Vector Hamiltonians in Nambu mechanics

V.N. Dumachev

Abstract

We give a generalization of the Nambu mechanics based on vector Hamiltonians theory. It is shown that any divergence-free phase flow in $\mathbb{R}^n$ can be represented as a generalized Nambu mechanics with $n - 1$ integral invariants. For the case when the phase flow in $\mathbb{R}^n$ has $n - 3$ or less first integrals, we introduce the Cartan concept of mechanics. As an example we give the fifth integral invariant of Euler top.

DOI: 10.3103/S1066369X18020044

Keywords: first integrals, integral invariants, splitting cohomology

Introduction

Consider phase flow of divergence-free type

$$\dot{x} = I(x). \quad (1)$$

In this paper, all objects are supposed to be smooth. We denote by $\Lambda^k(\mathbb{R}^n)$ the space of $k$-differential forms on $\mathbb{R}^n$, by $\Omega \in \Lambda^n(\mathbb{R}^n)$ a volume form, by $T^k(\mathbb{R}^n)$ the space of $k$-vector fields, by $H^k(\mathbb{R}^n)$ $k$th group of de Rham cohomologies, by $[\omega]$ the class of cohomologies of the form $\omega$. Henceforth we suppose $\dim H^{n-1}(\mathbb{R}^n) = 0$. Then $[\omega] \in H^k(\mathbb{R}^n)$ means that $\omega$ is closed, and $[\omega] = 0$ implies $\omega = d\nu$. Inner product of $X \in T(\mathbb{R}^n)$ and $\omega \in \Lambda^k(\mathbb{R}^n)$ is $X|\omega \in \Lambda^{k-1}(\mathbb{R}^n)$. Vector field $X_I$ on symplectic manifold $(\mathbb{R}^2, \Omega)$ is said to be Hamiltonian if the corresponding 1-form $\Theta = X_I|\Omega$ is closed. Due to Liouville's theorem, any Hamiltonian field preserves the form of phase volume $\Omega \in \Lambda^n(\mathbb{R}^n)$, i.e., the Lie derivative of the form $\Omega$ in vector field $X_I$ is zero:

$$L_{X_I} \Omega = X_I|d\Omega + d(X_I|\Omega) = 0.$$ 

In the classic case, $\Omega \in \Lambda^2(\mathbb{R}^2)$, i.e., $d\Omega = 0$, and provided the condition of Poincaré's lemma, from $d(X_I|\Omega) = 0$ follows $X_I|\Omega = dI \in \Lambda^1(\mathbb{R}^2)$, i.e., the function $I \in \Lambda^0(\mathbb{R}^2)$ is an invariant of dynamical system $X_I$. The phase flow (1) can be represented by means of Poisson bracket in the form

$$\dot{x} = \{I, x\}. \quad (1)$$
1 Nambu phase flows

In generalization of this case, Y. Nambu [1] supposes that \( n \)-dimensional phase flow is described by means of \( n - 1 \) invariants

\[ x = \{ I_1, I_2, \ldots, I_{n-1}, x \}. \tag{2} \]

From geometric point of view, the solution to system (2) is an integral curve \( l = l(\mathbb{R}^n) \), i.e., an object of dimension \( \dim(l) = 1 \). A curve is said to be algebraic, if it can be realized as complete intersection of \( n - 1 \) hypersurfaces, i.e., \( l = \bigcap_{k=1}^{n-1} I_k \). A hypersurface in algebraic geometry is an object with dimensionless by one than that of surrounding space, thus, any polynomial in \( \mathbb{R}^n \) forms an algebraic hypersurface [2]. If invariant \( I \) has a physical sense (e.g., energy integral), it is called Hamiltonian \( H \). For instance, for energy integral \( H = \frac{1}{2}(x_1^2 + x_2^2) \), from the preservation law

\[ \dot{H} = 0, \quad \text{i.e.,} \quad H_{x_1} \dot{x}_1 + H_{x_2} \dot{x}_2 = 0 \]

the Hamilton motion equations follow

\[ \dot{x}_1 = H_{x_2}, \quad \dot{x}_2 = -H_{x_1}, \]

which coincide with (1) in \( \mathbb{R}^2 \). Hence, invariants of Eq.(1) are hypersurfaces, whose complete intersection gives integral curve \( l \). For instance, Nambu phase flow in \( \mathbb{R}^3 \) is formed as tangent one to two invariants \( (I_1, I_2) \), i.e.,

\[ \dot{x} = l = N_1 \times N_2, \]

where \( N = \text{grad} I \). From the point of view of exterior differential algebra \( I \in \Lambda^0(\mathbb{R}^3) \), \( dI = (N \cdot dr) \in \Lambda^1(\mathbb{R}^3) \), \( \omega = dI_1 \wedge dI_2 \in \Lambda^2(\mathbb{R}^3) \). Denote \( \omega = (I, dS) \), and notice that \( \omega \) is always closed. It is straightforward that \( \omega = 1|\Omega \), and then \( d(1|\Omega) = 0 \), and hence, \( \text{div} l = 0 \). Increase in dimension leads to the following

**Theorem 1.** Nambu phase flow is divergence-free.

**Proof.** Consider differential form \( \omega = \bigwedge_{k=1}^{n-1} dI_k \). It is straightforward to see that it can be represented in the form \( \omega = (\dot{x} \cdot dS) \), where \( dS_k = (-)^k dx_1 \wedge dx_2 \wedge ... \wedge [dx_k] \wedge ... \wedge dx_n \) are Plücker coordinates of elementary hyperplatform, spanned on increment vectors \( (dx_1, \ldots, dx_n) \). In similar case further, we will say that we represent system (1) as differential form \( \omega \). Formula \( \omega = \bigwedge_{k=1}^{n-1} dI_k \) implies that \( \omega \) is closed and \( \text{div} x = 0 \). ∎

Hence due to Poincaré lemma \( \omega = dh \), \( h \in \Lambda^{n-2}(\mathbb{R}^n) \), and for Nambu phase flow

\[ dh = dI_1 \wedge ... \wedge dI_{n-1}. \]

The quantity \( h \) is called vector (tensor) Hamiltonian [3]. Further we will study system of the form

\[ dh = \bigwedge J^k, \quad \sum k = n - 1, \tag{3} \]

where \( J^k \in \Lambda^k(\mathbb{R}^n) \).

In mechanics, quantity \( I \in \Lambda^0(\mathbb{R}^n) \) is called the first integral, \( J \in \Lambda^1(\mathbb{R}^n) \) integral invariant, \( J \in \Lambda^k(\mathbb{R}^n) \) generalized integral invariant [4].
2 Splitting de Rham cohomologies

Endow de Rham complex

\[ 0 \to \Lambda^0(\mathbb{R}^n) \to \Lambda^1(\mathbb{R}^n) \to \cdots \to \Lambda^{n-1}(\mathbb{R}^n) \to \Lambda^n(\mathbb{R}^n) \to 0 \]

with differential module \( \{ C, d \} \), then

\[ Z(C) = \ker d = \{ x \in C \mid dx = 0 \} \]

are co-cycles of module \( \{ C, d \} \) (the space of closed forms),

\[ B(C) = \im d = dC = \{ x = dy \mid y \in C \} \]

are co-boundaries of module \( \{ C, d \} \) (the space of exact forms). In these notations, group of ith cohomologies of \( H^i \) is a quotient of ith co-cycles by ith co-boundaries \( H^i = Z^i/B^i \). Upper index of an arbitrary form will denote its order, i.e., \( \omega^k \in \Lambda^k(\mathbb{R}^n) \). For standard de Rham complex in \( \mathbb{R}^n \) we take \( \omega^k \in B^k \subset \Lambda^k(\mathbb{R}^n) \). Then \( \omega^k = db^{k-1} \) and

\[ H^k_k(\mathbb{R}^n) = Z^k/B^k. \]

Let us explain the origin of the lower index in \( H^k_k(\mathbb{R}^n) \). By definition,

\[ Z^k = \ker (d : \Lambda^k(\mathbb{R}^n) \to \Lambda^{k+1}(\mathbb{R}^n)) , \quad B^k = \im (d : \Lambda^{k-1}(\mathbb{R}^n) \to \Lambda^k(\mathbb{R}^n)). \]

In \( \Lambda^k(\mathbb{R}^n) \), there exists forms

\[ \omega^k = \bigwedge_{i=1}^{k} \omega^1, \quad \text{where} \quad \omega^1 \in \Lambda^1(\mathbb{R}^n) \]

such, that \( d\omega^1 = 0 \). Suppose that \( \omega^1 \in B^1 \) (i.e., \( \omega^1 = d\mu^0 \)). Then

\[ d\omega^k = 0, \quad \omega^k = \bigwedge_{i=1}^{k} \omega^0, \]

i.e., exact form \( \omega^k \in \Lambda^k(\mathbb{R}^n) \) is itself an exterior product of exact forms. We obtain the quotient space which we denote by

\[ H^k_{1,1,\ldots,1}(\mathbb{R}^n) = Z^k/B^{1,1,\ldots,1}, \]

where \( B^{1,1,\ldots,1} = \bigoplus_{i=1}^{k} \im_i (d : \Lambda^0(\mathbb{R}^n) \to \Lambda^1(\mathbb{R}^n)) \).

In similar way we get:

for \( \omega^k = d\mu^1 \wedge \bigwedge_{i=1}^{k-2} d\mu^0 \)

\[ H^k_{2,1,1,\ldots,1}(\mathbb{R}^n) = Z^k/B^{2,1,1,\ldots,1}, \]

where \( B^{2,1,1,\ldots,1} = \im (d : \Lambda^1(\mathbb{R}^n) \to \Lambda^2(\mathbb{R}^n)) \wedge \im_i (d : \Lambda^0(\mathbb{R}^n) \to \Lambda^1(\mathbb{R}^n)); \)

for \( \omega^k = d\mu^2 \wedge \bigwedge_{i=1}^{k-3} d\mu^0 \)

\[ H^k_{3,1,1,\ldots,1}(\mathbb{R}^n) = Z^k/B^{3,1,1,\ldots,1}, \]

where \( B^{3,1,1,\ldots,1} = \im (d : \Lambda^2(\mathbb{R}^n) \to \Lambda^3(\mathbb{R}^n)) \wedge \im_i (d : \Lambda^0(\mathbb{R}^n) \to \Lambda^1(\mathbb{R}^n)); \)

for \( \omega^k = d\mu^3 \wedge \bigwedge_{i=1}^{k-4} d\mu^0 \)

\[ H^k_{4,1,1,\ldots,1}(\mathbb{R}^n) = Z^k/B^{4,1,1,\ldots,1}, \]

where \( B^{4,1,1,\ldots,1} = \im (d : \Lambda^3(\mathbb{R}^n) \to \Lambda^4(\mathbb{R}^n)) \wedge \im_i (d : \Lambda^0(\mathbb{R}^n) \to \Lambda^1(\mathbb{R}^n)); \)
for \( \omega^k = d\mu^1 \wedge d\mu^2 \wedge \bigwedge_{i=1}^{k-4} d\mu_i \)

\[
H^k_{2,2,1,\ldots,1}(\mathbb{R}^n) = Z^k/B^{2,2,1,\ldots,1},
\]

where \( B^{2,2,1,\ldots,1} = \sum_{i=1}^{2} \Im_i \left( d : \Lambda^1(\mathbb{R}^n) \to \Lambda^2(\mathbb{R}^n) \right) \wedge_{i=1}^{k-4} \Im_i \left( d : \Lambda^0(\mathbb{R}^n) \to \Lambda^1(\mathbb{R}^n) \right) \);

... ...

etc.

Clearly, for any Nambu phase flow \( \omega \in \Lambda^{n-1}(\mathbb{R}^n) \), we have [\( \omega \) \( \in \) \( H^{n-1}_{1,1,\ldots,1}(\mathbb{R}^n) \). Moreover, by definition \( \dim H^{n-1}_{1,1,\ldots,1}(\mathbb{R}^n) = 0 \).

**Theorem 2.** Let \( \omega \in \Lambda^{n-1}(\mathbb{R}^n) \) be a divergence-free system. Then [\( \omega \) \( \in \) \( H^{n-1}_{1,1,\ldots,1}(\mathbb{R}^n) \).

**Proof.** Clearly, we need divergence-free condition only to have \( \omega \) being closed. Further proof is by induction.

For \( \omega^{n-1} \in \Lambda^{n-1}(\mathbb{R}^n) \), we have decomposition

\[
\omega^{n-1} = \sum_{i=1}^{n} A_i dS^n_i = \sum_{i=1}^{n-1} A_i dS_i^{n-1} \wedge \left( dx_n - \frac{A_n}{A_1} dx_1 \right) \quad \ldots \quad \omega^{n-1} = \omega^{n-2} \wedge \lambda^1,
\]

where \( \omega^{n-2} \in \Lambda^{n-2}(\mathbb{R}^{n-1}) \). Upper index of hypersurface \( S^n_i \) denotes the dimension of surround space. Descending in dimension till \( \omega^2 \in \Lambda^2(\mathbb{R}^3) \), we always have

\[
\sum_{i=1}^{3} A_i dS^3_i = (A_1 dx_2 - A_2 dx_1) \wedge \left( dx_3 - \frac{A_3}{A_1} dx_1 \right) \quad \ldots \quad \omega^2 = \lambda_1^1 \wedge \lambda_2^1.
\]

Theorem 2 provides the direct formula

\[
\sum_{i=1}^{n} A_i dS_i = A_1 \wedge_{i=2}^{n} \left( dx_i - \frac{A_i}{A_1} dx_1 \right), \quad \text{i.e.,} \quad \omega^{n-1} = \wedge_{i=1}^{n-1} \lambda_1^i.
\]

From the point of view of algebraic geometry, formula (4) can be considered as one of the invariants of real coordinate realization of of Birkhoff-Grothendieck theorem which states that on a projective curve, all holomorphic vector bundles split into direct sum of line bundles.

Further we will extensively use the analogy between (3) and (4).

**Definition.** Divergence-free phase flow

\[
\dot{x} = \{h, x\}.
\]

is called **Nambu mechanics**, if \([dh] \in H^{n-1}_{1,1,\ldots,1}(\mathbb{R}^n) \) wherein the class \([dh]\) = 0; **Poincaré mechanics**, if \([dh] \in H^{n-1}_{1,1,\ldots,1}(\mathbb{R}^n) \) (wherein the class \([dh]\) \( \neq 0 \)), or **Cartan mechanics** in remaining of the cases.
3 Non-integrable system

Non-integrability of the system of equations

\[
\begin{align*}
\dot{x} &= z^2 \\
y &= x^2 \\
\dot{z} &= y^2
\end{align*}
\]

was discussed, e.g., in [6]. This system can be rewritten in the form (5) with vector Hamiltonian

\[
h = \frac{1}{4} \begin{pmatrix} x^2 z - y^3 \\ x y^2 - z^3 \\ y z^2 - x^3 \end{pmatrix}.
\]

Due to (3), (4)

\[
\begin{align*}
\frac{dh}{dt} &= J_1 \wedge J_2 \\
J_1 &= z^2 dy - x^2 dx, \\
J_2 &= dz - y^2 dx,
\end{align*}
\]

i.e., \([dh] \in H^3_{1,1,1}(\mathbb{R}^3)\) is Poincaré mechanics.

4 Symplectic mechanics

The Hamiltonian

\[
H = \frac{1}{2} \sum_{i=1}^{2}(x_i^2 + p_i^2)
\]

generates integrable equations in \(\mathbb{R}^4\)

\[
\dot{x}_i = p_i, \quad \dot{p}_i = -x_i.
\]

From Nambu point of view, to be completely integrable, this phase flow should have three invariants

\[
\frac{dh}{dt} = J_1 \wedge J_2 \wedge J_3.
\]

The form of the first two integrals is due to exactness \(J_i = dI_i\), i.e.,

\[
I_i = \frac{1}{2}(x_i^2 + p_i^2), \quad i = 1, 2.
\]

The third invariant has the form

\[
J_3 = \frac{1}{2} \left( \frac{dx_1}{p_1} - \frac{dp_1}{x_1} - \frac{dx_2}{p_2} + \frac{dp_2}{x_2} \right).
\]

Then \(dh = dI_1 \wedge dI_2 \wedge J_3\), where

\[
h = \frac{1}{4} \left( (p_2^2 + x_2^2) dx_1 \wedge dp_1 + (x_1^2 + p_1^2) dx_2 \wedge dp_2 - (p_2 p_1 + x_1 x_2) dx_2 \wedge dp_1 \\
+ (x_1 p_2 - x_2 p_1) dp_1 \wedge dp_2 - (p_2 p_1 + x_1 x_2) dx_1 \wedge dp_2 + (x_1 p_2 - x_2 p_1) dx_1 \wedge dx_2 \right).
\]

Since \(dJ \neq 0\), then \([dh] \in H^3_{1,1,1}(\mathbb{R}^4)\) and \(\text{dim } H^3_{1,1,1}(\mathbb{R}^4) \neq 0\).

For symplectic mechanics in \(\mathbb{R}^6\), we get \(dh = \bigwedge_{i=1}^{3} dI_i \wedge J\), where

\[
J = \sum_{i>j} K_i \wedge K_j, \quad K_i = \frac{1}{2} \left( \frac{dx_i}{p_i} - \frac{dp_i}{x_i} \right), \quad i = 1, 2, 3.
\]
In general case for symplectic mechanics in \( \mathbb{R}^{2n} \), we get \( dh = \bigwedge_{i=1}^{n} dI_i \wedge J \), where
\[
J = \sum_{i} (-1)^i K_1 \wedge \ldots \wedge [K_i] \wedge \ldots \wedge K_n.
\]
Since \( dJ \neq 0 \), then \( dh \in H^{2n-1}_{n-1,1,\ldots,1}(\mathbb{R}^{2n}) \), and \( \dim H^{2n-1}_{n-1,1,\ldots,1}(\mathbb{R}^{2n}) \neq 0 \), i.e., symplectic mechanics in \( \mathbb{R}^{2n} \) is the Cartan mechanics.

5 Euler top

The motion of a solid body around a fixed point is described by the following equations in \( \mathbb{R}^6 \) [7]:
\[
\begin{align*}
\dot{x}_i &= \varepsilon_{ijk} \frac{x_j}{J_j} x_k + \varepsilon_{ijk} x_j y_k \\
\dot{y}_i &= \varepsilon_{ijk} y_j x_k, \quad i, j, k = 1, 2, 3.
\end{align*}
\]
For all \( X \) and \( j \), this system has three invariants:
\[
I_1 = y_1^2 + y_2^2 + y_3^2, \quad I_2 = x_1 y_1 + x_2 y_2 + x_3 y_3, \quad I_3 = \frac{x_1^2}{J_1} + \frac{x_2^2}{J_2} + \frac{x_3^2}{J_3} + x_1 y_1 + x_2 y_2 + x_3 y_3.
\]
In the case of \( X_1 = X_2 = X_3 = 0 \), we get another two invariants:
\[
I_4 = x_1^2 + x_2^2 + x_3^2, \quad J_5 = \frac{1}{\Delta} \left( \frac{x_1 dy_1}{J_1} + \frac{x_2 dy_2}{J_2} + \frac{x_3 dy_3}{J_3} \right),
\]
where \( \Delta = x \cdot y = \varepsilon_{ijk} y_i \frac{x_j}{J_j} x_k \). Thus
\[
dh = \bigwedge_{i=1}^{4} dI_i \wedge J_5,
\]
i.e., \( dh \in H^{5}_{1,1,1,1,1}(\mathbb{R}^{6}) \), and Euler top is Poincaré mechanics.

References

[1] Nambu Y. *Generalized Hamiltonian Dynamics*, Phys. Rev. D. 7, No. 8, pp. 5405-5412 (1973).
[2] Shafarevich I. R, Basic algebraic geometry, v. 1, Varieties in projective space, ed. Ed. 2, (Springer-Verlag, Berlin, 1994)
[3] Dumachev V. N. *Phase Flows and Vector Hamiltonians*, Russian Mathematics, Vol. 55, No. 3, pp. 1-7 (2011).
[4] Kozlov V. V. Integral Invariants After Poincaré and Cartan (URSS,Moscow, 1998) [in Russian].
[5] Dumachev V. N. *On splitting of exact differential forms*, International Journal of Pure and Applied Mathematics. Vol. 63, No. 2, pp. 213-222 (2010). arXiv:1010.2003
[6] Goriely A, Integrability and nonintegrability of dynamical systems (World Scientific, Singapore, 2001).
[7] Borisov A. V., Mamaev I. S. Rigid Body Dynamics (RKhD,Moscow, 2001) [in Russian].

Translated by P.I.Troshin