Random complex zeroes, III.
Decay of the hole probability

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Abstract
The ‘hole probability’ that a random entire function

$$\psi(z) = \sum_{k=0}^{\infty} \zeta_k \frac{z^k}{\sqrt{k!}},$$

where $$\zeta_0, \zeta_1, \ldots$$ are Gaussian i.i.d. random variables, has no zeroes in the disc of radius $$r$$ decays as $$\exp(-cr^4)$$ for large $$r$$.

We consider the (random) set of zeroes of a random entire function

$$\psi_\omega : \mathbb{C} \rightarrow \mathbb{C},$$

(0.1)

$$\psi(z, \omega) = \sum_{k=0}^{\infty} \zeta_k(\omega) \frac{z^k}{\sqrt{k!}},$$

where $$\zeta_k, k = 0, 1, 2, \ldots$$ are independent standard complex-valued Gaussian random variables, that is the distribution $$\mathcal{N}_\mathbb{C}(0, 1)$$ of each $$\zeta_k$$ has the density $$\pi^{-1} \exp(-|w|^2)$$ with respect to the Lebesgue measure $$m$$ on $$\mathbb{C}$$. This model is distinguished by invariance of the distribution of zero points with respect to the motions of the complex plane

$$z \mapsto az + b, \quad |a| = 1, \ b \in \mathbb{C},$$

see [6] for details and references.

Given large positive $$r$$, we are interested here in the ‘hole probability’ that $$\psi$$ has no zeroes in the disc of radius $$r$$

$$p(r) = \mathbb{P}(\psi(z, \cdot) \neq 0, \ |z| \leq r).$$

It is not difficult to show that $$p(r) \leq \exp(-c_\text{const} r^2)$$, see the Offord-type estimate in [3]. Yuval Peres told one of us that the recent work [4] led to conjecture that the actual hole probability might have a faster decay. In this note, we confirm this conjecture and prove
Theorem 1. \( \exp(-Cr^4) \leq p(r) \leq \exp(-cr^4) \).

Throughout, by \( c \) and \( C \) we denote various positive numerical constants whose values can be different at each occurrence.

It would be interesting to check whether there exists the limit

\[
\lim_{r \to \infty} \frac{\log p(r)}{r^4},
\]

and (if it does) to find its value.

The lower bound in Theorem 1 will be obtained in Section 1 by a straightforward construction. The upper bound in Theorem 1 follows from a large deviation estimate which has an independent interest.

Theorem 2. Let \( n(r) \) be a number of random zeroes in the disc \( \{ |z| \leq r \} \). Then for any \( \delta \in (0, \frac{1}{4}] \) and \( r \geq 1 \)

\( \Pr \left( \left| \frac{n(r)}{r^2} - 1 \right| \geq \delta \right) \leq \exp(-c(\delta)r^4). \)

Throughout, by \( c(\delta) \) we denote various positive constants which depend on \( \delta \) only. Since our argument is too crude to find a sharp constant \( c(\delta) \) in (1.2), we freely change the values of \( c(\delta) \) from line to line.

There is a fruitful analogy between random zero sets and one component Coulomb system which consists of charged particles of one sign in \( \mathbb{R}^2 \) embedded in a uniform background of the opposite sign (see [2] and references therein). Theorems 1 and 2 are consistent with the corresponding results for Coulomb systems [3].

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1 Proof of the lower bound in Theorem 1

In what follows, we frequently use two elementary facts: if \( \zeta \) is a standard complex Gaussian variable, then

(1.1) \( \Pr(|\zeta| \geq \lambda) = \frac{1}{\pi} \int_{|w| \geq \lambda} e^{-|w|^2} \, dm(w) = \int_{\lambda^2}^{\infty} e^{-t} \, dt = e^{-\lambda^2}, \)

and for \( \lambda \leq 1 \)

(1.2) \( \Pr(|\zeta| \leq \lambda) = 1 - e^{-\lambda^2} = \lambda^2 - \frac{\lambda^4}{2!} + \ldots \in \left[ \frac{\lambda^2}{2}, \lambda^2 \right]. \)
By $\Omega_r$ we denote the following event: (i) $|\zeta_0| \geq 2$; (ii) $|\zeta_k| \leq \exp(-2r^2)$ for $1 \leq k \leq 48r^2$; and (iii) $|\zeta_k| \leq 2^k$ for $k > 48r^2$. Since $\zeta_k$ are independent,

$$
P(\Omega_r) = P(i) \cdot P(ii) \cdot P(iii).
$$

Evidently, the first and third factors on the RHS are $\geq \text{const.}$ By (1.2), the probability of the event $|\zeta_k| \leq \exp(-2r^2)$ is $\geq \frac{1}{2} \exp(-4r^2)$. Since the events within the second group are independent, the probability of all of them to happen is $\geq \left(\frac{1}{2} \exp(-4r^2)\right)^{48r^2} = \exp(-192r^4 - Cr^2)$. Thus, $P(\Omega_r) \geq \exp(-Cr^4)$.

Now, we show that for $\omega \in \Omega_r$ the function $\psi$ does not vanish in the disc $\{|z| \leq r\}$. For such $z$ and $\omega$ we have

$$
|\psi(z)| \geq |\zeta_0| - \sum_{1 \leq k \leq 48r^2} |\zeta_k| \frac{r^k}{\sqrt{k!}} - \sum_{k > 48r^2} |\zeta_k| \frac{r^k}{\sqrt{k!}} = |\zeta_0| - \sum' - \sum''.
$$

Then

$$
\sum' \leq e^{-2r^2} \sum_{1 \leq k \leq 48r^2} \frac{r^k}{\sqrt{k!}} \leq e^{-2r^2} \sqrt{48r^2} \cdot \sqrt{\sum_{1 \leq k \leq 48r^2} \frac{r^{2k}}{k!}} < 7r e^{-2r^2 + 0.5r^2} < e^{-r^2} < \frac{1}{2},
$$

if $r$ is sufficiently large. At the same time,

$$
\sum'' \leq \sum_{k > 48r^2} \frac{2^k}{\sqrt{k!}} \left(\frac{k}{48}\right)^{k/2} < \sum_{k > 48r^2} \left(\frac{k}{12} \cdot \frac{e}{k}\right)^{k/2} < \sum_{k \geq 1} 2^{-k} = \frac{1}{2}
$$

(we used inequality $k! > \left(\frac{k}{e}\right)^k$ which follows from Stirling’s formula). Putting both estimates together, we get

$$
|\psi(z)| \geq |\zeta_0| - 1 \geq 1, \quad |z| \leq r,
$$

proving that $\psi$ does not vanish in the closed disc $\{|z| \leq r\}$ for $\omega \in \Omega_r$.

2 Large deviations of $\log M(r, \psi) - r^2/2$

Let $\psi$ be the random entire function (0.1) and let $M(r, \psi) = \max_{|z| \leq r} |\psi(z)|$. In this section we shall prove the following
Lemma 1. Given $\delta \in (0, \frac{1}{4}]$ and $r \geq 1$,
\[
\mathbb{P}\left( \left| \log \frac{M(r, \psi)}{r^2} - \frac{1}{2} \right| \geq \delta \right) \leq \exp(-c(\delta)r^4).
\]

The proof is naturally split into two parts. First we show that
\[
\mathbb{P}\left( \frac{\log M(r, \psi)}{r^2} \geq \frac{1}{2} + \delta \right) \leq \exp(-c(\delta)r^4),
\]
and then that
\[
\mathbb{P}\left( \frac{\log M(r, \psi)}{r^2} \leq \frac{1}{2} - \delta \right) \leq \exp(-c(\delta)r^4).
\]

Proof of (2.1). We use an argument similar to the one used in Section 1. We have
\[
M(r, \psi) \leq \left( \sum_{0 \leq k < 4er^2} |\zeta_k| \frac{r^k}{\sqrt{k!}} \right) = \sum_1 + \sum_2.
\]
Consider the event $A_r$ which consists of such $\omega$’s that (i) $|\zeta_k| \leq \exp(2\delta r^2/3)$ for $0 \leq k < 4er^2$; (ii) $|\zeta_k| \leq (\sqrt{2})^k$ for $k \geq 4er^2$. If $A_r$ occurs and $r$ is sufficiently large, then
\[
\sum_1 \leq \left( \sum_{0 \leq k < 4er^2} |\zeta_k|^2 \right) \cdot \left( \sum_{0 \leq k < 4er^2} \frac{r^{2k}}{k!} \right)
\leq 4er^2 \cdot \exp(4\delta r^2/3 + r^2) < \exp \left( \left( 1 + \frac{5}{3} \right) r^2 \right),
\]
and
\[
\sum_2 \leq \sum_{k \geq 4er^2} |\zeta_k| \left( \frac{k}{4e} \cdot \frac{e}{k} \right)^{k/2} \leq \sum_{k \geq 4er^2} \left( \frac{\sqrt{2}}{2} \right)^k \leq 1.
\]
Thus
\[
M(r, \psi) \leq \exp \left( \left( \frac{1}{2} + \delta \right) r^2 \right).
\]

It remains to estimate the probability of the complementary set $A_r^c = \Omega \setminus A_r$. If $A_r^c$ occurs, then at least one of the following happens: $\exists k \in [0, 4er^2)$: $|\zeta_k| \geq \exp(2\delta r^2)$, or $\exists k \in [4er^2, \infty)$: $|\zeta_k| \geq (\sqrt{2})^k$. Therefore
\[
\mathbb{P}(A_r^c) \leq 4er^2 \exp \left( - \exp \left( \frac{4}{3} \delta r^2 \right) \right) + \sum_{k \geq 4er^2} \exp(-2^k) < \exp \left( - \exp(\delta r^2) \right)
\]
provided that $r \geq r_0(\delta)$. This is much stronger than (2.1).
Proof of (2.2). Suppose that

\[(2.3) \quad \log M(r, \psi) \leq \left(\frac{1}{2} - \delta\right)r^2.\]

Then we use Cauchy’s inequalities and Stirling’s formula:

\[
|\zeta_k| = \left|\frac{\psi^{(k)}(0)}{\sqrt{k!}}\right| \leq \sqrt{k!} \frac{M(r, \psi)}{r^k}
\leq Ck^{1/4} \exp \left(\frac{k}{2} \log k - \frac{k}{2} + \left(\frac{1}{2} - \delta\right)r^2 - k \log r\right).
\]

Observe that the exponent equals

\[
\frac{k}{2} \left(1 - 2\delta\right)\frac{r^2}{k} - \log \frac{r^2}{k} - 1.
\]

We note that \((1 - 2\delta)\frac{r^2}{k} - \log \frac{r^2}{k} - 1 < -\delta\) when \(r^2/k\) is close enough to 1. Whence, for \((1 - \epsilon)r^2 \leq k \leq r^2,

\[
|\zeta_k| \leq Ck^{1/4} \exp \left(-\frac{k\delta}{2}\right).
\]

By (1.2), the probability of this event is \(\leq \exp \left(-c(\delta)k\right)\). Since \(\zeta_k\) are independent, multiplying these probabilities, we see that

\[
\exp \left(-c(\delta) \sum_{(1-\epsilon)r^2 \leq k \leq r^2} k\right) = \exp \left(-c_1(\delta)r^4\right)
\]

is an upper bound for the probability that event (2.3) occurs. \(\square\)

3 Mean lower bound for \(\log |\psi(z)| - |z|^2/2\)

Lemma 1 gives us a sharp upper bound for the ‘random potential’ \(\log |\psi(z)| - \frac{1}{2}|z|^2\) when \(\omega\) does not belong to an exceptional set in the probability space. Here, we give a mean lower bound for this potential.

**Lemma 2.** Given \(\delta \in (0, \frac{1}{4}]\) and \(r \geq 1,

\[
P\left(\frac{1}{r^2} \int_{rT} \log |\psi| \, d\mu \leq \frac{1}{2} - \delta\right) \leq \exp(-c(\delta)r^4).
\]
Here, we denote by $r \mathbb{T}$ the circle $\{|z| = r\}$, $\mu$ is a normalized angular measure on $r \mathbb{T}$.

The proof uses the following

**Claim 1.** Given $\delta \in (0, \frac{1}{4}]$, $r \geq 1$, and $z_0, \frac{1}{2}r \leq |z_0| \leq r$, there exists $\zeta \in z_0 + \delta r \mathbb{D}$ such that

$$\log |\psi(\zeta)| > \left(\frac{1}{2} - 3\delta\right)|z_0|^2,$$

unless $\omega$ belongs to an exceptional set of probability $\exp(-c(\delta)r^4)$.

**Proof of the claim.** The distribution (of probabilities) of the random potential $\log |\psi(z)| - \frac{1}{2}|z|^2$ is shift-invariant (see [6, Introduction]). Writing the lower bound (2.2) in Lemma 1 as

$$\mathbb{P}\left(\max_{z \in r \mathbb{D}} \log |\psi(z)| - \frac{1}{2}|z|^2 \leq -\delta r^2\right) \leq \exp(-c(\delta)r^4)$$

we can apply it to the function $z \mapsto \log |\psi(z)| - \frac{1}{2}|z|^2$ on $r \mathbb{D}$. We get

$$\mathbb{P}\left(\max_{z \in \delta r \mathbb{D}} \log |\psi(z) + z_0| - \frac{1}{2}|z_0 + z|^2 \leq -\delta(\delta r)^2\right) \leq \exp(-c(\delta)(\delta r)^4).$$

Assuming that $\omega$ does not belong to the exceptional set we obtain $z \in \delta r \mathbb{D}$ such that

$$\log |\psi(z + z_0)| - \frac{1}{2}|z + z_0|^2 \geq -\delta^3 r^2.$$

Taking into account that $|z| \leq 2\delta|z_0|$ we get

$$\frac{1}{2}|z_0 + z|^2 \geq \frac{1}{2}|z_0|^2(1 - 2\delta)^2;$$

$$\log |\psi(z + z_0)| \geq \frac{1}{2}|z_0|^2(1 - 2\delta)^2 - \delta^3 r^2 \geq \frac{1}{2}|z_0|^2 - 2\delta|z_0|^2 - \left(\frac{1}{2}\right)^2\delta(2|z_0|)^2 \geq \frac{1}{2}|z_0|^2 - 3\delta|z_0|^2,$$

which yields the claim. \(\square\)

**Proof of Lemma 2.** Now, we choose $\kappa = 1 - \delta^{1/4}$, take $N = [2\pi\delta^{-1}]$, and consider $N$ discs (see Fig. 1)

$$z_j + \delta r \mathbb{D}, \quad z_j = \kappa r \exp\left(\frac{2\pi ij}{N}\right), \quad j = 0, 1, \ldots, N - 1.$$

Claim 1 implies that if $\omega$ does not belong to an exceptional set of probability
Figure 1: Small discs near the large circle

\[ N \exp(-c(\delta)r^4) = \exp(-c_1(\delta)r^4), \] then we can choose \( N \) points \( \zeta_j \in z_j + \delta r \mathbb{D} \) such that

\[ \log |\psi(\zeta_j)| \geq \left( \frac{1}{2} - 3\delta \right) |z_j|^2 \geq \left( \frac{1}{2} - C\delta^{1/4} \right) r^2. \]

Let \( P(z, \zeta) \) be the Poisson kernel for the disc \( r \mathbb{D}, \) \( |z| = r, \) \( |\zeta| < r. \) We set \( P_j(z) = P(z, \zeta_j). \) Then

\[
\left( \frac{1}{2} - C\delta^{1/4} \right) r^2 \leq \frac{1}{N} \sum_{j=0}^{N-1} \log |\psi(\zeta_j)| \leq \int_{r^T} \left( \frac{1}{N} \sum_{j=0}^{N-1} P_j \right) \log |\psi| \, d\mu
\]

\[ = \int_{r^T} \log |\psi| \, d\mu + \int_{r^T} \left( \frac{1}{N} \sum_{j=0}^{N-1} P_j - 1 \right) \log |\psi| \, d\mu. \]

We have

\[
\int_{r^T} \left( \frac{1}{N} \sum_{j=0}^{N-1} P_j - 1 \right) \log |\psi| \, d\mu \leq \max_{z \in r^T} \left| \frac{1}{N} \sum_{j=0}^{N-1} P_j - 1 \right| \cdot \int_{r^T} |\log |\psi|| \, d\mu.
\]

The next two claims finish the job.

\[ \Box \]

Claim 2.
\[ \max_{z \in r^T} \left| \frac{1}{N} \sum_{j=0}^{N-1} P_j - 1 \right| \leq C\delta^{1/2}. \]

Claim 3.
\[ \int_{r^T} |\log |\psi|| \, d\mu \leq 10r^2 \]
provided that \( \omega \text{ does not belong to an exceptional set of probability } \exp(-cr^4). \)
Proof of Claim 2. We start with
\[ \int_{\kappa r \mathbb{T}} P(z, \zeta) \, d\mu(\zeta) = 1, \]
and split the circle \( \kappa r \mathbb{T} \) into a union of \( N \) disjoint arcs \( I_j \) of equal angular measure \( \mu(I_j) = \frac{1}{N} \) centered at \( z_j \). Then
\[ 1 = \frac{1}{N} \sum_{j=0}^{N-1} P(z, \zeta_j) + \sum_{j=0}^{N-1} \int_{I_j} (P(z, \zeta) - P(z, \zeta_j)) \, d\mu(\zeta), \]
and
\[ |P(z, \zeta) - P(z, \zeta_j)| \leq \max_{\zeta \in I_j} |\zeta - \zeta_j| \cdot \max_{z, \zeta} |\nabla_{\zeta} P(z, \zeta)| \]
\[ \leq C_1 \delta r \cdot \frac{C_2 r}{(r - |\zeta|)^2} = \frac{C \delta}{\delta^{1/2}} = C \delta^{1/2}, \]
proving the claim.

Proof of Claim 3. By Lemma 1, we know that unless \( \omega \) belongs to an exceptional set of probability \( \exp(-cr^4) \), there is a point \( \zeta \in \frac{1}{2} r \mathbb{T} \) such that \( \log |\psi(\zeta)| \geq 0 \). Fix such a \( \zeta \). Then
\[ 0 \leq \int_{r \mathbb{T}} P(z, \zeta) \log |\psi(z)| \, d\mu(z), \]
and hence
\[ \int_{r \mathbb{T}} P(z, \zeta) \log^- |\psi(z)| \, d\mu(z) \leq \int_{r \mathbb{T}} P(z, \zeta) \log^+ |\psi(z)| \, d\mu(z). \]
It remains to recall that for \( |z| = r \) and \( |\zeta| = \frac{1}{2}r \),
\[ \frac{1}{3} \leq P(z, \zeta) \leq 3, \]
and that
\[ \int_{r \mathbb{T}} \log^+ |\psi| \, d\mu \leq \log M(r, \psi) \leq r^2 \]
(provided \( \omega \) is non-exceptional). Hence
\[ \int_{r \mathbb{T}} \log^- |\psi| \, d\mu \leq 9r^2, \]
and
\[ \int_{r \mathbb{T}} |\log |\psi|| \, d\mu \leq 10r^2, \]
proving the claim.
4 Proof of Theorem 2

We shall prove that

\[
\mathbb{P}\left( \frac{n(r)}{r^2} \leq 1 + \delta \right) \leq \exp\left( -c(\delta)r^4 \right).
\]

The proof of the lower bound for \( n(r) \) is practically the same and is left to the reader.

Fix \( \kappa = 1 + \sqrt{\delta} \). Then by Jensen’s formula [1, Chapter 5, Section 3.1]

\[
n(r) \log \kappa \leq \int_{r}^{\kappa r} \frac{n(t)}{t} dt = \left( \int_{\kappa rT}^{T} - \int_{rT}^{T} \right) \log |\psi| d\mu,
\]

whence by Lemmas 1 and 2

\[
\frac{n(r)}{r^2} \leq \frac{1}{\log \kappa} \left( \kappa^2 \left( \frac{1}{2} + \delta \right) - \left( \frac{1}{2} - \delta \right) \right) = \frac{1}{2} \frac{\kappa^2 - 1}{\log \kappa} + \delta \frac{\kappa^2 + 1}{\log \kappa} \leq 1 + C\sqrt{\delta},
\]

provided that \( \omega \) does not belong to an exceptional set of probability \( \exp\left( -c(\delta)r^4 \right) \). This proves estimate (4.1).

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