Efficient analytical method to obtain the responses of a gear model with stochastic load and stochastic friction

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Abstract. The friction, which is widely existed in practical, is seldom considered when modelling the gear system with stochastic load. Due to the variation of the temperature and lubrication condition, friction is a stochastic factor to a gear model. In this paper, a gear model with stochastic load, stochastic friction, and some other deterministic factors is considered. Due to the effects of stochastic factors (i.e., load and friction), the gear system faces more vibration and noise than the case with all deterministic factors. Thus, to analyse the variation of responses in the gear dynamical model, the corresponding dynamic equation needs to be solved. However, the statistic characteristics of dynamic responses are hard to obtain by numerical methods. Thus, an efficient analytical method is proposed, and then, an approximate analytical solution of the dynamic equation can be obtained in this paper. By the obtained solution, the vibration and noise of gear systems can be well investigated. Simulation results are provided to demonstrate the superior performance of the proposed method.

1. Introduction

Gear system which is one of the most widely adopted mechanical parts shows significant important to the development of mechanical theory [1]. Therefore, the research about dynamics and vibrations of gear systems attracts lots of research attention. The gear models under deterministic domain have been investigated in decades. However, stochastic load is one of the main sources of gear vibration and noise in practical [2]. The gear system dynamic modelling with stochastic load attracts more and more attention. In [3], a gear model under stochastic load with transmission error, backlash, and periodic gear mesh stiffness was introduced. Wen et al. [4] investigated a gear dynamic model under stochastic load with backlash, time-varying mesh stiffness (TVMS), and constant damping coefficient. However, besides the load, the internal factors (e.g., friction, backlash, etc.) of a gear system may also have random variation and affect the system’s dynamic behaviour greatly [5]. For example, friction, as a main cause of vibration on gear system’s transient state, is observed as stochastic in lots of scenarios [6]. Therefore, the stochastic internal factor should be considered when modelling gear systems.

Under a gear system model, there are two kinds of forms, analytical expression and numerical solutions, to demonstrate the responses of the gear system. Therefore, some methods have been proposed to obtain the responses of a gear system [7]. For example, Runge Kutta-Monte Carlo (MC) method and statistical Newmark method are proposed to obtain the numerical solutions. Stochastic averaging method and path integration (PI) method are proposed to obtain the analytical expression [7]. Obtaining numerical solution is costly since a large number of samples are required for reasonable accuracy. By contrast, it would save much time if we could obtain the analytical solution [8]. Therefore, we focus on obtaining the analytical solution in this paper.
Several researchers have explored the analytical solution to a gear dynamic model under stochastic load. Naess et al. [9] derived the analytical solution to a gear system considering constant stiffness, constant damping coefficient, and backlash under the excitation of white noise. In [4], the analytical solution to a gear system considering constant damping coefficient, TVMS, and backlash under stochastic load was derived. However, only deterministic internal factors are considered in these works. A gear model with a stochastic internal factor under stochastic load cannot be solved by existing analytical methods.

In this paper, we consider the gear dynamic model with stochastic load, stochastic friction (i.e., the stochastic internal factor), and other deterministic internal factors (e.g., damping, TVMS, backlash, etc.). Then, a method is proposed to derive an approximate analytical solution of the corresponding dynamic equations. By the proposed method, the dynamic characteristics (e.g., vibration, noise, etc.) of gear systems can be well investigated. Compared with the MC method, our proposed method can achieve similar accuracy responses with much less time cost.

The remaining parts of this paper are organized as follows. The gear system with stochastic load and stochastic friction is modelled in Section 2. The method to obtain the responses of the gear model is proposed in Section 3. The simulation results are given in Section 4. Section 5 draws conclusions.

2. Gear dynamic model

In this work, a gear model considering TVMS, backlash, and friction is introduced and is shown in figure 1. In this model, both the load and the friction (i.e., the single stochastic internal factor) are stochastic. Therefore, the considered model is formulated as:

\[ J_1 \ddot{\theta}_1 = T_1 - FR_{b1} - F_f X_1 \]  
\[ J_2 \ddot{\theta}_2 = T_2 + FR_{b2} + F_f X_2 \]

where \( J_1, T_1, X_1, \) and \( R_{b1} \) are the moment of inertia, external torque, arm length of friction, and base circle of gear \( i \in \{1, 2\} \), respectively, \( F \) represents the total force between the contact teeth, \( F_f \) is the sliding friction [2]. Let \( x, \dot{x}, \) and \( \ddot{x} \) denote the relative angular displacement, the relative angular velocity, and the relative angular acceleration, respectively. Thus, we have

\[ x = \theta_1 R_{b1} - \theta_2 R_{b2} \]

where \( \theta_i \) is the angular displacement of gear \( i \in \{1, 2\} \).

Figure 1. Model of the gear system

Simplify equation (1) and equation (2), and then, a normalized equation is obtained and shown as:
\[ \ddot{x} + cL(\theta_1, \mu, t)\dot{x} + kL(\theta_1, \mu, t)g(x) = f(t) \]  
(4)

where \( c \) denotes the damping coefficient which is considered as a constant, \( k \) denotes the TVMS, \( \mu \)
represents the friction coefficient, \( g(x) = \begin{cases} x - b, & x > b \\ 0, & -b \leq x \leq b \\ x + b, & x < -b \end{cases} \) is the function of backlash, \( f(t) \) is the
external load, and \( L(\theta_1, \mu, t) \) is a function which is caused by friction. The expression of \( L(\theta_1, \mu, t) \)
can be obtained as [2]:

\[ L(\theta_1, \mu, t) = \chi_i + \mu(\chi_2\theta_1 + \chi_3) \]  
(5)

where \( \chi_j (j = 1, 2, 3) \) relates to gear design parameters. Note that \( \chi_j (j = 1, 2, 3) \) can be considered as
constant. \( \mu \) is a summation of a deterministic part \( \mu_0 \) and a random part \( \xi(t) \). The external load
\( f(t) \) is modeled as a combination of a constant deterministic part \( f_0 \), a periodical deterministic part
\( f_1 \cos(\varphi t) \), and a random part \( \xi(t) \) [4]. About the random part, it is generally set to a Gaussian white
noise. Therefore, the expressions of the external load are given as follows.

\[ f(t) = f_0 + f_1 \cos(\varphi t) + \xi(t) \]  
(6)

\[ E(\xi(t)) = 0 \]  
(7)

\[ E[\xi(t)\xi(t + \tau)] = r\delta(\tau) \]  
(8)

where \( \varphi \) is a constant frequency, \( t \) denotes time, \( r \) is the variance of the random part \( \xi(t) \), and \( \delta(\tau) \)
is the Dirac Delta function.

From equation (5), we can see that \( L(\theta_1, \mu, t) \) depends on \( \theta_1 \). Considering the deterministic
nonlinear and time-varying characteristics, \( \theta_1 \) is also stochastic because of \( f(t) \) as given in equation
(6). To obtain the responses of the gear system, we need to derive the analytical solution of the
stochastic differential equation (SDE), i.e., equation (4).

### 3. The proposed method to derive the analytical solution

The existing methods can obtain the analytical solution of the differential equation which \( L(\theta_1, \mu, t) \) is
independent with \( \theta_1 \). Note that \( L(\theta_1, \mu, t) \) depends on \( \theta_1 \) in our work. Therefore, equation (4) cannot
be solved by existing methods directly. To obtain the analytical solution of equation (4), a method is
proposed in this paper. The basic idea of the proposed method contains the following two steps:

1) Obtain the tentative analytical solution. We first introduce a method to transform the SDE
equation (4) into a form that can be solved by PI method. Then, we derive a tentative
analytical solution using PI method.

2) Adjust results by adding a modification function. Due to the previous transformation, errors
may be brought into the tentative analytical solution, and thus, a modification function is
applied to adjust the tentative analytical solution. Supervised learning is used to obtain the
modification function.

#### 3.1. Obtaining the tentative analytical solution

By considering the deterministic part of the load, equation (9) is obtained.

\[ \ddot{x} + cL(\theta_1, \mu, t)\dot{x} + kL(\theta_1, \mu, t)g(x) = \hat{f}(t) \]  
(9)

where \( \hat{f}(t) = f_0 + f_1 \cos(\varphi t) \) is the deterministic part of the load. The solution of equation (9), which
are \( x \) (considered as \( \mu_1 \)) and \( \dot{x} \) (considered as \( \mu_2 \)), can be obtained by solving equation (9). Note
that, according to equation (3), \( \dot{\theta}_1 \) and \( \dot{\theta}_2 \) can be obtained according to the obtained \( x \) in equation (9).
After that, we get \( \lambda_c(t) \) as expressed in equation (10) by substituting \( \dot{\theta}_1 \) to equation (3).

\[
\lambda_c(t) = \chi_1 + \mu \left( \chi_2 \dot{\theta}_1 + \chi_3 \right)
\]  

(10)

By replacing \( L(\theta_1, \mu, t) \) by \( \lambda_c(t) \) in equation (4), we can obtain

\[
\ddot{x} + c \lambda_c(t) \dot{x} + k \lambda_c(t) g(x) = f(t)
\]  

(11)

Based on this transformation, all parameters in equation (11) become deterministic except for load, and thus, it can be solved by PI method now.

For the case that \( g(x) \neq 0 \), the system equation is shown in equation (11). Under the excitation of \( \delta(t) \), an impulse function \( h_c(t) \) satisfies the following equation.

\[
\frac{d^2h_c(t)}{dt^2} + c \lambda_c(t) \frac{dh_c(t)}{dt} + k \lambda_c(t) h_c(t) = 0
\]  

(12)

The general solution to the homogeneous equation (12) can be obtained as

\[
h_c(t) = \frac{\nu}{\beta} e^{\alpha t} \sin(\beta t)
\]  

(13)

where \( \alpha = -\frac{c \lambda_c(t)}{2} \) and \( \beta = \sqrt{4k \lambda_c(t) - (c \lambda_c(t))^2} \). Let \( \sigma_1^2 \) and \( \sigma_2^2 \) denote the variances of \( x \) and \( \dot{x} \), respectively. And \( \sigma_{12} \) denotes the covariance of \( x \) and \( \dot{x} \). Then, we can obtain \( \sigma_1^2 \), \( \sigma_{12} \), and \( \sigma_2^2 \) as follows, where \( G_0 = 2\pi r \).

\[
\sigma_1^2 = \frac{G_0\nu^2}{4\beta^2 (\alpha^2 + \beta^2)} \left[ \frac{\beta^2}{\alpha} e^{2\alpha t} \left( \frac{\alpha^2 + \beta^2}{\alpha} + \alpha \cos(2\beta t) + \beta \sin(2\beta t) \right) \right]
\]  

(14)

\[
\sigma_{12} = \frac{G_0\nu^2}{4\beta^2} e^{2\alpha t} \left[ 1 - \cos(2\beta t) \right]
\]  

(15)

\[
\sigma_2^2 = -\frac{G_0\nu^2}{4\alpha \beta} \left( \beta^2 - e^{2\alpha t} \left[ \alpha^2 + \beta^2 + \alpha \beta \sin(2\beta t) - \alpha^2 \cos(2\beta t) \right] \right)
\]  

(16)

For the case that \( g(x) = 0 \), the system equation is shown as

\[
\ddot{x} + cL(\theta_1, \mu, t) \dot{x} = f(t)
\]  

(17)

Similarly, we have equation (18) by eliminating the random term of load \( f(t) \) in equation (17).

\[
\ddot{x} + cL(\theta_1, \mu, t) \dot{x} = \ddot{f}(t)
\]  

(18)

Then, we can obtain the results by solving equation (18). By substituting the obtained results to equation (10), we can obtain \( \lambda_c(t) \). The following equation is obtained by replacing \( L(\theta_1, \mu, t) \) to \( \lambda_c(t) \) in equation (17).

\[
\ddot{x} + c \lambda_c(t) \dot{x} = f(t)
\]  

(19)

The general solution to the homogeneous equation (19) is given as:
\[ h^{(2)}_a(t) = \frac{v}{a_x} \left(1 - e^{a_x v t} \right) \]

where \( a_x = c \dot{x}_a(t) \) and \( v \) represents the mean of initial velocity. Then, \( \sigma_1^2 \), \( \sigma_{12} \), and \( \sigma_2^2 \) can be obtained as:

\[ \sigma_1^2 = \frac{G_0 v^2}{a_x^2} \left( \Delta t - \frac{1}{a_x^2} \left[ \frac{3}{2} - 2 e^{-a_x \Delta t} + \frac{1}{2} e^{-2a_x \Delta t} \right] \right) \]

\[ \sigma_{12} = \frac{G_0 v^2}{a_x^2} \left[ \frac{1}{2} e^{-a_x \Delta t} + \frac{1}{2} e^{-2a_x \Delta t} \right] \]

\[ \sigma_2^2 = \frac{G_0 v^2}{2a_x} \left(1 - e^{-2a_x \Delta t} \right) \]

### 3.2. Updating the tentative analytical solution

Due to the transformation in Section 3.1, the obtained tentative analytical solution is not the accurate solution of equation (4). Therefore, we try to derive the modification function and then apply it to the tentative analytical solution to obtain the accurate solution of equation (4).

Generally, the exact analytical solution of equation (4) is hard to derive. The numerical responses obtained by MC method are regarded as accurate results. Therefore, a modification function is required to demonstrate the rule of errors according to the samples obtained by MC method and the tentative analytical solution. A supervised learning algorithm is applied to obtain the modification function. Finally, we adjust the tentative analytical solution by the modification function and then obtain the final probability distributed function (PDF). The obtained final PDF can be regarded as the analytical solution (i.e., PDF) of the SDE equation (4).

Therefore, a modification to \( \mu_1 \), \( \mu_2 \), \( \sigma_1 \), \( \sigma_2 \), and \( \sigma_{12} \) should be taken. The modification function, denoted \( \eta(t) \), is based on the results of MC method and the tentative analytical solution. For different cases, \( \eta(t) \) may have different forms, such as polynomial function, exponential function, and so on.

Supervised learning is used to find \( \eta(t) \). Supervised learning is used to find a mapping between a set of input samples and the corresponding output and this mapping is then applied to predict the outputs under other input data [10]. The samples obtained by the tentative analytical solution would be the input data and the samples obtained by MC method would be the corresponding output. We define the input samples and the corresponding output as \( \chi(t) \) and \( \zeta(t) \), respectively. Then, we try to seek a function \( \eta(t) : \chi(t) \rightarrow \zeta(t) \). To evaluate \( \eta(t) \), the risk function \( \varepsilon \) is defined as [11]:

\[ \varepsilon = \int_0^t \left| \frac{\zeta(t) - \eta(t) \chi(t)}{\zeta(t)} \right| dt \]  

To get a proper \( \eta(t) \), the value of \( \varepsilon \) should be minimized. In this paper, the structural risk minimization method is adopted to get \( \eta(t) \). After obtaining the modification function \( \eta(t) \), the modified results can be obtained by:

\[ \sigma_1(t) = \sigma_1(t) \eta_1(t) \]

\[ \sigma_2(t) = \sigma_2(t) \eta_2(t) \]

\[ \sigma_{12}(t) = \sigma_{12}(t) \eta_{12}(t) \]
where \( \tilde{\sigma}_1(t) \), \( \tilde{\sigma}_2(t) \), and \( \tilde{\sigma}_{12}(t) \) are the modified values of \( \sigma_1(t) \), \( \sigma_2(t) \), and \( \sigma_{12}(t) \), respectively, and \( \tilde{\eta}_1(t) \), \( \tilde{\eta}_2(t) \), and \( \tilde{\eta}_{12}(t) \) are the corresponding modification functions. Then, the improved PDF expressions of \( x \) and \( \dot{x} \) with a high accuracy are obtained.

4. Simulation results

The parameters of the gear system used in this validation are from [12] and friction coefficient parameters are set as \( \mu_0 = 0.04 \) and \( r_1 = 0.08 \). For the stochastic load in equation (6), the parameters are set as \( f_0 = 2 \times 10^4 \), \( f_1 = 2 \times 10^3 \), \( \varphi = 100 \), and \( r = 2 \times 10^3 \).

4.1. Accuracy evaluation

The instantaneous marginal PDF of \( x \) and \( \dot{x} \) at 0.3s (chosen by random) are shown in figure 2. According to the simulation results, the PDFs of the two methods are very close. This agreement validates the analytical solution which is obtained by the proposed method is accurate. The joint PDF of \( x \) and \( \dot{x} \) are shown in figure 2. It illustrates the marginal PDFs of \( x \) and \( \dot{x} \) at \( t = 0.76s \) (chosen by random). It is observed that the joint PDF from the proposed method agrees well with that from MC method.

4.2. Accuracy evaluation

Suppose \( H \) is the total computational cost for single simulation under stochastic load, \( H_1 \) is the time cost by solving the equation under deterministic load (obtaining \( \mu_1 \) and \( \mu_2 \)), \( H_2 \) is the time cost of obtaining a learning sample by MC method, and \( N_m \) denotes the number of samples. Note that \( H_2 \) is much smaller than \( H \). Table 1 gives the total computational cost comparison between MC method and the proposed method. The time cost of obtaining analytical solution from \([0, 1.6]\) s by the proposed method and MC method is also given in table 1.
Table 1. The total computation cost comparison.

| Method          | Calculation | Cost (s)   |
|-----------------|-------------|------------|
| MC method       | $H \times N_m$ | 127836.4 |
| Proposed method | $H_1 + H_2 \times N_m$ | 22337.1 |

From Table 1, the proposed method saves 83.5% computational cost. In addition, once the modification function is obtained, it can be applied to the longer responses. Therefore, with the increasing of simulation length of gear system, our proposed method could save more time in calculation.

5. Conclusions

This paper proposes an analytical solution to a spur gear dynamic model with stochastic load and stochastic friction. In the proposed method, a tentative analytical solution is derived using PI method and then updated using a supervised learning algorithm. The simulation results show that our proposed method can achieve a similar accuracy results under a much smaller time cost compared with the MC method. This analytical solution can be a useful tool in investigating gear systems’ vibration and noise.

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