Coulomb correction to the spectral bremsstrahlung rate in the quantum Migdal theory of the Landau–Pomeranchuk effect

O Voskresenskaya
Joint Institute for Nuclear Research, 141980, Dubna, Moscow region, Russia
E-mail: voskr@jinr.ru

Abstract. High-energy Coulomb corrections (CCs) to some quantities of the quantum Migdal theory of the Landau–Pomeranchuk–Migdal (LPM) effect are obtained analytically and numerically for the regimes of small and strong LPM suppression on the basis of a refined Molière multiple scattering theory. Numerical calculations are presented in the ranges of the nuclear charge number of a scatterer $6 \leq Z \leq 92$ and the expansion parameter of Molière $4.5 \leq B \leq 8.5$. It is shown that the relative CC to the spectral bremsstrahlung rate can reach a value of the order of $-19\%$ in the ranges considered and must be borne in mind, e.g., in the Monte-Carlo analysis of electromagnetic cascade LPM showers in extremely high-energy region.

1. Introduction

The theory of the multiple scattering of charged particles has been treated by several authors. However, the most widely used at present is the multiple scattering theory of Molière [1]. The multiple scattering theory is of interest for numerous applications related to particle transport in matter [2] and is widely used in the cascade shower theory [3]. The neutrino-oscillation experiments [4], the astronomical cosmic-ray observations [5], etc., face a problem of taking into account the multiple scattering effects.

As the Molière theory can be currently used in ultrahigh-energy region, the role of the high-energy CCs to the parameters of this theory becomes significant. Of especial importance is the Coulomb correction to the screening angular parameter, as this parameter enters into other important quantities of the Molière theory and the Migdal theory of the LPM effect [6, 7]. In [7] it is shown that the CC to the spectral bremsstrahlung rate of the classical LPM effect theory [8] for the regime of small LPM suppression allows completely eliminating the discrepancy between the predictions of the LPM effect theory and its measurement at least for high-$Z$ targets and also to further improve the agreement between the predictions of the LPM effect theory analogue for a thin layer of matter and data of SLAC E-146 experiment [9]. The aim of the presented work is a preliminary estimation of CCs to some important quantities in the quantum LPM effect theory, especially to the Migdal function $G(s)$ and the spectral bremsstrahlung rate which are of special interest in describing the shower production at energies $E$ exceeding $10^{13}$ eV [10, 11].

Qualitative evidence of the LPM effect for bremsstrahlung in nuclear-emulsion study of cosmic-ray-produced electromagnetic cascades over the range $10^{11} \leq E \leq 10^{13}$ eV was first reported in ref. [12]. The quantitative confirmation of this LPM predicted bremsstrahlung
suppression at Serpukhov experiment gives [13]. A direct evidence for reduction of the bremsstrahlung cross section due to the LPM effect in the measurements of cosmic ray electrons at \( E \sim 10^{12} \) eV was found in [14]. In ref. [15] it is point out a significant reduction of the Migdal cross section due to LPM effect for neutrino-induced electromagnetic showers in ice over the range \( 10^{16} \leq E \leq 10^{20} \) eV; the other mechanisms of its reduction are specified in [16].

The significance of LPM effect in studying the average behavior of electromagnetic cascade showers at extremely high energies (the so-called “LPM showers” [17]) was shown for energies \( E > 3.5 \cdot 10^{13} \) eV in lead [17, 18], \( E > 2.2 \cdot 10^{15} \) eV in standard rock [19], \( E > 10^{17} \) eV in air [20, 21], etc. The big difference between the averaged LPM shower and usual (“Bethe–Heitler” [17]) showers over the range \( 10^{15} \leq E \leq 10^{21} \) eV in lead, water and standard rock is explained in [17, 19]. A series of reports [22] is devoted to applications of the LPM effect to extremely high-energy cosmic ray research in the cosmic rays physics.

The first attempt to apply the LPM effect to the consideration of the electromagnetic cascade shower at superhigh energies was carried out in [23]. The first correct results obtained for the LPM cascade showers through Monte-Carlo simulation techniques with taking into account the LPM effect for one-dimensional case presents [24]. For both the one-dimensional and the three-dimensional cases, they were given in [25]. Calculations of LPM showers by a hybrid method, which combines a Monte-Carlo simulation with analytical calculations, were performed in [18, 26]. In [27] a matrix method for calculation of LPM showers was proposed. Characteristics of individual cascade LPM showers have been studied in [28, 29]. There have been shown the multi-peak structure of individual LPM showers [28] and a strong diversity among them [29].

Careful investigation of fluctuations in LPM showers by simulation technique, which takes into account the LPM effect, requires a sufficiently high accuracy of its theoretical description. Since the Migdal theory does not include all the corrections that should in principle be included (in particular, the CCs) [30], the refinement of the quantities of the conventional Migdal LPM effect theory by means of the high-energy CCs is of considerable interest. The present work is the first step in this direction. In Section 2 we present briefly our results for the high-energy CCs to the parameters of the Migdal LPM multiple scattering theory [6]. Then, in Section 3 we report the results of applying this improved Migdal multiple scattering theory for obtaining CCs to the quantities of the classical and quantum LPM effect theories by Migdal [8, 10]. Finally, in Sec. 4 we briefly sum up our results and discuss some perspectives.

### 2. Coulomb corrections to the parameters of the Molière theory

In this section we present an exact analytical result for the Molière screening angular parameter \( \theta_a (\theta_a' = \sqrt{1 + 167/\theta_a} \) ), valid to all orders in the Born parameter \( \xi = Z\alpha/\beta \)

\[
\Delta_{CC} \ln(\theta_a') \equiv \ln(\theta_a') - \ln(\theta_a')^0 = f(\xi)
\]

(1)

instead of an approximate one for the Molière \( \theta_a^M \) and first-order Born \( \theta_a^B \) values of screening angle, valid to second order in the parameter \( \xi \),

\[
\theta_a^M = \theta_a^0 \sqrt{1 + 3.34 \xi^2}
\]

(2)

that was obtained in the original paper of Molière [11]. Here, \( \alpha \) denotes the fine structure constant, \( \beta = v/c \) is the velocity of a projectile in units of the velocity of light, \( \theta_a^0 = 1.20 \alpha Z^{1/3} \) and \( f(\xi) = \xi^2 \sum_{n=1}^{\infty} [n(n^2 + \xi^2)]^{-1} \) represents the Bethe–Maximon function.

We also present the analytical and numerical results for the Coulomb corrections to the parameters \( b, B \) and \( \overline{d} \) of the Molière expansion method for the angular distribution function

\[
\bar{w}_n(\vartheta, L) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^\infty \frac{1}{B^n} w_n(\vartheta, L)
\]

\[
= \left[ \frac{y^2}{4} \ln \left( \frac{y^2}{4} \right) \right]^n,
\]

(3)
where \( \vec{\vartheta} \) is a two-dimensional particle scattering angle in the plane orthogonal to the incident particle direction, the relation \( \vec{\vartheta}^2 = \theta_c^2 B \) defines the average square value of \( \vec{\vartheta} \), \( \theta_c \) is the characteristic angle, \( L \) denotes the thickness of an absorber, \( y = \theta_c \eta \), \( \eta \) signifies the Fourier–Bessel transform variable corresponding to \( \vartheta \), \( J_0(\vartheta \eta) \) represents the zero-order Bessel function of the first kind and the expansion parameter \( B \) is defined by the transcendental equation \( B - \ln B = b \) with \( b = \ln(\theta_c/\theta'_a)^2 \).

For the Coulomb corrections to the Born parameters \( b^B \), \( B^B \) and \( \vec{\vartheta}^2^B \), we get

\[
\Delta_{cc}[b] = -f(\xi), \quad \Delta_{cc}[B] = f(\xi)/(1/B^B - 1), \quad \Delta_{cc}[\vec{\vartheta}^2] = \theta_c^2 \Delta_{cc}[B].
\]

(4)

The relative Coulomb corrections become

\[
\delta_{cc}[\vec{\vartheta}^2] = \delta_{cc}[B] = f(\xi)/(1 - B^B), \quad \delta_{cc}[\theta_a] = \exp [f(\xi)] - 1.
\]

(5)

The relative difference between the approximate \( \theta_a^M \) and exact \( \theta_a \) results reads

\[
\delta_{ccM}[\theta_a] \equiv (\theta_a - \theta_a^M)/\theta_a^M = \theta_a/\theta_a^M - 1 = R_{ccM}[\theta_a] - 1.
\]

(6)

The \( Z \) dependence of these corrections and differences presents figure 1. It shows that while the modules of the quantities \( \delta_{cc}[\vec{\vartheta}^2] \) and \( \Delta_{ccM}[\theta_a] \) reach about 6% for high-\( Z \) targets (dashdotted and dashed lines), the maximal value of \( \delta_{cc}[\theta_a] \) amounts approximately to 50% for \( Z = 92 \) (upper full line). The modules of CCs to the parameters \( b \) and \( B \) are: \( |\Delta_{cc}[B]| \sim 0.45 \) (lower full line) and \( |\Delta_{cc}[b]| \sim 0.40 \) (lower broken line), such as \( \Delta_{cc}[\ln(\theta'_a)] \sim 0.40 \) for \( Z = 92 \) (upper broken line). These corrections are also valid for such modifications of the Molière theory as those proposed in [2]. They will also be used below.

Figure 1. The \( Z \) dependence of the Coulomb corrections to some parameters of the Molière multiple scattering theory and the differences between exact and approximate results [6].
3. Applying the improved Molière theory to the description of the LPM effect

3.1. Coulomb corrections to the quantities of classical LPM effect theory

Based on the Coulomb corrections found in the previous section, we first get the analytical and numerical results for the Coulomb corrections to the quantities of the classical Migdal theory of the LPM effect for sufficiently thick targets, the basic formulae of which in the units \( h = c = 1, \ e^2 = 1/137 \) read

\[
\langle \frac{dI}{d\omega} \rangle = \Phi(s) \left( \frac{dI}{d\omega} \right)_0, \quad \left( \frac{dI}{d\omega} \right)_0 = \frac{2e^2}{3\pi} \gamma^2 q L, \quad \Phi(s) = 24s^2 \left[ \int_0^\infty dx e^{-2sx} \text{csch}(x) \sin(2sx) - \frac{\pi}{4} \right],
\]

\[
q = \frac{\vartheta^2}{L}, \quad s^2 = \frac{\lambda^2}{\vartheta^2}, \quad \lambda^2 = \gamma^{-2}.
\]

Here, \( \langle dI/d\omega \rangle \) is the electron spectral bremsstrahlung intensity averaged over various trajectories of electron motion, \( (dI/d\omega)_0 \) signifies the spectral bremsstrahlung intensity without accounting the multiple scattering effects in the radiation and \( \gamma \) denotes the Lorentz factor of the scattered particle.

The analytical result for the Coulomb correction \( \Delta_{CC} \) to the Born spectral bremsstrahlung rate \( (dI/d\omega)_0 \) is as follows:

\[
\Delta_{CC} \left[ \left( \frac{dI}{d\omega} \right)_0 \right] = \frac{2e^2}{3\pi} \gamma^2 L \Delta_{CC}[q], \quad \Delta_{CC}[q] = \frac{1}{L} \Delta_{CC} \left[ \vartheta^2 \right],
\]

where \( \Delta_{CC} \left[ \vartheta^2 \right] \) is given by [4]. In doing so, \( \Delta_{CC}[(dI/d\omega)_0] \) and \( \delta_{CC}[(dI/d\omega)_0] \) become

\[
\Delta_{CC} \left[ \left( \frac{dI}{d\omega} \right)_0 \right] = \frac{2(e\gamma\theta_c)^2}{3\pi(1/B^a - 1)} f(\xi), \quad \delta_{CC} \left[ \left( \frac{dI}{d\omega} \right)_0 \right] = \frac{f(\xi)}{1 - B^a}.
\]

For the regime of small LPM suppression, \( \Phi(s) \approx 1 - 0.012/s^4 \), the analytical result for the relative Coulomb correction to the Migdal function \( \Phi(s) \) in the entire range \( 1 \leq s \leq \infty \) reads

\[
\delta_{CC}[\Phi(s)] = \frac{0.012}{s^4} \delta_{CC} \left[ s^4 \right] \left( \frac{(s^4)^a}{(s^4)^a - 0.012} \right),
\]

where

\[
\delta_{CC} \left[ s^4 \right] = 1 - \left( \frac{\vartheta^2}{s^2} \right)^2 = 1 - \frac{1}{\left( \delta_{CC}[\vartheta^2] + 1 \right)^2}.
\]

Table I presents the sample means \( \delta_{CC}[(dI/d\omega)] \) (%) of the relative CCs for some targets of the experiment \[9\] and also the population mean \( \delta_{CC}[(dI/d\omega)]_{\mu} \) (%) over the entire range \( 1.0 \leq s \leq \infty \) of the parameter \( s \). It shows that \( \delta_{CC}[(dI/d\omega)] = (-4.50 \pm 0.05)\% \) (\( Z = 82 \)) and \( \delta_{CC}[(dI/d\omega)] = (-5.35 \pm 0.06)\% \) (\( Z = 92 \)) coincide within the experimental error with the values of the normalization corrections \((-4.5 \pm 0.2)\% \) for 2\%\text{L}_R lead target and \((-5.6 \pm 0.3)\% \) for 3\%\text{L}_R uranium target, respectively; \( \delta_{CC}[(dI/d\omega)]_{\mu} = (-4.70 \pm 0.49)\% \) excellently agrees with the weighted average \((-4.7 \pm 2)\% \) of the normalization correction obtained for 25 GeV data of the SLAC E-146 experiment \[9\].

This means that applying the improved multiple scattering theory by Molière allows one to avoid multiplying the results of predictions of the conventional Migdal LPM effect theory by a normalization factor and leads to agreement between the improved Migdal theory of the LPM effect and experimental data. We believe that this allows one to understand the origin of the normalization problem for high-\( Z \) targets discussed in \[9\].
Table 1. The dependence of the relative Coulomb correction $-\delta_{CC}[\langle dI/d\omega \rangle]$ (%) on the Migdal parameter $s$ in the regime of small LPM suppression for some high-$Z$ targets.

| Target | $Z$ | $s=1.0$ | $s=1.1$ | $s=1.2$ | $s=1.3$ | $s=1.5$ | $s=2.0$ | $s=\infty$ |
|--------|-----|--------|--------|--------|--------|--------|--------|----------|
| Au     | 79  | 4.32   | 4.28   | 4.26   | 4.24   | 4.22   | 4.21   | 4.19     |
| Pb     | 82  | 4.58   | 4.54   | 4.51   | 4.49   | 4.47   | 4.46   | 4.45     |
| U      | 92  | 5.45   | 5.41   | 5.36   | 5.34   | 5.33   | 5.31   | 5.30     |

$^a\delta_{CC}[\langle dI/d\omega \rangle] = (-4.50 \pm 0.05)\% \ (Z = 82), \ \delta_{CC}[\langle dI/d\omega \rangle] = (-5.35 \pm 0.06)\% \ (Z = 92), \ \delta_{CC}[\langle dI/d\omega \rangle]_{\mu} = (-4.70 \pm 0.49)\%.$

3.2. Applying the improved Molière theory to the description of an analogue of the LPM effect theory for a thin target

Application of the Molière multiple scattering theory to the analysis of experimental data [9] for a thin target is based on the use of the following expression for the spectral radiation rate [32]:

$$\langle \frac{dI}{d\omega} \rangle = \int w_{\mu}(\vartheta) \frac{dI(\vartheta)}{d\omega} d^2\vartheta,$$

$$\frac{dI(\vartheta)}{d\omega} = \frac{2e^2}{\pi} \left[ \frac{2\chi^2 + 1}{\chi\sqrt{\chi^2 + 1}} \ln \left( \chi + \sqrt{\chi^2 + 1} \right) - 1 \right], \quad (13)$$

which has two simple asymptotes at the small and large values of the parameter $\chi = \gamma\vartheta/2$,

$$\langle \frac{dI}{d\omega} \rangle = \frac{2e^2}{3\pi} \left\{ \begin{array}{ll} \gamma^2\vartheta^2, & \gamma^2\vartheta^2 \ll 1, \\ 3 \left[ \ln(\gamma^2\vartheta^2) - 1 \right], & \gamma^2\vartheta^2 \gg 1. \end{array} \right. \quad (14)$$

In the first case $\gamma^2\vartheta^2 \ll 1$, we have

$$\delta_{CC}[\langle \frac{dI}{d\omega} \rangle] = \delta_{CC}[\langle \frac{dI}{d\omega} \rangle_0] = \frac{f(\xi)}{1 - B^\vartheta}. \quad (15)$$

In the second case $\gamma^2\vartheta^2 \gg 1$, we find

$$\Delta_{cc}[\ln(\gamma^2\vartheta^2) - 1] = \Delta_{cc}[\ln(\vartheta^2)] = \Delta_{cc}[\ln(B)], \quad \Delta_{cc}[\ln(B)] = \Delta_{cc}[B] + f(Z\alpha) = \delta_{cc}[B]. \quad (16)$$

The Coulomb correction (16) then becomes

$$\Delta_{cc}[\ln(\gamma^2\vartheta^2) - 1] = \frac{\delta_{cc}[B]}{[\ln(\gamma^2\vartheta^2)^\vartheta - 1]}, \quad (17)$$

and we arrive at a result:

$$\delta_{cc}[\langle \frac{dI}{d\omega} \rangle] = \frac{\Delta_{cc}[\ln(\vartheta)\mu]}{[\ln(\gamma^2\vartheta^2)^\vartheta - 1] \left( 1 - B^\vartheta \right)}. \quad (18)$$

The numerical values of our corrections additionally improve the agreement between the predictions of the LPM effect theory analogue for a thin layer of matter and experiment [9] (figure 2, the dashed line "VKT") [7].
3.3. Coulomb corrections to some quantities of quantum LPM effect theory

We have also obtained analytical and numerical results for the Coulomb corrections to the quantities of the quantum Migdal LPM effect theory [10, 31]

$$
\langle \frac{dI}{d\omega} \rangle = \frac{1}{4} \left( \frac{dI}{d\omega} \right)_0 \left\{ \varepsilon^2 G(s) + 2[1 + (1 - \varepsilon)^2]\Phi(s) \right\}
$$

(19)

with the energy of radiation $\varepsilon = \omega/E$ in units of the incident particle energy $E$, Migdal’s parameter $s = \sqrt{(\omega/q)}/8\gamma^2$ and the Migdal functions $G(s)$ and $\Phi(s)$

$$
G(s) = 12\pi s^2 - 48s^2 \sum_{k=0}^{\infty} \frac{1}{(k + s + 1/2)^2 + s^2},
$$

$$
\Phi(s) = 6s - 6\pi s^2 + 24s^2 \sum_{k=1}^{\infty} \frac{2}{(k + s)^2 + s^2}.
$$

(20)

For the relative Coulomb correction to the Born spectral bremsstrahlung rate (19) one can get

$$
\delta_{CC} \left[ \langle \frac{dI}{d\omega} \rangle \right] = \delta_{CC} \left[ \left( \frac{dI}{d\omega} \right)_0 \right] + \frac{\Delta_{CC}[G(s)]}{G^B(s) + \Phi^B(s)} \delta_{CC}[G] + \frac{\Phi^B}{G^B + \Phi^B} \delta_{CC}[\Phi],
$$

$$
\delta_{CC} \left[ \left( \frac{dI}{d\omega} \right)_0 \right] = \frac{f(\xi)}{1 - B^B}.
$$

(21)

Equations (19) and (20) generalize [7] and [8] of the classical LPM effect theory for bremsstrahlung to the case when the radiation recoil effect becomes significant.
in accordance with (5) and (12), we have obtained:

\[ \delta \]

Thereby, we have

\[ \delta_{11} \]

\( \delta \) LPM suppression for some high-

\( \bar{Z} \) targets.

For the regime of small LPM suppression [10, 31]

Table 2. The dependence of the relative Coulomb correction \( -\delta_{cc}[dI/d\omega] (%) \) on the Migdal parameter \( s \) in the regime of small LPM suppression for some high-\( Z \) targets.

1. \( B^B = 8.46, s = 1.5 \) and \( \beta = 1 \).

| Target | \( Z \) | \( \delta_{cc}[s^4] \) | \( \delta_{cc}[G(s)] \) | \( \delta_{cc}[\Phi(s)] \) | \( \delta_{cc}\left[\left(\frac{dI}{d\omega}\right)_{0}\right] \) | \( \delta_{cc}\left[\left\langle\frac{dI}{d\omega}\right\rangle\right] \) |
|--------|-----|----------------|----------------|----------------|----------------|----------------|
| W      | 74  | -0.0799        | -0.0004        | -0.0002        | -0.0377        | -0.0380        |
| Au     | 79  | -0.0896        | -0.0005        | -0.0002        | -0.0419        | -0.0422        |
| Pb     | 82  | -0.0953        | -0.0005        | -0.0002        | -0.0445        | -0.0448        |
| U      | 92  | -0.1148        | -0.0006        | -0.0002        | -0.0529        | -0.0533        |

2. \( B^B = 4.50, s = 1.5 \) and \( \beta = 1 \).

| Target | \( Z \) | \( \delta_{cc}[s^4] \) | \( \delta_{cc}[G(s)] \) | \( \delta_{cc}[\Phi(s)] \) | \( \delta_{cc}\left[\left(\frac{dI}{d\omega}\right)_{0}\right] \) | \( \delta_{cc}\left[\left\langle\frac{dI}{d\omega}\right\rangle\right] \) |
|--------|-----|----------------|----------------|----------------|----------------|----------------|
| W      | 74  | -0.1822        | -0.0009        | -0.0004        | -0.0803        | -0.0810        |
| Au     | 79  | -0.2060        | -0.0009        | -0.0004        | -0.0894        | -0.0901        |
| Pb     | 82  | -0.2207        | -0.0010        | -0.0004        | -0.0949        | -0.0956        |
| U      | 92  | -0.2707        | -0.0012        | -0.0005        | -0.1129        | -0.1138        |

For the regime of small LPM suppression [10, 31]

\[ G(s)_{s \to \infty} \to 1 - 0.029/s^4, \]

\[ \Phi(s)_{s \to \infty} \to 1 - 0.012/s^4, \]

in accordance with (5) and (12), we have obtained:

\[ \delta_{cc}[G(s)] = 0.029 \delta_{cc}[s^4] \left( \frac{1}{(s^4)^2} - 0.029 \right), \]

\[ \delta_{cc}[\Phi(s)] = 0.012 \delta_{cc}[s^4] \left( \frac{1}{(s^4)^2} - 0.012 \right), \]

\[ \delta_{cc}[s^4] = 1 - \frac{1}{\delta_{cc}[\bar{G}^2] + 2\delta_{cc}[\bar{\bar{G}}^2] + 1}, \]

\[ \delta_{cc}[\bar{\bar{G}}^2] = \frac{\Delta_{cc}[\ln(q')]}{1 - B^B}. \]

Table 2 listed the results of numerical estimation of the relative Coulomb corrections \( \delta_{cc}[s^4], \delta_{cc}[G(s)], \delta_{cc}[\Phi(s)] \) [23], \( \delta_{cc}[dI/d\omega]_0 \) [10] and \( \delta_{cc}[dI/d\omega] \) [21] in the regime of small LPM suppression for some high-\( Z \) targets and \( B^B \) values from 8.46 [9] to 4.5 at \( s = 1.5 \). It shows that while the modules of corrections \( \delta_{cc}[s^4] \) for high-\( Z \) absorbers reach the values about 11.5% for \( B^B = 8.46 \) and 27% for \( B^B = 4.5 \), the modules of \( \delta_{cc}[G(s)] \) and \( \delta_{cc}[\Phi(s)] \) do not exceed 0.1%. The dominant contribution to the correction \( \delta_{cc}[dI/d\omega] \) gives \( \delta_{cc}[dI/d\omega]_0 \). It is seen from table 2 that \( \delta_{cc}[dI/d\omega]_0 \approx 5.33\% \) while \( \delta_{cc}[dI/d\omega] \approx 5.29\% \) for \( Z = 92 \) and \( B^B = 4.5; \delta_{cc}[dI/d\omega] \approx 11.4\% \) whereas \( \delta_{cc}[dI/d\omega]_0 \approx 11.3\% \) for \( Z = 92 \) and \( B^B = 4.5. \)

Thereby, we have \( \delta_{cc}[dI/d\omega] \approx 2\delta_{cc}[\Phi(s)] \) in the regime of small LPM suppression.
Table 3. Coulomb corrections to the quantities $\delta_{cc}[G(s)]$, $\delta_{cc}[\Phi(s)]$, $\delta_{cc}[(dI/d\omega)_{0}]$ and $\delta_{cc}[(dI/d\omega)]$ of the quantum Migdal LPM in the regime of strong LPM suppression.

1. $B^B = 8.46$, $s = 0.05$ and $\beta = 1$.

| Target | $Z$ | $f(\xi)$ | $\delta_{cc}[G(s)]$ | $\delta_{cc}[\Phi(s)]$ | $\delta_{cc}[(dI/d\omega)_{0}]$ | $\delta_{cc}[(dI/d\omega)]$ |
|--------|----|----------|------------------|------------------|------------------|------------------|
| C      | 6  | 0.0041   | -0.0005          | -0.0003          | -0.0005          | -0.0008          |
| Al     | 13 | 0.0107   | -0.0014          | -0.0007          | -0.0014          | -0.0023          |
| Fe     | 26 | 0.0430   | -0.0058          | -0.0029          | -0.0058          | -0.0094          |
| W      | 74 | 0.2810   | -0.0392          | -0.0194          | -0.0377          | -0.0621          |
| Au     | 79 | 0.3130   | -0.0438          | -0.0216          | -0.0419          | -0.0698          |
| Pb     | 82 | 0.3320   | -0.0466          | -0.0230          | -0.0445          | -0.0738          |
| U      | 92 | 0.3950   | -0.0560          | -0.0276          | -0.0530          | -0.0887          |

2. $B^B = 4.50$, $s = 0.05$ and $\beta = 1$.

| Target | $Z$ | $f(\xi)$ | $\delta_{cc}[G(s)]$ | $\delta_{cc}[\Phi(s)]$ | $\delta_{cc}[(dI/d\omega)_{0}]$ | $\delta_{cc}[(dI/d\omega)]$ |
|--------|----|----------|------------------|------------------|------------------|------------------|
| C      | 6  | 0.0041   | -0.0012          | -0.0006          | -0.0012          | -0.0019          |
| Al     | 13 | 0.0107   | -0.0031          | -0.0015          | -0.0031          | -0.0050          |
| Fe     | 26 | 0.0430   | -0.0125          | -0.0062          | -0.0123          | -0.0201          |
| W      | 74 | 0.2810   | -0.0873          | -0.0427          | -0.0803          | -0.1349          |
| Au     | 79 | 0.3130   | -0.0982          | -0.0479          | -0.0894          | -0.1507          |
| Pb     | 82 | 0.3320   | -0.1049          | -0.0511          | -0.0949          | -0.1603          |
| U      | 92 | 0.3950   | -0.1273          | -0.0617          | -0.1129          | -0.1921          |

For the regime of strong LPM suppression

$$G(s)_{s \to 0} \to 12\pi s^2, \; \Phi(s)_{s \to 0} \to 6s,$$

we find

$$\delta_{cc}[G(s)] = 1 - \frac{1}{\delta_{cc}[\Phi^2] + 1},$$

$$\delta_{cc}[\Phi(s)] = 1 - \frac{1}{\sqrt{\delta_{cc}[\Phi^2] + 1}}.$$

Table 3 and figure 3 demonstrate the $Z$ dependence of the corrections $\delta_{cc}[G(s)]$, $\delta_{cc}[\Phi(s)]$ $^{[25]}$, $\delta_{cc}[(dI/d\omega)_{0}]$ $^{[10]}$ and $\delta_{cc}[(dI/d\omega)]$ $^{[21]}$ in the regime of strong LPM suppression over the ranges $0 \leq Z \leq 100$ and $4.5 \leq B^B \leq 8.5$ at $s = 0.05$. It can be seen from table 3 that $\delta_{cc}[G(s)]$ is comparable with $\delta_{cc}[(dI/d\omega)_{0}]$ in the case of strong suppression. The value of $|\delta_{cc}[G(s)]|$ is twice as large than $|\delta_{cc}[\Phi(s)]|$ and reaches about 4.7% at $B^B = 8.46$ and 10.5% at $B^B = 4.50$ for $Z = 82$ (Pb). In other words, $\delta_{cc}[(dI/d\omega)_{0}] \approx \delta_{cc}[G(s)] \approx 2\delta_{cc}[\Phi(s)]$, and $\delta_{cc}[(dI/d\omega)] \approx 1.7\delta_{cc}[(dI/d\omega)_{0}]$ at $s \ll 1$. The $\delta_{cc}[(dI/d\omega)]$ module amounts to 7.4% at $B^B = 8.46$ and 16.0% at $B^B = 8.46$ for $Z = 82$ ($s = 0.05$). Its upper limit is about 19% over the ranges considered.
Figure 3. The $Z$ dependence of relative CCs to the quantities of the quantum LPM effect theory for the regime of strong LPM suppression: (a) $\delta_{CC}[\Phi(s)]$, (b) $\delta_{CC}[G(s)]$ and (c) $\delta_{CC}[\langle dI/d\omega \rangle]$ at $s = 0.05$ and $B^B = 8.46$.

Thus, we can conclude that such corrections as $\delta_{CC}[\langle dI/d\omega \rangle]$, $\delta_{CC}[G(s)]$ and $\delta_{CC}[\langle dI/d\omega \rangle_0]$ become significant in the regime of strong LPM suppression and must be borne in mind, e.g., in the investigations of the LPM showers in extremely high-energy region.

4. Summary and outlook

- Within the quantum Migdal theory of the LPM effect, we have obtained the analytical results for the Coulomb corrections to the Born bremsstrahlung rate $\langle dI/d\omega \rangle$ together with CCs to the Migdal functions $G(s)$ and $\Phi(s)$ in the regimes of small and strong LPM suppression based on results of the refined Molière multiple scattering theory.

- We evaluated the above correction in the regime of small LPM suppression over the ranges $6 \leq Z \leq 92$, $4.5 \leq B^B \leq 8.46$, and $1.5 \leq s \leq \infty$ and showed that while the modules of $\delta_{CC}[G(s)]$ and $\delta_{CC}[\Phi(s)]$ do not exceed 0.1%, the modulus of relative Coulomb correction to the Born bremsstrahlung rate $\delta_{CC}[\langle dI/d\omega \rangle]$ can reach a value of about 11% at $B^B = 4.5$ and $Z = 92$.

- We have performed analogous calculations for the regime of strong LPM suppression over the ranges $6 \leq Z \leq 92$ and $4.5 \leq B^B \leq 8.46$ at $s = 0.05$ and found that $\delta_{CC}[\langle dI/d\omega \rangle]$ has a sufficiently large value that ranges from around $-8\%$ for $B^B = 8.46$ up to $-19\%$ for $B^B = 4.5$ ($Z = 92$). The contribution of this correction should be appropriately considered, for instance, in the accurate modeling of electromagnetic cascade LPM showers in ultrahigh-energy region.

- The further development of this approach involves primarily taking into account together with the bremsstrahlung also pair production. On the basis of this analytical approach and the Monte-Carlo methods including refined quantities of the quantum LPM effect theory, we will try to contribute to the clarification for the structure of three dimensional cascade showers and also to solving the problem of diversity among them [33].

- The developed approach can also be useful for the analysis of cosmic-ray experiments in ultrahigh-energy region, where the LPM effect becomes significant (e.g., in applications motivated by a research of the superhigh-energy IceCubes neutrino-induced showers over the energy region $10^{16} \leq E \leq 10^{19}$ [34], in exploring the properties of the extremely high-energy LPM showers, for computing their characteristics, etc. [28]).
Acknowledgments
The author would like to thank Prof. Akeo Misaki (Saitama University, Japan) for his interest to the work and stimulating discussions.

References
[1] Molière G 1947 Z. Naturforsch. 2 a 133; 1948 Z. Naturforsch. 3 a 78; 1955 Z. Naturforsch. 10 a 177
[2] Nakatsuka T and Nishimura J 2008 Phys. Rev. E 78 021136
Nakatsuka T, Okei K, Iyono A and Bielajew A F 2015 Eur. Phys. J. A 51 161
[3] Kamata K and Nishimura J 1958 Suppl. Prog. Theor. Phys. 6 53
[4] Ambrosio M et al 2002 Nucl. Instrum. Methods Phys. Res. A 492 376; 2003 Phys. Lett. B 566 35
[5] Aoki S 2008 Nucl. Phys. (Proc. Suppl.) B 196 50
Okei K, Takahashi N and Nakatsuka T 2007 Proc. 30th Int. Cosmic Ray Conf, vol 4 (Merida, Mexico: Yucatan Autonoma Univ.) p 157
[6] Kuraev E, Voskresenskaya O and Torosyan H 2014 Phys. Rev. D 89 116016
Nakatsuka T, Okei K and Torosyan H 2014 Phys. Part. Nucl. Lett. 11 366
[7] Migdal A B 1954 Dokl. Akad. Nauk SSSR 96 49
[8] Anthony P L et al 1995 Phys. Rev. Lett. 75 1949; 1997 Phys. Rev. D 56 1373
[9] Klein S 2013 private communication
[10] Varfolomeev A A, Gerasimova R I, Gurevich I I et al 1960 Sov. Phys.–JETP 11 23
Varfolomeev A A, Glebov V I, Denisov E I et al 1976 Sov. Phys.–JETP 42 218
[11] Yoshida K, Kobayashi T and Nishimura J 2018 26th Extended European Cosmic Ray Symposium and 35th Russian Cosmic Ray Conference. Book of Abstracts ed A A Lagutin, R I Raikin et al (Barnaul: Altay State University Press) p 208
Kobayashi T, Yoshida K, Komori Y et al 2009 Proc. 31th Int. Cosmic Ray Conf, vol 2 (Łódź, Poland: University of Łódź) p 1039
[12] Gerhardt L and Klein S R 2010 Phys. Rev. D 82 074017
Klein S 1999 Rev. Mod. Phys. 71 1501
Misaki A 1989 Phys. Rev. D 40 3086
[13] Staniev T, Vankov Ch, Streitmatter R W et al 1982 Phys. Rev. D 25 1291
Misaki A 1990 Fort. Phys. 38 413
[14] Streitmatter R W and Stephen S A 1985 Proc. 19th Int. Cosmic Ray Conf, vol 7 (La Jolla, USA: Conference Papers) p 227
[15] Dedenko L G, Kolomatsky S G, Letyugin V G et al 1987 Proc. 20th Int. Cosmic Ray Conf, vol 5 (Moscow: Nauka) p 409
[16] 2018 26th Extended European Cosmic Ray Symposium and 35th Russian Cosmic Ray Conference. Book of Abstracts ed A A Lagutin, R I Raikin et al (Barnaul: Altay State University Press) p 206
Pomansky A 1947 Izv. Akad. Nauk SSSR, Ser.Fiz. 32 497
Klein S, Shibata M, Yuda T et al 1974 Proc. Int. Cosmic Ray Symp. on High Energy Phenomena (Tokyo: Cosmic Ray Laboratory, University of Tokyo) p 142
[17] Konishi E, Misaki A and Fujimaki N 1978 Nuovo Cim. A 44 509
[18] Dedenko L G, Matsushko V L, Stern B E et al 1981 Proc. 17th Int. Cosmic Ray Conf, vol 6 (Paris: CEN Saclay) p 159
[19] Misaki A 1988 ICR-Report 169-88-15 (Tanashi: Institute for Cosmic Ray Research, University of Tokyo)
[20] Konishi E, Adachi A, Takahashi N and Misaki A 1991 J. Phys. G 17 719
Misaki A 2018 26th Extended European Cosmic Ray Symposium and 35th Russian Cosmic Ray Conference. Book of Abstracts ed A A Lagutin, R I Raikin et al (Barnaul: Altay State University Press) p 207
[21] Cillis A N, Fanchiotti H, Garcia Canal C A and Sciutto S J 1999 Phys. Rev. D 59 113012
Aartsen M G et al (IceCube Collaboration) 2018 Phys. Rev D 98 062003 [Preprint astro-ph.HE/1807.01820]