Corrigendum: The Plebanski sectors of the EPRL vertex

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Abstract
We correct what amounts to a sign error in the proof of part (i) of theorem 3. The Plebanski sectors isolated by the linear simplicity constraints do not change—they are still the three sectors (deg), (II+) and (II−). What changes is the characterization of the continuum Plebanski two-form corresponding to the first two terms in the asymptotics of the EPRL vertex amplitude for Regge-like boundary data. These two terms do not correspond to the Plebanski sectors (II+) and (II−), but rather to the two possible signs of the product of the sign of the sector—+1 for (II+) and −1 for (II−)—and the sign of the orientation $\epsilon_{IJKL} B^{IJ} \wedge B^{KL}$ determined by $B^{IJ}$. This is consistent with what one would expect, as this is exactly the sign which classically relates the BF action, in the Plebanski sectors (II+) and (II−), to the Einstein–Hilbert action, whose discretization is the Regge action appearing in the asymptotics.

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The error and the corrected final result

The error lies in part (i) of theorem 3 of the paper. In order to state this error, let us define a numbered 4-simplex to be a geometrical 4-simplex with vertices numbered arbitrarily, and each tetrahedron numbered by the vertex it does not contain. An ‘ordered 4-simplex’ as defined in definition 3 is then a numbered 4-simplex that additionally satisfies a condition relating the numbering to orientation. In order for the argument for part (i) of theorem 3 to be valid, the numbered 4-simplex guaranteed by the reconstruction theorem must be ‘ordered’, because it is then used to calculate the Plebanski sector of the geometrical bivectors, whose well-definition requires this. But, in general, the reconstructed 4-simplex will not be ordered.

This is the error in the paper. As we will see, it can be easily corrected, and upon correction, the interpretation of the terms in the asymptotics of the vertex amplitude will no longer involve
only Plebanski sectors, but also the orientation $\epsilon_{ijkl}B_{ij}^k \wedge B_{kl}^j$ determined by the continuum two-form $B_{\mu\nu}^I$, reconstructed from the discrete data at the critical points. Specifically, let

$$\omega(B_{\mu\nu}) := \text{sgn} (\epsilon^{a\beta\gamma\delta} \epsilon_{ijkl} B_{\alpha\beta}^j B_{\gamma\delta}^k),$$

where $\epsilon^{a\beta\gamma\delta}$ is the fixed orientation on $M \cong \mathbb{R}^4$, and let $\nu(B_{\mu\nu}) = +1$, $-1$ if $B_{\mu\nu}^I$ is in the Plebanski sector (II+) or (II$^-$), respectively, and let $\nu(B_{\mu\nu}) = 0$ otherwise. Then the first and second terms in the asymptotics of equation (3.10) correspond to critical points, where $\omega v = +1$ or $-1$, respectively.

Note that this modified result is exactly what one would expect. The first and second terms in equation (3.10) are respectively $\epsilon^S_{\text{Regge}}$ and $-\epsilon^S_{\text{Regge}}$, where $S_{\text{Regge}}$ is the Regge action. The Regge action is a discretization of the Einstein–Hilbert action $S_{\text{EH}}$, and the relation of the BF action $S_{\text{BF}}$ to the Einstein–Hilbert action, in the Plebanski sectors (II+) and (II$^-$), is precisely $S_{\text{BF}} = \omega v S_{\text{EH}}$.

**Details of the correction**

In the following, $\{B_{ab}\}$ shall always denote a ‘discrete Plebanski field’ in the sense of [1]—that is, a set of $\mathfrak{s}(4)$ algebra elements $B_{ab}^I = -B_{ba}^I$ satisfying closure ($\sum_{a\beta\gamma\delta} B_{ab}^I = 0$) and orientation ($B_{\mu\nu}^I = -B_{\nu\mu}^I$). The algebra indices $IJ$ will usually be suppressed. The algebra elements $B_{ab}$ are also referred to as *bivectors* due to the antisymmetry of the algebra indices. Let $B_{\mu\nu}((B_{ab}), \sigma)$ denote the unique $\mathfrak{s}(4)$-valued two-form, constant with respect to $\partial_\sigma$, such that its integral over each triangle $\Delta_{ab}(\sigma)$ of the numbered 4-simplex $\sigma$ is equal to the algebra element $B_{ab}$. The existence and uniqueness of the two-form $B_{\mu\nu}$ satisfying these conditions is ensured by lemma 1 of [1]. The proof of lemma 1 does not depend on $\sigma$ being ordered; see also the related work in [2]. When defining the Plebanski sector and the orientation of a set of algebra elements $\{B_{ab}\}$, however, we will see that it is necessary to restrict $\sigma$ to be ordered, but for the mere reconstruction of $B_{\mu\nu}$ itself, we can and do omit this restriction.

We begin by noting that the proof of theorem 1 in [1] actually succeeds in proving the following much stronger statement.

**Theorem 1. Stronger statement.** For any numbered 4-simplex $\sigma$, $B_{\mu\nu}(B_{\text{Regge}}(\sigma), \sigma)$ is in the Plebanski sector (II+) and has orientation $\omega = +1$.

Let us next prove two lemmas, which will make the corrected proof of part (i) of theorem 3 a single line. For these two lemmas, let $P$ denote any orientation-reversing diffeomorphism such that $P \circ P$ is the identity.

**Lemma 3.** Given any discrete Plebanski field $\{B_{ab}\}$ and any numbered 4-simplex $\sigma$,

$$B_{\mu\nu}((B_{ab}), P\sigma) = -P^* B_{\mu\nu}((B_{ab}), \sigma).$$

**Proof.** As mentioned in [1], the only background structures used in the construction of the continuum two-form $B_{\mu\nu}((B_{ab}), \sigma)$ are the flat connection $\partial_\sigma$ and the fixed orientation $\epsilon_{a\beta\gamma\delta}$. We begin by making the fixed orientation an explicit argument in the construction $B_{\mu\nu}(B_{ab}, \sigma, \epsilon)$, so that, given $\{B_{ab}^I\}, \langle \sigma, \epsilon \rangle \mapsto B_{\mu\nu}^I$ is covariant under the symmetry group of $\partial_\sigma$, that is, under all of $\text{GL}(4)$. In particular, for $P \in \text{GL}(4)$, it follows that

$$B_{\mu\nu}((B_{ab}), P\sigma, P\epsilon) = P^* B_{\mu\nu}((B_{ab}), \sigma, \epsilon).$$


Furthermore, by definition of the reconstructed two-forms (and introducing the orientation $\hat{e}$ as an explicit argument also of each oriented triangle $\Delta_{ab}(\sigma, \hat{e})$), one has

$$\int_{\Delta_{ab}((P)\sigma, \hat{e})} B\left(B_{ab}'\right), Pb, \hat{e} := B_{ab} =: \int_{\Delta_{ab}((P)\sigma, \hat{e})} B\left(B_{ab}'\right), Pb, \hat{e} \hspace{1cm} \leftarrow \hspace{1cm} B_{ab}, Pb, \hat{e}$$

$$\hspace{1cm} = -\int_{\Delta_{ab}((P)\sigma, \hat{e})} B\left(B_{ab}'\right), Pb, \hat{e}$$

$$\hspace{1cm} = -\int_{\Delta_{ab}((P)\sigma, \hat{e})} P^* B\left(B_{ab}'\right), \sigma, \hat{e}$$

where the second to last equality holds because the sole effect of replacing $P\hat{e}$ with $\hat{e}$ in the argument for triangle $\Delta_{ab}$ is to reverse the orientation of the triangle and hence negate the value of the integral, and the last equality holds because of (2). Because the continuum two-forms are constant with respect to $\hat{a}$ and are completely determined by the values of the above integrals for all $a, b$ [1], it follows that the integrands of the first and last expressions are equal, which, combined with $B_{\mu\nu}(\sigma, \sigma) := B_{\mu\nu}(\sigma, \sigma, \hat{e})$, implies the claimed result (1). □

In order to understand the significance of the above lemma, we first note that, for $B_{\mu\nu}$ in the Plebanski sector (II+) or (II−), the action of $P$ on $B_{\mu\nu}$ flips the orientation of $B_{\mu\nu}$ while leaving its Plebanski sector unchanged and negation of $B_{\mu\nu}$ flips its Plebanski sector while leaving its orientation unchanged. These facts, together with the above lemma imply

$$\omega(B_{\mu\nu}(\sigma), Pb) = -\omega(B_{\mu\nu}(\sigma), Pb) \quad \text{and} \quad v(B_{\mu\nu}(\sigma), Pb) = -v(B_{\mu\nu}(\sigma), Pb)$$

Because of the above equations, if we wish to use $B_{\mu\nu}(\sigma)$ to define a Plebanski sector and orientation for a given set of algebra elements $\{B_{ab}\}$, a restriction must be placed on the numbered 4-simplex $\sigma$ such that it not possible to use both a 4-simplex $\sigma'$ and its parity reversal $P\sigma$; otherwise, the Plebanski sector and the orientation of $\{B_{ab}\}$ will be ill-defined. The restriction used is precisely that $\sigma$ be ordered in the sense of [1]. Once this restriction is made, $v(B_{\mu\nu}(\sigma), Pb) \quad \text{and} \quad \omega(B_{\mu\nu}(\sigma), Pb)$ are independent of the remaining freedom in $\sigma$. This was proven for $v(B_{\mu\nu}(\sigma), Pb)$ in lemma 2 of [1]. For $\omega(B_{\mu\nu}(\sigma), Pb)$, the proof follows from the same argument, together with the fact that, for any orientation-preserving diffeomorphism $\varphi$, $\omega(\varphi B_{\mu\nu}) = \omega(B_{\mu\nu})$. Thus, one may define $\varphi(B_{\mu\nu}(\sigma), Pb)$ and $\omega(B_{\mu\nu}(\sigma), Pb)$ where any ordered $\sigma$ may be used. (The significance of the ordering condition on $\sigma$ in this context is essentially that, by imposing a certain compatibility between the numbering and the orientation, the ordering condition endows the numbering of vertices with orientation information which turns out to be essential in extracting the Plebanski sector and dynamical orientation from the algebra elements $\{B_{ab}\}$.)

**Lemma 5.** For any numbered 4-simplex $\sigma$, $\omega(B_{ab}(\sigma)) = 1$.

**Proof.** Case 1, $\sigma$ is ordered: then $v(B_{ab}(\sigma)) := v(B_{ab}(\sigma), Pb) = +1$ and $\omega(B_{ab}(\sigma)) := \omega(B_{ab}(\sigma), Pb) = +1$ where, in each of these equations, the first equality follows by definition and the final equality is implied by the above stronger version of theorem 1.

Case 2, $\sigma$ is not ordered: then $P\sigma$ is an ordered 4-simplex, so that

$$v(B_{ab}(\sigma)) := v(B_{ab}(\sigma), Pb) = -v(B_{ab}(\sigma), Pb) = -1$$

and

$$\omega(B_{ab}(\sigma)) := \omega(B_{ab}(\sigma), Pb) = -\omega(B_{ab}(\sigma), Pb) = -1$$
where, in each of the above equations, the first equality follows by definition, the second equality follows from equation (3) and the final equality is implied by the above stronger version of theorem 1.

In both cases, one has $\omega(B_{ab}^{\text{geom}}(\sigma))\nu(B_{ab}^{\text{geom}}(\sigma)) = 1$, as claimed. □

The corrected statement and proof of part (i) of theorem 3 are then as follows.

**Theorem 3, Part (i), corrected.** Suppose $\{A_{ab}, n_{ab}\}$ is a set of non-degenerate reduced boundary data satisfying closure and $\{X_{\pm a}\}$ are such that the orientation is satisfied. If $\{X_{-a}\} \neq \{X_{+a}\}$, then $\{B_{ab}^{\text{phys}}(A_{ab}, n_{ab}, X_{\pm a})\}$ is either in the Plebanski sector (II+) or (II−). Furthermore, the sign $\mu$ in the reconstruction theorem equals $\omega \nu$.

**Proof.** Let $\sigma$ denote the numbered 4-simplex guaranteed by the reconstruction theorem, unique up to translation, rotation and inversion. Using the relation $B_{ab}^{\text{phys}} = \mu B_{ab}^{\text{geom}}(\sigma)$ between the physical and geometrical bivectors in the reconstruction theorem, and using lemma 5, one has

$$\omega(B_{ab}^{\text{phys}})\nu(B_{ab}^{\text{phys}}) = \omega(B_{ab}^{\text{geom}}(\sigma))(\mu \cdot \nu(B_{ab}^{\text{geom}}(\sigma))) = \mu.$$ □

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