FUNDAMENTAL POLYHEDRA OF PROJECTIVE ELEMENTARY
GROUPS

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Abstract. For $\mathcal{O}$ an imaginary quadratic ring, we compute a fundamental polyhedron of $\text{PE}_2(\mathcal{O})$, the projective elementary subgroup of $\text{PSL}_2(\mathcal{O})$. This allows for new, simplified proofs of theorems of Cohn, Nica, Fine, and Frohman. Namely, we obtain a presentation for $\text{PE}_2(\mathcal{O})$, show that it has infinite-index and is its own normalizer in $\text{PSL}_2(\mathcal{O})$, and split $\text{PSL}_2(\mathcal{O})$ into a free product with amalgamation that has $\text{PE}_2(\mathcal{O})$ as one of its factors.

1. Introduction

Given an order $\mathcal{O}$ in an imaginary quadratic field $K$, let $\text{PE}_2(\mathcal{O})$ denote the projective elementary group generated by the elementary (diagonal, triangular, or permutation) matrices in $\text{PSL}_2(\mathcal{O})$. The action of $\text{PE}_2(\mathcal{O})$ on the upper half-space model of hyperbolic 3-space, $\mathbb{H} = \{(\zeta, t) | \zeta \in \mathbb{C}, t \in (0, \infty)\}$, admits a fundamental polyhedron—a polyhedron whose orbit under $\text{PE}_2(\mathcal{O})$ tessellates $\mathbb{H}$. (See Sections 1.1, 2.2, and 7.3 of [5] for background.) Our aim is to compute one in particular:

Definition 1.1. Given a fundamental polygon $F \subset \mathbb{C}$ for $\mathcal{O}$, the Ford domain of $\Gamma \leq \text{PSL}_2(\mathcal{O})$ is

$$\mathcal{F} = \{P \in \mathbb{H} | \zeta(P) \in \mathcal{F}, t(P) \geq t(gP) \text{ for any } g \in \Gamma\},$$

where $\zeta(P)$ and $t(P)$ denote the first and second coordinates of $P$.

The image of $P \in \mathbb{H}$ under some $g \in \text{PSL}_2(\mathcal{O})$ has second coordinate

$$t(gP) = \frac{t(P)}{|\alpha - \beta \zeta(P)|^2 + |\beta|^2 t(P)^2}, \quad g = \begin{bmatrix} \delta & -\gamma \\ -\beta & \alpha \end{bmatrix}. \quad (1.1)$$

Therefore if $\beta = 0$ and $\alpha$ is a unit, $t(P) = t(gP)$ for all $P \in \mathbb{H}$. In this case, assuming $\alpha = \pm 1$, upper triangular matrices shift $\mathcal{F}$ just as they shift its projection $\overline{\mathcal{F}}$. If $\beta \neq 0$, (1.1) shows that points $P$ satisfying $t(P) = t(gP)$ form an open Euclidean hemisphere centered at $\alpha/\beta \in \partial \mathbb{H}$ with radius $1/|\beta|$. We call this the isometric hemisphere of $g$, denoted $S_g$. It follows from (1.1) that $t(P) \geq t(gP)$ if and only if $P$ is outside $S_g$. Thus, given a choice of $F$, the Ford domain of some $\Gamma \leq \text{PSL}_2(\mathcal{O})$ consists of $P \in \mathbb{H}$ above $\overline{F}$ that lie outside or on $S_g$ for every $g \in \Gamma$. Figure 1 shows a hollowed out Ford domain of $\text{PSL}_2(\mathbb{Z}[i\sqrt{10}])$ from above. The imaginary axis runs left to right.

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Theorem 1.2. The only Euclidean hemispheres that contribute a face to the Ford domain of $\text{PE}_2(O)$ are those of radius 1 centered on elements of $O$.

The projective elementary group always contains the reflection and shifts
\[
r = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad s(\alpha) = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}
\] (1.2)
for $\alpha \in O$. In particular, $rs(-\alpha) \in \text{PE}_2(O)$, which has isometric hemisphere with radius 1 and center $\alpha$. Theorem 1.2 asserts there is nothing else for the Ford domain of $\text{PE}_2(O)$. We could turn Figure 1 into a Ford domain for $\text{PE}_2(\mathbb{Z}[i\sqrt{10}])$ by erasing all hemispheres that lie completely below the white, translucent plane.

Let $\Delta$ denote the discriminant of $O$. When $|\Delta| > 12$ there is a gap between rows of unit hemispheres. The infinitude of points $\lambda/\mu$ with $(\lambda,\mu) = O$ that lie in this gap provides a new proof of Nica’s theorem [11] [14], which is a strengthened form of Cohn’s theorem [2].

Corollary 1.3. For $|\Delta| > 12$, $\text{PE}_2(O)$ is an infinite-index subgroup of $\text{PSL}_2(O)$. Furthermore, $\text{PE}_2(O)$ is its own normalizer.

(The proofs of Theorem 1.2 and Corollary 1.3 appear as the proof of Theorem 3.1.5 in the author’s dissertation [9].)

Notably, if $O$ is any ring of integers other than non-Euclidean, imaginary quadratic [17] or if $n$ is anything other than 2 [1], $\text{PE}_n(O) = \text{PSL}_n(O)$.

Nica proves non-normality of $\text{PE}_2(O)$ but does not compute the normalizer. Also, the approach in this paper is simpler. There is no need for the “special unimodular pairs” or solutions to Pell-type equations used in [11].

Stange has also reproved the infinite-index assertion of Corollary 1.3 by way of her Schmidt arrangements [16]. And another proof of both the infinite-index and non-normal assertions has been given recently in [15]. Sheydvasser computes a fundamental polyhedron for the group generated by integer translations and reflections over unit hemispheres in $\mathbb{H}$ centered on integers from an imaginary quadratic field or certain quaternion algebras. By showing that this group is commensurable with $\text{PSL}_2(O)$, Sheydvasser recovers Nica’s theorem over a larger set of rings.

A Ford domain for $\text{PE}_2(O)$ also gives a group presentation via Poincaré polyhedron theorem. Generators are face-pairing matrices and relations are defined by reflections and edge cycles. (See Section 6.2 of [7] for a description and [10] for a proof.) This produces the following presentation of $\text{PE}_2(O)$, originally due to Fine [6] (derived largely from Cohn’s main lemma in [3]).

Corollary 1.4. Let $\tau = \sqrt{\Delta}/2$ or $(1 + \sqrt{\Delta})/2$ depending on $\Delta \mod 2$. If $|\Delta| > 12$, $\text{PE}_2(O) = \langle r, s(1), s(\tau) \mid s(1)s(\tau)s(1)^{-1}s(\tau)^{-1}, r^2, (rs(1))^3 \rangle$.

Finally, we use the Ford domain of $\text{PE}_2(O)$ to prove that $\text{PSL}_2(O)$ factors non-trivially as a free product with amalgamation. In 1970 [12], Serre showed that $\text{PSL}_2(O)$ does not have the property FA as defined in [13] (see the comment just before Chapter 2). For finitely generated groups, not satisfying FA is strictly weaker than being a nontrivial amalgam, and it was conjectured that the stronger property holds (as discussed in Chapter 1 of [7]). This was confirmed by Fine and Frohman in 1988. It is a result which Fine describes in Section 1.3 of [7] as answering one of the foremost questions addressed in his research monograph. Their proof relies on the theory of fundamental groups of factor manifolds. They find a nicely
embedded, incompressible, separating two-sided surface in the factor manifold of PSL$_2(\mathbb{O})$, which gives an amalgam splitting by the Seifert-Van Kampen theorem. We are able to avoid such complexity. Matrices that pair faces lying at least partially above the Euclidean plane $t = 2/3$ (shown translucent, white in Figure 1) generate PE$_2(\mathcal{O})$ by Theorem 1.2. Those from faces lying at least partially below $t = 2/3$ form a group we call $\Gamma$. It is straightforward to check that the resulting amalgam is nontrivial.

**Corollary 1.5.** If $|\Delta| > 12$, then PSL$_2(\mathcal{O}) \simeq$ PE$_2(\mathcal{O}) * N \Gamma$ is a nontrivial amalgam, where $N$ is generated by $r$, $s(1)$, and $s(r)s(r)^{-1}$ and $\Gamma$ depends on $\mathcal{O}$.

This is the same amalgam splitting given in [8].

Remark that [6] and [8] only mention Corollaries 1.3 and 1.4 in the context of maximal orders, but it appears that their proofs apply without modification when $\mathcal{O}$ is a non-maximal order.

2. Computing the Ford Domain

We do not contribute to the study of PE$_2(\mathcal{O})$ or PSL$_2(\mathcal{O})$ when $|\Delta| \leq 12$, the cases where the two groups coincide. Our aim is to compute $\mathcal{F}$ for PE$_2(\mathcal{O})$ when $|\Delta| > 12$. Two lemmas are needed. The first is well-known and straightforward to check.

**Lemma 2.1.** If $|\Delta| > 12$ and $\alpha \in \mathcal{O}$ is not 0, 1, or $-1$, then $|\alpha| \geq 2$. □

Recall notation in (1.2). In all studies of PE$_2(\mathcal{O})$ of which the author is aware, its elements are written in the form $s(\alpha_n)r \cdots s(\alpha_1)r$ where $\alpha_2, \ldots, \alpha_{n-1} \neq 0, 1, -1$. This may have originated in [2] as Nica calls it Cohn’s standard form. It differs slightly, though nontrivially, from Lemma 2.2 below. Our proof of Theorem 1.2 does not work when Cohn’s form is used in place of the following.

**Lemma 2.2.** If $|\Delta| > 4$, every matrix in PE$_2(\mathcal{O})$ is equal to a product of the form $s(\alpha_n)r \cdots s(\alpha_1)r s(\alpha_0)$ for some $\alpha_0, \ldots, \alpha_n \in \mathcal{O}$ with $\alpha_1, \ldots, \alpha_{n-1} \neq 0, 1, -1$.

**Proof.** If $|\Delta| > 4$, the only units in $\mathcal{O}$ are 1 and $-1$. In particular, $s(\alpha)$ for $\alpha \in \mathcal{O}$ account for all upper triangular matrices in PSL$_2(\mathcal{O})$, and $rs(\alpha)r$ account for all lower triangular matrices. Thus $r$ and $s(\alpha)$ generate PE$_2(\mathcal{O})$ for $\alpha \in \mathcal{O}$ (or just $\alpha$ from a $\mathbb{Z}$-basis for $\mathcal{O}$). Moreover, since $r$ has order two and $s(\alpha)s(\beta) = s(\alpha + \beta)$, any matrix in PE$_2(\mathcal{O})$ can be written as $s(\alpha_n)r \cdots s(\alpha_1)r s(\alpha_0)$ for some $\alpha_0, \ldots, \alpha_n \in \mathcal{O}$. To see that $\alpha_i \neq 0, 1, -1$ is possible if $0 < i < n$, observe that $s(\alpha_{i+1})rs(0)s(\alpha_{i-1}) = s(\alpha_{i+1} + \alpha_{i-1})$ when $\alpha_i = 0$, and $s(\alpha_{i+1})rs(\pm 1)s(\alpha_{i-1}) = s(\alpha_{i+1} \pm 1)rs(\alpha_{i-1} \mp 1)$ when $\alpha_i = \pm 1$. □
We can now prove Theorem 1.2.

**Theorem 2.3.** The only Euclidean hemispheres that contribute a face to the Ford domain of $\text{PE}_2(\Theta)$ are those of radius 1 centered on elements of $\Theta$.

**Proof.** Let $\zeta \in \mathbb{C}$ lie outside each closed unit disc centered on an integer in $\Theta$. The claim follows if we can show $\zeta$ is not under any isometric hemisphere from $\text{PE}_2(\Theta)$.

Fix $g \in \text{PE}_2(\Theta)$ with nonzero bottom-left entry. By Lemma 2.2, we may write $g = s(\alpha_n)r \cdots s(\alpha_1)r s(\alpha_0)$ for some $\alpha_0, \ldots, \alpha_n \in \Theta$ with $\alpha_1, \ldots, \alpha_{n-1} \neq 0, 1, -1$. Let $\zeta_1 = \zeta + \alpha_0$. Note that $\zeta_1$ lies outside the unit disc centered on 0, so $|\zeta_1| > 1$. For $i = 1, \ldots, n-1$, let $\zeta_{i+1} = \alpha_i - 1/\zeta_i$. This gives

$$s(\alpha_i) r \begin{bmatrix} \zeta_i \\ 1 \end{bmatrix} = \zeta_i \begin{bmatrix} \zeta_{i+1} \\ 1 \end{bmatrix}. \quad (2.1)$$

Assume $|\zeta_i| > 1$ for induction. Then $\zeta_{i+1}$ lies inside the open unit disc centered on $\alpha_i$. Since $\alpha_i \neq 0, 1, -1$, this implies $|\zeta_{i+1}| > 1$ by Lemma 2.1.

Let $-\beta$ and $\alpha$ be the bottom-row entries of $g$. Our goal is to show $|\alpha/\beta - \zeta| > 1/|\beta|$. First observe that $\beta \neq 0$ forces $n \geq 1$. Next, the bottom-row entries of $s(\alpha_n)r$ are 1 and 0, so

$$\alpha - \beta \zeta = [-\beta \ \alpha] \begin{bmatrix} \zeta_i \\ 1 \end{bmatrix} = [1 \ 0] s(\alpha_{n-1})r \cdots s(\alpha_1)r s(\alpha_0) \begin{bmatrix} \zeta_i \\ 1 \end{bmatrix} = \zeta_n \cdots \zeta_1,$$

where the last equality uses (2.1). Since $|\zeta_n \cdots \zeta_1| > 1$, we are done. \hfill $\square$

**Corollary 2.4.** For $|\Delta| > 12$, $\text{PE}_2(\Theta)$ is an infinite-index subgroup of $\text{PSL}_2(\Theta)$. Furthermore, $\text{PE}_2(\Theta)$ is its own normalizer.

**Proof.** Let $g \in \text{PSL}_2(\Theta)$ have top- and bottom-left entries $\lambda$ and $\mu$. If $\lambda/\mu$ is not under the closure of any isometric hemisphere of $\text{PE}_2(\Theta)$, then $g$ is the unique matrix in $\text{PE}_2(\Theta)M$ that has minimal bottom-left entry magnitude $|\mu|$. Since there are infinitely many such $\lambda/\mu$ when $|\Delta| > 12$, there are infinitely many right cosets.

The same idea proves the stabilizer claim: Take any $g \in \text{PSL}_2(\Theta)$ that is not in $\text{PE}_2(\Theta)$, with $\lambda$ and $\mu$ as before. Assume without loss of generality that $\lambda/\mu$ is not under any (open) isometric hemisphere from $\text{PE}_2(\Theta)$. The top- and bottom-left entries of $gs(\alpha)g^{-1}$ are $-\alpha \lambda / \mu + 1$ and $-\alpha \mu / \lambda$. Their ratio is $\lambda/\mu - 1/\alpha \mu^2$, which is not under any isometric hemisphere of $\text{PE}_2(\Theta)$ for the appropriate choice of $\alpha$. Thus $g\text{PE}_2(\Theta)g^{-1} \neq \text{PE}_2(\Theta)$. \hfill $\square$

Next we recover Fine’s presentation of $\text{PE}_2(\Theta)$ [6].

**Corollary 2.5.** Let $\tau = \sqrt{\Delta}/2$ or $(1 + \sqrt{\Delta})/2$ depending on $\Delta \mod 2$. If $|\Delta| > 12,$

$$\text{PE}_2(\Theta) = \langle r, s(1), s(\tau) \mid s(1)s(\tau)s(1)^{-1}s(\tau)^{-1}, r^2, (rs(1))^3 \rangle.$$

**Proof.** Let $F$ from Definition 1.1 be the Voronoi cell around 0 in the lattice $\Theta$—a rectangle when $\Delta$ is even and a hexagon when $\Delta$ is odd—so that only the hemisphere centered at 0 contributes a face of $\mathcal{F}$. Then the face-pairing matrices are just $r$, which self-pairs the hemisphere, and $s(1)$ and $s(\tau)$, which pair parallel vertical walls. When $\Delta$ is even, the relations are $r^2$ from the self-paired hemisphere, $s(1)s(\tau)s(1)^{-1}s(\tau)^{-1}$ from the edge cycle of length four among the four vertical walls, and $(rs(1))^2$ from the edge cycle of length two where the hemisphere meets the two vertical walls $\mathcal{R}(\zeta) = \pm 1/2$. The only difference when $\Delta$ is odd occurs among the vertical edges, of which there are now six. They split into two cycles of length three. Both cycles give the same relation, $s(1)s(\tau)s(1)^{-1}s(\tau)^{-1}$. \hfill $\square$
We are now ready to prove that $\text{PSL}_2(\mathcal{O})$ is a free product with amalgamation.

**Corollary 2.6.** If $|\Delta| > 12$, then $\text{PSL}_2(\mathcal{O}) \simeq \text{PE}_2(\mathcal{O}) \ast_N \Gamma$ is a nontrivial amalgam, where $N$ is generated by $r, s(1)$, and $s(\tau)r(\tau)^{-1}$ and $\Gamma$ depends on $\mathcal{O}$.

**Proof.** Take $F$ from Definition 1.1 to be the rectangle centered at $\sqrt{\Delta}/4$, and let $\mathcal{F}$ be a Ford domain for $\text{PSL}_2(\mathcal{O})$ as shown in Figure 1. Write $\text{PSL}_2(\mathcal{O}) = \langle G \mid R \rangle$ using Poincaré's polyhedron theorem. By Theorem 2.3, the faces and edges that lie at least partially above the Euclidean plane $t = 2/3$ define a presentation of $\text{PE}_2(\mathcal{O})$, say $\langle G_a \mid R_a \rangle$. Let $G_b \subset G$ be the generators from faces that lie at least partially below $t = 2/3$, and let $R_b$ be the relations among them from $R$. Since paired faces are either both at least partially above or both at least partially below $t = 2/3$, we have $G = G_a \cup G_b$. The same is true of edges in a cycle, so $R = R_a \cup R_b$.

Let $N$ denote the subgroup of $\text{PE}_2(\mathcal{O})$ generated by $G_a \cap G_b$, which consists of $s(1)$ to pair the two vertical walls that intersect $t = 2/3$, $r$ to self-pair the unit hemisphere centered at 0, and $s(\tau)r(\tau)^{-1}$ to self-pair or pair the remaining unit hemisphere or hemispheres depending on $\Delta \mod 2$. If necessary, add relations to $R_b$ so that the inclusion map $N \hookrightarrow \langle G_b \mid R_b \rangle$ is well-defined. (It is not necessary, but proving this is not useful.) Then $\text{PSL}_2(\mathcal{O}) = \langle G_a \cup G_b \mid R_a \cup R_b \rangle \simeq \text{PE}_2(\mathcal{O}) \ast_N \langle G_b \mid R_b \rangle$. Let us show $N \neq \text{PE}_2(\mathcal{O})$ to verify that our amalgam is nontrivial. By Corollary 2.5, the map $r, s(1) \mapsto \text{Id}$ and $s(\tau) \mapsto s(\tau)$ extends to a homomorphism $\text{PE}_2(\mathcal{O}) \to \langle s(\tau) \rangle$ since all three group relations map to the identity. This homomorphism is surjective with kernel containing $N$. \qed

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