Analysis of Maxwell-Stefan systems for heat conducting fluid mixtures

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ANALYSIS OF MAXWELL–STEFAN SYSTEMS FOR HEAT CONDUCTING FLUID MIXTURES

CHRISTOPH HELMER AND ANSGAR JÜNGEL

Abstract. The global-in-time existence of weak solutions to the Maxwell–Stefan–Fourier equations in Fick–Onsager form is proved. The model consists of the mass balance equations for the partial mass densities \( \rho_i \) and the energy balance equation for the total energy. The diffusion and heat fluxes depend linearly on the gradients of the thermochemical potentials and the gradient of the temperature and include the Soret and Dufour effects. The cross-diffusion system exhibits an entropy structure, which originates from the consistent thermodynamic modeling. The lack of positive definiteness of the diffusion matrix is compensated by the fact that the total mass density is constant in time. The entropy estimate also allows for the proof of the a.e. positivity of the partial mass densities and temperature.

1. INTRODUCTION

Maxwell–Stefan equations describe the dynamics of multicomponent fluids by accounting for the gradients of the chemical potentials as driving forces. The global existence analysis is usually based on the so-called entropy or formal gradient-flow structure. Up to our knowledge, almost all existence results are concerned with the isothermal setting. Exceptions are the local-in-time existence result of [19] and the coupled Maxwell–Stefan and compressible Navier–Stokes–Fourier systems analyzed in [17, 24], where no temperature gradients in the diffusion fluxes (Soret effect) have been taken into account. In this paper, we suggest and analyze for the first time Maxwell–Stefan–Fourier systems in Fick–Onsager form, including Soret and Dufour effects.

1.1. Model equations. We consider the evolution of the partial mass densities \( \rho_i(x,t) \) and temperature \( \theta(x,t) \) in a fluid mixture, governed by the equations

\[
\partial_t \rho_i + \text{div } J_i = r_i, \quad J_i = -\sum_{j=1}^n M_{ij}(\rho, \theta) \nabla q_j - M_i(\rho, \theta) \nabla \frac{1}{\theta},
\]
\( \partial_t (\rho \theta) + \text{div} J_e = 0, \quad J_e = -\kappa(\theta) \nabla \theta - \sum_{j=1}^{n} M_j(\rho, \theta) \nabla q_j \quad \text{in} \quad \Omega, \quad i = 1, \ldots, n, \)

where \( \Omega \subset \mathbb{R}^3 \) is a bounded domain, \( \rho = (\rho_1, \ldots, \rho_n) \) is the vector of mass densities, and \( q_i = \log(\rho_i/\theta) \) is the thermo-chemical potential of the \( i \)-th species. The diffusion fluxes are denoted by \( J_i \), the reaction rates by \( r_i \), the energy flux by \( J_e \), and the heat conductivity by \( \kappa(\theta) \). The functions \( M_{ij} \) are the diffusion coefficients, and the terms \( M_i \nabla (1/\theta) \) and \( \sum_{j=1}^{n} M_j \nabla q_j \) describe the Soret and Dufour effect, respectively.

We prescribe the boundary and initial conditions

\[ \begin{align*}
J_i \cdot \nu &= 0, \\
J_e \cdot \nu + \lambda (\theta_0 - \theta) &= 0 \quad \text{on} \quad \partial \Omega, \quad t > 0, \\
\rho_i(\cdot, 0) &= \rho_i^0, \\
(\rho_i\theta)(\cdot, 0) &= \rho_i^0 \theta^0 \quad \text{in} \quad \Omega, \quad i = 1, \ldots, n,
\end{align*} \]

where \( \nu \) is the exterior unit normal vector to \( \partial \Omega \), \( \theta_0 > 0 \) is the constant background temperature, and \( \lambda \geq 0 \) is a relaxation parameter. Equations (3) mean that the fluid cannot leave the domain \( \Omega \), while heat transfer through the boundary is possible (if \( \lambda \neq 0 \)).

In Maxwell–Stefan systems, the driving forces \( \nabla (\rho_i \theta) \) are usually given by linear combinations of the diffusion fluxes [5, Sec. 14]:

\[ \partial_t \rho_i + \text{div} J_i = r_i, \quad \nabla (\rho_i \theta) = -\sum_{j=1}^{n} b_{ij} \rho_i \rho_j \left( \frac{J_i}{\rho_i} - \frac{J_j}{\rho_j} \right), \quad i = 1, \ldots, n, \]

where \( b_{ij} = b_{ji} \geq 0 \) for \( i, j = 1, \ldots, n \). We show in Section 2 that this formulation can be written as (1) for a specific choice of \( M_{ij} \) and \( M_i \).

We say that the diffusion fluxes in (1) are in Fick–Onsager form. As the heat flux is given by Fourier’s law, we call system (1)-(2) the Maxwell–Stefan–Fourier equations in Fick–Onsager form. We refer to Section 2 for details of the modeling.

To fulfill mass conservation, the sum of the diffusion fluxes and the sum of the reaction terms should vanish, i.e. \( \sum_{i=1}^{n} J_i = 0 \) and \( \sum_{i=1}^{n} r_i = 0 \) (see Section 2). Then, summing (1) over \( i = 1, \ldots, n \), we see that the total mass density \( \rho(\cdot, t) := \sum_{i=1}^{n} \rho_i(\cdot, t) = \rho^0 \) is constant in time (but generally not in space). Another consequence of the identity \( \sum_{i=1}^{n} J_i = 0 \) is that the diffusion matrix has a nontrivial kernel, and we assume that

\[ \sum_{i=1}^{n} M_{ij} = 0 \quad \text{for} \quad j = 1, \ldots, n, \quad \sum_{i=1}^{n} M_i = 0. \]

Moreover, we suppose that the matrix \( (M_{ij}) \) is positive semi-definite in the sense that there exists \( c_M > 0 \) such that

\[ \sum_{i=1}^{n} M_{ij}(\rho, \theta) z_i z_j \geq c_M |\Pi z|^2 \quad \text{for} \quad z \in \mathbb{R}^n, \quad \rho \in \mathbb{R}^n, \quad \theta \in \mathbb{R}_+, \]

where \( \Pi = I - 1 \otimes 1/n \) is the orthogonal projection on \( \text{span}\{1\}^\perp \). For a discussion of this assumption, we refer to Section 1.4.
Notation. We write $z$ for a vector of $\mathbb{R}^n$ with components $z_1, \ldots, z_n$ and similarly for other variables. In particular, $1 = (1, \ldots, 1) \in \mathbb{R}^n$. Furthermore, we set $\mathbb{R}_+ = [0, \infty)$ and $\Omega_T = \Omega \times (0, T)$.

1.2. Mathematical ideas. The mathematical difficulties of system (1)–(2) are the cross-diffusion structure, the lack of coerciveness of the diffusion operator, and the temperature terms. In particular, it is not trivial to verify the positivity of the temperature. These difficulties are overcome by exploiting the entropy structure of the equations. More precisely, we introduce the mathematical entropy

$$h(\rho, \theta) = \sum_{i=1}^n \rho_i (\log \rho_i - 1) - \rho \log \theta.$$ 

A formal computation (which is made precise for an approximate scheme; see (24)) shows that

$$\frac{d}{dt} \int_\Omega h(\rho, \theta) dx + c_M \int_\Omega |\nabla \Pi q|^2 dx + \int_\Omega \kappa(\theta) |\nabla \log \theta|^2 dx + \lambda \int_{\partial \Omega} \left( \frac{\theta_0}{\theta} - 1 \right) ds \leq \sum_{i=1}^n \int_\Omega r_i q_i dx.$$ 

Under suitable conditions on the heat conductivity and the reaction rates, this so-called entropy inequality provides gradient estimates for $\Pi q$, $\log \theta$, and $\theta$, but not for the full vector $q$. This problem was overcome in [9] for a more general (but stationary) multicomponent Navier–Stokes–Fourier system by using tools from mathematical fluid dynamics (effective viscous flux identity and Feireisl’s oscillations defect measure). For our model, the situation is much simpler. Indeed, an elementary computation, detailed in the proof of Lemma 5, shows that

$$\frac{1}{n} \sum_{i=1}^n \nabla q_i = \nabla \log \rho^0 - \nabla \log \theta - \sum_{i=1}^n \frac{\rho_i}{\rho} \nabla (\Pi q)_i$$

and consequently $\nabla q_i = \nabla (\Pi q)_i + \sum_{i=1}^n \nabla q_i / n$ is bounded in $L^2$. If $\rho^0$ is bounded, the total mass density is bounded too, and $\rho_i$ lies in $L^\infty$ for any $i = 1, \ldots, n$. This provides an $L^2$ estimate for $\nabla \rho_i = \rho_i (\nabla \log \theta + \nabla q_i)$. Together with a bound for the (discrete) time derivative of $\rho_i$, we deduce the strong convergence of $\rho_i$ from the Aubin–Lions compactness lemma.

The positivity of the temperature is a consequence of the $L^1$ estimate for $-\log \theta$ coming from (8). Still, there remains a difficulty. The estimate for $\kappa(\theta)^{1/2} \nabla \log \theta$ in $L^2$ from (8) is not sufficient to define $\kappa(\theta) \nabla \theta$ in the weak formulation. In the Navier–Stokes–Fourier equations, this difficulty is handled by replacing the local energy balance by the local entropy inequality and the global energy balance [15]. We choose another approach. The idea is to derive better estimates for the temperature by using $\theta$ as a test function in the weak formulation of (2). If $\kappa(\theta) \geq c_\alpha \theta^2$ for some $c_\alpha > 0$ and $M_j / \theta$ is assumed to be
bounded, then a formal computation, which is made precise in Lemma 3, gives

\[\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho^0 \theta^2 dx + c_\kappa \int_{\Omega} \theta^2 |\nabla \theta|^2 dx - \lambda \int_{\partial \Omega} (\theta_0 - \theta) \theta ds \]

\[= \sum_{j=1}^{n} \int_{\Omega} \frac{M_j}{\theta} \nabla q_j \cdot \nabla \theta dx \leq \frac{c_\kappa}{2} \int_{\Omega} \theta^2 |\nabla \theta|^2 dx + C \sum_{j=1}^{n} \int_{\Omega} |\nabla q_j|^2 dx.\]

Since \(\nabla q_j\) is bounded in \(L^2\), this yields uniform bounds for \(\theta^2\) in \(L^\infty(0,T;L^1(\Omega))\) and \(L^2(0,T;H^1(\Omega))\). These estimates are sufficient to treat the term \(\kappa(\theta)\nabla \theta\). The delicate point is to choose the approximate scheme in such a way that estimates (8) and (10) can be made rigorous; for details see Section 3.

1.3. State of the art. Before we state our main result, we review the state of the art of Maxwell–Stefan models. The isothermal equations were derived from the multi-species Boltzmann equations in the diffusive approximation in [2, 8]. The Fick–Onsager form of the Maxwell–Stefan equations was rigorously derived in Sobolev spaces from the multi-species Boltzmann system in [3]. The Maxwell–Stefan equations in the Fick–Onsager form, coupled with the momentum balance equation, can be identified as a rigorous second-order Chapman–Enskog approximation of the Euler (–Korteweg) equations for multicomponent fluids; see [18] for the Euler–Korteweg case and [23] for the Euler case. The work [7] is concerned with the friction limit in the isothermal Euler equations using the hyperbolic formalism developed by Chen, Levermore, and Liu. A formal Chapman–Enskog expansion of the stationary non-isothermal model was presented in [25]. Another non-isothermal Maxwell–Stefan system was derived in [1], but the energy flux is different from the expression in (2).

The existence analysis of (isothermal) Maxwell–Stefan equations started with the paper [16], where the existence of global-in-time weak solutions near the constant equilibrium was proved. A proof of local-in-time classical solutions to Maxwell–Stefan systems was given in [4]. In [22], the entropy or formal gradient-flow structure was revealed, which allowed for the proof of global-in-time weak solutions with general initial data. Maxwell–Stefan systems, coupled to the Poisson equation for the electric potential, were analyzed in [21].

All the mentioned results hold if the barycentric velocity vanishes. For non-vanishing fluid velocities, the Maxwell–Stefan equations need to be coupled to the momentum balance. The Maxwell–Stefan equations were coupled to the incompressible Navier–Stokes equations in [10], and the global existence of weak solutions was shown. A similar result can be found in [11], where the incompressibility condition was replaced by an artificial time derivative of the pressure and the limit of vanishing approximation parameters was performed. Coupled Maxwell–Stefan and compressible Navier–Stokes equations were analyzed in [6], and the local-in-time existence analysis was performed. A global existence analysis for a general isothermal Maxwell–Stefan–Navier–Stokes system was performed in [13]. For the existence analysis of coupled stationary Maxwell–Stefan and compressible Navier–Stokes–Fourier systems, we refer to [9, 17, 24]. In [9], temperature gradients were included in the partial mass fluxes, but only the stationary model was investigated. The
1.4. **Main result.** We impose the following assumptions:

(H1) **Domain:** $\Omega \subset \mathbb{R}^3$ is a bounded domain with a Lipschitz continuous boundary.

(H2) **Data:** $\theta^0 \in L^\infty(\Omega)$, $\inf_\Omega \theta^0 > 0$, $\theta_0 > 0$, $\lambda \geq 0$; $\rho_i^0 \in L^\infty(\Omega)$ satisfies $0 < \rho_* \leq \rho_i^0 \leq \rho^*$ in $\Omega$ for some $\rho_*$, $\rho^* > 0$.

(H3) **Diffusion coefficients:** For $i, j = 1, \ldots, n$, the coefficients $M_{ij}$, $M_j \in C^0(\mathbb{R}_+^n \times \mathbb{R}_+)$ satisfy (6)–(7) and $M_{ij}$, $M_i/\theta$ are bounded functions.

(H4) **Heat conductivity:** $\kappa \in C^0(\mathbb{R}_+)$ and there exist $c_\kappa$, $C_\kappa > 0$ such that for all $\theta \geq 0$,

$$c_\kappa (1 + \theta^2) \leq \kappa(\theta) \leq C_\kappa (1 + \theta^2).$$

(H5) **Reaction rates:** $r_1, \ldots, r_n \in C^0(\mathbb{R}^n \times \mathbb{R}_+) \cap L^\infty(\mathbb{R}^n \times \mathbb{R}_+)$ satisfies $\sum_{i=1}^n r_i = 0$ and there exists $c_r > 0$ such that for all $\mathbf{q} \in \mathbb{R}^n$ and $\theta > 0$,

$$\sum_{i=1}^n r_i(\Pi \mathbf{q}, \theta)q_i \leq -c_r|\Pi \mathbf{q}|^2.$$

The bounds on $\rho_i^0$ in Hypothesis (H2) are needed to derive positivity and boundedness of the partial mass densities. The uniform positive definiteness of the diffusion matrix on the orthogonal complement of span{$1$} in Hypothesis (H3) provides a control on the thermo-chemical potentials, but it excludes the dilute limit, i.e. situations when the mass densities vanish (see Section 2). This situation is included in the recent work [14], which deals with the isothermal case. The boundedness of $M_{ij}$ (and $r_i$) is imposed for convenience; suitable growth conditions may be imposed instead but they complicate the proofs. The growth condition for the heat conductivity in Hypothesis (H4) is used to derive higher integrability of the temperature, see (10), which allows us to treat the heat flux term. If $\lambda = 0$, we can impose the weaker condition $\kappa(\theta) \geq c_\kappa \theta^2$. Hypothesis (H5) is satisfied for the reaction terms used in [13]. The bound for $\sum_{i=1}^n r_i q_i$ gives a control on the $L^2(\Omega)$ norm of $\Pi \mathbf{q}$. Together with the estimates for $\nabla (\Pi \mathbf{q})$ from (8), we are able to infer an $H^1(\Omega)$ estimate for $\Pi \mathbf{q}$. Hypothesis (H5) may be replaced by a Robin boundary condition providing an $L^1(\partial \Omega)$ estimate for $\Pi \mathbf{q}$, but such a condition seems to be artificial. We note that Hypothesis (H5) was also used in [9].

We say that $(\mathbf{\rho}, \theta)$ is a weak solution to (1)–(4) if $\rho_i > 0$, $\theta > 0$ a.e. in $\Omega_T$,  

$$\rho_i \in L^\infty(\Omega_T) \cap L^2(0, T; H^1(\Omega)) \cap H^1(0, T; H^2(\Omega)'), \quad \nabla q_i \in L^2(0, T; L^2(\Omega)),
$$

$$\theta \in L^2(0, T; H^1(\Omega)) \cap W^{1,1}(0, T; W^{1,\infty}(\Omega)'), \quad \kappa(\theta) \nabla \theta \in L^1(\Omega_T);$$

where $q_i = \log(\rho_i/\theta)$; it holds that

$$\int_0^T \langle \partial_t \rho_i, \phi_i \rangle dt + \int_0^T \int_\Omega \left( \sum_{j=1}^n M_{ij} \nabla q_j + M_i \nabla \frac{1}{\theta} \right) \cdot \nabla \phi_i \, dx \, dt = \int_0^T \int_\Omega r_i \phi_i \, dx \, dt,$$

$$\int_0^T \langle \partial_t (\rho \theta), \phi_0 \rangle dt + \int_0^T \int_\Omega \kappa(\theta) \nabla \theta \cdot \nabla \phi_0 \, dx \, dt = \int_0^T \int_\Omega \sum_{j=1}^n M_j \nabla q_j \cdot \nabla \phi_0 \, dx \, dt.$$
\[
\lambda \int_0^T \int_{\partial \Omega} (\theta_0 - \theta) \phi_0 \, dx \, ds
\]
for all \( \phi_1, \ldots, \phi_n \in L^2(0, T; H^1(\Omega)) \), \( \phi_0 \in L^\infty(0, T; W^{1,\infty}(\Omega)) \) with \( \nabla \phi_0 \cdot \nu = 0 \) on \( \partial \Omega \), and \( i = 1, \ldots, n \); and the initial conditions (4) are satisfied in the sense of \( H^1(\Omega)' \) and \( W^{1,\infty}(\Omega)' \), respectively.

Our main result is as follows.

**Theorem 1 (Existence).** Let Hypotheses (H1)–(H5) hold. Then there exists a weak solution \((\rho, \theta)\) to (1)–(4) satisfying (11)–(14) and additionally

\[
\theta \in L^{16/3}(\Omega_T) \cap W^{1,16/15}(0, T; W^{1,16}(\Omega))', \quad \kappa(\theta) \nabla \theta \in L^{16/9}(\Omega_T).
\]

The proof is based on a suitable approximate scheme, uniform bounds coming from entropy estimates, and \( H^1(\Omega) \) estimates for the partial mass densities. More precisely, we use two levels of approximations. First, we replace the time derivative by an implicit Euler discretization to overcome issues with the time regularity. Second, we add higher-order regularizations for the thermo-chemical potentials and the logarithm of the temperature \( w = \log \theta \) to achieve \( H^2(\Omega) \) regularity for these variables. Since we are working in three space dimensions, we conclude \( L^\infty(\Omega) \) solutions, which are needed to define properly \( \rho_i = \exp(w + q_i) \).

A priori estimates are deduced from a discrete version of the entropy inequality (8). They are derived from the weak formulation by using \( q_i \) and \( -e^{-w} \) as test functions. The entropy structure is only preserved if we add additionally a \( W^{1,4}(\Omega) \) regularization and some lower-order regularization in \( w \). The properties for the heat conductivity allow us to obtain estimates for \( \theta \) in \( H^1(\Omega) \) and for \( \nabla \log \theta \) in \( L^2(\Omega) \). As already mentioned before, the semi-definiteness property (7) only provides estimates for \( \Pi q \) in \( H^1(\Omega) \), which are not sufficient to deduce strong convergence of the mass densities. By exploiting the boundedness of \( \rho_i \), identity (9) provides an estimate for \( q_i \) in \( H^1(\Omega) \) and for \( \rho_i = \theta \exp(q_i) \) in \( H^1(\Omega) \).

The paper is organized as follows. We explain the thermodynamical modeling of (1)–(2) in Section 2 and show that the Maxwell–Stefan formulation (5) can be always written as (1) with specific diffusion coefficients \( M_{ij} \) and \( M_i \). Theorem 1 is proved in Section 3.

**2. Modeling**

We consider an ideal fluid mixture consisting of \( n \) components with the same molar mass in a fixed container \( \Omega \subset \mathbb{R}^3 \). The balance equations for the partial mass densities \( \rho_i \) are given by

\[
\partial_t \rho_i + \text{div}(\rho_i v_i) = r_i, \quad i = 1, \ldots, n,
\]

where \( v_i \) are the partial velocities and \( r_i \) the reaction rates. Introducing the total mass density \( \rho = \sum_{i=1}^n \rho_i \), the barycentric velocity \( v = \rho^{-1} \sum_{i=1}^n \rho_i v_i \), and the diffusion fluxes \( J_i = \rho_i(v_i - v) \), we can reformulate the mass balances as

\[
\partial_t \rho_i + \text{div}(\rho_i v + J_i) = r_i, \quad i = 1, \ldots, n.
\]

By definition, we have \( \sum_{i=1}^n J_i = 0 \), which means that the total mass density satisfies \( \partial_t \rho + \text{div}(\rho v) = 0 \). We assume that the barycentric velocity vanishes, \( v = 0 \), i.e., the
barycenter of the fluid is not moving. Consequently, the total mass density is constant in time.

Then the non-isothermal dynamics of the mixture is given by the balance equations

$$\partial_t \rho_i + \text{div} \ J_i = r_i, \quad \partial_t E + \text{div} \ J_e = 0, \quad i = 1, \ldots, n,$$

where $J_e$ is the energy flux, and $E$ the total energy. We assume that the diffusion fluxes are proportional to the gradients of the thermo-chemical potentials $q_j$ and the temperature gradient (Soret effect) and that the energy flux is linear in the temperature gradient and the gradients of $q_j$ (Dufour effect):

$$J_i = -\sum_{j=1}^{n} M_{ij} \nabla q_j - M_i \nabla \frac{1}{\theta}, \quad i = 1, \ldots, n, \quad J_e = -\kappa(\theta) \nabla \theta - \sum_{j=1}^{n} M_j \nabla q_j.$$

The proportionality factor $\kappa(\theta)$ between the heat flux and the temperature gradient is the heat (or thermal) conductivity.

The thermo-chemical potentials and the total energy are determined in a thermodynamically consistent way from the Gibbs free energy

$$G = \theta \sum_{i=1}^{n} \rho_i \left( \log \rho_i - 1 \right) - \rho \theta \left( \log \theta - 1 \right).$$

For simplicity, we have set the heat capacity in the free energy equal to one. The physical entropy $s$, the chemical potentials $\mu_i$, and the total energy $E$ are defined by the free energy according to

$$s = -\frac{\partial G}{\partial \theta} = \sum_{i=1}^{n} \rho_i \left( \log \rho_i - 1 \right) + \rho \log \theta,$$

$$\mu_i = -\theta \frac{\partial s}{\partial \rho_i} = \theta \log (\rho_i / \theta), \quad i = 1, \ldots, n,$$

$$E = G - \theta \frac{\partial G}{\partial \theta} = \rho \theta.$$

We introduce the mathematical entropy $h := -s$ and the thermo-chemical potentials $q_j = \mu_j / \theta = \log (\rho_j / \theta)$ for $j = 1, \ldots, n$. These definitions lead to system (1)–(2). The Gibbs–Duham relation yields the pressure $p = -G + \sum_{i=1}^{n} \rho_i \mu_i = 0$. Note that we do not need a pressure blow-up at $\rho = 0$ to exclude vacuum or a superlinear growth in $\theta$ to control the temperature.

If the molar masses $m_i$ of the components are not the same, we need to modify the Gibbs free energy according to [9, Remark 1.2]

$$G = \theta \sum_{i=1}^{n} \frac{\rho_i}{m_i} \left( \log \frac{\rho_i}{m_i} - 1 \right) - c_W \rho \theta (\log \theta - 1),$$

where $c_W > 0$ is the heat capacity. For simplicity, we have set $m_i = 1$ and $c_W = 1.$
We already mentioned that the Maxwell–Stefan equations are usually formulated as

\begin{equation}
\partial_t \rho_i + \text{div} J_i = r_i, \quad d_i = -\sum_{j=1}^{n} b_{ij} \rho_i \rho_j \left( \frac{J_i}{\rho_i} - \frac{J_j}{\rho_j} \right), \quad i = 1, \ldots, n.
\end{equation}

According to [5, (176)], the driving forces are given by

\[ d_i = \rho_i \theta \nabla q_i + D_i \theta - 1 \nabla \theta. \]

The function \( D_i \) is the sum of the partial enthalpy [5, (85)] and the phenomenological thermal diffusivity [5, (106)]. The partial enthalpy of the \( i \)th species is the sum of the partial internal energy \( \rho_i \theta \) and the partial pressure \( \rho_i \theta \).

Thus, choosing a vanishing thermal diffusivity, we find that \( D_i = 2 \rho_i \theta \), and this gives the expression \( d_i = \nabla (\rho_i \theta) \) also found in [19]. If \( (b_{ij}) \) is symmetric, we see that \( \sum_{i=1}^{n} d_i = 0 \).

We show that (15) implies (1) for a specific choice of \( M_{ij} \) and \( M_i \). This result is well-known in the thermodynamic community (see, e.g., [5, (183)]); here, we make explicit the hypotheses to achieve this result. Namely, we need a property for the coefficients \( b_{ij} \) in (5). We suppose that the homogeneous system

\begin{equation}
\sum_{j=1}^{n} b_{ij} \rho_i \rho_j (u_i - u_j) = 0, \quad i = 1, \ldots, n,
\end{equation}

has only solutions in \( \text{span}\{1\} \), where \( 1 = (1, \ldots, 1) \in \mathbb{R}^n \). This holds true if, for instance, \( b_{ij} = b_{ji} > 0 \) for \( i, j = 1, \ldots, n \).

**Proposition 2.** Let \( b_{ij} = b_{ji} \geq 0 \) for \( i, j = 1, \ldots, n \) and let the linear system (16) has only solutions in \( \text{span}\{1\} \). Then (5) can be written as (1) with

\begin{equation}
M_{ij} = N_{ij} \rho_j \theta, \quad M_i = -2 \sum_{k=1}^{n} N_{ik} \rho_k \theta^2 \quad \text{for} \ i, j = 1, \ldots, n,
\end{equation}

where \( N_{ij} \) depends only on \( (b_{ij}) \) and \( \rho \), and \( M_{ij} \) and \( M_i \) satisfy (6).

**Proof.** It follows from the property on the linear system (16) that

\begin{equation}
-\sum_{j=1}^{n} b_{ij} \rho_i \rho_j \left( \frac{J_i}{\rho_i} - \frac{J_j}{\rho_j} \right) = d_i, \quad i = 1, \ldots, n,
\end{equation}

subject to \( \sum_{i=1}^{n} J_i = 0 \), has the unique solution [18, Lemma 1]

\begin{equation}
J_i = \rho_i u_i = -\sum_{j,k=1}^{n-1} \left( \delta_{ij} \rho_i - \frac{\rho_i \rho_j}{\rho} \right) c_{jk} d_k, \quad i = 1, \ldots, n - 1,
\end{equation}

and \( J_n = -\sum_{i=1}^{n-1} J_i \). The coefficients \( c_{jk} \) depend only on \( (b_{ij}) \). The diffusion fluxes can be reformulated as \( J_i = -\sum_{k=1}^{n-1} D_{ik} d_k \) after setting

\[ D_{ik} = \sum_{j=1}^{n-1} (\delta_{ij} \rho_i - \rho_i \rho_j / \rho) c_{jk}, \quad i, k = 1, \ldots, n - 1. \]
The aim is to formulate $J_i = -\sum_{k=1}^n N_{ik} d_k$ for some coefficients $N_{ik}$. Define

$$N_{ik} = D_{ik} - \frac{1}{n} \sum_{j=1}^{n-1} D_{ij}, \quad i, j = 1, \ldots, n - 1.$$ 

Then $(N_{ik})$ solves the linear system

$$N_{ij} + \sum_{k=1}^{n-1} N_{ik} = D_{ij}, \quad j = 1, \ldots, n - 1,$$

for each fixed $i \in \{1, \ldots, n - 1\}$. We extend $(N_{ij})$ to an $n \times n$ matrix by defining

$$N_{nj} = -\sum_{k=1}^{n-1} N_{kj}, \quad j = 1, \ldots, n - 1,$$

$$N_{in} = -\sum_{k=1}^{n-1} N_{ik}, \quad i = 1, \ldots, n.$$ 

It follows from $d_n = -\sum_{i=1}^{n-1} d_i$ that

$$J_i = -\sum_{k=1}^{n-1} D_{ik} d_k = -\sum_{k=1}^{n-1} \left( N_{ik} + \sum_{j=1}^{n-1} N_{ij} \right) d_k = -\sum_{k=1}^{n-1} (N_{ik} - N_{in}) d_k$$

$$= -\sum_{k=1}^{n} N_{ik} d_k - N_{in} d_n = -\sum_{k=1}^{n} N_{ik} d_k.$$

Since $d_k = \nabla (\rho_k \theta) = \rho_k \theta (\nabla q_k - 2 \theta \nabla (1/\theta))$, we find that

$$J_i = -\sum_{k=1}^{n} N_{ik} \rho_k \theta \nabla q_k + 2 \sum_{k=1}^{n} N_{ik} \rho_k \theta^2 \nabla \frac{1}{\theta}.$$ 

Setting $M_{ij} = N_{ij} \rho_j \theta$ for $i, j = 1, \ldots, n$ and $M_i = -2 \sum_{k=1}^{n} N_{ik} \rho_k \theta^2$, we arrive to the second expression in (1). The coefficients satisfy for $j = 1, \ldots, n$, 

$$\sum_{i=1}^{n} M_{ij} = \left( \sum_{i=1}^{n} N_{ij} \right) \rho_j \theta = 0, \quad \sum_{i=1}^{n} M_i = -2 \sum_{k=1}^{n} \left( \sum_{i=1}^{n} N_{ik} \right) \rho_k \theta^2 = 0.$$ 

This shows the proposition. 

We observe that the diffusion fluxes in (1) can be written as (19) under the conditions $\sum_{i=1}^{n} \nabla (\rho_i \theta) = 0$ and

$$\sum_{j=1}^{n} M_{ij} = -\frac{M_i}{\theta}, \quad \sum_{j=1}^{n} \frac{M_{ij}}{\rho_j} = 0 \quad \text{for } i = 1, \ldots, n.$$ 

Equations (20) are valid for the Maxwell–Stefan equations in Fick–Onsager form that are derived from the Boltzmann equation in the limit of small Knudsen numbers; see formulas (A13) and (A15) in [25]. To derive formulation (19), we set $d_j := \nabla (\rho_j \theta)$ and...
\( N_{ij} := M_{ij}/(\rho_j \theta) \). Then \( \sum_{j=1}^n N_{ij} = 0 \). Using \( \nabla q_j = d_j/(\rho_j \theta) + 2\rho_j \theta^2 \nabla (1/\theta) \), condition (20), and \( \sum_{i=1}^n d_i = 0 \), we compute

\[
J_i = -\sum_{j=1}^n N_{ij} d_j - \left( \sum_{j=1}^n N_{ij} \rho_j \theta^2 + M_i \right) \nabla \frac{1}{\theta} = -\sum_{j=1}^n N_{ij} d_j
\]

Thus, setting \( D_{ij} = N_{ij} + \sum_{k=1}^{n-1} N_{ik} \) for \( i, j = 1, \ldots, n-1 \), we obtain \( J_i = -\sum_{j=1}^{n-1} D_{ij} d_j \).

The matrix with coefficients \( R_{ij} = \delta_{ij} \rho_i - \rho_i \rho_j / \rho \) is invertible with inverse \( Q_{ij} = \delta_{ij} \rho_i^{-1} + \rho_n^{-1} \) for \( i, j = 1, \ldots, n-1 \); see the proof of [18, Lemma 1]. We introduce the matrix \( c_{ik} = \sum_{j=1}^{n-1} Q_{ik} D_{jk} \). Then \( D_{ik} = \sum_{j=1}^{n-1} R_{ij} c_{jk} \) and

\[
J_i = -\sum_{k=1}^{n-1} D_{ik} d_k = -\sum_{j,k=1}^{n-1} R_{ij} c_{jk} d_k = -\sum_{j,k=1}^{n-1} \left( \delta_{ij} \rho_i - \frac{\rho_i \rho_j}{\rho} \right) c_{jk} d_k,
\]

which equals (19).

3. Proof of Theorem 1

The idea of the proof is to approximate equations (1)–(4) by an implicit Euler scheme and to add some higher-order regularizations in space for the variables \( q \) and \( w = \log \theta \). The de-regularization limit is based on the compactness coming from the entropy estimates.

Set \( u_0 = \log \theta_0, \; \varepsilon > 0, \; N \in \mathbb{N} \), and \( \tau = T/N > 0 \). Let \((\bar{q}, \bar{w}) \in L^\infty(\Omega; \mathbb{R}^{n+1}) \) be given and define the approximate scheme

\[
0 = \frac{1}{\tau} \int_\Omega (\rho_i - \bar{\rho}_i) \phi_i dx + \int_\Omega \left( \sum_{j=1}^n M_{ij}(\rho, e^w) \nabla q_j - M_i(\rho, e^w) e^{-w} \nabla w \right) \cdot \nabla \phi_i dx
\]

\[
+ \varepsilon \int_\Omega (D^2 q_i : D^2 \phi_i + q_i \phi_i) dx - \int_\Omega r_i(\rho, e^w) \phi_i dx,
\]

(22)

\[
0 = \frac{1}{\tau} \int_\Omega (E - \bar{E}) \phi_0 dx + \int_\Omega \left( \kappa(\theta) \nabla \theta + \sum_{j=1}^n M_j(\rho, e^w) \nabla q_j \right) \cdot \nabla \phi_0 dx
\]

\[
- \lambda \int_{\partial \Omega} (\theta_0 - \theta) \phi_0 ds + \varepsilon \int_\Omega e^w (D^2 w : D^2 \phi_0 + |\nabla w|^2 \nabla w \cdot \nabla \phi_0) dx
\]

\[
+ \varepsilon \int_\Omega (e^{\bar{w}_0} + e^w)(w - w_0) \phi_0 dx
\]

for test functions \( \phi_i \in H^2(\Omega), \; i = 0, \ldots, n \). Here, \( D^2 u \) is the Hessian matrix of the function \( u \), \( : \) denotes the Frobenius matrix product, and \( E = \rho \theta \), \( \bar{E} = \bar{\rho} \bar{\theta} \). The lower-order regularization \( \varepsilon (e^{\bar{w}_0} + e^w)(w - w_0) \) yields an \( L^2(\Omega) \) estimate for \( w \). Furthermore, the higher-order regularization guarantees that \( q_i, w \in H^2(\Omega) \) if \( L^\infty(\Omega) \), while the \( W^{1,4}(\Omega) \)}
regularization term for $w$ allows us to estimate the higher-order terms when using the test function $e^{-w_0} - e^{-w}$.

**Step 1: solution of the linearized approximate problem.** In order to define the fixed-point operator, we need to solve a linearized problem. To this end, let $y^* = (q^*, w^*) \in W^{1,4}(\Omega; \mathbb{R}^{n+1})$ and $\sigma \in \{0, 1\}$ be given. We want to find the unique solution $y = (q, w) \in H^2(\Omega; \mathbb{R}^{n+1})$ to the linear problem

\begin{equation}
(23) \quad a(y, \phi) = \sigma F(\phi) \quad \text{for all } \phi = (\phi_0, \ldots, \phi_n) \in H^2(\Omega; \mathbb{R}^{n+1}),
\end{equation}

where

\begin{align*}
a(y, \phi) &= \int_{\Omega} \sum_{i,j=1}^n M_{ij}(\rho^*, e^{w^*}) \nabla q_j \cdot \nabla \phi_i dx + \int_{\Omega} \kappa(e^{w^*})e^{w^*} \nabla w \cdot \nabla \phi_0 dx \\
&\quad + \varepsilon \int_{\Omega} \sum_{i=1}^n (D^2 q_i : D^2 \phi_i + q_i \phi_i) dx \\
&\quad + \varepsilon \int_{\Omega} e^{w^*} (D^2 w : D^2 \phi_0 + |\nabla w^*|^2 \nabla w \cdot \nabla \phi_0) dx,
\end{align*}

\begin{align*}
F(\phi) &= -\frac{1}{\tau} \int_{\Omega} \sum_{i=1}^n (\rho_i^* - \bar{\rho}_i) \phi_i dx - \frac{1}{\tau} \int_{\Omega} (E^* - \bar{E}) \phi_0 dx + \lambda \int_{\partial \Omega} (e^{w_0} - e^{w^*}) \phi_0 dx \\
&\quad + \int_{\Omega} \sum_{i=1}^n M_i(\rho^*, e^{w^*}) e^{-w^*} \nabla w^* \cdot \nabla \phi_i dx - \int_{\Omega} \sum_{j=1}^n M_j(\rho^*, e^{w^*}) \nabla q_j^* \cdot \nabla \phi_j dx \\
&\quad + \int_{\Omega} \sum_{i=1}^n r_i(\rho^*, e^{w^*}) \phi_i dx - \varepsilon \int_{\Omega} (e^{w_0} + e^{w^*})(w^* - w_0) \phi_0 dx
\end{align*}

and $\rho_i^* = e^{w^* + \xi_i}$, $\rho^* = \sum_{i=1}^n \rho_i^*$, $E^* = \rho^* e^{w^*}$. By Hypothesis (H3) and the generalized Poincaré inequality [26, Chap. 2, Sec. 1.4], we have

\begin{equation*}
a(y, y) \geq \varepsilon \int_{\Omega} (|D^2 q|^2 + |q|^2) dx + \varepsilon \int_{\Omega} (|D^2 w|^2 + w^2) dx \geq \varepsilon C(\|q\|_{H^2(\Omega)}^2 + \|w\|_{H^2(\Omega)}^2).
\end{equation*}

Thus, $a$ is coercive. Moreover, $a$ and $F$ are continuous on $H^2(\Omega; \mathbb{R}^{n+1})$. The Lax–Milgram lemma shows that (23) possesses a unique solution $(q, w) \in H^2(\Omega; \mathbb{R}^{n+1})$.

**Step 2: solution of the approximate problem.** The previous step shows that the fixed-point operator $S : W^{1,4}(\Omega; \mathbb{R}^{n+1}) \times [0, 1] \to W^{1,4}(\Omega; \mathbb{R}^{n+1})$, $S(y^*, \sigma) = y$, where $y = (q, w)$ solves (23), is well defined. It holds that $S(y, 0) = 0$, $S$ is continuous, and since $S$ maps to $H^2(\Omega; \mathbb{R}^{n+1})$, which is compactly embedded into $W^{1,4}(\Omega; \mathbb{R}^{n+1})$, it is also compact. It remains to determine a uniform bound for all fixed points $y$ of $S(\cdot, \sigma)$, where $\sigma \in (0, 1]$. Let $y$ be such a fixed point. Then $y \in H^2(\Omega; \mathbb{R}^{n+1})$ solves (23) with $(q^*, w^*)$ replaced by $y = (q, w)$. With the test functions $\phi_i = q_i$ for $i = 1, \ldots, n$ and $\phi_0 = e^{-w_0} - e^{-w}$ (we need this test function since $\phi_0 = -e^{-w}$ does not allow us to control the lower-order term), we
obtain

\[
0 = \frac{\sigma}{\tau} \int_\Omega \sum_{i=1}^n (\rho_i - \bar{\rho}_i) q_i \, dx + \frac{\sigma}{\tau} \int_\Omega (E - \bar{E})(-e^{-w}) \, dx + \frac{\sigma}{\tau} \int_\Omega (E - \bar{E}) e^{-w_0} \, dx
\]

\[
+ \int_\Omega \sum_{i,j=1}^n M_{ij} \nabla q_i \cdot \nabla q_j \, dx + \int_\Omega \kappa(e^w)e^w \nabla w \cdot \nabla (-e^{-w}) \, dx - \sigma \int_\Omega \sum_{i=1}^n r_i q_i \, dx
\]

\[
- \sigma \int_\Omega \sum_{j=1}^n M_j e^{-w} \nabla w \cdot \nabla q_j \, dx + \sigma \int_\Omega \sum_{j=1}^n M_j \nabla q_j \cdot \nabla (-e^{-w}) \, dx
\]

\[
- \sigma \lambda \int_{\partial \Omega} (e^{w_0} - e^w)(e^{-w_0} - e^{-w}) \, dx + \varepsilon \int_\Omega \sum_{i=1}^n (|D^2 q_i|^2 + q_i^2) \, dx
\]

\[
+ \varepsilon \int_\Omega (|D^2 w|^2 + D^2 w : \nabla w \otimes \nabla w + |\nabla w|^4) \, dx
\]

\[
=: I_1 + \cdots + I_{12}.
\]

We see immediately that \(I_7 + I_8 = 0\). Furthermore,

\[
I_1 + I_2 = \frac{\sigma}{\tau} \int_\Omega \left( \sum_{i=1}^n (\rho_i - \bar{\rho}_i) \frac{\partial h}{\partial \rho_i} + (\theta - \bar{\theta}) \frac{\partial h}{\partial \theta} \right) \, dx.
\]

The function \((\rho, \theta) \mapsto h(\rho, \theta)\) is convex since, by the Cauchy–Schwarz inequality, for any \(z = (z, z_{n+1}) = (z_1, \ldots, z_n, z_{n+1}) \in \mathbb{R}^{n+1}\),

\[
z^T D^2 h z = \sum_{i=1}^n \frac{z_i^2}{\rho_i} - \frac{2}{\bar{\theta}} \sum_{i=1}^n z_i z_{n+1} + \frac{\rho}{\bar{\theta}^2} z_{n+1}^2 \geq \sum_{i=1}^n \left( \frac{1}{\rho_i} - \frac{1}{\rho} \right) z_i^2 \geq 0.
\]

This shows that

\[
h(\rho, \theta) - h(\bar{\rho}, \bar{\theta}) \leq \sum_{i=1}^n \frac{\partial h}{\partial \rho_i}(\rho, \theta)(\rho_i - \bar{\rho}_i) + \frac{\partial h}{\partial \theta}(\rho, \theta)(\theta - \bar{\theta})
\]

and consequently,

\[
I_1 + I_2 \geq \frac{\sigma}{\tau} \int_\Omega (h(\rho, \theta) - h(\bar{\rho}, \bar{\theta})) \, dx.
\]

By Hypotheses (H3) and (H5),

\[
I_4 \geq c_M \int_\Omega |\nabla \Pi q|^2 \, dx, \quad I_6 \geq \sigma c_r \int_\Omega |\Pi q|^2 \, dx.
\]

This gives an \(H^1(\Omega)\) bound for \(\Pi q\). Next, we have

\[
I_5 = \int_\Omega \kappa(e^w)|\nabla w|^2 \, dx, \quad I_9 = 2\sigma \lambda \int_{\partial \Omega} (\cosh(w_0 - w) - 1) \, ds \geq 0,
\]

...
\[ I_{11} = 2\varepsilon \int_{\Omega} (w - w_0) \sinh(w - w_0) dx \geq \varepsilon \int_{\Omega} (w - w_0)^2 dx, \]
\[ I_{12} = \varepsilon \int_{\Omega} \left( |D^2 w|^2 + |D^2 w - \nabla \otimes \nabla w|^2 + |\nabla w|^4 \right) dx. \]

Summarizing these estimates and applying the generalized Poincaré inequality, we arrive at the \textit{discrete entropy inequality}

\[
\frac{\sigma}{\tau} \int_{\Omega} \left( h(\rho, \theta) + e^{-\omega_0 E} \right) dx + \sigma \int_{\Omega} \left( c_M |\nabla q|^2 + c_r |\Pi q|^2 \right) dx + \int_{\Omega} \kappa(e^w) |\nabla w|^2 dx
+ \varepsilon C \left( \|q\|_{H^2(\Omega)}^2 + \|w\|_{H^2(\Omega)}^2 + \|w\|_{W^{1,4}(\Omega)}^4 \right) \leq \frac{\sigma}{\tau} \int_{\Omega} \left( h(\rho, \theta) + e^{-\omega_0 E} \right) dx.
\]

This estimate gives a uniform bound for \((q, w)\) in \(H^2(\Omega; \mathbb{R}^{n+1})\) and consequently also in \(W^{1,4}(\Omega; \mathbb{R}^{n+1})\), which proves the claim. We infer from the Leray–Schauder fixed-point theorem that there exists a solution \((q, w)\) to (21)–(22).

\textbf{Step 3: temperature estimate.} We need a better estimate for the temperature. The following result gives a conditional estimate. We prove below that \(\nabla q\) is uniformly bounded in \(L^2(\Omega)\), yielding an unconditional estimate.

\textbf{Lemma 3.} Let \((\rho, w)\) be a solution to (21)–(22) and set \(\theta = e^w\). Then there exists a constant \(C > 0\) independent of \(\varepsilon\) and \(\tau\) such that
\[
\frac{1}{\tau} \int_{\Omega} \rho^0 \theta^2 dx + \frac{1}{2} \int_{\Omega} \kappa(\theta) |\nabla \theta|^2 dx \leq C + \frac{1}{\tau} \int_{\Omega} \rho^0 \theta^2 dx + C \sum_{j=1}^{n} \int_{\Omega} |\nabla q_j|^2 dx.
\]

\textbf{Proof.} We use the test function \(\theta\) in (22). Observing that \((E - \bar{E}) \theta = \rho^0 (\theta - \bar{\theta}) \theta \geq (\rho^0 / 2)(\theta^2 - \bar{\theta}^2)\) and that \(\kappa(\theta) \geq c_\kappa (1 + \theta^2)\) by Hypothesis (H4), we find that
\[
\frac{1}{2\tau} \int_{\Omega} \rho^0 (\theta^2 - \bar{\theta}^2) dx + \frac{1}{2} \int_{\Omega} \kappa(\theta) |\nabla \theta|^2 dx + \frac{c_\kappa}{2} \int_{\Omega} \theta^2 |\nabla \theta|^2 dx - \lambda \int_{\partial \Omega} (\theta_0 - \theta) \theta dx
\]
\[
\leq - \sum_{j=1}^{n} \int_{\Omega} M_j \nabla q_j \cdot \nabla \theta dx - \varepsilon \int_{\Omega} \theta (D^2 \log \theta : D^2 \theta + |\nabla \log \theta|^2 \nabla \log \theta \cdot \nabla \theta) dx
\]
\[
- \varepsilon \int_{\Omega} (\theta_0 + \theta) (\log \theta - \log \theta_0) \theta dx
\]
\[
=: J_1 + J_2 + J_3.
\]

Since \(M_j / \theta\) is assumed to be bounded,
\[
J_1 \leq \frac{c_\kappa}{2} \int_{\Omega} \theta^2 |\nabla \theta|^2 dx + C \sum_{j=1}^{n} \int_{\Omega} |\nabla q_j|^2 dx.
\]
Furthermore,
\[
J_2 = -\varepsilon \int_{\Omega} \left( \frac{1}{\theta} \nabla \theta \cdot D^2 \theta \nabla \theta + |D^2 \theta|^2 + \frac{1}{\theta^2} |\nabla \theta|^4 \right) dx
\]
since MJ. A tedious but straightforward computation shows that

\[ \alpha J = \int_\Omega \left( |D^2 \theta|^2 + \frac{1}{\theta^2} |\nabla \theta|^4 + \left| D^2 \theta - \frac{1}{\theta} \nabla \theta \otimes \nabla \theta \right|^2 \right) dx \leq 0. \]

The last integral \( J_3 \) is bounded since \(-\theta^2 \log \theta\) is the dominant term. The last term on the left-hand side of (25) is bounded from below by \(-(\lambda/2) \int_{\Omega} \theta_\delta^2 dx\), which finishes the proof. \( \square \)

**Remark 4.** Better estimates can be derived if we assume that \( \kappa(\theta) \geq c_\kappa (1 + \theta^{\alpha + 1}) \) for \( \alpha \in (1, 2) \). Indeed, using \( \theta^\alpha \) as a test function in (22), we find that

\[
\frac{1}{\tau} \int_\Omega \rho^\theta (\theta - \bar{\theta}) \theta^\alpha dx + \alpha c_\kappa \int_\Omega \theta^{2\alpha} |\nabla \theta|^2 dx - \lambda \int_{\Omega} (\theta_0 - \theta) \theta^\alpha dx \\
\leq -\alpha \sum_{j=1}^n \int_\Omega M_j \theta^{\alpha - 1} \nabla q_j \cdot \nabla \theta dx - \varepsilon \int_\Omega (\theta_0 + \theta) (\log \theta - \log \theta_0) \theta^\alpha dx \\
- \varepsilon \int_\Omega \theta (D^2 \log \theta : D^2 \theta^\alpha + |\nabla \log \theta|^2 |\nabla \theta \cdot \nabla \theta^\alpha|) dx
\]

\((26)\) \( =: J_4 + J_5 + J_6. \)

A tedious but straightforward computation shows that \( J_6 \geq 0 \) if \( \alpha \in (1, 2) \). Furthermore, since \( M_j/\theta \) is bounded,

\[
J_4 \leq \frac{\alpha c_\kappa}{2} \int_\Omega \theta^{2\alpha} |\nabla \theta|^2 dx + C \sum_{j=1}^n \int_\Omega |\nabla q_j|^2 dx.
\]

The first integral on the right-hand side is controlled by the left-hand side of (26). This yields a bound for \( \theta^{\alpha + 1} \in L^\infty(0, T; L^1(\Omega)) \cap L^2(0, T; H^1(\Omega)) \subset L^{8/3}(\Omega_T) \) (see Lemma 7) and consequently \( \theta \in L^{8(\alpha + 1)/3}(\Omega_T) \), which is better than the result in Lemma 7. \( \square \)

**Step 4: uniform estimates.** Let \((q^k, w^k)\) be a solution to (21)–(22) for given \( q^{k-1} = \bar{q} \) and \( w^{k-1} = \bar{w} \), where \( k \in \mathbb{N} \). We set \( \theta^k = \exp(w^k) \), \( \rho_i^k = \exp(w^k + q^k_i) \) for \( i = 1, \ldots, n \), and \( E^k = \rho^k \theta^k \). Recall that \( \rho^\theta = \sum_{i=1}^n \rho_i^\theta \) does not depend on \( k \in \mathbb{N} \). We introduce piecewise constant functions in time. For this, let \( \rho_i^{\tau}(x, t) = \rho_i^k(x) \), \( \theta^{\tau}(x, t) = \theta^k(x) \), and \( q_i^{\tau}(x, t) = q_i^k(x) \) for \( x \in \Omega \), \( t \in ((k - 1)\tau, k\tau) \), \( k = 1, \ldots, N \). At time \( t = 0 \), we set \( \rho_i^{\tau}(x, 0) = \rho_i^0(x) \) and \( \theta^{\tau}(x, 0) = \theta^0(x) \) for \( x \in \Omega \). Furthermore, we introduce the shift operator \((\sigma \rho_i^{\tau})(x, t) = \rho_i^{k-1}(x) \) for \( x \in \Omega \), \( t \in ((k - 1)\tau, k\tau) \). Let \( \rho^{\tau} = (\rho_1^{\tau}, \ldots, \rho_n^{\tau}) \). Then \((\rho^{\tau}, \theta^{\tau})\) solves (see (21)–(22))

\[
0 = \frac{1}{\tau} \int_0^T \int_\Omega (\rho_i^{\tau}) - \sigma \rho_i^{\tau}) \phi_i dx dt + \int_0^T \int_\Omega \left( \sum_{j=1}^n M_j (\rho^{\tau}, \theta^{\tau}) \nabla q_j^{\tau} + M_i (\rho^{\tau}, \theta^{\tau}) \nabla \frac{1}{\theta^{\tau}} \right) \cdot \nabla \phi_i dx dt \\
+ \varepsilon \int_0^T \int_\Omega (D^2 q_i^{\tau} : D^2 \phi_i + q_i \phi_i) dx dt - \int_0^T \int_\Omega r_i (\rho^{\tau}, \theta^{\tau}) \phi_i dx dt,
\]
\begin{equation}
0 = \frac{1}{\tau} \int_0^T \int_\Omega \left( E^{(\tau)} - \sigma_{\tau} E^{(\tau)} \right) \phi_0 dx dt - \lambda \int_0^T \int_{\partial \Omega} \left( \theta_0 - \theta^{(\tau)} \right) \phi_0 ds dt
\end{equation}
\begin{align*}
+ \int_0^T \int_\Omega \left( \kappa(\theta^{(\tau)}) \nabla \theta^{(\tau)} + \sum_{j=1}^n \mathcal{M}_j(\rho^{(\tau)}, \theta^{(\tau)}) \nabla q_j^{(\tau)} \right) \cdot \nabla \phi_0 dx dt
\end{align*}
\begin{align*}
+ \varepsilon \int_0^T \int_\Omega \theta^{(\tau)} \left( D^2 \log \theta^{(\tau)} : D^2 \phi_0 + |\nabla \log \theta^{(\tau)}|^2 \nabla \log \theta^{(\tau)} \cdot \nabla \phi_0 \right) dx dt
\end{align*}
\begin{align*}
+ \varepsilon \int_0^T \int_\Omega \left( \theta_0 + \theta^{(\tau)} \right) (\log \theta^{(\tau)} - \log \theta_0) \phi_0 dx dt.
\end{align*}

The discrete entropy inequality (24), the $L^\infty$ bound for $\rho_i^{(\tau)}$, and the property $E^{(\tau)} = \rho^{(\tau)} \theta^{(\tau)} = \rho^0 \theta^{(\tau)} \geq \rho^0 \theta^{(\tau)}$ imply the following uniform bounds:
\begin{align*}
\| \rho_i^{(\tau)} \|_{L^\infty(0,T;L^\infty(\Omega))} + \| \theta^{(\tau)} \|_{L^\infty(0,T;L^1(\Omega))} \leq C, \\
\| \Pi q_i^{(\tau)} \|_{L^2(0,T;H^1(\Omega))} + \| \kappa(\theta^{(\tau)})^{1/2} \nabla \log \theta^{(\tau)} \|_{L^2(\Omega_T)} \leq C, \\
\varepsilon^{1/2} \| q_i^{(\tau)} \|_{L^2(0,T;H^2(\Omega))} + \varepsilon^{1/2} \| \log \theta^{(\tau)} \|_{L^2(0,T;H^2(\Omega))} \leq C, \\
\varepsilon^{1/4} \| \log \theta^{(\tau)} \|_{L^4(0,T;W^{1,4}(\Omega))} \leq C,
\end{align*}
for all $i = 1, \ldots, n$, where $C > 0$ is independent of $\varepsilon$ and $\tau$. Hypothesis (H4) yields
\begin{align*}
\| \nabla \theta^{(\tau)} \|_{L^2(\Omega_T)} + \| \nabla \log \theta^{(\tau)} \|_{L^2(\Omega_T)} \leq C.
\end{align*}

Then the $L^\infty(0,T;L^1(\Omega))$ bound for $\theta^{(\tau)}$ and the Poincaré–Wirtinger inequality show that
\begin{align*}
\| \theta^{(\tau)} \|_{L^2(0,T;H^1(\Omega))} \leq C.
\end{align*}

We proceed by proving more uniform estimates. The following lemma is the key result.

**Lemma 5.** There exists $C > 0$ independent of $\varepsilon$ and $\tau$ such that
\begin{equation}
\| \rho_i^{(\tau)} \|_{L^2(0,T;H^1(\Omega))} + \| \nabla q_i^{(\tau)} \|_{L^2(0,T;L^2(\Omega))} \leq C.
\end{equation}

**Proof.** We infer from $\rho^0 = \sum_{i=1}^n \rho_i^{(\tau)}$ and $\rho_i^{(\tau)} = \theta^{(\tau)} \exp(q_i^{(\tau)})$ that
\begin{align*}
\nabla \rho^0 &= \sum_{i=1}^n \nabla \rho_i^{(\tau)} = \sum_{i=1}^n \rho_i^{(\tau)} \left( \nabla \log \theta^{(\tau)} + \nabla q_i^{(\tau)} \right) \\
&= \rho^0 \nabla \log \theta^{(\tau)} + \sum_{i=1}^n \rho_i^{(\tau)} \left( \left( q_i^{(\tau)} - \frac{1}{n} \sum_{j=1}^n q_j^{(\tau)} \right) + \frac{1}{n} \sum_{i=1}^n \rho_i^{(\tau)} \sum_{j=1}^n \nabla q_j^{(\tau)} \right) \\
&= \rho^0 \nabla \log \theta^{(\tau)} + \sum_{i=1}^n \rho_i^{(\tau)} \left( \nabla \Pi q_i^{(\tau)} \right)_i + \frac{1}{n} \rho^0 \sum_{j=1}^n \nabla q_j^{(\tau)}.
\end{align*}
Since \((\rho_i^{(r)})\) is bounded in \(L^\infty(\Omega_T)\) and \((\nabla \log \theta^{(r)})\) and \((\nabla \Pi q^{(r)})\) are bounded in \(L^2(\Omega_T)\), we conclude that
\[
\frac{1}{n} \sum_{j=1}^n \nabla q_j^{(r)} = \nabla \log \rho^0 - \nabla \log \theta^{(r)} - \frac{1}{\rho^0} \sum_{i=1}^n \rho_i^{(r)} \nabla (\Pi q^{(r)})_i
\]
is uniformly bounded in \(L^2(\Omega_T)\). It follows that
\[
\nabla q_i^{(r)} = \nabla (\Pi q^{(r)})_i + \frac{1}{n} \sum_{j=1}^n \nabla q_j^{(r)}
\]
is uniformly bounded in \(L^2(\Omega_T)\) too. This shows that \(\nabla \rho_i^{(r)} = \rho_i^{(r)} (\nabla \log \theta^{(r)} + \nabla q_i^{(r)})\) is uniformly bounded in \(L^2(\Omega_T)\), finishing the proof. \(\square\)

**Lemma 6.** There exists \(C > 0\) independent of \(\varepsilon\) and \(\tau\) such that
\[
\|\theta^{(r)}\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla (\theta^{(r)})^2\|_{L^2(\Omega_T)} \leq C.
\]

**Proof.** Lemma 3 shows that for \(t \in [0,T]\),
\[
\int_\Omega \rho^0 ((\theta^{(r)})^2 - \theta^0)^2 \, dx + \int_0^t \int_\Omega (\theta^{(r)})^2 |\nabla \theta^{(r)}|^2 \, dx \, dt
\]
is bounded. Using \(\rho^0 \geq \rho_* > 0\), the lemma is proved. \(\square\)

**Lemma 7.** There exists \(C > 0\) independent of \(\varepsilon\) and \(\tau\) such that \(\theta^{(r)}\) is bounded in \(L^{16/3}(\Omega_T)\).

**Proof.** We know from Lemma 6 that \(\nabla (\theta^{(r)})^2\) is uniformly bounded in \(L^2(\Omega_T)\) and \((\theta^{(r)})^2\) is uniformly bounded in \(L^2(0,T;L^4(\Omega))\). We deduce from the Poincaré–Wirtinger inequality that \((\theta^{(r)})^2\) is uniformly bounded in \(L^2(\Omega_T)\) and consequently also in \(L^2(0,T;H^1(\Omega)) \subseteq L^2(0,T;L^6(\Omega))\). This shows that \((\theta^{(r)})\) is bounded in \(L^4(0,T;L^12(\Omega))\). By interpolation with \(1/r = \alpha/12 + (1 - \alpha)/2\) and \(r\alpha = 4\),
\[
\|\theta^{(r)}\|_{L^r(\Omega_T)} = \int_0^T \|\theta^{(r)}\|_{L^r(\Omega)} \, dt \leq \int_0^T \|\theta^{(r)}\|_{L^{12}(\Omega)}^\alpha \|\theta^{(r)}\|_{L^2(\Omega)}^{r(1-\alpha)} \, dt
\]
is bounded. The solution of \(1/r = \alpha/12 + (1 - \alpha)/2\) and \(r\alpha = 4\) is \(\alpha = 3/4\) and \(r = 16/3\). \(\square\)

**Lemma 8.** There exists \(C > 0\) independent of \(\varepsilon\) and \(\tau\) such that
\[
\tau^{-1} \|ho_i^{(r)} - \sigma_r \rho_i^{(r)}\|_{L^2(0,T;H^2(\Omega))} + \tau^{-1} \|	heta^{(r)} - \sigma_r \theta^{(r)}\|_{L^{16/3}(0,T;W^{1,6}(\Omega))} \leq C.
\]
Proof. Let $\phi_0 \in L^{16}(0, T; W^{1,16}(\Omega))$, $\phi_1, \ldots, \phi_n \in L^2(0, T; H^2(\Omega))$ and set $M_i^{(\tau)} = M_i(\rho^{(\tau)}, \theta^{(\tau)})$, $r_i^{(\tau)} = r_i(\rho^{(\tau)}, \theta^{(\tau)})$ for $i = 1, \ldots, n$. It follows from (27)–(28) and Hypotheses (H3)–(H5) that

$$
\frac{1}{\tau} \int_0^T \int_\Omega (\rho_i^{(\tau)} - \sigma_i^{(\tau)}) \phi_i dx dt \leq C \left\| \nabla q_i^{(\tau)} \right\|_{L^2(\Omega_T)} \| \nabla \phi \|_{L^2(\Omega_T)} 
$$

$$
+ \sum_{i=1}^n \left\| M_i^{(\tau)} \right\|_{L^\infty(\Omega_T)} \left\| \nabla \log \theta^{(\tau)} \right\|_{L^2(\Omega_T)} \left\| \nabla \phi \right\|_{L^2(\Omega_T)} 
$$

$$
+ \varepsilon \| q_i^{(\tau)} \|_{L^2(0, T; H^2(\Omega))} \| \phi \|_{L^2(0, T; H^2(\Omega))} + \| r_i^{(\tau)} \|_{L^2(\Omega_T)} \| \phi \|_{L^2(\Omega_T)} 
$$

$$
\leq C \| \phi \|_{L^2(0, T; H^2(\Omega))},
$$

and

$$
\frac{1}{\tau} \int_0^T \int_\Omega (E^{(\tau)} - \sigma_i^{(\tau)} E^{(\tau)}) \phi_0 dx dt \leq C + C \left\| \theta^{(\tau)} \right\|_{L^{8/3}(\Omega_T)} \left\| \nabla (\theta^{(\tau)})^2 \right\|_{L^2(\Omega_T)} \| \nabla \phi_0 \|_{L^8(\Omega_T)} 
$$

$$
+ \sum_{j=1}^n \left\| M_j^{(\tau)} \right\|_{L^\infty(\Omega_T)} \left\| \theta^{(\tau)} \right\|_{L^{8/3}(\Omega_T)} \left\| \nabla q_j^{(\tau)} \right\|_{L^2(\Omega_T)} \| \nabla \phi_0 \|_{L^8(\Omega_T)} 
$$

$$
+ \lambda \| \theta_0 - \theta^{(\tau)} \|_{L^{8/7}(0, T; L^{8/7}(\partial\Omega))} \| \phi_0 \|_{L^8(0, T; L^8(\partial\Omega))} 
$$

$$
+ \varepsilon \| \theta^{(\tau)} \|_{L^3(\Omega_T)} \log \theta^{(\tau)} \|_{L^2(0, T; H^2(\Omega))} \| \nabla \phi_0 \|_{L^8(\Omega_T)} 
$$

$$
+ \varepsilon \| \theta^{(\tau)} \|_{L^{16/3}(\Omega_T)} \left\| \nabla \log \theta^{(\tau)} \right\|_{L^4(\Omega_T)} \| \nabla \phi_0 \|_{L^{16}(\Omega_T)} 
$$

$$
+ \varepsilon \| \theta^{(\tau)} \| \log \theta^{(\tau)} \|_{L^2(\Omega_T)} \| \phi_0 \|_{L^2(\Omega_T)} \leq C \| \phi_0 \|_{L^{16}(0, T; W^{1,16}(\Omega))}.
$$

Since $|E^{(\tau)} - \sigma_i^{(\tau)} E^{(\tau)}| = \rho_0^0 |\theta^{(\tau)} - \sigma_i \theta^{(\tau)}| \geq \rho_0^0 |\theta^{(\tau)} - \sigma_i \theta^{(\tau)}|$, this concludes the proof. \qed

**Step 4: limit** $(\varepsilon, \tau) \to 0$. Estimates (29)–(31) allow us to apply the Aubin–Lions lemma in the version of [12]. Thus, there exist subsequences that are not relabeled such that as $(\varepsilon, \tau) \to 0$,

$$
\rho_i^{(\tau)} \to \rho_i, \quad \theta^{(\tau)} \to \theta \quad \text{strongly in } L^2(\Omega_T), \quad i = 1, \ldots, n.
$$

The $L^\infty(\Omega_T)$ bound for $(\rho_i^{(\tau)})$ and the $L^{16/3}(\Omega_T)$ bound for $(\theta^{(\tau)})$ imply the stronger convergences

$$
\rho_i^{(\tau)} \to \rho_i \quad \text{strongly in } L^r(\Omega_T) \text{ for all } r < \infty,
$$

$$
\theta^{(\tau)} \to \theta \quad \text{strongly in } L^\eta(\Omega_T) \text{ for all } \eta < 16/3.
$$

The uniform bounds also imply that, up to subsequences,

$$
\rho_i^{(\tau)} \to \rho_i \quad \text{weakly in } L^2(0, T; H^1(\Omega)),
$$

$$
\theta^{(\tau)} \to \theta \quad \text{weakly in } L^2(0, T; H^1(\Omega)),
$$

$$
\nabla q_i^{(\tau)} \to \nabla q_i \quad \text{weakly in } L^2(0, T; L^2(\Omega)),
$$
Using the compact embedding $H$, this shows that the sequences are converging in $L^2(0,T;H^2(\Omega))$. Moreover, as $(\varepsilon, \tau) \to 0$,

$$
\varepsilon \log \theta^{(\tau)} \to 0, \quad \varepsilon q_i^{(\tau)} \to 0 \quad \text{strongly in } L^2(0,T;H^2(\Omega)).
$$

We deduce from the linearity and boundedness of the trace operator $H^1(\Omega) \to H^{1/2}(\partial\Omega)$ that

$$
\theta^{(\tau)} \to \theta \quad \text{weakly in } L^2(0,T;H^{1/2}(\partial\Omega)).
$$

Using the compact embedding $H^{1/2}(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$, this gives

$$
\theta^{(\tau)} \to \theta \quad \text{strongly in } L^2(0,T;L^2(\partial\Omega)).
$$

Next, we prove that $\rho_i$ and $\theta$ are positive a.e. The functions $\rho_i^{(\tau)} = \exp(w_i^{(\tau)} + q_i^{(\tau)})$ and $\theta^{(\tau)} = \exp(w_i^{(\tau)})$ are positive in $\Omega_T$ and therefore, the limits $\rho_i$ and $\theta$ are nonnegative. We claim that $\rho$ and $\theta$ is even positive a.e. Indeed, by the Chebyshev inequality, we have for any $\delta \in (0,1)$,

$$
\text{meas}\{(x,t) : \rho_i^{(\tau)}(x,t) \leq \delta\} = \text{meas}\{(x,t) : -\log \theta^{(\tau)}(x,t) \geq -\log \delta\} \leq \frac{C}{-\log \delta} \int_0^T \int_{\{\theta^{(\tau)} \leq \delta\}} (-\log \theta^{(\tau)}) \, dx \, dt \leq \frac{C}{-\log \delta}.
$$

It follows in the limit $\delta \to 0$ and $(\varepsilon, \tau) \to 0$ that $\text{meas}\{(x,t) : \theta(x,t) = 0\} = 0$. Hence, $\theta > 0$ a.e. in $\Omega_T$. Furthermore, since $q_i^{(\tau)}$ is integrable and $\log \rho_i^{(\tau)} = q_i^{(\tau)} + \log \theta^{(\tau)}$, the same argument shows that

$$
\text{meas}\{(x,t) : \rho_i^{(\tau)}(x,t) \leq \delta\} = \text{meas}\{(x,t) : -\log \rho_i^{(\tau)}(x,t) \geq -\log \delta\} \leq \frac{C}{-\log \delta} \int_0^T \int_{\{\rho_i^{(\tau)} \leq \delta\}} (-\log \rho_i^{(\tau)}) \, dx \, dt \leq \frac{C}{-\log \delta},
$$

and in the limit $\delta \to 0$ and $(\varepsilon, \tau) \to 0$, we infer again that $\rho_i > 0$ a.e. in $\Omega_T$.

By assumption, $M_{ij}(\rho^{(\tau)}, \theta^{(\tau)})$ and $M_j(\rho^{(\tau)}, \theta^{(\tau)})/\theta^{(\tau)}$ are bounded. Then the strong convergences imply that these sequences are converging in $L^q(\Omega_T)$ for $q < \infty$, and the limits can be identified. Thus,

$$
M_{ij}(\rho^{(\tau)}, \theta^{(\tau)}) \to M_{ij}(\rho, \theta) \quad \text{strongly in } L^2(\Omega_T),
$$

$$
M_j(\rho^{(\tau)}, \theta^{(\tau)})/\theta^{(\tau)} \to M_j(\rho, \theta)/\theta \quad \text{strongly in } L^q(\Omega_T) \text{ for all } q < \infty.
$$

This shows that

$$
M_j(\rho^{(\tau)}, \theta^{(\tau)}) = \frac{1}{\theta^{(\tau)}} M_j(\rho^{(\tau)}, \theta^{(\tau)}) \theta^{(\tau)} \to \frac{1}{\theta} M_j(\rho, \theta) \theta = M_j(\rho, \theta).
$$
strongly in $L^\eta(\Omega_T)$ for $\eta < 16/3$ and
\[
\frac{1}{(\theta^{(r)})^2} M_j(\rho^{(r)}, \theta^{(r)}) \to \frac{1}{\theta^2} M_j(\rho, \theta)
\]
strongly in $L^\eta(\Omega_T)$ for $\eta < 8/3$.

The a.e. convergence of $(\rho_i^{(r)})$, $(\theta^{(r)})$ implies that
\[
q_i^{(r)} = \log \rho_i^{(r)} - \log \theta^{(r)} \to \log \rho_i - \log \theta =: q_i \quad \text{a.e. in } \Omega_T
\]
and consequently, $\nabla q_i^{(r)} \to \nabla q_i$ weakly in $L^2(\Omega_T)$. We conclude that
\[
M_{ij}(\rho^{(r)}, \theta^{(r)}) \nabla q_j^{(r)} \to M_{ij}(\rho, \theta) \nabla q_j \quad \text{weakly in } L^\eta(\Omega_T), \; \eta \leq 2,
\]
\[
M_j(\rho^{(r)}, \theta^{(r)}) \nabla q_j^{(r)} \to M_j(\rho, \theta) \nabla q_j \quad \text{weakly in } L^\eta(\Omega_T), \; \eta \leq 2,
\]
\[
M_j(\rho^{(r)}, \theta^{(r)}) \nabla \frac{1}{\theta^{(r)}} \to -\frac{1}{\theta^2} M_j(\rho, \theta) \nabla \theta \quad \text{weakly in } L^\eta(\Omega_T), \; \eta \leq 2.
\]
Moreover, by Hypothesis (H5),
\[
\left\| r_i (\Pi q^{(r)}, \theta^{(r)}) - r_i (\Pi q, \theta) \right\| \to 0 \quad \text{strongly in } L^\eta(\Omega_T), \; \eta < \infty.
\]
These convergences allow us to perform the limit $(\varepsilon, \tau) \to 0$ in (27)–(28) to obtain (13)–(14). Finally, we can show as in [20, p. 1980f] that the linear interpolant $\tilde{\rho}_i^{(r)}$ of $\rho_i^{(r)}$ and the piecewise constant function $\rho_i^{(r)}$ converge to the same limit, which leads to $\rho_i^0 = \tilde{\rho}_i^{(r)}(0) \to \rho_i(0)$ weakly in $H^1(\Omega)'$. Thus, the initial datum $\rho_i(0) = \rho_i^0$ is satisfied in the sense of $H^2(\Omega)'$. Similarly, $(\rho \theta)(0) = \rho^0 \theta^0$ in the sense of $W^{1,16}(\Omega)'$. This finishes the proof.

REFERENCES

[1] B. Anwasia, M. Bisi, F. Salvarani, and A. J. Soares. On the Maxwell–Stefan diffusion limit for a reactive mixture of polyatomic gases in non-isothermal setting. Kinetic Related Models 13 (2020), 63–95.
[2] A. Bondesan and M. Briant. Stability of the Maxwell–Stefan system in the diffusion asymptotics of the Boltzmann multi-species equation. Submitted for publication, 2019. arXiv:1910.08357.
[3] M. Briant and B. Grec. Rigorous derivation of the Fick cross-diffusion system from the multi-species Boltzmann equation in the diffusive scaling. Submitted for publication, 2020. arXiv:2003.07891.
[4] D. Bothe. On the Maxwell–Stefan equations to multicomponent diffusion. In: J. Escher et al. (eds). Parabolic Problems. Progress in Nonlinear Differential Equations and their Applications, pp. 81–93. Springer, Basel, 2011.
[5] D. Bothe and W. Dreyer. Continuum thermodynamics of chemically reacting fluid mixtures. Acta Mech. 226 (2015), 1757–1805.
[6] D. Bothe and P.-E. Druet. Mass transport in multicomponent compressible fluids: Local and global well-posedness in classes of strong solutions for general class-one models. Submitted for publication, 2020. arXiv:2001.08970.
[7] L. Boudin, B. Grec, and V. Pavan. Diffusion models for mixtures using a stiff dissipative hyperbolic formalism. J. Hyperbol. Eqs. 16 (2019), 293–312.
[8] L. Boudin, B. Grec, M. Pavić, and F. Salvarani. Diffusion asymptotics of a kinetic model for gaseous mixtures. Kinetic Related Models 6 (2013), 137–157.
[9] M. Buliček, A. Jüngel, M. Pokorný, and N. Zamponi. Existence analysis of a stationary compressible fluid model for heat-conducting and chemically reacting mixtures. Submitted for publication, 2020. arXiv:2001.06082.

[10] X. Chen and A. Jüngel. Analysis of an incompressible Navier–Stokes–Maxwell–Stefan system. *Commun. Math. Phys.* 340 (2015), 471–497.

[11] M. Dolce and D. Donatelli. Artificial compressibility method for the Navier–Stokes–Maxwell–Stefan system. *J. Dyn. Diff. Eqs.*, 2019. https://doi.org/10.1007/s10884-019-09808-4.

[12] M. Dreher and A. Jüngel. Compact families of piecewise constant functions in $L^p(0, T; B)$. *Nonlin. Anal.* 75 (2012), 3072–3077.

[13] W. Dreyer, P.-E. Druet, P. Gajewski, and C. Guhilke. Analysis of improved Nernst–Planck–Poisson models of compressible isothermal electrolytes. Part I. Derivation of the model and survey of the results. To appear in Z. *Angew. Math. Phys.*., 2020.

[14] P.-E. Druet. A theory of generalised solutions for ideal gas mixtures with Maxwell–Stefan diffusion. Submitted for publication, 2020. WIAS Preprint no. 2700, WIAS Berlin, Germany.

[15] E. Feireisl and A. Novotný. *Singular Limits in Thermodynamics of Viscous Flows*. Birkhäuser, Basel, 2009.

[16] V. Giovangigli and M. Massot. The local Cauchy problem for multicomponent flows in full vibrational non-equilibrium. *Math. Meth. Appl. Sci.* 21 (1998), 1415–1439.

[17] V. Giovangigli, M. Pokorný, and E. Zatorska. On the steady flow of reactive gaseous mixture. *Analysis (Berlin)* 35 (2015), 319–341.

[18] X. Huo, A. Jüngel, and A. Tzavaras. High-friction limits of Euler flows for multicomponent systems. *Nonlinearity* 32 (2019), 2875–2913.

[19] H. Hutridurga and F. Salvarani. Existence and uniqueness analysis of a non-isothermal cross-diffusion system of Maxwell–Stefan type. *Appl. Math. Lett.* 75 (2018), 108–113.

[20] A. Jüngel. The boundedness-by-entropy method for cross-diffusion systems. *Nonlinearity* 28 (2015), 1963–2001.

[21] A. Jüngel and O. Leingang. Convergence of an implicit Euler Galerkin scheme for Poisson–Maxwell–Stefan systems. *Adv. Comput. Math.* 45 (2019), 1469–1498.

[22] A. Jüngel and I. V. Stelzer. Existence analysis of Maxwell–Stefan systems for multicomponent mixtures. *SIAM J. Math. Anal.* 45 (2013), 2421–2440.

[23] L. Ostrowski and C. Rohde. Compressible multi-component flow in porous media with Maxwell–Stefan diffusion. To appear in *Math. Meth. Appl. Sci.*, 2020. arXiv:1905.08496.

[24] T. Piasecki and M. Pokorný. Weak and variational entropy solutions to the system describing steady flow of a compressible reactive mixture. *Nonlin. Anal.* 159 (2017), 365–392.

[25] S. Takata and K. Aoki. Two-surface problems of a multicomponent mixture of vapors and noncondensable gases in the continuum limit in the light of kinetic theory. *Phys. Fluids* 11 (1999), 2743–2756.

[26] R. Temam. *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, 2nd edn. Springer, New York, 1997.