Low-Memory Implementations of Ridge Solutions for Broad Learning System with Incremental Learning

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Abstract—The existing low-memory BLS implementation proposed recently avoids the need for storing and inverting large matrices, to achieve efficient usage of memories. However, the existing low-memory BLS implementation sacrifices the testing accuracy as a price for efficient usage of memories, since it can no longer obtain the generalized inverse or ridge solution for the output weights during incremental learning, and it cannot work under the very small ridge parameter (i.e., $\lambda = 10^{-8}$) that is utilized in the original BLS. Accordingly, it is required to develop the low-memory BLS implementations, which can work under very small ridge parameters and compute the generalized inverse or ridge solution for the output weights in the process of incremental learning.

In this paper, firstly we propose the low-memory implementations for the recently proposed recursive and square-root BLS algorithms on added inputs and the recently proposed square-root BLS algorithm on added nodes, by simply processing a batch of inputs or nodes in each recursion. Since the recursive BLS implementation includes the recursive updates of the inverse matrix that may introduce numerical instabilities after a large number of iterations, and needs the extra computational load to decompose the inverse matrix into the Cholesky factor when cooperating with the proposed low-memory implementation of the square-root BLS algorithm on added nodes, we only improve the low-memory implementations of the square-root BLS algorithms on added inputs and nodes, to propose the full low-memory implementation of the square-root BLS algorithm.

All the proposed low-memory BLS implementations compute the ridge solution for the output weights in the process of incremental learning, and most of them can work under very small ridge parameters. When the ridge parameter is not too small, the proposed low-memory implementations for the recursive and square-root BLS algorithms on added inputs and nodes for added inputs of the proposed full low-memory BLS implementation usually achieve better testing accuracies than the existing low-memory BLS implementation on added inputs. More importantly, when the ridge parameter is very small (i.e., $\lambda = 10^{-8}$) as in the original BLS, the existing low-memory BLS implementation on added inputs cannot work in any update (of the incremental learning), the proposed low-memory implementation for the recursive BLS algorithm on added inputs cannot work in the last update, while the proposed low-memory implementation for the square-root BLS algorithm on added inputs and the proposed full low-memory implementation of the square-root BLS algorithm (on added inputs and nodes) can work in all updates.

With respect to the existing low-memory BLS implementation on added inputs, the proposed low-memory implementation for the recursive BLS algorithm on added inputs and the part for added inputs of the proposed full low-memory implementation of the square-root BLS algorithm require nearly the same training time, while the proposed low-memory implementation for the square-root BLS algorithm on added inputs requires much more training time. On the other hand, the part for added nodes of the proposed full low-memory implementation of the square-root BLS algorithm speeds up the proposed low-memory implementation for the square-root BLS algorithm on new added nodes by a factor in the range from 3.77 to 23.73.

The proposed full low-memory implementation of the square-root BLS algorithm on added inputs and nodes can work when the ridge parameter is very small (i.e., $\lambda = 10^{-8}$) as in the original BLS. The part for added inputs of the proposed full low-memory implementation of the square-root BLS algorithm takes nearly the same training time to achieve better testing accuracies with respect to the existing low-memory BLS implementation on added inputs, is numerically more stable than the proposed low-memory implementation for the recursive BLS algorithm on added inputs, and is much faster than the proposed low-memory implementation for the square-root BLS algorithm on added inputs. On the other hand, the part for added nodes of the proposed full low-memory implementation of the square-root BLS algorithm is obviously faster than the proposed low-memory implementation for the square-root BLS algorithm on added nodes. Accordingly, it can be concluded that the proposed full low-memory implementation of the square-root BLS algorithm on added inputs and nodes is a good low-memory implementation of the original BLS on added nodes and inputs.

Index Terms—Broad learning system (BLS), incremental learning, added inputs, added nodes, inverse of a sum of matrices, random vector functional-link neural networks (RVFLNN), single layer feedforward neural networks (SLFN), low-memory implementations, partitioned matrix, inverse Cholesky factorization, ridge inverse, ridge solution.

I. INTRODUCTION

For the classification and regression problems, Single layer feedforward neural networks (SLFN) have been widely applied [1]–[3]. SLFN can be trained by traditional gradient-descent algorithms [4], [5], which are usually time-consuming and require huge processing power. Accordingly, the random vector functional-link neural network (RVFLNN) has been utilized to overcome the need for gradient-descent training [2].

To model time-variety data with moderate size, a dynamic step-wise updating algorithm was proposed in [6] for the RVFLNN model, which includes the concept of “incremental learning,” i.e., remodelling the network in an incremental way without a complete retraining process. When a new input is added or a new node is inserted, the step-wise updating algorithm in [6] remodels the network by only computing the pseudoinverse of that added input or node. Recently to deal with time-variety big data with high dimension, Broad Learning System (BLS) was proposed in [7], which improves
the step-wise updating algorithm in [6]. In BLS, the input data is transformed into the feature nodes to reduce the data dimensions, while the output weights are computed by the generalized inverse with the ridge regression approximation to achieve a better generalization performance, which assumes the ridge parameter $\lambda \rightarrow 0$ in the ridge inverse [9].

To improve the original BLS on new added inputs in [7], the inverse of a sum of matrices [10] was utilized in [11], to accelerate a step in the generalized inverse of a row-partitioned matrix. Moreover, two improved BLS algorithms were proposed in [12] to further accelerate the BLS algorithm on new added inputs, which compute the ridge solution [9] for the output weights to achieve a better generalization performance, instead of the generalized inverse solution with the ridge regression approximation utilized in the original BLS [7]. Accordingly in [12], it is no longer required to assume the ridge parameter $\lambda \rightarrow 0$ in the ridge inverse, and $\lambda$ can be any positive real number. To reduce the computational complexity, the recursive BLS algorithm and the square-root BLS algorithm proposed in [12] compute the output weights from the inverse and the inverse Cholesky factor of the Hermitian matrix in the ridge inverse, respectively, which are usually smaller than the ridge inverse. On the other hand, the square-root BLS algorithms on added nodes have been proposed in [13], [14] to improve the original BLS on added nodes in [7]. The square-root BLS algorithm proposed in [14] still computes the generalized inverse solution for the output weights, while in [13], the square-root BLS algorithm has been proposed to compute the ridge solution for the output weights from the inverse Cholesky factor of the Hermitian matrix in the ridge inverse.

Recently, a low-memory implementation of BLS was proposed in [15] for big-data scenarios, which avoids the need for storing and inverting large matrices, and finds a good tradeoff between efficient usage of memories and required iterations. However, the low-memory incremental learning algorithm proposed in [15] can no longer obtain the generalized inverse or ridge solution for the output weights, and then it sacrifices the testing accuracy as a price for efficient use of memories, with respect to the generalized inverse solution (with the ridge regression approximation) in [7] and the ridge solution in [12], [13]. On the other hand, the low-memory implementation in [15] has not provided the simulations under very small ridge parameters as the original BLS [1] and the efficient BLS algorithms proposed in [12], [13]. Accordingly, it is still required to develop the low-memory BLS implementations, which can work under very small ridge parameters and compute the generalized inverse or ridge solution for the output weights in the process of incremental learning.

In this paper, firstly we propose the low-memory implementations for both BLS algorithms on added inputs proposed in [12] and the BLS algorithm on added nodes proposed in [13], by simply processing a batch of inputs or nodes in each recursion. Then we propose the full low-memory implementation of the square-root BLS algorithm for added inputs and nodes, which is better than the above-mentioned low-memory BLS implementations. All the proposed low-memory BLS implementations compute the ridge solution for the output weights in the process of incremental learning, and most of them can work under very small ridge parameters.

This paper is organized as follows. Section II introduces the existing incremental BLS algorithms on added inputs and nodes. In Section III, we propose the low-memory implementations of the BLS algorithms proposed in [12], [13]. Then in Section IV, we propose the full low-memory implementation of the square-root BLS algorithm on added inputs and nodes that is better than the implementations proposed in Section III. The presented low-memory BLS implementations are compared by numerical experiments in Section V, and conclusions are given in Section VI finally.

II. EXISTING INCREMENTAL BLS ON ADDED INPUTS AND NODES

In this section, we will introduce the model of BLS, the generalized inverse solution and the ridge solution for BLS, and the incremental BLS algorithms on added inputs and nodes proposed in [12] and [13], respectively.

A. Broad Learning Model

In the BLS, the original input data $X \in \mathbb{R}^{N \times q}$ with $l$ training samples is projected by

$$Z_i = \phi(XW_e + \beta_e),$$

(1)
to become the $i$-th group of mapped features $Z_i$, where the weights $W_e$ and the biases $\beta_e$ are randomly generated and then fine-tuned by applying the linear inverse problem [7]. All the $n$ groups of mapped features can be concatenated into

$$Z^n = [Z_1 \cdots Z_n],$$

(2)

which are then enhanced by

$$H_j = \xi(Z^n W_h + \beta_h),$$

(3)
to become the $j$-th group of enhancement nodes $H_j$, where $W_h$ and $\beta_h$ are randomly generated. All the $m$ groups of enhancement nodes can be concatenated into

$$H^m = [H_1, \cdots, H_m].$$

(4)

The above described procedure to initialize BLS is summarized in Algorithm 1 by the function

$$\left(Z^n, H^m, W^n_e, \beta^n_e, W^n_h, \beta^n_h\right) = \text{Initialize}(X).$$

(5)

The $n$ groups of feature nodes $Z^n$ and the $m$ groups of enhancement nodes $H^m$ form the expanded input matrix

$$A^{n,m} = [Z^n | H^m],$$

(6)

where the superscripts $n, m$ in $A^{n,m}$ denote the numbers of feature and enhancement node groups, respectively. Finally, the connections of all the nodes are fed into the output by

$$\hat{Y} = A^{n,m} W^{n,m},$$

(7)
Algorithm 1 The Procedure to Initialize BLS

function Initialize(X)
    Compute $Z_i = \phi(XW_{e_i} + \beta_{e_i})$ with fine-tuned random $W_{e_i}$ and $\beta_{e_i}$ for $i = 1 : n$;
    Set $Z^n \equiv [Z_1 \cdots Z_n]$, $W_e^n \equiv [W_{e_1} \cdots W_{e_n}]$, $\beta_e^n \equiv [\beta_{e_1} \cdots \beta_{e_n}]$;
    Compute $H_j = \xi(Z^n W_{h_j} + \beta_{h_j})$ with random $W_{h_j}$ and $\beta_{h_j}$ for $j = 1 : m$;
    Set $H^m \equiv [H_1, \cdots, H_m]$, $W_h^m \equiv [W_{h_1} \cdots W_{h_m}]$, $\beta_h^m \equiv [\beta_{h_1} \cdots \beta_{h_m}]$;
    return $Z^n, H^m, W_e^n, \beta_e^n, W_h^m, \beta_h^m$
end function

where the output weight $W^{n,m} \in \mathbb{R}^{k \times c}$, the output $\hat{Y} \in \mathbb{R}^{l \times c}$, and $c$ is the size of the output. Assume that there are $k$ nodes totally. Then it can be seen that the expanded input matrix $A^{n,m} \in \mathbb{R}^{n \times k}$ includes $l$ training samples and $k$ nodes. In this paper, sometimes we add the subscript to a matrix to indicate the number of nodes, e.g., $A^{n,m}_l$ and $W^{n,m}_k$, or add the dotted subscript to a matrix to indicate the number of training samples, e.g., $A^{n,m}_l$ and $W^{n,m}_k$. Moreover, sometimes we also omit the superscripts for simplicity. For example, we may write $A^{n,m}_k, W^{n,m}_k, A^{n,m}_l$ and $W^{n,m}_k$ as $A_k, W_k, A_l$ and $W_f$, respectively.

B. Generalized Inverse Solution and Ridge Solution for BLS

The least-square solution [6] of (7) is the generalized inverse solution [9]

$$W = A^+ Y,$$

where the output label $Y \in \mathbb{R}^{l \times c}$ corresponds to the input $X$, and the generalized inverse $A^+ \in \mathbb{R}^{k \times l}$ satisfies

$$A^+ = (A^T A)^{-1} A^T.$$

The BLS algorithm in [7] computes the generalized inverse $A^+$ by

$$A^+ = \lim_{\lambda \to 0} (A^T A + \lambda I)^{-1} A^T,$$

which is the ridge regression approximation [7] of the generalized inverse [9].

It is required to assume the ridge parameter $\lambda \to 0$ in the original BLS algorithm [7] that is based on the ridge regression approximation of the generalized inverse, i.e., (10). To achieve a better generalization performance, instead of (10), the BLS algorithms proposed in [12] and [13] adopt the ridge solution [9]

$$\tilde{W} = A^+ Y,$$

where $A^+$ is the ridge inverse [9] of $A$, i.e.,

$$A^+ = (A^T A + \lambda I)^{-1} A^T.$$

Accordingly in [12] and [13], the assumption of $\lambda \to 0$ (for the generalized inverse with the ridge regression approximation in the existing BLS) is no longer required, and $\lambda$ can be any positive real number.

In the original BLS [7], the output weights are updated easily for any number of new added inputs or nodes, by computing the generalized inverse of those added inputs or nodes in just one iteration. The efficient incremental BLS algorithms on added inputs and nodes [12], [13] can obtain the ridge solution for the updated output weights, which include the recursive BLS algorithm on added inputs and the square-root BLS algorithm on added inputs and nodes [9], as will be introduced in the following two subsections.

C. The Recursive Algorithm and the Square-Root Algorithm for the Incremental BLS on Added Inputs

The BLS includes the incremental learning for the additional input training samples. When encountering new input samples with the corresponding output labels, the modeled BLS can be remodeled in an incremental way without a complete retraining process. It updates the output weights incrementally, without retraining the whole network from the beginning.

Denote the additional input training samples as $\tilde{X}_p$ with $p$ training samples. The incremental $p$ samples for feature nodes corresponding to $\tilde{X}_p$ are computed by

$$\tilde{Z}_i = \phi(X W_{e_i} + \beta_{e_i}) \quad (13)$$

for $i = 1, 2, \cdots, n$, and concatenated into

$$\tilde{Z}^n = [\tilde{Z}_1 \cdots \tilde{Z}_n]. \quad (14)$$

Then the incremental $p$ samples for enhancement nodes are computed by

$$\tilde{H}_j = \xi(\tilde{Z}^n W_{h_j} + \beta_{h_j}) \quad (15)$$

for $j = 1, 2, \cdots, m$, and concatenated into

$$\tilde{H}^m = [\tilde{H}_1, \cdots, \tilde{H}_m]. \quad (16)$$

The above described procedure to add inputs in BLS is summarized in Algorithm 2 by the function

$$(\tilde{Z}^n, \tilde{H}^m) = \text{AddInputs}(\tilde{X}_p, W_{e_p}^{n}, \beta_{e_p}^{n}, W_{h_p}^{m}, \beta_{h_p}^{m}). \quad (17)$$

Algorithm 2 The Procedure to Add Inputs in BLS

function AddInputs(X, $W_{e_p}^{n}, \beta_{e_p}^{n}, W_{h_p}^{m}, \beta_{h_p}^{m}$)
    Compute $\tilde{Z}_i = \phi(X W_{e_i} + \beta_{e_i})$ for $i = 1, 2, \cdots, n$, where $W_{e_i}$ and $\beta_{e_i}$ are in $W_e^n$ and $\beta_e^n$, respectively;
    Set $\tilde{Z}^n \equiv [\tilde{Z}_1 \cdots \tilde{Z}_n]$;
    Compute $\tilde{H}_j = \xi(\tilde{Z}^n W_{h_j} + \beta_{h_j})$ for $j = 1, 2, \cdots, m$, where $W_{h_j}$ and $\beta_{h_j}$ are in $W_h^m$ and $\beta_h^m$, respectively;
    Set $\tilde{H}^m \equiv [\tilde{H}_1, \cdots, \tilde{H}_m]$;
    return $\tilde{Z}^n, \tilde{H}^m$
end function

The incremental samples for feature nodes and enhancement nodes form the expanded input matrix $A_p \in \mathbb{R}^{p \times k}$, i.e.,

$$\tilde{A}_p = [\tilde{Z}^n | \tilde{H}^m]. \quad (18)$$

Accordingly, the expanded input matrix $A_p$ is updated into

$$A_i + p = [A_i \tilde{A}_p^T]^T, \quad (19)$$

As in [12], we follow the naming method in [16], [17], where the recursive algorithm updates the inverse matrix recursively, and the square-root algorithm updates the square-root (including the Cholesky factor) of the inverse matrix.
and the corresponding output labels \( Y_{i} \) is updated into
\[
Y_{i+p} = \begin{bmatrix} Y_{i}^T & Y_{p}^T \end{bmatrix}^T,
\]
(20)
where the output labels \( \tilde{Y}_{p} \in \mathbb{R}^{p \times c} \) correspond to the added input \( \tilde{X}_{p} \). From \( A_{i+p} \) and \( Y_{i+p} \), the output weights can be computed by (8) or (11).

In each iteration, the original incremental BLS algorithm in [7] updates the generalized inverse (with the ridge regression approximation) \( A_{i}^+ \) into \( A_{i+p}^+ \), and then utilizes the submatrix in \( A_{i+p}^+ \) to update the output weights \( W_{i} \) into \( W_{i+p} \). The \( l \times k \) expanded input matrix \( A_{i} \) has more rows than columns, i.e., \( l > k \), since usually there are more training samples than nodes in the neural networks [6], [7]. Accordingly, the recursive and square-root BLS algorithms on added inputs proposed in (12) update \( Q_{i} \in \mathbb{R}^{k \times k} \) defined by
\[
Q_{i} = (A_{i}^T A_{i} + \lambda I)^{-1}
\]
and the upper-triangular \( F_{i} \in \mathbb{R}^{k \times k} \) satisfying
\[
F_{i}F_{i}^T = Q_{i} = (A_{i}^T A_{i} + \lambda I)^{-1},
\]
respectively, from which the output weights are computed. Moreover, when there are more rows than columns in the newly added \( p \times k \) input matrix \( \tilde{A}_{p} \) (corresponding to the added inputs \( \tilde{X}_{p} \)), i.e., \( p > k \), the inverse of a sum of matrices [10] is utilized by both BLS algorithms proposed in (12) to compute the intermediate variables by a smaller matrix inverse or inverse Cholesky factorization.

In the recursive BLS algorithm on added inputs [12], \( Q_{i} \) and \( \tilde{W}_{i} \) are updated into \( Q_{i+p} \) and \( \tilde{W}_{i+p} \) by
\[
\begin{cases}
\tilde{B} = Q_{i}A_{p}^T(I + \tilde{A}_{p}Q_{i}A_{p}^T)^{-1} \\
Q_{i+p} = Q_{i} - \tilde{B}A_{p}Q_{i} \\
\tilde{W}_{i+p} = \tilde{W}_{i} + \tilde{B}(\tilde{Y}_{p} - \tilde{A}_{p}\tilde{W}_{i})
\end{cases}
\]
(23)
when \( p < k \), or by
\[
\begin{cases}
Q_{i+p} = (I + Q_{i}A_{p}^T\tilde{A}_{p})^{-1}Q_{i} \\
\tilde{W}_{i+p} = \tilde{W}_{i} + Q_{i+p}A_{p}(\tilde{Y}_{p} - \tilde{A}_{p}\tilde{W}_{i})
\end{cases}
\]
(24)
when \( p > k \). On the other hand, the square-root BLS algorithm on added inputs [12] computes the intermediate matrix \( S \in \mathbb{R}^{k \times p} \) by
\[
S = F_{i}^T\tilde{A}_{p},
\]
(25)
and updates \( F_{i} \) into \( F_{i+p} \) by
\[
F_{i+p} = F_{i}V.
\]
To compute the upper-triangular intermediate matrix \( V \) in (26) and update \( \tilde{W}_{i} \) into \( \tilde{W}_{i+p} \), the square-root BLS algorithm utilizes
\[
\begin{cases}
VV^T = I - S(I + S^TS)^{-1}S^T \\
\tilde{W}_{i+p} = \tilde{W}_{i} + F_{i}S(I + S^TS)^{-1}(\tilde{Y}_{p} - \tilde{A}_{p}\tilde{W}_{i})
\end{cases}
\]
(27)
when \( p < k \), or utilizes
\[
\begin{cases}
VV^T = (I + SS^T)^{-1} \\
\tilde{W}_{i+p} = \tilde{W}_{i} + F_{i+p}F_{i+p}^T\tilde{A}_{p}(\tilde{Y}_{p} - \tilde{A}_{p}\tilde{W}_{i})
\end{cases}
\]
(28)
when \( p \geq k \).

D. The Square-Root Algorithm for the Incremental BLS on Added Nodes

The BLS also includes the incremental learning for the additional nodes. When the network does not reach the desired accuracy, additional nodes can be inserted to achieve a better performance, by fast remodeling in broad expansion without a retraining process. Denote the additional \( q \) nodes as \( \tilde{A}_{q} \) with \( q \) columns. Then the expanded input matrix should be updated into
\[
A_{k+q} = [A_{k} | \tilde{A}_{q}].
\]
(29)
The square-root BLS algorithm on added nodes in [13] defines
\[
R_{k} = A_{k}^T A_{k} + \lambda I,
\]
and then \( R_{k+q} \) can be written as a \( 2 \times 2 \) block Hermitian matrix
\[
R_{k+q} = \begin{bmatrix} R_{k} & \tilde{R}_{k,q} \\
\tilde{R}_{k,q}^T & R_{q} \end{bmatrix},
\]
(31)
where \( \tilde{R}_{k,q} \in \mathbb{R}^{k \times q} \) and \( R_{q} \in \mathbb{R}^{q \times q} \) satisfy
\[
\begin{cases}
\tilde{R}_{k,q} = A_{k}^T\tilde{A}_{q} \\
R_{q} = \tilde{A}_{q}^T\tilde{A}_{q} + \lambda I.
\end{cases}
\]
(32)
An efficient inverse Cholesky factorization was proposed in [16] to compute the upper-triangular inverse Cholesky factor of \( R_{k} \), i.e., \( F_{k} \) satisfying
\[
F_{k}F_{k}^T = R_{k}^{-1} = (A_{k}^T A_{k} + \lambda I)^{-1}.
\]
(33)
Then in [13], the inverse Cholesky factorization [16] was extended to compute the upper-triangular inverse Cholesky factor of the \( 2 \times 2 \) block matrix \( R_{k+q} \) described in (31). The block inverse Cholesky factorization proposed in [13] obtain \( F_{k+q} \) from \( F_{k} \) by
\[
F_{k+q} = \begin{bmatrix} F_{k} & \tilde{F}_{k,q} \\
0 & F_{q} \end{bmatrix},
\]
(34)
where \( \tilde{F}_{q} \in \mathbb{R}^{q \times q} \). Notice that \( \tilde{F}_{q} \) in (35) is the upper-triangular inverse Cholesky factor of \( \tilde{F}_{q} - \tilde{R}_{k,q}^T F_{k} \tilde{F}_{k,q}^{T} \tilde{R}_{k,q} \), which can be computed by the inverse Cholesky factorization [16], or by inverting and transposing the lower-triangular Cholesky factor.

The above-described block inverse Cholesky factorization has been applied to develop the the square-root BLS algorithm on added nodes in [13]. The BLS algorithm in [12] computes the initial \( F_{k} \) from \( A_{k} \) by (33), and then obtains \( F_{k+q} \) from \( F_{k} \) by (32), (35) and (34), and utilizes the sub-matrices \( \tilde{F}_{q} \) and \( \tilde{F}_{k,q} \) in \( F_{k+q} \) (defined by (34)) to compute the ridge solution for the output weights by
\[
\tilde{W}_{k+q} = \begin{bmatrix} \tilde{W}_{k} + \tilde{F}_{k,q} \tilde{F}_{k,q}^T \tilde{A}_{q}^T \tilde{Y} - \tilde{R}_{k,q}^T \tilde{W}_{k} \\
\tilde{F}_{q} \tilde{F}_{q}^T \tilde{A}_{q}^T \tilde{Y} - \tilde{R}_{k,q}^T \tilde{W}_{k} \end{bmatrix},
\]
(36)
where the initial \( \tilde{W}_{k} \) is computed by
\[
\tilde{W}_{k} = F_{k}F_{k}^T A_{k}^T \tilde{Y}.
\]
(37)
III. LOW-MEMORY IMPLEMENTATIONS OF THE EXISTING BLS ALGORITHMS BY PROCESSING A BATCH OF INPUTS OR NODES IN EACH RECURSION

In [15], it has been mentioned that out-of-memory problems occur in the original BLS due to inverting too large matrices, and are easily faced in standard desktop PCs (Intel Core i5 Quad-Core with 8 GB DDR4, and BLS implemented in MATLAB) when \( l, k \) and \( c \) (i.e., the numbers of samples, nodes and outputs) satisfy \( l > 10000, k > 10000 \) and \( c > 100 \), respectively. Then to avoid the need to invert too large matrices, the hybrid recursive BLS implementation proposed in [15] divides \( l \) training samples into small batches and processes only one batch in each recursion. When there are \( b \) training samples in each small batch, only a \( b \times b \) matrix inversion is required in each recursion. In this section, we will propose the low-memory implementations for the recursive and square-root BLS Algorithms on added inputs proposed in [12], and the square-root BLS Algorithm on added nodes proposed in [13]. The proposed implementations also process only one batch of \( b \) inputs or nodes in each recursion by a \( b \times b \) matrix inversion or inverse Cholesky factorization, to avoid the inversion of too large matrices that causes out-of-memory problems.

A. The Proposed Low-Memory Implementation of the Recursive BLS Algorithm on Added Inputs

To process a batch of \( b \) samples (by only a \( b \times b \) matrix inversion or inverse Cholesky factorization) in each recursion, let us divide the expanded input matrix \( A_i \) with \( l \) training samples into \( l/b \) batches and each batch includes \( b \) samples. Accordingly, we have

\[
A_i = \begin{bmatrix} A_{b_0}^T & A_{b_2}^T & \cdots & A_{b_0}^T & \cdots & A_{b_l}^T \end{bmatrix}^T, \tag{38}
\]

where \( A_{b_j} \) (\( i = b, 2b, \ldots, (l/b)b \)) denotes the \( b_j \) rows of \( A_i \) from row \( i-b+1 \) to row \( i \). Accordingly, we can process \( A_{b_i} \) to update \( Q_{i-b} \) and \( \tilde{W}_{i-b} \) into \( Q_i \) and \( \tilde{W}_i \), respectively, by simply setting \( l = i-b \) and \( p = b \) in (23) to obtain

\[
\begin{align*}
\tilde{B} &= Q_{i-b} A_{b_i}^T (I + A_{b_i} Q_{i-b} A_{b_i}^T)^{-1} \tag{39a} \\
Q_i &= Q_{i-b} - \tilde{B} A_{b_i} Q_{i-b} \\
\tilde{W}_i &= \tilde{W}_{i-b} + \tilde{B} (Y_{b_i} - A_{b_i} \tilde{W}_{i-b}), \tag{39c}
\end{align*}
\]

where \( Y_{b_i} \) denotes the \( b_i \) rows of \( Y_i \) from row \( i-b+1 \) to row \( i \).

When \( i = b \), \( Q_0 \) and \( \tilde{W}_0 \) are required to compute \( Q_b \) and \( \tilde{W}_b \) by (39). To deduce \( Q_0 \) and \( \tilde{W}_0 \), set \( l = 0 \) to write (21) and (41) as

\[
Q_0 = (\lambda I)^{-1} = (1/\lambda)I \tag{40}
\]

and

\[
\tilde{W}_0 = \Phi, \tag{41}
\]

respectively, where \( \Phi \) denotes the empty matrix. Then (40) and (41) can be substituted into (39a), (39b) and (39c) with \( i = b \) to obtain

\[
\tilde{B} = \left[ A_{b_0} \right] \frac{1}{\lambda} \left( I + \frac{A_{b_0} A_{b_0}^T}{\lambda} \right)^{-1} = A_{b_0} (A_{b_0} A_{b_0}^T + \lambda I)^{-1}, \tag{42}
\]

and

\[
Q_b = (1/\lambda)I - \tilde{B} A_{b_0} (1/\lambda)I = (I - \tilde{B} A_{b_0})/\lambda \tag{43}
\]

and

\[
\tilde{W}_b = \Phi + \tilde{B} (Y_{b_0} - A_{b_0} \Phi) = \tilde{B} Y_{b_0}, \tag{44}
\]

respectively, which can be written as

\[
\begin{align*}
\tilde{B} &= A_{b_0}^T (A_{b_0} A_{b_0}^T + \lambda I)^{-1} \tag{45a} \\
Q_b &= (I - \tilde{B} A_{b_0})/\lambda \tag{45b} \\
\tilde{W}_b &= \tilde{B} Y_{b_0}. \tag{45c}
\end{align*}
\]

Now we can compute \( Q_b \) and \( \tilde{W}_b \) by (45) firstly, and then compute (39) recursively for \( i = 2b, 3b, \ldots, (l/b)b \), to obtain \( Q_i \) and \( \tilde{W}_i \). When new input samples are encountered, we can obtain \( Q_i \) and \( \tilde{W}_i \) by computing (39) iteratively for \( i = l, \ldots, (l/b)b \), where \( A_{b_i} \) denotes the \( b_i \) rows of \( A_{i+p} \) (described in (19)) from row \( i-b+1 \) to row \( i \).

We summarize the above-described low-memory implementation of the recursive BLS algorithm for added inputs [12] in the following Algorithm 3, where the functions \( \text{Initialize} \) and \( \text{AddInputs} \) are described in Algorithm 1 and Algorithm 2, respectively.

**Algorithm 3 : The Low-Memory Implementation of the Recursive BLS Algorithm on Added Inputs: Computation of Output Weights and Increment of New Inputs**

**Input:** Inputs \( X_i \) with labels \( Y_i \), added inputs with labels;

**Output:** Output weights: the ridge solution \( \tilde{W} \);

1. Get \((Z^n, H^m, \tilde{W}_n, \beta_e, \tilde{W}_h, \beta_h) = \text{Initialize} (X_i)\);
2. Set \( \hat{A}_i = [Z^n [H^m] \];
3. Utilize \( \hat{A}_i \) and \( Y_i \) to compute \( Q_i \) and \( \tilde{W}_i \) by (45), and then obtain \( Q_i \) and \( \tilde{W}_i \) by computing (39) iteratively for \( i = 2b, 3b, \ldots, (l/b)b \);
4. While new inputs \( \hat{X}_p \) and labels \( \hat{Y}_p \) are added do
5. Get \((\tilde{Z}_p, \tilde{H}_p) = \text{AddInputs}(\hat{X}_p, \hat{W}_n, \hat{W}_h, \beta_e, \tilde{W}_h, \beta_h)\);
6. Set \( \hat{A}_p = [\tilde{Z}_p [\tilde{H}_p] \ ];
7. Utilize \( \hat{A}_p \) and \( \hat{Y}_p \) to update \( Q_i \) and \( \tilde{W}_i \) into \( Q_{i+p} \) and \( \tilde{W}_{i+p} \), respectively, by computing (39) iteratively for \( i = l, \ldots, (l/b)b \);
8. Set \( \hat{A}_{i+p} = [A_i^T A_p^T]^T \) and \( l = l + p \);
9. End while
10. Set output weights \( \tilde{W} = \tilde{W}_i \);

B. The Proposed Low-Memory Implementation of the Square-Root BLS Algorithm on Added Inputs

As in the last subsection, let us simply set \( l = i-b \) and \( p = b \) in (25), (26a), (27a) and (26) to obtain

\[
\begin{align*}
S &= F_i^T A_{b_i}^T \tag{46a} \\
VV^T &= I - S (I + S^T S)^{-1} S^T \tag{46b} \\
\tilde{W}_i &= \tilde{W}_{i-b} + F_i - S (I + S^T S)^{-1} \times (Y_{b_i} - A_{b_i} \tilde{W}_{i-b}) \tag{46c} \\
F_i &= F_i - b V_i, \tag{46d}
\end{align*}
\]

which process \( A_{b_i} \) in the \( i^{th} \) \( (i = 1, 2, \ldots, l/b) \) recursion to update \( F_{i-b} \) and \( \tilde{W}_{i-b} \) into \( F_i \) and \( \tilde{W}_i \), respectively.
When \( i = b \), \( F_b \) and \( \tilde{W}_b \) are computed from \( F_0 \) and \( \tilde{W}_0 \) by (46). To deduce \( F_0 \), let us substitute (22) into (40) to obtain 
\[ F_0 = \frac{1}{\sqrt{\lambda}} I, \] (47)

To obtain \( F_b \), the above (47) is substituted into (46a) and (46d) (with \( i = b \)) to obtain 
\[ S = \frac{A^T_b}{\sqrt{\lambda}} \] (48)
and 
\[ V = \sqrt{\lambda} F_b, \] (49)
respectively, which are then substituted into (46b) to obtain 
\[ \sqrt{\lambda} F_b \sqrt{A^T F^T_b} = I - \frac{A^T}{\sqrt{\lambda}} \left( I + \frac{A^T}{\sqrt{\lambda}} \right)^{-1} A \], i.e., 
\[ F_b F^T_b = \frac{1}{\lambda} I - \frac{A^T}{\lambda} (A_b A^T_b + \lambda I)^{-1} A_b, \] (50)

Finally, to obtain \( \tilde{W}_b \), let us substitute (47), (48) and (41) (i.e., \( \tilde{W}_0 = \Phi \)) into (45c) (with \( i = b \)) to obtain 
\[ \tilde{W}_b = \frac{1}{\sqrt{\lambda}} \left( I + \frac{A^T}{\sqrt{\lambda}} \right)^{-1} (Y_b - A_b \Phi), \] i.e., 
\[ \tilde{W}_b = A^T_b (A_b A^T_b + \lambda I)^{-1} Y_b. \] (51)

Now we can compute the upper-triangular \( F_b \) by (50), and compute \( \tilde{W}_b \) by (51). Then we compute (46) iteratively for \( i = 2b, 3b, \ldots, (l/b)b \), to obtain \( F_i \) and \( \tilde{W}_i \). When new input samples are encountered, we can compute (46) iteratively for \( i = l + b, l + 2b, \ldots, l + (p/b)b \), where \( A_k \) denotes the \( b \) rows of \( A_{l+p} \) from row \( i - b + 1 \) to row \( i \), to obtain \( F_{l+p} \) and \( \tilde{W}_{l+p} \) finally.

In Algorithm 4, we summarize the above-described low-memory implementation of the square-root BLS algorithm for added inputs in [12], where the functions Initialize (described in Algorithm 2) and AddInputs (described in Algorithm 3) are utilized.

### C. The Proposed Low-Memory Implementation of the Square-Root BLS Algorithm on Added Nodes

To process a batch of \( b \) nodes in each recursion, we can divide the expanded input matrix \( A_k \) with \( k \) nodes into \( k/b \) batches and each batch includes \( b \) nodes, i.e., \( k = (k/b)b \).

Accordingly, let us write 
\[ A_k = \left[ A_{b_1} \ A_{b_2} \ \cdots \ A_{b_k} \right], \] (52)
where \( A_{b_j} \) (\( j = b, 2b, \ldots, k \)) denotes the \( b \) columns of \( A_k \) from column \( j - b + 1 \) to column \( j \). Then we can process \( A_{b_j} \) to update \( F_{j-b} \) and \( \tilde{W}_{j-b} \) into \( F_j \) and \( \tilde{W}_j \), respectively, by simply setting \( k = j - b \) and \( q = b \) in (31), (32), (35), (34) and (50) to obtain 
\[ R_j = \left[ R_{j-b} \ \tilde{R}_{j-b,b} \right], \] (53)
\[ \tilde{R}_{j-b,b} = A^T_{j-b} A_{b_j}, \] (54a)
\[ R_{b_j} = A^T_{b_j} A_{b_j} + \lambda I, \] (54b)

#### Algorithm 4: The Low-Memory Implementation of the Square-Root BLS Algorithm on Added Inputs: Computation of Output Weights and Increment of New Inputs

**Input:** Inputs \( X_i \) with labels \( Y_i \), added inputs with labels;

**Output:** Output weights: the ridge solution \( \tilde{W} \);

1. Get \((Z^m, H^m, \tilde{W}^m, \beta^m, \gamma^m) = \text{Initialize} (X_j)\);
2. Set \( A_I = [Z^m | H^m] \);
3. Utilize \( A_i \) and \( Y_i \) to compute \( F_i \) and \( \tilde{W}_i \) by (50) and (51), respectively, and then obtain \( F_I \) and \( \tilde{W}_I \) by computing (46) iteratively for \( i = 2b, 3b, \ldots, (l/b)b \);
4. While New inputs \( X_p \) and labels \( Y_p \) are added do
5. Get \((Z^m_p, H^m_p) = \text{AddInputs} (X_p, \tilde{W}^m_p, \beta^m_p, \gamma^m_p) \);
6. Set \( A_p = [Z^m_p | H^m_p] \);
7. Utilize \( A_p \) and \( Y_p \) to update \( F_i \) and \( \tilde{W}_i \) into \( F_{i+p} \) and \( \tilde{W}_{i+p} \), respectively, by computing (46) iteratively for \( i = l + b, l + 2b, \ldots, l + (p/b)b \);
8. Set \( A_{i+p} = [A_i^T \ A_p^T ]^T \) and \( l = l + p \);
9. End while
10. Set output weights \( \tilde{W} = \tilde{W}_I \);

\[
\begin{align*}
F_{b_j} F^T_{b_j} &= \left( R_{b_j} - \tilde{R}_{j-b,b} F_{j-b} F^T_{j-b} \tilde{R}_{j-b,b} \right)^{-1} \\
\tilde{F}_{j-b,b} &= -F_{j-b} F^T_{j-b} \tilde{R}_{j-b,b} F_{b_j}, \\
F_j &= \begin{bmatrix} F_{j-b} & \tilde{F}_{j-b,b} \\ 0 & F_{b_j} \end{bmatrix}
\end{align*}
\] (55a), (55b), (56)

and 
\[
\begin{align*}
\tilde{W}_j &= \begin{bmatrix} \tilde{W}_{j-b} + \tilde{F}_{j-b,b} F_{b_j} \left( A^T_{b_j} Y - \tilde{R}_{j-b,b} \tilde{W}_{j-b} \right) \\ F_{b_j} F^T_{b_j} \right) \end{bmatrix}, \\
\end{align*}
\] (57)

respectively, where \( F_{b_j} \) is the \( b \times b \) sub-matrix from the \((j - b + 1)^{th}\) row and column to the \( j^{th}\) row and column in \( F_j \).

Let us compute the initial \( F_b \) and \( \tilde{W}_b \) by (35) and (37), respectively, where \( k \) is set to \( b \). Then we can compute \( F_j \) and \( \tilde{W}_j \) from \( F_{j-b} \) and \( \tilde{W}_{j-b} \) iteratively by (54), (55), (56) and (57), for \( j = 2b, 3b, \ldots, (k/b)b \) to obtain \( F_k \) and \( \tilde{W}_k \) from \( F_b \) and \( \tilde{W}_b \), and for \( j = k+b, k+2b, \ldots, k+p \) to obtain \( F_{k+p} \) and \( \tilde{W}_{k+p} \) from \( F_k \) and \( \tilde{W}_k \). The above described algorithm can be summarized in the following Algorithm 5, which gives the low-memory implementation of the square-root BLS algorithm on added nodes proposed in [13].

### IV. The Proposed Full Low-Memory Square-Root BLS Implementation Based on Inverse Cholesky Factorizations with a Batch Size \( b \)

In the last section, we have developed the low-memory implementations for the recursive and square-root BLS algorithms on added inputs proposed in [12] and the square-root BLS algorithm on added nodes proposed in [13]. It is well known that in the processor units with limited precision, the recursive updates of the inverse matrix may introduce numerical instabilities after a large number of iterations [17].

\[ \text{Here } A_{b_j} \text{ and } A_{j-b} \text{ are in } A_{k+q}. \]
Algorithm 5: The Low-Memory Implementation of the Square-Root BLS Algorithm on Added Nodes: Computation of Output Weights and Increment of Nodes

**Input:** training sample $X$ and the corresponding label $Y$;  
**Output:** Output weights: the ridge solution $\tilde{W}$;  
1. Compute $(Z_n, H_n, W_n^b, \beta_n, W_k^b, \beta_n^m) = \text{Initialize} (X)$;  
2. Set $A_k^{n,m} = [Z_n^T H_n^T]$;  
3. Utilize $A_k^{n,m}$ and $Y$ to compute the initial $F_b$ and $\tilde{W}_b$ by (53) and (57), respectively, and then obtain $F_{n,m}^k$ and $W_{n,m}^k$ by computing (54), (55) and (57) iteratively for $j = 2b, 3b, \cdots, k$;  
4. While the target training error is not reached do  
5. If only enhancement nodes are added then  
6. Random $W_{h_{m+1}}$ and $\beta_{h_{m+1}}$;  
7. Compute $\tilde{A}_q = H_{h_{m+1}} + \xi (Z_n^T W_{h_{m+1}} + \beta_{h_{m+1}})$;  
8. Obtain $F_{k,p}^{n,m+1}$ and $W_{k+p}^{n,m+1}$ by computing (54), (55), (56) and (57) for $j = k + b, k + 2b; \cdots, k + p$;  
9. Set $A_{k+q}^{n,m+1} = [A_{k+q}^{n,m}]_q$;  
10. Set $m = m + 1$ and $k = k + p$;  
11. else if [feature nodes are added] then  
12. Fine-tune random $W_{e_{m+1}}$ and $\beta_{e_{m+1}}$;  
13. Compute $Z_{m+1} = \phi(X(W_{e_{m+1}} + \beta_{e_{m+1}}))$;  
14. Set random $W_{e_{m+1}}$ and $\beta_{e_{m+1}}$ for $i = 1, 2, \cdots, m$ and compute $H_{e_{m+1}} = (Z_{m+1} W_{e_{m+1}} + \beta_{e_{m+1}})$;  
15. Set $A_q = [Z_{m+1} H_{e_{m+1}}]$;  
16. Obtain $F_{k+p}^{n,m+1}$ and $W_{k+p}^{n,m+1}$ by computing (54), (55), (56) and (57) for $j = k + b, k + 2b; \cdots, k + p$;  
17. Set $A_{k+q}^{n,m+1} = [A_{k+q}^{n,m}]_q$;  
18. Set $n = n + 1$ and $k = k + p$;  
19. end if  
20. end while  
21. Set output weights $\tilde{W} = \tilde{W}_{k,m}^n$.

as will be verified by the numerical experiments in the next section. On the other hand, when cooperating with the low-memory implementation of the square-root BLS algorithm on added nodes [13], the low-memory implementation of the recursive BLS algorithm on added inputs needs the extra computational load to decompose the inverse matrix into the Cholesky factor. Accordingly, in this section, we only discuss the low-memory implementations of the square-root BLS algorithms on added inputs and nodes, to propose the full low-memory implementation of the square-root BLS algorithm.

### A. Inverse Cholesky Factorizations with a Batch Size $b$

To obtain $F_{l,k}^{n,m}$ and $\tilde{W}_{l,k}^{n,m}$, it requires $l/b - 1$ iterations to compute (46) iteratively for $i = 2b, 3b, \cdots, (l/b)b$ in Algorithm 4, and it requires $k/b - 1$ iterations to compute (55), (56) and (57) iteratively for $j = 2b, 3b, \cdots, k$ in Algorithm 5. Usually there are always more training samples than nodes in BLS [6], [7], [12]. i.e., $l > k$. Then it can be seen that Algorithm 4 requires more iterations to compute $F_{l,k}^{n,m}$ and $\tilde{W}_{l,k}^{n,m}$ than Algorithm 5. Furthermore, it is required to update all the $\frac{l}{b}k^2$ entries of $F_{i-l}^{b} \in \mathbb{R}^{k \times k}$ to compute $F_{i}^{b} \in \mathbb{R}^{k \times k}$ from $F_{i-l}^{b}$ by (46) in Algorithm 4, while it is only required to add a $j \times b$ sub-matrix to the right side of $F_{j-b}^{l} \in \mathbb{R}^{(j-b) \times (j-b)}$ to update $F_{j-b}$ into $F_{j}^{l} \in \mathbb{R}^{j \times j}$ by (56) for $j = 2b, 3b, \cdots, k$ in Algorithm 5. Accordingly, it can be concluded that Algorithm 4 requires more computational complexities to compute $F_{l,k}^{n,m}$ and $\tilde{W}_{l,k}^{n,m}$ than Algorithm 5.

Since Algorithm 5 is more efficient than Algorithm 4, we utilize (53) (with $k = b$), (55), (56) and (57) that are relevant to the computation of $F_b$ (i.e., $F_{l,k}^{n,m}$) in Algorithm 5, to develop the function

$$F_b = \text{InvChol}(R_b, b)$$  
(58)

described by Algorithm 6. It can be seen that Algorithm 6 obtains $F_b = \text{InvChol}(R_b, b)$ satisfying (33) (which is $F_b F_b^T = R_b^{-1} = (A_b^T A_b + \lambda I)^{-1}$), i.e., Algorithm 6 computes the inverse Cholesky factor of a Hermitian Matrix $R_b$. It can be seen that in Algorithm 6 with a batch size of $b$, only the inverse Cholesky factorizations of $b \times b$ matrices are required, and the inversions of large matrices are not required.

**Algorithm 6** The Proposed Inverse Cholesky Factorization of a Hermitian Matrix $R_b$ with a Batch Size of $b$

**function** $\text{InvChol}(R_b, b)$  
1. Compute the initial inverse Cholesky factor $F_b$ by $F_b F_b^T = R_b^{-1}$;  
2. for $j = 2b, 3b, \cdots, (k/b)b$ do  
3. Obtain $R_{j-b}$ and $\tilde{R}_{j-b}$ in $R_j$ by $R_j = \begin{bmatrix} \tilde{R}_{j-b} & R_{j-b-b} \\ \tilde{R}_{j-b} & R_{j-b-b} \end{bmatrix}$;  
4. Compute the inverse Cholesky factor $F_{j-b}$ by $F_{j-b} F_{j-b}^T = \tilde{R}_{j-b}^{-1}$;  
5. Compute $\tilde{F}_{j-b} = -F_{j-b} F_{j-b}^T \tilde{R}_{j-b}^{-1}$;  
6. Obtain $F_j = \begin{bmatrix} F_{j-b} & \tilde{F}_{j-b} \\ 0 & F_{j-b} \end{bmatrix}$;  
7. end for  
8. return $F_{(k/b)b} = F_k$;

**end function**

Since we are developing low-memory implementations in this paper, let us save the memories for storing $R_b$ in Algorithm 6, by substituting (53) into (55) to obtain

$$ \begin{align} 
F_{b} F_{b}^T &= \left( A_b^T A_b + \lambda I - A_b^T \frac{A_b^T \lambda - A_b^T}{A_b^T F_{j-b} A_b} \right)^{-1} \\
\tilde{F}_{j-b} &= -F_{j-b} F_{j-b}^T \tilde{R}_{j-b}^{-1} 
\end{align}$$

(59a)  
(59b)

Then (59), (33) (with $k = b$) and (56) are utilized to develop the function

$$F_k = \Phi_{chol}(A_k, \lambda, b)$$

(60)

described in Algorithm 7, which utilizes $A_k$ to compute the upper-triangular inverse Cholesky factor of the $k \times k$ Hermitian matrix $A_k^T A_k + \lambda I$. 

---

*Note: The text contains mathematical expressions and algorithms that are not fully rendered in this format. The documents should be viewed in a PDF or third-party PDF reader for proper rendering.*
Algorithm 7 The Proposed Inverse Cholesky Factorization of \( A_k^T A_k + \lambda I \)

\[
\text{function } \Phi_{\text{chol}}(A_k, \lambda, b) \\
\quad \text{Compute the initial inverse Cholesky factor } F_b \text{ by} \\
\quad F_b F_b^T = (A_k^T A_k + \lambda I)^{-1}; \\
\quad \text{for } j = 2b, 3b, \cdots , (k/b)b \text{ do} \\
\quad \quad \text{Compute the inverse Cholesky factor } F_{b_j} \text{ by} \\
\quad \quad F_{b_j} = \left( \begin{array}{c} A_{b_j}^T A_{b_j} + \lambda I - A_{b_j}^T \times \\
F_{b_j} F_{b_j}^T A_{b_j} F_{b_j}^T A_{b_j} \end{array} \right)^{-1} \\
\quad \quad \text{Compute } \tilde{F}_{j-b} = -F_{j-b} F_{j-b} A_{j-b} A_{j-b} F_{j-b}; \\
\quad \quad \text{Obtain } F_j = \begin{bmatrix} F_{j-b} & 0 \end{bmatrix}; \\
\quad \text{end for} \\
\quad \text{return } F_{(k/b)b} = F_k; \\
\text{end function}
\]

B. The Proposed Full Low-Memory Square-Root BLS Implementation

To avoid the out-of-memory problems caused by the inversions of large matrices, we can utilize (58) or (60) to compute the inverse Cholesky factorizations in the square-root BLS algorithm on added nodes [13] and that on added inputs [12], as will be introduced in what follows.

In the square-root BLS algorithm on added nodes [13], we utilize (60) to compute \( F_k \) from \( A_k \) directly, and then we can save the memories for storing \( R_k \). We can also save the memories for storing \( R_{k+q} \) by substituting (32) into (33) to obtain

\[
\begin{align}
\tilde{F}_q & = \left( A_q^T A_q + \lambda I - A_q^T \right)^{-1} A_q F_k F_k^T A_k \tilde{A}_q, \\
\tilde{F}_{k,q} & = -F_k F_k^T A_k \tilde{A}_q \tilde{F}_q.
\end{align}
\]

(61a, 61b)

To obtain the upper-triangular inverse Cholesky factor \( \tilde{F}_q \) satisfying (61a), we can utilize (58) to compute

\[
\tilde{F}_q = \text{InvChol} \left( \left( A_q^T A_q + \lambda I - A_q^T \right)^{-1} A_q F_k F_k^T A_k \tilde{A}_q, b \right).
\]

(62)

In the square-root BLS algorithm on added inputs [12], the initial \( \tilde{F}_{l,k} \) satisfying (22) can also be computed from \( A_{l,k} \) directly by (60). When \( p < k \), we utilize (60) to compute

\[
\Gamma = \Phi_{\text{chol}}(S, 1, b)
\]

satisfying \( \Gamma T^T = (I + S T^T S)^{-1} \), which is then substituted into (27) to obtain

\[
\begin{align}
V V^T & = I - S \Gamma T^T S^T, \\
W_{l+p} & = \tilde{W}_l + F_j S \Gamma T^T (Y_p - \tilde{A}_p \tilde{W}_l).
\end{align}
\]

(64a, 64b)

When \( p \geq k \), we utilize (60) to compute

\[
V = \Phi_{\text{chol}}(S^T, 1, b)
\]

satisfying (88a) (i.e., \( V V^T = (I + S S^T)^{-1} \)).

The above-described low-memory implementations of the square-root BLS algorithm on added nodes [13] and that on added inputs [12] are summarized in the following Algorithm 8, which are based on (58) and (60), the inverse Cholesky factorizations with a batch size \( b \). It can be seen that Algorithm 8 gives the full low-memory implementation of the square-root BLS Algorithm, which includes the increment of feature nodes, enhancement nodes, and new inputs.

V. Numerical Experiments

Numerical experiments will be carried out in this section, to compare the presented low-memory BLS implementations. The presented implementations on added inputs include the existing one proposed in [15], the part for added inputs in Algorithm 8, and the low-memory implementations of the recursive and square-root BLS algorithms in [12] (i.e., Algorithm 3 and Algorithm 4). On the other hand, the presented implementations on added nodes include the low-memory implementation of the square-root BLS algorithm in [13] (i.e., Algorithm 5) and the part for added nodes in Algorithm 8. Notice that Algorithm 8 is the proposed full low-memory implementation of the square-root BLS Algorithm, which includes the part for added inputs and that for added nodes.

We simulate the presented low-memory BLS implementations on MATLAB software platform under a standard desktop PC (Intel Core i5 Quad-Core with 8 GB DDR4). As in [7] and [15], we give the experimental results on the Modified National Institute of Standards and Technology (MNIST) dataset [18] with 60000 training images and 10000 testing images. We follow the simulations in the original BLS [7]: for the feature nodes \( Z_i \) in (1), we fine-tune the random \( W_{e_i} \) and \( \beta_{e_i} \) by the linear inverse problem (7), and for the enhancement nodes \( H_j \) in (3), we choose tansig for the activation function \( \xi \). The weights \( W_{h_j} \) and the biases \( \beta_{h_j} \) (\( j = 1, 2, \cdots, m \)) in (5) are drawn from the standard uniform distributions on the interval \([-1, 1]\), and so are the initial random \( W_{e_i} \) and \( \beta_{e_i} \) in (1).

As Table V in [7], Table I, Table III and Table IV give the simulation results for the incremental BLS on added inputs. In Table I and Table III, we set the total node number of the network as \( k = 11100 \) with \( 10 \times 10 \) feature nodes and 11000 enhancement nodes, train the initial network under the first \( l = 15000 \) training samples, and increase \( p = 9000 < k \) training samples in each update, until all the 60000 training samples are fed. In Table IV, we set the total node number of the network as \( k = 3100 \) with \( 10 \times 10 \) feature nodes and 3000 enhancement nodes, train the initial network under the first \( l = 10000 \) training samples, and increase \( p = 10000 > k \) training samples in each update, until all the 60000 training samples are fed. For each of the above-described update, the snapshot results are given in Table I, Table III and Table IV.

On the other hand, as Table IV in [7], Table II and Table V give the simulation results for the incremental BLS on added nodes. We set the initial network as \( 10 \times 6 \) feature nodes and 3000 enhancement nodes. The feature nodes are dynamically increased from 60 to 100, and the enhancement nodes are dynamically increased from 3000 to 11000. In each update, 10 feature nodes are added, and 2000 enhancement nodes are added, which include 750 enhancement nodes corresponding to the added feature nodes and 1250 additional enhancement nodes. For each of the above-described update, the snapshot results are given in Table II and Table V.
I and Table II show the testing accuracy with the batch size $b = 500$. From Table I, it can be seen that when the ridge parameter is set to $\lambda = 1/128$ as in [15], the proposed Algorithm 3, Algorithm 4 and the part for added inputs in Algorithm 8 usually achieve better testing accuracies.
than the existing low-memory BLS implementation on added inputs proposed in [15]. More importantly, Table I shows that when the ridge parameter is very small (i.e., $\lambda = 10^{-8}$) as in the original BLS [7], the existing low-memory BLS implementation on added inputs proposed in [15] cannot work in any update, the proposed recursive implementation (i.e., Algorithm 3) cannot work in the last update, and the proposed square-root implementations (i.e., Algorithm 4 and the part for added inputs in Algorithm 8) can work in all updates. This can be explained by the fact that both the proposed recursive BLS implementation and the BLS implementation in [15] include the recursive updates of the inverse matrix, which may introduce numerical instabilities after a large number of iterations in the processor units with limited precision [7]. On the other hand, Table II shows that both the proposed low-memory square-root BLS implementations on added nodes can work for a small ridge parameter $\lambda = 10^{-7}$, while the part for added nodes in Algorithm 8 can work for the very small ridge parameter $\lambda = 10^{-8}$ utilized in the original BLS [7].

Table III, Table IV and Table V include the training times in seconds with the batch size $b = 50$. Table III and Table IV show that with respect to the existing low-memory BLS implementation on added inputs proposed in [15], Algorithm 3 and the part for added nodes in Algorithm 8 require nearly the same training time, while Algorithm 4 requires much more training time. On the other hand, Table V shows that the part for added nodes in Algorithm 8 speeds up Algorithm 5 by a factor in the range from 3.77 to 23.73.

From Tables I, II, III, IV and V, it can be concluded that the proposed Algorithm 8 is a good low-memory implementation of the original BLS on new added nodes and inputs [7], which can work when the ridge parameter is very small (i.e., $\lambda = 10^{-8}$) as in the original BLS. The part for added inputs in Algorithm 8 spends nearly the same training time to achieve better testing accuracies with respect to the existing low-memory BLS implementation on added inputs [15], is numerically more stable than Algorithm 3 (i.e., the proposed recursive implementation), and is much faster than Algorithm 4. On the other hand, the part for added nodes in Algorithm 8 is obviously faster than Algorithm 5.

VI. CONCLUSION

In this paper, firstly we propose the low-memory implementations for the recursive and square-root BLS algorithms on new added inputs in [12] and the square-root BLS algorithm on new added nodes in [13], which simply process a batch of $b$ inputs or nodes in each recursion by a $b \times b$ matrix inversion or inverse Cholesky factorization. Since the recursive BLS implementation includes the recursive updates of the inverse matrix that may introduce numerical instabilities after a large number of iterations [7], and needs the extra computational load to decompose the inverse matrix into the Cholesky factor when cooperating with the proposed low-memory implementation of the square-root BLS algorithm on added nodes in [13], we only improve the low-memory implementations of the square-root BLS algorithms on added nodes and inputs, to propose the full low-memory implementation of the square-root BLS algorithm.

When the ridge parameter is set to $\lambda = 1/128$ as in [15], the proposed low-memory implementations for the recursive and square-root BLS algorithms on added inputs in [12] and the part for added inputs of the proposed full low-memory implementation of the square-root BLS algorithm usually achieve better testing accuracies than the existing low-memory BLS implementation on added inputs proposed in [15]. More importantly, when the ridge parameter is very small (i.e., $\lambda = 10^{-8}$) as in the original BLS [7], the existing low-memory BLS implementation on added inputs proposed in [15] cannot work in any update, the proposed low-memory implementation for the recursive BLS algorithm on added inputs in [12] cannot work in the last update, while the proposed low-memory implementation for the square-root BLS algorithm on added inputs in [12] and the proposed full low-memory implementation of the square-root BLS algorithm on added inputs and nodes can work in all updates.

With respect to the existing low-memory BLS implementation on added inputs proposed in [15], the proposed low-memory implementation for the recursive BLS algorithm on added inputs in [12] and the part for added inputs of the proposed full low-memory implementation of the square-root BLS algorithm require nearly the same training time, while the proposed low-memory implementation for the square-root BLS algorithm on added inputs in [12] requires much more training time. On the other hand, the part for added nodes of the proposed full low-memory implementation of the square-root BLS algorithm speeds up the proposed low-memory implementation for the square-root BLS algorithm on new added nodes in [13] by a factor in the range from 3.77 to 23.73.

The proposed full low-memory implementation of the
Algorithm 8: The Proposed Full Low-Memory Implementation of the Square-Root BLS Algorithm: Computation of Output Weights and Increment of Feature Nodes, Enhancement Nodes and New Inputs

Input: Inputs $X_l$ with labels $Y_l$, added inputs with labels; Output: Output weights: the ridge solution $\tilde{W}$;
1: Get $(\tilde{Z}^n, \tilde{H}^m, \tilde{W}_n^m, \beta_n^m, \tilde{W}_n^m, \beta_n^m) = \text{Initialize} (X_l)$;
2: Set $A_{n,k}^{m} = [Z^n_i [H]^m_j]$;
3: Compute $F_{n,k}^{m} = \phi_{\text{chol}} (A_{n,k}^{m}, \lambda, h)$;
4: Compute $W_{n,k}^{m} = F_{n,k}^{m} (F_{n,k}^{m})^T A_{n,k}^{m} Y_l$;
5: While the target training error is not reached do
6: if Only enhancement nodes are added then
7: Random $W_{h_{m+1}}$ and $\beta_{h_{m+1}}$;
8: Compute $\tilde{A}_{q} = H_{m+1} = \xi (Z^n W_{h_{m+1}} + \beta_{h_{m+1}})$;
9: Utilize $A_{n,k}^{m}, \tilde{A}_{q}$ and $F_{n,k}^{m}$ to update $F_{n,k}^{m}$ into $F_{n,k}^{m+1}$ by (62), (61b) and (32);
10: Update $W_{n,k}^{m+1}$ into $W_{n,k}^{m+1} + F_{n,k}^{m+1}$ by (36);
11: Set $m = m + 1$ and $k = k + 1$;
12: else if Feature nodes are added then
13: Fine-tuned random $W_{e_{n+1}}$ and $\beta_{e_{n+1}}$;
14: Compute $Z_{n+1} = \phi(X_{l} W_{e_{n+1}} + \beta_{e_{n+1}})$;
15: Set random $W_{e_{n+1}}$ for $i = 1, 2, \cdots, m$ and compute $h_{e_{n+1}} = (\xi Z_{n+1} W_{e_{n+1}} + \beta_{e_{n+1}})$;
16: Compute $\tilde{A}_{q} = [Z_{n+1} H_{e_{n+1}}]$;
17: Utilize $A_{n,k}^{m}, \tilde{A}_{q}$ and $F_{n,k}^{m}$ to update $F_{n,k}^{m}$ into $F_{n,k}^{m+1}$ by (62), (61b) and (32);
18: Set $n = n + 1$ and $k = k + 1$;
19: else if New inputs $\tilde{X}_{p}$ and labels $\tilde{Y}_{p}$ are added then
20: $(Z^n, H^m + \tilde{Z}^n, H^m + \tilde{H}^m) = \text{AddInputs} (\tilde{X}_{p}, \tilde{Y}_{p}, \beta_{c}^{n}, \tilde{W}_{e}^{m}, \beta_{c}^{m})$;
21: Set $\tilde{A}_{p} = [Z^n_i [H]^m_j]$
22: Compute the intermediate result $S$ by (25);
23: if $p < k$ then
24: Compute $\Gamma$ and $V$ by (63) and (64); Compute $\hat{V}$
25: Update $W_{n,k}^{m+1}$ into $W_{n,k}^{m+1} + p$ by (63) and (64);
26: Update $F_{n,k}^{m+1}$ into $F_{n,k}^{m+1} + p$ by (25);
27: else $p \geq k$
28: Compute $V$ by (65); Update $F_{n,k}^{m+1}$ into $F_{n,k}^{m+1} + p$ by (25); Update $W_{n,k}^{m+1}$ into $W_{n,k}^{m+1} + p$ by (28)
29: end if
30: Set $A_{n,k}^{m} + p, k = (A_{n,k}^{m})^T \tilde{A}_{p}^T$, and $l = l + p$;
31: end if
32: $W = W_{n,k}^{m}$.

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