COHERENT CATEGORICAL STRUCTURES FOR LIE BIALGEBRAS, MANIN TRIPLES, CLASSICAL $r$-MATRICES AND PRE-LIE ALGEBRAS

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Abstract. The broadly applied notions of Lie bialgebras, Manin triples, classical $r$-matrices and $\mathcal{O}$-operators of Lie algebras owe their importance to the close relationship among them. Yet these notions and their correspondences are mostly understood as classes of objects and maps among the classes. To gain categorical insight, this paper introduces, for each of the classes, a notion of homomorphisms, uniformly called coherent homomorphisms, so that the classes of objects become categories and the maps among the classes become functors or category equivalences. For this purpose, we start with the notion of an endo Lie algebra, consisting of a Lie algebra equipped with a Lie algebra endomorphism. We then generalize the above classical notions for Lie algebras to endo Lie algebras. As a result, we obtain the notion of coherent endomorphisms for each of the classes, which then generalizes to the notion of coherent homomorphisms by a polarization process. The coherent homomorphisms are compatible with the correspondences among the various constructions, as well as with the category of pre-Lie algebras.

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1. Introduction

This paper gives a categorical structure to each of the classes of Lie bialgebras, Manin triples, classical $r$-matrices, $\mathcal{O}$-operators and pre-Lie algebras by introducing their morphisms. The morphisms are compatible with natural correspondences among these classes, so that these correspondences become functors.

1.1. Lie bialgebras, Manin triples and the related structures. The Lie bialgebra is the algebraic structure corresponding to a Poisson-Lie group. It is also the classical structure of a quantized universal enveloping algebra $[5, 7]$.

The great importance of Lie bialgebras is reflected by its close relationship with several other fundamental notions. First Lie bialgebras are characterized by Manin triples and matched pairs of Lie algebras $[8]$. In fact, there is a one-one correspondence between Lie bialgebras and Manin triples of Lie algebras. The same holds for Lie bialgebras and matched pairs of Lie algebras associated to coadjoint representations.

Furthermore, solutions of the classical Yang-Baxter equation, or the classical $r$-matrices, naturally give rise to coboundary Lie bialgebras $[5, 18]$. Furthermore, such solutions are provided by $\mathcal{O}$-operators, which in turn are provided by pre-Lie algebras. See $[1, 2, 4, 12]$ for more details. These close relations can be summarized in the following diagram where some of the correspondences go both ways, showing by arrows in both directions, and some are even one-one correspondences, showing by two-directional double arrows.

\[
\begin{array}{cccccc}
\text{matched pairs of} & \text{Lie algebras} \\
\text{pre-Lie} & \text{O-operators on} & \text{solutions of} & \text{Lie bialgebras} \\
\text{algebras} & \text{Lie algebras} & \text{CYBE} & \\
\end{array}
\]

\[
\begin{array}{cccc}
& & & \\
& & & \\
\text{Manin triples of} & \text{Lie algebras} & & \\
& & & \\
& & & \\
\end{array}
\]

1.2. Existing morphisms of the structures. For further studies and applications of the important structures and their relations in the above diagram, it is fundamental to understand them in the context of categories. The first step in this direction is to define suitable morphisms for these structures to make them into categories, so that their relations can be made precise as functors and equivalences among these categories. Unfortunately, the understanding of their morphisms is quite limited and can be summarized as follows.

1.2.1. Morphisms of Lie bialgebras and Manin triples. A notion of homomorphisms of Lie bialgebras has been defined in analog to homomorphisms of associative bialgebras and has been applied in the important quantization of Lie bialgebras $[5, 9, 10, 11]$. The notion of isomorphisms of two Manin triples is also defined and agrees with the isomorphisms of Lie bialgebras. However, the expected extension of isomorphisms of Manin triples to homomorphisms do not agree with the homomorphisms of their corresponding Lie bialgebras $[5]$.

Explicitly, let $f$ be a homomorphism between two Lie bialgebras $(\mathfrak{g}_1, \delta_1)$ and $(\mathfrak{g}_2, \delta_2)$. Then $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a homomorphism of Lie algebras and $f^* : \mathfrak{g}_2^* \to \mathfrak{g}_1^*$ is a homomorphism of the dual Lie algebras. It is natural to expect that $f$ should also give a homomorphism of the corresponding Manin triples $(\mathfrak{g}_1 \bowtie \mathfrak{g}_1^*, \mathfrak{g}_1, \mathfrak{g}_1^*)$ and $(\mathfrak{g}_2 \bowtie \mathfrak{g}_2^*, \mathfrak{g}_2, \mathfrak{g}_2^*)$. In fact, as the most natural choice, the map $f + f^*$ should be a homomorphism between the two Lie algebras $\mathfrak{g}_1 \bowtie \mathfrak{g}_1^*$ and $\mathfrak{g}_2 \bowtie \mathfrak{g}_2^*$. However, this does not hold, even in the case when $\mathfrak{g}_1 = \mathfrak{g}_2$. 
1.2.2. **Morphisms of classical $r$-matrices, $O$-operators and pre-Lie algebras.** Recently, a notion of weak homomorphisms of classical $r$-matrices was introduced in [20] in associate with a notion of homomorphisms of the corresponding $O$-operators. But this notion of weak homomorphisms is defined only for the skew-symmetric classical $r$-matrices (hence only for triangular Lie bialgebras). It is natural to ask what to expect without the skew-symmetric condition, so that the anticipated homomorphisms of classical $r$-matrices are compatible with a suitably defined homomorphism of Lie bialgebras.

Homomorphisms of pre-Lie algebras are naturally defined, but it is not known how they could be preserved under the correspondence between pre-Lie algebras and $O$-operators.

1.3. **Our approach.** Thus so far our understanding of the important classes of objects in (1) and their relationship mostly remains at the level of sets and set correspondences. Morphisms for these objects are either not defined, or are defined but not compatible with the correspondences.

The goal of this paper is to introduce morphisms for all the classes in the diagram so that the diagram gives functors and natural equivalences of the resulting categories. Since the homomorphisms we will define for the various classes of objects are compatible with the correspondences among the classes, we use the uniform term of **coherent homomorphisms** for the new homomorphisms in all the classes.

To reach our goal, we utilize two strategies. The first strategy is a polarization process that allows us to first consider endomorphisms instead of homomorphisms. The second strategy is a change of order of the operations which equip Lie algebras with two extra structures: bialgebras and endomorphisms.

1.3.1. **Polarization.** Our strategy of polarization is in a sense similar to polarizing a homogeneous polynomial in one variable to a multilinear form in multivariable by linear substitutions and derivations [16]. The depolarization is then evaluating along a diagonal line. For us depolarization of morphisms in a category gives endomorphisms. So for each of the classes of constructions such as Lie bialgebras, instead of attempting to define morphisms between any two of them, we focus on the special case of endomorphisms on any given object. Taking Lie bialgebras as an example, we first define endomorphisms of a Lie bialgebra, instead of homomorphisms between any two Lie bialgebras.

This depolarization does not reduce the level of difficulty by itself, but allows us to look the endomorphisms of a Lie bialgebra for instance from a different angle and apply the strategy of changing of operation orders.

1.3.2. **Endomorphisms of Lie algebras and bialgebras of Lie algebras with endomorphisms.** With the polarization reduction, our goal is to define endomorphisms of Lie bialgebras and the related structures in diagram (1). To define endomorphisms for a Lie bialgebra, we can regard it as equipping an extra endomorphism structure to the Lie bialgebra which, by itself, is obtained from equipping a bialgebra structure to a Lie algebra. Thus we are looking at the composition of two processes of equipping two extra structures to a given structure, beginning with a Lie algebra.

Our second strategy is to switch the order of these two processes. This is shown in the following diagram for the instance of Lie bialgebras. Each of the other classes in Diagram (1) is also obtained from equipping a Lie algebra with an extra structure, so can be treated in the same way.
In this diagram, the vertical arrows are equipping a given structure by a suitable bialgebra structure and the horizontal arrows are equipping a given structure with suitable endomorphisms. Our goal for a notion of endomorphisms of Lie bialgebras, which amounts to endomorphisms of the bialgebra structures on Lie algebras, is starting from the Lie algebras in the diagram, going downward and then right. Instead of attacking this apparently challenging task, we go first right and then downward, that is, we first equip Lie algebras with endomorphisms, called **endo Lie algebras**. We then attempt to equip a bialgebra structure for endo Lie algebras. The notion of an endo Lie algebra is interesting on its own right: an associative algebra, in particular a commutative associative algebra equipped with an injective endomorphism is called a difference algebra [6, 13] as an algebraic study of difference equations and a close analogy of differential algebras [17].

The same strategy is applied to the other structures in Diagram (1): instead of defining endomorphisms of a Lie algebra with the various extra structures in the diagram, we define the various extra structures on an endo Lie algebra, resulting in the following diagram.

\[
\begin{array}{ccc}
\text{endo Lie algebras} & \text{endo \ pre-Lie algebras} & \text{O-operators on Lie algebras} \\
\text{endo \ pre-Lie bialgebras} & \text{endo \ Lie bialgebras} & \text{endo CYBE}
\end{array}
\]

Once constructed, each of the structures on endo Lie algebras naturally gives rise to a notion of endomorphisms of this structure on Lie algebras. Once this is done, then as noted above, a polarization process extends the notion of endomorphisms to a notion of homomorphisms, equipping each class of objects with a category structure. Furthermore, the correspondences among the classes are naturally functors and, in the case of one-one correspondences, equivalences. These are summarized in the following enriched diagram of (1). Here the labels on the arrows indicate the results where the functors and equivalences are established. It is remarkable that the natural categorical structure of pre-Lie algebras is compatible with the category of O-operators hereby introduced. In fact, there is a pair of adjoint functors between the two categories.
1.4. **Outline of the paper.** We first introduce in Section 2 the notion of an endo Lie algebra and formulate a bialgebra theory for endo Lie algebras together with their equivalences to matched pairs and Manin triples of endo Lie algebras. We then interpret a bialgebra of endo Lie algebras as an endomorphism of a Lie bialgebra, which then naturally generalizes to a homomorphism of Lie bialgebras that is compatible with that of Manin triples (and matched pairs) of Lie algebras, showing that the correspondences among Lie bialgebras, matched pairs of Lie algebras, and Manin triples of Lie algebras are equivalences of categories (Theorem 2.36).

We then extend in Section 3 the classical relations of Lie bialgebras with the classical Yang-Baxter equation as well as classical \( r \)-matrices to the context of endo Lie algebras. This naturally gives rise to a notion of coherent homomorphisms for any \( r \)-matrices, not just the skew-symmetric ones. This notion is shown to be compatible with the coherent homomorphisms of Lie bialgebras, leading to a functor of the corresponding categories (Theorem 3.13).

Finally in Section 4, we give the notion of \( O \)-operators on endo Lie algebras and apply it to define coherent homomorphisms of \( O \)-operators in such a way that is compatible with the coherent homomorphisms of classical \( r \)-matrices in Section 3 (Theorem 4.5 and Corollary 4.10). This notion of coherent homomorphisms of \( O \)-operators is moreover compatible with the natural homomorphism of pre-Lie algebras, giving rise to a pair of adjoint functors between the corresponding two categories (Propositions 4.19 and 4.20). We also consider a case where all the constructions can be given explicitly, providing natural examples of coherent isomorphisms of Lie bialgebras that are not the previously defined isomorphisms. This further justifies the significance of the coherent homomorphisms of Lie bialgebras introduced in this paper.

**Notations.** Throughout this paper, all vector spaces, tensor products, and linear homomorphisms are over a fixed field \( K \). Let \( \text{End}(V) \) denote the space of linear operators on a vector space \( V \). The vector spaces and Lie algebras are finite dimensional unless otherwise specified.

## 2. Endo Lie algebras and their bialgebras

In this section we introduce the notion of endo Lie algebras and give the equivalent structures of bialgebras, matched pairs and Manin triples for endo Lie algebras.

### 2.1. Endo Lie algebras and their representations.

We first introduce the notion of a representation of an endo Lie algebra characterized by a semi-direct product. We then introduce the notion of a dual representation of an endo Lie algebra in order to construct a reasonable representation on the dual space.

**Definition 2.1.** An **endo Lie algebra** is a triple \((g, [ , ], \phi)\), or simply \((g, \phi)\), where \((g, [ , ])\) is a Lie algebra and \(\phi : g \to g\) is a Lie algebra endomorphism.

As an analogy of a Lie algebra representation, we have

**Definition 2.2.** A **representation** of an endo Lie algebra \((g, \phi)\) is a triple \((V, \rho, \alpha)\) where \((V, \rho)\) is a representation of the Lie algebra \(g\) and \(\alpha \in \text{End}(V)\) such that

\[
\alpha(\rho(x)(v)) = \rho(\phi(x))(\alpha(v)), \quad \forall x \in g, v \in V.
\]
Two representations \((V_1, \rho_1, \alpha_1)\) and \((V_2, \rho_2, \alpha_2)\) of an endo Lie algebra \((g, \phi)\) are called **equivalent** if there exists a linear isomorphism \(\varphi : V_1 \to V_2\) such that
\[
\varphi(\rho_1(x)(v)) = \rho_2(x)(\varphi(v)), \quad \varphi \alpha_1(v) = \alpha_2 \varphi(v), \quad \forall x \in g, v \in V_1.
\]

With \(\text{ad}\) denoting the adjoint representation of the Lie algebra \(g\), the triple \((g, \text{ad}, \phi)\) is naturally a representation of the endo Lie algebra \((g, \phi)\), called the **adjoint representation** of \((g, \phi)\).

For vector spaces \(V_1\) and \(V_2\), and linear maps \(\phi_1 : V_1 \to V_1\) and \(\phi_2 : V_2 \to V_2\), we abbreviate \(\phi_1 + \phi_2\) for the linear map
\[
\phi_{V_1 \oplus V_2} : V_1 \oplus V_2 \to V_1 \oplus V_2, \quad \phi_{V_1 \oplus V_2}(v_1 + v_2) := \phi_1(v_1) + \phi_2(v_2), \quad \forall v_1 \in V_1, v_2 \in V_2. \tag{7}
\]

For a Lie algebra \(g\), a linear space \(V\) and a linear map \(\rho : g \to \text{End}(V)\), define a multiplication \([\cdot, \cdot]_\kappa\) on \(g \oplus V\) by
\[
[x + u, y + v]_\kappa := [x, y] + \rho(x)v - \rho(y)u, \quad \forall x, y \in g, u, v \in V. \tag{8}
\]

As is well-known, \(g \oplus V\) is a Lie algebra if and only if \((V, \rho)\) is a representation of \(g\). The Lie algebra is denoted by \(g \ltimes_\rho V\) and is called the **semi-direct product** of \(g\) and \(V\). Similarly for an endo Lie algebra, we have

**Proposition 2.3.** Let \((g, \phi)\) be an endo Lie algebra, \((V, \rho)\) a representation of the Lie algebra \(g\) and \(\alpha\) a linear operator on \(V\). Then \((g \ltimes_\rho V, \phi + \alpha)\) is an endo Lie algebra if and only if \((V, \rho, \alpha)\) is a representation of \((g, \phi)\). The resulting endo Lie algebra \((g \ltimes_\rho V, \phi + \alpha)\) is called the **semi-direct product** of the endo Lie algebra \((g, \phi)\) and its representation \((V, \rho, \alpha)\).

This result follows as a special case of matched pairs of endo Lie algebras in Theorem 2.10 in which the Lie algebra \(\mathfrak{h} := V\) is the abelian Lie algebra.

We next turn our attention to dual representations of endo Lie algebras. Denote the usual pairing between the dual space \(V^*\) and \(V\) by
\[
\langle \cdot, \cdot \rangle : V^* \times V \to K, \quad \langle w^* , v \rangle := w^*(v), \quad \forall v \in V, w^* \in V^*. \tag{9}
\]

For a linear map \(\varphi : V \to W\), the transpose of \(\varphi\) is defined by
\[
\varphi^* : W^* \to V^*, \quad \varphi^*(w^*)(v) := w^*(\varphi(v)), \quad \forall w^* \in W^*, v \in V. \tag{10}
\]

For a representation \((V, \rho)\) of a Lie algebra \(g\), its **dual representation** is the linear map defined by
\[
\rho^* : g \to \text{End}(V^*), \quad \rho^*(x) := -\rho(x)^*, \quad \forall x \in g. \tag{11}
\]

To obtain the dual representation of an endo Lie algebra, an extra condition is needed.

**Lemma 2.4.** Let \((g, \phi)\) be an endo Lie algebra. Let \((V, \rho)\) be a representation of the Lie algebra \(g\). For \(\beta \in \text{End}(V)\), the triple \((V^*, \rho^*, \beta^*)\) is a representation of \((g, \phi)\) if and only if \(\beta\) satisfies
\[
\beta(\rho(\phi(x))(v)) = \rho(x)(\beta(v)), \quad \forall x \in g, v \in V. \tag{12}
\]

**Proof.** By Eq. (5), the triple \((V^*, \rho^*, \beta^*)\) is a representation of \((g, \phi)\) means that
\[
\beta^*(\rho^*(x)) = \rho^*(\phi(x))\beta^*, \quad \forall x \in g.
\]

Then the lemma follows from Eqs. (10), (11) and the nondegenerate pairing in Eq. (9). \qed

We reserve a notion for this property due to its pivotal role in our study.
Definition 2.5. Let \((g, \phi)\) be an endo Lie algebra. Let \((V, \rho)\) be a representation of the Lie algebra \(g\) and let \(\beta \in \text{End}(V)\). We say that \(\beta\) **dually represents the endo Lie algebra** \((g, \phi)\) on \((V, \rho)\) if \((V^*, \rho^*, \beta^*)\) is a representation of \((g, \phi)\), that is, Eq. (12) holds. When \((V, \rho)\) is taken to be the adjoint representation \((g, \text{ad})\) of the Lie algebra \(g\), we simply say that \(\beta\) **dually represents** \((g, \phi)\).

For later applications, we display some direct consequences. By Lemma 2.4 we have

**Corollary 2.6.** Let \((g, \phi)\) be an endo Lie algebra. A linear operator \(\psi\) on \(g\) dually represents \((g, \phi)\) if and only if

\[
\psi[\phi(x), y] = [x, \psi(y)], \quad \forall x, y \in g.
\]

(13)

By Proposition 2.3 and Definition 2.5, we also have

**Corollary 2.7.** Let \((g, \phi)\) be an endo Lie algebra, \((V, \rho)\) be a representation of \(g\) and \(\beta \in \text{End}(V)\). If \(\beta\) dually represents \((g, \phi)\) on \((V, \rho)\), then we have the semi-direct product endo Lie algebra \((g \ltimes\rho^*, \phi + \beta^*)\).

2.2. **Matched pairs of endo Lie algebras.** We first recall the concept of a matched pair of Lie algebras [14, 19].

**Definition 2.8.** A matched pair of Lie algebras is a quadruple \((g, h, \rho, \mu)\), where \(g := (g, [\ , \ ]_g)\) and \(h := (h, [\ , \ ]_h)\) are Lie algebras, \(\rho : g \to \text{End}(h)\) and \(\mu : h \to \text{End}(g)\) are linear maps such that

(a) \((g, \rho)\) is a representation of \((h, [\ , \ ]_h)\),
(b) \((h, \rho)\) is a representation of \((g, [\ , \ ]_g)\) and
(c) the following compatibility conditions hold: for any \(x, y \in g\) and \(a, b \in h\),

\[
\rho(x)[a, b]_h - [\rho(x)a, b]_h - [a, \rho(x)b]_h + \rho(\mu(a)x)b - \rho(\mu(b)x)a = 0, \quad (14)
\]

\[
\mu(a)[x, y]_g - [\mu(a)x, y]_g - [x, \mu(a)y]_g + \mu(\rho(x)a)y - \mu(\rho(y)a)x = 0. \quad (15)
\]

For Lie algebras \((g, [\ , \ ]_g)\), \((h, [\ , \ ]_h)\) and linear maps \(\rho : g \to \text{End}(h)\), \(\mu : h \to \text{End}(g)\), define a multiplication on the direct sum \(g \oplus h\) by

\[
[x + a, y + b]_{\infty} := [x, y]_g + \mu(a)y - \mu(b)x + \rho(x)b - \rho(y)a + [a, b]_h, \quad \forall x, y \in g, a, b \in h. \quad (16)
\]

Then by [19], \((g \oplus h, [\ , \ ]_{\infty})\) is a Lie algebra if and only if \((g, h, \rho, \mu)\) is a matched pair of \(g\) and \(h\). We denote the resulting Lie algebra \((g \oplus h, [\ , \ ]_{\infty})\) by \(g \bowtie \rho^* h\) or simply \(g \bowtie h\). Further, for any Lie algebra \(l\) whose underlying vector space is a vector space direct sum of two Lie subalgebras \(g\) and \(h\), there is a matched pair \((g, h, \rho, \mu)\) such that there is an isomorphism from the resulting Lie algebra \((g \oplus h, [\ , \ ]_{\infty})\) via Eq. (16) to the Lie algebra \(l\) and the restrictions of the isomorphism to \(g\) and \(h\) are the identity maps.

**Definition 2.9.** A matched pair of endo Lie algebras is a quadruple \(((g, \phi_g), (h, \phi_h), \rho, \mu)\), where \((g, \phi_g)\) and \((h, \phi_h)\) are endo Lie algebras, \(\rho : g \to \text{End}(h)\) and \(\mu : h \to \text{End}(g)\) are linear maps such that

(a) \((g, \mu, \phi_g)\) is a representation of the endo Lie algebra \((h, \phi_h)\),
(b) \((h, \rho, \phi_h)\) is a representation of the endo Lie algebra \((g, \phi_g)\),
(c) \((g, h, \rho, \mu)\) is a matched pair of Lie algebras.

We have the following characterization of matched pairs of endo Lie algebras.

**Theorem 2.10.** Let \((g, \phi_g)\) and \((h, \phi_h)\) be endo Lie algebras and let \((g, h, \rho, \mu)\) be a matched pair of the Lie algebras \(g\) and \(h\). Then the pair \((g \bowtie h, \phi_g + \phi_h)\) is an endo Lie algebra if and only if \(((g, \phi_g), (h, \phi_h), \rho, \mu)\) is a matched pair of endo Lie algebras.
As noted after Proposition 2.3, the proposition follows from the theorem as the special case when the linear space \( V \) is regarded as an abelian Lie algebra \( \mathfrak{h} \).

**Proof.** Let \( x, y \in \mathfrak{g} \) and \( a, b \in \mathfrak{h} \). Then we have

\[
(\phi_{\mathfrak{g}} + \phi_{\mathfrak{h}})([x + a, y + b]_{\omega}) = \phi_{\mathfrak{g}}([x, y]_{\mathfrak{g}}) + \phi_{\mathfrak{h}}(\rho(x)(b)) - \phi_{\mathfrak{g}}(\rho(y)(a)) + \phi_{\mathfrak{h}}([a, b]_{\mathfrak{h}}),
\]

\[
([\phi_{\mathfrak{g}} + \phi_{\mathfrak{h}}](x + a), (\phi_{\mathfrak{g}} + \phi_{\mathfrak{h}})(y + b)]_{\omega} = [\phi_{\mathfrak{g}}(x), \phi_{\mathfrak{g}}(y)]_{\mathfrak{g}} + \rho(\phi_{\mathfrak{g}}(x))\phi_{\mathfrak{h}}(b) - \mu(\phi_{\mathfrak{h}}(b))\phi_{\mathfrak{g}}(x) + \mu(\phi_{\mathfrak{h}}(a))\phi_{\mathfrak{g}}(y) - \rho(\phi_{\mathfrak{g}}(y))\phi_{\mathfrak{h}}(a) + [\phi_{\mathfrak{h}}(a), \phi_{\mathfrak{h}}(b)]_{\mathfrak{h}}.
\]

Note that the equality of the left hand sides means that \((\mathfrak{g} \simeq \mathfrak{h}, \phi_{\mathfrak{g}} + \phi_{\mathfrak{h}})\) is an endo Lie algebra, while taking \( x = b = 0 \) and then \( a = y = 0 \) in the equality of the right hand sides yields the first two conditions for \(((\mathfrak{g}, \phi_{\mathfrak{g}}), (\mathfrak{h}, \phi_{\mathfrak{h}}), \rho, \mu)\) to be a matched pair of endo Lie algebras. Thus the conclusion follows. \( \Box \)

### 2.3. Manin Triples of Endo Lie Algebras

We recall the concept of a Manin triple of Lie algebras. See [5] for details.

**Definition 2.11.** A bilinear form \( \mathcal{B} \in (\mathfrak{g} \otimes \mathfrak{g})^* \) on a Lie algebra \( \mathfrak{g} \) is called **invariant** if

\[
\mathcal{B}([x, y], z) = \mathcal{B}(x, [y, z]), \quad \forall x, y, z \in \mathfrak{g}.
\]

As an analog of the notion of a Frobenius (associative) algebra, a Lie algebra with a nondegenerate symmetric invariant bilinear form is called a **quadratic Lie algebra**.

Let \((\mathfrak{g}, [\ , \ ]_{\mathfrak{g}})\) be a Lie algebra. Suppose that there is a Lie algebra structure \([\ , \ ]_{\mathfrak{g}^*}\) on its dual space \( \mathfrak{g}^* \) and a Lie algebra structure on the vector space direct sum \( \mathfrak{g} \oplus \mathfrak{g}^* \) which contains both \((\mathfrak{g}, [\ , \ ]_{\mathfrak{g}})\) and \((\mathfrak{g}^*, [\ , \ ]_{\mathfrak{g}^*})\) as Lie subalgebras. Define a bilinear form on \( \mathfrak{g} \oplus \mathfrak{g}^* \) by

\[
\mathcal{B}_d(x + a^*, y + b^*) := \langle x, b^* \rangle + \langle a^*, y \rangle, \quad \forall a^*, b^* \in \mathfrak{g}^*, \ x, y \in \mathfrak{g}.
\]

If \( \mathcal{B}_d \) is invariant, then \((\mathfrak{g} \oplus \mathfrak{g}^*, \mathcal{B}_d)\) is a quadratic Lie algebra and the triple \((\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)\) of Lie algebras is called a (standard) **Manin triple of Lie algebras** associated to \((\mathfrak{g}, [\ , \ ]_{\mathfrak{g}})\) and \((\mathfrak{g}^*, [\ , \ ]_{\mathfrak{g}^*})\). This Lie algebra on \( \mathfrak{g} \oplus \mathfrak{g}^* \) comes from a matched pair of \( \mathfrak{g} \) and \( \mathfrak{g}^* \) in Eq. (16), and hence will also be denoted by \( \mathfrak{g} \bowtie \mathfrak{g}^* \). Indeed, we have

**Theorem 2.12.** ([5]) Let \((\mathfrak{g}, [\ , \ ]_{\mathfrak{g}})\) be a Lie algebra. Suppose that there is a Lie algebra structure \([\ , \ ]_{\mathfrak{g}^*}\) on its dual space \( \mathfrak{g}^* \). Then there is a Manin triple of Lie algebras associated to \((\mathfrak{g}, [\ , \ ]_{\mathfrak{g}})\) and \((\mathfrak{g}^*, [\ , \ ]_{\mathfrak{g}^*})\) if and only if \((\mathfrak{g}, \mathfrak{g}^*, \text{ad}^*_{\mathfrak{g}}, \text{ad}^*_{\mathfrak{g}^*})\) is a matched pair of Lie algebras.

For endo Lie algebras, we give the following definition.

**Definition 2.13.** A **quadratic endo Lie algebra** is a triple \((\mathfrak{g}, \phi, \mathcal{B})\) where \((\mathfrak{g}, \phi)\) is an endo Lie algebra and \((\mathfrak{g}, \mathcal{B})\) is a quadratic Lie algebra.

Let \( \widehat{\phi} : \mathfrak{g} \to \mathfrak{g} \) denote the **adjoint linear transformation** of \( \phi \) under the nondegenerate bilinear form \( \mathcal{B} \):

\[
\mathcal{B}(\phi(x), y) = \mathcal{B}(x, \widehat{\phi}(y)), \quad \forall x, y \in \mathfrak{g}.
\]

**Proposition 2.14.** Let \((\mathfrak{g}, \phi, \mathcal{B})\) be a quadratic endo Lie algebra. Let \( \widehat{\phi} \) be the adjoint of \( \phi \). Then \( \widehat{\phi} \) dually represents \((\mathfrak{g}, \phi)\). In other words, \((\mathfrak{g}^*, \text{ad}^*_{\mathfrak{g}}, \widehat{\phi}^*)\) is a representation of the endo Lie algebra \((\mathfrak{g}, \phi)\). Furthermore, as representations of \((\mathfrak{g}, \phi), (\mathfrak{g}, \text{ad}, \phi)\) and \((\mathfrak{g}^*, \text{ad}^*_{\mathfrak{g}}, \widehat{\phi}^*)\) are equivalent.
Conversely, let \((\mathfrak{g}, \phi)\) be an endo Lie algebra and \(\psi \in \text{End}(\mathfrak{g})\) dually represent \((\mathfrak{g}, \phi)\). If the representation \((\mathfrak{g}^*, \text{ad}^*, \psi^*)\) of \((\mathfrak{g}, \phi)\) is equivalent to \((\mathfrak{g}, \text{ad}, \phi)\), then there exists a nondegenerate invariant bilinear form \(\mathcal{B}\) on \(\mathfrak{g}\) such that \(\hat{\phi} = \psi\).

**Proof.** For any \(x, y, z \in \mathfrak{g}\), we have

\[
0 = \mathcal{B}([\phi(x), \phi(y)], z) - \mathcal{B}(\phi[x, y], z) = \mathcal{B}(\phi(x), [\phi(y), z]) - \mathcal{B}([x, y], \hat{\phi}(z)) = \mathcal{B}(x, \hat{\phi}([\phi(y), z])) - \mathcal{B}(x, [y, \hat{\phi}(z)]).
\]

Thus \(\hat{\phi}([\phi(y), z]) = [y, \hat{\phi}(z)]\). Hence \((\mathfrak{g}^*, \text{ad}^*, \hat{\phi}^*)\) is a representation of \((\mathfrak{g}, \phi)\). Define a linear map \(\varphi : \mathfrak{g} \to \mathfrak{g}^*\) by

\[
\varphi(x)(y) := \mathcal{B}(x, y), \quad \forall x, y \in \mathfrak{g}.
\]

Since the bilinear form \(\mathcal{B}\) is nondegenerate, the linear map \(\varphi\) is a linear isomorphism. Moreover, for any \(x, y, z \in A\), we have

\[
\langle \varphi(\text{ad}(x)y), z \rangle = \mathcal{B}(x, y) = \mathcal{B}(z, x) = \langle \varphi(y), [z, x] \rangle = \langle \text{ad}^* (x) \varphi(y), z \rangle,
\]

\[
\langle \varphi(\phi(x)), y \rangle = \mathcal{B}(\phi(x), y) = \mathcal{B}(x, \hat{\phi}(y)) = \langle \varphi(x), \hat{\phi}(y) \rangle = \langle \hat{\phi}^* (\varphi(x)), y \rangle.
\]

Hence \((\mathfrak{g}, \text{ad}, \phi)\) is equivalent to \((\mathfrak{g}^*, \text{ad}^*, \hat{\phi}^*)\) as representations of \((\mathfrak{g}, \phi)\).

Conversely, suppose that \(\varphi : \mathfrak{g} \to \mathfrak{g}^*\) is a linear isomorphism giving the equivalence between \((\mathfrak{g}, \text{ad}, \phi)\) and \((\mathfrak{g}^*, \text{ad}^*, \psi^*)\). Define a bilinear form \(\mathcal{B}\) on \(\mathfrak{g}\) by

\[
\mathcal{B}(x, y) := \langle \varphi(x), y \rangle, \quad \forall x, y \in \mathfrak{g}.
\]

Then by a similar argument as above, we show that \(\mathcal{B}\) is a nondegenerate invariant bilinear form on \(\mathfrak{g}\) such that \(\hat{\phi} = \psi\). \(\square\)

We now extend the notion of Manin triples from the context of Lie algebras to that of endo Lie algebras.

**Definition 2.15.** Let \((\mathfrak{g}, \phi)\) be an endo Lie algebra. Suppose that \((\mathfrak{g}^*, \psi^*)\) is also an endo Lie algebra. A Manin triple of endo Lie algebras associated to \((\mathfrak{g}, \phi)\) and \((\mathfrak{g}^*, \psi^*)\) is a Manin triple \((\mathfrak{g} \bowtie \mathfrak{g}^*, \mathfrak{g}^*, \mathfrak{g}^*)\) of Lie algebras such that \((\mathfrak{g} \bowtie \mathfrak{g}^*, \phi + \psi^*, \mathcal{B}_d)\) is a quadratic endo Lie algebra. We use \(((\mathfrak{g} \bowtie \mathfrak{g}^*, \phi + \psi^*), (\mathfrak{g}, \phi), (\mathfrak{g}^*, \psi))\) to denote this Manin triple.

**Lemma 2.16.** Let \((\mathfrak{g} \bowtie \mathfrak{g}^*, \phi + \psi^*, \mathcal{B}_d)\) be a quadratic endo Lie algebra.

(a) The adjoint \(\widehat{\phi + \psi^*}\) of \(\phi + \psi^*\) with respect to \(\mathcal{B}_d\) is \(\psi + \phi^*\). Further \(\psi + \phi^*\) dually represents the endo Lie algebra \((\mathfrak{g} \bowtie \mathfrak{g}^*, \phi + \psi^*)\).

(b) \(\psi\) dually represents the endo Lie algebra \((\mathfrak{g}, \phi)\).

(c) \(\phi^*\) dually represents the endo Lie algebra \((\mathfrak{g}^*, \psi^*)\).

**Proof.**

(a) For any \(x, y \in \mathfrak{g}, a^*, b^* \in \mathfrak{g}^*\), we apply Eq. (18) to give

\[
\mathcal{B}_d((\phi + \psi^*)(x + a^*), y + b^*) = \langle \phi(x), b^* \rangle + \langle \psi^*(a^*), y \rangle = \langle x, \phi^*(b^*) \rangle + \langle a^*, \psi(y) \rangle = \mathcal{B}_d(x + a^*, (\psi + \phi^*)(y + b^*)).
\]

Hence the adjoint \(\widehat{\phi + \psi^*}\) of \(\phi + \psi^*\) with respect to \(\mathcal{B}_d\) is \(\psi + \phi^*\). By Proposition 2.14, for the quadratic Lie algebra \((\mathfrak{g} \bowtie \mathfrak{g}^*, \phi + \psi^*, \mathcal{B}_d)\), the linear map \(\widehat{\phi + \psi^*} = \psi + \phi^*\) dually represents \((\mathfrak{g} \bowtie \mathfrak{g}^*, \phi + \psi^*)\).
(b) By Item (a), \(\psi + \phi^*\) dually represents \((g \triangleright g^*, \phi + \psi^*)\). By Eq. (13), this is the case if and only if for any \(x, y \in g, a^*, b^* \in g^*\),

\[
(\psi + \phi^*)((\phi + \psi^*)(x + a^*), y + b^*)_{\infty} = [x + a^*, (\psi + \phi^*)(y + b^*)]_{\infty}.
\]

Now taking \(a^* = b^* = 0\) in the above equation, we have \(\psi[\phi(x), y]_g = [x, \psi(y)]_g\), that is, \(\psi\) dually represents \((g, \phi)\).

(c) Likewise, taking \(x = y = 0\) in the above equation yields \(\phi^*[\psi^*(a^*), b^*]_g = [a^*, \phi^*(b^*])_g\), that is, \(\phi^*\) dually represents \((g^*, \psi^*)\).

Enriching Theorem 2.12 to the context of endo Lie algebras, we obtain

**Theorem 2.17.** Let \((g, \phi)\) be an endo Lie algebra. Suppose that there is an endo Lie algebra structure \((g^*, \psi^*)\) on its dual space \(g^*\). Then there is a Manin triple of endo Lie algebras \(((g \triangleright g^*, \phi + \psi^*), (g, \phi), (g^*, \psi^*))\) associated to \((g, \phi)\) and \((g^*, \psi^*)\) if and only if \(((g, \phi), (g^*, \psi^*), \text{ad}^*_g, \text{ad}^*_g)\) is a matched pair of endo Lie algebras.

**Proof.** \((\Rightarrow)\) By the assumption, there is a Manin triple of endo Lie algebras \(((g \triangleright g^*, \phi + \psi^*), (g, \phi), (g^*, \psi^*))\) associated to \((g, \phi)\) and \((g^*, \psi^*)\). Then in particular \((g \triangleright g^*, g, g^*)\) is a Manin triple of Lie algebras associated to \(g\) and \(g^*\). Hence by Theorem 2.12, \((g, g^*, \text{ad}^*_g, \text{ad}^*_g)\) is a matched pair of Lie algebras for which the Lie algebra on \(g \oplus g^*\) is the Lie algebra \(g \triangleright g^*\). Since the homomorphism on \(g \triangleright g^*\) is \(\phi + \psi^*\), by Lemma 2.16, \((g^*, \text{ad}^*_g, \psi^*)\) and \((g, \text{ad}^*_g, \phi)\) are representations of the endo Lie algebras \((g, \phi)\) and \((g^*, \psi^*)\) respectively. Hence \(((g, \phi), (g^*, \psi^*), \text{ad}^*_g, \text{ad}^*_g)\) is a matched pair of endo Lie algebras.

\((\Leftarrow)\) If \(((g, \phi), (g^*, \psi^*), \text{ad}^*_g, \text{ad}^*_g)\) is a matched pair of endo Lie algebras, then \((g, g^*, \text{ad}^*_g, \text{ad}^*_g)\) is a matched pair of Lie algebras. Hence by Theorem 2.12 again, \(B_g\) is invariant on \(g \triangleright g^*\). By Theorem 2.10, the matched pair of endo Lie algebras also equips the Lie algebra \(g \triangleright g^*\) with the endomorphism \(\phi + \psi^*\), giving us a quadratic endo Lie algebra. This is exactly what we need. \(\square\)

### 2.4. Endo Lie bialgebras

With our previous preparations, we are ready to introduce the notion of endo Lie bialgebras, as an enrichment of the notion of Lie bialgebras that we now recall and refer the reader to [5] for details.

**Theorem 2.18.** Let \((g, [\cdot, \cdot]_g)\) be a Lie algebra. Suppose that there is a Lie algebra \((g^*, [\cdot, \cdot]_{g^*})\) on the linear dual \(g^*\) of \(g\). Let \(\delta : g \rightarrow g \otimes g\) denote the linear dual of the multiplication \([\cdot, \cdot]_g : g^* \otimes g^* \rightarrow g^*\). Then the quadruple \((g, g^*, \text{ad}^*_g, \text{ad}^*_g)\) is a matched pair of Lie algebras if and only if, for any \(x, y \in g\), we have

\[
\delta [x, y]_g = (\text{ad}^*_g(x) \otimes \text{id} + \text{id} \otimes \text{ad}^*_g(x))\delta(y) - (\text{ad}^*_g(y) \otimes \text{id} + \text{id} \otimes \text{ad}^*_g(y))\delta(x). \tag{20}
\]

Under our assumption of finite dimension, the Lie algebra structure \((g^*, [\cdot, \cdot]_{g^*})\) is equivalent to the condition that \((g, \delta)\) is a **Lie coalgebra** [15] which is defined without the dimensional restriction.

**Definition 2.19.** A linear space \(g\) with a linear map \(\delta : g \rightarrow g \otimes g\) is called a **Lie coalgebra** if \(\delta\) is coantisymmetric, in the sense that \(\delta = -\tau\delta\) for the flip map \(\tau : g \otimes g \rightarrow g \otimes g\), and satisfies the co-Jacobian identity:

\[
(id + \sigma + \sigma^2)(id \otimes \delta)\delta = 0, \tag{21}
\]

where \(\sigma(x \otimes y \otimes z) := z \otimes x \otimes y\) for \(x, y, z \in g\).

Combining a Lie algebra and a Lie coalgebra gives
Definition 2.20. A Lie bialgebra is a triple \((\mathfrak{g}, [\cdot, \cdot], \delta)\), where \(\mathfrak{g} := (\mathfrak{g}, [\cdot, \cdot])\) is a Lie algebra, \((\mathfrak{g}, \delta)\) is a Lie coalgebra (that is, \(\mathfrak{g}^*\delta^*\) is a Lie algebra when \(\mathfrak{g}\) is finite dimensional) and Eq. (20) holds.

The notion of a Lie bialgebra applies to Lie algebras of any dimension. Under the finite-dimension condition, the notion is characterized by matched pairs of Lie algebras because of Theorem 2.18, and then by Manin triples of Lie algebras thanks to Theorem 2.12. With the extra structure of endomorphisms, we also have the following equivalent characterizations.

Theorem 2.21. Let \((\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g}, \phi)\) be an endo Lie algebra. Suppose that there is an endo Lie algebra \((\mathfrak{g}^*, [\cdot, \cdot]_{\mathfrak{g}^*}, \psi^*)\) on the linear dual \(\mathfrak{g}^*\) of \(\mathfrak{g}\). Let \(\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}\) denote the linear dual of the multiplication \([\cdot, \cdot]_{\mathfrak{g}^*} : \mathfrak{g}^* \otimes \mathfrak{g}^* \rightarrow \mathfrak{g}^*\) on \(\mathfrak{g}^*\). Then the following statements are equivalent.

1. The quadruple \(((\mathfrak{g}, \phi), (\mathfrak{g}^*, \psi^*), \text{ad}^*_\mathfrak{g}, \text{ad}^*_{\mathfrak{g}^*})\) is a matched pair of endo Lie algebras;
2. There is a Manin triple of endo Lie algebras associated to the endo Lie algebras \((\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g}, \phi)\) and \((\mathfrak{g}^*, [\cdot, \cdot]_{\mathfrak{g}^*}, \psi^*)\);
3. The triple \((\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g}, \delta)\) is a Lie bialgebra. Furthermore, the linear operators \(\phi^*\) and \(\psi\) dually represent \((\mathfrak{g}^*, \psi^*)\) and \((\mathfrak{g}, \phi)\) respectively.

Proof. The equivalence (a) \(\iff\) (b) is given in Theorem 2.17.

By definition, Item (a) means that \(((\mathfrak{g}, \phi), (\mathfrak{g}^*, \psi^*), \text{ad}^*_\mathfrak{g}, \text{ad}^*_{\mathfrak{g}^*})\) is a matched pair of Lie algebras and that the dual representation conditions in Item (c) hold. Since the matched pair condition is equivalent to the triple \(((\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g}, \delta)\) being a Lie bialgebra by Theorem 2.18, we obtain the equivalence (a) \(\iff\) (c).

Definition 2.22. An endo Lie coalgebra is a Lie coalgebra \((\mathfrak{g}, \delta)\) together with a Lie coalgebra endomorphism \(\psi : \mathfrak{g} \rightarrow \mathfrak{g}\), that is, a \(\psi \in \text{End}(\mathfrak{g})\) such that

\[(\psi \otimes \psi)\delta = \delta \psi.\]  

(22)

Under the finite-dimension condition, Eq. (22) is equivalent to the condition that \(\psi^* : \mathfrak{g}^* \rightarrow \mathfrak{g}^*\) is an endomorphism of the Lie algebra \(\mathfrak{g}^*\).

Note that the dual representation conditions in Theorem 2.21. (c) are either defined or can be rephrased as follows without referring to the dual space \(\mathfrak{g}^*\) and its operations:

\[\text{id} \otimes \phi \delta = (\psi \otimes \text{id})\delta \phi,\]  

(23)

\[\psi[\phi(x), y]_\mathfrak{g} = [x, \psi(y)]_\mathfrak{g}, \quad \forall x, y \in \mathfrak{g}.\]  

(24)

We are thus led to the key notion of endo Lie bialgebras that applies to vector spaces of any dimensions and indeed to any modules, just like its classical counterpart of Lie bialgebras.

Definition 2.23. An endo Lie bialgebra is a quintuple \(((\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g}, \delta, \phi, \psi)\) or simply a triple \(((\mathfrak{g}, \phi), \delta, \psi)\) in which

1. \(((\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g}, \delta)\) is a Lie bialgebra,
2. \(((\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g}, \phi)\) is an endo Lie algebra,
3. \((\mathfrak{g}, \delta, \psi)\) is an endo Lie coalgebra,
4. the compatibility conditions in Eqs. (23) – (24) are satisfied.
Returning to the finite dimensional case, we immediately have

**Corollary 2.24.** Consider a quintuple \((g, [ , ]_g, \delta, \phi, \psi)\) where \((g, \phi)\) is an endo Lie algebra. Then the quintuple is an endo Lie bialgebra if and only if anyone (and hence all) of the equivalent conditions in Theorem 2.21 is satisfied.

### 2.5. Homomorphisms of Lie bialgebras and Manin triples

As both the main motivation and application of our study of endo Lie bialgebras, we introduce a new notion of homomorphisms of Lie bialgebras that is compatible with those of Manin triples and matched pairs. Then we compare this notion with the existing notion of Lie bialgebra homomorphisms.

#### 2.5.1. New homomorphisms for Lie bialgebras

We can rewrite Definition 2.23 in terms of morphisms of Lie bialgebras.

**Definition 2.25.** A coherent endomorphism on a Lie bialgebra \((g, [ , ]_g, \delta)\) consists of a Lie algebra homomorphism \(\phi : g \to g\) and a Lie coalgebra homomorphism \(\psi : g \to g\) satisfying Eqs. (23) – (24).

Then we immediately have

**Proposition 2.26.** The quintuple \((g, [ , ]_g, \delta, \phi, \psi)\) is an endo Lie bialgebra if and only if \((\phi, \psi)\) is a coherent endomorphism of the Lie bialgebra \((g, [ , ]_g, \delta)\).

Definition 2.25 motivates us to give the following notion of homomorphisms between any two Lie bialgebras.

**Definition 2.27.** Let \((g, [ , ]_g, \delta_g)\) and \((h, [ , ]_h, \delta_h)\) be Lie bialgebras. A coherent homomorphism of Lie bialgebras from \((g, [ , ]_g, \delta_g)\) to \((h, [ , ]_h, \delta_h)\) is a pair \((\phi, \psi)\) of linear maps such that

\[
\text{(a)} \quad \phi : g \to h \text{ is a homomorphism of Lie algebras,} \\
\text{(b)} \quad \psi : h \to g \text{ a homomorphism of Lie coalgebras,} \\
\text{(c)} \quad \text{the polarizations of Eq. (23) – (24) hold:}
\]

\[
(id_g \otimes \phi)\delta_g = (\psi \otimes id_h)\delta_h \phi, \quad (25) \\
\psi([\phi(x), y])_h = [x, \psi(y)]_g, \quad \forall x \in g, y \in h. \quad (26)
\]

If both \(\phi\) and \(\psi\) are bijective, the pair is called a coherent isomorphism of Lie bialgebras. Let LB denote the category of Lie bialgebras with its morphisms thus defined.

The benefit of coherent homomorphisms of Lie bialgebras is that it is compatible with the following naturally defined morphisms of Manin triples derived from Manin triples of endo Lie algebras.

**Definition 2.28.** Let \((g \bowtie g^*, g, g^*)\) and \((h \bowtie h^*, h, h^*)\) be Manin triples of Lie algebras. A coherent homomorphism between them is a Lie algebra homomorphism

\[
f : g \bowtie g^* \to h \bowtie h^*
\]

that restricts to Lie algebra homomorphisms

\[
f|_g : g \to h, \quad f|_{g^*} : g^* \to h^*.
\]

If \(f\) is bijective, it is called a coherent isomorphism of Manin triples. Let MT denote the category of Manin triples with the morphisms thus defined.

This notion is justified by the equivalence in the case of endomorphisms.
Proposition 2.29. Let \((\mathfrak{g} \bowtie \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)\) be a Manin triple of Lie algebras. Then a linear map \(f \in \text{End}(\mathfrak{g} \bowtie \mathfrak{g}^*)\) is a coherent endomorphism of the Manin triple if and only if \(((\mathfrak{g} \bowtie \mathfrak{g}^*, f), (\mathfrak{g}, f|_{\mathfrak{g}}), (\mathfrak{g}^*, f|_{\mathfrak{g}^*}))\) is a Manin triple of endo Lie algebras associated to \((\mathfrak{g}, f|_{\mathfrak{g}})\) and \((\mathfrak{g}^*, f|_{\mathfrak{g}^*})\).

Due to the correspondence between endo Lie bialgebras and Manin triples of endo Lie algebras given in Theorem 2.21, we obtain

Proposition 2.30. Let \((\mathfrak{g}, [\ , \ ]_{\mathfrak{g}}, \delta)\) be a Lie bialgebra and \((\mathfrak{g} \bowtie \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)\) be the corresponding Manin triple. Then \((\phi, \psi)\) is a coherent endomorphism of the Lie bialgebra \((\mathfrak{g}, [\ , \ ]_{\mathfrak{g}}, \delta)\) if and only if \(\phi + \psi^*\) is a coherent endomorphism of the Manin triple \((\mathfrak{g} \bowtie \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)\).

Now we show that the correspondence of Lie bialgebras with Manin triples established by Theorems 2.12 and 2.18 gives rise to an equivalence of the corresponding categories \(\text{LB}\) and \(\text{MT}\).

Proposition 2.31. Assume that all the spaces are finite dimensional.

(a) Let \((\mathfrak{g}, [\ , \ ]_{\mathfrak{g}}, \delta_{\mathfrak{g}})\) and \((\mathfrak{h}, [\ , \ ]_{\mathfrak{h}}, \delta_{\mathfrak{h}})\) be Lie bialgebras. Let \((\mathfrak{g} \bowtie \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)\) and \((\mathfrak{h} \bowtie \mathfrak{h}^*, \mathfrak{h}, \mathfrak{h}^*)\) be the corresponding Manin triples of Lie algebras. There is a bijection between the set \(\text{Hom}_{\text{LB}}(\mathfrak{g}, \mathfrak{h})\) of coherent homomorphisms between the Lie bialgebras and the set \(\text{Hom}_{\text{MT}}(\mathfrak{g} \bowtie \mathfrak{g}^*, \mathfrak{h} \bowtie \mathfrak{h}^*)\) of coherent homomorphisms between the Manin triples. The bijection is given by sending \((\phi, \psi)\) to \(f \coloneqq \phi + \psi^*\) and sending \(f\) to \((f|_{\mathfrak{g}},(f|_{\mathfrak{g}^*})^*)\).

(b) The correspondence in (a) gives an equivalence from the category \(\text{LB}\) of Lie bialgebras to the category \(\text{MT}\) of Manin triples.

Proof. (a). Assume that \((\phi, \psi)\) is a coherent homomorphism of the Lie bialgebras. Let \(x, y \in \mathfrak{g}, a^*, b^* \in \mathfrak{g}^*\). Then for \(f \coloneqq \phi + \psi^* : \mathfrak{g} \bowtie \mathfrak{g}^* \to \mathfrak{h} \bowtie \mathfrak{h}^*\) we have

\[
f([x + a^*, y + b^*]_{\mathfrak{g}^*}) = \phi([x, y]_{\mathfrak{g}}) + \phi(\text{ad}_{\mathfrak{g}^*}(a^*)y - \text{ad}_{\mathfrak{g}^*}(b^*)x) + \psi^*(\text{ad}_{\mathfrak{g}^*}^*(x)b^* - \text{ad}_{\mathfrak{g}^*}^*(y)a^*) + \psi^*([a^*, b^*]_{\mathfrak{g}^*}).
\]

\[
[f(x + a^*), f(y + b^*)]_{\mathfrak{h}^*} = [\phi(x), \phi(y)]_{\mathfrak{h}} + \text{ad}_{\mathfrak{h}^*}^*(\psi^*(a^*))\phi(y) - \text{ad}_{\mathfrak{h}^*}^*(\psi^*(b^*))\phi(x) + \text{ad}_{\mathfrak{h}^*}^*(\phi(x))\psi^*(b^*) - \text{ad}_{\mathfrak{h}^*}^*(\phi(y))\psi^*(a^*) + [\psi^*(a^*), \psi^*(b^*)]_{\mathfrak{h}^*}.
\]

Since \(\phi : \mathfrak{g} \to \mathfrak{h}\) and \(\psi^* : \mathfrak{g}^* \to \mathfrak{h}^*\) are homomorphisms of Lie algebras, we have

\[\phi([x, y]_{\mathfrak{g}}) = [\phi(x), \phi(y)]_{\mathfrak{h}}, \quad \psi^*([a^*, b^*]_{\mathfrak{g}^*}) = [\psi^*(a^*), \psi^*(b^*)]_{\mathfrak{h}^*}.
\]

By Eq. (25), we have

\[\phi(\text{ad}_{\mathfrak{g}^*}^*(a^*)y) = \text{ad}_{\mathfrak{h}^*}^*(\psi^*(a^*))\phi(y), \quad \phi(\text{ad}_{\mathfrak{g}^*}^*(b^*)x) = \text{ad}_{\mathfrak{h}^*}^*(\psi^*(b^*))\phi(x).
\]

By Eq. (26), we have

\[\psi^*(\text{ad}_{\mathfrak{h}^*}^*(x)b^*) = \text{ad}_{\mathfrak{h}^*}^*(\phi(x))\psi^*(b^*), \quad \psi^*(\text{ad}_{\mathfrak{h}^*}^*(y)a^*) = \text{ad}_{\mathfrak{h}^*}^*(\phi(y))\psi^*(a^*).
\]

Therefore \(f\) is a coherent homomorphism of Manin triples.

Conversely, by a similar argument, if \(f\) is a coherent homomorphism of Manin triples, then \((f|_{\mathfrak{g}}, (f|_{\mathfrak{g}^*})^*)\) is a coherent homomorphism of the corresponding Lie bialgebras.

(b). It follows from (a) directly. \(\square\)

Due to the correspondence between matched pairs and Manin triples of Lie algebras, we can define the coherent homomorphism of matched pairs of Lie algebras directly from Definition 2.28.
Definition 2.32. Let \((g, g^*, \text{ad}^*_g, \text{ad}^*_g)\) and \((h, h^*, \text{ad}^*_h, \text{ad}^*_h)\) be matched pairs of Lie algebras. A coherent homomorphism between them is a Lie algebra homomorphism

\[ f : g \rightarrow h \]

that restricts to Lie algebra homomorphisms

\[ f|_g : g \rightarrow h, \quad f|_{g^*} : g^* \rightarrow h^*. \]

If \(f\) is bijective, it is called a coherent isomorphism of matched pairs. Let \(MP\) denote the category of such matched pairs of Lie algebras with the morphisms thus defined.

Remark 2.33. We only define the coherent homomorphisms for matched pairs of the form \((g, g^*, \text{ad}^*_g, \text{ad}^*_g)\). This is enough for our purpose. A coherent homomorphism between any two matched pairs can be defined in the same way.

On the other hand, by Theorem 2.10 we also have the following compatibility of coherent homomorphisms of matched pairs.

Proposition 2.34. Let \((g, g^*, \text{ad}^*_g, \text{ad}^*_g)\) be a matched pair of Lie algebras. Then a linear map \(f : g \wedge g^* \rightarrow g \wedge g^*\) is a coherent endomorphism of this matched pair of Lie algebras if and only if \(((g, f|_g), (g^*, f|_{g^*}), \text{ad}^*_g, \text{ad}^*_g)\) is a matched pair of endo Lie algebras.

We further have

Proposition 2.35. (a) Let \((g \wedge g^*, g, g^*)\) and \((h \wedge h^*, h, h^*)\) be Manin triples of Lie algebras and let \((g, g^*, \text{ad}^*_g, \text{ad}^*_g)\) and \((h, h^*, \text{ad}^*_h, \text{ad}^*_h)\) be the corresponding matched pairs of Lie algebras. Then a linear map \(f : g \wedge g^* \rightarrow h \wedge h^*\) is a coherent homomorphism of Manin triples if and only if \(f\) is a coherent homomorphism of matched pairs.

(b) The correspondence in (a) gives an equivalence from the category \(MP\) of matched pairs of the form \((g, g^*, \text{ad}^*_g, \text{ad}^*_g)\) to the category \(MT\) of Manin triples.

Proof. (a) follows directly from Definitions 2.28 and 2.32.

(b) By (a), there is a bijection between the set of coherent homomorphisms \(f : g \wedge g^* \rightarrow h \wedge h^*\) of Manin triples and the set of coherent homomorphisms \(f : g \wedge g^* \rightarrow h \wedge h^*\) of matched pairs by sending \(f\) to \(f\) itself. This gives the desired equivalence. \(\square\)

Combining Propositions 2.31 and 2.35, we have the following three way equivalence of categories.

Theorem 2.36. Under the finite-dimensional assumption, the following categories are equivalent.

(a) the category \(LB\) of Lie bialgebras;
(b) the category \(MT\) of Manin triples;
(c) the category \(MP\) of matched pairs of the form \((g, g^*, \text{ad}^*_g, \text{ad}^*_g)\).

2.5.2. Comparison with the existing notion of morphisms of Lie bialgebras. We now compare the notion of coherent homomorphisms of Lie bialgebras in Definition 2.27 with the existing notions of homomorphisms of Lie bialgebras and Manin triples [5].

Definition 2.37. ([5]) A homomorphism of Lie bialgebras from \((g, [\ , \ ]_g, \delta_g)\) to \((h, [\ , \ ]_h, \delta_h)\) is a linear map \(f : g \rightarrow h\) that is both a Lie algebra homomorphism and a Lie coalgebra homomorphism: \(\delta_h f = (f \otimes f) \delta_g\). If \(f\) is also bijective, then \(f\) is called an isomorphism of Lie bialgebras.
The following result shows that in the bijective case, our notion of coherent homomorphisms coincides with the usual notion of isomorphisms of Lie bialgebras.

**Proposition 2.38.** Let \((g, [, ], \delta_g)\) and \((h, [, ], \delta_h)\) be two Lie bialgebras. Then \((g, [, ], \delta_g)\) is isomorphic to \((h, [, ], \delta_h)\) if and only if there exists a Lie algebra isomorphism \(\phi: g \to h\) such that \((\phi, \phi^{-1})\) is a coherent isomorphism.

**Proof.** Take \(\psi = \phi^{-1}\). Then Eqs. (25) and (26) hold automatically. Moreover, \(\phi^*\) is an isomorphism of Lie algebras if and only if \(\phi^{-1}\) is an isomorphism of Lie algebras. \(\square\)

**Remark 2.39.** However, in general, homomorphisms of Lie bialgebras and coherent homomorphisms of Lie bialgebras are not related. For example, when \(g = h\), \(f + f^*\) is usually not an endomorphism of the Lie algebra \(g \triangleright\triangleright g^*\).

Next we consider another notion of homomorphisms of Manin triples of Lie algebras, originated from the notion of isomorphisms of Manin triples [5].

**Definition 2.40.** Let \((g \triangleright\triangleright g^*, g, g^*)\) and \((h \triangleright\triangleright h^*, h, h^*)\) be Manin triples of Lie algebras. A **strong homomorphism** between them is a Lie algebra homomorphism \(f: g \triangleright\triangleright g^* \to h \triangleright\triangleright h^*\) that restricts to Lie algebra homomorphisms \(f|_g: g \to h\) and \(f|_{g^*}: g^* \to h^*\) and is compatible with the bilinear forms from the Manin triples:

\[
\mathcal{B}_{g,d}(x, y) = \mathcal{B}_{h,d}(f(x), f(y)), \quad \forall x, y \in g \triangleright\triangleright g^*.
\] (27)

A bijective strong homomorphism between two Manin triples is exactly the known notion of an **isomorphism** between two Manin triples [5]. In general, a strong homomorphism of Manin triples is a coherent homomorphism plus the compatibility condition in Eq. (27). This extra condition has a quite significant consequence.

**Proposition 2.41.** Let \((g \triangleright\triangleright g^*, g, g^*)\) and \((h \triangleright\triangleright h^*, h, h^*)\) be two Manin triples of Lie algebras. Let \((\phi, \psi)\) be a strong homomorphism between them. Then \(\psi\phi = \text{id}\). In particular \(\phi\) is injective and \(\psi\) is surjective.

**Proof.** By Eq. (27), we have \(\mathcal{B}_{g,d}(x, a) = \mathcal{B}_{h,d}(\phi(x), \psi^*(a)) = \mathcal{B}_{g,d}(\psi\phi(x), a)\) for all \(x \in g, a \in g^*\). Hence \(\psi\phi = \text{id}\). \(\square\)

We make the following remarks on the various notions of homomorphisms of Lie bialgebras and of Manin triples.

**Remark 2.42.**

(a) It is also known that an isomorphism of Lie bialgebras amounts to an isomorphism of the corresponding Manin triples.

(b) In general, the homomorphisms of Lie bialgebras in Definition 2.37 do not correspond to strong homomorphisms of Manin triples in Definition 2.40. Indeed, since the underlying Lie algebra \(g\) in a Lie bialgebra corresponds to the Lie subalgebra in a Manin triple \((g \triangleright\triangleright g^*, g, g^*)\), it is naturally expected that the homomorphism of the Lie bialgebra is given by \(f|_g\) in the homomorphism \(f = f|_g + f|_{g^*}\) of the Manin triple. Then by Proposition 2.41, the homomorphism \(f|_g\) of Lie bialgebras will need to be injective. This is an unusually strong restriction.
(c) The injectivity condition is due to the compatibility condition in Eq. (27). Eliminating this condition leaves us with the notion of coherent homomorphisms of Manin triples in Definition 2.32. As shown in Proposition 2.31, this notion is compatible with the notion of coherent homomorphisms of Lie bialgebras in Definition 2.27.

To finish the discussion in this section, we compare the notion of coherent homomorphisms of Lie bialgebras with the notion of weak homomorphisms introduced in [20].

Definition 2.43. Let \((\mathfrak{g}, [\ , \ ], \delta_1)\) and \((\mathfrak{g}, [\ , \ ], \delta_2)\) be Lie bialgebras. A weak homomorphism from \((\mathfrak{g}, [\ , \ ], \delta_2)\) to \((\mathfrak{g}, [\ , \ ], \delta_1)\) consists of a Lie algebra homomorphism \(\phi : \mathfrak{g} \to \mathfrak{g}\) and a Lie coalgebra homomorphism \(\psi : (\mathfrak{g}, \delta_2) \to (\mathfrak{g}, \delta_1)\) such that

\[
\psi(\phi(x), y) = [x, \psi(y)], \quad \forall x, y \in \mathfrak{g}.
\] (28)

If in addition, both \(\phi\) and \(\psi\) are linear isomorphisms, then \((\phi, \psi)\) is called a weak isomorphism from \((\mathfrak{g}, [\ , \ ], \delta_2)\) to \((\mathfrak{g}, [\ , \ ], \delta_1)\).

Note that the above notions of weak homomorphisms and weak isomorphisms are defined only when the two Lie bialgebras have the same underlying Lie algebra \(\mathfrak{g}\). Further they are mainly available for the triangular Lie bialgebras in [20], that is, they are constructed from skew-symmetric classical \(r\)-matrices. In this case, Eq. (26) is the same as Eq. (28), and Eq. (25) holds automatically, which implies that the two notions of coherent and weak homomorphisms of Lie bialgebras coincide.

3. Coboundary endo Lie bialgebras and homomorphisms of classical \(r\)-matrices

In this section, we study coboundary endo Lie bialgebras and introduce the notions of coherent homomorphisms of classical \(r\)-matrices. A coherent homomorphism between two classical \(r\)-matrices gives a coherent homomorphism of their corresponding Lie bialgebras.

3.1. Coboundary endo Lie bialgebras. Let \(\mathfrak{g}\) be a Lie algebra. For a given \(r \in \mathfrak{g} \otimes \mathfrak{g}\), define \(\delta_r : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}\) by

\[
\delta_r(x) := (\text{id} \otimes \text{ad}(x) + \text{ad}(x) \otimes \text{id})(r), \quad \forall x \in \mathfrak{g}.
\] (29)

The following is an important construction of Lie bialgebras.

Proposition 3.1. ([5]) Let \((\mathfrak{g}, [\ , \ ])\) be a Lie algebra and \(r \in \mathfrak{g} \otimes \mathfrak{g}\). Then \((\mathfrak{g}, [\ , \ ], \delta_r)\) is a Lie bialgebra, which is called a coboundary Lie bialgebra, if and only if for all \(x \in \mathfrak{g}\),

\[
(\text{ad}(x) \otimes \text{id} + \text{id} \otimes \text{ad}(x))(r + \tau(r)) = 0,
\] (30)

\[
(\text{ad}(x) \otimes \text{id} \otimes \text{id} + \text{id} \otimes \text{ad}(x) \otimes \text{id} + \text{id} \otimes \text{id} \otimes \text{ad}(x))(\delta_{[r_{12}, r_{13}]} + \delta_{[r_{13}, r_{23}]} + \delta_{[r_{12}, r_{23}]}) = 0.
\] (31)

Here \(\tau : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}\) is the flip map and, writing \(r = \sum_i a_i \otimes b_i\), we denote

\[
[r_{12}, r_{13}] = \sum_{i,j} [a_i, a_j] \otimes b_i \otimes b_j, \quad [r_{13}, r_{23}] = \sum_{i,j} a_i \otimes [a_j, b_j], \quad [r_{23}, r_{12}] = \sum_{i,j} a_j \otimes [a_i, b_j] \otimes b_i.
\]

For endo Lie bialgebras, we similarly define

Definition 3.2. An endo Lie bialgebra \(((\mathfrak{g}, \phi), \delta, \psi)\) is called coboundary if \(\delta = \delta_r\) for some \(r \in \mathfrak{g} \otimes \mathfrak{g}\).

Then we obtain
Theorem 3.3. Let \((g, \phi)\) be an endo Lie algebra and \(\psi\) dually represent \((g, \phi)\). Let \(r \in g \otimes g\). Then the linear map \(\delta_r\) induces an endo Lie bialgebra \(((g, \phi), \delta_r, \psi)\) if and only if \(r\) satisfies Eqs. (30) and (31), and for any \(x \in g\), the following conditions hold:

\[
(\psi \text{ad}(x) \otimes \text{id})(\text{id} \otimes \psi - \phi \otimes \text{id})(r) + (\text{id} \otimes \psi \text{ad}(x))(\psi \otimes \text{id} - \text{id} \otimes \phi)(r) = 0, \quad (32)
\]

\[
(\text{id} \otimes \psi \text{ad}(x))(\text{id} \otimes \psi - \phi \otimes \text{id})(r) = 0. \quad (33)
\]

Proof. By Definition 2.23 and Proposition 3.1, \(((g, \phi), \delta_r, \psi)\) is an endo Lie bialgebra if and only if \(r\) satisfies Eqs. (30) and (31), and Eqs. (22) and (23) hold. Let \(r = \sum_i a_i \otimes b_i\). For \(x \in g\), we have

\[
(\psi \otimes \psi)\delta_r(x) - \delta_r\psi(x)
\]

\[
= \sum_i (\psi([x, a_i]) \otimes \psi(b_i) + \psi(a_i) \otimes \psi([x, b_i]) - [\psi(x), a_i] \otimes b_i - a_i \otimes [\psi(x), b_i])
\]

\[
= (\psi \text{ad}(x) \otimes \text{id})(\text{id} \otimes \psi)(r) + (\text{id} \otimes \psi \text{ad}(x))(\psi \otimes \text{id})(r)
\]

\[
- \sum_i (\psi(x, \phi(a_i)) \otimes b_i + a_i \otimes \psi(x, \phi(b_i))
\]

\[
= (\psi \text{ad}(x) \otimes \text{id})(\text{id} \otimes \psi)(r) + (\text{id} \otimes \psi \text{ad}(x))(\psi \otimes \text{id})(r)
\]

\[
- (\psi \text{ad}(x) \otimes \text{id})(\psi \otimes \text{id})(r) - (\text{id} \otimes \psi \text{ad}(x))(\text{id} \otimes \phi)(r)
\]

\[
= (\text{id} \otimes \phi \text{ad}(x))(\text{id} \otimes \psi)(r) + (\text{id} \otimes \phi \text{ad}(x))(\text{id} \otimes \phi)(r)
\]

Thus Eq. (22) holds if and only if Eq. (32) holds.

Similarly we have

\[
(\text{id} \otimes \phi)\delta_r(x) - (\psi \otimes \phi)\delta_r\phi(x)
\]

\[
= \sum_i ([x, a_i] \otimes \phi(b_i) + a_i \otimes [\phi(x), b_i]) - \psi[\phi(x), a_i] \otimes b_i - \psi(a_i) \otimes [\phi(x), b_i]
\]

\[
= (\text{id} \otimes \phi \text{ad}(x))(\text{id} \otimes \phi)(r) + (\text{id} \otimes \phi \text{ad}(x))(\text{id} \otimes \phi)(r)
\]

\[
- (\text{id} \otimes \phi \text{ad}(x))(\text{id} \otimes \phi)(r) - (\text{id} \otimes \phi \text{ad}(x))(\text{id} \otimes \text{id})(r)
\]

\[
= (\text{id} \otimes \phi \text{ad}(x))(\text{id} \otimes \text{id})(r) + (\text{id} \otimes \phi \text{ad}(x))(\text{id} \otimes \phi)(r)
\]

Thus Eq. (23) holds if and only if Eq. (33) holds. Therefore the conclusion holds.

Consequently we have the following conclusion on coherent homomorphisms of Lie bialgebras.

Corollary 3.4. Let \((g, [\ , \ ])\) be a Lie algebra and \(\phi : g \to g\) be a Lie algebra endomorphism. Let \(\psi : g \to g\) be a linear map satisfying Eq. (24). Let \(r \in g \otimes g\). Then \(((g, [\ , \ ]), \delta_r, \psi)\) is a Lie bialgebra and \((\phi, \psi)\) is a coherent endomorphism of Lie bialgebras if and only if Eqs. (30)–(33) are satisfied.

As an application of Theorem 3.3, we obtain the following doubles from endo Lie bialgebras which are analogues of doubles from Lie bialgebras.

Theorem 3.5. Let \(((g, \phi), \delta, \psi)\) be an endo Lie bialgebra. Let \(\alpha : g^* \to g^* \otimes g^*\) be the linear dual of the multiplication on \(g\). Then \(((g^*, \psi^*), -\alpha, \phi^*)\) is also an endo Lie bialgebra. Further there is an endo Lie bialgebra structure on the direct sum \(g \oplus g^*\) of the underlying vector spaces of \(g\) and \(g^*\) which contains the two endo Lie bialgebras as endo Lie sub-bialgebras.
Proof. Denote the product on the Lie algebra $\mathfrak{g}^*$ by $[\ , \ ]_{\mathfrak{g}^*}$. By [5], $(\mathfrak{g}^*, [\ , \ ]_{\mathfrak{g}^*}, -\alpha)$ is a Lie bialgebra. Moreover, $\psi$ dually represents the endo Lie algebra $(\mathfrak{g}, \phi)$ whose algebra structure is given by $-\alpha$ if and only if $\psi$ dually represents the endo Lie algebra $(\mathfrak{g}, \phi)$ whose algebra structure is given by $\alpha$. Therefore with the fact that $\phi^*$ dually represents $(\mathfrak{g}^*, \psi^*)$, we obtain that $((\mathfrak{g}^*, \psi^*), -\alpha, \phi^*)$ is an endo Lie bialgebra.

Let $\{e_1, e_2, \cdots, e_n\}$ be a basis of $\mathfrak{g}$ and $\{e^1, e^2, \cdots, e^n\}$ its dual basis. Let $r = \sum_{i=1}^n e_i \otimes e^i$. Consider the Lie algebra $\mathfrak{g} \rtimes \mathfrak{g}^*$ induced by the matched pair $(\mathfrak{g}, \mathfrak{g}^*, \text{ad}_{\psi}, \text{ad}_{\phi}^*)$. Define

$$
\delta_r(u) = (\text{id} \otimes \text{ad}_{\mathfrak{g} \rtimes \mathfrak{g}^*}^\ast(u) + \text{ad}_{\mathfrak{g} \rtimes \mathfrak{g}^*}^\ast(u) \otimes \text{id})(r), \forall u \in \mathfrak{g} \rtimes \mathfrak{g}^*.
$$

By Lemma 2.16, $(\mathfrak{g} \rtimes \mathfrak{g}^*, \phi + \psi^*)$ is an endo Lie algebra that is dually represented by $(\psi + \phi^*)$. Hence Eq. (24) holds. Since

$$
((\phi + \psi^*) \otimes \text{id} - \text{id} \otimes (\psi + \phi^*))(r) = \sum_{i=1}^n (\phi(e_i) \otimes e^i - e_i \otimes \phi^*(e^i)) = 0,
$$

$$
((\psi + \phi^*) \otimes \text{id} - \text{id} \otimes (\phi + \psi^*))(r) = \sum_{i=1}^n (\psi(e_i) \otimes e^i - e_i \otimes \psi^*(e^i)) = 0,
$$

Eqs. (32)–(33) hold. By [5], we know that $r$ satisfies Eqs. (30) and (31) and hence $(\mathfrak{g} \rtimes \mathfrak{g}^*, [\ , \ ]_{\mathfrak{g} \rtimes \mathfrak{g}^*}, \delta_r)$ is a Lie bialgebra containing $(\mathfrak{g}, [\ , \ ]_{\mathfrak{g}}, \delta)$ and $(\mathfrak{g}^*, [\ , \ ]_{\mathfrak{g}^*}, -\alpha)$ as Lie subbialgebras. Thus $(\mathfrak{g} \rtimes \mathfrak{g}^*, \phi + \psi^*, \delta_r, \psi, \phi^*)$ is an endo Lie bialgebra. It is obvious that it contains $((\mathfrak{g}, \phi), \delta, \psi)$ and $((\mathfrak{g}^*, \psi^*), -\alpha, \phi^*)$ as endo Lie subbialgebras. This completes the proof.

Therefore there is the following construction of coherent homomorphisms on the doubles of Lie bialgebras.

**Corollary 3.6.** Let $(\mathfrak{g}, [\ , \ ], \delta)$ be a Lie bialgebra and $(\phi, \psi)$ be a coherent endomorphism. Let $\alpha : \mathfrak{g} \rightarrow \mathfrak{g}^* \otimes \mathfrak{g}^*$ be the linear dual of the multiplication on $\mathfrak{g}$. Then $(\mathfrak{g}^*, -\alpha^*)$ is a Lie bialgebra and $(\psi^*, \phi^*)$ is a coherent endomorphism. Furthermore, there is a Lie bialgebra structure on the direct sum $\mathfrak{g} \oplus \mathfrak{g}^*$ of the underlying vector spaces of $\mathfrak{g}$ and $\mathfrak{g}^*$ which contains the two Lie bialgebras as Lie subbialgebras and $(\phi + \psi^*, \psi + \phi^*)$ is a coherent homomorphism.

### 3.2. Homomorphisms of classical $r$-matrices

We now lift the relation between classical $r$-matrices and bialgebras to the level of categories. Theorem 3.3 immediately gives

**Corollary 3.7.** Let $(\mathfrak{g}, \phi)$ be an endo Lie algebra and $\psi$ dually represent $(\mathfrak{g}, \phi)$. Let $r \in \mathfrak{g} \otimes \mathfrak{g}$. Then the linear map $\delta_r$ given by Eq. (29) induces an endo Lie bialgebra $((\mathfrak{g}, \phi), \delta_r, \psi)$ if Eq. (30) and the following equations hold:

$$
[r_{12}, r_{13}] + [r_{13}, r_{23}] + [r_{12}, r_{23}] = 0, \quad (34)
$$

$$
(\phi \otimes \text{id} - \text{id} \otimes \psi)(r) = 0, \quad (35)
$$

$$
(\psi \otimes \text{id} - \text{id} \otimes \phi)(r) = 0. \quad (36)
$$

Eq. (34) is just the classical Yang-Baxter equation (CYBE) in a Lie algebra and a solution of the CYBE in a Lie algebra is also called a classical $r$-matrix. A Lie bialgebra $(\mathfrak{g}, [\ , \ ], \delta)$ is called quasi-triangular if it is obtained from a classical $r$-matrix $r$ by Eq. (29) and is called triangular if it is obtained from a skew-symmetric classical $r$-matrix $r$ (i.e. $r = -\tau(r)$). Moreover, it is straightforward to show that if $r$ is skew-symmetric, then Eq. (35) holds if and only if Eq. (36) holds.
Definition 3.8. Let \((\mathfrak{g}, \phi)\) be an endo Lie algebra. Let \(r \in \mathfrak{g} \otimes \mathfrak{g}\) and \(\psi \in \text{End}(\mathfrak{g})\). Then Eq. (34) with conditions given by Eqs. (35) and (36) is called the \(\psi\)-classical Yang-Baxter equation (\(\psi\)-CYBE) in \((\mathfrak{g}, \phi)\).

As in the case of endo Lie bialgebras, solutions of the \(\psi\)-CYBE in an endo Lie algebra motivates us to the notion of morphisms of classical \(r\)-matrices that is compatible with coherent homomorphisms of Lie bialgebras.

Definition 3.9. Let \(\mathfrak{g}, \mathfrak{h}\) be Lie algebras and \(r_\mathfrak{g}, r_\mathfrak{h}\) be classical \(r\)-matrices in \(\mathfrak{g}\) and \(\mathfrak{h}\) respectively. A coherent homomorphism from \(r_\mathfrak{g}\) to \(r_\mathfrak{h}\) consists of a Lie algebra homomorphism \(\phi: \mathfrak{g} \rightarrow \mathfrak{h}\) and a linear map \(\psi: \mathfrak{h} \rightarrow \mathfrak{g}\) satisfying

\[
(\psi \otimes \text{id}_\mathfrak{h})(r_\mathfrak{h}) = (\text{id}_\mathfrak{g} \otimes \phi)(r_\mathfrak{g}),
\]

\[
(\text{id}_\mathfrak{h} \otimes \psi)(r_\mathfrak{h}) = (\phi \otimes \text{id}_\mathfrak{g})(r_\mathfrak{g}),
\]

\[
\psi[\phi(x), y]_\mathfrak{h} = [x, \psi(y)]_\mathfrak{g}, \quad \forall x \in \mathfrak{g}, y \in \mathfrak{h}.
\]

If \(\phi\) and \(\psi\) are also linear isomorphisms, then \((\phi, \psi)\) is called a coherent isomorphism from \(r_\mathfrak{g}\) to \(r_\mathfrak{h}\). Let \(\text{Cr}\) denote the category of classical \(r\)-matrices with the morphisms thus defined.

Then by Definitions 3.8 and 3.9, we obtain

Proposition 3.10. Let \((\mathfrak{g}, \phi)\) be an endo Lie algebra and \(\psi \in \text{End}(\mathfrak{g})\) dually represent \((\mathfrak{g}, \phi)\). Then \(r \in \mathfrak{g} \otimes \mathfrak{g}\) is a solution of the \(\psi\)-CYBE in the endo Lie algebra \((\mathfrak{g}, \phi)\) if and only if \(r \in \mathfrak{g} \otimes \mathfrak{g}\) is a classical \(r\)-matrix and \((\phi, \psi)\) is a coherent endomorphism on \(r\).

Recall from [5] that two classical \(r\)-matrices \(r_1\) and \(r_2\) in a Lie algebra \(\mathfrak{g}\) are said to be equivalent if there is a Lie algebra isomorphism \(\phi: \mathfrak{g} \rightarrow \mathfrak{g}\) such that \((\phi \otimes \phi)(r_1) = r_2\).

Corollary 3.11. Let \(\mathfrak{g}\) be a Lie algebra and \(r_1, r_2\) be classical \(r\)-matrices in \(\mathfrak{g}\). Then \(r_1\) is equivalent to \(r_2\) if and only if there exists a Lie algebra isomorphism \(\phi: \mathfrak{g} \rightarrow \mathfrak{g}\) such that \((\phi, \phi^{-1})\) is a coherent isomorphism from \(r_1\) to \(r_2\).

Proof. If \(\phi: \mathfrak{g} \rightarrow \mathfrak{g}\) is an equivalence from \(r_1\) to \(r_2\), then it is straightforward to check that \((\phi, \phi^{-1})\) satisfies Eqs. (37)–(39). Conversely, Eqs. (37) implies \((\phi \otimes \phi)(r_1) = r_2\).

Remark 3.12. When both \(r_1\) and \(r_2\) are skew-symmetric in the same Lie algebra \(\mathfrak{g}\), then Eq. (37) holds if and only if Eq. (38) holds. On the other hand, there is a notion of weak homomorphism between two skew-symmetric \(r\)-matrices in a Lie algebra \(\mathfrak{g}\) given in [20] defined by Eqs. (37) and (39) only, that is, without Eq. (38). The two notions coincide in the skew-symmetric case. Note that Definition 3.9 is valid without the skew-symmetric restriction and in different Lie algebras.

Recall that for a Lie algebra \(\mathfrak{g}\) and a classical \(r\)-matrix \(r\) of in \(\mathfrak{g}\) satisfying Eq. (30), the triple \((\mathfrak{g}, [\cdot, \cdot], \delta_r)\) is a quasi-triangular Lie bialgebra. On the level of categories, we obtain

Theorem 3.13. Let \(\mathfrak{g}, \mathfrak{h}\) be Lie algebras and \(r_\mathfrak{g}, r_\mathfrak{h}\) be classical \(r\)-matrices in \(\mathfrak{g}\) and \(\mathfrak{h}\) respectively satisfying Eq. (30). If \((\phi, \psi)\) is a coherent homomorphism of the classical \(r\)-matrices from \(r_\mathfrak{g}\) to \(r_\mathfrak{h}\), then \((\phi, \psi)\) is a coherent homomorphism of the corresponding Lie bialgebras from \((\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g}, \delta_{r_\mathfrak{g}})\) to \((\mathfrak{g}, [\cdot, \cdot]_\mathfrak{h}, \delta_{r_\mathfrak{h}})\). This correspondence defines a functor from the category \(\text{Crs}\) of classical \(r\)-matrices satisfying Eq. (30), as a full sub-category of \(\text{Cr}\) to the category \(\text{QTLB}\) of quasi-triangular Lie bialgebras, as a full sub-category of \(\text{LB}\).
Proof. For the given pair \((\phi, \psi)\), an argument similar to the proof of Theorem 3.3, yields

(a) \((\psi \otimes \psi)\delta_{r_{\mathfrak{g}}} = \delta_{r_{\mathfrak{g}}}\phi\) if and only if for any \(x \in \mathfrak{h}\),

\[
\begin{align*}
& (\psi_{\text{ad}}(x) \otimes \text{id}_\mathfrak{g})((\text{id}_\mathfrak{g} \otimes \psi)(r_{\mathfrak{g}}) - (\phi \otimes \text{id}_\mathfrak{g})(r_{\mathfrak{g}})) \\
& + (\text{id}_\mathfrak{g} \otimes \psi_{\text{ad}}(x))((\psi \otimes \text{id}_\mathfrak{g})(r_{\mathfrak{g}}) - (\text{id}_\mathfrak{g} \otimes \phi)(r_{\mathfrak{g}})) = 0.
\end{align*}
\]

(b) \((\text{id}_\mathfrak{g} \otimes \phi)\delta_{r_{\mathfrak{g}}} = (\psi \otimes \text{id}_\mathfrak{g})\delta_{r_{\mathfrak{g}}}\phi\) if and only if for any \(x \in \mathfrak{g}\),

\[
(\text{ad}_\mathfrak{g}(x) \otimes \text{id}_\mathfrak{g} + \text{id}_\mathfrak{g} \otimes \text{ad}_\mathfrak{g}\phi(x))((\psi \otimes \text{id}_\mathfrak{g})(r_{\mathfrak{g}}) - (\text{id}_\mathfrak{g} \otimes \phi)(r_{\mathfrak{g}})) = 0.
\]

Thus \((\phi, \psi)\) is a coherent homomorphism of Lie bialgebras from \((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \delta_{\mathfrak{g}})\) to \((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{h}}, \delta_{\mathfrak{h}})\). The rest of the proof is straightforward.

Specializing to the skew-symmetric case gives the conclusion [20, Proposition 7.17], obtained by a different approach using \(\mathcal{O}\)-operators.

4. Homomorphisms of \(\mathcal{O}\)-operators and pre-Lie algebras

We define \(\mathcal{O}\)-operators on endo Lie algebras and endo pre-Lie algebras. Then an analysis similar to the previous sections motivates us to define coherent homomorphisms of \(\mathcal{O}\)-operators and of pre-Lie algebras, giving rise to the categories of \(\mathcal{O}\)-operators and of pre-Lie algebras. Moreover, there are natural functors between these categories and further to the category of triangular Lie bialgebras, completing Diagram (4). As in the previous sections, underneath these functors among categories, there are correspondences among the \(\mathcal{O}\)-operator and pre-Lie algebra structures on endo Lie algebras. The discussion is similar and will not be elaborated further.

4.1. \(\mathcal{O}\)-operators on endo Lie algebras and homomorphisms of \(\mathcal{O}\)-operators. For a vector space \(\mathfrak{g}\), through the isomorphism \(\mathfrak{g} \otimes \mathfrak{g} \cong \text{Hom}(\mathfrak{g}^*, K) \otimes \mathfrak{g} \cong \text{Hom}(\mathfrak{g}^*, \mathfrak{g})\), any \(r \in \mathfrak{g} \otimes \mathfrak{g}\) is identified with a map from \(\mathfrak{g}^*\) to \(\mathfrak{g}\) which we denote by \(r^\sharp\). Explicitly, writing \(r = \sum_i a_i \otimes b_i\), then

\[
r^\sharp : \mathfrak{g}^* \to \mathfrak{g}, \quad r^\sharp(a^*) = \sum_i \langle a^*, a_i \rangle b_i, \quad \forall a^* \in \mathfrak{g}^*.
\]

Note that \(r\) is skew-symmetric if and only if

\[
\langle r^\sharp(a^*), b^* \rangle + \langle a^*, r^\sharp(b^*) \rangle = 0, \quad \forall a^*, b^* \in \mathfrak{g}^*.
\]

Recall that an \(\mathcal{O}\)-operator on a Lie algebra \(\mathfrak{g}\) associated to a representation \((V, \rho)\) is a linear map \(T : V \to \mathfrak{g}\) satisfying

\[
[T(u), T(v)] = T(\rho(T(u))v - \rho(T(v))u), \quad \forall u, v \in V.
\]

For an endo Lie algebra, the corresponding notion is

**Definition 4.1.** Let \((\mathfrak{g}, \phi)\) be an endo Lie algebra. Let \((V, \rho)\) be a representation of the Lie algebra \(\mathfrak{g}\) and \(\alpha : V \to V\) be a linear map. A linear map \(T : V \to \mathfrak{g}\) is called an \(\mathcal{O}\)-operator on \((\mathfrak{g}, \phi)\) associated to \((V, \rho)\) and \(\alpha\) if \(T\) satisfies Eq. (44) and

\[
\phi T = T \alpha.
\]

If in addition, \((V, \rho, \alpha)\) is a representation of \((\mathfrak{g}, \phi)\), then \(T\) is called an \(\mathcal{O}\)-operator associated to \((V, \rho, \alpha)\).
We have the following relationship between $\mathcal{O}$-operators for endo Lie algebras and solutions of the CYBE in endo Lie algebras.

Proposition 4.2. Let $(\mathfrak{g}, \phi)$ be an endo Lie algebra and $\psi : \mathfrak{g} \to \mathfrak{g}$ be a linear map. Suppose that $r \in \mathfrak{g} \otimes \mathfrak{g}$ is skew-symmetric. Then $r$ is a solution of the $\psi$-CYBE in $(\mathfrak{g}, \phi)$ if and only if $r^\sharp$ is an $\mathcal{O}$-operator associated to $(\mathfrak{g}^*, \text{ad}^*)$ and $\psi^*$. If in addition, $\psi$ dually represents $(\mathfrak{g}, \phi)$, then $r$ is a solution of the $\psi$-CYBE in $(\mathfrak{g}, \phi)$ if and only if $r^\sharp$ is an $\mathcal{O}$-operator on $(\mathfrak{g}, \phi)$ associated to the representation $(\mathfrak{g}^*, \text{ad}^*, \psi^*)$.

Proof. By [12], $r$ is a solution of the CYBE in the Lie algebra $\mathfrak{g}$ if and only if

$$[r^\sharp(a^*), r^\sharp(b^*)] = r^\sharp(\text{ad}^*(r^\sharp(a^*))b^* - \text{ad}^*(r^\sharp(b^*))a^*), \quad \forall a^*, b^* \in \mathfrak{g}^*,$$

that is, $r^\sharp : \mathfrak{g}^* \to \mathfrak{g}$ is an $\mathcal{O}$-operator on $\mathfrak{g}$ associated to the representation $(\mathfrak{g}^*, \text{ad}^*)$.

Moreover, let $r = \sum a_i \otimes b_i$ and for any $a^* \in \mathfrak{g}^*$, we have

$$r^\sharp(\psi^*(a^*)) = \sum_{i=1}^n \langle \psi^*(a^*), a_i \rangle b_i = \sum_{i=1}^n \langle a^*, \psi(a_i) \rangle b_i, \quad \phi(r^\sharp(a^*)) = \sum_{i=1}^n \langle a^*, a_i \rangle \phi(b_i).$$

So, $\phi r^\sharp = r^\sharp \psi^*$ if and only if Eq. (35) holds. This completes the proof. □

We next show that the notion of $\mathcal{O}$-operators for endo Lie algebras naturally gives the following notion of morphisms of $\mathcal{O}$-operators for Lie algebras.

Definition 4.3. Let $T_\mathfrak{g}$ and $T_\mathfrak{h}$ be $\mathcal{O}$-operators on Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ associated to representations $(V_\mathfrak{g}, \rho_\mathfrak{g})$ and $(V_\mathfrak{h}, \rho_\mathfrak{h})$ respectively. A (coherent) homomorphism of $\mathcal{O}$-operators from $T_\mathfrak{g}$ to $T_\mathfrak{h}$ consists of a Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ and a linear map $\alpha : V_\mathfrak{g} \to V_\mathfrak{h}$ such that for all $x \in \mathfrak{g}, v \in V_\mathfrak{g}$,

$$\alpha \rho_\mathfrak{g}(x)(v) = \rho_\mathfrak{h}(\phi(x))(\alpha(v)), \quad T_\mathfrak{h} \alpha = \phi T_\mathfrak{g}.$$

In particular, if $\phi$ and $\alpha$ are invertible, then $(\phi, \alpha)$ is called an isomorphism from $T_\mathfrak{g}$ to $T_\mathfrak{h}$. Let $\mathcal{OP}$ denote the category of $\mathcal{O}$-operators with the morphisms thus defined.

Indeed, we immediately have

Corollary 4.4. Let $(\mathfrak{g}, \phi)$ be an endo Lie algebra. Let $(V, \rho)$ be a representation of the Lie algebra $\mathfrak{g}$ and $\alpha : \mathfrak{g} \to \mathfrak{g}$ be a linear map. Then $(V, \rho, \alpha)$ is a representation of $(\mathfrak{g}, \phi)$ and $T$ is an $\mathcal{O}$-operator on $(\mathfrak{g}, \phi)$ associated to $(V, \rho, \alpha)$ if and only if $T$ is an $\mathcal{O}$-operator on the Lie algebra $\mathfrak{g}$ associated to the representation $(V, \rho)$ and $(\phi, \alpha)$ is an endomorphism on the $\mathcal{O}$-operator $T$.

We now show that the notion of coherent homomorphisms of $\mathcal{O}$-operators is the correct one to be compatible with classical $r$-matrices.

Theorem 4.5. Let $r_\mathfrak{g}$, $r_\mathfrak{h}$ be skew-symmetric classical $r$-matrices in Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Let $\phi : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism and $\psi : \mathfrak{h} \to \mathfrak{g}$ be a linear map. Then $(\phi, \psi)$ is a coherent homomorphism of classical $r$-matrices from $r_\mathfrak{g}$ to $r_\mathfrak{h}$ if and only if $(\phi, \psi^*)$ is a homomorphism between the corresponding $\mathcal{O}$-operators $r_\mathfrak{g}^\sharp$ and $r_\mathfrak{h}^\sharp$.

This correspondence defines an equivalence from the category $\mathcal{SCr}$ of skew-symmetric classical $r$-matrices, as a full subcategory of $\mathcal{Cr}$ of classical $r$-matrices satisfying Eq. (30), to the category $\mathcal{SOP}_{\text{coad}}$ of $\mathcal{O}$-operators on Lie algebras associated to the coadjoint representations satisfying Eq. (43), as a full subcategory of $\mathcal{OP}$ of $\mathcal{O}$-operators.
Lemma 4.7. Let \( (g, \rho) \) be a representation. Let \( T : V \to g \) be a linear map which is identified as an element in \( (g \ltimes \rho, V^*) \otimes (g \ltimes \rho, V^*) \) (through \( \text{Hom}(V, g) \cong V^* \otimes g \subseteq (g \ltimes \rho, V^*) \otimes (g \ltimes \rho, V^*) \)). Then

\[
T_{r_T} := T - \sigma(T)
\]

is a skew-symmetric classical \( r \)-matrix in the Lie algebra \( g \ltimes \rho, V^* \) if and only if \( T \) is an \( \mathcal{O} \)-operator on \( g \) associated to \( (V, \rho) \).

Remark 4.6. When \( g = h \), the above conclusion is exactly [20, Proposition 7.10].

4.2. From \( \mathcal{O} \)-operators to classical \( r \)-matrices. In Section 4.1, we see that a skew-symmetric solution of the CYBE gives an \( \mathcal{O} \)-operator associated to the adjoint representation. Going in the opposite direction, an \( \mathcal{O} \)-operator gives rises to a solution of the CYBE in the semi-direct product Lie algebra as follows.

Lemma 4.7. ([1]) Let \( g \) be a Lie algebra and \( (V, \rho) \) be a representation. Let \( T : V \to g \) be a linear map which is identified as an element in \( (g \ltimes \rho, V^*) \otimes (g \ltimes \rho, V^*) \) (through \( \text{Hom}(V, g) \cong V^* \otimes g \subseteq (g \ltimes \rho, V^*) \otimes (g \ltimes \rho, V^*) \)). Then

\[
T_{r_T} := T - \sigma(T)
\]

is a skew-symmetric classical \( r \)-matrix in the Lie algebra \( g \ltimes \rho, V^* \) if and only if \( T \) is an \( \mathcal{O} \)-operator on \( g \) associated to \( (V, \rho) \).

We now lift Lemma 4.7 to the level of morphisms, that is, to use a coherent homomorphism of \( \mathcal{O} \)-operators on Lie algebras to induce a coherent homomorphism of classical \( r \)-matrices in the corresponding semi-direct product Lie algebras. Since the latter Lie algebras are much larger, extra restraints are needed to give a well-defined correspondence.

Theorem 4.8. Let \( g \) and \( h \) be Lie algebras, and \( (g, \rho_g), (V, \rho_h) \) be representations of \( g \) and \( h \) respectively. Let \( T_g : V_g \to g \) and \( T_h : V_h \to h \) be \( \mathcal{O} \)-operators, and \( r_{T_g} \) and \( r_{T_h} \) be the corresponding skew-symmetric classical \( r \)-matrices defined in Lemma 4.7. Let \( (\phi, \alpha) \) be a homomorphism of \( \mathcal{O} \)-operators from \( T_g \) to \( T_h \). Then for linear maps \( \psi : h \to g \) and \( \beta : V_h \to V_g \), the pair \( (\phi + \beta^*, \psi + \alpha^*) \) is a coherent homomorphism from \( r_{T_g} \) to \( r_{T_h} \) if and only if Eq. (39) and the following equations hold

\[
T_g \beta = \psi T_h,
\]

\[
\beta(\rho_h(\phi(x))b) = \rho_g(x)(\beta(b)), \quad \forall x \in g, b \in V_h,
\]

\[
\beta(\rho_g(y)\alpha(a)) = \rho_h(\psi(y))a, \quad \forall y \in h, a \in V_g.
\]

Proof. It is straightforward to deduce that the linear map \( \phi + \beta^* : g \ltimes \rho_g V^* \to h \ltimes \rho_h V^* \) is a homomorphism of Lie algebras if and only if \( \phi \) is a homomorphism of Lie algebras and Eq. (51) holds. Let \( \{e_1, e_2, \ldots, e_n\} \) be a basis of \( V_g \) and \( \{e^1, e^2, \ldots, e^n\} \) be its dual basis. Let \( \{f_1, f_2, \ldots, f_m\} \) be a basis of \( V_h \) and \( \{f^1, f^2, \ldots, f^m\} \) be its dual basis. Then the 2-tensors of the \( \mathcal{O} \)-operators \( T_g \) and \( T_h \) are \( \sum_{i=1}^{n} T_g(e_i) \otimes e^i \) and \( \sum_{j=1}^{m} T_h(f_j) \otimes f^j \) respectively. Hence
\[ r_{T_g} = \sum_{i=1}^{n} (T_g(e_i) \otimes e^i - e^i \otimes T_g(e_i)) \]
\[ r_{T_h} = \sum_{j=1}^{m} (T_h(f_j) \otimes f^j - f^j \otimes T_h(f_j)) \]
giving
\[ ((\phi + \beta^*) \otimes \text{id}_{g \ltimes r_0^* V^*_g})(r_{T_g}) = \sum_{i=1}^{n} (\phi T_g(e_i) \otimes e^i - \beta^*(e^i) \otimes T_g(e_i)), \]
\[ (\text{id}_{h \ltimes r_0^* V^*_h} \otimes (\psi + \alpha^*))(r_{T_h}) = \sum_{j=1}^{m} (T_h(f_j) \otimes \alpha^*(f^j) - f^j \otimes \psi T_h(f_j)). \]
Further,
\[ \sum_{i=1}^{n} \beta^*(e^i) \otimes T_g(e_i) = \sum_{i=1}^{n} \sum_{j=1}^{m} (\beta^*(e^i), f^j) f^j \otimes T_g(e_i) = \sum_{j=1}^{m} f^j \otimes \sum_{i=1}^{n} (\beta(f^j), e^i) e_i = \sum_{j=1}^{m} f^j \otimes T_h(\sum_{i=1}^{n} (\beta(f^j), e^i) e_i), \]
and similarly, \[ \sum_{j=1}^{m} T_h(f_j) \otimes \alpha^*(f^j) = \sum_{j=1}^{m} T_h(\alpha(e_i) \otimes e^i). \]

Then we obtain
\[ ((\phi + \beta^*) \otimes \text{id}_{g \ltimes r_0^* V^*_g})(r_{T_g}) = (\text{id}_{h \ltimes r_0^* V^*_h} \otimes (\psi + \alpha^*))(r_{T_h}) \]
if and only if Eqs. (48) and (50) hold. One similarly derives that
\[ (\text{id}_{g \ltimes r_0^* V^*_g} \otimes (\phi + \beta^*))(r_{T_g}) = ((\psi + \alpha^*) \otimes \text{id}_{h \ltimes r_0^* V^*_h})(r_{T_h}) \]
if and only if Eqs. (48) and (50) hold.

Finally, it is straightforward to deduce that Eq. (39) holds (where \( g \) is replaced by \( g \ltimes r_0^* V^*_g \), \( h \) by \( h \ltimes r_0^* V^*_h \), \( \phi \) by \( \phi + \beta^* \) and \( \psi \) by \( \psi + \alpha^* \)) if and only if Eqs. (39), (47) and (52) hold. Therefore the proof is completed. \( \square \)

Applying Theorems 4.8 and 3.13, we obtain a large supply of coherent homomorphisms of Lie bialgebras.

**Corollary 4.9.** Under the assumption of Theorem 4.8, if the linear maps \( \phi, \alpha, \psi, \beta \) satisfy Eqs. (39) and (50)–(52), then the pair \( (\phi + \beta^*, \psi + \alpha^*) \) is a coherent homomorphism between the Lie bialgebras \( (g \ltimes r_0^* V^*_g, \delta_{T_g}) \) and \( (h \ltimes r_0^* V^*_h, \delta_{T_h}) \).

To obtain from Theorem 4.8 a functor from the category of \( \mathcal{O} \)-operators and the category of classical \( r \)-matrices, there needs to be a consistent choice of \( \psi \) and \( \beta \).

**Corollary 4.10.** Under the assumption of Theorem 4.8, the assignment of objects
\[ (T : V_g \rightarrow g) \mapsto r_T, \]
and the assignment of morphisms
\[ ((\phi, \alpha) : T_g \rightarrow T_h) \mapsto ((\phi, \alpha^*) : r_{T_g} \rightarrow r_{T_h}) \]
define a functor from the category \( \mathcal{OP} \) of \( \mathcal{O} \)-operators to the category \( \text{SCr} \) of skew-symmetric classical \( r \)-matrices.

**Proof.** Under the assumption of Theorem 4.8, it is obvious that \( \beta = \psi = 0 \) satisfies Eqs. (39) and (50)–(52). Hence the conclusion holds. \( \square \)

The following two examples give other choices for \( \beta \) and \( \psi \) in Theorem 4.8.
Example 4.11. In Theorem 4.8, take \( g = h, V_g = V_h = V \) and \( \rho_g = \rho_h = \rho \). Further take \( \beta = \pm \alpha \) and \( \psi = \pm \phi \). According to Theorem 4.8, assume that \( T : V \to g \) is an \( O \)-operator on \( g \) associated to \((V, \rho)\) satisfying \( T \alpha = \phi T \) and the following equations

\[
\alpha \rho(x) = \rho(\phi(x)) \alpha, \quad \rho(x) \alpha = \alpha \rho(\phi(x)),
\]

\[
[\phi^2(x), \phi(y)] = [x, \phi(y)], \quad \alpha \rho(x) \alpha = \rho(\phi(x)), \quad \forall x \in g.
\]

Then \( r_T = T - \sigma(T) \) is a skew-symmetric classical \( r \)-matrix in the Lie algebra \( g \otimes_r V^* \) and \((\phi \pm \alpha^*, \pm \phi + \alpha^*)\) is a coherent isomorphism. Moreover, there is a Lie bialgebra \((g \otimes_r V^*, \delta_{r_T})\) and \((\phi \pm \alpha^*, \pm \phi + \alpha^*)\) is a coherent isomorphism.

Example 4.12. Let Lie algebras \( g, h \), representations \((V_g, \rho_g), (V_h, \rho_h)\), linear maps \( \phi, \alpha, T_g, T_h \) be as in Theorem 4.8. Further assume that \((\phi, \alpha)\) is an isomorphism of \( O \)-operators from \( T_g \) to \( T_h \), that is, \( \phi \) and \( \alpha \) are linear bijections. For \( 0 \neq \theta \in K \), take \( \beta = \theta \alpha^{-1} \) and \( \psi = \theta \phi^{-1} \). Then Eq. (39) holds automatically since it is equivalent to the fact that \( \phi \) is an isomorphism of Lie algebras from \( g \) to \( h \). Also Eqs. (50), (51) and (52) hold since \((\phi, \alpha)\) is an isomorphism of \( O \)-operators from \( T_g \) to \( T_h \). Thus by Theorem 4.8, \((\phi + \theta \alpha^{-1}, \theta \phi^{-1} + \alpha^*)\) is a coherent isomorphism between the skew-symmetric classical \( r \)-matrices \( r_{r_g} \) and \( r_{r_h} \). Furthermore, \((\phi + \theta \alpha^{-1}, \theta \phi^{-1} + \alpha^*)\) is a coherent isomorphism between the Lie bialgebras \((g \otimes_{r_g} V^*, \delta_{r_T})\) and \((h \otimes_{r_h} V^*, \delta_{r_T})\) in Corollary 4.9.

4.3. Functors among \( O \)-operators, pre-Lie algebras and Lie bialgebras. Now we consider the category of pre-Lie algebras and obtain an adjoint pair of functors from it to the category of \( O \)-operators.

Definition 4.13. Let \( A \) be a vector space with a bilinear product denoted by \( \cdot \). Then \((A, \cdot)\) is called a \textit{pre-Lie algebra} if

\[
x \cdot (y \cdot z) - (x \cdot y) \cdot z = y \cdot (x \cdot z) - (y \cdot x) \cdot z, \quad \forall x, y, z \in A.
\]

A \textit{homomorphism} \( f \) from a pre-Lie algebra \((A, \cdot_A)\) to \((B, \cdot_B)\) is defined as usual, that is, \( f \) is a linear map satisfying \( f(x \cdot_A y) = f(x) \cdot_B f(y) \) for all \( x, y \in A \). Let \textbf{PL} denote the category of pre-Lie algebras with the morphisms thus defined.

For any \( a \) in a pre-Lie algebra \( A \), let \( L(a) \) denote the left multiplication operator. Furthermore, define the linear map

\[
L : A \to \text{End}(A), \quad a \mapsto L(a), \quad \forall a \in A.
\]

As is well known, for a pre-Lie algebra \((A, \cdot)\), the multiplication

\[
[a, b] = a \cdot b - b \cdot a, \quad \forall a, b \in A
\]

defines a Lie algebra \((g(A), [ \cdot, \cdot ])\), called the \textit{sub-adjacent Lie algebra} of \((A, \cdot)\). Moreover, \((A, L)\) is a representation of the Lie algebra \((g(A), [ \cdot, \cdot ])\) \([1]\).

Definition 4.14. An \textit{endo pre-Lie algebra} is a triple \((A, \cdot, \phi)\) consisting of a pre-Lie algebra \((A, \cdot)\) and a pre-Lie algebra endomorphism \( \phi \) on \( A \).

Proposition 4.15. Let \((A, \cdot, \phi)\) be an endo pre-Lie algebra. Then \((g(A), [ \cdot, \cdot ], \phi)\) is an endo Lie algebra, where \([ \cdot, \cdot ]\) is given by Eq. (56). Moreover, \((A, L, \phi)\) is a representation of the endo Lie algebra \((g(A), [ \cdot, \cdot ], \phi)\). Conversely, let \((g(A), [ \cdot, \cdot ], \phi)\) be an endo Lie algebra. Suppose that there is a bilinear product denoted by \( \cdot \) such that Eq. (56) holds and \((A, L, \phi)\) is a representation of \((g(A), [ \cdot, \cdot ], \phi)\). Then \((A, \cdot, \phi)\) is an endo pre-Lie algebra.

Proof. It follows from a simple checking from Eq. (5) and Definition 4.14. \( \square \)
Remark 4.16. From the viewpoint of operads, due to the second half part of the above conclusion, the operad of endo pre-Lie algebras is the bisuccessor (2-splitting) of the operad of endo Lie algebras, which is consistent with the fact that the operad of pre-Lie algebras is the bisuccessor of the operad of Lie algebras [3].

Definition 4.17. Let \((A, \cdot, \phi)\) be an endo pre-Lie algebra. Let \([,]\) be the product given by Eq. (56). The triple \((g(A),[,],\phi)\) is called the sub-adjacent endo Lie algebra of \((A,\cdot,\phi)\) and \((A,\cdot,\phi)\) is called a compatible endo pre-Lie algebra structure on the endo Lie algebra \((g(A),[,],\phi)\).

Then Proposition 4.15 and Corollary 4.4 give the following conclusion.

Corollary 4.18. Let \((A,\cdot,\phi)\) be an endo pre-Lie algebra. Then the identity map \(id_A : A \rightarrow g(A)\) on \(A\) is an \(\mathcal{O}\)-operator of the sub-adjacent endo Lie algebra \((g(A),[,],\phi)\) associated to the representation \((A,L,\phi)\). Equivalently, let \(A\) be a pre-Lie algebra and \(\phi : A \rightarrow A\) be a pre-Lie homomorphism. Then the identity map \(id_A\) on \(A\) is an \(\mathcal{O}\)-operator of the sub-adjacent Lie algebra \(g(A)\) associated to the representation \((A,L)\) and \((\phi,\phi)\) is an endomorphism of \(\mathcal{O}\)-operators on \(id\).

The second half part of the above conclusion can be generalized as follows.

Proposition 4.19. Let \((A,\cdot_A)\) and \((B,\cdot_B)\) be pre-Lie algebras and \(\phi : A \rightarrow B\) be a pre-Lie algebra homomorphism. The assignments

\[
\begin{align*}
(A,\cdot) & \mapsto (id_A : A \rightarrow g(A)), \\
\phi & \mapsto ((\phi,\phi) : id_A \rightarrow id_B)
\end{align*}
\]

defines a functor \(F\) from the category PL of pre-Lie algebras to the category OP of \(\mathcal{O}\)-operators.

Proof. Since \(\phi\) is also a homomorphism between the sub-adjacent Lie algebras and \(id_B \phi = \phi id_A\), the pair \((\phi,\phi)\) is a (coherent) homomorphism between the \(\mathcal{O}\)-operators \(id_A\) and \(id_B\). The other axioms of functors are easy to verify. \(\square\)

In the other direction, by [1], for a representation \((V,\rho)\) of a Lie algebra \(g\) and an \(\mathcal{O}\)-operator \(T : V \rightarrow g\) on \(g\) associated to \((V,\rho)\), the operation \(u \cdot_T v := \rho(T(u))v, u,v \in V\), defines a pre-Lie algebra \((V,\cdot_T)\).

Proposition 4.20. Let \((g,[,]_g), (h,[,]_h)\) be Lie algebras and \((V_g,\rho_g), (V_h,\rho_h)\) be representations of \((g,[,]_g)\), \((h,[,]_h)\) respectively. Let \(T_g : V_g \rightarrow g\) be an \(\mathcal{O}\)-operator associated to \((V_g,\rho_g)\) and \(T_h : V_h \rightarrow h\) be an \(\mathcal{O}\)-operator associated to \((V_h,\rho_h)\). Suppose that \((\phi,\alpha)\) is a homomorphism of \(\mathcal{O}\)-operators from \(T_g\) to \(T_h\). Then \(\alpha\) is a homomorphism of pre-Lie algebras from \((V_g,\cdot_g)\) to \((V_h,\cdot_h)\). The assignments

\[
\begin{align*}
T_g & \mapsto (V_g,\cdot_g), \\
(\phi,\alpha) & \mapsto \alpha,
\end{align*}
\]

define a functor \(G\) from the category OP of \(\mathcal{O}\)-operators to the category PL of pre-Lie algebras.

Furthermore, the functor \(G\) is right adjoint to the functor \(F\) in Proposition 4.19.

Proof. For any \(u,v \in V_g\), we have

\[
\alpha(u \cdot_T g v) = \alpha(\rho_g(T_g(u))v) = \rho_h(\phi(T_g(u)))\alpha(v) = \rho_h(T_h(\alpha(u)))\alpha(v) = \alpha(u) \cdot_T h \alpha(v).
\]
Hence $\alpha$ is a homomorphism of pre-Lie algebras from $(V_{g}, T_{g})$ to $(V_{h}, T_{h})$. The other axioms of functors are easily verified.

To prove the adjointness of the functors $F$ and $G$, we only need to show that, for any $(A, \cdot) \in \mathbf{PL}$ and $T : W \rightarrow h$ in $\mathbf{OP}$, there is a bijection

$$\text{Hom}_{\mathbf{OP}}(F(A, \cdot), T) \cong \text{Hom}_{\mathbf{PL}}((A, \cdot), GT)$$

that is natural in both arguments. The left hand side consists of pairs $(\phi, \alpha)$ with $\phi : g(A) \rightarrow h$ and $\alpha : A \rightarrow W$ such that $\phi \text{id}_{A} = T\alpha$, that is, $\phi = T\alpha$, and the right hand side consists of pre-Lie algebra homomorphisms $\alpha : (A, \cdot) \rightarrow (W, \cdot T)$. Then we see that the natural bijection can be given by sending $(\phi, \alpha)$ to $\alpha$ whose inverse is sending $\alpha$ to $(T\alpha, \alpha)$. \hfill \Box

Utilizing Proposition 4.19, we apply homomorphisms of pre-Lie algebras to obtain three explicitly constructed examples of coherent homomorphisms of classical $r$-matrices and hence of Lie bialgebras.

**Proposition 4.21.** Let $(A, \cdot)$ be a pre-Lie algebra. Then $r_{\text{id}}$ is a skew-symmetric classical $r$-matrix in the Lie algebra $g(A) \ltimes_{L} A^{*}$ and $(g(A) \ltimes_{L} A^{*}, \delta_{r_{\text{id}}})$ is a Lie bialgebra. Furthermore, let $\phi : A \rightarrow A$ be an endomorphism of pre-Lie algebras satisfying

$$(\phi^{2} - \text{id})(x) \cdot \phi(y) = \phi(x) \cdot (\phi^{2} - \text{id})(y) = 0, \quad \forall x, y \in A.$$  \hfill (57)

Then $(\phi \pm \phi^{*}, \pm \phi + \phi^{*})$ is a coherent endomorphism on both the triangular $r$-matrix $r_{\text{id}}$ and the Lie bialgebra $(g(A) \ltimes_{L} A^{*}, \delta_{r_{\text{id}}})$.

**Proof.** Note that in this case, Eq. (57) is equivalent to Eqs. (53)–(54). Therefore the conclusion follows from Example 4.11. \hfill \Box

Now let $(A, \cdot A), (B, \cdot B)$ be pre-Lie algebras. Then $r_{\text{id}}_{A}$ and $r_{\text{id}}_{B}$ are skew-symmetric classical $r$-matrices in the Lie algebras $g(A) \ltimes_{L_{A}} A^{*}$ and $g(B) \ltimes_{L_{B}} B^{*}$ respectively. Moreover, $(g(A) \ltimes_{L_{A}} A^{*}, \delta_{r_{\text{id}}_{A}})$ and $(g(B) \ltimes_{L_{B}} B^{*}, \delta_{r_{\text{id}}_{B}})$ are triangular Lie bialgebras. We can thus give the following two results.

**Proposition 4.22.** Let $\phi : (A, \cdot A) \rightarrow (B, \cdot B)$ be a pre-Lie algebra homomorphism. Then $(\phi, \phi^{*})$ is both a coherent homomorphism of classical $r$-matrices from $r_{\text{id}}_{A}$ to $r_{\text{id}}_{B}$ and a coherent homomorphism of Lie bialgebras from $(g(A) \ltimes_{L_{A}} A^{*}, \delta_{r_{\text{id}}_{A}})$ to $(g(B) \ltimes_{L_{B}} B^{*}, \delta_{r_{\text{id}}_{B}})$.

**Proof.** The conclusion follows from Corollaries 4.10 and 4.9. \hfill \Box

**Proposition 4.23.** Let $\phi : (A, \cdot A) \rightarrow (B, \cdot B)$ and $\psi : (B, \cdot B) \rightarrow (A, \cdot A)$ be pre-Lie algebra homomorphisms such that $\psi \phi = \text{id}_{A}$. Then for any $0 \neq \theta \in K$, the pair $(\phi + \theta \psi^{*}, \theta \psi + \phi^{*})$ is both a coherent homomorphism of skew-symmetric classical $r$-matrices from $r_{\text{id}}_{A}$ to $r_{\text{id}}_{B}$ and a coherent homomorphism of Lie bialgebras from $(g(A) \ltimes_{L_{A}} A^{*}, \delta_{r_{\text{id}}_{A}})$ to $(g(B) \ltimes_{L_{B}} B^{*}, \delta_{r_{\text{id}}_{B}})$. In particular, if $\phi$ is a linear bijection, then $(\phi + \theta \phi^{-1}^{*}, \theta \phi^{-1} + \phi^{*})$ is both a coherent isomorphism of skew-symmetric classical $r$-matrices from $r_{\text{id}}_{A}$ to $r_{\text{id}}_{B}$ and a coherent isomorphism of Lie bialgebras from $(g(A) \ltimes_{L_{A}} A^{*}, \delta_{r_{\text{id}}_{A}})$ to $(g(B) \ltimes_{L_{B}} B^{*}, \delta_{r_{\text{id}}_{B}})$.

**Proof.** If $\psi$ is a homomorphism of pre-Lie algebras and $\psi \phi = \text{id}_{A}$, then $\theta \psi$ satisfies $\text{id}_{A} \theta \psi = \theta \psi \text{id}_{B}$, Eqs. (39), (51) and (52) where $\psi$ is replaced by $\theta \psi$ and $\beta$ by $\theta \psi$. Therefore the first conclusion follows from Theorem 4.8. The special case when $\phi$ is an isomorphism then follows directly or from Example 4.9. \hfill \Box
Remark 4.24. Note that when $\phi$ is invertible and $\theta \neq 0$, the inverse of $\phi + \theta \phi^{-1}^*$ is $\phi^{-1} + \theta^{-1} \phi^*$. Thus by Proposition 2.38, $\phi + \theta \phi^{-1}^*$ gives an isomorphism of Lie bialgebras from $(\mathfrak{g}(A) \ltimes_{L_A^\delta} A^*, \delta_{\text{id}_A})$ to $(\mathfrak{g}(B) \ltimes_{L_B^\delta} B^*, \delta_{\text{id}_B})$ if and only if $\theta = 1$. Therefore, when $\theta \neq 1$, the above isomorphism of pre-Lie algebras provide non-trivial examples of coherent isomorphisms of Lie bialgebras which are not the usual isomorphisms of Lie bialgebras.

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