Matching Impatient and Heterogeneous Demand and Supply

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Abstract

Service platforms must determine rules for matching heterogeneous demand (customers) and supply (workers) that arrive randomly over time and may be lost if forced to wait too long for a match. Our objective is to maximize the cumulative value of matches, minus costs incurred when demand and supply wait. We develop a fluid model, that approximates the evolution of the stochastic model, and captures explicitly the nonlinear dependence between the amount of demand and supply waiting and the distribution of their patience times. The fluid model invariant states approximate the steady-state mean queue-lengths in the stochastic system, and, therefore, can be used to develop an optimization problem whose optimal solution provides matching rates between demand and supply types that are asymptotically optimal (on fluid scale, as demand and supply rates grow large). We propose a discrete review matching policy that asymptotically achieves the optimal matching rates. We further show that when the aforementioned matching optimization problem has an optimal extreme point solution, which occurs when the patience time distributions have increasing hazard rate functions, a state-independent priority policy, that ranks the edges on the bipartite graph connecting demand and supply, is asymptotically optimal. A key insight from this analysis is that the ranking critically depends on the patience time distributions, and may be different for different distributions even if they have the same mean, demonstrating that models assuming, e.g., exponential patience times for tractability, may lack robustness. Finally, we observe that when holding costs are zero, a discrete review policy, that does not require knowledge of inter-arrival and patience time distributions, is asymptotically optimal.

Keywords: matching; two-sided platforms; bipartite graph; discrete review policy; high-volume setting; fluid model; reneging

1 Introduction

Service platforms exist to match demand (customers) and supply (workers); see, for example, Hu (2019) for a broader perspective on the intermediary role of platforms in the sharing economy. The challenge in operating these platforms is that demand and supply often come from heterogeneous customers and workers that arrive randomly over time, and do not like to wait. Then, there is a trade-off between making valuable matches quickly, and waiting in order to enable better matches. Our goal is to develop matching policies that optimize a long-run objective function consisting of the total value of matches made minus holding costs incurred when demand and supply wait.

The dislike of waiting can lead to demand and supply abandoning before being matched, e.g., as occurs in a ride hailing application, in which customers must be matched to drivers Wang et al.
(a) The network.  
(b) Exponential patience times.  
(c) Uniform patience times.

Figure 1: An asymptotically optimal matching policy for a network with one supply node and two demand nodes, having matching values $v_{11} = v_{21} = 1$ (i.e., matching a supply unit with a demand unit from either node gives a value of one), identical patience time distributions for all types having mean equal to one, holding costs per job per unit time $c_1^D = 4$, $c_1^S = 1$, and Poisson arrivals with rates per unit time $\lambda_1 = \mu_1 = 1$.

Moreover, the willingness of demand and supply to wait can change over time, with some people becoming more patient (for example, as a result of having already paid a waiting cost), and some becoming less patient (for example, as a result of becoming more and more frustrated while waiting). In a transplant application setting, where donors must be matched with needy patients, the survival time of a patient depends on the history, and not only on the current state of health Bertsimas et al. (2019). Moreover, in a call center setting, empirical evidence does not usually support using an exponential distribution to fit customer patience times Brown et al. (2005), Mandelbaum and Zeltyn (2013). As a result, using an exponential distribution to model demand and supply patience times may be too restrictive. Still, the exponential distribution is more analytically tractable, and, if optimal or near-optimal matching policies are robust to distributional characteristics, then the exponential distribution is a good modeling choice.

Figure 1 shows that the choice of distribution used to model demand and supply patience times can change the asymptotically optimal matching policy (where the limit regime is one in which the arrival rates become large). For the simple matching network in Figure 1a, an asymptotically optimal matching policy is a static priority (so state-independent) policy that makes matches at (appropriately chosen) discrete review time points, and does nothing at all other times. At each discrete review time point, either matches on the edge (1,1) are prioritized and the leftover supply (if any) at the supply node is used for making matches on the edge (2,1) or vice versa. Figures 1b and 1c show that the same parameters lead to different priority orderings for the exponential and uniform patience time distributions. As a result, we are motivated to develop methodology that allows us to analyze as general a class of patience time distributions as possible; in particular, the only assumptions we require are that the distributions have a density and finite mean.

The matching model we consider generalizes that shown in Figure 1 to allow for an arbitrary number of demand and supply types, each with a possibly different patience time distribution and arrival rate. The matching values between demand and supply types are the edge values on the underlying bipartite graph, which could be positive or zero. Since different edges have different values, some matches are preferable to others. Since different demand and supply types have different holding costs and patience time distributions, waiting is more costly for some types than others. Our objective captures the trade-off between these opposing priorities by incorporating both matching values and holding costs.
The generality of the patience times considered means that an exact analysis is intractable, because the state space required to have a Markovian description of the model is very complex. Our approach is to develop a deterministic fluid model to approximate the evolution of the stochastic model. The invariant states of the fluid model (that is, the fixed points of the fluid model equations) approximate the steady-state mean amount of demand and supply waiting to be matched, and depend nonlinearly on the patience time distributions. We use the aforementioned invariant states to define an optimization problem, termed the matching problem (MP), whose optimal solution gives matching rates between demand and supply types that are asymptotically optimal, as arrival rates grow large. The matching policies we propose for the stochastic model are discrete review policies that are motivated by MP optimal solutions. A discrete review policy makes matches at given discrete review time points, and does nothing at all other times. The time between review points must be small enough to prevent both high holding costs and the loss of demand and supply waiting to be matched, but also large enough to allow enough arrivals to be able to make high value matches.

1.1 Contributions of this paper

We contribute a number of theoretical results and practical insights for matching on service platforms. First, the MP provides an asymptotic upper bound on the objective function value as arrival rates become large (Theorem 3). We then develop a discrete review matching policy whose matching rates mimic an MP optimal solution (Theorem 4), and is asymptotically optimal as arrival rates become large. As a consequence, because an MP optimal solution depends on the patience distributions, so does the proposed matching policy. While this policy is applicable for general patience time distributions, our analysis provides further insight for the important special case when there exists an optimal extreme point solution to the MP (this is ensured when the patience time hazard rates are increasing, i.e., supply and demand become more impatient the longer they wait), which we discuss next.

When the patience time hazard rate functions are increasing, the MP becomes a convex maximization problem, and so has an optimal extreme point solution (Theorem 1). In this case, we propose a discrete review policy that is based on a static ranking of the edges, and which makes matches by prioritizing the use of demand and supply according to that ranking. We show that this policy also achieves the matching rates given by the MP (Theorem 5) and is asymptotically optimal as arrival rates become large. The static ranking depends on the patience time distribution, and can be different for different distributions with the same mean, as observed earlier in Figure 1. This demonstrates that while service platforms may focus on simple matching policies (i.e., priority ranking), they should carefully consider their users’ patience time behavior to calibrate these policies appropriately.

In the case that holding costs are zero, we observe that an MP optimal solution does not depend on the patience time distribution. Then, a discrete review policy based on a linear problem (LP), that does not need to know information about the arrival rates or the patience time distribution, is asymptotically optimal as arrival rates become large (Theorem 6 and Proposition 4).

The MP is based on the invariant states of a fluid model (Definition 2), as mentioned earlier. We show that the fluid model starting from any initial state converges to an invariant state as time becomes large (Theorem 2), which requires characterizing all the invariant states (Proposition 3). Our convergence proof is non-trivial when viewed in light of the fact that such a result for the fluid model approximating the single class many-server queue with reneging was only shown recently.
in Atar et al. (2021) (despite being worked on for many years), because dealing with the measure
tracking the age-in-service is complicated. In our analysis, we observe that the aforementioned
difficulty is eliminated in matching systems with reneging, allowing a complete convergence proof
for this general setting.

The remainder of the paper is organized as follows. We end Section 1 by reviewing related
literature. In Section 2, we provide a detailed model description. In Section 3, we propose our
matching policies. A fluid model is presented in Section 4. In particular, we present asymptotic
approximations for the stationary mean queue-lengths, and we show the impact of the patience
time distributions. In Section 5, we introduce our high-volume setting and we study the asymptotic
behavior of our proposed matching policies. In Section 6, we study a platform without holding costs
and we propose a matching policy that does not require the knowledge of the system parameters.
Concluding remarks are contained in Section 7. All the proofs are gathered in Appendices A–C.

1.2 Literature review

We focus on on-demand service platforms that aim to facilitate matching. We refer the reader to
Benjaafar and Hu (2020), Chen et al. (2020), and Hu (2020) for excellent higher-level perspectives
on how such platforms fit into the sharing economy and to Hu (2019) for a survey of recent sharing-
economy research in operations management. Three important research questions for such platforms
identified in Benjaafar and Hu (2020) are how to price services, pay workers, and match requests.
These decisions are ideally made jointly; however, because the joint problem is difficult, the questions
are often attacked separately. In this paper, we focus on the matching question.

Our basic matching model is a bipartite graph with demand on one side and supply on the
other side. There is a long history of studying two-sided matching problems described by bipartite
graphs, beginning with the stable matching problem introduced in the groundbreaking work of Gale
and Shapley (1962) and continuing to this day; see Abdulkadiroglu and Sönmez (2013) and Roth
and Sotomayor (1990) for later surveys and Ashlagi et al. (2020) for more recent work. In the
aforementioned literature, much attention is paid to eliciting agent preferences because the outcomes
of the matching decisions, made at one prearranged point in time, can be life-changing events (for
example, the matches between medical schools and potential residents). In contrast, many platform
matching applications are not life-changing events, so there is less need to focus on eliciting agent
preference. Moreover, supply and demand often arrive randomly and continuously over time, and
must be matched dynamically over time, as the arrivals occur (as opposed to at one prearranged
point in time).

Recent work has used dynamic two-sided matching models in the context of organ allocation Ding
et al. (2018), Khademi and Liu (2020), online dating and labor markets Arnosti et al. (2020), Kanoria
and Saban (2020), and ridesharing Banerjee et al. (2020), Özkan and Ward (2020). Joint pricing
and matching decisions are recently explored Özkan (2020), Varma et al. (2019, 2020) while learning
agent types and payoffs are considered by Hsu et al. (2021), Johari et al. (2019). Simultaneously
and motivated by the aforementioned applications, there have been many works that begin with
a more loosely motivated modeling abstraction, as ours does Akbarpour et al. (2020), Arnosti
(2020), Baccara et al. (2015), Blanchet et al. (2020), Büke and Chen (2017), Hu and Zhou (2018),
Kerimov et al. (2021), Leshno (2020). The aforementioned works focus on a range of issues; however,
most of them assume that agents’ waiting times are either infinite, deterministic, or exponentially
distributed. Allowing for more general willingness-to-wait distributions is important because in
practice the time an agent is willing to wait to be matched may depend on how long that agent has
already waited. The difficulty is that tracking how the system evolves over time requires tracking
the remaining time each agent present in the system will continue to wait to be matched, resulting
in an infinite-dimensional state space. We do not have stability issues (studied in Jonckheere et al.
2020, Moyal and Perry 2017) because agents’ waiting times are finite.

Taking an approach similar in spirit to how stochastic processing networks are controlled in the
queueing literature developed in Harrison (1996, 2006), Maglaras (1999, 2000), we use a high-volume
asymptotic regime to prove that our matching policies are asymptotically optimal. In particular,
we begin by solving a static matching problem and we focus on fluid-scale asymptotic optimality
results. Adopting an approach reminiscent of the one Gurvich and Ward (2014) use in a dynamic
multipartite matching model that assumes agents will wait forever to be matched, we use a discrete
review policy to balance the trade-off between having agents wait long enough to build up matching
flexibility but not so long that many will leave.

Our priority order matching policy is similar in spirit to a greedy policy studied in Kerimov
et al. (2021) for a matching problem without reneging. We show that our policy is effective for
general reneging distributions, and that the priority ordering critically depends on the reneging
distribution. We also do not require a unique non-degenerate solution of our fluid problem (i.e., the
“general position” condition of Kerimov et al. 2021), and are able to show that our priority ordering
is asymptotically optimal for any optimal extreme point solution.

We consider an objective function that involves the queue-lengths and is an infinite horizon
objective. As a result, we need to consider the steady-state performance of the system. This is
the focus of the infinite bipartite matching queueing models considered for example in Adan et al.
(2018), Adan and Weiss (2012, 2014), Afeche et al. (2019), Bušić et al. (2013), Caldentey et al.
(2009), Diamant and Baron (2019), Fazel-Zarandi and Kaplan (2018).

2 Model description

We begin by giving a high-level description of the stochastic matching model in Section 2.1 where we
include the primitive inputs to our model, the objective, and the basic evolution equations. Because
of the generally distributed patience times, a detailed formulation of the state space is needed to
understand the evolution of the system and study its performance. This is done in Section 2.2. By
convention, all vectors are denoted by bold letters and we let \( \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \).

2.1 The matching model

**Primitive inputs:** The set of demand nodes \( \mathcal{J} := \{1, \ldots, J\} \) (representing customer types) and
supply nodes (representing worker types) \( \mathcal{K} := \{1, \ldots, K\} \) form a bipartite graph, as shown in
Figure 2. The value of matching demand type \( j \in \mathcal{J} \) and supply type \( k \in \mathcal{K} \) is \( v_{jk} \geq 0 \). The holding
costs \( c^D_j \geq 0 \) and \( c^S_k \geq 0 \) are incurred for each unit of time demand type \( j \in \mathcal{J} \) or supply type \( k \in \mathcal{K} \)
waits.

Customers and workers arrive according to renewal processes, denoted by \( D_j(\cdot), j \in \mathcal{J} \) and \( S_k(\cdot),
k \in \mathcal{K} \), having respective rates \( \lambda_j > 0 \) and \( \mu_k > 0 \). Upon arrival, each type \( j \in \mathcal{J} \) customer and
type \( k \in \mathcal{K} \) worker independently samples their patience time from respective distributions \( G_j^D(\cdot) \)
and \( G_k^S(\cdot) \) having support on \([0, H_j^D]\) and \([0, H_k^S]\) for some \( H_j^D \in [0, \infty] \) and \( H_k^S \in [0, \infty] \), which
represents the maximum amount of time they are willing to wait to be matched. Customers and
workers that wait longer than their patience time leave the system without being matched. We
assume the patience times are absolutely continuous random variables with density functions $g^D_j(\cdot)$ and $g^S_k(\cdot)$ that have finite means $1/\theta^D_j > 0$ and $1/\theta^S_k > 0$.

**Objective and admissible policies:** The objective is to maximize the long-run average value of matches made, minus holding costs incurred for waiting. A matching policy is a $J \times K$ dimensional stochastic process $M(\cdot) := \{M_{jk}(\cdot), j \in J, k \in K\}$ that specifies the cumulative number of matches made between types $j \in J$ and $k \in K$ in the time interval $[0, t]$, denoted by $M_{jk}(t)$, for each $t \geq 0$, under the assumption that matches are made first-come-first-served (FCFS) within each node. Then, if $Q_j(t)$ represents the number of type $j \in J$ customers waiting and $I_k(t)$ represents the number of type $k \in K$ workers waiting at time $t \geq 0$, we want to maximize

$$V_M := \liminf_{t \to \infty} \frac{1}{t} V_M(t),$$

where

$$V_M(t) := \sum_{j \in J, k \in K} v_{jk} dM_{jk}(t) - \sum_{j \in J} \int_0^t c^D_j Q_j(s) ds - \sum_{k \in K} \int_0^t c^S_k I_k(s) ds.$$

The objective (1) captures the trade-off between making matches quickly and waiting for better matches. The difficulty when deciding whether or not to incur holding costs for a short period of time in order to enable a potential future higher value match is that the matching policy does not know how much longer each customer and worker currently waiting will remain waiting without being matched.

The class of matching policies over which we optimize (1) (that is, the admissible matching policy class) satisfy some natural assumptions. First, if a match occurs, then it cannot be taken back. That is, a matching process is non-decreasing, which creates a trade off of making a match immediately or waiting for a possible higher value match. Second, customers and workers must be present in the system to be matched, which is equivalent to having non-negative customer and worker queue-lengths. Third, a matching policy cannot know the future. We summarize these properties in the following definition.
Definition 1. A matching policy given by a stochastic process $M(\cdot)$ is called an admissible policy if $M_{jk}(t)$ is nondecreasing and right-continuous with left limits everywhere, nonanticipating, $M_{jk}(t) \in \mathbb{Z}_+$, and $Q_j(\cdot), I_k(\cdot)$ are almost surely nonnegative for each $j \in \mathbb{J}, k \in \mathbb{K}$.

A rigorous mathematical specification of the nonanticipating property requires a precise definition of the system state and system evolution equations, and we do this in Appendix B; see Definition 5.

Queue evolution equations: For a given matching policy, $M(\cdot)$, the demand and supply queue-lengths at time $t \geq 0$ are uniquely determined by

$$Q_j(t) := Q_j(0) + D_j(t) - \sum_{k=1}^{K} M_{jk}(t) \geq 0 \quad (2)$$

and

$$I_k(t) := I_k(0) + S_k(t) - \sum_{j=1}^{J} M_{jk}(t) \geq 0, \quad (3)$$

for all $j \in \mathbb{J}$ and $k \in \mathbb{K}$, where $R^D_j(\cdot)$ and $R^S_k(\cdot)$ denote the cumulative number of type $j \in \mathbb{J}$ customers and type $k \in \mathbb{K}$ workers that left the system without being matched in $[0, t]$ and will be formally defined in Section 2.2. We refer to $R^D(\cdot)$ and $R^S(\cdot)$ as the reneging processes, because in the queueing literature leaving the system before being served is commonly known as reneging. The equations (2) and (3) are balance equations that say that the number of type $j \in \mathbb{J}$ customers and type $k \in \mathbb{K}$ workers waiting to be matched at time $t \geq 0$ equals those present initially plus the cumulative arrivals minus the cumulative reneging minus the cumulative matches.

2.2 State space and reneging processes construction

Here, we present a detailed state space that will be needed to see the connection with the fluid model in Section 4 and in the proofs. However, Section 3 can be read without the information of the current section.

For $H \in [0, \infty)$, let $\mathcal{M}(0, H)$ denote the set of finite, nonnegative Borel measures on $[0, H)$ endowed with the topology of weak convergence. A state of our model at time $t \geq 0$ can be described from $Q_j(t) \in \mathbb{Z}_+, j \in \mathbb{J}$, $I_k(t) \in \mathbb{Z}_+, k \in \mathbb{K}$, and the following measure-valued processes:

- the measures $\eta_j^D(t) \in \mathcal{M}(0, H^D_j)$, $j \in \mathbb{J}$, which stores the amount of time that has passed between each type $j$ customer’s arrival time up until that customer’s sampled patience time;

- the measures $\eta_k^S(t) \in \mathcal{M}(0, H^S_k)$, $k \in \mathbb{K}$, which stores the amount of time that has passed between each type $k$ worker’s arrival time up until that worker’s sampled patience time.

The potential measures $\eta_j^D(\cdot)$ and $\eta_k^S(\cdot)$ count the number of potential customers and workers in the queue, because they do not account for whether or not the customers and workers have been matched. The fact that the measures $\eta_j^D(t)$ and $\eta_k^S(t)$, $j \in \mathbb{J}$ and $k \in \mathbb{K}$, are independent of the scheduling policy is helpful for analytic tractability. The definition of the potential measures involves additional notation, and it is given in Appendix B.

Since customers and workers are matched FCFS within their type, the head-of-line (HIL) customer or worker in a given type divides those that are actually in queue (because their waiting time
is less than the HL customer or worker) from those that have already been matched (because their waiting time is longer than the HL customer or worker). The waiting times of the type \( j \in \mathbb{J} \) HL customer and \( k \in \mathbb{K} \) HL worker are

\[
\chi^D_j(t) := \inf \left\{ x \in \mathbb{R}^+ : \int_{[0,x]} \eta^D_j(t)(dy) \geq Q_j(t) \right\}
\]

(4)

and

\[
\chi^S_k(t) := \inf \left\{ x \in \mathbb{R}^+ : \int_{[0,x]} \eta^S_k(t)(dy) \geq I_k(t) \right\},
\]

(5)

which are zero if no one is waiting. Then, the queue-lengths can be expressed in terms of the potential queue measures, for each \( t \geq 0 \), as

\[
Q_j(t) = \int_{[0,\chi^D_j(t)]} \eta^D_j(t)(dx) \quad \text{and} \quad I_k(t) = \int_{[0,\chi^S_k(t)]} \eta^S_k(t)(dx).
\]

(6)

We now move to the construction of the reneging processes. In order to define them, we require notation for customer and worker arrival times, patience times, and matching times for those that are matched. The arrival time of the \( l \)th type \( j \in \mathbb{J} \) customer and the arrival time of the \( l \)th type \( k \in \mathbb{K} \) worker can be expressed as

\[
e^D_{jl} = \inf \{ t \geq 0 : D_j(t) \geq l \} \quad \text{and} \quad e^S_{kl} = \inf \{ t \geq 0 : S_k(t) \geq l \},
\]

for \( l \in \mathbb{N} \), where it is assumed that they have finite fifth moments. The last assumption can be reduced to the same as in (Ata and Kumar 2005, Inequality 4), i.e., \( (2 + 2\epsilon) \) finite moments for some \( \epsilon > 0 \). Here, we choose \( \epsilon = 3/2 \) for simplicity. We denote the patience time of the \( l \)th type \( j \) customer and the patience time of the \( l \)th type \( k \) worker by \( r^D_{jl} \) and \( r^S_{kl} \), respectively, for \( l \in \mathbb{N} \). The matching times of the \( l \)th type \( j \) customer and the \( l \)th type \( k \) worker are denoted by \( m^D_{jl} \in [e^D_{jl}, e^D_{jl} + r^D_{jl}] \) and \( m^S_{kl} \in [e^S_{kl}, e^S_{kl} + r^S_{kl}] \) for those that are matched, and are set to \( m^D_{jl} = \infty \) and \( m^S_{kl} = \infty \) for those that renege. The \( l \)th type \( j \in \mathbb{J} \) customer reneges in \([0, l]\) if his potential waiting time

\[
w^D_{jl}(t) := \min \left\{ [t - e^D_{jl}]^+, r^D_{jl} \right\}
\]

changes from linearly increasing to constant before their matching time \( m^D_{jl} \), and, similarly, the \( l \)th type \( k \in \mathbb{K} \) worker reneges in \([0, l]\) if his potential waiting time

\[
w^S_{kl}(t) := \min \left\{ [t - e^S_{kl}]^+, r^S_{kl} \right\}
\]

changes from linearly increasing to constant before their matching time \( m^S_{kl} \). The abandonment of a customer can be expressed with the help of the following events: \( \frac{dw^D_{jl}}{dt}(s-) > 0 \) which means that the \( l \)th type \( j \) customer is still in the system just before time \( s \) and the event \( \frac{dw^S_{kl}}{dt}(s+) = 0 \) which means that he/she abandons the system exactly after time \( s \). Analogous events hold for the workers. Then, the indicator function

\[
1\left\{ \text{there exists } s \in [0, l] \text{ such that } s \leq m^D_{jl} \text{ and } \frac{dw^D_{jl}}{dt}(s-)>0 \text{ and } \frac{dw^D_{jl}}{dt}(s+)=0 \right\}
\]
determines if the $l$th type $j \in J$ customer reneges in $\left[0, t\right]$, and, similarly, the indicator function

$$1\left\{ \text{there exists } s \in [0,t] \text{ such that } s \leq m_{kl}^j \text{ and } \frac{dw_k^S(s-)}{dt} > 0 \text{ and } \frac{dw_k^S(s+)}{dt} = 0 \right\}$$

determines if the $l$th type $k \in K$ worker reneges in $\left[0, t\right]$. We have defined the potential waiting times only for the newly arriving customers and workers (i.e., $l \geq 1$). The definition of the potential waiting times for the initial number of customers and workers (i.e., $l < 1$) requires a bit more effort and we do this in Appendix B. Now, the cumulative number of type $j \in J$ customers reneging in $\left[0, t\right]$ is

$$R^D_j(t) := \sum_{i=1}^{\left\lfloor \frac{t}{l} \right\rfloor} Q^j_{il-} + D^j_{il} - D^j_{(i-1)l} - R^D_j((i-1)l) + R^D_j(il)$$

and the cumulative number of type $k \in K$ workers reneging in $\left[0, t\right]$ is

$$R^S_k(t) := \sum_{i=1}^{\left\lfloor \frac{t}{l} \right\rfloor} I^k_{il-} + S^k_{il} - S^k_{(i-1)l} - R^S_k((i-1)l) + R^S_k(il).$$

### 3 Proposed matching policies

In this section, we develop matching policies with the goal of maximizing our objective (1). To do so, we first define an optimization problem that determines the optimal matching rates (Section 3.1) and we refer to it as the matching problem (MP). Then, we propose a discrete review matching policy that (asymptotically) achieves these rates (Section 3.2) and we call this policy the matching-rate-based policy. When the existence of an optimal extreme point solution is ensured, we are able to propose a discrete review priority-ordering policy that prioritize the edges and also (asymptotically) achieves the optimal matching rates (Section 3.3).

A discrete review policy decides on matches at review time points and does nothing at all other times (see, for example, Gurvich and Ward 2014). Longer review periods allow more flexibility in making matches but risk losing impatient customers and workers. Shorter review periods prevent customer/worker loss but may not have sufficient numbers of customers/workers of each type to ensure the most valuable matches can be made.

Let $t > 0$. We let $l, 2l, 3l, \ldots$ be the discrete review time points, where $l \in (0, t)$ is the review period length. Moreover, let for $j \in J$, $k \in K$, and $i \in \{1, \ldots, \left\lfloor t/l \right\rfloor\}$,

$$Q^j_{il-} := Q^j_{i-1,l} + D^j_{il} - D^j_{(i-1)l} - R^D_j((i-1)l) + R^D_j(il)$$

and

$$I^k_{il-} := I^k_{i-1,l} + S^k_{il} - S^k_{(i-1)l} - R^S_k((i-1)l) + R^S_k(il).$$

The quantities $Q^j_{il-}$ and $I^k_{il-}$ are the maximum type $j$ demand and type $k$ supply that can be matched at time $il$. These follow by (2) and (3) and the fact that we make matches at discrete review periods and no matches are made between $(i-1)l$ and $il.$
3.1 A matching problem

Suppose we ignore the discrete and stochastic nature of customer and worker arrivals, and assume that these flow at their long-run average rates. Then, an upper bound on the long-run average matching value in (1) follows by solving an optimization problem to find the optimal instantaneous matching values, which we refer to as the matching problem. For \( m = (m_{jk} : j \in J \text{ and } k \in K) \) that denotes the instantaneous matching rate, the MP can be defined through functions \( q^*_j(m) \) and \( i^*_k(m) \) (defined in detail below), that approximate the steady-state mean queue-lengths, as follows:

\[
\begin{align*}
\max \sum_{j \in J, k \in K} v_{jk}m_{jk} & - \sum_{j \in J} c^D_j q^*_j(m) - \sum_{k \in K} c^S_k i^*_k(m) \\
\text{s.t.} \quad \sum_{j \in J} m_{jk} & \leq \mu_k, \quad k \in K, \\
\sum_{k \in K} m_{jk} & \leq \lambda_j, \quad j \in J, \\
&m_{jk} \geq 0, \quad j \in J, \quad k \in K. 
\end{align*}
\]

The objective function in (9) is the instantaneous version of the long-run average matching value in (1) and the constraints in (9) prevent us from matching more demand or supply than is available.

An optimal solution to the MP (9), \( m^* = (m^*_{jk} : j \in J \text{ and } k \in K) \), can be interpreted as the optimal instantaneous rate of matches between demand \( j \in J \) and supply \( k \in K \). An admissible matching policy that satisfies Definition 1 should maximize the long-run average matching value in (1) if its associated matching rates equal \( m^* \).

Unfortunately, having closed form expressions for the exact steady-state mean queue-lengths appears intractable. Fortunately, when demand and supply arrival rates become large, we can approximate the steady-state mean queue-lengths by developing a fluid model to approximate the evolution of the stochastic model defined in Section 2, and finding its invariant states, which we do in Section 4, in order to develop the approximating functions

\[
\begin{align*}
q^*_j(m) &= \begin{cases} \lambda_j, & \text{if } \sum_{k \in K} m_{jk} = 0, \\ \frac{\lambda_j}{\mu_j} G^D_{e,j}((G^D_{e,j})^{-1}
(1 - \sum_{k \in K} m_{jk})), & \text{otherwise} \end{cases}, \\
i^*_k(m) &= \begin{cases} \mu_k, & \text{if } \sum_{j \in J} m_{jk} = 0, \\ \frac{\mu_k}{\lambda_k} G^S_{e,k}((G^S_{e,k})^{-1}(1 - \sum_{j \in J} m_{jk})), & \text{otherwise} \end{cases}
\end{align*}
\]

where

\[
G^D_{e,j}(x) = \int_0^x \theta^D_j (1 - G^D_{e,j}(u))du \quad \text{and} \quad G^S_{e,k}(x) = \int_0^x \theta^S_k (1 - G^S_{e,k}(u))du \quad \text{for } j \in J, k \in K, x \in \mathbb{R}_+.
\]

are the excess life distributions of the patience times.

An optimal solution to the MP (9) provides a good approximation of the optimal instantaneous matching values if \( q^*(m) \) and \( i^*(m) \) in (10) and (11) provide good approximations to the mean steady-state queue-lengths. Figure 3 provides supporting evidence for this claim in a network with one demand and one supply node that makes every possible match.
Gamma distributed patience times with unit mean and variance $1/x$, $x = 0.7, 2, 5$.  
Exponential distributed patience times with rate $\theta = 1, 1.5, 2$.  

Figure 3: The solid lines represent the average queue-lengths and the dotted/triangle marker lines represent the invariant queue-lengths. Figure 3a in based on simulation and Figure 3b is based on exact calculations (see Appendix C). The parameters used are $\lambda = 1$ per unit time, $t = 100$, and the patience time distributions for both nodes is the same.

In general the MP is non-convex because the approximations of the queue-lengths depend on the patience time distributions through a nonlinear relationship. The only time the relationship is linear is when the patience times follow an exponential distribution, namely

$$q^*_j(m) = \frac{1}{\theta_j^D} \left( \lambda_j - \sum_{k \in K} m_{jk} \right)$$
$$i^*_k(m) = \frac{1}{\theta_k^S} \left( \mu_k - \sum_{j \in J} m_{jk} \right).$$

The last formulas have an intuitive explanation: the expressions in the parentheses represent the expected demand (supply) that is not matched, and so to find the expected demand (supply) waiting to be matched, one needs to multiply by the mean patience time. The relationship is quadratic when the patience times are uniformly distributed in $[0, \frac{2}{\theta_j^D}]$ and $[0, \frac{2}{\theta_k^S}]$, and is

$$q^*_j(m) = \frac{\lambda_j}{\theta_j^D} \left( 1 - \left( \sum_{k \in K} m_{jk} \right) \frac{2^2}{\lambda_j^2} \right)$$
$$i^*_k(m) = \frac{\mu_k}{\theta_k^S} \left( 1 - \left( \sum_{j \in J} m_{jk} \right) \frac{2^2}{\mu_k} \right).$$

In some cases, they cannot even be written in a closed form, for instance when the patience times follow a gamma distribution. Despite (9) being non-convex in general, we are able to characterize the structure of the objective function of (9) for a rather wide class of patience time distributions.

**Theorem 1.** If the hazard rate functions of the patience time distributions are (strictly) increasing (decreasing), then $q^*_j(m)$ and $i^*_k(m)$ are (strictly) concave (convex) functions of the vector of matching rates $m$, for all $j \in J$ and $k \in K$.

The following corollary is a direct consequence of the last theorem.

**Corollary 1.** If the hazard rate functions of the patience time distributions are increasing (decreasing), then the objective function of (9) is a convex (concave) function.

Having introduced the MP and studied its properties, we are now ready to introduce our proposed matching policies.
3.2 A matching-rate-based policy

Here, we define a policy that can mimic the optimal matching rates given by (9) as closely as possible, subject to the available stochastic demand and supply. Even though we shall show later that this policy is asymptotically optimal for an optimal solution to (9), we define it in a bit more general framework, i.e., for any feasible point. To this end, let \( m \) be a feasible point of (9). Then, to mimic the matching rates defined by \( m \), let the amount of type \( j \in \mathbb{J} \) demand we match with type \( k \in \mathbb{K} \) supply in review period \( i \in \{1, \ldots, \lfloor t/l \rfloor \} \) be

\[ M_{ijk}^r := \lfloor m_{jk} \min \left( l, \frac{Q_j(i(l-i))}{\lambda_j}, \frac{I_k(i(l-i))}{\mu_k} \right) \rfloor \geq 0, \tag{12} \]

which implies the cumulative number of matches made for \( j \in \mathbb{J}, k \in \mathbb{K}, \) and \( t \geq 0 \) is

\[ M_{jk}^r(t) := \sum_{i=1}^{\lfloor t/l \rfloor} M_{ijk}^r. \tag{13} \]

The matching-rate-based policy is defined for each feasible point of (9) and for all patience time distributions, and tries to achieve the feasible matching rates given by \( m \). The targeted rate during a review period between demand \( j \in \mathbb{J} \) and supply \( k \in \mathbb{K} \) is \( m_{jk}l \). Further, the policy takes into account the fraction of available demand and supply matched to ensure admissibility as the following result states.

**Proposition 1.** The proposed matching policy given by (12) and (13) is admissible for any feasible point \( m \) to (9).

The matching-rate-based policy tries to mimic the matching rates at each edge and discrete review point by proportionally splitting the available demand (supply) at a node between the available supply (demand) at other nodes in the same proportions as the feasible matching rates in \( m \). While we show that this is an effective policy for achieving the optimal matching rates, it does not give any insight about the order in which edges should be prioritized for matching. The issue can be illustrated in a simple matching model shown in Figure 4 for a given optimal MP solution, \( m^* \), where the matching-rate-based policy can easily be implemented, but does not distinguish the priority edges. This is in contrast to Figure 1, where a simple static ordering prioritizes the edges. Prioritizing edges can be useful for platforms wishing to implement simple rules for matching, i.e., each demand (supply) type has a fixed ranking of the supply (demand) types which it checks for availability to match at each discrete review point. Such a static priority ordering is appealing because the ordering is state-independent, and easy to describe to a practitioner. Thus, we would like to understand when the matching-rate-based policy reduces to a static priority ordering of the edges. In the next section, we show how to interpret an optimal extreme point solution as a static priority ordering of the edges (and use the system in Figure 4 to illustrate how to construct the priority order).
3.3 A priority-ordering policy

In this section, we propose a priority-ordering matching policy, which defines an order of edges over which matches are made in a greedy fashion at each discrete review point using available demand and supply. We will show that, with a properly chosen priority-ordering, such a policy is able to imitate the matching rates determined by an optimal extreme point solution to the MP when there exists one. For example, the existence of an optimal extreme point solution is guaranteed when the patience time distributions have increasing hazard rates. In other words, if supply and demand become more impatient the longer they wait in queue, then a simple priority ordering is enough to balance the trade-off between matching values and holding costs. This result is intuitive from the perspective of the steady-state queue behavior as characterized by Theorem 1: with increasing hazard rates the steady-state queues are concave functions of the matching rates. Thus, when deciding between making a match on two adjacent edges (i.e., edges that share a supply node or a demand node), the steady-state objective of (9) is a convex function (Corollary 1) of these two matching variables (since the queue-lengths enter the objective negatively), and hence an optimal solution will prioritize one over the other.

To formalize this intuition, our proof shows that a properly constructed priority-ordering can implement the matching rates determined by an optimal extreme point solution to the MP. In this section, we construct such a priority-ordering and show that it is able to replicate an optimal extreme point solution to the MP. In the rest of this section, we assume the existence of an optimal extreme solution to the MP (9) which we denote by \( m^\star \).

Before constructing the priority-ordering, we need to establish a few properties of an optimal extreme point solution \( m^\star \). It will be helpful to define the following: we say a node \( j \in J \) (\( k \in K \)) is slack if \( \sum_{k \in K} m^\star_{jk} < \lambda_j (\sum_{j \in J} m^\star_{jk} < \mu_k) \), and tight if \( \sum_{k \in K} m^\star_{jk} = \lambda_j (\sum_{j \in D} m^\star_{jk} = \mu_k) \); let \( E_0 = \{(j,k) : m^\star_{jk} > 0\} \) denote the set of edges with positive matching rates, and let \( (\{J,K\}, E_0) \) denote the induced graph of \( m^\star \), i.e., the bipartite graph between node sets \( J \) and \( K \) with edges corresponding to positive matching rates associated with \( m^\star \). Moreover, a tree is a connected acyclic undirected graph and a forest is an acyclic undirected graph; see Gross et al. (2018). Then, we have the following properties of the induced graph of an optimal extreme point solution.

**Lemma 1.** Let \( m^\star \) be an optimal extreme point solution to (9). Then, the induced graph of \( m^\star \) has the following properties:

1. It has no cycles, i.e., it is a forest, or collection of trees;
2. Each distinct tree has at most one node with a slack constrain;
3. Any node with a slack constraint is connected only to nodes with tight constraints.

With Lemma 1 in hand we are now ready to describe the intuition for the priority-ordering we construct. Lemma 1 guarantees that the set of edges, $E_0$, on which $\mathbf{m}^*$ makes matches can be partitioned into disjoint trees, each of which have almost all their nodes (i.e., all but at most one) with a tight constraint (i.e., all their capacity is used up by $\mathbf{m}^*$). Intuitively, this allows us to construct the priority sets by iteratively following a path traversing the tight nodes of each tree, removing an edge at each iteration, and updating the remaining capacities by deducting the match that $\mathbf{m}^*$ made on that edge. The edges removed in the $h$th iteration become the $h$th highest priority in our ordering, and the process results in a smaller tree after each iteration so works through all edges in finite time. This process continues until it reaches the final priority set that contains all the edges with $m^*_{jk} = 0$. Before we move to the general algorithm that constructs the priority sets, we present an example.

**Example 1.** Consider a system with two demand nodes and three supply nodes, with arrival rates as depicted in Figure 4. The patience time distributions, values, and holding costs are such that an optimal extreme point solution to the MP for this system sets $m^*_{11} = m^*_{12} = m^*_{22} = 1$ and $m^*_{23} = 0.5$, as depicted in the figure, with all other edges having $m^*_{jk} = 0$. The edges with positive matching rate, i.e., $E_0$, are drawn with dotted lines in the figure, note that these edges form a tree. For this optimal extreme point solution, we construct the following priority sets: $\mathcal{P}_0(\mathbf{m}^*) = \{(1,1),(2,3)\}$, $\mathcal{P}_1(\mathbf{m}^*) = \{(1,2)\}$, $\mathcal{P}_2(\mathbf{m}^*) = \{(2,2)\}$, and $\mathcal{P}_3(\mathbf{m}^*) = \{(1,3),(2,1)\}$. Note that in the first iteration, the algorithm selects two edges with tight leaf nodes from the tree (i.e., $(1,1)$ and $(2,3)$), since they are disjoint, while in the second iteration the algorithm picks only one of two available edges with tight leaf nodes (i.e., $(1,2)$ and $(2,2)$), since they share the end point $k = 2$, leaving the second edge until the next iteration.

We formally present the construction of the priority sets in Algorithm 1, and here we provide a more detailed explanation of the intuition of the algorithm. The order we traverse the trees in $E_0$ is determined by the following observation. In each tree, at least one of the tight nodes must be a “leaf” of the tree, i.e., it is connected to only one edge in $E_0$ (or, equivalently only one element of $\mathbf{m}^*$ involving this node is positive). For an example, assume that $j$ is such a leaf node in $E_0$ with a tight constraint, i.e., $\sum_k m^*_{jk} = \lambda_j$, and $m^*_{jk'} > 0$ for some $k'$, while $m^*_{jk} = 0$ for all other $k \neq k'$. Then, together this implies that $m^*_{jk'} = \lambda_j$, i.e., there is only one edge, $(j,k')$, that uses up all of node $j$’s capacity. Our algorithm gives this edge highest priority, since greedily matching all available supply and demand along this edge will match $\min(\lambda_j, \mu_{k'}) = \lambda_j$ (since we must have had $\lambda_j \leq \mu_{k'}$ for the initial value of $m^*_{jk'}$ to be feasible), and this replicates the value of $m^*_{jk'} = \lambda_j$. Then, removing this edge and reducing the capacities by $m^*_{jk'}$, results in a smaller tree with the same properties as $E_0$, so we can repeat the process until we work through the whole tree. This is exactly what Algorithm 1 does, with the following adjustment to allow for a potentially fewer number of priority sets in the final construction: more than one leaf node with a tight constraint can be put in the same priority set in a given iteration as long as their associated edges are disjoint (i.e., they do not share any endpoints).

Let $H + 1$ denote the final value of the counter $h$ in Algorithm 1, so that $\mathcal{P}_H(\mathbf{m}^*)$ denotes the last priority set with edges $(j,k)$ such that $m^*_{jk} > 0$, and $\mathcal{P}_{H+1}(\mathbf{m}^*)$ denotes the final priority set with all edges $m^*_{jk} = 0$.

Note that Algorithm 1 returns non-overlapping priority sets and it terminates as the following result states.
Lemma 2. Algorithm 1 runs in finite time $O(|E_0|^2)$.

The sets $\mathcal{P}_h(m^*)$ form a partition of the edges of the bipartite graph. Let $\mathcal{Q}_h(m^*) = \bigcup_{l=0}^{h} \mathcal{P}_l(m^*)$ denote the set of all priority edges chosen up to iteration $h$. Given this partition, define the following solution to (9) recursively for $(j, k) \in \mathcal{P}_h(m^*)$ as

$$y^p_{jk}(m^*) = \min \left( \lambda_j - \sum_{k':(j,k') \in \mathcal{Q}_{h-1}(m^*)} y^p_{j'k'}(m^*), \mu_k - \sum_{j':(j',k) \in \mathcal{Q}_{h-1}(m^*)} y^p_{j'k}(m^*) \right),$$

where an empty sum is defined to be zero.

Proposition 2. For $m^*$ an optimal extreme point solution to (9), we have $y^p_{jk}(m^*) = m^*_{jk}$ for all $j \in \mathbb{J}$ and $k \in \mathbb{K}$.

We are ready now to define recursively the priority-ordering matching policy. The amount of type $j \in \mathbb{J}$ demand we match with type $k \in \mathbb{K}$ supply for $(j, k) \in \mathcal{P}_0$ at the end of review period $i \geq 1$ is

$$M^p_{ijk} := \min (Q_j(il-), I_k(il-)).$$

(14)

For $(j, k) \in \mathcal{P}_h$, $h = 1, \ldots, JK$ and $i \in \{1, \ldots, \lfloor t/l \rfloor \}$, define recursively

$$M^p_{ijk} := \min \left( Q_j(il-) - \sum_{k':(j,k') \in \mathcal{Q}_{h-1}(m^*)} M^p_{ijk'}, I_k(il-) - \sum_{j':(j',k) \in \mathcal{Q}_{h-1}(m^*)} M^p_{ij'k} \right).$$

(15)
The aggregate number of matches at time $t \geq 0$ is given by

$$M_{jk}^p(t) := \sum_{i=1}^{\lfloor t/l \rfloor} M_{ijk}^p.$$  \tag{16}

Note that the admissibility of this policy is straightforward since the queue-lengths cannot be negative by its definition. Further, the implementation of the priority-ordering policy does not require the knowledge of the optimal extreme point solution once the priority sets have been constructed. In other words, a computer program needs to know only the priority sets to be able to implement the aforementioned policy. Furthermore, the following remark states that our policy can be seen as an extension of the $c\mu/\theta$ rule that has been extensively studied.

**Remark 1 (Connection to the $c\mu/\theta$ rule.).** The priority ordering studied in this section can be connected to the well-known $c\mu/\theta$ rule studied for multiclass queueing systems where $\mu$ denotes the service rate; Atar et al. (2011), Smith (1956). To see this, observe that for a system with a single supply node and exponential patience time distributions, the objective function of the MP takes the form:

$$\sum_j a_j m_{j1}$$

where $a_j = v_{j1} + c_{jP}/\theta_{j} + c_{jS}/\theta_{j}$. Hence, its optimal solution is given by a priority rule that orders the weights $a_j$. These weights do not depend on the arrival rates, and are similar to that of the $c\mu/\theta$ rule but are modified to include the matching values and the fact that there is not service rate in our model. However, in the general case, the priority ordering cannot be written as a simple priority rule because of the dimension of the model and the general patience time distributions.

We have proposed matching policies that use the information of an optimal solution to the MP, which involves an approximation of the demand and supply queue-lengths. The study of the asymptotic performance of the aforementioned policies requires first to show a connection between (1), which involves the stochastic queue-lengths, and the objective of the MP. In other words, we first need to show how the approximation of the queue-lengths is derived before we are able to analyze the asymptotic performance of the policies. We do this by developing a fluid approximation of the demand and supply queue-lengths in Section 4, and, then, using the fluid approximation, we show that our proposed matching policies are asymptotically optimal in Section 5.

4 The fluid model and its convergence to invariant states

We develop a measure-valued fluid model in Section 4.1 that arises as a accumulation point of the state descriptor under an appropriate scaling, i.e., a fluid or law of large number scaling as the arrival rates grow large. In Section 4.2, we characterize the invariant states (10) and (11) of the fluid model that are fixed points of the fluid model equations, and are used to approximate the steady-state queue-lengths. We further show in a rigorous way that an invariant point arises as a steady-state limit of the fluid model equations. This is an important result and one of the keys to show asymptotic optimality of our proposed matching policies.

4.1 The fluid model

Let $H_D^j$, $H_S^k$ be the right edges of the support (i.e., the essential supremum) of the cumulative patience time distribution functions for any $j \in J$ and $k \in K$. For a measure $\nu \in \mathcal{M}[0,L)$ and a Borel measurable function $f$ define $(f, \nu) := \int_{[0,L)} f(x)\nu(dx)$. Further, define $\mathcal{Y} := \mathbb{R}_+^J \times \mathbb{R}_+^K \times$
\( (\times_{j=1}^{J} M(0, H^{D}_j)) \times (\times_{k=1}^{K} M(0, H^{S}_k)) \) and recall that \( h^{D}_j(\cdot) \), \( h^{S}_k(\cdot) \) are the hazard rate (or failure) functions of the demand and supply patience time distributions, respectively. Roughly speaking, a fluid model solution is represented by a vector \((\mathcal{Q}, \mathcal{I}, \mathbf{\eta}^{D}, \mathbf{\eta}^{S}) \in C(\mathbb{R}_+, \mathbb{Y})\). This is the analogue of the state descriptor introduced in Section 2.2. The first two functions represent the fluid queue-lengths and the measure-valued functions \( \mathbf{\eta}^{D}(\cdot), \mathbf{\eta}^{S}(\cdot) \) approximate the potential fluid queue measures at any time. Below, we present the conditions and evolution equations that allow us to give a rigorous definition of the fluid model.

Let \( \overline{X}^{D}_j(t) \) and \( \overline{X}^{S}_k(t) \) denote the fluid analogues to (4) and (5), i.e., the waiting time of the type \( j \in \mathbb{J} \) HL customer and type \( k \in \mathbb{K} \) HL worker at \( t \geq 0 \), respectively. These can be written as follows for any \( t \geq 0 \),

\[
\overline{X}^{D}_j(t) = \inf \left\{ x \in \mathbb{R}_+ : \int_{[0,x]} \pi^{D}_j(t)(dy) \geq Q_j(t) \right\}
\]

and

\[
\overline{X}^{S}_k(t) = \inf \left\{ x \in \mathbb{R}_+ : \int_{[0,x]} \pi^{S}_k(t)(dy) \geq I_k(t) \right\},
\]

where \( \inf \emptyset = \infty \). Further, assuming that the potential measures do not have atoms at zero, the fluid queue-lengths and the potential measures are connected through the following relations for \( t \geq 0 \),

\[
\int_{[0,\overline{X}^{D}_j(t)]} \pi^{D}_j(t)(dx) = Q_j(t) \leq \int_{[0,H^{D}_j]} \pi^{D}_j(t)(dx)
\]

and

\[
\int_{[0,\overline{X}^{S}_k(t)]} \pi^{S}_k(t)(dx) = I_k(t) \leq \int_{[0,H^{S}_k]} \pi^{S}_k(t)(dx),
\]

where these are similar relations to (6).

We now move to define the fluid reneging functions. Observe that \( h^{D}_j(x) \) with \( x \in [0, \overline{X}^{D}_j(t)] \) represents the probability that fluid of type \( j \) demand that has waited \( x \) time units reneges in the next time instant. Additionally, the measure \( \eta^{D}_j(u) \) encodes how much fluid has waited \( x \) time units until time \( u \geq 0 \). Thus, \( \int_{[0,\overline{X}^{D}_j(t)]} h^{D}_j(x) \eta^{D}_j(u)(dx) \) is the instantaneous reneging rate at time \( u \geq 0 \). In a similar way, \( \int_{[0,\overline{X}^{S}_k(t)]} h^{S}_k(x) \eta^{S}_k(u)(dx) \) is the instantaneous reneging rate of fluid of type \( k \) supply at time \( u \geq 0 \). To derive the cumulative reneging fluid at \( t \geq 0 \), one integrates the instantaneous reneging rates in time interval \([0,t]\); namely,

\[
\overline{R}^{D}_j(t) = \int_{0}^{t} \int_{0}^{\overline{X}^{D}_j(t)} h^{D}_j(x) \eta^{D}_j(u)(dx)du \quad \text{and} \quad \overline{R}^{S}_k(t) = \int_{0}^{t} \int_{0}^{\overline{X}^{S}_k(t)} h^{S}_k(x) \eta^{S}_k(u)(dx)du.
\]

The rigorous connection between the fluid reneging functions and (7) and (8) follows from the compensator function used to define an appropriate martingale, similar to (Kang and Ramanan 2010, Proposition 5.1). The relations in (19) are directly connected with the fluid potential measures that do not depend on the control policy, and hence they are more tractable to work with than (7) and (8). When the patience times are exponentially distributed with rates \( \theta^{D}_j \) and \( \theta^{S}_k \), we have that

\[
\overline{R}^{D}_j(t) = \int_{0}^{t} \theta^{D}_j Q_j(u)du \quad \text{and} \quad \overline{R}^{S}_k(t) = \int_{0}^{t} \theta^{S}_k I_k(u)du,
\]
where these last formulas agree with the known results in the literature, e.g., see Pang et al. (2007). The conditions

\[ \int_0^t \langle h_j^D, \eta_j^D(u) \rangle \, du < \infty \quad \text{and} \quad \int_0^t \langle h_j^S, \eta_j^S(u) \rangle \, du < \infty \]  

(20)

ensure that the amount of reneging fluid of demand and supply in any finite time interval is finite.

The evolution equations for the fluid queue-lengths are analogous to (2) and (3) in the stochastic system

\[ \bar{Q}_j(t) = \bar{Q}_j(0) + \lambda_j t - \bar{R}_j^D(t) - \sum_{k \in \mathbb{K}} \bar{M}_{jk}(t), \]  

(21)

\[ T_k(t) = T_k(0) + \mu_k t - \bar{R}_k^S(t) - \sum_{j \in \mathbb{J}} \bar{M}_{jk}(t), \]  

(22)

where the functions \( \bar{M}_{jk}(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+ \) are the fluid analogues of the matching processes.

**Definition 2.** Given \( \lambda \in \mathbb{R}^J \) and \( \mu \in \mathbb{R}^K \), we say that \((\bar{Q}, \bar{T}, \bar{\eta}_D, \bar{\eta}_S) \in C(\mathbb{R}_+, \mathbb{Y})\) is a **fluid model solution** if there exist nondecreasing and absolutely continuous functions \( \bar{M}_{jk}(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+ \) with \( \bar{M}_{jk}(0) = 0 \), such that (17)–(22) hold for all \( j \in \mathbb{J} \) and \( k \in \mathbb{K} \). Further, for any continuous and bounded function \( f \in C_b(\mathbb{R}_+, \mathbb{R}_+) \) the following evolution equations hold for any \( j \in \mathbb{J} \), \( k \in \mathbb{K} \), and \( t \geq 0 \),

\[ \langle f, \bar{\eta}_j^D(t) \rangle = \int_0^t f(x + u) \frac{1 - G_j^D(x + t)}{1 - G_j^D(x)} \bar{\eta}_j^D(0) \, dx + \lambda_j \int_0^t f(t - u)(1 - G_j^D(t - u)) \, du, \]  

(23)

\[ \langle f, \bar{\eta}_k^S(t) \rangle = \int_0^t f(x + u) \frac{1 - G_k^S(x + t)}{1 - G_k^S(x)} \bar{\eta}_k^S(0) \, dx + \mu_k \int_0^t f(t - u)(1 - G_k^S(t - u)) \, du. \]  

(24)

The first term of the evolution equations for measures \( \bar{\eta}_j^D(\cdot) \) and \( \bar{\eta}_k^S(\cdot) \) in the last definition tracks when the patience time of fluid demand/supply initially present in the system at time zero expires. The second term has an analogous meaning for the newly arriving fluid. Analogous relations hold for the stochastic system, similar to those in (Kang and Ramanan 2010, Theorem 2.1), but we did not fully specify them because of the methodological overhead.

A fluid model solution associated with a particular initial state is not unique. Uniqueness requires a more precise specification of the matching policy. Hence, the fluid model provides the framework to compare the performance of different matching policies, and choose the one that is best.

Objective (1) is an infinite horizon objective function, which motivates understanding the steady-state performance. To do that, we must characterize all the invariant states of the fluid model, which serve as approximations for the mean steady-state queue-lengths that can be achieved under admissible matching policies. This is the topic of the next section.

### 4.2 Invariant analysis

Having written a fluid model for the matching model, we now move to characterize its invariant states. The first step is to present a formal definition of an invariant state.

**Definition 3.** A point \((q^*, i^*, \eta_{D,*}, \eta_{S,*}) \in \mathbb{Y}\) is said to be an **invariant state** for given \( \lambda \) and \( \mu \) if the constant function \((\bar{Q}, \bar{T}, \bar{\eta}_D, \bar{\eta}_S)\) given by \((\bar{Q}(t), \bar{T}(t), \bar{\eta}_D(t), \bar{\eta}_S(t)) = (q^*, i^*, \eta_{D,*}, \eta_{S,*})\) for all \( t \geq 0 \) is a fluid model solution. We denote the set of invariant states by \( \mathcal{I}_{\lambda, \mu} \).
Define the set of all possible matching rates

$$\mathcal{M} := \left\{ \bm{m} \in \mathbb{R}^{J \times K} : \sum_{k \in \mathbb{K}} m_{jk} \leq \lambda_j \text{ and } \sum_{j \in \mathbb{J}} m_{jk} \leq \mu_k \right\}$$

and recall that $G_{e,j}^D(x)$, $G_{e,k}^S(x)$ for $j \in \mathbb{J}, k \in \mathbb{K}, x \in \mathbb{R}_+$ are the excess life distributions of the patience times. Below we provide a characterization of the invariant points for our model and give again the formulas (10) and (11) for convenience.

**Proposition 3** (Characterization of the invariant states). Suppose that the patience time distributions are invertible and let $\lambda \in \mathbb{R}^J_+$ and $\mu \in \mathbb{R}^K_+$. A point $(q^*, i^*, \eta^{D,*}, \eta^{S,*})$ lies in the invariant manifold $\mathcal{I}_{\lambda, \mu}$ if and only if it satisfies the following relations for $j \in \mathbb{J}$ and $k \in \mathbb{K}$ with $\bm{m} \in \mathcal{M}$:

$$
\eta_j^{D,*}(dx) = \lambda_j (1 - G_j^D(x))dx, \quad \eta_k^{S,*}(dx) = \mu_k (1 - G_k^S(x))dx,
$$

$$
q_j^*(\bm{m}) = \begin{cases} 
\frac{\lambda_j}{\lambda_j}, & \text{if } \sum_{k \in \mathbb{K}} m_{jk} = 0, \\
\frac{\lambda_j}{\lambda_j} G_{e,j}^D\left((G_j^D)^{-1}\left(1 - \frac{\sum_{k \in \mathbb{K}} m_{jk}}{\lambda_j}\right)\right), & \text{if } \sum_{k \in \mathbb{K}} m_{jk} \in (0, \lambda_j],
\end{cases}
$$

$$
i_k^*(\bm{m}) = \begin{cases} 
\frac{\mu_k}{\mu_k}, & \text{if } \sum_{j \in \mathbb{J}} m_{jk} = 0, \\
\frac{\mu_k}{\mu_k} G_{e,k}^S\left((G_k^S)^{-1}\left(1 - \frac{\sum_{j \in \mathbb{J}} m_{jk}}{\mu_k}\right)\right), & \text{if } \sum_{j \in \mathbb{J}} m_{jk} \in (0, \mu_k].
\end{cases}
$$

There is in general a nonlinear relationship between the invariant states and the vector of the matching rates $\bm{m}$ that depends on the patience time distributions. Proposition 3 provides a characterization of the invariant states but it does not state how they arise. The next result explicitly explains the connection between a fluid model solution and an invariant state, i.e., a limit of a fluid model solution lies in the invariant manifold. We shall need this result when we show the connection of the MP and the proposed matching policies. This is also of independent interest since it is the key step to derive a fluid approximation, and it is usually the hardest step. For instance, the convergence of a fluid model solution remains an open question in a multiclass many-server queueing model. However, the absence of the service times in the matching model help us to show the convergence.

Before we state the convergence result, we need to characterize the fluid matching functions and the initial conditions. Having constant arrival rates, the cumulative input in the system for the fluid model is linear in time, and so one naturally expects that the desired cumulative amount of matches is also linear in time. Otherwise, a matching policy leaves unmatched demand and supply, and so is unlikely to maximize (1). Hence, we focus on linear fluid matching functions to show the convergence of a fluid model solution. Moreover, we do it for a wide class of initial conditions including random ones. Define the set of initial conditions:

$$
\mathcal{P} = \left\{ (\bm{Q}(0), \bm{I}(0), \eta^D(0), \eta^S(0)) \in \mathbb{R}^J_+ \times \mathbb{R}^K_+ \times \mathcal{J} \times \mathcal{K} : \mathbb{E}[\bm{Q}_j(0)] < \infty, \mathbb{E}[\bm{I}_k(0)] < \infty, \mathbb{E}[\langle 1, \eta^D_j(0) \rangle] < \infty, \mathbb{E}[\langle 1, \eta^S_k(0) \rangle] < \infty, \langle 1_{\{x\}}, \eta^D_j(0) \rangle = 0, \langle 1_{\{x\}}, \eta^S_k(0) \rangle = 0, \text{ for each } x \in \mathbb{R}_+, \text{ and } j \in \mathbb{J}, k \in \mathbb{K} \right\}.
$$
Theorem 2. Fix an $m \in \mathbb{M}$ and assume that $M_{jk}(t) = m_{jk}t$ for all $j \in \mathbb{J}$ and $k \in \mathbb{K}$. Further, we assume that the patience time distributions are invertible. If $P\left((Q(0), I(0), \eta^D(0), \eta^S(0)) \in \mathcal{P}\right) = 1$, then
\[ \lim_{t \to \infty} (Q(t), I(t), \eta^D(t), \eta^S(t)) = (q^*(m), i^*(m), \eta^D^*, \eta^S^*), \]
almost surely.

5 Performance analysis

In this section, we study the performance of the proposed matching policies in a high-volume setting when the goal is to optimize (1). We first derive an upper bound on the objective function in a high-volume setting in Section 5.1, then show that the matching-rate-based and priority-ordering policies achieve this upper bound asymptotically in Section 5.2. The fluid approximation of Section 4 and the convergence of a fluid model solution to an invariant point play an important role in the proofs. Last, we present simulations about the performance of our proposed policies in Section 5.3.

5.1 High-volume setting

We consider a high volume setting where the arrival rates of demand and supply grow large. In particular, consider a family of systems indexed by $n \in \mathbb{N}$, where $n$ tends to infinity, with the same basic structure as that of the system described in Section 2. To indicate the position in the sequence of systems, a superscript $n$ will be appended to the system parameters and processes.

We start by describing our high-volume setting (or asymptotic regime). In particular, we scale the arrival rates in such a way that they increase linearly with $n \in \mathbb{N}$; namely,
\[ \lambda^n_j = n\lambda_j \text{ for all } j \in \mathbb{J} \text{ and } \mu^n_k = n\mu_k \text{ for all } k \in \mathbb{K}. \]
In other words, we study the system when a large number of customers and workers is expected to arrive. Note that we do not scale the reneging parameters of the system, which is consistent with the existing literature; see, for example, Kang and Ramanan (2010). The high-volume scaled processes are defined by multiplying the stochastic processes by a factor of $1/n$; for example, given a policy $M^n(\cdot)$ for the $n$th system, the scaled number of matches is defined as $M^n(\cdot)/n$. Recall that $V_{M^n}(\cdot)$ is given in (1). In the high-volume setting, the next result shows that the objective function for any admissible policy cannot be higher than the optimal objective of the MP.

Theorem 3. Let $m^*$ be an optimal solution to (9). For any admissible matching policy in the $n$th system, $M^n(\cdot)$, we have that
\[ \lim_{t \to \infty} \lim_{n \to \infty} \frac{1}{nt} V_{M^n}(t) \leq \sum_{j \in \mathbb{J}, k \in \mathbb{K}} v_{jk}^* m_{jk}^* - \sum_{j \in \mathbb{J}} c_j^D q_j^*(m^*) - \sum_{k \in \mathbb{K}} c_k^S i_k^*(m^*), \]
almost surely.

Having established an upper bound, we investigate the existence of a policy that achieves this upper bound in the high-volume setting.
Definition 4. Let $m^*$ be an optimal solution to (9). An admissible policy $M^n(\cdot)$ is asymptotically optimal if
\[
\lim_{t \to \infty} \lim_{n \to \infty} \frac{1}{nt} \sum_{j \in J, k \in K} v_{jk} m_{jk}^* - \sum_{j \in J} c_j^D q_j^*(m^*) - \sum_{k \in K} c_k^S i_k^*(m^*) = 1,
\]
in probability.

In the remainder of Section 5, we study the asymptotic behavior of our proposed discrete review matching policies. To do so, we also need to scale the review period length as follows
\[
l^n = \frac{1}{n^{2/3}} l.
\]
The intuition behind the scaling of the length of the review period is that it should be long enough to allow us to make the most valuable matches as dictated by an MP optimal solution, but also short enough that the reneging does not hurt. Further, we note that the scaling of the review period is not unique. In fact, by (Ata and Kumar 2005, Equation 57), we can use any scaling in the form
\[
l^n = \frac{1}{n^\delta l} \text{ with } 1 - \epsilon/3 < \delta < 1, \text{ where } \epsilon \text{ is used in the assumption of the arrival rates moments.}
\]
Recalling that we choose $\epsilon = 3/2$, any $\delta$ in the interval $(1/2, 1)$ can be chosen. Here, we choose $\delta = 2/3$ for simplicity.

5.2 Asymptotic optimality of the matching policies

The goal of this section is to show that the proposed matching policies defined in Section 3 are asymptotically optimal. We start the analysis by examining the matching-rate-based policy which was introduced in Section 3.2.

Given a feasible point $m$ to (9) and $t \geq 0$, the number of matches for the matching-rate-based policy in the $n$th system at discrete review period $i \in \{1, \ldots, \lfloor t/l^n \rfloor \}$ is given by
\[
M_{r,n}^{jk}(t) := \lfloor nm_{jk} \min \left( l^n, \frac{Q_j^n(i^{p-n})}{\lambda_j^n}, \frac{I_j^n}{\mu_k^n} \right) \rfloor = \lfloor m_{jk} \min \left( n^{1/3} l, \frac{Q_j^n(i^{p-n})}{\lambda_j}, \frac{I_j^n}{\mu_k} \right) \rfloor.
\]
Note that the instantaneous matching rates $m_{jk}$ are scaled by $n$ in (28) because they remain feasible to (9) when the arrival rates are scaled by a factor of $n$. The cumulative number of matches until time $t$ in the $n$th system is given by
\[
M_{r,n}^{jk}(t) := \sum_{i=1}^{\lfloor t/l^n \rfloor} M_{r,n}^{jk}.
\]

Our goal is to show that the matching-rate-based policy is asymptotically optimal. The following result shows that the scaled number of matches approaches asymptotically any feasible point of MP, and it is one of the keys to prove the asymptotic optimality.

Theorem 4. Let $m$ be a feasible point to (9). Under the matching-rate-based policy (28) and (29), as $n \to \infty$,
\[
\sup_{0 \leq t \leq T} \left| \frac{M_{r,n}^{jk}(t)}{n} - m_{jk} t \right| \to 0,
\]
in probability for all $T \geq 0$, $j \in J$, and $k \in K$. 

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Note that the last result gives exactly the condition required in Theorem 2, i.e., the cumulative matches under the fluid scaling build up linearly in time. Although the detailed proof is given in the appendix, we provide the reader with a brief outline here to establish the basic template used in the proofs of this section. The proof proceeds in three basic steps: i) establish an upper bound on the amount of reneging during a review period, ii) use the reneging upper bound to derive a lower bound on the number of matches made during a review period, and iii) show that this lower bound on matches made approaches a feasible point to the MP.

Obviously the last result holds for an optimal solution to (9) as well. Moreover, a fluid model solution associated with a linear matching function approaches an invariant point by Theorem 2. In other words, the objective function (1) approaches the optimal objective function of the MP. Hence, the asymptotic optimality of the matching-rate-based policy, which is stated in the next corollary, follows by Theorem 3.

**Corollary 2.** The matching-rate-based policy is asymptotically optimal for an optimal solution, $m^*$, to (9).

Having studied the asymptotic performance of the matching-rate-based policy, we now move to the asymptotic optimality of the priority-ordering policy introduced in Section 3.3. In this case, the number of matches made in the $n$th system between type $j \in \mathbb{J}$ demand and type $k \in \mathbb{K}$ supply for $(j,k) \in P_h$, $h = 0, 1, \ldots, H + 1$ at the end of review period $i \in \{1, \ldots, \lfloor t/l_n \rfloor \}$ is given by

$$M_{p,n}^{jk} := \min \left( Q^p_j(i l_n -) - \sum_{k' : (j,k') \in Q_{h-1}} M_{i,j,k'}^n I_k^p(i l_n -) - \sum_{j' : (j',k) \in Q_{h-1}} M_{i,j'k}^n \right), \quad (30)$$

where an empty sum is defined to be zero. Hence, the aggregate number of matches at time $t \geq 0$ is given by

$$M_{p,n}^{jk}(t) := \sum_{i=1}^{\lfloor t/l_n \rfloor} M_{i,jk}^n. \quad (31)$$

The following result is analogous to Theorem 4 and it states that the priority-ordering policy asymptotically achieves the optimal matching rates as well.

**Theorem 5.** Suppose that there exists an optimal extreme point solution to (9) denoted by $m^*$ and let the priority sets be given by Algorithm 1. Under the priority-ordering matching policy (30) and (31), as $n \to \infty$,

$$\sup_{0 \leq t \leq T} \left| \frac{M_{p,n}^{jk}(t)}{n} - m^*_{jk} \right| \to 0,$$

in probability for all $T \geq 0$, $j \in \mathbb{J}$, and $k \in \mathbb{K}$.

The proof of the last theorem is based on a two level induction. We first proceed in an induction on the priority sets, and then for any set we proceed in a second induction on the review periods inside this set. For each review period, we show that the number of matches made is close to the optimal matching rates given by the MP. Now, the asymptotic optimality of the priority-ordering policy follows in the same way as in the matching-rate-based policy.

**Corollary 3.** The priority-ordering policy is asymptotically optimal for an optimal extreme point solution, $m^*$, to (9).
Theorems 4 and 5 suggest that the quantity $m^j_k t$ can be seen as an approximation of the mean number of matches at time $t$ for $j \in J$ and $k \in K$. Knowing the optimal matching rates is helpful for the system manager because the platform has a consistent target for matching rates in the system so that users can come to know what to expect in terms of matching priorities. Moreover, an optimal solution to the MP can be used to identify the most valuable matches in the network, which is useful for forecasting purposes.

5.3 Simulations

In this section, we present simulation results on the behavior of the aforementioned matching policies. We compare the objective (1) of the matching policies with the upper bound given by the optimal objective of the MP (9) plotting their ratio as a function of various parameters.

For our simulation, we consider a model with four demand and four supply nodes and Poisson arrivals with rates $\lambda = (3, 2, 1, 3)$ and $\mu = (2, 2, 2, 2)$. Further, we assume that the system is initially empty and all the nodes have the same patience time distribution with the same mean and variance, i.e., $\theta^D_j = \theta^S_k = \theta$. We consider two different distributions of the patience times both having an increasing hazard rate function; uniform in $[0, 2/\theta]$ and gamma with shape parameter equal to 3 and scale parameters equal to $1/3 \theta$, i.e., the mean patience time is $1/\theta$. We fix the time horizon $t = 100$ in (1) and we denote the value vector for each demand node by $v^j_j = (v^j_1, \ldots, v^j_K)$ for each $j \in J$. The values at each edge are given by the vectors: $v_1 = (1, 2, 3, 1)$, $v_2 = (1, 1, 1, 1)$, $v_3 = (2, 1, 1, 2)$, and $v_4 = (3, 3, 2, 1)$. The holding costs for demand and supply nodes are $c^D = (1, 2, 1, 2)$ and $c^S = (2, 1, 2, 1)$, respectively. Figures 5 and 6 show the ratio of average objective of the matching policy to the optimal objective of (9) for the two policies as a function of the scaling parameter $n$ for various values of the discrete review period $l^n$. Figures 7 and 8 show the aforementioned ratio as a function of the review period $l^n$ for various values of the mean patience times. To illustrate the impact of changing either the review period length, $l^n$, or the scaling parameter, $n$, all the figures of this section hold one of these parameters constant while varying the other (i.e., we do not let the review period length grow with the arrival rate as it did in Section 5.1). Below, we summarize the main observations from the simulations.

**The performance of the matching policies is good for small arrival rates.** We notice in Figures 5 and 6 that all the policies behave well for relatively small scaling parameter $n$ that can be translated to small arrival rates, even though the asymptotic optimality results (Theorem 4 and 5) hold for large arrival rates. Moreover, we need to decrease the review period $l$, especially when the mean patience times are small, in order to see better behavior from the matching policies.

**The priority-ordering policy seems to perform better than the matching-rate-based policy.** In all figures, the priority-ordering policy behaves better than the matching-rate-based policy. This can be explained intuitively as follows: First, the priority-ordering policy makes matches at each review period according to the priority of the edges that identifies the most valuable edges taking into account the holding costs as well. Second, the priority-ordering policy exhausts the demand or supply at each review period in contrast to the matching-rate-based policy.

**The review period depends on the mean patience times.** We have seen that the discrete review period is chosen in terms of the scaling parameter $n$ and becomes smaller as $n$ increases. However, Figures 7 and 8 suggest that in practice the performance for both policies is reasonably good for a much larger review period. In particular, the bigger the mean patience times, the bigger the review period can be chosen. This can be explained intuitively as follows: When customers and workers have small patience times, we need to decide on matches in a faster manner in order not
to lose the most valuable matches. On the other hand, in order to make matches, a review period should allow enough demand and supply to enter into the system. In other words, the discrete review length should strike a balance between the arrival and reneging rates.

(a) $l^n = 0.3$ for all $n$.  
(b) $l^n = 0.1$ for all $n$.  
(c) $l^n = 0.01$ for all $n$.  

Figure 5: Gamma distributed patience times with mean $1/3$ and variance $1/27$.

(a) $l^n = 0.3$ for all $n$.  
(b) $l^n = 0.1$ for all $n$.  
(c) $l^n = 0.01$ for all $n$.  

Figure 6: Uniform distributed patience times with mean $1/3$ and variance $1/27$.

(a) Mean patience times 0.1.  
(b) Mean patience times 0.5.  
(c) Mean patience times 1.  

Figure 7: Gamma distributed patience with $n = 100$.  

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A matching policy for a model without holding costs

The matching policies studied until now use the knowledge of the arrival rate parameters and the patience time distributions. This is because both policies use the information provided by the MP (9) which naturally depends on the invariant queue-lengths. The latter uses the information of the arrival rate parameters and the patience time distributions.

However, there is an insensitivity property on the fraction of reneging customers and workers. Table 1 shows the fraction of reneging customers and workers obtained by the simulations and the corresponding values calculated by the fluid model equations (denoted by $r^*_S$ and $r^*_D$) in a network with one demand and one supply node that makes every possible match. We observe that for exponential and gamma patience times the fraction of reneging customers and workers remains almost the same, which suggests an insensitivity property, similar to ones that appear in Atar et al. (2019) and Puha and Ward (2019).

Table 1: The fraction of reneging population for exponential (right) and gamma (left, rate parameter=0.7) distributions with unit means. The parameters are $T = 100$, $\lambda = 1$, and $n = 100$.

| $\mu$ | $R^D(T)/D(T)$ | $r^*_D$ | $R^S(T)/S(T)$ | $r^*_S$ | $R^D(T)/D(T)$ | $r^*_D$ | $R^S(T)/S(T)$ | $r^*_S$ |
|-------|----------------|---------|----------------|---------|----------------|---------|----------------|---------|
| 0.5   | 0.4942         | 0.5     | 1.4e-5         | 0       | 0.4847         | 0.5     | 0              | 0       |
| 0.9   | 0.1193         | 0.1     | 0.0022         | 0       | 0.0959         | 0.1     | 0.0221         | 0       |
| 1.0   | 0.0586         | 0       | 0.0585         | 0       | 0.0130         | 0       | 0.0581         | 0       |
| 1.2   | 0.0066         | 0       | 0.173          | 0.2     | 0              | 0       | 0.1698         | 0.2     |
| 1.5   | 4.3e-5         | 0       | 0.3314         | 0.5     | 0              | 0       | 0.3328         | 0.5     |

In this section, we sacrifice the generality of the objective function (1) to take advantage of the insensitivity property observed in Table 1. If we set $c^D_j = c^S_k = 0$ for each $j \in J$ and $k \in K$, then (1) becomes the cumulative matching value of the platform, and hence it is not affected by the number of customers/workers that renege. In other words, it is expected that the cumulative matching value is not affected by the patience time distribution due to the insensitivity property. This will allow us to propose a discrete review matching policy that does not use the arrival rate and patience time distributions information.
In the case without holding costs, the MP takes the following form:

\[
\begin{align*}
\max & \quad \sum_{j \in J, k \in K} v_{jk} m_{jk} \\
\text{s.t.} & \quad \sum_{j \in J} m_{jk} \leq \mu_k, \quad k \in K, \\
& \quad \sum_{k \in K} m_{jk} \leq \lambda_j, \quad j \in J, \\
& \quad m_{jk} \geq 0, \quad j \in J, \quad k \in K.
\end{align*}
\]

(32)

The number of matches made at each discrete review time point can be decided by solving an optimization problem with an objective to maximize the matching value and constraints that respect the amount of demand and supply available. For any \(i \geq 1\), let \(\mathcal{M}_{ijk}^{b, \star} : j \in J \text{ and } k \in K\) be given by an optimal solution to the following optimization problem:

\[
\begin{align*}
\max & \quad \sum_{j \in J, k \in K} v_{jk} \mathcal{M}_{ijk}^b \\
\text{s.t.} & \quad \sum_{j \in J} \mathcal{M}_{ijk}^b \leq I_k (il -), \quad k \in K, \\
& \quad \sum_{k \in K} \mathcal{M}_{ijk}^b \leq Q_j (il -), \quad j \in J, \\
& \quad \mathcal{M}_{ijk}^b \in \mathbb{Z}_+, \text{ for all } j \in J, k \in K.
\end{align*}
\]

(33)

The quantity \(\mathcal{M}_{ijk}^{b, \star}\) is the number of matches between type \(j\) demand and type \(k\) supply, and the cumulative number of matches for \(j \in J, k \in K, t \geq 0\), is given by

\[
\mathcal{M}_{jk}(t) = \sum_{i=1}^{t/l} \mathcal{M}_{ijk}^{b, \star}
\]

(34)

where we refer to the above policy as the LP-based matching policy.

The main difference between (33) and (32) is that the right-hand side of the constraints in (32) are replaced by the queue-lengths \(Q_j (il -) \in \mathbb{N}\) and \(I_k (il -) \in \mathbb{N}\). The proposed policy is myopic in the sense that the matching decisions made at each review time point are optimal given the available demand and supply but disregard the impact of future arrivals. The hope is that when discrete review points are well-placed, the aforementioned myopicity will not have too much negative impact, and the resulting total value of matches made can be close to the optimal value to (32).

The implementation of the LP-based matching policy does not require the knowledge of the arrival rates \(\lambda, \mu\) and the patience time distributions but it requires an optimal solution to (33) at each review point. It is helpful to observe that (33) can effectively be solved as an LP on any sample path, since the queue-lengths are always integer valued. This solution arises because the constraint matrix of (33) is totally unimodular, which implies that an optimal extreme point solution is integer valued if the right-hand sides of the constraints are integer valued; see Lemma 8 in the appendix.

Next we consider the asymptotic behavior of the LP-based policy introduced above in the same asymptotic regime as in Section 5. The number of matches in the \(n\)th system at a discrete review period is given by an optimal solution to (33) replacing the right-hand sides of the constraints
by \( Q_n^i(i^n) \) and \( F_n^k(i^n) \), respectively. The next theorem states that the matching policy is asymptotically optimal.

**Theorem 6.** The LP-based matching policy given by (33) and (34) in the \( n \)th system is asymptotically optimal for an optimal solution, \( m^* \), to (32).

We prove Theorem 6 using the same three basic steps as Theorem 4. However establishing a lower bound on the matches made during a review period (step ii) requires more effort since we cannot compare directly to the MP optimal solution. We overcome this challenge by leveraging a monotonicity property of the MP (32).

Having shown that the LP-based policy is asymptotically optimal, a question that arises is whether the matches made under this policy also achieve the optimal matching rates. The answer is nuanced in the case when there are multiple optimal solutions to the MP, since the limit of \( \frac{M_{b,n}(t)}{n} \) may oscillate between them. However, we are able to show that the matching rates must approach the set of optimal MP solutions asymptotically. To this end, for a real vector \( x \) and a set \( A \) in Euclidean space, denote the distance between them by \( d(x,A) \), e.g., one could consider \( d(x,A) := \inf_{z \in A} \|x - z\|^2 \).

**Proposition 4.** Let \( S \) be the set of all optimal solutions of (32). In other words, we have that \( S := \{m^* : m^* \text{ is an optimal solution to (32)}\} \). Then, for each \( t > 0 \) as \( n \to \infty \),

\[
d\left( \frac{M_{b,n}(t)}{nt}, S \right) \to 0,
\]

in probability.

A consequence of the last theorem is that an analogous result to Theorem 4 holds for the LP-based policy in the special case that (32) has a unique optimal solution.

**Corollary 4.** Assume (32) has a unique optimal solution \( m^* \). We have that for each \( t \geq 0 \) as \( n \to \infty \),

\[
\frac{M_{jk,n}(t)}{n} \to m^*_{jt},
\]

in probability, for all \( j \in J \) and \( k \in K \).

7 Conclusion

In this paper, we proposed and analyzed a model that takes into account three main features of service platforms: (i) demand and supply heterogeneity, (ii) the random unfolding of arrivals over time, (iii) the non-Markovian impatience of demand and supply. These features result in a trade-off between making a less good match quickly and waiting for a better match. The model is too complicated to solve for an optimal matching policy, and so we developed an approximating fluid model, that is accurate in high volume (when demand and supply arrival rates are large). We used the invariant states of the fluid model to define a matching optimization problem, whose solution gave asymptotically optimal matching rates, that depend on customer and worker patience time distributions. We proposed a discrete review policy to track those asymptotically optimal matching rates, and further established conditions under which a static priority ordering policy also resulted
in asymptotically optimal matching rates. Finally, we observed that when holding costs are zero, there is an insensitivity property that allows us to propose an LP-based matching policy, that also achieves asymptotically optimal matching rates, but does not depend on demand and supply patience time distributions.

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A Proofs

A.1 Proofs for Section 3

**Proof of Theorem 1.** We show the result when the hazard rate functions are increasing. The case of decreasing hazard rate functions shares the same machinery.

Assume that the hazard rate functions are increasing. By (Puha and Ward 2019, Lemma 1), we know that $q_j^D(m)$ and $i_k^S(m)$ are concave functions of $m_j^D := \sum_{j \in J} m_{jk} \leq \lambda_j$ and $m_k^S :=$
\[ \sum_{k \in \mathbb{K}} m_{jk} \leq \mu_k, \text{ respectively. By (Sydsaeter and Hammond 1995, Theorem 17.7), we have that for any } x_j = \sum_{j \in \mathbb{J}} x_{jk} \leq \lambda_j, \]
\[ q^*_j(m_j^D) - q^*_j(x_j) \leq \frac{dq^*_j(x_j)}{dx_j}(m_j^D - x_j). \]  \hspace{1cm} (35)

Furthermore, taking into account that \( \frac{d}{dx_j}(1 - \frac{x_j}{x_j'}) = \frac{\partial}{\partial x_{jk}}(1 - \frac{x_j}{x_j'}) = -\frac{1}{x_j} \) for all \( k \in \mathbb{K}, \) we have that
\[ dq^*_j(x_j) \quad \begin{array}{c}
\frac{\partial q^*_j(x)}{dx_{jk}} \\
\end{array} = \frac{1}{h_j^D((G_j^D)^{-1}(1 - \frac{x_j}{x_j'})} = \frac{\partial q^*_j(x)}{dx_{jk}}. \]

Now, by (35), we obtain
\[ q^*_j(m) - q^*_j(x) = q^*_j(m_j^D) - q^*_j(x_j) \leq \frac{dq^*_j(x_j)}{dx_j}(m_j^D - x_j) \leq \sum_{k \in \mathbb{K}} \frac{\partial q^*_j(x)}{dx_{jk}}(m_j^D - x_{jk}), \]

and hence \( q^*_j(m) \) is a concave function by (Sydsaeter and Hammond 1995, Theorem 17.7). The proof for \( i_k^*(m) \) follows the same machinery. \hfill \Box

**Proof of Lemma 1.** Let \( m_{jk}, j \in \mathbb{J}, k \in \mathbb{K} \) denote an optimal extreme point solution. By definition, an optimal extreme point solution cannot be written as a convex combination of any two distinct feasible solutions.

**Property 1.** Assume the induced graph has a cycle, and renumber the nodes involved in the cycle so that it can be represented by the set of edges \( C = \{(1, 1), (1, 2), \ldots, (i, i), (i, i + 1), \ldots, (r, r), (r, 1)\}, \) where the length of the cycle is assumed to be \( 2\epsilon \) for some \( \epsilon \) without loss of generality because the underlying graph is bipartite. Then, we will show that solution \( m_{jk}, j \in \mathbb{J}, k \in \mathbb{K} \) can be written as a convex combination of two distinct feasible solutions, contradicting the fact that it is an extreme point.

First, for edges in the cycle, let \( m_{ii}^1 = m_{ii}' + \epsilon, i = 1, \ldots, r \) and \( m_{ii+1}^1 = m_{ii+1}' - \epsilon, i = 1, \ldots, r - 1 \) and \( m_{ii}^1 = m_{ii}' - \epsilon \) for \( \epsilon > 0, \) while for edges not in the cycle, i.e., \( (j, k) \notin C, \) let \( m_{jk}^1 = m_{jk}'. \) For \( \epsilon \leq \min_{(j, k) \in C} (m_{jk}') \), all variables remain nonnegative, and we next argue that this solution remains feasible for the other constraints as well. Consider a demand node \( i = 1, \ldots, r - 1 \) involved in the cycle \( C. \) We have \( m_{ii}^1 + m_{ii+1}^1 = m_{ii}' + \epsilon + m_{ii+1}' - \epsilon = m_{ii}' + m_{ii+1}' \), so that the total contribution of edges \((i, i)\) and \((i, i + 1)\) to the flow entering node \( i \) is unchanged, hence the associated demand constraint for \( i \) is satisfied (since \( m' \) was assumed feasible). Similarly, for demand node \( r \) in the cycle \( C, \) we have \( m_{rr}^1 + m_{r1}^1 = m_{rr}' + \epsilon + m_{r1}' - \epsilon = m_{rr}' + m_{r1}' \) so that the associated demand constraint for \( r \) is satisfied. An identical argument demonstrates that the supply constraints are also satisfied.

Next, using the same \( \epsilon \), for edges in the cycle let \( m_{ii}^2 = m_{ii}' - \epsilon, i = 1, \ldots, r \) and \( m_{ii+1}^2 = m_{ii+1}' + \epsilon, i = 1, \ldots, r - 1 \) and \( m_{ii}^2 = m_{ii}' + \epsilon \) for \( \epsilon > 0, \) while for edges not in the cycle, i.e., \( (j, k) \notin C, \) let \( m_{jk}^2 = m_{jk}'. \) Again, this solution is feasible, and we have \( m' = 1/2 m^1 + 1/2 m^2, \) hence \( m' \) is not an extreme point of (9).

**Property 2.** Assume that in the induced graph, a given tree has more than one node with a slack constraint. Then, pick any two of these slack nodes, consider the path between them, and renumber the nodes involved in the path so that it can be represented by the set of edges \( P = \{(1, 1), (1, 2), \ldots, (i, i), (i, i + 1), \ldots, (r, r)\}. \) (This representation implicitly assumes that the path begins on a demand node and ends on a supply node, but the following argument applies for
the analogous path representation for any combination of begining/ending node classification). By the assumption that both endpoints of the path have slack constraints, we have \( \lambda_i > \sum_k m'_{ik} \) and \( \mu_r > \sum_j m'_{jr} \). Then, we will show that solution \( m'_{jk}, j \in J, k \in K \) can be written as a convex combination of two distinct feasible solutions, contradicting the fact that it is an extreme point.

First, for edges in the path, let \( m'_{ij} = m_{ii} + \epsilon, i = 1, \ldots, r \) and \( m'_{i+1} = m_{i+1} - \epsilon, i = 1, \ldots, r - 1 \) for \( \epsilon > 0 \), while for edges not in the path, i.e., \( (j, k) \notin P \), let \( m'_{jk} = m'_{jk} \). For all variables remain nonnegative, we do not violate the constraints on the endpoint nodes of the paths, and feasibility of this solution for the remaining constraints along the path follows from an identical argument to that posed in the proof of Property 1.

Next, using the same \( \epsilon \), for edges in the path let \( m_{ij}^2 = m_{ij} + \epsilon, i = 1, \ldots, r \) and \( m_{i+1}^2 = m_{i+1} + \epsilon, i = 1, \ldots, r - 1 \), while for edges not in the path, i.e., \( (j, k) \notin P \), let \( m_{jk}^2 = m'_{jk} \). Again, this solution is feasible, and we have \( m' = 1/2m^1 + 1/2m^2 \), hence \( m' \) is not an extreme point of (9).

**Property 3.** This follows as a direct consequence of Property 2, since if it were false, then there would be more than one node in the same tree with a slack constraint. \( \square \)

**Proof of Lemma 2.** In the proof we drop the dependence on \( m^* \) for the set definitions since this dependence is clear from the context. The inner while loop beginning in Step 6 always removes at least the element \((j, k)\) chosen in Step 7, so this while loop terminates. The outer while loop beginning in Step 3 removes the elements of \( P_h \subseteq E \) in Step 15, so as long as \(|P_h| > 0\), this while loop terminates also. Step 9 adds the element \((j, k) \in C \) to \( P_h \) as long as the condition in Step 8 is met. Thus, we must show that at least one element in \( C \) satisfies the condition in Step 8 at the start of each iteration of the outer while loop (which are indexed by the counter \( h \)).

To this end, consider the situation at the start of iteration \( h \) of the outer while loop where \(|E| > 0\). Then, we have \( E = \{(j, k) : m^*_{jk} > 0\} \setminus Q_{h-1} \), and at the start of iteration \( h \) we have \( C = E \) and for all \((j, k) \in E\) we have \( d_j = \lambda_j - \sum_{k;j(k) \in Q_{h-1}} m^*_{jk} \) and \( s_k = \mu_k - \sum_{j;j(k) \in Q_{h-1}} m^*_{jk} \) by Lemma 3. Next, let \( E_0 = \{(j, k) : m^*_{jk} > 0\} \) denote the induced graph of \( m^* \), which is an extreme point of (9), and thus Lemma 1 guarantees that \( G \) has the following two properties: i) it is a collection of trees (equivalently contains no cycles), and ii) each distinct tree in \( G \) has at most one node with a slack constraint. Then, observe that removing the edges \( Q_{h-1} \) from \( E_0 \) to create the new graph \( F = \{(j, k) : m^*_{jk} > 0\} \setminus Q_{h-1} \) will preserve both these properties: for i) \( F \) must still contain no cycles since we only removed edges (and hence is still a collection of trees), and for ii) if any tree in \( F \) has more than one slack node, then these two nodes were also in the same tree in \( E_0 \), contradicting Lemma 1. Then, note that \( F \) must contain a tree with at least two nodes, since \( E \) is its edge set and \(|E| > 0\), denote the tree by \( T \). Since \( T \) is a tree with at least two nodes, it contains at least two nodes with degree one. By property ii) of \( F \), at most one of these degree one nodes has a slack constraint, so therefore at least one of them must have a tight constraint. Without loss of generality, assume this is node \( j \in J \) (if it is in \( K \) an identical argument applies) and let \((j, k)\) denote the edge connected to node \( j \) in graph \( F \). Since \( j \) is tight we have \( \sum_k m^*_{jk'} = \lambda_j \), and since \( j \) is degree one in \( F \) we have \( \sum_{k' : (j, k') \notin Q_{h-1}} m^*_{jk'} = 0 \), which together imply that \( m^*_{jk} = \lambda_j - \sum_{k' : (j, k') \in Q_{h-1}} m^*_{jk'} = d_j \), and thus \((j, k)\) meets the condition in Step 8.

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Finally, for the running time, each iteration of the outer while loop performs a number of operations linear in $|C| \leq |E_0|$ (since it has to consider each element of $C$, either to check the condition in Step 8, or to remove the element from $C$ in step 11). The outer while loop removes at least one element from $E$ in each iteration, so the number of iterations is less than the initial size of $E$, which is $|E_0|$.

**Lemma 3.** For Algorithm 1, throughout iteration $h$ of the outer while loop beginning in Step 3, we maintain the following for each edge $(j, k) \in E$: either $d_j = \lambda_j - \sum_{k' : (j, k') \in Q_{h-1}(m^*)} m_{jk}'$ and $s_k = \mu_k - \sum_{j' : (j, j') \in Q_{h-1}(m^*)} m_{j'k}'$, or $(j, k) \notin C$.

**Proof of Lemma 3.** In the proof we drop the dependence on $m^*$ for the set definitions since this dependence is clear from the context. First we claim that at the beginning of iteration $h$, for all $(j, k) \in E$ we have $d_j = \lambda_j - \sum_{k : (j, k) \in Q_{h-1}} m_{jk}^*$ and $s_k = \mu_k - \sum_{j' : (j', k) \in Q_{h-1}} m_{j'k}^*$. This holds because the only time we change $d_j$ or $s_k$ is in Step 10, and each change to $d_j$ (resp. $s_k$) corresponds to a decrease of $m_{jk}'$ (resp. $m_{j'k}'$) for some $(j, k') \in Q_{h-1}$ (resp. $(j', k) \in Q_{h-1}$). Then, during iteration $h$, as long as $(j, k) \in C$, these values of $d_j$ and $s_k$ don’t change, since Step 11 implies that if $(j, k) \in C$, then no edge $(j', k')$ with either $j' = j$ or $k' = k$ has satisfied the condition in Step 8 yet, so neither $d_j$ nor $s_k$ have been updated in Step 10.

**Proof of Proposition 2.** We proceed by induction on $h$ up to set $H$, then argue for $P_{H+1}$ separately. For $h = 0$, consider an arc $(j, k) \in \mathcal{P}_0(m^*)$, i.e., in iteration 0 of the outer while loop, $(j, k)$ was chosen in Step 7 and satisfied the condition in Step 8. Without loss of generality, assume the condition satisfied in Step 8 was $m_{jk}^* = d_j$ (an identical argument holds if $m_{jk}^* = \mu_k$). By Lemma 3, since $(j, k) \in C$ when it was chosen, we have $m_{jk}^* = d_j = \lambda_j - \sum_{k' : (j, k') \in Q_{h-1}} m_{jk}'$. Further, by feasibility of $m^*$ for (9), we have $m_{jk}^* \leq \mu_k$, which implies that $\lambda_j \leq \mu_k$. Thus, by the definition of $y_{jk}^p(m^*)$ we have

$$y_{jk}^p(m^*) = \min(\lambda_j, \mu_k) = \lambda_j = m_{jk}^*.$$  

Now assume that $y_{jk}^p(m^*) = m_{jk}^*$ for all $(j, k) \in Q_{h-1}(m^*)$ and consider $h$. For $(j, k) \in \mathcal{P}_h(m^*)$ we know that in iteration $h$ of the outer while loop, $(j, k)$ was chosen in Step 7 and satisfied the condition in Step 8. Without loss of generality, assume the condition satisfied in Step 8 was $m_{jk}^* = d_j$ (an identical argument holds if $m_{jk}^* = \mu_k$). By Lemma 3, since $(j, k) \in C$ when it was chosen, we have $m_{jk}^* = d_j = \lambda_j - \sum_{k' : (j, k') \in Q_{h-1}} m_{jk}'$. Further, by feasibility of $m^*$ to (9), we have $\sum_{j' \in \mathbb{Z}} m_{jk}' \leq \mu_k$, and since $m_{jk}' \geq 0$ for all $j'$, we also have $m_{jk}^* + \sum_{j' : (j', k) \in Q_{h-1}(m^*)} m_{jk}' \leq \mu_k$, or equivalently $m_{jk}^* \leq \mu_k - \sum_{j' : (j', k) \in Q_{h-1}(m^*)} m_{jk}'$. From this it follows that $\lambda_j - \sum_{k' : (j, k') \in Q_{h-1}(m^*)} m_{jk}' \leq \mu_k - \sum_{j' : (j', k) \in Q_{h-1}(m^*)} m_{jk}'$ and by the definition of $y_{jk}^p(m^*)$ and the induction hypothesis, we have

$$y_{jk}^p(m^*) = \min \left( \lambda_j - \sum_{k' : (j, k') \in Q_{h-1}(m^*)} m_{jk}' \mu_k - \sum_{j' : (j', k) \in Q_{h-1}(m^*)} m_{jk}' \right)$$

$$= \lambda_j - \sum_{k' : (j, k') \in Q_{h-1}(m^*)} m_{jk}' = m_{jk}^*.$$  

Finally, for $h = H + 1$, for $(j, k) \in \mathcal{P}_{H+1}(m^*)$, we must have either $\sum_{k' : (j, k') \in Q_{h-1}(m^*)} m_{jk}' = \lambda_j$ or $\sum_{j' : (j', k) \in Q_{h-1}(m^*)} m_{jk}' = \mu_k$, since otherwise $m^*$ was not optimal (since the objective is

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strictly increasing in $m^*$. Thus, by definition we have $y_{jk}^P(m^*) = 0 = m_{jk}^*$.

**Proof of Proposition 1.** We shall show that the stochastic process $M_{jk}^*(\cdot)$ defined in (13) satisfies the properties of Definition 1.

By definition of the matching process $M_{jk}^*(\cdot)$ is nonanticipating. Next, we prove that the queue-lengths are nonnegative. We only show it for the process $Q_j(\cdot)$, the proof for $I_k(\cdot)$ follows in the same way. We have that for $j \in J$ and $m \in \{1, \ldots, \lfloor t/l \rfloor\}$,

$$Q_j(ml) = Q_j((m-1)l) + D_j(ml) - D_j((m-1)l) - R_j^D(ml) + R_j^D((m-1)l) - \sum_{k=1}^{K} M_{mjk}^r.$$ 

Now, using the inequalities $M_{mjk}^r \leq \left[ m_{jk} \frac{Q_j(ml-\epsilon)}{\lambda_j} \right] \leq m_{jk} \frac{Q_j(ml-\epsilon)}{\lambda_j}$ and $\sum_{k=1}^{K} \frac{m_{jk}}{\lambda_j} \leq 1$, we have that

$$Q_j(ml) \geq Q_j(ml-) - \sum_{k=1}^{K} M_{mjk}^r \geq Q_j(ml-) - Q_j(ml-) = 0.$$ 

The fact that $M_{jk}^*(\cdot)$ is nondecreasing follows by (13) and the fact that $Q_j(\cdot) \geq 0$ and $I_k(\cdot) \geq 0$.

**A.2 Proofs for Section 4**

**Proof of Proposition 3.** We first show the forward direction. Let

$$(Q(t), I(t), \eta^D(t), \eta^S(t)) = (q^*, i^*, \eta^{D,*}, \eta^{S,*}) \in I_{\lambda, \mu}$$

for all $t \geq 0$. We shall show that $(q^*, i^*, \eta^{D,*}, \eta^{S,*})$ satisfies (10), (11), (25), and (26) with $m \in M$. Relations (25) and (26) follow in the same way as in the proof of (Puha and Ward 2019, Theorem 1). Below we show (10) and (11) in the same way. Adapting (Puha and Ward 2019, Theorem 1), let $\chi_j^D \in [0, H_j^D]$ be the unique (because the patience time distributions are invertible) solution to

$$q_j^* = \int_{[0, \chi_j^D]} \eta_j^{D,*}(dx).$$

If $\chi_j^D = H_j^D$, then by (25) and (36), we obtain

$$q_j^* = \lambda_j \int_0^{H_j^D} (1 - G_j^D(u))du = \frac{\lambda_j}{\theta_j^D}.$$ 

Further, by (19), (25), and the definition of the hazard rate, we have that

$$R_j^D(t) = \int_0^t \int_0^{\chi_j^D} h_j^D(x) \eta_j^{D,*}(dx)du = \lambda_j \int_0^t \int_0^{H_j^D} h_j^D(x)(1 - G_j^D(x))dxdu = \lambda_j t.$$
By (21), we obtain for any $t \geq 0$,

$$\sum_{k \in \mathbb{K}} \bar{M}_{jk}(t) = \lambda_j t - \overline{R}^D_j(t) = \lambda_j t - \lambda_j t = 0,$$

which yields $\sum_{k \in \mathbb{K}} m_{jk} = 0$ for all $j \in \mathcal{J}$. Hence, (10) is satisfied with $\sum_{k \in \mathbb{K}} m_{jk} = 0$.

If $\chi^D_j < H^D_j$, then by (25) and (36), we have that $q^*_j = \frac{\lambda_j}{\theta_j} G^D_{e,j}(\chi^D_j)$, or equivalently

$$\chi^D_j = (G^D_{e,j})^{-1}\left(\frac{\theta_j}{\lambda_j} q^*_j\right).$$

In this case, (19) and (25) lead to

$$\overline{R}^D_j(t) = \lambda_j \int_0^t \int_0^{H^D_j} h^D_j(x)(1 - G^D_j(x)) dx du = \lambda_j G^D_j \left((G^D_{e,j})^{-1}\left(\frac{\theta_j}{\lambda_j} q^*_j\right)\right) t.$$

By last equation and (21), we obtain for any $t \geq 0$,

$$\sum_{k \in \mathbb{K}} M_{jk}(t) = \lambda_j t - \overline{R}^D_j(t) = \lambda_j \left(1 - G^D_j \left((G^D_{e,j})^{-1}\left(\frac{\theta_j}{\lambda_j} q^*_j\right)\right)\right) t. \quad (37)$$

Set $m_{jk} = \frac{M_{jk}(t)}{t}$ and observe that $\sum_{k \in \mathbb{K}} m_{jk} \in (0, \lambda_j]$ because $0 \leq G^D_j(\cdot) < 1$. By (37), we obtain

$$q^*_j = \frac{\lambda_j}{\theta_j} G^D_{e,j} \left((G^D_{e,j})^{-1}\left(1 - \frac{\sum_{k \in \mathbb{K}} m_{jk}}{\lambda_j}\right)\right),$$

and so (10) is satisfied with $\sum_{k \in \mathbb{K}} m_{jk} \in (0, \lambda_j]$.

We now move to the proof of the converse direction. Let $m \in \mathcal{M}$ and $(q^*, i^*, \eta^{D,*}, \eta^{S,*}) \in \mathcal{Y}$ that satisfies (10), (11), (25), and (26). For all $t \geq 0$ let $(Q(t), \overline{T}(t), \overline{R}^D(t), \overline{S}(t)) = (q^*, i^*, \eta^{D,*}, \eta^{S,*})$. We shall show that $(q^*, i^*, \eta^{D,*}, \eta^{S,*})$ is a fluid model solution. We do it for $(q^*, \eta^{D,*})$ and the proof is similar for $(i^*, \eta^{S,*})$. By (25), we have that

$$\int_{[0,H^D_j]} h^D_j(x) \eta^{D,*}(dx) = \lambda_j \int_0^{H^D_j} h^D_j(x)(1 - G^D_j(x)) dx = \lambda_j < \infty,$$

and so (20) is satisfied. Now define $\bar{M}_{jk}(t) := m_{jk}t$ for all $t \geq 0$, $j \in \mathcal{J}$, and $k \in \mathbb{K}$. Clearly, $\bar{M}_{jk}(\cdot)$ are nondecreasing and absolutely continuous for all $j \in \mathcal{J}$ and $k \in \mathbb{K}$. Further, by (10), (19), and (25), we obtain

$$\overline{R}^D_j(t) = \begin{cases} \lambda_j t, & \text{if } \sum_{k \in \mathbb{K}} m_{jk} = 0, \\ (\lambda_j - \sum_{k \in \mathbb{K}} m_{jk})t, & \text{if } \sum_{k \in \mathbb{K}} m_{jk} \in (0, \lambda_j], \end{cases}$$

which leads to

$$q^*_j = q^*_j + \lambda_j t - \overline{R}^D_j(t) - \sum_{k \in \mathbb{K}} m_{jk} t.$$
Proof of Theorem 2. Define \( \mathcal{P}_0 = \left\{ \omega \in \Omega : (Q(0), I(0), \eta^D(0), \eta^S(0)) \in \mathcal{P} \right\} \) and
\[
\mathcal{P}_1 = \left\{ \omega \in \Omega : \lim_{t \to \infty} (\overline{Q}(t), \overline{I}(t), \overline{\eta}^D(t), \overline{\eta}^S(t)) = (q^*(m), i^*(m), \eta^{D,*}, \eta^{S,*}) \right\}.
\]
We shall show that if \( \omega \in \mathcal{P}_0 \), then \( \omega \in \mathcal{P}_1 \). That is, \( 1 = \mathbb{P}(\mathcal{P}_0) \leq \mathbb{P}(\mathcal{P}_1) \) which proves the result. Throughout the proof, we assume that \( \omega \in \mathcal{P}_0 \).

Let \( f \in C_b(\mathbb{R}_+, \mathbb{R}_+) \) for \( j \in \mathbb{J} \) and \( k \in \mathbb{K} \). By definition of set \( \mathcal{P}_1 \), it follows \( \left\langle 1, \overline{\eta}^D_j(0) \right\rangle < \infty \) almost surely. Taking now the limit as \( t \) goes to infinity of the right-hand side of (23), we have that
\[
\int_0^{H^D_j} f(x + t) \frac{1 - G^D_j(x + t)}{1 - G^D_j(x)} \overline{\eta}^D_j(0)(dx) \to 0,
\]
by dominated convergence theorem. Furthermore,
\[
\lambda_j \int_0^t f(t - u)(1 - G^D_j(t - u))du \to \lambda_j \int_0^\infty f(u)(1 - G^D_j(u))du.
\]
These lead to
\[
\left\langle f, \overline{\eta}^D_j(t) \right\rangle \to \lambda_j \int_0^\infty f(u)(1 - G^D_j(u))du,
\]
and hence \( \overline{\eta}^D_j(t)(dx) \to \eta^{D,*}(dx) \) as \( t \to \infty \). In a similar way, we obtain \( \overline{\eta}^S_j(t)(dx) \to \eta^{S,*}(dx) \) as \( t \to \infty \).

We now move to the steady-state limit of the queue-lengths. We show the result for the demand queue-length and the convergence for the supply queue-length follows with similar arguments. Fix \( j \in \mathbb{J} \) and define \( m_j := \sum_{k \in \mathbb{K}} m_{jk} \). We distinguish three different cases: (i) \( 0 < m_j < \lambda_j \), (ii) \( m_j = \lambda_j \), and (iii) \( m_j = 0 \).

Case (i): We start by deriving convenient expressions for the queue-length and the reneging process. By (Kang and Ramanan 2010, Corollary 4.2), (23), and (24) are satisfied for every bounded Borel measurable function and \( t \geq 0 \). Further, the standard monotone convergence argument implies that (23) and (24) hold for every nonnegative Borel measurable function. For \( t > 0 \), recall that
\[
\overline{\eta}^D_j(t) = \inf\{x \in \mathbb{R}_+ : \langle 1_{[0,x]}, \overline{\eta}^D_j(t) \rangle \geq \overline{Q}_j(t) \}
\]
and note that \( \overline{\eta}^D_j(t) \leq H^D_j \) by (17). The function \( f(x) = 1_{\{[0,\overline{\eta}^D_j(t)]\}}(x) \) is a nonnegative Borel measurable function because \( [0,\overline{\eta}^D_j(t)] \) is a Borel set. Replacing \( f(x) \) in (23) and taking into account that \( \overline{Q}_j(t) = \langle 1_{\{[0,\overline{\eta}^D_j(t)]\}}, \overline{\eta}^D_j(t) \rangle \) by continuity of the fluid model solution, we have that
\[
\overline{Q}_j(t) = \int_0^{H^D_j} 1_{\{[0,\overline{\eta}^D_j(t)]\}}(x + t) \frac{1 - G^D_j(x + t)}{1 - G^D_j(x)} \overline{\eta}^D_j(0)(dx) + \lambda_j \int_0^t 1_{\{[0,\overline{\eta}^D_j(t)]\}}(u)(1 - G^D_j(u))du
\]
\[
= \int_0^{H^D_j \wedge \overline{\eta}^D_j(t)+} \frac{1 - G^D_j(x + t)}{1 - G^D_j(x)} \overline{\eta}^D_j(0)(dx) + \lambda_j \int_0^{\overline{\eta}^D_j(t)+} (1 - G^D_j(u))du
\]
\[
= A^D_j(t) + \lambda_j \int_0^{\overline{\eta}^D_j(t)+} (1 - G^D_j(u))du,
\]
36
where \( A_j^D(t) := \int_0^{H_j^D(\pi_j^D(t)-t)} \frac{1-G_j^D(x+t)}{1-G_j^D(x)} \eta_j^D(0)(dx) \). That is,
\[
\chi_j^D(t) \wedge t = (G_{e,j}^D)^{-1} \left( \frac{\theta_j^D(\overline{Q}_j(t) - A_j^D(t))}{\lambda_j} \right).
\] (38)

We now move to the rekening process. By the fact that the initial conditions lie in \( \mathcal{P}_1 \), we have that
\[
\overline{R}_j^D(t) = \int_0^t \int_0^{H_j^D(\pi_j^D(u)-u)} h_j^D(x) \eta_j^D(0)(dx) du.
\]
Replacing \( f(x) = h_j^D(x)1_{\{[0,\pi_j^D(0)]\}}(x) \) in (23) and taking into account (38), we have that
\[
\overline{R}_j^D(t) = \int_0^t \int_0^{H_j^D(\pi_j^D(u)-u)} h_j^D(x + u) \frac{1-G_j^D(x + u)}{1-G_j^D(x)} \eta_j^D(0)(dx) du
\[
+ \lambda_j \int_0^t \int_0^{H_j^D(\pi_j^D(u)-u)} h_j^D(x)(1 - G_j^D(u)) du
\]
\[= B_j^D(t) + \lambda_j \int_0^t G_j^D(1 - G_j^D(0))(dx) du,
\]
where \( B_j^D(t) := \int_0^t \int_0^{H_j^D(\pi_j^D(u)-u)} h_j^D(x + u) \frac{1-G_j^D(x + u)}{1-G_j^D(x)} \eta_j^D(0)(dx) du \) and
\[
H_j^D(\overline{Q}_j(u)) := 1 - G_j^D(1 - G_j^D(0))(dx) du.
\]

By (21), we have that
\[
\overline{Q}_j(t) = \overline{Q}_j(0) + \lambda_j \int_0^t H_j^D(\overline{Q}_j(u)) du - B_j^D(t) - \sum_{k \in \mathbb{K}} M_{jk}(t).
\] (39)

Now, define the function \( K_j(t) = (\overline{Q}_j(t) - q_j^*(m))^2 \). By (39), we have that for \( t \geq 0 \),
\[
K_j'(t) = 2(\overline{Q}_j(t) - q_j^*(m))\overline{Q}_j'(t) = 2(\overline{Q}_j(t) - q_j^*(m))(\lambda_j H_j^D(\overline{Q}_j(t)) - \sum_{k \in \mathbb{K}} m_{jk} t - (B_j^D)'(t)).
\] (40)

Note that
\[
0 \leq A_j^D(t) \leq \int_0^{H_j^D} \frac{1-G_j^D(x+t)}{1-G_j^D(x)} \eta_j^D(0)(dx) \to 0,
\]
and hence, \( A_j^D(t) \to 0 \) as \( t \to \infty \). Further, we have that
\[
0 \leq (B_j^D)'(t) = \int_0^{H_j^D} \frac{1-G_j^D(x+t)}{1-G_j^D(x)} \eta_j^D(0)(dx)
\]
\[= \int_0^{H_j^D} \frac{G_j^D(x+t)}{1-G_j^D(x)} \eta_j^D(0)(dx)
\]
\[\leq \int_0^{H_j^D} \frac{G_j^D(x+t)}{1-G_j^D(x)} \eta_j^D(0)(dx) \to 0
\]
as \( t \to \infty \) by dominated convergence theorem because \( g^D_j(x + t)^{t \to \infty} \). That is, \( (B^D_j)'(t) \to 0 \) as \( t \to \infty \). Let \( \epsilon > 0 \) and choose large enough \( t \) such that \( A^D_j(t) \leq \epsilon/2, \delta/4 \leq (B^D_j)'(t) \leq \delta/4 \). If \( \mathcal{Q}^*_j(t) - q^*_j(m) \geq \epsilon \), then \( \mathcal{Q}^*_j(t) - A^D_j(t) \geq q^*_j(m) + \epsilon/2 \). By the fact that \( G^D_j(\cdot) \) and \( G^D_j(e^j(\cdot)) \) are strictly increasing, we have that for large enough \( t \),

\[
\lambda_j H^D_j(\mathcal{Q}^*_j(t)) = \lambda_j - \lambda_j G^D_j \left( (G^D_j(e^j))^\prime \left( \frac{\theta^D_j(\mathcal{Q}^*_j(t) - A^D_j(t))}{\lambda_j} \right) \right) \\
\leq \lambda_j - \lambda_j G^D_j \left( (G^D_j(e^j))^\prime \left( \frac{\theta^D_j(q^*_j(m) + \epsilon/2)}{\lambda_j} \right) \right) \\
= \lambda_j - \lambda_j G^D_j \left( (G^D_j(e^j))^\prime \left( \frac{\theta^D_j q^*_j(m)}{\lambda_j} \right) \right) - \delta \\
= \lambda_j - \lambda_j (1 - \frac{m_j}{\lambda_j}) - \delta = m_j - \delta,
\]

where the last equation follows by (10) and we set

\[
\delta = \lambda_j G^D_j \left( (G^D_j(e^j))^\prime \left( \frac{\theta^D_j q^*_j(m)}{\lambda_j} \right) \right) - \lambda_j G^D_j \left( (G^D_j(e^j))^\prime \left( \frac{\theta^D_j q^*_j(m)}{\lambda_j} \right) \right) > 0.
\]

Note that \( \delta > 0 \) because \( G^D_j(\cdot) \) and \( G^D_j(e^j(\cdot)) \) are strictly increasing. In other words, for any \( \epsilon > 0 \) and large enough \( t \) there exists \( \delta > 0 \) such that \( \lambda_j H^D_j(\mathcal{Q}^*_j(t)) \leq m_j - \delta \) if \( \mathcal{Q}^*_j(t) - q^*_j(m) \geq \epsilon \). In the same way, \( \lambda_j H^D_j(\mathcal{Q}^*_j(t)) \geq m_j + \delta \) if \( \mathcal{Q}^*_j(t) - q^*_j(m) \leq -\epsilon \). If \( |\mathcal{Q}^*_j(t) - q^*_j(m)| \geq \epsilon \), then by (40), we have that for large enough \( t \),

\[
K^j(t) \leq 2\epsilon(m_j - \delta - m_j + \frac{\delta}{4} + \frac{\delta}{4} = -\epsilon\delta.
\]

That is, there exists large enough \( t_0 \) such that \( |\mathcal{Q}^*_j(t) - q^*_j(m)| < \epsilon \) for any \( \epsilon > 0 \) and \( t \geq t_0 \).

**Case (ii):** Note that (39) still holds in this case. Now, define \( K^D_j(t) = \mathcal{Q}^*_j(t)^2 \). The fact that \( \lambda_j = m_j \) leads to

\[
(K^D_j)'(t) = 2\mathcal{Q}^*_j(t)(-\lambda_j + \lambda_j H^D_j(\mathcal{Q}^*_j(t)) - (B^D_j)'(t)).
\]

As in the first case, for any \( \epsilon > 0 \) there exists \( \delta > 0 \) such that \( \lambda_j H^D_j(\mathcal{Q}^*_j(t)) \leq \lambda_j - \delta \) if \( \mathcal{Q}^*_j(t) \geq \epsilon \). Let \( \epsilon > 0 \) and choose large enough \( t \) such that \( A^D_j(t) \leq \epsilon/2, \delta/4 \leq (B^D_j)'(t) \leq \delta/4 \). We have that \( (K^D_j)'(t) \leq -\epsilon\delta \) for large enough \( t \). In other words, for large enough \( t \), \( \mathcal{Q}^*_j(t) < \epsilon \). That is, \( \mathcal{Q}^*_j(t) \to 0 = q^*_j(m) \) as \( t \to \infty \).

**Case (iii):** Let a \( j \in J \) such that \( m_j = 0 \). By (76) and (77), we have that then following relation hold for the fluid model for each \( t \geq 0 \),

\[
\langle 1, \eta^D_j(0) \rangle + \lambda_j t = \langle 1, \eta^D_j(t) \rangle + \mathcal{S}^D_j(t),\]

where \( \mathcal{S}^D_j(t) = \int_0^t \langle h^D_j, \eta^D_j(u) \rangle du \). By (21) and the fact that \( m_j = 0 \), we have that

\[
\langle 1, \eta^D_j(0) \rangle + \mathcal{Q}^*_j(t) = \langle 1, \eta^D_j(t) \rangle + \mathcal{R}^D_j(t) - \mathcal{S}^D_j(t) = \langle 1, \eta^D_j(t) \rangle.
\]
Using (17), we obtain

\[ \mathcal{Q}_j(t) \leq \langle 1, \eta^D_j(0) \rangle + \overline{Q}_j(t) - \mathcal{Q}_j(0) + \mathbf{R}^D_j(t) - S^D_j(t) = \langle 1, \eta^D_j(t) \rangle , \]

which leads to

\[ 0 \leq \langle 1, \eta^D_j(0) \rangle - \mathcal{Q}_j(0) + \mathbf{R}^D_j(t) - S^D_j(t). \tag{42} \]

Because there are not matches by time 0, we have \( \mathcal{Q}_j(0) = \langle 1, \eta^D_j(0) \rangle \). Furthermore, by (19) and the definition of \( S^D_j(t) \), we derive

\[ \mathbf{R}^D_j(t) = \int_0^t \left\langle 1_{\{[0, \bar{\eta}^D_j(u)]\}} h^D_j, \eta^D_j(u) \right\rangle \, du \leq \int_0^t \left\langle h^D_j, \eta^D_j(u) \right\rangle \, du = \overline{S}_j^D(t). \]

Combining the last inequality and (42), we have that

\[ 0 \leq \overline{R}^D_j(t) - \overline{S}_j^D(t) \leq 0, \]

which yields \( \overline{R}^D_j(t) = \overline{S}_j^D(t) \) for each \( t \geq 0 \). Replacing the last equation and \( \mathcal{Q}_j(0) = \langle 1, \eta^D_j(0) \rangle \) in (41), yield \( \mathcal{Q}_j(t) = \langle 1, \eta^D_j(t) \rangle \). Now, replacing \( f(x) = 1 \) in (23) leads to

\[ \mathcal{Q}_j(t) = \langle 1, \eta^D_j(t) \rangle = \int_0^{H^D_j} \frac{1 - G^D_j(x + t)}{1 - G^D_j(x)} \eta^D_j(0)(dx) + \lambda \int_0^t (1 - G^D_j(u)) \, du, \]

and hence \( \mathcal{Q}_j(t) = \frac{\lambda}{\eta^*_j} \) as \( t \to \infty \) because \( \int_0^{H^D_j} \frac{1 - G^D_j(x + t)}{1 - G^D_j(x)} \eta^D_j(0)(dx) \to 0 \) as \( t \to \infty \) by dominated convergence theorem.

**A.3 Proofs for Section 5**

**Proof of Theorem 3.** Consider the following optimization problem:

\[
\begin{align*}
\max & \sum_{j \in J, k \in K} v_{jk} M^*_j k(t) - \sum_{j \in J} \int_0^t c^D_j Q^*_j(s) ds - \sum_{k \in K} \int_0^t c^*_k T^*_k(s) ds \\
\text{s.t.} & \quad (\mathcal{Q}(\cdot), T(\cdot), \overline{\eta}^D(\cdot), \overline{\eta}^S(\cdot)) \text{ is a fluid model solution,} \\
& \quad \mathbf{M}(\cdot) \text{ is an admissible policy.}
\end{align*}
\]

(43)

First, we shall show that

\[
\limsup_{n \to \infty} \frac{V^{\leq_n}(t)}{n} \leq \sum_{j \in J, k \in K} v_{jk} M^*_j k(t) - \sum_{j \in J} \int_0^t c^D_j Q^*_j(s) ds - \sum_{k \in K} \int_0^t c^*_k T^*_k(s) ds,
\]

(44)

almost surely, where \((\overline{M}^*(t), \overline{Q}^*(t), \overline{T}^*(t))\) is an optimal solution to (43) at time \( t \geq 0 \). Let us fix an arbitrary admissible matching policy \( \mathbf{M}^n(\cdot) \). By Aveklouris et al. (2021), we have that the fluid-scaled state descriptor \((\overline{Q}^*(\cdot), \overline{T}^*(\cdot), \overline{\eta}^{D,n}(\cdot), \overline{\eta}^{S,n}(\cdot)) := \left( \frac{1}{n} Q^n(\cdot), \frac{1}{n} T^n(\cdot), \frac{1}{n} \eta^{D,n}(\cdot), \frac{1}{n} \eta^{S,n}(\cdot) \right)\) is
tight assuming that the fluid-scaled initial conditions are tight. That is, there exists a subsequence denoted by \((\bar{Q}^{n_k}(\cdot), \bar{T}^{n_k}(\cdot), \bar{\eta}^{D,n_k}(\cdot), \bar{\eta}^{S,n_k}(\cdot))\) such that
\[
\limsup_{n \to \infty} \left( \bar{Q}^{n_k}(t), \bar{T}^{n_k}(t), \bar{\eta}^{D,n_k}(t), \bar{\eta}^{S,n_k}(t) \right) = \lim_{z \to \infty} \left( \bar{Q}^{nz}(t), \bar{T}^{nz}(t), \bar{\eta}^{D,nz}(t), \bar{\eta}^{S,nz}(t) \right).
\]
Furthermore, by Aveklouris et al. (2021), there exists a further subsequence (which with an abuse of notation, we denote again by \(n_k\)) such that
\[
(\bar{Q}^{n_k}(\cdot), \bar{T}^{n_k}(\cdot), \bar{\eta}^{D,n_k}(\cdot), \bar{\eta}^{S,n_k}(\cdot)) \to (\bar{Q}(\cdot), \bar{T}(\cdot), \bar{\eta}^{D}(\cdot), \bar{\eta}^{S}(\cdot)),
\]
as \(z \to \infty\) where \((\bar{Q}(\cdot), \bar{T}(\cdot), \bar{\eta}^{D}(\cdot), \bar{\eta}^{S}(\cdot))\) is a fluid model solution. Using the Skorokhod’s representation theorem in a standard way, we conclude that there exists a probability space such that for any \(t \geq 0\),
\[
(\bar{Q}^{n_k}(t), \bar{T}^{n_k}(t), \bar{\eta}^{D,n_k}(t), \bar{\eta}^{S,n_k}(t)) \to (\bar{Q}(t), \bar{T}(t), \bar{\eta}^{D}(t), \bar{\eta}^{S}(t)),
\]
after almost surely as \(z \to \infty\). Further, by the nonnegativity of the queue-lengths, we have that for each \(j \in \mathbb{J}\) and \(k \in \mathbb{K}\), \(\bar{Q}^{n_k}_j(t) \leq \sup_n (\bar{Q}^n_j(t) + \bar{Q}^{n_k}_j(0)) < \infty\) and \(\bar{T}^{n_k}_k(t) \leq \sup_n (\bar{S}^n_k(t) + \bar{T}^n_k(0)) < \infty\) almost surely. Inequality (44) follows by applying the dominated convergence theorem and by observing that a fluid model solution is a feasible point to (43).

Next, we are going to show the following inequality:
\[
\limsup_{t \to \infty} \frac{1}{t} \left( \sum_{j \in \mathbb{J}, k \in \mathbb{K}} v_{jk} M_{jk}(t) - \sum_{j \in \mathbb{J}} \int_0^t c^D_j \bar{Q}_j(s) ds + \sum_{k \in \mathbb{K}} \int_0^t c^S_k \bar{T}_k(s) ds \right) \\
\leq \sum_{j \in \mathbb{J}, k \in \mathbb{K}} v_{jk} m^*_{jk} - \sum_{j \in \mathbb{J}} c^D_j q^*_j(m^*) - \sum_{k \in \mathbb{K}} c^S_k i^*_k(m^*),
\]
almost surely. By the nonnegativity of the fluid queue-lengths, we have that \(\sum_{k \in \mathbb{K}} \bar{M}_{jk}(t) \leq \bar{Q}_j(0) + \lambda_j t\) and \(\sum_{j \in \mathbb{J}} \bar{M}_{jk}(t) \leq \bar{T}_k(0) + \mu_k t\). That is, there exists \(m \in \mathbb{M}\) such that \(\bar{M}_{jk}(t) \leq m_{jk} t + \min(\bar{Q}_j(0), \bar{T}_k(0))\) for all \(j \in \mathbb{J}\) and \(k \in \mathbb{K}\). Let \((\bar{Q}^m(t), \bar{T}^m(t), \bar{\eta}^{D,m}(t), \bar{\eta}^{S,m}(t))\) denote a fluid model solution with fluid matching function \(m t\). Notice that by (21) and (22) the following inequalities hold: \(\bar{Q}^m_j(t_z) \leq \bar{Q}_j(t_z)\) and \(\bar{T}^m_k(t_z) \leq \bar{T}_k(t_z)\) with \(\bar{Q}^m_j(0) = \bar{T}^m_k(0) = 0\). That is,
\[
\sum_{j \in \mathbb{J}, k \in \mathbb{K}} v_{jk} M_{jk}(t) - \sum_{j \in \mathbb{J}} \int_0^t c^D_j \bar{Q}_j(s) ds + \sum_{k \in \mathbb{K}} \int_0^t c^S_k \bar{T}_k(s) ds \\
\leq U + \sum_{j \in \mathbb{J}, k \in \mathbb{K}} v_{jk} m^*_{jk} - \sum_{j \in \mathbb{J}} \int_0^t c^D_j \bar{Q}^m_j(s) ds - \sum_{k \in \mathbb{K}} \int_0^t c^S_k \bar{T}^m_k(s) ds,
\]
where \(U = J K \max_{jk}(v_{jk} \min_{jk}(\bar{Q}_j(0), \bar{T}_k(0)))\). Let now \((\bar{Q}^m(t_z), \bar{T}^m(t_z), \bar{\eta}^{D,m}(t_z), \bar{\eta}^{S,m}(t_z))\) be a subsequence such that
\[
\limsup_{t \to \infty} \left( \bar{Q}^m(t), \bar{T}^m(t), \bar{\eta}^{D,m}(t), \bar{\eta}^{S,m}(t) \right) = \lim_{z \to \infty} \left( \bar{Q}^m(t_z), \bar{T}^m(t_z), \bar{\eta}^{D,m}(t_z), \bar{\eta}^{S,m}(t_z) \right).
\]
By Theorem 2, there exists a further subsequence (which with an abuse of notation, we denote again by \(t_z\)) such that
\[
(\bar{Q}^m(t_z), \bar{T}^m(t_z), \bar{\eta}^{D,m}(t_z), \bar{\eta}^{S,m}(t_z)) \to (q^*(m), i^*(m), \eta^{D,*}, \eta^{S,*}),
\]
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almost surely as $z \to \infty$. Further, by Theorem 2, we have that for each $j \in J$ and $\epsilon > 0$, there exists $T > 0$ such that $|Q_j^m(tz) - q_j^*(m)| < \frac{\epsilon}{2}$ for all $tz > T$. Hence,

$$\left| \frac{1}{tz} \int_0^{tz} Q_j^m(s) - q_j^*(m) ds \right| \leq \frac{1}{tz} \int_0^{tz} |Q_j^m(s) - q_j^*(m)| ds$$

$$= \frac{1}{tz} \int_0^T |Q_j^m(s) - q_j^*(m)| ds + \frac{1}{tz} \int_t^{tz} |Q_j^m(s) - q_j^*(m)| ds$$

$$\leq \frac{1}{tz} \int_0^T |Q_j^m(s) - q_j^*(m)| ds + \frac{tz - T}{tz} \epsilon$$

$$\leq \frac{1}{tz} \int_0^T |Q_j^m(s) - q_j^*(m)| ds + \frac{\epsilon}{2}.$$ 

Choosing large enough $tz$, we have that $\frac{1}{tz} \int_0^T |Q_j^m(s) - q_j^*(m)| ds \leq \frac{\epsilon}{2}$. That is,

$$\left| \frac{1}{tz} \int_0^{tz} Q_j^m(s) - q_j^*(m) ds \right| \to 0,$$

almost surely as $z \to \infty$. In the same way, we obtain for each $k \in K$,

$$\left| \frac{1}{tz} \int_0^{tz} T_k^m(s) - i_k^*(m) ds \right| \to 0,$$

almost surely as $z \to \infty$. By (45), we have that

$$\limsup_{t \to \infty} \frac{1}{t} \left( \sum_{j \in J, k \in K} v_{jk} \mathbb{M}_{jk}(t) - \sum_{j \in J} \int_0^t c_j^D Q_j(s) ds - \sum_{k \in K} \int_0^t c_k^S T_k(s) ds \right)$$

$$\leq \limsup_{t \to \infty} \frac{1}{t} \left( U + \sum_{j \in J, k \in K} v_{jk} m_{jk} t - \sum_{j \in J} \int_0^t c_j^D Q_j^m(s) ds - \sum_{k \in K} \int_0^t c_k^S T_k^m(s) ds \right)$$

$$= \sum_{j \in J, k \in K} v_{jk} m_{jk} - \sum_{j \in J} c_j^D q_j^*(m) - \sum_{k \in K} c_k^S i_k^*(m),$$

because $\frac{U}{t} \to 0$, as $t \to \infty$. The proof is completed by observing that $m \in M$ is a feasible point to (9).

**A.3.1 Proof of Theorems 4 and 5**

Before we present the proof of Theorems 4 and 5, we show some preliminary results that are also used in the proofs in Section 6.

**Proposition 5.** For any $j \in J$, $k \in K$, and $i \geq 1$, the following inequalities hold almost surely

$$R_j^D(il) - R_j^D((i-1)l) \leq Q_j((i-1)l) + \sum_{m=D_j((i-1)l)+1}^{D_j(il)} 1\{r_{jm}^D \leq l\},$$

$$R_k^S(il) - R_k^S((i-1)l) \leq I_k((i-1)l) + \sum_{m=S_k((i-1)l)+1}^{S_k(il)} 1\{r_{mk}^S \leq l\}.$$
Proof. We show only the first inequality, the second one follows in the same way using (8). By (7), we have that

\[
R_j^D(il) - R_j^D((i - 1)l) = \sum_{h = D_j((i - 1)l) + 1}^{D_j(il)} \sum_{s \in [(i - 1)l, il]} 1 \left\{ s \leq m_{jh}^{Dh} \frac{dw_{j}^{Dh}}{dt}(s-) > 0, \frac{dw_{j}^{Dh}}{dt}(s+) = 0 \right\}
\]

\[
+ \sum_{h = D_j((i - 1)l)}^{D_j((i - 1)l)l} \sum_{s \in [(i - 1)l, il]} 1 \left\{ s \leq m_{jh}^{Dh} \frac{dw_{j}^{Dh}}{dt}(s-) > 0, \frac{dw_{j}^{Dh}}{dt}(s+) = 0 \right\}.
\]

Now, we have that

\[
D_j(il) \sum_{h = D_j((i - 1)l) + 1}^{D_j(il)} \sum_{s \in [(i - 1)l, il]} 1 \left\{ s \leq m_{jh}^{Dh} \frac{dw_{j}^{Dh}}{dt}(s-) > 0, \frac{dw_{j}^{Dh}}{dt}(s+) = 0 \right\} \leq \sum_{h = D_j((i - 1)l) + 1}^{D_j(il)} 1 \{ r_{jh}^{Dh} \leq 1 \}.
\]

To see this, observe that if \(1 \left\{ s \leq m_{jh}^{Dh} \frac{dw_{j}^{Dh}}{dt}(s-) > 0, \frac{dw_{j}^{Dh}}{dt}(s+) = 0 \right\} = 1\), then \(\frac{dw_{j}^{Dh}}{dt}(s+) = 0\). That is, \(s - e_{jh}^{Dh} > r_{jh}^{Dh}\). This yields \(l \geq r_{jh}^{Dh}\) by the fact that \(s \leq il\) and \(e_{jh}^{Dh} \geq (i - 1)l\). In other words, \(1 \{ r_{jh}^{Dh} \leq 1 \} = 1\).

To finish the proof it remains to show that

\[
D_j((i - 1)l) \sum_{h = 1}^{D_j((i - 1)l)} \sum_{s \in [(i - 1)l, il]} 1 \left\{ s \leq m_{jh}^{Dh} \frac{dw_{j}^{Dh}}{dt}(s-) > 0, \frac{dw_{j}^{Dh}}{dt}(s+) = 0 \right\} \leq Q_j((i - 1)l).
\]

To this end, the left hand side of the last inequality equals to

\[
D_j((i - 1)l) \sum_{h = 1}^{D_j((i - 1)l)} \sum_{s \in [0, il]} 1 \left\{ s \leq m_{jh}^{Dh} \frac{dw_{j}^{Dh}}{dt}(s-) > 0, \frac{dw_{j}^{Dh}}{dt}(s+) = 0 \right\} - R_j((i - 1)l)
\]

\[
= D_j((i - 1)l) \sum_{h = 1}^{D_j((i - 1)l)} \sum_{s \in [0, il]} 1 \left\{ s \leq m_{jh}^{Dh} \frac{dw_{j}^{Dh}}{dt}(s-) > 0, \frac{dw_{j}^{Dh}}{dt}(s+) = 0 \right\} - D_j((i - 1)l) + \sum_{k = 1}^{K} M_{jk}((i - 1)l) + Q_j((i - 1)l).
\]

Now observe that

\[
D_j((i - 1)l) \sum_{h = 1}^{D_j((i - 1)l)} \sum_{s \in [0, il]} 1 \left\{ s \leq m_{jh}^{Dh} \frac{dw_{j}^{Dh}}{dt}(s-) > 0, \frac{dw_{j}^{Dh}}{dt}(s+) = 0 \right\} + \sum_{k = 1}^{K} M_{jk}((i - 1)l)
\]

\[
= \sum_{h = 1}^{D_j((i - 1)l)} \left( \sum_{s \in [0, (i - 1)l]} 1 \left\{ s \leq m_{jh}^{Dh} \frac{dw_{j}^{Dh}}{dt}(s-) > 0, \frac{dw_{j}^{Dh}}{dt}(s+) = 0 \right\} + \sum_{s \in [(i - 1)l, il]} 1 \left\{ s \leq m_{jh}^{Dh} \frac{dw_{j}^{Dh}}{dt}(s-) > 0, \frac{dw_{j}^{Dh}}{dt}(s+) = 0 \right\} \right)
\]

\[
+ \sum_{k = 1}^{K} M_{jk}((i - 1)l)
\]

\[
\leq D_j((i - 1)l).
\]
To see the last inequality, note that the left-hand side counts the number of customers than arrive in time interval \([0, (i-1)l]\) that depart (due to matching or reneging) until time \((i-1)l\) and renege in time interval \(((i-1)l, il]\). Hence it cannot exceed the number of customers than arrive until time \((i-1)l\) which is \(D_j((i-1)l)\).

Moreover, we need the following lemma, which is taken from Ata and Kumar (2005), and adapted to the below form found in Plambeck and Ward (2006).

**Lemma 4.** *(Ata and Kumar)* For any finite constant \(\alpha > 0\), there exists \(\beta > 0\) such that

\[
\mathbb{P}\left( \max_{i \in \{1, \ldots, \lfloor T/l \rfloor \}} \max_{j \in J} \left| D_j^n(i\ell^n) - D_j^n((i-1)\ell^n) - n\lambda_j \ell^n \right| < \alpha n^{1/3} \right) \geq 1 - \beta n^{-1/6}
\]

and

\[
\mathbb{P}\left( \max_{i \in \{1, \ldots, \lfloor T/l \rfloor \}} \max_{k \in K} \left| S_k^n(i\ell^n) - S_k^n((i-1)\ell^n) - n\mu_k \ell^n \right| < \alpha n^{1/3} \right) \geq 1 - \beta n^{-1/6}.
\]

A consequence of Lemma 4 is that with high probability the following hold for each \(j \in J, k \in K\), and \(i \in \{1, \ldots, \lfloor T/l \rfloor \} \)

\[
n\lambda_j \ell^n - \alpha n^{1/3} < D_j^n(i\ell^n) - D_j^n((i-1)\ell^n) < n\lambda_j \ell^n + \alpha n^{1/3} \tag{46}
\]

and

\[
n\mu_k \ell^n - \alpha n^{1/3} < S_k^n(i\ell^n) - S_k^n((i-1)\ell^n) < n\mu_k \ell^n + \alpha n^{1/3} \tag{47}
\]

**Proof of Theorem 4.** Let \(\epsilon > 0\). Fix \(n\) large enough so that \(n^{2/3} > 2 \max_{j,k} (m_{jk})/\epsilon\). Denote by \(\Omega_2^n\) the events such that \((46)\) and \((47)\) hold and by \(\Omega_2^n\) the events such both inequalities in Proposition 5 hold. Let \(\omega \in \Omega_1^n \cap \Omega_2^n\) and \(\alpha := \min_{j,k} (\lambda_j, \mu_k)\). We first derive a lower bound for the number of reneging customers and workers at any discrete review period. By Proposition 5, the following upper bound of the reneging customers in the discrete review period \(((i-1)\ell^n, i\ell^n)\) holds almost surely: for any \(j \in J\),

\[
R_j^{D,n}(i\ell^n) - R_j^{D,n}((i-1)\ell^n) \leq Q_j^n((i-1)\ell^n) + \sum_{m = D_j^n((i-1)\ell^n) + 1}^{D_j^n(i\ell^n)} \mathbb{1}_{\{r_{jm} < i\ell^n\}}, \tag{48}
\]

where \(r_{jm}^D\) denotes the patience time of the \(m\)th customer arriving at node \(j \in J\). Further, for the reneging workers in the discrete review period \(((i-1)\ell^n, i\ell^n)\), we have that for any \(k \in K\),

\[
R_k^{S,n}(i\ell^n) - R_k^{S,n}((i-1)\ell^n) \leq I_k^n((i-1)\ell^n) + \sum_{m = S_k^n((i-1)\ell^n) + 1}^{S_k^n(i\ell^n)} \mathbb{1}_{\{r_{km} < i\ell^n\}}, \tag{49}
\]

where \(r_{km}^S\) denotes the patience time of the \(m\)th worker arriving at node \(k \in K\). In the sequel, we show that the second term of the right-hand side of \((48)\) and \((49)\) converge to zero under the high-volume setting. Denote by \(\Omega_1^n\) the events such that \((46)\) holds and by \((\Omega_1^n)^c\) its complement.
For simplicity, let \( \Gamma_n = \frac{1}{n!} \sum_{m=D_p^n((i-1)!)+1}^{D_p^n(i!)} 1 \{ r_{jm} \leq l^n \} \). For any \( \delta > 0 \), we have that

\[
\begin{align*}
\mathbb{P}(\Gamma_n \geq \delta) &\leq \mathbb{P}(\{\Gamma_n \geq \delta\} \cap \Omega_1^n) + \mathbb{P}(\{\Gamma_n \geq \delta\} \cap (\Omega_1^n)^c) \\
&\leq \mathbb{E}\left[ \frac{\Gamma_n | \Omega_1^n}{\delta} \right] + \beta n^{-1/6} \\
&\leq \frac{\mathbb{E}\left[ \Gamma_n | \Omega_1^n \right]}{\delta(1 - \beta n^{-1/6})} + \beta n^{-1/6},
\end{align*}
\]

(50)

where the third inequality follows by the conditional Markov’s inequality and Lemma 4. To complete the proof that \( \Gamma_n \) converges to zero in probability, we show that \( \mathbb{E}\left[ \Gamma_n 1_{\Omega_1^n} \right] \) goes to zero. To this end,

\[
0 \leq \mathbb{E}\left[ \frac{1}{n!} \sum_{m=D_p^n((i-1)!)+1}^{D_p^n(i!)} 1 \{ r_{jm} \leq l^n \} 1_{\Omega_1^n} \right] \leq \frac{1}{n!} \mathbb{E}\left[ (n \lambda_j l^n + \alpha n^{1/3}) 1 \{ r_{j1} \leq l^n \} \right]
\]

\[
= \left( \lambda_j + \frac{\alpha}{n^{1/3}} \right) \mathbb{P}(r_{j1} \leq l^n) = \lambda_j 6 n^{1/3} (\max_{j,k} m_{jk}) \epsilon.
\]

(51)

It follows that

\[
0 \leq \mathbb{E}\left[ \frac{1}{n!} \sum_{m=D_p^n((i-1)!)+1}^{D_p^n(i!)} 1 \{ r_{jm} \leq l^n \} 1_{\Omega_1^n} \right] \to 0,
\]

as \( n \to \infty \) because \( l^n \to 0 \). Now, by (50), we have that \( \lim_{n \to \infty} \mathbb{P}(\Gamma_n \geq \delta) = 0 \), for any \( \delta > 0 \). By the last convergence and for large enough \( n \), we have that

\[
\frac{1}{n!} \sum_{m=D_p^n((i-1)!)+1}^{D_p^n(i!)} 1 \{ r_{jm} \leq l^n \} \leq \frac{\lambda_j}{6 \max_{j,k} (m_{jk})} \epsilon.
\]

(51)

In a similar way using (47), we obtain for large enough \( n \),

\[
\frac{1}{n!} \sum_{m=S_p^n((i-1)!)+1}^{S_p^n(i!)} 1 \{ r_{km} \leq l^n \} \leq \frac{\mu_k}{6 \max_{j,k} (m_{jk})} \epsilon.
\]

(52)

Define \( \Omega_3^n \) the events such that (51) and (52) hold and note that \( \lim_{n \to \infty} \mathbb{P}(\Omega_1^n \cap \Omega_2^n \cap \Omega_3^n) = 1 \). In the sequel, we take \( \omega \in \Omega_1^n \cap \Omega_2^n \cap \Omega_3^n \).

Now, we move to the second step of proof of Theorem 4. We derive a desirable lower bound on the number of matches at any discrete review period. To this end, by (46), (48), and (51), we have

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that for $i \in \{1, \ldots, [T/l^n]\}$,

$$Q^n_j(i^n- \neg) = Q^n_j((i-1)^n) + D^n_j(i^n) - D^n_j((i-1)l) - R^n_{j,n}D(i^n) + R^n_{j,n}D((i-1)^n) \geq D^n_j(i^n) - D^n_j((i-1)l) - \sum_{m=D^n_j((i-1)l)+1}^{\infty} \{ r_{j,m} \leq l^n \}$$

$$\geq D^n_j(i^n) - D^n_j((i-1)l) - \frac{\lambda_j}{6T \max_{j,k} (m_{jk})} nl^n \epsilon$$

$$\geq n\lambda_j l^n - \alpha n^{1/3} - \frac{\lambda_j}{6T \max_{j,k} (m_{jk})} nl^n \epsilon$$

$$= n\lambda_j l^n \left(1 - \frac{\alpha}{l \lambda_j} - \frac{\epsilon}{6T \max_{j,k} (m_{jk})}\right)$$

$$\geq n\lambda_j l^n \left(1 - \frac{\alpha}{l \min_{j,k} (\lambda_j, \mu_k)} - \frac{\epsilon}{6T \max_{j,k} (m_{jk})}\right),$$

and by (47), (49), and (52), we have that

$$I^n_k(i^n- \neg) = I^n_k((i-1)^n) + S^n_k(i^n) - S^n_k((i-1)^n) - R^n_{k,n}S(i^n) + R^n_{k,n}S((i-1)^n) \geq n\mu_k l^n \left(1 - \frac{\alpha}{l \min_{j,k} (\lambda_j, \mu_k)} - \frac{\epsilon}{6T \max_{j,k} (m_{jk})}\right).$$

Now, the inequality $|x| \geq x - 1$ and the bounds for the quantities $Q_j(i^n- \neg)^n$, $I_k(i^n- \neg)^n$ yield

$$\frac{M_{i,j}^{r,n}}{nl^n} = \frac{1}{nl^n} \left[ m_{jk} \min \left( \frac{Q^n_j(i^n- \neg)}{\lambda_j}, \frac{I^n_k(i^n- \neg)}{\mu_k} \right) \right] \geq m_{jk} \left(1 - \frac{\alpha}{l \min_{j,k} (\lambda_j, \mu_k)} - \frac{\epsilon}{6T \max_{j,k} (m_{jk})}\right) - \frac{1}{nl^n}$$

$$\geq m_{jk} - \frac{\epsilon}{6T} - \frac{\epsilon}{6T} = m_{jk} - \frac{\epsilon}{2T},$$

for large enough $n$. The second inequality from the definition of $\alpha$ and the fact that $\frac{1}{nl^n} = \frac{1}{n^{1/3}t} \to 0$, $n \to \infty$. That is, for any $t \geq 0$ and any feasible point $m$,

$$\frac{M_{i,j}^{r,n}(t)}{n} = \frac{1}{n} \sum_{i=1}^{[T/l^n]} M_{i,j}^{r,n} \geq \frac{|t/l^n|}{n} l^n (m_{jk} - \frac{\epsilon}{2T}) \geq \frac{|t/l^n|}{n} l^n m_{jk} - \frac{\epsilon}{2} \geq m_{j,k} t - m_{jk} l^n - \frac{\epsilon}{2} \geq m_{j,k} t - \epsilon,$$

where the last inequality follows by $n^{2/3} > 2 \max_{j,k} (m_{jk}) l / \epsilon$.

On the other hand, we have $\frac{M_{i,j}^{r,n}(t)}{n} \leq m_{j,k} t - \epsilon$ by definition of the matching-rate-based policy for any $0 \leq t \leq T$ and $\epsilon$ is arbitrary small independent of time $t$. This concludes the proof. \qed

**Proof of Theorem 5.** Let $\epsilon > 0$. Denote by $\Omega^n_1$ the events such that (46) and (47) hold, by $\Omega^n_2$ the events such both inequalities in Proposition 5 hold, and by $\Omega^n_3$ the events such that (51) and (52) hold. Note that $\lim_{n \to \infty} P(\Omega^n_1 \cap \Omega^n_2 \cap \Omega^n_3) = 1$ and consider $\omega \in \Omega^n_1 \cap \Omega^n_2 \cap \Omega^n_3$. To show the result is enough to prove that for each $j \in J$, $k \in K$, $i \geq 1$, and large enough $n \in \mathbb{N}$,

$$nl^n (\frac{m_{j,k}}{T} - \frac{\epsilon}{T}) \leq M_{i,j}^{r,n} \leq nl^n (\frac{m_{j,k}}{T} + \frac{\epsilon}{T}).$$

(53)
Then, we have that for $n \in \mathbb{N}$,
\[
\frac{M^{p,n}_{jk}(t)}{n} = \frac{1}{n} \sum_{i=1}^{\lceil t/l \rceil} M^{p,n}_{ijk} \geq \frac{t}{n} \left( m^*_j k - \epsilon \right) \geq \left( \frac{t}{l} - 1 \right) \frac{t}{T} \left( m^*_j k - \frac{\epsilon}{T} \right) \geq m^*_j k t - \epsilon,
\]
because $l^n = 1/n^{1/3} \to 0$ as $n \to \infty$. On the other hand, we have that
\[
\frac{M^{p,n}_{jk}(t)}{n} = \frac{1}{n} \sum_{i=1}^{\lceil t/l \rceil} M^{p,n}_{ijk} \leq \frac{t}{n} \left( m^*_j k + \epsilon \right) \leq m^*_j k t + \epsilon.
\]
We next focus on proving (53). Define for any $(j, k) \in \mathcal{P}_h(m^*)$ with $h = 0, \ldots, H$,
\[
\delta^h_{jk} := \left| \lambda_j - \sum_{k':(j,k') \in \mathcal{Q}_{h-1}(m^*)} m^*_{jk'} - \mu_k + \sum_{j':(j',k) \in \mathcal{Q}_{h-1}(m^*)} m^*_{j'k} \right|
\]
and
\[
\delta^+ := \begin{cases} \min(\delta^h_{jk}, \delta^h_{jk} > 0), & \text{if } \exists \delta^h_{jk} > 0, \\ 0, & \text{otherwise} \end{cases}.
\]
Further, define
\[
\delta := \begin{cases} \min(\frac{\delta^+}{2n+1}, \frac{\epsilon}{2n+1 T}), & \text{if } \delta^+ > 0, \\ \frac{\epsilon}{2n+1 T}, & \text{otherwise} \end{cases}.
\]
We shall show that for $(j, k) \in \mathcal{P}_h(m^*)$ and $i \geq 1,
\[
nl^n(m^*_j k - 2^{h+1} \delta) \leq M^{p,n}_{ijk} \leq nl^n(m^*_j k + 2^{h+1} \delta),
\]
then (53) follows by noticing that $2^{h+1} \delta \leq 2^{h+1} \frac{\epsilon}{2n+1 T} \leq \frac{\epsilon}{T}$. We proceed in a two level induction: i) on the priority sets $\mathcal{P}_h(m^*)$, $h = 0, \ldots, H + 1$ and ii) on the review periods $i \in \{1, \ldots, \lfloor T/l^n \rfloor \}$. In particular, we first proceed in an induction on the priority sets. Then, for any set we proceed in a second induction on the review periods inside this set.

Let $(j, k) \in \mathcal{P}_0(m^*)$ and without loss of generality assume that $\lambda_j \leq \mu_k$. That is, $m^*_jk = \lambda_j$ and $m^*_{j'k'} = 0$ for $k' \neq k$. By definition of the first priority set, we have that for any $i \geq 1$, $M^{p,n}_{ijk} = \min(Q^n_j(i^n) - I^n_k(i^n) - I^n_k(i^n))$. Also, notice that node $j$ can only be connected to nodes $k' \neq k$ such that $m^*_{j'k'} = 0$ because otherwise $(j, k)$ cannot be a priority arc. In other words, $(j, k') \in \mathcal{P}_{H+1}(m^*)$ for $k' \neq k$. First, using the same ideas as in the proof of Theorem 4, we derive for large enough $n$ and each $i \geq 1$,
\[
Q^n_j(i^n) - I^n_k(i^n) \geq nl^n(\lambda_j - \delta),
\]
\[
I^n_k(i^n) \geq nl^n(\mu_k - \delta).
\]
The latter inequalities yield a lower bound of the number of matches for each $i \geq i$.
\[
M^{p,n}_{ijk} \geq nl^n(\min(\lambda_j, \mu_k) - \delta) = nl^n(m^*_jk - \delta) \geq nl^n(m^*_jk - 2\delta).
\]

Next, we shall show that the opposite inequality holds as well. By (2), (3), Lemma 4 choosing $\alpha = \delta$, and observing that $R^n(i^n) - R^n((i - 1)l^n) \geq 0$, we have that for large $n$,
\[
Q^n_j(i^n) - Q^n_j((i - 1)l^n) + D^n_j(i^n) - D^n_j((i - 1)l^n) + nl^n(\lambda_j + \delta) \leq Q^n_j((i - 1)l^n) + nl^n(\lambda_j + \delta)
\]
(57)
and

\[ I_k^n(i^n) \leq I_k^n((i-1)^n) + S_k^n(i^n) - S_k^n((i-1)^n) \leq I_k^n((i-1)^n) + nl^n(\mu_k + \delta). \] (58)

Furthermore, by (2) and (3), we get that

\[
Q^n_j(i^n) = Q^n_j(0) + D^n_j(i^n) - R_jD^n(i^n) - \sum_{h=1}^{i} \sum_{k' \in K} M_{hjk'}^{p,n} \\
= Q^n_j(0) + D^n_j(i^n) - D^n_j((i-1)^n) - R_jD^n(i^n) + R_jD^n((i-1)^n) \\
+ D^n_j((i-1)^n) - R_jD^n((i-1)^n) - \sum_{h=1}^{i-1} \sum_{k' \in K} M_{hjk'}^{p,n} - \sum_{k' \in K} M_{ijk'}^{p,n} \\
= Q^n_j((i-1)^n) + D^n_j(i^n) - D^n_j((i-1)^n) - R_jD^n(i^n) + R_jD^n((i-1)^n) - \sum_{k' \in K} M_{ijk'}^{p,n} \\
= Q^n_j(i^n-1) - \sum_{k' \in K} M_{ijk'}^{p,n}.
\]

Hence, we have that

\[ Q^n_j(i^n) = Q^n_j(i^n-1) - \sum_{k' \neq k} M_{ijk'}^{p,n} - \min(Q^n_j(i^n-1), I^n_k(i^n-1)). \] (59)

In a similar way, we obtain

\[ I^n_k(i^n) = I^n_k(i^n-1) - \sum_{j' \neq \delta} M_{ij'k'}^{p,n}. \] (60)

By the last equations, we have that for each \( i \geq 1 \) at least one of the two queue-lengths is zero. To see this, note that if \( \min(Q^n_j(i^n-1), I^n_k(i^n-1)) = Q^n_j(i^n-1) \), then \( M_{ijk'}^{p,n} = 0 \) for all \( k' \neq k \) and hence \( Q^n_j(i^n) = 0 \). If \( \min(Q^n_j(i^n-1), I^n_k(i^n-1)) = I^n_k(i^n-1) \), then \( M_{ijk}^{p,n} = I_k^n \) and hence \( M_{ijk'}^{p,n} = 0 \) for \( j' \neq j \). That is, \( I_k^n(i^n) = 0 \) and \( \min(Q^n_j(i^n), I^n_k(i^n)) = 0 \) for any \( i \geq 1 \). Further, by (57) and (58), we have that for large \( n \),

\[
M_{ijk}^{p,n} = \min(Q^n_j(i^n-1), I^n_k(i^n-1)) \\
\leq \min(Q^n_j((i-1)^n) + nl^n\lambda_j, I^n_k((i-1)^n) + nl^n\mu_k) + nl^n\delta. \] (61)

Further, by the assumptions for the initial conditions, we have that \( Q^n_j(0) \to 0 \) and \( I^n_k(0) \to 0 \) as \( n \to \infty \). That is, \( Q^n_j(0) \leq nl^n\delta \) and \( I^n_k(0) \leq nl^n\delta \). If \( \lambda_j = \mu_k \), then (61) yields

\[
M_{ijk}^{p,n} \leq \min(Q^n_j(0), I^n_k(0)) + nl^n(\lambda_j + \delta) \leq nl^n(\lambda_j + 2\delta) = nl^n(m_{jk}^* + 2\delta)
\]

and for \( i > 1 \),

\[
M_{ijk}^{p,n} \leq \min(Q^n_j((i-1)^n), I^n_k((i-1)^n)) + nl^n(\lambda_j + \delta) \leq nl^n(\lambda_j + \delta) \leq nl^n(m_{jk}^* + 2\delta),
\]

because \( \min(Q^n_j((i-1)^n), I^n_k((i-1)^n)) = 0 \). Let \( \lambda_j < \mu_k \). We shall use induction in the review period \( i \) to show that \( Q^n_j(i^n-1) \leq I^n_k(i^n-1) \) and \( Q^n_j(i^n) = 0 \) for each \( i \geq 1 \) and for large enough \( n \).
We start by proving this for $i = 1$ (i.e., the first step of the induction). Replacing $i = 1$ in (57) and (58), it follows that $Q^n_j(l^n) \leq n^m(\lambda_j + 2\delta)$ and $I^n_k(l^n) \leq n^m(\mu_k + 2\delta)$. By the last inequalities, (55), (56), and observing that $\delta \leq \frac{\mu_k - \lambda_j}{3}$, we get

$$Q^n_j(l^n) \leq n^m(\lambda_j + 2\delta) \leq n^m(\mu_k - \delta) \leq I^n_k(l^n),$$

and by (59), $Q^n_j(l^n) = 0$. Replacing now $i = 2$ in (57), we obtain $Q^n_j(2l^n) \leq n^m(\lambda_j + 2\delta)$. Using again (56) and (59), we have that

$$Q^n_j(2l^n) \leq n^m(\lambda_j + 2\delta) \leq n^m(\mu_k - \delta) \leq I^n_k(2l^n),$$

and $Q^n_j(2l^n) = 0$. Let now $Q^n_j(zl^n) \leq I^n_k(zl^n)$ and $Q^n_j(zl^n) = 0$ for all $1 < z \leq i$ for some $i > 1$ be the hypothesis induction (for the induction in priority sets). Assume that (54) is true, and large enough $n$, we get

$$Q^n_j(zl^n) \leq n^m(\lambda_j + 2\delta) \leq n^m(m^{*}_{jk} + 2\delta)$$

and hence (54) holds.

Now, we make the hypothesis induction for the induction in priority sets. Assume that (54) holds for $(j, k) \in P_r(m^*)$ for all $r = 0, \ldots, h - 1$, with $h < H + 1$. We shall show (54) holds for $h$ as well. For $(j, k) \in P_h(m^*)$, by definition of the priority ordering, we have that

$$M^n_{ij} = \min \left( Q^n_j(i^n) - \sum_{k': (j, k') \in Q_{h-1}(m^*)} M^n_{ijk}, I^n_k(i^n) - \sum_{j': (j', k') \in Q_{h-1}(m^*)} M^n_{ij'k} \right).$$

Without loss of generality assume that $(j, k)$ makes node $j$ tight. As in the first induction step, we observe that node $j$ can only be connected through edges $(j, k') \notin Q_{h-1}(m^*)$ such that $m^{*}_{jk'} = 0$ because otherwise $(j, k)$ cannot be a priority arc. In other words, $(j, k') \in P_{H+1}(m^*)$. Furthermore, the optimal solution to the MP becomes

$$m^{*}_{jk} = \min \left( \lambda_j - \sum_{k': (j, k') \in Q_{h-1}(m^*)} m^{*}_{jk'}, \mu_k - \sum_{j': (j', k') \in Q_{h-1}(m^*)} m^{*}_{j'k} \right) = \lambda_j - \sum_{k': (j, k') \in Q_{h-1}(m^*)} m^{*}_{jk'}.$$

By (54) and (55), we derive for large enough $n$ and each $i \geq 1$,

$$Q^n_j(i^n) - \sum_{k': (j, k') \in Q_{h-1}(m^*)} M^n_{ijk} \geq n^m(\lambda_j - \delta - \sum_{k': (j, k') \in Q_{h-1}(m^*)} m^{*}_{jk'} - \sum_{z=0}^{h-1} 2^{z+1} \delta) \geq n^m(\lambda_j - \sum_{k': (j, k') \in Q_{h-1}(m^*)} m^{*}_{jk'} - (2^{h+1} - 1)\delta),$$

where we use that $\sum_{z=0}^{h-1} 2^{z+1} = 2^{h+1} - 2$. In a similar way,

$$I^n_k(i^n) - \sum_{j': (j', k) \in Q_{h-1}(m^*)} M^n_{ijk} \geq n^m(\mu_k - \sum_{j': (j', k) \in Q_{h-1}(m^*)} m^{*}_{jk} - (2^{h+1} - 1)\delta).$$
The last two inequalities yield a lower bound of the number of matches for each $i \geq 1$,

$$\mathcal{M}_{ij}^{p,n} \geq n l^n (m_{jk}^* - (2^{h+1} - 1)\delta) \geq n l^n (m_{jk}^* - 2^{h+1} \delta).$$

We now move in proving the opposite inequality. First observe that (57) and (58) continue to hold. Using a similar way as in the first induction step, we obtain

$$Q_j^n(i l^n) = Q_j^n(\tilde{i} l^n) - \sum_{k':(j,k') \in Q_{h-1}(m^*)} \mathcal{M}_{ijk'}^{p,n} - \mathcal{M}_{ij}^{p,n}$$

(65)

and

$$I_j^n(i l^n) = I_j^n(\tilde{i} l^n) - \sum_{j':(j',k) \in Q_{h-1}(m^*)} \mathcal{M}_{ij'}^{p,n} - \mathcal{M}_{ij}^{p,n}.$$ 

(66)

Further, by (65), (66), and the definition of the priority ordering, we have that $\min(Q_j^n(i l^n), I_j^n(i l^n)) = 0$ for any $i \geq 1$. By (54), (57), and (58), we get the following inequality for large enough $n$,

$$\mathcal{M}_{ij}^{p,n} = \min \left( Q_j^n(i l^n) - \sum_{k':(j,k') \in Q_{h-1}(m^*)} \mathcal{M}_{ijk'}^{p,n}, I_j^n(i l^n) - \sum_{j':(j',k) \in Q_{h-1}(m^*)} \mathcal{M}_{ij'}^{p,n} \right)$$

$$\leq \min \left( Q_j^n((i - 1) l^n) + n l^n (\lambda_j + \delta) - n l^n \left( \sum_{k':(j,k') \in Q_{h-1}(m^*)} m_{jk'}^* - (2^{h+1} - 1)\delta \right), \right.$$ 

$$I_j^n((i - 1) l^n) + n l^n (\mu_k + \delta) - n l^n \left( \sum_{j':(j',k) \in Q_{h-1}(m^*)} m_{jk'}^* - (2^{h+1} - 1)\delta \right) \right)$$

(67)

If $\lambda_j - \sum_{k':(j,k') \in Q_{h-1}(m^*)} m_{jk'}^* \mu_k = \lambda_j - \sum_{j':(j',k) \in Q_{h-1}(m^*)} m_{jk'}^*$, then (67) and the assumptions for the initial conditions yield

$$\mathcal{M}_{ij}^{p,n} \leq \min(Q_j^n((i - 1) l^n), I_j^n((i - 1) l^n)) + n l^n (m_{jk}^* + 2^{h+1} \delta) = n l^n (m_{jk}^* + 2^{h+1} \delta).$$

Let $\lambda_j - \sum_{k':(j,k') \in Q_{h-1}(m^*)} m_{jk'}^* \mu_k < \lambda_j - \sum_{j':(j',k) \in Q_{h-1}(m^*)} m_{jk'}^*$. We shall use the same ideas as in the first induction step to show the upper bound in (54). Assume that

$$Q_j^n(z l^n) - \sum_{k':(j,k') \in Q_{h-1}(m^*)} \mathcal{M}_{ijk'}^{p,n} \leq I_j^n(z l^n) - \sum_{j':(j',k) \in Q_{h-1}(m^*)} \mathcal{M}_{ij'}^{p,n}$$

and $Q_j^n(z l^n) = 0$ for all $z \leq i$ for some $i > 1$. By (54) for $(j,k') \in Q_{h-1}(m^*)$, (57), and (54), and
observing that \( \delta \leq \frac{\mu_k - \sum_{j' \in J} |m_j^{*} - \lambda_j + \sum_{j'' \in J} |m_{j''}^{*} - \lambda_j|_2}{2^{n+1} + 1} \), we have that

\[
Q^n_j((i + 1)l^n) - \sum_{k', (j,k') \in \mathcal{Q}_{h-1}(m^*)} m_{k'}^{(i+1)l^n} \leq nl^n(\lambda_j - \sum_{k', (j,k') \in \mathcal{Q}_{h-1}(m^*)} m_{k'}^{*}) + 2^{h+1}\delta
\]

\[
\leq nl^n(\mu_k - \sum_{j' \in J} |m_{j'}^{*} - (2^{h+1} - 1)\delta)
\]

\[
\leq l^n_q((i + 1)l^n) - \sum_{j' \in J} |m_{j'}^{*} - (2^{h+1} - 1)\delta)
\]

and hence \(Q^n_j((i + 1)l^n) = 0\) by (65). That is, by (67) for each \(i \geq 1, (j, k) \in \mathcal{P}_h(m^*)\), and large enough \(n\),

\[
M_{ijk}^{p,n} \leq nl^n(\lambda_j - \sum_{k', (j,k') \in \mathcal{Q}_{h-1}(m^*)} m_{k'}^{*}) + 2^{h+1}\delta = nl^n(m_{j}^{*} + 2^{h+1}\delta)
\]

and hence (54) holds.

Last, it is clear from the previous steps that (54) holds for \((j, k) \in \mathcal{P}_{h+1}(m^*)\) since \(M_{ijk}^{p,n} = m_{jk}^{*} = 0\) for large \(n\). That is, (54) holds for any \((j, k) \in \mathcal{P}_h(m^*)\) with \(h = 0, \ldots, H + 1\).

\[\square\]

### A.4 Proofs for Section 6

Before we move to the main part of the proof of Theorem 6 and Proposition 4, we show some helpful properties of the value of matching problem (32).

Let \(v := \max_{j} v_{jk}\). For two vectors \(\mathbf{x} \in \mathbb{R}^J\) and \(\mathbf{b} \in \mathbb{R}^K\), let \(m_{jk}^{*}(\mathbf{x}, \mathbf{b})\) be an optimal solution to (32) if we replace \(\lambda\) and \(\mu\) by \(\mathbf{x}\) and \(\mathbf{b}\), respectively. Define the value of the matching problem

\[
F(\mathbf{x}, \mathbf{b}) := \sum_{j,k} v_{jk}m_{jk}^{*}(\mathbf{x}, \mathbf{b})
\]

**Lemma 5.** Let \(\lambda \in \mathbb{R}_+^J\) and \(\mu \in \mathbb{R}_+^K\). There exists a Lipschitz continuous mapping \(m^* : \mathbb{R}_+^J \times \mathbb{R}_+^K \rightarrow \mathbb{R}_+^J \times \mathbb{R}_+^K\) such that \(m^*(\lambda, \mu)\) is an optimal solution to (32)\(^1\).

**Proof.** We begin by observing that the feasible region of (32) is nonempty, closed, and convex. The remainder of the proof follows by using the Lipschitz selection theorem (Aubin and Frankowska 1990, Theorem 9.4.3) and (Schrijver 1986, Theorem 10.5) exactly as in (Bassamboo et al. 2006, Proposition 2).

**Lemma 6.** The value of (32) is nondecreasing in each of its arguments and the following inequalities hold, for any \(\alpha \geq 0\),

\[
F(\mathbf{x} + \alpha e_j, \mathbf{b}) - F(\mathbf{x}, \mathbf{b}) \in [0, \bar{v}JK\alpha], \quad \forall j \in J,
\]

\[
F(\mathbf{x}, \mathbf{b} + \alpha e_k) - F(\lambda, \mu) \in [0, \bar{v}JK\alpha], \quad \forall k \in K,
\]

where the constant \(C\) is the Lipschitz constant in Lemma 5.

\(^1\)Lemma 5 implies that there exists an optimal solution to (32) that is Lipschitz continuous. The later holds for any norm \(\|\cdot\|\) in \(\mathbb{R}_+^J \times \mathbb{R}_+^K\) as all norms in this space (and in general in any vector space with finite dimension) are equivalent. In the sequel, we freely use the norm that suits our approach.
Clearly, $F(\cdot, \cdot)$ is nondecreasing in each of its arguments by the definition of the MP. We shall show that the rate of change is bounded. To this end, observe that by Lemma 5 there exists a Lipschitz continuous optimal solution to (32) such that

$$\|m^*(x + \alpha e_j, b) - m^*(x, b)\|_\infty \leq C \|(x + \alpha e_j, b) - (x, b)\|_\infty \leq C \alpha.$$  

The latter leads to

$$\left| m^*_{jk}(x + \alpha e_j, b) - m^*_{jk}(x, b) \right| \leq C \alpha,$$

for any $j \in J$ and $k \in K$. Now, using the last inequality, we have that

$$0 \leq F(x + \alpha e_j, b) - F(x, b) = \sum_{j,k} v_{jk} (m^*_{jk}(x + \alpha e_j, b) - m^*_{jk}(x, b)) \leq \sum_{j,k} v_{jk} \left| m^*_{jk}(x + \alpha e_j, b) - m^*_{jk}(x, b) \right| \leq \bar{v}JKC \alpha.$$

The second inequality follows analogously. Note that both inequalities do not depend on the particular optimal solution $m^*(\cdot, \cdot)$, that is Lipschitz continuous, as any solution achieves the same value.

**Lemma 7.** For any $\epsilon > 0$, and any $A \geq x - \epsilon e$ and $B \geq b - \epsilon e$,

$$F(A, B) \geq F(x, b) - C_1 \epsilon,$$

where $C_1 := (J + K)JKC \bar{v}$.

**Proof.** First, by a telescoping sum (and assuming an empty sum evaluates to zero), we observe that

$$F(x, b) - F(x - \epsilon e, b - \epsilon e) = \sum_{j=1}^{J} [F(x - \epsilon \sum_{s=1}^{j-1} e_s, b) - F(x - \epsilon \sum_{s=1}^{j} e_s, b)] + \sum_{k=1}^{K} [F(x - \epsilon e, b - \epsilon \sum_{t=1}^{k-1} e_t) - F(x - \epsilon e, b - \epsilon \sum_{t=1}^{k} e_t)] \leq (J + K)JKC \bar{v} \epsilon,$$

where the inequality follows from the upper bound of Lemma 6 for each incremental change in the objective value. Then, since the lower bound in Lemma 6 implies $F(\cdot, \cdot)$ is monotone nondecreasing, we have

$$F(A, B) \geq F(x - \epsilon e, b - \epsilon e) \geq F(x, b) - (J + K)JKC \bar{v} \epsilon,$$

completing the proof.

**Lemma 8.** The constraint matrix of (33) is totally unimodular, which implies that an optimal extreme point solution is integer valued if the right hand side constraints of (33) are integer valued.
Proof. Totally unimodular matrices with elements $a_{il}$ are characterized by Tamir (1976) as those where every subset of rows $R$ can be partitioned into two subsets, $R_1 \cup R_2 = R$, such that $\sum_{i \in R_1} a_{il} - \sum_{i \in R_2} a_{il} \in \{-1, 0, 1\}$ for all columns $l$. The constraint matrix of (33) has a row for each $k \in \mathbb{K}$ and each $j \in \mathbb{J}$ and a column for every variable $y_{jk}$. For a given $j \in \mathbb{J}$ representing a row and $j' \in \mathbb{J}, k' \in \mathbb{K}$ representing a column, the corresponding element of the constraint matrix is given by

$$a_{(j):(j'k')} = \begin{cases} 1, & j' = j \vspace{1pt} \\ 0, & j' \neq j \end{cases}$$

representing that only those $y_{j'k'}$ with $j' = j$ are included in the sum for the constraint corresponding to $j$. Similarly, for a given $k \in \mathbb{K}$ and $j' \in \mathbb{J}, k' \in \mathbb{K}$, the corresponding constraint matrix element is

$$a_{(k):(j'k')} = \begin{cases} 1, & k' = k \vspace{1pt} \\ 0, & k' \neq k \end{cases}$$

Any subset of the rows of the constraint matrix corresponds to the union of some subsets of $\mathbb{J}$ and $\mathbb{K}$, i.e. $J' \cup K'$ where $J' \subseteq \mathbb{J}$ and $K' \subseteq \mathbb{K}$. Therefore, given a subset of rows $J' \cup K'$, define the required partition as $R_1 = J'$ and $R_2 = K'$. Then, for a given variable $y_{j'k'}$,

$$\sum_{j \in J'} a_{(j):(j'k')} - \sum_{k \in K'} a_{(k):(j'k')} = \begin{cases} -1, & j' \notin J' \text{ and } k' \in K' \vspace{1pt} \\ 0, & (j' \in K' \text{ and } k' \in K') \text{ or } (j' \notin J' \text{ and } k \notin K') \vspace{1pt} \\ 1, & j' \notin J' \text{ and } k' \notin K' \end{cases}$$

This completes the proof that the constraint matrix of (33) is totally unimodular. Then, by Tamir (1976), if the right hand side constraints of (33) are integer valued, any optimal extreme point solution to (33) is also integer valued. A similar unimodularity property is proven in DeValve et al. (2021). \qed

Proof of Theorem 6. Let $\epsilon > 0$ and $\alpha = \frac{\epsilon}{C_1 T}$, where $C_1$ is given in Lemma 7. Adapting the proof of Theorem 4, we choose large enough $n$, such that

$$\frac{1}{n l^n} \sum_{m = D_n^p ((i-1) l^n) + 1}^{D_n^p (i l^n)} 1\{r_{jm} \leq l^n\} \leq \frac{\epsilon}{C_1 T} \quad (69)$$

and

$$\frac{1}{n l^n} \sum_{m = S_n^p ((i-1) l^n) + 1}^{S_n^p (i l^n)} 1\{r_{km} \leq l^n\} \leq \frac{\epsilon}{C_1 T} \quad (70)$$

Define $\Omega_4^n$ the events such that (69) and (70) hold. In the sequel, we take $\omega \in \Omega_4^n \cap \Omega_2^n \cap \Omega_4^n$.

The next step is to show the following inequality for large enough $n$,

$$V_{M^{k,n}}(t) \geq \sum_{j \in \mathbb{J}, k \in \mathbb{K}} v_{jk} m_{jk}^* t - \epsilon. \quad (71)$$
Follow similar steps as in proof of Theorem 4, we obtain for \( i \in \{1, \ldots, \lfloor t/l \rfloor \}, \)
\[
\frac{Q^n_j(\ell^n_\text{i} - \ell^n)}{n} \geq \lambda_j - \frac{\epsilon}{C_1 t}
\]
and
\[
\frac{I^n_k(\ell^n_\text{i} - \ell^n)}{n} \geq \mu_k - \frac{\epsilon}{C_1 t}.
\]
Applying now Lemma 7, we have that for \( i \in \{1, \ldots, \lfloor t/l \rfloor \}, \)
\[
F \left( \frac{Q^n_j(\ell^n_\text{i} - \ell^n)}{n}, \frac{I^n_k(\ell^n_\text{i} - \ell^n)}{n} \right) \geq F(\lambda, \mu) - \frac{\epsilon}{t}.
\]
By the properties of linear programming, we have that \( F(\cdot, \cdot) \) is homogenous of degree one. That is,
\[
F \left( \frac{Q^n_j(\ell^n_\text{i} - \ell^n)}{n}, \frac{I^n_k(\ell^n_\text{i} - \ell^n)}{n} \right) = \frac{1}{n} F \left( Q^n_j(\ell^n_\text{i} - \ell^n), I^n_k(\ell^n_\text{i} - \ell^n) \right).
\]
Now observe that the optimization problem (33), which decides the matches for the LP-based policy, and the MP (32) have the same objective function and the same constraint matrix, which is totally unimodular by Lemma 8. This implies the well known fact that these programs have the same optimal value if they have the same right hand side constraints, and if these constraints are integer valued (i.e., the objective value of the integer program (33) is equal to the objective value of its LP relaxation (32); see for example (Papadimitriou and Steiglitz 1998, Theorem 13.2). Combining all the above together, we derive
\[
\sum_{j \in J, k \in K} v_{jk} M^{b,n}_{jk}(t) = \sum_{i=1}^{\lfloor t/l \rfloor} \frac{F \left( Q^n_j(\ell^n_\text{i} - \ell^n), I^n_k(\ell^n_\text{i} - \ell^n) \right)}{n} \geq \sum_{i=1}^{\lfloor t/l \rfloor} \frac{1}{n} F \left( Q^n_j(\ell^n_\text{i} - \ell^n), I^n_k(\ell^n_\text{i} - \ell^n) \right) \geq \frac{l^n}{t} \left( F(\lambda, \mu) - \frac{\epsilon}{t} \right) \geq tF(\lambda, \mu) - F(\lambda, \mu) l^n.
\]
By definition of \( F(\cdot, \cdot) \) and for \( n \) such that \( n^{2/3} > F(\lambda, \mu) l/\epsilon \), we have that
\[
\frac{V^{b,n}_{M}(t)}{n} = \sum_{j \in J, k \in K} v_{jk} M^{b,n}_{jk}(t) = \sum_{j \in J, k \in K} v_{jk} m^{*}_{jk} t - \epsilon,
\]
and hence (71) follows.

The last step is to show the opposite inequality of (71). To this end, observe that \( M^n(t) \) is clearly an admissible policy. The desirable inequality follows by Theorem 3 setting \( c_j^D = c_k^S = 0 \). The proof is completed by applied the squeeze theorem.

**Proof of Proposition 4.** Without loss of generality assume that \( t = 1 \) otherwise fix a \( t > 0 \) and appropriately scale (32) by \( t \). First, we note that the set of optimal solutions \( S \) is a closed set.
To see this observe that if it is not, then we can always increase one of the components of \( y^* \) and achieve a better optimal value.

In the sequel, we proceed to the proof of the result by contradiction. Fix a sample path such that Theorem 6 holds and assume that for all \( j \in J \) and \( k \in K \), we have that \( \frac{M_{jk}^{b,n}(1)}{n} \leq \frac{D_j(1)}{n} \) and \( \frac{D_j(1)}{n} \to \lambda \) almost surely. In other words, \( \frac{M_{jk}^{b,n}(1)}{n} \) is a bounded sequence in \( \mathbb{R}^{J \times K} \). Now, by Bolzano–Weierstrass theorem there exists a convergent subsequence, which with abuse of notation we denote by \( \frac{M_{jk}^{b,n}(1)}{n} \to x \), with \( x \notin S \). Note that we can assume the last without loss of generality since if all the subsequential limits lies in the close set \( S \), then the distance of the whole sequence from \( S \) must convergence to zero which yields a contradiction.

Now, consider \( \frac{M_{jk}^{b,n}(1)}{n} \) along the previous subsequence. First, note that by the admissibility of the LP-based policy any limit should be feasible to (32). To see this, observe that for all \( j \in J \) and \( k \in K \) the following hold almost surely by the nonnegativity of the queue-lengths

\[
\sum_{k \in K} \frac{M_{jk}^{b,n}(1)}{n} \leq \frac{D_j(1)}{n} \quad \text{and} \quad \sum_{j \in J} \frac{M_{jk}^{b,n}(1)}{n} \leq \frac{S_k(1)}{n},
\]

and by law of large numbers \( \lim_{n \to \infty} \frac{D_j(1)}{n} = \lambda_j, \lim_{n \to \infty} \frac{S_k(1)}{n} = \mu_k \). Now, we have that

\[
\sum_{j \in J, k \in K} v_{jk} \frac{M_{jk}^{b,n}(1)}{n} \to \sum_{j \in J, k \in K} v_{jk} x_{jk},
\]

and by Theorem 6 \( \sum_{j \in J, k \in K} v_{jk} x_{jk} = \sum_{j \in J, k \in K} v_{jk} m^*_j \). In other words, the vectors \( x \) and \( m^* \) achieve the same optimal value of (32) and so \( x \) is an optimal solution to (32). On the other hand, \( x \notin S \) which yields a contradiction.

\[\square\]

## B The state space

### B.1 Notation

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and \(Y\) be a metric space. Denote by \(\mathcal{C}_b(Y,Y)\) the space of functions \(f: Y \to Y\) that are continuous and bounded and by \(\mathcal{D}(Y,Y)\) the Skorokhod space; i.e., the space of functions \(f: Y \to Y\) that are right continuous with left limits endowed with the \(J_1\) topology; cf. Chen and Yao (2001). It is well-known that the space \((\mathcal{D}(Y,Y), J_1)\) is a complete and separable metric space (i.e., a Polish metric space); Billingsley (1999). We denote by \(\mathcal{B}(Y)\) the Borel \(\sigma\)-algebra of \(\mathcal{D}(Y,Y)\). We assume that all the processes are defined from \((\Omega, \mathcal{F}, \mathbb{P})\) to \((\mathcal{B}(\mathcal{D}[0,T]^m), \mathcal{D}[0,T]^m)\). Further, we write \(X(\cdot) := \{X(t), t \geq 0\}\) to represent a stochastic process and \(X(\infty)\) to represent a stochastic process in steady-state. For \(H \in [0, \infty]\), let \(\mathcal{M}[0,H]\) denote the set of finite, non-negative Borel measures on \([0,L]\) endowed with the topology of weak convergence. Given a measure \(\nu \in \mathcal{M}[0,H]\) and a Borel measurable function \(f\) define \(\langle f, \nu \rangle := \int_{[0,H]} f(x)\nu(dx)\).

### B.2 Potential measures

A state in our model is a vector \(Y := (Q,I,\eta^D,\eta^S) \in \mathcal{Y}\) where

\[
\mathcal{Y} := \mathbb{Z}_+^{J \times K} \times (\times_{j=1}^J \mathcal{M}[0,H_j^0]) \times (\times_{j=1}^K \mathcal{M}[0,H_k^S]),
\]
and $H_j^D$, $H_k^S$ are the right edges of the support of the cumulative patience time distribution functions for any $j \in \mathbb{J}$ and $k \in \mathbb{K}$.

Here, we focus on defining the measures $\eta^D(\cdot)$ and $\eta^S(\cdot)$. First, we need to define the potential waiting time of the $l$th type $j$ potential customer and type $k$ potential worker at time $t \geq 0$. We have already done this for $l \in \mathbb{N}$ in Section 2.2, and so here we focus on case $l < 1$. For each $j \in \mathbb{J}$ and $k \in \mathbb{K}$, there are $\langle 1, \eta_j^D(0) \rangle$ and $\langle 1, \eta_k^S(0) \rangle$ type $j$ potential customers and type $k$ potential workers that arrived at or prior to time zero whose potential abandonment time is after time zero. Let $0 \leq w_{jl}^D(0) < H_j^D$ (resp. $0 \leq w_{kl}^S(0) < H_k^S$) for $l = -\langle 1, \eta_j^D(0) \rangle + 1, \ldots, 0$ (resp. $l = -\langle 1, \eta_k^S(0) \rangle + 1, \ldots, 0$) be the amount of time that has elapsed since type $j$ zero potential customer $l$ (type $k$ zero potential worker $h$) arrived. We assume that the sequences of $w_{jl}^D(0)$ and $w_{kl}^S(0)$ are non-increasing in $l$, respectively, and set $e_{jl} = -w_{jl}^D(0)$ and $e_{kl} = -w_{kl}^S(0)$. For $l = -\langle 1, \eta_j^D(0) \rangle + 1, \ldots, 0$ and $h = -\langle 1, \eta_k^S(0) \rangle + 1, \ldots, 0$, the patience times of type $j$ zero potential customer $l$ and type $k$ zero potential worker $l$ are given by

$$r_{jl}^D = \inf\left\{ t > 0 : G_{jl}^D(t > U_{jl}) \right\} + w_{jl}^D(0)$$

and

$$r_{kl}^S = \inf\left\{ t > 0 : G_{kl}^S(t > U_{kl}) \right\} + w_{kl}^S(0),$$

where $U_{jl}^D$ and $U_{kl}^S$ are i.i.d uniform $(0, 1)$ random variables. Now, for each $l \geq -\langle 1, \eta_j^D(0) \rangle + 1$ and $t \geq 0$, the potential waiting time of type $j$ potential customer $l$ at time $t$ is given by

$$w_{jl}^D(t) := \min\left\{ [t - e_{jl}]^+, r_{jl}^D \right\},$$

and in an analogous way we define the potential waiting time of the $l$th type $k$ potential worker at time $t \geq 0$,

$$w_{kl}^S(t) := \min\left\{ [t - e_{kl}]^+, r_{kl}^S \right\}.$$  

The type $j$ potential customer queue process and the type $k$ potential worker queue process are given by the following measure-valued processes, respectively: for any $t \geq 0$ and any Borel set $B \in \mathcal{B}(\mathbb{R}_+)$,

$$\eta_j^D(t)(B) := \sum_{l = -\langle 1, \eta_j^D(0) \rangle + 1}^{D_j(t)} \delta_{w_{jl}^D(t)}(B) 1_{\{0 \leq t - e_{jl}^D < r_{jl}^D\}},$$

$$\eta_k^S(t)(B) := \sum_{l = -\langle 1, \eta_k^S(0) \rangle + 1}^{S_k(t)} \delta_{w_{kl}^S(t)}(B) 1_{\{0 \leq t - e_{kl}^S < r_{kl}^S\}},$$

where $\delta$ is the dirac measure. Then, for each $t \geq 0$ and $j \in \mathbb{J}$, $\langle 1, \eta_j^D(t) \rangle$ is the number of type $j$ potential customers in the queue that arrived by time $t$ and whose potential waiting time is less than their patience time. Note that at time $t$ such customers may be in queue or may have been matched. For each $t \geq 0$ and $k \in \mathbb{K}$, $\langle 1, \eta_k^S(t) \rangle$ has an analogous meaning.

It is useful to define the potential reneging processes that are independent of the matching control policy and do not take into account if customers and workers are matched in contrast to the reneging processes. To this end, for any $j \in \mathbb{J}$, $k \in \mathbb{K}$ and $t \geq 0$ the potential reneging processes are given by

$$S_j^D(t) := \sum_{l = -\langle 1, \eta_j^D(0) \rangle + 1}^{D_j(t)} 1_{\{there exists s \in [0,t] such that \frac{dw_{jl}^D(s-)}{dt} > 0 and \frac{dw_{jl}^D(s+)}{dt} = 0\}}$$
and
\[ S_k^S(t) := \sum_{h=-\langle 1, \eta_k^S(0) \rangle + 1}^{S_k(t)} 1 \{ \text{there exists } s \in [0,t] \text{ such that } \frac{d}{dt} S_k^S(s-) > 0 \text{ and } \frac{d}{dt} S_k^S(s+) = 0 \} . \] (75)

The potential measures satisfy the following balance equations for each \( j \in \mathbb{J}, k \in \mathbb{K}, \) and \( t \geq 0 , \)
\[ \langle 1, \eta_j^D(t) \rangle + D_j(t) = \langle 1, \eta_j^D(0) \rangle + S_j^D(t) \] (76)
and
\[ \langle 1, \eta_k^S(t) \rangle + S_k(t) = \langle 1, \eta_k^S(0) \rangle + S_k^D(t) . \] (77)

Having defined the state descriptor, we can now formally define the nonanticipating property that appears in definition of an admissible policy, i.e., Definition 1. In other words, we require a matching process to be adapted to the filtration of the history of the state descriptor as the following definition states:

**Definition 5.** An admissible matching policy \( M(\cdot) \) is **nonanticipating** if it is \( G_t \)-adapted where \( G_t \) is the following filtration: for \( t \geq 0 , \)
\[ G_t = \sigma (Y(s) = (Q(s), I(s), \eta^D(s), \eta^S(s)) , \ 0 \leq s \leq t) . \]

### C Exact calculations of the mean queue-lengths in Figure 3b

Let \( X(\infty) \) be a Markov chain with transition rates are shown in Figure 9.

![Figure 9: Transition rates of the Markov chain.](image)

Its probability distribution \( \pi(\cdot) \) is the (unique) solution of the balance equations that are written as follows: for \( x > 0 \)
\[ (\lambda + \mu + x\theta)\pi(x) = \lambda\pi(x-1) + (\mu + (x+1)\theta)\pi(x+1), \]
\[ (\lambda + \mu + x\theta)\pi(-x) = \mu\pi(-x+1) + (\lambda + (x+1)\theta)\pi(-x-1), \]
and \( x = 0 , \)
\[ (\lambda + \mu)\pi(0) = (\lambda + \theta)\pi(-1) + (\mu + \theta)\pi(1). \]

The solution to the balance equations in given by
\[ \pi(x) = \begin{cases} \frac{\lambda^x}{\prod_{j=1}^{\infty} (\mu + j\theta)} \pi(0) & \text{if } x > 0, \\ \frac{\mu^x}{\prod_{j=1}^{\infty} (\lambda + j\theta)} \pi(0) & \text{if } x < 0, \end{cases} \] (78)

where
\[ \pi(0) = \left( 1 + \sum_{x=1}^{\infty} \frac{\lambda^x}{\prod_{j=1}^{\infty} (\mu + j\theta)} + \sum_{x=1}^{\infty} \frac{\mu^x}{\prod_{j=1}^{\infty} (\lambda + j\theta)} \right)^{-1} . \]
To see this, observe that for $x > 0$,

$$
\lambda \pi(x - 1) + (\mu + (x + 1)\theta)\pi(x + 1) = \left(\frac{\lambda^{x-1}}{\prod_{j=1}^{x}(\mu + j\theta)} + (\mu + (x + 1)\theta)\frac{\lambda^{x+1}}{\prod_{j=1}^{x+1}(\mu + j\theta)}\right)\pi(0)
$$

$$
= \left(\frac{\lambda x}{\prod_{j=1}^{x}(\mu + j\theta)} + \lambda\frac{\lambda^x}{\prod_{j=1}^{x+1}(\mu + j\theta)}\right)\pi(0)
$$

$$
= (\lambda + \mu + x\theta)\pi(x).
$$

In a similar way, for $x > 0$ we have that

$$
\mu \pi(-x + 1) + (\lambda + (x + 1)\theta)\pi(-x - 1) = \left(\frac{\mu^{x-1}}{\prod_{j=1}^{x}(\lambda + j\theta)} + (\lambda + (x + 1)\theta)\frac{\mu^{x+1}}{\prod_{j=1}^{x+1}(\lambda + j\theta)}\right)\pi(0)
$$

$$
= \left(\frac{\mu x}{\prod_{j=1}^{x}(\lambda + j\theta)} + \mu\frac{\mu^x}{\prod_{j=1}^{x+1}(\lambda + j\theta)}\right)\pi(0)
$$

$$
= (\lambda + \mu + x\theta)\pi(-x).
$$

Last, we have that

$$
(\lambda + \theta)\pi(-1) + (\mu + \theta)\pi(1) = \left(\frac{(\lambda + \theta)}{\lambda + \theta} + (\lambda + \theta)\frac{\lambda}{\mu + \theta}\right)\pi(0)
$$

$$
= (\lambda + \mu)\pi(0).
$$

In other words, (78) satisfies the balance equations. Now, adapting Ward and Glynn (2003), we derive the following formulas

$$
\prod_{j=1}^{x}(\mu + j\theta) = \frac{1}{\mu} \prod_{j=0}^{x}(\mu + j\theta) = \frac{\theta^{x+1}\Gamma\left(\frac{\theta}{\theta} + x + 1\right)}{\mu\Gamma\left(\frac{\theta}{\theta}\right)} ,
$$

$$
\prod_{j=1}^{x}(\lambda + j\theta) = \frac{1}{\lambda} \prod_{j=0}^{x}(\lambda + j\theta) = \frac{\theta^{x+1}\Gamma\left(\frac{\theta}{\theta} + x + 1\right)}{\lambda\Gamma\left(\frac{\theta}{\theta}\right)} ,
$$

$$
\sum_{x=1}^{\infty} \frac{\lambda^x}{\prod_{j=1}^{x}(\mu + j\theta)} = \gamma\left(\frac{\lambda}{\theta}, \frac{\mu}{\theta}\right)\left(\frac{\lambda}{\theta}\right)^{-\mu/\theta} \frac{\mu}{\theta} e^{\lambda/\theta} - 1 ,
$$

$$
\sum_{x=1}^{\infty} \frac{\mu^x}{\prod_{j=1}^{x}(\lambda + j\theta)} = \gamma\left(\frac{\mu}{\theta}, \frac{\lambda}{\theta}\right)\left(\frac{\mu}{\theta}\right)^{-\lambda/\theta} \frac{\lambda}{\theta} e^{\mu/\theta} - 1 ,
$$

where $\gamma(x, y) := \int_{0}^{x} t^{y-1} e^{-t} dt$ and $\Gamma(y) := \gamma(\infty, y)$. The first two equations follow directly by Ward and Glynn (2003). We show the third equation and the forth one follows by symmetry. By the
power series expansion of the incomplete gamma function we have that

\[ \gamma\left(\frac{\lambda}{\theta}, \frac{\mu}{\theta}\right) = \left(\frac{\lambda}{\theta}\right)^{\mu/\theta} e^{-\lambda/\theta} \sum_{x=0}^{\infty} \frac{(\lambda/\theta)^x}{\sum_{j=0}^{\infty} \Pi_{j=0}^{x} (\mu/\theta + j)} \]

\[ = \left(\frac{\lambda}{\theta}\right)^{\mu/\theta} e^{-\lambda/\theta} \sum_{x=0}^{\infty} \frac{\lambda^x}{\prod_{j=0}^{x} (\mu + j\theta)} \]

\[ = \left(\frac{\lambda}{\theta}\right)^{\mu/\theta} e^{-\lambda/\theta} \frac{\theta}{\mu} \sum_{x=0}^{\infty} \frac{\lambda^x}{\prod_{j=0}^{x} (\mu + j\theta)} \left(\frac{\sum_{x=1}^{\infty} \lambda^x}{\prod_{j=1}^{x} (\mu + j\theta)} + 1\right), \]

where we define an empty product to be one.

Let \( \pi^n(\cdot) \) denote the distribution of \( X^n(\infty) \) with rates \( n\lambda \) and \( n\mu \). The means of \( X^{+,n}(\infty) \) and \( X^{-,n}(\infty) \) are given by the following expressions

\[ \frac{1}{n} \mathbb{E}[X^{+,n}(\infty)] = \frac{1}{n} \sum_{x=-\infty}^{\infty} \max(x, 0) \pi^n(x) = \frac{1}{n} \sum_{x=1}^{\infty} x \pi^n(x) \]

\[ = \frac{\mu}{\theta} \Gamma\left(\frac{n\mu}{\theta}\right) \pi^n(0) \sum_{x=1}^{\infty} \frac{(\lambda/\theta)^x}{\Gamma\left(\frac{n\mu}{\theta} + x + 1\right)}, \]

and

\[ \frac{1}{n} \mathbb{E}[X^{-,n}(\infty)] = \frac{\lambda}{\theta} \Gamma\left(\frac{n\lambda}{\theta}\right) \pi^n(0) \sum_{x=1}^{\infty} \frac{(\mu/\theta)^x}{\Gamma\left(\frac{n\lambda}{\theta} + x + 1\right)}, \]

where

\[ \pi^n(0) = \left( \gamma\left(\frac{n\lambda}{\theta}, \frac{n\mu}{\theta}\right), \frac{n\lambda}{\theta}\right)^{-n\mu/\theta} e^{n\lambda/\theta} + \gamma\left(\frac{n\mu}{\theta}, \frac{n\lambda}{\theta}\right) \left( \frac{n\mu}{\theta}\right)^{-n\lambda/\theta} e^{n\mu/\theta} - 1 \right)^{-1}. \]

When \( \mu/\theta := \alpha \in \mathbb{N} \) and \( \lambda/\theta := \beta \in \mathbb{N} \), we have that

\[ \sum_{x=1}^{\infty} \frac{x \pi^n(x)}{\Gamma\left(\frac{n\lambda}{\theta} + x + 1\right)} = \frac{\beta - \alpha n - an + 1}{(an)!} \left( -ae^{\alpha n} \gamma(an + 1, b\mu) + b^\alpha \mu e^{\beta n} \gamma(an + 1, b\mu) + b^{\alpha n + 1} n \pi^n\right). \]

In the special case where \( \lambda = \mu = \theta \), by observing that \( \Gamma(n) = (n-1)! \) and \( \sum_{x=1}^{\infty} \frac{x^n}{(n+x)!} = \frac{1}{(n-1)!} \), the above expressions can be simplified further as follows

\[ \frac{1}{n} \mathbb{E}[X^{+,n}(\infty)] = \frac{1}{n} \mathbb{E}[X^{-,n}(\infty)] = (n-1)! \pi^n(0) \sum_{x=1}^{\infty} \frac{x \pi^n(x)}{(n+x)!} = \pi^n(0), \]

where \( \pi^n(0) = \left(2 \gamma(n, n) n^{-n+1} e^n - 1\right)^{-1} \) and \( \gamma(x, n) := \int_{0}^{x} t^{n-1} e^{-t} dt = (n-1)! \left( 1 - e^{-x} \sum_{k=0}^{n-1} \frac{x^k}{k!} \right). \)

Moreover, the mean queue-lengths are exactly \( \frac{1}{n} \mathbb{E}[Q^n(\infty)] = \frac{1}{n} \mathbb{E}[X^{+,n}(\infty)] \) and \( \frac{1}{n} \mathbb{E}[I^n(\infty)] = \frac{1}{n} \mathbb{E}[X^{-,n}(\infty)]. \)