A REMARK ON A PRIORI ESTIMATE FOR THE NAVIER-STOKES EQUATIONS WITH THE CORIOLIS FORCE

HIROKI ITO AND JUN KATO

Abstract. The Cauchy problem for the Navier-Stokes equations with the Coriolis force is considered. It is proved that a similar a priori estimate, which is derived for the Navier-Stokes equations by Lei and Lin [11], holds under the effect of the Coriolis force. As an application existence of a unique global solution for arbitrary speed of rotation is proved, as well as its asymptotic behavior.

1. Introduction

In this note, we consider the initial value problem of the Navier-Stokes equations with the Coriolis force in $\mathbb{R}^3$,

\begin{align*}
\begin{cases}
\partial_t u - \nu \Delta u + \Omega e_3 \times u + (u, \nabla) u + \nabla p = 0, & \text{in } (0, \infty) \times \mathbb{R}^3, \\
\text{div } u = 0, & \text{in } (0, \infty) \times \mathbb{R}^3, \\
| u |_{t=0} = u_0, & \text{in } \mathbb{R}^3,
\end{cases}
\end{align*}

(\text{NS}_\Omega)

where $u = u(t, x) = (u_1(t, x), u_2(t, x), u_3(t, x))$ denotes the unknown velocity field, and $p = p(t, x)$ denotes the unknown scalar pressure, while $u_0 = u_0(x) = (u_0^1(x), u_0^2(x), u_0^3(x))$ denotes the initial velocity field. The constant $\nu > 0$ denotes the viscosity coefficient of the fluid, and $\Omega \in \mathbb{R}$ represents the speed of rotation around the vertical unit vector $e_3 = (0, 0, 1)$, which is called the Coriolis parameter.

Recently, this problem gained some attention due to its importance in applications to geophysical flows, see e.g. [12, 3]. Mathematically, (\text{NS}_\Omega) also have a interesting feature that there exists a global solution for arbitrary large data provided the speed of rotation $\Omega$ is large enough, see e.g. [1, 3, 7]. There are another type of results which shows the existence of a global solution uniformly in $\Omega$ provided the data is sufficiently small, see e.g. [4, 6, 10, 8]. The purpose of this note is, concerning to the latter, to relax the smallness condition of the data, based on the idea for the Navier-Stokes equations, $\Omega = 0$ in (\text{NS}_0), by [11].

Before stating our main results, we give a definition of function spaces. For $m \in \mathbb{R}$, we define

$$
\chi^m(\mathbb{R}^3) := \{ f \in \mathcal{S}' \mid \hat{f} \in L^1_{\text{loc}}, \| f \|_{\chi^m} := \int_{\mathbb{R}^3} | \xi |^m | \hat{f}(\xi) | d\xi < \infty \}.
$$

In particular, we only use spaces $\chi^{-1}, \chi^0, \text{ and } \chi^1$ below, so we summarize elementary estimates concerning the spaces we will use later.
Lemma 1. (1) For $s > 1/2$, \(|f|_{\chi^{-1}(\mathbb{R}^3)} \leq C \|f\|_{L^2}^{1-s/2} \|f\|_{\dot{H}^s}^{1/2}.
\)

(2) \(|f|_{\chi^0} \leq \|f\|_{\chi^{-1}}^{1/2} \|f\|_{\chi^1}^{1/2}.
\)

(3) \(|\nabla f|_{L^\infty} \leq \|f\|_{\chi^1}.
\)

Proof. (1) We take $R > 0$, which is determined later, to divide the integral
\[
\|f\|_{\chi^{-1}} = \int_{|\xi| \leq R} |\xi|^{-1} |\hat{f}(\xi)| d\xi + \int_{|\xi| > R} |\xi|^{-1} |\hat{f}(\xi)| d\xi
\]
\[
\leq \left( \int_{|\xi| \leq R} |\xi|^{-2} d\xi \right)^{1/2} \|f\|_{L^2} + \left( \int_{|\xi| > R} |\xi|^{-2-2s} \right)^{1/2} \|f\|_{\dot{H}^s}
\]
\[
= |S^2|^{1/2} \left( R^{1/2} \|f\|_{L^2} + \frac{1}{\sqrt{2s-1}} R^{-s+1/2} \|f\|_{\dot{H}^s} \right).
\]

Then, choosing $R = \|f\|_{L^2}^{-1/2} \|f\|_{\dot{H}^s}^{1/2}$, we obtain the desired result.

(2) This estimate is easily derived by the Hölder inequality,
\[
\|f\|_{\chi^0} = \int |\xi|^{-1/2} |\hat{f}(\xi)|^{1/2} |\xi|^{1/2} |\hat{f}(\xi)|^{1/2} d\xi \leq \|f\|_{\chi^{-1}} \|f\|_{\chi^1}.
\]

(3) This is also easily derived from the Fourier inversion formula and the Hausdorff-Young inequality. \qed

Now we state our main results.

Theorem 1. Let $\Omega \in \mathbb{R}$, and let $u_0 \in \chi^{-1}$ satisfy $\text{div} \ u_0 = 0$ and $\|u_0\|_{\chi^{-1}} < (2\pi)^3 \nu$. For $T > 0$, assume that $u \in C([0, T); \chi^{-1})$ is a solution to (NS$_\Omega$) in the distribution sense satisfying
\[
u \in L^1(0, T; \chi^{-1}), \quad \partial_t u \in L^1(0, T; \chi^{-1}).
\]

Then, $u$ satisfies
\[
\|u(t)\|_{\chi^{-1}} + (\nu - (2\pi)^{-3}\|u_0\|_{\chi^{-1}}) \int_0^t \|u(\tau)\|_{\chi^1} d\tau \leq \|u_0\|_{\chi^{-1}}, \quad 0 \leq t < T.
\]

Remark 2. (1) This a priori estimate is first derived in the case $\Omega = 0$ in [\S, Proof of Theorem 1.1]. Here, Theorem \S states that the same estimate also holds under the effect of the Coriolis force.

(2) In this note, we define the Fourier transform of $f$ by
\[
\hat{f}(\xi) = \mathcal{F}[f](\xi) := \int e^{-ix\xi} f(x) dx.
\]

The constant $(2\pi)^3$ in the theorem appears from the following formula:
\[
\mathcal{F}[fg](\xi) = (2\pi)^{-3}(\hat{f} * \hat{g})(\xi),
\]
where $f * g$ denotes the convolution of $f$ and $g$. 

From the a priori estimate (1.1), we especially obtain
\[ \|u\|_{L^\infty(0,T;\chi^{-1})} \leq \|u_0\|_{\chi^{-1}}, \quad \|u\|_{L^1(0,T;\chi^1)} \leq \frac{\|u_0\|_{\chi^{-1}}}{\nu - (2\pi)^{-3}\|u_0\|_{\chi^{-1}}} \]

As an application of Theorem 1 we obtain a unique global solution to (NSΩ).

**Theorem 2.** Let Ω ∈ R. Assume that \( u_0 \in \chi^{-1}(\mathbb{R}^3) \) satisfy div \( u_0 = 0 \) and \( \|u_0\|_{\chi^{-1}} < (2\pi)^3\nu \). Then, there exists a unique global solution \( u \in C([0, \infty); \chi^{-1}) \) to (NSΩ) satisfying
\[ u \in L^1(0, \infty; \chi^{-1}), \quad \partial_t u \in L^1_{\text{loc}}(0, \infty; \chi^{-1}), \]
and
\[ \sup_{t>0} \{\|u(t)\|_{\chi^{-1}} + (\nu - (2\pi)^{-3}\|u_0\|_{\chi^{-1}}) \int_0^t \|u(\tau)\|_{\chi^1} d\tau\} \leq \|u_0\|_{\chi^{-1}}. \]

**Remark 3.** (1) There are several results which treats the existence of a unique global solution to (NSΩ), see [8] and reference therein. In particular, the spaces \( FM^{-1}_0 \), which is considered by Giga, Inui, Mahalov, and Saal [4], and \( B_{1,1}^{-1} \) by [8], are larger than \( \chi^{-1} \). However, the advantage of this result is that the condition of the size of the data is merely \( \|u_0\|_{\chi^{-1}} < (2\pi)^3\nu \).

(2) In the Navier-Stokes equations, the case \( \Omega = 0 \), the corresponding result is proved in [11] Theorem 1.1. We notice that there is also the another approach by [13] Theorem 1.3. In our forthcoming paper we will consider that approach for (NSΩ).

As a byproduct, we also obtain the following.

**Theorem 3.** Let \( s > 3/2 \) and \( \Omega \in \mathbb{R} \). Assume that \( u_0 \in H^s(\mathbb{R}^3) \) satisfy div \( u_0 = 0 \) and \( \|u_0\|_{\chi^{-1}} < (2\pi)^3\nu \). Then, there exists a unique global solution \( u \in C([0, \infty); H^s) \) to (NSΩ) satisfying
\[ u \in AC([0, \infty); H^{s-1}) \cap L^1_{\text{loc}}(0, \infty; H^{s+1}) \]
and
\[ \sup_{t>0} \{\|u(t)\|_{\chi^{-1}} + (\nu - (2\pi)^{-3}\|u_0\|_{\chi^{-1}}) \int_0^t \|u(\tau)\|_{\chi^1} d\tau\} \leq \|u_0\|_{\chi^{-1}}. \]

**Remark 4.** Since \( s > 3/2 \), we have \( H^s \hookrightarrow \chi^{-1} \) by Lemma [11]. The condition \( s > 3/2 \) follows from the local well-posedness by Proposition [4] which we employ for the proof. For a interval \( I \) and a Banach space \( X \), \( AC(I; X) \) denotes the space of \( X \)-valued absolutely continuous functions.

Next theorem states the asymptotic behavior of a given global solution to (NSΩ) in the framework of Sobolev spaces.
Theorem 4. Let $s > 1/2$ and $\Omega \in \mathbb{R}$. Assume that $u \in C([0, \infty); H^s(\mathbb{R}^3))$ is a global solution to (NS$_\Omega$) satisfying

$$(1.2) \quad u \in AC([0, \infty); H^{s-1}(\mathbb{R}^3)) \cap L^1_{\text{loc}}(0, \infty; H^{s+1}(\mathbb{R}^3)).$$

Then, $\lim_{t \to \infty} \|u(t)\|_{\chi^{-1}} = 0$.

Remark 5. In the Navier-Stokes case $\Omega = 0$, this result corresponds to the result in [2]. In that result, the assumption is only $u \in C([0, \infty); \chi^{-1})$ is a global solution. Compared with that result, additional assumptions (1.2) are imposed for the uniqueness of solutions.

As an application of Theorem 4 we obtain the following.

Corollary 5. The global solution to (NS$_\Omega$) derived in Theorem 3 satisfies

$$\lim_{t \to 0} \|u(t)\|_{\chi^{-1}} = 0.$$
we observe that \( \text{Re}[(e_3 \times \hat{u}) \cdot \tilde{u}] = 0 \). Also, we have \((i \xi \hat{p}) \cdot \tilde{u} = 0 \), since \( \text{div} \ u = 0 \). Moreover, we notice that

\[
\mathcal{F}[(u, \nabla)u]_j(\xi) = \sum_{k=1}^{3} (2\pi)^{-3} \hat{u}_k * \hat{\partial_k u}_j(\xi)
\]

\[
= \sum_{k=1}^{3} (2\pi)^{-3} \int \hat{u}_k(\xi - \eta) i\eta \hat{u}_j(\eta) \, d\eta
\]

\[
= \sum_{k=1}^{3} (2\pi)^{-3} i\xi_k \int \hat{u}_k(\xi - \eta) \hat{u}_j(\eta) \, d\eta,
\]

since \( \sum_{k=1}^{3} (\xi_k - \eta_k) \hat{u}_k(\xi - \eta) = 0 \). Therefore, we obtain

\[
\partial_t |\hat{u}|^2 + 2\nu |\xi|^2 |\hat{u}|^2 \leq 2(2\pi)^{-3} \sum_{j=1}^{3} |\xi_k| \langle |\hat{u}_k| * |\hat{u}_j| \rangle |u_j|
\]

\[
\leq 2(2\pi)^{-3} |\xi| |\hat{u}| \langle |\hat{u}| * |\hat{u}| \rangle.
\]

Then, for \( \varepsilon > 0 \), we observe that

\[
\partial_t (|\hat{u}|^2 + \varepsilon)^{1/2} = \frac{\partial_t |\hat{u}|^2}{2(|\hat{u}|^2 + \varepsilon)^{1/2}}
\]

\[
\leq \frac{\nu |\xi|^2 |\hat{u}|^2}{(|\hat{u}|^2 + \varepsilon)^{1/2}} + (2\pi)^{-3} \frac{|\xi| |\hat{u}|}{(|\hat{u}|^2 + \varepsilon)^{1/2}} \langle |\hat{u}| * |\hat{u}| \rangle.
\]

Integrating with respect to \( t \), we obtain

\[
(|\hat{u}(t, \xi)|^2 + \varepsilon)^{1/2} + \int_0^t \frac{\nu |\xi|^2 |\hat{u}(\tau, \xi)|^2}{(|\hat{u}(\tau, \xi)|^2 + \varepsilon)^{1/2}} \, d\tau
\]

\[
\leq (|\hat{u}_0(\xi)|^2 + \varepsilon)^{1/2} + (2\pi)^{-3} \int_0^t \frac{|\xi| |\hat{u}(\tau, \xi)|}{(|\hat{u}(\tau, \xi)|^2 + \varepsilon)^{1/2}} \langle |\hat{u}(\tau)| * |\hat{u}(\tau)| \rangle(\xi) \, d\tau.
\]

Then, letting \( \varepsilon \to 0 \), we get

\[
|\hat{u}(t, \xi)| + \int_0^t \nu |\xi|^2 |\hat{u}(\tau, \xi)| \, d\tau \leq |\hat{u}_0(\xi)| + (2\pi)^{-3} \int_0^t |\xi| \langle |\hat{u}(\tau)| * |\hat{u}(\tau)| \rangle(\xi) \, d\tau.
\]

Finally, dividing by \( |\xi| \), and then integrating over \( \mathbb{R}^n \), we obtain

\[
\|u(t)\|_{\chi^{-1}} + \nu \int_0^t \|u(\tau)\|_{\chi^1} \, d\tau \leq \|u_0\|_{\chi^{-1}} + (2\pi)^{-3} \int_0^t \|u(\tau)\|_{\chi^0}^2 \, d\tau.
\]

By applying Lemma \[2\] (2), we obtain,

\[
(2.1) \quad \|u(t)\|_{\chi^{-1}} + \nu \|u\|_{L^1((0,t); \chi^1)} \leq \|u_0\|_{\chi^{-1}} + (2\pi)^{-3} \|u\|_{L^\infty((0,t); \chi^{-1})} \|u\|_{L^1((0,t); \chi^1)}.
\]

To derive the desired estimate \[1\], it suffices to prove that

\[
\|u\|_{L^\infty((0,T); \chi^{-1})} \leq \|u_0\|_{\chi^{-1}}, \quad 0 \leq t < T.
\]
For the proof, we first show that
\[(2.2) \quad \|u(t)\|_{\chi^{-1}} < (2\pi)^3 \nu, \quad 0 \leq t < T\]
holds by contradiction. From the assumption \(\|u_0\|_{\chi^{-1}} < (2\pi)^3 \nu\) and \(u \in C([0, T); \chi^{-1})\),
we observe that there exists \(\delta > 0\) such that \(2.2\) holds on \([0, \delta]\). Now assume that there
exists \(t_0 \in (0, T)\) such that \(\|u(t)\|_{\chi^{-1}} < (2\pi)^3 \nu\) for \(0 < t < t_0\) and
\[\|u(t_0)\|_{\chi^{-1}} = (2\pi)^3 \nu,\]
then by (2.1) we reach the contradiction
\[(2\pi)^3 \nu = \|u(t_0)\|_{\chi^{-1}} < \|u_0\|_{\chi^{-1}} < (2\pi)^3 \nu,\]
since \(\|u\|_{L^\infty((0,t_0);\chi^{-1})} = (2\pi)^3 \nu\). Therefore, we obtain (2.2). Finally, applying (2.2) to
estimate on the right hand side of (2.1), we obtain
\[\|u(t)\|_{\chi^{-1}} < \|u_0\|_{\chi^{-1}}, \quad 0 \leq t < T.\]
This completes the proof. \(\square\)

3. PROOF OF THEOREM 3

Below we fix \(\Omega \in \mathbb{R}\). For the existence of local solutions, we employ the following result.

**Proposition 6.** Let \(s > 3/2\). For \(u_0 \in H^s(\mathbb{R}^3)\) with \(\text{div } u_0 = 0\), there exists \(T = T(|\Omega|, s, \|u_0\|_{H^s}) > 0\) such that \((\text{NS}_\Omega)\) admits a unique strong solution \(u \in C([0, T]; H^s(\mathbb{R}^3))\)
satisfying
\[u \in AC([0, T]; H^{s-1}(\mathbb{R}^3)) \cap L^1(0, T; H^{s+1}(\mathbb{R}^3)).\]

**Remark 7.** (1) For the proof, we refer to [9, Lemma 3.1]. The idea is based on to construct
the solution to the integral equation
\[u(t) = e^{\nu \Delta} u_0 - \int_0^t e^{\nu(t-\tau)\Delta} \mathbb{P}(e_3 \times u)(\tau) d\tau - \int_0^t e^{\nu(t-\tau)\Delta} \mathbb{P}(u, \nabla u)(\tau) d\tau\]
by the contraction mapping argument, where \(\mathbb{P} = (\delta_{ij} + R_i R_j)_{i,j}\) is the Helmholtz projection.
We notice that the condition in [9, Lemma 3.1] is \(s > 3/2 + 1\), because their main subject is the Euler equation. For the above statement, \(s > 3/2\) is sufficient.

(2) In this proposition, the size of \(T\) is characterized by
\[(3.1) \quad C_0 |\Omega| T + C_1 \|u_0\|_{H^s} (T + T^{1/2} \nu^{-1/2}) \leq \frac{1}{2}.\]

(3) Since \(s > 3/2\), the solution constructed by Proposition 6 satisfies the assumptions in
Theorem [11]. In particular, since
\[\partial_t u = \nu \Delta u - \Omega \mathbb{P}(e_3 \times u) - \mathbb{P}(u, \nabla u) \quad \text{in } H^{s-1}\]
holds for a.e. \(t \in (0, T)\), we easily observe that \(\partial_t u \in L^1(0, T; \chi^{-1})\).
We will use the following energy estimate.

**Proposition 8.** Let \( s \geq 0 \) and \( T > 0 \). Assume that \( u \in C([0,T); H^s(\mathbb{R}^3)) \) is a solution to \((NS_{\Omega})\) satisfying

\[
u \in AC([0,T); H^{s-1}(\mathbb{R}^3) \cap L^1(0,T; H^{s+1}(\mathbb{R}^3)).
\]

Then, \( u \) satisfies

\[
\|u(t)\|_{H^s} \leq \|u_0\|_{H^s} e^{C \int_0^T \|\nabla u(\tau)\|_{L^\infty} d\tau}, \quad 0 \leq t < T.
\]

**Remark 9.** For the proof of this proposition, we also refer to [9, Proof of Theorem 4.1]. There, we easily observe that

\[
\frac{d}{dt}\|u(t)\|_{H^s} \leq C\|\nabla u(t)\|_{L^\infty}\|u(t)\|_{H^s}
\]

holds for \( s \geq 0 \). We notice that the term concerning \( \Omega \varepsilon_3 \times u \) vanishes due to

\[
\Omega(\varepsilon_3 \times u) \cdot u = 0.
\]

Now we are in a position to prove Theorem 3.

**Proof of Theorem 3.** Let \( T^* \) be the maximal existence time of a unique solution derived by applying Proposition 6 repeatedly. Now assume \( T^* < \infty \). Then, by (3.1), we must have

\[
(3.2) \quad \lim_{t \to T^*} \|u(t)\|_{H^s} = \infty.
\]

Since this solution satisfies the energy estimate in Proposition 8, we have

\[
\|u(t)\|_{H^s} \leq \|u_0\|_{H^s} e^{C \int_0^T \|\nabla u(\tau)\|_{L^\infty} d\tau}, \quad 0 \leq t < T^*.
\]

Then, since \( \|u_0\|_{\chi^{-1}} < (2\pi)^3 \nu \), applying Theorem 1 we obtain

\[
\int_0^{T^*} \|\nabla u(\tau)\|_{L^\infty} d\tau \leq \|u\|_{L^1(0,T^*; \chi^1)} \leq \frac{\|u_0\|_{\chi^{-1}}}{\nu - (2\pi)^3 \|u_0\|_{\chi^{-1}}}.
\]

This implies \( \sup_{0 < t < T^*} \|u(t)\|_{H^s} < \infty \), which contradicts to (3.2). \( \square \)

4. **Proof of Theorem 2**

In this section we give a proof of Theorem 2.

For \( u_0 \in \chi^{-1} \) and \( R > 0 \), we set

\[
D_R = \{ \xi \in \mathbb{R}^3 \mid |\xi| \leq R, \  \widehat{u_0}(\xi) \leq R \}, \quad u_0^R = F^1[\chi_{D_R} \widehat{u_0}],
\]

where \( \chi_{D_R} \) denotes the characteristic function of \( D_R \). Then, we observe that

\[
u_0^R \in H^\infty, \quad \|u_0^R\|_{\chi^{-1}} \leq \|u_0\|_{\chi^{-1}}.
\]
and from Lebesgue’s dominant convergence theorem,
\[ \|u_0^R - u_0\|_{\chi^{-1}} = \int |x|^{-1} (\chi_{\partial R}(x) - 1) |\hat{u}_0(x)| \, dx \to 0, \quad R \to \infty, \]
since \( u_0 \in \chi^{-1} \).

Now we apply Theorem 3 for the data \( u_0^R \) to derive a unique global solution \( u^R \in C([0, \infty); H^s) \) satisfying
\[
u = \nu - (2\pi)^{-3}\|u_0\|_{\chi^{-1}},
\]
for \( s > 3/2 \), and
\[ \|u^R\|_{L^\infty(0, \infty; \chi^{-1})} \leq \|u_0\|_{\chi^{-1}}, \quad \|u^R\|_{L^1(0, \infty; \chi^1)} \leq \frac{\|u_0\|_{\chi^{-1}}}{\nu - (2\pi)^{-3}\|u_0\|_{\chi^{-1}}}. \]

Below we first show that \( \{u^R\} \) is a Cauchy sequence in \( L^\infty(0, \infty; \chi^{-1}) \). If we set \( w = u^R - u^R' \), then \( w \) satisfies
\[
\partial_t w - \nu \Delta w + \Omega e_3 \times w + (u^R, \nabla)w + (w, \nabla)u^R + \nabla(f - p - p') = 0.
\]

Then, from the argument in the proof of Theorem 1 we obtain
\[
\|w(t)\|_{\chi^{-1}} + \nu \int_0^t \|w(\tau)\|_{\chi^1} \, d\tau \leq \|w(0)\|_{\chi^{-1}} + (2\pi)^{-3} \int_0^t \left( \|u^R(\tau)\|_{\chi^0} + \|u^{R'}(\tau)\|_{\chi^0} \right) \|w(\tau)\|_{\chi^0} \, d\tau.
\]
Here, applying Lemma 1 (2) we have
\[
\|u^R\|_{\chi^0} \|w\|_{\chi^0} \leq \|u^R\|_{\chi^{-1}}^{1/2} \|u^R\|_{\chi^1}^{1/2} \|w\|_{\chi^{-1}}^{1/2} \|w\|_{\chi^1}^{1/2} \leq \frac{1}{2} \left( \|u^R\|_{\chi^{-1}} \|w\|_{\chi^{-1}} + \|u^R\|_{\chi^1} \|w\|_{\chi^{-1}} \right).
\]
Therefore, combining (4.2) we obtain
\[
\|w(t)\|_{\chi^{-1}} + (\nu - (2\pi)^{-3}\|u_0\|_{\chi^{-1}}) \int_0^t \|w(\tau)\|_{\chi^1} \, d\tau \leq \|w(0)\|_{\chi^{-1}} + \int_0^t a(\tau) \|w(\tau)\|_{\chi^{-1}} \, d\tau,
\]
where
\[
a(\tau) = \frac{1}{2(2\pi)^3} \left( \|u^R(\tau)\|_{\chi^1} + \|u^{R'}(\tau)\|_{\chi^1} \right).
\]
Note that by (4.2) we have a uniform bound
\[
\int_0^\infty a(\tau) \, d\tau \leq \frac{\|u_0\|_{\chi^{-1}}}{(2\pi)^3\nu - \|u_0\|_{\chi^{-1}}}.
\]
Thus, applying Gronwall’s inequality to (4.4) we obtain
\[
\|w(t)\|_{\chi^{-1}} \leq \|w(0)\|_{\chi^{-1}} e^{\int_0^t a(\tau) \, d\tau},
\]
which implies
\[ \|u^R - u^{R'}\|_{L^\infty(0, \infty; \chi^{-1})} \leq \|u_0^R - u_0^{R'}\|_{\chi^{-1}} e^{\int_0^\infty a(\tau) \, d\tau} \to 0, \quad R, R' \to \infty. \]
Therefore, there exists \( u \in L^\infty(0, \infty; \chi^{-1}) \) such that \( u^R \to u \) in \( L^\infty(0, \infty; \chi^{-1}) \).

We next show the convergence in \( L^1(0, \infty; \chi^1) \). The convergence in \( L^\infty(0, \infty; \chi^{-1}) \) implies there exists a subsequence \( \{u^{R_i}\} \) such that for a.e. \((t, \xi), \)
\[ \mathcal{F}[u^{R_i}](t, \xi) \to \hat{u}(t, \xi), \quad R \to \infty. \]

Therefore, by Fatou’s lemma and the estimate derived from (4.4) and (4.5),
\[ \|w\|_{L^1(0, \infty; \chi^1)} \leq \frac{\|w(0)\|_{\chi^{-1}}}{\nu - (2\pi)^3\|u_0\|_{\chi^{-1}}} \left(1 + \int_0^\infty a(\tau) \, d\tau e^{\int_0^\infty a(\tau) \, d\tau}\right), \]
we conclude that
\[ \|u^R - u\|_{L^1(0, \infty; \chi^1)} \leq \liminf_{R \to 0} \|u^R - u^{R_i}\|_{L^1(0, \infty; \chi^1)} \to 0, \quad R \to \infty. \]

From convergence in \( L^\infty(0, \infty; \chi^{-1}) \cap L^1(0, \infty; \chi^1) \) we observe that the limit \( u \) satisfies the integral equation
\[ u(t) = e^{\nu t \Delta} u_0 - \Omega \int_0^t e^{\nu(t-\tau) \Delta} \mathbb{P}(e_3 \times u)(\tau) \, d\tau - \int_0^t e^{\nu(t-\tau) \Delta} \mathbb{P} \cdot (u \otimes u)(\tau) \, d\tau, \]
which \( u^R \) also satisfies for the data \( u_0^R \). In fact, we are able to estimate
\[ \|e^{\nu t \Delta} u_0^R - e^{\nu t \Delta} u_0\|_{\chi^{-1}} \leq \|u_0^R - u_0\|_{\chi^{-1}}, \]
\[ \left\| \int_0^t e^{\nu(t-\tau) \Delta} \mathbb{P}(e_3 \times u^R)(\tau) \, d\tau - \int_0^t e^{\nu(t-\tau) \Delta} \mathbb{P}(e_3 \times u)(\tau) \, d\tau \right\|_{\chi^{-1}} \leq t \|u^R - u\|_{L^\infty(0, \infty; \chi^{-1})}, \]
and
\[ \left\| \int_0^t e^{\nu(t-\tau) \Delta} \mathbb{P}(u^R, \nabla u^R)(\tau) \, d\tau - \int_0^t e^{\nu(t-\tau) \Delta} \mathbb{P}(u, \nabla u)(\tau) \, d\tau \right\|_{\chi^{-1}} \leq \int_0^t \left( \|u^R(\tau)\|_{\chi^0} + \|u(\tau)\|_{\chi^0} \right) \|u^R(\tau) - u(\tau)\|_{\chi^0} \, d\tau \leq C \left( \|u^R - u\|_{L^\infty(0, \infty; \chi^{-1})} + \|u^R - u\|_{L^1(0, \infty; \chi^1)} \right), \]
where we applied the estimate like (4.3) and the uniform bound (4.2).

We next show \( \partial_t u \in L^1(0, T; \chi^{-1}) \) for any \( T > 0 \), which implies \( u \in C([0, \infty); \chi^{-1}) \).
To prove this, we consider to apply \( \partial_t \) to the right hand of the integral equation. We first
notice that for the first term
\[
\|\partial_t e^{\nu \Delta t} u_0\|_{L^1(0, \infty; \chi^{-1})} = \|e^{\nu \Delta t} u_0\|_{L^1(0, \infty; \chi^{-1})}
= \nu \int_0^\infty \int |\xi| e^{-\nu |\xi|^2} |\hat{\alpha}_0(\xi)| \, d\xi \, dt
= \int |\xi|^{-1} |\hat{\alpha}_0(\xi)| \, d\xi = \|u_0\|_{\chi^{-1}}
\]
holds by changing the order of the integrals. This type of argument can be found in \([10]\) Lemma 3.5. (See also \([5]\) Theorem 2.5) in relation with the \(L^1\)-maximal regularity.) So, it suffices to show that \(\partial_t \Phi \in L^1(0, T; \chi^{-1})\), where
\[
\Phi(t) = \Omega \int_0^t e^{\nu(t-s)\Delta} P(e_3 \times u)(\tau) \, d\tau + \int_0^t e^{\nu(t-s)\Delta} P \nabla \cdot (u \otimes u)(\tau) \, d\tau.
\]
Since
\[
\partial_t \Phi(t) = \Delta \Phi(t) + P(e_3 \times u)(t) + P \nabla \cdot (u \otimes u)(t),
\]
we will check each term on the right hand side belongs to \(L^1(0, T; \chi^{-1})\). It is easy to see that
\[
\int_0^T \|P(e_3 \times u)\|_{\chi^{-1}} \, dt \leq T \|u\|_{L^\infty(0, T; \chi^{-1})},
\]
\[
\int_0^T \|P \nabla \cdot (u \otimes u)(t)\|_{\chi^{-1}} \, dt \leq \int_0^T \|u\|^2_{\chi^0} \, dt \leq \|u\|_{L^\infty(0, T; \chi^{-1})} \|u\|_{L^1(0, T; \chi^1)},
\]
\[
\int_0^T \left\| \Omega \int_0^t e^{\nu(t-s)\Delta} P(e_3 \times u)(\tau) \, d\tau \right\|_{\chi^{-1}} \, dt \leq \|\Omega\| T \|u\|_{L^1(0, T; \chi^1)}.
\]
And applying the argument the above again,
\[
\int_0^T \left\| \Omega \int_0^t e^{\nu(t-s)\Delta} P \nabla \cdot (u \otimes u)(\tau) \, d\tau \right\|_{\chi^{-1}} \, dt
\]
\[
\leq \int_0^T \left( \int_0^t \int |\xi|^2 e^{-\nu(t-s)|\xi|^2} (|\hat{\alpha}(\tau)| \ast |\hat{u}(\tau)|)(\xi) \, d\xi \, d\tau \right) \, dt
\]
\[
= \int_0^T \int_\tau^T \int |\xi|^2 e^{-\nu(t-s)|\xi|^2} \, dt (|\hat{\alpha}(\tau)| \ast |\hat{u}(\tau)|)(\xi) \, d\xi \, d\tau
\]
\[
\leq \int_0^T \|u\|^2_{\chi^0} \, d\tau \leq \|u\|_{L^\infty(0, T; \chi^{-1})} \|u\|_{L^1(0, T; \chi^1)}.
\]
Finally, we notice that (4.5) implies the uniqueness of solutions.

5. PROOF OF THEOREM \[\text{[4]}\]

In this section we give a proof of Theorem \[\text{[4]}\]
We take $\varepsilon > 0$ arbitrary small. Since $u_0 \in H^s \hookrightarrow \chi^{-1}$, we are able to choose $R_0 > 0$ such that
\[
\int_{|\xi| > R_0} |\xi|^{-1} |\hat{u}_0(\xi)| d\xi < \frac{\varepsilon}{2}.
\]
Now we set
\[
v_0 = F^{-1}[\chi_{\{ |\xi| \leq R_0 \}} \hat{u}_0], \quad w_0 = F^{-1}[\chi_{\{ |\xi| > R_0 \}} \hat{u}_0].
\]
Then, we observe that $v_0 \in H^\infty$, $w_0 \in H^s$, $u_0 = v_0 + w_0$, and
\[
\|w_0\|_{\chi^{-1}} < \frac{\varepsilon}{2}.
\]
By applying Theorem 3 for the initial data $w_0$ we obtain the solution $(w, p_w)$ to (NS$\Omega$). Then, $w \in C([0, \infty); H^s) \cap L^1(0, \infty; H^{s+1})$ satisfies
\[
\|w(t)\|_{\chi^{-1}} + (\nu - (2\pi)^{-3})\|w_0\|_{\chi^{-1}} \int_0^t \|w(\tau)\|_{\chi^{-1}} d\tau \leq \|w_0\|_{\chi^{-1}} < \frac{\varepsilon}{2}, \quad t > 0.
\]
Now we set $v := u - w$. Then, $v \in C([0, \infty); H^s)$ satisfies
\[
v \in AC([0, \infty); H^{s-1}(\mathbb{R}^3)) \cap L^1(0, \infty; H^{s+1}(\mathbb{R}^3))
\]
and
\[
\begin{aligned}
\partial_t v + \nu \Delta v + \Omega \epsilon_3 \times v + (v, \nabla)v + (w, \nabla)v + (v, \nabla)w + \nabla(p - p_w) &= 0, \\
\text{div } v &= 0, \\
v|_{t=0} &= v_0.
\end{aligned}
\]
Taking $L^2$-inner product with $v$, the equation becomes
\[
\frac{d}{dt} \|v(t)\|_{L^2}^2 + \nu \|\nabla v(t)\|_{L^2}^2 = \langle (v, \nabla)w, v \rangle_{L^2}.
\]
Since
\[
\langle (v, \nabla)w, v \rangle_{L^2} = -\langle w, (v, \nabla)v \rangle_{L^2},
\]
we obtain
\[
\|((v, \nabla)w, v)\|_{L^2} \leq \|w\|_{L^\infty} \|v\|_{L^2} \|\nabla v\|_{L^2}
\]
\[
\leq C \|w\|_{\chi^0} \|v\|_{L^2} \|\nabla v\|_{L^2}
\]
\[
\leq C \nu \|w\|_{\chi^0}^2 \|v\|_{L^2}^2 + \frac{\nu}{2} \|\nabla v\|_{L^2}^2
\]
Therefore, we obtain
\[
\frac{d}{dt} \|v(t)\|_{L^2}^2 + \nu \|\nabla v(t)\|_{L^2}^2 = C \nu \|w(t)\|_{\chi^0}^2 \|v(t)\|_{L^2}^2.
\]
Then, by Gronwall’s inequality,
\[
\|v(t)\|_{L^2}^2 + \nu \int_0^t \|\nabla v(t)\|_{L^2}^2 \leq \|v(0)\|_{L^2}^2 e^{C \nu \int_0^t \|w(\tau)\|_{\chi^0}^2 d\tau}.
\]

\[\text{(5.2)}\]
Therefore, by Lemma 1 (1), (5.2), (5.3) we obtain

\begin{equation}
(5.3) \quad \int_0^t \|w(\tau)\|_{\chi_0}^2 d\tau \leq \|w\|_{L^\infty(0, t; \chi^{-1})} \|w\|_{L^1(0, t; \chi^1)} \leq \frac{\|w_0\|_{\chi^{-1}}^2}{\nu - (2\pi)^{-3} \|w_0\|_{\chi^{-1}}}.
\end{equation}

Therefore, by Lemma 1 (1), (5.2), (5.3) we obtain

\[ \int_0^\infty \|v(t)\|_{\chi^{-1}}^4 d\tau \leq \int_0^\infty \|v(t)\|_{L^2}^2 \|\nabla v(t)\|_{L^2}^2 \leq \frac{2}{\nu} \|v_0\|_{L^2}^4 \exp \left( \frac{C\nu}{\nu - (2\pi)^{-3} \|w_0\|_{\chi^{-1}}} \right). \]

Since \( v \in C([0, \infty); \chi^{-1}) \), we observe that there exists \( t_0 > 0 \) such that \( \|v(t_0)\|_{\chi^{-1}} < \varepsilon/2 \), and thus we have \( \|u(t_0)\|_{\chi^{-1}} \leq \|v(t_0)\|_{\chi^{-1}} + \|w(t_0)\|_{\chi^{-1}} < \varepsilon \). So, applying Theorem 3 for the data \( u(t_0) \) we obtain

\[ \|u(t)\|_{\chi^{-1}} \leq \|u(t_0)\|_{\chi^{-1}} < \varepsilon, \quad t > t_0, \]

which implies \( \lim_{t \to 0} \|u(t)\|_{\chi^{-1}} = 0 \).

Here, we notice that in the final part of the proof we need the uniqueness of solutions, which is assured in our class of solutions. In fact, if \( u_1 \), and \( u_2 \in C([0, \infty); H^s) \) are two solutions to (NS\( \Omega \)) satisfying

\[ u_1, u_2 \in AC([0, \infty); H^{s-1}(\mathbb{R}^3)) \cap L^1_{\text{loc}}(0, \infty; H^{s+1}(\mathbb{R}^3)), \]

then, \( \tilde{u} := u_1 - u_2 \) satisfies \( \text{div} \tilde{u} = 0 \) and

\[ \partial_t \tilde{u} + \nu \Delta \tilde{u} + \Omega \varepsilon_3 \times \tilde{u} + (\tilde{u}, \nabla) \tilde{u} + (u_1, \nabla) \tilde{u} + (\tilde{u}, \nabla) u_2 + \nabla (p_1 - p_2) = 0, \]

and thus we obtain

\[ \frac{d}{dt} \|\tilde{u}(t)\|_{L^2}^2 + \frac{\nu}{2} \|\nabla \tilde{u}(t)\|_{L^2}^2 = \|\langle (\tilde{u}, \nabla) u_2, \tilde{u} \rangle\|_{L^2} \leq \|\nabla u_2(t)\|_{L^\infty} \|\tilde{u}(t)\|_{L^2}^2. \]

Therefore, we have

\[ \frac{d}{dt} \|\tilde{u}(t)\|_{L^2}^2 = C_{H^{s+1}} \|u_2(t)\|_{H^{s+1}} \|\tilde{u}(t)\|_{L^2}^2 \]

and Gronwall’s inequality implies \( \tilde{u}(t) = 0 \) for \( t > 0 \).

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Graduate School of Mathematics, Nagoya University, Nagoya 464-8602, Japan

E-mail address, J. Kato: jkato@math.nagoya-u.ac.jp