On the domain of attraction for the lower tail in Wicksell’s corpuscle problem

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Abstract

We consider the classical Wicksell corpuscle problem with spherical particles in \( \mathbb{R}^n \) and investigate the shapes of lower tails of distributions of ‘sphere radii’ in \( \mathbb{R}^n \) and ‘sphere radii’ in a \( k \)-dimensional section plane \( E_k \). We show in which way the domains of attraction are related to each other.

1 Wicksell’s corpuscle problem

Suppose that a collection of spheres is randomly scattered in the Euclidean space \( \mathbb{R}^n \) and we intersect the collection with a \( k \)-dimensional section plane \( E_k \), \( k \in \{1, \ldots, n-1\} \). Then, under suitable model assumptions, the relation between the radii distribution \( F \) of spheres in \( \mathbb{R}^n \) and the radii distribution \( F^{(n,k)} \) of corresponding spheres in the section plane is given by Wicksell’s integral equation,

\[
F^{(n,k)}(x) = 1 - \frac{(n-k)}{M_{n-k}} \int_x^\infty u(u^2-x^2)^{\frac{n-k-2}{2}} (1-F(u)) \, du, \quad x > 0,
\]

(1.1)

where \( M_{n-k} \in (0, \infty) \) denotes the \( (n-k) \)th moment of \( F \). Relation (1.1) has been discovered in the mid-twenties by S.D. Wicksell \[6\] for the special case \( n = 3 \) and \( k = 2 \). A proof of (1.1) can be found in \[4\, Ch. 4.3\]. Note that the density of \( F^{(n,k)} \) always exists and is given by

\[
f^{(n,k)}(x) = \frac{x(n-k)}{M_{n-k}} \int_x^\infty (u^2-x^2)^{\frac{n-k-2}{2}} dF(u), \quad x > 0,
\]

(1.2)

which can easily be seen by differentiating (1.1). From (1.1) and (1.2) we see that the representations of \( F^{(n,k)} \) and \( f^{(n,k)} \) only depend on the difference \( n-k \). In the following we will therefore write \( F^{(r)} \) for \( F^{(n,k)} \), \( f^{(r)} \) for \( f^{(n,k)} \), and \( M_{r} \) for \( M_{n-k} \), where \( r = n-k \).

In recent years there has been some effort in creating a theory of stereology of extremes. Drees and Reiss \[2\] have investigated the shapes of upper tails of distributions of ‘sphere radii’ and ‘circle radii’ that are connected by (1.1). In this paper we investigate the lower tail behaviour.

2 Extreme value theory and regular variation

Let \( \{X_i, i \geq 1\} \) be a sequence of independent and identically distributed real valued random variables with common distribution function \( F \). Denote the sample minimum by

\[
W_n = \min(X_1, \ldots, X_n), \quad n \geq 1.
\]
If there are sequences of normalizing constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that the normalized minima $(W_n - b_n)/a_n$ converge in distribution to a nondegenerate distribution function $H$, then $F$ lies in the domain of attraction of $H$. We denote this by $F \in \mathcal{D}(H)$. There are only three types of possible limiting distributions $H$, see e.g. [3], namely

$$H_{i,\alpha}(x) = \begin{cases} 1 - \exp(-(x)^{-\alpha}) & x < 0, \quad i = 1, \\ 1 - \exp(-x^{\alpha}) & x > 0, \quad i = 2, \end{cases} \quad \alpha > 0,$$

and

$$H_3(x) = 1 - \exp(-e^x), \quad x \in \mathbb{R}.$$  

$H_{1,\alpha}$, $H_{2,\alpha}$ and $H_3$ are called Fréchet, Weibull and Gumbel distributions respectively. Recall the following condition for $F \in \mathcal{D}(H_{2,\alpha})$. Let $\eta = \inf\{x \in \mathbb{R} : F(x) > 0\}$ denote the lower endpoint of $F$. Then $F \in \mathcal{D}(H_{2,\alpha})$ if and only if $\eta > -\infty$ and for all $x > 0$,

$$\lim_{s \downarrow 0} \frac{F(\eta + xs)}{F(\eta + s)} = x^\alpha. \quad (2.1)$$

In the following we recall some basic facts from the theory of regular variation, see e.g. [5]. A function $R$ is regularly varying at infinity with exponent $\rho \in \mathbb{R}$ if it is real-valued, positive and measurable on $[x_0, \infty)$, for some $x_0 > 0$, and if for each $x > 0$

$$\lim_{t \to \infty} \frac{R(tx)}{R(t)} = x^\rho.$$  

We then write $R \in \text{RV}_\infty(\rho)$. A function $R(\cdot)$ is regularly varying at 0 if $R(\frac{1}{x})$ is regularly varying at infinity with exponent $(-\rho)$ and this is denoted by $R \in \text{RV}_0(\rho)$. If $\rho = 0$, $R$ is said to be slowly varying. A function $R(\cdot)$ is regularly varying if and only if it can be written in the form

$$R(x) = x^\rho L(x), \quad (2.2)$$

where $\rho \in \mathbb{R}$ and $L(\cdot)$ is slowly varying. Note that the sum of two slowly varying functions is again slowly varying. For the investigation of the lower tail in Wicksell’s corpuscle problem we need the following lemmas.

**Lemma 2.1.** If $R_1 \in \text{RV}_0(\beta_1)$ and $R_2 \in \text{RV}_0(\beta_2)$ then $R_1 + R_2 \in \text{RV}_0(\min\{\beta_1, \beta_2\})$.

**Lemma 2.2 (cf. Theorem 2.7 in [5]).** Let $L$ be slowly varying at 0 on $(0, \infty)$ and bounded on each finite subinterval of $(0, \infty)$. Suppose that for $\beta > 0$ the integral $\int_\beta^\infty t^\beta f(t) dt$ is well-defined for some given real function $f$ and a given number $\delta \geq 0$. Then as $u \downarrow 0$

$$\int_\beta^\infty f(t)L(ut) dt \sim L(u) \int_\beta^\infty f(t) dt$$

for $\delta > 0$, and also for $\delta = 0$ provided that $L$ is non-increasing on $(0, \infty)$.

**Lemma 2.3 (cf. Exercise 1.1.1. in [3]).** Let $F \in \mathcal{D}(H_3)$ and $\eta = \inf\{x \in \mathbb{R} : F(x) > 0\} > -\infty$. Then for all $n \geq 1$,

$$\lim_{x \downarrow \eta} (x - \eta)^{-n} F(x) = 0.$$
Lemma 2.4. Let \( f(\cdot, \cdot) \) and \( g(\cdot, \cdot) \) be positive functions such that
\[
\int_0^\omega f(s, t) \, dt < \infty, \quad \int_0^\omega g(s, t) \, dt < \infty
\]
for some \( \omega \in (0, \infty) \). Furthermore assume for \( s \leq t < \omega \)
\[
\lim_{s \uparrow \omega} \frac{f(s, t)}{g(s, t)} = c, \quad c \in [0, \infty].
\]
Then
\[
\lim_{s \uparrow \omega} \int_s^\omega f(s, t) \, dt \int_s^\omega g(s, t) \, dt = c.
\]

3 Tail behaviour in Wicksell’s corpuscle problem

In recent years the main theoretical tools for estimating stereologically the tail of a particle size distribution have been stability properties of the domain of attraction. Consider the spherical Wicksell corpuscle problem with corresponding distribution functions \( F \) and \( F^{(r)} \), \( r \geq 1 \), as described in Section 1.

Note that there is no need to investigate the behaviour for \( F \in \mathcal{D}(H_{1, \alpha}) \) since in this case \( \eta = \inf\{x \in \mathbb{R} : F(x) > 0\} = -\infty \). Let us first assume that \( r = 1 \) and \( F \in \mathcal{D}(H_{2, \alpha}) \). The radius of the spheres in \( \mathbb{R}^n \) is denoted by \( \xi \). We have

**Theorem 3.1.** Let \( F \in \mathcal{D}(H_{2, \alpha}) \) and \( \eta = 0 \). Then \( F^{(1)} \in \mathcal{D}(H_{2, \beta}) \), where
\[
\beta = \begin{cases} 
2, & \alpha > 1, \\
\alpha + 1, & 0 < \alpha \leq 1.
\end{cases}
\]
If \( \eta > 0 \), then \( F^{(1)} \in \mathcal{D}(H_{2, 2}) \).

**Proof.** Let us first assume that \( \eta = 0 \). It is easily seen from (1.2) that
\[
F^{(1)}(t) = \frac{1}{M_1} \left[ \int_0^t u \, dF(u) + \int_t^\infty (u - \sqrt{u^2 - t^2}) \, dF(u) \right] = \frac{1}{M_1} [I_1(t) + I_2(t)],
\]
where
\[
I_1(t) = TF(t) - \int_0^t F(u) \, du.
\]
Using integration by parts in the first summand of (3.1) we get
\[
I_1(t) = tF(t) - \int_0^t F(u) \, du.
\]
For all \( \alpha > 0 \) and \( x > 0 \) we show that
\[
\frac{I_1(tx)}{I_1(t)} \to x^{\alpha+1} \quad \text{as} \quad t \downarrow 0,
\]
i.e. \( I_1 \in RV_0(\alpha + 1) \). For that consider
\[
\frac{I_1(tx)}{I_1(t)} = \frac{tx F(tx) - \int_0^{tx} F(u) \, du}{tF(t) - \int_0^{t} F(u) \, du} = \frac{xF(tx)}{F(t)} \cdot \frac{1 - (tx F(tx))^{-1} \int_0^{tx} F(u) \, du}{1 - (tF(t))^{-1} \int_0^{t} F(u) \, du}.
\]
Define \( G(s) = s^{-2} F(\frac{1}{s}) \in RV_\infty(-\alpha - 2) \) and apply Karamata’s theorem [3, Th. 0.6] to obtain
\[
\lim_{t \downarrow 0} \frac{\int_0^t F(u) \, du}{tF(t)} = \lim_{z \to \infty} \frac{1}{zG(z)} = \frac{1}{\alpha + 1},
\]
and
\[
\lim_{t \downarrow 0} \frac{\int_t^\infty F(u) \, du}{t} = \frac{1}{\alpha + 1},
\]
and
\[
\lim_{t \downarrow 0} \frac{\int_0^t F(u) \, du}{t} = \frac{1}{\alpha + 1}.
\]
which proves (3.2).

For the second summand $I_2(t)$ we consider three cases.

Case I, $\alpha > 1$: Since $\alpha > 1$, $E(\xi^{-1}) < \infty$. Rewriting $I_2(t)$ as

$$I_2(t) = E \left( (\xi - \sqrt{\xi^2 - t^2}) I_{(\xi > t)} \right) = t^2 E \left( (\xi + \sqrt{\xi^2 - t^2})^{-1} I_{(\xi > t)} \right)$$

and applying the dominated convergence theorem we get

$$\lim_{t \downarrow 0} E \left( (\xi + \sqrt{\xi^2 - t^2})^{-1} I_{(\xi > t)} \right) = \frac{1}{2} E(\xi^{-1}) \in (0, \infty),$$

(3.3)

which implies $I_2 \in RV_0(2)$. By Lemma 2.1, $F^{(1)} \in RV_0(2)$, whence $F^{(1)} \in D(H_{2,2})$.

Case II, $\alpha \in (0, 1)$: We show that $I_2 \in RV_0(\alpha + 1)$, which implies $F^{(1)} \in RV_0(\alpha + 1)$ by Lemma 2.1. For that we use again integration by parts to arrive at

$$I_2(t) = -t F(t) - \int_t^\infty F(u) \left( 1 - \frac{u}{\sqrt{u^2 - t^2}} \right) du.$$  \hspace{1cm} (3.4)

Writing $F(t) = t^\alpha L(t)$ where $L$ is slowly varying at 0, and substituting $u = tv$ in the integral of (3.4) yields that

$$I_2(t) = -t^{\alpha+1} L(t) + t^{\alpha+1} \int_1^\infty L(tv)v^\alpha g(v) dv,$$

where

$$g(v) = \frac{v}{\sqrt{v^2 - 1}} - 1, \quad v > 1,$$  \hspace{1cm} (3.5)

is a probability density function. For $\delta \in (0, 1 - \alpha)$ we have $\int_1^\infty t^\delta t^\alpha g(t) dt < \infty$ and therefore Lemma 2.2 implies

$$h(t) := \int_1^\infty L(tv)v^\alpha g(v) dv \to 1$$  \hspace{1cm} as $t \downarrow 0,$

where $c := \int_1^\infty v^\alpha g(v) dv \in (1, \infty)$. Thus

$$I_2(t) = t^{\alpha+1} L(t) [ch(t) - 1] \in RV_0(\alpha + 1).$$

Case III, $\alpha = 1$: We show that $I_2 \in RV_0(2)$. Choose $A > 1$ such that $c(A) := \int_1^A v g(v) dv > 1$. From (3.4) we get

$$I_2(t) = t \left[ \int_1^A F(vt) g(v) dv - F(t) + \int_A^\infty F(vt) g(v) dv \right],$$  \hspace{1cm} (3.6)

where $g$ is given by (3.5). Consider

$$J_1(t) = \int_1^A F(vt) g(v) dv - F(t) = t \int_1^A L(vt) g(v) dv - tL(t),$$

where $L$ is slowly varying at 0. Using the uniform convergence theorem for slowly varying functions we get

$$h^*(t) := \frac{\int_1^A L(tv) g(v) dv}{c(A)L(t)} \to 1$$  \hspace{1cm} as $t \downarrow 0,$

whence

$$J_1(t) = tL(t)[c(A)h^*(t) - 1] \in RV_0(1).$$
The second summand in (3.6) can be written as

$$J_2(t) = \int_A F(vt)g(v)\,dv = \int_A P(v^{-1}\xi \leq t)g(v)\,dv = P(\xi^{-1}\xi \leq t, \xi > A),$$

where $\zeta$ is a random variable with probability density function $g$ independent of $\xi$. Furthermore

$$P(\xi^{-1}\xi \leq t, \zeta > A) = P(\xi^{-1}\xi \leq t | \xi > A)P(\zeta > A) = F_{\xi^{-1}\xi}(t)P(\zeta > A),$$

where $\tilde{\zeta}$ is a random variable with probability density function $\tilde{g} : (A, \infty) \to (0, \infty)$ given by

$$\tilde{g}(u) = c^{-1}g(u)$$

for $c = \int_A g(u)\,du$. It is easy to check that $F_{\tilde{\xi}^{-1}} \in RV_0(1)$. By assumption $F_{\tilde{\xi}} \in RV_0(1)$ and since $\xi$ and $\tilde{\zeta}^{-1}$ are independent we have $F_{\tilde{\xi}^{-1}\xi} \in RV_0(1)$ (see e.g. [1], Theorem 3 and Corollary therein) and therefore $J_2 \in RV_0(1)$. Hence

$$I_2(t) = t[J_1(t) + J_2(t)] \in RV_0(2).$$

Now let $\eta > 0$. For sufficiently small $t$ we get from (3.1)

$$F^{(1)}(t) = \frac{1}{M_1} \int_0^\infty (u - \sqrt{u^2 - t^2})\,dF(u) = \frac{t^2}{M_1} E(\xi + \sqrt{\xi^2 - t^2})^{-1}.$$

For all $\alpha > 0$ we have $E(\xi^{-\alpha}) < \infty$, whence $F^{(1)} \in D(H_{2,2})$, using the same argument as in (3.3). The proof is complete. \hfill \Box

The case of a Gumbel limiting distribution is considered in the next theorem.

**Theorem 3.2.** If $F \in D(H_3)$, then $F^{(1)} \in D(H_{2,2})$.

**Proof.** The case $\eta > 0$ is covered by Theorem 3.1. So let $\eta = 0$. From (3.1) we see that

$$F^{(1)}(t) = \frac{t}{M_1} [F_{\xi/\eta_1}(t) - F_{\xi/\eta_2}(t)],$$

where $\eta_1$ is a random variable with density $g$ given by (3.3) and $\eta_2$ is uniformly distributed on $[0, 1]$ and $\eta_1, \eta_2, \xi$ are independent. It suffices to show that $F_{\xi/\eta_1} \in RV_0(1)$ and $F_{\xi/\eta_1}(t)/F_{\xi/\eta_2}(t) \to 0$ as $t \downarrow 0$.

Let us first prove that $F_{\xi/\eta_1} \in RV_0(1)$. For that we write

$$F_{\xi/\eta_1}(t) = P(\xi \leq t\eta_1, \eta_1 \leq 1/t) + P(\xi \leq t\eta_1, \eta_1 > 1/t)$$

$$= \frac{1}{t} \int_0^1 F(u)g(u/t)\,du + \frac{1}{t} \int_1^\infty F(u)g(u/t)\,du$$

$$= J_1^*(t) + J_2^*(t). \tag{3.7}$$

For the second summand in (3.7) we can apply Lemma 2.3 and get

$$\frac{J_2^*(tx)}{J_2^*(t)} = x^{-1} \frac{\int_1^\infty F(tw) \left(\frac{w}{\sqrt{w^2-x^2}} - 1\right)\,dw}{\int_1^\infty F(tw) \left(\frac{w}{\sqrt{w^2-1}} - 1\right)\,dw} \to x^{-1} x^2 = x \quad \text{as } t \downarrow 0.$$

Since $g \in RV_\infty(-2)$ the first summand in (3.7) can be written as

$$J_1^*(t) = t \int_0^1 F(u)u^{-2}L(u/t)\,du,$$
where \(L(1/t)\) is slowly varying at 0.

Because of Lemma \[2.3\] \(\int_0^1 u^{-\delta-2}F(u)\,du < \infty\) for some \(\delta > 0\) and therefore we can apply \[5\] Theorem 2.7] and get

\[
h^{**}(t) = \frac{\int_0^1 F(u)u^{-2}L(u/t)\,du}{L(1/t)\int_0^1 F(u)u^{-2}\,du} \to 1 \quad \text{as } t \downarrow 0,
\]

i.e.

\[
J^{*}_1(t) = tL(1/t)h^{**}(t) \int_0^1 F(u)u^{-2}\,du \in RV_0(1).
\]

Next we show that \(F_{\xi/\eta_2}(t)/F_{\xi/\eta_1}(t) \to 0\) as \(t \downarrow 0\). Since \(t^{-2}F_{\xi/\eta_1}(t) \in RV_0(-1)\), we have

\[
\lim_{t \downarrow 0} t^{-2}F_{\xi/\eta_1}(t) = \infty.
\]

Furthermore Lemma \[2.3\] implies

\[
\lim_{t \downarrow 0} t^{-2}F_{\xi/\eta_2}(t) = \lim_{t \downarrow 0} t^{-2} \int_0^1 F(tu)\,du = 0.
\]

The result follows.

The sectioning of the system of spheres with a \(k\)-dimensional plane can be obtained by an iterated intersection procedure. First we intersect the system with an \((n-1)\)-dimensional plane \(E_{n-1}\), then the obtained system of spheres in \(E_{n-1}\) is intersected with an \((n-2)\)-dimensional plane and so on. This leads to

**Corollary 3.3.** Let \(F \in \mathcal{D}(H_{2,\alpha})\). Then \(F^{(r)} \in \mathcal{D}(H_{2,2})\) for \(r \geq 2\). If \(F \in \mathcal{D}(H_{3})\), then \(F^{(r)} \in \mathcal{D}(H_{2,2})\) for all \(r \geq 1\).

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