Partial hyperbolicity and central shadowing

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Abstract. We study shadowing property for a partially hyperbolic diffeomorphism $f$. It is proved that if $f$ is dynamically coherent then any pseudotrajectory can be shadowed by a pseudotrajectory with “jumps” along the central foliation. The proof is based on the Tikhonov-Shauder fixed point theorem.

Keywords: partial hyperbolicity, central foliation, Lipschitz shadowing, dynamical coherence.

1 Introduction

The theory of shadowing of approximate trajectories (pseudotrajectories) of dynamical systems is now a well developed part of the global theory of dynamical systems (see, for example, monographs [12], [13]). This theory is of special importance for numerical simulations and the classical theory of structural stability.

It is well known that a diffeomorphism has the shadowing property in a neighborhood of a hyperbolic set [2], [4] and a structurally stable diffeomorphism has the shadowing property on the whole manifold [11], [17], [19].

There are a lot of examples of non-hyperbolic diffeomorphisms, which have shadowing property (see for instance [14], [21]) at the same time this phenomena is not frequent. More precisely the following statements are correct. Diffeomorphisms with $C^1$-robust shadowing property are structurally stable [18]. In [11] Abdenur and Diaz conjectured that $C^1$-generically shadowing is equivalent to structural stability, and proved this statement for so-called tame diffeomorphisms. Lipschitz shadowing is equivalent to structural stability [15] (see [21] for some generalizations).

In present article we study shadowing property for partially hyperbolic diffeomorphisms. Note that due to [7] one cannot expect that in general shadowing holds for partially hyperbolic diffeomorphisms. We use notion of central pseudotrajectory and prove that any pseudotrajectory of a partially hyperbolic diffeomorphism can be shadowed by a central pseudotrajectory. This result might be considered as a generalization of a classical shadowing lemma for the case of partially hyperbolic diffeomorphisms.
2 Definitions and the main result

Let $M$ be a compact $n$–dimensional $C^\infty$ smooth manifold, with a Riemannian metric dist. Let $|\cdot|$ be the Euclidean norm at $\mathbb{R}^n$ and the induced norm on the leaves of the tangent bundle $TM$. For any $x \in M$, $\varepsilon > 0$ we denote

$$B_\varepsilon(x) = \{ y \in M : \text{dist}(x, y) \leq \varepsilon \}.$$ 

Below in the text we use the following definition of partial hyperbolicity (see for example [6]).

**Definition 1.** A diffeomorphism $f \in \text{Diff}^1(M)$ is called *partially hyperbolic* if there exists $m \in \mathbb{N}$ such that the mapping $f^m$ satisfies the following property. There exists a continuous invariant bundle

$$T_xM = E^s(x) \oplus E^c(x) \oplus E^u(x), \quad x \in M$$

and continuous positive functions $\nu, \hat{\nu}, \gamma, \hat{\gamma} : M \to \mathbb{R}$ such that

$$\nu, \hat{\nu} < 1, \quad \nu < \gamma < \hat{\gamma} < \hat{\nu}^{-1}$$

and for all $x \in M$, $v \in \mathbb{R}^n$, $|v| = 1$

$$|Df^m(x)v| \leq \nu(x), \quad v \in E^s(x);$$

$$\gamma(x) \leq |Df^m(x)v| \leq \hat{\gamma}(x), \quad v \in E^c(x);$$

$$|Df^m(x)v| \geq \hat{\nu}^{-1}(x), \quad v \in E^u(x). \quad (1)$$

Denote

$$E^{cs}(x) = E^s(x) \oplus E^c(x), \quad E^{cu}(x) = E^c(x) \oplus E^u(x).$$

For further considerations we need the notion of dynamical coherence.

**Definition 2.** We say that a $k$–dimensional distribution $E$ over $TM$ is *uniquely integrable* if there exists a $k$–dimensional continuous foliation $W$ of the manifold $M$, whose leaves are tangent to $E$ at every point. Also, any $C^1$–smooth path tangent to $E$ is embedded to a unique leaf of $W$.

**Definition 3.** A partially hyperbolic diffeomorphism $f$ is *dynamically coherent* if both the distributions $E^{cs}$ and $E^{cu}$ are uniquely integrable.
If \( f \) is dynamically coherent then distribution \( E_c \) is also uniquely integrable and corresponding foliation \( W^c \) is a subfoliation of both \( W^{cs} \) and \( W^{cu} \). For a discussion how often partially hyperbolic diffeomorphisms are dynamically coherent see [5], [9].

In the text below we always assume that \( f \) is dynamically coherent.

For \( \tau \in \{s, c, u, cs, cu\} \) and \( y \in W^\tau(x) \) let \( \text{dist}_\tau(x, y) \) be the inner distance on \( W^\tau(x) \) from \( x \) to \( y \). Note that
\[
\text{dist}(x, y) \leq \text{dist}_\tau(x, y), \quad y \in W^\tau(x).
\]

Denote
\[
W^\tau_\varepsilon(x) = \{y \in W^\tau(x), \text{dist}_\tau(x, y) < \varepsilon\}.
\]

Let us recall the definition of the shadowing property.

**Definition 4.** A sequence \( \{x_k : k \in \mathbb{Z}\} \) is called \( d \)-pseudotrajectory \((d > 0)\) if \( \text{dist}(f(x_k), x_{k+1}) \leq d \) for all \( k \in \mathbb{Z} \).

**Definition 5.** Diffeomorphism \( f \) satisfies the *shadowing property* if for any \( \varepsilon > 0 \) there exists \( d > 0 \) such that for any \( d \)-pseudotrajectory \( \{x_k : k \in \mathbb{Z}\} \) there exists a trajectory \( \{y_k\} \) of the diffeomorphism \( f \) such that
\[
\text{dist}(x_k, y_k) \leq \varepsilon, \quad k \in \mathbb{Z}.
\]

**Definition 6.** Diffeomorphism \( f \) satisfies the *Lipschitz shadowing property* if there exist \( \mathcal{L}, d_0 > 0 \) such that for any \( d \in (0, d_0) \), and any \( d \)-pseudotrajectory \( \{x_k : k \in \mathbb{Z}\} \) there exists a trajectory \( \{y_k\} \) of the diffeomorphism \( f \), satisfying (3) with \( \varepsilon = \mathcal{L}d \).

As was mentioned before in a neighborhood of a hyperbolic set diffeomorphism satisfies the Lipschitz shadowing property [2], [4], [13].

We suggest the following generalization of the shadowing property for partially hyperbolic dynamically coherent diffeomorphisms.

**Definition 7** (see for example [10]). An \( \varepsilon \)-pseudotrajectory \( \{y_k\} \) is called *central* if for any \( k \in \mathbb{Z} \) the inclusion \( f(y_k) \in W^c_\varepsilon(y_{k+1}) \) holds (see Fig. 1).

**Definition 8.** A partially hyperbolic dynamically coherent diffeomorphism \( f \) satisfies the *central shadowing property* if for any \( \varepsilon > 0 \) there exists \( d > 0 \) such that for any \( d \)-pseudotrajectory \( \{x_k : k \in \mathbb{Z}\} \) there exists an \( \varepsilon \)-central pseudotrajectory \( \{y_k\} \) of the diffeomorphism \( f \), satisfying (3).
Definition 9. A partially hyperbolic dynamically coherent diffeomorphism \( f \) satisfies the \textit{Lipschitz central shadowing property} if there exist \( d_0, \mathcal{L} > 0 \) such that for any \( d \in (0, d_0) \) and any \( d \)-pseudotrajectory \( \{x_k : k \in \mathbb{Z}\} \) there exists an \( \epsilon \)-central pseudotrajectory \( \{y_k\} \), satisfying (3) with \( \epsilon = \mathcal{L}d \).

Note that the Lipschitz central shadowing property implies the central shadowing property.

We prove the following analogue of the shadowing lemma for partially hyperbolic diffeomorphisms.

Theorem 1. Let diffeomorphism \( f \in C^1 \) be partially hyperbolic and dynamically coherent. Then \( f \) satisfies the Lipschitz central shadowing property.

Note that for Anosov diffeomorphisms any central pseudotrajectory is a true trajectory.

Let us also mention the following related notion [10].

Definition 10. Partially hyperbolic, dynamically coherent diffeomorphism \( f \) is called \textit{plaque expansive} if there exists \( \epsilon > 0 \) such that for any \( \epsilon \)-central pseudotrajectories \( \{y_k\}, \{z_k\} \), satisfying

\[ \text{dist}(y_k, z_k) < \epsilon, \quad k \in \mathbb{Z} \]

hold inclusions

\[ z_0 \in W^c_\epsilon(y_0), \quad k \in \mathbb{Z}. \]

In the theory of partially hyperbolic diffeomorphisms the following conjecture plays important role [3], [10].

Conjecture 1 (Plague Expansivity Conjecture). Any partially hyperbolic, dynamically coherent diffeomorphism is plaque expansive.
Let us note that if the diffeomorphism \( f \) in Theorem \( \text{[1]} \) is additionally plaque expansive then leaves \( W^c(y_k) \) are uniquely defined (see Remark \( \text{[1]} \) below).

Among results related to Theorem \( \text{[1]} \) we would like to mention that partially hyperbolic dynamically coherent diffeomorphisms, satisfying plaque expansivity property are leaf stable (see \([10\), Chapter 7], \([10\) for details).

### 3 Proof of Theorem \( \text{[1]} \)

In what follows below we will use the following statement, which is consequence of transversality and continuity of foliations \( W^s, W^c \).

**Statement 1.** There exists \( \delta_0 > 0, L_0 > 1 \) such that for any \( \delta \in (0, \delta_0] \) such that for any \( x,y \in M \) satisfying \( \text{dist}(x,y) < \delta \) there exists unique point \( z = W^s_\varepsilon(x) \cap W^c_\varepsilon(y) \) for \( \varepsilon = L_0\delta \).

Note that for a fixed diffeomorphism \( f \), satisfying the assumptions of the theorem, it suffices to prove that its fixed power \( f^m \) satisfies the Lipschitz central shadowing property. Since foliations \( W^\tau, \tau \in \{s,u,c,cs,cu\} \) of \( f^m \) coincide with the corresponding foliations of the initial diffeomorphism \( f \) we can assume without loss of generality that conditions \( \text{(1)} \) hold for \( m = 1 \). Note that a similar claim can be done using adapted metric, see \([8\).

Denote
\[
\lambda = \min_{x \in M} (\min(\hat{\nu}^{-1}(x), \nu^{-1}(x))) > 1.
\]

Let us choose \( l \) so big that
\[
\lambda^l > 2L_0.
\]

Arguing similarly to previous paragraph it is sufficient to prove that \( f^l \) has the Lipschitz central shadowing property and hence, we can assume without loss of generality that \( l = 1 \).

Decreasing \( \delta_0 \) if necessarily we conclude from inequalities \( \text{(1)} \) that
\[
\text{dist}_s(f(x), f(y)) \leq \frac{1}{\lambda} \text{dist}_s(x, y), \quad y \in W^s_{\delta_0}(x) \quad \text{(4)}
\]
and
\[
\text{dist}_u(f(x), f(y)) \geq \lambda \text{dist}_u(x, y), \quad y \in W^u_{\delta_0}(x).
\]
Denote

\[ I^\tau_r(x) = \{ z^\tau \in E^\tau(x), \ |z^\tau| \leq r \}, \quad \tau \in \{ s, u, c, cs, cu \}, \quad r > 0, \]

\[ I_r(x) = \{ z \in T_x M, \ |z| \leq r \}, \quad r > 0. \]

Consider standard exponential mappings \( \exp_x : T_x M \to M \) and \( \exp_x^\tau : T_x W^\tau(x) \to W^\tau(x) \), for \( \tau \in \{ s, c, u, cs, cu \} \). Standard properties of exponential mappings imply that there exists \( \varepsilon_0 > 0 \), such that for all \( x \in M \) maps \( \exp_x, \exp_x^\tau \) are well defined on \( I_{\varepsilon_0}(x) \) and \( I_{\varepsilon_0}^\tau(x) \) respectively and \( D \exp_x(0) = \text{Id}, \ D \exp_x^\tau(0) = \text{Id} \). Those equalities imply the following.

**Statement 2.** For \( \mu > 0 \) there exists \( \varepsilon \in (0, \varepsilon_0) \) such that for any point \( x \in M \), the following holds.

**A1** For any \( y, z \in B_\varepsilon(x) \) and \( v_1, v_2 \in I^\varepsilon_x(x) \) the following inequalities hold

\[
\frac{1}{1 + \mu} \text{dist}(y, z) \leq |\exp_x^{-1}(y) - \exp_x^{-1}(z)| \leq (1 + \mu) \text{dist}(y, z),
\]

\[
\frac{1}{1 + \mu} |v_1 - v_2| \leq \text{dist}(\exp_x(v_1), \exp_x(v_2)) \leq (1 + \mu) |v_1 - v_2|.
\]

**A2** Conditions similar to **A1** hold for \( \exp_x^\tau \) and \( \text{dist}^\tau, \ \tau \in \{ s, c, u, cs, cu \} \).

**A3** For \( y \in W^\varepsilon_\tau(x), \ \tau \in \{ s, c, u, cs, cu \} \) the following holds

\[
\text{dist}^\tau(x, y) \leq (1 + \mu) \text{dist}(x, y).
\]

**A4** If \( \xi < \varepsilon \) and \( y \in W^\varepsilon^s_\xi(x) \cap W^\varepsilon^{cu}_\xi(x) \) then

\[
\text{dist}^c(x, y) \leq (1 + \mu) \xi.
\]

Consider small enough \( \mu \in (0, 1) \) satisfying the following inequality

\[
(1 + \mu)^2 L_0 / \lambda < 1. \quad (5)
\]

Choose corresponding \( \varepsilon > 0 \) from Statement 2. Let \( \delta = \min(\delta_0, \varepsilon / L_0) \).

For a pseudotrajectory \( \{ x_k \} \) consider maps \( h_k^* : U_k \subset E^s(x_k) \to E^s(x_{k+1}) \) defined as the following:

\[
h_k^*(z) = (\exp_{x_{k+1}}^*)^{-1}(p)
\]
where
\[ p = W_{\omega_0b_0}^cL(f(\exp_{x_k}(z))) \cap W_{\omega_0b_0}^sL(x_{k+1}) \]  
and \( U_k \) is the set of points for which map \( h_k^s \) is well-defined (see Fig. 2). Note that maps \( h_k^s(z) \) are continuous. The following lemma plays a central role in the proof of Theorem 1.

**Lemma 1.** There exists \( d_0 > 0, L > 1 \) such that for any \( d < d_0 \) and \( d \)-pseudotrajectory \( \{x_k\} \) maps \( h_k^s \) are well-defined for \( z \in I_{Ld}^s(x_k) \) and the following inequalities hold

\[ |h_k^s(z)| \leq Ld, \quad k \in \mathbb{Z}. \]  

**Proof.** Inequality (5) implies that there exists \( L > 0 \) such that

\[ L_0(1 + L(1+\mu)/\lambda)(1 + \mu) < L. \]  

Let us choose \( d_0 < \delta_0/2L \). Fix \( d < d_0 \), \( d \)-pseudotrajectory \( \{x_k\}, k \in \mathbb{Z} \) and \( z \in I_{Ld}^s(x_k) \).

Condition A2 of Statement 2 implies that

\[ \text{dist}_{s}(x_k, \exp_{x_k}(z)) \leq Ld(1 + \mu). \]

Inequality (4) implies the following

\[ \text{dist}_{s}(f(x_k), \exp_{x_k}(z)) \leq \frac{1}{\lambda}Ld(1 + \mu). \]

Inequalities (2) and \( \text{dist}(f(x_k), x_{k+1}) < d \) imply (see Fig. 3 for illustration)

\[ \text{dist}(x_{k+1}, f(\exp_{x_k}(z))) \leq \text{dist}(x_{k+1}, f(x_k)) + \text{dist}(f(x_k), f(\exp_{x_k}(z))) \leq d \left( 1 + \frac{1}{\lambda}L(1 + \mu) \right) < Ld < \delta_0. \]
Statement 1 implies that point $p$ from relation (6) is well-defined and inequality (8) implies the following

$$\text{dist} \left( p, x_{k+1} \right), \text{dist}_{cu} \left( p, f(\exp^{s}_{x_{k}}(z)) \right) < dL_0 \left(1 + \frac{1}{\lambda} L \left(1 + \mu \right) \right) < \frac{Ld}{1 + \mu}.$$  

This inequality and Statement 2 imply

$$\text{dist}_{cu} \left( f(\exp^{s}_{x_{k+1}}(z)), \exp^{s}_{x_{k}}(h^s_k(z)) \right) < Ld, \quad (9)$$

$$|h^s_k(z)| < Ld,$$

which completes the proof.

Let $d_0, L > 0$ are constants provided by Lemma 1. Let $d < d_0$ and $\{x_k\}$ is a $d$-pseudotrajectory. Denote

$$X^s = \prod_{k=-\infty}^{\infty} I^s_{Ld}(x_k).$$

This set endowed with the Tikhonov product topology is compact and convex.

Let us consider map $H : X^s \rightarrow X^s$ defined as following

$$H(\{z_k\}) = \{z'_{k+1}\}, \quad \text{where} \quad z'_{k+1} = h^s_k(z_k).$$

By Lemma 1 this map is well-defined. Since $z'_{k+1}$ depends only on $z_k$ map $H$ is continuous. Due to the Tikhonov-Schauder theorem [20], the mapping $H$
has a (maybe non-unique) fixed point \( \{ z^*_k \} \). Denote \( y^s_k = \exp^s_{z_k}(z^*_k) \). Since \( z^*_k + 1 = h^s_k(z^*_k) \), inequality (9) implies that
\[
y^s_{k+1} \in W^c_{Ld}(f(y^s_k)), \quad k \in \mathbb{Z}.
\] (10)
Since \( |z^*_k| < Ld \) we conclude
\[
dist(x_k, y^s_k) \leq dist_s(x_k, y^s_k) < (1 + \mu)Ld < 2Ld, \quad k \in \mathbb{Z}.
\]

Similarly (decreasing \( d_0 \) and increasing \( L \) if necessarily) one may show that there exists a sequence \( \{ y^u_k \in W^u_{2Ld}(x_k) \} \) such that
\[
y^u_{k+1} \in W^c_{Ld}(f(y^u_k)), \quad k \in \mathbb{Z}.
\]
Hence \( dist(y^s_k, y^u_k) < dist(y^s_k, x_k) + dist(x_k, y^u_k) < 4Ld \). Decreasing \( d_0 \) if necessarily we can assume that \( 4L_0Ld < \delta_0 \). Then there exists an unique point \( y_k = W^c_{Ld}(y^s_k) \cap W^s_{Ld}(y^u_k) \) and inclusion (10) implies that for all \( k \in \mathbb{Z} \) the following holds
\[
dist_{cu}(y_{k+1}, f(y_k)) < \ dist_{cu}(y_{k+1}, y^s_{k+1}) + dist_{cu}(y^s_{k+1}, f(y^s_k)) + dist_{cu}(f(y^s_k), f(y_k)) < 4L_0Ld + Ld + 4RL_0Ld = L_{cu}d,
\]
where \( R = \sup_{x \in M} |Df(x)| \) and \( L_{cu} > 1 \) do not depends on \( d \). Similarly for some constant \( L_{cs} > 1 \) the following inequalities hold
\[
dist_{cs}(y_{k+1}, f(y_k)) < L_{cs}d, \quad k \in \mathbb{Z}.
\]
Reducing \( d_0 \) if necessarily we can assume that points \( y_{k+1}, f(y_k) \) satisfy assumptions of condition A4 of Statement 2 hence
\[
dist_{c}(y_{k+1}, f(y_k)) < (1 + \mu) \max(L_{cs}, L_{cu})d, \quad k \in \mathbb{Z}
\]
and sequence \( \{ y_k \} \) is an \( L_{1}d \)-central pseudotrajectory with
\[
L_1 = (1 + \mu) \max(L_{cs}, L_{cu}).
\]

To complete the proof let us note that
\[
dist(x_k, y_k) < dist(x_k, y^s_k) + dist(y^s_k, y_k) < 2Ld + 4L_0Ld, \quad k \in \mathbb{Z}.
\]

Taking \( L = \max(L_1, 2L + 4L_0) \) we conclude that \( \{ y_k \} \) is an \( Ld \)-central pseudotrajectory which \( Ld \) shadows \( \{ x_k \} \). □

**Remark 1.** Note that we do not claim uniqueness of such sequences \( \{ y^s_k \} \) and \( \{ y^u_k \} \). In fact it is easy to show (we leave details to the reader) that uniqueness of those sequences is equivalent to the plaque expansivity conjecture.
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References

[1] F. Abdenur, L. Diaz, *Pseudo-orbit shadowing in the C¹ topology*, Discrete Contin. Dyn. Syst., 7 (2003), 223-245.

[2] D. V. Anosov, *Geodesic flows on closed Riemannian manifolds of negative curvature*, Trudy Mat. Inst. Steklov., 90 (1967), 3-210.

[3] Ch. Bonatti, L. J. Diaz, M. Viana, *Dynamics beyond uniform hyperbolicity. A global geometric and probabilistic perspective*, Springer, Berlin, 2004.

[4] R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*, Lecture Notes Math., 470, Springer, Berlin, 1975.

[5] M. Brin, *On dynamical coherence*, Ergodic Theory Dynam. Systems, 23 (2003), 395-401.

[6] K. Burns, A. Wilkinson, *Dynamical Coherence and Center Bunching*, Discrete and Continuous Dynamical Systems, 22 (2008), 89-100.

[7] Ch. Bonatti, L. Diaz, G. Turcat, *There is no shadowing lemma for partially hyperbolic dynamics*, C. R. Acad. Sci. Paris Ser. I Math. 330 (2000), 587-592.

[8] N. Gourmelon, *Adapted metric for diffeomorphisms with dominated splitting*, Ergod. Theory Dyn. Syst. 27 (2007), 1839-1849.
[9] F. Rodriguez-Hertz, M. A. Rodriguez-Hertz, R. Ures, *A survey of partially hyperbolic dynamics*, Fields Institute Communications, Partially Hyperbolic Dynamics, Laminations and Teichmuller Flow, 51 (2007), 35-88.

[10] M. W. Hirsch, C. C. Pugh, M. Shub, *Invariant Manifolds*, Lecture Notes in Math., 583, Springer-Verlag, Berlin-Heidelberg, 1977.

[11] A. Morimoto, *The method of pseudo-orbit tracing and stability of dynamical systems*, Sem. Note, 39 (1979), Tokyo Univ.

[12] K. J. Palmer, *Shadowing in Dynamical Systems, Theory and Applications*, Kluwer, Dordrecht, 2000.

[13] S. Yu. Pilyugin, *Shadowing in Dynamical Systems*, Lecture Notes in Math., 1706, Springer, Berlin, 1999.

[14] S. Yu. Pilyugin, *Variational shadowing*. Discrete Contin. Dyn. Syst. Ser. B 14 (2010), 733-737.

[15] S. Yu. Pilyugin, S. B. Tikhomirov, *Lipschitz shadowing imply structural stability*, Nonlinearity 23 (2010), 2509-2515.

[16] C. C. Pugh, M. Shub, A. Wilkinson, *Hölder foliations, revisited*, arXiv:1112.2646v1.

[17] C. Robinson, *Stability theorems and hyperbolicity in dynamical systems*, Rocky Mount. J. Math., 7 (1977), 425-437.

[18] K. Sakai, *Pseudo orbit tracing property and strong transversality of diffeomorphisms of closed manifolds*, Osaka J. Math., 31 (1994), 373-386.

[19] K. Sawada, *Extended f-orbits are approximated by orbits*, Nagoya Math. J., 79 (1980), 33-45.

[20] J. Schauder, *Der Fixpunktsatz in Funktionalräumen*, Stud. Math., 2 (1930), 171-180.

[21] S. B. Tikhomirov, *The Hölder shadowing property*, arXiv:1106.4053v1.