COALEScing SYSTEMS OF BROWNian PARTICLES ON THE SIERPINSKI GASKET AND STABLe PARTICLES ON THE LINE OR CIRCLE

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Abstract. A well-known result of Arratia shows that one can make rigorous the notion of starting an independent Brownian motion at every point of an arbitrary closed subset of the real line and then building a set-valued process by requiring particles to coalesce when they collide. Arratia noted that the value of this process will be almost surely a locally finite set at all positive times, and a finite set almost surely if the initial value is compact: the key to both of these facts is the observation that, because of the topology of the real line and the continuity of Brownian sample paths, at the time when two particles collide one or the other of them must have already collided with each particle that was initially between them. We investigate whether such instantaneous coalescence still occurs for coalescing systems of particles where either the state space of the individual particles is not locally homeomorphic to an interval or the sample paths of the individual particles are discontinuous. We show that Arratia’s conclusion is valid for Brownian motions on the Sierpinski gasket and for stable processes on the real line with stable index greater than one.

1. Introduction

A construction due to Richard Arratia [Arr79, Arr81] shows that it is possible to make rigorous sense of the informal notion of starting an independent Brownian motion at each point of the real line and letting particles coalesce when they collide.

Arratia proved that the set of particles remaining at any positive time is locally finite almost surely. Arratia’s argument is based on the simple observation that at the time two particles collide, one or the other must have already collided with each particle that was initially between them. The same argument shows that if we start an independent circular Brownian motion at each point of the circle and let particles coalesce when they collide, then, almost surely, there are only finitely many particles remaining at any positive time.

Arratia established something even stronger: it is possible to construct a flow of random maps \( (F_{s,t})_{s < t} \) from the real line to itself in such a way that for each fixed \( s \) the process \( (F_{s,s+u})_{u \geq 0} \) is given by the above particle system. Arratia’s flow has since been studied by several authors such as [TW98, STW00, SW02, LJR04, Ts04, FINR04, HW09] for purposes as diverse as giving a rigorous definition of a one-dimensional self-repelling Brownian motion to providing examples of noises...
that are, in some sense, completely “orthogonal” to those produced by Poisson processes or Brownian motions.

Coalescing systems of more general Markov processes have been investigated because of their appearance as the duals of models in genetics of the stepping stone type, see, for example, [Kle96, EF96, Eva97, DEF00+00, Zho03, XZ05, HT05, MRTZ06, Zho08].

Arratia’s “topological” argument for instantaneous coalescence to a locally finite set fails when one considers Markov processes on the line or circle with discontinuous sample paths or Markov processes with state spaces that are not locally like the real line. We show, however, that analogous conclusions hold for coalescing Brownian motions on the “finite” and “infinite” (that is, compact and non-compact) Sierpinski gaskets and stable processes on the circle and line – provided, of course, that the stable index is greater than 1, so that an independent pair of such motions collides with positive probability. Similar methods will apply to Markov processes on more general state spaces when the process and the state space have suitable local self-similarity properties, but we have not pursued a result with more encompassing hypotheses. As well as providing an interesting test case of a process with continuous sample paths on a state space that is not locally one-dimensional but is such that two independent copies of the process will collide with positive probability, the Brownian motion on the Sierpinski gasket was introduced as a model for diffusion in disordered media and it has since attracted a considerable amount of attention. The reader can get a feeling for this literature by consulting some of the earlier works such as [BP88, Lin90, Bar98] and more recent papers such as [HK03, KS05] and the references therein.

We give a sketch of the argument for the simplest case of stable processes on the circle in order to motivate some of the estimates that we must develop before we can present proofs in each of the various settings.

Observe from the scaling property of stable processes on the line that if \( X' \) and \( X'' \) are independent copies of a stable processes of index \( \alpha > 1 \) on the unit circle \( T \) starting from two distinct points that are \( 2\pi \varepsilon \) apart, then there exist positive constants \( \beta \) and \( p \) such that the probability the processes will collide by time \( \beta \varepsilon^\alpha \) is bounded below by \( p \).

Suppose we start with \( n + 1 \) stable particles in some configuration on \( T \). By the pigeonhole principle, there will be (at least) one pair of particles that are distance at most \( 2\pi / n \) apart. Therefore, with probability at least \( p \), these two particles in isolation would collide with each other by time \( \beta n^{-\alpha} \). Hence, in the coalescing system the probability that there is at least one collision between some pair of particles within the time interval \( [0, \beta n^{-\alpha}] \) is certainly at least \( p \) (either the two distinguished particles collide with each other or no others or some other particle(s) collides with one or both of the distinguished particles). Moreover, if there is no collision between any pair of particles after time \( \beta n^{-\alpha} \), then we can again find at time \( \beta n^{-\alpha} \) a possibly different pair of particles that are within distance \( 2\pi / n \) from each other, and the probability that this pair of particles will collide within the time interval \( [\beta n^{-\alpha}, 2\beta n^{-\alpha}] \) is again at least \( p \). By repeating this argument and using the Markov property, we see that if we let \( \tau_n^{n+1} \) be the first time there are \( n \) surviving particles starting from \( (n + 1) \) particles, then, regardless of the particular initial configuration of the \( n + 1 \) particles,

\[
P \{ \tau_n^{n+1} \geq k\beta n^{-\alpha} \} \leq (1 - p)^k.
\]
In particular, the expected time needed to reduce the number of particles from \( n + 1 \) to \( n \) is bounded above by \( C n^{-\alpha} \) for a suitable constant \( C \).

Thus, if we start with \( N \) particles somewhere on \( T \), then the probability that after some positive time \( t \) the number of particles remaining is greater than \( m \) is, by Markov’s inequality, bounded above by

\[
\frac{1}{t} \sum_{n=m}^{N-1} \mathbb{E}\left[\tau_{n+1}^n\right] \leq \frac{C}{t} \sum_{n=m}^{N-1} n^{-\alpha} \leq \frac{C'}{t} m^{1-\alpha}
\]

for some constant \( C' \). Letting \( N \to \infty \) and then letting \( m \to \infty \), we conclude that by time \( t \) there are only finitely many particles almost surely.

The above reasoning uses the compactness of \( T \) in a crucial way and it cannot be applied as it stands to deal with, say, coalescing stable processes on the real line. The primary difficulty is that the argument bounds the time to coalesce from some number of particles to a smaller number by considering a particular sequence of coalescent events, and while waiting for such an event to occur the particles might spread out to such an extent that the pigeon hole argument can no longer be applied. We overcome this problem by using a somewhat more sophisticated pigeonhole argument to assign the bulk of the particles to a collection of suitable disjoint pairs (rather than just selecting a single suitable pair) and then employing a simple large deviation bound to ensure that with high probability at least a certain fixed proportion of the pairs will have collided over an appropriate time interval.

2. Countable systems of coalescing Feller processes

In this section we develop some general properties of coalescing systems of Markov processes that we will apply later to Brownian motions on the Sierpinski gasket and stable processes on the line or circle.

2.1. Vector-valued coalescing process. Fix \( N \in \mathbb{N} \cup \{\infty\} \), where, as usual, \( \mathbb{N} \) is the set of positive integers. Write \([N]\) for the set \( \{1, 2, \ldots, N\} \) when \( N \) is finite and for the set \( \mathbb{N} \) when \( N = \infty \).

Fix a locally compact, second-countable, Hausdorff space \( E \). Note that \( E \) is metrizable. Let \( d \) be a metric giving the topology on \( E \). Denote by \( D := D(\mathbb{R}_+, E) \) the usual Skorokhod space of \( E \)-valued c\ ’adl\’ag paths. Fix a bijection \( \sigma : [N] \to [N] \).

We will call \( \sigma \) a ranking of \([N]\). Define a mapping \( \Lambda_\sigma : D_N \to D_N \) by setting \( \Lambda_\sigma \xi = \zeta \) for \( \xi = (\xi_1, \xi_2, \ldots) \in D_N \), where \( \zeta \) is defined inductively as follows. Set \( \zeta_{\sigma(1)}(t) = \zeta_{\sigma(1)}(t) \). For \( i > 1 \), set

\[
\tau_i := \inf \left\{ t \geq 0 : \xi_{\sigma(i)}(t) \in \{\zeta_{\sigma(1)}(t), \ldots, \zeta_{\sigma(i-1)}(t)\} \right\},
\]

with the usual convention that \( \inf \emptyset = \infty \). Put

\[
J_i := \min \left\{ j \in \{1, 2, \ldots, i-1\} : \xi_{\sigma(i)}(\tau_i) = \zeta_{\sigma(j)}(\tau_i) \right\} \quad \text{if} \; \tau_i < \infty.
\]

For \( t \geq 0 \), define

\[
\zeta_{\sigma(i)}(t) := \begin{cases} 
\xi_{\sigma(i)}(t) & \text{if } t < \tau_i \\
\zeta_{\sigma(J_i)}(t) & \text{if } t \geq \tau_i.
\end{cases}
\]

We call the map \( \Lambda_\sigma \) a collision rule. It produces a vector of “coalescing” paths from a vector of “free” paths: after the free paths labeled \( i \) and \( j \) collide, the corresponding coalescing paths both subsequently follow either the path labeled \( i \)
or the path labeled \( j \), according to whether \( \sigma(i) < \sigma(j) \) or \( \sigma(i) > \sigma(j) \). Note for each \( n < N \) that the value of \( (\xi_{\sigma(i)})_{1 \leq i \leq n} \) is unaffected by the value of \( (\xi_{\sigma(i)})_{j > n} \).

Suppose from now on that the paths \( \xi_1, \xi_2, \ldots \) are realizations of independent copies of a Feller Markov process \( X \) with state space \( E \).

A priori, the distribution of the finite or countable coalescing system \( \xi = \Lambda \sigma \xi \) depends on the ranking \( \sigma \). However, we have the following result, which is a consequence of the strong Markov property of \( \xi \) and the observation that if we are given a bijection \( \pi : [N] \to [N] \) and define a map \( \Sigma : D^N \to D^N \) by \( (\Sigma \xi)_i = \xi_{\pi(i)} \), \( i \in [N] \), then \( \Sigma \Lambda \sigma = \Lambda \sigma \pi \Sigma \).

**Lemma 2.1** ([Arr79][Arr81]). The distribution of \( \xi = \Lambda \sigma \xi \) is the same for all bijections \( \sigma : [N] \to [N] \).

From now on, we will, unless we explicitly state otherwise, take \( \sigma = \text{id} \), where \( \text{id} : [N] \to [N] \) is the identity bijection. To simplify notation, we will write \( \Lambda \) for the collision rule \( \Lambda \sigma \).

It is intuitively clear that the coalescing system \( \xi \) is Markov. For the sake of completeness, we establish this formally in the next lemma, the proof of which is essentially an argument from [Arr79][Arr81].

Define the right-continuous filtration \( (\mathcal{F}_t)_{t \geq 0} \) by

\[
\mathcal{F}_t := \bigcap_{\varepsilon > 0} \sigma \{ \xi_i(s) : s \leq t + \varepsilon, i \geq 1 \}.
\]

**Lemma 2.2.** The stochastic process \( \xi = \Lambda \xi \) is strong Markov with respect to the filtration \( (\mathcal{F}_t)_{t \geq 0} \).

**Proof.** Define maps \( m : \{1, 2, \ldots, N\} \times E^N \to \{1, 2, \ldots, N\} \) and \( \Pi : E^N \times E^N \to E^N \) by setting \( m(i, x) := \min \{ j : x_j = x_i \} \) and \( \Pi(x, y)_i := y_{m(i, x)} \). Note that

\[
\Pi(\Lambda \eta(t), \eta(t)) := \Lambda \eta(t), \quad \eta \in D^N, \ t \geq 0.
\]

Define a map \( \tilde{\Pi} : E^N \times D^N \to D^N \) by

\[
\tilde{\Pi}(x, \eta)(t) = \Pi(x, \eta(t)), \quad x \in E^N, \ \eta \in D^N, \ t \geq 0.
\]

Writing \( \{ \theta_s \}_{t \geq 0} \) for the usual family of shift operators on \( D^N \), that is, \( (\theta_s \eta)(t) = \eta(s + t) \), we have

\[
\theta_s \Lambda \eta = \Lambda \tilde{\Pi}(\Lambda \eta(s), \theta_s \eta), \quad \eta \in D^N, \ s \geq 0.
\]

Fix a bounded measurable function on \( f : D^N \to \mathbb{R} \) and set

\[
g(x, y) = \mathbb{E}^x \left[ f \left( \Lambda \tilde{\Pi}(x, \xi) \right) \right].
\]

Note that since the components of \( \xi \) are independent, if \( \Pi(x, y) = x \), then \( g(x, y) = g(x, x) \). Thus, for a finite \( (\mathcal{F}_t)_{t \geq 0} \) stopping time \( S \) we have from the strong Markov property of \( \xi \) that

\[
\mathbb{E}^x \left[ f(\theta_S \Lambda \xi) \mid \mathcal{F}_S \right] = \mathbb{E}^x \left[ f \left( \Lambda \tilde{\Pi}(\Lambda \xi(S), \theta_S \xi(S)) \right) \mid \mathcal{F}_S \right]
\]

\[
= g(\Lambda \xi(S), \xi(S))
\]

\[
= g(\Lambda \xi(S), \Lambda \xi(S))
\]

\[
= \mathbb{E}^x [f(\Lambda \xi)],
\]

as required. \( \square \)
2.2. Set-valued coalescing process. Write $K = K(E)$ for the set of nonempty compact subsets of $E$ equipped with the usual Hausdorff metric $d_H$ defined by

$$d_H(K_1, K_2) := \inf\{\varepsilon > 0 : K_1^\varepsilon \supseteq K_2^\varepsilon \text{ and } K_2^\varepsilon \supseteq K_1^\varepsilon\},$$

where $K^\varepsilon := \{y \in E : \exists x \in K, d(y, x) < \varepsilon\}$. The metric space $(K, d_H)$ is complete. It is compact if $E$ is.

If the locally compact space $E$ is not compact, write $C = C(E)$ for the set of nonempty closed subsets of $E$. Identify the elements of $C$ with their closures in the one-point compactification $\overline{E}$ of $E$. Write $d_C$ for the metric on $C$ that arises from the Hausdorff metric on the compact subsets of $\overline{E}$ corresponding to some metric on $\overline{E}$ that induces the topology of $\overline{E}$.

Let $\Xi \subseteq E$ denote the closure of the set $\{\zeta_i(t) : i = 1, 2, \ldots\}$ in $E$, where $\zeta = \Lambda \xi$.

The following result is an almost immediate consequence of Lemma 2.1.

**Lemma 2.3.** If $\mathbf{x}'$, $\mathbf{x}'' \in E^N$ are such that the sets $\{x_i' : i \in [N]\}$ and $\{x_i'' : i \in [N]\}$ are equal, then the distributions of the process $\Xi$ under $\mathbb{P}^{\mathbf{x}'}$ and $\mathbb{P}^{\mathbf{x}''}$ are also equal.

For the remainder of this section, we will make the following assumption.

**Assumption 2.4.** The Feller process $X$ is such that if $X'$ and $X''$ are two independent copies of $X$, then, for all $t_0 > 0$ and $\mathbf{x}' \in E$,

$$\lim_{\mathbf{x}'' \to \mathbf{x}'} \left[ \mathbb{P}^{\mathbf{x}', \mathbf{x}''} \{X_i' = X_i'' \text{ for some } t \in [0, t_0]\} = 1 \right].$$

**Proposition 2.5.** Let $\mathbf{x}'$, $\mathbf{x}'' \in E^N$ be such that the sets $\{x_i' : i \in [N]\}$ and $\{x_i'' : i \in [N]\}$ have the same closure. Then, the process $\Xi$ has the same distribution under $\mathbb{P}^{\mathbf{x}'}$ and $\mathbb{P}^{\mathbf{x}''}$.

**Proof.** We will consider the case where $E$ is compact. The non-compact case is essentially the same, and we leave the details to the reader.

We need to show for any finite set of times $0 < t_1 < \ldots < t_k$ that the distribution of $(\Xi_{t_1}, \ldots, \Xi_{t_k})$ is the same under $\mathbb{P}^{\mathbf{x}'}$ and $\mathbb{P}^{\mathbf{x}''}$.

We may suppose without loss of generality that $x_1', x_2', \ldots$ (resp. $x_1'', x_2'', \ldots$) are distinct.

Fix $n \in [N]$ and $\delta > 0$. Given $\varepsilon > 0$ that will be specified later, choose $y_i'' \in \{x_i'' : i \in [N]\}$ such that $d(x_i', y_i'') \leq \varepsilon$ for $1 \leq i \leq n$. Let $\mathbf{\eta}'$ (resp. $\mathbf{\eta}''$) be an $E^n$-valued process with coordinates that are independent copies of $X$ started at $(x_1', \ldots, x_n')$ (resp. $(y_1'', y_2'', \ldots, y_n'')$).

By the Feller property, there is a time $0 < t_0 < t_1$ that depends on $x_1', \ldots, x_n'$ such that for all $\varepsilon$ sufficiently small

$$\mathbb{P}\{\eta_i''(t) = \eta_j''(t) \text{ for some } 1 \leq i \neq j \leq n \text{ and } 0 < t \leq t_0\} \leq \frac{\delta}{2}.$$

By our standing Assumption 2.4, if we take $\varepsilon$ sufficiently small, then

$$\mathbb{P}\{\eta_i'(t) \neq \eta_i''(t) \text{ for all } 0 < t \leq t_0\} \leq \frac{\delta}{2n}, \quad 1 \leq i \leq n.$$

Write $\Xi'$ (resp. $\Xi''$, $\hat{\Xi}$, $\hat{\Xi}$) for the set-valued processes constructed from $\mathbf{\eta}'$ (resp. $\mathbf{\eta}''$, $(\mathbf{\eta}'', \mathbf{\eta}'', \mathbf{\eta}'')$, $(\mathbf{\eta}', \mathbf{\eta}'')$) in the same manner that $\Xi$ is constructed from $\xi$. We have

$$\mathbb{P}\{\hat{\Xi}_t = \Xi''_t \text{ for all } t \geq t_0\} \geq 1 - \delta,$$

$$\Xi'_t \subseteq \hat{\Xi}_t, \quad \text{for all } t \geq 0,$$

essentially the same, and we leave the details to the reader.
and, by Lemma 2.3, \( \hat{\Xi} \equiv \hat{\Xi} \).

For each \( z \in E \), define a continuous function \( \phi_z : K \to \mathbb{R}_+ \) by

\[
\phi_z(K) := \inf\{d(z, w) : w \in K\}.
\]

Note that \( K' \subseteq K'' \) implies that \( \phi_z(K') \geq \phi_z(K'') \) for any \( z \in E \). It follows that for points \( z_{\ell p} \in E \), \( 1 \leq p \leq q, 1 \leq \ell \leq k \),

\[
E \left[ \prod_{\ell=1}^{k} \prod_{p=1}^{q} \phi_{z_{\ell p}}(\hat{\Xi}_{t_\ell}) \right] \geq \prod_{\ell=1}^{k} \prod_{p=1}^{q} \phi_{z_{\ell p}}(\hat{\Xi}_{t_\ell})
\]

\[
= \prod_{\ell=1}^{k} \prod_{p=1}^{q} \phi_{z_{\ell p}}(\hat{\Xi}_{t_\ell})
\]

\[
\geq E \left[ \prod_{\ell=1}^{k} \prod_{p=1}^{q} \phi_{z_{\ell p}}(\Xi'_{t_\ell}) \right] - \delta(\sup\{d(z, w) : z, w \in E\}) \prod_{\ell=1}^{k} q_{\ell}
\]

Observe that

\[
E^{x'} \left[ \prod_{\ell=1}^{k} \prod_{p=1}^{q} \phi_{z_{\ell p}}(\Xi_{t_\ell}) \right] = \lim_{n \to \infty} E^{x'} \left[ \prod_{\ell=1}^{k} \prod_{p=1}^{q} \phi_{z_{\ell p}}(\Xi'_{t_\ell}) \right]
\]

and

\[
E \left[ \prod_{\ell=1}^{k} \prod_{p=1}^{q} \phi_{z_{\ell p}}(\Xi''_{t_\ell}) \right] \geq E^{x''} \left[ \prod_{\ell=1}^{k} \prod_{p=1}^{q} \phi_{z_{\ell p}}(\Xi_{t_\ell}) \right].
\]

Since \( \delta \) is arbitrary,

\[
E^{x'} \left[ \prod_{\ell=1}^{k} \prod_{p=1}^{q} \phi_{z_{\ell p}}(\Xi_{t_\ell}) \right] \geq E^{x''} \left[ \prod_{\ell=1}^{k} \prod_{p=1}^{q} \phi_{z_{\ell p}}(\Xi_{t_\ell}) \right].
\]

Moreover, we see from interchanging the roles of \( x' \) and \( x'' \) that the last inequality is actually an equality.

It remains to observe from the Stone-Weierstrass theorem that the algebra of continuous functions generated by the constants and the set \( \{\phi_z : z \in E\} \) is uniformly dense in the space of continuous functions on \( E \). \( \square \)

With Proposition 2.5 in hand, it makes sense to talk about the distribution of the process \( \Xi \) for a given initial state \( \Xi_0 \). The following result follows immediately from Dynkin’s criterion for a function of Markov process to be also Markov.

**Corollary 2.6.** The process \( (\Xi_t)_{t \geq 0} \) is strong Markov with respect to the filtration \( (\mathcal{F}_t)_{t \geq 0} \).

### 3. Brownian motion on the Sierpinski gasket

#### 3.1. Definition and properties of the gasket

Let

\[
G_0 := \{(0,0), (1,0), (1/2, \sqrt{3}/2)\}
\]

be the vertices of the unit triangle in \( \mathbb{R}^2 \) and denote by \( H_0 \) the closed convex hull of \( G_0 \). The **Sierpinski gasket**, which we also call the **finite gasket**, is a fractal subset of the plane that can be constructed via the following Cantor-like cut-out procedure. Let \( \{b_0, b_1, b_2\} \) be the midpoints of three sides of \( H_0 \) and let \( A \) be the interior of
the triangle with vertices \( \{b_0, b_1, b_2\} \). Define \( H_1 := H_0 \setminus A \) so that \( H_1 \) is the union of 3 closed upward facing triangles of side length \( 2^{-1} \). Now repeat this operation on each of the smaller triangles to obtain a set \( H_2 \), consisting of 9 upward facing closed triangles, each of side \( 2^{-2} \). Continuing this fashion, we have a decreasing sequence of closed non-empty sets \( \{H_n\}_{n=0}^{\infty} \) and we define the Sierpinski gasket as

\[
G := \bigcap_{n=0}^{\infty} H_n.
\]

We call each of the \( 3^n \) triangles of side \( 2^{-n} \) that make up \( H_n \) an \( n \)-triangle of \( G \). Denote by \( T_n \) the collection of all \( n \)-triangles of \( G \). Let \( V_n \) be the set of vertices of the \( n \)-triangles.

We call the unbounded set

\[
\tilde{G} := \bigcup_{n=0}^{\infty} 2^n G
\]

the \textit{infinite gasket} (where, as usual, we write \( cB := \{ cx : x \in B \} \) for \( c \in \mathbb{R} \) and \( B \subseteq \mathbb{R}^2 \)) . The concept of \( n \)-triangle, where \( n \) may now be a negative integer, extends in the obvious way to the infinite gasket. Denote the set of all \( n \)-triangles of \( \tilde{G} \) by \( \tilde{T}_n \). Let \( \tilde{V}_n \) be the vertices of \( \tilde{T}_n \).

Given a pathwise connected subset \( A \subseteq \mathbb{R}^2 \), let \( \rho_A \) be the shortest-path metric on \( A \) given by

\[
\rho_A(x,y) := \inf \{ |\gamma| : \gamma \text{ is a path between } x \text{ and } y \text{ and } \gamma \subseteq A \},
\]

where \( |\gamma| \) denote the length (that is, the 1-dimensional Hausdorff measure) of \( \gamma \). For the finite gasket \( G \), \( \rho_G \) is comparable to the usual Euclidean metric \( | \cdot | \) (see, for example, [Bar98] Lemma 2.12)) with the relation,

\[
|x - y| \leq \rho_G(x,y) \leq c|x - y|, \quad \forall x, y \in G,
\]

for a suitable constant \( 1 < c < \infty \). It is obvious that the same is also true for the metric \( \rho_{\tilde{G}} \) on the infinite gasket.

Let \( \mu \) denote the \( d_f \)-dimensional Hausdorff measure on \( \tilde{G} \) where \( d_f := \log 3 / \log 2 \) is the fractal or mass dimension of the gasket. For the finite gasket \( G \) we have \( 0 < \mu(G) < \infty \) and, with a slight abuse of notation, we will also use the notation \( \mu \) to denote the restriction of this measure to \( G \). Moreover, we have the following estimate on the volume growth of \( \mu \)

\[
(3.1) \quad \mu(B(x,r)) \leq Cr^{d_f} \quad \text{for } x \in \tilde{G}, \ 0 < r < 1,
\]

where \( B(x,r) \subseteq \tilde{G} \) is the open ball with center \( x \) and radius \( r \) in the Euclidean metric and \( C \) is a suitable constant (see [BP88]).

### 3.2. Brownian motions

We construct a graph \( G_n \) (respectively, \( \tilde{G}_n \)) embedded in the plane with vertices \( V_n \) (resp. \( \tilde{V}_n \)) by adding edges between pairs of vertices that are distance \( 2^{-n} \) apart from each other. Let \( X^n \) (resp. \( \tilde{X}^n \)) be the natural random walk on \( G_n \) (resp. \( \tilde{G}_n \)); that is, the discrete time Markov chain that at each step chooses uniformly at random from one of the neighbors of the current state. It is known (see [BP88, Bar98]) that the sequence \( (X^n_{\lfloor 5^n t \rfloor})_{t \geq 0} \) (resp. \( (\tilde{X}^n_{\lfloor 5^n t \rfloor})_{t \geq 0} \)) converges in distribution as \( n \to \infty \) to a limiting process \( (X_t)_{t \geq 0} \) (resp. \( (\tilde{X}_t)_{t \geq 0} \)) that is a \( G \)-valued (resp. \( \tilde{G} \)-valued) strong Markov process (indeed, a Feller process) with continuous sample paths. The processes \( X \) and \( \tilde{X} \) are called, for obvious
reasons, the Brownian motion on the finite and infinite gaskets, respectively. The Brownian motion on the infinite gasket has the following scaling property:

\[(2\tilde{X}_t)_{t \geq 0} \text{ under } \mathbb{P}^x \text{ has same law as } (\tilde{X}_{5t})_{t \geq 0} \text{ under } \mathbb{P}^{2x}.\]

The process \(\tilde{X}\) has a family \(\tilde{p}(t, x, y), x, y \in \tilde{G}, t > 0,\) of transition densities with respect to the measure \(\mu\) that is jointly continuous on \((0, \infty) \times \tilde{G} \times \tilde{G}\). Moreover, \(\tilde{p}(t, x, y) = \tilde{p}(t, y, x)\) for all \(x, y \in \tilde{G}\) and \(t > 0\), so that the process \(\tilde{X}\) is symmetric with respect to \(\mu\).

Let \(d_w := \log \frac{5}{\log 2}\) denote the walk dimension of the gasket. The following crucial “heat kernel bound” is established in [BPSS8]

\[(3.3) \quad \tilde{p}(t, x, y) \leq c_1 t^{-d_t/d_w} \exp \left( -c_2 \left( \frac{|x-y|^{d_w}}{t} \right)^{1/(d_w-1)} \right).\]

Because the infinite gasket \(\tilde{G}\) and the associated Brownian motion \(\tilde{X}\) both have re-scaling invariances that \(G\) and \(X\) do not, it will be convenient to work with \(\tilde{X}\) and then use the following observation to transfer our results to \(X\).

**Lemma 3.1** (Folding lemma). There exists a continuous mapping \(\psi: \tilde{G} \to G\) such that \(\psi\) restricted to \(G\) is the identity, \(\psi\) restricted to any 0-triangle is an isometry, and \(|\psi(x) - \psi(y)| \leq |x-y|\) for arbitrary \(x, y \in \tilde{G}\). Moreover, if the \(\tilde{G}\)-valued process \(\tilde{X}\) is started at an arbitrary \(x \in \tilde{G}\), then the \(G\)-valued process \(\psi \circ \tilde{X}\) has the same distribution the process \(X\) started at \(\psi(x)\).

**Proof.** Let \(L\) be the subset of the plane formed by the set of points of the form \(n_1(1, 0) + n_2(1/2, \sqrt{3}/2)\), where \(n_1, n_2\) are non-negative integers, and the line segments that join such points that are distance 1 apart. It is easy to see that there is a unique labeling of the vertices of \(L\) by \(\{1, \omega, \omega^2\}\) that has the following properties.

- **Label** \((0, 0)\) with 1.
- **If** vertex \(v\) is labeled \(a \in \{1, \omega, \omega^2\}\), then the vertex \(v + (1, 0)\) are labeled with \(a\omega^2\).

Indeed, the label of the vertex \(n_1(1, 0) + n_2(1/2, \sqrt{3}/2)\) is \(\omega^{n_1+2n_2}\).

Given a vertex \(v \in L\), let \(\iota(v)\) be the unique vertex in \(\{(0, 0), (1, 0), (1/2, \sqrt{3}/2)\}\) that has the same label as \(v\). If the vertices \(v_1, v_2, v_3 \in L\) are the vertices of a triangle with side length 1, then \(\iota(v_1), \iota(v_2), \iota(v_3)\) are all distinct.

With the above preparation, let us now define the map \(\psi\). Given \(x \in \tilde{G}\), let \(\Delta \in \tilde{T}_0\) be a triangle with vertices \(v_1, v_2, v_3\) that contains \(x\) (if \(x\) belongs to \(\tilde{V}_n\), then there may be more than one such triangle, but the choice will not matter). We may write \(x\) as a unique convex combination of the vertices \(v_1, v_2, v_3,\)

\[x = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3, \quad \sum_{i=1}^{3} \lambda_i = 1, \lambda_i \geq 0.\]

The triple \((\lambda_1, \lambda_2, \lambda_3)\) is the vector of barycentric coordinates of \(x\). We define \(\psi(x)\) by

\[\psi(x) := \lambda_1 \iota(v_1) + \lambda_2 \iota(v_2) + \lambda_3 \iota(v_3).\]

It is clear that \(\psi: \tilde{G} \to G\) is well-defined and has the stated properties.
Lemma 3.2 (Maximal inequality). (a) Let \( \tilde{X}^i, 1 \leq i \leq n \), be \( n \) independent Brownian motions on the infinite gasket \( \tilde{G} \) starting from the initial states \( x^i, 1 \leq i \leq n \). For any \( t > 0 \),

\[
\mathbb{P} \left\{ \sup_{0 \leq s \leq t} |\tilde{X}^i_s - x^i| > r, \text{ for some } 1 \leq i \leq n \right\} \leq 2nc_1 \exp \left( -c_2 (r^{d_w} / t)^{1/(d_w - 1)} \right),
\]

where \( c_1, c_2 > 0 \) are constants and \( d_w = \log 5 / \log 2 \) is the walk dimension of the gasket.

(b) The same estimate holds for the case of \( n \) independent Brownian motions \( X^i, 1 \leq i \leq n \), on the finite gasket \( G \) starting from the initial states \( x^i, 1 \leq i \leq n \).

Proof. (a) Let \( \tilde{X} = (\tilde{X}_t)_{t \geq 0} \) be a Brownian motion on \( \tilde{G} \). Then for \( x \in \tilde{G}, t > 0 \), and \( r > 0 \),

\[
\mathbb{P}^x \left\{ \sup_{0 \leq s \leq t} |\tilde{X}_s - x| > r \right\} \leq \mathbb{P}^x \{|\tilde{X}_t - x| > r / 2\} \]

\[
+ \mathbb{P}^x \{|\tilde{X}_t - x| \leq r / 2, \sup_{0 \leq s \leq t} |\tilde{X}_s - x| > r \}.
\]

Writing \( S := \inf\{s > 0 : |\tilde{X}_s - x| > r\} \), the second term above equals

\[
\mathbb{E}^x \left[ 1_{\{S < t\}} \mathbb{P}^x \{|\tilde{X}_{t-s} - x| \leq r / 2\} \right] \leq \sup_{y \in \partial B(x,r)} \sup_{s \leq t} \mathbb{P}^y \{|\tilde{X}_{t-s} - y| > r / 2\},
\]

so that

\[
\mathbb{P}^x \left\{ \sup_{0 \leq s \leq t} |\tilde{X}_s - x| > r \right\} \leq 2 \sup_{y \in \tilde{G}} \sup_{s \leq t} \mathbb{P}^y \{|\tilde{X}_s - y| > r / 2\}
\]

\[
\leq 2c_1 \exp \left( -c_2 (r^{d_w} / t)^{1/(d_w - 1)} \right),
\]

where the last estimate is taken from \cite{Bar98} Theorem 2.23(e)]. The lemma now follows by a union bound.

(b) This is immediate from part (a) and Lemma 3.1

3.3. Collision time estimates. We first show that two independent copies of \( \tilde{X} \) collide with positive probability.

Proposition 3.3. Let \( \tilde{X}' \) and \( \tilde{X}'' \) be two independent copies of \( \tilde{X} \). Then,

\[
\mathbb{P}^{(x',x'')}\{ \exists t > 0 : \tilde{X}'_t = \tilde{X}''_t \} > 0
\]

for all \((x', x'') \in \tilde{G} \times \tilde{G}\).

Proof. Note that \( \tilde{X} = (\tilde{X}', \tilde{X}'') \) is a Feller process on the locally compact separable metric space \( \tilde{G} \times \tilde{G} \) that is symmetric with respect to the Radon measure \( \mu \otimes \mu \) and has transition densities \( \tilde{p}(t, x', y') \times \tilde{p}(t, x'', y'') \). The corresponding \( \alpha \)-potential density is

\[
u_{\alpha}(x,y) := \int_{0}^{\infty} e^{-\alpha t} \tilde{p}(t, x_1, y_1) \times \tilde{p}(t, x_2, y_2) dt \quad \text{for } \alpha > 0,
\]
where \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \). A standard potential theoretic result says that a compact set \( B \subseteq \tilde{G} \times \tilde{G} \) is non-polar if there exists a non-zero finite measure \( \nu \) that is supported on \( B \) and has finite energy, that is,

\[
\int \int u^\alpha(x, y) \nu(dx) \nu(dy) < \infty.
\]

Take \( B = \{(x', x'') \in G \times G : x' = x''\} \) and \( \nu \) to be the ‘lifting’ of the Hausdorff measure \( \mu \) on the finite gasket onto \( B \). We want to show that

\[
\int_G \int_G \int_0^\infty e^{-at} \tilde{p}^2(t, x, y) \mu(dx) \mu(dy) < \infty.
\]

It will be enough to show that

\[
\int_G \int_G \int_0^\infty \tilde{p}^2(t, x, y) \mu(dx) \mu(dy) < \infty.
\]

It follows from the transition density estimate (3.3) and Lemma 3.4 below that

\[
\int_0^\infty \tilde{p}^2(t, x, y) dt \leq C |x - y|^{-\gamma}
\]

for some constant \( C \), where \( \gamma := 2d_f - d_w \). Thus,

\[
\int_G \int_G \int_0^\infty \tilde{p}^2(t, x, y) dt \mu(dx) \mu(dy)
\]

\[
\leq C \int_G \int_G |x - y|^{-\gamma} \mu(dx) \mu(dy)
\]

\[
\leq C \int_G \int_0^\infty |x - y|^{-s} \mu(dx) ds \mu(dy)
\]

\[
\leq C + C_1 \int_G \int_1^\infty s^{-d_f/\gamma} ds \mu(dy) \quad \text{[By (3.1)]}
\]

\[
\leq C + C_2 \int_1^\infty s^{-d_f/\gamma} ds.
\]

It remains to note that \( \gamma - d_f = (2 \log 3/\log 2 - \log 5/\log 2 - (\log 3/\log 2) = (\log 3 - \log 5)/\log 2 < 0 \), and so \( d_f/\gamma < 1 \).

This shows that \( \mathbb{P}^{(x', x'')}\{X \text{ hits the diagonal} \} > 0 \) for some \( (x', x'') \in \tilde{G} \times \tilde{G} \). Because \( \tilde{p}^2(t, x, y) > 0 \) for all \( x, y \in \tilde{G} \) and \( t > 0 \), we even have \( \mathbb{P}^{(x', x'')}\{X \text{ hits the diagonal} \} > 0 \) for all \( (x', x'') \in \tilde{G} \times \tilde{G} \). \( \square \)

We needed the following elementary result in the proof of Proposition 3.3.

**Lemma 3.4.** For \( \alpha > 1, \beta > 0 \) and \( A > 0 \),

\[
\int_0^\infty t^{-\alpha} \exp(-A/t^\beta) dt = \Gamma\left(\frac{\alpha - 1}{\beta}\right) \beta^{-1} A^{-\frac{\alpha - 1}{\beta}} < \infty.
\]

**Proof.** Make the change of variables \( u = At^{-\beta} \) in the integration. \( \square \)
We next establish a uniform lower bound on the collision probability of a pair of independent Brownian motions on the infinite gasket as long as the distance between their starting points remains bounded.

**Theorem 3.5.** There exist constants $\beta > 0$ and $\underline{p} > 0$ such that if $\tilde{X}'$ and $\tilde{X}''$ are two independent Brownian motions on $\tilde{G}$ starting from any two points $x, y$ belonging to the same $n$-triangle of $\tilde{G}$, then

$$\mathbb{P}^{(x,y)}\{\tilde{X}'_t = \tilde{X}''_t \text{ for some } t \in (0, \beta 5^{-n})\} \geq \underline{p}.$$  

This result will require a certain amount of work, so we first note that it leads easily to an analogous result for the finite gasket.

**Corollary 3.6.** If $X'$ and $X''$ are two independent Brownian motions on $G$ starting from any two points $x, y$ belonging to the same $n$-triangle of $G$, then

$$\mathbb{P}^{(x,y)}\{X'_t = X''_t \text{ for some } t \in (0, \beta 5^{-n})\} \geq \underline{p},$$

where $\beta > 0$ and $\underline{p} > 0$ are the constants given in Theorem 3.5.

**Proof.** The proof follows immediately from Lemma 3.1, because if $\tilde{X}'_t = \tilde{X}''_t$ for some $t$, then it is certainly the case that $\psi \circ \tilde{X}'_t = \psi \circ \tilde{X}''_t$. □

**Definition 3.7.** (Extended triangles for the infinite gasket). Recall that $\tilde{T}_n$ is the set of all $n$-triangles of $\tilde{G}$. Given $\Delta \in \tilde{T}_0$ such that $\Delta$ does not have the origin as one of its vertices, we define the corresponding extended triangle $\Delta^e \subset \tilde{G}$ as the interior of the union of the original 0-triangle $\Delta$ with the three neighboring 1-triangles in $\tilde{G}$ which share one vertex with $\Delta$ and are not contained in $\Delta$. Note that for the (unique) triangle $\Delta$ in $\tilde{T}_n$ having the origin as one of its vertices, there are two neighboring 1-triangles in $\tilde{G}$ that share one vertex with it which are not contained in $\Delta$. In this case, by $\Delta^e$, we mean the interior of the union of $\Delta$ and these two triangles.

Fix some $\Delta \in \tilde{T}_0$. Let $\tilde{Z}$ be the Brownian motion on $\Delta^e$ killed when it exits $\Delta^e$. It follows from arguments similar to those on page 590 in [Doo01], that $\tilde{Z}$ has transition densities $\tilde{p}_K(t, x, y)$, $t > 0$, $x, y \in \Delta^e$, with respect to the restriction of $\mu$ to $\Delta^e$, and these densities have the following properties:

- $\tilde{p}_K(t, x, y) = \tilde{p}_K(t, y, x)$ for all $t > 0$, $x, y \in \Delta^e$.
- $\tilde{p}_K(t, x, y) \leq \tilde{p}(t, x, y)$, for all $t > 0$, $x, y \in \Delta^e$.
- $y \mapsto \tilde{p}_K(t, x, y)$ is continuous for all $t > 0$, $x \in \Delta^e$, and $x \mapsto \tilde{p}_K(t, x, y)$ is continuous for all $t > 0$, $y \in \Delta^e$.

It follows that the process $\tilde{Z}$ is Feller and symmetric with respect to the measure $\mu$.

**Lemma 3.8.** Let $\tilde{Z}', \tilde{Z}''$ be two independent copies of the killed Brownian motion $\tilde{Z}$. Given any $\epsilon > 0$, there exists $0 < \delta < \epsilon$ such that the set of $(x, y) \in \Delta^e \times \Delta^e$ for which

$$\mathbb{P}^{(x,y)}\{\tilde{Z}'_t = \tilde{Z}''_t \text{ for some } t \in (\delta, \epsilon)\} > 0$$

has positive $\mu \otimes \mu$ mass.

**Proof.** An argument similar to that in the proof of Proposition 3.3 shows that

$$\mathbb{P}^{(x_0,y_0)}\{\tilde{Z}'_t = \tilde{Z}''_t \text{ for some } t > 0\} > 0$$
for some \((x_0, y_0) \in \Delta^c \times \Delta^c\).

Thus, for any \(\epsilon > 0\), we can partition the interval \((0, \infty)\) into the subintervals \((0, \epsilon), [i\epsilon, (i+1)\epsilon), i \geq 0\) and use the Markov property to deduce that there exists a point \((x_1, y_1) \in \Delta^c \times \Delta^c\) such that

\[
P^{(x_1, y_1)} \{\tilde{Z}_t' = \tilde{Z}_t'' \text{ for some } t \in (0, \epsilon)\} > 0.
\]

By continuity of probability, we can find \(0 < \eta < \epsilon < \infty\) such that

\[
P^{(x_1, y_1)} \{\tilde{Z}_t' = \tilde{Z}_t'' \text{ for some } t \in (\eta, \epsilon)\} > 0.
\]

By the Markov property,

\[
0 < P^{(x_1, y_1)} \{\tilde{Z}_t' = \tilde{Z}_t'' \text{ for some } t \in (\eta, \epsilon)\}
= \int_{\Delta^c} \int_{\Delta^c} \tilde{p}_K(\eta/2, x_1, x)\tilde{p}_K(\eta/2, y_1, y)
\times P^{(x, y)} \{\tilde{Z}_t' = \tilde{Z}_t'' \text{ for some } t \in (\eta/2, \epsilon - \eta/2)\} \mu(dx) \mu(dy).
\]

Therefore, the initial points \((x, y) \in \Delta^c \times \Delta^c\) for which the probability

\[
P^{(x, y)} \{\tilde{Z}_t' = \tilde{Z}_t'' \text{ for some } t \in (\eta/2, \epsilon - \eta/2)\}
\]

is positive form a set with positive \(\mu \otimes \mu\) measure. The proof now follows by taking \(\delta = \eta/2\).

We record the following result for the reader’s ease of reference.

**Lemma 3.9** (Lemma 3.35 of [Bar98]). There exists a constant \(c_1 > 1\) such that if \(x, y \in \Delta^c, r = |x - y|\), then

\[
P^x \{\tilde{X}_t = y \text{ for some } t \in (0, r^{d_w}) \text{ and } |\tilde{X}_t - x| \leq c_1 r \text{ for all } t \leq r^{d_w}\} > 0.
\]

**Lemma 3.10.** There exists a constant \(c > 0\) such that for each point \(x \in \Delta\), each open subset \(U \subset \Delta^c\), and each time \(0 < t \leq c\)

\[
P^x \{\tilde{Z}_t \in U\} > 0.
\]

In particular, \(\tilde{p}_K(t, x, y) > 0\) for all \(x, y \in \Delta^c\) and \(0 < t \leq c\).

**Proof.** The following three steps combined with the strong Markov property establish the lemma.

**Step 1.** There exists a constant \(c > 0\) such that starting from \(x \in \Delta^c\), the unskilled Brownian motion on the infinite gasket \(\tilde{X}\) will stay within \(\Delta^c\) up to time \(c\) with positive probability.

**Step 2.** Fix \(y \in U\). For all sufficiently small \(\eta > 0\),

\[
P^y \{\tilde{X} \text{ does not exit } U \text{ before time } \eta\} > 0.
\]

**Step 3.** For any \(\delta > 0, z, y \in \Delta^c\)

\[
P^z \{\tilde{Z} \text{ hits } y \text{ before } \delta\} > 0.
\]

Consider Step 1. Note that if \(x \in G\), then (see [Bar98, Equation 3.11]) there exists a constant \(c > 0\) such that for the unskilled process \(\tilde{X}\), we have,

\[
P^x \{|\tilde{X}_t - x| \leq 1/4 \text{ for } t \in [0, c]\} > 0.
\]

But if \(x \in \Delta^c\), then

\[
P^x \{\tilde{X}_t \in \Delta^c \text{ for } t \in [0, c]\} \geq P^x \{|\tilde{X}_t - x| \leq 1/4 \text{ for } t \in [0, c]\},
\]
and the claim follows.

Step 2 is obvious from the right continuity of the paths of the killed Brownian motion \( \tilde{Z} \) at time 0.

Consider Step 3. Fix \( z, y \in \Delta^c \) and \( 0 < \delta \leq |z - y| \). Let \( S_n \) be the \( n \)-th approximating graph of \( G \) with the set of vertices \( \mathcal{V}_n \). Choose \( n \) large enough so that we can find points \( z_0 \) and \( y_0 \) in \( \mathcal{V}_n \), close to \( z \) and \( y \) respectively so that

\[
|z - z_0| \leq \frac{\delta}{3}, \quad |y - y_0| \leq \frac{\delta}{3}
\]

and

\[
B(z, c_1|z - z_0|) \subseteq \Delta^c, \quad B(y_0, c_1|y - y_0|) \subseteq \Delta^c
\]

where \( c_1 \) is as in Lemma 3.9 and the notation \( B(u, r) \) denotes the intersection with the infinite gasket \( G \) of the closed ball in the plane of radius \( r \) around the point \( u \).

The length of a shortest path \( \gamma \) lying \( S_n \) between \( z_0 \) and \( y_0 \) is the same as their distance in the original metric \( \rho_G(z_0, y_0) \). Moreover, for any two points \( p \) and \( p' \) on \( \gamma \), the length of the segment of \( \gamma \) between \( p \) and \( p' \) is the same as their distance in the original metric \( \rho_G(p, p') \).

Thus, we can choose \( m + 1 \) equally spaced points \( z_0, z_1, \ldots, z_m = y_0 \) on \( \gamma \) such that

\[
\rho_G(z_{i+1}, z_i) = \frac{1}{m} \rho_G(z_0, y_0) \quad \text{for each } i.
\]

Since \( \gamma \) is compact, \( \text{dist}(\gamma, \partial \Delta^c) > 0 \). Thus we can choose \( m \) large so that

\[
B(z_i, c_1|z_{i+1} - z_i|) \subseteq \Delta^c \quad \text{for each } i.
\]

By repeated application of Lemma 3.9 and the strong Markov property, we conclude that the probability that \( \tilde{Z} \) hits \( y \) starting from \( z \) before the time

\[
T_m := |z - z_0|^d + |y - y_0|^d + \sum_{i=0}^{m-1} |z_{i+1} - z_i|^d
\]

is strictly positive. Step 3 follows immediately since

\[
T_m \leq \left( \frac{\delta}{3} \right)^d + \left( \frac{\delta}{3} \right)^d + \text{constant} \times m \times \frac{1}{m^d} |z_0 - y_0|^d \leq \delta
\]

for \( m \) sufficiently large, because \( d_w > 1 \). \( \square \)

Lemma 3.11. Let \( \tilde{Z}' \) and \( \tilde{Z}'' \) be two independent copies of the killed Brownian motion \( \tilde{Z} \). For any \( 0 < \delta < \beta \), the map

\[
(x, y) \mapsto \mathbb{P}^{(x,y)}(\tilde{Z}'_t = \tilde{Z}''_t \text{ for some } t \in (\delta, \beta)]
\]

is continuous on \( \Delta^c \times \Delta^c \).

Proof. We have

\[
\mathbb{P}^{(x,y)}(\tilde{Z}'_t = \tilde{Z}''_t \text{ for some } t \in (\delta, \beta)]
\]

\[
= \int_{\Delta^c} \int_{\Delta^c} \mathbb{p}_K(\delta, x, x') \mathbb{p}_K(\delta, y, y')
\]

\[
\times \mathbb{P}^{(x', y')}(\tilde{Z}'_t = \tilde{Z}''_t \text{ for some } t \in (0, \beta - \delta)] \mu(dx') \mu(dy'),
\]

and the result follows from the continuity of \( z \mapsto \mathbb{p}_K(\delta, z, z') \) for each \( z' \in \Delta^c \). \( \square \)
Proof of Theorem 3.5. For any \( x, y \in \Delta \),

\[
\mathbb{P}^{(x,y)} \{ \tilde{Z}_t' = \tilde{Z}_t'' \text{ for some } t \in (\delta, \beta) \} = \int_{\Delta} \int_{\Delta} K(\delta/2, x, x') K(\delta/2, y, y') \times \mathbb{P}^{(x', y')}(\tilde{Z}_t' = \tilde{Z}_t'' \text{ for some } t \in (\delta/2, \beta - \delta/2)) \mu(dx') \mu(dy') > 0,
\]

by Lemmas 3.8, 3.10, and 3.11.

Applying Lemma 3.11 and equation (3.5) and the fact that a continuous function achieves its minimum on a compact set, we have for any \( \Delta \in \tilde{T}_0 \) that

\[
q(\Delta) := \inf_{x, y \in \Delta} \mathbb{P}^{(x,y)} \{ \tilde{Z}_t' = \tilde{Z}_t'' \text{ for some } t \in (0, \beta) \} > 0.
\]

Note that for any two \( \Delta_1, \Delta_2 \in \tilde{T}_0 \) which do not contain the origin, there exists a local isometry between the corresponding extended triangles \( \tilde{\Delta}_1, \tilde{\Delta}_2 \). Since the unskilled Brownian motion \( \tilde{X} \) in \( \tilde{G} \) is invariant with respect to local isometries,

\[
q(\Delta_1) = q(\Delta_2).
\]

Given two independent copies \( \tilde{X}' \) and \( \tilde{X}'' \) of \( \tilde{X} \), set

\[
p := \inf_{\Delta \in \tilde{T}_0} \inf_{x, y \in \Delta} \mathbb{P}^{(x,y)} \{ \tilde{X}'_t = \tilde{X}''_t \text{ for some } t \in (0, \beta) \}.
\]

The above observations enable us to conclude that \( p > 0 \).

For the infinite gasket, if \( \Delta \in \tilde{T}_n \), then \( 2^n \Delta \in \tilde{T}_0 \) and the scaling property of Brownian motion on the infinite gasket gives us that for any \( \Delta \in \tilde{T}_n \)

\[
\inf_{x, y \in \Delta} \mathbb{P}^{(x,y)} \{ \tilde{X}'_t = \tilde{X}''_t \text{ for some } t \in (0, 5^{-n} \beta) \} = \inf_{x, y \in 2^n \Delta} \mathbb{P}^{(x,y)} \{ \tilde{X}'_t = \tilde{X}''_t \text{ for some } t \in (0, \beta) \}.
\]

Therefore, for any \( \Delta \in \tilde{T}_n \) and any \( x, y \in \Delta \),

\[
\mathbb{P}^{(x,y)} \{ \tilde{X}'_t = \tilde{X}''_t \text{ for some } t \in (0, 5^{-n} \beta) \} \geq p.
\]

Corollary 3.12. The Brownian motions \( \tilde{X} \) and \( X \) on the infinite and finite gaskets both satisfy Assumption 2.4.

Proof. By Theorem 3.5 and the Blumenthal zero-one law, we have for two independent Brownian motions \( \tilde{X}' \) and \( \tilde{X}'' \) on \( \tilde{G} \) and any point \( (x, x) \in \tilde{G} \times \tilde{G} \) that

\[
\mathbb{P}^{(x,x)} \{ \text{there exists } 0 < t < \epsilon \text{ such that } \tilde{X}'_t = \tilde{X}''_t \text{ for all } \epsilon > 0 \} = 1.
\]

Lemma 3.11 then gives the claim for \( \tilde{X} \). The proof for \( X \) is similar. \( \square \)

4. Instantaneous coalescence on the gasket

We will establish the following three results in this section after obtaining some preliminary estimates.
Theorem 4.1 (Instantaneous Coalescence). (a) Let \( \Xi \) be the set-valued coalescing Brownian motion process on \( \tilde{G} \) with \( \Xi_0 \) compact. Almost surely, \( \Xi_t \) is a finite set for all \( t > 0 \).

(b) The conclusion of part (a) also holds for the set-valued coalescing Brownian motion process on \( G \).

Theorem 4.2 (Continuity at time zero). (a) Let \( \Xi \) be the set-valued coalescing Brownian motion process on \( \tilde{G} \) with \( \Xi_0 \) compact. Almost surely, \( \Xi_t \) converges to \( \Xi_0 \) as \( t \downarrow 0 \).

(b) The conclusion of part (a) also holds for the set-valued coalescing Brownian motion process on \( G \).

Theorem 4.3 (Instantaneous local finiteness). Let \( \Xi \) be the set-valued coalescing Brownian motion process on \( \tilde{G} \) with \( \Xi_0 \) a possibly unbounded closed set. Almost surely, \( \Xi_t \) is a locally finite set for all \( t > 0 \).

Lemma 4.4 (Pigeon hole principle). Place \( M \) balls in \( m \) boxes and allow any two balls to be paired off together if they belong to the same box. Then, the maximum number of disjoint pairs of balls possible is at least \( (M - m)/2 \).

Proof. Note that in an optimal pairing there can be at most one unpaired ball per box. It follows that the number of paired balls is at least \( M - m \) and hence the number of pairs is at least \( (M - m)/2 \). \( \square \)

Define the \( \varepsilon \)-fattening of a set \( A \subseteq \tilde{G} \) to be the set \( A^\varepsilon := \{ y \in \tilde{G} : \exists x \in A, |y - x| < \varepsilon \} \). Define the \( \varepsilon \)-fattening of a set \( A \subseteq G \) in \( G \) similarly. Recall the constants \( \beta \) and \( \beta \) from Theorem 3.5. Set \( \gamma := 1/(1 - p/5) > 1 \). Given a finite subset \( A \) of \( \tilde{G} \) or \( G \) and a time-interval \( I \subseteq \mathbb{R}_+ \), define the random variable \( R(A; I) \) to be the range of the set-valued coalescing process \( \Xi \) in the finite or the infinite gasket during time \( I \) with initial state \( A \); that is, \( R(A; I) := \bigcup_{s \in I} \Xi_s \).

Define a stopping time for the same process \( \Xi \) by \( \tau^A_m := \inf \{ t : \#\Xi_t \leq m \} \).

Lemma 4.5. (a) Let \( \Xi \) be the set-valued coalescing Brownian motion process in the infinite gasket with \( \Xi_0 = A \), where \( A \subseteq \tilde{G} \) of cardinality \( n \) such that \( A^\varepsilon \) for some \( \varepsilon > 0 \) is contained in an extended triangle \( \Delta^\varepsilon \) of \( G \). Then, there exist constants \( C_1 \) and \( C_2 \) which may depend on \( \varepsilon \) but are independent of \( A \) such that

\[
P \left\{ \tau^A_{\lceil n^{\gamma-1} \rceil} > 253n^{-1/3} \text{or } R(A; [0, \tau^A_{\lceil n^{\gamma-1} \rceil}]) \not\subseteq A^{\varepsilon n^{-1/3}} \right\} \leq C_1 \exp(-C_2 n^{1/3}).
\]

(4.1)

(b) The same inequality holds for the set-valued coalescing Brownian motion process in the finite gasket.

Proof. (a) For any integer \( b \geq 1 \), the set \( A \) can be covered by at most \( 2 \times 3^b \) \( b \)-triangles. Put

\[ b_n := \max \{ b : 2 \times 3^b \leq n/2 \}, \]

or, equivalently,

\[ b_n = \lceil \log_3 (n/4) \rceil. \]
By Lemma 4.4, at time $t = 0$ it is possible to form at least $n/2 - n/4 = n/4$ disjoint pairs of particles, where two particles are deemed eligible to form a pair if they belong to the same $b_n$-triangle. Fix such an (incomplete) pairing of particles. Define a new “partial” coalescing system involving $n$ particles, where a particle is only allowed to coalesce with the one it has been paired up with and after such a coalescence occurs the two partners in the pair both follow the path of the particle having the lower rank among the two. The number of surviving distinct particles in this partial coalescing system is at least as great as the number of surviving particles in the original coalescing system.

By Theorem 3.3, the probability that at least $n/4$ independent Bernoulli trials with common success probability $p$ are independent of the other pairs. Thus, the number of coalescence by time $t_n$ in the partial coalescing system stochastically dominates a random variable that is distributed as the number of successes in $n/4$ independent Bernoulli trials with common success probability $p$. By Hoeffding’s inequality, the probability that a random variable with the latter distribution takes a value $np/5$ or greater is at least $1 - e^{-C_1 n}$ for some constant $C_1 > 0$. Thus, the probability that the number of surviving particles in the original coalescing system drops below $[(1 - p/5)n] = [n\gamma^{-1}]$ by time $t_n \leq 25\beta n^{-\log_3 5}$ is at least $1 - e^{-C_1 n}$.

From Lemma 3.2(a) and the fact that during a fixed time interval the maximum displacement of independent particles starting from the same initial configuration, the probability that over a time interval of length $25\beta n^{-\log_3 5}$ of the coalescing particles has moved more than a distance $\varepsilon n^{-(1/6)\log_3 5}$ from its original position is bounded by

$$2n c_1 \exp \left( - c_2 ((\varepsilon n^{-(1/6)\log_3 5})^{d_w} (25\beta n^{-\log_3 5})^{-1})^{1/(d_w - 1)} \right)$$

$$\leq 2 \exp \left( \log n - C_2 (n^{1/2})^{1/(d_w - 1)} \right)$$

$$\leq C_1 \exp (-C_2 n^{1/3}).$$

(b) The proof is identical to part (a). It uses Corollary 3.6 in place of Theorem 3.3 and Lemma 3.2(b) in place of Lemma 3.2(a).

Lemma 4.6. (a) Let $\Xi$ be the set-valued coalescing Brownian motion process in the infinite gasket with $\Xi_0 = A$. Fix $\varepsilon > 0$. Set $\nu_i := \varepsilon \gamma^{-i(1/6)\log_3 5\times 4}$ and $\eta_i = 25\beta \gamma^{-i\log_3 5}$ for $i \geq 1$. There are positive constants $C_1 = C_1(\varepsilon)$ and $C_2 = C_2(\varepsilon)$ such that

$$\mathbb{P} \left\{ \tau_{\gamma^k} > \sum_{i=k+1}^m \eta_i \text{ or } R(A; [0, \tau_{\gamma^k}^A]) \not\subseteq (A) \sum_{i=k+1}^m \nu_i \right\} \leq \sum_{i=k+1}^m C_1 \exp (-C_2 \gamma^{1/3}),$$

uniformly for all sets $A$ of cardinality $[\gamma^m]$ such that the fattening $A \sum_{i=k+1}^m \nu_i$ is contained in some extended triangle $\Delta^c$ of $G$.

(b) The analogous inequality holds for the set-valued coalescing Brownian motion process in the finite gasket.

Proof. Fix an extended triangle $\Delta^c$ of the infinite gasket and a set $A$ such that $\# A = [\gamma^m]$ and $A \sum_{i=k+1}^m \nu_i \subseteq \Delta^c$. We will prove the bound by induction on
m. By the strong Markov property and Lemma 4.1, we have, using the notation
\[ A_{\tau,m-1} := \Xi_{\tau,\gamma_{m-1}^A}, \]

\[ P \left\{ \tau_A^{\gamma_{m-1}} > \sum_{i=k+1}^{m} \eta_i \lor R(A;\eta_{m-1},A_{\tau,\gamma_{m-1}^A}) \subseteq (A)_{\sum_{i=k+1}^{m} \nu_i} \right\} \]

\[ \leq P \left\{ \tau_A^{\gamma_{m-1}} > \eta_m \lor R(A;\eta_{m-1},A_{\tau,\gamma_{m-1}^A}) \subseteq A^{\nu_m} \right\} \]

\[ + E \left\{ \mathbf{1} \left\{ A_{\tau,m-1} \subseteq A^{\nu_m} \right\} \right\} \times P \left\{ \tau_{A^{\nu_m}}^{\gamma_{m-1}} > \sum_{i=k+1}^{m-1} \eta_i \lor R(A_{\tau,m-1};\eta_{m-1},A_{\tau,\gamma_{m-1}^A}) \subseteq A_{\sum_{i=k+1}^{m-1} \nu_i} \right\} \]

\[ \leq C_1 \exp(-C_2 \gamma^{m/3}) \]

\[ + \sup_{A_1,A_1=\gamma_{m-1},A_{\tau,\gamma_{m-1}^A}} P \left\{ \tau_{A_1}^{\gamma_{m-1}} > \sum_{i=k+1}^{m-1} \eta_i \lor R(A_{\tau,\gamma_{m-1}^A}) \subseteq A_{\sum_{i=k+1}^{m-1} \nu_i} \right\} \]

Since \( A^{\nu_m} \subseteq A^{\nu_m+\nu_{m-1}} \subseteq \Delta^\gamma \), the second term on the last expression can be bounded similarly as

\[ \sup_{A_1,A_1=\gamma_{m-1},A_{\tau,\gamma_{m-1}^A}} P \left\{ \tau_{A_1}^{\gamma_{m-1}} > \sum_{i=k+1}^{m-1} \eta_i \lor R(A_{\tau,\gamma_{m-1}^A}) \subseteq A_{\sum_{i=k+1}^{m-1} \nu_i} \right\} \]

\[ \leq C_1 \exp(-C_2 \gamma^{(m-1)/3}) \]

\[ + \sup_{A_2,A_2=\gamma_{m-2},A_{\tau,\gamma_{m-1}^A}} P \left\{ \tau_{A_2}^{\gamma_{m-1}} > \sum_{i=k+1}^{m-2} \eta_i \lor R(A_{\tau,\gamma_{m-1}^A}) \subseteq A_{\sum_{i=k+1}^{m-2} \nu_i} \right\} \]

Iterating the above argument, the assertion follows.

(b) Same as part (a).

Proof of Theorem 4.1 (a) We may assume that \( Q := \Xi_0 \) is infinite, since otherwise there is nothing to prove. By scaling, it is enough to prove the theorem when \( Q \) is contained in \( G \). Let \( Q_1 \subseteq Q_2 \subseteq \ldots \subseteq Q \) be a sequence of finite sets such that \( \#Q_m = \gamma^{-m} \) and \( Q \) is the closure \( \bigcup_{m=1}^{\infty} Q_m \). By assigning suitable rankings to a system of independent particles starting from each point in \( \bigcup_{m=1}^{\infty} Q_m \), we can obtain coupled set-valued coalescing processes \( \Xi_1, \Xi_2, \ldots \) and \( \Xi \) with the property that \( \Xi_0 = Q_m, \Xi_0 = Q, \) and for each \( t > 0 \),

\[ \Xi_1 \subseteq \Xi_2 \subseteq \ldots \subseteq \Xi_t \]

and \( \Xi_t \) is the closure of \( \bigcup_{m=1}^{\infty} \Xi_t^m \).

Fix \( \varepsilon > 0 \) so that \( Q^t \sum_{i=0}^{\varepsilon - (1/6) \log_3 5 \times i} \) is contained in the extended triangle corresponding to \( G \). Set \( \varepsilon \eta := \varepsilon^{-\gamma^{-1} \log_3 5} \) and \( \eta := 25\beta \gamma^{-1} \log_3 5 \). Fix \( t > 0 \). Choose \( k_0 \) so that \( \sum_{i=k_0+1}^{\infty} \eta_i \leq t \). By Lemma 3.1, and the fact that \( s \mapsto \#\Xi_s^m \) is non-increasing, we have, for each \( k \geq k_0 \),

\[ P \left\{ \#\Xi_{\gamma^k}^m \leq [\gamma^k] \right\} \geq 1 - \sum_{i=k+1}^{m} C_1 \exp(-C_2 \gamma^k / 3). \]
By the coupling, the sequence of events \( \{ \#\Xi_t^m \leq [\gamma^k] \} \) decreases to the event \( \{ \#\Xi_t \leq [\gamma^k] \} \). Consequently, letting \( m \to \infty \), we have, for each \( k \geq k_0 \),

\[
\mathbb{P} \{ \#\Xi_t \leq [\gamma^k] \} \geq 1 - \sum_{i=k+1}^{\infty} C_1 \exp(-C_2 \gamma^{i/3}).
\]

Finally letting \( k \to \infty \), we conclude that

\[
\mathbb{P} \{ \#\Xi_t < \infty \} = 1.
\]

(b) Same as part (a). \(\square\)

Proof of Theorem 4.2. (a) Assume without loss of generality that \( Q := \Xi_0 \) is infinite and contained in the 1-triangle that contains the origin. By Theorem 4.1, \( \Xi_t \) is almost surely finite and hence it can be considered as a random element in \( (\mathcal{K}, d_H) \).

It is enough to prove that

\[
\lim_{t \downarrow 0} d_H(\Xi_t^m, Q) = 0 \text{ a.s.}
\]

Let \( Q_1 \subseteq Q_2 \subseteq \cdots \) be a nested sequence of finite approximating sets of \( Q \) chosen as in the proof of Theorem 4.1 and let \( \Xi^m_t \) be the corresponding coupled sequence of set-valued processes.

Fix \( \delta > 0 \). Choose \( m \) sufficiently large that \( Q \subseteq Q_{m/2}^{\delta/2} \). By the right-continuity of the finite coalescing process, we have

\[
\lim_{t \downarrow 0} d_H(\Xi^m_t, Q) = 0 \text{ a.s.}
\]

Thus, with probability one, \( (\Xi^m_t)^{\delta/2} \supseteq Q_m \) when \( t \) is sufficiently close to 0. But, by the choice of \( Q_m \), with probability one,

\[
(\Xi^m_t)^{\delta} \supseteq (Q_m)^{\delta/2} \supseteq Q
\]

for \( t \) sufficiently close to 0.

Conversely, choose \( \varepsilon > 0 \) sufficiently small so that \( \sum_{i=1}^{\infty} \nu_i < \delta/2 \) where \( \nu_i \) is defined as in Lemma 4.6. Set \( s_k := \sum_{i=k+1}^{\infty} \eta_i \sim C\gamma^{-k} \log s_k \). From Lemma 4.6, we have

\[
\Pr \left\{ \mathcal{R}(Q_m; [0, s_k]) \not\subseteq (Q)^{\delta} \right\}
\]

\[
\leq \Pr \left\{ \sum_{i=k+1}^{m} \eta_i \text{ or } \mathcal{R}(Q_m; [0, \tau_{[\gamma^k]}]) \not\subseteq (Q)^{\delta/2} \right\}
\]

\[
+ \Pr \left\{ \text{max displacement of } [\gamma^k] \text{ independent particles in } [0, s_{k-1}] > \delta/2 \right\}
\]

\[
\leq \sum_{i=k+1}^{m} C_1 \exp(-C_2 \gamma^{i/3}) + C_1' [\gamma^k] \exp(-C_2' \gamma^{k/3})
\]

(4.3)

By Theorem 4.1 \#\Xi_t < \infty almost surely, and hence \( \Xi_t^m = \Xi_t \) for all \( m \) sufficiently large almost surely. Therefore, by letting \( m \to \infty \) in (4.3), we obtain

\[
\Pr \left\{ \mathcal{R}(Q; [0, s_k]) \not\subseteq (Q)^{\delta} \right\} \leq C_3 \exp(-C_2 \gamma^{k/3}).
\]

Letting \( k \to \infty \), we deduce that, with probability one,

\[
\Xi_t \subseteq Q^\delta
\]

for \( t \) sufficiently close to 0. Combined with (4.2), this gives the desired claim. \(\square\)
Proof of Theorem 4.3. By scaling, it suffices to show that almost surely, the set \( \Xi_t \cap G \) is finite for all \( t > 0 \). Fix any \( 0 < t_1 < t_2 \). We will show that almost surely, the set \( \Xi_t \cap G \) is finite for all \( t \in [t_1, t_2] \).

Set \( J_{0,1} := G \). Now for \( r \geq 1 \), the set \( 2^r G \setminus 2^{r-1} G \) can be covered by exactly \( 2 \times 3^{r-1} \) many 0-triangles that we will denote by \( J_{r,\ell} \) for \( 1 \leq \ell \leq 2 \times 3^{r-1} \). The collection \( \{ J_{r,\ell} \} \) forms a covering of the infinite gasket.

Put \( Q := \Xi_0 \) and let \( D \) be a countable dense subset of \( Q \). Associate each point of \( D \) with one of the (at most two) 0-triangles to which it belongs. Denote by \( D_{r,\ell} \) the subset of \( D \) consisting of particles associated with \( J_{r,\ell} \). Construct a partial coalescing system starting from \( D \) such that two particles coalesce if and only if they collide and both of their initial positions belonged to the same set \( D_{r,\ell} \). Let \( (\Xi_t)_{t \geq 0} \) denote the set-valued coalescing process consisting of the (possibly empty) subset of the particles associated with \( J_{r,\ell} \). If we preserve the original ranking of the particles that started from \( D_{r,\ell} \), then, for each \( t > 0 \),

\[
\Xi_t \subseteq \bigcup_{(r,\ell)} \Xi_t^{r,\ell}
\]

and it thus suffices to prove that almost surely, the set \( G \cap \bigcup_{(r,\ell)} \Xi_t^{r,\ell} \) is finite for all \( t \in [t_1, t_2] \).

Fix \( \Delta = J_{r,\ell} \in \mathcal{F}_0 \). Recall the notation of Lemma 4.6. Find \( \varepsilon > 0 \) such that \( \Delta \Sigma_{i=0}^\nu_i \subseteq \Delta^\varepsilon \). Let \( A_1 \subseteq A_2 \subseteq \ldots \) be an increasing sequence of sets such that \( \bigcup_{m} A_m = D_{r,\ell} \). Construct coupled set-valued coalescing processes \( \tilde{\Xi}^1 \subseteq \tilde{\Xi}^2 \subseteq \ldots \subseteq \Xi^{r,\ell} \) such that \( \tilde{\Xi}^m_0 = A_m \). Note that by Lemma 4.6,

\[
\mathbb{P}\{ \Xi_t^{r,\ell} \cap G \neq \emptyset \text{ for some } t \in [t_1, t_2] \}
\]

\[
= \lim_{m \to \infty} \mathbb{P}\{ \tilde{\Xi}_t^m \cap G \neq \emptyset \text{ for some } t \in [t_1, t_2] \}
\]

\[
\leq \limsup_{m \to \infty} \mathbb{P}\{ \tau^{A_m}_{[\gamma^r]} > \sum_{i=r+1}^{\infty} \eta_i \text{ or } \tilde{\Xi}^{A_m}_{[\gamma^r]} \notin \Delta^\varepsilon \text{ or max displacement of the remaining } [\gamma^r] \text{ coalescing particles in } [\tau^{A_m}_{[\gamma^r]}, t_2] > (r - 3/2) \}
\]

\[
\leq \limsup_{m \to \infty} \mathbb{P}\{ \tau^{A_m}_{[\gamma^r]} > \sum_{i=r+1}^{\infty} \eta_i \text{ or } \tilde{\Xi}^{A_m}_{[\gamma^r]} \notin \Delta^\varepsilon \}
\]

\[
+ \mathbb{P}\{ \text{max displacement of } [\gamma^r] \text{ independent particles in } [0, t_2] > (r - 3/2) \}
\]

\[
\leq C_1 \exp(-C_2 \gamma^{r/3}) + 2c_1 [\gamma^r] \exp \left( -c_2 ((r - 3/2) d_w/t_2)^{1/(d_w-1)} \right)
\]

\[
\leq C_4 \exp(-C_4 \gamma^{r/3})
\]

for some constants \( C_3, C_4 > 0 \) that may depend on \( t_2 \) but are independent of \( r \) and \( \ell \). The first of the above inequalities follows from the fact that

\[
\inf_{x \in J_{r,\ell}, y \in G} |x - y| \geq 2^{r-1} - 1 \geq r - 1,
\]

which implies that \( \Delta^\varepsilon \) is at least at a distance \( (r - 3/2) \) away from \( G \).

Now by a union bound,

\[
\mathbb{P}\{ \Xi_t^{r,\ell} \cap G \neq \emptyset \text{ for some } t \in [t_1, t_2] \text{ and for some } \ell \} \leq 2 \times 3^{r-1} C_3 \exp(-C_4 \gamma^{r/3}).
\]
By the Borel-Cantelli lemma, the events $\Xi_t^r \cap G \neq \emptyset$ for some $t \in [t_1, t_2]$ happen for only finitely many $(r, \ell)$ almost surely. This combined with the fact that $\#\Xi_t^r < \infty$ for all $t > 0$ almost surely gives that

$$\# \bigcup_{(r, \ell)} (G \cap \Xi_t^r) < \infty$$

for all $t \in [t_1, t_2]$ almost surely. $\square$

5. Instantaneous coalescence of stable particles

5.1. Stable processes on the real line and unit circle. Let $X = (X_t)_{t \geq 0}$ be a (strictly) stable process with index $\alpha > 1$ on $\mathbb{R}$. The characteristic function of $X_t$ can be expressed as $\exp(-\Psi(\lambda)t)$ where $\Psi(\cdot)$ is called the characteristic exponent and has the form

$$\Psi(\lambda) = c|\lambda|\alpha (1 - i\text{sgn}(\lambda)\tan(\pi\alpha/2)), \quad \lambda \in (-\infty, \infty), i = \sqrt{-1}.$$ 

where $c > 0$ and $\nu \in [-1, 1]$. The Lévy measure of $\Pi$ is absolutely continuous with respect to Lebesgue measure, with density

$$\Pi(dx) = \begin{cases} 
    c^+ x^{-\alpha-1}dx & \text{if } x > 0, \\
    c^- |x|^{-\alpha-1}dx & \text{if } x < 0,
\end{cases}$$

where $c^+, c^-$ are two nonnegative real numbers such that $\nu = (c^+ - c^-)/(c^+ + c^-)$. The process is symmetric if $c^+ = c^-$ or equivalently $\nu = 0$. The stable process has the scaling property

$$X \overset{d}{=} (c^{-1/\alpha} X_{ct})_{t \geq 0}$$

for any $c > 0$. If we put $Y_t := e^{2\pi i X_t}$, then the process $(Y_t)_{t \geq 0}$ is the stable process with index $\alpha > 1$ on the unit circle $\mathbb{T}$.

We define the distance between two points on $\mathbb{T}$ as the length of the shortest path between them and continue to use the same notation $|\cdot|$ as for the Euclidean metric on the real line.

Theorem 5.1 (Instantaneous Coalescence). (a) Let $\Xi$ be the set-valued coalescing stable process on $\mathbb{R}$ with $\Xi_0$ compact. Almost surely, $\Xi_t$ is a finite set for all $t > 0$. (b) The conclusion of part (a) holds for the set-valued coalescing stable process on $\mathbb{T}$.

Theorem 5.2 (Continuity at time zero). (a) Let $\Xi$ be the set-valued coalescing stable process on $\mathbb{R}$ with $\Xi_0$ compact. Almost surely, $\Xi_t$ converges to $\Xi_0$ as $t \downarrow 0$. (b) The conclusion of part (a) holds for the set-valued coalescing stable process on $\mathbb{T}$.

Theorem 5.3 (Instantaneous local finiteness). Let $\Xi$ be the set-valued coalescing stable process on $\mathbb{R}$ with $\Xi_0$ a possibly unbounded closed set. Almost surely, $\Xi_t$ is a locally finite set for all $t \geq 0$.

We now proceed to establish hitting time estimates and maximal inequalities for stable processes that are analogous to those established for Brownian motions on the finite and infinite gaskets in Section 3. With these in hand, the proofs of Theorem 5.1 and Theorem 5.2 follow along similar, but simpler, lines to those in the proofs of the corresponding results for the gasket (Theorem 4.1 and Theorem 4.2), and so we omit them. However, the proof of Theorem 5.3 is rather different from
that of its gasket counterpart (Theorem 4.3), and so we provide the details at the end of this section.

**Lemma 5.4.** Let $Z = X' - X''$ where $X'$ and $X''$ are two independent copies of $X$, so that $Z$ is a symmetric stable process with index $\alpha$. For any $0 < \delta < \beta$,

$$\mathbb{P}^x \{ Z_t = 0 \text{ for some } t \in (\delta, \beta) \} > 0.$$  

**Proof.** The proof follows from [Ber96, Theorem 16] which says that the single points are not essentially polar for the process $Z$, the fact that $Z$ has a continuous symmetric transition density with respect to Lebesgue measure, and the Markov property of the $Z$. □

It is well-known that symmetric stable process $Z$ on $\mathbb{R}$ hits points (see, for example, [Ber96, Chapter VIII, Lemma 13]). Thus there exists a $0 < \beta < \infty$ so that

$$0 < \mathbb{P}^1 \{ Z_t = 0 \text{ for some } t \in (0, \beta) \} =: p \text{ (say).}$$

By scaling,

$$\mathbb{P}^c \{ Z_t = 0 \text{ for some } t \in (0, \beta \varepsilon^a) \} = p.$$

**Lemma 5.5.** Suppose that $X'$ and $X''$ are two independent stable processes on $\mathbb{R}$ starting at $x'$ and $x''$. For any $\varepsilon > 0$,

$$\inf_{|x'-x''| \leq \varepsilon} \mathbb{P} \{ X'_t = X''_t \text{ for some } t \in (0, \beta \varepsilon^a) \} = p.$$

Since $X'_t = X''_t$ always implies that $\exp(2\pi i X'_t) = \exp(2\pi i X''_t)$ (but converse is not true), we have the following corollary of the above lemma.

**Corollary 5.6.** If $Y'$ and $Y''$ are two independent stable processes on $\mathbb{T}$ starting at $y'$ and $y''$, then for any $\varepsilon > 0$

$$\inf_{|y'-y''| \leq 2\pi \varepsilon} \mathbb{P} \{ Y'_t = Y''_t \text{ for some } t \in (0, \beta \varepsilon^a) \} \geq p.$$

**Lemma 5.7** ([Ber96]). Suppose that $X$ is an $\alpha$-stable process on the real line. There exists a constant $C > 0$ such that

$$\mathbb{P}^0 \left\{ \sup_{0 \leq s \leq 1} |X_s| > u \right\} \leq Cu^{-\alpha}, \quad u \in \mathbb{R}_+.$$

**Corollary 5.8.** (a) Let $X^1, X^2, \ldots, X^n$ be independent stable processes of index $\alpha > 1$ on $\mathbb{R}$ starting from $x^1, x^2, \ldots, x^n$ respectively. Then for each $x \in \mathbb{R}_+$ and $t > 0$,

$$\mathbb{P} \left\{ \sup_{0 \leq s \leq t} |X^i_s - x| > u \text{ for some } 1 \leq i \leq n \right\} \leq Cntu^{-\alpha}.$$  

(b) The same bound holds for $n$ independent stable processes on $\mathbb{T}$ when $u < \pi$.

Again we set $\gamma := 1/(1 - p/5) > 1$. Fix $(\alpha - 1)/2 < \eta < \alpha - 1$ and define $h := 1 - (1 + \eta)/\alpha > 0$. Recall the definitions of $\tau_n^A$ and $\mathcal{R}(A; I)$.

**Lemma 5.9.** Fix $0 < \varepsilon \leq 1/2$.

(a) There is a constant $C_1 = C_1(\varepsilon)$ such that $\Xi$ be a set-valued coalescing stable process in $\mathbb{R}$ with $\Xi_0 = A$, then

$$\mathbb{P} \left\{ \tau_{[n \gamma - 1]}^A > \beta(2\ell/n)^\alpha \text{ or } \mathcal{R}(A, [0, \tau_{[n \gamma - 1]}]) \not\subset A^{\ell n^{-\eta}} \right\} \leq C_1 n^{-\eta},$$

where $n = \#A$ and $\ell/2$ is the diameter of $A$. 

(b) Let $Ξ$ be the set-valued coalescing process in $T$ with $Ξ_0 = A$, where $A$ has cardinality $n$. Then there exists constant $C_1 = C_1(ε)$, independent of $A$, such that
\[
P \left\{ \tau_{[n\gamma^{-1}]}^A > (2/n)^α \text{ or } R(A; [0, t_{[n\gamma^{-1}]}]) \not\subseteq A^{\varepsilon n^{-h}} \right\} \leq C_1 n^{-η}.
\]

Proof. (a) Note that $A^\varepsilon \subseteq [a - \ell/2, a + \ell/2]$ for some $a \in \mathbb{R}$, and this interval can be divided into $n/2$ subintervals of length $2\ell/n$. We follow closely the proof of Lemma 4.5. By considering a suitable partial coalescing particle system consisting of at least $n/4$ pairs of particles where a pair can only coalesce if they have started from the same subinterval, we have that the number of surviving particles in the original coalescing system is at most $[\gamma^{-1} n]$ within time $t_n := (2\ell/n)^α$ with error probability bounded by $\exp^{-C_1 n}$.

By Lemma 5.8, the maximum displacement of $n$ independent stable particles on $\mathbb{R}$ within time $t_n$ is at most
\[
ε(t_n)^{1/α}(1+n)/α = 2β^{1/α}ε\varepsilon n^{-1+1/α} = 2β^{1/α}ε\varepsilon n^{-h}
\]
with error probability at most $ε2n^{-η}$.

(b) The proof for part (b) is similar. \square

Using strong Markov property and Lemma 5.9 repetitively as we did in the proof of Lemma 4.6 we can obtain the following lemma. We omit the details.

**Lemma 5.10.** Let $0 < ε ≤ 1/2$, $\ell > 0$ be given. Let $ν_i := ε\gamma^{-hi}$ and $η_i := 2β^2\gamma^{-αi}$.

(a) Given a finite set $A \subseteq \mathbb{R}$, let $Ξ$ denote the set-valued coalescing stable process in $\mathbb{R}$ with $Ξ_0 = A$. Then, there exist constants $C_2 = C_2(ε)$ such that
\[
P \left\{ \tau_{[r\gamma^k]} > \ell^α \sum_{i=k+1}^{m} η_i \text{ or } R(A; [0, \tau_{[r\gamma^k]}]) \not\subseteq A^{\varepsilon m^{α} \sum_{i=k+1}^{m} ν_i} \right\} \leq C_2 γ^{-ηk},
\]
uniformly over all sets $A$ such that $A \subseteq [a - \ell/4, a + \ell/4]$ for some $a \in \mathbb{R}$ and $#A = [\gamma^m]$.

(b) Given a finite set $A \subseteq T$, let $Ξ$ denote the set-valued coalescing stable process in $T$ with $Ξ_0 = A$. Then, there exist constants $C_2 = C_2(ε)$ such that
\[
P \left\{ \tau_{[r\gamma^k]} > \sum_{i=k+1}^{m} η_i \text{ or } Ξ_{[r\gamma^k]} \not\subseteq A^{\varepsilon \sum_{i=k+1}^{m} ν_i} \right\} \leq C_2 γ^{-ηk},
\]
uniformly over all sets $A \subseteq T$ such that $#A = [\gamma^m]$.

**Proof of Theorem 5.3** By scaling, it is enough to show that for each $0 < t_1 < t_2 < \infty$, almost surely, the set $Ξ_t \cap [−1, 1]$ is finite for each $t \in [t_1, t_2]$. Set $d := 2/η$. For $r ≥ 1$, define
\[
J_{r,1} := \left\lceil -\sum_{j=1}^{r} j^d, -\sum_{j=1}^{r-1} j^d \right\rceil \quad \text{and} \quad J_{r,2} := \left\lceil \sum_{j=1}^{r-1} j^d, \sum_{j=1}^{r} j^d \right\rceil.
\]
Then the collection $\{J_{r,i}\}_{r \geq 1, i = 1, 2}$ forms a partition of the real line into bounded sets. Note that $\inf_{x \in [−1, 1] \cap J_{r,i}} |x - y| > r^{d+1}$ as $r \to \infty$.

Let $D$ be a countable dense subset of $Q$. Run a partial coalescing system starting from $D$ such that two particles coalesce if and only if they collide and both belonged initially to the same $J_{r,i}$. Let $(Ξ_t^{r,i})_{t \geq 0}$ denote the set-valued coalescing process
consisting of the (possibly empty) subset of the particles starting from \(D \cap J_{r,i}\). By a suitable coupling, for each \(t > 0\),
\[
\Xi_t \subseteq \bigcup_{r,i} \Xi_t^{r,i}
\]
and it thus suffices to prove that the set \([-1, 1] \cap \Xi_t^{r,i}\) is empty for all \(t \in [t_1, t_2]\) for all but finitely many pairs \((r, i)\) almost surely.

Fix a pair \((r, i)\). Find \(\varepsilon > 0\) such that \(\sum_{i=0}^{\infty} \nu_i \leq 1/2\) which implies that \((J_{r,i})^\nu \subseteq (J_{r,i})^{\varepsilon}\). Let \(A_1 \subseteq A_2 \subseteq \ldots\) be an increasing sequence of finite sets such that for \(\bigcup_m A_m = D \cap J_{r,i}\). Let \(\Xi^{m}\) be a coalescing set-valued stable processes such that \(\Xi^{m}_0 = A_m\) and couple these processes together so that \(\Xi^{1}_t \subseteq \Xi^{2}_t \subseteq \cdots \subseteq \Xi^{m}_t\). Set \(b = b(r) := (2/\eta) \log r\). Note that by Lemma 5.11 Corollary 5.8 and the fact that there exists \(c_1 > 0\) such that for all \(r\) sufficiently large
\[
\inf_{x \in [-1, 1], y \in J_{r,i}} |x - y| - r^d \geq cr^{d+1},
\]
we can write
\[
P\left\{\Xi_t^{r,i} \cap [-1, 1] \neq \emptyset \text{ for some } t \in [t_1, t_2]\right\} = \lim_{m \to \infty} P\left\{\Xi^{m}_t \cap [-1, 1] \neq \emptyset \text{ for some } t \in [t_1, t_2]\right\}
\]
\[
\leq \limsup_{m \to \infty} \left\{\sum_{i=b+1}^{\infty} \eta_i \text{ or } \Xi^{m}_t \not\subseteq (J_{r,i})^{\varepsilon}\right\}
\]
of the remaining \([\gamma]^b\) coalescing particles in \([\tau_{[\gamma]^b}]_m, t_2\] \(> cr^{d+1}\)
\[
\leq \limsup_{m \to \infty} \left\{\sum_{i=r+1}^{\infty} \eta_i \text{ or } \Xi^{m}_t \not\subseteq (J_{r,i})^{\varepsilon}\right\} + P\left\{\max \text{ displacement of } [\gamma]^b \text{ independent particles in } [0, t_2] > cr^{d+1}\right\}
\]
\[
\leq C_2 \gamma^{-\eta b} + C_3 [\gamma]^b c r^{-\alpha(d+1)} \leq C_2^r r^{-2} + C_3 r^{-\alpha}
\]
for suitable constants \(C_2^r, C_3^r > 0\). The proof now follows from the Borel-Cantelli lemma. \(\square\)

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24 STEVEN N. EVANS, BEN MORRIS, AND ARNAB SEN

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