WTO - A DETERMINISTIC APPROACH TO 4-FERMION PHYSICS

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The program WTO, which is designed for computing cross sections and other relevant observables in the $e^+ e^-$ annihilation into four fermions, is described. The various quantities are computed over both a completely inclusive experimental set-up and a realistic one, i.e. with cuts on the final state energies, final state angles, scattering angles and final state invariant masses. Initial state QED corrections are included by means of the structure function approach while final state QCD corrections are applicable in their naive formulation. A gauge restoring mechanism is included according to the Fermion-Loop scheme. The program structure is highly modular and particular care has been devoted to computing efficiency and speed.
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1 Program summary

Title of program: WTO

Computer: Alpha AXP-2100 (VAXstation·4000·90); Installation: INFN, Sezione di Torino, via P.Giuria 1, 10125 Torino, Italy

Operating system: OpenVMS(VMS)

Programming language used: FORTRAN 77

Memory required to execute with typical data: Peak working set size: 3000-3500

No. of bits in a word: 64(32)

No. of lines in distributed program: 35313

subprograms used: NAGLIB [1]
**Keywords**: $e^+e^-$ annihilation into four fermions, LEP 2, properties of the $W$ vector boson, Higgs boson, initial and final state QED radiation, QCD corrections, Minimal Standard Model, cross sections and moments of distributions, gauge invariance, theoretical error, deterministic integration.

**Nature of physical problem** An accurate description of the process $e^+e^- \rightarrow 4$ fermions is needed in order to fully describe the physics available at LEP 2 and at higher energies. In particular the properties of the $W$ boson can be correctly analyzed around and above the threshold only when the full gauge-invariant set of diagrams contributing to a given final state is included. Similarly the $Z-Z$ production and the Higgs boson production can be studied within the minimal standard model. Indeed from a field theoretical point of view both the $W$ and the $Z$ bosons (and the Higgs boson too) are unstable particles and a full description of their production can only proceed through the complete set of matrix elements for the 4-fermion processes. Although at typical LEP 2 energies the difference between the full calculation and the double resonant approximation for a process like $e^+e^- \rightarrow \mu^- \nu_\mu \nu_\mu \tau^+$ is totally negligible, the same is certainly not true anymore when we consider $e^+e^- \rightarrow e^- \nu_e \nu_\mu \mu^+$ or any of the neutral current processes.

**Method of solution** The helicity amplitudes for each given process are given, according to the formalism of ref. [2], in terms of the 7 independent invariants which characterize the phase space. The phase space itself, including all realistic kinematical cuts, is also described in terms of invariants. Initial state QED radiation is included by means of the structure function approach. Upon initialization the final state QCD corrections are included by adopting a naive approach (NQCD). The numerical integration, with complete cut-availability, is performed with the help of a deterministic integration routine which makes use of quasi-random, deterministic number sets, the shifted Korobov sets. The boundaries of the phase space, with kinematical cuts, are reconstructed through a backwards propagation of constraints.

**Restrictions on the complexity of the problem** The theoretical formulation is specifically worked out for massless fermions, although this is not a limitation of principle. Emission of photons is strictly collinear and no $p_t$ is therefore included. No interface exists with the standard packages for hadronization.

**Typical running time** Dependent on the process and on the required accuracy. For example a 0.1% accuracy for a CC11 process requires something of the order of 440 CPU seconds for Alpha AXP-2100. On the other extreme a 0.5% accuracy for some (but not all) of the NC processes requires approximately 12 CPU hours on the same computer.
2 Long Write-up

2.1 Introduction

WTO is a quasi-analytical, deterministic code for computing observables related to the process

\[ e^+ e^- \rightarrow f_1 f_2 f_3 f_4. \]

If one neglects the fermion masses there are 32 distinct processes of this kind (classified in ref. [3]), 29 of which are at present accessible with WTO. The only exclusion is at present given by \( M^1 M \) with \( M = E \otimes E, E = (\nu_e, e) \).

We have several codes, documented in the literature [4]-[5]-[6], which perform a similar task and all of them can be classified into three broad families, i.e. semi-analytical, Monte Carlo (MC) integrators or classical event generator. A semi-analytical code will perform as much as possible of the 7(9 with initial state QED radiation included through structure function approach) phase space integrations leaving at most 2(3) of them for some kind of numerical evaluation. The advantage is represented by computational speed and high numerical accuracy but only very few and selected cuts can be applied, typically on two of the final state invariant masses. From LEP 1 experience we know that semi-analytical calculations are relevant also to experimentalists and not only as the ideal source of benchmark. A MC approach clearly represents the other extreme giving full cut-availability but with no intermediate analytical step taking place in the process of obtaining the final result out of the matrix elements. Kinematical cuts are, to a large extent, treated with the help of IF statements and the efficiency of the integration procedure is dominated by the adaptivity of the integration routine.

WTO has been designed to lay somehow in between these two approaches. Basically the matrix elements are generated through the helicity formalism of ref. [2] and they are compact expressions completely given in terms of the invariants which describe the process. Also the momenta of the final states are, component by component, given in terms of the invariants used in the integration over the phase space, thus allowing to implement the kinematical cuts with an analytical control (almost completely). There are very few parts of the code where this is not implemented, having to do with the fact that in any \( 2 \rightarrow 4 \) process there is a non-linear constraint among the invariants. Although this is not a limitation of principle we have nevertheless verified that with an analytical treatment of the non-linear constraint the balance between CPU time and efficiency in the integration is not in favor of the former and, for this reason, some numerical control of the phase space boundaries has survived in WTO.

Having explained the quasi-analytical treatment of the calculation in WTO we spend now few words on the deterministic approach to the integration. The integration over the phase space is performed in WTO with the help of the NAG routine D01GCF. This routine uses the Korobov-Conroy number theoretic approach with a MC error estimate arising from converting the number theoretic formula for the \( n \)-cube \([0, 1]^n\) into a stochastic integration rule. This allows a ‘standard error’ to be estimated. There is no adaptive strategy at work since the routine D01GCF, being a deterministic one, will use a fixed grid. The evaluation of the specified observable will be repeated \( NRAND \) times to give the
final answer, however there is no possibility to examine the partial results but only the
average and the resulting standard error will be printed. The error in evaluating, say a
cross section, satisfies \( E < CK p^{-\alpha} \log^\beta p \), where \( p = \text{NPTS} \), \( \alpha \) and \( C \) are real numbers
depending on the convergence rate of the Fourier series, \( \beta \) is a constant depending on the
dimensionality \( n \) of the integral and \( K \) is a constant depending on \( \alpha \) and \( n \).

2.2 The phase space

The process under consideration is specified by

\[ e^+(p_+) e^-(p_-) \to f_1(q_1) f_2(q_2) f_3(q_3) f_4(q_4), \]  

(2)

and it is described in terms of 14 invariants (plus the c.m.s. energy)

\[ s = -(p_+ + p_-)^2, \]

\[ x_{1i} s = -(p_+ + q_{i-2})^2, \quad i = 3, \ldots, 6, \]

\[ x_{2i} s = -(p_- + q_{i-2})^2, \quad i = 3, \ldots, 6, \]

\[ x_{ij} s = -(q_{i-2} + q_{j-2})^2, \quad i \neq j = 3, \ldots, 6. \]  

(3)

Out of these we select 7 linearly independent combinations of invariants, i.e.

\[ m_2^2, \quad m_+^2, \quad M_0^2, \quad m_0^2, \quad m^2, \quad t_w, \quad t_1, \]  

(4)

such that

\[
\begin{align*}
x_{13} &= t_1, & x_{14} &= t_w - t_1, & x_{15} &= t_3, & x_{16} &= 1 - t_w - t_3, \\
x_{23} &= m_-^2 + m_0^2 + m^2 - t_1, & x_{24} &= 1 - m_+^2 - m_0^2 - m^2 - t_w + t_1, \\
x_{25} &= m_+^2 + M_0^2 + m^2 - t_3, & x_{26} &= -m_-^2 - M_0^2 - m^2 + t_w + t_3, \\
x_{34} &= m_-^2, & x_{35} &= m_+^2, & x_{36} &= m_0^2, \\
x_{45} &= M_0^2, & x_{46} &= 1 - m_-^2 - m_+^2 - M_0^2 - m_0^2 - m^2, \\
x_{56} &= m_+^2. 
\end{align*}
\]  

(5)

An auxiliary variable has been introduced, \( t_3 \), which is fixed by the non-linear constraint. Additional auxiliary variables to be considered are also

\[
\begin{align*}
e_1 &= m_-^2 + m_0^2 + m^2, & e_2 &= 1 + m_-^2 - m_+^2 - e_1, \\
e_3 &= m_+^2 + M_0^2 + M^2, & e_4 &= 1 + m_+^2 - m_-^2 - e_3, \\
t_2 &= t_w - t_1, & t_4 &= 1 - t_w - t_3. 
\end{align*}
\]  

(6)

This choice is far from being arbitrary and reflects the need of selecting those combinations of invariants which characterize the peak structure of the integrand. With the help of the previous quantities we can reconstruct completely the phase space. If \( M_{ij}, E_i, \theta_i \) and \( \psi_{ij} \)
(i,j=1,4) are the final state invariant masses, the final state energies, the scattering angles and the final state angles respectively, then in the $e^+e^-$ c.m.s. we can write

\begin{align*}
M_{12}^2 &= m_1^2, \quad M_{13}^2 = m_2^2, \quad M_{14}^2 = m_0^2, \\
M_{23}^2 &= M_0^2, \quad M_{24}^2 = 1 - m_2^2 - m_3^2 - M_0^2 - m_0^2 - m^2, \\
M_{34}^2 &= m_3^2,
\end{align*}

(7)

and

\begin{align*}
E_i &= \frac{1}{2} e_i \sqrt{s}, \quad i = 1, \ldots, 4, \\
\cos \theta_i &= 1 - \frac{t_i}{e_i}, \quad i = 1, \ldots, 4, \\
\cos \psi_{12} &= 1 - 2 \frac{m_2}{e_1 e_2}, \\
\cos \psi_{13} &= 1 - 2 \frac{m_2}{e_1 e_3}, \quad \text{etc.}
\end{align*}

(8)

Clearly we want to take into account the QED radiation from initial states which is done through the formalism of the structure function [8]-[10]. In this way the hard process becomes

\[ e^+(x+p_+)e^-(x-p_-) \to f_1(q_1) f_2(q_2) f_3(q_3) f_4(q_4), \]

(9)

and the corresponding kernel cross section is afterwards convoluted with the standard structure functions. Since we want to impose kinematical cuts in the laboratory frame, all the quantities must be reconstructed in presence of e.m. radiation. We have

\begin{align*}
E_i &= \frac{1}{2} (x e_i - \Delta t_i) \sqrt{s}, \quad \Delta = x_+ - x_-, \\
\cos \theta_i &= 1 - 2 \frac{x_- t_i}{x_+ e_i - \Delta t_i}, \\
\cos \psi_{12} &= 1 - \frac{1}{2} \frac{m_2^2}{E_1 E_2}, \quad \text{etc.}
\end{align*}

(10)

Formally any distribution is written as

\[ F = \int dx_+ dx_- \int dPS \Theta_{cut} D(x_+, s) D(x_-, s) f, \]

(11)

where $f$ refers to the kernel-distribution and $D$ to the structure function. Thus $f$ will depend on

\begin{align*}
x_+ p_+ \cdot q_i &= -\frac{1}{2} x_+ x_{1,i+2}s, \\
x_- p_- \cdot q_i &= -\frac{1}{2} x_- x_{2,i+2}s, \\
q_i \cdot q_j &= -\frac{1}{2} x_+ x_- x_{i+2,j+2}s.
\end{align*}

(12)
In presence of radiation we write

\[ x_{34} = x_+ x_- m_-, \ldots, x_{56} = x_+ x_- m_+ , \]
\[ x_{13} = x_- t_1, \text{ etc.,} \]
\[ x_{23} = x_+ (e_1 - t_1), \text{ etc.} \]  
\( (13) \)

Consequently

\[ x_+ p_+ \cdot q_1 = - \frac{1}{2} x_+ x_{13} s = - \frac{1}{2} x_+ x_- t_1 s = - \frac{1}{2} t_1 \hat{s}, \text{ etc.}, \]  
\( (14) \)

where \( \hat{s} = x_+ x_- s \). \( f \) can be given in terms of the invariants \( \{ I \} = m_-^2, \ldots, t_1 \) if we normalize everything with respect to \( \hat{s} \). Indeed we obtain

\[ \frac{d^7 F}{d m_-^2 \ldots d t_1} = \int dx_+ dx_- \Theta_{\text{cut}} D(x_+, s) D(x_-, s) f (\{ I \}, \hat{s}) \int dPS \delta_I, \]  
\( (15) \)

where \( \delta_I \) is the collection of delta-functions giving the invariants \( \{ I \} \) in terms of scalar products.

We now come back to the auxiliary invariant \( t_3 \). The easiest way of obtaining the non-linear constraint is to start in the c.m.s of \( P_\pm = x_\pm p_\pm \) with \( P_\pm \) along the z-axis. Then \( q_1 \), in the \( x-z \) plane, is constructed, component by component in terms of invariants. Obviously \( q_4 \) is fixed by energy-momentum conservation and \( q_2, q_3 \) will be constructed recursively but for \( q_3 \) we allow for a fifth component. The request that this extra component be zero gives the non-linear constraint and therefore \( t_3 \) in terms of the remaining independent invariants. The non-linear constraint, being quadratic in \( t_3 \), gives rise to two distinct solutions which must be properly interpreted. The phase space integral can be written as

\[ \int dPS = \int \Pi_i d^4 q_i \delta^+ \left( q_i^2 \right) \delta^4 \left( P - \sum_i q_i \right), \]  
\( (16) \)

with \( P^2 = -\hat{s} \). After some manipulations we obtain

\[ \int dPS = \frac{1}{8} \hat{s}^{3/2} \int \Pi_i de_i \Pi_j dt_j d^2 m_+ d^2 m_- \Pi_i \theta(e_i) \theta(2 - \sum_i e_i) \]
\[ \times \int \Pi_i d^3 q_i \delta \left( q_i^2 - \frac{1}{4} e_i^2 \hat{s} \right) \delta(X_1) \delta(X_2) \delta(X_{12}) J, \]  
\( (17) \)

where

\[ X_1 = t_1 + \frac{1}{2} (c_1 - 1) e_1, \]
\[ X_2 = t_2 + \frac{1}{2} (c_2 - 1) e_2, \]
\[ X_{12} = m_-^2 + \frac{1}{2} (c_{12} - 1) e_1 e_2. \]  
\( (18) \)
Here \( c_i \) denotes \( \cos \theta_i \) in the c.m.s. (with \( P_+ \) along the positive \( z \)-axis) and \( c_{12} \) is the cosine of the angle between \( q_1 \) and \( q_2 \) in the same system. Moreover

\[
J = \int d^3q_3 \delta \left( q_3^2 - \frac{1}{4} e_3^2 s \right) \delta \left( m^2 + \frac{1}{2} (c_{13} - 1)e_1 e_3 \right)
\times \delta \left( \sum_i e_i - m_+ - m^2 + \frac{1}{2} (c_{23} - 1)e_2 e_3 \right). \tag{19}
\]

To compute \( J \) we choose now another reference frame (\( II \)) given by

\[
q_1 = \frac{1}{2} e_1 \sqrt{s} (0, 0, 1, 1), \quad q_2 = \frac{1}{2} e_2 \sqrt{s} (s_{12}, 0, c_{12}, 1), \quad q_3 = \frac{1}{2} e_3 \sqrt{s} (s_{13} \cos \phi, s_{13} \sin \phi, c_{13}, 1). \tag{20}
\]

Let \( \Phi = s_{13} s_{12} \cos \phi + c_{13} c_{12} - 1 \) computed for \( c_{13} = 1 - 2m^2/e_1 e_3 \). We get

\[
J = \frac{1}{2 e_1 \sqrt{s}} \left[ \int_0^\pi d\phi + \int_\pi^{2\pi} d\phi \right]
\times \delta \left( \left( \sum_i e_i - m_+ - m^2 - 1 + \frac{1}{2} \Phi \right) \hat{s} \right)
\times \delta \left( \sum_i e_i - m_+ - m^2 - 1 + \frac{1}{2} \Phi \right) \hat{s}.
\]

\[
J = \frac{1}{2 e_1 \sqrt{s}} \frac{\theta_> + \theta_<}{R_4}. \tag{21}
\]

Before carrying on the full integration we can now connect the two \( \theta \) functions, \( \theta_> \) and \( \theta_< \), with the non-linear constraint. Let \( t_+ \) be the two solutions and moreover let \( q_3^\pm(I) \) be \( q_3 \) in the c.m.s computed according to the choice \( t_3 = t_3^+ \) or \( t_3 = t_3^- \). Let finally \( q_3(II) \) be \( q_3 \) as given in system \( II \) with

\[
c_{13} = 1 - 2 \frac{m^2}{e_1 e_3}, \quad s_{13}^2 = 1 - c_{13}^2. \tag{22}
\]

To fix \( \cos \phi \) we first transform \( q_3(I) \) from system \( I \) to system \( II \) and compare \( q_3^+(I) \) with \( q_3(II) \). The root for \( t_3 \) is chosen to match the two vectors, one root will correspond to \( \sin \phi > 0 (\theta_> \) and the other to \( \sin \phi < 0 (\theta_< \) for fixed \( \cos \phi \). Moreover

\[
R_4^2 = m^2 e_2^2 \left( e_1 e_3 - m^2 \right) s_{12}^2
- \left[ \frac{1}{2} e_2 \left( e_1 e_3 - 2 m^2 \right) c_{12} + e_1 \left( \sum_i e_i - m_+ - m^2 - 1 - \frac{1}{2} e_2 e_3 \right) \right]^2. \tag{23}
\]

where
\[ c_{12} = 1 - 2 \frac{m^2}{e_1 e_2}. \] (24)

Thus

\[
\int dPS = \frac{1}{8} \hat{s} \int d\{I\} \Pi_i \theta(e_i) \theta \left( 2 - \sum_i e_i \right) \theta \left( R_4^2 \right) (\theta > + \theta <) \times \int d^3 q_1 \delta \left( q_1^2 - \frac{1}{4} e_1^2 \hat{s} \right) \delta \left( t_1 + \frac{1}{2} e_1 ((c_1 - 1) \right) H, \] (25)

where \( \{I\} \) denotes collectively all the invariants and

\[
H = R_4^{-1} \int d^3 q_2 \delta \left( q_2^2 - \frac{1}{4} e_2^2 \hat{s} \right) \delta \left( t_1 + t_w + \frac{1}{2} e_2 (c_2 - 1) \right) \times \delta \left( m_2^2 + \frac{1}{2} e_1 e_2 (c_{12} - 1) \right). \] (26)

As a final step we compute \( H \) by going back to system I. Here

\[
q_1 = \frac{1}{2} e_1 \sqrt{\hat{s}} (s_1, 0, c_1, 1),
\]

\[
q_2 = \frac{1}{2} e_1 \sqrt{\hat{s}} (s_2 \cos \psi, s_2 \sin \psi, c_2, 1), \] (27)

and

\[
H = \frac{\sqrt{\hat{s}}}{R_2 R_4}. \] (28)

### 2.3 Boundaries of the phase space

In this section we consider the problem of reconstructing the full phase space in terms of our independent invariants. There are natural limits on our integration variables which can be derived by clustering the final state momenta \( q_1, \ldots, q_4 \) into pairs, i.e. \( (q_1+q_2), (q_3+q_4) \) or \( (q_1+q_3), (q_2+q_4) \) or \( (q_1+q_4), (q_2+q_3) \) and by imposing the phase space constraints relative to a \( 2 \rightarrow 2 \) process. To give an example we introduce

\[
P_\pm = x_\pm p_\pm, \quad Q_- = q_1 + q_2, \quad Q_+ = q_3 + q_4. \] (29)

We find

\[
\hat{s} = - (P_+ + P_-)^2, \]

\[
\hat{t} = - (P_+ - Q_-) = \left( m_2^2 - t_w \right)^2 \hat{s}. \] (30)
The condition to be fulfilled is that $\hat{t}$ must be in the physical region, therefore $X \geq 0$ with $X$ given by

$$X = -t_w^2 + \left(1 + m_-^2 + m_+^2\right) t_w - m_-^2.$$  \hspace{1cm} (31)

This fixes the natural limits of the variable $t_w$, i.e.

$$T_- \leq t_w \leq T_+, \quad T_\pm = \frac{1}{2} \left[1 + m_-^2 - m_+^2 \pm \lambda^{1/2} \left(1, m_-^2, m_+^2\right)\right],$$  \hspace{1cm} (32)

where we have introduced the corresponding Källen’s $\lambda$-function, whose positivity in turns implies

$$0 \leq m_-^2 \leq 1, \quad 0 \leq m_+^2 \leq (1 - m_-)^2.$$  \hspace{1cm} (33)

Similarly we obtain $-t_1 + T_-^* \leq t_4 \leq -t_1 + T_+^*$ with

$$T_\pm^* = \frac{1}{2} \left[e_1 + e_3 \pm \lambda^{1/2} \left(1, m_-^2, 1 + m^2 - e_1 - e_2\right)\right],$$  \hspace{1cm} (34)

with the further constraint that

$$\lambda \left(1, m_-^2, 1 + m^2 - e_1 - e_2\right) = (e_1 + e_3)^2 - 4 m_-^2 \geq 0,$$  \hspace{1cm} (35)

or $-t_4 + T_-^* \leq t_4 \leq -t_4 + T_+^*$ with

$$T_\pm^* = \frac{1}{2} \left[e_1 - e_4 \pm \lambda^{1/2} \left(1, e_1 - m_-^2 - m^2, e_3 - m_+^2 - m^2\right)\right],$$  \hspace{1cm} (36)

with the additional condition that

$$\lambda \left(1, e_1 - m_-^2 - m^2, e_3 - m_+^2 - m^2\right) \geq 0.$$  \hspace{1cm} (37)

The other constraints will be described directly in the presence of kinematical cuts. First of all cuts on the final state invariant masses are very easy to implement thanks to our choice of integration variables, eq. (3). Next we introduce a threshold for all energies. Thus if we require

$$E_i \geq l_i \sqrt{s},$$  \hspace{1cm} (38)

this will be the same as

$$\Delta t_i \leq x_i e_i - 2 l_i.$$  \hspace{1cm} (39)

Let us discuss explicitly the case of $\Delta$ positive, we get

$$t_i \leq X e_i - L_i, \quad X = \frac{x_+}{\Delta}, \quad L_i = 2 \frac{l_i}{\Delta}.$$  \hspace{1cm} (40)
Requiring consistency among these four conditions we obtain

\[
\begin{align*}
1 - t_w - Xe_4 + L_4 & \leq t_3 \leq Xe_3 - L_3, \\
t_w - Xe_2 + L_2 & \leq t_1 \leq Xe_1 - L_1, \\
1 - X(e_3 + e_4) + L_3 + L_4 & \leq t_w \leq X(e_1 + e_2) - L_1 - L_2, 
\end{align*}
\] (41)

with a derived condition on the radiated energy,

\[
x_+ - x - \geq \frac{1}{2} \left(1 + \sum_i L_i \right).
\] (42)

Next let us require that \( c_i \leq \cos \theta_i \leq C_i \), where \( \theta_i \) is the i-th scattering angle. First we define

\[
\gamma_i = \frac{1}{2} (1 - c_i), \quad \Gamma_i = \frac{1}{2} (1 - C_i). 
\] (43)

In terms of invariants the constraints become

\[
\sigma_i e_i \leq t_i \leq \Sigma_i e_i, 
\] (44)

with the following definition

\[
\frac{1}{\sigma_i} = 1 + \frac{\gamma_i}{1 - \gamma_i} \frac{x_-}{x_+}, \\
\frac{1}{\Sigma_i} = 1 + \frac{\Gamma_i}{1 - \Gamma_i} \frac{x_-}{x_+}. 
\] (45)

Eq. (42) will translate into another set of constraints

\[
\begin{align*}
t_3 & \geq \max (\sigma_3 e_3, 1 - \Sigma_4 e_4 - t_w), \\
t_3 & \leq \min (\Sigma_3 e_3, 1 - \sigma_4 e_4 - t_w), \\
t_1 & \geq \max (\sigma_1 e_1, t_w - \Sigma_2 e_2), \\
t_1 & \leq \min (\Sigma_1, e_1, t_w - \sigma_2 e_2), \\
t_w & \geq \max (1 - \Sigma_3 e_3 - \Sigma_4 e_4, \sigma_1 e_1 + \sigma_2 e_2), \\
t_w & \leq \min (1 - \sigma_3 e_3 - \sigma_4 e_4, \Sigma_1 e_1 + \Sigma_2 e_2). 
\end{align*}
\] (46)

The optimal use of a deterministic integration routine (fixed grid) is largely based on the possibility of backward propagation of the constraints. Typically we deal with an integral

\[
\int dx_1 \theta (b_1 - x_1) (x_1 - a_1) \ldots \int dx_n \theta [b_n (x_1, \ldots, x_{n-1}) - x_n] \theta [x_n - a_n (x_1, \ldots, x_{n-1})]. 
\] (47)

Solving the constraint \( b_n \geq a_n \) in terms of \( x_{n-1} \) gives \( c_n(x_1, \ldots, x_{n-2}) \leq x_{n-1} \leq d_n(x_1, \ldots, x_{n-2}) \), so that we have now to solve
and to iterate until the most external integration is reached. In our case all the constraints, including the non-linear one, are at most quadratic in the integration variables so that a complete solution of the procedure is available. However the rapid growth in the number of the constraints, especially when several kinematical cuts are imposed, and the eventual occurrence of some splitting of the integrals ([a ≤ x_i ≤ b] ∪ [c ≤ x_i ≤ d]) suggests an intermediate, hybrid, procedure where some of the constraints are left for a numerical control. Sometimes it is more convenient, in terms of computational speed, to increase the number of points in the fixed grid and to lose some of them instead of having a smaller number of points and full efficiency. This is also based on the fact that computing the amplitudes is usually faster than implementing the next level of backward propagation of the constraints.

2.4 The Matrix Elements

The generic process that we want to compute is characterized by six external fermionic legs and by a considerable number of diagrams, up to 64 diagrams for final state identical particles not including electrons or Higgs boson exchange. Already long time ago several authors realized that it is extremely unrealistic to write down the full matrix element M, to square it and to sum over polarizations. As a consequence of this fact various groups have developed their own way of evaluating the helicity amplitudes, each as a single complex number to be squared numerically. Our approach is essentially based on the request that everything has to be expressed in terms of the invariants which specify the process. To briefly summarize the main features of the method we first define

\[
\begin{align*}
    u_\lambda(p) &= \Pi_\lambda u(p), & \bar{u}_\lambda(p) &= \bar{u}(p) \Pi_{-\lambda}, \\
    v_\lambda(p) &= \Pi_{-\lambda} v(p), & \bar{v}_\lambda(p) &= \bar{v}(p) \Pi_\lambda, \\
    \Pi_\lambda &= \frac{1}{2} \left( 1 + \lambda \gamma^5 \right), & \lambda &= \pm 1,
\end{align*}
\]

where u, v are Dirac (massless) spinors. The key ingredient in evaluating helicity amplitudes is given by the following set of formulas where all the relevant operators are explicitly given.

\[
\begin{align*}
    v_\lambda(p) \bar{u}_\sigma(q) &= - (2 p \cdot q)^{-1/2} \Pi_{-\lambda} [\delta_{\lambda,\sigma} \bar{p}_\parallel q + \delta_{\lambda,-\sigma} \bar{p}_\perp q], \\
    u_\lambda(p) \bar{v}_\sigma(q) &= - (2 p \cdot q)^{-1/2} \Pi_\lambda [\delta_{\lambda,\sigma} \bar{p}_\parallel q + \delta_{\lambda,-\sigma} \bar{p}_\perp q], \\
    v_\lambda(p) \bar{v}_\sigma(q) &= - (2 p \cdot q)^{-1/2} \Pi_{-\lambda} [\delta_{\lambda,\sigma} \bar{p}_\parallel q + \delta_{\lambda,-\sigma} \bar{p}_\perp q], \\
    u_\lambda(p) \bar{u}_\sigma(q) &= - (2 p \cdot q)^{-1/2} \Pi_\lambda [\delta_{\lambda,\sigma} \bar{p}_\parallel q + \delta_{\lambda,-\sigma} \bar{p}_\perp q],
\end{align*}
\]
where we have introduced an auxiliary vector $n_\mu$ which satisfies $n \cdot n = 1$, $n \cdot p = n \cdot q = 0$ and it is otherwise arbitrary. In the previous relation an overall phase has been constantly neglected, i.e. these relations are exact modulus an arbitrary phase, for instance the phase of $\mathbf{\pi}_\lambda(p)u_\lambda(q)$. This fact alone will be of no consequence as long as one remembers to organize the calculation of all the diagrams contributing to a certain helicity amplitude (for a given process) in such a way that the unknown phase remains as an overall phase.

Thus in WTO all the processes $e^+(p_+)e^-(p_-) \rightarrow f_1(q_1)f_2(q_2)f_3(q_3)f_4(q_4)$ are organized in such a way that the requested operators are always relative to the pairs $(p_-, q_3)$, $(p_+, q_2)$ and $(q_1, q_4)$. Following this strategy and given that we still want a final answer completely expressible in terms of invariants, we have made a very specific choice of the three auxiliary vectors which we need. They are

$$l_\mu = \frac{1}{N_l} \varepsilon_{\mu\nu\alpha\beta} q_3^\nu q_2^\alpha p_+^\beta,$$

$$m_\mu = \frac{1}{N_m} \varepsilon_{\mu\nu\alpha\beta} q^\nu q_4^\alpha q_1^\beta,$$

$$n_\mu = \frac{1}{N_n} \varepsilon_{\mu\nu\alpha\beta} q_4^\nu p_+^\alpha q_3^\beta,$$  \hspace{1cm} (51)

where the proper normalization has been included, i.e.

$$N_l = \frac{1}{2} (x_{14}x_{15}x_{45})^{1/2},$$

$$N_m = \frac{1}{2} (x_{34}x_{36}x_{46})^{1/2},$$

$$N_n = \frac{1}{2} (x_{24}x_{25}x_{45})^{1/2}. \hspace{1cm} (52)$$

Actually only certain combinations of the auxiliary vectors appear explicitly in the calculation. For instance $\slashed{p}_-\slashed{n}_3$ etc. We make use of the relation

$$\gamma^\mu \varepsilon_{\mu\nu\alpha\beta} = \gamma_\nu \gamma_\alpha \gamma_\beta \gamma_5^5 + \gamma^5 (\delta_\nu\alpha \gamma_\beta - \delta_\nu\beta \gamma_\alpha + \delta_\alpha\beta \gamma_\nu),$$ \hspace{1cm} (53)

to obtain

$$\slashed{q}_2\slashed{p}_+ = \left(\frac{-x_{14}}{x_{15}x_{45}}\right)^{1/2} \gamma^5 \slashed{q}_2\slashed{n}_3, \hspace{1cm} (54)$$

where again all particles are strictly massless. It is relatively easy to show that no singularity will be introduced by the normalization factors at the boundaries of the phase

$$\gamma^\mu \varepsilon_{\mu\nu\alpha\beta} = \gamma_\nu \gamma_\alpha \gamma_\beta \gamma_5^5 + \gamma^5 (\delta_\nu\alpha \gamma_\beta - \delta_\nu\beta \gamma_\alpha + \delta_\alpha\beta \gamma_\nu),$$ \hspace{1cm} (53)

to obtain

$$\slashed{q}_2\slashed{p}_+ = \left(\frac{-x_{14}}{x_{15}x_{45}}\right)^{1/2} \gamma^5 \slashed{q}_2\slashed{n}_3, \hspace{1cm} (54)$$

where again all particles are strictly massless. It is relatively easy to show that no singularity will be introduced by the normalization factors at the boundaries of the phase.
space. However from a numerical point of view strong cancellation may occur for particular processes, especially when both \( l \) and \( n \) will appear giving a factor \( x_{45}^{-1} \) in the amplitude. To avoid numerical instabilities we have made a systematic use of the freedom in choosing the auxiliary vectors, namely

\[
\begin{align*}
  l_\mu &= \frac{1}{N_l} \varepsilon_{\mu \alpha \beta} \, k_\alpha q^\alpha_2 \, p^\beta_+, \\
  m_\mu &= \frac{1}{N_m} \varepsilon_{\mu \alpha \beta} \, k_\alpha q^\alpha_4 q^\beta_1, \\
  n_\mu &= \frac{1}{N_n} \varepsilon_{\mu \alpha \beta} \, k_\alpha p^\alpha_3 q^\beta_3, 
\end{align*}
\]

(55)

where \( k_\alpha \) is any linear combination of \( p_- \), \( q_1 \), \( q_3 \), \( q_4 \), \( k_\beta \) of \( p_+, p_- \), \( q_2 \), \( q_3 \) and \( k_\gamma \) of \( p_+, q_1 \), \( q_2 \), \( q_4 \).

Whenever a region of the phase-space is examined, which is dangerously closed to some portion of the boundary, then we have switched to a choice of the auxiliary vectors free of numerical instabilities in that region. Let us discuss this point in more details since it is connected with another important issue. Whenever we change auxiliary vectors for a particular diagram we must also remember that our amplitudes are defined modulus a phase, thus to implement this procedure we have to compute the relative phase. For instance we redefine

\[
\begin{align*}
  L_\mu &= \frac{1}{N_L} \varepsilon_{\mu \alpha \beta} \, p^\gamma q^\alpha_2 p^\beta_+, \\
  M_\mu &= \frac{1}{N_M} \varepsilon_{\mu \alpha \beta} \, q^\alpha_3 q^\beta_4 q^\gamma_1, \\
  N_\mu &= \frac{1}{N_N} \varepsilon_{\mu \alpha \beta} \, p^\alpha_+ p^\beta_3 q^\gamma_3, 
\end{align*}
\]

(56)

and assume that for a given diagram we need to switch from

\[
\begin{align*}
  \nabla_\lambda(q_2) \lambda_\lambda(p_+) &\rightarrow \nabla_\lambda(q_2) \lambda_\lambda(p_+), \\
  \nabla_\lambda(q_4) \mu_\mu_-(q_1) &\rightarrow \nabla_\lambda(q_4) \mu_\mu_-(q_1), \\
  \nabla_\mu(q_4) \mu_\mu_-(q_3) &\rightarrow \nabla_\mu(q_4) \mu_\mu_-(q_3). 
\end{align*}
\]

(57)

First of all we introduce

\[
\begin{align*}
  \nabla_\lambda(q_2) \lambda_\lambda(p_+) &= e^{i \phi_1} X_1, \\
  \nabla_\lambda(q_2) \lambda_\lambda(p_+) &= e^{i \phi_1} X_1, 
\end{align*}
\]

(58)

and we compute

\[
\begin{align*}
  e^{i(\phi_1-\phi_2)} X_1 X_L &= tr \left( \gamma^\lambda_+ \gamma^\lambda_2 \right), \\
  X_1^2 (X_L^2) &= tr \left( \gamma^\lambda_+ \gamma^\lambda_2 \right). 
\end{align*}
\]

(59)
In this context it is easy to show that

\begin{align*}
& e^{i(\phi_i - \phi_L)} = e^{-i(\phi_n - \phi_N)} = \frac{1}{2} \frac{1}{(x_{15} x_{24} x_{45})^{1/2}} (x_{15} x_{24} + x_{45} - x_{14} x_{25} - 4 i \lambda \varepsilon_4), \\
& e^{i(\phi_m - \phi_M)} = \frac{1}{2} \frac{1}{(x_{34} x_{35} x_{46} x_{56})^{1/2}} (x_{34} x_{35} + x_{46} x_{56} - x_{36} x_{45} + 4 i \lambda \varepsilon_{15}), \\
& \varepsilon_{15} = \varepsilon(q_1, q_2, q_3, q_4).
\end{align*}

A typical example of a diagram to evaluated will be

\begin{equation}
D(\lambda, \rho, \sigma) = \overline{\nu}_\lambda(p_+) \Gamma u_{-\lambda}(p_-) \overline{\nu}_\rho(q_1) \Gamma' v_{-\rho}(q_2) \overline{\nu}_\xi(q_3) \Gamma'' v_{-\xi}(q_4),
\end{equation}

where \(\Gamma, \ldots, \Gamma''\) are strings of \(\gamma\)-functions. This will be computed as

\begin{equation}
D(\lambda, \rho, \sigma) = tr \{ \Gamma' [v_{-\rho}(q_2) \overline{\nu}_\lambda(p_+)] \Gamma [u_{-\lambda}(p_-) \overline{\nu}_\xi(q_3)] \Gamma'' [v_{-\xi}(q_4) \overline{\nu}_\rho(q_1)] \}
= \sum_{\lambda, \rho, \xi = -1}^{+1} A_{\lambda, \rho, \xi} + i \sum_{i=1,5} b_{\lambda, \rho, \xi} \varepsilon_i.
\end{equation}

The presence of saturated \(\varepsilon\)-tensors seems to represent an obstacle towards giving the full amplitude in terms of invariants. Our strategy therefore is to express all \(|\varepsilon|\) in terms of invariants. For instance if we define \(\varepsilon_1 = \varepsilon(p_+, p_-, q_1, q_2)\) then we obtain

\begin{equation}
|\varepsilon|^2 = 2 \left( x_{13} x_{14} x_{23} x_{24} + x_{13} x_{24} x_{34} + x_{14} x_{23} x_{34} \\
- x_{13}^2 x_{24}^2 - x_{14}^2 x_{23}^2 - x_{34}^2 \right)
\end{equation}

and similar expressions for the other 5 independent saturated \(\varepsilon\)-tensors.

\begin{align*}
\varepsilon_1 &= \varepsilon(p_+, p_-, q_1, q_2), \\
\varepsilon_2 &= \varepsilon(p_+, p_-, q_1, q_3), \\
\varepsilon_3 &= \varepsilon(p_+, p_-, q_2, q_3), \\
\varepsilon_4 &= \varepsilon(p_+, q_1, q_2, q_3), \\
\varepsilon_5 &= \varepsilon(p_-, q_1, q_2, q_3).
\end{align*}

Moreover the sign of \(\varepsilon_1 \varepsilon_2, \ldots, \varepsilon_1 \varepsilon_5\) can all be determined in terms of invariants. Let \(\eta\) be the unknown sign of \(\varepsilon_1\), then we end up with some expression of the form

\begin{equation}
d(\lambda, \rho, \xi) = \left[ A_{\lambda, \rho, \xi} + i \eta \sum_{i=1,5} b_{\lambda, \rho, \xi} \text{sign}(\varepsilon_1 \varepsilon_i) |\varepsilon_i| \right].
\end{equation}

Therefore, as long as we only need \(|d|^2\), there will be no problem at all with the unknown \(\eta\) since \(\eta^2 = 1\). Of course we can have additional imaginary parts in the amplitude, due to
the width of the vector bosons. Suppose that in a diagram some internal line can be either a photon or a Z. Schematically we denote with \( a_\gamma + ib_\gamma \varepsilon, V = \gamma, Z \) the corresponding amplitude and with \( \alpha + i \beta \) the Z propagator. The overall expression for the amplitude is therefore

\[
d = (a_\gamma + \alpha a_\gamma) - \beta b_\gamma \varepsilon + i (b_\gamma + \beta a_\gamma) \varepsilon + i \beta a_\gamma,
\]

(66)

In computing \(|d|^2\) all terms which are linear in \( \varepsilon \) do not contribute after integration, therefore

\[
|d|^2 = (a_\gamma + \alpha a_\gamma)^2 + (\beta b_\gamma)^2 + (b_\gamma + \beta a_\gamma)^2 \varepsilon^2 + (\beta a_\gamma)^2.
\]

(67)

We give an example of how the helicity amplitudes are worked out explicitly. Let us start from some diagram whose expression is

\[
\omega (\lambda, \rho, \xi) = \overline{\sigma}_-(p_+ \gamma^\mu (a + b \gamma^5) u_\lambda(p_-) \times \overline{\sigma}_\rho (q_3) \gamma^\nu \gamma_+ (p_+ + p_- - \slashed{q}_4) \gamma^\rho (a + b \gamma^5) v_- (q_4) \times \overline{\sigma}_\xi (q_1) \gamma^{\mu \nu} \gamma_+ \varepsilon (q_2),
\]

(68)

where \( \gamma_\pm = 1 \pm \gamma^5 \). It is easy to show that

\[
\omega (\lambda, \lambda, \lambda) = 16 g_L^2 \delta_{\lambda, 1} (x_{15} x_{24} x_{34} x_{45} x_{46})^{-1/2}
\times \text{tr} \left[ \Pi_{-\lambda} \gamma^\mu \gamma^\rho \gamma_+ (p_+ + p_- - \slashed{q}_4) \gamma^\rho \gamma^5 \slashed{q}_2 \gamma^1 \gamma^5 \slashed{q}_3 \gamma_+ \right],
\]

\[
= 16 g_L^2 \delta_{\lambda, 1} (x_{15} x_{24} x_{34} x_{45} x_{46})^{-1/2}
\times \text{tr} \left[ \Pi_{-\lambda} \gamma^\mu \gamma_+ (p_+ + p_- - \slashed{q}_4) \gamma^5 \slashed{q}_2 \gamma^1 \gamma^5 \slashed{q}_3 \gamma_+ \right],
\]

\[
\omega (\lambda, \lambda, -\lambda) = \omega (\lambda, -\lambda, \lambda) = 0,
\]

\[
\omega (\lambda, -\lambda, -\lambda) = 16 g_L g_R \delta_{\lambda, -1} (x_{14} x_{25} x_{34} x_{46})^{-1/2}
\times \text{tr} \left[ \Pi_{-\lambda} \gamma^\mu \gamma_+ (p_+ + p_- - \slashed{q}_4) \gamma^5 \slashed{q}_2 \gamma^1 \gamma^5 \slashed{q}_3 \right],
\]

\[
= 16 g_L g_R \delta_{\lambda, -1} (x_{14} x_{25} x_{34} x_{46})^{-1/2}
\times \text{tr} \left[ \Pi_{-\lambda} \gamma^\mu \gamma_+ (p_+ + p_- - \slashed{q}_4) \gamma^5 \slashed{q}_2 \gamma^1 \gamma^5 \slashed{q}_3 \gamma_+ \right],
\]

(69)

where

\[
g_L = \frac{1}{2} (a + b), \quad g_R = \frac{1}{2} (a - b),
\]

(70)

after which the Kahane identities can be applied and the trace can be taken. For instance we find

\[
\omega (\lambda, -\lambda, -\lambda) = -32 g_L g_R \delta_{\lambda, -1} (x_{14} x_{25} x_{34} x_{46})^{-1/2}
\times \text{tr} \left[ \Pi_{-\lambda} \gamma^\mu \gamma_+ (p_+ + p_- - \slashed{q}_4) \slashed{q}_2 \gamma^5 \gamma^5 \slashed{q}_3 \gamma_+ \right],
\]

\[
= 64 g_L g_R \delta_{\lambda, -1} (x_{14} x_{25} x_{34} x_{46})^{-1/2}
\times \text{tr} \left[ \Pi_{-\lambda} \slashed{q}_4 \gamma_2 \slashed{q}_1 \gamma_3 \gamma_+ (p_+ + p_- - \slashed{q}_4) \slashed{q}_2 \gamma^5 \gamma^5 \slashed{q}_3 \right].
\]

(71)
There is another place where we must worry about relative phases. Consider the NC64 processes, typically $e^+e^- \to f\bar{f}f\bar{f}$. The 64 diagrams arise from 4 permutations of 16 different topologies, which we denote by

\[
\mathcal{P}_1\{q_1, q_2, q_3, q_4\} = \{q_1, q_2, q_3, q_4\},
\]
\[
\mathcal{P}_2\{q_1, q_2, q_3, q_4\} = \{q_3, q_2, q_1, q_4\},
\]
\[
\mathcal{P}_3\{q_1, q_2, q_3, q_4\} = \{q_1, q_4, q_3, q_2\},
\]
\[
\mathcal{P}_4\{q_1, q_2, q_3, q_4\} = \{q_3, q_4, q_1, q_2\}.
\]

In the massless case there are 12 distinct sets of helicity states which we denote by labels $h = a - m$. For instance the effect of $\mathcal{P}_2$ can be inferred from table 1.

| $p_+$ | $p_-$ | $q_1$ | $q_2$ | $q_3$ | $q_4$ | $h$ | $\mathcal{P}_2 \to$ | $p_+$ | $p_-$ | $q_1$ | $q_2$ | $q_3$ | $q_4$ | $h$ |
|-------|-------|-------|-------|-------|-------|-----|----------------|-------|-------|-------|-------|-------|-------|-----|
| $+$   | $-$   | $+$   | $+$   | $+$   | $+$   | $+$ | $a$            | $+$   | $-$   | $+$   | $+$   | $+$   | $-$   | $b$ |
| $-$   | $+$   | $+$   | $+$   | $+$   | $+$   | $a$ | $b$            | $-$   | $+$   | $+$   | $+$   | $+$   | $+$   | $+$ |
| $+$   | $-$   | $+$   | $+$   | $+$   | $+$   | $b$ | $c$            | $+$   | $-$   | $+$   | $+$   | $+$   | $-$   | $c$ |
| $-$   | $+$   | $+$   | $+$   | $+$   | $+$   | $c$ | $d$            | $-$   | $+$   | $+$   | $+$   | $+$   | $-$   | $d$ |
| $+$   | $-$   | $+$   | $+$   | $+$   | $+$   | $d$ | $e$            | $-$   | $-$   | $+$   | $+$   | $+$   | $+$   | $+$ |
| $-$   | $+$   | $+$   | $+$   | $+$   | $+$   | $+$ | $f$            | $-$   | $-$   | $+$   | $+$   | $+$   | $-$   | $f$ |
| $+$   | $-$   | $+$   | $+$   | $+$   | $+$   | $g$ | $-$   | $+$   | $+$   | $+$   | $+$   | $-$   | $-$   | $g$ |
| $-$   | $+$   | $+$   | $+$   | $+$   | $+$   | $-$   | $h$            | $-$   | $+$   | $+$   | $+$   | $+$   | $-$   | $-$   | $h$ |

Table 1: The effect of the $\mathcal{P}_2$ permutation on the non-zero helicity sets for the 16 fundamental topologies contributing to a typical NC64 process.

Our strategy has been to compute explicitly the 16 diagrams corresponding to the standard ordering $\{q_1, q_2, q_3, q_4\}$ and to perform the relative permutations of invariants and helicity sets for the remaining diagrams. This however requires the knowledge of all the relative phases which we have computed explicitly. Since the relative phases depend on the particular assignment of momenta and helicities but not on the particular topology under considerations it is enough to compute them for one diagram per permutation.

As we have already stressed all fermions in WTO are kept rigorously massless. This fact however should not be confused with a limitation of the helicity formalism. Indeed we briefly sketch how fermion masses can be incorporated into our approach. We consider spinors $u(p, n, \lambda)$, $\lambda = \pm 1$ defined to be the eigenstates of the operator $P_+(p, n, \lambda)$ corresponding to eigenvalue 1.

\[
P_+(p, n, \lambda) = \Lambda_+(p) \frac{1}{2} \left(1 + i\lambda \gamma^5 \not{p}\right) \Lambda_+(p),
\]
\[
\Lambda_+(p) = \frac{1}{2p_0} \left(-i \not{p} + m\right) \gamma^4,
\]

where $n$ is the polarization vector and $p \cdot n = 0$, $n^2 = 1$. If we consider the typical CC20 process, $e^+e^- \to e^-\bar{\nu}_e\mu^+$ and we are interested in the finite $m_e$ effects in the limit of
zero scattering angle then things can be organized in such a way that we only need to compute

$$u(p, n, \sigma) \overline{u}(q, \lambda), \text{ etc.,}$$

where $p^2 = -m^2$ while $q^2 = 0$. In this case we easily obtain

$$u(p, n, \sigma) \overline{u}(q, \lambda) = \frac{1}{8} (mp_0 q_0)^{-1/2} \times \sum_{a=\pm} N_a(p, q) (-i \not{p} + m) (1 + i a \not{q}) \not{q} \left(1 + \lambda \gamma^5 \right) \mathcal{P}^a(\lambda, \sigma),$$

$$m N_a^{-2} = am q \cdot n - p \cdot k, \quad \mathcal{P}^a(\lambda, \sigma) = \frac{1}{2} (1 + a \lambda \sigma).$$

Moreover if both fermionic lines are massive we can repeat the whole procedure by using

$$u(p, n, -\lambda) \overline{u}(q, n, \lambda) = \frac{1}{N^2} (-i \not{p} + m) \left(1 - i \lambda \gamma^5 \not{q}\right) \left(-i \not{q} + m\right) \left(1 + i \lambda \gamma^5 \not{p}\right),$$

$$u(p, n, \lambda) \overline{u}(q, n, \lambda) = \frac{1}{N^2} (-i \not{p} + m) \left(1 + i \lambda \gamma^5 \not{q}\right) \left(-i \not{q} + m\right) \left(1 + i \lambda \gamma^5 \not{p}\right),$$

with

$$N^2 = (p + q)^2.$$ 

For NC32 and non-leptonic NC64 or Mix43 the exchange of gluons must be taken into account. This raises the basic question of what to use for $\alpha_s$ or, alternatively, at which scale to compute it. Here the situation is far from satisfactory, especially as compared to the well established predictions for 2-fermion production at the $Z$ resonance. In the present version of WTO we have adopted the *ad hoc* prescription of using $\alpha_s(M_W)$ which becomes therefore an input parameter. However a more satisfactory choice would imply to consider a *running* $\alpha_s$. Ideally for each sub-process $g^*(s) \rightarrow \bar{q}q$ we should include a factor $\alpha_s(s)$ and subsequently integrate over the invariant mass $s$, however this can only be done when a reasonable cut is applied, $s \geq s_0$ to avoid entering non-perturbative regions where the correction factor would go out of control. Clearly more refined analyses are needed to reach a satisfactory level of description. We end this section with a very short description of what it is meant by NQCD. Consider the CC11 process $e^+e^- \rightarrow \mu^- \overline{\nu}_\mu u\overline{d}$ which effectively is a CC10 process. One would like to include final state QCD corrections, even when kinematical cuts are imposed. By *naive* QCD we mean a simple recipe where the total $W$-width is corrected by a factor

$$\Gamma_W \rightarrow \Gamma_W \left(1 + \frac{2}{3} \frac{\alpha_s(M_W)}{\pi}\right),$$

and where the cross section gets multiplied by

$$\sigma_{CC10,C} \rightarrow \sigma_{CC10,C} \left(1 + \frac{\alpha_s(M_W)}{\pi}\right).$$
This naive approach, consequence of our ignorance about the complete result, would be correct only for $\sigma_{CC03,ex}$, the double-resonant approximation with fully extrapolated setup. For $\sigma_{CC10,C}$ it is instead only a rough approximation because of two reasons. First of all in CC10 we have not only a virtual QCD correction to the $Wud$ vertex but also a box diagram. Moreover QED and QCD radiation are quite different if cuts are imposed, especially in presence of severe cuts. Thus any inclusion of final state QCD corrections is, at present, only a very crude approximation which moreover can become quite bad whenever stringent kinematical cuts are applied to the process.

2.5 Mappings of the phase space

For a total cross section we need a 7-dimensional (9-dimensional when initial state radiation is included) integration. In order to achieve the requested precision we must examine the structure of the integrand which will show a complicated peaking structure due to propagators (both in the time-like and in the space-like region), to Jacobians and to square-integrability but not integrability of the structure functions. As a consequence we must perform a careful analysis and find out, as much as possible, those isomorphisms of the phase space which are needed to cure the peaks of the differential distribution. First of all by transforming the integration over the phase space into an integration over the kinematically independent invariants we have introduced a Jacobian which contains two inverse square roots of polynomials. This will always be cured as follows. The part of the Jacobian denoted by $R_2$ is a quadratic form in $t_1$ so that our cross section can be schematically written as

$$\sigma = \int dm^2 \ldots dw \int_{t_l}^{u} dt_1 f(\ldots, t_1) \left(-at_1^2 + 2bt_1 + c\right)^{-1/2},$$

where $a, b, c$ are functions of the other invariants and moreover $a > 0$ over all the phase space. We introduce the following mapping

$$t_1 \rightarrow t'_1 = -\frac{1}{\sqrt{a}} \arccos \frac{b - at_1}{D}, \quad D^2 = b^2 + ac.$$ (81)

which gives

$$\sigma = \int dm^2 \ldots dw \int_{t'_l}^{u} dt'_1 f(\ldots, b/a + D/a \sin(\sqrt{a}t'_1)),
$$

$$t'_l = -\frac{1}{\sqrt{a}} \arcsin \left[1 + \frac{a}{D} (t_+ - t_l)\right],
$$

$$t'_u = \frac{1}{\sqrt{a}} \arcsin \left[1 + \frac{a}{D} (t_u - t_-)\right],
$$

$$t_\pm = \frac{b}{a} \mp \frac{D}{a}.$$ (82)

There is another square root in the Jacobian which is slightly more difficult to treat since it is a polynomial of order $> 2$ in all invariants. Actually it can be seen as a quartic form in the variable $z = m^2$ defined in eq.(4), i.e.
\[ R_4^2 = -\left[ z^2 + (\Sigma - 1) z + \lambda_+ \right] \left[ z^2 + (\Sigma - 1) z + \lambda_- \right], \]  
\tag{83}

where we have introduced
\[ \Sigma = m_-^2 + m_+^2 + M_0^2 + m_0^2, \]
\[ \lambda_\pm = M_0 m_0 \pm m_- m_+. \]  
\tag{84}

Given
\[ r_{1,2} = \frac{1}{2} \left( 1 - \Sigma \mp \sqrt{\Delta_-} \right), \]
\[ s_{1,2} = \frac{1}{2} \left( 1 - \Sigma \mp \sqrt{\Delta_+} \right), \]
\[ \Delta_\pm = (1 - \Sigma)^2 - 4 \lambda_\pm^2, \]  
\tag{85}

we have the following three possibilities
\[ \Delta_- \leq 0, \quad \Delta_- \geq 0 \geq \Delta_+, \quad \Delta_+ \geq 0, \]  
\tag{86}

which correspond to
\[ R_4^2 \leq 0, \]
\[ r_1 \leq z \leq r_2, \]
\[ r_1 \leq z \leq s_1, \quad \text{or} \quad s_2 \leq z \leq r_2, \]  
\tag{87}

Only the second one will be discussed here in some details. Again schematically we have
\[ I = \int_{r_1}^{r_u} \frac{dz f(z, \ldots)}{R_4}, \]
\[ r_1 \geq r_1, \quad r_u \leq r_2, \quad r_{1,2} = r \mp \frac{1}{2} \sqrt{\Delta_-}. \]  
\tag{88}

Next we introduce a new variable
\[ \cos \phi = 2 \frac{r - z}{\sqrt{\Delta_-}}, \]  
\tag{89}

and distinguish among three alternatives. If \( r_u \leq r \) then
\[ I = \int_{\mu_1}^{\mu_u} d\mu f \left( r - \frac{1}{2} \sqrt{\Delta_-} \cos \phi, \ldots \right), \quad \mu = \frac{\delta}{\gamma} F(\phi, K). \]  
\tag{90}

If instead \( r_1 \geq r \) then
\[ I = \int_{\mu_u}^{\mu_1} d\mu f \left( r + \frac{1}{2} \sqrt{\Delta_-} \cos \phi, \ldots \right), \quad \mu = \frac{\delta}{\gamma} F(\phi, K). \]  
\tag{91}
Finally if \( r_1 \leq r \leq r_u \) we first introduce
\[
\bar{\mu} = -\frac{\delta}{\gamma} F\left(\frac{\pi}{2}, K\right),
\] (92)
and subsequently we obtain
\[
I = \int_{\bar{\mu}}^{\mu} d\mu f \left( r - \frac{1}{2} \sqrt{\Delta_-} \cos \phi, \ldots \right) + \int_{\mu_u}^{\mu} d\mu f \left( r + \frac{1}{2} \sqrt{\Delta_-} \cos \phi, \ldots \right).
\] (93)
In the last equations \( F \) denotes the elliptic function
\[
F(\phi, K) = \int_0^\phi d\psi \left( 1 - K^2 \sin^2 \psi \right)^{-1/2},
\] (94)
and
\[
R^2_4 = -(z - r_1)(z - r_2) \left[ (z - r)^2 + s^2 \right],
A = \frac{s^2 + (r_2 - r)(r_1 - r)}{s \sqrt{\Delta_-}},
\]
\[
K_1 = A + \left( A^2 + 1 \right)^{1/2},
K^2 = \frac{1}{1 + K_1^2},
\]
\[
\frac{\delta}{\gamma} = -2 \frac{K_1 K}{\left( 2 s K_1 \sqrt{\Delta_-} \right)^{1/2}}.
\] (95)

The case of four real roots will not be considered explicitly here but it can be treated along the same lines, i.e. by introducing elliptic functions.

Whenever initial state radiation is included we have to consider the following two integrations
\[
\sigma = \int dx_+ dx_- D(x_+, s), D(x_-, s) \int dPS \Theta_{\text{cut}} f,
\] (96)
where the structure functions are
\[
D(x, s) = \frac{\beta}{2} (1 - x)^{\beta/2 - 1} \left\{ \exp \left[ \frac{1}{2} \beta \left( \frac{3}{4} - \gamma_e \right) \right] + D_r(x, s) \right\},
\] (97)
and the \( \beta \)-factor is
\[
\beta = 2 \frac{\alpha}{\pi} \left( \ln \frac{s}{m^2_e} - 1 \right).
\] (98)

In order to have a proper treatment of the \( x_+/x_- \) integration we first change variables to
\[
x_- = u, \quad x_+ = \frac{v}{u}.
\] (99)
The behavior near \( x = 1 \) is cured by introducing two new variables, \( 0 \leq U, V \leq 1 \) which are related to \( u, v \) through

\[
\begin{align*}
    u &= 1 - (1 - u_t) (1 - U)^{2/\beta}, \\
    v &= u - (u - u_t) (1 - V)^{2/\beta},
\end{align*}
\]

where we assume that the final state invariant masses are bounded by below, i.e. \( m_-^2 \geq M_1^2, \ldots, m_+^2 \geq M_6^2 \) and

\[
\begin{align*}
    u_t &= \max(u_a, u_b, u_c), \\
    u_a &= (M_1 + M_6)^2, \\
    u_b &= (M_2 + M_5)^2, \\
    u_c &= (M_3 + M_4)^2.
\end{align*}
\]

Finally most of the diagrams belonging to a given process receive resonating contributions by vector boson propagators. The resonating mapping is not always mandatory, especially for those processes where the final answer is not dominated by \( W \) or \( Z \) peaks, like for instance in NC processes where we can have large effects from photon exchange or from \( t \)-channel diagrams. However for CC processes we have a double-resonating \( W \)-exchange which will show up as

\[
\sigma = \int \ldots \int_{a_-}^{b_-} dm_-^2 \int_{a_+}^{b_+} dm_+^2 \frac{1}{\left[(vm_-^2 - \mu^2_w)^2 + (vm_+^2 - \mu^2_w)^2 + (vm_w^2 - \gamma_w^2)^2\right]^2} \times \int \ldots f \left(m_-^2, m_+^2, \ldots\right),
\]

where \( \mu^2_w = M_w^2/s \) and \( \gamma_w = \Gamma_w/M_w \). The lower and upper limits of integrations for \( m_{\pm}^2 \) may differ from the natural limits because of kinematical cuts. Let us deal with one specific integration and change variable, e.g. \( z = vm_-^2 \). The typical integral will be

\[
I = \frac{1}{v} \int_{z_-}^{z_+} dz \frac{f(z, \ldots)}{(z - \mu_a^2)^2 + (z \gamma_w)^2}.
\]

We first transform

\[
z = \mu_w^2 \left\{ 1 + \frac{\Gamma_w}{M_w} \tan \left[(z_b - z_a) Z + z_a\right] \right\},
\]

with

\[
z_{a,b} = \arctan \frac{z_{\pm} - \mu_w^2 + \left(\frac{\Gamma_w}{M_w}\right)^2 z_\pm}{\gamma_w \mu_w^2},
\]

\[
\mu^2_w = \frac{M_w^2}{s} \text{ and } \gamma_w = \frac{\Gamma_w}{M_w}.
\]
and where $0 \leq Z \leq 1$. In this way we get

$$I = \frac{z_b - z_a}{\gamma_\mu \mu_f} \int_0^1 dZ f (z(Z), \ldots).$$

(106)

Concerning the resonating mapping we stress that indeed there are 4-fermion processes which receive the largest contribution from double-resonating diagrams. A typical example is given by $e^+e^- \rightarrow \mu^+ \tau-u\bar{d}d$ at LEP 2 energies where the percentage difference between CC03 and CC11 is totally negligible. In this and similar situations the isomorphism of the phase space that we have described is mandatory. The other cases where the mapping is strongly advised are those where very stringent cuts are applied around the double resonance. For instance

$$e^+e^- \rightarrow \bar{b}b\bar{f}f, \quad M_Z - \Delta \leq M(fj) \leq M_Z + \Delta, \quad M(\bar{b}b) \geq M_\tau.$$  

(107)

On the contrary there are examples where double-resonating diagrams only give a small fraction of the total answer, either because there are $\gamma$-exchanges not suppressed by kinematical cuts or because of large $t$-channel contributions. In all these cases it turns out to be more convenient to deal with a flat phase-space, i.e. no mapping at all.

### 2.6 Gauge invariance

Any calculation for $e^+e^- \rightarrow 4$-fermions is only nominally a tree level approximation because of the presence of charged and neutral, unstable vector bosons and of their interaction with photons. Unstable particles require a special care and their propagators, in some channels, must necessarily include an imaginary part or in other words the corresponding S-matrix elements will show poles shifted into the complex plane. In any field-theoretical approach these imaginary parts are obtained by performing the proper Dyson resummation of the relative two-point functions, which at certain thresholds will develop the requested imaginary component. It is well known that the bosonic parts of vector boson self-energies are not separately gauge independent, or stated differently they will show a gauge-parameter dependence. This problem will arise in the context of a full one loop calculation, while here we can take into account only the fermionic parts which are obviously free of ambiguities. Thus the correct recipe seems representable by a Dyson resummation of fermionic self-energies where only the imaginary parts are actually included. As a result the vector boson propagators will be inserted into the corresponding tree level amplitudes with a $p^2$-dependent width. Its has already been noticed by several authors that even this simple idea gives rise to a series of inconsistencies, which sometimes may give results completely inconsistent even from a numerical point. The fact is that the introduction of a width into the propagators will inevitably result, in some cases, into a breakdown of the relevant Ward identities of the theory with a consequent violation of some well understood cancellation mechanism. Sometimes the effect of spoiling a cancellation among diagrams can result into a numerical catastrophe.

In the present version WTO is including reparation of gauge invariance according to the fermion-loop scheme [12] for the CC20 family, while an extension to other processes in the TeV region is currently in preparation.

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Here we briefly sketch the procedure adopted for CC20. In this case there is a violation of gauge invariance induced by the so called fusion diagram which contributes to $e^+e^- \rightarrow e^-\overline{u}d$ through a $\gamma(Z)$ and $W$ bremsstrahlung followed by $\gamma(Z) + W \rightarrow W^* \rightarrow u\overline{d}$. In WTO the width of a vector boson is always zero for a $t$-channel exchange thus for CC20 gauge invariance will be violated because of the $W$ width in the $s$-channel. The fermion-loop scheme includes in the calculation the imaginary part of the two triangle diagrams (with opposite charge flow) with a fermionic loop which represents the first order correction to the $VWW$ vertex. In the fermionic loop we include leptons, neutrinos and $u, d, c, s$ quarks, i.e. everything which is allowed in the $W$ decay. In principle and for $m_+^2 \geq (m_t + m_b)^2$ there are two additional imaginary parts for the vertex which correspond to cutting a $t$-line and a $b$-line. These extra, $m_t$-dependent, contributions have not been included so far in our calculation. To be more specific we take into account the corrections to

$$V_\nu(q) + W_\mu(p) \rightarrow W_\alpha(k), \quad V = \gamma, Z \quad p = p_- + q_1, \quad p = p_+ - q_2, \quad k = q_3 + q_4. \quad (108)$$

The correction factor becomes

$$V^{\gamma(Z)}_{\mu\nu\alpha} = (2\pi)^4 i g_s \left( c_w \right) \left[ -\frac{9}{16} i \frac{g^2}{\pi} V_{\mu\nu\alpha} \right], \quad (109)$$

where we have explicitly factorized the lowest order coupling. The corrected vertex $V$ does not factorize into the corresponding lowest order amplitude but instead three additional form factors must be included. They give

$$V_{\nu\alpha\mu} = \sum_{i=1}^{4} C_i V_{\nu\alpha\mu}^i,$$

$$C_i = \frac{y}{\lambda} \left( \left( a_i + \frac{b_i}{\lambda^2} \right) I_0 + \left( c_i + \frac{d_i}{\lambda} \right) L_0 \right), \quad i = 1, \ldots 3$$

$$C_4 = \frac{y}{\lambda} \left( \left( a_4 + \frac{b_4}{\lambda^2} + \frac{e_4}{\lambda^2} \right) I_0 + \left( c_4 + \frac{d_4}{\lambda} + \frac{f_4}{\lambda^2} \right) L_0 \right). \quad (110)$$

with

$$V_{\nu\alpha\mu}^1 = -\left[ \delta_{\alpha\mu} k_\nu + \delta_{\nu\mu} q_\alpha - \delta_{\nu\alpha} k_\mu \right],$$

$$V_{\nu\alpha\mu}^2 = (\delta_{\nu\mu} q_\alpha - \delta_{\nu\alpha} k_\mu),$$

$$V_{\nu\alpha\mu}^3 = (\delta_{\nu\mu} q_\alpha + \delta_{\nu\alpha} k_\mu),$$

$$V_{\nu\alpha\mu}^4 = -\frac{1}{k^2} k_\nu q_\alpha k_\mu. \quad (111)$$

Moreover
\[ \lambda = (p \cdot q)^2 - p^2 q^2, \]
\[ I_0 = \frac{1}{2}, \quad L_0 = \frac{1}{4 \sqrt{\lambda}} \log \frac{p \cdot q - \sqrt{\lambda}}{p \cdot q + \sqrt{\lambda}}. \] (112)

and

\[ a_1 = 6x - \frac{4}{3}z, \]
\[ a_2 = 5x, \]
\[ a_3 = -3x - \frac{2}{3}(y - z), \]
\[ a_4 = \frac{8}{3}y, \]
\[ b_1 = -x(yz - 4xy + xz), \]
\[ b_2 = \frac{3}{2}x(-yz + 4xy - xz), \]
\[ b_3 = \frac{1}{2}x(5yz - 4y^2 - 8xy + xz), \]
\[ b_4 = -\frac{1}{2}y(-\frac{176}{3}xy + 16xz - 16x^2 + \frac{4}{3}yz), \]
\[ c_1 = -4x(y - z + 2x), \]
\[ c_2 = 2x(-y + z - 4x), \]
\[ c_3 = 2x(3y - z + 2x), \]
\[ c_4 = -16xy, \]
\[ d_1 = -x^2(-3yz + 2y^2 + 6xy - xz), \]
\[ d_2 = \frac{3}{2}x^2(3yz - 2y^2 - 6xy + xz), \]
\[ d_3 = \frac{1}{2}x(-2y^2z - 9xyz + 14yx^2 + 10x^2y - x^2z), \]
\[ d_4 = -4xy(-3yz + 2y^2 + 18xy - 3xz + 2x^2), \]
\[ e_4 = -5xy^2(yz + 3xz - 6xy - 2x^2), \]
\[ f_4 = -10x^2y^2(-2yz + y^2 - 2xz + 6xy + x^2), \] (113)

with

\[ x = q^2, \quad y = k^2, \quad z = k^2 + q^2 - p^2. \] (114)

We have verified that the inclusion of the full correction in WTO accounts for an increase of the CPU time corresponding to a factor of 1.3, which we consider still very reasonable. It should be stressed here that we include the imaginary part for both \( \gamma W W \) and \( Z W W \) vertices in order to preserve the full \( SU(2) \otimes U(1) \) gauge invariance. This can be most easily seen by considering the sub-process \( e^- W^+ \rightarrow e^- u d \) and by writing the corresponding Ward identity \( p_\mu A^\mu_w + i \xi^2 M_w A_\phi = 0 \) in the \( R_\xi \) gauge (\( \phi \) being the Higgs ghost). The
relevance of the fermion-loop scheme for CC processes with electrons in the final state becomes more and more pronounced for smaller and smaller electron scattering angles. We have explicitly verified that a selection cut of $10^\circ$ on the $e^-\bar{\nu}_e\nu\mu\mu$ is enough to avoid any problem. Partial results are shown in table 2, where $\theta_m \leq \theta_e \leq \pi - \theta_m$. In both cases the W width is running. Moreover $E_{e,\mu} \geq 1$ GeV and $\theta(e^-,\mu^+) \geq 5^\circ$.

| $\theta_m$ (deg) | $\sigma$ (nb) | $\sigma_{FL}$ (nb) | $\delta_{FL}$ (%) |
|------------------|--------------|-------------------|-------------|
| $10^\circ$       | 0.20140(4)   | 0.20143(4)        | +0.01       |
| $5^\circ$        | 0.20653(3)   | 0.20643(3)        | -0.05       |
| $1^\circ$        | 0.21554(21)  | 0.21014(3)        | -2.57       |

Table 2: Inclusion of the gauge restoring terms at 190 GeV in $e^+e^- \rightarrow e^-\bar{\nu}_e\nu\mu\mu$, for $\theta_m \leq \theta_e \leq \pi - \theta_m$. In both cases the W width is running. Moreover $E_{e,\mu} \geq 1$ GeV and $\theta(e^-,\mu^+) \geq 5^\circ$.

Needless to say the fully extrapolated cross section for $e^+e^- \rightarrow e^-\bar{\nu}_e\nu\mu\mu(ud\bar{d})$ requires both gauge reparation and non zero electron mass. Thus strictly speaking one cannot set $\theta_m$ to zero in the present version of the program. For a better understanding of the effect of restoring gauge invariance in CC20 we have shown in Fig. 1 the angular distribution $d\sigma/d\cos \theta_l$ where $\theta_l$ is defined by

$$t_1 = x_+ e_1 \frac{1 + \cos \theta_l}{x_- + x_+ - (x_- - x_+) \cos \theta_l}, \quad (115)$$

i.e. $\theta_l$ is the angle between the incoming $e^-$ and the outgoing $l^-$. In Fig. 1 we have shown the angular distribution for $e^+e^- \rightarrow e^-\bar{\nu}_\mu\nu\mu$ with and without the fermion loop corrections as compared with the similar distribution for a CC11 process $e^+e^- \rightarrow \mu^-\bar{\nu}_\mu\nu\tau^+\tau^-$.  

### 2.7 Final state QED radiation

In this rather short section we would like to give an idea of the problematic connected with QED radiation. Because of gauge invariance there is no meaningful splitting of the QED corrections between initial state and final state, splitting which was instead applicable for LEP 1 physics. There are various attempts to circumvent the problem, most noticeably the current-splitting of GENTLE [13], but as a matter of principle even this technique cannot guarantee an unambiguous answer.

From this point of view only the full $O(\alpha)$ calculation makes sense and correctly reproduces the $O(\alpha \times \text{const})$ terms of the QED corrections. As in any other calculation of this type we would end up with a correction $\delta_{QED}$ factor of the following structure

$$\delta_{QED} = \delta_{\text{soft}} + \delta_{\text{virt}} + \delta_{\text{hard}}, \quad (116)$$

after which the proper exponentiation can be performed. What we know of $\delta_{QED}$ are the universal, leading logarithmic terms but a calculation of the virtual corrections, including
up to pentagon and hexagon diagrams, is missing. In such a situation it is extremely risky to partially account for final state QED radiation, even though any realistic estimate of cross sections and of energy losses should include at least some reasonable guess.

To this end we have preliminary investigated the inclusion of QED final state radiation to CC11 processes. We have made use of our knowledge of the universal soft terms and moreover of the general and simple result of ref. [14] for the emission of hard collinear radiation by charged particles. The latter allows to include hard photons emitted within a cone of fixed half-opening angle $\delta_c$, in the limit $\delta_c \ll 1$. This result is based on a rigorous gauge invariant procedure at $\delta_c \ll 1$ so that our correction factor for each charged fermionic line will be

$$\delta_{FSR} = \exp \left( \frac{\alpha}{\pi} \delta_{soft} \right) \left( 1 + \frac{\alpha}{\pi} \delta_{coll} \right).$$

(117)

In this way we can only make a very rough estimate of the effect of final state radiation since all the hard constant from virtual corrections are still missing. Moreover the procedure of exponentiation is also far from unique since we could decide to exponentiate not only the leading logarithms but also numerically relevant terms from $\delta_{coll}$. Let $E_i, m_i$ be the energy and the mass of the emitting fermion, moreover we denote by $E_0 = e_0 \sqrt{s}$ the energy threshold, i.e $E \geq E_0$. In our formulation

$$E_i = \frac{1}{2} \eta_i \sqrt{s}, \quad \eta_i = x_+ e_i + (x_+ - x_-) t_i.$$  

(118)

According to ref. [14] and to the subsequent generalization [15] we define

$$\rho_i = \frac{\delta_c \eta_i}{2 m_i},$$

$$1 - \varepsilon_i = \frac{2 e_0}{\eta_i},$$  

(119)

and compute $\delta_{coll}$ as $Q_i^2 C$ with $C$ given by eqs.(5-6) of ref. [15].

Our preliminary results show very little effect on the cross section for $e^+ e^- \rightarrow \mu^- \bar{\nu}_\mu u \bar{d}$, for instance we obtain $\sigma = 0.59192(4)$ nb and $\sigma_{FSR} = 0.59165(4)$ nb at $\sqrt{s} = 190$ GeV and for $\delta_c = 5^o$. The effect of QED final state radiation must certainly be included for any reliable determination of the physical observables at LEP 2 but we have decided for not including it in the present version of WTO since a more detailed theoretical investigation is needed.

### 2.8 Program structure

We start this section with a brief description of the general features of the program. With WTO it is possible to access 29 out of 32 4-fermion (massless) processes. The most relevant INPUT parameters are given in the BLOCK DATA INIT. They are:

1. \texttt{WM}. The $W$ mass (GeV).
2. \texttt{ZM}. The $Z$ mass (GeV).
3. \(ZG\). The \(Z\) total width (GeV).

Other quantities like the \(W\) width are not external input parameters but rather they are computed internally as given by the standard model. There is no possibility of changing the weak mixing angle independently from the others input parameters, thus \(s_w^2\) is always a derived quantity. In addition to the 29 processes which include gluon exchange in the non-leptonic case there are 4 processes \((e^+e^- \rightarrow f\bar{f}bb)\) which can be computed and where the Higgs boson exchange is included. All processes receive and internal number according to the scheme given in appendix. Moreover each process belong to a particular class, but it is important to know only the highest class in a particular family, thus CC11, CC20, NC21, NC24, NC32, Mix43, NC48, NC64. WTO will always perform a call to the NAG routine X02AJF which returns the machine precision (ZRM). Some of the internal controls are based on this quantity so one should be aware of the fact that, for instance, on a VAXstation \(\cdot 4000\cdot 90\) \(ZRM = 0.14E-16\) while on a Alpha AXP-2100 \(ZRM = 0.11E-15\). It is perhaps appropriate to give here a brief summary of the computational speed versus requested precision for some of the processes. In table 3 we have indicated both entries for a process representative of each family.

| Family | Process | Rel. error | CPU time |
|--------|---------|------------|----------|
| CC11   | All     | \(\leq 0.01\%\) | 3:38:53 |
| CC20   | All     | \(\leq 0.02\%\) | 2:42:57 |
| NC24   | \(\mu\mu^-\tau^+\tau^-\) | 0.25% | 11:47:47 |
| NC32   | \(\bar{u}u\bar{c}c\) | 0.08% | 12:30:52 |
| NC19   | \(\mu^+\mu^-\nu_{\mu}\nu_{\mu}\) | 0.48% | 12:06:14 |
| Mix43  | \(\mu^+\mu^-\nu_{\mu}\nu_{\mu}\) | 0.08% | 7:59:26 |
| NC48   | \(e^+e^-\bar{u}u\) | 0.17% | 2:10:21 |
| NC64   | \(\nu_{\mu}\nu_{\mu}\nu_{\mu}\nu_{\mu}\) | 0.03% | 5:24:42 |

Table 3: CPU time needed on a Alpha AXP-2100 computer for a given relative error (in percent) and for the some representative processes in each family.

As it can be seen from table 3 there are considerable differences in CPU time between different processes. The reason is not only connected with the number of diagrams contributing but also with the number of helicity sets required by each process. Always working in the massless fermion limit we have only 2(3) sets of non zero helicity states for CC11(CC20) but up to 8(12) for NC24(NC48).

2.9 Input

The meaning of the input parameters is the following.

\(\text{OPRT(CHARACTER*1)}\)

There is the possibility of printing some additional information about the calls to the various subroutines. This however requires a detailed knowledge of the internal structure.
of the calculation. Thus, by default, OPRT='N' and it can be set to 'Y' only by changing the corresponding line in BLOCK DATA INIT.

NPTS(INTEGER*4)
The actual number of points for the integration. Built-in choices are NPTS=1−10. If more integration points are needed then the array VK(NDIM) must be re-initialized through a call to the NAG routine D01GZF. The optimal coefficients for p-point integration over the n-cube \([0, 1]^n\) require that \(p\) is a prime number or \(p\) is a product of 2 primes, \(p_2\) and \(p_1\) chosen to be the nearest prime integer to \(p_2^2\). The built-in choices are shown in table 4.

| NPTS | \(p = p_1 \times p_2\) |
|------|------------------------|
| 1    | 2129                   |
| 2    | 5003                   |
| 3    | 10007                  |
| 4    | 20011                  |
| 5    | 40009                  |
| 6    | 80021                  |
| 7    | 99991                  |
| 8    | 10193 \times 101       |
| 9    | 22807 \times 151       |
| 10   | 32771 \times 181       |

Table 4: Built-in choices for \(p\), needed for \(p\)-point integration over the \(n\)-cube \([0, 1]^n\).

NRAND(INTEGER*4)
The number of times where the integral is computed in order to give an estimate of the standard error (usually NRAND ≤ 6). A ‘fast’ estimate of the result with standard error around 1% is already achieved with NPTS=4, NRAND=4. A more accurate but still intermediate answer will require (8, 6) and a very precise, but also very time consuming, estimate will proceed with (10, 6). There is no intrinsic limit to NRAND while NPTS ≤ 10 <, unless an independent call to NAG routine D01GZF is performed.

IPR(INTEGER*4)
The catalog number of the process \([1 \ldots 33]\). The internal structure of WTO is organized in such a way that the user does not have to bother about quantum numbers of the final state.

IPRO(INTEGER*4)
If IPR ≤ 3 there is the additional possibility of taking into account only the CC03 part of the CC11 family.

RS(REAL*8)
The c.m.s energy in GeV.
**OPEAK (CHARACTER*1)**

In general we are dealing with double resonant, single resonant and non-resonant diagrams. **OPEAK=’Y[N]’** will select the corresponding mapping.

Let us assume that we want to reach the same level of accuracy for all possible cases. As a consequence we observe that different processes under different sets of kinematical cuts will require values of **NPTS** and **NRAND** which may differ substantially. To illustrate this point we consider a particular example, \( e^+e^- \rightarrow \mu^+\mu^-\tau^+\tau^- \). Moreover we only include simple kinematical cuts on the invariant mass of the two fermion-antifermion pair, which can be coupled to a photon. The background to \( Z-Z \) production is therefore large, especially when we allow for small cuts, \( M(\mu^+\mu^-), M(\tau^+\tau^-) \geq M_0 \) with \( M_0 \) much smaller than \( M_Z \). To show our point we fix **NPTS**=8 and **NRAND**=6 and vary \( M_0 \).

Table 5 clearly illustrates as the numerical accuracy becomes better and better for growing \( M_0 \) and fixed number of points.

| \( M_0 \) (GeV) | \( \sigma \) (fb) | Rel. error(%) |
|-----------------|------------------|--------------|
| 5               | 10.29(8)         | 0.77         |
| 10              | 6.61(3)          | 0.43         |
| 30              | 3.130(2)         | 0.07         |
| 50              | 2.226(1)         | 0.03         |

Table 5: The effect of varying the cut on fermion-antifermion pair in \( e^+e^- \rightarrow \mu^+\mu^-\tau^+\tau^- \), inclusive of QED radiation.

In this example we have used **OPEAK=’N’** for \( M_0 = 5, 10 \) GeV and **OPEAK= ’Y’** for \( M_0 = 30, 50 \) GeV. High precision for very loose cuts requires therefore a much higher number of points while it is considerably easier to reach 0.1% or better for more realistic cuts. It goes without saying that a more stringent cut on the uninteresting boundaries of the phase space implies high precision achievable with high computational speed. Finally we observe that for some of the processes with very loose cuts a repetition with an increasing sequence of high values of **NPTS** can eventually yield erratic results.

**ALS (REAL*8)**

For NC32 and non-leptonic NC64, Mix43 the value of \( \alpha_s(M_W) \) must be initialized. Moreover for CC processes it is possible to include NQCD effects (Naive QCD). This is controlled by the variable **OQCD**.

**OQCD (CHARACTER*1)**

Inclusion of NQCD both in the total \( W \) width and in the final state, **ALS** must be initialized. Thus

\[
\Gamma_w = \Gamma_w^0 \left( 1 + \frac{2\alpha_s}{3\pi} \right). \quad (120)
\]

**OFL (CHARACTER*1)**
Whenever a CC20 process is considered an option is available for introducing the gauge restoring terms according to the FL-scheme. The $W^-$-width is always taken to be running. The $e^-$ scattering angle cannot be set to zero even in the presence of gauge reparation due to the approximation of $m_e = 0$.

**OFSR (CHARACTER*1)**

In the present version of the program this flag has been set to its default, OFSR='N' and it should not be changed. It is responsible for the inclusion of QED final state radiation for CC11 processes, a branch which deserves a more detailed theoretical analysis.

**ITCM (INTEGER*4)**

For CC11 and CC20 several observables can be computed:

- 0 The cross-section
- 1 unrenormalized moments of
  \[ E_\gamma = \left(1 - \frac{x_+ + x_-}{2}\right) \sqrt{s}. \] (121)
- 2 for CC10. Unrenormalized moments of $E_\mu$ as defined in eq.(10).
- 3 For CC10. Unrenormalized $T_n(\cos \theta_\mu)$, where $T_n$ is the $n$-th Chebyshev polynomial and $\cos \theta_\mu$ is defined in eq.(10) with $i = 1$.
- 4 unrenormalized $T_n(\cos \theta_{W^-})$. The $W^-$ scattering angle is defined by
  \[
  \cos \theta_{W^-} = \frac{1}{\beta_{W^-}} \left(1 - x_- \frac{t_w}{E_{W^-}}\right), \\
  E_{W^-} = \frac{1}{2} \left[x_+ \left(1 + m^2_+ - m^2_-\right) - (x_+ - x_-) t_w\right], \\
  \beta_{W^-}^2 = 1 - \frac{v m^2_2}{E^2_{W^-}}. \] (122)
- 5 unrenormalized $T_n(\cos \theta_{W^+})$. The $W^+$ scattering angle is defined by
  \[
  \cos \theta_{W^+} = \frac{1}{\beta_{W^+}} \left(1 - x_- \frac{1 - t_w}{E_{W^+}}\right), \\
  E_{W^+} = \frac{1}{2} \left[x_+ \left(1 + m^2_+ - m^2_-\right) - (x_+ - x_-) (1 - t_w)\right], \\
  \beta_{W^+}^2 = 1 - \frac{v m^2_2}{E^2_{W^+}}. \] (123)
• 6 unrenormalized $W$ virtuality defines as

$$V_W = \sqrt{v} (m_- + m_+) - 2 \frac{M_W}{s}.$$  

(124)

• 7 $M(W^+) + M(W^-)$ distribution

• 8 $M(W^-) - M(W^+)$ distribution

• 9 $M(W^+)$ distribution

• 10 $c = \cos \theta(l^-)$ distribution where

$$t_1 = x_+ e_1 \frac{1 + c}{x_- + x_+ - (x_- - x_+)c},$$

(125)

and $\theta$ is the angle between the incoming $e^-$ and the outgoing $l^-$. while for all other classes only the cross-section will be available. Please consult the WW/eg report [3] for more detailed informations concerning this quantities. Here unrenormalized means not divided by the corresponding cross-section.

**ITCNM(INTEGER*4)**

If $1 \leq \text{ITCM} \leq 6$ the order of the moments can be chosen.

**DIST(REAL*8)**

For $\text{ITCM}=7,8,9$ gives the value of $M_{W^+} + M_{W^-}$, $M_{W^-} - M_{W^+}$ or $M_{W^+}$ at which the differential cross section is required. For $\text{ITCM}=10$ DIST is $\theta \angle (e^-, l^-)$.

Moreover for $\text{ITCM}=0$ there is the additional possibility of binning the $M(W^-) + M(W^+)$ distributions by requiring the calculation of the cross-section with $M_1 \leq M(W^-) + M(W^+) \leq M_2$.

**OBIN(CHARACTER*1)**

OBIN='N'['P', 'M'] selects no binning or $\pm$ binning. For the latter we initialize

**ABP(M), BBP(M)(REAL*8)**

the limits $M_1, M_2$.

**OCOUL(CHARACTER*1)**

For CC processes the Coulomb correction factor can be included.

**IOS(INTEGER*4)**

Refers to the choice of the weak mixing angle, $s^2_W$ and of the $SU(2)$ coupling constant $g$. The user is advised to adopt IOS=1 which is the choice of a large variety of programs and corresponds to ($G_\mu$ being the Fermi coupling constant)
\[ s_w^2 = \frac{\pi \alpha}{\sqrt{2} G_{\mu} M_w^2}, \]
\[ g^2 = \frac{4\pi \alpha}{s_w^2} \]  
\[ (126) \]

although a more sensible choice (connected with the Ward identities) is given by

\[ s_w^2 = 1 - \frac{M_w^2}{M_Z^2}, \]
\[ g^2 = 4\sqrt{2} G_{\mu} M_w^2. \]  
\[ (127) \]

**ORAL** (CHARACTER*1)
If we adopt IOS=1 then the scale must be chosen for evaluating \( \alpha \). If ORAL='F' then it is possible to enter any value for ALWI= 1/\( \alpha \) while for ORAL='R' the program will compute \( \alpha(s) \) with a call to subroutine PSELF and HADR5. The two routines evaluate DER= DERL+DERH, DERL being the perturbative contribution to \( \Delta r \) from leptons and from the top quark while DERH evaluates the light hadron contribution using fits to the QED vacuum polarization from \( e^+e^- \) data (courtesy of F. Jegerlehner [7]).

\[ \alpha(s) = \frac{\alpha(0)}{1 - \Delta r}. \]  
\[ (128) \]

**IOSF** (INTEGER*4)
Refers to the choice of the Structure Function for Initial State QED radiation. Born observables require IOSF=0, for the corrected ones the user is advised to have IOSF=1. Indeed this choice is the default adopted by the WW/eg working group and corresponds to having \( D(x,s) \) of eq.(97) entirely in terms of \( \beta \) [8],

\[ D_r(x,s) = -\frac{1}{2} (1-x)^{-\beta/2} \left(1 - x^2\right) + \mathcal{O}(\beta). \]  
\[ (129) \]

Instead IOSF=2 corresponds to the so-called \( \eta \)-scheme [9]

\[ D_r(x,s) = -\frac{1}{2} \frac{\eta}{\beta} (1-x)^{-\beta/2} \left(1 - x^2\right) + \mathcal{O}\left(\frac{\eta^2}{\beta}\right), \]

\[ \eta = 2 \frac{\alpha}{\pi} \log \frac{s}{m_e^2}, \]  
\[ (130) \]

which respects gauge invariance. Finally IOSF=3 gives a mixed treatment as described in [10]. There is no way in which one can present any realistic estimate of the theoretical error for 4-fermion production with the actual level of knowledge that we have. The only pragmatic alternative is to examine the variation induced in our results by changing RS and SF. The first option roughly simulates (probably underestimate) the uncertainty.
connected with our ignorance of the $\mathcal{O}(\alpha)$ electroweak corrections while the second one reflects the uncertainty related to our ignorance about the complete $\mathcal{O}(\alpha \times \text{const})$ QED corrections which, at the moment, only control correctly the leading logarithmic parts. This we have done by allowing $\text{IOS}=1, 2$ and $\text{IOSF}=1, 2, 3$ in the calculation of the representative process $e^+e^- \rightarrow \mu^-\bar{\nu}_\mu, \nu_\tau \tau^+$. Results for the cross section at $\sqrt{s} = 190$ GeV and for canonical cuts are shown in Table 6.

$$\begin{array}{c|c|c|c|}
\text{IOS/IOSF} & 1 & 2 & 3 \\
\hline
1 & 0.19411(2) & 0.19395(2) & 0.19464(2) \\
2 & 0.19394(2) & 0.19378(2) & 0.19447(2) \\
\end{array}$$

Table 6: $\sigma (\text{nb})$ for $e^+e^- \rightarrow \mu^-\bar{\nu}_\mu, \nu_\tau \tau^+$ at $190$ GeV with $E_{\mu,\tau} \geq 1$ GeV and $10^o \leq \theta_{\mu,\tau} \leq 170^o$. Moreover $\theta_{\mu,\tau} \geq 5^o$.

Thus naively one could fix the central value at $\text{IOS}=2$ and $\text{IOSF}=2$ which preserve gauge invariance and assign a quite asymmetric theoretical error of

$$\sigma = 0.19378 \pm 0.0002(\text{num.})^{+0.00086}_{-0.00000}(\text{theor.}) \quad \text{(131)}$$

Moreover we have also show in tables 7–8 the effect of varying IOS/IOSF in the process $e^+e^- \rightarrow \mu^-\bar{\nu}_\mu, ud$ both for $\sigma$ and for $<E_\gamma>$. In this case we have included the Coulomb correction factor, $\text{OCOUL}=\text{Y}$, and the NQCD corrections, $\text{OQCD}=\text{Y}$. Canonical cuts are applied. The largest uncertainty for $<E_\gamma>$ is of $1.9, 3.2, 9.9, 20.5$ MeV for $E_{cm} = 161, 175, 190, 205$ GeV respectively.

$$\begin{array}{c|c|c|c|c|c|c|}
E_{cm} \text{(GeV)}/\text{IOS-IOSF} & 1-1 & 1-2 & 1-3 & 2-1 & 2-2 & 2-3 \\
\hline
161 & 0.4688 & 0.4673 & 0.4674 & 0.4685 & 0.4669 & 0.4670 \\
175 & 1.1250 & 1.1219 & 1.1221 & 1.1251 & 1.1220 & 1.1222 \\
190 & 2.1579 & 2.1484 & 2.1489 & 2.1583 & 2.1488 & 2.1493 \\
205 & 3.2317 & 3.2119 & 3.2129 & 3.2324 & 3.2126 & 3.2135 \\
\end{array}$$

Table 7: $<E_\gamma>$ (GeV) for $e^+e^- \rightarrow \mu^-\bar{\nu}_\mu, ud$ with $E_\mu \geq 1$ GeV, $E_{u,d} \geq 3$ GeV, $M_{ud} \geq 5$ GeV, $10^o \leq \theta_\mu \leq 170^o$, $\theta_{\mu,u}, \theta_{\mu,d} \geq 5^o$.

**OCUTS** (CHARACTER*2) It is the main decision branch for kinematical cuts.

There are few internal options.

1. **OCUTS** = ’EX’. Fully extrapolated set-up, use it only for processes free of singularities, e.g. never for CC20.

2. **OCUTS** = ’CC’. The Canonical Cuts used within the WW/eg Working Group. In this case cuts are automatically selected. Canonical Cuts are also available for
$$E_{cm} \text{ (GeV)}/\text{IOS-IOSF}$$

|       | 1-1     | 1-2     | 1-3     | 2-1     | 2-2     | 2-3     |
|-------|---------|---------|---------|---------|---------|---------|
| 161   | 0.13206 | 0.13204 | 0.13250 | 0.13201 | 0.13198 | 0.13244 |
| 175   | 0.49207 | 0.49186 | 0.49358 | 0.49177 | 0.49156 | 0.49329 |
| 190   | 0.60240 | 0.60192 | 0.60404 | 0.60188 | 0.60139 | 0.60352 |
| 205   | 0.62828 | 0.62754 | 0.62977 | 0.62764 | 0.62691 | 0.62913 |

Table 8: $\sigma \text{ (nb)}$ for $e^+e^- \rightarrow \mu^-\bar{\nu}_\mu u\bar{d}$. The cuts are the same as in table reftab7.

NC21, NC25 and NC50 processes. For completeness we make a list of the chosen canonical cuts [3]. Let $l$ be any final state charged lepton and $q$ any final state quark. Then

- $E_l \geq 1 \text{ GeV}, E_q \geq 3 \text{ GeV}$.
- $M(q_i,q_j), M(q_i,\bar{q}_j), M(\bar{q}_i,\bar{q}_j) \geq 5 \text{ GeV}$.
- $10^\circ \leq \theta_l \leq 170^\circ$.
- $\theta(l_i,q_j), \theta(l_i,\bar{q}_j) \geq 5^\circ$.

3. OCUTS='SC'. No cuts at all but whenever a $f\bar{f}$ pair appears in the final state a cut is applied to prevent singularity due to $\gamma^* \rightarrow f\bar{f}$. Again not to be used for CC20 etc. Actually it can ONLY be used for CC11, NC19(NC21), NC24(NC25), NC50 processes. The ‘security’ cut for CC11, NC19 and NC24 can be initialized through the variable STHG. For Higgs physics, NC21, NC25 and NC50 processes, the ‘simple’ cuts can be initialized through the variables ZCUT, BCUT.

4. OCUTS='FC'. You adopt your own cuts. Several variables must be initialized. The important thing to pay attention to is the ‘ordering’ of the final states. One should have in mind the classification given in appendix, therefore if IPR=18 is selected one must attribute cuts according to the fact that $1 = u, 2 = \bar{u}, 3 = \nu_e, 4 = \bar{\nu}_e$. The internal coding will attribute the following labels: for CC(Mix) processes $e^+e^- \rightarrow d(1)\bar{u}(2)u'(3)d'(4)$, where $u = \nu_l, u, c$ and $d = l, d, s, b$. For NC processes $e^+e^- \rightarrow f(1)\bar{f}(2)f'(3)\bar{f}'(4)$. Boundaries for the scattering angles or final state energies are stored as $1-4$. Boundaries for final state angles $\theta_{ij}, i, i = 1, 4$ or final state invariant masses $M_{ij}, i, i = 1, 4$ are stored as a vector of dimension 6, with entries $1-2/1-3/1-4/2-3/2-4/3-4$.

STHG(REAL*8)
Selects $M(\bar{f}f) \geq \text{STHG}$.

ZCUT, BCUT(REAL*8)
In $e^+e^- \rightarrow \bar{f}ffb$ selects $M_z - \text{ZCUT} \leq M(\bar{f}f) \leq M_z + \text{ZCUT}$ and $M(\bar{b}b) \geq \text{BCUT}$.

AIMM(I)(REAL*8)
Min cuts in GeV on FS invariant masses 1 – 6.

**BIMM(I)** (REAL*8)
Max cuts in GeV on FS invariant masses 1 – 6.

**AEM(I)** (REAL*8)
Lower cuts on FS energies (GeV) 1 – 4.

**ASAM(I)** (REAL*8)
Min cuts on the cosine of the scattering angles 1 – 4.

**BSAM(I)** (REAL*8)
Max cuts on the cosine of the scattering angles 1 – 4.

**AFSAM(I)** (REAL*8)
Min cuts on the cosine of the FS angles 1 – 6.

**BFSAM(I)** (REAL*8)
Max cuts on the cosine of the FS angles 1 – 6.

Whenever the Higgs signal has to be investigated few additional flags must be initialized. They are:

**HMI** (REAL*8)
The Higgs boson mass in GeV.

**ALSM** (REAL*8)
The value of $\alpha_s(M_W)$.

**OQCD** (CHARACTER*1)
The option for final state ‘naive’ QCD corrections.

**OGLU** (CHARACTER*1)
The inclusion of the $H \rightarrow gg$ channel in the total Higgs width. The most complete treatment will therefore evolve $\alpha_s$ to the scale $\mu = M_H$, evaluate the running $b,c$-quark masses and compute

$$\Gamma_H = \frac{G_G M_H}{4\pi} \left\{ 3 \left[ m_b^2(M_H) + m_c^2(M_H) \right] \left[ 1 + 5.67 \frac{\alpha_s}{\pi} + 42.74 \left( \frac{\alpha_s}{\pi} \right)^2 \right] + m_t^2 \right\} + \Gamma_{gg},$$

$$\Gamma_{gg} = \frac{G_G M_H^3}{36\pi} \frac{\alpha_s^2}{\pi^2} \left( 1 + 17.91667 \frac{\alpha_s}{\pi} \right).$$

Again the ‘ordering’ of final states is extremely important when one chooses cuts. Please
notice that for internal reasons WTO attributes the following labels to $\text{IPR}=30-33$: $1 = b, 2 = \bar{b}, 3 = \nu_l/l^-, 4 = \bar{\nu}_l/l^+$ but $1 = \mu^-, 2 = \mu^+, 3 = b, 4 = \bar{b}$.

2.10 Test run output

The typical calculations that can be performed with the program are illustrated in three different examples.

Sample 1. Evaluation of the total cross section for the process $e^+e^- \rightarrow \mu^-,\mu^+,\nu_l,u,\bar{d}$ inclusive of QED initial state radiation.

Sample 2. Evaluation of the total cross section for the process $e^+e^- \rightarrow e^+e^-\bar{d}d$ without QED initial state radiation.

Sample 3. Evaluation of the total cross section for the process $e^+e^- \rightarrow \mu^+\mu^-,\nu_l,\nu_l$ with QED initial state radiation.

In sample 1 – 3 we choose $E_{cm} = 190\text{ GeV}$, $M_W = 80.23\text{ GeV}$, $M_Z = 91.1888\text{ GeV}$ and $\Gamma_Z = 2.4974\text{ GeV}$. The various options are as follows.

For sample 1 we have:

OPEAK='Y'
The double-resonating mapping is requested.

ITC=0
The total cross section is computed.

OCOUL='N'
The Coulombic correction factor is not included.

IOS = 1, IOSF = 1
The default choices for the renormalization scheme and the structure functions.

ORAL='F', ALWI = 128.07D0
We have fixed $\alpha$ to be $1/\alpha = 128.07D0$.

OCUT = 'CC'
Canonical Cuts are applied.

IPR = 2
The internal process number is selected.

For sample 2 we have:
OPEAK='N'
The double-resonating mapping is NOT requested.

OCOUL='N'
The Coulombic correction factor does not apply for this process.

IOS = 1, IOSF = 0
The default choice for the renormalization scheme and NO QED radiation.

ORAL='F', ALWI = 128.07D0
We have fixed $\alpha$ to be $1/\alpha = 128.07D0$.

OCUT = 'CC'
Canonical Cuts are applied.

IPR = 25
The internal process number is selected.

For sample 3 we have:

OPEAK='Y'
The double-resonating mapping is requested.

OCOUL='N'
The Coulombic correction factor is NOT not requested.

IOS = 1, IOSF = 1
The default choices for the renormalization scheme and the structure functions.

ORAL='F', ALWI = 128.07D0
We have fixed $\alpha$ to be $1/\alpha = 128.07D0$.

OCUT = 'CC'
Canonical Cuts are applied.

IPR = 20
The internal process number is selected.

2.11 Program implementation
In this section a short description of each routine implemented in WTO is given.

BLOCK DATA INIT
IMPLICIT REAL*8 (A-H,O-Z)

38
Masses and numerical constants are initialized. As it can be seen the top quark mass is also initialized. At the moment this is only due to the following facts. Whenever $\alpha$ is evolved at the scale $s$ then the top quark enters the photon two point function. Moreover in NC21, i.e. $e^+e^- \rightarrow b\bar{b}\nu_e\nu_e$, there is a multiperipheral diagram with an internal top line.
| COMMON/FLS/OFL          | COMMON/QCD/ALS          | COMMON/NPR/IPR          | COMMON/HIGGS/HM         |
|------------------------|-------------------------|-------------------------|-------------------------|
| COMMON/PRT/OPRT        | COMMON/FSR/OFSR         | COMMON/OPA/DELC         | COMMON/DIS/DIST         |
| COMMON/DIS/KOUNT/IK    | COMMON/ISTRF/ISF        | COMMON/AQCD/OQCD        | COMMON/BME/BFACT        |
| COMMON/COUTL/DCOUL     | COMMON/TC/ITC,ITCN     | COMMON/SHEL/OTRANS      | COMMON/SP/PSG(4)        |
| COMMON/KOUNT/IK        | COMMON/ICUTS/IAC(4)     | COMMON/DIS/DIST         | COMMON/RM/RBM2,RCM2     |
| COMMON/TOPT/IOS,IOSF   | COMMON/ISA/ISAA,ISAF   | COMMON/COUL/OCOUL       | COMMON/CH/CH(36)        |
| COMMON/OCHANNEL/OTYPE   | COMMON/NCC/CHF2,CHFP2,CONC(10) | DIMENSION VK(NDIMMX) | DIMENSION IACM(4),AIMM(6),BIMM(6),AEM(4),ASAM(4),BSAM(4),AFSA(6),BFSAM(6),AFSAM(6),BFSAM(6),CUTS4(4,4) |
This subroutine will decide, according to the input parameters, which process to consider, e.g. which class and family, and which observable to compute. For each diagram all the couplings are computed, i.e. given a current
\[ \gamma^\mu \left( a + b \gamma^5 \right), \]
the chiral couplings constants
\[ g_L = \frac{1}{2} (a + b), \quad g_R = \frac{1}{2} (a - b), \]
are stored and all the relevant products \( \Pi_i g^i_L \Pi_j g^j_R \) are computed once for all.

All the basic parameters which control the boundaries of the phase space are initialized in TOW. All the I/O is controlled by this subroutine after a call to the subroutine D01GCF which performs the integration. The following functions give the differential cross sections according to the family to which the process belong. All of them essentially set the ingredients for the phase space and the requested mappings. Once the vector \( X(\text{NDIM}) \) is initialized, with \( 0 \leq x_i \leq 1, i = 1, \ldots, \text{NDIM} \), all independent invariants \( u, v, m_2^2, \ldots, t \) are constructed and from them all the combinations \( x_{ij}, i < j = 1, \ldots, 6 \). Next all non-zero helicity amplitudes are computed and squared. Within these subroutines the operations among complex numbers are always executed in terms of real and imaginary parts.

```
REAL*8 FUNCTION XSC(NDIM,X)
IMPLICIT REAL*8 (A-H,O-Z)
CHARACTER*1,OCOUL,OPEAK,OQCD,OFL
CHARACTER*10,OTYPE
DIMENSION X(NDIM)
```

With \( \text{XSC} \) all the CC3-CC11-CC20 processes are computed. As explained in the long write-up this function will allow different choices for the observables to be computed, including the total cross section and the moments of several distributions. Moreover the variable \( \text{OFL} = 'Y[N]' \) controls the inclusion of the imaginary part of the \( VWW, V = \gamma, Z \) one loop vertex in CC20.

```
REAL*8 FUNCTION XSNH24(NDIM,X)
IMPLICIT REAL*8 (A-H,O-Z)
CHARACTER*1,OQCD
COMMON/HIGGS/HM
COMMON/AQCD/OQCD
DIMENSION X(NDIM)
```
With this function the NC25 processes are computed by means of calls to functions XSH24 and XSN. If OQCD='Y' then $\alpha_s(M_H)$ is computed through a call to function RALPHAS and corrections are applied.

```
REAL*8 FUNCTION XSNH19(NDIM,X)
IMPLICIT REAL*8 (A-H,O-Z)
CHARACTER*1,OQCD
COMMON/AQCD/OQCD
DIMENSION X(NDIM)
```

The same as for function XSNH24 but for NC21 processes.

```
REAL*8 FUNCTION XSNH49(NDIM,X)
IMPLICIT REAL*8 (A-H,O-Z)
CHARACTER*1,OQCD
DIMENSION X(NDIM)
```

The same as for function XSNH24 but for NC50 processes.

```
REAL*8 FUNCTION XSN32(NDIM,X)
IMPLICIT REAL*8 (A-H,O-Z)
DIMENSION X(NDIM)
```

This function will compute all NC32 processes by first calling function XSN, i.e. the NC24 sub-class (no gluons), and then by calling function XSNG which will add the remaining 8 diagrams. This is made possible by the fact that there is no interference between NC diagrams and QCD diagrams. For historical reasons the corresponding Born cross sections (IOSF=0) are computed without gluon exchange diagrams for non-leptonic processes.

```
REAL*8 FUNCTION XSM43(NDIM,X)
IMPLICIT REAL*8 (A-H,O-Z)
CHARACTER*2,OFS
COMMON/FS/OFS
DIMENSION X(NDIM)
```

This function will compute both leptonic and non-leptonic Mix43 processes by first calling function XS35, i.e. the 35 diagrams without gluons but with both CC and NC contributions. Next function XS35G is called which includes the QCD diagrams and the interference $CC \otimes QCD$. For historical reasons the corresponding Born cross sections (IOSF=0) are computed without gluon exchange diagrams for non-leptonic processes.
REAL*8 FUNCTION XSN(NDIM,X)
IMPLICIT REAL*8 (A-H,O-Z)
CHARACTER*1,OPEAK
CHARACTER*10,OTYPE
COMMON/PS/OPEAK
COMMON/OCHANNEL/OTYPE
DIMENSION X(NDIM)

With this function all available NC processes are computed but without Higgs exchange. Thus it includes the NC19, NC24 and NC48 families. Only the total cross section is available.

REAL*8 FUNCTION XSH24(NDIM,X)
IMPLICIT REAL*8 (A-H,O-Z)
DIMENSION X(NDIM)

This function will include the Higgs boson exchange diagram to the NC24 processes. It should be stressed that in the limit of massless fermions there is no interference between NC24 and Higgs so that one can compute signal and background separately. If OQCD='Y' then $\alpha_s(M_W)$ is evolved to the $\alpha_s(M_H)$ and the Higgs boson signal is multiplied by

$$
\delta_{QCD} = 1 + \frac{5.67}{\pi} \alpha_s + 42.74 \left( \frac{\alpha_s}{\pi} \right)^2,
\alpha_s = \alpha_s(M_H).
$$

(135)

REAL*8 FUNCTION XSH19(NDIM,X)
IMPLICIT REAL*8 (A-H,O-Z)
DIMENSION X(NDIM)

With this function the Higgs boson exchange is added to the NC19 family thus obtaining NC21. The two additional diagrams again do not interfere with NC19 in the massless limit.

REAL*8 FUNCTION XSNH49(NDIM,X)
IMPLICIT REAL*8 (A-H,O-Z)
CHARACTER*1,OQCD
COMMON/QCD/ALS
COMMON/HIGGS/HM
COMMON/AQCD/OQCD
DIMENSION X(NDIM)
With this function the Higgs boson exchange is added to the NC48 family thus obtaining NC50. The two additional diagrams again do not interfere with NC48 in the massless limit.

\[ \text{REAL*8 FUNCTION XSNG(NDIM,X)} \]
\[ \text{IMPLICIT REAL*8 (A-H,O-Z)} \]
\[ \text{CHARACTER*10,OTYPE} \]
\[ \text{COMMON/OCHANNEL/OTYPE} \]
\[ \text{DIMENSION X(NDIM)} \]

The gluon exchange diagrams for non-leptonic NC processes are included. As explained in the previous sections we use a fixed \( \alpha_s \), i.e. \( \alpha_s(M_W) \).

\[ \text{REAL*8 FUNCTION XS35(NDIM,X)} \]
\[ \text{IMPLICIT REAL*8 (A-H,O-Z)} \]
\[ \text{CHARACTER*1,OCOUL,OPEAK} \]
\[ \text{CHARACTER*2,OFS} \]
\[ \text{CHARACTER*10,OTYPE} \]
\[ \text{COMMON/FS/OFS} \]
\[ \text{COMMON/PS/OPEAK} \]
\[ \text{COMMON/COUL/OCOUL} \]
\[ \text{COMMON/OCHANNEL/OTYPE} \]
\[ \text{DIMENSION X(NDIM)} \]

This function computes the diagrams for CC\( \otimes \)NC leptonic processes and for the non-leptonic ones where gluon exchange will be included separately.

\[ \text{REAL*8 FUNCTION XS35G(NDIM,X)} \]
\[ \text{IMPLICIT REAL*8 (A-H,O-Z)} \]
\[ \text{CHARACTER*1,OCOUL,OPEAK} \]
\[ \text{CHARACTER*2,OFS} \]
\[ \text{CHARACTER*10,OTYPE} \]
\[ \text{COMMON/FS/OFS} \]
\[ \text{COMMON/PS/OPEAK} \]
\[ \text{COMMON/COUL/OCOUL} \]
\[ \text{COMMON/OCHANNEL/OTYPE} \]
\[ \text{DIMENSION X(NDIM)} \]

Whenever CC\( \otimes \)NC non-leptonic processes are to be computed this function will return QCD\( \otimes \)(QCD+CC).

\[ \text{REAL*8 FUNCTION XSN64(NDIM,X)} \]
\[ \text{IMPLICIT REAL*8 (A-H,O-Z)} \]
\[ \text{CHARACTER*1,OPEAK} \]
\[ \text{CHARACTER*2,OFS} \]
\[ \text{COMMON/FS/OFS} \]
\[ \text{COMMON/PS/OPEAK} \]
\[ \text{DIMENSION X(NDIM)} \]
Finally this function allows to compute processes with identical particles in the final state, with the exclusion of $e^\pm$ and of $\nu_e$. For the non-leptonic case QCD is included and the proper color factor relative to each diagram is automatically computed.

```
SUBROUTINE REGION(N,X,J,A,B)
IMPLICIT REAL*8(A-H,O-Z)
DIMENSION X(N)
```

This subroutine is requested by D01GCF in order to set the limits of integrations. In WTO its content is trivial since all variables are in the range $[0,1]$.

```
SUBROUTINE CORRQCD(SCAL,ALS,BQM2,CQM2,RBM2,RCM2)
IMPLICIT REAL*8(A-H,O-Z)
```

Computes the QCD quark running masses. Given ALS, the value of $\alpha_s(M_W)$, and the pole $b, c$-quark masses this routine evaluates their running at a scale $\mu = $SCAL.  

```
REAL*8 FUNCTION RALPHAS(RS0,RS,ALS,NF)
IMPLICIT REAL*8(A-H,O-Z)
```

Given ALS, the value of $\alpha_s$ at a scale RS0, this routine returns $\alpha_s$ at the scale $\mu = RS$ with NF active flavors.

```
REAL*8 FUNCTION QCDLAM(NF,ALS,RS,X1,X2,XACC)
IMPLICIT REAL*8(A-H,O-Z)
PARAMETER (JMAX=50,NOUT=21)
```

```
REAL*8 FUNCTION QCDSCALE(NF,ALS,RS,X)
IMPLICIT REAL*8(A-H,O-Z)
```

Compute $\Lambda_{MS}^{n_f}$ in the $\overline{MS}$-scheme from $\alpha_s$ fixed at the scale RS0.

```
SUBROUTINE HADR5(E,DER,EDER)
******************************************************************
* * SUBROUTINE FOR THE EVALUATION OF THE LIGHT HADRON        *
* * CONTRIBUTIONS TO DELTA_R    *                            *
* * F. JEGGERLEHNER, PAUL SCHERRER INSTITUTE, CH-5232 VILIGEN  *
* *                                                                    *
******************************************************************
```
SUBROUTINE PSELF(P2X,PGGF)
IMPLICIT REAL*8(A-H,O-Z)
* COMMON/PARAM/EPS,QDELTA
COMMON/FMASS/EM,RM,TM,TQM,UQM,QCM,QMQ,BQM
COMMON/BPAR/WM,ZM,ZG,GF,PI,CFCT,FCNT,GE,ALPHAI,ALWI

PSELF computes the perturbative contribution to the running of $\alpha(s)$.

SUBROUTINE RBFF0(RM12,RM22,B0,B1,B21)
IMPLICIT REAL*8(A-H,O-Z)
* COMMON/PARAM/EPS,QDELTA
COMMON/BPAR/WM,ZM,ZG,GF,PI,CFCT,FCNT,GE,ALPHAI,ALWI
* DIMENSION ARG(2),CLN(2),FR(3)
SUBROUTINE RBFF(P2,RM12,RM22,RB0,RB1,RB21)
IMPLICIT REAL*8(A-H,O-Z)
* COMMON/PARAM/EPS,QDELTA
COMMON/BPAR/WM,ZM,ZG,GF,PI,CFCT,FCNT,GE,ALPHAI,ALWI
* DIMENSION CMP2(2)
DIMENSION CLNMP2(2)
DIMENSION GFPR(3),GFMR(3)
SUBROUTINE ROOTS(P2,RM12,RM22,RPR,RPI,RMR,OMRPR,OMRMR)
IMPLICIT REAL*8(A-H,O-Z)
* COMMON/PARAM/EPS,QDELTA
COMMON/BPAR/WM,ZM,ZG,GF,PI,CFCT,FCNT,GE,ALPHAI,ALWI
SUBROUTINE CQLNX(ARG,RES)
IMPLICIT REAL*8(A-H,O-Z)
* COMMON/BPAR/WM,ZM,ZG,GF,PI,CFCT,FCNT,GE,ALPHAI,ALWI

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These routines are required by PSELF and compute the relevant *perturbative* contributions to the photon two point function.

3 Test run output

3.1 Sample 1

This run is with:

NPTS = 8
NRAND = 6

\(1/\alpha_{\text{QED}}(s) = 0.12807E+03\) Coupling constants are \((\alpha-G_F)\) :

\[
\begin{align*}
g_{\text{v}}_{\text{e}} & = -0.140972545532E-01 & -0.140972545532E-01 \\
g_{\text{a}}_{\text{e}} & = -0.1857939837526E+00 & -0.1857939837526E+00 \\
g_{\text{w}}_{\text{f}} & = 0.2304098503535E+00 & 0.2304098503535E+00 \\
g_{\text{z}}_{\text{w}} & = -0.5714792058132E+00 & -0.5714792058132E+00 \\
\end{align*}
\]

\(G-L\) factor = \(0.104114673716395E+01\)

\(E_{\text{cm}}\) (GeV) = 0.19000E+03
\(\beta\) = 0.11453E+00 \(\sin^2\) = 0.23103E+00
\(M_{\text{W}}\) (GeV) = 0.80230E+02 \(M_{\text{Z}}\) (GeV) = 0.91189E+02
\(G_{\text{W}}\) (GeV) = 0.20337E+01 \(G_{\text{Z}}\) (GeV) = 0.24974E+01

alpha RS
0(\(\alpha^2\)) beta SF
CS not included
FSR not included
NQCD not included

Process is e+ e- \(\rightarrow\) mu- bnu_mu u bd
with the following cuts

E(GeV), SA(deg) cuts

\[
\begin{array}{cccc}
\text{mu-} & \text{bnu\_mu} & u & bd \\
E_{th} & 1.0000 & 0.0000 & 3.0000 & 3.0000 \\
C_{max} & 10.0000 & 0.0000 & 0.0000 & 0.0000 \\
C_{min} & 170.0000 & 180.0000 & 180.0000 & 180.0000 \\
\end{array}
\]

lower IM cuts (GeV)

\[
\begin{array}{cccc}
\text{mu-} & \text{bnu\_mu} & u & bd \\
\end{array}
\]

upper IM cuts (GeV)

\[
\begin{array}{cccc}
\text{mu-} & \text{bnu\_mu} & u & bd \\
\end{array}
\]

min FS angle(deg) cuts

\[
\begin{array}{cccc}
\text{mu-} & \text{bnu\_mu} & u & bd \\
\end{array}
\]

max FS angle(deg) cuts

\[
\begin{array}{cccc}
\text{mu-} & \text{bnu\_mu} & u & bd \\
\end{array}
\]

cc11-diagrams : charges -1.0000 0.0000 0.6667 0.3333

Cross-Section On exit IFAIL = 0

CPU time  4 h 15 min  5 sec

sec per call = 0.249E-02

sigma = 0.5919248E+00 +- 0.4040678E-04

Rel. error of  0.007 %
3.2 Sample 2

This run is with:

NPTS  = 10
NRAND = 5

\(1/\alpha_{\text{QED}}(s) = 0.12807 \times 10^3\) Coupling constants are (\(\alpha_G\F\)):

\[
\begin{align*}
gv_e & = -0.1409725455532 \times 10^{-1} & \text{and} & -0.1409725455532 \times 10^{-1} \\
ga_e & = -0.1857939837526 \times 10^0 & \text{and} & -0.1857939837526 \times 10^0 \\
g_{\text{wf}} & = 0.2304098503535 \times 10^0 & \text{and} & 0.2304098503535 \times 10^0 \\
g_{\text{zw}} & = -0.5714792058132 \times 10^{-1} & \text{and} & -0.5714792058132 \times 10^{-1} \\
\end{align*}
\]

G-L factor = 0.104114673716395 \times 10^1

\(E_{\text{cm}}(\text{GeV}) = 0.19000 \times 10^3\)

\(\beta = 0.11453 \times 10^0\)

\(\sin^2 = 0.23103 \times 10^0\)

\(M_W(\text{GeV}) = 0.80230 \times 10^2\)

\(M_Z(\text{GeV}) = 0.91189 \times 10^2\)

\(G_w(\text{GeV}) = 0.20337 \times 10^1\)

\(G_Z(\text{GeV}) = 0.24974 \times 10^1\)

alpha RS
No QED Radiation

Process is \(e^+e^- \rightarrow d\ bd\ e^- e^+\)
with the following cuts

\[
\begin{align*}
\text{E(GeV), SA(deg) cuts} & \\
d & bd & e^- & e^+ \\
E_{\text{th}} & 3.0000 & 3.0000 & 1.0000 & 1.0000 \\
C_{\text{max}} & 0.0000 & 0.0000 & 10.0000 & 10.0000 \\
C_{\text{min}} & 180.0000 & 180.0000 & 170.0000 & 170.0000 \\
\end{align*}
\]

\[\text{lower IM cuts (GeV)}\]

\[
\begin{align*}
d & bd & e^- & e^+ \\
\text{d} & 5.0000 & 0.0000 & 0.0000 \\
\text{bd} & 0.0000 & 0.0000 & 0.0000 \\
e^- & 0.0000 & & \text{e}^+ \\
e^+ & & & \text{e}^- \\
\end{align*}
\]

\[\text{upper IM cuts (GeV)}\]

\[
\begin{align*}
d & bd & e^- & e^+ \\
\text{d} & 190.0000 & 190.0000 & 190.0000 \\
\text{bd} & 190.0000 & 190.0000 & 190.0000 \\
e^- & 190.0000 & & \text{e}^+ \\
e^+ & & & \text{e}^- \\
\end{align*}
\]

\[\text{min FS angle(deg) cuts}\]

\[
\begin{align*}
d & bd & e^- & e^+ \\
\text{d} & 180.0000 & 180.0000 & 180.0000 \\
\text{bd} & 180.0000 & 180.0000 & 180.0000 \\
e^- & 180.0000 & & \text{e}^+ \\
e^+ & & & \text{e}^- \\
\end{align*}
\]
max FS angle(deg) cuts

|   | d   | bd  | e-  | e+  |
|---|-----|-----|-----|-----|
| d | 0.0000 | 5.0000 | 5.0000 | 5.0000 |
| bd| 5.0000 | 5.0000 | 5.0000 | 5.0000 |
| e-| 5.0000 | 5.0000 | 5.0000 | 5.0000 |

nc48-diagrams : charges -0.3333 -1.0000
isospin -0.5000 -0.5000

Cross-Section On exit IFAIL = 0

CPU time 9 h 9 min 3 sec
sec per call = 0.111E-02

sigma = 0.4304695E-01 +- 0.5671371E-04
Rel. error of 0.132 %

3.3 Sample 3

This run is with:

\[ NPTS = 9 \]
\[ NRAND = 6 \]

\[ 1/\alpha_{QED}(s) = 0.12807E+03 \]
Coupling constants are (alpha-G_F):

\[ g_{v_e} = -0.1409725455532E-01 \]
\[ g_{a_e} = -0.185793837526E+00 \]
\[ g_{w_f} = 0.2304098503535E+00 \]
\[ g_{zww} = -0.5714792058132E+00 \]

G-L factor = 0.104114673716395E+01

\[ E_{cm} \text{ (GeV)} = 0.19000E+03 \]
\[ \beta = 0.11453E+00 \]
\[ \sin^2 = 0.23103E+00 \]
\[ M_W \text{ (GeV)} = 0.80230E+02 \]
\[ M_Z \text{ (GeV)} = 0.91189E+02 \]
\[ G_W \text{ (GeV)} = 0.20337E+01 \]
\[ G_Z \text{ (GeV)} = 0.24974E+01 \]

\[ \alpha_{RS} \]
\[ 0(\alpha^2) \text{ beta SF} \]
\[ \text{CS not included} \]

Process is e+ e- -> mu- bnu_mu nu_mu mu+
with the following cuts

\[ E(\text{GeV}), \text{SA(deg)} \text{ cuts} \]

|   | mu- | bnu_mu | nu_mu | mu+ |
|---|-----|--------|-------|-----|
| E_th | 1.0000 | 0.0000 | 0.0000 | 1.0000 |
| C_max | 10.0000 | 0.0000 | 0.0000 | 10.0000 |
|-------|---------|--------|--------|---------|
| C_min | 170.0000| 180.0000| 180.0000| 170.0000 |

**lower IM cuts (GeV)**

|     | mu-     | bnu_mu | nu_mu | mu+     |
|-----|---------|--------|-------|---------|
| mu- | 0.0000  | 0.0000 | 0.0000|         |
| bnu_mu | 0.0000 | 0.0000 |     |         |
| nu_mu | 0.0000  |        |       |         |
| mu+  |         |        |       |         |

**upper IM cuts (GeV)**

|     | mu-     | bnu_mu | nu_mu | mu+     |
|-----|---------|--------|-------|---------|
| mu- | 190.0000| 190.0000| 190.0000|         |
| bnu_mu | 190.0000| 190.0000|     |         |
| nu_mu | 190.0000|        |       |         |
| mu+  |         |        |       |         |

**min FS angle(deg) cuts**

|     | mu-     | bnu_mu | nu_mu | mu+     |
|-----|---------|--------|-------|---------|
| mu- | 180.0000| 180.0000| 180.0000|         |
| bnu_mu | 180.0000| 180.0000|     |         |
| nu_mu | 180.0000|        |       |         |
| mu+  |         |        |       |         |

**max FS angle(deg) cuts**

|     | mu-     | bnu_mu | nu_mu | mu+     |
|-----|---------|--------|-------|---------|
| mu- | 0.0000  | 0.0000 | 0.0000| 5.0000  |
| bnu_mu | 0.0000 | 0.0000 |     |         |
| nu_mu | 0.0000  |        |       |         |
| mu+  |         |        |       |         |

Cross-Section On exit IFAIL = 0

CPU time 7 h 59 min 2 sec
sec per call = 0.139E-02

\[ \sigma = 0.2046758E+00 \pm 0.1704597E-03 \]

Rel. error of 0.083 \%

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4 Appendix

• CC11

1. $\mu^-, \bar{\nu}_\mu, \nu_\tau, \tau^+$
2. $\mu^-, \bar{\nu}_\mu, u, \bar{d}$
3. $d, \bar{u}, c, \bar{s}$

• CC20

4. $\mu^-, \bar{\nu}_\mu, u, \bar{d}$
5. $\mu^-, \bar{\nu}_\mu, \nu_\tau, \bar{\nu}_\tau$

• NC24

6. $\mu^-, \mu^+, \nu_\rho, \bar{\nu}_\rho$
7. $d, \bar{d}, \nu_\mu, \bar{\nu}_\mu$
8. $u, \bar{u}, \nu_\mu, \bar{\nu}_\mu$
9. $\mu^-, \mu^+, \tau^-, \tau^+$
10. $\mu^-, \mu^+, d, \bar{d}$
11. $\mu^-, \mu^+, u, \bar{u}$
12. $\nu_\mu, \bar{\nu}_\mu, \nu_\tau, \bar{\nu}_\tau$

• NC32

13. $s, \bar{s}, u, \bar{u}$
14. $d, \bar{d}, s, \bar{s}$
15. $u, \bar{u}, c, \bar{c}$

• NC19

16. $\nu_\mu, \bar{\nu}_\mu, \nu_e, \bar{\nu}_e$
17. $\mu^-, \mu^+, \nu_e, \bar{\nu}_e$
18. $u, \bar{u}, \nu_e, \bar{\nu}_e$
19. $d, \bar{d}, \nu_e, \bar{\nu}_e$

• NC64

20. $\mu^-, \bar{\nu}_\mu, \nu_\mu, \mu^+$
21. $d, \bar{u}, u, \bar{d}$

• NC48

22. $\mu^-, \mu^+, \nu_e, \nu_\mu, \mu^+$
23. $\nu_\mu, \bar{\nu}_\mu, \nu_\rho, \bar{\nu}_\rho$
24. $u, \bar{u}, e^-, \nu_e$
25. $d, \bar{d}, e^-, \nu_e$

• NC25(NC24+1)

26. $\mu^-, \mu^+, \mu^-, \mu^+$
27. $\nu_\mu, \bar{\nu}_\mu, \nu_\mu, \bar{\nu}_\mu$
28. $u, \bar{u}, u, \bar{u}$
29. $d, \bar{d}, d, \bar{d}$

• NC21(NC19+1)

30. $b, \bar{b}, \nu_\mu, \bar{\nu}_\mu$
31. $\mu^-, \mu^+, b, \bar{b}$

• NC50(NC48+2)

32. $b, \bar{b}, \nu_e, \bar{\nu}_e$
33. $b, \bar{b}, e^-, e^+$

• Mix43
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