The closed state space of affine Landau-Ginzburg B-models

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Abstract

We show that the Hochschild homology of the category of perfect modules over a curved algebra is equal to the Jacobi ring of the corresponding affine singularity. This implies that the closed state space of an affine Landau-Ginzburg B-model is the universal closed state space compatible with its open sector. As an application we derive mathematically Kapustin and Li’s formula for the open-sector correlators over discs. We also extend our results to affine orbifolds, proving along the way an orbifold generalization of the Hochschild-Kostant-Rosenberg isomorphism.

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1 Introduction

It is well known that the open sector of the B-twisted $\Sigma$-model on a Calabi-Yau manifold $X$ is described by the derived category $D^b(X)$ of sheaves on $X$. Physically, $\Sigma$-models are a special case of a more general supersymmetric theory known as a Landau-Ginzburg model, which is specified by $X$ together with a choice of holomorphic function $W$ on $X$ called the superpotential. Calabi-Yau Landau-Ginzburg models also admit a B-twist, so the derived category should be a special case of a more general construction depending on the pair $(X, W)$.

The basic idea of this general construction was suggested by Kontsevich, it is a category whose objects are ‘twisted complexes’, i.e. instead of carrying differentials they carry endomorphisms $d$ such that $d^2 = W$. When $X = \mathbb{C}^n$ these are classical objects called ‘matrix factorizations’ that arise in algebraic singularity theory. Orlov $[12]$ has given an equivalent description of this category as the ‘triangulated category of singularities’ of the zero locus of $W$. However, the physics predicts that there should be rather more structure than just a triangulated category, there should really be a Topological Conformal Field Theory (TCFT). This is a complicated algebraic structure that is essentially a representation of the topology of the moduli space of Riemann surfaces.

Thus we should try and construct a TCFT associated to a Calabi-Yau LG B-model. However, if we really want to describe the physicists theory then our construction must agree with those pieces of the theory that have already been physically predicted. Probably the simplest such prediction is of the vector space of closed states of the theory. When $X$ is affine this is the Jacobi ring $J_W = O_X/(dW)$.

Problem 1.1. Show that to an affine Calabi-Yau LG B-model we can associate a TCFT whose closed state space is $J_W$.

The results in this paper are an advance towards this result. Our starting point is the paper $[2]$ by Costello, which proves two results about the structure of TCFTs. The first result is that the open sector of a TCFT is given by the data of a Calabi-Yau dg- (or $A_\infty$-) category. The second is that, given the open sector of a TCFT, there is a universal way (under one additional hypothesis) to construct a compatible closed sector, it is given by the Hochschild chain-complex of the dg-category that defines the open sector.

To an affine LG B-model we associate two pieces of data, a dg-category $Br(X, W)$, and a chain complex that we call the off-shell closed state space that has homology equal to $J_W$. We then prove (Theorem 3.4) that the off-shell closed state
space is quasi-isomorphic to the Hochschild chain-complex of $Br(X, W)$. This does not solve Problem 1.1 but it does reduce it to the problem of showing that $Br(X, W)$ is Calabi-Yau.

The extra data that turns a dg-category into Calabi-Yau dg-category is the existence of a cyclic trace map that induces a homologically non-degenerate pairing. Physically this trace map comes from the 1-point open correlator over a disc. In a TCFT, this trace map factors through the closed sector, i.e. it can be written as a composition of a map that sends open states to closed states (the boundary-bulk propagator) with an element of the dual of the closed states (the 1-point closed correlator). Our theorem produces the boundary-bulk map explicitly, it is the universal boundary-bulk map from $Br(X, W)$ to its Hochschild chain-complex composed with our quasi-isomorphism.

We also have a natural element of the dual of the off-shell closed state space (at least when $W$ has isolated singularities), it is given by the classical residue map. It was argued by Vafa [13] that this agrees with the physical 1-point closed correlator. This means we can write down the open 1-point correlator explicitly using our boundary-bulk map, this confirms (and provides higher-order off-shell corrections to) the formula derived physically by Kapustin and Li [6].

This means we have a candidate trace map for the dg-category $Br(X, W)$. It is not cyclic, however by construction it factors through the Hochschild chain-complex, and this is the homotopy-invariant notion of ‘cyclic’ developed by Kontsevich and Soibelman [9]. In particular they prove that, provided that the induced pairing is homologically non-degenerate, we may perform an $A_\infty$-automorphism of $Br(X, W)$ that maps our trace map to a genuinely cyclic one.

**Problem 1.2.** Show that the Kapustin-Li pairing is homologically non-degenerate.

A solution to this would solve Problem 1.1. We would then have a Calabi-Yau $A_\infty$-category which would be quasi-isomorphic to $Br(X, W)$ and hence would have Hochschild homology equal to $J_W$. We should remark that the homotopy category of $Br(X, W)$ is known to be abstractly Calabi-Yau because of Auslander-Reiten duality [11], but this does not solve Problem 1.2.

It is also natural to ask the same questions when the space $X$ is not affine. The relevant dg-category $Br(X, W)$ does exist (and will be discussed further by the author in upcoming work) but the proof of Theorem 3.4 does not work in the global setting and new ideas will be needed.

The outline of this paper is as follows:

In the remainder of the introduction we describe, very informally and at a completely non-technical level, the proof of Costello’s theorem.
In Section 2 we discuss curved algebras, which are the mathematical objects describing affine Landau-Ginzburg B-models, and prove the main technical result (Theorem 2.8) of the paper.

In Section 3 we discuss Landau-Ginzburg models and their closed state spaces, and recover the Kapustin-Li formula. In Section 3.3 we show how to extend our results to orbifolds, the main difficulty is to prove an orbifold analogue of the Hochschild-Konstant-Rosenberg theorem.

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1.1 The universal closed state space of an open TFT

In this section we explain, using a toy model, some of the ideas behind the following theorem of Costello which is the main motivation for this current paper:

**Theorem 1.3.** [2, Theorem A] Let \( \Phi \) be an open TCFT, and let \( C_H(\Phi) \) be its Hochschild chain-complex. If the natural pairing on \( C_H(\Phi) \) is homologically non-degenerate then the pair \((C_H(\Phi), \Phi)\) form the homotopy-universal open-closed TCFT whose open sector is \( \Phi \).

A TCFT is an abstraction of a particular kind of 2-dimensional quantum field theory, it depends on the full structure of the topology of the moduli space of Riemann surfaces. We will describe a much simpler object called a strictly topological 2d TFT, which uses only topological surfaces (i.e. \( \pi_0 \) of the moduli space). The extra ideas and technicalities necessary for the full TCFT case as they can be found in the above paper.

Fix a set \( \Lambda \) whose elements will be called branes. A (strictly topological) string world-sheet is the following data:

- A 2d oriented manifold with boundary \( \Sigma \).
- An ordered set of oriented components of the boundary \( \partial \Sigma \). These are the closed strings.
- An ordered set of disjoint oriented intervals embedded in the remaining part of \( \partial \Sigma \). These are the open strings.
- The pieces of \( \partial \Sigma \) that are neither open nor closed strings are called free boundaries, they can be either intervals or circles. Each free boundary is labelled with a brane.
The closed and open strings may be oriented in a way that is either compatible with or opposite to the orientation on $\Sigma$. The former are called incoming and the latter outgoing. Notice that every open string starts at one brane and ends at another (or possibly the same) brane.

Figure 1 shows an example world-sheet. It has three incoming strings, of which one is closed and two are open (shown on the left), and two outgoing strings, one closed and one open (shown on the right). There are three free boundaries labelled by branes $E, F$ and $G$.

String world-sheets form the morphisms of a category $\text{Strng}$. An object of $\text{Strng}$ is a pair consisting of an ordered set of circles and an ordered set of oriented intervals, each of which is labelled at both ends by a brane. A world-sheet is a morphism between the object formed by its incoming strings and the object formed by its outgoing strings. Composition of morphisms in $\text{Strng}$ is given by gluing the outgoing strings of one world-sheet to the incoming strings of another.

Taking the disjoint union of world-sheets gives $\text{Strng}$ a monoidal product. There is an object $\{\phi\}$ in $\text{Strng}$ consisting of a single circle, and there is a family of objects $\{E \to F\}$ indexed by $\Lambda \times \Lambda$ which consist of a single interval going between two branes. Obviously these objects together freely generate the objects of $\text{Strng}$ under the monoidal product.

We can also write down a set of world-sheets that generate $\text{Strng}$ as a monoidal category. Firstly note that any world-sheet $\Sigma$ has a dual $\Sigma^\dagger$ where we flip the orientations on all the strings. It is clear (e.g. by Morse theory) that the following set of world-sheets, together with their duals, generate $\text{Strng}$:
• $P_c$, the genus-zero surface with two incoming and one outgoing closed strings (the ‘pair-of-pants’).

• $D_c$, the disc whose boundary is an incoming closed string.

• $I_c$, the cylinder with one incoming and one outgoing closed string.

• $P_o$, the disc with two incoming and one outgoing open strings embedded in its boundary.

• $I_o$, the disc with one incoming and one outgoing open string in its boundary.

• $T$, the cylinder with one end an outgoing closed string and one incoming open string in the other end (the ‘penny-whistle’).

The world-sheets in this list that have free boundaries come in families indexed by products of $\Lambda$. We also have to add ‘re-ordering’ morphisms which consist of products of $I_c$ and $I_o$ but with different orders on the sets of strings at each end. Notice that the disc $D_o$ with one incoming string is not in this list, since it can be obtained by gluing $T$ to $D_c$.

As an example, the world-sheet in Fig. 1 has source $\{F \to E, E \to F, \circ\}$, target $\{\varphi \to \varphi, \circ\}$, and can be written as the composition

$$\Sigma = (I_c \sqcup T \uparrow P_c) \circ (P_c^{-1} \sqcup T P_o)$$

A 2d (strictly) topological field theory is a monoidal functor

$$\Psi : \text{Strng} \to \text{Vect}$$

Since $\Psi$ is monoidal, it is determined on objects by its value on the single closed string

$$V := \Psi(\circ)$$

which is called the closed state space, and its value on each labelled open string

$$A(\mathcal{E}, \mathcal{F}) := \Psi(\mathcal{E} \to \mathcal{F})$$

which is the open state space between $\mathcal{E}$ and $\mathcal{F}$. Applying $\Psi$ to our generating set of morphisms gives us various linear maps between these vector spaces, for example we have a bilinear product

$$\Psi(P_c) : V^\otimes 2 \to V$$

These linear maps are subject to various relations coming from relations amongst world-sheets. For example, it is clear that

$$P_c \circ (P_c \sqcup I_c) \cong P_c \circ (I_c \sqcup P_o)$$

\[1\text{In Dutch: ‘Fluitje van een cent’, which means something easy or having no content. This worldsheet will used later as the basis of a formal adjoint construction.}\]
which implies that the product $\Psi(P_c)$ is associative. The totality of these relations mean that a TFT is equivalent to the following data:

- A commutative Frobenius algebra $V$ with product $\Psi(P_c)$, unit $\Psi(I_c)$, and trace map $\Psi(D_c)$.
- A linear category $A$, whose set of objects is $\Lambda$, with composition
  \[ \Psi(P_o) : A(\mathcal{E}, \mathcal{F}) \otimes A(\mathcal{F}, \mathcal{G}) \to A(\mathcal{E}, \mathcal{G}) \]
  and identity morphisms $\Psi(I_o) \in A(\mathcal{E}, \mathcal{E})$.
- A map (the ‘boundary-bulk map’)
  \[ \Psi(T) : A(\mathcal{E}, \mathcal{E}) \to V \]
  for each brane $\mathcal{E}$, such that
  1. The bilinear pairing $\Psi(D_c) \circ \Psi(T) \circ \Psi(P_o)$ on $A$ is non-degenerate and cyclically-invariant.
  2. The adjoint $\Psi(T)^\dagger$ (with respect to the pairings on $A$ and $V$) is a map of algebras
  3. For any two branes $\mathcal{E}$ and $\mathcal{F}$
  \[ \Psi(P_c) \circ (\Psi(T) \otimes \Psi(T)) = \sum_i \Psi(T) \circ \Psi(P_o) \circ (\Psi(P_o)(-, e_i) \otimes \Psi(P_o)(-, \epsilon_i)) \]
  where $\{e_i\}$ and $\{\epsilon_i\}$ are dual bases of $A(\mathcal{E}, \mathcal{F})$ and $A(\mathcal{F}, \mathcal{E})$.

Condition (1) says that $A$ is a Calabi-Yau category (it is the same as the structure that makes $V$ a Frobenius algebra, unfortunately ‘Frobenius category’ means something else). Condition (3) is called the Cardy condition and maybe needs a little explanation. It comes from the topological equivalence of the two worldsheets in Fig. 2 plus the fact that the copairing on $A$ (the adjoint to the pairing) is given by $\sum_i e_i \otimes \epsilon_i$.

A little more physics terminology: $V$ and $A$ are called the closed and open sectors of the theory, and if $\Sigma$ is a world-sheet with no outgoing strings then $\Psi(\Sigma)$ is called a correlator. For example $\Psi(D_c)$ is the 1-point closed correlator on a disc. Notice that the 1-point open correlator on a disc $\Psi(D_o)$ factors through the boundary-bulk map. We will use this in Section 3.2.

Now we come to the key construction from the point of view of this paper. There is a subcategory $\text{String}_o \subset \text{String}$ where we don’t allow any closed strings, a functor from this subcategory to $\text{Vect}$ is called an open TFT, by the above this is the same thing as a Calabi-Yau category $A$. We will show how, given one extra hypothesis, we can canonically extend any open TFT to a full TFT.
Let $A$ be a TFT, and consider the vector space $V = A / [A, A]$

We define a pairing on $V$ by

$$\langle [\alpha], [\beta] \rangle := \text{Tr}(x \mapsto \alpha x \beta)$$

where $\alpha \in A(E, E)$, $\beta \in A(F, F)$, and the trace is of the given endomorphism of $A(E, F)$. It is easy to check that this is well-defined.

**Lemma 1.4.** Let $A$ be an open TFT, and let $V$ be the above vector space. Assume that the pairing $\langle , \rangle$ is non-degenerate. Then $V$ is the canonical closed state space for $A$.

By ‘canonical closed state space’, we mean that $(V, A)$ form a TFT, and that taking $A$ to $(V, A)$ gives an adjoint to the forgetful functor that sends a TFT to its open sector. As we explained above this is just a baby version of Costello’s theorem (Thm. [1.3]).

**Proof.** Let $\mathcal{V}$ be the set of world-sheets whose incoming strings are all open, and that have exactly one outgoing closed string and no outgoing open strings. This set carries a left action of the category $\text{Strng}_o$ by gluing on purely open worldsheets. Note that by definition $A$ carries a right action of $\text{Strng}_o$. We define

$$V = A \otimes_{\text{Strng}_o} \mathcal{V}$$

To get $V$ we take the free vector space spanned by world-sheets in $\mathcal{V}$ with their open strings labelled by elements of $A$, then quotient by relations coming from the action of $\text{Strng}_o$ on either side. An example relation is shown in Fig. [3(a)]

It is clear that by applying successive relations we can reduce everything to the
form $\alpha \otimes T$ for some $\alpha \in A$ (as in the RHS of Fig. 3(a)). However there is still the residual relation shown in Fig. 3(b) so in fact $V = A / [A, A]$.

Now consider the cylinder with two incoming closed strings $(P_c \circ D_c)$. This induces a pairing on $V$, since given two elements of $V$ we can glue them into the ends of the cylinder and get a correlator which we can evaluate in the open sector. This is the pairing $\langle \cdot, \cdot \rangle$, it is well-defined by the definition of $V$. By the same reasoning, we can take any correlator, label it with elements of $A$ and $V$ and then evaluate it using just the structure of $A$. Since by assumption the pairing on $V$ is non-degenerate, this in fact defines a linear map for all world-sheets, because we can identify $V$ and $V^\vee$. Thus $(V, A)$ forms a TFT.

Finally we will say a few words about the more sophisticated case of TCFTs dealt with in [2]. Firstly, we let our world-sheets be Riemann surfaces (with boundary) instead of topological surfaces, and since these come in moduli spaces this produces a topological category. Applying a suitable chain functor (one that computes topological homology and is compatible with the gluing structure) we get a dg-category. A TCFT is a functor from this dg-category to $\text{dgVect}$.

The open sector $A$ of a TCFT is in general a Calabi-Yau $A_\infty$-category, but up to homotopy we can represent it as a Calabi-Yau dg-category. Since this is just a ‘derived’ analogue of a CY linear category it is then plausible that the canonical closed sector should be a derived version of $A / [A, A]$. This is precisely the Hochschild chain complex.
2 Curved algebras

In this section we prove that the Hochschild complex of the category of perfect dg-modules over a curved algebra is quasi-isomorphic to the Hochschild complex of the curved algebra itself.

We work over an arbitrary ground field $k$, and category means a $k$-linear category. A (bi)module over a $k$-linear category $\mathcal{A}$ means a (bi)module over the total algebra of $\mathcal{A}$, which is the same thing as a (bi)functor from $\mathcal{A}$ to $\text{Vect}$.

2.1 Curved algebras and cdg-modules

Definition 2.1. A curved dg-algebra (or cdga) is a triple $(A, d, W)$ where $A$ is a graded associative algebra, $d$ is a degree 1 derivation of $A$, and $W \in A$ is a degree 2 element such that $dW = 0$ and $d^2 = [W, -]$

A curved algebra is a curved dg-algebra where $d = 0$, and hence $W$ is central. The definitions of curved dg-category and curved category are similar.

We may choose to work with either a $\mathbb{Z}$-grading or just a $\mathbb{Z}_2$-grading. When we come to discuss Landau-Ginzburg models we will see that it is easier to find examples having just a $\mathbb{Z}_2$-grading, but that $\mathbb{Z}$-graded examples are more physically interesting since the additional grading corresponds to the vector R-charge. The definitions and proofs in this paper work identically for either choice, except for the arguments involving spectral sequences where we describe the special measures needed. Of course, in the $\mathbb{Z}_2$-graded version ‘$W$ is a degree 2 element’ means $W$ is a degree 0 element.

The name comes from thinking of $W$ as the ‘curvature’ of the ‘connection’ given by $d$. In this paper we will be mostly concerned with curved algebras and (non-curved) dg-categories. We will generally drop the $d$, but not the $W$, from the notation.

Definition 2.2. A cdg-module over a curved dg-algebra or -category $(A, W)$ is a pair $(M, d_M)$ where $M$ is a graded $A$-module and $d_M$ is a degree 1 linear endomorphism of $M$ such that such that

$$d_M(am) = (da)m + (-1)^{|a|}a(d_Mm)$$

and

$$d_M^2 = W$$

If $W = 0$ this is just the usual definition of a dg-module over a dga, but having a non-zero $W$ ‘twists’ the differentials.
Given two cdg-modules \((M,d_M)\) and \((N,d_N)\), we have a graded vector space \(\text{Hom}_A(M,N)\) with a degree 1 endomorphism
\[
d_{M,N}(f) := (d_N \circ f) - (-1)^{|f|} (f \circ d_M)
\]
As in the case \(W = 0\), this is in a fact a differential, even though neither \(d_M\) nor \(d_N\) is. The two copies of \(W\) that occur in the square of this expression have opposite signs and cancel. This means that the category of cdg-modules is a dg-category.

When \(A\) is commutative (or we are using bimodules), this definition can be seen as part of a larger structure. Given a cdg-module \(M\) over \((A,W_M)\), and another \(N\) over \((A,W_N)\), we can form their tensor product \(M \otimes_A N\), which is a cdg-module over \((A,W_M + W_N)\). This defines a monoidal product on the the category of cdg-\(A\)-modules where we allow all possible \(W\)s, and the Hom complex defined above is an internal Hom functor. This larger category is a curved dg-category with curvature \(W_M\) at each object \(M\). It is relevant to field theory with defects, and was used to great effect by Khovanov and Rozansky [8]. Arguably even when considering the sub-category of cdg-modules over a fixed \(W\) one should keep this (now central) curvature term, but we shall not do so in this paper.

In the ordinary case of a (non-curved) dg-algebra \(A\), one often studies not the category of dg-modules but rather the derived category \(D(A)\), which is its localization at quasi-equivalences. This is a triangulated category, and we wish to work instead with dg-categories. Now the category of dg-modules over \(A\) is of course a dg-category, but it is not the correct dg-category to study since its homotopy category is not \(D(A)\). We can correct it by taking only the sub-category of projective modules, this is a dg-category that does have \(D(A)\) as its homotopy category. Or we can take some smaller dg-category such as the category of perfect (i.e. finitely-generated projective) complexes, which has a slightly different homotopy category that may be better-behaved.

We will work with an analogue of the category of perfect complexes over a curved algebra. The derived category of modules over a curved algebra (in fact over a curved \(A_{\infty}\)-algebra) has been constructed in [11], but by analogy with the non-curved case one should not expect it to agree with the homotopy category of our category, except maybe in special cases such as \(A\) commutative and smooth.

**Definition 2.3.** A cdg-module \((M,d_M)\) over a cdga \((A,W)\) is perfect if \(M\) is a finitely-generated and projective module over the underlying algebra \(A\).

**Definition 2.4.** The dg-category \(\text{Perf}(A,W)\) is the full sub-category of the category of cdg-\((A,W)\)-modules with objects the perfect cdg-modules.

In fact we need to adjust this definition slightly. We need that for every perfect cdg-module \(M\) the underlying module is not just abstractly projective but is
explicitly given as a direct summand of $A^{\oplus r_M}$ for some $r_M \in \mathbb{N}$. This means that any morphism
$$\alpha \in \text{Hom}_{\text{perf}}(A,W)(M,N)$$
is explicitly a matrix over $A$ of size $r_M \times r_N$. Obviously this defines an equivalent category, but it is necessary for the next construction, which will be used in Theorem 2.8.

**Definition 2.5.** Let $\{M_0, ..., M_t\}$ be a set of perfect cdg-$\text{(A,W)}$-modules. We define
$$\text{Tr} : \text{Hom}(M_0, M_1) \otimes ... \otimes \text{Hom}(M_{t-1}, M_t) \otimes \text{Hom}(M_t, M_0) \to A^{\otimes (t+1)}$$
with
$$\text{Tr}(\alpha_0 \otimes ... \otimes \alpha_t) = \sum (\alpha_0)_{i_0i_1} \otimes ... \otimes (\alpha_t)_{i_ti_0}$$
and
$$\text{Tr}_{M_0,M_1} : \text{Hom}(M_0, M_1) \otimes ... \otimes \text{Hom}(M_{t-1}, M_t) \to \text{Hom}_A(M_0, A) \otimes A^{\otimes t-2} \otimes \text{Hom}_A(A, M_t)$$
with
$$\text{Tr}_{M_0,M_1}(\alpha_0 \otimes ... \otimes \alpha_{t-1}) = \sum (\alpha_0)_{i_1} \otimes (\alpha_1)_{i_1i_2} \otimes ... \otimes (\alpha_{t-2})_{i_{t-2}i_{t-1}} \otimes (\alpha_{t-1})_{i_{t-1}i_0}$$
where in each case every $i_k$ ranges from 1 to $r_{M_k}$.

We can also define a right-inverse to $\text{Tr}_{M,N}$. Given $\beta \in \text{Hom}_A(M, A)$ and $\gamma \in \text{Hom}_A(A, N)$ we define matrices
$$\bar{\beta} = (\beta, \delta_{1j}) \in \text{Hom}(M, M)$$
and
$$\bar{\gamma} = (\delta_{k1}, \gamma_l) \in \text{Hom}(M, N)$$
then note that
$$\text{Tr}_{M,N}(\bar{\beta} \otimes a_11_M \otimes ... \otimes a_t1_M \otimes \bar{\gamma}) = \beta \otimes a_1 \otimes ... \otimes a_t \otimes \gamma$$

**Caveat.** The signs in these formulas (and hence the signs in Theorem 2.8) are not yet correct. In particular the first component of $\text{Tr}$ should be a graded trace.

### 2.2 The Hochschild Complex

**Definition 2.6.** The bar resolution $B(A,W)$ of a curved dg-algebra $(A,W)$ is the complex
$$\prod_{k \geq 0} A^{\otimes k+2}[k]$$
with differential
$$d = d_2 + d_1 + d_W$$
where
\[ d_2(r_0 \otimes \ldots \otimes r_{k+1}) = \sum_{i=0}^{k} (-1)^i r_0 \otimes \ldots \otimes r_i r_{i+1} \otimes \ldots \otimes r_{k+1} \]

and
\[ d_1(r_0 \otimes \ldots \otimes r_{k+1}) = \sum_{i=0}^{k+1} (-1)^i r_0 \otimes \ldots \otimes dr_i \otimes \ldots \otimes r_{k+1} \]

The bar resolution of a curved dg-category \((A,W)\) is defined similarly, where
\[ A^\otimes k := \bigoplus_{x_0, \ldots, x_k \in \text{Ob}(A)} A(x_0, x_1) \otimes A(x_1, x_2) \otimes \ldots \otimes A(x_{k-1}, x_k) \]

Note that \(B(A,W)\) is a bimodule over \(A\), and that
\[ d^2(r_0 \otimes \ldots \otimes r_{k+1}) = W(r_0 \otimes \ldots \otimes r_{k+1}) - (r_0 \otimes \ldots \otimes r_{k+1})W \]

so \(B(A,W)\) is a cdg-module over \((A \otimes A^\text{op}, W \otimes 1 - 1 \otimes W)\). This is the natural notion of a cdg-bimodule over \((A,W)\). There is a map
\[ B(A,W) \to A \]

given by multiplication on the final summand
\[ A \otimes A \to A \]

and zero on the other summands. This is a map of cdg-bimodules, and when \(W = 0\) it is a quasi-isomorphism (which is why we call it the bar resolution). In the curved case we cannot say this, but there may be a sensible statement using the model category structure from [11].

**Definition 2.7.** The Hochschild complex of a curved dg-algebra or -category \((A,W)\) is the chain-complex
\[ CH(A,W) := A \otimes_{A \otimes A^\text{op}} B(A,W) \]

Note that this is an honest chain-complex, since the curvature becomes zero on tensoring with \(A\). The underlying vector space is
\[ \prod_{k \geq 0} A \otimes A^\otimes k \]
when $A$ is an algebra, or

$$
\prod_{k \geq 0} \bigoplus_{x_0, \ldots, x_k \in \text{Ob}(A)} A(x_k, x_0) \otimes A(x_0, x_1) \otimes \cdots \otimes A(x_{k-1}, x_k)
$$

when $A$ is a category.

In the case $W = 0$ these definitions are classical (though it’s more usual to take the bar resolution to be an infinite sum rather than product), and

$$
C_H(A, d) = A \otimes A^{op} A
$$

since $B(A)$ is a free resolution of $A$ as an $A$-dg-bimodule. The definition of the Hochschild complex for a curved dg-algebra is a special case of the definition for a curved $A_\infty$-algebra, which can be found in e.g. [10].

**Theorem 2.8.** Let $(A, W)$ be a curved algebra, and $\text{Perf}(A, W)$ be its dg-category of perfect cdg-modules. Then there is a quasi-isomorphism

$$
\Psi : C_H(\text{Perf}(A, W)) \longrightarrow C_H(A, W)
$$

given by

$$
\Psi(\alpha_0 \otimes \cdots \otimes \alpha_k) = \sum \text{Tr}(\alpha_0 \otimes d_{M_1}^{\otimes s_1} \otimes \cdots \otimes \alpha_k \otimes d_{M_0}^{\otimes s_0})
$$

where each $s_i$ ranges over $[0, \infty)$.

This formula was constructed essentially by guesswork, and it would be nice to have a better of understanding of it. In particular it does not work when $A$ has non-zero differential, so further insight will be necessary to extend it to that case, or to a global geometric version when we replace $A$ by a scheme. As it is we can only remark that it appears to be related to the Chern character.

In the case $W = 0$ it is certainly folklore that the two sides are quasi-isomorphic, but as far as we know this explicit formula is still new.

**Proof.** For notational convenience let $P = \text{Perf}(A, W)$. The Hochchild complex of $P$ computes $P \otimes_{P \otimes P^{op}} P$, the idea of the proof is to show that we can compute this instead using a flat resolution of $P$ constructed from the bar resolution of $(A, W)$, and that using this resolution instead produces $C_H(A, W)$.

Consider the $P$-bimodule $T$ given by

$$
T = \bigoplus_{M, N \in \text{Ob}(P)} \text{Hom}(M, B(A, W) \otimes_A N)
$$
which as a vector space is

\[ \prod_{k \geq 0} \bigoplus_{M,N \in \text{Ob}(P)} \text{Hom}_A(M, A) \otimes A^\otimes k \otimes \text{Hom}_A(A, N) \]

We claim that \( T \) is flat. By Lemma 2.9 below it is sufficient to check the flatness of the underlying bimodule \( \hat{T} \) over the underlying linear category \( \hat{P} \) of \( P \). The category \( \hat{P} \) is simply an (infinite-rank) matrix algebra over the algebra \( A \), and the bimodule \( \hat{T} \) is a direct sum of copies of the \( \hat{P} \)-bimodule

\[ S = \bigoplus_{M,N \in \text{Ob}(P)} \text{Hom}_A(M, A) \otimes \text{Hom}_A(A, N) \]

which as a left (respectively, right) module is just the canonical representation of a matrix algebra by column (row) vectors. But the canonical representation of a matrix algebra is flat, since it is isomorphic to the projective module generated by one of the canonical idempotents.

The natural map \( B(A, W) \to A \) induces a map of \( P \)-bimodules from \( T \) to the diagonal \( P \)-bimodule

\[ P = \bigoplus_{M,N \in \text{Ob}(P)} \text{Hom}_A(M, N) \]

We will show that \( T \) is a resolution of \( P \), by showing that it is quasi-isomorphic to \( B(P) \). We define the following two \( P \)-bimodule maps:

\[ \hat{\Psi} : B(P) \to T \]

and

\[ \hat{\Psi}^{-1} : T \to B(P) \]

which have components

\[ \hat{\Psi}(\alpha_0 \otimes \ldots \otimes \alpha_t) = \sum_{s_1, \ldots, s_t \geq 0} \text{Tr}_{M_0, M_{t+1}}(\alpha_0 \otimes d_{M_1}^{s_1} \otimes \alpha_1 \otimes d_{M_2}^{s_2} \otimes \ldots \otimes \alpha_t \otimes d_{M_t}^{s_t} \otimes \alpha_{t+1}) \]

and

\[ \hat{\Psi}^{-1}(\beta \otimes \alpha_1 \otimes \ldots \otimes \alpha_t \otimes \gamma) = \sum_{s_0, \ldots, s_t \geq 0} (-1)^{s_1} \beta \otimes d_{M}^{s_0} \otimes \alpha_1 1_M \otimes d_{M}^{s_2} \otimes \ldots \otimes \alpha_t 1_M \otimes d_{M}^{s_t} \otimes \gamma \]

(we are using Definition 2.5 here). It is straightforward to check that

\[ \hat{\Psi} \hat{\Psi}^{-1} = 1_T \]
and by chasing the definitions of the differentials on each side one can check that both $\hat{\Psi}$ and $\hat{\Psi}^{-1}$ are chain maps. Furthermore, composing $\hat{\Psi}$ with the natural map from $T$ to $P$ gives the natural quasi-isomorphism from $B(P)$ to $P$. This implies that $\hat{\Psi}$ and $\hat{\Psi}^{-1}$ are mutually inverse quasi-isomorphisms. Since $T$ is flat and $B(P)$ is free we get an induced quasi-isomorphism

$$\Psi := 1 \otimes \hat{\Psi} : C_H(P) \xrightarrow{\sim} P \otimes_{P \odot P^{\text{op}}} T$$

It remains to show that the latter complex is $C_H(A,W)$. As a vector space it’s a quotient, under the actions of $P \otimes P^{op}$, of the vector space

$$\prod_{k \geq 0} \bigoplus_{M,N \in \text{Ob}(P)} \text{Hom}_A(N,M) \otimes \text{Hom}_A(M,A) \otimes A^\otimes k \otimes \text{Hom}_A(A,N)$$

We can map this to $C_H(A,W)$ by sending

$$f \otimes g \otimes a_1 \otimes \ldots \otimes a_k \otimes h \mapsto hfg \otimes a_1 \otimes \ldots \otimes a_k$$

This is clearly well-defined on the quotient, and it is straight-forward to check that it is a chain-map. To see that it defines an isomorphism from $P \otimes_{P \odot P^{op}} T$ to $C_H(A,W)$ we just need to check the equality

$$S \otimes_{P \odot P^{op}} \hat{P} = \left( \bigoplus_{N \in \text{Ob}(P)} \text{Hom}_A(A,N) \right) \otimes_{\hat{P}} \hat{P} \otimes_{\hat{P}} \left( \bigoplus_{M \in \text{Ob}(P)} \text{Hom}_A(M,A) \right)$$

which is an elementary statement about matrix algebras.

Lemma 2.9. Let $A$ be a dga, and $M$ a right dgm over $A$. Let $\hat{A}$ and $\hat{M}$ denote the underlying algebra of $A$ and the underlying $\hat{A}$-module of $M$. If $\hat{M}$ is flat over $\hat{A}$ then $M$ is flat over $A$.

Proof. Let $N$ be any other dgm over $A$, and $\hat{N}$ be its underlying $\hat{A}$-module. Then $\text{Tor}_A(M,N)$ is the homology of $M \otimes_A B(A) \otimes_A N$. We can view this as a bicomplex with bigraded pieces $(M \otimes A^\otimes p \otimes N)_q$, this gives a spectral sequence whose zeroeth page is $\hat{M} \otimes_{\hat{A}} B(A) \otimes_{\hat{A}} \hat{N}$. The result follows. Note that if $A$ is only $\mathbb{Z}_2$-graded then the bicomplex is $N \times \mathbb{Z}_2$-graded, but this does not affect the argument.
3 Landau-Ginzburg B-models

3.1 The closed state-space is universal

**Definition 3.1.** An affine Landau-Ginzburg B-model is the following data:

- A smooth $n$-dimensional affine variety $X$ over $\mathbb{C}$.
- A choice of function $W \in \mathcal{O}_X$ (the ‘superpotential’).
- An action of $\mathbb{C}^*$ on $X$ (the ‘vector R-charge’).

such that

1. $-1 \in \mathbb{C}^*$ acts trivially.
2. $W$ has weight (‘R-charge’) equal to 2.

This means that $R := \mathcal{O}_X$ is a regular commutative algebra graded by the even integers, and $W \in R$ is an element of degree 2. Thus $R$ is a curved algebra (with no odd graded part).

There is a weaker definition of vector R-charge where we keep only the trivial action of the subgroup $\mathbb{Z}_2 \subset \mathbb{C}^*$. This corresponds to working with $\mathbb{Z}_2$-graded curved algebras.

**Definition 3.2.** The (off-shell) closed state space of an affine LG B-model $(X, W)$ is the graded vector space

$$\bigoplus \Omega^k_X[-k]$$

of holomorphic forms on $X$, with differential

$$\alpha \mapsto dW \wedge \alpha$$

Note that since $W$ has degree 2 the total degree of the differential is indeed 1.

The homology of this complex is

$$\Omega^k_X[-n] / (dW)$$

which is the Jacobi ring (times a volume form). It is well known in the physics literature that this is the space of physical closed states.
**Definition 3.3.** The category of B-branes for an affine LG model \((X,W)\) is the dg-category

\[ Br(X,W) := \text{Perf}(R,W) \]

of free cdg-modules over the curved algebra \((R,W)\). Note that a finitely-generated projective \(R\)-module is just a finite-rank vector bundle on \(X\).

Since \(R\) has no odd graded part it follows that a brane \(M\) splits as a direct sum \(M_{\text{ev}} \oplus M_{\text{od}}\) where \(M_{\text{ev}}\) (respectively \(M_{\text{od}}\)) is the sum of all factors that are shifted by an even (respectively odd) integer, and \(d_M\) exchanges these two factors.

If \(X = \mathbb{C}^n\), and we work with a \(\mathbb{Z}_2\)-grading, then since all vector bundles are trivial a B-brane \(M\) is described by a pair of polynomial matrices \(d_{\text{od}}^M\) and \(d_{\text{ev}}^M\) such that \(d_{\text{od}}^M d_{\text{ev}}^M = d_{\text{ev}}^M d_{\text{od}}^M = W\). This is a ‘matrix factorization’ of \(W\).

As discussed in the introduction, \(Br(X,W)\) is supposed to be the open sector of the B-model TCFT constructed from \((X,W)\). If it is indeed a TCFT, then it has a universal closed sector given by its Hochschild complex. The following theorem says that our definition of the off-shell state space is equivalent to the universal one.

**Theorem 3.4.** The closed state space \((\Omega^\bullet, dW \wedge)\) is quasi-isomorphic to \(C_H(\text{Br}(R,W))\).

The quasi-isomorphism is constructed in two stages, the first is the quasi-isomorphism

\[ \Psi : C_H(\text{Br}(R,W)) \simto C_H(R,W) \]

from Theorem 2.8 and the second is the following:

**Lemma 3.5.** There is a quasi-isomorphism

\[ \phi : C_H(R,W) \simto (\Omega_X^\bullet, dW \wedge) \]

**Proof.** Recall (Def. 2.6) that the differential on \(C_H(R,W)\) is the sum of two pieces, \(d_2\) and \(d_W\). Both terms are differentials, and they commute. If we forget the \(d_W\) term we are left with the usual Hochschild chain-complex \(C_H(R)\) of the algebra \(R\).

There is a well-known theorem of Hochschild, Kostant and Rosenberg [5] that the map

\[ \phi : C_H(R) \to \Omega_X^\bullet \]

(3.1)

\[ \phi(r_0 \otimes \ldots \otimes r_k) := \frac{1}{k!} r_0 dr_1 \wedge \ldots \wedge dr_k \]
is a quasi-isomorphism, where we put the zero differential on $\Omega_X^\bullet$. We also have that

$$(dW \wedge) \circ \phi = \phi \circ dW$$

If $R$ is $\mathbb{Z}$-graded then both $C_H(R)$ and $\Omega_X^\bullet$ are bicomplexes, and $\phi$ is a map of bicomplexes inducing an isomorphism on page one of the corresponding spectral sequences. These spectral sequences converge, since they degenerate at page two for degree reasons, so $\phi$ is a quasi-isomorphism of the total complexes.

If $R$ is $\mathbb{Z}_2$-graded then (ignoring the $\mathbb{Z}_2$-grading entirely) $C_H(R,W)$ is $\mathbb{Z}$-graded, with $d_2$ having degree $+1$ and $d_W$ having degree $-1$. The machinery of spectral sequences still works in this context, and the above argument holds.

In any TCFT there is a boundary-bulk map sending open string states (i.e. morphisms in the category of branes) to closed states. If the closed sector is the universal one the bulk-boundary map is tautological, it’s the inclusion of the category of branes as a summand of its Hochschild complex. For an affine LG B-model, we can compose this tautological map with our quasi-isomorphism $\phi \circ \Psi : C_H(\text{Br}(R,W)) \sim \rightarrow (\Omega_X^\bullet, dW \wedge)$ to give the boundary-bulk map. Explicitly, its the map whose component at a brane $(M,d_M)$ is

$$\tau_M : \text{End}_{\text{Br}(X,W)}(M) \rightarrow (\Omega_X^\bullet, dW \wedge)$$

$$\alpha \mapsto \sum_{k=0}^{\infty} \frac{1}{k!} \text{Tr}(\alpha(\partial d_M)^{\wedge k})$$

### 3.2 The Kapustin-Li formula

In [6] the authors give a formula for the 1-point correlator of open string states in a (particular class of) LG B-model, valid for physical (i.e. closed) states. In this section we derive an off-shell (i.e. chain-level) version of their formula.

In any TCFT, the open sector 1-point correlator can be factored as the composition of the boundary-bulk map and the closed 1-point correlator. We have already described the boundary-bulk map in the preceding section, so it remains to give the closed 1-point correlator.

Assume for the rest of this section that the hypersurface $W = 0$ has a single isolated singularity. Let $Z \subset X$ be the singular scheme, i.e. the scheme-theoretic zero-locus of $dW$, and let the dimension of $X$ be $n$.

**Definition 3.6.** The 1-point correlator on the closed state space $(\Omega_X^\bullet, dW \wedge)$ is the residue [4] III.9]

$$\langle - \rangle_c := \text{Res}_Z : \Omega_X^\bullet \rightarrow \mathbb{C}$$
A standard fact (proved in the above reference) is that the residue vanishes on the image of $\wedge dW$, so this correlator is closed.

If $X = \mathbb{C}^n$ then we can write the residue map as a contour integral

$$\langle \omega \rangle_c = \frac{1}{(2\pi i)^n} \oint \frac{\omega}{\prod_i \partial_i W}$$

where the integral is taken over a Lagrangian torus enclosing the singularity. Composing this with the boundary-bulk map we get the open 1-point correlator over a disc

$$\langle - \rangle_{\text{disk}} : \bigoplus_{M \in \text{Ob}(\text{Br}(X, W))} \text{End}_{\text{Br}(X, W)}(M)$$

$$\langle \alpha \rangle_{\text{disk}} = \langle \tau_M(\alpha) \rangle_c = \frac{1}{n!(2\pi i)^n} \oint \frac{\text{Tr}(\alpha(\partial d_M)^\wedge n)}{\prod_i \partial_i W}$$

which is the Kapustin-Li formula.

### 3.3 Orbifolds

We obtain more interesting and important examples of affine Landau-Ginzburg B-models if we allow the underlying space $X$ to be an orbifold, i.e. we take a quotient stack

$$X = [Y/G]$$

where $G$ is a finite group acting on a smooth affine variety $Y$, and add a superpotential $W$ which is a $G$-invariant function on $Y$. In this section we show how to adapt our results to this setting.

The category of B-branes on $(X, W)$ is just the category of $G$-equivariant B-branes on $(Y, W)$, but we can recast this definition. Let $R = \mathcal{O}_Y$. Recall that the twisted group ring

$$A := R \rtimes \mathbb{C}[G]$$

is the vector space $R \otimes \mathbb{C}[G]$ with multiplication

$$(r \otimes g) \circ (s \otimes h) = (rg(s) \otimes gh)$$

$A$ inherits a grading and a superpotential $W \otimes 1$ from $R$, making it a non-commutative curved algebra (the curvature is central since $W$ is invariant).

**Lemma 3.7.**

$$\text{Br}(X, W) := \text{Perf}(A, W)$$

**Proof.** This is elementary. □
Example 3.8. Let $X = [\mathbb{C}^2/\mathbb{Z}_2]$ where $\mathbb{Z}_2$ acts with weight 1 on each co-ordinate. We define a $\mathbb{C}^*$ R-charge action by letting $\mathbb{C}^*$ also act with weight 1 on each co-ordinate. Notice that $-1 \in \mathbb{C}^*$ does indeed act trivially on the orbifold (although not on $\mathbb{C}^2$). Let $x$ and $y$ be the two co-ordinates, and let $W = x^2 - y^2$.

The twisted group ring is $A = \mathbb{C}[x, y] \rtimes \mathbb{C}[\mathbb{Z}_2]$. Let $\tau$ be the generator of $\mathbb{Z}_2$, then we have a complete pair of orthogonal idempotents

$$e_0 = \frac{1}{2}(1 + \tau) \quad e_1 = \frac{1}{2}(1 - \tau) \quad e_i e_j = \delta_{ij}$$

This means we can write $A$ as a quiver algebra (with relations) using $e_1, e_2$ as nodes. Every equivariant vector bundle on $\mathbb{C}^2$ is a direct sum of the two line bundles $\mathcal{O}$ and $\mathcal{O}(1)$ associated to the two characters of $\mathbb{Z}_2$. These correspond to the projective $A$-modules $Ae_0$ and $Ae_1$. One example of a brane is given by $\mathcal{O} \oplus \mathcal{O}(1)$ with endomorphism

$$\left( \begin{array}{cc} 0 & x + y \\ x - y & 0 \end{array} \right)$$

This corresponds to the trivial $A$-module $A$ with endomorphism

$$e_0(x + y)e_1 + e_1(x - y)e_0 = x - y\tau$$

Lemma 3.7 and Theorem 2.8 imply that we have a quasi-isomorphism between $C_H(\text{Br}(X, W))$ and $C_H(A, W)$. We may view $A$ as a bimodule over the semi-simple algebra $\mathbb{C}[G]$. This leads to a different (and smaller) version of the bar resolution, i.e. we define

$$B_{\mathbb{C}[G]}(A, W) = \prod_{k \geq 0} A^{\otimes_{\mathbb{C}[G]}(k+2)}[k]$$

with the same differential $d = d_2 + d_W$ as in Definition 2.6. This differential is well-defined since $W$ is $G$-invariant. We get a corresponding version of the Hochschild complex

$$C_H(A, W)_{\mathbb{C}[G]} = B_{\mathbb{C}[G]}(A, W) \otimes_{A \otimes A^{op}} A$$

Lemma 3.9. The natural quotient map

$$C_H(A, W) \to C_H(A, W)_{\mathbb{C}[G]}$$

is a quasi-isomorphism.

Proof. If $W = 0$ the result is immediate, since $B_{\mathbb{C}[G]}(A)$ is a projective resolution of $A$ over $A \otimes A^{op}$. If $W \neq 0$ then, arguing as in Lemma 3.5, the map is a quasi-isomorphism between the zeroth pages of the corresponding spectral sequences. Furthermore, these spectral sequences converge since their first pages are isomorphic to $HH(A)$ which is bounded in degree since $Y$ is smooth. Therefore the map is a quasi-isomorphism. \qed
We would now like to produce a more geometric model for this chain-complex (i.e. an analogue of Lemma 3.5). To do this we have to restrict to the case when \( Y \) is a vector space \( \mathbb{C}^n \) (but we make no assumptions on \( G \) other than finiteness).

We also need to work at the level of the bar resolution rather than defining a quasi-isomorphism directly from the Hochschild complex as we did before. Recall that the bar resolution is a bimodule over \( A \), i.e. a \( G \times G^{op} \)-equivariant sheaf on \( Y \times Y \), and that in the \( W = 0 \) case it is a resolution of the diagonal bimodule \( A \). If we consider the diagonal \( A \) as an \( R \otimes R \)-module, i.e. as a (non-equivariant) sheaf on \( Y \times Y \), then it splits as a direct sum

\[
A = \bigoplus_{g \in G} R_g
\]

The summand at the identity \( e \in G \) is the sky-scraper sheaf along the diagonal \( \Delta \subset Y \times Y \). The other summands are the sky-scraper sheaves along the subvarieties

\[
\Delta_g := \{ (y, gy) ; y \in Y \} \subset Y \times Y
\]

which are the orbits of the diagonal under \( G \times G^{op} \). Specifically, \( R_g \) is the pull-back of \( R_e \) under the automorphism \((1_y \times g)\) of \( Y \times Y \).

We can write down a similar splitting of the bar resolution \( B_{C[G]}(A,W) \) by identifying

\[
A^{\otimes_{C[G]}(k+2)} \cong \bigoplus_{g \in G} R_g \otimes R_e^\otimes k \otimes R_e \subset A^\otimes(k+2)
\]

Note the \( G \times G^{op} \) action:

\[
f \times h : R_g \otimes R_e^\otimes k \otimes R_e \to R_{fgh} \otimes R_e^\otimes k \otimes R_e
\]

\[
r_0 \otimes r_1 \otimes ... \otimes r_k \otimes r_{k+1} \mapsto f(r_0) \otimes h^{-1}(r_1) \otimes ... \otimes h^{-1}(r_k) \otimes h^{-1}(r_{k+1})
\]

The differential preserves this splitting, so we have a direct sum of cdg-(\( R, W \))-bimodules

\[
B_{C[G]}(A,W) = \bigoplus_{g \in G} B_{C[G]}(A,W)_g
\]

The summand \( B_{C[G]}(A,W)_e \) is just \( B(R,W) \), i.e. the bar resolution for the un-orbifolded LG model on \( Y \). If we forget the \( d_W \) part of the differential we get the complex \( B(R) \) of \( R \)-bimodules, which is a free resolution of the diagonal \( R \).

For each other group element \( g \), the summand \( B_{C[G]}(A,W)_g \) is the pull-back of \( B(R,W) \) under the automorphism \((1_Y \times g)\). If again we forget the \( d_W \) part of the differential we get a complex \( B(R)_g \) of \( R \)-bimodules, which a free resolution of the ‘twisted diagonal’ \( R_g \).
Recall that $Y = \mathbb{C}^n$. Let $y^i$ be a set of co-ordinates on $Y$. We can identify $Y \times Y$ as the tangent bundle to the diagonal $\Delta$. Let $\pi : Y \times Y \to \Delta$ be the corresponding projection, and let $\tau$ be the tautological section of $\pi^* T_Y$. The Koszul complex is the complex of sheaves

$$K := (\pi^* \Omega^*_Y, -\tau)$$

which is

$$\bigoplus_{k \geq 0} R \otimes \wedge^k Y^\vee \otimes R$$

with differential

$$r \otimes dy^{i_1} \wedge \ldots \wedge dy^{i_k} \otimes s \mapsto \sum_i (-1)^{i+1} (ry^{i_1} \otimes dy^{i_1} \wedge \ldots \wedge dy^{i_k} \otimes s - r \otimes dy^{i_1} \wedge \ldots \wedge dy^{i_k} \otimes y^{i_1} s)$$

The Koszul complex is a free resolution of the diagonal, i.e. of the $R \otimes R$-module $R$, as such it must be quasi-isomorphic to our other free resolution $B(R)$. We can also get a similar resolution $K_g$ of each twisted diagonal $R_g$ by identifying $Y \times Y$ with the tangent bundle to $\Delta_g$. This is a free complex of $R \otimes R$-modules, and so is quasi-isomorphic to $B(R)_g$. The differential is given by contracting with the twisted tautological section

$$\tau_g = \sum_i (g(y^i) \otimes 1 - 1 \otimes y^i) \partial_{y^i}$$

We can stick all the $K_g$ together to make a $G \times G^{op}$-equivariant sheaf (i.e. $A$-bimodule)

$$K_G := \bigoplus_{g \in G} K_g$$

by letting

$$(f \times h) : K_g \rightarrow K_{fg h}$$

$$r \otimes dy^{i_1} \wedge \ldots \wedge dy^{i_k} \otimes s \mapsto f(r) \otimes h^{-1}(dy^{i_1}) \wedge \ldots \wedge h^{-1}(dy^{i_k}) \otimes h^{-1}(s)$$

$K_G$ is a resolution of $\oplus R_g = A$, and is $G \times G^{op}$-equivariantly quasi-isomorphic to $\oplus B(R)_g = B_{\mathbb{C}[G]}(A)$. We can write this quasi-isomorphism explicitly, as follows.

Given any $g \in G$, and any function $r$ on $Y$, the function $g(r) \otimes 1 - 1 \otimes r$ vanishes with multiplicity 1 along $\Delta_g$, so we can uniquely decompose

$$g(r) \otimes 1 - 1 \otimes r = \sum_i \delta^g_i(r) (g(y_i) \otimes 1 - 1 \otimes y_i)$$

for some elements $\delta^g_i(r) \in R \otimes R$. 

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Lemma 3.10. The $R \otimes R$-bilinear map

$$\tilde{\phi} : B_{C[G]}(A) \to K_G$$

with components

$$\tilde{\phi}_g : B(R)_g \to K_g$$

$$1 \otimes r_1 \otimes \ldots \otimes r_k \otimes 1 \mapsto \sum_{i_1, \ldots, i_k} \delta^g_{i_1}(r_1) \ldots \delta^g_{i_k}(r_k) dy^{i_1} \wedge \ldots \wedge dy^{i_k}$$

is a quasi-isomorphism of complexes of $A$-bimodules.

Proof. Both sides are resolutions of $A$, and $\tilde{\phi}$ is the identity on their final terms $\oplus_g R \otimes R$, so provided that $\tilde{\phi}_g$ is $G \times G^{\text{op}}$-equivariant and is a chain-map then it is a quasi-isomorphism of $A$-bimodules. Both facts are easy to check, using the properties

$$\delta^g_f(rs) = (g(r) \otimes 1)\delta^g_f(s) + \delta^g_f(r)(1 \otimes s)$$

and

$$(f \times h)\delta^g_f(r) = \sum_j \delta^{fgh}_j(h^{-1}(r))h_{ij}$$

where $h_{ij}$ are the components of the linear map corresponding to $h \in G$ in our chosen co-ordinates. \[\square\]

We get an induced quasi-isomorphism

$$1 \otimes \tilde{\phi} : C_H(A)_{C[G]} \xrightarrow{\sim} A \otimes_{A \otimes A^{\text{op}}} K_G$$

Lemma 3.11. We have a quasi-isomorphism

$$\psi : A \otimes_{A \otimes A^{\text{op}}} K_G \xrightarrow{\sim} (\bigoplus_{g \in G} \Omega^*_{Y^g}) / G$$

where $Y^g \subset Y$ is the hyperplane of fixed points of $g$, and $f \in G$ acts as

$$f^{-1} : \Omega^*_{Y^g} \to \Omega^*_{Y^{f^{-1}g}}$$

There is no differential on the right-hand side. The composition $\phi := \psi(1 \otimes \tilde{\phi})$ is the orbifold generalization of the HKR isomorphism \[\square\].

Proof. We can form the tensor product $A \otimes_{A \otimes A^{\text{op}}} K_G$ in two stages - first tensor over $R \otimes R$, then quotient by the $G$-actions on each side. Note that there is a slight clash of conventions here. When tensoring bimodules it is usual to tensor the left action on one bimodule with the right action on the other and vice-versa (this is necessary to get the standard definition of the Hochschild differential). On the other hand, if we view them as modules over the commutative ring $R \otimes R$
We resolve this clash using the isomorphism $g \mapsto g^{-1}$ of $A$ with $A^{\mathrm{op}}$. This means that

$$A \otimes_{A \otimes A^{\mathrm{op}}} K_G = \left( \bigoplus_{g,h \in G} R_{g^{-1}} \otimes_{R \otimes R} K_h \right) / G \times G^{\mathrm{op}}$$

with $G \times G^{\mathrm{op}}$ action (recalling 3.2) given by

$$f \times \hat{f} : R_{g^{-1}} \otimes_{R \otimes R} K_h \to R_{f_{g^{-1}}f} \otimes_{R \otimes R} K_{f_{h}f}$$

$$r \otimes \omega \mapsto f^{-1}(r \otimes \omega)$$

As chain-complexes,

$$R_{g^{-1}} \otimes_{R \otimes R} K_h \cong \Omega^{ullet}_Y$$

with differential

$$-\sum_i (h(y^i) - g^{-1}(y^i)) \partial_{y^i}$$

However since $K_h$ is a free resolution of $O_{\Delta_h}$ we are really calculating the derived intersection $O_{\Delta_{g^{-1}}} \otimes R \otimes R O_{\Delta_h}$, but since $\Delta_{g^{-1}}$ and $\Delta_h$ are just hyperplanes in a vector space it is easy to see that this is

$$\Lambda^\bullet (\Delta_{g^{-1}} + \Delta_h) \otimes O_{\Delta_{g^{-1}} \cap \Delta_h}$$

Also, since $N_{\Delta_g} = T_{\Delta_g}$ for all $g$,

$$N_{(\Delta_{g^{-1}} + \Delta_h)} = T_{\Delta_{g^{-1}} \cap \Delta_h}$$

The intersection $\Delta_{g^{-1}} \cap \Delta_h$ is isomorphic to $Y^{gh}$. We conclude that the map

$$\psi_{g,h} : R_{g^{-1}} \otimes_{R \otimes R} K_h \cong \Omega^{ullet}_Y \to \Omega^{ullet}_{Y^{gh}} \quad (3.3)$$

given by restricting differential forms from $Y$ to $Y^{gh}$ is a quasi-isomorphism, where we put no differential on the right-hand side. Summing these maps and quotienting by $G \otimes G^{\mathrm{op}}$ we get a quasi-isomorphism (since taking co-invariants by a finite group is exact in characteristic zero)

$$\psi : A \otimes_{A \otimes A^{\mathrm{op}}} K_G \xrightarrow{\sim} \left( \bigoplus_{g,h \in G} \Omega^{ullet}_{Y^{gh}} \right) / G \times G^{\mathrm{op}}$$

with components (we insert a numerical factor to make our formulas agree with those of the previous sections)

$$\frac{1}{k!} \sum_{g,h} \psi_{g,h} : A \otimes_{A \otimes A^{\mathrm{op}}} K_G^k \to \left( \bigoplus_{g,h} \Omega^k_{Y^{gh}} \right) / G \times G^{\mathrm{op}}$$
The $G^{op}$ factor simply identifies the summands over the fibres of the map $(g, h) \mapsto gh$, so the quotient is

$$\left( \bigoplus_{g \in G} \Omega^\bullet_{Y^g} \right) / G$$

where $f \in G$ acts as

$$f^{-1} : \Omega^\bullet_{Y^g} \to \Omega^\bullet_{Y^{f^{-1}gf}}$$

We have now produced a geometric model for $C_H(A)[G]$. What remains is to add a differential that will make it a model for $C_H(A, W)[G]$.

Let $d^K = \oplus_g \tau_g$ denote the differential on $K_G$. We can deform this, making $K_G$ into a cdg-bimodule $\hat{K}_G$, by adding the following term:

$$d^K_W : K^k_g \to K^k_g$$

$$d^K_W = k \sum_i \delta^i_g (W) dy^i \wedge$$

It is straight-forward to check that $d^K_W$ is equivariant, and that the deformed differential $d^K + d^K_W$ squares to $W \otimes 1 - 1 \otimes W$. This is the orbifold generalization of the 'diagonal brane' considered in [7]. It is also easy to check that the map $\tilde{\phi}$ becomes a map of cdg-bimodules

$$\tilde{\phi} : B(A, W)[G] \to \hat{K}_G$$

The tensor product $A \otimes_{A \otimes A^{op}} \hat{K}_G$ is still a chain-complex, since the superpotential becomes zero along the diagonal.

Tensoring $\hat{K}_G$ with $A$ and then passing through the map $\psi$ means (on each summand) restricting differential forms to $\Delta_{g^{-1}} \cap \Delta_h$, then taking their class in $\Omega^\bullet_{\Delta_{g^{-1}} \cap \Delta_h} = \Omega^\bullet_{Y^{gh}}$. For any function $r \in R$, we have

$$\left( \sum_i \delta^i_g (r) dy^i \right) |_{\Delta_s} = dr$$

so for any form $\omega \in R_{f^{-1}} \otimes_{R \otimes R} K_{f^{-1}g}$ we have

$$\psi(1 \otimes d^K_W)(\omega) = dW_g \wedge \psi(\omega)$$

where $W_g$ denotes the restriction of $W$ to $Y^g$.  

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Definition 3.12. The (off-shell) closed state space of the affine orbifold LG B-model \((|Y/G|, W)\) is the chain-complex of co-invariants
\[
\left( \bigoplus_{g \in G} (\Omega^\bullet_{Y^g}, dW_g \wedge) \right) / G
\]
where \(f \in G\) acts as
\[
f^{-1} : \Omega^\bullet_{Y^g} \to \Omega^\bullet_{Y^{f^{-1}g}}
\]
Let \(J_g\) denote the Jacobi ring of \(W_g\) on \(Y^g\). Then the homology of the closed state space (i.e. the physical state space) is isomorphic to
\[
\bigoplus_{[g] \subset G} \left( \bigoplus_{h \in [g]} J_h \right) / G \cong \bigoplus_{[g] \subset G} J_g / C_g
\]
where the second direct sum runs over one representative \(g\) of each conjugacy class \([g] \subset G\) and \(C_g\) is the centralizer of \(g\). Presumably this fact is known in the physics literature. Note that the second direct sum decomposition is canonical if and only if \(G\) is abelian,

Theorem 3.13. The closed state space of \((|Y/G|, W)\) is quasi-isomorphic to the universal closed state space \(C_H(\text{Perf}(|Y/G|, W))\).

Proof. By Theorem 2.8 and Lemmas 3.7 and 3.9 we know that \(C_H(\text{Perf}(|Y/G|, W))\) is quasi-isomorphic to \(C_H(A, W)_{\mathbb{C}[G]}\). Now we proceed as we did in Lemma 3.5: we have a map
\[
\phi := \psi(1 \otimes \tilde{\phi}) : C_H(A, W)_{\mathbb{C}[G]} \to \left( \bigoplus_g (\Omega^\bullet_{Y^g}, dW_g \wedge) \right) / G
\]
which is a quasi-isomorphism on page zero of the corresponding spectral sequences by Lemma 3.10 and Lemma 3.11. Thus it is a quasi-isomorphism of the total complexes. \(\square\)

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