Lorentz Covariance and Internal Space-time Symmetry of Relativistic Extended Particles

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Abstract

The difference between Lorentz invariance and Lorentz covariance is discussed in detail. A covariant formalism is developed for the internal space-time symmetry of extended particles, especially in connection with the insightful observations Feynman made during the period 1969-72. A Lorentz-group formalism is presented for the harmonic oscillator model of Feynman, Kislinger and Ravndal, which was originally based on hadronic mass spectra. This covariant version allows us to construct a parton distribution function by Lorentz-boosting the oscillator wave function of a hadron at rest. The role of the time-separation variable is discussed in detail. It is shown that, due to our inability to make measurements on this variable, it belongs to Feynman's rest of the universe. Our failure to observe the rest of the universe leads to an increase in entropy.

I. INTRODUCTION

Ever since the present form of quantum mechanics was formulated in 1927, the most pressing problem has been and still is how to combine quantum mechanics with special relativity. During this process, the success of quantum electrodynamics led many physicists to believe that quantum field theory is the ultimate answer to this question. On the other hand, in field theory, we produce measurable numbers by calculating scattering matrix elements using Feynman diagrams, where initial and final particles are all free particles. Indeed, the present form of quantum field theory, although covariant, can accommodate only scattering problems.

How about bound states? The calculation of the Lamb shift is one of the triumphs of quantum field theory. In order to calculate this shift, we need wave functions of the hydrogen atom whose behavior at the origin depends on their angular momentum states. Can we then calculate the hydrogen wave function and the Rydberg formula within the framework of quantum field theory? The answer to this question is clearly No. The Rydberg levels come from the localization condition for probability distribution. Does field theory say anything about localized probability distributions? The answer to this question is again No.

The hydrogen atom appears as a localized probability entity when it is at rest. Its angular momentum state determines how the distribution will appear to the observer in a rotated coordinate system. This is of course one of the fundamental problems in atomic spectra [1]. We can now consider how the wave function would look to observers in different Lorentz
frames. This had remained only as an academic question until Hofstadter [2] observed in 1955 that the proton is not a point particle, but its charge has a distribution. Hofstadter detected this from the relativistic recoil effects in the scattering of electrons by protons.

In spite of many laudable efforts to explain this form factor within the framework of quantum field theory, the workable model for the form factors did not emerge until after Gell-Mann’s formulation of the quark model, in which all hadrons are bound states of quarks and/or anti-quarks [3]. The question then is whether we can use the existing models of quantum mechanics, such as the nuclear shell model, to explain hadronic mass spectra [4]. For the mass spectra, one of the most effective models has been and still is the model based on harmonic oscillator wave functions [4-6]. The basic advantage of the oscillator model is that its mathematics is simple and transparent, and it does not bury physics in mathematics even though it does not always produce the most accurate numerical results. The oscillator model will prevail if there are no other models capable of producing better numerical results. This appears to be the present status of the quark model.

In the quark model, the charge distribution within the proton comes from the distribution of the charged particles inside the hadron. The success of the oscillator model for static or slow-moving hadrons does not necessarily mean that the model can be extended to the relativistic regime. The calculation of the form factor with Gaussian wave functions results in an exponential decrease for large momentum-transfer variables. However, this wrong behavior comes from the use of non-relativistic wave functions for relativistic problems. Feynman et al. made an attempt to construct a covariant oscillator model [5]. Even though they did not achieve this goal in their paper, Feynman et al. quote the work of Fujimura et al. [6] who calculated the nucleon form factor by taking into account the effect of the Lorentz-squeeze on the oscillator wave functions.

After studying these original papers, we can raise our level of abstraction. We observe first that the spherical harmonics can represent the three-dimensional rotation group, while serving as wave functions for the angular variables. Then, we can ask whether there are wave functions which can represent the Poincaré group. We can specifically ask whether it is possible to construct a set of normalizable harmonic oscillator wave functions to represent the Poincaré group. If Yes, the wave functions can be Lorentz-boosted. These wave functions then have to go through another set of tests. Are they consistent with the existing laws of quantum mechanics. If Yes, they then have to be exposed to the most cruel test in physics. Do they explain what we observe in high-energy laboratories?

The purpose of this paper is to show that we can use the oscillator wave functions to answer the question of whether quarks are partons. While the quark model is valid for static hadrons, Feynman’s parton picture works only in the Lorentz frame where the hadronic speed is close to that of light [7]. The quark model appears to be quite different from the parton model. On the other hand, they are valid in two different Lorentz frames. The basic question is whether the quark picture and the parton picture are two different manifestations of the same covariant entity. We shall discuss first the internal space-time symmetries of relativistic particles in terms of appropriate representations of the Poincaré group [8]. We then construct the oscillator wave functions satisfying the above-mentioned theoretical criterions [9-13]. This oscillator formalism will explains both the quark and the parton pictures in two separate Lorentz frames. This formalism produces all the peculiarities of Feynman’s original form of the parton picture including the incoherence of parton cross
sections.

In addition, Feynman’s parton picture raises an important question in measurement theory. Let us consider again a hadron consisting of two quarks. Then there is a Bohr-like radius measuring the space-like separation between the quarks. There is also a time-like separation between the quarks, and this variable becomes mixed with the longitudinal spatial separation as the hadron moves with a relativistic speed \[14\]. There are at present no quantum measurement theories to deal with the above-mentioned time-like separation. Because we do not know, we do not mention it.

We can afford to do this in nonrelativistic quantum mechanics. However, in the relativistic regime, the time-separation variable becomes as important as the longitudinal separation because they become mixed when the system is Lorentz-boosted. Can we still pretend not to know anything about this variable? The best solution to this problem is to put this variable into Feynman’s rest of the universe \[15,16\]. The net result is then that this unobservable variable leads to an increase in entropy.

For many years, the author has been trying to discuss these problems with his colleagues, but he has to confess that there has been a difficulty in communication. The difficulty seems to come from the time-separation variable which does not exist in the present form of non-relativistic quantum mechanics, but which plays the pivotal role in the relativistic world. As for the communication problem, the fundamental issue appears to be the difference between Lorentz invariance and Lorentz covariance. This difference is not clearly understood by most physicists these days. For this reason, we start this paper with a discussion of this problem.

In Sec. II, we discuss the historical origin of the concept of covariance. It is pointed out that, in formulating the theory of relativity, Immanuel Kant played the pivotal role before Einstein. Kant’s cultural background is also discussed in detail. We explain why the author is in a better position to appreciate the Kantian view of the world. In Sec. II, it is noted that relativistic particles have their internal space-time structures. It was Eugene Wigner who formulated this problem in terms of the subgroups of the Poincaré group known as the little groups \[8\]. In this section, we present a brief history of applications of the little groups to internal space-time symmetries of relativistic particles. In Sec. III, we construct representations of the little group for massive particles using harmonic-oscillator wave functions. In Sec. IV, it is shown that the Lorentz-boosted oscillator wave functions exhibit the peculiarities of Feynman’s parton model in the infinite-momentum limit. In Sec. V, we point out first that Feynman’s parton picture contains Feynman’s rest of the universe. It is then shown that our failure in measuring the time separation variable leads to an increase in entropy.

Much of the concept of Lorentz-squeezed wave function is derived from elliptic deformation of a sphere resulting in a mathematical technique group called contractions \[17\]. In Appendix A, we discuss the contraction of the three-dimensional rotation group to the two-dimensional Euclidean group. In Appendix B, we discuss the little group for a massless particle as the infinite-momentum/zero-mass limit of the little group for a massive particle.

II. COVARIANCE AND ITS HISTORICAL BACKGROUND

The words “Lorentz invariance” and “Lorentz covariance” are frequently used in physics. Are they the same word or are they different? Why is this question so serious? Unlike classi-
cal physics, modern physics depends heavily on observer’s state of mind or environment. In special relativity, observers in different Lorentz frames see the same physical system differently. The importance of the observer’s subjective viewpoint was emphasized by Immanuel Kant in his book entitled *Kritik der reinen Vernunft* whose first and second editions were published in 1781 and 1787 respectively. However, using his own logic, he ended up with a conclusion that there must be the absolute inertial frame, and that we only see the frames dictated by our subjectivity.

Einstein’s special relativity was developed along Kant’s line of thinking: things depend on the frame from which you make observations. However, there is one big difference. Instead of the absolute frame, Einstein introduced an extra dimension. Let us illustrate this using a Coca-Cola can. It appears like a circle if you look at it from the top, while it appears as a rectangle from the side. The real thing is a three-dimensional circular cylinder. While Kant was obsessed with the absoluteness of the real thing, Einstein was able to observe the importance of the extra dimension.

I was fortunate enough to be close to Eugene Wigner, and enjoyed the privilege of asking him many questions. I once asked him whether he thinks like Immanuel Kant. He said *Yes*. I then asked him whether Einstein was a Kantianist in his opinion. Wigner said very firmly *Yes*. I then asked him whether he studied the philosophy of Kant while he was in college. He said *No*, and said that he realized he had been a Kantianist after writing so many papers in physics. He added that philosophers do not dictate people how to think, but their job is to describe systematically how people think [18]. Wigner told me that I was the only one who asked him this question, and asked me how I knew the Kantian way of reasoning was working in his mind. I gave him the following answer.

I never had any formal education in oriental philosophy, but I know that my frame of thinking is affected by my Korean background. One important aspect is that Immanuel Kant’s name is known to every high-school graduate in Korea, while he is unknown to Americans, particularly to American physicists. The question then is whether there is in Eastern culture a “natural frequency” which can resonate with one of the frequencies radiated from the Kantian school of thought developed in Europe.

I would like to point out this question in the following way. Koreans absorbed a bulk of Chinese culture during the period of the Tang dynasty (618-907 AD). At that time, China was the center of the world as the United States is today. This dynasty’s intellectual life was based on Taoism which tells us, among others, that everything in this universe has to be balanced between its plus (or bright) side and its minus (or dark) side. This way of thinking forces us to look at things from two different or opposite directions. This aspect of Taoism could constitute a “natural frequency” which can be tuned to the Kantian view of the world where things depend how they are observed.

I would like to point out that Hideki Yukawa was quite fond of Taoism and studied systematically the books of Laotse and Chuangtse who were the founding fathers of Taoism [19]. Both Laotse and Chuangtse lived before the time of Confucius. It is interesting to note that Kantianism is also popular is Japan, and it is my assumption that Kant’s books were translated into Japanese by Japanese philosophers first, and Koreans of my father’s age learned about Kant by reading the translated versions. My publication record will indicate that I studied Yukawa’s papers before becoming seriously interested in Wignerism. I picked up a signal of possible connection between Kantianism and Taoism while reading Yukawa’s
papers carefully, and this led to my bold venture to ask Wigner whether he was a Kantianist.

Kant wrote his books in German, but he was born and spent his entire life in a Baltic enclave now called Kaliningrad located between Poland and Lithuania. Historically, this place was dominated by several different countries with different ideologies. However, Kant’s view was that the people there may appear differently depending on who looks at them, but they remain unchanged. At the same time, they had to entertain different ideologies imposed by different rulers. Kant translated this philosophy into physics when he discussed the absolute and relative frames. He was obsessed with the absolute frame, and this is the reason why Kant is not regarded as a physicist in Einstein’s world in which we live.

The people of Kant’s land stayed in the same place while experiencing different ideological environments. Almost like Kant, I was exposed to two different cultural environments by moving myself from Asia to the United States. Thus, I often had to raise the question of absolute and relative values. Let us discuss this problem using one concrete example.

About 4,500 years ago, there was a king named Yao in China. While he was looking for a man who could serve as the prime minister, he heard from many people that a person named Shiyu was widely respected and had a deep knowledge of the world. The king then sent his messengers to invite Shiyu to come to his palace and to run the country. After hearing the king’s message, Shiyu without saying anything went to a creek in front of his house and started washing his ears. He thought he heard the dirtiest story in his life.

Shiyu is still respected in the Eastern world as one of the wisest men in history. We do not know whether this person existed or is a made-up personality. In either case, we are led to look for a similar person in the Western world. In ancient Greece, each city was run by its city council. As we experience even these days, people accomplish very little in committee meetings. Thus, it is safe to assume that the city councils in ancient Greece did not handle matters too efficiently. For this reason, there was a well-respected wiseman like Shiyu who never attended his city council meetings. His name was Idiot. Idiot was a wiseman, but he never contributed his wisdom to his community. His fellow citizens labeled him as a useless person. This was how the word idiot was developed in the Western world.

We can now illustrate the difference between Lorentz invariance and Lorentz covariance. Idiot and Shiyu had the same personality if they were not the same person. The fact that they are the same person is an illustration of the invariance. On the other hand, they look quite different in the worlds with two different cultural backgrounds. This is what the covariance is about.

Idiot is a useless person in state-centered societies like Sparta. The same person is regarded as the ultimate wiseman in a self-centered society like Korea. I cannot say that I know everything about other Asian countries, but I have a deep knowledge of Korea where I was born and raised. The same person looks quite differently to observers in different cultural frames. While doing research in the United States with my Eastern background, I was frequently forced to find a common ground for two seemingly different views. This cultural background strongly influenced me in producing an expanded content of Einstein’s $E = mc^2$ tabulated in Table I.

Let us go back to the question of relative values. For Taoists, those two opposite faces of the same person is like “yang” (plus) versus “ying” (minus). Finding the harmony between these two opposite points of view is the ideal way to live in this world. We cannot always live
TABLE I. Further contents of Einstein’s \( E = mc^2 \). Massive and massless particles have different energy-momentum relations. Einstein’s special relativity gives one relation for both. Wigner’s little group unifies the internal space-time symmetries for massive and massless particles which are locally isomorphic to \( O(3) \) and \( E(2) \) respectively. It is a great challenge for us to find another unification. In this note, we present a unified picture of the quark and parton models which are applicable to slow and ultra-fast hadrons respectively.

| Massive, Slow | COVARIANCE | Massless, Fast |
|---------------|------------|---------------|
| Energy-Momentum | \( E = p^2/2m \) | \( E = [p^2 + m^2]^{1/2} \) | \( E = cp \) |
| Internal space-time symmetry | \( S_3 \) | \( S_1, S_2 \) | \( S_3 \) |
| Relativistic Extended Particles | Quark Model | Covariant Model of Hadrons | Partons |

like Shiyu, nor like Idiot. The key to happiness is to find a harmony between the individual and the society to which he/she belongs. The key word here seems to be “harmony.”

To Kantianists, however, it is quite natural for the same character to appear differently in two different environments. The problem is to find the absolute value from these two different faces. Does this absolute value exist? According to Kant, it exists. To most of us, it is very difficult to find it if it exists.

Let us finally visit Einstein. He avoids the question of the existence of the absolute value. Instead, he introduces a new variable. The variable is the ratio between the individual’s ability to contribute and the community’s need for his service. The best way to live in this world is to adjust this variable to the optimal value. Einstein’s approach is to a quantification of Taoism by introducing a new variable.

If Taoism is so close to Einsteinism, why do we have to mention Kant at all? We have to keep in mind that Kant was the first person who formulated the idea that observers can participate in drawing the picture of the world. It is not clear whether Einstein could have formulated his relativity theory without Kant. Kant spent many years for studying physics, namely observer-dependent physics. However, because of his obsession toward the absolute thing, he spent all of his time for finding the absolute frame. If one has a Taoist background, he/she is more likely to appreciate the concept of relativistic covariance.

I would like to stress that Taoism is not confined to the ancient Eastern world. It is practiced frequently in the United States. Let us look at American football games. The offensive strategy does not rely on brute force, but is aimed at breaking the harmony of the
defense. For instance, when the offensive team is near the end zone, the defense becomes very strong because it covers only a small area. Then, it is not uncommon for the offense to place four wide-receivers instead of two. This will divide the defense into two sides while creating a hole in the middle. Then the quarter-back can carry the ball to the end zone. The key word is to destroy the balance of the defense.

Taoism forms the philosophical base for Sun Tzu’s classic book on military arts \[22\]. When I watch the football games, I watch them as Sun-Tzu games. My maternal grandfather was fluent in the Chinese classic literature, and he was particularly fond of Sun Tzu. He told me many stories from Sun Tzu’s books. This presumably was how I inherited some of the Taoist tradition. Needless to say, my research life was influenced by my Asian background. Many of my Asian friends complain that they are handicapped to do original research because of the East-West cultural difference. I disagree with them. This difference could be the richest source of originality.

III. LITTLE GROUPS OF THE POINCARÉ GROUP

The Poincaré group is the group of inhomogeneous Lorentz transformations, namely Lorentz transformations preceded or followed by space-time translations. In order to study this group, we have to understand first the group of Lorentz transformations, the group of translations, and how these two groups are combined to form the Poincaré group. The Poincaré group is a semi-direct product of the Lorentz and translation groups. The two Casimir operators of this group correspond to the (mass)\(^2\) and (spin)\(^2\) of a given particle. The particle mass and its spin magnitude are Lorentz-invariant quantities. Then what are the Lorentz-covariant entities.

The question is how to construct the representations of the Lorentz group which are relevant to physics. With this point in mind, Wigner in 1939 studied the subgroups of the Lorentz group whose transformations leave the four-momentum of a given free particle \[8\]. The maximal subgroup of the Lorentz group which leaves the four-momentum invariant is called the little group. Since the little group leaves the four-momentum invariant, it governs the internal space-time symmetries of relativistic particles. Wigner shows in his paper that the internal space-time symmetries of massive and massless particles are dictated by the \(O(3)\)-like and \(E(2)\)-like little groups respectively.

The \(O(3)\)-like little group is locally isomorphic to the three-dimensional rotation group, which is very familiar to us. For instance, the group \(SU(2)\) for the electron spin is an \(O(3)\)-like little group. The group \(E(2)\) is the Euclidean group in a two-dimensional space, consisting of translations and rotations on a flat surface. We are performing these transformations everyday on ourselves when we move from home to school. The mathematics of these Euclidean transformations are also simple. However, the group of these transformations are not well known to us. In Appendix A, we give a matrix representation of the \(E(2)\) group.

The group of Lorentz transformations consists of three boosts and three rotations. The rotations therefore constitute a subgroup of the Lorentz group. If a massive particle is at rest, its four-momentum is invariant under rotations. Thus the little group for a massive particle at rest is the three-dimensional rotation group. Then what is affected by the rotation?
The answer to this question is very simple. The particle in general has its spin. The spin orientation is going to be affected by the rotation!

If the rest-particle is boosted along the $z$ direction, it will pick up a non-zero momentum component. The generators of the $O(3)$ group will then be boosted. The boost will take the form of conjugation by the boost operator. This boost will not change the Lie algebra of the rotation group, and the boosted little group will still leave the boosted four-momentum invariant. We call this the $O(3)$-like little group. If we use the four-vector coordinate $(x, y, z, t)$, the four-momentum vector for the particle at rest is $(0, 0, 0, m)$, and the three-dimensional rotation group leaves this four-momentum invariant. This little group is generated by

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

(3.1)

and

$$J_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

(3.2)

which satisfy the commutation relations:

$$[J_i, J_j] = i\epsilon_{ijk}J_k.$$ 

(3.3)

It is not possible to bring a massless particle to its rest frame. In his 1939 paper [8], Wigner observed that the little group for a massless particle moving along the $z$ axis is generated by the rotation generator around the $z$ axis, namely $J_3$ of Eq.(3.2), and two other generators which take the form

$$N_1 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}.$$ 

(3.4)

If we use $K_i$ for the boost generator along the i-th axis, these matrices can be written as

$$N_1 = K_1 - J_2, \quad N_2 = K_2 + J_1,$$ 

(3.5)

with

$$K_1 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}.$$ 

(3.6)

The generators $J_3, N_1$ and $N_2$ satisfy the following set of commutation relations.
\[ [N_1, N_2] = 0, \quad [J_3, N_1] = iN_2, \quad [J_3, N_2] = -iN_1. \] (3.7)

In Appendix A, we discuss the generators of the \( E(2) \) group. They are \( J_3 \) which generates rotations around the \( z \) axis, and \( P_1 \) and \( P_2 \) which generate translations along the \( x \) and \( y \) directions respectively. If we replace \( N_1 \) and \( N_2 \) by \( P_1 \) and \( P_2 \), the above set of commutation relations becomes the set given for the \( E(2) \) group given in Eq. (A7). This is the reason why we say the little group for massless particles is \( E(2) \)-like. Very clearly, the matrices \( N_1 \) and \( N_2 \) generate Lorentz transformations.

It is not difficult to associate the rotation generator \( J_3 \) with the helicity degree of freedom of the massless particle. Then what physical variable is associated with the \( N_1 \) and \( N_2 \) generators? Wigner was the one who discovered the existence of these generators, but did not give any physical interpretation to these translation-like generators. For this reason, for many years, only those representations with the zero-eigenvalues of the \( N \) operators were thought to be physically meaningful representations [23]. It was not until 1971 when Janner and Janssen reported that the transformations generated by these operators are gauge transformations [24, 25]. The role of this translation-like transformation has also been studied for spin-1/2 particles, and it was concluded that the polarization of neutrinos is due to gauge invariance [26, 27].

Another important development along this line of research is the application of group contractions to the unifications of the two different little groups for massive and massless particles. We always associate the three-dimensional rotation group with a spherical surface. Let us consider a circular area of radius 1 kilometer centered on the north pole of the earth. Since the radius of the earth is more than 6,450 times longer, the circular region appears flat. Thus, within this region, we use the \( E(2) \) symmetry group for this region. The validity of this approximation depends on the ratio of the two radii.

In 1953, Inonu and Wigner formulated this problem as the contraction of \( O(3) \) to \( E(2) \) [17]. How about then the little groups which are isomorphic to \( O(3) \) and \( E(2) \)? It is reasonable to expect that the \( E(2) \)-like little group be obtained as a limiting case for of the \( O(3) \)-like little group for massless particles. In 1981, it was observed by Ferrara and Savoy that this limiting process is the Lorentz boost [28]. In 1983, using the same limiting process as that of Ferrara and Savoy, Han et al. showed that transverse rotation generators become the generators of gauge transformations in the limit of infinite momentum and/or zero mass [29]. In 1987, Kim and Wigner showed that the little group for massless particles is the cylindrical group which is isomorphic to the \( E(2) \) group [30]. This completes the second row in Table I, where Wigner’s little group unifies the internal space-time symmetries of massive and massless particles.

We are now interested in constructing the third row in Table I. As we promised in Sec. I, we will be dealing with hadrons which are bound states of quarks with space-time extensions. For this purpose, we need a set of covariant wave functions consistent with the existing laws of quantum mechanics, including of course the uncertainty principle and probability interpretation.

With these wave functions, we propose to solve the following problem in high-energy physics. The quark model works well when hadrons are at rest or move slowly. However, when they move with speed close to that of light, they appear as a collection of infinite-number of partons [7]. As we stated above, we need a set of wave functions which can be Lorentz-boosted. How can we then construct such a set? In constructing wave functions for
any purpose in quantum mechanics, the standard procedure is to try first harmonic oscillator wave functions. In studying the Lorentz boost, the standard language is the Lorentz group. Thus the first step to construct covariant wave functions is to work out representations of the Lorentz group using harmonic oscillators [9,13].

IV. COVARIANT HARMONIC OSCILLATORS

If we construct a representation of the Lorentz group using normalizable harmonic oscillator wave functions, the result is the covariant harmonic oscillator formalism [13]. The formalism constitutes a representation of Wigner’s O(3)-like little group for a massive particle with internal space-time structure. This oscillator formalism has been shown to be effective in explaining the basic phenomenological features of relativistic extended hadrons observed in high-energy laboratories. In particular, the formalism shows that the quark model and Feynman’s parton picture are two different manifestations of one covariant entity [13,21]. The essential feature of the covariant harmonic oscillator formalism is that Lorentz boosts are squeeze transformations [31,32]. In the light-cone coordinate system, the boost transformation expands one coordinate while contracting the other so as to preserve the product of these two coordinate remains constant. We shall show that the parton picture emerges from this squeeze effect.

Let us consider a bound state of two particles. For convenience, we shall call the bound state the hadron, and call its constituents quarks. Then there is a Bohr-like radius measuring the space-like separation between the quarks. There is also a time-like separation between the quarks, and this variable becomes mixed with the longitudinal spatial separation as the hadron moves with a relativistic speed. There are no quantum excitations along the time-like direction. On the other hand, there is the time-energy uncertainty relation which allows quantum transitions. It is possible to accommodate these aspect within the framework of the present form of quantum mechanics. The uncertainty relation between the time and energy variables is the c-number relation [14], which does not allow excitations along the time-like coordinate. We shall see that the covariant harmonic oscillator formalism accommodates this narrow window in the present form of quantum mechanics.

For a hadron consisting of two quarks, we can consider their space-time positions $x_a$ and $x_b$, and use the variables

$$X = (x_a + x_b)/2, \quad x = (x_a - x_b)/2\sqrt{2}.$$  \hspace{1cm} (4.1)

The four-vector $X$ specifies where the hadron is located in space and time, while the variable $x$ measures the space-time separation between the quarks. In the convention of Feynman et al. [3], the internal motion of the quarks bound by a harmonic oscillator potential of unit strength can be described by the Lorentz-invariant equation

$$\frac{1}{2} \left\{ x^2 - \frac{\partial^2}{\partial x^2} \right\} \psi(x) = \lambda \psi(x).$$  \hspace{1cm} (4.2)

It is now possible to construct a representation of the Poincaré group from the solutions of the above differential equation [13].
The coordinate \( X \) is associated with the overall hadronic four-momentum, and the space-time separation variable \( x \) dictates the internal space-time symmetry or the \( O(3) \)-like little group. Thus, we should construct the representation of the little group from the solutions of the differential equation in Eq. (4.2). If the hadron is at rest, we can separate the \( t \) variable from the equation. For this variable we can assign the ground-state wave function to accommodate the c-number time-energy uncertainty relation \([14]\). For the three space-like variables, we can solve the oscillator equation in the spherical coordinate system with usual orbital and radial excitations. This will indeed constitute a representation of the \( O(3) \)-like little group for each value of the mass. The solution should take the form

\[
\psi(x, y, z, t) = \psi(x, y, z) \left( \frac{1}{\pi} \right)^{1/4} \exp \left( -\frac{t^2}{2} \right),
\]

where \( \psi(x, y, z) \) is the wave function for the three-dimensional oscillator with appropriate angular momentum quantum numbers. The above wave function constitutes a representation of Wigner’s \( O(3) \)-like little group for a massive particle \([13]\).

Since the three-dimensional oscillator differential equation is separable in both spherical and Cartesian coordinate systems, \( \psi(x, y, z) \) consists of Hermite polynomials of \( x, y, \) and \( z \). If the Lorentz boost is made along the \( z \) direction, the \( x \) and \( y \) coordinates are not affected, and can be temporarily dropped from the wave function. The wave function of interest can be written as

\[
\psi^n(z, t) = \left( \frac{1}{\pi} \right)^{1/4} \exp \left( -\frac{t^2}{2} \right) \phi_n(z),
\]

where \( \phi_n(z) \) is the one-dimensional oscillator wave function for the \( n \)-th excited states, which takes the form

\[
\phi_n(z) = \left( \frac{1}{\pi n! 2^n} \right)^{1/2} H_n(z) \exp(-z^2/2).
\]

The full wave function \( \psi^n(z, t) \) is

\[
\psi^n_0(z, t) = \left( \frac{1}{\pi n! 2^n} \right)^{1/2} H_n(z) \exp \left\{ -\frac{1}{2} \left( z^2 + t^2 \right) \right\}.
\]

The subscript 0 means that the wave function is for the hadron at rest. The above expression is not Lorentz-invariant, and its localization undergoes a Lorentz squeeze as the hadron moves along the \( z \) direction \([13]\).

It is convenient to use the light-cone variables to describe Lorentz boosts. The light-cone coordinate variables are

\[
u = (z + t)/\sqrt{2}, \quad v = (z - t)/\sqrt{2}.
\]

In terms of these variables, the Lorentz boost along the \( z \) direction,

\[
\begin{pmatrix} z' \\ t' \end{pmatrix} = \begin{pmatrix} \cosh \eta & \sinh \eta \\ \sinh \eta & \cosh \eta \end{pmatrix} \begin{pmatrix} z \\ t \end{pmatrix},
\]

takes the simple form
\[ u' = e^\eta u, \quad v' = e^{-\eta}v, \]  
(4.9)

where \( \eta \) is the boost parameter and is \( \tanh^{-1}(v/c) \). The \( u \) variable becomes expanded while the \( v \) variable becomes contracted. This is the squeeze mechanism illustrated and discussed extensively in the literature [31,32]. This squeeze transformation is also illustrated in Fig. 1.

The wave function of Eq. (4.10) can be written as

\[ \psi_0^n(z, t) = \left( \frac{1}{\pi n! 2^n} \right)^{1/2} H_n \left( \frac{(u + v)/\sqrt{2}}{\sqrt{2}} \right) \exp \left\{ -\frac{1}{2} \left( u^2 + v^2 \right) \right\}. \]  
(4.10)

If the system is boosted, the wave function becomes

\[ \psi_\eta^n(z, t) = \left( \frac{1}{\pi n! 2^n} \right)^{1/2} H_n \left( \frac{(e^{-\eta}u + e^{\eta}v)/\sqrt{2}}{\sqrt{2}} \right) \times \exp \left\{ -\frac{1}{2} \left( e^{-2\eta}u^2 + e^{2\eta}v^2 \right) \right\}. \]  
(4.11)

In both Eqs. (4.10) and (4.11), the localization property of the wave function in the \( uv \) plane is determined by the Gaussian factor, and it is sufficient to study the ground state only for the essential feature of the boundary condition. The wave functions in Eq. (4.10) and Eq. (4.11) then respectively become

\[ \psi_0(z, t) = \left( \frac{1}{\pi} \right)^{1/2} \exp \left\{ -\frac{1}{2} \left( u^2 + v^2 \right) \right\}. \]  
(4.12)

If the system is boosted, the wave function becomes

\[ \psi_\eta(z, t) = \left( \frac{1}{\pi} \right)^{1/2} \exp \left\{ -\frac{1}{2} \left( e^{-2\eta}u^2 + e^{2\eta}v^2 \right) \right\}. \]  
(4.13)

We note here that the transition from Eq. (4.12) to Eq. (4.13) is a squeeze transformation. The wave function of Eq. (4.12) is distributed within a circular region in the \( uv \) plane, and thus in the \( zt \) plane. On the other hand, the wave function of Eq. (4.13) is distributed in an elliptic region. This ellipse is a “squeezed” circle with the same area as the circle, as is illustrated in Fig. 1.

---

\[ \beta = 0 \quad \text{and} \quad \beta = 0.8 \]

**FIG. 1.** Effect of the Lorentz boost on the space-time wave function. The circular space-time distribution at the rest frame becomes Lorentz-squeezed to become an elliptic distribution.
V. FEYNMAN'S PARTON PICTURE

It is safe to believe that hadrons are quantum bound states of quarks having localized probability distribution. As in all bound-state cases, this localization condition is responsible for the existence of discrete mass spectra. The most convincing evidence for this bound-state picture is the hadronic mass spectra which are observed in high-energy laboratories. However, this picture of bound states is applicable only to observers in the Lorentz frame in which the hadron is at rest. How would the hadrons appear to observers in other Lorentz frames? More specifically, can we use the picture of Lorentz-squeezed hadrons discussed in Sec. IV.

Proton’s radius is $10^{-5}$ of that of the hydrogen atom. Therefore, it is not unnatural to assume that the proton has a point charge in atomic physics. However, while carrying out experiments on electron scattering from proton targets, Hofstadter in 1955 observed that the proton charge is spread out. In this experiment, an electron emits a virtual photon, which then interacts with the proton. If the proton consists of quarks distributed within a finite space-time region, the virtual photon will interact with quarks which carry fractional charges. The scattering amplitude will depend on the way in which quarks are distributed within the proton. The portion of the scattering amplitude which describes the interaction between the virtual photon and the proton is called the form factor.

Although there have been many attempts to explain this phenomenon within the framework of quantum field theory, it is quite natural to expect that the wave function in the quark model will describe the charge distribution. In high-energy experiments, we are dealing with the situation in which the momentum transfer in the scattering process is large. Indeed, the Lorentz-squeezed wave functions lead to the correct behavior of the hadronic form factor for large values of the momentum transfer.

While the form factor is the quantity which can be extracted from the elastic scattering, it is important to realize that in high-energy processes, many particles are produced in the final state. They are called inelastic processes. While the elastic process is described by the total energy and momentum transfer in the center-of-mass coordinate system, there is, in addition, the energy transfer in inelastic scattering. Therefore, we would expect that the scattering cross section would depend on the energy, momentum transfer, and energy transfer. However, one prominent feature in inelastic scattering is that the cross section remains nearly constant for a fixed value of the momentum-transfer/energy-transfer ratio. This phenomenon is called “scaling”.

In order to explain the scaling behavior in inelastic scattering, Feynman in 1969 observed that a fast-moving hadron can be regarded as a collection of many “partons” whose properties do not appear to be identical to those of quarks. For example, the number of quarks inside a static proton is three, while the number of partons in a rapidly moving proton appears to be infinite. The question then is how the proton looking like a bound state of quarks to one observer can appear different to an observer in a different Lorentz frame? Feynman made the following systematic observations.

a). The picture is valid only for hadrons moving with velocity close to that of light.

b). The interaction time between the quarks becomes dilated, and partons behave as free independent particles.

c). The momentum distribution of partons becomes widespread as the hadron
FIG. 2. Lorentz-squeezed space-time and momentum-energy wave functions. As the hadron’s speed approaches that of light, both wave functions become concentrated along their respective positive light-cone axes. These light-cone concentrations lead to Feynman’s parton picture.

The number of partons seems to be infinite or much larger than that of quarks. Because the hadron is believed to be a bound state of two or three quarks, each of the above phenomena appears as a paradox, particularly b) and c) together. We would like to resolve this paradox using the covariant harmonic oscillator formalism.

For this purpose, we need a momentum-energy wave function. If the quarks have the four-momenta $p_a$ and $p_b$, we can construct two independent four-momentum variables

$$P = p_a + p_b, \quad q = \sqrt{2}(p_a - p_b). \quad (5.1)$$

The four-momentum $P$ is the total four-momentum and is thus the hadronic four-momentum. $q$ measures the four-momentum separation between the quarks.

We expect to get the momentum-energy wave function by taking the Fourier transformation of Eq.(4.13):
Let us now define the momentum-energy variables in the light-cone coordinate system as

$$q_u = (q_0 - q_z)/\sqrt{2}, \quad q_v = (q_0 + q_z)/\sqrt{2}. \quad (5.3)$$

In terms of these variables, the Fourier transformation of Eq.(5.2) can be written as

$$\chi_{\eta}(q_z, q_0) = \left(\frac{1}{2\pi}\right) \int \psi_{\eta}(z, t) \exp \{-i(q_z z - q_0 t)\} dx dt. \quad (5.4)$$

The resulting momentum-energy wave function is

$$\chi_{\eta}(q_z, q_0) = \left(\frac{1}{\pi}\right)^{1/2} \exp \left\{-\frac{1}{2} \left(e^{-2\eta q_u^2} + e^{2\eta q_v^2}\right)\right\}. \quad (5.5)$$

Because we are using here the harmonic oscillator, the mathematical form of the above momentum-energy wave function is identical to that of the space-time wave function. The Lorentz squeeze properties of these wave functions are also the same, as are indicated in Fig. 2.

When the hadron is at rest with $\eta = 0$, both wave functions behave like those for the static bound state of quarks. As $\eta$ increases, the wave functions become continuously squeezed until they become concentrated along their respective positive light-cone axes. Let us look at the z-axis projection of the space-time wave function. The width of the quark distribution increases as the hadronic speed approaches that of the speed of light. The position of each quark appears widespread to the observer in the laboratory frame, and the quarks appear like free particles.

Furthermore, interaction time of the quarks among themselves become dilated. Because the wave function becomes wide-spread, the distance between one end of the harmonic oscillator well and the other end increases as is indicated in Fig. 2. This effect, first noted by Feynman [7], is universally observed in high-energy hadronic experiments. The period is oscillation is increases like $e^\eta$. On the other hand, the interaction time with the external signal, since it is moving in the direction opposite to the direction of the hadron, it travels along the negative light-cone axis. If the hadron contracts along the negative light-cone axis, the interaction time decreases by $e^{-\eta}$. The ratio of the interaction time to the oscillator period becomes $e^{-2\eta}$. The energy of each proton coming out of the Fermilab accelerator is $900 GeV$. This leads the ratio to $10^{-6}$. This is indeed a small number. The external signal is not able to sense the interaction of the quarks among themselves inside the hadron.

The longitudinal momentum distribution becomes wide-spread as the hadron moves, and if we insist on Heisenberg’s uncertainty relation, is Planck’s constant dependent on the hadronic velocity?
b. Is this apparent contradiction related to another apparent contradiction that the number of partons is infinite while there are only two or three quarks inside the hadron?

The answer to the first question is No, and that for the second question is Yes. Let us answer the first question which is related to the Lorentz invariance of Planck’s constant. If we take the product of the width of the longitudinal momentum distribution and that of the spatial distribution, we end up with the relation

\[<z^2><q_z^2> = \frac{1}{4}[\cosh(2\eta)]^2.\] (5.6)

The right-hand side increases as the velocity parameter increases. This could lead us to an erroneous conclusion that Planck’s constant becomes dependent on velocity. This is not correct, because the longitudinal momentum variable \(q_z\) is no longer conjugate to the longitudinal position variable when the hadron moves.

In order to maintain the Lorentz-invariance of the uncertainty product, we have to work with a conjugate pair of variables whose product does not depend on the velocity parameter. Let us go back to Eq. (5.3) and Eq. (5.4). It is quite clear that the light-cone variable \(u\) and \(v\) are conjugate to \(q_u\) and \(q_v\) respectively. It is also clear that the distribution along the \(q_u\) axis shrinks as the \(u\)-axis distribution expands. The exact calculation leads to

\[<u^2><q_u^2> = \frac{1}{4}, \quad <v^2><q_v^2> = \frac{1}{4}.\] (5.7)

Planck’s constant is indeed Lorentz-invariant.

Let us next resolve the puzzle of why the number of partons appears to be infinite while there are only a finite number of quarks inside the hadron. As the hadronic speed approaches the speed of light, both the \(x\) and \(q\) distributions become concentrated along the positive light-cone axis. This means that the quarks also move with velocity very close to that of light. Quarks in this case behave like massless particles.

We then know from statistical mechanics that the number of massless particles is not a conserved quantity. For instance, in black-body radiation, free light-like particles have a widespread momentum distribution. However, this does not contradict the known principles of quantum mechanics, because the massless photons can be divided into infinitely many massless particles with a continuous momentum distribution.

Likewise, in the parton picture, massless free quarks have a wide-spread momentum distribution. They can appear as a distribution of an infinite number of free particles. These free massless particles are the partons. It is possible to measure this distribution in high-energy laboratories, and it is also possible to calculate it using the covariant harmonic oscillator formalism. We are thus forced to compare these two results. According to Hussar’s calculation [34], the Lorentz-boosted oscillator wave function produces a reasonably accurate parton distribution.

VI. FEYNMAN’S REST OF THE UNIVERSE

We have seen in Sec. V that the time-separation variable plays the pivotal role in the light-cone formulation and Feynman’s parton picture. Then, do measure this variable or distribution along this variable? The answer to this question is clearly No. Does the present
form of quantum mechanics accommodate this non-measurable variable? The answer is Yes, and is an increase in entropy.

The entropy is a measure of our ignorance and is computed from the density matrix. The density matrix is needed when the experimental procedure does not analyze all relevant variables to the maximum extent consistent with quantum mechanics. The purpose of the present note is to discuss a concrete example of the entropy arising from our ignorance in relativistic quantum mechanics.

As was discussed in the literature for several different purposes, this wave function can be expanded as:

$$\psi_\eta(z, t) = \frac{1}{\cosh \eta} \sum_k (\tanh \eta)^k \phi_k(z) \phi_k^*(t);$$  (6.1)

where $\phi_k(z)$ is the $k$-th excited oscillator wave function given in Eq.(4.5). From this expression, we can construct the pure-state density matrix

$$\rho_\eta(z, t; z', t') = \psi_\eta(z, t) \psi_\eta^*(z', t'),$$  (6.2)

which satisfies the condition $\rho^2 = \rho$:

$$\rho_\eta^n(z, t; z', t') = \int \rho_\eta^n(z, t; z'', t'') \rho_\eta^n(z'', t''; z', t') dz'' dt''.$$  (6.3)

However, there are at present no measurement theories which accommodate the time-separation variable $t$. Thus, we can take the trace of the $\rho$ matrix with respect to the $t$ variable. Then the resulting density matrix is

$$\rho_\eta(z, z') = \int \psi_\eta(z, t) \psi_\eta^*(z', t) dt = \frac{1}{\cosh \eta} \sum_k (\tanh \eta)^{2k} \phi_k(z) \phi_k^*(z').$$  (6.4)

The trace of this density matrix is one, but the trace of $\rho^2$ is less than one, as

$$Tr \left( \rho^2 \right) = \int \rho_\eta(z, z') \rho_\eta(z', z) dz' dz = \frac{1}{\cosh \eta} \sum_k (\tanh \eta)^{4k},$$  (6.5)

which is less than one. This is due to the fact that we do not know how to deal with the time-like separation in the present formulation of quantum mechanics. Our knowledge is less than complete.

The standard way to measure this ignorance is to calculate the entropy defined as

$$S = Tr \left( \rho \ln(\rho) \right).$$  (6.6)

If we pretend to know the distribution along the time-like direction and use the pure-state density matrix given in Eq.(6.2), then the entropy is zero. However, if we do not know how to deal with the distribution along $t$, then we should use the density matrix of Eq.(6.4) to calculate the entropy, and the result is

$$S = 2 \left\{ (\cosh \eta)^2 \ln(\cosh \eta) - (\sinh \eta)^2 \ln(\sinh \eta) \right\}.$$  (6.7)

Let us go back to the wave function given in Eq.(4.6). From the wave function, we can derive the density matrix by performing the integral of Eq.(6.4). The result is
\[
\rho(z, z') = \left(\frac{1}{\pi \cosh 2\eta}\right)^{1/2} \exp\left\{-\frac{1}{4}[(z + z')^2 / \cosh 2\eta + (z - z')^2 \cosh 2\eta]\right\}.
\]

The diagonal elements of the above density matrix is
\[
\rho(z, z') = \left(\frac{1}{\pi \cosh 2\eta}\right)^{1/2} \exp\left(\frac{-z^2}{\cosh 2\eta}\right).
\]

The width of the distribution becomes \((\cosh \eta)^{1/2}\), and becomes wide-spread as the hadronic speed increases. Likewise, the momentum distribution becomes wide-spread. This simultaneous increase in the momentum and position distribution widths is called the parton phenomenon in high-energy physics. The position-momentum uncertainty becomes \(\cosh \eta\). This increase in uncertainty is due to our ignorance about the physical but unmeasurable time-separation variable.

The use of an unmeasurable variable as a “shadow” coordinate is not new in physics and is of current interest. Feynman’s book on statistical mechanics contains the following paragraph.

*When we solve a quantum-mechanical problem, what we really do is divide the universe into two parts - the system in which we are interested and the rest of the universe. We then usually act as if the system in which we are interested comprised the entire universe. To motivate the use of density matrices, let us see what happens when we include the part of the universe outside the system.*

In this section, we have identified Feynman’s rest of the universe as the time-separation coordinate in a relativistic two-body problem. Our ignorance about this coordinate leads to a density matrix for a non-pure state, and consequently to an increase of entropy.

## CONCLUDING REMARKS

The phenomenological aspects of the covariant oscillator formalism have been extensively discussed in the literature. In this paper, we used this formalism to illustrate the truth that the Lorentz covariance is quite different from the Lorentz invariance. The covariance we discussed in this paper is powerful enough to resolve the question of whether quarks are partons.

The oscillator formalism constitutes a representation of Wigner’s little group governing the internal space-time symmetries of relativistic particles. For this purpose, we have given a comprehensive review of the little groups for massive and massless particles. We have discussed also the contraction procedure in which the \(E(2)\)-like little group for massless particles is obtained from the \(O(3)\)-like little group for massive particles. In so doing, we have explained the contents of Table I.

In addition, we discussed the fundamental issue of the time-separation variable. It does not appear to be a significant variable in non-relativistic quantum mechanics. However, it plays the central role in covariant quantum mechanics. If we do not make observations on this variable, it constitutes Feynman’s rest of the universe. The failure to make measurements in this time-like direction results in an increase in entropy.

The harmonic oscillator is the natural language for physicists. If combined with the Lorentz group, it can provide a powerful theoretical framework in many different branches.
One noteworthy development has been that the oscillator representation of the Lorentz group forms the basis for the theory of coherent and squeezed states of light [32]. It has also been shown that the six-parameter Lorentz group constitutes the basic language for polarization optics [40]. It has been shown that the Jones vector and the Stokes parameters can be regarded as the two-component $SL(2, c)$ spinor and the Minkowskian four-vector respectively.

As is seen in Fig. 1, the Lorentz boost is a squeeze transformation. This geometric property is not yet widely known among physicists, but may play many important roles in the future. For instance, most of the soluble models in physics is based on diagonalization of coupled oscillators, including the Lee model in quantum field theory, the Bogoliubov transformation in superconductivity, and the covariant oscillator model discussed in this paper. It is well known that this diagonalization requires coordinate rotations, but it also requires squeeze transformations. In order to understand fully the system of two coupled oscillators, we need the group $O(3, 3)$ which is the Lorentz group applicable to a six-dimensional space consisting of three space-like and three time-like directions [41]. Indeed, coupled harmonic oscillators and the Lorentz group cannot be separated from each other.

APPENDIX A: CONTRACTION OF O(3) TO E(2)

In this Appendix, we explain what the $E(2)$ group is. We then explain how we can obtain this group from the three-dimensional rotation group by making a flat-surface or cylindrical approximation. This contraction procedure will give a clue to obtaining the $E(2)$-like symmetry for massless particles from the $O(3)$-like symmetry for massive particles by making the infinite-momentum limit.

The $E(2)$ transformations consist of rotation and two translations on a flat plane. Let us start with the rotation matrix applicable to the column vector $(x, y, 1)$:

$$ R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (A1) $$

Let us then consider the translation matrix:

$$ T(a, b) = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}. \quad (A2) $$

If we take the product $T(a, b)R(\theta)$,

$$ E(a, b, \theta) = T(a, b)R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta & a \\ \sin \theta & \cos \theta & b \\ 0 & 0 & 1 \end{pmatrix}. \quad (A3) $$

This is the Euclidean transformation matrix applicable to the two-dimensional $xy$ plane. The matrices $R(\theta)$ and $T(a, b)$ represent the rotation and translation subgroups respectively. The above expression is not a direct product because $R(\theta)$ does not commute with $T(a, b)$. The translations constitute an Abelian invariant subgroup because two different $T$ matrices commute with each other, and because
\[ R(\theta)T(a,b)R^{-1}(\theta) = T(a',b'). \]

The rotation subgroup is not invariant because the conjugation
\[ T(a,b)R(\theta)T^{-1}(a,b) \]
does not lead to another rotation.

We can write the above transformation matrix in terms of generators. The rotation is generated by
\[ J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]

The translations are generated by
\[ P_1 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix}. \]

These generators satisfy the commutation relations:
\[ [P_1, P_2] = 0, \quad [J_3, P_1] = iP_2, \quad [J_3, P_2] = -iP_1. \]

This \( E(2) \) group is not only convenient for illustrating the groups containing an Abelian invariant subgroup, but also occupies an important place in constructing representations for the little group for massless particles, since the little group for massless particles is locally isomorphic to the above \( E(2) \) group.

The contraction of \( O(3) \) to \( E(2) \) is well known and is often called the Inomu-Wigner contraction [17]. The question is whether the \( E(2) \)-like little group can be obtained from the \( O(3) \)-like little group. In order to answer this question, let us closely look at the original form of the Inomu-Wigner contraction. We start with the generators of \( O(3) \). The \( J_3 \) matrix is given in Eq.(3.2), and
\[ J_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]

The Euclidean group \( E(2) \) is generated by \( J_3, P_1 \) and \( P_2 \), and their Lie algebra has been discussed in Sec. [4].

Let us transpose the Lie algebra of the \( E(2) \) group. Then \( P_1 \) and \( P_2 \) become \( Q_1 \) and \( Q_2 \) respectively, where
\[ Q_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & i & 0 \end{pmatrix}. \]

Together with \( J_3 \), these generators satisfy the same set of commutation relations as that for \( J_3, P_1 \), and \( P_2 \) given in Eq.(A7):
FIG. 3. Contraction of O(3) to E(2) and to the cylindrical group, and contraction of the O(3)-like little group to the E(2)-like little group. The correspondence between E(2) and the E(2)-like little group is isomorphic but not identical. The cylindrical group is identical to the E(2)-like little group. The Lorentz boost of the O(3)-like little group for a massive particle is the same as the contraction of O(3) to the cylindrical group.

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & R
\end{pmatrix}
\]  

These matrices generate transformations of a point on a circular cylinder. Rotations around the cylindrical axis are generated by \( J_3 \). The matrices \( Q_1 \) and \( Q_2 \) generate translations along the direction of \( z \) axis. The group generated by these three matrices is called the cylindrical group [30,42].

We can achieve the contractions to the Euclidean and cylindrical groups by taking the large-radius limits of

\[
P_1 = \frac{1}{R} B^{-1} J_2 B, \quad P_2 = -\frac{1}{R} B^{-1} J_1 B, \tag{A11}
\]

and

\[
Q_1 = -\frac{1}{R} B J_2 B^{-1}, \quad Q_2 = \frac{1}{R} B J_1 B^{-1}, \tag{A12}
\]

where

\[
B(R) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & R
\end{pmatrix}.
\]  

21
The vector spaces to which the above generators are applicable are \((x, y, z/R)\) and \((x, y, Rz)\) for the Euclidean and cylindrical groups respectively. They can be regarded as the north-pole and equatorial-belt approximations of the spherical surface respectively [30]. These two different contraction procedures are illustrated in Fig. 3.

APPENDIX B: CONTRACTION OF O(3)-LIKE LITTLE GROUP TO E(2)-LIKE LITTLE GROUP

In this appendix, we shall discuss the contraction of \(O(3)\)-like little group to the \(E(2)\)-like little group as the infinite-momentum or zero-mass limit of the Lorentz-boosted \(O(3)\)-like little group. The relation of this procedure to the contraction of \(O(3)\) is illustrated in Fig. 3.

Let us go back to Eq.(A11) and Eq.(A12). Since \(P_1 P_2\) commutes with \(Q_2\), we can consider the following combination of generators.

\[
F_1 = P_1 + Q_1, \quad F_2 = P_2 + Q_2.
\]  

Then these operators also satisfy the commutation relations:

\[
[F_1, F_2] = 0, \quad [J_3, F_1] = iF_2, \quad [J_3, F_2] = -iF_1.
\]

However, we cannot make this addition using the three-by-three matrices for \(P_i\) and \(Q_i\) to construct three-by-three matrices for \(F_1\) and \(F_2\), because the vector spaces are different for the \(P_i\) and \(Q_i\) representations. We can accommodate this difference by creating two different \(z\) coordinates, one with a contracted \(z\) and the other with an expanded \(z\), namely \((x, y, Rz, z/R)\). Then the generators become

\[
P_1 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

\[
Q_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

Then \(F_1\) and \(F_2\) will take the form

\[
F_1 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

The rotation generator \(J_3\) takes the form of Eq.(3.2). These four-by-four matrices satisfy the \(E(2)\)-like commutation relations of Eq.(B2).

Now the \(B\) matrix of Eq.(A13), can be expanded to
\[
B(R) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & R & 0 \\
0 & 0 & 0 & 1/R
\end{pmatrix}.
\] (B6)

If we make a similarity transformation on the above form using the matrix
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \\
0 & 0 & 1/\sqrt{2} & 1/\sqrt{2}
\end{pmatrix},
\] (B7)

which performs a 45-degree rotation of the third and fourth coordinates, then this matrix becomes
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \cosh \eta & \sinh \eta \\
0 & 0 & \sinh \eta & \cosh \eta
\end{pmatrix},
\] (B8)

with \( R = e^{i\eta} \). This form is the Lorentz boost matrix along the \( z \) direction. If we start with the set of expanded rotation generators \( J_3 \) of Eq. (3.2), and perform the same operation as the original Inomu-Wigner contraction given in Eq. (A11), the result is
\[
N_1 = \frac{1}{R} B^{-1} J_2 B, \quad N_2 = -\frac{1}{R} B^{-1} J_1 B,
\] (B9)

where \( N_1 \) and \( N_2 \) are given in Eq. (3.4). The generators \( N_1 \) and \( N_2 \) are the contracted \( J_2 \) and \( J_1 \) respectively in the infinite-momentum/zero-mass limit.

This contraction procedure constitutes the second row of Table I which contains the expanded content of Einstein’s \( E = mc^2 \). This row states that the rotational degree of freedom for a massive particle remains unchanged, and it becomes the helicity degree of freedom for the massless particle. The rotational degrees of freedom along the two transverse direction collapse into one gauge degree of freedom.
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[18] Humboldt University in Berlin has a brilliant history. In front of the the main building, there is a statue of Hermann von Helmholtz. In the lobby of the main hall, there is a marble plate engraved with a quotation “Die Philosophen haben die Welt nur verschieden interpretiert; es kommt aber darauf an, wie sie verändern.” – Karl Marx.” In English, this quotation could say “Philosophers interpret this world in various ways. There comes the question of changing the world.” It is my recollection that Wigner never liked Marx, and he did not quote from Marx when he told me about philosophers. Yet, what Wigner said coincides with the first half of what Marx said in the marble plate at Humboldt University.

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