Bayesian and maximum likelihood estimations of the inverted exponentiated half logistic distribution under progressive Type II censoring

Kyeongjun Lee and Youngseuk Cho

Department of Statistics, Pusan National University, Busan, Korea

ABSTRACT
In this paper, the estimation of parameters, reliability and hazard functions of a inverted exponentiated half logistic distribution (IEHLD) from progressive Type II censored data has been considered. The Bayes estimates for progressive Type II censored IEHLD under asymmetric and symmetric loss functions such as squared error, general entropy and lindex loss function are provided. The Bayes estimates for progressive Type II censored IEHLD parameters, reliability and hazard functions are also obtained under the balanced loss functions. However, the Bayes estimates cannot be obtained explicitly, Lindley approximation method and importance sampling procedure are considered to obtain the Bayes estimates. Furthermore, the asymptotic normality of the maximum likelihood estimates is used to obtain the approximate confidence intervals. The highest posterior density credible intervals of the parameters based on importance sampling procedure are computed. Simulations are performed to see the performance of the proposed estimates. For illustrative purposes, two data sets have been analyzed.

1. Introduction
In life-testing experiment and reliability studies, unfortunately, the failure times of test items may not be recorded exactly. There are also situations wherein the withdrawal of items previous to failure is pre-designed in order to decrease the time or cost associated with experiment. Type I and Type II censoring schemes are the most common censoring schemes. However, these censoring schemes do not allow for items to be withdrew from the experiment at times other than the termination time. In the case of loss of contact with individuals under experiment or accidental breakage of items, that is, the withdrawal of items at times other than the termination time may be unavoidable. These motivations and reasons lead theoreticians and reliability practitioners directly into the area of progressive censoring [6].

Progressive Type II censoring can be explained as follows. Directly following the first discovered failure, $R_1$ surviving items are removed from the experiment at random.
Similarly, following the second discovered failure, $R_2$ surviving items are withdrow from the experiment at random. This process continues, directly following the $m$th discovered failure, all the remaining $R_m = n - R_1 - \cdots - R_{m-1} - m$ items are withdrow from the experiment. Then the $m$ ordered failure times are denoted by $X_1; m: n$, $X_2; m: n$, $\ldots$, $X_m; m: n$. In this experiment, the progressive Type II censoring scheme $R = (R_1, R_2, \ldots, R_m)$ is pre-set. The $X_1; m: n$, $X_2; m: n$, $\ldots$, $X_m; m: n$ are referred to as progressive Type II censored data. The joint probability density function (pdf) of progressive Type II censored data is given by

$$f(x_1; m: n, x_2; m: n, \ldots, x_m; m: n) = c' \prod_{i=1}^{m} f(x_i; m: n)[1 - F(x_i; m: n)]^{R_i},$$

where $c' = n(n - R_1 - 1)(n - R_1 - R_2 - 2) \cdots (n - \sum_{i=1}^{m-1}(R_i + 1))$. In the last few years, progressive Type II censoring has received considerable attention in the applied statistics. Recent papers based on progressive Type II censoring have been published by following authors: Cho et al. [8,9], Rastogi and Tripathi [24], Ahmed [1], Huang and Wu [11], and Wu [26].

The half logistic distribution (HLD) obtained by folding the logistic distribution (LD). Balakrishnan [5] introduced the HLD as a life-testing experiment model with increasing hazard rate. It can be applied as a failure model in life-testing studies. Some papers have been published in the literature on estimating the parameters of an HLD. Kim and Han [14] considered the Bayes estimation of the parameter of an HLD based on progressive Type II censoring. They computed the Bayes estimates under square error loss function (SF), and compared their performances with the maximum likelihood estimate (MLE) and approximate MLE. Jang et al. [12] considered the Bayes estimation of the parameter of an HLD based on multiply Type II censoring. They computed the Bayes estimates under SELF, and compared their performances with the MLE and approximate MLE.

The generalized versions of HLD was considered along with estimation of parameters and stress strength reliability based on complete sample by Arora et al. [2]. Azimi et al. [4] considered the Bayes estimation of the parameter and reliability of an generalized HLD based on progressively Type II censoring. They computed the Bayes estimates under precautionary loss function, linek loss function (LF) and general entropy loss function (GELF), and compared their performances. Azimi [3] considered the Bayes estimation of the shape parameter and reliability of an generalized HLD based on doubly Type II censoring. They computed the Bayes estimates under SELF, LF and GELF, and compared their performances. Rastogi and Tripathi [25] considered the Bayes estimation of the parameters of an generalized HLD based on progressive Type II censoring. They computed the Bayes estimates under SELF, LF and GELF, and compared their performances with the MLE.

Modification to generalized HLD has been done using its inverted, known as the inverted exponentiated half logistic distribution (IEHLD) and was studied by Potdar and Shirke [23]. They obtained the MLEs and confidence intervals (CIs) for the parameters using complete data. The cumulative distribution function (cdf), pdf, reliability and hazard function of IEHLD are as follows,

$$F(x; \alpha, \lambda) = 1 - \left[ \frac{1 - \exp \left( -\frac{1}{\lambda x} \right)}{1 + \exp \left( -\frac{1}{\lambda x} \right)} \right]^\alpha, \quad x > 0,$$  \hfill (1)
\[ f(x; \alpha, \lambda) = \frac{2\alpha}{\lambda x^2} \exp\left(-\frac{1}{\lambda x}\right) \frac{1 - \exp\left(-\frac{1}{\lambda x}\right)^{\alpha-1}}{1 + \exp\left(-\frac{1}{\lambda x}\right)^{\alpha+1}}, \quad x > 0, \quad (2) \]

\[ R(t) = \left[ \frac{1 - \exp\left(-\frac{1}{\lambda t}\right)}{1 + \exp\left(-\frac{1}{\lambda t}\right)} \right]^\alpha, \quad t > 0, \quad (3) \]

and

\[ H(t; \alpha, \lambda) = \frac{2\alpha}{\lambda t^2} \exp\left(-\frac{1}{\lambda t}\right) \left[ 1 - \exp\left(-\frac{2}{\lambda t}\right) \right]^{-1}, \quad t > 0, \quad (4) \]

where \( \lambda > 0 \) and \( \alpha > 0 \) are the scale and shape parameters. For \( \alpha = 1 \), we observe that IEHLD corresponds to the usual HLD. In this paper, we assume that both parameters are to be unknown.

In many reliability and life testing studies, it is not unusual that hazard rate is non-monotone. For example, in dosage mortality test on animals, it may happen during the treatment that mortality rate increases first and then starts decreasing. From Equation (4), hazard rate of IEHLD is nonmonotone. Hazard rate of the IEHLD show similar behavior to some well-known lifetime models: inverse Weibull, generalized inverted rayleigh and generalized inverted exponential distribution. So in this respect IEHLD can be considered as an alternative model to these distributions.

While generalized HLD has been used for several applications, a few work has been done on the inverted of the generalized HLD. In this paper, we consider the estimation of parameters, reliability and hazard functions of IEHLD based on progressive Type II censored sample.

The objective of the paper is two-fold. The first is to consider the MLEs of the parameters and reliability of IEHLD based on progressive Type II censoring. The observed information matrix is used to obtain the asymptotic CIs for the parameters. Next, the second is to consider the Bayes estimation of the parameters and reliability under the asymmetric and symmetric loss functions such as SELF, LF and GELF. Also, the Bayes estimates of the parameters and reliability of the IEHLD are obtained under the balanced SELF, LF and GELF. Since the Bayes estimates cannot be obtained explicitly, Lindley approximation method and importance sampling procedure are used to obtain the Bayes estimates of the parameters and reliability. Moreover, importance sampling procedure is used to obtain the highest posterior density (HPD) credible intervals for the parameters.

The rest of the paper is organized as follows. The MLEs of the parameters, reliability and hazard functions are presented in next section. In Section 3, the asymptotic CIs for the parameters, reliability and hazard functions are presented. In Section 4, we develop the Bayes estimates under the SELF, LF and GELF. In the same section, we have developed the HPD credible intervals of the parameters. In Section 5, simulated data and real data set has been analyzed and an optimal criterion was also presented. A Monte Carlo simulation of inferential procedures is carried out in Section 6, and finally in Section 7, we conclude the paper.
2. Maximum likelihood estimation

Given a progressive Type II censored data from a IEHLD, the likelihood can be constructed as follows:

\[
L(\alpha, \lambda | X) = c' \prod_{i=1}^{m} \frac{2\alpha}{\lambda x_{i;m:n}^{2}} \exp \left( -\frac{1}{\lambda x_{i;m:n}} \right) \left[ 1 - \exp \left( -\frac{1}{\lambda x_{i;m:n}} \right) \right]^{\alpha(1+R_{i})-1} \left[ 1 + \exp \left( -\frac{1}{\lambda x_{i;m:n}} \right) \right]^{\alpha(1+R_{i})+1}.
\]

Ignoring the constant, then, the log-likelihood of the data is

\[
l(\alpha, \lambda | X) \propto m \log \alpha - m \log \lambda - \frac{1}{\lambda} \sum_{i=1}^{m} x_{i;m:n} + \alpha \sum_{i=1}^{m} (1 + R_{i}) \log \frac{1}{H_{1,1}^{X}(\lambda)} - m \log \left[ 1 - \exp \left( -\frac{2}{\lambda x_{i;m:n}} \right) \right],
\]

where \( H_{a,b}^{X}(\lambda) = [1 + \exp\{-a/(\lambda x_{i;m:n})\}]/[1 - \exp\{-b/(\lambda x_{i;m:n})\}] \).

The corresponding likelihood equations are

\[
\frac{\partial l}{\partial \alpha} = m + \sum_{i=1}^{m} (1 + R_{i}) \log \frac{1}{H_{1,1}^{X}(\lambda)} = 0 \tag{5}
\]

and

\[
\frac{\partial l}{\partial \lambda} = -\frac{m}{\lambda} + \frac{1}{\lambda^2} \sum_{i=1}^{m} \frac{1}{x_{i;m:n}} - \frac{2\alpha}{\lambda^2} \sum_{i=1}^{m} \left( \frac{1 + R_{i}}{x_{i;m:n}} \right) T_{1,2}^{X}(\lambda) + \frac{2}{\lambda^2} \sum_{i=1}^{m} \frac{T_{2,2}^{X}(\lambda)}{x_{i;m:n}} = 0, \tag{6}
\]

where \( T_{a,b}^{X}(\lambda) = \exp\{-a/(\lambda x_{i;m:n})\}/[1 - \exp\{-b/(\lambda x_{i;m:n})\}] \).

From Equation (5), we get the MLE of \( \alpha \) as

\[
\hat{\alpha}(\lambda) = \frac{m}{\sum_{i=1}^{m} (1 + R_{i}) \log H_{1,1}^{X}(\lambda)}.
\]

Putting the \( \hat{\alpha}(\lambda) \) into Equation (6), we get

\[
\lambda = z(\lambda),
\]

where

\[
z(\lambda) = \left[ \sum_{i=1}^{m} \frac{1}{x_{i;m:n}} - 2m \left\{ \frac{\sum_{i=1}^{m} \left( \frac{1+R_{i}}{x_{i;m:n}} \right) T_{1,2}^{X}(\lambda)}{\sum_{i=1}^{m} (1 + R_{i}) \log H_{1,1}^{X}(\lambda)} \right\} \right] \left[ m + 2 \sum_{i=1}^{m} \frac{T_{2,2}^{X}(\lambda)}{x_{i;m:n}} \right]^{-1}.
\]

From above equation, we propose a simple iterative scheme to solve for \( \lambda \). This has been proposed in the literature by Kundu [15]. Start with an initial guess of \( \lambda \), say \( \lambda^{(0)} \), then obtain \( \lambda^{(1)} = z(\lambda^{(0)}) \) and proceed in this way iteratively to obtain \( \lambda^{(n+1)} = z(\lambda^{(n)}) \). Stop the iterative procedure, when \( |\lambda^{(n+1)} - \lambda^{(n)}| < \epsilon \), some pre-assigned tolerance limit. Once we obtain the MLE of \( \lambda \), say \( \hat{\lambda} \), then the MLE of \( \alpha \) can be obtained as \( \hat{\alpha} = \hat{\alpha}(\hat{\lambda}) \).
Using \( \hat{\alpha} \) and \( \hat{\lambda} \), the MLEs of reliability and hazard are obtained as
\[
\hat{R}(t) = \left[ 1 - \exp\left( -\frac{1}{\hat{\lambda}} \right) \right] \frac{1}{1 + \exp\left( -\frac{1}{\hat{\lambda}} \right)} \quad \text{and} \quad \hat{H}(t) = \frac{2\hat{\alpha}\exp\left( -\frac{1}{\hat{\lambda}} \right)}{1 - \exp\left( -\frac{2}{\hat{\lambda}} \right)}.
\]

3. Confidence intervals

3.1. Normal approximation of the MLE (NA)

The 100\((1 - \beta)\)% CIs for \( \alpha \) and \( \lambda \) can be constructed from the asymptotic normality of the MLEs with \( \text{Var}(\hat{\alpha}) \) and \( \text{Var}(\hat{\lambda}) \) estimated from the inverse of the observed Fisher information matrix.

From the log-likelihood in Equation (4), the second derivatives of log-likelihood with respect to \( \alpha \) and \( \lambda \) are obtained by
\[
l_{20} = \frac{\partial^2 l}{\partial \alpha^2} = -\frac{m}{\alpha^2}, \quad l_{11} = \frac{\partial^2 l}{\partial \alpha \partial \lambda} = -\frac{2}{\lambda^2} \sum_{i=1}^{m} \left( 1 + \frac{R_i}{x_{i;m:n}} \right) T_{1,2}^X(\lambda)
\]
and
\[
l_{02} = \frac{\partial^2 l}{\partial \lambda^2} = \frac{m}{\lambda^2} - \frac{2}{\lambda^3} \sum_{i=1}^{m} \frac{1}{x_{i;m:n}} + \frac{2\alpha}{\lambda^3} \sum_{i=1}^{m} \left( 1 + \frac{R_i}{x_{i;m:n}} \right) T_{1,2}^X(\lambda) \left[ 2 - \frac{1}{\lambda x_{i;m:n} H_{2,2}^X(\lambda)} \right]
\]
\[
- \frac{4}{\lambda^3} \sum_{i=1}^{m} \frac{T_{2,2}^X(\lambda)}{x_{i;m:n}} \left[ 1 - \frac{1}{\lambda x_{i;m:n} \left[ 1 - \exp\left( -\frac{2}{\lambda x_{i;m:n}} \right) \right]} \right].
\]

Let \( I(\alpha, \lambda) \) means the Fisher information matrix of the \( \alpha \) and \( \lambda \). Then, the Fisher information matrix is obtained by taking expectations of minus Equations (7) and (8).

Under some mild regularity conditions, \((\hat{\alpha}, \hat{\lambda})\) is approximately bivariately normal with mean \( (\alpha, \lambda) \) and covariance matrix \( I^{-1} = I^{-1}(\alpha, \lambda) \). In practice, we usually estimate \( I^{-1}(\alpha, \lambda) \) by \( I^{-1}(\hat{\alpha}, \hat{\lambda}) \).

A simpler and equally valid procedure is to use the approximation
\[
(\hat{\alpha}, \hat{\lambda}) \sim N\left[ (\alpha, \lambda), I^{-1}(\hat{\alpha}, \hat{\lambda}) \right],
\]
where
\[
I^{-1}(\hat{\alpha}, \hat{\lambda}) = \begin{pmatrix} -l_{20} & -l_{11} \\ -l_{11} & -l_{02} \end{pmatrix}_{(\alpha, \lambda) = (\hat{\alpha}, \hat{\lambda})}^{-1} = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix}.
\]

Therefore, based on the NA, a 100\((1 - \beta)\)% CIs for \( \alpha \) and \( \lambda \) are
\[
\hat{\alpha} \pm Z_{\beta/2} \sqrt{\tau_{11}} \quad \text{and} \quad \hat{\lambda} \pm Z_{\beta/2} \sqrt{\tau_{22}},
\]
where \( Z_{\beta/2} \) is the percentile of the standard normal distribution with right-tail probability \( \beta/2 \).
In order to find the approximate estimates of the variance of reliability and hazard functions, we use the delta method discussed in Greene [10]. Let
\[
\Psi'_{R} = \left( \frac{\partial R(t)}{\partial \alpha}, \frac{\partial R(t)}{\partial \lambda} \right) \quad \text{and} \quad \Psi'_{H} = \left( \frac{\partial H(t)}{\partial \alpha}, \frac{\partial H(t)}{\partial \lambda} \right),
\]
where
\[
\frac{\partial R(t)}{\partial \alpha} = -aH_{1,1}^t(\lambda)^{-a-1}S_{1,1}(\lambda), \quad \frac{\partial R(t)}{\partial \lambda} = -H_{1,1}^t(\lambda)^{-a}\log H_{1,1}^t(\lambda),
\]
\[
\frac{\partial H(t)}{\partial \alpha} = \frac{2}{\lambda t^2} T_{1,2}^t(\lambda), \quad \frac{\partial H(t)}{\partial \lambda} = \frac{2\alpha}{\lambda^2 t^2} T_{1,2}^t(\lambda) \left[-1 + \frac{1}{\lambda t} H_{2,2}^\prime(\lambda) \right],
\]
\[
H_{a,b}^\prime(\lambda) = \frac{1 + \exp[-a/(\lambda t)]}{1 - \exp[-b/(\lambda t)]}, \quad T_{a,b}^t(\lambda) = \frac{\exp[-a/(\lambda t)]}{1 - \exp[-b/(\lambda t)]},
\]
\[
S_{1,1}(\lambda) = \frac{2T_{1,1}^t(\lambda)}{\lambda^2 t[1 - \exp[-1/(\lambda t)]]}.
\]

The approximate estimates of \( \widehat{\text{var}}(\hat{R}) \) and \( \widehat{\text{var}}(\hat{H}) \) are given, respectively, by
\[
\widehat{\text{var}}(\hat{R}) = [\Psi'_{R}^{-1}(\hat{\alpha}, \hat{\lambda})\Psi_{R}] \quad \text{and} \quad \widehat{\text{var}}(\hat{H}) = [\Psi'_{H}^{-1}(\hat{\alpha}, \hat{\lambda})\Psi_{H}].
\]
Thus,
\[
\frac{\hat{R} - R}{\sqrt{\widehat{\text{var}}(\hat{R})}} \sim N(0,1) \quad \text{and} \quad \frac{\hat{H} - H}{\sqrt{\widehat{\text{var}}(\hat{H})}} \sim N(0,1),
\]
asymptotically. These results yield the asymptotic 100(1 - \( \beta \))% CIs for reliability and hazard functions are given, respectively, by
\[
\hat{R} \pm Z_{\beta/2}\sqrt{\text{var}(\hat{R})} \quad \text{and} \quad \hat{H} \pm Z_{\beta/2}\sqrt{\text{var}(\hat{H})}.
\]

### 3.2. Normal approximation of the log-transformed MLE (NL)

Meeker and Escobar [21] reported that the asymptotic CIs based on log-transformed MLE has better coverage probability. A 100(1 - \( \beta \))% normal approximate CIs for log-transformed MLE are
\[
\log \hat{\alpha} \pm Z_{\beta/2}\sqrt{\tau_{11}(\log \hat{\alpha})} \quad \text{and} \quad \log \hat{\lambda} \pm Z_{\beta/2}\sqrt{\tau_{22}(\log \hat{\lambda})},
\]
where \( \tau_{11}(\log \hat{\alpha}) \) and \( \tau_{22}(\log \hat{\lambda}) \) are the estimated variance of \( \log \hat{\alpha} \) and \( \log \hat{\lambda} \).

Therefore, based on the NL, a 100(1 - \( \beta \))% CIs for \( \alpha \) and \( \lambda \) are
\[
\hat{\alpha} \exp \left[ \pm \frac{Z_{\beta/2}\sqrt{\tau_{11}}}{\hat{\alpha}} \right] \quad \text{and} \quad \hat{\lambda} \exp \left[ \pm \frac{Z_{\beta/2}\sqrt{\tau_{22}}}{\hat{\lambda}} \right].
\]

Also, based on the NL, a 100(1 - \( \beta \))% CIs for reliability and hazard functions are
\[
\hat{R} \exp \left[ \pm \frac{Z_{\beta/2}\sqrt{\text{var}(\hat{R})}}{\hat{R}} \right] \quad \text{and} \quad \hat{H} \exp \left[ \pm \frac{Z_{\beta/2}\sqrt{\text{var}(\hat{H})}}{\hat{H}} \right].
\]
4. Bayes estimation

Choice of loss function is an important part in the Bayes estimation. In this section, therefore, we derive the Bayes estimates for parameters \(\alpha\) and \(\lambda\) of a IEHLD under both the symmetric and asymmetric loss functions. A very well known symmetric loss function is the SELF. The most commonly used asymmetric loss functions are the LF and GELF. The SELF, LF and GELF are defined by

\[
L_1(\theta, \hat{\theta}) = (\hat{\theta} - \theta)^2,
\]
\[
L_2(\theta, \hat{\theta}) = \exp(h(\hat{\theta} - \theta)) - h(\hat{\theta} - \theta) - 1,
\]
\[
L_3(\theta, \hat{\theta}) = \left(\frac{\theta}{\hat{\theta}}\right)^q - q\log\left(\frac{\theta}{\hat{\theta}}\right) - 1,
\]

where \(\hat{\theta}\) means an estimate for the \(\theta\), \(h \neq 0\) and \(q \neq 0\).

In LF and GELF, the sign of \(h\) and \(q\) mean the direction of asymmetry and its magnitude indicates the degree of asymmetry. For \(h < 0\) and \(q < 0\), the positive error is better than the negative error. For \(h > 0\) and \(q > 0\), the negative error is better than the positive error. For \(h\) close to zero, the LF is approximately the SELF. For \(q = -1\), the Bayes estimate under the GELF coincides with the Bayes estimate under the SELF.

The Bayes estimates of \(\theta\) under above loss functions can be obtained as

\[
\hat{\theta}_S = [E_\theta(\theta|X)],
\]
\[
\hat{\theta}_L = -\frac{1}{h}\log[E_\theta(\exp(-h\theta)|X)],
\]
\[
\hat{\theta}_E = [E_\theta(\theta^{-q}|X)]^{-1/q},
\]

provided the above expectation exists.

Moreover, we derive the Bayes estimates for parameters \(\alpha\) and \(\lambda\) of a IEHLD under balanced loss functions. A generalized loss function called the balanced loss function [13] of the form

\[
L_{\rho, w, \theta_0}(\theta, \hat{\theta}) = w\rho(\theta, \theta_0) + (1 - w)\rho(\theta, \hat{\theta}),
\]

where \(\rho\) is an arbitrary loss function, while \(\theta_0\) is a chosen a prior target estimate of \(\theta\), obtained for instance using the criterion of MLE and the weight \(w\) takes values in \([0,1)\).

By choosing SELF, Equation (10) reduced to the balanced SELF, LF and GELF as follow.

\[
L_4(\theta, \hat{\theta}) = w(\theta_0 - \hat{\theta})^2 + (1 - w)(\theta - \hat{\theta})^2,
\]
\[
L_5(\theta, \hat{\theta}) = w[\exp(h(\theta_0 - \hat{\theta})) - h(\theta_0 - \hat{\theta}) - 1]
\]
\[
+ (1 - w)[\exp(h(\theta - \hat{\theta})) - h(\theta - \hat{\theta}) - 1]
\]

and

\[
L_6(\theta, \hat{\theta}) = w\left[\left(\frac{\theta_0}{\hat{\theta}}\right)^q - q\log\left(\frac{\theta_0}{\hat{\theta}}\right) - 1\right] + (1 - w)\left[\left(\frac{\theta}{\hat{\theta}}\right)^q - q\log\left(\frac{\theta}{\hat{\theta}}\right) - 1\right].
\]
The corresponding Bayes estimate under above balanced loss functions can be obtained as
\[
\hat{\theta}_{BS} = w\theta_0 + (1 - w)E[\theta|X],
\]
\[
\hat{\theta}_{BL} = -\frac{1}{h}\log\left[w\exp(-h\theta_0) + (1 - w)E[\exp(-h\theta)|X]\right]
\]
and
\[
\hat{\theta}_{BE} = \{w\theta_0^{-q} + (1 - w)E[\theta^{-q}|X]\}^{-1/q}.
\]

It is clear that the Bayes estimates under balanced loss functions are more general, which include the both MLE and Bayes estimates as special cases.

Let \(X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}\) be a progressive Type II censored sample of size \(m\) from IEHLD. There does not exist any conjugate prior distribution for \(\alpha\) and \(\lambda\). Consequently, we assumed that the parameters follow the gamma \((c, d)\) and gamma \((a, b)\) prior distributions, respectively. Gamma distribution can accommodate variety of shapes depending upon parameter values. Thus the family of gamma distributions is highly flexible in nature and can be considered as suitable priors \(\alpha\) and \(\lambda\) [17]. Also, it is assumed that the parameters are independent. Therefore, the joint prior distribution of parameters is obtained as
\[
\pi(\alpha, \lambda) \propto \alpha^{c-1}e^{-d\alpha}\lambda^{a-1}e^{-b\lambda}, \alpha > 0, \lambda > 0.
\]

Then, the joint density of the parameters and data is obtained as follows.
\[
\pi(\alpha, \lambda, X) \propto \alpha^{m+c-1}\lambda^{m+a-1}
\]
\[
\times \prod_{i=1}^{m} \frac{1}{x_{i:m:n}^2} \exp\left(-\frac{1}{\lambda x_{i:m:n}} - b\lambda - d\alpha\right) \left[1 - \exp\left(-\frac{1}{\lambda x_{i:m:n}}\right)\right]^{\alpha(1+R_i)-1} \left[1 + \exp\left(-\frac{1}{\lambda x_{i:m:n}}\right)\right]^{\alpha(1+R_i)+1}.
\]

Then the posterior distribution of parameters, given data is obtained as
\[
\pi(\alpha, \lambda|X) \propto \frac{\pi(\alpha, \lambda, X)}{\int_0^{\infty} \int_0^{\infty} \pi(\alpha, \lambda, X) \, d\alpha \, d\lambda}.
\]

Now, we obtain Bayes estimates of parameters, reliability and hazard function under SELF, LF and GELF. Bayes estimate of \(\alpha\) under LF is obtained as
\[
\tilde{\alpha}_L = -\frac{1}{h}\log\left[\frac{\int_0^{\infty} e^{-h\alpha} \pi(\alpha, \lambda, X) \, d\alpha \, d\lambda}{\int_0^{\infty} \int_0^{\infty} \pi(\alpha, \lambda, X) \, d\alpha \, d\lambda}\right].
\]
Similarly, we can obtain the Bayes estimate of \(\lambda\) under LF. Also, we derive Bayes estimate of the reliability function under LF. It is obtained as
\[
\tilde{R}_L(t) = -\frac{1}{h}\log\left[\frac{\int_0^{\infty} e^{-hR(t)} \pi(\alpha, \lambda, X) \, d\alpha \, d\lambda}{\int_0^{\infty} \int_0^{\infty} \pi(\alpha, \lambda, X) \, d\alpha \, d\lambda}\right].
\]

The Bayes estimate for the hazard function can be evaluated likewise.
Next, Bayes estimate of $\alpha$ under GELF is obtained as

$$\tilde{\alpha}_E = \left[ \int_0^\infty \frac{\alpha^{-q} \pi(\alpha, \lambda) \, d\alpha \, d\lambda}{\int_0^\infty \int_0^\infty \pi(\alpha, \lambda) \, d\alpha \, d\lambda} \right]^{-1/q}.$$ 

Similarly, we can obtain the Bayes estimate of $\lambda$ under GELF. Also, we derive Bayes estimate of the reliability function under GELF. It is obtained as

$$\tilde{R}_E(t) = \left[ \int_0^\infty \frac{R(t)^{-q} \pi(\alpha, \lambda) \, d\alpha \, d\lambda}{\int_0^\infty \int_0^\infty \pi(\alpha, \lambda) \, d\alpha \, d\lambda} \right]^{-1/q}.$$ 

The Bayes estimate for the hazard function can be evaluated likewise.

### 4.1. Lindley’s approximation

In above subsection, the Bayes estimates are in the form of ratio of two integrals for which closed forms are not available. Therefore, we use Lindley approximation method [20] to obtain the Bayes estimates. For the $(\theta_1, \theta_2)$, the Lindley approximate Bayes estimate can be written as

$$\hat{g} = g(\hat{\theta}_1, \hat{\theta}_2) + 0.5(A + l_{30}B_{12} + l_{03}B_{21} + l_{21}C_{12} + l_{12}C_{21}) + p_1A_{12} + p_2A_{21}, \quad (14)$$

where

$$A = \sum_{i=1}^2 \sum_{j=1}^2 u_{ij} \tau_{ij}, \quad l_{ij} = \frac{\partial^2 g}{\partial \theta_i \partial \theta_j}, \quad A_{ij} = u_i \tau_{ii} + u_j \tau_{jj}, \quad B_{ij} = (u_i \tau_{ii} + u_j \tau_{jj}) \tau_{ij},$$

$$C_{ij} = 3u_i \tau_{ii} \tau_{ij} + u_j (\tau_{ii} \tau_{jj} + 2 \tau_{ij}^2), \quad p_i = \frac{\partial p}{\partial \theta_i}, \quad u_i = \frac{\partial g}{\partial \theta_i},$$

$$u_{ij} = \frac{\partial^2 g}{\partial \theta_i \partial \theta_j}, \quad p = \log \pi(\theta_1, \theta_2).$$

Here, note that $\tau_{ij}$ means the $(i, j)$th element of the matrix $[-\partial^2 l/\partial \theta_i^j \partial \theta_j^j]^{-1}$ and $l$ means the log-likelihood. Now, we compute $\hat{g}$ for estimation of $(\theta_1, \theta_2) = (\alpha, \lambda)$. For our problem, we have

\[
\begin{align*}
l_{03} & = \frac{\partial^3 l}{\partial \lambda^3} = -2 \frac{m}{\lambda^3} + 6 \frac{m}{\lambda^4} \sum_{i=1}^m \frac{1}{x_{i:m:n}} - 4 \frac{m}{\lambda^4} \sum_{i=1}^m \left\{ \left( \frac{1 + R_i}{x_{i:m:n}} \right) T_{1,2}(\lambda) \right\} \left\{ 3 - \frac{1}{\lambda x_{i:m:n}} H^X_{2,2}(\lambda) \right\} \\
& \quad + 2 \frac{m}{\lambda^5} \sum_{i=1}^m \left\{ \left( \frac{1 + R_i}{x_{i:m:n}} \right) T_{1,2}(\lambda) \right\} \left\{ 4 H^X_{2,2}(\lambda) - \frac{H^X_{2,2}(\lambda)^2}{\lambda x_{i:m:n}} - 4 T_{2,2}(\lambda) \frac{1}{\lambda x_{i:m:n}} \left[ 1 - \exp \left( -\frac{2}{\lambda x_{i:m:n}} \right) \right] \right\} \\
& \quad + 4 \frac{2}{\lambda^4} \sum_{i=1}^m \left\{ \left( \frac{1 + R_i}{x_{i:m:n}} \right) T_{1,2}(\lambda) \right\} \left\{ 2 - \frac{H^X_{2,2}(\lambda)}{\lambda x_{i:m:n}} \right\} \left\{ T_{2,2}(\lambda) \frac{x_{i:m:n}^2}{\lambda x_{i:m:n}} \frac{1 - \exp \left( -\frac{2}{\lambda x_{i:m:n}} \right) }{1 - \exp \left( -\frac{2}{\lambda x_{i:m:n}} \right) } \right\} .
\end{align*}
\]
\[ l_{12} = \frac{\partial^2 l}{\partial \alpha^2 \partial \lambda^2} = \frac{2}{\lambda^3} \sum_{i=1}^{m} \left\{ \left( \frac{1 + R_i}{x_{i,m,n}} \right) T_{1,2}^X(\lambda) \left[ 2 - \frac{H_{2,2}^X(\lambda)}{\lambda x_{i,m,n}} \right] \right\}, \]
\[ l_{21} = \frac{\partial^3 l}{\partial \alpha^2 \partial \lambda} = 0, \quad l_{30} = \frac{\partial^3 l}{\partial \alpha^3} = \frac{2m}{\alpha^3}, \quad p_1 = \frac{c - 1}{\alpha} - d, \quad p_2 = \frac{a - 1}{\alpha} - b, \]
and \( \tau_{ij} \) are given in Equation (9).

Now, we compute the Bayes estimate using Lindley approximation method. First of all, we estimate \( \alpha \) under LF, we observe that
\[ g(\alpha, \lambda) = e^{-h\alpha}, \quad u_1 = -he^{-h\alpha}, \quad u_{11} = h^2 e^{-h\alpha}, \quad u_2 = u_{22} = u_{12} = u_{21} = 0. \]

Using Equation (14), the Bayes estimate of \( \alpha \) under LF is obtained as
\[ \hat{\alpha}_L = -\frac{1}{h} \log[e^{-h\hat{\alpha}} + 0.5[h^2 e^{-h\hat{\alpha}} \tau_{11} + (-he^{-h\hat{\alpha}})(l_{30} \tau_{11}^2 + l_{30} \tau_{22} \tau_{21} + l_{12}(\tau_{22} \tau_{11} + 2\tau_{21}^2) + 2p_1 \tau_{11} + 2p_2 \tau_{12})]]. \]

The Bayes estimate of \( \lambda \) is computed likewise. Also, we compute the Bayes estimate of reliability function under LF. Here, we observe that
\[ g(\alpha, \lambda) = e^{-hR(t)}, \quad u_1 = he^{-hR(t)} \frac{\log H_{1,1}^t(\lambda)}{H_{1,1}^t(\lambda)^\alpha}, \quad u_2 = h\alpha e^{-hR(t)} \frac{S_{1,1}(\lambda)}{H_{1,1}^t(\lambda)^{\alpha+1}}, \]
\[ u_{11} = he^{-hR(t)} \frac{[\log H_{1,1}^t(\lambda)]^2}{H_{1,1}^t(\lambda)^\alpha} \left[-1 + \frac{h}{H_{1,1}^t(\lambda)^\alpha} \right], \]
\[ u_{12} = u_{21} = he^{-hR(t)} \frac{S_{1,1}(\lambda)}{H_{1,1}^t(\lambda)^{\alpha+1}} \left[1 - \alpha \log H_{1,1}^t(\lambda) + h\alpha \frac{\log H_{1,1}^t(\lambda)}{H_{1,1}^t(\lambda)^\alpha} \right], \]
\[ u_{22} = h\alpha e^{-hR(t)} \frac{S_{1,1}(\lambda)}{H_{1,1}^t(\lambda)^{\alpha+2}} \times \left[ \frac{H_{1,1}^t(\lambda)}{\lambda} \left\{ \frac{H_{1,1}^t(\lambda)}{\lambda t} - 2 \right\} - (\alpha + 1)S_{1,1}(\lambda) + h\alpha \frac{S_{1,1}(\lambda)}{H_{1,1}^t(\lambda)^\alpha} \right]. \]

Using Equation (14), the Bayes estimate of reliability function under LF is obtained as
\[ \hat{R}_L(t) = -\frac{1}{h} \log[e^{-h\hat{R}(t)} + 0.5[(u_{11} \tau_{11} + 2u_{12} \tau_{12} + u_{22} \tau_{22}) + l_{30}(u_1 \tau_{11} + u_2 \tau_{12}) \tau_{11} + l_{30}(u_2 \tau_{22} + u_1 \tau_{21}) \tau_{22} + l_{12}(3u_2 \tau_{22} \tau_{21} + u_1 (\tau_{22} \tau_{11} + 2\tau_{21}^2))] + p_1(u_1 \tau_{11} + u_2 \tau_{21}) + p_2(u_2 \tau_{22} + u_1 \tau_{12})]. \]

We can derive the bayes estimate of hazard function in a similar manner. Moreover, using Equation (11), we compute the Bayes estimates under balanced LF of the parameters, reliability and hazard functions.
Next, we estimate \( \alpha \) under GELF, we observe that

\[
g(\alpha, \lambda) = \alpha^{-q}, \quad u_1 = -q\alpha^{-q-1}, \quad u_{11} = q(q + 1)\alpha^{-q-2}, \quad u_2 = u_{22} = u_{12} = u_{21} = 0.
\]

Using Equation (14), the Bayes estimate of \( \alpha \) under GELF is obtained as

\[
\hat{\alpha}_E = \left[ \hat{\alpha}^{-q} + 0.5[q(q + 1)\alpha^{-q-2}\tau_{11} + (-q\alpha^{-q-1})l_{30}\tau_{11}^2 \\
+ l_{03}\tau_{22}\tau_{21} + l_{12}(\tau_{22}\tau_{11} + 2\tau_{21}^2) + 2p_1\tau_{11} + 2p_2\tau_{12}] \right]^{-1/q}.
\]

In a similar manner, we can obtain the Bayes estimate of \( \lambda \) under GELF.

Also, we compute the Bayes estimate of reliability function under GELF. Here, we observe that

\[
g(\alpha, \lambda) = H_{1,1}^t(\lambda)\alpha^q, \quad u_1 = qH_{1,1}^t(\lambda)\alpha^q\log H_{1,1}^t(\lambda), \quad u_2 = \alpha q S_{1,1}(\lambda)H_{1,1}^t(\lambda)\alpha^q-1,
\]

\[
u_{11} = q^2 H_{1,1}^t(\lambda)\alpha^q[\log H_{1,1}^t(\lambda)]^2, \quad u_{12} = q S_{1,1}(\lambda)H_{1,1}^t(\lambda)\alpha^q-1[1 + \alpha q \log H_{1,1}^t(\lambda)],
\]

\[
u_{22} = \alpha q S_{1,1}(\lambda)H_{1,1}^t(\lambda)\alpha^q-2 \left[ (\alpha q - 1)S_{1,1}(\lambda) + \frac{H_{1,1}^t(\lambda)}{\lambda}\left\{-2 + \frac{H_{1,1}^t(\lambda)}{\lambda t}\right\} \right].
\]

Using Equation (14), the Bayes estimate of reliability function under GELF is obtained as

\[
\hat{R}_E(t) = \left[ \hat{R}(t) \right]^{-q} + 0.5[(u_{11}\tau_{11} + 2u_{12}\tau_{12} + u_{22}\tau_{22}) + l_{30}(u_1\tau_{11} + u_2\tau_{12})\tau_{11} \\
+ l_{03}(u_2\tau_{22} + u_1\tau_{21})\tau_{22} + l_{12}(3u_2\tau_{22}\tau_{21} + u_1(\tau_{22}\tau_{11} + 2\tau_{21}^2))] \\
+ p_1(u_1\tau_{11} + u_2\tau_{21}) + p_2(u_2\tau_{22} + u_1\tau_{12}) \right]^{-1/q}.
\]

Similarly, we can derive the bayes estimate of hazard function under GELF. Using Equation (12), we can also obtain the Bayes estimates under balanced GELF of the parameters, reliability and hazard functions.

### 4.2. Importance sampling

Using Lindley approximation method, in the previous section, we obtain the Bayes estimates of the parameters, reliability and hazard functions. Although Lindley approximation method gives Bayes estimates of the parameters, it cannot be used to get HPD credible intervals. Therefore, we propose to use the importance sampling procedure to obtain Bayes estimates and also to get HPD credible intervals [17].

To implement the importance sampling, we re-write Equation (13) as follows

\[
\pi(\alpha, \lambda | X) \propto \alpha^{m+c-1}\lambda^{-m+a-1} \prod_{i=1}^{m} \left\{ \exp \left(-\frac{1}{\lambda x_{i,m,n}} - b\lambda - d\alpha \right) \right\} \left\{ \frac{1 - \exp \left(-\frac{1}{\lambda x_{i,m,n}} \right)}{1 + \exp \left(-\frac{1}{\lambda x_{i,m,n}} \right)} \right\}^{\alpha(1+R_i)-1} \left\{ \frac{1 + \exp \left(-\frac{1}{\lambda x_{i,m,n}} \right)}{1 - \exp \left(-\frac{1}{\lambda x_{i,m,n}} \right)} \right\}^{\alpha(1+R_i)+1}
\]
\[ \alpha \sim \alpha_0^{(m+c)-1} \exp \left[ -\alpha \left\{ d + \sum_{i=1}^{m} (1 + R_i) \log H_{1,1}^X (\lambda) \right\} \right] \lambda^{-(m-a)-1} \exp \left[ -\frac{1}{\lambda} \sum_{i=1}^{m} \frac{1}{X_{i,m:n}} \right] \]

\[ \times \exp \left[ -\left\{ b\lambda + \sum_{i=1}^{m} \log \left\{ 1 - \exp \left( -\frac{2}{\lambda X_{i,m:n}} \right) \right\} \right\} \right] \]

\[ \propto GA \left( \alpha; m + c, d + \sum_{i=1}^{m} (1 + R_i) \log H_{1,1}^X (\lambda) \right) \cdot IGA \left( \lambda; m - a, \sum_{i=1}^{m} \frac{1}{X_{i,m:n}} \right) k(\alpha, \lambda), \]

where

\[ k(\alpha, \lambda) = \frac{\left\{ d + \sum_{i=1}^{m} (1 + R_i) \log H_{1,1}^X (\lambda) \right\}^{m-c}}{\exp \left[ b\lambda + \sum_{i=1}^{m} \log \left\{ 1 - \exp \left( -\frac{2}{\lambda X_{i,m:n}} \right) \right\} \right]}. \]

Data are generated using the following three steps.

- **Step 1.** Generate \( \lambda \) from IGA \((m - a, \sum_{i=1}^{m} X_{i,m:n}^{-1})\).
- **Step 2.** Given \( \lambda \) generated in Step 1, generate \( \alpha \) from GA \((m + c, d + \sum_{i=1}^{m} (1 + R_i) \log H_{1,1}^X (\lambda))\).
- **Step 3.** Repeat Steps 1 and 2 to generate \((\alpha_1, \lambda_1), (\alpha_2, \lambda_2), \ldots, (\alpha_N, \lambda_N)\).

Then, we observe that the Bayes estimates of \( g(\alpha, \lambda) \) under LF is obtained as

\[ \hat{g}_{LF}(\alpha, \lambda) = -\frac{1}{h} \log \left[ \frac{\sum_{i=1}^{N} e^{-g(\alpha, \lambda_i) (\alpha, \lambda_i)}}{\sum_{i=1}^{N} k(\alpha, \lambda_i)} \right]. \]

The respective Bayes estimates \( \hat{g}_{SE} \) and \( \hat{g}_{GE} \) under SELF and GELF can be obtained similarly.

Next, we obtain HPD credible intervals of parameters using method of Chen and Shao [7]. First, we consider the HPD credible interval for \( \alpha \). The same procedure will be applicable to \( \lambda \). Suppose \( a_p \) is such that \( P(\alpha \leq a_p | X) = p \). Then the Bayes estimate of \( a_p \) under the SELF can be obtained from the \((\alpha_1, \lambda_1), (\alpha_2, \lambda_2), \ldots, (\alpha_N, \lambda_N)\) as follows.

Define

\[ v_i = \frac{k(\alpha_i, \lambda_i)}{\sum_{i=1}^{N} k(\alpha, \lambda_i)}, \quad i = 1, 2, \ldots, N. \]

Order \((\alpha_1, v_1), (\alpha_2, v_2), \ldots, (\alpha_N, v_N)\) as \((\alpha_{(1)}, v_{(1)}), (\alpha_{(2)}, v_{(2)}), \ldots, (\alpha_{(N)}, v_{(N)})\), where \( \alpha_{(1)} < \cdots < \alpha_{(N)} \). Here, \( v_{(i)} \)'s are not ordered. \( v_{(i)} \)'s are just associated with \( \alpha_{(i)} \). Then the Bayes estimate of \( a_p \) is \( \hat{a}_p = \alpha_{(N_p)} \), where \( N_p \) is the integer satisfying

\[ \sum_{i=1}^{N_p} v_{(i)} \leq p < \sum_{i=1}^{N_p+1} v_{(i)}. \]

Hence, using the above procedure a 100(1 - \( \beta \))% credible interval of \( \alpha \) can be obtained as \((\hat{\alpha}_\delta, \hat{\alpha}_{\delta+1-\beta})\), for \( \delta = v_{(1)}, v_{(1)} + v_{(2)}, \ldots, \sum_{i=1}^{N_\delta} v_{(i)} \). Therefore, a 100(1 - \( \beta \))% HPD credible interval of \( \alpha \) becomes \((\hat{\alpha}_{\delta^*}, \hat{\alpha}_{\delta^*+1-\beta})\), where \( \delta^* \) is such that \( \hat{\alpha}_{\delta^*+1-\beta} - \hat{\alpha}_{\delta^*} \leq \hat{\alpha}_{\delta+1-\beta} - \hat{\alpha}_\delta \) for all \( \delta \).
5. Examples

In this section, real and simulated data sets are analyzed for illustration. We analyze simulated and data set in Examples 1 and 2.

5.1. Simulated data

Example 1: In this example, using Monte Carlo simulation method, we generate progressive Type II censored data from IEHLD ($\alpha = 1.5, \lambda = 2.5$) of $n = 20$ and $m = 15$ with progressive Type II censoring scheme as $R_1 = 2, R_{15} = 3$ and $R_i = 0$ for $i \neq 1, 15$. In Table 1, we have tabulated a progressive Type II censored data.

All Bayes estimates are computed with respect to the noninformative and proper prior distribution. We denote prior 1 for noninformative Bayes estimates in which case hyperparameters are given values as $a = b = c = d = 0$. Also, we denote prior 2 for proper Bayes estimates in which case hyperparameters are given values as $a = c = 2$ and $b = d = 3$. Bayes

Table 1. Simulated progressive Type II censored data.

| $i$ | $x_{init}$ | $R_i$ | $x_i$ | $m_i$ |
|-----|------------|-------|-------|-------|
| 1   | 0.0619     | 2     | 0.0619 | 0.0986 |
| 2   | 0.0986     | 0     | 0.1259 | 0.1262 |
| 3   | 0.1259     | 0     | 0.1262 | 0.1497 |
| 4   | 0.1262     | 0     | 0.1497 | 0.1607 |
| 5   | 0.1497     | 0     | 0.1607 | 0.1989 |
| 6   | 0.1607     | 0     | 0.1989 | 0.2114 |
| 7   | 0.1989     | 0     | 0.2114 | 0.2114 |
| 8   | 0.2114     | 0     | 0.2114 | 0.2114 |

Table 2. Estimates of parameters for Example 1.

| Prior | $w$ | $\hat{\alpha}_L$ | $\hat{\alpha}_S$ | $\hat{\lambda}_L$ | $\hat{\lambda}_S$ | $\hat{\alpha}_E$ | $\hat{\lambda}_E$ |
|-------|-----|------------------|------------------|-------------------|-------------------|------------------|------------------|
| 1     | 1.6062 | 1.6382 | 1.6893 | 1.5355 | 1.4442 | 1.5737 | 1.4556 | 1.4074 |
| 2     | 1.8044 | 1.8489 | 1.7028 | 1.5966 | 1.7425 | 1.6103 | 1.5478 |
| 3     | 2.1676 | 2.1822 | 2.1286 | 2.0723 | 2.1460 | 2.0790 | 2.0279 |
| 4     | 2.0638 | 2.1008 | 1.9771 | 1.8767 | 2.0158 | 1.8938 | 1.8193 |
| 5     | 1.6286 | 1.6646 | 1.5565 | 1.4901 | 1.5834 | 1.4984 | 1.4617 |
| 6     | 1.6222 | 1.6480 | 1.5705 | 1.5219 | 1.5899 | 1.5281 | 1.5002 |
| 7     | 1.6158 | 1.6313 | 1.5847 | 1.5548 | 1.5964 | 1.5586 | 1.5409 |
| 8     | 1.7449 | 1.7776 | 1.6733 | 1.5995 | 1.7010 | 1.6091 | 1.5648 |
| 9     | 1.7053 | 1.7294 | 1.6539 | 1.6014 | 1.6736 | 1.6082 | 1.5764 |
| 10    | 1.6656 | 1.6805 | 1.6347 | 1.6033 | 1.6465 | 1.6074 | 1.5882 |

| Prior | $w$ | $\hat{\alpha}_L$ | $\hat{\alpha}_S$ | $\hat{\lambda}_L$ | $\hat{\lambda}_S$ | $\hat{\alpha}_E$ | $\hat{\lambda}_E$ |
|-------|-----|------------------|------------------|-------------------|-------------------|------------------|------------------|
| 1     | 2.6769 | 2.9679 | 3.0193 | 2.8508 | 2.7312 | 2.9272 | 2.8406 | 2.7967 |
| 2     | 2.7623 | 2.8262 | 2.6874 | 2.6415 | 2.7341 | 2.6888 | 2.6704 |
| 3     | 2.9083 | 2.9459 | 2.8260 | 2.7424 | 2.8793 | 2.8189 | 2.7884 |
| 4     | 2.8685 | 2.8962 | 2.8096 | 2.7500 | 2.8476 | 2.8045 | 2.7828 |
| 5     | 2.8288 | 2.8458 | 2.7933 | 2.7576 | 2.8161 | 2.7903 | 2.7773 |
| 6     | 2.8401 | 2.8974 | 2.7219 | 2.6151 | 2.7973 | 2.7123 | 2.6721 |
| 7     | 2.7518 | 2.7786 | 2.7170 | 2.6960 | 2.7389 | 2.7183 | 2.7101 |
| 8     | 2.8188 | 2.8593 | 2.7360 | 2.6589 | 2.7888 | 2.7292 | 2.7005 |
| 9     | 2.8046 | 2.8338 | 2.7454 | 2.6892 | 2.7832 | 2.7405 | 2.7197 |
| 10    | 2.7904 | 2.8080 | 2.7548 | 2.7204 | 2.7776 | 2.7519 | 2.7393 |
estimates under LF and GELF are obtained for three distinct values of \( h = -0.5, 0.5, 1.0 \) and \( q = -0.5, 0.5, 1.0 \). Also, the Bayes estimates under balanced loss function are obtained for three distinct values of \( w = 0.3, 0.5, 0.7 \).

The MLEs and Bayes estimates of the parameters are tabulated in Table 2. In the third column, the MLEs of parameters are tabulated. All other columns have uniformly five values. The first two values are Bayes estimate and MCMC estimate of \( \alpha \), respectively, and the third, fourth and fifth values are Bayes estimates under balanced loss function of \( \alpha \). The last five values have similar illustrations for the \( \lambda \).

The MLEs and Bayes estimates of the reliability and hazard functions are tabulated in Table 3. In the third column, the MLEs of reliability and hazard functions are tabulated. All other columns have uniformly four values. The first values are Bayes estimate of reliability function, and the second, third and fourth values are Bayes estimates under balanced loss function of reliability function. The last four values have similar illustrations for the hazard function.

CIs and HPD credible intervals of parameters, reliability and hazard functions are tabulated in Table 4. In CIs and HPD credible intervals, the noninformative and proper prior has been assumed on parameters.

| Prior | \( w \) | \( \hat{R} \) | \( \hat{R}_L \) | \( \hat{R}_E \) |
|-------|--------|--------|--------|--------|
| 1     | 0.0048 | 0.0123 | 0.0129 | 0.0129 |
| 0.3   | 0.0101 | 0.0105 | 0.0104 | 0.0104 |
| 0.5   | 0.0086 | 0.0088 | 0.0088 | 0.0088 |
| 0.7   | 0.0071 | 0.0072 | 0.0072 | 0.0072 |
| 2     | 0.0129 | 0.0135 | 0.0135 | 0.0135 |
| 0.3   | 0.0105 | 0.0109 | 0.0109 | 0.0109 |
| 0.5   | 0.0089 | 0.0092 | 0.0092 | 0.0091 |
| 0.7   | 0.0073 | 0.0074 | 0.0074 | 0.0074 |

| Prior | \( w \) | \( \hat{H} \) | \( \hat{H}_L \) | \( \hat{H}_E \) |
|-------|--------|--------|--------|--------|
| 1     | 0.3210 | 0.3270 | 0.3290 | 0.3228 |
| 0.3   | 0.3252 | 0.3266 | 0.3223 | 0.3194 |
| 0.5   | 0.3240 | 0.3250 | 0.3219 | 0.3198 |
| 0.7   | 0.3228 | 0.3234 | 0.3215 | 0.3203 |
| 2     | 0.3359 | 0.3380 | 0.3318 | 0.3277 |
| 0.3   | 0.3314 | 0.3329 | 0.3286 | 0.3257 |
| 0.5   | 0.3284 | 0.3295 | 0.3264 | 0.3243 |
| 0.7   | 0.3254 | 0.3261 | 0.3242 | 0.3230 |

| Prior | \( w \) | \( \Lambda_L \) | \( \Lambda_E \) |
|-------|--------|--------|--------|
| 1     | 0.0095 | 0.0023 | 0.0017 |
| 0.3   | 0.0079 | 0.0028 | 0.0022 |
| 0.5   | 0.0070 | 0.0032 | 0.0026 |
| 0.7   | 0.0061 | 0.0037 | 0.0031 |
| 2     | 0.0104 | 0.0025 | 0.0018 |
| 0.3   | 0.0085 | 0.0030 | 0.0022 |
| 0.5   | 0.0074 | 0.0034 | 0.0027 |
| 0.7   | 0.0063 | 0.0039 | 0.0032 |

| Prior | \( w \) | \( \Lambda_L \) | \( \Lambda_E \) |
|-------|--------|--------|--------|
| 1     | 0.3145 | 0.2911 | 0.2815 |
| 0.3   | 0.3164 | 0.2996 | 0.2923 |
| 0.5   | 0.3177 | 0.3055 | 0.2999 |
| 0.7   | 0.3190 | 0.3115 | 0.3080 |
| 2     | 0.3235 | 0.2990 | 0.2886 |
| 0.3   | 0.3227 | 0.3053 | 0.2976 |
| 0.5   | 0.3222 | 0.3097 | 0.3039 |
| 0.7   | 0.3217 | 0.3141 | 0.3105 |

| Parameter | NA | NL | HPD (prior 1) | HPD (prior 2) |
|-----------|----|----|---------------|---------------|
| \( \alpha \) | (0.3424, 2.8699) | (0.7313, 3.5278) | (0.2309, 2.4154) | (0.2271, 1.7910) |
| \( \lambda \) | (1.4184, 4.1198) | (1.7002, 4.5101) | (2.4558, 4.1049) | (2.7077, 3.7077) |
| \( R \) | (0.0000, 0.0223) | (0.0001, 0.1824) | (2.4558, 4.1049) | (2.7077, 3.7077) |
| \( H \) | (0.0686, 0.5733) | (0.1462, 0.7045) | (2.4558, 4.1049) | (2.7077, 3.7077) |
5.2. Real data

Example 2: In this example, we consider a real data set and illustrate the proposed estimates. The real data set is given by Lawless [18] and it represents the number of million revolutions before failure for each of 23 ball bearings in a life test as follows.

| 17.88 | 28.92 | 33   | 41.52 | 42.12 | 45.6  | 48.4  | 51.84 |
|-------|-------|------|-------|-------|-------|-------|-------|
| 51.96 | 54.12 | 55.56| 67.8  | 68.64 | 68.64 | 68.88 | 84.12 |
| 93.12 | 98.64 | 105.12| 105.84| 127.92| 128.04| 173.4 |

Potdar and Shirke [23] fitted six models IEHLD, generalized inverted exponential distribution (GIED), inverted exponential (IED), exponential (EXP), gamma (GA) and weibull (WEI). To goodness-of-fit test, Potdar and Shirke [23] considered negative log-likelihood, Akaike’s information criterion (AIC), Bayesian information criterion (BIC) and Kolmogorov–Smirnov (K–S) statistic. The values of MLEs, four goodness-of-fit test statistics are given in Table 5. From Table 5, we can see that an IEHLD fit the data quite well as compared with the GIED, IED, EXP, GA and WEI. Therefore, we can see that the real data comes from an IEHLD population and we make estimation on the parameters, reliability and hazard functions. In Table 6, we consider the case when the data are progressive Type II censoring.

All Bayes estimates are computed with respect to the noninformative and proper prior distribution. Bayes estimates under LF and GELF are obtained for three distinct values of $h = -0.5, 0.5, 1.0$ and $q = -0.5, 0.5, 1.0$. Also, the Bayes estimates under balanced loss function are obtained for three distinct values of $w = 0.3, 0.5, 0.7$.

The MLEs and Bayes estimates of the parameters are tabulated in Table 7. In the third column, the MLEs of parameters are tabulated. All other columns have uniformly five values. The first two values are Bayes estimate and MCMC estimate of $\alpha$, respectively, and the third, fourth and fifth values are Bayes estimates under balanced loss function of $\alpha$. The last five values have similar illustrations for the $\lambda$.

The MLEs and Bayes estimates of the reliability and hazard functions are tabulated in Table 8. In the third column, the MLEs of reliability and hazard functions are tabulated. All

| Distribution | $\hat{\alpha}$ | $\hat{\lambda}$ | $-\log L$ | AIC  | BIC  | K–S  |
|--------------|----------------|-----------------|-----------|------|------|------|
| IEHLD        | $3.3735$       | $0.0071$        | $113.8679$| $231.7358$ | $234.0068$ | $0.0609$ |
| GIED         | $5.3076$       | $129.9959$      | $113.5490$| $231.0980$ | $233.3690$ | $0.0703$ |
| IED          | $55.0551$      | $0.0138$        | $121.7259$| $245.4519$ | $246.5874$ | $0.3060$ |
| EXP          | $0.0138$       |                 | $121.4338$| $244.8675$ | $246.0030$ | $0.2622$ |
| GA           | $0.0557$       | $4.0244$        | $113.0298$| $230.0596$ | $232.3306$ | $0.1233$ |
| WEI          | $2.1018$       | $81.8745$       | $113.6920$| $231.3839$ | $233.6549$ | $0.1510$ |

| $i$ | $k_{i,mn}$ | $R_i$ |
|-----|------------|------|
| 1   | 17.88      | 3    |
| 2   | 28.92      | 0    |
| 3   | 33         | 0    |
| 4   | 41.52      | 0    |
| 5   | 51.84      | 0    |
| 6   | 42.12      | 0    |
| 7   | 45.6       | 0    |
| 8   | 48.4       | 0    |
| 9   | 51.84      | 0    |
| 10  | 84.12      | 0    |

| $i$ | $k_{i,mn}$ | $R_i$ |
|-----|------------|------|
| 1   | 68.64      | 0    |
| 2   | 68.88      | 0    |
| 3   | 84.12      | 0    |
| 4   | 93.12      | 0    |
| 5   | 98.64      | 0    |
| 6   | 105.12     | 0    |
| 7   | 105.84     | 0    |
| 8   | 127.92     | 0    |
| 9   | 128.04     | 0    |
| 10  | 173.4      | 0    |
Table 7. Estimates of parameters for Example 2.

| Prior | w   | $\hat{\alpha}_L$ | $\hat{\alpha}_E$ | $h = -0.5$ | $h = 0.5$ | $h = 1.0$ | $q = -0.5$ | $q = 0.5$ | $q = 1.0$ |
|-------|-----|------------------|------------------|------------|------------|------------|------------|------------|------------|
| 1     | 3.1146 | 3.1954 | 3.3756 | 2.8375 | 2.5987 | 3.0741 | 2.5987 | 2.7565 | 2.7565 |
| 0.3   | 3.1711 | 3.2991 | 2.9167 | 2.7276 | 2.5987 | 3.0741 | 2.7565 | 2.7565 | 2.7565 |
| 0.5   | 3.1550 | 3.2473 | 2.9713 | 2.8238 | 2.5987 | 3.0741 | 2.7565 | 2.7565 | 2.7565 |
| 0.7   | 3.1388 | 3.1947 | 3.0274 | 2.9302 | 2.5987 | 3.0741 | 2.7565 | 2.7565 | 2.7565 |
| 2     | 2.8574 | 3.0458 | 2.5630 | 2.4149 | 2.5987 | 3.0741 | 2.7565 | 2.7565 | 2.7565 |
| 0.3   | 2.9346 | 3.0666 | 2.7131 | 2.5786 | 2.5987 | 3.0741 | 2.7565 | 2.7565 | 2.7565 |
| 0.5   | 2.9860 | 3.0835 | 2.8199 | 2.7048 | 2.5987 | 3.0741 | 2.7565 | 2.7565 | 2.7565 |
| 0.7   | 3.0375 | 3.0941 | 2.9361 | 2.8492 | 2.5987 | 3.0741 | 2.7565 | 2.7565 | 2.7565 |

Table 8. Estimates of reliability ($t = 50$) and hazard functions ($t = 50$) for Example 2.

| Prior | w   | $\hat{R}_L$ | $\hat{R}_E$ | $h = -0.5$ | $h = 0.5$ | $h = 1.0$ | $q = -0.5$ | $q = 0.5$ | $q = 1.0$ |
|-------|-----|-------------|-------------|------------|------------|------------|------------|------------|------------|
| 1     | 0.6906 | 0.6665 | 0.6880 | 0.6853 | 0.6834 | 0.6844 | 0.6791 | 0.6766 | 0.6766 |
| 0.3   | 0.6737 | 0.6888 | 0.6868 | 0.6855 | 0.6834 | 0.6844 | 0.6791 | 0.6766 | 0.6766 |
| 0.5   | 0.6785 | 0.6893 | 0.6879 | 0.6870 | 0.6855 | 0.6844 | 0.6791 | 0.6766 | 0.6766 |
| 0.7   | 0.6834 | 0.6898 | 0.6890 | 0.6884 | 0.6855 | 0.6844 | 0.6791 | 0.6766 | 0.6766 |
| 2     | 0.5790 | 0.5995 | 0.5998 | 0.6001 | 0.6001 | 0.6001 | 0.6001 | 0.6001 | 0.6001 |
| 0.3   | 0.6125 | 0.6271 | 0.6266 | 0.6264 | 0.6264 | 0.6264 | 0.6264 | 0.6264 | 0.6264 |
| 0.5   | 0.6348 | 0.6453 | 0.6447 | 0.6443 | 0.6443 | 0.6443 | 0.6443 | 0.6443 | 0.6443 |
| 0.7   | 0.6571 | 0.6635 | 0.6629 | 0.6626 | 0.6626 | 0.6626 | 0.6626 | 0.6626 | 0.6626 |

other columns have uniformly four values. The first values are Bayes estimate of reliability function, and the second, third and fourth values are Bayes estimates under balanced loss function of reliability function. The last four values have similar illustrations for the hazard function.
CIs and HPD credible intervals of parameters, reliability and hazard functions are tabulated in Table 9. In CIs and HPD credible intervals, the noninformative and proper prior has been assumed on parameters.

5.3. Optimal progressive Type II censoring scheme

In the previous subsections, we have discussed point and interval estimations of unknown parameters of IEHLD when samples are obtained using progressive Type II censoring. So, to conduct a life-testing experiment under progressive Type II censoring, the values of \( n, m \) and \((R_1, R_2, \ldots, R_m)\) must be known in advance. However, in various reliability and life testing studies, it is desirable for practical considerations to select the optimum progressive Type II censoring scheme from a class of possible schemes. The problem of comparing two different censoring schemes has received much interest among various researchers, see for example, Ng et al. [22], Kundu [16], and Lee et al. [19]. To determine the optimum progressive Type II censoring, we consider an information measure as following criteria.

**Criterion I**: Minimizing the determinant of the variance–covariance matrix of the MLEs.

**Criterion II**: Minimizing the trace of the variance–covariance matrix of the MLEs.

Using the previous Example 2, we illustrate how we can find the optimal censoring scheme.

The calculated values of determinant and trace of the variance–covariance matrix of the MLEs are presented in Table 10. It is clear from these values that the optimal censoring scheme is the one with the minimum determinant or trace of the variance - covariance matrix of the MLEs. From Table 10, therefore, the optimum scheme is \((n, m, (R_1, \ldots, R_m)) = (23, 20, (0 \ast 19, 3))\).

6. Numerical results

To know the performance of proposed estimates based on progressive Type II censoring, we perform a numerical study. We simulate progressive Type II censored samples of size \( m \)

### Table 9. Interval estimates of parameters for Example 2.

|         | NA       | NL       | HPD (prior 1) | HPD (prior 2) |
|---------|----------|----------|---------------|---------------|
| \( \alpha \) | (0.7047, 5.5245) | (1.4367, 6.7521) | (0.3697, 5.4223) | (0.3497, 4.9107) |
| \( \lambda \) | (0.0042, 0.0099) | (0.0047, 0.0106) | (0.0034, 0.0102) | (0.0101, 0.0163) |
| \( R \)  | (0.5210, 0.8601) | (0.5402, 0.8827) |
| \( H \)  | (0.0115, 0.0304) | (0.0134, 0.0329) |

### Table 10. Comparison of different censoring scheme for Example 2.

| \((N, m)\) | \(R\) | \(\hat{\alpha}\) | \(\hat{\lambda}\) | Criterion 1 | Criterion 2 |
|------------|-------|----------------|----------------|-------------|-------------|
| (23, 20)   | (0\ast19, 3) | 1.7377 | 0.0083 | 1,041,381.0 | 760,618.8 |
|            | (3, 0\ast19) | 3.1146 | 0.0071 | 1,081,748.4 | 1,484,189.0 |
|            | (2, 0\ast18, 1) | 2.4506 | 0.0076 | 1,333,375.0 | 1,120,552.0 |
|            | (1, 0, 1, 0, 1, 0\ast15) | 3.1761 | 0.0069 | 1,042,573.0 | 1,673,170.0 |
|            | (1, 0\ast3, 2, 0\ast15) | 3.1763 | 0.0068 | 1,058,558.0 | 1,733,626.0 |
|            | (0\ast15, 1, 0\ast3, 2) | 1.8976 | 0.0080 | 1,868,302.0 | 880,998.7 |
|            | (0\ast15, 1, 0, 1, 0, 1) | 2.0217 | 0.0078 | 1,744,143.0 | 979,265.0 |
|            | (0\ast10, 3, 0\ast9) | 2.9928 | 0.0068 | 1,161,869.0 | 1,766,828.0 |
|            | (0\ast7, 2, 0\ast7, 1, 0\ast4) | 2.8543 | 0.0069 | 1,225,936.0 | 1,623,579.0 |
|            | (0\ast4, 1, 0\ast4, 1, 0\ast4, 1, 0\ast5) | 2.8563 | 0.0069 | 1,219,702.0 | 1,635,676.0 |
from a given sample of size \( n \) with given progressive Type II censoring scheme. For convenience, short notation is used to represent different progressive Type II censoring schemes, for instance, progressive Type II censoring scheme \((1, 0, 0, 0, 0, 0)\) is denoted by \((1, 0 \ast 5)\). Comparison between proposed estimates is made in terms of MSEs. The MLEs of parameters, reliability and hazard functions are obtained using a Kundu’s [15] method. Bayes estimates of parameters, reliability and hazard functions are derived with respect to SELF, LF, GELF, balanced SELF, balanced LF and balanced GELF. The Lindley approximation method has then been used to derive explicit expressions for Bayes estimates.

For now, we describe how we generate progressive Type II censored data for a given sample size \( n, m \) and progressive Type II censoring scheme. We use the following transformation introduced in Balakrishnan and Aggarwala [6].

\[
\begin{align*}
Z_1 &= nY_{1:m:n}, \\
Z_2 &= (n - R_1 - 1)(Y_{2:m:n} - Y_{1:m:n}), \\
\vdots & \ \\
Z_m &= (n - R_1 - \cdots - R_{m-1} - m - 1)(Y_{m:m:n} - Y_{m-1:m:n}).
\end{align*}
\]

(15)

It is known that if random variables \( Y_{i:m:n} \)'s are iid standard EXP, the spacings \( Z_i \)'s are also iid standard EXP random variables. From Equation (15) it follows that

\[
\begin{align*}
Y_{1:m:n} &= \frac{1}{n}Z_1, \\
Y_{2:m:n} &= \frac{1}{n - R_1 - 1}Z_2 + \frac{1}{n}Z_1, \\
\vdots & \ \\
Y_{m:m:n} &= \frac{1}{n - R_1 - \cdots - R_{m-1} - m - 1}Z_m + \cdots + \frac{1}{n}Z_1.
\end{align*}
\]

Next, we set \( X_{i:m:n} = F^{-1}(1 - \exp(Y_{i:m:n})) \), for \( i = 1, 2, \ldots, m \), where \( F^{-1}(\cdot) \) is the inverse cdf of the IEHLD. Then \( X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n} \) is the progressively Type II censored data from the IEHLD.

We replicated 10,000 times progressive Type II censored data of size \( n, m \) drawn from an IEHLD. The true value of \( \alpha \) and \( \lambda \) is taken as 0.5. We denote prior 1 for noninformative Bayes estimates in which case hyperparameters are given values as \( a = b = c = d = 0 \). Also, we denote prior 2 for proper Bayes estimates in which case hyperparameters are given values as \( a = c = 2 \) and \( b = d = 3 \). Bayes estimates under LF and GELF are obtained for three distinct values of \( h = -0.5, 0.5, 1.0 \) and \( q = -0.5, 0.5, 1.0 \). Also, the Bayes estimates under balanced loss function are obtained for three distinct values of \( w = 0.3, 0.5, 0.7 \). Finally, different schemes have been taken into consideration to obtain MSEs and biases of proposed estimates. A simulation results for various schemes are presented in the supplementary file. The MSEs and biases of parameters are presented in Table 11. In addition, the MSEs and biases of reliability and hazard functions are obtained as two distinct values of \( t = (5, 10) \), and these values are presented in Tables 12 and 13. We also obtain the average confidence lengths and the corresponding coverage percentages of reliability and hazard functions,
and these are presented in Tables 14 and 15. We present following discussions based on the MSEs and coverage percentages of estimates.

In Table 11, MSEs biases of proposed estimates of parameters are tabulated for different choices of sample size \( n, \) and progressive Type II censoring schemes. We have tabulated MSEs and biases of the respective MLEs in the fourth column of the table. Uniformly, all other columns contain 10 values. The first value represents the MSE of Bayes estimates by using the Lindley approximation method and the second value represents the corresponding bias. The third and fourth values correspond to the MSE and bias of parameters derived using importance sampling procedure. The last six values correspond to the MSE and bias of parameters derived Bayes estimates under balanced loss function. In general, it can be seen that the MSEs of all estimates reduce with an increasing in sample size \( n. \) For fixed sample size \( n, \) the MSEs of all estimates reduce with an decreasing in number of progressive Type II censoring \( R_i. \)

We can observe that the MLEs and noninformative Bayes estimates of \( \lambda \) under SELF and LF behave almost similar in terms of MSEs. This holds true for all presented schemes. However, Bayes estimates of \( \lambda \) under LF is marginally good. We can observe that noninformative Bayes estimates of \( \lambda \) under GELF are better than the MLEs and noninformative Bayes estimates of \( \lambda \) under SELF and LF in terms of MSEs. Similar pattern is observed in case of the estimation of \( \alpha. \) It is also seen that all proper Bayes estimates of \( \lambda \) perform better than its MLEs in terms of MSEs. Among proper Bayes estimates of \( \lambda, \) we can observe that proper Bayes estimates obtained using the GELF have overall lower MSEs. We observe that Lindley’s approximation method perform better compared to the other estimates. Similar behavior is observed in case of estimation of \( \alpha. \)

Also, the problem of selecting a suitable loss function is concerned, it can be seen that LF and GELF emerges as the best loss function. Among noninformative Bayes estimates of parameters, we observed that Bayes estimates obtained using the LF for the choice \( h = 1 \) have overall lower MSEs. Also, we observed that noninformative Bayes estimates obtained using the GELF for the choice \( q = 1 \) have overall lower MSEs. Therefore, the choice \( h = 1 \) and \( q = 1 \) seem to be a reasonable choice under LF and GELF. Also, the choice \( w = 0.3 \) seems to be a reasonable choice under balanced LF and balanced GELF, while \( w = 0.7 \) is a good choice under balanced SELF. Similar behavior is observed in case of proper Bayes estimation.

In Table 12, MSEs and biases of proposed estimates of reliability function are tabulated for two distinct values \( t = 5 \) and \( t = 10. \) The fourth and the twelfth columns represent MSEs and biases for MLEs of reliability function for \( t = 5 \) and \( t = 10. \) Uniformly, all other columns contain eight values, respectively. The first value represents the MSE of Bayes estimates by using the Lindley approximation method and the second value represents the corresponding bias. The last six values correspond to the MSE and bias of reliability function derived Bayes estimates under balanced loss function. It can be seen that the MSEs of all estimates reduce with an increasing in sample size \( n. \) For fixed sample size \( n, \) the MSEs of all estimates reduce with an decreasing in number of progressive Type II censoring \( R_i. \)

We can observe that the MLEs and noninformative Bayes estimates of reliability function under SELF, LF and GELF behave almost similar in terms of MSEs. This holds true for all presented schemes. However, Bayes estimates of reliability function under Bayes estimates is marginally good. We can observe that noninformative Bayes estimates of reliability
function under LF are better than the MLEs and noninformative Bayes estimates of reliability function under SELF and GELF in terms of MSEs. It is also seen that all proper Bayes estimates of reliability function perform better than its MLEs in terms of MSEs. Among proper Bayes estimates of reliability function, we can observe that proper Bayes estimates obtained using the LF have overall lower MSEs. We observed that Lindley’s approximation method perform better compared to the other estimates.

Also, the problem of selecting a suitable loss function is concerned, it can be seen that LF and GELF emerges as the best loss function. Among noninformative Bayes estimates of parameters, we observed that Bayes estimates obtained using the LF for the choice \( h = 1 \) have overall lower MSEs. Also, we observed that noninformative Bayes estimates obtained using the GELF for the choice \( q = -0.5 \) have overall lower MSEs. Therefore, the choice \( h = 1 \) and \( q = -0.5 \) seem to be a reasonable choice under LF and GELF, respectively. Also, the choice \( w = 0.3 \) seems to be a reasonable choice under balanced SELF, balanced LF and balanced GELF. Similar behavior is observed in case of proper Bayes estimation.

In Table 13, MSEs and biases of proposed estimates of hazard function are tabulated for two distinct values \( t = 5 \) and \( t = 10 \). We have presented MSEs and biases of all estimates of hazard function where each column is interpreted in a manner exactly similar to Table 12. We can observe that the MLEs and noninformative Bayes estimates of hazard function under SELF behave almost similar in terms of MSEs. We can observe that noninformative Bayes estimates of hazard function under GELF are better than the MLEs and noninformative Bayes estimates of hazard function under SELF and LF in terms of MSEs. It is also seen that all proper Bayes estimates of hazard function perform better than its MLEs in terms of MSEs. Among noninformative Bayes estimates of hazard function, we observed that Bayes estimates obtained using the LF for the choice \( h = 1 \) have overall lower MSEs. Also, we observed that noninformative Bayes estimates obtained using the GELF for the choice \( q = -0.5 \) have overall lower MSEs. Therefore, the choice \( h = 1 \) and \( q = -0.5 \) seem to be a reasonable choice under LF and GELF. Also, the choice \( w = 0.3 \) seems to be a reasonable choice under balanced SELF, balanced LF and balanced GELF. Similar behavior is observed in case of proper Bayes estimation.

In Table 14, interval estimates for the parameters are tabulated for different choices of sample size \( n, m \) and progressive Type II censoring schemes. Asymptotic CIs are computed using the normality property of MLEs. Estimates of HPD credible intervals using noninformative (prior 1) and proper (prior 2) prior are also tabulated in Table 14. In general, the average lengths of NA, NL and HPD credible intervals using noninformative and proper prior tends to decrease with the effective increase in sample sizes. We observed that the average lengths of NL CIs for parameters is wider than the corresponding lengths of NA CIs. Also, we observe that the average lengths of HPD (prior 1) credible intervals for parameters is wider than the corresponding lengths of NA CIs. We further observed that the HPD (prior 2) credible intervals using proper prior show superior than NA, NL and HPD (prior 1). In Table 14, we have also tabulated the coverage probabilities for the interval estimates of parameters. Coverage probabilities of NA and NL CIs are mostly below the nominal level. In case of HPD credible intervals, however, in some cases coverage probabilities lie below the nominal level and for some schemes it lies above the nominal level.

In Table 15, interval estimates for the reliability and hazard functions are tabulated for different choices of sample size \( n, m \) and progressive Type II censoring schemes. Asymptotic CIs are computed using the normality property of MLEs. In general, the average
lengths of NA and NL CIs tends to decrease with the effective increase in sample sizes. We observed that the average lengths of NL CIs for reliability and hazard functions is wider than the corresponding lengths of NA CIs. In Table 15, we have also tabulated the coverage probabilities for the interval estimates of reliability and hazard functions. Coverage probabilities of NA and NL CIs are mostly below the nominal level.

7. Conclusion

In this paper, the estimation of the parameters, reliability and hazard functions of an IEHLD based on progressive Type II censoring scheme has been considered. Both the classical and Bayes estimation of the parameters, reliability and hazard functions are provided. Since the MLEs of the parameters, reliability and hazard functions cannot be obtained in closed form, Kundu’s [15] method have been considered. Based on the asymptotic normality of the MLEs the approximate CIs are considered. Also, we consider the Bayes estimates of the parameters, reliability and hazard functions based on SELF, LF, GELF, balanced SELF, balanced LF and balanced GELF. Since they cannot be obtained in explicit forms, Lindley approximation method has been considered. In this paper, although we have mainly considered progressive Type II censoring and IEHLD, the same method can be extended for other distribution and censoring schemes also.

Disclosure statement

No potential conflict of interest was reported by the authors.

ORCID

Kyeongjun Lee http://orcid.org/0000-0002-0754-5172

References

[1] E.A. Ahmed, Bayesian estimation based on progressive Type-II censoring from two-parameter bathtub-shaped lifetime model: An Markov chain Monte Carlo approach, J. Appl. Stat. 41 (2014), pp. 752–768.
[2] S.H. Arora, G.C. Bihmani, and M.N. Patel, Some results on maximum likelihood estimators of parameters of generalized half logistic distribution under Type I progressive censoring with changing failure rate, Int. J. Contemp. Math. Sci. 5 (2010), pp. 685–698.
[3] R. Azimi, Bayesian estimation of generalized half logistic Type II doubly censored data, Int. J. Sci. World 1 (2013), pp. 57–60.
[4] R. Azimi, F. Yaghmaei, and M. Babanezhad, Bayesian estimation of generalized half logistic distribution under progressive Type II censored, Appl. Math. Sci. 6 (2012), pp. 5253–5261.
[5] N. Balakrishnan, Order statistics from the half logistic distribution, J. Stat. Comput. Simul. 20 (1985), pp. 287–309.
[6] N. Balakrishnan and R. Aggarwala, Progressive Censoring: Theory, Methods and Applications, Birkhauser, Boston, 2000.
[7] M.H. Chen and Q.M. Shao, Monte Carlo estimation of Bayesian credible and HPD intervals, J. Comput. Graph. Statist. 8 (1999), pp. 69–92.
[8] Y. Cho, H. Sun, and K. Lee, Estimating the entropy of a Weibull distribution under generalized progressive hybrid censoring, Entropy 17 (2015), pp. 101–122.
[9] Y. Cho, H. Sun, and K. Lee, Exact likelihood inference for an exponential parameter under generalized progressive hybrid censoring scheme, Stat. Methodol. 23 (2015), pp. 18–34.
[10] W.H. Greene, Econometric Analysis, 4th ed., Prentice-Hall, New York, 2000.
[11] S.-R. Huang and S.-J. Wu, Bayesian estimation and prediction for Weibull model with progressive censoring, J. Stat. Comput. Simul. 82 (2012), pp. 1607–1620.

[12] D.-H. Jang, J. Park, and C. Kim, Estimation of the scale parameter of the half-logistic distribution with multiply Type II censored sample, J. Korean Statist. Soc. 40 (2011), pp. 291–301.

[13] M. Joani, E. Marchand, and A. Parsian, Bayes and robust Bayesian estimation under a general class of balanced loss functions, Statist. Papers 53 (2012), pp. 51–60.

[14] C. Kim and K. Han, Estimation of the scale parameter of the half-logistic distribution under progressively Type II censored sample, Statist. Papers 51 (2010), pp. 375–387.

[15] D. Kundu, On hybrid censored Weibull distribution, J. Statist. Plann. Inference 137 (2007), pp. 2127–2142.

[16] D. Kundu, Bayesian inference and life testing plan for Weibull distribution in presence of progressive censoring, Technometrics 50 (2008), pp. 144–154.

[17] D. Kundu and B. Pradhan, Bayesian inference and life testing plans for generalized exponential distribution, Sci. China Ser. A. 52 (2009), pp. 1373–1388.

[18] J. Lawless, Statistical Models and Methods for Lifetime Data, John Wiley and Sons, New York, 1982.

[19] K. Lee, H. Sun, and Y. Cho, Exact likelihood inference of the exponential parameter under generalized Type II progressive hybrid censoring, J. Korean Statist. Soc. 45 (2015), pp. 123–136.

[20] D. Lindley, Approximate Bayesian methods, Trabajos Estadisticas 31 (1980), pp. 223–237.

[21] W.Q. Meeker and L.A. Escobar, Statistical Methods for Reliability Data, Wiley, New York, 1998.

[22] H.K.T. Ng, P.S. Chan, and N. Balakrishnan, Optimal progressive censoring plans for the Weibull distribution, Technometrics 46 (2004), pp. 470–481.

[23] K. Potdar and D. Shirke, Inference for the parameters of generalized inverted family of distributions, ProbStat. Forum 6 (2013), pp. 18–28.

[24] M.K. Rastogi and Y.M. Tripathi, Estimating the parameters of a Burr distribution under progressive Type II censoring, Stat. Methodol. 9 (2012), pp. 381–391.

[25] M.K. Rastogi and Y.M. Tripathi, Parameter and reliability estimation for an exponentiated half logistic distribution under progressive Type II censoring, J. Stat. Comput. Simul. 84 (2014), pp. 1711–1727.

[26] S.-J. Wu, Estimation of the two parameter bathtub-shaped lifetime distribution with progressive censoring, J. Appl. Stat. 35 (2008), pp. 1139–1150.