A quantum system with discrete and continuous evolution spectrum is studied. A final pointer basis is found, that can be defined in a precisely mathematical way. This result is used to explain the quantum measurement in the system.

I. INTRODUCTION

In the last three years we have devoted some papers to the study of a method to deal with quantum systems endowed with a continuous evolution spectrum [1] [2] [3] [4] [5] [6]. One of the results of this research was paper [7] (that can be considered as the part one of this paper) where decoherence was found in a system with continuous energy spectrum $0 < \omega < \infty$ and a sole discrete eigenvalue $\omega_0 < 0$. It was demonstrated that this system decoheres in a precisely defined final pointer basis. The corresponding Wigner functions were calculated and the classical limit was found. But in this example there was just one discrete eigenvalue non overlapping with the continuous spectrum, and therefore it was impossible to use systems of this kind to explain quantum measurements.

On the contrary in this paper we will solve the problem for a system with a free Hamiltonian with discrete spectrum with an arbitrary number of eigenvalues overlapping the continuous spectrum. Again now we will find a well defined final pointer basis that will be used to explain quantum measurements in a very easy way. In paper [8] we have explained the philosophy of the method, which we do not repeat here for the sake of conciseness.

The mathematical method that we will use is a generalization of the perturbation method for systems with continuous spectrum introduced in paper [8]. This generalization will be explained in section II. In section III we will solve the system and find its final pointer basis. Section IV will be devoted to the use of the system to explain quantum measurement. We will state our conclusion in section V. An appendix, to show complementary results, completes the paper.

II. PERTURBATIVE DIAGONALIZATION OF A LIOUVILLE OPERATOR

We will generalize section III of paper [8] for the case when the liouvillian has many discrete eigenvalues. This generalization is quite straightforward so we will only sketch the main calculations. But let us begin by reviewing the corresponding Hamiltonians.

We will have

$$ (1) \quad H = H^0 + H^1 $$

where the free Hamiltonian reads

$$ (1') \quad H^0 = \sum_{i=1}^{N} \Omega_i |i\rangle\langle i| + \int_0^\infty d\omega |\omega\rangle\langle \omega| . $$

The first term of the r.h.s. will play the role of what is usually called the “system” and the second one the role of the “environment”, $\Omega_i > 0$ are the discrete eigenvalues and $0 < \omega < \infty$ is the continuous spectrum. In order to follow the literature as close as possible we will use the review paper [9] [9] as a base, so we will choose an interaction

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1 The numbers to the left of the equation corresponds to the equations of paper [8] and can be used if the reader would like to follow both papers.

2 Also see bibliography of paper [9].
similar to \( V = (\sum_n \lambda_n q_n) x \), where \( q_n \) are a set of environment oscillators and \( x \) is the position of a particle (the “system”). Then in this \( V \) there is neither interaction among the environment modes nor the particle modes. There is only interaction between the environment and the particle. These properties, translated to our model, give the following interaction Hamiltonian

\[
\begin{align*}
H^1 &= \sum_{i=1}^{N} \int_{0}^{\infty} d\omega \ V_{x,i} |i\rangle \langle \omega| + \sum_{i=1}^{N} \int_{0}^{\infty} d\omega \ V_{i} \ |i\rangle \langle \omega| ,
\end{align*}
\]

where \( V_{x,i} = \frac{V}{i\omega} \). For simplicity we will take \( V_{x,i} \) real and \( V_{i} = V_{i}\omega \). The basis of the operators of the system we will use is the following set of vectors basis

\[
|i\rangle \equiv |i\rangle \langle i|, \quad |\omega\rangle \equiv |\omega\rangle \langle \omega| , \quad |i\rangle \langle \omega| \equiv |i\rangle \langle \omega| ,
\]

\[
|\omega\rangle \equiv |\omega\rangle \langle i|, \quad |\omega\rangle \langle \omega'| \equiv |\omega\rangle \langle \omega'| ,
\]

as in papers [2] [3]. The corresponding co-basis for the state will be

\[
(i\rangle |, \quad (\omega\rangle |, \quad (i\omega\rangle |, \quad (i\omega\rangle | , \quad (\omega\omega\rangle | , \quad (\omega\omega\rangle | , etc.
\]

Using these bases the unit superoperator reads

\[
I = \sum_{i,j} |i\rangle \langle i| + \int_{0}^{\infty} d\omega |\omega\rangle \langle \omega| + \int_{0}^{\infty} d\omega |i\rangle \langle \omega| +
\]

\[
+ \sum_{i} \int_{0}^{\infty} d\omega |\omega\rangle \langle i| + \int_{0}^{\infty} d\omega \int_{0}^{\infty} d\omega' |\omega\rangle \langle \omega'| .
\]

Also the free liouvillian reads

\[
L^0 = \sum_{i_j} (\Omega_i - \Omega_j) |i\rangle \langle i| + \sum_{i} \int_{0}^{\infty} d\omega (\Omega_i - \omega) |i\rangle \langle i| +
\]

\[
+ \sum_{i} \int_{0}^{\infty} d\omega (\omega - \Omega_i) |\omega\rangle \langle i| + \int_{0}^{\infty} d\omega \int_{0}^{\infty} d\omega' (\omega - \omega') |\omega\rangle \langle \omega'| .
\]

and the interaction liouvillian is

\[
L^1 = \sum_{i,j} \int_{0}^{\infty} d\omega [V_{x,i} |\omega\rangle - V_{x,j} |i\rangle] |i\rangle +
\]

\[
+ \sum_{i} \left[ \int_{0}^{\infty} d\omega \int_{0}^{\infty} d\omega' V_{x,i} |\omega\rangle \langle \omega' | - \int_{0}^{\infty} d\omega \int_{0}^{\infty} d\omega' V_{x,j} |i\rangle \langle \omega' | \right] |i\rangle +
\]

\[
+ \sum_{j} \left[ \int_{0}^{\infty} d\omega \int_{0}^{\infty} d\omega' V_{i,j} |i\rangle \langle \omega' | - \int_{0}^{\infty} d\omega \int_{0}^{\infty} d\omega' V_{i,j} |\omega\rangle \langle \omega' | \right] |\omega\rangle +
\]

\[
\int_{0}^{\infty} d\omega \int_{0}^{\infty} d\omega' \sum_{i} [V_{i,j} |i\rangle \langle \omega' | - V_{i,j} |\omega\rangle \langle i|] |\omega'\rangle +
\]

\[
\int_{0}^{\infty} d\omega \sum_{i} [V_{i,j} |i\rangle \langle \omega| - V_{i,j} |\omega\rangle \langle i|] |\omega| .
\]

Now we can repeat all the computations of paper [8] from eq. (62) to eq. (69) with no changes. The first change appears in eq. (70) when we must define the projector on the invariance space of the extended evolution liouvillian \( L^0_{ext} \), that now reads

\[
(70) \quad P_0 = \sum_{i} |i\rangle \langle i| + \int_{0}^{\infty} d\omega |\omega\rangle \langle \omega| .
\]
Now we can extend $I$, $L^0$ and $L^1$ to the complex plane as in paper [8]

\begin{equation}
I_{ext} = \sum_{i,j} |i \; j \rangle \langle i \; j| + \int_0^\infty d\omega |\omega \rangle \langle \omega | + \sum_i \int d\omega' |i \omega' \rangle \langle i \omega'| + \sum_i \int d\omega' |\omega' i \rangle \langle \omega' |i|.
\end{equation}

(59)

\begin{equation}
L^0_{ext} = (\Omega_i - \Omega_j) |i \; j \rangle \langle i \; j| + \sum_i \int d\omega' (\Omega_i - \Omega') |\omega' i \rangle \langle \omega'| + \sum_i \int d\omega' |\omega' |i\rangle \langle \omega'| + \sum_i \int d\omega' (\omega - \omega') |\omega' i \rangle \langle \omega'|,
\end{equation}

(60)

\begin{equation}
L^1_{ext} = \sum_{i,j} \left[ \int d\omega V_{zi} |z \; j \rangle - \int \frac{d\omega'}{\Gamma} V_{z'j} |i \; z' \rangle \right] (i \; j) + \sum_i \int d\omega' \sum_j V_{jz} |i \; j \rangle - \int \frac{d\omega}{\Gamma} V_{jz} |i \; j \rangle (i \; j) + \sum_i \int d\omega' \sum_j V_{jz} |i \; j \rangle - \int \frac{d\omega}{\Gamma} V_{jz} |i \; j \rangle (i \; j + \sum_i \int d\omega' \sum_j V_{jz} |i \; j \rangle - \int \frac{d\omega}{\Gamma} V_{jz} |i \; j \rangle (i \; j)
\end{equation}

(61)

and we can define the “non-diagonal” projectors:

\begin{equation}
P_{nd} = \sum_{i \neq j} |i \; j \rangle \langle i \; j|
\end{equation}

(14)

Then

\begin{equation}
I_{ext} = P_0 + P_{nd} + P_{\Gamma} + P_{\Gamma^*} + P_{\Gamma \Gamma^*}
\end{equation}

(15)

where the last three projectors correspond to the last three integrals of eq. [14].

Now we are ready to compute the discrete diagonal spectrum and the corresponding eigenvectors. Everything will be the same as in paper [8] e.g. $\lambda_i^{(0)} = \lambda_i^{(1)} = 0$ until we reach eq. (74) that now reads

\begin{equation}
\lambda_i^{(2)}|\phi_i^{(0)}\rangle = [P_{I_{ext}}L^1_{ext}Q_0^{-1}L^0_{ext}Q_0^{-1}L^1_{ext}P_0]|\phi_i^{(0)}\rangle
\end{equation}

(16)

where

\begin{equation}
Q_0 = I_{ext} - P_0 = P_{nd} + P_{\Gamma} + P_{\Gamma} + P_{\Gamma \Gamma}.
\end{equation}

(17)

From the definition of $P_0$ [10] we see that only discrete diagonal terms with vectors $(ii)$ remain between the square brackets of eq. (16). Then this eigenvalue equation reads

\begin{equation}
2\pi i \left[ \sum_i V_{\Omega i}^2 |i \; i \rangle \langle i \; i| - V_{\Omega i}^2 |i \; i \rangle \langle i \; i| (\omega - \Omega_i)(\phi^{(0)}_i) \right] = \lambda_i^{(2)} \left[ \sum_i |i \; i \rangle \langle i \; i| + \int d\omega |\omega \rangle \langle \omega| \phi^{(0)}_i \right]
\end{equation}

(18)

and it has the solutions

\begin{equation}
\lambda_i^{(2)} = 2\pi i V_{\Omega i}^2 \quad |\phi^{(0)}_i \rangle = |i \; i \rangle
\end{equation}

(19)

\begin{equation}
\lambda_i^{(2)} = 0 \quad |\phi^{(0)}_\omega \rangle = \sum_i \delta(\omega - \Omega_i)|i \; i \rangle + |\omega \rangle
\end{equation}

(19)
Then, as in paper [13], the degeneration of the discrete eigenvalues of \( L^0 \) (namely \( \infty \times \infty \)) have being partially removed since now we have just an \( \infty \) degeneration for eigenvalue \( \lambda_\infty = 0 \). Going to higher orders it can be shown that this degeneration always remains. We could have foreseen the solution (19), since there is no interaction among the discrete modes, there is only interaction between the discrete and continuous modes, so solution (19) must be just the sum of the corresponding solution of paper (see eq. (77)).

In analogous way we can find the corresponding eigenbras

\[
\lambda_{d ij}^{(2)} = 2\pi i V_{\Omega_i}^2 \quad (\psi_{d ij}^{(0)}) = (i \, i) - (\omega = \Omega_i) \quad (\omega = \Omega_i)
\]

(84)

\[
\lambda_{\infty}^{(2)} = 0 \quad (\psi_{\infty}^{(0)}) = (\omega)
\]

(20)

The computation of the rest of the spectrum and the eigenbasis follow the same lines and we obtain

\[
I_{ext} = \int_0^\infty d\omega |\phi_\omega)(\psi_\omega| + \sum_{i,j} |\phi_{d ij})(\psi_{d ij}| + \sum_i \int_\Gamma du |\phi_{u i})(\psi_{u i}|
\]

\[
+ \sum_i \int_\Gamma du' |\phi_{u' i})(\psi_{u' i}| + \int_\Gamma \int_\Gamma du \lambda_{u' i} |\phi_{u' i})(\psi_{u' i}|
\]

(21)

\[
I_{ext} = \sum_{i,j} \lambda_{d ij} |\phi_{d ij})(\psi_{d ij}| + \sum_i \int_\Gamma du \lambda_{u i} |\phi_{u i})(\psi_{u i}|
\]

\[
+ \sum_i \int_\Gamma du' \lambda_{u' i} |\phi_{u' i})(\psi_{u' i}| + \int_\Gamma \int_\Gamma du \lambda_{u' i} |\phi_{u' i})(\psi_{u' i}|
\]

(22)

where up to second order for the eigenvalues:

\[
\lambda_{d ij} = \Omega_i - \Omega_j + i\pi (V_{\Omega_i}^2 - V_{\Omega_j}^2) + \int_0^\infty d\omega \left[ -V_{\omega i}^2 P \frac{1}{\omega - \Omega_i} + V_{\omega j}^2 P \frac{1}{\omega - \Omega_j} \right]
\]

(23)

and for \( i = j \) the corresponding eigenvectors are

\[
|\phi_{d ii}) = (i \, i) + \int_0^\infty d\omega \frac{V_{\omega i}^2}{\Omega_i - \omega} |i \omega\rangle + \int_0^\infty d\omega \frac{V_{\omega i}^2}{\Omega_i - \omega} |i \omega\rangle
\]

(24)

\[
(\psi_{d ii}) = (i \, i) - (\omega = \Omega_i)
\]

(25)

these are the only new results (see appendix for the corresponding calculations). The rest is a simple straight generalization. Precisely

\[
\lambda_{u i} = u - \Omega_i
\]

(26)

\[
|\phi_{u i}) = |u i\rangle + \frac{V_{u i}}{u - \Omega_i} |i \, i\rangle - \int_\Gamma du' \frac{V_{u' i}}{u' - \Omega_i} |u \, u'\rangle
\]

(27)

\[
(\psi_{u i}) = (u \, i) + \frac{V_{u i}}{u - \Omega_i} [(i \, i) - (u)]
\]

(28)

\[
\lambda_{i u'} = \Omega_i - u' + i\pi V_{\Omega i}^2 - \int_0^\infty d\omega V_{\omega i}^2 P \frac{1}{\omega - \Omega_i}
\]

(29)

\[
|\phi_{i u'}) = |i \, u') + \frac{V_{i u'}}{u' - \Omega_i} |i \, i\rangle - \int_\Gamma du \frac{V_{i u'}}{u' - \Omega_i} |u \, u'\rangle
\]

(30)

\[
(\psi_{i u'}) = (i \, u') + \frac{V_{i u'}}{u' - \Omega_i} [(i \, i) - (u')] - \int_\Gamma du \frac{V_{i u'}}{u' - \Omega_i} |u \, u'|
\]

(31)
\[ \lambda_{u, u'} = u - u' \]  
(32)

\[ |\phi_{u, u'}\rangle = |u u'\rangle + \sum_i \left[ \frac{V_{u, i}}{u - \Omega_i} |i u'\rangle + \frac{V_{u', i}}{u' - \Omega_i} |u i\rangle \right] \]  
(33)

\[ (\psi_{u, u'}) = (u u') + \sum_i \left[ \frac{V_{u, i}}{u - \Omega_i} |i u'\rangle + \frac{V_{u', i}}{u' - \Omega_i} |u i\rangle \right] \]  
(34)

So now that the generalization is completed we will see the physical consequences of these, somehow unfamiliar, equations in the next section.

### III. DECOHERENCE AND THE FINAL POINTER BASIS

Let us first study the decoherence in the energy and then the one in all the observable of a CSCO that contains the Hamiltonian.

#### A. Decoherence in the energy

From paper \[7\] we know that a generalized state (let say at \( t = 0 \)) reads

\[
|\rho(0)\rangle = \int d\omega \rho_{\omega} |\psi_{\omega}\rangle + \sum_{i,j} \rho_{d_{ij}} |\psi_{d_{ij}}\rangle + \\
\sum_i \left[ \int_{\Gamma} d\omega \rho_{\omega i} |\psi_{\omega i}\rangle + \int_{\Gamma'} d\omega' \rho_{i\omega'} |\psi_{i\omega'}\rangle \right] + \\
+ \int_{\Gamma} d\omega \int_{\Gamma'} d\omega' \rho_{\omega\omega'} |\psi_{\omega\omega'}\rangle
\]

where the \( \rho \) satisfies the conditions stated in the quoted paper \( \rho_{\omega} = \overline{\rho_{\omega}} \geq 0; \rho_{d_{ij}} = \overline{\rho_{d_{ij}}}; \rho_{d_{ii}} \geq 0; \rho_{\omega i} = \overline{\rho_{i\omega}}; \rho_{\omega i} = \overline{\rho_{\omega'\omega'}} \) and

\[
\int_0^\infty d\omega \rho_{\omega} + \sum_i \rho_{d_{ii}} = 1.
\]

Using the diagonalized liouvillian of eq. \[22\] we can write the same state at \( t \) (up to second perturbation order in the eigenvalues) as:

\[
|\rho(t)\rangle = \int d\omega \rho_{\omega} |\omega\rangle + \\
+ \sum_{i,j} \rho_{d_{ij}} |\psi_{d_{ij}}\rangle \exp \left\{ i \left[ -\pi \left( V_{d_{ij}}^2 - V_{d_{ij}}^2 \right) + i \left( \Omega_i - \Omega_j \right) + \\
+ i \frac{V_{\omega i}^2}{\omega - \Omega_i} + V_{\omega j}^2 \frac{1}{\omega - \Omega_j} \right] \right\} + \\
+ \sum_i \int_{\Gamma} du \rho_{ui} |\psi_{ui}\rangle e^{i(u - \Omega_i) t} + \\
+ \sum_i \int_{\Gamma'} du' \rho_{i u'} |\psi_{i u'}\rangle e^{i(\Omega_i - u') t} e^{-iV_{\omega i}^2 t} \exp \left( -i \int_0^\infty d\omega \frac{V_{\omega i}^2}{\omega - \Omega_i} t \right) + \\
+ \int_{\Gamma} du' \int_{\Gamma'} du \ e^{i(u - u') t} |\psi_{u u'}\rangle
\]

\[3\] In paper \[2\] for the sake of simplicity was postulated that \( \rho_{\omega} \) was a regular function. Now we will change this assumption and allow to \( \rho_{\omega} \) to contain Diracs deltas as \( \delta(\omega - \Omega_i) \).
Taking into account the Riemann-Lebesgue theorem and the dumping factors we have:

\[
\lim_{t \to \infty} (\rho(t)) = \int d\omega \rho_{\omega}(\omega) + \sum_i \rho_{d\omega i} |\psi_{d\omega i}| = \rho^* \tag{37}
\]

begin \(\rho^*\) the equilibrium final state.

So, as in paper [7], the state \(\rho(t)\) decoheres into a diagonal state of the continuous part of the spectrum. Thus, we can ask ourselves where the terms are corresponding to the discrete part of the spectrum of the free liouvillian

\[
(\rho(0)) = \sum_{ij} \rho_{d\omega i}^0 (i \ j) \tag{38}
\]

when \(t \to \infty\). They are dissolved in the continuous spectrum but they are still there as we will see. While the off-diagonal terms \(i \neq j\) have disappeared as it should, dumped by their evolution factor, the diagonal terms \(i = j\) are in \(\rho_{\omega}\). To understand what is going on let us begin by a simple case. Let us suppose that \(t = 0\) there are only discrete terms, and let us write the diagonal terms of \((\rho(0))\) in the eigenbasis of \(L\)

\[
\sum_i \rho_{d\omega i}^0 (i \ i) = \int d\omega \rho_{\omega}(\omega) + \sum_i \rho_{d\omega i} [(i \ i) - (\omega = \Omega_i)] \tag{39}
\]

where we have used eq. (25) to compute \(|\psi_{d\omega i}|\). Then we necessarily have

\[
\rho_{d\omega i} = \rho_{ii}^0 \\
\rho_{\omega} = \sum_i \rho_{ii}^0 \delta(\omega - \Omega_i) \tag{40}
\]

as it can easily be verified. Thus from eq. (37) we have

\[
\lim_{t \to \infty} (\rho(t)) = \int d\omega \sum_i \rho_{i i}^0 \delta(\omega - \Omega_i)(\omega) = \sum_i \rho_{i i}^0 (\omega = \Omega_i) = \rho^* \tag{41}
\]

So we have found the **final pointer basis** \(\{(\omega = \Omega_i)\}\) in this case i.e. the pointer basis for the discrete part of the spectrum.

If the initial conditions would be completely general, with discrete and continuous diagonal components, etc., i.e.\n
\[
(\rho(0)) = \sum_{ij} \rho_{d\omega i}^0 (i \ j) + \int d\omega \rho_{\omega}^0 (\omega) + ... \tag{42}
\]

(where \(ij\) and \(\omega\) are now considered as eigenvectors of \(L_0\) and the dots symbolize other of diagonal terms) we would have in the basis of \(L\)

\[
(\rho(0)) = \sum_i \rho_{d\omega i} [(i \ i) - (\omega = \Omega_i)] + \int d\omega \rho_{\omega} (\omega) + ... \tag{43}
\]

so

\[
\rho_{d\omega i} = \rho_{ii}^0 \tag{44}
\]

\[
\rho_{\omega} = \sum_i \rho_{ii}^0 \delta(\omega - \Omega_i) + \rho_{\omega}^0 \tag{45}
\]

So in this general case we have

\[
\lim_{t \to \infty} (\rho(t)) = \sum_i \rho_{i i}^0 (\omega = \Omega_i) + \int d\omega \sum_i \rho_{\omega}^0 (\omega) = \rho^* \tag{46}
\]

since all the off-diagonal terms (the dots) disappear as explained so the **whole final pointer basis** is \(\{(\omega), (\omega = \Omega_i)\}\).

As an illustration we can compute the time evolution of the diagonal terms (discrete and continuous) for the simple initial condition (39). The evolution equation is
\( (\rho(t)| = (\rho(0)| e^{iLt} \) \tag{47}

and all its elements must be projected under the diagonal projector

\[ P = \int d\omega |\phi_\omega\rangle \langle \psi_\omega| + \sum_i |\phi_{dii}\rangle \langle \psi_{dii}| \] \( (48) \)

We will add a subindex \( d \) to the projected diagonal states. Then if

\[ (\rho(0)|_d = \sum_i \rho_{dii}(\psi_{dii}) + \int d\omega \rho_{\omega} (\psi_\omega) \]

it turns out that

\[ (\rho(t)|_d = \sum_i \rho_{dii}(\psi_{dii}) \exp(i\lambda_{dii} t) + \int d\omega \rho_{\omega} (\psi_\omega) = \]

\[ = \sum_i \rho^0_{dii} (i i) \exp(-2\pi V_{\Omega_i}^2 i t) + \sum_i \rho^0_{dii} [1 - \exp(-2\pi V_{\Omega_i}^2 i t)] (\omega = \Omega_i) \] \( (49) \)

where we have used eqs. (40), (19) and (20). Now we can clearly see how the factor \( \exp(-2\pi V_{\Omega_i}^2 i t) \) produces the decay of the discrete states \(| i i \rangle\) while the factors \([1 - \exp(-2\pi V_{\Omega_i}^2 i t)]\) make grow the states in the final discrete pointer basis: \( \{|\omega = \Omega_i\rangle\} \) as it should be.

### B. Decoherence in the other dynamical variables of the CSCO

If, as in paper [7] we could have other observable \( \{O_m\} \) in our CSCO (where \( m = 1, 2, ..., M \) and we consider that the spectra of these \( O_m \) observable are discrete for simplicity). Then the initial state (that generalizes the one in eq. (42)) would be

\[ (\rho(0)| = \sum_{i j m m'} \rho_{i j m m'} (i j m m') + \sum_{m m'} \int_0^\infty d\omega \rho_{\omega m m'} (\omega m m') + ... \] \( (50) \)

where the dots symbolize terms in the other vector of the cobasis. Then we would end as in (41) with

\[ (\rho_\star| = \sum_{i m m'} \rho_{i m m'} (\omega = \Omega_i m m') + \sum_{m m'} \int_0^\infty d\omega \rho_{\omega m m'} (\omega m m') + ... \] \( (51) \)

Now, following, step-by-step, section IIB of paper [7] we can diagonalize the matrices \( \rho_{i m m'} \) and \( \rho_{\omega m m'} \) to obtain

\[ \rho_\star = \sum_{i r} \rho_{i r r} (\omega = \Omega_i r r) + \sum_r \int_0^\infty d\omega \rho_{\omega r r} (\omega r r) + ... \] \( (52) \)

where \( r = 1, 2, ..., M \) and where \( \{|\omega r r|, (\omega = \Omega_i r r)\} \) is the final pointer basis that now corresponds to the final pointer CSCO \( \{H, P_1, ..., P_M\} \) where the \( P_i \) have only discrete spectra by the assumptions made at the beginning of the section, and as in eq. (22) of paper [7] they read

\[ P_j = \sum_i P_{i r}^{(j)} (\omega = \Omega_i r r) + \int_0^\infty d\omega P_{\omega r}^{(j)} (\omega r r) \] \( (53) \)

For the sake of simplicity, as in paper [7], we can make

\[ P_{i r}^{(j)} = P_{\omega r}^{(j)} = r^{(j)} = (r_1^{(j)}, r_2^{(j)}, ..., r_M^{(j)}) \] \( (54) \)

in such a way that the eigenvalues \( r^{(j)} \) label the eigenstates of \( P_j \). In this way the final pointer CSCO is defined.
C. Wigner function

Always following paper [3] it can be proved that the Wigner function corresponding to \( \rho_s \) reads

\[
\rho_s^W (q, p) = \sum_{i r} \rho_{i r}^W (q, p) + \sum_r \int_0^\infty d \omega \rho_{\omega r}^W (q, p)
\]

(55)

where

\[
\rho_{i r}^W (q, p) = C \delta [H^W(q, p) - \Omega_i] \delta [P^W_1(q, p) - r_1] ... \delta [P^W_M(q, p) - r_M]
\]

(56)

\[
\rho_{\omega r}^W (q, p) = C \delta [H^W(q, p) - \omega] \delta [P^W_1(q, p) - r_1] ... \delta [P^W_M(q, p) - r_M]
\]

(57)

Therefore we see that the classical analogue of the final pointers basis, \( \rho_{\omega r}^W \), are densities which are peaked along classical trajectories defined by the constant of motion \( (\omega r_1...r_M) \) and \( (\Omega_i r_1...r_M) \) respectively. Moreover they are \( \geq 0 \).

So everything that was said in paper [3] for just one discrete eigenvalue \( \omega_0 \) can be now repeated by the finite set of discrete eigenvalues \( \Omega_1...\Omega_N \), e.g. the histories-decoherence version of the theory ( [3] Appendix C) can be repeated word by word.

IV. QUANTUM MEASUREMENT THEORY

In this section, as an application we will sketch the elements and results of quantum measurement theory using the results of the previous sections.

As explained in review paper [4], that we take as a guide, let us consider a system with eigenvectors \( |s_i \rangle \) and a measurement apparatus with eigenvectors \( |A_i \rangle \), where now the index \( i \) represents the set of indices \( i, r \) or better \( i, r_1, ..., r_M \) of the previous section. Now we know that the measurement evolution brings a generic state of the system and the apparatus \( |\psi_0 \rangle = |\psi \rangle |A_0 \rangle \) to a new one \( |\psi_t \rangle \), precisely

\[
|\psi_0 \rangle = |\psi \rangle |A_0 \rangle = \left( \sum_i a_i |s_i \rangle \right) |A_0 \rangle \rightarrow \sum_i a_i |s_i \rangle |A_i \rangle = |\psi_t \rangle
\]

(58)

So this premeasurement process just correlates the state of the system with those of the apparatus. The corresponding density matrix evolves under this process as

\[
\rho_0 = |\psi_0 \rangle \langle \psi_0 | \rightarrow |\psi_t \rangle \langle \psi_t | = \rho_t
\]

(59)

So the final state is as pure as the initial one. This is precisely the quantum measurement problem: we must explain how this pure state \( \rho_t \) evolves to a diagonal density matrix, in such a way to allow classical-boolean measurements. The problem is readily solved if we call \( |s_i \rangle , |A_i \rangle = |i \rangle \) and we consider that really the Hamiltonian at the whole “system” (i.e. system, plus apparatus) is the first term of the r.h.s. of eq. (2) and to which we have added an “environment” with an Hamiltonian corresponding to the second term. Essentially the environment is the universe, so it is natural to endow this system with a continuous spectrum, since the universe contains at least the electromagnetic and the gravitational fields (both with continuous spectra). To this \( H_0 \) we can add the \( H^1 \) of eq. (3) that would be the simplest coupling possible.

Then, as proved in eq. (41) the matrix

\[
\rho_0 = |\psi_0 \rangle \langle \psi_0 | = \sum_{i j} \bar{a}_i a_j |i \rangle \langle j | = \sum_{i j} \bar{a}_i a_j |i \rangle \langle j |
\]

(60)

evolves to the final diagonal density matrix

\[
\rho_s = \sum_i |a_i|^2 (\omega = \Omega_i)
\]

(61)

in the discrete final pointer basis \( \{(\omega = \Omega_i)\} \). Moreover the corresponding Wigner functions of the vectors of this pointer basis are the ones displayed in eq. (56) which capture their classical nature. In this way the measurement quantum process and its classical output is completely explained.
V. CONCLUSION

As systems with continuous spectrum are very frequent in the universe if is clear that they cannot be excluded from the environment. Moreover an environment endowed with a continuous spectrum is very useful since it allows the use of Riemann-Lebesgue theorem. Nevertheless a big problem was that in such a system only the continuous diagonal states are stationary and therefore the only candidates for final equilibrium states are there. What is the ultimate fate of the states of the discrete spectrum is an intriguing question that deserved an answer. The answer is given in this paper. Such states dissolve themselves in the continuous diagonal but they remain as Dirac’s deltas originating the pointer basis \{(|\omega = \Omega_i|)\}. With this answer in hand it is easy to foresee the evolution of the system and to introduce a simple explanation for the quantum measurement process.

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APPENDIX A: PERTUBATION

The only really new characters in this paper with respect to those of paper [8] are the discrete off-diagonal elements with \( i \neq j \). Then let us compute their eigenvalues and eigenvectors with the perturbation method of this paper.

At zero order we have

\[
\lambda_{ij}^{(0)} = \Omega_i - \Omega_j \tag{85}
\]

\[
|\phi_{d_{ij}}^{(0)}\rangle = |i,j\rangle \tag{A1}
\]

At first order we must solve

\[
(\lambda^{(0)} - \mathbb{L}_{ext}^0)|\phi^{(1)}\rangle = (\mathbb{L}_{ext}^1 - \lambda^{(1)})|\phi^{(0)}\rangle \tag{A2}
\]

that in this case reads

\[
(\Omega_i - \Omega_j - \mathbb{L}_{ext}^0)|\phi^{(1)}\rangle = (\mathbb{L}_{ext}^1 - \lambda^{(1)})|i,j\rangle \tag{A3}
\]

which multiplied by \((i' j')\) gives

\[
(\Omega_i - \Omega_j - \Omega_{i'} + \Omega_{j'})(i' j' |\phi^{(1)}\rangle) = -\lambda^{(1)} \delta_{i i'} \delta_{j j'} \tag{A4}
\]

because from (3) \((i' j' |\mathbb{L}_{ext}^1 i j\rangle) = 0\) since there are no term \(|i' j'\rangle(i j\rangle\) in \(\mathbb{L}_{ext}^1\). So if \(i \neq i'\), \(j \neq j'\) (even if \(i \neq j\)) we have \(\lambda^{(1)} = 0\).

Now from eq. (3) we have

\[
(\Omega_i - \Omega_j - \mathbb{L}_{ext}^0)|\phi^{(1)}\rangle = \int_0^\infty d\omega \, V_{\omega i} |\omega j\rangle - \int_0^\infty d\omega \, V_{\omega j} |i \omega\rangle \tag{A5}
\]

so

\[
|\phi^{(1)}_{ij}\rangle = \int_\Gamma \frac{du \, V_{u i}}{\Omega_i - u} |u j\rangle - \int_\Gamma \frac{du' \, V_{u' j}}{u' - \Omega_j} |i u'\rangle. \tag{A6}
\]

We now can go to the second order where eq. (64) of paper [8] gives

\[
(\lambda^{(0)} - \mathbb{L}_{ext}^0)|\phi^{(2)}\rangle = (\mathbb{L}_{ext}^1 - \lambda^{(1)})|\phi^{(1)}\rangle - \lambda^{(2)}|\phi^{(0)}\rangle \tag{A7}
\]

which premultiplied by \((i' j')\) reads
\begin{align}
\langle \Omega_i - \Omega_j - \Omega_i' + \Omega_j' \rangle (i' j') \langle \phi^{(2)}_{i' j'} \rangle &= \langle i' j' | \mathbb{L}_{\text{ext}}^{1} | \phi^{(1)}_{i' j'} \rangle - \lambda^{(2)} \delta_{i i'} \delta_{j j'}.
\end{align}

\begin{align}
(i' j' | \mathbb{L}^{1}_{\text{ext}} | \phi^{(1)}_{i' j'}) &= \sum_{i j} \int_{\Gamma} \frac{d u'}{u' - \Omega_{j'}'} \frac{d u''}{u'' - \Omega_{i}'} \frac{d u'''}{u''' - \Omega_{j}''} \frac{d u'\prime}{u'\prime - \Omega_{i}'} \frac{d u''\prime}{u''\prime - \Omega_{j}''} \frac{d u''\prime\prime}{u''\prime\prime - \Omega_{j}''} \frac{d u'}{u'} \frac{d u''}{u''} \delta(u' - u'\prime) + \sum_{i j} \int_{\bar{\Gamma}} \frac{d u'}{u' - \Omega_{j'}'} \frac{d u''}{u'' - \Omega_{i}'} \frac{d u'''}{u''' - \Omega_{j}''} \frac{d u'\prime}{u'\prime - \Omega_{i}'} \frac{d u''\prime}{u''\prime - \Omega_{j}''} \frac{d u''\prime\prime}{u''\prime\prime - \Omega_{j}''} \frac{d u'}{u'} \frac{d u''}{u''} \delta(u'' - u').
\end{align}

\begin{align}
\int_{\Gamma} \frac{d u}{u' - \Omega_{j'}} &= \int_{0}^{\infty} d \omega \frac{V_{\omega}^{2}}{\omega - i \theta - \Omega_{j'}} = i \pi V_{\Omega_{j'}}^{2} + \int_{0}^{\infty} d \omega V_{\omega}^{2} P \left( \frac{1}{\omega - \Omega_{j'}} \right)
\end{align}

\begin{align}
\int_{\bar{\Gamma}} \frac{d u}{u - \Omega_{i'}} &= i \pi V_{\Omega_{i}}^{2} - \int_{0}^{\infty} d \omega V_{\omega}^{2} P \left( \frac{1}{\omega - \Omega_{i}} \right)
\end{align}

(cfr. \[8\] eqs. (89) and (91)). Then

\begin{align}
\lambda^{(2)} &= i \left[ V_{\Omega_{j}}^{2} + V_{\Omega_{i}}^{2} \right] + \int_{0}^{\infty} d \omega \left[ V_{\omega}^{2} P \left( \frac{1}{\omega - \Omega_{j}} \right) - V_{\omega}^{2} P \left( \frac{1}{\omega - \Omega_{i}} \right) \right]
\end{align}

and the computation is finished.

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