Solitary waves and modulation instability with the influence of fractional derivative order in nonlinear left-handed transmission line

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Abstract

The resolution of the reduced fractional nonlinear Schrödinger equation obtained from the model describing the wave propagation in the left-handed nonlinear transmission line presented by Djidere et al recently, allowed us in this work through the Adomian decomposition method (ADM) to highlight the behavior and to study the propagation process of the dark and bright soliton solutions with the effect of the fractional derivative order as well as the Modulation Instability gain spectrum (MI) in the LHNLTL. By inserting fractional derivatives in the sense of Caputo and in order to structure the approximate soliton solutions of the fractional nonlinear Schrödinger equation reduced, ADM is used. The pipe is obtained from the bright and dark soliton by the fractional derivatives order. By the bias of MI gain spectrum the instability zones occur when the value of the fractional derivative order tends to 1. Furthermore, when the fractional derivative order takes small values, stability zones appear. These results could bring new perspectives in the study of solitary waves in left-handed metamaterials as the memory effect could have a better future for the propagation of modulated waves because in this paper the stabilization of zones of the dark and bright solitons which could be described by a fractional nonlinear Schrödinger equation with small values of fractional derivatives order has been revealed. In addition, the obtained significant results are new and could find applications in many research areas such as in the field of information and communication technologies.

Keywords Solitary waves · Modulation instability · Fractional derivative order · Nonlinear left-handed transmission line

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1 Introduction

The evolution of digital electronics and communication has made enormous advances, particularly in the transmission of information. In previous years, coaxial cables were mainly used, which transported signals with a lot of losses. To overcome this difficulty, researchers have introduced a recent class of materials, called "metamaterials", to simplify the size of components and thus reduce maximum losses by making them more efficient. As a result, the micro line ribbons (El amine CHAIB 2012) were born. The expression “metamaterial” was first mentioned in the field of optics by the Russian physicist (Veselago 1968) when he theoretically introduced the concept of left-handed materials and its realization by Pendry et al. (1999).

In this current work, we focus on the study of the propagation of dark and bright solitons in the left-handed nonlinear transmission line (LHNLTL) by the Adomian decomposition method with the effect of the fractional derivative order \( (\alpha) \). Recently in literature fractional differential equations have been of great interest. Nowadays, there has been considerable attention in non linear evolution equations having fractional differential equations. At the very beginning, there were hardly any practical applications of fractional calculus, and it was highly regarded as an obscure field with only rarely or not at all advantageous mathematical manipulations. For almost 30 years, the paradigm began to move from simple mathematical formulations to applications in several fields. For previous years, fractional calculus was used and applied to almost all areas of science, engineering and mathematics. Fields of application of fractional differentiation and fractional integration are already well established, others have just begun and will continue. Fractional computation applications are found in turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological systems, astrophysics (Kilbas et al. 2006; Podlubny 1999; Samko et al. 1993; El-Sayed 1996; Herzallah et al. 2010, 2011; Magin 2006; West et al. 2003; Jesus and Machado 2008; Agrawal and Baleanu 2007; Tarasov 2008). Nevertheless, some historical summaries of the developments of the fractional calculation developments used can be found in the works of Kilbas et al. (2006), Podlubny (1999) and Samko et al. (1993). Several methods to obtain exact solutions or even soliton solutions to nonlinear partial differential equations have been proposed, such as the Bäcklund transformation method (Deng 2005), the Hirota’s bilinear method (Pashaev and Tanoğlu 2005), the inverse scattering transform method (Vakhnenko et al. 2003), the extended tanh method (De-Sheng et al. 2004), the Adomian approximation (Abassy et al. 2004), the variational method (Liu 2005), the variational iteration method (He 1999), the various Lindstedt–Poincare methods (He 2002), the Adomian decomposition method (ADM) (Wazwaz 1998; Zayed et al. 2008; Adomian 2013, 1980; Ray 2020, the F-expansion method (Wang and Zhang 2005), the exp-function method (Wu and He 2007), the homotopy perturbation method (He 2006; Gepreel 2011) and the results on the acquired solitary waves were successful thanks to the different mathematical methods recently used such as the Darboux transformation (Lan and Su 2019; Lan et al. 2019; Lan 2019a, b), Hirota–Riemann method (Lan 2020), numerical simulation (Lan and Guo 2020), Hirota method (Lan 2019c).

The main purpose of this work is to investigate the behavior of the dark and bright solitons by a nonlinear evolution equation with fractional order describing the waves propagation in nonlinear left-handed electrical transmission line giving by Ahmadou et al. (2020) and Abdoulkary et al. (2015),
Thus, in this article we apply the Adomian decomposition method (ADM) to this Eq. (1) of Ahmadou et al. (2020) and Abdoulkary et al. (2015) to study the solutions of dark and bright solitons with the effect of fractional derivatives order and initial conditions. The details of the set of nonlinear evolution equations are specified (Ahmadou et al. 2020; Abdoulkary et al. 2015). Here $\alpha$ is the fractional derivative order and $0 < \alpha \leq 1$. Similar essential nonlinear models have been subject to numerous previous studies. For instance, Hosseini et al. (2020) used the transformation of Bäcklund and the model of the bilinear form of Hirota to obtain a series of rational type solutions thanks to a new bilinear equation of Hirota. In another research, Hosseini et al. (2021) obtained solutions of the bright and dark solitons of the weakly nonlocal Schrödinger equation involving the nonlinearity of the parabolic law. Moreover a number of solutions of solitons and Jacobi elliptic functions to the Heisenberg ferromagnetic spin chain equation has been obtained by Hosseini et al. (2021). A new approach of optical solitons with a high order of dispersion for nonlinear perturbed Schrödinger equations of the fourth, sixth, eighth, tenth and twelfth order has been explored by Kudryashov (2020). Different wave structures of the perturbed nonlinear Schrödinger equation with the conformable fractional order have been investigated in Nestor et al. (2021) by adopting two types of integration architecture, namely sine-Gordon expansion and the modified expansion function methods $\text{exp}(-\psi(\zeta))$-expansion. Ultimately more interesting methods have been used and can be accessed in these following studies González-Gaxiola et al. (2020), Hosseini et al. (2019), Sylvere et al. (2021), Darvishi et al. (2018) and Korkmaz et al. (2020). The first section of this paper will give an overview of the fractional derivative order. The next section will give the preliminaries and the description of the Adomian decomposition method (ADM). Furthermore the behavior of the bright and dark solitons with the effect of derivative order will be examined. Some graphical representation will follow to highlight physical phenomena of the nonlinear left-handed transmission line. The last section will conclude the work.

1.1 Overview of the fractional derivative order

In recent years, several properties aimed at facilitating the use of the fractional derivative have been the subject of various scientific publications, followed by concordant results. It should be remembered that some of the usual properties have not known exclusive salvation. Among the current properties, here are few that will promote the use of the system to be learned with the appropriate derivatives of the order $\alpha$ (Nestor et al. 2020; Adomian 1980; Ray 2020) below

$$
\frac{d^\alpha k(t)}{dt^\alpha} = \lim_{\epsilon \to +\infty} \frac{k(t + \epsilon t^{1-a}) - k(t)}{\epsilon}, \quad k : (0, \infty) \to \mathbb{R}.
$$

(2)

With $t > 0$, $\alpha \in (0, 1]$.

Then if $k$ is $\alpha$-differentiable in some $(0, c)$, $c > 0$ and $\lim_{t \to 0^+} D^\alpha(k)(t)$ exists, however by accuracy

- $D^\alpha(g)(0) = \lim_{t \to 0^+} D^\alpha(k)(t)$. 

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Taking into account (Li et al. 2011; Khalil et al. 2014; Adomian 1980; Ray 2020), the following relevant properties of fractional derivatives are presented as follows:

- $D_t^\alpha (cg + ek) = cD_t^\alpha g + eD_t^\alpha k$, and $c, e \in \mathbb{R}$.
- $D_t^\alpha (t^\mu) = \mu t^{\mu-1}$ and $\mu e \mathbb{R}$.
- $D_t^\alpha (gk) = kD_t^\alpha (g) + fD_t^\alpha (k)$.
- $D_t^\alpha \left( \frac{g}{k} \right) = \frac{kD_t^\alpha (g) - fD_t^\alpha (k)}{g^2}$.

So if $g$ is differentiable, $D_t^\alpha (g)(t) = t^{1-a} \frac{dg}{dt}$.

- if $g, k : (0, \infty) \rightarrow \mathbb{R}$ is differentiable and $\alpha$ the differentiable functions: $D_t^\alpha (gok)(t) = t^{1-a}k'(t)g'(k(t))$.

### 2 Preliminaries and Adomian decomposition method (ADM)

#### 2.1 Preliminaries

Considering Eq. (1), we can obtain the following nonlinear Schrödinger equation with non-local operator of Riemann–Liouville as follows

$$\beta_2 \frac{j^{-3a}}{(2\alpha)!} ID_t^{2a} U - \frac{\partial U}{\partial z} + (j + j^{-a})\beta_0 U - T_4 |U|^2 U = 0, \quad (3)$$

with respectively $\beta_0 = 1/(\omega_0^3 \sqrt{Lc})$ the phase velocity, $\beta_2 = 2/(\omega_0^3 \sqrt{Lc})$ the second order dispersion constant and $T_4 = \rho^2/4(\omega_0^3 \sqrt{Lc})^3$ the Kerr nonlinear part of the left-handed nonlinear transmission line (LHNMTL), where $(\rho)$ is a constant of the polarization, and $0 < \alpha \leq 1$ (Ahmadou et al. 2020), while the nonlocal operator $ID_t^{2a}$ in the sense of Riemann–Liouville read as

$$ID_t^{2a} U(z, t) = \frac{1}{\Gamma(2 - \sigma)} \frac{d^2}{ds^2} \int_0^t (t - s)^{1-\sigma} \left[U(z, s) - U(z, 0) - s \frac{d}{ds} U(z, 0)\right] ds. \quad (4)$$

For simplification purpose, we set: $A = \beta_2 \frac{j^{-3a}}{(2\alpha)!}$, $B = (j + j^{-a})\beta_0$, $\sigma = 2\alpha$, so that Eq. (3) becomes

$$AID_t^{2a} U - \partial_z U + BU - T_4 |U|^2 U = 0, \quad U(z, 0) = f(z), \quad j = \sqrt{-1}. \quad (5)$$

This part can be found in several chapters of physics, especially plasma physics, nonlinear optics, superconductivity, etc. However, cubic nonlinearity is the most common nonlinearity in applications. Nevertheless, this is a simplified model for examining Bose–Einstein condensates, Kerr medium and nonlinear optical anomalous waves in the ocean (Kanth and Aruna 2009; Gross 1963; Dysthe 2000; Henderson et al. 1999; Kelley 1965; Talanov 1966).

Thus, Kanth and Aruna (2009) studied a nonlinear Schrödinger equation such as the Eq. (1) of Ahmadou et al. (2020) by applying the differential transformation method to obtain its approximate solution.

Let’s consider Eq. (1) in this work and we grant several definitions and basic properties of the theory of fractional calculus that are further applied in these papers (Podlubny 1999; Samko et al. 1993). Using fractional calculus in the sense of Riemann-Liouville and then
the Adomian decomposition method with the effect of the fractional derivative order, and we get this fractional nonlinear Schrödinger equation of the form Eq. (3) then Eq. (4).

For the purpose of simplification, Eq. (3) can be reduced to Eq. (5). It is worthwhile mentioning that Eq. (5) is a complex differential equation, with complex modulus term |U|², which is nonlinear. Therefore an appropriate method to sort-out solution of Eq. (5) is needed. We shall use in the following the Adomian decomposition method.

2.2 Adomian decomposition method (ADM)

The Adomian decomposition method is an iteration method for solving linear, nonlinear, algebraic equations. The main advantage of this ADM is to provide solutions of an infinite series, which converges quickly to the solution exact. The ADM consists in dividing a given equation into linear and nonlinear parts, inverting the higher-order derivative operator.

2.2.1 Description of the Adomian decomposition method

Consider a general nonlinear equation in canonical form as

\[ \mathcal{L}U + VU + WU = 0, \tag{6} \]

where the linear part of a general nonlinear Eq. (6) is \( \mathcal{L}(U) + V(U) \), \( \mathcal{L}(U) \) is a linear operator easily invertible, \( V(U) \) is the part equation and the nonlinear term is represented by \( W(U) \).

The technique consists of decomposition the linear and nonlinear parts of Eq. (6) to apply linear operator of the highest-ordered derivation of terms. Notice here, \( \mathcal{L} \) is a linear operator given as \( \mathcal{L} = \frac{d}{dt}(\cdot) \). Its inverse \( \mathcal{L}^{-1} \) is defined as \( \mathcal{L}^{-1} = \int_0^t(\cdot)ds \).

If we apply inverse operator \( \mathcal{L}^{-1} \) in all sided of Eq. (6) and using the initial condition \( U(z, 0) = f(z) \), we get the following equivalent equation

\[ \mathcal{L}^{-1}\mathcal{L}(U) = -\mathcal{L}^{-1}V(U) - \mathcal{L}^{-1}W(U), \tag{7} \]

this \( \mathcal{L}^{-1}\mathcal{L}(U) = U(z, t) - U(z, 0) \).

Next we shall write the linear terms broken up by an infinite series components of the functions \( U(z, t) \) defined as

\[ U(z, t) = \sum_{n=0}^{\infty} U_n(z, t), \tag{8} \]

the nonlinear term \( W(U) \) can be decomposed into an infinite series of polynomials

\[ W(U) = \sum_{n=0}^{\infty} A_n, \tag{9} \]

where \( A_n \) is called Adomian Polynomial of \( U_n(z, t) \). Substituting Eqs. (8) and (9) into Eq. (6) we have

\[ \sum_{n=0}^{\infty} U_n = U_0(z, 0) - \mathcal{L}^{-1}V \sum_{n=0}^{\infty} U_n - \mathcal{L}^{-1} \sum_{n=0}^{\infty} A_n, \tag{10} \]
We compute the elements of $U_n(z, t)$, $n = 0, 1, 2, \ldots$, using recursive relations:

\begin{align*}
U_0(z, 0) &= f(z), \\
U_{n+1}(z, t) &= -\xi^{-1} V U_n - \xi^{-1} A_n.
\end{align*}

(11)

It consists to:

- Show the initial condition as the first term of the solution in infinite solutions of series;
- break down the nonlinear function, in terms of special, polynomials which is called Adomian’s polynomials;
- provide successive terms of series solution by recurrence using Adomian’s polynomials as

$$A_n(U_0, U_1, U_2, \ldots, U_n) = \frac{1}{n!} \left\{ \frac{d^n}{d\lambda^n} \left[ (\sum_{k=0}^{n} \lambda^k U_k) (\sum_{k=0}^{n} \lambda^k U_k^*) (\sum_{k=0}^{n} \lambda^k U_k) \right]_{\lambda=0} \right\}. \quad (15)$$

3 Dark and bright solitons solutions

In this section it is assumed the following dark and bright expression as solutions of Eq. (5):

\begin{align*}
\begin{cases}
U(z, t) &= \tanh(kz - \omega t) e^{i(z-vt)}, \\
U(z, t) &= D \text{sech}(kz - \omega t) e^{i(z-vt)}.
\end{cases}
\end{align*}

(16)

Here $D$ is a constant to be determined later. At the initial condition $(t = 0, U_0 = \tanh(kz)e^{i\xi}, U_0 = D \text{sech}(kz)e^{i\xi})$.

The next section will emphasize the behavior of the dark and bright soliton solutions by employing the Adomian decomposition method.

3.1 Application of the Adomian decomposition method

3.1.1 Fractional dark soliton solutions

We now consider Eq. (5) which is a fractional nonlinear Schrödinger equation. Applying the fractional Riemann-Liouville integral of order $\sigma$ to Eq. (5) we get:

$$I_{t}^\sigma [\text{AiD}_x^\sigma U - \partial_x U + BU - T_4 |U|^2 U] = 0,$$

i.e,
\[ I_t^\sigma (\text{ID}_t^\sigma U) = \frac{1}{A} I_t^\sigma [\partial_z U - BU + T_4|U|^2 U]. \]  

(18)

Now, taking into account the properties of the inverse of the fractional derivative in the sense of Riemann–Liouville, we obtain

\[ (I_t^\sigma \text{ID}_t^\sigma U)(t) = U(z, t) - U(z, 0) - t \frac{\partial U(z, 0)}{\partial t}, \]

(19)

setting \( U(z, 0) = \tanh(kz)e^{i\varphi} = U_0(z) \) and \( \frac{\partial U(z, 0)}{\partial t} = 0 \), we get:

\[ U(z, t) = U(z, 0) + \frac{1}{A} I_t^\sigma \left[ \partial_z U - BU + T_4|U|^2 U \right]. \]

(20)

Using the ADM on Eq. (20), we divide the given equation into linear and nonlinear parts, the higher-order derivative operator contained in the linear operator on both sides and show the initial conditions and the first term of the series solution, it is obtained:

\[
\begin{align*}
U_0(z, 0) &= \tanh(kz)e^{i\varphi}, \\
U_{n+1}(z, t) &= \frac{1}{A} I_t^\sigma \left[ \partial_z U_n - BU_n + T_4A_n \right], \quad n \geq 0.
\end{align*}
\]

(21)

Where \( A_n = |U|^2 U \) and \( U(z, 0) = f(z) \). As in the Adomian decomposition method (Wazwaz 1998; Zayed et al. 2008; Adomian 2013; Takens 1981), if we consider that a serial solution of the function \( U(z, t) \) is provided by Eq. (8) and the nonlinear term \( A_n \) can be dissociated into an infinite range of polynomials yielded by Eq. (9). However, components \( U_n(z, t) \) will be calculated recursively, the \( A_n \) are considered as the adomial polynomials of the nonlinear terms of Eq. (15).

So, we have

\[ U_{n+1}(z, t) = \frac{1}{A} I_t^\sigma \left[ \partial_z U_n - BU_n + T_4A_n \right], \quad n \geq 0. \]

(22)

Equation (15) leads to:

- \( A_0(U_0) = U_0|U_0|^2 \),
- \( A_1(U_0, U_1) = 2U_1|U_0|^2 + U_1^*U_0^2 \),
- \( A_2(U_0, U_1, U_2) = 2U_2|U_0|^2 + 2|U_1|^2 U_0 + U_2^*U_0 + U_1^*U_0^2 + U_2^2U_0^* \),
- \( A_3(U_0, U_1, U_2, U_3) = 2U_3|U_0|^2 + 2U_2U_1^*U_0 + 2U_2^*U_0U_1 + 2U_1^*U_2U_0 + U_1|U_1|^2 + U_0^2U_3^* \),
- \( \ldots \)

and so forth.

With the application of the above recursive relations in Eq. (22), and using Kilbas et al. (2006), Podlubny (1999), Adomian (2013), Adomian and Rach (1989) and Ray (2020) Eq. (21).

We structure the approximate solutions of \( U(z, t) \) as below:

\[ U_1(z, t) = C_1 e^{i\varphi} \frac{t^\sigma}{A\Gamma(\sigma + 1)}, \]

(23)

with \( C_1 = ksech^2(kz) + (j - B)tanh(kz) + T_4tanh^3(kz) \).
\[ U_2(z, t) = C_2 e^{i\epsilon t^2/2} A^2 \Gamma(2\sigma + 1), \]  

(24)

- ...and so forth.

With
\[ C_2 = (j - B)k \sech^2(kz) + (j - B)^2 \tanh(kz) + (j - B)T_4 \tanh^2(kz) + (j - 2B)T_4 \tanh^3(kz) + 3T_4^2 \tanh^5(kz) + (2kT_4 - 2k^2) \tanh(kz) \sech^2(kz) + 3kT_4 \tanh^2(kz) \sech^2(kz). \]

The approximate solution for the fractional nonlinear Schrödinger equation from Eq. (5) is obtained by adding all solutions obtained above Eqs. (23), (24), and then adding the expression of the initial condition \( U_0 = \tanh(kz) e^{i\epsilon} \) of the dark soliton, we obtain

\[ U_{app}(z, t) = U_0 + U_1 + U_2 + \cdots \]

(25)

and the exact solution \( U_{exact}(z, t) \) is obtained when \( \alpha \rightarrow 2 \) in Eq. (25), which means that \( \sigma \rightarrow 2 \) because \( \sigma = 2\alpha \).

3.1.2 Fractional bright soliton solutions

In this subsection, we proceed in the same way as in the previous subsection to point out the behavior of the bright soliton with the effect of the fractional derivative order by Adomian decomposition method. Following the same way in the subsection given the dark soliton above with the same initial condition.

To this end, we structure the approximate solutions of \( U(z, t) \) as below:

\[ U_1(z, t) = C_1 e^{i\epsilon t^\sigma/\Gamma(\sigma + 1)}, \]  

(26)

with \( C_1 = (j - B)D\sech(kz) + T_4 D^3 \sech^3(kz) - D \tanh(kz) \sech(kz) \),

\[ U_2(z, t) = C_2 e^{i\epsilon t^{2\sigma}/A^2 \Gamma(2\sigma + 1)}, \]  

(27)

- ...and so forth.

With
\[ C_2 = (j - B)^2 D\sech(kz) + (T_4 D^3 (2j - 3B) - D k^2) \sech^3(kz) + 3T_4^2 D^5 \sech^5(kz) + 2(B - j)D \tanh(kz) \sech(kz) - 6T_4 D^3 \tanh(kz) \sech^3(kz) + D k^2 \tanh^2(kz) \sech(kz). \]

The approximate solution of the Schrödinger fractional nonlinear equation of the Eq. (5) is obtained by adding all the solutions obtained above the Eq. (26), the Eq. (27) and then adding the expression of the initial condition \( U_0 = D\sech(kz) e^{i\epsilon} \) of the bright soliton, we obtain
and the exact solution $U_{\text{exact}}(z, t)$ is obtained when $\alpha \to 1$ in Eq. (28), which means that $\sigma \to 2$ because $\sigma = 2\alpha$.

### 3.2 Modulation instability gain

The modulation instability (MI) is the one important phenomenon which happens when the nonlinear and dispersion terms are present in a nonlinear evolution equation. Usually modulation instability is present in a nonlinear system because of the low perturbations enforce on the continuous wave. Now, we consider the steady-state solution of Eq. (1) as follows

$$U(z, t) = \sqrt{P} e^{jT_4Pz},$$

where $P$ is the power incident. Suppose the perturb solution of Eq. (1) in the following expression

$$U(z, t) = (\sqrt{P} + \varepsilon(z, t)) e^{jT_4Pz}.$$  \hspace{1cm} (30)

Taking in to account the impact of the perturb field, we obtain the linearized equation

$$\frac{\partial \varepsilon(z, t)}{\partial z} - \beta_2 \frac{j^{-3\alpha}}{(2\alpha)!} \frac{\partial^{2\alpha} \varepsilon(z, t)}{\partial t^{2\alpha}} - 2\beta_2 \frac{j^{-2\alpha}}{(2\alpha)!} \frac{\partial^\alpha \varepsilon(z, t)}{\partial t^\alpha} + T_4 P (\varepsilon(z, t) + \varepsilon^*(z, t)) = 0. \hspace{1cm} (31)$$

Where $\varepsilon^*(z, t)$ is the complex conjugate of $\varepsilon(z, t)$. Now we suppose the solution for the perturbed expression as follows

$$\varepsilon(z, t) = a_1 e^{j(Kz + \Omega t)} + a_2 e^{-j(Kz + \Omega t)},$$ \hspace{1cm} (32)

where $a_1$ and $a_2$ are reals, while $K$ and $\Omega$ are the wave number and modulation frequency, respectively. Let’s now introduce Eq. (32) into Eq. (31) gives

$$\begin{pmatrix} d_1 \\ 2T_4P \\ d_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \hspace{1cm} (33)$$

with $d_1 = jK - (\Omega)^{2\alpha} \beta_2 \frac{j^{-\alpha}}{(2\alpha)!} - 2\beta_2 (\Omega)^{\alpha} \frac{j^{-\alpha}}{(2\alpha)!}$, $d_2 = -jK - (-\Omega)^{2\alpha} \beta_2 \frac{j^{-\alpha}}{(2\alpha)!} - 2(-\Omega)^{\alpha} \frac{j^{-\alpha}}{(2\alpha)!}$, the obtained $2 \times 2$ matrix has a non trivial solution when the determinant vanishes. So, the determinant linked with this matrix can read as

$$K^2 + B - 4T_4^2 P^2 = 0, \hspace{1cm} (34)$$

and we show that the general solution is valid unless:
with $B = ((\Omega)^{2}\beta_2 \frac{j^{n}}{(2\alpha)!} + 2\beta_2(\Omega)^{\alpha} \frac{j^{n}}{(2\alpha)!})((-\Omega)^{2}\beta_2 \frac{j^{-n}}{(2\alpha)!} + 2\beta_2(-\Omega)^{\alpha} \frac{j^{-n}}{(2\alpha)!})$.

\[ K = \pm 2T_4P \sqrt{1 - \frac{B}{4P^2T_4^2}}, \]  

(35)
4 Physical explanation of the results

Figures 1 and 2 show the pipe of the dark soliton solutions with the effect of the fractional derivative order in nonlinear left-handed transmission line for $\omega_0 = 0.032\ \text{rad s}^{-1}$, $C_0 = 100.045\ \text{pH}$, $C_L = 78.65\ \text{pH}$, $L_L = 200.054\ \text{nH}$, $\rho = 0.01$, $k = 0.98$. The effect of the fractional order are pointed out when $\alpha = 0.5$, $\alpha = 0.45$, $\alpha = 0.4$ and $\alpha = 0.35$ (see Fig. 2). In addition the shape of the wave undergoes deformations as $\alpha$ takes small values and this reflects the phenomenon of the memory effect which is one of the advantages linked to the fractional derivative order. On the other hand, for values of $\alpha$ tending towards 1, the result obtained is similar to the dark soliton solution known in the left-handed metamaterials (Fig. 3). We also emphasized the behavior of the bright soliton solutions with the effect of the fractional derivative order in Figs. 4, 5, and 6 for $\omega_0 = 0.032\ \text{rad s}^{-1}$, $C_0 = 100.045\ \text{pH}$, $C_L = 78.65\ \text{pH}$, $L_L = 200.054\ \text{nH}$, $\rho = 0.01$, $k = 0.7$, $k = 0.95$, $D = 0.084$. Moreover, the effect of fractional order on the width of the bright soliton is obtained (see Figs. 5, 6). More precisely, the solitons obtained took into account the terms of nonlinearity and of dispersion for their existence. As it is mentioned in several works that the nonlinearity and dispersion terms involve the modulation instability, we have set out the dynamic of the variation of MI gain spectra in the anomalous group velocity dispersion on the one hand and on the other hand the nonlinear term kerr. It is observed the instability zones when the derivative order $\alpha$ is near to 1 [see Fig. 8 ($R_3, R_4$)]. It should be noted that the instability is linked to MI due to unbidden temporal modulation of the continuous wave (CW) stack and change it into a pulse train. However, for the small value of the fractional derivative order stability zones appeared [see Fig. 8 ($R_1, R_2$)]. Enough but not the last, we use MATLAB calculation to dig out the gap between the approximate analytical results and the exacts.

![Fig. 3](image-url) The approximate dark soliton solution obtained by Adomian decomposition method of the Eq. (25) when $(\alpha = 0.5, \alpha = 0.55, \alpha = 0.58, \alpha = 0.6)$, $\omega_0 = 0.032\ \text{rad s}^{-1}$, $C_0 = 100.045\ \text{pH}$, $C_L = 78.65\ \text{pH}$, $L_L = 200.054\ \text{nH}$, $\rho = 0.01$, $k = 0.7$
From Tables 1 and 4 it is observed that when the value of derivative order tends to 1 the
\[ U_{app} \approx U_{exact} \]. It is also observed that the difference is greater between the exact obtained
and the results of the estimation when the values of the order of the derivatives are too
small (see Tables 2, 3, 5, 6). So, these results obtained are new and significant because dark
and bright solitons were obtained by the adomian decomposition method in a nonlinear left-handed transmission line with the variable \(t\) less than, greater than or equal to zero. We deduce that the behavior of the approximate bright solutions is practically the same as the behavior of its exact solution with different values of \(\alpha\) and we obtain small errors between the approximate solution and the exact solution for certain values of \(\alpha\) with the number of steps. On the other hand, the behavior of dark approximate solutions and the behavior of its exact solution, leads us to large errors for some \(\alpha\) values with the number of steps. Therefore, we infer that the bright soliton is more convergent than the dark soliton. This work could therefore be very interesting.

5 Summary

In this work, the Adomian decomposition method (ADM) has been successfully applied to obtain semi-analytic solutions of the fractional nonlinear Schrödinger equation with initial conditions. Efficiency, power, reliability and reduced calculations of this approach give it a wider applicability. Figures 1, 2 and 3 show the behavior of the dark soliton with the influence of the fractional derivative order. More precisely, the shape and amplitude of the wave are obtained with deeply small values of the derivative order. Concerning the bright soliton solutions it is also pointed out the same phenomenon observed previously to the dark solitons. As mentioned above, the divergence between kerr nonlinearity and group velocity terms is favorable to MI, then the MI gain spectrum versus anomalous group velocity dispersion and nonlinearity coefficient is also shown (see Figs. 7, 8). Several instability zones are obtained under the effect of the derivative order. Beside these
results, we obtain the errors between the approximate and the exacts dark (bright) solitons solutions (see Tables 1, 2, 3, 4, 5, 6). Therefore, we deduce that the approximate solution is convergent series as an exact solution (Ray 2020). The Adomian decomposition method is a very efficient and powerful technique for finding soliton solutions. However,
interpretation of the results reveals that the propagation of the wave is considerably disturbed when moving away with the effect of the derivative order, so that the evolution of the system can no longer be controlled. Nevertheless, these results are more precise

| Table 2 | Comparison between the absolute value of the approximate dark soliton solution obtained by Adomian decomposition method of the Eq. (25) and the absolute value of the exact solution when $(\alpha = 0.5), \omega_0 = 0.032 \text{ rad s}^{-1}, C_0 = 500.045 \text{ pH}, C_L = 478.65 \text{ pH}, L_L = 200.0540 \text{ nH}, \rho = 0.01, k = 0.75$ |
|---------|--------------------------------------------------|
| $N = 1, 2, \ldots, 10$ | $\text{abs (}U_{\text{app}}\text{)}$ | $\text{abs (}U_{\text{exact}}\text{)}$ | $\text{Error } |(U_{\text{app}}) - (U_{\text{exact}})|$ |
| 1       | 0.7531 | 0.9951 | 0.2420 |
| 2       | 0.7528 | 0.9920 | 0.2392 |
| 3       | 0.7526 | 0.9890 | 0.2364 |
| 4       | 0.7523 | 0.9860 | 0.2337 |
| 5       | 0.7520 | 0.9830 | 0.2310 |
| 6       | 0.7517 | 0.9800 | 0.2282 |
| 7       | 0.7514 | 0.9770 | 0.2256 |
| 8       | 0.7511 | 0.9740 | 0.2229 |
| 9       | 0.7508 | 0.9710 | 0.2203 |
| 10      | 0.7504 | 0.9681 | 0.2177 |

| Table 3 | Comparison between the absolute value of the approximate dark soliton solution obtained by Adomian decomposition method of the Eq. (25) and the absolute value of the exact solution when $(\alpha = 0.2), \omega_0 = 0.032 \text{ rad s}^{-1}, C_0 = 500.045 \text{ pH}, C_L = 478.65 \text{ pH}, L_L = 200.0540 \text{ nH}, \rho = 0.01, k = 0.75$ |
|---------|--------------------------------------------------|
| $N = 1, 2, \ldots, 10$ | $\text{abs (}U_{\text{app}}\text{)}$ | $\text{abs (}U_{\text{exact}}\text{)}$ | $\text{Error } |(U_{\text{app}}) - (U_{\text{exact}})|$ |
| 1       | 0.7531 | 0.9951 | 0.2420 |
| 2       | 2.4983 | 3.1348 | 0.6365 |
| 3       | 4.7712 | 6.0274 | 1.2562 |
| 4       | 6.9457 | 8.7256 | 1.7799 |
| 5       | 9.0417 | 11.2942 | 2.2526 |
| 6       | 11.0746 | 13.7666 | 3.1075 |
| 7       | 13.0553 | 16.1628 | 3.5044 |
| 8       | 14.9918 | 18.4963 | 3.8865 |
| 9       | 16.8900 | 20.7764 | 4.2561 |
| 10      | 18.7544 | 23.0105 | 4.2561 |

| Table 4 | Comparison between the absolute value of the approximate bright soliton solution obtained by Adomian decomposition method of the Eq. (28) and the absolute value of the exact solution when $(\alpha = 1), \omega_0 = 0.032 \text{ rad s}^{-1}, C_0 = 500.045 \text{ pH}, C_L = 478.65 \text{ pH}, L_L = 200.0540 \text{ nH}, \rho = 0.01, k = 0.75$ |
|---------|--------------------------------------------------|
| $N = 1, 2, \ldots, 10$ | $\text{abs (}U_{\text{app}}\text{)}$ | $\text{abs (}U_{\text{exact}}\text{)}$ | $\text{Error } |(U_{\text{app}}) - (U_{\text{exact}})|$ |
| 1       | 0.0038 | 0.0039 | 0.1622e−03 |
| 2       | 0.0038 | 0.0039 | 0.1622e−03 |
| 3       | 0.0038 | 0.0039 | 0.1622e−03 |
| 4       | 0.0038 | 0.0039 | 0.1622e−03 |
| 5       | 0.0038 | 0.0039 | 0.1622e−03 |
| 6       | 0.0038 | 0.0039 | 0.1622e−03 |
| 7       | 0.0038 | 0.0039 | 0.1622e−03 |
| 8       | 0.0038 | 0.0039 | 0.1622e−03 |
| 9       | 0.0038 | 0.0039 | 0.1622e−03 |
| 10      | 0.0038 | 0.0039 | 0.1622e−03 |
than that in Ahmadou et al. (2020) and Abdoulkary et al. (2015) because in this work, we obtain both dark and bright solitons with the effect of fractional derivatives order by applying the Adomian decomposition method (ADM) to this same Eq. (1) of Ahmadou et al. (2020) and Abdoulkary et al. (2015) and we also evaluated the modulational instability of these solitons. Since, analytical and numerical studies of the metamaterial transmission line have proven the presence of Schrödinger solitons, which has led us to formation of dark Schrödinger solitons in this nonlinear left-handed transmission line by applying the Adomian decomposition method (ADM) to the Eq. (1) of Ahmadou et al. (2020) and Abdoulkary et al. (2015) with the effect of fractional derivatives order. Finally, this nonlinear left-handed transmission line with the effect of the fractional derivative order also allowed us to generate bright envelope solitons by making an appropriate choice of the values of the fractional derivative order and the initial condition.

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