Explicit representation of Green function for 3D dimensional exterior Helmholtz equation

J. P. Cruz* E. L. Lakshtanov†

Abstract

We have constructed a sequence of solutions of the Helmholtz equation forming an orthogonal sequence on a given surface. Coefficients of these functions depend on an explicit algebraic formulae from the coefficient of the surface. Moreover, for exterior Helmholtz equation we have constructed an explicit normal derivative of the Dirichlet Green function. In the same way the Dirichlet-to-Neumann operator is constructed. We proved that normalized coefficients are uniformly bounded from zero.

Keywords: explicit solution, Helmholtz exterior problem, Green function, Dirichlet-to-Neumann operator.

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1 Introduction

Consider $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary $\partial \Omega$ and $k > 0$. The scattered field is given by Helmholtz equation and radiation condition

\[ \Delta \Psi (r) + k^2 \Psi (r) = 0, \quad r \in \Omega' = \mathbb{R}^3 \setminus \Omega, \]  
\[ \int_{|r|=R} \left| \frac{\partial \Psi (r)}{\partial |r|} - ik \Psi (r) \right|^2 dS = o(1), \quad R \to \infty, \]  

*Mathematics Department, University of Aveiro, Portugal, e-mail: pedrocruz@ua.pt
†Mathematics Department, University of Aveiro, Portugal, e-mail: lakshtanov@rambler.ru
with Dirichlet boundary conditions,
\[ \Psi(r) \equiv u_0(r), \quad r \in \partial \Omega, \quad u_0 \in C(\partial \Omega). \]  
(1.3)

For example, in [1] is proved the existence and uniqueness of the solution of (1.1)-(1.3). A
function \( \Psi(r) \) which satisfy mentioned conditions has asymptotics
\[ \Psi(r) = \frac{e^{ik|r|}}{|r|} f(q) + o \left( \frac{1}{|r|} \right), \quad r \to \infty, \quad q = r/|r| \in S^2, \]  
(1.4)

where function \( f(\theta, \varphi) = f(\theta, \varphi, k, u_0) \) is called \textit{scattering amplitude} and observable
\[ \sigma_T = \int_{S^2} |f(q)|^2 d\sigma(q) \]

is called Total Cross Section, \( \sigma \) is a square element of the unit sphere.

A very important particular case of the boundary condition is
\[ u_0 = e^{ik\langle r, \theta_0 \rangle}, \quad r \in \partial \Omega, \]
which is the scattering of a plane wave with incident angle \( \theta_0 \in S^2 \). The total momentum
transmitted to the obstacle is given by observable called Transport Cross Section (in a large
volume normalization)
\[ R = \int_{S^2} (1 - \langle q, \theta_0 \rangle) |f(q)|^2 d\sigma(q). \]

Unfortunately, analytical expressions of these observables for certain \( k > 0 \) exist only
for few bodies of simple shapes (see [16, 18]). Moreover, the scattering happens not only
by plane or spherical wave, but it could be caused, for example, by arbitrary secondary
radiation.

From another point of view there exists some numerical methods for direct scattering
calculation. One of them is based on a numerical solution of integral equation (see [15]).
Another method, developed by A. Ramm and S. Gutman in [3]-[6], allow to construct
the Green function and therefore to obtain solutions for arbitrary boundary condition \( u_0 \in C(\partial \Omega) \). The ground analytical achievement by A. Ramm and S. Gutman is the so called
Modified Rayleigh Conjecture. In particular, it follows that functions \( Y_{lm}(\theta, \varphi)h_l(k|r|)|_{\partial \Omega} \)
(spherical harmonics and spherical Hankel functions (see (1.7)) correspondingly) form a basis
in the space \( L_2(\partial \Omega, dS) \).

The aim of current work is to express explicitly coefficients of physical observables and
also for the normal derivative of the Dirichlet-Green function. We found a constant uniform
lower bound for normalization coefficients (denominators) and we prove convergence of all
produced series.

Greens function of the Laplacian in \( \Omega' \) is given by
\[ \Delta_r G(r, t) + k^2 G(r, t) = -\delta(r - t), \quad r, t \in \Omega', \quad G \equiv 0, \quad r \in \partial \Omega, \]
\[ \lim_{r \to \infty} \int_{|t| = R} \left| \frac{\partial G(r, t)}{\partial n_t} - ikG(r, t) \right|^2 dS = o(1), \quad R \to \infty. \]
As it is known (evidence from Green formula)

\[
\Psi(r) = \int_{\partial\Omega} \frac{\partial G(r, t)}{\partial n} u_0(t) dS(t), \quad r \in \overline{\Omega}
\]

In this paper we shall express kernel \( \frac{\partial G(r, t)}{\partial n} \) explicitly through coefficients of the surface \( \partial \Omega \) and also as measures \( C, \sigma_T, R \) and \( \{ A_{lm} \} \), where \( A_m, 0 \leq |m| \leq l \) are coefficients of representation

\[
f(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{|m| \leq l} A_{ml} Y_{lm}(\theta, \varphi), \quad (1.5)
\]

where \( Y_{ml}(\theta, \varphi) \) are spherical harmonics.

Let \( \overline{\mathbb{N}} = \mathbb{N} \cup \{0\} \). Let \( \mathcal{L} \) be the set of indexes

\[
\mathcal{L} = \{(l, m) : l, m \in \overline{\mathbb{N}}, |m| \leq l\}
\]

and let \( l \in \mathcal{L} \) with \( l = (l(1), l(2)) \). We also use the notation \( \overline{l} = (l(1), -l(2)) \) and set the order: \( (l, m) > (p, r) \) \( \iff \) \( l > p \) \& \( [l = p] \lor (|m| > |r|) \) \& \( [(l = p) \lor (m > 0)] \lor (m = -r) \), so \( \mathcal{L} = \{(0, 0), (1, 0), (1, -1), (1, 1), (2, 0), (2, -1), (2, 1), (2, -2), (2, 2), (3, 0), \ldots\} \). Let \( o = (0, 0) \) and let operations + and − have the natural definition in \( \mathcal{L} \times \mathbb{Z} \rightarrow \mathcal{L} \) correspondingly to the introduced order.

1.1 Surfaces with inverse radius-vector represented as finite combination of harmonics

Let \( \mathcal{F} \) be a subset of functions (multiindex) \( \mathcal{L}^{\overline{\mathbb{N}}} \) which have finite support and let capacity be defined by

\[
|d| = \sum_l d(l), \quad \text{Supp} \ d = \max\{l : d(l) \neq 0\}.
\]

Let \( e_l \in \mathcal{F} \) be defined as \( e_l(m) = \delta_{lm} \) (evaluated 1 only when \( l = m \)). Also set

\[
C^d = \frac{|d|!}{\prod_l d(l)!}, \quad a^d = \prod_l a^d_l,
\]

\[
I^d = \int_0^\pi \int_0^{2\pi} \prod_l (Y_l(\theta, \varphi)) d(l) d\theta d\varphi.
\]

Theorem 1.1. Let a star shaped surface \( \partial \Omega \) be given as a set \( \{ r = r(\theta, \varphi) \in \mathbb{R}^3, \theta \in [0, \pi], \varphi \in [0, 2\pi) \} \) where

\[
|r(\theta, \varphi)| = \frac{1}{\sum_{l \leq (N, N)} a_l Y_{l}(\theta, \varphi)} \quad (1.6)
\]

with \( N \geq 0 \) and where \( \{ a_{lm} \} \) are coefficients. Then

1. Functions

\[
\hat{\Psi}_n(r) = \sum_{k \leq n} c_{nk} Y_k(\theta, \varphi) h_{k(l)}(k|r|), \quad n \in \mathcal{L}
\]
satisfy (1.1), (1.2) and their restrictions \( \{ \hat{\Psi}_n|_{\partial \Omega}, \quad n \in \mathcal{L} \} \) form an orthonormal basis in \( L_2(\partial \Omega, d\theta d\varphi) \).

Here

\[
c_{nn} = 1/\lambda_n, \quad n \in \mathbb{N}
\]

\[
\lambda_o^2 = g_{oo} > 0, \quad \lambda_n^2 = g_{nn} - \sum_{k=0}^{n-1} \left| \sum_{p=0}^{\infty} c_{kp} g_{np} \right|^2 > 0, \quad n > o.
\]

We now define \( g_{ij}, \hat{h}_{nm}, c_{nm} \). Let

\[
g_{ij} = (-1)^{j(2)} \sum_{m=0}^{i(1)+j(1)} \frac{1}{k^m} \left( \sum_{l=0}^{m} \hat{h}_{i(1)l} \hat{h}_{j(1)(m-l)} \right) \sum_{d:|d|=m+2, \text{Supp } d \leq (N,N)} C^d a^d I^{d+e_i+e_j}
\]

where coefficients \( \hat{h}_{nj} \) are defined from the well known representation for Hankel spherical functions [19],

\[
h_n(t) = e^{ikt} t^n = \sum_{j=0}^{n} \hat{h}_{nj} t^n, \quad \hat{h}_{n0} = 1, \quad t \neq 0,
\]

with

\[
\hat{h}_{nm} = \frac{i^m}{2^m} \prod_{p=1}^{m} (n+p) \cdot \prod_{p=1}^{m} \frac{(n-m+p)}{p}, \quad 0 < m \leq n;
\]

for \( m < n \) we have

\[
c_{nm} = \frac{1}{\lambda_n} \left( \sum_{k=m}^{n-1} \sum_{p=0}^{\infty} c_{kp} c_{km} g_{np} \right).
\]

2. Consider an arbitrary function \( u_0 \in L_2(\partial \Omega) \), then we have

\[
\sigma_T = \frac{1}{k^2} \sum_{n=0}^{\infty} \sum_{m \leq n} \left[ \tau_{nm} \left( \sum_{p \leq n} c_{np} \overline{u}_p \right) \left( c_{nm} \sum_{p \leq n} \tau_{np} u_p + 2 \sum_{m<l<n} c_{lm} \sum_{p \leq l} \tau_{lp} u_p \right) \right],
\]

and coefficients of the scattering amplitude

\[
A_m = \frac{1}{k} \sum_{n \geq m} c_{nm} \sum_{p \leq n} \overline{c}_{np} u_p, \quad m \in \mathcal{L},
\]

where

\[
u_p = \int_{\theta_0}^{\pi} \int_{\varphi_0}^{2\pi} u_0(\theta, \varphi) \overline{Y}_p(\theta, \varphi) \overline{h}_p(k|\theta, \varphi|) d\theta d\varphi.
\]

3. Moreover, exists numbers \( C_i = C_i(k, \Omega), i = 1, 2 \) such that

\[
c_{nk} \leq \frac{C_1}{k(1)!}, \quad 0 \leq k \leq n, \quad k, n \in \mathcal{L}.
\]
Also

\[
\sum_{p=0}^{n} c_{np} u_p < \frac{C_1}{m(1)!} |\hat{u}_n|, \quad \hat{u}_n = (u_0, \hat{\Psi}_n|_{\partial \Omega})_{L_2(\partial \Omega, d\theta d\varphi)} = \sum_{k \leq n} c_{nk} u_k \to 0, \quad (1.11)
\]

when \( n \to \infty \) and

\[\lambda_n > C_2, \quad n \in \mathcal{L}. \quad (1.12)\]

4. We have weak convergence of \( \frac{\partial G}{\partial n} \):

\[
\frac{\partial G}{\partial n_t}(r, t) = \sum_n \hat{\Psi}_n(r) \hat{\Psi}_n(t), \quad r, t \in \Omega'.
\]

**Note 1.** Define \( \{A^L_m, m, L \in \mathcal{L}\} \) where \( A^L_m = 0 \) for \( L < m \) and

\[A^L_m = \frac{1}{k} \sum_{L \geq n \geq m} c_{nm} \sum_{p \leq n} c_{np} \hat{u}_p, \quad L \geq m.
\]

or, recursively

\[A^L_m = A^{L-1}_m + \frac{1}{k} c_{Lm} \sum_{p \leq L} c_{Lp} \hat{u}_p.
\]

From Theorem 1.1 and using the orthogonality of basis \( \{\hat{\Psi}_n|_{\partial \Omega}\} \) we have

\[|A_m - A^L_m| \leq \frac{C}{m(1)!} \left\| u_0 - \sum_{p \leq L} \hat{u}_p \right\| = o(1), \quad L \to \infty.
\]

Then

\[\sigma^L_T = \sum_{n \leq L} |A^L_n|^2,
\]

\[|\sigma_T - \sigma^L_T| \leq C \left\| u_0 - \sum_{p \leq L} \hat{u}_p \right\| = o(1), \quad L \to \infty.
\]

For the case \( u_0 = e^{ikz} \), i.e. \( \theta_0 = (0, 0, 1) \), we have

\[R^L = \sum_{n \leq L} \sqrt{\frac{(n(1) + n(2))(n(1) - n(2))}{(2n(1) + 1)(2n(1) - 1)}} Re(A_n \overline{A_{n+1}}_{(1,0)}) \quad (1.13)
\]

\[|R - R^L| \leq \left\| u_0 - \sum_{p \leq L} \hat{u}_p \right\| = o(1), \quad L \to \infty.
\]

Formula (1.13) is a simple corollary of equality \( xP_n(x) = \frac{n+1}{2n+1} P_{n+1}(x) + \frac{n}{2n+1} P_{n-1}(x) \) for Legendre polynomials.

**Note 2.** The Dirichlet-to-Neumann operator \( \mathcal{N} \) (which correspond the boundary condition \( u_0 \) to the normal derivative of the field \( u(r) \)) could be constructed in the following way:

\[(\mathcal{N}u_0)(r) = \sum_n \left( \int_0^\pi \int_0^{2\pi} u_0(\theta, \varphi) \overline{\hat{\Psi}_n(r(\theta, \varphi))} d\theta d\varphi \right) \frac{\partial \hat{\Psi}_n}{\partial n}(r).
\]
1.2 Comparison with other explicit representations

In spite of Theorem 1.1 be stated for star shaped bodies clearly all representations of the Green function will be the same for arbitrary, smooth enough, body. We just should put

\[ u_p = \int_0^\pi \int_0^{2\pi} u_0(\theta, \varphi) \bar{Y}_p(\theta, \varphi) \bar{h}_p(1)(k|\mathbf{r}(\theta, \varphi)|) d\theta d\varphi, \quad p \in \mathcal{L}, \]

\[ g_{ij} = \int_0^\pi \int_0^{2\pi} Y_i(\theta, \varphi) \bar{Y}_j(\theta, \varphi) h_{i,1}(k|\mathbf{r}(\theta, \varphi)|) \bar{h}_{j,1}(k|\mathbf{r}(\theta, \varphi)|) d\theta d\varphi, \quad i, j \in \mathcal{L}. \]

It is possible, like we mentioned above, functions \{\bar{Y}_p(\theta, \varphi) \bar{h}_p(1)(k|\mathbf{r}(\theta, \varphi)|)\} do form a basis for arbitrary, smooth enough body.

Our representation holds for all values of \( k > 0 \), but exists other explicit representations for the Green’s function which holds in a neighborhoods of points \( k = 0 \) and \( k = \infty \). We consider of interest to cite the following

**Theorem 1.2.** R. E. Kleinman [8, Th. 4.1] There exists \( \alpha > 0 \) such that when \( |k| < \alpha \), the Green’s function \( G(x, r) \) exists uniquely in \( \Omega' \) and is given explicitly by

\[ G(r, t) = -\frac{e^{ik|r-t|}}{4\pi |r-t|} + e^{ikr} \sum_{n=0}^{\infty} K^n U_0, \quad (1.14) \]

where

\[ K(U_0) = -2ik \int_{\Omega'} dv(t_1) \frac{G_0(r, t_1)}{|t_1|} \frac{\partial}{\partial |t_1|} [t_1 |U_0|], \]

\[ U_0 = U_0(t_1, t) = \int_{\partial \Omega} dS(t_0) \frac{e^{-ik|t_0|} + ik|t_0-t|}{4\pi |t_0-t|} \frac{\partial}{\partial t_0} G_0(t_1, t_0). \]

Here \( v \) is a volume element, function \( G_0(r, t_1) \) is the static Dirichlet-Green function.

As we can see, all summands here are determined recursively as in the Theorem 1.1. Also, is important to note, that the first approximation is a static Green function, which is supposed to be known. So, in the some sense the constructed solution is a pertubated static solution, that explain the quick, exponential, convergence of the approximation (1.14).

In the high-frequency exists the following representation for the Green’s function which holds for obstacles which satisfy non-trapping condition and single impact condition (see [12] for details)

\[ G(r, t) = -\frac{e^{ik|r-t|}}{4\pi |r-t|} + \sum_{m=0}^{l} \left( \frac{i}{k} \right) z_m(r, t) + R_l(r, t), \quad (1.15) \]

where function \( S(r, t) \) satisfy eikonal equation

\[ |\nabla S|^2 = 1, \]

and function \( z_m(r, t) \) are defined recursively through differential (transfer) equations:

\[ 2\nabla S \cdot \nabla z_m + (\nabla S)z_m = -\nabla z_{m-1}, \quad m \geq 0, z_{-1} \equiv 0. \]
If $r$ and $t$ don’t lie on the tangential rays, we have the estimate

$$|R_i(r, t)| = o(k^{1-\epsilon}), \quad k \to \infty.$$ 

In that case, the good, exponential, convergence exists since solution at high values of $k$ are close to the “classical” limit solution.

$\sigma_T$ representation (1.8) holds for all values of $k > 0$, but we don’t have any estimates about the rate of convergence. For the sphere, our representation is the well known solution (for ex. [19]) with very good convergence (as a $1/n!$). Of course, good convergence properties should be kept in a some neighborhood of the sphere. Moreover, we are sure that it will be possible to do it with developing of local limit theorems for sign distributions, really if $1/r$ is a characteristic function of a random variable with a sign distribution then elements of a Gram matrix $g_{ij}$ are in the sense of the probability of sum of $(i+j+2)$ of such variables (see [11] for integral theorem). By now it is out of our possibilities to describe the neighborhood of at least exponential convergence. From another point of view we want to remind that representation (1.8), in contrast to representations (1.15, 1.14), converges for all values of $k > 0$.

To finish our discussion we want to note that representation like (1.8) cannot be done for the 2-dimensional model, since Bessel functions of integer index do not allow a finite polynomial representation.

### 1.3 Polyhedrons and bodies which are revolutions of polylines

Calculation of the Gram matrix $G = \{g_{ij}\}$ for polyhedron is very natural and elements $g_{ij}$ need only $N (i+j)^2$ sums, where $N$ is the number of sides.

Let’s set $N+1$ points on the segment $[0, \pi]$

$$0 = \theta_0 \leq \theta_1 \leq \ldots \leq \theta_{N-1} \leq \theta_N = \pi,$$

and consider the set of bodies $\mathcal{R}$ which contains arbitrary bodies of revolution of the polyline with vertexes in points $\{\theta_i, i = 0, \ldots, N\}$.

**Theorem 1.3.** Consider polyline $r = \sum_i r_i(\theta)I_i(\theta)$, where $I_i, i = 1, \ldots, N$ is a characteristic function of the segment $[\theta_{i-1}, \theta_i]$ and coefficients $\{a_i, b_i, c_i, i = 1, \ldots, N\}$ determine parts of the polyline $r_i$:

$$a_i r_i \cos \theta + b_i r_i \sin \theta = f_i, \quad i = 1, \ldots, N. \quad (1.16)$$

Then Gram matrix coefficients $g_{ij}$ could be calculated as

$$g_{ij} = \sum_{p=1}^{N} \frac{1}{(f_p)^{i+j+2}} \sum_{l=0}^{i+j+2} C^l_{i+j+2} a_l^p b_l^{i+j+2}-l T_{ij,il(i+j)},$$

where

$$T_{ij,lm} = \int_{\theta_{p-1}}^{\theta_p} Y_i(\theta) Y_j(\theta) \cos^l \theta \sin^{m+2-l} \theta \sin d\theta, \quad l \leq m+2;$$

$$g_{ij} = \sum_{p=1}^{N} \frac{1}{k^{m+2}} \left( \sum_{n=0}^{m} \hat{h}_{in} \hat{h}_{jm-n} \frac{1}{(f_p)^{m+2}} \right) \left( \sum_{l=0}^{m+2} C^l_{m+2} a_l^p b_l^{m+2-l} T_{ij,lm} \right).$$
1.4 Explicit dependence of Gram matrix on frequency

We can represent elements of the Gram matrix \( \{ g_{ij}, i, j \in \mathcal{L} \} \) for arbitrary smooth enough surface as a polynomial of the inverse frequency \( k^{-1} \),

\[
g_{ij} = (-1)^{j(2)} \sum_{m=0}^{i(1)+j(1)} \frac{1}{k^{m+2}} p_{ij}^m, \tag{1.17}
\]

where

\[
p_{ij}^m = \sum_{l=0}^{m} \hat{h}_{i(l)j(l)(m-l)} \int_0^\pi d\theta \int_0^{2\pi} d\varphi \frac{Y_{i}(\theta, \varphi)Y_{j}(\theta, \varphi)}{|r(\theta, \varphi)|^{m+2}}, \quad 0 \leq m \leq i(1) + j(1). \tag{1.18}
\]

2 Proofs

Denote

\[ \Psi_n(r) = Y_n(\theta)h_n(k|r|), \quad n \in \mathcal{L}, \quad r \in \Omega', \]

where functions \( \Psi_n(r) \) satisfy (1.1) and (1.2). Moreover, they have asymptotics at the infinity (see (1.7))

\[ \Psi_n(r) \sim \frac{1}{k} Y_n(\theta) e^{ik|r|/|r|}, \quad |r| \to \infty. \]

Let \( A \) be a linear operator acting from \( L_2(\partial \Omega, dS) \) to \( L_2(S^2, d\sigma) \) which corresponds \( u_{\partial \Omega} \) to scattering amplitude \( f(q) \) (see (1.4)). Here \( dS \) and \( d\sigma \) are standard metrics on \( \partial \Omega \) and unit sphere \( S^2 \). In \[4, 2.12,2.16\], \[3, 2.10,2.13\] it is proved that \( A \) is a bounded operator and in particular it is proved that functions \( \Psi_n_{\partial \Omega} \) form a basis in \( L_2(\partial \Omega, dS) \). So we have the transformation

\[ A \left( \sum_{n \in \mathcal{L}} c_n \Psi_n \right) = \frac{1}{k} \sum_{n \in \mathcal{L}} c_n Y_n, \quad \sum_{n \in \mathcal{L}} c_n \Psi_n \in L_2(\partial \Omega, dS). \]

Let us prove that exists a constant \( C_1 = C_1(k, \Omega) \) such that

\[ |c_n| \leq \frac{C_1}{n(1)!}, \quad n \in \mathcal{L}. \tag{2.1} \]

Consider sphere \( S_R := \{|r| = R\} \) such that the body \( \Omega \) is strictly embedded in that sphere. Define also the \( A_1 : L_2(\partial \Omega, dS) \to L_2(S_R, d\sigma_R) \), where \( d\sigma_R = R^2 \sin \theta d\theta d\varphi \), as

\[ (A_1 u_0)(x) = \int_{\partial \Omega} \frac{\partial G(x, y)}{\partial n} u_0(y) dS(y), \quad x \in S_R. \]

Since function \( \frac{\partial G}{\partial n} \) has singularities only for \( x = y \) (for example, see [8]) operator \( A_1 \) is bounded. Note that functions \( h_n(k|r|) \) are constant on the \( S_R \),

\[ 4\pi R^2 |h_n(kR)| \cdot |c_n| = |(A_1 u_0, \nabla_n)_{L_2(S_R)}| \leq \| A_1 u_0 \|_{L_2(S_R)} \| \nabla_n \|_{L_2(S_R)} = 4\pi R^2 \| A_1 u_0 \|_{L_2(S_R)} \leq 4\pi R^2 \| A_1 \| \| u_0 \|_{L_2(\partial \Omega)}. \]
\[ |c_n| \leq \frac{\|A_1\|\|u_0\|_{L^2(\partial\Omega)}}{|h_n(kR)|} \]
taking into account the asymptotic (see [17, pp. 358-364]) of \( h_n(R) \sim \frac{n^l}{(R/2)^{m_0}} \), \( n \to \infty \), and since \( R \) could be chosen to satisfy \( kR \geq 2 \), we obtain (2.1) (that corresponds to (1.10 in Theorem 1)).

Let \( g(\theta, \varphi) \) be a density
\[
dS = [g(\theta, \varphi)]^2 d\sigma = |r| \sqrt{\left[ |r|^2 + \left( \frac{\partial |r|}{\partial \theta} \right)^2 \right] \sin^2 \theta + \left( \frac{\partial |r|}{\partial \varphi} \right)^2} d\theta d\varphi \quad (2.2)
\]
and for simplicity we shall use the same notation \( d\sigma = d\theta d\varphi \) for measure on the \( \partial\Omega \). It is evident that if function \( f \in L^2(\partial\Omega, d\sigma) \) then \( f/g \in L^2(\partial\Omega, dS) \).

Denote \( \tilde{\Psi}_n^0 = \Psi_n|_{\partial\Omega}/g \) and construct an orthonormal basis \( \hat{\Psi}_n^0 \) in \( L^2(\partial\Omega, dS) \). We shall construct it in form:
\[
\hat{\Psi}_n^0 = \sum_{k=0}^n c_{nk} \Psi_k, \quad n \in \mathcal{L} \tag{2.3}
\]
where
\[
\lambda_n = \left[ \|\Psi_n^0\|^2 - \sum_{k=0}^{n-1} |(\Psi_n^0, \hat{\Psi}_k^0)|^2 \right]^{1/2}.
\]
Using
\[
(\Psi_n^0, \hat{\Psi}_k^0) = \sum_{p=0}^k \tau_{kp}(\Psi_n^0, \hat{\Psi}_p^0)
\]
we have
\[
c_{nn} = 1/\lambda_n, \quad c_{nm} = \left[ \sum_{k=m}^{n-1} \sum_{p=0}^k \tau_{kp} c_{km}(\Psi_n^0, \Psi_p^0) \right] /\lambda_n, \quad m < n.
\]
Denote
\[
\hat{\Psi}_n(r) = g\hat{\Psi}_n^0 = \sum_{k=0}^n c_{nk} \Psi_k, \quad n \in \mathcal{L}, \quad r \in \Omega'.
\]
and let \( u_0 \) be a function in \( L^2(\partial\Omega, d\sigma) \) and also \( u_0/g = \sum_{n=0}^\infty \hat{u}_n \hat{\Psi}_n^0 \). Then function
\[
u(r) = \sum_{n=0}^\infty \hat{u}_n \hat{\Psi}_n(r), \quad r \in \mathbb{R}^3 \setminus \Omega
\]
satisfy (1.1)-(1.3).

Using (2.3) we obtain the corresponding scattering frequencies (see [15]),
\[
A_m = \frac{1}{k} \sum_{n=m}^\infty c_{nm} \left( \sum_{p=0}^n \tau_{np} \nu_p \right),
\]
Theorem 1.1 is proved.

Due to the representation of where

Next, we calculate (Ψ₀, Ψ). \( \sum \) Let us prove now (1.12). Set

\[
\lambda_n = \| F_n \|_{L_2(\partial \Omega, dS)}, \quad \text{where} \quad F_n = \Psi_n^0 - \sum_{k=o}^{n-1} (\Psi_n^0, \hat{\psi}_k) \hat{\psi}_k.
\]

By construction \( (AF_n, Y_n) = \frac{1}{k} \), therefore \( \| AF_n \| \geq \frac{1}{k} \), so

\[
\frac{1}{k} \leq \| AF_n \| \leq \| A \| \| F_n \|_{L_2(\partial \Omega, dS)}.
\]

Due to the representation of \( r(\theta, \varphi) \) and (2.2) exists \( S = \max_{\theta, \varphi} |g(\theta, \varphi)|^2 < \infty \), so we have

\[
\lambda_n = \| F_n \|_{L_2(\partial \Omega, dS)} \geq \frac{1}{S} \| F_n \|_{L_2(\partial \Omega, dS)} \geq (S\|A\|)^{-1}.
\]

Now, let us prove (1.8). We write (see (1.9))

\[
|A_m|^2 = \frac{1}{k^2} \sum_{n \geq m} \left[ c_{nm} \sum_{p \leq n} \tau_{np} u_p \left( c_{nm} \sum_{p \leq n} \tau_{np} u_p + 2 \sum_{m < l < n} c_{lm} \sum_{p \leq l} \tau_{lp} u_p \right) \right],
\]

\[
\sigma_T = \frac{1}{k^2} \sum_{n=m}^{\infty} \sum_{l=m}^{\infty} \left[ \tau_{nm} \left( \sum_{p \leq n} \tau_{np} u_p \right) \left( \sum_{m < l < n} c_{tm} \sum_{p \leq l} \tau_{tp} u_p \right) \right].
\]

Next, we calculate \( (\Psi_i^0, \Psi_j^0) \),

\[
g_{ij} = (\Psi_i^0, \Psi_j^0) = \int_0^\pi d\theta \int_0^{2\pi} d\varphi Y_i(\theta, \varphi) \tilde{Y}_j(\theta, \varphi) h_{ij}(1)(k|r(\theta, \varphi)||h_{ij}(1)(k|r(\theta, \varphi)|)\]

\[
(-1)^{j(2)} \int_0^\pi d\theta \int_0^{2\pi} d\varphi Y_i(\theta, \varphi) \tilde{Y}_j(\theta, \varphi) \left( \sum_{p=0}^{i(1)} \frac{h_{ij}(1)p}{(k|r|)^{p+1}} \right) \left( \sum_{p=0}^{j(1)} \frac{\tilde{h}_{ij}(1)p}{(k|r|)^{p+1}} \right)
\]

\[
(-1)^{j(2)} \int_0^\pi d\theta \int_0^{2\pi} d\varphi Y_i(\theta, \varphi) \tilde{Y}_j(\theta, \varphi) \frac{1}{k^{m+2}} \left( \sum_{l=0}^{m} \frac{h_{ij}(1)l}{h_{ij}(1)(m-l)} \right) \frac{1}{|r(\theta, \varphi)|^{m+2}}
\]

\[
(-1)^{j(2)} \sum_{m=0}^{i(1)+j(1)} \frac{1}{k^{m+2}} \sum_{l=0}^{m} \frac{1}{h_{ij}(1)l} \int_0^\pi d\theta \int_0^{2\pi} d\varphi Y_i(\theta, \varphi) \tilde{Y}_j(\theta, \varphi) \frac{1}{|r(\theta, \varphi)|^{m+2}} = (2.4)
\]

\[
(-1)^{j(2)} \sum_{m=0}^{i(1)+j(1)} \frac{1}{k^{m+2}} \left( \sum_{l=0}^{m} \frac{1}{h_{ij}(1)l} \right) \int_0^\pi d\theta \int_0^{2\pi} d\varphi \frac{Y_i(\theta, \varphi) \tilde{Y}_j(\theta, \varphi)}{|r(\theta, \varphi)|^{m+2}}
\]

\[
(-1)^{j(2)} \sum_{m=0}^{i(1)+j(1)} \frac{1}{k^{m+2}} \left( \sum_{l=0}^{m} \frac{1}{h_{ij}(1)l} \right) \sum_{d|d|=m+2, \text{Supp } d \subseteq (N,N)} C^{d^*} d_0^{d^*} d_1^{d_1+e_1+e_j}.
\]

Theorem 1.1 is proved.

Proof for Polyhedrons — Coefficients \( g_{ij} \). From (1.16) we have

\[
\frac{1}{r_i} = a_i \cos \theta + b_i \sin \theta \frac{f_i}{f_i}, \quad i = 1, \ldots, N.
\]
First of all note that

\[ g_{ij} = \sum_{p=1}^{N} g^p_{ij}, \]  
with \( g^p_{ij} := \int_{\theta_{p-1}}^{\theta_p} \Psi_i(r_p(\theta), \theta) \overline{\Psi_j(r_p(\theta), \theta)} \sin \theta d\theta. \)

Develop \( g^p_{ij}, \)

\[ g^p_{ij} = \int_{\theta_{p-1}}^{\theta_p} Y_i(\theta) \overline{Y_j(\theta)} h_i(k r(\theta)) h_j(k |r(\theta)|) \sin \theta d\theta = \]

\[ = \sum_{m=0}^{l+m} \frac{1}{m!} \sum_{l=0}^{m} \hat{h}_{i(l)} \overline{h}_{j(m-l)} \int_{0}^{\pi} \frac{Y_i(\theta, \phi) Y_j(\theta, \phi)}{|r(\theta, \phi)|^{m+2}} d\theta = \]

\[ = \sum_{m=0}^{l+m} \frac{1}{m!} \sum_{l=0}^{m} \hat{h}_{i(l)} \overline{h}_{j(m-l)} \frac{1}{(f_p)_{m+2}} \sum_{l=0}^{m+2} \hat{C}^l_{m+2} \hat{f}_p^{m+2-l} T^p_{ij,lm}, \]

where

\[ T^p_{ij,lm} = \int_{\theta_{p-1}}^{\theta_p} Y_i(\theta) \overline{Y_j(\theta)} \cos^l \theta \sin^{m+2-l} \theta \sin \theta d\theta, \quad l \leq m + 2. \]

Proof of the representation (1.17) follows explicitly from (2.4).

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