Accelerating Universe, Cosmological Constant and Dark Energy

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Abstract

Most of the calculations done to obtain the value of the cosmological constant \( \Lambda \) use methods of quantum gravity, a theory that has not been established as yet, and a variety of results are usually obtained. The numerical value of \( \Lambda \) is then supposed to be inserted in the Einstein field equations, hence the evolution of the universe will depend on the calculated value of \( \Lambda \). Here we present a fundamental approach to the problem. The theory presented here uses a Riemannian four-dimensional presentation of gravitation in which the coordinates are those of Hubble, i.e. distances and velocity rather than space and time. We solve these field equations and show that there are three possibilities for the universe to expand but only the accelerating universe is possible. We extract from the theory the cosmological constant and show that \( \Lambda = 2.036 \times 10^{-35} s^{-2} \). This value of \( \Lambda \) is in excellent agreement with the measurements obtained by the High-Z Supernova Team and the Supernova Cosmology Project. Finally it is shown that the three-dimensional space of the universe is flat, as the Boomerang experiment shows.
I. INTRODUCTION

The problem of the accelerating universe, the cosmological constant and the vacuum energy associated with it is of extremely high interest these days. There are many questions related to these problems, especially with respect to the cosmological constant at the quantum level, all of which are related to quantum gravity. For example, why there exists the critical mass density and why the cosmological constant has this or that value? Trying to answer these questions and others were recently the subject of many publications [1-18]. Most of the calculations done to obtain the value of the cosmological constant $\Lambda$ use methods of quantum gravity. But there is no such a theory as yet and hence a variety of results for $\Lambda$ are obtained. The numerical value of $\Lambda$ is subsequently supposed to be inserted in the Einstein field equations, thus determining the evolution of the universe and the vacuum energy associated with it.

In this paper we present a fundamentally different approach in which the cosmological constant is not included \textit{a priori} in the theory, but $\Lambda$ can be extracted from the theory without the use of any quantum gravity theory. The value of $\Lambda$ will emerge from the gravitational field equations. Our theory employs a Riemannian four-dimensional presentation of gravity in which the coordinates are those of Hubble, namely distances and (cosmological) velocity instead of the traditional space and time. The gravitational field equations so constructed are subsequently solved and different kinds of expansion are obtained. It is shown that the accelerating universe is the only acceptable possibility. We then extract the value of $\Lambda$ from the theory and show that it is given by $\Lambda = 2.036 \times 10^{-35} s^{-2}$, which is in excellent agreement with the recently obtained measurements by the \textit{High-Z Supernova Team} and the \textit{Supernova Cosmology Project} [19-25]. Furthermore we determine the kind of the three-dimensional spatial subspace of the four-dimensional curved space to be flat, which is in agreement with the recent Boomerang experiment [26,27].

In Section 2 a brief review of previous work is given. Section 3 is devoted to the accelerated universe, whereas in Section 4 it is shown how the Tolman metric can be considered
as an expanding universe. In Section 5 the value of the cosmological constant is calculated, and the spatial three-dimensional subspace is determined to be flat. Section 6 is devoted to the concluding remarks.

II. REVIEW OF PREVIOUS WORK

As in classical general relativity we start our discussion in flat space which will then be generalized to curved space.

The Flat Space Case. The flat-space cosmological metric is given by

$$ds^2 = \tau^2 dv^2 - \left( dx^2 + dy^2 + dz^2 \right).$$ (2.1)

Here $\tau$ is Hubble’s time, the inverse of Hubble’s constant, as given by Freedman’s measurements in the limit of zero distances and thus zero gravity [28,29]. As such, $\tau$ is a constant, in fact a universal constant (its numerical value is given in Section 5). Its role in cosmology theory resembles that of $c$, the speed of light in vacuum, in ordinary special relativity. The velocity $v$ is used here in the sense of cosmology, as in Hubble’s law, and is usually not the time-derivative of the distance.

The universe expansion is obtained from the metric (2.1) as a null condition, $ds = 0$. Using spherical coordinates $r, \theta, \phi$ for the metric (2.1), and the fact that the universe is spherically symmetric ($d\theta = d\phi = 0$), the null condition then yields $dr/dv = \tau$, or upon integration and using appropriate initial conditions, gives $r = \tau v$ or $v = H_0 r$, i.e. the Hubble law in the zero-gravity limit.

Based on the metric (2.1) a cosmological special relativity (CSR) was developed [30-36] (for applications to inflation in the universe see Refs. [37-39]). In this theory the receding velocities of galaxies and the distances between them in the Hubble expansion are united into a four-dimensional pseudo-Euclidean manifold, similarly to space and time in ordinary special relativity. The Hubble law is assumed and is written in an invariant way that enables one to derive a four-dimensional transformation which is similar to the Lorentz
transformation. The parameter in the new transformation is the ratio between the cosmic
time to $\tau$ (in which the cosmic time is measured backward with respect to the present time).
Accordingly, the new transformation relates physical quantities at different cosmic times in
the limit of weak or negligible gravitation.

The transformation between the four variables $x, y, z, v$ and $x', y', z', v'$ (assuming
$y' = y$ and $z' = z$) is given by

$$
x' = \frac{x - tv}{\sqrt{1 - t^2/\tau^2}}, \quad v' = \frac{v - tx/\tau^2}{\sqrt{1 - t^2/\tau^2}}, \quad y' = y, \quad z' = z.
$$

Equations (2.2) are the cosmological transformation and very much resemble the well-known
Lorentz transformation. In CSR it is the relative cosmic time which takes the role of the
relative velocity in Einstein’s special relativity. The transformation (2.2) leaves invariant
the Hubble time $\tau$, just as the Lorentz transformation leaves invariant the speed of light in
vacuum $c$.

**Generalization to Curved Space.** A cosmological general theory of relativity, suitable
for the large-scale structure of the universe, was subsequently developed [40-43]. In the
framework of cosmological general relativity (CGR) gravitation is described by a curved
four-dimensional Riemannian spacevelocity. CGR incorporates the Hubble constant $\tau$ at
the outset. The Hubble law is assumed in CGR as a fundamental law. CGR, in essence,
extends Hubble’s law so as to incorporate gravitation in it; it is actually a distribution theory
that relates distances and velocities between galaxies. The theory involves only measured
quantities and it takes a picture of the Universe as it is at any moment. The following is a
brief review of CGR as was originally given by the author in 1966 in Ref. 40.

The foundations of any gravitational theory are based on the principles of equivalence and
general covariance [44]. These two principles lead immediately to the realization that gravita-
tion should be described by a four-dimensional curved spacetime, in our theory spacevelocity,
and that the field equations and the equations of motion should be written in a generally
covariant form. Hence these principles were adopted in CGR also. Use is made in a four-
dimensional Riemannian manifold with a metric $g_{\mu\nu}$ and a line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. 
The difference from Einstein’s general relativity is that our coordinates are: $x^0$ is a velocitylike coordinate (rather than a timelike coordinate), thus $x^0 = \tau v$ where $\tau$ is the Hubble time in the zero-gravity limit and $v$ the velocity. The coordinate $x^0 = \tau v$ is the comparable to $x^0 = ct$ where $c$ is the speed of light and $t$ is the time in ordinary general relativity. The other three coordinates $x^k$, $k = 1, 2, 3$, are spacelike, just as in general relativity theory.

An immediate consequence of the above choice of coordinates is that the null condition $ds = 0$ describes the expansion of the universe in the curved spacevelocity (generalized Hubble’s law with gravitation) as compared to the propagation of light in the curved spacetime in general relativity. This means one solves the field equations (to be given in the sequel) for the metric tensor, then from the null condition $ds = 0$ one obtains immediately the dependence of the relative distances between the galaxies on their relative velocities.

As usual in gravitational theories, one equates geometry to physics. The first is expressed by means of a combination of the Ricci tensor and the Ricci scalar, and follows to be naturally either the Ricci trace-free tensor or the Einstein tensor. The Ricci trace-free tensor does not fit gravitation, and the Einstein tensor is a natural candidate. The physical part is expressed by the energy-momentum tensor which now has a different physical meaning from that in Einstein’s theory. More important, the coupling constant that relates geometry to physics is now also different.

Accordingly the field equations are

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu},$$

(2.3)

exactly as in Einstein’s theory, with $\kappa$ given by $\kappa = 8\pi k/\tau^4$, (in general relativity it is given by $8\pi G/c^4$), where $k$ is given by $k = G\tau^2/c^2$, with $G$ being Newton’s gravitational constant, and $\tau$ the Hubble constant time. When the equations of motion will be written in terms of velocity instead of time, the constant $k$ will replace $G$. Using the above equations one then has $\kappa = 8\pi G/c^2\tau^2$.

The energy-momentum tensor $T^{\mu\nu}$ is constructed, along the lines of general relativity theory, with the speed of light being replaced by the Hubble constant time. If $\rho$ is the
average mass density of the universe, then it will be assumed that \( T^{\mu \nu} = \rho u^\mu u^\nu \), where \( u^\mu \) is the four-velocity. In general relativity theory one takes \( T^0_0 = \rho \). In Newtonian gravity one has the Poisson equation \( \nabla^2 \phi = 4\pi G \rho \). At points where \( \rho = 0 \) one solves the vacuum Einstein field equations and the Laplace equation \( \nabla^2 \phi = 0 \) in Newtonian gravity. In both theories a null (zero) solution is allowed as a trivial case. In cosmology, however, there exists no situation at which \( \rho \) can be zero because the universe is filled with matter. In order to be able to have zero on the right-hand side of Eq. (2.3) one takes \( T^0_0 \) not as equal to \( \rho \), but to \( \rho_{\text{eff}} = \rho - \rho_c \), where \( \rho_c \) is the critical mass density, a constant given by \( \rho_c = 3/8\pi G r^2 \), whose value is \( \rho_c \approx 10^{-29} \text{g/cm}^3 \), a few hydrogen atoms per cubic meter. Accordingly one takes

\[
T^{\mu \nu} = \rho_{\text{eff}} u^\mu u^\nu; \quad \rho_{\text{eff}} = \rho - \rho_c
\]  

(2.4)

for the energy-momentum tensor.

In the next sections we apply CGR to obtain the accelerating expanding universe and related subjects.

**III. THE ACCELERATING UNIVERSE**

In previous works [40-43] it was shown that CGR predicts in a natural way that the universe is accelerating. In this section we show that result in still another way. To this end we solve the gravitational field equations (2.3) with an energy-momentum tensor that includes pressure,

\[
T^{\mu \nu} = [\rho_{\text{eff}} + p] u^\mu u^\nu - pg^{\mu \nu},
\]  

(3.1)

where \( p \) is the pressure. The gravitational field that is sought is assumed to be both static and spherically symmetric and is therefore given by

\[
ds^2 = e^\nu r^2 dv^2 - e^\lambda dv^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2)
\]  

(3.2)

with \( \nu, \lambda \) being functions of \( r \) alone, and \( u^0 = u_0^{-1} = (g_{00})^{-1/2} \); other components of \( u^\alpha \) are zero. As has been described in the last section, the universe expansion is obtained by the
null requirement, $ds = 0$. Since, moreover, the universe expands in a spherically symmetric way, one also has $d\theta = d\phi = 0$. As a result, Eq. (3.2) reduces to
\[ e^\nu \tau^2 dv^2 - e^\lambda dr^2 = 0, \tag{3.3} \]
which yields for the universe expansion the very simple formula
\[ \frac{dr}{dv} = \frac{\tau e^{(\nu - \lambda)/2}}. \tag{3.4} \]

The nonvanishing components of the mixed Einstein tensor $G^\nu_\mu$ gives the following for the gravitational field equations:
\[ G^0_0 = -e^{-\lambda} \left( \frac{1}{r^2} - \frac{\lambda'}{r} \right) + \frac{1}{r^2} = \kappa T^0_0, \tag{3.5a} \]
\[ G^1_0 = -e^{-\lambda} \frac{\dot{\lambda}}{r} = \kappa T^1_0, \tag{3.5b} \]
\[ G^1_1 = -e^{-\lambda} \left( \frac{\nu' + 1}{r} + \frac{1}{r^2} \right) + \frac{1}{r^2} = \kappa T^1_1, \tag{3.5c} \]
\[ G^2_2 = -\frac{1}{2} e^{-\lambda} \left( \nu'' + \frac{\nu'}{r} - \frac{\lambda'}{r} \right) + \frac{1}{2} e^{-\nu} \left( \ddot{\lambda} + \frac{\dot{\lambda}^2}{2} - \frac{\dot{\nu} \dot{\lambda}}{2} \right) = \kappa T^2_2, \tag{3.5d} \]
\[ G^3_3 = G^2_2 = \kappa T^3_3. \tag{3.5e} \]

All other components of the Einstein tensor vanish identically, a prime denotes differentiation with respect to $r$, and a dot denotes differentiation with respect to $v$.

The above equations then yield
\[ e^{-\lambda} \left( \frac{1}{r^2} - \frac{\lambda'}{r} \right) - \frac{1}{r^2} = -\kappa \rho_{eff}, \tag{3.6} \]
\[ e^{-\lambda} \left( \frac{1}{r^2} + \frac{\nu'}{r} \right) - \frac{1}{r^2} = \kappa p, \tag{3.7} \]
\[ e^{-\lambda} \left( \nu'' + \frac{1}{2} \nu' - \frac{\lambda'}{r} - \frac{1}{2} \nu' \lambda' \right) = \kappa p. \tag{3.8} \]

The conservation law $\nabla_\nu T^{\mu\nu} = 0$ yields
\[ p' = -\frac{1}{2} \nu' (p + \rho_{eff}). \tag{3.9} \]
Equation (3.9) is not independent of Eqs. (3.6)-(3.8) since it is a consequence of the contracted Bianchi identities. One therefore has three equations for the four unknown functions \( \nu, \lambda, \rho, p \). One assumes a functional dependence of \( \rho \) on \( r \), calculate \( \nu, \lambda \) from this knowledge, and finally calculate \( p \).

The solution of Eq. (3.6) is given by

\[
e^{-\lambda} = 1 - \frac{\kappa m(r)}{4\pi r}, \tag{3.10}
\]

where

\[
m(r) = 4\pi \int_0^r \rho_{\text{eff}}(r') r'^2 dr'. \tag{3.11}
\]

is the mass of the fluid contained in a ball of radius \( r \). The solution given by Eq. (3.10) is chosen so that \( g_{\mu\nu} \) is regular at \( r = 0 \) and goes to the Schwarzschild form

\[
e^{-\lambda} = 1 - \frac{r_s}{r}, \tag{3.12}
\]

where \( r_s = 2Gm \) (divided by \( c^2 \)) and \( m = m(r_0) \), if \( \rho_{\text{eff}}(r) = 0 \) for \( r > r_0 \).

We now assume that \( \rho \) is a constant for \( r \leq r_0 \). We then obtain from Eqs. (3.9), (3.6), (3.7) and (3.10) the following:

\[
e^{-\lambda} = 1 - \frac{r^2}{R^2}, \tag{3.13}
\]

\[
e^{\nu/2} = A - B \left(1 - \frac{r^2}{R^2}\right)^{1/2}, \tag{3.14}
\]

\[
p = \frac{1}{\kappa R^2} \left[3B \left(1 - \frac{r^2}{R^2}\right)^{1/2} - A\right] \tag{3.15}
\]

where \( A \) and \( B \) are constants, and

\[
R^2 = \frac{3}{\kappa \rho_{\text{eff}}}. \tag{3.16}
\]

The constants \( A \) and \( B \) can be fixed by the requirements that \( p = 0 \) and \( e^\nu \) join smoothly the Schwarzschild field on the surface of the sphere. One obtains

\[
A = \frac{3}{2} \left(1 - \frac{r_0^2}{R^2}\right)^{1/2}, \quad B = \frac{1}{2}, \tag{3.17}
\]
\[ e^{\nu/2} = \frac{3}{2} \left( 1 - \frac{r_0^2}{R^2} \right)^{1/2} - \frac{1}{2} \left( 1 - \frac{r^2}{R^2} \right)^{1/2}, \tag{3.18} \]

\[ p = \rho \left[ \frac{(1 - \frac{2}{R^2})^{1/2} - (1 - \frac{2}{R^2})^{1/2}}{3 (1 - \frac{2}{R^2})^{1/2} - (1 - \frac{2}{R^2})^{1/2}} \right], \tag{3.19} \]

with the condition that \( r_0^2 < R^2 \). If one assumes that pressure inside the fluid is everywhere finite, one obtains from Eq. (3.19) the more restrictive condition

\[ r_0^2 < \frac{8}{9} R^2. \tag{3.20} \]

The spacetime-coordinate version of the solutions presented above are due to K. Schwarzschild [45].

Using the above results in the equation for the universe expansion (3.4) we obtain

\[ \frac{dr}{dv} = \tau \left[ A \left( 1 - \frac{r^2}{R^2} \right)^{1/2} - B \left( 1 - \frac{r^2}{R^2} \right) \right]. \tag{3.21} \]

We now confine ourselves to the linear approximation, getting

\[ \frac{dr}{dv} = \tau \left( \tilde{A} + \tilde{B} \frac{r^2}{R^2} \right), \tag{3.22} \]

where \( \tilde{A} = A - B \) and \( \tilde{B} = B - A/2 \), or

\[ \frac{dv}{dr} = \tau^{-1} \left( \tilde{A} - \tilde{B} \frac{r^2}{R^2} \right). \tag{3.23} \]

A simplification is obtained if we also confine to the linear approximation of \( A \) and \( B \), hence \( A = 3/2, B = 1/2 \), thus \( \tilde{A} = 1, \tilde{B} = -1/4 \). Using now the standard notation \( \Omega = \rho/\rho_c \), we obtain

\[ \frac{dr}{dv} = \tau \left[ 1 + \frac{(1 - \Omega) r^2}{4c^2 \tau^2} \right] \tag{3.24} \]

for the equation of the expansion of the universe. Equation (3.24), except for the factor 4, is identical to Eq. (15) of Ref. 40 and Eq. (5.10) of Ref. 41.

The second term in the square bracket in the above equation represents the deviation from the standard Hubble law due to gravity. For without that term, Eq. (3.24) reduces to \( dr/dv = \tau \), thus \( r = \tau v + const \). The constant can be taken zero if one assumes, as usual,
that at $r = 0$ the velocity should also vanish. Thus $r = \tau v$, or $v = H_0 r$ (since $H_0 = 1/\tau$). Accordingly, the equation of motion (3.24) describes the expansion of the universe when $\Omega = 1$, namely when $\rho = \rho_c$, the equation coincides with the standard Hubble law.

The equation of motion (3.24) can easily be integrated exactly by the substitutions

$$\sin \chi = (\Omega - 1)^{1/2} r/2c\tau; \Omega > 1, \quad (3.25a)$$

$$\sinh \chi = (1 - \Omega)^{1/2} r/2c\tau; \Omega < 1. \quad (3.25b)$$

One then obtains, using Eqs. (3.24) and (3.25),

$$dv = c d\chi / (\Omega - 1)^{1/2} \cos \chi; \Omega > 1, \quad (3.26a)$$

$$dv = c d\chi / (1 - \Omega)^{1/2} \cosh \chi; \Omega < 1. \quad (3.26b)$$

We give below the exact solutions for the expansion of the universe for each of the case, $\Omega > 1$ and $\Omega < 1$. As will be seen, the case of $\Omega = 1$ can be obtained at the limit $\Omega \to 1$ from both cases.

**The case $\Omega > 1$.** From Eq. (3.26a) we have

$$\int dv = \frac{c}{\sqrt{\Omega - 1}} \int \frac{d\chi}{\cos \chi}, \quad (3.27)$$

where $\sin \chi = r/a$, and $a = c\tau \sqrt{\Omega - 1}$. A simple calculation gives [47]

$$\int \frac{d\chi}{\cos \chi} = \ln \left| \frac{1 + \sin \chi}{\cos \chi} \right|. \quad (3.28)$$

A straightforward calculation then gives

$$v = \frac{a}{2\tau} \ln \left| \frac{1 + r/a}{1 - r/a} \right|. \quad (3.29)$$

As is seen, when $r \to 0$ then $v \to 0$ and using the L’Hospital lemma, $v \to r/\tau$ as $a \to 0$ (and thus $\Omega \to 1$).

**The case $\Omega < 1$.** From Eq. (3.26b) we now have

$$\int dv = \frac{c}{\sqrt{1 - \Omega}} \int \frac{d\chi}{\cosh \chi}. \quad (3.30)$$
where \( \sinh \chi = r/b \), and \( b = c\tau \sqrt{1 - \Omega} \). A straightforward calculation then gives [47]

\[
\int \frac{d\chi}{\cosh \chi} = \arctan e^\chi. \tag{3.31}
\]

We then obtain

\[
\cosh \chi = \left(1 + \frac{r^2}{b^2}\right)^{1/2}, \tag{3.32}
\]

\[
e^\chi = \sinh \chi + \cosh \chi = r/b + \left(1 + \frac{r^2}{b^2}\right)^{1/2}. \tag{3.33}
\]

Equations (3.30) and (3.31) now give

\[
v = \frac{2c}{\sqrt{1 - \Omega}} \arctan e^\chi + K, \tag{3.34}
\]

where \( K \) is an integration constant which is determined by the requirement that at \( r = 0 \) then \( v \) should be zero. We obtain

\[
K = -\frac{\pi c}{2\sqrt{1 - \Omega}}, \tag{3.35}
\]

and thus

\[
v = \frac{2c}{\sqrt{1 - \Omega}} \left( \arctan e^\chi - \frac{\pi}{4} \right). \tag{3.36}
\]

A straightforward calculation then gives

\[
v = \frac{b}{\tau} \left\{ 2 \arctan \left( \frac{r}{b} + \sqrt{1 + \frac{r^2}{b^2}} \right) - \frac{\pi}{2} \right\}. \tag{3.37}
\]

As for the case \( \Omega > 1 \) one finds that \( v \to 0 \) when \( r \to 0 \), and again, using L'Hospital lemma, \( r = \tau v \) when \( b \to 0 \) (and thus \( \Omega \to 1 \)).

**Physical meaning.** To see the physical meaning of these solutions, however, one does not need the exact solutions. Rather, it is enough to write down the solutions in the lowest approximation in \( \tau^{-1} \). One obtains, by differentiating Eq. (3.24) with respect to \( v \), for \( \Omega > 1 \),

\[
d^2r/dv^2 = -kr; \quad k = (\Omega - 1)/c^2, \tag{3.38}
\]

the solution of which is

\[
r(v) = A \sin \frac{v}{c} + B \cos \frac{v}{c}, \tag{3.39}
\]
where $\alpha^2 = \Omega - 1$ and $A$ and $B$ are constants. The latter can be determined by the initial condition $r(0) = 0 = B$ and $dr(0)/dv = \tau = A\alpha/c$, thus

$$r(v) = \frac{c\tau}{\alpha} \sin \frac{\alpha v}{c}, \quad (3.40)$$

This is obviously a closed universe, and presents a decelerating expansion.

For $\Omega < 1$ we have

$$d^2r/dv^2 = (1 - \Omega) r/c^2, \quad (3.41)$$

whose solution, using the same initial conditions, is

$$r(v) = \frac{c\tau}{\beta} \sinh \frac{\beta v}{c}, \quad (3.42)$$

where $\beta^2 = 1 - \Omega$. This is now an open accelerating universe.

For $\Omega = 1$ we have, of course, $r = \tau v$.

We finally determine which of the three cases of expansion is the one at present epoch of time. To this end we have to write the solutions (3.40) and (3.42) in ordinary Hubble’s law form $v = H_0 r$. Expanding Eqs. (3.40) and (3.42) into power series in $v/c$ and keeping terms up to the second order, we obtain

$$r = \tau v \left( 1 - \frac{\alpha^2 v^2}{6c^2} \right) \quad (3.43a)$$

$$r = \tau v \left( 1 + \frac{\beta^2 v^2}{6c^2} \right) \quad (3.43b)$$

for $\Omega > 1$ and $\Omega < 1$, respectively. Using now the expressions for $\alpha$ and $\beta$, Eqs. (3.43) then reduce into the single equation

$$r = \tau v \left[ 1 + (1 - \Omega) \frac{v^2}{6c^2} \right]. \quad (3.44)$$

Inverting now this equation by writing it as $v = H_0 r$, we obtain in the lowest approximation

$$H_0 = h \left[ 1 - (1 - \Omega) \frac{v^2}{6c^2} \right], \quad (3.45)$$

where $h = \tau^{-1}$. To the same approximation one also obtains

$$H_0 = h \left[ 1 - (1 - \Omega) \frac{z^2}{6} \right] = h \left[ 1 - (1 - \Omega) \frac{r^2}{6c^2 \tau^2} \right], \quad (3.46)$$
where $z$ is the redshift parameter. As is seen, and it is confirmed by experiments, $H_0$ depends on the distance it is being measured; it has physical meaning only at the zero-distance limit, namely when measured *locally*, in which case it becomes $h = 1/\tau$.

It follows that the measured value of $H_0$ depends on the “short” and “long” distance scales [48]. The farther the distance $H_0$ is being measured, the lower the value for $H_0$ is obtained. By Eq. (3.46) this is possible only when $\Omega < 1$, namely when the universe is accelerating.

The possibility that the universe expansion is accelerating was first predicted using CGR by the author in 1996 [40] before the supernovae experiments results became known.

In the next section it is shown how the familiar Tolman metric can be looked upon as an expanding universe.

**IV. TOLMAN’S METRIC AS AN EXPANDING UNIVERSE**

In the four-dimensional spacevelocity the Tolman metric is given by

$$ds^2 = \tau^2 dv^2 - e^\mu dr^2 - R^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right), \quad (4.1)$$

where $\mu$ and $R$ are functions of $v$ and $r$ alone, and comoving coordinates $x^\mu = (x^0, x^1, x^2, x^3) = (\tau v, r, \theta, \phi)$ have been used. With the above choice of coordinates, the zero-component of the geodesic equation becomes an identity, and since $r, \theta$ and $\phi$ are constants along the geodesics, one has $dx^0 = ds$ and therefore

$$u^a = u_\alpha = (1, 0, 0, 0). \quad (4.2)$$

The metric (4.1) shows that the area of the sphere $r = \text{constant}$ is given by $4\pi R^2$ and that $R$ should satisfy $R' = \partial R/\partial r > 0$. The possibility that $R' = 0$ at a point $r_0$ should be excluded since it would allow the lines $r = \text{constants}$ at the neighboring points $r_0$ and $r_0 + dr$ to coincide at $r_0$, thus creating a caustic surface at which the comoving coordinates break down.
As has been shown in the previous sections the universe expands by the null condition $ds = 0$, and if the expansion is spherically symmetric one has $d\theta = d\phi = 0$. The metric (4.1) then yields

$$\tau^2 dv^2 - e^\mu dr^2 = 0,$$

thus

$$\frac{dr}{dv} = \tau e^{-\mu/2}. \quad (4.4)$$

This is the differential equation that determines the universe expansion. In the following we solve the gravitational field equations in order to find out the function $\mu(r)$.

The gravitational field equations (2.3), written in the form

$$R_{\mu\nu} = \kappa \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right), \quad (4.5)$$

where

$$T_{\mu\nu} = \rho_{\text{eff}} u_\mu u_\nu \quad (4.6)$$

with $\rho_{\text{eff}} = \rho - \rho_c$ and $T = T_{\mu\nu} g^{\mu\nu}$, are now solved. Using Eq. (4.2) one finds that the only nonvanishing components of $T_{\mu\nu}$ is $T_{00} = \rho_{\text{eff}}$ and that $T = \rho_{\text{eff}}$.

The only nonvanishing components of the Ricci tensor are (dots and primes denote differentiation with respect to $v$ and $r$, respectively):

$$R_{00} = -\frac{1}{2} \ddot{\mu} - \frac{2}{R} \ddot{R} - \frac{1}{4} \dot{\mu}^2, \quad (4.7a)$$

$$R_{01} = \frac{1}{R} R' \ddot{\mu} - \frac{2}{R} \ddot{R}', \quad (4.7b)$$

$$R_{11} = e^\mu \left( \frac{1}{2} \ddot{\mu} + \frac{1}{4} \dot{\mu}^2 + \frac{1}{2} \ddot{R} \ddot{\mu} \right) + \frac{1}{R} (\dot{\mu}' R' - 2 \ddot{R}''), \quad (4.7c)$$

$$R_{22} = R \ddot{R} + \frac{1}{2} R R' \ddot{\mu} + \dot{R}^2 + 1 - e^{-\mu} \left( RR'' - \frac{1}{2} R R' \mu' + R'^2 \right), \quad (4.7d)$$

$$R_{33} = \sin^2 \theta R_{22}, \quad (4.7e)$$

whereas the Ricci scalar is given by

$$R = 2e^{-\mu} \left[ \frac{2}{R} R'' + \left( \frac{R'}{R} \right)^2 - \frac{1}{R} R' \dot{\mu} \right] - \frac{2}{R} R \ddot{\mu} - 2 \left( \frac{\dot{R}}{R} \right)^2 - \frac{2}{R^2} - \frac{4}{R} \ddot{R} - \ddot{\mu} - \frac{1}{2} \dot{\mu}^2. \quad (4.8)$$
The field equations obtained for the components 00, 01, 11, and 22 (the 33 component contributes no new information) are given by

\[-\ddot{\mu} - \frac{4}{R} \ddot{R} - \frac{1}{2} \dot{\mu}^2 = \kappa \rho_{eff} \] (4.9)

\[2 \dot{R}' - R' \dot{\mu} = 0 \] (4.10)

\[\ddot{\mu} + \frac{1}{2} \dot{\mu}^2 + \frac{2}{R} \dot{R} \dot{\mu} + e^{-\mu} \left( \frac{2}{R} R' \mu' - \frac{4}{R} \dot{R}' \right) = \kappa \rho_{eff} \] (4.11)

\[\frac{2}{R} \ddot{R} + 2 \left( \frac{\dot{R}}{R} \right)^2 + \frac{1}{R} \dot{R} \ddot{\mu} + 2 + e^{-\mu} \left[ \frac{1}{R} R' \mu' - \frac{2}{R} \left( \frac{R'}{R} \right)^2 - \frac{2}{R^2} \dot{R}' \right] = \kappa \rho_{eff} \] (4.12)

It is convenient to eliminate the term with the second velocity derivative of \( \mu \) from the above equations. This can easily be done, and combinations of Eqs. (4.9)–(4.12) then give the following set of three independent field equations:

\[e^\mu \left( 2 R \ddot{R} + \dot{R}^2 + 1 \right) - R'^2 = 0 \] (4.13)

\[2 \dot{R}' - R' \dot{\mu} = 0 \] (4.14)

\[e^{-\mu} \left[ \frac{1}{R} R' \mu' - \left( \frac{R'}{R} \right)^2 - \frac{2}{R^2} \dot{R}' \right] + \frac{1}{R} \dot{R} \ddot{\mu} + \left( \frac{\ddot{R}}{R} \right)^2 + \frac{1}{R^2} = \kappa \rho_{eff} \] (4.15)

other equations being trivial combinations of (4.13)–(4.15).

The solution of Eq. (4.14) satisfying the condition \( R' > 0 \) is given by

\[e^\mu = \frac{R'^2}{1 + f(r)} \] (4.16)

where \( f(r) \) is an arbitrary function of the coordinate \( r \) and satisfies the condition \( f(r) > -1 \).

Substituting (4.16) in the other two field equations (4.13) and (4.15) then gives

\[2 R \ddot{R} + \dot{R}^2 - f = 0 \] (4.17)

\[\frac{1}{R R'} \left( 2 R \ddot{R}' - f' \right) + \frac{1}{R^2} \left( \dot{R}^2 - f \right) = \kappa \rho_{eff}, \] (4.18)

respectively.

The integration of these equations is now straightforward. From Eq. (4.17) we obtain the first integral

\[\dot{R}^2 = f(r) + \frac{F(r)}{R} \] (4.19)
where $F(r)$ is another arbitrary function of $r$. Substituting now (4.19) in Eq. (4.18) gives

$$\frac{F'}{R^2 R'} = \kappa \rho_{\text{eff}}.$$

The two Eqs. (4.19) and (4.20) are now integrated for the case for which $f$ equals to zero, and Eq. (4.19) consequently reduces to

$$\dot{R}^2 = \frac{F(r)}{R}.$$  \hspace{1cm} (4.21)

The integration of Eq. (4.21) gives

$$R(v, r) = \left[ R^{3/2} (r) \pm \frac{3}{2} F^{1/2} (r) v \right]^{2/3},$$  \hspace{1cm} (4.22)

where

$$R(r) = R(0, r),$$  \hspace{1cm} (4.23)

namely, $R(v, r)$ at $v = 0$. Differentiating Eq. (4.22) with respect to $r$ and using Eq. (4.20) we also obtain

$$R(v, r) = (\kappa \rho_{\text{eff}})^{-2/3} \left[ \frac{R^{1/2} (r) R' (r)}{F' (r)} \pm \frac{v}{2 F^{1/2} (r)} \right]^{-2/3}. $$  \hspace{1cm} (4.24)

Finally, from Eq. (4.20) we obtain

$$\frac{\partial}{\partial v} \left( \rho_{\text{eff}} R^2 R' \right) = 0,$$  \hspace{1cm} (4.25)

and accordingly one has

$$\frac{dr}{dv} = \tau e^{-\mu/2} = \tau (R')^{-1/2}. $$  \hspace{1cm} (4.26)

If the function $f$ is not zero, the integration of Eq. (4.19) then yields for $f > 0$,

$$\tau v = f^{-1} \left( f R^2 + FR \right)^{1/2} - F f^{-3/2} \arcsinh \left( f R/F \right)^{1/2} + \Phi (r),$$  \hspace{1cm} (4.27a)

and for $f < 0$,

$$\tau v = f^{-1} \left( f R^2 + FR \right)^{1/2} - F (-f)^{-3/2} \arcsin \left( -f R/F \right)^{1/2} + \Phi (r),$$  \hspace{1cm} (4.27b)

where $\Phi (r)$ is an arbitrary function of $r$. The solutions (4.27) were given in spacetime coordinates by Datt [49].
V. VALUE OF $\Lambda$: THEORY VERSUS EXPERIMENT

The Einstein gravitational field equations with the added cosmological term are [45]:

$$ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, $$  \hspace{1cm} (5.1)

where $\Lambda$ is the cosmological constant, the value of which is supposed to be determined by experiment. In Eq. (5.1) $R_{\mu\nu}$ and $R$ are the Ricci tensor and scalar, respectively, $\kappa = 8\pi G$, where $G$ is Newton’s constant and the speed of light is taken as unity.

Recently the two groups (the Supernovae Cosmology Project and the High-Z Supernova Team) concluded that the expansion of the universe is accelerating [19-25]. Both teams obtained

$$ \Omega_M \approx 0.3, \quad \Omega_{\Lambda} \approx 0.7, $$  \hspace{1cm} (5.2)

and ruled out the traditional $(\Omega_M, \Omega_{\Lambda}) = (1, 0)$ universe. Their value of the density parameter $\Omega_{\Lambda}$ corresponds to a cosmological constant that is small but, nevertheless, nonzero and positive,

$$ \Lambda \approx 10^{-52} \text{m}^{-2} \approx 10^{-35} \text{s}^{-2}. $$  \hspace{1cm} (5.3)

In Sections 2 and 3 a four-dimensional cosmological theory was presented. The theory predicts that the universe accelerates and hence it is equivalent to having a positive value for a cosmological constant in it. In the framework of this theory the zero-zero component of the field equations (2.3) is written as

$$ R^0_0 - \frac{1}{2} \delta^0_0 R = \kappa \rho_{\text{eff}} = \kappa (\rho - \rho_c), $$  \hspace{1cm} (5.4)

where $\rho_c = 3/\kappa \tau^2$ is the critical mass density and $\tau$ is Hubble’s time in the zero-gravity limit.

Comparing Eq. (5.4) with the zero-zero component of Eq. (5.1), one obtains the expression for the cosmological constant,

$$ \Lambda = \kappa \rho_c = 3/\tau^2. $$  \hspace{1cm} (5.5)
To find out the numerical value of $\tau$ we use the relationship between $h = \tau^{-1}$ and $H_0$ given by Eq. (3.46) (CR denote values according to Cosmological Relativity):

$$H_0 = h \left[1 - \left(1 - \Omega_M^{CR}\right) z^2 / 6\right],$$  \hfill (5.6)

where $z$ is the redshift and $\Omega_M^{CR} = \rho_M / \rho_c$ where $\rho_c = 3h^2 / 8\pi G$. (Notice that $\rho_c$ is different from the standard $\rho_c$ defined with $H_0$.) The redshift parameter $z$ determines the distance at which $H_0$ is measured. We choose $z = 1$ and take for

$$\Omega_M^{CR} = 0.245$$  \hfill (5.7)

(roughly corresponds to 0.3 in the standard theory), Eq. (5.6) then gives

$$H_0 = 0.874h.$$  \hfill (5.8)

At the value $z = 1$ the corresponding Hubble parameter $H_0$ according to the latest results from HST can be taken [28] as $H_0 = 72$km/s-Mpc, thus $h = (72/0.874)$km/s-Mpc or

$$h = 82.380$$km/s-Mpc,  \hfill (5.9)

and

$$\tau = 12.16 \times 10^9\text{years.}$$  \hfill (5.10)

What is left is to find the value of $\Omega_\Lambda^{CR}$. We have $\Omega_\Lambda^{CR} = \rho_\Lambda^{ST} / \rho_c$, where $\rho_\Lambda^{ST} = 3H_0^2 / 8\pi G$ and $\rho_c = 3h^2 / 8\pi G$. Thus $\Omega_\Lambda^{CR} = (H_0 / h)^2 = 0.874^2$, or

$$\Omega_\Lambda^{CR} = 0.764.$$  \hfill (5.11)

As is seen from Eqs. (5.7) and (5.11) one has

$$\Omega_M^{CR} + \Omega_\Lambda^{CR} = 1.009 \approx 1,$$  \hfill (5.12)

which means the universe is flat.

As a final result we calculate the cosmological constant according to Eq. (5.5). One obtains

$$\Lambda = 3/\tau^2 = 2.036 \times 10^{-35}\text{s}^{-2}.$$  \hfill (5.13)
Our results confirm those of the supernovae experiments and indicate on existance of the dark energy as has recently received confirmation from the Boomerang cosmic microwave background experiment [26,27], which showed that the universe is flat.

VI. CONCLUDING REMARKS

In this paper it has been shown that application of the relativistic theory in spacevelocity to the problem of the expansion of the universe yields on accelerating expansion. Furthermore, the cosmological constant that was extracted from the theory agrees with the experimental result. Finally, it has also been shown that the three-dimensional spatial space of the universe is flat, again in agreement with observations.
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