Simply and multiply scaled diffusion limits for continuous time random walks

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Abstract. First a survey is presented on how space-time fractional diffusion processes can be obtained by well-scaled limiting from continuous time random walks under the sole assumption of asymptotic power laws (with appropriate exponents for the tail behaviour of waiting times and jumps). The spatial operator in the limiting pseudo-differential equation is the inverse of a general Riesz-Feller potential operator. The analysis is carried out via the transforms of Fourier and Laplace. Then mixtures of waiting time distributions, likewise of jump distributions, are considered, and it is shown that correct multiple scaling in the limit yields diffusion equations with distributed order fractional derivatives (fractional operators being replaced by integrals over such ones, with the order of differentiation as variable of integration). It is outlined how in this way super-fast and super-slow diffusion can be modelled.

Keywords: continuous time random walk, anomalous diffusion, fractional diffusion processes, power laws, pseudo-differential equations, fractional derivatives.

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1. Introduction: space-time fractional diffusion

In recent years methods of fractional calculus have met more and more interest and found increasing use in modelling anomalous diffusion processes, namely in diffusion processes deviating essentially from Gaussian behaviour which is characterized by evolution of the second centered moment like the first power of time. Nowadays popular models are obtained by replacing in the classical diffusion equation the partial derivatives with respect to space and/or time by integro-differential operators, concretely by derivatives of non-integer order, in such a way that the resulting Green function can still be interpreted as a probability density evolving in time differently from the Gaussian type. Diffusion processes so behaving are called anomalous.

A more general approach to anomalous diffusion is provided by the so-called continuous time random walk (CTRW) which in essence is a random walk subordinated to a renewal process.

One of the purposes of this paper is to show that under appropriate assumptions on waiting times and jumps this integral equation reduces to a fractional diffusion equation with simple or distributed orders of differentiation, via a properly scaled transition to the diffusion limit.

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We begin by considering the Cauchy problem for the (spatially one-dimensional) space-time fractional diffusion equation

\[ t D_s^\beta u(x,t) = x D_\theta^\alpha u(x,t), \quad -\infty < x < +\infty, \quad t \geq 0, \quad u(x,0) = \delta(x), \quad (1.1) \]

where \( \alpha, \theta, \beta \) are real parameters restricted to the ranges

\[ 0 < \alpha \leq 2, \quad |\theta| \leq \min\{\alpha, 2 - \alpha\}, \quad 0 < \beta \leq 1, \quad (1.2) \]

in order to ensure the solution \( u(x,t) \) having the properties of a probability density in the spatial variable \( x \), evolving in time \( t \). Here \( t D_s^\beta \) denotes the Caputo fractional derivative of order \( \beta \), acting on the time variable \( t \), and \( x D_\theta^\alpha \) denotes the Riesz-Feller fractional derivative of order \( \alpha \) and skewness \( \theta \), acting on the space variable \( x \). Let us note that the term “skewness” is used (improperly) to account for asymmetry in the probability density.

Writing the transforms of Laplace and Fourier as

\[ \mathcal{L} \{ f(t); s \} = \mathcal{F} \{ f(x) \} = \int_{0}^{\infty} e^{-st} f(t) \, dt, \quad \mathcal{F} \{ f(x); \kappa \} = \mathcal{F} \{ g(x) \} = \int_{-\infty}^{\infty} e^{i\kappa x} g(x) \, dx, \]

we have the corresponding transforms of \( t D_s^\beta f(t) \) and \( x D_\theta^\alpha g(x) \) as

\[ \mathcal{L} \{ t D_s^\beta f(t) \} = s^\beta \mathcal{F} \{ f(x) \} - s^\beta - 1 f(0), \quad \mathcal{F} \{ x D_\theta^\alpha g(x) \} = -|\kappa|^\alpha i^{\theta \text{sign} \kappa} \mathcal{F} \{ g(\kappa) \}. \quad (1.3) \]

Notice that \( i^{\theta \text{sign} \kappa} = \exp[i (\text{sign} \kappa) \theta \pi / 2] \). For the mathematical details the interested reader is referred to [29, 64] on the Caputo derivative, and to [66] on the Feller potentials.

To our knowledge an equation of type (1.1), fractional both in space and time, appears first in Zaslavsky’s paper [83] of 1992. Observe that (1.1), in the special case \( \beta = 1, \theta = 0, \alpha = 2 \), is just the Cauchy problem for the classical diffusion equation

\[ \frac{\partial}{\partial t} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t), \quad -\infty < x < +\infty, \quad t \geq 0, \quad u(x,0) = \delta(x), \quad (1.4) \]

whose solution is the Gaussian density, evolving in time with variance \( \sigma^2 = 2t \),

\[ u(x,t) = \frac{1}{2\sqrt{\pi t}} \exp \left( -\frac{x^2}{4t} \right). \quad (1.5) \]

Other special cases are obtained when \( \beta = 1 \) and \( 0 < \alpha < 2 \) (space fractional diffusion equation) and \( \alpha = 2 \) and \( 0 < \beta < 1 \) (time fractional diffusion equation).

In the space fractional diffusion case the fundamental solution is a Lévy stable probability density evolving in time, as first discussed by Feller [19] in his pioneering 1952 paper. Feller’s parametrization of stable densities, see also [20], has been here adapted (i.e. slightly modified and extended) according to the 1998 paper by Gorenflo & Mainardi [30] on the analysis of random walk models associated to the space fractional operator. In the special case \( \alpha = 1 \) it is essentially distinct from that introduced originally by Lévy, see e.g. [47], and followed also nowadays by many authors.

There is a large literature on stable probability distributions that we cannot survey here: let us simply quote, in addition to the classical treatise by Gnedenko & Kolmogorov [22], the review paper by Schneider [70] and the recent treatises by Meerschaert & Scheﬄer [58], Sato [67], Steutel & Van Harn [76], Uchaikin & Zolotarev [79]. For the general theory of pseudo-differential operators and Markov processes the interested reader is referred to the excellent volumes by Jacob [45].
The time fractional diffusion case was investigated by Schneider & Wyss [71] in their pioneering 1989 paper where they adopted the equivalent integral form

\[
u(x, t) = \nu(x, 0) + \frac{1}{\Gamma(\beta)} \int_0^t \left[ \frac{\partial^2}{\partial x^2} \nu(x, \tau) \right] \frac{d\tau}{(t - \tau)^{1-\beta}}. \tag{1.6}
\]

The time fractional diffusion equation in the present form with the Caputo derivative has been adopted and investigated by several authors. From the former contributors let us quote Mainardi, see e.g. [48, 49, 50, 51] (see also [27, 28] and references therein), who has expressed the fundamental solution in terms of a special function (of Wright type) of which he has studied the analytical properties and provided plots also for \(1 < \beta < 2\) (intermediate phenomenon between diffusion and wave propagation).

In recent years several relevant papers have appeared, that have discussed and reviewed analytical properties and/or applications of the space-time fractional diffusion equations. Without pretending to be exhaustive let us quote some papers from different research groups: [2, 4, 7, 18, 26, 41, 52, 53, 54, 56, 57, 59, 60, 61, 65, 75, 78, 82, 85], where the interested reader can get further references.

For our purposes let us here limit ourselves to recall the representation in the Laplace-Fourier domain of the (fundamental) solution of (1.1) as it results from the application of the transforms of Laplace and Fourier. Using \(\tilde{\delta}(\kappa) \equiv 1\) we have from (1.1)

\[
s^\beta \tilde{u}(\kappa, s) - s^\beta - 1 = -|\kappa|^\alpha i^\theta \text{sign} \kappa \tilde{u}(\kappa, s),
\]

hence

\[
\tilde{u}(\kappa, s) = \frac{s^\beta - 1}{s^\beta + |\kappa|^\alpha i^\theta \text{sign} \kappa}. \tag{1.7}
\]

For explicit expressions and plots of the fundamental solution of (1.1) in the space-time domain we refer the reader to [52]. There, starting from the fact that the Fourier transform \(\tilde{u}(\kappa, t)\) can be written as a Mittag-Leffler function with complex argument, the authors have derived a Mellin-Barnes integral representation of \(u(x, t)\) with which they have proved the non-negativity of the solution for values of the parameters \(\{\alpha, \theta, \beta\}\) in the range (1.2) and analysed the evolution in time of its moments. The representation of \(u(x, t)\) in terms of Fox \(H\)-functions can be found in [54].

In the present paper it is our first intention to show how fractional diffusion (1.1) can be obtained from continuous time random walks by well-scaled transition to the diffusion limit. However, we do this only for a parameter range restricted a little more severely than (1.2). We assume \(0 < \beta \leq 1\) and either (i) or (ii),

\[
(i) \quad 0 < \alpha < 1, \quad -\alpha < \theta < \alpha,
(ii) \quad 1 < \alpha < 2, \quad -(2 - \alpha) < \theta < 2 - \alpha,
\]

leaving the non-treated border cases for \(\theta\) and the singular case \(\alpha = 1\) to a systematic study in a following paper in which we will also include the special case \(\alpha = 2\) which is the easy one. We will show that the method described by Gorenflo & Mainardi in [33, 34] and in a more general form by Gorenflo & Abdel-Rehim in [24] can be extended here to take into account the skewness.

Sections 2 - 5 are devoted to this first intention. We outline in Section 2 the basic theory of continuous time random walks and present in Section 3 some theorems of Tauber type that we use for passing to the diffusion limit. This passage is carried out in Section 4 under the assumption of appropriate power laws where, in contrast to earlier papers, we allow the spatial power laws to have different positive coefficients in the positive and the negative infinite. Assuming these coefficients given, we can express the resulting skewness parameter \(\theta\) as a function of these coefficients. In Section 5 we show how for given values of the parameters \(\alpha\) and \(\theta\) approximating continuous time random walks can be constructed.

Our second intention (see Section 6) is to sketch the principles of distributed order fractional diffusion with approximation by multiply scaled CTRW.
2. Continuous time random walks

A continuous random walk (CTRW) is generated by a sequence of independent identically distributed (iid) positive random waiting times $T_1, T_2, T_3, \ldots$, each having the same probability density function $\phi(t), \ t > 0$, and a sequence of iid random jumps $X_1, X_2, X_3, \ldots$, in $\mathbb{R}$, each having the same probability density $w(x), \ x \in \mathbb{R}$.

Let us remark that, for ease of language, we use the word density also for generalized functions in the sense of Gel'fand & Shilov [21], that can be interpreted as probability measures. Usually the $X_n$ in which the $\delta(0) = 0$ satisfies the initial condition $\Phi(+) = 1$ holds at points where they are continuous.

As a consequence $\Phi(t)$ is a non-decreasing function in $\mathbb{R}^+$ with $\Phi(0) = 0$, $\Phi(+\infty) = 1$ and $W(x)$ is a non-decreasing function in $\mathbb{R}$ with $W(-\infty) = 0$, $W(+\infty) = 1$. As these cumulative functions may have points of discontinuity, we agree on the provision that equations in which they occur are meant to hold at points where they are continuous.

Setting $t_0 = 0, t_n = T_1 + T_2 + \ldots T_n$ for $n \in \mathbb{N}$, the wandering particle makes a jump of length $X_n$ in instant $t_n$, so that its position is $x_0 = 0$ for $0 \leq t < T_1 = t_1$, and $x_n = X_1 + X_2 + \ldots X_n$, for $t_n \leq t < t_{n+1}$. We require the distribution of the waiting times and that of the jumps to be independent of each other. So, we have a compound renewal process (a renewal process with reward), compare [16].

By natural probabilistic arguments we arrive at the integral equation for the probability density $p(x,t)$ (a density with respect to the variable $x$) of the particle being in point $x$ at instant $t$, see e.g. [34, 38, 55, 68, 69],

$$p(x,t) = \delta(x) \Psi(t) + \int_0^t \phi(t') p(x, t-t') \left[ \int_{-\infty}^{+\infty} w(x-x') p(x', t-t') \, dx' \right] \, dt', \quad (2.3)$$

in which the survival function

$$\Psi(t) = \int_t^{+\infty} \phi(t') \, dt' \quad (2.4)$$

denotes the probability that at instant $t$ the particle still is sitting in its starting position $x = 0$. Clearly, (2.3) satisfies the initial condition $p(x,0) = \delta(x)$. In the Laplace-Fourier domain Eq. (2.3) reads as

$$\tilde{p}(\kappa, s) = \tilde{\Psi}(s) + \tilde{\phi}(\kappa) \tilde{w}(s) \tilde{\phi}(\kappa, s),$$

and using $\tilde{\Psi}(s) = (1 - \tilde{\phi}(s))/s$, explicitly

$$\tilde{p}(\kappa, s) = \frac{1 - \tilde{\phi}(s)}{s} \frac{1}{1 - \tilde{w}(\kappa) \tilde{\phi}(s)}. \quad (2.3)$$

This is known in physics as the the Montroll-Weiss equation, so named from the authors, see [62], who derive it in 1965 as the basic equation for the CTRW. From that time many papers have appeared in the literature of applied sciences, where the authors have treated several aspects of the CTRW, including the transition from the CTRW equation to generalized diffusion equations of fractional order. Without pretending to be exhaustive let us quote some papers from different research groups: [5, 6, 8, 35, 38, 43, 44, 46, 55, 53, 54, 56, 59, 60, 68, 69, 73, 75, 77, 81, 85], where the interested reader can get further references.
3. Tauberian lemmata

In view of the diffusion limit we now state some lemmata of Tauberian type by using the probability distribution functions. The advantage of stating the lemmata for probability distribution functions lies in the fact that so not only genuine CTRWs but also fully discrete random walks are covered (compare [39]), and also mixed situations. As corollaries the lemmata will be simplified for convenient use of density functions: though they are less general, they are often easier to check for their conditions and to apply.

**Lemma 1: Master Lemma for waiting times.** Assume that the waiting time probability distribution function \( \Phi(t) \) satisfies either (A) or (B):

(A) \[ \rho := \int_0^\infty t \Phi(t) < \infty, \] labelled as \( \beta = 1 \).

(B) \[ \int_t^\infty d\Phi(t') \sim c^{-1} \beta^{-1} t^{-\beta}, \] for \( t \to \infty \), \( \beta \in (0, 1) \) and \( c > 0 \).

Then with

\[ \lambda = \rho \quad \text{in case (A),} \quad \lambda = \frac{c\pi}{\Gamma(\beta + 1) \sin(\beta\pi)} \quad \text{in case (B),} \] we have the asymptotics (with \( s > 0 \)):

\[ 1 - \tilde{\phi}(s) \sim \lambda s^\beta \quad \text{for} \quad s \to 0. \] (3.3)

**Comment:** This lemma, referred to in the following also as the *temporal lemma*, is a special case of Karamata’s theorem of 1931. A proof can be found in [9].

**Corollary of Lemma 1:** Assume that the waiting time probability density function \( \phi(t) \) satisfies either (A) or (B)

(A) \[ \rho := \int_0^\infty t \Phi(t) < \infty, \] labelled as \( \beta = 1 \);

(B) \[ \int_t^\infty d\Phi(t') \sim c^{-1} \beta^{-1} t^{-\beta}, \] for \( t \to \infty \), \( \beta \in (0, 1) \) and \( c > 0 \).

Then, with \( \lambda \) as in (3.2), we have the asymptotics (3.3).

In the following we assume (if not said otherwise)

(i) \( 0 < \alpha < 1 \), \( -\alpha < \theta < \alpha \);

or

(ii) \( 1 < \alpha < 2 \), \( -(2 - \alpha) < \theta < 2 - \alpha \).

**Lemma 2: Master Lemma for jumps with skewed probability and infinite variance.** Assume that the jump probability distribution function \( W(x) \) obeys with constants \( b_+ > 0 \), \( b_- > 0 \) the asymptotics

\[ \int_x^\infty dW(x') \sim b_+ \alpha^{-1} x^{-\alpha}, \] as \( x \to +\infty \),

\[ \int_x^{-\infty} dW(x') \sim b_- \alpha^{-1} |x|^{-\alpha}, \] as \( x \to -\infty \).
Set
\[ b = b_+ + b_-, \quad d = b_+ - b_-, \quad q = -\frac{d}{b} = \frac{b_- - b_+}{b_- + b_+.} \] (3.6)

Then we have the asymptotics (with \( \kappa \in \mathbb{R} \))
\[ 1 - \hat{w}(\kappa) \sim -\mu |\kappa|^{\alpha} i^{\theta} \text{sign} \kappa \quad \text{as} \quad \kappa \to 0 \] (3.7)
with
\[ \mu = \frac{b\pi}{2\Gamma(\alpha + 1) \sin(\alpha \pi/2)} |1 + iq\tan(\alpha \pi/2)|, \] (3.8)

and
\[ \theta = \frac{2}{\pi} \arctan[q \tan(\alpha \pi/2)] = \frac{2}{\pi} \arctan \left[ \frac{d}{b} \tan(\alpha \pi/2) \right] \quad \text{in case (i)}, \] (3.9)

\[ \theta = -\frac{2}{\pi} \arctan[q \tan((2 - \alpha) \pi/2)] = \frac{2}{\pi} \arctan \left[ \frac{d}{b} \tan((2 - \alpha) \pi/2) \right] \quad \text{in case (ii)}. \]

**Corollary of Lemma 2:** Assume that the jump probability density \( w(x) \) obeys with constants \( b_+ > 0, b_- > 0 \) the asymptotics
\[ w(x) \sim b_+ x^{-\alpha - 1}, \quad \text{as} \quad x \to +\infty, \quad w(x) \sim b_- |x|^{-\alpha - 1}, \quad \text{as} \quad x \to -\infty. \] (3.10)

Then, with \( \mu \) as in (3.8), we have the asymptotics (3.7).

**Comment:** This lemma, referred to in the following also as the skew lemma, is essentially a re-formulation (with a constant corrected) of parts of the Gnedenko theorem on the domains of attraction of stable probability laws, see [22]. It can also be distilled from considerations carried out in many later treatises, e.g. in [9].

For use in Section 6 we give a lemma for the particular case \( \alpha = 2 \) in which necessarily \( \theta = 0 \).

**Lemma 3: Master Lemma for symmetric jumps with finite variance.** Assume that the jump probability distribution function \( W(x) \) is symmetric with finite variance,
\[ \int_{-\infty}^{-x} dW(x') = \int_{x}^{+\infty} dW(x') \quad \text{for} \quad x \geq 0, \quad \text{with} \quad \mu = \frac{1}{2} \int_{-\infty}^{+\infty} x^2 dW(x) < \infty. \] (3.11)

Then, we have the asymptotics (with \( \kappa \in \mathbb{R} \)):
\[ 1 - \hat{w}(\kappa) \sim \mu \kappa^2 \quad \text{for} \quad \kappa \to 0. \] (3.12)

**Corollary of Lemma 3:** Assume the jump probability density function \( w(x) \) to be symmetric with finite variance,
\[ w(-x) = w(x) \quad \text{for} \quad x \in \mathbb{R}, \quad \text{with} \quad \mu = \frac{1}{2} \int_{-\infty}^{+\infty} x^2 w(x) \, dx < \infty \] (3.13)

Then, we have the asymptotics (3.12).
4. Passage to the diffusion limit

Let us now consider a scale of random walks by multiplying the jumps $X_k$ by a positive factor $h$, the waiting times $T_k$ by a positive factor $\tau$. We get a rescaled random walk $x_n(h) = \sum_{k=1}^{n} h X_k$ with jump instants $t_n(\tau) = \sum_{k=1}^{n} \tau T_k$ that we investigate with the aim of passing to the limit $h \to 0$, $\tau \to 0$, under a scaling relation between $h$ and $\tau$ yet to be established, assuming the conditions of the temporal lemma (on waiting times) and of the skew lemma (on jumps) fulfilled. As it is convenient to work in the Fourier-Laplace domain we note that the density of the rescaled waiting times $\tau T_k$ and that of the rescaled jumps $h X_k$ are $\phi_\tau(s) = \phi(t/\tau)/\tau$ and $\phi_h(x) = w(x/h)/h$, respectively. The corresponding transforms are $\hat{\phi}_\tau(\kappa) = \hat{\phi}(\kappa s)$, $\hat{\phi}_\tau(\kappa) = \hat{\phi}(h \kappa)$. We are interested in the sojourn probability density $p_{h,\tau}(x, t)$ of the particle subject to the re-scaled random walk. In analogy to the Montroll-Weiss equation (2.3) we get

$$
\hat{\tilde{p}}_{h,\tau}(\kappa, s) = \frac{1 - \phi(\tau s)}{s} \frac{1}{1 - \tilde{\phi}(h \kappa) \phi(\tau s)}.
$$

Fixing now $s$ and $\kappa$ different from zero, we find for $h \to 0$, $\tau \to 0$ from the Lemmata of Section 3 (replacing there $s$ by $\tau s$ and $\kappa$ by $h \kappa$) by trivial calculations, omitting asymptotically negligible terms, the asymptotics (4.2) with (4.3)

$$
\tilde{p}(\kappa, s) \sim \frac{s^\beta - 1}{s^\beta + r(h, \tau) |\kappa| \alpha i \theta \text{sign} \kappa},
$$

and

$$
r(h, \tau) = \frac{\mu h}{\alpha \lambda \tau}. \tag{4.3}
$$

So we see that, for every fixed real $\kappa \neq 0$ and positive $s$, we have the limit relation

$$
\tilde{p}(\kappa, s) \sim \frac{s^\beta - 1}{s^\beta + |\kappa| \alpha i \theta \text{sign} \kappa} = \tilde{u}(\kappa, s),
$$

as $h$ and $\tau$ tend to zero under the scaling

$$
r(h, \tau) \equiv 1. \tag{4.5}
$$

Comparing with (1.7) we recognize here $\tilde{u}(\kappa, s)$ as the combined Fourier-Laplace transform of the solution to the Cauchy problem (1.1). Invoking now the continuity theorems of probability theory, see e.g. [20], we see that the time-parametrized sojourn probability density of the rescaled random walk converges weakly or in law to the solution of the Cauchy problem (1.1). We state this result as a Theorem.

**Theorem:** Assume the probability laws for the waiting times $T_k$ and for the jumps $X_k$ to fulfil the conditions of the Master Lemmata of Section 3. Replace the waiting times by $h T_k$, the jumps by $h X_k$ and observe the scaling relation $\mu h = \lambda \tau$. Then, for $h$ (and consequently $\tau$) tending to zero, the rescaled sojourn probability density $p_{h,\tau}(x, t)$ converges weakly to the soliton of the Cauchy problem (1.1), in other words to the fundamental solution of the space-time fractional diffusion equation

$$
t D_x^\beta u(x, t) = x D_x^\theta u(x, t),
$$

the skewness parameter $\theta$ being given by one of Eqs. (3.9).
5. Construction of probability densities

In earlier papers we have presented random walk models for approximating the processes described by (1.1) in the case \( \theta = 0 \) of symmetry. Here in Section 3 we can, for given \( \alpha \in (0, 1) \cup (1, 2) \) and admissible jump probability distribution function, calculate the resulting skewness parameter \( \theta \) from one of Eqs. (3.9). The inverse problem is to find a suitable jump probability law if \( \alpha \) and \( \theta \) are given, restricted to case (i) or case (ii) of Section 1, namely

(i) \( 0 < \alpha < 1, \quad -\alpha < \theta < \alpha \),

(ii) \( 1 < \alpha < 2, \quad -(2 - \alpha) < \theta < 2 - \alpha \).

Let us first observe that the formula for \( q \) in (3.6) implies

\[
q = \frac{\rho - 1}{\rho + 1} \quad \text{with} \quad \rho = \frac{b_-}{b_+}. \tag{5.1}
\]

We have \( -1 < q < 1 \), each such \( q \) attainable, and \( \rho = (1 - q)/(1 + q), \ 0 < \rho < \infty \).

Then for the skewness parameter \( \theta \) we deduce from Eqs (3.9)

\[
-\alpha < \theta < \alpha \quad \text{in case (i),} \quad -(2 - \alpha) < \theta < (2 - \alpha) \quad \text{in case (ii).} \tag{5.2}
\]

This means that the border cases \( \theta = \pm \alpha \) in case (i), \( \theta = \pm (2 - \alpha) \) in case (ii) are not covered. We leave them aside, intending to discuss them on another occasion. En passant let us remark that we also bypass the border case \( \alpha = 2 \) for which only \( \theta = 0 \) is admissible and so is covered by our papers [33] and [24].

Now assume \( \alpha \in (0, 1) \cup (1, 2) \) and \( \theta \) with the restrictions (5.2) are given. Then Eqs (3.9) imply

\[
q = \frac{\tan(\theta \pi/2)}{\tan(\alpha \pi/2)} \quad \text{in case (i),} \quad q = -\frac{\tan(\theta \pi/2)}{\tan((2 - \alpha) \pi/2)} \quad \text{in case (ii),} \tag{5.3}
\]

and for the ratio \( \rho = b_-/b_+ \) of the weight coefficients occurring in the Skew Lemma we find \( \rho = (1 - q)/(1 + q) \).

A possible way to proceed further is to take a non-negative integrable function \( v(x) \) which vanishes for \( x < 0 \), is normalized to \( \int_{-\infty}^{\infty} v(x) \, dx = 1 \) and obeys the asymptotics \( v(x) \sim b^* x^{-\alpha - 1} \) as \( x \to +\infty \) with a known positive constant \( b^* \). Then define, with weight coefficients \( b_- \) and \( b_+ \) still to be determined,

\[
w(x) = \frac{1}{b^*} [b_- v(-x) + b_+ v(x)]. \tag{5.4}
\]

Observe that this function behaves for \( x \to \pm \infty \) as required in the Skew Master Lemma. Normalization \( \int_{-\infty}^{\infty} w(x) \, dx = 1 \) requires \( b_-/b^* + b_+/b^* = 1 \), and from this relation and \( b_- = \rho b_+ \) we find with \( q \) known from (5.3)

\[
b_- = \frac{\rho b^*}{\rho + 1} = \frac{(1 - q)b^*}{2}, \quad b_+ = \frac{b^*}{\rho + 1} = \frac{(1 + q)b^*}{2}. \tag{5.5}
\]

If we choose a function \( v(x) \) with invertible primitive \( V(x) = \int_0^x v(x') \, dx' \) (for \( x \geq 0 \)), then generation of random jumps can be carried out by a standard Monte-Carlo technique, see [13, 32].

As a suitable function let us offer

\[
v(x) = \frac{\alpha x^{\alpha - 1}}{(1 + x^{\alpha})^2} \quad \text{for} \quad x > 0, \quad v(x) = 0 \quad \text{for} \quad x < 0. \tag{5.6}
\]

Then \( b^* = \alpha \) and the function

\[
V(x) = \frac{x^\alpha}{1 + x^\alpha} \quad \text{for} \quad x \geq 0 \tag{5.7}
\]
is invertible. In fact, for $0 \leq y < 1$ the equation $V(x) = y$ has the unique solution

$$x = \left( \frac{y}{1 - y} \right)^{1/\alpha}.$$

For completeness let us offer as easily invertible waiting time distribution function

$$\Phi(t) = \begin{cases} 
\exp(-t), & \text{if } \beta = 1, \\
1 - \frac{1}{1 + \Gamma(1 - \beta) t^\beta}, & \text{if } 0 < \beta < 1.
\end{cases}$$

(5.9)

6. Distributed order fractional diffusion

The authors of [14, 15] and [74], see also [35], have discussed the Cauchy problems for the distributed order time-fractional diffusion equation\(^2\)

$$\int_0^1 b(\beta) t D^\beta_\ast u(x, t) \, d\beta = \frac{\partial^2}{\partial x^2} u(x, t), \quad -\infty < x < +\infty, \quad t \geq 0,$$

(6.1)

with $u(x, 0) = \delta(x)$, and $b(\beta) \geq 0$, $\int_0^1 b(\beta) \, d\beta = 1$.

Here we use the notation adopted in [35], doing our analysis in dimensionless form. It was shown that the evolving function $u(x, t)$ is non-negative and normalized, so allowing interpretation as density with respect to $x$ of the probability at time $t$ of a diffusing particle to be in the point $x$. For simulation of the trajectories of diffusing particles (particle paths) the authors used a discrete backward-oriented Grünwald-Letnikov approximation of the temporal fractional derivative. For the Grünwald-Letnikov approximation to the fractional derivatives the reader is referred e.g. to the book by Podlubny [64]; for random walk models based on this approximation we refer to several papers of our research group, see e.g. [23, 25, 30, 31, 32, 36, 37, 39, 40]. Our aim now is to show how such distributed order fractional diffusion processes can be obtained as multiply scaled diffusion limits of continuous time random walks. In our notation we closely follow the notations applied in [33] and [24]. We concentrate our attention to the spatially one-dimensional situation and to the case $\alpha = 2$, indicating later possible generalizations.

First, consider the problem (6.1) in the very special case

$$b(\beta) = b_1 \delta(\beta - \beta_1) + b_2 \delta(\beta - \beta_2),$$

(6.2)

with $0 < \beta_1 < \beta_2 \leq 1$, $b_1 > 0$, $b_2 > 0$, $b_1 + b_2 = 1$, which designates the distributed order time-fractional diffusion equation

$$\left( b_1 t D_{\ast}^{\beta_1} + b_2 t D_{\ast}^{\beta_2} \right) u(x, t) = \frac{\partial^2}{\partial x^2} u(x, t),$$

(6.3)

with initial condition $u(x, 0) = \delta(x)$.

We introduce now the concept of a CTRW doubly scaled in time. Take two waiting time laws as in the Lemma 1 (Temporal Lemma) or in its Corollary, two laws decorated with indices 1 and 2, corresponding to the parameters $\beta_1, \beta_2$, the weights $b_1$ and $b_2$ of (6.2) and a jump law as in Lemma 3, or its Corollary. We then have (with $s > 0$)

$$\hat{\phi}_1(s) = 1 - \lambda_1 s^{\beta_1} + o(s^{\beta_1}), \quad \hat{\phi}_2(s) = 1 - \lambda_2 s^{\beta_2} + o(s^{\beta_2}), \quad \text{for } s \to 0,$$

(6.4)

\(^2\) We find a former idea of fractional derivative of distributed order in time in the 1969 book by Caputo [10], that was later developed by Caputo himself, see [11, 12] and by Bagley & Torvik, see [3]. A former application of this kind of derivative to chaotic dynamics (multifractional kinetics) has been developed by Zaslavsky [84].
Under the double-scaling relation \( r \tau \)

Here whose Laplace transform is

we obtain, setting

\[
\tilde{\phi}_r(t) = b_1 \frac{\phi_1(t/\tau_1)}{\tau_1} + b_2 \frac{\phi_2(t/\tau_2)}{\tau_2},
\]

whose Laplace transform is

\[
\tilde{\phi}_r(s) = b_1 \tilde{\phi}_1(s\tau_1) + b_2 \tilde{\phi}_2(s\tau_2).
\]

Here \( \tau := (\tau_1, \tau_2) \) is a vector of scaling factors.

Fixing now, as required by the continuity theorems of probability theory, \( \kappa \) and \( s \), and inserting the asymptotics into the equation (which is analogous to (4.1))

\[
\hat{p}_{h,\tau}(\kappa, s) = \frac{1}{s} \frac{1}{1 - \hat{w}(h\kappa) \tilde{\phi}_r(s)},
\]

we obtain, setting

\[
r_j = \frac{\lambda_j \tau_j^{\beta_j}}{\mu h^2} \quad \text{for} \quad j = 1, 2,
\]

by careful calculation,

\[
\hat{p}_{h,\tau}(\kappa, s) = \frac{b_1 \lambda_1 \tau_1^{\beta_1} s^{\beta_1-1} + b_2 \lambda_2 \tau_2^{\beta_2} s^{\beta_2-1} + o(\tau_1^{\beta_1}) + o(\tau_2^{\beta_2})}{\mu h^2 \kappa^2 + b_1 \lambda_1 \tau_1^{\beta_1} s^{\beta_1} + b_2 \lambda_2 \tau_2^{\beta_2} s^{\beta_2} + h^2 \mathcal{O}(\tau_1^{\beta_1} + \tau_2^{\beta_2})}
\]

\[
= \frac{b_1 r_1 s^{\beta_1-1} + b_2 r_2 s^{\beta_2-1} + o(1)}{\kappa^2 + b_1 r_1 s^{\beta_1} + b_2 r_2 s^{\beta_2} + o(1)}
\]

Under the double-scaling relation \( r_1 \equiv 1 \) and \( r_2 \equiv 1 \) we find

\[
\hat{p}_{h,\tau}(\kappa, s) \to \frac{B(s)/s}{\kappa^2 + B(s)} = \hat{u}(\kappa, s), \quad \text{with} \quad B(s) := b_1 s^{\beta_1} + b_2 s^{\beta_2}.
\]

The double scaling relation can be re-written as

\[
\tau_j = (\mu/\lambda_j)^{1/\beta_j} (h)^{2/\beta_j} ; \quad j = 1, 2.
\]

Recognizing here \( \hat{u}(\kappa, s) \) as Fourier-Laplace solution of (6.3) we can say that our time-doubly scaled CTRW tends weakly to this distributed order time fractional diffusion process.

Resisting the temptation to work out an explicit form of the solution of the Cauchy problem (6.3) in terms of Fox \( H \)-functions, we content ourselves with its Fourier-Laplace form given in (6.11). From this form we will deduce the asymptotic behaviours of the variance of the process,

\[
\langle x^2 \rangle(t) := \int_{-\infty}^{+\infty} x^2 u(x, t) \, dx = -\frac{\partial^2}{\partial \kappa^2} \hat{u}(\kappa = 0, t),
\]

for small \( t \) and for large \( t \). Using (6.13) and from

\[
\hat{u}(\kappa, s) = \frac{1/s}{1 + \kappa^2/B(s)} = \frac{1}{s} \left( 1 - \kappa^2/B(s) + \ldots \right)
\]
for fixed \( s > 0 \) and \( \kappa \to 0 \) (so that \( \kappa^2 / B(s) < 1 \)), we obtain

\[
\mathcal{L} \left\{ \langle x^2 \rangle(t); s \right\} = -\frac{\partial^2}{\partial \kappa^2} \hat{u}(\kappa = 0, s) = \frac{2}{s B(s)}.
\]

(6.15)

Now, looking at \( B(s) \) in (6.11), we see that for \( s \to \infty \) (which by Tauber theory corresponds to \( t \to 0 \)) \( s^{\beta} \) becomes negligible, whereas for \( s \to 0 \) (corresponding to to \( t \to \infty \)) \( s^{\beta_2} \) becomes negligible. This means

\[
-\frac{\partial^2}{\partial \kappa^2} \hat{u}(\kappa = 0, s) \sim \frac{2}{b_2 s^{\beta_2 + 1}} \quad \text{for } s \to \infty, \quad -\frac{\partial^2}{\partial \kappa^2} \hat{u}(\kappa = 0, s) \sim \frac{2}{b_1 s^{\beta_1 + 1}} \quad \text{for } s \to 0,
\]

(6.16)

hence by Laplace inversion

\[
\langle x^2 \rangle(t) \sim \frac{2t^{\beta_2}}{b_2 \Gamma(\beta_2 + 1)} \quad \text{for } t \to 0, \quad \langle x^2 \rangle(t) \sim \frac{2t^{\beta_1}}{b_1 \Gamma(\beta_1 + 1)} \quad \text{for } t \to \infty.
\]

(6.17)

**Comment:** The mean square deviation as a function of \( t \) has a smaller exponent for large time than for small time, so we have subdiffusion with retardation.

We can formally generalize. Taking a non-negative weight function \( b(\beta) \) as in (6.1) we define the set \( C := \{ \beta \mid b(\beta) > 0 \} \). For \( \beta \in C \) we assume that the waiting time densities \( \phi_{\beta}(t) \) obey asymptotics (with positive \( \lambda(\beta) \))

\[
\tilde{\phi}_{\beta}(s) = 1 - \lambda(\beta) s^{\beta} + o(s^{\beta}), \quad \text{for } s \to 0.
\]

(6.18)

Introducing positive scaling factors \( \tau(\beta) \) for \( \beta \in C \), denoting the set of all these \( \tau(\beta) \) simply by \( \tau \), we define the (rescaled) waiting time density

\[
\phi_{\tau}(t) = \int_C b(\beta) \frac{\phi_{\beta}(t/\tau(\beta))}{\tau(\beta)} d\beta.
\]

(6.19)

As generalization of the double-scaling relation we now use the \( \beta \)-parametrized scaling relation

\[
\frac{\lambda(\beta) \tau(\beta)^{\beta}}{\mu \kappa^2} \equiv 1,
\]

(6.20)

for \( \beta \in C \). Surely, the functions \( b(\beta) \) and \( \lambda(\beta) \) must satisfy appropriate regularity conditions for the existence of the integral (6.19) and transition to the diffusion limit being possible with a reasonable result. If everything works well we obtain in the limit the Cauchy problem (6.1).

We ask now, in this general case, for the asymptotics of \( \langle x^2 \rangle(t) \). Letting, in analogy to (6.11),

\[
\tilde{p}_{h,\tau}(\kappa, s) \sim \frac{B(s)/s}{\kappa^2 + B(s)} = \hat{u}(\kappa, s), \quad \text{with } B(s) := \int_0^1 b(\beta) s^{\beta} d\beta,
\]

(6.21)

by the Laplace-Fourier method, for fixed \( s > 0 \) and \( \kappa^2 / B(s) < 1 \), we have the expansion again (6.14) and (6.15).

As in [14, 15], let us consider the special case

\[
b(\beta) \equiv 1, \quad 0 < \beta \leq 1.
\]

(6.22)

We get

\[
B(s) = \int_0^1 s^{\beta} d\beta = \frac{s - 1}{\log s},
\]

(6.23)
and, by using (6.15), the asymptotics
\[ L\left\{ \langle x^2 \rangle (t); s \rangle \right\} \sim -\frac{2\log s}{s^2} \text{ for } s \to \infty , \quad L\left\{ \langle x^2 \rangle (t); s \rangle \right\} \sim \frac{2\log (1/s)}{s} \text{ for } s \to 0 . \] (6.24)

Consequently, see e.g. [1] for the needed Laplace pairs,
\[ \langle x^2 \rangle (t) \sim 2t \log (1/t) \text{ for } t \to 0 , \quad \langle x^2 \rangle (t) \sim 2 \log t \text{ for } t \to \infty . \] (6.25)

This means slightly anomalous super diffusion for small \( t \), but ultraslow diffusion for large \( t \).

If, more generally, see for details [15],
\[ b(\beta) = \nu \beta^{\nu-1} , \quad 0 < \beta \leq 1 , \quad \nu > 0 , \] (6.26)
then we have
\[ \langle x^2 \rangle (t) \sim \frac{2}{\nu} t \log (1/t) \text{ for } t \to 0 , \quad \langle x^2 \rangle (t) \sim \frac{2}{\Gamma(\nu+1)} \log^{\nu} t \text{ for } t \to \infty . \] (6.27)

By analogous methods one can by cleverly chosen time-multiply scaled CTRW obtain in the diffusion limit fractional diffusion processes of the form
\[ \int_0^1 b(\beta) \int^\beta \alpha^\alpha u(x, t) d\alpha = \int^\beta \alpha^\alpha u(x, t) d\alpha , \quad -\infty < x < +\infty , \quad t \geq 0 ; \quad u(x, 0) = \delta(x) , \] (6.28)
with \( b(\beta) \geq 0 \), \( \int_0^1 b(\beta) d\beta = 1 \), and \( \nu, \theta \) in the ranges (i) and (ii) as demanded in Section 1.

Furthermore, by applying analogous multiple scaling in space, one can obtain in the diffusion limit fractional diffusion processes of the type
\[ \int_0^2 a(\alpha) \alpha^{\alpha} u(x, t) d\alpha = \int^\beta \alpha^\alpha u(x, t) d\alpha , \quad -\infty < x < +\infty , \quad t \geq 0 ; \quad u(x, 0) = \delta(x) , \] (6.29)
with \( \int_0^2 a(\alpha) d\alpha = 1 \), \( a(\alpha) \geq 0 \), and \( 0 < \beta \leq 1 \).

Let us finally remark that we were inspired to multiple scaling of random walks by Hilfer’s paper [42].

7. Concluding comments
Our attention in this paper has been focused on the space-time fractional diffusion equation (1.1) and on the distributed order time fractional diffusion equation (6.1). Specifically, we have worked out how the stochastic processes modelled by these equations can be obtained as properly scaled diffusion limits of continuous time random walks, such random walks being useful for approximate simulation.

Concerning equation (1.1) these random walks are characterized by asymptotic power law behaviour of the tails of the probability laws for the jumps (the exponent being \(-\alpha\)) and the waiting time (the exponent being \(-\beta\)) between jumps. If the the relevant powers of \(|x|\) has different coefficients for \( x \) tending to \( \pm \infty \), then we will arrive at non-zero skewness \( \theta \), a situation rarely treated in the literature on applications. We display formulae for calculating this skewness, and conversely formulae for calculating the above-mentioned coefficients in case of a give value of the skewness. Essential for our analysis are some lemmata of Tauber type, in space distilled from the famous Gnedenko theorem on the domain of attraction of stable probability laws, in time by Karamata’s theorem on Laplace transforms. We have left aside some extreme and singular cases of the parameters, furthermore (in order not to overload our formulas) the possible decoration of the power laws by multiplication with slowly varying functions. Some extreme cases are trivial, others deserve indeed separate investigation in more detail.
For equation (6.1) we offer approximation by a continuous time random walk with a mixture of waiting times, each of which individually set in a scaling relation to the jump probability law. We discuss in detail the situation of a mixture of two waiting times which in the diffusion limit leads to a linear combination of two delta functions for the function \( b(\beta) \) weighing the individual fractional time derivatives. The behaviour of the process can be characterized as retarded subdiffusion visible in the different asymptotics of the second (centred) moment of the density for small and large times. Then, as a special case of a continuous weight function, we consider, simplifying the analysis applied in [13] and [14], the case of a weight function equal to 1 in the whole interval \( 0 < \beta \leq 1 \), and obtain slightly anomalous super-diffusion for small time, but ultraslow diffusion for large time \( t \). The field of distributed order fractional diffusion processes and the related integro-differential equations is highly fascinating, and we hope that our paper will inspire further research, concerning applications in modelling phenomena of strange kinetics\(^3\) as well as challenging analytical and numerical aspects of fractional calculus; let us quote [63] and [80] for the analytical, and [17] for the numerical side.

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\(^3\) This term was first introduced in 1993 by Shlesinger, Zaslavsky and Klafter [72].
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