Deconfinement Transition in Large $N$ Lattice Gauge Theory

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Abstract

We study analytically the phase diagram of the pure $SU(N)$ lattice gauge theory at finite temperature, and we attempt to estimate the critical deconfinement temperature. We apply large $N$ techniques to the Wilson and to the Heat Kernel action, and we study the resulting models both in the strong coupling and in the weak coupling limits. Using the Heat Kernel action, we establish an interesting connection between the Douglas-Kazakov phase transition of two-dimensional QCD and the deconfining transition in $d$ dimensions. The analytic results obtained for the critical temperature compare well with Montecarlo simulations of the full theory in $(2 + 1)$ and in $(3 + 1)$ dimensions.
1 Introduction

Finding a precise characterization of the finite temperature deconfining transition is a long standing problem of Lattice Gauge Theory (LGT). Since the seminal work of Svetitsky and Yaffe [1], it is understood that all the relevant properties of the transition can be encoded in a suitable effective action for the order parameter, the Polyakov loop, and a qualitatively satisfactory description of the phase diagram can be obtained in this way. Such an approach clearly involves an enormous simplification of the model, through a drastic reduction of its degrees of freedom; the non-trivial task is then the construction of an appropriate effective action. The simplest and most popular proposal for such an action is [2]-[8]

$$S_{\text{eff}}(J) = \sum_{\bar{x}} \text{Re} \left\{ J \sum_i \text{Tr}_f(V(\bar{x})) \text{Tr}_f(V^\dagger(\bar{x} + i)) \right\},$$

where $\text{Tr}_f(V(\bar{x}))$ is the Polyakov loop.

This action has been widely studied in the literature, both at finite $N$ [2, 3] and in the large $N$ limit [4, 5], with strong coupling [6, 7] and mean field [3] approximations, and with Montecarlo simulations [3, 8]. In particular, in the large $N$ limit it is exactly solvable, and leads to a peculiar phase diagram, with a first order point and a line of third order transitions of the Gross-Witten type [4, 5]. In the literature, the action (1) has been mainly used as a tool to understand the general features of the phase diagram, without really hoping to obtain precise determinations of, say, the location of the critical deconfinement temperature $T_c$. This is due to the fact that the approximations involved in going from the complete LGT to eq. (1) are rather drastic, and the correspondence between the coupling $J$ in eq. (1) and the coupling $\beta$ of the complete LGT (and the consequent possibility of finding a meaningful result for the critical temperature in the continuum limit) is rather ambiguous.

One of the goals of the present paper is to show that, by using new techniques which have become available in these last years for the large $N$ analysis of gauge theories, one can go a bit further and extract from a suitable generalization of the action (1) some reliable information on the deconfinement temperature. Even if the values obtained are still affected by large uncertainties, they essentially agree with those extracted from Montecarlo simulations. This is a significant progress, because the results are obtained analitycally and, what is more important, our approach opens the way to further improvement. In fact a better control on the approximations involved may be obtained within the framework of the hot Eguchi-Kawai model, and also the extension to the case of finite $N$ may be studied.

In this paper, we shall show that the action (1) can be considered as an approximation of a much more accurate effective action which, despite its complexity, can still be precisely studied in the large $N$ limit, in weak coupling as well as in strong coupling. Besides describing the phase diagram of the model, we will make an effort to extract as precisely as possible the value of the critical temperature in terms of the coupling of the original model, studying its scaling behaviour and...
making contact with the known results from Montecarlo simulations both in (2+1) and (3+1) dimensions. As we shall see in the last section, the two cases require very different analysis, and only in the (2+1)-dimensional case the approximations are really under control, the value that we obtain has the correct scaling behaviour, and the continuum limit can be correctly approached. In (3+1) dimensions things are significantly more complicated, and our results should be better taken as strong coupling estimates, valid for small values of the $N_t$ (the lattice size in the time direction); nevertheless, also in this case, some interesting information can be extracted, and a rather successful comparison with known Montecarlo estimates can be made.

To describe QCD on the lattice, we will use both the Wilson action and the action defined in terms of the Heat Kernel on the group manifold $[9]$, already employed in this context by $[10]$. We will study the approximations that are needed to reduce these actions to the model $[11]$, namely a Migdal-Kadanoff renormalization and the decoupling of the space-like plaquettes. We observe that the Heat Kernel action has definite advantages from the point of view of analytic calculations, and, upon decoupling the space-like plaquettes, allows an exact integration over the space-like degrees of freedom. The resulting model can be studied, in the large $N$ limit, both in weak coupling (where it is directly equivalent to the Wilson action), and in strong coupling, where a non-trivial transformation is needed to make the connection with the Wilson action.

In the process, we shall make contact with some interesting results recently obtained in the context of two-dimensional QCD (QCD2). In particular, we shall discuss the relation between the deconfinement transition in $(d + 1)$ LGT and the so-called Douglas-Kazakov transition $[11]$ in QCD2.

Our analysis starts with a review of the various approximations needed to obtain $S_{eff}$. This will be done in the next section, where we shall also fix our notations in the general framework of finite temperature LGT. Section 3 will be devoted to a discussion of how the same approximations, applied to the Heat Kernel, link the effective action for the Polyakov loop to the two-dimensional QCD on a cylinder. We then review some of the features of QCD2 which are relevant in this case. Sections 4 and 5 will be devoted to a weak coupling and a strong coupling expansion respectively, while in the last section we shall make contact with the known results on the deconfinement temperature of LGT, and we shall make some concluding remarks.

2 Finite Temperature LGT

Let us consider a pure gauge theory with gauge group $SU(N)$, defined on a $d + 1$ dimensional cubic lattice. In order to describe a finite temperature LGT, we have to impose periodic boundary conditions in one direction (which we shall call from now on “time-like” direction), while the boundary conditions in the other $d$ direction (which we shall call “space-like”) can be chosen freely. We take a lattice of $N_t$ ($N_s$)
spacings in the time (space) direction, and we work with the pure gauge theory, containing only gauge fields described by the link variables $U_{ni} \in SU(N)$, where $n \equiv (\vec{x}, t)$ denotes the space-time position of the link and $i$ its direction. It is useful to choose different bare couplings in the time and space directions. Let us call them $\beta_t$ and $\beta_s$ respectively. The Wilson action is then

$$S_W = \sum_n \frac{1}{N} \text{Re} \left\{ \beta_t \sum_i \text{Tr}_f(U_{n,0i}) + \beta_s \sum_{i<j} \text{Tr}_f(U_{n,ij}) \right\},$$

(2)

where $\text{Tr}_f$ denotes the trace in the fundamental representation and $U_{n,0i}$ ($U_{n,ij}$) are the time-like (space-like) plaquette variables, defined as usual by

$$U_{n,ij} = U_{ni} U_{n+\hat{i}j} U_{n+\hat{i}j}^\dagger U_{n,ij}^\dagger.$$  

(3)

In the following we shall call $S_s$ ($S_t$) the space-like (time-like) part of $S_W$. $\beta_s$ and $\beta_t$ are related to the (bare) gauge coupling $g$ and to the temperature $T$ by the usual relations

$$\frac{2N}{g^2} = a^{3-d} \sqrt{\beta_s \beta_t}, \quad T = \frac{1}{N_t a} \sqrt{\frac{\beta_t}{\beta_s}},$$

(4)

where $a$ is the space-like lattice spacing, while $\frac{1}{N_t T}$ is the time-like spacing. The two are related by the dimensionless ratio $\epsilon \equiv \frac{1}{N_t a}$. We can solve the above equations in terms of $\epsilon$, as

$$\beta_t = \frac{2N}{g^2 \epsilon} a^{d-3},$$

(5)

$$\beta_s = \frac{2N \epsilon}{g^2} a^{3-d}.$$  

(6)

In a finite temperature discretization it is possible to define gauge invariant observables which are topologically non-trivial, as a consequence of the periodic boundary conditions in the time directions. The simplest choice is the Polyakov loop, defined in terms of link variables as

$$P(\vec{x}) \equiv \text{Tr}_f V(\vec{x}) = \text{Tr}_f \prod_{t=1}^{N_t} (U_{\vec{x},\epsilon;0}).$$

(7)

As it is well known, the finite temperature theory has a new global symmetry (unrelated to the gauge symmetry), with symmetry group the center $C$ of the gauge group (in our case $Z_N$). The Polyakov loop is a natural order parameter for this symmetry.

In $d > 1$, finite temperature gauge theories admit a deconfinement transition at $T = T_c$, separating the high temperature, deconfined, phase ($T > T_c$) from the low temperature, confining domain ($T < T_c$). In the following we shall be interested in the phase diagram of the model as a function of $T$, and shall make some attempt
to locate the critical point $T_c$. The high temperature regime is characterized by the breaking of the global symmetry with respect of the center of the group. In this phase the Polyakov loop has a non-zero expectation value, and it is an element of the center of the gauge group (see for instance [1]).

The Wilson action eq.(3) is clearly too complex to be handled exactly, and some approximate effective action is needed. As we shall see, at least three different approximations are needed, in order to reach a solvable model. The first step is to identify the Polyakov loops as relevant dynamical variables. This can be done by dimensional reduction, as described in the next subsection (2.1). Second, in order to integrate out the spacelike degrees of freedom, one must decouple the spacelike plaquettes, and this will be described in subsection (2.2). The third and last step is to take the large $N$ limit, which allows to solve exactly the model, through a large $N$ factorization of the Polyakov loops (subsection 2.3); the large $N$ limit is important even when it does not lead to exact solvability, because in that limit a translational invariant master field is expected, and the model can be reduced to an effective one-plaquette model.

In the remaining part of this section we shall describe in some more detail these steps, and discuss their reliability. Since the approximations are well known, and have been already widely discussed in the literature, we shall mostly state the results, and refer the reader to some of the original papers.

### 2.1 Migdal-Kadanoff approximation

By means of an approximate renormalization group transformation, within the framework of a Migdal-Kadanoff bond-moving scheme, it is possible to reduce the original $(d+1)$ dimensional Wilson action, eq. (4) to a $d$ dimensional gauge theory coupled to the Polyakov lines, which play the role of a Higgs field [2, 3]. The resulting action is

$$
\exp(S_1) = \prod_{\vec{x}} \left( \prod_{i=1}^{d} \left( 1 + \sum_{r \neq 0} I_r(\beta_i) \frac{N_i}{I_0(\beta_i)} \right)^{N_i} \right) \frac{d_r}{d} \chi_r(V(\vec{x})U_{\vec{x};i}V(\vec{x}+i)U_{\vec{x};i}^\dagger) \right) \right) \times \\
\times \exp \left[ \frac{1}{N} \text{Re} \left( \sum_{i<j} \hat{\beta}_s \text{Tr}_l(U_{\vec{x};ij}) \right) \right] \quad .
$$

Here the sum is limited to the $d$ spacelike directions, $I_r(\beta_i)$ is the coefficient of representation $r$ in the character expansion of the Wilson action, and the new spacelike coupling is related to the old one by

$$
\hat{\beta}_s = N_i \beta_s \quad .
$$

In the literature the timelike part of this action is then further approximated by truncating the sum over representations at the first term, so as to reconstruct once more a Wilson-type action.
\[ S_2 = \sum_x \frac{1}{N} \operatorname{Re} \left\{ \hat{\beta}_t \sum_{i=1}^d \text{Tr}_f(V(x)U_{x,i}V^\dagger(\vec{x} + i)U_{x,i}^\dagger) + \hat{\beta}_s \sum_{i<j} \text{Tr}_f(U_{x,ij}) \right\}, \quad (10) \]

As we shall see, this truncation is not necessary if one works with the Heat Kernel action, to be discussed in the next section. In the present framework, the Migdal-Kadanoff approximation is expected to be a good approximation both at strong and at weak coupling (see for instance [12]) while it might be inadequate at intermediate couplings. In particular, it is easy to see that the timelike part of \( S_1 \), eq. (8), could have been obtained also within the framework of a strong coupling expansion.

The truncation of eq. (8) to eq. (10) is similarly expected to be valid only for large or small values of \( \beta_t \), the difference between the two regimes being only in the definition of the new timelike coupling

\[ \hat{\beta}_t \sim \frac{\beta_t}{N_t}, \quad \beta_t \text{ large}, \quad (11) \]
\[ \hat{\beta}_t \sim \frac{\beta_t^{N_t}}{(2N^2)^{N_t-1}}, \quad \beta_t \text{ small}. \quad (12) \]

In the large \( \beta_t \) limit the Migdal-Kadanoff approximation coincides with the approximation scheme for high temperature QCD known in the literature as “complete dimensional reduction”. This consists in the assumption of complete decoupling of the non-static modes in the compactified time direction. The resulting theory is a \( d \)-dimensional gauge theory coupled to a scalar field in the adjoint representation which represents the fluctuations of the Polyakov loop around the minimum \( V(\vec{x}) = 1 \). The corresponding action is the sum of the purely space-like part of \( S_W \) and a new term \( S_h \) which is the remnant of the timelike part of \( S_W \)

\[ S_h(m_0) = \frac{\hat{\beta}_h}{N} \text{Tr}_f \sum_x \left( -m_0^2 \phi(\vec{x})^2 + \sum_{i=1}^d U_{x,i} \phi(\vec{x}) U_{x,i}^\dagger \phi(\vec{x} + i) \right) \quad (13) \]

where \( \phi(x) \) is an Hermitian \( N \times N \) matrix and \( m_0^2 = d \).

This action is of the type studied by Kazakov and Migdal in the context of “induced QCD” [13] and in the present context by the authors of [14]. It is however well known that complete dimensional reduction does not take place in general and that non-static modes induce at one loop level some well defined interaction in the static sector. We expect this type of corrections to be the main source of error also in the Migdal-Kadanoff approximation, however it can be shown that they are of higher order in large \( N \) limit and therefore we shall not take them into account in the present paper.

Let us conclude by noticing that the effect of the Migdal-Kadanoff approximation is to reduce the lattice size in the time direction \( N_t \) to its extreme value \( N_t = 1 \). It will be important to keep in mind this fact in the last section, where we shall compare our prediction with the results of the Montecarlo simulations.
2.2 Decoupling of spacelike plaquettes

An interesting feature of eq.(8) is that its timelike and spacelike sectors are only very weakly coupled. Let us try to make this statement more precise. First of all one can see that in the framework of a standard mean field approximation the decoupling of timelike and spacelike degrees of freedom in (8) is complete \[3\]. This result appears to be confirmed beyond the mean field approximation by montecarlo simulations \[3\]. The decoupling was also checked in \[14\], where the contribution of timelike degrees of freedom to the space-like string tension at high temperature was evaluated in (2+1) dimensions within a strong coupling expansion and found to be neglected. This result turned out to be in agreement with high precision Montecarlo simulations \[16\].

In this paper we are addressing the problem of finding the critical deconfinement temperature and the relevant degrees of freedom appear only in the timelike part of the action. So, in agreement with the arguments given above, and following the literature on the subject \[3, 4, 8\], we shall from now on neglect the spacelike part of the action, eq.(8). It is of course far from obvious to what extent such a crude approximation still holds at temperatures of the order of the deconfinement transition, so an independent check of its validity (such as results from Montecarlo simulations) is going to be important.

The main outcome of this approximations is that in the resulting model the spacelike degrees of freedom can be explicitly integrated out, to obtain an effective action for the Polyakov loops only. This integration can be made (see section 5) by using a generalization of the Itzykson-Zuber-Harish-Chandra integral \[17\]. If one keeps only the lowest order in the coupling one recovers the action \(S_{\text{eff}}\), eq.(1) discussed in the introduction.

\[
S_{\text{eff}}(J) = \sum_{\vec{x}} \text{Re} \left\{ J \sum_i \text{Tr}_f(V(\vec{x}))\text{Tr}_f(V^\dagger(\vec{x}+i)) \right\} .
\]

Here we have introduced the new coupling \(J\) defined in terms of \(\beta\) as:

\[
J = \hat{\beta}_t/N^2 .
\]

\(J\) has a smooth large \(N\) behaviour and so it is the natural coupling in the large \(N\) limit. In the following two sections we shall sistematicaly use \(J\), referring to eq.(15) for the connection with \(\beta\), and shall come back to the original coupling \(\beta\) only in the last section, where the comparison with the Montecarlo results is made.

Higher order corrections to eq. (14) are discussed in Section 5 where a similar analysis for the Heat Kernel action is also carried out. As we shall see, a phase diagram much richer than that of \(S_{\text{eff}}\) emerges if one keeps the whole result of the integration.
2.3 Large N Limit

The model described by $S_{\text{eff}}$ can be solved exactly in the large $N$ limit \cite{4}, for any value of the space dimensions $d$, leading to a phase diagram with a first order phase transition located at $J = 1/d$. It is even possible to solve exactly the more general model in which the Polyakov loop is also coupled to an external “magnetic” field $h$,

$$S_{\text{eff}}(J, h) = \sum_{\vec{x}} \text{Re} \left\{ J \sum_i \text{Tr}_f(V(\vec{x}))\text{Tr}_f(V^\dagger(\vec{x} + i)) \right\} + hN \sum_{\vec{x}} \left[ \text{Tr}_f(V(\vec{x}) + V^\dagger(\vec{x})) \right]. \quad (16)$$

In this more general framework it becomes apparent that the first order phase transition is the end point of a line of third order phase transitions of the Gross-Witten type, located along the line

$$J = \frac{1 - 2h}{d}, \quad h, J \geq 0. \quad (17)$$

2.4 Comments

There are two interesting observations which must be made at this point. First it is possible to show that the three steps of approximation described above commute, in the sense that the order in which they are made can be changed, but the same phase diagram is found at the end. For instance, if we neglect from the beginning the spacelike plaquettes, then the Migdal Kadanoff result eq. (8) can be obtained exactly within a strong coupling expansion (namely integrating all the spacelike links except the ones in the lowest slice). Similarly one could start from the beginning within the framework of a large $N$ Eguchi-Kawai model, then make a Migdal Kadanoff approximation, and find again the same phase diagram (see \cite{5} and this same observation in \cite{4}). This denotes an internal consistency of the set of approximations used which suggests that the value $J_c = 1/d$ for the deconfinement temperature should be taken seriously. Indeed, such value is rather stable under the various refinements of eq. (1) that will be discussed in the rest of this paper; at the end our results will never differ from it more than 20%.

A second observation is that the magnetic field introduced in eq. (16) has a relevant physical meaning. It encodes the effect of the coupling of fermions to the pure gauge theory \cite{18, 19}. To be precise, one has to choose Wilson fermions and then make an expansion in the hopping parameter $K$. It is possible to see that at the lowest order in the $K$ expansion the $Z_N$ symmetry of the pure gauge theory is broken and that the fermions contribute with a magnetic term $h = 2N_f(2K)^{N_f}$, where $N_f$ is the number of fermions introduced.
3 The Heat Kernel Action

The Heat Kernel action is best defined in terms of the character expansion
\[ \exp(S_H) = \prod_{n, (i < j)} \left\{ \sum_r d_r \chi_r(U_{n,ij}) \exp\left(-\frac{NC_r^{(2)}}{2}\frac{s^2}{\beta_s}H_s\right) \right\} \times \prod_{n,i} \left\{ \sum_r d_r \chi_r(U_{n,0i}) \exp\left(-\frac{NC_r^{(2)}}{2}\frac{t^2}{\beta_t}H_t\right) \right\} \]

where the sum is over all the inequivalent, irreducible representations, labelled by \( r \), \( d_r \) is their dimension, \( \chi_r(U) \) is the character of \( U \) in the representation \( r \) and \( C_r^{(2)} \) is the value of the quadratic Casimir operator in the representation \( r \). The index \( s \) and \( t \) label, as before, the spatial and temporal couplings respectively. The Heat Kernel action in eq.(18) is normalized so as to coincide in the continuum limit (\( \beta_s, \beta_t \to \infty \)) with the Wilson action eq.(2). Hence in this limit eqs.(4,5,6) still hold with the substitution \( \beta \to \beta^H \).

In the case of the Wilson action, both the Migdal Kadanoff approximation (section 2.1) and the decoupling and subsequent integration of the spatial degrees of freedom (sect. 2.2) require a truncation of the resulting actions so as to obtain the final form of the effective action eq.(16). The effect of these truncations can be neglected if \( \beta_t \) is very large or very small, but it is important in the intermediate region in which we expect the transition to occur. The situation is different for the Heat Kernel action. In fact due to the well known properties of the Heat Kernel action the strong coupling expansion which leads to eq.(8) is exact in the timelike direction and the scaling of the Heat Kernel coupling is trivial:
\[ \beta_t^H \to \tilde{\beta}_t^H = \frac{\beta_t^H}{N_t} \] (19)

for all values of \( \beta_t^H \) and no truncation needed. As mentioned above, near the continuum limit (large \( \beta \)) the Heat Kernel and the Wilson action must coincide, in fact the same scaling law appears in eq. (11) and (19). On the contrary in the strong coupling regime the two behave very differently (as can be seen by comparing eq.(19) and eq.(12)). This can be checked by a direct comparison of the strong coupling expansion of the Wilson and the Heat Kernel action (see section 5). The strong coupling expansion provides a relation between \( \beta^H \) and \( \beta \), which, as expected compensates exactly the difference between the two scaling laws eqs. (19), (12). This not trivial relation must be kept into account if one wants to make contact with say, results coming from Monte Carlo simulations which are always made with the standard Wilson action.

The decoupling of the spatial plaquettes, which we shall assume also in this context, makes it possible to integrate exactly on the spatial links by using standard properties of the characters. We have
\[ Z_H = \]
\[
\begin{align*}
&= \int \prod_{\vec{x}} dV(\vec{x}) \prod_{\vec{x},i} dU_i(\vec{x}) \prod_{\vec{x},i} \left\{ \sum_r d_r \chi_r \left( U_i(\vec{x})V(\vec{x} + i)U_i^+(\vec{x})V^+(\vec{x}) \right) \exp \left( -\frac{C^{(2)}_r}{2NJ_H} \right) \right\} \\
&= \int \prod_{\vec{x}} dV(\vec{x}) \prod_{\vec{x},i} \sum_r \chi_r \left( V(\vec{x} + i) \right) \chi_r \left( V^+(\vec{x}) \right) \exp \left( -\frac{C^{(2)}_r}{2NJ_H} \right)
\end{align*}
\]

(20)

The effective action for the Polyakov loop given by (20), although much more complicated than that of eq.(16), can still be studied in the large N limit. This was done, both in the weak and in the strong coupling region, by K. Zarembo in a recent paper [10]. A rather puzzling result of [10] is the logarithmic dependence on the dimensions of the critical point, more precisely

\[ J_c = \frac{1}{2 \log(2d - 1)} \]

(21)

This is a strong coupling result, obtained by looking at the point where the symmetric vacuum becomes unstable, and it is apparently in disagreement with the results of the previous section which give a 1/d behaviour for the critical coupling. Such disagreement is completely eliminated if one takes into account the relation between \( \beta_H \) and \( \beta \).

In the weak coupling region the action (20) can be studied by reducing it to a Kazakov-Migdal model with a quadratic potential, whose solution is given by a semicircular distribution of eigenvalues for the Polyakov loop. In ref. [10] it was remarked that an instability in the solution appears for the value of the coupling constant where the radius of the distribution develops an imaginary part, presumably a signal of some type of phase transition. Here we will produce a better understanding of this phenomenon, which will also allow a more precise determination of the critical coupling. In order to do so we notice that the coupling between \( V(\vec{x}) \) and \( V(\vec{x} + i) \) at the r.h.s of eq. (20) is just the partition function of QCD2 on cylinder whose area \( A \) is \( \frac{1}{\beta_H} \) and whose fixed holonomies at the edges are \( V(\vec{x}) \) and \( V(\vec{x} + i) \).

In a recent paper [20] we have shown that QCD2 on a cylinder has a third order phase transition of the same type of that found by M. Douglas and V.Kazakov in the spherical geometry. Such a transition can be seen in the language of 2d QCD as an istanton driven delocalization of the trivial semicircle distribution of eigenvalues. We will show in the next section that in the present context of Finite Temperature LGT this phase transition is responsible for the instability of the weak coupling solution. Before proceeding to do that however, we shall end the present section by reviewing and completing the results of ref. [20] on the Douglas-Kazakov phase transition on a cylinder.

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1As usual in the context of QCD2, the area \( A \) of the cylinder is expressed in units of \( \frac{1}{g_2^2} \), where \( g_2 \) is the QCD2 coupling.
3.1 Douglas-Kazakov phase transition on a cylinder

As previously discussed the Heat Kernel effective theory for the Polyakov loop is given in terms of the partition function \( K_2(g_1, g_2; A) \equiv K_2(\phi, \theta; A) \) of QCD2 on a cylinder, with fixed holonomies (Polyakov loops) \( g_1 \) and \( g_2 \) (whose invariant angles are \( \{\phi_i\} \) and \( \{\theta_i\} \) respectively) at the two boundaries.

This partition function is the solution of the Heat Kernel equation on the \( SU(N) \) group manifold:

\[
\left( N \frac{\partial}{\partial A} - \frac{1}{2} \mathcal{J}^{-1}(\phi) \sum_i \frac{\partial^2}{\partial \phi_i^2} \mathcal{J}(\phi) - \frac{1}{24} N (N^2 - 1) \right) K_2(\phi, \theta; A) = 0
\]

uniquely determined by the condition:

\[
\lim_{A \to 0} K_2(g_1, g_2; A) = \hat{\delta}(g_1, g_2^{-1})
\]

where, in the notations of ref. [21], \( \mathcal{J}(\phi) \) is the Vandermonde determinant for a unitary matrix:

\[
\mathcal{J}(\phi) = \prod_{i<j} 2 \sin \frac{\phi_i - \phi_j}{2}
\]

and \( \hat{\delta} \) is the invariant delta function on the group manifold. An explicit form of \( K_2(\phi, \theta; A) \) is usually given by the standard character expansion:

\[
K_2(g_1, g_2; A) = \sum_r \exp \left[ -\frac{A}{2N} C_r^{(2)} \right] \chi_r(\phi) \chi_r(\theta)
\]

As pointed out in [21, 22], upon redefinition of the Kernel by \( K_2(\phi, \theta; A) \to \hat{K}_2 = \mathcal{J}(\phi) \mathcal{J}(\theta) K_2, \) eq. (22) becomes the (euclidean) free Schrodinger equation for \( N \) fermions on the circle, where \( A \) plays the role of the (euclidean) time. This means that, because of the condition (23), we can interpret \( K_2(\phi, \theta; A) \) as the euclidean transition amplitude of this system of fermions from the configuration \( \{\phi_i\} \) at the zero time to the configuration \( \{-\theta_i\} \) at the time \( A \).

By using the results of ref. [23] and [21] the above expression can be rewritten as a sum over the \( N \) integers \( l_i \) labelling the co-root lattice of \( SU(N) \):

\[
K_2(g_1, g_2; A) = \left( \frac{N}{4\pi} \right)^{1/2} \exp \left( \frac{A}{24} (N^2 - 1) \right) \sum_P \frac{(\mathcal{J}(\phi))^{(1-N)/2}}{\mathcal{J}(\theta) \mathcal{J}(\phi)} (-1)^{\sigma(P)} \frac{N(N-1)}{2} \sum_{\{l_i\}} \exp \left[ -\frac{N}{2A} \sum_{i=1}^{N} \left( \phi_i + \theta_{P(i)} + 2\pi l_i \right)^2 \right]
\]

where \( P \) denotes a permutation (of sign \( \sigma(P) \)) of the indices. The meaning of the modular transformation relating this last expression to the character expansion is clear: the integers labelling the unitary representations in the character expansion
correspond to discretized momenta of the fermions on the circle, while eq. (26) gives the corresponding coordinate representation and the integers in the co-root lattice are interpreted as the winding numbers of the fermions.

The large $N$ behaviour of the partition function (25) in terms of the eigenvalues distributions $\rho_0(x)$ and $\rho_1(x)$ corresponding respectively to $\{\phi_i\}$ and $\{\theta_i\}$ can be obtained starting from the differential equation (22) and then applying the same procedure used by Matytsin in [24] to find the large $N$ limit of the Itzykson-Zuber integral. The time (area) evolution of the eigenvalue distribution $\rho(x)$ is given by a Das-Jevicki hamiltonian [25]

$$
H [\rho(x), \Pi(x)] = \frac{1}{2} \int dx \rho(x) \left\{ \left( \frac{\partial \Pi(x)}{\partial x} \right)^2 - \frac{\pi^2}{3} \rho^2(x) \right\}
$$

(27)

where $\Pi(x)$ is the canonical momentum conjugate to $\rho(x)$. The only difference with respect to ref. [24] is that the density $\rho(x)$ is here a periodic function of period $2\pi$.

In terms of the complex quantity

$$ f(x, t) = \frac{\partial \Pi(x,t)}{\partial x} + i\pi \rho(x, t) $$

(28)

the equations of motion become the Hopf equation of motion for an ideal fluid:

$$ \frac{\partial f}{\partial t} + f \frac{\partial}{\partial x} f = 0, $$

(29)

with the boundary conditions

$$ \rho(x, t = 0) = \pi \rho_0(x), \quad \rho(x, t = \mathcal{A}) = \pi \rho_1(x) $$

(30)

A solution to eq. (29) can be obtained from the ansatz

$$ \rho(x, t) = \frac{2}{\pi r^2(t)} \sqrt{r^2(t) - x^2}, \quad |x| < r^2(t) $$

$$ \rho(x, t) = 0, \quad r^2(t) < |x| < \pi $$

(31)

The ansatz (31) provides a solution of the Hopf equation if the time dependence of the radius $r$ of the distribution has the form

$$ r(t) = 2 \sqrt{\frac{(t + \alpha)(\beta - t)}{\alpha + \beta}}. $$

(32)

The arbitrary constants $\alpha, \beta$ are determined by the boundary conditions (30). Naturally the initial and final distributions in (30) have consistently to be semicircular with radii $r(0) = r_0, r(\mathcal{A}) = r_1$. Given $r_0$ and $r_1$, the radius of the distribution is determined at any section of the cylinder by eq. (32). However because of the periodicity condition on the eigenvalue distribution the density (31,32) is a solution of the saddle-point equations (29,30) only if $r(t) < \pi$ for any $t$ on the trajectory.
For any given value of $r_0$ and $r_1$, the maximum value of $r(t)$ increases as the area $A$ increases, so the solution (29,32) is valid only if the area $A$ is smaller than a critical value $A_c$, where the maximum radius equals $\pi$ and the eigenvalues fill up the whole circle. The critical value of $A$ at which such phase transition occurs can be easily calculated and is given by

$$A_c = \sqrt{\frac{1}{2} \left( \pi^4 - \pi^2 r_0^2 + r_1^2 \right) + \sqrt{\left( \pi^4 - \pi^2 r_0 r_1 \right)^2 - \pi^6 (r_0 - r_1)^2}}$$

Note that the partition function of QCD on a sphere is a particular case corresponding to $r_0 = r_1 = 0$. Indeed in this case the critical value (33) becomes $A_c = \pi^2$, which is just the value found by Douglas and Kazakov [11]. Hence the large $N$ phase transition between the small area (gaussian) phase and the large area phase is the generalization to the cylinder of the Douglas-Kazakov phase transition on the sphere.

The meaning of the phase transition was discussed in [20]: in the gaussian phase the eigenvalues are confined in the $(-\pi, \pi)$ interval, namely the configurations where a fraction of the eigenvalues wind around their configuration space do not contribute in the large $N$ limit. Correspondingly all integers $l_i$ in eq. (26) can be set to zero. Such topologically non trivial configurations contribute instead in the large $N$ limit of the large area phase.

The role of instantons in inducing the Douglas-Kazakov phase transition on the sphere was further investigated by Gross and Matytsin [26].

4 Weak Coupling

As a result of the approximation schemes discussed in the previous sections, we can now write an effective action for the Polyakov loop starting either from the Wilson action or from the Heat Kernel action. Moreover in the large $N$ limit, we can assume that the saddle point solution is translational invariant and, as a consequence, the action reduces to a one plaquette integral, both in the Wilson and in the Heat Kernel case. The partition functions in the two cases are given by

$$Z_W = \int dV \prod_{i=1}^d dU_i e^{NJ \sum_{i=1}^d \text{Re} \text{Tr} U_i V U_i^\dagger V^\dagger}$$

and

$$Z_H = \int dV \left[ \mathcal{K}_2(V, V^\dagger; 1/J^H) \right]^d,$$
to the coupling $\beta_t$ of the theory defined on a lattice with $N_t$ links in the time
direction by eqs. (11, 12, 13) and keeping into account the redefinition (15). We
shall study the phase diagrams resulting from (34) and (35) keeping in mind that,
in the region where the two actions describe the same physical system, the same
critical phenomena should occur at values of $J$ and $J^H$ corresponding to the same
value of $\beta_t$. On the other hand, we know that the rescaling of $J^H$ given by eq.(19)
is exact after decoupling the spatial plaquettes. Thus the comparison between the
two partition functions can be used to improve eqs (11) and (12). In the case of
eq.(11), for instance, we will find that the two $J$’s must differ by a constant term
for the two theories to match in the weak coupling region.

4.1 Weak coupling expansion for the Heat Kernel action

The partition function (35) was studied in detail by Zarembo in [10], and part of
our analysis will overlap with his work. We will, however, be able to interpret the
critical phenomena occurring in the weak coupling region in terms of the Douglas-
Kazakov phase transition in the underlying QCD2 structure. Let us first rewrite
eq. (35) explicitly, as an integral over the invariant angles of $V$.

$$Z_H = \int \prod d\theta_i \left[ J^2(\theta) \right]^{(1-d)} \left[ \sum_{k_i} \sum_P (-1)^{\sigma(P)} e^{-\frac{1}{2}N J^H \sum_i (\theta_i - \theta_P(i) + 2\pi k_i)^2} \right]^d .$$  \hspace{1cm} (36)

As discussed in the previous section, if $J^H$ is larger than a critical value (to be
determined), the contributions of the winding configurations can be neglected in
the large $N$ limit, and all $k_i$’s in eq.(36) can be set to zero. Moreover, for small $\theta$,
we have

$$J^2(\theta) = \Delta^2(\theta) e^{-\frac{1}{2}N \sum_i \theta_i^2 + O(\theta^4)} .$$  \hspace{1cm} (37)

Inserting these results in eq.(36) we obtain

$$\int \prod d\theta_i \left[ \Delta^2(\theta) \right]^{(1-d)} e^{-N[dJ^H - \frac{4}{J^H} \sum_i \theta_i^2] \sum_P (-1)^{\sigma(P)} e^{N J^H \theta_P(i)} \theta_P(i)} \right]^d . \hspace{1cm} (38)$$

Provided $J^H$ is larger than its critical value, the only approximation needed to
go from eq.(35) to eq.(38) consists in neglecting $O(\theta^4)$ terms in (37). The r.h.s.
of eq.(38) is a Kazakov-Migdal model with quadratic potential in the mean field
approximation, which was solved exactly in the large $N$ limit, for any number of
space dimension, by Gross [27]. The eigenvalue distribution is a standard Wigner
distribution

$$\rho(\theta) = \frac{2}{\pi r^2} \sqrt{r^2 - \theta^2} , \hspace{1cm} (39)$$

where the radius $r$ of the distribution is given by

$$r^2 = \frac{4(2d-1)}{J^H (m^2(d-1) + d\sqrt{m^4 - 4(2d-1)})} . \hspace{1cm} (40)$$
The parameter $m$ which appears in (40) is the usual mass term of the Kazakov-Migdal model. Following the standard normalization we have that the coefficient of the quadratic term in $\theta$ in eq.(38) must be $m^2$. This implies

$$m^2 = 2d - \frac{d - 1}{6J^H}.$$  \hspace{1cm} (41)

In [10], Zarembo argues that a phase transition occurs at a value of $J^H$ equal to (or larger than) the one for which $r^2$ becomes complex. Up to the effect of terms of higher order in $\theta$, we can in fact determine the exact value of $J^H$ at which the phase transition occurs, and we will further argue that the phase transition is of third order. From the analysis of the previous section, we know that the kernel $K_2(V, V^\dagger; 1/J^H)$ undergoes a phase transition of the Douglas-Kazakov type when

$$\left(\frac{1}{J^H}\right)^2 = \pi^4 - \pi^2 r^2,$$  \hspace{1cm} (42)

where $r^2$ is given by (40). The results for the critical value of $J^H$, and for the value $r_c$ of the radius at the critical point, are given for different space dimensions in Table (I). The critical values of $J^H$ obtained by Zarembo are also listed for comparison.

| $d$  | $r_c$ | $J^H_c, \text{DK}$ | $J^H_c, \text{Z}$ |
|------|-------|----------------|----------------|
| 2    | 2.96  | 0.321          | 0.311          |
| 3    | 2.80  | 0.226          | 0.218          |
| 4    | 2.66  | 0.192          | 0.184          |
| $\infty$ | 0. | $1/\pi^2 \sim 0.101$ | $1/12 \sim 0.083$ |

**Tab.I.** Value of $J^H$ below which the Wigner distribution becomes unstable, from the Douglas-Kazakov phase transition (third column) and from ref [10]. The radius of the distribution at the critical point is given in the second column.

We observe that our values are consistently just slightly higher than Zarembo’s, which clearly indicates that we are dealing with the same instability and not with two different physical phenomena, and on the other hand shows that the present one is a better approximation. As a consistency check, we notice that the critical radius is less than $\pi$ for any number of dimensions, whereas if we take the solution of [10] we see that it exceeds $\pi$ for $d = 2$ and $d = 3$. In spite of the small value of $J^H$ we expect the weak coupling solution to be very reliable above the Douglas-Kazakov phase transition particularly for large $d$, where the small value of $r_c$ ensures that the quartic terms which have been neglected in eq. (38) are indeed small.

To sum up, we have established that the Wigner distribution of eigenvalues becomes unstable at a value of the coupling constant given by (42), and the transition is driven by the winding modes in the configuration space of the eigenvalues. This analogy with the Douglas-Kazakov phase transition on a sphere gives an almost
compelling argument that the corresponding phase transition is of the third order. The appearance at the critical point of new classical trajectories corresponding to winding eigenvalues has presumably the effect of spreading the distribution, especially at the extremes. On the other hand, for low dimensions, this transition occurs at a radius of the eigenvalue distribution very near to $\pi$. This probably means that the maximum corresponding to the broken, deconfined phase becomes unstable, and the distribution would collapse into the uniform one. However this does not mean that the Douglas-Kazakov transition can be identified with the deconfinement transition. This is not the case for high dimensions, where the critical radius is small, approaching zero as $d$ increases. In this case the Douglas-Kazakov phase transition is a transition from a classical Wigner distribution to another one, so far unknown, but still presumably peaked around the origin.

### 4.2 Weak coupling expansion for the Wilson action

The partition function $Z_W$ of eq. (34) can be written as an integral over the invariant angles of the Polyakov loop $V$,

$$Z_W = \int \prod_i d\theta_i J^2(\theta) \left( I(\theta, J) \right)^d ,$$

where

$$I(\theta, J) = \int [dU] e^{NJ \text{Re} \text{Tr} (U \cos \theta U^\dagger \cos \theta + U \sin \theta U^\dagger \sin \theta)}$$

$$= \int [dU] e^{NJ \sum_{i,j} |U_{ij}| \cos(\theta_i - \theta_j)} ,$$

and we denote by $\cos \theta$ and $\sin \theta$ diagonal matrices with eigenvalues $\cos \theta_i$ and $\sin \theta_i$ respectively. In the large $N$ limit this integral can be evaluated as a sum of saddle point contributions \[28\]. The result is

$$I(\theta, J) = \sum_P \exp \left[ NJ \sum_i \cos(\theta_i - \theta_{P(i)}) \right] \left( 1 + O\left( \frac{1}{NJ} \right) \right) ,$$

where

$$H_P(\theta) = \prod_{i<j} \left[ (\cos \theta_i - \cos \theta_j)(\cos \theta_{P(i)} - \cos \theta_{P(j)}) + (\sin \theta_i - \sin \theta_j)(\sin \theta_{P(i)} - \sin \theta_{P(j)}) \right] .$$

The denominator $H_P(\theta)$ in eq. \[15\] has a double pole whenever two $\theta$'s coincide, so it can be written as $\Delta^2(\theta) \exp \left( h_P(\theta) \right)$, and $h_P(\theta)$ can be easily calculated for small $\theta$, with the result

$$h_P(\theta) = -\frac{N}{3} \sum_i \theta_i^2 + \frac{N}{4} \sum_i \theta_i \theta_{P(i)} + O(\theta^3) .$$
For small $\theta$, the partition function can thus be written as

$$Z_W = \int \prod_i d\theta_i \left[ \Delta^2(\theta) \right]^{1-d} e^{-N[J-J^H+1/4]} \sum_i \theta_i^2 \left[ \sum_P (-1)^{\sigma(P)} e^{N(J-J^H)} \sum_i \theta_i \theta_{P(i)} \right]^d.$$  \hspace{1cm} (48)

Once again, this is a Kazakov-Migdal model with quadratic potential, and it coincides with (eq.(38)) if we identify

$$J = J^H + \frac{1}{4}. \hspace{1cm} (49)$$

We have thus identified the leading correction to eq. (11) away from the continuum (weak coupling) limit: the two lattice theories defined by the Wilson and the Heat Kernel actions begin to differ along the respective renormalization group trajectories by a constant shift of their couplings. As $J$ decreases higher order terms in the angles $\theta$ will become relevant and the two theories will become increasingly different.

As for the phase diagram, one cannot apply to the Wilson action the same precise estimate based on the Douglas-Kazakov phase transition, but only the simpler argument given by Zarembo. This is due to the fact that the Douglas-Kazakov transition crucially depends on the precise form of the Heat Kernel action, and in particular on the winding numbers $k_i$ which are neglected in the Kazakov-Migdal approximation. In any case, as mentioned above, the phase diagram of the two theories is the same as long as the relation (49) is duly taken into account.

5 Strong coupling

While the inverse coupling $\beta_t/N^2$ in the lattice theory with $N_t$ links in the time direction becomes very large as we approach the continuum limit, the effective coupling $J$ in the reduced $N_t = 1$ theory (34) remains relatively small, typically of order $1/d$, even at temperatures approaching the deconfining transition. An expansion in powers of $J$ is therefore appropriate to study the critical phenomena in the effective models for the Polyakov loop considered in the previous section.

As a preliminary step let us establish some notations. We are interested in the large $N$ limit, so the fundamental quantity is the distribution $\rho(\theta)$ of the eigenvalues of the Polyakov loop. It is convenient to expand $\rho(\theta)$ in its Fourier modes

$$\rho(\theta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \rho_n e^{in\theta} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} x_n e^{i\alpha_n + in\theta}, \hspace{1cm} (50)$$

where $\alpha_n \in (-\pi/2, \pi/2)$ is the argument of $\rho_n$ modulo $\pi$, and $x_n$ coincides with the modulus of $\rho_n$ up to a sign. The reality of $\rho(\theta)$ requires $\rho_{-n} = \rho^*_n$, while the normalization of $\rho(\theta)$ to 1 in the interval $(-\pi, \pi)$ fixes $\rho_0 = 1$.

\footnote{It is convenient for the following discussion to restrict the range for $\alpha_n$ and have $x_n$ taking also negative values}
The inverse formula
\[ \rho_n = \int_{-\pi}^{\pi} \rho(\theta) e^{-in\theta} \] (51)
shows that \( \rho_n \) corresponds to the large \( N \) limit of the loop winding \( n \) times in the time-like direction with the given eigenvalue distribution. In particular, \( \rho_{\pm 1} \) corresponds to the large \( N \) limit of the Polyakov loop.

The \( Z_N \) invariance of the effective theory becomes, in the large \( N \) limit, a \( U(1) \) invariance under the shift \( \theta \rightarrow \theta + \delta \), that is \( \alpha_n \rightarrow \alpha_n + n\delta \). If this symmetry is unbroken the eigenvalue distribution is simply given by \( \rho(\theta) = \frac{1}{2\pi} \), namely \( x_n = 0 \) for \( n \neq 0 \). In the broken phase the \( U(1) \) symmetry connects the different vacua. If we choose the vacuum peaked at \( \theta = 0 \), then the symmetry of the action for \( \theta_i \rightarrow -\theta_i \) will force the vacuum distribution \( \rho(\theta) \) to be even in \( \theta \), thus fixing all \( \alpha_n \) to zero. In this situation the eigenvalue distribution takes the form
\[ \rho(\theta) = \frac{1}{2\pi} \left[ 1 + \sum_{n=1}^{\infty} x_n \cos(n\theta) \right], \] (52)
with \( x_n \) real.

The next ingredient is the integration measure \( J^2(\theta) \), which in the large \( N \) limit can be expressed as a function of the loop variables \( \rho_n \), as
\[ J^2(\theta) = \exp \left[ \lim_{y \rightarrow 1} \frac{N^2}{2} \int_{-\pi}^{\pi} d\theta \ d\varphi \ \rho(\theta) \ \rho(\varphi) \ \log(1 - y \cos(\theta - \varphi)) \right], \] (53)
where the double integral, which would be divergent at \( \theta = \varphi \), has been regularized by the inclusion of the parameter \( y \), and terms in the exponent suppressed by powers of \( N \) have been neglected. Eq. (53) can be calculated by expanding in powers of \( \theta \) and \( \varphi \) and resumming the resulting expression. The result is
\[ J^2(\theta) = \exp \left[ \lim_{y \rightarrow 1} N^2 \left( C_0(y) + \sum_{k=1}^{\infty} C_k(y)x_k^2 \right) \right], \] (54)
where \( C_0(y) \) is an irrelevant divergent expression and \( C_k(y) \) is given by
\[ C_k(y) = \frac{1}{k} \left[ 1 - \frac{\sqrt{1 - y^2}}{y} \right]^k. \] (55)
After removing the divergence, the limit \( y \rightarrow 1 \) can be taken, and gives
\[ J^2(\theta) = \exp \left[ N^2 \sum_{k=1}^{\infty} \frac{1}{k}x_k^2 \right]. \] (56)

Let us consider now the partition function \( Z_W \), given in eq. (43). We are going to show that the integral \( I(\theta, J) \) can be written in the large \( N \) limit as
\[ I(\theta, J) = e^{N^2 \sum_{k=1}^{\infty} j^k F_k(x,\alpha)} \] (57).
where the functions $F_k(x, \alpha)$ can be shown to depend only on $x_i$ and $\alpha_i$ with $i \leq k$, and can be determined in principle by using Schwinger-Dyson equations.

Correspondingly, the large $N$ limit of the partition function $Z_W$ becomes

$$\frac{1}{N^2} \ln Z_W = \sum_{k=1}^{\infty} \left( d \int F_k(x, \alpha) - \frac{x_k^2}{k} \right). \quad (58)$$

It is already obvious from (58) that, for small enough $J$, the free energy has a maximum for $x_k = 0$, that is for a uniform distribution of eigenvalues. The first order approximation of (58) is well known (see for example [4]), and is simply $F_1(x, \alpha) = x_1^2$. The free energy to this order becomes

$$\frac{1}{N^2} \ln Z_W \approx (Jd - 1)x_1^2, \quad (59)$$

with all other $x_k$ set to zero. The phase structure of (59) is very simple: for $J < \frac{1}{d}$ the maximum of the free energy occurs at $x_1 = 0$ and we are in the unbroken phase. Above the critical point $J = \frac{1}{d}$ we have instead $x_1 = \frac{1}{2}$, which is the maximum value allowed by the positivity condition on $\rho(\theta)$. This is exactly the phase diagram of the large $N$ limit of the effective action (1) discussed at the beginning of the introduction. In fact it is easy to see that at this first order approximation the two actions are actually equivalent. This allows to better understand in which sense we expect, going to higher order in eq.(58), to improve the effective action (1).

Let us compare the first order expansion of the Wilson action with the corresponding one for the Heat Kernel action $Z_{HK}$, given in eq. (35) and (36). The strong coupling expansion for the kernel on the cylinder coincides with the character expansion, which is a power series in $e^{-\frac{1}{2JH}}$. The truncation of the series to the first order corresponds to a truncation of the character expansion to fundamental representation, namely

$$\sum_r \chi_r(\theta)\chi_r(-\theta)e^{-\frac{\phi(2)}{2NJH}} \approx 1 + 2\chi_f(\theta)\chi_f(-\theta)e^{-\frac{1}{2JH}}$$

$$\approx \exp \left[ N^2 e^{-\frac{1}{2JH}} x_1^2 \right], \quad (60)$$

where in the last step we have used the fact that $\chi_f(\theta) = N x_1 e^{i\alpha_1}$. At first order in the strong coupling expansion $Z_W$ and $Z_{HK}$ coincide, provided we make the identification

$$J = 2e^{-\frac{1}{2JH}}, \quad (61)$$

in agreement with eq.s (11) and (12). Indeed, we have just rederived, comparing the first order strong coupling expansions of $Z_W$ and $Z_{HK}$, the well known rescaling properties of the coupling constant $J$.  

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5.1 Strong coupling expansion for the Wilson action

The functions $F_k(x, \alpha)$, which characterize the strong coupling expansion of $Z_W$, can be determined by solving Schwinger-Dyson type of equations. Let us consider the integral

$$I(J; \Phi, \Psi) = \int [dU] e^{NJTr(U\Phi_1 U^\dagger \Psi_1 + U\Phi_2 U^\dagger \Psi_2)} ,$$

(62)

where $\Phi_\alpha$ and $\Psi_\alpha$ are hermitian matrices. This integral is more general than the one in eq. (44), which is obtained from $I(J; \Phi, \Psi)$ by imposing the constraints

$$\sum_{\alpha=1}^{2} \Phi_\alpha^2 = \sum_{\alpha=1}^{2} \Psi_\alpha^2 = 1$$

(63)

and by identifying $\Phi$ with $\Psi$. Notice that, for $\Phi_2 = \Psi_2 = 0$, eq. (62) reduces to the well known Itzykson-Zuber integral. The Schwinger-Dyson equations for $I(J; \Phi, \Psi)$ can easily be proved; they read

$$Tr \left[ \frac{1}{N} \frac{\partial}{\partial \Phi_{\alpha_1}} \frac{1}{N} \frac{\partial}{\partial \Phi_{\alpha_2}} \ldots \frac{1}{N} \frac{\partial}{\partial \Phi_{\alpha_s}} \right] I(J; \Phi, \Psi) = J^s Tr (\Psi_{\alpha_1} \Psi_{\alpha_2} \ldots \Psi_{\alpha_s}) I(J; \Phi, \Psi) .$$

(64)

These equation could, in principle, be solved order by order in $J$. It is however more convenient to make use of the fact that $I(J; \Phi, \Psi)$ reduces for $\Phi_2 = \Psi_2 = 0$ to the Itzykson-Zuber integral, whose “strong coupling” expansion has been explicitly derived in [17] up to the eighth order in $J$. In fact it is not difficult to verify that, given a term in the expansion of the Itzykson-Zuber integral, containing only $\Phi_1$ and $\Psi_1$, there is a unique extension of it to include the dependence on $\Phi_2$ and $\Psi_2$ which satisfies the following requirements: i) it is invariant under the $O(2)$ symmetry of the integral (62) and of the constraints (63); ii) it contains in each term $\Phi_i$ and $\Psi_i$ with the same power, and thus is invariant when $\Phi_i$ and $\Psi_i$ are rescaled respectively by a factor $\lambda$ and $1/\lambda$, which is a symmetry of the integral before imposing the constraints (63). Using this property, it is easy to generalize the known results concerning the Itzykson-Zuber integral, and calculate the functions $F_k(x, \alpha)$ appearing in eq. (67). Up to $k = 4$ they are given by

$$F_1(x, \alpha) = x_1^2 ,$$

(65)

$$F_2(x, \alpha) = \frac{1}{4} \left( 1 - 2x_1^2 + 2x_1^4 + x_2^2 - 2x_1^2 x_2 \cos(2\alpha_1 - \alpha_2) \right) ,$$

(66)

$$F_3(x, \alpha) = \frac{1}{12} \left( 3x_1^2 - 12x_1^4 + 16x_1^6 + 12x_1^2 x_2^2 + x_3^2 + 6x_1^2 x_2 \cos(2\alpha_1 - \alpha_2) ight.$$  

$$-24x_1^4 x_2 \cos(2\alpha_1 - \alpha_2) + 4x_1^2 x_3 \cos(3\alpha_1 - \alpha_3)$$  

$$-6x_1 x_2 x_3 \cos(\alpha_1 + \alpha_2 - \alpha_3) \right) ,$$

(67)
\[
F_4(x, \alpha) = \frac{1}{32} \left( -1 + 34x_1^4 - 144x_1^6 + 192x_1^8 - 40x_1^2x_2^2 + 226x_1^4x_2^2 \\
+4x_2^4 + 20x_2^2x_3^2 + x_4^2 + 34x_1^4x_2^2 \cos(4\alpha_1 - 2\alpha_2) \\
-12x_1^2x_2 \cos(2\alpha_1 - \alpha_2) + 152x_1^4x_2 \cos(2\alpha_1 - \alpha_2) \\
-384x_1^6x_2 \cos(2\alpha_1 - \alpha_2) - 48x_1^2x_2^2 \cos(2\alpha_1 - \alpha_2) \\
-16x_1^3x_3 \cos(3\alpha_1 - \alpha_3) + 80x_1^5x_3 \cos(3\alpha_1 - \alpha_3) \\
+8x_1x_2x_3 \cos(\alpha_1 + \alpha_2 - \alpha_3) - 120x_1^3x_2x_3 \cos(\alpha_1 + \alpha_2 - \alpha_3) \\
+16x_1x_2^2x_3 \cos(\alpha_1 - 2\alpha_2 + \alpha_3) - 10x_1^4x_4 \cos(4\alpha_1 - \alpha_4) \\
+20x_1^2x_2x_4 \cos(2\alpha_1 + \alpha_2 - \alpha_4) - 4x_2^2x_4 \cos(2\alpha_2 - \alpha_4) \\
-8x_1x_3x_4 \cos(\alpha_1 + \alpha_3 - \alpha_4) \right). \tag{68}
\]

Notice the invariance of the functions \( F_k(x, \alpha) \) under \( \alpha_k \to \alpha_k + k\delta \), as a result of the \( U(1) \) invariance of the theory in the large \( N \) limit.

We can now substitute the expressions \((55) - (68)\) into \((58)\), and study the free energy up to fourth order in \( J \). We have already noticed that at first order a phase transition occurs at \( J = \frac{1}{d} \). The higher order terms in \( J \) can be regarded then as higher order terms in \( \frac{1}{d} \) and the strong coupling expansion as a large \( d \) expansion.

Consider for example the coefficient of the quadratic term in \( x_1 \). The vanishing of this coefficient signals an instability of the symmetric vacuum \( x_k = 0 \). This leads us to study the equation

\[
dJ - \frac{d}{2}J^2 + \frac{d}{4}J^3 - 1 + O(J^5) = 0, \tag{69}
\]

which can be solved order by order in \( \frac{1}{d} \), leading to

\[
J = \frac{1}{d} \left[ 1 + \frac{1}{2d^2} + \frac{1}{4d^3} + O\left(\frac{1}{d^4}\right) \right]. \tag{70}
\]

It is instructive to compare this result with the result obtained by Zarembo \cite{14} using the Heat Kernel action. He finds that the exact value at which the instability occurs is given by \( J^H = \frac{1}{2\log(2d-1)} \). When inserted in \((61)\) this expression gives \( J = \frac{1}{d-1/2} \), whose expansion in powers of \( 1/d \) coincides with the one found in \((70)\) up to the third order. This is more that what one would have naively expected, as eq. \((71)\) was derived by equating the expansions of the Wilson and the Heat Kernel actions only at first order.

While eq. \((70)\) gives an upper limit for the critical value of \( J \), finding the precise value at which the deconfinement transition occurs, requires a more detailed analysis. As discussed at the beginning of this section, when searching for the broken vacuum we can set all \( \alpha_n \) to zero, and regard the free energy as a function of the \( x_n \) only. By using eq. \((55) - (68)\) we obtain

\[
\frac{1}{N^2} \ln Z_W = dJx_1^2 + \frac{dJ^2}{4} \left( 1 - 2x_1^2 + 2x_1^4 + x_2^2 - 2x_1^2x_2 \right)
\]

21
\[ J = 0 \] is shown in Fig. 1, where a contour plot of the free energy is shown for \( x \) as a function of \( x \) given by the equation

\begin{align*}
+ \frac{dJ^3}{12} & \left( 3x_1^3 - 12x_1^4 + 16x_1^6 + 12x_1^2 x_2^2 + x_3^2 \right) \\
+ 6x_1^2 x_2 & - 24x_1^4 x_2 + 4x_1^3 x_3 - 6x_1 x_2 x_3 \\
+ \frac{dJ^4}{32} & \left( -1 + 34x_1^4 - 144x_1^6 + 192x_1^8 - 40x_1^2 x_2^2 \right) \\
+ 226x_1^4 x_2^2 & + 4x_1^4 + 20x_1^2 x_3^2 + x_4^2 + 34x_1^4 x_2^2 \\
- 12x_1^2 x_2 & + 152x_1^4 x_2 - 384x_1^6 x_2 - 48x_1^2 x_3^2 - 16x_1^3 x_3 \\
+ 80x_1^5 x_3 & + 8x_1 x_2 x_3 - 120x_1^3 x_2 x_3 + 16x_1 x_2^2 x_3 \\
- 10x_1^4 x_4 & + 20x_1^2 x_2 x_4 + 4x_2^2 x_4 - 8x_1 x_3 x_4 \\
- x_1^2 - \frac{1}{2} x_2^2 & - \frac{1}{3} x_3^2 - \frac{1}{4} x_4^2 .
\end{align*}

(71)

We must now look for the maximum of (71), within the domain where \( \rho(\theta) \) is positive or zero for any \( \theta \). The equations for such a domain can be obtained in principle by requiring that, for some \( \theta_0 \),

\[ \rho(\theta_0) = \frac{d}{d\theta} \rho(\theta)|_{\theta=\theta_0} = 0 \]  

(72)

and by eliminating \( \theta_0 \) from the equations. The resulting equations for the \( x_n \)'s give the boundaries of the physical region. In order to have an intuitive picture of how the deconfining transition takes place let us set \( x_3 = x_4 = 0 \) in (71) and plot the free energy as a function of \( x_1 \) and \( x_2 \) at various values of the coupling \( J \). This is shown in Fig. 1, where a contour plot of the free energy is shown for \( d = 2 \) and \( J = 0.55, 0.60, 0.66 \).

The physical region in the \((x_1, x_2)\) plane can be easily determined, and it is represented by the region inside the thick line. The straight edge on the left is given by the equation \( x_1 - x_2 = 1/2 \) and corresponds to density distributions vanishing at \( \theta = \pi \), whereas the the straight edge on the opposite side corresponds to distributions vanishing at \( \theta = 0 \). The plot at \( J = 0.55 \) clearly shows the maximum at \( x_1 = x_2 = 0 \): the system is in the unbroken phase dominated by a constant distribution of eigenvalues. In the next plot, at \( J = 0.6 \), a local maximum has appeared at the edge of the physical region, and it is becoming competitive with the unbroken maximum. Notice that there is a symmetry \( x_1 \rightarrow -x_1 \), so that a symmetric maximum appears on the other edge of the physical region. This symmetry is accidental, and it is removed when \( x_3 \) and \( x_4 \) are switched on. In the last plot, at \( J = 0.66 \), the maximum at \( x_1 = x_2 = 0 \) has disappeared and the system is clearly in the broken phase.

When the dependence of (71) on \( x_3 \) and \( x_4 \) is taken into account, the boundary of the physical region is given by fourth order algebraic equations, together with the hyperplanes \( x_1 - x_2 + x_3 - x_4 = 1/2 \) and \( x_1 + x_2 + x_3 + x_4 = -1/2 \). The phase transition occurs when the maximum value of the free energy on the boundary becomes larger than the value at the symmetric point. The critical point can be
Fig. 1 Contour plots of the free energy for: (a), $J = 0.55$; (b), $J = 0.61$; (c), $J = 0.66$. The numbers 1 and 2 mark respectively the presence of the symmetric maximum ($x_1 = x_2 = 0$) and the “broken” one. The thick line encloses the “physical” region.
determined numerically, and it is given for different dimensions by the values listed in Table II.

| $d$ | $J_c$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|-----|-------|-------|-------|-------|-------|
| 2   | 0.601 | 0.50  | 0.06  | −0.03 | −0.03 |
| 3   | 0.379 | 0.52  | 0.03  | −0.03 | −0.02 |
| 4   | 0.275 | 0.53  | 0.02  | −0.03 | −0.02 |
| 100 | 0.010 | 0.49  | 0.004 | 0.004 | 0.005 |

Tab.II. Values of the critical coupling $J_c$ at the fourth order in the strong coupling expansion of the Wilson action. In the last four columns the corresponding values of $x_i$, $(i = 1 − 4)$ are reported.

The values of $J_c$ depends of course on the order at which the strong coupling expansion is truncated, the lowest order being, as already mentioned, $1/d$. The second and third order give for $d = 2$ $J = 0.66$ and $J = 0.62$ respectively, showing that the value reported in Tab. II is likely to be an upper bound. Higher dimensions are much less sensitive, in agreement with the idea that strong coupling expansions is also a large $d$ expansion.

It would be interesting then to discuss these results also from the point of view of the $1/d$ expansion, along the lines followed to study the instability of the symmetric vacuum. However, as the vacuum corresponding to the broken phase occurs on the boundary of the physical region, the constraints (72) must be taken into account, giving an extra effective potential of order $1$ in $1/d$. This means that already in the zeroth order of the expansion the Fourier components $x_n$ with $n > 1$ are not zero in general, and $x_1$ does not coincide with its naïve asymptotic value $x_1 = 0.5$, as clearly indicated by Table II. The zero order approximation is already far from trivial when dealing with the broken maximum near the boundary, and we will not attempt its solution here.

5.2 Strong coupling expansion for the Heat Kernel action

We consider now the problem of obtaining a strong-coupling expansion for the effective theory of the Polyakov loops using the Heat Kernel action instead of the Wilson action.

The starting point is naturally the character expansion

$$K_2(g_1, g_2^{-1}; 1/J^H) = \sum_R e^{-\frac{C_R}{2N^2J^H}} \chi_R(g_1) \chi_R(g_2^{-1}) \quad .$$

Considering the translational invariance of the saddle point solution for the effective theory, we are actually concerned with the case $g_1 = g_2 = V$, Tr$V$ being the
Polyakov loop. However, we shall consider the more general case $g_1 \neq g_2$ and put $g_1 = g_2 = V$ only at the very end.

The large $N$ limit of the kernel (73) was studied in Section 4 for large $J^H$. The "gaussian" solution which arises in that situation is valid only down to a critical value of $J^H$, where a third order Douglas-Kazakov phase transition takes place. It would be interesting to find out the exact expression in the large $N$ limit of the kernel (73), also in the strong coupling phase, extending the results of ref. [11] for the sphere to the cylinder.

Unfortunately, such an exact expression is not yet available. We shall instead give an expansion of the free energy $F(g_1, g_2^{-1}; J^H)$ corresponding to the kernel (73) in powers of the relevant coupling, which in this case must be taken to be

$$J_w = 2e^{-1/2J^H}$$

(74)

(notice that at the first order $J_w$ coincides with $J$). Define the free energy in the large $N$ limit by

$$K_2(g_1, g_2^{-1}; \frac{1}{J^H}) = e^{N^2F(g_1, g_2^{-1}; J^H)}$$

(75)

where as usual terms suppressed by powers of $N$ in the exponent will be neglected. We can write then

$$F(g_1, g_2^{-1}; J^H) = 2e^{-1/2J^H}F_1^H(g_1, g_2^{-1}; J^H) + 4e^{-1/2J^H}F_2^H(g_1, g_2^{-1}; J^H) + \ldots$$

(76)

where the residual dependence on $J^H$ in the individual terms is, as we shall see, polynomial in $1/J^H$.

The expansion (74) is of the same type of the one obtained when using the Wilson action; it is therefore what we need in order to compare the two cases. At lowest order, in fact, the two expansions coincide after the identification (74).

We leave to the Appendix some technical remarks about the way to obtain such a strong coupling expansion, and about its meaning and its potential connection with a string representation for the gauge theory. Presently, we quote the result of the expansion of the free energy $F(V, V^\dagger; J^H)$ up to fourth order in $J_w$. The result is expressed in terms of the Fourier modes of the density of eigenvalues $\rho(\theta)$, that is of the quantities $x_n$ and $\alpha_n$, introduced in the previous subsection. We find, in the notations of the previous section,

$$F_1^H(x, \alpha; J^H) = x_1^2$$

(77)

$$F_2^H(x, \alpha; J^H) = \frac{1}{4} \left[ -2x_1^2 + \frac{1}{2J^H}x_1^4 + x_2 - \frac{2}{J^H}x_1^2x_2\cos(2\alpha_1 - \alpha_2) \right]$$

(78)

$$F_3^H(x, \alpha; J^H) = \frac{1}{24} \left[ 6x_1^2 - \frac{6}{J^H}x_1^4 + \frac{1}{J^H}x_1^6 + \frac{12}{J^H}x_1^2x_2^2 + 2x_3^2 + \frac{12}{J^H}x_3^2x_2\cos(3\alpha_1 - \alpha_3) - \frac{8}{J^H}x_1^2x_2\cos(2\alpha_1 - \alpha_2) + \frac{6}{J^H}x_3^2\cos(3\alpha_1 - \alpha_3) - \frac{12}{J^H}x_1x_2x_3\cos(\alpha_1 + \alpha_2 - \alpha_3) \right]$$

(79)
The type of numerical analysis that leads to Table II using the Wilson action now gives the results listed in Table III. The same critical values, in terms of the original Heat Kernel coupling, are listed in the fourth column of Table IV. The two models reflect the critical values of the expansion (80) coincides with the corresponding one for the Wilson action, the higher orders are different, although the general structure remains the same. It is interesting to see how the differences of the two models reflect in the critical values of the symmetric vacuum becomes unstable at \( J_w \).

This can be checked by looking at the coefficients of \( x_1^2 \) in (80), and following the arguments leading to (74) in the previous subsection.

While the lowest order in \( J_w \) of the expansion (80) coincides with the corresponding one for the Wilson action, the higher orders are different, although the general structure remains the same. It is interesting to see how the differences of the two models reflect in the critical values of \( J_w \) at various dimensions. The same type of numerical analysis that lead to Table II using the Wilson action now gives the results listed in Table III. The same critical values, in terms of the original Heat Kernel coupling \( J^H \), are listed in the fourth column of Tab. IV.

| \( d \) | \( J_w \) | \( x_1 \) | \( x_2 \) | \( x_3 \) | \( x_4 \) |
|---|---|---|---|---|---|
| 2 | 0.601 | 0.41 | -0.13 | 0.03 | 0.07 |
| 3 | 0.339 | 0.45 | -0.10 | -0.01 | 0.05 |
| 4 | 0.238 | 0.50 | -0.07 | -0.05 | 0.02 |
| 100 | 0.0096 | 0.64 | 0.07 | -0.10 | -0.03 |

Tab.III. Same as Tab.II, but for the Heat Kernel action. Notice the coupling \( J_w \) is related to the Heat Kernel coupling through eq. (74).
The coincidence of the critical value of $J_w$ for $d = 2$ with the one given in Table II is probably accidental. However, it can be seen from eq. (80) that some relevant coefficients in the first three terms of the expansion coincide with the corresponding ones for the Wilson action when $J^H = 0.5$, that is when $J_w$ is around 0.75. This leads to expect a better agreement between the two theories in that region of $J_w$. In dimensions larger than 2 the discrepancies are of the order of $10^{-15} \%$, and they probably give an idea of the dependence of the results on the chosen regularization.

### 5.3 Summary of the results

The results of both the weak and the strong coupling expansions are summarized in Tab.IV.

| $d$ | $J^H_{DK}$ | $J^H_{DK}$ | $J^H_c$ | $J_c$ | $J^H_{s.v.i.}$ | $J_{s.v.i.}$ |
|-----|------------|------------|--------|------|---------------|-------------|
| 2   | 0.321      | 0.561      | 0.416  | 0.601| 0.455         | 0.667       |
| 3   | 0.226      | 0.468      | 0.282  | 0.379| 0.310         | 0.400       |
| 4   | 0.192      | 0.434      | 0.235  | 0.275| 0.257         | 0.286       |
| $\infty$ | $\frac{1}{\tau} \sim 0.101$ | 0.333 | $\frac{1}{2 \log(2d)}$ | $\frac{1}{7}$ | $\frac{1}{2 \log(2d)}$ | $\frac{1}{7}$ |

**Tab.IV.** Summary of our results. D.K refers to the Douglas-Kazakov point, s.v.i. to the symmetric vacuum instability point and $J_c, J^H_c$ are our best estimates for the deconfinement phase transition. See text for detailed explanation.

In the second and third columns the weak coupling results are listed, namely the value of $J^H$ and $J$ for which the semicircular distribution becomes unstable (as discussed in sect.4), in the Heat Kernel and Wilson model respectively.

In the two middle columns the best estimate for the phase transition, using the fourth order strong coupling expansion, is given for both models. In the last two columns the values of $J_H$ and $J$ at which the symmetric vacuum becomes unstable are given. They are obtained from the equations $J^H = 1/(2 \log(2d - 1))$ and $J = 1/(d - 1/2)$, the former is an exact result [10], the latter is valid up to fourth order corrections in $1/d$ (see discussion in sect. 5.1).

These results are better understood with reference to Fig.2, where the relation between $J$ and $J^H$ is plotted in the weak coupling regime (curve (a)) and in the strong coupling regime (curve (b)). Notice the large overlapping region around $J^H = 0.7$, where the strong and weak coupling definitions essentially coincide.

The Douglas-Kazakov phase transition, marked by the points whose coordinates are in the second and third columns of Tab.IV, gives for each dimension $d$ the extreme of the curve (a) below which the weak coupling solution is not anymore valid. For very large $d$ this region goes as far down as $J^H = 0.1$ into what one would naively expect to be a strong coupling regime. Indeed, as already remarked, it is really at large $d$ that the weak coupling solution can be already trusted just above
the Douglas-Kazakov transition, as the radius of the Wigner distribution is then small, and the approximations made in neglecting quartic terms in $\theta$’s are reliable.

The instability of the symmetric vacuum on the other hand, marks the end of the region where the strong coupling expansion can be relied upon. Beyond that point in fact the system is forced into a configuration where presumably all the Fourier modes $x_n$ of the eigenvalue distribution are excited (see discussion at the end of sect. 5.1) and hence non perturbative effects in $J$ take place. For lower dimensions (e.g. $d = 2, 3, 4$) there is an interval of values of $J^H$ for which both curves are admitted. In that range the system has two classically stable configurations, one corresponding to the unbroken phase and one to the broken phase.

As $J^H$ increases the system evolves along the curve (b), then before it reaches the point of instability for the symmetric vacuum it jumps onto the curve (a) above the Douglas-Kazakov point. The exact point of this transition has been estimated (earlier in this section) in the fourth order strong coupling expansion and it has been marked in Fig.2 for $d = 2$ and $d = 3$ with respectively a square and a triangle.

For large values of $d$ there is no overlap between the curves (a) and (b). This points to the existence of an intermediate phase where the symmetry is already broken, but the eigenvalue distribution is different from the semicircular one.

6 Conclusions

Let us now try to compare our predictions on the critical deconfinement temperature with the known results obtained with Montecarlo simulations. Let us stress again that we must expect two kinds of systematic deviations. The first one is due to the large $N$ approximation. Indeed, the values that we have obtained in the large $N$ limit should be considered as an upper bound, and in the cases $N = 2$ and $N = 3$, in which most of the simulations have been performed, rather large deviations should be expected. However, this is not an unsolvable problem. The techniques described in this paper can in fact be used also at finite $N$, and work is in progress to do so. Preliminary numerical tests show that the deviations in the predicted value of the critical temperature are exactly in the desired direction.

The second problem is that, in order to solve the model, we had to decouple space-like and time-like plaquettes in the original action. A direct consequence of this approximation is that the rule with which the coupling $\beta$ changes as a function of $N_t$ is essentially that of the strong coupling expansion. Once the spacelike plaquettes are decoupled, the theory becomes effectively two dimensional from the point of view of the strong coupling expansion, hence there is in principle no obstruction to extend the expansion to arbitrarily high values of $\beta$ and $N_t$, and the continuum limit can be taken. The scaling law of $N_t$ in this limit is completely fixed by eq. (11) if we use the Wilson regularization, or equivalently by eq. (14) in the Heat
Fig. 2. Redefinitions of the Heat Kernel inverse coupling \( J^H \) necessary to make contact with the Wilson case (a) in the weak coupling regime (large \( J^H \)), and (b) in the strong coupling one (small \( J^H \)). In the region \( 0.4 < J^H < 1 \) there is substantial agreement of the two redefinitions.
Kernel case. We see that $\hat{\beta}/\beta = 1/N_t$, or, equivalently, that

$$T_c = \frac{k}{\hat{\beta}},$$

with $k$ a constant to be determined. As a direct consequence of the decoupling approximation, this scaling law holds in any dimensions and for any gauge group. The relevant point is that, in the (2+1) dimensional case, eq. (81) in fact coincides with the correct scaling behaviour of the whole theory. This is probably due to the fact that in this case the decoupled spatial theory is a two dimensional gauge model, and, as such, essentially trivial. Notice that a similarly good behaviour of the decoupling approximation in (2+1) dimensions was already noticed in [13] in the complementary problem of describing the spacelike string tension.

The second important point is that our value of the deconfinement temperature (which is effectively extracted as if we had $N_t = 1$) must be, already at $N_t = 1$, in the weak coupling regime of the theory, unless we would have an additional uncertainty in making the extrapolation to $\beta \to \infty$. Remarkably enough, in (2+1) dimensions the values that we find fulfills also this second condition, both in the Wilson and in the Heat Kernel case. Notice however that these values are at the border of this region, a fact that will require some further caution, and will be better discussed in the following subsection.

This must be contrasted with the (3+1) dimensional case. First, in the full theory a completely different scaling law is expected. Second, the critical point, at least in the Wilson case, is below the threshold of validity of the weak coupling expansion (see Tab. IV). These are actually two aspects of the same problem, and they are warning us that the result obtained within our approach must be definitely considered as strong coupling result. These considerations show that a more refined analysis, for instance by using the techniques of the (hot, twisted) Eguchi-Kawai model, would be important in this case.

It should be noticed, however, that an incorrect scaling of dimensional quantities does not necessarily imply a total loss of predictive power. In fact, as recently noticed in [29], while even the most precise and recent simulations of SU(2) and SU(3) models in (3+1) dimensions still show strong deviations from asymptotic scaling, scaling of dimensionless ratios is fulfilled even at small couplings, and very small values of $N_t$ (down to $N_t = 3$). This opens the way to a possible improvement in our predictions, if other dimensional quantities besides the critical temperature, say the string tension, could be evaluated, for instance in the framework outlined in [13].

Let us now discuss, in turn, the (2+1) and the (3+1) dimensional cases.

### 6.1 2+1 dimensions

In (2+1) dimensions our best prediction for the large $N$ value of the critical coupling $J_c = \hat{\beta}_c/N^2$ is $J_c = 0.601$. Remarkably enough, this result follows both from the
Wilson and the Heat Kernel regularizations, once the relation between Wilson and Heat Kernel couplings is used. The result should be considered as the fourth order improved version of the standard mean-field result \( J_c = 1/d = 0.5 \). Comparing with the second and third order results we see that this should be considered as an upper bound, and that further orders would probably decrease the value. Keeping in mind the various factors of \( N \), we obtain

\[
\frac{T_c}{g^2} = \frac{\beta_c}{N^2 N_t} = 0.601 \quad .
\] (82)

Looking at Tab. IV and Fig. 2 we see that this value is at the border of the weak coupling region, hence its extrapolation toward the continuum limit \( (N_t \to \infty, \beta_c \to \infty) \) could be affected by systematic errors. A simple way to estimate these possible deviations is to notice that, had we chosen from the beginning a Heat Kernel regularization for the whole theory, then, as discussed in the introduction, the scaling law eq. (81) would have been exact. In that case we could not have used eq. (81) to map the Heat Kernel coupling to the Wilson one; instead, we would have had to keep the true Heat Kernel critical coupling, which turns out to be (see sect.5 and Tab.IV) \( J_{cH} = 0.416 \). Since the simulations with which we shall compare our results were performed with the Wilson action we shall use \( J_c = 0.601 \) in our comparison, but the difference between \( J_c \) and \( J_{cH} \) gives us an idea of size of the possible systematic errors involved in the continuum limit.

To the best of our knowledge, Montecarlo estimates of the deconfinement temperature in (2+1) dimensions only exist for the \( N = 2 \) and \( N = 3 \) models.

In the case of SU(2), very precise and careful estimates of the critical temperature exist \([16]\) for the lattice sizes 2, 3, 4. The values are reported in Tab. V, together with an estimate of the continuum limit \( (\beta \to \infty) \), using only the leading \( O(1/\beta) \) corrections (see \([16]\) for details). While the deviations from scaling are still rather strong in the \( T_c/g^2 \) data they are, as expected, much more under control for the dimensionless ratio \( T_c/\sqrt{\sigma} \), whose continuum limit requires the inclusion of only \( O(1/\beta^2) \) corrections.

| \( N \) | \( N_t \) | \( T_c/g^2 \) | \( T_c/\sqrt{\sigma} \) |
|------|-----|------------|------------|
| 2    | 2   | ~0.433     | ~1.015     |
| 2    | 3   | ~0.417     | ~1.063     |
| 2    | 4   | ~0.412     | ~1.088     |
| 2    | \( \infty \) | 0.385(10) | 1.121(8)   |
| 3    | 2   | ~0.454     |            |

\( \infty, J_c \)

\( \infty, J_{cH} \)

\( 0.477 \)

\( 0.977 \)

\( 0.601 \)

\( 0.416 \)

**Tab.V.** The deconfinement temperature \( T_c \) and the dimensionless ratio \( T_c/\sqrt{\sigma} \) as a function of the lattice size in the t direction \( N_t \) in the (2+1) dimensional SU(2)
and SU(3) LGT. Taken from [16] and [30]. In the sixth row we report the estimate of ref. [14], while our predictions are reported in the last two rows.

In the case of SU(3) there is, to the best of our knowledge, only one numerical result. It is the critical coupling for the $N_t = 2$ lattice, which turns out to be $\beta_c(N_t = 2) \sim 8.17$ [30]. This corresponds to $T_c/g^2 \sim 0.454$. No extrapolation to the continuum limit is possible in this case, however it is interesting to see that the value is definitely higher than those of the $N = 2$ case. This is a signature of the fact that as $N$ increases the $T_c/g^2$ values move toward our large $N$ limit.

Finally, let us mention the result obtained in [15] (also reported in Tab. V) by using as additional input the dimensionless ratio $\sigma = \pi T_c^2/3$, which comes from the effective string approach to the interquark potential in LGT’s. The resulting prediction for the critical temperature is

$$T_c = \frac{3(N^2 - 1)}{2\pi \beta},$$

which means, in the large $N$ limit, $T_c/g^2 = 0.477$, a result which again is in the same region of our large $N$ estimates.

### 6.2 3+1 dimensions

In this case we can test our results with a much larger sample of data. The main problem is obviously that the scaling laws (namely the dependence of $\beta_c$ on the value of $N_t$) of our simplified model and of the whole theory are completely different. This essentially means that, due to the decoupling of the space-like plaquettes (and to the fact that in this case the purely spacelike theory is non trivial), the lattice spacing in the time direction, with which we measure $N_t$ in our decoupled model, is related in a non trivial way to the lattice spacing of the starting theory. The only way to use our results is then to compare our decoupled theory and the Montecarlo simulations of the full theory for small values of $N_t$, where $J_c$ is small and we can trust our strong coupling expansion. In this case we must definitely study the Wilson action only, and no universality argument can be used. The scaling law is given by eq. (12)

Our best estimate for the critical coupling in $(3 + 1)$ dimensions is $J_c = 0.378$. Very precise data on SU(2) and SU(3) in the range $N_t = 2 - 16$ can be found in [29]. Their values in terms of the $\Lambda$ scale defined as

$$\Lambda = \left(\frac{24\pi^2 \beta}{11N^2}\right)^{51/121} \exp\left(-\frac{12\pi^2 \beta}{11N^2}\right)$$

(84)

together with the corresponding values of $J_c \equiv \beta_c/N^2$ are shown in Tables VIa and VIb.
The critical coupling $\beta_c/N^2$ and the corresponding deconfinement temperature $T_c/\Lambda$ as a function of the lattice size in the $t$ direction $N_t$ in the $(3+1)$ dimensional $SU(2)$ LGT. The data are taken from [29].

| $N_t$ | $\beta_c/N^2$ | $T_c/\Lambda$ |
|-------|----------------|----------------|
| 2     | 0.4700(8)      | 29.7(2)        |
| 3     | 0.5442(8)      | 41.4(3)        |
| 4     | 0.5746(2)      | 42.1(1)        |
| 5     | 0.5932(11)     | 40.6(5)        |
| 6     | 0.6066(8)      | 38.7(3)        |
| 8     | 0.6279(10)     | 36.0(4)        |
| 16    | 0.6849(25)     | 32.0(8)        |

The critical coupling $\beta_c/N^2$ and the corresponding deconfinement temperature $T_c/\Lambda$ as a function of the lattice size in the $t$ direction $N_t$ in the $(3+1)$ dimensional $SU(3)$ LGT. The data for $N_t \geq 3$ are taken from [29], the single one at $N_t = 2$ is taken from [11].

| $N_t$ | $\beta_c/N^2$ | $T_c/\Lambda$ |
|-------|----------------|----------------|
| 2     | $\sim 0.568$  | $\sim 78.8$   |
| 3     | 0.6167(11)     | 85.7(1.0)     |
| 4     | 0.63250(2)     | 75.41(2)      |
| 6     | 0.65490(6)     | 63.05(4)      |
| 8     | 0.6673(3)      | 53.34(1.5)    |
| 10    | 0.6844(8)      | 51.05(40)     |
| 12    | 0.6964(13)     | 48.05(65)     |
| 14    | 0.7092(11)     | 46.9(5)       |
| 16    | 0.716(6)       | 44.3(2.5)     |

The values of $J_c$ from Tables VIa and VIb corresponding to small values of $N_t$, together with some results for a large value of $N$ ($N = 81$) obtained through Montecarlo simulations [32] of the hot twisted Eguchi-Kawai model are compared with our predictions in Table VII. Our values are obtained using eq.(12) and the value ($N_t = 1, \ J_c = 0.378$) as input.
Tab. VII. The critical coupling $\beta_c/N^2$ as a function of $N_t$ and $N$. In the last column our predictions, according to the scaling law eq. (12) and $\beta_c/N^2 = 0.378$ for $N_t = 1$. The data for $N = 2$ and 3 are taken from Tab. VIa and VIb. Those for $N = 81$ are taken from [32].

We may observe that

a] the scaling law eq. (12) is definitely different from the scaling behaviour eq. (84). However the Montecarlo estimates of the critical couplings seem to be still far from the asymptotic scaling region and the values of $T_c/\Lambda$ keep decreasing as $\beta_c$ and $N_t$ increase. It is interesting to notice that the critical couplings for small values of $N_t$ have a dependence from $N_t$ which is not too different from our strong coupling expectation eq. (12).

b] As expected, the best agreement between data and predictions is obtained for our lowest value of $N_t$: $N_t = 2$.

c] At fixed $N_t$, as $N$ increases also $J_c$ systematically increases, and our large $N$ estimates seem to be the upper limit of this behaviour.

As we have seen, our results, obtained analytically in the large $N$ limit, show a more than qualitative agreement with Montecarlo simulations performed on the full theory; having used two different lattice regularizations gives us some control on the systematic errors associated with the continuum limit; the uncertainties due to the large $N$ approximation and to the decoupling of the space-like degrees of freedom can be further reduced by studying the model at finite $N$, and by using the Eguchi-Kawai approach. We believe that a combination of these methods may allow in future further improvements in the analytic determination of the deconfinement critical temperature.

Appendix
We are interested in the large-$N$ limit of the Heat Kernel on the cylinder

$$K_2(g_1, g_2^{-1}; J^H) = \sum_R e^{-\frac{C_R^{(2)}}{2\pi N} \chi_R(g_1) \chi_R(g_2^{-1})}$$  \tag{A.1}$$

In sec. (3) we utilized this kernel as a building block for the effective theory of Polyakov loops. We were therefore with the case $g_1 = g_2 = V$, $\text{Tr} V$ being the Polyakov loop and we gave in eq. (77-80) an expansion of the free energy corresponding to the partition function (A.1) in powers of $2e^{-1/2\pi}$, up to $4^{th}$ order.

We consider here the general case, sketching the procedure to obtain explicitly such an expansion and discussing a little its meaning.

The kernel (A.1) is a particular case of the QCD2 partition function on manifolds $M_{G,b}$ of genus $G$ with $b$ boundary components, the case $M_{0,2}$. As it is well known after [33] the character expansion of the QCD2 partition function on $M_{G,b}$ can be interpreted as a weighted sum over a suitable set of maps from world-sheets of all possible genera to $M_{G,b}$, that is, as a string theory.

The “string coupling constant” of this peculiar string theory is $1/N$ - there’s a factor of $N^{2-2g}$ in front of the contributions of the maps from genus $g$ world-sheet. Since we confine ourself to the $N = \infty$ limit, i.e. we do not care about $O(1/N)$ corrections in eq. (75), the expansion we are looking for corresponds to consider just the string tree level ($g = 0$) contributions.

The string interpretation of the QCD2 partition function arises technically limiting the sum over the $SU(N)$ representations to the representations obtained by symmetrizing tensor products of the fundamental representation and of its conjugate. As shown by the Douglas-Kazakov exact solution at $N = \infty$ in the case of the sphere this is the correct way to carry out the sum in the strong-coupling (small-area) regime. We too will therefore sum in eq. (A.1) over these representations.

They can be in general labeled by means of two sets of integers $\{l\}$ and $\{m\}$ satisfying

$$l_1 \geq l_2 \geq \ldots \geq l_p \geq 0 \geq m_1 \geq m_2 \geq \ldots \geq m_q \tag{A.2}$$

The $\{l\}$ and the negative of the $\{m\}$ identify two Young tableaux, $L$ and $M$, of order $l = \sum_i l_i$ and $m = -\sum_i m_i$; the representation is then called a “composite representation” in [33] and denoted as $(\bar{M}L)$. Its order can be defined as $l + m$. When the $\{m\}$ are all zero, the Casimir is given by

$$C_L^{(2)} = lN + l + \sum_i l_i(l_i - 2i) - \frac{l^2}{N}. \tag{A.3}$$

For the representation $(\bar{M}L)$ the Casimir is

$$C_{(\bar{M}L)}^{(2)} = C_L^{(2)} + C_M^{(2)} + 2 \frac{lm}{N}. \tag{A.4}$$
Equations (A.3, A.4) are the key point allowing to obtain an expansion in powers of $e^{-1/2JH}$ for the kernel (A.1). Indeed eq. (A.1) becomes

$$K_2(g_1, g_2^{-1}; JH) = \sum_{l,m=0}^{\infty} e^{-\frac{l+m}{2JH}} \sum_{L \in Y_l} \sum_{M \in Y_m} \chi_{(ML)}(g_1) \chi_{(ML)}(g_2^{-1}) \cdot e^{-\frac{1}{2JH} \left[ \frac{l+m}{N} + \frac{1}{N} \sum_i (l_i - 2m_i) + \frac{2lm}{N^2} \right]}.$$  \hfill (A.5)

$Y_l$ denotes here the set of Young tableaux made of $l$ boxes.

The characters $\chi_{(ML)}(g_i)$ can be handled in the large-$N$ limit rewriting them in terms of the quantities

$$\rho_k = \frac{1}{N} \text{Tr} g_1^k \quad ; \quad \sigma_k = \frac{1}{N} \text{Tr} g_2^k$$  \hfill (A.6)

and of their conjugates $\rho^*_k (= \frac{1}{N} \text{Tr} (g_1^*)^k)$ and $\sigma^*_k$. These quantities remain finite as $N \to \infty$, turning into the Fourier modes of the eigenvalue distributions $\rho(\theta)$ and $\sigma(\phi)$ corresponding to $g_1$ or $g_2$:

$$\rho_k = \int d\theta \rho(\theta) e^{-ik\theta}$$  \hfill (A.7)

This rewriting of the characters is accomplished by means of the Frobenius representation, extended to accomodate the case of composite representations $(\bar{M}L)$ [33, 34].

Having made explicit all the dependences from $N$ it is possible to expand explicitly eq. (A.5) up to a certain power (call it $P$) of $e^{-1/2JH}$, which means to consider all the representations $(\bar{M}L)$ such that $l+m \leq P$, to write the resulting expression as the exponential of a free energy and to neglect the subleading terms as $N \to \infty$ in the latter, obtaining finally an expression of the form (76). The expansion of the free energy can of course be obtained by subtracting from the expansion (A.5) the “disconnected” contributions (for instance at order $e^{-1/2JH}$ the contributions of products two rep.s of order $l + m = 1$).

Expanding the last exponentials in eq. (A.5) on gets power series in $1/JH N$ and $1/JH N^2$. However, when extracting the $O(N^2)$ terms only in the free energy, at fixed order $l + m$ only a polynomial in $1/JH$ survive [3]. The resulting expression makes thus sense as a strong coupling expansion.

Now we give the free energy, eq. (76) up to $3^{rd}$ order. We use the trigonometric notation for the complex modes of the eigenvalue distributions $\rho$ and $\sigma$:

$$\rho_k = r_k e^{i\alpha_k} \quad ; \quad \sigma_k = s_k e^{i\gamma_k} \quad (r_k, s_k \in \mathbb{R}).$$  \hfill (A.8)

3This fact has a nice interpretation in the QCD2 string language; it is related to the existence of a limiting value for the Euler characteristic of the covering space at fixed characteristic of the target.
\[ F(g_1, g_2^{-1}; J^H) = \]
\[ e^{-1/2J^H} r_1 s_1 \cos(\alpha_1 - \gamma_1) + \left( e^{-1/2J^H} \right)^2 \left[ -r_1^2 - s_1^2 - \frac{1}{J^H} r_2 s_1^2 \cos(\alpha_2 - 2\gamma_1) \right. \]
\[ - \frac{1}{J^H} r_1^2 s_2 \cos(2\alpha_1 - \gamma_2) - \frac{2}{J^H} r_2 s_1^2 \cos(\alpha_1 - \gamma_1) + r_2 s_2 \cos^2(\alpha_2 - \gamma_2) \]
\[ + \left. \left( 1 + 4J^H \right) r_1^2 s_1^2 \cos(2(\alpha_1 - \gamma_1)) \right] \]
\[ + \frac{(e^{-1/2J^H})^3}{24} \left[ \frac{-4 - 6J^H}{J^H^3} r_1 r_2 s_1^3 \cos(\alpha_1 + \alpha_2 - 3\gamma_1) + \frac{3}{J^H^2} r_3 s_1^3 \cos(\alpha_3 - 3\gamma_1) \right. \]
\[ + (6r_1 s_1 - \frac{3}{J^H^2} (r_1^3 s_1 + r_1 s_1^3) - \frac{1 + J^H}{J^H^3} r_1^3 s_1^3) \cos(\alpha_1 - \gamma_1) \]
\[ + \frac{1 + 3J^H + 3J^H^2}{J^H^4} r_1^3 s_1^3 \cos(3(\alpha_1 - \gamma_1)) + \left( \frac{6}{J^H} r_1 r_2 s_1 + \frac{6}{J^H^2} r_1 r_2 s_1^3 \right) \cos(\alpha_1 - \alpha_2 + \gamma_1) \]
\[ + \frac{-4 - 6J^H}{J^H^3} r_1^3 s_1^2 s_2 \cos(3\alpha_1 - \gamma_1 - \gamma_2) + \frac{12 + 6J^H}{J^H^2} r_1 r_2 s_1 s_2 \cos(\alpha_1 + \alpha_2 - \gamma_1 - \gamma_2) \]
\[ - \frac{6}{J^H} r_1 r_2 s_1 s_2 \cos(\alpha_3 - \gamma_1 - \gamma_2) + \left( \frac{6}{J^H} r_1 r_2 s_1 s_2 + \frac{6}{J^H^2} r_1^3 s_1 s_2 \right) \cos(\alpha_1 + \gamma_1 - \gamma_2) \]
\[ - \frac{6}{J^H} r_1 r_2 s_1 s_2 \cos(\alpha_1 - \alpha_2 - \gamma_1 + \gamma_2) + \frac{3}{J^H^2} r_1^3 s_3 \cos(3\alpha_1 - \gamma_3) \]
\[ - \frac{6}{J^H} r_1 r_2 s_3 \cos(\alpha_1 + \alpha_2 - \gamma_3) + 2r_3 s_3 \cos(\alpha_3 - \gamma_3) \]

(A.9)

In the case \( g_1 = g_2 \) the free energy up to 4\(^{th} \) order has been given in eq. \([80]\).

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