COLEMAN-GROSS HEIGHT PAIRINGS AND THE $p$-ADIC SIGMA FUNCTION

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ABSTRACT. We give a direct proof that the Mazur-Tate and Coleman-Gross heights on elliptic curves coincide. The main ingredient is to extend the Coleman-Gross height to the case of divisors with non-disjoint support and, doing some $p$-adic analysis, show that, in particular, its component above $p$ gives, in the special case of an ordinary elliptic curve, the $p$-adic sigma function.

We use this result to give a short proof of a theorem of Kim characterizing integral points on elliptic curves in some cases under weaker assumptions. As a further application, we give new formulas to compute double Coleman integrals from tangential basepoints.

1. INTRODUCTION

Let $p$ be a prime number, let $F$ be a number field and let $E$ be an elliptic curve over $F$. In this paper we relate two constructions of $p$-adic heights for $E$.

Suppose $E$ has ordinary reduction at all primes above $p$. Then, the construction of Mazur and Tate [MT83, MTT86, MT91, MST06] uses the $p$-adic sigma function constructed in [MT91] to define the $p$-adic height of a point, the resulting function being quadratic resulting in a height pairing.

On the other hand, Coleman and Gross [CG89] defined for any curve of positive genus with good reduction above $p$ a $p$-adic height pairing for divisors of degree 0 with disjoint support. This factors via the Jacobian of the curve and thus extends to any pair of divisors without the disjoint support assumption.

Thus, there is a certain range in which both constructions apply and then they are known to be equal by either the work of Coleman [Col91] tying the Coleman-Gross height pairing with the $p$-adic height defined via bi-extensions as in [MTS], or by the work of the second author [Bes04] tying it with the general height pairing defined by Nekovár [Nek93]. This proof is ultimately global in nature, even though both heights are sums of local terms.

Both constructions have subsequently been implemented numerically: the Mazur-Tate construction by Mazur, Stein, and Tate [MST06] and the Coleman-Gross construction, in the case of hyperelliptic curves, by the authors [BB11].

It is thus interesting to prove the equality of these two height pairings by comparing the local terms. Note that a priori the local terms are defined for two divisors with disjoint support in the Coleman-Gross construction and for a divisor with itself in the Mazur-Tate construction. Our modest goal is to provide such a comparison.

What we in fact do is to stretch both definitions somewhat, and we compare them on the extended range. From the Coleman-Gross side, we discuss the extension of the height pairing to divisors with non-disjoint support, which is suggested by the work of Gross in the complex case [Gro86]. This involves choosing a tangent vector...
at each of the points in the common support of the divisors. The height pairing decomposes as a sum of local terms, which depend on the tangent vectors but their sum does not. The terms away from $p$ are already discussed by Gross and the extension to places above $p$ is rather straightforward. We note that the procedure is rather easy to implement numerically, and there may well be some interest in such an implementation.

There is little trouble in comparing the height pairing away from $p$, so the interest lies in the $p$-adic theory. We analyze it using tools from $p$-adic Arakelov theory [Bes05]. For a curve $C$ over the algebraic closure of the $p$-adic numbers, fixing a point $x_0$ and a tangent vector $t_0$ at $x_0$, the local height $h(x - x_0, x - x_0)$ becomes a function on the tangent bundle to $C$. We analyze this function for a general curve. For an elliptic curve, the choice of an invariant differential provides a canonical choice of tangent vectors, and we prove that the local height, computed using these choices, is essentially the logarithm of the $p$-adic sigma function. From this it is standard to deduce the equality of the heights.

As a corollary, we give a short proof for a recent theorem of Kim [Kim10] (corrected by the first author, Kedlaya and Kim in [BKK11]), giving a $p$-adic characterization of integral points of some elliptic curves over $\mathbb{Q}$. In particular, we remove the assumption there about the finiteness of the $p$-part of the Tate-Shafarevich group.

We conclude by giving some numerical examples illustrating these results, as well as new formulas to compute double Coleman integrals from tangential basepoints.

We would like to thank J. Ellenberg for providing the motivation for this work.

2. An extension of the Coleman-Gross height to non-disjoint support

In this section we recall what we need about the Coleman-Gross height pairing [CG89] and we make the easy extension to divisors with non-disjoint support suggested in the complex case by [Gro86].

The height pairing is defined for divisors of degree 0 on a smooth complete curve $X/F$ defined over a number field $F$.

Remark 2.1. Since we will be using, as in [Bes05], the version of Coleman integration developed by Vologodsky [Vol03], we do not assume that $X$ has good reduction at primes above $p$. However, the resulting height pairing is only assured to coincide with standard definitions of height pairings under this additional assumption [Col91] [Bes04].

To define the height pairing one needs the following data [Bes04, Section 2]:

- A “global log” - a continuous idele class character
  \[ \ell : \mathbb{A}_K^* / K^* \to \mathbb{Q}_p \]
  such that above $p$ the local components are ramified

- For each $v | p$ a choice of a subspace $W_v \subset H^1_{dR}(X \otimes K_v / K_v)$ complementary to the space of holomorphic forms.

To use $p$-adic Arakelov geometry we need to impose the following

Assumption 2.2. Each subspace $W_v$ is isotropic with respect to the cup product.

This assumption guarantees that the height pairing is symmetric [CG89, Proposition 5.2]. It is also automatically satisfied for elliptic curves, as the dimension of $W_v$ is 1.
One obtains from $\ell$, for each $v|p$, $\mathbb{Q}_p$-linear maps $\text{tr}_v : K \to \mathbb{Q}_p$ as well as a choice of the $p$-adic logarithm $\log_v : K_v^* \to K_v$ such that $\ell_v = \text{tr}_v \circ \log_v$.

Suppose now that $D_1$ and $D_2$ are $F$-rational divisors on $X$ of degree 0 and make the additional assumption that $D_1 = \sum_i n_i(x_i)$, with $x_i$ rational over $F$.

This assumption is eventually removed by going to a field extension where $D_1$ has this property and using the functoriality properties of the height. We do not assume that $D_1$ and $D_2$ have disjoint support. The price we have to pay for that is that the height pairing decomposes as usual

$$h(D_1, D_2) = \sum_v h_v(D_1, D_2)$$

over all finite places $v$, but the local components depend in addition on the choice of $F$-rational tangent vectors at the points of the common support. As we will see, the behavior of the local heights, with respect to these vectors, is such that the global height is independent of these choices.

The local component $h_v$ depends only on the completion at $v$, so let $K = F_v$ and $C = X_v = X \otimes_F K$. Let $\chi = \ell_v$ be the local component of $\ell$ at $v$. When $v|p$ let $\log = \log_v$, and $\text{tr} = \text{tr}_v$ be the branch of the $p$-adic logarithm and the trace map deduced from it, and let $W = W_v$ be the complementary subspace.

To compute the local height pairings we need tangent vectors. For ease of presentation, let us assume that we have chosen, at each $K$-rational point $x$ of $C$, a tangent vector $t = t_x$. For each such $x$, we further choose a local parameter $z = z_x$ normalized in such a way that $\partial_t z = 1$, where $\partial_t$ is the derivation associated to $t$. Using this parameter, we can define a “value” for any rational function $f$, defined over $K$, at $x$ by

$$f[x] = f(x, t) = \frac{f(x)}{z^m},$$

where $m = \text{ord}_x(f)$ is the order of $f$ at $x$. Note that this clearly depends only on $t$ and not on the particular choice of parameter $z$. Given a divisor $D = \sum n_i(x_i)$, with all $x_i$ defined over $K$, we may define the value of $f$ at $D$ by the rule

$$f[D] := \prod_i f(x_i)^{n_i}.$$
(3) The dependence on the choice of tangent vectors is as follows: if $h'$ is the height function obtained from $h$ by changing $t_x$ to $\alpha t_x$, then

$$h'(D_1, D_2) = h(D_1, D_2) + \chi(\alpha) \cdot \ord_x D_1 \cdot \ord_x D_2.$$ 

**Proposition 2.4.** If $v$ does not divide $p$, then there exists a unique local height pairing on $C$.

*Proof.* For divisors with disjoint support, see [CG89, Prop 1.2] for the uniqueness. Condition (2) then clearly extends the definition uniquely to all divisors. The local height pairing for divisors with disjoint support can be constructed [CG89, (1.3)] as

$$h(D_1, D_2) = \ell_v(\pi_v) \cdot (D_1, D_2).$$

Here, $(D_1, D_2)$ denotes intersection multiplicity on a regular model of $C$ over $\mathcal{O}_K$ of extensions of $D_1$ and $D_2$ to this model. To make this have the required properties one of these extensions has to have zero intersection with all components of the special fiber. The same formula can be used for divisors with non-disjoint support provided that all the tangent vectors are chosen to be integral [Gro86, Section 5] and then extended to arbitrary tangent vectors using part (3) of Definition 2.3. □

We now turn to the case $v|p$, where a local height pairing is not unique, but is constructed (when the support is disjoint) in [CG89]. In this section we only describe the local height pairing for divisors with disjoint support, deferring treatment of the general case to Section 3.

Coleman and Gross describe their local height pairing in terms of Coleman integration of differentials of the third kind. We will need, however, the description provided in [Bes05]. To the complementary subspace $W$ one associates a $p$-adic Green function, which is a certain Coleman function on $C \times C - \Delta$, where $\Delta$ is the diagonal. This Green function is defined only up to an additive constant. However, for each divisor

$$D = \sum m_j(y_j)$$

of degree 0 on $C$ the function

$$G_D(x) = \sum_j m_j G(y_j, x)$$

is defined without ambiguity outside the support of $D$. It coincides with the Coleman integral used to define the height pairing in Coleman and Gross’s work [Bes05, Theorem 7.3]. In particular, if $f$ is a rational function on $C$ with divisor $(f)$, then

$$G(f) = \log(f) + \text{a constant.}$$

(2.2)

Note that with this definition, $D$ should a priori be in $Z^0(C)$, but we may define $G_D$ for any $D \in \text{Div}^0(C)$ by extending scalars. If $D_1$ and $D_2 = \sum_i n_i(x_i)$ have disjoint support, we may define

$$G_{D_1}(D_2) = \sum_i n_i G_{D_1}(x_i),$$

and the local height pairing for such divisors is given by

$$h(D_1, D_2) = \text{tr}(G_{D_1}(D_2)).$$
The local height pairing may be extended to divisors with non-disjoint support in the sense of Definition 2.3. Explicit formulas for this extension will be given in Section 3.

Going back to the global situation, for any finite place $v$ we have, by change of base, $F_v$-rational divisors $D_1, D_2$ of degree 0 on $X_v$, such that $D_1 = \sum n_i(x_i)$ with $x_i$ $F_v$-rational. The local height $h_v = h$ on $X_v$ depends on choices of tangent vectors, but condition (3) in Definition 2.3 implies that they are only required at the points of the common support of $D_1$ and $D_2$, and there we have made a global choice of a tangent vector. We can therefore define

$$h(D_1, D_2) = \sum_v h_v(D_1, D_2).$$

Condition (3) in Definition 2.3 and the fact that $\ell$ is an idele class character imply now that $h$ is independent of the choices of tangent vectors and it furthermore factors via the Jacobian of $X$.

### 3. The local height as a Coleman function

In this section we deal with the $p$-adic analysis of the local height pairing at primes above $p$. We will be using some facts from $p$-adic Arakelov theory, as developed in [Bes05]. We therefore assume now that $C$ is a complete curve over the algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$.

We will be using the theory of higher Coleman integration, as developed by Vologodsky [Vol03], and discussed further in [Bes15]. We will recall necessary facts when needed. For now, suffice it to say that the theory, dependent on the choice of a branch of the $p$-adic logarithm, provides a ring $\mathcal{O}_{\text{Col}}(U)$ of so-called Coleman functions on any smooth algebraic variety $U/\bar{\mathbb{Q}}_p$ containing the ring of regular functions $\mathcal{O}(U)$ on $U$, whose members are locally analytic, and that the differential induces a short exact sequence

$$0 \to \bar{\mathbb{Q}}_p \to \mathcal{O}_{\text{Col}}(U) \to (\mathcal{O}_{\text{Col}}(U) \otimes \mathcal{O}(U) \Omega^1(U))^{d=0} \to 0$$

(in this section differential forms and de Rham cohomologies are over $\bar{\mathbb{Q}}_p$). Finally, there is a compatible action of $\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ on this short exact sequence.

We begin by recalling the definition of the constant term of a Coleman function [BdJ10], [BF06, Definition 3.1]. If $H$ is a Coleman function on $C$, then, locally, near a point $x \in C$ with local parameter $z$, $H$ can be written as a polynomial $\sum_{i \geq 0} h_i \log(z)^i$, with $h_i$ meromorphic at $x$.

**Definition 3.1.** The constant term in a Laurent expansion of $h_0$ with respect to $z$ is called the constant term of $H$ with respect to the parameter $z$, denoted $c_z(H)$.

**Lemma 3.2.** Suppose $dF$ is meromorphic in a neighborhood of $x$ with a simple pole at $x$ with residue $r$. Let $t$ be a tangent vector at $x$. Let $z$ and $z'$ be two local parameters at $x$ with the relation $\partial_t z' = \beta \partial_t z$. Then we have

$$c_z(H) = c_{z'}(H) + r \log(\beta).$$

In particular, the constant term with respect to $t$,

$$H(x, t) := c_z(H), \text{ with } z \text{ such that } \partial_t z = 1$$

is well-defined, independent of the particular choice of $z$, and in addition we have, for any $\alpha \neq 0$

$$H(x, \alpha t) = H(x, t) + r \log(\alpha).$$
Proof. For a local parameter $z$, we have near $x$, $H = h + r \log(z)$, with $h$ analytic, hence $c_z(H) = h(x)$. We have

$$z'/z = \beta \cdot (1 + g(z))$$

with $g$ analytic vanishing at $x$. Thus we have

$$H = h' + r \log(z') = h' + r(\log(z) + \log(\beta) + \log(1 + g(z))).$$

Since $\log(1 + g(z))$ is analytic near $x$ and vanishes at $x$, we find

$$h = h' + r \log(\beta) + r \log(1 + g(z))$$

and

$$h(x) = h'(x) + r \log(\beta),$$

from which the result is clear. □

For any $x \in C$ consider the function $G_x(y) := G(x, y)$ on $C$.

Lemma 3.3. The form $dG_x$ is meromorphic near $x$, with a simple pole of residue 1. In other words, it satisfies the condition of Lemma 3.2 with $r = 1$.

Proof. This is essentially obvious from the theory in [Bes05] but is not stated there explicitly. From the definition of $G$ it follows that $G_x = \log(1)$, with 1 being the canonical section of $O(x)$ (for the terminology of log functions see below). If $z$ is a local parameter at $x$, then $z^{-1} \cdot 1$ extends to a non-vanishing section of $O(x)$ near $x$. From the properties of log functions it follows that $G_x - \log(z)$ extends to an analytic function near $x$, proving the result. □

At this point we can construct the extension of the Coleman-Gross local height pairing to divisors with non-disjoint support.

Proposition 3.4. The formulas, for $D_1$ and $D_2 = \sum_i n_i(x_i)$,

$$G_{D_1}[D_2] = \sum_i n_iG_{D_1}(x_i, t_{x_i})$$

and

$$h(D_1, D_2) = \text{tr}(G_{D_1}[D_2]),$$

define a local height pairing in the sense of Definition 2.3.

Proof. Everything is essentially clear. One just needs to observe that as a consequence of (2.2) we have $G(f)[D_2] = \log(f[D_2])$, and that condition 3 in Definition 2.3 is an immediate consequence of Lemma 3.3 and Lemma 3.2. □

We now turn to a more detailed analysis of the local height pairing.

Definition 3.5. For a point $x \in C$ and a tangent vector $t$ at $x$ define

$$\log_T(x, v) = G_x(x, v).$$

Lemmas 3.3 and 3.2 imply that $\log_T$ is a quasi-log function, in the sense of [Bes05, Definition 4.1], on the tangent line bundle $T$ on $C$. Recall that a quasi-log function on a line bundle is a function on the total space minus the zero section which on every fiber is the $p$-adic logarithm (as chosen) plus a constant. A quasi-log function is a log-function if it is furthermore a Coleman function with some additional properties. In [Bes05, Theorem 5.10] we constructed a log-function $\log_{O(\Delta)}$ unique up to an additive constant, on the line bundle $O(\Delta)$, with certain properties. The Green function $G$ is then

$$G = \log_{O(\Delta)}(1)$$

on $C \times C - \Delta$, where 1 is the canonical section of $O(\Delta)$ on $C \times C - \Delta$. 
For a line bundle \( \mathcal{L} \) let \( \mathcal{L}^{-1} := \text{Hom}(\mathcal{L}, \mathcal{O}) \) be the inverse line bundle. Given a log function on \( \mathcal{L} \) there is an induced one on \( \mathcal{L}^{-1} \), which can be described by the condition that the sum of the logs of two dual vectors (two vectors pairing to 1) is 0.

Recall that \( \mathcal{O}(-\Delta)\mid_\Delta \) may be identified with the canonical bundle \( \omega_C \) of \( C \) as follows: let \( z \) be a local parameter at the neighborhood \( U \) of some point \( x_0 \) on \( C \). The function \( z(x) - z(y) \) has a simple zero on \( \Delta \) at \( U \times U \), hence defines a non-vanishing section of \( \mathcal{O}(-\Delta) \). Its restriction to \( \Delta \) gets mapped to the section \( dz \) of \( \omega_C \). By duality, \( \mathcal{O}(\Delta)\mid_\Delta \) is identified with \( T \).

**Proposition 3.6.** The restriction of \( \log_{\mathcal{O}(\Delta)} \) to \( \Delta \) is equal, via the above identification, to the quasi-log function \( \log_T \) defined in (3.5), which is therefore a log function.

**Proof.** Fix a point \( x_0 \in C \) and a tangent vector \( t \) at \( x_0 \). Let \( z \) be a local parameter at a point \( x_0 \) such that \( \partial_z z = 1 \). As above, \( z(x) - z(y) \) defines a non-vanishing section of \( \mathcal{O}(-\Delta) \). We find, for \( x \neq y \) in a neighborhood of \( x_0 \),

\[
u(x, y) := \log_{\mathcal{O}(-\Delta)}(z(x) - z(y)) = \log(z(x) - z(y)) + \log_{\mathcal{O}(-\Delta)}(1) \quad \text{by the behavior of log = log(z(x) - z(y)) - log_{\mathcal{O}(\Delta)}(1) \quad \text{since 1 \( \in \mathcal{O}(\Delta) \) is dual to 1 \( \in \mathcal{O}(-\Delta) \)}
\]

\[
= \log(z(x) - z(y)) - G(x, y) \quad \text{by (3.2)}.
\]

The function \( u \) extends to \( (x_0, x_0) \), and by the identification above, its value there is \( \log_{\omega_C}(dz)(x_0) \). Fixing \( y = x_0 \) we find

\[
G_{x_0}(x) = -u(x, x_0) + \log(z),
\]

and by Definition 3.5 we find

\[
\log_T(x_0, t) = -u(x_0, x_0) = -\log_{\omega_C}(dz)(x_0),
\]

and the proof is complete upon noting that \( dz \) at \( x_0 \) is dual, by definition, to \( t \). \( \square \)

We next recall [Bes05, Proposition 4.4] that a log function on a line bundle on a variety \( U \) has (not always, but for example if \( U \) is a complete smooth curve with positive genus) a curvature, which is an element of \( H_{\text{dR}}^1(U) \otimes \Omega^1(U) \). It is defined as the unique such element whose pullback to the total space of the line bundle minus the zero section gives the \( p \)-adic \( \bar{\partial} \) operator ([Bes02, Section 6] and [Bes05, p. 323]) applied to the differential of the log function (we will recall later the definition of the \( p \)-adic \( \bar{\partial} \) operator in an affine situation when we need it).

**Remark 3.7.** When \( U \) is affine, the curvature is also \( \bar{\partial} \circ d \) of the log of any non-vanishing section.

We now identify the curvature for \( \log_T \). To do so, recall first that as part of the data for the height pairing we are given a decomposition \( H_{\text{dR}}^1(C) = W \otimes \Omega^1(C) \). If \( \{\omega_i, i = 1, \ldots, g\} \) is a basis for \( \Omega^1(C) \) there exists a unique choice of a basis \( \{\bar{\omega}_i, i = 1, \ldots, g\} \) for \( W \) which is dual to it via the cup product, i.e., \( \bar{\omega}_i \cup \omega_j = \delta_{ij} \). We have
Proposition 3.8. For any choice of a basis \( \{ \omega_i, i = 1, \ldots, g \} \) as above the curvature of \( \log T \) is given by

\[
\text{curve}(\log T) = (2 - 2g)\mu, \quad \text{with} \quad \mu = \frac{1}{g} \sum_{i=1}^{g} \bar{\omega}_i \otimes \omega_i.
\]

Proof. According to [Bes05, Definition 5.1 and Theorem 5.10], the curvature of \( \log O(\Delta) \) is

\[
\pi_1^\ast \nu + \pi_2^\ast \nu - g \sum_{i=1}^{g} (\pi_1^\ast \bar{\omega}_i \otimes \pi_2^\ast \omega_i + \pi_2^\ast \bar{\omega}_i \otimes \pi_1^\ast \omega_i),
\]

where \( \pi_i \) are the two projections from \( C \times C \) to \( C \). Since the curvature behaves nicely with respect to restriction, the result follows immediately by pulling back to the diagonal. \( \square \)

Remark 3.9. Note that the corollary is compatible with [Bes05, Proposition 4.4] in the sense that applying the cup product to the curvature one is expected to get the degree of the line bundle, and applying it to \( \mu \) one gets 1.

We next fix \( x_0 \in C \) and study the local height function \( h_T(x - x_0, x - x_0) \) itself. As in the introduction we fix a tangent vector \( t_0 \) at \( x_0 \) and the local height function depends on \( x \) and on a tangent vector \( t \) at \( x \). It is easily seen to be a quasi-log function on \( T|_{C - x_0} \), and we denote it by \( h_T \). We have

\[
h_T(x, t) = G_x(x, t) - 2G_{x_0}(x) + G_{x_0}(x_0, t_0).
\]

Since \( G_{x_0} \) is the log of the section 1 of \( O(x_0) \), we can interpret \( h_T \), ignoring the last summand which is a constant, as the pullback of the log function on \( T \otimes O(-2x_0) \) under the isomorphism of the above with \( T \) on \( C - x_0 \) provided by the canonical section 1. We have the following information about \( h_T \), which we know how to use to fully characterize it only in the genus 1 case.

Proposition 3.10. The curvature of \( h_T \) is \(-2g\mu|_{C - x_0}\). If \( t(x) \) is a section of \( T \) near \( x_0 \) whose value at \( x_0 \) is \( t_0 \), then the pullback of \( h_T \) under \( t \) has constant term 0 with respect to \( t_0 \).

Proof. The first statement follows because the curvature of \( O(x_0) \) is \( \mu \) by [Bes05, Corollary 5.12 and Definition 5.7]. For the second statement we observe that \( G_x(x, t(x)) \) has value \( G_{x_0}(x_0, t_0) \) at \( x_0 \), and that this is by definition also the constant term of \( G_{x_0}(x) \) with respect to \( t_0 \). \( \square \)

4. The Local Height for Elliptic Curves

In this section we specialize the consideration in the previous section to the case of an elliptic curve \( E \). We fix \( x_0 \) to be the identity element 0 of the group law. We furthermore fix an invariant differential \( \omega \). This provides us by duality a canonical invariant section \( v \) of the tangent bundle. We may use this to pullback the log function \( h_T \) to obtain a Coleman function \( \tau \) on \( E - 0 \). Let \( \bar{\omega} \in W \) be the dual class.

Theorem 4.1. Let \( \eta \) be the unique one-form of the second kind on \( E \) with a double pole at 0 and no other poles, representing the cohomology class \( \bar{\omega} \). Then \( \tau \) is the unique Coleman integral of the form

\[
\tau = -2 \int \left( \omega \times \int \eta \right).
\]
which is in addition a symmetric function and \( \tau(0, v) = 0 \).

Proof. The function \( \tau \) is symmetric by functoriality of its construction with respect to the involution \( x \to -x \) of \( E \), which induces \(-1\) on the first cohomology, hence preserves \( W \). We have \( \tau(0, v) = 0 \) by Proposition 3.10. From this Proposition, and Remark 3.7 it follows that \( \partial d \tau = -2g\mu = -2\omega \otimes \omega \). We now recall [Bes05, Proposition 2.7] that for an affine \( U \), the \( \partial \) operator sends \( \omega \times \int \omega \) to \( \omega \int \omega \), where \([\omega] \) is the cohomology class of \( \omega \) in \( H^1_{dR}(U) \). This means that on the affine \( E-0 \) we have \( d\tau = -2\omega \times \int \eta' \) where the class of \( \eta' \) in \( H^1_{dR}(E-0/K) \) is exactly the restriction of \( \omega \). Clearly this means that \( \eta' = \eta \). This determines \( \tau \) up to an addition of a Coleman integral of a constant multiple of \( \omega \). Two functions satisfying all the requirements therefore differ by such an integral which is in addition symmetric and vanishes at 0. It is therefore identically 0. \( \Box \)

Corollary 4.2. Suppose that \( E \) is ordinary and \( W \) is the unit root subspace, then \( \tau \), restricted to the residue disc of 0, is just \(-2 \log(\sigma_p) \), where \( \sigma_p \) is the \( p \)-adic sigma function, as defined in [MT91].

Proof. We recall a definition for the \( p \)-adic sigma function. The differential \( \omega \) determines a Weierstrass model of \( E \) over the ring of integers \( \mathcal{O}_K \),

\[
y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6
\]

in such a way that \( \omega \) becomes the differential \( dx/(2y + a_1x + a_3) \) and it determines a parameter at 0, \( t = -x/y \). We have the following local expansions:

\[
\begin{align*}
x &= t^{-2} + \cdots \\
y &= -t^{-3} + \cdots \\
\omega &= dt(1 + \cdots)
\end{align*}
\]

In particular, the parameter \( t \) is normalized with respect to the dual of \( \omega \) at 0. Let \( \eta \) be the form described in Theorem 4.1, which in particular satisfies \( [\eta] \cup [\omega] = 1 \).

Using the description of cup products in terms of residues and integrals, one finds the local expansion for \( \eta \) is

\[
\eta = dt(-t^{-2} + \cdots).
\]

According to [MST06, Theorem 1.3], \( \sigma \) is the unique odd function of the form \( \sigma = t + \cdots \) satisfying the differential equation

\[
x(t) + c = -\frac{d}{\omega} \left( \frac{1}{\sigma} \frac{d\sigma}{\omega} \right)
\]

with

\[
c = \frac{a_1^2 + 4a_2}{12} - \frac{E_2(E, \omega)}{12}.
\]

Thus, we can write, in a neighborhood of 0,

\[
\log(\sigma) = -\int \left( \omega \times \int \eta' \right) \text{ with } \eta' = (x + c)\omega.
\]

Clearly, \( \eta' \) is a form of the second kind with a double pole at 0 and no other poles. Its local expansion is \( dt(t^{-2} + \cdots) \). We claim that \( \eta' \) represents a cohomology class in \( W \) (the unit root subspace). Indeed, by switching to a standard model
\[ y^2 = 4x^3 - g_2x - g_3, \] noting that in this model \( \omega = dx/y \), letting \( \eta_0 = 2dx/y \) (this is what is usually denoted \( \eta \)) and \( u \) a generator of \( W \), we have, by [Kat73, A.2.4.1]

\[ E_2(E; \omega) = 12 \frac{\eta_0 \cup u}{\omega \cup u} \]

so

\[ \eta' = \eta_0 + c\omega = \eta_0 - \frac{\eta_0 \cup u}{\omega \cup u} \omega = \text{const} \cdot (\omega \cup u)\eta_0 - (\eta_0 \cup u)\omega, \]

and this is clearly a constant multiple of \( u \). Thus, \( \eta' = -\eta \). We therefore find that \(-2\log(\sigma)\) and \( \tau \) satisfy the same differential equation and they therefore differ by an integral of a constant multiple of \( \omega \). The local expansion of \( \sigma \) implies that \( \log(\sigma)(0,v) = 0 \). The same argument showing the uniqueness of \( \tau \) finishes the proof. \( \square \)

We are now ready to compare the Coleman-Gross construction to the Mazur-Tate construction. Suppose now that \( E \) is an elliptic curve defined over \( \mathbb{Q} \). Let \( \ell \) be the idele class character given as follows: the component \( \ell_p \) is the standard branch \( \log p \) of \( p \)-adic logarithm, normalized so that \( \log p(p) = 0 \). For \( q \neq p \) the character \( \ell_q \) is unramified, with \( \ell_q(q) = -\log p(q) \). With this choice we have the following:

**Corollary 4.3.** Fix a minimal Weierstrass equation for \( E \). Suppose the point \( x \in E(\mathbb{Q}) \) has coordinates \( (a/d^2, b/d^3) \), with \( a, b, d \in \mathbb{Z} \) and \( d \) prime to both \( a \) and \( b \), and does not reduce to a singular point in any of the bad reduction fibers of \( E \). Then the Coleman-Gross height of \( x \) is given by

\[ h((x)-(0), (x)-(0)) = \tau(x) + 2\log p(d). \]

In particular, if \( E \) is ordinary at \( p \) and \( W \) is taken to be the unit root subspace, then the Coleman-Gross height coincides with the Mazur-Tate height.

**Proof.** We have already identified the local height \( h_p \). The formula follows because the intersection pairing at any other primes is \( 2\text{ord}_p(d) \) (see for example [Sil88, (26)]). To compare with the Mazur-Tate height, assume first that the point \( x \) reduces to 0 above \( p \). In this case, our formula matches precisely [MST06, (1.1)]

\[ \frac{1}{p} \log p \left( \frac{\sigma_p(x)}{d} \right) \]

given that their height is given by \(-h((x)-(0), (x)-(0))/2\) and that their character is \( \ell/p \). Since both heights are quadratic, they are equal everywhere. \( \square \)

5. **Application to Kim’s theorem on integral points on elliptic curves**

The paper [Kim10], as corrected in [BKK11] contains a characterization of integral points on elliptic curves, under certain hypotheses. By removing some of these hypotheses, we prove a more general theorem, based on the main results of the previous section. We view this as an indication that the height pairing has some anabelian source, a fact we hope to expand on in later work.

**Theorem 5.1.** Let \( E \) be an elliptic curve over \( \mathbb{Q} \), with rank \( E(\mathbb{Q}) = 1 \), given by a minimal Weierstrass equation (4.1). Let \( E_{\text{good}} \) be the set of non-torsion rational points \( x \in E(\mathbb{Q}) \) with integral coordinates satisfying the additional condition that \( x \)
meets each bad reduction fiber at a non-singular point. Let \( \omega = dx/(2y+a_1x+a_3) \) be an invariant differential on \( E \) and let \( \eta_0 = x\omega \). Let

\[
D(z) := \int \left( \omega \times \int \eta_0 \right)
\]

be the unique such integral with \( \int \eta_0 \) and \( D \) having constant term 0 with respect to the tangent vector dual to \( \omega \) at 0. Then the function

\[
\frac{D(z)}{(\int_0^z \omega)^2}
\]

is constant on \( E_{\text{good}} \subset E(\mathbb{Q}) \subset E(\mathbb{Q}_p) \).

**Proof.** The condition on \( \int \eta_0 \) makes \( \int \eta_0 \) anti-symmetric, hence \( D \) symmetric, so, by choosing \( W \) appropriately we may assume that \( D \) is the function \( \tau \) from Theorem 4.1. By Corollary 4.3 the function \( D \) coincides with the global height \( h \) on \( E_{\text{good}} \), so the function which we wish to prove is constant is just \( h(x)/(\int_0^x \omega)^2 \). But this is obvious as both the numerator and the denominator are quadratic functions on the rank 1 group of rational points of \( E \). \( \square \)

**Remark 5.2.** The above is indeed a generalization of the results of Kim. He makes the assumption that at all the primes of bad reduction the Néron model has just one component, and it is easy to see that in this case \( E_{\text{good}} \) is the set of all (non-torsion) integral points. Kim’s formulation is slightly different but clearly equivalent. Thus, we obtain an extension of Kim’s result when there can be more than one component in the Néron model at some primes, and we further remove the condition about the \( p \)-part of the Tate-Shafarevich group of \( E \) being finite. From the description of the local height at primes of bad reduction it is further easy to see, using the description of the local heights at primes of bad reduction \([\text{Sil}88]\) that given the reduction types at the bad primes, and given the height at one integral point, one can write down a finite number of possibilities for \( D(z)/(\int_0^z \omega)^2 \) at the integral points.

### 6. Examples and another application

In \([\text{BKK}11]\) several numerical examples of Kim’s theorem were given. These call for the computation of iterated Coleman integrals \( \int_0^v \omega \eta \) where \( v \) is a tangential base point at 0. Although iterated Coleman integrals on elliptic and hyperelliptic curves can be explicitly computed \([\text{Bal}12]\), integrating directly from a tangential basepoint is more subtle than integrating from a finite Weierstrass point because one has to be careful with normalization with respect to the constant term. The examples in \([\text{BKK}11]\) were all done by using an integral 2-torsion point, or by a manipulation using two integral points that relies on Kim’s theorem, as well as the fact that the elliptic curve has rank 1.

Using Theorem 4.1 we can describe an alternative method for computing such integrals. We follow this with several examples.

In this section we change the notation slightly. For an elliptic curve \( E \) as in (4.1), we denote by \( \omega, \eta \) the forms \( dx/(2y+a_1x+a_3) \) and \( xdx/(2y+a_1x+a_3) \) respectively, so that \( \omega \) is the same as in Theorem 5.1 while \( \eta \) is what was \( \eta_0 \) there. We also use standard notation for iterated integrals \( \int_0^y \alpha \beta \), which is \( \int (\alpha \times \beta) \) evaluated at \( y \), where both the inner and outer integrals are fixed to vanish at \( x \). This also applies to \( x \) replaced by a tangential base point at 0.
Let $T$ be an auxiliary point. Using the following two formulas from \cite{BKK11}
\[
\int_v \omega \eta = \int_T \omega \eta + \int_T \omega \int_v \eta + \int_v \omega \eta,
\]
\[
\int_v \eta = \frac{1}{2} \int_{-T} \eta.
\]
we see that the integral $\int_v \omega \eta$ may be computed using integrals between points and the integral $\int_T \omega \eta$.

For any fixed $T$ we can use Theorem 4.1 to compute the integral above using an extension of the local height computation in \cite{BB11} to divisors with non-disjoint support, along the lines described in Sections 2 and 3. However, if $T$ is a torsion point, a more direct computation is possible.

**Proposition 6.1.** Let $E$, $\omega$ and $\eta$ be as above and let $v$ be the tangential base point at 0 dual to $\omega$. Let $T = (A, B) \neq 0$ be either a 2- or 3-torsion point. Then
\[
\int_T \omega \eta = \begin{cases}
\frac{1}{4} \log(f'(A) - a_1 B) & \text{if } 2T = 0, \\
\frac{1}{3} \log(2B + a_1 A + a_3) & \text{if } 3T = 0.
\end{cases}
\]

**Proof.** Let $n$ be the order of $T$. Consider the divisor $D = (T) - (0)$. Then $nD$ is the divisor of a rational function $g$ on $E$ and by Theorem 4.1 and Definition 2.3 we have
\[
2 \int_v \omega \eta = \frac{1}{n} \log(g[T]/g[0]),
\]
where the square bracket notation stands for the normalized value with respect to the chosen parameter at the point. It remains to compute the right hand side in both cases.

Recall that the normalized local parameter at 0 is $t = -x/y$. If $T = (A, B)$ is 2-torsion, then $2B + a_1 A + a_3 = 0$, and a local parameter at $T$ is $y_0 = y + \frac{1}{2}(a_1 x + a_3)$. To normalize it, we observe that $y_0^2 = f_0(x)$, with $f_0(x) = f(x) + \frac{1}{4}(a_1 x + a_3)^2$ so that $2y_0 dy_0 = f_0'(x) dx$,
\[
\omega = \frac{dx}{2y_0} = \frac{dy_0}{f_0'(x)},
\]
and evaluating at $T$ we see that the normalized parameter is
\[
t_T = \frac{y_0}{f_0'(A)} = \frac{y_0}{f'(A) - a_1 B}.
\]
We can take $g = x - A$. It is easy to see from the local expansion \cite{BB11} that $gt^2$ has value 1 at 0. At $T$ we have
\[
\frac{g}{t_T^2} = \frac{(f'(A) - a_1 B)^2 (x - A)}{y_0^2} = \frac{(f'(A) - a_1 B)^2 (x - A)}{f_0(x)},
\]
and the value at $T$ is going to be
\[
\frac{(f'(A) - a_1 B)^2}{f_0'(A)} = f'(A) - a_1 B.
\]
This proves the case of 2-torsion.

When $T$ is 3-torsion we proceed as follows: write the equation of the tangent at $T$ as $y = \alpha x + \beta = \alpha(x - A) + B$. We can then take $g = y - \alpha x - \beta$ as it has a
pole of order 3 at infinity, and the fact that $T$ is 3-torsion exactly means that it vanishes to order 3 at $T$. It will turn out that $\alpha, \beta$ are not relevant.

We compute the normalized values of $g$ at 0, which is the value of $gt^3$ there, which is 1 using the expansion (4.2).

At $T$ we have the parameter $x - A$. The normalization factor is the value at $T$ of

$$\frac{\omega}{d(x - A)} = \frac{\omega}{dx} = \frac{1}{2y + a_1x + a_3}$$

which is $1/(2B + a_1A + a_3)$. The normalized value of $g$ at $T$ is thus $(2B + a_1A + a_3)^3$ times the value of $g/(x - A)^3$ at $T$. To get this value, write $g = c(x - A)^3 + \cdots$ and substitute this into the equation of the curve

$$(g + \alpha(x-A)+B)^2 + a_1(x-A)(g+\alpha(x-A)+B)+(a_3+a_1A)(g+\alpha(x-A)+B) = (x-A)^3 + \cdots$$

Now we solve for the coefficient of $(x - A)^3$. The result is

$$2cB + c(a_3 + a_1A) = 1,$$

and so the normalized value is $(2B + a_1A + a_3)^2$. The final result is obtained by taking a log and multiplying by the required factors of 1/3 and 1/2.

\[\square\]

**Example 6.2** (Two-torsion). Consider the elliptic curve $E : y^2 = x^3 - 16x + 16$ with minimal model $y^2 + y = x^3 - x$ (Cremona label “37a1”). Let $P = (0, 4)$ be an integral point on $E$ (arising from an integral point on the minimal model) and let $\omega = \frac{dx}{y}, \eta = x \frac{dy}{2y}$ on $E$. Note that this curve has no integral Weierstrass points over $\mathbb{Q}$.

Nevertheless, one can consider Weierstrass points on $E$ over $\mathbb{Q}_p$ for various $p$. For example, at $p = 13$, we have the point

$$W = (7 + 7 \cdot 13 + 4 \cdot 13^2 + 7 \cdot 13^3 + 6 \cdot 13^4 + O(13^5), O(13^5))$$

on the short Weierstrass model, which corresponds to the point

$$W' = (5+8\cdot13+7\cdot13^2+11\cdot13^3+4\cdot13^4+O(13^5), 6+6\cdot13+6\cdot13^2+6\cdot13^3+6\cdot13^4+O(13^5))$$

on the minimal model. Using previous methods, we can compute the invariant ratio

$$\frac{\int_b^P \omega \eta}{(\int_b^P \omega)^2} = 11 \cdot 13 + 6 \cdot 13^2 + 7 \cdot 13^3 + 6 \cdot 13^5 + O(13^6),$$

as well as the auxiliary integrals

$$\int_W^P \omega \eta = 12 \cdot 13 + 7 \cdot 13^2 + 11 \cdot 13^3 + 7 \cdot 13^4 + 11 \cdot 13^5 + O(13^6)$$

$$\int_b^P \omega = 4 \cdot 13 + 2 \cdot 13^2 + 2 \cdot 13^3 + 10 \cdot 13^4 + O(13^6)$$

to deduce that

$$\int_b^W \omega \eta = 13 + 5 \cdot 13^2 + 8 \cdot 13^3 + 4 \cdot 13^4 + 13^5 + 9 \cdot 13^6.$$
Alternatively, using the proposition above, one can directly compute, via the minimal model, that
\[
\int_{W}^{W'} \omega' \eta' = \frac{1}{4} \log(f'(x(W')))
\]
\[
= 13 + 5 \cdot 13^2 + 8 \cdot 13^3 + 4 \cdot 13^4 + 13^5 + 9 \cdot 13^6,
\]
where \( \omega', \eta' \) denote the pullbacks of \( \omega, \eta \) to the minimal model.

**Example 6.3 (Two-torsion).** Consider the elliptic curve \( E : y^2 = x(x - 1)(x + 9) \) with 1-forms \( \omega = \frac{dx}{2y}, \eta = \frac{dx}{2y} \) and the points \( W_1 = (1, 0), W_2 = (0, 0) \). \( E \) has minimal model \( E' : y^2 = x^3 - x^2 - 30x + 72 \) (Cremona label “480f1”), and note that in particular \( W_1 \) and \( W_2 \) are integral on the minimal model but that the Tamagawa numbers of this curve are not 1.

The prime \( p = 7 \) is good and ordinary. Letting \( \omega', \eta' \) denote the pullbacks of \( \omega, \eta \), we compute \( \int_{W_1}^{W_2} \omega' \eta' \) by using the interpretation of the height pairing (via a tangential basepoint \( b \)), we have
\[
\int_{W_1}^{W_2} \omega' \eta' = \int_{b}^{W_2} \omega' \eta' - \int_{b}^{W_1} \omega' \eta'
\]
\[
= \frac{1}{4} \log \left( \frac{f'(x(W_2))}{f'(x(W_1))} \right)
\]
\[
= 6 \cdot 7 + 3 \cdot 7^2 + 3 \cdot 7^3 + 2 \cdot 7^5 + O(7^6).
\]

As a consistency check, we can compare this calculation to that described in [Bal12], where using near-boundary points in each Weierstrass disc, we can compute (via an auxiliary non-Weierstrass point \( Q = (-1, 4) \))
\[
\int_{W_1}^{Q} \omega \eta = 6 \cdot 7 + 5 \cdot 7^2 + 4 \cdot 7^3 + 6 \cdot 7^4 + O(7^6)
\]
\[
\int_{W_2}^{Q} \omega \eta = 2 \cdot 7^2 + 7^3 + 6 \cdot 7^4 + 5 \cdot 7^5 + O(7^6),
\]
and we see
\[
\int_{W_1}^{W_2} \omega \eta = \int_{W_1}^{Q} \omega \eta - \int_{W_2}^{Q} \omega \eta
\]
\[
= 6 \cdot 7 + 3 \cdot 7^2 + 3 \cdot 7^3 + 2 \cdot 7^5 + O(7^6).
\]

**Remark 6.4.** We note that if one is after the value \( \int_{P}^{Q} \omega \eta \) where \( P \) is an arbitrary point, then for the purposes of computation, it is much faster to use an intermediate 3-torsion point \( T \) to break up the path rather than using a 2-torsion point. This is because computing iterated integrals from a Weierstrass endpoint requires computations over very highly ramified extensions \( \mathbb{Q}_p(p^{1/d}) \), which is quite slow in existing implementations.

**Example 6.5 (Three-torsion).** We again consider \( E : y^2 = x^3 - 16x + 16 \) (minimal model 37a), this time over \( \mathbb{Q}_7 \), where it has a \( \mathbb{Q}_7 \)-rational 3-torsion point. As before, let \( P = (0, 4) \), which is integral on the minimal model. Again, using the method
in [BKK11], we compute the invariant ratio

$$\frac{\int_b^P \omega \eta}{(\int_b^P \omega)^2} = 7^{-1} + 1 + 3 \cdot 7 + 6 \cdot 7^2 + 5 \cdot 7^4 + 6 \cdot 7^5 + 6 \cdot 7^6 + O(7^7),$$

as well as the single integral

$$\int_b^P \omega = 2 \cdot 7 + 4 \cdot 7^3 + 5 \cdot 7^4 + 4 \cdot 7^5 + 7^6 + 2 \cdot 7^7 + 7^8 + 7^9 + O(7^{10})$$

to deduce the value of

$$\int_b^P \omega \eta = 4 \cdot 7 + 4 \cdot 7^2 + 4 \cdot 7^5 + 7^6 + 2 \cdot 7^7 + 5 \cdot 7^8 + O(7^9).$$

Meanwhile, we compute the invariant ratio

$$(\text{Three-torsion})$$

Example 6.6

We compute

$$\int_b^P \omega \eta = 2 \cdot 7 + 4 \cdot 7^3 + 5 \cdot 7^4 + 4 \cdot 7^5 + 7^6 + 2 \cdot 7^7 + 7^8 + 7^9 + O(7^9)$$

to deduce

$$\int_b^T \omega \eta = \int_b^P \omega \eta - \int_b^T \int_T^P \omega - \int_T^P \omega \eta$$

$$= 2 \cdot 7 + 2 \cdot 7^2 + 3 \cdot 7^3 + 6 \cdot 7^4 + 2 \cdot 7^5 + 6 \cdot 7^6 + 7^7 + 5 \cdot 7^8 + O(7^9)$$

Meanwhile, the corresponding 3-torsion point $T$ with

$$x(T) = 3 + 5 \cdot 7 + 3 \cdot 7^2 + 3 \cdot 7^3 + 6 \cdot 7^4 + 6 \cdot 7^5 + 6 \cdot 7^6 + 5 \cdot 7^7 + 5 \cdot 7^8 + 5 \cdot 7^9 + O(7^{10}),$$

$$y(T) = 3 + 3 \cdot 7 + 5 \cdot 7^2 + 4 \cdot 7^3 + 2 \cdot 7^5 + 2 \cdot 7^7 + 6 \cdot 7^8 + 2 \cdot 7^9 + O(7^{10})$$

We compute

$$\int_b^T \omega = 1 + 2 \cdot 7 + 3 \cdot 7^2 + 2 \cdot 7^3 + 3 \cdot 7^4 + 6 \cdot 7^5 + 7^6 + 5 \cdot 7^7 + 5 \cdot 7^8 + 4 \cdot 7^9 + O(7^{10})$$

$$\int_T^P \omega = 2 \cdot 7 + 4 \cdot 7^3 + 5 \cdot 7^4 + 4 \cdot 7^5 + 7^6 + 2 \cdot 7^7 + 7^8 + 7^9 + O(7^{10})$$

$$\int_T^P \omega \eta = 5 \cdot 7^2 + 4 \cdot 7^4 + 7^5 + 2 \cdot 7^6 + 2 \cdot 7^7 + 5 \cdot 7^8 + O(7^9)$$

to deduce

$$\int_b^T \omega \eta = \int_b^P \omega \eta - \int_b^T \int_T^P \omega - \int_T^P \omega \eta$$

$$= 2 \cdot 7 + 2 \cdot 7^2 + 3 \cdot 7^3 + 6 \cdot 7^4 + 2 \cdot 7^5 + 6 \cdot 7^6 + 7^7 + 5 \cdot 7^8 + O(7^9)$$

Meanwhile, the corresponding 3-torsion point $T_1$ on the minimal model has coordinates

$$x(T_1) = 6 + 4 \cdot 7 + 2 \cdot 7^2 + 4 \cdot 7^3 + 7^4 + 5 \cdot 7^5 + 7^6 + 3 \cdot 7^7 + 7^8 + 3 \cdot 7^9 + O(7^{10})$$

$$y(T_1) = 3 \cdot 7 + 5 \cdot 7^2 + 3 \cdot 7^3 + 2 \cdot 7^4 + 2 \cdot 7^5 + 4 \cdot 7^6 + 4 \cdot 7^9 + O(7^{10})$$

and we compute directly using the formula that

$$\int_b^{T_1} \omega' \eta' = \frac{1}{3} \log(2y(T_1) + a_3 x(T_1) + a_3))$$

$$= 2 \cdot 7 + 2 \cdot 7^2 + 3 \cdot 7^3 + 6 \cdot 7^4 + 2 \cdot 7^5 + 6 \cdot 7^6 + 7^7 + 5 \cdot 7^8 + O(7^{10})$$

**Example 6.6** (Three-torsion). We consider $E : y^2 = x^3 + 405x + 16038$ (minimal model 53a) over $\mathbb{Q}_7$, where it has a $\mathbb{Q}_7$-rational 3-torsion point. Let $P = (-9, 108)$, which is integral on the minimal model. Using the method in [BKK11], we compute the invariant ratio
\[
\frac{\int_b^P \omega \eta}{(\int_b^P \omega)^2} = 6 \cdot 7^{-1} + 6 + 7 + 6 \cdot 7^2 + 4 \cdot 7^3 + 2 \cdot 7^4 + 5 \cdot 7^5 + 4 \cdot 7^6 + O(7^7)
\]
as well as the single integral
\[
\int_b^P \omega = 6 \cdot 7 + 7^2 + 4 \cdot 7^3 + 5 \cdot 7^4 + 2 \cdot 7^5 + 5 \cdot 7^6 + 3 \cdot 7^7 + 6 \cdot 7^8 + 4 \cdot 7^9 + O(7^{10})
\]
to deduce the value of
\[
\int_b^P \omega \eta = 6 \cdot 7 + 3 \cdot 7^2 + 6 \cdot 7^3 + 6 \cdot 7^4 + 7^5 + 4 \cdot 7^6 + 7^7 + O(7^9).
\]
Meanwhile, \(E\) has the 3-torsion point \(T\) with
\[
x(T) = 3 + 6 \cdot 7 + 6 \cdot 7^2 + 3 \cdot 7^3 + 6 \cdot 7^4 + 5 \cdot 7^5 + 6 \cdot 7^6 + 7^8 + 5 \cdot 7^9 + O(7^{10})
y(T) = 2 + 5 \cdot 7 + 5 \cdot 7^2 + 3 \cdot 7^3 + 5 \cdot 7^4 + 4 \cdot 7^5 + 6 \cdot 7^6 + 3 \cdot 7^7 + 4 \cdot 7^9 + O(7^{10}).
\]
We compute
\[
\int_T^P \eta = 1 + 7 + 4 \cdot 7^2 + 3 \cdot 7^3 + 3 \cdot 7^4 + 6 \cdot 7^5 + 7^6 + 2 \cdot 7^7 + 7^8 + 2 \cdot 7^9 + O(7^{10})
\]
\[
\int_T^P \omega = 6 \cdot 7 + 7^2 + 4 \cdot 7^3 + 5 \cdot 7^4 + 2 \cdot 7^5 + 5 \cdot 7^6 + 3 \cdot 7^7 + 6 \cdot 7^8 + 4 \cdot 7^9 + O(7^{10})
\]
\[
\int_T^P \omega \eta = 3 \cdot 7 + 2 \cdot 7^2 + 3 \cdot 7^3 + 3 \cdot 7^5 + 2 \cdot 7^6 + 3 \cdot 7^7 + 6 \cdot 7^8 + O(7^9)
\]
to deduce
\[
\int_b^T \omega \eta = \int_b^P \omega \eta - \int_b^T \eta \int_T^P \omega - \int_T^P \omega \eta
\]
\[
= 4 \cdot 7 + 7^3 + 6 \cdot 7^4 + 5 \cdot 7^5 + 7^7 + 2 \cdot 7^8 + O(7^9).
\]
Meanwhile, the corresponding 3-torsion point \(T_1\) on the minimal model has coordinates
\[
x(T_1) = 5 + 3 \cdot 7 + 2 \cdot 7^2 + 5 \cdot 7^3 + 2 \cdot 7^5 + 3 \cdot 7^6 + 5 \cdot 7^7 + 7^8 + 3 \cdot 7^9 + O(7^{10})
y(T_1) = 6 + 5 \cdot 7 + 7^4 + 5 \cdot 7^5 + 4 \cdot 7^6 + 6 \cdot 7^7 + O(7^{10}).
\]
and we compute directly using the formula that
\[
\int_b^{T_1} \omega' \eta' = \frac{1}{3} \log(2y(T_1) + a_1 x(T_1) + a_3))
\]
\[
= 4 \cdot 7 + 7^3 + 6 \cdot 7^4 + 5 \cdot 7^5 + 7^7 + 2 \cdot 7^8 + 7^9 + O(7^{10}).
\]

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