Stochastic Samples versus Vacuum Expectation Values in Cosmology

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ABSTRACT

Particle theorists typically use expectation values to study the quantum back-reaction on inflation, whereas many cosmologists stress the stochastic nature of the process. While expectation values certainly give misleading results for some things, such as the stress tensor, we argue that operators exist for which there is no essential problem. We quantify this by examining the stochastic properties of a noninteracting, massless, minimally coupled scalar on a locally de Sitter background. The square of the stochastic realization of this field seems to provide an example of great relevance for which expectation values are not misleading. We also examine the frequently expressed concern that significant back-reaction from expectation values necessarily implies large stochastic fluctuations between nearby spatial points. Rather than viewing the stochastic formalism in opposition to expectation values, we argue that it provides a marvelously simple way of capturing the leading infrared logarithm corrections to the latter, as advocated by Starobinsky.

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1 Introduction

Schrödinger was the first to suggest that spacetime expansion can lead to particle production by ripping virtual particles out of the vacuum [1]. Following early work by Imamura [2], the first quantitative results were obtained by Parker [3]. He found that the effect is maximized during accelerated expansion, and for massless particles which are not conformally invariant [4], such as massless, minimally coupled scalars and (as noted by Grishchuk [5]) gravitons. Precisely this process is responsible for the primordial spectra of scalar and tensor perturbations which are believed to arise from inflation [6], the scalar contribution of which has been imaged [7].

Inflationary particle production results from the background gravitational field acting on quantum matter (and graviton) fluctuations. It is natural to wonder about the complementary process of back-reaction in which the newly produced particles modify the background gravitational field, either directly or through their self-interactions. People who approach this from the perspective of particle physics typically attempt to quantify back-reaction using expectation values or in-out matrix elements [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19].

Many cosmologists dismiss the use of expectation values and in-out matrix elements as giving an unreliable average over vastly different portions of a quantum wave function which has actually decohered [1]. They fear that particle theorists are falling victim to a sort of cosmological Schrödinger Cat Paradox based on a fictitious, mean geometry which bears no relation to what any observer would experience. Cosmologists prefer instead to study back-reaction using a stochastic formalism in which the super-horizon modes of various fields are regarded as classical, random variables [21, 22, 23, 24, 25]. Whereas expectation values in a homogeneous and isotropic state necessarily produce a homogeneous and isotropic geometry, cosmologists assert that the actual universe is not even approximately homogeneous on super-horizon scales [23]. They also fear that if the back-reaction inferred from expectation values ever becomes significant then the resulting universe would show unacceptable spatial fluctuation.

There is no question that cosmologists are right about certain operators being poorly described by their expectation values. For example, the vacuum

\[\text{For an excellent recent study of cosmological decoherence, which does not necessarily endorse the anti-VEV position, see [20].}\]
expectation value of the stress-energy tensor is homogeneous and isotropic, whereas we perceive inhomogeneities and anisotropies. What isn’t clear is whether or not expectation values are in all cases misleading. Such an extreme position would be embarrassing for cosmologists (in spite of the fact that some do advocate it) because the primordial power spectra are defined by taking expectation values [26].

We suspect that the reliability of expectation values depends upon the operator under study. For operators which average to zero, such as the density perturbation, the entire result arises from the decoherence effect, so one makes an enormous mistake by ignoring it. Other operators — for example, the square of a scalar field — acquire a significant homogeneous expectation value upon which spatial variations are superimposed. Any quantum fluctuation drives this sort of operator positive, so one might happen to inhabit a special region of the universe in which there is little effect for a long time, but there will sooner or later be a large effect. The expectation value of such an operator can correctly reflect the long-term trend everywhere in space, even though it misses variations from one region to another.

The purpose of this paper is to use the noninteracting, massless, minimally coupled scalar on de Sitter background to give a quantitative assessment of the two key issues under dispute between particle theorists and cosmologists:

1. How unreliable are expectation values? and
2. How much spatial variation should one expect?

In section 2 we review the de Sitter geometry, the scalar field model, and our stochastic realization of it. In section 3 we demonstrate that the square of the stochastic field follows a $\chi^2$ distribution whose mean grows with the number of e-foldings. Although fluctuations about this mean are significant, they do not contradict the picture provided by expectation values. In section 4 we address the issue of spatial fluctuations by studying the difference of the stochastic scalar at two points held at a fixed physical distance from one another. Our conclusions comprise section 5.

2 The Theoretical Context

We work on the spacetime manifold $T^3 \times R$. We assume each of the three spatial coordinates lies in the range, $0 \leq x^i < H^{-1}$, where $H$ is the Hubble
constant. We also assume that the geometry is locally de Sitter,
\[ ds^2 = -dt^2 + a^2(t) d\vec{x} \cdot d\vec{x} , \quad a(t) \equiv e^{Ht} . \] (1)

Note that the scale factor \( a(t) \) is normalized to unity at \( t = 0 \), rather than at the current time. Because the geometry (1) is homogeneous and isotropic we can expand any function of spacetime in a spatial Fourier series,
\[ f(t, \vec{x}) = \sum_{\vec{n}} f_{\vec{n}}(t) e^{i\vec{k} \cdot \vec{x}} , \quad \vec{k} = 2\pi H \vec{n} . \] (2)

A key event for any mode of wave number \( k \equiv \|\vec{k}\| \) is the time \( t_k \) of horizon crossing when its physical wave number redshifts down to the Hubble constant,
\[ \frac{k}{a(t_k)} = H . \] (3)

The Lagrangian for a massless, minimally coupled scalar is,
\[ \mathcal{L} = -\frac{1}{2} \partial_\mu \phi \partial^\nu \phi g^{\mu\nu} \sqrt{-g} = \frac{1}{2} a^3 \dot{\phi}^2 - \frac{1}{2} a \|\nabla \phi\|^2 . \] (4)

For all nonzero wave numbers \( k \) the canonically normalized, Bunch-Davies mode functions are,
\[ u(t, k) = \frac{H}{\sqrt{2k^3}} \left[ 1 - \frac{ik}{Ha(t)} \right] \exp \left[ \frac{ik}{Ha(t)} \right] \Rightarrow uu^* - \dot{u} u^* = \frac{i}{a^3} . \] (5)

For the \( k = 0 \) mode the two solutions are a constant and \( 1/a^3(t) \). Hence the field operator can be expanded as,
\[ \varphi(t, \vec{x}) = H^{\frac{3}{2}} \left\{ Q - \frac{P}{3Ha^3(t)} + \sum_{\vec{n} \neq 0} \left[ u(t, k)e^{i\vec{k} \cdot \vec{x}} A_{\vec{n}} + u^*(t, k)e^{-i\vec{k} \cdot \vec{x}} A_{\vec{n}}^\dagger \right] \right\} , \] (6)
where the nonvanishing commutators are,
\[ [Q, P] = i , \quad [A_{\vec{m}}, A_{\vec{n}}^\dagger] = \delta_{\vec{m}, \vec{n}} . \] (7)

The state \( |\Omega\rangle \) which is annihilated by all \( A_{\vec{n}} \) is known as Bunch-Davies vacuum. It does not matter very much what we assume about its dependence upon the 0-mode coordinate.
The quantity $\varphi(t, \vec{x})$ is a quantum field operator which obeys the Uncertainty Principle,

$$\left[\varphi(t, \vec{x}), \dot{\varphi}(t, \vec{y})\right] = i\delta^3(\vec{x} - \vec{y}).$$  \hfill (8)

Because the free field expansion (6) contains arbitrarily large wave numbers, expectation values of coincident products of $\varphi(t, \vec{x})$ can harbor ultraviolet divergences. All of these features are absent in the stochastic realization of $\varphi(t, \vec{x})$ which we construct by taking the infrared limit of the mode functions,

$$\lim_{k \ll H_a} u(t, k) = \frac{H}{\sqrt{2k^3}};$$  \hfill (9)

and retaining only the super-horizon modes. We denote this stochastic field as $\hat{\varphi}(t, \vec{x})$ and its definition is [17],

$$\hat{\varphi}(t, \vec{x}) \equiv \sum_{\vec{n} \neq 0} \sqrt{\frac{H}{2k^3}} \theta(Ha(t) - k) \left[ e^{i\vec{k} \cdot \vec{x}} \hat{A}_{\vec{n}} + e^{-i\vec{k} \cdot \vec{x}} \hat{A}_{\vec{n}}^\dagger \right].$$  \hfill (10)

Instead of being creation and annihilation operators, $\hat{A}_{\vec{n}}$ and $\hat{A}_{\vec{n}}^\dagger$ are considered to be (complex conjugate) stochastic random variables which follow a normal distribution with mean zero and standard deviation one. That is, one can express $\hat{A}_{\vec{n}}$ as the sum of two real, independent random variables,

$$\hat{A}_{\vec{n}} \equiv \alpha_{\vec{n}} + i\beta_{\vec{n}} \quad \text{where} \quad \rho(\alpha_{\vec{n}} = x, \beta_{\vec{n}} = y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2 + y^2)}. \hfill (11)$$

### 3 Fluctuations about the Mean

It is useful to express $\hat{\varphi}(t, \vec{x})$ in terms of the real stochastic variables $\alpha_{\vec{n}}$ and $\beta_{\vec{n}}$ which were introduced in (11),

$$\hat{\varphi}(t, \vec{x}) = \sum_{\vec{n} \neq 0} \sqrt{\frac{H}{2k^3}} \theta(Ha(t) - k) \left[ \cos(\vec{k} \cdot \vec{x}) \alpha_{\vec{n}} - \sin(\vec{k} \cdot \vec{x}) \beta_{\vec{n}} \right].$$  \hfill (12)

In this form one can recognize $\hat{\varphi}(t, \vec{x})$ as the sum of a vast number $N(t)$ of independent Gaussian random variables,

$$N(t) = 2 \sum_{\vec{n} \neq 0} \theta(Ha(t) - k) \simeq 2 \times 4\pi \int_{\frac{a}{2\pi}}^{a/2\pi} dn \Delta n^2 \simeq \frac{a^3(t)}{3\pi^2}. \hfill (13)$$
Of course the sum of any number of independent Gaussian random variables gives another Gaussian random variable whose mean is the sum of the means and whose variance is the sum of the variances. Because the mean of each variable is zero, the mean of \( \hat{\varphi}(t, \vec{x}) \) vanishes. On the other hand, its variance grows,

\[
\sigma^2(t) = \sum_{\vec{n} \neq 0} \frac{H^5}{2k^3} \theta(Ha(t) - k) \left[ \cos^2(\vec{k} \cdot \vec{x}) + \sin^2(\vec{k} \cdot \vec{x}) \right], \tag{14}
\]

\[
\simeq 4\pi \int_{a/2\pi}^{a/2\pi} dn^2 \frac{H^5}{2(2\pi Hn)^3} = \frac{H^2}{4\pi^2} \ln[a(t)]. \tag{15}
\]

Hence we can say that, at any given spacetime point, the stochastic field \( \hat{\varphi}(t, \vec{x}) \) follows a normal distribution with mean zero and variance \( \sigma^2(t) \),

\[
\rho(\hat{\varphi}(t, \vec{x}) = Z) = \frac{1}{\sqrt{2\pi\sigma^2(t)}} \exp\left[-\frac{Z^2}{2\sigma^2(t)}\right]. \tag{16}
\]

The fields at different spacetime points are not statistically independent, but we will not need to worry about that until the next section.

The variable \( \hat{\varphi}(t, \vec{x}) \) provides a classic example of why cosmologists distrust expectation values. Although its expectation value vanishes, the stochastic field experiences very significant and growing fluctuations, as its variance reveals. Someone interested in the behavior of \( \hat{\varphi}(t, \vec{x}) \) would derive a completely misleading picture from its expectation value.

On the other hand, consider the variable \( \hat{\varphi}^2(t, \vec{x}) \). Because it is the square of a Gaussian with mean zero it follows a \( \chi^2 \) distribution whose mean is the variance of the original Gaussian,

\[
\rho(\hat{\varphi}^2(t, \vec{x}) = Z) = \frac{1}{\sqrt{2\pi\sigma^2(t)Z}} \exp\left[-\frac{Z^2}{2\sigma^2(t)}\right]. \tag{17}
\]

The variance of \( \hat{\varphi}^2(t, \vec{x}) \) might seem to vindicate the extreme cosmologist position that expectation values are never reliable,

\[
\left\langle \left( \hat{\varphi}^2(t, \vec{x}) - \langle \hat{\varphi}^2(t, \vec{x}) \rangle \right)^2 \right\rangle = \int_0^\infty dZ \frac{(Z - \sigma^2)^2}{\sqrt{2\pi\sigma^2Z}} \exp\left[-\frac{Z}{2\sigma^2}\right] = 2\sigma^4(t). \tag{18}
\]

However, this only implies that that stochastic fluctuations about the mean are significant, even when considered as a fraction of the mean.

What the expectation value \( \langle \hat{\varphi}^2(t, \vec{x}) \rangle = \sigma^2(t) \) really tells us is:
That $\hat{\varphi}^2(t, \vec{x})$ grows without bound; and

- That this growth is proportional to the number of e-foldings, $\ln[a(t)]$.

We can gain a quantitative assessment of the reliability of the first conclusion by using (17) to compute the probability that $\hat{\varphi}^2(t, \vec{x})$ remains less than some constant value $\Phi^2$,

$$\text{Prob}(\hat{\varphi}^2(t, \vec{x}) < \Phi^2) = \int_0^{\Phi^2} dZ \frac{1}{\sqrt{2\pi \sigma^2(t) Z}} \exp\left[-\frac{Z}{2\sigma^2(t)}\right], \quad (19)$$

$$= \frac{2\Phi^2}{\pi \sigma^2(t)} \left\{ 1 + O\left(\frac{\Phi^2}{\sigma^2(t)}\right) \right\}. \quad (20)$$

Because $\sigma^2(t)$ grows like the number of e-foldings, we see that the probability for $\hat{\varphi}^2(t, \vec{x})$ to fall below any fixed value $\Phi^2$ goes to zero at late times. Of course that vindicates the inference of growth without bound. The second inference can be tested by computing the probability for $\hat{\varphi}^2(t, \vec{x})$ to be above some time dependent value $\Phi^2(t)$ which grows faster than $\sigma^2(t)$,

$$\text{Prob}(\hat{\varphi}^2(t, \vec{x}) > \Phi^2(t)) = \int_{\Phi^2}^\infty dZ \frac{1}{\sqrt{2\pi \sigma^2(t) Z}} \exp\left[-\frac{Z}{2\sigma^2(t)}\right], \quad (21)$$

$$= \frac{2\sigma^2(t)}{\pi \Phi^2(t)} \exp\left[-\frac{\Phi^2(t)}{2\sigma^2(t)}\right] \left\{ 1 + O\left(\frac{\sigma^2(t)}{\Phi^2(t)}\right) \right\}. \quad (22)$$

Under the assumption that $\sigma^2(t)/\Phi^2(t)$ goes to zero we see that this probability also approaches zero. Hence the second inference is equally valid, and we conclude that no serious error arises from using expectation values to study $\hat{\varphi}^2(t, \vec{x})$.

One might object that there is still a substantial disagreement between quantum field theoretic expectation values and stochastic samples because the former contain an ultraviolet divergent constant in addition to the infrared logarithm [27],

$$\langle \Omega | \varphi^2(t, \vec{x}) | \Omega \rangle = \left( \text{Divergent Constant} \right) + \frac{H^2}{4\pi^2} \ln[a(t)]. \quad (23)$$

The form of this divergence depends upon the regularization technique; with dimensional regularization in $D$ spacetime dimensions one finds [28],

$$\langle \Omega | \varphi^2(t, \vec{x}) | \Omega \rangle = \frac{H^{D-2}}{(4\pi)^{D/2}} \frac{\Gamma(D-1)}{\Gamma(D/2)} \left\{ 2 \ln[a(t)] - \psi\left(1 - \frac{D}{2}\right) \right\}.$$
\[ +\psi\left(\frac{D-1}{2}\right) + \psi(D-1) + \psi(1) + O\left(a^{-2}\right) \]. \quad (24)

However, many fully dimensionally regulated and renormalized computations have been done involving massless, minimally coupled scalars \[29, 30, 31, 32, 33, 34\] and gravitons \[35, 36, 37\] with various interactions. What always happens is that counterterms absorb the ultraviolet divergence and leave the infrared logarithm as the dominant contribution to the final result. That is just what one gets from \[\langle \hat{\varphi}^2(t, \vec{x}) \rangle\].

One might also object that the example of \[\hat{\varphi}^2(t, \vec{x})\] is contrived because it represents two fields at the same spacetime point. However, it is exactly these terms which are most responsible for the growth of the vacuum energy in \(\lambda \varphi^4\) theory \[29\] and for the photon developing a mass in scalar quantum electrodynamics \[38, 30\]. Note also that the use of expectation values gives precisely the correct results for the leading logarithm effects in each case \[29, 38, 30\],

\[
\langle \Omega | T_{\mu\nu}(t, \vec{x}) | \Omega \rangle \rightarrow -\frac{\lambda}{4!} \langle \hat{\varphi}^4(t, \vec{x}) \rangle g_{\mu\nu} = -\frac{\lambda}{8} \left[ \frac{H^2}{4\pi^2} \ln[a(t)] \right]^2 g_{\mu\nu}, \quad (25)
\]

\[
M^2_\gamma \rightarrow +e^2 \langle \hat{\varphi}^2(t, \vec{x}) \rangle = \frac{e^2 H^2}{4\pi^2} \ln[a(t)]. \quad (26)
\]

This is not an accident, nor is the coincidence restricted to lowest order perturbative results such as those given above. For scalar potential models one can show that Starobinsky’s formalism \[21\] captures the leading infrared logarithms of quantum field theoretic expectation values to all orders \[39\]. The best way of viewing the stochastic formalism is not as an alternative to expectation values but rather as marvelously simple way of deriving the most important contributions to them.

In some cases the stochastic formalism can do even more. Starobinsky and Yokoyama have shown how it can be used to sum the series of leading infrared logarithms to derive nonperturbative results for the late time limit \[40\]. For example, these results explicitly disprove the two simplest implementations of the common notion that cosmological evolution can be viewed as a renormalization group flow \[41\]. The Starobinsky-Yokoyama technique has recently been extended to Yukawa theory \[33\] and to scalar quantum electrodynamics \[42\]. It has not yet been extended to gravity but there are reasons for believing that some version of it can be \[43, 44\].
4 Fluctuations in Space

We can study spatial variation by taking the difference of the fields at points on the surface of simultaneity,

$$\Delta \hat{\varphi}(t, \Delta \vec{x}) \equiv \varphi(t, \vec{0}) - \varphi(t, \Delta \vec{x}) ,$$

(27)

$$= \sum_{\vec{n} \neq 0} \sqrt{\frac{2H^5}{k^3}} \theta(Ha(t) - k) \sin \left( \frac{k \cdot \Delta \vec{x}}{2} \right) \left[ \sin \left( \frac{k \cdot \Delta \vec{x}}{2} \right) \alpha_{\vec{n}} + \cos \left( \frac{k \cdot \Delta \vec{x}}{2} \right) \beta_{\vec{n}} \right].$$

(28)

Just like $\hat{\varphi}(t, \vec{x})$, this is a sum of independent Gaussians, so $\Delta \hat{\varphi}(t, \Delta \vec{x})$ is itself a Gaussian. Because the mean of each constituent is zero, the mean of $\Delta \hat{\varphi}(t, \Delta \vec{x})$ also vanishes. Its variance is the sum of the variance of each constituent,

$$\sigma^2(t, \Delta \vec{x}) = \sum_{\vec{n} \neq 0} \frac{2H^5}{k^3} \theta(Ha(t) - k) \sin^2 \left( \frac{k \cdot \Delta \vec{x}}{2} \right),$$

(29)

$$= 4\pi \int_{\frac{a}{2\pi}}^{a/\pi} \frac{d\vec{n}^2}{(2\pi \vec{n})^3} \left[ 1 - \frac{\sin(k \Delta \vec{x})}{k \Delta \vec{x}} \right],$$

(30)

$$= \frac{H^2}{2\pi^2} \int_{H \Delta \vec{x}}^{a H \Delta \vec{x}} \frac{dz}{z^2} \left[ 1 - \frac{\sin(z)}{z} \right],$$

(31)

$$= \frac{H^2}{2\pi^2} \left\{ \frac{\sin([a(t)H \Delta x])}{a(t)H \Delta x} - \frac{\sin([H \Delta x])}{H \Delta x} + \ln[a(t)] - \text{ci}[a(t)H \Delta x] + \text{ci}[H \Delta x] \right\}. (32)$$

The symbol $\text{ci}(x)$ stands for the cosine integral whose definition and expansion for small $x$ are,

$$\text{ci}(x) \equiv - \int_{x}^{\infty} dt \frac{\cos(t)}{t} = \ln(x) + \gamma + \int_{0}^{x} dt \left[ \frac{\cos(t) - 1}{t} \right],$$

(33)

$$= \ln(x) + \gamma + \sum_{n=1}^{\infty} \frac{(-1)^2 x^{2n}}{2n \cdot 2n!} .$$

(34)

Note the appearance of Euler’s constant, $\gamma \approx 0.577215665$.

Quantum gravitational perturbation theory breaks down when the expectation value of $G\hat{\varphi}^2(t, \vec{x})$ grows to be of order one [44],

$$\langle G\hat{\varphi}^2(t, \vec{x}) \rangle = \frac{GH^2}{4\pi^2} \times \ln[a(t)] \sim 1 \quad \implies \quad \ln[a(t)] \sim \frac{1}{GH^2} \gg 1 .$$

(35)

One might expect that this is also when back-reaction becomes significant. At this point the fluctuation of each of the two fields in $\Delta \hat{\varphi}(t, \Delta \vec{x})$ is enormous,
as one can see from expression (15). Cosmologists frequently express the worry that, if back-reaction of this sort ever becomes significant, it must induce similarly large fluctuations in \( \Delta \hat{\varphi}(t, \Delta \vec{x}) \). That would lead to an unacceptable level of inhomogeneity in the post-inflationary universe.

We can test the cosmologists’ fear by choosing \( \Delta x \) so as to keep the physical distance a constant fraction \( K \) of the Hubble length,

\[
a(t)\Delta x = \frac{K}{H}.
\]

From expression (32) and the asymptotic expansion (34) we see that the variance rapidly approaches a not especially large constant,

\[
\sigma^2(t, Ka^{-1}) = \frac{H^2}{2\pi^2} \left\{ \frac{\sin(K)}{K} - \frac{\sin(Ka^{-1})}{Ka^{-1}} + \ln(a) - \text{ci}(K) + \text{ci}(Ka^{-1}) \right\},
\]

\[
= \frac{H^2}{2\pi^2} \left\{ \frac{\sin(K)}{K} - 1 + \ln(K) + \gamma - \text{ci}(K) + O(a^{-2}) \right\}.
\]

This should be compared with the variance of \( \sigma^2(t) = \frac{H^2}{4\pi^2} \times \ln[a(t)] \) of each field in the difference. So it is false that spatial fluctuations between nearby points become enormous whenever back-reaction is significant.

Although our result (38) might seem surprising, there is a very simple and general reason for it based upon causality. The stochastic field acquires its fluctuations one instant at a time, as each new complement of modes experiences horizon crossing and contributes to the stochastic jitter. The result at any spacetime point \((t, \vec{x})\) depends upon what happened in the past light-cone of that point. After a long period of inflation, two fields a fixed distance apart very largely share the same past light-cone, so they experience almost the same fluctuations. This remains true even if the faction \( K \) is much greater than one, because significant back-reaction requires the staggering number of \( 1/GH^2 \sim 10^6 \) e-foldings.

Let us also note that the level (38) of inhomogeneity we see is about right to explain the primordial power spectra [44]. So far from the stochastic formalism invalidating theories of inflation which are based on back-reaction, it provides an essential ingredient.

5 Epilogue

We have employed a very simple scalar model on de Sitter background to examine the chief criticisms against using expectation values and in-out matrix
elements to infer back-reaction:

1. That stochastic fluctuations about the mean value change the entire picture; and

2. That significant back-reaction necessarily implies an unacceptable level of inhomogeneity.

Neither criticism is supported by our study. In section 3 we found that although there is significant stochastic fluctuation in the quantity \( \hat{\varphi}^2(t, \vec{x}) \), there is zero probability either for this quantity to remain bounded, or for it to grow at a faster rate than that predicted by its mean value. In section 4 we showed that there is no growth in the variance of the difference between two stochastic fields held at a fixed physical distance from one another.

The cosmologists’ objection that expectation values give a misleading picture for some operators is certainly correct for \( \hat{\varphi}(t, \vec{x}) \). On the other hand, expectation values give a fair representation of \( \hat{\varphi}^2(t, \vec{x}) \). So the picture that emerges is more complex than either a total refutation of expectation values or their complete vindication. The fair conclusion would seem to be that the sensitivity of a particular mechanism of back-reaction to stochastic fluctuations should always be checked, but there are no grounds for rejecting the mechanism prior to such a check.

We do not view the stochastic formalism of Starobinsky [21] as an alternative to quantum field theory but rather as a wonderful tool for isolating the most significant corrections [39]. The really intriguing thing is that the stochastic formalism can sometimes be used to obtain nonperturbative results [40, 33, 42]. We believe this can be done for quantum gravity [43, 14] and that it is worthwhile attempting to anticipate the result [45].

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References

[1] E. Schrödinger, Physica 6 (1939) 899.
[2] T. Imamura, Phys. Rev. 118 (1960) 1430.

[3] L. Parker, Phys. Rev. Lett. 21 (1968) 562; Phys. Rev. 183 (1969) 1057; Phys. Rev. D3 (1971) 346.

[4] L. H. Ford and L. Parker, Phys. Rev. D16 (1977) 1601.

[5] L. P. Grishchuk, Sov. Phys. JETP 40 (1975) 409.

[6] A. A. Starobinsky, JET Lett. 30 (1979) 682; V. F. Mukhanov and G. V. Chibisov, JETP Lett. 33 (1981) 532.

[7] E. Komatsu et al., arXiv:1001.4538.

[8] A. M. Polyakov, Sov. Phys. 25 (1982) 187; Nucl. Phys. B797 (2008) 199, arXiv:0709.2899; Nucl. Phys. B834 (2010) 316, arXiv:0912.5503.

[9] N. P. Myhrvold, Phys. Rev. D28 (1983) 2439.

[10] E. Calzetta, S. Habib and B. L. Hu, Phys. Rev. D37 (1988) 2901; S. Habib and H. Kandrup, Annals Phys. 191 (1989) 335; Phys. Rev. D39 (1989) 2871; S. Habib, Phys. Rev. D46 (1992) 2408, gr-qc/9208006.

[11] L. H. Ford, Phys. Rev. D31 (1985) 710.

[12] E. Mottola, Phys. Rev. D31 (1985) 754; Phys. Rev. D33 (1986) 1616; Phys. Rev. D33 (1986) 2136; P. Mazur and E. Mottola, Nucl. Phys. B278 (1986) 694; F. Cooper and E. Mottola, Phys. Rev. D40 (1989) 456; I. Antoniadis and E. Mottola, Phys. Rev. D45 (1992) 2013; Phys. Lett. B323 (1994) 284, hep-th/9301002; I. Antoniadis, P. O. Mazur and E. Mottola, Nucl. Phys. B388 (1992) 627, hep-th/9205015; New J. Phys. 9 (2007) 11, gr-qc/0612068.

[13] I. Antoniadis, J. Iliopoulos and T. N. Tomaras, Phys. Rev. Lett. 56 (1986) 1319; Nucl. Phys. B462 (1996) 437, hep-th/9510112.

[14] N. C. Tsamis and R. P. Woodard, Phys. Lett. B301 (1993) 351; Annals Phys. 238 (1995) 1; Nucl. Phys. B474 (1996) 235, hep-ph/9602315; Annals Phys. 253 (1997) 1, hep-ph/9602316.
[15] V. F. Mukhanov, L. R. Abramo and R. H. Brandenberger, Phys. Rev. Lett. 78 (1997) 1624, gr-qc/9609026; J. P. Zibin, R. H. Brandenberger and D. Scott, Phys. Rev. D63 (2001) 043511, hep-ph/0007219; M. Li, W. Lin, X. Zhang and R. H. Brandenberger, Phys. Rev. D65 (2002) 023519, hep-ph/0107160; G. Geshnizjani and R. H. Brandenberger, Phys. Rev. D66 (2002) 123507, gr-qc/0204074; JCAP 0504 (2005) 006, hep-th/0310265; R. H. Brandenberger and J. Martin, Phys. Rev. D71 (2005) 023504, hep-th/0410223; P. Martineau and R. H. Brandenberger, Phys. Rev. D72 (2005) 023507, astro-ph/0505236 astro-ph/0510523.

[16] L. R. W. Abramo and R. P. Woodard, Phys. Rev. D60 (1999) 044010, astro-ph/9811430; Phys. Rev. D60 (1999) 044011, astro-ph/9811431; Phys. Rev. D65 (2002) 043507, astro-ph/0109271; Phys. Rev. D65 (2002) 063516, astro-ph/0109273.

[17] L. R. W. Abramo and R. P. Woodard, Phys. Rev. D65 (2002) 063515, astro-ph/0109272.

[18] L. Parker and D. A. T. Vanzella, Phys. Rev. D69 (2004) 104009, gr-qc/0312108; L. Parker and A. Raval, Phys. Rev. D60 (1999) 063512, gr-qc/9905031; Phys. Rev. D60 (1999) 123502, gr-qc/9908013; Phys. Rev. D62 (2000) 083503, gr-qc/0003103; Phys. Rev. Lett. 86 (2001) 749.

[19] A. Ghosh, R. Madden and G. Veneziano, Nucl. Phys. B570 (2000) 207, hep-th/9908024; M. Gasperini, G. Marozzi and G. Veneziano, JCAP 1002 (2010) 009, arXiv:0912.3244; JCAP 0903 (2009) 011, arXiv:0901.1303.

[20] C. Kiefer, I. Lohmar, D. Polarski and A. A. Starobinsky, Class. Quant. Grav. 24 (2007) 1699, astro-ph/0610700.

[21] A. A. Starobinsky, “Stochastic de Sitter (inflationary) stage in the early universe,” in Field Theory, Quantum Gravity and Strings, ed. H. J. de Vega and N. Sanchez (Springer-Verlag, Berlin, 1986) pp. 107-126.

[22] A. Vilenkin, Phys. Rev. D27 (1983) 2848; Y. Nambu and M. Sasaki, Phys. Lett. B219 (1989) 240.
[23] A. S. Goncharov, A. D. Linde and V. F. Mukhanov, Int. J. Mod. Phys. A2 (1987) 561; A. D. Linde and A. Mezhlumian, Phys. Lett. B307 (1993) 25, gr-qc/9304015.

[24] G. I. Rigopoulos, E. P. S. Shellard and B. J. W. van Tent, Phys. Rev. D72 (2005) 083507, astro-ph/0410486; Phys. Rev. D73 (2006) 083521, astro-ph/0504508; Phys. Rev. D73 (2006) 083522, astro-ph/05067004.

[25] S. J. Rey, Nucl. Phys. B284 (1987) 706; M. Sasaki, Y. Nambu and K. I. Nakao, Nucl. Phys. B308 (1988) 868; S. Winitzki and A. Vilenkin, Phys. Rev. D61 (2000) 084008, gr-qc/9911029; J. Martin and M. Musso, Phys. Rev. D73 (2006) 043517, hep-th/0511292; K. Enqvist, S. Nurmi, D. Podolsky and G. I. Rigopoulos, JCAP 0804 (2008) 025, arXiv:0802.0395.

[26] V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, Phys. Rep. 215 (1992) 203; A. R. Liddle and D. H. Lyth, Phys. Rep. 231 (1993) 1, astro-ph/9393910.

[27] A. Vilenkin and L. H. Ford, Phys. Rev. D26 (1982) 1231; A. D. Linde, Phys. Lett. 116B (1982) 335; A. A. Starobinsky, Phys. Lett. 117B (1982) 175.

[28] N. C. Tsamis and R. P. Woodard, Class. Quant. Grav. 11 (1994) 2969; T. M. Janssen, S. P. Miao, T. Prokopec and R. P. Woodard, Class. Quant. Grav. 25 (2008) 245013, arXiv:0808.2449; S. P. Miao, N. C. Tsamis and R. P. Woodard, arXiv:1002.4037.

[29] V. K. Onemli and R. P. Woodard, Class. Quant. Grav. 19 (2002) 4607, gr-qc/0204065; Phys. Rev. D70 (2004) 107301, gr-qc/0406098.

[30] T. Prokopec, O. Tornkvist and R. P. Woodard, Phys. Rev. Lett. 89 (2002) 101301, astro-ph/0205331; Ann. Phys. 303 (2003) 251, gr-qc/0205130.

[31] T. Prokopec and R. P. Woodard, JHEP 0310 (2003) 059, astro-ph/0309593; B. Garbrecht and T. Prokopec, Phys. Rev. D73 (2006) 064036, gr-qc/0602011.
[32] T. Brunier, V. K. Onemli and R. P. Woodard, Class. Quant. Grav. 22 (2005) 59, gr-qc/0408080; E. O. Kahya and V. K. Onemli, Phys. Rev. D76 (2007) 043512, gr-qc/0612026.

[33] S. P. Miao and R. P. Woodard, Phys. Rev. D74 (2006) 044019, gr-qc/0602110.

[34] T. Prokopec, N. C. Tsamis and R. P. Woodard, Class. Quant. Grav. 24 (2007) 201, gr-qc/0607094; Phys. Rev. D78 (2008) 043523, arXiv:0802.3673.

[35] N. C. Tsamis and R. P. Woodard, Annals Phys. 321 (2006) 875, gr-qc/0506056.

[36] S. P. Miao and R. P. Woodard, Class. Quant. Grav. 23 (2006) 1721, gr-qc/0511140; Phys. Rev. D74 (2006) 024021, gr-qc/0603135.

[37] E. O. Kahya and R. P. Woodard, Phys. Rev. D76 (2007) 124005, arXiv:0709.0536; Phys. Rev. D77 (2008) 084012, arXiv:0710.5282.

[38] A. C. Davis, K. Dimopoulos, T. Prokopec and O. Törnkvist, Phys. Lett. B501 (2001) 165, astro-ph/0007214; Phys. Rev. D65 (2002) 063505, astro-ph/0108093.

[39] N. C. Tsamis and R. P. Woodard, Nucl. Phys. B724 (2005) 295, gr-qc/0506056.

[40] A. A. Starobinsky and J. Yokoyama, Phys. Rev. D50 (1994) 6357, astro-ph/9407016.

[41] R. P. Woodard, Phys. Rev. Lett. 101 (2008) 081301, arXiv:0805.3089.

[42] T. Prokopec, N. C. Tsamis and R. P. Woodard, Annals Phys. 323 (2008) 1324, arXiv:0707.0847.

[43] S. P. Miao and R. P. Woodard, Class. Quant. Grav. 25 (2008) 145009, arXiv:0803.2373.

[44] N. C. Tsamis and R. P. Woodard, Class. Quant. Grav. 26 (2009) arXiv:0807.5006.

[45] N. C. Tsamis and R. P. Woodard, Phys. Rev. D80 (2009) 083512, arXiv:0904.2368; Phys. Rev. D81 (2010) 103509, arXiv:1001.4929.