Coordinate Representation of the Two-Spinon wavefunction and Spinon Interaction

B. A. Bernevig *, D. Giuliano † and R. B. Laughlin *

*Department of Physics, Stanford University, Stanford, California 94305
† Istituto Nazionale di Fisica della Materia (INFM), Unità di Napoli, Napoli, Italy.

(Paper submitted March 21, 2022)

By deriving and studying the coordinate representation for the two-spinon wavefunction, we show that spinon excitations in the Haldane-Shastry model interact. The interaction is given by a short-range attraction and causes a resonant enhancement in the two-spinon wavefunction at short separations between the spinons. We express the spin susceptibility for a finite lattice in terms of the resonant enhancement, given by the two-spinon wavefunction at zero separation. In the thermodynamic limit, the spinon attraction turns into the square-root divergence in the dynamical spin susceptibility.

PACS numbers: 75.10.Jm, 75.40.Gb, 75.50.Ee

1. INTRODUCTION

One of the most important issues in contemporary physics is spin fractionalization, which takes place in strongly-interacting one dimensional (1D) antiferromagnets. At odds with what one would expect, the elementary excitation of these systems is not a spin-1 spin wave, but a gapless spin-1/2 excitation: the spinon.

Since their discovery within the framework of the Heisenberg model (HM), by Fadeev and Takhtajan [1], spinons are believed to be noninteracting particles since, according to the Bethe-Ansatz solution of the HM [2], the energy of a many-spinon solution is apparently given by the sum of the energies of each isolated spinon. In this paper we challenge the non-interacting spinons idea through a careful analysis of the two-spinon dynamics in an exact solution, which shows that they do actually interact.

Extensive amount of theoretical studies proved that spin fractionalization is a generic phenomenon in one-dimensional spin-1/2 interacting antiferromagnets [3]. Indeed, the large-scale physics of all these systems is always the same, given by “spinon gas” dynamics [4]. Therefore, without any loss of generality, one can choose to analyze a particular model, where the excitations are easier to visualize.

The simplest model that describes the properties of spinons is the Haldane-Shastry model (HSM) [5,6], where spins-1/2 located at the sites of a circular lattice antiferromagnetically interact and the interaction is inversely proportional to the square of the chord between the two sites. In this paper we investigate the basic features of the HSM by employing a formalism based on analytic variables on the unit radius circle [6]. By using real space coordinates, the spin-1/2 excitations become easier to construct and visualize than by making use of plane waves [7]. The formalism can be easily generalized to the study of other models. Such a formalism allows us to write a “real space” representation of the two-spinon wavefunction.

By analyzing the real space two-spinon wavefunction, we show that spinons scatter by means of a short-ranged attractive potential and analyze in detail the physical consequences of the existence of this potential. The short-rangeness of the interaction makes spinons free when they are widely separated. However, from the exact solution of the Schrödinger equation for two spinons we find that the amplitude of the wavefunction is greatly enhanced when they are on top of each other, phenomenon which we refer to as “resonant enhancement”. While the density of states is uniform at low energy, resonant enhancement causes the overlap between the wavefunction for the localized spin wave and that for the spinon pair to be significant, but not enough to create a two-spinon bound state. The corresponding matrix element is enhanced so as to make the spin-1 excitation absolutely unstable.

Physical consequences of the instability of the spin-wave appear in the functional form of the dynamical spin susceptibility (DSS), $\chi_q(\omega)$. The DSS is the Fourier transform of the spin-spin correlation function. Its functional form can be experimentally tested by means, for instance, of neutron scattering experiments, the probed quantity being the spectral density of states, $1/\pi \text{Im} \chi_q(\omega)$ [10]. A system with a stable spin-1 excitation would show a sharp pole in $\text{Im} \chi_q(\omega)$ at the corresponding dispersion relation, $\omega = \omega(q)$. On the other hand, absolute instability of the spin wave against decay into spinons will generate a branch cut in $\text{Im} \chi_q(\omega)$ at the threshold energy for the creation of a spinon pair, which is a signal of the opening of a decay channel, corresponding to the lack of spin wave integrity. Consequently, a sharp square-root singularity shows up at the threshold for the creation of a spinon pair, on top of the broadening in the spectral density of states. Experiments performed onto quasi one-dimensional antiferromagnets provide clear evidence for broad spectra, while no sharp spin-1 resonance has been
An exact calculation of the DSS cannot, in general, be performed, even for models exactly solvable with the Bethe-Ansatz. However, the HSM has the remarkable property that the wavefunction for a spin-1 excitation is fully decomposed in the basis of the two-spinon eigenstates [11]. This allows us to write an exact expression for the dynamical Spin Singlet. We discuss at length several properties of the two spinon solution. We derive the energy eigenvalues, the corresponding eigenvectors, and their norm. A discussion about spinon statistics is provided at the end of the Section; Sections VI and VII contain the key results of our work. In Section VI we write the Schrödinger equation for the two-spinon wavefunction, whose solutions are hypergeometric polynomials. From the behavior of the two-spinon wavefunction, we infer the nature of the interaction between spinons: a short-range attraction. The physical consequences of such an interaction are discussed at length in Section VII, where we derive an exact closed-form expression for the dynamical Spin Susceptibility in terms of the two-spinon wavefunctions and rigorously prove that the DSS is fully determined by spinon interaction. In the thermodynamic limit spinon interaction turns into the square root divergence in the DSS; In Section VIII we provide our main conclusions.

II. HALDANE-SHAISTRY HAMILTONIAN

The Haldane-Shastry model [12] is defined on a lattice with periodic boundary conditions. Let $N$ be the number of sites. Let $z_\alpha$, with $z_\alpha^N = 1$, be a complex number representing a lattice site on which a spin-1/2 electron resides, and let $\vec{S}_\alpha$ be a Heisenberg spin operator acting on that electron. The Haldane-Shastry Hamiltonian takes the form:

$$\mathcal{H}_{HS} = J \left( \frac{2\pi}{N} \right)^2 \sum_{\alpha<\beta}^{N} \frac{\vec{S}_\alpha \cdot \vec{S}_\beta}{|z_\alpha - z_\beta|^2} ,$$

where $J$ is the coupling strength. The interaction is an analytic function of the coordinates. This is related to the property of a complex variable $z$ laying on the unit circle: $z^* = z^{-1}$ which implies:

$$1 \frac{1}{|z_\alpha - z_\beta|^2} = - \frac{z_\alpha z_\beta}{(z_\alpha - z_\beta)^2} .$$

The representation in terms of the analytic variables $z_\alpha$, which we will use throughout the paper, comes out to be very useful for describing the properties of spinons in real space. The Hamiltonian in Eq.(1) is clearly invariant under spin rotations generated by the total spin:

$$[\mathcal{H}_{HS}, \vec{S}] = 0 \quad \vec{S} = \sum_{\alpha}^{N} \vec{S}_\alpha . \tag{2}$$

It also possesses an additional symmetry generated by a vector operator independent on $\vec{S}$:

$$[\mathcal{H}_{HS}, \vec{\Lambda}] = 0 \quad \vec{\Lambda} = \frac{i}{2} \sum_{\alpha \neq \beta} \left( \frac{z_\alpha + z_\beta}{z_\alpha - z_\beta} \right) (\vec{S}_\alpha \times \vec{S}_\beta) . \tag{3}$$

That $\vec{\Lambda}$ commutes with $\mathcal{H}_{HS}$ can be seen as follows:

$$[\mathcal{H}_{HS}, \vec{\Lambda}] = \sum_{j \neq k} \sum_{\alpha \neq \beta} \frac{z_j + z_k}{z_j - z_k} \frac{1}{|z_\alpha - z_\beta|^2} \left[ (\vec{S}_j \times \vec{S}_k), (\vec{S}_\alpha \times \vec{S}_\beta) \right]$$

$$= 4i \sum_{j \neq k} \sum_{\alpha \neq \beta} \frac{z_j + z_k}{z_j - z_k} \frac{1}{|z_j - z_k|^2} \left[ (\vec{S}_j \cdot \vec{S}_k) \vec{S}_\ell - (\vec{S}_\ell \cdot \vec{S}_k) \vec{S}_j \right]$$

$$+ i \sum_{j \neq k} \sum_{\alpha \neq \beta} \frac{z_j + z_k}{z_j - z_k} \frac{1}{|z_j - z_k|^2} \left( \vec{S}_j - \vec{S}_k \right) = 0 . \tag{4}$$

Although they both commute with $\mathcal{H}_{HS}$, $\vec{S}$ and $\vec{\Lambda}$ do not commute between themselves, being that $\vec{\Lambda}$ is a vector, as shown by the commutation relations:

$$[S^a, S^b] = i \epsilon^{abc} S^c \quad [S^a, \Lambda^b] = i \epsilon^{abc} \Lambda^c , \tag{5}$$

From the commutators in Eq.(3) it follows that $\mathcal{H}_{HS}$, $\vec{S}^2$, and $(\vec{\Lambda} \cdot \vec{S})$ all commute with each other. The extra symmetry of $\mathcal{H}_{HS}$ is the reason for the exceptional degeneracy of the energy eigenstates, as pointed out and discussed in [13]. The algebra generated by the two vector symmetries of $\mathcal{H}_{HS}$ is referred to as Yangian and is discussed in [14]. $\vec{\Lambda}$ can be physically interpreted as the spin current operator for the HSM, as we show in Appendix C.

Starting from next section we will review the properties of the ground state and of the one- and two-spinon excited states of the HSM. This will allow us to to define the formalism we will use in order to describe the relevant physical properties of the model.
III. GROUND STATE

Let \( N \) be even. We proceed by first giving the representation of the ground state \( |Ψ_{GS}\rangle \) in terms of the \( z \)-coordinates and then proving that it is the actual ground state of \( \mathcal{H}_{HS} \). \( |Ψ_{GS}\rangle \) is defined in terms of its projection onto the set of states with \( M = N/2 \) spins up and the remaining spins down. If \( z_1, \ldots, z_M \) are the coordinates of the up spins, one defines the state \(|z_1, \ldots, z_M\rangle\) as: 
\[
|z_1, \ldots, z_M\rangle = \prod_{j=1}^{M} S_j^+ \prod_{\alpha=1}^{N} \alpha^J \{0\}
\]
where \(|0\rangle\) is the empty state. The projections are given by:
\[
Ψ_{GS}(z_1, \ldots, z_M) = \prod_{j<k} (z_j - z_k)^2 \prod_{j=1}^{M} z_j
\]
where \( z_1, \ldots, z_M \) denote the locations of the \( \uparrow \) sites all others being \( \downarrow \). We can imagine the spin system as a 1-dimensional string of boxes populated by hard-core bosons, the \( \downarrow \) spin state corresponding to an empty box and the \( \uparrow \) spin state corresponding to an occupied one. The total number of bosons is conserved, as it is physically the same thing as the eigenvalue of \( S^z \). Let us, now, review the main properties of \( Ψ_{GS}(z_1, \ldots, z_M) \).

A. The norm of \( Ψ_{GS} \)

\( Ψ_{GS}(z_1, \ldots, z_M) \) is a homogeneous polynomial of degree \( N - 1 \) in the variables \( z_1, \ldots, z_M \). Its norm can be computed by using the following identity:
\[
C_M = \sum_{z_1, \ldots, z_M} \prod_{i<j} |z_i - z_j|^4
\]
\[
= \left( \frac{N}{2\pi i} \right)^M \oint_{z_1} \cdots \oint_{z_M} \prod_{i<j} (1 - \frac{z_i}{z_j})^2
\]
where the integrals are calculated on the circle of radius 1. The integral in Eq.6 has been evaluated by Wilson [12]. The result is:
\[
\left( \frac{1}{2\pi i} \right)^M \oint_{z_1} \cdots \oint_{z_M} \prod_{i<j} (1 - \frac{z_i}{z_j})^2 = \frac{(2M)!}{2^M}
\]
therefore:
\[
C_M = \frac{(2M)!}{2^M} N^M
\]

B. Singlet Sum Rule

We shall prove that the ground state is a spin singlet by showing that \(|Ψ_{GS}\rangle\) is annihilated by both \( S^z \) and \( S^- \). \( S^z |Ψ_{GS}\rangle = 0 \) because \(|Ψ_{GS}\rangle\) has an equal number of \( \uparrow \) and \( \downarrow \) spins
\[
[S^- Ψ_{GS}](z_1, \ldots, z_M) = \sum_{\alpha=1}^{N} \langle z_1, \ldots, z_M | S^- |Ψ_{GS}\rangle
\]
\[
= \lim_{z_1 \to 0} \sum_{\ell=1}^{N-1} \frac{1}{\ell!} \left( \sum_{\alpha=1}^{N} \alpha^\ell \right) \frac{\partial^\ell}{\partial z_1^\ell} Ψ_{GS}(z_1, \ldots, z_M) = 0
\]
since \( \sum_{\alpha=1}^{N} \alpha^\ell = N \delta_{\ell0} \pmod{N} \).

C. Coordinate Invariance

Spin rotational invariance implies that \( Ψ_{GS} \) is invariant under interchange of \( \uparrow \) and \( \downarrow \) coordinates. More generally, the quantization axis can be taken to be an arbitrary direction in spin space. Denoting the sites complementary to \( z_1, \ldots, z_M \) by \( η_1, \ldots, η_M \), so that
\[
\prod_{k<k}^{M} (z_k - η_k) = z^N - 1
\]
we have for fixed \( j \)
\[
\prod_{k \neq j}^{M} (z_j - η_k) = \lim_{z \to z_j} \frac{z^N - 1}{z - z_j} = N z_j^{N-1}
\]
and thus
\[
\prod_{j<k}^{M} (z_j - η_k)^2 \prod_{j}^{M} z_j = N \prod_{j<k}^{M} \frac{1}{z_j - η_k}
\]
\[
= (-1)^M \prod_{j<k}^{M} (η_j - η_k)^2 \prod_{j}^{M} η_j
\]

D. Reality

The ground state is its own complex conjugate, and therefore is a real number:
\[
Ψ_{GS}^*(z_1, \ldots, z_M) = \prod_{j<k} (z_j^* - z_k^*) \prod_{j} z_j^*
\]
\[
= \prod_{j<k} (z_k - z_j)^2 \prod_{j} z_j^{1-N} = Ψ_{GS}(z_1, \ldots, z_M)
\]
E. Translational Invariance

The crystal momentum of the state, $q$, is defined (mod $2\pi$) by the equation:

$$\Psi_{GS}(z_1, \ldots, z_M) = e^{iq}\Psi_{GS}(z_1, \ldots, z_M), \quad (15)$$

where $z = \exp(2\pi i/N)$. From Eq. (15) it comes out that $q$ can be either 0 or $\pi$, according to whether $N$ is divisible by four or not. $\Psi_{GS}$ equals itself, up to an overall minus sign, when translated by one lattice constant.

F. Disordered State

$|\Psi_{GS}\rangle$ is a disordered state. The way spin-spin correlations fall off with the distance defines whether a state of a magnetic system takes order or not. The relevant quantity is the spin-spin correlation function, $\chi(z_\alpha) = \langle \Psi_{GS}| \sum_{\alpha} S_\alpha^+ S_\beta^- |\Psi_{GS}\rangle/|\Psi_{GS}\rangle|\Psi_{GS}\rangle$, which can be expressed in terms of two-spinon wavefunctions only, as we show in Section VII.

\begin{center}
\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{Spin-spin correlation function decay for $N = 60$.}
\end{figure}
\end{center}

One-dimensional systems do not break continuous symmetries, so they are not alleged to order. However, there is a substantial difference between half-integer spin chains and integer spin ones \[2\]. Both have a disordered ground state, but the former have excitations above the ground state which are gapless in the thermodynamic limit while the latter have a gap that survives the thermodynamic limit and is given by $\Delta = \hbar v/\xi$, where $v$ is the spin-wave velocity of a nearby ordered state and $\xi$ is the correlation length. The consequence of this is that the fall-off of the spin correlations in the ground state of half-odd spin chains is not as abrupt as for integer-spin chains, where the correlations are suppressed within one or two lattice spacings (“Haldane’s conjecture”). Fig. (1) shows that the behavior of the HSM is the one expected for half-odd spin chains. Correlations decay as $(-1)^{x/2}/x$, according to Haldane’s conjecture.

G. Ground State Energy

$|\Psi_{GS}\rangle$ is an eigenstate of $\mathcal{H}_{HS}$ with eigenvalue:

$$\mathcal{H}_{HS}|\Psi_{GS}\rangle = -J \left( \frac{\pi^2}{24} \right) \left( N + \frac{5}{N} \right) |\Psi_{GS}\rangle. \quad (16)$$

We trade sums over spins on the lattice for derivative operators that are understood to act onto the analytic extension of $\Psi_{GS}(z_1, \ldots, z_M)$, in which the $z_j$’s are allowed to take any value on the unit circle. After computing the derivatives, we constrain them again to lattice sites. We begin by observing that $[S_\alpha^+ S_\beta^- |\Psi_{GS}\rangle(z_1, \ldots, z_M)$ is identically zero unless one of the arguments $z_1, \ldots, z_M$ equals $z_\alpha$. We have

$$\left\{ \sum_{\beta \neq \alpha} S_\alpha^+ S_\beta^- \right\}_{z_\alpha}, \quad \Psi_{GS}(z_1, \ldots, z_M)$$

$$= \sum_{j=1}^{M} \sum_{\beta \neq j}^{N} \frac{1}{|z_j - z_\beta|^2} \Psi_{GS}(z_1, \ldots, z_{j-1}, z_\beta, z_{j+1}, \ldots, z_M)$$

$$= \sum_{j=1}^{M} \sum_{\beta \neq j}^{N-2} \left( \sum_{|\beta| \neq j}^{N} \frac{z_\beta(z_\beta - z_j)^{2\ell}}{|z_j - z_\beta|^2} \right) \frac{\partial^{\ell}}{\partial z_j^{\ell}} \left\{ \Psi_{GS}(z_1, \ldots, z_M) \right\}_{z_j}$$

$$= \sum_{j=0}^{N-2} \sum_{\ell=0}^{M} \frac{z_j^{\ell+1}}{\ell!} A_{\ell} \frac{\partial^{\ell}}{\partial z_j^{\ell}} \left\{ \Psi_{GS}(z_1, \ldots, z_M) \right\}_{z_j} \quad (17)$$

The coefficients $A_{\ell}$ are evaluated in appendix B. Their remarkable property is that they are zero for $N > l > 2$. Hence, Eq. (17) can be rewritten as:

$$\sum_{j=1}^{M} \left\{ \frac{(N-1)(N-5)}{12} z_j - \frac{N-3}{2} \frac{2z_j}{z_j - z_k} \frac{\partial}{\partial z_j} \right\}$$

$$+ \sum_{j \neq k}^{M} \frac{z_j^2}{(z_j - z_k)(z_j - z_m)} + \sum_{j \neq k}^{M} \frac{z_j^2}{(z_j - z_k)^2} \Psi_{GS}(z_1, \ldots, z_M)$$

$$= \left\{ \frac{N}{8} - \sum_{j \neq k}^{M} \frac{1}{|z_j - z_k|^2} \right\} \Psi_{GS}(z_1, \ldots, z_M). \quad (18)$$
In Eq.(18) we have made use of the rule:

\[ \frac{z_\alpha^2}{(z_\alpha - z_\beta)(z_\alpha - z_\gamma)} + \frac{z_\beta^2}{(z_\beta - z_\alpha)(z_\beta - z_\gamma)} + \frac{z_\gamma^2}{(z_\gamma - z_\alpha)(z_\gamma - z_\beta)} = 1 . \] (19)

We also have:

\[ \left\{ \sum_{\beta \neq \alpha} \frac{S_\alpha^z S_\beta^z}{|z_\alpha - z_\beta|^2} \right\} \Psi_{GS}(z_1, \ldots, z_M) = \left\{ \frac{-N(N^2-1)}{48} + \sum_{j \neq k} \frac{1}{|z_j - z_k|^2} \right\} \times \Psi_{GS}(z_1, \ldots, z_M) . \] (20)

This completes the proof, since

\[ \mathcal{H}_{HS} = J \frac{2\pi}{N} \left\{ \sum_{\alpha \neq \beta} \frac{S_\alpha^z S_\beta^z}{|z_\alpha - z_\beta|^2} + \sum_{\alpha} \frac{N(N^2+5)}{48} + \frac{N+1}{12} S_\alpha^2 \right\} . \] (21)

The wavefunction \( \Psi_{GS}(z_1, \ldots, z_M) \) was first introduced by Haldane and Shastry in analogy to the exact Sutherland solution of the continuum limit of the problem. The proof that this wavefunction is the actual ground state of \( \mathcal{H}_{HS} \) is a consequence of the factorization of the HS Hamiltonian, as we are going to discuss next.

**H. Factorization of \( \mathcal{H}_{HS} \)**

In Appendix D we prove that \( \mathcal{H}_{HS} \) can be written as:

\[ \mathcal{H}_{HS} = J \frac{2\pi}{N} \left\{ \frac{2}{9} \sum_{\alpha} D_\alpha \cdot \bar{D}_\alpha - \frac{N(N^2+5)}{48} + \frac{N+1}{12} S^2 \right\} . \] (22)

The operators \( D_\alpha \) are given by:

\[ D_\alpha = \frac{1}{2} \sum_{\beta \neq \alpha} \frac{z_\alpha + z_\beta}{z_\alpha - z_\beta} \left[ i(S_\alpha \times \bar{S}_\beta) + S_\beta \right] . \] (23)

and they annihilate \( \langle \Psi_{GS} \rangle \) (see Appendix D). Eq.(22) implies that \( \langle \Psi_{GS} \rangle \) is the ground state of the HSM because \( \mathcal{H}_{HS} \) can be written as a constant plus nonnegative definite operators, and the only state satisfying the requirement of minimum energy is \( \langle \Psi_{GS} \rangle \).

**I. Degeneracy**

The HSM ground state is not degenerate, but is nearly so. We already pointed out that half-odd spin magnets have a gapless spectrum. In the next sections we will see that elementary excitations above the ground state are spinons and that their spectrum is relativistic. In particular, at the endpoints of the Brillouin zone, the energy becomes the same as the ground state energy, modulo corrections that are sub-leading in the thermodynamic limit. This means that, in principle, one can have many states with the same energy as the ground state which are distinguished from one another by their number of spinons.

An example is provided by the singlet state of two spinons with total momentum \( \pi \). It is given by:

\[ \Psi_S(z_1, \ldots, z_M) = \prod_{j<k} (z_j - z_k)^2 \left[ 1 - \prod_{j=1}^M z_j^2 \right] . \] (24)

Its energy is given by:

\[ \mathcal{H}_{HS} \Psi_S = -J \left( \frac{\pi^2}{24} \right) \left( N - \frac{7}{N} \right) \langle \Psi_S \rangle . \] (25)

and is the energy of the ground state plus corrections that go to zero in the thermodynamic limit.

**J. Spin current**

We now show that \( \langle \Psi_{GS} \rangle \) is an eigenstate of \( \Lambda^z \) belonging to the 0-eigenvalue. The action of \( \Lambda^z \) on the ground state gives:

\[ \Lambda^z \langle \Psi_{GS} \rangle = 0 . \] (26)

Eq.(26) can be proved as follows:

\[ \langle \Lambda^z \Psi_{GS} \rangle(z_1, \ldots, z_M) = \left[ \frac{1}{2} \sum_{j=1}^M \sum_{\beta \neq j} z_\beta \left( \frac{z_j + z_\beta}{z_j - z_\beta} \right) \right. \]

\[ \times \sum_{l=0}^{N-1} \frac{z_j^l}{l!} \frac{\partial}{\partial z_j} \left\{ \Psi_{GS}(z_1, \ldots, z_M) \right\} \]

\[ \times \Psi_{GS}(z_1, \ldots, z_M) = 0 , \] (27)

where we have made use of the results of Appendix B and of the technique described in detail in subsection III.F.
IV. ONE-SPINON WAVEFUNCTION.

At odds with the naive idea that the elementary excitations for interacting magnets are integer spin states (spin flips), Faddev and Takhtajan [1] first conjectured that one-dimensional half odd spin chains exhibit excitations given by spinons carrying half-odd spin. For a chain with an even number of sites, the ground state is a disordered spin singlet but, if the number of sites is odd, the minimum possible value for the total spin is $1/2$. In the thermodynamic limit, it makes no difference whether one begins with an odd or an even number of sites. The minimum possible value for the total spin is $1/2$. In order to prove that it is a spin-$1/2$ state, we need to show that $S^-$ annihilates it. Indeed, per Eq.(13) we have:

$$\sum_{\beta \neq \alpha} S^-_\beta \Psi_\alpha = 0 \ ,$$

which proves that $\Psi_\alpha$ is the spin-$1/2$ component of a spin doublet.

Let the number of sites $N$ be odd and let

$$\Psi_\alpha (z_1, \ldots, z_M)$$

where $M = (N - 1)/2$. This is a $\downarrow$ spin on site $\alpha$ surrounded by an otherwise featureless singlet sea. It is worth stressing that Eq.(31) makes perfect sense for any $z_\alpha$ on the unit circle. Nevertheless, as $z_\alpha$ coincides with a lattice site, it represents a spin $\downarrow$ localized at the corresponding site. The spin density of the corresponding state, plotted as a function of the spinon position, will be uniformly zero, as appropriate for the disordered spin singlet, except for an abrupt dip centered at $z = z_\alpha$ (see Fig.(2)). Such a dip is what we refer to as “real space representation” of a spinon at $z_\alpha$. Hence, a spinon can be visualized as a local defect in an otherwise featureless singlet sea. This defect behaves like a real quantum mechanical particle, as we will show in the following.

![FIG. 2. Spin and charge profiles of the localized spinon $| \Psi_\alpha \rangle$ defined by Eq.(31). The dotted lines are a guide to the eye.](image)

A. One-Spinon Spin Doublet

We look for one-spinon and two-spinon wavefunctions in the functional form given by Eq.(28). Here we analyze the one-spinon wavefunction.
B. One-Spinon Energy

Eq. (31) corresponds to a particular choice of $\Phi$ in Eq. (28), given by:

$$\Phi(z_1, \ldots, z_M) = \Phi_\alpha(z_1, \ldots, z_M) = \prod_{j=1}^M (z_\alpha - z_j) \ . \quad (33)$$

Eq. (30), once written for the state $\Phi_\alpha$, takes the form:

$$\left\{ M(M-1) - z_\alpha^2 \frac{\partial^2}{\partial z_\alpha^2} - \frac{N - 3}{2} \left[ M - z_\alpha \frac{\partial}{\partial z_\alpha} \right] \right\} \Phi_\alpha = \lambda \Phi_\alpha \ . \quad (34)$$

The eigenstate of $H_{HS}$ is given by:

$$\Psi_m(z_1, \ldots, z_M) = \frac{1}{N} \sum_{\alpha=1}^N (z_\alpha^*)^m \Phi_\alpha(z_1, \ldots, z_M) \quad (35)$$

and the energy eigenvalue is

$$H_{HS}\vert \Psi_m \rangle = \left\{ -J\left( \frac{\pi^2}{24} \right)(N - \frac{1}{N}) \right. + \left. \frac{J}{2} \left( \frac{2\pi}{N} \right)^2 m(N - \frac{1}{2} - m) \right\} \vert \Psi_m \rangle \ , \quad (36)$$

with $0 \leq m \leq (N - 1)/2$ and $\lambda = m((N - 1)/2 - m)$.

C. Crystal Momentum

The state $\vert \Psi_m \rangle$ is a propagating $\downarrow$ spinon with crystal momentum

$$q = \frac{\pi}{2} N - \frac{\pi}{N} (m + \frac{1}{4}) \quad (\text{mod } 2\pi) \ , \quad (37)$$

per the definition

$$\Psi_m(z_1z, \ldots, z_Mz) = \exp(iq) \Psi_m(z_1, \ldots, z_M) \ , \quad (38)$$

where $z = \exp(2\pi i/N)$. Rewriting the eigenvalue as

$$H\vert \Psi_m \rangle = \left\{ - J\left( \frac{\pi^2}{24} \right)(N + \frac{5}{N} - \frac{3}{N^2}) + E_q \right\} \vert \Psi_m \rangle \ , \quad (39)$$

we obtain the dispersion relation

$$E(q) = \frac{J}{2} \left( \frac{\pi^2}{2} - q^2 \right) \quad (\text{mod } \pi) \ , \quad (40)$$

plotted in Fig. (3). Note that the momenta available to the spinon span only the inner or outer half of the Brillouin zone, depending on whether $N - 1$ is divisible by 4 or not. The loss of half of the states available for a regular fermion is a peculiar property of spinon spectrum. No negative energy states appear, i.e., there is nothing like an “antispinon”. One can picture a spinon as either an electron or a hole whose charge has been pulled out by the interaction. According to such a picture, a spinon can arise either from an electron with the same spin or from a hole with the opposite spin, which explains the halving of the Brillouin zone.

![Spinon Dispersion](image)

FIG. 3. Top: Spinon dispersion given by Eq. (40). Bottom: Allowed values of $q$ for adjacent odd $N$.

The spinon dispersion at low energies is linear in $q$ with a velocity

$$v_{\text{spinon}} = \frac{\pi}{2} J \quad , \quad (41)$$

The half-band of single elementary excitations for odd $N$ are the only $S = 1/2$ states without extra degeneracies. The ground state of the odd-N spin chain is 4-fold degenerate and is given by $\vert \Psi_m \rangle$ for $m = 0$ and $(N - 1)/2$ and their $\uparrow$ counterparts. This corresponds physically to a “left-over” spinon with momentum $\pm \pi$.

D. Spin-Current

We now study the action of $\Lambda^z$ on the state for one propagating spinon. Working as for the ground state one gets:

$$\Lambda^z\vert \Psi_m \rangle = \left\{ \frac{N - 1}{4} - m \right\} \vert \Psi_m \rangle \ , \quad (42)$$

and the eigenvalue of $\Lambda^z$ comes out to be proportional to the spinon velocity

$$\frac{dE(q)}{dq} = -\frac{2\pi J}{N} \left\{ \frac{N - 1}{4} - m \right\} \ . \quad (43)$$
Eq. (12) is proven by first letting $\Lambda^2$ act on the state $\Psi_{\alpha}$ defined in Eq. (11):

$$[\Lambda^2 \Psi_{\alpha}](z_1, \ldots, z_M) = \frac{1}{2} \sum_{j=1}^{M} \sum_{\beta \neq j}^{N} z_{\beta} \left( \frac{z_{j} + z_{\beta}}{z_{j} - z_{\beta}} \right) \sum_{l=0}^{N-1} \left( \frac{z_{j} - z_{\beta}}{l!} \right) \times \frac{\partial}{\partial z_{j}} \left( \frac{\Psi_{\alpha}(z_1, \ldots, z_M)}{z_{j}} \right)$$

$$= \frac{1}{2} \left\{ -M(N-2) + 2 \sum_{j=1}^{M} \sum_{i \neq j}^{M} \frac{z_{j}}{z_{j} - z_{i}} + 2 \sum_{j=1}^{M} \frac{z_{j}}{z_{j} - z_{\alpha}} \right\} \Psi_{\alpha}(z_1, \ldots, z_M) . \quad (44)$$

After taking $M = (N-1)/2$ one gets:

$$[\Lambda^2 \Psi_{\alpha}](z_1, \ldots, z_M) = \left\{ \frac{N-1}{4} - z_{\alpha} \frac{\partial}{\partial z_{\alpha}} \right\} \Psi_{\alpha}(z_1, \ldots, z_M) , \quad (45)$$

which, on the basis of the states $|\Psi_{m}\rangle$, gives the result quoted in Eq. (12).

**E. The Norm**

The squared norm of $\Psi_{m}$ is defined as:

$$\langle \Psi_{m} | \Psi_{m} \rangle = \sum_{z_1, \ldots, z_M} |\Psi_{m}(z_1, \ldots, z_M)|^2 . \quad (46)$$

By means of a simple algebraic procedure, we generated a recursion relation between $|\Psi_{m}| |\Psi_{m}\rangle$ and $|\Psi_{m+1}| |\Psi_{m+1}\rangle$ such a procedure can be straightforwardly extended to the norm of the multiple-spinon states. We discuss it at length in Appendix E. The induction relation is:

$$\frac{\langle \Psi_{m} | \Psi_{m} \rangle}{\langle \Psi_{m+1} | \Psi_{m+1} \rangle} = \frac{(m - \frac{1}{2})(M + m + 1)}{m(M - m + \frac{1}{2})} . \quad (47)$$

This recursively gives:

$$\langle \Psi_{m} | \Psi_{m} \rangle = \frac{\Gamma[M+1] \Gamma[m + \frac{1}{2}] \Gamma[M - m + \frac{1}{2}]}{\Gamma[M] \Gamma[\frac{1}{2}] \Gamma[M + \frac{1}{2}] \Gamma[M - m + 1]} C_{M} \quad (48)$$

where $C_{M}$ is the over-all constant we have introduced in Eq. (7).

**V. TWO-SPINON WAVEFUNCTION**

Let us now focus on the two-spinons state. Spinons maintain their integrity when many of them are present. This does not mean that spinons are noninteracting. They can be separated at large distances, therefore being asymptotic states of the system. However, they also scatter strongly off each other by means of a short-ranged attractive potential. The interaction between spinons is not enough to create two-spinon bound states, but it generates a peculiar “piling-up” of the relative wavefunction when the two spinons are on top of each other (Fig. (1)). This is the reason for the huge decay amplitude for a spin wave into a pair of spinons, that is the spin fractionalization.

In this section we derive the two-spinon eigenstates, their norm and the corresponding value of the spin-current. Moreover, we show that the appropriate statistics they obey is neither fermionic nor bosonic. They are semions, i.e., particles with 1/2 fractional statistics.

**A. Two-spinon Energy**

Two ↓-spinons can be pictured as two ↓-spins within an otherwise featureless disordered sea. The state with two ↓-spinons centered at $z_{\alpha}$ and $z_{\beta}$, respectively, is given by $(N$ is even and $M = N/2 - 1)$:

$$\Psi_{\alpha \beta}(z_1, \ldots, z_M) = \prod_{j=1}^{M} (z_{\alpha} - z_{j}) (z_{\beta} - z_{j}) \prod_{j<k}^{M} (z_{j} - z_{k})^2 \prod_{j=1}^{M} z_{j} . \quad (49)$$

As for the one-spinon case, $z_{\alpha}$ and $z_{\beta}$ are not necessarily lattice sites. If they are, Eq. (13) represents a pair of spinons at $z_{\alpha}$ and $z_{\beta}$. To derive the eigenvalue equation, we start from a wavefunction in the form of Eq. (28), where we take the function $\Phi$ to be equal to:

$$\Phi_{\alpha \beta} = \prod_{j=1}^{M} (z_{\alpha} - z_{j}) (z_{\beta} - z_{j}) \quad (50)$$

Eq. (30) can be rewritten for $\Phi_{\alpha \beta}$, yielding:

$$\frac{1}{2} \left\{ \sum_{j=1}^{M} z_{j}^2 \frac{\partial^2 \Phi_{\alpha \beta}}{\partial z_{j}^2} + \sum_{j \neq k}^{M} \frac{4z_{j}^2}{z_{j} - z_{k}} \frac{\partial \Phi_{\alpha \beta}}{\partial z_{j}} \right\} - \frac{N - 3}{2} \sum_{j=1}^{M} z_{j} \frac{\partial \Phi_{\alpha \beta}}{\partial z_{j}}$$

$$= \left\{ \frac{2}{z_{\alpha} - z_{\beta}} \frac{\partial}{\partial z_{\alpha}} - \frac{2}{z_{\beta} - z_{\alpha}} \frac{\partial}{\partial z_{\beta}} - \frac{z_{\alpha}^2}{z_{\alpha}^2} - \frac{z_{\beta}^2}{z_{\beta}^2} \right\} \Phi_{\alpha \beta}$$

$$+ \left\{ \frac{N - 3}{2} \left[ z_{\alpha} \frac{\partial}{\partial z_{\alpha}} + z_{\beta} \frac{\partial}{\partial z_{\beta}} \right] + \left[ 2M^2 - M(N - 2) \right] \right\} \Phi_{\alpha \beta}$$
and, in deriving Eq.(53) we used the identity

\[ (z_\beta^n) = \lambda \Phi_{\alpha \beta} . \]  \hspace{1cm} (51)

Let us now define the states \( \Psi_{mn} \) as follows:

\[ \Psi_{mn}(z_1, \ldots, z_M) = \sum_{\alpha, \beta} \frac{N}{N} (z_\beta^m) \frac{N}{N} (z_\beta^n) \Psi_{\alpha \beta}(z_1, \ldots, z_M) \]  \hspace{1cm} (52)

A set of linearly independent states may be constructed by taking only the \( \Psi_{mn} \) with \( M \geq m \geq n \geq 0 \), which shows the overcompleteness of the set of states \( \Psi_{mn} \). On such a set of states Eq.(51) becomes:

\[
\frac{1}{2} \left\{ \sum_{j=1}^{M} z_j^2 \frac{\partial^2}{\partial z_j^2} + \sum_{j \neq k} \frac{4z_j^2 - z_j - z_k}{z_j - z_k} \frac{\partial}{\partial z_j} -(N-3) \sum_{j=1}^{M} z_j \frac{\partial}{\partial z_j} \right\} \Psi_{mn} \\
= \left\{ \frac{N^2}{48} (N - 19 \frac{N}{N^2}) + m(\frac{N}{2} - 1 - m) + n(\frac{N}{2} - 1 - n) + \frac{m - n}{2} \right\} \Psi_{mn} - \sum_{\ell=0}^{\ell_M} (m - n + 2\ell) \Psi_{m+n, \ell, \ell} , \hspace{1cm} (53)
\]

where \( \ell_M = n \) if \( m + n < M \), \( \ell_M = M - m \) if \( m + n \geq M \) and, in deriving Eq.(53) we used the identity

\[
\frac{x + y}{x - y} (x^m y^n - x^n y^m) = 2 \sum_{\ell=0}^{m-n} x^{m-\ell} y^{n+\ell} - (x^m y^n + x^n y^m) . \]  \hspace{1cm} (54)

We look for solutions to Eq.(53) which are linear combinations of the states \( \Psi_{m+n, \ell, \ell} \):

\[
\Phi_{mn} = \sum_{\ell=0}^{\ell_M} a_{\ell}^{mn} \Psi_{m+n, \ell, \ell} . \]  \hspace{1cm} (55)

The coefficients \( a_\ell \) are found to be: basis of the states \( |\Psi_m\rangle \)

\[
a_{\ell}^{mn} = \frac{-(m - n + 2\ell)}{2(\ell + m + n + 1/2)} \sum_{k=1}^{\ell} a_{k}^{mn} ; \hspace{0.5cm} (a_0 = 1) \]  \hspace{1cm} (56)

and the corresponding two-spinon energies are given by:

\[
E_{mn} = -J \frac{\pi^2}{24} (N - 19 \frac{N}{N^2}) + \frac{J}{2} \frac{2\pi}{N^2} \left[ m(\frac{N}{2} - 1 - m) + n(\frac{N}{2} - 1 - n) - \frac{m - n}{2} \right] . \]  \hspace{1cm} (57)

In terms of spinon momenta, the expression of the energy is:

\[
E_{mn} = -J \frac{\pi^2}{24} (N + \frac{5}{N}) + |E(q_m) + E(q_n) - \pi J [q_m - q_n]| \hspace{0.5cm} (q_m \leq q_n) . \]  \hspace{1cm} (58)

\( E_{mn} \) is the sum of the ground-state contribution, \( E_{GS} = -J(\pi^2/24)(N + 5/N) \), and \( E(q_m, q_n) \), which is the two-spinon energy above the ground state. \( E(q_m, q_n) \) is the sum of the energies of two isolated spinons plus a negative interaction contribution that becomes negligibly small in the thermodynamic limit.

Such a simple solution for the two-spinon problem is possible because the matrix to which Eq.(53) corresponds is lower triangular, i.e. takes the form

\[
\text{Matrix} = \begin{bmatrix}
E_0 & 0 & 0 & 0 & \cdots \\
v_{10} & E_1 & 0 & 0 & \cdots \\
v_{20} & v_{21} & E_2 & 0 & \cdots \\
v_{30} & v_{31} & v_{32} & E_3 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix} . \]  \hspace{1cm} (59)

where:

\[ E_j = \lambda_{m+j, n-j} \]  \hspace{1cm} and  \hspace{1cm} \[ v_{pq} = -(m - n + 2p + 2q) \]

and:

\[
\lambda_{mn} = m(\frac{N}{2} - 1 - m) + n(\frac{N}{2} - 1 - n) - \frac{m - n}{2} . \]  \hspace{1cm} (60)

The eigenvalues of such a matrix are its diagonal elements, and the corresponding eigenvectors are generated by recursion.

The transformation in Eq.(55) can be inverted and it takes the form:

\[
\Psi_{mn} = \sum_{\ell=0}^{\ell_M} b_{\ell}^{mn} \Phi_{m+n, \ell, \ell} . \]  \hspace{1cm} (61)

The coefficients \( b_{\ell}^{mn} \) can be expressed in a closed-form formula in terms of the coefficients of the two-spinon wavefunctions. We provide their expression in Section VI.

**B. The Norm**

The squared norm of the state \( \Phi_{mn} \) is defined as:

\[
\langle \Psi_{mn} | \Phi_{mn} \rangle = \sum_{z_1, \ldots, z_M} |\Phi_{mn}(z_1, \ldots, z_M)|^2 . \]  \hspace{1cm} (62)
As for the one-spinon wavefunction, we calculate the norm of the two-spinon states by means of mathematical induction. The details of our calculation are discussed in Appendix E. The basic induction relations are given by:

\[
\frac{\langle \Phi_{mn} | \Phi_{mn} \rangle}{\langle \Phi_{m,n-1} | \Phi_{m,n-1} \rangle} = \frac{(n - \frac{1}{2})(M - n + \frac{3}{2})(m - n + 1)^2}{n(M - n + 1)(m - n + \frac{3}{2})(m - n + \frac{1}{2})} \tag{63}
\]

From Eq. (63) one gets the formula for the squared norm:

\[
\langle \Phi_{mn} | \Phi_{mn} \rangle = C_M \frac{M + \frac{1}{2}}{\pi} \frac{\Gamma[m - n + \frac{1}{2}]\Gamma[m - n + \frac{3}{2}]}{\Gamma^2[m - n + 1]} \times \frac{\Gamma[m + 1]\Gamma[M - m + \frac{3}{2}]\Gamma[n + \frac{1}{2}]\Gamma[M - n + 1]}{\Gamma^2[M - m + 1]\Gamma[n + 1]\Gamma[M - n + \frac{1}{2}]} \tag{64}
\]

Eq. (64) basically agrees with the result quoted in [14], although we derived it by making direct use of the operator \(\mathcal{H}_{HS}\) (See Appendix E).

C. Spin Current

The \(\Psi_{mn}\) are eigenstates of \(\Lambda^x\). Indeed, a manipulation similar to the one-spinon case yields:

\[
\Lambda^x |\Psi_{mn}\rangle = \left\{ \frac{N - 2}{2} - m - n \right\} |\Psi_{mn}\rangle \tag{66}
\]

with the eigenvalue given by the sum of the two spinon velocities. We will just skip the proof of Eq. (66) which works exactly like the proof of Eq. (62).

D. Spinon Statistics

Spinons are semions, i.e. particles obeying 1/2 fractional statistics. Since the 2-spinon wavefunction \(\Psi_{\alpha\beta}\) has the property

\[
\Psi_{\alpha\beta}^*(z_1, ..., z_{N/2-1}) = (z_{\alpha} z_{\beta})^{1-N/2} \Psi_{\alpha\beta}(z_1, ..., z_{N/2-1}) \tag{72}
\]

the Berry phase vector potential for adiabatic motion of spinon \(\alpha\) in the presence of \(\beta\) is

\[
\frac{1}{2} \left[ \langle \psi{\alpha\beta} | \frac{\partial}{\partial z_{\alpha}} \psi{\alpha\beta} \rangle + \langle z_{\alpha} \frac{\partial}{\partial z_{\alpha}} \psi{\alpha\beta} | \psi{\alpha\beta} \rangle \right] = \frac{1}{2} (1 - \frac{N}{2}) . \tag{67}
\]

The phase to “exchange” the spinons by moving \(\alpha\) all the way around the loop is thus

\[
\Delta \phi = \oint \frac{1}{2} (1 - \frac{N}{2}) dz_{\alpha} = \pm \frac{\pi}{2} i \quad (\text{mod } 2\pi) . \tag{68}
\]

This number is 0 or \(\pi\) for bosons or fermions. The number of states available to \(\ell \downarrow\) spinons, determined by counting the number of distinct symmetric polynomials of the form

\[
\Phi_{z_1, ..., z_{N/\ell}} (z_1, ..., z_{(N-\ell)/2})
\]

is

\[
N^\text{fermi}_\ell = \binom{N/2 + \ell/2}{\ell} \tag{70}
\]

This is just halfway between the numbers

\[
N^\text{fermi}_\ell = \binom{N/2}{\ell} \quad N^\text{bose}_\ell = \binom{N/2 + \ell}{\ell} \tag{71}
\]

likewise calculated assuming that the number of states available for one particle is \(N/2\).

VI. SCATTERING RESONANCE

In this Section we provide one of the key results of our study: the analysis of the interaction between two spinons. First, we properly define the real space representation for the two-spinon relative wavefunction. Then, we study the behavior of the corresponding amplitude as a function of the spinon separation. Here we construct the real space wavefunction for a spinon pair and we show that our results provide a clear evidence for spinons being interacting particles.

The real space representation for the two-spinon wavefunctions corresponding to the energy eigenstate \(|\Phi_{mn}\rangle\),

\[
z_{\alpha}^m z_{\beta}^n p_{mn} (z_{\alpha}/z_{\beta}),
\]

is defined by the decomposition of the state of two localized spinons at \(z_{\alpha}\) and \(z_{\beta}\), \(|\Psi_{\alpha\beta}\rangle\), in the basis of the fully polarized two-spinon eigenstates:

\[
\Psi_{\alpha\beta} = \sum_{m=0}^{M} \sum_{n=0}^{m} (-1)^{m+n} z_{\alpha}^m z_{\beta}^n p_{mn} (z_{\alpha}/z_{\beta}) |\Phi_{mn}\rangle \tag{72}
\]

\(|\Phi_{mn}\rangle\) is an eigenstate of \(\mathcal{H}_{HS}\) with eigenvalue \(E_{mn}\). This implies
\[ \langle \Phi_{mn} | \mathcal{H}_{HS} | \Psi_{\alpha\beta} \rangle = E_{mn} \langle \Phi_{mn} | \Psi_{\alpha\beta} \rangle \quad . \tag{73} \]

From Eq.(51) we see that \( \langle \Phi_{mn} | \mathcal{H}_{HS} | \Psi_{\alpha\beta} \rangle \) can be written as a differential operator acting on \( \langle \Phi_{mn} | \Psi_{\alpha\beta} \rangle \). \( \Psi_{\alpha\beta} \) is perfectly defined for any \( z_\alpha, z_\beta \) on the unit circle, so, the differential operator acts on the analytic extension of \( \langle \Phi_{mn} | \Psi_{\alpha\beta} \rangle \) as

\[ \langle \Phi_{mn} | \mathcal{H}_{HS} | \Psi_{\alpha\beta} \rangle = E_{GS} \langle \Phi_{mn} | \Psi_{\alpha\beta} \rangle + \]

\[ \frac{J}{2} \left( \frac{2\pi}{N} \right)^2 \left\{ (M - z_\alpha \frac{\partial}{\partial z_\alpha}) z_\alpha \frac{\partial}{\partial z_\alpha} + (M - z_\beta \frac{\partial}{\partial z_\beta}) z_\beta \frac{\partial}{\partial z_\beta} \right\} \langle \Phi_{mn} | \Psi_{\alpha\beta} \rangle . \tag{74} \]

In the differential operator in Eq.(74) we recognize the sum of the energies of the two free spinons and a velocity-dependent interaction, which diverges at small spinon separations. Eq.(77) allows for determination of the exact expression of \( p_{mn}(z_\alpha/z_\beta) \). Indeed, by using Eqs.(72,74), we find the following equation for \( p_{mn}(z) \) (\( z = z_\beta/z_\alpha \)):

\[ z(1 - z) \frac{d^2 p_{mn}}{dz^2} + \left[ \frac{1}{2} - m + n \right] \]

\[ - \left( -m + n + \frac{3}{2} \right) \frac{dp_{mn}}{dz} + \frac{m - n}{2} p_{mn} = 0 . \tag{75} \]

This equation is a special case of the hypergeometric equation \( 15 \) where the parameters \( c, b, a \) are given by

\[ c = \frac{1}{2} - m + n \quad b = \frac{1}{2} \quad a = -m + n . \]

The solution is a hypergeometric series whose regular solution stops at a power of \( z \) given by \( z^{m-n} \), thus becoming the “hypergeometric polynomial”

\[ p_{mn}(z) = \frac{\Gamma[m - n + 1]}{\Gamma[\frac{1}{2}] \Gamma[m - n + \frac{1}{2}]} \]

\[ \times \sum_{k=0}^{m-n} \frac{\Gamma[k + \frac{1}{2}] \Gamma[m - n - k + \frac{1}{2}]}{\Gamma[k + 1] \Gamma[m - n - k + 1]} z^k . \tag{76} \]

The value of the spinon wavefunction at zero separation between spinons can be computed by means of general identities among hypergeometric series \( 15 \). It is given by:

\[ p_{mn}(1) = \Gamma[1/2] \Gamma[m - n + 1] / \Gamma[m - n + 1/2] . \tag{77} \]

According to Eq.(76), \( |p_{mn}(z)|^2 \) is the density of probability for two spinons as a function of the distance between them.

![FIG. 4. Left panel: square modulus of the two-spinon wavefunction as a function of the separation between the two spinons for \( N = 1000 \). Right panel: the same plot on a log-log scale. The dashed line is a guide to the eye. It is a plot of \( 1/x \) with an appropriate offset (\( z = \exp(\theta) \)).](image)

The interpretation of our results is straightforward. Spinons do actually behave like real particles. Indeed, we have been able to determine a differential equation for \( p_{mn} \), which for the two-spinon wavefunction is the same as the Schrödinger equation for a pair of ordinary particles. The interaction between spinons is clearly shown in Fig.(4), where we plot \( |p_{mn}(z)|^2 \). At large separations, the probability density oscillates and averages to 1, independently on the distance between the spinons. This is a typical feature of noninteracting particles. Indeed, at large separations spinons are in fact noninteracting. However, at small separations, \( |p_{mn}|^2 \) shows a resonant enhancement, which corresponds to a huge increase of the probability of configurations with the two spinons on top one of each other. The enhancement is highest at relative spinon momentum of \( \pi \). Such features are clear evidence of the interaction among spinons, which can be characterized as follows:

1. It is attractive: it favors configurations with spinons on top one of each other;

2. It is short ranged: spinons are free at large distances.

As \( N \) gets larger, the resonant enhancement peaks up \( 14 \). Hence, the resonant enhancement safely survives the thermodynamic limit, even though, in this limit, the energy for the two-spinon solution is the sum of the energies of the two isolated spinons. However, the attraction is not strong enough to create a two spinon bound state, even in the thermodynamic limit. This corresponds to the absence of a low-energy stable spin-one excitation, which is a typical feature of 1D spin-1/2 antiferromagnets.

The attractive force may also be inferred from the energy eigenvalue if we rewrite it as

\[ \mathcal{H}_{HS} |\Phi_{mn}\rangle \]

\[ = \left\{ -J \frac{z^2}{24} (N + \frac{5}{N}) + E_{pq} + E_{q_0} + V_{q_0 - q_0} \right\} |\Phi_{mn}\rangle = \]

11
\[ \left\{ -J \frac{\pi^2}{24} (N + \frac{5}{N}) + E(q_m, q_n) \right\} |\Phi_{mn}\rangle , \]  

(78)

where

\[ V_q = -J \frac{\pi}{N} |q| \]  

(79)

Note that this potential vanishes as \( N \to \infty \), as expected for particles that interact only when they are close together. However, as we already pointed out, the vanishing of the interaction potential in the thermodynamic limit does not mean that no residual effects survive such a limit. The resonant enhancement when the two spinons are on the same site does survive the thermodynamic limit and it is the main reason of the instability of the spin-1 spin-wave, as we discuss at length in the next section.

Before concluding this Section, we provide the expression of the coefficients \( b^\ell_{\ell^\prime} \) in Eq. (61). From Eq. (76) it is straightforward to prove that:

\[ b^\ell_{\ell^\prime} = \frac{\Gamma[m - n + 2\ell + 1]}{\Gamma[\frac{1}{2} + m - n + 2\ell + \frac{1}{2}]} \times \frac{\Gamma[\ell + \frac{1}{2} \Gamma[m - n + \ell + \frac{1}{2}]}{\Gamma[\ell + 1] \Gamma[m - n + \ell + 1]} . \]  

(80)

### VII. SPIN SUSCEPTIBILITY

In this Section, we work out the dynamical spin susceptibility for the HSM. We show that the DSS depends only on the \( p_{mn} \)'s calculated at \( z = 1 \), which allows us to obtain for any finite \( N \) a simple closed-form expression for the DSS and to relate it to the spinon interaction. By carefully taking the thermodynamic limit of our result, we obtain Haldane-Zirnbauer formula for the DSS in the thermodynamic limit [11]. Haldane-Zirnbauer formula shows that there is no low-energy spin-1 pole in the DSS, but the function takes a sharp square-root singularity at the two-spinon threshold on top of a branch cut, corresponding to the lack of integrity of the spin-one excitation. Our analysis definitely proves that the square root sharp edge on top of the broad spectrum is nothing but the interaction between spinons. The resonant enhancement is the square root singularity in the spin-susceptibility. This result is of the utmost importance, since it represents a way to experimentally test interaction among spinons in 1D. We will come back to such a point in the concluding remarks.

Let us begin with the calculation of the spin susceptibility for a finite lattice. The DSS is the dynamical propagator for a spin-1 spin flip. A spin flip with momentum \( q \) is created by acting on \( |\Psi_{GS}\rangle \) with \( S^-_q \), defined as:

\[ S^-_q = \sum_\alpha (z^*_\alpha)_k (S^z_\alpha - iS^y_\alpha) \quad (q = 2\pi k/N) . \]  

(81)

A peculiar property of the HSM is that a spin flip at \( z_\alpha \) is the same as a spinon pair at the same site [11]. Therefore, we can fully decompose \( S^-_q \) \( |\Psi_{GS}\rangle \) in the basis of the two-spinon eigenstates:

\[ S^-_q |\Psi_{GS}\rangle = \sum_\alpha (z^*_\alpha)_k |\Psi_\alpha\rangle \]  

(82)

The susceptibility is given by

\[ \chi_q(\omega) = \sum_X \frac{|\langle X|S^-_q|\Psi_{GS}\rangle|^2}{\langle X|X\rangle \langle \Psi_{GS}|\Psi_{GS}\rangle} \times \frac{2(E_X - E_{GS})}{(\omega + i\eta)^2 - (E_X - E_{GS})^2} , \]  

(83)

( \( |X\rangle \) is an exact eigenstate of \( H_{HS} \) with energy \( E_X \)). Then, from Eqs. (82,83) we have that \( \chi_q(\omega) \) takes a nonzero contribution only if \( |X\rangle = |\Phi_{mn}\rangle \). Then, Eq. (83) becomes:

\[ \chi_q(\omega) = N^2 \sum_{m=0}^M \sum_{n=0}^m \frac{\langle \Phi_{mn}|\Phi_{mn}\rangle}{\langle \Psi_{GS}|\Psi_{GS}\rangle} p_{mn}(1) \times \delta(m + n - k) \frac{2(E_{mn} - E_{GS})}{(\omega + i\eta)^2 - (E_{mn} - E_{GS})^2} . \]  

(84)

Eq. (84) is another relevant result of our work. It shows that only the \( p_{mn}(z) \)'s at \( z = 1 \) determine \( \chi_q(\omega) \). Therefore, the spin susceptibility is completely determined by spinon interaction.

Let us analyze the thermodynamic limit of eq. (84). In the thermodynamic limit the gamma functions can be approximated by using Stirling’s formula:

\[ \Gamma[z] \approx \sqrt{\pi}(z - 1)^{(z - \frac{1}{2})}e^{-(z - 1)} \]  

(85)

From Eqs. (77,83) we get, in the thermodynamic limit:

\[ N^2 \frac{\Phi_{mn}|\Phi_{mn}\rangle}{\langle \Psi_{GS}|\Psi_{GS}\rangle} p_{mn}(1) = \frac{\pi N(m - n + \frac{1}{2})}{\sqrt{n(M - n)(m + \frac{1}{2})(M - m - \frac{1}{2})}} . \]  

(86)

Since the joint two-spinon density of states is flat, the sums over \( m \) and \( n \) become integrals over the (halved) one-spinon Brillouin zone:
\[
\sum_{m} \to -M \int_{-\pi}^{\pi} dq \quad \text{(87)}
\]

From Eq.(82), we see that Eq.(81) turns, in the thermodynamic limit, into the Haldane-Zirnbauer formula for the DSS [13]

\[
\chi_{q}(\omega) = \frac{J}{2} \int_{-\pi}^{\pi} dq_{1} \int_{-\pi}^{\pi} dq_{2} \left[ \frac{|q_{1} - q_{2}| \delta(q_{1} + q_{2} - q)}{E(q_{1})E(q_{2})} \right] \times \frac{2E(q_{1}, q_{2})}{(\omega + q)^2 - E^2(q_{1}, q_{2})}, \quad \text{(88)}
\]

where \(E(q)\) and \(E(q_{1}, q_{2})\) are the one-spinon and the two-spinon energies, respectively. Integration over \(q_{1}, q_{2}\) in Eq.(82) provides:

\[
\chi_{q}(\omega) = \frac{J}{4} \Theta[\omega_{1}(q) - \omega] \frac{\Theta[\omega - \omega_{-1}(g)]}{\sqrt{\omega - \omega_{-1}(g)}} \times \frac{\Theta[\omega_{2}(g) - \omega] \Theta[\omega - \omega_{-1}(g)]}{\sqrt{\omega - \omega_{-1}(g)}} \quad \text{(89)}
\]

\(\omega_{-1}(q)\) and \(\omega_{+1}(q)\) are the threshold energies for a spinon pair with momentum \(q\), according to whether \(0 \leq q \leq \pi\) or \(\pi \leq q \leq 2\pi\), respectively. They are given by \(\omega_{-1}(q) = (J/2)q(\pi - q), \omega_{+1}(q) = (J/2)(2\pi - q)(q - \pi)\). \(\omega_{2}(q) = (J/2)q(2\pi - q)\) is the upper threshold for the spin-1 excitation. From Eq.(82), we see that the resonant enhancement, given by \(P_{mn}^2(1)\), has turned into a square-root singularity in \(\chi_{q}(\omega)\) vs. \(\omega\) at fixed \(q\), with the branch cut originating either at \(\omega_{-1}(q)\) or at \(\omega_{+1}(q)\), depending on the value of \(q\). Because the two-spinon joint density of states is uniform, the main conclusion we trace out from our calculation is that the branch cut in \(\chi_{q}(\omega)\), i.e., the broadness of the spectral density of states, is the spinon interaction.

A measurement of \(\chi_{q}(\omega)\) in one-dimensional spin-1/2 antiferromagnets can be performed by means of neutron scattering experiments [14]. The result of the measurements [14] does, in fact, show a sharp threshold followed by a broad spectrum, in good agreement with predictions of Eq.(89). In light of our present discussion, we conclude that what is actually seen in such an experiment is a direct consequence of the spinon interaction in 1-dimensional antiferromagnets. Hence, the experiments provide evidence that spinons do interact and that the spinon interaction is what determines the peculiar low-energy physics of spin-1/2 antiferromagnetic chains.

From Eq.(82) we also derive the formula for the two-spinon wavefunction \(\Psi_{GS}\) in terms of the two-spinon excitations \(S_{1}^{\pm}, S_{-}^{\pm}\) at \(z = 1\):

\[
\chi_{z_{a}}(\omega) = \frac{\langle \Psi_{GS}|S_{1}^{+}S_{-}^{-}|\Psi_{GS}\rangle}{\langle \Psi_{GS} | \Psi_{GS} \rangle} \quad \text{(90)}
\]

Eq.(10) is the formula we have plotted in Fig.1.

VIII. CONCLUSIONS

In this paper we developed a simple approach to the study of spinon excitations of the Haldane-Shastry model, based on the formalism of the analytic variables. Within out approach we picture spinons as local defects in the disordered sea. Our formalism allows for a consistent real-space representation of the wavefunction for two spinons. We construct the Schrödinger equation, whose solution is the two-spinon wavefunction, which shows that spinons behave as real quantum-mechanical particles. By means of a careful study of the real-space two-spinon wavefunction, we reveal the main result: the existence of spinon interaction and its survival in the thermodynamic limit. Spinon interaction is a short-range attraction, which generates a resonant enhancement of the probability for two spinons to be at the same site. Such an interaction determines the low-energy physics of 1D interacting antiferromagnets. Since the low-energy joint density of states is uniform, the broadness in the spectral density, is exclusively caused by the resonant enhancement, as we show from the finite-\(N\) expression for the spin susceptibility (Section VII). In the thermodynamic limit the resonant enhancement develops a square root singularity followed by a branch cut, which is the broadness in the spectral density of states. The branch cut reflects the absolute instability of the spin wave towards decay into a spinon pair. Then, we show that, even though in the thermodynamic limit the interaction is irrelevant, its main effect, the resonant enhancement, peaks up.

In conclusion, we analyzed spinon interaction in an exact solution of the Haldane-Shastry model and its consequences for the low-energy physics of 1-D spin-1/2 antiferromagnets.

ACKNOWLEDGMENTS

This work was supported primarily by the National Science Foundation under grant No. DMR-9813899. Additional support was provided by the U.S. Department of Energy under contract No. DE-AC03-76SF00515 and by the Bing Foundation.
In this Appendix we will prove some of the formulas we used throughout the paper. Since the lattice sites \( z_\alpha \) are roots of unity we have
\[
\prod_{\alpha}^{N} (z - z_\alpha) = z^N - 1 .
\] (A1)

Then for \( 0 \leq m \leq N \) we have
\[
\sum_{\alpha=1}^{N-1} \frac{z_m^m}{z_\alpha - 1} = \frac{N}{2\pi i} \oint_{C} \frac{z^{m-1} dz}{(z - 1)(z^N - 1)} = \frac{N}{2\pi i} \oint_{C'} \frac{z^{m-1} dz}{(z - 1)(z^N - 1)}
\]
\[
= -\frac{N}{2\pi i} \int \left\{ \frac{1 + \frac{m-1}{1} x + \frac{m}{2} x^2 + ...}{\left( \frac{N}{1} \right) + \left( \frac{N}{2} \right) x + \left( \frac{N}{3} \right) x^2 + ...} \right\} dx
\]
\[
= \frac{N + 1}{2} - m ,
\] (A2)

and
\[
\sum_{\alpha=1}^{N-1} \frac{z_m^m}{|z_\alpha - 1|^2} = -\sum_{\alpha=1}^{N-1} \frac{z_m^{m+1}}{(z_\alpha - 1)^2}
\]
\[
= -\frac{N}{2\pi i} \oint_{C} \frac{z^{m} dz}{(z - 1)^2(z^N - 1)} = \frac{N}{2\pi i} \oint_{C'} \frac{z^{m} dz}{(z - 1)^2(z^N - 1)}
\]
\[
= \frac{1}{2\pi i} \int \left\{ \frac{1 + \frac{m-1}{1} x + \frac{m}{2} x^2 + ...}{\left( \frac{N}{1} \right) + \left( \frac{N}{2} \right) x + \left( \frac{N}{3} \right) x^2 + ...} \right\} dx
\]
\[
= \frac{N^2 - 1}{12} - \frac{m(N-1)}{2} + \frac{m(m-1)}{2} .
\] (A3)

In this Appendix we work out the coefficients \( A_l \) that appear in the eigenvalue equations for the eigenfunctions of the HSM are defined as:
\[
A_l = -\sum_{\alpha=1}^{N-1} \frac{z_\alpha^2}{(z_\alpha - 1)^2} ,
\] (B1)

They can be computed by using the equations from Appendix A. In particular we have:
\[
A_0 = \sum_{\alpha=1}^{N-1} \frac{z_\alpha^2}{|z_\alpha - 1|^2} = \frac{(N-1)(N-5)}{12} ,
\]
\[
A_1 = -\sum_{\alpha=1}^{N-1} \frac{z_\alpha^2}{z_\alpha - 1} = -\frac{N-3}{2} ,
\]
\[
A_2 = -\sum_{\alpha=1}^{N-1} z_\alpha^2 = 1 ,
\]
\[
A_l = -\sum_{\alpha=1}^{N-1} z_\alpha^2(z_\alpha - 1)^{l-2} = 0 \ (l > 2) .
\] (B2)

In this Appendix we provide the physical interpretation of the operator \( \Lambda \), as the spin-current operator. In order to do so, we first construct the continuous interpolation of the lattice spin field, given by the spin density operator \( \bar{\rho}(z) \). Then, we define a current density on the unit circle, \( \bar{j}(z) \). We prove that \( \bar{\rho} \) and \( \bar{j} \) obey an equation which, once restricted to the lattice, takes the form of the continuity equation for the spin density. The operator \( \Lambda \) comes out to be the global operator whose density is given by \( \bar{j}(z) \).

The first step of such a construction is defining the field interpolating the spin operators into the interstices by means of the formula
\[
\bar{\sigma}(z) = \left[ \frac{z^N - z^{-N}}{2N} \right] \sum_{\beta} \left( \frac{z + z_\beta}{z - z_\beta} \right) \bar{\sigma}_\beta .
\] (C1)

Then we can associate to \( \bar{\sigma}(z) \) a “σ model-like” Hamiltonian given by:
\[
\frac{1}{2\pi i} \int \left [ \frac{d\sigma}{dz} \right ] \cdot \left [ \frac{d\bar{\sigma}}{dz} \right ] \frac{dz}{z} = -2 \sum_{\alpha \neq \beta} \frac{\vec{S}_\alpha \cdot \vec{S}_\beta}{|z_\alpha - z_\beta|^2}
\]
+ \frac{3}{8} (N - 1) + \frac{S^2}{8} \tag{C2}
\]

where the integral is performed over the unit circle. Eq. (22) gives the HS Hamiltonian plus an irrelevant constant and an operator which commutes with it.

We also have spin density and spin current density operators
\[
\rho(\vec{z}) = -i\vec{\sigma}(\vec{z}) \times \bar{\vec{\sigma}}(\vec{z})
\]

\[
\vec{j}(\vec{z}) = \frac{1}{2i} \left \{ \vec{\sigma} \times \left [ \frac{d\vec{\sigma}}{dz} \right ] - \left [ \frac{d\bar{\vec{\sigma}}}{dz} \right ] \times \bar{\vec{\sigma}} \right \} , \tag{C3}
\]

That \(\rho(\vec{z})\) is an appropriate definition of the spin density may be seen by taking the limit \(z \to z_\alpha\), being \(z_\alpha\) a site on the lattice. One gets
\[
\lim_{z \to z_\alpha} \rho(\vec{z}) = \vec{S}_\alpha . \tag{C4}
\]

That \(\vec{j}(\vec{z})\) is a proper spin current can be inferred from the continuity equation
\[
\lim_{z \to z_\alpha} \left \{ \frac{d\vec{j}}{dz} \left [ \frac{1}{z} \right ] \right \} = 0 . \tag{C5}
\]

The zero-momentum component of this conserved current density is
\[
\frac{1}{2\pi i} \int \vec{j} \frac{dz}{z} = \vec{\Lambda} . \tag{C6}
\]

The operator \(\vec{\Lambda}\) is then a scaled spin current. Its action on the state with a fixed number of propagating spinons, Eqs. (27, 42, 66), is definitely consistent with such an interpretation.

**APPENDIX D: FACTORIZABILITY OF \(\mathcal{H}_{HS}\)**

In this Appendix we will prove the factorization formula, Eq. (22). In order to do so, we split the proof in two steps. First, we will show that the operator \(\vec{D}_\alpha\) annihilates \(|\Psi_{GS}\rangle\), then, we will prove the factorization equation. Let us begin with the first proof. The operator
\[
\Omega_\alpha = \sum_{\beta \neq \alpha} \frac{z_\alpha}{z_\alpha - z_\beta} \left [ \frac{1}{2} (S_\alpha^z + S_\beta^z) - (S_\alpha^z + 1) (S_\beta^z + 1) \right ]
\]
\[
- \frac{N - 1}{2} (S_\alpha^z + 1) \]  
\[
\text{for all } \alpha . \tag{D2}
\]

Since \(|\Psi_{GS}\rangle\) is also its own time-reverse it must be destroyed by the time-reverse of the vector operator, i.e.
\[
\sum_{\beta \neq \alpha} \frac{z_\alpha}{z_\alpha - z_\beta} \left [ \frac{1}{2} (S_\alpha^z + S_\beta^z) + (S_\alpha^z + 1) (S_\beta^z + 1) \right ]
\]
\[
\sum_{\beta \neq \alpha} \frac{z_\alpha}{z_\alpha - z_\beta} \left [ \frac{1}{2} (S_\alpha^z + S_\beta^z) + S_\beta^z \right ] . \tag{C3}
\]

The difference of these is the trivial operator \(\vec{S}_\alpha \times \vec{S}_\beta\), and their sum is \(2\vec{D}_\alpha\).

We prove, now, the factorizability of \(\mathcal{H}_{HS}\). In order to do so, we need the following identities:
\[
\sum_{\alpha} \left | i(\vec{S} \times \vec{S}_\alpha) + \vec{S} \cdot \vec{D}_\alpha \right |
\]
\[
= \sum_{\alpha} \left [ i \vec{S} \cdot (\vec{S}_\alpha \times \vec{D}_\alpha) + \vec{S} \cdot \vec{D}_\alpha \right ] = \frac{3}{2} \vec{S} \cdot \vec{\Lambda} , \tag{D5}
\]

\[
\sum_{\alpha} \left [ i(\vec{S} \times \vec{S}_\alpha) + \vec{S} \cdot [i(\vec{S}_\alpha \times \vec{S}) + \vec{S}] \right ]
\]
\[
= \frac{3}{2} N (N - 1) S^2 \ . \tag{D6}
\]
\[
\sum_{\beta \neq \gamma \neq \alpha}^{N} \frac{\vec{S}_{\beta} \cdot \vec{S}_{\gamma}}{(z_{\alpha} - z_{\gamma})(z_{\alpha} - z_{\beta})} = -\frac{1}{2}S^2 + \frac{3}{8}N + \frac{3}{8} \sum_{\alpha \neq \beta}^{N} \frac{\vec{S}_{\alpha} \cdot \vec{S}_{\beta}}{|z_{\alpha} - z_{\beta}|^2} \cdot \tag{D7}
\]
\[
i \sum_{\alpha \neq \beta \neq \gamma}^{N} \frac{z_{\alpha} z_{\gamma}}{(z_{\alpha} - z_{\gamma})(z_{\alpha} - z_{\beta})} \vec{S}_{\alpha} \cdot (\vec{S}_{\gamma} \times \vec{S}_{\beta}) = i \sum_{\alpha \neq \beta \neq \gamma}^{N} \frac{z_{\alpha} + z_{\gamma}}{(z_{\alpha} - z_{\gamma})(z_{\alpha} - z_{\beta})} (\vec{S}_{\alpha} \times \vec{S}_{\gamma}) \cdot \vec{S}_{\beta} = \vec{K} \cdot \vec{S} \cdot \tag{D8}
\]
By putting together the identity
\[
\sum_{\alpha \neq \beta \neq \gamma}^{N} \sum_{\beta \neq \gamma \neq \alpha}^{N} \frac{i(\vec{S}_{\alpha} \times \vec{S}_{\beta}) + \vec{S}_{\alpha}}{(z_{\alpha} - z_{\gamma})(z_{\alpha} - z_{\beta})} \cdot \left[ i(\vec{S}_{\alpha} \times \vec{S}_{\beta}) + \vec{S}_{\beta} \right] = \sum_{\alpha}^{N} \vec{D}_{\alpha}^{\dagger} \cdot \vec{D}_{\alpha} + \frac{3}{2}S^2 \cdot \vec{K} + \frac{3}{8}(N - 1)S^2 \cdot \tag{D9}
\]
and the identity
\[
\sum_{\alpha \neq \beta \neq \gamma}^{N} \sum_{\beta \neq \gamma \neq \alpha}^{N} \frac{-i(\vec{S}_{\alpha} \times \vec{S}_{\beta}) + \vec{S}_{\alpha}}{(z_{\alpha} - z_{\gamma})(z_{\alpha} - z_{\beta})} \cdot \left[ i(\vec{S}_{\alpha} \times \vec{S}_{\beta}) + \vec{S}_{\beta} \right] = \frac{3}{2} \sum_{\alpha \neq \beta \neq \gamma}^{N} \sum_{\beta \neq \gamma \neq \alpha}^{N} (z_{\alpha} - z_{\gamma})(z_{\alpha} - z_{\beta}) \left[ \vec{S}_{\alpha} \cdot \vec{S}_{\beta} + i\vec{S}_{\gamma} \cdot (\vec{S}_{\alpha} \times \vec{S}_{\beta}) \right] = \frac{3}{2} \left[ \frac{1}{2} \sum_{\alpha \neq \beta}^{N} \frac{\vec{S}_{\alpha} \cdot \vec{S}_{\beta}}{|z_{\alpha} - z_{\beta}|^2} + \frac{N(N^2 + 5)}{16} \right. \nonumber \]
\[
\left. - \frac{S^2}{2} + \vec{S} \cdot \vec{K} \right] \cdot \tag{D10}
\]
the proof is complete.

Since \(\langle \Phi | \vec{D}_{\alpha}^{\dagger} \cdot \vec{D}_{\alpha} | \Phi \rangle\) is nonnegative for any wavefunction \(|\Phi\rangle\), this provides an explicit demonstration that \(|\Psi_{GS}\rangle\) is the true ground state. The annihilation operators and their equivalence to \(\mathcal{H}_{HS}\) when squared and summed were originally discovered by Shastry. They are lattice versions of the Knizhnik-Zamolodchikov operators known from studies of the Calogero-Sutherland model, the 1-dimensional Bose gas with inverse-square repulsions \(18\).
Thus, Eq. (E8) implies the identity:

\[ (m - 1)(M - m + 1)\langle \Psi_{m-1}|e_1|\Psi_m \rangle = \langle \Psi_{m-1}|[H - \frac{E_{GS}}{(J^22^N)}]e_1|\Psi_m \rangle = (M + m(M - m))\langle \Psi_{m-1}|e_1|\Psi_m \rangle + (M - m + 1)\langle \Psi_{m-1}|\Psi_m \rangle, \]  

(E8)

thus, Eq. (E8) implies the identity:

\[ \langle \Psi_{m-1}|e_1|\Psi_m \rangle = -\frac{M - m + 1}{2(M - m + 1)}\langle \Psi_{m-1}|\Psi_m \rangle. \]  

(E9)

In order to determine a suitable induction relation, let us introduce the operator

\[ e_M(z_1, \ldots, z_M) = z_1 \cdots z_M. \]

Clearly

\[ e_M(z_1, \ldots, z_M)e_M(\frac{1}{z_1}, \ldots, \frac{1}{z_M}) = 1. \]  

(E10)

Since all the \( \Psi_m \)'s are products of the ground state factor, \( \Psi_{GS} \), times a symmetric polynomial of degree less than 2 in each variable, we have

\[ \langle \Psi_{m-1}|e_1|\Psi_m \rangle = \langle [e_M^* \Psi_m]|e_1|e_M^* \Psi_{m-1} \rangle \]

\[ = \langle \Psi_{M-m}|e_1|\Psi_{M-m+1} \rangle. \]  

(E11)

At this point, we use again Eq. (E11) in order to write Eq. (E12) as

\[ \langle \Psi_{M-m}|e_1|\Psi_{M-m+1} \rangle = -\frac{m}{2(m - \frac{1}{2})}\langle \Psi_{M-m}|\Psi_{M-m} \rangle = \frac{m}{2(m - \frac{1}{2})}\langle \Psi_{m}|\Psi_{m} \rangle. \]  

(E12)

Eq. (E12) closes the induction relation:

\[ \frac{\langle \Psi_m|\Psi_m \rangle}{\langle \Psi_{m-1}|\Psi_{m-1} \rangle} = \frac{(m - \frac{1}{2})(M - m + 1)}{m(M - m + \frac{1}{2})}. \]  

(E13)

The formula generated by recursion is:

\[ \langle \Psi_m|\Psi_m \rangle = \prod_{j=1}^m \frac{(j - \frac{1}{2})(M - j + 1)}{j(M - m + j + \frac{1}{2})}C_M \]

where

\[ C_M = \frac{\Gamma[M + 1]\Gamma[m + \frac{1}{2}]\Gamma[M - m + \frac{1}{2}]}{\Gamma[M + \frac{1}{2}]\Gamma[m + 1]\Gamma[M - m + 1]} \]  

(E14)

The constant \( C_M \) is expressed in terms of Wilson’s integral as

\[ C_M = \frac{N^M(2M)!}{2^M}. \]  

(E15)

Eqs. (E14,E15) complete the proof.

Now we work out the formula for the two-spinon eigenstates. In this case \( M = N/2 - 1 \) and Eq. (E5) becomes:

\[ \langle [H - \frac{E_{GS}}{(J^22^N)}]e_1\Phi_{GS} \rangle = \times \Psi_{GS} \sum_{j=1}^M z_j^2 \frac{\partial}{\partial z_j} + (M - \frac{1}{2})e_1 \]  

\[ \times \Psi_{GS} \sum_{j=1}^M z_j^2 \frac{\partial}{\partial z_j} + (M - \frac{1}{2})e_1 \]  

where again we work with the Hamiltonian in scaled units. Eq. (E11) now becomes:

\[ \sum_{j=1}^M z_j^2 \frac{\partial}{\partial z_j} \Phi_{a\beta} = 2[e_1 \Phi_{a\beta}] \]

\[ + \left[Mz_\alpha - z_\alpha^2 \frac{\partial}{\partial z_\alpha} + Mz_\beta - z_\beta^2 \frac{\partial}{\partial z_\beta} \right] \Phi_{a\beta} \]  

(E17)

where the state \( \Phi_{a\beta} \) has been defined in Eq. (E5). Hence, by letting \( e_1 \) act onto the two-spinon eigenstate \( \Phi_{mn} \), we obtain

\[ \langle [H - \frac{E_{GS}}{(J^22^N)}]e_1\Phi_{mn} \rangle = \sum_{j=1}^M z_j^2 \frac{\partial}{\partial z_j} \Phi_{mn} = \sum_{k} \sum_{\alpha, \beta = 1} a_k^{mn} \sum_{\alpha, \beta = 1} \left[ \frac{z_\alpha^m(z_\beta^n)^n}{N^N} \right] [2[e_1 \Phi_{a\beta}] \]

\[ + \left[Mz_\alpha - z_\alpha^2 \frac{\partial}{\partial z_\alpha} + Mz_\beta - z_\beta^2 \frac{\partial}{\partial z_\beta} \right] \Phi_{a\beta} \]  

(E18)

which implies

\[ \langle [H - \frac{E_{GS}}{(J^22^N)}] - (\lambda_{mn} + M + \frac{1}{2})e_1\Phi_{mn} \rangle \]

\[ = \sum_{k} a_k^{mn} \{ [M-m-k+1]\sum_{j} b_{j}^{m-k-1,n-k} \Phi_{m+k-1+j,n-k-j} \]

"
\[ + [M - n + k + 1] \sum_j b_j^{m+k,n-k-1} \Phi_{m+k+j,n-k-1-j} \] 

(E19)

(See eqs. (56), (60) for the definition of the coefficients \(a_k^{mn}, b_k^{mn}\) and eq. (61) for the definition of \(\lambda_{mn}\).

Let us, now, take the scalar product of both sides of Eq. (E19) with \(\Phi_{m-1,n}\). The result can be recast in the form

\[ \frac{\langle \Phi_{m-1,n} | e_1 | \Phi_{mn} \rangle}{\langle \Phi_{m-1,n} | \Phi_{mn} \rangle} = - \frac{M - m + 1}{2(M - m + \frac{1}{2})} \] 

(E20)

On the other hand, by taking the scalar product of \(\Phi_{m,n-1}\) with both sides of Eq. (E19), we obtain:

\[ (\lambda_{m,n-1} - \lambda_{mn} - M - \frac{1}{2}) \langle \Phi_{mn} | e_1 | \Phi_{mn} \rangle = \sum_k a_k^{mn} \{ [M - m - k + 1] 
\times \sum_j b_j^{m+k-1,n-k} \langle \Phi_{mn} | \Phi_{m+k-1,j,n-k-j} \rangle 
+ [M - n + k + 1] \sum_j b_j^{m+n,k-1} 
\times \langle \Phi_{mn} | \Phi_{m+k+j,n-k-j} \rangle \} \] 

(E21)

which implies the relations:

\[ \frac{\langle \Phi_{mn} | e_1 | \Phi_{mn} \rangle}{\langle \Phi_{mn} | \Phi_{mn} \rangle} = - \frac{(M - n + \frac{3}{2})(m - n + 1)^2}{2(m - n + \frac{3}{2})(m - n + \frac{1}{2})(M - n + 1)} \] 

(E22)

In order to complete the proof, we need two more identities, which can be proved in the same way we did for Eq. (E13):

\[ \frac{\langle \Phi_{m-1,n} | e_1 | \Phi_{mn} \rangle}{\langle \Phi_{mn} | \Phi_{mn} \rangle} = \frac{\langle \Phi_{M-n,M-m} | e_1 | \Phi_{M-n,M-m+1} \rangle}{\langle \Phi_{mn} | \Phi_{mn} \rangle} = - \frac{(m + \frac{1}{2})(m - n)^2}{2(m - n + \frac{1}{2})m(m - n - \frac{1}{2})} \] 

(E23)

\[ \frac{\langle \Phi_{m,n-1} | e_1 | \Phi_{mn} \rangle}{\langle \Phi_{mn} | \Phi_{mn} \rangle} = \frac{\langle \Phi_{M-n,M-m} | e_1 | \Phi_{M-n+1,M-m} \rangle}{\langle \Phi_{mn} | \Phi_{mn} \rangle} = - \frac{n}{2(n - \frac{1}{2})} \] 

(E24)

Hence, the proof is given by the following induction relations:

\[ \frac{\langle \Phi_{mn} | \Phi_{mn} \rangle}{\langle \Phi_{m-1,n} | \Phi_{mn} \rangle} = \frac{(n - \frac{1}{2})(M - n + \frac{3}{2})(m - n + 1)^2}{n(M - n + 1)(m - n + \frac{3}{2})(m - n + \frac{1}{2})} \] 

(E25)

\[ \frac{\langle \Phi_{mn} | \Phi_{mn} \rangle}{\langle \Phi_{m,n-1} | \Phi_{mn} \rangle} = \frac{(M - m + 1)(m - n + \frac{1}{2})(m - n - \frac{1}{2})m}{(M - m + \frac{1}{2})(m - n)^2(m + \frac{1}{2})} \] 

(E26)

Eqs. (E25), (E27) imply

\[ \frac{\langle \Phi_{mn} | \Phi_{mn} \rangle}{\langle \Phi_{mn} | \Phi_{mn} \rangle} = C'_M \frac{\Gamma[m - n + \frac{1}{2}] \Gamma[m - n + \frac{3}{2}]}{\Gamma^2[m - n + 1]} \times \frac{\Gamma[m + 1] \Gamma[M - m + \frac{1}{2}] \Gamma[n + \frac{1}{2}] \Gamma[M - n + 1]}{\Gamma[m + \frac{1}{2}] \Gamma[M - m + 1] \Gamma[n + 1] \Gamma[M - n + \frac{1}{2}]} \] 

(E27)

and the constant \(C'_M\) is now given by:

\[ C'_M = N^M \frac{(2M)!}{2M} \frac{M + \frac{1}{2}}{\pi} \] 

(E28)

[1] L. D. Fadeev and L. A. Takhtajan, Russian Math. Surveys 34 11 (1979).
[2] H. A. Bethe, Z. Physik 71, 205 (1931).
[3] F. D. M. Haldane, Phys. Rev. Lett. 50, 1153 (1983); Phys Lett A93, 464 (1983).
[4] F. D. M. Haldane, Phys. Rev. Lett. 66, 1529 (1991).
[5] F. D. M. Haldane, Phys. Rev. Lett. 60, 635 (1988); ibid. 66, 1529 (1991).
[6] B. S. Shastry, Phys. Rev. Lett. 60, 639 (1988).
[7] R. B. Laughlin et al, Field Theory for Low-Dimensional Systems, ed. G. Morandi et al (Springer, Heidelberg, 1999).
[8] Z. N. C. Ha and F. D. N. Haldane, Phys. Rev. 34, 11 (1979).
[9] F. D. M. Haldane, Z. N. C. Ha, J. C. Talstra, D. Bernard and V. Pasquier, Phys. Rev. Lett., 69, 2021 (1992).
[10] D. A. Tennant et al, Phys. Rev. B 60, 13368 (1995); R. Coldea et al., Phys. Rev. Lett. 79, 151 (1997); R. Coldea, Journal of Magnetism and Magnetic Materials, vol. 177-181, 659 (1998).

[11] F. D. M. Haldane, M. R. Zirnbauer, Phys. Rev. Lett. 71, 4055 (1993).

[12] K. G. Wilson, Jour. Mat. Phys. 3, 1040 (1962).

[13] B. Sutherland, Phys. Rev. A 4, 2019 (1971); ibid. 5, 1372 (1972).

[14] Y. Kato, Phys. Rev. Lett. 81, 5402 (1998).

[15] M. Abramowitz, Handbook of Mathematical Functions, (United States. National Bureau of Standards. Applied mathematics series, 55, 1964).

[16] B. A. Bernevig, D. Giuliano and R. B. Laughlin, cond-mat/0011069.

[17] B. S. Shastry, Phys. Rev. Lett. 69, 1153 (1992).

[18] V. G. Knizhnik and A. B. Zamolodchikov, Nucl. Phys. B 247, 83 (1984).

[19] F. Calogero, J. Math Phys. 10, 2197 (1969).