A gravitational field can be seen as the anholonomy of the tetrad fields. This is more explicit in the teleparallel approach, in which the gravitational field-strength is the torsion of the ensuing Weitzenböck connection. In a tetrad frame, that torsion is just the anholonomy of that frame. The infinitely many tetrad fields taking the Lorentz metric into a given Riemannian metric differ by point-dependent Lorentz transformations. Inertial frames constitute a smaller infinity of them, differing by fixed-point Lorentz transformations. Holonomic tetrads take the Lorentz metric into itself, and correspond to Minkowski flat spacetime. An accelerated frame is necessarily anholonomic and sees the electromagnetic field strength with an additional term.

I. INTRODUCTION

Anholonomy — the property of a differential form which is not the differential of anything, or of a vector field which is not a gradient — is commonplace in many chapters of Physics. Heat and work, for instance, are typical anholonomic coordinates on the space of thermodynamic variables, and the angular velocity of a generic rigid body is a classical example of anholonomic velocity. In gravitation theory, however, anholonomy does not seem to have had its pervading role as emphasized as it should. We intend here to fill in that gap, by bringing to the forefront the anholonomic character of some well-known features.

We are going to use the notation \( \{e_a, e^a\} \) for general linear frames, and \( \{h_a, h^a\} \) for a generic tetrad field, with the Lorentz indices \( a, b, c, \ldots = 0, 1, 2, 3 \) raised and lowered by the Lorentz metric

\[
\eta = \eta^{-1} = \text{diag}(1,-1,-1,-1).
\]

Greek indices \( \mu, \nu, \rho, \ldots = 0, 1, 2, 3 \) will refer to the Riemannian spacetime. Curve parameters will be indicated by \( u \) and \( v \), with the correspondent tangent fields denoted by the respective capitals \( U \) and \( V \), as in

\[
U = \frac{d}{du} = U^\lambda \partial_\lambda = \frac{dx^\lambda}{du} \partial_\lambda.
\]

The notation \( i, j, k, \ldots = 1, 2, 3 \) is reserved for space indices. Parenthesis \( (\mu\nu\rho\ldots) \) and brackets \( [\mu\nu\rho\ldots] \) indicate symmetrization and antisymmetrization of included indices. Thus, \( \Gamma^\lambda_{(\mu\nu)} = \frac{1}{2}(\Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\nu\mu}) \) and \( \Gamma^\lambda_{[\mu\nu]} = \frac{1}{2}(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}) \) designate the symmetric and antisymmetric parts of \( \Gamma^\lambda_{\mu\nu} \).

A spacetime is a 4-dimensional Riemannian manifold whose tangent space at each point is a Minkowski spacetime \(^1\). Consider, on such a general spacetime, a coordinate system \( \{x^\mu\} \), and also a coordinate system \( \{y^a\} \) on the tangent Minkowski spacetime. Such coordinate systems define, on their domains of definition, local bases for vector fields, formed by the sets of gradients \( \{\frac{\partial}{\partial x^\mu}\}, \{\frac{\partial}{\partial y^a}\} \), as well as bases \( \{dx^\mu\}, \{dy^a\} \) for covector fields, or differentials. These bases are dual, in the sense that \( dx^\mu(\frac{\partial}{\partial x^\nu}) = \delta^\mu_\nu \) and \( dy^a(\frac{\partial}{\partial y^b}) = \delta^a_b \). On the respective domains of definition, any vector or covector can be expressed in terms of these bases, which can furthermore be extended by direct product to constitute bases for general tensor fields.

A “holonomic” base like \( \{\frac{\partial}{\partial x^\mu}\} \), related to coordinates, is a very particular case of linear base. Any set of four linearly independent fields \( \{e_a\} \) will form another base, and will have a dual \( \{e^a\} \) whose members are such that \( e^a(e_b) = \delta^a_b \). These frame fields are the general linear bases on the spacetime differentiable manifold whose set, under conditions making of it also a differentiable manifold, constitutes the bundle of linear frames. Of course, on the common domains they are defined, the members of a base can be written in terms of the members of the other: \( e_a = e_a^\mu \partial_\mu \), \( e^a = e^a_\mu dx^\mu \), and conversely. We can consider general transformations taking any base \( \{e_a\} \) into any other set \( \{e'_a\} \) of four linearly independent fields. These transformations constitute the linear group \( GL(4, \mathbb{R}) \) of all

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A tetrad field \( \{ h_\mu \} \) will be a linear base which relates \( g \) to the Lorentz metric \( \eta = \eta_{ab} dy^a dy^b \) by
\[
\eta_{ab} = g(h_a, h_b) = g_{\mu\nu} h_\mu h_\nu.
\]
This means that a tetrad field is a linear frame whose members \( h_\mu \) are (pseudo-)orthogonal by the metric \( g \). We shall see later how two of such bases are related by the Lorentz subgroup of the linear group \( GL(4, \mathbb{R}) \). The components of the dual base members \( \{ h^a = h_\mu^a dx^\mu \} \) satisfy
\[
h^a_\mu h^\mu_\nu = \delta^a_\nu \quad \text{and} \quad h^a_\mu h_\mu^\nu = \delta^a_\nu,
\]
so that Eq. (2) has the converse
\[
g_{\mu\nu} = \eta_{ab} h^a_\mu h^b_\nu.
\]
We shall be almost exclusively interested in tetrad fields. In consequence, though many of our later statements — such as those given in Eqs. (18-21) below — hold for general linear frames, we shall specialize them accordingly.

An important point we would like to stress is that anholonomy is related to the very existence of a gravitational field. Given a Riemannian metric as in (1), the presence or absence of a gravitational field is fixed by the anholonomic or holonomic character of the forms \( h^a = h^a_\mu dx^\mu \). We can think of a change of coordinates \( \{ y^a \} \leftrightarrow \{ x^\mu \} \) represented by
\[
 dy^a = \frac{\partial y^a}{\partial x^\mu} dx^\mu = dy^a \left( \frac{\partial}{\partial x^\mu} \right) dx^\mu.
\]
The 1-form \( dy^a \) is holonomic, just the differential of the coordinate \( y^a \), and the objects \( \{ \frac{\partial y^a}{\partial x^\mu} \} \) are the components of the holonomic form \( dy^a \) written in the base \( \{ dx^\mu \} \). Thus, such a coordinate change is just a change of holonomic bases of 1-forms. Take now a dual base \( \{ h^a \} \) such that \( dh^a \neq 0 \), which is not formed by differentials. Apply the anholonomic 1-forms \( h^a \) (such that \( dh^a \neq 0 \)) to \( \frac{\partial}{\partial x^\mu} \). The results, \( h^a_\mu = h^a( \frac{\partial}{\partial x^\mu} ) \), give the components of each \( h^a = h^a_\mu dx^\mu \) along \( dx^\mu \). The procedure can be inverted when the \( h^a_\mu \)'s are linearly independent, and defines vector fields \( h_\mu = h_\mu^a \frac{\partial}{\partial x^a} \) which are not gradients. Because closed forms are locally exact, holonomy/anholonomy can be given a trivial criterion: A form is holonomic \( \text{iff} \) its exterior derivative vanishes. A holonomic tetrad will always be of the form \( \{ h^a = dy^a \} \) for some coordinate set \( \{ y^a \} \). For such a tetrad, the metric tensor (4) would be simply the components of the Lorentz metric \( \eta \) transformed to the coordinate system \( \{ x^\mu \} \). The Levi-Civita connection, or Christoffel symbol,
\[
\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} [ \partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} ],
\]
leads to a Riemann curvature tensor — the gravitational field strength in General Relativity — which vanishes if \( \{ h_\mu \} \) is holonomic. A gravitational field is present only when the tetrad fields are anholonomic.

Teleparallelism \( \text{[3]} \) provides an approach to gravitation which is both alternative and equivalent to General Relativity. The teleparallel presentation of gravity is closer to the gauge-theoretical paradigm \( \text{[4]} \) and thereby stresses the similarities between gravitation and the other fundamental interactions \( \text{[5]} \). It stresses also their main difference: By putting the accent on the tetrad frames, it highlights the inertial character of the gravitational force. In teleparallel gravity, the Weitzenbök connection
\[
\Gamma^\lambda_{\mu\nu} = h_\lambda^a \partial_\mu h^a_\nu
\]
plays a central part: Its torsion will be the gravitational field strength. We shall for that reason pay special attention to the torsions of linear connections. It should be remarked that for holonomic tetrads \( \Gamma \) is torsionless.

Our policy will be to review well-known facts while emphasizing their anholonomic content. After some preliminaries on connections and their torsions in section \( \text{[H]} \) we proceed to a \textit{resumé} on three metric-related structures: The tetrad fields, the Levi-Civita connection, and the Weitzenböck connection. In section \( \text{[I]} \) we review the usual lore on tetrad fields as introduced through the metric they determine, and section \( \text{[IV]} \) is devoted to the Levi-Civita connection. Non-inertial frames are discussed in section \( \text{[V]} \) in which it is shown that accelerated frames are necessarily anholonomic. A synopsis on teleparallelism is given in section \( \text{[VI]} \). The last section sums it all up and adds some comments on remaining questions.
II. LINEAR CONNECTIONS

Linear connections have a great degree of intimacy with spacetime because they are defined on the bundle of linear frames, which is a constitutive part of its manifold structure. That bundle has some properties not found in the bundles related to gauge theories \[6\]. Mainly, it exhibits soldering, which leads to the existence of torsion for every connection \[2\]. Linear connections — in particular, Lorentz connections — always have torsion, while gauge potentials have not. The torsion \(T\) of a linear connection \(\Gamma\) in a linear frame is just the covariant derivative of the frame members.

In a holonomic base, the torsion components are essentially the antisymmetric parts of the connection components:

\[
T^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu} - \Gamma^\lambda_{\mu\nu} = -2 \Gamma^\lambda_{[\mu\nu]}.
\] (7)

When \(T^\lambda_{\mu\nu} \neq 0\) it will be impossible to make all the components \(\Gamma^\lambda_{\mu\nu}\) equal to zero in a holonomic base. Torsion has important consequences, even if vanishing: The property \(T^\lambda_{\mu\nu} = 0\), which holds for the Levi-Civita connection of a metric, is at the origin of the well-known cyclic symmetry of the Riemann tensor components.

The condition of metric compatibility is that the metric be everywhere parallel-transported by the connection, that is, \(\nabla_{\lambda} g_{\mu\nu} = 0\), or equivalently

\[
\partial_\lambda g_{\mu\nu} = 2 \Gamma^\lambda_{(\mu\nu)}\lambda.
\] (8)

where we have used the notation \(\Gamma^\lambda_{\mu\nu\lambda} = g_{\mu\rho} \Gamma^\rho_{\nu\lambda}\). A metric defines a Levi-Civita connection \(\overset{\circ}{\Gamma}\), which is that unique connection which satisfies this condition and has zero torsion. Its components in a holonomic base are the Christoffel symbols \(5\). If a connection \(\Gamma\) preserves a metric and is not its Levi-Civita connection, then it will have \(T^\lambda_{\mu\nu} \neq 0\).

The difference between two connections is a tensor. The expression

\[
K^\lambda_{\mu\nu} = \overset{\circ}{\Gamma}^\lambda_{\mu\nu} - \Gamma^\lambda_{\mu\nu}
\] (9)
defines the contorsion tensor \(K\) of \(\Gamma\). Using \(8\) both for \(\Gamma\) and \(\overset{\circ}{\Gamma}\), we have \(\Gamma_{(\lambda\mu)\nu} = \overset{\circ}{\Gamma}_{(\lambda\mu)\nu}\) and consequently

\[
K_{(\lambda\mu)\nu} = 0.
\] (10)

Metric compatibility gives one further constraint: Contorsion is fixed by the torsion tensor:

\[
K^\lambda_{\mu\nu} = \frac{1}{2} [T^\lambda_{\mu\nu} + T^\lambda_{\nu\mu} + T^\lambda_{\nu\mu}].
\] (11)

As both \(T\) and \(K\) are tensors, this relationship holds in any basis.

When we say that some field (vector, covector, tensor, spinor) is everywhere parallel-transported by a connection, we mean the vanishing of the corresponding covariant derivative all over the domain on which field and connection are defined. This is a very strong condition. Most frequently, the interest lies in parallel-transport along a curve. Thus, for example, the geodesic equation

\[
\frac{\nabla U^\lambda}{\nabla u} \equiv \frac{dU^\lambda}{du} + \Gamma^\lambda_{\mu\nu} U^\mu U^\nu = 0
\] (12)
defines a curve \(\gamma(u)\) whose velocity field \(U\) itself is parallel-transported by \(\Gamma\) along the curve. For a general connection, this equation defines a self-parallel curve. Each connection defines an acceleration which is given by the so-called equation of force

\[
\frac{\nabla U^\lambda}{\nabla u} = a^\lambda.
\] (13)

III. THE CLASS OF FRAME FIELDS OF A METRIC

The base \(\{h_a\}\) is far from being unique. There exists actually a six-fold infinity of tetrad fields \(\{h_a = h_a^\mu \frac{\partial}{\partial x^\mu}\}\), each one relating \(g\) to the Lorentz metric \(\eta\) by Eqs. \(2\). This comes from the fact that, at each point of the Riemannian spacetime, Eq. \(1\) only determines the tetrad field up to transformations of the six-parameter Lorentz group in the anholonomic indices. Suppose in effect another tetrad \(\{h'_a\}\) such that

\[
g_{\mu\nu} = \eta_{ab} h^a_\mu h^b_\nu = \eta_{cd} h'^c_\mu h'^d_\nu.
\] (14)
Contracting both sides with $h_{c\mu}h_{f\nu}$, we arrive at
\[ \eta_{ab} = \eta_{cd} (h^c_{\mu}h^d_{\nu})(h^i_{\mu}h^j_{\nu}). \]

This equation says that the matrix with entries
\[ \Lambda^a_b = h^a_{\mu} h^b_{\mu}, \] (15)
which gives the transformation
\[ h^a_{\mu} = \Lambda^a_b h^b_{\mu}, \] (16)
satisfies
\[ \eta_{cd} \Lambda^c_a \Lambda^d_b = \eta_{ab}. \] (17)

This is just the condition that a matrix $\Lambda$ must satisfy in order to belong to (the vector representation of) the Lorentz group.

Basis $\{h_a\}$ will be anholonomic — unrelated to any coordinate system — in the generic case. This means that, given the commutation table
\[ [h_a, h_b] = f^c_{ab} h_c, \] (18)
there will be non-vanishing structure coefficients $f^c_{ab}$ for some $a, b, c$. The frame $\{\frac{\partial}{\partial x^a}\}$ has been presented above as holonomic precisely because its members commute with each other. The dual expression of the commutation table above is the Cartan structure equation
\[ dh^c = -\frac{1}{2} f^c_{ab} h^a \wedge h^b = \frac{1}{2} (\partial_x h^c_{\nu} - \partial_{\nu} h^c_{\mu}) \, dx^\mu \wedge dx^\nu. \] (19)

The structure coefficients represent the curls of the base members:
\[ f^c_{ab} = h^c([h_a, h_b]) = h^a_{\mu} h^b_{\nu} (\partial_{\nu} h^c_{\mu} - \partial_{\mu} h^c_{\nu}) = h^c_{\mu} [h_a(h_b^{\mu}) - h_b(h_a^{\mu})]. \] (20)

If $f^c_{ab} = 0$, then $dh^a = 0$ implies the local existence of functions (coordinates) $y^a$ such that $h^a = dy^a$. The tetrads are gradients when the curls vanish.

Equation (14) tells us that the components of metric $g$, in the tetrad frame, are just those of the Lorentz metric. This does not mean that the frame is inertial, because the metric derivatives — which turn up in the expressions of forces and accelerations — are not tensorial. In order to define derivatives with a well-defined tensor behavior (that is, which are covariant), it is essential to introduce connections $\Gamma^\lambda_{\mu\nu}$, which are vectors in the last index but whose non-tensorial behavior in the first two indices compensates the non-tensoriality of the usual derivatives. Connections obey in consequence a special law: In the tetrad frame, a connection $\Gamma$ has components
\[ \omega^{ab}_{\lambda} = h^a_{\nu} \left[ h_c(h_b^{\lambda}) + h^a_{\lambda} \Gamma^\lambda_{\mu\nu} h^\mu_{\nu} h^c_{\nu} \right] = h^a_{\nu} \nabla^c(h_b^{\lambda}). \] (21)

This transformation law ensures the tensorial behavior of the covariant derivative: $\nabla_{\nu} V^\lambda = h^a_{\nu} h^b_{\lambda} \nabla_a V^b = h^a_{\nu} h^b_{\lambda} \nabla_a V^b$ and $\nabla_a V^b = \Lambda^c_a \Lambda^b_d \nabla^c V^d$. The antisymmetric part of $\omega^{ab}_{\lambda}$, in the last two indices can be computed by using Eqs. (17) and (20). The result shows that torsion, seen from the anholonomic frame, includes the anholonomy:
\[ T^{ab}_{bc} = -f^a_{bc} - (\omega^{ab}_{\lambda} - \omega^{ab}_{\nu})\] (22)

There is a constraint on the first two indices of $\omega^{ab}_{\lambda}$ if $\Gamma$ preserves the metric. In effect, Eqs. (8) and (20) lead to
\[ \omega_{abc} = -\omega_{bac}. \] (23)

This antisymmetry in the first two indices, after lowering with the Lorentz metric, says that $\omega$ is a Lorentz connection. This is to say that it is of the form
\[ \omega = \frac{1}{2} J^b_a \omega^{ab}_{\lambda} h^c, \]
with $J^b_a$ the Lorentz generators written in an appropriate representation. Therefore, any connection preserving the metric appears, when its components are written in the tetrad frame, as a Lorentz-algebra valued 1-form. If we use
are matrix representations in gauge theories, like the adjoint representation, but their members are not invertible. For such Lorentz connections, use of (22) for three combinations of the indices gives

$$\omega_{ab} = - \frac{1}{2} (f_{ab} + T_{ab}^c + T_{bc}^a + T_{cb}^a + T_{cb}^a).$$

(25)

The components of a velocity $U$ are given by the holonomic form $dx^\mu$ applied to the time-evolution vector field $\frac{d}{du}$, that is,

$$U^\mu = \frac{dx^\mu}{du} = dx^\mu \left( \frac{d}{du} \right).$$

The velocity $U^\mu$ represents, consequently, the variation of the coordinate $x^\mu$ in time $u$. In the tetrad frame $\{h_a\}$, $U$ has components

$$U^a = h^a_\mu U^\mu = h^a_\mu \frac{dx^\mu}{du} = h^a_\mu \left( \frac{d}{du} \right),$$

(26)

If $\{h_a\}$ is holonomic, then $h^a_\mu = \frac{\partial y^a}{\partial x^\mu} dx^\mu$ for some coordinates $\{y^a\}$, and $U^a$ measures the variation of coordinate $y^a$ in time $u$. If $\{h_a\}$ is not holonomic, however, $U^a$ will be an anholonomic velocity: Its components will be the variations of no coordinates with time (a classical non-relativistic example has been mentioned in the Introduction, the angular velocity of a rigid body in the general, non-planar case). We have said that the tetrad frame “sees” everything in terms of the flat, Minkowski space coordinates. The difference with respect to “native” special-relativistic objects lies in the anholonomic character of the frame. An usual holonomic velocity $U^\mu$ in Riemann spacetime, for example, becomes, in the tetrad frame, an anholonomic velocity, whose components $U^a$ in flat Minkowski space are not derivatives of any coordinate with respect to time. A “native” special-relativistic observer would see a holonomic velocity $V^a = dx^a/d\sigma$, with $d\sigma^2 = \eta_{ab} dx^a dx^b$. In the tetrad frame $\{h_a\}$, the equation of force (13) has the form

$$\frac{dU^a}{du} + \omega_{bc} U^b U^c = a^a,$$

(27)

where $\omega_{bc}$ and $U^a$ are given by (24) and (25) respectively.

The Riemannian metric $g = (g_{\mu\nu})$ is a Lorentz invariant, for which any two tetrad fields as $\{h_a\}$ and $\{h'_a\}$ in (14) are equivalent. A metric corresponds to an equivalence class of tetrad fields, the quotient of the set of all tetrads by the Lorentz group. The sixteen fields $h^a_\mu$ correspond, from the field-theoretical point of view, to ten degrees of freedom — like the metric — once the equivalence under the six-parameter Lorentz group is taken into account.

The tetrads belong to the carrier space of a matrix representation of the Lorentz group. They have, however, a very special characteristic: They are themselves invertible matrices. A group element taking some member of the representation space into another can in consequence be written in terms the initial and final members, as in (15). This establishes a deep difference with respect to the other fundamental interactions, described by gauge theories. There are matrix representations in gauge theories, like the adjoint representation, but their members are not invertible.

IV. A PREFERRED CONNECTION

A metric $g$ defines a preferred connection, the Levi-Civita connection $\Gamma$ given by (15) which is, we repeat, the single connection preserving $g$ which has zero torsion. Its curvature Riemann tensor,

$$R_{\rho^\lambda \mu^\nu} = \partial_\rho \Gamma_{\mu^\lambda \nu} - \partial_\nu \Gamma_{\rho^\lambda \mu} + \Gamma_{\rho^\sigma \nu} \Gamma_{\sigma^\mu \rho} - \Gamma_{\rho^\sigma \mu} \Gamma_{\sigma^\nu \rho},$$

is the covariant representative of the gravitational field in General Relativity. The Lorentz connection $\tilde{\omega}$ obtained via a tetrad field $h_a$ is, in this case, usually called “spin-connection”. It appears, for example, in the Dirac equation (7)

$$i\hbar \gamma^c \psi \left( \partial_\mu - i \frac{1}{4} \omega^{ab}_\mu \sigma_{ab} \right) \psi \equiv i\hbar \gamma^c \left( h_c - i \frac{1}{4} \omega^{ab}_c \sigma_{ab} \right) \psi = mc\psi,$$

(28)
Matrix $\Lambda$ has the form

$$\omega^a_{\mu b} = h^a_\lambda \hat{\Gamma}^\lambda_{\mu \nu} h^\nu_b + h^a_\mu \partial_\mu h_b^\nu.$$  (29)

This expression, combined with (20), gives

$$\omega^a_{\nu c} - \omega^a_{eb} = f^a_{eb}. $$  (30)

We see that, once looked at from the frame $\{h_a\}$, the symmetric connection $\hat{\Gamma}$ acquires an antisymmetric part, which has only to do with the anholonomy of the basis. That this is a mere artifact due to the frame anholonomy is better seen in an example in electromagnetism. In effect, a symmetric connection does not alter the expression $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ of the field strength in terms of the electromagnetic potential $A_\mu$. In frame $\{h_a\}$, however,

$$h^a_\mu h^\nu_b \left( \partial_\mu A_\nu - \hat{\Gamma}^\lambda_{\mu \nu} A_\lambda \right) = h^a_\mu A_\nu - \omega^\nu_{\nu c} A_c,$$

so that (30) leads to

$$F_{ab} = h_a(A_b) - h_b(A_a) - f^c_{ab} A_c.$$  (31)

On the other hand, this is exactly what comes out from a direct calculation of the invariant form $F = dA = d\langle A_a h^a \rangle$ by using (19) in the absence of any connection. Notice that the last term in the expression above is essential to the invariance of $F_{ab}$ under a $U(1)$ gauge transformation as seen from the frame $\{h_a\}$, which is $A_a \rightarrow A'_a = A_a + h_a \phi$:

$$F'_{ab} = h_a A'_b - h_b A'_a + f^c_{ab} A'_c = F_{ab} + h_a h_b \phi - h_b h_a \phi - f^c_{ab} h_c \phi = F_{ab}.$$  

The force equation (27) can be expressed, by using (25) with $T^a_{bc} = 0$, in terms of the anholonomy coefficients as

$$\frac{dU^a}{du} + f^a_{bc} U^b U^c = a^a.$$  (32)

The Riemann curvature tensor will have tetrad components

$$\hat{R}^a_{bcd} = h^a_\rho h^\rho_b h^\sigma_c h^\tau_d \hat{\Gamma}^{\rho \sigma \tau}_{\mu \nu},$$

which gives

$$\hat{R}^a_{bcd} = h_c \omega^a_{bd} - h_d \omega^a_{bc} + \omega^a_{ec} \omega^e_{bd} - \omega^a_{ed} \omega^e_{bc} - f^e_{cd} \omega^a_{be}. $$  (33)

V. NON-INERTIAL FRAMES

Another tetrad frame $\{h'_a\}$ will see another spin connection, that is, will see the connection $\hat{\omega}$ with other components, as given by (24). Suppose for a moment the frame $\{h'_a\}$ to be such that $\hat{\omega}^a_{\nu b} = 0$ (such a frame does exist at each point, and along a differentiable curve, see below). In that case $\hat{\omega}$ would be a pure gauge of the Lorentz group,

$$\hat{\omega}^a_{\mu b} = h^c_\nu \hat{\omega}^a_{\nu c} = (\Lambda^{-1})^a_c \partial_\nu A^c_b = (\partial_\nu \ln \Lambda)^a_b.$$  (34)

Matrix $\Lambda$ has the form

$$\Lambda = \exp W = \exp \left[ \frac{1}{2} J_{cd} \alpha^{cd} \right],$$

with $J_{cd}$ denoting the generators and $\alpha^{cd}$ the parameters of the Lorentz transformation. Therefore,

$$\hat{\omega}^a_{\nu b} = (\Lambda^{-1} d\Lambda)^a_b = (d \ln \Lambda)^a_b = (dW)^a_b = dW^a_b.$$  (35)

Furthermore, using the vector representation for $J_{cd}$,

$$W^a_{\nu b} = \frac{1}{2} (J_{cd})^a_{eb} \alpha^{cd} = \frac{1}{2} (\eta_{\nu d} \delta^a_c - \eta_{\nu b} \delta^a_d) \alpha^{cd} = \frac{1}{2} (\alpha^a_b - \alpha_b^a) = \alpha^a_b,$$
so that Eq. (34) is the same as

$$\hat{\omega}^{ab}_{\mu} = h_c (a^a b) \, .$$

Notice that A represents here that very special Lorentz transformation taking \( \{ h_a \} \) into a tetrad \( \{ h'_a \} \) in which the connection has vanishing components. From Eq. (24) written for \( h'_a \) in the form

$$\partial_\nu h'_a \lambda + \tilde{\varphi}^{\lambda}_{\mu \nu} h'_a \mu = h'_c \lambda \omega^{c}_{\alpha \nu},$$

the condition \( \hat{\omega}^{c}_{\alpha \nu} = 0 \), if valid on a general domain, would lead to vanishing curvature. Take however the integral curve \( \gamma \) of a vector field \( U \) with fixed initial values. Then, the condition

$$U^{\nu} \partial_\nu h'_a \lambda + \tilde{\varphi}^{\lambda}_{\mu \nu} h'_a \mu = h'_c \lambda \omega^{c}_{\alpha \nu} U^{\nu} = 0$$

is possible even in the presence of curvature: It means simply that the four tetrad vectors \( h'_a \) are parallel-transported along \( \gamma \). It is a deep result [8-11] that the connection \( \hat{\omega}^{c}_{\alpha \nu} \) can be made to vanish at a point of \( \gamma \) by a choice of \( \{ h'_a \} \), and that this frame can be propagated along it while preserving this property. Each vector \( h'_a \) will then feel no force along \( \gamma \), as \( \hat{\omega}^{c}_{\alpha \nu} U^{\nu} = 0 \) all along. This characterizes an inertial frame, in which Special Relativity applies. If the curve is timelike, an observer attached to this frame will be an inertial observer [12, 13, 14]. As every other frame can be got from it at each point by a Lorentz transformation, General Relativity appears as a gauge theory for the Lorentz group along the curve. Distinct curves require different frames, and one same frame cannot be parallel-transported along two distinct intersecting curves unless the Riemann curvature tensor vanishes. A clear statement of the equivalence principle along these lines can be found in Ref. [17].

The timelike member \( h_0 \) of a set \( \{ h_a \} \) of vector fields constituting a tetrad will define, for each set of initial conditions, an integral curve \( \gamma \). It is always possible to identify \( h_0 \) to the velocity of frame \( \gamma \) as \( \hat{\omega}^{a}_{\nu 0} \). The frame, as it is carried along that timelike curve, will be inertial or not, according to the corresponding force law. The force equation can be obtained by using, for example, Eq. (29) written for \( h_0 \):

$$\partial_\nu h_0 \lambda + \tilde{\varphi}^{\lambda}_{\mu \nu} h_0 \mu = h_a \lambda \omega^{a}_{\nu 0} \, .$$

This leads, with \( U = \frac{d}{du} h_0 \), to the expression

$$h_0 \nu \partial_\nu h_0 \lambda + \tilde{\varphi}^{\lambda}_{\mu \nu} h_0 \mu h_0 \nu = U^{\nu} \partial_\nu h_0 \lambda + \tilde{\varphi}^{\lambda}_{\mu \nu} U^{\mu} U^{\nu} = h_a \lambda \omega^{a}_{\nu 0} h_0 \nu ,$$

implying the frame acceleration

$$a^{\lambda} = h_a \lambda \omega^{a}_{\nu 0} \, .$$

(38)

The relation to anholonomy is given by Eq. (25), torsion turning up as an accelerating factor:

$$a^{\lambda} = h_a \lambda \omega^{a}_{\nu 0} = - h_a \lambda (f_{00}^{\alpha} + T_0^{\alpha}) = - \eta_{00} h_b \lambda (f^{c}_{0b} + T^{c}_{0b}) \, .$$

(39)

Let us examine what happens in the absence of torsion. The acceleration is then measured by the timelike component of the tetrad commutators involving the timelike member,

$$a^{\lambda} = h_k \lambda \omega^{k}_{00} = h_k \lambda f^{0}_{0k} = h_k \lambda dh^0 ([h_0 , h_k]) \, .$$

(40)

It follows that an accelerated frame is necessarily anholonomic: It must have at least \( f^{0}_{0k} \neq 0 \). From Eq. (36), the transformation to an inertial frame involves only time-derivatives of boost parameters (essentially the relative velocity):

$$\omega^{k}_{00} = h_0 (\alpha^k_0) \, .$$

(41)

In the inertial frame \( h' \), the velocity of frame \( h \) will have for components the boost transformations: \( U^{\nu}_{0} = h'_{\nu} h_0^{\mu} = \Lambda^{\nu}_{\mu} \). Something about the behavior of the spacelike members of the tetrad along the curve \( \gamma \) can be obtained from Eq. (29) for \( h_1 \). Indicating by \( a_i^{(1)} \) the covariant change rate of \( h^1 \), we find

$$a_i^{(1)} = \nabla_U h_i^{(1)} = h_a \lambda \omega^{a}_{i0} = h_a \lambda h_1 (\alpha^k_0) = \frac{1}{2} h_c \lambda (f_{ic0} + f_{0ci} + f_{o0i}) = h_c \lambda h_0 (\alpha^c_1) \, .$$
\[ \nabla_U h_a^\lambda = h_c^\lambda \tilde{\omega}^c_{ab} U^\nu. \]

for any \( U \), the Fermi-Walker derivative will be

\[ \nabla_{(FW)}^U h_a^\lambda = \nabla_U h_a^\lambda + a_a U^\lambda - U^a a^\lambda. \]

The particular case

\[ \nabla_{(FW)}^U h_0^\lambda = \nabla_U h_0^\lambda - U^a_0 a^\lambda = 0 \]

implies that \( h_0 \) is kept tangent along the curve. The other tetrad members, however, rotate with angular velocity

\[ \omega^k = \frac{1}{2} \epsilon^{kij} \omega_{ij} = \frac{1}{2} \epsilon^{kij} f^i_{\ j0}, \quad (42) \]

which shows the \( \tilde{\omega}_a^b \)'s in their role of Ricci’s coefficient of rotation \[ \text{[16]} \].

As another example, by Eq. (31), the electromagnetic field, when looked at from a non-inertial frame, will forcibly include extra, anholonomy-related, terms:

\[ F_{0k} = h_0(A_k) - h_k(A_0) - f^a_{\ 0k} A_a = h_0(A_k) - h_k(A_0) + a_k A_0 + \epsilon_{kij} \omega^l A^l \]

\[ F_{jk} = h_j(A_k) - h_k(A_j) - f^a_{\ jk} A_a. \]

In the simplest gauge \((A^0 = 0)\), the electric field reduces to the Euler derivative

\[ E = \frac{dA}{du} + \omega \wedge A. \quad (43) \]

### VI. TELEPARALLELISM

Each tetrad \( \{h_a\} \) defines a special connection, the Weitzenböck connection given by \[ \text{[10]} \]. That connection has some very interesting properties:

1. It has vanishing components in the tetrad frame \( \{h_a\} \) itself \[ \text{[2]} \]:

\[ \tilde{\omega}_\lambda^a h_\nu = h_a^\lambda \tilde{\Gamma}_\mu^\lambda h_\nu^\mu + h_\nu^\rho \partial_\nu h_\rho^\mu = 0. \quad (44) \]

2. Justifying the name “teleparallelism”, it parallel-transports each vector of the tetrad \( \{h_a\} \) everywhere:

\[ \nabla^\lambda h^\mu_\nu \equiv \partial^\lambda h^\mu_\nu + \tilde{\Gamma}_\mu^\rho h^\lambda_\rho = 0. \quad (45) \]

3. In consequence, it preserves the metric \( g \):

\[ \nabla^\lambda g_\mu_\nu = 0. \]

4. It has vanishing Riemann curvature tensor:

\[ R^\sigma_\sigma^\mu_\nu = 0. \]

5. It has a non-vanishing torsion \( T \):

\[ T^\lambda_\nu_\mu = h_a^\lambda (\partial_\nu h^a_\mu - \partial_\mu h^a_\nu) = h_a^\lambda f^a_\mu h^b_\nu. \quad (46) \]

In the frame \( \{h_a\} \) itself, this torsion is pure anholonomy and, consequently, a measure of the non-triviality of the metric \( g \).

6. The Levi-Civita and the Weitzenböck connections are related by

\[ \tilde{\Gamma}_\lambda_\mu_\nu = \tilde{\Gamma}_\lambda_\mu_\nu - T^\lambda_\mu_\nu \tilde{\Gamma}_\lambda. \quad (47) \]

7. In consequence the geodesic equation of General Relativity acquires, in terms of \( \tilde{\Gamma} \), the form of a force equation:

\[ \frac{dU^\lambda}{du} + \tilde{\Gamma}_\lambda_\mu_\nu U^\mu U^\nu = - K^\lambda_\mu_\nu U^\mu U^\nu. \quad (48) \]
8. Use of Eq. 14 for the specific case of the Weitzenböck connection gives its contorsion as

$$\tilde{K}^\lambda_{\mu\nu} = h_a^\lambda \tilde{\omega}^a_{\mu\nu} h^b_\mu.$$  

(49)

This means that $\tilde{\omega}$ is the Weitzenböck contorsion of a tetrad $h$ seen from the frame $\{h_a\}$:

$$\tilde{\omega}^a_{bc} = \tilde{K}^a_{bc}.$$  

9. The index symmetries give to the force equation (18) the form

$$\frac{dU^\lambda}{du} + \bar{\Gamma}^\lambda_{\mu\nu} U^\mu U^\nu = h_a^\lambda f_b^a c U^b U^c.$$  

(50)

The right-hand side "force" is one more measure of the tetrad non-holonomy.

The Weitzenböck connection is a kind of (curvature) "vacuum" of every other connection. In fact, a general connection with holonomic components $\Gamma^\lambda_{\mu\nu}$ will be related to its non-holonomic components $\omega^a_{\mu
u}$ by

$$\Gamma^\lambda_{\mu\nu} = h_a^\lambda \omega^a_{\mu\nu} h^b_\mu + \bar{\Gamma}^\lambda_{\mu\nu},$$

which is actually the inverse of (21), with a further substitution of Eq. (6). Suppose then we look at the Weitzenböck connection of a tetrad $h^a_\mu$ from another tetrad $h'^a_\mu$. It will have the expression

$$\omega'^a_{\mu\nu} = h'^a_\lambda \Gamma^\lambda_{\mu\nu} h'_b_\mu + h'_a_\mu \partial_b h'_b_\mu = (\Lambda \partial_b \Lambda^{-1})^a_b,$$

(51)

with $\Lambda$ as given by Eq. (15). The Weitzenböck connection of a tetrad $h^a$, when looked at from another tetrad $h'^a$, is the vacuum of a gauge theory for the Lorentz group (whose corresponding field strength would be the curvature tensor). The "gauge", or the group element, is just that relating the two connections.

We can also consider the difference between the Weitzenböck connections of two different tetrad fields. If $\bar{\Gamma}$ is related to $h^a$ and $\Gamma'$ to $h'^a$, then

$$\bar{\Gamma}^\lambda_{\mu\nu} - \Gamma'^\lambda_{\mu\nu} = h'^a_\lambda h^b_\mu [\Lambda \partial_b \Lambda^{-1}]^a_b,$$

(52)

which tells us that two distinct tetrads, $h^a$ and $h'^a$, can have the same Weitzenböck connection. In that case, they differ by a point-independent Lorentz transformation. Along a curve of parameter $u$, the accelerations defined by these connections will differ as

$$\bar{\alpha}^\lambda - \alpha'^\lambda = h'^a_\lambda h^b_\mu U^\mu U^\nu [\Lambda \partial_b \Lambda^{-1}]^a_b.$$  

(53)

VII. FINAL COMMENTS

There is a functional six-fold infinity of tetrad fields determining a given metric as in Eq. (11). This six-foldedness is "functional" because such tetrad fields differ by point-dependent (that is, local) Lorentz transformations. Anholonomy is essential to the presence of a gravitational field: All holonomic tetrads correspond to Minkowski flat space. Each tetrad field defines also a Weitzenböck flat connection, whose torsion measures its anholonomy and represents, in the teleparallel approach, the gravitational field strength. There is a (non-functional) six-fold infinity of tetrad fields with the same Weitzenböck connection, differing from each other by point-independent (global) Lorentz transformations. As each result of General Relativity can be stated in terms of the tetrad anholonomy, gravitation reduces to frame effects. In General Relativity the absence or presence of gravitation is signaled by the vanishing or not of a covariant derivative, the curvature tensor. The field is a "covariant" anholonomy. In teleparallelism, the presence of field is signaled by a simple anholonomy, that of the tetrad field itself. In the tetrad frame, everything happens in Minkowski space, but the frame will be, we insist, necessarily anholonomic. A holonomic velocity in Riemann space becomes, once written with components in the tetrad frame, an anholonomic velocity in flat tangent Minkowski space.

A better understanding of the relationship between the standard formulation of General Relativity and teleparallelism is still necessary. In particular, it should be decided which field is fundamental — metric or tetrad. The equivalence of both approaches may come to disappear at the quantum level. If an interaction is mediated by a spin-2 field, matter can attract both matter and antimatter, but mediating vector (spin-1) fields would give opposite signs for matter-matter and matter-antimatter interactions 21. Antimatter produced by high-energy matter collisions, however small its amount, would produce a cosmic repulsion. Whether or not the exchange of constrained four-vectors can be equivalent to that of a spin-2 field is an open question.

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