The infinite dimensional symmetry groups of the Friedmann equations

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Abstract

We find the symmetry generators for the Friedman equations emanating from a perfect fluid source, in the presence of a cosmological constant term. The relevant dynamics is seen to be governed by two coupled, first order ODE’s, the continuity and the quadratic constraint equation. Arbitrary functions appear in the components of the symmetry vector, indicating the infinity of the group. When the equation of state is considered as arbitrary but ab initio given, previously known results are recovered and/or generalized. When the pressure is considered among the dynamical variables, solutions for models with different equations of state are mapped to each other; thus enabling the presentation of solutions to models with complicated equations of state starting from simple known cases.

I Introduction

An interesting discussion on the appearance of symmetry in nature and in various motifs of art is presented by H. Weyl in [1]. A symmetric structure usually represents an economical process in Nature and an aesthetically pleasing object to human mind. In Art, usually the symmetry is an expression of invariance under some form of translation, rotation or reflection. The (Ancient) Greek word for Art, “τέχνη”, can be translated today as Science. In the latter, the concept of symmetry is appealing for it leads to a characterization of the system under consideration. The invariance associated with the symmetry produces a simplification which makes understanding easier.

The concept of symmetry played an important role in the development of physical theory. Kepler attempted to impose the idea of symmetry for the description of the motion of the planets. Furthermore, Newton’s laws of mechanics embodied symmetry principles, such are the Galilean invariance or the principle of equivalence of inertial frames [2]. Another example is Maxwell’s equations which describe...
the electromagnetic theory: while they possess the Lorentz invariance and the gauge invariance, these
symmetries were explicitly developed a few decades after Maxwell’s work [3].

The first systematic approach for the description of symmetry in modern science was constructed
by Sophus Lie towards the end of 19th century [4–6]. The novelty of Lie’s work was to consider the
infinitesimal representations of the finite transformations induced on a manifold by continuous groups;
thereby moving from the group to a local algebraic representation in terms of vector fields on the tangent
space of the manifold. The study of the invariance properties of various geometrical objects is then
performed using the Lie Derivative with respect to the afore mentioned fields (generators). This resulted in
a linearization of all equations describing the symmetry conditions. The original work of Lie was motivated
by geometric considerations; he commenced with point transformations, and so point symmetries, and
then extended his work by use of contact transformations [7].

Emmy Noether in her revolutionary work on the invariance of the Action Integral of the Calculus of
Variations under infinitesimal transformations [8] introduced routine dependence of the transformation
on the derivatives, without the requirement that the transformation be contact, and thus the use of
generalised transformations has become well-established as a tool in the study of differential equations,
particularly partial differential equations. In addition, Noether’s work provides a systematic method for
the construction of conservation laws for dynamical systems which follow from a variation principle. In
particular there is a direct relation of the infinitesimal transformations which leave form invariant the
Action Integral with conservation laws for the equations of motion; for a recent review and discussion on
Noether’s work we refer the reader to [9].

Since Lie’s theory provides a systematic way for the treatment of nonlinear differential equations, by
determining exact and analytic solutions or conservation laws, it plays a significant role in the development
of the fundamental modern physics, such as Analytic Mechanics, Quantum Mechanics, General Relativity
and Cosmology [10–14].

In gravitational physics, specifically in General Relativity, the context of symmetry is essential for
the determination of exact solutions to the Einstein field equations [15]. The assumption of existence
of a collineation, such as an isometry, affects the Einstein tensor so that to simplify the corresponding
nonlinear Einstein field equations. For instance, in Bianchi models the gravitational field equations reduce
from partial differential equations to ordinary differential equations because of the existence of the three-
dimensional isometry group for the metric tensor. Indeed, there are exact solutions for the Einstein field
equations without any isometry [16]; however other kind of symmetries can exist for these spacetimes [17].

Other types of symmetries have also been used before in order to study and solve the Einstein field
equations: The infinitesimal transformations of the phase space of the dynamical variables for the field
equations appear in the literature [18–29], while in the recent review [30] the various approaches have
been categorized and a discussion is given for the geometric character of the symmetries of the field
equations. The invariants which follow from the latter analysis have been used for the quantization
process in quantum cosmology and the solution of the corresponding Wheeler-DeWitt equation [31].

A group of infinitesimal transformations which leave invariant a given dynamical system can be
used to determine invariants and conservation laws. The later are applied to simplify the dynamical
system and reduce it to a known integrable class or to write the differential equations in an algebraic
form. Moreover, if a solution for a given dynamical system is known, then symmetries can be applied to
construct new solutions. For instance, the Gasperini-Veneziano duality property of the dilaton field [32], in
a spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) background space, is a discrete symmetry
which is directly related to a continuous symmetry [33]. Specifically the $O(d, d)$ invariance of the two-
dimensional linear system of Lorentzian signature can be transformed to the Gasperini-Veneziano duality
invariance [34].

In [35] a group of point transformations under which the Einstein field equations are invariant have
been applied for the construction of new solutions from known ones in the case of spatially flat FLRW
spacetime. While a family of transformations which relate various inflationary behaviours in the presence
of a scalar field were discussed in [36]. An alternative reconstruction method of inflationary solutions
from other known solutions was studied in [37]. In particular, in the case of spatially flat FLRW a
map was found which transforms solutions into solutions, while the different physical models result into
equivalent systems which are characterized by the admitted $SL(3,R)$ Lie algebra as invariant infinitesimal transformations. Recently, in [38], there appeared a new point transformation which keeps invariant the cosmological field equations of a spatially flat FLRW background space where the matter source is that of an ideal gas.

In this work we consider the gravitational field equations of FLRW spacetime, without imposing the vanishing of spatial curvature, while for the matter source we consider the existence of a cosmological constant and of a perfect fluid with an arbitrary equation of state. We find three different infinite families of symmetries which leave invariant the field equations, when a corresponding constraint equation is satisfied by the unknown functions appearing in the generators. We recover and generalize previous results from the literature while we demonstrate how the new results can be used to determine new solutions from old.

In particular we show how the symmetry generators found can be used to connect models with different equations of state and thus to collectively describe the main epochs of the cosmological evolution, starting from the early inflation era, passing to the radiation and then to matter eras; these solutions can also be used to describe the late-time acceleration phase of the universe. Moreover, starting from the exact solution of a massless scalar field - which is described by a stiff fluid- we show that under a specific transformation, the exact solution which correspond to a modified Chaplygin gas model is recovered. This later model is known to be describable by a quintessence scalar field with non-zero potential. Last but not least we show how an exact solution of a parametric dark energy model can be constructed by starting from the matter dominated era. The plan of the paper is as follows.

In Section 2, we define the gravitational model of our consideration, we discuss the concept of symmetry in the resulting ordinary differential equations, and we perform a detailed derivation of the symmetry vectors in the most generic scenario. The generators of the transformations which leave invariant the field equations are determined. In Section 3, we recover previous results of the literature, which can be seen as special cases of our analysis. Our general results are applied in Section 4. Finally in Section 5, we discuss our results and draw our conclusions.

II Cosmological field equations and their symmetries

We adopt the Friedmann–Lemaître–Robertson–Walker line element

$$ds^2 = -N(t)^2 dt^2 + a(t)^2 \left[ \frac{dr^2}{1-kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right]$$

and consider a perfect fluid with energy density $\rho$ and pressure $P$. The energy momentum tensor is

$$T_{\mu\nu} = \left( \rho + P \right) u_\mu u_\nu + P g_{\mu\nu},$$

where $u_\mu = (N,0,0,0)$ is the comoving velocity, $u_\mu u_\mu = -1$.

The system to be solved is composed by Einstein’s field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu},$$

together with the continuity equation $\nabla_\mu T^{\mu\nu} = 0$ (we adopt the units $G = c = 1$). The latter, together with the temporal component of (II.3) are first order ordinary differential equations which can be written as

$$E_1 := \Lambda - \frac{3k}{a^2} + 8\pi \rho - 3 \left( \frac{\dot{a}}{Na} \right)^2 = 0$$

$$E_2 := \frac{3}{a} \left( P + \rho \right) + \dot{\rho} = 0$$
The spatial components of (II.3) comprise the unique (due to the high symmetry) equation

\[ E_3 := \Lambda - \frac{k}{a^2} - 8\pi P - 3 \left( \frac{\dot{a}}{Na} \right)^2 - 2 \frac{d}{dt} \left( \frac{\dot{a}}{Na} \right) = 0 \]  

which, however, does not constitute an independent relation, i.e. it can be obtained by use of (II.4), (II.5) and the time derivative of the first.

Since the system is described by purely first order equations, namely (II.4) and (II.5), we expect it to admit an infinite number of Lie-point symmetries [39,40]. We briefly review the basic aspects of obtaining the Lie-point symmetries of a system of differential equations; for the interested reader we refer to the textbooks [39, 40]. Assume a set of ordinary differential equations of the form \( E_I(t, x, \dot{x}, \ddot{x}) = 0 \) like the ones we have here.1

A vector

\[ X = \chi(t, x) \frac{\partial}{\partial t} + \xi^i(t, x) \frac{\partial}{\partial x^i} \]  

in the space of the dependent and independent variables, \( x \) and \( t \) respectively, is called a Lie-point symmetry of the system \( E_I = 0 \) if it satisfies the infinitesimal criterion of invariance

\[ \text{pr}^2 X(E_I) = 0, \mod E_I = 0, \text{ for all } I, J. \]  

The \( \text{pr}^2 X \) is the second (due to considering up to second order equations) prolongation of the vector \( X \), i.e. an extension of the original \( X \) vector to the space of the derivatives \( \dot{x} \) and \( \ddot{x} \) and it is given by the formula:

\[ \text{pr}^2 X = X + \left( \frac{d\xi^i}{dt} - \dot{x}^i \frac{d\chi}{dt} \right) \frac{\partial}{\partial \dot{x}^i} + \left[ \frac{d}{dt} \left( \frac{d\xi^i}{dt} - \dot{x}^i \frac{d\chi}{dt} \right) - \ddot{x} \frac{d\chi}{dt} \right] \frac{\partial}{\partial \ddot{x}^i}. \]  

The extended coefficients of \( \text{pr}^2 X \) in the space of \( (\dot{x}, \ddot{x}) \) are such so that it is guaranteed that the transformation law maps functions to functions [41]: Given a generator \( X \) that sets a transformation law in the space of \( (t, x) \) and the fact that the \( x \) are functions of \( t \) at the level of the equations, the \( \dot{x} = \frac{dx}{dt} \) and \( \ddot{x} = \frac{d^2x}{dt^2} \) obviously cannot transform in a random manner. The correct transformation law for the derivatives is assured by the formula (II.9).

Let us now proceed into applying this general theory onto the Friedmann equations. We are going to deal with two distinct situations: In the first we assume some generic, but supposedly specific equation of state during the transformation. In the second case we treat the pressure as a dynamical variable and we allow it to transform together with the rest of the variables, thus enabling connections between models described by different equations of state.

### II.1 Invariance that preserves the equation of state

We begin our study by considering a fixed equation of state for \( P \) in (II.4)-(II.6). This means that we do not allow for \( P \) to variate under the transformation. Thus, our dependent variables are \( x = (N, a, \rho) \) and the generator has the general form

\[ X = \chi(t, N, a, \rho) \frac{\partial}{\partial t} + \xi^i(t, N, a, \rho) \frac{\partial}{\partial N} + \xi^2(t, N, a, \rho) \frac{\partial}{\partial a} + \xi^3(t, N, a, \rho) \frac{\partial}{\partial \rho}. \]  

As we mentioned our system is completely characterized by the first order equations \( E_1 = 0 \) and \( E_2 = 0 \). The symmetry condition (II.8) states that the prolongation of the vector has to annihilate the equations modulo the equations themselves. We realize it in the following manner: we demand that the

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1For the needs of the general context we assume second order of ordinary differential equations, even though for our main result we need only consider first order ones: \( E_1 = 0 \) and \( E_2 = 0 \).
action of the prolongation vector returns multiples of the equations, which of course are bound to be zero
and thus satisfy (II.8). In other words we require

\[ \text{pr}^2 X(E_1) = (\sigma_1 \dot{a} + \sigma_2 \dot{\rho} + \sigma_3) E_1 + (\sigma_4 \dot{a}^2 + \sigma_5 \dot{a} + \sigma_6) E_2 \]  
\[ \text{pr}^2 X(E_2) = (\sigma_7 \dot{a} + \sigma_8 \dot{\rho} + \sigma_9) E_1 + (\sigma_{10} \dot{a}^2 + \sigma_{11} \dot{a} + \sigma_{12} \dot{\rho} + \sigma_{13} \dot{\dot{N}} + \sigma_{14}) E_2, \]

where all the \( \sigma_i, i = 1, \ldots, 14 \) are functions of \( t, a, \rho \) and \( N \). The specific dependence on the derivatives \( \dot{a}, \dot{\rho} \) and \( \dot{\dot{N}} \) in the right hand side of (II.11) has been chosen so that it is in accordance with the terms produced by the left hand side, hence avoiding any trivial multipliers \( \sigma_i \). Since the \( \chi \) and \( \xi \) functions of the vector (II.10) have no dependence on derivatives (we search for point symmetries), the coefficients of the terms involving derivatives in (II.11) must be separately set equal to zero. This forms a set of partial differential equations for \( \chi, \xi \)'s and algebraic in the \( \sigma_i \) to be solved.

Refraining from presenting the specific solutions for the \( \sigma_i \) - since the latter are multipliers of zeros
and are not important in our considerations - the general symmetry vector satisfying (II.11) can be of
the following three general categories:

1. The vector

\[ X_1 = \chi_1(t) \chi_2(a, \rho) \frac{\partial}{\partial t} - N \chi_1(t) \chi_2(a, \rho) \frac{\partial}{\partial N}, \]

with \( \chi_1(t) \) an arbitrary function and \( \chi_2(a, \rho) \) satisfying the condition

\[ a \frac{\partial \chi_2}{\partial a} - 3(P(a, \rho) + \rho) \frac{\partial \chi_2}{\partial \rho} = 0. \]

The special solution \( \chi_2 = \text{constant} \) results in the parametrization invariance generator

\[ X_{\text{par}} = \chi_1(t) \frac{\partial}{\partial t} - N \chi_1(t) \frac{\partial}{\partial N}. \]

It describes the well known property of (II.4), (II.5) and (II.6) being invariant under arbitrary
changes in the time variable; a symmetry that is present in all cosmological systems [42].

2. The second case we distinguish is characterized by

\[ X_2 = \frac{4\pi a^2 N f(a, \rho)}{3k - a^2(\Lambda + 8\pi \rho)} \frac{\partial}{\partial N} + f(a, \rho) \frac{\partial}{\partial \rho}, \]

under the condition that \( f(a, \rho) \) satisfies

\[ 3f(a, \rho) \left( \frac{\partial P}{\partial \rho} + 1 \right) - 3 \frac{\partial f}{\partial \rho} (P(a, \rho) + \rho) + a \frac{\partial f}{\partial a} = 0. \]

3. Finally we have the symmetry generator

\[ X_3 = N \left[ \frac{\partial \xi}{\partial a} - \frac{3}{a} (P(a, \rho) + \rho) \frac{\partial \xi}{\partial \rho} + \frac{a (12\pi P(a, \rho) + 4\pi \rho - \Lambda)}{a^2(\Lambda + 8\pi \rho) - 3k} \xi(a, \rho) \right] \frac{\partial}{\partial N} \]

\[ + \xi(a, \rho) \frac{\partial}{\partial a} - \frac{3(P(a, \rho) + \rho)}{a} \xi(a, \rho) \frac{\partial}{\partial \rho}, \]

with \( \xi(a, \rho) \) an arbitrary function. Unlike the \( \chi_2(a, \rho) \) and \( f(a, \rho) \) we introduced previously, the
(II.17) \( \xi(a, \rho) \) here does not depend on the equation of state \( P = P(a, \rho) \).

By enforcing the resulting generators on the third equation we derive

\[ \text{pr}^2 X_i(E_3) = \Sigma^i E_i, \quad i = 1, 2, 3, \]

(II.18)
where the $\Sigma^i$ are multiplying functions. Thus, the resulting vector fields are automatically symmetries of the third equation since they satisfy condition (II.8).

We thus have three distinct infinite dimensional groups. It can be easily checked, with the help of conditions (II.13) and (II.16), that the subalgebras spanned by each of the $X_1$, $X_2$ and $X_3$ vectors separately are closed. For example, the commutator of two type $X_1$ vectors produces again a type $X_1$ vector, etc. In what regards cross commutation relations, it can be shown that $[X_1, X_2]$ is an $X_1$ vector, $[X_2, X_3]$ is a type $X_3$ vector, while $[X_1, X_3] = 0$ by virtue of (II.16).

Especially in what regards the vector $X_3$, involving the free function $\xi(a, \rho)$, we can directly see why it leaves invariant the continuity equation $E_2 = 0$ of (II.5). The latter is sensitive only in changes in $a$ and $\rho$\footnote{Since a change in $t$ for example transforms $\dot{a}$ and $\dot{\rho}$ in the same manner.} By calculating the integral curve of $X_3$ in the $(a, \rho)$ surface we obtain

$$
\frac{da}{\xi} = \frac{d\rho}{-\frac{3(P(a, \rho) + \rho)}{a}} \Rightarrow \frac{d\rho}{da} = -\frac{3(P(a, \rho) + \rho)}{a},
$$

(II.19)

but this is none other than equation (II.5) itself.

In the above setting we considered $P = P(a, \rho)$. However, the process can be generalized to include an equation of state that also incorporates the Hubble function. To this end we may assume that we have such a pressure of the form $P = P(a, \rho, \frac{\dot{a}}{a})$.

With such an expression, by considering $P_3 \propto a^{-\mu}$, we can create a dependence on the $\mu$-th power of the Hubble function. But, irrespectively of the latter, and avoiding the meticulous details, it can be verified that the following generators are symmetries of the Friedmann equations when expression (II.20) is considered:

1. The generator $X_1$ of (II.12) under the condition

$$
a \frac{\partial y_2}{\partial a} - 3 \left[ \rho + P_1(a, \rho) + 3P_2(a, \rho)\Sigma(a, \rho)^{\mu/2} \right] \frac{\partial y_2}{\partial \rho} = 0.
$$

(II.21)

2. The $X_2$ of (II.15) whenever

$$
3f(a, \rho) \left[ \frac{\partial P_1}{\partial \rho} + 1 + \Sigma(a, \rho)^{\mu/2} \frac{\partial P_2}{\partial \rho} + \frac{4\mu \pi}{3} \Sigma(a, \rho)^{\mu/2} P_2(a, \rho) \right]
- 3 \left[ P_1(a, \rho) + \rho + P_2(a, \rho)\Sigma(a, \rho)^{\mu/2} \right] \frac{\partial f}{\partial \rho} + \frac{\partial f}{\partial a} = 0.
$$

(II.22)

The comparison to the previous cases is direct, since setting $P_2 = 0$ in (II.21) and (II.22) leads to (II.13) and (II.16) respectively.

3. Finally we have the symmetry generator

$$
X_3 = N \left[ \frac{\partial \xi}{\partial a} - \frac{3}{a} \left( P_1 + P_2\Sigma(a, \rho)^{\mu/2} + \rho \right) \frac{\partial \xi}{\partial \rho} + a \left( \frac{12\pi}{3\Sigma(a, \rho)} \right) \left( P_1 + P_2\Sigma(a, \rho)^{\mu/2} + 4\pi \rho - \Lambda \right) \xi(a, \rho) \frac{\partial}{\partial \xi} \right]
$$

$$
+ \xi(a, \rho) \frac{\partial}{\partial a} - \frac{3}{a} \left( P_1 + P_2\Sigma(a, \rho)^{\mu/2} + \rho \right) \xi(a, \rho) \frac{\partial}{\partial \rho}
$$

(II.23)

with $\xi(a, \rho)$ an arbitrary function and where for simplification we have set $\Sigma(a, \rho) = \frac{1}{3}(\Lambda + 8\pi \rho) - k$ in all the previous relations. Obviously setting $P_2 = 0$ again we return to the previous case and specifically to generator (II.17).
It is easy to verify that all of the above generators satisfy \( pr^2X_i(E_j) = 0 \), \( i, j = 1, 2, 3 \), whenever \( E_1 = 0 \), \( E_2 = 0 \) and \( E_3 = 0 \). Once more, the invariance of the two first leads directly to the invariance of the third.

### II.2 Invariance connecting different models

Up to now we discussed invariance of our equations for a given (even though not explicitly specified) equation of state. It is more interesting however to allow for \( P \) to be dynamical and thus change under a symmetry transformation. In this manner we can use a symmetry vector in order to map known solutions corresponding to simple equations of state, to other theories involving more complicated expressions for the latter.

To this end let us consider the following possible generator

\[
X = \chi(t, N, a, \rho, P) \frac{\partial}{\partial t} + \xi^1(t, N, a, \rho, P) \frac{\partial}{\partial N} + \xi^2(t, N, a, \rho, P) \frac{\partial}{\partial a} + \xi^3(t, N, a, \rho, P) \frac{\partial}{\partial \rho} + \xi^4(t, N, a, \rho, P) \frac{\partial}{\partial P}.
\]

So we extended the vector by assuming a \( \partial_P \) component, while all the coefficients may additionally depend on \( P \). The symmetry condition (II.8) is now expressed as

\[
pr^2X(E_1) = (\sigma_1 \dot{u} + \sigma_2 \dot{\rho} + \sigma_3) E_1 + (\sigma_4 \dot{a}^2 + \sigma_5 \dot{a} + \sigma_6) E_2
\]

\[
pr^2X(E_2) = (\sigma_7 \dot{u} + \sigma_8 \dot{\rho} + \sigma_9) E_1 + (\sigma_{10} \dot{a}^2 + \sigma_{11} \dot{a} + \sigma_{12} \dot{\rho} + \sigma_{13} \dot{N} + \sigma_{14} \dot{P} + \sigma_{15}) E_2,
\]

the difference being just in the addition of a linear in \( \dot{P} \) term on the right hand side of the second equation.

The above system is solved in terms of the functions \( \sigma_i \) and the coefficients of the generator \( X \) and yields (again we refrain from giving the multipliers \( \sigma \))

1. The parametrization invariance vector \( X_{par} \) of (II.14).

2. The symmetry generator

\[
X_4 = X_2 + \left[ (P + \rho) \frac{\partial f}{\partial \rho} - \frac{a}{3} \frac{\partial f}{\partial a} - f(a, \rho) \right] \frac{\partial}{\partial P}
\]

where the \( X_2 \) is the one of (II.15), but with the difference that now the \( f(a, \rho) \) appearing here is not bound by condition (II.16); it is an arbitrary function.

3. The generator

\[
X_5 = N \left[ \frac{\partial \sigma}{\partial a} = \frac{3(P + \rho)}{a} \frac{\partial \sigma}{\partial \rho} - \frac{a(L + 8\pi \rho)}{a^2(L + 8\pi \rho) - 3k} \sigma(a, \rho) \right] \frac{\partial}{\partial N} + \sigma(a, \rho) \frac{\partial}{\partial a}
\]

\[
+ \left[ \frac{3(P + \rho)^2}{a} \frac{\partial \sigma}{\partial \rho} - (P + \rho) \frac{\partial \sigma}{\partial a} + \frac{(P + \rho)}{a} \sigma(a, \rho) \right] \frac{\partial}{\partial P},
\]

where \( \sigma(a, \rho) \) is another free function.

Thus, we obtain two new point symmetry generators that involve transformations at the level of the pressure \( P \) as well. That is two new infinite dimensional symmetry groups characterized by the free functions \( f(a, \rho) \) and \( \sigma(a, \rho) \). As far as the algebra properties are concerned it is easy to see that \( [X_4, X_4] \) is also a \( X_4 \) vector, \( [X_5, X_5] \) belongs to the same class as \( X_5 \), while \( [X_4, X_5] \) gives a composition of the two vectors. In particular, for two vectors \( X_4^{(f)} \) and \( X_5^{(\sigma)} \) corresponding to some particular \( f(a, \rho) \) and \( \sigma(a, \rho) \) respectively, we obtain:

\[
[X_4^{(f)}, X_5^{(\sigma)}] = - \left( X_4^{(f)} + X_5^{(\sigma)} \right)
\]

where \( \tilde{f}(a, \rho) \) and \( \tilde{\sigma}(a, \rho) \) are functions related to \( f(a, \rho) \) and \( \sigma(a, \rho) \) by the differential equations

\[
\sigma(a, \rho) \frac{\partial f}{\partial a} = \tilde{f}(a, \rho), \quad f(a, \rho) \frac{\partial \sigma}{\partial \rho} = -\tilde{\sigma}(a, \rho).
\]
III Relation to previous results

In this section we explore how our result is connected to what is previously encountered in the literature and how it generalizes some known symmetry properties of the Friedmann equations.

Recently, a new symmetry for the Friedmann equations in the spatially flat FLRW case without cosmological constant and with $P = w\rho$, was reported in [35]. Let us demonstrate how the latter is a particular case of generator $X_3$ as expressed by (II.17). The symmetry transformation introduced in [35] is:

$$\ddot{a} = a^s, \quad \ddot{\rho} = a^{-3(w+1)(s-1)}\rho, \quad d\bar{t} = sa\dot{a}^{2(w+1)(s-1)}dt,$$  \hspace{1cm} (III.1)

with $s = $constant being the parameter of the transformation. The above transformation is mentioned as a symmetry of the Friedmann equations in the cosmological time gauge, where $\bar{N} = 1$. In our language, where we use a generic time gauge, the last of the three relations can be viewed as a change in the lapse function while $t$ itself remains unchanged. In this case this implies that we may take

$$\bar{N} = sa^{\frac{2}{3}(w+1)(s-1)}N, \quad \bar{t} = t. \hspace{1cm} (III.2)$$

With this consideration the symmetry generator of the above transformation reads

$$\left(\frac{d\bar{t}}{ds}_{|s=1} \right) \frac{\partial}{\partial \bar{t}} + \left(\frac{d\bar{N}}{ds}_{|s=1} \right) \frac{\partial}{\partial \bar{N}} + \left(\frac{d\ddot{a}}{ds}_{|s=1} \right) \frac{\partial}{\partial a} + \left(\frac{d\ddot{\rho}}{ds}_{|s=1} \right) \frac{\partial}{\partial \rho} =$$

$$N(1 + \frac{3}{2}(w+1)\ln(a)) \frac{\partial}{\partial N} + a\ln(a) \frac{\partial}{\partial a} - 3(w+1)\rho \ln(a) \frac{\partial}{\partial \rho}. \hspace{1cm} (III.3)$$

But this is exactly the generator you obtain from $X_3$ when you set in (II.17), $P = w\rho, k = \Lambda = 0$ and $\xi(a, \rho) = a\ln(a)$.

We can see now how the symmetry, reported in [35] for the $k = 0$ case, can now be extended in a spatially curved universes. By considering $k \neq 0 \neq \Lambda$, from (II.17) we derive

$$X_3 = N \left[ 1 + \frac{3\ln(a)}{3k - a^2(\Lambda + 8\pi \rho)} \right] \frac{\partial}{\partial N} + a\ln(a) \frac{\partial}{\partial a} - 3(w+1)\rho \ln(a) \frac{\partial}{\partial \rho}, \hspace{1cm} (III.4)$$

which leads to the transformation

$$\ddot{a} = a^s, \quad \ddot{\rho} = a^{-3(w+1)(s-1)}\rho, \quad \bar{N} = \frac{sNa^{\frac{2}{3}(3w+5)-1} \sqrt{3k - a^2(\Lambda + 8\pi \rho)}}{\sqrt{3ka^{3(w+1)} - 8\pi \rho a^{2w+3}}}, \quad \bar{t} = t. \hspace{1cm} (III.4)$$

Alternatively if we want to consider just invariance starting from the time gauge $\bar{N} = 1$ and considering a time transformation, the above should be interpreted as

$$\ddot{a} = a^s, \quad \ddot{\rho} = a^{-3(w+1)(s-1)}\rho, \quad d\bar{t} = \frac{sa^{\frac{2}{3}(3w+5)-1} \sqrt{3k - a^2(\Lambda + 8\pi \rho)}}{\sqrt{3ka^{3(w+1)} - 8\pi \rho a^{2w+3}}} dt, \hspace{1cm} (III.5)$$

which is the generalization of (III.1) in the case of non zero spatial curvature and cosmological constant.

In [35] another symmetry property of the Friedmann equations in the spatially flat case and without cosmological constant is given. It involves an arbitrary function and thus is related to an infinite dimensional symmetry group. Specifically, it is stated that the transformation

$$\tilde{\rho} = \tilde{\rho}(\rho) \hspace{1cm} (III.6a)$$

$$\dot{H} = \left(\frac{\dot{\rho}}{\rho}\right)^{1/2} \dot{H} \hspace{1cm} (III.6b)$$

$$\tilde{P} = -\tilde{\rho} + \left(\frac{\tilde{\rho}}{\rho}\right)^{1/2} (\rho + P) \frac{d\tilde{\rho}}{d\rho}. \hspace{1cm} (III.6c)$$

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where $H = \frac{\dot{a}}{a}$ is the Hubble function in the gauge $N = 1$ leaves the corresponding equations

$$3H^2 - 8\pi \rho = 0 \quad \text{and} \quad \dot{\rho} + 3H(P + \rho) = 0 \quad \text{(III.7)}$$

invariant.

Surprisingly, this symmetry transformation does not have an exact equivalent in the study we did in the previous section and this is because the treatment of $H$ as a basic variable changes the formalism. But let us proceed to see in detail what happens and how the aforementioned symmetry can also be recovered.

The generator $X_4$ of (II.26) can induce a change in the pressure like the one implied by (II.6a) if we consider the particular case $k = 0 = \Lambda$ and $f = f(\rho)$; then,

$$X_4 = -\frac{N f(\rho)}{2\rho} \frac{\partial}{\partial N} + f(\rho) \frac{\partial}{\partial \rho} + [(P + \rho)f'(\rho) - f(\rho)] \frac{\partial}{\partial P}. \quad \text{(III.8)}$$

The infinitesimal change implied by the latter up to first order in the parameter of the transformation, say $\lambda$, is

$$\bar{\rho} \sim \rho + \lambda f(\rho) \quad \text{(III.9)}$$

$$\bar{N} \sim N - \lambda \frac{N f(\rho)}{2\rho} \quad \text{(III.10)}$$

$$\bar{P} \sim P + \lambda [(P + \rho)f'(\rho) - f(\rho)] \quad \text{(III.11)}$$

If we use (III.9) and (III.10) we may write

$$\bar{H} = \frac{\dot{\bar{a}}}{\bar{N}\bar{a}} \sim \frac{1}{N} \frac{\dot{a}}{N(1 - \lambda \frac{\dot{a}}{2\rho})} \sim \left(1 + \lambda \frac{f(\rho)}{2\rho}\right) H. \quad \text{(III.12)}$$

which is exactly the infinitesimal change emanating from (III.6b) for the $\bar{\rho}(\rho)$ given by (III.9), since

$$\bar{H} = \left(\frac{\mu}{\rho}\right)^{1/2} H \sim H \left(1 + \lambda \frac{f(\rho)}{2\rho}\right). \quad \text{(III.13)}$$

So the necessary transformation of $H$ that keeps invariant (II.4) is attributed to a change of the gauge variable $N$ (or equivalently, as we said before, it can also be seen as a change in $dt$). The situation however for the pressure is not the same: By introducing (III.9) into (III.6c) we obtain

$$\bar{P} = -\bar{\rho} + \left(\frac{\rho}{\rho}\right)^{1/2} (\rho + P) \frac{\text{d}\bar{\rho}}{\text{d}\rho} \sim P + \lambda [(P + \rho)f'(\rho) - f(\rho)] + \lambda(P + \rho)f'(\rho) \quad \text{(III.14)}$$

which is different from what we see in (III.11).

The reason for the above discrepancy rests in the difference of considering $H$ in place of $a$ and $N$, or equivalently $dt$, as the basic variable. The Hubble function in a generic time gauge is $H = \frac{\dot{a}}{N\dot{a}}$. Any transformation of $H$, when seen at the level of its components, can be attributed to $a$ or to $N$ and $dt$ interchangeably. In the case at hand, of (III.8), the corresponding change in $H$ is owed to a transformation in $N$, or equivalently the time in the $a$-formalism if we want to keep $N$ fixed (e.g. $N = 1$). For maximum compatibility, let us turn to equations (III.7) written in the gauge $N = 1$ and thus consider that the change in $H$ is owed to a transformation in the time variable. When you perform such a transformation, the $\dot{\rho}$ that we see in the second of (III.7) is also going to be transformed. On the other hand, if you forget that $H$ is made up by some specific components and treat it as the basic variable, you have no reason to enforce a transformation of the $\dot{\rho}$ in equation (III.7). Hence, the observed difference between the two situations is passed on to the pressure of each case that now needs to be different in order to keep the equations invariant during these two distinct scenarios. This is the essence of the “discrepancy” we see between (III.14) and (III.11).
As we did in the previous sections, we can proceed and derive the general symmetry generator whose special case is transformation (III.6) in the $H$-formalism of the Friedmann equations. For this to happen we need to consider a generator of the form

$$X = \chi(t, H, \rho, P) \frac{\partial}{\partial t} + \xi^1(t, H, \rho, P) \frac{\partial}{\partial H} + \xi^2(t, H, \rho, P) \frac{\partial}{\partial \rho} + \xi^3(t, H, \rho, P) \frac{\partial}{\partial P}$$  \hspace{1cm} (III.15)

which we require to be a symmetry of the equations

$$E^{H}_1 := -3H^2 + 8\pi\rho + \Lambda = 0$$  \hspace{1cm} (III.16)

$$E^{H}_2 := 3H(P + \rho) + \dot{\rho} = 0$$  \hspace{1cm} (III.17)

$$E^{H}_3 := -3H^2 - 2H + \Lambda - 8\pi P = 0.$$  \hspace{1cm} (III.18)

We additionally include the cosmological constant in our considerations.

Since the first equation is algebraic, we start from the last two demanding, i.e.

$$\text{pr}^1 X(E^{H}_3) = 0, \mod E^{H}_1 = 0, E^{H}_2 = 0,$$  \hspace{1cm} (III.19)

$$\text{pr}^1 X(E^{H}_3) = 0, \mod E^{H}_1 = 0, E^{H}_2 = 0.$$  \hspace{1cm} (III.20)

The resulting symmetry generator has the additional property $X(E_1) = 0$ and it can be split into two parts:

1. The first is related to the effect that transformations in the time variable produce to the rest of the quantities and it has the rather complicated form

$$X_6 = \chi \frac{\partial}{\partial t} - 4\pi \int (\rho + P) \frac{\partial \chi}{\partial P} dP \frac{\partial}{\partial H} - 3H \int (\rho + P) \frac{\partial \chi}{\partial P} dP \frac{\partial}{\partial \rho} + \left[ 3H \int (\rho + P) \frac{\partial \chi}{\partial P} dP \right]$$

$$- 4\pi (\rho + P) \int (\rho + P) \frac{\partial^2 \chi}{\partial P \partial t} dP + \int (\rho + P) \frac{\partial^2 \chi}{\partial P \partial H} dP - 3H(P + \rho) \int (\rho + P) \frac{\partial^2 \chi}{\partial P \partial H} dP$$

$$- (\rho + P) (\partial H + 3H\chi) + (\rho + P)^2 \left( 3H \frac{\partial \chi}{\partial \rho} + 4\pi \frac{\partial \chi}{\partial H} \right) \frac{\partial}{\partial P}.$$  \hspace{1cm} (III.21)

where $\chi = \chi(t, H, \rho, P)$ is an arbitrary function.

2. The second is the vector

$$X_7 = h(t, H, \rho) \frac{\partial}{\partial H} + \frac{3H}{4\pi} h(t, H, \rho) \frac{\partial}{\partial \rho} + \frac{1}{4\pi} \left[ (\rho + P) \left( 3H \frac{\partial h}{\partial \rho} + 4\pi \frac{\partial h}{\partial H} \right) - \frac{\partial h}{\partial t} - 3H h(t, H, \rho) \right] \frac{\partial}{\partial P},$$  \hspace{1cm} (III.22)

where $h(t, H, \rho)$ is an arbitrary function of its arguments.

The particular case of the symmetry transformation (III.6) arises when one considers $h(t, H, \rho) = \frac{4\pi f(\rho)}{3H}$ which yields

$$X_7 = \frac{4\pi f(\rho)}{3H} \frac{\partial}{\partial H} + f(\rho) \frac{\partial}{\partial \rho} + \left[ (P + \rho) f'(\rho) - f(\rho) - \frac{4\pi (P + \rho)}{3H^2} f(\rho) \right] \frac{\partial}{\partial P}. $$  \hspace{1cm} (III.23)

Now this generator is completely compatible with transformation (III.6) in the pressure, with the additional use of the $H^2 = \frac{8\pi\rho}{3}$, since the first was written in [35] for the $\Lambda = 0$ case. However, simply by mere application of $H^2 = \frac{8\pi\rho}{3} + \frac{\Lambda}{8\pi}$ in (III.23), the generalization of (III.6) in the $\Lambda \neq 0$ case can be derived. We thus see how using $H$ as the basic variable leads to different symmetry generators than when applying the theory with the equations expressed in terms of the scale factor.
IV Generation of new solutions from known ones

In this section, the generators \( X_4, X_5 \) will be used to connect models with different equations of state and hence different cosmological epochs. The method of generating new solutions from old ones \([39, 40]\) will be implemented. Let us outline the method. Suppose the following Lie point symmetry generator is known for some system of partial differential equations

\[
Z = \xi^i(x^j, u^\beta) \frac{\partial}{\partial x^i} + \eta^\alpha(x^j, u^\beta) \frac{\partial}{\partial u^\alpha},
\]  

with \( \xi^i(x^j, u^\beta), \eta^\alpha(x^j, u^\beta) \) known functions. The finite transformation can be obtained through the integral curves of the system

\[
\xi^i(\tilde{x}^j(\lambda), \tilde{u}^\beta(\lambda)) = \frac{d\tilde{x}^i}{d\lambda}, \quad \eta^\alpha(\tilde{x}^j(\lambda), \tilde{u}^\beta(\lambda)) = \frac{d\tilde{u}^\alpha}{d\lambda},
\]

with the initial conditions

\[
\tilde{x}^i(0) = x^i, \quad \tilde{u}^\alpha(0) = u^\alpha.
\]

The result will read in general

\[
\tilde{x}^i = \tilde{x}^i(x^j, u^\beta; \lambda), \quad \tilde{u}^\alpha = \tilde{u}^\alpha(x^j, u^\beta; \lambda),
\]

and since the transformation is invertible,

\[
x^i = x^i(\tilde{x}^j, \tilde{u}^\beta; \lambda), \quad u^\alpha = u^\alpha(\tilde{x}^j, \tilde{u}^\beta; \lambda),
\]

where \( \lambda \) the parameter.

Some simple special solution of the differential equation will have the form

\[
u^\alpha = f^\alpha(x^i), \text{ or } u^\alpha - f^\alpha(x^i) = 0 \iff F(u^\alpha, x^i) = 0.
\]

Hence, the solution is a surface described by \([IV.0]\) in the space with variables \((x^i, u^\alpha)\). Due to \([IV.5]\) we get

\[
F(u^\alpha, x^i) = 0 \quad \overset{[IV.0]}{\Rightarrow} \quad F[x^i(\tilde{x}^j, \tilde{u}^\beta; \lambda), u^\alpha(\tilde{x}^j, \tilde{u}^\beta; \lambda)] = 0 \quad \Rightarrow \quad \tilde{F}(\tilde{x}^j, \tilde{u}^\beta; \lambda) = 0.
\]

The surface \([IV.7]\) will represent a new form of solution and hopefully we will be able to bring it in the form

\[
\tilde{u}^\beta = \tilde{f}^\beta(\tilde{x}^j; \lambda),
\]

where \( \lambda \) will act as an integration constant.

When do we expect to find a different form of solution? The surface that represents the original solution should not be invariant under the action of the generator, that is

\[
Z[F(u^\alpha, x^i)] \neq 0.
\]

If the above does not hold, then we get the same form of solution in the variables \( \tilde{x}^j, \tilde{u}^\alpha \).
IV.1 Cosmological epochs

The starting point is the solution of the Einstein’s plus perfect fluid equations for flat FLRW \((k = 0)\), in the absence of cosmological constant \((\Lambda = 0)\). The assumed equation of state is \((P = w_1 \rho)\) with \(w_1 < -\frac{1}{3}\), corresponding to the inflation epoch. Furthermore, we are going to choose the gauge \((N = 1)\). To this end, the equations become

\[
8\pi \rho - 3 \left(\frac{\dot{a}}{a}\right)^2 = 0, \tag{IV.10}
\]

\[
3(1 + w_1) \frac{\dot{a}}{a} \rho + \dot{\rho} = 0, \tag{IV.11}
\]

with solutions

\[
P = w_1 \rho, \tag{IV.12}
\]

\[
N = 1, \tag{IV.13}
\]

\[
\rho = \rho_0 a^{-3(1+w_1)}, \tag{IV.14}
\]

\[
a(t) = \left[\frac{6\pi (1 + w_1)^2 \rho_0}{(1 + w_1)^{1/3(1+w_1)}}\right]^{1/2} \left[t^{3/2} e^{3(1+w_1)}\right]. \tag{IV.15}
\]

To achieve our goal we form the generator

\[
Z = X_4 + X_5. \tag{IV.16}
\]

The functions \(\sigma(a, \rho), f(a, \rho)\) are arbitrary, hence we can choose them properly. We demand the \(\partial_N\) component of the generator to be equal to zero in order to remain at the same gauge \(\tilde{N} = 1\), where \(\tilde{N}\) the transformed lapse function. This implies that the observers remain co-moving in the transformed system as well. Additionally, we choose the component \(\partial_\rho\) to be equal to \(\rho\), which also leads the component of \(\partial_P\) to be independent of the scale factor \(a\). Thus, the transformed equation of state will remain only a function of \(\tilde{\rho}, \tilde{P}\). These requirements reduce the form of the functions and the generator to

\[
\sigma(a, \rho) = \frac{a}{2} \ln a, f(a, \rho) = \rho, \tag{IV.17}
\]

\[
Z = \frac{a}{2} \ln a \partial_a + \rho \partial_\rho + \frac{P - \rho}{2} \partial_P. \tag{IV.18}
\]

The equations \(\text{(IV.2)}\) for the generator \(Z\) read

\[
\left.\frac{dt}{d\lambda}\right|_{\lambda=0} = 0, \quad \left.\frac{d\tilde{N}}{d\lambda}\right|_{\lambda=0} = 0, \quad \left.\frac{d\tilde{a}}{d\lambda}\right|_{\lambda=0} = \frac{a}{2} \ln a, \quad \left.\frac{d\tilde{\rho}}{d\lambda}\right|_{\lambda=0} = \tilde{\rho}, \quad \left.\frac{d\tilde{P}}{d\lambda}\right|_{\lambda=0} = \frac{\tilde{P} - \tilde{\rho}}{2}, \tag{IV.19}
\]

where \(\lambda\) the parameter of the curves, chosen so that at \(\lambda = 0\) the transformation is the identity. The transformation as well as the inverse are

\[
\tilde{t} = t, \quad \tilde{N} = N, \quad \tilde{a} = a e^{\lambda/2}, \quad \tilde{\rho} = e^\lambda \rho, \quad \tilde{P} = -e^\lambda \rho + e^{3/2} (P + \rho), \tag{IV.20}
\]

\[
t = \tilde{t}, \quad N = \tilde{N}, \quad a = \tilde{a} e^{-\lambda/2}, \quad \rho = e^{-\lambda} \tilde{\rho}, \quad P = e^{-\lambda} \left[-\tilde{\rho} + e^{3/2} \left(\tilde{P} + \tilde{\rho}\right)\right]. \tag{IV.21}
\]

All we have to do now is to use \(\text{IV.21}\) in \(\text{IV.12} - \text{IV.15}\) and solve with respect to the new variables

\[
\tilde{P} = \left(-1 + (1 + w_1) e^{-\lambda/2}\right) \tilde{\rho}, \tag{IV.22}
\]

\[
\tilde{N} = 1, \tag{IV.23}
\]

\[
\tilde{\rho} = e^\lambda \rho_0 \tilde{a}^{-3(1+w_1)} e^{-\lambda/2}, \tag{IV.24}
\]

\[
\tilde{a}(\tilde{t}) = \left[6\pi (1 + w_1)^2 \rho_0\right]^{1/3(1+w_1)} \left[t^{2/3} e^{3(1+w_1)}\right]. \tag{IV.25}
\]
By the choice $\lambda = 2 \ln \frac{3(1+w_1)}{4}$ and the redefinition $\rho_0 = \frac{16}{9(1+w_1)^2} \tilde{\rho}_0$, the inflation transforms to the radiation epoch solution.

\begin{align*}
\tilde{P} &= \frac{1}{3} \tilde{\rho}, \\
\tilde{N} &= 1, \\
\tilde{\rho} &= \tilde{\rho}_0 \tilde{a}^{-4}, \\
\tilde{a}(\tilde{t}) &= 2 \left( \frac{2\pi}{3} \tilde{\rho}_0 \right)^{1/4} \tilde{t}^{1/2}.
\end{align*}

(IV.26) (IV.27) (IV.28) (IV.29)

Now we can repeat the transformation (IV.21) with a different parameter $\lambda$ and as known solutions the (IV.26)-(IV.29). There exist a parameter $\lambda$ which can transform the radiation epoch solution into the dust epoch. In the following table we present the values of the parameters $\lambda$ and the corresponding epoch’s, as well as the equations of state.

| Transitions                  | $\lambda$ | Equations of state                |
|------------------------------|-----------|-----------------------------------|
| Inflation → Radiation,       | $2 \ln \frac{3(1+w_1)}{4}$ | $w_i < -\frac{1}{3} \rightarrow w_r = \frac{1}{3}$ |
| Radiation → Dust,            | $\ln \frac{16}{9}$ | $w_r = \frac{1}{3} \rightarrow w_d = 0$ |
| Dust → De Sitter,            | $\rightarrow \infty$ | $w_d = 0 \rightarrow w_{dS} = -1$ |

Table 1: Mappings between solutions for various values of $\lambda$.

As we can see, the last transition can be obtained only as a limit of $\lambda$.

IV.2 Stiff plus generalized (Anti-) Chaplygin fluid from stiff equation of state

Let us assume an equation of state describing stiff matter $P = \rho$ and $k = 0$ with arbitrary $\Lambda$. The solution is

\begin{align*}
\tilde{P} &= \tilde{\rho}, \\
\tilde{N} &= 1, \\
\tilde{\rho} &= \tilde{\rho}_0 \tilde{a}^{-6}, \\
\tilde{a}(\tilde{t}) &= \sqrt{2} \left( \frac{\pi \tilde{\rho}_0}{\Lambda} \right)^{1/6} \sinh^{1/3} \left( \sqrt{3\Lambda t} \right).
\end{align*}

(IV.30) (IV.31) (IV.32) (IV.33)

For this case, only the $X_4$ generator will be used, with the function $f(a, \rho)$ chosen to have the form

\begin{align*}
f(a, \rho) &= \rho^{-\nu}, \\
X_4 &= -\frac{4\pi N \rho^{-\nu}}{\Lambda + 8\pi \rho} \partial_N + \rho^{-\nu} \partial_\rho - \rho^{-1-\nu} [\nu P + (1 + \nu) \rho] \partial_P,
\end{align*}

(IV.34) (IV.35)

where $\nu$ some arbitrary constant. The transformed solutions read

\begin{align*}
\tilde{P} &= \tilde{\rho} - \frac{2\lambda(1 + \nu)}{\tilde{\rho}^\nu}, \\
\tilde{N} &= \frac{\sqrt{\Lambda + 8\pi \rho}}{\sqrt{\Lambda + 8\pi \left[ (1 + \nu) \Lambda + \tilde{\rho}^{1+\nu} \right]^{1+\nu}}}, \\
\tilde{\rho} &= \left[ (\rho_0 \tilde{a}^{-6})^{1+\nu} + (1 + \nu) \Lambda \right]^{1/(1+\nu)}, \\
\tilde{a} &= \sqrt{2} \left( \frac{\pi \tilde{\rho}_0}{\Lambda} \right)^{1/6} \sinh^{1/3} \left( \sqrt{3\Lambda t} \right).
\end{align*}

(IV.36) (IV.37) (IV.38) (IV.39)
From (IV.36) the generalized Anti-Chaplygin matter can be obtained if \( \lambda < 0 \) while the Chaplygin for \( \lambda > 0 \) (assuming of course for the power of \( \tilde{\rho} \) in the second term \( \nu > 0 \)). Notice however, that in this case the observers are not co-moving any more since there is a lapse function \( \tilde{N} \neq 1 \).

**IV.3 Scale-factor dependent equation of state from dust solution**

The only difference with the first subsection is to choose different functions \( \sigma(a, \rho), f(a, \rho) \) and our starting point is the dust solution corresponding to \( P = 0 \). Again, we want to maintain the lapse gauge but there is no other restriction. The functions and the generator are

\[
\sigma(a, \rho) = a^2, \quad f(a, \rho) = 2a\rho, \tag{IV.40}
\]

\[
Z = a^2 \partial_a + 2a\rho \partial_\rho + a \left( P - \frac{5}{3} \rho \right) \partial_P. \tag{IV.41}
\]

The induced transformation reads

\[
t = \tilde{t}, \quad N = \tilde{N}, \quad a = \frac{\tilde{a}}{1 + \lambda \tilde{a}}, \quad \rho = \frac{\tilde{\rho}}{(1 + \lambda \tilde{a})^2}, \quad P = \frac{3\tilde{P}(1 + \lambda \tilde{a}) + 5\lambda \tilde{a} \tilde{\rho}}{3(1 + \lambda \tilde{a})^2}, \tag{IV.42}
\]

and the new solution are

\[
\tilde{P} = -\frac{5\lambda \tilde{a}}{3(1 + \lambda \tilde{a})} \tilde{\rho}, \tag{IV.43}
\]

\[
\tilde{N} = 1, \tag{IV.44}
\]

\[
\tilde{\rho} = \frac{(1 + \lambda \tilde{a})^5}{\tilde{a}^3} \rho_0, \tag{IV.45}
\]

\[
\tilde{a}(\tilde{t}) = \frac{(6\pi \rho_0)^{1/3} 2^{1/3}}{1 - \lambda (6\pi \rho_0)^{1/3} 2^{1/3}}. \tag{IV.46}
\]

As we can observe from (IV.43) the equation of state depends on the scale factor. Furthermore, by use of (IV.46) and the redefinition of the parameter \( \lambda \) as follows \( \lambda = -\frac{3^{2/3}}{5(2\pi \rho_0)^{1/3} 2^{1/3}} w \) the equation of state may be written as

\[
\tilde{P} = w \tilde{t}^{2/3} \tilde{\rho}. \tag{IV.47}
\]

The equation of state is time dependent and this dependence is similar to the dependence of the original’s solution, scale factor.

The infinite symmetry freedom offers infinite possibilities. The ability to obtain different equations of state from the simplest original one \( (P = 0) \) is due to the assumption that \( \rho \) and \( P \) are independent variables.

**V Conclusion**

Symmetries of differential equations are of wide interest since they provide an effective method of simplifying and solving a given set of equations. Among other things, they can be used to reduce the order of the equations, find symmetry invariant solutions or utilize already known solutions to acquire new ones.

Given the continually rising interest in cosmological solutions, especially in what regards the nature of dark energy, we thoroughly studied the point symmetries of the Friedmann equations. Due to the fact that the system basically comprises of first order equations, namely (II.4) and (II.5), the resulting symmetry groups are infinite dimensional. In order to be as generic as possible we considered the equations in their original form, prior to assuming any gauge fixing condition.

We separately examined two main scenarios: In the first case a specific equation of state is given and substituted inside the equations of motion. As a result the emanating symmetry vector has the property
of leaving the equation of state invariant as well. The second, more intriguing, possibility involved taking the pressure as being a separate dynamical variable which is allowed to take part in the transformation. Performing such a change of variables can lead to a set of Friedmann equations corresponding to a fluid governed by a different equation of state than that of the initial system.

In the first scenario - where we assumed some ab initio given equation of state - we derived the general symmetry vectors corresponding to dependences of the form $P = P(a, \rho)$; plus an additional possibility where we considered dependence from some power of the first derivative of the scale factor $a$. To the above general forms of equations of state there corresponds a wide class of fluids that can be considered in various cosmological configurations.

However, the most interesting situation emerges when the pressure is allowed to be affected by the transformation. This leads to a distinct class of symmetry vectors which can be used to map solutions among models with different equations of state. Again infinite dimensional symmetry groups are involved, whose generators we derived. To demonstrate their possible applications, we used a transformation to “scan” through the solutions of the various epochs of the universe (see the example of section IV.1). We also started from a simple solution corresponding to a typical linear equation of state, like $P = \rho$, and showed how - through an appropriate symmetry transformation - it can be turned into the corresponding solution of a more complicated type of fluid of stiff matter plus Chaplygin gas (see section IV.2). We even used this approach to pass into a solution linked to a time dependent equation of state parameter (see example IV.3). Considering that the aforementioned symmetry groups are infinite dimensional there is an immensely large set of possibilities to be explored in this respect, where simple solutions can be used to derive those of more perplexing configurations.

In the case of spatially flat space-times, $k = 0$, where the equations can be formulated purely in terms of the Hubble parameter $H$, we observed that the symmetry generator changes when $H$ is to be considered as the basic variable of the transformation in place of the scale factor $a$. As we saw this genuinely results into distinct symmetry transformations.

To underline the generalization that our work offers, we compared with previously known results of symmetry transformations of the Friedmann equations. We showed how these known symmetries emerge as special cases from the generators we derived and how they can be extended to a wider class of cases (e.g. non-vanishing $k$ and/or $\Lambda$) since they are part of more general transformations.

Lastly, it is noteworthy to mention that even larger symmetry groups for the system under consideration can be achieved: If one allows $N$ to be simply defined by the quadratic constraint $E_2$, then the initial vector will not have a $\frac{\partial}{\partial N}$ component and it would only have to satisfy one symmetry condition, namely that for $E_1$. The application of this idea to this, as well as other cosmological systems, is currently under investigation.

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