RENORMALIZATION CONDITIONS AND NON-DIAGRAMMATIC APPROACH TO RENORMALIZATIONS

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Abstract

The representation of the bare parameters of Lagrangian in terms of total vertex Green's functions is used to obtain the general form of renormalization conditions. In the framework of our approach renormalizations can be carried out without treatment to Feynman diagrams.
1 Introduction

The representation of the bare parameters of Lagrangian in terms of Green’s functions is useful for investigation of a great variety of subjects in the quantum field theory and statistical physics. For example, this approach was applied to the theory of self-generating interactions [1], critical phenomena [2], study of the second (third, fourth etc.) Legendre transformation [3] and so on. We want to show that the representation like this may be used successfully in the renormalization theory, specifically, to obtain the renormalization conditions.

The renormalization is the redefinition of the bare parameters of Lagrangian through inserting the infinities connected with the loop integrals into the bare parameters [4]. But to divide each infinite integral into the finite and infinite parts we must carry out the subtraction procedure which has ambiguities owing to the different choices of the subtraction point (or the mass parameter $\mu$ in the dimensional regularization). Therefore we have to impose some renormalization conditions. Usually these conditions are postulated on the strength of some general considerations [5], [6], [7]. We propose a method which enables us to get these conditions. Besides, as it will be shown, within our formulation we can deal with usual (ultraviolet) infinities without treatment to Feynmann diagrams.

The purpose of this article is to look into the old approach - the representation of the bare parameters in terms of Green’s functions - in the light of the renormalizations and generalise it in such a way that the bare parameters can be expressed consistently in terms of renormalized vertex functions $\Gamma^{(n)}$ which themselves depend on a renormalized mass and coupling constant. This program leads to the usual renormalization conditions automatically.

The article is organized as follows: in Sec.2 we introduce a method which allows us to represent the bare mass and coupling constant in terms of the total vertex Green’s functions and obtain the corresponding expressions. In Sec.3 the renormalization conditions are derived in the most general form. We shall discuss them thoroughly at the second order of $\hbar$ and show how to apply our approach to higher orders.
2 Bare mass and coupling constant in terms of the vertex functions

For definiteness we shall consider scalar fields. Let us introduce a generating functional of connected Green’s functions by the path integral expression

\[
\exp \left( \frac{i}{\hbar} W[J] \right) = N_0^{-1} \int D\varphi \exp \left( \frac{i}{\hbar} S[\varphi] + \frac{i}{\hbar} \int dx J(x) \varphi(x) \right),
\]

where

\[
N_0 = \int D\varphi \exp \left( \frac{i}{\hbar} S[\varphi] \right).
\]

The generating functional \( \Gamma[\Phi] \) of the vertex functions is determined as

\[
\Gamma[\Phi] = W[J] - \int dx J(x) \Phi(x),
\]

\[
\Phi(x) = \frac{\delta W[J]}{\delta J(x)}.
\]

The expression (3) is the functional Legendre transformation which introduce new functional argument \( \Phi \) instead of the functional argument \( J \). It follows from (3) that

\[
\frac{\delta \Gamma[\Phi]}{\delta \Phi(x)} = -J(x).
\]

We can expand \( \Gamma[\Phi] \) in the Taylor series

\[
\Gamma[\Phi] = \sum \frac{1}{n!} \int dx_1 \cdots dx_n \Gamma^{(n)}(x_1, \cdots, x_n) \Phi(x_1) \cdots \Phi(x_n),
\]

where

\[
\Gamma^{(n)}(x_1, \cdots, x_n) = \left. \frac{\delta^n \Gamma[\Phi]}{\delta \Phi(x_1) \cdots \delta \Phi(x_n)} \right|_{\Phi(x_1) = \cdots = \Phi(x_n) = 0}
\]

are the vertex functions.

In momentum space \( \Gamma^{(n)} \) is presented as

\[
\Gamma^{(n)}(x_1, \cdots, x_n) = \int \frac{dk_1}{(2\pi)^4} \cdots \frac{dk_n}{(2\pi)^4} (2\pi)^4 \delta(k_1 + \cdots + k_n) \times
\]

\[
\times \exp i(k_1 x_1 + \cdots + k_n x_n) \Gamma^{(n)}(k_1, \cdots, k_n).
\]
Noting that the integral (1) is the functional Fourier transformation one can write the formula for the inverse transformation

$$\exp \left( \frac{i}{\hbar} S[\varphi] \right) = N \int D(\frac{1}{\hbar} J) \exp \left( \frac{i}{\hbar} W[J] - \frac{i}{\hbar} \int dx J(x) \varphi(x) \right),$$

(9)

where \( N = (2\pi)^{-\nu} N_0 \) (\( \nu \) is the dimensionality of the space \( \varphi \)).

Using (3) and replacing of \( J \) by \( \Phi \) we get

$$\exp \left( \frac{i}{\hbar} S[\varphi] \right) = N \int D\Phi \det \left[ -\frac{1}{\hbar} \frac{\delta^2 \Gamma}{\delta \Phi^2} \right] \times 
\times \exp \left( \frac{i}{\hbar} \Gamma[\Phi] - \frac{i}{\hbar} \int dx \frac{\delta \Gamma}{\delta \Phi(x)} (\Phi(x) - \varphi(x)) \right).$$

(10)

It is convenient for us to introduce the following notation

$$F[\Phi, \varphi] = \det \left[ -\frac{1}{\hbar} \frac{\delta^2 \Gamma}{\delta \Phi^2} \right] \times 
\times \exp \left( \frac{i}{\hbar} \Gamma[\Phi] - \frac{i}{\hbar} \int dx \frac{\delta \Gamma}{\delta \Phi(x)} (\Phi(x) - \varphi(x)) \right).$$

(11)

Let us differentiate the both sides of (10) with respect to \( \varphi(x) \)

$$\frac{\delta S}{\delta \varphi(x)} \exp \left( \frac{i}{\hbar} S[\varphi] \right) = N \int D\Phi \frac{\delta \Gamma}{\delta \Phi(x)} F[\Phi, \varphi].$$

(12)

Now let us expand \( \delta \Gamma / \delta \Phi \) in the integrand in the Taylor series

$$\frac{\delta S}{\delta \varphi(x)} \exp \left( \frac{i}{\hbar} S[\varphi] \right) =
= N \Gamma^{(1)}(x) \int D\Phi F[\Phi, \varphi] + N \int dy_1 \Gamma^{(2)}(x, y_1) \int D\Phi F[\Phi, \varphi] + 
+ \frac{1}{2!} N \int dy_1 dy_2 \Gamma^{(3)}(x, y_1, y_2) \int D\Phi F[\Phi, \varphi] + 
+ \frac{1}{3!} N \int dy_1 dy_2 dy_3 \Gamma^{(4)}(x, y_1, y_2, y_3) \int D\Phi F[\Phi, \varphi] + \cdots.$$  

(13)

Due to the invariance of the measure \( DJ \) in (9) with respect to the translation \( J \rightarrow J + \varepsilon \) where \( \varepsilon \) is well diminishing function [4], we have

$$0 = N \int D(\frac{1}{\hbar} J) \left( \frac{\delta W}{\delta J(x)} - \varphi(x) \right) \exp \left( \frac{i}{\hbar} W[J] - \frac{i}{\hbar} \int dx J(x) \varphi(x) \right),$$

(14)
or, in terms of $\Phi$ and $\Gamma[\Phi]$

$$N \int D\Phi \Phi(x) F[\Phi, \varphi] = \varphi(x) \exp \left( \frac{i}{\hbar} S[\varphi] \right).$$  \hfill (15)

Using such a technique we can obtain the following relations

$$N \int D\Phi \Phi(y_1) \cdots \Phi(y_3) F[\Phi, \varphi] = \varphi(y_1) \cdots \varphi(y_3) \exp \left( \frac{i}{\hbar} S[\varphi] \right) +$$

$$+ i \hbar N \int D\Phi \left\{ - \frac{\delta^2 \Gamma}{\delta \Phi(y_1) \delta \Phi(y_2)} \right\}^{-1} \Phi(y_3) + \cdots \right\} F[\Phi, \varphi] + O(\hbar^2),$$

$$N \int D\Phi \Phi(y_1) \cdots \Phi(y_5) F[\Phi, \varphi] = \varphi(y_1) \cdots \varphi(y_5) \exp \left( \frac{i}{\hbar} S[\varphi] \right) +$$

$$i \hbar N \int D\Phi \left\{ - \frac{\delta^2 \Gamma}{\delta \Phi(y_1) \delta \Phi(y_2)} \right\}^{(-1)} \Phi(y_3) \cdots \Phi(y_5) + \cdots \right\} F[\Phi, \varphi] + O(\hbar^2)$$

(16) \hfill (17)

(The dots in the figure brackets in both expressions means the terms with permutations of arguments).

The expression like $(\delta^2 \Gamma/\delta \Phi^2)^{-1}$ in the functional integrals must be expanded in series in powers of $\Phi$. Then there will be the terms like $\hbar \int D\Phi \cdots \Phi^2 \exp \left( \frac{i}{\hbar} S[\varphi] \right)$ in the expressions (16) and (17). Ones may be replaced by the terms $\hbar \varphi \cdots \varphi \exp \left( \frac{i}{\hbar} S[\varphi] \right)$. This substitution gives the mistake of the order $\hbar^2$.

The classical action for scalar $\lambda \varphi^4$ model at 4-dimensional space-time is presented as follows

$$S[\varphi] = - \frac{1}{2} \int d^4 x d^4 y \varphi(x) K(x - y) \varphi(y) - \frac{\lambda}{4!} \int d^4 x \varphi^4(x),$$

$$\hfill (18)$$

where

$$K(x - y) = (\partial^2 + m^2) \delta^4(x - y),$$

and $m$ and $\lambda$ are bare quantities. Thus we have

$$\frac{\delta S[\varphi]}{\delta \varphi(x)} = - \int dy K(x - y) \varphi(y) - \frac{\lambda}{3!} \varphi^3(x).$$

Further we shall omit index 4 in all the integrals.
Substituting (15)-(17) and (19) into (13) and equating the expressions in the fronts of the same powers of \( \varphi \) at both sides we get (up to the first order in \( \bar{\hbar} \))

\[
-(\partial^2 + m^2)\delta(x - y) = \Gamma^{(2)}(x, y) - \frac{i\bar{\hbar}}{2} \int dy_1 dy_2 \Gamma^{(4)}(x, y_1, y_2, y) \left[ \Gamma^{(2)}(y_1, y_2) \right]^{-1},
\]

\[
-\lambda \delta(x - y_1)\delta(x - y_2)\delta(x - y_3) = \Gamma^{(4)}(x, y_1, y_2, y_3) + \frac{3}{2!} i\bar{\hbar} \int dz_1 dz_2 du_1 du_2 \Gamma^{(4)}(x, z_1, z_2, y_3) \left[ \Gamma^{(2)}(z_1, u_1) \right]^{-1} \left[ \Gamma^{(2)}(z_2, u_2) \right]^{-1} \times \Gamma^{(4)}(y_1, y_2, u_1, u_2) - \frac{3!}{5!} 10 i\bar{\hbar} \int dz_1 dz_2 \Gamma^{(6)}(x, z_1, z_2, y_1, y_2, y_3) \left[ \Gamma^{(2)}(z_1, z_2) \right]^{-1}
\]

\[
0 = \frac{1}{5!} \Gamma^{(6)}(x, y_1, y_2, y_3, y_4, y_5) + \frac{i\bar{\hbar}}{24} \int dz_1 dz_2 du_1 du_2 \Gamma^{(6)}(x, z_1, z_2, y_3, y_4, y_5) \times \left[ \Gamma^{(2)}(z_1, u_1) \right]^{-1} \left[ \Gamma^{(2)}(z_2, u_2) \right]^{-1} \Gamma^{(4)}(y_1, y_2, u_1, u_2) + \cdots.
\]

(in (22) we have written down all the terms of the order of \( \bar{\hbar}^0 \) but not all terms of the order of \( \bar{\hbar} \)). Due to the absence of terms with the odd powers of \( \varphi \) in \( L_{int} \) the Green’s functions \( \Gamma^{(n)} \) with odd \( n \) are absent too, i.e. \( \Gamma^{(1)} = \Gamma^{(3)} = \cdots = 0 \).

We can rewrite the three latter expressions in more compact form (the sense of these condensed notations becomes clear from the comparison the old and the new expressions)

\[
-(\partial^2 + m^2)\delta_{ij} = \Gamma^{(2)}_{ij} - \frac{i\bar{\hbar}}{2} \Gamma^{(4)}_{ijkl} \left[ \Gamma^{(2)} \right]^{-1}_{kl},
\]

\[
-\lambda \delta_{ij} \delta_{ik} \delta_{il} = \Gamma^{(4)}_{ijkl} + \frac{3}{2!} i\bar{\hbar} \Gamma^{(4)}_{imlp} \left[ \Gamma^{(2)} \right]^{-1}_{mn} \left[ \Gamma^{(2)} \right]^{-1}_{pr} \Gamma^{(4)}_{jknr} - \frac{3!}{5!} 10 i\bar{\hbar} \Gamma^{(6)}_{imnjkld} \left[ \Gamma^{(2)} \right]^{-1}_{mn}
\]

\[
0 = \frac{1}{5!} \Gamma^{(6)}_{ijklmn} + \frac{i\bar{\hbar}}{24} \Gamma^{(6)}_{iprlnm} \left[ \Gamma^{(2)} \right]^{-1}_{ps} \left[ \Gamma^{(2)} \right]^{-1}_{rt} \Gamma^{(4)}_{jknr} + \cdots.
\]

If we use the similar notations for \( S \)

\[
S[\varphi] = \sum A^m \varphi^m
\]
then the general form of our expansions will be

\[ A^m = \sum_{\beta=0}^{\infty} \sum_{n, \beta=2^n, 2^n} s h^l \int \left( \prod_{n_i} \left[ \Gamma^{(n_i)} \right]^{\alpha_i} \right) \left[ \Gamma^{(2)} \right]^{-\beta} (d^4x)^{2\beta}, \]

(27)

where \( n_i > 2, \alpha_i > 0; \) the symmetry factor \( s \) and the index \( l \) can be found directly from the expression (13).

So the "bare" quantities are expressed in terms of the total vertex Green's functions. This representation allows us to analyse the renormalization conditions by the most natural way. Such a programm will be discussed in the next section.

3 The renormalization conditions

The relations (20), (21), (22) enable us to get the renormalization conditions. Let us expand \( \Gamma^{(2)} (x, y) \) and \( \Gamma^{(4)} (x, y_1, y_2, y_3) \) over \( \bar{\hbar} \):

\[ \Gamma^{(2)} = \Gamma^{(2)}_0 + \bar{\hbar} \Gamma^{(2)}_1 + \bar{\hbar}^2 \Gamma^{(2)}_2 + \cdots, \]

(28)

\[ \Gamma^{(4)} = \Gamma^{(4)}_0 + \bar{\hbar} \Gamma^{(4)}_1 + \bar{\hbar}^2 \Gamma^{(4)}_2 + \cdots. \]

(29)

Substituting (28) and (29) into (20) and confining ourselves up to the first order in \( \bar{\hbar} \) we have

\[-(\partial^2 + m^2)\delta(x - y) = (\Gamma^{(2)}_0 (x, y) + \bar{\hbar} \Gamma^{(2)}_1 (x, y) + \cdots) -
\]

\[-\frac{i\bar{\hbar}}{2} \int dy_1 dy_2 \left( \Gamma^{(4)}_0 (x, y_1, y_2, y) + \cdots \right) \left[ \Gamma^{(2)}_0 (y_1, y_2) + \cdots \right]^{-1} + O(\bar{\hbar}^2). \]

(30)

Let us expand the parameters \( m \) and \( \lambda \) in powers of \( \bar{\hbar} \):

\[ m^2 = m_0^2 + \bar{\hbar} m_1^2 + \bar{\hbar}^2 m_2^2 + \cdots, \]

\[ \lambda = \lambda_0 + \bar{\hbar} \lambda_1 + \bar{\hbar}^2 \lambda_2 + \cdots, \]

(31)

here \( m_0^2, \lambda_0, m_1^2, \lambda_1 \) etc. are unknown so far; below we shall determine them. In the all integrals below we shall suppose some regularization to be introduced.

Comparing (30) and (31) we find

\[ \Gamma^{(2)}_0 (x, y) = -(\partial^2 + m_0^2)\delta(x - y), \]

(32)
\( \Gamma^{(2)}_1(x, y) - \frac{i}{2} \int dy_1 dy_2 \Gamma^{(4)}_0(x, y_1, y_2, y) \left[ \Gamma^{(2)}_0(y_1, y_2) \right]^{-1} = -m_1^2 \delta(x - y). \)  

(33)

From (21) we have

\( \Gamma^{(4)}_0(x, y_1, y_2, y_3) = -\lambda_0 \delta(x - y_1) \delta(x - y_2) \delta(x - y_3), \)

which leads to

\( \Gamma^{(2)}_1(x, y) + \frac{i}{2} \lambda_0 \delta(x - y) \left[ \Gamma^{(2)}_0(x, x) \right]^{-1} = -m_1^2 \delta(x - y). \)  

(35)

From the integral representation of \( \left[ \Gamma^{(2)}(x, y) \right]^{-1} \)

\[ \left[ \Gamma^{(2)}_0(x, y) \right]^{-1} = \frac{1}{(2\pi)^4} \int dk \frac{1}{k^2 - m_0^2} e^{-ik(x-y)}, \]

(36)

and from (35) we get

\[-m_1^2 \delta(x - y) = \Gamma^{(2)}_1(x, y) + \frac{i}{2} \lambda_0 \delta(x - y) \frac{1}{(2\pi)^4} \int dk \frac{1}{k^2 - m_0^2}. \]  

(37)

Performing Fourier transformation we have

\[-m_1^2 \delta(p_1 + p_2) = \Gamma^{(2)}_1(p_1, p_2) \delta(p_1 + p_2) + \frac{i\lambda_0}{2} \frac{\delta(p_1 + p_2)}{(2\pi)^4} \int dk \frac{1}{k^2 - m_0^2}. \]  

(38)

From (32) it follows that

\[ p^2 - m_0^2 = \Gamma^{(2)}_0(p). \]

(39)

Thus, (21) in momentum space (up to the first order in \( \hbar \)) has the form:

\[ p^2 - m_0^2 - \hbar m_1^2 = \Gamma^{(2)}_0(p) + \]

\[ + \hbar \left[ \Gamma^{(2)}_1(p) + \frac{i\lambda_0}{2} \frac{1}{(2\pi)^4} \int dk \frac{1}{k^2 - m_0^2} \right]. \]  

(40)

From (21) we get
\[ -\lambda_0 \delta(p_1 + p_2 + p_3 + p_4) = \Gamma^{(4)}_0(p_1, p_2, p_3, p_4) \delta(p_1 + p_2 + p_3 + p_4), \quad (41) \]

\[ -\lambda_1 \delta(p_1 + p_2 + p_3 + p_4) = \Gamma^{(4)}_1(p_1, p_2, p_3, p_4) \delta(p_1 + p_2 + p_3 + p_4) + \frac{3}{2} i \lambda_0^2 \delta(p_1 + p_2 + p_3 + p_4) \int \frac{dk}{(2\pi)^4} \frac{1}{k^2 - m_0^2} \frac{1}{(k - p_1 - p_2)^2 - m_0^2}, \quad (42) \]

and the final form of (21) (up to the first order in \( \hbar \))

\[-\lambda_0 - \hbar \lambda_1 = \Gamma^{(4)}_0(p_1, p_2, p_3, p_4) + \hbar \left[ \Gamma^{(4)}_1(p_1, p_2, p_3, p_4) + \frac{3}{2} i \lambda_0^2 \int \frac{dk}{(2\pi)^4} \frac{1}{k^2 - m_0^2} \frac{1}{(k - p_1 - p_2)^2 - m_0^2} \right]. \quad (43)\]

Let us introduce the demand for the total ("dressed") vertex functions to be finite. It's following from this requirement that

(i) \( m_0 \) and \( \lambda_0 \) are finite quantities;

(ii) \( m_1 \) and \( \lambda_1 \) are infinite ones and they cancel out the infinite parts of integrals in r.h.s. of (40) and (43); for example, if we make use of the dimensional regularization then infinite parts are equal to

\[-\frac{1}{(2\pi)^4} \frac{\lambda_0 m_0^2}{32\pi^2} \frac{1}{4 - n} \quad and \quad -\frac{1}{(2\pi)^4} \frac{\lambda_0^2}{32\pi^2} \frac{3}{4 - n}.\]

Now \( \Gamma^{(2)}_0, \Gamma^{(2)}_1, \Gamma^{(4)}_0, \Gamma^{(4)}_1 \) etc. depend only on finite quantities - \( m_0 \) and \( \lambda_0 \). Acting the same manner we can have \( \Gamma^{(n)} \) expressed in term of renormalized mass and coupling constant.

Thus we have

\[ \Gamma^{(2)}(p) = p^2 - m_0^2 + \text{finite parts}, \quad (44) \]

\[ \Gamma^{(4)}(p_1, \cdots, p_4) = -\lambda_0 + \text{finite parts}. \quad (45) \]

where "finite parts" mean the finite addition of the above-mentioned integrals; they give contributions to the total vertex Green’s functions.

As the extraction of infinite parts of any integrals has ambiguities the finite parts of (44) and (43) are also ambiguous. Hence, formulas (44) and (45) represent a general form of renormalization conditions. The finite parts in (44) and (45) can be specified only by specifications of \( m_0 \) and \( \lambda_0 \). The
requirement that \( m_0 \) and \( \lambda_0 \) are physical mass and coupling constant is equivalent to putting finite parts in (44) and (45) equal to zero. Putting

\[
\Gamma^{(2)}(p = 0) = -m_0^2, \tag{46}
\]

\[
\Gamma^{(4)}(p_1 = \cdots = p_4 = 0) = -\lambda_0 \tag{47}
\]

we introduce some new constants \( m_0 \) and \( \lambda_0 \) which are related to the previous ones through finite renormalization [6], [7], [8].

Let us write down the terms of the order \( \hbar^2 \) from (13)

\[
-m_2^2 \delta(x - y) = \Gamma^{(2)}_2(x, y) + 
-\frac{i}{4} \lambda_0 m_1^2 \delta(x - y) \int dz \left[ \Gamma^{(2)}_0(x, z) \right]^{-1} \left[ \Gamma^{(2)}_0(z, x) \right]^{-1} - 
-\frac{1}{2} \lambda_1 \delta(x - y) \left[ \Gamma^{(2)}_0(x, x) \right]^{-1} - 
-\frac{1}{4} \lambda_0 \delta(x - y) \int dz \left[ \Gamma^{(2)}_0(x, z) \right]^{-1} \left[ \Gamma^{(2)}_0(z, z) \right]^{-1} \left[ \Gamma^{(2)}_0(z, x) \right]^{-1} - 
-\frac{1}{6} \lambda_2 \left[ \Gamma^{(2)}_0(x, y) \right]^{-3}. \tag{48}
\]

Substituting \( \left[ \Gamma^{(2)}_0(x, y) \right]^{-1} \) into (48) we get

\[
-m_2^2 \delta(x - y) = \Gamma^{(2)}_2(x, y) + 
-\frac{i}{4} \lambda_0 m_1^2 \delta(x - y) C_1 - \frac{i}{2} \lambda_1 \delta(x - y) C_2 - 
-\frac{1}{4} \lambda_0 \delta(x - y) C_3 - \frac{1}{6} \lambda_2 \left[ \Gamma^{(2)}_0(x, y) \right]^{-3} \tag{49}
\]

where

\[
C_1 = \frac{1}{(2\pi)^4} \int \frac{dk}{[k^2 - m_0^2]^2},
\]

\[
C_2 = \frac{1}{(2\pi)^4} \int \frac{dk}{k^2 - m_0^2},
\]

\[
C_3 = \frac{1}{(2\pi)^8} \int \frac{dk_1 dk_2}{[k_1^2 - m_0^2]^2 [k_2^2 - m_0^2]}.
\]

Performing Fourier transformation and restoring all the terms of lower orders we have

\[
p^2 - m_0^2 - \hbar m_1^2 - \hbar^2 m_2^2 = \Gamma^{(2)}_0(p) + \hbar \left[ \Gamma^{(2)}_1(p) + \frac{i}{2} \lambda_0 \int \frac{dk}{k^2 - m_0^2} \right] + 
+ \hbar^2 \left[ \Gamma^{(2)}_2(p) + \frac{i}{4} \lambda_0 m_1^2 C_1 - \frac{i}{2} \lambda_1 C_2 - \frac{1}{4} \lambda_0 C_3 - \frac{1}{6} \lambda_2 f \right], \tag{50}
\]

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where
\[ I = I(p) = \frac{1}{(2\pi)^8} \int \frac{dk_1 dk_2}{(k_1^2 - m_0^2)(k_2^2 - m_0^2)((p - k_1 - k_2)^2 - m_0^2)}. \]

and \( m_1^2 \) and \( \lambda_1 \) are determined by (11) and (43). Therefore we can write

\[ p^2 - m_0^2 - \hbar^2 m_2^2 = \Gamma_0^{(2)}(p) + \hbar \Gamma_1^{(2)}(p) + \hbar^2 \Gamma_2^{(2)}(p) + \]

\[ + \hbar^2 \left( -\frac{i}{4} \lambda_0 \frac{1}{(2\pi)^4} \frac{1}{32\pi^2} \frac{1}{4 - n} C_1 + \frac{i}{2} \frac{1}{(2\pi)^4} \frac{\lambda_0^2}{32\pi^2} \frac{1}{4 - n} C_2 - \frac{1}{4} \lambda_0 C_3 - \frac{1}{6} \lambda_0^2 I \right) + \text{finite const.}, \]

We can require \( m_2^2 \) to be infinite and equal to the constant infinite parts (including \( I(m_0^2) \)) of the bracket in rhs. But for the cancellation of the remaining infinity (which itself depend on \( p \)) we must require

\[ (p^2 - m_0^2)(1 - \frac{d\tilde{\Gamma}^{(2)}}{dp^2}(m_0^2) + \frac{1}{6} \lambda_0 \hbar^2 \frac{dI(m_0^2)}{dp^2}) = 0, \]

i.e.

\[ \frac{d\tilde{\Gamma}^{(2)}}{dp^2}(m_0^2) = 1 + \frac{1}{6} \lambda_0 \hbar^2 \frac{dI(m_0^2)}{dp^2} = Z_\varphi, \]

where

\[ \tilde{\Gamma}^{(2)} = \Gamma_0^{(2)} + \hbar \Gamma_1^{(2)} + \hbar^2 \Gamma_2^{(2)}. \]

If we introduce the renormalized (finite) 2 - points vertex \( \Gamma_R^{(2)} \) as follows

\[ \Gamma^{(2)} = Z_\varphi \Gamma_R^{(2)}; \]

then we can see that (48) leads to

\[ \frac{d\Gamma_R^{(2)}}{dp^2}(m_0^2) = 1. \]

Expanding (52) in another point \( m_0^2 \) (i.e., using the ambiguity in finite parts in (11) and (43)) we shall have \( Z_\varphi \) and (56) defined at the point \( m_0^2 \).

From now on we have to regard \( \Gamma^{(2)} \) in (28) as \( \Gamma_R^{(2)} \), but our formulas (32) - (41) will not change because they were obtained in the first order in \( \hbar \) and the distinction between \( \Gamma^{(2)} \) and \( \Gamma_R^{(2)} \) appears at least in the second order.
Nevertheless if we wanted to study higher orders of $\hbar$ we should take into consideration the fact that $\Gamma^{(n)}$ are products of two series in $\hbar$. On the other hand, according to (52) we can consider $m^2$ to be the product of two types of infinities. Hereby, in point of fact, we have introduced the wavefunction renormalization.

Now our main purpose is to prove the renormalizability of our model and investigate the question whether our renormalization conditions change in higher orders or not.

It is obvious, that if there were no terms like $I(p)$ which lead to the infinities depending on $p$, then the general form of the conditions (Eq.(39),(40)) would not change, because all constant infinities (i.e. terms without depending on $p$) may be cancelled out by the corresponding parts of the bare parameters $m^2$ and $\lambda$ from (31) in the same manner as it was done in (40),(43).

Moreover, having chosen the same subtraction point in all the orders of $\hbar$ we can always obtain conditions like (46),(47). The only problem can arise because of the necessity to introduce requirement like (55).

For the detailed investigation of this problem we should return to our main expression (27) and rewrite it in momentum representation:

$$A_p^{(m)} = \sum_{\beta=0}^{\infty} \sum_{\sum n_i\alpha_i - 2\beta = m} \left( \frac{d^4 p}{(2\pi)^4} \right)^{2\beta} \left( \prod_{n_i} \left[ \tilde{\Gamma}_p^{(n_i)} \right]^{\alpha_i} \right) \left[ \tilde{\Gamma}_p^{(2)} \right]^{-1} \beta, \quad (57)$$

where $\delta$-functions are put into $\tilde{\Gamma}$

$$\tilde{\Gamma}_p^{(m)}(p_1, \ldots, p_m) = \Gamma_p^{(m)}(p_1, \ldots, p_m) \delta(\sum_{i=1}^m p_i), \quad (58)$$

and integrations must be carried out over all the momenta in all $(\tilde{\Gamma}_p^{(2)})^{-1}$.

Acting the same manner as it was done in (32)-(43) and (48)-(50) we can have $\tilde{\Gamma}_p^{(m)}$ expressed in term of $(\tilde{\Gamma}_0^{(2)})^{-1}$ and $\tilde{\Gamma}_0^{(4)}$, i.e.

$$\tilde{\Gamma}_p^{(m)} = \Gamma_p^{(m)} \left[ (\tilde{\Gamma}_0^{(2)})^{-1}, \tilde{\Gamma}_0^{(4)} \right]. \quad (59)$$

As in our new notations

$$(\tilde{\Gamma}_0^{(2)})^{-1}(p_1, p_2) = \frac{\delta(p_1 + p_2)}{p_1^2 - m_0^2}, \quad (60)$$
\begin{equation}
\tilde{\Gamma}_0^{(4)}(p_1, \ldots, p_4) = -\lambda_0 \delta(p_1 + \ldots + p_4)
\end{equation}

we find from (57)

\begin{equation}
A_p^m = \sum_{\beta=0}^{\infty} K \int \left( \frac{d^4 p}{(2\pi)^4} \right)^{2\beta} \prod_{n_i} \left[ \tilde{\Gamma}_0^{(4)} \right]^{\frac{1}{4}(m+2\beta)} \left[ (\tilde{\Gamma}_0^{(2)})^{-1} \right]^{\beta},
\end{equation}

where \( K \) contains both the symmetry factor and \( \bar{\hbar} \). The representation like (62) i.e. the expansion in series over \( [\Gamma^{(2)}]^{-1} \) is more convenient for analysing the indices of overall divergency of infinite integrals.

Now we can find the index of divergency \( N_m \) of the common term in (62) (we mean, certainly, the index of \( p \) in integrand)

\begin{equation}
N_m = 4 \cdot 2\beta - 2\beta - 4 \left( \beta + \frac{1}{4}(m + 2\beta) - 1 \right) = 4 - m.
\end{equation}

Here

- \( 4 \cdot 2\beta \) appears from integrals over momenta,
- \(-2\beta \) from \( (\tilde{\Gamma}_0^{(2)})^{-1} \),
- \(-4 \left( \beta + \frac{1}{4}(m + 2\beta) - 1 \right) \) from all of the \( \delta \)-functions in \( (\tilde{\Gamma}_0^{(2)})^{-1} \), \( \tilde{\Gamma}_0^{(4)} \) and \(-1 \) corresponds to the common \( \delta \)-function (providing the conversation of 4-momentum) which must be took out from the integrand. Hence, in the expansion of any divergent integral in series of \( p^2 \) no terms exept possibly terms in front of \( (p^2)^0 \) (they have been already discussed) and \( (p^2)^1 \) are infinite because \( N_m = 2, 0, -2, \ldots \). The terms in front of \( (p^2)^1 \) can be always dealt with as well as it was done in (52) - (55). Thus, the obtained conditions have the same form irrespective of the order in \( \bar{\hbar} \) we want to study. In other words, having proved that the renormalization conditions have the same form in all orders of \( \bar{\hbar} \) we proved thereby, that our model is renormalizable.

Let us discuss briefly the unrenormalizable theories. In this case the infinities stand not only in front of the two first terms in expansion in \( p^2 \) but also in front of the higher terms. Therefore our reasons used for obtaining (14), (15), (53) will not be enough to cancel out the terms of orders older than \( (p^2)^1 \). In other words, no redefinition \( m^2 \) and \( \lambda \) and no conditions like (53) will be able to remove the infinities in the terms of order \( (p^2)^2, (p^2)^3 \) etc. That is why the renormalization conditions will make no sense any more as themselves will contain infinities.

So we have studied completely the three renormalization conditions which are usually postulated to renormalize the \( \lambda \varphi^4 \) model or to obtain the finite
effective action for this model. Although our method is applicable to derive
the conditions like (46), (47) for any other vertices (Γ(6), Γ(8) etc.) they are
scarcely useful in renormalization and we shall not discuss them in details.
We only note that these conditions at \( p_i = 0 \) have the form
\[
\Gamma^{(n)}(p_i = 0) = 0, \quad n > 4.
\]
(64)
This fact can be easily established the same way as it was done for Γ(2) and
Γ(4).

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