Effects of Bose-Einstein Condensation on forces among bodies sitting in a boson heat bath

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Abstract

We explore the consequences of Bose-Einstein condensation on two-scalar-exchange mediated forces among bodies that sit in a boson gas. We find that below the condensation temperature the range of the forces becomes infinite while it is finite at temperatures above condensation.
Van der Waals type forces where two photons are being exchanged \([1]\) or the extremely feeble forces generated by 2-neutrino exchange \([2, 3]\) provide examples of forces among two static bodies in a vacuum produced by the exchange of two quanta in the t-channel. Spin independent interactions arising from double (pseudo)scalar exchange \([4, 5, 6]\) (such as axions and/or more bizarre specimens of modern completions of the Standard Model (SM)) provide further examples for these so-called dispersion forces \([7]\). When the objects that feel such forces are placed in a heat bath at a temperature \(T\), the forces get modified. Indeed, in the case of molecules in the relic photon background, the long-range Casimir-Polder forces among them are strongly affected for distances much larger than \(T^{-1}\) \([8]\) and, for the 2-neutrino forces, the cosmic neutrino background completely screens off the interaction at large distances \([9]\) (again, large meaning much larger than \(T^{-1}\)).

In the present paper we shall deal with a gas of scalar bosons carrying an abelian charge and non-zero chemical potential. As mentioned before, their double exchange between fermions has been studied in vacuum. The case of a non-charged scalar bath in a classical Boltzmann distribution was briefly discussed in \([6]\). However we are not aware of discussions on the effects resulting from placing the interacting system in a charged scalar heat bath displaying genuine quantum statistical effects such as Bose-Einstein condensation. Because in the previously reported instances, interesting effects did result, we think it is worthwhile to raise this issue here. Admittedly, light scalar bosons have a much different status than photons and neutrinos and their nature is entirely speculative. Nonetheless, in almost any extension of the Standard Model, scalars are present and some have been furthermore suggested as candidates for dark matter so that might be part of the cosmic relic background. Not to speak about axions needed to solve the serious CP problem of QCD and sometimes advertised also as galactic halo dark matter or even as dark matter on a cosmic scale.

Since we do not pretend to resemble any particular model of matter scalar interactions we will construct a very simple toy model that avoids unessential complications and that might mimic more "realistic" interactions of matter and light scalar fields. Consider the interaction lagrangean,

\[
L_{\text{int}} = g \Phi^2 \varphi^2
\]  

\(\Phi\) is a heavy scalar field of mass \(M\) and \(\varphi\) is a light scalar field of mass \(m\). Let us now put two such heavy particles \(\Phi\) in vacuum at a distance \(r\). Their lowest order interaction is given by the Feynman amplitude in figure 1. The potential is obtained from the NR limit of this amplitude via Fourier transformation. That is,

\[
V(r) = i \int \frac{d^3Q}{(2\pi)^3} e^{iQ \cdot r} T \left( q \simeq (0, Q) \right) \frac{T(q)}{4M^2}
\]
Figure 1: *Feynman diagram corresponding to the interaction which gives rise to the long-range force.*

where

$$T(q) = g^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} \frac{1}{(q - k)^2 - m^2 + i\epsilon}. \quad (3)$$

Taken at face value the previous integral diverges. It has to be regulated and the piece which leads to a long-range interaction extracted. We are left with a function of log $Q^2$, with $Q \equiv |Q|$, after dropping any polynomial in $Q$ that would only yield contact interactions [3, 10]. Introducing a convergence factor $e^{-\eta Q}$ to evaluate the Fourier transform and taking the limit $\eta \to 0$ we obtain:

$$V_{\text{vac}}(r) = -\frac{g^2 m}{64\pi^3 r^2 M^2} K_1(2mr) \quad (4)$$

for the potential. The same result can be derived using the dispersion theoretic techniques of Feinberg and Sucher [7].

For small $\varphi$ mass, eq. (4) becomes:

$$V_{\text{vac}}(r) = -\frac{g^2 m}{128\pi^3 r^3 M^2} \quad (5)$$

valid for $r \ll 1/m$. Beyond this range the Bessel function gives rise to the characteristic Yukawa factor $e^{-2mr}$.

Incidentally, identical $r$ behaviour obtains for the spin independent potential arising in double exchange of pseudoscalars coupled to matter fermions via Yukawa couplings [5, 6].

Next we introduce our system in a heat reservoir made of an ideal relativistic $\varphi$ gas at temperature $T$ ($T > m$). We further assume that the particles in the gas carry a conserved quantum number (which we will refer to as "charge") corresponding to a quantum mechanical operator $Q$. We may use real time finite temperature field theory [11] to calculate the effect of the heat bath on the potential between the two massive particles. So, we take for the
$T$-dependent $\varphi$-propagator,

$$D_F(k, T) = \frac{1}{k^2 - m^2 + i\epsilon} - \frac{2\pi i}{k^2 - m^2} \left[ \theta(k^0) n(k^0, T) + \theta(-k^0) \bar{n}(k^0, T) \right]$$  \hspace{0.5cm} (6)$$

where

$$n(\omega, T) = \left( e^{(\omega - \mu)/T} - 1 \right)^{-1} \quad \text{and} \quad \bar{n}(\omega, T) = \left( e^{(\omega + \mu)/T} - 1 \right)^{-1}$$  \hspace{0.5cm} (7)$$

are the B-E distribution functions for particles (charge +1) and antiparticles (charge -1), respectively. $\mu$ is the chemical potential associated to the conserved charge $Q$.

The amplitude of figure 1 now generalizes to

$$\mathcal{T}(q) = g^2 \int \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} \delta(4)(k + k' - q) D_F(k, T) D_F(k', T).$$  \hspace{0.5cm} (8)$$

This amplitude generates two distinct contributions to the potential. The first one arises from the first piece in $D_F(k, T)$ and is the vacuum potential just derived. The other corresponds to the situation where one of the scalars in the double exchange process is supplied by the thermal bath. This effect is described by the crossed terms in the amplitude involving the thermal piece of one $\varphi$-propagator along with the vacuum piece of the other propagator. This thermal component of the Feynman amplitude can be written as

$$\mathcal{T}_T(q) = -ig^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\sqrt{k^2 + m^2}} \left[ \frac{1}{(k - q)^2 - m^2} + \frac{1}{(k + q)^2 - m^2} \right] (n + \bar{n})$$  \hspace{0.5cm} (9)$$

where use has been made of energy-momentum conservation and of the $\delta(k^2 - m^2)$ in $D_F(k, T)$. In the static limit, i.e. momentum transfer $q \simeq (0, Q)$, where matter is supposed to be at rest in the frame of the heat reservoir, the piece in square brackets in eq. (9) reads:

$$-2 \left( Q^2 - 4k^2 \hat{Q} \cdot \hat{k} \right)^{-1}.$$  \hspace{0.5cm} (10)$$

So, finally the $T$-dependent amplitude to be Fourier transformed is

$$\mathcal{T}_T(q \simeq (0, Q)) = ig^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{k^2 + m^2}} \frac{1}{Q^2 - 4k^2Q \cdot k} (n + \bar{n})$$  \hspace{0.5cm} (11)$$

i.e. it has been reduced to an integral over the phase space of the real particles (and antiparticles) in the heat bath.

The reservoir is thermodynamically characterised by a temperature $T$, a volume $V$, and a fixed charge $Q$. Then, the chemical potential $\mu(T)$ is determined from the relation

$$Q = \sum_k (n - \bar{n}).$$  \hspace{0.5cm} (12)$$

\text{For definiteness we take $Q > 0$, i.e. particles outnumber antiparticles.}
For a Bose-Einstein gas, the sum over states in the previous formula can be converted to an integral like the one in eq. (11) as long as its temperature is above a critical temperature $T_c$. Below that temperature, if one makes the replacement

$$\sum_k \to \mathcal{V} \int \frac{d^3k}{(2\pi)^3}$$

in eq. (12), the result is less than $Q$. This is because below $T_c$ a large macroscopic fraction of the charge resides in the lowest energy state and the density of states $\mathcal{V}k^2/2\pi^2$ in the continuous representation of the sum over states gives a zero weight to the zero mode. On the contrary if the gas is above $T_c$ then the charge is thinly distributed over the states and no individual state is populated by a macroscopic fraction of the total charge so that by passing to the continuum essentially only an infinitesimally small error is done. Let us discuss both cases in turn. First start with the non degenerate case, i.e. when $T$ is above the condensation temperature $T_c$. In this instance, the phase space integrals in the formulas above correctly describe the physics of the problem. Therefore we can take eq. (11) and plug it in the expression for the potential (2)

$$V_T(r) = -\frac{1}{2\pi^2} \frac{1}{r} \frac{g^2}{4M^2} \int_0^\infty dQ Q \sin Qr \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{k^2 + m^2}} \frac{1}{Q^2 - 4k^2 Q \cdot k} (n + \bar{n}).$$

(14)

After trivial integration over $Q$ and over the polar angle in $k$-space we get

$$V_T(r) = -\frac{g^2}{64M^2\pi^3} \frac{1}{r^2} \int_0^\infty \frac{k^2 d|k|}{\sqrt{k^2 + m^2}} \frac{\sin 2|k|r}{|k|} \left[ \frac{1}{e^{(\sqrt{k^2 + m^2} - \mu)/T} - 1} + \frac{1}{e^{(\sqrt{k^2 + m^2} + \mu)/T} - 1} \right].$$

(15)

This equation by itself is not sufficient to determine $V_T(r)$ for the functions $n$ and $\bar{n}$ contain the chemical potential $\mu(T)$ which has to be obtained through

$$\rho \equiv \frac{Q}{\mathcal{V}} = \frac{1}{2\pi^2} \int k^2 d|k| (n - \bar{n})$$

(16)

which, by the way, also determines the critical temperature $T_c$ via the implicit equation [12]:

$$\rho = \rho (T = T_c, \mu = m).$$

(17)

So eqs. (13) and (16) give the solution to our problem. We can use the high-temperature expansion of eq. (16) derived in [12] to obtain the chemical potential as a function of $T$. To leading order, $\mu(T) = m(T_c/T)^2$, and we introduce it in eq. (15) to get our potential above the condensation temperature. The integral in eq. (15) can be easily split in three pieces upon using the following identity:

$$\frac{1}{e^y - 1} = \frac{1}{y} - \frac{1}{2} + 2 \sum_{k=1}^\infty \frac{y}{y^2 + (2\pi k)^2}$$

(18)
The resulting integrals can be found in [13]. The $-\frac{1}{2}$ piece exactly cancels the vacuum potential (4) so that the total contribution to the potential is finally:

$$V_{\text{total}}^{T \geq T_c} = -\frac{g^2}{64M^2\pi^2} \frac{1}{r^2} T e^{-2mr\sqrt{1-\xi^4}} + 2 \sum_{k=1}^{\infty} e^{-rT\sqrt{2(\alpha + \beta)}} \cos \left( rT \sqrt{2(\alpha - \beta)} \right)$$  \hspace{1cm} (19)

with $\alpha \equiv \sqrt{(4k\pi mT_c^2/T^3)^2 + [m^2(1 - \xi^4)/T^2 + 4k^2\pi^2]^2}$, $\beta \equiv 4k^2\pi^2 + m^2(1 - \xi^4)/T^2$, and $\xi \equiv T_c/T$.

Notice in the first term of eq. (19) the typical Yukawa damping factor that cuts off the interaction at distances long compared to the Compton wavelength of $\varphi$. Furthermore, since

$$T \sqrt{2(\alpha + \beta)} > 2m\sqrt{1 - \xi^4}$$  \hspace{1cm} (20)

all modes in the second term of eq. (19) are even more suppressed at large distances. This makes sense physically and is very convenient when trying to do a numerical evaluation.

Below $T_c$, a macroscopic fraction of the charge carried by particles in the reservoir piles up in the zero mode state (the condensate) and the integrals in eqs. (15) and (16) no longer correctly describe the physical situation. Indeed, eq. (16) gives the density of charge in excited states $\rho^*$, i.e. the thermal modes [12]. For a relativistic boson gas,

$$\rho^* = \frac{1}{3} mT^2$$  \hspace{1cm} (21)

where we used $\mu = m$ in this temperature regime since $\mu$ has to be always less or equal than $m$ and it monotonically increases as the temperature decreases until it reaches $m$ at $T_c$ (and stays fixed in the macroscopic sense thereafter). Because the definition of the condensation temperature eq. (17) implies in this case

$$\rho = \frac{1}{3} mT_c^2$$  \hspace{1cm} (22)

the charge density in the ground state is

$$\rho_0 = \rho \left( 1 - \left( \frac{T}{T_c} \right)^2 \right).$$  \hspace{1cm} (23)

The Feynman amplitude eq. (11), which involves also a sum over states, should be split accordingly in two parts. The zero mode term on the one hand and on the other hand the integral over thermal modes. The zero momentum mode contributes to the amplitude

$$\mathcal{T}_T (Q)|_{k=0} = \frac{ig^2}{mQ^2} \frac{1}{V} \left( n + \bar{n} \right)|_{k=0}$$  \hspace{1cm} (24)

where the distribution function factor can be rewritten as

$$\frac{1}{V} \left( n + \bar{n} \right)|_{k=0} = \frac{Q}{V} \left( 1 - \left( \frac{T}{T_c} \right)^2 \right) + \frac{1}{V} \frac{2}{e^{2m/T} - 1}$$  \hspace{1cm} (25)
since $\mu = m - \mathcal{O}(T/Q)$.

As long as the net charge $Q$ is a macroscopically large number many orders of magnitude larger than $T/m$, this factor coincides essentially with the condensate contribution to the density $\rho_0$. One may gain intuition on how charge is distributed among states by making a few numerical exercises with our formulae. By way of example, we take a fiducial volume of $10 m^3$ filled with 400 units of charge per cubic cm (i.e. numerically equal to the photons in the MWBR). Then, for $m = 10^{-6} eV$, $4 \times 10^{11}$ particles and 15 antiparticles populate the ground state, while $4.6 \times 10^8$ particles and $4.2 \times 10^8$ fill the excited states, at $T = 0.01 T_c = 3 \times 10^{-4} eV$. Clearly, the statement following eq. (23) is correct and it allows us to calculate $T(Q)|_{k=0}$ in terms of the fixed quantity $\rho$ (eq. (22)):

$$T(Q)|_{k=0} = \frac{ig^2}{3Q^2} (T^2_c - T^2). \tag{26}$$

The Fourier transform of this equation gives the contribution of the condensate to the potential. It is:

$$V_0(r) = -\frac{g^2}{48M^2\pi} \frac{1}{r} \left( T^2_c - T^2 \right). \tag{27}$$

The thermal contribution (i.e. from the excited states) is just eq. (15) with the chemical potential held fixed at the constant value $\mu = m$. The same expansion, eq. (18), and the same integration techniques can be used now to generate the thermal potential below the condensation temperature. Again, we find a piece that exactly cancels the vacuum contribution to the interaction, and what finally turns out to be the total potential, i.e. the thermal contribution, plus the condensate contribution (27), plus the vacuum contribution (4) is:

$$V_{total}^{T \leq T_c} = -\frac{g^2}{64M^2\pi^2} \left[ \frac{4\pi(T^2_c - T^2)}{3} + \frac{1}{r^2} T \left( 1 + 2 \sum_{k=1}^{\infty} e^{-2rT\sqrt{2(\gamma + \eta)}} \cos \left( 2\sqrt{2(\gamma - \eta)} \right) \right) \right] \tag{28}$$

where $\gamma \equiv k\pi \sqrt{k^2\pi^2 + m^2/T^2}$ and $\eta \equiv k^2\pi^2$.

It is easy to check that the potential (28) for $T \leq T_c$ and the potential (19) for $T \geq T_c$ coincide for $T = T_c$ ($\xi = 1$).

Inspection of these results leads us immediately to realize an important consequence of Bose-Einstein condensation. All terms in the infinite sum in eq. (28) decay faster than $e^{-2mr}$ for $T > m$. Therefore, for distances much larger than the Compton wavelength of $\varphi$, i.e. $r \gg m^{-1}$ and hence $r \gg T^{-1}_c$, $V_{total}^{T \leq T_c}$ is given by:

$$V_{total}^{T \leq T_c} \approx -\frac{g^2}{48M^2\pi} \frac{T^2_c}{r} \left[ 1 + \mathcal{O} \left( \frac{T^2}{T^2_c}, \frac{r}{rT^2_c} \right) \right]. \tag{29}$$

Namely, at low temperature (i.e. below $T_c$) the force, that was finite-ranged at high temperature (i.e. above $T_c$), becomes infinite-ranged. This comes about because the medium
absorbs and restores 3-momentum in the scattering process so that the 4-momentum squared of the other \( \varphi \) quantum exchanged in the \( t \)-channel can reach the mass-shell in the physical region of the scattering process. For the \( k = 0 \) mode in the bath, in particular, the propagator of the second particle (see eq. (10)) takes the form of the Coulomb propagator, and becomes singular at the edge of the \( Q \)-integration region, exactly as in the Coulomb case. For \( T > T_c \) this infinite wavelength mode has zero measure, and it does not contribute to the potential. However, in the condensed phase, the infinite-range potential arises as a collective phenomenon essentially because all charge piles up in the ground state.

What we would like to do now is to show graphically the transition of the potential as we vary the temperature from \( T > T_c \) to \( T < T_c \). In the first temperature regime, i.e. above condensation, we compute eq. (19) numerically and below \( T_c \) we evaluate eq. (28) where again numerical methods are used. Figure 2 displays our results.

![Figure 2: Total potential, divided by \(-g^2/128M^2\pi^3r^3\), for a relativistic bose gas made of particles of \( m \sim 10^{-6}\text{eV} \) and having a density \( \rho \sim 40 \text{ cm}^{-3} \). Behaviour above and below \( T_c \) is shown. Note the Yukawa exponential damping for \( T > T_c \) starting at \( r \sim 1/m \Rightarrow r T_c \sim 10^3 \).](image)

Let us briefly summarize our findings. Light scalars are basic ingredients of many completions of the Standard model. They may carry a new conserved quantum number. If ordinary matter is neutral with respect to this new charge, then these scalars should couple to ordinary matter pairwise. But double (pseudo) scalar exchange generates long range spin independent forces among bulk matter exactly as 2-neutrino exchange and 2-photon exchange (van der Waals forces) do. All these dispersion forces are modified when matter is introduced in a
heat bath. The MWBR effect on van der Waals type potentials and the relic neutrino cosmic background effect on neutrino forces have been recently surveyed. The present paper presents an investigation of the effects of a relativistic ideal Bose gas on potentials generated by 2-scalar exchange. An example for such a heat bath could be provided by hot dark matter (i.e relativistic at decoupling) made of hypothetic relic light scalars. For this purpose we use a very simple model for matter-scalar interactions for we do not want to commit ourselves to any specific extension of the SM. We do not expect that the particular \( r \)-behaviour is in any sense realistic but the phenomena produced by Bose-Einstein condensation of the heat reservoir are totally independent of the form of the interaction chosen. What we find is a very dramatic effect: below the critical temperature, the finite-range force that we had above this temperature becomes an infinite-range force. The phenomenon arises as a combination of kinematics (3-momentum exchange of the matter system with the medium) and the collective effect of condensation of charge. In the particular model studied in this paper, a potential of the form \( \sim \exp(-2mr)/r^2 \) at \( T > T_c \) converts to a \( \sim 1/r \) potential at \( T < T_c \). Should hot relic scalars populate our Universe with a present density such that their temperature is below the threshold for Bose-Einstein condensation, then the effect described above would provide an excellent opportunity for experiments searching for forces weaker than gravity for in this case no exponential decay with distance occurs and, furthermore, a milder power law fall-off with distance ensues.

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References

[1] H. B. G. Casimir and P. Polder, Phys. Rev. 73 (1948) 360; E.M. Lifschitz, JETP Lett. 2 (1956) 73; G. Feinberg and J. Sucher, J. Chem. Phys. 48 (1968) 3333; J. Soffer and J. Sucher, Phys. Rev. 161 (1967) 1664; G. Feinberg and J. Sucher, Phys. Rev. A2 (1970) 2395.

[2] G. Feinberg and J. Sucher, Phys. Rev. 166 (1968) 1638; J. A. Grifols, E. Masso and R. Toldra, Phys. Lett. B389 (1996) 363; E. Fischbach, Ann. Phys. (N.Y.) 247 (1996) 213.

[3] S. P. H. Hsu and P. Sikivie, Phys. Rev. D49 (1994) 4951.

[4] V.M. Mostepanenko and I.Yu. Sokolov, Sov. J. Nucl. Phys. 46 (1987) 685; J. A. Grifols and S. Tortosa, Phys. Lett. B328 (1994) 98.

[5] F. Ferrer and J. A. Grifols, Phys. Rev. D58 (1998) 096006.

[6] F. Ferrer and M. Nowakowski, Phys. Rev. D59 (1999) 075009.

[7] G. Feinberg, J. Sucher and C.-K. Au, Phys. Rep. 180 (1989) 83; G. Feinberg and J. Sucher, in Long-Range Casimir Forces: Theory and Recent Experiments in Atomic Systems, edited by Frank S. Levin and David A. Micha (Plenum, New York, 1993).

[8] F. Ferrer and J. A. Grifols, Phys. Lett. B460 (1999) 371.

[9] C. J. Horowitz and J. Pantaleone, Phys. Lett. B319 (1993) 186; F. Ferrer, J. A. Grifols and M. Nowakowski, Phys. Lett. B446 (1999) 111; F. Ferrer, J. A. Grifols and M. Nowakowski, hep-ph/9906463, Phys. Rev. D to be published.

[10] M. Consoli and P.M. Stevenson, hep-ph/9711449.

[11] See for instance, R. L. Kobes, G. W. Semenoff and N. Weiss, Z. Phys. C29 (1985) 371.

[12] H. E. Haber and H. A. Weldon, Phys. Rev. Lett. 46 (1981) 1497; H. E. Haber and H. A. Weldon, Phys. Rev. D25 (1982) 502.

[13] I. S. Gradshteyn, I. M. Ryzhik, Table of Integrals, Series and Products, corrected and enlarged edition (Academic Press, INC 1980).

[14] E. Fischbach and C. L. Talmadge, The Search for Non-newtonian Gravity (Springer-Verlag, 1999).