ON NON-INVARIANT HYPERSURFACES OF AN $\varepsilon$-PARA SASAKIAN MANIFOLD

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Abstract. Non-invariant hypersurfaces of an $\varepsilon$–para Sasakian manifold of an induced structure $(f, g, u, v, \lambda)$ have been studied in this paper. Some properties followed by this structure have been obtained. A necessary and sufficient condition for totally umbilical non-invariant hypersurfaces equipped with $(f, g, u, v, \lambda)$ – structure of $\varepsilon$–para Sasakian manifold to be totally geodesic has also been explored.

Keywords: $\varepsilon$–Para Sasakian Manifold, totally umbilical, totally geodesic.

1. Introduction

In 1976, Sato [1] introduced a structure $(\phi, \xi, \eta)$ satisfying $\phi^2 = I - \eta \otimes \xi$ and $\eta(\xi) = 1$ on a differentiable manifold, which is now well known as an almost paracontact structure. The structure is an analogue of the almost contact structure [2, 3] and is closely related to almost product structure (in contrast to almost contact structure, which is related to almost complex structure). An almost contact manifold is always odd-dimensional but an almost paracontact manifold could be even-dimensional as well. In 1969, T. Takahashi [4] introduced almost contact manifolds equipped with associated pseudo Riemannian metrics. In particular, he studied Sasakian manifolds equipped with an associated pseudo– Riemannian metric. These indefinite almost contact metric manifolds and indefinite Sasakian manifolds are also known as $\varepsilon$–almost contact metric manifolds and $\varepsilon$–Sasakian manifolds respectively [5, 6, 7].

Also, in 1989, K. Matsumoto [8] replaced the structure vector field $\xi$ by $-\xi$ in an almost paracontact manifold and associated a Lorentzian metric with the resulting structure, calling it a Lorentzian almost paracontact manifold. In a Lorentzian almost paracontact manifold given by Matsumoto, the semi–Riemannian metric has only index 1 and the structure vector field $\xi$ is always timelike. These circumstances motivated the authors in [9] to associate a semi–Riemannian metric, not necessarily Lorentzian, with an almost paracontact structure, and they called this indefinite
almost paracontact metric structure an \( \varepsilon \)-almost paracontact structure, where the structure vector field \( \xi \) is spacelike or timelike according as \( \varepsilon = 1 \) or \( \varepsilon = -1 \).

In [9] the authors studied \( \varepsilon \)-almost paracontact manifolds, and in particular, \( \varepsilon \)-para Sasakian manifolds. They gave basic definitions, some examples of \( \varepsilon \)-almost paracontact manifolds and introduced the notion of an \( \varepsilon \)-para Sasakian structure. The basic properties, some typical identities for curvature tensor and Ricci tensor of the \( \varepsilon \)-para Sasakian manifolds were also studied in [9]. The authors in [9] proved that if a semi-Riemannian manifold is one of flat, proper recurrent or proper Ricci-recurrent, then it can not admit an \( \varepsilon \)-para Sasakian structure. Also. they showed that, for an \( \varepsilon \)-para Sasakian manifold, the conditions of being symmetric, semi-symmetric or of constant sectional curvature are all identical.

On the other hand In 1970, S. I. Goldberg et. al [10] introduced the notion of a non-invariant hypersurface of an almost contact manifold in which the transform of a tangent vector of the hypersurface by the \((1, 1)\) structure tensor field \( \phi \) defining the almost contact structure is never tangent to the hypersurface.

The notion of \((f, g, u, v, \lambda)\) - structure was given by K.Yano [11]. It is well known [12, 13] that a hypersurface of an almost contact metric manifold always admits a \((f, g, u, v, \lambda)\) - structure. Authors [10] proved that there always exists a \((f, g, u, v, \lambda)\) - structure on a non-invariant hypersurface of an almost contact metric manifold. They also proved that there does not exist invariant hypersurface of a contact manifold. R. Prasad [14] studied the non-invariant hypersurfaces of trans-Sasakian manifolds. Non-invariant hypersurfaces of nearly Trans-Sasakian manifold have been studied by S. Kishor et. al [15]. The present paper is devoted to the study of non-invariant hypersurfaces of \( \varepsilon \)-para Sasakian manifolds. The contents of the paper are organized as follows:

In section-2 some preliminaries are given. Section-3 deals with the study of non-invariant hypersurfaces of \( \varepsilon \)-para Sasakian manifolds. A necessary and sufficient condition for a totally umbilical non-invariant hypersurface of an \( \varepsilon \)-para Sasakian manifold to be totally geodesic is found.

2. Preliminaries

Let \( \tilde{M} \) be an almost contact metric manifold with almost contact metric structure \((\phi, \xi, \eta, g)\) where \( \phi \) is \((1, 1)\) tensor field, \( \eta \) is \(1\)-form and \( g \) is a compatible Riemannian metric such that

\begin{align*}
(2.1) \quad & \phi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta \phi = 0, \\
(2.2) \quad & g(\phi X, \phi Y) = g(X, Y) - \epsilon \eta (X) \eta(Y), \\
(2.3) \quad & g(X, \phi Y) = g(\phi X, Y), \quad g(X, \xi) = \epsilon \eta(X)
\end{align*}
On Non−Invariant Hypersurfaces

for all \(X, Y \in T\tilde{M}\).

An almost contact metric manifold is an \(\varepsilon\)–para Sasakian manifold if

\[
(\nabla_X \phi)(Y) = -g(\phi X, \phi Y) - \varepsilon \eta(Y) \phi^2 X
\]

for all vector fields \(X, Y\) on \(\tilde{M}\) where \(\nabla\) is the operator of covariant differentiation with respect to \(g\). From (2.4), we have

\[
\tilde{\nabla}_X \xi = \varepsilon \phi X
\]

A hypersurface of an almost contact metric manifold \(\tilde{M} (\phi, \xi, \eta, g)\) is called a non-invariant hypersurface, if the transform of a tangent vector of the hypersurface under the action of \((1, 1)\) tensor field \(\phi\) defining the contact structure is never tangent to the hypersurface. Let \(X\) be a tangent vector on a non-invariant hypersurface of an almost contact metric manifold \(\tilde{M}\), then \(X\phi\) is never tangent to the hypersurface.

Let \(M\) be a non-invariant hypersurface of an almost contact metric manifold, then defining

\[
\phi X = f X + u(X)\hat{N},
\]

\[
\phi \hat{N} = -U,
\]

\[
\xi = V + \lambda\hat{N}, \quad \lambda = n(\hat{N});
\]

\[
\eta(X) = \nu(X),
\]

where \(f\) is a \((1, 1)\) tensor field, \(u, v\) are 1−forms, \(\hat{N}\) is a unit normal to the hypersurface, \(X \in TM\) and \(u(X) \neq 0\), then we get an induced \((f, g, u, v, \lambda)\) structure on \(M\) satisfying the conditions \([11, 12]\) :

\[
f^2 = -I + u \otimes U + v \otimes V,
\]

\[
fU = -\lambda V, \quad fV = \lambda U,
\]

\[
u_{of} = \lambda v, \quad vo_{f} = -\lambda u,
\]

\[
u(U) = 1 - \lambda^2, \quad u(V) = v(U) = 0, \quad v(V) = 1 - \lambda^2,
\]

\[
g(fX, fY) = g(X, Y) - u(X)u(Y) - v(X)v(Y),
\]
for all \(X, Y \in TM\), where \(\lambda = n(\mathring{N})\).

The Gauss and Weingarten formulae are given by

\[
\tag{2.16}
\nabla_XY = \nabla_XY + \sigma(X,Y)\mathring{N},
\]

\[
\tag{2.17}
\nabla_X\mathring{N} = -A_X\mathring{N},
\]

for all \(X, Y \in TM\), where \(\nabla\) and \(\nabla\) are the Riemannian and induced Riemannian connections on \(\mathring{M}\) and \(M\) respectively and \(\mathring{N}\) is the unit normal vector in the normal bundle \(T^\perp M\).

The second fundamental form \(\sigma\) on \(M\) is related to \(A_X\mathring{N}\) by

\[
\tag{2.18}
\sigma(X,Y) = g(A_X\mathring{N},Y), \quad \text{for all } X, Y \in TM.
\]

\section{Non-invariant Hypersurfaces}

\begin{lemma}
Let \(M\) be a non-invariant hypersurface with \((f, g, u, v, \lambda)\) – structure of an \(\varepsilon\)–para Sasakian manifold \(\mathring{M}\), then
\end{lemma}

\[
\tag{3.1}
(\nabla_X\phi)Y = (\nabla_Xf)Y - u(Y)A_X\mathring{N}X + \sigma(X,Y)U + ((\nabla_Xu)Y + \sigma(X,fY))\mathring{N}
\]

\[
\tag{3.2}
(\nabla_X\eta)Y = (\nabla_Xu)Y - \lambda\sigma(X,Y)
\]

\[
\tag{3.3}
\nabla_X\xi = \left(\nabla_XV - \lambda A_X\mathring{N}\right) + (\sigma(X,V) + X\lambda)\mathring{N}
\]

for all \(X, Y \in TM\).

\begin{proof}
Consider
\[
(\nabla_X\phi)Y = \nabla_X(\phi Y) - \phi(\nabla_XY)
\]
\[
= \nabla_X(fY + u(Y)\mathring{N}) - \phi(\nabla_XY + \sigma(X,Y)\mathring{N})
\]
\[
= \nabla_X(fY) + \nabla_X(u(Y)\mathring{N}) - f(\nabla_XY) - u(\nabla_XY)\mathring{N} - \sigma(X,Y)(-U)
\]
\[
= \nabla_X(fY) + \sigma(X,fY)\mathring{N} + u(Y)(-A_X\mathring{N}) + \nabla_X(u(Y))\mathring{N} - f(\nabla_XY)
\]
\[
-u(\nabla_XY)\mathring{N} + \sigma(X,Y)U
\]
\end{proof}
which gives,

\[
(\nabla_X \phi)Y = (\nabla_X f)Y - u(Y)(\nabla_X X) + \sigma(X, Y)U + (\nabla_X u)Y + \sigma(X, fY)\hat{N}
\]

Also,

\[
(\nabla_X \eta)Y = \nabla_X \eta(Y) - \eta(\nabla_X Y) = \nabla_X (v(Y)) - v(\nabla_X Y) - \sigma(X, Y)\eta(\hat{N}).
\]

Therefore

\[
(\nabla_X \eta)Y = (\nabla_X u)Y - \lambda \sigma(X, Y)
\]

Now consider

\[
\nabla_X \xi = \nabla_X \xi + \sigma(X, \xi)\hat{N} = \nabla_X V + \nabla_X \lambda\hat{N} + \sigma(X, V)\hat{N} = \nabla_X V - \lambda \nabla_X \hat{N} + (X \lambda)\hat{N} + \sigma(X, V)\hat{N}
\]

which gives

\[
\nabla_X \xi = (\nabla_X V - \lambda A_{\xi} X) + (\sigma(X, V) + X \lambda)\hat{N}.
\]

\[\square\]

**Theorem 3.1.** Let \( M \) be a non-invariant hypersurface with \((f, g, u, v, \lambda)\) – structure of an \( \varepsilon \)–para Sasakian manifold \( \tilde{M} \), then

\[(3.4)\quad \sigma(X, \xi)U = -\varepsilon f^2 X + \varepsilon u(X)U + f(\nabla_X \xi), \]

and

\[(3.5)\quad u(\nabla_X \xi) = -\varepsilon u(fX)\]

**Proof.** Consider

\[
\left(\nabla_X \phi\right) \xi = \nabla_X (\phi \xi) - \phi(\nabla_X \xi) = -\varepsilon \phi(\phi X)
\]

or

\[(3.6)\quad \left(\nabla_X \phi\right) \xi = -\varepsilon \phi(fX + u(X))\hat{N}\]

Also we know that

\[(3.7)\quad \left(\nabla_X \phi\right) \xi = -\phi(\nabla_X \xi) + \sigma(X, \xi)U\]
From (3.6) and (3.7), we have

\[-\phi(\nabla_X \xi) + \sigma(X, \xi)U = -\varepsilon \phi(fX + u(X)) \hat{N} = -\varepsilon \phi(fX) + \varepsilon u(X)U\]

Now from (2.6) & (2.7), we have

\[-f(\nabla_X \xi) - u(\nabla_X \xi) \hat{N} + \sigma(X, \xi)U = -\varepsilon (fX + u(fX) \hat{N}) + \varepsilon u(X)U\]

Equating tangential and normal parts, we get

\[\sigma(X, \xi)U = -\varepsilon f^2 X + \varepsilon u(X)U + f(\nabla_X \xi),\]

and

\[u(\nabla_X \xi) = -\varepsilon u(fX)\]

\[\square\]

**Theorem 3.2.** Let \(M\) be a non-invariant hypersurface with \((f, g, u, v, \lambda)\) – structure of an \(\varepsilon\)-para Sasakian manifold \(\widetilde{M}\), then

\[(\nabla_X f) Y = -g(X, Y)V + \varepsilon v(Y)X + \sigma(X, Y)U + u(Y)A_N X\]

\[(\nabla_X u) Y = -\lambda g(X, Y) - \sigma(X, fY)\]

**Proof.** From equations (3.1) & (2.4), we have

\[(\nabla_X f) Y - u(Y)A_N X + \sigma(X, Y)U + ((\nabla_X u) Y + \sigma(X, fY)) \hat{N}\]

\[= -g(X, Y)V - \lambda g(X, Y) \hat{N} + \varepsilon v(Y)X\]

Equating tangential and normal parts in the above equation, we get (3.8) & (3.9) respectively. \[\square\]

**Theorem 3.3.** Let \(M\) be a non-invariant hypersurface with \((f, g, u, v, \lambda)\) – structure of an \(\varepsilon\)-para Sasakian manifold \(\widetilde{M}\), then

\[\left(\nabla_X \phi\right) Y = -g(X, Y)V - \lambda g(X, Y) \hat{N} + \varepsilon v(Y)X + 2\sigma(X, Y)U\]

**Proof.** Consider

\[\left(\nabla_X \phi\right) Y = \nabla_X (\phi Y) - \phi \left(\nabla_X Y\right)\]

\[= \nabla_X (fY) + \nabla_X \left(\varepsilon (Y) \hat{N}\right) - f(\nabla_X Y) - u(\nabla_X Y) \hat{N},\]
This implies

\[(3.11) \quad (\tilde{\nabla}_X \phi) Y = (\nabla_X f) Y - u(Y) A_\xi X + \sigma(X, Y) U + ((\nabla_X u) Y + \sigma(X, fY)) \hat{N}\]

Using (3.8) & (3.9), above equation reduces to

\[(\tilde{\nabla}_X \phi) Y = -g(X, Y) V - \lambda g(X, Y) \hat{N} + \epsilon v(Y) X + 2\sigma(X, Y) U\]

\[\square\]

Furthur, we proceed for some results on totally geodesic non-invariant hypersurfaces.

**Theorem 3.4.** Let \(M\) be a totally umbilical non-invariant hypersurface with \((f, g, u, v, \lambda)\) – structure of an \(\epsilon\)-para Sasakian manifold \(\tilde{M}\), then it is totally geodesic if and only if

\[(3.12) \quad \epsilon u(X) = X\lambda = 0\]

**Proof.** Consider

\[\tilde{\nabla}_X \xi = \nabla_X \xi + \sigma(X, \xi) \hat{N}\]
\[= \nabla_X (V + \lambda \hat{N}) + \sigma(X, V) \hat{N}\]
\[= \nabla_X V + \nabla_X \lambda \hat{N} + \sigma(X, V) \hat{N}\]
\[= \nabla_X V + \lambda \nabla_X \hat{N} + (X\lambda) \hat{N} + \sigma(X, V) \hat{N}\]

or

\[(3.13) \quad \tilde{\nabla}_X \xi = \left(\nabla_X V - \lambda A_\xi X\right) + (\sigma(X, V) + X\lambda) \hat{N},\]

Now from (2.5), the above equation is reduced to

\[\epsilon(fX + u(X)) \hat{N} = \left(\nabla_X V - \lambda A_\xi X\right) + (\sigma(X, V) + X\lambda) \hat{N},\]

Equating normal parts on both the sides, we get

\[(3.14) \quad \sigma(X, V) + X\lambda = \epsilon u(X)\]

Now if \(M\) is totally umbilical, then \(A_\xi = \zeta I\), \(\zeta\) is Kahlerian metric and equation (2.18) reduces to \(\sigma(X, Y) = g(\hat{A}_\xi X, \hat{Y}) = g(\zeta X, \zeta Y) = \zeta g(X, Y)\).

Therefore

\[\sigma(X, Y) = \zeta g(X, Y),\]
and equation (3.14) implies
\[ \zeta g(X, Y) + X\lambda = \epsilon u(X) \]
or
\[ (3.15) \quad \epsilon u(X) - X\lambda - \zeta g(X, Y) = 0 \]
Now if \( M \) is totally geodesic i.e. \( \zeta = 0 \), then (3.15) gives
\[ \epsilon u(X) - X\lambda = 0. \]

\( \square \)

**Theorem 3.5.** Let \( M \) be a non-invariant hypersurface with \((f, g, u, v, \lambda) - \) structure of an \( \epsilon-\)para Sasaki manifold \( \hat{M} \). If \( U \) is parallel, then we have
\[ (3.16) \quad \epsilon \lambda X + f \left( A_N X \right) - g(\phi X, U)V - \epsilon \lambda v(X) V = 0 \]

**Proof.** Consider
\[ (\tilde{\nabla}_X \phi) \hat{N} = \tilde{\nabla}_X \phi \hat{N} - \phi \left( \tilde{\nabla}_X \hat{N} \right) \]
\[ = -\tilde{\nabla}_X U - \phi(-A_N X) \]
\[ = -\tilde{\nabla}_X U - (-f(-A_N X) - u(A_N X) \hat{N}) \]
This gives
\[ (3.17) \quad (\tilde{\nabla}_X \phi) \hat{N} = -\nabla_X U - f(A_N X) \]
From equation (2.4), we have
\[ (3.18) \quad (\tilde{\nabla}_X \phi) Y = -g(\phi X, \phi Y) V - \lambda g(\phi X, \phi Y) \hat{N} + \epsilon \eta (Y) X - \epsilon \eta (X) \eta (Y) \xi \]
Substituting \( Y = \hat{N} \), we have
\[ (3.19) \quad (\tilde{\nabla}_X \phi) \hat{N} = g(\phi X, U) V + \lambda g(\phi X, U) \hat{N} - \epsilon \lambda X + \epsilon \lambda v(X) \xi \]
Now from (3.17) and (3.19), we have
\[ -\nabla_X U + f(A_N X) = g(\phi X, U) V + \lambda g(\phi X, U) \hat{N} - \epsilon \lambda X + \epsilon \lambda v(X) \xi \]
Equating tangential parts on both the sides, we have

\[(3.20) \quad \nabla_X U = f(A_N X) - g(\phi X, U)V - \epsilon \lambda X + \epsilon \lambda V(X)V\]

Now if \(U\) is parallel, then

\[\epsilon \lambda X - f(A_N X) + g(\phi X, U)V - \epsilon \lambda V(X)V = 0\]

\[\square\]

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