On convexity and solution concepts in cooperative interval games

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Abstract. Cooperative interval game is a cooperative game in which every coalition gets assigned some closed real interval. This models uncertainty about how much the members of a coalition get for cooperating together.
In this paper we study convexity, core and the Shapley value of games with interval uncertainty.
Our motivation to do so is twofold. First, we want to capture which properties are preserved when we generalize concepts from classical cooperative game theory to interval games. Second, since these generalizations can be done in different ways, mainly with regard to the resulting level of uncertainty, we try to compare them and show their relation to each other.

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1 Introduction

Uncertainty and inaccurate data are an everyday issue in real-world situations. Therefore it is important to be able to make decisions even when the exact data are not available and only bounds on them are known.

In classical cooperative game theory, every group of players (coalition) knows the precise reward for their cooperation; in cooperative interval games, only the worst and the best possible outcome is known. Such situations can be naturally modeled with intervals encapsulating these outcomes.

Cooperative games under interval uncertainty were first considered by Branzei, Dimitrov and Tijs in 2003 to study bankruptcy situations [11] and later further extensively studied by Alparslan Gök in her PhD thesis [1] and in follow-up papers (see the references section of [10]).

We note that there are several other models incorporating a different level of uncertainty, namely fuzzy cooperative games, multichoice games, crisp games (see [12] for more), or games under bubbly uncertainty [20].

There are several reasons why it is interesting to study cooperative interval game. From the aforementioned models of cooperative games, it is a quite simple model but it is easier to analyze and it is suitable for situations where we do not have any other assumptions on data we get. There are already a few applications
of this model, namely on forest situations [5], airport problems [4], bankruptcy [11] or network design [15].

We continue in the line of research started in [8] (there is also an updated version on arXiv [9]). We focus on selections, that is on possible outcomes of interval games.

**Our results.** Here is a summary of our results and also how our paper is organized.

- Section 3 is about convexity in interval games. We characterize selection convex interval games in a style of Shapley [26].
- Section 4 investigates a problem of core coincidence, i.e. when the two different versions of generalized core for interval games coincide. We partially solve this problem.
- Section 5 is about the Shapley value for interval games. We present a different axiomatization of the interval Shapley extension than the one by [17] and also show some important properties of this function.

2 Preliminaries

2.1 Classical cooperative games

Comprehensive sources on classical cooperative game theory are for example [12,14,16,22]. For more on applications, see e.g. [7,13,18]. Here we present only the necessary background theory for studying interval games. We examine only the games with transferable utility (TU) and therefore by a cooperative game or a game we mean a cooperative TU game.

**Definition 1.** (Cooperative game) A cooperative game is an ordered pair \((N, v)\), where \(N = \{1, 2, \ldots, n\}\) is a set of players and \(v : 2^N \rightarrow \mathbb{R}\) is a characteristic function of the cooperative game. We further assume that \(v(\emptyset) = 0\).

The set of all cooperative games with a player set \(N\) is denoted by \(G^N\). Subsets of \(N\) are called coalitions and \(N\) itself is called the grand coalition. We often write \(v\) instead of \((N, v)\) because we can identify a game with its characteristic function.

**Solution concepts.** To further analyze players’ gains, we need a payoff vector which can be interpreted as a proposed distribution of rewards between players.

**Definition 2.** (Payoff vector) A payoff vector for a cooperative game \((N, v)\) is a vector \(x \in \mathbb{R}^N\) with \(x_i\) being a reward given to the \(i\)th player.

**Definition 3.** (Imputation) An imputation of \((N, v) \in G^N\) is a vector \(x \in \mathbb{R}^N\) such that \(\sum_{i \in N} x_i = v(N)\) and \(x_i \geq v(\{i\})\) for every \(i \in N\).

The set of all imputations of a given game \((N, v)\) is denoted by \(I(v)\).
Definition 4. (Core) The core of \((N, v) \in G^N\) is the set
\[
C(v) = \left\{ x \in I(v); \sum_{i \in S} x_i \geq v(S), \forall S \subseteq N \right\}.
\]

The last solution concept we will write about is the Shapley value. It was introduced by Lloyd Shapley in 1952 [25]. It has many interesting properties; namely, it is a one-point solution concept, it always exists and it can be axiomatized by very natural axioms. We refer to [22] for a survey of results on the Shapley value.

Theorem 1. (Shapley, 1952, [26]) There exists a unique function \(f : G^N \rightarrow \mathbb{R}^N\), satisfying the following properties for every \((N, v) \in G^N\).

- (Efficiency) It holds that \(\sum_{i \in N} f_i(v) = v(N)\).
- (Dummy player) It holds \(f_i(v) = 0\) for every \(i \in N\), such that for every \(S \setminus \{i\} \subseteq N\), equality \(v(S \cup \{i\}) = v(S)\) holds.
- (Symmetry) If for every \(S \subseteq N \setminus \{i, j\}\),
  \[
  v(S \cup i) - v(S) = v(S \cup j) - v(S)
  \]
  holds, then \(f_i(v) = f_j(v)\).
- (Additivity) For every two games \(u, v \in G^N\), \(f_i(u + v) = f_i(u) + f_i(v)\) holds.

This unique function is called the Shapley value \((\phi)\) and it is defined as
\[
\phi_i(v) := \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n - |S| - 1)!}{n!(v(S \cup i) - v(S))}.
\]

Classes. There are many important classes of cooperative games. Here we show the most important ones.

Definition 5. (Monotonic game) A game \((N, v)\) is monotonic if for every \(T \subseteq S \subseteq N\) we have
\[
v(T) \leq v(S).
\]

Informally, in monotonic games, bigger coalitions are stronger.

Another important type of game is a convex game.

Definition 6. (Convex game) A game \((N, v)\) is convex if its characteristic function is supermodular. The characteristic function is supermodular if for every \(S \subseteq T \subseteq N\),
\[
v(T) + v(S) \leq v(S \cup T) + v(S \cap T).
\]

Clearly, supermodularity implies superadditivity. The class of convex games is maybe the most prominent class, it has many applications and theoretical properties. We present the most important one for this paper.

Theorem 2. (Shapley, 1971, [26]) Every convex game has a nonempty core.
2.2 Interval analysis

Definition 7. (Interval) An interval \( X \) is a set
\[
X := [\underline{X}, \overline{X}] = \{ x \in \mathbb{R} : \underline{X} \leq x \leq \overline{X} \}.
\]
with \( \underline{X} \) being the lower bound and \( \overline{X} \) being the upper bound of the interval. The length of an interval \( X \) is defined as \( |X| := |\overline{X} - \underline{X}| \).

From now on, by an interval we mean a closed interval. The set of all real closed intervals is denoted by \( \mathbb{I} \).

The following definition (from [19]) shows how to do basic arithmetics with intervals.

Definition 8. (Interval arithmetics) For every \( X, Y, Z \in \mathbb{I} \), and \( 0 \notin Z \), define
\[
X + Y := [\underline{X} + \underline{Y}, \overline{X} + \overline{Y}],
\]
\[
X \ominus Y := [\underline{X} - \overline{Y}, \overline{X} - \underline{Y}],
\]
\[
X \cdot Y := [\min S, \max S], \quad S = \{ \underline{X}\underline{Y}, \underline{X}\overline{Y}, \overline{X}\underline{Y}, \overline{X}\overline{Y} \}, \text{ and}
\]
\[
X / Z := [\min S, \max S], \quad S = \{ \underline{X}/\underline{Z}, \underline{X}/\overline{Z}, \overline{X}/\underline{Z}, \overline{X}/\overline{Z} \}.
\]

For our purpose, we need to have a slightly different definition of subtraction. The aforementioned subtraction operator is known as Moore’s subtraction operator.

Definition 9. (Partial subtraction operator) For every \( I, J \in \mathbb{I} \), such that \( \underline{I} - \overline{J} \leq \overline{I} - \underline{J} \), define
\[
I - J := [\underline{I} - \overline{J}, \overline{I} - \underline{J}].
\]

In other words, the length of the subtracted interval has to be lesser or equal to the length of the interval we subtract from.

Example 1. Take two intervals \([1, 4]\) and \([3, 5]\). Then \([1, 4] - [3, 5] = [-2, -1]\). Notice, however, that \([3, 5] - [1, 4]\) is undefined.

We note that this notation is not common in interval analysis. The minus sign is used for Moore’s subtraction operator there. Also, in our previous paper [8,9] we used minus sign for Moore’s subtraction.

Alparslan Gök [1] choose to compare intervals in the following way, using the weakly better operator. That was inspired by Hinojosa et al. [24].

Definition 10. (Weakly better operator \( \succeq \)) An interval \( I \) is weakly better than interval \( J \) (\( I \succeq J \)) if \( \underline{I} \geq \underline{J} \) and \( \overline{I} \geq \overline{J} \). Interval \( I \) is better than \( J \) (\( I \succ J \)) if and only if \( I \succeq J \) and \( I \neq J \).

Naturally, we also use \( A \prec B \) and \( C \preceq D \) for \( B \succ A \) and \( D \succeq C \), respectively.
2.3 Cooperative interval games

Now we review basics of cooperative games with interval uncertainty. The following is the main definition of this paper.

**Definition 11.** (Cooperative interval game) A cooperative interval game is an ordered pair $(N, w)$, where $N = \{1, 2, \ldots, n\}$ is a set of players and $w : 2^N \to \mathbb{R}$ is the characteristic function of the cooperative game. We further assume that $w(\emptyset) = [0, 0]$. The set of all interval cooperative games on a player set $N$ is denoted by $IG^N$.

**Note 1.** We often write $w(i)$ instead of $w(\{i\})$ and $w(i, j)$ instead of $w(\{i, j\})$.

**Note 2.** Every cooperative interval game in which its characteristic function maps to degenerate intervals only can be associated with a classical cooperative game. The converse holds as well.

**Definition 12.** (Border games) For every $(N, w) \in IG^N$, border games $(N, \underline{w}) \in G^N$ (lower border game) and $(N, \overline{w}) \in G^N$ (upper border game) are given by $\underline{w}(S) := \inf_S w(S)$ and $\overline{w}(S) := \sup_S w(S)$ for every $S \in 2^N$.

**Definition 13.** (Length game) The length game of $(N, w) \in IG^N$ is the game $(N, |w|) \in G^N$ with $|w|(S) := \overline{w}(S) - \underline{w}(S)$, $\forall S \in 2^N$.

**Definition 14.** (Degenerated game) We call a game $(N, w) \in IG^N$ degenerated if its length game is everywhere zero, that is, $|w|(S) = 0$ for every $S \in 2^N$. A non-degenerated game is a game which is not degenerated.

The basic notion of our approach will be a selection and consequently a selection imputation and a selection core.

**Definition 15.** (Selection) A game $(N, v) \in G^N$ is a selection of $(N, w) \in IG^N$ if for every $S \in 2^N$ we have $v(S) \in w(S)$. The set of all selections of $(N, w)$ is denoted by $\text{Sel}(w)$.

Note that border games are examples of selections and also of degenerated games.

**Solution concepts.** There are many possibilities how to define imputations and core for interval games. We present the following two. The first one is based on selections, the second one on the weakly better operator.

**Definition 16.** The set of interval selection imputations (or just selection imputations) of $(N, w) \in IG^N$ is defined as $SI(w) = \bigcup \{I(v) \mid v \in \text{Sel}(w)\}$.
Definition 17. The interval selection core (or just selection core) of \((N, w) \in IG^N\) is defined as 
\[
SC(w) = \bigcup \{ C(v) \mid v \in Sel(w) \}.
\]

In an analogous way as in classical games, we have a term for games with nonempty selection core for all selections.

Definition 18. An interval game is called strongly balanced if every selection of this game has a nonempty core. The set of all strongly balanced games on a player set \(N\) is denoted by \(BIG^N\).

Definition 19. The set of interval imputations of \((N, w) \in IG^N\) is defined as
\[
I(w) := \{(I_1, I_2, \ldots, I_N) \in IR^N \mid \sum_{i \in N} I_i = w(N), I_i \succeq w(i), \forall i \in N\}.
\]

Definition 20. The interval core of \((N, w) \in IG^N\) is defined as
\[
C(w) := \{(I_1, I_2, \ldots, I_N) \in I(w) \mid \sum_{i \in S} I_i \succeq w(S), \forall S \in 2^N \setminus \{\emptyset\}\}.
\]

An important difference between the definitions of interval and selection core and imputation is that selection concepts yield payoff vectors from \(R^N\), while \(I\) and \(C\) yield vectors from \(IR^N\). Thus they both possess a different degree of uncertainty.

Classes of interval games.

Definition 21. (Size monotonic interval game) A game \((N, w) \in IG^N\) is size monotonic if for every \(T \subseteq S \subseteq N\) we have 
\[
|w|(T) \leq |w|(S).
\]
That is, its length game is monotonic. The class of size monotonic games on a player set \(N\) is denoted by \(SMIG^N\).

As we can see, size monotonic games capture situations in which an interval uncertainty grows with the size of a coalition.

We should be careful with the following analogy of a convex game since unlike for classical games, supermodularity is not the same as convexity.

Definition 22. (Supermodular interval game) An interval game \((N, w) \in IG^N\) is supermodular interval if for every \(S \subseteq T \subseteq N\) holds
\[
w(T) + v(S) \preceq w(S \cup T) + w(S \cap T).
\]

We get immediately that an interval game is supermodular interval if and only if its border games are convex.
Definition 23. (Convex interval game) An interval game \((N, w)\) is convex interval if its border games and length game are convex. We write \(\text{CIG}^N\) for a set of convex interval games on a player set \(N\).

A convex interval game is supermodular as well but the converse does not hold in general. See [2] for characterizations of convex interval games and discussion of their properties.

Finally, we define selection based classes of interval games. The paper [8] discusses their properties and relations with the previous classes.

Definition 24. (Selection monotonic interval game) An interval game \((N, v)\) is selection monotonic if all its selections are monotonic games. The class of such games on a player set \(N\) is denoted by \(\text{SeMIG}^N\).

Definition 25. (Selection convex interval game) An interval game \((N, v)\) is selection convex if all its selections are convex games. The class of such games on a player set \(N\) is denoted by \(\text{SeCIG}^N\).

2.4 Notation

We will use \(\leq\) relation on real vectors. For every \(x, y \in \mathbb{R}^N\) we write \(x \leq y\) if \(x_i \leq y_i\) holds for every \(1 \leq i \leq N\).

We do not use symbol \(\subset\) in this paper. Instead, \(\subseteq\) and \(\subsetneq\) are used for the subset and the proper subset relation, respectively, to avoid ambiguity.

We also use \(x(S)\) instead of \(\sum_{i \in S} x_i\) occasionally.

Throughout the papers on cooperative interval games, notation, especially of core and imputations, is not unified. It is, therefore, possible to encounter different notation from ours. Also, in some papers the selection core is called the core of interval game. We consider that confusing and that is why we use the term selection core instead. The term selection imputation is then used because of its connection with the selection core.

3 Convexity

We present a characterization of the interval games in the class \(\text{SeCIG}\), analogous to a classical result of Shapley on convex games [26] and to Theorem 3.1 on convex interval games in [2].

Theorem 3. For every interval game \((N, w)\), the following assertions are equivalent.

1. The game \((N, w)\) is a selection convex interval game.
2. For every nonempty \(S, T \in 2^N\), such that \(S \cap T \neq T\), and \(S \cap T \neq S\),

\[
\overline{w}(S) + \overline{w}(T) \leq \underbar{w}(S \cup T) + \underbar{w}(S \cap T).
\]
3. For every coalition \( U_1, U_2, U \in 2^N \), such that \( U_1 \subseteq U_2 \subseteq N \setminus U \), and \( U \) is nonempty,
\[
\overline{w}(U_1 \cup U) - w(U_1) \leq \overline{w}(U_2 \cup U) - \overline{w}(U_2).
\]

4. For every coalition \( T_1, T_2 \in 2^N \), and for every \( i \in N \), such that \( T_1 \subseteq T_2 \subseteq N \setminus \{i\} \),
\[
\overline{w}(T_1 \cup \{i\}) - w(T_1) \leq \overline{w}(T_2 \cup \{i\}) - w(T_2).
\]

**Proof.**

(1) \( \leftrightarrow \) (2) : This proof is very similar to the proof of Theorem 2 in [9].

(2) \( \rightarrow \) (3) : Suppose for a contradiction that there exist \( U_1, U_2, U \in 2^N \), \( U \) nonempty, such that \( U_1 \subseteq U_2 \subseteq N \setminus U \), and
\[
\overline{w}(U_1 \cup U) - w(U_1) > \overline{w}(U_2 \cup U) - \overline{w}(U_2).
\]

Define \( S := U_1 \cup U \), and \( T := U_2 \). Both \( S \) and \( T \) are nonempty sets and they are incomparable. Furthermore:
\[
\overline{w}(U_1 \cup U) - w(U_1) > \overline{w}(U_2 \cup U) - \overline{w}(U_2),
\]
\[
\overline{w}(S) - w(U_1) > \overline{w}(T \cup U) - \overline{w}(T),
\]
\[
\overline{w}(S) + \overline{w}(T) > \overline{w}(T \cup U) + w(U_1),
\]
\[
\overline{w}(S) + \overline{w}(T) > \overline{w}(S \cup T) + w(S \cap T).
\]

And we obtained a contradiction.

(3) \( \rightarrow \) (4) : Straightforward; take \( U_1 := T_1, U_2 := T_2, U := \{i\} \).

(4) \( \rightarrow \) (3) : Suppose that (4) holds and (3) does not. Take \( U \) that violates (3) of minimal cardinality. If \( |U| = 1 \), we get a contradiction. If \( |U| > 1 \), we can construct \( U' \), with \( |U'| = |U| - 1 \), such that it violates (3) as well. This contradicts the minimality of \( U \).

(3) \( \rightarrow \) (2) : For a contradiction, take \( S \) and \( T \) which violate (2). Define \( U := S \setminus T \); this must be nonempty since \( S \) and \( T \) are nonempty and incomparable. Define \( U_1 := S \cap T \) and \( U_2 := T \). As for the conditions on \( U_1 \) and \( U_2 \), we see that \( U_1 \subseteq U_2 \), since \( U \) is nonempty and \( (S \cap T) \subseteq T \). Now:
\[
\overline{w}(S \cup U) - \overline{w}(T) > \overline{w}(S \cup T) + w(S \cap T)
\]
\[
\overline{w}(U_1 \cup U) + \overline{w}(T) > \overline{w}(U_1 \cup U \cup T) + w(U_1)
\]
\[
\overline{w}(U_1 \cup U) - w(U_1) > \overline{w}(U_2 \cup U) - \overline{w}(U_2)
\]

A contradiction. \( \Box \)

4 Core coincidence

In Alparslan-Gök’s PhD thesis [1] and in paper [6], the following question is suggested:
“A difficult topic might be to analyze under which conditions the set of payoff vectors generated by the interval core of a cooperative interval game coincides with the core of the game in terms of selections of the interval game.”

The main thing to notice is that while the interval core gives us a set of interval vectors, selection core gives us a set of real numbered vectors. To be able to compare them, we need to assign to a set of interval vectors a set of real vectors generated by these interval vectors. That is exactly what the following function gen does.

**Definition 26.** The function \( \text{gen} : 2^{\mathbb{R}^N} \rightarrow 2^{\mathbb{R}^N} \) maps to every set of interval vectors a set of its selections. It is defined as

\[
\text{gen}(S) = \bigcup_{s \in S} \{ (x_1, x_2, \ldots, x_n) \mid x_i \in s_i \}.
\]

The core coincidence problem can be formulated in the following way.

**Problem 1.** (Core coincidence problem) What are the necessary and sufficient conditions so that an interval game satisfies \( \text{gen}(C(w)) = \mathcal{SC}(w) \)?

To avoid a cumbersome notation we define the following property.

**Definition 27.** Let \((N, w)\) be a cooperative interval game. We call the game core-coincident if \( \text{gen}(C(w)) = \mathcal{SC}(w) \). Also, we say that a set of interval games is core-coincident if all games in this set are core-coincident.

Our results in this section are an important step towards a complete classification of core-coincident games.

4.1 Positive results

**Proposition 1.** Every cooperative interval game with empty selection core is core-coincident.

*Proof.* This easily follows from [8, Theorem 7].

**Proposition 2.** Every degenerated cooperative interval game is core-coincident.

*Proof.* It is easy to check that definitions of selection core (Definition 17) and interval core (Definition 20) coincide for degenerate games.

We present the following example, showing there exist infinitely many core-coincident non-degenerated games with a nonempty player set. But first, we need one more result.
Theorem 4. (Core coincidence technical lemma. [3]) For every interval game \((N, w)\) we have \(\text{gen}(C(w)) = SC(w)\), if and only if for every \(x \in SC(w)\), there exist nonnegative vectors \(l^{(x)}\) and \(u^{(x)}\), such that

\[
\begin{align*}
x(N) - l^{(x)}(N) &= w(N), \\
x(N) + u^{(x)}(N) &= \pi(N), \\
x(S) - l^{(x)}(S) &\geq w(S), \forall S \in 2^{N}, \\
x(S) + u^{(x)}(S) &\geq \pi(S), \forall S \in 2^{N}.
\end{align*}
\]

Theorem 5. There are infinitely many non-degenerated core-coincident interval games.

Proof. Define a game \((N, w_A)\), \(w_A(S) := 1/|S|\), if \(S \neq N\), and further \(w_A(N) := [\lfloor N \rfloor, |N| + b], b > 0, b \in \mathbb{R}\).

Clearly, \(C(w_A)\) consists exactly of vectors \(x\), such that \(x(N) \in w_A(N)\), and \(x_i \geq 1, \forall i \in N\).

Take any such vector \(x\). Define \(l_i^{(x)} := x(i) - 1\), and \(u_1^{(x)} := x(1) + \pi(N) - x(N)\), and \(u_i^{(x)} := x(i), \text{ for every } i \in N, i \neq 1\). It is now straightforward to check that all inequalities of Theorem [3] hold and, therefore, this game is core-coincident.

\(\square\)

4.2 Negative results

Theorem 6. Let \((N, w)\) be an interval game such that:

- a game \((N, u)\), defined by

\[
u(S) := \begin{cases} \pi(S) & \text{if } S = N, \\ \underline{w}(S) & \text{if } S \neq N, \end{cases}
\]

has a nonempty core, and

- \(C(\underline{w}) \neq \emptyset\).

Then \((N, w)\) is not core-coincident.

Proof. We define an excess function as \(e(x, S) := x(S) - w(S)\).

If for every \(x \in C(u)\), and every player \(i \in N\), there is a coalition \(S \in 2^N \setminus N, i \in S\), such that \(e(x, S) = 0\), then we claim that the core of the upper border game \(\pi\) of \(w\) is empty.

To see this, observe that \(C(\underline{w}) \subseteq C(u)\). But, if every \(x \in C(u)\) has the aforementioned property then, by Theorem [3] none of those \(x\) can be in \(C(\underline{w})\); a contradiction with \(C(\underline{w}) \neq \emptyset\).

So the other option is that there exists a vector \(y \in C(u)\), and a player \(j \in N\), such that for every \(S \in 2^N \setminus N, j \in S, e(x, S) > 0\). We define

\[
m := \min_{x \in 2^{N} \setminus N, j \in S} e(x, S),
\]
and $M$ the set on which this minimum is attained. We pick an arbitrary player $j' \in N \setminus M$. Such player must exist. Then we construct a new vector $y'$:

$$y'_k := \begin{cases} 
  y_k - m & \text{if } k = j, \\
  y_k + m & \text{if } k = j', \\
  y_k & \text{else.}
\end{cases}$$

It can be checked that $y' \in C(u)$ and by a similar argument as in the previous case, $y'$ does not satisfy the mixed system of inequalities in Theorem 4 and we are done. 

This theorem has several important corollaries.

**Corollary 1.** Every interval game $(N, w) \in \text{BIG}^N$ with $|w|(S) > 0$ for every $S \in 2^N$, is not core-coincident.

**Corollary 2.** Classes $\text{SeCIG}^N$ and $\text{CIG}^N$ are not core-coincident for $|N| > 1$. Furthermore, every game in $\text{SeCIG}^N \cup \text{CIG}^N$ with every interval non-degenerated and $|N| > 1$ is not core-coincident.

**Proof.** Theorem 2 implies that selection convex games are totally balanced. In [2], it is proved that a game is convex interval game if and only if its lower border game and its length game are convex. This completes the proof.

Observe that $\text{SeCIG}^N \subseteq \text{SeSIG}^N \subseteq \text{SeMIG}^N$, so we immediately obtain that all these sets are not core-coincident as well for nontrivial player sets. Also, $\text{CIG}^N \subseteq \text{SIG}^N$, so superadditive interval games are not core-coincident either.

From this we conclude that selection core and interval core behave differently on many important and widely used classes with nontrivial uncertainty. Therefore, to further develop theory and solve problems regarding both versions of cores of interval games is an important task.

## 5 The Shapley value

**Preliminaries and definitions.** Before we list the axioms we need in this section, a few definitions are needed.

Every function $f : \text{IG}^N \to \mathbb{R}^N$ is called **interval value function**. We omit **interval** when context is clear.

- Two intervals $I, J$ are said to be **indifferent** if $\frac{I + J}{2} = \frac{J + I}{2}$. We denote it by $I \sim J$.
- Let $(N, w) \in \text{IG}^N$ and $i \in N$. Then, $i$ is called a **null player** in $w$ if $w(S) = w(S \cup i)$ for every $S \subseteq N \setminus \{i\}$.
- Let $(N, w) \in \text{IG}^N$ and $i \in N$. Then, $i$ is called a **total null player** in $w$ if $w(S) \ominus w(S \cup i) = [0, 0]$ for every $S \subseteq N \setminus \{i\}$. In other words, $i$ is total null player if it is a dummy player in every selection.
Let \((N, w) \in IG^N\) and \(i, j \in N\). Then, \(i\) and \(j\) are symmetric players in \((N, w)\), if \(w(S \cup i) = w(S \cup j)\) for every \(S \subseteq N \setminus \{i, j\}\).

We can state a few axioms.

1. Indifference efficiency (IEFF): \(\sum_{i \in N} \Psi_i(w) \sim w(N)\) for all \((N, w) \in IG^N\).
2. Efficiency (EFF): \(\sum_{i \in N} \Psi_i(w) = w(N)\) for all \((N, w) \in IG^N\).
3. Indifference null player property (INP): There exists a unique \(t \geq 0\) such that \(\Psi_i(w) = [-t, t]\) for any \((N, w) \in IG^N\) and all null players \(i\) in \((N, w)\).
4. Total null player property (TNP): For every total null player \(i\) in a game \((N, w)\), \(\Psi_i(w) = [0, 0]\).
5. Symmetry (SYM): \(\Psi_i(w) = \Psi_j(w)\) for all \((N, w) \in IG^N\) and all symmetric players \(i\) and \(j\) in \((N, w)\).
6. Additivity (ADD): \(\Psi(v + w) = \Psi(v) + \Psi(w)\) for all \((N, v), (N, w) \in IG^N\) with \((N, v + w) \in IG^N\).

**Definition 28.** The interval Shapley value extension is a value function \(\Phi^* : IG^N \rightarrow IR^N\),

\[\Phi^*_i(w) := \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n - |S| - 1)!}{n!}(w(S \cup i) \ominus w(S)).\]

**Theorem 7.** \([17]\) The function \(\Phi^*\) satisfies axioms IEFF, INP, SYM, and ADD. Furthermore, it is the only function satisfying these axioms.

We now prove an important, yet never noted and proved property of the interval Shapley value extension.

**Theorem 8.** For every interval game \((N, w) \in IG^N\), we have

\[\Phi^*_i(w) = \{\phi_i(v) | v \in Sel(w)\}.\]

**Proof.** Every resulting value from \(\Phi^*_i(w)\) corresponds to some selection of \(w\) by Definition 28 and interval arithmetics, more precisely because of Moore’s subtraction (Definition 8). The converse holds as well. \(\square\)

In other words, the interval Shapley value extension contains exactly all possible Shapley values that can be attained when uncertainty is settled. We find this property very important.

However, as is noted in [17], efficiency is not always satisfied. Let us explain this issue. From properties of interval arithmetics, we see that \(X - X\) is not equal to \([0, 0]\) in general for \(X \in IR\). In fact, for every interval \(A, B \in IR\), \(|A + B| \geq \min\{|A|, |B|\}\). An analogous fact holds for Moore’s subtraction as well. Since, by definition of the interval Shapley value extension, in \(\sum_{i \in N} \Phi^*_i(w)\) are some intervals added and subtracted multiple times, the resulting value does not satisfy efficiency if we first compute \(\Phi^*_i(w)\) for every \(i\), and only then add them together. This is the reason why EFF is not satisfied in general. We can first simplify \(\sum_{i \in N} \Phi^*_i(w)\) and only then add it together. Then we would get the efficiency by the same reasoning as we get efficiency for the Shapley value in classical games.
Another axiomatization. The following theorem shows a different axiomatization of the interval Shapley value extension than [17]. We show that the axiom TNP can be interchanged with the axiom INP, which is, from our point of view, more natural.

Theorem 9. There is a unique value function satisfying axioms IEFF, TNP, SYM and ADD. Furthermore, it equals $\Phi^\ast$.

Proof. If a value function satisfies IEFF, INP, SYM, and ADD, then it is equal to $\Phi^\ast$. From its formula, we conclude that TNP is satisfied.

Now in the other direction, if a value function satisfies IEFF, TNP, SYM and ADD we want to show that it satisfies INP as well.

Our goal is to prove that in every game $(N', w')$ with a null player $h$, $\phi^\ast_h(w')$ is an interval symmetric around zero.

It suffices to prove that:

- If $k \in \Phi^\ast_h(w')$, then $-k \in \Phi^\ast_h(w')$, and
- If $a < b$, and $a, b \in \Phi^\ast_h(w')$, then $[a, b] \subseteq \Phi^\ast_h(w')$.

Both of these claims can be proved by using ADD axiom and the fact, that on degenerated game, $\Phi^\ast$ coincides with $\phi$. We omit technical details here.

We know that every null player gets a symmetrical interval under a value function $\phi^\ast$ satisfying IEFF, TNP, SYM and ADD. So the only remaining option is that there must exist a game in which two null players get a different symmetrical interval. Let us denote such game as $(N'', w'')$ and the two null players as $i$ and $j$.

Observe from the definition of null player that

$$w''(S \cup i) = w''(S)$$

holds for every $S \subseteq N \setminus \{i\}$, and thus, specially, for every $S \subseteq N \setminus \{i, j\}$. Following the same reasoning, we arrive on conclusion that

$$w''(S \cup j) = w''(S)$$

holds for every $S \subseteq N \setminus \{i, j\}$. Combining this, we get that

$$w''(S \cup j) = w''(S \cup i), \forall S \subseteq N \setminus \{i, j\}.$$ 

That means that $i$ and $j$ are symmetrical and from the axiom SYM, $\phi^\ast_i(w'')$ should be equal to $\phi^\ast_j(w'')$, a contradiction.

Finally, we note that the independence of properties IEFF, TNP, SYM, and ADD follows from Theorem 7 and from [23].

Theorem 10. For every $(N, w) \in \text{SeCIG}$, we have $\text{gen}(\Phi^\ast(w)) \subseteq \mathcal{SC}(w)$.

Proof. From Theorem 8 the Shapley value of every selection is in $\text{gen}(\Phi^\ast(w))$. Since every selection of $(N, w)$ is a convex game, its Shapley value lies in its core and thus also in $\mathcal{SC}(w)$. \qed
On the improved interval Shapley-like value. In Han et al. [17], an improved Shapley-like value satisfying EFF is presented.

**Definition 29.** (The improved interval Shapley-like value) For any \((N, w) \in IG^N\) with \(\sum_{i \in N} \Phi^*_i(w) \neq w(N)\), the improved interval Shapley-like value \(I\Phi^*(w)\) is defined by

\[
I\Phi^*_i(w) := \Phi^*_i(w_m) + \frac{|\Phi^*_i(w_u)|}{\sum_{i \in N} |\Phi^*_i(w_u)|} \left[ -\frac{1}{2} |w(N)|, \frac{1}{2} |v(N)| \right],
\]

where \(w_m(S) := \frac{w(S) + w(S)}{2}\).

**Theorem 11.** For every interval game \((N, w) \in IG^N\), we have

\[
I\Phi^*_i(w) \neq \{ \phi_i(v) | v \in \text{Sel}(w) \}.
\]

**Proof.** We use Theorem 8 and Definition 28. \(\square\)

We believe that this is a big downside of the improved interval Shapley-like value. We borrow a game from [17] to illustrate the theorem.

**Example 2.** Let \((N, v)\) be a three-person interval game where \(v(1) = [0, 2], v(2) = [1/2, 3/2], v(3) = [1, 2], v(1, 2) = [2, 3], v(2, 3) = [4, 4], v(1, 3) = [3, 4],\) and \(v(1, 2, 3) = [6, 7].\) Then \(\Phi^*(v) = ([11/12, 31/12], [7/6, 17/6], [23/12, 43/12]).\) However, \(I\Phi^*(v) = ([19/12, 23/12], [11/6, 13/6], [31/12, 35/12]).\)

By Theorem 11 there must be a selection \(v'\) of \((N, v)\), such that \(\phi_1(v') = 11/12.\) But this value is not contained in \(I\Phi^*_1(v)\).

6 Conclusion and future research

We investigated convexity in interval games, core coincidence problem and interval Shapley value. To this end, we would like to summarize our results.

- We showed a Shapley-like characterization of selection convex interval games in Theorem 8.
- We tried to characterize all core-coincident games. Our main contribution is Theorem 6 saying that a large class of interval games is not core-coincident. This result implies that many classes, including \(\text{CIG}^N, \text{ScIG}^N,\) and strongly balanced games are not core-coincident.
- We analyzed interval Shapley value extension for interval games. We emphasized several facts which speak in favor of using this solution concept. Also, we showed a different, from our point of view more natural axiomatization of this value function in Theorem 9.

Apart from the open problems presented in the papers from the references we think it could be interesting to define prekernel for interval games and axiomatically characterize it, analogously to Peleg [21]. Also, interval games with communication structures were not studied yet. See Bilbao’s book [7] for a theoretical background.
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