MULTIPOLE MOMENTS IN KALUZA-KLEIN THEORIES

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ABSTRACT. This paper contains discussion of the problem of motion of extended
i.e. non point test bodies in multidimensional space. Extended bodies are described
in terms of so called multipole moments. Using approximated form of equations of
motion for extended bodies deviation from geodesic motion is derived. Results are
applied to special form of space-time.

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1. Introduction

Kaluza-Klein type theories have regained a lot of interest [1] over recent years.
It was mainly due to the construction of gauge theories that the old idea of adding
a compactified dimension introduced by Kaluza and Klein [2, 3] has been developed
and utilised in a variety of theories. All of them are based on an assumption that
the standard four-dimensional world is a subset of a higher-dimensional space time,
with extra space dimensions dynamically compactified.

However these types of Kaluza–Klein theories leave an intriguing question unan-
swered: where the distinction between the observed space–like dimensions and other
compactified ones comes from. One possible solution to this problem was given by
Kopczyński [4]. His idea was to assume that this is the form of matter that gives
us this distinction. Constituents of matter would be not points in a \( m \)-dimensional
space-time but rather \((k-1)\)-dimensional objects — strings, membranes etc. There
exists a cosmological solution [5] such that we have expanding \( m-k+1 \) space like
dimensions and contracting \( k \) dimensions filled with matter.

Then the question arises whether that extendness in additional space dimensions
has any influence on macroscopic properties of matter. The purpose of this paper
is to look at behaviour of test bodies — extended in compactified dimensions —
moving in the observed \((m-k+1)\)-dimensional space-time. We will show that the
non point structure of these objects modifies their equations of motion, which results
in deviations from the geodesic motion. Furthermore, these deviations depend on
the topological structure of the compactified space.

It is highly nontrivial to obtain any exact solution of such a problem. Therefore
we will follow another way. We will describe extended bodies in terms of multipole
moments and write equation of motion in that approximate way [6].

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The structure of that paper is following. In section 2. we briefly discuss equations of motion for extended bodies and introduce some basic mathematical tools. In section 3. we define the problem and discuss initial assumptions that will build the model of space-time to be considered afterwards. In sections 4. and 5. we derive equation of motion in dipole and multipole approximation. In section 6. we estimate form of multipole moments applying results from previous sections to a special form of space-time.

2. Equations of motion for extended bodies

In order to describe extended bodies first we choose some unique world line laying inside the world tube of the body that represents its dynamical properties. Although in General Relativity there are many existing possible unique choices we follow the ideas of Dixon [6] and Beiglböck [7]. Thus we imply that the best such a line, called the centre-of-mass world line, is the one such that at at every point of it momentum \( p^\kappa \) and angular momentum \( S^{\kappa\lambda} \) are perpendicular:

\[
   p_\kappa S^{\kappa\lambda} = 0 .
\]

We parametrise this world line affinely as \( z(s) \). Then we introduce coordinate system \( X^\kappa \) at the given point \( z \) of the centre-of-mass world line in the way that any point \( x \) of space-time is given by a vector:

\[
   X^\kappa = -\sigma^\kappa(z, x) ,
\]

where \( \sigma(z, x) \) is the biscalar world-function [8]. Properties of the world function imply that \( X^\kappa \) is a two-point vector defined at points \( z \) and \( x \) such that its length is equal to geodesic distance between that points.

On the centre-of-mass world line we define tensorial multipole moments of the energy-momentum tensor. Because those moments describe inner properties of matter we would rather like to separate them from external space-time properties. As ordinary energy-momentum tensor describes both internal properties of matter and external geometry we introduce another quantity: energy-momentum skeleton \( \hat{T}^{\kappa\lambda} \). It is obtained from energy-momentum tensor \( T^{\kappa\lambda} \) by subtraction from \( T^{\kappa\lambda} \) information on geometry of space-time. Therefore it describes only properties of the given body and not external space-time. Itself it is in some sense tensor-valued quantity i.e. a function on the space-time manifold \( M \) whose values at any point \( z \) is a tensor valued distribution on the tangent space \( T_z M \) to \( M \). Energy-momentum skeleton \( \hat{T}^{\kappa\lambda} \) is symmetric in its indices \( \kappa \) and \( \lambda \) and is zero unless \( z \) lies on the centre-of-mass line. Relation between energy-momentum skeleton \( \hat{T}^{\kappa\lambda} \) and energy-momentum tensor \( T^{\kappa\lambda} \) is derived and discussed in [6].

Definition of the \( 2^n \)-pole moment at the given point \( z \) of the centre-of-mass world line is following:

\[
   I^{\kappa_1 \cdots \kappa_n\lambda\mu}(z(s)) = \int X^{\kappa_1} \cdots X^{\kappa_n} \hat{T}^{\lambda\mu}(z, X) DX ,
\]

where \( DX \) is a volume element. The monopole and the dipole moments are the momentum \( p^\kappa \) and the angular momentum \( S^{\kappa\lambda} \) respectively.
Equations of motion express absolute derivatives along the world line $\delta p^\lambda/\text{ds}$ and $\delta S^{\lambda\mu}/\text{ds}$ in terms of curvature tensor and higher multipole moments of the body. Generally equations of motion in $2^n$-pole approximation are following [6]:

\begin{equation}
\frac{\delta}{\text{ds}} p_k = \frac{1}{2} v^\lambda S^{\mu\nu} R_{\kappa\lambda\mu\nu} + \frac{1}{2} \sum_{n=2}^{N} \frac{1}{n!} I^{\nu_1\ldots\nu_n,\lambda\mu} \nabla_\kappa g_{\lambda\mu,\nu_1\ldots\nu_n},
\end{equation}

\begin{equation}
\frac{\delta}{\text{ds}} S^{\kappa\lambda} = 2p^{[\kappa[v,\lambda]} + \sum_{n=1}^{N-1} \frac{1}{n!} g^{\sigma[\kappa I^{\lambda]}\rho_1\ldots\rho_n} g_{(\sigma v,\mu)\rho_1\ldots\rho_n}.
\end{equation}

where $g_{\lambda\mu,\nu_1\ldots\nu_n}$ denotes $n$th tensorial extension of metric tensor $g_{\lambda\mu}$ (see Schouten [9]) and $v^\lambda \equiv \dot{z}^\lambda$ is a tangent vector to the centre-of-mass world line.

Neglecting higher than dipole moments we obtain Mathisson-Papapetrou equations [10, 11]:

\begin{equation}
\frac{\delta}{\text{ds}} p^k = \frac{1}{2} v^\lambda S^{\mu\nu} R^\kappa_{\lambda\mu\nu},
\end{equation}

\begin{equation}
\frac{\delta}{\text{ds}} S^{\kappa\lambda} = 2p^{[\kappa[v,\lambda]}.
\end{equation}

that describe motion of spinning particles.

3. Space-time structure in Kaluza-Klein theories

We want to find equations of motion of $(k-1)$-dimensional body moving in $m$-dimensional space-time $\mathcal{M}$. We assume its structure being in the form of $\mathcal{M}_I \times \mathcal{M}_E$ and the signature of the metric tensor of $(+,-,\ldots,\ldots,-)$. We introduce symbols:

$\mathcal{M}_E$ — $(m-k+1)$-dimensional external space-time,

$\mathcal{M}_I$ — $(k-1)$-dimensional internal space.

Metric is of the form:

\begin{equation}
d^2 s = g^2(x^\alpha)ds^2_E - f^2(x^\alpha)ds^2_I, \quad \alpha = 0, 1, \ldots, m-1,
\end{equation}

where $ds^2_E$ and $ds^2_I$ are external and internal quadratic length elements, written explicitly as:

\begin{equation}
ds^2_E = h_{ab}(x^a)dx^a dx^b; \quad a, b = 0, 1, \ldots, m-k,
\end{equation}

\begin{equation}
ds^2_I = h_{kl}(x^k)dx^k dx^l; \quad k, l = m-k+1, \ldots, m-1.
\end{equation}

We have introduced conventions of denoting external coordinates by initial letters of Latin alphabet — $a, b, c \ldots$ and internal coordinates by central ones — $k, l, m \ldots$.

We assume, that characteristic length of internal space is very small and metric tensor of external space-time does not depend on internal coordinates. Moreover we postulate existence of vectors $\xi$ with only internal components such that Lie derivative of the external metric tensor along the corresponding vector field vanishes:

\begin{equation}
\mathcal{L}_\xi g_{ab} = 0.
\end{equation}
Using the definition of the Lie derivative we write this expression explicit and obtain the condition:

\[ \xi_l g_{ab,l} + g_{al} \xi_l^b + g_{bl} \xi_l^a = 0. \]

But from our assumption follows that partial derivatives of internal components of Killing vector over external indices vanish:

\[ \xi^l_{,b} = 0, \quad \xi^l_{,a} = 0. \]

Thus we obtain conditions on partial derivatives of components of the metric tensor: and it follows that external components of metric tensor \( g_{ab} \) depend only on external coordinates:

\[ g_{ab}(x^\alpha) \equiv g(x^\alpha) h_{ab} = g_{ab}(x^\alpha). \]

By imposing some symmetries on internal space — e.g. by choice of the Killing vector \( \xi^l \) — we would have obtained next restrictions on metric tensor \( g_{\alpha\beta} \).

Now we assume that internal scale factor \( f(x^\alpha) \) depends only on external coordinates: \( f(x^\alpha) = f(x^\alpha) \). Thus coordinates from \( M_E \) could only introduce additional scale factor at \( ds_I \), but they cannot change internal geometry of \( M_I \).

Summarising the metric on \( M \) is of the form:

\[ ds^2 = g_{ab} dx^a dx^b - f^2(x^\alpha) ds_I^2. \]

Internal space \( M_I \) has symmetries connected with some symmetry group \( G \) acting on that. So there exist Killing vectors in amount equal to the number of generators of the group \( G \).

Because of lack of internal space boundary we cannot fully use Beiglbock theorem on uniqueness of choice of the world line of the centre of mass. From this and due to existence of internal symmetries we can only specify equivalence class of the centre of mass world line. Points \( x^k \) and \( x^{k'} \) of internal space are connected by equivalence relation if they lay on the same orbit of action of group \( G: [x^k] \). Centre of mass world line in the internal space is obtained by intersection of the centre of mass world sheet by surface \( x^k = \text{const.} \). Thus the centre of mass world sheet is of the form of \( l \times M_I \). Due to assumed symmetries the world line in external space should not depend on given choice of coordinates \( x^k \).

Thus it makes sense to use coordinates \( v^\alpha \) of the world line only in correspondence to external components. Internal components could be set arbitrary; in calculations we will put \( v^k = 0 \).

Therefore we impose conditions on external components of momentum and angular momentum only. The equation: \( p_\alpha S^{\alpha\beta} = 0 \) takes then the following form:

\[ p_\alpha S^{\alpha\beta} = 0. \]

For mixed components we have the constraint:

\[ p_k S^{ka} = 0. \]

Thus in 5-dimensional theory, assuming \( p_5 \neq 0 \), we obtain on mixed components condition: \( S^{5\alpha} = 0 \).

We do not impose any restrictions on internal components of momentum and angular momentum.
4. Dipole approximation

We will now determine equations of motion for \((k-1)\)-dimensional body in the dipole approximation. This approximation should give us some insight into the form of modified equations of motion. From some 4-dimensional consideration [12] we can interpret the dipole moment as resulting only from space-like extendness. Thus dipole solution correspond to extended spinning particles.

Dipole term contains components of the Riemann tensor \(R_{\alpha\beta\gamma\delta}\), so first we will determine Christoffel symbols in the coordinates introduced in the previous section. We obtain the least amount of independent components of curvature coefficients for the triples of indices: \((abc)\), \((abk)\), \((ckl)\), \((akl)\), \((cbk)\), \((klm)\). Thus we have that Christoffel symbols are:

\[
\Gamma^c_{ab} = \frac{1}{2} g^{cd} (\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}) , \quad \Gamma^m_{kl} = \frac{1}{2} g^{mn} (\partial_k g_{ln} + \partial_l g_{kn} - \partial_n g_{kl}) ,
\]

\[
\Gamma^k_{ab} = 0 , \quad \Gamma^c_{kb} = 0 .
\]

Now we are able to calculate components of curvatures tensor with internal and external indices. We divide different sets of indices into such that contain only external components \((abcd)\):

\[
R^d_{cba} = \partial_b \Gamma^d_{ac} - \partial_a \Gamma^d_{bc} + \Gamma^c_{ac} \Gamma^d_{eb} - \Gamma^c_{bc} \Gamma^d_{ea} ,
\]

and that containing only internal components \((klmn)\):

\[
R^m_{mlk} = \partial_l \Gamma^m_{km} - \partial_k \Gamma^m_{lm} + \Gamma^p_{kl} \Gamma^m_{pm} - \Gamma^p_{lm} \Gamma^m_{pk} .
\]

We are left with the rest mixed components. Almost all of them vanish as follows:

\[
R^l_{kba} = \frac{1}{2} g^l_k (\partial_b (\frac{1}{f^2} \partial_a f^2) - \partial_a (\frac{1}{f^2} \partial_b f^2)) + \frac{1}{4} f^4 g^l_k \partial_a f^2 \partial_b f^2 +
\]

\[
- \frac{1}{4} f^4 g^l_k \partial_a f^2 \partial_b f^2 = 0 ,
\]

\[
R^m_{lka} = \partial_a (\frac{h^{mn}}{f^2} f^2 (\partial_k h_{nl} + \partial_l h_{nk} - \partial_n h_{kl})) = 0 ,
\]

\[
R^k_{eba} = 0 ,
\]

except the one type of components of the Riemann tensor with indices \((albk)\):

\[
R^l_{bka} = \frac{1}{4} g^l_k \frac{1}{f^4} \partial_a f^2 \partial_b f^2 - \frac{1}{2} g^l_k \frac{1}{f^2} \partial_a \partial_b f^2 + \frac{1}{2} g^l_k \frac{1}{f^2} \Gamma^c_{ab} \partial_c f^2 .
\]

Putting obtained elements of \(R^a_{\beta\gamma\delta}\) into equation (6) we derive equations of evolution along the world line of the mass centre for external components of momentum \(\frac{δ}{ds} P^a\) and internal ones \(\frac{δ}{ds} K^k\).

First we evaluate equation for external momentum evolution:

\[
(18) \quad \frac{δ}{ds} P^a (z^a, [z^k]) = \frac{1}{2} g^{\beta\gamma} S^{\gamma\delta} R^a_{\beta\gamma\delta} .
\]
Dividing summation over indices into the one over external indices and the one over internal indices we have:

\[ \frac{\delta}{ds} p^a = \frac{1}{2} v^b S^{cd} R^a_{bcd} + \frac{1}{2} v^k S^{cd} R^a_{kcd} + v^l S^{cl} R^a_{lk} + v^k S^{kl} R^a_{bl} + v^l S^{cm} v^l R^a_{klm} . \]

Now we can use calculated elements of curvature tensor obtaining formula:

\[ \frac{\delta}{ds} p^a(z^a, [z^k]) = \frac{1}{2} v^b S^{cd} R^a_{bcd} . \]

Thus the equation of external momentum evolution does not show any influence of assumed extendness of the particles.

Now we can repeat the same procedure for equation of internal components of momentum:

\[ \frac{\delta}{ds} p^k(z^a, [z^k]) = \frac{1}{2} v^b S^{\gamma\delta} R^a_{\gamma\delta} = v^a S^{bl} R^k_{abl} + \frac{1}{2} v^l S^{mn} R^k_{lmn} = \]

\[ = \frac{1}{2} v^l S^{mn} R^k_{lmn} - v^a S^{bk} \left( \frac{1}{4} f^2 \partial_a f^2 \partial_b f^2 - \frac{1}{2} f^2 \partial_b \partial_a f^2 + \frac{1}{2} f^2 \Gamma_{bc}^{\alpha} \partial_\alpha f^2 \right) , \]

and we obtain formula:

\[ \frac{\delta}{ds} p^k(z^a, [z^k]) = -v^a S^{bk} \left( \frac{1}{4} f^2 \partial_a f^2 \partial_b f^2 - \frac{1}{2} f^2 \partial_b \partial_a f^2 + \frac{1}{2} f^2 \Gamma_{bc}^{\alpha} \partial_\alpha f^2 \right) . \]

The second of equations of motion (7) remains unchanged:

\[ \frac{\delta S_{\kappa\lambda}}{ds} = 2p^{[\kappa, \nu]} , \]

thus conditions imposed on metric do not change the structure of this equation.

Internal components of momentum \( p^k \) correspond to generalised charge of the particles, thus the law of the charge conservation requires \( \delta p^k / ds = 0 \). In 5-dimensional space-time with one additional internal dimension we have identically \( S^{kl} = 0 \), i.e. mixed components of spin tensor vanish. So we have then \( \delta p^4 / ds = 0 \) that corresponds to conservation of electric charge in dipole approximation.

5. Multipole approximation

Now we want to consider equations of motion in better than dipole approximation. Equations of motion are given by expression (4) and (5). First we will estimate multipole moments using assumed smallness of characteristic length of additional dimensions.

The \( 2^n \)-pole moment reads:

\[ I^{\kappa_1 \cdots \kappa_n, \lambda\mu}(z(s)) = \int_{T_s M} X^{\kappa_1} \cdots X^{\kappa_n} \hat{T}^{\lambda\mu}(z, x) DX . \]
According to our previous assumption the metric tensor $g_{\alpha \beta}$ is a block-diagonal matrix hence its determinant is given by a product of external metric tensor determinant $g_E$ and internal metric tensor determinant $g_I$:

$$\det g_{\alpha \beta} = \det g_{ab} \cdot \det g_{kl} = g_E \cdot g_I = (-1)^{(k-1)} f^{2(k-1)} g_E \cdot h_I.$$  

We have assumed that the whole internal space is the orbit of symmetry group action hence the energy-momentum skeleton $\hat{T}^{\kappa \lambda}$ is constant on the orbit, i.e.:

$$\hat{T}(z, X^a, X^k) = \hat{T}(z, X^a),$$

Using this property we can separate integral in expression (23) defining $2^n$-pole moment:

$$I^{\kappa_1 \cdots \kappa_n, \lambda \mu}(z(s)) = f^{(k-1)}(z^a) \left( \int_{T_z \mathcal{M}_E} DX_E \sqrt{-g_E(z^a)} X^{\kappa_1}_E \cdots X^{\kappa_n}_E \hat{T}^{\lambda \mu} \right) \cdot \left( \int_{T_z \mathcal{M}_I} DX_I \sqrt{h_I(z^k)} X^{\kappa_1}_I \cdots X^{\kappa_n}_I \right).$$

According to the kind of components $\lambda \mu$ we will have different expression for multipole moments. Let us deal with the cases:

- $\lambda, \mu$ are external components $a, b$,
- $\lambda, \mu$ are mixed components $a, k$,
- $\lambda, \mu$ are internal components.

Thus in the first case we have simple formula:

$$I^{c \cdots d k \cdots l a b}(z^a, [z^k]) = f^{(k-1)}(z^a) \left( \int_{T_z \mathcal{M}_E} DX_E \sqrt{-g_E(z^a)} X^c \cdots X^d \hat{T}^{a b} \right) \cdot \left( \int_{T_z \mathcal{M}_I} DX_I \sqrt{h_I(z^k)} X^k \cdots X^l \right),$$

This gives that the external moment $I^{c \cdots d a b}_E$ is additionally multiplied by the constant $\int DX_I \sqrt{h_I} X^k \cdots X^l$ and a conformal factor $f^{(k-1)}(z^a)$.

In the other cases we do not have so unique interpretation of obtained multipole terms and we have to consider what is the interpretation of the integral:

$$\int_{T_z \mathcal{M}_E} DX_E \sqrt{-g_E} X^{\kappa_1}_E \cdots X^{\kappa_n}_E \hat{T}^{k l}$$

appearing in the expression on multipole moments::

$$I^{\kappa_1 \cdots \kappa_n, k l}(z(s)) = f^{(k-1)}(z^a) \left( \int_{T_z \mathcal{M}_E} DX_E \sqrt{-g_E} X^{\kappa_1}_E \cdots X^{\kappa_n}_E \hat{T}^{k l} \right) \cdot \left( \int_{T_z \mathcal{M}_I} DX_I \sqrt{h_I} X^{\kappa_1}_I \cdots X^{\kappa_n}_I \right).$$

Therefore we have to consider properties of internal components of the energy-momentum skeleton $\hat{T}^{k l}$. For energy-momentum tensor it would be energy density,
but we have subtracted from it some quantities to obtain energy-momentum skeleton containing less information.

We can write the second integral in expression (29) in the form:

\[
\int_{T_zM_I} DX_I \sqrt{h_I(z^a)} X^{k_1} \cdots X^{k_n} = \sqrt{h_I(z^a)} C^{k_1 \cdots k_n} .
\]

Here \( C^{k_1 \cdots k_n} \) are numbers dependent on internal space geometry and the choice of the components themselves; \( g(z^a) \) is taken on group symmetry action corresponding to given point \( z^a \). We can use this quantity due to isometries of internal space:

\[ \mathcal{L} g_I = 0 \text{ on given orbit.} \]

Now let us try to estimate multipole moments in case the test body is point-like in external dimensions. Then external moments vanish: \( I^E = 0 \) and we could look for pure influence of existence of internal dimensions on equations of motion.

Only moments with internal first \( n \) indices: \( m_1 \cdots m_n \) have influence on equations of motion and they read:

\[
I^{m_1 \cdots m_n \lambda \mu} (z(s)) = \int_{T_zM_E} DX_E \sqrt{-g_E} \hat{T}^{\lambda \mu} \int_{T_zM_I} DX_I \sqrt{g_I} X^{m_1} \cdots X^{m_n} .
\]

Because the body is point-like in external dimensions we can write the energy-momentum skeleton \( \hat{T}^{\lambda \mu} \) as distribution around area \( X^a = 0 \):

\[
\hat{T}^{\lambda \mu} = M \delta^{m-k} (X^a) ,
\]

i.e. \( \hat{T}^{\lambda \mu} \) is concentrated only at the point \( X^a = 0 \) and some compact area of internal space. The quantity \( M \) is a dimensional constant. Hence we have integral:

\[
\int_{T_zM_I} DX_E \hat{T}^{\lambda \mu} = M ,
\]

and the form of multipole moments in that case is following:

\[
I^{k_1 \cdots k_n \lambda \mu} (z(s)) = M \sqrt{-g(z)} C^{k_1 \cdots k_n} .
\]

As we see that moments are functions only of external coordinates.

5.1 Momentum evolution.

We will determine now equation of motion for momentum in two cases: for external and internal components.

(a) Equation of motion for external components of momentum.

We put into equation of momentum evolution (4) calculated form of multipole moments (34) and we have the following equation:

\[
\frac{\delta}{\delta s} p_a (z^a, [z^k]) = \frac{1}{2} \rho^b S^{abcd} R_{abcd} + \frac{1}{2} \rho^b S^{abcd} R_{abcd} + \sum_{n=2}^{N} \frac{1}{n!} M R^{n+k-1} C^{k_1 \cdots k_n} \nabla_a g_{\lambda \mu, k_1 \cdots k_n} (z^a, [z^k]) .
\]
From the definition of the covariant derivative it follows that it could be written in terms of the partial derivatives and Christoffel symbols:

\[
\nabla_a g_{\lambda\mu,k_1\cdots k_n} = \partial_a g_{\lambda\mu,k_1\cdots k_n} + \Gamma_{a\lambda}^{\nu} g_{\nu\mu,k_1\cdots k_n} - \cdots - \Gamma_{ak_n}^{\nu} g_{\lambda\nu,k_1\cdots\mu}.
\]

We have calculated general form of Christoffel symbols according to our assumptions on the properties of the space-time and we have that \(\Gamma_{ak}^{\nu} = 0\) and \(\Gamma_{mn}^{ak} = \frac{1}{2} g^{mn} \partial_a g_{kn}\), and we can write formula (35) as following:

\[
(36) \quad \frac{\delta d^s}{ds} p_a(z^a, [z^k]) = \frac{1}{2} v^b S_{abcd} R_{abcd} + \sum_{n=2}^{N} \frac{1}{n!} M R^{n+k-1} C^{k_1\cdots k_n\lambda\mu} \cdot \left( \partial_a g_{\lambda\mu,k_1\cdots k_n} - \cdots - \frac{1}{2} g^{mn} (\partial_a g_{kn}) g_{\lambda\mu,k_1\cdots m}(z^a, [z^k]) \right).
\]

That gives us equation of external momentum evolution along the centre-of-mass world line in 2\(n\)-pole approximation.

(b) Equation of motion for internal momentum components

We repeat the same procedure as in the point (a) and the equation on internal momentum evolution reads:

\[
(37) \quad \frac{\delta d^s}{ds} p_m(z^a, [z^k]) = v^a S_{m}^{ab} \left( \frac{1}{4} f^2 \partial_a f^2 \partial_b f^2 - \frac{1}{2} f^2 \partial_b \partial_a f^2 + \frac{1}{2} f^2 \Gamma_{a}^{\nu} \partial_c f^2 \right) + \sum_{n=2}^{N} \frac{1}{n!} M R^{n+k-1} C^{k_1\cdots k_n\lambda\mu} \nabla_m g_{\lambda\mu,k_1\cdots k_n}(z^a, [z^k]).
\]

In general choice of coordinate system this derivative does not vanish as we could expect in standard point-like Kaluza-Klein theories.

5.2 Angular momentum evolution.

Now let us consider equation for angular momentum. Equation of motion are of the following form:

\[
(38) \quad \frac{\delta d^s}{ds} S^a_{\kappa\lambda}(z^a, [z^k]) = 2 p^a [\kappa^a \nu^\lambda] + \sum_{n=1}^{N-1} \frac{1}{n!} g^{\sigma[\kappa^a \nu^\lambda]} g_{(\sigma\nu,\mu_1\cdots\mu_n)} g_{\rho_1\cdots\rho_n}(z^a, [z^k]).
\]

We will evaluate this equation in three cases of indices \(\kappa, \lambda\).

(a) Equations of external components \(S^{ab}\).

In that case \(\kappa, \lambda = a, b\) and we have additional terms:

\[
(37) \quad g^{c[a \mu_1\cdots\mu_n]} g_{(cl,k)r_1\cdots r_n}.
\]

We can expand expression in brackets such that \(g_{(cl,k)} = g_{cl,k} - g_{lk,c} + g_{kcl}\). Thus the only term different from zero is \(g_{lk,c,\cdots r_n}\). We also have that 2\(n\)-moment with
mixed first \( n \) indices vanishes: 

\[ I^{br_1 \cdots r_n \mu \nu} = 0, \]

and we are left with the equation:

\[ \frac{\delta}{ds} S^{ab} = 2p^{[a,b]} . \]

Here we do not have higher than dipole terms due to our assumption that we deal with particles being point-like in four standard dimensions and extended in compactified additional dimensions. That extendness gives contribution in external space only to intrinsic angular momentum and as spin effects appear only in the dipole term we do not have the higher ones.

(b) Equations of mixed components \( S^{am} \).

Here we have additional terms as the following:

\[ g^{ca} I^{mr_1 \cdots r_n \lambda \mu} g_{(d,k) r_1 \cdots r_n} - g^{nm} I^{ar_1 \cdots r_n \lambda \mu} g_{(n,l,k) r_1 \cdots r_n} \]

and \( I^{ar_1 \cdots r_n \mu \nu} = 0 \), hence we have angular momentum equation in the form:

\[ \frac{\delta}{ds} S^{am} = -p^k v^a - \sum_{n=1}^{N-1} \frac{1}{n!} g^{ca} I^{mr_1 \cdots r_n \lambda \mu} g_{(d,k) r_1 \cdots r_n} (z^a, [z^k]). \]

(c) Equations of mixed components \( S^{nm} \)

In that case we just have following equation:

\[ \frac{\delta}{ds} S^{nm} = - \sum_{n=1}^{N-1} \frac{1}{n!} g^{pm} I^{mr_1 \cdots r_n \lambda \mu} g_{(\lambda \mu,p) r_1 \cdots r_n} (z^a, [z^k]), \]

that is rather of general form.

All right hand sides of obtained equations (36), (37), (40), (42), (43) give us deviation from geodesic motion. In the other case momentum would be parallely propagated:

\[ \frac{\delta}{ds} p^\kappa = 0, \]

as well as angular momentum:

\[ \frac{\delta}{ds} S^{\kappa \lambda} = 0. \]

6. Conclusions

We have derived equations of motion for particles extended in additional dimensions characteristic for Kaluza-Klein type theories. There are extra terms containing multipole moments coming form extendness of the bodies. Thus we have deviation from geodesic motion.

We can estimate this deviation considering the inner space being in the simplest form of the \((k-1)\)-dimensional sphere of the radius \( R \) parametrised by \( k-1 \) angles
\[ C_{k_1 \cdots k_n} = \int_0^{2\pi} R d\phi \int_0^{\pi} R d\theta \cdots \int_0^{\pi} R d\chi (R^i \phi^i)(R^j \theta^j) \cdots (R^l \chi^l) = \]
\[ = \frac{2^{j+1} \pi^{n+k-1}}{(i+1)(j+1) \cdots (l+1)} R^{n+k-1}, \]

where \( i + j + \cdots + l = n \).

Thus we see that multipole moments are of order \( R^{n+k-1} \). The radius of internal space is of the Planck scale hence additional terms in equation on momentum and angular momentum evolution give only slight contribution to equation of motion unless the space-time is quite regular.

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