Rabi–Josephson oscillations and self-trapped dynamics in atomic junctions with two bosonic species

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Received 23 August 2010, in final form 21 November 2010
Published 17 January 2011
Online at stacks.iop.org/JPhysB/44/035301

Abstract

We investigate the dynamics of two-component Bose–Einstein condensates, composed of atoms in two distinct hyperfine states, which are linearly coupled by two-photon Raman transitions. The condensate is loaded into a double-well potential. A variety of dynamical behaviour, ranging from regular Josephson oscillations to mixed Rabi–Josephson oscillations and to regimes featuring increasing complexity are described in terms of a reduced Hamiltonian system with four degrees of freedoms, which are the numbers of atoms in each component in the left and right potential wells, whose canonically conjugate variables are phases of the corresponding wavefunctions. Using the system with four degrees of freedom, we study the dynamics of fractional imbalances of the two bosonic components and compare the results to direct simulations of the Gross–Pitaevskii equations describing the bosonic mixture. We perform this analysis when the fractional imbalance oscillates around a zero-time averaged value and in the self-trapping regime as well.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The study of Josephson oscillations and self-trapping both with a single bosonic component [1] and in bosonic binary mixtures [2–10] trapped in double-well potentials (DWP) has attracted much interest in the context of the current work on ultracold quantum gases. A specific ramification of this topic corresponds to the situation in which the two components of the mixture are different hyperfine states of the same bosonic atom [2, 3, 6], which may be linearly coupled by an external resonant field [11]. This setting suggests a possibility of studying the interplay between Josephson and Rabi oscillations, the latter being induced by the linear interconversion between the components [12, 13]. This subject was considered in several earlier works [14, 15, 17]. In particular, in [14] the authors analysed a crossover between the Josephson and Rabi dynamics, using a nonstationary model (with a linearly growing magnetic field), which, in the Josephson limit, was reduced to a system of two degrees of freedom. In [15], a two-degrees-of-freedom model was used too, with the objective of studying the quasiparticles’ spectrum in the symmetry-broken ground state. As suggested in [16], there exists a possibility of distinguishing between the Rabi and Josephson regimes by considering a beam-splitter model based on a nonstationary DWP. An experimental implementation of internal bosonic Josephson junctions with rubidium spinor Bose–Einstein condensate has been recently considered by Zibold et al [18] in connection with bifurcations occurring at the transition from Rabi to Josephson dynamics.

The natural minimum basis for the analysis of the Rabi–Josephson oscillations in two-component systems in the DWP should include four degrees of freedom [10]. The objective of...
this work is to analyse physically relevant dynamical regimes within the framework of the minimal system. The predictions will be verified via the comparison to direct simulations of the underlying Gross–Pitaevskii equations (GPEs).

The paper is organized as follows. The model is described in section 2, and the finite-mode approximation is derived in section 3, where we also verify its accuracy by comparison to direct simulations of the underlying GPEs. In section 4, we report the main results of the work, which demonstrate the interplay between the Josephson and Rabi oscillations, concluding that the oscillations become characterized by a high degree of complexity with the increase of the strength of Rabi coupling. In order to enlarge and complete the analysis presented in [9], the study of the self-trapping regime is faced as well. In fact, in section 5—within the self-trapping regime—we discuss the accuracy of our model following the same path as in section 4; we comment about the influence of the linear coupling constant on the onset of the self-trapping dynamics. The paper is concluded in section 6.

2. The model

We consider a binary Bose–Einstein condensate of two different species (with indices 1 and 2) of repulsively interacting bosons. The condensate is trapped in a DWP, which can be produced, for example, by a far off-resonance laser barrier separating each component into two regions, L (left) and R (right). These components may be, for example, two distinct hyperfine states, \(|F = 2, m_F = 1 \rangle\) and \(|F = 1, m_F = -1 \rangle\), of \(^{87}\text{Rb}\) [2, 3]. Note that since \(|\Delta m_F| = 2\) one needs two photons to couple the two levels. A weak external magnetic field gives rise to a small difference, \(\hbar \omega_n \equiv \hbar \omega \equiv \hbar \omega_0 - \delta\). Using the rotating-wave approximation (e.g. neglecting high-frequency terms in the atom–field interaction), in the mean-field approximation macroscopic wavefunctions \(\Psi_{n}(r, t)\) \((n = 1, 2)\) of the two components of the condensates obey the system of coupled Gross–Pitaevskii equations (GPEs) [12, 19]:

\[
\dot{\Psi}_n(r) = \frac{-\hbar^2}{2m} \nabla^2 \Psi_n + \left[ V_{\text{trap}}(r) + \frac{(-1)^n}{2} \hbar \delta + \frac{g_n}{2} |\Psi_n|^2 \right] \Psi_n + \frac{\Omega}{2} \Psi_{3-n},
\]

where \(V_{\text{trap}}(r)\) is the trapping potential and \(\Psi_{n}(r, t)\) is subject to the normalization condition

\[
\int d^3 r |\Psi_{n}(r, t)|^2 = N_n(t),
\]

with \(N_n(t)\) the number of bosons of the \(n\)th species. Similarly, \(m, a_n\) and \(g_n = 4\pi \hbar^2 a_n/m_n\) denote, respectively, the atomic mass, s-wave scattering length and intra-species nonlinearity coefficient of the \(n\)th species (the atomic mass is common for both species). The constant accounting for the linear interconversion between the bosonic components is expressed in terms of the respective Rabi frequency, \(\Omega\). Finally, \(g_{12} = 2\pi \hbar^2 a_{12}/m\) is the coefficient accounting for the nonlinear interaction between the species, \(a_{12}\) being the respective s-wave scattering length. Note that the total number of particles \(N_1(t) + N_2(t)\) is a conserved quantity. In the following, we consider both \(g_n\) and \(g_{12}\) as free parameters, due to the possibility of changing the scattering lengths by means of the Feshbach-resonance technique; see, e.g., [20] and references therein.

Equations (1) can also be derived in a different physical setting, by assuming that the two hyperfine states may be coupled by an external ac magnetic field \(B \cos(\omega t)\) of frequency \(\omega = \omega_0 - \delta\). In this case, the linear coupling term in equation (1) corresponds to the Rabi frequency \(\Omega = \mu \cdot B/\hbar\), where \(\mu\) is the dipole matrix element for the transition between the two hyperfine states [11].

The trapping potential for both components is taken to be the superposition of a strong harmonic confinement in the transverse \((x, y)\) plane and of a DWP in the axial \((z)\) direction, i.e.

\[
V_{\text{trap}}(r) = \frac{1}{2} m \omega_n^2 (x^2 + y^2) + V_{\text{DWP}}(z) .
\]

We proceed by writing the Lagrangian associated with the GPEs (1):

\[
\mathcal{L} = \int d^3 r \left[ \sum_{n=1,2} \dot{\Psi}_n |\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 \right] \Psi_n - \left( V_{\text{trap}}(r) |\Psi_n|^2 \right) + \left( \omega_n \frac{\hbar}{2} |\Psi_n|^2 + \frac{g_n}{2} |\Psi_n|^4 \right) - \left( \Omega/2 \right) \left( \Psi_1 \Psi_2 + \Psi_1 \Psi_2^* - g_{12} |\Psi_1|^2 |\Psi_2|^2 \right) ,
\]

where \(\Psi_n^*\) stands for the complex conjugate of \(\Psi_n\). To derive, at first, the 1D approximation, we adopt the usual ansatz

\[
\Psi_n(x, y, z, t) = \frac{1}{\sqrt{\pi a_{\perp,n}}} \exp \left( - \frac{x^2 + y^2}{2a_{\perp,n}^2} \right) f_n(z, t) ,
\]

where \(a_{\perp,n} = \sqrt{\hbar/(m \omega_n)}\) are the respective transverse confinement radii, with the 1D wavefunctions \(f_n(z, t)\) obeying normalization conditions \(\int dz |f_n(z, t)|^2 = N_n(t)\). Note that the factorized ansatz (5) is valid under the strong transverse confinement, namely when \(g_n |f_n|^2/4 \pi a_{\perp,n}^2 \ll \hbar \omega_n\) [21]. By inserting the ansatz (5) into the Lagrangian (4) and performing the integration in the transverse plane, we derive the effective Lagrangian for the 1D wavefunctions:

\[
\mathcal{L}_{1D} = \int dz \left[ \sum_{n=1,2} \dot{f}_n \left( \hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} \right) f_n - \left( \epsilon_n + V_{\text{DWP}}(z) \right) \left( \frac{-\hbar}{2} |\Psi_n|^2 - \frac{\hat{g}_n}{2} |f_n|^4 \right) - \left( \Omega/2 \right) \left( \dot{f}_1 f_2 + f_1 \dot{f}_2^* - g_{12} |f_1|^2 |f_2|^2 \right) \right] ,
\]

with the following constants introduced:

\[
\epsilon_n = \frac{1}{2} \left[ \frac{\hbar^2}{2m (a_{\perp,n})^2} + \frac{\hbar \omega_n}{2m} \right] ,
\]

\[
\hat{g}_n = g_n / \left( 2 \pi a_{\perp,n}^2 \right) ,
\]

\[
\hat{\Omega} = 2a_{\perp,1} a_{\perp,2} \left( a_{\perp,1}^2 + a_{\perp,2}^2 \right)^{-1} \Omega .
\]
and
\[ \tilde{g}_{12} = (g_{12}/\pi) \left(a_{1+}^2 + a_{1-}^2\right)^{-1}. \]  
(9)

By varying \( L_{1D} \) with respect to \( f_n \), we derive the effective 1D GPEs:
\[ i\hbar \frac{\partial f_n}{\partial t} = -\frac{\hbar^2}{2m_n} \frac{\partial^2 f_n}{\partial z^2} + [\epsilon_n + V_{\text{DWP}}(z) + (\pm 1)^n \hbar \delta] f_n + (\tilde{\Omega}/2) f_{3-m}. \]
\[ + \tilde{g}_n |f_n|^2 + \tilde{g}_{12} f_{3-m}^2 f_n + (\tilde{\Omega}/2) f_{3-m}. \]  
(10)

3. The finite-mode system

To approximate the dynamics by a finite-mode approximation, we make use of the two-mode decomposition for each wavefunction, as originally introduced in [22]:
\[ f_n(z, t) = \psi_n^L(t) \phi_n^L(z) + \psi_n^R(t) \phi_n^R(z). \]  
(11)

The orthonormal real functions \( \phi_n^\alpha(z) \) are constructed according to the same path as in [7] and as commented below. These functions are localized in the left and in the right wells, respectively (\( \alpha = L, R \)) [7], and
\[ \psi_n^\alpha(t) = \sqrt{N_n^\alpha(t)} e^{i\varphi_n^\alpha(t)}, \]  
(12)

with the total number of particles in the \( n \)th species being \( N_n^L(t) + N_n^R(t) = |\psi_n^L(t)|^2 + |\psi_n^R(t)|^2 = N_n(t) \). As described in the appendix, we derive explicit evolution equations for the temporal evolution of the fractional imbalances, \( z_n = (N_n^L - N_n^R)/N_n \) and intra-species relative phases, \( \theta_n = \theta_n^L - \theta_n^R \):
\[ \dot{z}_n = -\frac{2}{\hbar} \left( K_n - K_{n-1} N_n \right) \sqrt{1 - z_n^2} \sin \theta_n \]
\[ + \frac{V_n N_n}{\hbar} (1 - z_n^2) \sin \theta_n + \frac{1}{\hbar} \left( V_{12} \sqrt{1 - z_{3-n}^2} \cos \theta_{3-n} \right) \]
\[ + K_{c,12} N_{3-n} \sqrt{1 - z_{3-n}^2} \sin \theta_n \]
\[ \times \left( \sqrt{(1+z_n)(1+z_{2-n})} \sin \gamma_L - \sqrt{(1-z_n)(1-z_{2-n})} \sin \gamma_R \right), \]
\[ \dot{\theta}_n = \frac{U_n - V_n}{\hbar} z_n \sin \theta_n + \frac{2}{\hbar} \left( K_n - K_{n-1} N_n \right) \frac{z_n \cos \theta_n}{\sqrt{1 - z_n^2}} \]
\[ - \frac{V_n N_n}{\hbar} \frac{z_n \cos \theta_n}{\sqrt{1 - z_n^2}} + \frac{U_{12} - V_{12}}{\hbar} \frac{N_{3-n} z_n \cos \theta_n}{\sqrt{1 - z_n^2}} \]
\[ - 2 \left( \sqrt{V_{12} \sqrt{1 - z_{3-n}^2} \cos \theta_{3-n}} + K_{c,12} \right) \frac{N_{3-n} z_n \cos \theta_n}{\sqrt{1 - z_n^2}} \]
\[ + \frac{2}{\hbar} \sqrt{N_n} \left( \frac{1 + z_{3-n}}{1 - z_n} \sin \gamma_L - \frac{1 - z_{3-n}}{1 - z_n} \sin \gamma_R \right). \]  
(13)

Here the signs plus and minus pertain to \( n = 1 \) and \( n = 2 \), respectively. The total number of particles \( N_n \) of each component and the respective phases, \( \gamma_\alpha = \theta_1^\alpha - \theta_2^\alpha \) (recall \( \alpha = L, R \)), evolve according to
\[ N_n = \pm \left( \frac{\Omega}{2\hbar} \sqrt{N_1 N_2} \right) (\sqrt{(1+z_1)(1+z_2)} \sin \gamma_L \]
\[ + \sqrt{(1-z_1)(1-z_2)} \sin \gamma_R), \]
\[ \gamma_L = \frac{1}{2\hbar} \left( N_1(U_{12} - U)(1+z_1) - N_2(U_{12} - U)(1+z_2) + \Delta E \right) \]
\[ + \frac{1}{\hbar} \left( K_1 \sqrt{1 - z_1^2} \cos \theta_1 - K_2 \sqrt{1 - z_2^2} \cos \theta_2 \right) \]
\[ - \frac{\tilde{\Omega}}{2 \hbar \sqrt{N_1 N_2}} \frac{(N_1(1+z_2) - N_2(1+z_1))}{\sqrt{(1+z_1)(1+z_2)}} \cos \gamma_L \]
\[ + \frac{K_{c,12}}{\hbar} \left[ N_1 \sqrt{1 - z_1^2} \cos \theta_1 - N_2 \sqrt{1 - z_2^2} \cos \theta_2 \right] \]
\[ - \frac{V}{2\hbar} \left[ N_1(1-z_1)(2+\cos 2\theta_1) - N_2(1-z_2)(2+\cos 2\theta_2) \right] \]
\[ - \frac{1}{\hbar} \left[ K_{c,1} N_1(2 + z_1) + K_{c,12} N_2 + V_{12} N_2 \sqrt{1 - z_2^2} \cos \theta_2 \right] \]
\[ \times \sqrt{1 - z_1^2} \cos \gamma_L + \frac{1}{\hbar} \left[ K_{c,2} N_2(2 + z_2) + K_{c,12} N_1 \right] \]
\[ \times \sqrt{1 - z_1^2} \cos \gamma_L - \sqrt{1 - z_2^2} \cos \gamma_R \right), \]  
(14)

where \( \Delta E \equiv E_2 - E_1 = \hbar \delta \). The latter relation follows from the normalization of functions \( \phi_n \), and the first equation from equation (A.2). Obviously, in the absence of Rabi coupling, i.e. \( \tilde{\Omega} = 0 \), the total number of particles is conserved in each component. In that case, equations (13) reduce to the equations of motion for the coupled pendulums derived in [7]. We observe that due to the nonlinearity associated with the intra- and inter-species interactions, the system is nonintegrable also when \( \tilde{\Omega} = 0 \). On the other hand, in the absence of the Josephson coupling, i.e. when \( K_n = 0 \), the dynamics will be characterized by independent Rabi oscillations in each well; when the Josephson coupling is finite, i.e. \( K_n \neq 0 \), and much smaller than \( \tilde{\Omega} \), the aforementioned single-well independent Rabi oscillations will be weakly coupled by the Josephson tunnelling. Note that when both \( K_n = 0 \) and \( K_n \neq 0 \), the Rabi interconversions will be deformed by nonlinear effects due to the intra- and the inter-species interactions.
In our calculations the axial double-well potential is given by
\[ V_{\text{DWP}}(z) = -U_0 \left[ \tanh^2 \left( \frac{z - z_0}{A} \right) + \tanh^2 \left( \frac{z + z_0}{A} \right) \right] \]
with
\[ U_0 = \hbar \omega_\perp \left[ 1 + \tanh^2 \left( \frac{2z_0}{A} \right) \right]^{-1} \]
that is the combination of two Pöschl–Teller (PT) potentials separated by a potential barrier, the height of which may be changed by changing \( A \), and centred around the points \(-z_0\) and \(+z_0\). Then the wavefunctions of the eigenvalue problem in the presence of the only potential \( V_\alpha(z) (\alpha = L, R) \) are exactly known. In particular, the wavefunction of the ground state is [23]
\[ \phi_{\alpha,\text{PT}}^a(z) = B \left[ 1 - \tanh^2 \left( \frac{z - z_0}{A} \right) \right]^{C_n/2} \]
where
\[ C_n = -\frac{1}{2} + \frac{2mU_0A^2}{\hbar^2} + \frac{1}{4} \]
If \( \alpha = L \) (R), the function \( \phi_{\text{PT}}^a(z) \) is centred around \(-z_0\) (+\(z_0\)), and it is the ground state wavefunction of the PT potential centred around the point \(-z_0\) (\(+z_0\)). In equation (17), \( B \), equal for both sides, ensures the normalization of the above wavefunction \( \phi_{\alpha}^a(z) \) and \( \phi_{\alpha}^R(z) \). Then, proceeding from the functions (17), \( \phi_L(z) \) and \( \phi_R(z) \) can be determined following the same perturbative approach as in [7], where it is shown that, under certain hypothesis, the aforementioned functions can be written in terms of \( \phi_n^a,\text{PT}(z) \) and \( \phi_n^R,\text{PT}(z) \) given by (17).

We get [7]
\[ \phi_{\alpha}^L(z) = \frac{1}{2} \left[ \left( \frac{1}{\sqrt{1+s}} + \frac{1}{\sqrt{1-s}} \right) \phi_{\text{PT}}^{L,\text{PT}}(z) \right. \]
\[ + \left. \left( \frac{1}{\sqrt{1+s}} - \frac{1}{\sqrt{1-s}} \right) \phi_{\text{PT}}^{R,\text{PT}}(z) \right] \]
\[ \phi_{\alpha}^R(z) = \frac{1}{2} \left[ \left( \frac{1}{\sqrt{1+s}} - \frac{1}{\sqrt{1-s}} \right) \phi_{\text{PT}}^{L,\text{PT}}(z) \right. \]
\[ + \left. \left( \frac{1}{\sqrt{1+s}} + \frac{1}{\sqrt{1-s}} \right) \phi_{\text{PT}}^{R,\text{PT}}(z) \right] \]
where \( s = \int_{z_0}^{\infty} dz \phi_{\text{PT}}^{L,\text{PT}}(z)\phi_{\text{PT}}^{R,\text{PT}}(z) \).

The relations between the macroscopic parameters involved on the right-hand sides of equations (13)–(14) and the microscopic parameters of the problem are reported in equation (A.2) of the appendix. We observe, moreover, that to make the system fully symmetric we also assume that \( K_1 = K_2 \equiv K \), \( U_1 = U_2 \equiv U \), \( K_{z_1} = K_{z_2} \equiv K_z \), and \( V_1 = V_2 \equiv V \).

Obviously, in the absence of Rabi coupling, i.e. \( \Omega = 0 \), the total number of particles is conserved in each component. In that case, equations (13) reduce to the equations of motion for coupled pendulums derived in [7]. Note, moreover, that in our calculations we have assumed that both the components feel the same harmonic potential so that \( \omega_1 = \omega_2 \equiv \omega_\perp \).

To verify the reliability of the finite-mode approximation which leads to the ODEs (13) and (14), we compare the evolution of fractional imbalances \( z_n \), as predicted by this system, with results of direct numerical simulations of the 1D GPEs (10) both when the fractional imbalances \( z_n(t) \) oscillate around a zero-time averaged value and when the time-averaged value of \( z_n(t) \) is non-zero, that is, the self-trapping regime. For the oscillations characterized by \( \langle z_n(t) \rangle = 0 \), the results of the comparison are presented in figure 1. This figure shows good agreement, especially when the Rabi coupling, \( \Omega \), is small enough. At larger values of \( \Omega \), the finite-mode approximation demonstrates a deviation, which accumulates at sufficiently long times.

In figure 2, we report the above comparison between GPEs and ODEs when the fractional imbalances are both self-trapped. Also in this case the distance between the predictions from the two approaches increases for long times when \( \Omega \) is big enough. It is worth observing that the authors of [9] have performed the comparison between the predictions of GPEs and those of ODEs being derived from the two-mode approximation only when the fractional imbalances oscillate around zero. Julia-Diaz and co-workers, moreover, have...
integrated the aforementioned ODEs under the assumptions of small imbalances, and small intra- and inter-species phase differences [9].

4. Josephson and Rabi oscillation mixing and the self-trapped dynamics

One of the central themes of this work is the interplay of the Josephson oscillations and Rabi oscillations. We start with the situation in which only the Josephson coupling is present, i.e. \( \bar{\Omega} = 0 \), see the upper panels of figures 1, 3 and 4. In the other panels of these figures we report the behaviour of the system in the presence of finite values of \( \bar{\Omega} \). One can see that the greater the Rabi coupling between the two species (hyperfine atomic states), the greater the dynamical complexity exhibited by the system, as observed, especially, in the lower panels of figures 1, 3 and 4. Moreover, as shown in the lower panels of figure 1, when \( \bar{\Omega} = 20 \, \text{K} \), the behaviour of \( z_1(t) \) and \( z_2(t) \) is strongly asymmetric with respect to each other; this asymmetry is absent when \( E_2 - E_1 = 0 \).

In figure 3 we plot the dynamics of the finite-dimensional system in the planes of fractional imbalances \( z_n(t) \) and phases \( \theta_n(t) \), using the trapping and input parameters of figure 1.

The figure shows that the motion is fully periodic for \( \bar{\Omega} = 0 \), and then it becomes quasi-periodic and finally aperiodic by increasing the value of \( \bar{\Omega} \). To quantify the increasing of complexity with the growth of \( \bar{\Omega} \), in figure 4 we show the power spectrum of the oscillations, represented by the absolute value, \( |z_n(\omega)| \), of the Fourier transform of \( z_n(t) \), denoting by \( \bar{\omega}_1 \) the frequency associated with the maximum of \( |z_1(\omega)| \), and by \( \bar{\omega}_2 \) that associated with the maximum of \( |z_2(\omega)| \). The frequencies \( \bar{\omega}_n \) are the fundamental frequencies of \( z_n(t) \), i.e. the frequencies of the carrier waves of \( z_n(t) \).

Let us focus on table 1 obtained with \( N_1(0) = 200 > N_2(0) = 100 \). From the data reported there, we see that—whatever the value of the Rabi coupling \( \bar{\Omega} = \bar{\omega}_1 > \bar{\omega}_2 \). Then the multi-particle tunneling period associated with \( \bar{\omega}_1, \bar{T}_1 \), given by \( 2\pi/\bar{\omega}_1 \), is smaller than that associated with \( \bar{\omega}_2, \bar{T}_2 \), given by \( 2\pi/\bar{\omega}_2 \). This is due to the fact that—within a mechanical analogy—the ODEs (13)–(14) describe two coupled pendulums. At least in the presence of sufficiently weak inter-species interactions, the mass of each pendulum is related to the inverse of the particle number \( N_n \) of the \( n \)th species, as discussed for a single component in [1]. From table 1, it is possible to infer as well that the greater the \( \bar{\Omega} \) the greater the ratio \( \bar{T}_2/\bar{T}_1 \). On the other hand, if we keep fixed the Rabi coupling and increase \( N_1(0) \) with respect to \( N_2(0) \), the period \( \bar{T}_1 \) will be smaller and smaller compared with \( \bar{T}_2 \). Moreover, from table 1 again, we see that the relative changes \( r_n \) of \( \bar{\omega}_n \) with respect to their values at \( \bar{\Omega} = 0 \) increase with increasing \( \bar{\Omega} \).
Figure 4. Josephson’s regime. The absolute value of $|z_n(\omega)|$ of the Fourier transform of the fractional imbalances versus frequency $\omega$. Note that the vertical axis is on the logarithmic scale. The parameters, initial conditions and units as in figure 1.

Table 1. Spectra of figure 4. Frequencies $\tilde{\omega}_n$ of the maxima of $|z_n(\omega)|$ for different values of $\tilde{\omega}$ and their relative changes $r_n$ with respect to the absence of Rabi coupling.

| $\tilde{\omega}$ | $\tilde{\omega}_1$ | $\tilde{\omega}_2$ | $r_1$ | $r_2$ |
|---------------|-----------------|-----------------|------|------|
| 0             | 0.0268          | 0.0224          | 0    | 0    |
| $K$           | 0.0272          | 0.0220          | 0.015| 0.018|
| $3K$          | 0.0291          | 0.0211          | 0.086| 0.060|
| $20K$         | 0.0341          | 0.0153          | 0.27 | 0.32 |

From figure 4 it is possible to gain physical insight into the dynamics of the system, especially for large values of the Rabi coupling $\tilde{\omega}$. We can see that when $\tilde{\omega}$ is sufficiently large, $|z_n(\omega)|$ exhibits a multi-peak structure related to the appearance of frequencies different from the fundamental one. This is reflected in an increasing number of harmonics involved in $z_n(t)$ (see, in particular, the lower panels of figure 1 where $\tilde{\omega} = 20K$) and, accordingly, in an increasing degree of complexity. By analysing figure 4 again, one can conclude that the power spectrum approaches that of random oscillations as $\tilde{\omega}$ increases. This, immediately, reflects quite complicated dynamics when we focus on the plane $z_n(t) - \theta_n(t)$ (see, in particular, the lower panels in figure 3 where $\tilde{\omega} = 20K$). Finally, note that the complexity pertaining to high values of $\tilde{\omega}$ increases if the dynamics is observed on sufficiently long time scales as shown in the lower panels of figures 3 and 4.

For the investigation of the self-trapped dynamics let us consider again figure 2 and also figures 5 and 6. In figure 5 we report the dynamics in the planes $(z_n, \theta)$, while in figure 6, we show the absolute value $|z_n(\omega)|$ of the Fourier transform of $z_n(t)$. From these figures, we see that in correspondence to the high value of the linear coupling constant, increasing complexity is observed within the dynamics supported by the junction.
5. Conclusion

In this work, we have introduced a model which allows one to study the interplay of the Josephson and Rabi oscillations in a binary Bose–Einstein condensate trapped in the double-well potential structure. The Rabi coupling is provided by the interconversion between the two species of the condensate, which represent distinct hyperfine states of the same atom.

To capture core features of the dynamics, we have derived a finite-mode approximation with four degrees of freedom that represent the populations of the two species in the two symmetric potential wells. Comparison to full simulations of the underlying Gross–Pitaevskii system demonstrates that the truncated system provided a reasonable accuracy. Systematic simulations of the system reveal the transition from regular Josephson oscillations to complex dynamics with the increase of the Rabi interconversion rate. Within the framework of this analysis, we have discussed the possibility of inferring, at least at a qualitative level, the behaviour of the fractional imbalances and the dynamics in the plane $\zeta_n(t) - \theta_n(t)$ proceeding from the Fourier analysis of the fractional imbalances, especially for large Rabi couplings.

We have analysed the dynamics of the atomic Josephson junction when both the components are self-trapped as well. We have shown that also in this case the truncated system gives rise to reliable predictions with good agreement with the predictions of the associated Gross–Pitaevskii system. We have studied the influence of the linear coupling on the self-trapping onset.

Appendix

In this appendix, we discuss the path followed in deriving the evolution equations for the fractional imbalances $\zeta_n = (N^L_n - N^R_n)/N_n$ and the intra-species relative phases $\theta_n = \theta^R_n - \theta^L_n$, i.e. equations (13) and (14). We start by deriving the effective...
Lagrangian $L_{\text{eff}}$ in terms of variables $N_n^\alpha$ and $\theta_n^\alpha$:

$$
L_{\text{eff}} = \sum_{n=1,2} \left[ -\hbar \dot{N}_n^L N_n^L - \hbar \dot{N}_n^R N_n^R - E_n^L N_n^L - E_n^R N_n^R \\
+ 2K_n \sqrt{N_n^L N_n^R} \cos (\theta_n^R - \theta_n^L) \\
- \left( \frac{U_n^L}{2}(N_n^L)^2 + \frac{U_n^R}{2}(N_n^R)^2 \right) \\
- 2K_{e,n} \sqrt{N_n^L N_n^R} \cos (\theta_n^R - \theta_n^L) \\
- V_n N_n^L (2 + \cos 2(\theta_n^L - \theta_n^L)) \\
- U_{12}^L N_n^L N_n^R - U_{12}^R N_n^L N_n^R \\
- 2K_{e,12}(N_1^L + N_1^R) \sqrt{N_1^L N_1^R} \cos (\theta_1^R - \theta_1^L) \\
+ (N_1^L + N_1^R) \sqrt{N_1^L N_1^R} \cos (\theta_1^R - \theta_1^L) \\
- V_1(N_1^L N_1^R + N_2^L N_2^L) - 4V_{12} \\
\times \left( \sqrt{N_1^L N_1^R} \sqrt{N_2^L N_2^R} \cos(\theta_1^R - \theta_1^L) \cos (\theta_2^R - \theta_2^L) \right) \\
- 2(R_{12}^L \sqrt{N_1^L N_2^L} \cos (\theta_1^L - \theta_1^L)) \\
- 2(R_{12}^R \sqrt{N_1^R N_2^R} \cos (\theta_1^R - \theta_1^L)) \\
- 2(S_{12}^{LR} \sqrt{N_1^L N_2^R} \cos (\theta_1^L - \theta_1^R)) \\
- 2(T_{12}^{RL} \sqrt{N_1^L N_2^L} \cos (\theta_1^L - \theta_1^R)), \quad (A.1)
$$

where the following constants are introduced:

$$
E_n^\alpha = \int dz \left[ \frac{\hbar^2}{2m} \left( \frac{d\phi_n^\alpha}{dz} \right)^2 + \frac{\hbar^2}{2ma_n^2} \sum_{\alpha,n} + \right] \\
\cdot \left( \frac{V(z)}{2} \right) (\phi_n^\alpha)^2 \\
U_n^\alpha = \tilde{g}_n \int dz (\phi_n^\alpha)^4 \\
K_n = - \int dz \left[ \frac{\hbar^2}{2m} \left( \frac{d\phi_n^\alpha}{dz} \right)^2 + \right] \\
\cdot \left( \frac{V(z)}{2} \right) (\phi_n^\alpha)^2 \\
V_n = 2\tilde{g}_n \int dz (\phi_n^\alpha)^2 (\phi_n^\alpha)^2 \\
U_{12}^\alpha = \tilde{g}_{12} \int dz (\phi_1^\alpha(z))^3 (\phi_2^\alpha(z))^2 \\
K_{e,12} = \tilde{g}_{12} \int dz (\phi_1^\alpha(z))^3 (\phi_2^\alpha(z))^2 \\
V_{12} = \tilde{g}_{12} \int dz (\phi_1^\alpha(z))^3 (\phi_2^\alpha(z))^2 \\
R_{12}^\alpha = (\tilde{\Omega}/2) \int dz \phi_1^\alpha(z) \phi_2^\alpha(z) \\
S_{12}^{LR} = (\tilde{\Omega}/2) \int dz \phi_1^\alpha(z) \phi_2^\alpha(z) \\
T_{12}^{RL} = (\tilde{\Omega}/2) \int dz \phi_1^\alpha(z) \phi_2^\alpha(z),
$$

To analyse the finite-mode dynamics induced by the Lagrangian (A.1), we define the canonical momenta conjugate to the generalized coordinates $\hbar \dot{\theta}_n^\alpha$:

$$
p_{\theta_n^\alpha} = \frac{1}{\hbar} \frac{\partial L_{\text{eff}}}{\partial \theta_n^\alpha} = -N_n^\alpha. \quad (A.3)
$$

Accordingly, the Hamiltonian of the system is written in terms of the canonical coordinates and momenta as follows:

$$
H = \sum_{n=1,2} \left[ p_{\theta_n^L} E_n^L + p_{\theta_n^R} E_n^R \right] \\
- \sum_{n=1,2} \left[ -2K_n \sqrt{p_{\theta_n^L} p_{\theta_n^R}} \cos (\theta_n^R - \theta_n^L) \\
+ \left( \frac{U_n^L}{2} p_{\theta_n^L} + \frac{U_n^R}{2} p_{\theta_n^R} \right) \\
- 2K_{e,n} (p_{\theta_n^L} + p_{\theta_n^R}) \sqrt{p_{\theta_n^L} p_{\theta_n^R}} \cos (\theta_n^R - \theta_n^L) \\
+ V_n (2 + \cos 2(\theta_n^L - \theta_n^L)) p_{\theta_n^L} p_{\theta_n^R} \right] \\
+ U_{12} p_{\theta_n^L} p_{\theta_n^R} + U_{12} p_{\theta_n^R} p_{\theta_n^R} \\
- 2K_{e,12} (p_{\theta_1^L} + p_{\theta_1^R}) \sqrt{p_{\theta_1^L} p_{\theta_1^R}} \cos (\theta_1^R - \theta_1^L) \\
+ (p_{\theta_1^L} + p_{\theta_1^R}) \sqrt{p_{\theta_1^L} p_{\theta_1^R}} \cos (\theta_1^R - \theta_1^R) \\
+ V_{12} (p_{\theta_1^L} p_{\theta_1^R} + p_{\theta_1^R} p_{\theta_1^R}) \\
+ 4V_{12} \sqrt{p_{\theta_1^L} p_{\theta_1^R}} \cos (\theta_1^R - \theta_1^L) \cos (\theta_1^R - \theta_1^R) \\
+ 2(R_{12}^L \sqrt{p_{\theta_1^L} p_{\theta_1^R}} \cos (\theta_1^R - \theta_1^R)) \\
+ 2(R_{12}^R \sqrt{p_{\theta_1^R} p_{\theta_1^R}} \cos (\theta_1^R - \theta_1^R)) \\
+ 2(S_{12}^{LR} \sqrt{p_{\theta_1^L} p_{\theta_1^R}} \cos (\theta_1^R - \theta_1^R)) \\
+ 2(T_{12}^{RL} \sqrt{p_{\theta_1^L} p_{\theta_1^R}} \cos (\theta_1^R - \theta_1^R)). \quad (A.4)
$$

The evolution equations for populations $N_n^\alpha$ and phases $\theta_n^\alpha$ are derived, as the canonical equations, from the Hamiltonian (A.4):

$$
p_{\theta_n^\alpha} = \frac{1}{\hbar} \frac{\partial H}{\partial \theta_n^\alpha}, \quad \theta_n^\alpha = \frac{1}{\hbar} \frac{\partial H}{\partial p_{\theta_n^\alpha}}. \quad (A.5)
$$

We observe that due to the orthonormality of the decomposition basis, $R_{12}^\alpha = \tilde{\Omega}/2$ and $S_{12}^{LR} = T_{12}^{RL} = 0$. From the symmetry between the two wells in the DWB structure, it also follows that $E_n^L = E_n^R = E_n$. $U_{12}^L = U_{12}^R = U_{12}$, $U_{12}^L = U_{12}^R = U_{12}$. Using equation (A.5), we derive explicit evolution equations for equation (13) and the evolution equations for the total number of particles of each component $N_n$ and the respective phases, $\gamma_n = \theta_1^\alpha - \theta_2^\alpha$ (recall that $\alpha = L, R$):

$$
\dot{z}_n = \frac{2(K_n - K_{e,12} N_n)}{\hbar} (1 - z_n^2) \sin \theta_n \\
+ \frac{V_n N_n}{2\hbar} (1 - z_n^2) \sin 2\theta_n \\
+ \frac{2}{\hbar} (V_{12} \sqrt{1 - z_{12}^2} \cos \theta_{12} + K_{e,12}) N_{3-n} \sqrt{1 - z_n^2} \sin \theta_n \\
+ \frac{\tilde{\Omega}}{2\hbar} \sqrt{N_1 N_2} \sqrt{(1 + z_1)(1 + z_2)} \sin \gamma_L \\
- \sqrt{(1 - z_1)(1 - z_2)} \sin \gamma_R,
$$

where $\tilde{\Omega}$ is the angular frequency of the oscillating potential, and $z_n = N_n^L - N_n^R$ is the population difference.
\[
\dot{\theta}_n = \frac{U_n - V_n N_n z_n \cos \theta_n - K_{c2} N_2 z_n \cos \theta_n}{\hbar} + \frac{V_n N_n}{\hbar} z_n \cos \theta_n + \frac{\hbar}{2h} \left( U_{12} - V_{12} \right) N_{3-n} z_{3-n} + \frac{2}{h} \sqrt{V_{12} \Omega_{12} \left( 1 - z_{3-n}^2 \cos \theta_{3-n} + K_{c12} N_{3-n} z_n \cos \theta_n \right)} \sqrt{1 - z_n^2} \left( \Omega_{12} \right) \frac{\hbar}{2h} \sqrt{N_{3-n} N_n \left( \frac{1 + z_{3-n}}{1 + z_n} \cos \gamma_L - \frac{1 - z_{3-n}}{1 - z_n} \cos \gamma_R \right)}, \tag{A.6}
\]

\[
\dot{N}_n = \pm \left( \frac{\Omega}{2h} \sqrt{N_1 N_2} \right) \left( \sqrt{1 + z_1} - \sqrt{1 + z_2} \right) \sin \gamma_L,
\]

\[
\dot{\gamma}_L = \frac{1}{2h} \left( N_1 \left( U_{12} - U \right) \left( 1 + z_1 \right) - N_2 \left( U_{12} - U \right) \left( 1 + z_2 \right) \right) + \Delta E \right)
\]

\[
+ \frac{1}{h} \left( K_{c1} \sqrt{1 + z_1} \cos \theta_1 - K_{c2} \sqrt{1 + z_2} \cos \theta_2 \right) + \frac{\hbar}{2h} \sqrt{N_1 N_2} \left( N_2 \left( 1 + z_2 \right) - N_1 \left( 1 + z_1 \right) \right) \cos \gamma_L
\]

\[
+ \frac{K_{c12}}{h} \left[ N_1 \sqrt{1 - z_2^2} \cos \theta_1 - N_2 \sqrt{1 - z_2^2} \cos \theta_2 \right]
\]

\[
+ \frac{V_{12}}{h} \left[ N_1 \left( 1 - z_1 \right) \left( 2 + \cos 2 \theta_1 \right) - N_2 \left( 1 - z_2 \right) \left( 2 + \cos 2 \theta_2 \right) \right] \right)
\]

\[
+ \frac{K_{c12}}{h} \left[ N_1 \sqrt{1 - z_2^2} \cos \theta_1 - N_2 \sqrt{1 - z_2^2} \cos \theta_2 \right]
\]

\[
+ \frac{1}{h} \left( K_{c1} N_1 \left( 2 + z_1 \right) + K_{c12} N_2 + V_{12} N_2 \sqrt{1 - z_2^2} \cos \theta_2 \right) \right)
\]

\[
+ \frac{V_{12}}{2} \left( N_1 \left( 1 - z_1 \right) - N_2 \left( 1 - z_2 \right) \right),
\]

\[
\dot{\gamma}_R = \frac{1}{2h} \left( N_1 \left( U_{12} - U \right) \left( 1 - z_1 \right) - N_2 \left( U_{12} - U \right) \left( 1 - z_2 \right) \right) + \Delta E \right)
\]

\[
+ \frac{1}{h} \left( K_{c1} \sqrt{1 + z_1} \cos \theta_1 - K_{c2} \sqrt{1 + z_2} \cos \theta_2 \right) - \frac{\hbar}{2h} \sqrt{N_1 N_2} \left( N_2 \left( 1 - z_2 \right) - N_1 \left( 1 - z_1 \right) \right) \cos \gamma_R
\]

\[
+ \frac{K_{c12}}{h} \left[ N_1 \sqrt{1 - z_2^2} \cos \theta_1 - N_2 \sqrt{1 - z_2^2} \cos \theta_2 \right]
\]

\[
+ \frac{V_{12}}{2} \left( N_1 \left( 1 + z_1 \right) \left( 2 + \cos 2 \theta_1 \right) - N_2 \left( 1 + z_2 \right) \left( 2 + \cos 2 \theta_2 \right) \right] \right)
\]

\[
- \frac{1}{h} \left( K_{c1} N_1 \left( 2 - z_1 \right) + K_{c12} N_2 + V_{12} N_2 \sqrt{1 - z_2^2} \cos \theta_2 \right) \right)
\]

\[
\times \left( 1 + z_1 \right) \cos \theta_1 + \frac{1}{h} \left[ K_{c2} N_2 \left( 2 - z_2 \right) \right]
\]

\[
+ K_{c12} N_1 + V_{12} N_1 \sqrt{1 - z_1^2} \cos \theta_1 \right)
\]

\[
\times \left( 1 + z_2 \right) \cos \theta_2 + \frac{V_{12}}{2} \left( N_1 \left( 1 + z_1 \right) - N_2 \left( 1 + z_2 \right) \right) \right) \right)
\]

(A.7)

\[
\text{References}
\]

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