Airfoil shape optimization based on Non Uniform Rational B-spline and optimization algorithm

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Abstract. In this article, we propose an innovation method of airfoil shape optimization, on the premise of ensuring the fitting accuracy; try to improve the smoothness of the airfoil. Non Uniform Rational B-Spline and quadratic penalty function are employed to parametric modelling and smoothness optimization. The monotone curvature variation is used to design smooth curves and least squares method ensuring high fitting accuracy. This program has been tested for a standard NACA 2415 and S825 airfoil and optimized to improve its smoothness. The results of optimization examples show that this method has high efficiency and precision.

1. Introduction
The airfoil curve is generally expressed by a group of discrete data points, without specific function description. The existing parametric characterization methods of airfoil curve mainly include CST method, spline fitting method, parsec method and Hicks Henne parametric method. These methods have advantages and disadvantages in fitting accuracy smoothness and robustness. In order to express conic arc accurately and have the advantages of B-spline curve, non-uniform rational B-spline (NURBS) curve came into being and developed rapidly [1-2].

2. Mathematical model

2.1. Optimization problem
A Non Uniform Rational B-Spline curve is defined by [3]

\[
p(u) = \sum_{i=0}^{n} R_{i,k}(u)d_i = \begin{bmatrix} R_{i_0}(u) & R_{i_1}(u) & \cdots & R_{i_k}(u) \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_n \end{bmatrix}
\]

(1)

\[R_{i,k}(u)\] is called k-th rational basis function expressed as

\[ R_{i,k}(u) = \frac{a_i N_{i,k}(u)}{\sum_{j=0}^{n} a_j N_{j,k}(u)} \]

(2)
2.2. Objective function

In order to avoid the complexity of nonlinear problem solving, we set up a mathematical model of unknown control vertices after parameterizing the data points and determining the node vector. The constructed approximation curve is not exactly passing through the data points, but is close to the data points in a certain sense. The objective function $E$ is defined as the sum of the squares of the distances between the points with parameter values on the approximation curve and the data points, which satisfies the endpoint constraints $I$ is about $0.01$

\[ E = \sum_{i=0}^{m-1} \left\| q_i - p(u_i) \right\|^2 = \sum_{i=0}^{m} \left\| q_i - \sum_{j=0}^{q} d R_{ij}(u_i) \right\|^2 \]  

\[ \chi_i = q_i - q_0 N_{m,0}(\bar{u}) - q_m N_{m,m}(\bar{u}) \quad i = 1, 2, \ldots, m - 1 \]

Take the parameter value into the above formula to get

\[ E = \sum_{i=1}^{m} \left\{ \chi_i - \sum_{j=1}^{q} d R_{ij}(\bar{u}) \right\} \]

However, the least square method only reduces the approximation error value; that is to say, the generated curve meets the fitting accuracy requirements but may still have the problem of smoothness. FarIn et al [4] think that if the curve is smooth, its curvature is continuous and consists of some monotone segments. H.P. Moreton et al [5] put forward the curve fairing standard according to the curvature change. They minimize the integral of the square of the curvature derivative of the arc length to construct the minimum variation curves, as shown in the formula.

\[ MVC = \int \frac{1}{\kappa'(u)} \right| du \]

\[ \kappa'(u) = \left| \frac{p'(u) \times p''(u)}{p'(u) \times p''(u)} \right| \frac{p'(u) \times p''(u)}{p'(u)} \]

The monotone discriminant of curve is

\[ \lambda = ab - 3cd \]

\[ a = \left| p'(u) \times p''(u) \right| \]

\[ b = \left| p'(u) \times p''(u) \right| \]

\[ c = (p'(u) \times p''(u)) \]

\[ d = \left| p'(u) \times p''(u) \right| \]

If $\lambda > 0$ is satisfied, the curvature of the curve is monotonically increasing, otherwise, it is monotonically decreasing. At that time, it is the extreme point of curvature when $\lambda = 0$. The precondition to ensure the smoothness of the curve is that the curvature is monotonous, that is, the number of curvature extreme points should be as small as possible, so it is important to control the number of curvature extreme points for the smoothness of the curve. Yulin Wang [6] gives the monotone formula of curvature as
\[ \lambda_j(t) = \xi_{s_j} B_{4k-6}^j(t), \quad s = 0, \ldots, 4k - 6 \]  
\[ \xi_{s_j} = k^{(k - 1)}(k - 2)(\Delta h_{s_j} \times \Delta h_{s_j}) (\Delta h_{s_j} \times \Delta h_{s_j}) \alpha_{4k-6,4k-6} \Delta h_{s_j} \Delta h_{s_j} \alpha_{4k-6,4k-6} \]  

If the following relation is satisfied, the curvature of the curve is monotonic

\[ \xi_{s_j} \geq 0 \quad \text{or} \quad \xi_{s_j} \leq 0, \quad s = 0, \ldots, 4k - 6 \]  

2.3. Optimization method

The optimization of airfoil parametric fairing modeling is an approximate optimization problem under monotonic curvature constraints. There are many optimization methods, including simplex method, Newton method, and gradient descent method and so on. Due to the existence of constraints, it is more difficult to solve constrained optimization problems than unconstrained optimization problems. Therefore, one of the effective methods at present is to transform constrained optimization problems into unconstrained optimization problems, that is, to impose a penalty term to punish the infeasible iteration points. In the process of iteration, the penalty amount is constantly increased to drive the iteration points closer to the feasible area. This is the application of penalty function method, in which the most widely used penalty function is the quadratic penalty function [7-8].

Consider constrained optimization

\[ \min f(x) \]

\[ \text{s.t. } c_i(x) = 0, \quad i = 1, 2, \ldots, m \]

The quadratic penalty function is

\[ F(x, M) = f(x) + M \left\{ \sum_{i=1}^{m} c_i(x) + \sum_{i=m+1}^{p} \left[ \min(0, c_i(x)) \right]^2 \right\} \]

\[ M \] is penalty factor, \( P(x, M) \) is penalty function.

The optimization problem to be solved in this paper is to construct a cubic B-spline curve, which can approach the data points of the given fixed wing and satisfy the requirement of monotonic curvature of the curve. The mathematical solution model is defined as

\[ f(x) = \min \left( \sum_{i=1}^{p} \left\| q_i - p(u_i) \right\|^2 \right) \]

\[ \text{s.t. } \delta_{s_j} \xi_{s_j} \geq 0, \quad s = 0, \ldots, 4k - 6 \]

\( \delta_j \in \{1, -1\} \) is the curvature monotone factor. If \( \delta_j = 1 \) it represents monotonic increasing of curvature, \( \delta_j = -1 \) it is monotonic decreasing of curvature. The above formula is an inequality constrained optimization problem, for which the equivalent equality constraint is

\[ \min(0, \delta_{s_j} \xi_{s_j}) = 0, \quad s = 0, \ldots, 4k - 6 \]

Therefore, the above formula is transformed into an unconstrained problem, and the penalty function of the outer point is constructed as

\[ F(x, M) = \sum_{i=1}^{p} \left\| q_i - p(u_i) \right\|^2 + M \sum_{i=1}^{p} \left[ \min(0, \delta_{s_j} \xi_{s_j}) \right]^2 \]

Satisfy

\[ M \sum_{i=1}^{p} \left[ \min(0, \delta_{s_j} \xi_{s_j}) \right]^2 \geq 0 \]

\[ M \delta_j < M_1 < \cdots < M_p, \lim_{M_{\alpha \to \infty}} M_\alpha \to \alpha, M_\alpha = cM_{\alpha - 1} \]

The quasi Newton method is used to solve the minimum value of the function. The quasi Newton method can construct the curvature approximation of the objective function under the condition of satisfying the value and the first derivative of the objective function, so that the Hesse matrix can be
calculated without the second derivative. The efficiency of the quasi Newton method is much higher than that of the Newton method.

Iterative point of unconstrained optimization problem

\[ x_{k+1} = x_k - \alpha_k H_k \nabla f(x_k) \]  

where \( \alpha_k = 1, H_k = \left[ \nabla^2 f(x_k) \right]^{-1} \) is Newton's method, The central idea of quasi Newton method is to construct similar matrix.

The new iteration point is

\[ x_{k+1} = x_k - \alpha_k B_k^{-1} \nabla f(x_k) \]  

3. Optimization process
The optimization process is as shown in the figure2 and table 1.

![Figure 2. Optimization flow chart](image_url)

### Table 1 Fairing curve fitting algorithm of airfoil

| Algorithm: Pseudo code of fitting optimization algorithm |
|----------------------------------------------------------|
| **Input:** Initial knot vector, degree k of the curve(k=3), Initial solution \( x_0 = \{ x_{d_{i,a}}, y_{d_{i,a}}, \ldots, x_{d_{k,a}}, y_{d_{k,a}} \} \) \( \varepsilon = 10^{-6} \), \( M_1 < M_2 < \ldots < M_k \), \( \lim_{k \to \infty} M_k = cM_k, c = 5 \sim 10 \) |
| **Output:** Optimal positions of control points |
| **Step1:** Using Quasi Newton method to solve unconstrained problems \( \min F(x, M_k) \) |
| One: Set initial solution \( x_0, \varepsilon = 10^{-6}, k = 0 \). |
| Two: If \( \| \nabla f(x_k) \| \leq \varepsilon \), stop. |
| Three: Compute search direction \( d_k = -\nabla f(x_k) \). |
| Four: Solving step factor \( \alpha_k \) and set \( x_{k+1} = x_k + \alpha_k d_k \). |
| Five: Correcting \( B_k \) produces \( B_{k+1} \). |
| Six: Set \( k := k + 1 \), and turn to two. |
| **Step2:** Test iteration termination criteria |
| while \( \| x_k(M_k) - x(M_{k-1}) \| \leq \varepsilon \) and \( \left| \frac{f[x_k(M_k)] - f[x_k(M_{k-1})]}{f[x_k(M_{k-1})]} \right| \leq \varepsilon \) then |
| Output Optimal solution \( x_k \) |
| Else |


Set $M_{k+1} = cM_k$, $x_0 = x_kM_k$, $k = k + 1$
Using Quasi Newton method to solve unconstrained problems $\min F(x, M_k)$
End for

4. Results and analysis
Selected two airfoil as optimization cases, among them, naca2415 airfoil has smaller rate of curvature change, convergence after 30 steps of iteration, the S825 airfoil has more complex curves, especially the tail of airfoil, so his iteration steps are longer than naca2415.

![Figure 3. The iterative process](image1)

![Figure 4. Iterative error curve](image2)

![Figure 5. Curvature of naca2415airfoil and S825 curve](image3)

Table 2 compares the upper and lower surface accuracy after airfoil fitting, the fitting accuracy of two airfoils has been greatly improved. It can be seen from Figure 3-5 that the airfoil curve has good smoothness. The result of optimization reaches the goal.
Table 2 Fitting error

|          | Upper surface | Lower surface |
|----------|---------------|---------------|
| NACA2415 | 0.000552      | 0.000518      |
| S825     | 0.000788      | 0.000647      |

5. Conclusion

NURBS method has powerful function in shape definition; the objective of this article was to introduce a new optimization method for object reconstruction by means of smooth NURBS curves. In this optimization problem the nurb parameters are used to describe the shape of the airfoil, the control points and their respective weights are considered as design variables in the optimization process. The case study of airfoil fitting optimization shows that on the premise of ensuring the fitting accuracy, the smoothness of airfoil curve is improved.

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