Further results for the Dunkl Transform and the generalized Cesàro operator

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Abstract

In this paper, we consider Dunkl theory on $\mathbb{R}^d$ associated to a finite reflection group. This theory generalizes classical Fourier analysis. First, we give for $1 < p \leq 2$, sufficient conditions for weighted $L^p$-estimates of the Dunkl transform of a function $f$ using respectively the modulus of continuity of $f$ in the radial case and the convolution for $f$ in the general case. In particular, we obtain as application, the integrability of this transform on Besov-Lipschitz spaces. Second, we provide necessary and sufficient conditions on nonnegative functions $\varphi$ defined on $[0,1]$ to ensure the boundedness of the generalized Cesàro operator $C_\varphi$ on Herz spaces and we obtain the corresponding operator norm inequalities.

Keywords : Dunkl operators, Dunkl transform, Dunkl translations, Dunkl convolution, Besov-Lipschitz spaces, Herz spaces, Cesàro operators.

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1 Introduction

Dunkl theory generalizes classical Fourier analysis on $\mathbb{R}^d$. It started twenty years ago with Dunkl’s seminal work [9] and was further developed by several mathematicians (see [8, 11, 14, 16, 18]) and later was applied and generalized

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in different ways by many authors (see [3, 4, 5, 17]). A key tool in the study of special functions with reflection symmetries are Dunkl operators. These are commuting differential-difference operators \( T_i, 1 \leq i \leq d \) associated to an arbitrary finite reflection group \( W \) on an Euclidean space and to a non-negative multiplicity function \( k \). The Dunkl kernel \( E_k \) has been introduced by C.F. Dunkl in [10]. This kernel is used to define the Dunkl transform \( F_k \). K. Trimèche has introduced in [18] the Dunkl translation operators \( \tau_x, x \in \mathbb{R}^d \), on the space of infinitely differentiable functions on \( \mathbb{R}^d \). At the moment an explicit formula for the Dunkl translation \( \tau_x(f) \) of a function \( f \) is unknown in general. However, such formula is known when the function \( f \) is radial (see next section). In particular, the boundedness of \( \tau_x \) is established in this case. As a result one obtains a formula for the convolution \( \ast_k \). In the case \( k \equiv 0 \), the \( T_i \) reduce to the corresponding partial derivatives \( \frac{\partial}{\partial x_i} \). Therefore Dunkl analysis can be viewed as a generalization of classical Fourier analysis (see next section, Remark 2.1). An important motivation to study Dunkl operators originates in their relevance for the analysis of quantum many body systems of Calogero-Moser-Sutherland type. These describe algebraically integrable systems in one dimension and have gained considerable interest in mathematical physics (see [19]).

Let \( f \) be a function in \( L^p_k(\mathbb{R}^d), 1 \leq p < \infty \) where \( L^p_k(\mathbb{R}^d) \) denote the space \( L^p(\mathbb{R}^d, w_k(x)dx) \) with \( w_k \) the weight function associated to the Dunkl operators given by

\[
w_k(x) = \prod_{\xi \in \mathbb{R}^+} |\langle \xi, x \rangle|^{2k(\xi)}, \quad x \in \mathbb{R}^d \quad \text{(see next section)}.
\]

The modulus of continuity \( \omega_{p,k}(f) \) of first order of a radial function \( f \) in \( L^p_k(\mathbb{R}^d) \) is defined by

\[
\omega_{p,k}(f)(x) = \sup_{0 < t \leq x} \int_{S^{d-1}} \| \tau_{tu}(f) - f \|_{p,k} d\sigma(u), \quad x > 0,
\]

where \( S^{d-1} \) is the unit sphere on \( \mathbb{R}^d \) with the normalized surface measure \( d\sigma \).

We use \( || \) \( _{p,k} \) as a shorthand for \( ||_{L^p_k(\mathbb{R}^d)} \).

We set for \( f \in L^p_k(\mathbb{R}^d), \)

\[
\tilde{\omega}_{p,k}(f)(x) = \sup_{0 < t \leq x} || f \ast_k \phi_t ||_{p,k}, \quad x > 0,
\]

where \( \phi \) is a radial function in \( \mathcal{S}(\mathbb{R}^d) \) satisfying

\[
\exists \ c > 0; \ |F_k(\phi)(y)| > c, \ \forall \ y \in \left\{ z \in \mathbb{R}^d : \frac{1}{2} < \| z \| < 1 \right\}.
\]
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\[ \phi_t \text{ being the dilation of } \phi \text{ given by } \phi_t(y) = \frac{1}{t^{\gamma+\frac{d}{2}}} \phi\left(\frac{y}{t}\right), \text{ for all } t \in (0, +\infty) \]

and \( y \in \mathbb{R}^d \). Note that \( \mathcal{F}_k(\phi_t)(y) = \mathcal{F}_k(\phi)(ty) \).

Obviously, \( \omega_{p,k}(f)(x) \) and \( \tilde{\omega}_{p,k}(f)(x) \) are nondecreasing in \( x \).

For \( \beta > 0, 1 \leq p < +\infty \), we define the weighted Besov-Lipschitz spaces (see [7] for the classical case) denoted by \( BD_{\beta,k}^p \) as the subspace of radial functions \( f \) in \( L_k^p(\mathbb{R}^d) \) satisfying

\[ \sup_{x > 0} \frac{\omega_{p,k}(f)(x)}{x^\beta} < +\infty. \]

Let \( \varphi \) a nonnegative function defined on \([0, 1]\). For a measurable complex-valued function \( f \) on \( \mathbb{R}^d \), we define the generalized Cesàro operator \( C_{\varphi} \) by

\[ C_{\varphi}f(x) = \int_0^1 f\left(\frac{x}{t}\right) t^{-(\gamma+d)} \varphi(t) dt, \quad x \in \mathbb{R}^d, \]

where \( \gamma = \sum_{\xi \in R_+} k(\xi) \) with \( R_+ \) a fixed positive root system (see next section).

If \( k \equiv 0, \varphi \equiv 1 \) and \( d = 1 \), we obtain the classical Cesàro average operator \( C \) given by

\[ Cf(x) = \begin{cases} 
\int_x^{+\infty} \frac{f(y)}{y} dy, & \text{if } x > 0 \\
-\int_{-\infty}^x \frac{f(y)}{y} dy, & \text{if } x < 0.
\end{cases} \]

We define for \( \beta > 0 \) and \( 1 \leq p, q < +\infty \), the homogeneous Herz-type space for the Dunkl operators \( H_{\beta,k}^{p,q} \) (see [12] for the classical case) by the space of functions \( f \) in \( L_k^p(\mathbb{R}^d)_{\text{loc}} \) satisfying

\[ \left( \sum_{j=-\infty}^{+\infty} (2^j \beta \| f \chi_j \|_{p,k})^q \right)^{\frac{1}{q}} < +\infty, \]

where \( \chi_j \) is the characteristic function of the set

\[ A_j = \{ x \in \mathbb{R}^d ; 2^j-1 \leq \| x \| \leq 2^j \} \text{ for } j \in \mathbb{Z}, \]

and \( L_k^p(\mathbb{R}^d)_{\text{loc}} \) is the space \( L_{\text{loc}}^p(\mathbb{R}^d, w_k(x) dx) \).

In this paper, first we give for \( 1 < p \leq 2 \), sufficient conditions for weighted \( L^p \)-estimates of the Dunkl transform \( \mathcal{F}_k(f) \) of a function \( f \) using respectively the modulus of continuity \( \omega_{p,k}(f) \) of \( f \) in the radial case and
the convolution $\tilde{\omega}_{p,k}(f)$ for $f$ in the general case. In particular, we obtain as application, the integrability of this transform on the Besov-Lipschitz spaces $BD^{\beta,k}_{p}$. This is an extension to the Dunkl transform on higher dimension of the results obtained by F. Móricz in [13] for the Fourier transform on the real line. Second, we provide necessary and sufficient conditions for $C_{\varphi}$ to be bounded on the Herz spaces $H^{\beta,k}_{p,q}$ when $\beta > 0$, $1 < p < +\infty$, $1 \leq q < +\infty$ and we obtain the corresponding operator norm inequalities.

The contents of this paper are as follows:
In section 2, we collect some basic definitions and results about harmonic analysis associated with Dunkl operators.
In section 3, we give sufficient conditions for weighted $L^{p}$-estimates of the Dunkl transform of function which we apply on the Besov-Lipschitz spaces.
In section 4, we provide necessary and sufficient conditions for $C_{\varphi}$ to be bounded on Herz spaces.

Along this paper we denote by $\langle ., . \rangle$ the usual Euclidean inner product in $\mathbb{R}^{d}$ as well as its extension to $\mathbb{C}^{d} \times \mathbb{C}^{d}$, we write for $x \in \mathbb{R}^{d}$, $\|x\| = \sqrt{\langle x, x \rangle}$ and we use $c$ to denote a suitable positive constant which is not necessarily the same in each occurrence. Furthermore, we denote by

- $\mathcal{E}(\mathbb{R}^{d})$ the space of infinitely differentiable functions on $\mathbb{R}^{d}$.
- $\mathcal{S}(\mathbb{R}^{d})$ the Schwartz space of functions in $\mathcal{E}(\mathbb{R}^{d})$ which are rapidly decreasing as well as their derivatives.
- $\mathcal{D}(\mathbb{R}^{d})$ the subspace of $\mathcal{E}(\mathbb{R}^{d})$ of compactly supported functions.

2 Preliminaries

In this section, we recall some notations and results in Dunkl theory and we refer for more details to the surveys [15].

Let $W$ be a finite reflection group on $\mathbb{R}^{d}$, associated with a root system $R$. For $\alpha \in R$, we denote by $H_{\alpha}$ the hyperplane orthogonal to $\alpha$. Given $\beta \in \mathbb{R}^{d} \setminus \bigcup_{\alpha \in R} H_{\alpha}$, we fix a positive subsystem $R_{+} = \{ \alpha \in R : \langle \alpha, \beta \rangle > 0 \}$. We denote by $k$ a nonnegative multiplicity function defined on $R$ with the property that $k$ is $W$-invariant. We associate with $k$ the index

$$\gamma = \sum_{\xi \in R_{+}} k(\xi) \geq 0,$$
and the weight function \( w_k \) given by

\[
w_k(x) = \prod_{\xi \in \mathbb{R}^+} |\langle \xi, x \rangle|^{2k(\xi)}, \quad x \in \mathbb{R}^d.
\]

\( w_k \) is \( W \)-invariant and homogeneous of degree \( 2\gamma \).

Further, we introduce the Mehta-type constant \( c_k \) by

\[
c_k = \left( \int_{\mathbb{R}^d} e^{-\frac{||x||^2}{2}} w_k(x) dx \right)^{-1}.
\]

For every \( 1 \leq p \leq +\infty \), we denote by \( L^p_k(\mathbb{R}^d) \), the space \( L^p(\mathbb{R}^d, w_k(x) dx) \) and we use \( \| \|_{p,k} \) as a shorthand for \( \| \|_{L^p_k(\mathbb{R}^d)} \).

By using the homogeneity of degree \( 2\gamma \) of \( w_k \), it is shown in [14] that for a radial function \( f \) in \( L^1_k(\mathbb{R}^d) \), there exists a function \( F \) on \( [0, +\infty) \) such that

\[
f(x) = F(||x||), \quad x \in \mathbb{R}^d.
\]

The function \( F \) is integrable with respect to the measure \( r^{2\gamma+d-1} dr \) on \( [0, +\infty) \) and we have

\[
\int_{\mathbb{R}^d} f(x) dv_k(x) = \int_0^{+\infty} \left( \int_{S^{d-1}} f(ry) w_k(ry) d\sigma(y) \right) r^{d-1} dr = \int_0^{+\infty} \left( \int_{S^{d-1}} w_k(ry) d\sigma(y) \right) F(r) r^{d-1} dr = d_k \int_0^{+\infty} F(r) r^{2\gamma+d-1} dr, \tag{2.1}
\]

where \( S^{d-1} \) is the unit sphere on \( \mathbb{R}^d \) with the normalized surface measure \( d\sigma \) and

\[
d_k = \int_{S^{d-1}} w_k(x) d\sigma(x) = \frac{c_k^{-1}}{2^{\gamma+d} \Gamma(\gamma + \frac{d}{2})}. \tag{2.2}
\]

The Dunkl operators \( T_j, \quad 1 \leq j \leq d, \) on \( \mathbb{R}^d \) associated with the reflection group \( W \) and the multiplicity function \( k \) are the first-order differential-difference operators given by

\[
T_j f(x) = \frac{\partial f}{\partial x_j} (x) + \sum_{\alpha \in R_+} k(\alpha) \alpha_j \frac{f(x) - f(\rho_\alpha(x))}{\langle \alpha, x \rangle}, \quad f \in \mathcal{E}(\mathbb{R}^d), \quad x \in \mathbb{R}^d,
\]

where \( \rho_\alpha \) is the reflection on the hyperplane \( \mathbb{H}_\alpha \) and \( \alpha_j = \langle \alpha, e_j \rangle, \ (e_1, \ldots, e_d) \) being the canonical basis of \( \mathbb{R}^d \).
Remark 2.1 In the case $k \equiv 0$, the weighted function $w_k \equiv 1$ and the measure associated to the Dunkl operators coincide with the Lebesgue measure. The $T_j$ reduce to the corresponding partial derivatives. Therefore Dunkl analysis can be viewed as a generalization of classical Fourier analysis.

For $y \in \mathbb{C}^d$, the system

$$\begin{cases} T_j u(x, y) = y_j u(x, y), & 1 \leq j \leq d, \\ u(0, y) = 1. \end{cases}$$

admits a unique analytic solution on $\mathbb{R}^d$, denoted by $E_k(x, y)$ and called the Dunkl kernel. This kernel has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$. We have for all $\lambda \in \mathbb{C}$ and $z, z' \in \mathbb{C}^d$, $E_k(z, z') = E_k(z', z)$, $E_k(\lambda z, z') = E_k(z, \lambda z')$ and for $x, y \in \mathbb{R}^d$, $|E_k(x, iy)| \leq 1$.

The Dunkl transform $\mathcal{F}_k$ is defined for $f \in \mathcal{D}(\mathbb{R}^d)$ by

$$\mathcal{F}_k(f)(x) = c_k \int_{\mathbb{R}^d} f(y) E_k(-ix, y) w_k(y) dy, \quad x \in \mathbb{R}^d.$$ 

We list some known properties of this transform:

i) The Dunkl transform of a function $f \in L^1_k(\mathbb{R}^d)$ has the following basic property

$$\|\mathcal{F}_k(f)\|_{\infty,k} \leq \|f\|_{1,k}.$$ 

ii) The Dunkl transform is an automorphism on the Schwartz space $\mathcal{S}(\mathbb{R}^d)$.

iii) When both $f$ and $\mathcal{F}_k(f)$ are in $L^1_k(\mathbb{R}^d)$, we have the inversion formula

$$f(x) = \int_{\mathbb{R}^d} \mathcal{F}_k(f)(y) E_k(ix, y) w_k(y) dy, \quad x \in \mathbb{R}^d.$$ 

iv) (Plancherel’s theorem) The Dunkl transform on $\mathcal{S}(\mathbb{R}^d)$ extends uniquely to an isometric automorphism on $L^2_k(\mathbb{R}^d)$. Since the Dunkl transform $\mathcal{F}_k(f)$ is of strong-type $(1, \infty)$ and $(2, 2)$, then by interpolation, we get for $f \in L^p_k(\mathbb{R}^d)$ with $1 \leq p \leq 2$ and $p'$ such that $\frac{1}{p} + \frac{1}{p'} = 1$, the Hausdorff-Young inequality

$$\|\mathcal{F}_k(f)\|_{p',k} \leq c \|f\|_{p,k}.$$
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The Dunkl transform of a radial function in $L^1_k(\mathbb{R}^d)$ is also radial and could be expressed via the Hankel transform. More precisely, according to ([14], proposition 2.4), we have the following results:

$$\int_{S^{d-1}} E_k(ix,y)w_k(y)d\sigma(y) = d_k j_{\gamma+\frac{d}{2}-1}(\|x\|),$$

(2.3)

and

$$\mathcal{F}_k(f)(x) = \int_0^{+\infty} \left( \int_{S^{d-1}} E_k(-ix,y)w_k(y)d\sigma(y) \right) F(r)r^{2\gamma-d-1}dr$$

$$= c_k^{-1} \mathcal{H}_{\gamma+\frac{d}{2}-1}(F)(\|x\|), \quad x \in \mathbb{R}^d,$$

(2.4)

where $F$ is the function defined on $[0, +\infty)$ by $F(\|x\|) = f(x)$, $x \in \mathbb{R}^d$, $\mathcal{H}_{\gamma+\frac{d}{2}-1}$ is the Hankel transform of order $\gamma+\frac{d}{2}-1$ and $j_{\gamma+\frac{d}{2}-1}$ the normalized Bessel function of the first kind and order $\gamma+\frac{d}{2}-1$.

K. Trimèche has introduced in [18] the Dunkl translation operators $\tau_x$, $x \in \mathbb{R}^d$, on $\mathcal{E}(\mathbb{R}^d)$. For $f \in \mathcal{S}(\mathbb{R}^d)$ and $x, y \in \mathbb{R}^d$, we have

$$\mathcal{F}_k(\tau_x(f))(y) = E_k(ix,y)\mathcal{F}_k(f)(y).$$

(2.7)

Notice that for all $x, y \in \mathbb{R}^d$, $\tau_x(f)(y) = \tau_y(f)(x)$ and for fixed $x \in \mathbb{R}^d$

$$\tau_x$$

is a continuous linear mapping from $\mathcal{E}(\mathbb{R}^d)$ into $\mathcal{E}(\mathbb{R}^d)$.

As an operator on $L^2_k(\mathbb{R}^d)$, $\tau_x$ is bounded. A priori it is not at all clear whether the translation operator can be defined for $L^p$-functions with $p$ different from 2. However, according to ([17], Theorem 3.7), the operator $\tau_x$ can be extended to the space of radial functions in $L^p_k(\mathbb{R}^d)$, $1 \leq p \leq 2$ and we have for a radial function $f \in L^p_k(\mathbb{R}^d)$

$$\|\tau_x(f)\|_{p,k} \leq \|f\|_{p,k}.$$
The Dunkl convolution product \( *_k \) of two functions \( f \) and \( g \) in \( L^2_k(\mathbb{R}^d) \) is given by
\[
(f *_k g)(x) = \int_{\mathbb{R}^d} \tau_x(f)(-y)g(y)w_k(y)dy, \quad x \in \mathbb{R}^d.
\]
The Dunkl convolution product is commutative and for \( f, g \in D(\mathbb{R}^d) \), we have
\[
\mathcal{F}_k(f *_k g) = \mathcal{F}_k(f)\mathcal{F}_k(g). \tag{2.9}
\]
It was shown in ([17], Theorem 4.1) that when \( g \) is a bounded radial function in \( L^1_k(\mathbb{R}^d) \), then
\[
(f *_k g)(x) = \int_{\mathbb{R}^d} f(y)\tau_x(g)(-y)w_k(y)dy, \quad x \in \mathbb{R}^d,
\]
initially defined on the intersection of \( L^1_k(\mathbb{R}^d) \) and \( L^2_k(\mathbb{R}^d) \) extends to \( L^p_k(\mathbb{R}^d) \), \( 1 \leq p \leq +\infty \) as a bounded operator. In particular,
\[
\|f *_k g\|_{p,k} \leq \|f\|_{p,k}\|g\|_{1,k}. \tag{2.10}
\]

3 Weighted \( L^p \)-estimates for the Dunkl transform with sufficient conditions

In this section, we give sufficient conditions for weighted \( L^p \)-estimates of the Dunkl transform of function which we apply on the Besov-Lipschitz space.

Throughout this section, we denote by \( p' \) the conjugate \( \frac{p}{p-1} \) of \( p \) for \( 1 < p \leq 2 \). According to (2.8) and (2.10), we recall that :
- The modulus of continuity \( \omega_{p,k}(f) \) of first order of a radial function \( f \) in \( L^p_k(\mathbb{R}^d) \) is defined by
\[
\omega_{p,k}(f)(x) = \sup_{0 < t \leq x} \int_{S^{d-1}} \|\tau_{tu}(f) - f\|_{p,k}d\sigma(u), \quad x > 0.
\]
- We set for \( f \in L^p_k(\mathbb{R}^d) \),
\[
\overline{\omega}_{p,k}(f)(x) = \sup_{0 < t \leq x} \|f *_k \phi_t\|_{p,k}, \quad x > 0,
\]
where \( \phi \) is a radial function in \( S(\mathbb{R}^d) \) satisfying
\[
\exists \ c > 0; \ |\mathcal{F}_k(\phi)(y)| > c, \quad \forall \ y \in \left\{ z \in \mathbb{R}^d : \frac{1}{2} < \|z\| < 1 \right\}.
\]
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φ_t being the dilation of φ given by φ_t(y) = \frac{1}{t^{(\gamma + \frac{d}{2})}} \phi(\frac{y}{t}) \text{, for all } t \in (0, +\infty) \text{ and } y \in \mathbb{R}^d.

For θ ≥ 1, we introduce a class of nonnegative w_k-locally integrable radial functions on \{x \in \mathbb{R}^d : \|x\| ≥ 1\} which we denote by G_θ. We said that a function g belongs to the class G_θ if there exists a constant κ_θ ≥ 1 such that for η = 1, 2, ...

\[ \left( \int_{1 \leq \|z\| \leq 2} g(2^n z)^\theta w_k(z)dz \right)^\frac{1}{\theta} \leq κ_θ \int_{\frac{1}{2} \leq \|z\| \leq 1} g(2^n z)w_k(z)dz. \quad (3.1) \]

If we set
\[ C_η = \{y \in \mathbb{R}^d : 2^η \leq \|y\| < 2^{η+1}\}, \text{ for } η = 0, 1, 2, ... \]
then using (2.1) and the change of variables \( y = 2^η z \), we can write (3.1) in the form

\[ \left( \int_{C_η} g(y)^\theta w_k(y)dy \right)^\frac{1}{\theta} \leq κ_θ 2^{n(1-\frac{\theta}{p})} (2^{(γ + d)} + 2^η) \int_{C_{η-1}} g(y)w_k(y)dy. \quad (3.2) \]

Example 3.1 If \( g \equiv \text{const} \), then \( g \in G_θ \) where we choose

\[ κ_θ \geq 2^{2γ+d} \left( \frac{2^{2γ+d} - 1}{2γ + d} \right)^{\frac{1}{p} - 1}. \]

Theorem 3.1 Let \( 1 < p \leq 2 \) and \( f \) be a radial function in \( L_p^p(\mathbb{R}^d) \). Then for \( 1 \leq q \leq p' \) and \( g \in G_{\frac{p}{p-q+q}} \), we have

\[ \int_{\|y\| \geq 2} g(y)|\mathcal{F}_k(f)(y)|^q w_k(y)dy \leq C \int_{\|y\| \geq 1} g(y)\|y\|^{-\frac{q}{p}(2γ+d)} \omega^q_{p,k}(\|y\|)w_k(y)dy, \]

where \( C \) is a constant depending only on \( p \) and \( q \).

Proof. Let \( f \) be a radial function in \( L_p^p(\mathbb{R}^d) \) for \( 1 < p \leq 2 \), then by (2.7) we have

\[ \mathcal{F}_k(\tau_{ixu} f) - f)(y) = (E_k(ixu, y) - 1)\mathcal{F}_k f(y), \]
We can assert by Haussdorf-Young’s inequality that
\[ \|F_k(\tau x u(f) - f)\|_{p',k} = \left( \int_{\mathbb{R}^d} |F_k f(y)|^{p'} |E_k(i x u(y) - 1|^p w_k(y)dy \right)^{\frac{1}{p'}} \leq c \|\tau x u(f) - f\|_{p,k}. \] (3.3)

On the other hand, from (2.1) and (2.4), we get
\[ c_k \int_{\mathbb{R}^d} |F_k f(y)|^{p'} |E_k(i x u(y) - 1|^p w_k(y)dy \]
\[ = \int_0^{+\infty} \mathcal{H}_{\gamma + \frac{d}{2} - 1}(F)(r) \left( \int_{S^{d-1}} |E_k(i x u, z) - 1|^p w_k(z)d\sigma(z) \right) r^{2\gamma + d - 1}dr, \]
where \( r = \|y\| \) and \( F \) is the function defined on \((0, +\infty)\) by \( F(\|y\|) = f(y) \) for all \( y \in \mathbb{R}^d \).

By (2.2), (2.3) and Hölder’s inequality, we have
\[ d_k |j_{\gamma + \frac{d}{2} - 1}(x) - 1| \]
\[ = \left| \int_{S^{d-1}} |E_k(i x u, z) - 1| w_k(z)d\sigma(z) \right| \]
\[ \leq \left( \int_{S^{d-1}} |w_k(z)d\sigma(z) \right)^\frac{1}{p'} \left( \int_{S^{d-1}} |E_k(i x u, z) - 1|^p w_k(z)d\sigma(z) \right)^\frac{1}{p'} \]
\[ \leq d_k^{\frac{1}{p'}} \left( \int_{S^{d-1}} |E_k(i x u, z) - 1|^p w_k(z)d\sigma(z) \right)^\frac{1}{p'}. \]

According to (3.3), it follows that
\[ \int_0^{+\infty} \mathcal{H}_{\gamma + \frac{d}{2} - 1}(F)(r)|^p |j_{\gamma + \frac{d}{2} - 1}(x) - 1|^r r^{2\gamma + d - 1}dr \leq c \|\tau x u(f) - f\|_{p,k}. \] (3.4)

Integrating the two members of (3.4) over \( S^{d-1} \), this yields
\[ \left( \int_0^{+\infty} \mathcal{H}_{\gamma + \frac{d}{2} - 1}(F)(r)|^p |j_{\gamma + \frac{d}{2} - 1}(x) - 1|^r r^{2\gamma + d - 1}dr \right)^\frac{1}{p'} \leq c \omega_{p,k}(f)(x) \] (3.5)

From (2.5) and (2.6), we get
\[ |j_{\gamma + \frac{d}{2} - 1}(x) - 1| = \frac{4\Gamma(\gamma + \frac{d}{2})}{\sqrt{\pi}\Gamma(\gamma + \frac{d}{2} - \frac{1}{2})} \int_0^1 (1 - t^2)^{\gamma + \frac{d}{2} - \frac{1}{2}} \sin^2\left(\frac{rt}{2}\right)dt, \]
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then, we obtain

\[ \frac{4\Gamma\left(\gamma + \frac{d}{2}\right)}{\sqrt{\pi} \Gamma\left(\gamma + \frac{d}{2} - \frac{1}{2}\right)} \int_{\frac{1}{2}}^{1} (1 - t^2)^{\gamma + \frac{d}{2} - \frac{3}{2}} \sin^2\left(\frac{\pi x}{2}\right) dt \leq |j_{\gamma + \frac{d}{2} - 1}(rx) - 1|. \]

Let \( x = \frac{\pi}{2\eta}, \eta = 1, 2, \ldots \) and \( y \in C_\eta \), we have

\[ \frac{\pi}{8} \leq \frac{\pi x}{2} \leq \frac{\pi}{2}, \]

which gives that

\[ c = \frac{4\Gamma\left(\gamma + \frac{d}{2}\right)}{\sqrt{\pi} \Gamma\left(\gamma + \frac{d}{2} - \frac{1}{2}\right)} \int_{\frac{1}{2}}^{1} (1 - t^2)^{\gamma + \frac{d}{2} - \frac{3}{2}} dt \leq |j_{\gamma + \frac{d}{2} - 1}(r \frac{\pi}{2\eta + 1}) - 1|. \]

Hence from (3.5), we find that

\[ \left( \int_{2^\eta}^{2^{\eta+1}} |H_{\gamma + \frac{d}{2} - 1}(F)(r)|^{p} r^{2\gamma + d - 1} dr \right)^{\frac{1}{p'}} \leq c \omega_{p,k}(f)(\frac{\pi}{2\eta + 1}). \]  

(3.6)

Now, take \( g \in G_{\frac{p}{p-q}, q} \) and put \( \tilde{g}(\|y\|) = g(y) \), for \( y \in \mathbb{R}^d \). Applying Hölder’s inequality, it follows from (2.1), (3.2) and (3.6) that

\[ \int_{2^\eta}^{2^{\eta+1}} \tilde{g}(r)|H_{\gamma + \frac{d}{2} - 1}(F)(r)|^q r^{2\gamma + d - 1} dr \]

\[ \leq \left( \int_{2^\eta}^{2^{\eta+1}} |H_{\gamma + \frac{d}{2} - 1}(F)(r)|^{p} r^{2\gamma + d - 1} dr \right)^{\frac{q}{p'}} \left( \int_{2^\eta}^{2^{\eta+1}} \tilde{g}^{p'\eta/(p-q)} (r) r^{2\gamma + d - 1} dr \right)^{\frac{p-q}{p'}} \]

\[ \leq c \kappa^{p-q} \int_{2^\eta}^{2^{\eta+1}} \tilde{g}(r) |H_{\gamma + \frac{d}{2} - 1}(F)(r)|^q r^{2\gamma + d - 1} dr. \]

Then, we deduce

\[ \int_{2}^{+\infty} \tilde{g}(r)|H_{\gamma + \frac{d}{2} - 1}(F)(r)|^q r^{2\gamma + d - 1} dr \]

\[ \leq c \kappa^{p-q} \sum_{\eta=1}^{+\infty} 2^{-\frac{\eta p}{p-q} (2\gamma + d)} \omega_{p,k}(f)(\frac{\pi}{2\eta+1}) \int_{2^\eta}^{2^{\eta+1}} \tilde{g}(r) r^{2\gamma + d - 1} dr. \]

Using the monotonicity of \( \omega_{p,k}(f)(x) \) in \( x > 0 \) and the fact that \( \frac{\pi}{2\eta+1} \leq \frac{\pi}{r} \),

\[ 2^{-\frac{\eta p}{p-q} (2\gamma + d)} \leq r^{-\frac{\eta p}{p-q} (2\gamma + d)}, \]

we obtain
\[ \int_2^{+\infty} \tilde{g}(r)|H_{\gamma+\frac{d}{2}-1}(F)(r)|^{q'} r^{2\gamma+d-1} dr \]

\[ \leq c \kappa \frac{\nu'}{\nu} \sum_{\eta=1}^{+\infty} \int_{2^{\eta-1}}^{2^{\eta}} \tilde{g}(r)r^{-\frac{q'(2\gamma+d)}{\nu}} \omega_{p,k}^q(F)\left(\frac{\pi}{r}\right) r^{2\gamma+d-1} dr \]

\[ \leq c \kappa \frac{\nu'}{\nu} \int_1^{+\infty} \tilde{g}(r)r^{-\frac{q'(2\gamma+d)}{\nu}} \omega_{p,k}^q(F)\left(\frac{\pi}{r}\right) r^{2\gamma+d-1} dr. \]  

By (2.1), (2.4) and (3.7), we conclude that

\[ \int \|y\| \geq 2 \tilde{g}(y)|F_k(f)(y)|^q w_k(y) dy \leq C \int_{\|y\| \geq 1} g(y)\|y\|^{-\frac{q'(2\gamma+d)}{\nu}} \omega_{p,k}^q(F)\left(\frac{\pi}{\|y\|}\right) w_k(y) dy, \]

where \( C = c \kappa \frac{\nu'}{\nu} \) is a constant depending only on \( p \) and \( q \). Our theorem is proved. \( \Box \)

In particular, for the case \( g \equiv \text{const} \) and \( f \in BD_p^{\beta,k} \), we obtain as an application of Theorem 3.1, the following result which is of special interest and was proved in [1, Theorem 3.2].

Corollary 3.1 Let \( \beta > 0 \), \( 1 < p \leq 2 \) and \( f \in BD_p^{\beta,k} \), then

1. For \( 0 < \beta \leq \frac{2(\gamma+\frac{d}{2})}{p} \), we have

\[ F_k(f) \in L_k^p(\mathbb{R}^d) \] provided that \( \frac{2(\gamma+\frac{d}{2})p}{\beta p + 2(\gamma+\frac{d}{2})(p-1)} < q \leq p' \).

2. For \( \beta > \frac{2(\gamma+\frac{d}{2})}{p} \), we have \( F_k(f) \in L_k^1(\mathbb{R}^d) \).

Theorem 3.2 Let \( 1 < p \leq 2 \) and \( f \) be a function in \( L_k^p(\mathbb{R}^d) \). Then for \( 1 \leq q \leq p' \) and \( g \in G_{\frac{p}{p + q}} \), we have

\[ \int_{\|y\| \geq 2} g(y)|F_k(f)(y)|^q w_k(y) dy \leq C \int_{\|y\| \geq 1} g(y)\|y\|^{-\frac{q'(2\gamma+d)}{\nu}} \omega_{p,k}^q(F)\left(\frac{\pi}{\|y\|}\right) w_k(y) dy, \]

where \( C \) is a constant depending only on \( p \) and \( q \).

Proof. Let \( f \) be a function in \( L_k^p(\mathbb{R}^d) \) for \( 1 < p \leq 2 \), then by (2.9) we have for \( x \in (0, +\infty) \),

\[ F_k(f \ast_k \phi_x)(y) = F_k(f)(y)F_k(\phi_x)(y), \]

\[ = F_k(f)(y)F_k(\phi)(xy), \text{ a.e. } y \in \mathbb{R}^d. \]
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Using Haussdorf-Young’s inequality, we obtain

$$\|F_k(f *_k \phi_x)\|_{p',k} = \left( \int_{\mathbb{R}^d} |F_k f(y)|^{p'} |F_k(\phi)(xy)|^{p' w_k(y)} dy \right)^{\frac{1}{p'}} \leq c \|f *_k \phi_x\|_{p,k} \leq c \tilde{\omega}_{p,k}(f)(x).$$ (3.8)

Let $x = \frac{1}{2^\eta}$ and $\eta = 1, 2, \ldots$. For $y \in C_\eta$, we have

$$\frac{1}{2} \leq x \|y\| \leq 1,$$

this gives using (3.8) and the property of the function $\phi$,

$$\left( \int_{C_\eta} |F_k f(y)|^{p'} w_k(y) dy \right)^{\frac{1}{p'}} \leq c \tilde{\omega}_{p,k}^q(f)(\frac{1}{2^{\eta+1}}).$$ (3.9)

Now, take $g \in G_{\frac{p}{p'-q}+q}$. Applying Hölder’s inequality, it follows from (3.2) and (3.9) that

$$\int_{C_\eta} g(y)|F_k f(y)|^q w_k(y) dy$$

$$\leq \left( \int_{C_\eta} |F_k f(y)|^{p'} w_k(y) dy \right)^{\frac{1}{p'}} \left( \int_{C_\eta} g^{p'q-1}(y) w_k(y) dy \right)^{\frac{1}{p'q-1}}$$

$$\leq c \kappa \frac{p'}{p'-q} 2^{-\eta \frac{p'}{p'q} (2\gamma+d)} \tilde{\omega}_{p,k}^q(f)(\frac{1}{2^{\eta+1}}) \int_{C_{\eta-1}} g(y) w_k(y) dy.$$

Then, we deduce

$$\int_{\|y\| \geq 2} g(y)|F_k f(y)|^q w_k(y) dy$$

$$\leq c \kappa \frac{p'}{p'-q} \sum_{\eta=1}^{+\infty} 2^{-\eta \frac{p'}{p'q} (2\gamma+d)} \tilde{\omega}_{p,k}^q(f)(\frac{1}{2^{\eta+1}}) \int_{C_{\eta-1}} g(y) w_k(y) dy.$$

As in the proof of Theorem 3.1, using the monotonicity of $\tilde{\omega}_{p,k}(f)(x)$ in $x > 0$, we conclude that

$$\int_{\|y\| \geq 2} g(y)|F_k f(y)|^q w_k(y) dy \leq C \int_{\|y\| \geq 1} g(y) \|y\|^{-\frac{p'}{p'-q} (2\gamma+d)} \tilde{\omega}_{p,k}^q(f)(\frac{1}{\|y\|}) w_k(y) dy,$$

where $C = c \kappa \frac{p'}{p'-q}$ is a constant depending only on $p$ and $q$. This completes the proof. □
Remark 3.1 As consequence, from Theorem 3.2, we deduce in the particular case when \( g \equiv \text{const} \) and \( f \in L_{k}^{p}(\mathbb{R}^{d}) \) satisfying
\[
\sup_{x>0} \frac{\tilde{\omega}_{p,k}(f)(x)}{x^{\beta}} < +\infty,
\]
similar results to those obtained in corollary 3.1.

Remark 3.2 It was shown in [2, Theorem 3.1] that the Dunkl transform is a continuous map between a class of Besov spaces and Herz spaces.

4 Boundedness of the Cesàro operator on Herz spaces

In this section, we provide necessary and sufficient conditions for the generalized Cesàro operator \( C_{\varphi} \) to be bounded on Herz spaces. Before, we start with some useful lemmas.

Lemma 4.1 Let \( 1 \leq q < \infty \). If \( f \) be a non-negative measurable function on \([0,1] \), then
\[
\left( \int_{0}^{1} f \right)^{q} \leq \int_{0}^{1} f^{q}.
\]

Lemma 4.2 (see E. F. Beckenbach and R. Bellman [6]) Let \( 1 \leq q < \infty \). If \( f \) be a non-negative measurable and concave function on \([0,1] \), then
\[
\left( \int_{0}^{1} f \right)^{\frac{1}{q}} \leq \frac{1 + \frac{1}{q}}{2^{\frac{1}{q}}} \int_{0}^{1} f^{\frac{1}{q}}.
\]

Theorem 4.1 Let \( \beta > 0 \), \( 1 < p < +\infty \), \( 1 \leq q < +\infty \) and \( \varphi \) be a real-valued non-negative measurable function defined on \([0,1] \). Suppose that \( t \mapsto t^{-(2\gamma + d)(1 - \frac{1}{p})} \varphi(t) \) is a concave function on \([0,1] \), then the generalized Cesàro operator \( C_{\varphi} \) can be extended to a bounded operator from \( H_{p,q}^{\beta,k} \) into itself if and only if
\[
\int_{0}^{1} t^{\beta - (2\gamma + d)(1 - \frac{1}{p})} \varphi(t) dt < +\infty.
\]

Moreover, when (4.3) holds, the operator norm \( \| C_{\varphi} \| \) of \( C_{\varphi} \) on \( H_{p,q}^{\beta,k} \) satisfies the following inequality
\[
\int_{0}^{1} t^{\beta - (2\gamma + d)(1 - \frac{1}{p})} \varphi(t) dt \leq \| C_{\varphi} \| \leq c_{q,\beta} \int_{0}^{1} t^{\beta - (2\gamma + d)(1 - \frac{1}{p})} \varphi(t) dt,
\]
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with $c_{q,\beta} = 2^{(1-\frac{2}{q})}(1 + \frac{1}{q})(1 + 2^{\beta})$.

**Proof.** Let $\beta > 0$, $1 < p < +\infty$, $1 \leq q < +\infty$ and $f \in H_{p,q}^{\beta,k}$.

Suppose that (4.3) holds. Using the Minkowski inequality for integrals and the homogeneity of degree $2\gamma$ of $w_k$, we get

$$
\|\mathcal{C}_\varphi f\chi_j\|_{p,k} = \left( \int_{A_j} \left( \int_0^1 \left| f\left(\frac{x}{t}\right) t^{-(2\gamma+d)} \varphi(t) dt \right|^p w_k(x) dx \right)^{\frac{1}{p}} \right)^\frac{q}{p} 
\leq \int_0^1 \left( \int_{A_j} \left| f\left(\frac{x}{t}\right) t^{-(2\gamma+d)} \varphi(t) dt \right|^p w_k(x) dx \right)^{\frac{1}{p}} \psi(t) dt 
= \int_0^1 \left( \int_{\frac{2j-1}{t} \leq \|u\| \leq \frac{2j+1}{t}} \left| f(u) t^{\frac{p}{q}} w_k(u) du \right|^q \psi^q(t) dt \right)^{\frac{1}{q}} t^{-(2\gamma+d)(1-\frac{1}{p})} \varphi(t) dt.
$$

Put $\psi(t) = t^{-(2\gamma+d)(1-\frac{1}{p})} \varphi(t)$, $t \in (0, 1)$. Since for each $t \in (0, 1)$, there exists an integer $m$ such that $2^{m-1} \leq t \leq 2^m$, then we obtain

$$
\|\mathcal{C}_\varphi f\chi_j\|_{p,k} \leq \int_0^1 \left( \|f\chi_{j-m}\|_{p,k}^p + \|f\chi_{j-m+1}\|_{p,k}^p \right)^{\frac{1}{p}} \psi(t) dt 
\leq \int_0^1 \left( \|f\chi_{j-m}\|_{p,k} + \|f\chi_{j-m+1}\|_{p,k} \right) \psi(t) dt.
$$

Using (4.1), it follows that

$$
\left( \sum_{j=-\infty}^{+\infty} 2^{j\beta q} \|\mathcal{C}_\varphi f\chi_j\|_{p,k}^q \right)^{\frac{1}{q}} 
\leq \left[ \sum_{j=-\infty}^{+\infty} 2^{j\beta q} \left( \int_0^1 \left( \|f\chi_{j-m}\|_{p,k} + \|f\chi_{j-m+1}\|_{p,k} \right) \psi(t) dt \right)^q \right]^{\frac{1}{q}} 
\leq \left( \sum_{j=-\infty}^{+\infty} 2^{j\beta q} \int_0^1 \left( \|f\chi_{j-m}\|_{p,k} + \|f\chi_{j-m+1}\|_{p,k} \right)^q \psi^q(t) dt \right)^{\frac{1}{q}}.
$$
which gives
\[
\sum_{j=-\infty}^{+\infty} 2^{j\beta q} \| (C_{\varphi} f) \chi_j \|_{p,k}^q \left( \sum_{j=-\infty}^{+\infty} 2^{j\beta q} \| f \chi_{j-m} \|_{p,k}^q \right) \leq 2^{1-\frac{2}{q}} \left( \int_{0}^{1} \left( \sum_{j=-\infty}^{+\infty} 2^{j\beta q} \| f \chi_{j-m} \|_{p,k}^q \right) \left( \int_{0}^{1} \left( 2^{m\beta} + 2^{(m-1)\beta} \right) \psi(t) dt \right)^{\frac{1}{q}} \right).
\]

Then from (4.2), we have
\[
\sum_{j=-\infty}^{+\infty} 2^{j\beta q} \| (C_{\varphi} f) \chi_j \|_{p,k}^q \leq \left[ 2^{1-\frac{2}{q}} (1 + \frac{1}{q}) \left( \sum_{j=-\infty}^{+\infty} 2^{j\beta q} \| f \chi_{j-m} \|_{p,k}^q \right) \left( \int_{0}^{1} t^{\beta \psi(t) dt} \right) \right] \left( \sum_{j=-\infty}^{+\infty} 2^{j\beta q} \| f \chi_{j-m} \|_{p,k}^q \right)^{\frac{1}{q}}.
\]
Hence we deduce that
\[
\| C_{\varphi} \| \leq c_{q,\beta} \int_{0}^{1} t^{-2\gamma-d(1-\frac{1}{p})} \psi(t) dt,
\]
where \( c_{q,\beta} = 2^{1-\frac{2}{q}} (1 + \frac{1}{q})(1 + 2^\beta) \).

Conversely, assume that the generalized Cesàro operator \( C_{\varphi} \) is bounded, then for \( f \in H_{p,q}^{\beta,k} \),
\[
\left( \sum_{j=-\infty}^{+\infty} 2^{j\beta q} \| (C_{\varphi} f) \chi_j \|_{p,k}^q \right)^{\frac{1}{q}} \leq \| C_{\varphi} \left( \sum_{j=-\infty}^{+\infty} 2^{j\beta q} \| f \chi_{j} \|_{p,k}^q \right)^{\frac{1}{q}}. (4.4)
\]
For any \( \varepsilon \in (0, 1) \), we set
\[
f_{\varepsilon}(x) = \begin{cases} \|x\|^{-(\beta+\varepsilon+\frac{2\gamma+d}{p})} & \text{if } \|x\| > 1 \\ 0 & \text{otherwise,} \end{cases}
\]
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then for \( j = 0, -1, -2, \ldots \), \( \| f_\varepsilon \chi_j \|_{p,k} = 0 \), and for \( j \in \mathbb{N} \setminus \{ 0 \} \), it follows from (2.1) and (2.2) that

\[
\| f_\varepsilon \chi_j \|_{p,k}^p = \int_{2^j x \leq \| x \| \leq 2^{j+1}} \| x \|^{-(\beta + \varepsilon + 2\beta p) + d} + \pi^{-k} \omega_k(x) \, dx = \int_{S_{d-1}} w_k(y) \, d\sigma(y) \int_{2^j}^{2^{j+1}} r^{-(\beta + \varepsilon) - (\beta + \varepsilon) p} \, dr = C_\varepsilon 2^{-j(\beta + \varepsilon) p},
\]

where

\[
C_\varepsilon = \frac{2(\beta + \varepsilon)p - 1}{(\beta + \varepsilon)p} \int_{S_{d-1}} w_k(y) \, d\sigma(y) = \frac{2(\beta + \varepsilon)p - 1}{(\beta + \varepsilon)p} \, d_k.
\]

Hence, it yields

\[
\left( \sum_{j=-\infty}^{+\infty} 2^{j\beta q} \| f_\varepsilon \chi_j \|_{p,k}^q \right)^{\frac{1}{q}} = \left( \sum_{j=1}^{+\infty} 2^{j\beta q} \| f_\varepsilon \chi_j \|_{p,k}^q \right)^{\frac{1}{q}} = C_\varepsilon^{\frac{1}{p}} \left( \sum_{j=1}^{+\infty} 2^{-j\varepsilon} \right)^{\frac{1}{q}} \leq \frac{C_\varepsilon^{\frac{1}{p}}}{(1 - 2^{-\varepsilon})^{\frac{1}{q}}}, \quad (4.5)
\]

thus \( f_\varepsilon \in H_{p,q}^{\beta,k} \). Now, it’s easy to see that

\[
C_\varphi f_\varepsilon(x) = \int_0^{\| x \|} \frac{\| x \|^{-(\beta + \varepsilon + 2\beta d) p}}{t} \varphi(t) \, dt = \| x \|^{-(\beta + \varepsilon + 2\beta d) p} \int_0^{\| x \|} t^{\beta + \varepsilon - (2\gamma + d)(1 - \frac{1}{p})} \varphi(t) \, dt.
\]

Recalling that \( \psi(t) = t^{-(2\gamma + d)(1 - \frac{1}{p})} \varphi(t), \ t \in (0, 1) \), we get

\[
\sum_{j=-\infty}^{+\infty} 2^{j\beta q} \| (C_\varphi f_\varepsilon) \chi_j \|_{p,k}^q
\]

\[
= \sum_{j=-\infty}^{+\infty} 2^{j\beta q} \left[ \int_{A_j} \left( \| x \|^{-(\beta + \varepsilon + 2\beta d) p} \int_0^{\| x \|} t^{\beta + \varepsilon} \psi(t) \, dt \right)^p w_k(x) \, dx \right]^\frac{q}{p}
\]

\[
\geq \sum_{j=-\infty}^{+\infty} 2^{j\beta q} \left[ \int_{\| x \| > 1} \chi_j(x) \| x \|^{-(\beta + \varepsilon + 2\beta d) p} \left( \int_0^{\| x \|} t^{\beta + \varepsilon} \psi(t) \, dt \right)^p w_k(x) \, dx \right]^\frac{q}{p}
\]

\[
\geq \left( \int_0^1 t^{\beta + \varepsilon} \psi(t) \, dt \right)^q \sum_{j=1}^{+\infty} 2^{j\beta q} \left( \int_{A_j} \| x \|^{-(\beta + \varepsilon + 2\beta d) p} w_k(x) \, dx \right)^\frac{q}{p}.
\]
It follows from (4.5) that

\[ \left( \sum_{j=-\infty}^{+\infty} 2^{j\beta q} \| (C_\varphi f_\varepsilon) \chi_j \|_{p,k}^q \right)^{\frac{1}{q}} \geq C_{\varepsilon}^{\frac{1}{p}} \frac{2^{-\varepsilon}}{(1 - 2^{-q\varepsilon})^{\frac{1}{q}}} \left( \int_0^1 t^{\beta + \varepsilon} \psi(t) \, dt \right), \]

thus combining this inequality with (4.4) and (4.5), we can assert that

\[ \| C_\varphi \| \geq \int_0^1 t^{\beta + \varepsilon - (2\gamma + d)(1 - \frac{1}{p})} \varphi(t) \, dt, \]

which implies when \( \varepsilon \to 0 \)

\[ \| C_\varphi \| \geq \int_0^1 t^{\beta - (2\gamma + d)(1 - \frac{1}{p})} \varphi(t) \, dt. \]

This completes the proof of the theorem. \( \square \)

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