SECANT VARIETIES AND HIRSCHOWITZ BOUND ON VECTOR BUNDLES OVER A CURVE

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Abstract. For a vector bundle $V$ over a curve $X$ of rank $n$ and for each integer $r$ in the range $1 \leq r \leq n - 1$, the Segre invariant $s_r$ is defined by generalizing the minimal self-intersection number of the sections on a ruled surface. In this paper we generalize Lange and Narasimhan’s results on rank 2 bundles which related the invariant $s_1$ to the secant varieties of the curve inside certain extension spaces. For any $n$ and $r$, we find a way to get information on the invariant $s_r$ from the secant varieties of certain subvariety of a scroll over $X$. Using this geometric picture, we obtain a new proof of the Hirschowitz bound on $s_r$.

1. Introduction

Let $X$ be a smooth algebraic curve of genus $g \geq 2$. Let $V$ be a vector bundle over $X$ of rank $n$. For each $1 \leq r < n$, the $r$-th Segre invariant of $V$ is defined by

$$s_r(V) := \min\{r \deg(V) - n \deg(E)\},$$

where the minimum is taken over the subbundles $E$ of $V$ of rank $r$. If a subbundle $E$ realizes this minimum, that is, $s_r(V) = r \deg(V) - n \deg(E)$, then $E$ is called a maximal subbundle of $V$.

When $n = 2$ and $r = 1$, we have $s_1(V) = -e$ for the classical Segre invariant $e$ of the ruled surface $\mathbb{P}V$ defined by the minimal self-intersection number of the sections ([Ha], V §2). In general, the invariant $s_r(V)$ has an alternative definition in terms of the intersection numbers (see [La3]).

The invariant $s_r$ is lower semicontinuous: for any family $\{V_t : t \in T\}$ of rank $n$ bundles parameterized by a variety $T$, the sublocus

$$\{t \in T : s_r(V_t) \leq s\}$$

is closed for each $s$. Hence $s_r$ induces a natural stratification on the moduli space $U(n, d)$ of vector bundles over $X$ of rank $n$ and degree $d$. This stratification has been studied by several authors ([BrLa], [La1], [RuTe]).

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bound on ruled surfaces \[Na\] says that \[s_1(V) \leq g\] when \(n = 2\) and \(r = 1\). For arbitrary \(n\) and \(r\), Mukai and Sakai \[MuSa\] proved that \(s_r(V) \leq r(n - r)g\). The sharp upper bound of \(s_r\) was obtained by Hirschowitz:

**Proposition 1.1.** (\[Hi\], Théorème 4.4) Let \(V\) be a vector bundle of rank \(n\) over \(X\) of genus \(g\). For \(1 \leq r < n\), we have

\[
s_r(V) \leq r(n - r)(g - 1) + \varepsilon,
\]

where \(\varepsilon\) is the integer satisfying

\[
0 \leq \varepsilon \leq n - 1 \quad \text{and} \quad r(n - r)(g - 1) + \varepsilon \equiv r \deg(V) \mod n. \quad \square
\]

As the main result of this paper, we reprove this bound by relating the invariants \(s_r\) to the geometry of certain secant varieties. This yields generalizations of the results of Lange and Narasimhan for bundles of rank 2 (\[LaNa\], \[La2\]). Let us briefly review their results here.

One can show that every vector bundle \(V\) of rank 2 and degree \(d \gg 0\) fits into the exact sequence

\[
0 \to O_X \to V \to L \to 0
\]

for some line bundle \(L\). So the bundle \(V\) corresponds to a point \(v \in \mathbb{P}_L := \mathbb{P}H^1(X, L^{-1}) \cong \mathbb{P}H^0(X, K_X L)^\vee\).

The curve \(X\) maps into \(\mathbb{P}_L\) via the linear system \([K_X L]\). We consider the secant variety \(Sec^k X\) which is the closure of the union of \(\mathbb{P}^{k-1} \subset \mathbb{P}_L\) spanned by \(k\) general points of \(X\). The invariant \(s_1\) and the secant variety \(Sec^k X\) are related as follows.

**Proposition 1.2.** (\[LaNa\], Proposition 1.1) Suppose that \(s \equiv d \mod 2\) and \(4 - d \leq s \leq d\). Then \(s_1(V) \geq s\) if and only if \(v \not\in Sec^k X\), where \(k = \left\lceil \frac{d + g - 1}{2} \right\rceil\).

By using this, Lange \[La2\] reproved Nagata’s bound \(s_1(V) \leq g\): By Riemann-Roch, \(\dim \mathbb{P}H^1(X, L^{-1}) = d + g - 2\) for \(d \gg 0\). Since the curve \(X\) has no secant defect, \(\dim Sec^k X = 2k - 1\). Hence every \(v \in \mathbb{P}_L\) lies on \(Sec^m X\) for \(m = \left\lceil \frac{d + g - 1}{2} \right\rceil\). By Proposition 1.2, then,

\[
s_1(V) \leq 2m - d = 2 \left\lceil \frac{d + g - 1}{2} \right\rceil - d \leq g.
\]

In this paper, we reprove the Hirschowitz bound by establishing a statement which generalizes Proposition 1.2 to the case of rank \(r\) subbundles of rank \(n\) bundles for any pair \(r\) and \(n\) (Theorem 4.4). The key observation is that the locus of rank 1 vectors inside a certain scroll over \(X\) plays the role of the curve \(X\) in the rank 2 case. This idea recently appeared in the second author’s work on symplectic bundles of rank 4 over a curve. Since he studied the case of genus 2, the involved data was secant lines (\[Hit\], Lemma 8). Here we fully investigate the idea for curves of higher genus and higher secant varieties.
In Section 2, we arrange a basic setup for our study. In particular, the locus $\Delta$ of rank 1 vectors inside the scroll is defined.

In Section 3, we briefly recall the dictionary for interpreting the secants of the scrolls in terms of the data of elementary transformations.

In Section 4, the key technical result on the lifting of elementary transformations is proved. The criterion on the lifting is given in terms of the secant varieties of $\Delta$.

In Section 5, we apply this criterion to get the Hirschowitz bound. Once we have the criterion on the lifting, it is straightforward to get the expected bound on $s_r$. The only technical point is to show that the variety $\Delta$ has no secant defect. This requires some argument appealing to the Terracini lemma. The proof of the secant non-defectiveness of $\Delta$ is completed by applying Hirschowitz’ lemma saying that the tensor product of two general bundles is nonspecial.

2. The rank 1 locus

Let $X$ be a smooth algebraic curve of genus $g \geq 2$. Let $V$ be a vector bundle over $X$ of rank $n > 1$. Firstly, we establish a fact which justifies the validity of the forthcoming discussion on extension spaces.

**Lemma 2.1.** (i) Suppose that $V$ is a general stable bundle in $U(n, d)$ where $d > (2g - 1)n$. For every positive integer $r$ with $1 \leq r < n$, the bundle $V$ fits into an exact sequence

$$0 \to \mathcal{O}_X \otimes (n-r) \to V \to F \to 0$$

for some $F \in U(r, d)$.

(ii) Under the same assumption on $n$, $r$ and $d$, a general stable bundle $V$ is fitted into an exact sequence

$$0 \to E \to V \to F \to 0$$

for a general $E \in U(n-r, 0)$ and a general $F \in U(r, d)$.

**Proof.** (The following argument was kindly informed to us by Peter Newstead.)

The case when $r = 1$ was dealt with in the paper of Atiyah ([At] Theorem 3, where it is attributed to J.-P. Serre): Under the assumptions, we see that $h^0(X, V) - h^0(X, V(-x)) = n$ for each $x \in X$. Hence $V$ is generated by global sections and there is a surjection $H^0(X, V) \otimes \mathcal{O}_X \to V$. For each $x \in X$, let $N_x$ be the kernel of the evaluation map $H^0(X, V) \to V_x$. The union $N = \bigcup_{x \in X} N_x$ has dimension at most $h^0(X, V) - n + 1$, so there is an $(n-1)$-dimensional subspace $P \subset H^0(X, V)$ such that $P \cap N = \{0\}$. This means that there is an exact sequence

$$0 \to P \otimes \mathcal{O}_X \to V \to \det V \to 0.$$
For $r > 1$, the same argument yields an exact sequence

$$0 \to \mathcal{O}_X^{\oplus(n-r)} \to V \to F \to 0$$

for some $F$ of rank $r$. Here the stability of $F$ is not guaranteed, but $H^0(X,F^*) = 0$ by the stability of $V$. It is well-known that the bundles $F$ of fixed rank and degree satisfying $H^0(X,F^*) = 0$ form a bounded family. To get an irreducible family, apply the above Serre–Atiyah construction to those $F$: Consider the irreducible variety $T$ parameterizing all the extensions of the form

$$0 \to \mathcal{O}_X^{\oplus(r-1)} \to F \to L \to 0$$

where $L \in \text{Pic}^d(X)$. From the above construction, every $F$ satisfying $H^0(X,F^*) = 0$, in particular every stable bundle $F \in U(r,d)$, fits into some exact sequence in $T$. Now consider the family of extensions

$$0 \to \mathcal{O}_X^{\oplus(n-r)} \to V \to F \to 0$$

for all $F$ fitting into some exact sequence in $T$. This is again parameterized by an irreducible family and contains all stable bundles $V \in U(n,d)$. The proof of (i) is completed by noting that $F$ is stable for the general member of the family.

Statement (ii) follows from (i) by deforming $\mathcal{O}_X^{\oplus(n-r)}$ in $U(n-r,0)$ and $F$ in $U(r,d)$ respectively. \qed

Now we study the space $\mathbb{P} := \mathbb{P}H^1(X,Hom(F,E))$ associated to the exact sequence (2). We consider the ruled variety $\pi : \mathbb{P}Hom(F,E) \to X$. Let $\mathcal{O}(1)$ be the line bundle on $\mathbb{P}Hom(F,E)$ such that $\pi^*\mathcal{O}(1) \cong Hom(F,E)^*$. By Serre duality and the projection formula,

$$\mathbb{P} \cong \mathbb{P}H^0(X,K_X \otimes Hom(F,E)^*)^\vee \cong \mathbb{P}H^0(\mathbb{P}Hom(F,E),\pi^*K_X \otimes \mathcal{O}(1))^\vee.$$ 

Hence there is a rational map

$$\phi : \mathbb{P}Hom(F,E) \dashrightarrow \mathbb{P}$$

given by the complete linear system $|\pi^*K_X \otimes \mathcal{O}(1)|$.

**Lemma 2.2.** Let $E \in U(n-r,0)$ and $F \in U(r,d)$ where $d > (2g-1)r$. Then $\phi$ is an embedding.

**Proof.** It is easy to see that $\phi$ is an embedding if

$$h^0(X,K_X \otimes Hom(F,E)^*) - h^0(X,K_X(-x-y) \otimes Hom(F,E)^*) = 2r(n-r)$$

for any $x,y \in X$. Via Serre duality, this holds if

$$h^0(X,Hom(F,E)) = h^0(X,Hom(F,E) \otimes \mathcal{O}_X(x+y)) = 0.$$ 

By our assumptions, $Hom(F,E)$ is semistable of slope $-d/r$, which is smaller than $-2$. Therefore we get the above cohomology vanishing. \qed

Now we introduce the key object of our discussion: the locus $\Delta$ defined by rank 1 vectors, or decomposable vectors, in the scroll $\mathbb{P}Hom(F,E)$. 

Definition 2.3. The locus $\Delta$ is the subvariety of $\mathbb{P} \text{Hom}(F, E)$ defined by $\Delta = \bigcup_{x \in X} \Delta_x$, where $\Delta_x = \{ [\mu \otimes e] : \mu \in F^*_x, e \in E_x \} \subset \mathbb{P} \text{Hom}(F, E)|_x \cong \mathbb{P}(F^*_x \otimes E_x)$.

Remark 2.4. It is easily seen that $\dim \Delta = n - 1$. Also $\Delta \sim \mathbb{P} F^*$ if $r = n - 1$. So the locus $\Delta$ reduces to the curve $X$ when we consider line subbundles of rank 2 vector bundles.

3. Scrolls and elementary transformations

The map $\phi : \mathbb{P} \text{Hom}(F, E) \to \mathbb{P}$ in the previous section can be understood in terms of elementary transformations. In this section, we briefly recall this connection.

Let $W$ be a vector bundle over $X$. We consider two kinds of elementary transformations of $W$. Firstly, for any nonzero $\mu \in W^*_x$, we get an exact sequence

$$0 \to \tilde{W} \to W \to C_x \to 0$$

whose restriction to $x$ gives

$$0 \to C \to \tilde{W}_x \to W_x \to \mu \to C \to 0.$$  

The locally free sheaf or vector bundle $\tilde{W}$ is called the elementary transformation of $W$ associated to $\mu$.

Next, for any nonzero $w \in W_x$, there is a unique extension

$$0 \to W \to \hat{W} \to C_x \to 0$$

which restricts to $x$ as

$$0 \to C \cdot w \to W_x \to \hat{W}_x \to C \to 0.$$  

We call the vector bundle $\hat{W}$ the elementary transformation of $W$ at $w$.

These two processes are dual to each other: taking the dual of (3), we get

$$0 \to W^* \to (\hat{W})^* \to C_x \to 0.$$  

Here $(\hat{W})^*$ is nothing but the elementary transformation of $W^*$ at $\mu$: under the above notations,

$$\left( \hat{W} \right)^* \cong (\tilde{W}^*).$$

Now consider the map

$$\phi : \mathbb{P}W \to \mathbb{P} H^1(X, W)$$

given by the linear system $|\pi^* K_X \otimes \mathcal{O}(1)|$, where $\pi : \mathbb{P}W \to X$ is the projection and $\mathcal{O}(1)$ is the bundle on $\mathbb{P}W$ satisfying $\pi_\ast \mathcal{O}(1) \cong W^*$. As we noticed in Lemma 2.2, if $W$ is semistable of slope less than $-2$, then $\phi$ embeds $\mathbb{P}W$ into $\mathbb{P} H^1(X, W)$ as a scroll.

This embedding can be understood in terms of the elementary transformations as follows. For each $[w] \in \mathbb{P}W$, we consider the elementary transformation $\hat{W}$ of $W$ at $w$. From the long exact sequence associated to (4),
we have the 1-dimensional kernel of the surjection $H^1(X,W) \to H^1(X,\hat{W})$
and this gives the image point $\phi([w]) \in \mathbb{P}H^1(X,W)$. Accordingly, the secant line joining two points $\phi([w_1])$ and $\phi([w_2])$ of $\mathbb{P}W$ is given by
\[
\frac{[w_1][w_2]}{\mathbb{P} \ker[H^1(X,W) \to H^1(X,\hat{W})]}
\]
where $\hat{W}$ is the elementary transformation of $W$ at $w_1$ and $w_2$.

We will also need the description of the embedded tangent spaces of $\mathbb{P}W$.
For each nonzero $w \in W_x$, we consider the following diagram
\[
\begin{array}{ccccccccc}
\mathbb{W} \otimes O_X(x) & \longrightarrow & (C_x)^{rkW} \\
\uparrow & & \uparrow \\
0 & \longrightarrow & W & \longrightarrow & \mathbb{W} \otimes O_X(x) & \longrightarrow & \tau & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & W & \longrightarrow & \mathbb{W} & \longrightarrow & C_x & \longrightarrow & 0 \\
\end{array}
\]

The $(rk(W) + 1)$-dimensional dimensional kernel of the map
\[H^1(X,W) \to H^1(X,\mathbb{W} \otimes O_X(x))\]
gives the embedded tangent space $T_{[w]}\mathbb{P}W$ in $\mathbb{P}H^1(X,W)$. This can be seen from the blowing-up and blowing-down description of the elementary transformations. For details, we refer the reader to [Ma] and [Tj].

4. A CRITERION FOR LIFTING OF ELEMENTARY TRANSFORMATIONS

Consider an exact sequence
\[0 \to E \to V \to F \to 0,
\]
where $E \in U(n-r,0)$ and $F \in U(r,d)$. This corresponds to a point $v \in \mathbb{P}Hom(F,E) = \mathbb{P}$. In this section, we find a geometric criterion to determine when an elementary transformation $\tilde{F}$ of $F$ has a lifting $\tilde{F} \to V$ such that the following diagram commutes.

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & E & \longrightarrow & V & \longrightarrow & F & \longrightarrow & 0 \\
& & \searrow & & \nearrow \tilde{F} \\
& & & & \uparrow & & \uparrow & & \uparrow \\
& & & & 0 & & \tilde{F} & & 0 \\
\end{array}
\]

Let $\tilde{F}$ be an elementary transformation of $F$ given by the sequence
\[0 \to \tilde{F} \to F \to \tau \to 0,
\]
for some torsion sheaf $\tau$ of degree $k$. Dualizing this, we get
\[0 \to F^* \to \tilde{F}^* \to \tau^\prime \to 0.
\]

Tensoring by $E$, we obtain
\[0 \to Hom(F,E) \to Hom(\tilde{F},E) \to (\tau^\prime)^{(n-r)} \to 0.
\]
In this setting, we have the following cohomological criterion due to Narasimhan and Ramanan.

**Lemma 4.1.** ([NaRa], Lemma 3.1) The elementary transformation \( \tilde{F} \subset F \) lifts to a map \( \tilde{F} \to V \) so that the diagram (9) is commutative if and only if \( v \in \mathbb{P} = \mathbb{P}H^1(X, \text{Hom}(F, E)) \) lies in \( \mathbb{P}(\ker \beta) \) for the map

\[
\beta : H^1(X, \text{Hom}(F, E)) \to H^1(X, \text{Hom}(\tilde{F}, E))
\]

associated to the exact sequence (9).

For further discussions, it will be convenient to have an explicitly constructed parameter space of the elementary transformations of \( F \).

**Lemma 4.2.** For a vector bundle \( F \) and a fixed number \( k > 0 \), the elementary transformations \( \tilde{F} \leftarrow F \) with \( \text{deg}(F/\tilde{F}) = k \) are parameterized by a projective variety \( Q_k \) of dimension at most \( k \cdot \dim(F) \).

**Proof.** One can take the Quot scheme of \( F \) parameterizing the surjections \( F \to \tau \), where \( \tau \) runs over the space of torsion sheaves of degree \( k \). A more explicit parameter space can be constructed as follows.

If \( k = 1 \) then \( Q_1 = \mathbb{P}F^* \): Any linear functional \( \mu : F_x \to \mathbb{C} \) gives an elementary transformation \( \tilde{F} \) whose quotient \( F/\tilde{F} \) has degree 1. Conversely, any elementary transformation \( \tilde{F} \) with degree 1 quotient determines an element of \( F^* \) up to a constant. Furthermore, there is a bundle \( p_1 : \mathcal{F}_1 \to Q_1 \times X \) such that for each \( \mu \in Q_1 \), the bundle \( \tilde{F} = \mathcal{F}_1|_{[\mu] \times X} \) is the kernel of \( F \xrightarrow{\mu} \mathbb{C}_x \). Indeed, let \( \pi_X : Q_1 \to X \) be the projection and let \( \overline{Q}_1 \) be the copy of \( Q_1 \) embedded in \( Q_1 \times X \) as the graph of \( \pi_X \). Also let \( \pi_{\overline{Q}_1} : Q_1 \times X \to \overline{Q}_1 \) be the projection. Then the bundle \( \mathcal{F}_1 \) is defined by the following sequence over \( Q_1 \times X \):

\[
0 \to \mathcal{F}_1 \to \pi_X^* F \otimes \pi_{\overline{Q}_1}^* \mathcal{O}(-1) \to \mathcal{O}_{\overline{Q}_1} \to 0.
\]

Here the quotient map is given by the composition

\[
\pi_X^* F \otimes \pi_{\overline{Q}_1}^* \mathcal{O}(-1) \to \pi_X^* F \otimes \pi_{\overline{Q}_1}^* \mathcal{O}(-1) \otimes \mathcal{O}_{Q_1} \to \mathcal{O}_{Q_1}
\]

where the second map is the pairing \( f \otimes \mu \mapsto \mu(f) \) for \( f \in F_x \) and \( \mu \in F_x^* \) for some \( x \in X \).

By induction, assume that there is a variety \( Q_k \) parameterizing the elementary transformations of \( F \) with \( \text{deg}(F/\tilde{F}) = k \), together with a classifying bundle \( p_k : \mathcal{F}_k \to Q_k \times X \). Each \( \mu \in \mathbb{P}\mathcal{F}_k^* \) represents a bundle \( F_k \) in \( Q_k \), a point \( x \in X \) and a codimension 1 subspace of \( F_x \), or equivalently an elementary transformation \( \tilde{F}_k = \ker[F_k \to \mathbb{C}_x] \). Hence the space \( Q_{k+1} := \mathbb{P}\mathcal{F}_k^* \) parameterizes the elementary transformations \( \tilde{F} \) of \( F \) with quotient of degree \( k + 1 \).

Also, we have a classifying bundle \( \mathcal{F}_{k+1} \) over \( Q_{k+1} \times X \): let \( q_{X_k}^{k+1} \) and \( q_{X_{k+1}}^{k+1} \) be the compositions \( Q_{k+1} \to Q_k \times X \to Q_k \) and \( Q_{k+1} \to Q_k \times X \to X \) respectively. Let \( \overline{Q}_{k+1} \) be the copy of \( Q_{k+1} \) embedded in \( Q_{k+1} \times X \) as the
Let $D$ be given by the images $x_i$ such that $\pi_i$ under the map $\tau$ the support of $\phi$: spaces is a direct consequence of (i) and the definition of the embedding $\Delta$. Consider the diagram (6) and the associated map (10). For a general $F$ in $Q_k$:

(i) The support of $\tau = F/\tilde{F}$ consists of $k$ distinct points $x_1, x_2, \ldots, x_k$ of $X$. Moreover, $\tilde{F}$ is the intersection of the subsheaves $\ker [\mu_i : F \to \mathbb{C}_{x_i}]$ for some $\mu_i \in F_{x_i}^*$, $i = 1, 2, \ldots, k$.

(ii) The subspace $\mathbb{P}(\ker \beta) \subset \mathbb{P}$ is the join of the $k$ distinct linear spaces

$$\mathbb{P}(\mu_i \otimes E_{x_i}) \subset \mathbb{P}Hom(F, E)_{x_i}$$

for $i = 1, 2, \ldots, k$.

**Proof.** (i) In the description of $Q_k$ in Lemma 4.2, the support of $\tau$ of $\tilde{F} \subset Q_k$ is given by the images $x_1, x_2, \ldots, x_k$ of $\tilde{F}$ under the projections

$$\pi_1 = q_{X}^{k}, \quad \pi_2 = q_{X}^{k-1}q_{k-1}, \quad \ldots, \quad \pi_k = q_{X}^{2}q_{k-2}q_{k-1}.$$ 

Let $D$ be the big diagonal inside $X^k$ consisting of the $k$-tuples $(x_1, x_2, \ldots, x_k)$ such that $x_i = x_j$ for some $1 \leq i \neq j \leq k$. The inverse image of $X^k \setminus D$ under the map $(\pi_1, \pi_2, \ldots, \pi_k) : Q_k \to X^k$ parameterizes those $\tilde{F}$ such that the support of $\tau$ consists of $k$ distinct points.

(ii) The description of $\mathbb{P}(\ker \beta) \subset \mathbb{P}$ as a join of $k$ distinct linear subspaces is a direct consequence of (i) and the definition of the embedding $\phi : \mathbb{P}Hom(F, E) \to \mathbb{P}$, as was explained in Section 3. \qed

Now we find a geometric criterion for the lifting of $\tilde{F}$ as in (6).

**Theorem 4.4.** Assume $E \subset U(n-r, 0), F \subset U(r, d)$ where $d > (2g-1)r$ so that $\Delta$ is a subvariety of the scroll $\mathbb{P}Hom(F, E)$ in $\mathbb{P}$. Consider the diagram (6) and the associated map (10).

(i) If some elementary transformation $\tilde{F} \subset Q_k$ lifts to $V$ so that the diagram (6) is commutative, then $v \in \text{Sec}^k \Delta$, where $\Delta$ is the sublocus of the scroll $\mathbb{P}Hom(F, E)$ defined in (2.3).

(ii) If a point $v \in \text{Sec}^k \Delta$ is general, then $V$ admits an elementary transformation $\tilde{F} \subset Q_k$ as a subsheaf.
(iii) Either the condition that \( V \) admits an elementary transformation \( \tilde{F} \subset F \) as a subsheaf or the condition \( v \in \text{Sec}^k\Delta \) implies that \( s_r(V) \leq rd - n(d - k) \).

**Proof.** If an elementary transformation \( \tilde{F} \subset F \) has a lifting \( \tilde{F} \rightarrow V \), then \( v \in \mathbb{P}(\ker \beta) \) by Lemma 4.1. If \( \tilde{F} \subset Q_k \) is general, then by Lemma 4.3 the point \( v \) lies on the linear space spanned by \([\mu_i \otimes e_i] \in \Delta\) for some \( \mu_i \in F_{x_i}^* \) and \( e_i \in E_{x_i}, 1 \leq i \leq k \). Therefore \( v \in \text{Sec}^k\Delta \). By continuity, we get (i).

Now suppose that \( v \) is a general point of \( \text{Sec}^k\Delta \). Then \( v \) lies on some linear space spanned by \([\{\mu_i \otimes e_i \in \Delta_{x_i} : i = 1, 2, \ldots, k\}\) for some distinct points \( x_1, x_2, \ldots, x_k \in X \). This data defines an elementary transformation \( \tilde{F} \) of \( F \), which is general if we choose the vectors \( \mu_i \otimes e_i \) generally. Hence \( v \in \mathbb{P}(\ker \beta) \) by Lemma 4.3 and \( \tilde{F} \rightarrow F \) has a desired lifting \( \tilde{F} \rightarrow V \) by Lemma 4.1. This proves (ii).

Finally, consider (iii). The condition that \( V \) admits an elementary transformation \( \tilde{F} \subset Q_k \) as a subsheaf, obviously implies \( s_r(V) \leq rd - n(d - k) \).

Now consider the family of vector bundles parameterized by \( \text{Sec}^k\Delta \subset \mathbb{P} \). From (ii), a general bundle \( V \) with \( v \in \text{Sec}^k\Delta \) has a subsheaf \( \tilde{F} \) of degree \( d - k \), hence \( s_r(V) \leq rd - n(d - k) \). Since the invariant \( s_r \) is lower semicontinuous, this inequality holds for every bundle \( V \) with \( v \in \text{Sec}^k\Delta \). \( \square \)

**Remark 4.5.** (i) It can be asked if the inequality \( s_r(V) \leq rd - n(d - k) \) implies either the condition \( v \in \text{Sec}^k\Delta \) or the lifting of some \( \tilde{F} \subset Q_k \) to \( V \). The answer seems to be No, due to the possible existence of diagrams of the following form:

\[
\begin{array}{cccccc}
0 & \longrightarrow & E & \longrightarrow & V & \longrightarrow & F_d & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & E' & \longrightarrow & F_{d-k} & \longrightarrow & E'' & \longrightarrow & 0 \\
\end{array}
\]  

(11)

The point is that a subsheaf \( F_{d-k} \) of \( V \) of rank \( r \) and degree \( d - k \) may intersect \( E \) in some subsheaf \( E' \) of rank \( \geq 1 \). If one can prove that this kind of diagram does not exist for general \( V \), then we get the equivalence

\[
s_r(V) \leq rd - n(d - k) \iff v \in \text{Sec}^k\Delta
\]  

by the closedness of both spaces. But in general, we cannot exclude the possibility that every maximal subbundle \( F_{d-k} \) of \( V \) yields a diagram of the form (11).

In the special case when either \( r = 1 \) or \( r = n - 1 \), the diagram (11) does not exist, and we get the equivalence (12). When \( n = 2 \) and \( r = 1 \), this is exactly the statement in Proposition 4.2.

(ii) Also one may ask if the generality assumption in (ii) can be dropped. The answer is Yes if \( \dim(\ker \beta) \) is constant for every \( \tilde{F} \subset Q_k \). In this case, there is a projective bundle \( \mathcal{P} \) over \( Q_k \) whose fiber over \( \tilde{F} \) is the corresponding \( \mathbb{P}(\ker \beta) \). This fibration induces a morphism \( \mathcal{P} \rightarrow \mathbb{P} \) which maps onto \( \text{Sec}^k\Delta \).
But in general, $\dim(\ker \beta)$ may drop, since there may exist $\overline{F} \in Q_k$ with $h^0(X, \text{Hom}(\overline{F}, E)) \neq 0$. If $k < d$ then $\deg \text{Hom}(\overline{F}, E) < 0$, so such an $\overline{F}$ is necessarily unstable.

5. **Hirschowitz bound**

In this section, we reprove Hirschowitz bound in Proposition 1.1 by applying our geometric criterion on lifting of elementary transformations.

Let $V$ be a bundle in $U(n, d)$. Note that the upper bound of $s_r$ is attained by a general bundle in $U(n, d)$, due to the semi-continuity of $s_r$. Hence it suffices to prove the upper bound for general $V$. Moreover, since $s_r(V) = s_r(V \otimes L)$ for any line bundle $L$, we may assume that $d = \deg V > (2g - 1)n$. Then by Lemma 2.1, $V$ fits into the exact sequence

$$0 \to E \to V \to F \to 0$$

for some general bundles $E \in U(n - r, 0)$ and $F \in U(r, d)$. Let $v \in \mathbb{P} = \mathbb{P}H^1(X, \text{Hom}(F, E))$ be the point corresponding to this sequence.

Consider the subvariety $\Delta$ of the scroll $\mathbb{P} \text{Hom}(F, E) \subset \mathbb{P}$. Since $\dim \Delta = n - 1$, the expected dimension of $\text{Sec}^k \Delta$ is $nk - 1$ unless $\text{Sec}^k \Delta = \mathbb{P}$. Note that $\dim \mathbb{P} = (n - r)(d + r(g - 1)) - 1$ by Riemann-Roch formula. Thus if $\Delta \subset \mathbb{P}$ has no secant defect, then

$$v \in \text{Sec}^m \Delta \text{ for } m = \left\lfloor \frac{n - r}{n} (d + r(g - 1)) \right\rfloor.$$

We may write $(n - r)(d + r(g - 1)) = mn - \varepsilon$ for some $\varepsilon$ in the range $0 \leq \varepsilon \leq n - 1$. By Theorem 4.4 (iii),

$$s_r(V) \leq rd - n(d - m) = r(n - r)(g - 1) + \varepsilon.$$

Therefore the Hirschowitz bound is obtained from the following.

**Theorem 5.1.** For general bundles $E \in U(n - r, 0)$ and $F \in U(r, d)$ where $d > (2g - 1)n$, the subvariety $\Delta$ in $\mathbb{P}$ has no secant defect.

To prove this, we invoke the Terracini lemma (cf. [Te]).

**Proposition 5.2.** (Terracini lemma) Let $Z \subset \mathbb{P}^N$ be a projective variety, and let $z_1, \ldots, z_k$ be general points of $Z$. Then

$$\dim(\text{Sec}^k Z) = \dim < T_{z_1} Z, \ldots, T_{z_k} Z >,$$

where $T_{z_i} Z$ denotes the embedded tangent space to $Z$ in $\mathbb{P}^N$ at $z_i$. \(\square\)

Now we use the dictionary in Section 3 to describe the embedded tangent spaces of $\Delta \subset \mathbb{P} \text{Hom}(F, E)$ in terms of the elementary transformations.

**Lemma 5.3.** Let $[\mu \otimes e] \in \Delta_x$ in $\mathbb{P} \text{Hom}(F, E)|_x$. Let $\widehat{F}^*$ be the elementary transformation of $F^*$ at $\mu$. Also let $\widehat{E}$ be the elementary transformation of $E$ at $e$. Consider the elementary transformation

(13) \hspace{1cm} 0 \to F^* \otimes E \to \widehat{F}^* \otimes \widehat{E} \to \tau \to 0
where \( \tau \) is some torsion sheaf \( \tau \) of degree \( n \). Then
\[
T_{[\mu \otimes e]} \Delta = \mathbb{P} \ker [\gamma : H^1(X, F^* \otimes E) \to H^1(X, \widehat{F}^* \otimes \widehat{E})]
\]
for the map \( \gamma \) coming from the long exact sequence associated with \( (16) \).

**Proof.** Since we are considering a local problem, we consider trivialisations of \( F^* \) and \( E \) over a suitable neighbourhood \( U \) of \( x \). Then \( F^* \otimes E|_U \cong \mathcal{O}_U^{\oplus r(n-r)} \).

Taking elementary transformations, we obtain
\[
\widehat{F}^* \otimes \widehat{E} \cong \mathcal{O}_U(2x) \oplus \mathcal{O}_U(x)^{\oplus(n-2)} \oplus \mathcal{O}_U^{\oplus(r-1)(n-r-1)}.
\]

Therefore the skyscraper sheaf \( \tau \) is isomorphic to \( \mathbb{C}_{2x} \oplus \mathbb{C}_x^{\oplus(n-2)} \). The first term \( \mathbb{C}_{2x} \) corresponds to the line defined by the constant section \( \mu \otimes e : U \to \mathbb{P}(F^* \otimes E) \). The next term \( \mathbb{C}_x^{\oplus(n-2)} \) together with the subsheaf \( \mathbb{C}_x \subset \mathbb{C}_{2x} \) correspond to the join of two linear spaces
\[
\mathbb{P}(\mu \otimes E_x) \cong \mathbb{P}^{n-r-1} \quad \text{and} \quad \mathbb{P}(F_x^* \otimes e) \cong \mathbb{P}^{r-1}
\]
which intersect at \([\mu \otimes e]\). Since all of these are tangent data of \( \Delta \) and they span a linear space of dimension \( n - 1 \), we get the desired result. \( \square \)

**Proof of Theorem 5.1.** From the dimension computations in the beginning of this section, it suffices to show that
\[
\dim \text{Sec}^k \Delta = nk - 1
\]
for every \( k < m = \left[ \frac{2n}{m} (d + r(g - 1)) \right] \), and
\[
\dim \text{Sec}^m \Delta = \dim \mathbb{P} = nm - 1 - \varepsilon,
\]
where \( 0 \leq \varepsilon \leq n - 1 \). By the Terracini lemma,
\[
\dim \text{Sec}^k \Delta = \dim (\langle T_{[\mu_1 \otimes e_1]} \Delta, \ldots, T_{[\mu_k \otimes e_k]} \Delta \rangle)
\]
for general points \([\mu_i \otimes e_i] \in \Delta, 1 \leq i \leq k \).

By generality, we may assume that \( \mu_1 \otimes e_1, \ldots, \mu_k \otimes e_k \) lie over \( k \) distinct points of \( X \). In this case, by Lemma 5.3, the join of the spaces \( T_{[\mu_i \otimes e_i]} \Delta \) in \( \mathbb{P} \) can be expressed as the projectivised kernel of the map
\[
\Gamma : H^1(X, F^* \otimes E) \to H^1(X, \widehat{F}^* \otimes \widehat{E})
\]
where \( \widehat{F}^* \) (resp. \( \widehat{E} \)) are the elementary transformations of \( F^* \) (resp. \( E \)) at \( \mu_1, \ldots, \mu_k \) (resp. \( e_1, \ldots, e_k \)).

Note that
\[
\deg(\widehat{F}^* \otimes \widehat{E}) = r \deg \widehat{E} + (n - r) \deg \widehat{F}^* = rk + (n - r)(-d + k)
\]
\[
= rm + (n - r)(-d + m) - n(m - k)
\]
\[
= nm - (n - r)d - n(m - k)
\]
\[
= r(n - r)(g - 1) + \varepsilon - n(m - k),
\]
where \( r, m \) are the degrees and \( d \) is the dimension of \( X \).
where $0 \leq \varepsilon \leq n - 1$. If \( \hat{F}^* \otimes \hat{E} \) is nonspecial, then \( h^0(X, \hat{F}^* \otimes \hat{E}) = 0 \) for \( k < m \) and \( h^0(X, \hat{F}^* \otimes \hat{E}) = \varepsilon \) for \( k = m \). Thus
\[
\dim \mathbb{P} (\ker \Gamma) = - \deg(F^* \otimes E) + \deg(\hat{F}^* \otimes \hat{E}) - h^0(X, \hat{F}^* \otimes \hat{E}) - 1 = nk - h^0(X, \hat{F}^* \otimes \hat{E}) - 1 = \min\{nk - 1, \dim \mathbb{P}\}
\]
as expected. Therefore it remains to check that \( \hat{F}^* \otimes \hat{E} \) is nonspecial. But we assumed that \( E \) and \( F \) are general, and \( \hat{E} \) and \( \hat{F}^* \) are obtained from a general elementary transformation of \( E \) and \( F^* \) respectively, hence they are general. By Hirschowitz’ lemma ([Hi] 4.6, see also [RuTe] Theorem 1.2), the tensor product of two general bundles is nonspecial. \( \square \)

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