The Physical Role of Gravitational and Gauge Degrees of Freedom in General Relativity - I: Dynamical Synchronization and Generalized Inertial Effects.

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Abstract

This is the first of a couple of papers in which the peculiar capabilities of the Hamiltonian approach to general relativity are exploited to get both new results concerning specific technical issues, and new insights about old foundational problems of the theory. The first paper includes:

1) a critical analysis of the various concepts of symmetry related to the Einstein-Hilbert Lagrangian viewpoint on the one hand, and to the Hamiltonian viewpoint, on the other. This analysis leads, in particular, to a re-interpretation of active diffeomorphisms as passive and metric-dependent dynamical symmetries of Einstein’s equations, a re-interpretation which enables to disclose the (up to now unknown) connection of a subgroup of them to Hamiltonian gauge transformations on-shell;

2) a re-visitation of the canonical reduction of the ADM formulation of general relativity, with particular emphasis on the geometro-dynamical effects of the gauge-fixing procedure, which amounts to the definition of a global (non-inertial) space-time laboratory. This analysis discloses the peculiar dynamical nature that the traditional definition of distant simultaneity and clock-synchronization assume in general relativity, as well as the gauge relatedness of the”conventions” which generalize the classical Einstein’s convention.

3) a clarification of the physical role of Dirac and gauge variables, as their being related to tidal-like and generalized inertial effects, respectively. This clarification is mainly due to the fact that, unlike the standard formulations of the equivalence principle, the Hamiltonian formalism allows to define a generalized notion of ”force” in general relativity in a natural way;

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I. INTRODUCTION.

This is the first of a couple of papers in which we aim to show the peculiar capability of the Hamiltonian ADM formulation of metric gravity to grasp a series of conceptual and technical problems that appear to have not been directly discussed so far. Some of such problems, although possibly not of primary importance for the working relativist, are deeply rooted into the foundational level of the theory and seems particularly worth of clarification in connection with the alternative programs os string theory and loop quantum gravity. Some other problems are in fact new problems which give rise to interesting new solutions about general issues. Our two papers should be read in sequence, since the first contains various technical premises for the second. One of the main foundational issues we want to revisit in the second paper (hereafter referred to as II) is the well-known Hole Argument or Lochbetrachtung, raised by Einstein in 1915-1916 [1] and, after two years of struggle, dismissed by him mainly on pragmatic grounds. The deep conceptual content of the argument has been rebirth by a seminal paper by Stachel [2], and essentially seized since then by the philosophers of science. On the other hand, in the physical literature, the Hole Argument has been bypassed by the recognition that a 4-geometry does not correspond to a single tensor solution of Einstein’s equation but rather to a whole equivalence class of solutions in a definite sense (see, e.g., [3]). We believe, however, that the problem deserves further investigation even from a physicist’s point of view and in paper II we shall show indeed that there is still some beef to bite around the issue.

Previous partial accounts of the material of this and the following paper can be found in Refs. [4, 5, 6].

The first reason we have to adopt the Hamiltonian approach to general relativity is that all of the problems we are interested in are deeply entangled with the initial value problem of the theory. On the other hand, we do believe that the constrained ADM methodology is just the only proper way to analyze all the relevant aspects of such a problem. This is no surprise, after all, and it is not by chance that the modern treatment of the initial value problem within the Lagrangian configurational approach [7] must in fact mimic the Hamiltonian methods (see more in Section II).

Second, in the context of the Hamiltonian formalism, we can exploit the nearly unknown Bergmann and Komar’s theory of general coordinate group symmetries [8]. This help us in clarifying the various concepts of symmetry related to the Einstein-Hilbert Lagrangian viewpoint, on the one hand, and to the Hamiltonian viewpoint, on the other. In particular, it enables us to show that active diffeomorphisms, as dynamical symmetries of Einstein’s equations, admit a subgroup which can be interpreted in a passive way as the Legendre pull-back of Hamiltonian gauge transformations on shell. This is the first relevant outcome that will also turn out to be a crucial premise for the discussions given in II.

Third, only in the Hamiltonian approach can we isolate the gauge variables, which carry the descriptive arbitrariness of the theory, from the Dirac observables (DO), which are gauge invariant quantities carrying the intrinsic degrees of freedom of the gravitational field, and are subjected to hyperbolic (and therefore ”determinate”, or ”causal” in the customary sense) evolution equations. The superiority of the Hamiltonian approach is essentially due to the fact that it allows working off shell, i.e., avoiding immediate transition to the space of solutions of Einstein’s equations.

All of our results are obtained by working within a class of space-times of the Christodoulou-Klainermann type [9], which are globally hyperbolic space-times asymptot-
ically flat at spatial infinity, enjoying some other interesting properties. Such space-times can be foliated in Cauchy 3-hypersurfaces $\Sigma_\tau$ (where $\tau$ plays the role of parameter time) which play also the role of simultaneity surfaces and are the basic starting point of the ADM canonical formulation. These surfaces are mathematically described by an embedding $x^\mu = z^\mu(\tau, \vec{\sigma})$ ($\vec{\sigma}$ arbitrary 3-coordinates adapted to the $\Sigma_\tau$ surfaces). Once the embedding is given, one can evaluate the unit normals and the extrinsic curvature of $\Sigma_\tau$, and two specific congruences of time-like observers. The first, defined by the field of unit normals, is a surface-forming congruence; the second, defined by the field of $\tau$-gradients of the embedding functions, is in general a rotating congruence, viz. a non-surface-forming one. Starting from this mathematical background, the ADM formulation is realized by a multilevel circular procedure which, bringing to the solution of the Einstein-Hamilton equations in terms of 4 initial data for the DO on a given $\Sigma_{\tau_0}$, backfires to a dynamical identification of the initial chrono-geometrical 3+1 setting.

The procedure starts with the Hamiltonian transcription of Einstein’s equations in terms of 20 canonical variables, functions of the components of the 4-metric and their derivatives and adapted to the 3+1 splitting. Note, incidentally, that unlike such canonical variables, the initial embedding functions $x^\mu = z^\mu(\tau, \vec{\sigma})$ stay as external elements of the game until the canonical procedure reaches its aim with the solution of Einstein’s equations.

Since the original Einstein’s equations are not hyperbolic, it turns out that the canonical variables are not all functionally independent, but satisfy eight constraints, given as functions of the canonical variables that vanish on a 12-dimensional constraint surface (not a phase space!) to which the physically meaningful states are restricted. When used as generators of canonical transformations, the eight constraints map points on the constraint surface to points on the same surface; these transformations are known as Hamiltonian gauge transformations. If, following Dirac, we make the reasonable demand that the evolution of all physical variables be unique, then - barring subtler complications - the points of the constraint surface lying on the same gauge orbit, i.e. linked by gauge transformations, must describe the same physical state. Conversely, only the functions in phase space that are invariant with respect to gauge transformations can describe physical quantities. To eliminate this ambiguity and create a one-to-one mapping between points in the phase space and physical states, further constraints must be imposed, known as gauge conditions or gauge-fixings. The number of independent gauge-fixing must be equal to the number of independent constraints, i.e. 8. Such gauge-fixings can be implemented by arbitrary functions of the canonical variables, except that they must define a 4-dimensional reduced phase space that intersects each gauge orbit exactly once (orbit conditions) and is coordinatized by the above mentioned Dirac observables (DO). Technically, this coordinatization is carried through by the so-called Shanmugadhasan transformation[10] which (though almost implicitly) ends with the construction of a new array of 20 canonical variables in which the 4 canonically conjugate DO are separated from the eight (Abelianized) constraints and their conjugated variables$^1$. These latter are precisely the eight gauge variables that parametrize the gauge orbits generated by the constraints. The gauge-fixing of the gauge variables, together with the enforcement of the eight constraints, reduce the 20-dimensional phase space to the 4-dimensional phase space of the intrinsic degrees of freedom of the theory.

The analysis of the canonical reduction and of the geometro-dynamical meaning of the

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$^1$ As a matter of fact, things are subtler: see Section IIC
gauge fixings is instrumental to clarify an important issue. Actually, a complete gauge fixing, has the following implications: i) it removes all the gauge arbitrariness of the theory by determining the functional form of all the gauge variables in terms of the Dirac observables, which, at this stage of the procedure, are 4 arbitrary fields of \((\tau, \vec{\sigma})\); ii) given the geometrodynamical meaning of the gauge variables and of their variations (see Section IID), the determination of their functional form in terms of DO entails the implicit fixation of all the elements characterizing the 3+1 splitting of space-time, in particular: a) the form and 4-dimensional packing of the Cauchy surfaces, together with a standard of (mathematical) local proper time; b) the choice of the 3-coordinates on the Cauchy surfaces; c) the determination of the two congruences of time-like observers; and d) on-shell (i.e., on the solutions of Einstein’s equations) the unique fixation of a 4-coordinate system.

In physical terms this set of choices amount eventually to individuate a network of intertwined and synchronized local laboratories made up with test matter (obviously up to a coherent choice of chrono-geometric standards). We shall call such network a global (non-inertial) space-time laboratory. This interpretation shows that, unlike in ordinary gauge theories where the gauge variables are inessential degrees of freedom, the concept of reduced phase space is very abstract and not directly useful in general relativity: it is nothing else than the space of gravitational equivalence classes each of which is described by the set of all laboratory networks living in a gauge orbit.

These effects of the gauge-fixing procedure entail in turn a physically interesting consequence which typically characterizes the canonical description of metric gravity. Actually, once the complete gauge fixing has determined the functional form of the gauge variables in terms of DO, we are eventually left with the problem of solving the Einstein equations for the DO themselves, in terms of their initial values, on some Cauchy surface \(\Sigma_\tau\). It is only this fundamental step that brings to its end the whole ADM construction, for the solution determines in particular the extrinsic curvature of the surfaces \(\Sigma_\tau\), which, in its turn, can make explicit the embedding functions \(x^\mu = z^\mu(\tau, \vec{\sigma})\). This fixes, as it were, explicitly the space-time universe corresponding to the given initial values of the DO, including the definition of simultaneity, distant clock synchronization and gravito-magnetism.

It is important to stress, therefore, that the complete determination of the chronogeometry depends upon the solution of Einstein-Hamilton equations of motion i.e., once the Hamiltonian formalism is fixed by the gauge choices, upon the initial conditions for the Dirac observables. This implies that the admissible notions of distant simultaneity turn out to be dynamically determined. Every solution of Einstein equations with a given set of admissible initial data admits as many dynamical simultaneity notions as admissible on-shell 3+1 splittings of space-time. On-shell, each such splitting defines the synchronization of clocks in the family of complete Hamiltonian gauges differing only in the choice of the 3-coordinates on the simultaneity leaves and in the implied choice of the shift functions (namely in the gravito-magnetic properties) as shown in Subsection IID. These dynamically determined simultaneity notions are much less in number than those admissible in special relativity, where such notions are non-dynamical due to the absolute chrono-geometrical structure of Minkowski space-time. The upshot, however, is that, in canonical metric gravity in analogy to what happens in a non-dynamical way within the framework of parametrized Minkowski theories (see Ref.[11, 12] and Appendix A of Ref.[13]), different admissible conventions about distant simultaneity within the same universe are merely gauge-related conventions, corresponding to different complete gauge options. We believe that this result throws an interesting new light even on the old - and outdated - debate about the so-called conven-
tionality of distant simultaneity in special relativity, showing the trading of conventionality with gauge freedom. It is clear that this trading owes its consistency to the complete Hamiltonian gauge mechanism based on the 3+1 splitting of space-time. Of course, it rests to be shown how the above dynamical determination can be enforced in practice to synchronize actual clocks, i.e., essentially, how to generalize to the gravity case the formal structure of Einstein-Reichenbach’s convention. This discussion is given in all details in Ref.[12] for the case of special relativity and can easily be extended to general relativity.

Finally, the separation carried out by the Shanmugadhasan transformation (conjoined with the circumstance that the Hamiltonian point of view brings naturally to a re-reading of geometrical features in terms of the traditional concept of force), leads to a third result of our investigation which, again, would be extremely difficult to characterize within the Lagrangian viewpoint at the level of the Hilbert action or Einstein’s equations. This result, concerning the overall physical role of gravitational and gauge degrees of freedom, is something that should be added to the traditional wisdom of the equivalence principle asserting the local impossibility of distinguishing gravitational from inertial effects. Actually, the isolation of the gauge arbitrariness from the true intrinsic degrees of freedom of the gravitational field is instrumental to understand and visualize which aspects of the local effects, showing themselves, e.g., on test matter, have a genuine gravitational origin and which aspects depend solely upon the choice of the reference frame and could therefore even be named inertial in analogy with their non-relativistic Newtonian counterparts. Indeed, two main differences characterize the issue of inertial effects in general relativity with respect to the non-relativistic situation: the existence of autonomous degrees of freedom of the gravitational field independently of the presence of matter sources, on the one hand, and the local nature of any single general-relativistic reference system, on the other. We shall show that, although the very definition of inertial forces (and of gravitational force in general) appears to be rather arbitrary in general relativity, it seems natural to characterize first of all as genuine gravitational effects those which are directly correlated to the DO, while the gauge variables appear to be correlated to the general relativistic counterparts of Newtonian inertial effects. Another aspect of the Hamiltonian connection “gauge variables - inertial effects” is related to the 3+1 splitting of space-time mentioned above. Since a variation of the gauge variables modifies the foliation and thereby the identification of the global (non-inertial) space-time laboratory, a variation of gauge variables also modifies the generalized inertial effects that manifest themselves locally.

The only weakness of the analysis leading to the physical characterization of tidal-like and generalized inertial effects is that the separation of the two autonomous degrees of freedom of the gravitational field from the gauge variables is, as yet, a gauge (i.e. coordinate) - dependent concept. The known examples of pairs of conjugate DO are neither invariant under passive diffeomorphisms (PDIQ, i.e., coordinate-independent) nor tensors. In view of clarifying this point, in paper II we will discuss the relation between the notion of DO and that of the so-called Bergmann observables (BO)[14] which are defined (although rather ambiguously) to be, again, as uniquely predictable from the initial data, but also invariant under standard passive diffeomorphisms (PDIQ).

A possible starting point to attack the problem of the connection of DO with BO seems to be a Hamiltonian re-formulation of the Newman-Penrose formalism [15] (that contains only PDIQ) employing Hamiltonian null-tetrads carried by the surface-forming congruence of time-like observers. In view of this program, in paper II we will argue in favor of a main conjecture according to which special Darboux bases for canonical gravity should exist in which
the inertial effects (gauge variables) are described by PDIQ while the autonomous degrees of freedom (DO) are also BO. The hoped for validity of this conjecture, besides amounting in particular to state the internal consistency of Bergmann’s *multiple definition* (which is not fully evident as it stands), would render our distinction about *generalized inertial* and *tidal-like* effects an invariant statement, giving a remarkable contribution to the old-standing debate about the equivalence principle. Note in addition that, since the Newman-Penrose PDIQ are tetradic quantities, the validity of the conjecture would eliminate the existing difference between the observables for the gravitational field and the observables for matter, usually built by means of tetrads associated to some time-like observer. Furthermore, this would also provide a starting point for defining a metrology in general relativity in a generally covariant way\(^2\), replacing the empirical metrology \([16]\) used till now. Finally, it would also enable to replace the *test matter* of the axiomatic approach to measurement theory (see Appendix A of paper II) with dynamical matter.

The plan of the paper is the following. In Section II the Einstein-Hilbert Lagrangian viewpoint and the related local symmetries are summarized. Particular emphasis is given to the analysis of the most general group \(Q\) of dynamical symmetries of Einstein’s equations (Bergmann-Komar group), and the *passive view* of *active* diffeomorphisms is clarified. Finally, some remarks are given about the issue of the choice of coordinate systems and its relation to the *Lagrangian gauge fixings*. The ADM Hamiltonian viewpoint and its related canonical local symmetries are synthetically expounded in Section III. Building on the acquired knowledge about the structure of \(Q\), particular emphasis is given to a discussion of the general Hamiltonian gauge group and to the correspondence between *active diffeomorphisms* and *on-shell gauge transformations*. Furthermore, the analysis of the chrono-geometrical meaning of a complete gauge fixing and the particularities of the closure of the ADM construction are related to the issue of the *dynamical nature* of the *conventions* about distant simultaneity and gravito-magnetism in general relativity. As shown in Section IV, the results obtained in Section III about the canonical reduction lead naturally to the physical interpretation of the DO and the gauge variables as characterizing *tidal-like* and *inertial-like* effects, respectively. The up to now gauge-dependent status of this distinction is stressed at the end, as well as the possibility of further clarification of the issue to be discussed in paper II, together all concluding remarks. Finally, Appendix A contains a miscellanea of properties of the accelerated observers, extracted from various scattered sources.

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\(^2\) Recall that this is the main conceptual difference from the non-dynamical metrology of special relativity
II. THE EINSTEIN-HILBERT LAGRANGIAN VIEWPOINT AND THE RELATED LOCAL SYMMETRIES.

The basic assertion of the general covariance of general relativity amounts to the statement that Einstein’s equations have a tensor character. This is a statement of symmetry with many facets.

A) Local Noether Symmetries of the Einstein-Hilbert action.

Given a pseudo-Riemannian 4-dimensional manifold $M^4$ with its maximal coordinate atlas, the Einstein-Hilbert action for pure gravity without matter

$$S_H = \int d^4x \mathcal{L}(x) = \int d^4x \sqrt{4g} \, 4R,$$  \hspace{1cm} \text{(2.1)}

defines a variational principle for the metric 2-tensor over $M^4$ whose components, in the coordinate chart $x^\mu$, are $4g_{\mu\nu}(x)$. The associated Euler-Lagrange equations are Einstein’s equations

$$4G_{\mu\nu}(x) \overset{\text{def}}{=} R_{\mu\nu}(x) - \frac{1}{2} 4R(x) 4g_{\mu\nu}(x) = 0.$$ \hspace{1cm} \text{(2.2)}

As well known, the action (2.1) is invariant under general coordinate transformations (the passive diffeomorphisms $\text{Diff} M^4$), which are a subset of local Noether symmetries (second Noether theorem) of the action. This has the consequence that:

i) Einstein’s equations are form invariant under general coordinate transformations;

ii) the Lagrangian density $\mathcal{L}(x)$ is singular, namely its Hessian matrix has vanishing determinant.

This in turn entails that:

i) four of the ten Einstein equations are Lagrangian constraints, namely restrictions on the Cauchy data;

ii) four combinations of Einstein’s equations and their gradients vanish identically (contracted Bianchi identities).

In conclusion, there are only two dynamical second-order equations depending on the accelerations of the metric tensor. As a consequence, the ten components $4g_{\mu\nu}(x)$ of the metric tensor are functionals of two ”deterministic” dynamical degrees of freedom and eight further degrees of freedom which are left completely undetermined by Einstein’s equations even once the Lagrangian constraints are satisfied. This state of affairs makes the treatment of both the Cauchy problem of the non-hyperbolic system of Einstein’s equations and the definition of observables within the Lagrangian context [7] extremely complicated.

In modern terminology, general covariance is interpreted as the statement that a physical solution of Einstein’s equations properly corresponds to a 4-geometry, namely the equivalence class of all the 4-metric tensors, solutions of the equations, written in all possible 4-coordinate systems. This equivalence class is usually represented by the quotient $4\text{Geom} = 4\text{Riem}/\text{Diff} M^4$, where $4\text{Riem}$ denotes the space of metric tensors solutions.
of Einstein’s equations. Then, any two inequivalent Einstein space-times are different 4-geometries.

B) Invariance of Einstein’s Equations under active diffeomorphisms.

Let us recall the basic underlying mathematical concept of active diffeomorphism and its consequent action on the tensor fields defined on the differentiable manifold \( M^4 \) (see for instance Ref.[3]). Consider a (geometrical or active) diffeomorphism \( D_A \) which maps points of \( M^4 \) to points of \( M^4 \): \( D_A : p \rightarrow p' = D_A \cdot p \), and its tangent map \( D_A^* \) which maps tensor fields \( T \mapsto D_A^* \cdot T \) in such a way that \([T](p) \rightarrow [D_A^* \cdot T](p) \equiv [T'](p)\). Then \([D_A^* \cdot T](p) = [T](D_A^{-1} \cdot p)\). It is seen that the transformed tensor field \( D_A^* \cdot T \) is a new tensor field whose components in general will have at \( p \) values that are different from those of the components of \( T \). On the other hand, the components of \( D_A^* \cdot T \) have at \( p' \) - by construction - the same values that the components of the original tensor field \( T \) have at \( p \): \( T'(D_A \cdot p) = T(p) \) or \( T'(p) = T(D_A^{-1} \cdot p) \). The new tensor field \( D_A^* \cdot T \) is called the drag-along of \( T \). For later use it is convenient to recall that there is another, non-geometrical - so-called dual - way of looking at the active diffeomorphisms. This duality is based on the circumstance that in each region of \( M^4 \) covered by two or more charts there is a one-to-one correspondence between an active diffeomorphism and a specific coordinate transformation. The coordinate transformation \( T_{D_A} : x(p) \rightarrow x'(p) = [T_{D_A}x](p) \) which is dual to the active diffeomorphism \( D_A \) is defined such that \([T_{D_A}x](D_A \cdot p) = x(p)\). In its essence, this duality transfers the functional dependence of the new tensor field in the new coordinate system to the old system of coordinates. By analogy, the coordinates of the new system \([x']\) are said to have been dragged-along with the active diffeomorphism \( D_A \). It is important to note here, however, that the above dual view of active diffeomorphisms, as particular coordinate-transformations, is defined only implicitly (see more below).

In abstract coordinate-independent language, Einstein’s equations (2.2) can be written as \( G = 0 \), where \( G \) is the Einstein 2-tensor \( G_{\mu \nu} = \Gamma^\sigma_{\mu \nu}dx^\sigma \otimes dx^\nu \) in the coordinate chart \( x^\mu \). Under an active diffeomorphism \( D_A : M^4 \mapsto M^4 \), \( D_A \in _{A}Diff M^4 \), we get \( G = 0 \mapsto D_A^* G = 0 \) (\( D_A^* G \) is the drag-along or push-forward of \( G \)), which shows that active diffeomorphisms are symmetries of the tensor Einstein’s equations. \(^3\)

C) Dynamical symmetries of Einstein’s partial differential equations (PDE)

Einstein’s equations, considered as a set of partial differential equations in a given coordinate chart, conjoined with a choice of a function space for the solutions, have their own passive dynamical symmetries [17] which only partially overlap with the local Noether symmetries. Let us stress that:

i) A dynamical symmetry is defined only on the space of solutions of the equations of motion, namely it is an on-shell concept. As a consequence, the very definition of dynamical symmetries entails the study of the integrability of the equations of motion. In

\[^3\] Note that a subset of active diffeomorphisms are the conformal isometries, i.e. those conformal transformations which are also active diffeomorphisms, namely \( ^4g = \Omega^2 \, ^4g \equiv \phi^* \, ^4g \) for some \( \phi \in _ADiff M^4 \) with \( \Omega \) strictly positive. Since the Hilbert action is not invariant under the conformal transformations which are not ordinary isometries (i.e. conformal isometries with \( \Omega = 1 \) for which \( \mathcal{L}_X \, ^4g = 0 \), if \( X \) is the associated Killing vector field), only these latter are Noether dynamical symmetries.
particular, in the case of completely Liouville - integrable systems dynamical symmetries are re-interpretable as maps of the space of Cauchy data onto itself. Let us stress that in gauge theories, and especially in Einstein’s theory, the space of Cauchy data is partitioned in gauge-equivalent classes of data: all of the Cauchy data in a given class identify a single Einstein space-time (or 4-geometry). The dynamical symmetries of Einstein’s equations follow therefore in two classes: a) those mapping inequivalent Einstein space-times among themselves, and b) those acting within a single Einstein space-time mapping gauge-equivalent Cauchy data among themselves (actually, they are on shell gauge transformations).

i) Only a subset of such symmetries (called Noether dynamical symmetries) can be extended off-shell in the variational treatment of the action principle.

The passive diffeomorphisms \( pDiff M^4 \) are just an instantiation of Noether dynamical symmetries of Einstein’s equations.

Let us observe that in the physical literature on field theory one is mainly concerned with the natural Noether symmetries of the Hilbert action, i.e. with passive diffeomorphisms. On the other hand, according to Stachel [2], it is just the dynamical symmetry nature of passive diffeomorphisms that expresses the real physically relevant content of general covariance. This dualism active-passive has been a continuous source of confusion and ambiguity in the literature, which we would like to clarify presently.

Let us look preliminarily at some implications of points A) and B). Choose a reference coordinate chart \( x^\mu \), where the metric components are \( ^4 g_{\mu\nu}(x) \). Every passive diffeomorphism defines a new system of coordinates \( x^\mu \mapsto x'^\mu = f^\mu(x) \) [with inverse \( x^\mu \mapsto x^\mu = h^\mu(x') \)] where the new form of the metric components is given by the standard tensorial transformation rule

\[
^4 g'_{\mu\nu}(x'(x)) = \frac{\partial h^\alpha(x')}{\partial x'^\mu} \frac{\partial h^\beta(x')}{\partial x'^\nu}^4 g_{\alpha\beta}(x). \tag{2.3}
\]

On the contrary, an active diffeomorphism \( D_A p \mapsto p' \) defines both a coordinate transformation (the drag-along coordinate system) \( x^\mu \mapsto y^\mu_A(x) \) with \( y^\mu_A|_{p'} = x^\mu|_p \) and the drag-along \( D_A^4 g \) of the metric tensor, whose components are defined through the equation

\[
dy^\alpha_A dy^\beta_A (D_A^4 g)_{\mu\nu}(y_A)|_{p'} = dx^\mu dx^\nu^4 g_{\mu\nu}(x)|_{p'}. \tag{2.4}
\]

As a consequence, we have: i) the tensor components \( (D_A^4 g)_{\mu\nu}(y_A) \) are not the components of the metric tensor in the chart \( y^\mu_A \) implied by Eq.(2.3); ii) in the original coordinate chart \( (D_A^4 g)_{\mu\nu}(x) \neq ^4 g_{\mu\nu}(x) \).

The hints for a clarification of the active/passive ambiguity can be found in a nearly forgotten paper by Bergmann and Komar [8] [see, however, Ref.[18]] in which it is shown that the biggest group \( Q \) of passive dynamical symmetries of Einstein’s equations is not \( pDiff M^4 \) \([x^\mu = f^\mu(x')\]) but instead a larger group of transformations of the form

\[
Q : \quad x'^\mu = f^\mu(x',^4 g_{\alpha\beta}(x)),
\]

\[
^4 g'_{\mu\nu}(x'(x)) = \frac{\partial h^\alpha(x',^4 g(x'))}{\partial x'^\mu} \frac{\partial h^\beta(x',^4 g(x'))}{\partial x'^\nu}^4 g_{\alpha\beta}(x). \tag{2.4}
\]

It is clear that in this way we allow for metric dependent coordinate systems, whose associated 4-metrics are in general different from those obtainable from a given 4-metric solution
of Einstein’s equations by *passive diffeomorphisms*: actually, the transformations (2.4) map points to points, but associate with a given point \( x \) an image point \( x' \) that depends also on the metric field \(^4\). It is remarkable, however, that not only these new transformed 4-metric tensors are still solutions of Einstein’s equations, but that, at least for the subset \( Q' \subset Q \) which corresponds to mappings among gauge-equivalent Cauchy data, they belong indeed to the *same 4-geometry*, i.e., the same equivalence class generated by applying all *passive diffeomorphisms* to the original 4-metrics: \(^4\text{Geom} = \frac{4\text{Riem}}{p\text{Diff}} M^4 = \frac{4\text{Riem}}{Q'}\). Note, incidentally, that this circumstance is mathematically possible only because \( p\text{Diff} M^4 \) is a *non-normal* sub-group of \( Q \). The 4-metrics built by using passive diffeomorphisms are, as it were, only a dense sub-set of the metrics obtainable by means of the group \( Q \). The restricted set of active diffeomorphisms passively reinterpreted with Eq.(2.4) belongs to the set of local Noether symmetries of the Einstein-Hilbert action.

There is no clear statement in the literature about the dynamical symmetry status of the group \( _A\text{Diff} M^4 \) of *active diffeomorphisms* and their relationship with the group \( Q \), a point which is fundamental for our program. To clarify this point, let us consider an infinitesimal transformation of the type (2.4) connecting a 4-coordinate system \([x^\mu]\) to a new one \([x'^\mu]\) by means of metric-dependent infinitesimal descriptors:

\[
x'^\mu = x^\mu + \delta x^\mu = x^\mu + \xi^\mu(x, 4g).
\]  

(2.5)

This will induce the usual formal variation of the metric tensor \(^5\)

\[
\delta^4 g_{\mu\nu} = -\left( \xi_{\mu,\nu}(x, 4g) + \xi_{\nu,\mu}(x, 4g) \right).
\]  

(2.6)

If \( \delta^4 g_{\mu\nu}(x) \) is now identified with the local variation of the metric tensor induced by the *drag along* of the metric under an infinitesimal active diffeomorphism \( 4g \mapsto \tilde{4}g \) so that

\[
\tilde{\delta}^4 g_{\mu\nu} \equiv 4\mu\nu(x) - 4g_{\mu\nu}(x) = -\left( \xi_{\mu,\nu}(x, 4g) + \xi_{\nu,\mu}(x, 4g) \right),
\]  

(2.7)

the solution \( \xi_{\mu}(x, 4g) \) of these Killing-type equations identifies a corresponding *passive Bergmann-Komar dynamical symmetry* belonging to \( Q \). We see that the new system of coordinates \([x'^\mu]\) is identical to the *drag along* of the old coordinate system, so that here we have made explicit the merely implicit *dual view* quoted above.

This result should imply that all the *active diffeomorphisms* connected to the identity in \(_A\text{Diff} M^4 \) can be reinterpreted as elements of a *non-normal* sub-group of *generalized passive transformations* in \( Q \). Clearly this sub-group is disjoint from the sub-group \( p\text{Diff} M^4 \): again, this is possible because diffeomorphism groups do not possess a canonical identity. However, let us recall that, unfortunately, there is no viable mathematical treatment of the diffeomorphism group in the large.

In conclusion, what is known as *4-geometry*, or as *Einstein (or on-shell, or dynamical)* gravitational field, is also an equivalence class of solutions of Einstein’s equations *modulo* the dynamical symmetry transformations of \(_A\text{Diff} M^4 \). Therefore, usually one finds the following statement \([8]\)

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\(^4\) Strictly speaking, Eqs.(2.4) should be defined as transformations on the tensor bundle over \( M^4 \).

\(^5\) What is relevant here is the *local* variation \( \delta^4 g_{\mu\nu}(x) = \mathcal{L}_{\xi^\gamma} \partial^\gamma 4g_{\mu\nu}(x) = 4\mu\nu(x) - 4g_{\mu\nu}(x) \) which differs from the *total* variation by a *convective* term: \( \delta^4 \mu\nu(x) = 4\mu\nu(x) - \nabla^4 \mu\nu(x) = \delta^4 \mu\nu(x) + \delta^4 x^\gamma \partial_\gamma 4g_{\mu\nu}(x) \).
$^4\text{Geom} = ^4\text{Riem}/_P\text{Diff} M^4 = ^4\text{Riem}/_Q = ^4\text{Riem}/_A\text{Diff} M^4$. \hspace{1cm} (2.8)

It should be stressed, however, that the last two equalities hold in the previously explained weak sense.

It is clear that a parametrization of the 4-geometries should be grounded on the two independent dynamical degrees of freedom of the gravitational field. Within the framework of the Lagrangian dynamics, however, no algorithm is known for evaluating the observables of the gravitational field, viz. its two independent degrees of freedom. The only result we know of is given in Ref.[9] where, after a study of the index of Einstein’s equations, it is stated that the two degrees of freedom are locally associated to symmetric trace-free 2-tensors on two-planes, suggesting a connection with the Newman-Penrose formalism [15].

On the other hand, as we shall see in the next Section, it is the Hamiltonian framework which has the proper tools to attack these problems. Essentially, this is due to the fact that the Hamiltonian methods allow to work off-shell, i.e., without immediate transition to the space of solutions of Einstein’s equations. Thus the soldering to the above results is reached only at the end of the canonical reduction, when the on-shell restriction is made.

Let us now make some remarks about the choice of coordinate systems. On the one hand, it is clear from Eq.(2.8) that, given a solution of Einstein’s equations in a coordinate system, its form in any other system, either ordinary or extended, can be obtained by means of Eqs.(2.3) or (2.4). On the other hand, in practice one looks for the most convenient coordinate system for dealing with specific problems. This is always done by imposing some conditions to be satisfied by the metric tensor in the wanted coordinate system, so that such coordinate conditions amount to a complete or partial breaking of general covariance.

In the variational approach A) these conditions are named Lagrangian gauge fixings. If we start with Einstein’s equations in an arbitrary coordinate system $x^\mu$ of the atlas of $M^4$, the transition to the special coordinate system $x'^\mu$, identified by a set of conditions on the metric, may either correspond to an ordinary coordinate transformation (passive diffeomorphism) $x'^\mu = f^\mu(x)$ between two charts of the atlas of $M^4$ or, most likely, to an extended transformation of the type (2.4) (passive re-interpretation of an active diffeomorphism).

i) The usual search for exact solutions of Einstein’s equations relies on a choice of coordinates dictated by the assumed Killing symmetries of the metric tensor, which are special metric conditions.

ii) The Lagrangian gauge fixing procedure amounts to the determination of the inverse coordinate transformation $x^\mu = h^\mu(x')$ as a solution of Eq.(2.3) interpreted as a partial differential equation for $h^\mu(x')$, with the metric $^4g_{\mu\nu}(x')$ satisfying the required conditions. Since the group of passive diffeomorphisms as well as its extension (2.4) depend on four arbitrary functions, a choice of either a specific coordinate system or a family of coordinate

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6 Note nevertheless that even at the Lagrangian level one can define off-shell (or kinematical) gravitational fields defined as $^4\text{Riem}'/_P\text{Diff} M^4$, where $^4\text{Riem}'$ are all the possible metric tensors on $M^4$. Of course only the subset of solutions of Einstein equations are Einstein gravitational fields.

7 As we shall see, in the canonical formulation of general relativity one speaks of Hamiltonian gauge fixings, which correspond to a fixation of the coordinates of $M^4$ only on-shell. In particular, the fixation of the 3-coordinates on a Cauchy surface are made by imposing 3 gauge fixing constraints on the 3-metric.
systems has to be done by imposing $N$ suitable functional conditions on the metric tensor (either $N = 4$ or $N \leq 4$). Typical instantiations of this fact are the following:

a) Algebraic Lagrangian gauge fixings:
   a1) Family of synchronous coordinates: $\mathbf{4}g^{i'}_{\mu}(x') = 0$, $i = 1, 2, 3$; since $N = 3$, there is a residual gauge freedom, namely the solution $h^\mu$ depends upon an arbitrary function.
   a2) Family of 3-orthogonal coordinates: $\mathbf{4}g_{ij}(x') = 0$, $i \neq j$; again there is a residual gauge freedom depending upon an arbitrary function.

b) Non-algebraic Lagrangian gauge fixings, in which the metric $\mathbf{4}g_{\mu\nu}(x')$ is only restricted to be a solution of partial differential equations, so that there is an extra dependence upon new arbitrary functions:
   b1) Family of harmonic coordinates: they are associated to all the functional forms of $\mathbf{4}g_{\mu\nu}(x')$ which satisfy the four partial differential equations: $\Gamma^\alpha_{\mu\nu}[\mathbf{4}g'(x')]\mathbf{4}g^{\mu\nu}(x') = 0$.
   b2) Family of Riemann normal coordinates around a point [19]: they are defined by asking that the geodesics emanating from the point are straight lines.

Let us end this Section with a remark on general covariance that, with the exception of Kretschmann [20], is usually considered a genuine and fundamental feature of general relativity which can be extended to special relativity and Newton mechanics only in a formal and artificial way:

Let us remark that, in special relativity, the embedding $x^\mu = z^\mu(\tau, \vec{\sigma})$ is usually described with respect to the axes of an instantaneous inertial observer (see Appendix A for the terminology concerning time-like observers) chosen as origin of a global inertial reference frame, namely a congruence of time-like straight-lines parallel to the time axis of the instantaneous inertial observer. More generally, we can introduce (already in Minkowski space-time) a global non-inertial reference frame defined as a congruence of time-like world-lines, determined by a unit vector field, one of which is selected as an instantaneous non-inertial observer $X^\mu(\tau)$. This latter is then used as the centroid, origin of the curvilinear 3-coordinate system $\sigma^r$, $r = 1, 2, 3$, on the simultaneity $\tau = \text{const.}$ 3-surfaces $\Sigma_\tau$, so that the embeddings can be parametrized as $z^\mu(\tau, \vec{\sigma}) = X^\mu(\tau) + F^\mu(\tau, \vec{\sigma})$, $F^\mu(\tau, \vec{0}) = 0$. See Ref.[12] for the definition of the admissible embeddings in special relativity.

Obviously, in curved space-times, globally inertial reference frames do not exist (only local ones do, freely falling along 4-geodesics), but still we can safely use the notion of global non-inertial laboratory provided that the topology of $M^4$ is trivial. To every such frame a special global coordinate chart $x^\alpha$ in the atlas of $M^4$ can be associated.

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8 In Ref.[13] it is shown that within parametrized Minkowski theories it is possible to re-formulate the dynamics of isolated systems in special relativity on arbitrary space-like hyper-surfaces that are leaves of the foliation associated with an arbitrary 3+1 splitting and also define a surface-forming congruence of accelerated time-like observers. In these theories the embeddings $z^\mu(\tau, \sigma)$ of the space-like hyper-surfaces are new configuration variables at the Lagrangian level. However, they are gauge variables because the Lagrangian is invariant under separate $\tau$- and $\vec{\sigma}$-reparametrizations (which are diffeomorphisms). This form of special relativistic general covariance implies the existence of four first class constraints analogous to the super-hamitonian and super-momentum constraints of ADM canonical gravity, which assure the independence of the description from the choice of the 3+1 splitting.
III. THE ADM HAMILTONIAN VIEWPOINT AND THE RELATED CANONICAL LOCAL SYMMETRIES.

This Section provides the analysis of the Cauchy problem and the counting of degrees of freedom within the framework of the ADM canonical formulation of metric gravity [21]. Since we are interested in a model of general relativity able to incorporate the standard model of elementary particles and its extensions, and since these models are a chapter of the theory of representations of the Poincare’ group on Minkowski space-time, we will consider only non-compact, topologically trivial space-times. Moreover they must be globally hyperbolic pseudo-Riemannian 4-manifolds \( M^4 \) asymptotically flat at spatial infinity, because only in this case a Hamiltonian formulation is possible. Actually, unlike the Lagrangian formulation, the Hamiltonian formalism requires a 3+1 splitting of \( M^4 \) and a global mathematical time function \( \tau \). This entails in turn a foliation of \( M^4 \) by space-like hyper-surfaces \( \Sigma^\tau \) (simultaneity Cauchy surfaces, assumed diffeomorphic to \( \mathbb{R}^3 \) so that any two points on them are joined by a unique 3-geodesic), to be coordinatized by adapted 3-coordinates \( \vec{\sigma} \).

If \( \tau \) is the mathematical time labeling these 3-surfaces, \( \Sigma^\tau \), and \( \vec{\sigma} \) are 3-coordinates (with respect to an arbitrary observer, a centroid \( X^\mu(\tau) \), chosen as origin) on them, then \( \sigma^A = (\tau, \vec{\sigma}) \) can be interpreted as Lorentz-scalar radar 4-coordinates and the surfaces \( \Sigma^\tau \) are described by embedding functions \( z^\mu = z^\mu(\tau, \vec{\sigma}) = X^\mu(\tau) + \tilde{F}^\mu(\tau, \vec{0}) = 0 \). In these coordinates the metric is \( g_{AB}(\tau, \vec{\sigma}) = z_A^\mu(\tau, \vec{\sigma}) g_{\mu\nu}(z(\tau, \vec{\sigma})) z_B^\nu(\tau, \vec{\sigma}) \). Since the 3-surfaces \( \Sigma^\tau \) are equal time 3-spaces with all clocks synchronized, the spatial distance between two equal-time events will be \( dl_{12} = \int_{l_1}^{l_2} dl \sqrt{g_{rs}(\tau, \vec{\sigma}(l)) \frac{d\sigma^r(l)}{dl} \frac{d\sigma^s(l)}{dl}} \) [\( \vec{\sigma}(l) \) is a parametrization of the 3-geodesic \( \gamma_{12} \) joining the two events on \( \Sigma^\tau \)]. Moreover, by using test rays of light we can define the one-way velocity of light between events on different \( \Sigma^\tau \)'s. Therefore, the Hamiltonian description has naturally built in the tools (essentially the 3+1 splitting) to make contact with experiments in a relativistic framework, where simultaneity is a frame-dependent property. Let us note that the manifestly covariant description using Einstein’s equations is the natural one for the search of exact solutions, but is inadequate to describe experiments.

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9 The 3-surfaces \( \Sigma^\tau \) are instances of equal time surfaces corresponding to a convention of synchronization of distant clocks, a definition of 3-space and a determination of the one-way velocity of light, generalizing the customary Einstein convention valid only in the inertial systems of special relativity. For a discussion of this topic see Ref.[12]. The use of the parameter \( \tau \), labeling the leaves of the foliation, as an evolution parameter corresponds to the hyper-surface point of view of Ref.[22]. The threading point of view is instead a description involving only a rotating congruence of observers: since the latter is rotating, it is not surface-forming (non-zero vorticity) and in each point we can only divide the tangent space in the direction parallel to the 4-velocity and the orthogonal complement (the local rest frame). On the other hand, the slicing point of view, originally adopted in ADM canonical gravity, uses two congruences: the non-rotating one with the normals to \( \Sigma^\tau \) as 4-velocity fields and a second (rotating, non-surface-forming) congruence of observers, whose 4-velocity field is the field of time-like unit vectors determined by the \( \tau \) derivative of the embeddings identifying the leaves \( \Sigma^\tau \) (their so-called evolution vector field). Furthermore, as Hamiltonian evolution parameter it uses the affine parameter describing the world-lines of this second family of observers.

10 An improper vector notation is used throughout for the sake of simplicity.
As shown in Ref.[13], with this formulation the so-called problem of time can be treated in such a way that in presence of matter and in the special relativistic limit of vanishing Newton constant, one recovers the parametrized Minkowski theories, quoted at the end of the previous section, equipped with a global time $\tau$. A canonical formulation with well-defined Poisson brackets requires in addition the specification of suitable boundary conditions at spatial infinity and a definite choice of the functional space for the fields. While the problem of the boundary conditions constitutes an intriguing issue within the Lagrangian approach, the Hamiltonian one is more easy to treat. Even if we shall consider only metric gravity, let us remark that with the inclusion of fermions it is natural to resolve the metric tensor in terms of cotetrad fields [23] 

\[
E^{(\alpha)}(x) \eta^{(\beta)}(x); \ \eta^{(\alpha)(\beta)} \text{ is the flat Minkowski metric in Cartesian coordinates}
\]

and to reinterpret the gravitational field as a theory of time-like observers endowed with tetrads, whose dynamics is controlled by the ADM action thought as a function of the cotetrad fields.

Only the aspects important to our program will be reviewed here. The reader is referred to Ref.[13] for the relevant notations and the general technical development of the Hamiltonian description of metric gravity, which requires the use of Dirac-Bergmann [24, 25, 26, 27, 28] theory of constraints (see Refs.[29, 30] for updated reviews). We use a Lorentzian signature $\epsilon(+-)-$, with $\epsilon = \pm 1$ according to particle physics and general relativity conventions, respectively.

### A. ADM Action, Asymptotic Symmetries and Boundary Conditions.

We start off with replacement of the ten components $^{4}g_{\mu\nu}$ of the 4-metric tensor by the configuration variables of ADM canonical gravity: the lapse $N(\tau, \vec{\sigma})$ and shift $N_{r}(\tau, \vec{\sigma})$ functions and the six components of the 3-metric tensor on $\Sigma_{\tau}$, $^{3}g_{rs}(\tau, \vec{\sigma})$. We have $^{4}g_{AB} = \begin{pmatrix} 4g_{\tau\tau} = \epsilon(N^{2} - ^{3}g_{rs}N^{r}N^{s}) & 4g_{\tau r} = -\epsilon^{3}g_{su}N^{u} \\ 4g_{\tau r} = -\epsilon^{3}g_{ru}N^{u} & 4g_{rs} = -\epsilon^{3}g_{rs} \end{pmatrix}$. Einstein’s equations are then recovered as the Euler-Lagrange equations of the ADM action

\[
S_{ADM} = \int d\tau L_{ADM}(\tau) = \int d\tau d^{3}\sigma L_{ADM}(\tau, \vec{\sigma}) =
\]

\[
= -\epsilon k \int_{\Delta \tau} d\tau \int d^{3}\sigma \left\{ \sqrt{\gamma}N \left[ \frac{3}{2}R + \frac{3}{2}K^{rs}K_{rs} - \frac{1}{2}(3K)^{2} \right] \right\}(\tau, \vec{\sigma}),
\]

which differs from Einstein-Hilbert action (2.1) by a suitable surface term. Here $^{3}K_{rs}$ is the extrinsic curvature of $\Sigma_{\tau}$, $^{3}K$ its trace, and $^{3}R$ the 3-curvature scalar.

Besides the ten configuration variables listed above, the ADM functional phase space $\Gamma_{20}$ is coordinatized by ten canonical momenta $\tilde{\pi}^{N}(\tau, \vec{\sigma}), \tilde{\pi}^{r}(\tau, \vec{\sigma}), \tilde{\Pi}^{rs}(\tau, \vec{\sigma})$. Such canonical variables, however, are not independent since they are restricted to the constraint submanifold $\Gamma_{12}$ by the eight first class constraints $^{3}G_{rstw} = ^{3}g_{rt}^{3}g_{sw} + ^{3}g_{rw}^{3}g_{st} - ^{3}g_{rs}^{3}g_{tw}$ is the Wheeler-DeWitt super-metric]

14
\[ \tilde{\pi}^N(\tau, \vec{\sigma}) \approx 0, \]
\[ \tilde{\pi}_N^r(\tau, \vec{\sigma}) \approx 0, \]
\[ \tilde{\mathcal{H}}(\tau, \vec{\sigma}) = \epsilon [k \sqrt{\gamma}^3 R - \frac{1}{2k \sqrt{\gamma}}^3 G_{rsuv}^3 \tilde{\Pi}_{rs}^3 \tilde{\Pi}_{uv}^3] (\tau, \vec{\sigma}) \approx 0, \]
\[ ^3\mathcal{H}'(\tau, \vec{\sigma}) = -2 ^3\tilde{\Pi}_{rs}^3 s(\tau, \vec{\sigma}) = -2 [\partial_s ^3 \tilde{\Pi}_{rs}^s + ^3 \Gamma_{su}^r^3 \tilde{\Pi}_{su}^3] (\tau, \vec{\sigma}) \approx 0. \] (3.2)

While the first four are primary constraints, the remaining four are the super-hamiltonian and super-momentum secondary constraints arising from the requirement that the primary constraints be constant in \( \tau \). More precisely, this requirement guarantees that, once we have chosen the initial data inside the constraint sub-manifold \( \Gamma_{12}(\tau_0) \) corresponding to a given initial Cauchy surface \( \Sigma_{\tau_0} \), the time evolution does not take them out of the constraint sub-manifolds \( \Gamma_{12}(\tau) \), for \( \tau > \tau_0 \).

The evolution in \( \tau \) is ruled by the Hamilton-Dirac Hamiltonian

\[ H_{(D)ADM} = \int d^3\sigma \left[ N \tilde{\mathcal{H}} + N_r^3 \tilde{\mathcal{H}'r} + \lambda_N^N \tilde{\pi}_N^N + \lambda_r^N \tilde{\pi}_N^N \right] (\tau, \vec{\sigma}) \approx 0, \] (3.3)

where \( \lambda_N^N(\tau, \vec{\sigma}) \) and \( \lambda_r^N(\tau, \vec{\sigma}) \) are arbitrary Dirac multipliers in front of the primary constraints\(^{11}\). The resulting hyperbolic system of Hamilton-Dirac equations has the same solutions of the non-hyperbolic system of (Lagrangian) Einstein’s equations with the same boundary conditions. Let us stress that Hamiltonian hyperbolicity is explicitly paid by the arbitrariness of the Dirac multipliers\(^{12}\).

At this point let us see the further conditions to be required with respect to the above standard ADM formulation.

Additional requirements \(^{13}\) on the Cauchy and simultaneity 3-surfaces \( \Sigma_\tau \) induced by particle physics are:

i) Each \( \Sigma_\tau \) must be a Lichnerowitz 3-manifold \(^{31}\), namely it must admit an involution so that a generalized Fourier transform can be defined and the notion of positive and negative frequencies can be introduced (otherwise the notion of particle cannot be properly defined, like it happens in quantum field theory in arbitrary curved space-times \(^{32}\)).

ii) Both the metric tensor and the fields of the standard model of elementary particles must belong to the same family of suitable weighted Sobolev spaces so that there are no Killing vector fields on space-time (this avoids the cone-over-cone structure of singularities in the space of metrics) and no Gribov ambiguity (either gauge symmetries or gauge copies \(^{33}\)) in the particle sectors; in both cases no well defined Hamiltonian description could be available.

iii) Space-time must be asymptotically flat at spatial infinity and satisfying boundary conditions there in a way independent of the direction (in analogy to what is required for

\(^{11}\) These are four velocity functions (gradients of the metric tensor) which are not determined by Einstein’s equations.

\(^{12}\) Of course this is just the Hamiltonian counterpart of the ”indeterminateness” or the so-called ”indeterminism” surfacing in what Einstein called Hole Argument (”Lochbetrachtung”) in 1915-1916 \(^{1}\).
the defining non-Abelian charges in Yang-Mills theory [33]). This eliminates the supertranslations (i.e., the obstruction to define angular momentum in general relativity) and reduces the spi group of asymptotic symmetries to the ADM Poincare’ group. The constant ADM Poincare’ generators should become the standard conserved Poincare’ generators of the standard model of elementary particles when gravity is turned off and space-time (modulo a possible renormalization of the ADM energy to subtract an infinite term coming from its dependence on both G and 1/G) becomes Minkowskian.13 As a consequence, the admissible foliations of the space-time must have the simultaneity surfaces Στ tending in a direction-independent way to Minkowski space-like hyper-planes at spatial infinity, where they must be orthogonal to the ADM 4-momentum. Now, these are exactly the conditions satisfied by the Christodoulou-Klainermann space-times [9], which are near Minkowski space-time in a norm sense and have a rest-frame condition of zero ADM 3-momentum. The hypersurfaces Στ define the rest frame of the τ-slice of the universe and admit asymptotic inertial observers to be identified with the fixed stars (this also defines the standard of rotations the spatial precession of gyroscopes is referred to).14 Another interesting point is that this class of space-times admits an asymptotic Minkowski metric (asymptotic background) which allows to define weak gravitational field configurations and background-independent gravitational waves [36] that do not require splitting of the metric in a background term plus a perturbation (and without being a bimetric theory of gravity).

13 Incidentally, this is the first example of consistent deparametrization of general relativity. In presence of matter we get the description of matter in Minkowski space-time foliated with the space-like hyper-planes orthogonal to the total matter 4-momentum (Wigner hyper-planes intrinsically defined by matter isolated system). Of course, in closed space-times, the ADM Poincare’ charges do not exist and the special relativistic limit is lost.

14 These properties are concretely enforced [13] by using a technique introduced by Dirac [24] for the selection of space-times admitting asymptotically flat 4-coordinates at spatial infinity. Dirac’s method brings to an enlargement of the ADM phase space, subsequently reduced to the standard one by adding suitable constraints, as shown explicitly in Ref. [13]. As a consequence the admissible embeddings of the simultaneity leaves Στ have the following direction-independent limit at spatial infinity: 

\[ z^\mu(\tau, \vec{\sigma}) = X^\mu(\tau) + F^\mu(\tau, \vec{\sigma}) \rightarrow \vec{\sigma} \rightarrow \infty X^\mu(\infty)(0) + c_A^\mu A^A = X^\mu_A(\tau) + A^\mu A^\sigma \tau. \]

Here \( X^\mu_A(\infty)(\tau) = X^\mu_A(\infty)(0) + A^\mu A^\tau \) is just the world-line of an asymptotic inertial observer having τ as proper time and \( c_A^\mu \) denotes an asymptotic constant tetrad with \( c_A^\mu \) parallel to the ADM 4-momentum (it is orthogonal to the asymptotic space-like hyper-planes). Such inertial observers corresponding to the fixed stars can be endowed with a spatial triad \( e^\tau_A = \delta^\tau_A, \ a = 1, 2, 3 \). Then the asymptotic spatial triad \( e^\tau_A = \delta^\tau_A \) can be transported in a dynamical way (on-shell) by using the Sen-Witten connection [34] (it depends on the extrinsic curvature of the Στ’s) in the Frauendiener formulation [35] in every point of Στ, where it becomes a well defined triad \( e^\tau_A(WSW)(\tau, \vec{\sigma}) \). This defines a local compass of inertia, to be compared with the local gyroscopes (whether Fermi-Walker transported or not). The Wigner-Sen-Witten (WSW) local compass of inertia consists in pointing to the fixed stars with a telescope. It is needed in a satellite like Gravity Probe B to detect the frame-dragging (or gravito-magnetic Lense-Thirring effect) of the inertial frames by means of the rotation of a FW transported gyroscope.

Finally from Eq. (12.8) of Ref. [13] we get the following set of partial differential equations for the determination of the embedding \( x^\mu = z^\mu(\tau, \vec{\sigma}) \) (\( x^\mu \) is an arbitrary 4-coordinate system in which the asymptotic hyper-planes of the Στ’s have \( c_A^\mu \) as asymptotic tetrad)
As shown in Ref.[13], a consistent treatment of the boundary conditions at spatial infinity requires the explicit separation of the asymptotic part of the lapse and shift functions from their bulk part: \( N(\tau, \vec{\sigma}) = N_{(as)}(\tau, \vec{\sigma}) + n(\tau, \vec{\sigma}), \ N_r(\tau, \vec{\sigma}) = N_{(as)}r(\tau, \vec{\sigma}) + n_r(\tau, \vec{\sigma}), \) with \( n \) and \( n_r \) tending to zero at spatial infinity in a direction-independent way\(^{15}\). On the contrary, \( N_{(as)}(\tau, \vec{\sigma}) = -\lambda_r(\tau) - \frac{1}{2} \lambda_{ru}(\tau) \sigma^u \) and \( N_{(as)}r(\tau, \vec{\sigma}) = -\lambda_r(\tau) - \frac{1}{2} \lambda_{ru}(\tau) \sigma^u. \) In the Christodoulou-Klainermann space-times [9] we have \( N_{(as)}(\tau, \vec{\sigma}) = \epsilon, \ N_{(as)}r(\tau, \vec{\sigma}) = 0. \)

Recall that the evolution is parametrized by the mathematical parameter \( \tau \) of the adapted coordinate system \((\tau, \vec{\sigma})\) on \( M^4 \), which labels the surfaces \( \Sigma_\tau \). As shown in Ref.[13], the Hamiltonian ruling the evolution is the weak ADM energy ([37] (the volume form \( E_{ADM} \)). As shown by DeWitt [38], this is a consequence of the fact that in non-compact space-times the weakly vanishing ADM Dirac Hamiltonian (3.3) has to be modified with a suitable surface term in order to have functional derivatives, Poisson brackets and Hamilton equations mathematically well defined.

It follows, therefore, that the boundary conditions of this model of general relativity imply that the real Dirac Hamiltonian is\(^{16}\)

\[
H_D = E_{ADM} + H_{(D)ADM} \approx E_{ADM},
\]

and this entails that an effective evolution takes place in mathematical time \( \tau \)\(^{17}\), and that a non-vanishing Hamiltonian survives in the reduced phase space of the intrinsic degrees of freedom (no frozen reduced phase space picture).

The weak ADM energy, and also the other nine asymptotic weak Poincare’ charges \( \bar{P}_{ADM}, \ J_{AB}^{ADM}, \) are Noether constants of the motion whose numerical value has to be given as part of the boundary conditions. The numerical value of \( E_{ADM} \) is the mass of the \( \tau \)-slice of the

\[
z^\mu(\tau, \vec{\sigma}) = X^\mu_{(\infty)}(0) + F^A(\tau, \vec{\sigma}) \frac{\partial z^\mu(\tau, \vec{\sigma})}{\partial \sigma^A},
\]

\[
F^\tau(\tau, \vec{\sigma}) = \frac{-\epsilon \tau}{-\epsilon + n(\tau, \vec{\sigma})},
\]

\[
F^r(\tau, \vec{\sigma}) = \sigma^r + [\epsilon c^{(u,s,w)}_{(a)} (\tau, \vec{\sigma}) - \delta_{(a)}^r] \delta_{(a)}^u \sigma^u + \frac{\epsilon n^r(\tau, \vec{\sigma})}{-\epsilon + n(\tau, \vec{\sigma})}.
\]

\(^{15}\) We would like to recall that Bergmann [14] made the following critique of general covariance: it would be desirable to restrict the group of coordinate transformations (space-time diffeomorphisms) in such a way that it could contain an invariant sub-group describing the coordinate transformations that change the frame of reference of an outside observer; the remaining coordinate transformations would be like the gauge transformations of electromagnetism. This is just what is done here by the redefinition of the lapse and shift functions after separating out their asymptotic part. In this way, preferred inertial asymptotic coordinate systems are selected that can be identified as fixed stars.

\(^{16}\) As shown in Ref.[13], the correct treatment of the boundary conditions leads to rewrite Eqs.(3.3) and (3.4) in terms of \( n \) and \( n_r \). Moreover the momenta \( \tilde{\pi}_N, \ \tilde{\pi}_\sigma^r \) should be always replaced by \( \tilde{\pi}_N, \ \tilde{\pi}_\sigma^r \).

\(^{17}\) As we shall see, the super-hamiltonian constraint is only the generator of the gauge transformations connecting different admissible 3+1 splittings of space-time and has nothing to do with the temporal evolution (no Wheeler-DeWitt interpretation).
universe, while \( J_{\text{ADM}}^s \) gives the value of the spin of the universe. Since, in our case, space-time is of the Christodoulou-Klainermann type [9], the ADM 3-momentum has to vanish. This implies three first class constraints

\[
\vec{P}_{\text{ADM}} \approx 0,
\]

which identify the rest frame of the universe. As shown in Ref.[13], the natural gauge fixing to these three constraints is the requirement the the ADM boosts vanish: \( J_{\text{ADM}}^r \approx 0 \). In this way we decouple from the universe its 3-center of mass \(^{18}\) and only relative motions survive, recovering a Machian flavour.

### B. Hamiltonian Gauge Transformations.

At this point a number of important questions must be clarified. When used as generators of canonical transformations, the eight first class constraints will map points of the constraint surface to points on the same surface. We shall say that they generate the infinitesimal transformations of the \textit{off-shell Hamiltonian} gauge group \( G_8 \). The action of \( G_8 \) gives rise to a Hamiltonian gauge orbit through each point of the constraint sub-manifold \( \Gamma_{12} \). Every such orbit is parametrized by eight phase space functions - namely the independent \textit{off-shell Hamiltonian gauge variables} - conjugated to the first class constraints. We are left thereby with a pair of conjugate canonical variables, the \textit{off-shell DO}, which are the only Hamiltonian gauge-invariant and deterministically ruled quantities. The same counting of degrees of freedom of the Lagrangian approach is thus obtained. Finally, let us stress here, in view of the later discussion, that both the off-shell Christoffel symbols and the off-shell Riemann tensor can be read as functions of both the off-shell DO and the Hamiltonian gauge variables.

The eight infinitesimal off-shell Hamiltonian gauge transformations have the following interpretation[13]:

i) those generated by the four primary constraints modify the lapse and shift functions; these in turn determine how densely the space-like hyper-surfaces \( \Sigma_\tau \) are distributed in space-time and also the conventions to be \textit{pre-fixed} on each \( \Sigma_\tau \) about gravito-magnetism (see Section IV of Ref.[36] for its dependence upon the choice of gauge, i.e. on-shell of the 4-coordinates);

ii) those generated by the three super-momentum constraints induce a transition on \( \Sigma_\tau \) from a given 3-coordinate system to another one;

iii) that generated by the super-hamiltonian constraint induces a transition from a given 3+1 splitting of \( M^4 \) to another, by operating normal deformations [39] of the space-like hyper-surfaces\(^{20}\).

---

\(^{18}\) This is equivalent to a choice of the centroid \( X^\mu(\tau) \) [or of the asymptotic one \( X^\mu(\infty)(\tau) \)], origin of the 3-coordinates on each \( \Sigma_\tau \).

\(^{19}\) Note that the off-shell Hamiltonian gauge transformations are \textit{local Noether transformations} (second Noether theorem) under which the ADM Lagrangian (3.1) is \textit{quasi-invariant}.

\(^{20}\) Note that in \textit{compact} space-times the super-hamiltonian constraint is usually interpreted as generator of the evolution in some \textit{internal time}, either like York’s internal \textit{extrinsic} time or like Misner’s internal \textit{intrinsic} time. In this paper instead the super-hamiltonian constraint is the generator of those Hamiltonian
iv) those generated by the three rest-frame constraints (3.5) can be interpreted as a change of centroid to be used as origin of the 3-coordinates.

As a consequence, the whole set of Hamiltonian off-shell gauge transformations contains also a change of the global non-inertial space-time laboratory and its associated coordinates.

Making the quotient of the constraint hyper-surface with respect to the off-shell Hamiltonian gauge transformations by defining $\Gamma_4 = \Gamma_{12}/G_8$, we obtain the so-called reduced off-shell conformal super-space. Each of its points, i.e. a Hamiltonian off-shell (or kinematical) gravitational field, is an off-shell equivalence class, called an off-shell conformal 3-geometry, for the space-like hyper-surfaces $\Sigma_\tau$: note that, since it contains all the off-shell 4-geometries connected by Hamiltonian gauge transformations, it is not a 4-geometry.

An important digression is in order here. The space of parameters of the off-shell gauge group $G_8$ contains eight arbitrary functions. Four of them are the Dirac multipliers $\lambda_N(\tau, \vec{\sigma})$, $\lambda_N^\vec{N}(\tau, \vec{\sigma})$ of Eqs.(3.3), while the other four are functions $\alpha(\tau, \vec{\sigma})$, $\alpha_r(\tau, \vec{\sigma})$ which generalize the lapse and shift functions in front of the secondary constraints in Eqs.(3.3) 21. These arbitrary functions correspond to the eight local Noether symmetries under which the ADM action is quasi-invariant.

On the other hand, from the analysis of the dynamical symmetries of the Hamilton equations (equivalent to Einstein’s equations), it turns out (see Refs.[40, 41]) that on-shell only a sub-group $G_{4,\text{dyn}}$ of $G_8$ survives, depending on four arbitrary functions. But in the present context, a crucial result for our subsequent discussion is that a further subset, denoted by $G_{4,\text{P}} \subset G_{4,\text{dyn}}$, can be identified within the sub-group $G_{4,\text{dyn}}$: precisely the subset corresponding to the phase space counterparts of those passive diffeomorphisms which are projectable to phase space. On the other hand, as already said, Einstein’s equations have $Q$ as the largest group of dynamical symmetries and, even if irrelevant to the local Noether symmetries of the ADM action, the existence of this larger group is a fundamental mathematical premise to our second paper II. In order to take it into account in the present context, the parameter space of $G_8$ must be enlarged to arbitrary functions depending also on the 3-metric, $\lambda_N(\tau, \vec{\sigma}) \mapsto \lambda_N(\tau, \vec{\sigma}, g_{rs}(\tau, \vec{\sigma}))$, $\ldots$, $\alpha_r(\tau, \vec{\sigma}) \mapsto \alpha_r(\tau, \vec{\sigma}, g_{rs}(\tau, \vec{\sigma}))$. Then, the restriction of this enlarged gauge group to the dynamical symmetries of Hamilton equations defines an extended group $G_{4,\text{dyn}}$ which, under inverse Legendre transformation, defines a new non-normal sub-group $Q_{\text{can}}$ of the group $Q$ (see Ref.[8]). But now, the remarkable and fundamental point is that $Q_{\text{can}}$ contains both active and passive diffeomorphisms. In particular:

i) the intersection $Q_{\text{can}} \cap \text{Diff } M^4$ identifies the space-time passive diffeomorphisms which, respecting the 3+1 splitting of space-time, are projectable to $G_{4,\text{P}}$ in phase space;  

ii) the remaining elements of $Q_{\text{can}}$ are the projectable subset of active diffeomorphisms in their passive view.

This entails that, as said in Ref.[8], Eq.(2.8) may be completed with
gauge transformations which imply that the description is independent of the choice of the allowed 3+1 splitting of space-time: this is the proper answer to the criticisms raised against the phase space approach on the basis of its lack of manifest covariance.

21 In Ref. [8] they are called descriptors and written in the form $\alpha = N \xi$, $\alpha_r = g^{rs} \alpha_s = \xi^r \pm N^r \xi$. 

19
\[ ^4\text{Geom} = ^4\text{Riem}/Q_{\text{can}}. \] (3.6)

In conclusion, the real gauge group acting on the space of the solutions of the Hamilton-Dirac equations is the on-shell extended Hamiltonian gauge group \( \tilde{G}_{4,\text{dyn}} \) and the on-shell equivalence classes obtained by making the quotient with respect to it eventually coincide with the on-shell 4-geometries of the Lagrangian theory. Therefore, the Hamiltonian Einstein (or on-shell, or dynamical) gravitational fields coincide with the Lagrangian Einstein (or on-shell, or dynamical) gravitational fields.

Let us remark that, while it is known how to formulate an initial value problem for the partial differential equations of the Hamiltonian theory in a complete Hamiltonian gauge and how to connect the problems in different gauges by using on-shell transformations in \( Q_{\text{can}} \), no mathematical technique is known for dealing with active diffeomorphisms in \( Q' \) but not in \( Q_{\text{can}} \) in connection to the Cauchy problem within the framework of abstract differential geometry. As already said, for the configurational Einstein equations a technique, mimicking the Hamiltonian treatment, does exist and the Cauchy problems in different 4-coordinate systems are connected by transformations in \( p\text{Diff }M^4 \).

This is the way in which passive space-time diffeomorphisms, under which the Hilbert action is invariant, are reconciled on-shell with the allowed Hamiltonian gauge transformations adapted to the 3+1 splittings of the ADM formalism. Furthermore, our analysis of the Hamiltonian gauge transformations and their Legendre counterparts gives an extra bonus: namely that the on-shell phase space extended gauge transformations include also symmetries that are images of active space-time diffeomorphisms. The basic relevance of this result for a deep understanding of the so-called Hole Argument will appear fully in paper II.

C. The Shanmugadhasan Canonical Transformation and the Canonical Reduction.

Having clarified these important issues, let us come back to the canonical reduction. The off-shell freedom corresponding to the eight independent types of Hamiltonian gauge transformations is reduced on-shell to four types like in the case of \( p\text{Diff }M^4 \): precisely the transformations in \( [Q_{\text{can}} \cap p\text{Diff }M^4] \). At the off-shell level, this property is manifest by the circumstance that the original Dirac Hamiltonian contains only 4 arbitrary Dirac multipliers and that the correct gauge-fixing procedure [13, 42] starts by giving only the four gauge fixing to the secondary constraints. The gauge fixing functions must satisfy the orbit conditions ensuring that each gauge orbit is intersected only in one point by the gauge fixing surface (locally this requires a non-vanishing determinant of the Poisson brackets of the gauge functions with the secondary constraints). Then, the requirement of time constancy generates the four gauge fixing constraints to the primary constraints, while time constancy of such secondary gauge fixings leads to the determination of the four Dirac multipliers. Since the original constraints plus the above eight gauge fixing constraints form a second class set, it is possible to introduce the associated Dirac brackets and conclude the canonical

---

\[ ^{22} \text{This agrees with the results of Ref.}[43] \text{ according to which the projectable space-time diffeomorphisms depend only on four arbitrary functions and their time derivatives.} \]
reduction by realizing an off-shell reduced phase space $\Gamma_4$. Of course, once we reach a completely fixed Hamiltonian gauge (a copy of $\Gamma_4$), general covariance is completely broken. Finally, recall that a completely fixed Hamiltonian gauge is equivalent on-shell to a definite choice of the space-time 4-coordinates on $M^4$, within the Lagrangian viewpoint [40, 41].

In order to visualize the meaning of the various types of degrees of freedom we need the construction of a Shanmugadhasan canonical basis [10] of metric gravity having the following structure ($\bar{a} = 1, 2$ are non-tensorial indices of the DO $\pi_{\bar{a}}$) with

\begin{align*}
\begin{array}{c|c|c|c|c|c}
(\pi^n, \xi^r) & (\partial^n, \xi^r) & (\phi, r_{\bar{a}}) \\
\hline
\bar{a} & n & n_r & 3 g_{r\bar{a}} \\
\hline
\end{array}
\end{align*}

\begin{align*}
\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c}
consequence of the effect of finite gauge transformations (see Ref.[23] for the case of tetrad gravity).

b) The second canonical transformation would be instead a complete Shanmugadhasan transformation, where \( Q_H(\tau, \vec{\sigma}) \approx 0 \) would denote the Abelianization of the super-hamiltonian constraint\(^{26}\). The variables \( n, n_r, \xi^r \), \( \Pi_H \) are the final Abelianized Hamiltonian gauge variables, while \( r^{\prime}_a, \pi^{\prime}_a \) are the final DO.

In absence of explicit solutions of the Lichnerowicz equation, the best we can do is to construct the quasi-Shanmugadhasan transformation. On the other hand, such transformation has the remarkable property that, in the special gauge \( \pi_{\phi}(\tau, \vec{\sigma}) \approx 0 \), the variables \( r_a, \pi_a \) form a canonical basis of off-shell DO for the gravitational field even if the solution of the Lichnerowicz equation is not known.

Let us stress the important fact that the Shanmugadhasan canonical transformation is a highly non-local transformation\(^{27}\). Since it is not known how to build a global atlas of coordinate charts for the group manifold of diffeomorphism groups, it is not known either how to express the \( \xi^r \)'s, \( \pi_{\phi} \) and the DO in terms of the original ADM canonical variables\(^{28}\).

**D. The Gauge Fixings and their Chrono-Geometrical Interpretation.**

The four gauge fixings to the secondary constraints, when written in the quasi-Shanmugadhasan canonical basis, have the following meaning:

i) the three gauge fixings for the parameters \( \xi^r \) of the spatial passive diffeomorphisms generated by the super-momentum constraints correspond to the choice of a system of 3-coordinates on \( \Sigma_\tau \).\(^{29}\) The time constancy of these gauge fixings constrains the choice of a system of 3-coordinates on \( \Sigma_\tau \).\(^{29}\) The time constancy of these gauge fixings constrains the choice of a system of 3-coordinates on \( \Sigma_\tau \).\(^{29}\)

\(^{26}\) If \( \hat{\phi}[r_a, \pi_a, \xi^r, \pi_{\phi}] \) is the solution of the Lichnerowicz equation, then \( Q_H = \phi - \hat{\phi} \approx 0 \). Other forms of this canonical transformation should correspond to the extension of the York map \(^{[50]}\) to asymptotically flat space-times: in this case the momentum conjugate to the conformal factor would be just York time and one could add the maximal slicing condition as a gauge fixing. Again, however, nobody has been able so far to build a York map explicitly.

\(^{27}\) This feature has a Machian flavor, although in a non-Machian context: with or without matter, the whole 3-space is involved in the definition of the observables. Furthermore, these space-times allow the separation \(^{[13]}\) of the 4-center of mass of the universe (decoupled point particle clock) reminding the Machian statement that only relative motions are dynamically relevant.

\(^{28}\) This should be compared to the Yang-Mills theory in case of a trivial principal bundle, where the corresponding variables are defined by a path integral over the original canonical variables \(^{[29, 30, 33]}\).

\(^{29}\) Since the diffeomorphism group has no canonical identity, this gauge fixing has to be done in the following way. We choose a 3-coordinate system by choosing a parametrization of the six components \( 3 g_{rs}(\tau, \vec{\sigma}) \) of the 3-metric in terms of only three independent functions. This amounts to fix the three functional degrees of freedom associated with the diffeomorphism parameters \( \xi^r(\tau, \vec{\sigma}) \). For instance, a 3-orthogonal coordinate system is identified by \( 3 g_{rs}(\tau, \vec{\sigma}) = 0 \) for \( r \neq s \) and \( 3 g_{rr} = \phi^2 \exp(\sum_{a=1}^{2} \gamma_a r_{a}) \). Then, we impose the gauge fixing constraints \( \xi^r(\tau, \vec{\sigma}) - \sigma^r \approx 0 \) as a way of identifying this system of 3-coordinates with a conventional origin of the diffeomorphism group manifold.
for the shift functions $n_r$ (determination of gravo-magnetism) while the time constancy of the latter leads to the fixation of the Dirac multipliers $\lambda^a$. 

ii) the gauge fixing to the super-hamiltonian constraint determines $\pi_\phi$: it is a fixation of the form of $\Sigma_\tau$. It amounts to the choice of one particular 3+1 splitting of $M^4$ as well as to \textit{the choice of a notion of simultaneity}, namely of a convention for the synchronization of all the clocks lying on $\Sigma_\tau$. Since the time constancy of the gauge fixing on $\pi_\phi$ determines the gauge fixing for the lapse function $n$ (and then of the Dirac multiplier $\lambda_n$), it follows a connection with the choice of the standard of local proper time (see below).

Finally the gauge fixings to the rest-frame conditions (3.5) have the following meaning:

iii) they completely determine a \textit{global non-inertial space-time laboratory} associated to the embedding $\hat{z}^\mu(\tau, \vec{\sigma}) = X^\mu(\tau) + F^\mu(\tau, \vec{\sigma})$ describing the 3+1 splitting selected by i) and ii).

In conclusion, in a completely fixed Hamiltonian gauge \textit{all the gauge variables} $\xi^r$, $\pi_\phi$, $n$, $n_r$ \textit{become uniquely determined functions of the DO} $r^a(\tau, \vec{\sigma})$, $\pi_\alpha(\tau, \vec{\sigma})$, which at this stage are four arbitrary fields. Conversely, this entails that, after such a fixation of the gauge $G$, the functional form of the DO in terms of the original variables becomes gauge-dependent. At this point it is convenient to denote them as $r^G_\alpha$, $\pi^G_\alpha$.

As a consequence, a representative of a \textit{Hamiltonian kinematical or off-shell gravitational field}, in a given gauge equivalence class, is parametrized by $r_\alpha$, $\pi_\alpha$ and is an element of a \textit{conformal gauge orbit} (it contains all the 3-metrics in a conformal 3-geometry) spanned by the gauge variables $\xi^r$, $\pi_\phi$, $n$, $n_r$. Therefore, according to the gauge interpretation based on constraint theory, a \textit{Hamiltonian kinematical or off-shell gravitational field} is an equivalence class of 4-metrics modulo the Hamiltonian group of gauge transformations, which contains a well defined conformal 3-geometry. Clearly, this is a consequence of the different invariance properties of the ADM and Hilbert actions, even if they generate the same equations of motion.

Moreover, also the (unknown) solution $\phi(\tau, \vec{\sigma})$ of the Lichnerowicz equation becomes a uniquely determined functional of the DO, and this implies that all the geometrical tensors like the 3-metric $^3g_{rs}(\tau, \vec{\sigma})$, the extrinsic curvature $^3K_{rs}(\tau, \vec{\sigma})$ of the simultaneity surfaces $\Sigma_\tau$ (determining their \textit{final actual form}, see below), and the 4-metric $^4g_{AB}(\tau, \vec{\sigma})$ become uniquely determined functionals of the DO only.

This is true in particular for the \textit{weak ADM energy} $E_{ADM} = \int d^3\sigma \mathcal{E}_{ADM}(\tau, \vec{\sigma})$, since the energy density $\mathcal{E}_{ADM}(\tau, \vec{\sigma})$ depends not only on the DO but also on $\phi$ and on the gauge variables $\xi^r$ and $\pi_\phi$ (this is how the non-tensorial nature of the energy density in general relativity reveals itself in our approach). In a fixed gauge we get $E_{ADM} = \int d^3\sigma \mathcal{E}^G_{ADM}(\tau, \vec{\sigma})$ and this becomes the functional that rules the Hamilton equations [37] for the DO in the completely fixed gauge

$$
\frac{\partial r^G_\alpha(\tau, \vec{\sigma})}{\partial \tau} = \{r^G_\alpha(\tau, \vec{\sigma}), E_{ADM}\}^*, \quad \frac{\partial \pi^G_\alpha(\tau, \vec{\sigma})}{\partial \tau} = \{\pi^G_\alpha(\tau, \vec{\sigma}), E_{ADM}\}^*,
$$

where $E_{ADM}$ is intended as the restriction of the weak ADM energy to $\Gamma_4$ and where the $\{\cdot, \cdot\}^*$ are Dirac Brackets. By using the inversion of the first set of Eqs.(3.8) to get $\pi^G_\alpha = \pi^G_\alpha[r^G_b, \frac{\partial r^G_\alpha}{\partial \tau}]$, we arrive at the second order in time equations $\frac{\partial \pi^G_\alpha(\tau, \vec{\sigma})}{\partial \tau^2} = \{\pi^G_\alpha(\tau, \vec{\sigma}), \frac{\partial E_{ADM}}{\partial \tau}\}^*$. 

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\[ F^G_a [r^G_a (\tau, \vec{\sigma}), \frac{\partial r^G_a (\tau, \vec{\sigma})}{\partial \tau}] \] spatial gradients of \( r^G_a (\tau, \vec{\sigma}) \), where the \( F^G_a \)'s are effective forces whose functional form depends on the gauge \( G \).

Thus, once we have chosen any surface of the foliation as initial Cauchy surface \( \Sigma_{\tau_0} \) and assigned the initial data \( r_a (\tau_0, \vec{\sigma}), \pi_a (\tau_0, \vec{\sigma}) \) of the DO, we can calculate the solution of the Einstein-Hamilton equations corresponding to these initial data. Having found the solution in a completely fixed gauge, besides the values of the DO throughout space-time we get the value of the extrinsic curvature \( ^3K_{rs}(\tau, \vec{\sigma}) \) of the simultaneity surfaces \( \Sigma_{\tau} \) as an extra bonus. Therefore, on-shell, in the gauge \( G \) with given initial data for the DO on \( \Sigma_{\tau_0}^{(G)} \), the leaves \( \Sigma^{(G)}_{\tau} \) of the 3+1 splitting are dynamically determined in the adapted radar 4-coordinates \((\tau, \vec{\sigma})\) (which, as shown in II, also determine the point-events of \( M^4 \) in the gauge \( G \)). As said in footnote 14, the knowledge of the lapse and shift functions and of the extrinsic curvature in the gauge \( G \) allows to find the embedding \( z^\mu_G (\tau, \vec{\sigma}) \) of the simultaneity leaves and the 4-metric \( ^4g_{G, \mu \nu}(x_G) \) in the gauge \( G \) in an arbitrary 4-coordinate system \( x^\mu_G = z^\mu_G (\tau, \vec{\sigma}) \) of \( M^4 \). If we redo all the calculations in another complete Hamiltonian gauge \( G_1 \) with adapted radar 4-coordinates \((\tau_1, \vec{\sigma}_1)\), then we can find the embedding \( z^\mu_{G_1} (\tau_1, \vec{\sigma}_1) \) of the new simultaneity surfaces \( \Sigma_{\tau_1}^{(G_1)} \) and the 4-metric \( ^4g_{G_1, \mu \nu}(x_{G_1}) \) in another arbitrary 4-coordinate system \( x^\mu_{G_1} = z^\mu_{G_1} (\tau_1, \vec{\sigma}_1) \). The new initial data for the DO on the new Cauchy surface \( \Sigma_{\tau_0}^{(G_1)} \), corresponding to the same universe identified by the initial data on \( \Sigma_{\tau_0}^{(G)} \), have to be extracted from the requirements \( x^\mu_{G_1} = x^\mu_{G_1}(x_G) \) and \( ^4g_{G_1, \mu \nu}(x_{G_1}) = \frac{\partial x^\alpha_{G_1}}{\partial x^\alpha_{G}} \frac{\partial x^\beta_{G_1}}{\partial x^\beta_{G}} ^4g_{\alpha \beta}(x_G) \).

This circular setting brings the ADM procedure to its end by determining the "universe", corresponding to the given initial conditions for the DO in every gauge and including the associated admissible dynamical definitions of simultaneity, distant clocks synchronization and gravito-magnetism.

It is important to stress, therefore, that the complete determination of the chronogeometry clearly depends upon the solution of Einstein-Hamilton equations of motion i.e., once the Hamiltonian formalism is fixed by the gauge choices, upon the initial conditions for the DO. This implies that the admissible notions of distant simultaneity turn out to be dynamically determined as said in the Introduction. However, as stressed there, within the Hamiltonian approach to metric gravity, different admissible conventions about distant simultaneity within the same universe are merely gauge-related conventions, corresponding to different complete gauge options in analogy to what happens in a non-dynamical way within the framework of parametrized Minkowski theories\(^{30}\). The admissible dynamical simultaneity notions in our class of space-times are much less in number than the non-dynamical admissible simultaneity notions in special relativity: as shown in Section VIII of the second paper in Ref.[23], if Minkowski space-time is thought of as a special solution (with vanishing DO) of Einstein-Hamilton equations, then its allowed 3+1 splittings must have 3-conformally flat simultaneity 3-surfaces (due to the vanishing of the DO the Cotton-York tensor vanishes), a restriction absent in special relativity considered as an autonomous theory.

We believe that this result throws an interesting new light even on the old - and outdated - debate about the so-called conventionality of distant simultaneity in special relativity.

\(^{30}\)See Ref.[12] for a discussion of this point in special relativity and the gauge nature of the admissible notions of simultaneity in parametrized Minkowski theories.
showing the trading of *conventionality* with *gauge freedom*. It is clear that the mechanism of the complete Hamiltonian gauge based on the 3+1 splitting of space-time plays a crucial role here.

Of course, it rests to be shown how the above dynamical determination can be enforced *in practice* to synchronize actual clocks, i.e., essentially, how to generalize to the gravity case the formal structure of Einstein-Reichenbach’s convention. This discussion is given in all details in Ref.[12].
IV. ON THE PHYSICAL INTERPRETATION OF DIRAC OBSERVABLES AND GAUGE VARIABLES: TIDAL-LIKE AND INERTIAL-LIKE EFFECTS.

Let us now discuss with a greater detail the physical meaning of the Hamiltonian gauge variables and DO.

As shown in Section III, the 20 off-shell canonical variables of the ADM Hamiltonian description are naturally subdivided into two sets by the quasi-Shanmugadhasan transformation:

i) The first set contains seven off-shell Abelian Hamiltonian gauge variables whose conjugate momenta are seven Abelianized first class constraints. The eighth canonical pair comprises the variable in which the super-hamiltonian constraint has to be solved (the conformal factor of the 3-metric, $\phi = (3g)^{1/12}$) and its conjugate momentum as the eighth gauge variable. Precisely, the gauge variables are: $\xi^r$, $\pi_\phi$ (primary gauge variables), $n$, $n_r$ (secondary gauge variables). Note that a primary gauge variable has its arbitrariness described by a Dirac multiplier, while a secondary gauge variable inherits the arbitrariness of the Dirac multipliers through the Hamilton equations.

ii) The second set contains the off-shell gauge invariant (non-local and in general non-tensor) DO: $r_\alpha(\tau, \vec{\sigma})$, $\pi_\alpha(\tau, \vec{\sigma})$, $\bar{\alpha} = 1, 2$. They satisfy hyperbolic Hamilton equations.

Let us stress again that the above subdivision of canonical variables in two sets is a peculiar outcome of the quasi-Shanmugadhasan canonical transformation which has no simple counterpart within the Lagrangian viewpoint at the level of the Hilbert action and/or of Einstein’s equations: at this level the only clear statement is whether or not the curvature vanishes. As anticipated in the Introduction, this subdivision amounts to an extra piece of (non-local) information which should be added to the traditional wisdom of the equivalence principle asserting the local impossibility of distinguishing gravitational from inertial effects. Indeed, we shall presently see that it allows to distinguish and visualize which aspects of the local physical effects on test matter contain a genuine gravitational component and which aspects depend solely upon the choice of the global non-inertial space-time laboratory with the associated atlas of 4-coordinate systems in a topologically trivial space-time: these latter effects could then be named inertial, in analogy with what happens in the non-relativistic Newtonian case in global rigid non-inertial reference frames. Recall again that, when a complete choice of gauge is made, the gauge variables as well as any tensorial quantity become fixed uniquely by the gauge-fixing procedure to functions of DO in that gauge.

One should be careful in discussing this subject because the very definition of inertial force (and gravitational as well) seems rather unnatural in general relativity. We can take advantage, however, from the circumstance that the Hamiltonian point of view leads naturally to a re-reading of geometrical features in terms of the traditional concept of force.

First of all, recall that we are still considering here the case of pure gravitational field without matter. It is then natural first of all to characterize as genuine gravitational effects those which are directly correlated to the DO. It is also crucial to stress that such purely gravitational effects are absent in Newtonian gravity, where there are no autonomous gravitational fields, i.e., fields not generated by matter sources. It seems therefore plausible to trace inertial (much better than fictitious, in the relativistic case) effects to a pure off-shell
dependence on the Hamiltonian gauge variables\textsuperscript{31}. Recall also that, at the non-relativistic level, Newtonian gravity is fully described by action-at-a-distance forces and, in absence of matter, there are no tidal forces among test particles. Tidal-like forces are entirely determined by the variation of the action-at-a-distance force created by the Newton potential of a massive body on the test particles. In vacuum general relativity instead the geodesic deviation equation shows that tidal forces, locally described by the Riemann tensor, act on test particles even in absence of any kind of matter.

Indeed fixing the off-shell Hamiltonian gauge variables determines the weak ADM energy density $\mathcal{E}_{ADM}^G(\tau, \vec{\sigma})$ and the Hamilton equations (3.8). Therefore, from these equations the form of the effective inertial forces $F^G_a$ is uniquely determined: they describe the form in which physical gravitational effects determined by the DO show themselves. Such appearances undergo inertial changes upon going from one global non-inertial reference frame to another. Furthermore genuine gravitational effects are always necessarily dressed by inertial-like appearances. Thus, the situation is only vaguely analogous to the phenomenology of non-relativistic inertial forces. These latter describe purely apparent (or really fictitious) mechanical effects which show up in accelerated Galilean reference frames\textsuperscript{32} and can be eliminated by going to (global) inertial reference frames\textsuperscript{33}. Besides the existence of autonomous gravitational degrees of freedom, it is therefore clear that the further deep difference concerning inertial-like forces in the general-relativistic case with respect to Newtonian gravity rests upon the fact that now inertial reference frames exist only locally if freely falling along 4-geodesics.

For the sake of clarity, consider the non-relativistic Galilean framework in greater detail. If a global non-inertial reference frame has translational acceleration $\vec{u}(t)$ and angular velocity $\vec{\omega}(t)$ with respect to a given inertial frame, a particle with free motion ($\vec{a} = \ddot{\vec{x}} = 0$) in the inertial frame has the following acceleration as seen from the non-inertial frame

\textsuperscript{31} By introducing dynamical matter the Hamiltonian procedure leads to distinguish among action-at-a-distance, gravitational, and inertial effects, with consequent relevant implications upon concepts like gravitational passive and active masses and, more generally, upon the problem of the origin of inertia. See Ref.\textsuperscript{[51]} for other attempts of separating inertial from tidal effects in the equations of motion in configuration space for test particles, in a framework in which asymptotic inertial observers are refuted. In this reference one finds also the following version (named Mach 11) of the Mach principle: "The so-called inertial effects, occurring in a non-inertial frame, are gravitational effects caused by the distribution and motion of the distant matter in the universe, relative to the frame". Thus inertial means here non-tidal + true gravitational fields generated by cosmic matter. In the above reference it is also suggested that super-fluid Helium II may be an alternative to fixed stars as a standard of non rotation. Of course all these interpretations are questionable. On the other hand, the Hamiltonian framework offers the tools for making such a distinction while distant matter effects are hidden in the non-locality of DO and gauge variables. Since in a fixed gauge the gauge variables are functions of the DO in that gauge, tidal effects are clearly mixed with inertial ones. For a recent critical discussion about the origin of inertia and its connection with inertial effects in accelerated and rotating frames see Ref.\textsuperscript{[52]}.

\textsuperscript{32} With arbitrary global translational and rotational 3-accelerations.

\textsuperscript{33} See Ref.\textsuperscript{[53]} for the determination of quasi-inertial reference frames in astronomy as those frames in which rotational and linear acceleration effects lie under the sensibility threshold of the measuring instruments.
\[ \vec{a}_{NI} = -\vec{\omega}(t) + \vec{x} \times \dot{\vec{\omega}}(t) + 2 \dot{\vec{x}} \times \vec{\omega}(t) + \vec{\omega}(t) \times [\vec{x} \times \vec{\omega}(t)]. \] (4.1)

After multiplication of this equation by the particle mass, the second term on the right hand side is the Jacobi force, the third is the Coriolis force and the fourth the centrifugal force.

We have given in Ref.[54] a description of non-relativistic gravity which is generally covariant under arbitrary passive Galilean coordinate transformations \([t' = T(t), \vec{x}' = \vec{f}(t, \vec{x})]\). The analogue of Eq.(4.1) in this case contains more general apparent forces, which are reduced to those appearing in Eq.(4.1) in particular rigid coordinate systems. The discussion given in Ref.[54] is a good introduction to the relativistic case, just because in general relativity there are no global inertial reference frames.

Two different approaches have been considered in the literature in the general relativistic case concerning the choice of reference frames, namely using either

i) a single accelerated time-like observer with an arbitrary associated tetrad,

or

ii) a congruence of accelerated time-like observers with a conventionally chosen associated field of tetrads\textsuperscript{34}.

Usually, in both approaches the observers are test observers, which describe phenomena from their kinematical point of view without generating any dynamical effect on the system.

i) Consider first the case of a single test observer with his tetrad (see Ref.[55, 56]).

After the choice of the associated local Minkowskian system of (Riemann-Gaussian) 4-coordinates, the line element becomes\textsuperscript{35} \[ ds^2 = -\delta_{ij} dx^i dx^j + 2\epsilon_{ijk} x^j \frac{\dot{\phi}}{c} dx^k dx^i + [1 + 2 \frac{\vec{a}}{c^2} (dx^o)^2]. \] The test observer describes a nearby time-like geodesics \( y^\mu(\lambda) \) (\( \lambda \) is the affine parameter or proper time) followed by a test particle in free fall in a given gravitational field by means of the following spatial equation: \[ \frac{d^2 \vec{a}}{(dy^o)^2} = -\vec{a} - 2 \vec{\omega} \times \frac{dy^i}{dy^o} + \frac{2}{c^2} \left( \vec{a} \cdot \frac{dy^o}{dy^o} \right) \frac{dy^i}{dy^o}. \] Thus, the relative acceleration of the particle with respect to the observer with this special system of coordinates\textsuperscript{36} is composed by the observer 3-acceleration plus a relativistic correction and

\textsuperscript{34} The time-like tetrad field is the 4-velocity field of the congruence. The conventional choice of the spatial triad is equivalent to a choice of a specific system of gyroscopes (see footnote 45 in Appendix A for the definition of a Fermi-Walker transported triad). See the local interpretation in Ref.[22] of inertial forces as effects depending on the choice of a congruence of time-like observers with their associated tetrad fields as a reference standard for their description. Note that, in gravitational fields without matter, gravitomagnetic effects as described by \( ^4g_{\tau r} \) are purely inertial effects in our sense, since are determined by the shift gauge variables. While in metric gravity the tetrad fields are used only to rebuild the 4-metric, the complete theory taking into account all the properties of the tetrad fields is tetrad gravity [23].

\textsuperscript{35} If the test observer is in free fall (geodesic observer) we have \( \vec{a} = 0 \). If the triad of the test observer is Fermi-Walker transported (standard of non-rotation of the gyroscope) we have \( \vec{\omega} = 0 \).

\textsuperscript{36} It replaces the global non-inertial non-relativistic reference frame. With other coordinate systems, other terms would of course appear.
by a Coriolis acceleration\textsuperscript{37}. Note that, from the Hamiltonian point of view, the constants $\vec{a}$ and $\vec{\omega}$ are constant functionals of the DO of the gravitational field in this particular gauge.

As said above, different Hamiltonian gauge fixings on-shell, corresponding to on-shell variations of the Hamiltonian gauge variables, give rise to different appearances of the physical effects as gauge-dependent functionals of the DO in that gauge of the type $F_G(r_{\dot{a}}, \pi_{\dot{a}})$ (like $\vec{a}$ and $\vec{\omega}$ in the previous example).

In absence of matter, we can consider the zero curvature limit, which is obtained by putting the DO to zero. In this way we get Minkowski space-time (a solution of Einstein’s equations) equipped with those kinds of coordinates systems which are compatible with Einstein’s theory\textsuperscript{38}. In particular, the quantities $F_G = \lim_{r_{\dot{a}}, \pi_{\dot{a}} \to 0} F_G(r_{\dot{a}}, \pi_{\dot{a}})$ describe inertial effects in those 4-coordinate systems for Minkowski space-time which have a counterpart in Einstein general relativity.

In presence of matter Newtonian gravity is recovered with a double limit:

a) the limit in which DO are restricted to the solutions of the Hamilton equations (3.7) with matter, $r_{\dot{a}} \to f_{\dot{a}}(\text{matter})$, $\pi_{\dot{a}} \to g_{\dot{a}}(\text{matter})$, so that their insertion in the Hamilton equations for matter produces effective gauge-dependent action-at-a-distance forces;

b) the $c \to \infty$ limit, in which curvature effects, described by matter after the limit a), disappear, so that the final action-at-a-distance forces are the Newtonian ones.

This implies that the functionals $F_G(r_{\dot{a}}, \pi_{\dot{a}})$ must be restricted to the limit $F_{\text{Newton}} = \lim_{c \to \infty} \lim_{r_{\dot{a}} \to f_{\dot{a}}, \pi_{\dot{a}} \to g_{\dot{a}}} \left( F_G + \frac{1}{c} F_{G1} + \ldots \right) = F_{G0}|_{r_{\dot{a}}=f_{\dot{a}}, \pi_{\dot{a}}=g_{\dot{a}}}$. Then $F_{\text{Newton}}$, which may be coordinate dependent, becomes the \textit{Newtonian inertial force} in the corresponding general Galilean coordinate system.

ii) Consider then the more general case of a \textit{congruence of accelerated time-like observers} which is just the case with reference to our \textit{global non-inertial space-time laboratory}. In this way it is possible to get a much more accurate and elaborate description of the relative 3-acceleration, as seen in his own local rest frame by each observer of the congruence which intersects the geodesic of a test particle in free fall (see Ref.[22]). The identification of various types of 3-forces depends upon:

\textsuperscript{37} This is caused by the rotation of the spatial triad carried by the observer relative to a Fermi-Walker transported triad. The vanishing of the Coriolis term justifies the statement that for an observer which is not in free fall ($\dot{a} \neq 0$) a local coordinate system produced by Fermi-Walker transport of the spatial triad of vectors is the best possible realization of a non-rotating system.

\textsuperscript{38} As shown in Ref.[13] this implies the vanishing of the Cotton-York 3-conformal tensor, namely the condition that the allowed 3+1 splittings of Minkowski space-time compatible with Einstein’s equations have the leaves \textit{3-conformally flat} in absence of matter. This solution of Einstein’s equations, has been named \textit{void space-time} in Ref.[23]: Minkowski space-time in Cartesian 4-coordinates is just a gauge representative of it. Note that, even if Einstein always rejected this concept, a void space-time corresponds to the description of a \textit{special class} of 4-coordinate systems for Minkowski space-time without matter. As a consequence special relativity, considered as an autonomous theory, admits much more general inertial effects associated with the admissible 3+1 splittings of Minkowski space-time [12] whose leaves are not 3-conformally flat.
a) the gravitational field (the form of the geodesics obviously depends on the metric tensor; usually the effects of the gravitational field are classified as gravito-electric and gravito-magnetic, even if this is strictly valid only in harmonic coordinates),

b) the properties (acceleration, vorticity, expansion, shear) of the congruence of observers,

c) the choice of the time-parameter used to describe the particle 3-trajectory in the local observer rest frame.

There are, therefore, many possibilities for defining the relative 3-acceleration (see Ref.[22]) and its separation in various types of inertial-like accelerations (See Appendix A for a more complete discussion of the properties of the congruences of time-like observers).

Summarizing, once a local reference frame has been chosen, in every 4-coordinate system we can consider:

a) the genuine tidal gravitational effects which show up in the geodesic deviation equation: they are well defined gauge-dependent functionals of the DO associated to that gauge; DO could then be called non-local tidal-like degrees of freedom;

b) the fact that geodesic curves will have different geometrical descriptions corresponding to different gauges (i.e. different inertial forces), although they will be again functionally dependent only on the DO in the relevant gauge;

c) the issue of the description of the relative 3-acceleration of a free particle in free fall, as given in the local rest frame of a generic observer of the congruence, which will contain various terms. Such terms are identifiable with the general relativistic extension of the various non-relativistic kinds of inertial accelerations and all will again depend on the DO in the chosen gauge, both directly and through the Hamiltonian gauge variables of that gauge.

Three general remarks:

First of all, the picture we have presented is not altered by the presence of matter. The only new phenomenon besides the above purely gravitational, inertial and tidal effects, is that from the solution of the super-hamiltonian and super-momentum constraints emerge action-at-a-distance, Newtonian-like and gravito-magnetic effects among matter elements, as already noted in footnote 31.

Second, the reference standards of time and length correspond to units of coordinate time and length and not to proper times and proper lengths [16]: this is not in contradiction with general covariance, because an extended laboratory, in which one defines the reference standards, corresponds to a particular completely fixed on-shell Hamiltonian gauge plus a local congruence of time-like observers. For instance, in astronomy and in the theory of satellites, the unit of time is replaced by a unit of coordinate length (ephemerides time). This leads to the necessity of taking into account the theory of measurement in general relativity.

Third, as evident from the structure of the Shanmugadhasan transformation, the distinction between tidal-like and generalized non-inertial effects is a gauge (i.e., a coordinate) dependent concept. Although we deem this result to have physical interest as it stands, the possibility remains open of pushing our knowledge even further. Precisely, in the second paper we will exploit to this effect a discussion about the relation between the notion of DO
and that of the so-called *Bergmann observables* (BO)[14] which (although rather ambiguously) are defined to be uniquely *predictable* from the initial data, but also invariant under standard *passive diffeomorphisms* (PDIQ).

A possible starting point to attack the problem of the connection of DO with BO seems to be a Hamiltonian re-formulation of the Newman-Penrose formalism [15] (that contains only PDIQ) employing Hamiltonian null-tetrads carried by the surface-forming congruence of time-like observers. In view of this program, in paper II we will argue in favor of a *main conjecture* according to which special Darboux bases for canonical gravity should exist in which the inertial effects (gauge variables) are described by PDIQ while the autonomous degrees of freedom (DO) are also BO. The hoped for validity of this conjecture would amount - among other important consequences - to attributing to our separation between *tidal-like* and *generalized inertial effects* the status of an invariant statement. This would give, in our opinion, a remarkable contribution to the long standing debate about the equivalence principle.
APPENDIX A: TIME-LIKE ACCELERATED OBSERVERS.

In this Appendix we collect a number of scattered properties of time-like observers.

An inertial observer in Minkowski space-time $M^4$ is a time-like future-oriented straight line $\gamma$ [57]. Any point $P$ on $\gamma$ together with the unit time-like tangent vector $e^\mu_{(o)}$ at $P$ is an instantaneous inertial observer. Let us choose a point $P$ on $\gamma$ as the origin of an inertial system $I_P$ having $\gamma$ as time axis and three orthogonal space-like straight lines orthogonal to $\gamma$ in $P$, with unit tangent vectors $e^\mu_r$, $r = 1, 2, 3$ as space axes. Let $x^\mu$ be a Cartesian 4-coordinate system referred to these axes, in which the line element has the form $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ with $\eta_{\mu\nu} = \epsilon (++--)$, $\epsilon = \pm 1$. Associated to these coordinates there is a reference frame (or system of reference or platform [16]) given by the congruence of time-like straight lines parallel to $\gamma$, namely a unit vector field $u^\mu(x)$. Each of the integral lines of the vector field is identified by a fixed value of the three spatial coordinates $x^i$ and represent an observer: this is a reference point according to Møller [58]. A reference frame $l$, i.e. a time-like vector field $l^\mu(x) \frac{\partial}{\partial x^\mu}$ with its congruence of time-like world-lines and its associated 1+3 splitting of $TM^4$, admits the decomposition of Eq.(A3) (see below).

While in Newtonian physics an absolute reference frame is an imagined extension of a rigid body and a clock (with any coordinate systems attached), in general relativity [59] we must replace the rigid body either by a cloud of test particles in free fall (geodesic congruence) or by a test fluid (non-geodesic congruence for non-vanishing pressure). Therefore a reference frame is schematized as a future-pointing time-like congruence with all the possible associated 4-coordinate systems. This is called a platform in Ref.[57], where there is a classification of the possible types of platforms and the definition of the position vector of a neighboring observer in the local rest frame of a given observer of the platform. Then, the Fermi-Walker covariant derivative (applied to a vector in the rest frame it produces a new vector still in the rest frame [60]) is used to define the 3-velocity (and then the 3-acceleration) of a neighboring observer in the rest frame of the given observer, as the natural generalization of the Newtonian relative 3-velocity (and 3-acceleration). See Ref.[22] for a definition, based on these techniques, of the 3-acceleration of a test particle in the local rest frame of an observer crossing the particle geodesics, with the further introduction of the Lie and co-rotating Fermi-Walker derivatives.

Consider now the point of view of the special (non-rotating, surface-forming) congruence of time-like accelerated observers whose 4-velocity field is the field of unit normals to the space-like hyper-surfaces $\Sigma_\tau$.

We want to describe this non-rotating Hamiltonian congruence, by emphasizing its interpretation in terms of gauge variables and DO. The field of contravariant and covariant unit normals to the space-like hyper-surfaces $\Sigma_\tau$ are expressed only in terms of the lapse and shift gauge variables (as in Sections III and IV, we use coordinates adapted to the foliation: $l^A(\tau, \vec{\sigma}) = b^A_\mu(\tau, \vec{\sigma}) l^\mu(\tau, \vec{\sigma})$ with the $b^A_\mu(\tau, \vec{\sigma}) = \frac{\partial A^A}{\partial x^\mu}$ being the transition coefficients from adapted to general coordinates )

$$l^A(\tau, \vec{\sigma}) = \frac{1}{N(\tau, \vec{\sigma})} \left(1; -N^r(\tau, \vec{\sigma})\right),$$

$$l_A(\tau, \vec{\sigma}) = N(\tau, \vec{\sigma}) \left(1; 0\right), \quad l^A(\tau, \vec{\sigma}) l_A(\tau, \vec{\sigma}) = 1.$$

(A1)
Since this congruence is surface forming by construction, it has zero vorticity and is non-rotating (in the sense of congruences). As said in Section III, in Christodoulou-Klainermann space-times \cite{9} we have 
\[ N(\tau, \vec{\sigma}) = \epsilon + n(\tau, \vec{\sigma}), \]
\[ N^r(\tau, \vec{\sigma}) = n^r(\tau, \vec{\sigma}). \]
The specific time-like direction identified by the normal has inertial-like nature, in the sense of being dependent on Hamiltonian gauge variables only. Therefore the world-lines of the observers of this foliation\cite{39} change on-shell going from a 4-coordinate system to another. On the other hand, the embeddings \[ z^\mu_\tau(\tau, \vec{\sigma}) \]
of the leaves $\Sigma_{\tau}$ of the WSW foliation in space-time depend on both the DO and the gauge variables.

If $x^\mu_{\vec{\sigma}_o}(\tau)$ is the time-like world-line of the observer crossing the leaf $\Sigma_{\tau_o}$ at $\vec{\sigma}_o$, we have

\[ x^\mu_{\vec{\sigma}_o}(\tau) = z^\mu_\tau(\tau, \vec{\rho}_{\vec{\sigma}_o}(\tau)), \quad \text{with} \quad \vec{\rho}_{\vec{\sigma}_o}(\tau) = \vec{\sigma}_o, \quad \dot{x}^\mu_{\vec{\sigma}_o}(\tau) = \frac{dx^\mu_{\vec{\sigma}_o}(\tau)}{d\tau}, \]

\[ l^\mu_{\vec{\sigma}_o}(\tau) = l^\mu(\tau, \vec{\rho}_{\vec{\sigma}_o}(\tau)) = \frac{\dot{x}^\mu_{\vec{\sigma}_o}(\tau)}{\sqrt{4g_{\mu\nu}(x_{\vec{\sigma}_o}(\tau))\dot{x}^\mu_{\vec{\sigma}_o}(\tau)\dot{x}^\nu_{\vec{\sigma}_o}(\tau)}}, \]

\[ a^\mu_{\vec{\sigma}_o}(\tau) = \frac{dl^\mu_{\vec{\sigma}_o}(\tau)}{d\tau}, \quad a^\mu_{\vec{\sigma}_o}(\tau) l_{\vec{\sigma}_o\mu}(\tau) = 0. \]

(A2)

Here $a^\mu_{\vec{\sigma}_o}(\tau)$ is the 4-acceleration of the observer $x^\mu_{\vec{\sigma}_o}(\tau)$.

As for any congruence, we have the decomposition $(P_{\mu\nu} = \eta_{\mu\nu} - l_{\mu} l_{\nu})$

\[ 4\nabla_\mu l_\nu = l_\mu a_\nu + \frac{1}{3} \Theta P_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}, \]

\[ a^\mu = l^\nu 4\nabla_\nu l^\mu = \dot{l}^\mu, \]

\[ \Theta = 4\nabla_\mu l^\mu, \]

\[ \sigma_{\mu\nu} = \frac{1}{2} \left( a_\mu l_\nu + a_\nu l_\mu \right) + \frac{1}{2} \left( 4\nabla_\mu l_\nu + 4\nabla_\nu l_\mu \right) - \frac{1}{3} \Theta P_{\mu\nu}, \]

with magnitude $\sigma^2 = \frac{1}{2} \sigma_{\mu\nu} \sigma^{\mu\nu},$

\[ \omega_{\mu\nu} = -\omega_{\nu\mu} = \epsilon_{\mu\nu\alpha\beta} \omega^\alpha l^\beta = \frac{1}{2} \left( a_\mu l_\nu - a_\nu l_\mu \right) + \frac{1}{2} \left( 4\nabla_\mu l_\nu - 4\nabla_\nu l_\mu \right) = 0, \]

\[ \omega^\mu = \frac{1}{2} \epsilon^{\mu\alpha\beta\gamma} \omega_{\alpha\beta} l_\gamma = 0, \]

(A3)

where $a^\mu$ is the 4-acceleration, $\Theta$ the expansion (it measures the average expansion of the infinitesimally nearby world-lines surrounding a given world-line in the congruence), $\sigma_{\mu\nu}$ the shear (it measures how an initial sphere in the tangent space to the given world-line, which

\[ \text{Footnote 39: It is called the Wigner-Sen-Witten (WSW) foliation\cite{13} due to its properties at spatial infinity (see footnote 14). The associated observers are called Eulerian observers when a perfect fluid is present as dynamical matter.} \]

\[ \text{Footnote 40: Note that the mathematical time parameter} \ \tau \ \text{labeling the leaves of the foliation is not in general the proper time of any observer of the congruence.} \]

33
is Lie transported along $l^\mu$ \footnote{It has zero Lie derivative with respect to $l^\mu \partial_\mu$.}, is distorted towards an ellipsoid with principal axes given by the eigenvectors of $\sigma^\mu_\nu$, with rate given by the eigenvalues of $\sigma^\mu_\nu$ and $\omega_{\mu\nu}$ the twist or vorticity \footnote{The unit vector $\mathcal{N}^\mu(\tau, \bar{\sigma})$ contains a DO dependence in the overall normalizing factor. The existence of this space-like gauge direction seems to indicate that synchronous or time orthogonal 4-coordinates with $N_r(\tau, \bar{\sigma}) = -4 g_{rr}(\tau, \bar{\sigma}) = 0$ (absence of gravito-magnetism) have singular nature \cite{61}. Note that the evolution vector of the slicing point of view has $N(\tau, \bar{\sigma}) l^\mu(\tau, \bar{\sigma})$ and $|\vec{N}(\tau, \bar{\sigma})| N^\mu(\tau, \bar{\sigma})$ as projections along the normal and the plane tangent to $\Sigma_\tau$, respectively.} (it measures the rotation of the nearby world-lines infinitesimally surrounding the given one); $\sigma_{\mu\nu}$ and $\omega_{\mu\nu}$ are purely spatial ($\sigma_{\mu\nu} l^\nu = \omega_{\mu\nu} l^\nu = 0$). Due to the Frobenius theorem, the congruence is (locally) hyper-surface orthogonal if and only if $\omega_{\mu\nu} = 0$.

The equation $\frac{1}{3} \theta^\mu \partial_\mu l = \frac{1}{3} \Theta$ defines a representative length $l$ along the world-line of $l^\mu$, describing the volume expansion (or contraction) behaviour of the congruence.

While all these quantities depend on the Hamiltonian gauge variables, the expansion and the shear depend a priori also upon the DO, because the covariant derivative is used in their definition.

Yet, the ADM canonical formalism provides additional information. Actually, on each space-like hyper-surface $\Sigma_\tau$ of the foliation, there is a privileged contravariant space-like direction \footnote{This requires that $N_\mu dx^\mu$ is a closed 1-form, namely that in adapted coordinates we have $\partial_\tau \frac{\vec{N}}{|\vec{N}|} = \partial_\tau \frac{\vec{N}}{|\vec{N}|}$ and $\partial_r \frac{\vec{N}}{|\vec{N}|} = \partial_s \frac{\vec{N}}{|\vec{N}|}$. This requires in turn $\frac{\vec{N}}{|\vec{N}|} = \partial_\tau f$ with $\partial_\tau f = |\vec{N}| + \text{const.}$} identified by the lapse and shift gauge variables \footnote{The unit vector $N^\mu(\tau, \bar{\sigma})$ contains a DO dependence in the overall normalizing factor. The existence of this space-like gauge direction seems to indicate that synchronous or time orthogonal 4-coordinates with $N_r(\tau, \bar{\sigma}) = -4 g_{rr}(\tau, \bar{\sigma}) = 0$ (absence of gravito-magnetism) have singular nature \cite{61}. Note that the evolution vector of the slicing point of view has $N(\tau, \bar{\sigma}) l^\mu(\tau, \bar{\sigma})$ and $|\vec{N}(\tau, \bar{\sigma})| N^\mu(\tau, \bar{\sigma})$ as projections along the normal and the plane tangent to $\Sigma_\tau$, respectively.}.

\begin{align}
\mathcal{N}^\mu(\tau, \bar{\sigma}) &= \frac{1}{|\vec{N}(\tau, \bar{\sigma})|} \begin{pmatrix} 0; n^r(\tau, \bar{\sigma}) \end{pmatrix}, \\
\mathcal{N}_\mu(\tau, \bar{\sigma}) &= |\vec{N}(\tau, \bar{\sigma})| \begin{pmatrix} 1; \frac{N_r(\tau, \bar{\sigma})}{|\vec{N}(\tau, \bar{\sigma})|^2} \end{pmatrix}, \\
N^\mu(\tau, \bar{\sigma}) \mathcal{N}_\mu(\tau, \bar{\sigma}) &= 0, \quad N^\mu(\tau, \bar{\sigma}) \mathcal{N}_\mu(\tau, \bar{\sigma}) = -1, \\
|\vec{N}(\tau, \bar{\sigma})| &= \sqrt{3 g_{rs} N_r N_s}(\tau, \bar{\sigma}).
\end{align} \tag{A4}

If 4-coordinates, corresponding to an on-shell complete Hamiltonian gauge fixing, exist such that the vector field defined by $\mathcal{N}^\mu(\tau, \bar{\sigma})$ on each $\Sigma_\tau$ is surface-forming (zero vorticity\footnote{This requires that $N_\mu dx^\mu$ is a closed 1-form, namely that in adapted coordinates we have $\partial_\tau \frac{\vec{N}}{|\vec{N}|} = \partial_\tau \frac{\vec{N}}{|\vec{N}|}$ and $\partial_r \frac{\vec{N}}{|\vec{N}|} = \partial_s \frac{\vec{N}}{|\vec{N}|}$. This requires in turn $\frac{\vec{N}}{|\vec{N}|} = \partial_\tau f$ with $\partial_\tau f = |\vec{N}| + \text{const.}$}), then each $\Sigma_\tau$ can be foliated with 2-surfaces, and the 3+1 splitting of space-time becomes a (2+1)+1 splitting corresponding to the 2+2 splittings studied by Stachel and d’Inverno \cite{62}.

We have therefore a natural candidate for one of the three spatial vectors of each observer, namely: $E^\mu_{\bar{\sigma}, (\mathcal{N})}(\tau) = \mathcal{N}^\mu_{\bar{\sigma}}(\tau) = N^\mu(\tau, \bar{\sigma}) l^\mu(\tau, \bar{\sigma})$. By means of $l^\mu_{\bar{\sigma}}(\tau) = l^\mu(\tau, \bar{\sigma})$ and $\mathcal{N}^\mu_{\bar{\sigma}}(\tau)$, we can construct two null vectors at each space-time point.
\[ K^\mu_{\sigma_o}(\tau) = \sqrt{\frac{|N|}{2}} \left( l^\mu_{\sigma_o}(\tau) + N^\mu_{\sigma_o}(\tau) \right), \]
\[ L^\mu_{\sigma_o}(\tau) = \frac{1}{\sqrt{2|N|}} \left( l^\mu_{\sigma_o}(\tau) - N^\mu_{\sigma_o}(\tau) \right). \]

and then get a null tetrad of the type used in the Newman-Penrose formalism [15]. The last two axes of the spatial triad can be chosen as two space-like circular complex polarization vectors \( E^\mu_{\sigma_o}(\pm) (\tau) \), like in electromagnetism. They are built starting from the transverse helicity polarization vectors \( E^\mu_{\sigma_o(1,2)} (\tau) \), which are the first and second columns of the standard Wigner helicity boost generating \( K^\mu_{\sigma_o}(\tau) \) from the reference vector \( K^\mu_{\sigma_o}(\tau) = |N| \left( 1; 001 \right) \) (see for instance the Appendices of Ref.[63]).

Let us call \( E^{(ADM)\mu}_{\sigma_o(a)} (\tau) \) the ADM tetrad formed by \( l^\mu_{\sigma_o}(\tau), N^\mu_{\sigma_o}(\tau), E^\mu_{\sigma_o(1,2)} (\tau) \). This tetrad will not be in general Fermi-Walker transported along the world-line \( x^\mu_{\sigma_o}(\tau) \) of the observer\(^{44}\).

Another possible (but only on-shell) choice of the spatial triad together with the unit normal to \( \Sigma_\tau \) is the local WSW (on-shell) compass of inertia quoted in footnote 14, namely the triads transported with the Frauendiener-Sen-Witten transport (see footnote 73 and Eq.(12.2) of Ref.[13]) starting from an asymptotic conventional triad (choice of the fixed stars) added to the ADM 4-momentum at spatial infinity. As shown in Eq.(12.3) of Ref.[13], they have the expression \( E^{(WSW)\mu}_{\sigma_o(a)} (\tau) = \frac{\partial x^\mu_{\sigma_o(\tau)}}{\partial \tau} \sqrt{\det \Sigma_{\sigma_o(\tau)}} \) where the triad \( E^{(WSW)\mu}_{\sigma_o(a)} (\tau) \) is

\(^{44}\)It is a tetrad in adapted coordinates: if \( E^\mu_{(a)} = \frac{\partial x^\mu}{\partial \sigma^a} E^A_{(a)} \), then \( E^{(ADM)A}_{\sigma_o(a)} (\tau) \). Given the 4-velocity \( l^\mu_{\sigma_o}(\tau) = E^\mu_{\sigma_o(\tau)} (\tau) \) of the observer, the spatial triads \( E^\mu_{\sigma_o(a)} (\tau), a = 1,2,3 \), have to be chosen in a conventional way, namely by means of a conventional assignment of an origin for the local measurements of rotations. Usually, the choice corresponds to Fermi-Walker (FW) transported (gyroscope-type transport, non-rotating observer) tetrads \( E^{(FW)\mu}_{\sigma_o(a)} (\tau) \), such that

\[
\frac{D}{D\tau} E^{(FW)\mu}_{\sigma_o(a)} (\tau) = \Omega^{(FW)\mu\nu}_{\sigma_o} E^{(FW)\nu}_{\sigma_o(a)} (\tau) = l^\mu_{\sigma_o}(\tau) a_{\sigma_o}^\nu(\tau) E^{(FW)\nu}_{\sigma_o(a)} (\tau),
\]
\[
\Omega^{(FW)\mu\nu}_{\sigma_o} = \frac{\partial x^\mu_{\sigma_o(\tau)}}{\partial \tau} l^\nu_{\sigma_o}(\tau) - \frac{\partial x^\nu_{\sigma_o(\tau)}}{\partial \tau} l^\mu_{\sigma_o(\tau)}.
\]

The triad \( E^{(FW)\mu}_{\sigma_o(a)} (\tau) \) is the correct relativistic generalization of global Galilean non-rotating frames (see Ref.[54]) and is defined using only local geometrical and group-theoretical concepts. Any other choice of the triads (Lie transport, co-rotating-FW transport, ...) is obviously also possible [22]. A generic triad \( E^\mu_{\sigma_o(a)} (\tau) \) will satisfy \( \frac{D}{D\tau} E^\mu_{\sigma_o(a)} (\tau) = \Omega^\mu_{\sigma_o} E^\nu_{\sigma_o(a)} + \Omega^{(SR)}_{\sigma_o} \) with \( \Omega^\mu_{\sigma_o} = \Omega^{(FW)\mu\nu} + \Omega^{(SR)\mu\nu} \) with the spatial rotation part \( \Omega^{(SR)\mu\nu} = \epsilon_{\mu \nu \alpha \beta} J^{\alpha \beta}_{\sigma_o} \) with \( J^{\mu}_{\sigma_o} = 0 \), producing a rotation of the gyroscope in the local space-like 2-plane orthogonal to \( l^\mu_{\sigma_o} \) and \( J^\mu_{\sigma_o} \).
solution of the Frauendiener-Sen-Witten equation restricted to a solution of Einstein equations.

Given an observer with world-line $x^\mu_\sigma (\tau)$ and tetrad $E^\mu_\sigma (\alpha) (\tau)$, the geometrical properties are described by the Frenet-Serret equations [64]

$$
\frac{D}{D\tau} l^\mu_\sigma (\tau) = \kappa^\mu_\sigma (\tau) E^\mu_\sigma (1)(\tau),
$$

$$
\frac{D}{D\tau} E^\mu_\sigma (1)(\tau) = a^\mu_\sigma (\tau) = \kappa^\mu_\sigma (\tau) l^\mu_\sigma (\tau) + \tau^\sigma_\sigma (1) E^\mu_\sigma (2)(\tau),
$$

$$
\frac{D}{D\tau} E^\mu_\sigma (2)(\tau) = -\tau^\sigma_\sigma (1) E^\mu_\sigma (1)(\tau) + \tau^\sigma_\sigma (2) E^\mu_\sigma (3)(\tau),
$$

$$
\frac{D}{D\tau} E^\mu_\sigma (3)(\tau) = -\tau^\sigma_\sigma (2) E^\mu_\sigma (2)(\tau),
$$

(A6)

where $\kappa^\mu_\sigma (\tau)$, $\tau^\sigma_\sigma (\alpha) (\tau)$, $\alpha = 1, 2$, are the curvature and the first and second torsion of the world-line. $E^\mu_\sigma (\alpha)(\tau)$, $\alpha = 1, 2, 3$ are said the normal and the first and second bi-normal of the world-line, respectively.

Let us now look at the description of a geodesics $y^\mu (\tau)$, the world-line of a scalar test particle, from the point of view of those observers $y^\mu_{\sigma \nu} (\tau)$ of the congruence who intersect it, namely such that at $\tau$ it holds $x^\mu_{\sigma \nu} (\tau) (\tau) = y^\mu (\tau)$. The family of these observers is called a relative observer world 2-sheet in Ref. [22].

Since the parameter $\tau$ labeling the leaves $\Sigma_\tau$ of the foliation is not the proper time $s = s(\tau)$ of the geodesics $y^\mu (\tau) = Y^\mu (s(\tau))$, the geodesics equation

$$(\frac{d^2 Y^\mu(s)}{ds^2}) + 4 \Gamma^\mu_{\alpha\beta}(Y(s)) \frac{dy^\alpha(s)}{ds} \frac{dy^\beta(s)}{ds} = 0 \quad \text{(or } m a^\mu(s) = m \frac{d^2 Y^\mu(s)}{ds^2} = F^\mu(s), \text{ where } m \text{ is the mass of the test particle}),$$

becomes

$$
\frac{d^2 y^\mu(\tau)}{d\tau^2} + 4 \Gamma^\mu_{\alpha\beta}(y(\tau)) \frac{dy^\alpha(\tau)}{d\tau} \frac{dy^\beta(\tau)}{d\tau} - \frac{dy^\mu(\tau)}{d\tau} \frac{d2s(\tau)}{d\tau^2} \left( \frac{ds(\tau)}{d\tau} \right)^{-1} = 0,
$$

(A7)

or

$$
m a^\mu_y(\tau) = m \frac{d^2 y^\mu(\tau)}{d\tau^2} = f^\mu(\tau).
$$

(A8)

We see that the force $f^\mu(\tau)$ contains an extra-piece with respect to $F^\mu(s(\tau))$, due to the change of time parameter.

Let $U^\mu(\tau) = V^\mu(s(\tau)) = \frac{dY^\mu(s)}{ds} |_{s=s(\tau)} = \frac{\dot{y}^\mu(s)}{\sqrt{g_{\alpha\beta}(y(s)) \dot{y}^\alpha(s) \dot{y}^\beta(s)}}$ with $\dot{y}^\mu(s) = \frac{dy^\mu(s)}{d\tau}$ be the 4-velocity of the test particle and $ds = \sqrt{4 g_{\alpha\beta}(y(s)) \dot{y}^\alpha(s) \dot{y}^\beta(s)} d\tau$ be the relation between the two parameters. By using the intrinsic or absolute derivative along the geodesics parametrized with the proper time $s = s(\tau)$, the geodesics equation becomes

$$
\mathcal{A}^\mu(s) = \frac{DV^\mu(s)}{ds} = 0 \quad \text{or } \dot{\mathcal{A}}^\mu(\tau) = \frac{dU^\mu(\tau)}{d\tau} = \frac{dy^\mu(\tau)}{d\tau} \frac{d2s(\tau)}{d\tau^2} \left( \frac{ds(\tau)}{d\tau} \right)^{-1} = g^\mu(\tau).
$$
In non-relativistic physics spatial inertial forces are defined as minus the spatial relative accelerations, with respect to an accelerated global Galilean frame (see Ref.
[54]). In general relativity one needs the whole relative observer world 2-sheet to define an abstract 3-path in the quotient space of space-time by the observer-family world-lines, representing the trajectory of the test particle in the observer 3-space. Moreover, a well defined projected time derivative is needed to define a relative acceleration associated to such 3-path. At each point \( P(\tau) \) of the geodesics, identified by a value of \( \tau \), we have the two vectors \( U^{\mu}(\tau) \) and \( l^{\mu}_{\sigma_y(\tau)}(\tau) \). Therefore, each vector \( X^{\mu} \) in the tangent space to space-time in that point \( P(\tau) \) admits two splittings:

i) \( X^{\mu} = X_{U} U^{\mu} + P(U)^{\mu}_{\nu} X^{\nu} \), \( P^{\mu\nu}(U) = 4 g^{\mu\nu} - U^{\mu} U^{\nu} \), i.e., into a temporal component along \( U^{\mu} \) and a spatial transverse component, living in the local rest frame \( LRS_U \);

ii) \( X^{\mu} = X_{l} l^{\mu}_{\sigma_y(\tau)} + P(l_{\sigma_y(\tau)})^{\mu}_{\nu} X^{\nu} \), i.e., into a temporal component along \( l^{\mu}_{\sigma_y(\tau)}(\tau) \) and a spatial transverse component, living in the local rest frame \( LRS_l \), which is the plane tangent to the leave \( \Sigma_{\tau} \) in \( P(\tau) \) for our surface-forming congruence.

The measurement of \( X^{\mu} \) by the observer congruence consists in determining the scalar \( X_l \) and the spatial transverse vector. In adapted coordinates and after a choice of the spatial triads, the spatial transverse vector is described by the three (coordinate independent) tetradic components \( X_{(a)} = E_{(a)}^{\mu} X^{\mu} \). The same holds for every tensor. Moreover, every spatial vector like \( P(U)^{\mu}_{\nu} X^{\nu} \) in \( LRS_U \) admits a 2+1 orthogonal decomposition (relative motion orthogonal decomposition) into a component in the 2-dimensional rest subspace \( LRS_U \cap LRS_L \) transverse to the direction of relative motion and one component in the 1-dimensional (longitudinal) orthogonal complement along the direction of the relative motion in each such rest space.

At each point \( P(\tau) \), the tangent space is split into the relative observer 2-plane spanned by \( U^{\mu}(\tau) \) and \( l^{\mu}_{\sigma_y(\tau)}(\tau) \) and into an orthogonal space-like 2-plane. We have the 1+3 orthogonal decomposition

\[
U^{\mu}(\tau) = \gamma(U,l)(\tau) \left( l^{\mu}_{\sigma_y(\tau)}(\tau) + \nu^{\mu}(U,l)(\tau) \right),
\]

\[
\gamma(U,l) = U_{\mu} l^{\mu}_{\sigma_y(\tau)}; \quad \nu(U,l) = \sqrt{\nu^{\mu}(U,l) \nu_{\mu}(U,l)},
\]

\[
\nu^{\mu}(U,l) = \frac{\nu^{\mu}(U,l)}{\nu(U,l)}, \quad \text{relative 4-velocity tangent to } \Sigma_{\tau}.
\]

The equation of geodesics, written as \( m A^{\mu}(s) = 0 \), is described by the observers’ family as:

i) a temporal projection along \( l^{\mu}_{\sigma_y(\tau)} \), leading to the evolution equation \( m A_{\mu} l^{\mu}_{\sigma_y(\tau)} = 0 \), for the observed energy \( (E(U,l) = \gamma(U,l)) \) of the test particle along its world-line;

ii) a spatial projection orthogonal to \( l^{\mu}_{\sigma_y(\tau)} \) (tangent to \( \Sigma_{\tau} \)), leading to the evolution equation for the observed 3-momentum of the test particle along its world-line, with the kinematic quantities describing the motion of the family of observers entering as inertial forces. If, instead of writing \( m P(l)^{\mu}_{\nu} A^{\mu}(s) = 0 \) with \( P(l)^{\mu\nu} = 4 g^{\mu\nu} - l^{\mu}_{\sigma_y(\tau)} l^{\nu}_{\sigma_y(\tau)} \), we rescale the particle proper time \( s(\tau) \) to the sequence of observer proper times \( s_{U,l}(\tau) \) defined by
\( \frac{ds(U,l)}{ds} = \gamma(U,l) \), the spatial projection of the geodesics equation, re-scaled with the gamma factor \(^{46}\), can be written in the form

\[
m \left( \frac{D_{(FW)}(U,l)}{ds(U,l)} \right)_{\mu}^{\nu} v^{\nu}(U,l) = m a_{(FW)}^{\mu}(U,l) = F_{(FW)}^{(G)\mu}(U,l),
\]

\[
F_{(FW)}^{(G)\mu}(U,l) = -\gamma(U,l)^{-1} P_{\mu\nu}(l) \frac{Dl_\nu^{\mu}(\tau(s))}{ds} = -\left( \frac{D_{(FW)}(U,l)}{ds(U,l)} \right)_{\mu}^{\nu} \gamma^{\nu}(S(U,l)) = \gamma(U,l) \left[ -a_{\nu}(l) + \left( -\omega_{\nu\nu}(l) + \Theta_{\nu}(l) \right) v^\nu(U,l) \right],
\]

where \( v^\mu(U,l) = U^\mu - \gamma(U,l) l_{\sigma \nu}(\tau) = v(U,l) \dot{v}^\mu(U,l) \) with \( v(U,l) = \gamma(U,l) v(U,l) \), and \( P(l)^{\mu}_{\nu} \frac{Dl}{ds} = \left( \frac{D_{(FW)}(U,l)}{ds(U,l)} \right)^{\mu}_{\nu} \) is the spatial FW intrinsic derivative along the test world-line and \( a_{(FW)}^{\mu}(U,l) \) is the FW relative acceleration. The term \( F_{(FW)}^{(G)\mu}(U,l) \) can be interpreted as the set of inertial forces due to the motion of the observers themselves, as in the non-relativistic case. Such inertial forces depend on the following congruence properties:

i) the acceleration vector field \( a^\mu(l) \), leading to a gravito-electric field and a spatial gravito-electric gravitational force;

ii) the vorticity \( \omega_{\mu\nu}(l) \) and expansion + shear \( \Theta_{\mu\nu}(l) \) mixed tensor fields, leading to a gravito-magnetic vector field and tensor field and a Coriolis or gravito-magnetic force linear in the relative velocity \( v^\mu(U,l) \).

Then, by writing \( v^\mu(U,l) = v(U,l) \dot{v}^\mu(U,l) \), the FW relative acceleration can be decomposed into a longitudinal and a transverse relative acceleration

\[
a_{(FW)}^{\mu}(U,l) = \frac{D_{(FW)}(U,l)}{ds(U,l)} v(U,l) \dot{v}^\mu(U,l) + \gamma(U,l) a_{(FW)}^{(\perp)\mu}(U,l),
\]

\[
a_{(FW)}^{(\perp)\mu}(U,l) = v(U,l) \left( \frac{D_{(FW)}(U,l)}{ds(U,l)} \right)_{\nu}^{\mu} \dot{v}^{\nu}(U,l) = v^2(U,l) \left( \frac{D_{(FW)}(U,l)}{ds(U,l)} \right)_{\nu}^{\mu} \dot{v}^{\nu}(U,l) = \frac{d^2_{(FW)}(U,l)}{d\tau(U,l)} \gamma(U,l) \gamma^{\nu}(U,l).
\]

In the second expression of the transverse FW relative acceleration, the reparametrization \( \frac{d\tau(U,l)}{ds(U,l)} = \nu(U,L) \) to a spatial arc-length parameter has been done. Since \( \gamma(U,l) a_{(FW)}^{(\perp)\mu}(U,l) \) is the transverse part of the relative acceleration, i.e. the FW relative centripetal acceleration, \(-m \gamma(U,l) a_{(FW)}^{(\perp)\mu}(U,l)\) may be interpreted as a centrifugal force, so that the geodesics equation is rewritten as \( m \frac{D_{(FW)}(U,l)v(U,l)}{ds(U,l)} \dot{v}^\mu(U,l) = F_{(FW)}^{(G)\mu}(U,l) - m \gamma(U,l) a_{(FW)}^{(\perp)\mu}(U,l) \), with the first member called sometimes Euler force.

\(^{46}\) Namely \( m \gamma^{-1}(U,l) P(l)^{\mu}_{\nu} A^{\nu} = 0 \).
The 3-path in the abstract quotient space can be treated as an ordinary 3-curve in a 3-dimensional Riemann space. Its tangent is \( \hat{\nu}^\mu(U, l) \), while its normal and bi-normal are denoted \( \hat{\eta}_<(FW)(U, l) \) and \( \hat{\xi}_<(FW)(U, l) \) respectively. The 3-dimensional Frenet-Serret equations are then

\[
\left( \frac{D_{(FW)(U, l)}}{dr(U, l)} \right)^\mu_\nu \hat{\nu}^\nu(U, l) = \kappa_{(FW)}(U, l) \hat{\eta}_<(FW)(U, l),
\]

\[
\left( \frac{D_{(FW)(U, l)}}{dr(U, l)} \right)^\mu_\nu \hat{\eta}_<(FW)(U, l) = -\kappa_{(FW)}(U, l) \hat{\nu}^\mu(U, l) + \tau_{(FW)}(U, l) \hat{\xi}_<(FW)(U, l),
\]

\[
\left( \frac{D_{(FW)(U, l)}}{dr(U, l)} \right)^\mu_\nu \hat{\xi}_<(FW)(U, l) = -\tau_{(FW)}(U, l) \hat{\eta}_<(FW)(U, l),
\]  

(A12)

where \( \kappa_{(FW)}(U, l) = 1/\rho_{(FW)}(U, l) \) and \( \tau_{(FW)}(U, l) \) are the curvature and torsion of the 3-curve, respectively.

The main drawback of the 1+3 \((\text{threading})\) description, notwithstanding its naturalness from a locally operational point of view, is the use of a rotating congruence of time-like observers: this introduces an element of non-integrability and, as yet, no formulation of the Cauchy problem for the 1+3 re-formulation of Einstein’s equations has been worked out.
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