RATIONAL MORITA EQUIVALENCE FOR HOLOMORPHIC POISSON MODULES

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Abstract. We introduce a weak concept of Morita equivalence, in the birational context, for Poisson modules on complex normal Poisson projective varieties. We show that Poisson modules, on projective varieties with mild singularities, are either rationally Morita equivalent to a flat partial homomorphic sheaf, or a sheaf with a meromorphic flat connection or a co-Higgs sheaf. As an application, we study the geometry of meromorphic rank two \( \mathfrak{sl}_2 \)-Poisson modules which can be interpreted as a Poisson analogous to transversally projective structures for codimension one holomorphic foliations. Moreover, we describe the geometry of the symplectic foliation induced by the Poisson connection on the projectivization of the Poisson module.

1. INTRODUCTION

K. Morita in his celebrated work [38] introduced an equivalence in algebra proving that two rings have equivalent categories of left modules if and only if there exists an equivalence bimodule for the rings. Weinstein [50] and Xu [51] have introduced a geometric Morita equivalence in the context of integrable Poisson real manifolds having as one of the motivations the fact that symplectic realizations of Poisson manifolds is the analogous to representations of associative algebras. For non-integrable Poisson manifolds, an infinitesimal notion of Morita equivalence has been introduced by Crainic in [18] and by Ginzburg in [24] in order to study the invariance, respectively, of Poisson cohomology and Poisson Grothendieck groups.

In this work we introduce an weak concept of Morita equivalence in the birational context in order to reduce the study of Poisson modules, on normal complex projective varieties, either partial connections or meromorphic flat connection or co-Higgs sheaves. More precisely, we say that two Poisson normal projective variety \((X, \sigma_1)\) and \((Y, \sigma_2)\) are rationally Morita equivalents if there exists a normal variety \((S, \varrho)\), with a (possibly meromorphic) Poisson bivector \(\varrho\), and two diagrams

\[
\begin{array}{ccc}
(S, \varrho) & \xrightarrow{h} & (Y, \sigma_2) \\
\downarrow{f} & & \downarrow \\
(X, \sigma_1) & & 
\end{array}
\]

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such that $f$ and $h$ are dominant Poisson morphisms.

A. Polishchuk in [39] has studied the algebraic geometry of Poisson modules motivated by Bondal’s conjecture about the non-triviality of the degeneration locus of a Poisson structure, see also [5, 26, 20]. Poisson modules also appear in the context of generalized complex geometry introduced by N. Hitchin in [31] and developed by M. Gualtieri in [28]. Gualtieri’s concept of a generalized holomorphic bundles [27] in the Poisson case coincides with the notion of Poisson modules and if the Poisson structure is the trivial one we obtain a co-Higgs bundle [29]. Motivated by Hitchin’s work [29] the authors in [32] have established a Kobayashi–Hitchin correspondence for Poisson modules via bi-Hermitian geometry. Rayan in [43] has studied moduli spaces of stable co-Higgs bundles on the projective line. See also [8, 42, 14, 8, 6, 15] for more details on the study of co-Higgs sheaves and their moduli spaces.

A Poisson structure on a projective complex manifold induces a natural Poisson structure on its minimal model by pushing forward the Poisson bivector. Thus, it is natural to consider Poisson varieties with mild singularities motivated by the development of the minimal model program. We say that a Poisson variety $(X, \sigma)$ is klt if $X$ is a klt variety and the Poisson structure $\sigma$ is either generically symplectic or the associated symplectic foliation $\mathcal{F}_{\sigma}$ has canonical singularities. We say that $(X, \sigma)$ is transcendental, in the spirit of [2, 35], if the symplectic foliation $\mathcal{F}_{\sigma}$ has no positive-dimensional algebraic subvariety tangent to passing through a general point of $X$.

**Theorem 1.1.** Let $(E, \nabla)$ be a locally free Poisson module on a klt Poisson projective variety $(X, \sigma)$. Then one of the following holds up to rational Morita equivalence:

(a) $(E, \nabla)$ corresponds to a flat partial holomorphic sheaf on a transcendental Poisson variety;

(b) $(E, \nabla)$ corresponds to a meromorphic flat connection on a generically symplectic variety.

(c) $(E, \nabla)$ corresponds to a co-Higgs sheaf on a variety with trivial Poisson structure.

(d) $(E, \nabla)$ corresponds to a meromorphic co-Higgs sheaf $(E_0, \psi)$ on a transcendental Poisson variety $(Y, \sigma_0)$, there exist a rational map $\zeta : Y \dashrightarrow B$, over a variety $B$ with $\dim(B) = \dim(\mathcal{F}_{\sigma_0})$, the co-Higgs field $\psi$ is tangent to $TY|_B$ and satisfies $D_0(\psi) = 0$, where $D_0$ is a meromorphic extension of a Poisson connection on $TY|_B \otimes \text{End}(E_0)$.

As an application of Theorem 1.1 we provide a structure theorem for rank two $\mathfrak{sl}_2$-Poisson modules.
Corollary 1.2. Let \((E, \nabla)\) be a rank two \(\mathfrak{sl}_2\)-Poisson holomorphic module on a klt Poisson projective variety \((X, \sigma)\). Then there exist projective varieties \(Y\) and \(Z\) with klt singularities and a quasi-étale Poisson cover \(f : W \times Y \to X\) and one of the following holds:

(a) \((\pi_2)_* f^*(E, \nabla)\) is a \(\mathfrak{sl}_2\) partial holomorphic sheaf on \(Y\), where \(\pi_2\) denotes the projection on \(Y\).

(b) \(W\) and \(Y\) are generically symplectics, then \((\pi_2)_* f^*(E, \nabla)\) is a rank two locally free sheaf with a meromorphic flat connection with poles on the degeneracy Poisson divisor of \(Y\).

(c) \(W\) is symplectic and after a birational trivialization of \(f^*(E, \nabla)\) the Poisson connection is defined as

\[
\hat{\nabla} = \delta_W + \begin{pmatrix} f_1 & f_2 \\ f_0 & -f_1 \end{pmatrix} \otimes v,
\]

for some rational vector field \(v\) tangent to \((Y, 0)\), rational functions \(f_0, f_1, f_2 \in K(Y)\), and \(\delta_W\) denotes the Poisson differential on \(W\).

(d) There exist a rational map \(\zeta : Y \dashrightarrow B\), over a variety \(B\) with \(\dim(B) = \dim(\mathcal{F}_\alpha)\), such that \((\pi_2)_* f^*(E, \nabla)\) corresponds to a \(\mathfrak{sl}_2\)-Poisson meromorphic module \((E_0, \hat{\nabla})\), such that after a birational trivialization the Poisson connection defined on the trivial bundle as

\[
\hat{\nabla} = \delta + \begin{pmatrix} f_1 & f_2 \\ f_0 & -f_1 \end{pmatrix} \otimes v
\]

for some rational Poisson vector field \(v\) and rational functions \(f_0, f_1, f_2\) on \(X\) such that \(\{f_i, f_j\} = 0\), for all \(i, j\).

In the section 5 we point out that the study of rank two \(\mathfrak{sl}_2\)-Poisson modules is equivalent to the understanding of the following objects:

i) a triple of rational vector fields \((v_0, v_1, v_2)\) on \(X\) such that

\[
\begin{align*}
\delta(v_0) &= v_0 \wedge v_1 \\
\delta(v_1) &= 2v_0 \wedge v_2 \\
\delta(v_2) &= v_1 \wedge v_2,
\end{align*}
\]

(1)

ii) the symplectic foliation \(\mathcal{F}_\nabla\) correspondent to the Polishchuk’s Poisson structure induced by \(\nabla\) on \(\pi : \mathbb{P}(E, \nabla) \to (X, \sigma)\).

We can say that the study of the triples \((v_0, v_1, v_2)\) satisfying (2) is the Poisson analogous to transversely projective holomorphic foliations theory due to B. Scárdua [45]. We refer the works [17, 16, 36], where the authors have studied transversely projective foliations via meromorphic connections on rank two vector bundles [19].
From Corollary 1.2 we observe that the geometric study of the symplectic foliation \( \mathcal{F} \) reduces, up to a quasi-étale Poisson cover, to the foliation \( \mathcal{F}_{\nu_0} \) on \( \mathbb{P}(E_0, \nabla_0) \to (Y, \sigma_0) \). Our next result describes the geometry of such foliation.

**Corollary 1.3.** Setting as in Corollary 1.2. Let \( \mathcal{F}_{\nu_0} \) be the symplectic foliation induced on \( \mathbb{P}(E_0) \to (Y, \sigma_0) \). Then one of the following holds:

(a) \( \mathcal{F}_{\nu_0} \) is a dimension 2 foliation which is a pull-back of a foliations by curves on \( (Y, 0) \).

(b) \( \mathcal{F}_{\nu_0} \) is a Riccati foliation of codimension one on \( \mathbb{P}(E_0) \), if \( (Y, \sigma_0) \) is generically symplectic.

(c) \( \mathcal{F}_{\nu_0} \) is a Riccati foliation of codimension one on \( \mathbb{P}(E_0) \) which is given by a morphism \( \mathcal{A} \to \text{det}_{\nu_0}(\pi^*(T\mathcal{F}_{\nu_0}^*)) \subset \Omega^1_{\mathbb{P}(E_0)} \), where \( \mathcal{A} \) is a line bundle and \( \text{det}_{\nu_0} : \pi^*\Omega^1_Y \to \Omega^1_{\mathbb{P}(E_0)} \) is the pull-back morphism of reflexive forms.

(d) There exist a rational Poisson vector field \( v \) generically transversal to \( \mathcal{F}_{\sigma_0} \) such that \( \mathcal{F}_{\nu_0} \) has dimension \( 2k + 2 \) and it is the pull-back of the foliation induced by \( v \) and \( \mathcal{F}_{\sigma_0} \). In particular, if \( \dim(Y) = 2k + 1 \), then \( \mathbb{P}(E_0) \) is generically symplectic and there exist a rational Poisson map \( \zeta : Y \to B \) generically transversal to \( \mathcal{F}_{\sigma_0} \), where \( B \) is a generically symplectic variety with \( \dim(B) = 2k \) and the induced map \( \mathbb{P}(E_0) \to B \) is Poisson.

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## 2. Holomorphic foliations

Throughout this paper a variety is a scheme of finite type over \( \mathbb{C} \) and regular in codimension one. As usual, we denote by \( TX = \text{Hom}(\Omega^1_X, \mathcal{O}_X) \) the tangent sheaf of \( X \) and \( X_{\text{reg}} \) its smooth locus. Given \( p \in \mathbb{N} \), we denote by \( \Omega^p_X \) the sheaf \( (\Omega^p_X)^{**} \).

Let \( X \) be a normal variety and suppose that \( K_X \) is \( \mathbb{Q} \)-Cartier, i.e., some nonzero multiple of it is a Cartier divisor. Consider a resolution of singularities \( f : Z \to X \). There are uniquely defined rational numbers \( a(E, X, \mathcal{F}) \geq 0 \) such that

\[
K_Z = f^*K_X + \sum_i a(E_i, X)E_i,
\]

where \( E_i \) runs through all exceptional prime divisors for \( f \). We say that \( X \) is klt if all \( a(E_i, X) > -1 \) for every \( f \)-exceptional prime divisor \( E_i \), for some resolution \( f : Z \to X \). For more details we refer to [34 Section 2.3].
Definition 2.1. A morphism $f : Z \to X$ between normal varieties is called a quasi-étale morphism if $f$ is finite and étale in codimension one.

Definition 2.2. A foliation $\mathcal{F}$ on a normal variety $X$ is a coherent subsheaf $T\mathcal{F} \subset TX$ such that $\mathcal{F}$ is closed under the Lie bracket, and the dimension of $\mathcal{F}$ is the generic rank of $T\mathcal{F}$. The canonical bundle is defined by $K_{\mathcal{F}} = \det(T\mathcal{F})^{**}$.

We will denote by $a : T\mathcal{F} \to TX$ an injective morphism (anchor map) defining the foliation $\mathcal{F}$.

The singular set of $\mathcal{F}$ is defined by $\text{Sing}(\mathcal{F}) = \text{Sing}(N\mathcal{F})$, where $N\mathcal{F} = TX/T\mathcal{F}$ is the normal sheaf of the foliation. Hereafter we will suppose that $\text{cod}(\text{Sing}(\mathcal{F})) \geq 2$. We have an exact sequence of sheaves

$$0 \to T\mathcal{F} \to TX \to N\mathcal{F} \to 0.$$

Definition 2.3. Let $\mathcal{F}$ be a holomorphic foliation on a projective variety $X$ and $f : Y \to X$ a projective birational morphism. We say that $\mathcal{F}$ has canonical singularities if the divisor $K_{f^{-1}\mathcal{F}} - f^*K_{\mathcal{F}}$ is effective.

3. Poisson modules

A Poisson structure on a variety $X$ is a $\mathbb{C}$-linear Lie bracket

$$\{\cdot, \cdot\} : \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$$

which satisfies the Leibniz rule $\{f, gh\} = h\{f, g\} + f\{g, h\}$ and Jacobi identities

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

for all $f, g, h \in \mathcal{O}_X$. The bracket corresponds to a bivector field $\sigma \in H^0(X, \wedge^2TX)$ given by $\sigma(df \wedge dg) = \{f, g\}$, for all $f, g \in \mathcal{O}_X$.

We will denote a Poisson structure on $X$ as the pair $(X, \sigma)$, where $\sigma \in H^0(X, \wedge^2TX)$ is the correspondent Poisson bivector field. The bivector induces a morphism

$$\sigma^# : \Omega^1_X \to TX$$

which is called by anchor map and it is defined by $\sigma^#(\theta) = \sigma(\theta, \cdot)$, for all $\theta \in \Omega^1_X$.

Definition 3.1. The symplectic foliation associated to $\sigma$ is the foliation given by $\mathcal{F}_\sigma := \text{Ker}(\sigma^#)$, whose dimension is the rank the anchor map $\sigma^# : \Omega^1_X \to TX$. A Poisson variety $(X, \sigma)$ is called generically symplectic if the anchor map $\sigma^# : \Omega^1_X \to TX$ is generically an isomorphism. Then, the degeneracy loci of $\sigma^#$ is an effective anti-canonical divisor $D(\sigma) \in |-K_X|$.

A meromorphic Poisson bivector is a meromorphic section $\sigma$ of $\wedge^2TX$ such that $[\sigma, \sigma] = 0$, where $[,]$ denotes the Schouten bracket. Observe that in this case we have a Poisson structure outside of the poles divisor of $\sigma$. 
A rational vector field is a section \( v \in H^0(X, TX \otimes \mathcal{L}) \) for some invertible sheaf \( \mathcal{L} \). We say that \( v \) is a Poisson rational vector field with respect to \( \sigma \) if it is an infinitesimal symmetry of \( \sigma \), i.e., \( L_v(\sigma) = 0 \), where \( L_v(\cdot) \) denotes the Lie derivative.

We denote by \( \delta \) the corresponding Poisson differential. For instance, \( \delta(f) = -\sigma^#(df) \) and \( \delta(v) = L_v(\sigma) \), for all \( f \in \mathcal{O}_X \) and germs of vector field \( v \).

**Definition 3.2.** We say that a Poisson variety \((X, \sigma)\) is klt if \( X \) is a klt variety and the Poisson structure \( \sigma \) is either generically symplectic or the associated symplectic foliation \( F_\sigma \) has canonical singularities.

**Definition 3.3.** Let \((X, \sigma)\) be a Poisson projective variety. A Poisson connection on a sheaf of \( \mathcal{O}_X \)-module \( E \) is a \( \mathbb{C} \)-linear morphism of sheaves \( \nabla : E \to TX \otimes E \) satisfying the Leibniz rule
\[
\nabla(fs) = \delta(f) \otimes s + f \nabla(s)
\]
for all \( f \in \mathcal{O}_X \) and \( s \in E \). We say that \( E \) is a Poisson module if it admits a Poisson flat connection. Equivalently, a Poisson connection defines a \( \mathbb{C} \)-linear bracket \( \{ \cdot, \cdot \} : \mathcal{O}_X \times E \to E \) by
\[
\{f, s\} := \nabla(s)(df)
\]
for all \( f \in \mathcal{O}_X \) and \( s \in E \).

**Definition 3.4.** Let \( a : \mathcal{G} \to TX \) be a holomorphic foliation. A holomorphic partial connection on a sheaf of \( \mathcal{O}_X \)-module \( E \) is a \( \mathbb{C} \)-linear morphism of sheaves \( \nabla : E \to \mathcal{G}^* \otimes E \) satisfying the Leibniz rule
\[
\nabla(fs) = a^*(df) \otimes s + f \nabla(s)
\]
for all \( f \in \mathcal{O}_X \) and \( s \in E \) and \( a^* : \Omega_X^1 \to \mathcal{G}^* \) denotes the dual map.

**Example 3.5.** Let \((X, \sigma)\) be a generically symplectic variety of dimension \( 2n \) with degeneracy divisor \( \mathcal{D}(\sigma) = \mathcal{D} \). It follows from [40, Proposition 4.4.1] that \( \sigma \) induces a skew-symmetric morphism
\[
\sigma^# : \Omega_X^1(\log \mathcal{D}) \to TX(-\log \mathcal{D})
\]
which is an isomorphism if and only if \( \mathcal{D} \) is reduced [40, Proposition 4.4.2]. Therefore, if \( \mathcal{D} \) is reduced then the isomorphism \( \sigma^# \) gives an one-to-one correspondence between Poisson flat connections and logarithmic flat connections. If \( \mathcal{D} \) is not reduced, then a Poisson connection corresponds to a meromorphic flat connection with poles along \( \mathcal{D} \).

**Example 3.6.** Let \((X, \sigma)\) be a Poisson variety which is not generically symplectic. If \( E \) admits a holomorphic flat connection \( \nabla : E \to \Omega_X^1 \otimes E \). Then
\[
\sigma^# \circ \nabla : E \to \Omega_X^1 \otimes E \to T\mathcal{F}_\sigma \otimes E
\]
is a Poisson flat connection on \( E \) tangent to the symplectic foliation \( T\mathcal{F}_\sigma \).
Example 3.7. Let \((X, \sigma)\) be a Poisson projective variety whose the symplectic foliation \(\mathcal{F}_\sigma\) is algebraic, regular in codimension one, and its leaves are the fibers of a rational map \(\rho: X \to Y\). If \(\nabla : E \to \Omega^1_Y \otimes E\) is a holomorphic flat connection, then \(\rho^*E\) is a Poisson module on \(X\), with Poisson connection given by

\[
\omega \circ \rho^*\nabla : \rho^*E \to \Omega^1_{Y|X} \otimes \rho^*E \to T_{Y|X} \otimes \rho^*E,
\]

where \(\omega : \Omega^1_{Y|X} \to T_{Y|X}\) denotes the induced isomorphism given by \((\sigma^\#)|_{T\mathcal{F}_\sigma}\).

Example 3.8. Let \((X, 0)\) be a projective variety with the trivial Poisson structure. Let \((E, \phi)\) be a co-Higgs bundle on \(X\). Then \(E\) is a Poisson module on \(X\) whose Poisson structure is given by \(\nabla (fs) := \phi(df)s\).

Example 3.9. Let \((X, \sigma)\) be a Poisson projective manifold and \(\rho : X \to Y\) a rational map with connected fibers such that \(\rho_*\sigma = 0\), then \(\rho_*E\) is a co-Higgs sheaf on \(Y\). In general, \(\rho_*E\) is a Poisson module on \((Y, \rho_*\sigma)\), since \(f_*\mathcal{O}_X \simeq \mathcal{O}_Y\). See in the next section Proposition 4.3 due to Polishchuk.

4. Rational Morita equivalence

Let us recall the notion of Morita equivalence of real Poisson manifolds which was developed by Weinstein [50] and Xu [51] and it works verbatim in the complex context: we say that two Poisson manifold \((X, \sigma_1)\) and \((Y, \sigma_2)\) are Morita equivalents if there exists a symplectic manifold \((S, \omega)\) and two diagrams

\[
\begin{array}{ccc}
(S, \omega) & \xleftarrow{f} & (X, \sigma_1) \\
\downarrow{h} & & \downarrow{(Y, -\sigma_2)} \\
(X, \sigma_1) & \xrightarrow{h} & (Y, -\sigma_2),
\end{array}
\]

such that are Poisson submersions. This equivalence has the following important properties:

- there is a bijection between the leaves of the symplectic foliations of \((X_1, \sigma_1)\) and \((X_2, \sigma_2)\).
- the fields of Casimir functions of \((X_1, \sigma_1)\) and \((X_2, \sigma_2)\) are isomorphic.

Definition 4.1. We say that two Poisson normal projective varieties \((X, \sigma_1)\) and \((Y, \sigma_2)\) are rationally Morita equivalents if there exists a symplectic manifold \((S, \varrho)\) with a (possibly meromorphic) Poisson bivector \(\varrho\), and two diagrams

\[
\begin{array}{ccc}
(S, \varrho) & \xleftarrow{f} & (X, \sigma_1) \\
\downarrow{h} & & \downarrow{(Y, \sigma_2)} \\
(X, \sigma_1) & \xrightarrow{h} & (Y, \sigma_2),
\end{array}
\]

such that \(f\) and \(h\) are dominant Poisson morphisms, i.e., \(f_*\varrho = \sigma_1\) and \(h_*\varrho = \sigma_1\). We say that two Poisson modules \(E_1 \to (X, \sigma_1)\) and \(E_2 \to (Y, \sigma_2)\) are rationally
Morita equivalents if there is an equivalence \((X_1, \sigma_1) \leftarrow (S, \varrho_{12}) \rightarrow (X_2, \sigma_2)\) such that \(h_* f^* E_1\) and \(E_2\) are isomorphic as Poisson modules.

We say that
\[
(X_1, \sigma_1) \leftarrow (S, \varrho_{12}) \rightarrow (X_2, \sigma_2)
\]
is isomorphic to
\[
(X_3, \sigma_3) \leftarrow (Q, \varrho_{34}) \rightarrow (X_4, \sigma_4)
\]
if there is a birational map \(\zeta : (S, \varrho_{12}) \rightarrow (Q, \varrho_{34})\) such that \(\zeta_* \varrho_{12} = \varrho_{34}\). Therefore, for each \((X_1, \sigma_1) \leftarrow (S, \varrho_{12}) \rightarrow (X_2, \sigma_2)\) we can always take the resolution of singularities \(\tilde{S} \rightarrow S\) and the lifting of the meromorphic Poisson bivector field \(\tilde{\varrho}_{12}\) will give us that \(\zeta_* \varrho_{12} = \varrho_{12}\). Thus, we obtain an equivalence

\[
\begin{array}{ccc}
\tilde{S} & \overset{\zeta}{\longrightarrow} & S \\
\downarrow{f} & & \downarrow{g} \\
(X, \sigma_1) & & (Y, \sigma_2), \\
\end{array}
\]

with \(\tilde{S}\) smooth.

Since \(f^{-1} J_{\sigma_1} = h^{-1} J_{\sigma_2}\), we obtain the following.

**Proposition 4.2.** The following holds:

1) there is a bijection between the leaves of the symplectic foliations of \((X_1, \sigma_1)\) and \((X_2, \sigma_2)\),

2) the fields of Casimir rational functions of \((X_1, \sigma_1)\) and \((X_2, \sigma_2)\) are isomorphic,

3) Morita equivalence implies the rational Morita equivalence.

Let us list some known results:

**Proposition 4.3.** (Polischuk [39]) Let \(X\) be a Poisson variety. Let \(f : X \rightarrow Y\) be a morphism such that \(f_* \mathcal{O}_X \simeq \mathcal{O}_Y\). Then the Poisson structure on \(X\) induces canonically a Poisson structure on \(Y\) such that \(f\) is a Poisson morphism. Furthermore, if \(\mathcal{E}\) is a Poisson module on \(X\), then \(f_* \mathcal{E}\) is a Poisson module on \(Y\).

**Theorem 4.4.** (Kaledin [33]) Let \(X\) be a Poisson variety. The reduction, any completion and the normalization of \(X\) are again Poisson varieties.

Therefore, we can assume that the Poisson variety \(X\) is reduced and normal. Moreover, if there exist a morphism \(f : X \rightarrow Y\) with connected fibers, then by Proposition 4.3 we have that \(X\) is rationally Morita equivalent to \(Y\), since \(f_* \mathcal{O}_X \simeq \mathcal{O}_Y\) [22, Chapter 9]. Also, this also say us that a Poisson structure on a projective complex manifold \(X\) induces a natural Poisson structure on its minimal model by pushing forward the Poisson bivector.
Example 4.5. Let $S$ be a smooth Poisson surface with a Poisson structure $\sigma \in \mathcal{H}(S, \mathcal{O}_S(-K_S))$. Let $S^{[r]} = \text{Hilb}^r(S)$ the Hilbert scheme parametrizing 0-dimensional subschemes of $S$ of length $r$. Bottacin in [9] extended a Beauville’s construction of a symplectic structure on the Hilbert scheme $S^{[r]}$, let us say $\sigma^{[r]}$, of a symplectic surface $(S, \sigma)$ [4]. Consider $\text{Bl}_\Delta(S^2)$ the blowup along the diagonal $\Delta \subset S^2 = S \times S$. Ran in [41] showed that there is a diagram

\[
\begin{array}{c}
S^{[2]} \xleftarrow{p} \text{Bl}_\Delta(S^2) \xrightarrow{q} S^2,
\end{array}
\]

such that $q^*(\sigma \times \sigma)$ is a meromorphic Poisson bivector on $\text{Bl}_\Delta(S^2)$ with simple pole on $\mathbb{P}(\Omega_S^1)$ and $p_* q^*(\sigma \times \sigma)$ is the Poisson structure $\sigma^{[2]}$ on $S^{[2]}$. Therefore, $S^{[2]}$ is rationally Morita equivalent to $S^2$. In the general, we can conclude by Ran’s induction construction that $S^{[r]}$ is rationally Morita equivalent to $S^{[r-1]} \times S$, for all $r \geq 2$, see [41, Section 1.6]. Since $S^{[r-1]} \times S$ is clearly Morita equivalent to $S$, we conclude that $S^{[r]}$ is Morita equivalent to $S$, for all $r \geq 2$.

Definition 4.6. Let $(X, \sigma)$ be a Poisson variety which is not generically symplectic. We say that $(X, \sigma)$ is transcendental, in the spirit of [2, 35], if the symplectic foliation $\mathcal{F}_\sigma$ has no positive-dimensional algebraic subvariety tangent to passing through a general point of $X$.

Definition 4.7. A morphism $f : Z \to X$ between normal varieties is called a quasi-étale morphism if $f$ is finite and étale in codimension one.

Now, we will prove our main result.

Theorem 4.8. Let $(E, \nabla)$ be a locally free Poisson module on a klt Poisson projective variety $(X, \sigma)$. Then one of the following holds up to rational Morita equivalence:

(a) $(E, \nabla)$ corresponds to a flat partial holomorphic sheaf on a transcendental Poisson variety;

(b) $(E, \nabla)$ corresponds to a meromorphic flat connection on a generically symplectic variety.

(c) $(E, \nabla)$ corresponds to a co-Higgs sheaf on a variety with trivial Poisson structure.

(d) $(E, \nabla)$ corresponds to a meromorphic co-Higgs sheaf $(E_0, \psi)$ on a transcendental Poisson variety $(Y, \sigma_0)$, there exist a rational map $\zeta : Y \dashrightarrow B$, over a variety $B$ with $\dim(B) = \dim(\mathcal{F}_{\sigma_0})$, the co-Higgs field $\psi$ is tangent to $T_{Y|B}$ and satisfies $D_0(\psi) = 0$, where $D_0$ is a meromorphic extension of a Poisson connection on $T_{Y|B} \otimes \text{End}(E_0)$. 
Proof. Let \((X, \sigma)\) be a generically symplectic variety. Consider the degeneracy Poisson divisor \(D = \{ \sigma^n = 0 \} \in |- K_X|\). Then, it follows from Example 3.5 that \(\nabla\) induces a meromorphic connection with poles along \(D\).

Suppose that \((X, \sigma)\) is not generically symplectic and consider the associated the symplectic foliation \(\mathcal{F}_\sigma\). Since \(K_{\mathcal{F}_\sigma} \simeq O_X\) and \(\mathcal{F}_\sigma\) has canonical singularities, it follows from [21, Proposition 8.14] that there exist a diagram

\[
\begin{array}{ccc}
X & \leftarrow & Z = W \times Y \\
\downarrow f & & \downarrow \pi_2 \\
Y & \rightarrow & \end{array}
\]

such that \(f : Z \to X\) is a quasi-étale cover, \(\pi_2 : Z \to Y\) is the natural projection, \(Y\) and \(Z\) are normal klt projective varieties, and it there is a transcendental foliation \(\mathcal{H}\) on \(Y\) such that \(\pi_2^{-1}\mathcal{H} = f^{-1}\mathcal{F}_\sigma\).

Since \(f : Z \to X\) is a quasi-étale cover there is a Poisson bivector \(\tilde{\sigma} \in H^0(Z, \wedge^2 T_Z)\) such that \(f_*\tilde{\sigma} = \sigma\). That is, \(f\) is a Poisson morphism. Indeed, the map \(f : f^{-1}(X_{\text{reg}}) \to X_{\text{reg}}\) is a map between complex manifolds which is a local biholomorphism, so there is a well-defined pull back Poisson bivector field \((f^{-1})^*(\sigma|_{X_{\text{reg}}})\) which extends to a section \(\tilde{\sigma} \in H^0(Z, \wedge^2 T_Z)\). Similarly, one can also see that \(f^*E\) is a Poisson module, with respect to \(\tilde{\sigma}\), by lifting the local matrices of vector fields which represent the Poisson connection \(\nabla\). Let \(\tilde{\nabla}\) denote such induced Poisson connection on \(f^*E\). Thus, we have a Morita equivalence

\[
(X, \sigma) \leftarrow (Z, \tilde{\sigma}) \rightarrow (Y, \sigma_2),
\]

where \(\sigma_2 =: (\pi_2)_*\tilde{\sigma}\). In particular, we have that \(E_1\) is rationally Morita equivalent to \((\pi_2)_*(f^*E) := E_0\). Denote by \(\nabla_0\) the Poisson connection induced on \(E_0\).

Let us make the following simple but important observation:

Let \((G, \nabla) \to (W, \sigma)\) be the flat Poisson connection on a Poisson variety \((W, \sigma)\) such that \(\text{Sing}(\mathcal{F}_\sigma) \cup \text{Sing}(W)\) has codimension \(\geq 2\). Then, it is either:

(i) \(G\) correspond to a flat holomorphic sheaf along the symplectic foliation, or

(ii) \(\nabla\) induces a non-trivial section \(\phi \in H^0(W, N\mathcal{F}_\sigma \otimes \text{End}(G))\) such that \(\phi \wedge \phi = 0\).

In fact, if \((G, \nabla)\) is such that \(\nabla : G \to T\mathcal{F}_\sigma \otimes G\), then \(G\) correspond to a flat holomorphic sheaf along the symplectic foliation. Indeed, since \(\text{cod}(\text{Sing}(\mathcal{F}_\sigma)) \geq 2\), then the map

\[
\begin{array}{ccc}
T\mathcal{F}_\sigma & \xrightarrow{i} & \Omega^1_W \\
\downarrow \phi & & \downarrow \sigma^2_W \\
\Omega^1_W & \xrightarrow{\sigma^2_W} & T\mathcal{F}_\sigma
\end{array}
\]

is an isomorphism, where \(i : T\mathcal{F}_\sigma \to \Omega^1_W\) is the inclusion and \(\sigma^2_W : \Omega^1_W \to T\mathcal{F}_\sigma\) denotes the anchor map. Therefore, the Poisson connection
\[ \vartheta^{-1} \circ \nabla_0 : G \longrightarrow \mathcal{F}_\sigma^* \otimes G \]
is a partial flat connection on \( G \). If the Poisson connection \( \nabla_0 : G \to TW \otimes G \) does not factor through \( T\mathcal{F}_\sigma \), then it induces a non-trivial section \( \phi \in H^0(W, N\mathcal{F}_\sigma \otimes \text{End}(G)) \) such that \( \phi \land \phi = 0 \). In fact, we have the commutative diagram

\[
\begin{array}{ccc}
G & \xrightarrow{\nabla} & TW \otimes G \\
\downarrow{\phi} & & \downarrow{\pi \otimes \text{Id}} \\
N\mathcal{F}_\sigma \otimes G.
\end{array}
\]

where \( \phi = (\pi \otimes \text{Id}) \circ \nabla \) and \( \pi : TW \to N\mathcal{F}_\sigma \) denotes the projection. Now, let \( W^0 := W - \text{Sing}(\mathcal{F}_\sigma) \cup \text{Sing}(W) \) and consider \( \phi_0 \) the restriction of \( \phi \) on \( W^0 \). In order to show that \( \phi \land \phi = 0 \) we follow the argument in [47] and next we extend \( \phi_0 \). Indeed, let \( \theta \) be a local matrix representing the connection \( \nabla_0 \). By flatness we have that

\[ \delta(\theta) + \theta \land \theta = 0. \]

Then \( \pi(\delta(\theta)) = 0 \), since \( \delta(\theta) \) is tangent to \( \mathcal{F}_\sigma|_{W^0} \) and \( 0 = \pi(\theta \land \theta) = \pi(\theta) \land \pi(\theta) \). Since \( \pi(\theta) \) is the local matrix of \( \phi_0 \), we conclude that \( \phi_0 \land \phi_0 = 0 \), i.e., \( \phi \land \phi = 0 \), since \( \text{Sing}(\mathcal{F}_\sigma) \cup \text{Sing}(W) \) has codimension \( \geq 2 \).

If \( \sigma_2 = 0 \), then \( (\pi_2)_*[\tilde{\nabla}] \in H^0(Y, TY \otimes \text{End}(E_0)) \) is a co-Higgs field.

If \( (Y, \sigma_2) \) is generically symplectic, then the Poisson module \((E_0, \nabla_0)\) corresponds to a meromorphic flat connection.

From now on we suppose that \( \sigma_2 \neq 0 \) and \((Y, \sigma_2)\) is not generically symplectic.

If \( \mathcal{F}_\sigma \) is algebraic, i.e., the symplectic foliation \( f^*\mathcal{F}_\sigma \) is given by the fibration \( \pi_2 : Z = W \times Y \to Y \). Then, the connection \( \tilde{\nabla} \) corresponds either to a relative flat connection \( \tilde{\nabla} : f^*E \to \Omega^{\bullet}_{Z/Y} \otimes f^*E \), since we have the isomorphism \( \Omega^{\bullet}_{Z/Y} \simeq T_{Z/Y} \), or the induced co-Higgs field \( [\tilde{\nabla}] \in H^0(Z, \pi_2^*TY \otimes \text{End}(f^*E)) \) is such that \( (\pi_2)_*[\tilde{\nabla}] \in H^0(Y, TY \otimes \text{End}(E_0)) \).

If the symplectic foliation \( \mathcal{F}_\sigma \) is not algebraic, then \( \mathcal{X} \) correspond to the symplectic foliation of the Poisson structure \( \sigma_2 \), since \( \pi_2^{-1}\mathcal{F}_{\sigma_2} = \mathcal{F}_\sigma = f^{-1}\mathcal{F}_\sigma = \pi_2^{-1}\mathcal{X} \). Therefore, as we have seen above, it is either:

(i) \((E_0, \nabla_0)\) correspond to a flat holomorphic sheaf along the symplectic foliation, or

(ii) \( \nabla_0 \) induces a non-trivial section \( \phi \in H^0(Y, N\mathcal{X} \otimes \text{End}(E_0)) \) such that \( \phi \land \phi = 0 \).
Consider an embedding $Y \subset \mathbb{P}^N$ and let $k := \dim(\mathcal{K})$. We can take a generic projection $q : \mathbb{P}^N \dashrightarrow \mathbb{P}^n$, such that restricted to $Y$ we produce a finite surjective morphism $q_Y : Y \to \mathbb{P}^n$. We also can take a generic rational linear projection $p : \mathbb{P}^n \dashrightarrow \mathbb{P}^k$ in such way that the fibration $p \circ (q_Y) : Y \to \mathbb{P}^k$ is generically transversal to $\mathcal{K}$. After we take a Stein factorization we obtain a rational fibration $\zeta : Y \to B$, with connected fibers, which induces on $Y$ an algebraic foliation $\mathcal{F}$ of dimension equal to $n - k$. Consider the tangency loci between $\mathcal{K}$ and $\mathcal{F}$ given by

$$S := \{y \in Y; \omega(\xi)(y) = 0\},$$

where $\omega \in H^0(Y, \Omega_Y^{[n-k]} \otimes \det(N\mathcal{K}))$ and $\xi \in H^0(Y, \wedge^k TY \otimes \det(T\mathcal{F})^*)$ are the tensors inducing $\mathcal{K}$ and $\mathcal{F}$, respectively. Observe that $S$ contain the singular sets of $\mathcal{K}$ and $\mathcal{F}$. By transversality the induced map

$$\beta : T\mathcal{F} \to \pi^* TY$$

is an isomorphism on $Y^0 := Y - S$. This provide us a co-Higgs field $\tilde{\psi}_0 \in H^0(Y^0, (T\mathcal{F} \otimes \text{End}(E_0))|_{Y^0})$ given by $\tilde{\psi}_0 = \phi_0 \circ \beta^{-1}$. Considering

$$T\mathcal{F} \otimes \text{End}(E_0)(\ast S) = \lim_{r \to 0} \left( (T\mathcal{F} \otimes \text{End}(E_0) \otimes \mathcal{O}_X(r \cdot S) \right)$$

the correspondent sheaf of meromorphic sections with poles of arbitrary order on $S$, we obtain a meromorphic co-Higgs field

$$\psi_0 \in H^0(Y, T\mathcal{F} \otimes \text{End}(E_0) \otimes \mathcal{O}_Y(\ast S)).$$

Recall that $\mathcal{K}$ and $\mathcal{F}$ are regular on $Y^0$. Thus, it follows from [47, Corollary 3.3] that the Partial connection on $N\mathcal{K}|_{Y^0} \simeq T\mathcal{F}|_{Y^0}$ and the Poisson connection $\nabla_0$ induce a Poisson connection $\tilde{D}_0$ on

$$(N\mathcal{K}^* \otimes \text{End}(E_0))|_{Y^0} \simeq (\Omega_Y^1|_{B} \otimes \text{End}(E_0))|_{Y^0}$$

such that $\tilde{D}_0(\tilde{\psi}_0) = 0$. Denoting by $D_0$ its meromorphic extension we have that $D_0(\psi_0) = 0$.

\[\square\]

**Remark 4.9.** If $\psi$ denotes a local representation of $\psi_0$, then from [47, Proposition 3.2, equation 3.2] we conclude that $D_0(\psi_0) = 0$ implies that

$$\delta(\psi) = 0 \quad \psi \land \psi = 0.$$

Fixed a klt Poisson structure $(X, \sigma)$, consider the associated category of Poisson modules $\text{Rep}(X, \sigma)$. Also, denote by $\text{Co-Higgs}(X)$ the category of co-Higgs bundles and $\text{Conn}(X, D)$ the category of meromorphic connections along $D$. It follows from the proof of Theorem 4.8 that via the Morita equivalence between $(X, \sigma)$ and $(Y, \sigma_2)$ we obtain the following induced functors:

- $\pi_* f^* : \text{Rep}(X, \sigma) \to \text{Co-Higgs}(Y)$, if $\pi_* f^*(\sigma) = 0$. 
• \( \pi^* f^* : \text{Rep}(X, \sigma) \to \text{Conn}(Y, D) \), if \( \pi^* f^*(\sigma) \) is generically symplectic and \( D \) is the degeneracy Poisson Divisor.

5. Rank two \( \mathfrak{sl}_2 \)-Poisson modules

Let \((X, \sigma)\) be a Poisson projective variety with Poisson bivector \(\sigma\) and denote by \(\delta\) the Lie algebroid derivation induced by \(\sigma\). As we have seen in the Theorem \[1\] the presence of singularities of the Poisson structure forces in certain situations that the Poisson connections have poles along a divisor. In this section we will study the geometry of rank two meromorphic Poisson module which are trace free.

**Definition 5.1.** A rank two vector bundle \(E\) on a projective Poisson manifold \((X, \sigma)\) is called by meromorphic Poisson module if there exist a connection

\[ \nabla : E \to E \otimes T_X(D) \]

with effective polar divisor \(D\), such that \(\nabla^2 = 0\). If \(\text{tr}(\nabla) = 0\) we say that \((E, \nabla)\) is a meromorphic \(\mathfrak{sl}_2\)-Poisson module. If \(D = \emptyset\), then \((E, \nabla)\) is a holomorphic Poisson module.

Given two different \(\mathfrak{sl}_2\)-Poisson structure \((E, \nabla_1)\) and \((E, \nabla_2)\), with effective polar divisor \(D\), then \((E, \nabla_1 - \nabla_2)\) is a \(\mathfrak{sl}_2\) co-Higgs bundle with a co-Higgs field

\[ \nabla_1 - \nabla_2 : E \to E \otimes T_X(D) \]

Firstly, we prove the following Polishchuk’s result \[39\] in our context.

**Proposition 5.2.** Let \((E, \nabla)\) be a rank two meromorphic \(\mathfrak{sl}_2\)-Poisson module on a projective Poisson manifold \(X\). Then, there exist a triple of rational vector fields \((v_0, v_1, v_2)\) on \(X\) such that

\[ \begin{align*}
\delta(v_0) &= v_0 \wedge v_1 \\
\delta(v_1) &= 2v_0 \wedge v_2 \\
\delta(v_2) &= v_1 \wedge v_2
\end{align*} \]

where \(\delta(v) = [v, \sigma]\), where \(\sigma\) is the Poisson bivector of \(X\) and \([, ,]\) denotes the Schouten bracket. Moreover, \(\mathbb{P}(E, \nabla)\) has meromorphic Poisson structure induced by \(\nabla\).

**Proof.** Since \(X\) is projective we have that \(\mathbb{P}(E)\) birationally equivalent to \(X \times \mathbb{P}^1\). In the vector bundle \(X \times \mathbb{C}^2\) we have a trace free Poisson connection given by

\[ \nabla(Z) = \delta(Z) + \begin{pmatrix} v_1 & v_2 \\ v_0 & -v_1 \end{pmatrix} \cdot Z, \]

where \(Z = (z_1, z_2) \in \mathbb{C}^2\). The flatness condition is equivalent to

\[ \begin{align*}
\delta(v_0) &= v_0 \wedge v_1 \\
\delta(v_1) &= 2v_0 \wedge v_2 \\
\delta(v_2) &= v_1 \wedge v_2
\end{align*} \]


Consider the meromorphic bivector
\[ \Sigma_\sigma = \sigma + (v_0 + 2v_1z + v_2z^2) \wedge \frac{\partial}{\partial z}, \]
where \([1 : z] \in \mathbb{P}^1\) denotes the affine coordinate. Then \(\Sigma\) is Poisson if and only if
\[
[v_0, \sigma] = \delta(v_0) = v_0 \wedge v_1 \\
[v_1, \sigma] = \delta(v_1) = 2v_0 \wedge v_2 \\
[v_2, \sigma] = \delta(v_2) = v_1 \wedge v_2.
\]

We observe that the study of the triples \((v_0, v_1, v_2)\), satisfying (2) is the Poisson analogous to a transversely projective holomorphic foliations theory due to B. Scárdua [15]. Indeed, if the Poisson connection \((E, \nabla)\) is such that
\[ \nabla: E \to T\mathcal{F}(D) \otimes E \subset TX \otimes E(D), \]
then \(\nabla\) induces on each leaf of \(\mathcal{F}\) a transversely projective holomorphic foliation. Let \(p \in X \setminus \text{Sing}(\mathcal{F})\) and \(F_p\) the symplectic leaf of \(\mathcal{F}\) passing through \(p\). Since \((\sigma^\#)^{-1} \circ \delta|_{F_p} = d \circ (\sigma^\#)^{-1}|_{F_p}\), where \(d\) denotes the de Rham differential, we have that
\[
d \circ (\sigma^\#)^{-1}(v_0) = (\sigma^\#)^{-1} \circ \delta|_{F_p}(v_0) = (\sigma^\#)^{-1}(v_0) \wedge (\sigma^\#)^{-1}(v_1) \\
d \circ (\sigma^\#)^{-1}(v_1) = (\sigma^\#)^{-1} \circ \delta|_{F_p}(v_1) = 2(\sigma^\#)^{-1}(v_0) \wedge (\sigma^\#)^{-1}(v_2) \\
d \circ (\sigma^\#)^{-1}(v_2) = (\sigma^\#)^{-1} \circ \delta|_{F_p}(v_2) = (\sigma^\#)^{-1}(v_1) \wedge (\sigma^\#)^{-1}(v_2).
\]
Now, defining \((\sigma^\#)^{-1}|_{F_p}(v_i) = \omega_i\), we obtain the transversely projective structure on \(F_p\) given by the triple \((\omega_0, \omega_1, \omega_2)\). As we already have seen, a natural way to produce such Poisson structure is by considering a meromorphic connection \(\nabla: E \to E \otimes \Omega^1_X(D)\) and composing with the anchor map \(\sigma^\#: \Omega^1_X \to TX\) we get \(\nabla = \sigma^\# \circ \tilde{\nabla}: E \to E \otimes T\mathcal{F}(D)\). See [23] Proposition 1.8.2 for a more general consideration for principal bundles in the real category.

**Remark 5.3.** There is a connection between transversely projective holomorphic foliation and quantization of symplectic foliation. Biswas in [17] showed that for any regular transversely projective foliation \(\mathcal{F}\) there is a regular transversely symplectic foliation \(\tilde{\mathcal{F}}\) on its conormal bundle \(N\mathcal{F}^*\). Moreover, he proves that the restriction of \(\tilde{\mathcal{F}}\) to the complement of the zero section admits a canonical quantization.

**Example 5.4.** Consider a Poisson structure in \(\mathbb{P}^3\) induced in homogeneous coordinates by \(\sigma = v_0 \wedge v_1\), where \(v_0\) and \(v_1\) are degree one polynomial vector fields satisfying \([v_0, v_1] = 0\). In this case we have that
\[
\delta(v_i) = L_{v_i}(v_0 \wedge v_1) = 0
\]
for \(i = 1, 2\).
Example 5.5. Consider the Poisson structure in $\mathbb{P}^3$ induced in homogeneous coordinates by $\sigma = v_0 \wedge v_1$, where

$$v_0 = z_1 \frac{\partial}{\partial z_1} + 2z_2 \frac{\partial}{\partial z_2} + 3z_3 \frac{\partial}{\partial z_3}, \; v_1 = -4z_0 \frac{\partial}{\partial z_1} - 4z_1 \frac{\partial}{\partial z_2} - 4z_2 \frac{\partial}{\partial z_3}.$$ 

This Poisson structure correspond to the exceptional foliation appearing in the Cerveau–Lins Neto’s classification [13]. A computation gives us

$$\delta(v_0) = L_{v_0}(v_0 \wedge v_1) = v_0 \wedge v_1$$

$$\delta(v_1) = L_{v_1}(v_0 \wedge v_1) = 0,$$

since $[v_1, v_0] = -v_1$. We refer the reader to the [40, Proposition 8.9.2] for a more conceptual construction.

Remark 5.6. Let $(E, \nabla)$ be a holomorphic rank two $\mathfrak{sl}_2$-Poisson module on a normal projective Poisson variety $(X, \sigma)$. Then $P(E, \nabla)$ is rationally Morita equivalent to $(X, \sigma)$, since $\pi^* \Sigma \sigma = \sigma$. In particular, if $(E, \phi)$ is a rank two $\mathfrak{sl}_2$ holomorphic co-Higgs bundle, Then $(P(E), \Sigma \phi)$ is rationally Morita equivalent to $X$ with zero Poisson structure, since $\pi^* \Sigma \phi = 0$.

Proposition 5.7. Let $(E, \phi)$ be a rank two $\mathfrak{sl}_2$ co-Higgs meromorphic bundle on a normal projective Poisson variety $X$. Then after a birational trivialization of $(E, \phi)$ the co-Higgs fields is of the form

$$\begin{pmatrix} f_1 & f_2 \\ f_0 & -f_1 \end{pmatrix} \otimes v_\phi,$$

for some rational vector field $v_\phi \in H^0(X, TX \otimes L)$ and rational functions $f_0, f_1, f_2 \in K(X)$. Moreover, the symplectic foliation induced on $P(E)$ has dimension two and it is the pull-back of the one-dimensional foliation $\mathcal{H}_{v_0}$ on $X$ induced by $v_\phi$. In particular, if $\mathcal{H}_{v_0}$ has canonical singularities and $L$ is not pseudo-effective, then symplectic foliation on $P(E)$ is a foliation by rational surfaces.

Proof. We have that

$$v_0 \wedge v_1 = v_0 \wedge v_2 = v_1 \wedge v_2 = 0.$$ 

We assume without loss of generality that the rational vector field $v_0$ is not identically zero. Then, there exist rational functions $f, g, h$ such that $v_1 = f v_0$, $v_2 = g v_0$, and $v_2 = h v_1 = h f v_0$, so $g = h f_1$. Therefore, we get the rational co-Higgs fields

$$\Phi \otimes v = \begin{pmatrix} f & h f \\ 1 & -f \end{pmatrix} \otimes v_0.$$ 

Now, since the induced Poisson bivector is given by

$$\Sigma = (1 + 2fz + h f z^2)v_0 \wedge \frac{\partial}{\partial z},$$

we conclude that the symplectic $\mathcal{F}_\Sigma$ has dimension two and it is the pull-back of the foliation $\mathcal{H}_{v_0}$ tangent to $v_0$. Now, if $\mathcal{H}_{v_0}$ has canonical singularities and $L$ is
not pseudo-effective, then it follows from [11, 21] that \( \mathcal{H}_{v_0} \) is a foliation by rational curves. Hence, the leaves of \( \mathcal{F}_2 = \pi^* \mathcal{H}_{v_0} \) are rational surfaces. \( \square \)

The author showed in [14] that if \((E, \phi)\) is stable and nilpotent co-Higgs holomorphic bundle on a compact Kähler surface, then the symplectic foliation induced on \( \mathbb{P}(E) \) is algebraic with rational leaves.

**Example 5.8.** Rayan in [42] gave a complete description for the moduli spaces of Schwarzenberger’s co-Higgs holomorphic bundles. Consider a nonsingular conic \( C = \{ Q = 0 \} \) and a degree two covering \( f^Q : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2 \) branched over \( C \). The rank two vector bundle \( j^Q_2 \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, k) = V^Q_k \), for each \( k \geq 0 \) is called Schwarzenberger bundle [46]. Rayan in [42] proved that Schwarzenberger bundles are \( \mathfrak{sl}_2 \) co-Higgs bundles with co-Higgs fields of the form \( \Phi \otimes v \), where \( \Phi \in H^0(\mathbb{P}^2, \text{End}_{0}(V^Q_k)(1)) \) and \( v \in H^0(\mathbb{P}^2, TP^2(−1)) \).

**Example 5.9.** We consider in this example a Poisson interpretation due to Pym [40, Section 8.7] for degree two pull-back foliations on \( \mathbb{P}^3 \). Let \( E = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1) \) and a rational vector field \( v_0 : \mathcal{O}_{\mathbb{P}^2} \to TP^2(−1) \). Then the nilpotent co-Higgs field induces on \( \mathbb{P}(E) \) a symplectic foliation such that the contraction of \( \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(1)) \subset \mathbb{P}(E) \to \mathbb{P}^3 \) correspond a symplectic foliation given by \( v \wedge v_0 \), where \( v_0 \in H^0(\mathbb{P}^3, TP^3(−1)) \) is a rational vector field which is tangent to a linear projection \( \mathbb{P}^3 \to \mathbb{P}^2 \) and \( v \) is a global holomorphic vector field on \( \mathbb{P}^2 \).

**Definition 5.10.** Let \( Z \to X \) be a \( \mathbb{P}^1 \)-bundle. A codimension one holomorphic foliation \( \mathcal{G} \) on \( Z \) is called by *Riccati foliation* if it is generically transversal to the \( \mathbb{P}^1 \)-bundle \( Z \to X \).

In the next result, as a consequence of Theorem [11] we will give a geometric description for rank two \( \mathfrak{sl}_2 \)-Poisson holomorphic modules.

**Corollary 5.11.** Let \( (E, \nabla) \) be a rank two \( \mathfrak{sl}_2 \)-Poisson holomorphic module on a klt Poisson projective variety \((X, \sigma)\). Then there exist projective varieties \( Y \) and \( Z \) with klt singularities and a quasi-étale Poisson cover \( f : W \times Y \to X \) and one of the following holds:

(a) \( (\pi_2)_*f^*(E, \nabla) \) is a \( \mathfrak{sl}_2 \) partial holomorphic sheaf on \( Y \), where \( \pi_2 \) denotes the projection on \( Y \).

(b) \( W \) and \( Y \) are generically symplectics, then \( (\pi_2)_*f^*(E, \nabla) \) is a rank two locally free sheaf with a meromorphic flat connection with poles on the degeneracy Poisson divisor of \( Y \).
(c) \( W \) is symplectic and after a birational trivialization of \( f^*(E, \nabla) \) the Poisson connection is defined as

\[
\tilde{\nabla} = \delta_W + \begin{pmatrix} f_1 & f_2 \\ f_0 & -f_1 \end{pmatrix} \otimes v,
\]

for some rational vector field \( v \) tangent to \((Y, 0)\), rational functions \( f_0, f_1, f_2 \in K(Y) \), and \( \delta_W \) denotes the Poisson differential on \( W \).

(d) There exist a rational map \( \zeta : Y \dasharrow B \), over a variety \( B \) with \( \dim(B) = \dim(\mathcal{F}_\alpha) \), such that \((\pi_2)_* f^*(E, \nabla) \) corresponds to a \( \mathfrak{sl}_2 \)-Poisson meromorphic module \((E_0, \nabla)\), such that after a birational trivialization the Poisson connection defined on the trivial bundle as

\[
\tilde{\nabla} = \delta + \begin{pmatrix} f_1 & f_2 \\ f_0 & -f_1 \end{pmatrix} \otimes v,
\]

for some rational Poisson vector field \( v \) and rational functions \( f_0, f_1, f_2 \) on \( X \) such that \( \{f_i, f_j\} = 0 \), for all \( i, j \).

Proof. From the proof of Theorem 1.1 we have that there exist projective varieties \( Y \) and \( Z \) with klt singularities and a quasi-étale Poisson cover \( f : (W \times Y, \tilde{\sigma}) \rightarrow (X, \sigma) \) such that \( f^*(E, \nabla) \) is a \( \mathfrak{sl}_2 \)-Poisson module on \((W \times Y, \tilde{\sigma})\). Then, the result follows from Theorem 1.1. In fact, if \((Y, (\pi_2)_* f^*(\sigma))\) is generically symplectic, then \((\pi_2)_* f^*(E, \nabla) \) is a rank two locally free sheaf with a meromorphic flat connection with poles on the degeneracy Poisson divisor of \( Y \). If \((\pi_2)_* f^*(\sigma) = 0\), then \((\pi_2)_* f^*(E, \nabla) \) is a co-Higgs sheaf with co-Higgs field \((\pi_2)_* f^*(\nabla) = \phi \) and after we take a birational trivialization of \( f^*(E, \nabla) \) the Poisson connection is defined as \( \tilde{\nabla} = \delta_W + (\pi_2)_* \phi \). Hence, we conclude the part (c) from proposition 5.7. Finally, for the item (d), after we take a birational trivialization of \( f^*(E, \nabla) \) we use the remark 4.9 and Theorem 4.8, part (d), in order to conclude that

\[
\delta(\psi) = 0 \quad \psi \wedge \psi = 0
\]

with \( \psi \) tangent a rational map \( \zeta : Y \dasharrow B \). This implies that

\[
v_0 \wedge v_1 = v_0 \wedge v_2 = v_1 \wedge v_2 = 0.
\]

Then, there existe rational functions \( f_0, f_1, f_2 \) such that \( v_1 = f_1 v_0, v_2 = f_2 v_0 \), and \( v_1 = f_0 v_2 \). We may assume without loss of generality that \( v_0 \neq 0 \) and \( f_2 \neq 0 \), since the other cases follow similarly. By taking the Poisson derivation \( \delta \) in \( v_2 = f_2 v_0 \), we get that

\[
0 = \delta(v_2) = \delta(f_2) \wedge v_0
\]

which implies that \( v_0 = h_2 \delta(f_2) \), for some invertible rational function \( h_2 \). We also can conclude that \( v_0 = h_1 \delta(f_1) \). Then

\[
h_1 \delta(f_1) = h_2 \delta(f_2).
\]
Thus,
\[ \{f_1, f_2\} = \delta(f_1)(f_2) = \frac{h_2}{h_1} \cdot \delta(f_2)(f_2) = \frac{h_2}{h_1} \cdot \{f_2, f_2\} = 0. \]

Now, on the one hand, by using that \( v_1 = f_0v_2 \), we obtain that \( v_1 = h_0\delta(f_0) \). On the other hand, since \( v_1 = f_1h_1\delta(f_1) \) and \( v_2 = f_2h_2\delta(f_2) \), we have that
\[ \{f_0, f_1\} = \delta(f_1)(f_1) = \frac{f_1h_1}{h_0} \cdot \delta(f_0)(f_0) = 0. \]

and also \( \{f_0, f_2\} = 0. \)

As we have seen above, the geometric study of the symplectic foliation \( \mathcal{F}_\sigma \) reduces, up to a quasi-étale Poisson cover, to the foliation \( \mathcal{F}_{\sigma_0} \) on \( \mathbb{P}(E_0, \nabla_0) \to (Y, \sigma_0) \).

**Corollary 5.12.** Let \( \mathcal{F}_{\sigma_0} \) be the symplectic foliation induced on \( \pi : \mathbb{P}(E_0) \to (Y, \sigma_0) \). Then one of the following holds:

(a) \( \mathcal{F}_{\sigma_0} \) is a dimension 2 foliation which is a pull-back of a foliations by curves on \( Y \).

(b) \( \mathcal{F}_{\sigma_0} \) is a Riccati foliation of codimension one on \( \mathbb{P}(E_0) \), if \( (Y, \sigma_0) \) is generically symplectic.

(c) \( \mathcal{F}_{\sigma_0} \) is a Riccati foliation of codimension one on \( \mathbb{P}(E_0) \) which is given by a morphism \( \mathcal{A} \to d_{\text{refl}}\pi(\pi^*(T\mathcal{F}_{\sigma_0})) \subset \Omega^1_{\mathbb{P}(E_0)} \), where \( \mathcal{A} \) is a line bundle and \( d_{\text{refl}}\pi : \pi^*\Omega^1_Y \to \Omega^1_{\mathbb{P}(E_0)} \) is the pull-back morphism of reflexive forms.

(d) There exist a rational Poisson vector field \( v \) generically transversal to \( \mathcal{F}_{\sigma_0} \) such that \( \mathcal{F}_{\sigma_0} \) has dimension \( 2k + 2 \) and it is the pull-back of the foliation induced by \( v \) and \( \mathcal{F}_{\sigma_0} \). In particular, if \( \dim(Y) = 2k + 1 \), then \( \mathbb{P}(E_0) \) is generically symplectic and there exist a rational Poisson map \( \zeta : Y \to B \) generically transversal to \( \mathcal{F}_{\sigma_0} \), where \( B \) is a generically symplectic variety with \( \dim(B) = 2k \) and the induced map \( \mathbb{P}(E_0) \to B \) is Poisson.

**Proof.** If \( (Y, \sigma_0) \) is generically symplectic, then \( \nabla_0 \) corresponds to a meromorphic flat connection. Hence, \( \mathcal{F}_{\sigma_0} \) is a Riccati foliation of codimension one. If \( \nabla_0 \) corresponds to a partial flat meromorphic connection
\[ E_0 \to T\mathcal{F}_{\sigma_0}^*(D) \otimes E_0 \subset \Omega^1_Y \otimes E_0, \]
then \( \mathcal{F}_{\sigma_0} \) is a Riccati foliation of codimension one, which is given by the meromorphic 1-form
\[ \alpha = dz + \omega_0 + 2\omega_1z + \omega_2z^2 \]
where \( \omega_i \)'s are meromorphic sections of \( \pi^*(T\mathcal{F}_{\sigma_0}^*) \subset \pi^*\Omega^1_Y \). Therefore, \( \mathcal{F}_{\sigma_0} \) is induced by a morphism \( \mathcal{A} \to d_{\text{refl}}\pi(\pi^*(T\mathcal{F}_{\sigma_0})) \subset \Omega^1_{\mathbb{P}(E_0)} \), where \( \mathcal{A} \) is a line bundle and
\[ d_{\text{refl}}\pi : \pi^*\Omega^1_Y \to \Omega^1_{\mathbb{P}(E_0)} \]
is the pull-back morphism which there exist by [23] Theorem 1.4.
Now, recall from the proof of Proposition 5.2 that the foliation $F_{\nabla_0}$ is induced by the bivector

$$\Sigma_{\sigma_0} = \sigma_0 + (v_0 + 2v_1z + v_2z^2) \wedge \frac{\partial}{\partial z}.$$ 

If $\nabla_0$ corresponds to a co-Higgs field on $(Y, 0)$, then from Proposition 5.7 we have that $F_{\nabla_0}$ is a dimension 2 foliation which is a pull-back of a foliation by curves. This shows the part (c). Finally, for the case (d) we have the bivector

$$\Sigma_{\sigma_0} = \sigma_0 + v(z) \wedge \frac{\partial}{\partial z},$$

where $v(z) = \ell v$, with $\ell$ being the non-zero rational function $f_0 + 2f_1z + f_2z^2$. Thus, $F_{\nabla_0}$ is induced by

$$\Sigma_{\sigma_0}^{k+1} = \sum_{j=0}^{k+1} \binom{k+1}{j} \ell v_0^{k+1-j} \wedge \left( v \wedge \frac{\partial}{\partial z} \right)^j = (k+1)\ell \sigma_0^k \wedge v \wedge \frac{\partial}{\partial z} \neq 0$$

since $\sigma_0^{k+1} = 0$, $\sigma_0^k \neq 0$ and $v$ is transversal to $\sigma_0^k$. This show us that $F_{\nabla_0}$ is induced by the pull-back of the foliation generated by $v$ and $\sigma_0^k$. Now, if $\text{dim}(Y) = 2k + 1$, then the symplectic foliation has codimension one and the rational vector fields $v$ induces a foliation whose leaves are tangent to the fibers of $\zeta : Y \to B$. The condition $\delta(v) = L_v\sigma_0 = 0$, say us that there exist a non-trivial Poisson bivector $\sigma_B$ such that $\zeta_*\sigma_0 = \sigma_B$. \hfill $\square$

References

[1] C. Araujo, S. Druel, On Fano foliations. Adv. Math. 238 (2013), p. 70-118.
[2] C. Araujo, S. Druel, On codimension 1 del Pezzo foliations on varieties with mild singularities. Math. Ann. 360 (2014), no. 3-4, 769-798.
[3] E. Ballico, S. Huh, 2-Nilpotent co-Higgs structures, Manuscripta math. Vol. 159, Issue 1-2, (2019), 39-56.
[4] A. Beauville, Variétés Kähleriennes dont la première classe de Chern est nulle, J. Differential Geometry 18 (1983), 755-782.
[5] A. Beauville, Holomorphic symplectic geometry: a problem list. In: Complex and differential geometry, pp. 49-63, Springer Proc. Math., vol. 8, Springer, Heidelberg, 2011.
[6] I. Biswas, S. Rayan, A vanishing theorem for co-Higgs bundles on the moduli space of bundles, Geom. Dedicata 193 (2018), 145–154.
[7] I. Biswas, On the Quantization of a Transversely Symplectic Foliation., Southeast Asian Bulletin of Mathematics , Vol. 30, 6, 2006, 1029-1047.
[8] I. Biswas, O. Garcia-Prada, J. Hurtubise, S. Rayan, Principal co-Higgs bundles on $\mathbb{P}^1$, (2018), arXiv:1810.12376.
[9] F. Bottacin, Poisson structures on Hilbert schemes of points of a surface and integrable systems, Manuscripta math. 97 (1998), 517-527.
[10] C. Bartocci and E. Macri, Classification of Poisson surfaces, Commun. Contemp. Math. 7 (2005), 89-95.
[11] F. Bogomolov, M. McQuillan, Rational Curves on Foliated Varieties. Folation theory in algebraic geometry. Cham: Springer. Simons Symposia, 21-51 (2016).
[12] F. Campana, M. Paun, *Foliations with positive slopes and birational stability of orbifold cotangent bundles*. Publications mathématiques de l'IHÉS, Vol. 129, Issue 1, pp 1-49 (2019).

[13] D. Cerveau, A Lins, *Irreducible components of the space of holomorphic foliations of degree two in $\mathbb{C}P(n)$*. Annals of Math. 143, 577–612, 1996.

[14] M. Corrêa, *Rank two nilpotent co-Higgs sheaves on complex surfaces*, Geom. Dedicata 183 (2016), 25-31.

[15] A. V. Colmenares, *Moduli spaces of semistable rank-2 co-Higgs bundles over $\mathbb{P}^1 \times \mathbb{P}^1$*, Q. J. Math., Vol 68, 2017, 1139-1162.

[16] S. Druel, *Some remarks on regular foliations with numerically trivial canonical class*, EPIGA 1 (2017), Article Nr. 4.

[17] S. Druel, *Codimension one foliations with numerically trivial canonical class on singular spaces*, preprint arXiv:1809.06905, 2018.

[18] M. Crainic, *Differentiable and algebroid cohomology, van Est isomorphisms and characteristic classes*. Comment. Math. Helv., Tome 78 (2003) no. 4, pp. 681-721.

[19] P. Deligne, *Equations différentielles a points singuliers reguliers*, Lecture Notes in Mathematics, Vol. 163, Springer-Verlag, Berlin, 1970.

[20] S. Druel, *Generalized complex geometry*. Ann. of Math. (2) 174, 1 (2011), 75-123.

[21] N. Hitchin, *Poisson modules and degeneracy loci*, Inst. Hautes Etudes Sci. Publ. Math. 114 (2011), no. 1, 352-362.

[22] M. Gualtieri, B. Pym, *Poisson modules and degeneracy loci*, Proc. Lond. Math. Soc. (3) 107 (2013), no. 3, 627-654.

[23] M. Gualtieri, *Generalized Calabi-Yau manifolds*, Quart. J. Math. Oxford, 54 (2003) 281–308.

[24] N. Hitchin, *Generalized Calabi-Yau manifolds*, Quart. J. Math. Oxford, 54 (2003) 281–308.

[25] N. Hitchin, *Generalized holomorphic bundles and the B-field action*, Journal of Geometry and Physics 61 (2011), 352-362.
[34] J. Kollár, S. Mori, Birational geometry of algebraic varieties. In: Cambridge Tracts in Mathematics, vol. 134. Cambridge University Press, Cambridge (1998). With the collaboration of C. H. Clemens and A. Corti, translated from the 1998 Japanese original.
[35] F. Loray, J. Pereira, F. Touzet, Singular foliations with trivial canonical class. To appear in Inventiones Mathematicae. http://arxiv.org/abs/1107.1538v1, 2011.
[36] F. Loray, J. Pereira, F. Touzet, Representations of quasiprojective groups, flat connections and transversely projective foliations. Journal de l’École polytechnique 3 (2016), 263-308.
[37] M. McQuillan, Canonical models of foliations. Pure Appl. Math. Q. 4 (2008), no. 3, part 2, p. 877-1012.
[38] K. Morita, Duality for modules and its applications to the theory of rings with minimum condition. Sci. Rep. Tokyo Kyoiku Daigaku Sect. A 6 (1958), 83-142.
[39] A. Polishchuk, Algebraic geometry of Poisson brackets. Algebraic geometry, 7. J. Math. Sci. 84 (1997), no. 5, p. 1413-1444.
[40] B. Pym, Poisson Structures and Lie Algebroids in Complex Geometry, Phd. Thesis, University of Toronto, 2013.
[41] Z. Ran, Deformations of holomorphic pseudo-symplectic Poisson manifolds. Adv. Math. 304, 1156-1175 (2017).
[42] S. Rayan, Constructing co-Higgs bundles on $\mathbb{C}F^2$, Q. J. Math. 65 (2014), no. 4, 1437-1460
[43] S. Rayan, Co-Higgs bundles on $\mathbb{P}^1$, New York J. Math. 19 (2013), 925-945.
[44] F. Sakai, Anti-Kodaira dimension of ruled surfaces, Sci. Rep. Saitama Univ. 2 (1982), 1-7.
[45] B. Scárdua, Transversely affine and transversely projective holomorphic foliations, Ann. Sci. Ecole Norm. Sup. (4) 30 (1997), no. 2, 169-204.
[46] R. L. E., Schwarzenberger, Vector bundles on the projective plane. Proc. London Math. Soc. (3) 11 (1961), 623-640.
[47] Y. Wang, Generalized holomorphic structures, J. Geom. Phys. 86 (2014), 273-283.
[48] A. Weinstein, Coisotropic calculus and Poisson groupoids. J. Math. Soc. Jap. 40, 705-727 (1988)
[49] A. Weinstein, The local structure of Poisson manifolds, J. Differential Geometry , 18 (1983), 523-557
[50] A. Weinstein, Symplectic groupoids and Poisson manifolds, Bulletin of the American mathematical Society, 16 (1987), no. 1, 101-104.
[51] P. Xu, Morita equivalence of Poisson manifolds. Comm. Math. Phys. 142 (1991), 493-509
[52] P. Xu, Morita equivalent symplectic groupoids. In: Symplectic geometry, groupoids, and integrable systems (Berkeley, CA, 1989), 291-311. Springer, New York, 1991.

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