A Full Asymptotic Series of European Call Option Prices in the SABR Model with $\beta = 1$

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Preliminaries on Option Pricing
  Stochastic Alpha Beta Rho (SABR) Model
  The Black-Scholes Theory
  Generalization of Hull-White Formula
Preliminaries on Malliavin Calculus
  Exponential Formula
Option Pricing Formula for SABR Model
  Derivation for $G_s$
  Conditional Expectation of $\Lambda_s G_s$: $E[\Lambda_s G_s | \mathcal{F}_t]$
    A Formula by Marc Yor
  Full Expression of Option Price
  Approximation of Option Price
Another Look at The Correction Term
References
Appendix
The SABR model is an extension of the Black Scholes model in which the volatility parameter follows a stochastic process:

\[ dS_t = rS_t dt + \sigma_t S_t^\beta (\rho dW_t + \sqrt{1 - \rho^2} dZ_t), \]  

\[ d\sigma_t = \alpha \sigma_t dW_t. \]
Approximation for Implied Volatilities of SABR Model

Hagan et al. derived, with perturbation techniques, an approximating direct formula for this implied volatility under the SABR model in [5]:

\[
\sigma_{BS}(S_0, K) = \frac{\sigma_0}{(S_0 K)^{\frac{1-\beta}{2}} \left[ 1 + \frac{(1-\beta)^2}{24} \ln^2 \frac{S_0}{K} + \frac{(1-\beta)^4}{1920} \ln^4 \frac{S_0}{K} + \cdots \right]} \times z(x(z))
\]

\[
1 + \left( \frac{(1-\beta)^2}{24} \frac{\sigma_0^2}{(S_0 K)^{1-\beta}} + \frac{1}{4} \frac{\rho \beta \alpha \sigma_0}{(S_0 K)^{(1-\beta)/2}} + \frac{2 - 3 \rho^2}{24} \frac{\alpha^2}{\sigma_0^2} \right) \tau + O(\tau^2)
\]

where \( z := -\frac{\alpha}{\sigma_0}(S_0 K)^{\frac{1-\beta}{2}} \ln \left( \frac{S_0}{K} \right) \) and \( x(z) = \ln \left( \frac{\sqrt{1 - 2\rho z + z^2 + z - \rho}}{1 - \rho} \right) \).
In the special case $\beta = 1$, the SABR implied volatility formula reduces to

$$\sigma_{BS}(S_0, K) = \sigma_0 \frac{y}{f(y)} \left[ 1 + \left( \frac{1}{4} \rho \alpha \sigma_0 + \frac{2 - 3 \rho^2}{24} \alpha^2 \right) \tau + O(\tau^2) \right], \quad (4)$$

where $y := -\frac{\alpha}{\sigma_0} \ln\left( \frac{S_0}{K} \right)$ and $f(y) = \ln \left( \frac{\sqrt{1 - 2 \rho y + y^2} + y - \rho}{1 - \rho} \right)$.

European call: $BS(t, x, \sigma_{BS}) = e^x N(d_+) - Ke^{-r(T-t)} N(d_-)$. 
The Black-Scholes Theory

\[ dS_t = rS_t dt + \sigma S_t dW_t. \]  \hspace{1cm} (5)

Let \( X_t = \ln S_t \) denote the logarithm of stock price. The price of an European call option with payoff \((X_T - K)_+\) at time \( t \) satisfy the Black-Scholes-Merton equation:

\[ \mathcal{L}_{BS}(\sigma)BS(t, x, \sigma) = 0, \]  \hspace{1cm} (6)

where \( \mathcal{L}_{BS}(\sigma) = \mathcal{L}_{BS}(\sigma) = \frac{\partial}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} + \left( r - \frac{1}{2} \sigma^2 \right) \frac{\partial}{\partial x} - r \cdot \) is the Black-Scholes differential operator. And the closed-form solution of above PDE (6) is

\[ BS(t, x, \sigma) = e^x N(d_+) - Ke^{-r(T-t)} N(d_-). \]  \hspace{1cm} (7)
Consider the model under a risk-neutral probability:

$$dS_t = rS_t dt + \sigma_t S_t (\rho dW_t + \sqrt{1 - \rho^2} dZ_t), \ t \in [0, T]$$  \hspace{1cm} (8)$$

Where $W_t$ and $Z_t$ are independent standard Brownian motions defined in a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, $\mathcal{F}_t = \mathcal{F}_t^W \cup \mathcal{F}_t^Z := \sigma \{ W_s, Z_s, s \leq t \}$, and $\sigma_t$ is a square integrable process adapted to $\{\mathcal{F}_t^W\}$. 
Assume that hypotheses (H1) to (H4) in [1] by Alòs hold. Then, for all $t \in [0, T]$, 

$$V_t = E[BS(t, X_t, v_t)|\mathcal{F}_t] + \frac{\rho}{2} \int_t^T e^{-r(s-t)} E\left[ H(s, X_s, v_s) \Lambda_s \bigg| \mathcal{F}_t \right] ds$$  \hspace{1cm} (9)$$

where $v_s^2 = \frac{1}{T-s} \int_s^T \sigma_u^2 du$ is the future average volatility, and

$$H(s, X_s, v_s) := \left( \frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \right) BS(s, X_s, v_s),$$

$$\Lambda_s := \left( \int_s^T D_s^W \sigma_r^2 dr \right) \sigma_s.$$ 

We denote $V_{s,T} = v_s^2(T-s) = \int_s^T \sigma_u^2 du$ and $V_{t,s} = \int_t^s \sigma_u^2 du$. 

Z. Guo, H. Schellhorn

A Full Asymptotic Series of European Call Option Prices in the Stochastic Alpha Beta Rho(SABR) Model
Theorem

Suppose $F \in \mathbb{D}_\infty([0, T])$ satisfies the following condition:

$$
\frac{(T - t)^{2n}}{(2^n n!)^2} E \left[ \left( \sup_{u_1, \ldots, u_n \in (t, T)} |(D_{u_n} \cdots D_{u_1} F)(\omega^t)| \right)^2 \right] \xrightarrow{n \to \infty} 0,
$$

for fixed $t \in [0, T]$, then

$$
E[F|\mathcal{F}_t] = \sum_{n=0}^{\infty} \frac{1}{2^n n!} \int_{[t, T]^n} (D_{s_n}^2 \cdots D_{s_1}^2 F)(\omega^t) ds_n \cdots ds_1.
$$

(10)
Freezing Operator

Definition
Given \( \omega \in \Omega \), a freezing operator \( \omega^t \) is defined as:

\[
W(s, \omega^t(\omega)) = \begin{cases} 
W(s, \omega), & \text{if } s \leq t; \\
W(t, \omega), & \text{if } t \leq s \leq T. 
\end{cases}
\] (11)

The freezing operator \( \omega^t \) is a mapping from \( \Omega \) to \( \Omega \).
Apply Exponential Formula to $F = H(s, X_s, v_s) \Lambda_s$

Goal: $E[H(s, X_s, v_s) \Lambda_s | \mathcal{F}_t]$, recall that

$$V_t = E[BS(t, X_t, v_t) | \mathcal{F}_t] + \frac{\rho}{2} \int_t^T e^{-r(s-t)} E[H(s, X_s, v_s) \Lambda_s | \mathcal{F}_t] \, ds.$$  

Let $F = H(s, X_s, v_s) \Lambda_s$, using iterated conditioning:

$$E[F | \mathcal{F}_t] = E \left[ E \left[ H(s, X_s, v_s) \Lambda_s | \mathcal{F}_T^W \cup \mathcal{F}_T^Z \right] | \mathcal{F}_t \right] = E[\Lambda_s G_s | \mathcal{F}_t],$$  

where $G_s = G(s, X_s, v_s) = E[H(s, X_s, v_s) | \mathcal{F}_T^W \cup \mathcal{F}_T^Z]$ depends only on Brownian motion $\{Z_t\}_{t \geq 0}$. 

Z. Guo, H. Schellhorn  
A Full Asymptotic Series of European Call Option Prices in the
Apply Exponential Formula to $F = H(s, X_s, v_s)\Lambda_s$

Then the option price formula (9) becomes:

$$V_t = E[BS(t, X_t, v_t)|\mathcal{F}_t] + \frac{\rho}{2} \int_t^T e^{-r(s-t)} E[\Lambda_s G_s|\mathcal{F}_t] ds,$$

(14)

where

$$\Lambda_s := \int_s^T D_s^W \sigma_t^2 dr \sigma_s = \int_s^T 2\alpha \sigma_t^2 dr \sigma_s = 2\alpha \sigma_s V_s, T,$$

(15)

$$G_s = \sum_{n=0}^{\infty} \frac{1}{2^n n!} \omega^t_Z \circ \int_{[t, T]^n} D^{2n, Z}_{\tau \otimes n} H(s, X_s, v_s) d\tau \otimes n, \ t \leq s.$$

(16)

Goal: $E[G_s \Lambda_s|\mathcal{F}_t]$
The Faà di Bruno’s formula can be generalized to Malliavin derivative in the following way:
If $f$ and $g$ are functions with a sufficient number of derivatives, then for a random variable $F \in D^N([0, T])$ and $\forall n \leq N$, by chain rule and Faà di Bruno’s formula we have

$$D_t^n f(g(F)) = \sum_{k=1}^{n} f^{(k)}(g(F)) \cdot B_{n,k}(g'(F), \ldots, g^{n-k+1}(F)) D_t^n F,$$

where $B_{n,k}(x_1, \ldots, x_{n-k+1})$ are the incomplete exponential Bell polynomials.
Malliavin derivative of $H_s$: $D^{2n}_{\tau \otimes n} H(s, X_s, \nu_s)$

\[ H_s = \left( \frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \right) BS(s, X_s, \nu_s) = \frac{-d_-}{\sqrt{2\pi V_{s,T}}} e^{X_s - \frac{d_+^2}{2}}. \]

Define two real-valued functions $p(\cdot)$ and $q(\cdot)$ such that $q(p(s, X_s, \nu_s)) = H_s$,

\[ p(s, X_s, \nu_s) = X_s - \frac{d_+^2}{2} + \ln(-d_-), \quad (18) \]

\[ q(x) = \frac{1}{\sqrt{2\pi V_{s,T}}} e^{x}. \quad (19) \]
Malliavin derivative of $H_s$: $D_{n}^{2n,Z} H(s, X_s, v_s)$

\[
D_{\tau}^{Z} X_s = D_{\tau} \int_{t}^{s} \sigma_u \sqrt{1 - \rho^2} dZ_u = \sigma_{\tau} \sqrt{1 - \rho^2} \mathbb{1}_{\{\tau \leq s\}}, \quad \text{(20)}
\]

Then by Faà di Bruno’s formula,

\[
D_{\tau \bigotimes n}^{2n,Z} H_s = D_{\tau \bigotimes n}^{2n,Z} q(p(s, X_s, v_s))
= \sum_{k=1}^{2n} q^{(k)}(p(s, X_s, v_s)) \cdot B_{2n,k} \left( b_1, \ldots, b_{2n-k+1} \right) D_{\tau \bigotimes n}^{2n,Z} X_s
= (1 - \rho^2)^n H_s B_{2n}(b_1, \ldots, b_{2n}) \prod_{i=1}^{n} \sigma_{\tau_i}^2 \mathbb{1}_{\{\tau_i \leq s\}} \quad \text{(21)}
\]

where $b_k = p^{(k)}(s, X_s, v_s), \ k = 1, \ldots, 2n$
Expression of $G_s$

$$G_s = \sum_{n=0}^{\infty} \frac{1}{2^n n!} \omega_t^Z \circ \int_{[t,T]^n} D_{\tau \otimes n}^{2n,Z} H(s, X_s, v_s) d\tau \otimes n$$

$$= \sum_{n=0}^{\infty} \frac{(1 - \rho^2)^n}{2^n n!} H_s^\omega B_{2n}(b_1^\omega, \ldots, b_{2n}^\omega) \int_t^s \prod_{i=1}^n \sigma_{\tau_i}^2 d\tau \otimes n$$

$$= H_s^\omega \sum_{n=0}^{\infty} \frac{(1 - \rho^2)^n}{2^n n!} V_{t,s}^n B_{2n}(b_1^\omega, \ldots, b_{2n}^\omega).$$

$$b_j = \frac{1}{\sqrt{V_s, T d_-(s, X_s, v_s))j}} \begin{cases} (-1)^{j+1} - d_-(s, X_s, v_s), & j = 1, 2; \\ (-1)^{j+1}(j-1)!, & j \geq 3. \end{cases}$$
Interpretation of $\Lambda_s G_s$

Notice that

$$X^\omega_s := X_t + r(s-t) - \frac{1}{2} V_{t,s} + \frac{\rho}{\alpha} (\sigma_s - \sigma_t) + \omega^t_Z \circ \int_t^s \sigma_u \sqrt{1 - \rho^2} dZ_u$$

$$= X_t + r(s-t) - \frac{1}{2} V_{t,s} + \frac{\rho}{\alpha} (\sigma_s - \sigma_t) \quad (22)$$

$$d^\omega_\pm(s, X_s, \nu_s) := \omega^t_Z \circ d_\pm((s, X_s, \nu_s)) = d_\pm(s, X^\omega_s, \nu_s)$$

$$= \frac{X^\omega_s - \ln K + r(T-s) \pm \frac{1}{2} V_{s,T}}{\sqrt{V_{s,T}}} \quad (23)$$

and recall that $\Lambda_s = 2\alpha V_{s,T} \sigma_s$, thus $\Lambda_s G_s$ is a function that depends on $\sigma_s, V_{t,s} = \int_t^s \sigma_u^2 du$ and $V_{s,T} = \int_s^T \sigma_u^2 du$. Z. Guo, H. Schellhorn
Joint Density of \( \left( \int_0^t e^{\sigma W_s} ds, W_t \right) \)

Proposition 2 In [6] by Yor (1992): the joint density of \( \left( \int_0^t e^{\sigma W_s} ds, W_t \right) \) has been derived for the case \( \sigma = 2 \),

\[
\phi_{t,\sigma}(x, y) := \frac{1}{dx\,dy} \mathbb{P}\left( \int_0^t e^{\sigma W_s} ds \in dx, W_t \in dy \right) = \frac{\sigma}{2x} e^{-\frac{2}{\sigma^2 x} (1 + e^{\sigma y})} \cdot \theta\left( \frac{4e^{\sigma y}/2}{\sigma^2 x}, \frac{\sigma^2 t}{4} \right), \quad (24)
\]

for \( x > 0, y \in \mathbb{R}, t > 0 \), where

\[
\theta(r, t) = \frac{r}{\sqrt{2\pi^3 t}} e^{\frac{\pi^2}{2t}} \int_0^\infty e^{-\frac{\xi^2}{2t}} \cdot e^{-r \cosh \xi \sinh \xi} \sin \frac{\pi \xi}{t} d\xi, \quad r, t > 0.
\]

(25)
Joint Density of \( \left( \int_0^t e^{\sigma W_s - \mu s} ds, W_t \right) \)

A straightforward application of the Cameron-Martin-Girsanov theorem implies that the joint density of \( \left( \int_0^t e^{\sigma W_s - \mu s} ds, W_t \right) \), \( \sigma > 0, \mu \in \mathbb{R} \), which we denote by \( \phi_{t,\sigma,\mu}(x, y) \), \( x > 0, y \in \mathbb{R} \), can be connected with the density \( \phi_{t,\sigma,0}(x, y) = \phi_{t,\sigma,0}(x, y) \) through the formula

\[
\phi_{t,\sigma,\mu}(x, y) = e^{-\frac{\mu}{\sigma}y + \frac{\mu^2}{2\sigma^2} t} \phi_{t,\sigma,0}(x, y - \frac{\mu}{\sigma} t) \tag{26}
\]
Calculation of $E[\Lambda_s G_s | \mathcal{F}_t]$

Define $h(V_{t,s}, v_s, \sigma_s) = \Lambda_s G_s$, then $E[\Lambda_s G_s | \mathcal{F}_t]$ can be calculated as follows:

$$E[\Lambda_s G_s | \mathcal{F}_t] = E[h(V_{t,s}, v_s, \sigma_s) | \mathcal{F}_t] = E[E[h(V_{t,s}, v_s, \sigma_s) | \mathcal{F}_s] | \mathcal{F}_t]$$

$$= E \left[ \int_0^\infty h(V_{t,s}, \frac{v}{\sqrt{T-s}}, \sigma_s) F'_{V_s,T}(v) dv \bigg| \mathcal{F}_t \right]$$

$$= \int_0^\infty dx \int_{-\infty}^\infty dy \int_0^\infty dv \ h(\sigma_t^2 x, \frac{v}{\sqrt{T-s}}, \sigma_s(y)) F'_{V_s,T}(v) \phi_{s-t,2\alpha,\alpha^2}(x,y)$$

where $\sigma_s(y) = \sigma_t \exp(\alpha y - \frac{1}{2} \alpha^2(s-t))$. 
Marginal Density of $\int_0^t e^{\sigma W_s - \mu s} ds$

The conditional density of $V_{s,T}$ is $F_{V_{s,T}}'(v) = \frac{1}{\sigma_s^2} \psi_{V_{s,T}}(\frac{v}{\sigma_s^2})$, where

$$\psi_{V_{s,T}}(v) = \int_{\mathbb{R}} \phi_{T-s,2\alpha,\alpha^2}(v, z) dz,$$

and

$$F_{V_{s,T}}(v) = \mathbb{P}\left( V_{s,T} \leq v \mid \mathcal{F}_s \right) = \mathbb{P}\left( \int_s^T \sigma_u^2 du \leq v \mid \sigma_s \right)$$

$$= \mathbb{P}\left( \int_s^T \sigma_s^2 e^{2\alpha(W_u-W_s)-\alpha^2(u-s)} du \leq v \mid \sigma_s \right)$$

$$= \mathbb{P}\left( \int_0^{T-s} e^{2\alpha(W_u)-\alpha^2 u} du \leq \frac{v}{\sigma_s^2}, W_{T-s} < \infty \right)$$

$$= \int_0^{\frac{v}{\sigma_s^2}} \int_{-\infty}^{\infty} \phi_{T-s,2\alpha,\alpha^2}(x, z) dz dx. \quad (27)$$
Marginal Density of $\int_0^t e^{\sigma W_s - \mu s} ds$

One straightforward application of (27) is using the conditional density of $V_{t,T}$ to obtain the first conditional expectation in (9):

$$E[BS(t, X_t, v_t)|\mathcal{F}_t] = \int_0^\infty BS\left(t, X_t, \sqrt{\frac{v}{T-t}}\right) F'_{V_{t,T}}(v) dv$$

$$= \int_0^\infty BS\left(t, X_t, \sqrt{\frac{v}{T-t}}\right) \frac{1}{\sigma_t^2} \psi_{V_{t,T}}\left(\frac{v}{\sigma_t^2}\right) dv$$

$$= \int_0^\infty \int_{-\infty}^{\infty} \frac{1}{\sigma_t^2} BS\left(t, X_t, \sqrt{\frac{v}{T-t}}\right) \phi_{T-t, 2\alpha, \alpha^2} \left(\frac{v}{\sigma_t^2}, z\right) dz dv \quad (28)$$
A Formula for European Call Option Price

\[ V_t = E[BS(t, X_t, v_t)|F_t] + \frac{\rho}{2} \int_t^T e^{-r(s-t)} E[\Lambda_s G_s|F_t] ds. \]

\[ = \int_0^\infty \int_{-\infty}^{\infty} \frac{1}{\sigma_t^2} BS\left(t, X_t, \sqrt{\frac{v}{T-t}}\right) \phi_{T-t,2\alpha,\alpha^2} \left(\frac{v}{\sigma_t^2}, z\right) dzdv \]

\[ + \rho \alpha \int_t^T \int_0^\infty \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} l(s, v, z, x, y) dydxdzdvds, \quad (29) \]

\[ l(\cdot) = \frac{e^{-r(s-t)}}{\sigma_s(y)} \cdot f\left(s, X_s^{x,y}, \sqrt{\frac{v}{T-s}}\right) \cdot \phi_{T-s,2\alpha,\alpha^2} \left(\frac{v}{\sigma_s^2}, z\right) \cdot \phi_{s-t,2\alpha,\alpha^2}(x, y), \]

\[ f(s, X_s, v_s) = V_{s,T} H_s^{\omega} \sum_{n=0}^{\infty} \frac{(1 - \rho^2)V_{t,s}^n}{2^n n!} B_{2n}(b_1^{\omega}, \ldots, b_{2n}^{\omega}). \]

Z. Guo, H. Schellhorn

A Full Asymptotic Series of European Call Option Prices in the SABR Model with \( \beta = 1 \)
Parameters of Approximation Results

In the following tables we compare the values of the approximate option prices. We have chosen $T - t = 1$, $\ln X_t = 100$, $r = 0.1$, $\sigma_t = 0.3$, $\alpha = 1$, $\rho = 0, \pm 0.5$ and varying values for the strike price $K$.

- Column 1: Strike price $K$;
- Column 2: Monte Carlo Simulation with number of simulation times $N = 10^6$;
- Column 3: Approximated prices obtained by Black-Scholes formula with volatility approximated by (4);
- Column 4: Approximated prices obtained by formula (29) with $f(\cdot)$ approximated by (40).
\( \rho = 0 \)

| K  | Monte Carlo | Hagan    | formula (29) |
|----|-------------|----------|--------------|
| 90 | 23.573138   | 23.415000| 23.626726    |
| 95 | 20.440334   | 20.337570| 20.457574    |
| 100| 17.562962   | 17.624483| 17.594033    |
| 105| 15.066565   | 15.291032| 15.063452    |
| 110| 12.885739   | 13.322697| 12.875527    |
\[ \rho = -0.5 \]

| K  | Monte Carlo | Hagan    | 1st order approx. |
|----|-------------|----------|-------------------|
| 90 | 23.972526   | 22.025500| 23.762565         |
| 95 | 20.640584   | 19.229952| 20.539753         |
| 100| 17.500136   | 16.889528| 17.505670         |
| 105| 14.688296   | 14.952772| 14.836533         |
| 110| 12.121686   | 13.353472| 12.884976         |
\( \rho = 0.5 \)

| K   | Monte Carlo | Hagan    | 1st order approx. |
|-----|-------------|----------|------------------|
| 90  | 22.352943   | 24.228522| 22.979063        |
| 95  | 20.035690   | 20.836574| 20.304502        |
| 100 | 17.186214   | 17.691469| 17.555458        |
| 105 | 15.172375   | 14.842598| 14.965057        |
| 110 | 13.080356   | 12.333288| 12.802697        |

Z. Guo, H. Schellhorn

A Full Asymptotic Series of European Call Option Prices in the
Recall that 

\[ V_t = E[BS(t, X_t, v_t)|F_t] + J, \]

where

\[ J := \frac{\rho}{2} \int_t^T e^{-r(s-t)} E[H_s \wedge_s |F_t] ds \]

\[ = \frac{\rho}{2} \int_t^T e^{-r(s-t)} \frac{2\alpha}{\sqrt{2\pi}} E\left[ \sigma_s E\left[ -d_- e^{X_s - d_-^2} |F_T \cup F^{W}_t \cup F^{Z}_t \right] |F_t \right] ds \]

\[ = C_1 \int_t^T E\left[ \sigma_s E\left[ -d_- e^{-d_-^2} |F_T \cup F^{W}_t \cup F^{Z}_t \right] |F_t \right] ds \quad (30) \]

for \( d_\pm \) evaluated at \((s, X_s, v_s)\), where \( C_1 = \frac{1}{\sqrt{2\pi}} \rho \alpha K e^{-(T-t)} \).
Second Approach to Calculate $E[H_s \Lambda_s \mid \mathcal{F}_t]$ 

Denote $Q_s ::= E\left[ -d_- e^{-\frac{d_-^2}{2}} \mid \mathcal{F}_W^T \cup \mathcal{F}_Z^T \right]$, then the correction term can be written as $J = C_1 \int_t^T E\left[ \sigma_s Q_s \mid \mathcal{F}_t \right] ds$.

$$d_- (s, X_s, v_s) = \lambda(V_s, T) Z + \gamma(V_{t,s}, V_s, T, \sigma_s). \quad (31)$$

where $Z = \int_t^s \sigma_u dZ_u$ is conditional normal with variance $V_{t,s}$ i.e. $Z \sim \mathcal{N}(0, V_{t,s})$,

$$\gamma(V_{t,s}, V_s, T, \sigma_s) := \frac{\kappa + \frac{\rho}{\alpha} (\sigma_s - \sigma_t) - \frac{1}{2} (V_{t,s} + V_s, T)}{\sqrt{V_{s,T}}},$$

$$\lambda(V_s, T) := \sqrt{1 - \rho^2} \frac{1}{V_{s,T}}.$$
Calculation of $Q_s$

Goal: $E[R(s, X_s, \nu_s)|\mathcal{F}_t]$

$$Q_s = \int_{\mathbb{R}} -(\lambda z + \gamma)e^{-\frac{(\lambda z + \gamma)^2}{2}} \frac{1}{\sqrt{2\pi V_{t,s}}} e^{-\frac{z^2}{2V_{t,s}}} dz = C_2 \gamma e^{C_3 \gamma^2}$$

where $C_2 = -\frac{1}{(2-\rho^2)^{3/2}}$, $C_3 = -\frac{1}{2(2-\rho^2)}$. Thus we have

$$J = C_1 \int_t^T E\left[\sigma_s Q_s \middle| \mathcal{F}_t\right] ds = C_1 C_2 \int_t^T E\left[R_s \middle| \mathcal{F}_t\right] ds. \quad (32)$$

where $R_s := R(s, X_s, \nu_s) = \sigma_s \gamma e^{C_3 \gamma^2}$ is a random variable depends only on Brownian motion $\{W_t\}_{t \geq 0}$.
Goal: \( E[R(s, X_s, \nu_s)|\mathcal{F}_t] \).

Now we can apply exponential formula (10) to \( R(s, X_s, \nu_s) \) such that:

\[
E\left[R(s, X_s, \nu_s) \bigg| \mathcal{F}_t \right] = \sum_{n=0}^{\infty} \frac{1}{2^n n!} r_n(s, X_t, \nu_t), \quad t \leq s, \quad (33)
\]

where \( r_n(s, X_t, \nu_t) = \omega^t_\mathcal{W} \circ \int_{[t, T]^n} D^{2n, W}_\tau \otimes_n R(s, X_s, \nu_s) d\tau \otimes^n. \)
First Order Approximation of $E \left[ R(s, X_s, v_s) \mid \mathcal{F}_t \right]$ 

Let $f(x, y) = yxe^{C_3 x^2}$, then $R(s, X_s, v_s) = f(\gamma, \sigma_s)$, and

$$D^2_t W R_s = f_x(\gamma, \sigma_s) D^2_t W \gamma + f_{xx}(\gamma, \sigma_s)(D^t W \gamma)^2 + f_y(\gamma, \sigma_s) D^2_t W \sigma_s$$

By the structure of $\sigma_t$ for $t \in [0, T]$, we have the following results:

$$D^W_t \sigma_s = \alpha \sigma_s 1_{\{\tau \leq s\}}$$
$$D^W_t V_{s,T} = 2\alpha V_{\tau \wedge s, T}$$
$$D^W_t V_{t,s} = 2\alpha V_{\tau, s} 1_{\{\tau \leq s\}}$$

$$D^2_t W \sigma_s = \alpha^2 \sigma_s 1_{\{\tau \leq s\}}$$
$$D^2_t W V_{s,T} = 4\alpha^2 V_{\tau \wedge s, T}$$
$$D^2_t W V_{t,s} = 4\alpha^2 V_{\tau, s} 1_{\{\tau \leq s\}}.$$
Therefore, \( J = \frac{\rho}{2} \int_t^T e^{-r(s-t)} E[H_s \wedge s | \mathcal{F}_t] ds = C_1 C_2 \int_t^T E[R_s | \mathcal{F}_t] ds \)

\[
\approx C_1 C_2 \int_t^T \sum_{n=0}^{1} \frac{1}{2^n n!} \omega^t W \circ \int_{[t,T]^n} D_{2n}^2 W R(s, X_s, \nu_s) d\tau \otimes^n ds
\]

\[
= C_1 C_2 \int_t^T 1 + \frac{1}{2} \int_t^T D_{2}^2 W R^\omega d\tau ds
\]

\[
= \frac{1}{2} C_1 C_2 \left[ \int_t^T p_1(s) + p_2(s) ds + 2(T-t) \right]. \quad (34)
\]

where \( p_1(s) := \omega^t W \circ \int_t^s D_{2n}^2 W R_s d\tau, \quad p_2(s) := \omega^t W \circ \int_s^T D_{2n}^2 W R_s d\tau \)
Conclusion

- Convergence Analysis
- Stochastic Volatility F.B.M
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**Lemma: Faà di Bruno’s formula**

**Faà di Bruno’s formula.** If $f$ and $g$ are functions with a sufficient number of derivatives, then

$$
\frac{d^n}{dx^n} f(g(x)) = \sum \frac{n!}{\Pi_{i=1}^{n} m_i!} f(\sum_{k=1}^{n} m_k) (g(x))^\cdot \prod_{j=1}^{n} \left( \frac{g^{(j)}(x)}{j!} \right)^{m_j},
$$

subject that all nonnegative integers $(m_1, \ldots, m_n)$ satisfying the constraint $\sum_{k=1}^{n} km_k = n$. A simpler formula expressed in terms of Bell polynomials $B_{n,k}(x_1, \ldots, x_{n-k+1})$:

$$
\frac{d^n}{dx^n} f(g(x)) = \sum_{k=1}^{n} f^{(k)}(g(x)) \cdot B_{n,k} \left( g'(x), \ldots, g^{n-k+1}(x) \right).
$$

(35)
Exponential Bell polynomials

The partial or incomplete exponential Bell polynomials are a triangular array of polynomials given by

\[ B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum \frac{n!}{\prod_{i=1}^{n-k+1} j_i!} \prod_{i=1}^{n-k+1} (\frac{x_i}{j_i!})^{j_i}, \quad (37) \]

where the sum is taken over all sequences \( j_1, j_2, \ldots, j_{n-k+1} \) non-negative integers such that these two conditions are satisfied: \( \sum_{i=1}^{n-k+1} j_i = k \) and \( \sum_{i=1}^{n-k+1} i \cdot j_i = n \). The sum

\[ B_n(x_1, \ldots, x_n) = \sum_{k=1}^{n} B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) \quad (38) \]

is called the \( n \)th complete exponential Bell polynomials.
1st order approximation of \( f(\cdot) \) and option prices

Let \( m > 0 \), define \( L^\omega_s = v_s^2(T - s)H^\omega_s \) and

\[
f_m(s, X_s, v_s) := L^\omega_s \sum_{n=0}^{m} \frac{((1 - \rho^2)V_{t,s})^n}{2^n n!} B_{2n}(p'(X_s^\omega), \ldots, p^{2n}(X_s^\omega)) \tag{39}
\]

then the first order approximation \( f_1(s, v_s, X_s) \) is then calculated as following:

\[
f_1(s, X_s, v_s) = L^\omega_s \left(1 + \frac{(1 - \rho^2)V_{t,s}}{2} \left[ (p^{(1)}(X_s^\omega))^2 + p^{(2)}(X_s^\omega) \right] \right)
\]

\[
= -d^\omega_- e^{X_s^\omega} - \frac{d^{\omega^2}_-}{2} \left(1 + \frac{(1 - \rho^2)V_{t,s}}{2} \frac{d^{\omega^2}_- - 3}{V_{s,T}} \right) \tag{40}
\]

for \( d^\omega_\pm \) evaluated at \((s, X_s, v_s)\).
Convergence Analysis

- Conditions on the convergence of the series

\[
\frac{(T - t)^{2n}}{(2^n n!)^2} E \left[ \left( \sup_{u_1, \ldots, u_n \in (t, T)} |(D_{u_n}^2 \ldots D_{u_1}^2 F)(\omega^t)| \right)^2 \right] \xrightarrow{n \to \infty} 0, \\
\frac{c^{2n}}{n!^2} E \left[ \left( \sup_{\tau_i \in (t, T)} |H_s B_{2n}(b_1^\omega, \ldots, b_{2n}^\omega) \prod_{i=1}^n \sigma_{\tau_i}^2 \mathbb{1}_{\{\tau_i \leq s\}} \right)^2 \right] \xrightarrow{n \to \infty} 0,
\]

where \( c = \frac{(T-t)^{\sqrt{1-\rho}}}{\sqrt{2}} \), and \( b_j = p(j)(s, X_s, v_s) \) for \( j = 1, \ldots, 2n \).
Full expression of $p_1(s)$

\[
p_1(s) := \omega_t \circ \int_t^s D_{\tau}^2 W R_\tau^\omega d\tau
\]

\[
= R_\tau^\omega \left[ \alpha^2(s-t) + \left( \frac{1}{\alpha^2} + 2C_3 \gamma \right) \right] \left[ \frac{\rho \alpha^2(s-t)}{\sqrt{1-e^{-\alpha^2(T-t)}}} - \frac{2\alpha \sigma_t \left( \frac{1}{\alpha^2} (1-e^{-\alpha^2(s-t)}) - (s-t)e^{-\alpha^2(s-t)}) \right)}{\sqrt{e^{-\alpha^2(s-t)} - e^{-\alpha^2(T-t)}}} \right]
\]

\[
- 2\alpha^2 \left( \frac{\gamma^2}{\alpha^2} (e^{-\alpha^2(s-t)} - e^{-\alpha^2(T-t)}) + \gamma \right) \left[ \frac{\frac{1}{\alpha^2} (1-e^{-\alpha^2(s-t)}) - (s-t)e^{-\alpha^2(T-t)})}{e^{-\alpha^2(s-t)} - e^{-\alpha^2(T-t)}} \right] + (6C_3 + 4C_3^2 \gamma \omega) \cdot \left( \rho \alpha - \frac{1}{2} \alpha^2(s-t) + \sigma_t e^{-\alpha^2(s-t)} \right)^2 (s-t) - 2\sigma_t e^{-\alpha^2(T-t)} \left( \frac{1}{\alpha^2} (1-e^{-\alpha^2(s-t)}) - (s-t)e^{-\alpha^2(T-t)}) \right)
\]

\[
- 2\sigma_t^3 \rho e^{-\frac{1}{2} \alpha^2(s-t)} \left( \frac{1}{\alpha^2} (1-e^{-\alpha^2(s-t)}) - (s-t)e^{-\alpha^2(T-t)}) \right) A_1^\omega
\]

\[
+ \frac{4\sigma_t^4}{\alpha^4} \left[ \left( \frac{1}{2} (1+e^{-2\alpha^2(s-t)}) + (e^{-\alpha^2(T-t+s-t)} - e^{-\alpha^2(s-t)} - e^{-\alpha^2(T-t)}) + \alpha^2 e^{-2\alpha^2(T-t+s-t)}(s-t) \right) A_1^\omega
\]

\[
+ \left( \frac{1}{2} (1-e^{-2\alpha^2(s-t)}) + 2(e^{-\alpha^2(T-t+s-t)} - e^{-\alpha^2(T-t)}) + \alpha^2 e^{-2\alpha^2(T-t)(s-t)} \right) A_3^\omega \right]
\]

Z. Guo, H. Schellhorn  A Full Asymptotic Series of European Call Option Prices in the
Full expression of $p_2(s)$

$$p_2(s) := \omega^t_W \circ \int_s^T D^2_{\tau,W} R^\omega_{\tau} \, d\tau = D^2_{\tau,W} R^\omega_{\tau} \int_s^T d\tau$$

$$= R^\omega_{s} \left[ \left( \frac{1}{\gamma^\omega} + 2C_3 \gamma^\omega \right) \left( -2\alpha^2 \left( \sqrt{V^\omega_{s,T}} + \gamma^\omega \right) \right) + A^\omega_{3} \left( 2\alpha V^\omega_{s,T} \right)^2 \right] (T - s)$$

$$= R^\omega_{s} \left[ -2\alpha^2 \left( 2C_3 (\gamma^2 \omega + \sqrt{V^\omega_{s,T}} \gamma^\omega) + 1 + \frac{\sqrt{V^\omega_{s,T}}}{\gamma^\omega} \right) + \alpha^2 B^\omega_{3} \right] (T - s) \quad (42)$$

where

$$B^\omega_{3} = 4V^\omega_{s,T}^2 A^\omega_{3} = 4 \left( C_3^2 \gamma^3 + (2C_3^2 \sqrt{V^\omega_{s,T}} + 3C_3) \gamma^2 + (C_3^2 V^\omega_{s,T} + 4C_3 \sqrt{V^\omega_{s,T}}) \gamma^\omega \right)$$

$$+ 6C_3 V^\omega_{s,T} + 3 + \frac{2 \sqrt{V^\omega_{s,T}}}{\gamma^\omega}. \quad (43)$$
\[ A_1^\omega := \omega^t W \circ A_1 = \frac{4 C_3^2 \gamma^\omega^2 + (4 C_3^2 \sqrt{V_{s,T}^\omega} + 8 C_3)\gamma^\omega + 6 C_3 \sqrt{V_{s,T}^\omega}}{\sqrt{V_{s,T}^\omega}^3} + \frac{1}{\sqrt{V_{s,T}^\omega}^3 \gamma^\omega}, \]

\[ A_2^\omega := \omega^t W \circ A_2 = \frac{C_3 (2 C_3^2 \gamma^\omega^2 + (V_{s,T}^\omega + 2 C_3 \sqrt{V_{s,T}^\omega} + 3)\gamma^\omega + 3 \sqrt{V_{s,T}^\omega})}{\sqrt{V_{s,T}^\omega}^3} + \frac{1}{2 \sqrt{V_{s,T}^\omega} \gamma^\omega}, \]

\[ A_3^\omega := \omega^t W \circ A_3 = \frac{C_3^2 \gamma^\omega^3 + (2 C_3^2 \sqrt{V_{s,T}^\omega} + 3 C_3)\gamma^\omega^2 + (C_3^2 V_{s,T}^\omega + 4 C_3 \sqrt{V_{s,T}^\omega})\gamma^\omega}{V_{s,T}^\omega^2} \]

\[ + \frac{6 C_3 V_{s,T}^\omega + 3}{4 V_{s,T}^\omega^2} + \frac{1}{2 \sqrt{V_{s,T}^\omega}^3 \gamma^\omega}. \]