Iterative construction of $U_q(s\ell(n+1))$
representations and Lax matrix factorisation

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Abstract

The construction of a generic representation of $gl(n+1)$ or of the trigonometric deformation of its enveloping algebra known as algebraic induction is conveniently formulated in terms of Lax matrices. The Lax matrix of the constructed representation factorises into parts determined by the Lax matrix of a generic representation of the algebra with reduced rank and others appearing in the factorised expression of the Lax matrix of the special Jordan-Schwinger representation.

1 Introduction

Representations of $gl(n+1)$ as well as of the trigonometric deformation $U_q(gl(n+1))$ of its enveloping algebra can be obtained from representations of the corresponding algebras with rank smaller by one unit and a set of $n+1$ Heisenberg pairs $x_i, \partial_i, i = 1, ..., n+1$. This iterative procedure is known as algebraic induction method. The background of this method is the general method of induced representations, in particular the construction of $U(n)$ representations from their characters \cite{1,2}. Biedenharn and Lohe \cite{5} developed the method of algebraic induction in application to quantum groups, relying on earlier results \cite{3} and going back to Holstein and Primakoff \cite{4}. Representations constructed by this method in terms of Heisenberg operators have been considered in \cite{6}.

The $RLL$ relation with the Lax matrix $L$ composed of the generators of the considered algebra is a simple and compact expression of the algebra relations. We shall show that the Lax matrices provide a natural formulation of the algebraic induction, allowing an easy derivation and simplifying essentially the expression of the constructed $gl(n+1)$ representation in terms...
of a \( gl(n) \) representation and a set of \( n+1 \) Heisenberg pairs \( x_i, \partial_i \). The latter are used to construct first of all the special Jordan-Schwinger form of representations of \( gl(n+1) \) and \( U_q(gl(n+1)) \). The corresponding Lax matrices have simple factorization properties and have a point of degeneracy at the spectral parameter value \( u=1 \). The matrix product of Lax matrices of two representations as well as the Lax matrix of the tensor product representation obey the Yang-Baxter \( RLL \) relation. Then choosing one of the representations to be of the Jordan-Schwinger form we observe that constraints can be imposed reducing the other tensor factor to a \( gl(n+1) \) or \( U_q(gl(n+1)) \) representation without disturbing the algebra relation for the tensor product generators. Underlying relations appear similar to the classical fusion method \[13\]. This results in the algebraic induction, i.e. the construction of representations of \( gl(n+1) \) and \( U_q(gl(n+1)) \) in terms of a representation of \( gl(n) \) or \( U_q(gl(n)) \) and \( n+1 \) Heisenberg conjugated pairs. Further, the relation between the product of Lax matrices of the two representations and the tensor product Lax matrix results in triangular factorisation relations for the latter Lax matrix. This factorised expression provides a compact formulation of the algebraic induction.

Our motivation of reconsidering the algebraic induction method in relation to Lax matrices arises from the task of factorisation of the Yang-Baxter R-operator acting on the tensor product of two generic representations. In the case of of \( sl(2) \) this factorisation has been established in \[7\] by regarding the action of \( R \) in the \( RLL \) relation as the permutation of pairs of parameters built from the representation parameter \( \ell \) and the spectral parameter \( u \) by decomposing this permutation into more elementary ones. Solving the defining conditions for the factor operators is essentially simpler compared to the conditions for the complete R-operator. The method has been developed in application to the trigonometric and elliptic deformations of \( sl(2) \) \[9\], to \( sl(3) \) and its trigonometric deformation and also to \( sl(n) \) \[8\]. All cases rely on triangular factorization relations for the corresponding Lax matrices.

The triangular factorization of the \( sl(n) \) Lax matrix has been obtained in \[8\] by using the representation induced from the Borel subgroup of triangular matrices. An extension this approach to factorization to the trigonometric deformation case may be allowed using a formulation like in \[10, 11\].

We start from the tensor product of two representations of \( gl(n+1) \) or \( U_q(gl(n+1)) \) and consider the relation of the Lax matrix composed of the co-product generators to the product of Lax matrices of the tensor factors (Sect. 2). Then we analyse the factorisation properties of the Lax matrix of the special Jordan-Schwinger representations, where the \( gl(n+1) \) algebra generators are constructed from the mentioned Heisenberg conjugated pairs (Sect. 3). The first of the tensor factors is substituted in the Jordan-Schwinger form. The second tensor factor can be constrained from \( gl(n+1) \) to \( gl(n) \) without disturbing the algebra relations in the tensor product (Sect. 4). The constrained tensor product Lax matrix is expressed as a product of factors of the Jordan-Schwinger Lax matrix and a matrix involving essentially the \( gl(n) \) generators (Sect. 5). Together with the factorization properties of the Jordan-Schwinger Lax matrix this leads to the wanted factorisation formulae and a compact formulation of the algebraic induction.

2 Tensor product in terms of Lax matrices

We consider the co-product of the \( U_q(gl(n+1)) \) algebra. One of the factors will be later substituted in the restricted Jordan-Schwinger form. We define the co-product on the generators in the Chevalley form, \( e_i, f_i, h_i = \frac{1}{2}(N_i - N_{i+1}), i = 1, \ldots, n \) in the symmetric way,

\[
\Delta(e_i) = e_i \otimes q^\frac{1}{2}(N_{i+1} - N_i) + q^{-\frac{1}{2}}(N_i - N_{i+1}) \otimes e_i,
\] (2.1)
\[ \Delta(f_i) = f_i \otimes q^\frac{1}{2}(N_{i+1} - N_i) + q^\frac{1}{2}(N_i - N_{i+1}) \otimes f_i, \quad \Delta(N_i) = N_i \otimes 1 + 1 \otimes N_i. \]

Having in mind the representations \( \pi^{(1)}, \pi^{(2)} \) on linear spaces \( V^{(1)}, V^{(2)} \) and also the tensor product \( V^{(1)} \otimes V^{(2)} \) we shall use the notation by subscripts \( (1), (2), (12) \) (called Sweedler’s notation in [16]) and omit the symbol \( \otimes \).

\[ \Delta(e_i) = e_i^{(12)} = e_i^{(1)} q^\frac{1}{2}(N_{i+1} - N_i) + e_i^{(2)} q^\frac{1}{2}(N_i - N_{i+1}), \quad \Delta(f_i) = f_i^{(12)} = f_i^{(1)} q^\frac{1}{2}(N_{i+1} - N_i) + f_i^{(2)} q^\frac{1}{2}(N_i - N_{i+1}), \]
\[ \Delta(N_i) = N_i^{(1)} + N_i^{(2)}. \]

The Cartan-Weyl generators are defined iteratively as

\[ E_{i,i+1} = e_i, \quad E_{i+1,i} = f_i, \]
\[ E_{i,j} = \begin{cases} E_{i,j-1}, & i < j, \\ E_{j-1,j}, & i > j + 1. \end{cases} \]

Here we use an appropriate modification of the commutator notation defined as \( [A, B]_q = AB - qBA \).

We intend to write the co-product explicitly in the latter basis. The result can be compactly formulated in terms of the upper and lower triangular parts of the Lax matrices. Let us compose the Lax matrix according to Jimbo [12],

\[ L_{ij}(u) = q^{-(u - \frac{1}{2}) - \frac{1}{2}(E_{ij} + E_{ji})} E_{j,i}, \quad i > j \]
\[ L_{ij}(u) = q^{+(u - \frac{1}{2}) + \frac{1}{2}(E_{ij} + E_{ji})} E_{j,i}, \quad i < j \]
\[ L_{ii}(u) = [u + E_{ii}] \]

and consider the standard decomposition

\[ \lambda \ L(u) = q^u L_+ - q^{-u} L_- \]

where \( \lambda = q - q^{-1} \) and \( (L_+)_{ij} = 0, \ i > j, \ (L_-)_{ij} = 0, \ i < j. \) We use the standard notation \( [x] = (q^x - q^{-x})\lambda^{-1}. \)

Consider now the generators and Lax matrices for two representations of the algebra, \( E_{ij}^{(1)}, E_{ij}^{(2)}, L^{(1)}(u), L^{(2)}(u). \) Then the Lax matrix composed by the same rule (2.4) with the co-product Cartan-Weyl generators has the form

\[ \lambda L^{(12)}(u) = q^u L^{(12)}_+ - q^{-u} L^{(12)}_- \]

where

\[ L^{(12)}_\pm = L^{(1)}_\pm L^{(2)}_\pm \]

A proof of this known relation [14, 15, 17] is given in Appendix A.

Let us recall also the situation in the undeformed case.

\[ L_{ij}(u) = u \delta_{ij} + E_{ji} \]

Here the Lax matrix for the tensor product is composed according to the latter prescription with the trivial co-product

\[ \Delta_1(E_{ij}) = E_{ij}^{(12)} = E_{ij}^{(1)} + E_{ij}^{(2)}, \]
\[ L_{ij}^{(12)}(u) = u\delta_{ij} + E_{ji}^{(1)} + E_{ji}^{(2)}. \]

Consider now the Yang-Baxter relation involving the fundamental \((n+1) \times (n+1)\) R-matrix and the Lax matrices

\[ \tilde{R}(u - v)L_1(u)L_2(v) = L_1(v)L_2(u)\tilde{R}(u - v), \quad (2.7) \]

\[ L_1 = L \otimes I, \quad L_2 = I \otimes L. \]

This relation is also fulfilled if one substitutes the Lax matrix by the matrix product \(L^{(1)}(u + \delta_1)L^{(2)}(u + \delta_2)\) or by the Lax matrix composed from the co-product generators \(L^{(12)}(u)\), eqs. \(2.2, 2.6\).

**Proposition 1.** In both the rational (undeformed) and the trigonometric cases the following relation holds for the Lax matrices of representations \(\pi^{(1)}, \pi^{(2)}\) and of the tensor product \(\pi^{(12)}\)

\[ L^{(1)}(u + \delta_1)L^{(2)}(u + \delta_2) = [u + \delta_3] L^{(12)}(u + \delta_1) + L^{(1)}(\Delta_1)L^{(2)}(\Delta_2) \]

with the shifts related as

\[ \delta_1 + \delta_2 = \delta_3 + \delta_4, \quad \Delta_1 + \Delta_2 = -\delta_3 + \delta_4, \quad \Delta_1 - \Delta_2 = \delta_1 - \delta_2 \quad (2.8) \]

The proof is straightforward in both the undeformed and deformed cases by substituting the explicit forms of the Lax matrices and comparing terms with the same dependence on the spectral parameter \(u\).

3 Jordan-Schwinger representations

We construct generators of \(g\ell(n + 1)\) by taking \(n + 1\) Heisenberg pairs \(x_i, \partial_i, i = 1, ..., n + 1\). In the undeformed case

\[ E_{ij} = x_i\partial_j \]

obey the Lie algebra relations. The constraint

\[ \sum x_i\partial_i = 2\ell \quad (3.1) \]

can be imposed to fix a representation of \(s\ell(n + 1)\), irreducible for generic \(\ell\). We postpone the elimination of a degree of freedom by the constraint \(3.1\) and discuss its effect in Appendix B.

In the deformed case we start with

\[ E_{ij}^J = \frac{x_i}{x_j}[N_j], \quad N_j = x_j\partial_j \quad i, j = 1, ..., n + 1 \]

We check easily that for \(i, j, k\) pairwise different

\[ [E_{ij}^J, E_{jk}^J]_{q^{\pm 1}} = q^{\mp N_j}E_{ik}^J, \quad [E_{ij}^J, E_{ji}^J]_1 = [N_i - N_j]. \quad (3.2) \]

These operators can be related to the Chevalley basis of \(U_q(sl(n + 1))\) algebra as

\[ e_i = E_{i,i+1}^J, \quad f_i = E_{i+1,i}^J, \quad 2h_i = [N_i - N_{i+1}], \quad i = 1, ..., n \]

The algebra relations including Serre’s relations can be checked.
Alternatively one can extend the construction to the Cartan-Weyl generators by defining them by q-commutators iteratively (2.4). In our case this leads to

\[ E_{ij} = q^{-(N_{i+1} + \ldots + N_{j-1})} E_{ij}^J, \quad i < j, \] (3.3)

\[ E_{ij} = q^{(N_{i-1} + \ldots + N_{j+1})} E_{ij}^J, \quad i > j, \quad i, j = 1, \ldots, n + 1 \]

The algebra relations in the Cartan-Weyl form are fulfilled. This property is preserved after imposing the constraint \( \sum x_i \partial_i = 2\ell \).

Let us now compose the Lax matrix according to (2.4) substituting the generators in Jordan-Schwinger form,

\[ L_{ij}(u) = (q^{\pm 1})^{(u-\frac{1}{2})} + N_{ij} \frac{x_j}{x_i} [N_i], \quad i \neq j \] (3.4)

\[ L_{ii} = [u + N_i], \]

The sign + or − in the exponent of \( q \) stands in the cases \( i < j \) or \( i > j \), respectively. We have introduced the notation \( N_{ij} = N_{ji}, \quad i \neq j \) where for the case \( i < j \)

\[ N_{ij} = \frac{1}{2} N_i + N_{i+1} + \ldots + N_{j-1} + \frac{1}{2} N_j \]

The Lax matrix in the form before imposing the constraint (3.1) can be simplified by the following similarity transformation,

\[ D^{(x)} L(u) D^{(x)-1} = \bar{L}(u) \] (3.5)

where \( D^{(x)} \) is a diagonal matrix with

\[ D_{ii}^{(x)} = q^{-N_{i,n+1}} x_i, \quad i \leq n, \quad D_{n+1,n+1}^{(x)} = x_{n+1}. \] (3.6)

The simplified Lax matrix \( \bar{L} \) has the elements

\[ \bar{L}_{ij} = (q^{\pm 1})^{u-1}[N_i], \quad \bar{L}_{ii} = [u - 1 + N_i], \]

Again the sign + stands in the case \( i < j \) and the sign − in the case \( i > j \).

\[ \bar{L}(u) = \begin{pmatrix}
[q^{-u+1}[N_{i+1}] & q^{u-1}[N_i] & \ldots & q^{u-1}[N_1] \\
q^{-u+1}[N_2] & [u-1+N_2] & \ldots & q^{u-1}[N_2] \\
& \ldots & \ldots & \ldots \\
q^{-u+1}[N_{n+1}] & q^{-u+1}[N_{n+1}] & \ldots & q^{-u+1}[N_{n+1}] \\
\end{pmatrix} \] (3.7)

Note that \( u=1 \) is a singular point of this matrix.

\[ \bar{L}(1) = \begin{pmatrix}
[N_1] & [N_1] & \ldots & [N_1] & [N_1] \\
[N_2] & [N_2] & \ldots & [N_2] & [N_2] \\
& \ldots & \ldots & \ldots & \ldots \\
[N_{n+1}] & [N_{n+1}] & \ldots & [N_{n+1}] & [N_{n+1}] \\
\end{pmatrix} = D^N M_1, \]

\[ L(1) = D^{(x)-1} D^N M_1 D^{(x)}. \] (3.8)

\( M_1 \) denotes the \((n+1)\times(n+1)\) matrix with all elements equal to 1 and \( D^N \) denotes the diagonal matrix with \( D_{ii}^N = [N_i] \).
It is not difficult to see that the matrix \( \tilde{L}(u) \) can be transformed to an upper triangular matrix \( \tilde{K} \),

\[
\tilde{L} = \tilde{M}_L \tilde{K} \tilde{M}_R
\]

with special lower triangular matrices \( M_L, M_R \) having 1 on the diagonal and the only further non-zero elements

\[
(\tilde{M}_L)_{n+1,i} = -q^{\alpha_i}, \quad (\tilde{M}_R)_{n+1,i} = q^{2(1-u)}
\]

Later we shall find it useful to express these matrices in terms of the standard matrix \( m_1 \),

\[
m_1 = \begin{pmatrix}
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \ldots & 1 & 1 \\
\end{pmatrix}
\]

with diagonal and last-row elements equal to 1 and other elements zero and the diagonal matrices

\[
(D_L)_{ii}(u) = q^{\alpha_i}, \quad (D_R)_{ii}(u) = q^{2(1-u)}, \quad i \leq n,
\]

\[
(D_L)_{n+1,n+1}(u) = 1, \quad (D_R)_{n+1,n+1}(u) = 1.
\]

as

\[
\tilde{M}_L = D_L^{-1}m_1^{-1}D_L, \quad \tilde{M}_R = D_R^{-1}m_1D_R.
\]

Let us first write down the matrix \( \tilde{L} \tilde{M}_R^{-1} \)

\[
= \begin{pmatrix}
[u-1]q^{N_1} & \lambda[u-1][N_1] & \ldots & \lambda[u-1][N_1] & q^{u-1}[N_1] \\
0 & [u-1]q^{N_2} & \ldots & \lambda[u-1][N_2] & q^{u-1}[N_2] \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-u-1]q^{2(1-u)-N_{n+1}} & \ldots & \ldots & \ldots & [u-1+N_{n+1}]
\end{pmatrix}
\]

This matrix has vanishing elements below the diagonal besides of the last row.

\( \tilde{K} \) is obtained by multiplying this matrix by \( M_L^{-1} \) from the left in order to clean up the last row. Choosing

\[
\alpha_i = 2(1-u) - N_{n+1} - 2N_1 - \ldots - 2N_{i-1} - N_i, \quad i = 1, \ldots, n
\]

we obtain the wanted form

\[
\tilde{K} = \begin{pmatrix}
[u-1]q^{N_1} & \lambda[u-1][N_1] & \ldots & \lambda[u-1][N_1] & q^{u-1}[N_1] \\
0 & [u-1]q^{N_2} & \ldots & \lambda[u-1][N_2] & q^{u-1}[N_2] \\
\vdots & \vdots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & 0 & [u-1+N_1+\ldots+N_{n+1}]q^{-N_1-\ldots-N_n}
\end{pmatrix}
\]

We have obtained the factorized form of the Jordan-Schwinger Lax matrix

\[
L(u) = M_L(u)K(u)M_R(u)
\]

where \( M_L, M_R \) are lower triangular with 1 on the diagonal and the only further non-vanishing elements on the last row.

\[
K(u) = D^{(x)-1}\tilde{K}(u)D^{(x)}
\]
is upper triangular and $\tilde{K}$ involves on the operators $N_i, \ i = 1, ..., n+1$. The left and right lower triangular factors are calculated from the standard matrix $m_1$ by similarity transformation with diagonal matrices,

$$M_L(u) = D_L^{-1}(u)m_x^{-1}D_L(u), \ M_R(u) = D_R^{-1}(u)m_xD_R(u), \ m_x = D(x)^{-1}m_1D(x).$$  (3.12)

Notice that there is an alternative factorized form where lower triangular matrices appear instead of upper triangular and vice versa. In the above factorization we have given a distinguished role to the last column and last row in $\text{3.7}$. The mentioned alternative form is obtained by distinguishing instead the first column and first row. More forms can be obtained by distinguishing in the analogous way the row and column of number $i$. Further, one can start the first step of producing zero elements in $\text{3.7}$ with a row instead of a column. Then the roles of $M_L$ and $M_R$ are in interchanged. Whereas in the considered form $M_R$ has simpler elements than $M_L$ this will then appear oppositely.

The Jordan-Schwinger form of $g\ell(n+1)$ does not cover all representations. For some particular values of $\ell$ the finite-dimensional representations symmetric in the tensor indices are involved, whereas arbitrary Young tableaux of index permutation symmetry are not covered by this form. As explained in Appendix B the representation modules of lowest weight spanned by polynomials have weights of the restricted form with $n-1$ zero components and one component equal to $\ell$. The representation constraint $\text{3.1}$ commutes with the Lie algebra but not with all generators of the Heisenberg algebra from which the latter is composed. We would like to use this constraint to eliminate $N_{n+1}$ and $x_{n+1}$. Some changes are expected because $x_{n+1}$ does not commute with the constraint. In Appendix B we show that this leads to minor modifications of the resulting factorization. A remarkable point is that the dependence on the representation parameter $\ell$ introduced by this constraint can be localized in the left factor $M_L$.

In the undeformed case the corresponding formulæ for the Lax matrices and the factorization are obtained by taking the limit $q \rightarrow 1$. The factorization $\text{3.11}$ holds where the factors on r.h.s. simplify as

$$M_L \rightarrow m_x^{-1}, \ M_R \rightarrow m_x$$

where in this limit

$$m_x \rightarrow \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_1 & x_2 & \ldots & x_n & x_{n+1} \\ x_{n+1} & x_{n+1} & \ldots & x_{n+1} & 1 \end{pmatrix}$$

The central factor simplifies to

$$\tilde{K} \rightarrow \begin{pmatrix} u-1 & 0 & \ldots & 0 & N_1 \\ 0 & u-1 & \ldots & 0 & N_2 \\ \ldots & \ldots & \ddots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & u-1+N_1+\ldots+N_{n+1} \end{pmatrix}.$$  (3.13)

4 Reduction

Now we turn to the special case where one of the tensor factors is constructed by Jordan-Schwinger generators. We shall denote the corresponding operators by subscript $x$ instead of (1),

$$E_{ij}^{(1)} = E_{ij}^x, \ L^{(1)}(u) = L^x(u).$$
The second tensor factor is so far a generic $g\ell(n+1)$ or $U_q(g\ell(n+1))$ representation. We shall omit later the label (2). To prevent confusions we denote in the following the diagonal generators of the second factor $E_{ii} = N_{i}^{(2)}$ by $E_i$ and the diagonal generators of the first $E_{ii}^{x} = N_{i}^{(1)}$ by $N_i$. We recall that $u = 1$ is a point of degeneracy of $L^x(u)$ and consider the particular case of the relation (2.8)

$$L^x(u+1)L^{(2)}(u) = [u] L^{(12)}(u+1) + L^x(1)L^{(2)}(0)$$

(4.1)

We shall investigate the condition for vanishing of the remainder $L_r = L^x(1)L^{(2)}(0)$, and we shall see that this results in the first step of the reduction of the second tensor factor from $U_q(g\ell(n+1))$ to $U_q(g\ell(n))$ preserving the algebra relations of $U_q(g\ell(n+1))$ for the reduced generators involved in the Lax matrix $L^{(12)}(u)$ for the tensor product and the Yang-Baxter $RLL$ relations for both $L^x(u+1)L^{(2)}(u)$ and $L^{(12)}(u)$. This is reminiscent to the fusion procedure [13], where in a similar way higher representations are constructed starting from a tensor product and applying a projection preserving the algebra relations.

### 4.1 Undeformed case

We substitute $L^x(1)$ using (3.7) and establish as a sufficient condition for the vanishing of the second term on r.h.s. of (4.1) the vanishing of the following set of $n+1$ operators

$$\phi_i = \sum_{s=1}^{n+1} x_s E_{is}$$

(4.2)

We would like to see whether the constraints $\phi_i = 0$ are compatible with the original algebra relations. This can be done by analyzing the consequences of the RLL relation for $L$ substituted as $L^x(u+1)L^{(2)}(u)$ or as $L^{(12)}(u+1)$ and the relation (4.1). In the undeformed case it is easier to study the commutation relations.

Indeed, we have

$$[\phi_i, \phi_j] = x_i \phi_j - x_j \phi_i$$

and

$$[E_{ij}^x + E_{ij}, \phi_k] = \delta_{jk} \phi_i$$

(4.3)

This shows that the constraints generate an ideal in the tensor product algebra $g\ell^x(n+1) \otimes g\ell(n+1)$ (in the general sense that multiplication by polynomials in $x_i$ is allowed). In terms of the theory of the constrained systems it means that constraints $\phi_i$ are in involution, i.e. can be consistently set equal to zero. We can construct the factor (coset) algebra, this means that the constraints can be imposed preserving the algebra relations. With the constraints we have $L^x(u+1)L^{(2)}(u) = [u] L^{(12)}(u+1)$ and both Lax expressions obey the Yang-Baxter relation (2.7).

The constraints can be used to eliminate the generators $E_{i,n+1}$. After this first reduction we establish another ideal generated by $E_{n+1,j}$ now with respect to the reduced algebra. Indeed, we have

$$[E_{n+1,i}, E_{n+1,j}] = \delta_{i,n+1} E_{n+1,j} - \delta_{n+1,j} E_{n+1,i}$$

We may restrict to $i, j = 1, ..., n$, then the right hand side is just zero.

The commutators of the tensor product generators with these constraints are

$$[E_{ij}^x + E_{ij}, E_{n+1,k}] = -\delta_{ik} E_{n+1,j}$$

8
The notation \( \tilde{E}_{ij} \) means that in the case \( j = n + 1 \) the substitution according to the constraint \( \varphi_i = 0 \) has to be done. Again the constraints \( E_{n+1,i}, i = 1,...,n \) can be imposed without disturbing the original Lie algebra relations or the Yang-Baxter relations.

In this way we have eliminated in two steps the generators \( E_{i,n+1}, i = 1,...,n+1 \) and \( E_{n+1,j}, j = 1,...,n \). The second tensor factor is reduced to the algebra \( gl(n) \).

It is useful to draw the attention to the following point concerning the second reduction step. If one would consider the constraints \( E_{n+1,i} = 0, i = 1,...,n+1 \) before the first reduction one would find an obstacle.

No problem arises in the Borel subalgebra involving the generators \( E_{i,j}, i \geq j \). However, the commutators with the other generators result in terms not vanishing with these constraints, i.e. preventing from imposing the constraints.

\[
[E_{n+1,i}, E_{j,k}] = \delta_{ij}E_{n+1,k} - \delta_{k,n+1}E_{ji}
\]

The unwanted second term does not appear if \( E_{j,n+1} \) is replaced by a linear combination of \( E_{js} \),

\[
[E_{n+1,i}, \tilde{E}_{j,n+1}] = \delta_{ij}\tilde{E}_{n+1,k}, \quad \tilde{E}_{j,n+1} = \sum_{1}^{n} A_s E_{js}
\]

At this point the coefficients are arbitrary. Their relation to the operators in the other tensor factor arises because this replacement of \( E_{j,n+1} \) should be constructed by consistent reduction.

### 4.2 q-deformed case

Let us look first how the last discussed point about the second reduction appears in the deformed case. The constraints \( E_{n+1,i} = 0, i = 1,...,n+1 \) can be consistently imposed within the Borel subalgebra where these operators belong to. However, the commutation relations

\[
[E_{n+1,j}, E_{j,n+1}] = [E_{n+1} - E_{j}]
\]

are not compatible with these constraints.

We write a set of commutation relations of the deformed algebra in terms of the Lax matrix elements

\[
L_{ji}(0) = \mathcal{E}_{ij} = q^{\pm \frac{1}{4} E_{i}} E_{ij} q^{\pm \frac{1}{4} E_{i}}
\]

where the sign + stands if \( i > j \) and the sign − if \( i < j \).

\[
[\mathcal{E}_{ij}, \mathcal{E}_{jk}] = q^{\pm E_{j}} \mathcal{E}_{ik}
\]

Here the sign + stands if \( i > j > k \) or \( i < j, j > k \) and the sign − stands if \( i < j < k \) or \( i > j, j < k \). In Appendix C we outline proofs of (4.6).

The above relation (4.4) can be rewritten replacing l.h.s. by \( [E_{n+1,j}, \mathcal{E}_{j,n+1}] \). Now, similar to the undeformed case, one observes that the problem is removed, if \( \mathcal{E}_{j,n+1}, j = 1,...,n+1 \) are replaced by linear combinations as

\[
\mathcal{E}_{j,n+1} = \sum_{1}^{n} A_s \mathcal{E}_{j,s}.
\]

Indeed, all terms appearing if calculating the commutator with the above relations (4.6) for \( \mathcal{E}_{ij} \) are proportional to some \( E_{n+1,k} \). Therefore, the obstacle for imposing the constraints \( E_{n+1,i} = 0 \) is removed if this replacement can be done consistently.
On the other hand we shall see now that this replacement corresponds to the constraints emerging in the first step with particular coefficient $A_i$ related to the first tensor factor.

Consider the relation (4.1). The second term on r.h.s. has the form

$$ L_r = L^x(1)L(0) = D^{(x)}M_1D^{(x)} - L(0) $$  \hspace{1cm} (4.8)

The diagonal matrices $D^x, D^N$ are defined above (3.6, 3.8) and $M_1$ is the $n+1 \times n+1$ matrix with all elements equal to 1. This term $L_r$ can be written in terms of

$$ \varphi_i = \sum_{1}^{i-1} X_s E_{is} + X_i [E_i] + \sum_{i+1}^{n+1} X_s E_{is} = \sum_{1}^{n} X_s E_{is} $$  \hspace{1cm} (4.9)

Here we have introduced $X_i = q^{-N_i,n+1}x_i$ obeying $X_iX_j = qX_jX_i$ for $n \geq i > j$ and $X_{n+1} = x_{n+1}$. $E_{ij}$ are defined in (4.5). The second form in (4.9) is a short-hand notation assuming $E_{ii} = [E_i]$. The matrix elements of (4.8) are

$$ (L_r)_{ij} = (L^x(1)L(0))_{ij} = q^{N_i,n+1}x_i^{-1}[N_i]\varphi_j $$  \hspace{1cm} (4.10)

The algebraic relations involving $\varphi_i$, $i = 1, ..., n+1$ can be derived immediately from the fact that both the l.h.s. and the first term in the r.h.s of (4.1) obey the Yang Baxter RLL relation. Therefore,

$$ \hat{R}_{12}(u - v) \left( [u]L_1^{12}(u + 1)L_{r2} + [v]L_{r1}L_2^{12}(v + 1) + L_{r1}L_{r2} \right) = $$

$$ \left( [v]L_1^{12}(v + 1)L_{r2} + [u]L_{r1}L_2^{12}(u + 1) + L_{r1}L_{r2} \right) \hat{R}_{12}(u - v). $$

All relations contained here consist only of terms linear or bilinear in $\varphi_i$, and no term that would not vanish with $\varphi_i$ is involved. Therefore the constraints $\varphi_i = 0, i = 1, ..., n+1$ can be imposed preserving the original algebra relations. The analogous consequence of the RLL relation applies also in the undeformed case and leads to the relations (4.3) derived in the previous subsection in another way.

These constraints are imposed as the first reduction step and used to replace $E_{i,n+1}, i = 1, ..., n+1$. After this one can impose the constraints $E_{n+1,i} = 0, i = 1, ..., n$ as the second reduction step. In this way the second tensor factor being a representation of $U_q(gl(n+1))$ is reduced to $U_q(gl(n))$.

This procedure results in the iterative construction of $U_q(gl(n+1))$ representations from $U_q(gl_q(n))$ representations by combining the latter with the special Jordan-Schwinger representations of $U_q(gl(n+1))$ which can be formulated in terms of the Heisenberg pairs $x_i, \partial_i, i = 1, ..., n+1$. We have shown that this construction is conveniently formulated in terms of the Lax matrices, representing the algebras in question owing to the Yang-Baxter relation. In particular the reduced tensor product Lax matrix $L^{(12)}$ represents the resulting $U_q(gl(n+1))$ representation.

Here the reduction eliminated the Cartan-Weyl generators of the second tensor factor with indices $(n+1,i)$ or $(i,n+1)$, i.e. referring to the last row and last column of the Lax matrix. Obviously other versions can be obtained by choosing for elimination the generator related to the row and column of number $i$. 

10
5 Factorisation

After the reduction the relation (4.11) takes the form

\[ L^x(u + 1) L'(u) = [u] L^{(12')}(u + 1) \]  (5.1)

The Lax matrix of the co-product is modified to \( L^{(12')} \) by substituting zero for the generators of the second tensor factor with the first index equal to \( n \), \( E_{n+1,i} = 0, i = 1, ..., n + 1 \) and substituting the generators with the second index equal to \( n + 1 \) according to the constraints \( \varphi_i = 0 \) by expressions in terms of the remaining generators. The same substitution also modifies the Lax matrix of the second factor. As the result \( L'(u) \) has zeroes on the last column besides of the lowest entry, \( (L'(u))_{n+1,n+1} = [u] \). The other elements of the last row are to be calculated according to the constraints \( \varphi_i = 0 \) in terms of the generators of the remaining \( U_q(g\ell(n)) \),

\[ (L'(u))_{n+1,j} = q^{-u} \tilde{e}_{i,n+1} = -q^{-u} x_{n+1}^{-1} \sum_{i}^{n} X_i e_{i,s} \]

The remaining \( n \times n \) block in \( L'(u) \) coincides with the Lax matrix of a generic \( U_q(g\ell(n)) \) representation.

In the second section we have obtained factorized representations of the Jordan-Schwinger Lax matrix \( L^{(x)}(u) \) in terms of triangular matrices. The central factor involves only \( N_i = x_i \partial_i \) and \( x_i \) enter the left and right factors.

We shall see that the reduced tensor product Lax operator allows triangular factorized representations. This follows from the factorisation properties of the Jordan-Schwinger representations, the reduced relation (5.1) and from

\[ L'(0) = m^{-1} L^n(0) m_x, \quad m_x = D^{(x)} I m_D^{(x)}. \]

\( m_1 \) denotes the \( n + 1 \times n + 1 \) matrix with diagonal and last row elements equal to 1 and the other elements zero. \( L^n(0) \) denotes the \( n + 1 \times n + 1 \) matrix with zeros on the last row and the last column and the remaining \( n \times n \) block matrix coinciding with the generic \( U_q(g\ell(n)) \) Lax matrix at \( u = 0 \).

5.1 Undeformed case

Because of the simple dependence on the spectral parameter \( u \) we have in this case

\[ L'(u) = m^{-1} L^n(u) m_x, \quad m_x = D^{(x)} I m_D^{(x)} \]

\[ L'(u) = uI + L'(0), \quad L^n(u) = uI + L^n(0) \]

The factorization of the Jordan-Schwcinger representation Lax matrix (3.11) simplifies to

\[ L^x(u + 1) = m^{-1} K(u + 1) m_x \]

\[ K(u + 1) = D^{(x)} I K(u + 1) D^{(x)} \]

The Lax matrix of the reduced tensor product is proportional to the product of \( L^x(u + 1) \) and \( L'(u) \) and therefore factorises as well,

\[ u L^{(12')}(u + 1) = m^{-1} K(u + 1) L^n(u) m_x \]

The resulting factorisation formula provides a compact formulation of the algebraic induction. The algebra \( g\ell(n) \) is represented by \( L^n(u) \), because its upper block is the corresponding Lax matrix. The other factors are the ones contained in the triangular factorisation of the Jordan-Schwinger form \( g\ell(n + 1) \) Lax matrix. \( L^{(12')}(u) \) represents the constructed algebra \( g\ell(n + 1). \)
5.2 q-deformed case

The non-trivial dependence on the spectral parameter can be represented as

$$\lambda L(u) = q^u L_+ - q^{-u} L_-.$$ 

Applied to the reduced Lax matrix $L'(u)$, $L_+$ reduces to $L'^n_+$, the $L_-$ of the $U_q(g\ell(n))$ case supplemented with the $n+1$st row and $n+1$st column of zeros. Let $L'^n_-$ be the corresponding $L_-$ of $U_q(g\ell(n))$ supplemented by zeros in the same way. However, instead of the latter we need to substitute

$$L'_- = m_x^{-1} (L'^n_- - L'^n_+) m_x + L'^n_+,$$

and obtain

$$\lambda L'(u) = (q^u - q^{-u}) L'^n_- + m_x^{-1} q^{-u} (L'^n_- - L'^n_+) m_x$$

(5.2)

Here the diagonal matrices in the definition of $m_x$ are the ones introduced for $q \neq 1$ in (3.6).

The factorization of the Jordan-Schwinger representation Lax matrix now involves additionally the diagonal matrices $D_L$ and $D_R$ (3.9).

$$L^x(u + 1) = D_L^{-1}(u + 1) m_x^{-1} D_L(u + 1) K(u + 1) D_R^{-1}(u + 1) m_x D_R(u + 1),$$

$$K(u + 1) = D^{(x)-1}(u) K(u + 1) D^{(x)}.$$ 

The matrices $D_{L/R}$ (3.9) depend on the spectral parameter $u$. In particular, $D_R(u + 1)$ has $q^{-2u}$ on the diagonal besides of the last diagonal element, which is 1. In order to get the left factors in the second term for $L'(u)$ closer to the right factors in $L^x(u + 1)$ we transform (5.2) to

$$\lambda L'(u) = (q^u - q^{-u}) L'^n_- + D_R^{-1}(u + 1) m_x^{-1} D_R(u + 1) q^u (L'^n_- - L'^n_+) D_R^{-1}(u + 1) m_x D_R(u + 1)$$

$$= (q^u - q^{-u}) (L'^n_- - D_R^{-1}(u + 1) m_x^{-1} D_R(u + 1) L_- D_R^{-1}(u + 1) m_x D_R(u + 1))$$

$$+ D_R^{-1}(u + 1) m_x^{-1} D_R(u + 1) L'^n(u) D_R^{-1}(u + 1) m_x D_R(u + 1)$$

$L'^n(u)$ is the Lax matrix of $U_q(g\ell(n))$ supplemented by zeros in the $n+1$st row and $n+1$st column. Notice that the first term in the last expression has non-vanishing elements only on the last row. This allows to rewrite the sum into one factorised expression in terms of $L'^n(u)$ which differs from $L'^n(u)$ in the last diagonal element $(n + 1, n + 1)$ being now non-zero and equal to $[u]$.

$$L'(u) = M_R^{-1}(u + 1) L'^n(u) M'M_R(u + 1),$$

(5.3)

$$M_R(u + 1) = D_R^{-1}(u + 1) m_x D_R(u + 1), \quad m_x = D^{(x)-1}(u) m_1 D^{(x)}$$

$$M' = I + (L'^n_- - M_R^{-1}(u + 1) L_- M_R(u + 1)),$$

With this factorised form of $L'(u)$ it is now straightforward to write the factorization of the reduced tensor product Lax matrix.

**Proposition 2.** The Lax matrix $L'^{12}(u + 1)$ of a $U_q(g\ell(n + 1))$ representation can be constructed from the Lax matrix of a $U_q(g\ell(n))$ representation and $n + 1$ Heisenberg conjugated pairs $x_i, \theta_i, i = 1, \ldots, n + 1$ as

$$[u] L'^{12}(u + 1) = M_L(u + 1) K(u + 1) L'^{n}(u) M'M_R(u + 1)$$

(5.4)

$L'^{n}(u)$ is block diagonal with the upper $n \times n$ block being the Lax matrix of $g\ell_q(n)$ and the last diagonal element equal to $[u]$.
\( M_L, M_R \) are lower-triangular, \( K(u) \) is upper-triangular and they appear as factors in the Lax matrix of the Jordan-Schwinger form of \( U_q(g\ell(n+1)) \) constructed in terms of \( x_i \partial_i \),
\[
L^x(u) = M_L(u)K(u)M_R(u) \\
M_L(u) = D_L^{-1}(u)m_x^{-1}D_L(u), \quad M_R(u) = D_R^{-1}(u)m_xD_R(u), \\
m_x = D^{(x)}(u)m_1 D^{(x)}(u), \quad K(u) = D^{(x)}(u) \tilde{K}(u) D^{(x)}(u)
\]
\( \tilde{K} \) is given in (3.10) and the diagonal matrices \( D^{(x)}, D_L, D_R \) are defined in (3.6, 3.8). \( m_1 \) is lower triangular with elements equal to 1 on the diagonal and on the lowest row and all other elements zero. \( M' \) is lower-triangular with units on the diagonal and the only other non-vanishing elements on the last row. It is calculated from the lower-triangular part \( L_- \) of the \( U_q(g\ell(n)) \) Lax matrix and \( D_L, D_R, D^{(x)} \) as in (6.3)

An alternative factorised form can be obtained where the analog of \( K \) is lower triangular and of \( M_L, M_R, M' \) are upper triangular. Also the block structure of the analogon of \( L^n(u) \) is opposite with the \( U_q(g\ell(n)) \) Lax matrix appearing as the lower \( n \times n \) block. One arrives at this alternative form if one uses the constraints \( \varphi_i = 0 \) to eliminate \( E_{1,1} \) and proceeds with the constraints \( E_{1,i} = 0 \) to do the reduction to \( U_q(g\ell(n)) \) in a different way. The alternative factorisation of the Jordan-Schwinger Lax matrix interchanging the roles of upper and lower triangular matrices is then applied. Further forms exist corresponding to the reduction by consistent elimination of the row and column of number \( i \).

6 Discussion

The Lax matrices of the Jordan-Schwinger type representations of \( g\ell(n+1) \) or \( U_q(g\ell(n+1)) \) show a simple structure allowing useful factorised expressions. Although this form covers only a special class of representations it can be used as building block for constructing generic representations by the method of algebraic induction. A generic representation is obtained by combining a Jordan-Schwinger type representation formulated in terms of \( n + 1 \) Heisenberg pairs with a generic representation of the corresponding algebra with rank lower by one unit. The representation parameter \( \ell \) associated with the Jordan-Schwinger representation becomes the additional weight component. Iterating this procedure a generic representation is finally constructed in terms of \( \frac{1}{2}(n+1)n \) Heisenberg pairs; \( n \) pairs of them are eliminated by solving the corresponding representation constraints, specifying simultaneously the \( n \) weight components as the representation labels of \( s\ell(n+1) \) or of its trigonometric deformation.

In this paper we have shown how the algebraic induction is derived from the relation between the product of Lax matrices of two representations and the Lax matrix composed of the co-product generators. We have specified one of these representations in the Jordan-Schwinger form and made use of the simple factorisation properties of the latter. The other representation in the tensor product can be constrained to a representation of the corresponding algebra with reduced rank while preserving the algebra relations in the tensor product representation. In this way the constructed representation is formulated in terms of the constrained tensor product Lax matrix. The relation of the latter to the product of Lax matrices of the two representations, one being of the Jordan-Schwinger type and the other generic but reduced in rank, results in factorised expressions.

The factorised expression (5.4) provides a short and simple formulation of the iterative construction of \( U_q(g\ell(n+1)) \) representations from \( U_q(g\ell(n)) \) representations equivalent to the involved expressions of the algebraic induction known so far. We see that the Lax matrices provide the appropriate formulation.
We have pointed out that there are several forms of factorisation of the Jordan-Schwinger Lax matrix and also of the constructed generic representation Lax matrix. The form considered explicitly here selects the last row and last column of the matrices. Correspondingly the parameter $\ell$ in the representation constraint becomes the $n$th component of the weight of the constructed representation. The other forms, selecting instead the row and the column of number $i$, may be of interest because the comparison of different forms results in explicit representations of the intertwining operators relating the equivalent representations differing in the ordering of the weight components. Explicit intertwining operators are of interest in the factorisation method of constructing the Yang-Baxter $R$ operator for generic representations.

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Appendix A

We define the co-product on the generators in the Chevalley form as in (2.2). The Cartan-Weyl generators are defined iteratively as (2.3) for the generators on both tensor factors $E^{(1)}_{ij}, E^{(2)}_{ij}$ and the ones on the tensor product $E^{(12)}_{ij}$. We intend to write the co-product explicitly in the Cartan-Weyl basis. In the first step we obtain

$$E^{(12)}_{i,i+2} = E^{(1)}_{i,i+2} q^{\frac{1}{2}(N^{(2)}_{i+2} - N^{(1)}_{i})} + E^{(2)}_{i,i+2} q^{-\frac{1}{2}(N^{(1)}_{i+2} - N^{(2)}_{i})}$$

The generic case is obtained as

$$E^{(12)}_{i,i+k} = E^{(1)}_{i,i+k} q^{\frac{1}{2}(N^{(2)}_{i+k} - N^{(1)}_{i})} + E^{(2)}_{i,i+k} q^{-\frac{1}{2}(N^{(1)}_{i+k} - N^{(2)}_{i})} - \lambda \sum_{s=1}^{k-1} E^{(1)}_{i+s,i+k} q^{\frac{1}{2}(N^{(2)}_{i+k} - N^{(1)}_{i+s} - N^{(2)}_{i} + N^{(1)}_{i})} E^{(2)}_{i+1,i+k}$$

The iteration in the direction of $i > j$ results in

$$E^{(12)}_{j,j+k} = E^{(1)}_{j,j+k} q^{\frac{1}{2}(N^{(2)}_{j+k} - N^{(1)}_{j})} + E^{(2)}_{j,j+k} q^{-\frac{1}{2}(N^{(1)}_{j+k} - N^{(2)}_{j})} + \lambda \sum_{s=1}^{k-1} E^{(1)}_{j+s,j+k} q^{\frac{1}{2}(N^{(2)}_{j+k} - N^{(1)}_{j+s} - N^{(2)}_{j} + N^{(1)}_{j})} E^{(2)}_{j+1,j+k}$$

(6.1)
The Lax matrix (2.24) can be decomposed as

$$\lambda L(u) = q^u L_+ - q^{-u} L_-, \lambda = q - q^{-1}$$

where $L_+$ is upper triangular and $L_-$ is lower triangular with the elements in terms of the Cartan-Weyl generators,

$$(L_+).i,i = q^{-N_i}, \quad (L_+).i,j = \lambda q^{N_i} E_{j,i} q^{-\frac{1}{2}N_j}, \quad i < j,$$

$$(L_+).i,i = q^{N_i}$$

We rewrite the result for $E^{(12)}_{i,i+k}$ in order to obtain the corresponding relation for the Lax matrix elements.

$$q^{-\frac{1}{2}(E^{(1)}_{i+k} + E^{(2)}_{i+k})} E^{(12)}_{i,i+k} q^{-\frac{1}{2}E^{(1)}_{i+k}} E^{(2)}_{i,i+k} = \left( q^{-\frac{1}{2}E^{(1)}_{i+k}} E^{(1)}_{i,i+s+k} q^{-\frac{1}{2}E^{(1)}_{i+k}} \right) q^{-E^{(2)}_{i+k}} +$$

$$q^{-E^{(1)}_{i+k}} \left( q^{-\frac{1}{2}E^{(2)}_{i+k}} E^{(2)}_{i,i+k} q^{-\frac{1}{2}E^{(2)}_{i+k}} \right) - \lambda \sum_{s=1}^{k-1} \left( q^{-\frac{1}{2}E^{(1)}_{i+k}} E^{(1)}_{i,i+s+k} q^{-\frac{1}{2}E^{(1)}_{i+k}} \right) \left( q^{-\frac{1}{2}E^{(2)}_{i+k}} E^{(2)}_{i,i+s} q^{-\frac{1}{2}E^{(2)}_{i+k}} \right)$$

Up to a factor $-\lambda^{-1}$ this coincides term by term with

$$(L^{(12)}_{i+k,i}) = \sum_{s=0}^{k} (L^{(1)}_{i+k,i+s}) (L^{(2)}_{i,s,i})$$

The case $E^{(12)}_{j+k,j}$ is analogous. In this way we have checked that the relations (2.6) $L^{(12)}_+ = L^{(1)}_+ L^{(2)}_-$ are indeed equivalent to the coproduct rules (2.22).

**Appendix B**

First of all the constraint (3.11) fixes the representation of the $g\ell(1)$ subalgebra generated by $\sum_{1}^{n+1} N_i$. The constraint allows to eliminate one pair, e.g. $x_{n+1}, \partial_{n+1}$. It is solved for $N_{n+1}$ and the representation is restricted to functions of $\frac{x_{i}}{x_{n+1}}$, so we can set $x_{n+1} = 1$ for simplicity.

We have a lowest weight module spanned by polynomials of $x_i, i = 1, ..., n$. The constant 1 is the lowest weight vector, in particular an eigenvector of $h_i = \frac{1}{2}(N_i - N_{i+1})$, and the weight components are $(0, ..., 0, \ell)$.

In the situation after the constraint has been imposed the matrix elements with indices $i, j = 1, ..., n$ are still given by (3.4) and the remaining ones are

$$L_{n+1,n+1} = [u + 2\ell - \sum_{1}^{n} N_s], \quad L_{i,n+1} = q^{-u} \frac{1}{x_i} [N_i],$$

$$L_{n+1,j} = q^{-u+\frac{1}{2}N_{n+1,j}} x_j [2\ell - \sum_{1}^{n} N_s], \quad \tilde{N}_{i,n+1} = \tilde{N}_{n+1,i} = \ell - \frac{1}{2} \sum_{1}^{n} N_s + \frac{1}{2} \sum_{i+1}^{n} N_s$$
We observe the simplification of the Lax matrix by

\[ D_\ell^{(x)} L(u) D_\ell^{(x)-1} = \tilde{L}_\ell(u) \]

\[ D_\ell^{(x)} = \text{diag}(q^{-\tilde{N}_{1,n+1} x_1}, ..., q^{-\tilde{N}_{n,n+1} x_n}, q^{-1}) \]

Here \( \tilde{L}_\ell \) coincides with the simplified Lax matrix \( \tilde{L} \) \( \text{(3.7)} \) above besides of the substitution

\[ N_{n+1} \to \tilde{N}_{n+1} = 2\ell + 1 - \sum_{1}^{n} N_s. \] \( \text{(6.3)} \)

The remaining steps are now the same as above, only the latter substitution has to be done. Thus \( \tilde{M}_R \) is unchanged and the substitution turns \( \tilde{M}_L \) to \( \tilde{M}_L^{(\ell)} \) and \( \tilde{K} \) to \( \tilde{K}_\ell \). The upper triangular matrix \( \tilde{K}_\ell \) coincides (up to the substitution) \( \text{(6.3)} \) with \( \tilde{K} \) \( \text{(3.10)} \) up to the lowest diagonal element, which turns to

\[ (\tilde{K}_\ell)_{n+1,n+1} = [u + 2\ell] q^{-\sum_{1}^{n} N_s} \]

Notice that the representation parameter dependence in \( D_\ell^{(x)} \) can be easily absorbed by rescaling \( q^{-\tilde{N}_{i,n+1} x_i} \to x_i, i = 1, ..., n, \)

\[ D_\ell^{(x)} \to D_1^{(x)} = \text{diag}(..., q^\tilde{N}_{i,n+1} x_i, ..., 1) \cdot q^{-\frac{1}{2}} \]

leaving \( N_i \) unchanged. Further we write

\[ \tilde{K}_\ell = D_{1\ell} K_1, \quad D_{1\ell} = \text{diag}(1, ..., 1, [u + 2\ell]) \]

Then all the remaining representation parameter dependence resides in \( D_{1\ell} \) and \( D_L^{(\ell)} \) and is factorized to the left:

\[ L(u) = M_L(u) D_{1\ell} K_1(u) M_R(u) \]

\[ K = D_1^{(x)-1} \tilde{K}_1 D_1^{(x)}, \quad M_{L/R} = D_1^{(x)-1} \tilde{M}_{L/R} D_1^{(x)} \]

\[ \tilde{M}_R = D_R^{-1}(u) m_1 D_R(u), \quad \tilde{M}_L = D_L^{(\ell)-1}(u) m_1^{-1} D_L^{(\ell)}(u) \]

Therefore the results for reduction and factorization are not changed essentially besides of the same modification of the left-most factors.

**Appendix C**

The algebra relations of \( U_q(g\ell(n + 1)) \) are implicit in the Yang-Baxter RLL relation \( \text{(2.7)} \). Explicitly we have for the Lax matrix

\[ \lambda L(u) = q^u L_+ - q^{-u} L_- \]

\[ (L_-)_{ij} = \begin{cases} 0, & i < j \\ q^{-E_i}, & i = j \\ -\lambda E_{ji}, & i > j \end{cases} \]

\[ (L_+)_{ij} = \begin{cases} \lambda E_{ij}, & i < j \\ q^{E_i}, & i = j \\ 0, & i > j \end{cases} \]

\( \text{(6.4)} \)

where \( E_{ij} = q^{E_j} E_{ij} q^{E_i} \) for \( i < j \) or \( i > j \), respectively. Further, for the fundamental R matrix we have

\[ \lambda \tilde{R}_{12}(u) = q^u \tilde{R}_+ - q^{-u} \tilde{R}_-, \tilde{R}_- \mid q = \tilde{R}_+ \mid q^{-1} = (\tilde{R}_+ \mid q)^{-1} \]
\[(\tilde{R}_+)^{ij_{i_2}}_{ji_{j_2}} = \delta_j^{i_2} \delta_j^{i_2} q^{\delta_j^{i_2} \delta_j^{i_2}} + \lambda \delta_j^{i_2} \delta_j^{i_2} \theta(i_1 - i_2)\]

By separating the dependence on \(u + v\) and \(u - v\) the spectral parameter dependent RLL relation implies
\[
\tilde{R}_+ L_{1\pm} L_{2\pm} = L_{1\pm} L_{2\pm} \tilde{R}_+,
\]
\[
\tilde{R}_+ L_{1+} L_{2-} = L_{1-} L_{2+} \tilde{R}_+.
\]

As an example we pick up the case with all subscripts +,
\[
(\tilde{R}_+)^{ij_{i_2}}_{ji_{j_2}} L^k_{i_1+1} L^k_{i_2+1} = L^k_{i_1+1} L^k_{i_2+1} (\tilde{R}_+)^{i_1 i_2}_{j_1 j_2}
\]
and substitute the explicit form of \(R_+\). In the special case \(i_2 < i_1 = j_1 < j_2\) we find we find
\[
\lambda^2 [\mathcal{E}_{i_1 i_2}, \mathcal{E}_{j_2 i_1}] + \lambda q E_{i_1} E_{j_2} = 0
\]

By relabeling indices this results in the corresponding relation of (4.6).

The Jordan-Schwinger representation provides an alternative of checking algebra relations. Starting from (3.2) we derive easily
\[
[E_{ij}, E_{jk}]_q = E_{ik}, \quad i < j < k,
\]
\[
[E_{ij}, E_{jk}]_{q^{-1}} = E_{ik}, \quad i > j > k,
\]
\[
[E_{ij}, E_{ji}]_1 = [N_i - N_j], \quad i \neq j,
\]
\[
[E_{i+1,i}, E_{i,j}]_1 = E_{i+1,i} q^{N_i - N_{i+1}}, \quad i + 1 < j,
\]
\[
[E_{i-1,i}, E_{i,j}]_1 = E_{i-1,i} q^{N_{i-1} - N_i}, \quad i - 1 > j,
\]
\[
[E_{k,i}, E_{i+1}]_1 = E_{k,i+1} q^{N_{i+1} - N_k}, \quad k - 1 > i.
\]

The proofs in the Jordan-Schwinger form rely on the relations (3.2). Let us do the 4th relation as an example.
\[
[E_{i+1,i}, E_{i,j}]_1 = [E_{i+1,i}, q^{-\sum_{i=1}^{j-1} N_i} E_{i,j}]_1 = [E_{i+1,i}, E_{i,j}]_1 q^{-\sum_{i=1}^{j-1} N_i}
\]
\[
= q^{N_i} E_{i+1,i} q^{-\sum_{i=1}^{j-1} N_i} = E_{i+1,i} q^{N_i - N_{i+1}}
\]

Relying on the latter relations we can check (4.6). For example, let \(i < j < k\)
\[
[\mathcal{E}_{ij}, \mathcal{E}_{jk}]_1 = q^{-E_i} q^{-E_k} [E_{ij}, E_{jk}]_q q^{-E_i} q^{-E_k} = -q^{-E_i} \mathcal{E}_{ik}.
\]

In the last step the 1st relation of (6.5) has been applied.