RADÓ THEOREM AND ITS GENERALIZATION
FOR CR-MAPPINGS

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Abstract. A generalization of Radó’s theorem for CR-functions on locally Lipschitz hypersurfaces is obtained. It is proved also that closed pre images of pluripolar sets by CR-mappings are removable for bounded CR-functions.

1. Introduction

A well-known theorem of Radó [9] states that a continuous function $f$ defined on a domain in $\mathbb{C}$ and holomorphic on the complement of its zero set $f^{-1}(0)$ is holomorphic everywhere. The result is correct for holomorphic functions in $\mathbb{C}^n$ as well as in the plane. It is well-known that $f^{-1}(0)$ can be replaced by $f^{-1}(E)$, $E$ a closed subset of zero capacity in $\mathbb{C}$ (see [14]).

Recently J.-P.Rosay and E.L.Stout [10] have shown that an analogue of the classical Radó’s theorem take place for CR-functions on a $C^2$-hypersurface in $\mathbb{C}^n$ with nonvanishing Levi form. Then H.Alexander [1] has proved the removability in the same situation of closed sets of the type $f^{-1}(E)$, $E$ a closed polar set in $\mathbb{C}$. We improve here these results in the following theorem which can be considered as an extension of Radó’s theorem to bounded CR-mappings of hypersurfaces.

**Theorem 1.** Let $\Gamma$ be a locally Lipschitz hypersurface in $\mathbb{C}^n$ with one-sided extension property at each point, $\Sigma$ is a closed subset of $\Gamma$ and

$$f : \Gamma \setminus \Sigma \longrightarrow \mathbb{C}^m \setminus E$$

is a CR-mapping of class $L^{\infty}$ such that the cluster set of $f$ on $\Sigma$ along of Lebesgue points of $f$ is contained in a closed complete pluripolar set $E$. Then there is a CR-mapping $\tilde{f} : \Gamma \longrightarrow \mathbb{C}^m$ of class $L^{\infty}(\Gamma)$ such that $\tilde{f} |_{\Gamma \setminus \Sigma} = f$.

We say that $\Gamma$ has one-sided extension property at its point $a$ if for an arbitrary neighbourhood $U \ni a$ there is a (smaller) neighbourhood $V \ni a$ and a connected component $W$ of $V \setminus \Gamma$ such that $a \in W$ and every bounded CR-function on $\Gamma \cap U$ extends holomorphically into $W$. As it was shown by Trépreau [15] this property at each point has an arbitrary locally Lipschitz hypersurfaces in $\mathbb{C}^2$ which contains no analytic discs. The same is true for hypersurfaces of class $C^2$ in $\mathbb{C}^n$ containing no complex hypersurfaces (see [15]).

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We show indeed that the trivial extension of $f$ by a constant on $\Sigma$ is a $\text{CR}$-mapping on the whole $\Gamma$.

Theorem 1 is true also without the supposition of the one-sided extension property, but the general case is more complicated, and this is related with analytic discs belonging to $\Gamma$. We shall prove the general theorem in the next paper.

Theorem 1 is equivalent to the following new result on the removability of singularities for bounded $\text{CR}$-functions.

**Theorem 2.** Let $\Gamma, \Sigma, E$ and $f$ be as in Theorem 1. Then each $\text{CR}$-function of class $L^\infty$ on $\Gamma \setminus \Sigma$ extends to a $\text{CR}$-function of class $L^\infty$ on $\Gamma$.

Theorem 1 follows obviously from Theorem 2. In opposite direction, given a $\text{CR}$-function $g$ of class $L^\infty$ on $\Gamma \setminus \Sigma$ corresponds the mapping $(f, g) : \Gamma \setminus \Sigma \to \mathbb{C}^{m+1} \setminus E \times \mathbb{C}$

which cluster set on $\Sigma$ is contained in the closed complete pluripolar set $E \times \mathbb{C}$. By Theorem 1, the map $(f, g)$ extends to a $\text{CR}$-map of whole $\Gamma$, and thus its last component $g$ extends to there as a $\text{CR}$-function.

If $\Gamma$ is the boundary of a bounded domain or if $\Gamma$ admits one-sided holomorphic extension of $\text{CR}$-functions (say, if $\Gamma \in C^2$ contains no complex hypersurface, as in [15]) then it follows from Theorem 2 and a uniqueness theorem that the Hausdorff $(2n - 1)$-measure of $\Sigma$ vanishes, and $\Gamma \setminus \Sigma$ is locally connected.

We prefer to work here with the class $L^\infty$ instead of $C^\infty$, since $L^\infty$ is stable by considering extensions whereas the $\text{CR}$-extension of a bounded continuous $\text{CR}$-function from $\Gamma \setminus \Sigma$ onto $\Gamma$ is not continuous in general.

The proof of Radó theorem for $\text{CR}$-functions in [10] is based on results [8] on the holomorphic continuation of $\text{CR}$-functions from a part of the boundary of a domain in $\mathbb{C}^n$, $n \geq 2$ (see also [7]). In the proof of Theorem 1 we use instead of this a geometric extension of the graph in spirit of R.Harvey and H.B.Lawson [6]. Our starting point was a generalization of the Harvey - Lawson theorem [6] on boundaries of holomorphic chains for $\text{MC}$-cycles in the complement to a polynomially convex compact set [4, 1985]. It can be considered as a geometric version (for $\text{CR}$-functions of class $C^1$) of theorems on holomorphic continuation in [8],[10],[7].

Let us specify the terminology.

We say that a hypersurface $M$ in a smooth $k$-dimensional manifold $\mathcal{M}$ is **locally Lipschitz** if for every point $a \in M$ there is a coordinate chart $(U, x)$, $x = (x', x_k)$ on $\mathcal{M}$ such that $a \in U$ and $M \cap U$ is represented as the graph $x_k = h(x')$ of a function $h$ over the domain in $\mathbb{R}^{k-1}$ which satisfies there Lipschitz condition $|h(b) - h(c)| \leq C |b - c|$ with a constant $C$. Note that the Hausdorff $k-1$-measure (with respect to some fixed smooth metric on $\mathcal{M}$) restricted to such $M$ is locally finite, and $M$ has tangent planes in almost every point with respect to this measure (Rademacher’s theorem, see e.g. [5,3.1.6.]). Thus the integral on $M$ for a differential $(k-1)$-form $\varphi$ with Lipschitz coefficients and with compact $\text{supp}\varphi \cap M$ is well-defined.

A point $a$ in a locally Lipschitz $M \subset \mathcal{M}$ is called a **Lebesque point** for a given vector-function $f$ of class $L^1_{\text{loc}}(M)$ with values in $\mathbb{R}^N$ if there is a constant $\tilde{f}(a) \in \mathbb{R}^N$ such that

$$\lim_{r \to 0} \frac{1}{r^{k+1}} \int_{M \cap |x'| < r} |f(x) - \tilde{f}(a)| \, dx' = 0$$
as \( r \to 0 \) in the chart \((U, x)\) with \( x(a) = 0 \) described above. It is wellknown (see [5,2.9.8.]) that almost every point \( a \in M \) is such a point and \( f(a) = 0 \) almost everywhere. Thus we shall assume in the further that \( f \) is defined (as \( \tilde{f}(a) \)) on the set of its Lebesque points \( a \) only.

For a locally Lipschitz hypersurfacse \( M \) in a complex \( n \)-dimensional manifold \( \mathcal{M} \) the notion of \( CR\)-functions of class \( L^1_{loc}(M) \) is well-defined: a function \( f \) of this class is a \( CR\)-function on \( M \) if \( \int_M f \partial \varphi = 0 \) for every smooth form \( \varphi \) of bidegree \((n, n - 2)\) in \( M \) with compact \( \text{supp} \varphi \cap M \).

A set \( E \subset \mathbb{C}^m \) is called complete pluripolar, if there is a plurisubharmonic function \( \varphi \) in \( C^m \) such that \( E = \{ \zeta : \varphi(\zeta) = -\infty \} \).

2. One-sided holomorphic extension

The problem is local, so we can assume that \( \Gamma \ni 0 \) is represented as the graph \( v = h(z', u) \) of some Lipschitz function in a domain in the space of variables \((z_1, ..., z_{n-1}, Rez_n) = (z', u)\) (it is convenient to use the notation \( z_n = u + iv \)).

Fix a connected component \( \Gamma_0 \) of \( \Gamma \setminus \Sigma \), set \( f_1 = f \) on \( \Gamma_0, f_1 = 0 \) on \( \Gamma \setminus \Gamma_0 \) and denote by \( \Gamma_1 \supset \Gamma_0 \) the set of points \( a \in \Gamma \) such that \( f_1 \) is a \( CR\)-function in a neighbourhood of \( a \) on \( \Gamma \). We have to show that \( \Gamma_1 = \Gamma \).

By the one-sided extension property, for each point \( a \in \Gamma_1 \) there is a neighbourhood \( V_a \ni a \) and a connected component \( W_a \) of \( V_a \setminus \Gamma \) such that \( a \in \overline{W_a} \) and each \( CR\)-function on \( \Gamma \) extends holomorphically into \( W_a \). Shrinking \( V_a \) we can assume that the intersection of \( V_a \) with each line \((z', u) = \text{const} \) is an interval (i.e. \( V_a \) is convex in \( v \)-direction) intersecting \( \Gamma_1 \). Then the union of all \( W_a, a \in \Gamma_1 \) is an open set of the form \( W^+ \cup W^- \) where \( W^+ \) is plased over \( \Gamma \) (i.e. \( v > h(z', u) \) on \( W^+ \)) and \( W^- \) is contained in \( \{ v < h(z', u) \} \). It follows that the set

\[
W = W^+ \cup W^- \cup (\overline{W^+} \cap \overline{W^-} \cap \Gamma_1)
\]

is open, convex in \( v \)-direction, and \( \overline{W} \supset \Gamma_1 \). By a uniqueness theorem and a removable singularities theorem the holomorphic extensions of \( f_1 \) into \( W_a, a \in \Gamma_1 \), constitute holomorphic functions in \( W^+ \) and \( W^- \), and these functions extend to a holomorphic (vector-) function in \( W \) which we denote by the same symbol \( f_1 \).

By the construction, there is a Lipschitz function \( \epsilon(z', u) \) such that \( \epsilon = 0 \) outside of \((z', u)(\Gamma_1)\), the hypersurface \( \Gamma' : v = (h + \epsilon)(z', u) \) is contained in \( W \cup \Sigma_1 \), and \( \Gamma' \setminus \Sigma_1 \) is a smooth \((C^\infty \text{ or even } C^m \text{ if you want})\). The set \((f_1)^{-1}(E) \cap W \) is pluripolar, so we can assume that its intersection with \( \Gamma' \setminus \Sigma_1 \) has the Hausdorff dimension \( 2n - 3 \), in particular, it has the locally connected complement in \( \Gamma' \setminus \Sigma \). Set \( \Sigma^1_1 = \Sigma_1 \cup (\Gamma_1 \cap (f_1)^{-1}(E)) \) and \( \Gamma_1' = \Gamma' \setminus \Sigma^1_1 \). If \( \epsilon(z', u) \) is taken sufficiently small and rapidly tends to zero as \((z', u)\) approaches to \((z', u)(\Sigma_1)\), then

\[
f_1 \mid \Gamma_1' \longrightarrow \mathbb{C}^m \setminus E
\]

and the cluster set of \( f_1 \) on \( \Sigma^1_1 \) is contained in \( E \). Thus, substituting \( \Gamma \) onto \( \Gamma_1', \Sigma_1 \) onto \( \Sigma^1_1 \) and \( \Gamma_1 \) onto \( \Gamma_1' \) and then restoring old notations, we can assume that \( \Gamma_1 \) is smooth and the mapping \( f_1 \) is holomorphic in a neighbourhood of \( \Gamma_1 \).
3. Reducing to $n = 2$

We assume as above that $\Gamma \ni 0$ is represented as the graph of some Lipschitz function over a domain in the space of variables $z_1, ..., z_n, \text{Re}z_n$. Then the vector $(0, ..., 0, i)$ does not belong to $C_a\Gamma$, the tangent cone to $\Gamma$ at the point $a$, for all $a \in \Gamma$. Shrinking $\Gamma$ a little we can assume that the same is true for some $C$-linearly independent system of vectors $\xi_1, ..., \xi_n$, $\xi_j \notin C_a\Gamma$ for $a \in \Gamma, j = 1, ..., n$. Making a suitable $C$-linear changing of coordinates we obtain the situation when $i\xi_j \notin C_a\Gamma$ for all standard coordinate ors $e_j$ in $\mathbb{C}^n$. It follows then that for each $j$ there is a neighborhood $U_j \ni 0$ such that $\Gamma \cap U_j$ is represented as the graph of a Lipschitz function over a domain in the space of variables $z_k, k \neq j, \text{Re}z_j$. Set $U = \cap U_j$.

We have to show that $\int_{\Gamma} f_0 \partial \varphi = 0$ for an arbitrary smooth $(n, n - 2)$-form $\varphi$ with $\text{supp}\varphi \subset U$. This form is represented as $\sum_{j<k} \varphi_{jk} dz_j \wedge dz_k \wedge dV_{jk}$ where $\varphi_{jk}$ are smooth functions supported in $U$ and $dV_{jk} = \prod_{l \neq j,k} idz_l \wedge dz_l$.

By the construction, the projection $\Gamma_{jk}$ of $\Gamma \cap U_k$ into the space of variables \{z_l, l \neq j, k\} is an open set, and $\Gamma_{c(j,k)} = \Gamma \cap \{z_l = c_l, l \neq j, k\}$ is a Lipschitz hypersurface in \{z_l = c_l, l \neq j, k\} $\simeq \mathbb{C}^2$ for all $c(j,k) \in \Gamma_{jk}$. As $\partial \varphi = \sum_{j<k} \partial(\varphi_{jk} dz_j \wedge dz_k) \wedge dV_{jk}$, we have by Fubini theorem for differential forms (see e.g. [4, A4.4.]) that

$$\int_{\Gamma} f_0 \partial \varphi = \sum_{j<k} \int_{\Gamma_{jk}} (\int_{\Gamma_{z(j,k)}} f_0 \partial(\varphi_{jk} dz_j \wedge dz_k)) \wedge dV_{jk}.$$  

If $f_0 | \Gamma_{c(j,k)} \in CR(\Gamma_{c(j,k)})$ for almost every $c(j,k) \in \Gamma_{jk}$, then almost all inner integrals vanish, and the right hand side is zero.

Taking $\varphi$ in a dense sequence of such forms with compact $\text{supp}\varphi \cap \Gamma \setminus \Sigma$ we obtain from this representation that

$$f | \Gamma_{c(j,k)} \setminus \Sigma \longrightarrow \mathbb{C}^m \setminus E$$

are the mappings of the class $CR \cap L^\infty$ for almost every $c(j,k) \in \Gamma_{jk}$. As $\Gamma_{jk} \subset \{z_l = c_l, l \neq j, k\} \simeq \mathbb{C}^2$, we obtain that it is enough to prove Theorem 1 for the case $n = 2$.

4. Analytic extension of the graph

To show that $\Sigma_1$ is empty we assume that $0$ is the boundary point of $\Gamma_1$ in $\Gamma$ and come at last to a contradiction.

The base domain $G \subset \mathbb{C} \times \mathbb{R}$ can be taken bounded and convex, and the function $h(z, u)$ defined and with Lipschitz condition in a neighbourhood of $\bar{G}$. Then the graph $S : v = h(z, u)$ over $bG$ is a two-dimensional sphere in $\mathbb{C}^2$. As $0$ is limiting point for $\Gamma_1$, we can assume, that $S$ is not contained in $\Sigma_1$. By Shcherbina’s theorem [12, 13] the polynomially convex hull $\bar{S}$ of $S$ is the graph of a continuous function $h(z, u)$ over $\bar{G}$ foliated in a one-parametric family of analytic discs with boundaries on $S$.

The graph $M$ of the map $f_1$ over $\Gamma_1$ is a smooth maximally convex 3-dimensional manifold in $\mathbb{C}^2 \times \mathbb{C}^m$ which boundary $M \setminus M$ is contained in $(\bar{S} \times \mathbb{C}^m) \cup (\mathbb{C}^2 \times E)$. As $f_1$ is uniformly bounded, there are closed balls $B_2, B_m$ with centers in origins such that $M \setminus M$ is contained in $(\bar{S} \times B_m) \cup (B_2 \times E)$. This compact set is polynomially convex due to the following.
Lemma 1. Let $X_1 \subset X_2$ be polynomially convex compact sets and $Y$ is a complete pluripolar set in $\mathbb{C}^N$. Then the set $X = X_1 \cup (Y \cap X_2)$ is polynomially convex.

Proof. The set $Y$ is represented as $\{\zeta : \varphi(\zeta) = -\infty\}$ for some function $\varphi$ plurisubharmonic in $\mathbb{C}^N$. If $a \not\in X_1 \cup Y$ then there is a polynomial $p$ such that $p(a) = 1$ and $|p| < 1$ on $X_1$. Let $C = \sup(\varphi(\zeta) : \zeta \in X_1)$ and a positive integer $s$ is taken so big that $|p(\zeta)| e^C < e^{\varphi(a)}$ for all $\zeta \in X_1$ (it is possible because $a \not\in Y$). Then the function $\psi = |p|^s e^C$ is plurisubharmonic in $\mathbb{C}^N, \psi(\zeta) < \psi(a)$ for $\zeta \in X$, and the same is true for $\zeta \in Y$ because $\psi \mid Y = 0$. It follows from the maximum principle for plurisubharmonic functions on polynomially convex hulls (see, e.g. [3]) that $a$ is not contained in the hull of $X$. The rest follows from the inclusion $X \subset X_2$.

Thus, the polynomially convex hull of the set $(S \times B_m) \cup (B_2 \times E)$ for $B_2 \supset S$ is the compact set

$$K = (\tilde{S} \times B_m) \cup (B_2 \times E),$$

and the graph $M$ of $f_1$ is attached to this $K$. By a generalization of Harvey - Lawson theorem in [4, Theorem 19.6.2] there is a two-dimensional (complex) analytic subset $A$ in $\mathbb{C}^{2+m} \setminus (K \cup M)$ such that $A \cup K \cup M$ is compact and $M \setminus K \subset \tilde{A}$.

5. The projection of the extension

We show that $A$ is the graph of a holomorphic mapping over an open set in $\mathbb{C}^2$ with the boundary in $\Gamma \cup \tilde{S}$. (The main difficulty here is that the projection of $\tilde{A}$ is as well as $\Gamma \cup \tilde{S}$ contained in the ball $B_2$, the "shadow" of $B_2 \times E"). It is convenient to use in this Section coordinates $(z', z'')$ for $\mathbb{C}^2 \times \mathbb{C}^m$.

We essentially use the pluripolarity of $B_2 \times E$ and the following result due to E.Bishop [2, 11] on the removability of pluripolar singularities for analytic sets (see [4]).

Lemma 2. Let $Y$ be a closed complete pluripolar subset of a bounded domain $U = U' \times U'' \subset \mathbb{C}^{n+m}$, and $A$ is a pure $p$-dimensional analytic subset in $U \setminus Y$ without limit points on $U' \times bU''$. Suppose that $U'$ contains a nonempty subdomain $V'$ such that $A \cap (V' \times U'')$ is an analytic set. Then $A \cap U$ is analytic in $U$.

First of all, we apply this lemma to the unbounded component $U'$ of $\mathbb{C}^2 \setminus (\tilde{S} \cup \Gamma)$. Let $U'$ be an open ball in $\mathbb{C}^m$ containing $B_m$ and $Y = U \cap B_2 \times E$. Then $A \cap (U \setminus Y)$ is an analytic set satisfying the conditions of Lemma 2. By the maximum principle, $A$ is projected into $B_2$ because its boundary is contained in $K \cup M$. Thus, for $V' = U' \setminus B_2$, the set $A \cap (V' \times U'')$ is empty (hence analytic). It follows from Lemma 2 that $A \cap U$ is analytic in $U$. As $A \cap (V' \times U'')$ is empty and the projection of $A \cap U$ into $U'$ is proper, the set $A \cap U$ is also empty. Thus, we have proved that the projection of $A$ into $\mathbb{C}^2$ is contained in the closure of the union of all bounded component of $\mathbb{C}^2 \setminus (\Gamma \cup \tilde{S})$.

Take now an arbitrary point $a' \in \Gamma_1 \setminus \tilde{S}$ and show that the set $A \cap \{z' = a'\}$ is empty. This set is closed analytic in $a' \times (\mathbb{C}^m \setminus (E \cup a'))$ where $a' = f_1(a')$. As $E \cup a'$ is complete pluripolar, its intersection with $B_m$ is polynomially convex. As $A \cap \{z' = a'\}$ is compact, it follows from a maximum principle on analytic sets (see, e.g., [4, 6.3]) that the dimension of $A \cap \{z' = a'\}$ is zero, i.e. this set is discrete. Thus, given $b = (a', b') \in A$ there is a neighbourhood $U = U' \times U''$ such that the projection of $A \cap U$ into $U'$ is an analytic covering (see [4]). But $\dim A = 2$,
and there is no point in \( A \cap U \) over unbounded component of \( \mathbb{C}^2 \setminus (\Gamma \cup \tilde{S}) \) which has nonempty intersection with \( U' \). This contradiction shows that there is no such points \( b \), i.e. \( A \cap U \) is empty.

Let now \( U' \) be a bounded component of \( \mathbb{C}^2 \setminus (\Gamma \cup \tilde{S}) \) such that \( bU' \cap (\Gamma_1 \setminus \tilde{S}) \) is not empty, and \( a' \) is a point in this nonempty set. Then (see Sect.1) there is a neighbourhood \( V' \ni a' \) such that \( f_1 \) is holomorphic in \( V' \). We have in \((V' \times \mathbb{C}^m)\setminus M\) two analytic sets, \( A \cap (V' \times \mathbb{C}^m) \) and the graph of \( f_1 \) over \( V' \cap U' \), of pure dimension 2 with the same smooth boundary \( M \cap (V' \times \mathbb{C}^m) \). By a boundary uniqueness theorem for analytic sets (see [4, 10.2]), these sets coincide. Thus, the analytic covering \( A \cap (U' \times \mathbb{C}^m) \to U' \) is onesheeted over \( V' \cap U' \), which follows that it is one-sheeted over whole \( U' \). It means that \( A \) over \( U' \) is the graph of a bounded holomorphic map, and this map is a continuation of \( f_1 \) into \( U' \). In particular, we obtain that \( f_1 \) as the boundary value of this map is \( CR \) on \( bU' \cap (\Gamma \setminus \tilde{S}) \).

In terms of components of \( \Gamma \setminus \tilde{S} \), it means that there are only two possibilities: either this component is contained in \( \Gamma_1 \) or it is contained in \( \Sigma_1 \).

### 6. Removability of \( \Sigma \)

Return to notations \((z, w)\) for coordinates in \( \mathbb{C}^2 \) and denote by \( p \) the projection \((z, w) \mapsto (z, u)\) into \( \mathbb{C}^2 \times \mathbb{R} \).

Let \( \delta \subset \tilde{S} \) be an analytic disc with the boundary in \( S \) such that \( p(\delta) \cap p(\Gamma_1) \) is not empty. Show that \( \delta \subset p(\Gamma_1) \).

Suppose it is not. Then there is a point \( a \in \Sigma_1 \) such that \( p(a) \in p(\delta) \), and a convex domain \( G_1 \subset G \) containing \( p(a) \) such that \( bG_1 \cap p(\delta) \cap p(\Gamma_1) \) is not empty, say, it contains \( p(b) \) for some \( b \in \Gamma_1 \). By Sect.1, \( f_1 \) is holomorphic in a one-sided neighbourhood \( V \) of \( b \) convex in \( v \)-direction, as in Sect.1. Let \( h_1 \) be a Lipschitz function on \( bG_1 \) such that its graph \( S_1 : v = h_1(z, u) \) is contained in \( \Gamma \cup V \) but does not contain \( b \). We can assume \( v < h(z, u) \) on \( S_1 \cap V \) (changing \( w \) onto \(-w\) and shrinking \( V \) if it is necessary). Then for \( h_t = th + (1 - t)h_1, 0 < t < 1 \), we have \( h_t \leq h \) and \( h_t < h \) over \( p(V) \). By Shcherbina’s theorem [12, 13] polynomially convex hull \( \tilde{S}_t \) of \( S_t : v = h_t(z, u) \) is the graph of a continuous function \( h_t \) over \( G_1 \) foliated in a one-parametric family of analytic discs with boundaries in \( S_t \). As \( h_t \leq h \), we have \( h_t \leq h \), and \( h_t \to h \) as \( t \to 0 \) by continuity.

Thus, for \( t > 0 \) small enough there is a disc \( \delta_t \subset \tilde{S}_t \) such that \( p(\delta_t) \subset \delta_t \) and \( p(b\delta_t) \cap p(V) \) is not empty. As \( v \leq h(z, u) \) on \( \delta_t \) and \( v < h(z, u) \) on \( b\delta_t \cap V \), we have \( v < h(z, u) \) on the whole \( \delta_t \), i.e. the disc \( \delta_t \) is placed strongly under the hypersurface \( \tilde{S} \). (This is true because the discs \( \delta_t = (0, \epsilon) \) for \( \epsilon > 0 \) do not intersect \( \tilde{S} \), and we can apply the argument principle.)

As we proved above (with \( \tilde{S}_t \) instead of \( \tilde{S} \)), each component of \( p(\delta_t \setminus \Gamma) \) is either contained in \( p(\Gamma_1) \) or it is contained in \( p(\Sigma_1) \). But \( p(\delta_t \setminus \Gamma) \) and \( p(\tilde{S} \cap \Gamma) \) have no common point because \( \delta_t \cap \tilde{S} \) is empty, and the projection \( p \mid \Gamma \) is one-to-one. As \( p(\delta_t) \cap p(\Gamma_1) \) is not empty by the construction, the set \( p(\delta_t) \cap p(\Sigma_1) \) must be empty, in particular, \( p(a) \notin p(\Sigma_1) \). The contradiction (with choosing of \( a \)) shows that there is no such point \( a \), i.e. \( \delta \subset p(\Gamma_1) \).

If \( \Sigma_1 \) is not empty, it follows from the above that there is a disc \( \delta^0 \subset \tilde{S} \) such that \( p(\delta^0) \subset p(\Sigma_1) \cap \partial(p(\Gamma_1)) \). As \( \delta^0 \) is not contained in \( \Gamma \), there is a point \( c \in \Sigma_1 \) \( \tilde{S} \) such that \( p(c) \in p(\delta^0) \). The component \( \Gamma_c \) of \( \Gamma \setminus \tilde{S} \) containing \( c \) has nonempty intersection with \( \Gamma_1 \) because \( c \) is limiting point for \( \Gamma_1 \). As it was proved above, it
follows that $\Gamma_c$ is contained in $\Gamma_1$. This contradiction (with $c \in \Gamma_1$) shows that $\Sigma_1$ is empty.

Thus, we have proved that for each component $\Gamma_0$ of $\Gamma \setminus \Sigma$ the map $f_1$ (equal to $f$ on $\Gamma_0$ and to 0 outside of it) is $CR$ on the whole $\Gamma$. As $\Gamma$ contains no analytic discs, this $f_1$ extends holomorphically into one-sided neighbourhoods of each point of $\Gamma$. It follows from a boundary uniqueness theorem for holomorphic functions that $\Gamma \setminus \Gamma_0$ has zero Hausdorff 3-measure, in particular, $\Gamma_0$ is the single component of $\Gamma \setminus \Sigma$. In other words, the set $\Sigma$ has zero Hausdorff $(2n-1)$-measure for general $n$, and its complement in $\Gamma$ is locally connected (for every connected open set $\Gamma' \subset \Gamma$ the set $\Gamma' \setminus \Sigma$ is also connected).

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