Fluctuations in a Hořava-Lifshitz Bouncing Cosmology

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Hořava-Lifshitz gravity is a potentially UV complete theory with important implications for the very early universe. In particular, in the presence of spatial curvature it is possible to obtain a non-singular bouncing cosmology. The bounce is realized as a consequence of higher order spatial curvature terms in the gravitational action. Here, we extend the study of linear cosmological perturbations in Hořava-Lifshitz gravity coupled to matter in the case when spatial curvature is present. As in the case without spatial curvature, we find that there is no extra dynamical degree of freedom for scalar metric perturbations. We study the evolution of fluctuations through the bounce and show that the solutions remain non-singular throughout. If we start with quantum vacuum fluctuations on sub-Hubble scales in the contracting phase, and if the contracting phase is dominated by pressure-less matter, then for \( \lambda = 1 \) and in the infrared limit the perturbations at late times are scale invariant. Thus, Hořava-Lifshitz gravity can provide a realization of the “matter bounce” scenario of structure formation.

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I. INTRODUCTION

Hořava has proposed a simple model of quantum gravity [1,2] which is conservative in the sense that it is based on using the usual metric degrees of freedom in four space-time dimensions, but is radical in the sense that it abandons general covariance and local Lorentz invariance. Instead, the theory is based on a scaling symmetry in which space and time scale differently. Spatial diffeomorphism and space-independent time reparametrizations remain as symmetries of the theory. Hořava-Lifshitz gravity, as this theory is now called, has a free-field ultraviolet (UV) fixed point. It is argued that there is also an infrared fixed point in which local Lorentz symmetry and space-time diffeomorphism invariance emerge. There have been several general studies of Hořava-Lifshitz gravity [3] and a number of studies of cosmological aspects of the theory [4].

Since Hořava-Lifshitz gravity has the same number of fields as General Relativity but has a reduced symmetry, we should expect an extra physical mode [1]. This mode could be ghost-like [5], it could be strongly coupled [3,7], or it could be simply well-behaved but phenomenologically ruled out. In a previous paper [8] (see also [2]) we showed that, in the absence of spatial curvature, the extra mode is not propagating at all (this conclusion was later confirmed in [10]). Our analysis showed that the strong coupling instability discussed in [6] and the ghost-like evolution studied in [5] are regulated by taking into account the expansion of space which is inevitable in the presence of background matter 1.

Earlier, it had been shown [12] (see also [13]) that in the presence of spatial curvature it is possible to obtain a non-singular bouncing cosmology. At the bounce point the expansion rate of the universe vanishes and hence the question arises as to whether linear cosmological fluctuations are well-behaved in a bouncing Hořava-Lifshitz cosmology in the same way as they are well-behaved in a spatially flat expanding cosmology. In this paper we will firstly show that the presence of spatial curvature does not change the conclusion that there are no extra propagating degrees of freedom. Secondly, we show that the fluctuations pass through the bounce smoothly in spite of the fact that a term in the equations of motion blows up.

As argued in [12], Hořava-Lifshitz gravity in the presence of non-vanishing spatial curvature may yield a concrete realization of the “matter bounce” (see [14, 15, 16] for original works, [17] for more recent studies and [18] for a short review) scenario. In this scenario, fluctuations which originate as quantum vacuum perturbations of a matter scalar field on sub-Hubble scales in a matter-dominated contracting phase will evolve into a scale-invariant spectrum of curvature perturbations at later times in the expanding phase, with a special shape and distinguished amplitude of the three point function [19]. However, in [12] the evolution of fluctuations was considered in the context of the Einstein gravity equations, and without analyzing their propagation through the actual bounce. The results of the present paper show that the equations of General Relativity indeed provide an excellent approximation to the actual evolution for IR modes of interest to current cosmological observations.

The outline of this paper is as follows. We first briefly review Hořava-Lifshitz gravity. Next, we analyze the conditions which must be satisfied in order to obtain a non-singular bouncing cosmology. We find that in order to realize a non-singular bounce, non-trivial spatial curvature (either a closed or an open universe) is needed. We also specify the conditions on the matter content in the contracting phase which must be satisfied in order to obtain a bounce. In the next section we then extend the theory of linear adiabatic cosmological perturbations [8] to the case in which spatial curvature is present. We show that there is no extra propagating degree of freedom, as in the case studied in [8] 2. However, at the bounce point some of the coefficients in the equations of motion blow up. Thus, in section V we study the evolution of cosmological fluctuations through the bounce. We find that on IR scales relevant for current cosmological observations the evolution of fluctuations in the pre-bounce contracting phase is indistinguishable from what happens in General Relativity. Then, we show that the fluctuations evolve smoothly through the bounce. Finally, we show that a scale-invariant spectrum of curvature perturbations emerges in the case of a “matter bounce”, i.e. a bouncing cosmology in which the contracting phase is dominated by pressureless matter. There are corrections of order $\lambda - 1$, where $\lambda = 1$ is the IR fixed point at which the IR part of the action reduces to that of General Relativity. Finally, we discuss our results and give some conclusions.

II. BRIEF REVIEW OF HOŘAVA-LIFSHITZ THEORY

In Hořava-Lifshitz gravity space and time are treated differently. The space-time manifold has an extra structure, namely a given foliation of space-time into constant time hypersurfaces. Instead of full space-time diffeomorphism invariance, the symmetry of the Hořava-Lifshitz theory is foliation-preserving diffeomorphisms, which consists of (time-dependent) spatial dif-

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1 After the work reported in this paper was completed, a paper appeared [11] showing how the strong coupling instability can be resolved by adding extra terms to the original Hořava-Lifshitz action.

2 Note that we are considering the “non-projectable” version of Hořava-Lifshitz gravity.
feomorphisms and space-independent time reparametrizations. A key ingredient in the theory is the anisotropic scaling symmetry
\[ t \rightarrow t', \quad x^i \rightarrow l x^i. \tag{2.1} \]

In order to obtain a power-counting renormalizable theory of gravity in four space-time dimensions we set the scaling dimension \( z = 3 \). In this case, the theory in the UV region should flow to a free-field fixed point and is renormalizable by power counting. Meanwhile, in the IR region the theory is expected to flow to the General Relativity limit where \( \lambda = 1 \).

The basic variables are the spatial metric \( g_{ij} \), the shift vector \( N^i \) and the lapse function \( N \).

The action contains the terms consistent with the symmetries of the theory (in particular spatial diffeomorphism invariance) and \( \lambda \) the general case. We will consider the general case with the correct scaling dimension. The kinetic part is given by
\[ \text{The spatial metric and the shift vector are functions of space and time. For the lapse function there are two choices: either} \]
\[ \text{either} N \text{ depends only on time (when the so-called “projectability condition” is satisfied), or it is taken to depend on both space and time} \]
\[ \text{(the general case). We will consider the general case}. \tag{2.2} \]
\[ \text{The action of Horava-Lifshitz gravity contains a “kinetic” part and a “potential” part,} \]
\[ S^g = S^g_R + S^g_V. \tag{2.3} \]

The action contains the terms consistent with the symmetries of the theory (in particular spatial diffeomorphism invariance) and with the correct scaling dimension. The kinetic part is given by
\[ S^g_K = \frac{2}{\kappa^2} \int dt d^3 x \sqrt{N} \left( K_{ij} K^{ij} - \lambda K^2 \right), \tag{2.4} \]

where
\[ K_{ij} = \frac{1}{2N} \left( g_{ij} - \nabla_i N_j - \nabla_j N_i \right), \tag{2.5} \]
\[ \text{is the extrinsic curvature and} \]
\[ K = g^{ij} K_{ij}. \text{ In General Relativity, general covariance requires} \]
\[ \lambda = 1. \text{ The coupling constant} \lambda \text{ is} \]
\[ \text{dynamical and thus runs as the energy scale changes.} \]

The coupling constant \( \lambda \) is understood to be constructed from the spatial metric, and \( \lambda \) is the determinant of the spatial metric. The “detailed balance” condition reduces the number of terms in the potential. The most general potential is discussed in [21]. Choosing the simple form of the potential will simplify our equations (which are already complicated enough) but will not change our basic conclusions concerning the number of dynamical degrees of freedom and concerning the non-singular behavior of the solutions through the bounce.

We will take the potential part of the action to be of the “detailed-balance” form
\[ S^g_V = \int dt d^3 x \sqrt{g} N \left[ \frac{\kappa^2}{2w^2} C_{ijkl} C^{ijkl} + \frac{\kappa^2}{8} \frac{\mu^2}{R_{ij} R_{ij} R_{ij}} - \frac{\kappa^2}{8(1 - 3\lambda)} \left( \frac{1}{4} R^2 + \Lambda R - 3\Lambda^2 \right) \right], \tag{2.6} \]

where \( C_{ijkl} \) is the Cotton tensor defined by
\[ C^{ijkl} = \frac{\epsilon^{ijkl}}{\sqrt{g}} \nabla_k \left( R_{ij} - \frac{1}{4} R g_{ij} \right). \tag{2.7} \]

Here and in the above, tensors like \( R \) are understood to be constructed from the spatial metric, and \( g \) is the determinant of the spatial metric. The “detailed balance” condition reduces the number of terms in the potential. The most general potential is discussed in [21]. Choosing the simple form of the potential will simplify our equations (which are already complicated enough) but will not change our basic conclusions concerning the number of dynamical degrees of freedom and concerning the non-singular behavior of the solutions through the bounce.

We consider the simplest form of matter to be coupled to gravity, namely a scalar matter field \( \phi \). The general structure of the action of scalar-field matter in Hořava-Lifshitz theory contains two parts: a quadratic kinetic term invariant under foliation-preserving diffeomorphisms and a potential term:
\[ S^\phi = \int dt d^3 x \sqrt{g} N \left[ \frac{1}{2N^2} \left( \partial_i \phi \right)^2 + F(\phi, \partial_i \phi, g_{ij}) \right], \tag{2.8} \]

where \( F \) will contain higher order terms in spatial derivatives consistent with the symmetries and with power-counting renormalizability.

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\( ^3 \) As discussed in [2, 20], there might be problems in the general case when attempting to quantize the theory.

\( ^4 \) As is well known and as is reviewed at the beginning of Section V., a scalar field oscillating about the minimum of its potential yields a matter-dominated equation of state provided that the quadratic term in the expansion of the potential about the minimum does not vanish.
The speed of light in Hořava-Lifshitz theory can be obtained by comparing the action with that of General Relativity. The Einstein-Hilbert action in 3 + 1 dimensions is written in ADM form as

$$S_{EH} = \frac{c^3}{16\pi G} \int cdt d^3x \sqrt{g} N \left\{ \frac{1}{c^2} (K_{ij} K^{ij} - K^2) + R - 2 \frac{\Lambda_{GR}}{c^2} \right\}$$ (2.9)

The expressions for the gravitational constant and the speed of light in Hořava-Lifshitz gravity can be derived by comparing the coefficients in the action with those in General Relativity. In the infrared limit one obtains

$$c = \frac{\kappa^2 \mu}{4} \sqrt{\frac{\Lambda}{1 - 3\lambda}},$$ (2.10)

which can be seen from the ratio of the coefficients of the kinetic term and the $R$ term. In addition,

$$16\pi G = \frac{\kappa^4 \mu}{8} \sqrt{\frac{\Lambda}{1 - 3\lambda}},$$ (2.11)

and

$$\Lambda_{GR} = \frac{3\kappa^4 \mu^2 \Lambda^2}{32(1 - 3\lambda)} = \frac{3}{2} c^2 \Lambda.$$ (2.12)

Finally, it is easy to get the coefficient of the $R^2$ term:

$$\kappa^2 \mu^2 = \frac{8(1 - 3\lambda)c^3}{16\pi G \Lambda}.$$ (2.13)

## III. MATTER BOUNCE BACKGROUND

In this section we analyze the background cosmology of Hořava-Lifshitz gravity and study under which conditions a non-singular bounce will occur.

We take the background metric to be

$$ds^2 = -dt^2 + \bar{g}_{ij} dx^i dx^j,$$ (3.1)

with

$$\bar{g}_{ij} = a^2 \delta_{ij} = \frac{a^2}{1 + \frac{\bar{k} a^2}{4}} \delta_{ij},$$ (3.2)

where $r^2 \equiv \delta_{ij} x^i x^j$ and $\bar{k}$ is the spatial curvature which takes the values $\bar{k} = -1, 0, 1$. As we will see, in order to obtain a matter bounce in Hořava-Lifshitz gravity, $\bar{k} \neq 0$ is needed. Note that we are using units in which the spatial coordinates $x_i$ are dimensionless (with respect to the usual dimensions - not the anisotropic scaling dimension) but the scale factor carries dimension of length.

The background equations of motion take the form

$$\frac{6(3\lambda - 1)}{\kappa^2} H^2 = \rho - \frac{3\kappa^2 \mu^2}{8(3\lambda - 1)} \left( \frac{\bar{k}}{a^2} - \Lambda \right)^2,$$ (3.3)

$$\dot{\rho} + 3 H (1 + w) \rho = 0,$$

where $\rho$ and $p$ are the energy and pressure densities, respectively, and the equation of state parameter $w$ is $w \equiv p/\rho$. All other background equations are consistent with the above equations, for example

$$\frac{2(3\lambda - 1)}{\kappa^2} \dot{H} = -\frac{(1 + w)\rho}{2} + \frac{\kappa^2 \mu^2}{4(3\lambda - 1)} \left( \frac{\bar{k}}{a^2} - \Lambda \right) \frac{\bar{k}}{a^2}.$$ (3.4)

From (3.3) it follows that a bounce can only occur if $\bar{k} \neq 0$ (since otherwise $H = 0$ cannot be obtained).
In this work, we consider scalar field matter. The background energy \( \rho \) and pressure \( p \) for this matter take the form

\[
\rho = \frac{\dot{\phi}^2}{2} + V, \quad p = \frac{\dot{\phi}^2}{2} - V.
\] (3.5)

Since the cosmological constant must be tuned to be very small today, we will concentrate on the case when \( \frac{k}{a^2} \gg \Lambda \) is always satisfied.

When the equation of state for the scalar field satisfies \( w < \frac{1}{3} \), then in the contracting phase the higher order curvature term in (3.3) will eventually catch up with the energy density \( \rho \), resulting in a cosmological bounce, a time when \( H = 0 \) and \( \dot{H} > 0 \).

To take one step further, we would not like to have super-deflation or super-inflation around the bounce. For this purpose, we need \( \dot{H} \) to change sign twice, once before and once after the bounce time when \( H = 0 \). This is achieved if

\[
\left( \frac{k}{a^2} - \Lambda \right)^2 < \frac{4}{3(1 + w)} \left( \frac{k}{a^2} - \Lambda \right) \frac{k}{a^2}
\] (3.6)

which for negligible cosmological constant is realized is \( w < \frac{1}{3} \). Otherwise, the cosmology will either begin with a phase of super deflation leading to a bounce and then to deceleration, or with accelerated contraction followed by a bounce and then super inflation. Note that the “matter bounce” conditions \( k/a^2 \gg \Lambda \) and \( w = 0 \) yield a usual bounce without super deflation or super inflation.

There exist three different phases in a matter bounce cosmology: the first is the contracting phase during which the equation of state is dominated by pressure-less matter. This phase ends when the spatial curvature-induced higher derivative terms in the equations of motion become important. When this occurs, the second phase - the bouncing phase - begins during which the curvature-induced terms will allow the universe to evolve from contraction to expansion in a non-singular way. The last phase begins when the curvature-induced higher derivative terms cease to be important as the universe grows in size. At that point, the expanding phase that we observe today begins. In order to be consistent with late time cosmology, there needs to be entropy generation during or after the bounce such that we get an expanding radiation phase. How to generate the required amount of entropy is an issue we will not address here.

In the contracting and expanding phases, the scale factor can be parameterized as a power law:

\[
a = a_B \eta^{2(1 + 3w)},
\] (3.7)

which yields

\[
H = \frac{2}{(1 + 3w)(\eta - \eta_B)},
\] (3.8)

where \( \eta_B \) is the time of the bounce. Note that for a matter-dominated contracting phase the equation of state parameter is \( w = 0 \).

As in [17], we model the evolution of the Hubble parameter in the bouncing phase by linearly expanding in time about the bounce point:

\[
a(\eta) = \frac{a_B}{1 - y(\eta - \eta_B)^2}
\] (3.9)

which leads to

\[
H = \frac{2y(\eta - \eta_B)}{1 - y(\eta - \eta_B)^2}
\] (3.10)

In the following sections we will study the evolution of linear cosmological fluctuations in this background. We will assume that the bounce occurs at a radius which is large in Planck units (this is a natural assumption if the universe starts out cold and with a length scale related to the initial temperature by dimensional analysis). Later on in the text we will call this the “large bounce radius assumption”.

### IV. PERTURBATIONS IN THE PRESENCE OF CURVATURE

We will focus on scalar metric perturbations. In General Relativity, these fluctuations can be described in terms of four scalar functions of space and time \( \phi, \psi, B \) and \( E \) (see e.g. [22] for a comprehensive review of the theory of cosmological perturbations and [23] for a shorter overview):

\[
ds^2 = -(1 + 2\phi)dt^2 + 2\nabla_i B a(t)^2 dt dx^i + a(t)^2 \left[ (1 + 2\psi)\delta_{ij} + 2\nabla_i \nabla_j E \right] dx^i dx^j.
\] (4.1)
There are two scalar gauge degrees of freedom which allow us to eliminate two of these four functions. For example, in longitudinal
gauge one chooses to set \( E = B = 0 \). However, in Hořava-Lifshitz gravity one loses one of the gauge degrees of freedom, namely
the one corresponding to space-dependent time reparametrizations. Thus, one can only eliminate one of the scalar degrees of
freedom and one should expect an extra propagating mode.

We will follow [8] and use the remaining gauge freedom in the scalar sector to eliminate the function \( E \). Thus, we write the
perturbed spatial metric in the form

\[
g_{ij} \equiv (1 - 2\psi) \bar{g}_{ij} = a^2 \frac{(1 - 2\psi) \delta_{ij}}{1 + \frac{\bar{k}}{4} r^2} .
\]  

Due to the conformal properties of the Cotton tensor, for the perturbed metric (4.2) \( C_{ij} = 0 \) and \( \epsilon^{ijk} R_{il} \nabla_j R_{kk} = 0 \).

In addition to the fluctuation in the spatial metric, there are perturbations of the shift vector, the lapse function, and the matter
scalar field:

\[
N = 1 + \phi(t, x) ,
\]

\[
N_i = \nabla_i B(t, x) ,
\]

\[
\phi = \phi_0 + Q(t, x) .
\]  

Note that we are not enforcing the “projectability condition”. If we had enforced this condition, then \( \phi \) would be constrained
to be a function of time only, and we could use the residual gauge freedom of space-independent time reparametrizations to set
\( \phi = 0 \).

### A. Solving the Constraints

The equations of motion for \( N \) and \( N_i \) are:

\[
0 = -\frac{2}{\kappa^2} \left( K_{ij} K^{ij} - \lambda K^2 \right) - \frac{\kappa^2}{2w^2} C_{ij} C^{ij} + \frac{\kappa^2 \mu^2}{2w^2} \epsilon^{ijk} R_{il} \nabla_j R_{kj} - \frac{\kappa^2 \mu^2}{8} R_{ij} R^{ij}
\]

\[
+ \frac{\kappa^2 \mu^2}{8(1 - 3\lambda)} \left( \frac{1 - 4\lambda}{4} R^2 + \Lambda R - 3\Lambda^2 \right) - \frac{1}{2N^2} \left( \dot{\phi} - N^i \partial_i \phi \right)^2 + F ,
\]

\[
0 = \frac{4}{\kappa^2} \nabla_j \left( K^j_i - \lambda K \delta^j_i \right) - \frac{1}{N} \left( \dot{\phi} - N^i \partial_i \phi \right) \partial_i \phi .
\]  

For the background metric (4.2), \( C_{ij} = 0 \) and \( \epsilon^{ijk} R_{il} \nabla_j R_{kj} = 0 \).

At first-order, the energy constraint gives

\[
0 = \frac{4(1 - 3\lambda)H}{\kappa^2} \Delta B + \phi \left( \frac{12H^2 (1 - 3\lambda)}{\kappa^2} + \dot{\phi}_0^2 \right)
\]

\[
+ \frac{\kappa^4 (k - a^2 \Lambda) \mu^2 (a^2 \Delta \psi + 3k \psi) - 2a^4 (-1 + 3\lambda) \left( 12H (-1 + 3\lambda) \dot{\psi} + \kappa^2 \left( Q \dot{\phi}_0 + QY' \right) \right)}{2a^4 \kappa^2 (-1 + 3\lambda)} ,
\]

while the momentum constraint yields

\[
0 = \frac{4}{\kappa^2} \left[ (-1 + 3\lambda) \left( H \phi + \dot{\psi} \right) - \left( \frac{2k}{a^2} B + (1 - \lambda) \Delta B \right) \right] - \dot{\phi}_0 Q .
\]  

In the above \( \Delta \) is the Laplacian constructed using the background spatial metric \( \bar{g}_{ij} \).

As was done in the spatially flat model in [8], we can combine the above two constraint equations and solve (after choosing
The above solutions should be understood in momentum space where
into eigenfunctions which means that there is in fact only one dynamical degree of freedom in our system. This degree of freedom is precisely the Sasaki-Mukhanov \cite{24} combination of matter and metric perturbations, defined as

\[
\sum_{i,j} \bar{g} \frac{\partial^2}{\partial x_i \partial x_j} \phi_i \phi_j = \sum_{i,j} \frac{\partial^2 \phi_i}{\partial \xi_i \partial \xi_j} \left( \phi_i + \frac{H}{\varphi_0} Q \right) \phi_j \left( \phi_i + \frac{H}{\varphi_0} Q \right) = \sum_{i,j} \frac{\partial^2 \phi_i}{\partial \xi_i \partial \xi_j} \left( \phi_i + \frac{H}{\varphi_0} Q \right) \phi_j \left( \phi_i + \frac{H}{\varphi_0} Q \right)
\]

where

\[
\sum_{i,j} \frac{\partial^2 \phi_i}{\partial \xi_i \partial \xi_j} \left( \phi_i + \frac{H}{\varphi_0} Q \right) \phi_j \left( \phi_i + \frac{H}{\varphi_0} Q \right) = \sum_{i,j} \frac{\partial^2 \phi_i}{\partial \xi_i \partial \xi_j} \left( \phi_i + \frac{H}{\varphi_0} Q \right) \phi_j \left( \phi_i + \frac{H}{\varphi_0} Q \right)
\]

The above solutions should be understood in momentum space where \( \Delta \equiv -k^2/a^2 \). That is, we decompose the perturbations into eigenfunctions \( Q_\kappa(x) \) of the background spatial Laplacian:

\[
\left( \Delta + \frac{k^2}{a^2} \right) Q_\kappa(x) = 0
\]

with eigenvalues

\[
\begin{align*}
  k^2 &\geq 0, & \kappa &= 0 \\
  k^2 &= \ell(\ell + 2), & \kappa &= \pm 1
\end{align*}
\]

Note that there is no singularity at the bounce point because \( \varphi_0 \neq 0 \) at the bounce time except for a measure zero set of initial conditions on the phase of oscillation of \( \varphi_0 \).

**B. Quadratic Action**

Using \cite{4,7}, we get a quadratic action for the two variables \( \psi \) and \( Q \):

\[
S_2[\psi, Q] = \int dt d^3x \sqrt{g} \left[ c_\psi \dot{\psi}^2 + c_Q \dot{Q}^2 + c_\psi \dot{Q} \dot{\psi} \right. + f_\psi \dot{\psi} \dot{Q} + f_Q \dot{Q} \dot{\psi} + \left. f_\psi \dot{\psi} Q + f_Q \dot{Q} \psi + \omega_\psi \psi^2 + \omega_Q Q^2 + \omega_\psi \psi Q \right]
\]

where

\[
\bar{g} \equiv \det \bar{g}_{ij} = \frac{a^6}{(1 + \frac{k^2}{a^2})^6}
\]

and the various “coefficients” (whose explicit expressions are given in Appendix A1) should be understood in momentum space. From Appendix A1 we notice that

\[
c_\psi Q^2 + c_\psi \dot{\psi}^2 + c_\psi \dot{Q} \dot{\psi} \propto \left( \psi + \frac{H}{\varphi_0} Q \right)^2,
\]

which means that there is in fact only one dynamical degree of freedom in our system. This degree of freedom is precisely the Sasaki-Mukhanov \cite{2,3} combination of matter and metric perturbations, defined as

\[
-\zeta \equiv \psi + \frac{H}{\varphi_0} Q
\]
which is the gauge-invariant curvature perturbation on uniform-density hypersurfaces. From (4.13), we can express \( Q \) in terms of \( \psi \) and \( \zeta \),

\[
Q = -\frac{\dot{\phi}_0}{H}(\zeta + \psi),
\]

\[
\dot{Q} = -\left(\frac{\dot{\phi}_0 H - \dot{\phi}_0 \dot{H}}{H^2}\right)(\zeta + \psi) - \frac{\dot{\phi}_0}{H}\left(\dot{\zeta} + \dot{\psi}\right), \quad \text{etc.}
\]

(4.14)

After plugging the above relations into (4.10), using the background equations of motion and performing many integrations by part, we get a new action for the two variables \( \zeta \) and \( \psi \):

\[
S_2[\zeta, \psi] = \int dt d^3x \sqrt{\bar{g}} \left( c_\zeta \dot{\zeta}^2 + f_\zeta \dot{\zeta} + \bar{f}_\zeta \dot{\psi} + \bar{\omega}_\zeta \zeta^2 + \bar{\omega}_\psi \psi^2 \right)
\]

(4.15)

where the various coefficients can be found in Appendix A.

From (4.15), it follows that \( \psi \) is not an independent dynamical variable but rather a pure constraint, which can be solved for explicitly in terms of \( \zeta \)

\[
\psi = -\frac{\bar{\omega}_\zeta \zeta + \bar{f}_\zeta}{2\bar{\omega}_\psi}.
\]

(4.16)

Note that the coefficient \( \bar{\omega} \) does not vanish in our background. Hence, there is no strong coupling instability related to the constrained field \( \psi \).

Plugging (4.16) into (4.15), we get an effective second-order action for a single variable \( \zeta \)

\[
S_2[\zeta] = \int dt d^3x \sqrt{\bar{g}} \left( \Gamma \dot{\zeta}^2 + f \dot{\zeta} + \omega_{HL} \zeta^2 \right),
\]

(4.17)

with

\[
\Gamma = c_\zeta - \frac{f_\zeta}{4\bar{\omega}_\psi},
\]

\[
f = f_\zeta - \frac{\bar{f}_\zeta}{2\bar{\omega}_\psi},
\]

\[
\omega_{HL} = \omega_\zeta - \frac{\bar{\omega}_\zeta^2}{4\bar{\omega}_\psi}.
\]

(4.18)

The reader can verify that in the limit \( \lambda = 1 \) and for vanishing spatial curvature the coefficient \( \Gamma \) reduces to that in the general relativistic theory of cosmological perturbations.

After integrating by parts, we have

\[
S_2[\zeta] \approx \int dt d^3x \sqrt{\bar{g}} \left( \Gamma \dot{\zeta}^2 - \Omega \zeta^2 \right),
\]

(4.19)

where we made use of the definition

\[
\Omega \equiv -\left[\omega_{HL} - \frac{1}{2}\left(f + 3Hf\right)\right].
\]

(4.20)

In order to write the action in canonical form, we introduce the new variable

\[
u = z \zeta,
\]

(4.21)

Note that there is a singularity in the defining equation for \( \zeta \) at times when \( \dot{\phi}_0 = 0 \). This singularity is due to the fact that at these times the uniform density hypersurface becomes degenerate and hence \( \zeta \) ceases to be a good variable to describe the fluctuations. This problem also arises during reheating in inflationary cosmology and in that context was studied in detail in [25, 26] with the conclusion that \( \zeta \) continues through this singularity without any problem.
with $z = a\sqrt{2\Gamma}$. After changing to conformal time $\eta$ (which is defined by $dt = a d\eta$) we have

$$S_2[u] = \int d\eta d^3x \sqrt{\tilde{h}} \frac{1}{2} \left[ u'^2 + \left( \frac{z''}{z} - \frac{a^2 \Omega}{\Gamma} \right) u^2 \right],$$

(4.22)

where $\mathcal{H} \equiv a'/a$, $\bar{h} \equiv \det h_{ij}$, $h_{ij}$ is the background spatial metric without the factor $a^2$, and a prime indicates the derivative with respect to conformal time. Note that the above result should be understood in momentum space.

The classical equation of motion for the canonically normalized variable $u$ is simply

$$u''_k + \omega^2(\eta, k) u_k = 0,$$

(4.23)

with

$$\omega^2(\eta, k) \equiv \frac{a^2 \Omega}{\Gamma} - \frac{z''}{z} = \frac{a^2 \Omega}{\Gamma} - \left( \mathcal{H} + \frac{\Gamma'}{\Gamma} \right)^2 - \left( \mathcal{H} + \frac{\Gamma'}{\Gamma} \right)' \omega.$$  

(4.24)

As a consistency check, it can be verified that in the limit of General Relativity one obtains the usual equation of motion found e.g. in [22].

## V. EVOLUTION OF PERTURBATIONS DURING THE BOUNCE

In this section we will study the evolution of fluctuations from the time of their generation early in the contracting phase until late times in the expanding period. We will first review the evolution of fluctuations in a matter-dominated phase of contraction in General Relativity. Then, we study the changes to the evolution in the matter-dominated contracting phase which arise when the dynamics is studied using the equations of Hořava-Lifshitz gravity. The third step is to study the dynamics of the fluctuation modes in the bounce phase, the transition period between matter-dominated contraction and matter-dominated expansion. Finally, we need to match the solutions in the bouncing phase to those in the post-bounce matter period.

### A. Einstein Gravity Analysis of the Matter-Dominated Contracting Phase

Since we (for the sake of simplicity) model matter in terms of a scalar field, we describe the background matter in terms of a scalar field condensate $\varphi_0$.

In the limit $H^2 \ll m^2$ and making use of the WKB approximation, the equation of motion for $\varphi_0$

$$\varphi_0'' + 3H \varphi_0' + m^2 \varphi_0 = 0,$$

(5.1)

can be solved, and the solution is

$$\varphi_0 \propto m^{-1/2} a^{-3/2} \exp(i \int m dt),$$

(5.2)

and thus we see that the energy density is proportional to $a^{-3}$ and the time average of the pressure is approximately equal to zero. Thus, the oscillating scalar field condensate indeed gives us a matter-dominated contracting background cosmology.

We now turn to the description of the curvature fluctuations in the matter-dominated contracting phase, first making use of the perturbation equations from General Relativity. In this case, the equation of motion for the canonical fluctuation variable $u$ defined in (4.21) reduces to

$$u''_k + \omega^2(\eta, k) u_k = 0,$$

(5.3)

with

$$\omega^2 = k^2 - \frac{z''}{z},$$

(5.4)
where \( z \propto a \) as long as the equation of state of the background is unchanged. In a matter-dominated phase, then on scales much larger than the Hubble radius (where the first term on the right-hand side of (5.4) can be neglected) we have

\[
\omega^2 = -\frac{2}{\eta^2}.
\] (5.5)

If the fluctuations originate as quantum vacuum perturbations on sub-Hubble scales early in the contracting phase, then at Hubble radius crossing we have a vacuum power spectrum \( P_u \) for \( u \), i.e.

\[
u_k(\eta_{H}(k)) = \frac{1}{\sqrt{2k}},
\] (5.6)

where \( \eta_{H}(k) \) is the conformal time when the scale \( k \) exits the Hubble radius, and hence

\[P_u(\eta_{H}(k)) \sim k^2.\] (5.7)

To convert this vacuum spectrum into a scale-invariant one, we require a mechanism which boosts the amplitude of long wavelength modes relative to those of short wavelength ones.

In an expanding universe, the amplitude of the dominant mode of \( u \) on super-Hubble lengths grows as \( z(\eta) \) and hence the curvature fluctuation \( \zeta \) is constant. In contrast, in a contracting phase the dominant mode of \( u \) grows as \( \eta^{-1} \) (and hence \( \zeta \) scales as \( z^{-1} \)). This provides exactly the boost of long wavelength modes required to turn the initial vacuum spectrum of fluctuations into a scale-invariant one, as can be seen as follows:

\[
\rho(\eta_H(k)) = \frac{z(\eta)}{\eta}P_u(\eta_{H}(k)) = \frac{z(\eta)}{\eta_{H}(k)} = \frac{\eta_{H}(k)}{\eta_{H}(k)},
\] (5.8)

where in the second step we have made use of the time evolution of \( u \). Since in the matter phase

\[\eta_{H}(k) \sim k^{-1},\] (5.9)

the factors of \( k \) in (5.8) cancel and we indeed obtain a scale-invariant spectrum.

Another way to reach this conclusion is to consider the equation of motion of the curvature perturbation,

\[
\hat{\zeta}_{k} + 3H\hat{\zeta}_{k} - \frac{k^2}{a^2}\hat{\zeta}_{k} = 0,
\] (5.10)

which has a solution

\[
\zeta_k = A \frac{ic_s[1 - ic_s k(\eta - \eta_B)]}{\sqrt{2c_s^2 k^2(\eta - \eta_B) + 4}} \exp\left[i c_s k(\eta - \eta_B)\right].
\] (5.11)

The constant factor \( A \) is determined by the initial condition. Thus the spectrum of the curvature perturbation is scale-invariant in the contracting phase. The initial conditions yield

\[
A = \sqrt{\frac{3(1 + w)H_B}{2M_P}} \left(1 + 3w\right) \frac{3(1 + w)}{1 + 3w},
\] (5.12)

where \( H_B \) is the Hubble scale, \( \mathcal{H}_B \) is the conformal Hubble scale, and the subscript “\( B \)” denotes the momentum when the contracting phase ends.

Since the spectrum of fluctuations is scale-invariant and the fluctuations exit the Hubble radius with an amplitude smaller than 1, the amplitude of the fluctuations always remains perturbatively small.

### B. Hořava-Lifshitz Contracting Phase

In this subsection we follow the fluctuation modes during the Hořava-Lifshitz contracting phase between when they exit the Hubble radius with a vacuum spectrum until the end of the contracting phase, when the higher derivative terms scaling as \( a^{-4} \) in the action become important. The terms scaling as \( a^{-2} \) in the action which are induced by the spatial curvature are negligible throughout since we are starting the evolution in the matter-dominated contracting phase and therefore the curvature radius is
larger than the wavelength we considered. Thus, it is a good approximation to first solve the curvature perturbation in flat space (\( \hat{k} = 0 \)), and in the limit \( \lambda \to 1 \). In this limit, we can simplify the expression for \( \omega^2 \) and obtain

\[
\omega^2 = \left( c^2 k^2 - \frac{2}{\eta^2} \right) + \frac{1}{12} c^2 k^2 \left( -24 + c^2 k^2 \eta^2 - \frac{8k^2}{\eta^4 \Lambda a_B^2} \right) (\lambda - 1) \tag{5.13}
\]

If \( \lambda = 1 \), then

\[
\omega^2 = -\frac{2}{\eta^2} + c^2 k^2. \tag{5.14}
\]

For modes outside of the Hubble radius the \( k^2 \) term is negligible (note that Hubble radius crossing corresponds to \( \eta \approx 1 \)). Hence, the correction terms to the mode equation in Ho\v{r}ava-Lifshitz gravity compared to those in the Einstein theory are negligible in the contracting phase. The same conclusion can be reached if we keep the leading terms due to spatial curvature. Assuming that the universe is closed, the action for the background becomes

\[
S_B^0 = \int dt d^3x \sqrt{g} \left( \frac{1}{2} \eta^4 \right) \frac{k^3}{2\pi G} \left[ -\frac{3}{(1-3\lambda)} \frac{\pi y}{\kappa} + \frac{3}{2(1-3\lambda)} \frac{1}{\Lambda a^4} - \frac{3\Lambda}{8(1-3\lambda)} \right]. \tag{5.15}
\]

In the limit \( \Lambda a^2 \ll 1 \) which is relevant in our case when we follow modes in a phase in which the cosmological constant has a negligible effect, the \( R^2 \) terms dominate. (The limit \( \Lambda a^2 \gg 1 \) would correspond to a phase during which the cosmological constant is dominant. In this case the vacuum energy compensates the cosmological constant term in Ho\v{r}ava-Lifshitz gravity.) In our case, Eq. (5.13) can be expanded to second order of \( k \), yielding

\[
\omega^2 = -\frac{2}{\eta^2} + c^2 (3 - 2\lambda)k^2, \tag{5.16}
\]

from which it follows that the curvature perturbation is the same as in Eq. (5.11), except that \( c_s = c\sqrt{3 - 2\lambda} \).

To conclude, we see that the higher derivative terms in Ho\v{r}ava-Lifshitz gravity lead to correction terms in the equation of motion for super-Hubble scale fluctuations.

### C. Ho\v{r}ava-Lifshitz Bouncing Phase

The contracting phase transits into the bouncing phase when the term in the action scaling as \( a^{-4} \) becomes important. Inspection of the Friedmann equation (5.9) shows that this happens when

\[
T \sim T_0 \left( \frac{M_{pl}}{\mu} \right)^2. \tag{5.17}
\]

During the bouncing phase the scale factor is given by (5.9) with \( y \sim \eta_B^{-2} \). In this case, it follows from Eq. (4.24) that the mode frequency is given by

\[
\omega^2 = \left[ (-4 + \pi)(-4 + k^2)\kappa^8 \mu^4 - 2\kappa^4 \mu^2 (2\pi y (5 - 6\lambda - 27\lambda^2 + k^4 (1 - 4\lambda + 3\lambda^2) - 2k^2 (3 - 11\lambda + 6\lambda^2))) + (-5 + k^2)\kappa^4 \mu^2 \Lambda a_B^2 + (-4\pi \kappa^2 (k^4 (1 - 4\lambda + 3\lambda^2) - 9(-1 + 2\lambda + 3\lambda^2) - 2k^2 (4 - 15\lambda + 9\lambda^2)) \Lambda \mu^2 + (-6 + k^2)\kappa^8 \Lambda^2 \mu^4 a_B^2) \right] / \left[ 64(-3 + k^2)y(-1 + 3\lambda)\kappa^4 (k^4 (1 + k^2 (1 + \lambda) - 3\lambda) \mu^2 + (8\pi y (1 - 3\lambda) + k^4 (-1 - k^2 (1 + \lambda) + 3\lambda) \Lambda \mu^2 \Lambda a_B^2) \right] \eta_B^2 \right] \tag{5.18}
\]

where we already assumed that \( \hat{k} = 1 \). Setting \( \lambda = 1 \) yields

\[
\omega^2 = \frac{(-4 + \pi) (2\pi^2 (k^2 - 4) + (-2c^2 (k^2 - 6K) - 3\pi y) \Lambda a_B^2)}{16 (k^2 - 3) \eta^2 \Lambda a_B^2}, \tag{5.19}
\]
and after some approximations one obtains

$$\omega^2 = -\frac{(-4 + k^2)(-4 + \pi)\kappa^4\mu^2}{256(-3 + k^2)\eta^2}. \quad (5.20)$$

With this expression for the frequency, the solutions of the mode equation become

$$\zeta = c_1(\eta - \eta_B)^{(1 + \frac{\eta - \eta_B}{\frac{\pi}{2}\eta_B^2})/2} + c_2(\eta - \eta_B)^{(1 + \frac{\eta - \eta_B}{\frac{\pi}{2}\eta_B^2})/2}. \quad (5.21)$$

If we make the reasonable “fast bounce assumption” $\frac{\omega^2_s}{\mu^2} \gg 1$ then we find that for IR modes (i.e. modes with $k \ll 1$) the value of $\zeta$ is almost unchanged between the beginning and end of the bounce phase, i.e.

$$|\zeta| \simeq |\zeta_c|, \quad (5.22)$$

where $\zeta_c$ denotes the value of $\zeta$ at the end of the contracting phase and beginning of the expanding phase, respectively.

Note, in particular, that - as follows from the equation of the curvature perturbation in the bouncing phase - there is no singularity or instability in the solution for $\zeta$. Thus, we conclude that the fluctuations pass through the bounce without singularity and without change in the spectrum.

### D. Expanding Phase

In the expanding phase, the mode equation for $\zeta$ has two fundamental solutions. The dominant mode is constant in time on super-Hubble scales at late times is the same one as emerges after the bounce at the beginning of the expanding phase.

### E. Matching Condition

As we have seen, there are three phases in the matter bounce scenario. In each phase we have derived approximate analytical solutions of the mode equations. All that remains is to match them correctly. The matching conditions for cosmological perturbations across a space-like slice were discussed by Hwang and Vishniac [27] and by Deruelle and Mukhanov [28]. These works show that $\zeta$ and $\Phi$ must be continuous across the transition surface.

As stressed in [29], these matching conditions for fluctuations are only applicable if the background satisfies the continuity of both the induced metric and the extrinsic curvature on the matching surface. If one were to match across a singular transition between contraction and expansion as was done [30] in four dimensional toy models of the Ekpyrotic [31] scenario, then the background does not satisfy the matching conditions and hence the applicability of the matching conditions to the fluctuations is questionable. However, in a non-singular bouncing cosmology such as the one we are considering here we can apply the matching conditions consistently at the transition between the contracting matter phase and bounce phase, and between the bounce phase and the expanding matter phase. This procedure has already been applied in the case of the nonsingular mirage cosmology bounce of [32], the higher derivative gravity bounce [33], and in the quintom and Lee-Wick bounces [17].

Matching between the contracting phase and the bouncing phase implies that the spectrum of $\zeta$ at the beginning of the bounce phase is the same as it is at the end of the contracting phase, namely scale-invariant. Since the mode functions are the same at the beginning and end of the bounce phase it follows that the spectrum is scale-invariant at the end of the bounce phase. Matching at the transition between the bounce phase and the expanding phase preserves the scale-invariance of the spectrum. Hence, we conclude that the spectrum of cosmological perturbations is scale-invariant at late times.

More specifically, the values of the mode functions in the expanding phase are given by

$$\zeta_e = \frac{\sqrt{3}c_s H_B}{2M_p\sqrt{2c_s^3}} \exp(ic_sk\eta), \quad (5.23)$$

and hence the spectrum of curvature perturbation is

$$P_\zeta = \frac{k^3}{2\pi^2} |\zeta|^2 = \frac{3H_B^2}{4\pi^2 c_s M_p^2} \quad (5.24)$$

where $c_s = \sqrt{3 - 2\lambda}$, and $H_B$ is the value of $|H|$ at the end of the contracting phase (the maximal value of $|H|$).
VI. CONCLUSIONS

We have studied the evolution of linear cosmological perturbations in a bouncing Hořava-Lifshitz cosmology. We have seen that at linear order in perturbation theory there are no extra dynamical degrees of freedom, the same conclusion as was reached in an expanding Hořava-Lifshitz cosmology [2].

The equations of motion for the fluctuations contain a singularity at the bounce point. We have seen, however, that the solutions are non-singular and thus can be smoothly extended from the contracting to the expanding phase. We have derived approximate solutions of the equations of motion in the contracting and bounce phases. We have seen that the extra terms in the Hořava-Lifshitz action have a negligible effect on the evolution of fluctuations on super-Hubble scales. Thus, an initial vacuum spectrum of sub-Hubble fluctuations in the far past evolves into a scale-invariant spectrum of curvature fluctuations on super-Hubble scales at the end of the contracting phase. Because of the smooth matching of the fluctuations between the contracting phase and the bounce phase, and between the bounce phase and the expanding phase, and because of the fact that modes which are super-Hubble at the end of the contracting phase hardly change between the beginning and end of the bounce phase, the scale-invariance of the spectrum of cosmological perturbations is preserved during the bounce, as initially conjectured in [12].

We have seen that initial vacuum fluctuations lead to a power spectrum which is perturbatively small throughout the bounce phase. However, if one were to arrange the value of the spatial curvature and the cosmological constant such that a cyclic background would result, the non-trivial evolution of fluctuations on super-Hubble scales in the contracting phase would destroy the cyclicity of the evolution [5]. The fluctuations would no longer be perturbatively small during the second bounce. Thus, one should not consider values of the parameters in the Hořava-Lifshitz action which would lead to a cyclic background.

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APPENDIX A: VARIOUS COEFFICIENTS

1. Coefficients in (4.10)

The coefficients which appear in Eq. (4.10) are

$$c_{\psi} = \frac{4 (3k + a^2 \Delta) (-1 + 3\lambda) \dot{\varphi}_0^2}{8H^2 (3k + a^2 \Delta) (-1 + 3\lambda) + \kappa^2 (-2k + a^2 \Delta(-1 + \lambda)) \dot{\varphi}_0^2},$$

(A1)

$$c_Q = \frac{4H^2 (3k + a^2 \Delta) (-1 + 3\lambda)}{8H^2 (3k + a^2 \Delta) (-1 + 3\lambda) + \kappa^2 (-2k + a^2 \Delta(-1 + \lambda)) \dot{\varphi}_0^2},$$

(A2)

$$c_c = \frac{8H (3k + a^2 \Delta) (-1 + 3\lambda) \dot{\varphi}_0}{8H^2 (3k + a^2 \Delta) (-1 + 3\lambda) + \kappa^2 (-2k + a^2 \Delta(-1 + \lambda)) \dot{\varphi}_0^2}.$$

(A3)

$$f_{\psi} = -\frac{4H}{a^4 \kappa^2 (8H^2 (3k + a^2 \Delta) (-1 + 3\lambda) + \kappa^2 (-2k + a^2 \Delta(-1 + \lambda)) \dot{\varphi}_0^2) \times [(3k + a^2 \Delta) (24a^4 H^2 (1 - 3\lambda)^2 + (3k + a^2 \Delta) \kappa^4 (k - a^2 \Lambda) \mu^2) + 3a^4 \kappa^2 (-2k + a^2 \Delta(\lambda - 1)) (3\lambda - 1) \dot{\varphi}_0^2]},$$

(A4)

$$f_Q = \frac{\kappa^2 \dot{\varphi}_0 (a^2 H \Delta (1 - 3\lambda) \dot{\varphi}_0 + (2k - a^2 \Delta (-1 + \lambda)) V)}{8H^2 (3k + a^2 \Delta) (-1 + 3\lambda) + \kappa^2 (-2k + a^2 \Delta(-1 + \lambda)) \dot{\varphi}_0^2},$$

(A5)
\[ f_\psi = \frac{(-1 + 3\lambda) (-a^2\Delta \kappa^2 \phi_0^3 + 8H (3\hat{k} + a^2\Delta) V'')}{8H^2 (3k + a^2\Delta) (-1 + 3\lambda) + \kappa^2 (-2k + a^2\Delta (-1 + \lambda)) \phi_0^2}, \]  \hspace{1cm} (A6)

\[ \tilde{f}_\psi = -\frac{\phi_0}{2a^4(-1 + 3\lambda) (8H^2 (3k + a^2\Delta) (-1 + 3\lambda) + \kappa^2 (-2k + a^2\Delta (-1 + \lambda)) \phi_0^2)} \times \left[ (3k + a^2\Delta) (48a^4H^2 (1 - 3\lambda)^2 + \kappa^4 (2\hat{k} - a^2\Delta (1 - \lambda)) (\hat{k} - a^2\Lambda) \mu^2) + 6a^4\kappa^2 (-2k + a^2\Delta (1 - \lambda)) (3\lambda - 1) \phi_0^2 \right]. \]  \hspace{1cm} (A7)

\[ \omega_\psi = \frac{1}{8a^8(\kappa - 3\kappa\lambda)^2 (8H^2 (3k + a^2\Delta) (-1 + 3\lambda) + \kappa^2 (-2k + a^2\Delta (-1 + \lambda)) \phi_0^2)} \times \left\{ (3k + a^2\Delta) \left[ -1152a^8H^4 (1 - 3\lambda)^4 - 16a^6H^2 \kappa^4 (1 - 3\lambda)^2 (\hat{k}(\Delta(-4 + 3\lambda) - 6\lambda) + a^2\Delta (\Delta(-1 + \lambda) - \Lambda)) \mu^2 \\
+ (3k + a^2\Delta) \kappa^8 (2\hat{k} - a^2\Delta (-1 + \lambda)) (\hat{k} - a^2\Lambda)^2 \mu^4 \\
- 2a^6\kappa^2 (-1 + 3\lambda) \phi_0^2 \left[ 2a^2H^2 (1 - 3\lambda)^2 (-12\hat{k} + a^2\Delta (-5 + 3\lambda)) \\
- \kappa^4 (2\hat{k} - a^2\Delta (-1 + \lambda)) (\hat{k} - 4\Delta + 3\kappa\lambda - 6\lambda + a^2\Delta (\Delta(-1 + \lambda) - \Lambda)) \mu^2 \\
- 6a^2\kappa^2 (-2k + a^2\Delta (-1 + \lambda)) (-1 + 3\kappa) \phi_0^2 \right] \right\}, \]  \hspace{1cm} (A8)

\[ \omega_\psi = \frac{1}{64a^2H^2 (3k + a^2\Delta) (-1 + 3\lambda) - 8a^2 (2k - a^2\Delta (-1 + \lambda)) \phi_0^2} \times \left\{ -192H^2k^3 \delta_{3} g_3 - 64a^2H^2 \delta_{4} g_3 + 576H^2k^3 \lambda g_3 + 192a^2H^2 \delta_{4} \lambda g_3 + 12a^2H^2 \kappa^2 \phi_0^2 \\
- 36a^2H^2 \kappa^2 \lambda \phi_0^2 - 16k^3 \Delta \kappa^2 g_3 \phi_0^2 - 8a^2 \delta_{4} \kappa^2 g_3 \phi_0^2 + 8a^2 \delta_{4} \kappa^2 \lambda g_3 \phi_0^2 + 8a^2 \delta_{4} \kappa^4 \phi_0^4 \\
+ 8\Delta g_1 (-8H^2 (3k + a^2\Delta) (-1 + 3\lambda) + \kappa^2 (2\hat{k} - a^2\Delta (-1 + \lambda)) \phi_0^2) \\
+ 8\Delta^2 g_2 (8H^2 (3k + a^2\Delta) (-1 + 3\lambda) + \kappa^2 (-2k + a^2\Delta (-1 + \lambda)) \phi_0^2) \\
+ 8a^2H^2 \kappa^2 \mu \phi_0^2 V' - 24a^2H^2 \kappa^2 \phi_0^2 V' + 8k^3 \phi_0^4 (V')^2 + 4a^2 \delta_{4} \kappa^2 (V')^2 - 4a^2 \kappa^4 \lambda \phi_0^2 \\
+ 4 (-8H^2 (3k + a^2\Delta) (-1 + 3\lambda) + \kappa^2 (2\hat{k} - a^2\Delta (-1 + \lambda)) \phi_0^2) V'' \right\}, \]  \hspace{1cm} (A9)

\[ \omega_c = \frac{1}{2a^4(-1 + 3\lambda) (8H^2 (3k + a^2\Delta) (-1 + 3\lambda) + \kappa^2 (-2k + a^2\Delta (-1 + \lambda)) \phi_0^2)} \times \left\{ a^2H (3k + a^2\Delta) \kappa^4 (-1 + 3\lambda) (\hat{k} - a^2\Lambda) \mu^2 \phi_0 \\
+ \left[ (3k + a^2\Delta) (48a^4H^2 (1 - 3\lambda)^2 - \kappa^4 (2\hat{k} - a^2\Delta (-1 + \lambda)) (\hat{k} - a^2\Lambda) \mu^2) \\
+ 6a^4\kappa^2 (-2k + a^2\Delta (-1 + \lambda)) (-1 + 3\kappa) \phi_0^2 \right] V' \right\}, \]  \hspace{1cm} (A10)

2. Coefficients in (4.15)

The coefficients which appear in Eq. (4.15) are

\[ c_\xi = \frac{4 (3k + a^2\Delta) (-1 + 3\lambda) \phi_0^2}{8H^2 (3k + a^2\Delta) (-1 + 3\lambda) + \kappa^2 (-2k + a^2\Delta (-1 + \lambda)) \phi_0^2}. \]  \hspace{1cm} (A11)
\[
\omega_c = \frac{1}{16a^8H^3(1 - 3\lambda)^3 (8H^2 (k^2 - 3k) (-1 + 3\lambda) + \kappa^2 (2k + k^2(-1 + \lambda)) \hat{\phi}_0^2)} \\
\times (-2a^2H k^2 (-1 + \lambda)) (-1 + 3\lambda) (a^2k^4 (-k - a^2\Lambda) \mu^2 - 16(1 - 3\lambda)^2 (a^4g_1 + a^2k^2g_2 - k^4g_3)) \hat{\phi}_0 + 32a^4 (k^2 - 3\bar{k}) (H - 3H\lambda)^2 \left( 12a^4H^2 (1 - 3\lambda)^2 + k^4 (k - a^2\Lambda) \mu^2 - 8a^4 (1 - 3\lambda)^2 H \right) \hat{\phi}_0 V' + 4a^4k^2(-1 + \lambda) (4a^4 (H - 3H\lambda)^2 (18k + k^2(-7 + 3\lambda))) + k^4 (k + k^2(-1 + \lambda)) (k - a^2\Lambda) \mu^2 - 8a^4 (2k + k^2(-1 + \lambda)) (1 - 3\lambda)^2 H \hat{\phi}_0 V' - 8a^4k^2 (-1 + \lambda) (1 - 3\lambda)^2 \hat{\phi}_0^2 V' + 128a^8H^3 (k^2 - 3\bar{k}) (1 - 3\lambda)^4 g_1 + 128a^6H^2 k^2 (k^2 - 3\bar{k}) (1 - 3\lambda)^3 g_3 + a^6 \left( \kappa^2 (2k + k^2(-1 + \lambda)) (V')^2 - 8H^2 (k^2 - 3\bar{k}) (1 - 3\lambda) V'' \right) \right)
\]

\[
F_c = \frac{\Delta (3k + a^2\Delta) \kappa^4 (-1 + \lambda) (-k + a^2\Lambda) \mu^2 \hat{\phi}_0}{2a^2H(-1 + 3\lambda) (8H^2 (3k + a^2\Delta) (-1 + 3\lambda) + \kappa^2 (-2k + a^2\Delta(-1 + \lambda)) \hat{\phi}_0^2)} \\
\times \left\{ H (3\bar{k} + a^2\Delta) \left[ 24a^4H^2 (1 - 3\lambda)^2 + \bar{k}k^4 (k - a^2\Lambda) \right] \hat{\phi}_0 \right. \\
+ 3a^4Hk^2 (-2k + a^2\Delta(-1 + \lambda)) (-1 + 3\lambda) \hat{\phi}_0^3 + 8a^4H^2 (3k + a^2\Delta) (1 - 3\lambda)^2 V' \\
+ a^4k^2 (-2k + a^2\Delta(-1 + \lambda)) (-1 + 3\lambda) \hat{\phi}_0^2 V' \right\},
\]
\[\omega_\psi = \frac{1}{16a^6H^4\kappa^2(-1 + 3\lambda)^3} \]

\[
\left(4a^4(1 - 3\lambda)^2 \hat{H} (24a^4H^2(1 - 3\lambda)^2 + H (k^2 - 3k) \kappa^4 (\bar{k} + a^2\Lambda) \mu^2 + 2a^4\kappa^2(-1 + 3\lambda)\tilde{\psi}_0 (3H\tilde{\psi}_0 - 2V')) + \right.

\[(-k + \bar{a}^2\Lambda) \mu^2 - 8 (2k + k^2(-1 + \lambda)) \lambda (k - 3k\lambda)^2 (a^4 g_1 + a^2 k^2 g_2 - k^4 g_3) \tilde{\psi}_0^4 + 6 a^8H \kappa^4 (2k + k^2(-1 + \lambda)) (1 - 3\lambda)^2 \phi_0^3 V' - \]

\[2a^4\kappa^2(-1 + 3\lambda)\tilde{\psi}_0^3 \left(24a^4 (2k + k^2(-1 + \lambda)) (H - 3H\lambda)^2 \tilde{\phi}_0 + \right) \]

\[4a^2H^2(1 - 3\lambda)^2 (k^2 - 6\hat{k} + 3k^2\lambda) - \bar{k}\kappa^4 (2\bar{k} + k^2(-1 + \lambda)) \lambda (-k + a^2\Lambda) \mu^2) V') - \]

\[16a^4 (k^2 - 3k) (H - 3H\lambda)^2 \tilde{\phi}_0 \left(24a^4H^2(1 - 3\lambda)^2 \tilde{\phi}_0 + (12a^4H^2(1 - 3\lambda)^2 + \kappa^4 (-k + a^2\Lambda) \mu^2) V') + \]

\[2H^3 (k^2 - 3k) (-1 + 3\lambda) \left(16a^4H^2\kappa^2(1 - 3\lambda)^2 (-k^4(-1 + \lambda) + 3k^2k(-1 + \lambda) - 3k (\bar{k} + a^2\Lambda)) \mu^2 - \right) \]

\[(k^2 - 3k) \kappa^6 (2k + k^2(-1 + \lambda)) (\bar{k} - a^2\Lambda) \mu^2 + 32a^8(-1 + 3\lambda)^3 V' \left(\tilde{\phi}_0 + V'\right) + \]

\[H \tilde{\phi}_0^2 (-96a^6H^4 (k^2 - 3k) (1 - 3\lambda)^4 - 4a^4H^2\kappa^4(1 - 3\lambda)^2 (k^6(-1 + \lambda) - 30k^2 (k - a^2\Lambda) + \]

\[k^4 (k(-9 + (8 - 3\lambda)\lambda) + 4a^2\Lambda) - 3k^2 \bar{k} (k(-9 + \lambda) + a^2(7 + \lambda)\Lambda)) \mu^2 - \]

\[\left(k^2 + 3k_0 \kappa^4 (\bar{k} + k^2(-1 + \lambda)) (k - a^2\Lambda) \mu^2 + 8a^2(-1 + 3\lambda)^3 \left(16H^2 \bar{k} (k^2 - 3k) (-1 + 3\lambda) \right) \right) \]

\[\left(a^4 g_1 + a^2 k^2 g_2 - k^4 g_3 + a^6 \kappa^2 (2k + k^2(-1 + \lambda)) \lambda \right) \right) \tilde{\phi}_0 V' + a^6 \kappa^2 (2k + k^2(-1 + \lambda)) (V')^2)))) / (8H^2 (k^2 - 3k) (-1 + 3\lambda) + \kappa^2 (2k + k^2(-1 + \lambda)) \tilde{\phi}_0^2) \]

These expressions simplify dramatically if one sets \(\lambda = 1\) and \(\bar{k} = 0\). Setting, in addition, \(w = 0\) as is appropriate for the contracting phase \(w = 0\), one gets

\[c_\zeta = \frac{6}{\kappa^2} \]

\[f_\zeta = -\frac{36}{\eta^3\kappa^2 a_B} \]

\[\bar{f}_e = 0 \]

\[\bar{\omega}_e = \frac{6 \left(9 + k^2\eta^2\right)}{\eta^6\kappa^2 a_B^2} \]

\[\bar{\omega}_\lambda = \frac{3k^2 \left(\kappa^4 \Lambda M^2 + \frac{6\kappa_\lambda}{\eta^2 \kappa^2} \right)}{8k^2} \]

\[\bar{\omega}_\phi = \frac{3k^2 \left(\kappa^4 \Lambda M^2 + \frac{6\kappa_\lambda}{\eta^2 \kappa^2} \right)}{16k^2} \]

where \(M = \mu/a\)

Inserting these coefficients into the expression for the frequency \(\omega\) one obtains

\[\omega^2 = \frac{2}{\eta^2} + c^2 k^2. \quad (A14)\]

If the condition \(\lambda = 1\) is slightly relaxed and one expands to first order in \(\lambda - 1\) then one gets

\[\omega^2 = \left(c^2 k^2 - \frac{2}{\eta^2}\right) + \frac{1}{12} c^2 k^2 \left(-24 + c^2 k^2 \eta^2 - \frac{8k^2}{\eta^2 \Lambda a_B^2}\right) (\lambda - 1). \quad (A15)\]
APPENDIX B: USEFUL FORMULAE

1. Preliminaries

The connection for the perturbed metric \( (4.2) \) is (up to second-order):\[\Gamma_{ij}^{(0)} \equiv -\frac{k}{2 \left( 1 + \frac{k^2}{4} \right)} (x^i \delta_{jk} + x^j \delta_{ik} - x^k \delta_{ij}),\]
\[\Gamma_{ij}^{(1)} \equiv -\left( \partial_j \psi \delta_{ik} + \partial_k \psi \delta_{ij} - \partial_i \psi \delta_{jk} \right),\]
\[\Gamma_{ij}^{(2)} \equiv -2 \left( \psi \partial_j \psi \delta_{ik} + \psi \partial_k \psi \delta_{ij} - \psi \partial_i \psi \delta_{jk} \right).\]

The Laplacian for the metric \( (4.2) \) including fluctuations is \[\Delta \equiv \Delta^{(0)} + \Delta^{(1)} + \ldots,\]
with \[\Delta^{(0)} \equiv g_{ij}^{(0)} \partial_i \partial_j - g_{ij}^{(0)} \Gamma_{ij}^{(0)} \partial_k,\]
\[\Delta^{(1)} \equiv g_{ij}^{(1)} \partial_i \partial_j - g_{ij}^{(0)} \Gamma_{ij}^{(1)} \partial_k - g_{ij}^{(0)} \Gamma_{ij}^{(0)} \partial_k,\]

where \( g_{(0)ij} \equiv \bar{g}_{ij} \). For our purpose, we do not need higher-order terms such as \( \Delta^{(2)} \) etc. Obviously, \( \Delta^{(0)} \) is just the background Laplacian (acting on scalar fields) and \( \Delta^{(1)} \) is the first-order backreaction on the Laplacian due to the fluctuation \( \psi \).

2. Expansions of Various Quantities

For the metric \( (4.2) \), we have\[\mathcal{E} \equiv E_{ij} E^{ij} - \lambda E^2 = 3(1 - 3\lambda) \left( H - \frac{\dot{\psi}}{1 - 2\psi} \right)^2 - 2(1 - 3\lambda) \left( H - \frac{\dot{\psi}}{1 - 2\psi} \right) \Delta B + \left[ \nabla_i \nabla_j B \nabla^i \nabla^j B - \lambda (\Delta B)^2 \right],\]
and\[R_{ij} = \frac{1}{1 - 2\psi} \left[ \frac{2k}{a^2} + \Delta \psi + \frac{3 \partial_k \psi \partial^k \psi}{1 - 2\psi} \right] g_{ij} + \left( \nabla_i \nabla_j \psi + \partial_i \psi \partial_j \psi \right),\]

with \( \partial_i \psi \partial^i \psi \equiv g^{ij} \partial_i \psi \partial_j \psi \). The above results are exact.

Now it is straightforward to expand various quantities up to second-order in perturbations. In this appendix we simply collect the final results.

- \( \mathcal{E} \equiv E_{ij} E^{ij} - \lambda E^2 \) Denote \( \mathcal{E} \equiv E_{ij} E^{ij} - \lambda E^2 \) as a shorthand. Then, to second-order in perturbations, we have\[\mathcal{E}_{(0)} \equiv 3(1 - 3\lambda)H^2,\]
\[\mathcal{E}_{(1)} \equiv -2H(1 - 3\lambda) \left[ 3\psi + \Delta_{(0)} B \right],\]
\[\mathcal{E}_{(2)} \equiv (1 - 3\lambda) \left[ 3\psi^2 + 2 \psi \Delta_{(0)} B - 12H \dot{\psi} \psi - 6H \psi \Delta_{(0)} B \right] + \left[ 1 - \lambda \right] \left( \Delta_{(0)} B \right)^2 + \frac{2k}{a^2} B \Delta_{(0)} B \right].\]

- \( R \)
\[R_{(0)} \equiv \frac{\ddot{k}}{a^2},\]
\[R_{(1)} \equiv \frac{4}{a^2} \left( a^2 \Delta_{(0)} \psi + 3 \dot{k} \psi \right),\]
\[R_{(2)} \equiv 2 \left( 5 \psi \Delta_{(0)} \psi + \frac{12k}{a^2} \psi^2 \right).\]
• $R^2$

\[
\begin{align*}
(R^2)_{(0)} &= \frac{36k^2}{a^4}, \\
(R^2)_{(1)} &= 48k \left( a^2 \Delta_{(0)}^\psi + 3k^2 \psi \right), \\
(R^2)_{(2)} &= 8 \left[ 2(\Delta_{(0)}^\psi)^2 + \frac{54k^2}{a^4} \psi^2 + \frac{27k}{a^4} \psi \Delta_{(0)}^\psi \right].
\end{align*}
\]

(B7)

• $R_{ij}R^{ij}$

\[
\begin{align*}
(R_{ij}R^{ij})_{(0)} &= \frac{12k^2}{a^4}, \\
(R_{ij}R^{ij})_{(1)} &= \frac{16k}{a^4} \left( a^2 \Delta_{(0)}^\psi + 3k^2 \psi \right), \\
(R_{ij}R^{ij})_{(2)} &= 6 \left( \Delta_{(0)}^\psi \right)^2 + \frac{74k^2}{a^4} \psi \Delta_{(0)}^\psi + \frac{144k^2}{a^4} \psi^2.
\end{align*}
\]

(B8)

[1] P. Horava, “Membranes at Quantum Criticality,” JHEP 0903, 020 (2009) [arXiv:0812.4287 [hep-th]].
[2] P. Hořava, “Quantum Gravity at a Lifshitz Point,” Phys. Rev. D 79, 084008 (2009) [arXiv:0901.3775 [hep-th]].
[3] J. Kluson, “Branes at Quantum Criticality,” JHEP 0907, 079 (2009) [arXiv:0904.1345 [hep-th]].
[4] H. Nastase, “On IR solutions in Hořava gravity theories,” arXiv:0904.3604 [hep-th].
[5] C. Bogdanos and E. N. Saridakis, “Perturbative instabilities in Horava gravity,” arXiv:0907.1636 [hep-th].
[6] D. Blas, O. Pujolas and S. Sibiryakov, “Strong-coupling in Horava gravity,” JHEP 0907, 070 (2009) [arXiv:0905.0554 [hep-th]].
[7] G. Calcagni, “Cosmology of the Lifshitz universe,” JHEP 0909, 112 (2009) [arXiv:0904.0829 [hep-th]].
[8] E. Kiritsis and G. Kofinas, “Hořava-Lifshitz Cosmology,” Nucl. Phys. B 821, 467 (2009) [arXiv:0904.1334 [hep-th]].
[9] S. Mukohyama, “Dark matter as integration constant in Horava-Lifshitz gravity,” Phys. Rev. D 80, 064005 (2009) [arXiv:0905.3563 [hep-th]].
[10] S. Mukohyama, “Caustic avoidance in Horava-Lifshitz gravity,” JCAP 0909, 005 (2009) [arXiv:0906.5069 [hep-th]].
[11] N. Ashford, “Cuscuton and low energy limit of Horava-Lifshitz gravity,” Phys. Rev. D 80, 081502 (2009) [arXiv:0907.5201 [hep-th]].
[12] S. Carloni, E. Elizalde and P. J. Silva, “An analysis of the phase space of Horava-Lifshitz cosmologies,” arXiv:0909.2219 [hep-th].
[13] A. Kobakhidze, “On the infrared limit of Horava’s gravity with the global Hamiltonian constraint,” arXiv:0906.5401 [hep-th].
[14] F. W. Shu and Y. S. Wu, “Stochastic Quantization of the Hořava Gravity,” arXiv:0906.1645 [hep-th].
[10] A. Wang and R. Maartens, “Linear perturbations of cosmological models in the Horava-Lifshitz theory of gravity without detailed balance,” arXiv:0907.1748 [hep-th]; A. Wang, D. Wands and R. Maartens, “Scalar field perturbations in Horava-Lifshitz cosmology,” arXiv:0905.5167 [hep-th].

[11] D. Blas, O. Pujolas and S. Sibiryakov, “A healthy extension of Horava gravity,” arXiv:0909.3525 [hep-th]; B. Chen, S. Pi and J. Z. Tang, “Power spectra of scalar and tensor modes in modified Horava-Lifshitz gravity,” arXiv:0910.0338 [hep-th].

[12] R. Brandenberger, “Matter Bounce in Horava-Lifshitz Cosmology,” Phys. Rev. D 80, 043516 (2009) [arXiv:0904.2835 [hep-th]].

[13] Y. F. Cai and E. N. Saridakis, “Non-singular cosmology in a model of non-relativistic gravity,” JCAP 0910, 020 (2009) [arXiv:0906.1789 [hep-th]].

[14] D. Wands, “Duality invariance of cosmological perturbation spectra,” Phys. Rev. D 60, 023507 (1999) [arXiv:gr-qc/9809062].

[15] F. Finelli and R. Brandenberger, “On the generation of a scale-invariant spectrum of adiabatic fluctuations in cosmological models with a contracting phase,” Phys. Rev. D 65, 103522 (2002) [arXiv:hep-th/0112249].

[16] L. E. Allen and D. Wands, “Cosmological perturbations through a simple bounce,” Phys. Rev. D 70, 063515 (2004) [arXiv:astro-ph/0404441].

[17] Y. F. Cai, T. T. Qiu, R. Brandenberger and X. M. Zhang, “A Nonsingular Cosmology with a Scale-Invariant Spectrum of Cosmological Perturbations from Lee-Wick Theory,” Phys. Rev. D 80, 023511 (2009) [arXiv:0810.4677 [hep-th]]; Y. F. Cai, T. Qiu, R. Brandenberger, Y. S. Piao and X. Zhang, “On Perturbations of Quintom Bounce,” JCAP 0803, 013 (2008) [arXiv:0711.2187 [hep-th]].

[18] R. H. Brandenberger, “Alternatives to Cosmological Inflation,” arXiv:0902.4731 [hep-th].

[19] Y. F. Cai, W. Xue, R. Brandenberger and X. Zhang, “Non-Gaussianity in a Matter Bounce,” JCAP 0905, 011 (2009) [arXiv:0903.0631 [astro-ph.CO]].

[20] M. Li and Y. Pang, “A Trouble with Hoˇ rava-Lifshitz Gravity,” JHEP 0908, 015 (2009) [arXiv:0905.2751 [hep-th]].

[21] T. Sotiriou, M. Visser and S. Weinfurtner, “Phenomenologically viable Lorentz-violating quantum gravity,” Phys. Rev. Lett. 102, 251601 (2009) [arXiv:0904.5463 [hep-th]].

[22] T. P. Sotiriou, M. Visser and S. Weinfurtner, “Quantum gravity without Lorentz invariance,” JHEP 0910, 033 (2009) [arXiv:0905.2798 [hep-th]].

[23] V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, “Theory of cosmological perturbations. Part 1. Classical perturbations. Part 2. Quantum theory of perturbations. Part 3. Extensions,” Phys. Rept. 215, 203 (1992).

[24] R. H. Brandenberger, “Lectures on the theory of cosmological perturbations,” Lect. Notes Phys. 646, 127 (2004) [arXiv:hep-th/0306071].

[25] M. Sasaki, “Large Scale Quantum Fluctuations in the Inflationary Universe,” Prog. Theor. Phys. 76, 1036 (1986); V. F. Mukhanov, “Quantum Theory of Gauge Invariant Cosmological Perturbations,” Sov. Phys. JETP 67, 1297 (1988) [Zh. Eksp. Teor. Fiz. 94, 1 (1988)].

[26] F. Finelli and R. H. Brandenberger, “Parametric amplification of gravitational fluctuations during reheating,” Phys. Rev. Lett. 82, 1362 (1999) [arXiv:hep-ph/9809490].

[27] W. B. Lin, X. H. Meng and X. M. Zhang, “Adiabatic gravitational perturbation during reheating,” Phys. Rev. D 61, 121301 (2000) [arXiv:hep-ph/9912010].

[28] J. C. Hwang and E. T. Vishniac, “Gauge-invariant joining conditions for cosmological perturbations,” Astrophys. J. 382, 363 (1991).

[29] N. Demelle and V. F. Mukhanov, “On matching conditions for cosmological perturbations,” Phys. Rev. D 52, 5549 (1995) [arXiv:gr-qc/9503050].

[30] R. Durrer and F. Vernizzi, “Adiabatic perturbations in pre big bang models: Matching conditions and scale invariance,” Phys. Rev. D 66, 083503 (2002) [arXiv:hep-ph/0203275].

[31] D. H. Lyth, “The primordial curvature perturbation in the ekpyrotic universe,” Phys. Lett. B 524, 1 (2002) [arXiv:hep-ph/0106153].

[32] R. Brandenberger and F. Finelli, “On the spectrum of fluctuations in an effective field theory of the ekpyrotic universe,” JHEP 0111, 056 (2001) [arXiv:hep-th/0109004].

[33] D. H. Lyth, “The failure of cosmological perturbation theory in the new ekpyrotic scenario,” Phys. Lett. B 526, 173 (2002) [arXiv:hep-th/0112249].

[34] J. Khoury, B. A. Ovrut, P. J. Steinhardt and N. Turok, “The ekpyrotic universe: Colliding branes and the origin of the hot big bang,” Phys. Rev. D 64, 123522 (2001) [arXiv:hep-th/0103239].

[35] R. Brandenberger, H. Firouzjahi and O. Saremi, “Cosmological Perturbations on a Bouncing Brane,” JCAP 0711, 028 (2007) [arXiv:0707.4181 [hep-th]].

[36] S. Alexander, T. Biswas and R. H. Brandenberger, “On the Transfer of Adiabatic Fluctuations through a Nonsingular Cosmological Bounce,” arXiv:0707.4679 [hep-th].

[37] R. H. Brandenberger, “Processing of Cosmological Perturbations in a Cyclic Cosmology,” Phys. Rev. D 80, 023535 (2009) [arXiv:0905.1514 [hep-th]].