On a Gel’fand-Yaglom-Peres theorem for $f$-divergences

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Abstract. It is shown that the $f$-divergence between two probability measures $P$ and $R$ equals the supremum of the same $f$-divergence computed over all finite measurable partitions of the original space, thus generalizing results previously proved by Gel’fand and Yaglom and by Peres for the Information Divergence and more recently by Dukkipati, Bhatnagar and Murty for the Tsallis’ and Rényi’s divergences.

Keywords. Information divergence, Kullback-Leibler divergence, Hellinger’s discrimination, Relative Entropy, Rényi’s divergences, Tsallis’ divergences

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1. Introduction

The purpose of this short note is to generalize for arbitrary $f$-divergences a result proved by Gel’fand and Yaglom [1] and Peres [2] for the Information Divergence and, more recently, by Dukkipati et al [3] for Tsallis’ and Rényi’s divergences. Our method focuses on the fundamental notion of convexity of the generating function $f$ together with some standard integration results, thus stressing the fact that many properties of the Information Divergence can be extended to the general class of $f$-divergences (cf. [4, 5]).

The rest of the note is organized as follows. In this introduction we set up the basic definitions and notation and state the main result, which is then proved in Section 2.

Consider two probability measures $P$ and $R$ on a measurable space $(X, \mathcal{A})$ and let $p$ and $r$ be their Radon-Nykodim derivatives with respect to a common dominating measure $\mu$, which without loss of generality can be taken $\mu = P + Q$. The differential version of the Information or Kullback-Leibler divergence is $I(P||R) = \int_X \ln(p/r) \, p \, d\mu$. Gel’fand and Yaglom [1] and Perez [2] (see also [6, Theorem 2.4.2]) showed that

$$I(P||R) = \sup_{\pi} \sum_{k=1}^{m} P(E_k) \ln \frac{P(E_k)}{R(E_k)},$$

where the supremum is taken over all finite measurable partitions $\pi = \{E_1, \ldots, E_m\}$ ($m \geq 1$) of $X$. In other words, by discretizing both $P$ and $R$ and computing the corresponding divergence one can get as close as wanted to $I(P||R)$. Recently Dukkipati et al. [3] proved a similar result for the Rényi’s family of divergences

$$I_\alpha(P||R) = \frac{1}{\alpha - 1} \ln \int_X (p/r)^{\alpha - 1} \, p \, d\mu$$

and hence also for the Tsallis’ divergences

$$T_\alpha(P||R) = \frac{1}{\alpha - 1} \left[ \int_X (p/r)^{\alpha - 1} \, p \, d\mu - 1 \right] = \frac{1}{\alpha - 1} \left[ \exp\{((\alpha - 1) I_\alpha(P||R)) - 1 \} \right]$$

($\alpha > 0$). Their proof rely on measure theoretic considerations along with the inequality $[P(E)]^\alpha \leq \int_E (dP/dR)^\alpha \, dR \left[ R(E) \right]^{\alpha-1}$, which follows from Hölder’s Inequality.

Shortly, The $f$-divergence generated by $f$ is $D_f(P, R) = \int f(p/r) \, p \, d\mu$, where $f : [0, \infty) \to R$ is convex, $f(1) = 0$ and, to avoid undefined expressions, $f(0) = \lim_{u \to 0} f(u)$, $0 \cdot f(0/0) = 0$ and $0 \cdot f(a/0) = \lim_{e \to 0} f(a/e) = a \lim_{u \to \infty} f(u)/u$. The class of $f$-divergences was introduced by Csiszár [7, 8] and Ali and Silvey [4] and includes, besides the Information Divergence $I(P||R) = D_{\ln u}(P, R)$ and the family of Tsallis’ divergences $T_\alpha(P||R) = D_{[u^{\alpha - 1}]/(\alpha - 1)}(P, R)$, the variational distance ($f(u) = |u - 1|$), the $\chi^2$ divergence ($f(u) = (u - 1)^2$), the Hellinger discrimination ($f(u) = (\sqrt{u} - 1)^2$) and many other distances and discrepancy measures between probability measures. While Rényi’s divergences are not properly an $f$-divergence, they are functions of them (i.e. $I_\alpha(P||Q) = (\alpha - 1)^{-1} \ln[1 + (\alpha - 1) T_\alpha(P||R)]$).

Our main result, of which the case of the Information and the Tsallis’ divergences are special cases, is the following.

**Proposition 1.** Let $f$ and $D_f$ be as defined above. Then for any $P$ and $R$

$$D_f(P, R) = \sup_{\pi} \sum_{k=1}^{m} R(E_k) \ f \left( \frac{P(E_k)}{R(E_k)} \right),$$

(1)
where the supremum is taken over all finite measurable partitions \( \pi \) of \( X \).

Since the Rényi’s divergences \( I_\alpha(P||R) = (\alpha - 1)^{-1} \ln[1 + (\alpha - 1)T_\alpha(P||R)] \) are a continuous monotone function of Tsallis’ divergences, it follows from (1) that

\[
I_\alpha(P||R) = \sup_\pi \frac{1}{\alpha - 1} \ln \sum_{k=1}^m \frac{P(E_k)^\alpha}{R(E_k)^{\alpha-1}},
\]

which is Dukkipati et al. [3] main result.

2. Proof of Proposition 1

We begin with some preliminary considerations. First, note that both sides of (1) remain the same if we substitute \( f(u) \) by \( f(u) = f(u) - a(u - 1) \). By taking \( y = a(u - 1) \) to be a support line to the graph of \( f \) at \( u = 1 \) we see that we can assume without loss of generality that \( f(u) \) is nonnegative, non increasing for \( u < 1 \) and nondecreasing for \( u > 1 \). Second, since \( P(A) = \int_A (p/r) \, r \, d\mu \), if \( a \leq p(x)/r(x) \leq b \) on \( A \), then also \( a \leq P(A)/R(A) \leq b \). Finally, the left hand side of (1) is greater than or equal to the right hand side because, if \( \pi = \{E_j : j \in J\} \) is a finite partition of \( X \), Jensen’s inequality implies that

\[
\int f(p/r) r \, d\mu = \sum_{j \in J} R(E_j) \int_{E_j} f(p/r) \frac{r}{R(E_j)} \, d\mu \\
\geq \sum_{j \in J} R(E_j) f \left( \int_{E_j} (p/r) \frac{r}{R(E_j)} \, d\mu \right) = \sum_{j \in J} f \left( \frac{P(E_j)}{R(E_j)} \right) R(E_j).
\]

We will now prove (1) in the case that \( D_f(P, R) < \infty \). Due to the last consideration above, it will be enough to prove that the left hand side of (1) is less than or equal to the right hand side or, equivalently, that given any \( \epsilon > 0 \) there exists a partition \( \pi \) such that the difference between the leftmost and the rightmost sides of (2) is less than or equal to \( \epsilon \). To do this, consider \( 0 < H < K \) and define \( A_H = \{x \in X : p(x) < H \, r(x)\} \), \( C_K = \{x \in X : p(x) > K \, r(x)\} \) and \( B_{H,K} = X - (A_H \cup C_K) = \{x \in X : H \, r(x) \leq p(x) \leq K \, r(x)\} \). Since \( D_f(P, R) < \infty \), \( \int_{A_H} f(p/r) \, r \, d\mu \) must also be finite. Hence (i) \( \lim_{H \to 0} \int_{A_H} f(p/r) \, r \, d\mu = 0 \) by dominated convergence and (ii) also \( \lim_{H \to 0} f[P(A_H)/R(A_H)] R(A_H) = 0 \) because Jensen’s inequality implies that \( 0 \leq f[P(A_H)/R(A_H)] R(A_H) \leq \int_{A_H} f(p/r) \, r \, d\mu \). Therefore, for \( H_0 \) small enough, \( \int_{A_{H_0}} f(p/r) \, r \, d\mu - f[P(A_{H_0})/R(A_{H_0})] R(A_{H_0}) < \epsilon/3 \). A similar argument shows that, for \( K_0 \) large enough, \( \int_{C_{K_0}} f(p/r) \, r \, d\mu - f[P(C_{K_0})/R(C_{K_0})] R(C_{K_0}) < \epsilon/3 \). Next, since \( f \) is convex, it is continuous and hence absolutely continuous in \([H_0, K_0] \). Therefore, there exists a \( \delta > 0 \) such that \( |f(u) - f(u')| < \epsilon/3 \) whenever \( |u - u'| < \delta \). With this in mind, partition the interval \([H_0, K_0]\) in (say) \( m \) subintervals \( I_1, \ldots, I_m \), each having length less than \( \delta \), and define \( E_i = \{x \in X : p(x)/r(x) \in I_i\} \).

Since for \( x \in E_i \) we have that \( p(x)/r(x) \in I_i \) and hence also \( P(E_i)/R(E_i) \in I_i \),

\[
0 \leq \int_{E_i} f(p/r) \, r \, d\mu - f \left( \frac{P(E_i)}{R(E_i)} \right) R(E_i) \\
= \int_{E_i} \left( f(p/r) - f \left( \frac{P(E_i)}{R(E_i)} \right) \right) r \, d\mu \leq \int_{E_i} \frac{\epsilon}{3} r \, d\mu \leq (\epsilon/3) R(E_i).
\]
To finish this part of the proof, consider the partition $\pi = \{E_0 = A_{H_0}, E_1, \ldots, E_n, E_{n+1} = C_{K_0}\}$. The previous considerations imply that the difference between the leftmost and the rightmost terms in (2) is less than or equal than $\epsilon/3 + (\epsilon/3)R(B_{H_0,K_0}) + \epsilon/3 \leq \epsilon$.

Now suppose that $D_f(P,R) = \infty$. Then either $\int_{\{p>r\}} f(p/r) r \, d\mu$ or $\int_{\{p<r\}} f(p/r) r \, d\mu$ should be infinite. Suppose first that $\int_{\{p>r\}} f(p/r) r \, d\mu = \infty$. We will show that there is a sequence of disjoint subsets $D_n$ such that $\sum_{n=1}^{\infty} f[P(D_n)/R(D_n)] R(D_n) = \infty$. This would imply, of course, that the sets $D_n$ can be used to construct a partition of $X$ so that the rightmost term of (2) is as large as wanted, and this in turn that the right hand side of (1) is infinite. Indeed, let $D_n = \{x \in X: p(x) > r(x) \text{ and } (n-1) \leq f[p(x)/r(x)] < n\}$ and for $n \geq 1$ define $b_n = \inf\{u \in \mathbb{R} : u > 1 \text{ and } f(u) \geq n\}$. Since $f$ is continuous and (we are assuming wlog) nondecreasing for $u > 1$, it follows that $D_n = \{x \in X: p(x) > r(x) \text{ and } b_{n-1} \leq p(x)/r(x) < b_n\}$. Hence, $b_{n-1} \leq P(D_n)/R(D_n) < b_n$, $(n-1) \leq f[P(D_n)/R(D_n)] < n$ and $\{f(p/r) - f[P(D_n)/R(D_n)]\} \leq 1$ in $D_n$. Therefore,

$$\int_{\{p>r\}} f(p/r) r \, d\mu = \sum_{n=1}^{\infty} f\left(\frac{P(D_n)}{R(D_n)}\right) R(D_n) + \sum_{n=1}^{\infty} \int_{D_n} f(p/r) - f\left(\frac{P(D_n)}{R(D_n)}\right) r \, d\mu \leq \sum_{n=1}^{\infty} f\left(\frac{P(D_n)}{R(D_n)}\right) R(D_n) + 1.$$ 

This shows that if $\int_{\{p>r\}} f(p/r) r \, d\mu = \infty$, then so should be $\sum_{n=1}^{\infty} f\left(\frac{P(D_n)}{R(D_n)}\right) R(D_n)$. The case that $\int_{\{p<r\}} f(p/r) r \, d\mu = \infty$ is dealt with in a similar manner. \hfill \Box

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