Sparse Hanson–Wright inequality for a bilinear form of sub-Gaussian variables

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In this paper, we derive a new version of Hanson–Wright inequality for a sparse bilinear form of sub-Gaussian variables. Our results are a generalization of previous deviation inequalities that consider either sparse quadratic forms or dense bilinear forms. We apply the new concentration inequality to testing the cross-covariance matrix when data are subject to missing. Using our results, we can find a threshold value of correlations that controls the family-wise error rate. Furthermore, we discuss the multiplicative measurement error case for the bilinear form with a boundedness condition.

KEYWORDS
concentration inequality, covariance matrix, general missing dependency, measurement error, missing data

1 | INTRODUCTION

Let \((Z_{ij}, Z_{ik}), j = 1, \ldots, n\), be pairs of (possibly dependent) sub-Gaussian random variables with mean zero. Suppose \(\gamma_i\) is a random variable that corresponds to \(Z_i\) and is independent with \(Z_j\). We allow (general) dependency between \(\gamma_i\) and \(\gamma_j\). We write the random variables by vectors \(Z_i = (Z_{i1}, \ldots, Z_{in})^\top, \gamma_i = (\gamma_{i1}, \ldots, \gamma_{in})^\top, i \neq 2\). For a given \(n \times n\) non-random matrix \(A\), Rudelson and Vershynin (2013) dealt with a concentration inequality of a quadratic form \(Z_i^\top AZ_j\), which is known as the Hanson–Wright inequality. Recently, this result has been extended in two directions. On one hand, Zhou (2019) included Bernoulli random variables \(\gamma_i\) in the quadratic form, that is, \((Z_1^\top, Z_2^\top, \gamma_1)\) where \(\gamma_i\) is the Hadamard (element-wise) product of two vectors (or matrices). On the other hand, Park et al. (2021) proved that the sub-Gaussian vectors could be distinct \(Z_1 \neq Z_2\) and derived the deviation inequality of a bilinear form \(Z_1^\top A \gamma_1\).

In this paper, considering the above two extensions altogether, we aim to analyze the concentration of the bilinear form \((Z_1^\top \gamma_1, Z_2^\top \gamma_2)\) where \(\gamma_1, \gamma_2\) are vectors of Bernoulli random variables that may be dependent on each other. Furthermore, we generalize the Bernoulli variables to bounded random variables, which is the first attempt in the literature. Table 1 compares different types of bilinear forms. It should be pointed out that our results are completely new and not a special case of any other similar work including Zhou’s (2019). One may consider converting the bilinear form into a quadratic form by concatenating \(Z_1, Z_2\), that is,

\[
(Z_1^\top \gamma_1)^\top A(Z_2^\top \gamma_2) = (Z^\top \gamma)^\top \begin{bmatrix} 0 & A/2 \\ A/2 & 0 \end{bmatrix} (Z^\top \gamma)
\]

where \(Z = (Z_1^\top, Z_2^\top, \gamma_1, \gamma_2)^\top\). However, we cannot directly apply the previous results (e.g., Theorem 1 in Zhou, 2019) because entries in \(Z\) are not independent. There are other papers that discuss Hanson–Wright inequality with dependency on \(Z\) in (1); see Adamczak (2015) and Vu and Wang (2015). However, they assume the convex concentration property, which is not satisfied by sub-Gaussian \(Z\).

We apply the new version of Hanson–Wright inequality to estimating the high-dimensional cross-covariance matrix when data are subject to either missingness or measurement errors. We treat the problem as multiple testing of individual elements of the matrix and propose an estimator thresholding their sample estimates. To determine the cutoff value, we make use of the extended Hanson–Wright inequality and thus can control the family-wise error rate at a desirable level.
TABLE 1  Comparison of bilinear forms in the literature and their references

|                | Identical ($Z_1 = Z_2$)                              | Distinct ($Z_1 \neq Z_2$) |
|----------------|--------------------------------------------------------|----------------------------|
| Dense          | $Z_1^T A Z_1$,                                         | $Z_1^T A Z_2$,             |
|                | (Rudelson & Vershynin, 2013)                          | (Park et al., 2021)       |
| Sparse         | $(Z_1 * \gamma_1)^T A (Z_1 * \gamma_1)$,             | $(Z_1 * \gamma_1)^T A (Z_2 * \gamma_2)$ |
|                | (Zhou, 2019)                                          |                            |

The paper is organized as follows. In Section 2, we present the main result of $(Z_1 * \gamma_1)^T A (Z_2 * \gamma_2)$, Theorem 1, and apply it to the problem of testing the cross-covariance matrix in Section 3. In Section 4, we conclude this paper with a brief discussion.

2  | MAIN RESULT

We first define the sub-Gaussian (or $\psi_2$-) norm of a random variable $X$ in $\mathbb{R}$ by

$$||X||_{\psi_2} = \sup_{p \geq 1} \left( \frac{E|X|^p}{p} \right)^{\frac{1}{p}},$$

and $X$ is called sub-Gaussian if its $\psi_2$-norm is bounded. For a matrix $A$, its operator norm $||A||_2$ is defined by the square-root of the largest eigenvalue of $A^T A$. $||A||_p = \left( \sum_{i,j} a_{ij}^p \right)^{\frac{1}{p}}$. For a vector $x$, $||x||_2$ is a 2-norm and, $D(x)$ is a diagonal matrix having its diagonal entries by $x$.

We now describe the main theorem of this paper.

Theorem 1. Let $(Z_{ij}, Z_{2j}), i = 1, \ldots, n$, be independent pairs of (possibly correlated) random variables satisfying $E Z_{ij} = E Z_{2j} = 0$, and $||Z_{ij}||_{\psi_2} \leq K_i$, $i = 1,2$ and $j$. Also, suppose $(\gamma_{ij}, \gamma_{2j}), i = 1, \ldots, n$, are independent pairs of (possibly correlated) Bernoulli random variables such that $E \gamma_{ij} = \pi_{ij}$ and $E \gamma_{2j} = \pi_{2j}$ for $j = 1, \ldots, n, i = 1,2$. Assume $Z_1$ and $\gamma_1$ are independent for distinct pairs $(i,j) \neq (i',j')$. Then, we have that for every $t > 0$

$$P(||(Z_1 * \gamma_1)^T A (Z_2 * \gamma_2) - E(Z_1 * \gamma_1)^T A (Z_2 * \gamma_2)||_2 > t|| \leq 2 \exp \left\{ -c \min \left( \frac{t^2}{K_i^2 K_2^2 \max_{i,j} |a_{ij}|}, \frac{t}{K_i K_2 ||A||_2} \right) \right\},$$

for some numerical constant $c > 0$. For a matrix $A = (a_{ij}, 1 \leq i, j \leq n)$, we define $||A||_{\psi_2} = \sqrt{\sum_{i,j} a_{ij}^2 \pi_{ij} + \sum_{i \neq j} a_{ij}^2 \pi_{ij}}$.

Note that Theorem 1 is a combination of the results for diagonal and off-diagonal cases given below.

Lemma 1. Assume $Z_1, Z_2, \gamma_1, \gamma_2$ are defined as in Theorem 1. Let $A_{\text{diag}} = \text{diag}(a_{11}, \ldots, a_{nn})$ be a diagonal matrix. We denote $E \gamma_{ij} = \pi_{12j}$. Then, for $t > 0$, we have

$$P(||(Z_1 * \gamma_1)^T A_{\text{diag}} (Z_2 * \gamma_2) - E(Z_1 * \gamma_1)^T A_{\text{diag}} (Z_2 * \gamma_2)||_2 > t|| \leq 2 \exp \left\{ -c \min \left( \frac{t^2}{K_i^2 K_2^2 \sum_{j=1}^n \pi_{12j} a_{1j}^2}, \frac{t}{K_i K_2 \max_j |a_{1j}|} \right) \right\},$$

for some numerical constant $c > 0$.

Lemma 2. Assume $Z_1, Z_2, \gamma_1, \gamma_2$ are defined as in Theorem 1. Let $A_{\text{off}}$ be an $n \times n$ matrix with its diagonals zero. We denote $E \gamma_{ij} = \pi_{ij}$ for $i = 1,2$ and $j = 1, \ldots, n$. Then, for $t > 0$, we have

$$P(||(Z_1 * \gamma_1)^T A_{\text{off}} (Z_2 * \gamma_2)||_2 > t|| \leq 2 \exp \left\{ -c \min \left( \frac{t^2}{K_i^2 K_2^2 \sum_{i \neq j} \pi_{12j} a_{ij}^2}, \frac{t}{K_i K_2 ||A_{\text{off}}||_2} \right) \right\},$$

for some numerical constant $c > 0$. 
A complete proof of the two lemmas, in spirit of Zhou (2019), is provided in Appendix. It is noted that our theorem above can yield Theorem 1.1 in Zhou (2019) and Lemma 5 in Park et al. (2021) under the same setting of each.

To handle more general cases where we do not have information about mean, we provide the concentration inequality for non-centered sub-Gaussian variables.

**Theorem 2.** Let \((\tilde{Z}_i, \tilde{Z}_j)\), i = 1,...,n, be independent pairs of (possibly correlated) random variables with \(E\tilde{Z}_i = \mu_1, E\tilde{Z}_j = \mu_2\), and \(\|\tilde{Z}_i - \mu_i\|_{\infty} \leq K, i = 1,2\) and \(\nu_i\). Also, suppose \((\tilde{Y}_i, \tilde{Y}_j)\), i = 1,...,n, are independent pairs of (possibly correlated) Bernoulli random variables such that \(E\tilde{Y}_i = \pi_i\) and \(E\tilde{Y}_j = \pi_{12}\) for \(j = 1,...,n, i = 1,2\). Assume \(\tilde{Z}_i\) and \(\tilde{Y}_j\) are independent for distinct pairs \((i,j) \neq (i',j')\). Then, we have that for every \(t > 0\)

\[
P\left(\left\|\tilde{Z}_1 \times \gamma_1\right\|^T A(\tilde{Z}_2 \times \gamma_2) - E(\tilde{Z}_1 \times \gamma_1)^T A(\tilde{Z}_2 \times \gamma_2)\right| > t\} \leq d \exp\left\{-c\min\left\{\frac{t^2}{E_1}, \frac{t}{E_2}\right\}\right\},
\]

for some numerical constants \(c, d > 0\). \(E_1\) and \(E_2\) are defined by

\[
E_1 = \max\left\{K_1^2 K_2^2 |A|_2^2, V_1^2 V_2^2 |D(\mu_1 + \mu_2)|_2^2, K_1^2 (\mu_1 \times \pi_1)_2^2, V_2^2 |(A(\mu_1 \times \pi_1))_2^2, V_1^2 |(A(\mu_2 \times \pi_2))_2^2, V_2^2 |(A(\mu_2 \times \pi_2))_2^2, V_1^2 |(A(\mu_1 \times \pi_1))_2^2, V_2^2 |(A(\mu_1 \times \pi_1))_2^2, V_1^2 |(A(\mu_2 \times \pi_2))_2^2, V_2^2 |(A(\mu_2 \times \pi_2))_2^2\right\}
\]

\[
E_2 = \max\left\{K_1^2 K_2^2 |A|_2^2, V_1^2 V_2^2 |D(\mu_1 + \mu_2)|_2^2, K_1^2 (\mu_1 \times \pi_1)_2^2, V_2^2 |(A(\mu_1 \times \pi_1))_2^2, V_1^2 |(A(\mu_1 \times \pi_1))_2^2, V_2^2 |(A(\mu_1 \times \pi_1))_2^2, V_1^2 |(A(\mu_2 \times \pi_2))_2^2, V_2^2 |(A(\mu_2 \times \pi_2))_2^2, V_1^2 |(A(\mu_2 \times \pi_2))_2^2, V_2^2 |(A(\mu_2 \times \pi_2))_2^2\right\}
\]

For a matrix \(A = (a_{ij}, 1 \leq i, j \leq n)\), we define \(|A|_F = \sqrt{\sum_{i=1}^n \sum_{j=1}^n a_{ij}^2}\) and \(\sum_{i=1}^n \sum_{j=1}^n a_{ij}^2\). Also, we define \(V_1 = \max_{1 \leq i \leq n} (\sigma^{x_1}_i (1 - \sigma^{x_1}_i)\) and \(V_2\) similarly.

The bilinear form of \(\tilde{Z}_i\) in Theorem 2 can be decomposed into bilinear forms and linear combinations of centered random variables. Then, we can apply either Theorem 1 or Hoeffding's inequality (Theorem 5 in Appendix) to them. The detail of the proof can be found in Appendix B1.

## 3 Application to Testing the Cross-Covariance Matrix

A cross-covariance matrix \(\Sigma_{XY}\) between two random vectors \(X = (X_1, ..., X_q)^T\) and \(Y = (Y_1, ..., Y_q)^T\), with its \((k, \ell)\)-th entry being \(\sigma_{k\ell}^{XY} = \text{Cov}(X_k, Y_\ell)\), refers to the off-diagonal block matrix of the covariance matrix of the matrix \(Z = (X^T, Y^T)^T\). It is often considered a less important part in the covariance matrix of \(Z\). \(\Sigma_{22}\), and tends to be overpenalized by shrinkage estimators favoring an invertible estimate. However, it is a crucial statistical summary in some applications. For example, the study of multi-omics data, which aims to explain molecular variations at different molecular levels, receives much attention due to public availability of biological big data and the covariation between two different data sources is just as important as that within each data source. The main question here is to find pairs (or positions in the matrix) of \(X_k\)'s and \(Y_\ell\)'s that present a large degree of association, which can be treated by hypothesis testing:

\[
H_{0k\ell} : \sigma_{k\ell}^{XY} = 0 \text{ vs. } H_{1k\ell} : \text{not } H_{0k\ell},
\]

for \(1 \leq k \leq p, 1 \leq \ell \leq q\).

Testing the cross-covariance matrix has not been much explored in literature. Cai et al. (2019) directly address the problem of estimating the cross-covariance matrix. However, they vaguely assume \(q \ll p\) and consider simultaneous testing of hypotheses

\[
H_{0k} : \sigma_{k1}^{XY} = ... = \sigma_{kq}^{XY} = 0, \text{ vs. } H_{1k} : \text{not } H_{0k}
\]

for \(k = 1, ..., p\). They build Hotelling's \(T^2\) statistics for individual hypotheses and decide which rows of \(\Sigma_{XY}\) are not 0. Hence, the sparsity pattern in Cai et al. (2019) is not the same as considered in this paper. Moreover, their method cannot address missing data.

Considering that (2) is equivalent to \(H_{0k} : \sigma_{k1}^{XY} = 0\) where \(\rho_{k1}^{XY} = \text{Cor}(X_k, Y_1)\), one can instead analyze the sample correlation coefficient, denoted by \(\rho_{k1}^{XY}\), to test it. Bailey et al. (2019) analyzed a universal thresholding estimator based on its asymptotic normality. Though they are interested in estimation of a large correlation matrix, not a cross-correlation matrix directly, their method can be applied to estimation of the cross-
covariance matrix. For example, the proposed estimator would be a $p \times q$ matrix with its component $p_{kl}^{XY}$, and they aim to find a cutoff value $c(n,p,q)$ to control the family-wise error rate (FWER). However, again, if data are subject to missing, their method is no longer valid.

Here, we address the multiple testing problem for the cross-covariance matrix when some of the data are missing or measured with errors.

We apply the modified Hanson–Wright inequalities (Theorems 1, 2, 3, and 4) to choose a threshold value that controls FWER. More specifically, we derive the concentration results for an appropriate cross-covariance estimator $\hat{\sigma}_{kl}^{XY}$ in the following form:

$$P[\max_{k,l} |\hat{\sigma}_{kl}^{XY} - \sigma_{kl}^{XY}| > c(n,p,q)] \leq \alpha, \quad 0 < \alpha < 1$$

under some regularity conditions. We begin with the simplest case where data are completely observed and walk through more complicated cases later.

### 3.1 Complete data case

We begin with the complete data case.

**Assumption 1.** Assume each component of random vectors $X \in \mathbb{R}^p$ and $Y \in \mathbb{R}^q$ is sub-Gaussian; that is, it holds for some $K_X, K_Y > 0$

$$\max_{1 \leq k \leq p} |X_k - E X_k|_{\ell_2} \leq K_X, \quad \max_{1 \leq r \leq q} |Y_r - E Y_r|_{\ell_2} \leq K_Y.$$  \tag{3}

Let us denote the mean vector and cross-covariance matrix of $X$ and $Y$ as follows:

$$E X = \mu^X = (\mu^X_k, 1 \leq k \leq p)^\top,$$

$$E Y = \mu^Y = (\mu^Y_r, 1 \leq r \leq q)^\top,$$

$$\text{Cov}(X, Y) = \Sigma_{XY} = (\sigma_{kl}^{XY}, 1 \leq k \leq p, 1 \leq l \leq q).$$  \tag{4}

Suppose we observe $n$ independent samples $\{(X_i, Y_i)\}_{i=1}^n$ of $(X, Y)$ in Assumption 1. Then, the cross-covariance $\sigma_{kl}^{XY}$ can be estimated by

$$s_{kl} = \frac{1}{n-1} \sum_{i=1}^n (X_k - X_k)(Y_l - Y_l).$$

where $X_k, Y_l$ are the sample means corresponding to $\mu^X_k, \mu^Y_l$. We can analyze the concentration of each component of $S_{XY} = (s_{kl}, 1 \leq k \leq p, 1 \leq l \leq q)$ using Theorem 1 as its special case where all $x$'s are 1. We first notice that

$$s_{kl} = \frac{1}{n} \sum_{i=1}^n (X_k - \mu^X_k)(Y_l - \mu^Y_l) - \frac{1}{n(n-1)} \sum_{i \neq j} (X_k - \mu^X_k)(Y_l - \mu^Y_l),$$

where $\mu^X_k = E X_k$ and $\mu^Y_l = E Y_l$. Hence, we can rewrite the sample cross-covariance estimator in a matrix-form by

$$s_{kl} = X_k^{1:k} A X_l^{1:l}$$

where $X_{1:k}^1 = (X_{1k} - \mu^X_k, \ldots, X_{nk} - \mu^X_k)^\top, X_{2:l}^1 = (Y_{1l} - \mu^Y_l, \ldots, Y_{nl} - \mu^Y_l)^\top$, and $A = (n(n-1)^{-1}(nI - I I^\top)$. Note that $|A|_F = 1/\sqrt{n-1}$ and $|A|_2 = 1/(n-1)$. Then, by Theorem 1, the element-wise deviation inequality for the sample cross-covariance is

$$P[|s_{kl} - \sigma_{kl}^{XY}| > t] \leq 2 \exp \left\{ -\frac{c_1(n-1)t^2}{K_X^2 K_Y^2} \right\}, \quad t < K_X K_Y^2.$$

for some numerical constant $c_1 > 0$. Putting $t = K_X K_Y^2 \sqrt{\log(2pq/n)} / \sqrt{c_1(n-1)}$ and using the union bounds, we can get...
\[
P \left[ \max_{k,r} |s_{k,r} - \sigma_{k,r}^{XY}| > C_1 k x K Y \sqrt{\log(pq/\alpha) \over n - 1} \right] \leq \alpha, \tag{5} \]

if \(n/\log(pq/\alpha) > d_1\) for some numerical constants \(C_1, d_1 > 0\).

### 3.2 Missing data case

For the case where data are subject to missing, we introduce assumptions for missing indicators.

**Assumption 2.** Each component \(\delta_k^X\) of the indicator vector \(\delta^X = (\delta_k^X, 1 \leq k \leq p)^\top\) corresponding to \(X\) is 1 if \(X_k\) is observed and 0 otherwise. \(\delta^X\) is similarly defined. Their moments are given by

\[
\mathbb{E}[\delta^X] = (\pi_k^X, 1 \leq k \leq p)^\top, \quad \mathbb{E}[\delta^Y] = (\pi_k^Y, 1 \leq k \leq q)^\top, \quad \mathbb{E}[\delta^X(\delta^Y)^\top] = \pi_k^X \cdot 1 \leq k \leq p, 1 \leq \ell \leq q).
\]

Note that the above assumption does not mention independence between components of the indicator vector, which means that it allows \(\delta_k^X\) and \(\delta_{\ell}^Y, k \neq \ell\), to be dependent with each other. This implies that multiple components in different positions can be missing together under some dependency structure. Assumption 2 of Park et al. (2021) is equivalent to this, and they called it the general missing dependency assumption. For the missing mechanism, missing completely at random (MCAR) is assumed; that is, \((X, Y)\) is independent of \((\delta^X, \delta^Y)\). More generally, MCAR can be stated as follows.

**Assumption 3** Assumption 3 of Park et al. (2021). An event that an observation is missing is independent of both observed and unobserved random variables.

Suppose \(n\) independent samples are generated from the population model under Assumptions 1.2, and 3. Each sample consists of the observational data \((X_i, Y_i)\) and their missing indicators \((\delta^X_i, \delta^Y_i)\). However, due to missingness, we can only observe \(\tilde{X}_{ik} = \delta^X_i X_{ik}, \tilde{Y}_{ik} = \delta^Y_i Y_{ik}\), for \(i, j = 1, \ldots, n, k = 1, \ldots, p, \ell = 1, \ldots, q\). We can easily check that

\[
\mathbb{E}\left[ \sum_{i=1}^n \tilde{X}_{ik} \tilde{Y}_{ik} \right] = n \pi_{ik} (\sigma_{ik}^X + \mu_{ik}^Y), \quad \mathbb{E}\left[ \sum_{i=1}^n \tilde{X}_{ik} \tilde{Y}_{ik} \right] = n(n - 1) \alpha_{kd} \pi_{ik}^X \mu_{ik}^Y.
\]

From the above observation, we define an estimator of the cross-covariance as follows, which is unbiased for \(\sigma_{ik}^{XY}\).

\[
\hat{s}_{ik} = \frac{\sum_{i=1}^n \tilde{X}_{ik} \tilde{Y}_{ik} - \sum_{i=1}^n \tilde{X}_{ik} \tilde{Y}_{ik}}{n(n - 1) \sigma_{ik}^X \sigma_{ik}^Y} = (\hat{X}_{1(k)} + \delta_{1(k)})^\top A_{ik} (\hat{X}_{2(\ell)} + \delta_{2(\ell)})^\top.
\]

The last representation is a bilinear form of \(\hat{X}_{1(k)} \hat{X}_{2(\ell)}, \delta_{1(k)}, \delta_{2(\ell)}, \) and \(A_{ik}\) defined as below.

\[
\hat{X}_{1(k)} = (\tilde{X}_{1i}, \ldots, \tilde{X}_{1k})^\top, \quad \hat{X}_{2(\ell)} = (\tilde{Y}_{1i}, \ldots, \tilde{Y}_{1\ell})^\top, \\
\delta_{1(k)} = (\delta_{11}, \ldots, \delta_{1k})^\top, \quad \delta_{2(\ell)} = (\delta_{21}, \ldots, \delta_{2\ell})^\top, \\
A_{ik} = \begin{pmatrix} 1 & 1 \\ n(n - 1) & n(n - 1) \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ \alpha_k \sigma_{ik}^X \sigma_{ik}^Y \end{pmatrix}^\top.
\]

Thus, we apply Theorem 2 to \(\hat{s}_{ik}\) and get, for \(t < E_{1(k)}/E_{2(\ell)}\)

\[
P[|\hat{s}_{ik} - \sigma_{ik}^{XY}| > t] \leq \exp \left\{ -\frac{c_2 t^2}{E_{1(k)}} \right\},
\]

where \(c_2\) is a numerical constant.
where \(c_2, d > 0\) are some numerical constants and \(E_{1,k',e}, E_{2,k'}\) are defined below.

\[
E_{1,k,e} = \max \left\{ \begin{array}{l}
\max \left\{ (K_x^2)(K_y^2)(K_z^2)(K_t^2)[\mu_x^2][\mu_y^2][\mu_z^2][\mu_t^2] \left( \frac{1}{n_{c,k,e}} + \frac{1}{n(n-1)}\right) \right\}, \ni \frac{1}{n(n-1)} \right\}, \\
\max \left\{ (K_x^2)[\mu_x^2][\mu_y^2][\nu_x^2][\nu_y^2] \left( \frac{1}{n_{c,k,e}} + \frac{1}{n(n-1)}\right) \right\}
\right\}
E_{2,k,e} = \max \left\{ K_x K_y K_z [\mu_x^2][K_y^2][\mu_y^2][\mu_z^2] \right\} \left( \frac{1}{n_{c,k,e}} + \frac{1}{n(n-1)}\right)
\]

Putting \(t = \sqrt{E_{1,k,e}/\log(dpq/\alpha)c_2}\) and using the union bound argument to the indices \((k,e')\), we can get for some numerical constants \(C_2, d_2 > 0\)

\[
P\left[ \max_{k,e} |\tilde{\delta}_{ke} - \sigma_{ke}^2| > C_2 \sqrt{\log(dpq/\alpha)\max E_{1,k,e}} \right] \leq \alpha,
\]

if \(\sqrt{\log(dpq/\alpha)} < d_2 \min_{k,e} E_{1,k,e}/E_{2,k,e}\). A simple calculation leads to

\[
\max_{k,e} E_{1,k,e} \leq \frac{\{f_2(K_x, K_y, K_z, K_t, \mu_x, \mu_y, \mu_z, \nu_x, \nu_y, \nu_z, \mu_t, \nu_t)\}^2}{(n-1)(\sigma_{min}^j)^2} ,
\]

\[
\min_{k,e} \sqrt{E_{1,k,e}/E_{2,k,e}} \geq g_2(K_x, K_y, K_z, K_t, \mu_x, \mu_y, \mu_z, \nu_x, \nu_y, \nu_z, \mu_t, \nu_t, \mu_t, \nu_t) \geq 1(\sigma_{min}^j)^2
\]

where \(\sigma_{min}^j = \min_{k,e} \sigma_{ke}^j, \sigma_{max}^j = \min(\min_{k,e} \sigma_{ke}^j, \sigma_{min}^j, \sigma_{min}^j, \sigma_{min}^j)\), \(\sigma_{max}^j = \max_{k,e} \sigma_{ke}^j, f_2(K_x, K_y, K_z, K_t, \mu_x, \mu_y, \mu_z, \nu_x, \nu_y, \nu_z, \mu_t, \nu_t) = \max\{K_x K_y K_z K_t, K_x K_y K_t, K_x K_t, K_y K_t, \nu_x \nu_y \nu_z \nu_t, \nu_x \nu_y \nu_t, \nu_x \nu_t, \nu_y \nu_t, \nu_z \nu_t\}\), and \(g_2(K_x, K_y, K_z, K_t, \mu_x, \mu_y, \mu_z, \nu_x, \nu_y, \nu_z, \mu_t, \nu_t, \mu_t, \nu_t) = \min(1, \mu_x \mu_y / K_t, \mu_x \mu_z / K_t, \mu_x \mu_t / K_t, \mu_y \mu_t / K_t, \mu_z \mu_t / K_t, \mu_y \mu_t / K_t, \mu_z \mu_t / K_t, \mu_t \mu_t / K_t)\). The superscripts “\(j\)” and “\(M\)” stand for joint and marginal, respectively. Then, we conclude that for some numerical constants \(C_2, d_2 > 0\)

\[
P\left[ \max_{k,e} |\tilde{\delta}_{ke} - \sigma_{ke}^M| > C_2 f_2(K_x, K_y, K_z, K_t, \mu_x, \mu_y, \mu_z, \nu_x, \nu_y, \nu_z, \mu_t, \nu_t) \sqrt{\log(dpq/\alpha)\max E_{1,k,e}^{1/2}} \right] \leq \alpha,
\]

if \(\frac{n-1}{\log(dpq/\alpha)} > \frac{d_2}{g_2(K_x, K_y, K_z, K_t, \mu_x, \mu_y, \mu_z, \nu_x, \nu_y, \nu_z, \mu_t, \nu_t, \mu_t, \nu_t)} \) holds.

### 3.3 Measurement error case

The missing data problem is a special case of the multiplicative measurement error case if the multiplicative factor only takes either 1 or 0. Under some boundedness condition on the factors, we can extend the current framework to the multiplicative measurement error case, which is given below.

**Theorem 3.** Let \((Z_1, Z_2), j = 1, ..., n\), be independent pairs of (possibly correlated) random variables satisfying \(E(Z_1) = E(Z_2) = 0\), and \(\|Z_i\|_{\mu}^2 \leq K_i, i = 1, 2, \mbox{ and } \forall j\). Also, suppose \((\gamma_1, \gamma_2), j = 1, ..., n\), are independent pairs of (possibly correlated) non-negative random variables such that \(\gamma_i \leq B_0\) almost surely for some \(B_0 > 0, i = 1, 2, \mbox{ and } \forall j\). Assume \(Z_1\) and \(Z_2\) are independent for distinct pairs \((i,j) \neq (i',j')\). Then, we have that for every \(t > 0\)

\[
P\left[ |(A_1 \ast \gamma_1) \ast A_2 \ast \gamma_2 - E(A_1 \ast \gamma_1) \ast A_2 \ast \gamma_2| > t \right] \leq 2 \exp \left( -c \min \left( \frac{t^2}{\|K_2^{1/2} D(B_2) A(D(B_2)) \|_{\mu}^2} \right) \right)
\]

for some numerical constant \(c > 0, D(B_1)\) is a diagonal matrix with diagonal elements being from \(B_1 = (B_{1j}, 1 \leq j \leq n) \). \(D(B_2)\) is similarly defined.

The proof is straightforward, and thus, we outline the idea behind it and omit the detail. In the diagonal part, we need to modify (A1) as
For the off-diagonal case, a careful investigation into its proof shows that the missing indicators are conditioned in the analysis and the fact they are Bernoulli random variables is not used until Step 4 in Appendix A.3. The result of Lemma 5 (see Step 4 in Appendix A.3) can be extended to the bounded random errors as we can derive

\[ \mathbb{E}(\lambda|a_1Y_j|Z_1Z_2)^2 \leq \mathbb{E}(\lambda|B_1B_2|Z_1Z_2)^2, \quad \lambda > 0. \]

Furthermore, we state the result for the case with non-zero means.

**Theorem 4.** Let \((Z_i, \tilde{Z}_j), i = 1, \ldots, n\) be independent pairs of (possibly correlated) random variables with \(\mathbb{E}Z_i = \mu_1\), \(\mathbb{E}\tilde{Z}_j = \mu_2\), and \(\mathbb{E}|\tilde{Z}_j - Z_i|_2 \leq K_i, i = 1,2\) and \(\forall jl\). Also, suppose \((\gamma_{ij}, \gamma_{jl})\) are pairs of (possibly correlated) non-negative random variables such that \(\gamma_{ij} \leq B_i\) almost surely for some \(B_i > 0, i = 1,2\) and \(\forall j\). Assume \(\tilde{Z}_j\) and \(\gamma_{ij}\) are independent for distinct pairs \((i,j) \neq (i',j')\). Then, we have that for every \(t > 0\),

\[
P(\mathbb{E}(\tilde{Z}_i - Z_i = \mathbb{E}(\tilde{Z}_i + \gamma_{ij} = \mathbb{E}(\tilde{Z}_i + \gamma_{ij}) > t \leq \exp\left\{-\min\left(\frac{t^2}{E_1}, \frac{t}{E_2}\right)\right\},
\]

for some numerical constants \(c, d > 0\), \(E_1\) and \(E_2\) are defined by

\[
E_1 = \max\left\{K_1K_2||D(B_1)||_2, \max_{\relphantom{i=1} i=s,n} B_2^2||D(\mu_1 + \mu_2)||_2, \max_{\relphantom{i=1} i=s,n} K_1^2||D(B_1)||_2, \max_{\relphantom{i=1} i=s,n} B_1^2||D(\mu_2)||_2\right\},
\]

\[
E_2 = \max\left\{K_1K_2||A||_2, \max_{\relphantom{i=1} i=s,n} B_1^2||D(\mu_1)||_2, \max_{\relphantom{i=1} i=s,n} B_2^2||D(\mu_2)||_2\right\}.
\]

The rest of arguments are similar to Section 3.2. Assume we observe \(\tilde{X}_i = \delta_i^X + X_i\) and \(\tilde{Y}_i = \delta_i^Y + Y_i, i = 1, \ldots, n\). While \((X_i, Y_i)\) is an independent copy of \((X, Y)\) in Assumption 1, \((\delta_i^X, \delta_i^Y)\) is no longer a vector of binary variables but an independent copy of \((\delta^X, \delta^Y)\) in Assumption 4.

**Assumption 4.** Each component \(\delta_i^k\) of \(\delta^X = (\delta_i^k, 1 \leq k \leq p)^T\) is a measurement error corresponding to \(X_k\) of \(X\), which is a non-negative random variable satisfying \(\delta_i^k \leq B_i^k\) almost surely for each \(k\). \(\delta^Y\) is similarly defined. Their moments are given by

\[
\mathbb{E}\delta_i^X = u^X = (u^X_k, 1 \leq k \leq p)^T, \quad \mathbb{E}\delta_i^Y = u^Y = (u^Y_{k'}, 1 \leq k' \leq q)^T, \quad \mathbb{E}\delta_i^X = u^Y = (u^X_k, 1 \leq k \leq p, 1 \leq k' \leq q).
\]

Accordingly, the unbiased estimator for \(\sigma_{kk'}^{XY}\) is

\[
\hat{s}_{kk'} = \frac{\sum_{i=1}^n \tilde{X}_i \tilde{Y}_{k'}}{nu_{kk'}} - \frac{\sum_{i=1}^n \tilde{X}_i \tilde{Y}_{k'}}{n(n-1)u_{kk'}}.
\]

In this case, we can derive

\[
P(\hat{s}_{kk'} - \sigma_{kk'}^{XY}) > t \leq \exp\left\{-\min\left(\frac{t^2}{E_1}, \frac{t}{E_2}\right)\right\},
\]

where \(c, d > 0\) are some numerical constants and \(E_1, E_2\) are defined below.
\[
E_{1,k} = \max \left[ \max \left\{ K_x^2 |\mu_x^2 (B_x^e)^2 (B_x^e)^2, |\mu_x^2 | (B_x^e)^2 (B_x^e)^2 \right\}, \frac{1}{n u_{k,e}} \frac{1}{n(n-1)} \frac{1}{u_{k,e}^2} \left( u_{k,e}^2 \right)^2 \right], \right.
\]
\[
K_x^2 |\mu_x^2 (B_x^e)^2 (B_x^e)^2, |\mu_x^2 | (B_x^e)^2 (B_x^e)^2 \right\}, \frac{1}{n u_{k,e}} \frac{1}{n(n-1)} \frac{1}{u_{k,e}^2} \left( u_{k,e}^2 \right)^2 \right], \right.
\]
\[
E_{2,k} = \max \left\{ K_x K^2_x |\mu_x^2 | B_x^e, |\mu_x^2 | K_x B_x^e \right\} \frac{1}{u_{k,e}} \frac{1}{n(n-1)} \frac{1}{u_{k,e}^2} \left( u_{k,e}^2 \right)^2 \right].
\]

Moreover, we can observe
\[
\max_{k,c} E_{1,k} < \left\{ f_3(K_x, K^2_x, \mu_x, \mu_y, B_x, B_y) \right\}^2, \quad (n-1) \left( u_{c,r}^2 \right)^2 \right],
\]
\[
\min_{k,c} \sqrt{E_{1,k} / E_{2,k}} \geq g_5(K_x, K^2_x, \mu_x, \mu_y, B_x, B_y) \sqrt{n-1} \cdot \left( u_{c,r}^2 \right)^2, \right]
\]

where \( u_{c,r}^2 = \min_{k,c} u_{c,r}^2, u_{c,r}^2 = \min(\min_{k,c} u_{c,r}^2, \min_{c,r} u_{c,r}^2), \mu_x = \max_{k,c} |\mu_x^2|, \mu_y = \max_{c,r} |\mu_y|^2, B_x = \max_{k,c} B_x^e, B_y = \max_{c,r} B_y^e, \)

\[
f_5(K_x, K^2_x, \mu_x, \mu_y, B_x, B_y) = \frac{1}{B_x^e} = \frac{1}{B_y^e}, \quad g_5(K_x, K^2_x, \mu_x, \mu_y, B_x, B_y) = \min(K_x, (B_x^e), K_y^2(B_y^e), (B_y^e)).
\]

The superscripts “J” and “M” stand for joint and marginal, respectively.

Repeating the calculation as in the previous section, we conclude that for some numerical constants \( C_3, d_3 > 0 \)

\[
\mathbb{P} \left[ \max_{k,c} |\delta_{c,r}^2 | \frac{1}{n \log \left( \frac{pq}{q^2} \right)} \frac{\log \left( \frac{pq}{q^2} \right)}{\frac{1}{(n-1)} \cdot \left( u_{c,r}^2 \right)^2 \right. \left. \left( u_{c,r}^2 \right)^2 \right] \leq \alpha, \right.
\]

if \( \frac{n-1}{\log \left( \frac{pq}{q^2} \right)} \geq \frac{d_3}{f_5(K_x, K^2_x, \mu_x, \mu_y, B_x, B_y)} \sqrt{n-1} \cdot \left( u_{c,r}^2 \right)^2 \right]

holds.

4 | DISCUSSION

We discuss the generalized Hanson–Wright inequality where the sparse structure and bilinear form are considered for the first time. This extension facilitates an analysis of concentration of the sample (cross-)covariance estimator even when mean parameters are unknown and some of the data are missing. We apply this result to multiple testing of the cross-covariance matrix.

We further consider a measurement error case extended from the missing data case, which is limited to a bounded random variable. It would be interesting to consider more general multiplicative errors as future work; for example, the truncated normal distribution defined on \((0, \infty)\) can be a good example.

The concentration inequalities used for testing the cross-covariance matrix involve numerical constants (e.g., \( C_1 \) in (5), \( C_2 \) in (7)). In practice, they should be specified to control FWER. We can empirically search the constants through the cross-validation, or the sample splitting approach, by following Bailey et al. (2019), Bickel and Levina (2008), and Cai and Liu (2011). The constants are searched over a grid of real values such that they minimize the distance between a regularized estimator based on one sample set and the unregularized one based on another sample. One can simply apply the procedure in Section 3.7 of Bailey et al. (2019) to our case, where we need to permute variables and corresponding indicators together to keep the missing structure intact.

CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.
APPENDIX A: PROOF OF THEOREM 1

A.1 Proof of Lemma 1 (diagonal part)

Define $S_0$ by

$$S_0 = \sum_{i=1}^{n} a_{ii} Z_i Z_i - \mathbb{E}\left[ \sum_{i=1}^{n} a_{ii} Z_i Z_i \right].$$

For $|\lambda| < 1/(4eK_1 K_2 \max |a_{ii}|)$,

$$\mathbb{E}\exp(\lambda S_0) \leq \prod_{i=1}^{n} \left( 1 - \pi_{12} \right) + \pi_{12} \mathbb{E}_x \exp(\lambda a_{ii} Z_i Z_i)$$

$$\leq \prod_{i=1}^{n} \left( 1 - \pi_{12} \right) + \pi_{12} \left( 1 + \lambda K_1 K_2 \mathbb{E}Z_i Z_i + 16\lambda^2 K_1^2 K_2^2 \right)$$

$$\leq \prod_{i=1}^{n} \left( 1 - \pi_{12} \right) + \pi_{12} \left( 1 + \lambda K_1 K_2 \mathbb{E}Z_i Z_i + 16\lambda^2 K_1^2 K_2^2 \right)$$

$$\leq \prod_{i=1}^{n} \left( 1 - \pi_{12} \right) + \pi_{12} \left( 1 + \lambda K_1 K_2 \mathbb{E}Z_i Z_i + 16\lambda^2 K_1^2 K_2^2 \right)$$

where the first inequality uses Lemma 3 and the second holds since $1 + x \leq e^x$.

**Lemma 3.** Assume that $\|Z_i\|_{\psi_2} \leq K_1, \|Z_i\|_{\psi_2} \leq K_2$ for some $K_1, K_2 > 0$ and $|\lambda| < 1/(4eK_1 K_2 \max |a_{ii}|)$ for given $\{a_{ii}\}_{i=1}^{n} \subset \mathbb{R}$. Then, for any $i$, we have

$$\mathbb{E}\exp(\lambda a_{ii} Z_i Z_i) - 1 \leq \lambda a_{ii} \mathbb{E}Z_i Z_i + 16\lambda^2 a_{ii}^2 K_1^2 K_2^2.$$

**Proof.** First, we define the sub-exponential (or $\psi_1$-) norm of a random variable by
Since the product of sub-Gaussian random variables is sub-exponential, we define \( Y_i = Z_1Z_2/\|Z_1Z_2\|_{\nu_1} \). Setting \( t_i = \lambda a_i \|Z_1Z_2\|_{\nu_1} \), we get

\[
\mathbb{E}\exp(t_i Y_i) = 1 + t_i \mathbb{E}Y_i + \sum_{s \geq 2} \frac{t_i^s \mathbb{E}|Y_i|^s}{s!}
\leq 1 + t_i \mathbb{E}Y_i + \frac{2|t_i|^s s^s}{s!}
\leq 1 + t_i \mathbb{E}Y_i + \frac{|t_i|^s e^s}{\sqrt{s!}}
\]

In the last two inequalities, we use the following observations. First, note that for the subexponential variable satisfying \( \|Y_i\|_{\psi_1} = 1 \), it holds (see Prop 2.7.1, Vershynin, [2018])

\[
\mathbb{E}|Y_i|^s \leq 2s^s, \quad s \geq 1.
\]

Second, we use Stirling's formula for \( s \geq 2 \) that

\[
\frac{1}{s!} \leq e^s - 2s^s \sqrt{\pi}
\]

If \( |\lambda| < 1/(4eK_1K_2max|a_i|) \), then \( e|t_i| \leq 1/2 \), and thus, we get

\[
\sum_{s \geq 3} \sum_{i \leq 3} e|t_i|^s \leq (e|t_i|)^3 \sum_{i \geq 0} (1/2)^i \leq (e|t_i|)^2.
\]

Using the above, we have

\[
\mathbb{E}\exp(t_i Y_i) \leq 1 + t_i \mathbb{E}Y_i + 2e^2|t_i|^2/\sqrt{s!}
\]

\[
= 1 + \lambda_i \mathbb{E}Z_1Z_2 + 2e^2 \lambda_i^2 \mathbb{E}\|Z_1Z_2\|^2_{\nu_1} / \sqrt{s!}
\]

\[
\leq 1 + \lambda_i \mathbb{E}Z_1Z_2 + 16\lambda_i^2 K_1^2 K_2^2.
\]

Then, for \( x > 0 \) and \( 0 < \lambda < 1/(4eK_1K_2max|a_i|) \), we have

\[
P(S_0 > x) = P(\exp(\lambda S_0) > \exp(\lambda x))
\leq \mathbb{E}\exp(\lambda x)
\leq \exp\left\{ -\lambda x + 16\lambda^2 K_1^2 K_2^2 \sum_{i=1}^{n} |a_i|^2 \right\}.
\]

For the optimal choice of \( \lambda \), that is,

\[
\lambda = \min \left( \frac{x}{32eK_1^2 \sum_{i=1}^{n} |a_i|^2}, \frac{1}{2eK_1^2 K_2 max|a_i|} \right)
\]
we can obtain the concentration bound

\[
P(S_0 > x) \leq \exp\left\{-\min\left(x^2 \frac{1}{32K_1K_2^2 \sum_i \sigma_i^2 \sum_j \sigma_j^2 \max_i a_{ij}} \right)\right\}.
\]

Considering \(-A_{\text{diag}}\) instead of \(A_{\text{diag}}\) in the theorem, we have the same probability bound for \(P(S_0 < -x), x > 0\), which concludes the proof.

### A.2 Proof of Lemma 2 (off-diagonal part)

Define \(S_{\text{off}}\) by

\[
S_{\text{off}} = \sum_{1 \leq i \neq j \leq n} a_{ij} Z_i Z_j,
\]

whose expectation is zero due to independence across distinct \(i\) and \(j\). Then, we claim the lemma below:

**Lemma 4.** Assume \(||Z_1||_{v_2} = ||Z_2||_{v_2} = 1\) and let \(\{a_{ij}\}_{1 \leq i \neq j \leq n} \subset \mathbb{R}\) be given. For \(|\lambda| < 1/(2\sqrt{C_4}||A||_2)\) for some numeric constant \(C_4 > 0\), we have

\[
\mathbb{E}\exp(i S_{\text{off}}) \leq \exp\left(1.44C_4 \lambda^2 \sum_{i \neq j} a_{ij} \xi_i \xi_j\right).
\]

The proof is pending until Appendix A.3. Without loss of generality, we can assume \(||Z_1||_{v_2} = ||Z_2||_{v_2} = 1\): otherwise set \(Z_1 \leftarrow Z_1/||Z_1||_{v_2}, Z_2 \leftarrow Z_2/||Z_2||_{v_2}\). Using Lemma 4, we get for \(x > 0\) and \(0 < \lambda < 1/(2\sqrt{C_4}||A||_2)\)

\[
P(S_0 > x) = \mathbb{P}(\exp(i S_0) > \exp(i x)) \\
\leq \mathbb{E}\exp(i S_0) \\
\leq \exp\left\{-i x + 1.44C_4 \lambda^2 \sum_{i \neq j} a_{ij} \xi_i \xi_j\right\}.
\]

For the optimal choice of \(\lambda\), that is,

\[
\lambda = \min\left(x^2 \frac{1}{32K_1K_2^2 \sum_i \sigma_i^2 \sum_j \sigma_j^2 \max_i a_{ij}} \right)
\]

we can obtain the concentration bound

\[
P(S_0 > x) \leq \exp\left\{-c \min\left(x^2 \frac{1}{\sum_{i \neq j} a_{ij} \xi_i \xi_j} \frac{1}{||A||_2}\right)\right\}.
\]

Considering \(-A_{\text{off}}\) instead of \(A_{\text{off}}\) in the theorem, we have the same probability bound for \(P(S_0 < -x), x > 0\), which concludes the proof.
A.3 | Proof of Lemma 4

This proof follows the logic of the proof of (9) in Zhou (2019). We first decouple the off-diagonal sum $S_{\text{off}}$.

A.3.1 | Step 1. Decoupling

We introduce Bernoulli variables $\eta = (\eta_1, \ldots, \eta_n) \top$ with $\mathbb{E}\eta_k = 1/2$ for any $k$. They are independent with each other and also independent of $Z_1, Z_2$ and $\gamma_1, \gamma_2$. Given $\eta$, we define $Z_1^\eta$ by a subvector of $Z_1$ at which $\eta_1 = 1$ and $Z_2^\eta$ by a subvector of $Z_2$ at which $\eta_2 = 0$, respectively. Let $\mathbb{E}_0$ be the expectation with respect to a random variable $Q$ where $Q$ can be any of $Z_1, Z_2, \gamma_1, \gamma_2, \eta$. Define a random variable

$$S_q = \sum_{i,j: i \neq j} a_{ij}(1-\eta_i)\gamma_{ij} Z_i Z_j.$$  

Using $S_{\text{off}} = 4\mathbb{E}_0 S_q$ and Jensen's inequality with $x \mapsto e^{ax}$ for any $a \in \mathbb{R}$, we get

$$\mathbb{E}\exp(iS_{\text{off}}) = \mathbb{E}_0 \mathbb{E}_0 \exp(i\mathbb{E}_0 4iS_q) \leq \mathbb{E}_0 \mathbb{E}_0 \mathbb{E}_0 \exp(4iS_q).$$

We condition all the variables except $Z_{2j}^\eta = (Z_{2j}, \eta_j = 0)$ on $\exp(4iS_q)$ and consider its moment generating function denoted by $f(\gamma_1, \gamma_2, \eta, Z_{1j}^\eta)$

$$f(\gamma_1, \gamma_2, \eta, Z_{1j}^\eta) = \mathbb{E}(\exp(4iS_q)|\gamma_1, \gamma_2, \eta, Z_{1j}^\eta).$$

Note that $S_q$ can be seen a linear combination of sub-Gaussian variables $Z_{2j}$, for $j$ such that $\eta_j = 0$, that is,

$$S_q = \sum_{j = 0} Z_{2j} \left( \sum_{i = 1} a_{ij} \gamma_{ij} Z_i \right).$$

conditional on all other variables. Thus, the conditional distribution of $S_q$ is sub-Gaussian satisfying

$$||S_q||_{\psi_2} \leq C_0 \sigma_{\eta_j},$$

where $\sigma_{\psi_2}^2 = \sum_{j = 0} Z_{2j} = \sum_{i = 1} a_{ij} \gamma_{ij} Z_i)^2$. Therefore, we have that for some $C' > 0$

$$f(\gamma_1, \gamma_2, \eta, Z_{1j}^\eta) = \mathbb{E}(\exp(4iS_q)|\gamma_1, \gamma_2, \eta, Z_{1j}^\eta) \leq \exp(C' \lambda^2 \sigma_{\psi_2}^2).$$

(A2)

Taking the expectations on both sides, we get

$$\mathbb{E}_0 \mathbb{E}_0 f(\gamma_1, \gamma_2, \eta, Z_{1j}^\eta) \leq \mathbb{E}_0 \mathbb{E}_0 \exp(C' \lambda^2 \sigma_{\psi_2}^2) =: f_0.$$

A.3.2 | Step 2. Reduction to normal random variables

Assume that $\eta, \gamma_1, \gamma_2$, and $Z_{1j}^\eta$ are fixed. Let $g = (g_1, \ldots, g_n) \top$ be given where $g_i$ is i.i.d. from $N(0,1)$. Replacing $Z_{2j}$ by $g_j$ in $S_q$, we define a random variable $Z$ by

$$Z = \sum_{j = 0} g_j \left( \sum_{i = 1} a_{ij} \gamma_{ij} Z_i \right).$$

Due to the property of Gaussian variables, the conditional distribution of $Z$ follows $N(0, \sigma_{\psi_2}^2)$. Hence, its conditional moment generating function is given by
\[ E_x \exp(tZ) = E_\gamma \exp(tZ)|y_1, y_2, Z_1^n = \exp(t^2 \gamma^2 / 2). \] (A3)

Now, consider \( Z \) as a linear combination of \( \{Z_u : \eta = 1\} \) by rewriting it by

\[ Z = \sum_{u=1}^n Z_u \left( \sum_{\gamma=0}^r g_{u\gamma} y_1^{\gamma} \right), \]

where \( Z_u \)'s are regarded random variables and others fixed. Then, we can get for some \( C_3 > 0 \)

\[ E_{Z_1^2} \exp\left( \sqrt{2C_1 Z} \right) = E_\gamma \left( \exp\left( \sqrt{2C_1 Z} \right) \right) \leq \exp\left( C_3 \sum_{u=1}^n y_1^{2} \left( \sum_{\gamma=0}^r g_{u\gamma} y_1^{\gamma} \right)^2 \right). \] (A4)

We now get an upper bound of \( f_\gamma \) from step 1 by using (A3) and (A4). First, choosing \( t = \sqrt{2C_1 Z} \) at (A3) gives the same term in (A2):

\[ \tilde{f}_\gamma = E_x E_y E_\gamma \exp\left( \sqrt{2C_1 Z} \right) = E_x E_y E_\gamma \exp\left( \sqrt{2C_1 Z} \right). \]

Then, we apply (A4) to the right-hand side of the above:

\[ \tilde{f}_\gamma \leq E_x E_y \exp\left( C_3 \sum_{u=1}^n y_1^{2} \left( \sum_{\gamma=0}^r g_{u\gamma} y_1^{\gamma} \right)^2 \right) = E_\gamma \left( \exp\left( C_3 \sum_{u=1}^n (g^\top B_{\eta x} g) \right) \right), \] (A5)

where \( B_{\eta x} \) has its \( (k, \ell) \)-th element \( \sum_{i=1}^n y_1^{i} a_{k\ell} y_2 a_{k\ell}, \) if \( k, \ell \in \{ j : \eta = 0 \}; 0, \) otherwise.

A.3.3 | Step 3. Integrating out the normal random variables

For fixed \( y_1, y_2, \eta, \) the quadratic form \( g^\top B_{\eta x} g \) is identical in distribution with \( \sum_{i=1}^n s_i^2 g_i^2 \) due to the rotational invariance of \( g \) where \( s_i^2 \) is the eigenvalue of \( B_{\eta x}. \) Note that \( B_{\eta x} \) is symmetric and positive semi-definite. Symmetry is obvious and the latter holds true because

\[ x^\top B_{\eta x} x = \sum_{k=1}^n x_k y_1^{k} a_{k\ell} y_2 a_{k\ell} = \sum_{i=1}^n y_1^{i} \left( \sum_{\gamma=0}^r g_{i\gamma} y_1^{\gamma} \right)^2 \geq 0, \forall x \in \mathbb{R}^n. \]

The eigenvalues satisfy, for any realization of \( y_1, y_2, \eta, \)

\[ \max_{2 \leq s \leq n} s_i^2 = |B_{\eta x}|_2 \leq |A|_2 \]

\[ \sum_{1 \leq i \leq n} s_i^2 = \text{tr}(B_{\eta x}) = \sum_{u=1}^n y_1^{u} \sum_{\gamma=0}^r a_{u\gamma} y_1^{\gamma} g_{u\gamma}. \] (A6)

Now, using \( g_i^2 \sim \chi_2^2 \) and \( E_\gamma \left( \exp(tg_i^2) \right) = (1 - 2t)^{-1/2} \leq \exp(2t) \) for \( t < 1/4, \) we get from (A5) and (A6)

\[ f_\gamma \leq E_\gamma \left( \exp(C_3 \sum_{u=1}^n y_1^{u} \sum_{\gamma=0}^r a_{u\gamma} y_1^{\gamma} g_{u\gamma} \right) \left| \eta \right) \]

\[ = E_\gamma E_\gamma \left( \exp\left( C_3 \sum_{u=1}^n y_1^{u} g_{u\gamma} \right) \right) \left| \eta, y_1, y_2 \right) \]

\[ = E_\gamma E_\gamma \left( \exp\left( \sum_{u=1}^n g_{u\gamma}^2 \right) \right) \left| \eta, y_1, y_2 \right) \]

\[ \leq E_\gamma \left[ \prod_{u=1}^n \exp(2C_3 \gamma^2 s_i^2) \right] \left| \eta \right) \]

\[ = E_\gamma \left[ \exp\left( 2C_3 \gamma^2 \sum_{u=1}^n y_1^{u} \sum_{\gamma=0}^r a_{u\gamma} y_1^{\gamma} g_{u\gamma} \right) \right] \left| \eta \right). \]
for $\lambda^2 < 1/(4C_3|A|_2^2)$. It is worth noting that the Bernoulli variables $\gamma_1, \gamma_2$ are decoupled by $\eta$.

### A.3.4 | Step 4. Integrating out the Bernoulli random variables

We now integrate out $\gamma_1, \gamma_2$ from $\tilde{f}_n$ by using the following lemma. For $0 < \lambda^2 \leq 1/(8C_3|A|_2^2)$, we have

$$
\tilde{f}_n \leq \mathbb{E} \left[ \exp \left( 2C_3\lambda^2 \sum_{i=1}^{N} \sum_{j=0}^{n} a_i^2 \gamma_{ij} \right) \right] \leq \exp \left( 2.88C_3\lambda^2 \sum_{i \neq j} a_i^2 \sigma_{ij} \gamma_{ij} \right).
$$

**Lemma 5.** For $0 < \lambda^2 \leq 1/(4|A|_2^2)$, the conditional expectation satisfies

$$
\mathbb{E} \left[ \exp \left( \tau \sum_{i=1}^{N} \sum_{j=0}^{n} a_i^2 \gamma_{ij} \right) \right] \leq \exp \left( 1.44\tau \sum_{i \neq j} a_i^2 \sigma_{ij} \right).
$$

**Proof.** Due to the independence of $\gamma_{ij}, i = 1, 2, \ldots, n$, we get

$$
\mathbb{E} \left[ \exp \left( \tau \sum_{i=1}^{N} \sum_{j=0}^{n} a_i^2 \gamma_{ij} \right) \right] = \prod_{i=1}^{N} \mathbb{E} \left[ \exp \left( \tau \sum_{j=0}^{n} a_i^2 \gamma_{ij} \right) \right] = \prod_{i=1}^{N} \left[ (1 - \pi_{ij}) + \pi_{ij} \exp \left( \tau \sum_{j=0}^{n} a_i^2 \gamma_{ij} \right) \right].
$$

Then, we use the local approximation of the exponential function:

$$
e^{-x} - 1 \leq 1.2x, \ 0 \leq x \leq 0.35.
$$

To use it, $\tau$ should satisfies, for given $\eta$ and $\gamma_2$,

$$
\tau \sum_{i=0}^{n} a_i^2 \gamma_{ij} < 0.35,
$$

which leads to

$$
(1 - \pi_{ij}) + \pi_{ij} \exp \left( \tau \sum_{j=0}^{n} a_i^2 \gamma_{ij} \right) \leq 1 + 1.2\pi_{ij} \tau \sum_{j=0}^{n} a_i^2 \gamma_{ij} \leq \exp \left( 1.2\pi_{ij} \tau \sum_{j=0}^{n} a_i^2 \gamma_{ij} \right).
$$

where we use $1 + x \leq e^x$ in the last inequality. A sufficient condition for $\tau$ for any realization of $\eta$ and $\gamma_2$ in (A7) is $\tau \leq 1/(4|A|_2^2)$ since

$$
\tau \sum_{i=0}^{n} a_i^2 \gamma_{ij} \leq \max_i \left( \sum_{j=0}^{n} a_i^2 \right) \leq \tau|A|_2^2 \leq 0.25.
$$

Hence, for $\tau \leq 1/(4|A|_2^2)$,
\[
E \left[ \exp \left( r \sum_{i=0}^{\gamma_1} \sum_{j=0}^{\gamma_2} a_{ij}^2 \right) \eta \right] \\
= E_i E_j \left[ \exp \left( r \sum_{i=1}^{\gamma_1} \sum_{j=0}^{\gamma_2} a_{ij}^2 \right) \eta \right] \\
\leq E \left[ \exp \left( \sum_{i=1}^{\gamma_1} \sum_{j=0}^{\gamma_2} 1.2 \pi a_{ij}^2 \right) \eta \right] \\
= \prod_{i=1}^{\gamma_1} E \left[ \exp \left( 1.2 \pi a_{ij}^2 \right) \eta \right] \\
\leq \prod_{i=1}^{\gamma_1} \exp \left( 1.2^2 \pi a_{ij}^2 \right).
\]

We apply the similar argument used for \( \eta \) to \( \gamma_1 \) in the last inequality. Note that \( r \leq 1/(4|A|_2^2) \) is also sufficient since

\[
1.2r \sum_{i=1}^{\gamma_1} \pi a_{ij}^2 \leq 1.2 \max_j \sum_i a_{ij}^2 \leq 1.2 r |A|_2^2 \leq 0.3.
\]

Finally, we observe that

\[
\exp \left( 1.2^2 r \sum_{i=0}^{\gamma_1} \sum_{j=0}^{\gamma_2} a_{ij}^2 \right) \leq \exp \left( 1.2^2 r \sum_{i \neq j} a_{ij}^2 \right),
\]

which concludes the proof.

\[\square\]

### A.3.5 | Step 5. Combining things together

Assume \( |x| \leq 1/(\sqrt{8}C_3 |A|_2) \). Combining all the steps above, we get

\[
E \exp(\lambda S_{\text{off}}) \\
\leq E_{\eta} E_x \exp(4S_x) \\
\leq E_{\eta} E_x \mathbb{E}_{\gamma_1, \gamma_2} \exp(4S_{\gamma_1, \gamma_2, \eta, \gamma_2}) \\
\leq E_{\eta} \mathbb{I}_{\eta} \\
\leq \exp \left( 2.88 C_3 \lambda^2 \sum_{i \neq j} a_{ij}^2 \right),
\]

which proves Lemma 4.
APPENDIX APPENDIX B: PROOF OF THEOREM 2

Let us define the centered sub-Gaussian variable $Z_i = \tilde{Z}_i - \mu_i$. Then, it is easy to check that the bilinear form is decomposed into eight terms.

$$S \equiv (\tilde{Z}_1 + \gamma_1)^\top A(\tilde{Z}_2 + \gamma_2) - E(\tilde{Z}_1 + \gamma_1)^\top A(\tilde{Z}_2 + \gamma_2)$$

$$= \sum_{ij} a_{ij} (\gamma_1 Z_{1,j} - E[\gamma_1 Z_{1,j}]) + \sum_{ij} a_{ij} (\gamma_2 Z_{2,j} - E[\gamma_2 Z_{2,j}])$$

$$= \sum_{ij} a_{ij} (\gamma_1 Z_{1,j} - E[\gamma_1 Z_{1,j}]) + \sum_{ij} a_{ij} (\gamma_2 Z_{2,j} - E[\gamma_2 Z_{2,j}])$$

$$= \sum_{ij} a_{ij} (\gamma_1 Z_{1,j} - E[\gamma_1 Z_{1,j}]) + \sum_{ij} a_{ij} (\gamma_2 Z_{2,j} - E[\gamma_2 Z_{2,j}])$$

We will use Theorem 1 for the bilinear forms $S_{1,1}, S_{2,1}, S_{3,1}, S_{4,1}, S_{2,2}, S_{3,2}, S_{4,2}, S_{3,3}$. For $S_{2,1} = \sum_{ij} a_{ij} (\gamma_1 Z_{1,j} - E[\gamma_1 Z_{1,j}])$, we treat $\gamma_1 Z_{1,j}$ as the centered sub-Gaussian variables with $\psi_2$-norm at most 1 in which the $n$ sub-Gaussian are completely observed. Applying the union bound and combining the results of $S_{ij}$, we can derive the bound for $P(|S| > t)$.

Theorem 5: Hoeffding’s inequality. Let $Z_i, i = 1, ..., n$, be mean-zero independent sub-Gaussian variables satisfying $\max_{i} ||Z_i||_{\psi_2} \leq K$, and $\gamma_i, i = 1, ..., n$, be independent Bernoulli variables. Also, suppose $Z_i$ and $\gamma_i$ are independent for all $i$. Let $a = (a_i, 1 \leq i \leq n) \in \mathbb{R}^n$ be a given coefficient vector. Then, we have

$$E \exp \left( i \sum_{i=1}^{n} a_i \gamma_i Z_i \right) \leq \exp \left( C \psi_2^2 \sum_{i=1}^{n} a_i^2 \right), \quad i > 0,$$

for some numerical constant $C > 0$. Moreover, we have

$$P \left( \left| \sum_{i=1}^{n} a_i \gamma_i Z_i \right| > t \right) \leq 2 \exp \left( -\frac{ct^2}{K^2 \|a\|_2^2} \right), \quad t > 0,$$

for some numerical constant $c > 0$. 