DISTORTIONS OF THE HELICOID

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Abstract. Colding and Minicozzi have shown that an embedded minimal disk $0 \in \Sigma \subset B_R$ in $\mathbb{R}^3$ with large curvature at $0$ looks like a helicoid on the scale of $R$. Near $0$, this can be sharpened: on the scale of $|A|^{-1}(0)$, $\Sigma$ is close, in a Lipschitz sense, to a piece of a helicoid. We use surfaces constructed by Colding and Minicozzi to see this description cannot hold on the scale $R$.

In [3, 4, 5, 6], Colding and Minicozzi give a complete description of the structure of embedded minimal disks in a ball in $\mathbb{R}^3$. Roughly speaking, they show that any such surface is either modeled on a plane (i.e. is nearly graphical) or is modeled on a helicoid (i.e. is two multi-valued graphs glued together along an axis). In the latter case, the distortion may be quite large. For instance, in [8], Meeks and Weber “bend” the helicoid; that is, they construct minimal surfaces where the axis is an arbitrary $C^{1,1}$ curve (see Figure 2). A more serious example of distortion is given by Colding and Minicozzi in [2]. There they construct a sequence of minimal disks modeled on the helicoid, but where the ratio between the scales (a measure of the tightness of the spiraling of the multi-graphs) at different points of the axis becomes arbitrarily large (see Figure 1). Note, locally, near points of large curvature, the surface is close to a helicoid, and so the distortions are necessarily global in nature.

Figure 1. A cross section of one of Colding and Minicozzi’s examples. Here $R = 1$ and $(0, s)$ is a blow-up pair.
Following [4] we make the meaning of large curvature precise by saying a pair $(y, s) \in \Sigma \times \mathbb{R}^+$ is a (C) blow-up pair if $\sup_{B_{\epsilon}\cap \Sigma} |A|^2 \leq 4C^2 s^{-2} = 4|A|^2(y)$ (here $C$ is large and fixed and $\Sigma \subset \mathbb{R}^3$ minimal). For $\Sigma$ minimal with $\partial \Sigma \subset \partial B_R$ where $(0, s)$ is a blow-up pair, there are two important scales; $R$ the outer scale and $s$ the blow-up scale. The work of Colding and Minicozzi gives a value $0 < \Omega < 1$ so that the component of $\Sigma \cap B_{\Omega R}$ containing 0 consists of two multi-valued graphs glued together (see for instance Lemma 2.5 of [7] for a self-contained explanation). On the other hand, Theorem 1.5 of [1] shows that on the scale of $s$ (provided $R/s$ is large), $\Sigma$ is bi-Lipschitz to a piece of a helicoid with Lipschitz constant near 1. Using the surfaces constructed in [2] we show that such a result cannot hold on the outer scale and indeed fails to hold on certain smaller scales:

**Theorem 0.1.** Given $1 > \Omega, \epsilon > 0$ and $1/2 > \gamma \geq 0$ there exists an embedded minimal disk $0 \in \Sigma$ with $\partial \Sigma \subset \partial B_R$ and $(0, s)$ a blow-up pair so: the component of $B_{\Omega R^{-\gamma} s} \cap \Sigma$ containing 0 is not bi-Lipschitz to a piece of a helicoid with Lipschitz constant in $((1 + \epsilon)^{-1}, 1 + \epsilon)$.

First, we recall the surfaces constructed in [2]:

**Theorem 0.2.** (Theorem 1 of [2]) There is a sequence of compact embedded minimal disks $0 \in \Sigma_i \subset B_1 \subset \mathbb{R}^3$ with $\partial \Sigma_i \subset \partial B_1$ containing the vertical segment $\{(0, t): |t| \leq 1\} \subset \Sigma_i$ such that the following conditions are satisfied:

1. $\lim_{i \to \infty} |A_{\Sigma_i}|^2(0) \to \infty$
2. $\sup_{\Sigma_i} |A_{\Sigma_i}|^2 \leq 4|A_{\Sigma_i}|^2(0) = 8a_i^{-4}$ for a sequence $a_i \to 0$
3. $\sup_{\Sigma_i} \sup_{B_{\delta} \subset \Sigma_i} |A^{\Sigma_i}|^2 < K\delta^{-4}$ for all $1 > \delta > 0$ and $K$ a universal constant.
4. $\Sigma_i \setminus \{x_3 - \text{axis}\} = \Sigma_{1,i} \cup \Sigma_{2,i}$ for multi-valued graphs $\Sigma_{1,i}$ and $\Sigma_{2,i}$.
Proof. We proceed by contradiction, that is suppose there were a \( D > 0 \) extrinsic density ratio, i.e. \( D > 1 \). This also gives \( \Theta_i \) near the axis, whereas away from the axis use \( \Theta_i \) and Heinz’s curvature estimates.

Next introduce some notation. For a surface \( \Sigma \) (with a smooth metric) we denote intrinsic balls by \( \mathcal{B}_\gamma^\Sigma \) and define the (intrinsic) density ratio at a point \( p \) as: 
\[
\theta_\gamma(p, \Sigma) = (\pi s^2)^{-1} \text{Area}(\mathcal{B}_\gamma^\Sigma(p)) .
\]
When \( \Sigma \) is immersed in \( \mathbb{R}^3 \) and has the induced metric, \( \theta_\gamma(p, \Sigma) \leq \Theta_\gamma(p, \Sigma) = (\pi s^2)^{-1} \text{Area}(B_s(p) \cap \Sigma) \), the usual (extrinsic) density ratio. Importantly, the intrinsic density ratio is well-behaved under bi-Lipschitz maps. Indeed, if \( f : \Sigma \to \Sigma' \) is injective and with \( \alpha^{-1} < \text{Lip } f < \alpha \), then:
\[
\theta_{\alpha^{-1}}(p,\Sigma) \leq \theta_\gamma(f(p),\Sigma') \leq \alpha^4 \theta_{\alpha^{-1}}(p,\Sigma). \tag{0.1}
\]
This follows from the inclusion, \( \mathcal{B}_{\alpha^{-1}}(f^{-1}(p)) \subset f^{-1}(\mathcal{B}_{\gamma}^\Sigma(p)) \) and the behavior of area under Lipschitz maps, \( \text{Area}(f^{-1}(\mathcal{B}_{\gamma}^\Sigma(p))) \leq (\text{Lip } f^{-1})^2 \text{Area}(\mathcal{B}_{\gamma}^\Sigma(p)) \).

Note that by standard area estimates for minimal graphs, if \( \Sigma \cap B_s(0) \) is a minimal graph then \( \theta_\gamma(p, \Sigma) \leq 2 \). In contrast, for a point near the axis of a helicoid, for large \( s \) the density ratio is large. Thus, in a helicoid the density ratio for a fixed, large \( s \) measures, in a rough sense, the distance to the axis. More generally, this holds near blow-up pairs of embedded minimal disks:

**Lemma 0.4.** Given \( D > 0 \) there exists \( R > 1 \) so: If \( 0 \in \Sigma \subset B_{2R_s} \) is an embedded minimal disk with \( \partial \Sigma \subset \partial B_{2R_s} \) and \( (0, s) \) a blow-up pair then \( \theta_{R_s}(0, \Sigma) \geq D \).

**Proof.** We proceed by contradiction, that is suppose there were a \( D > 0 \) and embedded minimal disks \( 0 \in \Sigma_i \) with \( \partial \Sigma_i \subset \partial B_{2R_i} \) with \( R_i \to \infty \) and \( (0, s) \) a blow-up pair so that \( \theta_{R_i}(0, \Sigma_i) \leq D \). The chord-arc bounds of \([2]\) imply there is a \( 1 > \gamma > 0 \) so \( \mathcal{B}_{\gamma}^{\Sigma_i}(0) \supset \Sigma_i \cap B_{\gamma R_i} \). Hence, the intrinsic density ratio bounds the extrinsic density ratio, i.e. \( D \geq \theta_{R_i}(p, \Sigma_i) \geq \gamma^2 \Theta_{\gamma R_i}(p, \Sigma_i) \). Then, by a result of Schoen and Simon \([9]\) there is a constant \( K = K(D \gamma^{-2}) \), so \( |A_{\Sigma_i}|^2(0) \leq K(\gamma R_i)^{-2} \).

But for \( R_i \) very large this contradicts that \((0, s)\) is a blow-up pair for all \( \Sigma_i \). \( \square \)

**Remark 0.5.** Note that the above does not depend on the strength of chord-arc bounds. In fact, it is also an immediate consequence of the fact that intrinsic area bounds on a disk give total curvature bounds. In turn, the total curvature bounds again yield uniform curvature bounds. See Section 1 of \([3]\) for more detail.

To produce our counterexample, we exploit the fact that two points on a helicoid that are equally far from the axis must have the same density ratio. Assuming the existence of a Lipschitz map between our surface \( \Sigma \) and a helicoid, we get a contradiction by comparing the densities for two appropriately chosen points that map to points equally far from the axis of the helicoid.

**Proof.** (of Theorem 0.1) Fix \( 1 > \Omega, \epsilon > 0 \) and \( 1/2 > \gamma \geq 0 \) and set \( \alpha = 1 + \epsilon \).

Let \( \Sigma_i \) be the surfaces of Theorem 0.2, we claim for \( i \) large, \( \Sigma_i \) will be the desired example. Suppose this was not the case. Setting \( s_i = C a_i^2 / \sqrt{2} \), where \( a_i \) is as in \([2]\) and \( C \) is the blow-up constant, one has \((0, s_i)\) is a blow-up pair in \( \Sigma_i \), since \( \sup_{\Sigma_i \cap B_{s_i}} |A_{\Sigma_i}|^2 \leq 8a_i^{-4} = 4C^{-2}s_i^{-2} = 4|A_{\Sigma_i}|^2(0) \), moreover, \( s_i \to 0 \). Hence, with \( R_i = \Omega s_i^2 < 1 \), the component of \( B_{R_i} \cap \Sigma_i \) containing \( 0 \), \( \Sigma'_i \), is bi-Lipschitz to a piece of a helicoid with Lipschitz constant in \((\alpha^{-1}, \alpha)\). That is, there are subsets \( \Gamma_i \) of helicoids and diffeomorphisms \( f_i : \Sigma'_i \to \Gamma_i \) with \( \text{Lip } f_i \in (\alpha^{-1}, \alpha) \).

**Remark 0.3.** \([2]\) and \([3]\) are slightly sharper than what is stated in Theorem 1 of \([2]\), but follow easily. \([2]\) follows from the Weierstrass data (see Equation (2.3) of \([2]\)) and \( \Theta_i \) near the axis, whereas away from the axis use \( \Theta_i \) and Heinz’s curvature estimates.
We now begin the density comparison. First, Lemma 0.4 implies there is a constant $r > 0$ so for $i$ large $\theta_{rs_i}(0, \Sigma'_i) \geq 4\alpha^8$ and thus by (0.1) $\theta_{rs_i}(f_i(0), \Gamma_i) \geq 4\alpha^8$. We proceed to find a point with small density on $\Sigma_i$ that maps to a point on $\Gamma_i$ equally far from the axis as $f_i(0)$ (which has large density).

Let $U_i$ be the (interior) of the component of $B_{1/2R_i} \cap \Sigma_i$ containing 0. Note for $i$ large enough, as $s_i/R_i \to 0$, the distance between $\partial U_i$ and $\partial \Sigma_i'$ is greater than $4\alpha^2rs_i$. Similarly, for $p \in \partial U_i$ for $i$ large, $p' \in B_{\Sigma_i'}^{4\alpha^2rs_i}(p)$ implies $|p'| \geq \frac{1}{4}R_i$. Hence, property (3) gives that $\theta_{\alpha^2rs_i}(\Sigma_i', U_i) \leq \theta_{\alpha^2rs_i}(f_i(0), \Gamma_i) \geq 4\alpha^4$ so $2\alpha^4 \geq \alpha^4\theta_{rs_i}(p_i, \Sigma_i') \geq 4\alpha^4$.

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