ALGEBRAIC CONSTRUCTIONS
IN THE CATEGORY OF VECTOR BUNDLES

by
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Abstract
In this paper we present the category of generalized Lie algebroids. Important results (a theorem of Maurer-Cartan type, theorems of Cartan type,...) emphasize the importance and the utility of the objects of this new category.

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1 Introduction
The motivation for our researches was to extend the notion of Lie algebroid using the extension Mod-morphism associated to a Bv-morphism. Using this general framework, we get a panoramic view over classical concepts from mathematics. [1]

We introduced the notion of interior differential system (IDS) of a generalized Lie algebroid, in general, and of a Lie algebroid, in particular. We develop the exterior differential calculus for generalized Lie algebroids and, in this general framework, we establish the structure equations of Maurer-Cartan type and we characterize the involutivity of an IDS in a theorem of Cartan type. Finally, using the classical notion of exterior differential system(see: [2,4,6,7]) (EDS) of a generalized Lie algebroid, in general, and of a Lie algebroid, in particular, we characterize the involutivity of an IDS in a theorem of Cartan type. In particular, we can obtain similar results with classical results for Lie algebroids. (see: [3,8,9])
2 Preliminaries

In general, if \( \mathcal{C} \) is a category, then we denoted by \(|\mathcal{C}|\) the class of objects and for any \( A, B \in |\mathcal{C}| \), we denote by \( \mathcal{C}(A, B) \) the set of morphisms of \( A \) source and \( B \) target.

Let \( \text{Vect}, \text{Liealg}, \text{Mod}, \text{Man}, B \) and \( \text{Bv} \) be the category of real vector spaces, Lie algebras, modules, manifolds, fiber bundles and vector bundles respectively.

We know that if \( (E, \pi, M) \in |\text{Bv}| \), then \( (\Gamma(E, \pi, M), +, \cdot) \) is a \( F(M) \)-module.

In addition, if \( (E, \pi, M) \in |\text{Bv}| \) such that \( M \) is paracompact and if \( A \subseteq M \) is closed, then for any section \( u \) over \( A \) it exists \( \tilde{u} \in \Gamma(E, \pi, M) \) such that \( \tilde{u}|_A = u \).

Note: In the following, we consider only vector bundles with paracompact base.

**Proposition 2.1** If \( (\varphi, \varphi_0) \in \text{Bv}^\ast((E, \pi, M), (E', \pi', M')) \), then it exists a \( \text{Mod}- \)morphism \( \Gamma(\varphi, \varphi_0) \) of \( \Gamma(E, \pi, M) \) source and \( \Gamma(E', \pi', M') \) target.

**Proof.** For every \( y \in \varphi_0(M) \), we fixed \( x_y \in M \) such that \( \varphi_0(x_y) = y \). Then we obtain a section \( \Gamma(\varphi, \varphi_0)u \) over the closed set \( \varphi_0(M) \) defined by

\[
\Gamma(\varphi, \varphi_0)u(y) = \varphi(u_{x_y}).
\]

As \( M' \) is paracompact, then it results that the section \( \Gamma(\varphi, \varphi_0)u \) can be regarded as a section of \( (\Gamma(E', \pi', M') +, \cdot) \).

So, we obtain a \( \text{Mod}- \)morphism

\[
\begin{array}{ccc}
\Gamma(E, \pi, M) & \xrightarrow{\Gamma(\varphi, \varphi_0)} & \Gamma(E', \pi', M') \\
u & \mapsto & \Gamma(\varphi, \varphi_0)u
\end{array}
\]

defined by

\[
\Gamma(\varphi, \varphi_0)u(y) = \varphi(u_{x_y}),
\]

for any \( y \in \varphi_0(M) \).

**q.e.d.**

**Definition 2.1** A \( \text{Mod}- \)morphism given by the previous proposition is called the extension \( \text{Mod}- \)morphism associated to the \( \text{Bv}^\ast \)-morphism \( (\varphi, \varphi_0) \).

**Remark 2.1** The construction of the extension \( \text{Mod}- \)morphism associated to a \( \text{Bv}^\ast \)-morphism \( (\varphi, \varphi_0) \) is not unique, but any two extension \( \text{Mod}- \)morphisms associated to a \( \text{Bv}^\ast \)-morphism \( (\varphi, \varphi_0) \) has the same properties.

**Example 2.1** If \( (\varphi, \varphi_0) \in \text{Bv}^\ast((E, \pi, M), (E', \pi', M')) \) such that \( \varphi_0 \in \text{Diff}(M, M') \), then we obtain the unique \( \text{Mod}- \)morphism

\[
\begin{array}{ccc}
\Gamma(E, \pi, M) & \xrightarrow{\Gamma(\varphi, \varphi_0)} & \Gamma(E', \pi', M') \\
u & \mapsto & \Gamma(\varphi, \varphi_0)u
\end{array}
\]

defined by

\[
(\Gamma(\varphi, \varphi_0)u)(x') = \varphi\left(u_{\varphi_0^{-1}(x')}\right).
\]

\( \Gamma(\varphi, \varphi_0) \) is called the \( \text{Mod}- \)morphism associated to the \( \text{Bv}^\ast \)-morphism \( (\varphi, \varphi_0) \).

3 The category of generalized Lie algebroids

We know that a Lie algebroid is a vector bundle \( (F, \nu, N) \in |\text{Bv}| \) such that there exists

\( (\rho, Id_N) \in \text{Bv}^\ast((F, \nu, N), (TN, \tau_N, N)) \).

and an operation

\[ \Gamma (F, \nu, N) \times \Gamma (F, \nu, N) \xrightarrow{[\,]_F} \Gamma (F, \nu, N) \]

\[ (u, v) \mapsto [u, v]_F \]

with the following properties:

**LA_1.** the equality holds good

\[ [u, f \cdot v]_F = f [u, v]_F + \Gamma (\rho, Id_N)(u) f \cdot v, \]

for all \( u, v \in \Gamma (F, \nu, N) \) and \( f \in F(N) \),

**LA_2.** the 4-tuple \((\Gamma (F, \nu, N), +, \cdot, [\,]_F)\) is a Lie \(F(N)\)-algebra,

**LA_3.** the \(\text{Mod}\)-morphism \(\Gamma (\rho, Id_N)\) is a Lie\(\text{Alg}\)-morphism of \((\Gamma (F, \nu, N), +, \cdot, [\,]_F)\) source and \((\Gamma (TN, \tau_N, N), +, \cdot, [\,]_{TN})\) target.

Obviously, in the definition of the Lie algebroid we use essential the \(\text{Mod}\)-morphism \(\Gamma (\rho, Id_N)\) associated to the \(\text{B}^\nu\)-morphism \((\rho, Id_N)\).

So, we are interested to finding the answer to the following question:

- **Could we to extend the notion of Lie algebroid using the extension \(\text{Mod}\)-morphism associated to a \(\text{B}^\nu\)-morphism?**

**Definition 3.1** Let \(M, N \in |\text{Man}|\) and \(h \in \text{Man}(M, N)\) be surjective. If \((F, \nu, N) \in |\text{B}^\nu|\) such that there exists

\( \langle \rho, \eta \rangle \in \text{B}^\nu ((F, \nu, N), (TM, \tau_M, M)) \)

and an operation

\[ \Gamma (F, \nu, N) \times \Gamma (F, \nu, N) \xrightarrow{[\,]_{F,h}} \Gamma (F, \nu, N) \]

\[ (u, v) \mapsto [u, v]_{F,h} \]

with the following properties:

**GLA_1.** the equality holds good

\[ [u, f \cdot v]_{F,h} = f [u, v]_{F,h} + \Gamma (Th \circ \rho, h \circ \eta)(u) f \cdot v, \]

for all \( u, v \in \Gamma (F, \nu, N) \) and \( f \in F(N) \).

**GLA_2.** the 4-tuple \( \left( \Gamma (F, \nu, N), +, \cdot, [\,]_{F,h} \right) \) is a Lie \(F(N)\)-algebra,

**GLA_3.** the \(\text{Mod}\)-morphism \(\Gamma (Th \circ \rho, h \circ \eta)\) is a Lie\(\text{Alg}\)-morphism of \( \left( \Gamma (F, \nu, N), +, \cdot, [\,]_{F,h} \right) \) source and \( \left( \Gamma (TN, \tau_N, N), +, \cdot, [\,]_{TN} \right) \) target, then we will say that the triple

\[ \langle (F, \nu, N), [\,]_{F,h}, (\rho, \eta) \rangle \]

is a generalized Lie algebroid. The couple \( \langle [\,]_{F,h}, (\rho, \eta) \rangle \) will be called generalized Lie algebroid structure.
Definition 3.2 We define the set of morphisms of 
\[ \left( (F, \nu, N), [,]_{F,h}, (\rho, \eta) \right) \]
source and 
\[ \left( (F', \nu', N'), [,]_{F',h'}, (\rho', \eta') \right) \]
target as being the set 
\[ \{ (\varphi, \varphi_0) \in B^\nu ((F, \nu, N), (F', \nu', N')) \} \]
such that the \textbf{Mod}-morphism \( \Gamma (\varphi, \varphi_0) \) is a \textbf{LieAlg}-morphism of 
\[ \left( \Gamma (F, \nu, N), +, \cdot [,]_{F,h} \right) \]
source and 
\[ \left( \Gamma (F', \nu', N'), +, \cdot [,]_{F',h'} \right) \]
target.

We remark that we can discuss about the category \textbf{GLA} of generalized Lie algebroids.

Let \( (F, \nu, N) \) be a generalized Lie algebroid.

- Locally, for any \( \alpha, \beta \in \mathbb{T}_p \), we set 
  \[ [t_\alpha, t_\beta]_F = L^\gamma_{\alpha\beta} t_\gamma. \]
  We easily obtain that 
  \[ L^\gamma_{\alpha\beta} = -L^\gamma_{\beta\alpha}, \]
  for any \( \alpha, \beta, \gamma \in \mathbb{T}_p \).

The real local functions \( \{ L^\gamma_{\alpha\beta}, \alpha, \beta, \gamma \in \mathbb{T}_p \} \) will be called the structure functions of the generalized Lie algebroid \( (F, \nu, N), [,]_{F,h}, (\rho, \eta) \).

- We assume that \((F, \nu, N)\) is a vector bundle with type fibre the real vector space \( (\mathbb{R}^p, +, \cdot) \) and structure group a Lie subgroup of \( (\text{GL}(p, \mathbb{R}), \cdot) \).

We take \( (x^i, y^i) \) as canonical local coordinates on \( (TM, \tau_M, M) \), where \( i \in \mathbb{T}_m \).

Consider \( (x^i, y^i) \to (x^i(x^i), y^i(x^i, y^i)) \) a change of coordinates on \( (TM, \tau_M, M) \).

Then the coordinates \( y^i \) change to \( y'^i \) by the rule:
\begin{align*}
y'^i = & \frac{\partial x^i}{\partial x^j} y^j. \tag{3.2}
\end{align*}

We take \( (\varphi, \varphi^\alpha) \) as canonical local coordinates on \( (F, \nu, N) \), where \( \varphi^\alpha \in \mathbb{T}_p \), \( \alpha \in \mathbb{T}_p \).

Consider \( (\varphi^i, \varphi^\alpha) \to (\varphi^i, \varphi'^\alpha) \) a change of coordinates on \( (F, \nu, N) \). Then the coordinates \( \varphi^\alpha \) change to \( \varphi'^\alpha \) by the rule:
\begin{align*}
\varphi'^\alpha = & \Lambda^\alpha_{\alpha'} \varphi^\alpha. \tag{3.3}
\end{align*}

- We assume that \( (\theta, \mu) = \text{mut} (Th \circ \rho, h \circ \eta) \). If \( z^\alpha t_\alpha \in \Gamma (F, \nu, N) \) is arbitrary, then
\begin{align*}
\Gamma (Th \circ \rho, h \circ \eta) (z^\alpha t_\alpha) f (h \circ \eta (\varphi)) = & \left( \frac{\partial}{\partial \alpha} \varphi^\alpha \frac{\partial f}{\partial \varphi^\alpha} \right) (h \circ \eta (\varphi)) = \left( \left( \frac{\partial}{\partial \alpha} \varphi^\alpha \right) (z^\alpha t_\alpha) \frac{\partial f}{\partial \varphi^\alpha} \right) (\eta (\varphi)), \tag{3.4}
\end{align*}
for any \( f \in \mathcal{F} (N) \) and \( \varphi \in N \).
The coefficients $\rho_i^\alpha$ respectively $\hat{\theta}_i^\alpha$ change to $\rho_i^\alpha$ respectively $\hat{\theta}_i^\alpha$ by the rule:

\[(3.5)\]

\[\rho_i^\alpha \to \Lambda_\alpha^\beta \rho_i^\beta \frac{\partial x^\gamma}{\partial x^i},\]

respectively

\[(3.6)\]

\[\hat{\theta}_i^\alpha \to \Lambda_\alpha^\beta \hat{\theta}_i^\beta \frac{\partial x^\gamma}{\partial x^i},\]

where $\|\Lambda_\alpha^\beta\| = \|\Lambda_\alpha^\beta\|^{-1}$.

**Remark 3.1** The following equalities hold good:

\[(3.7)\]

\[\rho_i^\alpha \circ h \frac{\partial f \circ h}{\partial x^i} = \left(\hat{\theta}_i^\alpha \frac{\partial f}{\partial x^i}\right) \circ h, \forall f \in F(N).\]

and

\[(3.8)\]

\[\left(L^\gamma_{\alpha\beta} \circ h\right) \left(\rho^\gamma_k \circ h\right) = \left(\hat{\rho}_i^\alpha \circ h\right) \frac{\partial \left(\rho^\gamma_k \circ h\right)}{\partial x^i} - \left(\rho^\gamma_k \circ h\right) \frac{\partial \left(\rho^\gamma_k \circ h\right)}{\partial x^i}.\]

In the next we build some examples of objects of the category GLA.

**Theorem 3.1** Let $M \in |\text{Man}_m|$ and $g, h \in \text{Iso}_{\text{Man}}(M)$ be. Using the tangent $B^y$-morphism $(Tg, g)$ and the operation

\[
\begin{array}{c}
\Gamma (TM, \tau_M, M) \times \Gamma (TM, \tau_M, M) \\
(u, v)
\end{array}
\quad \xrightarrow{[\cdot]_{TM,h}} \quad
\begin{array}{c}
\Gamma (TM, \tau_M, M) \\
[u, v]_{TM,h}
\end{array}
\]

where

\[\Gamma (TM, \tau_M, M) \ni (u, v) \mapsto u \in \Gamma (TM, \tau_M, M), v \in \Gamma (TM, \tau_M, M),
\]

for any $u, v \in \Gamma (TM, \tau_M, M)$, we obtain that

\[\left(\Gamma (TM, \tau_M, M), [\cdot]_{TM,h}\right) \in |\text{GLA}|.\]

For any Man-isomorphisms $g$ and $h$ we obtain new and interesting generalized Lie algebroid structures for the tangent vector bundle $(TM, \tau_M, M)$. For any base \(\{t_\alpha, \alpha \in \mathbb{1, m}\}\) of the module of sections $(\Gamma (TM, \tau_M, M), +, \cdot)$ we obtain the structure functions

\[L^\gamma_{\alpha\beta} (g, h) = \left(\theta_i^\alpha \frac{\partial \theta_i^\beta}{\partial x^\gamma} - \theta_i^\beta \frac{\partial \theta_i^\alpha}{\partial x^\gamma}\right) \tilde{\theta}_j^\gamma, \alpha, \beta, \gamma \in \mathbb{1, m}\]

where $\theta_i^\alpha$, $i, \alpha \in \mathbb{1, m}$ are real local functions such that

\[\Gamma (T(h \circ g) + \{h \circ g\}) (t_\alpha) = \theta_i^\alpha \frac{\partial}{\partial x^i} \]

and $\tilde{\theta}_j^\gamma$, $i, \gamma \in \mathbb{1, m}$ are real local functions such that

\[\Gamma \left(T(h \circ g)^{-1} + \{h \circ g\}^{-1}\right) \left(\frac{\partial}{\partial x^i}\right) = \tilde{\theta}_j^\gamma t_\gamma.\]

In particular, using arbitrary basis for the module of sections and arbitrary isometries (symmetries, translations, rotations,..) for the Euclidean 3-dimensional space $\Sigma$, 

\[\text{GLA}.\]
we obtain a lot of generalized Lie algebroid structures for the tangent vector bundle $(T\Sigma, \tau_\Sigma, \Sigma)$.

We assume that $((F, \nu, N), [\cdot]_F, (\rho, Id_N))$ is a Lie algebroid and let $h \in \text{Man}(N, N)$ be a surjective application. Let $\mathcal{AF}_F$ be a vector fibred $(n+p)$-atlas for the vector bundle $(F, \nu, N)$ and let $\mathcal{AF}_{TN}$ be a vector fibred $(n+n)$-atlas for the vector bundle $(TN, \tau_N, N)$.

If $(U, \xi_U) \in \mathcal{AF}_{TN}$ and $(V, s_V) \in \mathcal{AF}_F$ such that $U \cap h^{-1}(V) \neq \emptyset$, then we define the application

$$
\tau_N^{-1}(U \cap h^{-1}(V))) \xrightarrow{\xi_{U \cap h^{-1}(V)}} (U \cap h^{-1}(V)) \times \mathbb{R}^n \quad (\xi, u(\xi)) \mapsto (\xi, \xi_{U \cap h^{-1}(V)} u(\xi)).
$$

**Proposition 3.1** The set

$$
\mathcal{AF}_{TN} = \bigcup_{(U, \xi_U) \in \mathcal{AF}_{TN}, (V, s_V) \in \mathcal{AF}_F} \left\{ \left( U \cap h^{-1}(V), \xi_{U \cap h^{-1}(V)} \right) \right\}
$$

is a vector fibred $n+n$-atlas for the vector bundle $(TN, \tau_N, N)$.

If $X = X^i \frac{\partial}{\partial \xi^i} \in \Gamma(TN, \tau_N, N)$, then we obtain the section

$$
\tilde{X} = \tilde{X}^i \circ h \frac{\partial}{\partial \xi^i} \in \Gamma(TN, \tau_N, N),
$$

such that $\tilde{X}(\tilde{x}) = X(h(\tilde{x}))$, for any $\tilde{x} \in U \cap h^{-1}(V)$.

The set $\left\{ \frac{\partial}{\partial \xi^i}, \ i \in \{1, \ldots, n\} \right\}$ is a base for the $\mathcal{F}(N)$-module $\left( \Gamma(TN, \tau_N, N), +, \cdot \right)$.

**Theorem 3.2** If we consider the operation

$$
\Gamma(F, \nu, N) \times \Gamma(F, \nu, N) \xrightarrow{[\cdot]_{F,h}} \Gamma(F, \nu, N)
$$

defined by

$$
[t_\alpha, t_\beta]_{F,h} = \left( L^{\alpha\beta} \circ h \right) t_\gamma,
$$

$$
[t_\alpha, ft_\beta]_{F,h} = f \left( L^{\alpha\beta} \circ h \right) t_\gamma + \rho_\alpha \circ h \frac{\partial f}{\partial \xi^i} t_\beta,
$$

$$
[f t_\alpha, t_\beta]_{F,h} = -[t_\beta, f t_\alpha]_{F,h},
$$

for any $f \in \mathcal{F}(N)$, then $\left( (F, \nu, N), [\cdot]_{F,h}, (\rho, Id_N) \right) \in |\text{GLA}|$.

The generalized Lie algebroid

$$
\left( (F, \nu, N), [\cdot]_{F,h}, (\rho, Id_N) \right)
$$

given by the previous theorem, will be called the **generalized Lie algebroid associated to the Lie algebroid** $((F, \nu, N), [\cdot]_F, (\rho, Id_N))$ and to the surjective application $h \in \text{Man}(N, N)$.

In particular, if $h = Id_N$, then the generalized Lie algebroid

$$
\left( (F, \nu, N), [\cdot]_{F,Id_N}, (\rho, Id_N) \right)
$$

will be called the **generalized Lie algebroid associated to the Lie algebroid** $((F, \nu, N), [\cdot]_F, (\rho, Id_N))$. 

6
3.1 The pull-back Lie algebroid of a generalized Lie algebroid

Let \((F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta)\) be a generalized Lie algebroid.

Let \(\mathcal{AF}_F\) be a vector fibred \((n + p)\)-atlas for the vector bundle \((F, \nu, N)\) and let \(\mathcal{AF}_{TM}\) be a vector fibred \((m + p)\)-atlas for the vector bundle \((TM, \tau_M, M)\).

Let \((h^* F, h^* \nu, M)\) be the pull-back vector bundle through \(h\).

If \((U, \xi_U) \in \mathcal{AF}_{TM}\) and \((V, s_V) \in \mathcal{AF}_F\) such that \(U \cap h^{-1}(V) \neq \phi\), then we define the application

\[
\begin{align*}
  & h^* \nu^{-1}(U \cap h^{-1}(V)) \xrightarrow{\tilde{s}_{U \cap h^{-1}(V)}} (U \cap h^{-1}(V)) \times \mathbb{R}^p \\
  & (\zeta, z(h(\zeta))) \mapsto (\zeta, t_{V,h(\zeta)}^{-1} z(h(\zeta))).
\end{align*}
\]

**Proposition 3.1.1** The set

\[
\mathcal{AF}_F^\text{pull} = \bigcup_{(U, \xi_U) \in \mathcal{AF}_{TM}, (V, s_V) \in \mathcal{AF}_F, U \cap h^{-1}(V) \neq \phi} \{(U \cap h^{-1}(V)), \tilde{s}_{U \cap h^{-1}(V)}\}
\]

is a vector fibred \(m + p\)-atlas for the vector bundle \((h^* F, h^* \nu, M)\).

If \(z = z^\alpha t_\alpha \in \Gamma(F, \nu, N)\), then we obtain the section

\[Z = (z^\alpha \circ h) T_\alpha \in \Gamma(h^* F, h^* \nu, M)\]

such that \(Z(x) = z(h(x))\), for any \(x \in U \cap h^{-1}(V)\).

**Theorem 3.1.1** Let \(\left(h^* F, \rho^*, \text{Id}_M\right)\) be the \(B^*\)-morphism of \((h^* F, h^* \nu, M)\) source and \((TM, \tau_M, M)\) target, where

\[
h^* F \xrightarrow{h^*} TM
\]

\[
Z^\alpha T_\alpha (x) \mapsto (Z^\alpha : \rho^*_\alpha \circ h) \frac{\partial}{\partial x^i}(x)
\]

Using the operation

\[
\Gamma(h^* F, h^* \nu, M) \times \Gamma(h^* F, h^* \nu, M) \xrightarrow{[\cdot, \cdot]_{h^* F}} \Gamma(h^* F, h^* \nu, M)
\]

defined by

\[
[T_\alpha, T_\beta]_{h^* F} = (L^\gamma_{\alpha\beta} \circ h) T_\gamma,
\]

\[
[T_\alpha, f T_\beta]_{h^* F} = f (L^\gamma_{\alpha\beta} \circ h) T_\gamma + (\rho^*_\alpha \circ h) \frac{\partial f}{\partial x^i} T_\beta,
\]

\[
[fT_\alpha, T_\beta]_{h^* F} = -[T_\beta, fT_\alpha]_{h^* F},
\]

for any \(f \in \mathcal{F}(M)\), it results that

\[
\left((h^* F, h^* \nu, M), [\cdot, \cdot]_{h^* F}, \left(h^* F, \rho^*, \text{Id}_M\right)\right)
\]

is a Lie algebroid which is called the pull-back Lie algebroid of the generalized Lie algebroid \(\left((F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta)\right)\).
3.2 Interior Differential Systems

Let \( \left( (h^*F, h^*\nu, M), [\cdot, \cdot]_h, (h^*F, h^*\rho, Id_M) \right) \) be the pull-back Lie algebroid of the generalized Lie algebroid \( \left( (F, \nu, N), [\cdot, \cdot]_F, (\rho, \eta) \right) \).

**Definition 3.2.1** Any vector subbundle \((E, \pi, M)\) of the vector bundle \((h^*F, h^*\nu, M)\) will be called interior differential system (IDS) of the generalized Lie algebroid \( \left( (F, \nu, N), [\cdot, \cdot]_F, (\rho, \eta) \right) \).

In particular, if \( h = Id_N = \eta \), then any vector subbundle \((E, \pi, N)\) of the vector bundle \((F, \nu, N)\) will be called interior differential system of the Lie algebroid \( \left( ((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N)) \right) \).

**Remark 3.2.1** If \((E, \pi, M)\) is an IDS of the generalized Lie algebroid \( \left( (F, \nu, N), [\cdot, \cdot]_F, (\rho, \eta) \right) \), then we obtain a vector subbundle \((E^0, \pi^0, M)\) of the vector bundle \((h^*F, h^*\nu, M)\) such that
\[ \Gamma (E^0, \pi^0, M) = \left\{ \Omega \in \Gamma \left( h^*F, h^*\nu, M \right) : \Omega (S) = 0, \forall S \in \Gamma (E, \pi, M) \right\}. \]

The vector subbundle \((E^0, \pi^0, M)\) will be called the annihilator vector subbundle of the IDS \((E, \pi, M)\).

**Proposition 3.2.1** If \((E, \pi, M)\) is an IDS of the generalized Lie algebroid \( \left( (F, \nu, N), [\cdot, \cdot]_F, (\rho, \eta) \right) \), such that \( \Gamma (E, \pi, M) = \langle S_1, ..., S_r \rangle \), then it exists \( \Theta^{r+1}, ..., \Theta^p \in \Gamma \left( h^*F, h^*\nu, M \right) \) linearly independent such that \( \Gamma (E^0, \pi^0, M) = \langle \Theta^{r+1}, ..., \Theta^p \rangle \).

**Definition 3.2.2** The IDS \((E, \pi, M)\) of the generalized Lie algebroid \( \left( (F, \nu, N), [\cdot, \cdot]_F, (\rho, \eta) \right) \) will be called involutive if \([S, T]_{h^*F} \in \Gamma (E, \pi, M), \) for any \( S, T \in \Gamma (E, \pi, M) \).

**Proposition 3.2.2** If \((E, \pi, M)\) is an IDS of the generalized Lie algebroid \( \left( (F, \nu, N), [\cdot, \cdot]_F, (\rho, \eta) \right) \), and \( \{S_1, ..., S_r\} \) is a base for the \( F (M) \)-submodule \( \Gamma (E, \pi, M), +, \cdot \) then \((E, \pi, M)\) is involutive if and only if \([S_a, S_b]_{h^*F} \in \Gamma (E, \pi, M), \) for any \( a, b \in 1, r \).
4 Exterior differential calculus

Let \( (F, \nu, N), [[\cdot, \cdot]_{F,h}, (\rho, \eta) \) be a generalized Lie algebroid. We denoted by \( \Lambda^q(F, \nu, N) \) the set of differential forms of degree \( q \). We remark that if

\[
\Lambda(F, \nu, N) = \bigoplus_{q \geq 0} \Lambda^q(F, \nu, N),
\]

then we obtain the exterior differential algebra \( (\Lambda(F, \nu, N), +, \cdot, \wedge) \).

**Definition 4.1** For any \( z \in \Gamma(F, \nu, N) \), the application

\[
\Lambda(F, \nu, N) \xrightarrow{L_z} \Lambda(F, \nu, N),
\]

defined by

\[
L_z(f) = \Gamma(T h \circ \rho, h \circ \eta) z(f),
\]

for any \( f \in \mathcal{F}(N) \) and

\[
L_z(\omega)(z_1, ..., z_q) = \Gamma(T h \circ \rho, h \circ \eta) z_1(\omega((z_1, ..., z_q))) - \sum_{i=1}^q \omega\big((z_1, ..., [z, z_i]_{F,h}, ..., z_q)\big),
\]

for any \( \omega \in \Lambda^q(F, \nu, N) \) and \( z_1, ..., z_q \in \Gamma(F, \nu, N) \), is called the covariant Lie derivative with respect to the section \( z \).

**Theorem 4.1** If \( z \in \Gamma(F, \nu, N) \), \( \omega \in \Lambda^q(F, \nu, N) \) and \( \theta \in \Lambda^r(F, \nu, N) \), then

\[
L_z(\omega \wedge \theta) = L_z\omega \wedge \theta + \omega \wedge L_z\theta.
\]

**Definition 4.2** For any \( z \in \Gamma(F, \nu, N) \), the application

\[
\Lambda(F, \nu, N) \xrightarrow{i_z} \Lambda(F, \nu, N)
\]

\[
\Lambda^q(F, \nu, N) \ni \omega \mapsto i_z\omega \in \Lambda^{q-1}(F, \nu, N),
\]

defined by \( i_z f = 0 \), for any \( f \in \mathcal{F}(N) \) and

\[
i_z\omega(z_2, ..., z_q) = \omega(z, z_2, ..., z_q),
\]

for any \( z_2, ..., z_q \in \Gamma(F, \nu, N) \), is called the interior product associated to the section \( z \).

**Theorem 4.2** If \( z \in \Gamma(F, \nu, N) \), \( \omega \in \Lambda^q(F, \nu, N) \) and \( \theta \in \Lambda^r(F, \nu, N) \) we obtain

\[
i_z(\omega \wedge \theta) = i_z\omega \wedge \theta + (-1)^q \omega \wedge i_z\theta.
\]

**Theorem 4.3** For any \( z, v \in \Gamma(F, \nu, N) \) we obtain

\[
L_v \circ i_z - i_z \circ L_v = i_{[z,v]_{F,h}}.
\]

**Theorem 4.4** The application

\[
\Lambda^q(F, \nu, N) \xrightarrow{d\omega} \Lambda^{q+1}(F, \nu, N)
\]
defined by $d^F f(z) = \Gamma (Th \circ \rho, h \circ \eta)(z) f$, for any $z \in \Gamma (F, \nu, N)$, and

$$d^F \omega (z_0, z_1, \ldots, z_q) = \sum_{i=0}^{q} (-1)^i \Gamma (Th \circ \rho, h \circ \eta) z_i (\omega ((z_0, z_1, \ldots, z_i, \ldots, z_q)))$$

$$+ \sum_{i<j} (-1)^{i+j} \omega \left( ([z_i, z_j]_{F,h}, z_0, z_1, \ldots, \hat{z}_i, \ldots, \hat{z}_j, \ldots, z_q) \right),$$

for any $z_0, z_1, \ldots, z_q \in \Gamma (F, \nu, N)$, is unique with the following property:

$$(4.4)\quad L_z = d^F \circ i_z + i_z \circ d^F, \forall z \in \Gamma (F, \nu, N).$$

This application will be called the exterior differentiation operator for the exterior differential algebra of the generalized Lie algebroid $((F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta)).$

**Theorem 4.5** The exterior differentiation operator $d^F$ given by the previous theorem has the following properties:

1. For any $\omega \in \Lambda^q (F, \nu, N)$ and $\theta \in \Lambda^r (F, \nu, N)$ we obtain

$$(4.5)\quad d^F (\omega \wedge \theta) = d^F \omega \wedge \theta + (-1)^q \omega \wedge d^F \theta.$$

2. For any $z \in \Gamma (F, \nu, N)$ we obtain $L_z \circ d^F = d^F \circ L_z.$

3. $d^F \circ d^F = 0.$

**Theorem 4.6** (of Maurer-Cartan type)

If $((F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta))$ is a generalized Lie algebroid and $d^F$ is the exterior differentiation operator for the exterior differential $F(N)$-algebra $(\Lambda(F, \nu, N), +, \cdot, \wedge)$, then we obtain the structure equations of Maurer-Cartan type

$$d^F t^\alpha = -\frac{1}{2} L^\alpha_{\beta\gamma} t^\beta \wedge t^\gamma, \alpha \in \Gamma_{1,p}$$

and

$$d^F \tilde{x}^i = \theta_0^i t^\alpha, \tilde{i} \in \Gamma_{1,n},$$

where $\{t^\alpha, \alpha \in \Gamma_{1,p}\}$ is the coframe of the vector bundle $(F, \nu, N)$.

This equations will be called the structure equations of Maurer-Cartan type associated to the generalized Lie algebroid $\left( (F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta) \right).$

**Proof.** Let $\alpha \in \Gamma_{1,p}$ be arbitrary. Since

$$d^F t^\alpha (t_\beta, t_\gamma) = -L^\alpha_{\beta\gamma}, \forall \beta, \gamma \in \Gamma_{1,p}$$

it results that

$$(1)\quad d^F t^\alpha = -\sum_{\beta<\gamma} L^\alpha_{\beta\gamma} t^\beta \wedge t^\gamma.$$ 

Since $L^\alpha_{\beta\gamma} = -L^\alpha_{\gamma\beta}$ and $t^\beta \wedge t^\gamma = -t^\gamma \wedge t^\beta$, for any $\beta, \gamma \in \Gamma_{1,p}$, it results that

$$(2)\quad \sum_{\beta<\gamma} L^\alpha_{\beta\gamma} t^\beta \wedge t^\gamma = \frac{1}{2} L^\alpha_{\beta\gamma} t^\beta \wedge t^\gamma.$$
Using the equalities (1) and (2) it results the structure equation \((C_1)\).

Let \(i \in \overline{1,n}\) be arbitrarily. Since \(d^F \omega^i (t_\alpha) = \theta^i_\alpha\), for any \(\alpha \in \overline{1,p}\), it results the structure equation \((C_2)\).

**Corollary 4.1** If \(\left( (h^* F, h^* \nu, M), [\cdot, \cdot], \left( h^* \rho, \text{Id}_M \right) \right)\) is the pull-back Lie algebroid associated to the generalized Lie algebroid \(\left( (F, \nu, N), [\cdot, \cdot], (\rho, \eta) \right)\) and \(d^{h^*F}\) is the exterior differentiation operator for the exterior differential \(F(M)\)-algebra \((\Lambda (h^* F, h^* \nu, M), +, \cdot, \wedge)\), then we obtain the following structure equations of Maurer-Cartan type

\[
(C_1') \quad d^{h^* F} T^\alpha = -\frac{1}{2} (L^\alpha_\beta \circ h) T^\beta \wedge T^\gamma, \quad \alpha \in \overline{1,p}
\]

and

\[
(C_2') \quad d^{h^* F} x^i = (\rho^i_\alpha \circ h) T^\alpha, \quad i \in \overline{1,m}.
\]

This equations will be called the **structure equations of Maurer-Cartan type associated to the pull-back Lie algebroid**

\[
\left( (h^* F, h^* \nu, M), [\cdot, \cdot], \left( h^* \rho, \text{Id}_M \right) \right).
\]

**Theorem 4.7** (of Cartan type) Let \((E, \pi, M)\) be an IDS of the generalized Lie algebroid \(\left( (F, \nu, N), [\cdot, \cdot], (\rho, \eta) \right)\). If \(\\{ \Theta^{+1}, ..., \Theta^p \}\) is a base for the \(F(M)\)-submodule \((\Gamma (E^0, \pi^0, M), +, \cdot)\), then the IDS \((E, \pi, M)\) is involutive if and only if it exists

\[
\Omega^\alpha_\beta \in \Lambda^1 (h^* F, h^* \nu, M), \quad \alpha, \beta \in \overline{r+1,p}
\]

such that

\[
d^{h^* F} \Theta^\alpha = \Sigma_{\beta \in \overline{1+p}} \Omega^\alpha_\beta \wedge \Theta^\beta \in \mathcal{I} \left( \Gamma (E^0, \pi^0, M) \right).
\]

**Proof.** Let \(\{ S_1, ..., S_r \}\) be a base for the \(F(M)\)-submodule \((\Gamma (E, \pi, M), +, \cdot)\).

Let \(\{ S_{r+1}, ..., S_p \}\) be a base for the \(F(M)\)-module \((\Gamma (h^* F, h^* \nu, M), +, \cdot)\).

Let \(\Theta^1, ..., \Theta^r \in \Gamma \left( h^* F, h^* \nu, M \right)\) such that \(\\{ \Theta^1, ..., \Theta^r, \Theta^{r+1}, ..., \Theta^p \}\) is a base for the \(F(M)\)-module \((\Gamma (h^* F, h^* \nu, M), +, \cdot)\).

For any \(a, b \in \overline{1,r}\) and \(\alpha, \beta \in \overline{r+1,p}\), we have the equalities:

\[
\Theta^a (S_b) = \delta^a_b
\]
\[
\Theta^a (S_\beta) = 0
\]
\[
\Theta^{\alpha} (S_b) = 0
\]
\[
\Theta^\alpha (S_\beta) = \delta^\alpha_\beta
\]

We remark that the set of the 2-forms

\[
\left\{ \Theta^a \wedge \Theta^b, \Theta^a \wedge \Theta^\beta, \Theta^\alpha \wedge \Theta^\beta, \quad a, b \in \overline{1,r} \wedge \alpha, \beta \in \overline{r+1,p} \right\}
\]
is a base for the $\mathcal{F}(M)$-module $(\Lambda^2 (h^*F, h^*\nu, M), +, \cdot)$.

Therefore, we have

$$d^{h^*F} \Theta^\alpha = \Sigma_{b < c} A^a_{bc} \Theta^b \wedge \Theta^c + \Sigma_{b, \gamma} B^a_{b\gamma} \Theta^b \wedge \Theta^{\gamma} + \Sigma_{\beta < \gamma} C^a_{\beta \gamma} \Theta^\beta \wedge \Theta^{\gamma},$$

where, $A^a_{bc}, B^a_{b\gamma}$ and $C^a_{\beta \gamma}$, $a, b, c \in \overline{1, r}$, $\alpha, \beta, \gamma \in \overline{r+1, p}$ are real local functions such that $A^a_{bc} = -A^a_{cb}$ and $C^a_{\beta \gamma} = -C^a_{\gamma \beta}$.

Using the formula

$$(d^{h^*F} \Theta^\alpha (S_b, S_c) = \Gamma \left( h^*F , Id_M \right) S_b \left( \Theta^\alpha (S_c) \right) - \Gamma \left( h^*F , Id_M \right) S_c \left( \Theta^\alpha (S_b) \right) - \Theta^\alpha ([S_b, S_c]_{h^*F}),$$

we obtain that

$$A^a_{bc} = -\Theta^\alpha ([S_b, S_c]_{h^*F}),$$

for any $b, c \in \overline{1, r}$ and $\alpha \in \overline{r+1, p}$.

We admit that $(E, \pi, M)$ is an involutive IDS of the generalized Lie algebroid \(((F, \nu, N), [\cdot]_{F,h}, (\rho, \eta))\).

As $[S_b, S_c]_{h^*F} \in \Gamma (E, \pi, M)$, for any $b, c \in \overline{1, r}$, it results that $\Theta^\alpha ([S_b, S_c]_{h^*F}) = 0$, for any $b, c \in \overline{1, r}$ and $\alpha \in \overline{r+1, p}$. Therefore, for any $b, c \in \overline{1, r}$ and $\alpha \in \overline{r+1, p}$, we obtain $A^a_{bc} = 0$ and

$$d^{h^*F} \Theta^\alpha = \Sigma_{b, \gamma} B^a_{b\gamma} \Theta^b \wedge \Theta^{\gamma} + \frac{1}{2} C^a_{\beta \gamma} \Theta^\beta \wedge \Theta^{\gamma} = \left( B^a_{b\gamma} \Theta^b + \frac{1}{2} C^a_{\beta \gamma} \Theta^\beta \right) \wedge \Theta^{\gamma}.$$

As

$$\Omega^\alpha_{\beta} = B^a_{b\gamma} \Theta^b + \frac{1}{2} C^a_{\beta \gamma} \Theta^\beta \in \Lambda^1 (h^*F, h^*\nu, M),$$

for any $\alpha, \beta \in \overline{r+1, p}$, it results the first implication.

Conversely, we admit that it exists

$$\Omega^\alpha_{\beta} \in \Lambda^1 (h^*F, h^*\nu, M), \: \alpha, \beta \in \overline{r+1, p},$$

such that

$$d^{h^*F} \Theta^\alpha = \Sigma_{\beta \in \overline{r+1, p}} \Omega^\alpha_{\beta} \wedge \Theta^\beta,$$

for any $\alpha \in \overline{r+1, p}$.

Using the affirmations (1), (2) and (4) we obtain that $A^a_{bc} = 0$, for any $b, c \in \overline{1, r}$ and $\alpha \in \overline{r+1, p}$.

Using the affirmation (3), we obtain $\Theta^\alpha ([S_b, S_c]_{h^*F}) = 0$, for any $b, c \in \overline{1, r}$ and $\alpha \in \overline{r+1, p}$.

Therefore, we have $[S_b, S_c]_{h^*F} \in \Gamma (E, \pi, M)$, for any $b, c \in \overline{1, r}$.

Using the Proposition 3.2.2, we obtain the second implication. \(q.e.d.\)

If \(\left( (F', \nu', N'), [\cdot]_{F', h'}, (\rho', \eta') \right) \) is another generalized Lie algebroid and \((\varphi, \varphi_0)\) is a \textbf{GLA}-morphism of

$$\left( (F, \nu, N), [\cdot]_{F,h}, (\rho, \eta) \right)$$
source and
\[
\left( (F', \nu', N'), [\cdot]_{F', h'}, (\rho', \eta') \right)
\]
target, then obtain the application
\[
\Lambda^q (F', \nu', N') \xrightarrow{(\varphi, \varphi_0)^*} \Lambda^q (F, \nu, N)
\]
where
\[
((\varphi, \varphi_0)^* \omega') (z_1, \ldots, z_q) = \omega' (\Gamma (\varphi, \varphi_0) (z_1), \ldots, \Gamma (\varphi, \varphi_0) (z_q)),
\]
for any \(z_1, \ldots, z_q \in \Gamma (F, \nu, N)\).

**Theorem 4.8** If \((\varphi, \varphi_0)\) is a GLA-morphism of
\[
\left( (F, \nu, N), [\cdot]_{F, h}, (\rho, \eta) \right)
\]
target and
\[
\left( (F', \nu', N'), [\cdot]_{F', h'}, (\rho', \eta') \right)
\]
then the following affirmations are satisfied:

1. For any \(\omega' \in \Lambda^q (F', \nu', N')\) and \(\theta' \in \Lambda^r (F', \nu', N')\) we obtain
   \[
   (\varphi, \varphi_0)^* (\omega' \wedge \theta') = (\varphi, \varphi_0)^* \omega' \wedge (\varphi, \varphi_0)^* \theta'.
   \]
2. For any \(z \in \Gamma (F, \nu, N)\) and \(\omega' \in \Lambda^q (F', \nu', N')\) we obtain
   \[
   i_z ((\varphi, \varphi_0)^* \omega') = (\varphi, \varphi_0)^* (i_{\varphi(z)} \omega').
   \]
3. If \(N = N'\) and \((Th \circ \rho, h \circ \eta) = (Th' \circ \rho', h' \circ \eta') \circ (\varphi, \varphi_0)\), then we obtain
   \[
   (\varphi, \varphi_0)^* \circ dF' = dF \circ (\varphi, \varphi_0)^*.
   \]

### 4.1 Exterior Differential Systems

Let \(\left( (h^* F, h^* \nu, M), [\cdot]_{h^* F}, \left( h^* F, Id_M \right) \right)\) be the pull-back Lie algebroid of the generalized Lie algebroid \(\left( (F, \nu, N), [\cdot]_{F, h}, (\rho, \eta) \right)\).

**Definition 4.1.1** Any ideal \((I, +, \cdot)\) of the exterior differential algebra of the pullback Lie algebroid \(\left( (h^* F, h^* \nu, M), [\cdot]_{h^* F}, \left( h^* F, Id_M \right) \right)\) closed under differentiation operator \(d^{h^* F}\), namely \(d^{h^* F} I \subseteq I\), will be called differential ideal of the generalized Lie algebroid \(\left( (F, \nu, N), [\cdot]_{F, h}, (\rho, \eta) \right)\).

In particular, if \(h = Id_N = \eta\), then any ideal \((I, +, \cdot)\) of the exterior differential algebra of the Lie algebroid \(\left( (F, \nu, N), [\cdot]_F, (\rho, Id_M) \right)\) closed under differentiation operator \(d^F\), namely \(d^F I \subseteq I\), will be called differential ideal of the Lie algebroid \(\left( (F, \nu, N), [\cdot]_F, (\rho, Id_M) \right)\).

**Definition 4.1.2** Let \((I, +, \cdot)\) be a differential ideal of the generalized Lie algebroid \(\left( (F, \nu, N), [\cdot]_{F, h}, (\rho, \eta) \right)\) or of the Lie algebroid \(\left( (F, \nu, N), [\cdot]_F, (\rho, Id_M) \right)\) respectively.
If it exists an IDS \((E, \pi, M)\) such that for all \(k \in \mathbb{N}^*\) and \(\omega \in \mathcal{I} \cap \Lambda^k (h^*F, h^*\nu, M)\) we have \(\omega (u_1, ..., u_k) = 0\), for any \(u_1, ..., u_k \in \Gamma (E, \pi, M)\), then we will say that \((\mathcal{I}, +, \cdot)\) is an exterior differential system (EDS) of the generalized Lie algebroid

\[
\left( (F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta) \right).
\]

In particular, if \(h = \text{Id}_N = \eta\) and it exists an IDS \((E, \pi, M)\) such that for all \(k \in \mathbb{N}^*\) and \(\omega \in \mathcal{I} \cap \Lambda^k (F, \nu, M)\) we have \(\omega (u_1, ..., u_k) = 0\), for any \(u_1, ..., u_k \in \Gamma (E, \pi, M)\), then we will say that \((\mathcal{I}, +, \cdot)\) is an exterior differential system (EDS) of the Lie algebroid

\[
\left( (F, \nu, N), [\cdot, \cdot]_F, (\rho, \eta) \right).
\]

**Theorem 4.1.1** (of Cartan type) The IDS \((E, \pi, M)\) of the generalized Lie algebroid \(\left( (F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta) \right)\) is involutive, if and only if the ideal generated by the \(\mathcal{F}(M)\)-submodule \((\Gamma (E^0, \pi^0, M), +, \cdot)\) is an EDS of the generalized Lie algebroid

\[
\left( (F, \nu, N), [\cdot, \cdot]_F, (\rho, \eta) \right).
\]

**Proof.** Let \((E, \pi, M)\) be an involutive IDS of the generalized Lie algebroid

\[
\left( (F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta) \right).
\]

Let \(\{ \Theta^{r+1}, ..., \Theta^p \} \) be a base for the \(\mathcal{F}(M)\)-submodule \((\Gamma (E^0, \pi^0, M), +, \cdot)\).

We know that

\[
\mathcal{I} (\Gamma (E^0, \pi^0, M)) = \bigcup_{q \in \mathbb{N}} \{ \Omega_\alpha \land \Theta^\alpha, \{ \Omega_{r+1}, ..., \Omega_p \} \} \subset \Lambda^q (h^*F, h^*\nu, M).
\]

Let \(q \in \mathbb{N}\) and \(\{ \Omega_{r+1}, ..., \Omega_p \} \subset \Lambda^q (h^*F, h^*\nu, M)\) be arbitrary.

Using the Theorems 4.5 and 4.7 we obtain

\[
d^{h^*F} (\Omega_\alpha \land \Theta^\alpha) = d^{h^*F} \Omega_\alpha \land \Theta^\alpha + (-1)^{q+1} \Omega_\beta \land d^{h^*F} \Theta^\beta
\]

\[
= \left(d^{h^*F} \Omega_\alpha + (-1)^{q+1} \Omega_\beta \land \Omega^\beta_\alpha \right) \land \Theta^\alpha.
\]

As

\[
d^{h^*F} \Omega_\alpha + (-1)^{q+1} \Omega_\beta \land \Omega^\beta_\alpha \in \Lambda^{q+2} (h^*F, h^*\nu, M)
\]

it results that

\[
d^{h^*F} \left( \Omega_\beta \land \Theta^\beta \right) \in \mathcal{I} (\Gamma (E^0, \pi^0, M))
\]

Therefore,

\[
d^{h^*F} \mathcal{I} (\Gamma (E^0, \pi^0, M)) \subseteq \mathcal{I} (\Gamma (E^0, \pi^0, M)).
\]

Conversely, let \((E, \pi, M)\) be an IDS of the generalized Lie algebroid

\[
\left( (F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta) \right)
\]

such that the \(\mathcal{F}(M)\)-submodule \((\mathcal{I} (\Gamma (E^0, \pi^0, M)), +, \cdot)\) is an EDS of the generalized Lie algebroid

\[
\left( (F, \nu, N), [\cdot, \cdot]_F, (\rho, \eta) \right).
\]

Let \(\{ \Theta^{r+1}, ..., \Theta^p \} \) be a base for the \(\mathcal{F}(M)\)-submodule \((\Gamma (E^0, \pi^0, M), +, \cdot)\). As

\[
d^{h^*F} \mathcal{I} (\Gamma (E^0, \pi^0, M)) \subseteq \mathcal{I} (\Gamma (E^0, \pi^0, M))
\]

it results that it exists

\[
\Omega^\alpha_\beta \in \Lambda^1 (h^*F, h^*\nu, M), \alpha, \beta \in r + 1, p
\]

such that

\[
d^{h^*F} \Theta^\alpha = \Sigma_{\beta \in r+1, p} \Omega^\alpha_\beta \land \Theta^\beta \in \mathcal{I} (\Gamma (E^0, \pi^0, M))
\]

Using the Theorem 4.7, it results that \((E, \pi, M)\) is an involutive IDS. \(q.e.d.\)
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ALGEBRAIC CONSTRUCTIONS
IN THE CATEGORY OF VECTOR BUNDLES

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Abstract
The category of generalized Lie algebroids is presented. We obtain an exterior
differential calculus for generalized Lie algebroids. In particular, we obtain similar
results with the classical and modern results for Lie algebroids. So, a new result of
Maurer-Cartan type is presented. Supposing that any vector subbundle of the pull-
back vector bundle of a generalized Lie algebroid is called interior differential system
(IDS) for that generalized Lie algebroid, a theorem of Cartan type is obtained.
Extending the classical notion of exterior differential system (EDS) to generalized
Lie algebroids, a theorem of Cartan type is obtained. Using the theory of linear
connections of Ehresmann type presented in the paper [1], the identities of Cartan
and Bianchi type are presented.

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Keywords: vector bundle, (generalized) Lie algebroid, interior differential system,
exterior differential calculus, exterior differential system, Cartan identities, Bianchi
identities.

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1 Introduction

Using the notion of generalized Lie algebroid introduced in the paper [1] we present the category of generalized Lie algebroids. In the framework of Lie algebroids (see [2]) we know the following

**Theorem** (of Maurer-Cartan type) If \( ((F, \nu, N), [,], (\rho, \text{Id}_N)) \) is a Lie algebroid and 
\[
d^F t^\alpha = -\frac{1}{2} L^\alpha_{\beta \gamma} t^\beta \wedge t^\gamma, \; \alpha \in \Gamma, p
\]
and
\[
d^F x^i = \rho^i_\alpha t^\alpha, \; i \in \Gamma, n,
\]
where \( \{ t^\alpha, \alpha \in \Gamma, p \} \) is the coframe of the vector bundle \((F, \nu, N)\).

These equations are called the structure equations of Maurer-Cartan type associated to the Lie algebroid \((F, \nu, N), [,], (\rho, \text{Id}_N))\).

In this paper we present an exterior differential calculus for generalized Lie algebroids. In particular, we obtain similar results with the classical results for Lie algebroids. (see: [4, 8, 9]) A new result of Maurer-Cartan type is presented.

We know (see [2]) that an interior differential system (IDS) of an Lie algebroid \((F, \nu, N), [,], (\rho, \text{Id}_N))\) is a vector subbundle \((E, \pi, N)\) of the vector bundle \((F, \nu, N)\). The IDS \((E, \pi, N)\) is called involutive if \( [S, T]_F \in \Gamma (E, \pi, N) \), for any \( S, T \in \Gamma (E, \pi, N) \).

If \((E, \pi, M)\) is an IDS of the Lie algebroid 
\[
((F, \nu, N), [,], (\rho, \text{Id}_N)),
\]
then we obtain a vector subbundle \((E^0, \pi^0, N)\) of the dual vector bundle \(\left( \hat{F}, \hat{\nu}, N \right)\) so that
\[
\Gamma(E^0, \pi^0, N) \overset{\text{def}}{=} \left\{ \Omega \in \Gamma \left( \hat{F}, \hat{\nu}, N \right) : \Omega(S) = 0, \; \forall S \in \Gamma (E, \pi, N) \right\}.
\]
The vector subbundle \((E^0, \pi^0, N)\) is called the annihilator vector subbundle of the IDS \((E, \pi, N)\).

A characterisation of the involutivity of an IDS (see [2]) is presented in the following

**Theorem** (of Cartan type) Let \((E, \pi, N)\) be an IDS of the Lie algebroid 
\[
((F, \nu, N), [,], (\rho, \text{Id}_N)),
\]
then we obtain a vector subbundle \((E^0, \pi^0, N)\) of the dual vector bundle \(\left( \hat{F}, \hat{\nu}, N \right)\) so that
\[
\Omega^0_\alpha \in \Lambda^1(F, \nu, N), \; \alpha, \beta \in \Gamma, p
\]
so that
\[
d^F \Theta^\alpha = \sum_{\beta \in \Gamma, p} \Omega^0_\beta \wedge \Theta^\beta \in \Gamma \left( E^0, \pi^0, N \right).
\]
In this paper we extend the notion of IDS for generalized Lie algebroids and we characterized the involutivity of an IDS in a new theorem of Cartan type.

The classical notion of exterior differential system (EDS) was studied in many papers. (see: [3, 5, 6, 7]) A new point of view in the framework of Lie algebroids is presented in the paper [2].

Any ideal \((\mathcal{I}, +, \cdot)\) of the exterior differential algebra of the Lie algebroid
\[
((F, \nu, N), [\cdot]_F, (\rho, Id_M))
\]
closed under differentiation operator \(d^F\), namely \(d^F \mathcal{I} \subseteq \mathcal{I}\), is called differential ideal. If \((\mathcal{I}, +, \cdot)\) is a differential ideal of the Lie algebroid \(((F, \nu, N), [\cdot]_F, (\rho, Id_N))\) so that it exists an IDS \((E, \pi, N)\) so that for all \(k \in \mathbb{N}^*\) and \(\omega \in \mathcal{I} \cap \Lambda^k (F, \nu, N)\) we have \(\omega(u_1, \ldots, u_k) = 0\), for any \(u_1, \ldots, u_k \in \Gamma (E, \pi, N)\), then we say that \((\mathcal{I}, +, \cdot)\) is an exterior differential system (EDS) of the Lie algebroid \(((F, \nu, N), [\cdot]_F, (\rho, Id_N))\).

In the paper [2] is presented the following

**Theorem** (of Cartan type) The IDS \((E, \pi, N)\) of the Lie algebroid
\[
((F, \nu, N), [\cdot]_F, (\rho, Id_N))
\]
is involutive, if and only if the ideal generated by the \(\mathcal{F} (N)\)-submodule \((\Gamma (E^0, \pi^0, N), +, \cdot)\) is an EDS of the Lie algebroid \(((F, \nu, N), [\cdot]_F, (\rho, Id_N))\).

In this paper we extend the notion of EDS to generalized Lie algebroids. The involutivity of an IDS in a theorem of Cartan type is characterized. Finally, using the theory of linear connections of Ehresmann type presented in the paper [1], the identities of Cartan and Bianchi type emphasize the utility of the exterior differential calculus for generalized Lie algebroids.

### 2 The category of generalized Lie algebroids

In general, if \(\mathcal{C}\) is a category, then we denote \(|\mathcal{C}|\) the class of objects and for any \(A, B \in |\mathcal{C}|\), we denote \(\mathcal{C} (A, B)\) the set of morphisms of \(A\) source and \(B\) target. Let **Vect**, **Liealig**, **Mod**, **Man** and \(\mathcal{B}^Y\) be the category of real vector spaces, Lie algebras, modules, manifolds and vector bundles respectively.

We know that if \((E, \pi, M) \in \mathcal{B}^Y\), \(\Gamma (E, \pi, M) = \{ u \in \text{Man} (M, E) : u \circ \pi = Id_M \}\) and \(\mathcal{F} (M) = \text{Man} (M, \mathbb{R})\), then \((\Gamma (E, \pi, M), +, \cdot)\) is a \(\mathcal{F} (M)\)-module. If \((\varphi, \varphi_0) \in \mathcal{B}^Y ((E, \pi, M), (E', \pi', M'))\) such that \(\varphi_0 \in \text{Iso}_{\text{Man}} (M, M')\), then, using the operation
\[
\mathcal{F} (M) \times \Gamma (E', \pi', M') \longrightarrow \Gamma (E', \pi', M') \quad (f, u') \longmapsto f \circ \varphi_0^{-1} \cdot u'
\]
it results that \((\Gamma (E', \pi', M'), +, \cdot)\) is a \(\mathcal{F} (M)\)-module and we obtain the **Mod**-morphism
\[
\Gamma (E, \pi, M) \quad u \quad \begin{array}{c}
\Gamma (\varphi, \varphi_0) \\
\downarrow \end{array} \Gamma (E', \pi', M') \quad \Gamma (\varphi, \varphi_0) \quad u
\]
defined by
\[
\Gamma (\varphi, \varphi_0) \quad u \quad (y) = \varphi \left( u_{\varphi_0^{-1} (y)} \right),
\]
for any \(y \in M'\).
We know that a Lie algebroid is a vector bundle \((F, \nu, N) \in \mathcal{B}^\nu\) such that there exists
\[
(\rho, \text{Id}_N) \in \mathcal{B}^\nu((F, \nu, N), (TN, \tau_N, N))
\]
and an operation
\[
\Gamma (F, \nu, N) \times \Gamma (F, \nu, N) \xrightarrow{[\cdot, \cdot]} \Gamma (F, \nu, N)
\]
\[
(u, v) \mapsto [u, v]_F
\]
with the following properties:

**LA1.** the equality holds good
\[
[u, f \cdot v]_F = f [u, v]_F + \Gamma (\rho, \text{Id}_N) (u) f \cdot v,
\]
for all \(u, v \in \Gamma (F, \nu, N)\) and \(f \in \mathcal{F}(N)\).

**LA2.** the 4-tuple \((\Gamma (F, \nu, N), +, \cdot, [\cdot, \cdot]_F)\) is a Lie \(\mathcal{F}(N)\)-algebra,

**LA3.** the \textbf{Mod}-morphism \(\Gamma (\rho, \text{Id}_N)\) is a \textbf{LieAlg}-morphism of \((\Gamma (F, \nu, N), +, \cdot, [\cdot, \cdot]_F)\) source and \((\Gamma (TN, \tau_N, N), +, \cdot, [\cdot, \cdot]_{TN})\) target.

**Definition 2.1** Let \(M, N \in \mathcal{Man}\), \(h \in \text{Iso}_{\mathcal{Man}}(M, N)\) and \(\eta \in \text{Iso}_{\mathcal{Man}}(N, M)\). If \((F, \nu, N) \in \mathcal{B}^\nu\) so that there exists
\[
(\rho, \eta) \in \mathcal{B}^\nu((F, \nu, N), (TM, \tau_M, M))
\]
and an operation
\[
\Gamma (F, \nu, N) \times \Gamma (F, \nu, N) \xrightarrow{[\cdot, \cdot]} \Gamma (F, \nu, N)
\]
\[
(u, v) \mapsto [u, v]_{F, h}
\]
with the following properties:

**GLA1.** the equality holds good
\[
[u, f \cdot v]_{F, h} = f [u, v]_{F, h} + \Gamma (Th \circ \rho, h \circ \eta) (u) f \cdot v,
\]
for all \(u, v \in \Gamma (F, \nu, N)\) and \(f \in \mathcal{F}(N)\).

**GLA2.** the 4-tuple \(\left(\Gamma (F, \nu, N), +, \cdot, [\cdot, \cdot]_{F, h}\right)\) is a Lie \(\mathcal{F}(N)\)-algebra,

**GLA3.** the \textbf{Mod}-morphism \(\Gamma (Th \circ \rho, h \circ \eta)\) is a \textbf{LieAlg}-morphism of
\[
\left(\Gamma (F, \nu, N), +, \cdot, [\cdot, \cdot]_{F, h}\right)
\]
source and
\[
\left(\Gamma (TN, \tau_N, N), +, \cdot, [\cdot, \cdot]_{TN}\right)
\]
target, then we will say that \textit{the triple} \(\left((F, \nu, N), [\cdot]_{F, h}, (\rho, \eta)\right)\) is a \textit{generalized Lie algebroid}. The couple \(\left([\cdot]_{F, h}, (\rho, \eta)\right)\) will be called \textit{generalized Lie algebroid structure}. 

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Definition 2.2 We define the set of morphisms of 

\[(F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta)\]

source and 

\[(F', \nu', N'), [\cdot, \cdot]_{F',h'}, (\rho', \eta')\]

target as being the set

\[\{(\varphi, \varphi_0) \in B^V((F, \nu, N), (F', \nu', N'))\}\]

such that \(\varphi_0 \in Iso_{\text{Man}}(N, N')\) and the \(\text{Mod}\)-morphism \(\Gamma (\varphi, \varphi_0)\) is a \(\text{LieAlg}\)-morphism of 

\[\left(\Gamma (F, \nu, N), +, \cdot, [\cdot, \cdot]_{F,h}\right)\]

source and 

\[\left(\Gamma (F', \nu', N'), +, \cdot, [\cdot, \cdot]_{F',h'}\right)\]

target.

So, we can discuss about the category \(\text{GLA}\) of generalized Lie algebroids. Examples of objects of this category are presented in the paper [1]. We remark that \(\text{GLA}\) is a subcategory of the category \(B^v\).

Let \(\left(\left(F, \nu, N\right), [\cdot, \cdot]_{F,h}, (\rho, \eta)\right)\) be an arbitrary object of the category \(\text{GLA}\).

• Locally, for any \(\alpha, \beta \in \overline{1, p}\), we set 

\[L^\gamma_{\alpha\beta} = -L^\gamma_{\beta\alpha}\]

for any \(\alpha, \beta, \gamma \in \overline{1, p}\). The real local functions \(L^\gamma_{\alpha\beta}\), \(\alpha, \beta, \gamma \in \overline{1, p}\) will be called the structure functions of the generalized Lie algebroid \(\left(\left(F, \nu, N\right), [\cdot, \cdot]_{F,h}, (\rho, \eta)\right)\).

• We assume the following diagrams:

\[
\begin{array}{cccc}
F & \xrightarrow{\rho} & TM & \xrightarrow{T_h} & TN \\
\downarrow \nu & & \downarrow \tau_M & & \downarrow \tau_N \\
N & \xrightarrow{\eta} & M & \xrightarrow{h} & N \\
(\chi^i, z^\alpha) & & (x^i, y^j) & & (\chi^i, z^\tilde{i})
\end{array}
\]

where \(i, \tilde{i} \in \overline{1, m}\) and \(\alpha \in \overline{1, p}\).

If

\[
(\chi^i, z^\alpha) \longrightarrow (\chi^i \nu (\chi^\tilde{i}), z^\alpha' (\chi^i, z^\alpha)),
\]

\[
(x^i, y^j) \longrightarrow (x^i \nu (x^\tilde{i}), y^j (x^i, y^j))
\]

and

\[
(\chi^i, z^\tilde{i}) \longrightarrow (\chi^i \nu (\chi^\tilde{i}), z^\nu (\chi^i, z^\tilde{i})),
\]

then

\[
z^{\alpha'} = A_{\alpha}^{\beta} z^\alpha,
\]

\[
y^i = \frac{\partial x^i}{\partial z^\alpha} y^j
\]

and

\[
z^{\nu} = \frac{\partial x^\nu}{\partial z^\alpha} z^\tilde{i}.
\]
We assume that \((\theta, \mu) = (Th \circ \rho, h \circ \eta)\). If \(z^\alpha t_\alpha \in \Gamma (F, \nu, N)\) is arbitrary, then

\[
\Gamma (Th \circ \rho, h \circ \eta) (z^\alpha t_\alpha) f (h \circ \eta (z)) = \left( \theta_\alpha^i z^\alpha \frac{\partial f}{\partial x^i} \right) (h \circ \eta (z)) = \left( (\rho_\alpha^j \circ h) (z^\alpha \circ h) \frac{\partial f \circ h}{\partial x^j} \right) (\eta (z)),
\]

for any \(f \in \mathcal{F} (N)\) and \(z \in \mathcal{N}\).

The coefficients \(\rho_\alpha^i\) respectively \(\theta_\alpha^i\) change to \(\rho_\alpha^i\) respectively \(\theta_\alpha^i\) according to the rule:

\[
\rho_\alpha^i = \Lambda_\alpha^\alpha \rho_\alpha^i \frac{\partial x^i}{\partial x^j},
\]

respectively

\[
\theta_\alpha^i = \Lambda_\alpha^\alpha \theta_\alpha^i \frac{\partial x^i}{\partial x^j},
\]

where

\[
\|\Lambda_\alpha^\alpha\| = \|\Lambda_\alpha^\alpha\|^{-1}.
\]

Remark 2.1 The following equalities hold good:

\[
\rho_\alpha^i \circ h \frac{\partial f \circ h}{\partial x^i} = \left( \theta_\alpha^i \frac{\partial f}{\partial \nu^i} \right) \circ h, \forall f \in \mathcal{F} (N).
\]

and

\[
(L_{\alpha \beta}^\gamma \circ h) \left( \rho_\alpha^k \circ h \right) = (\rho_\alpha^i \circ h) \frac{\partial (\rho_\alpha^k \circ h)}{\partial x^i} - (\rho_\alpha^j \circ h) \frac{\partial (\rho_\alpha^k \circ h)}{\partial x^j}.
\]

3 Interior Differential Systems

Let \(\left( (F, \nu, N), [\cdot, \cdot]_{F; h}, (\rho, \eta) \right)\) be an object of the category \(\text{GLA}\).

Let \(\mathcal{AF}_F\) be a vector fibred \((n + p)\)-atlas for the vector bundle \((F, \nu, N)\) and let \(\mathcal{AF}_{TM}\) be a vector fibred \((m + m)\)-atlas for the vector bundle \((TM, \tau_M, M)\).

Let \((h^*F, h^*\nu, M)\) be the pull-back vector bundle through \(h\).

If \((U, \xi_U) \in \mathcal{AF}_{TM}\) and \((V, s_V) \in \mathcal{AF}_F\) such that \(U \cap h^{-1} (V) \neq \phi\), then we define the application

\[
h^*\nu^{-1} (U \cap h^{-1} (V)) \xrightarrow{\bar{s}_{U \cap h^{-1} (V)}} (U \cap h^{-1} (V)) \times \mathbb{R}^p \rightarrow (\mathcal{X}, \bar{z} \circ h (\mathcal{X})),
\]

Proposition 3.1 The set

\[
\mathcal{AF} = \bigcup_{(U, \xi_U) \in \mathcal{AF}_{TM}, (V, s_V) \in \mathcal{AF}_F} \left\{ (U \cap h^{-1} (V), \bar{s}_{U \cap h^{-1} (V)}) \right\}
\]

is a vector fibred \(m + p\)-atlas for the vector bundle \((h^*F, h^*\nu, M)\).
If $z = z^\alpha t_\alpha \in \Gamma (F, \nu, N)$, then we obtain the section

$$Z = (z^\alpha \circ h) T_\alpha \in \Gamma (h^* F, h^* \nu, M)$$

such that $Z (x) = z (h (x))$, for any $x \in U \cap h^{-1} (V)$.

**Theorem 3.1** Let $\left( h^* F, Id_M \right)$ be the $BV$-morphism of $(h^* F, h^* \nu, M)$ source and $(TM, \tau_M, M)$ target, where

$$h^* F \xrightarrow{[\cdot, h^* F]} TM$$

$$Z^\alpha T_\alpha (x) \mapsto (Z^\alpha \cdot \rho^i_\alpha \circ h) \frac{\partial}{\partial x^i} (x)$$

Using the operation

$$\Gamma (h^* F, h^* \nu, M) \times \Gamma (h^* F, h^* \nu, M) \xrightarrow{[\cdot, h^* F]} \Gamma (h^* F, h^* \nu, M)$$

defined by

$$[T_\alpha, T_\beta]_{h^* F} = (L^\gamma_{\alpha\beta} \circ h) T_\gamma,$$

$$[T_\alpha, fT_\beta]_{h^* F} = f (L^\gamma_{\alpha\beta} \circ h) T_\gamma + (\rho^i_\alpha \circ h) \frac{\partial f}{\partial x^i} T_\beta,$$

$$[fT_\alpha, T_\beta]_{h^* F} = - [T_\beta, fT_\alpha]_{h^* F},$$

for any $f \in \mathcal{F} (M)$, it results that

$$\left( (h^* F, h^* \nu, M), [\cdot, h^* F], (\rho, \eta) \right)$$

is a Lie algebroid which is called the pull-back Lie algebroid of the generalized Lie algebroid $\left( (F, \nu, N), [\cdot, F, h], (\rho, \eta) \right)$.

**Definition 3.1** Any vector subbundle $(E, \pi, M)$ of the pull-back vector bundle $(h^* F, h^* \nu, M)$ will be called interior differential system (IDS) of the generalized Lie algebroid $\left( (F, \nu, N), [\cdot, F, h], (\rho, \eta) \right)$.

In particular, if $h = Id_N = \eta$, then we obtain the definition of the IDS of a Lie algebroid. (see [2])

**Remark 3.1** If $(E, \pi, M)$ is an IDS of the generalized Lie algebroid

$$\left( (F, \nu, N), [\cdot, F, h], (\rho, \eta) \right),$$

then we obtain a vector subbundle $(E^0, \pi^0, M)$ of the vector bundle $\left( h^* F, h^* \nu, M \right)$ such that

$$\Gamma (E^0, \pi^0, M)_{\text{put}} \left\{ \Omega \in \Gamma \left( h^* F, h^* \nu, M \right) : \Omega (S) = 0, \forall S \in \Gamma (E, \pi, M) \right\}.$$

The vector subbundle $(E^0, \pi^0, M)$ will be called the annihilator vector subbundle of the IDS $(E, \pi, M)$. 

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Proposition 3.1 If \((E, \pi, M)\) is an IDS of the generalized Lie algebroid
\[
\left( (F, \nu, N), [\cdot, \cdot]_{F, h}, (\rho, \eta) \right)
\]
such that \(\Gamma (E, \pi, M) = \{S_1, \ldots, S_r\}\), then it exists \(\Theta^{r+1}, \ldots, \Theta^{p} \in \Gamma \left(h^*F, h^*\nu, M\right)\)
linearly independent such that \(\Gamma (E^0, \pi^0, M) = \langle \Theta^{r+1}, \ldots, \Theta^{p} \rangle\).

Definition 3.2 The IDS \((E, \pi, M)\) of the generalized Lie algebroid
\[
\left( (F, \nu, N), [\cdot, \cdot]_{F, h}, (\rho, \eta) \right)
\]
will be called involutive if \([S, T]_{h^*F} \in \Gamma (E, \pi, M)\), for any \(S, T \in \Gamma (E, \pi, M)\).

Proposition 3.2 If \((E, \pi, M)\) is an IDS of the generalized Lie algebroid
\[
\left( (F, \nu, N), [\cdot, \cdot]_{F, h}, (\rho, \eta) \right)
\]
and \(\{S_1, \ldots, S_r\}\) is a base for the \(\mathcal{F}(M)\)-submodule \((\Gamma (E, \pi, M), +, \cdot)\) then \((E, \pi, M)\)
is involutive if and only if \([S_a, S_b]_{h^*F} \in \Gamma (E, \pi, M)\), for any \(a, b \in \overline{1, r}\).

4 Exterior differential calculus

We propose an exterior differential calculus in the general framework of generalized Lie algebroids. As any Lie algebroid can be regarded as a generalized Lie algebroid, in particular, we obtain a new point of view over the exterior differential calculus for Lie algebroids. Let \(\left( (F, \nu, N), [\cdot, \cdot]_{F, h}, (\rho, \eta) \right) \in |\text{GLA}|\) be.

Definition 4.1 For any \(q \in \mathbb{N}\) we denote by \((\Sigma_q, \circ)\) the permutations group of the set \(\{1, 2, \ldots, q\}\).

Definition 4.2 We denoted by \(\Lambda^q (F, \nu, N)\) the set of \(q\)-linear applications
\[
\Gamma (F, \nu, N)^q \stackrel{\omega}{\longrightarrow} \mathcal{F}(N)
\]
\[
(z_1, \ldots, z_q) \longmapsto \omega (z_1, \ldots, z_q)
\]
such that
\[
\omega (z_{\sigma(1)}, \ldots, z_{\sigma(q)}) = \text{sgn} (\sigma) \cdot \omega (z_1, \ldots, z_q)
\]
for any \(z_1, \ldots, z_q \in \Gamma (F, \nu, N)\) and for any \(\sigma \in \Sigma_q\).

The elements of \(\Lambda^q (F, \nu, N)\) will be called differential forms of degree \(q\) or differential \(q\)-forms.

Remark 4.1 If \(\omega \in \Lambda^q (F, \nu, N)\), then \(\omega (z_1, \ldots, z, \ldots, z, \ldots, z_q) = 0\). Therefore, if \(\omega \in \Lambda^q (F, \nu, N)\), then
\[
\omega (z_1, \ldots, z_i, \ldots, z_j, \ldots, z_{\ldots}, z_q) = -\omega (z_1, \ldots, z_j, \ldots, z_i, \ldots, z_q).
\]

Theorem 4.1 If \(q \in \mathbb{N}\), then \((\Lambda^q (F, \nu, N), +, \cdot)\) is a \(\mathcal{F}(N)\)-module.

Definition 4.3 If \(\omega \in \Lambda^q (F, \nu, N)\) and \(\theta \in \Lambda^r (F, \nu, N)\), then the \((q + r)\)-form \(\omega \wedge \theta\) defined by
\[
\omega \wedge \theta (z_1, \ldots, z_{q+r}) = \sum_{\sigma(1) \prec \cdots \prec \sigma(q)} \text{sgn} (\sigma) \cdot \omega (z_{\sigma(1)}, \ldots, z_{\sigma(q)}) \theta (z_{\sigma(q+1)}, \ldots, z_{\sigma(q+r)})
\]
\[
= \frac{1}{q! r!} \sum_{\sigma \in \Sigma_{q+r}} \text{sgn} (\sigma) \cdot \omega (z_{\sigma(1)}, \ldots, z_{\sigma(q)}) \theta (z_{\sigma(q+1)}, \ldots, z_{\sigma(q+r)}),
\]
for any $z_1, ..., z_{q+r} \in \Gamma (F, \nu, N)$, will be called the exterior product of the forms $\omega$ and $\theta$.

Using the previous definition, we obtain

**Theorem 4.2** The following affirmations hold good:

1. If $\omega \in \Lambda^q (F, \nu, N)$ and $\theta \in \Lambda^r (F, \nu, N)$, then

   \[
   \omega \wedge \theta = (-1)^{qr} \theta \wedge \omega.
   \]

2. For any $\omega \in \Lambda^q (F, \nu, N)$, $\theta \in \Lambda^r (F, \nu, N)$ and $\eta \in \Lambda^s (F, \nu, N)$ we obtain

   \[
   (\omega \wedge \theta) \wedge \eta = \omega \wedge (\theta \wedge \eta).
   \]

3. For any $\omega, \theta \in \Lambda^q (F, \nu, N)$ and $\eta \in \Lambda^s (F, \nu, N)$ we obtain

   \[
   (\omega + \theta) \wedge \eta = \omega \wedge \eta + \theta \wedge \eta.
   \]

4. For any $\omega \in \Lambda^q (F, \nu, N)$ and $\theta, \eta \in \Lambda^s (F, \nu, N)$ we obtain

   \[
   \omega \wedge (\theta + \eta) = \omega \wedge \theta + \omega \wedge \eta.
   \]

5. For any $f \in \mathcal{F} (N)$, $\omega \in \Lambda^q (F, \nu, N)$ and $\theta \in \Lambda^s (F, \nu, N)$ we obtain

   \[
   (f \cdot \omega) \wedge \theta = f \cdot (\omega \wedge \theta) = \omega \wedge (f \cdot \theta).
   \]

**Theorem 4.3** If

\[
\Lambda (F, \nu, N) = \bigoplus_{q \geq 0} \Lambda^q (F, \nu, N),
\]

then

\[
(\Lambda (F, \nu, N), +, \cdot, \wedge)
\]

is a $\mathcal{F} (N)$-algebra. This algebra will be called the exterior differential algebra of the vector bundle $(F, \nu, N)$.

**Remark 4.2** If $\{t^\alpha, \alpha \in \mathbb{T}_p\}$ is the coframe associated to the frame $\{t_\alpha, \alpha \in \mathbb{T}_p\}$ of the vector bundle $(F, \nu, N)$ in the vector local $(n+p)$-chart $U$, then

\[
t^{\alpha_1} \wedge ... \wedge t^{\alpha_q} (z_1^{\alpha_1} t_{\alpha_1}, ..., z_q^{\alpha_q} t_{\alpha_q}) = \frac{1}{q!} \det \begin{vmatrix}
  z_1^{\alpha_1} & \cdots & z_1^{\alpha_q} \\
  \cdots & \cdots & \cdots \\
  z_q^{\alpha_1} & \cdots & z_q^{\alpha_q}
\end{vmatrix},
\]

for any $q \in \mathbb{T}_p$.

**Remark 4.3** If $\{t^\alpha, \alpha \in \mathbb{T}_p\}$ is the coframe associated to the frame $\{t_\alpha, \alpha \in \mathbb{T}_p\}$ of the vector bundle $(F, \nu, N)$ in the vector local $(n+p)$-chart $U$, then, for any $q \in \mathbb{T}_p$ we define $C^q_{ext}$ exterior differential forms of the type

\[
t^{\alpha_1} \wedge ... \wedge t^{\alpha_q}
\]

such that $1 \leq \alpha_1 < ... < \alpha_q \leq p$.

The set

\[
\{t^{\alpha_1} \wedge ... \wedge t^{\alpha_q}, 1 \leq \alpha_1 < ... < \alpha_q \leq p\}
\]

is a base for the $\mathcal{F} (N)$-module

\[
(\Lambda^q (F, \nu, N), +, \cdot).
\]
Therefore, if $\omega \in \Lambda^q (F, \nu, N)$, then

$$\omega = \omega_{\alpha_1 \ldots \alpha_q} t^{\alpha_1} \wedge \ldots \wedge t^{\alpha_q}.$$ 

In particular, if $\omega$ is an exterior differential $p$-form $\omega$, then we can written

$$\omega = a \cdot t^1 \wedge \ldots \wedge t^p,$$

where $a \in \mathcal{F} (N)$.

**Definition 4.4** If

$$\omega = \omega_{\alpha_1 \ldots \alpha_q} t^{\alpha_1} \wedge \ldots \wedge t^{\alpha_q} \in \Lambda^q (F, \nu, N)$$

such that

$$\omega_{\alpha_1 \ldots \alpha_q} \in C^r (N),$$

for any $1 \leq \alpha_1 < \ldots < \alpha_q \leq p$, then we will say that the $q$-form $\omega$ is differentiable of $C^r$-class.

**Definition 4.5** For any $z \in \Gamma (F, \nu, N)$, the $\mathcal{F}(N)$-multilinear application

$$\Lambda (F, \nu, N) \xrightarrow{L_z} \Lambda (F, \nu, N),$$

defined by

$$L_z (f) = \Gamma (Th \circ \rho, h \circ \eta) z (f), \forall f \in \mathcal{F} (N)$$

and

$$L_z \omega (z_1, \ldots, z_q) = \Gamma (Th \circ \rho, h \circ \eta) z (\omega ((z_1, \ldots, z_q)))$$

$$- \sum_{i=1}^{q} \omega \left( (z_1, \ldots, [z, z_i]_{F,h}, \ldots, z_q) \right),$$

for any $\omega \in \Lambda^q (F, \nu, N)$ and $z_1, \ldots, z_q \in \Gamma (F, \nu, N)$, will be called the covariant Lie derivative with respect to the section $z$.

**Theorem 4.4** If $z \in \Gamma (F, \nu, N), \omega \in \Lambda^q (F, \nu, N)$ and $\theta \in \Lambda^r (F, \nu, N)$, then

(4.6) $$L_z (\omega \wedge \theta) = L_z \omega \wedge \theta + \omega \wedge L_z \theta.$$ 

**Proof.** Let $z_1, \ldots, z_{q+r} \in \Gamma (F, \nu, N)$ be arbitrary. Since

$$L_z (\omega \wedge \theta) (z_1, \ldots, z_{q+r}) = \Gamma (Th \circ \rho, h \circ \eta) z ((\omega \wedge \theta) (z_1, \ldots, z_{q+r}))$$

$$- \sum_{i=1}^{q+r} (\omega \wedge \theta) \left( (z_1, \ldots, [z, z_i]_{F,h}, \ldots, z_{q+r}) \right)$$

$$= \Gamma (Th \circ \rho, h \circ \eta) z \left( \sum_{\sigma(1)<\ldots<\sigma(q)} \text{sgn} (\sigma) \cdot \omega \left( z_{\sigma(1)}, \ldots, z_{\sigma(q)} \right) \right)$$

$$\cdot \theta \left( z_{\sigma(q+1)}, \ldots, z_{\sigma(q+r)} \right) - \sum_{i=1}^{q+r} (\omega \wedge \theta) \left( (z_1, \ldots, [z, z_i]_{F,h}, \ldots, z_{q+r}) \right)$$

$$= \sum_{\sigma(1)<\ldots<\sigma(q)} \text{sgn} (\sigma) \cdot \Gamma (Th \circ \rho, h \circ \eta) z \left( (z_{\sigma(1)}, \ldots, z_{\sigma(q)}) \right)$$

$$\cdot \theta \left( z_{\sigma(q+1)}, \ldots, z_{\sigma(q+r)} \right) + \sum_{\sigma(1)<\ldots<\sigma(q)} \text{sgn} (\sigma) \cdot \omega \left( z_{\sigma(1)}, \ldots, z_{\sigma(q)} \right)$$

$$\cdot \Gamma (Th \circ \rho, h \circ \eta) z \left( \theta \left( z_{\sigma(q+1)}, \ldots, z_{\sigma(q+r)} \right) \right) - \sum_{\sigma(1)<\ldots<\sigma(q)} \text{sgn} (\sigma)$$

$$\cdot \sum_{i=1}^{q} \omega \left( z_{\sigma(1)}, \ldots, [z, z_{\sigma(i)}]_{F,h}, \ldots, z_{\sigma(q+r)} \right) \cdot \theta \left( z_{\sigma(q+1)}, \ldots, z_{\sigma(q+r)} \right)$$
we obtain that for any \( z \) it results the conclusion of the theorem. q.e.d.

**Proof.** Let \( z \in \Gamma (F, \nu, N) \), the \( \mathcal{F} (N) \)-multilinear application

\[
\Lambda (F, \nu, N) \rightarrow \Lambda (F, \nu, N) \\
\Lambda^q (F, \nu, N) \ni \omega \mapsto i_\omega \in \Lambda^{q-1} (F, \nu, N),
\]

where

\[
i_{z} (\omega \wedge \theta) (z_2, ..., z_q) = \omega (z_2, ..., z_q),
\]

for any \( z_2, ..., z_q \in \Gamma (F, \nu, N) \), will be called the \textit{interior product associated to the section} \( z \).

For any \( f \in \mathcal{F} (N) \), we define \( i_\omega f = 0 \).

**Remark 4.4** If \( z \in \Gamma (F, \nu, N) \), \( \omega \in \Lambda^p (F, \nu, N) \) and \( U \) is an open subset of \( N \) such that \( z|_U = 0 \) or \( \omega|_U = 0 \), then \( (i_\omega \omega)|_U = 0 \).

**Theorem 4.5** If \( z \in \Gamma (F, \nu, N) \), then for any \( \omega \in \Lambda^q (F, \nu, N) \) and \( \theta \in \Lambda^r (F, \nu, N) \) we obtain

\[
i_{z} (\omega \wedge \theta) = i_{z} \omega \wedge \theta + (-1)^q \omega \wedge i_{z} \theta.
\]

**Proof.** Let \( z_1, ..., z_{q+r} \in \Gamma (F, \nu, N) \) be arbitrary. We observe that

\[
i_{z_1} (\omega \wedge \theta) (z_2, ..., z_{q+r}) = (\omega \wedge \theta) (z_1, z_2, ..., z_{q+r}) \\
= \sum_{\sigma(1)<...<\sigma(q)} \text{sgn} (\sigma) \cdot \omega (z_{\sigma(1)}, ..., z_{\sigma(q)}) \cdot \theta (z_{\sigma(q+1)}, ..., z_{\sigma(q+r)}) \\
= \sum_{\sigma(1)<...<\sigma(q)} \text{sgn} (\sigma) \cdot \omega (z_1, z_{\sigma(2)}, ..., z_{\sigma(q)}) \cdot \theta (z_{\sigma(q+1)}, ..., z_{\sigma(q+r)}) \\
+ \sum_{1=\sigma(1)<...<\sigma(q)} \text{sgn} (\sigma) \cdot \omega (z_{\sigma(1)}, ..., z_{\sigma(q)}) \cdot \theta (z_1, z_{\sigma(q+2)}, ..., z_{\sigma(q+r)}) \\
+ \sum_{\sigma(2)<...<\sigma(q)} \text{sgn} (\sigma) \cdot i_{z_1} \omega (z_{\sigma(2)}, ..., z_{\sigma(q)}) \cdot \theta (z_{\sigma(q+1)}, ..., z_{\sigma(q+r)}) \\
+ \sum_{\sigma(1)<...<\sigma(q)} \text{sgn} (\sigma) \cdot \omega (z_{\sigma(1)}, ..., z_{\sigma(q)}) \cdot i_{z_1} \theta (z_{\sigma(q+2)}, ..., z_{\sigma(q+r)}).
\]
In the second sum, we have the permutation
\[ \sigma = \left( \begin{array}{ccccccc} 1 & \ldots & q & q+1 & q+2 & \ldots & q+r \\ \sigma(1) & \ldots & \sigma(q) & 1 & \sigma(q+2) & \ldots & \sigma(q+r) \end{array} \right). \]

We observe that \( \sigma = \tau \circ \tau' \), where
\[ \tau = \left( \begin{array}{ccccccc} 1 & 2 & \ldots & q+1 & q+2 & \ldots & q+r \\ 1 & \sigma(1) & \ldots & \sigma(q) & \sigma(q+2) & \ldots & \sigma(q+r) \end{array} \right) \]
and
\[ \tau' = \left( \begin{array}{ccccccc} 1 & 2 & \ldots & q & q+1 & q+2 & \ldots & q+r \\ 2 & 3 & \ldots & q+1 & 1 & q+2 & \ldots & q+r \end{array} \right). \]
Since \( \tau(2) < \ldots < \tau(q+1) \) and \( \tau' \) has \( q \) inversions, it results that
\[ \text{sgn} (\sigma) = (-1)^q \cdot \text{sgn} (\tau). \]

Therefore,
\[ i z_1 (\omega \wedge \theta) (z_2, \ldots, z_{q+r}) = (i z_1 \omega \wedge \theta) (z_2, \ldots, z_{q+r}) \]
\[ + (-1)^q \sum_{\tau(2) < \ldots < \tau(q)} \text{sgn} (\tau) \cdot \omega (z_{\tau(2)}, \ldots, z_{\tau(q)}) \cdot i z_1 \theta (z_{\tau(q+2)}, \ldots, z_{\tau(q+r)}) \]
\[ = (i z_1 \omega \wedge \theta) (z_2, \ldots, z_{q+r}) + (-1)^q (\omega \wedge i z_1 \theta) (z_2, \ldots, z_{q+r}). \]

**Theorem 4.6** For any \( z, v \in \Gamma (F, \nu, N) \) we obtain
\[ (4.8) \quad L_v \circ i z - i z \circ L_v = i [z, v]_{F, h}. \]

**Proof.** Let \( \omega \in \Lambda^q (F, \nu, N) \) be arbitrary. Since
\[ i z (L_v \omega) (z_2, \ldots, z_q) = L_v \omega (z, z_2, \ldots, z_q) \]
\[ = \Gamma (Th \circ \rho, h \circ \eta) v (\omega (z, z_2, \ldots, z_q)) - \omega ([v, z]_{F, h}, z_2, \ldots, z_q) \]
\[ - \sum_{i=2}^{q} \omega \left( z, z_2, \ldots, [v, z_i]_{F, h}, \ldots, z_q \right) \]
\[ = \Gamma (Th \circ \rho, h \circ \eta) v (i z_1 \omega (z_2, \ldots, z_q)) - \sum_{i=2}^{q} i z_1 \omega \left( z_2, \ldots, [v, z_i]_{F, h}, \ldots, z_q \right) \]
\[ - i [v, z]_{F, h} (z_2, \ldots, z_q) = \left( L_v (i z_1 \omega) - i [v, z]_{F, h} \right) (z_2, \ldots, z_q), \]
for any \( z_2, \ldots, z_q \in \Gamma (F, \nu, N) \) it result the conclusion of the theorem. *q.e.d.*

**Definition 4.7** If \( f \in \mathcal{F} (N) \) and \( z \in \Gamma (F, \nu, N) \), then we define
\[ d^F f (z) = \Gamma (Th \circ \rho, h \circ \eta) (z) f. \]

**Theorem 4.7** The \( \mathcal{F} (N) \)-multilinear application
\[ \Lambda^q (F, \nu, N) \xrightarrow{d^F} \Lambda^{q+1} (F, \nu, N) \]
\[ \omega \mapsto d^F \omega \]

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defined by

\[ d^F \omega (z_0, z_1, \ldots, z_q) = \sum_{i=0}^{q} (-1)^i \Gamma (Th \circ \rho, h \circ \eta) z_i (\omega ((z_0, z_1, \ldots, \hat{z}_i, \ldots, z_q))) + \sum_{0 \leq i < j} (-1)^{i+j} \omega \left( \left[ z_i, z_j \right]_{F,h}, z_0, z_1, \ldots, \hat{z}_i, \ldots, \hat{z}_j, \ldots, z_q \right) \]

for any \( z_0, z_1, \ldots, z_q \in \Gamma (F, \nu, N) \), is unique with the following property:

\[ (4.9) \quad L_z = d^F \circ i_z + i_z \circ d^F, \quad \forall z \in \Gamma (F, \nu, N). \]

This \( \mathcal{F}(N) \)-multilinear application will be called the exterior differentiation operator for the exterior differential algebra of the generalized Lie algebroid \(((F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta)). \)

Proof. We verify the property (4.9) Since

\[
\begin{align*}
(i_{z_0} \circ d^F) \omega (z_1, \ldots, z_q) & = d\omega (z_0, z_1, \ldots, z_q) \\
& = \sum_{i=0}^{q} (-1)^i \Gamma (Th \circ \rho, h \circ \eta) z_i (\omega (z_0, z_1, \ldots, \hat{z}_i, \ldots, z_q)) \\
& + \sum_{0 \leq i < j} (-1)^{i+j} \omega \left( \left[ z_i, z_j \right]_{F,h}, z_0, z_1, \ldots, \hat{z}_i, \ldots, \hat{z}_j, \ldots, z_q \right) \\
& = \Gamma (Th \circ \rho, h \circ \eta) z_0 (\omega (z_1, \ldots, z_q)) + \sum_{i=1}^{q} (-1)^i \omega \left( \left[ z_0, z_i \right]_{F,h}, z_1, \ldots, \hat{z}_i, \ldots, z_q \right) \\
& + \sum_{1 \leq i < j} (-1)^{i+j} \omega \left( \left[ z_i, z_j \right]_{F,h}, z_0, z_1, \ldots, \hat{z}_i, \ldots, \hat{z}_j, \ldots, z_q \right) \\
& = \Gamma (Th \circ \rho, h \circ \eta) z_0 (\omega (z_1, \ldots, z_q)) - \sum_{i=1}^{q} \omega \left( z_1, \ldots, \left[ z_0, z_i \right]_{F,h}, \ldots, z_q \right) \\
& - \sum_{i=1}^{q} (-1)^{i-1} \Gamma (Th \circ \rho, h \circ \eta) z_i (i_{z_0} \omega ((z_1, \ldots, \hat{z}_i, \ldots, z_q))) \\
& - \sum_{0 \leq i < j} (-1)^{i+j-2} i_{z_0} \omega \left( \left[ z_i, z_j \right]_{F,h}, z_1, \ldots, \hat{z}_i, \ldots, \hat{z}_j, \ldots, z_q \right) \\
& = (L_{z_0} - d^F \circ i_{z_0}) \omega (z_1, \ldots, z_q),
\end{align*}
\]

for any \( z_0, z_1, \ldots, z_q \in \Gamma (F, \nu, N) \) it results that the property (4.9) is satisfied.

In the following, we verify the uniqueness of the operator \( d^F \).

Let \( d^F \) be an another exterior differentiation operator satisfying the property (4.9).

Let \( S = \{ q \in \mathbb{N} : d^F \omega = d^F \omega, \quad \forall \omega \in \Lambda^q (F, \nu, N) \} \) be.

Let \( z \in \Gamma (F, \nu, N) \) be arbitrary.

We observe that (4.9) is equivalent with

\[ (1) \quad i_z \circ (d^F - d^F) + (d^F - d^F) \circ i_z = 0. \]

Since \( i_z f = 0 \), for any \( f \in \mathcal{F}(N) \), it results that

\[ ((d^F - d^F) f) (z) = 0, \quad \forall f \in \mathcal{F}(N). \]

Therefore, we obtain that

\[ (2) \quad 0 \in S. \]
In the following, we prove that

\[ q \in S \implies q + 1 \in S \]

Let \( \omega \in \Lambda^{p+1} (F, \nu, N) \) be arbitrary. Since \( i_z \omega \in \Lambda^q (F, \nu, N) \), using the equality (1), it results that

\[ i_z \circ (d^F - d^{F}) \omega = 0. \]

We obtain that, \( (d^F - d^{F}) (z_0, z_1, \ldots, z_q) = 0 \), for any \( z_0, \ldots, z_q \in \Gamma (F, \nu, N) \).

Therefore \( d^F \omega = d^{F} \omega \), namely \( q + 1 \in S \).

Using the Peano’s Axiom and the affirmations (2) and (3) it results that \( S = \mathbb{N} \).

Therefore, the uniqueness is verified.

Note that if \( \omega = \omega_{\alpha_1 \ldots \alpha_q} t^{\alpha_1} \wedge \ldots \wedge t^{\alpha_q} \in \Lambda^q (F, \nu, N) \), then

\[
d^F \omega (t_{\alpha_0}, t_{\alpha_1}, \ldots, t_{\alpha_q}) = \sum_{i=0}^q (-1)^i \theta_{0}^k \partial \omega_{\alpha_0 \ldots \alpha_i \ldots \alpha_q} \frac{\partial}{\partial x^k} + \sum_{i<j} (-1)^{i+j} L_{\alpha_i \alpha_j} \cdot \omega_{\alpha_0 \alpha_0 \ldots \alpha_i \ldots \alpha_j \ldots \alpha_q}.
\]

Therefore, we obtain

\[
d^F \omega = \left( \sum_{i=0}^q (-1)^i \theta_{0}^k \partial \omega_{\alpha_0 \ldots \alpha_i \ldots \alpha_q} \frac{\partial}{\partial x^k} + \sum_{i<j} (-1)^{i+j} L_{\alpha_i \alpha_j} \cdot \omega_{\alpha_0 \alpha_0 \ldots \alpha_i \ldots \alpha_j \ldots \alpha_q} \right) t^{\alpha_0} \wedge t^{\alpha_1} \wedge \ldots \wedge t^{\alpha_q}.
\]

**Remark 4.5** If \( d^F \) is the exterior differentiation operator for the generalized Lie algebroid

\[
\left( (F, \nu, N), \left( [, ]_{F,h}, (\rho, \eta) \right) \right),
\]

\( \omega \in \Lambda^q (F, \nu, N) \) and \( U \) is an open subset of \( N \) such that \( \omega \mid_U = 0 \), then \( (d^F \omega) \mid_U = 0 \).

**Theorem 4.8** The exterior differentiation operator \( d^F \) given by the previous theorem has the following properties:

1. For any \( \omega \in \Lambda^q (F, \nu, N) \) and \( \theta \in \Lambda^p (F, \nu, N) \) we obtain

\[
d^F (\omega \wedge \theta) = d^F \omega \wedge \theta + (-1)^q \omega \wedge d^F \theta.
\]

2. For any \( z \in \Gamma (F, \nu, N) \) we obtain

\[
L_z \circ d^F = d^F \circ L_z.
\]

3. \( d^F \circ d^F = 0 \).

Proof.

1. Let \( S = \{ q \in \mathbb{N} : d^F (\omega \wedge \theta) = d^F \omega \wedge \theta + (-1)^q \omega \wedge d^F \theta, \forall \omega \in \Lambda^q (F, \nu, N) \} \) be. Since

\[
d^F (f \wedge \theta) (z, v) = d^F (f \cdot \theta) (z, v) = \Gamma (T(h \circ \rho, h \circ \eta) z (f (\omega (v))) - \Gamma (T(h \circ \rho, h \circ \eta) v (f \omega (z))) - f \omega \left( [z, v]_{F,h} \right) = \Gamma (T(h \circ \rho, h \circ \eta) z (f) \cdot \omega (v) + f \cdot \Gamma (T(h \circ \rho, h \circ \eta) z (\omega (v))) - \Gamma (T(h \circ \rho, h \circ \eta) v (f) \cdot \omega (z) - f \cdot \Gamma (T(h \circ \rho, h \circ \eta) v (\omega (z))) - f \omega \left( [z, v]_{F,h} \right) = d^F f (z) \cdot \omega (v) - d^F f (v) \cdot \omega (z) + f \cdot d^F \omega (z, v) = (d^F f \wedge \omega) (z, v) + (\omega \wedge f) (z, v) \}
\]

2. \( d^F \circ d^F = 0 \).
it results that

\[(1.1) \quad 0 \in S.\]

In the following we prove that

\[(1.2) \quad q \in S \implies q + 1 \in S.\]

Without restricting the generality, we consider that \(\theta \in \Lambda^r (F, \nu, N)\). Since

\[
d^F (\omega \land \theta) (z_0, z_1, \ldots, z_{q+r}) = i_{z_0} \circ d^F (\omega \land \theta) (z_1, \ldots, z_{q+r}) = (L_{z_0} \omega \land \theta + \omega \land L_{z_0} \theta) (z_1, \ldots, z_{q+r}) - [d^F \circ (i_{z_0} \omega \land \theta + (-1)^q \omega \land i_{z_0} \theta)] (z_1, \ldots, z_{q+r}) - \left( (-1)^{q+1} i_{z_0} \omega \land d^F \theta + (-1)^q d^F \omega \land i_{z_0} \theta \right) (z_1, \ldots, z_{q+r}) - \left( i_{z_0} \omega \land i_{z_0} \theta \right) (z_1, \ldots, z_{q+r}) + \omega \land (L_{z_0} \theta - d^F \circ i_{z_0} \theta) (z_1, \ldots, z_{q+r}) + \left( (-1)^q i_{z_0} \omega \land d^F \theta - (-1)^q d^F \omega \land i_{z_0} \theta \right) (z_1, \ldots, z_{q+r}) = \left( (i_{z_0} \circ d^F) \omega \land \theta + (-1)^q d^F \omega \land i_{z_0} \theta \right) (z_1, \ldots, z_{q+r})\]

for any \(z_0, z_1, \ldots, z_{q+r} \in \Gamma (F, \nu, N)\), it results (1.2).

Using the Peano’s Axiom and the affirmations (1.1) and (1.2) it results that \(S = \mathbb{N}\).

Therefore, it results the conclusion of affirmation 1.

2. Let \(z \in \Gamma (F, \nu, N)\) be arbitrary.

Let \(S = \{ q \in \mathbb{N} : (L_z \circ d^F) \omega = (d^F \circ L_z) \omega, \forall \omega \in \Lambda^q (F, \nu, N) \}\) be.

Let \(f \in \mathcal{F} (N)\) be arbitrary. Since

\[
\begin{align*}
(d^F \circ L_z) f (v) &= i_v \circ (d^F \circ L_z) f = (i_v \circ d^F) \circ L_z f \\
&= (L_v \circ L_z) f - \left( (d^F \circ i_v) \circ L_z \right) f \\
&= (L_v \circ L_z) f - L_{[z,v]} \circ F, h f + d^F \circ i_{[z,v]} \circ F, h f - d^F \circ L_z (i_v f) \\
&= (L_v \circ L_z) f - L_{[z,v]} \circ F, h f + d^F \circ i_{[z,v]} \circ F, h f - 0 \\
&= (L_v \circ L_z) f - L_{[z,v]} \circ F, h f + d^F \circ i_{[z,v]} \circ F, h f - L_z \circ d^F (i_v f) \\
&= (L_z \circ i_v) (d^F f) - L_{[z,v]} \circ F, h f + d^F \circ i_{[z,v]} \circ F, h f \\
&= (i_v \circ L_z) (d^F f) + L_{[z,v]} \circ F, h f - L_{[z,v]} \circ F, h f \\
&= i_v \circ (L_z \circ d^F) f = (L_z \circ d^F) f (v), \forall v \in \Gamma (F, \nu, N),
\end{align*}
\]

it results that

\[(2.1) \quad 0 \in S.\]

In the following we prove that

\[(2.2) \quad q \in S \implies q + 1 \in S.\]
Let \( \omega \in \Lambda^q (F, \nu, N) \) be arbitrary. Since

\[
(d^F \circ L_z) \omega (z_0, z_1, ..., z_q) = i_{z_0} \circ (d^F \circ L_z) \omega (z_1, ..., z_q)
\]

\[
= (i_{z_0} \circ d^F) \circ L_z \omega (z_1, ..., z_q)
\]

\[
= [(L_{z_0} \circ L_z) \omega - ((d^F \circ i_{z_0}) \circ L_z) \omega] (z_1, ..., z_q)
\]

\[
= [(L_{z_0} \circ L_z) \omega - L_{[z_0,z]_{F,h}} \omega] (z_1, ..., z_q)
\]

\[
+ [d^F \circ i_{[z_0,z]_{F,h}} \omega - d^F \circ L_z (i_{z_0} \omega)] (z_1, ..., z_q)
\]

\[
= [(L_{z_0} \circ L_z) \omega - L_{[z_0,z]_{F,h}} \omega] (z_1, ..., z_q)
\]

\[
+ [d^F \circ i_{[z_0,z]_{F,h}} \omega - L_z \circ d^F (i_{z_0} \omega)] (z_1, ..., z_q)
\]

\[
= [(L_{z_0} \circ L_z) \omega - L_{[z_0,z]_{F,h}} \omega + d^F \circ i_{[z_0,z]_{F,h}} \omega] (z_1, ..., z_q)
\]

\[
= [(i_{z_0} \circ L_z) (d^F \omega) + L_{[z_0,z]_{F,h}} \omega - L_{[z_0,z]_{F,h}} \omega] (z_1, ..., z_q)
\]

\[
i_{z_0} \circ (L_z \circ d^F) \omega (z_1, ..., z_q)
\]

\[
= (L_z \circ d^F) \omega (z_0, z_1, ..., z_q), \ \forall z_0, z_1, ..., z_q \in \Gamma (F, \nu, N),
\]

it results (2.2).

Using the Peano’s Axiom and the affirmations (2.1) and (2.2) it results that \( S = \mathbb{N} \).
Therefore, it results the conclusion of affirmation 2.

3. It is remarked that

\[
i_z \circ (d^F \circ d^F) = (i_z \circ d^F) \circ d^F = L_z \circ d^F - (d^F \circ i_z) \circ d^F
\]

\[
= L_z \circ d^F - d^F \circ L_z + d^F \circ (d^F \circ i_z) = (d^F \circ d^F) \circ i_z,
\]

for any \( z \in \Gamma (F, \nu, N) \).

Let \( \omega \in \Lambda^q (F, \nu, N) \) be arbitrary. Since

\[
(d^F \circ d^F) \omega (z_1, ..., z_{q+2}) = i_{z_{q+2}} \circ ... \circ i_{z_1} \circ (d^F \circ d^F) \omega = ...
\]

\[
i_{z_{q+2}} \circ (d^F \circ d^F) \circ i_{z_{q+1}} (\omega (z_1, ..., z_q))
\]

\[
i_{z_{q+2}} \circ (d^F \circ d^F) (0) = 0, \ \forall z_1, ..., z_{q+2} \in \Gamma (F, \nu, N),
\]

it results the conclusion of affirmation 3.

**Theorem 4.9** If \( d^F \) is the exterior differentiation operator for the exterior differential \( F(N) \)-algebra \( \Lambda (F, \nu, N), +, \cdot, \wedge \), then we obtain the structure equations of Maurer-Cartan type

\[
(C_1) \quad d^F t^\alpha = -\frac{1}{2} L^{\alpha}_{\beta \gamma} t^\beta \wedge t^\gamma, \ \alpha \in \overline{1, p}
\]

and

\[
(C_2) \quad d^F \xi^i = \theta^i_{\alpha} t^\alpha, \ i \in \overline{1, n},
\]

where \( \{ t^\alpha, \alpha \in \overline{1, p} \} \) is the coframe of the vector bundle \( (F, \nu, N) \).

This equations will be called the structure equations of Maurer-Cartan type associated to the generalized Lie algebroid \( ((F, \nu, N), [, ], [F,h], (\rho, \eta)) \).
Proof. Let $\alpha \in \overline{1, p}$ be arbitrary. Since
\[ d^F t^\alpha (t_\beta, t_\gamma) = -L^\alpha_{\beta\gamma}, \ \forall \beta, \gamma \in \overline{1, p} \]
it results that
\[ d^F t^\alpha = -\sum_{\beta < \gamma} L^\alpha_{\beta\gamma} t^\beta \wedge t^\gamma. \] (1)

Since $L^\alpha_{\beta\gamma} = -L^\alpha_{\gamma\beta}$ and $t^\beta \wedge t^\gamma = -t^\gamma \wedge t^\beta$, for any $\beta, \gamma \in \overline{1, p}$, it results that
\[ \sum_{\beta < \gamma} L^\alpha_{\beta\gamma} t^\beta \wedge t^\gamma = \frac{1}{2} L^\alpha_{\beta\gamma} t^\beta \wedge t^\gamma. \] (2)

Using the equalities (1) and (2) it results the structure equation (C1).

Let $\tilde{t} \in \overline{1, n}$ be arbitrarily. Since
\[ d^F \tilde{t} (t_\alpha) = \theta^\tilde{t}_\alpha, \ \forall \alpha \in \overline{1, p} \]
it results the structure equation (C2).

q.e.d.

Corollary 4.1 If $\left( (h^*F, h^*\nu, M), [\cdot, h^*F], h^*F, \rho, Id_M \right)$ is the pull-back Lie algebroid associated to the generalized Lie algebroid $\left( (F, \nu, N), [\cdot, F], (\rho, \eta) \right)$ and $d^{h^*F}$ is the exterior differentiation operator for the exterior differential $F(M)$-algebra
\[ (\Lambda (h^*F, h^*\nu, M), +, \cdot, \wedge), \]
then we obtain the following structure equations of Maurer-Cartan type

(C1) \[ d^{h^*F} T^\alpha = -\frac{1}{2} (L^\alpha_{\beta\gamma} \circ h) T^\beta \wedge T^\gamma, \ \alpha \in \overline{1, p} \]
and

(C2) \[ d^{h^*F} x^i = (\rho^i_\alpha \circ h) T^\alpha, \ i \in \overline{1, m}. \]

This equations will be called the structure equations of Maurer-Cartan type associated to the pull-back Lie algebroid
\[ \left( (h^*F, h^*\nu, M), [\cdot, h^*F], h^*F, \rho, Id_M \right). \]

Theorem 4.10 (of Cartan type) Let $(E, \pi, M)$ be an IDS of the generalized Lie algebroid $\left( (F, \nu, N), [\cdot, F], (\rho, \eta) \right)$. If $\{ \Theta^p \}$ is a base for the $\mathcal{F}(M)$-submodule $(\Gamma (E^0, \pi^0, M), +, \cdot)$, then the IDS $(E, \pi, M)$ is involutive if and only if it exists
\[ \Omega^\alpha_\beta \in \Lambda^1 (h^*F, h^*\nu, M), \ \alpha, \beta \in \overline{r + 1, p} \]
such that
\[ d^{h^*F} \Theta^\alpha = \sum_{\beta \in \overline{r + 1, p}} \Omega^\alpha_\beta \wedge (\Theta^\beta \in \mathcal{I} (\Gamma (E^0, \pi^0, M))). \]

Proof: Let $\{ S_1, ..., S_r \}$ be a base for the $\mathcal{F}(M)$-submodule $(\Gamma (E, \pi, M), +, \cdot)$
Let \( \{ S_{r+1}, \ldots, S_p \} \in \Gamma (h^*F, h^*\nu, M) \) such that
\[
\{ S_1, \ldots, S_r, S_{r+1}, \ldots, S_p \}
\]
is a base for the \( \mathcal{F}(M) \)-module
\[
(\Gamma (h^*F, h^*\nu, M), +, \cdot).
\]

Let \( \Theta^1, \ldots, \Theta^r \in \Gamma \left( h^*F, h^*\nu, M \right) \) such that
\[
\{ \Theta_1, \ldots, \Theta_r, \Theta_{r+1}, \ldots, \Theta_p \}
\]
is a base for the \( \mathcal{F}(M) \)-module
\[
\left( \Gamma \left( h^*F, h^*\nu, M \right), +, \cdot \right).
\]

For any \( a, b \in \overline{1, r} \) and \( \alpha, \beta \in \overline{r+1, p} \), we have the equalities:
\[
\Theta^a (S_b) = \delta^a_b,
\Theta^a (S_\beta) = 0,
\Theta^a (S_b) = 0,
\Theta^a (S_\beta) = \delta^a_\beta,
\]

We remark that the set of the 2-forms
\[
\left\{ \Theta^a \wedge \Theta^b, \Theta^a \wedge \Theta^\beta, \Theta^\alpha \wedge \Theta^\beta, a, b \in \overline{1, r} \wedge \alpha, \beta \in \overline{r+1, p} \right\}
\]
is a base for the \( \mathcal{F}(M) \)-module
\[
(\Lambda^2 (h^*F, h^*\nu, M), +, \cdot).
\]

Therefore, we have
\[
(1) \quad d^{h^*F} \Theta^a = \sum_{b \leq c} A_{bc}^\alpha \Theta^b \wedge \Theta^c + \sum_{b, \gamma} B_{b\gamma}^\alpha \Theta^b \wedge \Theta^\gamma + \sum_{\beta < \gamma} C_{\beta\gamma}^\alpha \Theta^\beta \wedge \Theta^\gamma,
\]
where, \( A_{bc}^\alpha, B_{b\gamma}^\alpha \) and \( C_{\beta\gamma}^\alpha, a, b, c \in \overline{1, r} \wedge \alpha, \beta, \gamma \in \overline{r+1, p} \) are real local functions such that \( A_{bc}^\alpha = -A_{cb}^\alpha \) and \( C_{\beta\gamma}^\alpha = -C_{\gamma\beta}^\alpha \).

Using the formula
\[
(2) \quad d^{h^*F} \Theta^a (S_b, S_c) = \Gamma \left( h^*F, Id_M \right) S_b (\Theta^a (S_c)) - \Gamma \left( h^*F, Id_M \right) S_c (\Theta^a (S_b)) - \Theta^a ([S_b, S_c]_{h^*F}),
\]
we obtain that
\[
(3) \quad A_{bc}^\alpha = -\Theta^a ([S_b, S_c]_{h^*F}), \quad \forall (b, c \in \overline{1, r} \wedge \alpha \in \overline{r+1, p}).
\]

We admit that \( (E, \pi, M) \) is an involutive IDS of the generalized Lie algebroid
\[
\left( (F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta) \right).
\]

As
\[
[S_b, S_c]_{h^*F} \in \Gamma (E, \pi, M), \quad \forall b, c \in \overline{1, r},
\]
it results that
\[ \Theta^\alpha ([S_b, S_c]_{h^*F}) = 0, \quad \forall (b, c \in \overline{1, r} \land \alpha \in \overline{r + 1, p}). \]

Therefore,
\[ A^\alpha_{bc} = 0, \quad \forall (b, c \in \overline{1, r} \land \alpha \in \overline{r + 1, p}) \]
and we obtain
\[ d^{h^*F} \Theta^\alpha = \sum_{b, \gamma} B^\alpha_{b\gamma} \Theta^b \land \Theta^\gamma + \frac{1}{2} C^\alpha_{b\gamma} \Theta^b \land \Theta^\gamma \]
\[ = \left( B^\alpha_{b\gamma} \Theta^b + \frac{1}{2} C^\alpha_{b\gamma} \Theta^b \right) \land \Theta^\gamma. \]

As
\[ \Omega^\alpha_{\gamma} = B^\alpha_{\gamma} \Theta^b + \frac{1}{2} C^\alpha_{\beta\gamma} \Theta^\beta \in \Lambda^1 (h^*F, h^*\nu, M), \quad \forall \alpha, \beta \in \overline{r + 1, p} \]
it results the first implication.

Conversely, we admit that it exists
\[ \Omega^\alpha_{\beta} \in \Lambda^1 (h^*F, h^*\nu, M), \quad \alpha, \beta \in \overline{r + 1, p} \]
such that
\[ d^{h^*F} \Theta^\alpha = \sum_{\beta \in \overline{r + 1, p}} \Omega^\alpha_{\beta} \land \Theta^\beta, \quad \forall \alpha \in \overline{r + 1, p}. \]

Using the affirmations (1), (2) and (4) we obtain that
\[ A^\alpha_{bc} = 0, \quad \forall (b, c \in \overline{1, r} \land \alpha \in \overline{r + 1, p}). \]

Using the affirmation (3), we obtain
\[ \Theta^\alpha ([S_b, S_c]_{h^*F}) = 0, \quad \forall (b, c \in \overline{1, r} \land \alpha \in \overline{r + 1, p}). \]

Therefore,
\[ [S_b, S_c]_{h^*F} \in \Gamma (E, \pi, M), \quad \forall b, c \in \overline{1, r}. \]

Using the Proposition 3.2, we obtain the second implication. \( q.e.d. \)

Let \( \left( (F', \nu', N'), [\cdot]_{F', h'}, (\rho', \eta') \right) \) be another generalized Lie algebroid.

**Definition 4.8** For any morphism \( (\varphi, \varphi_0) \) of
\[ \left( (F, \nu, N), [\cdot]_{F, h}, (\rho, \eta) \right) \]
source and
\[ \left( (F', \nu', N'), [\cdot]_{F', h'}, (\rho', \eta') \right) \]
target we define the application
\[ \Lambda^q (F', \nu', N') \xrightarrow{(\varphi, \varphi_0)^*} \Lambda^q (F, \nu, N) \]
\[ \omega' \xrightarrow{(\varphi, \varphi_0)^*} (\varphi, \varphi_0)^* \omega' , \]
where
\[ ((\varphi, \varphi_0)^* \omega') (z_1, ..., z_q) = \omega' (\Gamma (\varphi, \varphi_0) (z_1), ..., \Gamma (\varphi, \varphi_0) (z_q)), \]
for any \( z_1, ..., z_q \in \Gamma (F, \nu, N) \).
Remark 4.5 It is remarked that the $B^y$-morphism $(Th \circ \rho, h \circ \eta)$ is a $GLA$-morphism of
\[
\left( (F, \nu, N), [\cdot, \cdot]_{F, \nu}, (\rho, \eta) \right)
\]
source and
\[
\left( (TN, \tau_N, N), [\cdot]_{TN, Id_N}, (Id_{TN}, Id_N) \right)
\]
target.
Moreover, for any $i \in 1, n$, we obtain
\[
(Th \circ \rho, h \circ \eta)^* (dz^i) = dF z^i,
\]
where $d$ is the exterior differentiation operator associated to the exterior differential Lie $F(N)$-algebra
\[
(\Lambda (TN, \tau_N, N), \oplus, \cdot, \wedge).
\]

**Theorem 4.11** If $(\varphi, \varphi_0)$ is a morphism of
\[
\left( (F, \nu, N), [\cdot]_{F, \nu}, (\rho, \eta) \right)
\]
source and
\[
\left( (F', \nu', N'), [\cdot]_{F', \nu'}, (\rho', \eta') \right)
\]
target, then the following affirmations are satisfied:
1. For any $\omega^r \in \Lambda^q (F', \nu', N')$ and $\theta' \in \Lambda^r (F', \nu', N')$ we obtain
\[
(\varphi, \varphi_0)^* (\omega^r \wedge \theta') = (\varphi, \varphi_0)^* \omega^r \wedge (\varphi, \varphi_0)^* \theta'.
\]
2. For any $z \in \Gamma (F, \nu, N)$ and $\omega' \in \Lambda^q (F', \nu', N')$ we obtain
\[
i_z ((\varphi, \varphi_0)^* \omega') = (\varphi, \varphi_0)^* (i_{\Gamma (\varphi, \varphi_0) z} \omega').
\]
3. If $N = N'$ and
\[
(Th \circ \rho, h \circ \eta) = (Th' \circ \rho', h' \circ \eta') \circ (\varphi, \varphi_0),
\]
then we obtain
\[
(\varphi, \varphi_0)^* \circ dF' = dF \circ (\varphi, \varphi_0)^*.
\]

**Proof.** 1. Let $\omega' \in \Lambda^q (F', \nu', N')$ and $\theta' \in \Lambda^r (F', \nu', N')$ be arbitrary. Since
\[
(\varphi, \varphi_0)^* (\omega^r \wedge \theta') (z_1, ..., z_{q+r}) = (\omega^r \wedge \theta') (\Gamma (\varphi, \varphi_0) z_1, ..., \Gamma (\varphi, \varphi_0) z_{q+r})
\]
\[
= \frac{1}{(q+r)!} \sum_{\sigma \in \Sigma_{q+r}} sgn (\sigma) \cdot \omega^r (\Gamma (\varphi, \varphi_0) z_1, ..., \Gamma (\varphi, \varphi_0) z_{q+r})
\]
\[
\cdot \theta' (\Gamma (\varphi, \varphi_0) z_{q+1}, ..., \Gamma (\varphi, \varphi_0) z_{q+r})
\]
\[
= \frac{1}{(q+r)!} \sum_{\sigma \in \Sigma_{q+r}} sgn (\sigma) \cdot (\varphi, \varphi_0)^* \omega^r (z_1, ..., z_q) (\varphi, \varphi_0)^* \theta' (z_{q+1}, ..., z_{q+r})
\]
\[
= ((\varphi, \varphi_0)^* \omega^r \wedge (\varphi, \varphi_0)^* \theta') (z_1, ..., z_{q+r}),
\]
for any $z_1, ..., z_{q+r} \in \Gamma (F, \nu, N)$, it results the conclusion of affirmation 1.
2. Let \( z \in \Gamma (F, \nu, N) \) and \( \omega' \in \Lambda^q (F', \nu', N') \) be arbitrary. Since

\[
i_z ( (\varphi, \varphi_0)^* \omega') (z_2, \ldots, z_q) = \omega' (\Gamma (\varphi, \varphi_0) z, \Gamma (\varphi, \varphi_0) z_2, \ldots, \Gamma (\varphi, \varphi_0) z_q) = i_{\Gamma (\varphi, \varphi_0)^* \omega'} (\Gamma (\varphi, \varphi_0) z_2, \ldots, \Gamma (\varphi, \varphi_0) z_q) = (\varphi, \varphi_0)^* (i_{\Gamma (\varphi, \varphi_0)^* \omega'}) (z_2, \ldots, z_q),
\]

for any \( z_2, \ldots, z_q \in \Gamma (F, \nu, N) \), it results the conclusion of affirmation 2.

3. Let \( \omega' \in \Lambda^q (F', \nu', N') \) and \( z_0, \ldots, z_q \in \Gamma (F, \nu, N) \) be arbitrary. Since

\[
( (\varphi, \varphi_0)^* dF' \omega') (z_0, \ldots, z_q) = (dF' \omega') (\Gamma (\varphi, \varphi_0) z_0, \ldots, \Gamma (\varphi, \varphi_0) z_q) = \sum_{i=0}^q (-1)^i \Gamma (Th' \circ \rho', h' \circ \eta') (\Gamma (\varphi, \varphi_0) z_i) \\
\omega' \left( \left( \Gamma (\varphi, \varphi_0) z_0, \Gamma (\varphi, \varphi_0) z_1, \ldots, \Gamma (\varphi, \varphi_0) z_i, \ldots, \Gamma (\varphi, \varphi_0) z_q \right) \right)
\]

\[
+ \sum_{0 \leq i < j} (-1)^{i+j} \cdot \omega' \left( \left( \Gamma (\varphi, \varphi_0) [z_i, z_j]_{F,h}, \Gamma (\varphi, \varphi_0) z_0, \Gamma (\varphi, \varphi_0) z_1, \ldots, \Gamma (\varphi, \varphi_0) z_i, \ldots, \Gamma (\varphi, \varphi_0) z_q \right) \right) \\
\cdot (\varphi, \varphi_0) z_i, \ldots, \Gamma (\varphi, \varphi_0) z_j, \ldots, \Gamma (\varphi, \varphi_0) z_q)
\]

and

\[
dF ((\varphi, \varphi_0)^* \omega') (z_0, \ldots, z_q) = \sum_{i=0}^q (-1)^i \Gamma (Th \circ \rho, h \circ \eta) (z_i) \cdot ((\varphi, \varphi_0)^* \omega') (z_0, \ldots, \hat{z}_i, \ldots, z_q)
\]

\[
+ \sum_{0 \leq i < j} (-1)^{i+j} \cdot ((\varphi, \varphi_0)^* \omega') \left( [z_i, z_j]_{F,h}, z_0, \ldots, \hat{z}_i, \ldots, \hat{z}_j, \ldots, z_q \right)
\]

\[
= \sum_{i=0}^q (-1)^i \Gamma (Th \circ \rho, h \circ \eta) (z_i) \cdot \omega' \left( \left( \Gamma (\varphi, \varphi_0) z_0, \ldots, \Gamma (\varphi, \varphi_0) z_i, \ldots, \Gamma (\varphi, \varphi_0) z_q \right) \right)
\]

\[
+ \sum_{0 \leq i < j} (-1)^{i+j} \cdot \omega' \left( \left( \Gamma (\varphi, \varphi_0) [z_i, z_j]_{F,h}, \Gamma (\varphi, \varphi_0) z_0, \Gamma (\varphi, \varphi_0) z_1, \ldots, \Gamma (\varphi, \varphi_0) z_i, \ldots, \Gamma (\varphi, \varphi_0) z_q \right) \right)
\cdot (\varphi, \varphi_0) z_i, \ldots, \Gamma (\varphi, \varphi_0) z_j, \ldots, \Gamma (\varphi, \varphi_0) z_q)
\]

it results the conclusion of affirmation 3.

\[q.e.d.\]

**Definition 4.9** For any \( q \in \mathbb{N} \), we define

\[ Z^q (F, \nu, N) = \{ \omega \in \Lambda^q (F, \nu, N) : d\omega = 0 \} \]

the set of **closed differential exterior** \( q \)-**forms** and

\[ \mathcal{B}^q (F, \nu, N) = \{ \omega \in \Lambda^q (F, \nu, N) : \exists \eta \in \Lambda^{q-1} (F, \nu, N) \mid d\eta = \omega \} \]

the set of **exact differential exterior** \( q \)-**forms**.

### 5 Exterior Differential Systems

Let \( \left( (h^* F, h^* \nu, M), [\cdot]_{h^* F}, \left( h^* h, \text{Id}_M \right) \right) \) be the pull-back Lie algebroid of the generalized Lie algebroid \( \left( (F, \nu, N), [\cdot]_{F,h}, (\rho, \eta) \right) \).
Definition 5.1 Any ideal \((\mathcal{I}, +, \cdot)\) of the exterior differential algebra of the pull-back Lie algebroid \(\bigl( (h^* F, h^* \nu, \rho, \eta), \lbrack \cdot, \cdot \rbrack_{h^* F}, \bigl(h^* F, Id_M\bigr) \bigr)\) closed under differentiation operator \(d_{h^* F}\), namely \(d_{h^* F} \mathcal{I} \subseteq \mathcal{I}\), will be called differential ideal of the generalized Lie algebroid \(\bigl( (F, \nu, N), \lbrack \cdot, \cdot \rbrack_{F,h}, (\rho, \eta) \bigr)\).

In particular, if \(h = Id_N = \eta\), then we obtain the definition of the differential ideal of a Lie algebroid. (see [2])

Definition 5.2 Let \((\mathcal{I}, +, \cdot)\) be a differential ideal of the generalized Lie algebroid \(\bigl( (F, \nu, N), \lbrack \cdot, \cdot \rbrack_{F,h}, (\rho, \eta) \bigr)\).

If it exists an IDS \((E, \pi, M)\) such that for all \(k \in \mathbb{N}^*\) and \(\omega \in \mathcal{I} \cap \Lambda^k (h^* F, h^* \nu, M)\), we have \(\omega (u_1, ..., u_k) = 0\), for any \(u_1, ..., u_k \in \Gamma (E, \pi, M)\), then we will say that \((\mathcal{I}, +, \cdot)\) is an exterior differential system (EDS) of the generalized Lie algebroid \(\bigl( (F, \nu, N), \lbrack \cdot, \cdot \rbrack_{F,h}, (\rho, \eta) \bigr)\).

In particular, if \(h = Id_N = \eta\), then we obtain the definition of the EDS of a Lie algebroid. (see [2])

Theorem 5.1 (of Cartan type) The IDS \((E, \pi, M)\) of the generalized Lie algebroid \(\bigl( (F, \nu, N), \lbrack \cdot, \cdot \rbrack_{F,h}, (\rho, \eta) \bigr)\) is involutive, if and only if the ideal generated by the \(\mathcal{F}(M)\)-submodule \(\Gamma (E^0, \pi^0, M)\) is an EDS of the same generalized Lie algebroid.

Proof. Let \((E, \pi, M)\) be an involutive IDS of the generalized Lie algebroid \(\bigl( (F, \nu, N), \lbrack \cdot, \cdot \rbrack_{F,h}, (\rho, \eta) \bigr)\).

Let \(\{\Theta^0, ..., \Theta^p\}\) be a base for the \(\mathcal{F}(M)\)-submodule \(\Gamma (E^0, \pi^0, M)\).

We know that
\[ \mathcal{I} (\Gamma (E^0, \pi^0, M)) = \bigcup_{q \in \mathbb{N}} \{\Omega_\alpha \wedge \Theta^\alpha, \{\Omega_{r+1}, ..., \Omega_p\} \subset \Lambda^q (h^* F, h^* \nu, M)\} \].

Let \(q \in \mathbb{N}\) and \(\{\Omega_{r+1}, ..., \Omega_p\}\) be arbitrary.

Using the Theorems 4.8 and 4.10 we obtain
\[ d_{h^* F} (\Omega_\alpha \wedge \Theta^\alpha) = d_{h^* F} \Omega_\alpha \wedge \Theta^\alpha + (-1)^{q+1} \Omega_\beta \wedge d_{h^* F} \Theta^\beta \]
\[ = \bigl( d_{h^* F} \Omega_\alpha + (-1)^{q+1} \Omega_\beta \wedge \Omega^\beta_\alpha \bigr) \wedge \Theta^\alpha. \]

As
\[ d_{h^* F} \Omega_\alpha + (-1)^{q+1} \Omega_\beta \wedge \Omega^\beta_\alpha \in \Lambda^{q+2} (h^* F, h^* \nu, M) \]

it results that
\[ d_{h^* F} \bigl( \Omega_\beta \wedge \Theta^\beta \bigr) \in \mathcal{I} (\Gamma (E^0, \pi^0, M)) \]

Therefore,
\[ d_{h^* F} \mathcal{I} (\Gamma (E^0, \pi^0, M)) \subseteq \mathcal{I} (\Gamma (E^0, \pi^0, M)) \]

Conversely, let \((E, \pi, M)\) be an IDS of the generalized Lie algebroid
\[ \bigl( (F, \nu, N), \lbrack \cdot, \cdot \rbrack_{F,h}, (\rho, \eta) \bigr) \]
such that the \(\mathcal{F}(M)\)-submodule \(\mathcal{I} (\Gamma (E^0, \pi^0, M))\) is an EDS of the generalized Lie algebroid \(\bigl( (F, \nu, N), \lbrack \cdot, \cdot \rbrack_{F,h}, (\rho, \eta) \bigr)\).
Let $\{\Theta^r, \ldots, \Theta^p\}$ be a base for the $\mathcal{F}(M)$-submodule $(\Gamma (E^0, \pi^0, M), +, \cdot)$. As
\[
d^h* I (\Gamma (E^0, \pi^0, M)) \subseteq \mathcal{I} (\Gamma (E^0, \pi^0, M))
\]
it results that it exists
\[
\Omega^\alpha_\beta \in \Lambda^1 (h^* F, h^* \nu, M), \quad \alpha, \beta \in r + 1, p
\]
such that
\[
d^h* \Theta^\alpha = \sum_{\beta \in r + 1, p} \Omega^\alpha_\beta \wedge \Theta^\beta \in \mathcal{I} (\Gamma (E^0, \pi^0, M)).
\]
Using the Theorem 4.10, it results that $(E, \pi, M)$ is an involutive IDS. \textit{q.e.d.}

6 Torsion and curvature forms. Identities of Cartan and Bianchi type

Using the theory of linear connections of Eresmann type presented in [1] for the diagram:
\[
\begin{array}{ccc}
E & (F, [\cdot]_{F, h}, (\rho, Id_N)) \downarrow \\
\pi & \downarrow h^* \nu \\
M \longrightarrow h & N
\end{array}
\]
where $(E, \pi, M) \in |BY|$ and $((F, \nu, N), [\cdot]_{F, h}, (\rho, Id_N)) \in |GLA|$, we obtain a linear $\rho$-connection $\rho \Gamma$ for the vector bundle $(E, \pi, M)$ by components $\rho \Gamma^a_{\alpha \beta}$. Using the components of this linear $\rho$-connection, we obtain a linear $\rho$-connection $\rho \hat{\Gamma}$ for the vector bundle $(E, \pi, M)$ given by the diagram:
\[
\begin{array}{ccc}
E & (h^* F, [\cdot]_{h^* F}, (h^* F, \rho, Id_M)) \downarrow \\
\pi & \downarrow h^* \nu \\
M \longrightarrow Id_M & N
\end{array}
\]
If $(E, \pi, M) = (F, \nu, N)$, then, using the components of the same linear $\rho$-connection $\rho \Gamma$, we can consider a linear $\rho$-connection $\rho \hat{\Gamma}$ for the vector bundle $(h^* E, h^* \pi, M)$ given by the diagram:
\[
\begin{array}{ccc}
h^* E & (h^* E, [\cdot]_{h^* E}, (h^* E, \rho, Id_M)) \downarrow \\
h^* \pi & \downarrow h^* \pi \\
M \longrightarrow Id_M & M
\end{array}
\]

\textbf{Definition 6.1} If $(E, \pi, M) = (F, \nu, N)$, then the application
\[
\begin{array}{ccc}
\Gamma (h^* E, h^* \pi, M)^2 & (\rho, h)^T \\
(U, V) \longrightarrow \Gamma (h^* E, h^* \pi, M) & \rho T (U, V)
\end{array}
\]
defined by:
\[
(\rho, h) T (U, V) = \rho \hat{D}_U V - \rho \hat{D}_V U - [U, V]_{h^* E},
\]
for any $U, V \in \Gamma (h^* E, h^* \pi, M)$, will be called $(\rho, h)$-torsion associated to linear $\rho$-connection $\rho \Gamma$.

**Remark 6.1** In particular, if $h = Id_M$, then we obtain the application

$$(6.4') \quad \Gamma (E, \pi, M)^2 \xrightarrow{\rho \mathbb{T}} \Gamma (E, \pi, M)$$

defined by:

$$(6.5') \quad \rho \mathbb{T} (u, v) = \rho D_u v - \rho D_v u - [u, v]_E,$$

for any $u, v \in \Gamma (E, \pi, M)$, which will be called $\rho$-torsion associated to linear $\rho$-connection $\rho \Gamma$.

Moreover, if $\rho = Id_{TM}$, then we obtain the torsion $\mathbb{T}$ associated to linear connection $\Gamma$.

**Proposition 6.1** The $(\rho, h)$-torsion $(\rho, h) \mathbb{T}$ associated to linear $\rho$-connection $\rho \Gamma$ is $\mathbb{R}$-bilinear and antisymmetric.

If

$$(6.6) \quad (\rho, h) \mathbb{T}^c_{ab} (S_a, S_b) = (\rho, h) \mathbb{T}^c_{ab} S_c$$

then

$$(6.6') \quad (\rho, h) \mathbb{T}^c_{ab} = \rho \Gamma^c_{ab} - \rho \Gamma^c_{ba} - L^c_{ab} \circ h.$$

In particular, if $h = Id_M$ and $\rho \mathbb{T}^c (s_a, s_b) = (\rho \mathbb{T}^c_{ab} s_c)$, then

$$(6.6'') \quad \mathbb{T}^c_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}.$$

**Definition 6.2** If $(E, \pi, M) = (F, \nu, N)$, then the vector valued 2-form

$$(6.7) \quad (\rho, h) \mathbb{T} = ((\rho, h) \mathbb{T}^c_{ab} S_c) S^a \wedge S^b$$

will be called the vector valued form of $(\rho, h)$-torsion $(\rho, h) \mathbb{T}$.

In particular, if $h = Id_M$, then the vector valued 2-form

$$(6.7') \quad \rho \mathbb{T} = (\rho \mathbb{T}^c_{ab} s_c) s^a \wedge s^b$$

will be called the vector form of $\rho$-torsion $\rho \mathbb{T}$.

Moreover, if $\rho = Id_{TM}$, then the vector valued form $(6.7')$ becomes:

$$(6.7'') \quad \mathbb{T}^i_{jk} = (\mathbb{T}_{jk}^{\frac{\partial}{\partial x^j}}) dx^i \wedge dx^k.$$

**Definition 6.3** For each $c \in \mathbb{1}^n$ we obtain the scalar 2-form of $(\rho, h)$-torsion $(\rho, h) \mathbb{T}$

$$(6.8) \quad (\rho, h) \mathbb{T}^c = (\rho, h) \mathbb{T}^c_{ab} S^a \wedge S^b.$$
In particular, if $h = Id_M$, then, for each $c \in \Gamma M$, we obtain the scalar 2-form of $\rho$-torsion $\rho T$

\[
(6.8') \quad \rho T^c = \rho T^c \chi^a \wedge \chi^b.
\]

Moreover, if $\rho = Id_{TM}$, then the scalar 2-form (6.9') becomes:

\[
(6.8'') \quad T^i = T^i_{jk} dx^j \wedge dx^k.
\]

**Definition 6.4** The application

\[
(6.9) \quad (\Gamma (h^*F, h^*\nu, M)^2 \times \Gamma (E, \pi, M)) \overset{(\rho, h)\mathbb{R}}{\longrightarrow} \Gamma (E, \pi, M)
\]

defined by

\[
(6.10) \quad (\rho, h)\mathbb{R} (Z, V) u = \rho D_Z \left( \rho D_V u \right) - \rho D_V \left( \rho D_Z u \right) - \rho D_{[Z,V]}h^* u,
\]

for any $Z, V \in \Gamma (h^*F, h^*\nu, M)$, $u \in \Gamma (E, \pi, M)$, will be called $(\rho, h)$-curvature associated to linear $\rho$-connection $\rho \Gamma$.

**Remark 6.2** In particular, if $h = Id_M$, then we obtain the application

\[
(6.9') \quad \Gamma (F, \nu, M) \times \Gamma (E, \pi, M) \overset{\rho \mathbb{R}}{\longrightarrow} \Gamma (E, \pi, M)
\]

defined by

\[
(6.10') \quad \rho \mathbb{R} (z, v) u = \rho D_z (\rho D_v u) - \rho D_v (\rho D_z u) - \rho D_{[z,v]}u,
\]

for any $z, v \in \Gamma (F, \nu, M)$, $u \in \Gamma (E, \pi, M)$, which will be called $\rho$-curvature associated to linear $\rho$-connection $\rho \Gamma$.

Moreover, if $\rho = Id_{TM}$, then we obtain the curvature $\mathbb{R}$ associated to linear connection $\Gamma$.

**Proposition 6.2** The $(\rho, h)$-curvature $(\rho, h)\mathbb{R}$ associated to linear $\rho$-connection $\rho \Gamma$, is $\mathbb{R}$-linear in each argument and antisymmetric in the first two arguments.

If

\[
(\rho, h)\mathbb{R} (T_\beta, T_\alpha) s_b \overset{\text{put}}{=} (\rho, h)\mathbb{R}^a_{\alpha \beta} s_a,
\]

then

\[
(6.11) \quad \rho \mathbb{R}^a_{\alpha \beta} = \rho^i \partial h^{\alpha \beta} + \rho^{\alpha \beta} \Gamma e^\beta_{\alpha \beta} - \rho^i \partial h^{\alpha \beta} + \rho^{\alpha \beta} \nabla e^\beta_{\alpha \beta} \circ h.
\]

In particular, if $h = Id_M$ and $\rho \mathbb{R} (t_\beta, t_\alpha) s_b \overset{\text{put}}{=} \rho \mathbb{R}^a_{\alpha \beta} s_a$, then

\[
(6.11') \quad \rho \mathbb{R}^a_{\alpha \beta} = \rho^i \partial h^{\alpha \beta} + \rho^{\alpha \beta} \Gamma e^\beta_{\alpha \beta} - \rho^i \partial h^{\alpha \beta} + \rho^{\alpha \beta} \nabla e^\beta_{\alpha \beta} \circ h.
\]

Moreover, if $\rho = Id_{TM}$, then equality (6.11') becomes:

\[
(6.11'') \quad \mathbb{R}^a_{b \alpha \beta} = \frac{\partial \Gamma^a_{b \alpha \beta}}{\partial x^b} + \Gamma^a_{b \alpha \beta} \nabla e^\beta_{b \alpha \beta} - \frac{\partial \Gamma^a_{b \alpha \beta}}{\partial x^\beta} - \Gamma^a_{b \alpha \beta} \nabla e^\beta_{b \alpha \beta}.
\]
Definition 6.5 The vector mixed form

\[(\rho, h) \mathbb{R} = \left( (\rho, h) \mathbb{R}_{b}^{\alpha \beta} s_{a} \right) T^{a} \wedge T^{\beta} \]  

will be called the vector valued form of \((\rho, h)\)-curvature \((\rho, h) \mathbb{R}\).

In particular, if \(h = \text{Id}_{M}\), then the vector mixed form

\[(6.12') \quad \rho \mathbb{R} = \left( (\rho \mathbb{R}_{b}^{\alpha \beta} s_{a}) t^{a} \wedge t^{\beta} \right) s^{b} \]

will be called the vector valued form of \(\rho\)-curvature \(\rho \mathbb{R}\).

Moreover, if \(\rho = \text{Id}_{TM}\), then the vector form \((6.12')\) becomes:

\[(6.12'') \quad \mathbb{R} = \left( (\mathbb{R}_{b}^{a} h_{k} s_{a}) dx^{h} \wedge dx^{k} \right) s^{b}.\]

Definition 6.6 For each \(a, b \in \mathbb{I}, \mathbb{n}\) we obtain the scalar 2-form of \((\rho, h)\)-curvature \((\rho, h) \mathbb{R}\)

\[(6.13) \quad (\rho, h) \mathbb{R}_{b}^{a} = (\rho, h) \mathbb{R}_{b}^{a} T^{\alpha} \wedge T^{\beta}.\]

In particular, if \(h = \text{Id}_{M}\), then, for each \(a, b \in \mathbb{I}, \mathbb{n}\), we obtain the scalar 2-form of \(\rho\)-curvature \(\rho \mathbb{R}\)

\[(6.13') \quad \rho \mathbb{R}_{b}^{a} = \rho \mathbb{R}_{b}^{a} t^{\alpha} \wedge t^{\beta}.\]

Moreover, if \(\rho = \text{Id}_{TM}\), then the scalar form \((6.13')\) becomes:

\[(6.13'') \quad \mathbb{R}_{b}^{a} = \mathbb{R}_{b}^{a} h_{k} dx^{h} \wedge dx^{k}.\]

Theorem 6.1 The identities

\[(C_{1}) \quad (\rho, h) \mathbb{T}^{a} = d^{h*F} S^{a} + \Omega_{b}^{a} \wedge S^{b},\]

and

\[(C_{2}) \quad (\rho, h) \mathbb{R}_{b}^{a} = d^{h*F} \Omega_{b}^{a} + \Omega_{c}^{a} \wedge \Omega_{b}^{c}\]

hold good. These will be called the first respectively the second identity of Cartan type.

Proof. To prove the first identity we consider that \((E, \pi, M) = (F, \nu, M)\). Therefore, \(\Omega_{b}^{a} = \rho \Gamma_{bc}^{a} S^{c}\). Since

\[d^{h*F} S^{a}(U, V) S_{a} = ((\Gamma^{h*F} \rho, \text{Id}_{M}) U) S^{a}(V) - (\Gamma^{h*F} \rho, \text{Id}_{M}) V (S^{a}(U)) - S^{a}([U, V]_{h*F}) S_{a}\]

\[= (\Gamma^{h*F} \rho, \text{Id}_{M}) U (V^{a}) - (\Gamma^{h*F} \rho, \text{Id}_{M}) V (U^{a}) - S^{a}([U, V]_{h*F}) S_{a}\]

\[= \rho \tilde{D}_{U} V - V^{b} \rho \tilde{D}_{U} S_{b} - \rho \tilde{D}_{V} U - U^{b} \rho \tilde{D}_{V} S_{b} - [U, V]_{h*F}\]

\[= (\rho, h) \mathbb{T}(U, V) - (\rho \Gamma_{bc}^{a} V^{b} U^{c} - \rho \Gamma_{bc}^{a} U^{c} V^{b}) S_{a}\]

\[= ((\rho, h) \mathbb{T}(U, V) - \Omega_{b}^{a} \wedge S^{b}(U, V)) S_{a},\]

it results the first identity.
To prove the second identity, we consider that \((E, \pi, M) \neq (F, \nu, M)\). Since
\[
(\rho, h) \mathbb{R}^a_b (Z, W) s_a = (\rho, h) \mathbb{R} ((W, Z), s_b)
\]
\[
= \rho \mathcal{D}_Z (\rho \mathcal{D}_W s_b) - \rho \mathcal{D}_W (\rho \mathcal{D}_Z s_b) - \rho \mathcal{D}_{[Z, W]_{h^*F}} s_b
\]
\[
= \rho \mathcal{D}_Z (\Omega^a_b (W) s_a) - \rho \mathcal{D}_W (\Omega^a_b (Z) s_a) - \Omega^a_b ([Z, W]_{h^*F}) s_a
\]
\[
+ (\Omega^a_c (Z) \Omega^a_b (W) - \Omega^a_c (W) \Omega^a_b (Z)) s_a
\]
\[
= (d^{h^*F} \Omega^a_b (Z, W) + \Omega^a_b \wedge \Omega^a_c (Z, W)) s_a
\]
it results the second identity.

**Corollary 6.1** In particular, if \(h = \text{Id}_M\), then the identities \((C_1)\) and \((C_2)\) become

\[(C'_1)\]
\[
\rho \mathcal{T}^a = d^F s^a + \omega^a_b \wedge s^b,
\]

and

\[(C'_2)\]
\[
\rho \mathbb{R}^a_b = d^F \omega^a_b + \omega^a_b \wedge \omega^c_b
\]
respectively.

Moreover, if \(\rho = \text{Id}_{TM}\), then the identities \((C'_1)\) and \((C'_2)\) become:

\[(C''_1)\]
\[
\mathcal{T}^i = ddx^i + \omega^i_j \wedge dx^j = \omega^i_j \wedge dx^j
\]
and

\[(C''_2)\]
\[
\mathbb{R}^i_j = d\omega^i_j + \omega^i_h \wedge \omega^h_j
\]
respectively.

**Theorem 6.2** The identities

\[(B_1)\]
\[
d^{h^*F} (\rho, h) \mathcal{T}^a = (\rho, h) \mathcal{R}^a_b \wedge S^b - \Omega^a_c \wedge (\rho, h) \mathcal{T}^c
\]

and

\[(B_2)\]
\[
d^{h^*F} (\rho, h) \mathbb{R}^a_b = (\rho, h) \mathcal{R}^a_b \wedge \Omega^a_c - \Omega^a_c \wedge (\rho, h) \mathbb{R}^c_b
\]
hold good. We will called these the first respectively the second identity of Bianchi type.

If the \((\rho, h)\)-torsion is null, then the first identity of Bianchi type becomes:

\[(\tilde{B}_1)\]
\[
(\rho, h) \mathbb{R}^a_b \wedge s^b = 0.
\]

**Proof.** We consider \((E, \pi, M) = (F, \nu, M)\). Using the first identity of Cartan type and the equality \(d^{h^*F} \circ d^{h^*F} = 0\), we obtain:

\[
d^{h^*F} (\rho, h) \mathcal{T}^a = d^{h^*F} \Omega^a_c \wedge S^b - \Omega^a_c \wedge d^{h^*F} S^c.
\]

Using the second identity of Cartan type and the previous identity, we obtain:

\[
d^{h^*F} (\rho, h) \mathcal{T}^a = ((\rho, h) \mathbb{R}^a_b - \Omega^a_b \wedge \Omega^a_c) \wedge S^b - \Omega^a_c \wedge (\rho, h) \mathcal{T}^c - \Omega^a_c \wedge S^b.
\]

After some calculations, we obtain the first identity of Bianchi type.
Using the second identity of Cartan type and the equality $d^*F \circ d^*F = 0$, we obtain:

$$d^*F \Omega^a_c \wedge \Omega^b_b - \Omega^c_c \wedge d^*F \Omega^a_c = d^*F (\rho, h) \mathbb{R}^a_b.$$ 

Using the second of Cartan type and the previous identity, we obtain:

$$d^*F (\rho, h) \mathbb{R}^a_b = ((\rho, h) \mathbb{R}^a_c \wedge \Omega^a_c) \wedge \Omega^b_b - ((\rho, h) \mathbb{R}^c_b - \Omega^c_b \wedge \Omega^b_b).$$

After some calculations, we obtain the second identity of Bianchi type. 

q.e.d.

**Corollary 6.2** In particular, if $h = Id_M$, then the identities $(B_1)$ and $(B_2)$ become

$$(B'_1) \quad d^F \rho^a = \rho \mathbb{R}^a_b \wedge s^b - \omega^a_c \wedge \rho \mathbb{R}^c_e$$

and

$$(B'_2) \quad d^F \rho^a_b = \rho \mathbb{R}^a_c \wedge \omega^c_b - \omega^a_c \wedge \rho \mathbb{R}^c_e,$$

respectively.

Moreover, if $\rho = Id_{TM}$, then the identities $(B'_1)$ and $(B'_2)$ become:

$$(B''_1) \quad dT^a = \mathbb{T}^a_j \wedge dx^j - \omega^i_k \wedge T^k$$

and

$$(B''_2) \quad d\mathbb{T}^i_j = \mathbb{T}^i_h \wedge \omega^h_j - \omega^i_h \wedge \mathbb{T}^h_j,$$

respectively.

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