Comparison of the results of solving the problem of nano-beams contact interaction according to different theories of size-dependent behavior

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Abstract. A mathematical model of the package for two nano Bernoulli-Euler beams connected through boundary conditions is constructed. Used gradient theory of elasticity, non-local theory of elasticity and the theory of the account of contact interaction of B. Ya. Kantor. The influence of the length dimension parameters on the modes of beam vibrations is studied.

1. Introduction
Micro- and nano-beams, plates, and shells are widely applied in micro- and nano-electromechanical systems such as vibration sensors [1], micro-cables [2], and micro-switches [3]. Dependence of elastic behaviour on the body dimensions in the micro scale has been experimentally observed in metals [4, 5] and alloys [6] as well as polymers [7] and crystals [8], biomembranes [9].

The classical mechanics of a rigid body does allow neither for interpretation nor prediction of the micron- and submicron-scale behaviour due to lack of a parameter controlling the scale effects. In recent years, many efforts have been devoted to propose numerous theories allowing for modelling the scale effects in continua, including the couple stress theory [10, 11], non-local theory of elasticity [12], gradient theory of elasticity [13] as well as theory of surface elasticity [14].

Beams and beam designs are widely used as elements of numerous micro devices in modern engineering. In many cases, the structural elements of the devices are under the influence of complex external dynamic excitations. Investigations of non-linear dynamics and contact interactions of the beam structures belong to important (but unsolved) challenging problems in the field of fabricating various sensors and amplifiers.

2. Problem formulation
The studied mechanical structure consists of two beams occupying a 2D part of the space $\mathbb{R}^2$ with the rectangular system of coordinates introduced in the following way: in the body of beams the so-called reference line $z = 0$ is fixed, the axis $OX$ goes from left to right along the middle beams line, whereas the axis $OZ$ tends down and is perpendicular to $OX$. In the given system of coordinates the mentioned structure is defined as the 2D space: $\Omega = \left\{ x \in [0, a], -\frac{h}{2} \leq z \leq h + \frac{3h}{2} \right\}, 0 \leq t < \infty$ (figure 1).

The imposed fundamental hypotheses and assumptions are as follows: each beam consists of one layer; beams are isotropic, elastic and they obey the Hooke’s law; a beam longitudinal dimension
essentially over creases its transverse dimension and its unit thickness; the undeformed axis is a straight line; during the deformation of the beam, load acts along the OZ axis and the external forces do not change their directions; contact pressure is within the Kantor model [15]; normal stresses on the surfaces are parallel to each other and they are negligibly small.

To account for the contact between the beams, a Winkler coupling between the compression and the contact pressure between the two beams is used [15]:

$$\Psi = \frac{1}{2} \left[ 1 + \text{sign} \left( w_1 - h_k - w_2 \right) \right] \quad (1)$$

where $\Psi = 1$, if $w_1 > w_2 + h_k$ that is, there is contact between the plate and the beam, else $\Psi = 0$, $w_1, w_2, h_k$ - deflections of the first and second beams and the gap between them, respectively.

![Figure 1. The computation scheme of two beams with a clearance.](image1)

2.1. The Bernoulli-Euler model

Recall that the hypotheses for beams were proposed by J. Bernoulli [16] and L. Euler [17], whereas for the plates – by G. Kirchhoff [18], and for the shells – by Love [19]. Their main features are as follows: lines, which were straight and perpendicular to the midline (beams) or midsurface (plates and shells) before deformation, remain straight and perpendicular to the midline/midsurface after deformation and their length is also conserved. Bernoulli-Euler model (figure 2) considers only the beam deflection (without rotation and warping of the transverse cross section).

![Figure 2. Kinematic parameters and loadings of the Bernoulli-Euler beam.](image2)
2.2. Non-local theory of elasticity

The Eringen’s nonlocal elasticity involves spatial integrals which represent weighted averages of the contributions of strain tensors of all points in the body to the stress tensor at the given point while that of constitutive equation of classical elasticity is an algebraic relationship between the stress and strain tensors. Integral-partial differential Eringen’s constitutive equations can be converted to equivalent differential constitutive equations under certain conditions, because is difficult mathematically to obtain the solution of nonlocal elasticity problems due to the spatial integrals in Eringen’s constitutive equations. The simplified nonlocal constitutive equation for the normal stress and strain in a one-dimensional case is given by [4]

\[ \sigma_{xx} - l^2 \frac{d^2 \sigma_{xx}}{dx^2} = E \epsilon_{xx} \]  

(2)

where \( \sigma_{xx} \) is the normal stress, \( \epsilon_{xx} \) the normal strain, \( E \) the Young’s modulus and \( l^2 = (\epsilon_0 a)^2 \) the scale coefficient that incorporates the small scale effect, where \( \epsilon_0 \) - material constant, and \( a \) - internal length scale. The equations of the mathematical model are obtained using the Hamilton-Ostrogradsky principle. The mathematical model of contact between two nano-beams, based on the kinematic Bernoulli-Euler hypothesis, is described by a system of resolving equations:

\[ \frac{\partial^3 w_m}{\partial x^3} \left( \lambda + 2\mu \right) \frac{bh^3}{12} - l^2 \frac{\partial^6 w_m}{\partial x^6} - q_m + (-1)^m K (w_1 - w_2 - \tilde{h}) \Psi = \frac{\partial^3 w}{\partial t^3} + \epsilon \frac{\partial w}{\partial t}, \]  

(3)

where \( m \) – beam number (\( m=1,2 \)), \( \tilde{h} \) - the gap between the beams.

Boundary conditions are:

\[ w_m = 0; \quad \frac{\partial w_m}{\partial x} = 0; \quad \frac{\partial^3 w_m}{\partial x^3} = 0 \]  

(4)

Initial conditions are:

\[ w_m(x,0) = 0; \quad \frac{\partial w_m(x,0)}{\partial t} = 0 \]  

(5)

2.3. Gradient theory of elasticity

Based on the higher order stress theory of Mindlin [20] and Lam et al. [7] proposed the theory of elasticity of the deformation gradient, in which, in addition to the classical equations of equilibrium of forces and moments, a new additional equilibrium equation is introduced, which determines the behavior of stresses of higher orders and the equation of equilibrium of moments. Three parameters of the material length scale are introduced for isotropic linear elastic materials \( (l_0, l_1, l_2) \). According to this theory, the total strain energy density is a function of the symmetric strain tensor, the dilation gradient vector, the deviator tension gradient tensor, and the symmetric rotation gradient tensor.

The mathematical model of contact between two nano-beams, based on the kinematic Bernoulli-Euler hypothesis, is described by a system of resolving equations:
\[
\frac{\partial^4 \omega_m}{\partial x^4} \left( (\lambda + 2\mu) \frac{bh^3}{12} + tl^2 \mu bh + l^2 \omega^2 bh + \frac{96}{225} l^2 \mu bh \right) - \\
- \frac{\partial^4 \omega_m}{\partial x^4} \left( 2l^2 \mu bh \frac{bh^3}{12} + \frac{14}{225} l^2 \omega^2 bh \right) - q_m + \\
+ (\text{-1})^n K (w_1 - w_2 - \tilde{h}) \frac{d^2 \omega}{dt^2} + \varepsilon \frac{d \omega}{dt},
\]

Boundary conditions and initial conditions are (4), (5).

3. Results

For the considered tasks the following parameter values were used: \( a / h = 30, \ q_l = 10 \sin(5.3t), \ q_2 = 0, \ \varepsilon = 0.1, \ \tilde{h} = 0.01 \)

The research results for system (3) and (6) are shown in figures 3,4. For the first case (figure 3), coefficient \( l = 0.6 \). The second case (figure 4) is characterized by the values \( l_0 = 0.3; l_I = 0.3; l_2 = 0.3, \) i.e. all three dimensional factors are taken into account. The figures in the first row show the deflections of the first \( w_1 \) and second beams \( w_2 - a \). The second line shows the phase synchronization of oscillations \( \phi_1 - \phi_2 - b \), marked in dark color. The third and fourth lines show the Fourier spectrum \( S(\omega) - c, g \); the wavelet spectrum based on the Morlet wavelet \( -i, d \); phase portrait \(-e, i\); Poincaré section \(-f, j\) for the first and second beams, respectively.

Consider the case where both beams are described by nonlocal theory. The signal before time \( t \approx 300 \) has a complex asymmetric character, after that the time values of the beam oscillations begin to have the properties of mirror symmetry and the process of symmetry of the signals is completely completed at \( t \approx 450 \). At the same time, synchronization at frequency \( \omega \approx 2 \) disappears. The Fourier spectrum of both beams indicates the randomness of the system. However, the wavelet shows the inhomogeneity of the frequency spectrum over time. It should be noted that Fourier spectra and wavelet spectra are qualitatively similar. The phase portraits are mirror symmetrical along the axis of the signal. Poincaré sections for the first and second beams differ in location but have qualitative similarities.

Consider the case when both beams are described by the theory of the deformation gradient taking into account the values \( l_0 = 0.3; l_I = 0.3; l_2 = 0.3 \). The deflection(signal) graphs differ significantly from the previous case. Synchronicity of oscillations is present at the interval up to \( t \approx 200 \). The phase synchronization of the graph shows that at this time interval synchronization occurs at two frequencies: at the excitation frequency \( \omega_p = 5.3 \) and at the frequency \( \omega \approx 2 \), as in the previous case. Synchronization at the excitation frequency is uniform over the entire time interval, but unlike the previous case, the phase synchronization is weaker. The Fourier spectrum for both beams is qualitatively similar as for the previous case, but there are differences. The number of frequencies with the highest energy (peaks on the spectrum) is much larger and their distribution is significantly different from the previous case. Thus, the energy of the contact interaction of two nano-beams differs significantly from the beams according to the classical theory and therefore it is necessary to use the theories of size-dependent behavior in the study of the contact of NEMS elements. The wavelet spectrum also has a significant difference from the previous case. For the first beam, both the excitation frequency and all frequencies of the range are present at the entire time interval \( \omega \in (0; 3) \). The frequencies of the remaining ranges appear at the initial time \( t \in (0; 230) \) and at the end of the time interval of the contact interaction studies. Phase portraits of both beams in the same way as for the previous case of a "mirrored", but qualitatively different. They also differ quantitatively in the values of the rate of oscillations \( \dot{\omega} \).
Figure 3. Characteristics of dynamic contact interaction of beams

Concluding remarks. The paper presents a mathematical model of contact interaction of nano-beams. The model is based on the kinematic hypothesis of Bernoulli – Euler and two elasticity higher order theories. The study of the influence of the values of the dimensional parameters of the length of the material on the nature of vibrations. The results show that the oscillation characteristics during the contact interaction of two nano-beams differ significantly for different theory.

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**Figure 4.** Characteristics of dynamic contact interaction of beams

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