New results for a two-loop massless propagator-type Feynman diagram

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Abstract

We consider the two-loop massless propagator-type Feynman diagram with an arbitrary (non-integer) index on the central line. We analytically prove the equality of the two well-known results existing in the literature which express this diagram in terms of $3\,F_2$-hypergeometric functions of argument $-1$ and 1, respectively. We also derive new representations for this diagram which may be of importance in practical calculations.
I. INTRODUCTION

The exact computation of the two-loop massless propagator-type Feynman diagram has been the subject of extensive studies over the last decades, see Ref. [1] for a historical review. The diagram is defined as:

\[ J(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = \int \int \frac{[d^Dk_1]}{k_1^{2\alpha_1} k_2^{2\alpha_2} (k_2 - p)^{2\alpha_3} (k_1 - p)^{2\alpha_4} (k_2 - k_1)^{2\alpha_5}}, \] (1)

where \([d^Dk] = d^Dk/(2\pi)^D\), the \(\alpha_i\) (the five powers of the propagators) are the so-called indices of the diagram and \(p\) is the external momentum in an Euclidean space-time of dimensionality \(D\), see Fig. 1a for a graphical representation. When all five indices are integers, the diagram is easily computed, \(e.g.,\), with the help of the integration by parts (IBP) procedure [2,3]. On the other hand, for arbitrary (non-integer) indices an exact evaluation becomes highly non-trivial and peculiar cases have to be considered, see, \(e.g.,\), Refs. [2–18]. Among these peculiar cases, the simplest non-trivial diagram is the one with an arbitrary index on the central line:

\[ J(1, 1, 1, 1, \alpha) = \int \int \frac{[d^Dk_1]}{k_1^2 k_2^2 (k_2 - p)^2 (k_1 - p)^2 (k_2 - k_1)^{2\alpha}} \equiv J(\alpha), \] (2)

see Fig. 1b. The diagram (2) has been extensively studied in the past, see the review [1], and its exact computation has led to a number of important applications over the years. As a first well known example, let us mention the analytical evaluation of the 5-loop \(\beta\)-function of the \(\Phi^4\) model, see Refs. [6,7] and discussions and references therein. Secondly, the most complicated contributions to the values of critical exponents computed within a \(1/N\)-expansion, see Refs. [2,11] as well as Vasil’ev’s textbook [21], depend, at order \(1/N^2\), on the derivative with respect to the index of the central line of \(J(\alpha)\); the fully analytic computation of the derivative has been obtained in Ref. [13] based on the results of Ref. [8]. More recently, it was realized in Ref. [16] that general multiloop techniques such as those developed in Refs. [2,3] as well as the sophisticated developments brought by Refs. [6,8] were of crucial importance for the exact computation of interaction correction effects in brane world-like effective field theories describing some modern planar condensed matter physics systems, see also Refs. [17,18,22] for developments in this direction as well as Ref. [23] and references therein for a short review on some of these aspects.
A. Existing results for $J(\alpha)$

In what follows we shall use dimensional regularization and take $D = 4 - 2\varepsilon$. The dependence of $J(\alpha)$ on momentum is trivial as it follows from simple dimensional analysis. We may then write Eq. (2) as

$$J(\alpha) = \frac{p^{2d_F}}{(4\pi)^D} I(\alpha),$$

where

$$d_F = D - 4 - \alpha = -\alpha - 2\varepsilon,$$

is the dimension of the diagram and $I(\alpha)$ its so-called (dimensionless) coefficient function. It is the latter that is of interest as it is in general non-trivial to compute. For most diagrams, the coefficient function is known only in the form of the first few terms of the Laurent series in $\varepsilon$. In the case of $I(\alpha)$ exact results are available and we shall focus on them.

There are actually two different results which can be found in the literature for $I(\alpha)$. The first one, Ref. [7], has been derived by solving functional equations obtained with the help of a combination of the IBP procedure [2, 3] together with several other transformations [2, 9, 24]. The result contains a one-fold series and reads:

$$I(1 + \alpha) = -2 \frac{\Gamma^2(1 - \varepsilon)\Gamma(-\varepsilon - \alpha)\Gamma(\alpha + 2\varepsilon)}{\Gamma(2 - 2\varepsilon)} \left[ \frac{1}{\Gamma(1 + \alpha)\Gamma(1 - 3\varepsilon - \alpha)} \right] \times \sum_{n=1}^{\infty} (-1)^n \frac{\Gamma(n + 1 - \varepsilon)}{\Gamma(n + \varepsilon)} \left( \frac{1}{n + \alpha + \varepsilon} + \frac{1}{n - \alpha - 2\varepsilon} \right) + \cos[\pi\varepsilon].$$

Notice that the above one-fold series can be represented as a combination of two $_3F_2$-hypergeometric functions of argument $-1$.

The second result, Ref. [8], is based on an application of the Gegenbauer polynomial technique that in-turn is a generalization of earlier studies [5]. It expresses the function
\( I(1 + \alpha) \) in terms of a single \( _3F_2 \) function of argument 1 with the result reading:

\[
I(1 + \alpha) = \frac{-2 \Gamma(1 - \varepsilon) \Gamma(-\varepsilon - \alpha) \Gamma(\alpha + 2\varepsilon)}{\Gamma(2 - 2\varepsilon)} \times \left[ \frac{\Gamma(1 - \varepsilon)}{\Gamma(1 - \alpha - 3\varepsilon)} \sum_{n=0}^{\infty} \frac{\Gamma(n + 2 - 2\varepsilon)}{\Gamma(n + 2 + \alpha)} \frac{1}{n + 1 + \alpha + \varepsilon} - \pi \cot[\pi(\alpha + 2\varepsilon)] \right].
\] (6)

As it was discussed already in Ref. [8], the equality of the two results (5) and (6) provides the following relation between two \( _3F_2 \)-hypergeometric functions of argument \(-1\) and a single \( _3F_2 \)-hypergeometric function of argument 1:

\[
_3F_2(2A, B, 1; B + 1, 2 - A; -1) + \frac{B}{1 + A - B} \ 3F_2(2A, 1 + A - B, 1; 2 + A - B, 2 - A; -1) = B \cdot \frac{\Gamma(2 - A) \Gamma(B + A - 1) \Gamma(B - A) \Gamma(1 + A - B)}{\Gamma(2A) \Gamma(1 + B - 2A)} - \frac{1 - A}{B + A - 1} \ 3F_2(2A, B, 1; B + 1, A + B; 1),
\] (7)

where, hereafter \( A, B \) and \( C \) are arbitrary. Notice that such a relation does not appear in standard textbooks.

The purpose of the present short paper is to demonstrate analytically the equality of the two above results (5) and (6) and, hence, to recover analytically the relation (7) between \( _3F_2 \)-hypergeometric functions. The demonstration will be given in details as the method of calculation involved, or some generalization of it, may be useful to other physicists and mathematicians. Actually, the use of integral representations as done below is rather similar to manipulations of \( \Psi \)-functions and associated functions carried out earlier [19,20] in order to simplify the famous Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation at the first two orders of perturbation theory.

As a by product of our analysis, we obtain several other representations for the considered diagram \( I(\alpha) \) which, in-turn, should be useful for its high-order \( \varepsilon \)-expansion in the case where \( \alpha = 1 + a\varepsilon \). Such expansions are planned for our future investigations.

**B. Particular cases**

Before closing this section, it is convenient to consider two particular cases: \( \alpha = 0 \) and \( \alpha = -\varepsilon \), where \( I(1 + \alpha) \) can be represented in terms of a combination of \( \Gamma \)-functions.
In the case \(\alpha = 0\), an application of the IBP procedure yields:

\[
I(1) = \frac{1}{\varepsilon^3} \frac{\Gamma^2(1-2\varepsilon)}{(1-2\varepsilon)\Gamma^2(1-2\varepsilon)} \left[ 1 - \frac{\Gamma^2(1-2\varepsilon)\Gamma(1+2\varepsilon)}{\Gamma^2(1+\varepsilon)\Gamma(1-\varepsilon)\Gamma(1-3\varepsilon)} \right] = 1 \varepsilon^3 \frac{\Gamma(1)\Gamma(1-2\varepsilon)}{(1-2\varepsilon)\Gamma(1-2\varepsilon)} \left[ 1 - \frac{\Gamma(1-2\varepsilon)}{\Gamma(1+\varepsilon)\Gamma(1-3\varepsilon)} \frac{1}{\cos[\pi\varepsilon]} \right].
\]

In the case \(\alpha = -\varepsilon\), the diagram has been calculated by the method of uniqueness (see Ref. [2] and also discussions in Refs. [11,21]), and reads:

\[
I(1-\varepsilon) = \frac{3\Gamma(1-\varepsilon)\Gamma(\varepsilon)}{(2-2\varepsilon)\Gamma(2-2\varepsilon)} \left[ \psi'(1-\varepsilon) - \psi'(1) \right].
\]

A modern evaluation of the result can be found in Ref. [17]. Notice that, with respect to practical applications, the result (9) has been used in Ref. [17] in order to recover some of the results of Ref. [16] where the more general Eq. (6) was used in order to compute the most complicated part of the two-loop correction to electromagnetic current correlations in brane worlds / planar condensed matter physics systems.

II. TRANSFORMATION OF EQ. (5)

In this section, we shall demonstrate analytically how the result of Eq. (6) can be obtained from the result of Eq. (5).

As a starting point, let us note that Kazakov’s formula (5) was initially written under the following form:

\[
I(1+\alpha) = -2 \frac{\Gamma^2(1-\varepsilon)\Gamma(\varepsilon)(-\varepsilon - \alpha)\Gamma(\alpha + 2\varepsilon)}{\Gamma(2-2\varepsilon)\Gamma(1+\alpha)\Gamma(1-3\varepsilon - \alpha)} \times \left[ \sum_{n=1}^{\infty} (-1)^n \frac{\Gamma(n+1-2\varepsilon)}{\Gamma(n+\varepsilon)} \left( \frac{1}{n+\alpha+\varepsilon} + \frac{1}{n-\alpha-2\varepsilon} \right) \right. \\
- \cos[\pi\varepsilon] \times \sum_{n=1}^{\infty} (-1)^n \frac{\Gamma(n+1-3\varepsilon)}{\Gamma(n)} \left( \frac{1}{n+\alpha} + \frac{1}{n-\alpha-3\varepsilon} \right) \left. \right],
\]

and that the last sum in the rhs is equal to: \(-\Gamma(1+\alpha)\Gamma(1-3\varepsilon - \alpha)\).

A. Transformation of the last term in Eq. (10)

It is instructive to first evaluate the last term in Eq. (10). In order to do so, we can represent the factor \((1/(n+\alpha) + 1/(n-\alpha - 3\varepsilon))\) in the following integral form:

\[
\frac{1}{n+\alpha} + \frac{1}{n-\alpha-3\varepsilon} = \int_{0}^{1} dx \frac{x^{n-1}}{x^{\alpha} + x^{\alpha-3\varepsilon}}.
\]
Then, the last sum in the rhs of Eq. (10) can be summed as:

\[
\sum_{n=1}^{\infty} (-1)^n \frac{\Gamma(n + 1 - 3\varepsilon)}{\Gamma(n)} x^{n-1} = -\frac{\Gamma(2 - 3\varepsilon)}{(1 + x)^{2-3\varepsilon}},
\]  

(12)

which yields:

\[
\sum_{n=1}^{\infty} (-1)^n \frac{\Gamma(n + 1 - 3\varepsilon)}{\Gamma(n)} \left( \frac{1}{n + \alpha} + \frac{1}{n - \alpha - 3\varepsilon} \right) = -\Gamma(2 - 3\varepsilon) \int_0^1 dx \frac{x^{\alpha + x^{-\alpha - 3\varepsilon}}}{(1 + x)^{2-3\varepsilon}}. 
\]  

(13)

After the replacement \( x \to y = 1/x \), the last term in Eq. (13) reads:

\[
\int_0^1 dx \frac{x^{-\alpha - 3\varepsilon}}{(1 + x)^{2-3\varepsilon}} = \int_1^\infty dy \frac{y^\alpha}{(1 + y)^{2-3\varepsilon}}. 
\]  

(14)

The result (13) can then be expressed as:

\[
\sum_{n=1}^{\infty} (-1)^n \frac{\Gamma(n + 1 - 3\varepsilon)}{\Gamma(n)} \left( \frac{1}{n + \alpha} + \frac{1}{n - \alpha - 3\varepsilon} \right) = -\Gamma(2 - 3\varepsilon) \int_0^\infty dx \frac{x^{\alpha}}{(1 + x)^{2-3\varepsilon}}, 
\]  

(15)

where the last integral is evaluated in terms of \( \Gamma \)-functions because:

\[
\int_0^\infty dx \frac{x^{\alpha}}{(1 + x)^\beta} = \frac{\Gamma(\alpha + 1)\Gamma(\beta - \alpha - 1)}{\Gamma(\beta)}. 
\]  

(16)

We therefore come to the final result having the following advertised form:

\[
\sum_{n=1}^{\infty} (-1)^n \frac{\Gamma(n + 1 - 3\varepsilon)}{\Gamma(n)} \left( \frac{1}{n + \alpha} + \frac{1}{n - \alpha - 3\varepsilon} \right) = -\Gamma(1 + \alpha)\Gamma(1 - 3\varepsilon - \alpha). 
\]  

(17)

**B. Transformation of the first sum in the rhs of Eq. (10) to an \(_3F_2\) function of argument \( 1 \)**

Following the analysis of the previous subsection, it is convenient to represent the factor 

\( (1/(n + \alpha + \varepsilon) + 1/(n - \alpha - 2\varepsilon)) \) in the following form:

\[
\frac{1}{n + \alpha + \varepsilon} + \frac{1}{n - \alpha - 2\varepsilon} = \int_0^1 dx \, x^{n-1} \left[ x^{\alpha + \varepsilon} + x^{-\alpha - 2\varepsilon} \right]. 
\]  

(18)

The first series in the rhs of Eq. (10) can then be summed as:

\[
\sum_{n=1}^{\infty} (-1)^n \frac{\Gamma(n + 1 - 3\varepsilon)}{\Gamma(n + \varepsilon)} x^{n-1} = -\frac{\Gamma(2 - 2\varepsilon)}{\Gamma(1 + \varepsilon)} \, _2F_1(2 - 2\varepsilon, 1; 1 + \varepsilon; -x) 
\]

\[
= -\frac{\Gamma(2 - 2\varepsilon)}{\Gamma(1 + \varepsilon)} \frac{1}{(1 + x)^{2-3\varepsilon}} \, _2F_1(3\varepsilon - 1, \varepsilon; 1 + \varepsilon; -x),
\]
where we have used the standard transformation formula for the \( {}_2F_1 \)-hypergeometric function:

\[
{}_2F_1(A, B; C; z) = (1 - z)^{C - A - B} {}_2F_1(C - A, C - B; C; z) .
\]  

(19)

So, the first sum in the rhs of Eq. (10) can be represented as:

\[
J_1 \equiv - \sum_{n=1}^{\infty} (-1)^n \frac{\Gamma(n + 1 - 2\varepsilon)}{\Gamma(n + \varepsilon)} \left( \frac{1}{n + \alpha + \varepsilon} + \frac{1}{n - \alpha - 2\varepsilon} \right)
\]

\[
= \frac{\Gamma(2 - 2\varepsilon)}{\Gamma(1 + \varepsilon)} \int_0^1 dx \frac{x^\alpha + x^{-\alpha - 2\varepsilon}}{(1 + x)^{2 - 3\varepsilon}} {}_2F_1(3\varepsilon - 1, \varepsilon; 1 + \varepsilon; -x)
\]

\[
= \frac{\Gamma(2 - 2\varepsilon)}{\Gamma(\varepsilon)} \int_0^1 dx \frac{x^\alpha + x^{-\alpha - 2\varepsilon}}{(1 + x)^{2 - 3\varepsilon}} \int_0^1 dt \frac{t^{\varepsilon - 1}}{(1 + xt)^{3\varepsilon - 1}} \equiv \frac{\Gamma(2 - 2\varepsilon)}{\Gamma(\varepsilon)} \tilde{J}_1 ,
\]  

(20)

where we have used the integral representation of the \( {}_2F_1 \)-hypergeometric function:

\[
{}_2F_1(A, B; C; z) = \frac{\Gamma(C)}{\Gamma(B)\Gamma(C - B)} \int_0^1 dt \frac{t^{B-1}(1 - t)^{C-B-1}}{(1 - z t)^A} .
\]  

(21)

After some algebra (see Appendix A) we have:

\[
J_1 = \frac{\Gamma(2 - 2\varepsilon)}{\Gamma(1 + \varepsilon)} \int_0^1 dx \frac{(1 - x)\alpha}{x_2^{1 + 2\varepsilon}} {}_2F_1\left(-\alpha, \varepsilon; 1 + \varepsilon; \frac{x_2}{x_2 - 1}\right)
\]

\[
= \frac{\Gamma(2 - 2\varepsilon)}{\Gamma(1 + \varepsilon)} \int_0^1 dx \frac{1}{x_2^{1 + 2\varepsilon}} {}_2F_1(-\alpha, 1; 1 + \varepsilon; x_2) ,
\]  

(22)

where we have used another standard transformation formula for the \( {}_2F_1 \)-hypergeometric function:

\[
{}_2F_1(A, B; C; z) = (1 - z)^{-A} {}_2F_1\left(A, C - B; C; \frac{z}{z - 1}\right) .
\]  

(23)

Taking the standard series representation for the \( {}_2F_1 \)-hypergeometric functions, we have for \( J_1 \) the following final result:

\[
J_1 = \int_0^1 dx \frac{1}{x_2^{1 + 2\varepsilon}} \sum_{n=0}^{\infty} \frac{\Gamma(n - \alpha)\Gamma(2 - 2\varepsilon)}{\Gamma(-\alpha)\Gamma(n + 1 + \varepsilon)} x_2^n = \sum_{n=0}^{\infty} \frac{\Gamma(n - \alpha)\Gamma(2 - 2\varepsilon)}{\Gamma(-\alpha)\Gamma(n + 1 + \varepsilon)} \frac{1}{n + 1 - 2\varepsilon - \alpha} ,
\]  

(24)

which is expressed in terms of a single \( {}_3F_1 \)-hypergeometric function of argument 1.

However, the result does not coincide yet with the one of Eq. (6). Indeed, we have now:

\[
I(1 + \alpha) = 2 \frac{\Gamma^2(1 - \varepsilon)\Gamma(\varepsilon)\Gamma(-\varepsilon - \alpha)\Gamma(\alpha + 2\varepsilon)}{\Gamma(2 - 2\varepsilon)} \left[ \frac{1}{\Gamma(1 + \alpha)\Gamma(1 - 3\varepsilon - \alpha)} \right] \nonumber
\]

\[
\times \sum_{n=0}^{\infty} \frac{\Gamma(n - \alpha)\Gamma(2 - 2\varepsilon)}{\Gamma(-\alpha)\Gamma(n + 1 + \varepsilon)} \frac{1}{n + 1 - 2\varepsilon - \alpha} - \cos[\pi\varepsilon] \nonumber
\]

\[
, 
\]  

(25)

where the series as well as the term containing only \( \Gamma \)-functions differ from the corresponding ones in Eq. (6).
C. Transformation of the series in the rhs of Eq. (25) to the one in Eq. (6)

We now use the transformation formula for the 3\(F_2\)-hypergeometric function of argument 1:

\[
\begin{align*}
3F_2(A, B, C; E, F; 1) &= \frac{\Gamma(1 - A)\Gamma(E)\Gamma(F)\Gamma(C - B)}{\Gamma(E - B)\Gamma(F - B)\Gamma(1 + B - A)\Gamma(C)} \\
&\times 3F_2(B, B - E + 1, B - F + 1; B - C + 1, B - A + 1; 1) + \left( B \leftrightarrow C \right). 
\end{align*}
\]

(26)

For \( E = B + 1 \), the 3\(F_2\)-hypergeometric function can be represented as the sum of another 3\(F_2\)-hypergeometric function and a term containing only \( \Gamma \)-functions. In terms of series representations, the relation has the following form:

\[
\sum_{n=0}^{\infty} \frac{\Gamma(n + A)\Gamma(n + C)}{n!\Gamma(n + F)} \frac{1}{n + B} = \frac{\Gamma(1 - A)\Gamma(A)\Gamma(C - B)}{\Gamma(F - B)\Gamma(1 + B - A)} \\
- \frac{\Gamma(1 - A)\Gamma(A)\Gamma(C - F)}{\Gamma(F - C)\Gamma(1 + C - F)} \sum_{n=0}^{\infty} \frac{\Gamma(n + C - F + 1)\Gamma(n + C)}{n!\Gamma(n + C - A + 1)} \frac{1}{n + C - B}.
\]

(27)

Using this relation with: \( B = 1 - \alpha - 2\varepsilon \), \( F = 1 + \varepsilon \), \( A = -\alpha \) and \( C = 1 \), we can transform the series in Eq. (25) as follows:

\[
\sum_{n=0}^{\infty} \frac{\Gamma(n - \alpha)\Gamma(2 - 2\varepsilon)}{\Gamma(-\alpha)\Gamma(n + 1 + \varepsilon)} \frac{1}{n + 1 - 2\varepsilon - \alpha} = \frac{\Gamma(1 + \alpha)\Gamma(1 - \alpha - 2\varepsilon)\Gamma(\alpha + 2\varepsilon)}{\Gamma(\alpha + 3\varepsilon)} \\
- \frac{\Gamma(1 + \alpha)\Gamma(2 - 2\varepsilon)}{\Gamma(\varepsilon)\Gamma(1 - \varepsilon)} \sum_{n=0}^{\infty} \frac{\Gamma(n + 1 - \varepsilon)}{\Gamma(n + 2 + \alpha)} \frac{1}{n + 2\varepsilon + \alpha}.
\]

Focusing on the terms containing only products of \( \Gamma \)-functions, they can be simplified as follows:

\[
\frac{\Gamma(1 - \alpha - 2\varepsilon)\Gamma(\alpha + 2\varepsilon)}{\Gamma(1 - \alpha - 3\varepsilon)\Gamma(\alpha + 3\varepsilon)} - \cos[\pi\varepsilon] = \frac{\sin[\pi(\alpha + 3\varepsilon)]}{\sin[\pi(\alpha + 2\varepsilon)]} - \cos[\pi\varepsilon] \\
= \frac{\sin[\pi\varepsilon]\cos[\pi(\alpha + 2\varepsilon)]}{\sin[\pi(\alpha + 2\varepsilon)]} = \sin[\pi\varepsilon] \cot[\pi(\alpha + 2\varepsilon)].
\]

Then, \( I(1 + \alpha) \) can be written as:

\[
I(1 + \alpha) = 2 \frac{\Gamma(1 - \varepsilon)\Gamma(-\varepsilon - \alpha)\Gamma(\alpha + 2\varepsilon)}{\Gamma(1 - \alpha - 3\varepsilon)} \left[ -\sum_{n=0}^{\infty} \frac{\Gamma(n + 1 - \varepsilon)}{\Gamma(n + 2 + \alpha)} \frac{1}{n + 2\varepsilon + \alpha} \\
+ \frac{\Gamma(1 - \alpha - 3\varepsilon)\pi\cot[\pi(\alpha + 2\varepsilon)]}{\Gamma(2 - 2\varepsilon)} \right].
\]

(28)

At this point, we can see that the terms containing only products of \( \Gamma \)-functions are the same in Eqs. (6) and (28). Hence, the corresponding series should be identical too. To demonstrate
this, it is convenient to use the following transformation formula for the $3F_2$-hypergeometric function of argument 1:

$$
\sum_{n=0}^{\infty} \frac{\Gamma(n+1-\varepsilon)}{n!\Gamma(n+2+\alpha)} \frac{1}{n+2\varepsilon + \alpha} = \frac{\Gamma(1-\varepsilon)}{\Gamma(2-2\varepsilon)} \sum_{n=0}^{\infty} \frac{\Gamma(n+2-2\varepsilon)}{\Gamma(n+2+\alpha)} \frac{1}{n+1+\alpha + \varepsilon},
$$

(32)

which proves the equality of the rhs of Eqs. (6) and (28).

III. OTHER USEFUL REPRESENTATIONS FOR $I(1+\alpha)$

Eqs. (25) and (28) can be considered as new results for the diagram $I(1+\alpha)$. Other new results can be obtained with the help of the transformation formula (26) with $A = 1-\delta$ and $\delta \to 0$; the later will involve the replacement of a $\Gamma$-function by a $\Psi$-function which may be more convenient to expand the diagram in $\varepsilon$. Notice that similar representations, but for a two-loop massless Feynman diagram with two arbitrary non-adjacent indices, have been obtained in our paper [18], see App. B for a summary of our results.

In order to proceed with the derivation, we consider the sum:

$$
\sum_{n=0}^{\infty} \frac{\Gamma(n+1-\varepsilon)}{\Gamma(n+2+\alpha)} \frac{\Gamma(n+1-\delta)}{n!\Gamma(1-\delta)} \frac{1}{n+1+\alpha + \varepsilon},
$$

which coincides with the one in Eq. (6) when $\delta \to 0$. Applying the transformation formula
Eq. (26) with $A = 1 - \delta$, $C = 2 - 2\varepsilon$, $F = 2 + \alpha$ and $B = 1 + \alpha + \varepsilon$, yields:

$$\sum_{n=0}^{\infty} \frac{\Gamma(n + 2 - 2\varepsilon)}{\Gamma(n + 2 + \alpha)} \frac{\Gamma(n + 1 - \delta)}{n!\Gamma(1 - \delta)} \frac{1}{n + 1 + \alpha + \varepsilon} = \frac{\Gamma(\delta)\Gamma(1 + \alpha - 3\varepsilon)\Gamma(1 + \alpha + \varepsilon)}{\Gamma(1 - \varepsilon)\Gamma(1 + \alpha + \varepsilon + \delta)} \frac{\Gamma(\delta)}{\Gamma(1 - \alpha - 2\varepsilon)\Gamma(\alpha + 2\varepsilon)} \sum_{n=0}^{\infty} \frac{\Gamma(n + 1 - \alpha - 2\varepsilon)\Gamma(n + 2 - 2\varepsilon)}{n!\Gamma(n + 2 - 2\varepsilon + \delta)} \frac{1}{n + 1 - \alpha - 3\varepsilon}.$$ 

Taking the limit $\delta \to 0$, we can transform the rhs to the following form:

$$\frac{\Gamma(1 - \alpha - 3\varepsilon)}{\Gamma(1 - \varepsilon)} \left[ \frac{1}{\delta} + \Psi(1) - \Psi(1 + \alpha + \varepsilon) \right] - \frac{1}{\Gamma(1 - \alpha - 2\varepsilon)\Gamma(\alpha + 2\varepsilon)} \sum_{n=0}^{\infty} \frac{\Gamma(n + 1 - \alpha - 2\varepsilon)}{n!} \frac{1}{n + 1 - \alpha - 3\varepsilon} \left[ \frac{1}{\delta} + \Psi(1) - \Psi(n + 2 - 2\varepsilon) \right] + O(\delta).$$

The series appearing in factor of $1/\delta$ can be summed as:

$$\sum_{n=0}^{\infty} \frac{\Gamma(n + 1 - \alpha - 2\varepsilon)}{n!} \frac{1}{n + 1 - \alpha - 3\varepsilon} = \frac{\Gamma(1 - \alpha - 2\varepsilon)\Gamma(\alpha + 2\varepsilon)\Gamma(1 - \alpha - 3\varepsilon)}{\Gamma(1 - \varepsilon)},$$

which leads to the cancellation of all terms $\sim 1/\delta$. Hence, we obtain:

$$\sum_{n=0}^{\infty} \frac{\Gamma(n + 2 - 2\varepsilon)}{\Gamma(n + 2 + \alpha)} \frac{1}{n + 1 + \alpha + \varepsilon} = -\frac{\Gamma(1 - \alpha - 3\varepsilon)}{\Gamma(1 - \varepsilon)} \Psi(1 + \alpha + \varepsilon) + \frac{1}{\Gamma(1 - \alpha - 2\varepsilon)\Gamma(\alpha + 2\varepsilon)} \sum_{n=0}^{\infty} \frac{\Gamma(n + 1 - \alpha - 2\varepsilon)}{n!} \frac{\Psi(n + 2 - 2\varepsilon)}{n + 1 - \alpha - 3\varepsilon}.$$ 

Similar calculations can be repeated for the series appearing in the rhs of Eqs. (25) and (28). The corresponding results read:

$$\sum_{n=0}^{\infty} \frac{\Gamma(n - \alpha)}{\Gamma(n + 1 + \varepsilon)} \frac{1}{n + 1 - 2\varepsilon - \alpha} = -\frac{\Gamma(2\varepsilon - 1)}{\Gamma(\alpha + 3\varepsilon)} \Psi(1 - \alpha - 2\varepsilon) + \frac{1}{\Gamma(-\alpha - \varepsilon)\Gamma(1 + \alpha + \varepsilon)} \sum_{n=0}^{\infty} \frac{\Gamma(n - \alpha - \varepsilon)}{n!} \frac{\Psi(n - \alpha)}{n + 2\varepsilon - 1},$$

and

$$\sum_{n=0}^{\infty} \frac{\Gamma(n + 1 - \varepsilon)}{\Gamma(n + 2 + \alpha)} \frac{1}{n + \alpha + 2\varepsilon} = -\frac{\Gamma(1 - \alpha - 3\varepsilon)}{\Gamma(2 - 2\varepsilon)} \Psi(\alpha + 2\varepsilon) + \frac{1}{\Gamma(1 + \alpha + \varepsilon)\Gamma(-\alpha - \varepsilon)} \sum_{n=0}^{\infty} \frac{\Gamma(n - \alpha - \varepsilon)}{n!} \frac{\Psi(n + 1 - \varepsilon)}{n + 1 - \alpha - 3\varepsilon}.$$
IV. CONCLUSION

We have analytically proven the equality of two previously existing results (5) and (6), containing $3F_2$-hypergeometric functions of arguments $-1$ and $1$, respectively. As a by-product of our calculations, the relation (7) between these two types of $3F_2$-hypergeometric functions has been proven exactly. Such a relation should be useful for various applications because there are a lot of transformation formulas available for the $3F_2$-hypergeometric function of argument $1$ (see [25]). Thus, Eq. (7) gives a possibility to apply some of these transformation formulas to the $3F_2$-hypergeometric function of argument $-1$. Moreover, as already mentioned in the introduction, the way integral representations were used in Sec. II and App. A is rather similar to manipulations carried out with $\Psi$-functions and associated functions in [19,20] in order to simplify BFKL results at the first two orders of perturbation theory and to verify their conformal properties or a violation of them.

Moreover, using transformation formulas Eqs. (26) and (29) for $3F_2$-hypergeometric functions of argument 1 [25] we have found several new representations for the diagram $I(\alpha)$ under consideration. We hope to use them in our future $\varepsilon$-expansions of $I(\alpha)$ and $J(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ with values of the indices $\alpha$ and $\alpha_i (i = 1, ..., 5)$ close to 1.

In closing, our analysis was devoted to the simplest non-trivial two-loop massless propagator-type Feynman diagram. We hope that such an analysis may be generalized to allow the exact computation of more complicated diagrams such as those appearing upon studying $1/N$ corrections to dynamical fermion mass generation in QED$_3$ [26,27] and RQED$_4,5$ [28], see Figs. 2a and 2b.
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Appendix A: Evaluation of the integral $\tilde{J}_1$

In this Appendix, we give the details of the evaluation of the integral $\tilde{J}_1$ in Eq. (20).

Introducing the new variable $t_1 = xt$, we have for $\tilde{J}_1$:

$$\tilde{J}_1 = \int_0^1 dx \frac{x^\alpha + x^{-\alpha - 3\varepsilon}}{(1 + x)^{2 - 3\varepsilon}} \int_0^x dt_1 \frac{t_1^{\varepsilon - 1}}{(1 + t_1)^{3\varepsilon - 1}} = \int_0^1 dt_1 \frac{t_1^{\varepsilon - 1}}{(1 + t_1)^{3\varepsilon - 1}} \int_{t_1}^1 dx \frac{x^\alpha + x^{-\alpha - 3\varepsilon}}{(1 + x)^{2 - 3\varepsilon}}. \tag{A1}$$

Note that after the replacement $x \to y = 1/x$ the last term in the last integral of (A1) reads:

$$\int_{t_1}^1 dx \frac{x^{-\alpha - 3\varepsilon}}{(1 + x)^{2 - 3\varepsilon}} = \int_1^{1/t_1} dy \frac{y^\alpha}{(1 + y)^{2 - 3\varepsilon}}, \tag{A2}$$

and, thus, the result (A1) can be expressed as:

$$\tilde{J}_1 = \int_0^1 dt_1 \frac{t_1^{\varepsilon - 1}}{(1 + t_1)^{3\varepsilon - 1}} \int_{t_1}^1 dx \frac{x^\alpha}{(1 + x)^{2 - 3\varepsilon}}. \tag{A3}$$

Using the new variables $x_1 = 1/(1 + x)$, and later $x_1 = x_2/(1 + t_1)$, we have for the last integral in (A3):

$$\int_{t_1}^{1/t_1} dx \frac{x^\alpha}{(1 + x)^{2 - 3\varepsilon}} = \int_{t_1/(1+t_1)}^{1/(1+t_1)} dx_1 \frac{(1 - x_1)^\alpha}{x_1^{\alpha+3\varepsilon}} = \frac{1}{(1 + t_1)^{1-\alpha - 3\varepsilon}} \int_{t_1}^1 dx_2 \frac{(1 - x_2)}{x_2^{\alpha+3\varepsilon}}. \tag{A4}$$

Substituting the new result (A4) to the rhs of (A3), we obtain:

$$\tilde{J}_1 = \int_0^1 dt_1 \frac{1}{t_1^{1-\varepsilon}} \int_{t_1}^1 dx_2 \frac{(1 + t_1 - x_2)^\alpha}{x_2^{\alpha+3\varepsilon}} = \int_0^1 dx_2 \frac{1}{x_2^{\alpha+3\varepsilon}} \int_0^{x_2} dt_1 \frac{(1 + t_1 - x_2)^\alpha}{t_1^{1-\varepsilon}}. \tag{A5}$$

Using the new variable $t_2 = t_1/x_2$, we see that:

$$\tilde{J}_1 = \int_0^1 dx_2 \frac{1}{x_2^{\alpha+2\varepsilon}} \int_0^1 dt_2 \frac{(1 - x_2 + x_2 t_2)^\alpha}{t_1^{1-\varepsilon}} = \int_0^1 dx_2 \frac{(1 - x_2)^\alpha}{x_2^{\alpha+2\varepsilon}} \int_0^1 dt_2 \frac{(1 - x_2)}{t_2^{1-\varepsilon}} = \frac{\Gamma(\varepsilon)}{\Gamma(1 + \varepsilon)} \int_0^1 dx_2 \frac{(1 - x_2)^\alpha}{x_2^{\alpha+2\varepsilon}} 2F1\left(-\alpha, \varepsilon; 1 + \varepsilon; \frac{x_2}{x_2 - 1}\right), \tag{A6}$$

where, in the last step, we have applied the property (21).
Appendix B: Reminder of results for $J(\alpha, 1, \beta, 1, 1)$

In this appendix, we recall, for completeness, the exact results obtained in Ref. [18], based on the results of Ref. [8], for the diagram $J(\alpha, 1, \beta, 1, 1)$ with two arbitrary non-adjacent indices, see Fig. 3. As will be seen below, these results show that $J(\alpha, 1, \beta, 1, 1)$ can be expressed in terms of a linear combination of two $3F_2$-hypergeometric functions of argument 1. Moreover, as in the case of $J(\alpha)$ above, some representations involve $\Psi$-functions.

To start with, we extract the momentum dependence of the diagram which leads to:

$$J(\alpha, 1, \beta, 1, 1) = \frac{p^{2d_F}}{(4\pi)^D} G(\alpha, 1, \beta, 1, 1),$$

where $d_F = D - 3 - \alpha - \beta = 1 - \alpha - \beta - 2\varepsilon$ and we use the notation $G(\alpha, 1, \beta, 1, 1)$ for the dimensionless coefficient function. The later reads:

$$G(\alpha, 1, \beta, 1, 1) = \frac{1}{\alpha - 1 - \beta} \frac{\Gamma(\tilde{\alpha})\Gamma(3 - \tilde{\alpha} - \tilde{\beta})}{\Gamma(\alpha)\Gamma(\lambda - 2 + \tilde{\alpha} + \tilde{\beta})} \frac{\Gamma(2\lambda)}{\Gamma(2\lambda)} I(\tilde{\alpha}, \tilde{\beta}),$$

where $\tilde{\alpha} = D/2 - \alpha$, $\lambda = D/2 - 1$, $D = 4 - 2\varepsilon$. In Eq. (B2), the function $I(\tilde{\alpha}, \tilde{\beta})$ can be written in four different forms which read:

$$I(\tilde{\alpha}, \tilde{\beta}) = \frac{\Gamma(1 + \lambda - \tilde{\alpha})}{\Gamma(3 - \tilde{\alpha} - \tilde{\beta})} \frac{\pi \sin[\pi(\tilde{\beta} - \tilde{\alpha} + \lambda)]}{\sin[\pi(\tilde{\lambda} - 1 + \tilde{\beta})] \sin[\pi(\tilde{\alpha})]} + \sum_{n=0}^{\infty} \frac{\Gamma(n + 2\lambda)}{n!(n + \lambda + \tilde{\alpha} - 1)}$$

$$\times \left( \frac{\Gamma(n + 1)}{\Gamma(n + 2 + \lambda - \tilde{\beta})} - \frac{\Gamma(n - 2 + \lambda + \tilde{\alpha} + \tilde{\beta})\Gamma(2 - \tilde{\beta})\Gamma(\lambda)}{\Gamma(n - 1 + 2\lambda + \tilde{\alpha})\Gamma(3 - \tilde{\alpha} - \tilde{\beta})\Gamma(\lambda + \tilde{\alpha} - 1)} \frac{\sin[\pi(\tilde{\beta} + \lambda - 1)]}{\sin[\pi(\tilde{\alpha})]} \right),$$

$$I(\tilde{\alpha}, \tilde{\beta}) = \frac{\Gamma(1 + \lambda - \tilde{\alpha})}{\Gamma(3 - \tilde{\alpha} - \tilde{\beta})} \frac{\pi \sin[\pi(\tilde{\alpha})]}{\sin[\pi(\tilde{\lambda} - 1 + \tilde{\beta})] \sin[\pi(\tilde{\alpha} + \tilde{\beta} + \lambda - 1)]}$$

$$+ \sum_{n=0}^{\infty} \frac{\Gamma(n + 2\lambda)}{n!(n + \lambda + \tilde{\alpha} - 1)} \left( \frac{1}{\Gamma(n + 2 + \lambda - \tilde{\beta})} - \frac{\Gamma(n + 2 - \tilde{\alpha})\Gamma(2 - \tilde{\beta})\Gamma(\lambda)}{\Gamma(n + 2 + \lambda - \tilde{\beta})\Gamma(3 - \tilde{\alpha} - \tilde{\beta})\Gamma(\lambda + \tilde{\alpha} - 1)} \frac{\sin[\pi(\tilde{\beta} + \lambda - 1)]}{\sin[\pi(\tilde{\alpha} + \tilde{\beta} + \lambda - 1)]} \right).$$
\[ I(\tilde{\alpha}, \tilde{\beta}) = \frac{\Gamma(1 + \lambda - \tilde{\alpha})}{\Gamma(3 - \tilde{\alpha} - \tilde{\beta})} \frac{\pi \sin[\pi(\tilde{\beta} - \tilde{\alpha} + \lambda)]}{\sin[\pi(\tilde{\lambda} - 1 + \tilde{\beta})\sin[\pi\tilde{\alpha}]]} - \frac{\Gamma(1 + \lambda - \tilde{\alpha})}{\Gamma(3 - \tilde{\alpha} - \tilde{\beta})} \Psi(\lambda + \tilde{\alpha} - 1) \] (B5)

\[ + \sum_{n=0}^{\infty} \frac{\Gamma(n + \tilde{\beta} + \lambda - 1)}{n!(n + 1 - \tilde{\alpha})} \Psi(n + 2\lambda) \frac{\sin[\pi(\tilde{\beta} + \lambda - 1)]}{\pi} \]

\[ - \sum_{n=0}^{\infty} \frac{\Gamma(n + 2\lambda)}{n!(n + \lambda + \tilde{\alpha} - 1)} \frac{\Gamma(n - 2 + \lambda + \tilde{\alpha} + \tilde{\beta})\Gamma(2 - \tilde{\beta})\Gamma(\lambda)}{\Gamma(n - 1 + 2\lambda + \tilde{\alpha})\Gamma(3 - \tilde{\alpha} - \tilde{\beta})\Gamma(\lambda + \tilde{\alpha} - 1)\sin[\pi\tilde{\alpha}]} \]

\[ I(\tilde{\alpha}, \tilde{\beta}) = \frac{\Gamma(1 + \lambda - \tilde{\alpha})}{\Gamma(3 - \tilde{\alpha} - \tilde{\beta})} \frac{\pi \sin[\pi\tilde{\alpha}]}{\sin[\pi(\tilde{\lambda} - 1 + \tilde{\beta})\sin[\pi(\tilde{\alpha} + \tilde{\beta} + \lambda - 1)]]} - \frac{\Gamma(1 + \lambda - \tilde{\alpha})}{\Gamma(3 - \tilde{\alpha} - \tilde{\beta})} \Psi(\lambda + \tilde{\alpha} - 1)(B6) \]

\[ + \sum_{n=0}^{\infty} \frac{\Gamma(n + 2\lambda)}{n!(n + 1 - \tilde{\alpha})} \left( \frac{\Gamma(n + 2 - \tilde{\alpha})\Gamma(\lambda)}{\Gamma(n + 3 + \lambda - \tilde{\alpha} - \tilde{\beta})\Gamma(3 - \tilde{\alpha} - \tilde{\beta})\Gamma(\lambda + \tilde{\alpha} - 1)\sin[\pi(\tilde{\alpha} + \tilde{\beta} + \lambda - 1)]} \right) \]

\[ + \frac{\Gamma(n + \tilde{\beta} + \lambda - 1)}{\Gamma(n + 2\lambda)} \Psi(n + 2\lambda) \frac{\sin[\pi(\tilde{\beta} + \lambda - 1)]}{\pi} \]

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29 Two other results for this diagram can be found in Refs. [4] and [12], see the review [1] for a presentation of these results.