CHIMNEYS, LEOPARD SPOTS, AND THE IDENTITIES OF
BASMAJIAN AND BRIDGEMAN

DANNY CALEGARI

ABSTRACT. We give a simple geometric argument to derive in a common manner orthospectrum identities of Basmajian and Bridgeman. Our method also considerably simplifies the determination of the summands in these identities. For example, for every odd integer $n$, there is a rational function $q_n$ of degree $2(n - 2)$ so that if $M$ is a compact hyperbolic manifold of dimension $n$ with totally geodesic boundary $S$, there is an identity $\chi(S) = \sum_i q_n(l_i)$ where the sum is taken over the orthospectrum of $M$. When $n = 3$, this has the explicit form $\sum_i 1/(e^l_i - 1) = -\chi(S)/4$.

1. Orthospectrum identities

Let $M$ be a compact hyperbolic $n$-manifold with totally geodesic boundary $S$. An orthogeodesic is a properly immersed geodesic arc perpendicular to $S$ at either end. The orthospectrum is the set of lengths of orthogeodesics, counted with multiplicity.

Basmajian [1] and Bridgeman–Kahn [2, 3] derived identities relating the orthospectrum of $M$ to the area of $S$ and the volume of $M$ respectively. The following identity is implicit in [1]:

Basmajian’s Identity ([1]). There is a function $a_n$ depending only on $n$, so that if $M$ is a compact hyperbolic $n$-manifold with totally geodesic boundary $S$, and $l_i$ denotes the (ordered) orthospectrum of $M$, with multiplicity, there is an identity

$$\text{area}(S) = \sum_i a_n(l_i)$$

Basmajian’s identity is not well known; in fact, Bridgeman and Kahn were apparently unaware of Basmajian’s work when they derived the following by an entirely different method:

Bridgeman’s Identity ([2, 3]). There is a function $v_n$ depending only on $n$, so that if $M$ is a compact hyperbolic $n$-manifold with totally geodesic boundary $S$, and $l_i$ denotes the (ordered) orthospectrum of $M$, with multiplicity, there is an identity

$$\text{volume}(M) = \sum_i v_n(l_i)$$

In this paper, we show that both theorems can be derived from a common geometric perspective. In fact, the derivation gives a very simple expression for the functions $a_n$ and $v_n$, which we describe in §2. The derivation rests on a simple geometric decomposition.

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**Definition.** Let $\pi$ and $\pi'$ be totally geodesic $\mathbb{H}^{n-1}$'s in $\mathbb{H}^n$ with disjoint closure in $\mathbb{H}^n \cup S^{n-1}_\infty$. A chimney is the closure of the union of the geodesic arcs from $\pi$ to $\pi'$ that are perpendicular to $\pi$.

Thus, the boundary of the chimney consists of three pieces: the base, which is a round disk in $\pi$, the side, which is a cylinder foliated by geodesic rays, and the top, which is the plane $\pi'$. Note that the distance from the base to the top is realized by a unique orthogeodesic, called the core. The height of the chimney is the length of this orthogeodesic, and the radius is the radius of the base (these two quantities are related, and either one determines the chimney up to isometry).

**Chimney Decomposition.** Let $M$ be a compact hyperbolic $n$-manifold with totally geodesic boundary $S$. Let $M_S$ be the covering space of $M$ associated to $S$. Then $M_S$ has a canonical decomposition into a piece of zero measure, together with two chimneys of height $l_i$ for each number $l_i$ in the orthospectrum.

**Proof.** If $S$ is disconnected, the cover $M_S$ is also disconnected, and consists of a union of connected covering spaces of $M$, one for each component of $S$. The boundary of $M_S$ consists of a copy of $S$, together with a union of totally geodesic planes. Each such plane is the top of a chimney, with base a round disk in $S$, and these chimneys are pairwise disjoint and embedded. Since $M$ is geometrically finite, the limit set has measure zero, and therefore these chimneys exhaust all of $M_S$ except for a subset of measure zero. Every oriented orthogeodesic in $M$ lifts to a unique geodesic arc with initial point in $M_S$. Evidently this arc is the core of a unique chimney in the decomposition, and all chimneys arise this way. \(\square\)

Basmajian’s identity is immediate (in fact, though Basmajian does not express things in these terms, the argument we give is quite similar to his):

**Proof.** $S$ in $M_S$ is decomposed into a set of measure zero together with the union of the bases of the chimneys. Thus

$$\text{area}(S) = 2 \sum \text{area of the base of a chimney of height } l_i$$

\(\square\)

**Remark.** Thurston calls the chimney bases leopard spots; they arise in the definition of the skinning map (see e.g. [7]).

Bridgeman’s identity takes slightly more work, but is still elementary:

**Proof.** If $p$ is a point in $M$, and $\gamma$ is an arc from $p$ to $S$, there is a unique geodesic in the relative homotopy class of $p$ which is perpendicular to $S$. Thus, the unit tangent sphere to $p$ is decomposed into a set of measure zero, together with a union of round disks, one for each relative homotopy class of arc $\gamma$.

The area of the disk in $UT_p$ associated to $\gamma$ can be computed as follows. Let $\tilde{\gamma}$ be the unique lift of $\gamma$ to $M_S$ with one endpoint on $S$, and let $\tilde{p}$, a lift of $p$, be the other endpoint of $\tilde{\gamma}$. If $N$ is the complete hyperbolic manifold with $M$ as compact core and $N_S$ denotes the cover of $N$ associated to $S$ (so that $M_S$ is a convex subset of $N_S$), let $h_S$ be the harmonic function on $N_S$ whose value at every point $q$ is the probability that Brownian motion starting at $q$ exits the end associated to $S$. Note that $h_S = 1/2$ on $S$, and at every point $q$ depends only on the distance from $q$ to $S$. Then the area of the disk in $UT_p$ associated to $\gamma$ is $\Omega_{n-1} \cdot h_S(\tilde{p})$, where
$\Omega_{n-1} := 2\pi^{n/2}/\Gamma(n/2)$ denotes the area of a Euclidean sphere of dimension $n - 1$ and radius 1.

Since the volume of the unit tangent bundle of $M$ is $\Omega_{n-1} \cdot \text{volume}(M)$, it follows that the volume of $M$ is equal to the integral of $h_S$ over $M_S$. In each chimney, $h_S$ restricts to a harmonic function $h$, equal to $1/2$ on the base, and whose value at each point depends only on the distance to the base. Hence

$$\text{volume}(M) = 2 \sum_i \text{integral of } h \text{ over a chimney of height } l_i$$

□

\textbf{Remark.} In fact, precisely because our derivation is utterly unlike that of [3], we do not know whether Bridgeman’s function $v_n$ is equal to the integral of $h$ over an $n$-dimensional chimney of given height, only that there is such a function $v_n$ with the desired properties. If $n = 2$, our $v_2$ and Bridgeman’s $v_2$ agree, but the proof is not easy; one short derivation follows from [4], together with a geometric dissection argument.

2. Explicit formulae

In this section we show that the summands in the area and volume identities have a very nice explicit form. The expressions we obtain depend on the following elementary ingredients:

\textbf{quadrilateral:} A chimney is a solid of revolution, obtained by revolving a hyperbolic quadrilateral $Q$ with three right angles and one ideal vertex about the $S^{n-2}$ of directions perpendicular to one of the finite sides (which becomes the core of the chimney, the other finite side becoming the radius of the base). In a quadrilateral with three right angles and one ideal vertex, the length of one finite edge determines the other. If one finite edge has length $l$, let $\iota(l)$ denote the length of the other finite edge, so that $\iota$ is an involution on $(0, \infty)$. Then $\iota$ is defined implicitly by the fact that it is positive, and the identity

$$1/ \cosh^2(l) + 1/ \cosh^2(\iota(l)) = 1$$

or equivalently,

$$\sinh(\iota(l)) = 1/ \sinh(l)$$

If we write $\alpha = e^l$ and $\beta = e^{\iota(l)}$, then $\alpha$ and $\beta$ are related by

$$\beta + \beta^{-1} = 2 \left( \frac{\alpha + \alpha^{-1}}{\alpha - \alpha^{-1}} \right)$$

\textbf{hyperbolic volume:} If $B$ is a ball of radius $r$ in $n$-dimensional hyperbolic space, let $V_n^H(r)$ denote the volume of $B$. One has the following integral formula for $V_n^H$:

$$V_n^H(r) = \Omega_{n-1} \int_0^r \sinh^{n-1}(t)dt$$

The base of an $n$-dimensional chimney of height $l$ is just the volume of an $(n-1)$-dimensional ball in hyperbolic space of radius $\iota(l)$. For $n$ even, the integral $\int_0^{\iota(l)} \sinh^{n-1}(t)dt$ is a polynomial in $\beta + \beta^{-1}$, and therefore a rational function in $\alpha$ of degree $2(n-1)$. If the dimension of $M$ is at least 3, the set of numbers $e^l$ where $l$ runs over the orthospectrum are algebraic (by Mostow rigidity), and contained in a quadratic extension of the trace field of $M$. 


If $S$ has even dimension, then the area of $S$ is proportional to the Euler characteristic, by the Chern–Gauss–Bonnet theorem; in fact, for a hyperbolic manifold of dimension $n$ where $n$ is even, one has:

$$\text{area}(S) = (2\pi)^{n/2} \chi(S) r_n$$

where $r_n$ is a rational number depending on $n$.

The following corollary appears to be new:

**Rational Identity.** For every odd integer $n$, there is a rational function $q_n$ of degree $2(n - 2)$, with integral coefficients, so that if $M$ is a compact hyperbolic manifold of (odd) dimension $n$ with totally geodesic boundary $S$, there is an identity

$$\chi(S) = \sum_i q_n(e^{l_i})$$

where $\chi$ denotes Euler characteristic (which takes values in $\mathbb{Z}$) and $l_i$ denotes the orthospectrum of $M$ (with multiplicity). Note that for $n \geq 3$, the numbers $e^{l_i}$ are all contained in a fixed number field $K$ (depending on $M$).

**Example.** It is elementary to compute $q_n$ for small $n$. For example,

$$q_3(x) = \frac{4}{1 - x^2}$$

$$q_5(x) = \frac{5x^6 - 33x^4 + 63x^2 - 27}{8(x^2 - 1)^3}$$

The denominator is easily seen to be an integer multiple of $(x^2 - 1)^{n-2}$.

**Remark.** In the case of 3 dimensions, the identity has the following form. Let $M$ be a hyperbolic 3-manifold with totally geodesic boundary $S$. Then

$$\sum_i \frac{1}{e^{2l_i} - 1} = -\chi(S)/4$$

This is vaguely reminiscent of McShane’s identity [5], which says that for $S$ a hyperbolic once-punctured torus, there is an identity

$$\sum_i \frac{1}{1 + e^{l_i}} = 1/2$$

where the sum is taken over lengths $l_i$ of simple closed geodesics in the surface $S$.

If there is a simple relation between our identities and McShane’s identity, it is not obvious. However, Mirzakhani [6] showed how to derive and generalize McShane’s identity as a sum over embedded orthogeodesics on a surface with boundary. The appearance of orthogeodesics in yet another identity is quite suggestive of a more substantial connection, though we do not know what it might be.

To determine the summands in the volume identity, one needs the following additional ingredients:

**$\phi$-quadrilateral:** If $Q$ is a hyperbolic quadrilateral with three right angles and one vertex with angle $\phi$, then one of the lengths $l$ of the edges ending at right angles determines the other $e^l$, defined implicitly by the identity

$$\sinh(e^l) = \sinh(l)e^\phi = \cos(\phi)/\sinh(l)$$
spherical volume: If $B$ is a ball of radius $r$ in $n$-dimensional spherical space, let $V_n^S(r)$ denote the volume of $B$. One has the following integral formula for $V_n^S$:

$$V_n^S(r) = \Omega_{n-1} \int_0^r \sin^{n-1}(t)dt$$

harmonic: Let $h$ be the harmonic function on $\mathbb{H}^n$ equal to the indicator function of a round disk $D$ in $S^n_{\infty}$, so that $h = \frac{1}{2}$ on the plane $\pi$ bounded by $\partial D$. For $q$ bounded away from $D$ by $\pi$, if $t$ is the distance from $q$ to $\pi$, then $h(q)$ is $\Omega_{n-1}$ times the volume of a ball in $S^n_{\infty}$ of radius $\theta$, where $\sin(\theta) = \frac{1}{\cosh(t)}$.

level sets: Nearest point projection from an equidistant surface to a totally geodesic hyperplane multiplies distances by $\frac{1}{\cosh(t)}$. If $C$ is a chimney of height $l$ (and radius $\iota(l)$), let $C_t$ be the level set at distance $t$ from the base. Orthogonal projection of $C_t$ to the base of the chimney is surjective if $t \leq l$, and otherwise surjects onto an annulus with outer radius $\iota(l)$, and inner radius $\iota_\phi(l)$, where $\phi$ is defined implicitly by $\sin(\phi) = \cosh(l)/\cosh(t)$.

The area of $C_t$ is therefore

$$\text{area}(C_t) = \begin{cases} \cosh^{-1}(t)V_{n-1}^H(\iota(l)) & \text{if } t \leq l \\ \cosh^{-1}(t)(V_{n-1}^H(\iota(l)) - V_{n-1}^H(\iota_\phi(l))) & \text{if } t \geq l \end{cases}$$

Putting this all together, we get an explicit integral formula for $v_n$:

$$v_n(l)/2 = \int_0^l \cosh^{-1}(t)V_{n-1}^H(\iota(l))V_n^S(\arcsin(1/\cosh(t)))\Omega_{n-1}^{-1}dt$$

$$+ \int_l^\infty \cosh^{-1}(t)(V_{n-1}^H(\iota(l)) - V_{n-1}^H(\iota_\phi(l)))V_n^S(\arcsin(1/\cosh(t)))\Omega_{n-1}^{-1}dt$$

Notice when $n$ is even this can be expressed in closed form in terms of elementary functions (compare with the formulae and the derivation in [3], pp. 4–11).

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DEPARTMENT OF MATHEMATICS, CALTECH, PASADENA CA, 91125