MINIMAL MODELS, GT-ACTION AND FORMALITY OF THE LITTLE DISK OPERAD

DAN PETERSEN

Abstract. We give a new proof of formality of the operad of little disks. The proof makes use of an operadic version of a simple formality criterion for commutative differential graded algebras due to Sullivan. We see that formality is a direct consequence of the fact that the Grothendieck–Teichmüller group operates on the chain operad of little disks.

Introduction

Let $D_2$ be the topological operad of little disks. It was proven in [Tamarkin 03] that this operad is formal: there is a chain of quasi-isomorphisms of dg operads connecting the operad $\text{Chains}(D_2)$ and its homology $H(D_2)$. A different proof, which works for little disks of any dimension, was given in [Kontsevich 99], see also the improvements in [Lambrechts–Volic 08].

In this note we give a short proof of formality of $D_2$. We begin by recalling from Sullivan a simple characterization of when a cdga is formal, and explain why this characterization carries over without changes to dg operads. The crucial tool is the notion of a minimal model of a cdga or a dg operad, respectively. Using this one can immediately deduce from the action of $GT$ on $\text{Chains}(D_2)$ and the surjectivity of $GT(\mathbb{Q}) \to \mathbb{Q}^\times$, proven by Drinfel’d, that $D_2$ is a formal operad. Finally we give some motivation for the proof coming from the theory of weights in the cohomology of algebraic varieties.

I am grateful to Johan Alm for patient explanations, keen interest and stimulating conversations.

Formality of the little disk operad

Fix a base field $k$ of characteristic zero. If $V$ is a graded vector space, then we denote by $V^i$ its degree $i$ summand. We call $\phi_q \in \text{GL}(V)$ a grading automorphism if it has the form $\phi_q(v) = q^i v$ when $v \in V^i$, where $q \in k^\times$ is

Supported by the Göran Gustafsson foundation for scientific and medical research.
a fixed non-root of unity. In the same way there are grading automorphisms of any graded algebra or any operad in graded vector spaces. The following proposition is proven in [Sullivan 77, Theorem 12.7]. We recall Sullivan’s proof.

**Proposition.** Let $A$ be a nilpotent commutative differential graded algebra. If a grading automorphism of $H(A)$ lifts to an automorphism of $A$, then $A$ is formal.

**Proof.** Denote by $\sigma$ a lift to $A$ of the grading automorphism $\phi_q$ of $H(A)$. Let $p: M \rightarrow A$ be a minimal model. By comparing $p$ and $\sigma \circ p$, the uniqueness of the minimal model implies that $\sigma$ induces an automorphism $\tilde{\sigma}: M \rightarrow M$, well defined up to homotopy.

From the explicit inductive construction of the minimal model one can see that the eigenvalues of $\tilde{\sigma}$ on $M^i$ are products of eigenvalues on $H^{i_n}(A)$, with $\sum_n i_n \geq i$. Thus all eigenvalues of $\tilde{\sigma}$ on $M^i$ have the form $q^j$, where $j \geq i$. Define $M_j$ as the subspace of $M$ where $\tilde{\sigma}$ acts as multiplication by $q^j$. Define $\mathcal{I} = \bigoplus_{j>i} M^i_j$ and $S = \bigoplus_i M^i_i$.

By the preceding paragraph we see that $M = \mathcal{I} \oplus S$, $dS = 0$, and that $\mathcal{I}$ is an ideal. Hence

$$M \rightarrow M/(\mathcal{I}, d\mathcal{I}) = S/(S \cap d\mathcal{I}) = H(M)$$

makes sense and is easily seen to be a quasi-isomorphism. $\square$

We now assume that $P$ is a dg operad with $H(P)(0) = 0$ and $H(P)(1) \cong k$. This implies that $P$ has a minimal model, well defined up to homotopy, which may be constructed via an explicit inductive construction, see [Markl 96]. In the next proposition we assume that $P$ is cohomologically graded, but the result is of course valid also in the homological case. That Sullivan’s result is true for operads is also proven in [GNPR 05, Corollary 5.2.2]. They, like Sullivan, use this result for proving that formality descends to a smaller ground field.

**Proposition.** If a grading automorphism of $H(P)$ lifts to $P$, then $P$ is formal.

**Proof.** Repeat word for word the preceding proof, with the substitution $A \rightsquigarrow P$ and the tacit understanding that ‘minimal model’ now refers to the operadic minimal model, and ‘ideal’ refers to operadic ideal. $\square$

We can now prove formality of the little disk operad $D_2$. We first recall very briefly the Grothendieck–Teichmüller group $GT$ and its action on $\text{Chains}(D_2)$. See [Bar-Natan 98, Tamarkin 03] or the expositions in [Merkulov 11, Fresse 13] for more details.
There is an operad in groupoids $\mathbf{PaB}$, such that the objects of $\mathbf{PaB}(n)$ are parenthesized permutations of $\{1, \ldots, n\}$, and morphisms are braids on $n$ strands whose start and end must have the same label. There is a weak equivalence between $\mathbf{PaB}$ and the operad of fundamental groupoids of $D_2$. Since moreover $D_2(n)$ is a $K(\pi, 1)$ space for all $n$, we have an isomorphism $\text{Chains}(D_2) \cong \text{Chains}(\text{Nerve}(\mathbf{PaB}))$. If we take chains with $k$-coefficients, then we may as well replace $\mathbf{PaB}$ with its $k$-pro-unipotent completion $\hat{\mathbf{PaB}}$, as in rational homotopy theory. The completion is useful because whereas $\mathbf{PaB}$ itself does not have many automorphisms, it turns out that $\hat{\mathbf{PaB}}$ has a quite large automorphism group.

![Figure 1. The braiding $\tau$, the associator $\phi$, and the twist $\tau^2$.](image)

The operad $\mathbf{PaB}$ is generated by a morphism $\tau$ in $\mathbf{PaB}(2)$ (the braiding) and $\phi$ in $\mathbf{PaB}(3)$ (the associator), see Figure 1, and an automorphism of $\hat{\mathbf{PaB}}$ is determined by the images of $\tau$ and $\phi$. The image of $\tau$ can be described by a scalar $\lambda \in k^{\times}$: if we abusively denote by $\tau^2$ the ‘twist’ in Figure 1, then we must have $\tau^2 \mapsto (\tau^2)^\lambda$ for some such parameter, and $\lambda$ determines the image of $\tau$. The exponentiation makes sense because $\text{Hom}(\hat{\mathbf{PaB}}(2), ((12), (12)))$ is a pro-unipotent group. Describing the image of $\phi$ is more complicated, since we need to describe an element of a completion of a three-strand braid group. One finds that the image of $\phi$ can be described by an element $f$ in the pro-unipotent completion of the free group $F_2$, and that $f$ must satisfy a certain list of equations which we do not write down. One can then define an algebraic group $\text{GT}$ consisting of all such pairs $(\lambda, f)$, with group operation corresponding to compositions of automorphisms. This is the Grothendieck–Teichmüller group. By construction it acts on $\hat{\mathbf{PaB}}$ and hence on $\text{Chains}(D_2)$.

**Theorem.** The operad $D_2$ of little disks is formal over $\mathbb{Q}$.

**Proof.** Consider the map $\text{GT} \to \mathbb{G}_m$ which maps a pair $(\lambda, f)$ to $\lambda$. We claim that this sends an automorphism of $\text{Chains}(D_2)$ to the induced automorphism on homology, where $\mathbb{G}_m$ acts on homology via the grading action. The easiest way to see this is to use that the homology operad $H(D_2)$ (which is the operad of Gerstenhaber algebras) is generated in arity 2. In particular the automorphism induced on homology by $(\lambda, f)$ can not depend on $f$, since $f$ only affects
PaB(n) for n ≥ 3. The space \( D_2(2) \) is homotopic to a circle and its fundamental group is generated by the twist \( \tau^2 \). The map \( \tau^2 \mapsto (\tau^2)^\lambda \) induces the identity on \( H_0(D_2(2)) \) and multiplication by \( \lambda \) on \( H_1(D_2(2)) \), which proves the claim. Finally, \( GT(\mathbb{Q}) \to \mathbb{Q}^\times \) is surjective (in fact even split), as proven in [Drinfel'd 90, Section 5]. By the formality criterion established earlier, this shows that \( D_2 \) is formal.

\[ \square \]

**Remark.** It is a well established principle that a formality isomorphism for the little disks must in one way or another involve the choice of an associator, see [Kontsevich 99]. This principle holds true also for our proof: Drinfel’d deduces the surjectivity of \( GT(\mathbb{Q}) \to \mathbb{Q}^\times \) from the existence of a rational associator.

**Remarks on weights**

Deligne, Griffiths, Morgan and Sullivan [DGMS 75] proved that compact Kähler manifolds are formal. Their proof uses classical Hodge theory and the \( dd^c \)-lemma. However, in the introduction they explain that they originally conjectured the result for smooth projective varieties by thinking about (at the time conjectural) properties of étale cohomology and positive characteristic algebraic geometry. Namely, one expected to be able to give purely algebraic constructions of Massey products in the étale cohomology, which should in particular be equivariant with respect to the Frobenius map. But the \( n \)th Massey product \( \mu_n \) decreases cohomological degree by \( n - 2 \), and by the Weil conjectures all eigenvalues of Frobenius on \( H^i \) should have absolute value \( q^{i/2} \). Thus Frobenius equivariance should force a ‘uniform’ vanishing of \( \mu_n \) for all \( n > 2 \), and we expect the variety to be formal. This is an instance of the philosophy of ‘weights’ in cohomology, see e.g. Deligne’s 1974 ICM address [Deligne 75].

A proof of formality along these lines was later obtained by Deligne via the proof of the Weil conjectures [Deligne 80, (5.3)]: for \( X \) a smooth complex projective variety, one may choose a countable subfield \( k \) over which \( X \) is defined and use étale cohomology to obtain a dg algebra with an action of \( \text{Gal}(\overline{k}/k) \) computing \( H(X) \), and the Galois action can be used to define a ‘weight filtration’ which implies formality.

The topological space \( D_2(n) \) is homotopy equivalent to the configuration space of \( n \) points in the complex plane. This, in turn, is the complex points of the algebraic variety

\[
F_n = \mathbb{A}^n \setminus \text{(big diagonal)}
\]

which is defined over \( \mathbb{Z} \). This fact, as well as the actions of \( \text{Gal}(\overline{Q}/Q) \) on \( \text{Chains}(D_2) \otimes \mathbb{Q}_\ell \) for any prime \( \ell \) (via the embedding \( \text{Gal}(\overline{Q}/Q) \to \hat{GT} \) constructed in [Drinfel’d 90] and [Ihara 94], where \( \hat{GT} \) denotes the profinite version
of the Grothendieck-Teichmüller group), can lead one to speculate that the operad $D_2$ is actually (up to homotopy) the base change to $\mathbb{C}$ of some algebro-geometrically defined operad defined over $\mathbb{Q}$ (or perhaps even $\mathbb{Z}$). This was proposed in [Morava 07]. Note though that the spaces $F_n$ do not themselves form an operad in any natural sense. The $\ell$-adic Galois representation on the étale cohomology group $H^i_{\text{ét}}(F_n, \mathbb{Q}_\ell)$ is known: it is a sum of copies of the Tate object $\mathbb{Q}_\ell(-i)$ of weight $2i$, see [Kim 94]. This coincides with the Galois action on $H^i(D_2(n)) \otimes \mathbb{Q}_\ell$ defined via $\hat{GT}$, as one sees from the commutative diagram

$$
\begin{array}{ccc}
\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) & \longrightarrow & \hat{GT} \\
\downarrow & & \downarrow \\
\hat{GT}(\mathbb{Q}_\ell) & \longrightarrow & \hat{\mathbb{Z}}^x
\end{array}
$$

where the composition in the top row is the cyclotomic character.

We have explained that for smooth projective varieties the yoga of weights predicted vanishing of all Massey products. Something similar happens here. Suppose we did not know that $D_2$ is formal. By a Homotopy Transfer Theorem there is a structure of strong homotopy operad on $H(D_2)$ making it quasi-isomorphic to $\text{Chains}(D_2)$ [Granåker 07]. Just as for $A_\infty$-algebras this structure is encoded by an infinite sequence of higher order multilinear operations $\mu_n$ which in this case raise homological degree by $n - 2$. If these operations were compatible with the weights in cohomology, they would all need to vanish for $n > 2$ and $D_2$ would be formal.

In Deligne’s formality proof we needed a Galois action to define the weight filtration, and the Galois action was obtained from étale cohomology. But here we do not need any algebraic geometry or a realization of Morava’s proposal to get a Galois action on $\text{Chains}(D_2)$, since we already know that $GT$ acts on this chain operad. All in all, this suggests strongly that there should exist a proof of formality of $D_2$ using only the fact that $GT$ acts on its operad of chains. The present note is the result of this line of thinking.

**Remark.** By reasoning with weights exactly as above, one is led to conjecture that operads of smooth projective varieties are always formal. In fact the main theorem of [GNPR 05] is that operads of compact Kähler manifolds are formal. Just as in [DGMS 75] their proof uses classical Hodge theory and does not directly involve the theory of weights.

**References**

[Bar-Natan 98] Dror Bar-Natan. On associators and the Grothendieck-Teichmüller group. I. Selecta Math. (N.S.), 4(2):183–212, 1998.
[Deligne 75] Pierre Deligne. Poids dans la cohomologie des variétés algébriques. In Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1, pages 79–85. Canad. Math. Congress, Montreal, Que., 1975.

[Deligne 80] Pierre Deligne. La conjecture de Weil. II. Inst. Hautes Études Sci. Publ. Math., (52):137–252, 1980.

[DGMS 75] Pierre Deligne, Phillip Griffiths, John Morgan, and Dennis Sullivan. Real homotopy theory of Kähler manifolds. Invent. Math., 29(3):245–274, 1975.

[Drinfel’d 90] Vladimir Drinfel’d. On quasitriangular quasi-Hopf algebras and on a group that is closely connected with Gal(\(\overline{\mathbb{Q}}/\mathbb{Q}\)). Algebra i Analiz, 2(4):149–181, 1990.

[Fresse 13] Benoît Fresse. Homotopy of operads and Grothendieck–Teichmüller groups. Book in preparation. http://math.univ-lille1.fr/~fresse/OperadHomotopyBook/, 2013.

[Granärker 07] Johan Granärker. Strong homotopy properads. Int. Math. Res. Not. IMRN, (14):Art. ID rnm044, 26, 2007.

[GNPR 05] Francisco Guillén Santos, Vicente Navarro, Pere Pascual, and Agustí Roig. Moduli spaces and formal operads. Duke Math. J., 129(2):291–335, 2005.

[Ihara 94] Yasutaka Ihara. On the embedding of Gal(\(\overline{\mathbb{Q}}/\mathbb{Q}\)) into \(\hat{G}\). In The Grothendieck theory of dessins d’enfants (Luminy, 1993), volume 200 of London Math. Soc. Lecture Note Ser., pages 289–321. Cambridge Univ. Press, Cambridge, 1994.

[Kim 94] Minhyong Kim. Weights in cohomology groups arising from hyperplane arrangements. Proc. Amer. Math. Soc., 120(3):697–703, 1994.

[Kontsevich 99] Maxim Kontsevich. Operads and motives in deformation quantization. Lett. Math. Phys., 48(1):35–72, 1999.

[Lambrechts–Volic 08] Pascal Lambrechts and Ismar Volic. Formality of the little N-disks operad. Preprint. arXiv:0808.0457, 2008.

[Markl 96] Martin Markl. Models for operads. Comm. Algebra, 24(4):1471–1500, 1996.

[Merkulov 11] Sergei Merkulov. Grothendieck–Teichmüller group in algebra, geometry and quantization: a survey. Notes from a seminar at Stockholm University organized by Torsten Ekedahl and Sergei Merkulov. http://www2.math.su.se/~sm/ 2011.

[Morava 07] Jack Morava. The motivic Thom isomorphism. In Elliptic cohomology, volume 342 of London Math. Soc. Lecture Note Ser., pages 265–285. Cambridge Univ. Press, Cambridge, 2007.

[Sullivan 77] Dennis Sullivan. Infinitesimal computations in topology. Inst. Hautes Études Sci. Publ. Math., (47):269–331 (1978), 1977.

[Tamarkin 03] Dmitry E. Tamarkin. Formality of chain operad of little discs. Lett. Math. Phys., 66(1-2):65–72, 2003.

E-mail address: danpete@math.kth.se

Department of Mathematics, KTH Royal Institute of Technology, 100 44 Stockholm, Sweden