Reduced basis model order reduction for Navier–Stokes equations in domains with walls of varying curvature

Martin W. Hess\textsuperscript{a}, Annalisa Quaini\textsuperscript{b} and Gianluigi Rozza\textsuperscript{a}

\textsuperscript{a}SISSA Mathematics Area, mathLab, International School for Advanced Studies, Trieste, Italy; \textsuperscript{b}Department of Mathematics, University of Houston, Houston, TX, USA

ABSTRACT

We consider the Navier–Stokes equations in a channel with a narrowing and walls of varying curvature. By applying the empirical interpolation method to generate an affine parameter dependency, the offline-online procedure can be used to compute reduced order solutions for parameter variations. The reduced-order space is computed from the steady-state snapshot solutions by a standard POD procedure. The model is discretised with high-order spectral element ansatz functions, resulting in 4752 degrees of freedom. The proposed reduced-order model produces accurate approximations of steady-state solutions for a wide range of geometries and kinematic viscosity values. The application that motivated the present study is the onset of asymmetries (i.e. symmetry breaking bifurcation) in blood flow through a regurgitant mitral valve, depending on the Reynolds number and the valve shape. Through our computational study, we found that the critical Reynolds number for the symmetry breaking increases as the wall curvature increases.

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1. Introduction and motivation

We consider the flow of an incompressible fluid through a planar channel with a narrowing, where the walls creating the narrowing have variable curvature. An application that motivated the present study is the flow of blood through a regurgitant mitral valve. Mitral regurgitation (MR) is a valvular disease characterised by abnormal leaking of blood through the mitral valve from the left ventricle into the left atrium of the heart (see Figure 1). In certain cases, the regurgitant jet ‘hugs’ the wall of the heart’s atrium as shown in Figure 1 (right). These wall-hugging, non-symmetric regurgitant jets have been observed at low Reynolds numbers (Albers et al. 2004; Vermeulen et al. 2009) and are said to undergo the Coanda effect (Wille and Fernholz 1965). Such jets represent one of the biggest challenges in echocardiographic assessment of MR (Ginghina 2007). In Quaini, Glowinski, and Canic (2016), Pitton, Quaini, and Rozza (2017), Pitton and Rozza (2017), Wang et al. (2017), Hess et al. (2018) and Hess, Quaini, and Rozza (2018), we made a connection between the cardiovascular and bioengineering literature reporting on the Coanda effect in MR and the fluid dynamics literature with the goal of identifying and understanding the main features of the corresponding flow conditions.

In Quaini, Glowinski, and Canic (2016), Pitton, Quaini, and Rozza (2017), Pitton and Rozza (2017), Wang et al. (2017), Hess et al. (2018) and Hess, Quaini, and Rozza (2018), we studied what triggers the Coanda effect in a simplified setting by reformulating the problem in terms of the hydrodynamic stability of solutions of the incompressible Navier–Stokes equations in contraction-expansion channels with straight walls. Such channels have the same geometric features of MR and the wall-hugging effect is nothing but a symmetry breaking bifurcation. Here, we extend the study to contraction-expansion channels with curved walls as a first step towards more realistic geometries and eventually fluid–structure interaction.

For numerical treatment of the incompressible Navier–Stokes problem, we apply the spectral element method (SEM), which uses high-order polynomial ansatz functions such as Legendre polynomials. See, e.g. Patera (1984), Canuto et al. (2006, 2007), and Karniadakis and Sherwin (2005), Fick et al. (2018) for
applications in fluid dynamics. With a coarse partitioning of the computational domain into spectral elements, the high-order \textit{ansatz} functions are prescribed over each element. The \textit{ansatz} functions are modified for numerical stability and to enable continuity across element boundaries. Let \( p \) be the order of the polynomial. Typically, an exponential error decay under \( p \)-refinement can be observed, which provides computational advantages over more standard finite element methods.

Varying wall curvature and kinematic viscosity are considered for the parametric model order reduction. From a set of sampled high-order solves, a reduced-order model is generated, which approximates the high-order solutions and allows fast parameter sweeps of the two-dimensional parameter domain. The offline-online decomposition required for fast reduced-order parameter evaluations is established with the \textit{empirical interpolation method} (Barrault et al. 2004; Quarteroni and Rozza 2007; Rozza 2009; Chaturantabut and Sorensen 2010; Maday et al. 2015). The reduced-order modelling (ROM) techniques described in this work are implemented in open-source project ITHACA-SEM. This extends our previous work (Hess, Quaini, and Rozza 2018; Hess and Rozza 2018) to bifurcations in geometries with non-affine geometry variations.

The outline of the paper is as follows. In Section 2, the model problem is defined and the parametric variations are explained. Section 3 provides details on the spectral element discretisation and Section 4 explains the model order reduction with the empirical interpolation. Numerical results are presented in Section 5, while in Section 6 conclusions are drawn and future perspectives and developments are discussed.

2. Problem formulation

Let \( \Omega \in \mathbb{R}^2 \) be the computational domain. Incompressible, viscous fluid motion in spatial domain \( \Omega \) over a time interval \((0, T)\) is governed by the incompressible \textit{Navier–Stokes} equations:

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \Delta \mathbf{u} + \mathbf{f},
\]

\[
\nabla \cdot \mathbf{u} = 0,
\]

where \( \mathbf{u} \) is the vector-valued velocity, \( p \) is the scalar-valued pressure, \( \nu \) is the kinematic viscosity and \( \mathbf{f} \) is a body forcing. Boundary and initial conditions are prescribed as

\[
\mathbf{u} = \mathbf{d} \quad \text{on } \Gamma_D \times (0, T),
\]

\[
\nabla \mathbf{u} \cdot \mathbf{n} = \mathbf{g} \quad \text{on } \Gamma_N \times (0, T),
\]

\[
\mathbf{u} = \mathbf{u}_0 \quad \text{in } \Omega \times 0,
\]

with \( \mathbf{d}, \mathbf{g} \) and \( \mathbf{u}_0 \) given and \( \partial \Omega = \Gamma_D \cup \Gamma_N, \Gamma_D \cap \Gamma_N = \emptyset \). The \textit{Reynolds} number \( \text{Re} \), which characterises the flow (Holmes, Lumley, and Berkooz 1996), depends on \( \nu \), a characteristic velocity \( U \), and a characteristic length \( L \):

\[
\text{Re} = \frac{UL}{\nu}.
\]

We are interested in the steady states, i.e. solutions where \( \partial \mathbf{u} / \partial t \) vanishes. The high-order simulations are obtained through time-advancement, while the reduced-order solutions are computed through fixed-point iterations.

2.1. Nonlinear solver

The \textit{Oseen}-iteration is a secant modulus fixed-point iteration, which in general exhibits a linear rate of
convergence (Burger 2010). It solves for a steady-state solution, i.e. \( \frac{\partial u}{\partial t} = 0 \) is assumed. Given a current iterate (or initial condition) \( u^k \), the next iterate \( u^{k+1} \) is found by solving the following linear system:

\[
-\nu \Delta u^{k+1} + (u^k \cdot \nabla) u^{k+1} + \nabla p = f \quad \text{in} \quad \Omega, \\
\nabla \cdot u^{k+1} = 0 \quad \text{in} \quad \Omega, \\
\quad u^{k+1} = d \quad \text{on} \quad \Gamma_D, \\
\nabla u^{k+1} \cdot n = g \quad \text{on} \quad \Gamma_N.
\]

Iterations are stopped when the relative difference between iterates falls below a predefined tolerance in a suitable norm, like the \( L^2(\Omega) \) or \( H^1_0(\Omega) \) norm.

### 2.2. Model description

We consider the channel flow through a narrowing created by walls of varying curvature and with variable kinematic viscosity. See Figures 2 and 3 for the steady-state velocity components for \( \nu = 0.15 \) in a geometry with straight walls and curved walls, respectively. In all the cases under consideration, the spectral element expansion uses modal Legendre polynomials of order \( p = 10 \) for the velocity. The pressure ansatz space is chosen of order \( p-2 \) to fulfill the inf-sup stability condition (Quarteroni and Valli 1994; Boffi, Brezzi, and Fortin 2013). A parabolic inflow profile is prescribed at the inlet (i.e. \( x = 0 \)) with horizontal velocity component \( u_x(0, y) = y(3 - y) \) for \( y \in [0, 3] \). At the outlet (i.e. \( x = 18 \)) we impose a stress-free boundary condition, while everywhere else a no-slip condition is prescribed. We consider symmetric boundary conditions, because we want to study the symmetry breaking due to the nonlinearity in problem (1)–(2). For a more realistic setting one would have to account for different inlet velocity profiles and the pulsatility of the flow (i.e. include the Strouhal number among the parameters).

Each curved wall is defined by a second-order polynomial, interpolating three prescribed points. While the points at the domain boundary \( y = 0 \) and \( y = 3 \) are kept fixed, the inner points are moved towards \( x = 0 \) in order to create an increasing curvature. The viscosity varies in the interval \( \nu \in [0.15, 0.2] \). We recall that

![Figure 2. Full order, steady-state solution in the geometry with straight walls and for \( \nu = 0.15 \): velocity in x-direction (top) and y-direction (bottom).](image1)

![Figure 3. Full order, steady-state solution in the geometry with curved walls with the largest considered curvature and for \( \nu = 0.15 \): velocity in x-direction (top) and y-direction (bottom).](image2)
the Reynolds number \( \text{Re} (6) \) depends on the kinematic viscosity. As Re is varied for each fixed geometry, a supercritical pitchfork bifurcation occurs: for \( \text{Re} \) higher than the critical bifurcation point, three solutions exit. Two of these solutions are stable, one with a jet towards the top wall and one with a jet towards the bottom wall, and one is unstable. The unstable solution is symmetric to the horizontal centreline at \( y = 1.5 \), while the jet of the stable solutions is said to undergo the Coanda effect.

In this investigation, we do not deal with recovering all bifurcation branches, but limit our attention to the stable branch of solutions with jets hugging the bottom wall. However, we remark that recovering all bifurcating solutions with model reduction methods is also possible. See, e.g. Herrero, Maday, and Pla (2013).

3. Spectral element full order discretisation

The spectral/hp element software framework we use for the numerical solution of problem (2) is Nektar++, version 4.4.0.2. The large-scale discretised system that has to be solved at each step of the Oseen-iteration can be written as

\[
\begin{bmatrix}
A & -D_{\text{bnd}}^T & B \\
-D_{\text{bnd}} & 0 & -D_{\text{int}} \\
B^T & -D_{\text{int}}^T & C
\end{bmatrix}
\begin{bmatrix}
\mathbf{v}_{\text{bnd}} \\
\mathbf{p} \\
\mathbf{v}_{\text{int}}
\end{bmatrix}
= 
\begin{bmatrix}
\mathbf{f}_{\text{bnd}} \\
\mathbf{0} \\
\mathbf{f}_{\text{int}}
\end{bmatrix},
\]

for fixed parameter vector \( \mathbf{\mu} \), which denotes the geometrical and physical parameters. In (7), \( \mathbf{v}_{\text{bnd}} \) and \( \mathbf{v}_{\text{int}} \) denote the arrays of the velocity degrees of freedom on the boundary and in the interior of the domain, respectively. The array of the pressure degrees of freedom is denoted by \( \mathbf{p} \). The forcing terms on the boundary and interior are \( \mathbf{f}_{\text{bnd}} \) and \( \mathbf{f}_{\text{int}} \), respectively. Next, we explain the matrix blocks.

Matrix \( A \) assembles the boundary-boundary velocity coupling, \( B \) the boundary-interior velocity coupling, \( B^T \) the interior-boundary velocity coupling, and \( C \) assembles the interior-interior velocity degree of freedom coupling. The matrices \( D_{\text{bnd}} \) and \( D_{\text{int}} \) assemble the pressure-velocity boundary and pressure-velocity interior contributions. Due to the varying geometry, each matrix is dependent on the parameter \( \mathbf{\mu} \).

The linear system (7) is assembled in local degrees of freedom, i.e. \( \text{ansatz} \) functions with support extending over spectral element boundaries are treated separately for each spectral element. See Karniadakis and Sherwin (2005) for detailed explanations. As a result, matrices \( A, B, \tilde{B}, C, D_{\text{bnd}} \) and \( D_{\text{int}} \) have a block structure, with each block corresponding to a spectral element. This allows for an efficient matrix assembly since each spectral element is independent from the others, but the local degrees of freedom need to be gathered into the global degrees of freedom in order to obtain a non-singular system.

The boundary-boundary global element coupling is achieved with the rectangular assembly matrix \( M \), which gathers the local boundary degrees of freedom. Multiplication of the first row of (7) by \( M^T \) sets the boundary-boundary coupling in local degrees of freedom:

\[
\begin{bmatrix}
M^T MA & -M^T MD_{\text{bnd}}^T & M^T MB \\
-M_{\text{bnd}} & 0 & -D_{\text{int}} \\
\tilde{B}^T & -D_{\text{int}}^T & C
\end{bmatrix}
\begin{bmatrix}
\mathbf{v}_{\text{bnd}} \\
\mathbf{p} \\
\mathbf{v}_{\text{int}}
\end{bmatrix}
= 
\begin{bmatrix}
M^T M \mathbf{f}_{\text{bnd}} \\
\mathbf{0} \\
\mathbf{f}_{\text{int}}
\end{bmatrix}.
\]

The action of the matrix in (8) on the prescribed Dirichlet boundary conditions is computed and added to the source term. Since the Dirichlet boundary conditions are known, the corresponding equations are removed from the system. Let \( N_\delta \) denote the system size after removal of the known boundary conditions. The resulting system of high-order dimension \( N_\delta \times N_\delta \) is composed of the block matrices and depends on the parameter \( \mathbf{\mu} \). For simplicity of notation, we will write such system in compact form as:

\[
\mathcal{A}(\mathbf{\mu}) \mathbf{x}(\mathbf{\mu}) = \mathbf{f}(\mathbf{\mu}).
\]

4. Reduced-order space generation

The ROM computes an approximation to the full order model using a few modes of the POD as \( \text{ansatz} \) functions (Lassila et al. 2014). To achieve a computational speed-up, the matrix assembly for a new parameter of interest is independent of the large-scale discretisation size \( N_\delta \). The \textit{empirical interpolation method} (see Section 4.1) computes an affine parameter dependency, which enables an offline-online decomposition (see Section 4.2). After a time-intensive offline phase, reduced order solves can be evaluated quickly over the parameter range of interest.

The POD of \( N \) (typically small) uniformly sampled full-order solves, called \textit{snapshots}, is performed. The most dominant modes corresponding to 99.99%
of the POD energy (as suggested in Lassila et al. 2014) form the projection matrix \( U \in \mathbb{R}^{N_{x} \times N} \) and implicitly define the low-order space \( V_N = \text{span}(U) \). The large-scale system (9) is then projected onto the reduced-order space:

\[
U^T A(\mu) U x_N(\mu) = U^T f(\mu). \tag{10}
\]

The low order solution \( x_N(\mu) \) approximates the large-scale solution as \( x(\mu) \approx U x_N(\mu) \). The stability properties of the full-order model do not necessarily carry over to the reduced-order model, which can introduce instabilities. In particular, the reduced-order inf-sup stability constant might approach zero for some parameter value, while the full-order inf-sup stability constant is bounded away from zero. One way to alleviate this problem is by using inf-sup supremizers (Lassila et al. 2014) or considering space-time variational approaches (Yano and Patera 2013).

4.2. Offline-online decomposition

The offline-online decomposition (Hesthaven, Rozza, and Stamm 2016) enables the computational speed-up of the ROM approach in many-query scenarios. It relies on an affine parameter dependency, such that all computations depending on the high-order model size \( N_\delta \) can be performed in a parameter-independent offline phase. Then, the input–output evaluation performed online is independent of \( N_\delta \) and thus fast.

After applying the empirical interpolation method in the geometry parameter, the parameter dependency is cast in an affine form. Therefore, there exists an affine expansion of the system matrix \( A(\mu) \) in the parameter \( \mu \) given by (11). To achieve fast reduced order solves, the offline-online decomposition computes the parameter independent projections offline, which are stored as small-sized matrices of the order \( N \times N \). Since in an Oseen-iteration each matrix is dependent on the previous iterate, the submatrices corresponding to each basis function are assembled and then formed online using the affine expansion computed from the EIM and a fast evaluation of a single matrix entry as required by the EIM coefficient functions \( \tau_i \).

5. Numerical results

Snapshot solutions are sampled over a uniform \( 8 \times 9 \) grid from the full-order model, with 8 samples along the \( v \) parameter direction and 9 samples along the geometry parameter direction. The number of required snapshot computations might potentially be reduced when using a greedy sampling, which requires error indicators or error estimators. See Hesthaven, Rozza, and Stamm (2016). Error estimation does even allow a certification of the ROM accuracy, but it requires an estimation of the inf-sup constant as well as a bound on the empirical interpolation error.

The vertical velocity at the point \( (2, 1.5) \) is used to generate the bifurcation diagram reported in Figure 4. As expected, in a fixed geometry the symmetry breaking bifurcation occurs when the Re exceeds a critical value. It is very interesting to observe that such critical value increases as the wall curvature increases, i.e. as the walls get more curved, the stronger the inertial forces need to be to break the symmetry of the solution. This means that the estimates for the critical Re in 3D geometries with straight walls found in Pitton, Quaini, and Rozza (2017) and Wang et al. (2017)
provide a lower bound for the critical Re at which the Coanda effect is observed in vivo (see Figure 1 (right)). We remark that the 2D case can be seen as a limit of the 3D case for channel depth tending to infinity. In Pitton, Quaini, and Rozza (2017), the influence of the channel depth on the flow pattern is investigated. It is shown that non-symmetry jets appear at higher Re as the channel depth gets smaller. Thus, we expect that the critical Re for the symmetry breaking in a 3D channel with curved walls to be higher than the values reported here for a 2D channel. This indicates the Coanda effect occurs in mitral valves with elongated orifices (corresponding to deeper channels).

The accuracy of the ROM is assessed using $N = 72$ snapshots for the POD to recover the original snapshot data. Figure 5 shows the decay of the energy of the POD modes. To reach the typical threshold of 99.99% on the POD energy, $N = 33$ POD modes are required as RB ansatz functions.

Figure 6 shows the bifurcation diagram of the reduced-order model with $N = 20$ basis functions. The absolute error at the point value is less than 0.01 at 46 parameter locations and less than 0.1 at 63 parameter locations. This indicates, that the high-order solutions have been well-resolved at these configurations. There are a few outliers, where the iteration scheme did not converge and the value for the last iterate is shown. Most likely this can be resolved by taking a finer snapshot sampling into account or by using a localised reduced-order modelling approach (Hess et al. 2018).

The empirical interpolation method relies on the fast computation of a few matrix entries during the online phase. Since the spectral element ansatz functions have support over a whole spectral element, this operation cannot be performed as fast as with a finite element or finite volume method for instance, where ansatz functions have a local support. Nevertheless, the computational gain is significant after the affine form has been established. The time requirement for a single fixed point iteration step reduces from about 10 s to 0.1 s.

6. Conclusion and outlook

We proposed a reduced-order model that combines empirical interpolation method and a POD reduced basis technique to recover full-order solutions of the Navier–Stokes equations in domains with walls of varying curvature (non-affine variation). The non-linear geometry changes allow to simulate more realistic scenarios in the context of the Coanda effect in cardiology, but also require a fine sampling at the snapshot and empirical interpolation level. Since the model
problem studied here undergoes a supercritical pitchfork bifurcation, introducing non-unique solutions, further numerical techniques are required to recover all bifurcation branches. The spectral element method is a suitable method for these tasks. However, the computational gain that one can expect is not as significant as in the case of methods using ansatz functions with a local support, such as the finite element method.

As a next step, we will enhance the reduced-order model proposed here by using localises bases in order to recover every solutions in the considered parameter domain with high accuracy.

Notes
1. https://github.com/mathLab/ITHACA-SEM
   and https://mathlab.sissa.it/ITHACA-SEM
2. See www.nektar.info.

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No potential conflict of interest was reported by the authors.

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ORCID
Annalisa Quaini http://orcid.org/0000-0001-9686-9058
Gianluigi Rozza http://orcid.org/0000-0002-0810-8812

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