SPECIALIZATION OF THE TORSION SUBGROUP OF THE CHOW GROUP

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ABSTRACT. An example is given in which specialization is not injective.

Consider the diagram,

\[
\begin{array}{c c c c c c}
V_{\bar{s}} & \longrightarrow & V_s & \longrightarrow & V & \leftarrow & V_\eta & \leftarrow & V_{\bar{\eta}} \\
\downarrow & & \downarrow & & f & & \downarrow & & \downarrow \\
\bar{s} & \longrightarrow & s & \leftarrow & i_s & \leftarrow & S & \leftarrow & \eta & \leftarrow & \bar{\eta},
\end{array}
\]

(1)

in which all squares are Cartesian, \( S = Spec(A) \) is a finite type, regular, integral affine \( \mathbb{Z} \)-scheme (or a localization of such), \( g \) is the inclusion of the generic point, \( \eta = Spec(K) \), \( i_s \) is the inclusion of a non-generic point, \( s = Spec(\mathbb{F}) \), and \( f \) is a smooth, projective morphism with geometrically connected \( n \)-dimensional fibers. Furthermore, the map \( \nu \) (respectively \( \epsilon \)) corresponds to a choice of algebraic closure, \( \mathbb{F} \subseteq \bar{\mathbb{F}} \) (respectively \( K \subseteq \bar{K} \)).

Let \( l \) be a prime number. For any abelian group, \( B \), write \( B[l^\infty] \subset B \) for the subgroup consisting of elements annihilated by some power of \( l \). For \( r \geq 0 \) there is a specialization homomorphism of Chow groups,

\[
\sigma^r_{\bar{s}} : CH^r(V_{\bar{\eta}})[l^\infty] \rightarrow CH^r(V_{\bar{s}})[l^\infty].
\]

(2)

When \( l \) is distinct from the characteristic of \( \mathbb{F} \), \( \sigma^1_{\bar{s}} \) is injective. This fact has proved to be very useful for bounding torsion in \( CH^1(V_\eta) \subset CH^1(V_{\bar{\eta}})^{Gal(\bar{K}/K)} \) (cf. [Sil, VIII.7.3.2]). It is interesting to ask if injectivity continues to hold for \( r > 1 \). In fact it does for \( r \in \{2, n\} \) as will be recalled in the Proposition below. The purpose of this note is to show that injectivity may fail in the range \( 2 < r < n \).

1 The failure of injectivity when \( l = char(\mathbb{F}) \) and \( r \) is arbitrary is classical [Sil, III.6.4].

1991 Mathematics Subject Classification. 14C25.
Partial support by the NSF (DMS-0200012) gratefully acknowledged.

1 After this paper was written, C. Soulé and C. Voisin posted Torsion cohomology classes and algebraic cycles on complex projective manifolds AG/0403254 on the e-print archives lanl.arXiv.org. Their e-print is also concerned with non-injectivity of specialization of torsion in the Chow group among other issues. Both the differences and the similarities between the cycles considered in the e-print and the cycles considered here appear to be interesting.

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Before proving the injectivity and non-injectivity assertions, we describe the map, $\sigma^r_s$, in detail. Extra effort is required here to deal with the case $\text{codim}_S(s) > 1$, which plays an important role in later arguments.

**Construction of the specialization homomorphism.** By localizing we may assume that $S$ is the spectrum of a regular local ring and that $s$ is the closed point. Write $\tilde{S}, E \to S, s$ for the blow up of $S$ along $s$. The local ring at the exceptional divisor, $\mathcal{O}_{\tilde{S}, E}$, is the valuation ring of a discrete valuation, $v : \hat{K}^* \to \mathbb{Z}$. Choose a valuation $\bar{v}$ of $\bar{K}$ extending $v$ [Bou, VI.3.3]. In each intermediate field, $K \subset L \subset \bar{K}$, finite over $K$, $\bar{v}$ specifies a discrete valuation ring, $\mathcal{O}_{L, \bar{v}}$, finite over $\mathcal{O}_{\tilde{S}, E}$. Write $s_{L, \bar{v}}$ for the spectrum of its residue field. Base change $f$ by $\text{Spec}(\mathcal{O}_{L, \bar{v}}) \to S$. There is a specialization homomorphism [Fu, 20.3.1],

$$\sigma^r_{L, \bar{v}} : CH^r(V_L) \to CH^r(V_{s_{L, \bar{v}}}).$$

Let $L \subset L'$ be a finite extension of intermediate fields. Since specialization is functorial for flat morphisms [Fu, Proposition 20.3(b)], the maps $\sigma^r_{L, \bar{v}}$ and $\sigma^r_{L', \bar{v}}$ are related by a commutative diagram. Taking the limit over finite intermediate fields gives a map,

$$\sigma^r_{K, \bar{v}} : CH^r(V_{\bar{v}}) \to CH^r(V_{s_{K, \bar{v}}}),$$

where $s_{K, \bar{v}}$ is the spectrum of an algebraic closure of the residue field of $\mathcal{O}_{\tilde{S}, E}$. Now (2) is constructed from (3) by restricting to $l$-power torsion subgroups and composing with the inverse of the pullback isomorphism [Le],

$$CH^r(V_{\bar{v}})[l^{\infty}] \xrightarrow{\sim} CH^r(V_{s_{K, \bar{v}}})[l^{\infty}].$$

The construction is independent of the choice of $\bar{v}$: Any valuation on $\bar{K}$ extending $v$ has the form $\bar{v} \circ \gamma$ for $\gamma \in \text{Gal}(\bar{K}/K)$ [Bou, VI.8.6]. There is a commutative diagram,

$$\begin{array}{ccc}
CH^r(V_{\bar{v}})[l^{\infty}] & \xrightarrow{\sigma^r_{K, \bar{v}}(\cdot)} & CH^r(V_{s_{K, \bar{v}}})[l^{\infty}] \\
\downarrow & & \downarrow \\
CH^r(V_{\bar{v}})[l^{\infty}] & \xrightarrow{\sigma^r_{K, \bar{v} \circ \gamma}(\cdot)} & CH^r(V_{s_{K, \bar{v} \circ \gamma}})[l^{\infty}] \\
\end{array} \sim \leftarrow \sim \begin{array}{ccc}
CH^r(V_{\bar{v}})[l^{\infty}] & \xrightarrow{\sigma^r_{K, \bar{v} \circ \gamma}(\cdot)} & CH^r(V_{s_{K, \bar{v} \circ \gamma}})[l^{\infty}] \\
\downarrow & & \downarrow \\
CH^r(V_{\bar{v}})[l^{\infty}] & \xrightarrow{\sim} & CH^r(V_{s_{K, \bar{v} \circ \gamma}})[l^{\infty}]
\end{array},$$

with vertical maps isomorphisms induced by $\gamma$.

**Lemma 1.** Let $\mathcal{X} = \sum n_i \mathcal{X}_i \in Z^r(\mathcal{V})$ be a linear combination of integral subschemes which are flat over $S$. Write $X_i$ for the generic fiber of $\mathcal{X}_i$. Suppose that the image of $\sum n_i X_i$ in $CH^r(V_{\bar{v}})$ is annihilated by a power of $l$. Then $\sigma^r_s(\sum n_i X_i)$ is equal to the class of $\sum n_i (X_i \times_S \bar{s})$ in $CH^r(V_{\bar{s}})$.

**Proof.** This follows from standard intersection theory [Fu, 1.5, 6.1, 20.3] \hfill \square

**An injectivity result.** To place the main result of this note in perspective we recall
Proposition. In the situation of (1) suppose that \( l \in A^* \). If \( r \in \{1, 2, n\} \), then \( \sigma_\eta^r \) is injective.

Proof. For a smooth, projective variety, \( T \), over an algebraically closed field of characteristic prime to \( l \), Bloch has defined a cycle class map to etale cohomology \([Bl]\),

\[
\lambda_T^r : CH^r(T)[l^\infty] \to H^{2r-1}(T, \mathbb{Q}_l/\mathbb{Z}_l(r)).
\]

When \( r = 1 \) this map is an isomorphism \([Bl, 3.6]\). Roitman’s theorem \([Ro]\), \([Bl, 4.2]\) shows that \( \lambda_T^r \) is an isomorphism when \( n = \dim(T) \). As a consequence of the Merkuriev-Suslin theorem, \( \lambda_T^2 \) is injective \([Co-Sa-So, Corollaire 4]\).

Since \( f \) is smooth and projective, the cospecialization map on cohomology,

\[
c_s : H^{2r-1}(V_\eta, \mathbb{Q}_l/\mathbb{Z}_l(r)) \to H^{2r-1}(V_\zeta, \mathbb{Q}_l/\mathbb{Z}_l(r)),
\]

is an isomorphism \([Mi, VI.4.2]\). The assertion follows from the compatibility of Bloch’s cycle class map with specialization: \( c_s^{-1} \circ \lambda_{V_\eta} = \lambda_{V_\zeta} \circ \sigma_\zeta^r \) \([Bl, 3.8]\). \( \square \)

The main result.

Theorem. There exist diagrams (1) with \( S \) of finite type over \( \text{Spec}(\mathbb{Z}) \) and primes \( l \in A^* \) such that for all \( r \) in the range, \( 2 < r < n \), and all closed points \( s \in S \), \( \sigma_\zeta^r \) is not injective.

Proof. Clearly we need a variety \( V_\zeta \) for which \( \text{Ker}(\lambda_{V_\eta}^r) \neq 0 \) for \( r \) in the range, \( 2 < r < n \). In \([Sch]\) certain varieties with this property were constructed as products of a smooth projective variety, \( W_\zeta \), and an elliptic curve, \( Y_\zeta \). Cycles in \( \text{Ker}(\lambda^r) \) were constructed as exterior products, \( z \times \tau \), with \( z \in Z^{r-1}(W_\eta) \) and \( \tau \in CH^1(Y_\eta)[l^\infty] \). The idea of the proof is to arrange that \( z \) specialize to a torsion class and then to exploit the divisibility of \( \tau \) to conclude that \( z \times \tau \) specializes to zero. To implement this idea we begin with a presentation of the exterior product construction of \([Sch, \S 1]\) in a relative context.

Fix a prime number \( l \) and a finitely generated field, \( k_0 \), with \( l \neq \text{char}(k_0) \). Let \( k_0 \subset K \) be a finitely generated extension of transcendence degree one. Let \( Y/K \) be an elliptic curve whose \( j \)-invariant, \( J(Y) \in K \), is transcendental over \( k_0 \). Suppose that there is a torsion point, \( \zeta_K \in Y \), of exact order \( l^m \). Let \( W \) be a geometrically connected, smooth, projective variety over \( k_0 \) of dimension \( n - 1 \). Let \( A_0 \subset k_0 \) be a regular integral domain of finite type over \( \mathbb{Z} \) with fraction field \( k_0 \). By inverting an element in \( A_0 \) if necessary, we may arrange that \( l \in A_0^* \) and that \( W \) extends to smooth, projective morphism, \( h : W \to S_0 := \text{Spec}(A_0) \). Let \( A \) be a regular integral domain, flat and of finite type over \( A_0 \) whose fraction field may be identified with \( K \). By inverting an element in \( A \) if necessary, we arrange that \( Y/K \) extends to an abelian scheme, \( Y \to S := \text{Spec}(A) \). The identity section is denoted \( e \) and the \( l^m \)-torsion section which extends \( \zeta_K, \zeta \). The composition

\[
f : V := W \times_{S_0} Y \to Y \to S
\]

is projective and smooth of relative dimension \( n \) with connected fibers. The fibers have product structures:

\[
\begin{align*}
V_\eta &= W \times_{S_0} Y \times_S \eta \simeq W \times_{S_0} Y_\eta \simeq W \times_{k_0} Y_\eta \\
V_s &= W \times_{S_0} Y \times_S s \simeq W \times_{S_0} Y_s \simeq W_{s_0} \times_{s_0} Y_s,
\end{align*}
\]
where \( s_0 \in S_0 \) is the image of \( s \) and \( s \) is an arbitrary closed point of \( S \).

The elements of \( \text{CH}^r(V_{\bar{\eta}})[l^{\infty}] \) whose specialization we wish to study are constructed from subschemes of \( \mathcal{Y} \) (respectively \( \mathcal{W} \)) which are flat over \( S \) (respectively \( S_0 \)). Define groups of algebraic cycles

\[
Z^{-1}_{fl}(W) = \{ \sum n_i \mathcal{Z}_i \in Z^{-1}(W) : \text{each subscheme } \mathcal{Z}_i \text{ is flat over } S_0 \}\).
\[
Z^1_{fl}(\mathcal{Y}) = \{ \sum n_i \mathcal{U}_i \in Z^1(\mathcal{Y}) : \text{each subscheme } \mathcal{U}_i \text{ is flat over } S \}\).
\[
Z^r_{fl}(\mathcal{V}) = \{ \sum n_i \mathcal{X}_i \in Z^r(\mathcal{V}) : \text{each subscheme } \mathcal{X}_i \text{ is flat over } S \}\).

The image of \( Z^1_{fl}(\mathcal{Y}) \to \text{CH}^1(\mathcal{Y}) \) is denoted \( \text{CH}^1_{fl}(\mathcal{Y}) \). Define \( \text{CH}^{-1}_{fl}(W) \) and \( \text{CH}^r_{fl}(\mathcal{V}) \) analogously. Let \( \bar{k}_0 \) be an algebraic closure of \( k_0 \). We construct a diagram

\[
\begin{array}{ccc}
\text{CH}^r(W_{s_0}) \otimes \text{CH}^1(\mathcal{Y}_{\bar{s}}) & \xrightarrow{\times} & \text{CH}^r(W_{s_0} \times_{s_0} \mathcal{Y}_{\bar{s}}) \\
 i_W \otimes i_\mathcal{Y} & & i_W \otimes i_\mathcal{Y} \\
 \text{CH}^r(W) \otimes \text{CH}^1_{fl}(\mathcal{Y}) & \xrightarrow{\times} & \text{CH}^r_{fl}(W \times_{S_0} \mathcal{Y}) \\
 j_W \otimes j_{\mathcal{Y}} & & j_W \otimes j_{\mathcal{Y}} \\
 \text{CH}^r(W_{k_0}) \otimes \text{CH}^1(\mathcal{Y}_{\bar{\eta}}) & \xrightarrow{\times} & \text{CH}^r(W_{k_0} \times_{k_0} \mathcal{Y}_{\bar{\eta}}). \\
\end{array}
\]

The horizontal maps are exterior product maps [Fu, 1.10 and proof of Proposition 20.2] [Sch, 1.1]. In the middle row an element represented by \( \mathcal{Z} \otimes \mathcal{U} \) for subschemes \( \mathcal{Z} \subset \mathcal{W} \) flat over \( S_0 \) and \( \mathcal{U} \subset \mathcal{Y} \) flat over \( S \) is mapped to the class of the subscheme \( \mathcal{Z} \times_{S_0} \mathcal{U} \), which is flat over \( S \) [Ha, III.9.2b,c]. The maps labeled \( j^* \) are flat pullback maps, while the maps \( i^! \) are intersections with a geometric closed fiber. All of the maps in the diagram may be defined on the level of algebraic cycles (ie. without first passing to rational equivalence classes). The top square of the diagram commutes. In fact with \( \mathcal{Z} \) and \( \mathcal{U} \) as above,

\[
i_W'(\mathcal{Z} \otimes \mathcal{U}) = \mathcal{Z} \times_{S_0} \mathcal{U} \times_{S} \bar{s} \simeq \mathcal{Z}_{s_0} \times_{s_0} \mathcal{U}_{\bar{s}} \simeq \mathcal{Z}_{\bar{s}} \times_{\bar{s}} \mathcal{U}_{\bar{s}} = i_W'({\mathcal{Z}}) \times i_Y'(\mathcal{U}).
\]

The bottom square also commutes:

\[
j_W'(\mathcal{Z} \times_{S_0} \mathcal{U}) = \mathcal{Z} \times_{S_0} \mathcal{U} \times_{S} \bar{\eta} \simeq \mathcal{Z}_{k_0} \times_{k_0} \mathcal{U}_{\bar{\eta}} \simeq \mathcal{Z}_{k_0} \times_{\bar{k}_0} \mathcal{U}_{\bar{\eta}} = j_W'({\mathcal{Z}}) \times j_Y'(\mathcal{U}).
\]

Define \( T = \zeta - e \in \text{CH}^1_{fl}(\mathcal{Y}) \).

**Lemma 2.**

(i) If \( \mathcal{Z} \in \text{CH}^{-1}_{fl}(W) \) is such that \( j_W'({\mathcal{Z}}) \otimes 1/l^m \) has exact order \( l^m \) in \( \text{CH}^{-1}(W_{k_0}) \rightleq \mathbb{Q}/l^m \mathbb{Z}/l^m, \) then \( j_W'({\mathcal{Z}}) \times j_Y'(T) \) has exact order \( l^m \) in \( \text{CH}^r(V_{\bar{\eta}})[l^{\infty}] \).

(ii) If \( i_W({\mathcal{Z}}) \in \text{CH}^{-1}(W_{s_0})^{\text{tors}}, \) then \( \sigma_{\bar{s}}^* (j_W'({\mathcal{Z}}) \times j_Y'(T)) = 0 \) in \( \text{CH}^r(V_{\bar{\eta}})[l^{\infty}] \).

**Proof.**

(i) There is an injective group homomorphism, \( \mathbb{Q}/l^m \mathbb{Z}/l^m \to \text{CH}^1(\mathcal{Y}_{\bar{\eta}}), \) mapping \( 1/l^m \) to \( j_Y'(T), \) whose image is a direct summand of \( \text{CH}^1(\mathcal{Y}_{\bar{\eta}}) \). Thus \( j_W'({\mathcal{Z}}) \times j_Y'(T) \in \text{CH}^{-1}(W_{k_0}) \otimes \text{CH}^1(\mathcal{Y}_{\bar{\eta}}) \) has exact order \( l^m \). By [Sch, 0.2] the exterior product map,

\[
\text{CH}^{-1}(W_{k_0}) \otimes \text{CH}^1(\mathcal{Y}_{\bar{\eta}})[l^{\infty}] \xrightarrow{\times} \text{CH}^r(W_{k_0} \times_{k_0} \mathcal{Y}_{\bar{\eta}})[l^{\infty}],
\]
is injective.

(ii) \( j^*_W(Z) \times j^*_Y(T) = j^*_Y(Z \times T) \) and \( \sigma^*_Y(j^*_W(Z \times T)) = i^*_Y(Z \times T) \) by Lemma 1. Observe that \( i^*_W(Z) \otimes i^*_Y(T) = 0 \in CH^{l-1}(W_s) \otimes CH^1(Y_s)[l^\infty] \), since \( CH^1(Y_s)[l^\infty] \) is a divisible group and \( i^*_W(Z) \) is torsion. Now the commutativity of the upper square in the diagram implies \( \sigma^*_Y(j^*_W(Z \times T)) = 0. \)

To complete the proof of the theorem we give an example where the hypotheses of Lemma 2 are fulfilled.

**Jacobian of genus three curves.** Set \( k_0 = \mathbb{Q}(a) \),

\[
E : y^2z - [(a^2 - 4)x^3 + (2a^2 - 4a)x^2z + (a^2 - 4)xz^2] = 0,
\]

and \( W = E^3 \). Now \( W \) is isogenous to the Jacobian of the genus 3 curve,

\[
C : x^4 + y^4 + z^4 + 4(x^2y^2 + y^2z^2 + z^2x^2) = 0.
\]

A map, \( \pi : C \to E \), is given by the field extension, \( k_0(E) \to k_0(C) \) [Bu-Sch-Top, 4.2],

\[
x/z \mapsto (2(x/z)^2 + (y/z)^2 + a)^2, \quad y/z \mapsto (y/z)(2(x/z)^2 + (y/z)^2 + a).
\]

The element, \( \kappa \in Aut(\mathbb{P}^2) \), given by

\[
x \circ \kappa = y, \quad y \circ \kappa = z, \quad z \circ \kappa = x,
\]

stabilizes \( C \) and gives rise to an embedding,

\[
\varrho : C \to E^3, \quad \varrho = (\pi, \pi \circ \kappa, \pi \circ \kappa^2).
\]

Define a cycle,

\[
Z := \varrho(C) - (-1)_* \varrho(C) \in Z^2(W),
\]

where \((-1)\) denotes inversion in the group law on the abelian variety, \( W \). For certain values of \( a \) and \( l \) calculations based on [Bl-Es], and [Bu-Sch-Top] show that the class of \( Z \) in \( CH^2(W_{k_0}) \) is not torsion and its image in \( CH^2(W_{k_0})/l \) is not zero. For instance this holds when \( a = l = 5 \) [Sch2, 3.1], [Bu-Sch-Top, 4.8, 9.2] and when \( a \) is an indeterminant and \( l \in \{5, 7, 11, 13, 17\} \) [Sch2, 3.3]. Choose \( A_0 \subset k_0 \) as above, so that \( C \) and \( E \) extend to relative curves, smooth and projective over \( S_0 = Spec(A_0) \), and \( \rho \) extends as well. Write \( Z \in \mathbb{Z}_{\mathbb{Q}l}(W) \) for the obvious extension of \( Z \). Now \( j^*_W(Z) \otimes 1/l^m = Z \otimes 1/l^m \) has exact order \( l^m \) in \( CH^2(W_{k_0}) \otimes \mathbb{Q}l/\mathbb{Z}l \).

Consider \( i^*_W(Z) = \rho_{s_0}(C_{s_0}) - (-1)_* \rho_{s_0}(C_{s_0}) \in CH^2(W_{s_0}) \), where \( C_{s_0} \) is the fiber of the relative curve extending \( C \) over \( s_0 \). The cohomology class, \( [i^*_W(Z)] \in H^1(W_{s_0}, \mathbb{Q}l(2)) \), vanishes, since inversion on the abelian variety, \( W_{s_0} \), acts trivially on \( H^1(W_{s_0}, \mathbb{Q}l(2)) \). Since \( s_0 \in S_0 \) is a closed point, \( s_0 \) is the spectrum of an algebraic closure of a finite field. Soulé’s theorem [So, Théorème 4] says that nullhomologous, one dimensional cycles on an Abelian variety over an algebraic closure of a finite field are torsion, i.e. \( i^*_W(Z) \in CH^2(W_{s_0})_{\text{tors}} \). Thus the hypotheses of Lemma 2 are satisfied.

This proves the theorem when \( n = 4 \). It is easy to extend this to the case \( n > 4 \) by taking \( W = E^3 \times \mathbb{P}^{n-4} \) instead of \( W = E^3 \) (cf. [Sch, 3.3]).
Remark. The arguments in this paper do not yield results about the specialization map (2) when S is an open subset of the ring of integers of a number field. Indeed the assumption that \( J(Y) \in K \) is transcendental over \( k_0 \) is needed here because of the important role that this hypothesis plays in [Sch, 0.2]. Totaro [To, 7.2] has constructed elements in the kernel of Bloch’s cycle class map which are defined over number fields. It would be interesting to know how these cycles behave under specialization.

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