(p, q)–deformed Fibonacci and Lucas polynomials: characterization and Fourier integral transforms

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Abstract. A full characterization of (p, q)-deformed Fibonacci and Lucas polynomials is given. These polynomials obey non-conventional three-term recursion relations. Their generating functions and Fourier integral transforms are explicitly computed and discussed. Relevant results known in the literature are examined as particular cases.

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1. Introduction

The classical orthogonal polynomials (COPs) and the quantum orthogonal polynomials (QOPs), also called q–orthogonal polynomials, constitute an interesting set of special functions with potential applications in physics, in probability and statistics, in approximation theory and numerical analysis, to cite a few domains where they are involved. Since the birth of quantum mechanics, the COPs also made their appearance in the bound-state wavefunctions of exactly solvable potentials.

Depending on the set of parameters, each family of orthogonal polynomials occupies different levels within the Askey hierarchy [16]. For instance, the classical Hermite polynomials $H_n(x)$ are the ground level, the Laguerre $L_n^{(\alpha)}(x)$ and Charlier $C_n(x; a)$ polynomials are one level higher, the Jacobi $P_n^{(\alpha, \beta)}(x)$, the Meixner $M_n(x; \beta, c)$, the Krawtchouk $K_n(x; p, N)$ and the Meixner/Pollaczek $F_n^{(\lambda)}(x; \phi)$ polynomials are two levels higher, the Hahn $Q_n(x; \alpha, \beta, N)$, the dual Hahn $R_n(\lambda(x); \gamma, \delta, N)$ polynomials, etc. are three levels higher, and so on. Besides, all orthogonal polynomial families in this Askey scheme are characterized by a set of properties:

- they are solutions of second order differential or difference equations,
- they are generated by three-term recurrence relations,
- they are orthogonal with respect to weight functions,
- they obey the Rodrigues-type formulas.

The other polynomial families which do not obey the above characteristic properties, do not belong to the Askey $q$-scheme.
In this work, we deal with the study of \((p, q)\)-Fibonacci and \((p, q)\)-Lucas polynomials characterized by non-conventional recurrence relations. Their \(q\)-analogs were recently introduced by Atakishiyev et al [3].

The paper is organized as follows. In Section 2, we give a formulation of the \((p, q)\)-deformed Fibonacci and \((p, q)\)-deformed Lucas polynomials. Their generating functions are computed and discussed. We perform the computation of the associated Fourier integral transforms in Section 3 and end with some concluding remarks in the Section 4.

2. \((p, q)\)-deformed Fibonacci and Lucas polynomials

In this section, we study in details the \((p, q)\)-deformed Fibonacci and Lucas polynomials. The corresponding generating functions are computed and discussed.

2.1. \((p, q)\)-deformed Fibonacci polynomials

Start with the following definition of \((p, q)\)-analogs of Fibonacci polynomials [3, 7, 8],

\[
F_{n+1}(x, s) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}_p s^k x^{n-2k} = x^n 2F_1 \left( -\frac{n}{2}, -\frac{1-n}{2}; -\frac{4s}{x^2} \right), \quad n \geq 0, \tag{1}
\]

given by:

**Definition 2.1**

\[
F_{n+1}(x, s|p, q) := \sum_{k=0}^{\lfloor n/2 \rfloor} (pq)^{k(k+1)/2} \binom{n-k}{k}_{p, q} s^k x^{n-2k}, \tag{2}
\]

where the \((p, q)\)-binomial coefficients \(\binom{n}{k}_{p, q}\) are given by [15]

\[
\binom{n}{k}_{p, q} := \frac{((p, q); (p, q))_n}{((p, q); (p, q))_k((p, q); (p, q))_{n-k}}, \tag{3}
\]

and \((p, q)\)-shifted factorial \(((a, b); (p, q))_n\) is defined as

\[
((a, b); (p, q))_n := \prod_{k=0}^{n-1} (ap^k - bq^k) \quad \text{for} \quad n \geq 1, \quad ((a, b); (p, q))_0 := 1. \tag{4}
\]

When \(n \to \infty\),

\[
((a, b); (p, q))_\infty := \prod_{k=0}^{\infty} (ap^k - bq^k). \tag{5}
\]

**Remark 2.2** As expected, when the parameter \(p \to 1\), the \((p, q)\)-Fibonacci polynomials [2] are reduced to their \(q\)-version [3], i.e

\[
F_{n+1}(x, s|q) = \sum_{k=0}^{\lfloor n/2 \rfloor} q^{k(k+1)/2} \binom{n-k}{k}_q s^k x^{n-2k}, \tag{6}
\]
Proof. By using the \((p, q)\)-identities:

\[
(p, q); (p, q)_{n-k} = \frac{(p, q); (p, q)_n}{(p^n, q^n); (p, q)_n} (-1)^k (pq)^{\binom{k}{2}} - n^k,
\]

\[
((p^n, q^n); (p, q)_2k = ((q^2, q^n), (p^{1+n/-2}, q^{1+n/-2}), (p^{1+n/-2}, -q^{1+n/-2}), (p^{1+n/-2}, -q^{1+n/-2}); (p, q)_k),
\]

we get

\[
F_{n+1}(x, s | p, q) = \sum_{k=0}^{[n/2]} \frac{(pq)^{k(k+1)/2}((p, q); (p, q))_{n-k}}{(p, q); (p, q)_k ((p, q); (p, q)_{n-k}) s^k x^{n-2k}} = x^n \sum_{k=0}^{[n/2]} \frac{(p-1)^{k-1/2}((p^n, q^n); (p, q)_2k}{((p^n, q^n); (p, q); (p, q)_k) s^{k(p+1/2)^k - 2k}} x^{n-2k}
\]

with \(p^{(k+1)/2} = ((p, 0); (p, q)_k). \square

Remark 2.4 In the limit when \(p \to 1\), \((7)\) is reduced to \(q\)-hypergeometric function characterizing the \(q\)-Fibonacci polynomials investigated in \([3]\), i. e.

\[
F_{n+1}(x, s | p, q) = x^n {}_4\phi_1 \left( \begin{array}{c} q^{-n/2}, q^{(1-n)/2}, -q^{-n/2}, -q^{(1-n)/2} \\ q^{n} \end{array} \mid q; q^n s \right), \quad n \geq 0.
\]
Lemma 2.5 The \((p, q)\)-binomial coefficients
\[
\left[ \begin{array}{c} n-k \\ k \end{array} \right]_{p,q} = \frac{((p, q); (p, q))_{n-k}}{((p, q); (p, q))_{k}((p, q); (p, q))_{n-2k}},
\]
where \(0 \leq 2k \leq n\), \(n \in \mathbb{N}\) satisfy the following identities:
\[
\left[ \begin{array}{c} n-k \\ k \end{array} \right]_{p,q} = q^{k} \left[ \begin{array}{c} n-1-k \\ k \end{array} \right]_{p,q} + p^{n-2k} \left[ \begin{array}{c} n-1-k \\ k-1 \end{array} \right]_{p,q},
\]
\[
\left[ \begin{array}{c} n-k \\ k \end{array} \right]_{p,q} = p^{k} \left[ \begin{array}{c} n-1-k \\ k \end{array} \right]_{p,q} + q^{n-2k} \left[ \begin{array}{c} n-1-k \\ k-1 \end{array} \right]_{p,q},
\]
\[
\left[ \begin{array}{c} n-k \\ k \end{array} \right]_{p,q} = p^{k} \left[ \begin{array}{c} n-1-k \\ k \end{array} \right]_{p,q} + q^{-k} \left[ \begin{array}{c} n-1-k \\ k-1 \end{array} \right]_{p,q},
\]

\[ - (p^{n-2k+1} - q^{-2k+1}) q^{-k} \left[ \begin{array}{c} n-k \\ k-1 \end{array} \right]_{p,q} \]

Proof. Using the relations [7]:
\[
\left[ \begin{array}{c} n-k \\ k \end{array} \right]_{q} = q^{k} \left[ \begin{array}{c} n-1-k \\ k \end{array} \right]_{q} + \left[ \begin{array}{c} n-1-k \\ k-1 \end{array} \right]_{q}, \quad \left[ \begin{array}{c} n-k \\ k \end{array} \right]_{p/q} = p^{-k(n-k)} \left[ \begin{array}{c} n \\ k \end{array} \right]_{p,q},
\]
\[
\left[ \begin{array}{c} n-k \\ k \end{array} \right]_{q} = \left[ \begin{array}{c} n-1-k \\ k \end{array} \right]_{q} + q^{n-2k} \left[ \begin{array}{c} n-1-k \\ k-1 \end{array} \right]_{q},
\]
yields the required identities:
\[
\left[ \begin{array}{c} n-k \\ k \end{array} \right]_{p,q} = q^{k} \left[ \begin{array}{c} n-1-k \\ k \end{array} \right]_{p,q} + p^{n-2k} \left[ \begin{array}{c} n-1-k \\ k-1 \end{array} \right]_{p,q},
\]
\[
\left[ \begin{array}{c} n-k \\ k \end{array} \right]_{p,q} = p^{k} \left[ \begin{array}{c} n-1-k \\ k \end{array} \right]_{p,q} + q^{n-2k} \left[ \begin{array}{c} n-1-k \\ k-1 \end{array} \right]_{p,q},
\]
and
\[
\left[ \begin{array}{c} n-k \\ k \end{array} \right]_{p,q} = p^{k} \left[ \begin{array}{c} n-1-k \\ k \end{array} \right]_{p,q} + q^{-k} \left[ \begin{array}{c} n-1-k \\ k-1 \end{array} \right]_{p,q},
\]
\[ - (p^{n-2k+1} - q^{-2k+1}) q^{-k} \left[ \begin{array}{c} n-k \\ k-1 \end{array} \right]_{p,q} \]

\[ \square \]

Proposition 2.6 The \((p, q)\)-deformed Fibonacci polynomials satisfy the following non-standard three-term recursion relations:
\[
F_{n+1}(x, s|p, q) = x F_{n}(x, qs|p, q) + sq^{n-1} F_{n-1}(x, qp^{-1}s|p, q),
\]
\[
= x F_{n}(x, sp|p, q) + sp^{n-1} F_{n-1}(x, spq^{-1}|p, q),
\]
\[ = (x + sp(q - p)D_{(p,q)}) F_{n}(x, sp|p, q) + sp^{n} F_{n-1}(x, s|p, q), \quad n \geq 1, \]
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with the initial conditions $F_0(x,s|p,q) = 0$, $F_1(x,s|p,q) = 1$ and the (p,q)-Jackson’s derivative $D_{(p,q)}$ given by

$$D_{(p,q)}f(x) = \frac{f(px) - f(qx)}{(p - q)x}.$$ (25)

**Proof.** By multiplying the equations (13), (14) and (15) of the Lemma 2.5 by $(pq)^{k(k+1)/2}s^{k}x^{n-2k}$ and summing from $k = 0$ to $\lfloor n/2 \rfloor$, we get

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (pq)^{k(k+1)/2} \binom{n-k}{k}_{p,q} s^{k}x^{n-2k} = \sum_{k=0}^{\lfloor n/2 \rfloor} (pq)^{k(k+1)/2} \binom{n-1-k}{k}_{p,q} (qs)^{k}x^{n-2k},$$

and

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (pq)^{k(k+1)/2} \binom{n-k}{k}_{p,q} s^{k}x^{n-2k} = \sum_{k=0}^{\lfloor n/2 \rfloor} (pq)^{k(k+1)/2} \binom{n-1-k}{k}_{p,q} (ps)^{k}x^{n-2k},$$

and

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (pq)^{k(k+1)/2} \binom{n-k}{k}_{p,q} s^{k}x^{n-2k} = \sum_{k=0}^{\lfloor n/2 \rfloor} (pq)^{k(k+1)/2} \binom{n-1-k}{k}_{p,q} (ps)^{k}x^{n-2k},$$

which are equivalent to

$$F_{n+1}(x,s|p,q) = xF_n(x,qs|p,q) + sp^{n-1}F_{n-1}(x,qp^{-1}s|p,q),$$

and

$$F_{n+1}(x,s|p,q) = xF_n(x,sp|p,q) + sp^{n-1}F_{n-1}(x,sp^{-1}|p,q),$$

and

$$F_{n+1}(x,s|p,q) = (x + sp(q - p)D_{(p,q)})F_n(x,sp|p,q) + sp^{n}F_{n-1}(x,s|p,q), n \geq 1,$$ (28)

respectively, with $F_0(x,s|p,q) = 0$, $F_1(x,s|p,q) = 1$. □

**Remark 2.7** In the limit case when $p \to 1$, the equations (22)-(24) are reduced to their $q$-analog: [3,7], i.e

$$F_{n+1}(x,s|q) = xF_n(x,qs|q) + sqF_{n-1}(x,qs|q),$$

(29)

$$= xF_n(x,s|q) + sq^{n-1}F_{n-1}(x,qs^{-1}|q),$$

(30)

$$= xF_n(x,s|q) + (s - q)D_qF_n(x,s|q) + sF_{n-1}(x,s|q), n \geq 1,$$ (31)

with the initial values $F_0(x,s|q) = 0$ and $F_1(x,s|q) = 1$, where $D_q$ is the $q$-Jackson differential operator defined by

$$D_qf(x) := \frac{f(x) - f(qx)}{(1 - q)x}.$$ (32)
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**Definition 2.8** The generating function \( f_F(x, s; t|p, q) \) associated with the \((p, q)\)-Fibonacci polynomials is defined as follows:

\[
f_F(x, s; t|p, q) := \sum_{n=0}^{\infty} F_n(x, sp^{-n}|p, q)t^n. \tag{33}
\]

**Proposition 2.9** The generating functions (33) can be re-expressed in terms of the hypergeometric function \( 2\varphi_2 \):

\[
f_F(x, s; t|p, q) = \frac{t}{1 - xt} 2\varphi_2 \left( \begin{array}{c} (p, q), 0 \\ (p, xtq), (p, 0) \end{array} \bigg| (p, q); -qst^2 \right), \quad |t| < 1. \tag{34}
\]

**Proof.** From [2], we have

\[
f_F(x, s; t|p, q) := \sum_{n=0}^{\infty} F_n(x, sp^{-n}|p, q) t^n
\]

\[
= \sum_{n=0}^{\infty} \left\lfloor \frac{n-1}{2} \right\rfloor (pq)^{k(k+1)/2} \left[ \begin{array}{c} n-k \\ k \end{array} \right] _{p,q} s^n p^{-nk} x^n t^n
\]

\[
= \sum_{k=0}^{\infty} (pq)^{k(k+1)/2} s^k \sum_{n=0}^{\infty} \left\lfloor \frac{n-1}{2} \right\rfloor \left[ \begin{array}{c} n-1-k \\ n-1-2k \end{array} \right] _{p,q} x^n t^n p^{-nk}
\]

\[
= \sum_{k=0}^{\infty} (pq)^{k(k+1)/2} s^k \sum_{m=0}^{\infty} \left( (p, q); (p, q) \right)_m (st^2)^k
\]

\[
= t \sum_{k=0}^{\infty} \frac{(p, q); (p, q)_k (p^{1+k}, q^{1+k}); (p, q)_m (xtp^{-k})^m}{(p, q); (p, q)_m}.
\]

By using the equation (50) of [13], the expression (35) is transformed into

\[
f_F(x, s; t|p, q) = t \sum_{k=0}^{\infty} (pq)^{k(k+1)/2} p^{-2k-2k} (st^2)^k \frac{p^{k+2}}{(p, xtq); (p, q)_k+1}
\]

\[
= \frac{t}{1 - xt} \sum_{k=0}^{\infty} \frac{(p^{-1}q)^{k(k-1)/2} (st^2)^k}{((p, xtq), (p, 0); (p, q)_k)}.
\]

which achieves the proof. \( \square \)

**Remark 2.10** In the limit when \( p \to 1 \), the generating function (34) is reduced to its \( q \)–version provided by Atakishiyev et al [3]:

\[
f_F(x, s; t|q) := \sum_{n=0}^{\infty} F_n(x, s|q) t^n = \frac{t}{1 - xt} \phi_1 \left( \begin{array}{c} q \\ qxt \end{array} \bigg| q; -qst^2 \right), \quad |t| < 1. \tag{36}
\]
Remark 2.11 In the limit

(i) When $p, q \to 1$, the $(p, q)$–deformed Fibonacci polynomials are reduced to the classical case $F_n(x, s)$ given by the explicit sum formula

$$F_{n+1}(x, s) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} s^k x^{n-2k} = x^n 2F_1 \left( \frac{-n}{2}, \frac{1-n}{2} \middle| -\frac{4s}{x^2} \right), \quad n \geq 0,$$

where $\binom{n}{k} := \frac{n!}{k!(n-k)!}$ is the binomial coefficient and $2F_1$ is a hypergeometric function. They obey the following three-term recursion relation

$$F_{n+1}(x, s) = xF_n(x, s) + sF_{n-1}(x, s), \quad n \geq 1,$$

with initial values $F_0(x, s) = 0$ and $F_1(x, s) = 1$. Their generating function $f_F(x; s; t)$ is given by

$$f_F(x|s; t) := \sum_{n=0}^{\infty} F_n(x, s) t^n = \frac{t}{1 - x t - s t^2}, \quad |t| < 1.$$

The polynomials are monic and normalized so that for $s = 1$, one recovers the Fibonacci polynomials $f_n(x) = F_n(x, 1)$ introduced by Catalan (18), eq. (37.1) p. 443.

(ii) When $x = s = 1$, the $(p, q)$–Fibonacci polynomials furnish the $(p, q)$–Fibonacci number $F_n(1, 1|p, q) := F_n(p, q)$ given by

$$F_n(p, q) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \binom{pq}{k}^{(k+1)/2},$$

which is the $(p, q)$–extension of the $q$–Fibonacci number.

$$F_n(q) = \sum_{k=0}^{\lfloor n/2 \rfloor} q^{k(k+1)/2} \binom{n-k}{k}.$$
Finally, the limit case when \((p, q) \rightarrow (1, 1)\) yields the classical version of the generating function \(f_F(t)\) of the Fibonacci polynomials (see [10, 20, 22, 5] for more details), i.e.

\[
f_F(t) = \sum_{n=0}^{\infty} F_n t^n = \frac{t}{1 - t - t^2}, \quad |t| < 1.
\]

(45)

2.2. \((p, q)\)-deformed Lucas polynomials

Let us introduce the \((p, q)\)-analogs of the Lucas polynomials:

\[
L_n(x, s) := \sum_{k=0}^{[n/2]} \frac{n}{n-k} \binom{n-k}{k} s^{k} x^{n-2k} = x^n \, _2F_1 \left( \begin{array}{c} -\frac{n}{2}, \frac{1-n}{2} \\ 1-n \end{array} \right) - \frac{4s}{x^2}, \quad n \geq 0,
\]

(46)

as follows:

**Definition 2.12**

\[
L_n(x, s|p, q) := \sum_{k=0}^{[n/2]} (pq)^{\frac{k}{2}} \frac{[n]_{p,q}}{[n-k]_{p,q}} \binom{n-k}{k} s^{k} x^{n-2k},
\]

(47)

where the \((p, q)\)-number \([n]_{p,q}\) is given by

\[
[n]_{p,q} := \frac{p^n - q^n}{p - q}.
\]

(48)

In the limit when \(p \rightarrow 1\), the \((p, q)\)-Lucas polynomials are reduced to the \(q\)-Lucas polynomials introduced in [7]

\[
L_n(x, s|q) := \sum_{k=0}^{[n/2]} q^{\frac{k}{2}} \frac{[n]_{q}}{[n-k]_{q}} \binom{n-k}{k} s^{k} x^{n-2k},
\]

(49)

where the \(q\)-number \([n]_{q}\) is defined as

\[
[n]_{q} := \frac{1 - q^n}{1 - q}.
\]

(50)

The following proposition holds.

**Proposition 2.13** The \((p, q)\)-Lucas polynomials (47) can be defined as follows:

\[
L_n(x, s|p, q) = x^n \, _5\varphi_5 \left( \begin{array}{c} (p^{-\frac{n}{2}}, q^{-\frac{n}{2}}), (p^{1-n}, q^{1-n}), (p, q) \\ (p^{1-n}, q^{1-n}), (p, 0), (p, 0) \end{array} \right) - \frac{sq^n p^{1+n}}{x^2}
\]

(51)

**Proof.** The proof is the same as in the Proposition 2.3
Lemma 2.14 The \((p, q)\)-coefficients

\[
\frac{[n]_{p,q}}{[n-k]_{p,q}} \begin{bmatrix} n-k \\ k \end{bmatrix}_{p,q}
\]  

satisfy the following identities:

\[
\frac{[n]_{p,q}}{[n-k]_{p,q}} \begin{bmatrix} n-k \\ k \end{bmatrix}_{p,q} = q^k \begin{bmatrix} n-k \\ k \end{bmatrix}_{p,q} + p^{n-2k} \begin{bmatrix} n-1-k \\ k-1 \end{bmatrix}_{p,q}
\]  

(53)

and

\[
\frac{[n]_{p,q}}{[n-k]_{p,q}} \begin{bmatrix} n-k \\ k \end{bmatrix}_{p,q} = p^k \begin{bmatrix} n-k \\ k \end{bmatrix}_{p,q} + q^{n-k} \begin{bmatrix} n-1-k \\ k-1 \end{bmatrix}_{p,q}
\]  

(54)

Besides, in analogous way as for the Fibonacci polynomials, we can prove the following result.

Proposition 2.15 The \((p, q)\)-Lucas polynomials \((47)\) satisfy non-standard recursion relations for \(n \geq 1\):

\[
L_n(x, s|p, q) = F_{n+1}(x, p^{-1}s|p, q) + sp^{n-1}F_{n-1}(x, sp^{-1}|p, q),
\]

(55)

\[
L_n(x, sq^{-1}|p, q) = F_{n+1}(x, sp^{-1}|p, q) + sp^{-1}q^nF_{n-1}(x, p^{-1}s|p, q)
\]

(56)

with \(L_0(x, s|p, q) = 1\), \(L_1(x, s|p, q) = x\).

Remark 2.16 In the limit when \(p \to 1\), the polynomials \((47)\) are reduced to the well known \(q\)-Lucas polynomials \(L_{n}(x, s|q)\) studied by Atakishiyev et al [3]:

\[
L_n(x, s|q) := \sum_{k=0}^{[n/2]} q^{k(k-1)/2} \left( \frac{[n]_q}{[n-k]_q} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \right) s^k x^{n-2k}
\]

\[
= x^n \phi_1 \left( \frac{q^{-n/2}, q^{(1-n)/2}, -q^{-n/2}, -q^{(1-n)/2}}{q^{1-n}} \left| q^{-n} s, \frac{x^2}{x^2} \right. \right), \quad n \geq 0,
\]

(57)

where the \(q\)-number \([n]_q\) is defined as \([n]_q := (1-q^n)/(1-q)\), satisfying the following recursion relations

\[
L_n(x, s|q) = F_{n+1}(x, s|q) + sF_{n-1}(x, s|q),
\]

(58)

\[
L_n(x, sq|q) = F_{n+1}(x, s|q) + sq^nF_{n-1}(x, s|q)
\]

(59)

with \(L_0(x, s|q) = 1\), \(L_1(x, s|q) = x\).

The proof is similar to that previously performed for the Fibonacci polynomials.

Definition 2.17 The generating function \(f_L(x, s; t|p, q)\) associated with the \((p, q)\)-Lucas polynomials \(L_{n}(x, s|p, q)\) is defined by

\[
f_L(x, s; t|p, q) := \sum_{n=0}^{\infty} L_n(x, sp^{-n}|p, q)t^n.
\]

(60)

Proposition 2.18 The generating functions \((60)\) is explicitly given by

\[
f_L(x, s; t|p, q) = \frac{1 + spt^2}{1 - xpt} \phi_2 \left( \begin{array}{c} (p, q), 0 \\ (p, xtpq), (p, 0) \end{array} | (p, q); -qst^2 \right), \quad |t| < 1.
\]

(61)
Proof. The proof is immediate from the definition:

\[
 f_L(x, s; t|p, q) = \sum_{n=0}^{\infty} L_n(x, sp^{-n}|p, q)t^n
 = \sum_{n=0}^{\infty} F_{n+1}(x, p^{-1-n}s|p, q)t^n + s \sum_{n=0}^{\infty} F_{n-1}(x, sp^{-1-n}|p, q)p^{n-1}t^n. \tag{62}
\]

and the use of the Proposition 2.9. □

Remark 2.19 In the limit,

(i) When \((p, q) \to (1, 1)\) the polynomials (67) are reduced to the well known Lucas polynomials \(L_n(x, s)\) given by the following formula [3]:

\[
 L_n(x, s|q) := \sum_{k=0}^{[n/2]} \binom{n}{k} (n-k)k^k x^{n-2k} = x^n \binom{-n/2}{1-n} \frac{1}{x^2}, \quad n \geq 0 \tag{63}
\]
satisfying the following recursion relation

\[
 L_n(x, s|q) = F_{n+1}(x, s) + sF_{n-1}(x, s) \tag{64}
\]
and admitting the following generating function [3]

\[
 f_L(x, s; t) = \sum_{n=0}^{\infty} F_n(x, s)t^n = \frac{1 + st^2}{1 - xt - st^2}, \quad |t| < 1, \tag{65}
\]
which can be easily derived from the tree-term recursion relation (58) and (36). For \(s = 1\), we recover the normalized Lucas polynomials \(l_n(x) = L_n(x, 1)\) investigated by Bicknell [18].

(ii) For \(x = s = 1\), the \((p, q)\)-deformed Lucas polynomials become the \((p, q)\)-Lucas numbers:

\[
 L_n(p, q) := \sum_{k=0}^{[n/2]} \binom{pq}{k} \binom{n}{k} p_nq \left[ \begin{array}{c} n-k \\ k \end{array} \right]_{p,q}, \tag{66}
\]
generalizing the \(q\)-Lucas numbers [7]

\[
 L_n(q) := \sum_{k=0}^{[n/2]} q^{\binom{k}{2}} \binom{n}{k} n_q \left[ \begin{array}{c} n-k \\ k \end{array} \right]_q. \tag{67}
\]

When \((p, q) \to (1, 1)\) one obtains the well known Lucas number:

\[
 L_n = \frac{1}{2^{n-1}} \sum_{k=0}^{n} \binom{n}{2k} 5^k. \tag{68}
\]

The \((p, q)\)-generating function associated to the \((p, q)\)-Lucas number is given by

\[
 f_L(t|p, q) = \frac{1 + pt^2}{1 - pt} 2t^2 \varphi_2 \left( \begin{array}{c} (p, q), 0 \\ (p, tpq), (p, 0) \end{array} \right| (p, q); -qt^2), \quad |t| < 1 \tag{69}
\]
generalizing the \(q\)-Lucas number

\[
 f_L(t|p, q) = \frac{1 + t^2}{1 - t} 1 \varphi_1 \left( \begin{array}{c} q \\ t \end{array} \right| q; -qt^2), \quad |t| < 1 \tag{70}
\]
The Lucas polynomials $L_n$.

The proof stems from the observation that the Fibonacci polynomials $F_n(x, s)$ (resp. the Lucas polynomials $L_n(x, s)$) are essentially the Chebyshev polynomials of the second kind $U_n(x)$ (resp. of the first kind $T_n(x)$) (see [3] for more details) with $d_x T_n(x) = n U_{n-1}(x)$ [17].

**Proposition 2.20**

$$D_{(p,q)} L_n(n, s|p, q) = [n]_{p,q} F_n(x, s|p, q), \quad [n]_{p,q} = \sum_{k=0}^{n-1} p^{n-1-k} q^k. \quad (72)$$

The proof stems from the observation that the Fibonacci polynomials $F_n(x, s)$ (resp. the Lucas polynomials $L_n(x, s)$) are essentially the Chebyshev polynomials of the second kind $U_n(x)$ (resp. of the first kind $T_n(x)$) (see [3] for more details) with $d_x T_n(x) = n U_{n-1}(x)$ [17].

**3. Fourier transforms of $F_n(x, s|p, q)$ and $L_n(x, s|p, q)$**

In this section, we derive explicit formulas for the classical Fourier integral transforms of the $(p, q)$—deformed Fibonacci $F_n(x, s|p, q)$ and Lucas $L_n(x, s|p, q)$ polynomials.

**3.1. Fourier transforms of the $(p, q)$—Fibonacci polynomials**

Rewrite the $(p, q)$—deformed Fibonacci polynomials [2] in the following form:

$$F_{n+1}(x, s|p, q) = \sum_{k=0}^{\lfloor n/2 \rfloor} c_{n,k}^{(F)}(p, q) x^k s^{n-2k}, \quad (73)$$

where the associated $(p, q)$—deformed coefficients are given by

$$c_{n,k}^{(F)}(p, q) := (pq)^{k(k+1)/2} \binom{n-k}{k}_{p,q}. \quad (74)$$

Using the relations

$$\binom{n-k}{k}_{p^{-1}, q^{-1}} = (pq)^{k(2k-n)} \binom{n-k}{k}_{p,q} \quad (75)$$

we can express the $(p^{-1}, q^{-1})$—deformed coefficients from (74) as

$$c_{n,k}^{(F)}(p^{-1}, q^{-1}) = (pq)^{-k(n+1-k)} c_{n,k}^{(F)}(p, q), \quad (76)$$

allowing to define the $(p^{-1}, q^{-1})$—Fibonacci polynomials in the form:

$$F_{n+1}(x, s|p^{-1}, q^{-1}) = \sum_{k=0}^{\lfloor n/2 \rfloor} c_{n,k}^{(F)}(p^{-1}, q^{-1}) s^k x^{n-2k}$$

$$= x^n 4\varphi_3 \left((p^{-\frac{n}{2}}, q^{-\frac{n}{2}}), (p^{\frac{1-n}{2}}, q^{\frac{1-n}{2}}), (p^{-\frac{n}{2}}, q^{\frac{n}{2}}), (p^{\frac{1-n}{2}}, -q^{\frac{1-n}{2}}) \mid (p, q); -s \frac{x^2}{2} \right) \quad (77)$$
In the limit when $p \to 1$, we immediately obtain the $q^{-1}$–Fibonacci polynomials as follows:

$$F_{n+1}(x, s|q^{-1}) = \sum_{k=0}^{|n/2|} c_{n,k}^{(F)}(q^{-1}) s^k x^{n-2k},$$

leading to the formula

$$F_{n+1}(q^{-1}) = \sum_{k=0}^{|n/2|} c_{n,k}^{(F)}(q^{-1}) = 4\varphi_3 \begin{pmatrix} q^{-2}, q^{-\frac{1}{2}}, -q^{-\frac{3}{2}}, -q^{-\frac{1}{2}} \\ q^{-n}, 0, 0 \mid q; -1 \end{pmatrix}. \quad (78)$$

The associated $q^{-1}$–Fibonacci number is given by

$$F_{n+1}(q^{-1}) = \sum_{k=0}^{|n/2|} c_{n,k}^{(F)}(q^{-1}) = 4\varphi_3 \begin{pmatrix} q^{-2}, q^{-\frac{1}{2}}, -q^{-\frac{3}{2}}, -q^{-\frac{1}{2}} \\ q^{-n}, 0, 0 \mid q; -1 \end{pmatrix}. \quad (79)$$

**Theorem 3.1** The Fourier transform of the function $e^{-x^2/2} F_{n+1}(ae^{ix}, s|p, q)$ is given by

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} F_{n+1}(ae^{ix}, s|p, q)e^{ixy-x^2/2} dx = (pq)^{n/2} F_{n+1}(ae^{-\kappa y}, q^{-1}) e^{-y^2/2} \quad (80)$$

leading to the formula

$$F_{n+1}(a, s|p, q) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} F_{n+1}(ae^{ix}, s|p, q)e^{ixy-x^2/2} dxdy, \quad (81)$$

where $a$ is an arbitrary constant factor and $q = p^{-1} e^{-2\kappa^2}$.

**Proof.** Using (73) and (76), we obtain:

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} F_{n+1}(ae^{ix}, s|p, q)e^{ixy-x^2/2} dx$$

$$= \sum_{k=0}^{|n/2|} c_{n,k}^{(F)}(p, q)s^k a^{n-2k} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ixy+i(n-2k)\kappa x-x^2/2} dx$$

$$= \sum_{k=0}^{|n/2|} c_{n,k}^{(F)}(p, q)s^k a^{n-2k} e^{-\frac{1}{4}[\kappa(n-2k)+y]^2}$$

$$= (pq)^{n/2} F_{n+1}(ae^{-\kappa y}, q^{-1}) e^{-y^2/2},$$

where the Gauss integral transform $\int_{\mathbb{R}} e^{ixy-x^2/2} dx = \sqrt{2\pi} e^{-y^2/2}$ is used. The proof is achieved by integrating (80) with respect to $y$. □

In the limit when the parameter $p \to 1$, the Fourier transform (80) is reduced to the well-known results investigated by Atakishiyev et al [3],

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} F_{n+1}(ae^{ix}, s|q)e^{ixy-x^2/2} dx = q^{n/2} F_{n+1}(ae^{-\kappa y}, qs|q^{-1}) e^{-y^2/2}, \quad (82)$$

where $a$ is an arbitrary constant factor and $q = e^{-2\kappa^2}$.
(p, q)—deformed Fibonacci and Lucas polynomials: characterization and Fourier integral transforms

3.2. (p, q)—Lucas integral transform

Rewrite here also the (p, q)—deformed Lucas polynomials \( \{L_{n}\} \) as

\[
L_n(x, s|p, q) = \sum_{k=0}^{[n/2]} c_{n,k}^{(L)}(p, q)s^k x^{n-2k},
\]

where the coefficients \( c_{n,k}^{(L)}(p, q) \) are given by

\[
c_{n,k}^{(L)}(p, q) := (pq)^{k(k-1)/2} \frac{[n]_{p,q}}{[n-k]_{p,q}} \binom{n-k}{k}_{p,q}.
\]

By using \( (75) \), one can show that the \((p^{-1}, q^{-1})\)—coefficients \( [84] \) can be expressed as:

\[
c_{n,k}^{(L)}(p^{-1}, q^{-1}) = (pq)^{k(k-n)} c_{n,k}^{(L)}(p, q)
\]

permitting to define the \((p^{-1}, q^{-1})\)—deformed Lucas polynomials as follows:

\[
L_n(x, s|p^{-1}, q^{-1}) = \sum_{k=0}^{[n/2]} c_{n,k}^{(L)}(p^{-1}, q^{-1})s^k x^{n-2k}
\]

\[
= x^n \varphi_3 \left( \begin{array}{c}
(p^{-\frac{n}{2}}, q^{-\frac{n}{2}}), (p^{-\frac{n}{2}}, q^{\frac{n}{2}}), (p^{-\frac{n}{2}}, q^{\frac{n}{2}}), (p^{\frac{n}{2}}, q^{\frac{n}{2}}), (p^{\frac{n}{2}}, q^{\frac{n}{2}}) \\
(p^{-1-n}, q^{-1-n}), 0, 0
\end{array} \right) (p, q); -\frac{s_{pq}}{x^2} \right).
\]

The limit when \( p \to 1 \) yields the \( q^{-1} \)—Lucas polynomials \( [3] \):

\[
L_n(x, s|q^{-1}) = \sum_{k=0}^{[n/2]} c_{n,k}^{(L)}(q^{-1})s^k x^{n-2k} = x^n \varphi_3 \left( \begin{array}{c}
q^{-\frac{n}{2}}, q^{\frac{n}{2}}, q^{\frac{n}{2}}, q^{\frac{n}{2}}, q^{\frac{n}{2}} \\
q^{1-n}, 0, 0
\end{array} \right) (q, -\frac{s_{pq}}{x^2} \right).
\]

Their associated \( q^{-1} \)—Lucas numbers are found when \( x = s = 1 \) as:

\[
L_n(q^{-1}) = 4\varphi_3 \left( \begin{array}{c}
q^{-\frac{n}{2}}, q^{\frac{n}{2}}, q^{\frac{n}{2}}, q^{\frac{n}{2}} \\
q^{1-n}, 0, 0
\end{array} \right) (q, -\frac{s_{pq}}{x^2} \right).
\]

Finally, we have the following:

**Theorem 3.2** The Fourier transform of the function \( e^{-x^2/2}L_n(be^{ix}, s|p, q) \) is given by

\[
\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} L_n(be^{ix}, s|p, q)e^{isy-x^2/2}dx = (pq)^{\frac{n^2}{4}} L_n(be^{-k}, (pq)^{-1}s|p^{-1}, q^{-1})e^{-y^2/2} \quad (89)
\]

providing the formula

\[
L_n(b, s|p, q) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} L_n(be^{ix}, s|p, q)e^{isy-x^2/2}dxdy, \quad (90)
\]

where \( b \) is an arbitrary constant factor and \( q = p^{-1}e^{-2\kappa^2} \).

**Proof.** The proof is immediate from the Theorem 3.1 \( \Box \)

In the limit when the parameter \( p \to 1 \), the Fourier transform \( (89) \) is reduced to the well-known formula derived by Atakishiyev et al \( [3] \)

\[
\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} L_n(be^{ix}, s|q)e^{isy-x^2/2}dx = q^{\frac{n^2}{4}} L_n(be^{-k}, q^{-1}s|q^{-1})e^{-y^2/2}, \quad (91)
\]

where \( a \) is an arbitrary constant factor and \( q = e^{-2\kappa^2} \).
4. Conclusion

In the present work, a full characterization of \((p, q)\)–deformed Fibonacci and Lucas polynomials has been achieved. These polynomials obey non-conventional three-term recursion relations as previously shown for their \(q\)–analogs. Besides, the formulae for the computation of the associated Fourier integral transforms have been deduced. Previous known results have been recovered as particular cases and properly discussed.

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