Likelihood ratio-based policy gradient methods for distorted risk measures: A non-asymptotic analysis

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Abstract

We propose policy-gradient algorithms for solving the problem of control in a risk-sensitive reinforcement learning (RL) context. The objective of our algorithm is to maximize the distorted risk measure (DRM) of the cumulative reward in an episodic Markov decision process (MDP). We derive a variant of the policy gradient theorem that caters to the DRM objective. Using this theorem in conjunction with a likelihood ratio (LR) based gradient estimation scheme, we propose policy gradient algorithms for optimizing DRM in both on-policy and off-policy RL settings. We derive non-asymptotic bounds that establish the convergence of our algorithms to an approximate stationary point of the DRM objective.

1 Introduction

We consider a risk-sensitive reinforcement learning (RL) problem, where an optimal policy is learned by maximizing a risk measure of the cumulative rewards. We consider the distorted risk measure (DRM), which focuses on the entire distribution of the cumulative rewards, while a risk-neutral objective is concerned with only the mean of this distribution. A DRM uses a distortion function to alter the cumulative distribution, and calculate the expected value of the risk with respect to this distorted distribution [17].

For a risk measure, coherency is a desirable property. A risk measure is coherent if it is translation invariant, subadditive, positive homogeneous, and monotonic [11]. A DRM is coherent if the distortion function is concave [18]. Several risk measures, including the popular Conditional Value-at-Risk (CVaR), can be expressed as a DRM. For a random variable $X$ modeling the underlying rewards, CVaR of $X$ is the mean of the rewards below a certain quantile, usually referred to as Value-at-Risk (VaR). CVaR treats all rewards below VaR equally, while ignoring rewards beyond VaR. Using the distortion function in DRMs allows one to consider all possible reward values, and also account for very low rewards (i.e., those below VaR) by emphasizing them suitably. Some examples of the distortion functions that vary the emphasis placed as one goes from low to high reward values are the proportional hazard transform, and Wang’s transform (see [9] for more examples).

In this paper, we adopt the policy gradient approach to find a policy that optimizes the DRM of the cumulative reward in an episodic MDP. The basis for a policy gradient algorithm is the expression for the gradient of the performance objective. In the risk-neutral case, such an expression is derived using the likelihood ratio (LR) method [5]. We derive the policy gradient expression for the DRM objective using the LR method for both on-policy as well as off-policy RL settings. In the case of DRM, policy gradient estimation is challenging for the following two reasons: (i) DRM estimation requires knowledge of the underlying distribution; and (ii) Using the empirical distribution function (EDF) as a proxy for the true distribution in the DRM policy gradient theorem leads to a biased gradient estimate. In contrast, policy gradient estimation is considerably simpler in a risk-neutral setting as the task is to estimate the mean cumulative reward, and using a sample average leads to an unbiased gradient estimate. We characterize the bias in DRM policy gradient estimates. In particular, we establish that the bias is of order $O\left(\frac{1}{m}\right)$, where $m$ is the batch size (or the number of episodes).

Using the DRM policy gradient theorem, we propose two policy gradient algorithms for optimizing the DRM of the cumulative rewards in an episodic MDP. Both algorithms incorporate a LR-based gradient estimate scheme together with mini-batching. The first algorithm caters to an on-policy RL setting, while the second algorithm is devised for an off-policy RL setting. We derive non-asymptotic bounds that establish local convergence to an $\epsilon$-stationary point of the DRM objective (see Definition 1). These bounds show that an order $O\left(\frac{1}{\epsilon^2}\right)$ number of iterations of the DRM policy gradient algorithms (both on-policy and off-policy) are enough to find an $\epsilon$-stationary point. These bounds are derived by carefully handling the mini-batch size in each iteration of the DRM policy gradient algorithms.

Related work. Risk-sensitive RL has been studied widely in the literature, with focus on specific risk measures like expected exponential utility [3], variance related measures [13], CVaR [11,4]. In [12], the authors survey policy gradient algorithms for optimizing different risk measures in a constrained as well as an unconstrained RL setting. A cumulative prospect theory (CPT) based approach towards RL is studied in [8,14], while coherent risk measures have been considered in an RL setting in [16].

To the best of our knowledge, we are first to derive a policy gradient theorem under a DRM objective, and devise policy gradient algorithms to optimize DRM in a RL context. The
closest related work is \cite{8,14}, where authors consider CPT-based objective in an RL setting. CPT employs a distortion function as in DRMs. However, the distortion function underlying the CPT risk measure is not concave, and hence, CPT is non-coherent. In \cite{8}, the authors employ a simultaneous perturbation method for policy gradient estimation, and provide asymptotic convergence guarantees for their algorithm. In contrast, using concavity of the distortion function assists in adopting the LR method for gradient estimation, in turn leading to non-asymptotic bounds for our algorithms. In a non-RL context, the authors in \cite{6} study the sensitivity of DRM using the perturbation method for policy gradient estimation, and provide asymptotic convergence guarantees for their algorithm. We make the following assumptions to derive the DRM analogue to the policy gradient theorem:

(A1). The class of target policies \(\{\pi_\theta, \theta \in \Theta'\}\) is proper, i.e., there exists a positive constant \(M\) s.t. \(\forall \theta \in \Theta',\ max_{s \in S} \Prob(S_M \neq 0 \mid S_0 = s, \pi_\theta) < 1\).

(A2). \(\forall \theta \in \mathbb{R}^d, \|\nabla \log \pi_\theta(a \mid s)\| \leq M, \forall a \in A, s \in S\).

(A3). \(\exists M_\theta > 0, g'_+(0) \leq M_\theta\), where \(g'_+(0)\) is the right derivative of the distortion function \(g\) at 0.

The assumption \((A1)\) is a common requirement in the analysis of episodic MDPs (cf. \cite{2}), while the assumptions \((A2)\) \((A3)\) are required to ensure the boundedness of the gradient.

We now provide the DRM analogue to the policy gradient theorem below.

**Theorem 1. (DRM policy gradient)** Assume \((A1)\) \((A3)\) Then the gradient of the DRM in \((1)\) is

\[
\nabla \rho_\theta(x) = -\int_{-M_r}^{M_r} g'(1 - F_{R^\theta}(x)) \nabla F_{R^\theta}(x) dx. \tag{4}
\]

**Proof.** See Section 5. \(\square\)

In an RL setting, we do not have a direct measurement of \((4)\), so in each iteration of \((3)\), we estimate the value of \(\nabla \rho_\theta(x)\) for a given \(\theta_k\). In the following sections, we describe gradient algorithms that use \((4)\) to derive LR-based DRM gradient estimates.

### 3.1 DRM optimization in an on-policy RL setting

Using \(F_{R^\theta}(x) = \mathbb{E}\left[\mathbb{I}\{R^\theta \leq x\}\right]\), we obtain the following expression for \(\nabla F_{R^\theta}(x)\) (see Lemma 3 in Section 5 for a proof)

\[
\nabla F_{R^\theta}(x) = \mathbb{E}\left[\mathbb{I}\{R^\theta \leq x\} \sum_{t=0}^{T-1} \nabla \log \pi_\theta(A_t \mid S_t)\right]. \tag{5}
\]

We generate \(m\) episodes using the policy \(\pi_\theta\), and estimate \(F_{R^\theta}(\cdot)\) and \(\nabla F_{R^\theta}(\cdot)\) using sample averages. We denote by \(R^\theta_i\) the cumulative reward, and \(T^i\) the length of the \(i^{th}\) episode. Also, we denote by \(A^i_t\) and \(S^i_t\) the action and state at time \(t\) in episode \(i\), respectively. We form the estimate \(G^m_{R^\theta}(\cdot)\) of \(F_{R^\theta}(\cdot)\) as follows:

\[
G^m_{R^\theta}(x) = \frac{1}{m} \sum_{i=1}^{m} \mathbb{I}\{R^\theta_i \leq x\}. \tag{6}
\]
We form the estimate \( \hat{\nabla} G_{Re}^m(\cdot) \) of \( \nabla F_{Re}(\cdot) \) as follows:
\[
\hat{\nabla} G_{Re}^m(x) = \frac{1}{m} \sum_{i=1}^{m} \mathbb{1}\{R_i^b \leq x\} \sum_{t=0}^{T_i^b-1} \nabla \log \pi_\theta(A_i^t \mid S_i^t). \tag{7}
\]
Using (6) and (7), we estimate \( \nabla \rho_\theta(\theta) \) by
\[
\hat{\nabla} \rho_\theta(\theta) = - \int_{-M_r}^{M_r} g'(1 - G_{Re}^m(x)) \hat{\nabla} G_{Re}^m(x) dx. \tag{8}
\]
The gradient estimator in (8) is biased since \( \mathbb{E}[g'(1 - G_{Re}^m(\cdot)) \neq g'(1 - F_{Re}(\cdot)) \). However, we can control the bias by increasing the number of episodes \( m \), as the mean squared error of this estimator is of order \( O(1/m) \) (see Lemma 5 in Section 5 for a proof).

We solve (3) using the following update iteration:
\[
\theta_{k+1} = \theta_k + \alpha \hat{\nabla} G_{Re}^m(\theta_k). \tag{9}
\]
Algorithm 1 presents the pseudocode of DRM-onP.

**Algorithm 1 DRM-onP**
1. **Input**: Parameterized form of the policy \( \pi \), iteration limit \( N \), step-size \( \alpha \), and batch size \( m \);
2. **Initialize**: Policy parameter \( \theta_0 \in \mathbb{R}^d \), and \( \gamma \in (0, 1] \);
3. for \( k = 0, \ldots, N - 1 \) do
4. Generate \( m \) episodes each using \( \pi_{\theta_k} \);
5. Use (6) to estimate \( \hat{\nabla} \rho_\theta(\theta_k) \);
6. Use (9) to calculate \( \theta_{k+1} \);
7. end for
8. **Output**: Policy \( \theta_R \), where \( R \) is chosen uniformly at random from \( \{1, \ldots, N\} \).

3.2 DRM optimization in an off-policy RL setting

We consider a behavior policy \( b \), that is assumed to be proper (see (A7)). The cumulative discounted reward \( R^b \) is defined by \( R^b = \sum_{t=0}^{T_i^b} \gamma^t r(S_t, A_t, S_{t+1}) \), where \( A_t \sim b(\cdot, S_t), S_{t+1} \sim \pi(\cdot, S_t, A_t) \), and \( \gamma \in (0, 1] \). Let \( \psi_\theta \) denote the importance sampling ratio, and is defined by
\[
\psi_\theta = \prod_{t=0}^{T_i-1} \pi_\theta(A_t \mid S_t) / b(A_t \mid S_t). \tag{10}
\]
Using \( F_{Re}(x) = \mathbb{E}[\mathbb{1}\{R^b \leq x\} \psi_\theta] \), we obtain the following analogue of (5) for the off-policy setting (see Lemma 7 in Section 5 for a proof)
\[
\nabla F_{Re}(x) = \mathbb{E}[\mathbb{1}\{R^b \leq x\} \psi_\theta \sum_{t=0}^{T_i^b-1} \nabla \log \pi_\theta(A_t \mid S_t)]. \tag{11}
\]
We form the estimation \( H_{Re}^m(\cdot) \) of \( \nabla F_{Re}(\cdot) \) as follows:
\[
H_{Re}^m(x) = \min\{H_{Re}^m(x), 1\}, \tag{12}
\]
We form the estimate \( \hat{\nabla} H_{Re}^m(\cdot) \) of \( \nabla F_{Re}(\cdot) \) as follows:
\[
\hat{\nabla} H_{Re}^m(x) = \frac{1}{m} \sum_{i=1}^{m} \mathbb{1}\{R_i^b \leq x\} \psi_i. \tag{13}
\]
We form the estimate \( \hat{\nabla} H_{Re}^m(\cdot) \) of \( \nabla F_{Re}(\cdot) \) as follows:
\[
\hat{\nabla} H_{Re}^m(x) = \frac{1}{m} \sum_{i=1}^{m} \mathbb{1}\{R_i^b \leq x\} \psi_i \sum_{t=0}^{T_i^b-1} \nabla \log \pi_\theta(A_t^i \mid S_t^i). \tag{14}
\]
As in the on-policy case, the gradient estimator in (15) is biased, but the mean squared error of this estimator is of order \( O(1/m) \) (see Lemma 9 in Section 5 for a proof).

We solve (2) using the following update iteration:
\[
\theta_{k+1} = \theta_k + \alpha \hat{\nabla} H_{Re}^m(\theta_k). \tag{16}
\]

4 Main results

Our non-asymptotic analysis establishes a bound on the number of iterations of our proposed algorithms to find an \( \epsilon \)-stationary point (see Definition 1 below) of the DRM.

**Definition 1. (\( \epsilon \)-stationary point)** Let \( \theta_R \) be the output of an algorithm \( A \). Then, \( \theta_R \) is called an \( \epsilon \)-stationary point of problem (2), if
\[
\mathbb{E}\left[ \|\nabla \rho_\theta(\theta_R)\| \right] \leq \epsilon,
\]
where \( \|\cdot\| \) denotes the d-dimensional Euclidean norm.

In an RL setting, the objective need not be convex, and hence, it is common in literature to consider the convergence of the policy gradient algorithms to an approximate stationary point (cf. [10, 15]).

4.1 Non-asymptotic bounds for the algorithm DRM-onP

We make the following assumptions for our analysis:
\[
\text{(A4)} \quad \forall \theta_1, \theta_2 \in \mathbb{R}^d, \quad \| \nabla F_{Re}^{-1}(\theta_1) - \nabla F_{Re}^{-1}(\theta_2) \| \leq L_g \| \theta_1 - \theta_2 \| \quad \text{ a.s.},
\]
where \( F_{Re}^{-1}(s) = \inf\{x : F_{Re}(x) \geq s\} \) is the quantile of \( R^b \).
\[
\text{(A5)} \quad \exists L_g > 0, \ |g’(t_1) - g’(t_2)| \leq L_g |t_1 - t_2|, \ \forall t_1, t_2 \in [0, 1].
\]

The assumption (A4) is needed to ensure the smoothness of the DRM \( \rho_\theta \). Assumptions related to Lipschitzness are common in literature for the non-asymptotic analysis of the policy gradient algorithms (cf. [10, 15]). Further, the assumption (A5) is needed to ensure that the bias of the gradient estimator converges to zero.

The main result that provides a convergence rate for the algorithm DRM-onP is presented below.
Theorem 2. (DRM-onP) Assume \(\text{(AI)} (A5)\) Let \(\rho_g = \max_{\theta \in \mathbb{R}^d} \rho_g(\theta)\). Set \(\alpha = \frac{1}{\sqrt{N}}\), and \(m = \sqrt{N}\). Then,
\[
E \left[\left\|\nabla \rho_g(\theta_R)\right\|^2\right] \leq \frac{2\left(\rho_g^* - \rho_g(\theta_0)\right)}{\sqrt{N}} + \frac{4M_r^2M_e^2\rho_g^2M_1^2\left(M_g^3L_q + 8(e^2M_g^2 + L_g^2)\right)}{\sqrt{N}},
\]
where \(\theta_R\) is chosen uniformly at random from \(\{\theta_1, \ldots, \theta_N\}\). In the above, the constants \(L_g, L_q, M_g, \) and \(M_1\) are as defined in \(\text{(A2)} (A5)\) The constant \(M_e\) is an upper bound on the episode length (see (17)).

Proof. See Section\ref{sec:proofs}.

4.2 Non-asymptotic bounds for the algorithm DRM-offP

For the off-policy RL setting, we require the following additional assumptions:

(A6). For every \(\theta \in \mathbb{R}^d\), the target policy \(\pi_\theta\) is absolutely continuous with respect to the behavior policy \(b\), i.e., \(\forall \theta \in \mathbb{R}^d, b(a|s) = 0 \Rightarrow \pi_\theta(a|s) = 0, \forall a \in \mathcal{A}, \forall s \in \mathcal{S}\).

(A7). The behavior policy \(b\) is proper, i.e., \(\exists M > 0. \text{ s.t. } \max_{s \in \mathcal{S}} \mathbb{P}(S_M \neq 0 \mid S_0 = s, b) < 1\).

An assumption like (A6) is a standard requirement for off-policy evaluation, while (A7) is common to the analysis of episodic MDPs. The main result that provides a convergence rate for the algorithm DRM-offP is presented below.

Theorem 3. (DRM-offP) Assume \(\text{(AI)} (A7)\) Let \(\rho_g = \max_{\theta \in \mathbb{R}^d} \rho_g(\theta)\). Set \(\alpha = \frac{1}{\sqrt{N}}\), and \(m = \sqrt{N}\). Then,
\[
E \left[\left\|\nabla \rho_g(\theta_R)\right\|^2\right] \leq \frac{2\left(\rho_g^* - \rho_g(\theta_0)\right)}{\sqrt{N}} + \frac{4M_r^2M_e^2\rho_g^2M_1^2\left(M_g^3L_q + 8(e^2M_g^2 + L_g^2)\right)}{\sqrt{N}},
\]
where \(\theta_R\) is chosen uniformly at random from \(\{\theta_1, \ldots, \theta_N\}\). In the above, the constants \(L_g, L_q, M_g, \) and \(M_1\) are as defined in \(\text{(A2)} (A5)\) The constant \(M_e\) is an upper bound on the importance sampling ratio (see Lemma\ref{lem:is_ratio}), and \(M_e\) is an upper bound on the episode length (see (12)).

Proof. See Section\ref{sec:proofs}.

Remark 1. From Theorem 2 (resp. Theorem 3), we conclude that after \(N\) iterations of (9) (resp. (16)), algorithm DRM-onP (resp. DRM-offP) returns an iterate that satisfies 
\[
E \left[\left\|\nabla \rho_g(\theta_R)\right\|^2\right] = O\left(\frac{1}{N}\right). \text{In other words, an order } O\left(\frac{1}{N}\right) \text{ number of iterations are enough to find an } \epsilon \text{-stationary point for both algorithms.}
\]

5 Convergence analysis

Our analysis proceeds through a sequence of lemmas.

Lemma 1. \(0 \leq g'(t) \leq M_g, \forall t \in [0, 1]\).

Proof. Since \(g\) is a non-decreasing and concave, we obtain \(g'(t) \geq 0; \forall t \in [0, 1]\), and if \(t_1 \leq t_2\) then \(g'(t_1) \geq g'(t_2), \forall t_1, t_2 \in [0, 1]\).

Hence, from (A3) we obtain \(0 \leq g'(t) \leq M_g, \forall t \in [0, 1]\).

Proof. (Theorem 1) From (AI) we observe that the episode length \(T\) is bounded for any episode. So,
\[
\exists M_e > 0 \text{ s.t. } T \leq M_e \text{ a.s. (17)}
\]
From (A2) and (17), we obtain \(\forall x \in [-M_r, M_r], \)
\[
\left\|\nabla F_{\rho_g}(x)\right\| \leq E \left[\left\|\nabla F_{\rho_g}(x)\right\|\right] \leq M_e M_1, (19)
\]
where the last inequality follows from (18), and since the state and action spaces are finite.

\[
\nabla \rho_g(\theta) = \nabla \int_{-M_r}^{M_r} \{g(1 - F_{\rho_g}(x)) - 1\} \, dx + \int_{-M_r}^{M_r} \nabla g(1 - F_{\rho_g}(x)) \, dx
\]
\[
= \int_{-M_r}^{M_r} \nabla g(1 - F_{\rho_g}(x)) \, dx + \int_{-M_r}^{M_r} \nabla g(1 - F_{\rho_g}(x)) \, dx
\]
\[
= - \int_{-M_r}^{M_r} g'(1 - F_{\rho_g}(x)) \nabla F_{\rho_g}(x) \, dx.
\]

In the above, the equality in (20) follows by an application of the dominated convergence theorem to interchange the differentiation and the expectation operation. The aforementioned application is allowed since (i) \(\rho_g(\theta)\) is finite for any \(\theta \in \mathbb{R}^d\); (ii) \(g'(\cdot) \leq M_g\) from Lemma\ref{lem:is_ratio} and \(\nabla F_{\rho_g} (\cdot)\) is bounded from (19). The bounds on \(g'\) and \(\nabla F_{\rho_g}\) imply \(F_{\rho_g}^{M_r} \mid g'(1 - F_{\rho_g}(x)) \nabla F_{\rho_g}(x) \mid dx \leq 2M_r M_g M_e M_1\).

Lemma 2.
\[
\left\|\nabla \rho_g(\theta_1) - \nabla \rho_g(\theta_2)\right\| \leq M_g L_q \left\|\theta_1 - \theta_2\right\|, \forall \theta_1, \theta_2 \in \mathbb{R}^d.
\]

Proof. Let \(f_{\rho_g}\) be the probability density function of \(R^0\). We observe that \(\forall s \in [0, 1], \nabla F_{\rho_g}^{-1}(s) = f_{\rho_g}(F_{\rho_g}^{-1}(s)) \nabla F_{\rho_g}^{-1}(s)\), and from (A4) we obtain that
\( \nabla F_{R^e}^{-1}(s) \) exists, and hence, \( f_{R^e}(F_{R^e}^{-1}(s)) > 0 \). Using (19), we obtain \( \forall s \in [0, 1] \), for some constant \( C \),

\[
\| \nabla F_{R^e}^{-1}(s) \| = \left\| \frac{\nabla F_{R^e}(F_{R^e}^{-1}(s))}{f_{R^e}(F_{R^e}^{-1}(s))} \right\| \leq CM_t M_t. \tag{21}
\]

Using (8), Lemma 2.1], we can rewrite (11) as \( \rho_\theta(\theta) = \int_0^1 g'(1-s)F_{R^e}^{-1}(s)ds \), and form \( \nabla \rho_\theta(\theta) \) as follows:

\[
\nabla \rho_\theta(\theta) = \int_0^1 g'(1-s)F_{R^e}^{-1}(s)ds = \int_0^1 g'(1-s)\nabla F_{R^e}^{-1}(s)ds. \tag{22}
\]

In the above, the last equality in (22) follows by an application of the dominated convergence theorem to interchange the differentiation and the expectation operation. The aforementioned application is allowed since (i) \( \Omega \) is finite and the underlying measure is bounded, as we consider an MDP where the state and actions spaces are finite, and the policies are proper, (ii) \( \nabla \log \pi_\theta(A_t|S_t) \) is bounded from (A2).

**Lemma 4.** \( \mathbb{E} \left[ \| \hat{\nabla} G_{R^e}^m(x) \|^2 \right] \leq 4M_t^2 M^2_\sigma M^2_\varepsilon M^2_t. \)

**Proof.** From (7), using (A2) and (17), we obtain

\[
\| \hat{\nabla} G_{R^e}^m(x) \|^2 \leq M_t^2 M^2_\sigma \text{ a.s., } \forall x \in [-M_t, M_t]. \tag{24}
\]

From (8), and Lemma 1, we obtain

\[
\mathbb{E} \left[ \| \hat{\nabla} G_{R^e}^m(\theta) \|^2 \right] \leq M_t^2 \mathbb{E} \left[ \int_{-M_t}^{M_t} \| \hat{\nabla} G_{R^e}^m(\theta) \|^2 dx \right] \leq 2M_t^2 \mathbb{E} \left[ \int_{-M_t}^{M_t} \| \hat{\nabla} G_{R^e}^m(\theta) \|^2 dx \right]
\]

(from Cauchy-Schwarz inequality)

\[
\leq 2M_t^2 M^2_\sigma \int_{-M_t}^{M_t} \mathbb{E} \left[ \| \hat{\nabla} G_{R^e}^m(\theta) \|^2 \right] dx \leq 2M_t^2 M^2_\sigma \int_{-M_t}^{M_t} M^2_t dx = 4M_t^2 M^2_\sigma M^2_t,
\]

where the last inequality follows from (24).

**Lemma 5.**

\[
\mathbb{E} \left[ \| \hat{\nabla} G_{R^e}^m(\theta) - \nabla \rho_\theta(\theta) \|^2 \right] \leq \frac{32M_t^2 M^2_\sigma M^2_\varepsilon (e^2 M^2_\sigma + L^2)}{m}.
\]

**Proof.** Since \( \forall x \in [-M_t, M_t], |\{R^e_t \leq x\}| \leq 1 \) a.s., using Hoeffding’s inequality, we obtain \( \forall x \in [-M_t, M_t], \)

\[
\mathbb{P} \left( |G_{R^e}^m(x) - F_{R^e}(x)| > \varepsilon \right) \leq 2 \exp \left( \frac{-m \varepsilon^2}{2} \right),
\]

and

\[
\mathbb{E} \left[ \left| G_{R^e}^m(x) - F_{R^e}(x) \right|^2 \right] = \int_0^\infty \mathbb{P} \left( |G_{R^e}^m(x) - F_{R^e}(x)| > \sqrt{t} \right) dt \leq \int_0^\infty 2 \exp \left( \frac{-m \varepsilon^2}{2} \right) dt = \frac{4}{m}. \tag{25}
\]

From (18), and (7, Theorem 1.8-1.9)], we obtain \( \forall x \in [-M_t, M_t], \)

\[
\mathbb{P} \left( \left| \hat{\nabla} G_{R^e}^m(x) - \nabla F_{R^e}(x) \right| > \varepsilon \right) \leq 2 \varepsilon^2 \exp \left( \frac{-m \varepsilon^2}{2M_t^2 M^2_\varepsilon} \right),
\]

and

\[
\mathbb{E} \left[ \left| \hat{\nabla} F_{R^e}(x) - \hat{\nabla} G_{R^e}^m(x)^2 \right| \right] = \int_0^\infty \mathbb{P} \left( \left| \nabla F_{R^e}(x) - \hat{\nabla} G_{R^e}^m(x) \right| > \sqrt{t} \right) dt \leq \int_0^\infty 2 \varepsilon^2 \exp \left( \frac{-m \varepsilon^2}{2M_t^2 M^2_\varepsilon} \right) dt = \frac{4 \varepsilon^2 M_t^2 M^2_\varepsilon}{m}. \tag{26}
\]
Now, from (24), we obtain $\forall x \in [-M_r, M_r],$
\[
E \left[ \left\| (g' (1 - F_R(x)) - g'(1 - G_{R}^m(x))) \hat{\nabla} G_{R}^m (x) \right\|^2 \right] 
\leq M_r^2 M_r^2 E \left[ \left\| g' (1 - F_R(x)) - g'(1 - G_{R}^m(x)) \right\|^2 \right] 
\leq L_g^2 M_r^2 M_r^2 E \left[ G_{R}^m (x) - F_R(x) \right]^2 \leq \frac{4 L_g^2 M_r^2 M_r^2}{m}, \quad (27) \]
where the last two inequalities follow from [A5] and (25). From Lemma 1 we obtain $\forall x \in [-M_r, M_r],$
\[
E \left[ \left\| g' (1 - F_R(x)) \left( \nabla F_R(x) - \hat{\nabla} G_{R}^m (x) \right) \right\|^2 \right] 
\leq M_r^2 \left\| \nabla F_R(x) - \hat{\nabla} G_{R}^m (x) \right\|^2 \leq \frac{4 e^2 M_g^2 M_r^2 M_r^2}{m}, \quad (28) \]
where the last inequality follows from (26).

From (4), (8), and Cauchy-Schwarz inequality, we obtain
\[
E \left[ \left\| \hat{\nabla} G_{R}^m (\theta) - \nabla \rho_g (\theta) \right\|^2 \right] 
\leq 2 M_r E \left[ \int_{-M_r}^{M_r} \left\| g' (1 - F_R(x)) \nabla F_R(x) - g'(1 - G_{R}^m(x)) \hat{\nabla} G_{R}^m (x) \right\|^2 \, dx \right] 
\leq 2 M_r \int_{-M_r}^{M_r} E \left[ \left\| g' (1 - F_R(x)) \nabla F_R(x) - g'(1 - G_{R}^m(x)) \hat{\nabla} G_{R}^m (x) \right\|^2 \right] \, dx 
\quad \text{(from Fubini’s theorem)} 
\leq 2 M_r \int_{-M_r}^{M_r} E \left[ \left\| g' (1 - F_R(x)) \left( \nabla F_R(x) - \hat{\nabla} G_{R}^m (x) \right) \right\|^2 \right] \, dx 
\quad + \left( g' (1 - F_R(x)) - g'(1 - G_{R}^m(x)) \right) \hat{\nabla} G_{R}^m (x) \right\|^2 \right] \, dx 
\leq 4 M_r \int_{-M_r}^{M_r} \left[ \left\| g' (1 - F_R(x)) \left( \nabla F_R(x) - \hat{\nabla} G_{R}^m (x) \right) \right\|^2 \right] 
\quad + E \left[ \left\| g' (1 - F_R(x)) - g'(1 - G_{R}^m(x)) \right\|^2 \right] \, dx 
\quad \text{(since } 2(a, b) \leq \|a\|^2 + \|b\|^2) 
\leq 32 M_r^2 M_g^2 M_r^2 (e^2 M_g^2 + L_g^2) \frac{1}{m}, \quad (29) \]
where the last inequality follows from (27) and (28).

**Proof.** *(Theorem 2)* Using the fundamental theorem of calculus, we obtain
\[
\rho_g (\theta_k) - \rho_g (\theta_{k+1}) = \langle \nabla \rho_g (\theta_k), \theta_k - \theta_{k+1} \rangle 
+ \int_0^1 \langle \nabla \rho_g (\theta_{k+1} + \tau (\theta_k - \theta_{k+1})), \nabla \rho_g (\theta_k), \theta_k - \theta_{k+1} \rangle d\tau 
\leq \langle \nabla \rho_g (\theta_k), \theta_k - \theta_{k+1} \rangle 
+ \int_0^1 \| \nabla \rho_g (\theta_{k+1} + \tau (\theta_k - \theta_{k+1})) - \nabla \rho_g (\theta_k) \| \| \theta_k - \theta_{k+1} \| d\tau 
\leq \langle \nabla \rho_g (\theta_k), \theta_k - \theta_{k+1} \rangle 
+ M_g L_q |\theta_k - \theta_{k+1}|^2 \int_0^1 (1 - \tau) d\tau \quad \text{(from Lemma 2)} 
= \langle \nabla \rho_g (\theta_k), \theta_k - \theta_{k+1} \rangle + \frac{M_g L_q}{2} |\theta_k - \theta_{k+1}|^2 
= \alpha \left\langle \nabla \rho_g (\theta_k), -\hat{\nabla} G_{R}^m (\theta_k) \right\rangle \left\| \nabla \rho_g (\theta_k) \right\|^2 
\leq \frac{\alpha}{2} \left\| \nabla \rho_g (\theta_k) \right\|^2 + \frac{M_g L_q}{2} \alpha^2 \left\| \hat{\nabla} G_{R}^m (\theta_k) \right\|^2 
\leq \frac{\alpha}{2} \left\| \nabla \rho_g (\theta_k) \right\|^2 + \frac{M_g L_q}{2} \alpha^2 \left\| \hat{\nabla} G_{R}^m (\theta_k) \right\|^2 
\quad \text{(since } 2(a, b) \leq \|a\|^2 + \|b\|^2) 
= \frac{\alpha}{2} \left\| \nabla \rho_g (\theta_k) - \hat{\nabla} G_{R}^m (\theta_k) \right\|^2 
\quad - \frac{\alpha}{2} \left\| \nabla \rho_g (\theta_k) \right\|^2 + \frac{M_g L_q}{2} \alpha^2 \left\| \hat{\nabla} G_{R}^m (\theta_k) \right\|^2. \quad (30) \]

Taking expectations on both sides of (30), we obtain
\[
\alpha E \left[ \left\| \nabla \rho_g (\theta_k) \right\|^2 \right] 
\leq 2 E \left[ \rho_g (\theta_{k+1}) - \rho_g (\theta_k) \right] + M_g L_q \alpha^2 E \left[ \left\| \hat{\nabla} G_{R}^m (\theta_k) \right\|^2 \right] 
\quad + \alpha E \left[ \left\| \nabla \rho_g (\theta_k) - \hat{\nabla} G_{R}^m (\theta_k) \right\|^2 \right] 
\leq 2 E \left[ \rho_g (\theta_{k+1}) - \rho_g (\theta_k) \right] 
\quad + 4 M_r^2 M_g^2 M_r^2 \left( \alpha M_g^2 L_q + \frac{8}{m} (e^2 M_g^2 + L_g^2) \right), \quad (31) \]
where the last inequality follows from Lemmas 3, 4. Summing up (31) from $k = 0, \ldots, N - 1$, we obtain
\[
\alpha \sum_{k=0}^{N-1} E \left[ \left\| \nabla \rho_g (\theta_k) \right\|^2 \right] \leq 2 E \left[ \rho_g (\theta_N) - \rho_g (\theta_0) \right] 
\quad + N \alpha M_r^2 M_g^2 M_r^2 \left( \alpha M_g^2 L_q + \frac{8 (e^2 M_g^2 + L_g^2)}{m} \right) . \]

Since $P(R = k) = \frac{1}{N}$, we obtain
\[
E \left[ \left\| \nabla \rho_g (\theta_k) \right\|^2 \right] = \frac{1}{N} \sum_{k=0}^{N-1} E \left[ \left\| \nabla \rho_g (\theta_k) \right\|^2 \right] \leq 2 \left( \rho^*_g - \rho_g (\theta_0) \right) \frac{N}{N} 
\quad + 4 M_r^2 M_g^2 M_r^2 \left( \frac{\alpha M_g^2 L_q + \frac{8 (e^2 M_g^2 + L_g^2)}{m}}{\sqrt{N}} \right) 
\quad \leq 2 \left( \rho^*_g - \rho_g (\theta_0) \right) + 4 M_r^2 M_g^2 M_r^2 \left( \frac{\alpha M_g^2 L_q + \frac{8 (e^2 M_g^2 + L_g^2)}{m}}{\sqrt{N}} \right), \quad (32) \]
where $\alpha = \frac{1}{\sqrt{N}}$, and $m = \sqrt{N}$ justifies the last inequality. \qed
5.2 Proofs for DRM-offP

Lemma 6. For any episode generated using b, the importance sampling ratio \( \psi_0 \leq M_s, \forall \theta \in \mathbb{R}^d \).

Proof. From (A2) and (A6) we obtain \( \forall \theta \in \mathbb{R}^d, \pi_\theta(a|s) > 0 \) and \( b(a|s) > 0, \forall a \in A, \forall s \in \mathcal{S} \). From (A7) we obtain that the episode length is bounded for \( b \). So the importance sampling ratio \( \psi_0 \) is bounded by any episode. Hence WLOG, we say \( \psi_0 \leq M_s, \forall \theta \in \mathbb{R}^d \), for some constant \( M_s > 0 \).

Lemma 7. \( \forall x \in [-M_r, M_r] \).

\[
\nabla F_{\psi}(x) = \mathbb{E} \left[ \mathbb{P}(R^b \leq x) \psi_{\theta} \sum_{t=0}^{T-1} \nabla \log \pi_\theta(A_t | S_t) \right].
\]

Proof. Let \( \Omega \) be the set of sample episodes. For any episode \( \omega \in \Omega \), we denote by \( T(\omega) \), its length, and \( S_t(\omega) \) and \( A_t(\omega) \), the state and action at time \( t \in \{1, 2, \ldots \} \) respectively. Let \( \mathbb{P}_b(\omega) = \prod_{t=0}^{T(\omega)-1} b(A_t(\omega), S_t(\omega)) p(S_t+1(\omega), A_t(\omega), S_t(\omega)) \). From \( \nabla \mathbb{P}_b(\omega) = \psi_{\theta} \prod_{t=0}^{T(\omega)-1} \nabla \log \pi_\theta(A_t(\omega), S_t(\omega)) \), and (23), we obtain

\[
\nabla F_{\psi}(x) = \mathbb{E} \left[ \mathbb{P}(R^b \leq x) \psi_{\theta} \sum_{t=0}^{T-1} \nabla \log \pi_\theta(A_t | S_t) \right].
\]

Lemma 8. \( \mathbb{E} \left[ \left\| \nabla H_{\psi}(\theta) \right\|^2 \right] \leq 4M_r^2M_s^2M^2M^2.

Proof. From (A7) we observe that the episode length \( T \) is bounded for any episode. So,

\[
\exists M_e > 0 \text{ s.t. } T \leq M_e, \text{ a.s.}
\]

From (14), (A2) [2], and Lemma 6, we obtain

\[
\left\| \nabla H_{\psi}(x) \right\|^2 \leq M_r^2M_s^2M^2 \text{ a.s., } \forall x \in [-M_r, M_r].
\]

The result follows by using similar arguments as in Lemma 3 along with (33).

Lemma 9.

\[
\mathbb{E} \left[ \left\| \nabla H_{\psi}(\theta) - \nabla \rho_\theta(\theta) \right\|^2 \right] \leq \frac{32M_r^2M_s^2M^2M^2(e^2M_s^2 + L_0^2M_s^2)}{m}.
\]

Proof. We use parallel arguments to the proof of Lemma 5. From Lemma 6, we obtain \( \forall x \in [-M_r, M_r], \left\| \nabla H_{\psi}(x) \right\| \leq M_s \text{ a.s.} \) From Hoeffding inequality, we obtain \( \forall x \in [-M_r, M_r], \)

\[
\mathbb{P} \left( \left| \nabla H_{\psi}(x) - F_{\psi}(x) \right| > \epsilon \right) \leq 2 \exp \left( \frac{-m\epsilon^2}{2M^2} \right). \tag{34}
\]

From (12) and (13), we observe that \( \mathbb{P}(\left| \nabla H_{\psi}(x) - F_{\psi}(x) \right| > \epsilon) \leq \mathbb{P}(\left| \nabla H_{\psi}(x) - F_{\psi}(x) \right| > \epsilon). \) Hence, we obtain \( \forall x \in [-M_r, M_r], \)

\[
\mathbb{P}(\left| \nabla H_{\psi}(x) - F_{\psi}(x) \right| > \epsilon) \leq 2 \exp \left( \frac{-m\epsilon^2}{2M^2} \right). \tag{35}
\]

Using similar arguments as in (26) along with (35), we obtain \( \forall x \in [-M_r, M_r], \)

\[
\mathbb{P}(\left| \nabla H_{\psi}(x) - F_{\psi}(x) \right| > \epsilon) \leq \frac{4M_r^2M_s^2M^2}{m}. \tag{36}
\]

From (A2) [2], and Lemma 6, we obtain \( \forall x \in [-M_r, M_r], \)

\[
\left\| \mathbb{E}(\nabla F_{\psi}(x) - \nabla \rho_\theta(x)) \right\| \leq M_s M_e M_l \text{ a.s.} \tag{37}
\]

From (37), and [7], Theorem 1.8-1.9], we obtain \( \forall x \in [-M_r, M_r], \)

\[
\mathbb{P}(\left| \nabla H_{\psi}(x) - \nabla \rho_\theta(x) \right| > \epsilon) \leq 2 \exp \left( \frac{-m\epsilon^2}{2M^2} \right). \tag{38}
\]

Using similar arguments as in (26) along with (38), we obtain \( \forall x \in [-M_r, M_r], \)

\[
\mathbb{E}(\left\| \nabla F_{\psi}(x) - \nabla \rho_\theta(x) \right\|^2) = \frac{4M_r^2M_s^2M^2}{m}. \tag{39}
\]

Using similar arguments as in (27), along with (36), and (33), we obtain \( \forall x \in [-M_r, M_r], \)

\[
\mathbb{E}(\left\| \nabla F_{\psi}(x) - \nabla \rho_\theta(x) \right\|^2) \leq \frac{4M_r^2M_s^2M^2}{m}. \tag{40}
\]

Using similar arguments as in (28) along with (39), we obtain \( \forall x \in [-M_r, M_r], \)

\[
\mathbb{E}(\left\| \nabla F_{\psi}(x) - \nabla \rho_\theta(x) \right\|^2) \leq \frac{4M_r^2M_s^2M^2}{m}. \tag{41}
\]

Using similar arguments as in (29) along with (40) and (41), we obtain

\[
\mathbb{E}(\left\| \nabla H_{\psi}(\theta) - \nabla \rho_\theta(\theta) \right\|^2) \leq \frac{32M_r^2M_s^2M^2M^2(e^2M_s^2 + L_0^2M_s^2)}{m}.
\]
Proof. (Theorem 3) By using a completely parallel argument to the initial passage in the proof of Theorem 2 leading up to (41), we obtain

\[ \alpha \mathbb{E} \left[ \| \nabla \rho_g(\theta_k) \|^2 \right] \]

\[ \leq 2 \mathbb{E} [\rho_g(\theta_{k+1}) - \rho_g(\theta_k)] + M_g L_q \alpha^2 \mathbb{E} \left[ \| \nabla H \rho_g(\theta_k) \|^2 \right] + \alpha \mathbb{E} \left[ \| \nabla \rho_g(\theta_k) - \nabla H \rho_g(\theta_k) \|^2 \right] \]

\[ \leq 2 \mathbb{E} [\rho_g(\theta_{k+1}) - \rho_g(\theta_k)] + \alpha 4M_g^2 M_s^2 M_r^2 M_l^2 \left( \alpha M_g^2 L_q + \frac{8(e^2 M_g^2 + L_g^2 M_s^2)}{m} \right), \]

where the last inequality follows from Lemmas 9-8. Summing the above equation from \( k = 0, \cdots, N - 1 \), we obtain

\[ \alpha \sum_{k=0}^{N-1} \mathbb{E} \left[ \| \nabla \rho_g(\theta_k) \|^2 \right] \leq 2 \mathbb{E} [\rho_g(\theta_N) - \rho_g(\theta_0)] \]

\[ + N \alpha 4M_g^2 M_s^2 M_r^2 M_l^2 \left( \alpha M_g^2 L_q + \frac{8(e^2 M_g^2 + L_g^2 M_s^2)}{m} \right). \]

Since \( \mathbb{P}(R = k) = \frac{1}{N} \), we obtain

\[ \mathbb{E} \left[ \| \nabla \rho_g(\theta_R) \|^2 \right] = \frac{1}{N} \sum_{k=0}^{N-1} \mathbb{E} \left[ \| \nabla \rho_g(\theta_k) \|^2 \right] \]

\[ \leq 2 \frac{(\rho_g^* - \rho_g(\theta_0))}{N^2} \]

\[ + 4M_R^2 M_g^2 M_s^2 M_l^2 \left( \alpha M_g^2 L_q + \frac{8(e^2 M_g^2 + L_g^2 M_s^2)}{m} \right) \]

\[ \leq 2 \frac{(\rho_g^* - \rho_g(\theta_0))}{\sqrt{N}} \]

\[ + 4M_R^2 M_g^2 M_s^2 M_l^2 \left( \alpha M_g^2 L_q + \frac{8(e^2 M_g^2 + L_g^2 M_s^2)}{m} \right), \]

where \( \alpha = \frac{1}{\sqrt{N}} \), and \( m = \sqrt{N} \) justifies the last inequality. \( \square \)

6 Conclusions

We proposed policy gradient algorithms for risk-sensitive RL control using DRM. We incorporated a LR-based scheme for gradient estimation in on-policy as well as off-policy RL settings, and provided non-asymptotic bounds that establish convergence to an approximate stationary point of the DRM objective.

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