Electric and magnetic U(1) currents in lattice confinement studies

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Making use of an Ehrenfest-Maxwell relation we show that in Abelian projected SU(2), in the maximal Abelian gauge, the dynamical electric charge density generated by the coset fields, gauge fixing and ghosts shows antiscreening as in the case of the non-Abelian charge.

1 Introduction

Lattice studies based on Abelian projection have had considerable success identifying the dynamical variables relevant to the physics of quark confinement. There is no definitive way as yet of choosing the optimum variables, but in the maximal Abelian gauge the U(1) fields remaining after Abelian projection produce a heavy quark potential that continues to rise linearly\textsuperscript{1,2,3}. Further the string tension is almost, but not exactly, equal to the full SU(2) quantity; 92% in a recent study at $\beta = 2.5115$\textsuperscript{4}.

This suggests that we may be close to identifying an underlying principle governing confinement. All elements of a dual superconducting vacuum appear to be present\textsuperscript{5,1}; in the maximal Abelian gauge magnetic monopoles reproduce nearly all of the U(1) string tension\textsuperscript{6}. The spontaneous breaking to the U(1) gauge symmetry is signalled by the non-zero vacuum expectation value of monopole operator\textsuperscript{7,8}. The profile of the electric field and the persistent magnetic monopole currents in the vortex between quark and antiquark are well described by an effective theory, the Ginzburg–Landau, or equivalently a Higgs theory, giving a London penetration depth and Ginzburg–Landau coherence length\textsuperscript{9,10}.

Central to finding the effective theory is the definition of the field strength

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operator in the Abelian projected theory, entering not only in the vortex profiles but also in the formula for the monopole operator. All definitions should be equivalent in the continuum limit, but use of the appropriate lattice expression should lead to a minimization of discretization errors.

In Ref. [11] we exploit lattice symmetries to derive such an operator that satisfies Ehrenfest relations; Maxwell’s equations for ensemble averages irrespective of lattice artifacts.

The charged coset fields are normally discarded in Abelian projection, as are the ghost fields arising from the gauge fixing procedure. Since the remainder of the SU(2) infrared physics must arise from these, an understanding of their rôle is central to completing the picture of full SU(2) confinement. In the maximal Abelian gauge a localised cloud of like polarity charge is induced in the vacuum in the vicinity of a source, producing an effect reminiscent of the antiscreening of charge in QCD. In other gauges studied, the analogous current is weaker, and acts to screen the source [14]. (This a tentative result, however, without the benefit of the refined definition of flux.)

2 Maxwell Ehrenfest relation

We first introduce and review the method due to Zach et al. [4] in pure U(1) theories. Consider a shift in a U(1) link angle, \( \theta_\nu(x_0) \rightarrow \theta_\nu(x_0) + \theta^s(x_0) \), in the partition function containing a Wilson loop source term

\[
Z_W(\{\theta^s\}) = \int [d(\theta_\nu + \theta^s)] \sin \theta_W \ e^{\beta \sum \cos \theta_{\mu\nu}}.
\]

Since the Haar measure is invariant under this shift, \( Z_W \) is constant in these variables. Absorbing the shift into the integration variable and taking the derivative gives

\[
\frac{\partial}{\partial \theta^s(x_0)} Z_W = \int [d\theta] (\cos \theta_W - \sin \theta_W \beta \Delta_\mu \sin \theta_{\mu\nu}) \ e^{\beta \sum \cos \theta_{\mu\nu}} = 0.
\]

This can be cast into the form

\[
\langle \Delta_\mu F_{\mu\nu} \rangle_W \equiv \frac{\langle \sin \theta_W \Delta_\mu \frac{1}{2} \sin \theta_{\mu\nu} \rangle}{\langle \cos \theta_W \rangle} = e \delta_{x,x_W} = J_\nu.
\]

We use the term Ehrenfest-Maxwell relation because it is the expectation value of what is normally a classical extremum of the path integral — an Euler–Lagrange equation. If we define flux using \( \sin \theta_{\mu\nu} \) instead of \( \theta_{\mu\nu} \) for example, then we get a precise lattice definition of current.
Before addressing the full problem we first generalize from U(1) to SU(2) without the complication of gauge fixing, with a shift $U_\mu(x_0) \rightarrow U_\mu(x_0)U^*(x_0)$

$$Z_W(\{U^*\}) = \int \! \left[ d(UU^*) \right] W_3(U) \ e^{\beta S(U)}; \quad W_3 = \frac{1}{2} Tr(U^\dagger U^\dagger U U i \sigma_3).$$

The size of the source is irrelevant so we choose it to be the simplest case, i.e. a plaquette. We choose the shift to be in the 3 direction

$$\frac{d}{d \epsilon_\mu(x_0)} Z_W = 0; \quad U^*(x_0) = \left( 1 - \frac{i}{2} \epsilon_3(x_0) \sigma_3 \right),$$

giving the Ehrenfest relation

$$\beta \frac{\langle W_3 S_\mu \rangle}{\langle W \rangle} = \delta_{x,x_0}; \quad W = \frac{1}{2} Tr(U^\dagger U^\dagger U U).$$

The notation $(S)_\mu$ denotes an $\epsilon$ derivative.\[\]

For $\beta = 2.5$, $\beta \langle W_3(S)_\mu \rangle = 0.0815(2)$, and $\langle W \rangle = 0.0818(1)$, and the difference $= 0.0003(3)$; i.e. zero within statistical errors.

To cast this into the form of Maxwell’s equation we decompose the link into diagonal $D_\mu$ and off-diagonal parts $O_\mu$

$$U_\mu(x) = D_\mu(x) + O_\mu(x).$$

We then group terms involving the diagonal part into $div E$ and group all terms having at least one factor of the off-diagonal link into the current.

$$[\beta \langle (S)_\mu W \rangle]_{U=D} = \frac{1}{e} \langle \Delta F_{\mu\nu} \rangle W; \quad \langle \cdots \rangle_W \equiv \frac{\langle W_3 \cdots \rangle}{\langle W \rangle},$$

giving the final form of the Ehrenfest relation

$$\langle \Delta F_{\mu\nu} \rangle_W = \langle J^{dyn.}_\nu \rangle_W + J^{static}_\nu; \quad \delta_{x,x_0} = \frac{1}{e} J^{static}_\nu.$$

This then tells us how to choose a lattice definition of field strength that satisfies an Ehrenfest relation:

$$F_{\mu\nu} = \frac{1}{e^2} Tr(D^\dagger D^\dagger D D i \sigma_3)_{\mu\nu}.$$

The effect of gauge fixing gives

$$Z_W(\{U^*\}) = \int \! \left[ d(UU^*) \right] W_3(U) \ \Delta_{FP} \delta[F] \ e^{\beta S(U)},$$

3
where we have introduced

\[ 1 = \Delta_{FP} \int \prod_{j,y} dg_j(y) \prod_{i,x} \delta[F^g_i(U^{\{g_j(y)\}}; x)], \]

and integrated out the \( g \) variables in the standard way. So \( \Delta_{FP} = \det M \) where

\[ M_{ix,jy} = \frac{\partial F^g_i(x)}{\partial g_j(y)} \bigg|_{g=0} \]

In this case \( Z_W \) is not invariant. The shift is inconsistent with the gauge condition. It is invariant, however, under an infinitesimal shift together with an infinitesimal ‘corrective’ gauge transformation that restores the gauge fixing

\[ G(x) = \left( 1 - i \frac{1}{2} \eta(x) \cdot \sigma \right); \quad U^s(x_0) = \left( 1 - i \frac{1}{2} \epsilon_3(x_0) \sigma_3 \right). \]

Using the invariance of the measure under combination of a shift and a corrective gauge transformation we obtain

\[ \frac{\partial}{\partial \epsilon_\mu(z_0)} + \sum_{k,z} \frac{\partial \eta_k(z)}{\partial \epsilon_\mu(z_0)} \frac{\partial}{\partial \eta_k(z)} \right] Z_W = 0. \]

In shorthand notation, the Ehrenfest relation reads

\[ \left\langle (W_3)_{\mu}^{s} + (W_3)_{\mu}^{g} + W_3 \left( \frac{\Delta_{FP}}{\Delta_{FP}} \right)_{\mu}^{s} + \left( \frac{\Delta_{FP}}{\Delta_{FP}} \right)_{\mu}^{g} + \beta(S)_{\mu} \right\rangle = 0. \quad (1) \]

Gauge fixing has introduced three new terms:

- \( (W_3)_{\mu}^{g} \) comes from the corrective gauge transformation acting on the source which is U(1) invariant but not SU(2) invariant.
- \( \left( \frac{\Delta_{FP}}{\Delta_{FP}} \right)_{\mu}^{s} \) is the effect of the shift on the Faddeev-Popov determinant.
- \( \left( \frac{\Delta_{FP}}{\Delta_{FP}} \right)_{\mu}^{g} \) is due to the corrective gauge transformation of the Faddeev-Popov determinant.
The last two derivatives are subtle. The key is to first consider the constraint up to first order in the shift and the corrective gauge transformations. Imposing that it is still zero fixes the \( \{ \eta \} \).

\[
F_i(x) + \frac{\partial F_i(x)}{\partial \epsilon_\mu(z_0)} \epsilon_\mu(z_0) + \sum_{k,z} \frac{\partial F_i(x)}{\partial \eta_k(z)} \eta_k(z) \equiv 0.
\]

Then define the Faddeev-Popov matrix as a derivative of the corrected constraint with respect to a general gauge transformation

\[
M_{i;x;j;y} + \delta M_{i;x;j;y} = \left. \frac{\partial}{\partial g_{j(y)}} \right|_{g=0} \left\{ F_\mu^g(x) + \frac{\partial F_\mu^g(x)}{\partial \epsilon_\mu(z_0)} \epsilon_\mu(z_0) + \sum_{k,z} \frac{\partial F_\mu^g(x)}{\partial \eta_k(z)} \eta_k(z) \right\}.
\]

Finally we evaluate the derivative using

\[
\frac{(\Delta)_{\mu}}{\Delta} = Tr[M^{-1}(M)_{\mu}].
\]

A check of the Ehrenfest theorem is given in Table 1. Some of the individual terms on the right hand side require a \( 2N \times 2N \) matrix inversion, where \( N \) is the lattice volume. Hence to test the result numerically, we chose as small a lattice as possible. The exactness of the theorem does not involve the size of the lattice which is \( 4^4 \) in Table 1. The last column employs a different source. The links making up the plaquette are replaced by the diagonal parts only as a second test of the theorem.

Again by separating the links into diagonal and off-diagonal parts we get the final form of the Ehrenfest-Maxwell relation.

\[
\langle \Delta_\mu F_{\mu\nu} \rangle = \langle J^\text{dyn.}_\nu \rangle + J^\text{static}_\nu \bigg|_s + J^\text{static}_\nu \bigg|_g + \left( J^\text{FP}_\nu \right)_s + \left( J^\text{FP}_\nu \right)_g.
\]

The right hand side consists of a sum of conserved currents. The first term comes from the excitation of the charged coset fields, the static term has an extra non-local contribution coming from the corrective gauge transformation, and the last two contributions are from the ghost fields. These terms give a non vanishing charge density cloud around a static source. The left hand side can be used as a lattice operator to measure the total charge density and does not require the matrix inversions needed to measure the individual terms separately that limited the numerical tests to small lattices.
Table 1: Terms in the Ehrenfest relation, Eqn.(1), on a 4\(^4\) lattice at \(\beta = 2.5\). The column labeled \(W_3\) corresponds to the source described in the text. In the second column the source links are replaced by their diagonal parts of the links to test a second source. The theorem gives zero for the sum.

| Source: Ehrenfest term | \(W_3\)         | \(W_3(U \rightarrow D)\) |
|------------------------|------------------|--------------------------|
| \(\langle (W_3)_\mu s\rangle\) | 0.65468(10)      | 0.63069(20)               |
| \(\langle (W_3)_\mu g\rangle\) | 0.06095(7)       | 0.04463(4)                |
| \(\langle W_3\frac{(\Delta_{FF})_\mu}{\Delta_{FF}} g\rangle\) | 0.00127(21)      | 0.00132(50)               |
| \(\langle W_3\frac{(\Delta_{FF})_\mu}{\Delta_{FF}} s\rangle\) | 0.00529(3)       | 0.00564(3)                |
| \(\langle \beta(S)_\mu s\rangle\) | -0.72246(68)     | -0.68275(50)              |

As a simple application we use this definition of flux to calculate \(divE\) on a source and the total flux away from the source. In Table 2, we see that the total integrated flux on a plane between the charges plus the flux on a back plane of the torus is larger than the \(divE\) on the source. The interpretation is the bare charge is dressed with same polarity charge by the interactions and the neighborhood has a cloud of like charge. Hence there is antiscreening. This charge density has contributions from all terms in the Ehrenfest relation. Table 2 also shows that the interactions increase the charge on the source itself.

3 Summary

We have seen that the electric charge induced by Abelian Wilson loop must be reinterpreted. The coset fields renormalise the charge of the loop as measured by \(|\langle \Delta_\nu F_{\nu\mu}\rangle|\) and charge is also induced in the surrounding vacuum.
$\beta \quad \text{divE(cl.pt.charge)} \quad \text{divE (on source)} \quad \text{total flux}$

| $\beta$  | 0.1   | 0.1042(1) | 0.0910(8) (mid) |
|-----------|-------|-----------|------------------|
| (almost classical) | 0.148(8) (back) | 0.1092(8) (total) |
| 2.4       | 0.4166 | 0.5385(19) | 0.7455(70) (mid) |
|           |       |           | 0.0359(72) (back) |
|           |       |           | 0.7815(95) (total) |

Table 2: $\text{divE} \equiv \langle \Delta \nu F_{\nu 4} \rangle$, normalized to $\frac{1}{\beta}$ for a ‘classical’ point charge, measured on a $3 \times 3$ Wilson loop source on an $8^4$ lattice. Integrated electric flux is measured on the midplane centered on the Wilson loop and on a plane on the far side of the torus, and the sum being the total flux.

Full SU(2) has antiscreening/asymptotic freedom of color charge, and in the maximal Abelian gauge alone have we seen analogous behaviour, in that the source charge is increased and induces charge of like polarity in the neighboring vacuum. Whether this renormalization of charge can account for the reduction of the string tension upon Abelian projection in this gauge is not clear. The improved field strength expression defined by the Ehrenfest identity does not coincide with the lattice version of the ’t Hooft field strength operator.

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