Measuring lifetime of correspondence with classical decay of correlation in quantum chaos

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Abstract – A very weakly coupled linear oscillator is proposed as a detector for observing time-irreversible characteristics of a quantum system, and it is used to measure the lifetime during which a classically chaotic quantum system shows decay of correlation. Except for a particular case where the lifetime agrees with the conventional Heisenberg time, which is proportional to the Hilbert space dimension \(N\), it is in general much longer: the lifetime increases in proportion to the product of \(N\) and the number of superposed eigenstates, and is proportional to \(N^2\) in the case of full superposition.

A classical chaotic system can be the simplest origin of irreversibility [1]. Even a low-dimensional classical system exhibits mixing in a fully chaotic state, which means the loss of memory in the sense of decay of autocorrelation. Its quantum counterpart could be the minimal quantal unit exhibiting loss of memory and time-irreversibility. Indeed, normal diffusion [2,3], energy dissipation [4], energy spreading [5] and many apparent features of irreversible phenomena can be realized in classically chaotic quantum systems with a small number of degrees of freedom.

The appearance of the loss of memory in isolated quantum systems is also a severe problem limiting the performance of quantum computation algorithms [6]. If the chaotic region is bounded in the phase space and thus the effective dimension \(N_{\text{dim}}\) of the subspace of the Hilbert space relevant for the chaotic region is finite, the decay of correlation in a quantum system can be observed only on a finite time scale. Even if the system is unbounded and \(N_{\text{dim}}\) is infinite, persistent coherence and localization properties inherent in a quantum system may prevent the system from the complete decay of correlation [2,3]. However, the multidimensional unbounded quantum chaos systems can mimic classical chaotic dynamics [7] and recover classical normal diffusion implying the complete decay of correlation (the Anderson transition) [8,9].

Irreversibility in quantum system has been explored directly by numerical time-reversal experiments [10–12]. In particular, it has been extensively investigated by many investigators in the context of fidelity [13,14]. The time scale on which quantum chaos can show exponential sensitivity is quite short and is up to the Ehrenfest time proportional to \(\log(N_{\text{dim}})\) at the most [15], but the time scale on which the decay of correlation is observed is much longer and is said to be as long as the Heisenberg time, which is proportional to \(N_{\text{dim}}\) [14].

The main purposes of the present article are first to propose a general method for observing the characteristics of time-irreversibility related to the mixing and the decay of correlation in isolated quantum systems, and next, to apply the method to classically chaotic quantum systems by measuring the maximal time beyond which classical decay of correlation can no longer be observed. This time is the lifetime during which quantum systems can mimic classical time-irreversibility. We hereafter call it the lifetime of correspondence with classical decay of correlation, or simply lifetime.

Even in classical dynamics, the decay of correlation in chaotic systems is realized only for ideally chaotic systems...
such as a C-system or a K-system. We thus limit our consideration to quantum systems which show an ideal Markovian chaotic behavior in the classical limit. Even for such a limited situation, the lifetime introduced above has not been investigated quantitively, in particular, for systems which are finitely bounded in the phase space and so cannot exhibit explicit diffusive behavior.

The fidelity method is a powerful tool extracting directly time-irreversible characteristics of a quantum system [14]. However, the fidelity method is not convenient for the purpose of observing irreversible characteristics with adequate numerical accuracy over an extremely long time scale without disturbing the dynamics of the examined system.

We introduce a linear oscillator as a detector for observing time-irreversible characteristics of a quantum system. If the object system is classically chaotic, the oscillator “converts” the motion of the object system to a Brownian motion in the homogeneously and infinitely extended linear oscillator’s action space. Let the Hamiltonian of the motion in the homogeneously and infinitely extended linear oscillator’s action space is

\[ H(t) = \int \bar{X}(\theta, q, t) + e^{i\theta_{0}}(p, q) g(\theta) + \omega \bar{J}. \]

(1)

where \( \bar{v} \) is a Hermitian operator, and \( g(\theta) \) is a 2\( \pi \)-periodic function with null average [16], which is usually taken to be

\[ g(\theta) = \cos(\theta). \]

\( \bar{J} \) has the eigenvalue \( J = jh \) (\( j \in \mathbb{Z} \)) for the eigenstate \( \langle \theta | J \rangle = e^{-i\theta_{0}j/h} \) because of the 2\( \pi \)-periodicity in \( \theta \)-space. Technically it is convenient to impose the action periodic boundary condition identifying \( J = Lh \) with \( J = -Lh \), which quantizes \( \theta \) as \( \theta_{k} = 2\pi k/(2L) \) (\( k \in \mathbb{Z} \)), where \( -L < k \leq L \), and so \( \langle \theta_{k} | J \rangle = e^{-i\theta_{0}k/h} \sqrt{2L} \). From the Heisenberg equation of motion it immediately follows that \( \bar{J}(t) = \bar{J} - \eta \int_{s}^{t} \sin(\omega s + \theta) \bar{v}(p(s), q(s))ds \). Here we mainly consider the case of \( \omega \neq 0 \) in order to remove the DC component of \( \bar{v}(p(s), q(s)) \) which exists in general and hinders irreversibility-related information, but the particular case \( \omega = 0 \) is also important and it is also discussed later. If the fluctuating part of \( \bar{v} \) has a stationary autocorrelation with vanishing tail, \( \bar{J}(t) \) exhibits a Brownian motion in the action space. An ideal example of a dynamical system realizing the above feature is a classical chaotic system such as a C-system or a K-system, which have the ideal Markovian property. For such systems a stationary normal diffusion is realized in the \( J \)-space as a typical irreversible behavior.

To be concrete, we hereafter confine ourselves to the case that the system \( \bar{H}(\bar{p}, \bar{q}, t) \) is a kicked rotor (KR) whose classical version exhibits ideal chaos with the Markovian property. However, a one-dimensional quantum kicked rotor cannot mimic classical chaotic motion because of the quantum interference effect [8] and cannot be an ideal example. We therefore take as the object system \( S \) the coupled kicked rotor (CKR) with the Hamiltonian

\[ H(\vec{p}, \vec{q}, t) = \sum_{n=1}^{N} \left[ \frac{\vec{p}_{n}^{2}}{2} + V(\vec{q}_{n}) \right] + cV_{12}(\vec{q}_{1}, \vec{q}_{2})\Delta(t), \]

as a sample system \( S \), where \( \Delta(t) = \sum_{\ell}(\delta(t-\ell T)) \) represents the \( \delta \)-functional kick with the period \( T \). It can mimic well the classical chaotic motion [7], which is induced by the entanglement between the two constituent systems [17]. The CKR is observed at the integer multiple of the fundamental period \( T \) as \( t = \tau T \) where \( \tau \in \mathbb{Z} \). Then the one step evolution of the CKR from \( t(=\tau T + 0) \) to \( t + T \) is described by the unitary operator

\[ \hat{U} = e^{-i(V(\vec{q}_{1}) + V(\vec{q}_{2}) + cV_{12}(\vec{q}_{1}, \vec{q}_{2}))/\hbar} e^{-i(\vec{p}_{1}^{2} + \vec{p}_{2}^{2})T/2\hbar}. \]

(2)

Let the Heisenberg representation of the operator \( \hat{X} \) be \( \hat{U}^{-\tau} \hat{X} \hat{U}^{\tau} \), then the time evolution by a single operation of \( \hat{U} \) yields the mapping rule

\[ \hat{q}(\tau + 1) = \hat{q}(\tau) + \hat{p}(\tau) T, \]

(3)

\[ \hat{p}(\tau + 1) = \hat{p}(\tau) - \left[ \frac{\partial}{\partial \hat{q}} V(\hat{q}) + V_{12}(\hat{q}) \right] \bigg|_{\hat{q} = \hat{q}(\tau + 1)}. \]

(4)

Each KR, say KR1 and KR2, is defined in the bounded phase space \( \{q_{1}, p_{1}\} \in [0, 2\pi] \times [0, 2\pi] \) and \( h = 2\pi/N \), where \( N_{1} = N_{2} \) is a positive integer, and the dimension of the Hilbert space for the CKR is \( N = N_{1}N_{2} = N_{1}^{2} \). This is the most important parameter which controls the lifetime, as will be shown later on. Since our system is bounded, it cannot exhibit diffusive motion. But a diffusive motion may be realized in the additional degree of freedom of \( L \). As for the specific form of \( V \) we take mainly Arnold’s cat map \( V(\vec{q}) = -K_{c} q_{2}^{2} / 2 \) and also the standard map \( V(\vec{q}) = K \cos q_{1} \) (defined in the bounded phase space \( [0, 2\pi] \times [0, 2\pi] \)) with the interaction \( V_{12}(\vec{q}_{1}, \vec{q}_{2}) = \cos q_{1} - q_{2} \). Hereafter, we couple \( L \) with \( S \) via only the first KR (KR1) as \( \bar{v}(\bar{p}, \bar{q}) = \bar{v}(\bar{p}_{1}) = \sin(p_{1}) \) so that the entanglement between the KRMs may sensitively be reflected in the dynamics of \( L \), and take the initial states of \( L \) and \( S \) as \( |\theta, J = 0\rangle \) and \( |\Psi_{0}\rangle \), respectively.

Coupling \( L \) with \( S \) by the scheme of the Hamiltonian (1), the Heisenberg equation of motion for the canonical pair operator \( (\theta, J) \) yields the mapping rule for \( \hat{J}(\tau) \) as

\[ \hat{J}(\tau + 1) - \hat{J}(\tau) = 2\eta \sin(\omega T/2) \sin(\omega T\tau + \theta + \omega T/2) \tilde{v}_{\tau}, \]

(5)

where \( \tilde{v}_{\tau} = \hat{U}^{-\tau} \bar{v}(\bar{p}, \bar{q}) \hat{U}^{\tau} \). Recall that \( \tilde{v}_{\tau} \) varies only when the kick is applied at \( t = \tau T \), and in our case \( \tilde{v}_{\tau} = \sin(\tilde{p}_{\tau}(\tau)) \), and \( \bar{v}(\tau) = \theta + \omega T \). We reduce the coupling strength \( \eta \) weak enough such that the backaction from \( L \) to \( S \) is negligible, then the motion of \( \bar{v}(\tau) \) and \( \bar{q}(\tau) \) is independent of \( L \). The bracket \( \langle \ldots \rangle \) denotes the expectation value with respect to the initial condition \( |0\rangle = |\Psi_{0}\rangle \otimes |J = 0\rangle \). Then the mean square displacement (MSD) of action of \( L \), namely \( \Delta J^{2}(\tau) = \langle (\hat{J}^{2}(\tau) - \langle \hat{J}(\tau) \rangle)^{2} \rangle \)

is equal to \( \langle J^{2}(\tau) \rangle - \langle \hat{J}(0) \rangle = 0 \) (note that the RHS of eq. (5) vanishes by the summation over \( \theta \), and
that $\Delta J^2(\tau) = \langle (\hat{J}(\tau) - \hat{J}(0))^2 \rangle$. Therefore, the MSD can be expressed by using eq. (5) recursively to lead to

$$\Delta J^2(\tau) = \sum_{s=0}^{\tau-1} D_{\omega}(s),$$

where $D = 2\eta^2 \sin^2(\omega T/2)/\omega^2$ and $C_{\tau}(s) = \langle \langle \Psi_0 | \hat{c}_\tau \hat{c}_{\tau-s} | \Psi_0 \rangle \rangle + c.c./2$. (7)

The autocorrelation function satisfies $C_{\tau}(s) = C_{\tau}(s).$ Scaled by $\eta^2$ the result does not depend on $\eta$ if it is taken sufficiently small.

Equation (6) works in both quantum and classical dynamics. The MSD is the sum of $D_{\omega}(\tau)$, which is nothing more than the finite-time Fourier component of the autocorrelation function. For classical KR having the ideally chaotic property the autocorrelation function decays because of the Markovian nature of the classical chaotic dynamics, which is the indicator of chaotic irreversibility. The decay of the autocorrelation function makes the finite-time Fourier component $D_{\omega}(\tau)$ converge to the classical diffusion constant $D_{\omega}(\omega)$ as $\tau \rightarrow \infty$. The quantum motion can follow its classical counterpart at least in the initial stage, if we prepare the ensemble of classical initial points such that its probability distribution agrees with the quantum probability distribution of the initial state $|\Psi_0\rangle$. Under such situation, the quantum autocorrelation function decays following its classical counterpart temporally and $D_{\omega}(\tau)$ finally goes down to 0 on time average, because the quantum autocorrelation function is the sum of the finite number, i.e., $N$, of the trigonometric function. This means that $\Delta J^2$ saturates at a finite value $\Delta J^2_{cl}$ as $\tau \rightarrow \infty$. $D_{\omega}(\tau)$ deviates from the classical value $D_{\omega}(\omega)$ and eventually approaches toward 0. Therefore, the decay of the autocorrelation function mimicking the classical irreversibility disappears at a characteristic finite time which we define as the lifetime $\tau_L$.

In practice we define the lifetime $\tau_L$ as the time beyond which the stationarity of diffusion of $\hat{L}$ is lost: We introduce the temporal diffusion exponent $\alpha(\tau)$ at $\tau$ such that $\Delta J^2(s) \propto s^{\alpha(\tau)}$ for the appropriate interval of $s$ in the logarithmic scale. We define $\tau_L$ as the first time step at which $\alpha(\tau_L) < 1$, where the threshold value $r(1 > r)$ is chosen as follows: the diffusion exponent fluctuates in time around 1 from 0 to 2; $\alpha = 0$ implies the tendency toward the saturation, while $\alpha = 2$ means the tendency to exhibit a ballistic motion due to the temporal quantum resonance. Therefore, it is reasonable to choose $r = 0.5$.

The lifetime thus defined in general varies very widely with the choice of $|\Psi_0\rangle$. In order to eliminate such accidental fluctuations, we add a classically negligible small term such as $\xi_{IR}\cos(\hat{q}_I - \hat{q}_{IR})$ of $\xi_{IR} \sim O(\hbar)$ to the potential $V(\hat{q}_I)$, and take the average of $\tau_L$ with respect to the ensemble of the phase parameter $\theta (i = 1, 2)$ (see footnote 1). We refer to it hereafter as the average lifetime $\tau_L$ (see footnote 2).

Figure 1(a) shows a typical example of $\Delta J^2$ vs. $\tau$ at various values of $\epsilon$ increased very slightly from 0. These examples are obtained for the initial state $|\Psi_0\rangle = |p_1 = N_1/2\rangle \otimes |p_2 = N_2/2\rangle$, which is composed of all the eigenstates of the evolution operator $\hat{U}$ with almost equal weights and random phases. The classical result is shown by a black line exhibiting an ideal diffusion with almost the same diffusion constant. The quantum motions can follow the classical normal diffusion only within a finite time scale. They all deviate from the classical diffusion line and tend to saturate at finite levels. Note that the time at which the deviation from classical normal diffusion occurs increases with $\epsilon$. Indeed, as is shown in fig. 1(b), the averaged lifetime increases drastically as $\epsilon$ exceeds the classically negligible characteristic value $\epsilon_c$ which is proportional to $\hbar^2$. Figure 1 implies that $\epsilon_c$ is the threshold above which quantum-classical correspondence is well realized and the lifetime is maximally enhanced.

This situation corresponds to the recovery of classical chaotic diffusion in the coupled quantum standard map with unbounded momentum space [7]. We note that the lifetime ($\sim \Omega^2$) for $N = 64 \times 64$ greatly exceeds the Heisenberg time ($\sim N$). We are interested in the nature of the lifetime in such a situation that the quantum-classical correspondence is well achieved.

An advantage of our method is that it enables us to observe the irreversibility characteristics even for eigenstates of $\hat{U}$ which are invariant in time except for the

\footnote{Such additional potential term does not change classical chaotic dynamics any more, but the lifetime of the quantum counterpart fluctuates widely with the value of $\theta (i = 1, 2)$. Therefore its average is significant.}

\footnote{It might seem adequate to take an average over an ensemble of $\omega$. But we do not take such an average here, because the particular case $\omega = 0$ is demonstrated. The averaging with respect to $\omega$ results in a similar effect as the averaging over $\theta (i = 1, 2)$.}
phase. Examining the characteristics of eigenstates, we encounter a remarkable and nontrivial phenomenon. Let us construct the initial state $|\Psi_0\rangle$ by superposing $M$ ($\leq N$) eigenstates as $|\Psi_0\rangle = \sum_m^M |C_m| |m\rangle$ where $|m\rangle$ is the eigenstate of $\hat{U}$ and $C_m \sim 1/\sqrt{M}$. We show in fig. 2(a) examples of time evolution starting from such an initial state with increasing $M$. Apparently the lifetime at which the $\Delta J^2$ deviates from classical diffusion increases with $M$, although the variation is not systematic, implying a large fluctuation of $\tau_L$. To confirm the above observation we computed the average lifetime $\langle \tau_L \rangle$ and show in fig. 2(b) how it varies with $M$. The lifetime $\langle \tau_L \rangle$ is proportional to $N$ if $M = 1$, and it increases in proportion to $M$, which means $\langle \tau_L \rangle \propto N^2$ in the limit of full superposition $M = N$. Such a behavior is observed irrespective of the frequency $\omega$ if $\omega \neq 0$.

On the contrary, for the very particular choice of $\omega = 0$, the $M$-dependence of the averaged lifetime is very weak as is shown in fig. 2(c), which means that $\langle \tau_L \rangle \propto N$ for the full superposition $M = N$. The above results are hardly expected and they mean that there is a basic difference between the case of the Fourier component of quantal autocorrelation function at $\omega \neq 0$ and that of the Fourier component at $\omega = 0$. We therefore consider the reason closely. To this end we evaluate the saturation level $\Delta J^2_\infty$ of MSD by averaging eq. (6) over $\tau$. With the use of the eigenstate $|m\rangle$ and its eigenangle $\gamma_m$ satisfying $\hat{U}|m\rangle = e^{-i\gamma_m} |m\rangle$, a straightforward calculation yields

$$\frac{\Delta J^2_\infty}{\bar{D}} = \sum_{m=1}^M |C_m|^2 \sum_{n=1}^N |\langle m|\hat{v}|n\rangle|^2 = \frac{1}{4} \left[ \frac{1}{\sin^2(\frac{\delta_{mn}}{2})} + \frac{1}{\sin^2(\frac{\gamma_m + \gamma_n}{2})} \right],$$

where $\delta_{mn} = \gamma_m - \gamma_n \pm \omega T \mod 2\pi$. Here the interference terms between different $m$’s are neglected (the diagonal approximation) because their contribution is negligibly small in the regime of concern.

First, we consider the very particular case of $\omega = 0$ in order to compare with the general case discussed later. In our setting the diagonal element with respect to the eigenfunction vanishes because $|\langle p|n\rangle|^2 = |\langle -p|n\rangle|^2$, and thus the diffusion process is free from the ballistic explosion beyond the Heisenberg time, which is well known in the study of fidelity [14]. Now we begin with evaluating the matrix elements. In the maximal entanglement regime $\epsilon > \epsilon_c$, all the eigenfunctions lose their identity, and so the matrix elements $|\langle m|\hat{v}|n\rangle|^2$ between any pair of $|m\rangle$ and $|n\rangle$ are almost the same, which allows us to approximate them by a single parameter, say $\langle \langle |v|^2 | \rangle \rangle$, while the expression for the autocorrelation function at $s = 0$ is given by the relation $C_r(0) = \sum_m |C_m|^2 \sum_n |\langle m|\hat{v}|n\rangle|^2 \sim N \langle \langle |v|^2 | \rangle \rangle$ (the subscript $r$ of $\tau$ can now be omitted), and so

$$\langle \langle |v|^2 | \rangle \rangle \sim C_r(0)/N. \hspace{1cm} (9)$$

Next we consider the most important resonance factor $1/(4\sin^2(\delta_{mn}/2)) \sim 1/|\gamma_m - \gamma_n|^2$. Let us fix $m$, then in the summation over $n$, according to the Wigner surmise, the most dominant contribution comes from the nearest-neighboring levels on each side, which have the average distance $\sim 2\pi/N$. Therefore, the contribution from the resonance factor of each term is approximated by $4/(4\sin^2\pi/N) \sim N^2/\pi^2$ irrespective of $m$. With this and eq. (9) we evaluate the saturation level as $\Delta J^2_\infty \sim DC_r(0)/N\pi^2$. The lifetime is the time at which the classical diffusion $D_{\omega,cl}^{(cl)}\tau_L$ is suppressed by the finite saturation level, namely, $D_{\omega,cl}^{(cl)}\tau_L = \Delta J^2_\infty$, which leads to

$$\tau_L \sim \frac{C_r(0)N}{\pi^2 A_{\tilde{\omega}}^{(cl)}} \hspace{1cm} (10)$$

independent of $M$, where $A_{\tilde{\omega}}^{(cl)} = D_{\omega,cl}^{(cl)}/D = \sum_{s=-\infty}^{\infty} C_r(s)\cos(\omega Ts)$ is the Fourier component of the classical autocorrelation function. This shows fairly good agreement with the numerical result in fig. 2(c). This time scale almost coincides with the so-called Heisenberg time.

However, in the general case of $\omega \neq 0$ the lifetime is enhanced much more than in the case of $\omega = 0$. If $\omega \neq 0$ the statistical distribution of $|\delta_{mn}| = |\gamma_m - \gamma_n \pm \omega T \mod 2\pi|$ does not suffer from the restriction of level repulsion in

Fig. 2: (Colour online) (a) $\Delta J^2$ vs. $\tau$ for various $M$ in the regime above the threshold, where $\epsilon = 1.0$, $N = 32 \times 32$. (b) $\langle \langle \tau_L \rangle \rangle$ vs. $M$ for $\omega = \sqrt{2}$, where $N = 16 \times 16$, $32 \times 32$, $64 \times 64$ denoted by symbols. Theoretical results of eq. (11) are shown by lines without markers. (c) $\langle \langle \tau_L \rangle \rangle$ vs. $M$ for $\omega = 0$. Results of eq. (10) are indicated by lines without markers.
the vicinity of $|\delta_{mn}^\pm| \sim 0$. Hence, unlike the case of $\omega = 0$, $|\delta_{mn}^\pm|$ can be arbitrarily small, which will make the term $1/\sin^2(\delta_{mn}^+/2)$ larger. As $M$ increases, the chance to encounter smaller $|\delta_{mn}^\pm|$ increases. In the summation over $n$ in the RHS of eq. (8) the term with the smallest $|\delta_{mn}^\pm|$ and $2\pi - |\delta_{mn}^\pm|$ will be the most dominant. We evaluate the most dominant contribution. Firstly, we fix $m$, and we take the hypothesis that $w_n = |\delta_{mn}^\pm|$ ($1 \leq n \leq N$ and $n \neq m$) (and also $w_n = |\delta_{mn}^\pm|$) are independent stochastic variables uniformly distributed over the range $[0, 2\pi]$. The independency is a rather bold hypothesis considering that the level repulsion exists if $\delta_{mn}^\pm \sim \delta_{mn}^\pm$, but its effect is quite limited.

Under the above hypothesis, the probability that a $w = w$ is the minimum in the set $\{w_1, w_2, \ldots, w_N, 2\pi - w_1, 2\pi - w_2, \ldots, 2\pi - w_N\}$ is the probability that $w_l = w$ is in $0 \leq w \leq \pi$ and simultaneously all $w_j$ ($j \neq i$) satisfy $w \leq w_j \leq 2\pi - w$ which is

$$\frac{1}{(2\pi)^N} \prod_{j \neq i} \int_{w_0}^{2\pi-w} dw_j = \frac{(2\pi - 2w)^{N-1}}{(2\pi)^N}$$

for $0 \leq w \leq \pi$, which is equal to the probability of $2\pi - w_i$ being minimum and taking the value $w$. Thus the probability that the minimum value $\min\{w_1, w_2, \ldots, w_N, 2\pi - w_1, 2\pi - w_2, \ldots, 2\pi - w_N\}$ takes $w$ is $p(w) = 2N(2\pi - 2w)^{N-1}/(2\pi)^N$ which asymptotically approaches $Ne^{-Nw/\pi}$ in the limit of $N \to \infty$. Secondly, we have to take the summation over $m$. At this second stage it is quite plausible to suppose that $w_m$ is now a statistically independent variable; then after a similar evaluation as the above it follows that the probability of $\min\{w_1, \ldots, w_M\}$ taking a value $w$ is

$$P(w) = Mp(w) \left[ \int_w^\infty p(w')dw' \right]^{M-1} = \frac{MN}{\pi} e^{-MN/w}.$$ 

Thus the average of the minimal $|\delta_{mn}^\pm|$ is $\pi/MN$ and the most dominant term of $1/\sin^2(\delta_{mn}^+/2)$ in the RHS of eq. (8) is $(MN/\pi)^2$. Then by using eq. (9) and $|C_m|^2 \sim 1/M$ we may evaluate $\Delta J_2^M = 2DCr(0)MN/\pi^2$. The lifetime is decided by $D_0^{(2)} = \Delta J_2^M$ as before to yield

$$\tau_L \sim \frac{2DCr(0)MN}{\pi^2 A_0^{(2)}}.$$ 

This is our final result and it agrees well with the numerical results as depicted in fig. 2(b). In the short limit of correlation time i.e., $Cr(s) = \delta_0 Cr(0)$, eq. (11) becomes very simply $\tau_L \sim 2MN/\pi^2$, independent of the details of the system. We have confirmed that all the results discussed above are valid also for the coupled quantum standard maps. We here note that the lifetimes represented by eqs. (10) and (11) contain only parameters of the object system except for the frequency $\omega$. We can expect that these are general features of ideally chaotic quantum maps which are bounded in a finite region of phase space.

In conclusion, we proposed a method to observe a characteristic of time-irreversibility, namely the decay of correlation over an extremely long time scale, in quantum systems and in the associated quantum states. Applying it to coupled kicked rotors, the lifetime of correspondence with classical irreversible behavior is measured in the full entanglement regime, and it is found to be proportional to the number of eigenstates composing the examined state. In the case of full superposition of all eigenstates, it is proportional to the square of the Hilbert space dimension and is much longer than the conventional Heisenberg time, which comes from the basic difference in the asymptotic behavior of finite-time Fourier components of quantum autocorrelation at non-zero frequency compared to the component at zero frequency. This lifetime, in principle, should be observable directly in time-reversal fidelity experiments with an extremely weak perturbation (and so a very high precision is required for numerical computation).

So far, the localization length has been taken as the order parameter of the transition to irreversible diffusion in infinitely extended quantum chaos systems such as the standard map. However, it cannot be used in finitely bounded quantum systems. Our perspective is to use the lifetime measured by our method for the study of transition phenomena related to the time-irreversibility in bounded quantum systems [18] as demonstrated in fig. 1, combining with the finite-size scaling analysis of the critical behavior.

We finally comment that the linear oscillator $L$ can be replaced by a two-level system without any essential modifications, which means that experimental implementation could be possible with an optical lattice.

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REFERENCES

[1] PRIGOGINE I., From Being to Becoming: Time and Complexity in the Physical Sciences (Freeman, San Francisco) 1980.
[2] CASATI G. et al., Stochastic Behavior in Classical and Quantum Hamiltonian Systems, edited by CASATI G. and FORD J., Lect. Notes Phys., Vol. 93 (Springer-Verlag, Berlin) 1979, p. 334.
[3] FISHER S. et al., Phys. Rev. Lett., 49 (1982) 509.
[4] IKEDA K., Ann. Phys., 227 (1993) 1.
[5] COHEN D., Phys. Rev. Lett., 82 (1999) 4951; Ann. Phys., 283 (2000) 175.
[6] NIelsen M. A. and CHUANG I. L., Quantum Computation and Quantum Information (Cambridge University Press) 2001; MIQUEL C. et al., Phys. Rev. Lett., 78 (1997)
[7] Adachi S. et al., Phys. Rev. Lett., 61 (1988) 659; Gadway B. et al., Phys. Rev. Lett., 110 (2013) 190401.
[8] Casati G. et al., Phys. Rev. Lett., 62 (1989) 343.
[9] Lemarie G. et al., EPL, 87 (2009) 37007; Lopez M. et al., Phys. Rev. Lett., 108 (2012) 095701.
[10] Ikeda K., Quantum Chaos, edited by Casati G. and Chirikov B. V. (Cambridge University Press) 1996, p. 145; Yamada H. S. and Ikeda K. S., Phys. Rev. E, 82 (2010) 060102(R).
[11] Ballentine L. E. and Zibin J. P., Phys. Rev. A, 54 (1996) 3813.
[12] Benenti G. and Casati G., Phys. Rev. E, 79 (2009) 025201(R).
[13] Peres A., Phys. Rev. A, 30 (1984) 1610.
[14] Gorin T. et al., Phys. Rep., 435 (2006) 33; Jacquod Ph. and Petitjean C., Adv. Phys., 58 (2009) 67.
[15] Berman G. P. and Zaslavsky G. M., Physica A, 91 (1978) 450.
[16] Shepelyansky D. L., Physica D, 8 (1983) 208.
[17] Lakshminarayan A., Phys. Rev. E, 64 (2001) 036207; Fujisaki H. et al., Phys. Rev. E, 67 (2003) 066201; Demkowicz-Dobrzański R. and Kus M., Phys. Rev. E, 70 (2004) 066216.
[18] Matsui F. et al., unpublished.