Nonlinear evolution equation associated with hypergraph Laplacian

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Let $V$ be a finite set, $E \subset 2^V$ be a set of hyperedges, and $w : E \rightarrow (0, \infty)$ be an edge weight. On the (weighted) hypergraph $G = (V, E, w)$, we can define a multivalued nonlinear operator $L_{G,p}$ ($p \in [1, \infty)$) as the subdifferential of a convex function on $\mathbb{R}^V$, which is called “hypergraph $p$-Laplacian.” In this article, we first introduce an inequality for this operator $L_{G,p}$, which resembles the Poincaré–Wirtinger inequality in PDEs. Next, we consider an ordinary differential equation on $\mathbb{R}^V$ governed by $L_{G,p}$, which is referred as “heat” equation on the graph and used to study the geometric structure of the hypergraph in recent researches. With the aid of the Poincaré–Wirtinger type inequality, we can discuss the existence and the large time behavior of solutions to the ODE by procedures similar to those for the standard heat equation in PDEs with the zero Neumann boundary condition.

KEYWORDS
hypergraph, nonlinear evolution equation, ordinary differential equation, Poincaré–Wirtinger's inequality, p-Laplacian, subdifferential

1 Introduction

A weighted hypergraph $G$ is the triplet of a finite set $V$, a family $E \subset 2^V$ of subsets with more than one element of $V$, and a function $w : E \rightarrow (0, \infty)$, which is called the vertex set $V$, the set of hyperedges $E$, and the edge weight $w$, respectively (see Figure 1). In this paper, we consider the so-called hypergraph $p$-Laplacian $L_{G,p} : \mathbb{R}^V \rightarrow 2^{\mathbb{R}^V}$ defined on $G = (V, E, w)$ with $p \in [1, \infty)$ (see Section 2.2 for the precise definition) and the following ordinary differential equation associated with $L_{G,p}$:

$$\frac{d}{dt} x(t) + L_{G,p}(x(t)) \ni h(t), \quad (1)$$

where $x : [0, T] \rightarrow \mathbb{R}^V$ is an unknown function and $h : [0, T] \rightarrow \mathbb{R}^V$ is a given external force with $T > 0$. This equation is referred as a “heat” equation on the hypergraph.

When $G$ is a usual graph, namely, if $E$ consists of subsets with two elements of $V$, then $L_{G,2}$ becomes single-valued and coincides with a square matrix $D - A$, where $D$ and $A$ stand for the degree matrix and the adjacency matrix of $G$, respectively (see Remark 2.3 below). Then the random walk on the graph can be characterized by $L_{G,2}D^{-1} = I - AD^{-1}$, called the random walk normalized Laplacian matrix on $G$. In this case, the weight function $w$ in the Laplacian can be regarded as the conductance (reciprocal of resistance) of “current” or “flow” over each edge $e \in E$. Such a matrix can be used to investigate some properties of network represented by a graph. In particular, one of the important applications is PageRank, an algorithm to determine the importance of a Web site introduced by Brin–Page. Moreover, the linear differential equation $x'(t) = (I - AD^{-1})x(t)$ can be found in the definition of another pagerank, called the heat kernel.
### 2 | PROPERTIES OF HYPERGRAPH P-LAPLACIAN

#### 2.1 | Preliminary

We first fix some terms and definitions of maximal monotone operators and subdifferential operators (see, e.g., textbooks\textsuperscript{17–19}). Let $H$ be a real Hilbert space with the norm $| \cdot |$ and the inner product $(\cdot, \cdot)$ and $A$ be a (possibly) multivalued operator from $H$ into $2^H$, which stands for the power set of $H$. The domain and range of $A$ are denoted by $D(A)$ and $R(A)$, respectively. An operator $A$ is said to be monotone if $(y_1 - y_2, x_1 - x_2) \geq 0$ holds for any $x_j \in D(A)$ and $y_j \in Ax_j$ ($j = 1, 2$) and a monotone operator $A$ is said to be maximal monotone if $R(id + A) = H$, where $id$ is the identity map. It is well known that

- If $A$ is maximal monotone, then $Ax$ becomes a closed convex subset in $H$ for any $x \in D(A)$. Based on this fact, we define the minimal section of $A$ by $A^x := \left( Ax \right)^\circ$, where $C^\circ := \text{argmin}_{y \in C} |y| = \text{Proj}_C 0$ for a closed convex set $C \subset H$.
- The maximal monotone operator is demiclosed. That is, if $y_m \in Ax_m$, $x_m \to x$ strongly in $H$, and $y_m \to y$ weakly in $H$ as $m \to \infty$, then the limits satisfy $x \in D(A)$ and $y \in Ax$.

Let $\varphi : H \to (-\infty, +\infty]$ be a proper (i.e., $\varphi \not\equiv +\infty$) lower semicontinuous (l.s.c., for short) and convex functional. The set $D(\varphi) := \{ x \in H ; \varphi(x) < +\infty \}$ is called the effective domain of $\varphi$. Then we can define a (possibly) nonlinear...
multivalued mapping on $H$ by

$$\partial \varphi : x \mapsto \{ \eta \in H ; (\eta, \xi - x) \leq \varphi(\xi) - \varphi(x) \forall \xi \in D(\varphi) \},$$

known as the subdifferential of $\varphi$. As for the basic results in the convex analysis, we can see that

- The convex function $\varphi$ is continuous at the interior points of $D(\varphi)$. Especially, if $\varphi$ is defined on the whole space $H$ (i.e., $D(\varphi) = H$), then $\varphi$ is continuous on $H$.
- The subdifferential of a proper l.s.c. convex functional is always maximal monotone.
- Let $\varphi, \psi : H \to (-\infty, +\infty]$ be proper l.s.c. convex functions such that the intersection of $D(\varphi)$ and the interior of $D(\psi)$ is not empty. Then it follows that $\partial (\varphi + \psi) = \partial \varphi + \partial \psi$.

Moreover, assume that $\varphi$ is even; namely, $D(\varphi)$ is symmetric (i.e., $x \in D(\varphi)$ iff $-x \in D(\varphi)$) and $\varphi(-x) = \varphi(x)$. Then the subdifferential $\partial \varphi$ becomes an odd operator, that is, $D(\partial \varphi)$ is symmetric and $\partial \varphi(-x) = -\partial \varphi(x)$. Indeed, if $\eta \in \partial \varphi(x)$,

$$(\eta, \xi - x) \leq \varphi(\xi) - \varphi(x) \forall \xi \in D(\varphi) \iff (-\eta, -\xi - (-x)) \leq \varphi(-\xi) - \varphi(-x) \forall \xi \in D(\varphi),$$

which implies $-\eta \in \partial \varphi(-x)$.

### 2.2 Definition of hypergraph $p$-Laplacian

Let $V := \{ v_1, \ldots, v_n \}$ be a vertex set and $E \subset 2^V$ (the power set of $V$) be a set of hyperedges. Note that each $e \in E$ consists of more than one element of $V$. Moreover, a positive function $w : E \to (0, \infty)$ is defined as a weight on each hyperedge $e \in E$. Then the triplet $G = (V, E, w)$ is called the (weighted) hypergraph.

In this paper, we shall consider some ordinary differential equations over $\mathbb{R}^V$, the set of functions $x : V \to \mathbb{R}$. By letting $x_e := x(v_i)$, we can identify $\mathbb{R}^V$ with the $n$-dimensional Euclidean space $\mathbb{R}^n$. Define norms on $\mathbb{R}^V$ by

$$|x|_V^q := \begin{cases} \left( \sum_{v \in V} |x(v)|^q \right)^{1/q} & \text{if } q \in [1, \infty), \\ \max_{v \in V} |x(v)| & \text{if } q = \infty, \end{cases}$$

and the $\ell^2$-inner product on $\mathbb{R}^V$ by $x \cdot y := \sum_{v \in V} x(v)y(v)$ for each $x, y \in \mathbb{R}^V$. Recall that the standard inequality $|x|_V^q \leq |x|_V^q$ holds if $q < r$.

Henceforth, we express the indicator function on $S \subset V$ by $1_S \in \mathbb{R}^V$; that is, let $1_S(u) = 1$ if $u \in S$ and $1_S(u) = 0$ if $u \notin S$. Then we define the base polytope for the hyperedge $e \in E$ by

$$B_e := \text{conv} \{ 1_u - 1_v ; u, v \in e \},$$

(2)

where $1_v := 1_{\{v\}}$ and conv$Q$ denotes the convex hull of $Q \subset \mathbb{R}^V$.

We here define

$$f_e(x) := \max_{u,v \in e} (x(u) - x(v)) = \max_{u,v \in e} |x(u) - x(v)| = \max_{b \in B_e} b \cdot x.$$

Obviously, we have

- $f_e(x) \geq 0$ and $f_e(x) = f_e(-x)$ for any $x \in \mathbb{R}^V$.
- $f_e(x) = 0$ iff $x(u) = x(v)$ for any $u, v \in e$.

Moreover, since $f_e$ is convex and its domain coincides with the whole space $\mathbb{R}^V$ (which yields the continuity of $f_e$), we can define the subdifferential of $f_e$. Here, we recall the following maximum rule of subdifferential (see, e.g., Proposition 2.54 in Mordukhovich and Nam).
Lemma 2.1. Let \( g_j : \mathbb{R}^V \to \mathbb{R} \) \((j = 1, 2, \ldots, m)\) be convex functions satisfying \( D(g_j) = \mathbb{R}^V \). Then for every \( x \in \mathbb{R}^V \), it holds that

\[
\partial (\max_k g_k(x)) = \text{conv} \left( \bigcup_{j \in J(x)} \partial g_j(x) \right).
\]

where

\[
J(x) := \left\{ j \in \{1, 2, \ldots, m\} : g_j(x) = \max_k g_k(x) \right\}.
\]

Thanks to Lemma 2.1, the subdifferential of \( f_e \) can be represented by

\[
\partial f_e(x) = \text{argmax}_{b \in B_e} b \cdot x = \left\{ b_e \in B_e : b_e \cdot x = \max_{b \in B_e} b \cdot x \right\}.
\]

Obviously, \( b_e \cdot x = f_e(x) \) holds for every \( x \in \mathbb{R}^V \) and \( b_e \in \partial f_e(x) \). By the standard argument of convex analysis, we can derive the specific representation of the subdifferential of the following functional.

Proposition 2.2. Let \( g : [0, \infty) \to \mathbb{R} \) be a convex, nondecreasing, and \( C^1 \)-function. Then

\[
\varphi_{G,g}(x) := \sum_{e \in E} w(e)g(f_e(x))
\]

is continuous, convex, and even, and its domain \( D(\varphi_{G,g}) \) coincides with \( \mathbb{R}^V \). Moreover, its subdifferential is characterized by

\[
\partial \varphi_{G,g}(x) = L_{G,g}(x) := \sum_{e \in E} w(e)g'(f_e(x)) \partial f_e(x) = \left\{ \sum_{e \in E} w(e)g'(f_e(x))b_e : b_e \in \text{argmax}_{b \in B_e} b \cdot x \right\},
\]

which is an odd maximal monotone operator satisfying \( D(L_{G,g}) = \mathbb{R}^V \).

Proof. Evidently, \( D(\varphi_{G,g}) = \mathbb{R}^V \) and \( \varphi_{G,g} \) is even and convex. We here only show that \( \partial \varphi_{G,g}(x) \) can be explicitly represented by \( L_{G,g}(x) := \sum_{e \in E} w(e)g'(f_e(x)) \partial f_e(x) \). The subdifferential formula of the composition of convex functionals has already been obtained by Corollary 3.5 of Combari et al.\(^{21}\) within a fairly general setting. For the sake of completeness, however, we here give another proof via Brouwer’s fixed point theorem.

For any \( x, y \in \mathbb{R}^V \) and \( b_e \in \text{argmax}_{b \in B_e} b \cdot x \), we get

\[
\left( \sum_{e \in E} w(e)g'(f_e(x))b_e \right) \cdot (y - x) \leq \sum_{e \in E} w(e)g'(f_e(x))(f_e(y) - f_e(x)) \leq \sum_{e \in E} w(e)\left( g(f_e(y)) - g(f_e(x)) \right) = \varphi_{G,g}(y) - \varphi_{G,g}(x).
\]

Here, we use the fact that \( b_e \cdot y \leq \text{max}_{b \in B_e} b \cdot y = f_e(y) \). Hence, \( L_{G,g}(x) \subset \partial \varphi_{G,g}(x) \) holds, and it is enough to check the maximality of \( L_{G,g} \).

Let \( \{a_e\}_{e \in E} \) be a vector in \( \mathbb{R}^E \). If \( a_e \geq 0 \) for any \( e \in E \), then \( \psi(x) := \sum_{e \in E} a_e w(e)f_e(x) \) is a proper continuous convex function and its subdifferential coincides with \( \partial \psi(x) = \left\{ \sum_{e \in E} a_e w(e)b_e : b_e \in \text{argmax}_{b \in B_e} b \cdot x \right\} \) since \( D(f_e) = \mathbb{R}^V \) (recall the sum rule of the subdifferential, see, e.g., Theorem 2.10 of Barbu\(^{17}\)). Hence, the maximality of the subdifferential operator implies that

\[
x + \sum_{e \in E} a_e w(e)b_e = y, \ b_e \in \text{argmax}_{b \in B_e} b \cdot x
\]  

possesses a unique solution \( x \in \mathbb{R}^V \) for each given \( y \in \mathbb{R}^V \). Here, we define a mapping on \( \mathbb{R}^E \) by \( \Gamma : \{a_e\}_{e \in E} \to \{g'(f_e(x))\}_{e \in E} \), where \( x \) is a solution to (3).
Multiplying (3) by $x$ and recalling $b_e \cdot x = f_e(x) \geq 0$, we have

$$|x|_{\ell^2}^2 \leq |x|_{\ell^2} |y|_{\ell^2},$$

which yields

$$|x|_{\ell^\infty} \leq |x|_{\ell^2} \leq |y|_{\ell^2}.$$  

Since $f_e(x) \leq 2|x|_{\ell^\infty}$, we can see that $\Gamma$ maps a closed convex set

$$K := \left\{ \{a_e\}_{e \in E} \in \mathbb{R}^E; \ 0 \leq a_e \leq L := \max_{0 \leq |y|_{\ell^2} \leq |y|_{\ell^2}} g'(s) \right\}$$

into itself. Next, suppose that $a_e^m \to a_e$ as $m \to \infty$ for each $e \in E$. Let $x^m \in \mathbb{R}^V$ and $b_e^m \in \text{argmax}_{b \in B_e} b \cdot x^m$ satisfy (3) with $\{a_e^m\}_{e \in E}$; that is, $x^m + \sum_{e \in E} a_e^m w(e) b_e^m = y$. Testing this equation by $x^m$, we get $|x^m|_{\ell^\infty} \leq |y|_{\ell^2}$. Moreover, the definition of $B_e$ yields $|b_e^m|_{\ell^2} \leq 2$. Hence, we can extract convergent subsequences of $\{x^m\}$ and $\{b_e^m\}$. Let their limit be $x^\infty \in \mathbb{R}^V$ and $b_e^\infty \in \mathbb{R}^V$. Since the demiclosedness of maximal monotone operator leads to $b_e^\infty \in \partial f_e(x^\infty)$, we can see that $x^\infty$ is a solution to (3) with $\{a_e\}_{e \in E}$ by taking the limit as $m \to \infty$, which together with the continuity of $f_e$ and $g'$ assures the continuity of $\Gamma$ (remark that the original sequences $\{x^m\}$ and $\{b_e^m\}$ converge to $x^\infty$ and $b_e^\infty$, respectively, by the uniqueness of solution to (3)). Therefore Brouwer’s fixed point theorem is applicable to $\Gamma$ and the solvability of $x + LG(x) \ni y$ can be obtained for any $y \in \mathbb{R}^V$.

The hypergraph $p$-Laplacian is defined by $LG_p$ with $g(s) = \frac{1}{p} s^p \ (p \geq 1)$ in Proposition 2.2:

$$\varphi_{G,p}(x) := \frac{1}{p} \sum_{e \in E} w(e)(f_e(x))^p,$$

and

$$\partial \varphi_{G,p}(x) = LG_p(x) := \sum_{e \in E} w(e)(f_e(x))^{p-1} \partial f_e(x)$$

$$= \left\{ \sum_{e \in E} w(e)(f_e(x))^{p-1} b_e; \ b_e \in \text{argmax}_{b \in B_e} b \cdot x \right\}.$$  

Remark 2.3. If $p > 1$ and $G$ is a usual graph; that is, each $e \in E$ contains two elements, $LG_p(x)$ becomes a single-valued operator. Indeed, since $f_e(x) = |x(v_i) - x(v_j)|$ when $e = \{v_i, v_j\}$, we get

$$\varphi_{G,p}(x) = \frac{1}{2p} \sum_{i,j=1}^n w_{ij} |x_i - x_j|^p,$$

where $x(v_i)$ is abbreviated to $x_i$ and $w_{ij} := w(\{v_i, v_j\})$ if $\{v_i, v_j\} \in E$ (i.e., $v_i$ and $v_j$ are connected directly) and $w_{ij} := 0$ if $\{v_i, v_j\} \not\in E$ (i.e., $v_i$ and $v_j$ are disconnected). Clearly, this functional is totally differentiable except for $p = 1$ and its subgradient coincides with its derivative (see Ch. 1.2 of Barbu17). Especially, calculating partial derivatives for the case where $p = 2$, we have

$$\partial_{x_i} \varphi_{G,2}(x) = \sum_{j=1}^n w_{ij} (x_i - x_j) = d_i x_i - \sum_{j=1}^n w_{ij} x_j = (-w_{i1}, \ldots, d_i - w_{ii}, \ldots, w_{in}) \cdot x,$$

where $d_i := \sum_{i=1}^n w_{ij}$ denoting the (weighted) number of vertex connected to $v_i$. Hence, $LG_2 = \partial \varphi_{G,2}$ coincides with $D - A$, where the square matrix $D = \text{diag}(d_1, \ldots, d_n)$ and $A = (w_{ij})$ are called the (weighted) degree matrix and the (weighted) adjacency matrix.

On the other hand, when $G$ is a hypergraph, $LG_p(x)$ possibly returns a set-value on $\bigcup_{e \in E} \bigcup_{u \in E} \{x \in \mathbb{R}^V; x(u) = x(v)\}$ (union of hyperplanes) by the singularity of derivative of max function even if $p > 1$.  

2.3 | Poincaré–Wirtinger type inequality

We here decompose the vertex set \( V \) into “connected components” by the following manner. Define \( S_1 \subset V \) by the set of elements connected with \( v_1 \).

\[
S_1 := \left\{ v_1 \in V; \exists u_1, \ldots u_{k-1} \in V, \exists e_1, e_2, \ldots e_k \in E \text{ s.t.} u_{j-1}, u_j \in e_j \forall j = 1, 2, \ldots k, \text{ where } u_0 = v_1 \text{ and } u_k = v_1 \right\}.
\]

If \( S_1 \not\subset V \), let \( i_1 \) be the least index satisfying \( v_{i_1} \notin S_1 \) and define

\[
S_2 := \left\{ v_1 \in V; \exists u_1, \ldots u_{k-1} \in V, \exists e_1, e_2, \ldots e_k \in E \text{ s.t.} u_{j-1}, u_j \in e_j \forall j = 1, 2, \ldots k, \text{ where } u_0 = v_{i_1} \text{ and } u_k = v_1 \right\}.
\]

Continue this task inductively until \( S_1 \cup S_2 \cup \ldots \cup S_l = V \) holds. One can expect the “heat” is not delivered between two separated components. It is easy to see that

- \( S_j \cap S_k = \emptyset \) if \( j \neq k \).
- for any \( e \in E \), there exists \( j \in \{1, \ldots , l\} \) such that \( e \subset S_j \) and \( e \cap S_k = \emptyset \) if \( k \neq j \).

Here, we define \( \phi \in \mathbb{R}^V \) by

\[
\phi = \phi_{c_1, \ldots , c_l} = \sum_{j=1}^l c_j 1_{S_j},
\]

where \( c_1, \ldots , c_l \in \mathbb{R} \) are some constants, that is to say, \( \phi \) satisfies \( \phi(v) = c_j \) if \( v \in S_j \). We next show that all 0-eigenfunctions of \( L_{G,p} \) can be obtained by \( \phi \) for some \( c_1, \ldots , c_l \).

**Theorem 2.4.** Let \( p \geq 1 \). Then \( x \in \mathbb{R}^V \) satisfies \( 0 \in L_{G,p}(x) \) if and only if \( x = \phi_{c_1, \ldots , c_l} \) with some constant \( c_1, \ldots , c_l \in \mathbb{R} \).

Moreover, for every \( x \in \mathbb{R}^V \) and \( c_1, \ldots , c_l \in \mathbb{R} \), it holds that

\[
\varphi_{G,p}(x + \phi_{c_1, \ldots , c_l}) = \varphi_{G,p}(x), \quad L_{G,p}(x + \phi_{c_1, \ldots , c_l}) = L_{G,p}(x). \tag{4}
\]

**Proof.** We recall that the definition of the subdifferential yields

\[
0 \in L_{G,p}(x) = \partial \varphi_{G,p}(x) \iff \varphi_{G,p}(x) = \min_{y \in \mathbb{R}^V} \varphi_{G,p}(y) = 0.
\]

Since \( w(e) > 0 \) and \( f_e \geq 0 \), \( \varphi_{G,p}(x) = \frac{1}{p} \sum_{e \in E} w(e)(f_e(x))^p = 0 \) implies that \( f_e(x) = \max_{u,v \in e} |x(u) - x(v)| \) holds for every \( e \in E \), and vice versa. Therefore, it follows from 0 \( \in L_{G,p}(x) \) that \( x(u) = x(v) \) for every \( e \in E \), namely, \( x = \phi_{c_1, \ldots , c_l} \) with some constants \( c_1, \ldots , c_l \in \mathbb{R} \). Conversely, we can easily obtain 0 \( \in L_{G,p}(\phi_{c_1, \ldots , c_l}) \); that is, \( \varphi_{G,p}(\phi_{c_1, \ldots , c_l}) = 0 \) since \( f_e(\phi_{c_1, \ldots , c_l}) = |c_j - c_j| = 0 \) holds for every \( e \in E \) satisfying \( e \subset S_j \) and for every \( j = 1, \ldots , l \).

Since \( (1_u - 1_v) \cdot \phi_{c_1, \ldots , c_l} = c_j - c_j = 0 \) holds for any \( u,v \in e \subset S_j \), we obtain \( b \cdot \phi_{c_1, \ldots , c_l} = 0 \) for any \( b \in B_e = \text{conv}\{1_u - 1_v; \ u,v \in e\} \). Hence, we obtain

\[
f_e(x + \phi_{c_1, \ldots , c_l}) = f_e(x), \quad b \cdot (x + \phi_{c_1, \ldots , c_l}) = b \cdot x \quad \forall b \in B_e, \forall e \in E, \forall x \in \mathbb{R}^V,
\]

which leads to (4).

**Remark 2.5.** When \( p > 1 \), then \( L_{G,p}(\phi_{c_1, \ldots , c_l}) \) returns a single value. Indeed,

\[
L_{G,p}(\phi_{c_1, \ldots , c_l}) = \sum_{e \in E} w(e)(f_e(\phi_{c_1, \ldots , c_l}))^{p-1} \partial f_e(\phi_{c_1, \ldots , c_l}) = 0.
\]
We define \( \tilde{x} \in \mathbb{R}^V \) by averaging of \( x \in \mathbb{R}^V \) with respect to each connected component \( S_k \), which can be expected to be a stable state of a system according to Theorem 2.4:

\[
\tilde{x} := \phi_{c_1, \ldots, c_l}, \quad \text{where} \quad c_k = \frac{1}{\#S_k} \sum_{v \in S_k} x(v) \quad k = 1, 2, \ldots, l. \tag{6}
\]

Here and henceforth, \( \#S \) stands for the number of elements belonging to \( S \subset V \). Then we can obtain the Poincaré–Wirtinger type inequality:

**Theorem 2.6.** Let \( q \in [1, \infty] \) and \( p \geq 1 \). Then every \( x \in \mathbb{R}^V \) and \( y \in L_{G,p}(x) \) satisfy

\[
|x - \tilde{x}|_{p,q}^p \leq C x \cdot y = pC \varphi_{G,p}(x), \tag{7}
\]

where

\[
C = C_{G,p} := \frac{1}{\min_{e \in E} w(e)} \left( \sum_{j=1}^{l} \#S_j^{2-\frac{1}{p}} \right)^{\frac{1}{p}}. \tag{8}
\]

**Proof.** Recall that \( b_e \cdot x = f_e(x) \) holds for every \( x \in \mathbb{R}^V \) and \( b_e \in \partial f_e(x) \), which together with the representation of \( L_{G,p} \) implies \( x \cdot y = \sum_{e \in E} w(e)(f_e(x))^p = p\varphi_{G,p}(x) \).

We next demonstrate \( |x - \tilde{x}|_{p,q}^p \leq pC_{G,p}\varphi_{G,p}(x) \). Let \( u, v \in S_j \) (\( j = 1, \ldots, l \)). By the definition, there exist \( v_{j_1}, \ldots, v_{j(k-1)} \in S_j \) and \( e_{j_1}, \ldots, e_{j_k} \in E \) such that \( v_{j(i-1)}, v_j \in e_i \) for any \( i = 1, 2, \ldots, k \), where \( v_0 = u \) and \( v_k = v \). Then we obtain

\[
|x(u) - x(v)| \leq \sum_{i=1}^{k} |x(v_{j(i-1)}) - x(v_{j_i})| \leq \sum_{i=1}^{k} f_{e_i}(x) \leq \frac{\#S_j^{1/p'}}{\min_{e \in E} w(e)^{1/p}} \left( \sum_{e \in E} w(e)(f_e(x))^p \right)^{1/p},
\]

where \( p' \) is the Hölder conjugate exponent:

\[
p' := \begin{cases} 
\frac{p}{p-1} & \text{if } p > 1, \\
\infty & \text{if } p = 1.
\end{cases} \tag{9}
\]

Recalling (6), we obtain

\[
|x(u) - \tilde{x}(u)| \leq \frac{1}{\#S_j} \sum_{v \in S_j} |x(u) - x(v)| \leq \frac{\#S_j^{1/p'}}{\min_{e \in E} w(e)^{1/p}} \left( \sum_{e \in E} w(e)(f_e(x))^p \right)^{1/p},
\]

for each \( u \in S_j \), which leads to (7) with \( q = 1 \). From the general inequality \( |x|_{p,1} \leq |x|_{p,q} \), we can derive (7) for every \( q \in (1, \infty) \).

**Remark 2.7.** Let \( y_j \in L_{G,p}(x_j) \) (\( j = 1, 2 \)), then we can easily obtain

\[
(y_1 - y_2) \cdot (x_1 - x_2) \geq \sum_{e \in E} w(e) \left( (f_e(x_1))^{p-1} - (f_e(x_2))^{p-1} \right) (f_e(x_1) - f_e(x_2)). \tag{10}
\]

However, it seems to be difficult to establish an estimate of \( |x_1 - x_2|_{p,1} \) or \( |x_1 - \tilde{x}_1 - x_2 + \tilde{x}_2|_{p,1} \) from this inequality. For instance, let \( \#V = 4, E = \{V\}, w \equiv 1, \) and
In this section, we consider the ordinary differential equation associated with the hypergraph $p$-Laplace by using properties given above.

\[
(C) \left\{ \begin{array}{l}
\frac{d}{dt} x(t) + L_{G,p}(x(t)) \ni h(t) \quad t \in (0,T), \\
x(0) = x_0,
\end{array} \right.
\]

where $x : [0,T] \rightarrow \mathbb{R}^V$ is an unknown function and $h : [0,T] \rightarrow \mathbb{R}^V$ is a given external force. Henceforth, we write $x'(t) := \frac{d}{dt} x(t)$.

\[ L^q(0,T;\mathbb{R}^V) := \left\{ h : [0,T] \rightarrow \mathbb{R}^V; \int_0^T |h(t)|^q_{L^q} dt < \infty \right\}, \]

\[ W^{1,q}(0,T;\mathbb{R}^V) := \left\{ h : [0,T] \rightarrow \mathbb{R}^V; \int_0^T \left( |h(t)|^q_{L^q} + |h'(t)|^q_{L^q} \right) dt < \infty \right\}. \]

for $q \in [1,\infty)$, and $h \in L^\infty(0,T;\mathbb{R}^V)$ iff $\text{esssup}_{t \in [0,T]} |h(t)|_{L^\infty} < \infty$.

Since the maximal monotonicity of $L_{G,p}$ in the Hilbert space $\mathbb{R}^V$ endowed with $\ell^2$-norm is shown in Proposition 2.2, the abstract theory by Kömura (see also Theorem 3.6 and 3.7 in Brézis) can be applied and it holds that

**Theorem 3.1.** For every $x_0 \in \mathbb{R}^V$ and $h \in L^2(0,T;\mathbb{R}^V)$, (C) possesses a unique solution satisfying

\[ x \in W^{1,2}(0,T;\mathbb{R}^V), \]

\[ t \mapsto \varphi_{G,p}(x(t)) \text{ is absolutely continuous on } [0,T], \]

\[ \left( \int_0^t |x'(s)|_{L^2}^2 ds \right)^{1/2} \leq \left( \int_0^t |h(s)|_{L^2}^2 ds \right)^{1/2} + \sqrt{\varphi_{G,p}(x_0)} \quad \forall t \in [0,T]. \]

Moreover, if $t_0 \in (0,T)$ is a right-Lebesgue point of $h$, $x$ is right-differentiable at $t_0$ and the right-derivative of $x$ denoted by $\frac{d^+x}{dt}$ satisfies

\[ \frac{d^+x}{dt}(t_0) = \left( h(t_0 + 0) - L_{G,p}(x(t_0)) \right)^\circ, \quad (11) \]

where $h(t_0 + 0) := \lim_{\tau \rightarrow +0} \frac{1}{\tau} \int_{t_0}^{t_0 + \tau} h(s) ds$. If $h \in W^{1,1}(0,T;\mathbb{R}^V)$, the solution also fulfills
Remark 3.2. As for a significant property of nonlinear multivalued evolution equation, we here give an example of a solution that does not belong to $C^1$-class. Let $\#V = 4$, $E = \{V\}$, and $w \equiv 1$. We solve (C) with $p = 2$ and given data

$$h \equiv 0, x_0 = \begin{pmatrix} x_0(v_1) \\ x_0(v_2) \\ x_0(v_3) \\ x_0(v_4) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \\ -2 \end{pmatrix}.$$ 

When $t$ is sufficiently small, the order of initial data is preserved by the continuity; that is, $x_4(t) < x_3(t) < x_2(t) < x_1(t)$ holds (here and henceforth, we write $x_i(t) := x(t)(v_i)$). Then

$$L_{G,2}(x(t)) = f_\varepsilon(x(t)) \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} x_1(t) - x_4(t) \\ 0 \\ 0 \\ x_4(t) - x_1(t) \end{pmatrix}.$$ 

Hence, (C) is equivalent to

$$x_1'(t) = -x_1(t) + x_4(t), \quad x_2'(t) = x_1(t) - x_4(t), \quad x_3'(t) = x_2'(t) = 0,$$

which yields

$$x_1(t) = 2e^{-2t}, \quad x_2(t) = 1, \quad x_3(t) = -1, \quad x_4(t) = -2e^{-2t}$$

until $t \leq t_0 = \frac{1}{2} \log 2$.

In order to see the behavior of solution after $x_1$ and $x_4$ touches $x_2$ and $x_3$, respectively, that is, $t \geq t_0$, we have to specify the minimal section of

$$L_{G,2}(x(t)) = \begin{pmatrix} (x_1(t) - x_4(t)) \\ \lambda_1 \lambda_2 \\ -\mu_1 \\ -\mu_2 \end{pmatrix} = \begin{pmatrix} \lambda_1, \lambda_2, \mu_1, \mu_2 \geq 0, \\ \lambda_1 + \lambda_2 = 1, \\ \mu_1 + \mu_2 = 1 \end{pmatrix}.$$ 

Since $\sqrt{\lambda_1^2 + \lambda_2^2 + \mu_1^2 + \mu_2^2}$ attains its minimum at $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 1/2$, we get

$$L_{G,2}^\varepsilon(x(t)) = \frac{(x_1(t) - x_4(t))}{2} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}.$$ 

By (11), (C) with $t \geq t_0$ implies

$$x_1'(t) = -\frac{1}{2} (x_1(t) - x_4(t)), \quad x_2'(t) = -\frac{1}{2} (x_2(t) - x_3(t)), \quad x_3'(t) = \frac{1}{2} (x_2(t) - x_3(t)), \quad x_4'(t) = \frac{1}{2} (x_1(t) - x_4(t)).$$
Therefore, the solution of \((C)\) is
\[
x_1(t) = \begin{cases} 
2e^{-2t} & \text{if } t \leq \frac{1}{2} \log 2, \\
\frac{1}{2}e^{-t} & \text{if } t \geq \frac{1}{2} \log 2,
\end{cases}
\]
\[
x_2(t) = \begin{cases} 
1 & \text{if } t \leq \frac{1}{2} \log 2, \\
\frac{1}{2}e^{-t} & \text{if } t \geq \frac{1}{2} \log 2.
\end{cases}
\]

Analogously, we can construct a solution to \((C)\) for the same hypergraph as the above with \(p \neq 2\) only by replacing \(L_{G^2}(x) = (f_e(x))df_e(x)\) with \(L_{G,p}(x) = (f_e(x))^{p-1}df_e(x)\).

Multiplying the equation of \((C)\) by \(1_{S_k} (k = 1, 2, \ldots l)\), we have
\[
\frac{d}{dt} \sum_{v \in S_k} x(t)(v) = \sum_{v \in S_k} h(t)(v). \tag{12}
\]

We here use the fact that \(y \cdot 1_{S_k} = 0\) holds for any \(x \in \mathbb{R}^l, y \in L_{G,p}(x), \) and \(k = 1, \ldots, l\). Indeed, recall \((5)\); that is, every \(e \in E\) and \(b \in B_e\) satisfy \(b \cdot 1_{S_k} = 1 - 1 = 0\). Then from Proposition 2.2, we get \(y \cdot 1_{S_k} = \sum_{e \in E} w(e)(f_e(x))^{p-1}b_e \cdot 1_{S_k} = 0\).

Integrating \((12)\) over \([0, t]\) and collecting them with respect to \(k = 1, \ldots, l\), we obtain the following identity:
\[
\bar{x}(t) = \bar{x}_0 + \int_0^t h(s)ds.
\]

Viewing this mass conservation law derived and recalling Theorem 2.4 and Poincaré–Wirtinger's inequality \((7)\), one may expect to deal with the hypergraph Laplacian \(L_{G,p}\) and \((1)\) by treatments similar to those for the standard Laplacian \(-\Delta\) with the homogeneous Neumann boundary condition and parabolic equations governed by the Neumann Laplacian.

We here consider the large time behavior of the solution to \((C)\). Let \(h \equiv 0\), then \(\bar{x}(t) = \bar{x}_0\) holds for any \(t > 0\). By Theorem 2.6,
\[
\sum_{e \in E} w(e)(f_e(x(t)))^{p-1}b_e(t) \cdot (x(t) - \bar{x}_0) \geq \frac{1}{C_{G,p}} |x(t) - \bar{x}_0|^p,
\]
\[
\forall b_e(t) \in \text{argmax } b \cdot x(t).
\]

Note that \(b_e(t) \cdot \bar{x}_0 = 0\) holds since \(\bar{x}_0 = \sum_{j=1}^l c_j 1_{S_j}\) with some suitable \(c_1, \ldots, c_l \in \mathbb{R}\) (recall \((5)\) and \((6)\)) and \(b_e(t) \cdot 1_{S_j} = 0\).

Hence, multiplying \((C)\) by \(x(t) - \bar{x}_0\), we have
\[
\frac{1}{2} \frac{d}{dt} |x(t) - \bar{x}_0|^2 + \frac{1}{C_{G,p}} |x(t) - \bar{x}_0|^p \leq 0,
\]
which leads to the following estimate of solutions.

**Theorem 3.3.** Let \(h \equiv 0\). Then the solution to \((C)\) satisfies
\[
X(t) \leq \left( X(0)^{\frac{2}{p-1}} - \frac{(2 - p)t}{C_{G,p}} \right)^{\frac{1}{p-1}} \quad \text{if } 1 \leq p < 2,
\]
\[
X(t) \leq X(0) \exp \left( -\frac{2t}{C_{G,p}} \right) \quad \text{if } p = 2,
\]
\[
X(t) \leq \left( X(0)^{\frac{p-2}{p}} + \frac{(p - 2)t}{C_{G,p}} \right)^{\frac{1}{p-1}} \quad \text{if } p > 2,
\]
where \(X(t) := |x(t) - \bar{x}_0|^2, \) \((s)_+ := \max\{s, 0\}, \) and \(C_{G,p}\) is a constant defined by \((8)\).
Remark 3.4. Optimality of decay rate in Theorem 3.3 can be easily obtained as follows. Since \( y(u) = \bar{y}(v) \) holds for any \( y \in \mathbb{R}^V \) if \( u, v \in e \), we have

\[
f_\varepsilon(x(t)) = \max_{u, v \in e} |x(t)(u) - x(t)(v)| = \max_{u, v \in e} |x(t)(u) - \bar{x}_0(u) + \bar{x}_0(v) - x(t)(v)|
\]

\[
\leq \sum_{u \in V} |x(t)(u) - \bar{x}_0(u)| \leq \sqrt{n} |x(t) - \bar{x}_0|_{\ell^2}.
\]

Hence, testing (C) by \( x(t) - \bar{x}_0 \), we can get

\[
\frac{1}{2} \frac{d}{dt} |x(t) - \bar{x}_0|^2 \leq -p \varphi_{G, p}(x(t)) \geq -p \max_{e \in E} w(e) |x(t) - \bar{x}_0|^p,
\]

which yields the estimate of \( X(t) = |x(t) - \bar{x}_0|^2 \) from below.

3.2 Periodic problem

Next, we consider the following time-periodic problem:

\[
(P) \begin{cases}
\frac{d}{dt} x(t) + L_{G, p}(x(t)) \ni h(t) & t \in (0, T), \\
x(0) = x(T).
\end{cases}
\]

Note that the abstract results cannot be applied since \( \varphi_{G, p} \) is not coercive. Multiplying (P) by \( 1_{S_k} \), integrating over \([0, T]\), and using the periodicity, we have

\[
\int_0^T h(t) dt = 0 \tag{13}
\]

as a necessary condition of the existence of solutions. Recall (9); that is, \( p' \) stands for the Hölder conjugate of \( p \).

**Theorem 3.5.** Let \( h \in L^{\tilde{p}}(0, T; \mathbb{R}^V) \) with \( \tilde{p} := \max\{2, p'\} \) and assume (13). Then (P) possesses at least one solution \( x \in W^{1,2}(0, T; \mathbb{R}^V) \).

**Proof.** We first deal with the following approximation problem:

\[
(P)_\varepsilon \begin{cases}
\frac{d}{dt} x_\varepsilon(t) + \varepsilon x_\varepsilon(t) + L_{G, p}(x_\varepsilon(t)) \ni h(t) & t \in (0, T), \\
x_\varepsilon(0) = x_\varepsilon(T).
\end{cases}
\]

Remark that the main term of \((P)_\varepsilon\) coincides with the subdifferential of

\[
\varphi_{G, p}^\varepsilon(x) := \frac{\varepsilon}{2} |x|^2 + \varphi_{G, p}(x);
\]

that is, \( \partial \varphi_{G, p}^\varepsilon(x) = \varepsilon x + L_{G, p}(x) \). Since \( \varphi_{G, p}^\varepsilon \) is coercive, \((P)_\varepsilon\) possesses a unique periodic solution \( x_\varepsilon \in W^{1,2}(0, T; \mathbb{R}^V) \) for any given \( h \in L^2(0, T; \mathbb{R}^V) \) (see Corollary 3.4 of Brézis).

Testing \((P)_\varepsilon\) by \( 1_{S_k} \) \((k = 1, \ldots, l)\) and integrating over \([0, T]\), we get

\[
\sum_{i \in \Delta_k} \int_0^T x_\varepsilon(t)(v) dt = 0
\]

by the condition \( x_\varepsilon(0) = x_\varepsilon(T) \) and (13). Then the continuity of \( x_\varepsilon \) implies that there exists some \( t_k \in [0, T] \) such that \( \sum_{i \in \Delta_k} x_\varepsilon(t_k)(v) = 0 \). Multiplying \((P)_\varepsilon\) by \( 1_{S_k} \) again, we obtain
\[
\frac{d}{dt}\left(\sum_{x \in S_k} x_t(t)v(t)\right) + \varepsilon\left(\sum_{x \in S_k} x_t(t)(v)\right) = \left(\sum_{x \in S_k} h(t)(v)\right),
\]

which leads to

\[
\left(\sum_{x \in S_k} x_t(t)v(t)\right) = \int_t^e e^{-\varepsilon(t-s)}\left(\sum_{x \in S_k} h(s)(v)\right) ds
\]

and

\[
|\bar{x}_\varepsilon(t)|_{\varepsilon'} \leq |\bar{x}_\varepsilon(t)|_{\varepsilon} \leq \int_0^T |h(s)|_{\varepsilon'} ds
\]

for any \(q \in [1, \infty]\) and \(t \in [0, T]\).

Multiplying (P), by \(x_t\), and integrating over \([0, T]\), we have

\[
\varepsilon \int_0^T |x_t(t)|_{\varepsilon'}^2 dt + p \int_0^T \varphi_{G,p}(x_t(t)) dt
\]

\[
\leq \int_0^T |h(t)|_{\varepsilon'} |x_t(t)|_{\varepsilon} dt
\]

\[
\leq \left(\int_0^T |h(t)|_{\varepsilon'}^p dt\right)^{1/p} \left[\left(\int_0^T |x_t(t) - \bar{x}_\varepsilon(t)|_{\varepsilon'}^p dt\right)^{1/p} + \left(\int_0^T |\bar{x}_\varepsilon(t)|_{\varepsilon'}^p dt\right)^{1/p}\right].
\]

From this estimate together with (7) and (14), we can derive

\[
\varepsilon \int_0^T |x_t(t)|_{\varepsilon'}^2 dt + \int_0^T |x_t(t)|_{\varepsilon}^p dt \leq C,
\]

where \(C\) is some general constant independent of \(\varepsilon \in (0, 1]\). Let \(t_0 \in [0, T]\) attain the minimum of \(t \mapsto |x_t(t)|_{\varepsilon'}\). Clearly, \(|x_t(t_0)|_{\varepsilon'} \leq C\) by (15).

Testing (P), by \(x_t\), we get

\[
\int_0^T |x_t'(t)|_{\varepsilon'}^2 dt \leq \int_0^T |h(t)|_{\varepsilon'}^2 dt.
\]

This immediately yields

\[
\sup_{0 \leq t \leq T} |x_t(t)|_{\varepsilon'} \leq C
\]

by \(|x_t(t_0)|_{\varepsilon'} \leq C\) and

\[
\int_0^T |y_t(t)|_{\varepsilon}^2 dt \leq C,
\]

where \(y_t\) is the section of \(L_{G,p}(x_t)\) satisfying (P), i.e., that is, \(x_t'(t) + \varepsilon x_t(t) + y_t(t) = h(t)\) and \(y_t(t) \in L_{G,p}(x_t(t))\) for a.e. \(t \in (0, T)\).

By (16) and (17), we can apply Ascoli–Arzela’s theorem and extract a subsequence (we omit relabeling), which strongly converges in \(C([0, T]; \mathbb{R}^1)\). Let \(x\) be its limit, which evidently fulfills the periodic condition. Then (15) yields

\[
|\varepsilon x_t|_{L^2(0,T;\mathbb{R}^V)} \leq \sqrt{\varepsilon}C \to 0
\]

and (16) leads to

\[
x_t' \rightharpoonup x' \text{ weakly in } L^2(0,T;\mathbb{R}^V).
\]

Moreover, (18) implies that \(\{y_t\}_{\varepsilon>0}\) also possesses a subsequence, which weakly converges in \(L^2(0,T;\mathbb{R}^V)\). Thanks to the maximal monotonicity of \(L_{G,p}\), its limit \(y \in L^2(0,T;\mathbb{R}^V)\) satisfies \(y \in L_{G,p}(x)\), whence it follows Theorem 3.5. \(\square\)
Although the difference of two solutions can be hardly estimated (see Remark 2.7), we can show the uniqueness of periodic solution by virtue of Theorem 5 in Haraux\textsuperscript{23}:

**Theorem 3.6.** Let $x_1, x_2$ be two solutions to (P) with the same given $h$, then there exists some constant $\gamma \in \mathbb{R}^V$ such that $x_1 = x_2 + \gamma$.

**Remark 3.7.** We can easily see that if $x_1$ is a solution to (P), then $x_2 := x_1 + \phi_{c_1, c_2, \ldots, c_k}$ also satisfies (P) for any $c_1, c_2, \ldots, c_k \in \mathbb{R}$ and obtain $f(x_1(t)) = f(x_2(t))$ by (10). Still, $\gamma = \phi_{c_1, c_2, \ldots, c_k}$ dose not necessarily hold in Theorem 3.6. For instance, let $\alpha > 0$, $\beta \geq 0$ and

$$h(t) = \begin{pmatrix} 2\alpha \exp \left( 2 \left( t - \frac{T}{2} \right) \right) + 2\beta \\ 0 \\ -2\alpha \exp \left( 2 \left( t - \frac{T}{2} \right) \right) - 2\beta \end{pmatrix},$$

then

$$x(t) = \begin{pmatrix} \alpha \cosh \left( 2 \left( t - \frac{T}{2} \right) \right) + \beta \\ \frac{a}{b} \\ -\alpha \cosh \left( 2 \left( t - \frac{T}{2} \right) \right) - \beta \end{pmatrix}$$

becomes a periodic solution to (P) with $\#V = 4$, $E = \{V\}$, $w \equiv 1$, and $p = 2$ for arbitrary fixed $a, b \in (-\alpha - \beta, \alpha + \beta)$.

### 4 CONCLUSION

In this article, we study the hypergraph $p$-Laplacian from the viewpoint of nonlinear analysis and find the lack of coerciveness and the Poincaré–Wirtinger type inequality for this operator. We can see some validity of these tools in the treatment of the Cauchy problem and time-periodic problem of the evolution equation governed by the hypergraph $p$-Laplacian.

Interestingly, the multiplicity of $L_{G,p}$ implies that the ODE describes the diffusion of “heat” from the vertex with maximum to minimum belonging to the same hyperedge and the vertices with middle value halt until the maximum or minimum touches. This property might suggest a new PDE model describing competition of two groups. Namely, we can expect hypergraph Laplacian in reaction–diffusion system describes the effect of aid/replenishment/assistance to injured/suffering members (vertices) from others in each group (hyperedge).

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### CONFLICT OF INTEREST

The authors declare that there are no conflicts of interests in this work.

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