mu(S)^{d+1 \over d} |f(x)| \right) d\mu(x) + \mu(S)^{-1 \over d}
\]
\[
(2.19)
\]
where $\mu$ denotes the Lebesgue measure in $\mathbb{R}^d$.

Duality in Orlicz spaces: It is well known that the Orlicz space $\phi(L)(\mu)$ is defined to be the function space with the norm
\[
\|f\|_{\phi(L)} := \inf \left\{ s > 0 \mid \int_X \phi(s^{-1}|f|) d\mu \leq 1 \right\}
\]
where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is convex and increasing with $\phi(0) = 0$. And the dual of $\phi(L)(\mu)$ is the Orlicz space $\psi(L)(\mu)$ (with the same norm) where $\psi$ is the Legendre-Fenchel transform of $\phi$:
\[
\psi(y) := \sup \left\{ xy - \phi(x) \mid x \in \mathbb{R}^+ \right\} ,
\]
and vice versa. In particular, we set $\psi_*(x) = \ln(e^x + 1)$ and $\psi_*(x) = e^x - 1$. Note that the Legendre-Fenchel transform of $\phi_*$ is not $\psi_*$ but comparable to $e^x$, however, if we consider the restriction of Lebesgue measure $\mu$ on a set $S$ of finite measure with $\mu(S) \lesssim 1$, then $\phi_*(L)(\mu|_S)$ and $\psi_*(L)(\mu|_S)$ are mutually dual spaces.

Such duality result, together with the “quasi-boundedness” of CZO in $L \log L$ as shown by Proposition 2.13, yields the following:
Corollary 2.14. For any set $S_1, S_2$ with finite Lebesgue measures such that $\mu(S_1) = c\mu(S_2) \lesssim 1$ for some constant $c > 0$, the C-Z operator $T$ described in Lemma 2.12 and Proposition 2.13 is a bounded operator from $L^\infty(\mu|_{S_1})$ to $\psi_*(L)(\mu|_{S_2})$ where $\psi_*(x) = e^x - 1$ and $\mu|_{S_i}$ denotes the restriction of Lebesgue measure on $S_i$. Moreover,

$$\|T\|_{L^\infty(\mu|_{S_1}) \to \psi_*(L)(\mu|_{S_2})} \lesssim 1$$

which is independent of $S_1$ and $S_2$.

Proof. Without loss of generality we assume $\mu(S_1) = 1$ (In general we set $\|f\|_{\psi_*(L)(\mu|_{S_2})} = \mu(S_1)^{-\frac{1}{2}}$ for the argument below; also notice that the proof becomes trivial for the special case $\mu(S_1) = 0$). First, recall that the dual of $\psi_*(L)(\mu|_{S_2})$ is $\phi_*(L)(\mu|_{S_2})$ where $\phi_*(x) = x \ln(e + x)$. By Proposition 2.13, for any $f \in \phi_*(L)(\mu|_{S_2})$ with $\|f\|_{\phi_*(L)(\mu|_{S_2})} = 1$,

$$\int_{S_1} |T(f\mathbb{1}_{S_2})|d\mu(x) \lesssim \int_{\mathbb{R}^d} |f\mathbb{1}_{S_2}|d\mu(x) + \int_{\mathbb{R}^d} |f\mathbb{1}_{S_2}| \ln^+ (|f\mathbb{1}_{S_2}|) d\mu(x) + 1 \lesssim 2\|f\mathbb{1}_{S_2}\|_{\phi_*(L)(\mu)} \lesssim 2\|f\|_{\phi_*(L)(\mu|_{S_2})} \lesssim 1.$$ 

Then, by the above estimate and the self-adjointness of $T$, it follows that for any $g \in L^\infty(\mu|_{S_1})$ and $f \in \phi_*(L)(\mu|_{S_2})$ with $\|f\|_{\phi_*(L)(\mu|_{S_2})} = 1$,

$$|\int_{S_2} f(x)T(g\mathbb{1}_{S_1})(x)d\mu(x)| \lesssim \|g\mathbb{1}_{S_1}\|_{L^\infty} \int_{S_1} |T(f\mathbb{1}_{S_2})|d\mu(y) \lesssim \|g\mathbb{1}_{S_1}\|_{L^\infty}.$$ 

Then, by the duality $\psi_*(L)(\mu|_{S_2}) \approx (\phi_*(L)(\mu|_{S_2}))^*$, it follows that

$$\|T(g\mathbb{1}_{S_1})\|_{\psi_*(L)(\mu|_{S_2})} \lesssim \|g\mathbb{1}_{S_1}\|_{L^\infty},$$ 

in other words, $T : L^\infty(\mu|_{S_1}) \to \psi_*(L)(\mu|_{S_2})$ is bounded.

Proof of Lemma 2.12. First we prove for $k = 1$. Let $f_x(y)$ denote the translation $f(x - y)$, then for an open ball $B$ centered at 0 with radius $r_B$, we have the decomposition

$$|g_t \ast T f| = \left| \int_{\mathbb{R}^d} g_t(y)Tf_x(y)dy \right| \lesssim \int_{\mathbb{R}^d} |g_t||Tf_x|dy + \int_B |g_t| |T(f_x \mathbb{1}_{(3B)^c})|dy + \int_B |g_t| |T(f_x \mathbb{1}_{3B})|dy =: H + I + J$$

where $\kappa B$ is $\kappa$-multiple dilation of $B$ from the center. If $p > 1$, since $B$ is centered at 0, i.e. $c_B = 0$, by the assumption on $g$ and $\Phi$, Hölder’s inequality and $L^p$-boundedness of $CZO$,

$$H \lesssim \int_{3B^c} \Phi_t(y)|Tf_x(y)dy| \lesssim \|\Phi_t\|_{L^p(B^c)}\|Tf_x\|_{L^p(B^c)} \lesssim \|\Phi_t(r_B)\|_{L^p}^{\frac{1}{p-1}} (\|\Phi_t\|_{L^1})^{1/p'} \|f_x\|_{L^p} \lesssim \Phi(r_B/t)^{\frac{1}{p-1}} (\|\Phi\|_{L^1})^{1/p'} \|f\|_{L^p}.$$
If \( p = 1 \), pick a \( q \) such that \( p < q < \infty \), then by the above argument with \( L^p \) interpolations

\[
H \lesssim \int_{B^c} \Phi_t(y)|Tf_x(y)|dy \\
\lesssim |\Phi(r_B/t)|^{q-1}q^{-1} \left( \|\Phi\|_{L^q} \right)^1q \|f\|_{L^q} \\
\lesssim |\Phi(r_B/t)|^{q-1}q^{-1} \left( \|\Phi\|_{L^q} \right)^1q \|f\|_{L^{q-p}}^{q-p} ||f||_{L^p}^{p/q} .
\]

By the “size” condition of the \( C-Z \) kernel \( K \) and Hölder’s inequality

\[
|T \left( f_x 1_{(3B)^c} \right)(c_B)| \lesssim r_B^{-\alpha} ||\|f\||_{L^p} \lesssim r_B^{-\alpha} ||\|f\||_{L^p} .
\]

For any \( y \in B \), by the “smoothness” condition of \( K \),

\[
|T \left( f_x 1_{(3B)^c} \right)(y) - T \left( f_x 1_{(3B)^c} \right)(c_B)| \lesssim \int_{(3B)^c} |K(y,z) - K(c_B,z)| |f_x(z)|dz \\
\lesssim \int_{(3B)^c} |y - c_B| |z - c_B| |f_x(z)|dz \lesssim \|f\|_{L^\infty} .
\]

Therefore \( I \lesssim \|f\|_{L^\infty} + r_B^{-\alpha} \|f\|_{L^p} \). By the duality of the Orlicz spaces,

\[
J \lesssim \|g_t\|_{\Phi^*(\mathcal{L}(\mu|B))} \|T \left( f_x 1_{(3B)^c} \right)\|_{\psi^*(\mathcal{L}(\mu|B))}
\]

where \( \Phi^* \) and \( \psi^* \) are given in Corollary 2.14. Now, by the corollary (being applied with \( S_1 = 3B \) and \( S_2 = B \) as well as \( r_B \approx 1 \),

\[
J \lesssim \|g_t\|_{\Phi^*(\mathcal{L}(\mu|B))} \|f_x 1_{3B}\|_{\mathcal{L}\infty} \\
\lesssim \|f_x\|_{\mathcal{L}\infty} \int_{\mathbb{R}^d} |g_t(x)| \ln(e + |g_t(x)|)dx \\
\lesssim \ln(e + t^{-1}) \|f_x\|_{\mathcal{L}\infty} \int_{\mathbb{R}^d} \Phi(x) \ln(e + \Phi(x))dx .
\]

To sum up, we have shown that, for some constants \( \alpha, \beta, \gamma > 0 \),

\[
\|g_t \ast |Tf||_{L^\infty} \lesssim \left( \Phi(r_B/t)\right)^{1-\beta} ||\Phi||_{L^\infty}^{\beta} + \|\Phi\|_{\Phi^*(\mathcal{L})} \ln(e + t^{-1}) \\
\times \left( \|f\|_{L^\infty} + ||f||_{L^p} + ||f||_{L^p}^{2/\alpha} \|f\|_{L^p}^{1-\alpha} \right) .
\]

If \( t \leq r_B \approx 1 \), then, by the decreasing property of \( \Phi \),

\[
\|g_t \ast |Tf||_{L^\infty} \lesssim \left( 1 + \Phi(1)\right)^{1-\beta} ||\Phi||_{L^\infty}^{\beta} + \|\Phi\|_{\Phi^*(\mathcal{L})} \ln(e + t^{-1}) \\
\times \left( \|f\|_{L^\infty} + ||f||_{L^p} + ||f||_{L^p}^{2/\alpha} \|f\|_{L^p}^{1-\alpha} \right) ,
\]

which proves the lemma for \( k = 1 \). The proof for \( k \neq 1 \) is similar: We still do the decomposition in the first step, that is

\[
|g_t \ast |Tf|^k| \lesssim \int_{B^c} |g_t| |Tf_x|^k dy + \int_B |g_t| \left| T \left( f_x 1_{(3B)^c} \right)^k \right| dy + \int_B |g_t| |T \left( f_x 1_{3B} \right)^k| dy \\
=: H + I + J .
\]

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The estimations for $H$ and $I$ are completely the same as those for $k = 1$. In order to estimate $J$, we need to modify the Orlicz spaces we used in the proof, i.e. we take $\psi_k(x) = e^{x^2/k} - 1$ instead of $\psi_k(x) = e^x - 1$. Then the corresponding $\phi_k$ (Legendre-Fenchel transform of $\psi_k$) is still some function growing slightly faster than linear up to a logarithmic factor which is approximately $(\ln(e + x))^k$ as $x$ gets larger. Thus, Corollary 2.14 and the definition of $\psi_k$ yields similar estimates:

\[
J \lesssim \|g_t\|_{\phi_k(L(\mu_B))} \left\| T \left( f_x 1_{3B} \right) \right\|^k_{\psi_k(L(\mu_B))} \\
\lesssim \|\Phi_t\|_{\phi_k(L(\mu_B))} \left( \|T \left( f_x 1_{3B} \right)\|_{\psi_k(L(\mu_B))} \right)^k \\
\lesssim \int_{\mathbb{R}^d} \Phi_t(x) (\ln(e + \Phi_t(x)))^k dx \|f_x 1_{3B}\|_{L^\infty}^k \\
\lesssim_d (\ln(e + t^{-1}))^k \int_{\mathbb{R}^d} \Phi_t(x) (\ln(e + \Phi_t(x)))^k dx \|f\|_{L^\infty}^k . \\
\]

\[
\square
\]

3 Analyticity in BMO-type spaces

The main result of this paper is as follows:

**Theorem 3.1.** Assume $u_0 \in \text{bmo}(\mathbb{R}^d)$ and $f(\cdot, t)$ is divergence-free and real-analytic in the space variable with the analyticity radius at least $\delta_f$ for all $t \in [0, \infty)$, and the analytic extension $f + ig$ satisfies

\[
\Gamma(t) := \sup_{s < t} \sup_{|y| < \delta_f} \|f(\cdot, y, s)\|_{\text{bmo}} + \|g(\cdot, y, s)\|_{\text{bmo}} < \infty.
\]

Fix a $t_0 > 0$ and let

\[
T_* = \min \left\{ \frac{1}{C \|u_0\|_{\text{bmo}} \Phi_1(\|u_0\|_{\text{bmo}})}, \frac{\|u_0\|_{\text{bmo}} \Phi_1(\|u_0\|_{\text{bmo}})}{C \Phi_1(\Gamma(t_0))} \right\}
\]

where $C$ is a constant and $\Phi_1(r)$ is some function with logarithmic growth as $r \to \infty$, both of which are independent of $u_0$ and $f$. Then there exists a solution

\[
u \in C([0, T_*], \text{bmo}(\mathbb{R}^d)^d)
\]

of the NSE (1.1)-(1.3) such that for every $t \in (0, T_*)$, $u$ is a restriction of an analytic function $u(x, y, t) + iv(x, y, t)$ in the region

\[
\mathcal{D}_t =: \left\{ (x, y) \in \mathbb{C}^d \mid |y| \leq \min\{ct^{1/2} \Phi_2(t), \delta_f\} \right\} 
\]

where $\Phi_2(t)$ is an explicitly defined function with logarithmic growth as $t \to 0^+$ which will be given in the proof. Moreover,

\[
\sup_{t \in (0, T)} \sup_{y \in \mathcal{D}_t} \|u(\cdot, y, t)\|_{\text{bmo}} + \sup_{t \in (0, T)} \sup_{y \in \mathcal{D}_t} \|v(\cdot, y, t)\|_{\text{bmo}} < \infty , \\
\sup_{t \in (0, T)} \sup_{y \in \mathcal{D}_t} \phi_1(t) \|u(\cdot, y, t)\|_{L^\infty} + \sup_{t \in (0, T)} \sup_{y \in \mathcal{D}_t} \phi_1(t) \|v(\cdot, y, t)\|_{L^\infty} < \infty
\]

where $\phi_1(t) = (\ln(e + 1/t))^{-1}$. 

The Proof: We construct an approximating sequence as follows:

\[
\begin{aligned}
  u^{(0)} &= 0, \quad \pi^{(0)} = 0, \\
  \partial_t u^{(n)} - \Delta u^{(n)} &= - \left( u^{(n-1)} \cdot \nabla \right) u^{(n-1)} - \nabla \pi^{(n-1)} + f, \\
  u^{(n)}(x, 0) &= u_0(x), \quad \nabla \cdot u^{(n)} = 0, \\
  \Delta \pi^{(n)} &= -\partial_j \partial_k \left( u_j^{(n)} u_k^{(n)} \right).
\end{aligned}
\]

By induction and Lemma 2.10, we know there are \( a^{(n)}(t) \) such that

\[
a^{(n)}(t) \cdot u^{(n)}(t) \in C^\infty([\delta, T], C^\infty(\mathbb{R}^d)) \cap L^\infty([0, T], L^\infty(\mathbb{R}^d))
\]

for arbitrarily small \( \delta \) and each \( u^{(n)}(t) \) is real analytic for every \( t \). Let \( u^{(n)}(x, y, t) + iv^{(n)}(x, y, t) \) and \( \pi^{(n)}(x, y, t) + i\rho^{(n)}(x, y, t) \) be the analytic extensions of \( u^{(n)} \) and \( \pi^{(n)} \) respectively. Inductively we have analytic extensions for all approximate solutions and the real and imaginary parts satisfy

\[
\begin{aligned}
  \partial_t u^{(n)} - \Delta u^{(n)} &= - \left( u^{(n-1)} \cdot \nabla \right) u^{(n-1)} + \left( v^{(n-1)} \cdot \nabla \right) v^{(n-1)} - \nabla \pi^{(n-1)} + f, \\
  \partial_t v^{(n)} - \Delta v^{(n)} &= - \left( u^{(n-1)} \cdot \nabla \right) v^{(n-1)} - \left( v^{(n-1)} \cdot \nabla \right) u^{(n-1)} - \nabla \rho^{(n-1)} + g,
\end{aligned}
\]

where

\[
\Delta \pi^{(n)} = -\partial_j \partial_k \left( u_j^{(n)} u_k^{(n)} - v_j^{(n)} v_k^{(n)} \right), \quad \Delta \rho^{(n)} = -2\partial_j \partial_k \left( u_j^{(n)} v_k^{(n)} \right).
\]

Now define

\[
\begin{aligned}
  U^{\alpha}_n(x, t) &= u^{(n)}(x, \alpha t), \quad \Pi^{(n)}_\alpha(x, t) = \pi^{(n)}(x, \alpha t), \quad F_\alpha(x, t) = f(x, \alpha t), \\
  V^{\alpha}_n(x, t) &= v^{(n)}(x, \alpha t), \quad R^{\alpha}_\alpha(x, t) = \rho^{(n)}(x, \alpha t), \quad G_\alpha(x, t) = g(x, \alpha t),
\end{aligned}
\]

then the approximation scheme becomes (for simplicity we drop the subscript \( \alpha \))

\[
\begin{aligned}
  \partial_t U^{(n)} - \Delta U^{(n)} &= -\alpha \cdot \nabla V^{(n)} - \left( U^{(n-1)} \cdot \nabla \right) U^{(n-1)} + \left( V^{(n-1)} \cdot \nabla \right) V^{(n-1)} - \nabla \Pi^{(n-1)} + F, \\
  \partial_t V^{(n)} - \Delta V^{(n)} &= -\alpha \cdot \nabla U^{(n)} - \left( U^{(n-1)} \cdot \nabla \right) V^{(n-1)} - \left( V^{(n-1)} \cdot \nabla \right) U^{(n-1)} - \nabla R^{(n-1)} + G, \\
  \Delta \Pi^{(n)} &= -\partial_j \partial_k \left( U_j^{(n)} U_k^{(n)} - V_j^{(n)} V_k^{(n)} \right), \quad \Delta R^{(n)} = -2\partial_j \partial_k \left( U_j^{(n)} V_k^{(n)} \right).
\end{aligned}
\]

with initial conditions

\[
U^{(n)}(x, 0) = u_0(x), \quad V^{(n)}(x, 0) = 0 \quad \text{for all } x \in \mathbb{R}^d,
\]

for which we have the following iteration:

\[
\begin{aligned}
  U^{(n)}(x, t) &= e^{t\Delta} u_0 - \int_0^t e^{(t-s)\Delta} \left( U^{(n-1)} \cdot \nabla \right) U^{(n-1)} ds + \int_0^t e^{(t-s)\Delta} \left( V^{(n-1)} \cdot \nabla \right) V^{(n-1)} ds \\
  &\quad - \int_0^t e^{(t-s)\Delta} \nabla \Pi^{(n-1)} ds + \int_0^t e^{(t-s)\Delta} F ds - \int_0^t e^{(t-s)\Delta} \alpha \cdot \nabla V^{(n)} ds, \\
  V^{(n)}(x, t) &= -\int_0^t e^{(t-s)\Delta} \left( U^{(n-1)} \cdot \nabla \right) V^{(n-1)} ds - \int_0^t e^{(t-s)\Delta} \left( V^{(n-1)} \cdot \nabla \right) U^{(n-1)} ds \\
  &\quad - \int_0^t e^{(t-s)\Delta} \nabla R^{(n-1)} ds + \int_0^t e^{(t-s)\Delta} G ds - \int_0^t e^{(t-s)\Delta} \alpha \cdot \nabla U^{(n)} ds.
\end{aligned}
\]
where
\[ \Pi^{(n)}(x, t) = -(\Delta)^{-1} \sum_j \partial_j \partial_k \left( U_j^{(n)} U_k^{(n)} - V_j^{(n)} V_k^{(n)} \right), \]
\[ R^{(n)}(x, t) = -2(\Delta)^{-1} \sum_j \partial_j \partial_k \left( U_j^{(n)} V_k^{(n)} \right). \]

We state that, for some \( T \) depending only on \( \|u_0\|_{bmo}, \|F\|_{bmo} \) and \( \|G\|_{bmo} \), the sequences \( U^{(n)} \), \( V^{(n)} \) constructed as above have a common upper bound in the four types of function spaces: \( \dot{B}_0^{0,\infty} \), \( bmo \), \( \phi(t)L^\infty \) and \( \psi(t)\dot{B}_x^{1,1} \). More precisely, our claim is: There exists \( T \) such that for all \( n \)
\[ U^{(n)}(x), V^{(n)}(x) \in C([0, T]; bmo({\mathbb{R}^d})^d), \tag{3.8} \]
\[ \phi_1(t)U^{(n)}(x), \phi_1(t)V^{(n)}(x) \in C([0, T]; L_\infty^\infty({\mathbb{R}^d})^d), \tag{3.9} \]
\[ \phi_2(t)U^{(n)}(x), \phi_2(t)V^{(n)}(x) \in C([0, T]; \dot{B}_x^{1,1}({\mathbb{R}^d})^d) \tag{3.10} \]
where \( \phi_1(t) \) is given in Theorem 3.1 and \( \phi_2(t) = 1_T \). Moreover
\[ L_n := \sup_{t < T} \phi_1(t)\|U^{(n)}\|_{L_\infty^n} + \sup_{t < T} \phi_1(t)\|V^{(n)}\|_{L_\infty^n}, \quad L'_n := \sup_{t < T} \|U^{(n)}\|_{\dot{B}_0^{0,\infty}} + \sup_{t < T} \|V^{(n)}\|_{\dot{B}_0^{0,\infty}}, \]
\[ L''_n := \sup_{t < T} \|U^{(n)}\|_{bmo} + \sup_{t < T} \|V^{(n)}\|_{bmo}, \quad L'''_n := \sup_{t < T} \phi_2(t)\|U^{(n)}\|_{\dot{B}_x^{1,1}} + \sup_{t < T} \phi_2(t)\|V^{(n)}\|_{\dot{B}_x^{1,1}} \]
are all bounded by a constant only determined by \( \|u_0\|_{bmo}, \|F\|_{bmo} \) and \( \|G\|_{bmo} \).

Proof of the claim: At the initial step of the iteration, i.e.
\[ U^{(0)}(x, t) = e^{\Delta} u_0 - \int_0^t e^{(t-s)\Delta} F \, ds - \int_0^t e^{(t-s)\Delta} \alpha \cdot \nabla V^{(0)} \, ds, \tag{3.11} \]
\[ V^{(0)}(x, t) = \int_0^t e^{(t-s)\Delta} G \, ds - \int_0^t e^{(t-s)\Delta} \alpha \cdot \nabla U^{(0)} \, ds, \tag{3.12} \]
we have the following chain of estimates which follows by Lemma 2.2 and Lemma 2.9:
\[ \|U^{(0)}\|_{bmo} \lesssim \|e^{\Delta} u_0\|_{bmo} + \int_0^t \|e^{(t-s)\Delta} F\|_{bmo} ds + \int_0^t \|e^{(t-s)\Delta} \alpha \cdot \nabla V^{(0)}\|_{bmo} ds \]
\[ \lesssim \|u_0\|_{bmo} + \int_0^t \|F\|_{bmo} ds + |\alpha| \int_0^t \|\nabla e^{(t-s)\Delta} V^{(0)}\|_{L_\infty^n} ds \]
\[ \lesssim \|u_0\|_{bmo} + t \sup_{s < t} \|F\|_{bmo} + |\alpha| \int_0^t (t-s)^{-1/2} \|V^{(0)}\|_{BMO} ds \]
\[ \lesssim \|u_0\|_{bmo} + t \sup_{s < t} \|F\|_{bmo} + |\alpha| t^{1/2} \sup_{s < t} \|V^{(0)}\|_{bmo}. \tag{3.13} \]
And similarly,
\[ \|V^{(0)}\|_{bmo} \lesssim \|u_0\|_{bmo} + t \sup_{s < t} \|G\|_{bmo} + |\alpha| t^{1/2} \sup_{s < t} \|U^{(0)}\|_{bmo}. \tag{3.14} \]
If we assume \( \alpha \) is a vector such that \( C|\alpha|^{1/2} < 1/2 \) for all \( t < T \) with some proper choice of \( C \) according to the above estimations, then combining (3.13) and (3.14) gives
\[ \sup_{t < T} \|U^{(0)}\|_{bmo} + \sup_{t < T} \|V^{(0)}\|_{bmo} \lesssim \|u_0\|_{bmo} + T \left( \sup_{t < T} \|F\|_{bmo} + \sup_{t < T} \|G\|_{bmo} \right). \tag{3.15} \]
Since $bmo \hookrightarrow B^0_{\infty,\infty}$ (see Lemma 2.8), the above estimate implies

$$
sup_{t<T} \|U^{(0)}\|_{B^0_{\infty,\infty}} + sup_{t<T} \|V^{(0)}\|_{B^0_{\infty,\infty}} \lesssim \|u_0\|_{bmo} + T \left( sup_{t<T} \|F\|_{bmo} + sup_{t<T} \|G\|_{bmo} \right). \quad (3.16)
$$

The estimations in \( \phi_1(t) L^\infty \) are similar: By taking the \( L^\infty \)-norm of (3.11) and using the fact that \( B^0_{\infty,1} \hookrightarrow L^\infty \) (see Kozono et al. [18]), we obtain

$$
\|U^{(0)}\|_{L^\infty} \lesssim \|e^t \Delta u_0\|_{L^\infty} + \int_0^t \|e^{(t-s)} \Delta F\|_{L^\infty} ds + \int_0^t \|e^{(t-s)} \alpha \cdot \nabla V^{(0)}\|_{L^\infty} ds
$$

$$
\lesssim \|e^t \Delta u_0\|_B^{0,1} + \int_0^t \|e^{(t-s)} \Delta F\|_{B^{0,1}_{\infty,1}} ds + |\alpha| \int_0^t \|\nabla e^{(t-s)} \Delta V^{(0)}\|_{B^{0,1}_{\infty,1}} ds.
$$

Then, by Lemma 2.3

$$
\|U^{(0)}\|_{L^\infty} \lesssim \ln(e + 1/t) \|u_0\|_{B^0_{\infty,\infty}} + \int_0^t \ln(e + 1/(t - s)) \|F\|_{B^0_{\infty,\infty}} ds
$$

$$
+ |\alpha| \int_0^t (t - s)^{-\frac{1}{2}} \ln(e + 1/(t - s)) \|V^{(0)}\|_{B^0_{\infty,\infty}} ds
$$

$$
\lesssim \ln(e + 1/t) \|u_0\|_{B^0_{\infty,\infty}} + \psi_1(t) \sup_{s < t} \|F\|_{B^0_{\infty,\infty}} + |\alpha| t^\frac{1}{2} \psi_2(t) \|V^{(0)}\|_{B^0_{\infty,\infty}}.
$$

where \( \psi_1, \psi_2 \) are given explicitly by

$$
\psi_1(t) = t^{-1} \int_0^t \ln(e + 1/(t - s)) ds, \quad \psi_2(t) = t^{-\frac{1}{2}} \int_0^t (t - s)^{-\frac{1}{2}} \ln(e + 1/(t - s)) ds.
$$

Since \( L^\infty \hookrightarrow bmo \hookrightarrow B^0_{\infty,\infty} \) (see Lemma 2.8) we end up with

$$
\|U^{(0)}\|_{L^\infty} \lesssim \ln(e + 1/t) \|u_0\|_{bmo} + t \psi_1(t) \sup_{s < t} \|F\|_{bmo} + |\alpha| t^\frac{1}{2} \psi_2(t) \|V^{(0)}\|_{L^\infty}. \quad (3.17)
$$

It follows from the same reasoning that

$$
\|V^{(0)}\|_{L^\infty} \lesssim \ln(e + 1/t) \|u_0\|_{bmo} + t \psi_1(t) \sup_{s < t} \|G\|_{bmo} + |\alpha| t^\frac{1}{2} \psi_2(t) \|U^{(0)}\|_{L^\infty}. \quad (3.18)
$$

In the rest of the proof we will always write \( \phi_1(t) \) for \( \ln(e + 1/t)^{-1} \). If we assume \( \alpha \) is such that \( C|\alpha| t^{1/2} \psi_2(t) < 1/2 \) for all \( t < T \) with some proper choice of \( C \), then (3.17) and (3.18) imply that

$$
\sup_{t<T} \phi_1(t) \|U^{(0)}\|_{L^\infty} + \sup_{t<T} \phi_1(t) \|V^{(0)}\|_{L^\infty} \lesssim \|u_0\|_{bmo} + T \|\psi_3(T) \left( \sup_{t<T} \|F\|_{bmo} + \sup_{t<T} \|G\|_{bmo} \right) \right). \quad (3.19)
$$

where \( \psi_3(t) = \phi_1(t) \psi_1(t) \). For the estimations in \( \dot{B}^1_{1,1} \): In virtue of (2.3), we deduce

$$
\|U^{(0)}\|_{\dot{B}^1_{1,1}} \lesssim \|e^t \Delta u_0\|_{\dot{B}^1_{1,1}} + \int_0^t \|e^{(t-s)} \Delta F\|_{\dot{B}^1_{1,1}} ds + \int_0^t \|e^{(t-s)} \alpha \cdot \nabla V^{(0)}\|_{\dot{B}^1_{1,1}} ds
$$

$$
\lesssim t^{-1/2} \|u_0\|_{B^0_{\infty,\infty}} + \int_0^t (t - s)^{-\frac{1}{2}} \|F\|_{\dot{B}^0_{\infty,\infty}} ds + |\alpha| \int_0^t (t - s)^{-\frac{1}{2}} \|\nabla V^{(0)}\|_{\dot{B}^0_{\infty,\infty}} ds.
$$
Since $\dot{B}_{2,1}^1 \hookrightarrow \dot{B}_{2,\infty}^0$ (trivial fact by the definitions) and $\text{bmo} \hookrightarrow \text{BMO} \hookrightarrow \dot{B}_{2,\infty}^0$ (see Lemma 2.8)

$$
\|U^{(0)}\|_{\dot{B}_{2,1}^1} \lesssim t^{-1/2}\|u_0\|_{\text{BMO}} + \int_0^t (t-s)^{-\frac{1}{2}}\|F\|_{\text{BMO}} ds + |\alpha| \int_0^t (t-s)^{-\frac{1}{2}}\|\nabla V^{(0)}\|_{\dot{B}_{2,1}^1} ds
$$

$$
\lesssim t^{-1/2}\|u_0\|_{\text{bmo}} + t^{\frac{3}{2}} \sup_{s<t} \|F\|_{\text{bmo}} + |\alpha| \sup_{s<t} s^{\frac{3}{2}}\|V^{(0)}\|_{\dot{B}_{2,1}^1} \int_0^t (t-s)^{-\frac{1}{2}}s^{-\frac{1}{2}} ds .
$$

Thus, for all $t < T$,

$$
t^{\frac{3}{2}}\|U^{(0)}\|_{\dot{B}_{2,1}^1} \lesssim \|u_0\|_{\text{bmo}} + t \sup_{s<t} \|F\|_{\text{bmo}} + |\alpha| t^{\frac{3}{2}} \sup_{s<t} s^{\frac{3}{2}}\|V^{(0)}\|_{\dot{B}_{2,1}^1} . \tag{3.20}
$$

Similarly, for all $t < T$,

$$
t^{\frac{3}{2}}\|V^{(0)}\|_{\dot{B}_{2,1}^1} \lesssim \|u_0\|_{\text{bmo}} + t \sup_{s<t} \|F\|_{\text{bmo}} + |\alpha| t^{\frac{3}{2}} \sup_{s<t} s^{\frac{3}{2}}\|U^{(0)}\|_{\dot{B}_{2,1}^1} . \tag{3.21}
$$

Again, choosing $\alpha$ properly, combination of the above two estimates shows that

$$
\sup_{t<T} t^{\frac{3}{2}}\|U^{(0)}\|_{\dot{B}_{2,1}^1} + \sup_{t<T} t^{\frac{3}{2}}\|V^{(0)}\|_{\dot{B}_{2,1}^1} \lesssim \|u_0\|_{\text{bmo}} + T \left( \sup_{t<T} \|F\|_{\text{bmo}} + \sup_{t<T} \|G\|_{\text{bmo}} \right) . \tag{3.22}
$$

To sum up the estimates (3.15), (3.16), (3.19) and (3.22) we’ve shown that

$$
M_0 := \max\{L_0, L_0', L_0''\} \lesssim \|u_0\|_{\text{bmo}} + T\psi_3(T) \left( \sup_{t<T} \|F\|_{\text{bmo}} + \sup_{t<T} \|G\|_{\text{bmo}} \right) .
$$

To control the rest of $M_n$’s in the iteration scheme, estimations for the nonlinear terms play an essential role. In the following we will demonstrate the idea with the terms such that

$$
\int_0^t e^{(t-s)\Delta} (U^{(n)} \cdot \nabla) U^{(n)} ds , \quad \int_0^t e^{(t-s)\Delta} (U^{(n)} \cdot \nabla) V^{(n)} ds , \quad \int_0^t e^{(t-s)\Delta} \nabla II^{(n)} ds , \quad ...
$$

in those four spaces (we won’t write down details of the estimation for each term in each function space because some of the ideas and techniques are similar). First we will derive estimation in $L^\infty$; we will see the results for the other spaces essentially follow from it.

Recall that $\nabla \cdot U^{(n)} = 0$, so $(U^{(n)} \cdot \nabla) U^{(n)} = \nabla (U^{(n)} \otimes U^{(n)})$. For induction hypothesis, we suppose (3.8)-(3.10) holds true for $n$. Then, by Lemma 2.2

$$
\left\| \int_0^t e^{(t-s)\Delta} (U^{(n)} \cdot \nabla) U^{(n)} ds \right\|_{L^\infty} \lesssim \int_0^t \|e^{(t-s)\Delta} \nabla (U^{(n)} \otimes U^{(n)})\|_{L^\infty} ds
$$

$$
\lesssim \int_0^t \|\nabla e^{(t-s)\Delta} (U^{(n)} \otimes U^{(n)})\|_{L^\infty} ds
$$

$$
\lesssim \int_0^t (t-s)^{-1/2}\|U^{(n)} \otimes U^{(n)}\|_{\text{BMO}} ds
$$

$$
\lesssim \int_0^t (t-s)^{-1/2}\|U^{(n)} \otimes U^{(n)}\|_{L^\infty} ds
$$

$$
\lesssim \left( \sup_{s<t} \phi_1(s)\|U^{(n)}\|_{L^\infty} \right)^2 \int_0^t (t-s)^{-1/2}(\phi_1(s))^{-2} ds .
$$
Note that the estimates for $V^{(n)}$ follow in the same way, thus we have

$$\phi_1(t) \left\| \int_0^t e^{(t-s)\Delta} (U^{(n)} \cdot \nabla) U^{(n)} \, ds \right\|_{L^\infty} \lesssim t^{\frac{d}{4}} \psi_4(t) \left( \sup_{s < t} \phi_1(s) \| U^{(n)} \|_{L^\infty} \right)^2,$$

(3.23)

$$\phi_1(t) \left\| \int_0^t e^{(t-s)\Delta} (V^{(n)} \cdot \nabla) V^{(n)} \, ds \right\|_{L^\infty} \lesssim t^{\frac{d}{4}} \psi_4(t) \left( \sup_{s < t} \phi_1(s) \| V^{(n)} \|_{L^\infty} \right)^2,$$

(3.24)

where

$$\psi_4(t) = t^{-\frac{d}{2}} \phi_1(t) \int_0^t (t - s)^{-1/2} (\phi_1(s))^{-2} \, ds.$$ 

The bound for the pressure term is obtained in a similar way: For induction hypothesis, we suppose (3.8)-(3.10) hold true for $n$. For convenience we will denote by $P$ the projection operator, i.e.

$$P(f \otimes g) := -(\Delta)^{-1} \sum \partial_j \partial_k (f_j \cdot g_k).$$

Since $\|\nabla f\|_\infty \lesssim \|\nabla f\|_{\dot{B}^0_{\infty,1}}$ if $f \in BMO$ and $\nabla f \in \dot{B}^0_{\infty,1}$ (see Kozono et al. [18]), we have

$$\left\| \int_0^t e^{(t-s)\Delta} \nabla \Pi^{(n)} \, ds \right\|_{L^\infty} \lesssim \int_0^t \left\| \nabla e^{(t-s)\Delta} P (U^{(n)} \otimes U^{(n)} - V^{(n)} \otimes V^{(n)}) \right\|_{L^\infty} \, ds$$

$$\lesssim \int_0^t \left\| \nabla e^{(t-s)\Delta} P (U^{(n)} \otimes U^{(n)} - V^{(n)} \otimes V^{(n)}) \right\|_{\dot{B}^0_{\infty,1}} \, ds.$$ 

Note that $P$ is a bounded operator from $\dot{B}^0_{\infty,\infty}$ into itself (see Lemarié [19] and Han and Hofmann [16]). Then, by Lemma 2.6 and Lemma 2.8 it follows that

$$\left\| \int_0^t e^{(t-s)\Delta} \nabla \Pi^{(n)} \, ds \right\|_{L^\infty} \lesssim \int_0^t (t - s)^{-1/2} \left\| P (U^{(n)} \otimes U^{(n)} - V^{(n)} \otimes V^{(n)}) \right\|_{\dot{B}^0_{\infty,\infty}} \, ds$$

$$\lesssim \int_0^t (t - s)^{-1/2} \left\| U^{(n)} \otimes U^{(n)} - V^{(n)} \otimes V^{(n)} \right\|_{\dot{B}^0_{\infty,\infty}} \, ds$$

$$\lesssim \int_0^t (t - s)^{-1/2} \left( \| U^{(n)} \otimes U^{(n)} \|_{L^\infty} + \| V^{(n)} \otimes V^{(n)} \|_{L^\infty} \right) \, ds$$

$$\lesssim \left( \sup_{s < t} \phi_1(s) \| U^{(n)} \|_{L^\infty} \right)^2 + \left( \sup_{s < t} \phi_1(s) \| V^{(n)} \|_{L^\infty} \right)^2 \int_0^t (t - s)^{-1/2} (\phi_1(s))^{-2} \, ds.$$ 

Thus

$$\phi_1(t) \left\| \int_0^t e^{(t-s)\Delta} \nabla \Pi^{(n)} \, ds \right\|_{L^\infty} \lesssim t^{\frac{d}{4}} \psi_4(t) \left( \sup_{s < t} \phi_1(s) \| U^{(n)} \|_{L^\infty} \right)^2 + \left( \sup_{s < t} \phi_1(s) \| V^{(n)} \|_{L^\infty} \right)^2.$$ 

(3.25)
Following the same argument as in the estimation for $\|U^{(0)}\|_{L^\infty}$, we obtain
\[
\left\| \int_0^t e^{(t-s)\Delta} \alpha \cdot \nabla V^{(n+1)} ds \right\|_{L^\infty} \lesssim |\alpha| t^{3/2} \psi_2(t) \|V^{(n+1)}\|_{L^\infty},
\]
(3.26)
\[
\left\| \int_0^t e^{(t-s)\Delta} F ds \right\|_{L^\infty} \lesssim t \psi_1(t) \sup_{s < t} \|F\|_{bmo}, \quad \left\| e^{t\Delta} u_0 \right\|_{L^\infty} \lesssim \phi_1(t)^{-1} \|u_0\|_{bmo}.
\]
(3.27)

In summation, (3.6), together with (3.23)-(3.27), shows
\[
\sup_{t < T} \phi_1(t) \|U^{(n)}\|_{L^\infty} \lesssim \|u_0\|_{bmo} + T^{3/2} \psi_4(T) \left(L_n\right)^2 + T \psi_3(T) \sup_{t < T} \|F\|_{bmo} + |\alpha| t^{3/2} \psi_2(t)L_{n+1}.
\]
(3.28)

The estimations in “$\phi_1(t)L^\infty$” are similar but with special attention to “the mixed terms”: With the induction hypothesis $V^{(n)} \in \tilde{B}^{1}_{1,1}$ we have
\[
\left\| \int_0^t e^{(t-s)\Delta} (U^{(n)} \cdot \nabla) V^{(n)} ds \right\|_{L^\infty} \lesssim \int_0^t \|U^{(n)}\|_{L^\infty} \left\| e^{(t-s)\Delta} |\nabla V^{(n)}| \right\|_{L^\infty} ds
\]
\[
\lesssim \int_0^t \|U^{(n)}\|_{L^\infty} \|\nabla V^{(n)}\|_{\tilde{B}^0_{\infty,1}} ds
\]
\[
\lesssim \int_0^t \|U^{(n)}\|_{L^\infty} \|V^{(n)}\|_{\tilde{B}^1_{\infty,1}} ds
\]
\[
\lesssim \left( \sup_{s < t} \phi_1(s) \|U^{(n)}\|_{L^\infty} \right) \left( \sup_{s < t} s^{1/2} \|V^{(n)}\|_{\tilde{B}^1_{\infty,1}} \right)
\]
\[
\times \int_0^t s^{-1/2} \phi_1(s)^{-1} ds.
\]

Thus
\[
\phi_1(t) \left\| \int_0^t e^{(t-s)\Delta} (U^{(n)} \cdot \nabla) V^{(n)} ds \right\|_{L^\infty} \lesssim t^{1/2} \psi_1(t) \psi_2(t)L_n L''_n.
\]

Similar to the deduction for (3.25), we have
\[
\phi_1(t) \left\| \int_0^t e^{(t-s)\Delta} \nabla R^{(n)} ds \right\|_{L^\infty} \lesssim \phi_1(t) \int_0^t (t-s)^{-1/2} \left\| P \left(U^{(n)} \otimes V^{(n)}\right) \right\|_{\tilde{B}^0_{\infty,\infty}} ds
\]
\[
\lesssim \phi_1(t) \int_0^t (t-s)^{-1/2} \left\| U^{(n)} \otimes V^{(n)} \right\|_{L^\infty} ds
\]
\[
\lesssim t^{1/2} \psi_4(t) \left( \sup_{s < t} \phi_1(s) \|U^{(n)}\|_{L^\infty} \right) \left( \sup_{s < t} \phi_1(s) \|V^{(n)}\|_{L^\infty} \right).
\]

With the above estimates at hand, we have from (3.7) that
\[
\sup_{t < T} \phi_1(t) \|V^{(n)}\|_{L^\infty} \lesssim T^{3/2} \psi_4(T) \left(L_n\right)^2 + T^{3/2} \psi_5(T)L_n L''_n
\]
\[
+ T \psi_3(T) \sup_{t < T} \|G\|_{bmo} + |\alpha| t^{3/2} \psi_2(t)L_{n+1}.
\]
(3.29)
So, with $\alpha$ such that $C|\alpha|^{1/2}\psi_2(t) < 1/2$ for all $t < T$ with some constant $C$, (3.28) and (3.29) imply that
\[
L_{n+1} \lesssim \|u_0\|_{bmo} + T\psi_3(T) \left( \sup_{t < T} \|F\|_{bmo} + \sup_{t < T} \|G\|_{bmo} \right) + T^{\frac{1}{2}} \psi_4(T) \left( L_n \right)^2 + T^{\frac{1}{2}} \phi_1(T) \psi_2(T) L_n L''_n .
\]
(3.30)

Recall that $L^\infty \hookrightarrow bmo \hookrightarrow B^0_{\infty,\infty}$, it follows immediately
\[
L'_{n+1} \lesssim \|u_0\|_{bmo} + T\psi_1(T) \left( \sup_{t < T} \|F\|_{bmo} + \sup_{t < T} \|G\|_{bmo} \right) + T^{\frac{1}{2}} \psi_4(T) \left( \phi_1(T) \right)^{-1} \left( L_n \right)^2 + T^{\frac{1}{2}} \psi_2(T) L_n L''_n ,
\]
(3.31)
\[
L''_{n+1} \lesssim \|u_0\|_{bmo} + T\psi_1(T) \left( \sup_{t < T} \|F\|_{bmo} + \sup_{t < T} \|G\|_{bmo} \right) + T^{\frac{1}{2}} \psi_4(T) \left( \phi_1(T) \right)^{-1} \left( L_n \right)^2 + T^{\frac{1}{2}} \psi_2(T) L_n L''_n .
\]
(3.32)

For the estimations in $\dot{B}^1_{\infty,1}$, due to large amount of overlap with those previous arguments, it suffices to demonstrate the followings: By Lemma 2.8 and the boundedness of the projection $P$ in $B^0_{\infty,\infty}$ with the fact that $U^{(n)}$, $V^{(n)}$ are divergence free,
\[
\left\| \int_0^t e^{(t-s)\Delta} \nabla R^{(n)}(s) \right\|_{\dot{B}^1_{\infty,1}} \lesssim \int_0^t \left\| e^{(t-s)\Delta} \nabla P \left( U^{(n)} \otimes V^{(n)} \right) \right\|_{\dot{B}^1_{\infty,1}} ds \\
\lesssim \int_0^t (t-s)^{-\frac{1}{2}} \left\| P \left( \nabla (U^{(n)} \otimes V^{(n)}) \right) \right\|_{\dot{B}^0_{\infty,\infty}} ds \\
\lesssim \int_0^t (t-s)^{-\frac{1}{2}} \left\| (U^{(n)} \cdot \nabla)V^{(n)} + (V^{(n)} \cdot \nabla)U^{(n)} \right\|_{\dot{B}^0_{\infty,\infty}} ds .
\]

Then, with the induction hypothesis $U^{(n)}$, $V^{(n)} \in \dot{B}^1_{\infty,1}$, it follows that
\[
\left\| \int_0^t e^{(t-s)\Delta} \nabla R^{(n)}(s) \right\|_{\dot{B}^1_{\infty,1}} \lesssim \int_0^t (t-s)^{-\frac{1}{2}} \left( \|U^{(n)}\|_{L^\infty} \|\nabla V^{(n)}\|_{L^\infty} + \|V^{(n)}\|_{L^\infty} \|\nabla U^{(n)}\|_{L^\infty} \right) ds \\
\lesssim \int_0^t (t-s)^{-\frac{1}{2}} \left( \|U^{(n)}\|_{L^\infty} \|\nabla V^{(n)}\|_{\dot{B}^0_{\infty,1}} + \|V^{(n)}\|_{L^\infty} \|\nabla U^{(n)}\|_{\dot{B}^0_{\infty,1}} \right) ds \\
\lesssim \psi_5(t) \left( \sup_{s \leq t} \phi_1(s) \|U^{(n)}\|_{L^\infty} \right) \left( \sup_{s \leq t} s^{\frac{1}{2}} \|V^{(n)}\|_{L^\infty} \right) \\
\quad + \psi_5(t) \left( \sup_{s \leq t} \phi_1(s) \|V^{(n)}\|_{L^\infty} \right) \left( \sup_{s \leq t} s^{\frac{1}{2}} \|U^{(n)}\|_{L^\infty} \right)
\]
where
\[
\psi_5(t) = \int_0^t s^{-\frac{1}{2}} (t-s)^{-\frac{1}{2}} (\phi_1(s))^{-1} ds .
\]
The bounds in $\dot{B}^{1}_{\infty, 1}$ for all other nonlinear terms follow in a similar fashion. So
\[
L''_{n+1} \leq \epsilon \| u_0 \|_{bmo} + T (\sup_{t < T} \| F \|_{bmo} + \sup_{t < T} \| G \|_{bmo}) + T^{\frac{4}{bmo}} \psi_5(T) L_n L''_n. \tag{3.33}
\]

In conclusion, (3.30)-(3.33) yield uniform bound for all the function spaces:
\[
M_{n+1} := \max \{ L_{n+1}, L'_{n+1}, L''_{n+1}, L'''_{n+1} \}
\leq \| u_0 \|_{bmo} + T \Psi_1(T) \left( \sup_{t < T} \| F \|_{bmo} + \sup_{t < T} \| G \|_{bmo} \right) + T^{\frac{4}{bmo}} \Psi_2(T)(M_n)^2
\]
where
\[
\Psi_1(t) = \max\{1, \psi_1(t), \psi_3(t)\}, \quad \Psi_2(t) = \max\{\psi_2, \psi_4, \psi_5, \phi_1 \psi_2, \phi_1^{-1} \psi_4\}.
\]

If we take $T_*$ such that
\[
T_*^{\frac{4}{bmo}} \Psi_2(T_*) \leq \frac{1}{C \left( \left\| u_0 \right\|_{bmo} + T_* \Psi_1(T_*) \left( \sup_{t < T_*} \| F \|_{bmo} + \sup_{t < T_*} \| G \|_{bmo} \right) \right)},
\]
where the constant $C$ was generated in our iteration scheme, independent of $T$, $u_0$, $F$ and $G$. Then all the sequences are bounded by $\left(2CT_*^{\frac{4}{bmo}} \Psi_2(T_*)\right)^{-1}$. This completes the proof of the claim.

Now the standard converging argument with Lemma 2.11 (applied for each $t$ with $p = \infty$) shall complete the proof that the limit function $u$ (i.e. the complexified solution of the NSE (1.1)-(1.3)) exists and it is bounded locally uniformly in time (the time interval only depends on $\| u_0 \|_{bmo}$, $\| F \|_{bmo}$ and $\| G \|_{bmo}$) and uniformly in $y$-variables over the complex domain
\[
D_t := \{(x, y) \in \mathbb{C}^d \mid |y| \leq \min\{ct^{1/2} \min\{1, \psi_2(t)^{-1}\}, \delta_T\} \}
\]
in any of the four spaces (over $x \in \mathbb{R}^d$) with the upper bound only depending on $\| u_0 \|_{bmo}$, $\| F \|_{bmo}$ and $\| G \|_{bmo}$. The analyticity properties of $u$, i.e. the existence of the higher order space derivatives is justified by the uniform convergence on any compact subset of $D_t$, following from Lemma 2.11 (see Grujić and Kukavica [13] for more details). This ends the proof of Theorem 3.1.

In fact, by the existence of space derivatives and using the equation (1.1) (again with uniform convergence on compact subset in the complex space), it follows that the classical derivatives in time also exist. Therefore, we have

**Corollary 3.2.** The solution $u(t)$ stated in Theorem 3.1 is the classical solution of (1.1)-(1.3).

**Remark 3.3.** With some modification of the proof, one can show there exists some $T_*$ such that the equations (1.1)-(1.3) has an analytical solution
\[
u \in C([0, T_*), B^{0}_{\infty, \infty}(\mathbb{R}^d)^d)
\]
with the initial value $u_0 \in B^{0}_{\infty, \infty}(\mathbb{R}^d)$ (all the other assumptions remain the same). The work required is similar and easier, we omit the details.

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4 Analyticity of Vorticity

In this section, we formulate and prove a ‘twin’ theorem for estimating the oscillation of vorticity, i.e. we consider the vorticity-velocity formulation of the 3D Navier-Stokes equations:

$$\partial_t \omega - \Delta \omega = \omega \nabla u - u \nabla \omega, \quad \omega(0, x) = \omega_0$$

and we will show

**Theorem 4.1.** Assume \( \omega_0 \in bmo(\mathbb{R}^3) \cap L^p(\mathbb{R}^3) \) where \( 1 \leq p < 3 \). Fix a \( t_0 > 0 \) and let

$$T_\omega = \begin{cases} \min \left\{ \frac{C(\|\omega_0\|_{bmo} + \|\omega_0\|_{L^p})}{\Phi_1(1(\Gamma(t_0)))} \left( \frac{\|\omega_0\|_{bmo} + \|\omega_0\|_{L^p})}{\Phi_1(1(\Gamma(t_0)))} \right) \right\} & \text{if } p > 1 \\
\min \left\{ \frac{1}{C(\|\omega_0\|_{bmo} + \|\omega_0\|_{L^p})^2} \left( \frac{\|\omega_0\|_{bmo} + \|\omega_0\|_{L^p})^2}{\Phi_1(1(\Gamma(t_0)))} \right) \right\} & \text{if } p = 1 \end{cases}$$

(4.1)

where \( \Gamma(\cdot) \) is given in Theorem 3.1, \( C \) is a constant and \( \Phi_1(r) \) is some function with logarithmic growth as \( r \to \infty \), both of which are independent of \( \omega_0 \). Then there exists a solution

$$\omega \in C([0, T_\omega), bmo(\mathbb{R}^3^3))$$

for the NSE (1.1)-(1.3) such that for every \( t \in (0, T_\omega) \), \( u \) is a restriction of an analytic function \( \omega(x, y, t) + i\zeta(x, y, t) \) in the region

$$D_t = \left\{ (x, y) \in \mathbb{C}^3 \mid |y| \leq \min\{ct^{1/2}\Phi_2(t), \delta_f\} \right\}$$

where \( \delta_f \) is defined as before and \( \Phi_2(t) \) is an explicitly defined function with logarithmic growth as \( t \to 0^+ \) as given in Theorem 3.1. Moreover,

$$\sup_{t \in (0, T)} \sup_{\gamma \in D_t} \|\omega(\cdot, y, t)\|_{bmo} + \sup_{t \in (0, T)} \sup_{\gamma \in D_t} \|\zeta(\cdot, y, t)\|_{bmo} < \infty \ , \quad (4.2)$$

$$\sup_{t \in (0, T)} \sup_{\gamma \in D_t} \Phi_1(t) \|\omega(\cdot, y, t)\|_{L^\infty} + \sup_{t \in (0, T)} \sup_{\gamma \in D_t} \Phi_1(t) \|\zeta(\cdot, y, t)\|_{L^\infty} < \infty \quad (4.3)$$

where \( \Phi_1(t) = [\ln(e + 1/t)]^{-1} \).

**Proof.** We construct an approximating sequence as follows:

$$\partial_t \omega^{(n)} - \Delta \omega^{(n)} = \omega^{(n-1)} \nabla u^{(n-1)} - u^{(n-1)} \nabla \omega^{(n-1)}, \quad \omega^{(n)}(0, x) = \omega_0 ,$$

$$u^{(n-1)}_j(x, t) = c \int_{\mathbb{R}^3} \epsilon_{j,k,l} \partial_{yk} \frac{1}{|x - y|} \omega^{(n-1)}_k(y, t) dy .$$

We let \( u^{(n)} + i\omega^{(n)} \) and \( \omega^{(n)} + i\zeta^{(n)} \) be the analytic extension of the approximating sequence and let

$$U^{(n)}(x, t) = u^{(n)}(x, \alpha t, t) , \quad W^{(n)}(x, t) = u^{(n)}(x, \alpha t, t) ,$$

$$V^{(n)}(x, t) = v^{(n)}(x, \alpha t, t) , \quad Z^{(n)}(x, t) = \zeta^{(n)}(x, \alpha t, t) ,$$

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for which we have the iterations:

\[
W^{(n+1)}(x, t) = e^{t\Delta} \omega_0 + \int_0^t e^{(t-s)\Delta} W^{(n)} \nabla U^{(n)} ds - \int_0^t e^{(t-s)\Delta} Z^{(n)} \nabla V^{(n)} ds
- \int_0^t e^{(t-s)\Delta} U^{(n)} \nabla W^{(n)} ds + \int_0^t e^{(t-s)\Delta} V^{(n)} \nabla W^{(n)} ds + \int_0^t e^{(t-s)\Delta} \alpha \cdot \nabla Z^{(n+1)} ds
\]

\[
Z^{(n+1)}(x, t) = \int_0^t e^{(t-s)\Delta} Z^{(n)} \nabla U^{(n)} ds + \int_0^t e^{(t-s)\Delta} W^{(n)} \nabla V^{(n)} ds
- \int_0^t e^{(t-s)\Delta} V^{(n)} \nabla W^{(n)} ds - \int_0^t e^{(t-s)\Delta} U^{(n)} \nabla Z^{(n+1)} ds
\]

where

\[
U_j^{(n)}(x, t) = c \int_{\mathbb{R}^3} \epsilon_{j,k,\ell} \partial_{y_k} \frac{1}{|x-y|} W^{(n)}_\ell (y, t) dy , \quad (4.4)
\]

\[
V_j^{(n)}(x, t) = c \int_{\mathbb{R}^3} \epsilon_{j,k,\ell} \partial_{y_k} \frac{1}{|x-y|} Z^{(n)}_\ell (y, t) dy . \quad (4.5)
\]

Analogous to the proof of Theorem 3.1 we have the statement: There exists \( T \) such that for all \( n \)

\[
W^{(n)}, Z^{(n)} \in C([0, T); bmo(\mathbb{R}^3)^3) , \quad W^{(n)}, Z^{(n)} \in C([0, T); L^p(\mathbb{R}^3)^3) , \quad (4.6)
\]

\[
\phi_1(t)W^{(n)}, \phi_1(t)Z^{(n)}, \phi_1(t)U^{(n)}, \phi_1(t)V^{(n)} \in C([0, T); L^\infty(\mathbb{R}^3)^3) , \quad (4.7)
\]

where \( \phi_1(t) \) is given in Theorem 4.1. Moreover

\[
\sup_{t<T} \phi_1(t)\|W^{(n)}\|_{L^\infty} + \sup_{t<T} \phi_1(t)\|Z^{(n)}\|_{L^\infty} < K_n , \quad \sup_{t<T} \|W^{(n)}\|_{bmo} + \sup_{t<T} \|Z^{(n)}\|_{bmo} < K'_n ,
\]

\[
\sup_{t<T} \|W^{(n)}\|_{B_\infty,\infty} + \sup_{t<T} \|Z^{(n)}\|_{B_0,\infty} < K'_n , \quad \sup_{t<T} \|W^{(n)}\|_{L^p} + \sup_{t<T} \|Z^{(n)}\|_{L^p} < K''_n ,
\]

\[
\sup_{t<T} \phi_1(t)\|U^{(n)}\|_{L^\infty} + \sup_{t<T} \phi_1(t)\|V^{(n)}\|_{L^\infty} < Q_n ,
\]

where \( K_n, K'_n, K''_n, Q_n \) are all bounded by a constant determined by \( \max\{\|\omega_0\|_{bmo}, \|\omega_0\|_{L^p}\} \).

Prove of the Claim: The estimates for \( W^{(0)} \) and \( Z^{(0)} \) are very similar and easier compared to Theorem 3.1, so we get straight to the conclusion: With the choice of \( \alpha \) such that \( C|\alpha|t^{1/2}\psi_2(t) < 1/2 \) for all \( t < T \) and some constant \( C \), we have

\[
\mathcal{M}_0 := \max\{K_0, K'_0, K''_0, K'''_0, Q_0\} \leq \|\omega_0\|_{bmo} + \|\omega_0\|_{L^p} .
\]

The essence of binding the rest of \( \mathcal{M}_n \)’s still lies in the \( L^\infty \)-estimation (and \( L^p \)-estimation) of the nonlinear terms.

\[
\left\| \int_0^t e^{(t-s)\Delta} W^{(n)} \nabla U^{(n)} ds \right\|_{L^\infty} \leq \int_0^t \left\| e^{(t-s)\Delta} W^{(n)} \nabla U^{(n)} \right\|_{L^\infty} ds
\]

\[
\leq \int_0^t \|W^{(n)}\|_{L^\infty} \left\| e^{(t-s)\Delta} \nabla U^{(n)} \right\|_{L^\infty} ds .
\]
Notice that the map

$$(Tf)_j(x,t) := c \nabla \int_{\mathbb{R}^3} \epsilon_{j,k,l} \partial_{y_k} \frac{1}{|x-y|} f_\ell(y,t) \, dy$$

defines a $C-Z$ operator that satisfies the conditions in Lemma 2.12, so applying the Lemma

$$\left\| \int_0^t e^{(t-s)\Delta} W^{(n)} \nabla U^{(n)} \, ds \right\|_{L^\infty} \lesssim \int_0^t \left\| W^{(n)} \right\|_{L^\infty} \left\| e^{(t-s)\Delta} |TW^{(n)}| \right\|_{L^\infty} \, ds$$

$$\lesssim \int_0^t \phi_1(t-s)^{-1} \left\| W^{(n)} \right\|_{L^\infty} \left( \left\| W^{(n)} \right\|_{L^\infty} + \left\| W^{(n)} \right\|_{L^p} \right) \, ds$$

$$\lesssim K_n (K_n + K''_n) \int_0^t \phi_1(t-s)^{-1} \left( \phi_1(s)^{-1} + \phi_1(s)^{-2} \right) \, ds .$$

Again, by Lemma 2.12

$$\left\| \int_0^t \nabla e^{(t-s)\Delta} U^{(n)} W^{(n)} \, ds \right\| \lesssim \int_0^t \left\| W^{(n)} \right\|_{L^\infty} \left\| \nabla G_{t-s} (x-\cdot) U^{(n)}(\cdot) \right\|_{L^1} \, ds$$

$$\lesssim \int_0^t \left\| W^{(n)} \right\|_{L^\infty} \left( \left\| G_{t-s} (x-\cdot) \nabla |U^{(n)}(\cdot)| \right\|_{L^1} - 6 \left| U^{(n)}(x) \right| \right) \, ds$$

$$\lesssim \int_0^t \left\| W^{(n)} \right\|_{L^\infty} \sup_{x \in \mathbb{R}^3} \left\| G_{t-s} (x-\cdot) \nabla U^{(n)}(\cdot) \right\|_{L^1} \, ds$$

$$\lesssim \int_0^t \left\| W^{(n)} \right\|_{L^\infty} \left\| e^{(t-s)\Delta} \nabla U^{(n)} \right\|_{L^\infty} \, ds$$

$$\lesssim K_n (K_n + K''_n) \int_0^t \phi_1(t-s)^{-1} \left( \phi_1(s)^{-1} + \phi_1(s)^{-2} \right) \, ds .$$

Therefore

$$\left\| \int_0^t e^{(t-s)\Delta} U^{(n)} \nabla W^{(n)} \, ds \right\|_{L^\infty} \lesssim \left\| \int_0^t e^{(t-s)\Delta} W^{(n)} \nabla U^{(n)} \, ds \right\|_{L^\infty} + \left\| \int_0^t \nabla e^{(t-s)\Delta} U^{(n)} W^{(n)} \, ds \right\|_{L^\infty}$$

$$\lesssim K_n (K_n + K''_n) \int_0^t \phi_1(t-s)^{-1} \left( \phi_1(s)^{-1} + \phi_1(s)^{-2} \right) \, ds .$$

Let $W^{(n)}_x(y)$ denote the translation $W^{(n)}(x-y)$ and $B$ be the unit ball centered at 0. Then, from (4.4) we know

$$\left| U^{(n)}(x,t) \right| \lesssim \int_B \frac{1}{|y|^2} \left| W^{(n)}_x(y,t) \right| \, dy + \int_{B^c} \frac{1}{|y|^2} \left| W^{(n)}_x(y,t) \right| \, dy$$

$$\lesssim \| W^{(n)}_x \|_{L^\infty} \int_B |y|^{-2} \, dy + \| W^{(n)}_x \|_{L^p} \| |y|^{-2} 1_{B^c} \|_{L^{p'}} \lesssim \| W^{(n)} \|_{L^\infty} + \| W^{(n)} \|_{L^p} ,$$

where we used the fact $p' > \frac{3}{2}$ (since $p < 3$), so

$$\| U^{(n)} \|_{L^\infty} \lesssim \| W^{(n)} \|_{L^\infty} + \| W^{(n)} \|_{L^p} . \quad (4.8)$$
For the estimations in $L^p$, we divide the proof into two cases. For $p > 1$: by Young’s inequality and the $L^p$-boundedness of $T$, we have

$$\left\| \int_0^t e^{(t-s)\Delta} W^{(n)} \nabla U^{(n)} ds \right\|_{L^p} \lesssim \int_0^t \| W^{(n)} \|_{L^\infty} \left\| e^{(t-s)\Delta} |TW^{(n)}| \right\|_{L^p} ds$$

$$\lesssim \int_0^t \| W^{(n)} \|_{L^\infty} \| W^{(n)} \|_{L^p} ds \lesssim K_n K''_n \int_0^t \phi_1(t-s)^{-1} ds .$$

Similar to the argument for $L^\infty$-estimation, we deduce

$$\left\| \int_0^t \nabla e^{(t-s)\Delta} U^{(n)} W^{(n)} ds \right\|_{L^p} \lesssim \int_0^t \| W^{(n)} \|_{L^\infty} \left\| \nabla G_{t-s} \ast |U^{(n)}| \right\|_{L^p} ds$$

$$\lesssim \int_0^t \| W^{(n)} \|_{L^\infty} \left( \left\| G_{t-s} \ast |U^{(n)}| \right\|_{L^p} - 6\|U^{(n)}\|_{L^p} \right) ds$$

$$\lesssim \int_0^t \| W^{(n)} \|_{L^\infty} \left\| e^{(t-s)\Delta} |\nabla U^{(n)}| \right\|_{L^p} ds$$

$$\lesssim K_n K''_n \int_0^t \phi_1(s)^{-1} ds .$$

Combining the above two results,

$$\left\| \int_0^t e^{(t-s)\Delta} U^{(n)} \nabla W^{(n)} ds \right\|_{L^p} \lesssim K_n K''_n \int_0^t \phi_1(t-s)^{-1} ds .$$

For $p = 1$: by Young’s inequality, the $L^p$-boundedness of $T$ and interpolations in $L^p$, we have

$$\left\| \int_0^t e^{(t-s)\Delta} W^{(n)} \nabla U^{(n)} ds \right\|_{L^1} \lesssim \int_0^t \left\| e^{(t-s)\Delta} W^{(n)} \nabla U^{(n)} \right\|_{L^1} ds$$

$$\lesssim \int_0^t \| W^{(n)} \|_{L^2} \| \nabla U^{(n)} \|_{L^2} ds$$

$$\lesssim \int_0^t \| W^{(n)} \|_{L^\infty} \| W^{(n)} \|_{L^1} ds$$

$$\lesssim K_n K''_n \int_0^t \phi_1(t-s)^{-1} ds$$

as well as

$$\left\| \int_0^t \nabla e^{(t-s)\Delta} U^{(n)} W^{(n)} ds \right\|_{L^1} \lesssim \int_0^t \left\| \nabla e^{(t-s)\Delta} U^{(n)} W^{(n)} \right\|_{L^1} ds$$

$$\lesssim \int_0^t \| \nabla G_{t-s} \|_{L^1} \left\| U^{(n)} W^{(n)} \right\|_{L^1} ds$$

$$\lesssim \int_0^t (t-s)^{-1/2} \| U^{(n)} \|_{L^\infty} \| W^{(n)} \|_{L^1} ds$$

$$\lesssim Q_n K''_n \int_0^t (t-s)^{-1/2} \phi_1(s)^{-1} ds .$$
Combining the above two results,

\[ \left\| \int_0^t e^{(t-s)\Delta} U^{(n)} \nabla W^{(n)} \, ds \right\|_{L^p} \lesssim K''(K_n + Q_n) \int_0^t \left( \phi_1(t-s)^{-1} + (t-s)^{-1/2} \phi_1(s)^{-1} \right) \, ds. \]

After estimating all the other nonlinear terms in exactly the same way, one can conclude that:

If \( p > 1 \), then

\[ \mathcal{M}_{n+1} := \max\{K_{n+1}, K'_{n+1}, K''_{n+1}, K'''_{n+1}, Q_{n+1}\} \lesssim \|\omega_0\|_{bmo} + \|\omega_0\|_{L^p} + T \Psi_1^{\omega}(T) (\mathcal{M}_n)^2; \]

If \( p = 1 \), then

\[ \mathcal{M}_{n+1} \lesssim \|\omega_0\|_{bmo} + \|\omega_0\|_{L^p} + T^{1/2} \Psi_2^{\omega}(T) (\mathcal{M}_n)^2, \]

where \( \Psi_1^{\omega} \) and \( \Psi_2^{\omega} \) are some function with logarithmic blow-up at \( t = 0 \). If we take \( T^{\omega} \) such that

\[ T^{\omega} \Psi_1^{\omega}(T^{\omega}) \lesssim (\|\omega_0\|_{bmo} + \|\omega_0\|_{L^p})^{-1} \quad \text{for} \ p > 1; \]
\[ T^{1/2} \omega \Psi_2^{\omega}(T^{\omega}) \lesssim (\|\omega_0\|_{bmo} + \|\omega_0\|_{L^p})^{-1} \quad \text{for} \ p = 1, \]

then all the sequences are bounded by

\[ (T^{\omega} \Psi_1^{\omega}(T^{\omega}))^{-1} \quad \text{if} \ p > 1; \]
\[ (T^{1/2} \omega \Psi_2^{\omega}(T^{\omega}))^{-1} \quad \text{if} \ p = 1. \]

This ends the proof of the claim.

The rest of the proof is very similar to that of Theorem 3.1, we omit the details. \( \Box \)

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