TIME-FREQUENCY SHIFT INvarIANCE OF GABOR SPACES GENERATED BY INTEGER LATTICES

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Abstract. We study extra time-frequency shift invariance properties of Gabor spaces. For a Gabor space generated by an integer lattice, we state and prove several characterizations for its time-frequency shift invariance with respect to a finer integer lattice. The extreme cases of full translation invariance, full modulation invariance, and full time-frequency shift invariance are also considered. The results show a close analogy with the extra translation invariance of shift-invariant spaces.

Key words: Extra time-frequency shift invariance, Gabor space, Time-frequency analysis, Shift-invariant space

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1. Introduction

The time-frequency structured systems that are complete in the space of square integrable functions play a fundamental role in applied harmonic analysis. Systems that span a proper subspace are relevant, for example in communications engineering, and many aspects of these have been studied from an application oriented point of view. From a more mathematical, structure oriented point of view, many aspects remain to be explored.

An interesting question regarding subspaces spanned by time-frequency structured systems is whether they are invariant under time-frequency shifts other than those pertaining to their defining property. To state the question formally, we define unitary operators, translation $T_u : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, $T_u f(x) = f(x - u)$, modulation $M_\eta : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, $M_\eta f(x) = e^{2\pi i \eta \cdot x} f(x)$, and time-frequency shift $\pi(u, \eta) = M_\eta T_u$, where $u, \eta \in \mathbb{R}^d$. For $\varphi \in L^2(\mathbb{R}^d)$ and $\Lambda$ an additive closed subgroup of $\mathbb{R}^{2d}$, we define the time-frequency structured Gabor system $(\varphi, \Lambda) = \{ \pi(u, \eta) \varphi : (u, \eta) \in \Lambda \}$ and the respective Gabor space $G(\varphi, \Lambda) = \text{span}(\pi(u, \eta) \varphi : (u, \eta) \in \Lambda)$. Note that, by definition, $G(\varphi, \Lambda)$ is invariant under time-frequency shift by elements in $\Lambda$, that is, $\pi(u, \eta) f \in G(\varphi, \Lambda)$ for all $(u, \eta) \in \Lambda$ and $f \in G(\varphi, \Lambda)$. The question is then, given $(u_0, \eta_0) \notin \Lambda$, what conditions on $\varphi$ are necessary and sufficient for the space $G(\varphi, \Lambda)$ to be invariant under $\pi(u_0, \eta_0)$?

This question is motivated by the work [ACH+10] which treats the case of shift-invariant spaces. Extra translation invariance of shift-invariant spaces in $L^2(\mathbb{R}^d)$ is characterized for the single variable case ($d = 1$) in [ACH+10], and later, for the multivariable case ($d \geq 2$) in [ACP11].

While only translations are of concern for invariance of shift-invariant spaces, in the case of Gabor spaces one needs to consider translations, modulations and also their combinations (i.e., time-frequency shifts). What makes the invariance properties of Gabor spaces even more difficult to analyze is the fact that time-frequency shifts do not commute. In this paper, we restrict our attention to integer time-frequency lattices in which case all time-frequency shifts do commute.

Some related works are the following. In [Bow07], structural properties of Gabor spaces are studied in close analogy with those of shift-invariant spaces. In particular, characterizations for Gabor spaces are given in terms of range functions, analogously to the characterizations for shift-invariant spaces in [BDR94-2]. In [CMP16], time-frequency shift invariance of Gabor spaces is studied in the context of the Amalgam Balian-Low theorem. The Amalgam Balian-Low Theorem asserts that there is no Gabor system which is a Riesz basis for $L^2(\mathbb{R}^d)$ and at the same time its window function has good time-frequency localization. As a generalization of this theorem, [CMP16] showed that if a Gabor system

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generated by a rational lattice and a window function having good decay in time and frequency is a Riesz basis for the Gabor space it spans, then the Gabor space cannot be invariant under time-frequency shifts by elements not in the generating lattice.

In this paper, we mainly focus on extra invariance of Gabor spaces $G(\varphi, \Lambda)$ where $\varphi \in L^2(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^d$ is an integer lattice, i.e., a lattice contained in $\mathbb{Z}^d$. When $\Lambda \subseteq \hat{\Lambda} \subseteq \mathbb{Z}^d$, we give complete characterizations for the $\hat{\Lambda}$-invariance of $G(\varphi, \Lambda)$, which turn out to have close analogy with the case for shift-invariant spaces. A major difference from the shift-invariance space case is that, as often in time-frequency analysis, the Zak transform is employed in place of the Fourier transform. Through the Zak transform, time-frequency shifts are represented on the time-frequency plane and are therefore easier to access than when the Fourier transform is used. By scaling the Zak transform, the results obtained generalize to the case $\Lambda \subset \mathbb{Z}^d$.

For any $(u, \eta) \in (\mathbb{R}^d)^2$ and $f \in L^2(\mathbb{R}^d)$, we have

\[ f(x) = \sum_{k \in \mathbb{Z}^d} f(x + k) e^{-2\pi i k \cdot x} \quad \text{in} \quad L^2([0, 1)^{2d}), \]

which is quasi-periodic in the sense that

\[ f(x + k, \omega + \ell) = e^{2\pi i k \cdot \omega} f(x, \omega) \quad \text{for all} \quad k, \ell \in \mathbb{Z}^d. \]

The mapping $f \mapsto Zf$ is a unitary map from $L^2(\mathbb{R}^d)$ onto $L^2([0, 1)^{2d})$, where the functions in $L^2([0, 1)^{2d})$ are understood to be quasi-periodic on $\mathbb{R}^{2d}$.

For any $u, \eta \in \mathbb{R}^d$ and $f \in L^2(\mathbb{R}^d)$, we have

\[ (Z\pi(u, \eta)f)(x, \omega) = \sum_{k \in \mathbb{Z}^d} (\pi(u, \eta) f)(x + k) e^{-2\pi i k \cdot \omega} = e^{2\pi i \eta \cdot x} \int_{\mathbb{R}^d} f(x + k - u) e^{-2\pi i k \cdot \omega} \, dk. \]

By the quasi-periodicity of Zak transform, it follows that for $u, \eta \in \mathbb{Z}^d$,

\[ (Z\pi(u, \eta)f)(x, \omega) = e^{2\pi i (\eta \cdot x - u \cdot \omega)} Zf(x, \omega). \]

A (full rank) lattice $\Gamma$ in $\mathbb{R}^d$ is a discrete subgroup of $\mathbb{R}^d$ represented by $\Gamma = AZ^d$ for some $A \in GL(d, \mathbb{R})$, where $GL(d, \mathbb{R})$ denotes the general linear group of degree $d$ over $\mathbb{R}$. We will consider lattices in $\mathbb{R}^d$ for collections of time elements $u \in \mathbb{R}^d$, and lattices in $\mathbb{R}^{2d}$ for collections of time-frequency elements $(u, \eta) \in \mathbb{R}^d \times \mathbb{R}^d$. We reserve the letter $\Gamma$ for lattices in $\mathbb{R}^d$ and $\Lambda$ for lattices in $\mathbb{R}^{2d}$. In many cases, separable lattices of the form $\Lambda = AZ^d \times BZ^d \subset \mathbb{R}^{2d}$, where $A, B \in GL(d, \mathbb{R})$, are considered. We write $\Lambda = \alpha Z^d \times \beta Z^d$ in the case where $A = \alpha I$ and $B = \beta I$, $\alpha, \beta > 0$.

For $\varphi \in L^2(\mathbb{R}^d)$ and an additive closed subgroup $\Lambda \subset \mathbb{R}^{2d}$, let $\{ \pi(u, \eta) \varphi : (u, \eta) \in \Lambda \}$ and $\overline{G}(\varphi, \Lambda) = \overline{\text{span}}\{ \pi(u, \eta) \varphi : (u, \eta) \in \Lambda \}$ be the Gabor system and Gabor space, respectively. For $\varphi \in L^2(\mathbb{R}^d)$ and an additive closed subgroup $\Gamma \subset \mathbb{R}^d$, let $S(\varphi, \Gamma) = \{ \pi(u, \eta) \varphi : (u, \eta) \in \Gamma \}$, in particular, $S(\varphi, \mathbb{Z}^d)$ is called the shift-invariant space (SIS) generated by $\varphi$.
Let $V$ be a closed subspace of $L^2(\mathbb{R}^d)$. Given $(u, \eta) \in \mathbb{R}^d \times \mathbb{R}^d$, we say that $V$ is invariant under time-frequency shift by $(u, \eta)$ if $\pi(u, \eta)f \in V$ for all $f \in V$. Given a subset $\Lambda \subset \mathbb{R}^d$, we say that $V$ is $\Lambda$-invariant if $\pi(u, \eta)f \in V$ for all $(u, \eta) \in \Lambda$ and $f \in V$. Given a subset $\Gamma \subset \mathbb{R}^d$, we say that $V$ is $\Gamma$-invariant if it is $\Gamma \times \{0\}$-invariant. We say that $V$ is shift-invariant if it is $\mathbb{Z}^d$-invariant, i.e., $\mathbb{Z}^d \times \{0\}$-invariant.

We define the time invariance set of $V$ as

$$T(V) = \{u \in \mathbb{R}^d : T_u f \in V \text{ for all } f \in V\}.$$ 

If $V$ is shift-invariant, then $T(V)$ is an additive closed subgroup of $\mathbb{R}^d$ containing $\mathbb{Z}^d$ (Proposition 2.1 in [ACP11]). Similarly, we define the time-frequency invariance set of $V$ as

$$\mathcal{P}(V) = \{(u, \eta) \in \mathbb{R}^d \times \mathbb{R}^d : \pi(u, \eta)f \in V \text{ for all } f \in V\}.$$ 

If $V$ is $\Lambda$-invariant where $\Lambda \subset \mathbb{R}^{2d}$ is a lattice, then $\mathcal{P}(V)$ is an additive closed subgroup of $\mathbb{R}^{2d}$ containing $\Lambda$ (see Proposition 3.1 in Appendix I). Thus, if $\mathcal{P}(V)$ contains a lattice $\Lambda \subset \mathbb{R}^{2d}$ and a subset $S \subset \mathbb{R}^{2d}$, then $\mathcal{P}(V)$ contains the smallest additive closed subgroup of $\mathbb{R}^{2d}$ generated by $\Lambda$ and $S$.

3. Shift-Invariant Spaces

As preparation to our analysis on extra invariance of Gabor spaces, we collect some results in shift-invariant spaces. Extra invariance of shift-invariant spaces in $L^2(\mathbb{R}^d)$ is completely characterized in [ACH10] for $d = 1$ and in [ACP11] for $d \geq 2$. We remark that extending single variable results to the multivariate setting is not easily done: the variety of closed subgroups of $\mathbb{R}^d$ for $d \geq 2$ is more complex than in the case $d = 1$ where the only possible closed subgroups containing $\mathbb{Z}$, are $\mathbb{R}$ and $\frac{1}{n}\mathbb{Z}$, $n \in \mathbb{N}$.

3.1. Fourier transform characterization of shift-invariant spaces.

Functions belonging to a shift-invariant space can be characterized using the Fourier transform. For this we need to recall the notion of dual lattice. For an additive subgroup $\Gamma$ of $\mathbb{R}^d$, its annihilator is the additive closed subgroup of $\mathbb{R}^d$ given by

$$\Gamma^* = \{\omega \in \mathbb{R}^d : e^{-2\pi i \gamma \omega} = 1 \text{ for all } \gamma \in \Gamma\}.$$ 

Note that $(\Gamma^*)^* = \overline{\Gamma}$ (the closure of $\Gamma$ in the standard topology of $\mathbb{R}^d$) and that $(\Gamma')^* \subset \Gamma^*$ if $\Gamma \subset \Gamma'$. If $\Gamma \subset \mathbb{R}^d$ is a (full rank) lattice, then so is $\Gamma^*$ which is then called the dual lattice of $\Gamma$. If $\Gamma = A\mathbb{Z}^d$ where $A \in GL(d, \mathbb{R})$, then $\Gamma^* = (A^{-1})^T \mathbb{Z}^d$. In particular, $(c_1\mathbb{Z} \times \ldots \times c_d\mathbb{Z})^* = \frac{1}{c_1} \mathbb{Z} \times \ldots \times \frac{1}{c_d} \mathbb{Z}$ where $c_1, \ldots, c_d > 0$.

**Lemma 1** (Theorem 4.3 in [ACP11]). Let $\varphi \in L^2(\mathbb{R}^d)$ and let $\Gamma$ be an additive closed subgroup of $\mathbb{R}^d$. Then $f \in L^2(\mathbb{R}^d)$ belongs in $\mathcal{S}(\varphi, \Gamma)$ if and only if there exists a $\Gamma^*$-periodic measurable function $m(\xi)$ such that $\hat{f}(\xi) = m(\xi) \hat{\varphi}(\xi)$.

Note that $\Gamma \subset \mathbb{R}^d$ in Lemma 1 is not necessarily discrete. Lemma 1 was proved in [BDR94] for the case where $\Gamma$ is a lattice.

3.2. Extra invariance of shift-invariant spaces.

While invariance of shift-invariant spaces is concerned with translations only, invariance of Gabor spaces concerns with both translations and modulations. For this reason, invariance sets associated with shift-invariant spaces and Gabor spaces are subsets of $\mathbb{R}^d$ and $\mathbb{R}^{2d}$ respectively. To compare these sets, we need to match their ambient space dimensions. Thus, we will consider shift-invariant spaces in $L^2(\mathbb{R}^{2d})$ and Gabor spaces in $L^2(\mathbb{R}^d)$ so that their invariance sets are subsets of $\mathbb{R}^{2d}$.

In [ACP11], extra invariance of shift-invariant spaces in $L^2(\mathbb{R}^{2d})$ is completely characterized, more precisely, the paper characterizes the $\Gamma$-invariance of shift-invariant spaces where $\Gamma \subset \mathbb{R}^{2d}$ is an arbitrary closed subgroup containing $\mathbb{Z}^{2d}$. To compare with the case for Gabor spaces, we state the result when $\Gamma \subset \mathbb{R}^{2d}$ is a (full rank) lattice containing $\mathbb{Z}^{2d}$.

Note that a closed subgroup of $\mathbb{R}^{2d}$ which contains $\mathbb{Z}^{2d}$ and an element in $\mathbb{R}^{2d} \setminus \mathbb{Q}^{2d}$, is non-discrete. This implies that every lattice $\Gamma \subset \mathbb{R}^{2d}$ containing $\mathbb{Z}^{2d}$ is a rational lattice, which is a lattice consisting
of rational elements only. In fact, any lattice $\tilde{\Gamma} \subset \mathbb{R}^d$ containing $\mathbb{Z}^d$ satisfies $\mathbb{Z}^d \subseteq \tilde{\Gamma} \subseteq \frac{1}{m} \mathbb{Z}^d \times \frac{1}{n} \mathbb{Z}^d$ for some (possibly large) $m, n \in \mathbb{N}$. Note that its dual lattice $\Gamma^* \subset \mathbb{R}^d$ satisfies $m\mathbb{Z}^d \times n\mathbb{Z}^d \subseteq \Gamma^* \subseteq \mathbb{Z}^d$, and $|\mathbb{Z}^d/\Gamma^*| = |\tilde{\Gamma}/\mathbb{Z}^d| = |\tilde{\Gamma} \cap [0, 1)^d|$.

**Proposition 2** ([ACH+10], [ACP11]). Let $\varphi \in L^2(\mathbb{R}^d)$ and let $\tilde{\Gamma} \subset \mathbb{R}^d$ be a lattice satisfying $\mathbb{Z}^d \subseteq \tilde{\Gamma} \subseteq \frac{1}{m} \mathbb{Z}^d \times \frac{1}{n} \mathbb{Z}^d$ where $m, n \in \mathbb{N}$ (so that $m\mathbb{Z}^d \times n\mathbb{Z}^d \subseteq \Gamma^* \subseteq \mathbb{Z}^d$). We write $\mathbb{Z}^d/\tilde{\Gamma}^* = \{I_0 = \tilde{\Gamma}^*, I_1, \ldots, I_{N-1}\}$, where $N = |\mathbb{Z}^d/\tilde{\Gamma}^*|$ and the cosets $I_0, I_1, \ldots, I_{N-1}$ form a partition of $\mathbb{Z}^d$. For $\ell = 0, 1, \ldots, N-1$, let

$$\begin{align*}
B_\ell &= \bigcup_{(r,s) \in I_\ell} (r, s) + [0, 1)^d, \\
U_\ell &= \{ f \in L^2(\mathbb{R}^d) : \hat{f} = \hat{g} \cdot \chi_{B_\ell} \text{ for some } g \in S(\varphi, \mathbb{Z}^d) \}.
\end{align*}$$

The following are equivalent.

(a) $S(\varphi, \mathbb{Z}^d)$ is $\tilde{\Gamma}$-invariant, that is, $S(\varphi, \mathbb{Z}^d) = S(\varphi, \tilde{\Gamma})$.

(b) $U_\ell \subseteq S(\varphi, \mathbb{Z}^d)$ for all $\ell = 0, 1, \ldots, N-1$.

(c) $F^{-1}(\hat{\varphi} \cdot \chi_{B_\ell}) \subseteq S(\varphi, \mathbb{Z}^d)$ for all $\ell = 0, 1, \ldots, N-1$.

(d) For a.e. $(\xi, \omega), \hat{\varphi}(\xi, \omega) \neq 0$ implies that $\hat{\varphi}(\xi + r, \omega + s) = 0$ for all $(r, s) \in (\mathbb{Z}^d \times \mathbb{Z}^d) \backslash \tilde{\Gamma}^*$. Equivalently, for a.e. $(\xi, \omega)$, at most one of the sums $\sum_{(r,s) \in I_\ell} |\hat{\varphi}(\xi + r, \omega + s)|^2$, $\ell = 0, 1, \ldots, N-1$ is nonzero.

Moreover, if any one of the above holds, $S(\varphi, \mathbb{Z}^d)$ is the orthogonal direct sum

$$S(\varphi, \mathbb{Z}^d) = U_0 \oplus \cdots \oplus U_{N-1}$$

with each $U_\ell$ being a (possibly trivial) subspace of $S(\varphi, \mathbb{Z}^d)$ which is invariant under translations by $\tilde{\Gamma}$.

From the fact that $S(\varphi, \mathbb{Z}^d)$ is translation invariant if and only if it is $\frac{1}{m} \mathbb{Z}^d \times \frac{1}{n} \mathbb{Z}^d$-invariant for all $m, n \in \mathbb{N}$, we obtain the following.

**Proposition 3.** Let $\varphi \in L^2(\mathbb{R}^d)$. Then $S(\varphi, \mathbb{Z}^d)$ is invariant under all translations if and only if $\hat{\varphi}(\xi, \omega)$ vanishes a.e. outside a fundamental domain of the lattice $\mathbb{Z}^d$.

**Remark 4.** Proposition 2 hinges on the representations associated with $S(\varphi, \mathbb{Z}^d)$ and $S(\varphi, \tilde{\Gamma})$. If $S(\varphi, \mathbb{Z}^d) = S(\varphi, \tilde{\Gamma})$ where $\varphi \in L^2(\mathbb{R}^d)$ and $\tilde{\Gamma} \supseteq \mathbb{Z}^d$, then every function $f$ in $S(\varphi, \mathbb{Z}^d)$ can be expressed in two different ways (in the Fourier transform domain):

$$m(\xi, \omega) \hat{\varphi}(\xi, \omega) = \hat{f}(\xi, \omega) = \hat{m}(\xi, \omega) \hat{\varphi}(\xi, \omega) \quad \text{a.e.,}$$

where $m(\xi, \omega)$ is $\mathbb{Z}^d$-periodic and $\hat{m}(\xi, \omega)$ is $\tilde{\Gamma}^*$-periodic, and thus we have

$$m(\xi, \omega) = \hat{m}(\xi, \omega) \quad \text{for a.e. } (\xi, \omega) \text{ such that } \hat{\varphi}(\xi, \omega) \neq 0.$$ 

Picking $\hat{m}(\xi, \omega)$ a genuinely $\tilde{\Gamma}^*$-periodic function (e.g., $\hat{m}(\xi, \omega) = e^{-2\pi i \frac{a}{m} \xi + \frac{b}{n} \omega}$ if $f = T_{\xi, \omega} \varphi$ and $(\frac{a}{m}, \frac{b}{n}) \in \tilde{\Gamma}$ for some $a, b \in \mathbb{Z}$) and exploiting the fact that $\tilde{\Gamma}^* \subseteq \mathbb{Z}^d$, we get some restrictions on set $\{(\xi, \omega) : \hat{\varphi}(\xi, \omega) \neq 0\}$ which is defined up to a measure zero set. Clearly, it is impossible that $\hat{\varphi}(\xi, \omega) \neq 0$ a.e. This yields the condition (d) in Proposition 2.

## 4. Gabor spaces

When considering time-frequency shift invariant spaces, i.e., Gabor spaces, the Zak transform replaces the Fourier transform and adjoint lattice takes over the role of dual lattice (compare Lemma 1 with Lemma 3).
4.1. Zak transform representation for Gabor spaces.
Recall that Lemma 4.1 gives Fourier transform representation for shift-invariant spaces. In this section, we treat analogous representations for Gabor spaces using Zak transform.

For a (full rank) lattice \( \Lambda \subset \mathbb{R}^{2d} \), its adjoint lattice is defined by
\[
\Lambda^\circ = \{(x, \omega) \in \mathbb{R}^{2d} : \pi(u, \eta) \circ \pi(x, \omega) = \pi(x, \omega) \circ \pi(u, \eta) \text{ for all } (u, \eta) \in \Lambda\}.
\]

Using the relation (1), we immediately see that (\( \Lambda^\circ \subset \mathbb{R}^{2d} \)) is \( \Lambda \)-invariant if and only if there exists a function which is both \( \Lambda \)-invariant and \( \mathbb{R}^{2d} \)-periodic, which naturally suggests that \( \Lambda^\circ \subset \mathbb{R}^{2d} \).

Let \( \Lambda = A\mathbb{Z}^{2d} \) where \( A \in \text{GL}(2d, \mathbb{R}) \), then
\[
\Lambda^\circ = \{(x, \omega) \in \mathbb{R}^{2d} : e^{2\pi i (\eta \cdot x - \omega \cdot u)} = 1 \text{ for all } (u, \eta) \in \Lambda\}.
\]

If \( \Lambda = A\mathbb{Z}^{2d} \) where \( A \in \text{GL}(2d, \mathbb{R}) \), then
\[
\Lambda^\circ = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}(A^{-1})^T \mathbb{Z}^{2d}.
\]

If \( \Lambda \) is a separable lattice of the form \( \Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d \) where \( A, B \in \text{GL}(d, \mathbb{R}) \), then \( \Lambda^\circ = (A^{-1})^T \mathbb{Z}^d \times (A^{-1})^T \mathbb{Z}^d \) (cf. [Z88] p.154). In particular, \((A\mathbb{Z}^d \times \beta \mathbb{Z}^d)^\circ = \mathbb{Z}^d \times \frac{1}{\beta} \mathbb{Z}^d \) where \( \alpha, \beta > 0 \). It is easily seen that \( (\Lambda^\circ)^\circ = \Lambda \) for any lattice \( \Lambda \subset \mathbb{R}^{2d} \), and that the adjoint reverses the inclusions: \( (\Lambda^\circ)^\circ \subset \Lambda \).

When \( \Lambda \subset \mathbb{Z}^{2d} \), we have \( \Lambda^\circ \supset \mathbb{Z}^{2d} \) and in this case the functions in \( G(\varphi, \Lambda) \) are accessible through a simple expression using the Zak transform.

**Lemma 5.** Let \( \varphi \in L^2(\mathbb{R}^d) \) and let \( \Lambda \subset \mathbb{Z}^{2d} \) be a lattice. Then \( f \in L^2(\mathbb{R}^d) \) belongs to \( G(\varphi, \Lambda) \) if and only if there exists a \( \Lambda^\circ \)-periodic measurable function \( h(x, \omega) \) such that
\[
Zf(x, \omega) = h(x, \omega) Z\varphi(x, \omega).
\]

A proof of Lemma 5 is given in Appendix II. Below we describe the main mechanics of the proof, to help the reader understand the following results. Assume that \( (\varphi, \Lambda) \) is a frame for its closed linear span \( G(\varphi, \Lambda) \), so that every \( f \in G(\varphi, \Lambda) \) can be expressed in the form
\[
f = \sum_{(u, \eta) \in \Lambda} c_{u, \eta} \pi(u, \eta) \varphi, \quad \{c_{u, \eta}\}_{(u, \eta) \in \Lambda} \in \ell^2(\Lambda).
\]

Applying the Zak transform on both sides and using (3), we obtain the equation (5) with \( h(x, \omega) = \sum_{(u, \eta) \in \Lambda} c_{u, \eta} e^{2\pi i (\eta \cdot x - \omega \cdot u)} \in L^1_\text{loc}(\mathbb{R}^d \times \mathbb{R}^d) \). Note that the requirement \( \Lambda \subset \mathbb{Z}^{2d} \) enables the use of (3), and that \( h(x, \omega) \) is \( \Lambda^\circ \)-periodic, since for any \( (x_0, \omega_0) \in \Lambda^\circ \),
\[
h(x + x_0, \omega + \omega_0) = \sum_{(u, \eta) \in \Lambda} c_{u, \eta} e^{2\pi i (\eta \cdot x - \omega \cdot u)} e^{2\pi i (\eta \cdot x_0 - \omega \cdot \omega_0)} = \sum_{(u, \eta) \in \Lambda} c_{u, \eta} e^{2\pi i (\eta \cdot x - \omega \cdot u)} = h(x, \omega).
\]

As can be seen above, the condition \( \Lambda \subset \mathbb{Z}^{2d} \) plays a crucial role in Lemma 5 and therefore cannot be dropped. Conversely, assume that (5) holds for some \( \Lambda^\circ \)-periodic measurable function \( h(x, \omega) \) where \( \Lambda \subset \mathbb{R}^{2d} \) is a lattice. Then since \( Zf(x, \omega) \) and \( Z\varphi(x, \omega) \) are quasi-periodic, \( h(x, \omega) \) can be replaced with a function which is both \( \Lambda^\circ \)-periodic and \( \mathbb{Z}^{2d} \)-periodic. That is, \( h(x, \omega) \) can be always assumed to be \( \mathbb{Z}^{2d} \)-periodic, which naturally suggests that \( \Lambda^\circ \supset \mathbb{Z}^{2d} \), i.e., \( \Lambda \subset \mathbb{Z}^{2d} \). Hence, the requirement \( \Lambda \subset \mathbb{Z}^{2d} \) in Lemma 5 is not only essential but also very natural for (5) to hold.

Note that since both sides of (5) are quasi-periodic, it is sufficient to check the equality (5) only for a.e. \( (x, \omega) \) in \( [0, 1)^{2d} \).

4.2. Extra time-frequency shift invariance of Gabor spaces.

Equipped with the representation for Gabor spaces, we are ready to analyze extra invariance of Gabor spaces \( G(\varphi, \Lambda) \) where \( \varphi \in L^2(\mathbb{R}^d) \) and \( \Lambda \subset \mathbb{Z}^{2d} \) is a lattice.

Let \( \tilde{\Lambda} \subset \mathbb{R}^{2d} \) be a closed subgroup which contains \( \Lambda \) strictly, that is, \( \Lambda \subset \tilde{\Lambda} \subset \mathbb{R}^{2d} \). Then \( G(\varphi, \Lambda) \) is \( \tilde{\Lambda} \)-invariant if and only if \( G(\varphi, \Lambda) = G(\varphi, \tilde{\Lambda}) \), in which case every \( f \in G(\varphi, \Lambda) \) admits another representation as a function of \( G(\varphi, \tilde{\Lambda}) \).
4.2.1. The case $\Lambda \subseteq \tilde{\Lambda} \subseteq \mathbb{Z}^d$.

As our first main result, we characterize the $\tilde{\Lambda}$-invariance of $G(\varphi, \Lambda)$ when $\Lambda, \tilde{\Lambda} \subseteq \mathbb{R}^d$ are lattices such that $\Lambda \subseteq \tilde{\Lambda} \subseteq \mathbb{Z}^d$.

**Theorem 6.** Let $\varphi \in L^2(\mathbb{R}^d)$ and let $\Lambda, \tilde{\Lambda} \subseteq \mathbb{R}^d$ be lattices satisfying $\Lambda \subseteq \tilde{\Lambda} \subseteq \mathbb{Z}^d$ (so that $\Lambda^\circ \supseteq \tilde{\Lambda}^\circ \supseteq \mathbb{Z}^d$). We write the quotient $\Lambda^\circ / \tilde{\Lambda}^\circ$ as $\{I(0) = \tilde{\Lambda}^\circ, I(1), \ldots, I(N-1)\}$, where $N$ is the order of $\Lambda^\circ / \tilde{\Lambda}^\circ$ and the cosets $I(0), I(1), \ldots, I(N-1)$ all together forms a partition of $\Lambda^\circ$. Let $D \subset [0,1)^d$ be a fundamental domain of the lattice $\Lambda^\circ$. For $\ell = 0, 1, \ldots, N - 1$, let

\[ B(\ell) = \bigcup_{(u,\eta) \in I(\ell)} (u, \eta) + D, \]

\[ U(\ell) = \{ f \in L^2(\mathbb{R}^d) : Zf = Zg \cdot \chi_B(\ell) \text{ for some } g \in G(\varphi, \Lambda) \}. \]

The following are equivalent.

(a) $G(\varphi, \Lambda)$ is $\tilde{\Lambda}$-invariant, i.e., $G(\varphi, \Lambda) = G(\varphi, \tilde{\Lambda})$.

(b) $U(\ell) \subseteq G(\varphi, \Lambda)$ for all $\ell = 0, 1, \ldots, N - 1$.

(c) $Z^{-1}(Z\varphi \cdot \chi_B(\ell)) \in G(\varphi, \Lambda)$ for all $\ell = 0, 1, \ldots, N - 1$.

(d) For a.e. $(x, \omega)$,

\[ Z\varphi(x, \omega) \neq 0 \implies Z\varphi(x + u, \omega + \eta) = 0 \text{ for all } (u, \eta) \in \Lambda^\circ \setminus \tilde{\Lambda}^\circ. \]

Equivalently, for a.e. $(x, \omega)$, at most one of the sums $\sum_{(u,\eta) \in I(\ell) \cap [0,1)^d} |Z\varphi(x + u, \omega + \eta)|^2$, $\ell = 0, 1, \ldots, N - 1$ is nonzero.

Moreover, if any one of the above holds, $G(\varphi, \Lambda)$ is the orthogonal direct sum

\[ G(\varphi, \Lambda) = U(0) \oplus \cdots \oplus U(N-1) \]

with each $U(\ell)$ being a (possibly trivial) subspace of $G(\varphi, \Lambda)$ which is $\tilde{\Lambda}$-invariant.

**Remark 7.** (a) By scaling the Zak transform as $Z_\alpha f(x, \omega) = \sum_{k \in \mathbb{Z}^d} f(x + \alpha k) e^{-2\pi i \alpha k \cdot \omega}$ where $\alpha > 0$, Theorem 6 can be generalized to the case where $\Lambda \subseteq \tilde{\Lambda} \subseteq \alpha \mathbb{Z}^d \times 1/2 \mathbb{Z}^d$, $\alpha > 0$.

(b) It is easily seen that each of $I(\ell)$, $\ell = 0, 1, \ldots, N - 1$ is of the form $\{(u,\eta) \in \mathbb{R}^d : e^{2\pi i (u a - \eta)} = \zeta_N\}$ where $\zeta_N$ is an $N$ th root of unity. While proving Theorem 6 we will assume without loss of generality that

\[ I(\ell) = \{(u,\eta) \in \mathbb{R}^d : e^{2\pi i (b u - \alpha \eta)} = e^{2\pi i \ell / N} \text{ for all } (a, b) \in \tilde{\Lambda} \}, \quad \ell = 0, 1, \ldots, N - 1. \]

(c) Let $\mathcal{K} \subset \Lambda^\circ$ be a set of representatives of the quotient $\Lambda^\circ / \tilde{\Lambda}^\circ = \{I(0), I(1), \ldots, I(N-1)\}$, so that $\mathcal{K}$ consists of exactly $N$ elements each of which represents one $I(\ell)$. If $D \subset [0,1)^d$ is a fundamental domain of the lattice $\Lambda^\circ$, then the finite union $\tilde{D} = \bigcup_{(u,\eta) \in \mathcal{K}} (u, \eta) + D$ is a fundamental domain of the coarser lattice $\tilde{\Lambda}^\circ$. The $\tilde{\Lambda}^\circ$-periodization of $D$ is the set $B(\ell)$, while the $\Lambda^\circ$-periodization $\tilde{D}$ is $\mathbb{R}^d$.

For the proof of Theorem 6 we need the following lemma.

**Lemma 8** (cf. Lemma 4.3 in [ACH+10]). Under the same assumptions as in Theorem 6 if $U(\ell) \subseteq G(\varphi, \Lambda)$ for some $\ell$, then it is a $\tilde{\Lambda}$-invariant closed subspace of $G(\varphi, \Lambda)$.

**Proof.** The proof is similar to Lemma 4.3 in [ACH+10].

Assume that $U(\ell) \subseteq G(\varphi, \Lambda)$ for some $\ell$. To see that $U(\ell)$ is closed, suppose that $\{f_n\}_{n=1}^\infty \subset U(\ell)$ is a sequence that converges to some $f$ in $L^2(\mathbb{R}^d)$. Since $G(\varphi, \Lambda)$ is closed and $\{f_n\}_{n=1}^\infty \subset G(\varphi, \Lambda)$, it follows that $f \in G(\varphi, \Lambda)$. Further, since $Z$ is unitary we have

\[ \|f_n - f\|_{L^2(\mathbb{R}^d)}^2 = \|Z(f_n - f)\|_{L^2([0,1)^d)}^2 \]

\[ = \|Zf_n - Zf\chi_B(\ell)c\|_{L^2([0,1)^d)}^2 + \|Zf_n - Zf\chi_B(\ell)c\|_{L^2([0,1)^d)}^2 \]

\[ = \|Zf_n - Zf\chi_B(\ell)c\|_{L^2([0,1)^d)}^2 + \|Zf \cdot \chi_B(\ell)c\|_{L^2([0,1)^d)}^2. \]

Since the left hand side converges to zero, we must have $Zf_n \to Zf \cdot \chi_B(\ell)c$ in $L^2([0,1)^d)$ and $Zf \cdot \chi_B(\ell)c = 0$. Since $Zf_n \to Zf$ in $L^2([0,1)^d)$, we have $Zf = Zf \cdot \chi_B(\ell)c$ which together with $f \in G(\varphi, \Lambda)$ implies that $f \in U(\ell)$. Thus, $U(\ell)$ is a closed subspace of $G(\varphi, \Lambda)$.
Let us first see that $U^{(t)}$ is $Λ$-invariant. Fix any $f ∈ U^{(t)}$ and let $g ∈ G(ϕ, Λ)$ be such that $Zf = Zg · χ_{B^{(t)}}$. For any $(a, b) ∈ Λ (⊆ Z^{2d})$, we have

\[
Zπ(a, b)f(x, ω) = e^{i2π(bx - a·ω)}Zf(x, ω) = e^{i2π(bx - a·ω)}Zg(x, ω) · χ_{B^{(t)}}(x, ω) = \left(Zπ(a, b)g\right)(x, ω) · χ_{B^{(t)}}(x, ω),
\]

where $π(a, b)g ∈ G(ϕ, Λ)$, and thus $π(a, b)f ∈ U^{(t)}$. This shows that $U^{(t)}$ is $Λ$-invariant.

Next, to see that $U^{(t)}$ is in fact $\tilde{Λ}$-invariant, we fix any $(a, b) ∈ \tilde{Λ} (⊆ Z^{2d})$ and consider a $Λ^c$-periodic function given by

\[
h_{a, b}^{(t)}(x, ω) = \frac{1}{M} e^{−2πiℓ/N} \sum_{(u, η) ∈ Λ^c ∩ [0, 1)^{2d}} e^{2πi(b(x + u) - a·(ω + η))} χ_{B^{(t)}}(x + u, ω + η),
\]

where $M = |\tilde{Λ}^c / Z^{2d}| = |\tilde{Λ}^c ∩ [0, 1)^{2d}|$. Then

\[
h_{a, b}^{(t)}(x, ω) χ_{B^{(t)}}(x, ω) = \frac{1}{M} e^{−2πiℓ/N} \sum_{(u, η) ∈ Λ^c ∩ [0, 1)^{2d}} e^{2πi(b(x + u) - a·(ω + η))} χ_{B^{(t)}}(x + u, ω + η) χ_{B^{(t)}}(x, ω)
\]

For any $f ∈ U^{(t)}$, since $supp Zf ⊆ B^{(t)}$, we have

\[
\left(Zπ(a, b)f\right)(x, ω) = e^{2πi(bx - a·ω)}Zf(x, ω) = h_{a, b}^{(t)}(x, ω) Zf(x, ω).
\]

By Lemma 5 and since $U^{(t)}$ is $Λ$-invariant, it follows that $π(a, b)f ∈ G(ϕ, Λ) ⊆ U^{(t)}$. Therefore, $U^{(t)}$ is $\tilde{Λ}$-invariant.

**Proof of Theorem 2**  
(a) ⇒ (b): Assume that $G(ϕ, Λ) = G(ϕ, \tilde{Λ})$. Fix any $ℓ = 0, 1, \ldots, N - 1$. If $f ∈ U^{(t)}$, then exists $g ∈ G(ϕ, Λ)$ such that $Zf = Zg · χ_{B^{(t)}}$. Since $B^{(t)}$ is periodic with respect to $\tilde{Λ}^c$, it follows by Lemma 5 that $f ∈ G(ϕ, \tilde{Λ})$. Since $G(ϕ, Λ) ⊆ G(ϕ, \tilde{Λ}) = G(ϕ, Λ)$, we conclude that $U^{(t)} ⊆ G(ϕ, Λ)$.

(b) ⇒ (a): Assume that $U^{(t)} ⊆ G(ϕ, Λ)$ for all $ℓ = 0, 1, \ldots, N - 1$. Then Lemma 8 implies that all $U^{(t)}$, $ℓ = 0, 1, \ldots, N - 1$ are $Λ^c$-invariant closed subspaces of $G(ϕ, Λ)$. These subspaces are mutually orthogonal, since the sets $B^{(t)}$, $ℓ = 0, 1, \ldots, N - 1$ are disjoint. Moreover, every $f ∈ G(ϕ, Λ)$ can be decomposed as $f = f^{(0)} + \ldots + f^{(N-1)}$, where $f^{(t)} = Z^{-1}(Zf · χ_{B^{(t)}}) ∈ U^{(t)}$ for $ℓ = 0, 1, \ldots, N - 1$. Therefore, we have the orthogonal direct sum decomposition

\[
G(ϕ, Λ) = U^{(t)} ⊕ \cdots ⊕ U^{(N-1)}.
\]

Since all $U^{(t)}$ are $\tilde{Λ}$-invariant, so is $G(ϕ, Λ)$.

(b) ⇒ (c): This is trivial, since $ϕ ∈ G(ϕ, Λ)$.

(c) ⇒ (d): Assume that $Z^{-1}(Zϕ · χ_{B^{(t)}}) ∈ G(ϕ, Λ)$ for all $ℓ = 0, 1, \ldots, N - 1$. Then for each $ℓ$, Lemma 6 implies that there exists a $Λ^c$-periodic measurable function $h^{(t)}(x, ω)$ such that

\[
Zϕ(x, ω) · χ_{B^{(t)}}(x, ω) = h^{(t)}(x, ω) Zϕ(x, ω).
\]

By a standard periodization trick, we get

\[
\sum_{(u, η) ∈ Λ^c ∩ [0, 1)^{2d}} |Zϕ(x + u, ω + η)|^2 χ_{B^{(t)}}(x + u, ω + η) = |h^{(t)}(x, ω)|^2 \sum_{(u, η) ∈ Λ^c ∩ [0, 1)^{2d}} |Zϕ(x + u, ω + η)|^2,
\]

and

\[
\sum_{(u, η) ∈ Λ^c ∩ [0, 1)^{2d}} |Zϕ(x + u, ω + η)|^2 = |h^{(t)}(x, ω)|^2 \sum_{(u, η) ∈ Λ^c ∩ [0, 1)^{2d}} |Zϕ(x + u, ω + η)|^2.
\]
Note that the left hand sides of (7) and (8) coincide if \((x, \omega) \in B(0, \ell)\). Thus, for a.e. \((x, \omega) \in B(0, \ell)\),
\[
\sum_{(u, \eta) \in \Lambda^\circ \cap [0, 1)^d} |Z\varphi(x + u, \omega + \eta)|^2 = |h(0)(x, \omega)|^2 \sum_{(u, \eta) \in \Lambda^\circ \cap [0, 1)^d} |Z\varphi(x + u, \omega + \eta)|^2 = |h(0)(x, \omega)|^2 \sum_{(u, \eta) \in \Lambda^\circ \cap [0, 1)^d} |Z\varphi(x + u, \omega + \eta)|^2,
\]
from which we see that if \(\sum_{(u, \eta) \in \Lambda^\circ \cap [0, 1)^d} |Z\varphi(x + u, \omega + \eta)|^2 \neq 0\), then \(|h(0)(x, \omega)|^2 = 1\) and in turn, \(\sum_{(u, \eta) \in (\Lambda^\circ \\setminus \Lambda^\circ) \cap [0, 1)^d} |Z\varphi(x + u, \omega + \eta)|^2 = 0\). Since the sets \(B(0, \ell)\), \(\ell = 0, 1, \ldots, N - 1\) form a partition of \(\mathbb{R}^d\), we conclude that for a.e. \((x, \omega) \in \mathbb{R}^d\), \(\sum_{(u, \eta) \in \Lambda^\circ \cap [0, 1)^d} |Z\varphi(x + u, \omega + \eta)|^2 \neq 0\) implies \(\sum_{(u, \eta) \in (\Lambda^\circ \\setminus \Lambda^\circ) \cap [0, 1)^d} |Z\varphi(x + u, \omega + \eta)|^2 = 0\). Then (d) follows by observing that Zak transform is quasi-periodic.

(d) \(\Rightarrow\) (a): Assume that (d) holds, and fix any \((a, b) \in \tilde{\Lambda} (\subseteq \mathbb{Z}^d)\). We will show \(\pi(a, b) \varphi \in \mathcal{G}(\varphi, \Lambda)\) using Lemma 5 more precisely, by constructing a \(\Lambda^\circ\)-periodic measurable function \(h : \mathbb{R}^d \to \mathbb{C}\) such that \((Z\pi(a, b) \varphi)(x, \omega) = h(x, \omega) Z\varphi(x, \omega)\). Noting that \(D\) is a fundamental domain of the lattice \(\Lambda^\circ\), we will define \(h\) on \(D\) and extend it \(\Lambda^\circ\)-periodically to \(\mathbb{R}^d\). By assumption, the set of all \((x, \omega) \in D\) for which (b) is violated is a measure zero set which we denote by \(D_0 \subset D\). Define \(h(x, \omega) = 0\) for \((x, \omega) \in D_0\). Next, fix any \((x, \omega) \in D_0\).

- If \(\sum_{(u, \eta) \in I(0) \cap [0, 1)^d} |Z\varphi(x + u, \omega + \eta)|^2 = 0\) for all \(\ell = 0, 1, \ldots, N - 1\), equivalently, if \(Z\varphi(x + u, \omega + \eta) = 0\) for all \((u, \eta) \in \Lambda^\circ \cap [0, 1)^d\), then define \(h(x, \omega) = 0\).

- Otherwise, there exists a unique \(0 \leq \ell_0 \leq N - 1\) such that \(\sum_{(u, \eta) \in I(\ell_0) \cap [0, 1)^d} |Z\varphi(x + u, \omega + \eta)|^2 = 0\) for all except \(\ell_0\), equivalently, \(Z\varphi(x + u, \omega + \eta) = 0\) for all \((u, \eta) \in (\Lambda^\circ \setminus I(\ell_0)) \cap [0, 1)^d\). We define \(h(x, \omega) = e^{2\pi i (b \cdot x - a \cdot \omega)}\). Define \(\tilde{Z} \varphi(x + u, \omega + \eta) = h(x, \omega) Z\varphi(x, \omega)\). Observe that for any \((u, \eta) \in I(\ell_0) \cap [0, 1)^d\), we have \(e^{2\pi i (b \cdot (x-u) - a \cdot (\omega + \eta))} = e^{2\pi i \ell_0 / N} \cdot e^{2\pi i (b \cdot x - a \cdot \omega)} = h(x, \omega)\). Combining with the fact that \(Z\varphi(x + u, \omega + \eta) = 0\) for \((u, \eta) \in (\Lambda^\circ \setminus I(\ell_0)) \cap [0, 1)^d\), we obtain that for all \((u, \eta) \in \Lambda^\circ \cap [0, 1)^d\),
\[
e^{2\pi i (b \cdot (x+u) - a \cdot (\omega + \eta))} Z\varphi(x + u, \omega + \eta) = h(x, \omega) Z\varphi(x + u, \omega + \eta).
\]

With \(h(x, \omega)\) defined on \(D\) as above, it follows that for all \((u, \eta) \in \Lambda^\circ \cap [0, 1)^d\),
\[
e^{2\pi i (b \cdot (x+u) - a \cdot (\omega + \eta))} Z\varphi(x + u, \omega + \eta) = h(x, \omega) Z\varphi(x + u, \omega + \eta)\] a.e. \((x, \omega) \in D\). This in fact holds for all \((u, \eta) \in \Lambda^\circ\), since \(\Lambda^\circ \supseteq \mathbb{Z}^d\) and Zak transform is quasi-periodic. Therefore, with \(h(x, \omega)\) extended \(\Lambda^\circ\)-periodically from \(D\) to \(\mathbb{R}^d\), we have
\[
e^{2\pi i (b \cdot x - a \cdot \omega)} Z\varphi(x, \omega) = h(x, \omega) Z\varphi(x, \omega)\] a.e.,

From (b) and Lemma 5 we conclude that \(\pi(a, b) \varphi \in \mathcal{G}(\varphi, \Lambda)\). \(\Box\)

**Corollary 9.** Let \(\varphi \in L^2(\mathbb{R}^d)\) and let \(\Lambda \subseteq \mathbb{Z}^d\) be a lattice. Then \(\mathcal{G}(\varphi, \Lambda)\) is \(\mathbb{Z}^d\)-invariant if and only if \(Z\varphi(x, \omega)\) vanishes a.e. on \([0, 1)^d\) \(D\), where \(D \subset [0, 1)^d\) is a fundamental domain of the lattice \(\Lambda^\circ\). When \(\varphi\) is a Riesz basis for \(\mathcal{G}(\varphi, \Lambda)\), the latter condition is refined to: \(Z\varphi(x, \omega) \neq 0\) a.e. on \(D\) and \(Z\varphi(x, \omega) = 0\) a.e. on \([0, 1)^d \setminus D\).

**Proof.** By Theorem 5 and the quasi-periodicity of Zak transform, it follows that \(\mathcal{G}(\varphi, \Lambda)\) is \(\mathbb{Z}^d\)-invariant if and only if for a.e. \((x, \omega), Z\varphi(x, \omega) \neq 0\) implies that \(Z\varphi(x + u, \omega + \eta) = 0\) for all \((u, \eta) \in (\Lambda^\circ \cap [0, 1)^d) \setminus \{(0, 0)\}\). The latter is equivalent to that for a.e. \((x, \omega), \) we have \(Z\varphi(x + u, \omega + \eta) \neq 0\) for at most one \((u, \eta) \in \Lambda^\circ \cap [0, 1)^d\), which holds if and only if \(Z\varphi(x, \omega)\) vanishes a.e. on \([0, 1)^d \setminus D\) where \(D \subset [0, 1)^d\) is a fundamental domain of the lattice \(\Lambda^\circ\).

For the second part, observe that \((\varphi, \Lambda)\) is a Riesz basis for \(\mathcal{G}(\varphi, \Lambda)\) with Riesz bounds \(B \geq A > 0\) if and only if
\[
mA \leq \sum_{(u, \eta) \in \Lambda^\circ \cap [0, 1)^d} |Z\varphi(x + u, \omega + \eta)|^2 \leq mB\] a.e.,
where \(m = |\Lambda^\circ \cap [0, 1)^d| \geq 1\). In this case, we have \(Z\varphi(x, \omega) \neq 0\) a.e. at least on a fundamental domain of the lattice \(\Lambda^\circ\). The claim is then straightforward. \(\Box\)
Remark 4. In Proposition 2 extra invariance of $S(\varphi, \mathbb{Z}^{2d})$ is characterized through zeros of $\hat{\varphi}(\xi, \omega)$, while in Theorem 4 extra invariance of $G(\varphi, \Lambda)$ is characterized through zeros of $Z\varphi(x, \omega)$. Similar to Remark 4 we have following.

Assume that $G(\varphi, \Lambda) = G(\varphi, \tilde{\Lambda})$, where $\varphi \in L^2(\mathbb{R}^{2d})$ and $\Lambda, \tilde{\Lambda} \subseteq \mathbb{R}^{2d}$ are lattices satisfying $\Lambda \subseteq \tilde{\Lambda} \subseteq \mathbb{Z}^{2d}$ (so that $\Lambda^0 \supseteq \tilde{\Lambda}^0 \supseteq \mathbb{Z}^{2d}$). Then every $f \in G(\varphi, \Lambda)$ can be represented in two different ways (in the Zak transform domain):

$$h(x, \omega) Z\varphi(x, \omega) = Zf(x, \omega) = \tilde{h}(x, \omega) Z\varphi(x, \omega),$$

where $h(x, \omega)$ is $\Lambda^0$-periodic and $\tilde{h}(x, \omega)$ is $\tilde{\Lambda}^0$-periodic (see Lemma 4), and thus we have

$$h(x, \omega) = \tilde{h}(x, \omega) \text{ for a.e. } (x, \omega) \text{ such that } Z\varphi(x, \omega) \neq 0.$$

By picking $h(x, \omega)$ and $\tilde{h}(x, \omega)$ that are genuinely $\Lambda^0$-periodic and $\tilde{\Lambda}^0$-periodic respectively, and exploiting the fact that $\tilde{\Lambda}^0 \subseteq \Lambda^0$, we obtain some restrictions on the set $\{(x, \omega) : Z\varphi(x, \omega) \neq 0\}$ which is defined up to a measure zero set. Clearly, it is impossible that $Z\varphi(x, \omega) \neq 0$ for a.e. $(x, \omega)$ in $\mathbb{R}^{2d}$. This yields condition (d) in Theorem 6.

4.2.2. The case $\Lambda = \mathbb{Z}^{2d}$ with $\tilde{\Lambda} = \mathbb{R}^d \times \mathbb{Z}^d \subseteq \mathbb{R}^{2d}$.

To compare with shift-invariant spaces $S(\varphi, \mathbb{Z}^{2d})$, we consider the lattice $\Lambda = \mathbb{Z}^{2d}$. We will treat the extreme cases $\tilde{\Lambda} = \mathbb{R}^d \times \mathbb{Z}^d \subseteq \mathbb{R}^{2d}$. Note that since $\mathbb{R}^d \times \mathbb{Z}^d$ is the smallest closed subgroup of $\mathbb{R}^{2d}$ containing $\mathbb{R}^d \times \{0\}$ and $\mathbb{Z}^{2d}$, the space $G(\varphi, \mathbb{Z}^{2d})$ is $\mathbb{R}^d \times \mathbb{Z}^d$-invariant if and only if it is $\mathbb{R}^d \times \{0\}$-invariant. Similarly, the space $G(\varphi, \mathbb{Z}^{2d})$ is $\mathbb{R}^d \times \mathbb{R}^d$-invariant if and only if it is $\{0\} \times \mathbb{R}^d$-invariant.

Proposition 11. Let $\varphi \in L^2(\mathbb{R}^d)$.

(a) $G(\varphi, \mathbb{Z}^{2d})$ is invariant under all translations ($\mathbb{R}^d \times \{0\}$-invariant) if and only if there exists a measurable set $E \subseteq [0, 1)^d$ such that $Z\varphi(x, \omega) \neq 0 \text{ a.e. on } [0, 1)^d \setminus E \text{ and } Z\varphi(x, \omega) = 0 \text{ a.e. on } [0, 1)^d \times \{0\}$.

(b) $G(\varphi, \mathbb{Z}^{2d})$ is invariant under all modulations ($\{0\} \times \mathbb{R}^d$-invariant) if and only if there exists a measurable set $E \subseteq [0, 1)^d$ such that $Z\varphi(x, \omega) \neq 0 \text{ a.e. on } [0, 1)^d \times \{0\}$ and $Z\varphi(x, \omega) = 0 \text{ a.e. on } [0, 1)^d \times \{0\}$.

(c) $G(\varphi, \mathbb{Z}^{2d})$ is invariant under all time-frequency shifts ($\mathbb{Z}^{2d}$-invariant) if and only if $G(\varphi, \mathbb{Z}^{2d})$ is either $\{0\} \text{ or } L^2(\mathbb{R}^d)$. Consequently, a nontrivial proper Gabor subspace $G(\varphi, \mathbb{Z}^{2d})$ of $L^2(\mathbb{R}^d)$ cannot be invariant under all time-frequency shifts.

Proof. For any $(u, \eta) \in \mathbb{R}^d \times \mathbb{R}^d$, Lemma 3 together with 2 implies that $\pi(u, \eta)\varphi \in G(\varphi, \mathbb{Z}^{2d})$ if and only if there exists $\mathbb{Z}^{2d}$-periodic measurable function $h(x, \omega)$ satisfying $e^{2\pi i u \cdot x} Z\varphi(x-u, \omega) = h(x, \omega) Z\varphi(x, \omega)$ for a.e. $(x, \omega) \in [0, 1)^d$.

(a) ($\Rightarrow$) Assume that $E \subseteq [0, 1)^d$ is a measurable set such that $Z\varphi(x, \omega) \neq 0 \text{ a.e. on } [0, 1)^d \times E \text{ and } Z\varphi(x, \omega) = 0 \text{ a.e. on } [0, 1)^d \setminus E$. Fix any $u \in \mathbb{R}^d$. For a.e. $\omega_0 \in E$ fixed, we have $Z\varphi(-u, \omega_0) \neq 0$ a.e. and thus exists a measurable function $h(-u, \omega_0)$ such that $Z\varphi(-u, \omega_0) = h(-u, \omega_0) Z\varphi(-u, \omega_0)$ a.e. For a.e. $\omega_0 \in [0, 1)^d \times \{0\}$ fixed, we always have $Z\varphi(-u, \omega_0) = \omega_0 h(-u, \omega_0) Z\varphi(-u, \omega_0)$ a.e. so we may set $h(-u, \omega_0) = 0$. With $h(x, \omega)$ defined on $[0, 1)^d$ as above (and extended $\mathbb{Z}^{2d}$-periodically over $\mathbb{R}^d$), we have $Z\varphi(x-u, \omega) = h(x, \omega) Z\varphi(x, \omega)$ for a.e. $(x, \omega) \in [0, 1)^d \times \{0\}$.

($\Leftarrow$) Suppose to the contrary that $S \subseteq [0, 1)^d \times E_1$ is a measurable set with $0 < \mu(S) < \mu(E_1)$ such that $Z\varphi(x, \omega) \neq 0 \text{ a.e. on } S \text{ and } Z\varphi(x, \omega) = 0 \text{ a.e. on } ([0, 1)^d \times E_1 \setminus S).$ Here $E_1 \subseteq [0, 1)^d$ is a set of positive measure satisfying $0 < \mu\{x \in [0, 1)^d : Z\varphi(x, \omega) \neq 0\} < 1$ for every $\omega_0 \in E_1$. Here $\mu(\cdot)$ denotes the Lebesgue measure. Then there exist $u \in \mathbb{R}^d$ and an open set $U \subseteq [0, 1)^d \times E_1$ such that $\mu(U \cap S + (u, 0)) > \frac{1}{2} \mu(U)$ and $\mu(U \cap ([0, 1)^d \times E_1 \setminus S)) \geq \frac{1}{2} \mu(U)$). Note that the sets $S + (u, 0)$ and $([0, 1)^d \times E_1) \setminus S$ intersect on a set of Lebesgue measure at least $\mu(U)/2$ which we will denote by $W$. Since $\pi(u, 0)\varphi \in G(\varphi, \mathbb{Z}^{2d})$, there exists a $\mathbb{Z}^{2d}$-periodic measurable function $h(x, \omega)$ satisfying $Z\varphi(x-u, \omega) = h(x, \omega) Z\varphi(x, \omega)$ for a.e. $(x, \omega) \in [0, 1)^d$. However, for a.e. $(x, \omega) \in W$, we have $0 \neq Z\varphi(x-u, \omega) = h(x, \omega) Z\varphi(x, \omega) = 0$, which is a contradiction.

(b) The proof of (b) is similar to (a).

(c) The implication ($\Leftarrow$) is obvious.
(⇒) Assume that \( \mathcal{G}(\varphi, \mathbb{Z}^{2d}) \) is invariant under all time-frequency shifts. From (a) and (b), it follows that either (i) \( Z\varphi(x, \omega) = 0 \) a.e. on \([0, 1)^{2d}\) or (ii) \( Z\varphi(x, \omega) \neq 0 \) a.e. on \([0, 1)^{2d}\). The proof is complete by observing that each (i) and (ii) corresponds to \( \mathcal{G}(\varphi, \mathbb{Z}^{2d}) = \{0\} \) and \( \mathcal{G}(\varphi, \mathbb{Z}^{2d}) = L^2(\mathbb{R}^d) \), respectively (cf. [HW89, Theorem 4.3.3]).

□

Remark 12 (Shift-invariant spaces vs. Gabor spaces generated by integer lattices).

(i) There is no \( \varphi \in L^2(\mathbb{R}^d) \) such that \( \mathcal{S}(\varphi, \mathbb{Z}^d) = L^2(\mathbb{R}^d) \). Indeed, with \( \varphi \in L^2(\mathbb{R}^d) \) fixed, not every function \( \tilde{f}(\xi) \) of \( L^2(\mathbb{R}^d) \) can be expressed in the form \( \tilde{f}(\xi) = f(\xi) = m(\xi) \tilde{\varphi}(\xi) \) where \( m(\xi) \) is \( \mathbb{Z}^d \)-periodic, hence \( \mathcal{S}(\varphi, \mathbb{Z}^d) \neq L^2(\mathbb{R}^d) \).

(ii) There exists \( \varphi \in L^2(\mathbb{R}^d) \) such that \( \mathcal{G}(\varphi, \mathbb{Z}^{2d}) = L^2(\mathbb{R}^d) \). For example, \( \mathcal{G}(\chi_{[0,1]^d}, \mathbb{Z}^{2d}) = L^2(\mathbb{R}^d) \) where \( \chi_{[0,1]^d} \) is the characteristic function of \([0,1]^d\). In fact, since \( |Z\chi_{[0,1]^d}(x,\omega)| = 1 \) for all \((x,\omega) \in \mathbb{R}^d \times \mathbb{R}^d \), the Gabor system \((\chi_{[0,1]^d}, \mathbb{Z}^{2d})\) is an orthonormal basis for \( L^2(\mathbb{R}^d) \) (cf. [Gro81] Corollary 8.3.2).

(iii) There exists a nontrivial proper shift-invariant space \( \mathcal{S}(\varphi, \mathbb{Z}^d) \) of \( L^2(\mathbb{R}^d) \) which is invariant under all translations (cf. Proposition 3). For example, the shift-invariant space generated by \( \varphi(x) = \sin(\pi x)/\pi x \) is invariant under all translations. This space is also known as the Paley-Wiener space of signals bandlimited to \([-1/2, 1/2]\).

(iv) There is no nontrivial proper Gabor subspace \( \mathcal{G}(\varphi, \mathbb{Z}^{2d}) \) of \( L^2(\mathbb{R}^d) \) which is invariant under all time-frequency shifts (Proposition 11).

Example 1. We consider a Gabor space \( \mathcal{G}(\varphi, 4\mathbb{Z} \times 2\mathbb{Z}) \) where \( \varphi \in L^2(\mathbb{R}) \), which corresponds to the case \((d=1, p_1=4, p_2=2)\).

First, we pick a pair \((a, b)\) in \( \mathbb{Z}_4 \times \mathbb{Z}_2 \) and let \( \tilde{\Lambda} \subset \mathbb{R}^2 \) be the smallest closed subgroup of \( \mathbb{R}^2 \) containing \( 4\mathbb{Z} \times 2\mathbb{Z} \) and \((a, b)\). Then \( 4\mathbb{Z} \times 2\mathbb{Z} \subseteq \tilde{\Lambda} \subseteq \mathbb{Z}^2 \) so that \( \mathbb{Z}^2 \subseteq \tilde{\Lambda}^\circ \subseteq \frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z} \). For illustration of \( \tilde{\Lambda} \supseteq 4\mathbb{Z} \times 2\mathbb{Z} \), we observe \( \tilde{\Lambda} \) in the region \([0, 4) \times [0, 2) \) which is a fundamental domain of \( 4\mathbb{Z} \times 2\mathbb{Z} \); the generating element \((a, b) \in \tilde{\Lambda} \) is marked in red. Likewise, for illustration of \( \tilde{\Lambda}^\circ \supseteq \mathbb{Z}^2 \), we observe \( \tilde{\Lambda}^\circ \) in the region \([0, 1)^2 \) which is a fundamental domain of \( \mathbb{Z}^2 \); the complement of \( \tilde{\Lambda}^\circ \) in \( \frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z} \), i.e., \((\frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z}) \setminus \tilde{\Lambda}^\circ \), is marked as empty nodes. Overlapped with \( \tilde{\Lambda}^\circ \), we depict the set \( B(0) \) which is the \( \tilde{\Lambda}^\circ \)-periodic extension of \([0, \frac{1}{2}) \times [0, \frac{1}{2}) \) to \( \mathbb{R}^2 \). In all figures, the edges of \([0, 1)^2 \) are drawn in thick line.

(i) If \((a, b) = (0, 0)\), then \( \tilde{\Lambda} = 4\mathbb{Z} \times 2\mathbb{Z}, \tilde{\Lambda}^\circ = \frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z} \).

(ii) If \((a, b) = (1, 0) \) or \((3, 0)\), then \( \tilde{\Lambda} = \mathbb{Z} \times 2\mathbb{Z}, \tilde{\Lambda}^\circ = \frac{1}{2}\mathbb{Z} \times \mathbb{Z} \).

(iii) If \((a, b) = (2, 0)\), then \( \tilde{\Lambda} = 2\mathbb{Z} \times 2\mathbb{Z}, \tilde{\Lambda}^\circ = \frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z} \).

(iv) If \((a, b) = (0, 1)\), then \( \tilde{\Lambda} = 4\mathbb{Z} \times \mathbb{Z}, \tilde{\Lambda}^\circ = \mathbb{Z} \times \frac{1}{2}\mathbb{Z} \).
From Theorem 6, we have that $\mathcal{G}(\varphi, 4\mathbb{Z} \times 2\mathbb{Z})$ is $\tilde{\Lambda}$-invariant if and only if for a.e. $(x, \omega)$, $Z\varphi(x, \omega) \neq 0$ implies that

$$Z\varphi(x + \frac{5}{4}, \omega + \frac{1}{4}) = 0 \quad \text{for all } (\frac{5}{4}, \frac{1}{4}) \in (\frac{1}{4} \mathbb{Z} \times \frac{1}{4} \mathbb{Z}) \setminus \tilde{\Lambda}^\circ.$$ 

Moreover in this case, if $Z\varphi(x, \omega) \neq 0$ a.e. on $B(0) \cap [0, 1)^2$, then it follows that $Z\varphi(x, \omega) = 0$ a.e. on $[0, 1)^2 \setminus B(0)$.

Next, let $\tilde{\Lambda} \subseteq \mathbb{R}^2$ be any lattice such that $4\mathbb{Z} \times 2\mathbb{Z} \subseteq \tilde{\Lambda} \subseteq \mathbb{Z}^2$. There are two cases which are not treated above: $\tilde{\Lambda} = 2\mathbb{Z} \times \mathbb{Z}$ and $\tilde{\Lambda} = \mathbb{Z}^2$.

When $\tilde{\Lambda} = \mathbb{Z}^2$, Corollary 9 states that $\mathcal{G}(\varphi, 4\mathbb{Z} \times 2\mathbb{Z})$ is $\mathbb{Z}^2$-invariant if and only if there exists a fundamental domain $D \subset [0, 1)^2$ of the lattice $\frac{1}{4} \mathbb{Z} \times \frac{1}{4} \mathbb{Z}$ such that $Z\varphi(x, \omega) = 0$ a.e. on $[0, 1)^2 \setminus D$. For example, as depicted in (viii), if $Z\varphi(0, 1)$ is supported on $[0, \frac{1}{4}) \times [0, \frac{1}{4})$, equivalently, if $Z\varphi(x, \omega)$ is supported on $B(0)$, then $\mathcal{G}(\varphi, 4\mathbb{Z} \times 2\mathbb{Z})$ is $\mathbb{Z}^2$-invariant.

**Remark 13** (Shift-invariant spaces vs. Gabor spaces — Remarks 4 and 10 revisited).

(i) **Extra invariance of shift-invariant spaces**

Let $\varphi \in L^2(\mathbb{R}^2)$ and let $\tilde{\Gamma} \subseteq \mathbb{R}^2$ be a proper super-lattice of $\mathbb{Z}^2$, i.e., a lattice strictly containing $\mathbb{Z}^2$. Then $\mathbb{Z}^2 \subset \tilde{\Gamma} \subseteq \mathbb{Z}^d \times \frac{1}{m} \mathbb{Z}^d$ for some $m, n \in \mathbb{N}$ (cf. Section 3.2). Assume that $S(\varphi, \mathbb{Z}^2)$ is invariant under translations by $\tilde{\Gamma}$, that is, $S(\varphi, \mathbb{Z}^2) = S(\varphi, \tilde{\Gamma})$. Then every function $f \in S(\varphi, \mathbb{Z}^d)$ can be expressed in two different ways:

$$m(\xi, \omega) \hat{\varphi}(\xi, \omega) = \hat{f}(\xi, \omega) = \hat{m}(\xi, \omega) \hat{\varphi}(\xi, \omega),$$

where $m(\xi, \omega)$ is $\mathbb{Z}^d$-periodic and $\hat{m}(\xi, \omega)$ is $\tilde{\Gamma}^*$-periodic. Since the Fourier transforms $\hat{\varphi}(\xi, \omega), \hat{f}(\xi, \omega) \in L^2(\mathbb{R}^2)$ are non-periodic, there is no other periodicity involved in the equation. From the different periodicity of $m(\xi, \omega)$ and $\hat{m}(\xi, \omega)$, we get some constraints on the set of zeros of $\hat{\varphi}(\xi, \omega)$. Note that $\mathbb{Z}^2 \subset \tilde{\Gamma} \subseteq \frac{1}{m} \mathbb{Z}^d \times \frac{1}{n} \mathbb{Z}^d$ implies $m\mathbb{Z}^d \times n\mathbb{Z}^d \subseteq \tilde{\Gamma}^* \subseteq \mathbb{Z}^2$. With $m, n \in \mathbb{N}$ large, the lattice $\tilde{\Gamma}$ has a large density which corresponds to $\tilde{\Gamma}^*$ having a small density.

(ii) **Extra invariance of Gabor spaces**

Let $\varphi \in L^2(\mathbb{R}^2)$ and let $\Lambda, \tilde{\Lambda} \subseteq \mathbb{R}^2$ be lattices satisfying $\Lambda \subseteq \tilde{\Lambda} \subseteq \mathbb{Z}^2$ (so that $\mathbb{Z}^2 \subseteq \tilde{\Lambda}^\circ \subseteq \Lambda^\circ$). Assume that $\mathcal{G}(\varphi, \Lambda)$ is $\tilde{\Lambda}$-invariant, that is, $\mathcal{G}(\varphi, \Lambda) = \mathcal{G}(\varphi, \tilde{\Lambda})$. Then every function $f$ in $\mathcal{G}(\varphi, \Lambda)$ can be expressed in two different ways:

$$h(x, \omega) Z\varphi(x, \omega) = Zf(x, \omega) = \tilde{h}(x, \omega) Z\varphi(x, \omega),$$

where $m(\xi, \omega)$ is $\mathbb{Z}^d$-periodic and $\hat{m}(\xi, \omega)$ is $\tilde{\Gamma}^*$-periodic. Since the Fourier transforms $\hat{\varphi}(\xi, \omega), \hat{f}(\xi, \omega) \in L^2(\mathbb{R}^2)$ are non-periodic, there is no other periodicity involved in the equation. From the different periodicity of $m(\xi, \omega)$ and $\hat{m}(\xi, \omega)$, we get some constraints on the set of zeros of $\hat{\varphi}(\xi, \omega)$. Note that $\mathbb{Z}^2 \subset \tilde{\Gamma} \subseteq \frac{1}{m} \mathbb{Z}^d \times \frac{1}{n} \mathbb{Z}^d$ implies $m\mathbb{Z}^d \times n\mathbb{Z}^d \subseteq \tilde{\Gamma}^* \subseteq \mathbb{Z}^2$. With $m, n \in \mathbb{N}$ large, the lattice $\tilde{\Gamma}$ has a large density which corresponds to $\tilde{\Gamma}^*$ having a small density.
where \( h(x, \omega) \) is \( \Lambda^d \)-periodic and \( \tilde{h}(x, \omega) \) is \( \tilde{\Lambda}^d \)-periodic. Note that unlike the (non-periodic) Fourier transform, the Zak transform is quasi-periodic. Therefore, by replacement if necessary, \( h(x, \omega) \) and \( \tilde{h}(x, \omega) \) can be assumed to be \( Z^{2d} \)-periodic, which conforms to the condition that \( \Lambda, \tilde{\Lambda} \subseteq Z^{2d} \) (equivalently, \( \Lambda^d, \tilde{\Lambda}^d \supseteq Z^{2d} \)). This obviously limits the lattice \( \tilde{\Lambda} \) to be coarser than \( Z^{2d} \), whereas the lattice \( \Gamma \) discussed in (i) has no such restrictions and can be highly dense. Similarly as in (i), we obtain some constraints on the set of zeros of \( Z \varphi(x, \omega) \) using the different periodicity of \( h(x, \omega) \) and \( \tilde{h}(x, \omega) \).

| Space | Invariance Lattice (Dual/Adjoint Lattice) | Periodicity of Transform |
|-------|-----------------------------------------|-------------------------|
| \( \mathcal{S}(\varphi, Z^{2d}) \) | \( Z^{2d} \subseteq \tilde{\Gamma} \subseteq \frac{1}{2} Z^{d} \times \frac{1}{2} Z^{d} \) | \( \tilde{\varphi}(\xi, \omega) \) is non-periodic |
| \( \mathcal{G}(\varphi, \Lambda) \) | \( \Lambda \subseteq \tilde{\Lambda} \subseteq Z^{2d} \) | \( Z \varphi(x, \omega) \) is quasi-periodic |

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Appendix I

**Proposition A.1.** Let \( V \) be a closed subspace of \( L^2(\mathbb{R}^d) \) and \( \Lambda \subset \mathbb{R}^{2d} \) be a lattice. If \( V \) is \( \Lambda \)-invariant, then \( P(V) \) is an additive closed subgroup of \( \mathbb{R}^{2d} \) containing \( \Lambda \).

To prove Proposition A. 1, we will use similar arguments as in the proof of [ACP11] Proposition 2.1. Recall that an additive semigroup is a nonempty set with an associative additive operation. We need the following lemma which is proven for the case \( \Gamma = \mathbb{Z}^d \) in [ACP11] Lemma 2.2. [ACP11]
Lemma A.2. Let $\Gamma \subset \mathbb{R}^d$ be a lattice. If $H$ is a closed additive semigroup of $\mathbb{R}^d$ containing $\Gamma$, then $H$ is an additive group.

Proof. Let $\Pi$ be the quotient map from $\mathbb{R}^d$ onto $D = \mathbb{R}^d/\Gamma$. Here $D$ is a fundamental domain of the lattice $\Gamma$. Since $H$ is a semigroup containing $\Gamma$, we have $H + \Gamma = H$ where $H + \Gamma$ denotes the set $\{h + \gamma : h \in H, \gamma \in \Gamma\}$. Indeed, $H + \Gamma \subseteq H$ comes from the fact that $H$ is closed under addition, and $H \subseteq H + \Gamma$ is due to the fact that $0 \in \Gamma$. Therefore,
\[ \Pi^{-1}([\Pi(H)]) = \bigcup_{h \in \Pi(H)} h + \Gamma = \bigcup_{h \in H} h + \Gamma = H + \Gamma = H. \]

This implies that $\Pi(H)$ is closed in $D$ and is therefore compact.

Since a compact semigroup of $D$ is necessarily a group [HR63, Theorem 9.16], it follows that $\Pi(H) \subset D$ is a group and consequently $H$ is a group.

Proof of Proposition A.7. It is immediate from definition that $\Lambda \subseteq \mathcal{P}(V)$. To show that $\mathcal{P}(V)$ is closed, let $\{(u_n, \eta_n)\}_{n=1}^{\infty} \subset \mathcal{P}(V)$ and $(u_0, \eta_0) \in \mathbb{R}^d \times \mathbb{R}^d$ be such that $\lim_{n \to \infty}(u_n, \eta_n) = (u_0, \eta_0)$ in the usual product topology of $\mathbb{R}^d \times \mathbb{R}^d$. Then for every $f \in \mathcal{G}(\varphi, \Lambda)$, we have
\begin{align*}
\| \pi(u_n, \eta_n)f - \pi(u_0, \eta_0)f \|_2 &= \| M_{u_n} T_{u_n} f - M_{u_0} T_{u_0} f \|_2 \\
&\leq \| M_{u_n} T_{u_n} f - T_{u_n} f \|_2 + \| M_{u_0} - M_{u_0} T_{u_0} f \|_2 \\
&= \left( \int_{\mathbb{R}^d} |f(x - u_n) - f(x - u_0)|^2 dx \right)^{1/2} + \left( \int_{\mathbb{R}^d} |e^{2\pi i \eta_n \cdot x} - e^{2\pi i \eta_0 \cdot x}|^2 \cdot |f(x - u_0)|^2 dx \right)^{1/2} \\
&\to 0 \text{ as } n \to 0,
\end{align*}
which implies that $\pi(u_0, \eta_0)f \in \overline{V} = V$. Therefore, $\mathcal{P}(V)$ is closed.

Next, we show that $\mathcal{P}(V)$ is a semigroup of $\mathbb{R}^{2d}$. Let $(u, \eta), (u', \eta') \in \mathcal{P}(V)$. Then for any $f \in V$, we have $\pi(u, \eta)f \in V$ and in turn $\pi(u', \eta')[\pi(u, \eta)f] \in V$. Noting that $\pi(u + u', \eta + \eta') = e^{2\pi i \eta' \cdot u'} \pi(u', \eta') \circ \pi(u, \eta)$ (cf. (11)), we have $\pi(u + u', \eta + \eta')f = e^{2\pi i \eta' \cdot u'} \pi(u', \eta')[\pi(u, \eta)f] \in V$, therefore, $(u + u', \eta + \eta') \in \mathcal{P}(V)$. This shows that $\mathcal{P}(V)$ is closed under the additive operation given by $(u, \eta) + (u', \eta') = (u + u', \eta + \eta')$. It is easy to check that this operation is associative, thus $\mathcal{P}(V)$ is a semigroup of $\mathbb{R}^{2d}$.

Finally, since $\mathcal{P}(V)$ is a closed semigroup of $\mathbb{R}^{2d}$ containing a lattice $\Lambda$, we conclude from Lemma A.2 that $\mathcal{P}(V)$ is a group.

Appendix II - Proof of Lemma 5

To prove Lemma 5 we will use arguments similar to the proof of [BDR94, Theorem 2.14].

For $f, g \in L^2(\mathbb{R}^d)$ and a lattice $\Lambda \subset \mathbb{Z}^{2d}$, we define the $\Lambda^e$-periodic function
\[ [f, g]_{\Lambda^e}(x, \omega) := \sum_{(u, \eta) \in \Lambda^{e^1(0,1)^{2d}}} Zf(x + u, \omega + \eta) Zg(x + u, \omega + \eta). \]

It is clear that $[f, f]_{\Lambda^e}(x, \omega) \geq 0$ and by the Cauchy-Schwarz inequality, $||[f, g]_{\Lambda^e}(x, \omega)||^2 \leq [f, f]_{\Lambda^e}(x, \omega) \\
[g, g]_{\Lambda^e}(x, \omega)$.

Let $\varphi \in L^2(\mathbb{R}^d)$ and let $\Lambda \subset \mathbb{Z}^{2d}$ be a lattice. We denote by $P = P_{\mathcal{G}(\varphi, \Lambda)}$ the orthogonal projection from $L^2(\mathbb{R}^d)$ onto $\mathcal{G}(\varphi, \Lambda)$, and by $Q = P_{Z[\mathcal{G}(\varphi, \Lambda)]}$ the orthogonal projection from $L^2([0,1)^{2d})$ onto $Z[\mathcal{G}(\varphi, \Lambda)]$. Then $ZP = QZ$. Indeed, for any fixed $f \in L^2(\mathbb{R}^d)$, we have
\[ \| f - Pf \| \leq \| f - g \| \quad \text{for all } g \in \mathcal{G}(\varphi, \Lambda), \]
and since the Zak transform is unitary, this is equivalent to
\[ \| Zf - Z(Pf) \|_{L^2([0,1)^{2d})} \leq \| Zf - Zg \|_{L^2([0,1)^{2d})} \quad \text{for all } Zg \in Z[\mathcal{G}(\varphi, \Lambda)], \]
By the uniqueness of best approximation in $L^2([0,1)^{2d})$, it follows that $Z(Pf) = P_{Z[\mathcal{G}(\varphi, \Lambda)]}(Zf) = Q(Zf)$, and consequently, $ZP = QZ$. 


Proposition A.3. Let \( \varphi \in L^2(\mathbb{R}^d) \) and let \( \Lambda \subseteq \mathbb{Z}^{2d} \) be a lattice. For any \( f \in L^2(\mathbb{R}^d) \), we have \( Z(Pf)(x, \omega) = hf(x, \omega)Z\varphi(x, \omega) \), where
\[
h_f(x, \omega) := \begin{cases} 
\frac{|f|_\Lambda}{|\varphi|_\Lambda}(x, \omega)/|\varphi|_\Lambda(x, \omega) & \text{on } \supp \varphi \cap \Lambda^\circ, \\
0 & \text{otherwise}.
\end{cases}
\]

Proof. Note that \( h_f(x, \omega)Z\varphi(x, \omega) \in L^2([0,1]^{2d}) \) for any \( f \in L^2(\mathbb{R}^d) \). In fact, \( \Psi : L^2([0,1]^{2d}) \rightarrow L^2([0,1]^{2d}) \) defined by \( \Psi(Zf)(x, \omega) = h_f(x, \omega)Z\varphi(x, \omega) \), \( f \in L^2(\mathbb{R}^d) \), is a bounded linear operator with \( \| \Psi \| \leq 1 \). To see this, we compute
\[
\| h_f Z \varphi \|_{L^2([0,1]^{2d})}^2 = \iint_{[0,1]^{2d}} |h_f(x, \omega)Z\varphi(x, \omega)|^2 \, dx \, d\omega = \iint_{[0,1]^{2d}} \left| \frac{|f|_\Lambda}{|\varphi|_\Lambda}(x, \omega) \right|^2 |Z\varphi(x, \omega)|^2 \, dx \, d\omega
\]
\[
= \iint_D \left| \frac{|f|_\Lambda}{|\varphi|_\Lambda}(x, \omega) \right|^2 \sum_{(u, \eta) \in \Lambda \cap [0,1]^{2d}} |Z\varphi(x + u + \eta)|^2 \, dx \, d\omega
\]
\[
= \iint_D \sum_{(u, \eta) \in \Lambda \cap [0,1]^{2d}} |f, \varphi|_\Lambda^*(x, \omega) \chi_{\supp \varphi \cap \Lambda^\circ} (x, \omega) \, dx \, d\omega
\]
\[
\leq \iint_D |f, f|_\Lambda^*(x, \omega) \, dx \, d\omega = \iint_{[0,1]^{2d}} |Zf(x, \omega)|^2 \, dx \, d\omega = \| Zf \|_{L^2([0,1]^{2d})}^2,
\]
where we have used the fact that \( \supp Z\varphi \subseteq \supp [\varphi, \varphi]_\Lambda^\circ \). Here \( D \subset \mathbb{R}^{2d} \) is a fundamental domain of the lattice \( \Lambda^\circ \).

We will show that \( \Psi \) is the orthogonal projection from \( L^2([0,1]^{2d}) \) onto \( Z[G(\varphi, \Lambda)] \), i.e., \( \Psi = Q \). It then follows immediately that \( Z(Pf)(x, \omega) = Q(Zf)(x, \omega) = \Psi(Zf)(x, \omega) = h_f(x, \omega)Z\varphi(x, \omega) \) for any \( f \in L^2(\mathbb{R}^d) \).

First, we verify that \( \Psi = 0 \) on \( Z[G(\varphi, \Lambda)] \). Since the Zak transform is unitary, we have \( Z[G(\varphi, \Lambda)]^\perp = Z[G(\varphi, \Lambda)] \) so it is enough to check that \( h_f(x, \omega)Z\varphi(x, \omega) = 0 \) for all \( f \in G(\varphi, \Lambda) \). Using \( \Theta \), we obtain
\[
f \in G(\varphi, \Lambda) \leftrightarrow f = \pm \pi(u, \eta) \varphi \text{ for all } (u, \eta) \in \Lambda
\]
\[
\Leftrightarrow Zf \perp Z(\pi(u, \eta) \varphi) \text{ for all } (u, \eta) \in \Lambda
\]
\[
\Leftrightarrow 0 = \iint_{[0,1]^{2d}} Zf(x, \omega)Z\varphi(x, \omega) e^{-2\pi i (\eta \cdot x - u \cdot \omega)} \, dx \, d\omega \text{ for all } (u, \eta) \in \Lambda
\]
\[
\Leftrightarrow 0 = \iint_D |f, \varphi|_\Lambda^*(x, \omega) e^{-2\pi i (\eta \cdot x - u \cdot \omega)} \, dx \, d\omega \text{ for all } (u, \eta) \in \Lambda
\]
\[
\Leftrightarrow |f, \varphi|_\Lambda^*(x, \omega) = 0 \text{ a.e. on } D, \text{ (thus, a.e. on } \mathbb{R}^{2d}),
\]
where we have used the fact that \( \{ e^{-2\pi i (\eta \cdot x - u \cdot \omega)} \}_{(u, \eta) \in \Lambda} \) is a Fourier basis for \( L^2(D) \). This shows that for \( f \in G(\varphi, \Lambda) \), we have \( h_f(x, \omega) = 0 \) and therefore \( h_f(x, \omega)Z\varphi(x, \omega) = 0 \).

Next, in order to prove that \( \Psi = \text{id} \) on \( Z[G(\varphi, \Lambda)] \), it is enough to show that \( \Psi(Z(\pi(u, \eta) \varphi)) = Z(\pi(u, \eta) \varphi) \) for all \( (u, \eta) \in \Lambda \). Let us fix any \( (u, \eta) \in \Lambda \). Then for \( (x, \omega) \in \supp [\varphi, \varphi]_\Lambda^\circ \),
\[
h_{Z(\pi(u, \eta) \varphi)}(x, \omega) = \frac{|\pi(u, \eta) \varphi, \varphi|_\Lambda^*(x, \omega)}{|\varphi, \varphi|_\Lambda^*(x, \omega)}
\]
\[
= \sum_{(u', \eta') \in \Lambda \cap [0,1]^{2d}} e^{2\pi i (\eta \cdot x - u \cdot \omega)} Z\varphi(x + u, \omega + \eta) Z\varphi(x + u', \omega + \eta')
\]
\[
= e^{-2\pi i (\eta \cdot x - u \cdot \omega)} [\varphi, \varphi]_\Lambda^*(x, \omega)
\]
\[
equiv e^{-2\pi i (\eta \cdot x - u \cdot \omega)}
\]
where we have used \( \Theta \) and the fact that \( e^{2\pi i (\eta \cdot u' - u \cdot \eta')} = 1 \) for all \( (u', \eta') \in \Lambda^\circ \). Since \( \supp Z\varphi \subseteq \supp [\varphi, \varphi]_\Lambda^\circ \), we obtain
\[
h_{Z(\pi(u, \eta) \varphi)}(x, \omega) Z\varphi(x, \omega) = e^{-2\pi i (\eta \cdot x - u \cdot \omega)} Z\varphi(x, \omega) = Z(\pi(u, \eta) \varphi)(x, \omega),
\]
which means that \( \Psi(Z(\pi(u, \eta) \varphi)) = Z(\pi(u, \eta) \varphi) \). This completes the proof. \( \square \)
Proof of Lemma 3. If \( f \in \mathcal{G}(\varphi, \Lambda) \), then \( Pf = f \) so that Proposition A.3 gives \( Zf(x, \omega) = hf(x, \omega)Z\varphi(x, \omega) \), where \( hf(x, \omega) \) is \( \Lambda^\circ \)-periodic. Conversely, assume that \( f \in L^2(\mathbb{R}^d) \) is such that 
\[
Zf(x, \omega) = Z\varphi(x, \omega) \quad \text{for some} \quad \Lambda^\circ \text{-periodic function} \quad h(x, \omega).
\]
Then
\[
[f, \varphi]_{\Lambda^\circ}(x, \omega) = \sum_{(u, \eta) \in \Lambda^\circ \cap [0,1)^2} Zf(x + u, \omega + \eta)Z\varphi(x + u, \omega + \eta) = h(x, \omega) [\varphi, \varphi]_{\Lambda^\circ}(x, \omega),
\]
so that \( hf(x, \omega) = h(x, \omega) \) on the support of \([\varphi, \varphi]_{\Lambda^\circ}\). Since \( \text{supp } Z\varphi \subseteq \text{supp } [\varphi, \varphi]_{\Lambda^\circ} \), it follows that 
\[
Z(Pf)(x, \omega) = hf(x, \omega)Z\varphi(x, \omega) = h(x, \omega)Z\varphi(x, \omega) = Zf(x, \omega).
\]
Therefore, \( Pf = f \) which means that \( f \in \mathcal{G}(\varphi, \Lambda) \). This completes the proof. \( \square \)

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