The H/Q-correspondence and a generalization of the supergravity c-map

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Abstract

Given a hypercomplex manifold with a rotating vector field (and additional data), we construct a conical hypercomplex manifold. As a consequence, we associate a quaternionic manifold to a hypercomplex manifold of the same dimension with a rotating vector field. This is a generalization of the HK/QK-correspondence. As an application, we show that a quaternionic manifold can be associated to a conical special complex manifold of half its dimension. Furthermore, a projective special complex manifold (with a canonical c-projective structure) associates with a quaternionic manifold. The latter is a generalization of the supergravity c-map. We do also show that the tangent bundle of any special complex manifold carries a canonical Ricci-flat hypercomplex structure, thereby generalizing the rigid c-map.

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1 Introduction

The HK/QK-correspondence is a construction of a (pseudo-)quaternionic Kähler manifold from a (pseudo-)hyper-Kähler manifold of the same dimension with a rotating vector field (see Definition 3.1 and \[15, 2, 16, 4\]). This correspondence gives also the supergravity c-map, which associates a quaternionic Kähler manifold with a projective special Kähler manifold. The supergravity c-map was introduced in theoretical physics \[13\].

The inverse construction of the HK/QK-correspondence is called the QK/HK-correspondence. It has been generalized to a Q/H-correspondence, a construction of hypercomplex manifolds from quaternionic manifolds \[10\]. The purpose of this paper is to construct a quaternionic manifold from a hypercomplex manifold endowed with a rotating vector field and some extra data. We shall call this construction the hypercomplex/quaternionic-correspondence (H/Q-correspondence for short). We briefly explain how we obtain this correspondence. First we define the notion of a conical hypercomplex manifold (Definition 2.1). Next we construct a conical hypercomplex manifold \(\hat{M}\) for every hypercomplex manifold \(M\) with a rotating vector field \(Z\) (Theorem 3.9) and additional data: a two-form \(\Theta\) on \(M\), a U(1)-bundle over \(M\) whose curvature satisfies (3.1) and a function \(f\) on \(M\) such that \(df = -\iota_Z \Theta\). The manifold \(\hat{M}\) is endowed with a free action of the Lie algebra \(\text{Lie} \mathbb{H}^* \cong \mathbb{R} \oplus \text{su}(2)\) and its quotient space \(\bar{M}\) carries a quaternionic structure, provided that the quotient map \(\hat{M} \rightarrow \bar{M}\) is a submersion. The H/Q-correspondence is then defined as \(M \mapsto \bar{M}\) (Theorems 4.1 and 4.8).

In addition, we show that \(\bar{M}\) carries not only a quaternionic connection but also an (induced) affine quaternionic vector field (Proposition 4.7). Note that we give an example of our H/Q-correspondence from a hypercomplex Hopf manifold, which does not admit any hyper-Kähler structure (Example 5.3). Therefore the H/Q-correspondence is a proper generalization of the HK/QK-correspondence. Examples like hypercomplex or quaternionic Hopf manifolds show that hypercomplex and quaternionic manifolds arise naturally beyond the context of hyper-Kähler and quaternionic Kähler geometry. We refer to \[25, 18, 19\] for the theory of quaternionic manifolds and constructions of such manifolds.

The rigid c-map \[2\] allows to associate with a conical special Kähler manifold its cotangent bundle endowed with a hyper-Kähler structure with a rotating vector field \[2\]. In the absence of a metric, we show that the tangent bundle of a special complex manifold carries a canonical hypercomplex structure and that its Obata connection is Ricci flat (Theorem 6.5). In this way we establish a generalization of the rigid c-map which assigns a Ricci flat hypercomplex manifold to each special complex manifold. When the special complex manifold is conical, the resulting hypercomplex manifold is shown to admit a canonical rotating vector field (Lemma 8.1). The notion of a (conical) special complex manifold was introduced in \[3\]. It is a generalization of a (conical)
special Kähler manifold. We give a local example which does not arise as a special Kähler manifold (Example 8.9). In addition, we find many (different) quaternionic structures on the tangent bundle of a conical special complex manifold in this example (Example 8.9), using a generalization of the supergravity c-map.

As an application of our H/Q-correspondence, we indeed generalize the supergravity c-map by associating a quaternionic manifold with every conical special complex manifold and therefore with every projective special complex manifold (using the extra data involved in the H/Q-correspondence), see Theorem 8.3. It is shown in Proposition 7.3 that any projective special complex manifold possesses a canonical c-projective structure and in Theorem 7.10 that its c-projective Weyl curvature is of type (1, 1). So our generalized supergravity c-map can be formulated as associating a quaternionic manifold to a projective special complex manifold endowed with its canonical c-projective structure with c-projective Weyl curvature of type (1, 1). This addresses one of the questions raised in [6], where a different construction of quaternionic manifolds from c-projective structures was obtained, compare Remark 8.5.

In the special case of the HK/QK-correspondence, the two-form Θ, which is part of the data entering the H/Q-correspondence, is the Z-invariant Kähler form ω₁ in the hyper-Kähler-triple (ω₁, ω₂, ω₃). However, in general, we have a freedom in the choice of Θ in the H/Q-correspondence (see Section 5). In particular we find two choices of Θ in Example 5.4 which yield different quaternionic structures on the resulting space. This shows that our H/Q-correspondence is not an inverse construction of the Q/H-correspondence without a further specification of Θ. It is left for future studies to find a suitable choice of Θ which gives an inverse construction.

We summarize our constructions in this paper as the following commutative diagram.

\[ \begin{array}{c}
M: \text{conical hypercomplex} \\
\tilde{M}: \text{quaternionic} \\
N: \text{conical special complex} \\
\tilde{N}: \text{projective special complex}
\end{array} \]

\[ \begin{array}{c}
(P, \eta) \\
\xrightarrow{U(1)} \\
(M = TN, f, \Theta) \xrightarrow{\text{conification}} \tilde{M} = C_P(M)
\end{array} \]

\[ \begin{array}{c}
(N, J, \nabla, \xi) \xrightarrow{\text{rigid c-map}} (\tilde{N}, \tilde{J}, P_{\tilde{N}}) \\
\xrightarrow{\text{Theorem 5.9}} (\tilde{N}, \tilde{J}, P_{\tilde{N}}) \rightarrow \tilde{M} = \tilde{M}/\mathcal{D}
\end{array} \]

\[ \begin{array}{c}
\text{Proposition 4.2} \\
\text{Theorems 5.7 and 1.3} \\
\text{Theorem 5.3} \\
\text{H/Q-corresp.}
\end{array} \]

\[ \begin{array}{c}
\text{generalized supergravity c-map}
\end{array} \]

2 Preliminaries

Throughout this paper, all manifolds are assumed to be smooth and without boundary and maps are assumed to be smooth unless otherwise mentioned. The space of sections of a vector bundle $E \to M$ is denoted by $\Gamma(E)$. 
In this section we introduce hypercomplex and quaternionic structures and derive some properties of conical hypercomplex manifolds.

We say that $M$ is a quaternionic manifold with the quaternionic structure $Q$ if $Q$ is a subbundle of $\text{End}(TM)$ of rank 3 which at every point $x \in M$ is spanned by endomorphisms $I_1, I_2, I_3 \in \text{End}(T_x M)$ satisfying

$$I_1^2 = I_2^2 = I_3^2 = -\text{id}, \quad I_1 I_2 = -I_2 I_1 = I_3,$$

and there exists a torsion-free connection $\nabla$ on $M$ such that $\nabla$ preserves $Q$, that is, $\nabla_X \Gamma(Q) \subset \Gamma(Q)$ for all $X \in \Gamma(TM)$. Such a torsion-free connection $\nabla$ is called a quaternionic connection and the triplet $(I_1, I_2, I_3)$ is called an admissible frame of $Q$ at $x$. Note that we use the same letter $\nabla$ for the connection on $\text{End}(TM)$ induced by $\nabla$. The dimension of the quaternionic manifold $M$ is denoted by $4n$.

An almost hypercomplex manifold is defined to be a manifold $M$ endowed with 3 almost complex structures $I_1, I_2, I_3$ satisfying the quaternionic relations (2.1). If $I_1, I_2, I_3$ are integrable, then $M$ is called a hypercomplex manifold. There exists a unique torsion-free connection on a hypercomplex manifold for which the hypercomplex structures are parallel. It is called the Obata connection [22]. Obviously, hypercomplex manifolds are quaternionic manifolds with $Q = \langle I_1, I_2, I_3 \rangle$.

**Definition 2.1.** We say that a hypercomplex manifold $(M, (I_1, I_2, I_3))$ with a vector field $V$ is conical if $\nabla^0 V = \text{id}$ holds, where $\nabla^0$ is the Obata connection. The vector field $V$ is called the Euler vector field.

We state some lemmas for conical hypercomplex manifolds, which will be used later.

**Lemma 2.2.** Let $(M, (I_1, I_2, I_3), V)$ be a conical hypercomplex manifold. Then we have $L_V I_\alpha = 0$, $L_{I_\alpha V} I_\alpha = 0$ for $\alpha \in \{1, 2, 3\}$ and $L_{I_\alpha V} I_\beta = -2I_\gamma$ for any cyclic permutation $(\alpha, \beta, \gamma)$.

**Proof.** The formulas follow immediately from $L_V = \nabla^0 V - \nabla^0 V = \nabla^0 V - \text{id}$ and $L_{I_\alpha V} = \nabla^0_{I_\alpha V} - I_\alpha$. □

For a connection $\nabla$ and $X \in \Gamma(TM)$, we define

$$L_X \nabla Y Z := L_X (\nabla Y Z) - \nabla L_X Y Z - \nabla_Y (L_X Z),$$

where $Y, Z \in \Gamma(TM)$. Note that $L_X \nabla$ is a tensor.

**Lemma 2.3.** Let $(M, (I_1, I_2, I_3), V)$ be a conical hypercomplex manifold. Then we have $L_V \nabla^0 = 0$ and $L_{I_\alpha V} \nabla^0 = 0$.

**Proof.** By Lemma 2.2 $V$ and $I_\alpha V$ are quaternionic vector fields, namely $L_V \Gamma(Q) \subset \Gamma(Q)$ and $L_{I_\alpha V} \Gamma(Q) \subset \Gamma(Q)$, where $Q = \langle I_1, I_2, I_3 \rangle$. By [10] Proposition 4.2], it is enough to check $\text{Ric}^\nabla(V, \cdot) = 0$ and $\text{Ric}^\nabla(I_\alpha V, \cdot) = 0$. We have

$$\text{Ric}^\nabla(V, Y) = -\text{Ric}^\nabla(Y, V) = -\text{Tr} R^\nabla(\cdot, Y)V = 0.$$

Here we used the skew-symmetry of the Ricci tensor of the Obata connection. It follows that also $\text{Ric}^\nabla(I_\alpha V, \cdot) = -\text{Ric}^\nabla(V, I_\alpha \cdot) = 0$, by the hermitian property of the Ricci tensor of the Obata connection. □
Alternatively we could have used Lemma 2.2 and the explicit form of the Obata connection to check $L_{I_α}∇^0 = 0$. Note that $L_V∇^0 = 0$ follows from the uniqueness of the Obata connection, since the vector field $V$ preserves the hypercomplex structure.

**Example 2.4 (The Swann bundle).** The principal $\mathbb{R}^+ \times SO(3)$ bundle over a quaternionic manifold, whose fibers consist of all volume elements and admissible frames at each point, possesses a hypercomplex structure (see [24, 10]). It is conical and is called the Swann bundle. The fundamental vector field generated by $c(\neq 0) \in T_1\mathbb{R}^+ = \mathbb{R}$ is the Euler vector field, as can be easily checked from the explicit representation of the Obata connection (see [5] for example). In the notation of [10] with $ε = -1$ and $c = -4(n + 1)$, a basis of fundamental vector fields for the principal action is given by the vector fields $V = Z_0$ and $Z_α = -I_αZ_0$ with non-trivial commutators $[Z_α, Z_β] = -2Z_γ$ and Lie derivatives $L_{Z_α}I_β = -2I_γ$ for any cyclic permutation of $\{1, 2, 3\}$, where we have denoted by $(I_1, I_2, I_3)$ the hypercomplex structure of the Swann bundle. Specializing to the Swann bundle $\mathbb{H}^*/\{±1\}$ of a point, we see that $Z_0$ corresponds to $1$ and $(Z_1, Z_2, Z_3)$ to $(i, j, k)$ in $T_1(\mathbb{H}^*/\{±1\}) = T_1\mathbb{H} = \mathbb{H}$.

**Lemma 2.5.** On any conical hypercomplex manifold $(M, (I_1, I_2, I_3), V)$, the distribution $\mathcal{D} := \langle V, I_1V, I_2V, I_3V \rangle$ on $\{x \in M \mid V_x \neq 0\}$ is integrable.

**Proof.** This follows from Lemma 2.2.

## 3 Conification of hypercomplex manifolds

The main result of this section is a construction of conical hypercomplex manifolds $\hat{M}$ of dimension $\dim \hat{M} = \dim M + 4$ from hypercomplex manifolds $M$ with a rotating vector field.

Let $M$ be a hypercomplex manifold of dimension $4n$ with a hypercomplex structure $H = (I_1, I_2, I_3)$.

**Definition 3.1.** A vector field $Z$ on a hypercomplex manifold $(M, (I_1, I_2, I_3))$ is called rotating if $L_ZI_1 = 0$ and $L_ZI_2 = -2I_3$.

Note that if $Z$ is rotating, then $L_ZI_3 = 2I_2$. In this section we will essentially show that by choosing a (local) primitive of the one-form $ι_ZΘ$ we can construct a conical hypercomplex manifold $(\hat{M}, \hat{H}, V)$ for a hypercomplex manifold $(M, H)$ with a rotating vector field $Z$ and a closed two-form $Θ$ such that $L_ZΘ = 0$.

Let $f$ be a smooth function on $M$ such that $df = -ι_ZΘ$ and $f_1 := f - (1/2)Θ(Z, I_1Z)$ is nowhere vanishing. Consider a principal $U(1)$-bundle $π : P \to M$ with a connection form $η$ whose curvature form is

$$dη = π^* \left( Θ - \frac{1}{2}d((ι_ZΘ) \circ I_1) \right).$$

Since the curvature $dη$ is a basic form, we will usually identify it with its projection $Θ - \frac{1}{2}d((ι_ZΘ) \circ I_1)$ on $M$. With this understood we have the following lemma, which follows immediately from the definition of $f_1$. 

5
Lemma 3.2. \( df_1 = -t_Zdf \).

Define a vector field \( Z_1 \) on \( P \) by \( Z_1 = Z^{h_0} + (\pi^*f_1)x_P \), where \( Z^{h_0} \) is the \( \eta \)-horizontal lift and \( X_P \) is the fundamental vector field such that \( \eta(X_P) = 1 \). We will write \( f_1 \) for \( \pi^*f_1 \).

Remark 3.3. Note that \([X_P, Z_1] = 0\). Therefore if \( Z_1 \) generates a U(1)-action on \( P \), then its action commutes with the principal action of \( \pi : P \to M \).

Set \( \hat{M} = \mathbb{H}^* \times P \). Let \((e_0^R, e_1^R, e_2^R, e_3^R) \) (resp. \((e_0^L, e_1^L, e_2^L, e_3^L) \)) be the right-invariant (resp. the left-invariant) frame of \( \mathbb{H}^* \) which coincides with \((1, i, j, k) \) at \( 1 \in \mathbb{H}^* \). Note that \([e_1^R, e_2^R] = -2e_3^R \). We will use the same letter for vectors or vector fields canonically lifted to the product \( \hat{M} = \mathbb{H}^* \times P \) as for those on the factors \( \mathbb{H}^* \) and \( P \). Set

\[ V_1 := e_1^L - Z_1. \]

We denote the space of integral curves of \( V_1 \) by \( \hat{M} \). We assume that the quotient map \( \hat{\pi} : \hat{M} \to \hat{M} \) is a submersion. Note that “submersion” requires that the quotient space \( \hat{M} \) is smooth.

Lemma 3.4. We assume that the equation (3.7) holds. If \( L_ZI_1 = 0 \) and \( L_Z\Theta = 0 \), we have

\[ L_{V_1}Y^{h_0} = -[Z, Y]^{h_0} \]

for all \( Y \in \Gamma(TM) \).

Proof.

\[
-L_{V_1}Y^{h_0} = -[e_1^L - Z_1, Y^{h_0}] = [Z_1, Y^{h_0}]
= [Z^{h_0}, Y^{h_0}] + [f_1X_P, Y^{h_0}] = [Z^{h_0}, Y^{h_0}] - (Y^{h_0}f_1)X_P
= [Z, Y]^{h_0} + \eta([Z^{h_0}, Y^{h_0}])X_P - (Y^{h_0}f_1)X_P
= [Z, Y]^{h_0} - d\eta(Z, Y)X_P - (Yf_1)X_P
= [Z, Y]^{h_0},
\]

where we have used Lemma 3.2. \( \Box \)

Note that

\[
T_{(z, p)}\hat{M} \cong T_z\mathbb{H}^* \oplus T_pP = \langle e_0^R, e_1^R, e_2^R, e_3^R \rangle_z \oplus \langle X_P \rangle_p \oplus \text{Ker } \eta_p
= \langle V_1 \rangle_{(z, p)} \oplus \langle e_0^R, e_1^R, e_2^R, e_3^R \rangle_z \oplus \text{Ker } \eta_p
\]

for \((z, p) \in \mathbb{H}^* \times P \). We define three endomorphisms fields \( \tilde{I}_1, \tilde{I}_2, \tilde{I}_3 \) on \( \hat{M} \) of rank \( \tilde{I}_\alpha = 4n + 4 \) \((\alpha = 1, 2, 3) \) as follows:

\[
\tilde{I}_\alpha V_1 = 0, \quad \tilde{I}_\alpha e_\alpha^R = e_\alpha^R, \quad \tilde{I}_\alpha e_\alpha^L = -e_\alpha^L, \quad \tilde{I}_\alpha e_\beta^R = e_\gamma^R, \quad \tilde{I}_\alpha e_\gamma^R = -e_\beta^R,
\]

\[
(\tilde{I}_\alpha)_{(z, p)}((Y^{h_0})_{(z, p)}) = ((I_\alpha^p)_{(z, p)}(\pi_*(Y)))^{h_0}_{(z, p)}
\]
for \( Y \in T_pM \). Here \( I'_\alpha \) is defined by

\[
(3.2) \quad I'_\alpha = \sum_{\beta=1}^{3} A_{\alpha\beta} I_\beta,
\]

where \( A = (A_{\alpha\beta}) \in \text{SO}(3) \) is the representation matrix of \( \text{Ad}_z|_{\text{ImH}} \) with respect to the basis \((i, j, k)\). Note that \( \text{Ker} \tilde{I}_\alpha = \langle V_1 \rangle \), \( \text{Im} \tilde{I}_a = T\mathbb{H}^* \oplus \text{Ker} \eta (\alpha = 1, 2, 3) \) and that \( \tilde{I}_1, \tilde{I}_2, \tilde{I}_3 \) satisfy the quaternionic relations on \( T\mathbb{H}^* \oplus \text{Ker} \eta \).

**Lemma 3.5.** \( L_{e^R_0} \tilde{I}_\alpha = 0. \)

**Proof.** The flow \( \varphi_t : (z, p) \mapsto (e^t z, p) \) of \( e^R_0 \) preserves the decomposition \( \tilde{M} = \mathbb{H}^* \times P \) and acts trivially on the second factor. In particular, it preserves the distribution \( \text{Ker} \eta \). The action on the first factor is tri-holomorphic with respect to the (standard) hyper-complex structure induced by \( \tilde{I}_\alpha \) on \( \mathbb{H}^* \). Since \( \text{Ad}_z = \text{Ad}_z e \) for all \( r > 0 \), we also see that \( \varphi_t \) preserves the tensors \( \tilde{I}_\alpha|_{\text{Ker} \eta} \).

**Lemma 3.6.** If \( Z \) is rotating and \( L_Z \Theta = 0 \), then we have \( L_{V_1} \tilde{I}_\alpha = 0. \)

**Proof.** By the definition of \( \tilde{I}_\alpha \), it is easy to obtain \( (L_{V_1} \tilde{I}_\alpha)V_1 = 0 \) and \( (L_{V_1} \tilde{I}_\alpha)e^R_0 = 0 \) \((\delta = 0, \ldots, 3)\). Moreover, by Lemma 3.4 we have

\[
(L_{V_1} \tilde{I}_\alpha)(z, p)(Y^{h_0}) = [V_1, \tilde{I}_\alpha Y^{h_0}](z, p) - \tilde{I}_\alpha[V_1, Y^{h_0}](z, p)
\]

\[
= [e^t_1, \tilde{I}_\alpha Y^{h_0}](z, p) - [Z_1, \tilde{I}_\alpha Y^{h_0}](z, p) + \tilde{I}_\alpha[Z_1, Y]^{h_0}(z, p)
\]

\[
= [e^t_1, \tilde{I}_\alpha Y^{h_0}](z, p) - [Z^h, \tilde{I}_\alpha Y^{h_0}](z, p) - [f_1 X_P, \tilde{I}_\alpha Y^{h_0}](z, p) + (I'_{\alpha}[Z, Y])^{h_0}(z, p)
\]

\[
= [e^t_1, \tilde{I}_\alpha Y^{h_0}](z, p) - ([L_{V_1'} Y]Y^{h_0})(z, p),
\]

where we have used that \([Z^h_\eta, \tilde{I}_\alpha Y^{h_0}] + [f_1 X_P, \tilde{I}_\alpha Y^{h_0}] = [Z, I'_{\alpha} Y]^{h_0} + \eta([Z, I'_{\alpha} Y])X_P - (I'_{\alpha} Y)(f_1)X_P = [Z, I'_{\alpha} Y]^{h_0} \) at the point \((z, p)\), by Lemma 3.2 Taking the flow \( \varphi_t \) generated by \( e^t_1 \), we have

\[
[e^t_1, \tilde{I}_\alpha Y^{h_0}](z, p) = \sum_{\beta=1}^{3} \left( \frac{d}{dt} \bigg|_{t=0} A_{\alpha\beta}(t) \right) (I_\beta Y)^{h_0}(z, p),
\]

where

\[
A(t) = (A_{\alpha\beta}(t)) = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos 2t & \sin 2t \\
0 & -\sin 2t & \cos 2t
\end{pmatrix} \in \text{SO}(3),
\]

is the matrix associated with \( \varphi_t(z) \). On the other hand, we see that

\[
L_Z I'_1 = -2A_{12} I_3 + 2A_{13} I_2,
\]

\[
L_Z I'_2 = -2A_{22} I_3 + 2A_{23} I_2,
\]

\[
L_Z I'_3 = -2A_{32} I_3 + 2A_{33} I_2.
\]
and hence
\[
L_Z(I'_1, I'_2, I'_3) = (L_Z I'_1, L_Z I'_2, L_Z I'_3) = (I_1, I_2, I_3) \left( \frac{d}{dt} A(t) \right).
\]

Therefore we have \((LV_i I_\alpha)(z,\rho)(Y^{h_\alpha}) = 0\).

By Lemma \ref{lemma3.6} we can define an almost hypercomplex structure \((\hat{I}_1, \hat{I}_2, \hat{I}_3)\) on \(\hat{M}\) satisfying \(\pi_s \circ \hat{I}_\alpha = \hat{I}_\alpha \circ \pi_s\).

**Lemma 3.7.** The almost hypercomplex structure \(\hat{H} = (\hat{I}_1, \hat{I}_2, \hat{I}_3)\) is integrable, that is, \((\hat{M}, \hat{H})\) is a hypercomplex manifold.

**Proof.** Let \(\hat{X}\) and \(\hat{Y}\) be projectable vector fields on the total space of the submersion \(\pi : \hat{M} \to \hat{M}\) and denote by \(X = \pi_s \hat{X}, Y = \pi_s \hat{Y}\) their projections. Then we have \(\pi_s (N^I_\alpha (X, Y)) = N^I_\alpha (\hat{X}, \hat{Y})\), where \(N^I_\alpha\) and \(N^\hat{I}_\alpha\) are the Nijenhuis tensors of \(\hat{I}_\alpha\) and \(\hat{I}_\alpha\), respectively. Using that \(\hat{I}_\alpha V_1 = 0\) and \(L_{V_1} \hat{I}_\alpha = 0\) (Lemma \ref{lemma3.6}) we see that \(N^I_\alpha (V_1, \cdot) = 0\). Since \(N^I_\alpha\) and \(N^\hat{I}_\alpha\) are tensors, it is sufficient to show that the horizontal component of \(N^I_\alpha (A, B)\) vanishes for sections \(A\) and \(B\) of \(\langle e_0^R, e_1^R, e_2^R, e_3^R \rangle \oplus \text{Ker} \eta\). It is easy to see that \(N^I_\alpha (e_a^R, e_b^R) = 0\) and \(N^I_\alpha (e_a^R, X^{h_\alpha}) = 0\), for all \(a, b \in \{0, \ldots, 3\}\). So we only need to show that the horizontal component of \(N^I_\alpha (X^{h_\alpha}, Y^{h_\beta})\) vanishes, i.e. the component in \(\langle e_0^R, e_1^R, e_2^R, e_3^R \rangle \oplus \text{Ker} \eta\). It is given by

\[
([X, Y] + I'_\alpha [X, I'_\alpha Y] + I'_\alpha [I'_\alpha X, Y] - [I'_\alpha X, I'_\alpha Y])^{h_\alpha} = 0,
\]

since \((I'_1, I'_2, I'_3)\) is a hypercomplex structure on \(M\), for every \(z \in \mathbb{H}^*\).

Since \(L_{V_1} e_0^R = 0\), we can define a vector field \(V = \pi_s e_0^R\) on \(\hat{M}\). Let \(\hat{V}^0\) be the Obata connection with respect to \(\hat{H}\).

**Lemma 3.8.** We have \(\hat{V}^0 V = \text{id}\).

**Proof.** Using the explicit representation of the Obata connection (see \ref{section3} for example) and Lemma \ref{lemma3.5} we have

\[
12(\hat{V}^0_{\pi_s Y} \pi_s e_0^R) = \pi_s \left( \sum_{(\alpha, \beta, \gamma)} (\hat{I}_\alpha [\hat{I}_\beta Y, e_\gamma^R] + \hat{I}_\alpha [e_\beta^R, \hat{I}_\gamma Y]) + 2 \sum_{a=1}^3 \hat{I}_a [e_a^R, Y] \right),
\]

where \((\alpha, \beta, \gamma)\) indicates sum over cyclic permutations of \((1, 2, 3)\) and \(Y\) is a projectable vector field on \(\hat{M}\) commuting with \(e_0^R\). Evaluating the expression on \(Y = e_a^R\) and \(Y = U^{h_\alpha}\), we obtain \(12 \pi_s Y\).

As a consequence, by Lemmas \ref{lemma3.7} and \ref{lemma3.8} we can conclude
Theorem 3.9 (Conification). Let $M$ be a hypercomplex manifold with a hypercomplex structure $H = (I_1, I_2, I_3)$, a closed two-form $\Theta$ and a rotating vector field $Z$ such that $L_Z \Theta = 0$. Let $f$ be a smooth function on $M$ such that $df = -\iota_Z \Theta$ and assume $f_1 := f - (1/2)\Theta(Z, I_1 Z)$ does nowhere vanish. Consider a principal $U(1)$-bundle $\pi : P \to M$ with a connection form $\eta$ whose curvature form is

$$d\eta = \pi^* \left( \Theta - \frac{1}{2} d((\iota_Z \Theta) \circ I_1) \right).$$

If the quotient map $\tilde{\pi} : \tilde{M} \to \hat{M}$ is a submersion, then $(\hat{M}, \hat{H})$ is a conical hypercomplex manifold with the Euler vector field $V = \tilde{\pi}^* e^0_R$. 

Remark 3.10. The assumption that $\tilde{\pi}$ is a submersion is always satisfied locally by considering local 1-parameter subgroup generated by $V_1$, since the vector field $V_1$ has no zeros. Note that “submersion” requires that the quotient space is a smooth manifold.

We say that $(\hat{M}, \hat{H}, V)$ is the conification of $(M, H, Z, f, \Theta)$ associated with $(P, \eta)$ and denote it by $(\hat{M}, \hat{H}, V) = \mathcal{C}_P(M, H, Z, f, \Theta)$ (or simply $\hat{M} = \mathcal{C}_P(M)$ if there is no confusion).

4 The hypercomplex/quaternionic-correspondence

Building on the conification construction of the last section we will now construct a quaternionic manifold $\tilde{M}$ of dimension $\dim \tilde{M} = \dim M$ from a hypercomplex manifold $M$ with rotating vector field. The resulting quaternionic manifold is endowed with a torsion-free quaternionic connection and an affine quaternionic vector field $X$.

The space of leaves of the integrable distribution $\mathcal{D} := \langle V, I_1 V, I_2 V, I_3 V \rangle$ on $\hat{M}$ is denoted by $\tilde{M}$. We shall show that $\tilde{M} = \mathcal{C}_P(M)/\mathcal{D}$ is a quaternionic manifold, which is the main theorem of this paper. In addition, we show that $\hat{M}$ has a natural quaternionic connection $\tilde{\nabla}$ and an affine quaternionic vector field $X$ induced from the fundamental vector field $X_P$ of $P \to M$.

Using Theorem 3.9 and a similar argument as in [24, Theorem 2.1], we prove Theorem 4.1.

Theorem 4.1 (H/Q-correspondence). Let $M$ be a hypercomplex manifold with a hypercomplex structure $H = (I_1, I_2, I_3)$, a closed two-form $\Theta$ and a rotating vector field $Z$ such that $L_Z \Theta = 0$. Let $f$ be a smooth function on $M$ such that $df = -\iota_Z \Theta$ and assume that $f_1 := f - (1/2)\Theta(Z, I_1 Z)$ does nowhere vanish. Consider a principal $U(1)$-bundle $\pi : P \to M$ with a connection form $\eta$ whose curvature form is

$$d\eta = \pi^* \left( \Theta - \frac{1}{2} d((\iota_Z \Theta) \circ I_1) \right).$$

If both quotient maps $\tilde{\pi} : \tilde{M} \to \hat{M}$ and $\hat{\pi} : \hat{M} \to \tilde{M}$ defined above are submersions, then there exists an induced quaternionic structure $\hat{Q}$ on $\tilde{M}$.
Proof. As we proved in Theorem 3.9, \( \bar{M} = \mathcal{C}_p(M) \) is a conical hypercomplex manifold with the hypercomplex structure \( H = (I_1, I_2, I_3) \). Let \( \varphi = \sum_{a=0}^{3} \varphi_a i_a \) (\( (i_0, i_1, i_2, i_3) = (1, i, j, k) \)) be the right-invariant Maurer-Cartan form on \( \mathbb{H}^* \) and extend it with the same letter to \( \bar{M} \) as \( \varphi|_\mathbb{P} = 0 \). Set \( \bar{\theta}_0 = \varphi_0 \). Since \( L_{\bar{V}} \bar{\theta}_0 = 0 \), we can define the one-form \( \bar{\theta}_0 \) on \( \bar{M} \) such that \( \bar{\theta}_0 = \bar{\pi}^* \bar{\theta}_0 \). We define \( \bar{\theta}' = \bar{\theta}_0 + \sum_{a=1}^{3} (\bar{\theta}_0 \circ \bar{I}_a) i_a \) and take the Euler vector field \( \bar{V} \) on \( \bar{M} \) as in Theorem 3.9. Here define an \( \bar{I}_a \)-invariant distribution

\[
\bar{\mathcal{H}} := \text{Ker} \bar{\theta}'.
\]

It holds that \( T\bar{M} = \mathcal{D} \oplus \bar{\mathcal{H}} \). Since \( L_{\bar{V}} \bar{\theta}' = 0 \) and \( L_{\bar{I}_a V} \bar{\theta}' = 2(\bar{\theta}_0 \circ \bar{I}_a) i_a \beta - 2(\bar{\theta}_0 \circ \bar{I}_a) i_\beta \) for any cyclic permutation \( (\alpha, \beta, \gamma) \) (these are checked by straightforward calculations), the distribution \( \bar{\mathcal{H}} \) is invariant along leaves of \( \mathcal{D} \). Since \( \bar{\pi} \) is a submersion, there exist a neighborhood \( \mathcal{U} \subset \bar{M} \) of \( x \in \bar{M} \) and a section \( s : \mathcal{U} \to \bar{M} \). Then we can define

\[
\bar{I}_a(Y) := \bar{\pi}_s(\bar{I}_a(Y_{\bar{\pi}(y)})�,
\]

for \( y \in \mathcal{U} \), where \( Y \in T_y \bar{M} \) and \( Y^{h\psi} \) is the \( \bar{\theta}' \)-horizontal lift of \( Y \) with respect to \( \bar{\mathcal{H}} \). Although each \( \bar{I}_a \) depends on the sections, the subbundle \( \bar{\mathcal{Q}} = \langle \bar{I}_1, \bar{I}_2, \bar{I}_3 \rangle \subset \text{End}(T\bar{M}) \) is independent of the section by Lemma 2.2. This means that \((\bar{M}, \bar{\mathcal{Q}})\) is an almost quaternionic manifold.

Next we show that there exists a torsion-free connection which preserves \( \bar{\mathcal{Q}} \). We define a connection \( \bar{\nabla} \) on \( \bar{M} \) by

\[
(4.1) \quad \bar{\nabla}_Y W = \bar{\pi}_s(\bar{\nabla}_Y^{h\psi} W^{h\psi}), \quad Y, W \in \Gamma(T\bar{M}),
\]

where \( \bar{\nabla}^{h\psi} \) is the Obata connection of \( \bar{M} \). Note that \( \bar{\nabla} \) is well-defined by Lemma 2.3. Since the Obata connection is torsion-free, then so is \( \bar{\nabla} \). To show that \( \bar{\nabla} \) preserves \( \bar{\mathcal{Q}} \), we consider \( I \in \Gamma(\bar{\mathcal{Q}}) \). Then \( (IW)^{h\psi} = \sum_{a=1}^{3} a_{\alpha} \bar{I}_a W^{h\psi} \) for some functions \( a_{\alpha} \) with \( \sum_{a=1}^{3} a_{\alpha}^2 = 1 \), which implies

\[
(\bar{\nabla}_Y I)W = \bar{\pi}_s(\sum_{a=1}^{3} (Y^{h\psi} a_{\alpha}) \bar{I}_a W^{h\psi}),
\]

showing that \( \bar{\nabla} \) preserves \( \bar{\mathcal{Q}} \). Therefore \((\bar{M}, \bar{\mathcal{Q}}, \bar{\nabla})\) is a quaternionic manifold. \( \square \)

Remark 4.2. The assumption that \( \bar{\pi} \) is a submersion is always satisfied locally.

Next we shall show that our construction induces a vector field \( \bar{X} \) which is an affine quaternionic vector field of \((\bar{M}, \bar{\mathcal{Q}}, \bar{\nabla})\), where \( \bar{\nabla} \) is given by (4.1). \( \square \)

Lemma 4.3. We have \( L_{\bar{V}_1} X_P = 0 \) and \( L_{X_P} \bar{I}_a = 0 \).

Proof. The first equation can be checked by a straightforward calculation. The second follows from \([X_P, \bar{I}_a Y^{h\eta}] = [X_P, (\bar{I}_a Y)^{h\eta}] = 0\). \( \square \)

By Lemma 4.3 we can define a vector field \( \bar{X}_P := \bar{\pi}_s X_P \) on \( \bar{M} \). Moreover \( \bar{X}_P \) satisfies the following.
**Lemma 4.4.** We have \( L_{X_P} \hat{\theta}_0 = 0 \), in addition, \( L_{X_P} \hat{\nabla}^0 = 0 \).

*Proof.* The first claim follows from Lemma 4.3, as \((L_{X_P} \hat{\theta}_0) \circ \hat{\pi}_* = \hat{\pi}_* \circ (L_{X_P} \hat{\theta}_0)\). Since the Obata connection is uniquely determined by the hypercomplex structure, we have \( L_{X_P} \hat{\nabla}^0 = 0 \) by the invariance of the hypercomplex structure \((\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)\) under \( \hat{X}_P \). \( \square \)

The next two lemmas follow respectively from \([e^R_p, X_P] = 0\) and \( L_{X_P} \hat{\theta}_0 = 0 \) by projection.

**Lemma 4.5.** We have \( L_Y \hat{X}_P = 0 \) and \( L_{\hat{A}_V} \hat{X}_P = 0 \).

**Lemma 4.6.** We have \( L_{\hat{X}_P} \hat{\theta}_0 = 0 \) on \( \hat{M} \).

Lemma 4.5 allows us to define a vector field \( X := \hat{\pi}_* \hat{X}_P \) on \( \hat{M} \).

**Proposition 4.7.** Let \((\hat{M}, \hat{Q})\) be a quaternionic manifold obtained from a hypercomplex manifold \( M \) satisfying the assumptions in Theorem 4.1 and \( \nabla \) the quaternionic connection defined by (4.1). The vector field \( X \) is an affine quaternionic vector field of \((\hat{M}, \hat{Q}, \hat{\nabla})\), that is, satisfies \( L_X \Gamma(\hat{Q}) \subset \Gamma(\hat{Q}) \) and \( L_X \hat{\nabla} = 0 \).

*Proof.* It follows easily from Lemma 4.4 that \( X \) preserves the quaternionic structure \( \hat{Q} \). From Lemma 4.6 and the closure of \( \theta_0 \) we do also obtain that \( p_h[\hat{X}_P, Y^{h\theta}] = 0 \), where \( p_h \) and \( p_v \) denote the projections from \( T\hat{M} \) onto the horizontal and vertical subbundles, respectively. Using this, for any vector fields \( Y \) and \( W \) on \( \hat{M} \), we compute

\[
(L_X \hat{\nabla})_Y W = \hat{\pi}_* \left( [\hat{X}_P, \hat{\nabla}^0] Y^{h\theta} W^{h\phi} - \hat{\nabla}^0 p_h[\hat{X}_P, Y^{h\theta}] W^{h\phi} - \hat{\nabla}^0 Y^{h\theta} p_h[\hat{X}_P, W^{h\phi}] \right)
\]

We call the correspondence from a hypercomplex manifold \((M, H, Z, f, \Theta)\) to a quaternionic manifold \((\hat{M}, \hat{Q}, \hat{\nabla}, \hat{X})\) described in Theorem 4.1 (and Proposition 4.7 for the additional structure \( X \)) the hypercomplex/quaternionic-correspondence \((H/Q\)-correspondence\) for short. As we mentioned in Remarks 3.10 and 4.2, the global assumption in Theorem 4.1 (H/Q-correspondence) that \( \hat{\pi} \) and \( \hat{\pi} \) are submersions is always satisfied locally. Under stronger assumptions and by considering Swann’s twist [27], we have the following global result. We use the notation \( \zeta_A \) for the action induced from the group \( \langle A \rangle \) generated by a vector field \( A \) to distinguish \( U(1) \)-actions.

**Theorem 4.8** (H/Q-correspondence, second version). Let \( M \) be a hypercomplex manifold with a hypercomplex structure \( H = (I_1, I_2, I_3) \), a closed two-form \( \Theta \) and a rotating vector field \( Z \) such that \( L_Z \Theta = 0 \). Let \( f \) be a smooth function on \( M \) such that \( df = -\iota_Z \Theta \) and assume that \( f_1 := f - (1/2)\Theta(Z, I_1Z) \) does nowhere vanish. Consider a principal \( U(1) \)-bundle \( \pi : P \to M \) with a connection form \( \eta \) whose curvature form is

\[
d\eta = \pi^* \left( \Theta - \frac{1}{2} d((\iota_Z \Theta) \circ I_1) \right).
\]

If \( Z_1 = Z^{h\eta} + f_1 X_P \) generates a free \( U(1) \)-action on \( P \), then the conification \( \hat{M} \) of \( M \) is \( \mathbb{H}^n \times_{\langle V_1 \rangle} P \) and the quaternionic manifold \( \hat{M} \) coincides with the twist of \( M \) given by the twist data \( (\Theta - \frac{1}{2} d((\iota_Z \Theta) \circ I_1), Z, f_1) \) as manifolds.
Proof. By Lemma 3.2, we see \( \iota_Z d\eta = -df_1 \). It follows that \( L_Z \Theta = 0 \) and \( L_Z I_1 = 0 \). Therefore we obtain a twist \( M' := P/\langle Z_1 \rangle \) of \( M \) with the twist data \( (\Theta - \frac{1}{2} d((\iota_Z \Theta) \circ I_1), Z, f_1) \) since \( Z_1 = Z^{\theta_1} + f_1 X_F \) generates a free U(1)-action. Let \( \pi' : P \to M' \) be the quotient map by the action of \( \langle Z_1 \rangle \). We define an action of \( \langle V_1 \rangle (\cong U(1)) \subset \langle e_1^F \rangle \times \langle Z_1 \rangle \) on \( \mathbb{H}^* \times P \) by

\[
\zeta_{V_1}(u)(z, p) = (\zeta_{V_1}(u)z, \zeta_{Z_1}(u^{-1})p)
\]

for \((z, p) \in \mathbb{H}^* \times P\). We see that the conification \( \tilde{M} \) of \( M \) is a fiber bundle \( (\mathbb{H}^* \times P)/\langle V_1 \rangle \) over \( M' \), which is associated with \( \pi' : P \to M' \) and usually denoted by \( \mathbb{H}^* \times_{\langle V_1 \rangle} P \). Moreover the quotient of \( \tilde{M} \) by \( \mathbb{H}^* \) is \( M' \), that is, \( \tilde{M} = M' \).

\[
\tilde{M} = \mathbb{H}^* \times P \\
P \downarrow \pi_2 \downarrow \tilde{\pi} \\
\pi \\
\tilde{M} = \mathbb{H}^* \times_{\langle V_1 \rangle} P \\
P \downarrow \pi' \downarrow \tilde{\pi}' \\
\pi' \downarrow \tilde{\pi}' \\
\tilde{M} = \mathbb{H}^* \times P \\
\tilde{\pi} \downarrow \tilde{\pi} \\
\tilde{M} = \mathbb{H}^* \times P
\]

In the above diagram, \( \pi_2 \) is the projection onto the second factor \( P \). \( \square \)

Remark 4.9. Note that the bundle \( \tilde{\pi} : \tilde{M} \to \tilde{M} \) is associated to the principal \( U(1) \)-bundle \( P \to \tilde{M} = M' = P/\langle Z_1 \rangle \). Therefore sections of \( \tilde{\pi} \) are in one-to-one correspondence with equivariant maps \( P \to \mathbb{H}^* \). Let \( \lambda : P \to \mathbb{H}^* \) be such that \( \lambda(\zeta_{Z_1}(u)p) = \zeta_{e_1^F}(u^{-1})\lambda(p) \) for all \( u \in U(1) \) and \( p \in P \) and set \( F_{\lambda} := [\lambda, \text{id}]_{\langle V_1 \rangle} : P \to \tilde{M} \). If we consider a local section \( s : U(\subset \tilde{M} = M') \to P \), then \( s' := F_{\lambda} \circ s : U \to \tilde{M} \) is a local section of \( \tilde{\pi} : \tilde{M} \to \tilde{M} \) and the equivariance of \( \lambda \) implies that \( s' \) is independent of \( s \). As we observed in the proof of Theorem 4.1, the quaternionic structure \( \tilde{Q} = \langle \tilde{I}_1, \tilde{I}_2, \tilde{I}_3 \rangle \) on \( \tilde{M} \) is induced from the hypercomplex structure on \( \tilde{M} \) and a local section \( s' \). For \( Y \in T_x \tilde{M} \), we have

\[
\tilde{I}_\alpha(Y) = \hat{\pi}_s(\tilde{I}_\alpha Y_{s(X)}^{h_{\theta_1^F}}) = \hat{\pi}_s(\tilde{I}_\alpha s' Y),
\]

since the decomposition \( T\tilde{M} = D \oplus \mathcal{H} \) is \( \hat{I}_\alpha \)-invariant. From \( s' = F_{\lambda} \circ s = [\lambda \circ s, s]_{\langle V_1 \rangle} = \tilde{\pi} \circ (\lambda \circ s, s) \), it holds that

\[
\tilde{I}_\alpha(Y) = \hat{\pi}_s(\tilde{I}_\alpha s' Y) = \hat{\pi}_s(\tilde{I}_\alpha ((\lambda \circ s)_s(Y) + s_s Y)) = \hat{\pi}_s(\tilde{I}_\alpha ((\lambda \circ s)_s(Y) + s_s Y)) = \pi' (\pi' s(\tilde{I}_\alpha s_s Y)).
\]
Note that \((\lambda \circ s)_s(Y) + s_s Y \in T(\lambda(s(x)), s(x)) \tilde{M}\).

Next we consider the decomposition \(TP|_{s(U)} = (Z_1) \oplus s_*(TU)\). Let \(p^\vee\) be the projection from \(TP|_{s(U)}\) onto \(s_*(TU)\). Note that \(s_*(T_z U)\) is generated by the tangent vectors of the form \(p^\vee(W_{h_0}^\alpha) \coloneqq W^\vee\) at each point \(s(x)\), where \(W\) is a tangent vector of \(M\) at \(\pi(s(x))\) and \(\eta\) is the connection form on \(P\). We define (an almost hypercomplex structure) \(I^\vee_\alpha\) on \(s(U)\) by \(I^\vee_\alpha(W^\vee) = (I^\vee_\alpha W)^\vee\) for each \(W^\vee \in s_*(T_z U)\), where \(I^\vee_\alpha\) is given by (3.2) for \(z = \lambda(s(x))\). Since \(\tilde{I}_\alpha(Z_1) = \tilde{I}_\alpha(e_1^\prime) \in T\mathbb{H}^*\) (by \(\tilde{I}_\alpha V_1 = 0\)), we have
\[
(4.3) \quad p^\vee(\pi_2s(\tilde{I}_\alpha(W^\vee))) = p^\vee(\pi_2s(\tilde{I}_\alpha(W_{h_0} + aZ_1))) = p^\vee(\tilde{I}_\alpha W_{h_0}) = p^\vee((I^\vee_\alpha W)^{h_0}) = (I^\vee_\alpha W)^\vee = I^\vee_\alpha(W^\vee),
\]
where \(a \in \mathbb{R}\). Then it holds that
\[
\tilde{I}_\alpha(Y) = \pi_1s(\pi_2s(\tilde{I}_\alpha s_s Y)) = \pi_1s(p^\vee(\pi_2s(\tilde{I}_\alpha s_s Y))) = \pi_1s((I^\vee_\alpha s_s Y))
\]
from (4.2) and (4.3). Therefore \(\tilde{Q}\) can be identified with \((I^\vee_1, I^\vee_2, I^\vee_3)\) on \(s(U)\). Note that \((I^\vee_1, I^\vee_2, I^\vee_3)\) is independent of the choice of \(\lambda\), and hence it is shown again that \(\tilde{Q}\) is independent of the choice of \(\lambda\), which is identified with a section of \(\tilde{M}\).

Note that a quaternionic Kähler metric obtained by the HK/QK-correspondence is described directly in terms of the objects on \(P\) (instead of \(M\)) in [4, 21].

**Remark 4.10.** The conification \(\tilde{M}\) of \(M\) is locally isomorphic to the Swann bundle of \(M\), which is conical as discussed in Example 2.3. Note that the Swann bundle is an \(\mathbb{H}^*/\{\pm 1\}\)-bundle over a quaternionic manifold whereas \(\tilde{M}\) is the quotient of \(\tilde{M}\) by \(\mathbb{H}^*\) as above. Indeed, take an open set \(U\) of \(\tilde{M}\) and local sections \(s : U \rightarrow \tilde{M}\), \(s' : U \rightarrow \mathcal{U}(M)\), where \(\pi^{Sw} : \mathcal{U}(M) \rightarrow \tilde{M}\) is the Swann bundle of \(M\). For a local trivialization \(\Phi : \hat{\pi}^{-1}(U) \rightarrow U \times \mathbb{H}^*\) associated to \(s\) and given by \(\Phi(x) = (\hat{\pi}(x), \phi(x))\), we can define a double covering \(F : \hat{\pi}^{-1}(U) \rightarrow (\pi^{Sw})^{-1}(U)\) by
\[
F(x) = \Phi^{-1}(s'((\hat{\pi}(x)), p(\phi(x))).
\]
Here \(\Phi' : (\pi^{Sw})^{-1}(U) \rightarrow U \times \mathbb{H}^*/\{\pm 1\}\) is a local trivialization associated to \(s'\) and \(p : \mathbb{H}^* \rightarrow \mathbb{H}^*/\{\pm 1\}\) is the projection. See [21, 6] for the (twisted) Swann bundle.

## 5 Examples of the H/Q-correspondence

In this section, we give examples of the H/Q-correspondence.

**Example 5.1** (HK/QK-correspondence). Let \((M, g, H = (I_1, I_2, I_3))\) be a (possibly indefinite) hyper-Kähler manifold with a rotating Killing vector field \(Z\) and \(f\) a nowhere vanishing smooth function such that \(df = -i_\zeta \Theta\), where \(\Theta\) is the Kähler form with respect to \(g\) and \(I_1\). Set \(f_1 = f - (1/2)g(Z, Z)\) and assume that the functions \(g(Z, Z)\) and \(f_1\) are nowhere zero. From these data, we can obtain a (possibly indefinite) quaternionic Kähler manifold \((\tilde{M}, \tilde{g})\) [13, 21, 4]. The metric \(\tilde{g}\) is positive definite under the assumptions specified in [2, Corollary 2] for the signs of the functions \(f, f_1\) and for the signature of \(g\). Also the sign of the scalar curvature of \(\tilde{M}\) is determined by these choices.
In the HK/QK-correspondence, the initial data $\Theta$ is a non-degenerate 2-form. In our more general setting, we may also choose $\Theta = 0$, like in the following example.

**Example 5.2 (Conical hypercomplex manifold).** Let $(M, (I_1, I_2, I_3), V)$ be a conical hypercomplex manifold with the Euler vector field $V$. Choose $f_1 = f = 1$, $\Theta = 0$, and consider the trivial principal bundle $P = M \times U(1)$ with the connection $\eta = dt$, where $t$ is the angular coordinate of $U(1)$ such that $dt(X_P) = 1$ on the fundamental vector field $X_P$. We assume that $Z := I_1 V$ generates a free $U(1)$-action on $M$ and that the periods of $Z$, $X_P$ and $e^t_1$ are the same. It holds that $Z$ is rotating from Lemma [2.2]. Then $V_1$ generates a free $U(1)$-action on $\tilde{M} = H^* \times P = H^* \times M \times U(1)$ of the same period.

Therefore

$$\tilde{M}(= (H^* \times M \times U(1))/\langle V_1 \rangle) \ni [z, p, q] = [zq, \zeta (q^{-1})p, 1] \mapsto (zq, \zeta (q^{-1})p) \in H^* \times M$$

gives a diffeomorphism $\tilde{M} \cong H^* \times M$, and hence $\tilde{M} \cong M$ as smooth manifolds. In fact, we can define a diffeomorphism $\varphi : M \rightarrow M'(= \tilde{M})$ by $\varphi'(x) = \pi'(x, 1)$. A global section $\tilde{M} \rightarrow \tilde{M}$ gives rise to a hypercomplex structure $(\tilde{I}_1, \tilde{I}_2, \tilde{I}_3)$ on $\tilde{M}$ but the latter does not coincide with $(I_1, I_2, I_3)$ in general (under the diffeomorphism $\varphi'$). The quaternionic structure $\tilde{Q}$ on $\tilde{M}$, however, coincides with $(I_1, I_2, I_3)$. Note that $\tilde{Q}$ is independent of the section, as shown in the proof of Theorem [4.1] and Remark [4.3]. More explicitly, considering $\lambda_2 : M \times U(1) \rightarrow H^*$ defined by $\lambda_2(x, u) = z \cdot u^{-1}$ ($z \in H^*$) and the section $s : \tilde{M} \rightarrow P$ defined by $s(x) = ((\varphi')^{-1}(x), 1)$, we see that the section $F_{\lambda_1} \circ s$ gives the hypercomplex structure $(\tilde{I}_1, \tilde{I}_2, \tilde{I}_3)$ and, hence, the quaternionic structure $\langle \tilde{I}_1, \tilde{I}_2, \tilde{I}_3 \rangle$ on $\tilde{M} \cong M$.

The next example shows that our $H/Q$-correspondence is a proper generalization of the HK/QK-correspondence.

**Example 5.3 (Hypercomplex Hopf manifold).** Consider $H^n \cong R^{4n}$ as a right-vector space over the quaternions with the standard hypercomplex structure

$$\tilde{H} = (\tilde{I}_1 = t, \tilde{I}_2 = R_t, \tilde{I}_3 = \tilde{I}_1 \tilde{I}_2 = -R_t)$$

and the standard flat hyper-Kähler metric $\tilde{g}$ and set $\tilde{M} = H^n \setminus \{0\}$. Take $A \in Sp(n)Sp(1)$ and $\lambda > 1$. Then $\langle \lambda A \rangle$ is a group of homotheties which acts freely and properly discontinuously on the simply connected manifold $\tilde{M}$. The quotient space $\tilde{M}/\langle \lambda A \rangle$ inherits a quaternionic structure $\tilde{Q}$ and a quaternionic connection $\tilde{\nabla}$ which are invariant under the centralizer $G^Q$ of $\lambda A$ in $GL(n, H)Sp(1)$. In fact, the quaternionic structure $\tilde{Q}$ on $\tilde{M}$ is $GL(n, H)Sp(1)$-invariant and induces therefore an almost quaternionic structure $Q$ on $\tilde{M}/\langle \lambda A \rangle$, since $\langle \lambda A \rangle \subset GL(n, H)Sp(1)$. Moreover, the Levi-Civita connection $\tilde{\nabla}$ on $(\tilde{M}, \tilde{g})$, which coincides with the Obata connection with respect to $\tilde{H}$, is invariant under all homotheties of $\tilde{M}$. Since $\langle \lambda A \rangle$ acts by homotheties, we see that $\tilde{\nabla}$ induces a torsion-free connection $\nabla$ on $\tilde{M}/\langle \lambda A \rangle$, which preserves $Q$. This means that $Q$ is a quaternionic structure on $\tilde{M}/\langle \lambda A \rangle$. In particular, if $A \in Sp(n)$, then the quotient $\tilde{M}/\langle \lambda A \rangle$ inherits an induced hypercomplex structure $H = (I_1, I_2, I_3)$ from $\tilde{H}$, which is invariant under the centralizer $G^H$ of $\lambda A$ in $GL(n, H)$, since $\langle \lambda A \rangle$ preserves $\tilde{H}$. We say
that \((\tilde{M}/\langle \lambda A \rangle, Q)\) (resp. \((\tilde{M}/\langle \lambda A \rangle, H)\)) is a quaternionic (resp. hypercomplex) Hopf manifold. See [23, 10].

We start with a hypercomplex Hopf manifold \(M := \tilde{M}/\langle \lambda A \rangle\), where \(A \in \text{Sp}(n)\). Take \(q \in \text{Sp}(1)\) such that \(q \neq \pm 1\). The centralizer of \(q\) in \(\text{Sp}(1)\) is isomorphic to \(\text{Sp}(n)\). We consider a \(U(1)\)-action : \(z \mapsto ze^{-it}\) on \(\tilde{M}\) defined by the right multiplication of \(U(1) \cong U_q(1) \subset \text{Sp}(n)\). This action induces one on \(M\) and the corresponding vector field \(Z\) is rotating. Therefore we can apply the same procedure as in Example 5.2 under the setting \(P = M \times U_q(1)\) (resp. \(\tilde{P} = \tilde{M} \times U_q(1)\)) and \(\Theta = 0\), and we have the quaternionic manifold \(\tilde{M}(= M')\) (resp. \(\tilde{M}(= \tilde{M}')\)) by the \(H/Q\)-correspondence.

The quotient map of an action by a group \(G\) is denoted by \(\pi_G\) and the objects associated with \(\tilde{M}\) are denoted by the corresponding letters for \(M\) with \(\tilde{\phantom{M}}\), for example, the projection of the twist from \(\tilde{M} \times U_q(1)\) is denoted as \(\tilde{\pi}'\), where we use the notation of Theorem 4.8. Let \(R_q\) be the right multiplication by \(q\).

\[
\begin{align*}
\tilde{M}' &= \tilde{M} = \tilde{M} \quad \xrightarrow{\langle \lambda AR_q \rangle} \quad M' = \tilde{M} \\
\tilde{P} &= \tilde{M} \times U_q(1) \quad \xrightarrow{\langle \lambda A \rangle} \quad P = M \times U_q(1) \\
\tilde{M} &\xrightarrow{\langle \lambda A \rangle} \quad M
\end{align*}
\]

Since \(\pi' \circ \pi_{\langle \lambda A \rangle} = \pi_{\langle \lambda AR_q \rangle} \circ \tilde{\pi}'\), and \(\tilde{M}' = \tilde{M}\) is a manifold with an invariant quaternionic structure under the action of \(\langle \lambda AR_q \rangle\) (Example 5.2 and Proposition 4.7), we have

\[\tilde{M} = M' = \tilde{M}/\langle \lambda AR_q \rangle.\]

Therefore it holds that

\[M = \tilde{M}/\langle \lambda A \rangle \quad \xrightarrow{\text{H/Q}} \quad \tilde{M} = \tilde{M}/\langle \lambda AR_q \rangle.\]

In particular, we can choose \(A = E_n \in \text{Sp}(n)\). Then the centralizer \(G^n\) of \(\lambda = \lambda E_n\) is \(\mathbb{R}^{>0} \times \text{SL}(n, \mathbb{H})\). We take the subgroup \(\mathbb{R}^{>0} \times \text{Sp}(n)\) of \(G^n\), which acts transitively on \(M\). Then

\[M = (\mathbb{R}^{>0}/\langle \lambda \rangle) \times \frac{\text{Sp}(n)}{\text{Sp}(n - 1)}.\]

On the other hand, considering the subgroup \(\mathbb{R}^{>0} \times \text{Sp}(n)U_q(1)\) of the centralizer \(G^Q\) of \(\lambda R_q\), we see that

\[
(\mathbb{R}^{>0}/\langle \lambda \rangle) \times \frac{\text{Sp}(n)}{\text{Sp}(n - 1)} \quad \xrightarrow{\text{H/Q}} \quad (\mathbb{R}^{>0}/\langle \lambda \rangle) \times \frac{\text{Sp}(n)U(1)}{\text{Sp}(n - 1)\Delta U(1)}.
\]
where \( \Delta_{U(1)} \) is a diagonally embedded subgroup of \( \text{Sp}(n)U(1) \subset \text{Sp}(n)\text{Sp}(1) \) which is isomorphic to \( U(1) \). Considering the case of \( n = 2 \), we have an invariant quaternionic structure on the homogeneous space

\[
\tilde{M} = \mathbb{R}^+ \times \frac{\text{Sp}(2)U(1)}{\text{Sp}(1)\Delta_{U(1)}} = \frac{T^2 \cdot \text{Sp}(2)}{U(2)}
\]

by the H/Q-correspondence. Note that \( T^2 \times \text{Sp}(2) \) carries a hypercomplex structure and \( (T^2 \times \text{Sp}(2))/U(2) \) is a homogeneous quaternionic manifold considered in [19].

Since \( M \) is diffeomorphic to \( S^1 \times S^{4n-1} \), \( M \) cannot admit any hyper-Kähler structure. Therefore the HK/QK-correspondence cannot be applied to the hypercomplex Hopf manifold \( M \). The H/Q-correspondence is thus a proper generalization of the HK/QK one.

In the following example, the closed form \( \Theta \) is non-zero and degenerate.

**Example 5.4 (Lie group with left-invariant hypercomplex structure).** Consider \( G = \text{SU}(3) \). The Lie algebra \( \mathfrak{g} \) of \( G \) is decomposed as \( \mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1 \), where \( \mathfrak{g}_0 = \mathfrak{s}(u(1) \oplus u(2)) \cong u(1) \oplus \mathfrak{su}(2) \cong \mathbb{H} \) and \( \mathfrak{g}_1 \) is the unique complementary \( \mathfrak{g}_0 \)-module with the action of \( \mathbb{H} \) obtained from the adjoint action of \( \mathfrak{g}_0 \) [19]. Denote by \( V \in \mathfrak{g}_0 \) the vector which corresponds to \( 1 \in \mathbb{H} \). We use the same letters for left-invariant vector fields and corresponding elements of \( \mathfrak{g} \) in this example. Three complex structures \( I_1, I_2, I_3 \) on \( \mathfrak{g} \) can be defined as follows. They preserve the decomposition \( \mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1 \) and act on \( \mathfrak{g}_0 = \mathbb{H} \) by the standard hypercomplex structure \( (R_i, R_j, R_iR_j = -R_k) \). On \( \mathfrak{g}_1 \) they are defined by

\[
(5.1) \quad I_\alpha|_{\mathfrak{g}_1} = -\text{ad}_{I_\alpha V}|_{\mathfrak{g}_1}, \quad \alpha = 1, 2, 3.
\]

These structures extend to a left-invariant hypercomplex structure on \( G \) [19], which we denote again by \( (I_1, I_2, I_3) \).

Let \( G_0 \cong (U(1) \times SU(2))/\{\pm 1\} \cong U(2) \) be the subgroup of \( G \) corresponding to \( \mathfrak{g}_0 \). Note that \( G_0 \subset G \) is a hypercomplex submanifold and therefore totally geodesic with respect to the Obata connection \( \nabla^G \) of \( G \) [24]. The Obata connection \( \nabla^{G_0} \) of \( G_0 \) is given by \( \nabla^G_{X}Y = XY \) for \( X, Y \in \mathfrak{g}_0 = \mathbb{H} \), where \( XY \) denotes the product of the quaternions \( X \) and \( Y \). Indeed, \( \nabla^{G_0} \) is torsion-free and \( I_1, I_2, I_3 \) are parallel with respect to \( \nabla^{G_0} \). Then it holds \( \nabla^G_X V = \nabla^{G_0}_X V = X \) for \( X \in \mathfrak{g}_0 \). For \( X \in \mathfrak{g}_1 \), by (5.1) and the explicit expression of the Obata connection (see [3]), we also find that \( \nabla^G_X V = X \). Hence the hypercomplex manifold \((G, (I_1, I_2, I_3))\) is conical with the Euler vector field \( V \) (see also [26]).

Consider the right-action of \( U(2) \) on \( SU(3) \) given by

\[
AB := A \begin{pmatrix} B & 0 \\ 0 & \text{det}(B)^{-1} \end{pmatrix}
\]

for \( A \in SU(3) \) and \( B \in U(2) \). Let \( l : SU(3) \rightarrow SU(3)/U(2) \cong \mathbb{C}P^2 \) be the projection and \( k : S^5 \rightarrow \mathbb{C}P^2 \) the Hopf fibration. The pullback bundle \( P := l^#S^5 \) of \( k : S^5 \rightarrow \mathbb{C}P^2 \)
by \( l \) is a \( U(1) \)-bundle over \( SU(3) \). The usual identification between the Stiefel manifold \( V_2(\mathbb{C}^3) \) and \( SU(3) \) is given by
\[
V_2(\mathbb{C}^3) \ni (a_1, a_2) \leftrightarrow A = (a_1, a_2, \bar{a}_1 \times \bar{a}_2) \in SU(3).
\]
We can write
\[
P = \{ (A, u) \in SU(3) \times S^5 \mid l(A) = k(u) \}
= \{ (A, u) \in SU(3) \times S^5 \mid \langle c_3(A) \rangle = \langle u \rangle \in \mathbb{C}P^2 \}
= \{ (A, \alpha c_3(A)) \in SU(3) \times S^5 \mid \alpha \in U(1) \}
\cong SU(3) \times U(1),
\]
where \( c_3(A) \) denotes the third column of \( A \). This shows that \( P \) is a trivial bundle. Let \( l_\# : P \to S^5 \) be the bundle map given by \( l_\#(A, \alpha) = \alpha(\bar{a}_1 \times \bar{a}_2) = \alpha c_3(A) \). Consider the pullback connection \( l_\#^* \eta \) on \( P \) from the standard one \( \eta \) of \( k \) and take \( \Theta = l^* \omega \), where \( \omega \) is the Kähler form on \( \mathbb{C}P^2 \). Set \( Z := I_1 V \). We see that \( Z \) generates a \( U(1) \)-action on \( SU(3) \) and is rotating by Lemma 2.2. Since
\[
\langle Z \rangle \subset SU(2) \subset U(2),
\]
\( Z \) is tangent to the fiber of \( l \). Hence, we have \( l_\# Z \Theta = 0, L_\# Z \Theta = 0 \), and also have \( d\Theta = 0 \) by \( d\omega = 0 \). So we can choose \( f = f_1 = 1 \) (see Section 3 for the notation) and then see that \( Z_1 \) generates a free \( U(1) \)-action on \( P \) given by
\[
\zeta_{Z_1}(u)(A, \alpha) = (\zeta_Z(u)(A), u\alpha), \quad u \in U(1).
\]
To see this, it is sufficient to check that \( Z \) is horizontal with respect to the pullback connection. The vector field \( Z \) is lifted to \( SU(3) \times U(1) \) as \( Z_{(A, \alpha)} = (Z_A, 0) \in TSU(3) \times TU(1) \) for \( A \in SU(3) \) and \( \alpha \in U(1) \) with the same letter \( Z \). From \( Z \in su(2) \), it holds that
\[
l_\#_* Z_{(A, \alpha)} = \frac{d}{dt} l_\#_*((\zeta_{Z_1}(e^{it})(A, \alpha))) \bigg|_{t=0}
= \frac{d}{dt} l_\#((\zeta_{Z_1}(e^{it}))(A), \alpha)) \bigg|_{t=0}
= \frac{d}{dt} \alpha c_3(\zeta_{Z_1}(e^{it})(A)) \bigg|_{t=0}
= \frac{d}{dt} \alpha c_3(A) \bigg|_{t=0} = 0.
\]
In particular, \( (l_\#^* \eta)(Z) = 0 \), that is, \( Z \) is horizontal with respect to the pullback connection. So we see that \( Z_1 = Z + X_P \). Therefore, by applying the H/Q-correspondence to \( G = SU(3) \), we have a quaternionic manifold
\[
\bar{G} = P/\langle Z_1 \rangle = (SU(3) \times U(1))/U(1) \cong SU(3).
\]
The identification is given by

$$(SU(3) \times U(1))/U(1) \ni [(A, \alpha)]_{(Z_1)} = [(\zeta Z(\alpha^{-1})A, 1)]_{(Z_1)} \cong \zeta Z(\alpha^{-1})A \in SU(3).$$

Note that there exists no Riemannian metric $g$ on $G$ such that $g$ is hyper-Kählerian with respect to $(I_1, I_2, I_3)$ since $G$ is compact. The situation is summarized in the following diagram.

Note also that $SU(3) \times U(1)$ is a three-fold covering of $U(3) : (A, \alpha) \mapsto \alpha A$. The kernel is the cyclic group $\{(\zeta 1, \zeta^{-1}) \mid \zeta^3 = 1\}$. The principal bundle $P \to SU(3)$ induces a principal bundle $U(3) = P/\mathbb{Z}_3 \to PSU(3) = SU(3)/\mathbb{Z}_3$. The actions generated by $Z_1$ and $Z$ commutes with that of $\mathbb{Z}_3$. The vector field $Z$ (resp. $Z_1$) on $SU(3)$ (resp. $SU(3) \times U(1)$) induces one on $PSU(3)$ (resp. $U(3)$), which is denoted by the same letter $Z$ (resp. $Z_1$). We obtain the following diagram.

We can apply the $H/Q$-correspondence to the Lie group $G_1 = PSU(3)$ with the induced left-invariant hypercomplex structure and see that its resulting space is $SU(3)/\mathbb{Z}^3$. In fact, since the action of $\langle Z_1 \rangle$ on $U(3)$ is given by $\zeta Z_1(u)(\alpha A) = (ua)(\zeta Z(u)(A))$ and its orbit $\{(ua)(\zeta Z(u)(A)) \mid u \in \langle Z_1 \rangle\}$ of $\alpha A \in U(3)$ intersects $SU(3)$ at exactly three
points, then the resulting space $U(3)/\langle Z_1 \rangle$ is $SU(3)/Z^3$. Consequently, we have $G_1 \cong G_1$ again.

Next we compare the quaternionic structures on the resulting space(s) derived from the pullback connection $\eta_1$, which is not flat, and the trivial connection $\eta_0$ as in Example 5.2. Recall the notation in Remark 4.9. We claim that the two quaternionic structures are different. We label the objects obtained from $\eta_i$ by the symbol $\eta_i$ or just by the letter $i$ ($i = 0, 1$), when no confusion is possible. Since $Z^{h_{00}} = Z^{h_{10}}$, $i\bar{Z}\Theta_0 = i\bar{Z}0 = 0$ and $i\bar{Z}\Theta_1 = 0$, the vector field $Z_1$ on $P$ is $Z_1 = Z + X_P$ for both connections $\eta_0$ and $\eta_1$. Then the resulting spaces $\bar{G}^0$ and $\bar{G}^1$ coincide and we simply write $\bar{G}$ for both. Let $a$ be the 1-form on $\bar{G}$ such that $\eta_1 - \eta_0 = \pi^*a$. Consider a local section $s: \bar{G} \to P$. Since $W^{h_{01}} - W^{h_{00}} = -a(W)X_P$ for a tangent vector $W$ at $\pi(s(x)) \in \bar{G}$ (we omit the reference points of tangent vectors), we have

$$W^{\vee 1} - W^{\vee 0} = -a(W)\mathfrak{x}$$

where $\mathfrak{x} = p^\vee(X_P)$ and we recall that $W^{\vee i} = p^\vee(W^{h_{0i}})$. Therefore we see that

$$I^{\vee 1}_\alpha(W^{\vee 1}) = (I^{\vee}_\alpha W)^{\vee 1} = (I^{\vee}_\alpha W)^{\vee 0} - a(I^{\vee}_\alpha W)\mathfrak{x} = I^{\vee 0}_\alpha(W^{\vee 0}) - a(I^{\vee}_\alpha W)\mathfrak{x} = I^{\vee 0}_\alpha(W^{\vee 1}) + a(W)I^{\vee 0}_\alpha\mathfrak{x} - a(I^{\vee}_\alpha W)\mathfrak{x}.$$

On the other hand, since $W^{\vee 1} = W^{h_{01}} + cZ_1 = W^{h_{01}} + c(Z^{h_{01}} + X_P)$, we have $\eta_1(W^{\vee 1}) = c$ and $\pi_*(W^{\vee 1}) = W + cZ$. It holds that

$$(\pi^*a)(W^{\vee 1}) = a(W) + ca(Z) = a(W) + a(Z)\eta_1(W^{\vee 1}).$$

Hence we have

$$I^{\vee 1}_\alpha = I^{\vee 0}_\alpha + (\pi^*a - a(Z)\eta_1) \otimes (I^{\vee 0}_\alpha\mathfrak{x}) - ((\pi^*a - a(Z)\eta_1) \circ I^{\vee 1}_\alpha) \otimes \mathfrak{x}.$$ 

Set $\rho := \pi^*a - a(Z)\eta_1$ and $A := \rho \otimes (I^{\vee 0}_\alpha\mathfrak{x}) - (\rho \circ I^{\vee 1}_\alpha) \otimes \mathfrak{x}$. If $Q^{\vee 0} := \langle I^{\vee 1}_0, I^{\vee 0}_2, I^{\vee 0}_3 \rangle$, then $A^2 = -|A|^2 I$, where $| \cdot |$ is the norm induced from the metric on $Q^{\vee 0}$ such that $I^{\vee 0}_1, I^{\vee 0}_2, I^{\vee 0}_3$ are orthonormal. As the rank of $A$ is at most 2, this is only possible if $A = 0$. This implies $\rho = \pi^*a - a(Z)\eta_1 = 0$, which is equivalent to $a = 0$. By Remark 4.9, the quaternionic structure $Q^i$ can be identified with $Q^{\vee i}$ ($i = 0, 1$). Then we see that $\bar{Q}^0 \neq \bar{Q}^1$ since $\eta_0 \neq \eta_1$. This proves the claim.

6 The tangent bundle of a special complex manifold and a generalization of the rigid c-map

In this section, we consider a generalization of the rigid c-map \[9, 14, 8\]. The generalization associates a hypercomplex manifold $M$, the Obata connection of which is Ricci-flat, with a special complex manifold. In the case of a \textit{conical} special complex manifold, we
shall show that the hypercomplex manifold carries a canonical rotating vector field $Z^M$ (Lemma 6.1), such that we can apply our H/Q correspondence. Consequently, we shall construct a quaternionic manifold from a conical special complex manifold as the generalized supergravity c-map (Theorem 8.3). We start with defining a class of manifolds generalizing conical special Kähler manifolds [3, 21].

**Definition 6.1.** A special complex manifold $(N, J, \nabla)$ is a complex manifold $(N, J)$ endowed with a torsion-free flat connection $\nabla$ such that the $(1,1)$-tensor field $\nabla J$ is symmetric. A conical special complex manifold $(N, J, \nabla, \xi)$ is a special complex manifold $(N, J, \nabla)$ endowed with a vector field $\xi$ such that

- $\nabla \xi = \text{id}$ and
- $L_\xi J = 0$ or, equivalently, $\nabla_\xi J = 0$.

The connection $\nabla$ is called the **special connection**. To see that $L_\xi J = 0$ is equivalent to $\nabla_\xi J = 0$ it suffices to write $L_\xi \xi = \nabla \xi \xi = \nabla \xi - \text{id}$, using that $\nabla$ is torsion-free and $\nabla \xi = \text{id}$. We also note that the integrability of $J$ follows from the symmetry of $\nabla J$ since $\nabla$ is torsion-free. We set $A := \nabla J$.

**Lemma 6.2.** For every conical special complex manifold, we have $L_\xi J = A \xi = 0$.

**Proof.** Based on the symmetry of $\nabla J$, we compute

$$A_\xi A = A(J \xi) = -J(A \xi) = -JA \xi = 0.$$  

Using this and the properties listed in Definition 6.1, we then obtain

$$(L_\xi J)X = -A_{JX} \xi + JA_X \xi = 0$$

for all $X \in \Gamma(TN)$. Note that in the last step we have used the symmetry of $A = \nabla J$. \hfill $\square$

Next we consider the tangent bundle $TN =: M$ of a special complex manifold $(N, J, \nabla)$. We can define the $\nabla$-horizontal lift $X^{hv}$ and the vertical lift $X^v$ of $X \in \Gamma(TN)$. See [7] for example. The $C^\infty(M)$-module $\Gamma(TM)$ is generated by vector fields of the form $X^{hv} + Y^v$, where $X, Y \in \Gamma(TN)$. On $M$, we define a triple of $(1,1)$-tensors $(I_1, I_2, I_3)$ by

$$I_1(X^{hv} + Y^v) = (JX)^{hv} - (JY)^v,$$

$$I_2(X^{hv} + Y^v) = Y^{hv} - X^v,$$

$$I_3(X^{hv} + Y^v) = (JY)^{hv} + (JX)^v$$

for $X^{hv} + Y^v \in TM$. Note that $(I_1, I_2, I_3)$ is an almost hypercomplex structure. In fact, it is easy to see $I_2^2 = -\text{id}$ and

$$(I_1 \circ I_2)(X^{hv} + Y^v) = I_1(Y^{hv} - X^v) = (JY)^{hv} + (JX)^v = I_3(X^{hv} + Y^v),$$

$$(I_2 \circ I_1)(X^{hv} + Y^v) = I_2((JX)^{hv} - (JY)^v) = -(JY)^{hv} - (JX)^v = -I_3(X^{hv} + Y^v)$$

for $X^{hv} + Y^v \in TM$. Note that it holds

$$[X^{hv}, Y^{hv}] = [X, Y]^{hv}, [X^{hv}, Y^v] = (\nabla_X Y)^v, [X^v, Y^v] = 0$$

for $X, Y \in \Gamma(TN)$. 

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Lemma 6.3. For every special complex manifold \((N,J,\nabla)\), the canonical almost hypercomplex structure \((I_1, I_2, I_3)\) on \(M = TN\) is integrable, that is, \((M, (I_1, I_2, I_3))\) is a hypercomplex manifold.

Proof. Thanks to (6.4), the Nijenhuis tensors of \(I_1\) and \(I_2\) can be easily calculated and we find the following. Using that \(J\) is integrable, \(\nabla\) is flat and \(\nabla J\) is symmetric, we see that \(I_1\) is integrable. Because \(\nabla\) is flat and torsion-free, \(I_2\) is integrable. The integrability of \(I_3\) follows from that of \(I_1\) and \(I_2\) [5, Theorem 3.2].

We define a connection \(\nabla'\) by
\[
\nabla' := \nabla - \frac{1}{2}J(\nabla J) = \nabla - \frac{1}{2}JA.
\]
Then we see that \(\nabla'J = 0\) and \(\nabla'\) is torsion-free for every special complex manifold. Moreover, when the special complex manifold is conical, it holds that \(\nabla'\xi = \nabla\xi = \text{id}\).

Lemma 6.4. For every special complex manifold \((N,J,\nabla)\), we have
\[
R_{X,Y}^{\nabla'} = -\frac{1}{4}[A_X, A_Y]
\]
for \(X, Y \in TN\).

Proof. Set \(S = -(1/2)J(\nabla J)\). Since \(\nabla\) is flat, we see that the curvature \(R^{\nabla'}\) of \(\nabla'\) is given by
\[
R_{X,Y}^{\nabla'} = (\nabla_X S)_Y - (\nabla_Y S)_X + [S_X, S_Y]
\]
for \(X, Y \in TN\). By
\[
(\nabla_X S)_Y - (\nabla_Y S)_X = -\frac{1}{2}[A_X, A_Y] - \frac{1}{2}J(R_{X,Y}^{\nabla'}),
\]
\[
[S_X, S_Y] = \frac{1}{4}[A_X, A_Y],
\]
we have the conclusion. \(\square\)

Hence a special complex manifold admits the complex connection \(\nabla'\) such that \(R^{\nabla'}\) is of type \((1, 1)\). In fact, it follows from \(A_{JX} = -JA_X\) for all \(X \in TN\). The following theorem is a generalization of the rigid c-map in the absence of a metric.

Theorem 6.5 (Generalized rigid c-map). The tangent bundle of any special complex manifold \((N,J,\nabla)\) carries a canonical hypercomplex structure, defined by (6.1)-(6.3), and the Obata connection of the hypercomplex manifold \((M = TN, (I_1, I_2, I_3))\) is Ricci flat.

Proof. The integrability of the canonical almost hypercomplex structure defined by (6.1)-(6.3) was proven in Lemma 6.3. Let \(\tilde{\nabla}^0\) be its Obata connection. Using the explicit expression of the Obata connection, we have
\[
\tilde{\nabla}^0_{X^{hv}Y^{hv}} = (\nabla_X Y)^{hv}, \quad \tilde{\nabla}^0_{U^vX^{hv}} = -\frac{1}{2}(JA_X U)^v = -\frac{1}{2}(JA_UX)^v
\]
\[
\tilde{\nabla}^0_{X^{hv}}U^v = (\nabla_X U)^v, \quad \tilde{\nabla}^0_{U^vV^v} = \frac{1}{2}(JA_VU)^hv = \frac{1}{2}(JA_UV)^hv.
\]
for $X, Y, U, V, W \in \Gamma(TN)$. It can be also checked directly, using by (6.1)-(6.4), that the above formulas for $\tilde{\nabla}^0$ on horizontal and vertical lifts extend uniquely to a torsion-free connection $\tilde{\nabla}^0$ for which $I_1, I_2, I_3$ are parallel. We see that the bundle projection from $(TN, \tilde{\nabla}^0)$ onto $(N, \nabla^v)$ is an affine submersion [1]. Again, a straightforward calculation (or the fundamental equations of an affine submersion) gives

\[
R_{U^v, V^v}^{\tilde{\nabla}^0} W^v = -\frac{1}{4} (A_U A_V W)^v + \frac{1}{4} (A_V A_U W)^v = (R_{U^v, V^v}^{\nabla^v} W)^v,
\]

\[
R_{U^v, V^v}^{\tilde{\nabla}^0} X^{hv} = -\frac{1}{4} (A_U A_V X)^{hv} + \frac{1}{4} (A_V A_U X)^{hv} = (R_{U^v, V^v}^{\nabla^v} X)^{hv},
\]

\[
R_{U^v, X^{hv}}^{\tilde{\nabla}^0} Y^v = -\frac{1}{2} (J(H_{U^v, X}^\nabla) X)^{hv} - \frac{1}{4} (A_X A_U V)^{hv} - \frac{1}{4} (A_U A_X V)^{hv},
\]

\[
R_{U^v, X^{hv}}^{\tilde{\nabla}^0} Y^{hv} = \frac{1}{2} (J(H_{X, Y}^\nabla) U)^v + \frac{1}{4} (A_X A_U V)^v + \frac{1}{4} (A_U A_X Y)^v,
\]

\[
R_{X^{hv}, Y^{hv}}^{\tilde{\nabla}^0} U^v = (R_{X^{hv}, Y^{hv}}^{\nabla^v} U)^v,
\]

\[
R_{X^{hv}, Y^{hv}}^{\tilde{\nabla}^0} Z^{hv} = (R_{X^{hv}, Z}^{\nabla^v})^{hv}
\]

for $X, Y, Z, U, V, W \in TN$, where $H^\nabla$ is the Hessian (the second covariant derivative) with respect to $\nabla$ and we have used Lemma 6.4. Note that $(H_{X, Y}^\nabla)(Z) = (H_{X, Z}^\nabla)(Y)$ for all $X, Y, Z \in TN$, since $\nabla J$ is symmetric. Hence the flatness of $\nabla$ means that the Hessian of $J$ with respect to $\nabla$ is totally symmetric. By these equations, the Ricci tensor of $\tilde{\nabla}^0$ satisfies

\[
Ric^{\tilde{\nabla}^0}(X^{hv}, Y^{hv}) = \frac{1}{2} \text{Tr}(H_{X, Y}^\nabla) + \frac{1}{2} \text{Tr} A_X A_Y,
\]

\[
Ric^{\tilde{\nabla}^0}(X^{hv}, U^v) = Ric^{\tilde{\nabla}^0}(U^v, X^{hv}) = 0,
\]

\[
Ric^{\tilde{\nabla}^0}(U^v, V^v) = \frac{1}{2} \text{Tr}(H_{U^v, V^v}^\nabla) + \frac{1}{2} \text{Tr} A_U A_V
\]

for $X, Y, U, V \in TN$. From $(\nabla J)J = -J(\nabla J)$, it holds that

\[
\text{Tr}(H_{X, Y}^\nabla) + \text{Tr} A_X A_Y = 0
\]

for all $X, Y \in TN$. Therefore the Obata connection of the manifolds obtained from our hypercomplex version of the c-map is Ricci flat.

\[\square\]

**Remark 6.6.** The horizontal distribution on $M$ is integrable by [6.4] and each leaf is totally geodesic with respect to the Obata connection $\tilde{\nabla}^0$, since $\tilde{\nabla}^0_{X^{hv}} Y^{hv} = (\nabla^v_X Y)^{hv}$ for $X, Y \in \Gamma(TN)$.

**Remark 6.7.** In [12] Theorem A1, a hypercomplex structure was obtained on a neighborhood of the zero section of the tangent bundle of a complex manifold with a complex connection whose curvature is of type $(1,1)$. By contrast, our generalized rigid c-map gives a hypercomplex structure on the whole tangent bundle when the manifold is special complex.
7 The c-projective structure on a projective special complex manifold

In this section, we discuss projective special complex manifolds and obtain the c-projective Weyl curvature of a canonically induced c-projective structure. Let $(N, J, \nabla, \xi)$ be a conical special complex manifold. Since $L_\xi J = 0$ and $L_J \xi J = 0$, we obtain a complex structure $\bar{J}$ on the quotient $\bar{N} := N/\langle \xi, J\xi \rangle$ if $\bar{N}$ is a smooth manifold.

Lemma 7.1. We have $L_\xi \nabla' = 0$ and $L_J \xi \nabla' = 0$.

Proof. By Lemmas 6.4 and 6.2, we have

$$(L_\xi \nabla')_X Y = [\xi, \nabla'_X Y] - \nabla'_{[\xi, X]} Y - \nabla'_X [\xi, Y]$$

$$= \nabla'_\xi \nabla'_X Y - \nabla'_X \nabla'_Y \xi - \nabla'_{[\xi, X]} Y - \nabla'_X \nabla'_Y \xi + \nabla'_X \nabla'_Y \xi$$

$$= R_{\xi, X} Y = 0$$

and

$$(L_J \xi \nabla')_X Y = [J\xi, \nabla'_X Y] - \nabla'_{[J\xi, X]} Y - \nabla'_X [J\xi, Y]$$

$$= \nabla'_{J\xi} \nabla'_X Y - \nabla'_X \nabla'_Y J\xi - \nabla'_{[J\xi, X]} Y - \nabla'_X \nabla'_Y J\xi + \nabla'_X \nabla'_Y J\xi$$

$$= R_{J\xi, X} Y = 0$$

for all $X, Y \in \Gamma(TN)$.

Recall [17] that a smooth curve $t \mapsto c(t)$ on a complex manifold $(M, J)$ is called $J$-planar with respect to a connection $\nabla$ if $\nabla_c c' \in \text{span}\{c', Jc'\}$. We say that torsion-free complex connections $\nabla^1$ and $\nabla^2$ on a complex manifold $(M, J)$ are c-projectively related [8] if they have the same $J$-planar curves. It is known that $\nabla^1$ and $\nabla^2$ are c-projectively related if and only if there exists a one-form $\theta$ on $M$ such that

$$\nabla^1_X Y = \nabla^2_X Y + \theta(X) Y + \theta(Y) X - \theta(JX) JY - \theta(JY) JX$$

for $X, Y \in \Gamma(TM)$. See [17] for example. This defines an equivalence relation on the space of torsion-free complex connections on $M$. The equivalence classes are called c-projective structures.

Definition 7.2. We call the complex manifold $(\bar{N}, J)$ a projective special complex manifold if $p_N: (N, J, \nabla, \xi) \to (\bar{N}, \bar{J})$ is a principal $\mathbb{C}^\times$-bundle, where the principal $\mathbb{C}^\times$-action is generated by the holomorphic vector field $\xi - \sqrt{-1} J\xi$.

Note that a projective special Kähler manifold is a Kähler quotient of a conical special Kähler manifold. Similarly, a projective special complex manifold carries an induced c-projective structure as follows.

Proposition 7.3. Any projective special complex manifold $(\bar{N}, \bar{J})$ carries a canonical c-projective structure.
Proof. Consider a connection form \( \hat{\alpha} = \alpha - \sqrt{-1}(\alpha \circ J) \) of type (1, 0) on the principal \( \mathbb{C}^* \)-bundle \( p_N : N \to \hat{N} \). (Note that any \( \mathbb{C}^* \)-invariant real one-form \( \alpha \) such that \( \alpha(\xi) = 1 \) is the real part of such a connection.) We have \( TN = \text{Ker} \hat{\alpha} \oplus (\xi, J\xi) \), where \( \text{Ker} \hat{\alpha} \) is \( J \)-invariant. We denote the \( \hat{\alpha} \)-horizontal lift of \( X \in \Gamma(T\hat{N}) \) by \( X^{\hat{\alpha}} \). By Lemma 7.4 we can define \( \nabla^{\alpha} \) by

\[
\nabla^{\alpha}_{X}Y = p_{N^*}(\nabla'_{X^{\hat{\alpha}}}Y^{\hat{\alpha}})
\]

for \( X, Y \in \Gamma(T\hat{N}) \). We claim that \( \nabla^{\alpha}J = 0 \). In fact, using that \( JY^{\hat{\alpha}} = (JY)^{\hat{\alpha}} \) for \( Y \in T\hat{N} \) we have

\[
\nabla^{\alpha}_{X}(JY) = p_{N^*}(\nabla'_{X^{\hat{\alpha}}}JY^{\hat{\alpha}}) = p_{N^*}J(\nabla'_{X^{\hat{\alpha}}}Y^{\hat{\alpha}}) = J(p_{N^*}(\nabla'_{X^{\hat{\alpha}}}Y^{\hat{\alpha}})).
\]

To show that the \( c \)-projective structure \( [\nabla^{\alpha}] \) does not depend on \( \alpha \), we consider another connection form \( \hat{\beta} = \beta - \sqrt{-1}(\beta \circ J) \) of type \( (1, 0) \). Then there exist one-forms \( \theta_0 \) and \( \theta_1 \) on \( \hat{N} \) such that

\[
\hat{\beta} - \hat{\alpha} = (p_{N^*}\theta_0) + (p_{N^*}\theta_1)\sqrt{-1}.
\]

On the other hand, we can write \( X^{\hat{\alpha}} - X^{\hat{\beta}} = a\xi + bJ\xi \) for some functions \( a, b \) on \( N \). It is easy to see that

\[
a = \theta_0(X) \circ p_N, \ b = -\theta_0(JX) \circ p_N, \ \theta_1 = -\theta_0 \circ J
\]

for \( X \in T\hat{N} \). By the definition (7.1) of the induced connection on \( \hat{N} \), we have

\[
\nabla^{\alpha}_{X}Y = p_{N^*}(\nabla'_{X^{\hat{\alpha}}}Y^{\hat{\alpha}})
\]

for \( X, Y \in \Gamma(T\hat{N}) \), which means that \( \nabla^{\alpha} \) and \( \nabla^{\beta} \) are \( c \)-projectively related. Here we write \( \theta_0(X) \) for \( \theta_0(X) \circ p_N \) etc. \( \blacksquare \)

We denote the induced \( c \)-projective structure given in Proposition 7.3 by \( \mathcal{P}_{\nabla} \) (without a label \( \alpha \)). Next we prove that the \( c \)-projective Weyl curvature of \( \mathcal{P}_{\nabla} \), is of type \((1, 1)\) (see Theorem 7.10).

Note that \( \xi, J\xi \) are the fundamental vector fields generated by \( 1, \sqrt{-1} \in \mathbb{C} = \text{Lie} \mathbb{C}^* \), respectively. Recall that \( A = \nabla J \) and \( A_\xi = A_{J\xi} = 0 \). We also have that \( L_\xi A = 0 \), since \( L_\xi \nabla = 0 \) and \( L_\xi J = 0 \).

Lemma 7.4. \( L_{J\xi} \nabla = A, \ L_{J\xi} A = -2JA \) and \( L_{J\xi}(JA) = 2A \).
Let η be a connection form on the principal bundle \( p_N : N \to \tilde{N} \). As before, we assume that \( \text{Ker} \eta \) is \( J \)-invariant or, equivalently, that \( \eta \) is of type \((1,0)\) (but not necessarily holomorphic). Using \( \eta \) we can project the connection \( \nabla' \) on \( N \) to a connection \( \nabla'' \) on \( \tilde{N} \), which is complex with respect to \( \tilde{J} \), as shown in the proof of Proposition \[7.3\]. Note that the quotient \( p_N : (N, \nabla') \to (\tilde{N}, \nabla'') \) is an affine submersion with the horizontal distribution \( \mathcal{H} := \text{Ker} \eta \) (in the sense defined in \[1\]). From now on the \( \eta \)-horizontal lift of \( X \in TN \) is denoted by \( \tilde{X} \). Note that our sign convention for the curvature tensor is different from the one in \[1\]. Let \( h : TN \to \mathcal{H} \) and \( v : TN \to V \) be the projections with respect to the decomposition \( TN = \mathcal{H} \oplus V \), where \( V = \text{Ker} p_{N*} \). We define the fundamental tensors \( A_{\nabla'} \) and \( T_{\nabla'} \) by

\[
A_{\nabla'}^E F = v(\nabla'_{h_E}hF) + h(\nabla'_{h_E}vF)
\]

and

\[
T_{\nabla'}^E F = h(\nabla'_{v_E}vF) + v(\nabla'_{v_E}hF)
\]

for \( E, F \in \Gamma(TN) \).

**Lemma 7.5.** We have \( T_{\nabla'} = 0 \), \( A_{\nabla'}^X \xi = X \) and \( A_{\nabla'}^X J\xi = JX \) for any horizontal vector \( X \).

Let \( a \) and \( b \) be \((0,2)\)-tensors defined by

\[
A_{\nabla'}^X Y = a(X, Y)\xi + b(X, Y)J\xi
\]

for horizontal vectors \( X \) and \( Y \). Since \( \nabla' \) and the projections \( v, h \) are \( \mathbb{C}^* \)-invariant, \( A_{\nabla'} \) is \( \mathbb{C}^* \)-invariant, and hence, \( a = p_N^*\tilde{a} \) and \( b = p_N^*\tilde{b} \) for some tensors \( \tilde{a} \) and \( \tilde{b} \) on \( \tilde{N} \).

For any \((0,2)\)-tensor \( k \) on a complex manifold with a complex structure \( J \), define the \((0,2)\)-tensor \( k_J \) by \( k_J(X, Y) := k(X, JY) \).

**Lemma 7.6.** We have

\[
v((\nabla'_X J)\tilde{Y}) = (\tilde{a}(X, \tilde{J}Y) + \tilde{b}(X, Y)) \xi + (\tilde{b}(X, \tilde{J}Y) - \tilde{a}(X, Y)) J\xi
\]

for \( X, Y \in T\tilde{N} \).

**Lemma 7.7.** We have \( \tilde{b}(X, Y) = -\tilde{a}(X, \tilde{J}Y) = -\tilde{a}_J(X, Y) \) for tangent vectors \( X \) and \( Y \) on \( \tilde{N} \). Consequently, the fundamental tensor \( A_{\nabla'} \) satisfies

\[
(7.2) \quad A_{\nabla'}^X \tilde{Y} = \tilde{a}(X, Y)\xi - \tilde{a}_J(X, Y)J\xi
\]

for tangent vectors \( X, Y \) on \( \tilde{N} \).

**Proof.** By \( \nabla' J = 0 \) and Lemma \[7.6\] we have the conclusion. \( \square \)

Let \((r, \theta)\) be the polar coordinates with respect to a (smooth) local trivialization \( p_N^{-1}(U) \cong U \times \mathbb{C}^* \) of the principal \( \mathbb{C}^* \)-bundle \( p_N : N \to \tilde{N} \) such that \( \xi = r\partial/\partial r \) and \( J\xi = \partial/\partial \theta \). A principal connection \( \eta \) is locally given by

\[
\eta := \eta_1 \otimes 1 + \eta_2 \otimes \sqrt{-1} = p_N^*(\gamma_1 \otimes 1 + \gamma_2 \otimes \sqrt{-1}) + \left( \frac{dr}{r} \otimes 1 + d\theta \otimes \sqrt{-1} \right)
\]

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for a $\mathbb{C}$-valued one-form $\gamma_1 \otimes 1 + \gamma_2 \otimes \sqrt{-1}$ on $\tilde{U} \subset \tilde{N}$. For each local trivialization $p_N^{-1}(U) \cong \tilde{U} \times \mathbb{C}^*$, we set

$$B := e^{2\theta} A \ (e^{2\theta} = (\cos 2\theta)\text{id} + (\sin 2\theta)J).$$

The symmetric (1, 2)-tensor field $B$ is defined \textit{locally} and $B$ is projectable by Lemma 7.8, i.e. horizontal (i.e. $B_\xi = B_{J\xi} = 0$) and $\mathbb{C}^*$-invariant. Therefore we obtain an induced \textit{locally} defined symmetric tensor field $\tilde{B}$ on $\tilde{N}$.

\textbf{Lemma 7.8.} The tensor $B^2 : (X, Y) \mapsto B_X \circ B_Y$ is a globally defined tensor field on $N$, in particular, $[B, B]$ is so. As a consequence, we have the globally defined tensor fields $B^2$ and $[B, B]$ on $\tilde{N}$.

\textit{Proof.} It follows from $B^2 = A^2$. \hfill \qed

For a $(0, 2)$-tensor $a$ and a $(1, 1)$-tensor $K$, we define an $\text{End}(TN)$-valued 2-form $a \wedge K$ by $$(a \wedge K)_{X,Y} Z = a(X, Z)KY - a(Y, Z)KX$$

for tangent vectors $X, Y$ and $Z$.

\textbf{Proposition 7.9.} The curvature $R_{\tilde{\nabla}^n}$ of $\tilde{\nabla}^n$ is of the form

$$R_{\tilde{\nabla}^n} = -\frac{1}{4}[\tilde{B}, \tilde{B}] + 2\mathfrak{a}^a \otimes \text{Id} - 2(\tilde{a}_j)^a \otimes \tilde{J} + \tilde{a} \wedge \text{Id} - \tilde{a}_j \wedge \tilde{J},$$

where $(\cdot)^a$ denotes anti-symmetrization. Moreover we have $d\gamma_1 = -2\tilde{a}^a$ and $d\gamma_2 = 2(\tilde{a}_j)^a$.

\textit{Proof.} By the fundamental equation for an affine submersion [1], we have

$$(R_{\tilde{\nabla}^n} Z)_{\tilde{\nabla}^n} = \tilde{h}(B_{X,Y} \tilde{Z}) + \tilde{h}(\tilde{\nabla}'_{v[\tilde{X}, \tilde{Y}]} \tilde{Z}) + A_{X}^\nabla \cdot A_{\tilde{Y}}^\nabla \tilde{Z} - A_{X}^\nabla \cdot A_{\tilde{Y}}^\nabla \tilde{Z}$$

for $X, Y, Z \in \Gamma(TN)$. Since

$$v[\tilde{X}, \tilde{Y}] = \eta_1([\tilde{X}, \tilde{Y}]) \xi + \eta_2([\tilde{X}, \tilde{Y}]) J\xi$$

$$= -(d\eta_1)(\tilde{X}, \tilde{Y}) \xi - (d\eta_2)(\tilde{X}, \tilde{Y}) J\xi$$

$$= -(d\gamma_1)(X, Y) \xi - (d\gamma_2)(X, Y) J\xi,$$

we have

$$h(\tilde{\nabla}'_{v[\tilde{X}, \tilde{Y}]} \tilde{Z}) = h(\tilde{\nabla}'_{\tilde{Z}} v[\tilde{X}, \tilde{Y}])$$

$$= h(\tilde{\nabla}'_{\tilde{Z}}(-(d\gamma_1)(X, Y) \xi - (d\gamma_2)(X, Y) J\xi))$$

$$= -(d\gamma_1)(X, Y) \tilde{Z} - (d\gamma_2)(X, Y) (J\tilde{Z}).$$

Moreover, by

$$A_{\tilde{X}}^\nabla \tilde{Y} - A_{\tilde{Y}}^\nabla \tilde{X} = v[\tilde{X}, \tilde{Y}] = -d\gamma_1(X, Y) \xi - d\gamma_2(X, Y) J\xi,$$

we have $d\gamma_1 = -2\tilde{a}^a$ and $d\gamma_2 = 2(\tilde{a}_j)^a$. \hfill \qed
From these equations, it follows that we obtain

\begin{equation}
(7.5) \quad \bar{a} = \frac{1}{8(n+1)} \mathcal{B} - \frac{1}{2} P^{\nabla^c}. \tag{7.5}
\end{equation}

Now we set \( \dim N = 2(n+1) \). By Proposition 7.3 and \( \text{Tr} \bar{B}_X = 0 \) for all \( X \in TN \), we obtain

\begin{equation}
(7.3) \quad \text{Ric}^{\nabla^c}(Y, Z) = \frac{1}{4} \text{Tr} \bar{B}_Y \bar{B}_Z + (\bar{a}(Z, Y) - \bar{a}(Y, Z))
- (\bar{a}(\bar{J}Y, \bar{J}Z) + \bar{a}(Y, Z)) - 2n\bar{a}(Y, Z) + \bar{a}(Y, Z) - \bar{a}(\bar{J}Y, \bar{J}Z)
= \frac{1}{4} \text{Tr} \bar{B}_Y \bar{B}_Z - (2n+1)\bar{a}(Y, Z) + \bar{a}(Z, Y)
- \bar{a}(\bar{J}Y, \bar{J}Z) - \bar{a}(\bar{J}Z, JY).
\end{equation}

We define a \((0, 2)\)-tensor \( P^D \) on a complex manifold \((M, J)\), which is called the Rho tensor of a connection \( D \), by

\[
P^D(X, Y) = \frac{1}{m+1} \left( \text{Ric}^D(X, Y) + \frac{1}{m-1} \left( \left( (\text{Ric}^D)^s(X, Y) - (\text{Ric}^D)^s(JX, JY) \right) \right) \right),
\]

for \( X, Y \in TM \), where \( 2m = \dim M \geq 4 \), \( \text{Ric}^D \) is the Ricci tensor of \( D \) and \((\cdot)^s\) is the symmetrization of a \((0, 2)\)-tensor. The c-projective Weyl curvature \( W^{c, |\bar{D}|} \) of a c-projective structure \([\bar{D}]\) is given by

\[
(7.4) \quad W^{c, |\bar{D}|} = R^\bar{D} + (P^D)^a \otimes \text{Id} - (P^\bar{D})^a \otimes \bar{J} + \frac{1}{2} P^D \wedge \text{Id} - \frac{1}{2} P^\bar{D} \wedge \bar{J}.
\]

See [8]. We shall compute the c-projective Weyl curvature of \([\nabla^c]^\eta\). From (7.3) it holds

\[
(\text{Ric}^{\nabla^c})^s(Y, Z) = \frac{1}{4} \text{Tr} \bar{B}_Y \bar{B}_Z - 2n\bar{a}^s(Y, Z) - 2\bar{a}^s(\bar{J}Y, \bar{J}Z),
\]

\[
(\text{Ric}^{\nabla^c})^s(\bar{J}Y, \bar{J}Z) = \frac{1}{4} \text{Tr} \bar{B}_Y \bar{B}_Z - 2n\bar{a}^s(\bar{J}Y, \bar{J}Z) - 2\bar{a}^s(Y, Z)
\]

and hence

\[
(\text{Ric}^{\nabla^c})^s(Y, Z) - (\text{Ric}^{\nabla^c})^s(\bar{J}Y, \bar{J}Z) = -2(n-1) \left( \bar{a}^s(Y, Z) - \bar{a}^s(\bar{J}Y, \bar{J}Z) \right).
\]

From these equations, it follows that

\[
(n+1)P^{\nabla^c}(Y, Z) = \frac{1}{4} \text{Tr} \bar{B}_Y \bar{B}_Z - (2n+1)\bar{a}(Y, Z) + \bar{a}(Z, Y) - \bar{a}(\bar{J}Y, \bar{J}Z) - \bar{a}(\bar{J}Z, \bar{J}Y)
- 2(\bar{a}^s(Y, Z) - \bar{a}^s(\bar{J}Y, \bar{J}Z))
= \frac{1}{4} \text{Tr} \bar{B}_Y \bar{B}_Z - (2n+1)\bar{a}(Y, Z) + \bar{a}(Z, Y) - (\bar{a}(Y, Z) + \bar{a}(Z, Y))
= \frac{1}{4} \text{Tr} \bar{B}_Y \bar{B}_Z - 2(n+1)\bar{a}(Y, Z).
\]

Setting \( \bar{B}(Y, Z) = \text{Tr} \bar{B}_Y \bar{B}_Z \), which is a symmetric, \( \bar{J} \)-hermitian globally defined \((0, 2)\)-tensor on \( N \), we have

\[
(7.5) \quad \bar{a} = \frac{1}{8(n+1)} \bar{B} - \frac{1}{2} P^{\nabla^c}.
\]
Therefore the coefficients of the curvature form $d\eta = d\gamma_1 + \sqrt{-1}d\gamma_2 = -2\bar{a}^a + 2\sqrt{-1}(\bar{a}_j)^a$ are determined by

\begin{align}
\bar{a}^a &= -\frac{1}{2} (P^{\tilde{\nabla}'\eta})^a \left( -\frac{1}{2(n+1)} (Ric^{\tilde{\nabla}'\eta})^a \right), \\
(\bar{a}_j)^a &= \frac{1}{8(n+1)} \bar{B}_j - \frac{1}{2} (P^{\tilde{\nabla}'\eta})^a \left( \frac{1}{8(n+1)} \bar{B}_j - \frac{1}{2(n+1)} (Ric^{\tilde{\nabla}'\eta})^a \right).
\end{align}

By the above calculations we arrive at the following theorem.

**Theorem 7.10.** Let $(N, J, \nabla, \xi)$ be a conical special complex manifold which is the total space of a (holomorphic) principal $\mathbb{C}^*$-bundle $p_N : N \to \tilde{N}$, the base of which is a projective special complex manifold $\tilde{N}$ with $\dim \tilde{N} = 2n \geq 4$. The c-projective Weyl curvature $W^{c,\bar{P}\bar{\nabla}'}$ of the canonically induced c-projective structure $\bar{P}\bar{\nabla}'$ is given by

\[ W^{c,\bar{P}\bar{\nabla}'} = -\frac{1}{4} [\bar{B}, \bar{B}] - \frac{1}{4(n+1)} \bar{B}_j \otimes \bar{J} + \frac{1}{8(n+1)} \bar{B} \wedge \text{Id} - \frac{1}{8(n+1)} \bar{B}_j \wedge \bar{J}. \]

In particular, $W^{c,\bar{P}\bar{\nabla}'}_{\bar{J}_(-), \bar{J}_(-)} = W^{c,\bar{P}\bar{\nabla}'}$, that is, $W^{c,\bar{P}\bar{\nabla}'}$ is of type $(1, 1)$ as an $\text{End}(T\tilde{N})$-valued two-form.

**Proof.** Take a principal connection $\eta$ of type $(1, 0)$. By Proposition 7.3, the canonically induced c-projective structure is $[\bar{\nabla}'\eta]$. From Proposition 7.9 equation (7.4) and the symmetry of $\bar{B}$, it holds that

\[ W^{c,\bar{\nabla}'\eta} = -\frac{1}{4} [\bar{B}, \bar{B}] - \frac{1}{4(n+1)} \bar{B}_j \otimes \bar{J} + \frac{1}{8(n+1)} \bar{B} \wedge \text{Id} - \frac{1}{8(n+1)} \bar{B}_j \wedge \bar{J}. \]

Since $\bar{B}_j$, $[\bar{B}, \bar{B}]$ and $\bar{B} \wedge \text{Id} - \bar{B}_j \wedge \bar{J}$ are of type $(1, 1)$, $W^{c,\bar{P}\bar{\nabla}'}$ is of type $(1, 1)$. \hfill \Box

The following corollary is a direct consequence of Theorem 7.10.

**Corollary 7.11.** Any complex manifold $(\tilde{N}, \bar{J})$ with a c-projective structure $\bar{P}$ such that $W^{c,\bar{P}}$ is not of type $(1, 1)$ can not be realized as a projective special complex manifold whose canonical c-projective structure is $\bar{P}$.

### 8 A generalization of the supergravity c-map

The supergravity c-map associates a (pseudo-)quaternionic Kähler manifold with any projective special Kähler manifold. In this section, we give a generalization of the supergravity c-map by using the results in previous sections. Let $(N, J, \nabla, \xi)$ be a conical special complex manifold and set $Z := J\xi$.

**Lemma 8.1.** $2Z^{hv}$ is a rotating vector field on $TN$. 

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Proof. Since $L_Z J = 0$ and $\nabla_Z J = 0$ (cf. Lemma 6.2), we have $L_{Z^{hv}} I_1 = 0$. Moreover we have

$$(L_{Z^{hv}} I_2)(X^{hv} + Y^v) = [Z, Y]^{hv} - (\nabla_Z X)^v - (\nabla_Z Y)^{hv} + [Z, X]^v$$

$$= - (\nabla_Y Z)^{hv} - (\nabla_X Z)^v$$

$$= -(JY)^{hv} - (JX)^v$$

$$= -I_3(X^{hv} + Y^v)$$

for all $X, Y \in \Gamma(TN)$.

\[ \square \]

Remark 8.2. By the equations for $\tilde{\nabla}^0$ in the proof of Theorem 6.5, we have

$$\tilde{\nabla}^0_{X^{hv}} \xi^{hv} = (\nabla'_{X^{hv}} \xi)^{hv} = (\nabla_X \xi - \frac{1}{2} JA_X \xi)^{hv} = X^{hv},$$

$$\tilde{\nabla}^0_{X^{hv}} \xi^{hv} = -\frac{1}{2} (JA_X \xi)^v = 0$$

for $X \in TN$, when $(N, J, \nabla, \xi)$ is a conical special complex manifold.

We have the following theorem.

Theorem 8.3 (Generalized supergravity c-map). Let $(N, J, \nabla, \xi)$ be a $2(n+1)$-dimensional conical special complex manifold. Let $\Theta$ be a closed two-form on $M = TN$ such that $L_{Z^M} \Theta = 0$, where $Z^M = 2Z^{hv}$. Consider a $U(1)$-bundle $\pi : P \to M$ over $M$ and $\eta$ a connection form whose curvature form is

$$d\eta = \pi^* \left( \Theta - \frac{1}{2} d((\iota_{Z^M} \Theta) \circ I_1) \right).$$

Let $f$ be a smooth function on $M$ such that $df = -\iota_{Z^M} \Theta$ and $f_1 := f - (1/2) \Theta(Z^M, I_1 Z^M)$ does nowhere vanish. If $\tilde{\pi} : \tilde{M} \to \tilde{M}$ and $\tilde{\pi} : \tilde{M} \to \tilde{M}$ are submersions, we have an assignment from a $2n$-dimensional projective special complex manifold $(\tilde{N}, J, \mathcal{P}_{\tilde{\nabla}})$ whose c-projective Weyl curvature is of type $(1, 1)$ to a $4(n+1)$-dimensional quaternionic manifold

$$\tilde{M}(= \tilde{TN}) = C_{(P, \eta)}(M, \langle I_1, I_2, I_3 \rangle, Z^M, f, \Theta)/\mathcal{D}$$

foliated by $(2n+4)$-dimensional leaves such that $\tilde{N}$ coincides with the space of its leaves.

Proof. By Theorem 4.1, Lemma 8.1 and Proposition 7.3, we have an assignment from a $2n$-dimensional projective special complex manifold $(\tilde{N}, \tilde{J}, \mathcal{P}_{\tilde{\nabla}})$ to a $4(n+1)$-dimensional quaternionic manifold $\tilde{TN}$. By virtue of Theorem 7.10, the c-projective Weyl curvature of $\mathcal{P}_{\tilde{\nabla}}$, is of type $(1, 1)$. Next we give a foliation on $\tilde{TN}$ whose leaves space is $\tilde{N}$. Set $\mathcal{L} := \mathcal{V} \oplus \langle \xi^{hv}, Z^{hv} \rangle$, where $\mathcal{V}$ is the vertical distribution of $T(TN) \to TN$. The distribution $\mathcal{L}$ is $Z^M = 2Z^{hv}$-invariant and integrable by (6.4). Therefore each leaf $L$ of $\mathcal{L}$ is a $Z^M = 2Z^{hv}$-invariant submanifold of $TN$. Consider the pull-back $\iota^# P$ of $P$ by the inclusion $\iota : L \to TN$ with the bundle map $\iota^# : \iota^# P \to P$ and $\tilde{L} := \mathbb{H}^* \times \iota^# P$. Since $V_1$ is tangent to $\tilde{L}$, then $\tilde{L} := \tilde{L}/\langle V_1 \rangle$ is a submanifold $\tilde{M}$. Moreover $V, \tilde{I}_1(V), \tilde{I}_2(V)$,
\( \hat{I}_3(V) \) are tangent to \( \hat{L} \) because \( V \) is induced by \( e_0^R \). Taking the quotient again, we obtain a submanifold \( L := \hat{L}/(V, \hat{I}_1(V), \hat{I}_2(V), \hat{I}_3(V)) \) on a quaternionic manifold \( \overline{TN} \). Therefore the quaternionic manifold \( \overline{TN} \) is foliated by \((2n + 4)\)-dimensional leaves such that the space of its leaves \( \overline{L} \) is the projective special complex manifold \( \overline{N} \).

**Remark 8.4.** If we assume that \( Z_1 = (Z^M)^{h_\eta} + f_1 X_P \) generates a free \( U(1) \)-action on \( P \) instead of assuming that \( \tilde{\pi} : \tilde{M} \to M \) and \( \hat{\pi} : \hat{M} \to \hat{M} \) are submersions, we obtain the same result as in Theorem 8.3 (see Theorem 4.8).

**Remark 8.5.** Borówka and Calderbank have given a construction of a quaternionic manifold from a complex manifold of half the dimension with a c-projective structure, known as the quaternionic Feix-Kaledin construction [6]. Their construction generalizes the original construction [11, 20], which yields a hyper-Kähler structure on a neighborhood of any Kähler manifold. They also point out that this construction is a generalization of [12, Theorem A] (see [6, Proposition 5.4]). More precisely, the initial data of the quaternionic Feix-Kaledin construction are a complex manifold with a c-projective structure of type \( (1, 1) \) and a complex line bundle with a connection of type \( (1, 1) \). Note that this construction is different from our generalization of the supergravity c-map, in which the real dimension of the quaternionic manifold \( \overline{TN} \) is related to the real dimension of the projective special complex manifold \( \overline{N} \) by \( \dim(\overline{TN}) = 2 \dim(\overline{N}) + 4 \).

We consider a conical special complex manifold \( (N, J, \nabla, \xi) \), which we endow now with an additional structure. Let \( \psi \) be a \( J \)-hermitian, \( \nabla \)-parallel two-form on \( (N, J, \nabla, \xi) \). We consider a function \( \mu = (1/2)\psi(\xi, J\xi) \) on \( N \). Then we see \( d\mu = -\iota_Z \psi \). Set

\[
\begin{align*}
\Theta &= -\pi_{TN}^* \psi, \\
f &= -2\pi_{TN}^* \mu + c
\end{align*}
\]

for some constant \( c \). Then it holds that

\[
df = -\iota_{Z^M} \Theta, \quad f_1 = f - \frac{1}{2} \Theta(Z^M, I_1 Z^M) = 2\pi_{TN}^* \mu + c,
\]

where \( \pi_{TN} : TN \to N \) is the bundle projection. In fact, we have

\[
df = -2d(\pi_{TN}^* \mu) = 2\pi_{TN}^* (\iota_Z \psi) = -\iota_{Z^M} \Theta
\]

and

\[
f_1 = f - \frac{1}{2} \Theta(Z^M, I_1 Z^M) \\
= -\psi(\xi, J\xi) \circ \pi_{TN} - 2\Theta(Z^{h\nu}, I_1 Z^{h\nu}) + c \\
= \psi(J\xi, \xi) \circ \pi_{TN} + c = 2\pi_{TN}^* \mu + c.
\]

**Corollary 8.6.** Let \( (N, J, \nabla, \xi) \) be a \( 2n \)-dimensional conical special complex manifold and \( \psi \) a \( J \)-hermitian, \( \nabla \)-parallel two-form on \( N \). Consider a \( U(1) \)-bundle \( \pi : P \to M \) over \( M = TN \) and \( \eta \) a connection form whose curvature form is

\[
d\eta = (\pi_{TN} \circ \pi)^* \psi.
\]
If $\tilde{\pi} : \tilde{M} \to \hat{M}$ and $\hat{\pi} : \hat{M} \to \bar{M}$ are submersions and $\mu^{-1}(-c/2) = \emptyset$, then the generalized supergravity c-map of Theorem 8.3 can be specialized to this setting such that the data $\Theta$ and $f$ are related to $\psi$ by equations (8.1) and (8.2).

Proof. By a straightforward calculation, we have $d((\iota_Z \psi) \circ J) = 2\psi$. Then it is easy to check

$$d\eta = (\pi_{TN} \circ \pi)^* \psi = (\pi_{TN} \circ \pi)^* \left(-\psi + d((\iota_Z \psi) \circ J)\right) = (\pi_{TN} \circ \pi)^* \left(-\psi + \frac{1}{2} d((\iota_2 Z \psi) \circ J)\right) = \pi^* \left(\Theta - \frac{1}{2} d((\iota_Z M \Theta) \circ I_1)\right),$$

where $\Theta$ is the two-form given by (8.1). Since $d\psi = 0$ and $\iota_Z \psi = -d\mu$, it holds $L_Z M \Theta = 0$. The function $f_1 = f - (1/2)\Theta(Z_m^M, I_1 Z_m^M)$ does nowhere vanish by $\mu^{-1}(-c/2) = \emptyset$. Therefore Theorem 8.3 leads to the conclusion.

Therefore a conical special complex manifold $(N, J, \nabla, \xi)$ with a $J$-hermitian, $\nabla$-parallel two-form $\psi$ such that $(1/2\pi)\Theta(Z_m^M, I_1 Z_m^M) \in H^2_{DR}(N, \mathbb{Z})$ and $\mu = (1/2)\Theta(\xi, J\xi)$ is not surjective gives rise to a quaternionic manifold of dimension $2\dim N$ under a suitable choice of the constant $c$.

For $t \in \mathbb{R}/\pi\mathbb{Z}$, we define a connection $\nabla^t$ by $\nabla^t = e^{t J} \circ \nabla \circ e^{-t J}$, which is a special complex connection by [3, Proposition 1]. Moreover, by

$$\nabla^t = \nabla - (\sin t) e^{t J}(\nabla J)$$

([3, Lemma 1]), we see that $\nabla^t$ satisfies $\nabla^t \xi = \text{id}$. Therefore $\{\nabla^t\}_{t \in \mathbb{R}/\pi\mathbb{Z}}$ is a family of conical special complex connections if $\nabla J \neq 0$.

**Lemma 8.7.** If $\psi$ is $J$-hermitian and $\nabla$-parallel, then $\psi$ is $\nabla^t$-parallel.

Proof. Since $\nabla^t - \nabla$ is a linear combination of $\nabla J$ and $J(\nabla J) = -(\nabla J)J$, it suffices to remark that $\nabla \psi = 0$, $J \cdot \psi = 0$ and, hence, $(\nabla_X J) \cdot \psi = 0$ for all $X$. Here the dot stands for the action on the tensor algebra by derivations. □

Hence, Corollary 8.6 and Lemma 8.7 imply

**Corollary 8.8.** If $A(= \nabla J) \neq 0$, there exists an $(\mathbb{R}/\pi\mathbb{Z})$-family of quaternionic manifolds obtained from a conical special complex manifold with $\psi$ under the same assumptions of Corollary 8.6 by the $H/Q$-correspondence (for any chosen function $f$ in the construction).

Proof. By Lemma 8.7, $\nabla^t_X \psi = 0$. Since $(N, J, \nabla^t, \xi)$ are conical special complex manifolds, we have the conclusion. □
To give an example, we recall the (local) characterization of a conical special complex manifold [3]. Let \((\mathbb{C}^{n+1}, J)\) be the standard complex vector space and \(U\) an open subset in \(\mathbb{C}^{n+1}\) with the standard coordinate system \((z_0, \ldots, z_n)\). We consider a holomorphic one-form \(\alpha = \sum F_idz_i\) on \(U\), which is also viewed as a holomorphic map \(\phi = \phi_\alpha\) from \(U\) to \((T^*U = U \times \mathbb{C}^{n+1} \subset \mathbb{C}^{2(n+1)}\). If \(\text{Re} \phi : U \to \mathbb{R}^{2(n+1)}\) is an immersion, which is equivalent to \(\phi\) being totally complex [3], then we can find an affine connection \(\nabla\) such that \((U, J, \nabla)\) is a special complex manifold. In fact, we can take a local coordinate system
\[
(x_0 := \text{Re} z_0, \ldots, x_n := \text{Re} z_n, y_0 := \text{Re} F_0, \ldots, y_n := \text{Re} F_n)
\]
on \(U\) induced by \(\phi\) and a connection \(\nabla\) defined by the condition that \((x_0, \ldots, x_n, y_0, \ldots, y_n)\) is affine. Moreover, \(\sum\) is affine. Moreover, \(\sum\) is an immersion, which is equivalent to \(\phi\) being totally complex [3], then we can find an affine connection \(\nabla\) such that \((U, J, \nabla)\) is an affine coordinate system \((\alpha = dz_0)\) in that special case. In addition to being holomorphic and totally complex, we assume that \(\phi\) is conical, which is equivalent to the condition that functions \(F_0, \ldots, F_n\) are homogeneous of degree one, i.e. \(F_i(\lambda z) = \lambda F_i(z)\) for all \(\lambda\) near \(1 \in \mathbb{C}^*\) and \(z \in U\). Then \(U\) is conical, that is, any conical holomorphic one-form \(\phi\) such that \(\text{Re} \phi\) is an immersion on \(U\) defines a conical special complex (and symplectic) manifold structure of complex dimension \(n\). Conversely, any such manifold can be locally obtained in this way (see [3] Corollary 5).

If we choose \(\alpha = -\sum\sqrt{-1}z_i dz_i\) on \(\mathbb{C}^{n+1}\setminus\{0\}\), then the generalized c-map associates an open submanifold of \((\mathbb{H}^{n+1}, Q)\) with the standard quaternionic structure \(Q\) to the complex projective space \((\mathbb{C}P^n, J^s, [\nabla^{FS}])\), where \(J^s\) is the standard complex structure and \(\nabla^{FS}\) is the Levi-Civita connection of the Fubini-Study metric. Here we have chosen \(\Theta = 0\). We can also apply Corollary [8.6] by choosing the standard symplectic form as \(\psi\). More generally, we have the following example.

**Example 8.9.** For a holomorphic function \(g\) of homogeneous degree one, we consider the holomorphic 1-form
\[
\alpha = gdz_0 - \sqrt{-1}\sum_{i=1}^{n} z_i dz_i
\]
on \(U := \{(z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} \mid \text{Im} g_0 \neq 0\}\), where \(g_i = \frac{\partial g}{\partial z_i} (i = 0, 1, \ldots, n)\). [Comment Vicente: we should perhaps use a different symbol for \(F\) to avoid Note that \(d\alpha \neq 0\) if there exists \(i\) such that \(g_i \neq 0\) \((i \geq 1)\). Setting \(z_i = u_i + \sqrt{-1}v_i\) \((i = 0, 1, \ldots, n)\), we have
\[
(x_0, \ldots, x_n, y_0, y_1, \ldots, y_n) = \text{Re} \phi(u_0, \ldots, u_n, v_0, \ldots, v_n)
= (\text{Re} z_0, \ldots, \text{Re} z_1, \text{Re} g, \text{Re} (-\sqrt{-1}z_1), \ldots, \text{Re} (-\sqrt{-1}z_n))
= (u_0, \ldots, u_n, \text{Re} g, v_1, \ldots, v_n).
\]
Since its Jacobian matrix is given by

$$
\frac{\partial (x_0, \ldots, y_n)}{\partial (u_0, \ldots, v_n)} = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
\Re g_0 & \Re g_n & -\Im g_0 & -\Im g_1 & \ldots & -\Im g_n \\
0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & & 0 & 1 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \ddots & \ddots \\
0 & \ldots & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
$$

we see that $\Re \phi$ is an immersion and we obtain a conical special complex structure on $U$. The coordinate vector fields of $(x_0, \ldots, y_n)$ are given by

$$
\frac{\partial}{\partial x_i} = \frac{\partial}{\partial u_i} + \frac{\Re g_i}{\Im g_0} \frac{\partial}{\partial v_0} \quad (i \geq 0),
$$

$$
\frac{\partial}{\partial y_0} = -\frac{1}{\Im g_0} \frac{\partial}{\partial v_0}, \quad \frac{\partial}{\partial y_j} = -\frac{\Im g_j}{\Im g_0} \frac{\partial}{\partial v_0} + \frac{\partial}{\partial v_j} \quad (j \geq 1)
$$
on $U$. Let $\nabla$ (resp. $\nabla^{st}$) be the flat affine connection on $U$ such that $(x_0, \ldots, y_n)$ (resp. $(u_0, \ldots, v_n)$) is a $\nabla$ (resp. $\nabla^{st}$)-affine coordinate system. We define $S$ by $\nabla = \nabla^{st} + S$. Then we calculate

$$
0 = \nabla_X \frac{\partial}{\partial x_i} = (\nabla^{st}_X + S_X) \left( \frac{\partial}{\partial u_i} + \frac{\Re g_i}{\Im g_0} \frac{\partial}{\partial v_0} \right)
= X \left( \frac{\Re g_i}{\Im g_0} \right) \frac{\partial}{\partial v_0} + S_X \frac{\partial}{\partial u_i} + \frac{\Re g_i}{\Im g_0} S_X \frac{\partial}{\partial v_0} \quad (i \geq 0)
$$

and similarly we have

$$
-X \left( \frac{1}{\Im g_0} \right) \frac{\partial}{\partial v_0} - \frac{1}{\Im g_0} S_X \frac{\partial}{\partial v_0} = 0,
$$

$$
-X \left( \frac{\Im g_j}{\Im g_0} \right) \frac{\partial}{\partial v_0} - \frac{\Im g_j}{\Im g_0} S_X \frac{\partial}{\partial v_0} + S_X \frac{\partial}{\partial v_j} = 0 \quad (j > 0).
$$

From these equations, it holds that

$$
(8.3) \quad S_X \frac{\partial}{\partial u_i} = -X \Re g_i \frac{\partial}{\partial v_0}, \quad S_X \frac{\partial}{\partial v_i} = \frac{X \Im g_i}{\Im g_0} \frac{\partial}{\partial v_0} \quad (i \geq 0).
$$

Using $A_X Y = (\nabla_X J)(Y) = S_X J Y - JS_X Y$ and (8.3), we have the matrix representation

$$
(8.4) \quad A = \nabla J = \frac{1}{\Im g_0} \begin{pmatrix}
A_0 & \ldots & A_n \\
0_2 & \ldots & 0_2 \\
\vdots & \ddots & \vdots \\
0_2 & \ldots & 0_2
\end{pmatrix}
$$

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of $A$ with respect to the frame
\[
\left( \frac{\partial}{\partial u_0}, \frac{\partial}{\partial v_0}, \ldots, \frac{\partial}{\partial u_n}, \frac{\partial}{\partial v_n} \right),
\]
where
\[
A_i = \begin{pmatrix} -d\text{Re } g_i & d\text{Im } g_i \\ d\text{Im } g_i & d\text{Re } g_i \end{pmatrix}
\]
and $0_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Note that we change the order of the frame for simplicity. This means that $A \neq 0$ if there exists $i$ such that $g_i \neq$ constant. By Lemma 7.8 and (8.4), $A^2 = (\nabla J)^2$ induces a globally defined tensor on $\bar{U}$, in particular
\[
\operatorname{Tr} A^2 = \operatorname{Tr}_{A^2} = \frac{2}{(\text{Im } g_0)^2} (d\text{Re } g_0 \otimes d\text{Re } g_0 + d\text{Im } g_0 \otimes d\text{Im } g_0)
\]
also induces the the symmetric tensor $\mathcal{B}$ on $\bar{U}$. By Lemma 6.4 and (8.4), we see that
\[
R_{\nabla'} = -\frac{1}{4} A \wedge A = -\frac{1}{4(\text{Im } g_0)^2} \begin{pmatrix} A_0 \wedge A_0 & A_0 \wedge A_1 & \cdots & A_0 \wedge A_n \\ 0_2 & 0_2 & \cdots & 0_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0_2 & 0_2 & \cdots & 0_2 \end{pmatrix}
\]
as the matrix representation.

Since
\[
dx_i = du_i \quad (i \geq 0), \quad dy_0 = \sum_{i=0}^n \text{Re } g_i \, du_i - \text{Im } g_i \, dv_i,
\]
\[
(0_2 \wedge 0_2)(\frac{\partial}{\partial u_0}, \frac{\partial}{\partial u_i}) = \text{Re } g_i \quad \text{and} \quad (0_2 \wedge 0_2)(J \frac{\partial}{\partial u_0}, J \frac{\partial}{\partial u_i}) = 0.
\]
Moreover since
\[
\xi = \sum_{i=0}^n x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i} = \cdots + \sum_{i=1}^n u_i \frac{\partial}{\partial u_i} + v_i \frac{\partial}{\partial v_i},
\]
\[
J\xi = \cdots + \sum_{i=1}^n u_i \frac{\partial}{\partial v_i} - v_i \frac{\partial}{\partial u_i},
\]
we have $\mu = \psi(\xi, J\xi) = (1/2) \sum_{i=1}^n (u_i^2 + v_i^2)$. Take a $\text{U}(1)$-bundle $\pi : T\bar{U} \times \text{U}(1) \to T\bar{U}$ with a connection form
\[
\eta = (\pi_{TU} \circ \pi)^* \left( \sum_{i=1}^n u_i dv_i \right) + d\theta.
\]
where \( \theta \) is the angular coordinate of \( U(1) \). The special case Corollary 8.6 of Theorem 8.3 can be applied and then we obtain a quaternionic manifold.

We consider the horizontal subbundle of \( p_\mathcal{V} : U \to \bar{U} \) given by the kernel of \( \kappa = -(1/2s)d\mu \circ J \) on each level set \( \mu^{-1}(s) \subset U \) \((s \neq 0)\). We retake \( U \) as an open set in \( \bigcup_{s > 0} \mu^{-1}(s) \). For horizontal vector fields \( X \) and \( Y \) tangent to each level set \( \mu^{-1}(s) \), \( XY_\mu = 0 \) means that

\[
(p_\mathcal{V}^* \bar{a})(X, Y) = \frac{1}{2s} \psi(JX, Y),
\]

where \( \bar{a} \) is the \( \xi \)-component of the fundamental tensor of \( \mathcal{A}_\mathcal{V} \) as in Section 7. Here we used \( d\kappa = \psi/s \). This means that \( \bar{a} \) is symmetric and \( J \)-hermitian, and hence the Ricci tensor of the connection \( \bar{\nabla}^{\kappa} \) on \( \bar{U} \) induced from \( \kappa \) is symmetric and \( J \)-hermitian. Therefore it holds

\[
p_\mathcal{V}^* \bar{a} = -\frac{1}{\sum_{i=1}^n (u_i^2 + v_i^2)} \sum_{i=1}^n du_i \otimes dv_i + dv_i \otimes dv_i.
\]

Hence the Ricci tensor \( Ric^{\bar{\nabla}^{\kappa}} \) of \( \bar{\nabla}^{\kappa} \) satisfies

\[
-\frac{1}{\sum_{i=1}^n (u_i^2 + v_i^2)} \sum_{i=1}^n du_i \otimes dv_i + dv_i \otimes dv_i = \frac{1}{4(n+1)(\Im g_0)^2}(d\Re g_0 \otimes d\Re g_0 + d\Im g_0 \otimes d\Im g_0) - \frac{1}{2(n+1)} p_\mathcal{V}^*(Ric^{\bar{\nabla}^{\kappa}})
\]

by (7.3). In particular, we see that \( Ric^{\bar{\nabla}^{\kappa}} \geq 0 \). For example, when we choose \( g = -\sqrt{-1} z_1/z_0^{-1} \) for \( l(\neq 1) \in \mathbb{Z} \), we obtain

\[
\begin{align*}
d\Re g_0 &= \frac{\sqrt{-1}}{2}(-l + 1)l(-w^{l-1}dw + \bar{w}^{l-1}d\bar{w}), \\
d\Im g_0 &= -\frac{1}{2}(-l + 1)l(w^{l-1}dw + \bar{w}^{l-1}d\bar{w}), \\
d\Re g_1 &= \frac{\sqrt{-1}}{2}(-l + 1)l(w^{l-2}dw - \bar{w}^{l-1}d\bar{w}), \\
d\Im g_1 &= \frac{1}{2}(-l + 1)l(w^{l-2}dw + \bar{w}^{l-2}d\bar{w}), \\
d\Re g_j &= d\Im g_j = 0 \quad (j > 1),
\end{align*}
\]

where \( w = z_1/z_0 \). We denote the corresponding objects with subscript \( l \) for ones given by \( g = -\sqrt{-1} z_1^l/z_0^{-1} \). It holds that

\[
(8.5) \quad R^{\bar{\nabla}^{\kappa}} = -\frac{\sqrt{-1} l^2 |w|^{2(l-2)}}{(w^l + \bar{w}^l)^2} \begin{pmatrix} 0 & -|w|^2 & -\Im w & \Re w & 0 & \ldots & 0 \\ -|w|^2 & 0 & -\Re w & -\Im w & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \quad dw \wedge d\bar{w}
\]
and
\[
(8.6) \quad \text{Tr}(A^l)^2 = \text{Tr}(\nabla^l J)^2 = \frac{4l^2 |w|^2(l-1)}{(w^l + \bar{w}^l)^2} (dw \otimes d\bar{w} + d\bar{w} \otimes dw).
\]

Finally we consider the quaternionic Weyl curvature of \(TU\). Let \(W^q\) be the quaternionic Weyl curvature of the quaternionic structure \(Q = \langle I_1, I_2, I_3 \rangle\). In \([5]\), the explicit expression of \(W^q\) is given and it is shown that \(W^q\) is independent of the choice of the quaternionic connection. Since the Obata connection of the c-map is Ricci flat by Remark 8.11, we see that \(W^{q,l} = R^{q,l}\) for \(g = -\sqrt{-1}z_1/\bar{z}_0^{l-1}\). If \(l \neq 1\), then we see that
\[
W^{q,l}_{Xv,Yv} Z^v = R^{q,l}_{Xv,Yv} Z^v = \left( R^{q,l}_{X,Y} \right)^v.
\]

Because the vertical lift is determined by a differential manifold structure (not by a connection), we see that \(W^{q,l} \neq W^{q,k}\) on \(T(U_k \cap U_l)\) if \(l \neq k\), where \(U_j = \{(z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} \mid \text{Im} g_0 \neq 0\} = \{(z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} \mid \text{Re} (z_1/z_0) \neq 0\}\) for \(g = -\sqrt{-1}z_1/\bar{z}_0^{l-1}\). Here we used (8.5). So we can find different quaternionic structures \(Q^{\alpha_1}, \ldots, Q^{\alpha_n}\) on \(T(\cap_{i=1}^l U_{\alpha_i})\), where \(1 \neq \alpha_i \in \mathbb{Z}\). Note that \(Q^0\) is the flat quaternionic structure.

**Remark 8.10.** Since \(d\alpha \neq 0\) except the trivial case \(g = -\sqrt{-1}z_0\), Example 8.9 with \(g = -\sqrt{-1}z_1/\bar{z}_0^{l-1}\) (\(l \neq 0\)), which is local one, is not given by a local special Kählerian one.

**Remark 8.11.** For a conical special Kähler manifold \(N\), the particular twist data which yields the quaternionic Kähler structure of the supergravity c-map on \(T^*N \cong TN\) is given in [21, Lemma 5.1] in consistency with [4]. As we noted in the introduction, we also have a freedom in the choice of the data \(\Theta\) etc. for our generalized supergravity c-map. For illustration, the two form \(\Theta\) can be chosen as trivial (\(\Theta = 0\)) or as in equation (8.11). For illustration, we can give yet another possible choice of \(\Theta\). Assume that \(\dim N \geq 6\). Let \(\{\bar{U}_\alpha\}_{\alpha \in \Lambda}\) be an open covering of \(N\) with local trivializations \(U_\alpha := p_N^{-1}(\bar{U}_\alpha) \cong \bar{U}_\alpha \times \mathbb{C}^*\) and \(g_{\alpha\beta} : \bar{U}_\alpha \cap \bar{U}_\beta \to \mathbb{C}^*\) be the corresponding transition functions. Let \((r_\alpha, \theta_\alpha)\) be the polar coordinates with respect to a (smooth) local trivialization \(p_N^{-1}(\bar{U}_\alpha) \cong \bar{U}_\alpha \times \mathbb{C}^*\) for each \(\alpha \in \Lambda\). A principal connection \(\eta\) is locally given by

\[
\eta = p_N^*(\gamma_1^\alpha \otimes 1 + \gamma_2^\alpha \otimes \sqrt{-1}) + \left( \frac{dr_\alpha}{r_\alpha} \otimes 1 + d\theta_\alpha \otimes \sqrt{-1} \right)
\]

for a \(\mathbb{C}\)-valued one-form \(\gamma_1^\alpha \otimes 1 + \gamma_2^\alpha \otimes \sqrt{-1}\) on \(\bar{U}_\alpha \subset \bar{N}\) for each \(\alpha \in \Lambda\). If we write \(g_{\alpha\beta} = e^{f^1_{\alpha\beta} + f^2_{\alpha\beta} \sqrt{-1}}\), then

\[
\begin{align*}
f^1_{\alpha\beta} + f^1_{\beta\gamma} - f^1_{\alpha\gamma} & = 0, \\
f^2_{\alpha\beta} + f^2_{\beta\gamma} - f^2_{\alpha\gamma} & \in 2\pi \mathbb{Z}, \\
\gamma^1_\beta - \gamma^1_\alpha & = df^1_{\alpha\beta}, \\
\gamma^2_\beta - \gamma^2_\alpha & = df^2_{\alpha\beta}.
\end{align*}
\]
Therefore we obtain a principal U(1)-bundle \( p_S : S \to \tilde{\mathcal{N}} \) with transition functions 
\[ e^{f_{\alpha \beta} \sqrt{-1}} : U_\alpha \cap U_\beta \to U(1) \]
and connection \( \eta_S \) locally given by 
\[ p_S^*(\gamma_2^\alpha \otimes \sqrt{-1}) + d\theta_\alpha \otimes \sqrt{-1}. \]
In fact, the collection \( \{e^{f_{\alpha \beta} \sqrt{-1}}\} \) of local U(1)-valued functions satisfies the cocycle condition and the collection \( \{\gamma_\alpha\} \) of local \( \sqrt{-1}\mathbb{R} \)-valued one-forms satisfying \( \gamma_\beta^2 - \gamma_\alpha^2 = df_{\alpha \beta}^2 \)
defines a connection form \( \eta_S \). By Proposition [7.9] and (7.7), its curvature \( d\eta_S( = p_S^*(d\gamma_2^\alpha)) \)
is \( 2(\bar{a}_j)^\alpha \), where \( (\bar{a}_j)^\alpha \) is given by 
\[ (\bar{a}_j)^\alpha = \frac{1}{8(n+1)} \mathcal{B}_j - \frac{1}{2} (P^\alpha_j)^a. \]
On \( TN \), we choose the two-form \( \Theta = 2(p_N \circ \pi_{TN})^*(\bar{a}_j)^\alpha \) and consider the pull-back connection \( (p_N^\# \circ \pi_{TN^\#})^*\eta_S \) on the pull-back bundle \( P = \pi_{TN^\#}p_N^\#S \). Since \( \iota_{Z^M}\Theta = 0 \), we can see that the assumptions in Theorem [8.3] hold. It is left for future studies to find a canonical choice of \( \Theta \) for the generalized supergravity c-map, which allows to invert the H/Q-correspondence of [10].

As an application of Theorem [8.3], we have the following corollary by patching quaternionic manifolds locally constructed by the generalized supergravity c-maps.

**Corollary 8.12.** Let \((M, J, \sqrt{\nabla})\) be a complex manifold with a c-projective structure \( \sqrt{\nabla} \) and \( \dim M = 2n \). If \( 2n = \dim M \geq 4 \) and the harmonic curvature of its normal Cartan connection vanishes, then there exists a \( 4(n+1) \)-dimensional quaternionic manifold \((\tilde{M}, Q)\) with the vanishing quaternionic Weyl curvature foliated by \((n+2)\)-dimensional complex manifolds whose leaves space is \( M \).

**Proof.** Since \( \dim M \geq 4 \) and the harmonic curvature of its normal Cartan connection vanishes, \((M, J, \sqrt{\nabla})\) is locally isomorphic to \((\mathbb{C}P^n, J^c, [\nabla^{FS}])\) (see [8] for example). So we may assume that \( M = \bigcup_\alpha U_\alpha \), where \( U_\alpha \) is an open subset \( \mathbb{C}P^n \). Set \( V_\alpha := p^{-1}(U_\alpha) \), where \( p : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{C}P^n \) is the projection. We consider the standard complex structure and the standard flat connection induced from \( \mathbb{C}^{n+1} \) on each \( V_\alpha \). By Theorem [8.3] we have a quaternionic manifold \( W_\alpha := \varphi'(TV_\alpha) \subset \mathbb{H}^{n+1} \), where \( \varphi' \) is the diffeomorphism given in Example [5.2]. Here we have chosen the two-form \( \Theta = 0 \) and \( f = f_1 = 1 \) on \( TV_\alpha \) for each \( \alpha \). We set \( \tilde{M} := \bigcup_\alpha W_\alpha \). The induced quaternionic structure on each \( W_\alpha \) coincides with the standard one from \( \mathbb{H}^{n+1} \). Hence an almost quaternion structure \( Q \) on \( \tilde{M} \) can be obtained. Since there exists a quaternionic connection on each \( W_\alpha \), one can obtain a quaternionic connection on \( \tilde{M} \) by the partition of unity, that is, \( Q \) is a quaternionic structure with vanishing quaternionic Weyl curvature. For each \( p \in TV_\alpha \cap TV_\beta \), the leaf of \( L \) through \( p \) in \( TV_\alpha \) is denoted by \( L^\alpha \) and corresponding leaf in \( W_\alpha \) is denoted by \( \tilde{L}^\alpha \), that is \( \tilde{L}^\alpha = \varphi'(L^\alpha) \). Since \( \tilde{L}^\alpha = \tilde{L}^\beta \) in \( M \), we obtain leaves in \( \tilde{M} \) and see that its leaves space is \( M \). Since the subbundle \( L \) is an \( I_1 \)-invariant in \( T(TV_\alpha) \), each leaf \( L \) is a complex manifold with \( I := I_{L|L} \). Each leaf \( \tilde{L} \) on \( \tilde{M} \) is obtained by the Swann’s twist with an almost complex structure \( \tilde{I} \). By [27, Proposition 3.8] and \( \Theta = 0 \), \( \tilde{I} \) is integrable. \( \square \)
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