Soap Bubbles in Outer Space: Interaction of a Domain Wall with a Black Hole

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Abstract

We discuss the generalized Plateau problem in the 3+1 dimensional Schwarzschild background. This represents the physical situation, which could for instance have appeared in the early universe, where a cosmic membrane (thin domain wall) is located near a black hole. Considering stationary axially symmetric membranes, three different membrane-topologies are possible depending on the boundary conditions at infinity: 2+1 Minkowski topology, 2+1 wormhole topology and 2+1 black hole topology.

Interestingly, we find that the different membrane-topologies are connected via phase transitions of the form first discussed by Choptuik in investigations of scalar field collapse. More precisely, we find a first order phase transition (finite mass gap) between wormhole topology and black hole topology; the intermediate membrane being an unstable wormhole collapsing to a black hole. Moreover, we find a second order phase transition (no mass gap) between Minkowski topology and black hole topology; the intermediate membrane being a naked singularity.

For the membranes of black hole topology, we find a mass scaling relation analogous to that originally found by Choptuik. However, in our case the parameter $p$ is replaced by a 2-vector $\vec{p}$ parametrizing the solutions. We find that $\text{Mass} \propto |\vec{p} - \vec{p}_0|^\gamma$ where $\gamma \approx 0.66$. We also find a periodic wiggle in the scaling relation.

Our results show that black hole formation as a critical phenomenon is far more general than expected.
Cosmic strings and domain walls have played an important role in theoretical cosmology and astrophysics (for a review of topological defects, see for instance [1]). Most of the work has been devoted to cosmic strings, while domain walls have not attracted so much attention. In fact, it has been argued that stable domain walls are cosmologically disastrous. This was already pointed out by Zeldovich et. al. [2], who considered domain wall structures in models with spontaneous breaking of CP-symmetry. They argued that the energy density of the domain walls is so large, that they would dominate the universe completely, violating the observed approximate isotropy and homogeneity. So if domain walls were ever formed in the early universe, they were assumed to have somehow disappeared again, for instance by collapse, evaporation or simply by inflating away from our visible universe.

Much later however, Hill et. al. [3] introduced the so-called ”light” domain walls. They considered a late-time (post-decoupling) phase transition and found that light domain walls could be produced, that were not necessarily in contradiction with the observed large-scale structure of the universe.

Domain walls are formed in phase transitions where a discrete symmetry is broken. Already from this, one can argue that it is difficult to believe that domain walls should not have been formed sometime during the early evolution of the universe, where a number of phase transitions certainly took place. It is also worth mentioning that domain walls and other topological defects are now commonly seen experimentally in various areas of condensed matter physics (for a review, see for instance [4]).

In the leading approximation, a domain wall is described by the Dirac-Nambu-Goto action [5]

\[ S = \mu \int d^3 \zeta \sqrt{-\det G_{AB}} \]  

(1)

where the induced metric on the world-volume is

\[ G_{AB} = g_{\mu\nu} x^\mu_{A} x^\nu_{B} \]  

(2)

and \( \mu \) is the tension. Here \( x^\mu (\mu = 0, 1, 2, 3) \) denote spacetime coordinates while \( \zeta^A (A = 0, 1, 2) \) denote coordinates on the world volume. The assumption made here is that the dimensions of the domain walls are much greater than their thickness. The domain wall is thus approximated by a relativistic membrane, which in turn is assumed to be described by the action (1). Using
this model, the world-volume of the membrane is a minimal 2+1-surface embedded in a curved 3+1 dimensional spacetime with metric $g_{\mu\nu}$. We are thus dealing with a generalization of the classical Plateau problem in 3 Euclidean dimensions (see for instance [3]).

In this paper, we shall be interested in stationary axially symmetric membranes embedded in the background of a Schwarzschild black hole. Using the reparametrization invariances on the world-volume, a stationary axially symmetric membrane can be parametrized by

$$\begin{align*}
t &= \tau, \quad r = \lambda, \quad \phi = \sigma, \quad \theta = \theta(\lambda) \\
\end{align*}$$

where $(\tau, \sigma, \lambda)$ are the three coordinates on the world-volume, and we use standard Schwarzschild coordinates in target space. Then the action (1) reduces to

$$S_{\text{eff}} = 2\pi\mu \Delta t \int r \, dr \, \sin \theta \sqrt{1 + r^2(\theta')^2 \left(1 - \frac{2M}{r}\right)}$$

where a prime denotes derivative with respect to $r \,(= \lambda)$. The corresponding equation of motion determining $\theta(r)$ is

$$\theta'' + (2r - 3M)(\theta')^3 - \frac{1}{\tan \theta}(\theta')^2 + \frac{3r - 4M}{r(r - 2M)} \theta' - \frac{1}{r(r - 2M) \tan \theta} = 0 \quad (5)$$

The line element on the world-volume, as obtained from (2), is given by

$$d\Sigma^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(r^2(\theta')^2 + \left(1 - \frac{2M}{r}\right)^{-1}\right) dr^2 + r^2 \sin^2 \theta \, d\phi^2 \quad (6)$$

with $\theta(r)$ determined by (3), while the scalar curvature of the world-volume becomes

$$(^{(3)}R) = -2 \frac{\theta' \left(1 - \frac{3M}{2r}\right) - \frac{1}{r \tan \theta}^2 + \left(\frac{\sqrt{3}M \theta'}{2r}\right)^2}{1 + r^2(\theta')^2 \left(1 - \frac{2M}{r}\right)} \quad (7)$$

In the following, we consider the membrane world-volume as a 2+1 dimensional spacetime embedded in the background of the 3+1 dimensional Schwarzschild spacetime.

In flat Minkowski spacetime ($M = 0$), equation (3) is solved by

$$Z = \pm a \, \text{Arccosh} \left(\frac{R}{a}\right) + b \quad (8)$$
where \((a, b)\) are arbitrary constants \((a \geq 0)\) and
\[
(R, Z) = (r \sin \theta, -r \cos \theta)
\] (9)
are the standard cylinder coordinates chosen such that the north pole corresponds to \(\theta = \pi\).

In 3 dimensional Euclidean geometry, the solution (8) is called a catenoid. In the relativistic setting here, it is more properly described as a 2+1 dimensional wormhole. More precisely, the corresponding world-volume line-element (3) is
\[
d\Sigma^2 = -dt^2 + \frac{R^2}{R^2 - a^2}dR^2 + R^2d\phi^2
\] (10)
Thus, for \(a \neq 0\) the membrane world-volume is a 2+1 dimensional wormhole, while for \(a = 0\) it is 2+1 dimensional Minkowski space.

We now consider equation (5) in the general case \(M \neq 0\). We have solved it numerically using the fourth order Runge-Kutta method. It is convenient to introduce proper cylinder coordinates
\[
(R_p, Z_p) = (l \sin \theta, -l \cos \theta)
\] (11)
where \(l\) is the proper radial distance
\[
l = 2M + \sqrt{r(r - 2M)} + 2M \ln \left( \sqrt{\frac{r - 2M}{2M}} + \sqrt{\frac{r}{2M}} \right)
\] (12)
In these coordinates, the horizon of the 3+1 dimensional black hole corresponds to
\[
l_h = \sqrt{R_p^2 + Z_p^2} = 2M
\] (13)

For the numerical integrations, we impose the following sets of boundary conditions:

I) \(R_p = 0\) , \(Z_p > 2M\) , \(\frac{dZ_p}{dR_p} = 0\) (14)

II) \(R_p > 2M\) , \(Z_p = 0\) , \(\frac{dR_p}{dZ_p} = 0\) (15)

III) \(\sqrt{R_p^2 + Z_p^2} = 2M\) , \(\frac{d\theta}{dl} = 0\) (16)
These will describe all types of stationary axially symmetric and $Z_2$-symmetric (with respect to the equatorial plane) membranes in the Schwarzschild background. Some examples of solutions are shown in Fig. 1.

The first set of solutions (I) describe membranes which are always outside the 3+1 black hole. The boundary conditions (14) are chosen to ensure axial symmetry. The corresponding membranes have the topology of 2+1 Minkowski space, and are deformed versions of the $a = 0$ membranes (8) in flat 3+1 Minkowski space.

The second set of solutions (II) are also always outside the 3+1 black hole. The boundary conditions (15) are chosen to ensure $Z_2$-symmetry with respect to the equatorial plane of the 3+1 dimensional black hole. The corresponding membranes have the topology of a 2+1 wormhole, and are deformed versions of the $b = 0$ membranes (8) in flat 3+1 Minkowski space. For these 2+1 wormholes, the 3+1 black hole is located in the middle of the throat in the embedding diagram.

The third set of solutions (III) describe membranes entering the 3+1 black hole. Notice that the boundary conditions (16) were chosen such that the membranes cross the horizon of the 3+1 black hole orthogonally. This condition actually follows directly from the equation of motion (5). More precisely, assuming that $\theta'$ is regular at the horizon, the equation of motion (5) gives the boundary condition

$$\theta' = \frac{1}{2M \tan \theta}, \quad r = 2M$$

(17)

Using $(R, Z)$-coordinates (9), this condition is equivalent to $dZ/dR = 0$, while in proper cylinder coordinates (11) it becomes $d\theta/dl = 0$. Thus in flat space cylinder coordinates, the membranes cross the horizon \textit{horizontally} while in proper cylinder coordinates, they cross the horizon \textit{orthogonally}. The picture is thus analogue to that of magnetic field lines crossing the 3+1 black hole horizon \cite{7}. Obviously this third family of membranes entering the 3+1 black hole have no counterpart in flat 3+1 Minkowski space. These membranes have themselves the topology of a 2+1 dimensional black hole, as follows from equations (8) and (10): They have a spacetime singularity at $r \sin \theta = 0$ hidden behind the horizon located at $r_h = 2M$.

Each solution from any of the three families of membranes is uniquely specified by its asymptotic behaviour. Asymptotically, the solution to equa-
tion (5) is of the same form as (8):

\[ Z_p = \pm a_p \text{Arccosh} \left( \frac{R_p}{a_p} \right) + b_p \]  

(18)

where \((a_p, b_p)\) are constants \((a_p \geq 0)\). That is, a membrane is specified by a 2-vector

\[ \vec{p} = \left( \begin{array}{c} a_p \\ b_p \end{array} \right) \]  

(19)

Numerically we can then compute \(\vec{p} = (a_p, b_p)\) corresponding to the 3 sets of boundary conditions (14)-(16). The result is shown in Fig.2. In this plot, each point \(\vec{p} = (a_p, b_p)\) corresponds to a stationary axially symmetric and \(Z_2\)-symmetric membrane embedded in the 3+1 dimensional Schwarzschild background. Thus, the two components of \(\vec{p}\) are not independent.

We are particularly interested in the "phase transitions" between the different membrane topologies, as discussed above. At the point \(\vec{p}_0 = (0,0)\) there is a transition between wormhole topology and black hole topology. The limiting membrane is an "unstable" wormhole which collapses in the \(Z\)-direction and becomes what is formally a 2+1 dimensional Schwarzschild spacetime:

\[ d\Sigma^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2d\phi^2 \]  

(20)

with horizon at \(r_h = 2M\) and singularity at \(r = 0\). This membrane of course just corresponds to the equatorial plane of the 3+1 dimensional Schwarzschild black hole.

The other transition happens at the point

\[ \vec{p}_* = (0.3048..., 2.0457...) \]

This is a transition between Minkowski topology and black hole topology. It is an interesting observation that the transition point in parameter space is approached by infinite logarithmic spirals from both sides. This can be seen by doing a conformal transformation that blows up the region near \(\vec{p}_*\)

\[ \vec{p} \rightarrow \vec{p}^\prime = \left( \begin{array}{c} a_p' \\ b_p' \end{array} \right) = \frac{\vec{p} - \vec{p}_*}{|\vec{p} - \vec{p}_*| \ln|\vec{p} - \vec{p}_*|} \]  

(21)

The result of this transformation is shown in Fig.3. Moreover, the limiting membrane corresponding to \(\vec{p}_*\) is a 2+1 dimensional naked singularity as
follows from equation (8): The singularity of this membrane is located at 
\((r, \theta) = (2M, \pi)\), and it is not hidden behind a horizon.

These results are very similar to analogue results, first obtained by Chop-
tuik [8], for the spherical collapse of scalar or Yang-Mills fields in 3+1 di-
mensions (for a review, see [9]).

In particular, in the case of Yang-Mills collapse [10] or massive scalar fi eld 
collapse [11], two different types of phase transitions occur at the threshold of 
black hole formation: A first order phase transition (finite mass gap) where 
the limiting solution is an unstable soliton star and a second order phase 
transition (no mass gap) where the limiting solution is a naked singularity.

For our membranes of 2+1 dimensional black hole topology, the mass 
inside the apparent horizon \(S\) can be defined by (up to normalization) [12]
\[
\text{Mass} = -\frac{1}{4\pi} \int_S \varepsilon_{ABC} \nabla^B \xi^C d\zeta^A
\] (22)

where \(\nabla_B\) is the covariant derivative with respect to the metric (2), and \(\xi^C\) is 
the timelike-at-infinity Killing vector on the world-volume. Using (2) and 
(3) we get:
\[
\text{Mass} = \frac{\sin \theta_0}{2}
\] (23)

where \(\theta_0\) is the polar angle at which the membrane crosses the horizon of the 
3+1 dimensional black hole. It should be stressed that (23) is the mass inside 
the apparent horizon of the membrane world-volume. Since the membrane 
world-volume (6) is not a vacuum solution in 2+1 dimensions, the mass 
(23) does not equal the mass measured at infinity (for a discussion of the 
different mass definitions, see for instance [12]). Notice also that our units 
and conventions are such that the mass (23) is dimensionless. This corre-
responds to units where the 3-dimensional gravitational constant equals unity.

From (23) follows that the transition between wormhole topology and 
black hole topology \(\left(\theta_0 = \frac{\pi}{2}\right)\) is of first order (finite mass gap) while the 
transition between 2+1 Minkowski topology and black hole topology \(\left(\theta_0 = \pi\right)\) 
is of second order (no mass gap).

A generic result of the investigations of scalar field collapse [8] (see [9] for 
a review) is a mass scaling relation of the form \(M_{BH} \propto |p - p_*|^\gamma\), where \(p\) 
parametrizes the solutions and \(p_*\) is the critical parameter defined such that a 
black hole is formed for \(p > p_*\). In our case of stationary membranes, the

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parameter \( p \) is replaced by the 2-vector \( \vec{p} \), c.f. eqs.(18)-(19), while the mass of the black hole is given by (23).

In Fig.4., we show a double-logarithmic plot of (Mass) versus \(|\vec{p} - \vec{p}_*|\). It corresponds to a relation of the form

\[
\ln(Mass) = \gamma \ln |\vec{p} - \vec{p}_*| + \text{periodic function}
\]

(24)

where an additive constant has been absorbed in the periodic function. This is a mass scaling relation analogous to that of Choptuik, including the periodic wiggle with period \( \omega \) [13, 14, 15] (\( \omega \) is the period in \( \ln|\vec{p} - \vec{p}_*| \)). Numerically we find the following values of the parameters

\[
\gamma \approx 0.66 \\
\omega \approx 3.56
\]

The periodic function reflects the periodic self-similarity of the critical solution [13, 14, 15], already present in the original investigation [8]. It should also be mentioned that more precise numerical computations indicate that \( \gamma = 2/3 \), but at the present moment we have no analytical proof of this.

It is also interesting to compare with the case of stationary cosmic strings in the background of a black hole [16]. In this case, the world-sheet of the string can be considered a 1+1 dimensional spacetime. Depending on the boundary conditions at infinity, the topology of the string world-sheet is either that of 1+1 Minkowski spacetime or that of a 1+1 black hole [17, 18]. Also in this case, there is a phase transition between the two topologies. However, this phase transition is of first order, that is to say, there is a finite mass gap. Thus in the stationary string case, there is no phase transition of second order and no mass scaling relation of the type originally discovered by Choptuik.

In conclusion, using analytical and numerical methods, we have considered the interaction of a domain wall with a Schwarzschild black hole. As a result we have shown that, although our physical setup is completely different, the phenomena concerning black hole formation in 2+1 dimensions are very similar to those observed for gravitational collapse of various fields in 3+1 dimensions. This again confirms the generality of black hole formation as a critical phenomenon [8], involving different types of phase transitions. And most importantly, our results show that black hole formation as a critical phenomenon is far more general than expected.
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Fig. 1. Some examples of membranes in the Schwarzschild background. To obtain the full spatial membrane geometry, the curves must be rotated around the $Z_p$-axis.
Fig. 2. Plot showing the parameters $(a_p, b_p)$ determining the asymptotic behaviour of the membranes, c.f. eq.(18). We only show the pairs $(a_p, b_p)$ corresponding to membranes in the northern hemisphere. The complete plot is obtained by reflection in the $a_p$-axis.
Fig. 3. Conformal magnification of the region in parameter space (Fig. 2) describing the transition from Minkowski topology membranes to black hole topology membranes.
Fig. 4. $\ln(Mass)$ plotted versus $\ln |\vec{p} - \vec{p}^*|$. The result is the sum of a linear function and a periodic function, c.f. equation (24).