The Derived Category Analogue of the Hartshorne-Lichtenbaum Vanishing Theorem

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Abstract. Let $a$ be an ideal of a local ring $(R, m)$ and $X$ a $d$-dimensional homologically bounded complex of $R$-modules whose all homology modules are finitely generated. We show that $H^d_a(X) = 0$ if and only if $\dim \hat{R}/a\hat{R} + \nu > 0$ for all prime ideals $p$ of $\hat{R}$ such that $\dim \hat{R}/p - \inf X \otimes_R \hat{R}_p = d$.

1. Introduction

The Hartshorne-Lichtenbaum Vanishing Theorem is one of the most important results in the theory of local cohomology modules. There are several proofs known now of this result; see e.g. [BH], [CS] and [Sc]. Also, there are several generalizations of this result. The second named author, Naghipour and Tousi [DNT] have extended it to local cohomology with support in stable under specialization subsets. Takahashi, Yoshino and Yoshizawa [TYY] have extended it to local cohomology with respect to pairs of ideals. Also, more recently, the Hartshorne-Lichtenbaum Vanishing Theorem is extended to generalized local cohomology modules; see [DH]. Our aim in this paper is to establish a generalization of the Hartshorne-Lichtenbaum Vanishing Theorem which contains all of these generalizations. We do this by establishing the derived category analogue of the Hartshorne-Lichtenbaum Vanishing Theorem. For giving the precise statement of this result, we need to fix some notation.

Throughout, $R$ is a commutative Noetherian ring with nonzero identity. The derived category of $R$-modules is denoted by $D(R)$. We use the symbol $\simeq$ for denoting isomorphisms in $D(R)$. For a complex $X \in D(R)$, its supremum and infimum are defined, respectively, by $\sup X := \sup \{i \in \mathbb{Z} | H_i(X) \neq 0 \}$ and $\inf X := \inf \{i \in \mathbb{Z} | H_i(X) \neq 0 \}$, with the usual convention that $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$. Also, amplitude of $X$ is defined by $\text{amp} X := \sup X - \inf X$. Recall that $\dim_R X$ is defined by $\dim_R X := \sup \{\dim R/p - \inf X_p | p \in \text{Spec} R \}$.

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and we define $\text{Assh}_R X$ by

$$\text{Assh}_R X := \{ p \in \text{Spec } R \mid \dim R/p - \inf X_p = \dim R X \}.$$ 

Any $R$-module $M$ can be considered as a complex having $M$ in its $0$-th spot and $0$ in its other spots. We denote the full subcategory of homologically left bounded complexes by $\mathcal{D}_<^L(R)$. Also, we denote the full subcategory of complexes with finitely generated homology modules that are homologically bounded (resp. homologically left bounded) by $\mathcal{D}_<^L(R)$ (resp. $\mathcal{D}_<^L(R)$).

Let $a$ be an ideal of $R$ and $X \in \mathcal{D}_<^L(R)$. A subset $Z$ of Spec $R$ is said to be stable under specialization if $V(p) \subseteq Z$ for all $p \in Z$. For any $R$-module $M$, $\Gamma_Z(M)$ is defined by

$$\Gamma_Z(M) := \{ x \in M \mid \text{Supp}_RX \subseteq Z \}.$$ 

The right derived functor of the functor $\Gamma_Z(-)$ exists in $\mathcal{D}(R)$ and the complex $R\Gamma_Z(X)$ is defined by $R\Gamma_Z(X) := \Gamma_Z(I)$, where $I$ is any injective resolution of $X$. Also, for any integer $i$, the $i$-th local cohomology module of $X$ with respect to $Z$ is defined by $H_i^Z(X) := H_{\I}(R\Gamma_Z(X))$. To comply with the usual notation, for $Z := V(a)$, we denote $R\Gamma_Z(-)$ and $H_i^Z(-)$ by $\Gamma_a(-)$ and $H_i^a(-)$, respectively. By [F3, Corollary 3.7 and Proposition 3.14 d], for any complex $X \in \mathcal{D}_<^L(R)$, we know that

$$\sup \{ i \in Z \mid H_i^a(X) \neq 0 \} \leq \dim_R X$$

with equality if $R$ is local and $a$ is its maximal ideal. Denote the set of all ideals $b$ of $R$ such that $V(b) \subseteq Z$ by $F(Z)$. Since for any $R$-module $M$, $\Gamma_Z(M) = \bigcup_{b \in F(Z)} \Gamma_b(M)$, one can easily check that $H_i^Z(X) \cong \lim_{\longrightarrow b \in F(Z)} H_i^b(X)$ for all integers $i$. Hence $H_i^Z(X) = 0$ for all $i > \dim_R X$.

Let $(R, m)$ be a local ring, $Z$ a stable under specialization subset of Spec $R$ and $X \in \mathcal{D}_<^L(R)$. We prove that $H_{\dim_R X}^Z(X) = 0$ if and only if for any $p \in \text{Assh}_R(X \otimes_R \hat{R})$, there is $q \in Z$ such that $\dim \hat{R}/q\hat{R} + p > 0$. Yoshino and Yoshizawa [YY, Theorem 2.10] have showed that for any abstract local cohomology functor $\delta : \mathcal{D}_<^L(R) \rightarrow \mathcal{D}_<^L(R)$, there is a stable under specialization subset $Z$ of Spec $R$ such that $\delta \cong R\Gamma_Z$. Thus our result may be considered as the largest generalization possible of the Hartshorne-Lichtenbaum Vanishing Theorem. In fact, we show that it includes all known generalizations of the Hartshorne-Lichtenbaum Vanishing Theorem.

2. Results

Let $Z$ be a stable under specialization subset of Spec $R$ and $X \in \mathcal{D}(R)$. The Propositions 2.1 and 2.3 below determine some situations where the local cohomology modules $H_i^Z(X)$ are Artinian. Recall that $\text{Supp}_RX$ is defined by $\text{Supp}_RX := \{ p \in \text{Spec } R \mid X_p \neq 0 \}$ ($= \bigcup_{i \in Z} \text{Supp}_RX_i(X)$).
Proposition 2.1. Let $Z$ be a stable under specialization subset of $\text{Spec } R$ and $X \in D_f^I(R)$. Assume that $\text{Supp}_R X \cap Z$ consists only of finitely many maximal ideals. Then $H^i_Z(X)$ is Artinian for all $i \in \mathbb{Z}$.

Proof. Let $p$ be a prime ideal and $E(R/p)$ denote the injective envelope of $R/p$. Since, $p$ is the only associated prime ideal of $E(R/p)$, it turns out

$$\Gamma_Z(E(R/p)) = \bigoplus_{p \in \text{Spec } R} (p, p \notin Z).$$

For each integer $i$, the $R_p$-module $\text{Ext}^i_{R_p}(R_p/pR_p, X_p)$ is finitely generated, and so

$$\mu^i(p, X) := \text{Vdim}_{R_p}(\text{Ext}^i_{R_p}(R_p/pR_p, X_p)) < \infty.$$

By [F2, Proposition 3.18], $X$ possesses an injective resolution $I$ such that $I_i \cong \bigoplus_{p \in \text{Spec } R} E(R/p)^{(\mu^i(p, X))}$ for all integers $i$. Let $i \in \mathbb{Z}$. Then

$$\Gamma_Z(I_i) = \bigoplus_{p \in \text{Spec } R} \Gamma_Z(E(R/p)^{(\mu^i(p, X))}) = \bigoplus_{p \in \text{Supp}_R X \cap Z} E(R/p)^{(\mu^i(p, X))}.$$

By the assumption, $\text{Supp}_R X \cap Z$ consists only of finitely many maximal ideals. This yields that $\Gamma_Z(I_i)$ is an Artinian $R$-module, and so $H^i_Z(X) = H_{-i}(\Gamma_Z(I))$ is Artinian too.

We record the following immediate corollary which extends [Z, Theorem 2.2]. We first recall some definitions. The left derived tensor product functor $- \otimes^L_R -$ is computed by taking a projective resolution of the first argument or of the second one. Also, the right derived homomorphism functor $R \text{Hom}_R(-, \sim)$ is computed by taking a projective resolution of the first argument or by taking an injective resolution of the second one. Let $a$ be an ideal of $R$ and $M, N$ two $R$-modules. The notion of generalized local cohomology modules $H^i_a(M, N) := \lim_{\rightarrow} \text{Ext}^i_R(M/a^nM, N)$ was introduced by Herzog in his Habilitationsschrift [He]. When $M$ is finitely generated, [Y, Theorem 3.4] yields that $H^i_a(M, N) \cong H_{-i}(R \Gamma_a(R \text{Hom}_R(M, N)))$ for all integers $i$.

Corollary 2.2. Let $a$ be an ideal of $R$ and $M$ and $N$ two finitely generated $R$-modules. Assume that $\text{Supp}_R M \cap \text{Supp}_R N \cap V(a)$ consists only of finitely many maximal ideals. Then $H^i_a(M, N)$ is Artinian for all $i \in \mathbb{Z}$.

Proposition 2.3. Let $Z$ be a stable under specialization subset of $\text{Spec } R$. Assume that for any finitely generated $R$-module $M$ of finite dimension, $H^\text{dim}_Z M(M)$ is Artinian. Then for any finite dimensional complex $X \in D^I_R(R)$, $H^\text{dim}_Z X(X)$ is Artinian.
PROOF. Set $d := \dim_R X$ and $s := \sup X$. Clearly, we may assume that $X \not\cong 0$, and so $n := \amp X$ is a non-negative integer. We argue by induction on $n$. Let $n = 0$. Then $X \cong \Sigma^s H_s(X)$, and so

$$H^d_\mathbb{Z}(X) = H^d_\mathbb{Z}(\Sigma^s H_s(X)) = H^{d+s}_\mathbb{Z}(H_s(X)).$$

On the other hand, by [F3, Proposition 3.5], $d = \sup \{\dim_R H_i(X) - i | i \in \mathbb{Z}\}$. Hence $\dim_R H_s(X) = d + s$, and so $H^{d+s}_\mathbb{Z}(H_s(X))$ is Artinian by our assumption. Now, assume that $n \geq 1$ and let $W := \tau_{\geq n} X$ and $Y := \tau_{s-1} X$ be truncated complexes of $X$; see [C, A.1.14]. Since $\amp W = 0$ and $\amp Y \leq n - 1$, these complexes satisfy the induction hypothesis. Next, one has

$$\dim_R X = \sup \{\dim_R H_i(X) - i | i \in \mathbb{Z}\} = \max \{\sup \{\dim_R H_i(X) - i | i \in \mathbb{Z} - \{s\}\}, \dim_R H_s(X) - s\} = \max \{\dim_R Y, \dim_R W\}.$$

Thus by Grothendieck’s Vanishing Theorem and induction hypothesis, we deduce that $H^d_\mathbb{Z}(W)$ and $H^d_\mathbb{Z}(Y)$ are Artinian. Now, by [F1, Theorem 1.41], there is a short exact sequence

$$0 \rightarrow W \rightarrow X \rightarrow Y \rightarrow 0$$

of complexes which induces a long exact sequence

$$H^{d-1}_\mathbb{Z}(Y) \rightarrow H^d_\mathbb{Z}(W) \rightarrow H^d_\mathbb{Z}(X) \rightarrow H^d_\mathbb{Z}(Y) \rightarrow 0.$$

It implies that $H^d_\mathbb{Z}(X)$ is Artinian. \hfill \Box

COROLLARY 2.4. Let $Z$ be a stable under specialization subset of $\text{Spec } R$, a an ideal of $R$ and $X \in D^f_\square (R)$.

i) If $R$ is local, then $H^\dim_R X(X)$ is Artinian.

ii) If $\dim_R X$ is finite, then $H^\dim_R X(X)$ is Artinian.

PROOF. In view of the above proposition, i) follows by [DNT, Theorem 2.6 and Lemma 3.2] and ii) follows by [BS, Exercise 7.1.7]. \hfill \Box

Let $A$ be an Artinian $R$-module. Recall that the set of attached prime ideals of $A$, $\text{Att}_R A$, is the set of all prime ideals $p$ of $R$ such that $p = \text{Ann}_R L$ for some quotient $L$ of $A$. Clearly, $A = 0$ if and only if $\text{Att}_R A$ is empty. If $R$ is local with the maximal ideal $m$, then $\text{Att}_R A = \text{Ass}_R (\text{Hom}_R (A, E(R/m)))$. Also, for an exact sequence $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$ of Artinian $R$-modules, one can see $\text{Att}_R U \subseteq \text{Att}_R V \subseteq \text{Att}_R U \cup \text{Att}_R W$. For proving our theorem, we need to the following lemmas.

LEMMA 2.5. Let $(R, m)$ be a local ring and $X \in D^f_\square (R)$. Then $\text{Att}_R (H^\dim_R m X(X)) = \text{Ass}_R X$. 

**Proof.** Set \( d := \dim_R X \). By Proposition 2.1, \( H^d_m(X) \) is an Artinian \( R \)-module. Hence, we have a natural isomorphism \( H^d_m(X) \cong H^d(X) \otimes_R \hat{R} \), and so [L, Corollary 3.4.4] provides a natural \( \hat{R} \)-isomorphism \( H^d_m(X) \cong H^d(X) \otimes_R \hat{R} \). From the definition of attached prime ideals, it follows that

\[
\text{Att}_R(H^d_m(X)) = \{ q \cap R | q \in \text{Att}_R(H^d_m(X) \otimes_R \hat{R}) \}.
\]

Let \( q \) be a prime ideal of \( \hat{R} \), \( p := q \cap R \) and \( M \) an \( R \)-module. We have the natural isomorphism \( M_p \otimes_R (\hat{R})_q \cong (M \otimes_R \hat{R})_q \). Since, the natural ring homomorphism \( R_p \rightarrow (\hat{R})_q \) is faithfully flat, \( M_p = 0 \) if and only if \( (M \otimes_R \hat{R})_q = 0 \). This implies that \( \inf_X = \inf(X \otimes_R \hat{R})_q \). On the other hand, one can easily check that \( \dim_X = \dim_X \otimes_R \hat{R} \). Thus, we can immediately verify that

\[
\text{Assh}_R X = \{ q \cap R | q \in \text{Assh}_R(X \otimes_R \hat{R}) \}.
\]

Therefore, we may and do assume that \( R \) is complete, and so it possesses a normalized dualizing complex \( D \). By [Ha, Chapter V, Theorem 6.2], there is a natural isomorphism

\[
H^i_m(X) \cong \text{Hom}_R(\text{Ext}_R^{-d}(X, D), E(R/m))
\]

for all integers \( i \). Since all homology modules of \( X \) and of \( D \) are finitely generated, \( X \) is homologically bounded and the injective dimension of \( D \) is finite, it follows that \( \text{RHom}_R(X, D) \in D^f(R) \). In particular, \( \text{Ext}_R^{-d}(X, D) \) is a finitely generated \( R \)-module for all \( i \in \mathbb{Z} \). Thus we have

\[
\text{Att}_R(H^d_m(X)) = \text{Att}_R(\text{Hom}_R(\text{Ext}_R^{-d}(X, D), E(R/m)))
\]

\[
= \text{Ass}_R(\text{Hom}_R(\text{Ext}_R^{-d}(X, D), E(R/m))), E(R/m))
\]

\[
= \text{Ass}_R(\text{Ext}_R^{-d}(X, D))
\]

\[
= \text{Ass}_R(\text{H}(\text{RHom}_R(X, D)))
\]

[F1, Theorem 16.20] implies that \( \sup(\text{RHom}_R(X, D)) = d \). Let \( p \in \text{Spec } R \). By [F1, Theorem 12.26], \( p \in \text{Ass}_R(\text{H}_d(\text{RHom}_R(X, D))) \) if and only if \( \text{depth}_{R_p} \text{RHom}_R(X, D)_p = -d \). But, [C, Lemma A.6.4 and A.6.32] and [F1, Theorem 15.17], yield that

\[
\text{depth}_{R_p} \text{RHom}_R(X, D)_p = \text{depth}_{R_p} \text{RHom}_R(X_p, D_p)
\]

\[
= \text{depth}_{R_p} D_p + \inf X_p
\]

\[
= - \dim \frac{R}{p} + \inf X_p.
\]

Therefore, \( p \in \text{Ass}_R(\text{H}_d(\text{RHom}_R(X, D))) \) if and only if \( \text{dim} \frac{R}{p} - \inf X_p = \dim_X X \). This means \( \text{Att}_R(H^d_m(X)) = \text{Assh}_R X \), as desired. \( \square \)
LEMMA 2.6. Let \((R, m)\) be a local ring, \(Z\) a stable under specialization subset of \(\text{Spec } R\) and \(X \in D^f(R)\). Then \(H^\dim(X) \otimes_R Z(X)\) is a homomorphic image of \(H^\dim(X) \otimes_R Z(X)\).

PROOF. Let \(a\) be an ideal of \(R\) and \(x \in m\). Let \(I\) be an injective resolution of \(X\). Then \(I_x\), the localization of \(I\) at \(x\), provides an injective resolution of \(X_x\) in \(D^f(R_x)\). Now, [BS, Lemma 8.1.1] yields the following exact sequence of complexes

\[0 \to \Gamma_{a+(x)}(I) \to \Gamma_a(I) \to \Gamma_{a}(I_x) \to 0\]

where the maps are the natural ones. Set \(d := \dim R X\). We deduce the long exact sequence

\[\cdots \to H^d_{a+(x)}(X) \to H^d_a(X) \to H^d_{a,R_x}(X_x) \to 0\]

By Corollary 2.4, \(H^d_a(X)\) is Artinian. Hence \(H^d_a(X)\) is supported at most at \(m\), and so

\[H^d_{a,R_x}(X_x) \cong H^d_a(X)_x = 0\]

Hence, the natural homomorphism \(H^d_{a+(x)}(X) \to H^d_a(X)\) is epic.

We may choose \(x_1, x_2, \ldots, x_n \in R\) such that \(m = a + (x_1, x_2, \ldots, x_n)\). Set \(a_i := a + (x_1, \ldots, x_i-1)\) for \(i = 1, \ldots, n + 1\). By the above argument, the natural homomorphism \(H^d_{a_{i+1}}(X) \to H^d_{a_i}(X)\) is epic for all \(1 \leq i \leq n\). Hence \(H^d_a(X)\) is a homomorphic image of \(H^d_m(X)\). This completes the proof, because \(H^d_a(X) \cong \varprojlim_b H^d_b(X)\), where the direct limit is over all ideals \(b\) of \(R\) such that \(V(b) \subseteq Z\).

LEMMA 2.7. Let \(M\) be a finitely generated \(R\)-module and \(X \in D^f(R)\).

i) \(\dim_R(M \otimes_R L X) \leq \dim_R X\).

ii) If \(\text{Supp}_R M \cap \text{Assh}_R X \neq \emptyset\), then \(\dim_R(M \otimes_R L X) = \dim_R X\) and

\[\text{Assh}_R(M \otimes_R L X) = \text{Supp}_R M \cap \text{Assh}_R X\]

PROOF. For any Noetherian local ring \(S\) and any two complexes \(V, W \in D^f(S)\), Nakayama’s Lemma for complexes asserts that \(\inf(V \otimes_R^L W) = \inf V + \inf W\); see e.g. [C, Corollary A.4.16]. In particular, this yields that \(\text{Supp}_R(V \otimes_R^L W) \subseteq \text{Supp}_R V \cap \text{Supp}_R W\). Now, by noting that for any complex \(Y \in D(R)\), we have

\[\dim_R Y = \sup \{\dim_R/P - \inf_Y P | P \in \text{Supp}_R Y\}\]

both assertions follow immediately.

Next, we conclude our theorem.

THEOREM 2.8. Let \((R, m)\) be a local ring, \(Z\) a stable under specialization subset of \(\text{Spec } R\) and \(X \in D^f(R)\). Then \(\text{Att}_R(H^\dim(X) \otimes_R Z(X)) = \{p \in \text{Assh}_R(X \otimes_R \hat{R}) | \dim \hat{R}/q \hat{R} + p = 0\text{ for all } q \in Z\}\).
PROOF. Set \( d := \dim_R X \) and \( s := \sup X \). We may assume that \( n := \amp X \) is a non-negative integer. First, by induction on \( n \), we prove the inclusion \( \subseteq \). If \( n = 0 \), then \( X \cong \Sigma^s H_s(X) \), and so

\[
H^d_\Z(X) = H^d_\Z(\Sigma^s H_s(X)) = H^{d+s}_\Z(H_s(X)) .
\]

In the proof of Proposition 2.3, we saw that \( \dim H_s(X) = d + s \), hence [DNT, Corollary 2.7] implies that

\[
\Att_\hat{\R}(H^d_\Z(X)) = \Att_\hat{\R}(H^{d+s}_\Z(H_s(X)))
\]

= \{ p \in \Assh_\hat{\R}(H_s(X) \otimes_\R \hat{\R}) | \dim \hat{\R}/q \hat{\R} + p = 0 \text{ for all } q \in \Z \}.

Now, assume that \( n \geq 1 \) and \( p \in \Att_\hat{\R}(H^d_\Z(X)) \). By Lemma 2.6, \( H^d_\Z(X) \) is an homomorphic image of \( H^d_m(X) \), and so Lemma 2.5 yields that

\[
\Att_\hat{\R}(H^d_\Z(X)) \subseteq \Att_\hat{\R}(H^d_m(X)) = \Assh_\hat{\R}(X \otimes_\R \hat{\R}) .
\]

Thus \( p \in \Assh_\hat{\R}(X \otimes_\R \hat{\R}) \). Let \( W := \tau_{\geq n} X \) and \( Y := \tau_{s-1} X \) be truncated complexes of \( X \). We have a short exact sequence

\[
0 \rightarrow W \rightarrow X \rightarrow Y \rightarrow 0
\]

of complexes and from the proof of Proposition 2.3, we know that \( \dim_R X = \max\{ \dim_R W, \dim_R Y \} \). From the long exact sequence

\[
\cdots \rightarrow H^d_\Z(W) \rightarrow H^d_\Z(X) \rightarrow H^d_\Z(Y) \rightarrow 0 ,
\]

we deduce that

\[
\Att_\hat{\R}(H^d_\Z(X)) \subseteq \Att_\hat{\R}(H^d_\Z(W)) \cup \Att_\hat{\R}(H^d_\Z(Y)) .
\]

Thus, either \( p \in \Att_\hat{\R}(H^d_\Z(W)) \) or \( p \in \Att_\hat{\R}(H^d_\Z(Y)) \). By Grothendieck’s Vanishing Theorem, the first case implies that \( \dim_R W = d \) and the second case implies that \( \dim_R Y = d \). Since \( \amp W = 0 \) and \( \amp Y \leq n - 1 \), in both cases, the induction hypothesis yields that \( \dim \hat{\R}/q \hat{\R} + p = 0 \) for all \( q \in \Z \).

Now, we prove the inclusion \( \supseteq \). Let \( p \in \Assh_\hat{\R}(X \otimes_\R \hat{\R}) \) be such that \( \dim \hat{\R}/q \hat{\R} + p = 0 \) for all \( q \in \Z \). We have to show that \( p \in \Att_\hat{\R}(H^d_\Z(X)) \). Since \( H^d_\Z(X) \) is an Artinian \( R \)-module, we have the natural isomorphism \( H^d_\Z(X) \cong H^d_\Z(X) \otimes_\R \hat{\R} \). On the other hand, by [L, Corollary 3.4.4], for any ideal \( a \) of \( R \), there is a natural \( \hat{\R} \)-isomorphism \( H^d_a(X) \otimes_\R \hat{\R} \cong H^d_{a\hat{\R}}(X \otimes_\R \hat{\R}) \). Let \( \hat{\Z} := \{ q \in \Spec \hat{\R} | q \cap R \in \Z \} \), which can be easily checked that is a stable under specialization subset of \( \Spec \hat{\R} \). It is straightforward to see that the two families \( \{ a\hat{\R} | a \text{ is an ideal of } R \text{ with } V(a) \subseteq \Z \} \) and \( \{ b\hat{\R} | b \text{ is an ideal of } \hat{\R} \text{ with } V(b) \subseteq \hat{\Z} \} \) are
cofinal. This implies that \( H^d_\text{d}(X) \cong H^d_\text{d}(X \otimes_R \hat{R}) \). Also, we have \( \dim_{\hat{R}}(X \otimes_R \hat{R}) = \dim_R X \) and \( \dim \hat{R}/q + p = 0 \) for all \( q \in \hat{Z} \). Therefore, we may and do assume that \( R \) is complete.

Since \( R \) is complete, there is a complete regular local ring \((T, n)\) and a surjective ring homomorphism \( f : T \rightarrow R \). One can easily check that \( X \in D^f_{\square}(T) \) and \( \dim_T X = \dim_R X \). Set \( \hat{Z} := \{ f^{-1}(q) | q \in \mathcal{Z} \} \), which is clearly a stable under specialization subset of \( \text{Spec } T \).

By [L, Corollary 3.4.3], for any ideal \( b \) of \( T \), there is a natural \( T \)-isomorphism \( H^d_\text{d}(X) \cong H^d_\text{d}(\hat{X}) \). From this, we can conclude a natural \( T \)-isomorphism \( H^d_\text{d}(X) \cong H^d_\text{d}(X) \). For any Artinian \( R \)-module \( A \) and any \( q \in \text{Spec } R \), it turns out that \( A \) is also Artinian as a \( T \)-module and \( q \in \text{Att}_T A \) if and only if \( f^{-1}(q) \in \text{Att}_T A \). Finally, we have \( \dim T/\bar{q} + f^{-1}(p) = 0 \) for all \( \bar{q} \in \hat{Z} \) and \( \text{Assh } X = \{ f^{-1}(q) | q \in \text{Assh}_R X \} \). Thus from now on, we can assume that \( R \) is a complete regular local ring.

Lemma 2.7 yields that \( \dim_R(p \otimes^L_R X) \leq \dim_R X \) and \( \dim_R(R/p \otimes^L_R X) = \dim_R X \). Let \( P \) be a projective resolution of \( X \). Applying \( - \otimes_R P \) to the short exact sequence

\[
0 \rightarrow p \rightarrow R \rightarrow R/p \rightarrow 0,
\]
yields the following exact sequence of complexes

\[
0 \rightarrow p \otimes^L_R X \rightarrow X \rightarrow R/p \otimes^L_R X \rightarrow 0.
\]

It yields the following exact sequence

\[
\cdots \rightarrow H^d_\text{d}(p \otimes^L_R X) \rightarrow H^d_\text{d}(X) \rightarrow H^d_\text{d}(R/p \otimes^L_R X) \rightarrow 0.
\]

As \( R \) is regular, the projective dimension of any \( R \)-module is finite, and so for any finitely generated \( R \)-module \( M \), one has \( M \otimes^L_R X \in D^f_{\square}(R) \). Since \( \dim R/q + p = 0 \) for all \( q \in \mathcal{Z} \), it follows that \( \Gamma_\mathcal{Z}(\Gamma_p(M)) = \Gamma_m(M) \) for all \( R \)-modules \( M \). Let \( I \) be an injective resolution of \( R/p \otimes^L_R X \). Since

\[
\text{Supp}_R I = \text{Supp}_R(R/p \otimes^L_R X) \subseteq V(p),
\]
by [L, Corollary 3.2.1], \( \Gamma_p(I) \cong I \), and so

\[
\Gamma_\mathcal{Z}(I) \cong \Gamma_\mathcal{Z}(\Gamma_p(I)) = \Gamma_m(I).
\]

In particular, there is an isomorphism \( H^d_\text{d}(R/p \otimes^L_R X) \cong H^d_\text{d}(R/p \otimes^L_R X) \). Therefore, by Lemmas 2.7 and 2.5, we deduce that \( p \in \text{Att}_R(H^d_\text{d}(R/p \otimes^L_R X)) \subseteq \text{Att}_R(H^d_\text{d}(X)) \).

Now, we are ready to establish the derived category analogue of the Hartshorne-Lichtenbaum Vanishing Theorem.

**Corollary 2.9.** Let \((R, m)\) be a local ring, \( \mathcal{Z} \) a stable under specialization subset of \( \text{Spec } R \) and \( X \in D^f_{\square}(R) \). The following are equivalent:

\[ \text{i) } H^d_{\dim \mathcal{Z}} X(X) = 0. \]
ii) For any $p \in \text{Assh}_{\hat{R}}(X \otimes_{\hat{R}} \hat{R})$, there is $q \in \mathbb{Z}$ such that $\dim \hat{R}/q\hat{R} + p > 0$.

**Corollary 2.10.** Let $a$ be an ideal of the local ring $(R, m)$ and $X \in \mathcal{D}_{f\square}(R)$.

1) $\text{Att}_{\hat{R}}(\dim_{\hat{R}} X) = \{ p \in \text{Assh}_{\hat{R}}(X \otimes_{\hat{R}} \hat{R}) \mid \dim \hat{R}/a\hat{R} + p = 0 \}$.

2) The following are equivalent:
   i) $\dim_{\hat{R}} X = 0$.
   ii) $\dim \hat{R}/a\hat{R} + p > 0$ for all $p \in \text{Assh}_{\hat{R}}(X \otimes_{\hat{R}} \hat{R})$.

**Corollary 2.11.** Let $(R, m)$ be a local ring, $\mathcal{Z}$ a stable under specialization subset of $\text{Spec } R$ and $M, N$ two finitely generated $R$-modules. Assume that $R\text{Hom}_{\hat{R}}(M, N) \in \mathcal{D}_{f\square}(R)$ and set $d := \dim_R(R\text{Hom}_{\hat{R}}(\hat{M}, \hat{N}))$. The following are equivalent:
   i) $H^d_{\mathcal{Z}}(M, N) = 0$.
   ii) For any $p \in \text{Assh}_{\hat{R}}(R\text{Hom}_{\hat{R}}(\hat{M}, \hat{N}))$, there is $q \in \mathbb{Z}$ such that $\dim \hat{R}/q\hat{R} + p > 0$.

**Proof.** Note that $R\text{Hom}_{\hat{R}}(M, N) \otimes_{\hat{R}} \hat{R} \cong R\text{Hom}_{\hat{R}}(\hat{M}, \hat{N})$, and so the result follows by Corollary 2.9. \qed

**Remark 2.12.** Let $\mathcal{Z}$ be a stable under specialization subset of $\text{Spec } R$ and $X \in \mathcal{D}_{\square}(R)$.

1) Suppose that dimension of $X$ is finite. Then $H^d_{\mathcal{Z}}(X) = \lim_{\rightarrow a} H^d_{a\mathcal{Z}}(X)$, where the direct limit is over all ideals $a$ of $R$ such that $V(a) \subseteq \mathcal{Z}$. But, $H^d_{\mathcal{Z}}(X)$ is not Artinian in general. To this end, let $R$ be a finite dimensional Gorenstein ring such that the set $\mathcal{Z} := \{ m \in \text{Max } R \mid \text{ht } m = \dim R \}$ is infinite. Clearly, $\mathcal{Z}$ is a stable under specialization subset of $\text{Spec } R$. The minimal injective resolution of $R$ has the form

$$0 \rightarrow \bigoplus_{\text{ht } p = 0} E(R/p) \rightarrow \bigoplus_{\text{ht } p = 1} E(R/p) \rightarrow \cdots \rightarrow \bigoplus_{\text{ht } p = \dim R} E(R/p) \rightarrow 0.$$ 

Hence $H^d_{\mathcal{Z}}(R) = \bigsqcup_{m \in \mathcal{Z}} E(R/m)$, which is not Artinian.

2) Suppose that $R$ is local with the maximal ideal $m$ and $I, J$ two ideals of $R$. In [TYY], Takahashi, Yoshino and Yoshizawa considered the following stable under specialization subset of $\text{Spec } R$

$$W(I, J) = \{ p \in \text{Spec}(R) \mid I^n \subseteq p + J \text{ for a natural integer } n \}.$$ 

For each integer $i$, they called $H^i_{I, J}(-) := H^i_{W(I, J)}(-)$, $i$-th local cohomology functor with respect to $(I, J)$. For the ring $R$ itself, they extended the Hartshorne-Lichtenbaum Vanishing Theorem; see [TYY, Theorem 4.9]. Namely, they showed that $H^d_{I, J}(R) = 0$ if and only if for any prime ideal $p \in \text{Assh}_{\hat{R}} \hat{R} \cap V(J \hat{R})$, we
have \( \dim \hat{R}/I \hat{R} + p > 0 \). On the other hand by [DNT, Theorem 2.8], \( H^{\dim}_{\hat{R},J}(R) = 0 \) if and only if for any prime ideal \( p \in \text{Assh}_{\hat{R}} \hat{R} \), there is \( q \in W(I, J) \) such that \( \dim \hat{R}/q \hat{R} + p > 0 \). Hence the following statements are equivalent:

i) For any prime ideal \( p \in \text{Assh}_{\hat{R}} \hat{R} \cap V(J \hat{R}) \), we have \( \dim \hat{R}/I \hat{R} + p > 0 \).

ii) For any prime ideal \( p \in \text{Assh}_{\hat{R}} \hat{R} \), there is \( q \in W(I, J) \) such that \( \dim \hat{R}/q \hat{R} + p > 0 \).

As Takahashi, Yoshino and Yoshizawa [TYY, Remark 4.10] have mentioned, it is not so easy to check the equivalence of these statements directly. Here, we do this under the extra assumption that \( R \) is complete. (In fact this assumption is not needed for the implication \( \text{ii) } \implies \text{i) } \).) Suppose \( \text{ii) } \) holds and let \( p \in \text{Assh}_{\hat{R}} \hat{R} \cap V(J) \). By the assumption there is \( q \in W(I, J) \) such that \( \dim \hat{R}/q \hat{R} + p > 0 \). Since \( q \in W(I, J) \), there is a natural integer \( n \), such that \( I^n \subseteq q + J \). This yields that \( I^n + p \subseteq q + p \), and so

\[
\dim R/I + p = \dim R/I^n + p \geq \dim R/q + p > 0.
\]

Conversely, suppose that \( \text{i) } \) holds and let \( p \in \text{Assh}_{\hat{R}} \hat{R} \). First, assume that \( J \subseteq p \). Then by the assumption, \( \dim R/I + p > 0 \), and so there is \( q \in V(I + p) \) such that \( \dim R/q > 0 \). Then \( I + J \subseteq I + p \subseteq q \). Hence \( q \in W(I, J) \) and \( \dim R/q + p = \dim R/q > 0 \). Thus \( \text{ii) } \) follows when \( J \subseteq p \). Now, assume that \( J \nsubseteq p \).

3) Suppose that \( R \) is local and \( F(Z) \) denote the set of all ideals \( b \) of \( R \) such that \( V(b) \subseteq Z \). As we mentioned in the introduction \( H^i_Z(X) \cong \lim_{\rightarrow b \in F(Z)} H^i_b(X) \) for all integers \( i \). The relationship between \( H^{\dim R}_{Z}(X) \) and \( H^{\dim R}_{b}(X) \)'s is more deeper. In fact by Theorem 2.8, we have

\[
\text{Att}_{\hat{R}}(H^{\dim R}_{Z}(X)) = \bigcap_{b \in F(Z)} \text{Att}_{\hat{R}}(H^{\dim R}_{b}(X)).
\]

This implies that \( H^{\dim R}_{Z}(X) = 0 \) if and only if \( H^{\dim R}_{b}(X) = 0 \) for an ideal \( b \in F(Z) \).

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References

[BH] M. Brodmann and C. Huneke, A quick proof of the Hartshorne-Lichtenbaum vanishing theorem, *Algebraic geometry and its applications*, (West Lafayette, IN, 1990), 305–308, Springer, New York, 1994.

[BS] M. Brodmann and R. Y. Sharp, *Local cohomology: An algebraic introduction with geometric applications*, Cambridge Studies in Advanced Mathematics 60, Cambridge University Press, Cambridge, 1998.

[CS] F. W. Call and R. Y. Sharp, A short proof of the local Lichtenbaum-Hartshorne theorem on the vanishing of local cohomology, Bull. London Math. Soc. 18(3), (1986), 261–264.

[C] L. W. Christensen, *Gorenstein dimensions*, Lecture Notes in Mathematics, 1747, Springer-Verlag, Berlin, 2000.

[DH] K. Divaani-Aazar and A. Hajikarimi, *Generalized local cohomology modules and homological Gorenstein dimensions*, Comm. Algebra 39(6), (2011), 2051–2067.

[DNT] K. Divaani-Aazar, R. Naghipour and M. Tousi, The Lichtenbaum-Hartshorne theorem for generalized local cohomology and connectedness, Comm. Algebra 30(8), (2002), 3687–3702.

[F1] H-B. Foxby, Hyperhomological algebra & commutative rings, in preparation.

[F2] H-B. Foxby, A homological theory of complexes of modules, Preprint Series no. 19 a & 19 b, Department of Mathematics, University of Copenhagen, 1981.

[F3] H-B. Foxby, Bounded complexes of flat modules, J. Pure Appl. Algebra 15(2), (1979), 149–172.

[Ha] R. Hartshorne, *Residues and duality*, Lecture Notes in Mathematics 20, Springer-Verlag, Berlin-New York, 1966.

[He] J. Herzog, *Komplex Auflösungen und Dualität in der lokalen Algebra*, Habilitationsschrift, Universität Regensburg, (1974).

[L] J. Lipman, Lectures on local cohomology and duality, Local cohomology and its applications (Guanajuato, 1999), Lecture Notes in Pure and Appl. Math. 226, Dekker, New York, (2002), 39–89.

[Sc] P. Schenzel, Explicit computations around the Lichtenbaum-Hartshorne vanishing theorem, Manuscripta Math. 78(1), (1993), 57–68.

[TYY] R. Takahashi, Y. Yoshino and T. Yoshizawa, Local cohomology based on a nonclosed support defined by a pair of ideals, J. Pure Appl. Algebra 213(4), (2009), 582–600.

[Y] S. Yassemi, Generalized section functors, J. Pure Appl. Algebra 95(1), (1994), 103–119.

[YY] Y. Yoshino and T. Yoshizawa, Abstract local cohomology functors, Math. J. Okayama Univ. 53 (2011), 129–154.

[Z] N. Zamani, On graded generalized local cohomology, Arch. Math. (Basel) 86(4), (2006), 321–330.

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