Finite Horizon Time Inhomogeneous Singular Control Problem of One-dimensional Diffusion via Dynkin Game

Yipeng Yang*

October 14, 2014

Abstract

The Hamilton-Jacobi-Bellman equation (HJB) associated with the time inhomogeneous singular control problem is a parabolic partial differential equation, and the existence of a classical solution is usually difficult to prove. In this paper, a class of finite horizon stochastic singular control problems of one dimensional diffusion is solved via a time inhomogeneous zero-sum game (Dynkin game). The regularity of the value function of the Dynkin game is investigated, and its integrated form coincides with the value function of the singular control problem. We provide conditions under which a classical solution to the associated HJB equation exists, thus the usual viscosity solution approach is avoided. We also show that the optimal control policy is to reflect the diffusion between two time inhomogeneous boundaries. For a more general terminal payoff function, we showed that the optimal control involves a possible impulse at maturity.

Key words. stochastic singular control, time inhomogeneous zero-sum game, HJB equation

AMS subject classifications. 93E20, 60G40, 91A23

1 Introduction

The stochastic singular control problem is one of the classical research topics in control theory that keeps receiving a lot of interest in years. A typical such problem has an objective function as a functional of an underlying stochastic process over a finite or infinite horizon that needs to be minimized or maximized. This objective function often involves a functional of the control actions, which gives its name singular control or impulsive control depending on the form of the objective function. Its well known deterministic counterpart is the minimum fuel control problem. The application of the stochastic singular control is certainly broad,
such as financial engineering, resource management, or mechanical system control, see, e.g., [31].

The goal of the stochastic singular control is to characterize the optimal control policy, if exists, and find the optimal value of the objective function. In a typical setting, the value function is known to satisfy a partial differential equation, referred to as the Hamilton-Jacobi-Bellman (HJB) equation. However, the existence and regularity of its solution always remain a big challenge. Even in the time homogeneous one dimensional case, efforts are needed and special forms are treated in order to characterize the regularity of the optimal value function, see, e.g., [21][30]. The form of optimal policies in multi-dimensional or time inhomogeneous singular control problems could be much more complicated. Soner and Shreve [32] studied a two dimensional singular stochastic control problem and showed the existence of a smooth solution to the associated HJB equation, where the underlying process is a two dimensional Brownian motion with no drift. How to extend this method to a general multi-dimensional diffusion is still not clear.

Besides the viscosity solution technique on the study of HJB equations and stochastic control problems, e.g., [9][12], which often yields a less regular solution, a lot of literature tackle the singular stochastic control problem through the optimal stopping and game theory. The standard optimal stopping problem is often referred to as the one obstacle problem, and double obstacles problem is referred to as a game. The connection between singular control and optimal stopping is a well known fact. To name a few, Karatzas and Shreve [19] studied the connection between optimal stopping and singular stochastic control of one dimensional Brownian motion, and showed that the region of inaction in the control problem is the optimal continuation region for the stopping problem. Baldursson and Karatzas [1] established and exploited the duality between the myopic investor’s problem (optimal stopping) and the social planning problem (stochastic singular control), where an integral form and change of variable formula were also presented on this connection. Ma [21] dealt with a one dimensional stochastic singular control problem where the drift term is assumed to be linear and the diffusion term is assumed to be smooth, and he showed that the value function is convex and $C^2$ and the controlled process is a reflected diffusion over an interval. Guo and Tomecek [17] solved a one dimensional singular control problem via a switching problem, and showed, using the smooth fit property [30], that under some conditions the value function is continuously differentiable ($C^1$). This connection in a finite horizon case can also be found in [5], where the regularity of the value function is not fully investigated.

To be brief, there exists a double obstacle problem (game) associated with a singular control problem, and the derivative (with respect to state variable) of the value function of the singular control problem is just the value of this game. The optimal continuation region of this game coincides with the continuation region of the singular control problem, and the boundary of the continuation region turns out to be the reflecting boundary in the optimal singular control so that the controlled process is a reflected diffusion within this boundary. Therefore, it is possible to pass on the regularity of the value function of the game to the regularity of the value function of the singular control, with extra smoothness add-on. For example, Fukushima and Taksar [14] used this idea to show the existence of a
classical solution to the HJB equation associated with a general one dimensional singular control problem. See also [36] for an extension and a correction. Therefore, the regularity of the value function of optimal stopping and game becomes critical for the study of the singular stochastic control.

There are several major approaches to the study of optimal stopping and game, see [24][20] for a general mathematical framework, and [38] for an early survey. On the study of regularity of the value function, method of convex analysis can be found in [4], time-discretization method can be found in [23], and the penalty method was introduced by Stettner and Zabczyk [33]. Another major approach is via the variational inequalities pioneered by Bensoussan and Lions [3]. Since the variational inequalities involve Dirichlet form, it sparks the research interest in the study of Dirichlet form and its connection to Markov processes, see, e.g., [22][16]. The application of Dirichlet form to optimal stopping was studied in [25]. Zabczyk [39] extended this result to a zero-sum game. Fukushima and Menda [15] later investigated the refined solutions under absolute continuity condition on the transition function of the underlying process.

However, most of these work dealt with the time homogeneous case. In this case, the associated HJB equation is an elliptic PDE, and theories guarantee the existence of a smooth solution under some conditions, see, e.g., [14]. For the time inhomogeneous optimal stopping and game, however, the associated HJB equations are parabolic PDEs, and the existence and regularity of the solutions are harder to analyze. For example, Oshima [20] applied the time inhomogeneous Dirichlet form to this problem, and showed the fine and cofine continuity of the value function, with possible existence of exceptional sets in the result. Refined solutions to time inhomogeneous optimal stopping and game via Dirichlet form was studied in [37]. Palczewski and Stettner [27][28] used the penalty method to characterize the continuity of the value functions of a time inhomogeneous optimal stopping problem under weak Feller condition on the underlying process. Stettner [34] then extended this method to finite horizon double obstacle problem (game). But only continuity of the value functions was able to prove.

Dai and Yi [10] studied a one dimensional finite horizon optimal investment problem with transaction costs, where the terminal CRRA utility function is maximized. This problem is indeed a singular stochastic control problem, and by constructing its connection with a parabolic two obstacle problem, the authors proved the $C^{1.2}(C^{2.1}$ in their paper due to different order of $x,t)$ regularity of the value function. The risky asset in their model follows a standard geometric Brownian motion, and the objective function only involves a function of the terminal wealth. Furthermore, that work lacks the verification theorem to show optimality.

In this paper we consider a more general model which certainly includes the one in [10]. We provide conditions under which a classical solution to the associated HJB equation exists, thus the usual viscosity solution approach is avoided. The organizations of this paper is as follows. In the next section, we introduce the time inhomogeneous singular stochastic control problem that we are concerned with. Then its associated zero-sum game problem is investigated in Section 3 where conditions and regularity of the value function are proved.
This result is passed on to the value function of the singular control in Section 4 where a verification theorem and the optimal control policy are shown. In Section 5 we study the finite horizon singular control in the situation that the terminal payoff function is more general, and show that the optimal policy involves a possible jump at maturity. The discussion on the regularity of the free boundaries of the continuation region in the optimal control is provided in Section 6, followed by concluding remarks.

2 Problem Formulation

Given a probability space \((\Omega, \mathcal{F}_t, X, P_x)\), where \(P_x\) is a family of measures under which \(X(t) = X_t\) is a one dimensional Itô diffusion

\[
dX_t = \mu(X_t)dt + \sigma(X_t)dB_t, \quad X_s = x,
\]

in which \(B_t\) is a standard Brownian motion, we consider the model \(\mu(x) = cx + d\) for some constants \(c, d\), and here \(\sigma\) is a differentiable bounded function such that \(\sigma(x) \geq \epsilon > 0, \forall x\), for some constant \(\epsilon\). Let \((A^{(1)}_t, A^{(2)}_t) = \mathcal{S}\) be two right continuous and nondecreasing \(\mathcal{F}_t\) adapted processes. We call \((A^{(1)}_t, A^{(2)}_t)\) the admissible controls on \(X_t\) and the controlled process then follows

\[
dX_t = \mu(X_t)dt + \sigma(X_t)dB_t + dA^{(1)}_t - dA^{(2)}_t, \quad X_s = x. \tag{1}
\]

Here we assume that \(A^{(1)}_t - A^{(2)}_t\) is the minimal decomposition of a bounded variation process into a difference of two nondecreasing processes, and \(\mathcal{S}\) is the set of all admissible controls.

Consider the process \(Z_t = (t, X_t)\) with time inhomogeneous cost function

\[
k_{\mathcal{S}}(z) = k_{\mathcal{S}}(s, x) = E_{(s,x)} \left\{ \int_s^T H(t, X_t)dt + \int_s^T e^{-ct}f_1(t, X_t)dA^{(1)}_t + \int_s^T e^{-ct}f_2(t, X_t)dA^{(2)}_t + G(X_T) \right\}, \tag{2}
\]

where \(H(\cdot)\) is called the holding cost, \(f_1(\cdot), f_2(\cdot)\) are the control costs and \(G(\cdot)\) is the terminal cost. We discount \(f_1, f_2\) to time 0 for convenience. If one likes, the whole parts \(e^{-ct}f_1(t, x), e^{-ct}f_2(t, x)\) can be taken as the time inhomogeneous control costs.

One looks for a control policy \(\mathcal{S}\) that minimizes \(k_{\mathcal{S}}(z)\), i.e.,

\[
W(z) = \min_{\mathcal{S} \in \mathcal{S}} k_{\mathcal{S}}(z). \tag{3}
\]

This problem is called the time inhomogeneous singular control problem. We solve this problem through a related time inhomogeneous zero-sum game (Dynkin game).
3 Time Inhomogeneous Dynkin Game

Now consider the objective function of a time inhomogeneous zero-sum game

\[ J_z(\tau, \sigma) = J_{(s,x)}(\tau, \sigma) = E_{(s,x)} \left\{ \int_{s}^{\tau \wedge \sigma \wedge T} e^{ct} h(t, Y_t)dt \right. \]

\[ + 1_{(\sigma < \tau \wedge T)}(-f_1(\sigma, Y_\sigma)) + 1_{(\tau < \sigma \wedge T)}f_2(\tau, Y_\tau) \]

\[ + 1_{(T \leq \tau \wedge \sigma)}g(Y_T) \} \quad \tau \wedge \sigma \geq s, \quad (4) \]

where \( Y_t \) follows

\[ dY_t = (\sigma(Y_t)\sigma'(Y_t) + \mu(Y_t))dt + \sigma(Y_t)dB_t, \quad Y_s = x. \quad (5) \]

Two players, \( P_1, P_2 \), observe the process \( Z_t = (t, Y_t) \) and each of them can stop the process at any time \( \tau, \sigma \) respectively before \( T \). If \( P_1 \) stops the process at \( \tau \), he pays \( P_2 \) the amount

\[ \int_{s}^{\tau} e^{ct} h(t, Y_t)dt + f_2(\tau, Y_\tau). \]

For convenience, we call the part \( \int_{s}^{\tau} e^{ct} h(t, Y_t)dt \) the dividend. If \( P_2 \) stops the process at \( \sigma \), he receives from \( P_1 \) the amount

\[ \int_{s}^{\sigma} e^{ct} h(t, Y_t)dt + (-f_1(\sigma, Y_\sigma)). \]

If no one stops the game before \( T \), \( P_1 \) pays \( P_2 \) the amount

\[ \int_{s}^{T} e^{ct} h(t, Y_t)dt + g(Y_T). \]

Therefore \( P_1 \) tries to minimize his payment and \( P_2 \) tries to maximize his income. The value of this game is thus given by

\[ V(z) = V(s, x) = \inf_{\tau} \sup_{\sigma} J_z(\tau, \sigma) = \sup_{\sigma} \inf_{\tau} J_z(\tau, \sigma), \quad Y_s = x. \quad (6) \]

Assumption 3.1. The functions \( f_1, f_2, g, h \) are all bounded and continuous, \( M > f_2(t, y) > 0 > -f_1(t, y) > -M, \) \( f_2(T, y) \geq g(y) \geq -f_1(T, y), \forall y, \) for some constant \( M. \)

The following is a version of Theorem 1 in [34].

Theorem 3.1. Under Assumption 3.1, the function \( V(z) \) is a continuous function and is the value of the zero sum game. The saddle point of this game has the form

\[ \hat{\tau} = \inf\{t \geq s : V(t, Y_t) = f_2(t, Y_t)\} \wedge T \]

\[ \hat{\sigma} = \inf\{t \geq s : V(t, Y_t) = -f_1(t, Y_t)\} \wedge T, \quad Y_s = x. \quad (7) \]
Very often in this paper, \((s, x)\) denotes a starting point of the process, and \((t, y)\) denotes an arbitrary point in the interested region. Define \(E = \{(t, y): 0 \leq t \leq T, -f_1(t, y) < V(t, y) < f_2(t, y)\}\), then this is the continuation region of this zero sum game. Also define the sets \(E_1 = \{(t, y): 0 \leq t \leq T, -f_1(t, y) = V(t, y)\}\) and \(E_2 = \{(t, y): 0 \leq t \leq T, f_2(t, y) = V(t, y)\}\), and we notice that \(|V(t, y)| \leq M, \forall (t, y) \in [0, T] \times \mathbb{R}\).

Consider the infinitesimal generator of \(Y_t^L := \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} + (\sigma \sigma' + \mu) \frac{\partial}{\partial x} + \frac{\partial}{\partial t}, \quad \left(\frac{\partial}{\partial x} \right)
\)
then it is uniformly parabolic in our settings. We also denote \(L := \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} + (\sigma \sigma' + \mu) \frac{\partial}{\partial x}\). If the variable \(y\) is used in places of \(x\), we use \(\frac{\partial}{\partial y}\) in \((8)\), and similarly if \(s\) is used in places of \(t\), \(\frac{\partial}{\partial s}\) is then used.

**Assumption 3.2.** \(h(t, y) \in C([0, T] \times \mathbb{R}), f_1, f_2 \text{ are } C^{1,2}([0, T] \times \mathbb{R}) \text{ functions. } h(t, y) \text{ is strictly increasing in } y, f_1(t, y) \text{ is nondecreasing in } y, f_2(t, y) \text{ is nonincreasing in } y, \forall t \in [0, T]. \text{ Furthermore, there exist continuous curves } a(t) \text{ and } b(t) \text{ with } a(t) < 0 < b(t), \forall t \in [0, T], \text{ such that}

\[
\begin{align*}
L(-f_1(t, a(t))) + e^{ct}h(t, a(t)) &= 0, \\
L f_2(t, b(t)) + e^{ct}h(t, b(t)) &= 0, \quad \forall t \in (0, T),
\end{align*}
\]

and \(L(-f_1(t, y)) + e^{ct}h(t, y)\) and \(L f_2(t, y) + e^{ct}h(t, y)\) are both strictly increasing in \(y\).

Without loss of generality, we set \(h(t, 0) = 0, \forall t \in [0, T], g(0) = 0\).

\[
\begin{array}{c}
\begin{tikzpicture}
\fill[black] (0,0) circle (5pt); \node at (0,0) {$a(t)$};
\fill[black] (0,0) circle (5pt); \node at (0,0) {$b(t)$};
\draw (-2,0) -- (2,0); \draw (0,-2) -- (0,2);
\end{tikzpicture}
\end{array}
\]

**Figure 1:** Curves \(a(t), b(t)\)

It is obvious that \(\forall t \in (0, T),\)

\[
\begin{align*}
L(-f_1(t, y)) + e^{ct}h(t, y) &< 0, \quad y < a(t), \\
L f_2(t, y) + e^{ct}h(t, y) &> 0, \quad y > b(t), \\
L(-f_1(t, y)) + e^{ct}h(t, y) &> 0, \quad y > a(t), \\
L f_2(t, y) + e^{ct}h(t, y) &< 0, \quad y < b(t).
\end{align*}
\]
Define the sets $Dab := \{(t, y) : 0 \leq t < T, a(t) < y < b(t)\}$, $Da := \{(t, y) : 0 \leq t \leq T, y < a(t)\}$ and $Db := \{(t, y) : 0 \leq t \leq T, y > b(t)\}$, then we have

**Proposition 3.1.** For any point $(t, y) \in Dab$, it is not optimal for either player to stop the game immediately.

**Proof.** Consider the optimal stopping strategy for player $P_2$, who wants to maximize the income. If he stops the game immediately, the payoff would be $-f_1(t, y)$ plus the expected dividend up to $t$. Construct a small ball $B_r(t, y)$ centered at $(t, y)$ such that $B_r(t, y) \subset Dab$, and consider the strategy to stop the game at the first exit time $\tau_B$ of the ball $B_r(t, y)$. By Dynkin’s formula, the newly future expected payoff would be

$$E_{(t,y)} \left( \int_t^{\tau_B} e^{cu}h(u,Y_u)du - f_1(\tau_B, Y_{\tau_B}) \right)$$

$$= E_{(t,y)} \left( \int_t^{\tau_B} e^{cu}h(u,Y_u) + \mathcal{L}(-f_1)(u,Y_u)du \right) - f_1(t, y) > -f_1(t, y), \quad (9)$$

since $e^{cu}h(u,Y_u) + \mathcal{L}(-f_1)(u,Y_u) > 0$ in this region, and this is a contradiction. Therefore, it is not optimal for $P_2$ to stop this game in $Dab$. Similarly we can show that it is not optimal for $P_1$ either to stop the game in $Dab$. \qed

By a similar argument, it is easy to see that it is not optimal for $P_2$ to stop the game in the region $Db$, and it is not optimal for $P_1$ to stop the game in the region $Da$. Thus $Dab \subset E$, $E_1 \subset Da$, $E_2 \subset Db$.

In what follows, $H^m$ denotes the $m$-th order Sobolev space, and $H^0_m$ denotes the $m$-th order Sobolev space with compact support.

**Assumption 3.3.** There exist two points $A, B$ such that $A \leq a(T), B \geq b(T)$, $g(y)$ is in $H^1((A, B))$. Furthermore, $-f_1(T, y) = g(y), \forall y \leq A$ and $f_2(T, y) = g(y), \forall y \geq B$.

It is obvious that $A < 0 < B$ in our settings. Let $\tau_{AB}$ be the first exit time if the process exits the interested region through the line segment $\{T\} \times [A, B]$. Obviously $\tau_{AB} = T$, but we use the notation $\tau_{AB} < T$ to denote the event that the process exits the interested region through the line segment $\{T\} \times [A, B]$, and $T < \tau_{AB}$ denotes the event that the process exits the region but not through the line segment $\{T\} \times [A, B]$, see Figure 2.

**Assumption 3.4.** There exist constant curves $\bar{A}, \bar{B}$ with $\bar{A} < A, \bar{B} > B$, such that for any $y < A$,

$$E_{(t,y)} \left( \int_t^{\tau_{\bar{A}} \wedge T \wedge \tau_{AB}} (e^{cu}h(u,Y_u) + \mathcal{L}(-f_1)(u,Y_u)))du + 1_{(\tau_{\bar{A}} \wedge \tau_{AB} < T)}(2M) \right) \leq 0,$$

and for any $y > \bar{B}$,

$$E_{(t,y)} \left( \int_t^{\tau_{\bar{B}} \wedge T \wedge \tau_{AB}} (e^{cu}h(u,Y_u) + \mathcal{L}f_2(u,Y_u))du - 1_{(\tau_{\bar{B}} \wedge \tau_{AB} < T)}(2M) \right) \geq 0,$$

where $\tau_a, \tau_b$ are the first hitting times to the curves $a, b$, respectively.
Notice that $1_{(\tau_a \land \tau_{AB} < T)}$ denotes the event that the process either hits the curve $a$ first or exits the interested region through the line segment $\{T\} \times [A, B]$.

**Proposition 3.2.** It is optimal for $P_2$ to stop the game in the region $[0, T] \times (-\infty, \tilde{A}]$ and it is optimal for $P_1$ to stop the game in the region $[0, T] \times [\tilde{B}, \infty)$.

**Proof.** It suffices to prove the first half. Consider the starting point $(s, x) \in [0, T] \times (-\infty, \tilde{A}]$. If $s = T$, the result automatically holds. Now suppose $s < T$. Since $\mathcal{L}(-f_1(t, y)) + e^{ct}h(t, y) < 0, y < a(t)$, and if $P_2$ stops the game in the region $(s, T) \times (-\infty, a]$, the payoff would be

$$E_{(s, x)} \left( \int_s^T (\mathcal{L}(-f_1(t, Y_t)) + e^{ct}h(t, Y_t))dt - f_1(s, x) \right) < -f_1(s, x),$$

which is certainly not optimal. Now suppose $P_2$ would stop the game beyond the region $(s, T) \times (-\infty, a]$, the payoff will be less than (since $M$ is an upper bound of $V$)

$$E_{(s, x)} \left( \int_s^{\tau_a \land \tau_{AB} < T} (\mathcal{L}(-f_1(t, Y_t)) + e^{ct}h(t, Y_t))dt - f_1(s, x) \right) \leq -f_1(s, x)$$

by Assumption 3.4. Therefore the optimal strategy for $P_2$ at this starting point is to stop the game immediately. \qed
Now it is clear that $[0, T] \times (-\infty, \tilde{A}] \subset E_1$, $[0, T] \times [\tilde{B}, \infty) \subset E_2$. In fact we can say something more about the termination region of this game. Let $\tilde{E}_1$ be the largest connected region in $E_1$ which contains the set $[0, T] \times (-\infty, \tilde{A}]$, and $\tilde{E}_2$ the largest connected region in $E_2$ which contains $[0, T] \times [\tilde{B}, \infty)$, then we have

**Proposition 3.3.** $\tilde{E}_1, \tilde{E}_2$ are both simply connected.

This proposition says there are no holes in $\tilde{E}_1$ or $\tilde{E}_2$.

**Proof.** By the continuity of $V$ (Theorem 3.1) and the properties of $f_1, f_2$, we know that the upper boundary (in $y$) of $\tilde{E}_1$ and the lower boundary (in $y$) of $\tilde{E}_2$ are continuous curves. Suppose there is a point $(s, x)$ below the the upper boundary of $\tilde{E}_1$ with $V(s, x) > -f_1(s, x)$, then again by the continuity of $V$ and $f_1$, there is an open connected region $B_{(s, x)}$ containing $(s, x)$ such that $V(t, y) > -f_1(t, y)$, $\forall (t, y) \in B_{(s, x)}$ and furthermore $\partial B_{(s, x)} \subset \tilde{E}_1$. Since in the region $B_{(s, x)}$ the inequality $\mathcal{L}(-f_1(t, y)) + e^{ct}h(t, y) < 0$ holds while on the boundary $\partial B_{(s, x)}$ we have $V(t, y) = -f_1(t, y)$, Dynkin’s formula and an argument similar to (10) easily imply that $V(s, x) < -f_1(s, x)$, hence a contradiction. The rest of this proposition can be proved analogously. 

Denote $\tilde{a}$ the upper boundary (in $y$) of $\tilde{E}_1$, and $\tilde{b}$ the lower boundary of (in $y$) $\tilde{E}_2$. We call a directed curve a $t_+$ curve if the points $(t, y)$ on this curve is in a nondecreasing order in $t$ as the curve being lined up from the initial point to the terminal point. A $t_-$ curve is defined similarly. Notice that $\tilde{a}(T) = A$, $\tilde{b}(T) = B$.

**Proposition 3.4.** If there is a point $(t, y)$ with $y < a(t)$ such that $V(t, y) > -f_1(t, y)$, then this point is connected to $Dab$ along a $t_+$ curve in $E$. If there is a point $(t, y)$ with $y > b(t)$ such that $V(t, y) < -f_2(t, y)$, then this point is connected to $Dab$ along a $t_+$ curve in $E$.

**Proof.** We just need to prove the first half as usual. If this point $(t, y)$ is not connected to $Dab$ along any $t_+$ curve in $E$, that means starting from this point, the game should be stopped before the process $Y_t$ hits the curve $a(t)$. Since in this region $\mathcal{L}(-f_1(t, y)) + e^{ct}h(t, y) < 0$ holds and a similar argument to (10) easily implies that $V(t, y) \leq -f_1(t, y)$, hence a contradiction.

Let $\bar{U} = [0, T] \times [m, n]$ be any simply connected region such that $m(s) \leq a(s), n(s) \geq b(s), s \in [0, T]$, are functions of $s$ with possible jumps (in this case we can still draw a continuous curve as the boundary of $\bar{U}$), and further $m(T) = A, n(T) = B$. Denote $\tau_{\bar{U}}$ the first exit time of $\bar{U}$, and define

$$F_{\bar{U}}(s, x) = E_{(s, x)} \left( \int_s^{\tau_{\bar{U}}} e^{cu}h(u, Y_u)du + R(\tau_{\bar{U}}, Y_{\tau_{\bar{U}}}) \right), (s, x) \in \bar{U},$$

(11)

where $R(\tau_{\bar{U}}, Y_{\tau_{\bar{U}}}) = g(Y_T)$ if $\tau_{\bar{U}} = T$, $R(\tau_{\bar{U}}, Y_{\tau_{\bar{U}}}) = -f_1(\tau_{\bar{U}}, Y_{\tau_{\bar{U}}})$ if $\tau_{\bar{U}} < T$ and $Y_{\tau_{\bar{U}}} \leq a(\tau_{\bar{U}})$, and $R(\tau_{\bar{U}}, Y_{\tau_{\bar{U}}}) = f_2(\tau_{\bar{U}}, Y_{\tau_{\bar{U}}})$ if $\tau_{\bar{U}} < T$ and $Y_{\tau_{\bar{U}}} \geq b(\tau_{\bar{U}})$. 

9
For any point \((s, x) \in U\) with \(x < a(s)\), define an associated cone \(C_{(s,x)}\) as
\[
C_{(s,x)} = \left\{ (t, y) \in U : s \leq t \leq T, x < y \leq a(t), \frac{t - s}{y - x} \leq \eta \right\},
\]
where \(\eta > 0\) is a constant. Let \(\bar{U} = U \cup C_{(s,x)}\), and define the function \(F_{\bar{U}}(s, x)\) similarly as in (11), we put a joint assumption on the process \(Y_t\) and the functions \(f_1, f_2, h, g\) as follows:

**Assumption 3.6.** For any \((s, x) \in U\) with \(x < a(s)\), \(F_{\bar{U}}(s, x) \geq F_U(s, x)\).

Similarly for any point \((s, x) \in U\) with \(x > b(s)\), we may define an associated cone \(C_{(s,x)}\) as
\[
C_{(s,x)} = \left\{ (t, y) \in U : s \leq t \leq T, b(t) \leq x < y, \frac{t - s}{x - y} \leq \eta \right\}.
\]

**Assumption 3.6.** For any \((s, x) \in U\) with \(x > b(s)\), \(F_{\bar{U}}(s, x) \leq F_U(s, x)\).

**Remark 3.1.** In the one dimensional time homogeneous case, this assumption holds true due to the a priori given conditions on \(f_1, f_2, h\). But in the case of time inhomogeneous or multiple dimensional case, the situation is much more complicated and more conditions are needed.

**Proposition 3.5.** Under Assumptions 3.5, 3.6, the continuation region \(E\) is simply connected, and its lower boundary (in \(y\)) is the curve \(\bar{a}\), its upper boundary (in \(y\)) is the curve \(\bar{b}\).

**Proof.** We just prove the first half as usual. For any point \((s, x) \in \bar{E}\) with \(x < a(s)\), consider the new continuation region \(E \cup C_{(s,x)}\), then by Assumption 3.5 we have \(F_{\bar{E} \cup C_{(s,x)}} \geq F_{E(a,s)} = V(s, x)\). Since in the region below the curve \(a\), only player \(P_2\) wants to stop the game who wants to maximize his payoff, the new continuation region \(E \cup C_{(s,x)}\) is certainly better than \(E\), if not identical, and this is a contradiction since \(E\) is assumed to be optimal. Thus \(\bar{E} \cup C_{(s,x)} = \bar{E}\). Since this holds for any point \((s, x) \in \bar{E}\) with \(x < a(s)\), we know that the region between the curve \(\bar{a}\) and the curve \(\bar{b}\) (since \(E\) contains the region \(Dab\)) is simply connected. \(\square\)

Now it can be seen that \(\bar{E}_1 = E_1, \bar{E}_2 = E_2\). Furthermore, Proposition 3.5 implies that \(\bar{a}, \bar{b}\) can be taken as functions of \(t\), with possible jumps (in this case we line up these points and still get continuous curves). It is worth noticing that \(\bar{a}\) and the curve \(a\) are not necessarily identical, and \(\bar{b}\) and the curve \(b\) are not necessarily identical either. Denote \(D\bar{a}\bar{b}\) the open connected region between the two curves \(\bar{a}\) and \(\bar{b}\), \(D\bar{a}\) the region to the left (in \(y\)) of \(\bar{a}\) and \(D\bar{b}\) the region to the right (in \(y\)) of \(\bar{b}\), then \(D\bar{a}\bar{b} = E\) excluding the line segment \([A, B]\), \(D\bar{a} = E_1\), \(D\bar{b} = E_2\), see Figure 3

Certainly in the one dimensional time homogeneous case, \(\bar{a}, \bar{b}\) are both constants, but in the time inhomogeneous case, a \(C^1\) regularity on \(\bar{a}, \bar{b}\) is needed. Without loss of generality, \([0, T] \times [\bar{A}, \bar{B}]\) is assumed to be large enough to contain the curves \(\bar{a}\) and \(\bar{b}\).
Figure 3: Curves $\tilde{a}(t), \tilde{b}(t)$

Now we can write $V(z) = J_z(\tau, \delta)$. Let $\phi(t, y) = -f_1(t, y)$ on the curve $y = \tilde{a}(t)$, $\phi(t, y) = f_2(t, y)$ on the curve $y = \tilde{b}(t)$, $s \leq t \leq T$, and $\phi(T, y) = g(y)$ on the line $t = T, \tilde{a}(T) \leq y \leq \tilde{b}(T)$, we can further rewrite $V$ as

$$V(s) = E(s, x) \left[ \int_s^\tau e^{ct} h(t, Y_t) dt + \phi(\tau_{D\tilde{a}}, Y_{\tau_{D\tilde{a}}}) \right], \quad Y_s = x,$$

(12)

where $\tau_{D\tilde{a}}$ is the first exit time of the region $D\tilde{a}$ defined as $\tau_{D\tilde{a}}(s, x) = \inf\{t > s : (t, Y_t) \notin D\tilde{a}\}$.

**Theorem 3.2.** Assume Assumptions 3.1 – 3.6. If $\tilde{a}(t), \tilde{b}(t)$ are $C^1$ functions of $t$, then the function $V$ in (12) is bounded and continuous and is the unique weak solution of the following problem:

$$\mathcal{L}V(t, y) + e^{ct} h(t, y) = 0, \quad (t, y) \in D\tilde{a},$$

$$\mathcal{L}V(t, y) + e^{ct} h(t, y) < 0, \quad (t, y) \in D\tilde{a},$$

$$\mathcal{L}V(t, y) + e^{ct} h(t, y) > 0, \quad (t, y) \in D\tilde{b},$$

$$-f_1(t, y) < V(t, y) < f_2(t, y), \quad (t, y) \in D\tilde{a},$$

$$V(t, y) = -f_1(t, y), \quad (t, y) \in D\tilde{a},$$

$$V(t, y) = f_2(t, y), \quad (t, y) \in D\tilde{b},$$

$$V(T, y) = g(y).$$

Furthermore, by considering $V(t, y)$ as a mapping $V : [0, T] \rightarrow H^2(U)$, where $U$ is the open interval $(A, B)$, then $V \in L^2(0, T : H^2(U)) \cap L^\infty(0, T : H^1(U))$, $\frac{\partial V}{\partial t} \in L^2(0, T; L^2(U))$.

**Proof.** Let $\tilde{V}(t, y) = V(t, y) + B(t, y)$ on $D\tilde{a}$, where

$$B(t, y) = f_1(t, \tilde{a}(t)) + \frac{y - \tilde{a}(t)}{b(t) - \tilde{a}(t)}(-f_1(t, \tilde{a}(t)) - f_2(t, \tilde{b}(t))), \quad t \in [0, T], \tilde{a}(t) \leq y \leq \tilde{b}(t),$$

(13)
and consider the following partial differential equation on the extended rectangular region 
\([0, T] \times \bar{U} = [0, T] \times [\bar{A}, \bar{B}]\):

\[
\begin{align*}
\mathcal{L} \tilde{V}(t, y) &= 1_{((t, y) \in D\tilde{a}\tilde{b})} \left( \mathcal{L}B(t, y) - e^{ct}h(t, y) \right) \triangleq \tilde{f}(t, y), \\
\tilde{V}(t, y) &= 0, \quad (t, y) \notin D\tilde{a}\tilde{b}, \\
\tilde{V}(T, y) &= 1_{(y \in [A, B])} (g(y) + B(T, y)) \triangleq \tilde{g}(y).
\end{align*}
\]

By the conditions on the functions \(f_1, f_2, g, h\) and the \(C^1\) property of curves \(\tilde{a}, \tilde{b}\), it can be seen that (for each \(t\))

\[
\tilde{f} \in L^2(0, T; L^2(U)),
\]

and \(\tilde{g} \in H^1_0(U)\). If we change the variable \(t\) to \(T - t\), the terminal condition of (14) becomes initial condition, and Theorem 5 in Chapter 7 of [11] can be applied. Thus there exists a unique weak solution \(\tilde{V}\) to the problem (14). Furthermore, \(\tilde{V} \in L^2(0, T; H^2(U)) \cap L^\infty(0, T; H^1_0(U))\), \(\frac{d}{dt} \tilde{V} \in L^2(0, T; L^2(U))\). Now it is easy to recover \(V(t, y)\) by writing

\[
V(t, y) = \begin{cases} \\
\tilde{V}(t, y) - B(t, y), & (t, y) \in D\tilde{a}\tilde{b}, \\
-f_1(t, y), & (t, y) \in D\tilde{a}, \\
f_2(t, y), & (t, y) \in D\tilde{b}.
\end{cases}
\]

thus the regularity result on \(V(t, y)\) follows.

Now expression (12) shows that \(V\) is \(h\)-harmonic, and if \(V\) is a \(C^{1,2}(D\tilde{a}\tilde{b})\) function (that is, \(C^{1,0} \cap C^{0,2}\)), (13) should hold on \(V\). By the uniqueness of the weak solution, we know that this solution is given by (12). The rest of this theorem has been proved in Theorem 3.1.

Since \(V\) is a weak solution of the PDE (13), we can not assure that \(V\) is \(C^{1,2}(D\tilde{a}\tilde{b})\), however, we can conclude the following regularity result.

**Corollary 3.1.** Assuming the conditions in Theorem 3.2, \(V(t, y)\) in (13) is in \(C^{0,1}([0, T] \times U) \cap C([0, T] \times U)\).

**Proof.** This fact is due to the regularity results \(V \in L^2(0, T : H^2(U))\) in Theorem 3.2, and of course the continuous version is chosen in this case. That is, if a function is in \(H^2(U)\), then the \(C^1\) version of this function is chosen.

With this result, it is straight forward to conclude the following.

**Corollary 3.2.** Assuming the conditions in Theorem 3.2, then for any \(t \in [0, T]\),

\[
\frac{\partial V}{\partial y}(t, \tilde{a}(t)) = -\frac{\partial f_1}{\partial y}(t, \tilde{a}(t)), \quad \frac{\partial V}{\partial y}(t, \tilde{b}(t)) = \frac{\partial f_2}{\partial y}(t, \tilde{b}(t)).
\]

Since \(V(t, y) \geq -f_1(t, y)\) and \(V(t, y) \leq f_2(t, y)\), \(\forall (t, y) \in (0, T) \times U\), it is easy to see that
Lemma 3.1. For any $t \in [0, T)$,
\[
\begin{align*}
\lim_{\epsilon \to 0^+} \inf \frac{V(t + \epsilon, \tilde{a}(t)) - V(t, \tilde{a}(t))}{\epsilon} &\geq -f_1(t, \tilde{a}(t)), \\
\lim_{\epsilon \to 0^+} \sup \frac{V(t, \tilde{a}(t)) - V(t - \epsilon, \tilde{a}(t))}{\epsilon} &\leq -f_2(t, \tilde{a}(t)),
\end{align*}
\]
and
\[
\begin{align*}
\lim_{\epsilon \to 0^+} \sup \frac{V(t + \epsilon, \tilde{b}(t)) - V(t, \tilde{b}(t))}{\epsilon} &\leq f_2(t, \tilde{a}(t)), \\
\lim_{\epsilon \to 0^+} \inf \frac{V(t, \tilde{b}(t)) - V(t - \epsilon, \tilde{b}(t))}{\epsilon} &\geq f_2(t, \tilde{a}(t)).
\end{align*}
\]

Furthermore, we can prove the following:

Lemma 3.2. There is a constant $K > 0$ such that for any $t \in [0, T)$,
\[
\begin{align*}
\lim_{\epsilon \to 0^+} \sup \frac{V(t + \epsilon, \tilde{a}(t)) - V(t, \tilde{a}(t))}{\epsilon} &\leq K, \\
- K &\leq \lim_{\epsilon \to 0^+} \inf \frac{V(t, \tilde{a}(t)) - V(t - \epsilon, \tilde{a}(t))}{\epsilon},
\end{align*}
\]
and
\[
\begin{align*}
- K &\leq \lim_{\epsilon \to 0^+} \inf \frac{V(t + \epsilon, \tilde{b}(t)) - V(t, \tilde{b}(t))}{\epsilon}, \\
\lim_{\epsilon \to 0^+} \sup \frac{V(t, \tilde{b}(t)) - V(t - \epsilon, \tilde{b}(t))}{\epsilon} &\leq K.
\end{align*}
\]

Proof. It suffices to prove the first inequality and the rest can be done in a similar manner. By the $C^1$ property of $\tilde{a}$, we can find a small interval $(t, t + \epsilon)$ such that for all the points $(s, \tilde{a}(t))$ with $s \in (t, t + \epsilon)$, either $(s, \tilde{a}(t)) \in D\tilde{a}$, or $(s, \tilde{a}(t)) \in D\tilde{a}\tilde{b}$. In the former case, $V(s, \tilde{a}(t)) = -f_1(s, \tilde{a}(t))$ and the result automatically holds.

Now let us suppose $(s, \tilde{a}(t)) \in D\tilde{a}\tilde{b}, \forall s \in (t, t + \epsilon)$. We can write
\[
\tilde{a}(t + \epsilon) = \tilde{a}(t) + \tilde{a}'(t)\epsilon + o(\epsilon).
\]
Since $V \in C^{0,1}([0, T] \times U) \cap C([0, T] \times U)$ by Corollary 3.1, $V(t, y)$ is uniformly Lipschitz in $y$, thus there is a constant $M_1$ such that
\[
|V(t + \epsilon, \tilde{a}(t)) - V(t + \epsilon, \tilde{a}(t + \epsilon))| \leq M_1|\tilde{a}'(t)\epsilon| + |o(\epsilon)|.
\]
But $V(t + \epsilon, \tilde{a}(t + \epsilon)) = -f_1(t + \epsilon, \tilde{a}(t + \epsilon))$, hence
\[
V(t + \epsilon, \tilde{a}(t)) \leq -f_1(t + \epsilon, \tilde{a}(t + \epsilon)) + M_1|\tilde{a}'(t)\epsilon| + |o(\epsilon)|.
\]
Therefore
\[
\limsup_{\epsilon \to 0^+} \frac{V(t + \epsilon, \tilde{a}(t)) - V(t, \tilde{a}(t))}{\epsilon} \leq \limsup_{\epsilon \to 0^+} \frac{-f_1(t + \epsilon, \tilde{a}(t + \epsilon)) + M_1|\tilde{a}'(t)| + |o(\epsilon)| - (-f_1(t, \tilde{a}(t)))}{\epsilon} \\
= \lim_{\epsilon \to 0^+} \frac{-f_1(t + \epsilon, \tilde{a}(t + \epsilon)) - (-f_1(t, \tilde{a}(t)))}{\epsilon} + M_1|\tilde{a}'(t)| < \infty
\]
by the smoothness of \( f_1 \).

\[ \square \]

4 Time Inhomogeneous Singular Control

In this section we assume the conditions in Theorem 3.2. Let the functions \( V, h \) be given in Theorem 3.2 and define the function \( W(s, x) \) on \([0, T] \times U\) as

\[
W(s, x) = \int_0^x e^{-cs}V(s, y)dy,
\]
(16)

we shall investigate the properties of \( W \). Firstly it is easy to see that \( W \in C^{0,2}([0, T] \times U) \cap C^{0,1}([0, T] \times U) \) by Corollary 3.1 and on \([0, T] \times U\),

\[
\frac{\partial W}{\partial x} = e^{-cs}V, \quad \frac{\partial^2 W}{\partial x^2} = \frac{\partial e^{-cs}V}{\partial x}.
\]

What is not very obvious is the following proposition:

**Proposition 4.1.** \( W \in C^{1,0}([0, T] \times U) \).

**Proof.** For any \((s, x) \in D\tilde{a}, \) by (16) and (13) and using integration by parts, we get

\[
\frac{\partial W}{\partial s}(s, x) = -c \int_0^x e^{-cs}V(s, y)dy + \int_0^x e^{-cs}V_x(s, y)dy \\
= -c \int_0^x e^{-cs}V(s, y)dy \\
- \int_0^x e^{-cs} \left( \frac{1}{2} \sigma^2(y)V_{xx}(s, y) + (\sigma(y)\sigma'(y) + \mu(y))V_x(s, y) + e^{cs}h(s, y) \right)dy \\
= -cW(s, x) - e^{-cs} \left( \frac{1}{2} \sigma^2(x)V_x(s, x) - \frac{1}{2} \sigma^2(0)V_x(s, 0) \right) \\
- \int_0^x \left( e^{-cs}\mu(y)V_x(s, y) + h(s, y) \right)dy,
\]
which is bounded and continuous.
Now we can send $x \to \tilde{a}(s)$, and by the continuity of $W$ and $V_x$, we know that $\frac{\partial W}{\partial s}(s, \tilde{a}(s)^+)$ exists which is
\[
\frac{\partial W}{\partial s}(s, \tilde{a}(s)^+) = -cW(s, \tilde{a}(s)) - e^{-cs} \left( \frac{1}{2} \sigma^2(\tilde{a}(s)) V_x(s, \tilde{a}(s)) - \frac{1}{2} \sigma^2(0) V_x(s, 0) \right) - \int_0^{\tilde{a}} \left( e^{-cs} \mu(y)V_x(s, y) + h(s, y) \right) dy.
\]
Similarly $\frac{\partial W}{\partial s}(s, \tilde{b}(s)^-) \exists$.

If $(s, x)$ is in the interior of $D\tilde{a}$, $V(s, x) = -f_1(s, x)$, and if $(s, x)$ is in the interior of $D\tilde{b}$, $V(s, x) = f_2(s, x)$. Now let us consider the quantity $\frac{\partial W}{\partial s}(s, \tilde{a}(s))$ (and similarly $\frac{\partial W}{\partial s}(s, \tilde{b}(s)))$. We have shown that $\frac{\partial W}{\partial s}(s, \tilde{a}(s)^+)$ is well defined, and the difference between $\frac{\partial W}{\partial s}(s, \tilde{a}(s)^+)$ and $\frac{\partial W}{\partial s}(s, \tilde{a}(s))$ happens in the following integral over a set of zero Lebesgue measure
\[
\lim_{\epsilon \to 0} \int_{\tilde{a}(s)^+}^{\tilde{a}(s)} \frac{e^{-c(s+\epsilon)}V(s + \epsilon, y) - e^{-cs}V(s, y)}{\epsilon} dy
\]
\[
= -c \int_{\tilde{a}(s)^+}^{\tilde{a}(s)} e^{-cs}V(s, y) dy + e^{-cs} \lim_{\epsilon \to 0} \int_{\tilde{a}(s)^+}^{\tilde{a}(s)} \frac{V(s + \epsilon, y) - V(s, y)}{\epsilon} dy
\]
\[
= e^{-cs} \lim_{\epsilon \to 0} \int_{\tilde{a}(s)^+}^{\tilde{a}(s)} V(s + \epsilon, y) - V(s, y) dy.
\]
Here Lemma 3.1 and 3.2 come into play and we conclude that the above quantity is zero. Thus $\frac{\partial W}{\partial s}(s, \tilde{a}(s)^+) = \frac{\partial W}{\partial s}(s, \tilde{a}(s))$ the continuity of $W_s$ follows.

**Remark 4.1.** As a conclusion, by integrating $V(s, y)$ as in (16), we not only obtain the $C^{0, 2}$ regularity of $W(s, x)$, but also the $C^{1, 0}$ regularity of $W(s, x)$.

For each $s \in [0, T)$, define the function
\[
C(s) = -\frac{1}{2} \sigma^2(\tilde{a}(s)) \frac{\partial^2 W}{\partial x^2}(s, \tilde{a}(s)) - \mu(\tilde{a}(s)) \frac{\partial W}{\partial x}(s, \tilde{a}(s)) - \frac{\partial W}{\partial s}(s, \tilde{a}(s)) - \int_0^{\tilde{a}(s)} h(s, y) dy,
\]
then obviously $C(s)$ is a continuous function of $s$. If we define
\[
H(s, x) = \int_0^x h(s, y) dy + C(s),
\]
then since $h(s, y)$ can be arbitrarily chosen (under the assumptions in Theorem 3.2), $H$ can also be an arbitrary function being taken as the holding cost in (2). Similarly, in order to link the terminal costs of the singular control and the Dynkin game, we put
\[
G(x) = \int_0^x e^{-cT} g(y) dy.
\]
Now that on $D\ddot{a}, V$ satisfies

$$\frac{1}{2} \sigma^2(x) \frac{\partial^2 V}{\partial x^2} + (\sigma(x) \sigma'(x) + \mu(x)) \frac{\partial V}{\partial x} + \frac{\partial V}{\partial s} + e^{cs} h(s, x) = 0,$$

which is equivalent to

$$\frac{1}{2} \sigma^2(x) e^{-cs} \frac{\partial^2 V}{\partial x^2} + (\sigma(x) \sigma'(x) + \mu(x)) e^{-cs} \frac{\partial V}{\partial x} + e^{-cs} \frac{\partial V}{\partial s} + h(s, x) = 0. \quad (17)$$

We may now rewrite (17) as

$$\frac{1}{2} \sigma^2(x) \frac{\partial^2 e^{-cs} V}{\partial x^2} + \sigma(x) \sigma'(x) \frac{\partial e^{-cs} V}{\partial x} + \mu(x) \frac{\partial e^{-cs} V}{\partial x} + \frac{\partial e^{-cs} V}{\partial s} + e^{-cs} V + h(s, x) = 0. \quad (18)$$

Since $\mu'(x) = c$, we further get

$$\frac{1}{2} \sigma^2(x) \frac{\partial^2 e^{-cs} V}{\partial x^2} + \sigma(x) \sigma'(x) \frac{\partial e^{-cs} V}{\partial x} + \mu(x) \frac{\partial e^{-cs} V}{\partial x} + e^{-cs} V + h(s, x) = 0. \quad (19)$$

Integrating (18) from 0 to $x$ we get

$$\frac{1}{2} \sigma^2(x) \frac{\partial^2 W}{\partial x^2} + \mu(x) \frac{\partial W}{\partial x} + \frac{\partial W}{\partial s} + H = 0, \quad (19)$$

which is the HJB equation of the value function of the singular control problem.

Firstly (19) certainly holds at $x = \tilde{a}(s)$ by construction. That is, if we construct, for each $s$, a function $U_s(x) = \frac{1}{2} \sigma^2(x) \frac{\partial^2 W}{\partial x^2} + \mu(x) \frac{\partial W}{\partial x} + \frac{\partial W}{\partial s} + H$, then $U_s(\tilde{a}(s)) = 0$. Furthermore, $U'_s(x) = 0$ for $x \in (\tilde{a}(s), \tilde{b}(s))$ by (18), so (19) holds for $x \in (\tilde{a}(s), \tilde{b}(s))$. Actually we can say something more about $W$. By Theorem 3.2, we see that $U'_s(x) < 0$ for $x < \tilde{a}(s)$, and $U'_s(x) > 0$ for $x > \tilde{b}(s)$, thus

$$\frac{1}{2} \sigma^2(x) \frac{\partial^2 W}{\partial x^2} + \mu(x) \frac{\partial W}{\partial x} + \frac{\partial W}{\partial s} + H > 0, \quad x \in [\tilde{a}, \tilde{a}(s)) \cup (\tilde{b}(s), \tilde{B}]. \quad (20)$$

As a summary, we have the following theorem:

**Theorem 4.1.** Assume Assumptions 3.1 – 3.6. If $\tilde{a}(s), \tilde{b}(s)$ are $C^1$ functions of $s$, then there exists a $C^{1,2}([0,T] \times \mathbb{U}) \cap C^0,1([0,T] \times \mathbb{U})$ function $W(s, x)$ which satisfies

$$\frac{1}{2} \sigma^2(x) \frac{\partial^2 W}{\partial x^2}(s, x) + \mu(x) \frac{\partial W}{\partial x}(s, x) + \frac{\partial W}{\partial s}(s, x) + H(s, x) = 0, \quad x \in (\tilde{a}(s), \tilde{b}(s)), \quad (21)$$

$$\frac{1}{2} \sigma^2(x) \frac{\partial^2 W}{\partial x^2}(s, x) + \mu(x) \frac{\partial W}{\partial x}(s, x) + \frac{\partial W}{\partial s}(s, x) + H(s, x) > 0, \quad x \in [\tilde{a}, \tilde{a}(s)) \cup (\tilde{b}(s), \tilde{B}]. \quad (22)$$

$$\frac{\partial W}{\partial x}(s, x) = -e^{-cs} f_1(s, x), \quad x \leq \tilde{a}(s), \quad \frac{\partial W}{\partial x}(s, x) = e^{-cs} f_2(s, x), \quad x \geq \tilde{b}(s), \quad (23)$$

$$-e^{-cs} f_1(s, x) < \frac{\partial W}{\partial x}(s, x) < e^{-cs} f_2(s, x), \quad x \in (\tilde{a}(s), \tilde{b}(s)), \quad (24)$$

$$W(T, x) = G(x), \quad x \in \mathbb{U}. \quad (25)$$
Notice that \( W \) is \( C^{1,2} \) on \([0, T) \times \mathbb{R}\), and \( f_1, f_2 \) are smooth functions, more properties can be derived from [23], for example, we have

\[
\frac{\partial^2 W}{\partial x^2}(s, x) = -e^{-cs} \frac{\partial f_1}{\partial x}(s, x), \quad x \leq \tilde{a}(s), \quad \frac{\partial^2 W}{\partial x^2}(s, x) = e^{-cs} \frac{\partial f_2}{\partial x}(s, x), \quad x \geq \tilde{b}(s).
\]

Let \( S^* \) be the control policy to reflect the process \( X_t \) within the region \( D\tilde{a}\tilde{b} \). That is, whenever the process touches the curve \( \tilde{a} \), \( A_t^{(1)} \) in (11) increases and pushes \( X_t \) back to \( D\tilde{a}\tilde{b} \), with smallest possible effort; whenever the process touches the curve \( \tilde{b} \), \( A_t^{(2)} \) increases and pushes \( X_t \) back to \( D\tilde{a}\tilde{b} \), with smallest possible effort.

We call \( S = (A_t^{(1)}, A_t^{(2)}) \) an admissible control if

1. There is a filtered measurable space \((\Omega, \{\mathcal{F}_t\}_{t \geq 0})\) subject to usual conditions and a probability measure \(P_x\) on it such that \(\{X_t\}_{t \geq 0}\) is an \(\mathcal{F}_t\)-adapted process; \((A_t^{(1)}, A_t^{(2)})\) are right continuous \(\mathcal{F}_t\) measurable processes with bounded variation, and \(A_t^{(1)} - A_t^{(2)}\) is the minimal decomposition of a bounded variation process into a difference of two nondecreasing processes.

2. There is \(\mathcal{F}_t\) adapted Brownian motion \(B_t\) such that the following equation

\[
dX_t = \mu(X_t)dt + \sigma(X_t)dB_t + dA_t^{(1)} - dA_t^{(2)}, \quad X_s = x,
\]

holds \(P_x\) a.s., \(\forall s \in [0, T)\), and the controlled process \(X_t\) is a reflected \(\mathcal{F}_t\) measurable process within a compact region containing \((s, x)\) in \([0, T) \times \mathbb{R}\) with continuous boundary.

**Remark 4.2.** The probability space \(\Omega\) with the filtration \(\{\mathcal{F}_t\}\) is not fixed a priori. It is part of an admissible policy. The filtration \(\mathcal{F}_t\) is assumed to be right continuous and \(\mathcal{F}_0\) is assumed to contain every \(P_x\)-negligible set.

In what follows we shall prove that the control policy \(S^*\) is optimal. Firstly, it can be seen that the reflected SDE in (11) has a unique solution for each \((s, x) \in D\tilde{a}\tilde{b}\), see, e.g., Theorem 2.6 in [6]. And obviously the controlled reflected diffusion \(X_t\) is \(\mathcal{F}_t\) measurable.

If the control \((A_t^{(1)}, A_t^{(2)})\) involves possible jumps at time \(t\), we use

\[
\Delta A_t^{(i)} := A_t^{(i)} - A_{t-}^{(i)}, \quad t \in [0, T), \quad i = 1, 2,
\]

to denote the jumps in the control, and use \(A_t^{(i),c}(i = 1, 2)\) to denote the continuous part of the processes \(A_t^{(i)}, i = 1, 2\). Since \(A_t^{(1)}, A_t^{(2)}\) are the minimal decomposition of a bounded variation process into a difference of two nondecreasing processes, we have \(\Delta A_t^{(1)} \Delta A_t^{(2)} = 0, \forall t \in [0, T)\).

In a similar manner we define

\[
\Delta X_t := X_t - X_{t-}, \quad \Delta W(t, X_t) := W(t, X_t) - W(t-, X_{t-}), \quad \forall t \in [0, T).
\]

We assume that the definition of \(W(s, x)\) in [16] still holds for \((s, x) \notin D\tilde{a}\tilde{b}\) by noticing that \(V(s, x) = -f_1(s, x), x < \tilde{a}(s)\) and \(V(s, x) = f_2(s, x), x > \tilde{b}(s)\). Then we have the following theorem.
Theorem 4.2. Assume Assumptions 3.1 – 3.6 and that \( \tilde{a}(s), \tilde{b}(s) \) are \( C^1 \) functions of \( s \in [0, T] \). Let \( k_S(z) = k_S(s, x) \) be given by the following

\[
k_S(z) = E(s, x) \left( \int_s^T H(t, X_t) dt + G(X_T) \right) \\
+ E(s, x) \left( \int_s^T e^{-ct} \left( f_1(t, X_t) dA_t^{(1),c} + f_2(t, X_t) dA_t^{(2),c} \right) \right) \\
+ E(s, x) \left( \sum_{s \leq t \leq T} e^{-ct} \left( \int_{X_{nt}}^{X_{nt}+\Delta A_t^{(1)}} f_1(t, y) dy \right. \right. \\
+ \left. \left. \int_{X_{nt}^\Delta A_t^{(2)},c}^{X_{nt}^\Delta A_t^{(2)}} f_2(t, y) dy \right) \right),
\]

then

1. For any admissible policy \( S \), \( W(s, x) \leq k_S(s, x) \), \( \forall (s, x) \in [0, T] \times \mathbb{R} \).

2. \( W(s, x) = k_S^*(s, x) \), \( \forall (s, x) \in [0, T] \times \mathbb{R} \).

Proof. Applying the generalized Ito's formula to \( W(s, x) \) yields, for any stopping time \( \tau \in [s, T] \) and admissible control \( S \),

\[
W(\tau, X_\tau) = W(s, x) + \int_s^\tau \left( \frac{\partial W}{\partial t}(t, X_t) + \mu(X_t) \frac{\partial W}{\partial x}(t, X_t) + \frac{1}{2} \sigma^2(X_t) \frac{\partial^2 W}{\partial x^2}(t, X_t) \right) dt \\
+ \int_s^\tau \frac{\partial W}{\partial x}(t, X_t) \sigma(X_t) dB_t \\
+ \int_s^\tau \frac{\partial W}{\partial x}(t, X_t)(dA_t^{(1),c} - dA_t^{(2),c}) + \sum_{s \leq t < \tau} \Delta W(t, X_t).
\]

by taking expectation of both sides of (27) we get

\[
W(s, x) = E(s, x) W(\tau, X_\tau) \\
- E(s, x) \left( \int_s^\tau \left( \frac{\partial W}{\partial t}(t, X_t) + \mu(X_t) \frac{\partial W}{\partial x}(t, X_t) + \frac{1}{2} \sigma^2(X_t) \frac{\partial^2 W}{\partial x^2}(t, X_t) \right) dt \right) \\
- E(s, x) \int_s^\tau \frac{\partial W}{\partial x}(t, X_t)(dA_t^{(1),c} - dA_t^{(2),c}) - E(s, x) \sum_{s \leq t < \tau} \Delta W(t, X_t).
\]
We can rewrite \( k_S(s, x) \) as

\[
k_S(s, x) = E_{(s,x)} \left( \int_s^\tau H(t, X_t) dt + k_S(\tau, X_\tau) \right)
\]

\[
+ E_{(s,x)} \left( \int_s^\tau e^{-ct} \left( f_1(t, X_t) dA_t^{(1),c} + f_2(t, X_t) dA_t^{(2),c} \right) \right)
\]

\[
+ E_{(s,x)} \left( \sum_{s \leq t < \tau} e^{-ct} \left( \int_{X_{nt-}^-}^{X_{nt-}^+} f_1(t, y) dy \right.ight.
\]

\[
+ \left. \int_{X_{nt-}^-}^{X_{nt-}^+} f_2(t, y) dy \right) \right), \quad \tau \in [s, T).
\]

Therefore,

\[
k_S(s, x) - W(s, x)
\]

\[
= E_{(s,x)} \left( k_S(\tau, X_\tau) - W(\tau, X_\tau) \right)
\]

\[
+ E_{(s,x)} \int_s^\tau \left( H(t, X_t) + \frac{\partial W}{\partial t}(t, X_t) + \mu(X_t) \frac{\partial W}{\partial x}(t, X_t) + \frac{1}{2} \sigma^2(X_t) \frac{\partial^2 W}{\partial x^2}(t, X_t) \right) dt
\]

\[
+ E_{(s,x)} \int_s^\tau e^{-ct} f_1(t, X_t) dA_t^{(1),c}
\]

\[
+ E_{(s,x)} \int_s^\tau e^{-ct} f_2(t, X_t) dA_t^{(2),c}
\]

\[
+ E_{(s,x)} \sum_{s \leq t < \tau} \Delta W(t, X_t)
\]

\[
+ E_{(s,x)} \left( \sum_{s \leq t < \tau} \left( \int_{X_{nt-}^-}^{X_{nt-}^+} e^{-ct} f_1(t, y) dy + \int_{X_{nt-}^-}^{X_{nt-}^+} e^{-ct} f_2(t, y) dy \right) \right).
\]

By Theorem 4.1, the second, third and fourth expectations in (29) are all nonnegative, and this holds true as \( \tau \to T \).

Define the sets

\[
\Gamma_+ = \{ t \in [s, T] : \Delta A_t^{(1)} > 0 \}, \quad \Gamma_- = \{ t \in [s, T] : \Delta A_t^{(2)} > 0 \},
\]

then \( \Gamma_+ \cap \Gamma_- = \emptyset \). If we send \( \tau \to T \), \( k_S(T-, X_{T-}) \) can be written as \( G(X_T) \) plus the possible jump of control at \( T \). Thus by sending \( \tau \to T \) and the fact that \( k_S(T, X_T) = G(X_T) = W(T, X_T) \) (after a possible jump at \( T \)), we can rewrite the remaining parts in (29).
as
\[ E_{(s,x)}(G(X_T) - W(T, X_T)) + E_{(s,x)} \sum_{s \leq t \leq T} \Delta W(t, X_t) \]
\[ + E_{(s,x)} \left( \sum_{s \leq t \leq T} \left( \int_{X_{nt}^- + \Delta A_t^{(1)}(t)} e^{-ct} f_1(t, y)dy + \int_{X_{nt}^- - \Delta A_t^{(2)}(t)} e^{-ct} f_2(t, y)dy \right) \right) \]
\[ = E_{(s,x)} \left[ \sum_{t \in \Gamma_+} \int_{X_{nt}^- + \Delta A_t^{(1)}(t)} \left( \frac{\partial W}{\partial x}(t, y) + e^{-ct} f_1(t, y) \right) dy \right] \]
\[ + E_{(s,x)} \left[ \sum_{t \in \Gamma_-} \int_{X_{nt}^- - \Delta A_t^{(2)}(t)} \left( -\frac{\partial W}{\partial x}(t, y) + e^{-ct} f_2(t, y) \right) dy \right] . \tag{30} \]

Once again by Theorem 4.1, these two integrals are nonnegative, hence \( k_S(s, x) \geq W(s, x) \).

If \( S = S^* \), except a possible jump at time \( s \) and the controlled process is the reflected diffusion in \( D\tilde{a}\tilde{b} \), then the second expectation in (29) is zero. Since \( A_t^{(1)} \) only increases when \( X_t \) hits \( \tilde{a} \), and \( A_t^{(2)} \) only increases when \( X_t \) hits \( \tilde{b} \), by (23) the third and fourth expectations in (29) are both zeros. The remaining parts in (29) is expressed in (30) which indicates that under \( S^* \) it is equal to zero. Therefore \( W(s, x) = k_{S^*}(s, x) \), \( \forall (s, x) \in [0, T] \times \mathbb{R} \).

The proof of Theorem 4.2 also implies that the optimal admissible control is unique.

**Remark 4.3.** The term “a possible jump at time \( s \)” means if the initial state of the process is outside of the region \( D\tilde{a}\tilde{b} \), apply a control \( \Delta A_t^{(1)} \) or \( \Delta A_t^{(2)} \) to immediately bring it into \( D\tilde{a}\tilde{b} \).

## 5 Optimal Control with A More General Terminal Cost

Consider again the cost function of the singular control problem
\[
k_S(z) = k_S(s, x) = E_{(s,x)} \left\{ \int_s^T H(t, X_t)dt + \int_s^T e^{-ct} f_1(t, X_t)dA_t^{(1)} + \int_s^T e^{-ct} f_2(t, X_t)dA_t^{(2)} + G(X_T) \right\} , \tag{31} \]
where \( G'(x) = e^{-ct} g(x) \). In Assumption 3.1 we have the condition
\[ -f_1(T, x) \leq g(x) \leq f_2(T, x) , \forall x . \tag{32} \]

In this section, we shall consider a more general terminal cost \( G(x) \) such that (32) does not necessarily hold. We shall see that the optimal control involves a jump at the terminal \( T \). In this section the functions \( f_1, f_2, g, h \) are still assumed to be bounded and continuous with \( f_2 > 0 > -f_1 \), and we further assume Assumptions 3.2 and 3.4.

The following is a relaxed condition on \( g(x) \) which provides a more general terminal cost function \( G(x) = \int_0^x e^{-ct} g(y)dy \).
Assumption 5.1. There exist two points $A, B$ such that $A < 0 < B$, $g(x)$ is in $H^1((A, B))$. Furthermore, $\forall x < A$, $-f_1(T, x) \geq g(x)$, $\forall x > B$, $f_2(T, x) \leq g(x)$, and $\forall x \in [A, B]$, $-f_1(T, x) \leq g(x) \leq f_2(T, x)$.

Comparing this assumption with Assumption 3.3, it can be seen that we no longer require the conditions $A \leq a(T)$ or $B \geq b(T)$.

Define $G(x)$ as

$$G(x) = \min_{y_1 \geq 0, y_2 \geq 0} \left[ G(x + y_1) + \int_x^{x+y_1} e^{-cT} f_1(T, u) du, \ G(x - y_2) + \int_x^{x-y_2} e^{-cT} f_2(T, u) du \right],$$

then we have

Proposition 5.1. For any $x$ with $g(x) < -f_1(T, x)$, $G(x) < G(x)$ and $	ilde{g}(x) := e^{cT} \tilde{G}'(x) = -f_1(T, x)$, and for any $x$ with $g(x) > f_2(T, x)$, $G(x) < G(x)$ and $\tilde{g}(x) = f_2(T, x)$.

Proof. Firstly it is obvious that $G(x) \leq G(x)$, $\forall x$. If $g(x) < -f_1(T, x)$, then by continuity we can find an interval $[x, x + \Delta x]$ such that for any $y \in [x, x + \Delta x]$, $g(y) < -f_1(T, y)$. Therefore, $G(x + \Delta x) = G(x) + \int_x^{x+\Delta x} e^{-cT} g(y) dy < G(x) - \int_x^{x+\Delta x} e^{-cT} f_1(T, y) dy$, which implies that $G(x) < G(x + \Delta x)$ and $\tilde{g}(x) = -e^{-cT} f_1(T, x)$ for small $\Delta x$, it can be easily derived that $\tilde{G}'(x) = -e^{-cT} f_1(T, x)$.

The rest of the proposition can be proved in a similar way. \hfill \Box

Proposition 5.2. $\tilde{g}(x)$ is continuous.

Proof. Let $I = (-\infty, A)$ and $II = (B, \infty)$, then certainly $g(A) = -f_1(T, A)$, $g(B) = f_2(T, B)$ and $A < B$ by Assumption 5.1. We have shown that $G(x) < G(x)$ and $\tilde{g}(x) = -f_1(T, x)$, $\forall x \in I$, and $G(x) < G(x)$, $\tilde{g}(x) = f_2(T, x)$, $\forall x \in II$. Since for any $x \geq A$, $g(x) \geq -f_1(T, x)$, then it is easy to see that $\min_{y_1 \geq 0} [G(x + y_1) + \int_x^{x+y_1} e^{-cT} f_1(T, u) du] = G(x)$, i.e., $y_1 = 0$. Similarly for any $x \leq B$, $g(x) \leq f_2(T, x)$, so $\min_{y_2 \geq 0} [G(x - y_2) + \int_x^{x-y_2} e^{-cT} f_2(T, u) du] = G(x)$, i.e., $y_2 = 0$. As a conclusion, on $[A, B]$ we have $G(x) = G(x)$ and $g(x) = g(x)$, hence the continuity of $g(x)$.

In fact we can tell that $\tilde{g}(x)$ satisfies Assumption 3.3 and in particular,

$$\tilde{g}(x) \in H^1(\min\{a(T), A\}, \max\{b(T), B\}).$$

Now we are ready to consider the newly modified zero-sum game

$$\tilde{J}_s(\tau, \sigma) = \tilde{J}_{(s,x)}(\tau, \sigma) = E_{(s,x)} \left\{ \int_0^\tau e^{cT} h(t, Y_t) dt + 1_{(\sigma < \tau \wedge T)}(-f_1(\sigma, Y_\sigma)) + 1_{(\tau < \sigma \wedge T)} f_2(\tau, Y_\tau) + 1_{(T < \tau \wedge \sigma)} \tilde{g}(Y_T) \right\}, \ \tau \wedge \sigma \geq s,$$ (34)
where \( Y_t \) follows

\[
dY_t = (\sigma(Y_t)\sigma'(Y_t) + \mu(Y_t))dt + \sigma(Y_t)dB_t, \quad Y_s = x. \tag{35}
\]

The value of this game is thus given by

\[
V(z) = V(s, x) = \inf_{\tau} \sup_{\sigma} \tilde{J}_z(\tau, \sigma) = \sup_{\tau} \inf_{\sigma} \tilde{J}_z(\tau, \sigma), \quad Y_s = x. \tag{36}
\]

The following is a direct result from Section 3. Notice that the two free boundary curves \( \tilde{a}, \tilde{b} \) satisfy \( \tilde{a}(T) = \min \{a(T), A\} \) and \( \tilde{b}(T) = \max \{b(T), B\} \).

**Theorem 5.1.** Assume Assumptions 3.2, 3.4, 3.5 3.6 and 5.1. If \( \tilde{a}(t), \tilde{b}(t) \) are \( C^1 \) functions of \( t \), then the function \( V \) in (36) is bounded and continuous and is the unique weak solution of the following problem:

\[
\begin{align*}
\mathcal{L}V(t, y) + e^{ct}h(t, y) &= 0, & (t, y) &\in D\tilde{a}\tilde{b}, \\
\mathcal{L}V(t, y) + e^{ct}h(t, y) &< 0, & (t, y) &\in D\tilde{a}, \\
\mathcal{L}V(t, y) + e^{ct}h(t, y) &> 0, & (t, y) &\in D\tilde{b}, \\
-f_1(t, y) < V(t, y) &< f_2(t, y), & (t, y) &\in D\tilde{a}\tilde{b}, \\
V(t, y) &= -f_1(t, y), & (t, y) &\in D\tilde{a}, \\
V(t, y) &= f_2(t, y), & (t, y) &\in D\tilde{b}, \\
V(T, y) &= \tilde{g}(y).
\end{align*}
\]

By considering \( V(t, y) \) as a mapping \( V : [0, T] \to H^2(U) \), where \( U \) is the open interval \((\tilde{A}, \tilde{B})\), then \( V \in L^2(0, T ; H^2(U)) \cap L^\infty(0, T ; H^1(U)), \frac{dV}{dt} \in L^2(0, T; L^2(U)) \). Furthermore, \( V(t, y) \) is in \( C^{0,1}([0, T] \times U) \cap C([0, T] \times U) \).

For the related singular control problem, if we define

\[
\begin{align*}
W(s, x) &= \int_0^x e^{-cs}V(s, y)dy, \\
H(s, x) &= \int_0^x h(s, y)dy + C(s), \\
G(x) &= \int_0^x e^{-cT}g(y)dy,
\end{align*}
\]

where

\[
C(s) = -\frac{1}{2}\sigma^2(\tilde{a}(s)) \frac{\partial^2 W}{\partial x^2}(s, \tilde{a}(s)) - \mu(\tilde{a}(s)) \frac{\partial W}{\partial x}(s, \tilde{a}(s)) - \frac{\partial W}{\partial s}(s, \tilde{a}(s)) - \int_0^{\tilde{a}(s)} h(s, y)dy,
\]

and define \( \tilde{G}(x) \) as in (33), then we immediately get the following result from Section 4
Theorem 5.2. Assume Assumptions 3.2, 3.4, 3.5, 3.6 and 5.1. If \( \tilde{a}(s), \tilde{b}(s) \) are \( C^1 \) functions of \( s \), then there exists a \( C^{1,2}([0,T] \times U) \cap C^0([0,T] \times U) \) function \( W(s,x) \) which satisfies

\[
\frac{1}{2} \sigma^2(x) \frac{\partial^2 W}{\partial x^2}(s,x) + \mu(x) \frac{\partial W}{\partial x}(s,x) + \frac{\partial W}{\partial s}(s,x) + H(s,x) = 0, \quad x \in (\tilde{a}(s), \tilde{b}(s)),
\]

\[
\frac{1}{2} \sigma^2(x) \frac{\partial^2 W}{\partial x^2}(s,x) + \mu(x) \frac{\partial W}{\partial x}(s,x) + \frac{\partial W}{\partial s}(s,x) + H(s,x) > 0, \quad x \in [\tilde{a}(s), \tilde{b}(s)] \cup (\tilde{A}, \tilde{b}(s)).
\]

\[
\frac{\partial W}{\partial x}(s,x) = -e^{-cs} f_1(s,x), \quad x \leq \tilde{a}(s),
\]

\[
\frac{\partial W}{\partial x}(s,x) = e^{-cs} f_2(s,x), \quad x \geq \tilde{b}(s),
\]

\[
e^{-cs} f_1(s,x) < \frac{\partial W}{\partial x}(s,x) < e^{-cs} f_2(s,x), \quad x \in (\tilde{a}(s), \tilde{b}(s)),
\]

\[
W(T,x) = \tilde{G}(x), \quad x \in U.
\]

Let \( k_S(z) = k_S(s,x) \) be given by

\[
k_S(z) = E_{(s,x)} \left( \int_s^T H(t,X_t)dt + G(X_T) \right)
\]

\[
+ E_{(s,x)} \left( \int_s^T e^{-ct} \left( f_1(t,X_t)dA^{(1),c}_t + f_2(t,X_t)dA^{(2),c}_t \right) \right)
\]

\[
+ E_{(s,x)} \left( \sum_{s \leq t \leq T} e^{-ct} \left( \int_{X_{nt}^- + \Delta A^{(1)}_t} f_1(t,y)dy \right. \right.
\]

\[
\left. + \int_{X_{nt}^- - \Delta A^{(2)}_t} f_2(t,y)dy \right) \right),
\]

then

1. For any admissible policy \( S \), \( W(s,x) \leq k_S(s,x), \forall (s,x) \in [0,T] \times \mathbb{R} \).

2. \( W(s,x) = k_{S^*}(s,x), \forall (s,x) \in [0,T] \times \mathbb{R} \), where \( S^* \) is the control policy to reflect the process \( X_t \) within the region \( D\tilde{a}\tilde{b} \).

Proof. We only need to verify that \( W \) as given is optimal. Following the proof of Theorem
we arrive at
\[
\begin{align*}
    k_S(s, x) - W(s, x) &= E(s, x) (k_S(\tau, X_\tau) - W(\tau, X_\tau)) \\
    &= E(s, x) \left( E(\tau, X_\tau) - W(\tau, X_\tau) \right) \\
    &+ E(s, x) \int_{s}^{\tau} \left( H(t, X_t) + \frac{\partial W}{\partial t} (t, X_t) + \mu(X_t) \frac{\partial W}{\partial x} (t, X_t) + \frac{1}{2} \sigma^2(X_t) \frac{\partial^2 W}{\partial x^2} (t, X_t) \right) dt \\
    &+ E(s, x) \int_{s}^{\tau} \left( e^{-ct} f_1(t, X_t) + \frac{\partial W}{\partial x} (t, X_t) \right) dA^{(1)}_t \\
    &+ E(s, x) \int_{s}^{\tau} \left( e^{-ct} f_2(t, X_t) - \frac{\partial W}{\partial x} (t, X_t) \right) dA^{(2)}_t \\
    &+ E(s, x) \sum_{s < t < \tau} \Delta W(t, X_t) \\
    &+ E(s, x) \left( \sum_{s \leq t < \tau} \left( \int_{X_{nt-}}^{X_{nt+}+\Delta A^{(1)}_t} e^{-ct} f_1(t, y) dy + \int_{X_{nt-}+\Delta A^{(1)}_t}^{X_{nt+}} e^{-ct} f_2(t, y) dy \right) \right) \\
    &\text{By Theorem 4.1, the second, third and fourth expectations in (39) are all nonnegative, and} \\
    &\text{this holds true as } \tau \to T. \\
    \text{Define the sets} \\
    \Gamma_+ &= \{ t \in [s, T] : \Delta A^{(1)}_t > 0 \}, \\
    \Gamma_- &= \{ t \in [s, T] : \Delta A^{(2)}_t > 0 \}, \\
    \text{then once again } \Gamma_+ \cap \Gamma_- = \phi. \\
    \text{If we send } \tau \to T, k_S(T^-, X_{T^-}) \text{ can be written as } G(X_T) \text{ plus the possible jump of} \\
    \text{control at } T. \text{ Thus by sending } \tau \to T \text{ and the fact that } k_S(T, X_T) = G(X_T), W(T^-, X_{T^-}) = \\
    W(T, X_{T^-}) = \tilde{G}(X_{T^-}) \text{ by the continuity of } W, \text{ we can rewrite the remaining parts in (39) as} \\
    E(s, x) (G(X_T) - \tilde{G}(X_{T^-})) + E(s, x) \sum_{s \leq t < T} \Delta W(t, X_t) \\
    &+ E(s, x) \left( \sum_{s \leq t < T} \left( \int_{X_{nt-}}^{X_{nt+}+\Delta A^{(1)}_t} e^{-ct} f_1(t, y) dy + \int_{X_{nt+}+\Delta A^{(2)}_t}^{X_{nt+}} e^{-ct} f_2(t, y) dy \right) \right) \\
    &= E(s, x) (G(X_T) - \tilde{G}(X_{T^-})) \\
    &+ E(s, x) \left( \int_{X_{T^-}}^{X_{T+}+\Delta A^{(2)}_t} e^{-ct} f_1(t, y) dy + \int_{X_{T+}+\Delta A^{(2)}_t}^{X_{T+}} e^{-ct} f_2(t, y) dy \right) \\
    &+ E(s, x) \left[ \sum_{s \leq t < T} \int_{X_{nt-}}^{X_{nt+}+\Delta A^{(1)}_t} \left( \frac{\partial W}{\partial x} (t, y) + e^{-ct} f_1(t, y) \right) dy \right] \\
    &+ E(s, x) \left[ \sum_{s \leq t < T} \int_{X_{nt-}+\Delta A^{(2)}_t}^{X_{nt+}} \left( -\frac{\partial W}{\partial x} (t, y) + e^{-ct} f_2(t, y) \right) dy \right] \quad (40)
\end{align*}
\]
As similar to the proof of Theorem 4.2, the last two integrals are nonnegative. By the definition of $\tilde{G}$ in (33), we can write

$$\tilde{G}(X_{T^-}) = \min_{\Delta A^{(1)}, \Delta A^{(2)}} \left( G(X_T) + \int_{X_{T^-} + \Delta A^{(1)}_T} e^{-ct} f_1(t, y) dy + \int_{X_{T^-} - \Delta A^{(2)}_T} e^{-ct} f_2(t, y) dy \right),$$

and now it can be seen that the quantity (40) is nonnegative, hence $k_S(s, x) \geq W(s, x)$.

The rest of the proof is similar to that of Theorem 4.2.

6 Regularity of the Free Boundaries

It should be noticed that the $C^1$ regularity of the two free boundaries $\tilde{a}(t)$ and $\tilde{b}(t)$ are crucial in showing the regularity of the value function of the Dynkin game, see, e.g., the proofs of Theorem 3.2 and Lemma 3.2. In [10], the authors claimed that one free boundary is continuous, and the other is $C^\infty$ by the same arguments as in Friedman [13]. However, in [13] only the $C^1$ regularity was proved, under some conditions. Karatzas studied a particular one dimensional singular stochastic control problem in [18], also through a game approach, and he claimed that the free boundary is continuously differentiable on $[\epsilon, T]$ for any $0 < \epsilon < T$. However, for a zero-sum game of a general diffusion process with general obstacles and general terminal payoff, the regularity of the value function and the regularity of the free boundaries is still not fully understood. The closely related problem is the one-sided optimal stopping problem such as the American option problem, as discussed in [29], where the author characterized the free boundary as the unique solution to a free-boundary integral equation. Further result can be found in [7]. Later Yang, Jiang and Bian [35] proved that the free boundary is continuously differentiable under a particular condition, which was further relaxed by Bayraktar and Xing [2]. But in that work, “...it is essential to have the value function $V(S, t)$ as the classical solution of the free boundary problem”. The point is, it is often hard to tell which result comes first. If one is familiar with closed-form solutions of PDE with free boundaries, once the solution to the PDE is found, the free boundaries are obtained simultaneously, and vice versa. A very recent result on the regularity of the free boundary for the American put option can be found in [8] where the author proved the $C^\infty$ regularity of the free boundary for American put options with dividend payment. This work may shed some light on the regularity of the free boundaries of two-obstacle problem with general terminal payoff function, and the author shall leave this problem as an interesting future research topic.

Concluding Remarks

In this paper, we studied a type of time inhomogeneous stochastic singular control problems of one dimensional diffusion. We first investigated the optimal policy and the regularity of the value function $V$ of a time inhomogeneous zero-sum game (Dynkin game), then the integrated form of this value function, which is $C^{1,2}$, coincides with the optimal value function
of the singular control problem. Thus the existence of a classical solution to the HJB equation is proved. We also characterized the optimal control policy as to reflect the diffusion between two time inhomogeneous curves, which are the free boundaries of the HJB equation. It can be concluded that the $C^1$ property of the free boundaries is critical for obtaining the smoothness of the solution of the HJB equation.

It is well known that the time inhomogeneous stochastic singular control problem and zero-sum game have numerous variations, and very often the difficulty arises in finding the optimal control policies and the regularity of the value functions. This paper aims to set a basis for further searches of the form of optimal control policies, as well as the existence and regularity of the solutions of the HJB equations that are associated with more general time inhomogeneous singular control problems.

References

[1] F.M. Baldursson and I. Karatzas, *Irreversible Investment and Industry Equilibrium*, Finance and Stochastics, 1 (1997) pp. 69–89.

[2] E. Bayraktar and H. Xing, *Analysis of the optimal exercise boundary of American options for jump diffusions*, SIAM J. Math. Anal., 41(2) (2009) pp. 825–860.

[3] A. Bensoussan and J.L. Lions, *Applications des Inéquations Variationnelles en contrôle Stochastique*, Dunod, Paris, 1978.

[4] J.M. Bismut, *Convex inequalities in stochastic control*, J. Funct. Anal., 42 pp. 226–270, 1981.

[5] F. Boetius and M. Kohlmann, *Connections between optimal stopping and singular stochastic control*, Stochastic Processes and their Applications 77 (1998) pp. 253–281.

[6] K. Burdzy, W. Kang and K. Ramanan, *The Skorokhod problem in a time-dependent interval*, Stochastic Processes and their Applications, 119 pp. 428–452, 2009.

[7] X. Chen and J. Chadam, *A mathematical analysis of the optimal exercise boundary for American put options*, SIAM J. Math. Anal., 38(5) (2007) pp. 1613–1641.

[8] X. Chen and H. Cheng, *Regularity of the free boundary for the American put option*, Discrete and Continuous Dynamical Systems, Series B, 17(6) (2012) pp. 1751–1759.

[9] M.G. Crandall, H. Ishii and P. Lions, *User’s guide to viscosity solutions of second order partial differential equations*, American Mathematical Society. Bulletin. New Series 27 (1) pp. 1–67, 1992.

[10] M. Dai and F. Yi, *Finite horizon optimal investment with transaction costs: a parabolic double obstacle problem*, J. Differential Equations, 246(2009) pp. 1445–1469.
[11] L.C. Evans, *Partial Differential Equations: Second Edition*, Graduate Studies in Mathematics, AMS 2010.

[12] W.H. Fleming and H.M Soner, *Controlled Markov Processes and Viscosity Solutions*, Springer, 2nd edition, 2006.

[13] A. Friedman, *Parabolic Variational Inequalities in One Space Dimension and Smoothness of the Free Boundary*, J. Func. Anal., 18 pp. 151–176, 1975.

[14] M. Fukushima and M. Taksar, *Dynkin Games Via Dirichlet Forms and Singular Control of One-Dimensional Diffusion*, SIAM J. Control Optim., 41(3)(2002) pp. 682–699.

[15] M. Fukushima and K. Menda, *Refined Solutions of Optimal Stopping Games for Symmetric Markov Processes*, Technology Reports of Kansai University, 48(2006) pp. 101–110.

[16] M. Fukushima, Y. Oshima and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes, 2nd Edn.*, Walter de Gruyter, Berlin, New York, 2011.

[17] X. Guo and P. Tomecek, *A Class of Singular Control Problems and the Smooth Fit Principle*, SIAM J. Control Optim., 47(6)(2009) pp. 3076–3099.

[18] I. Karatzas, *A Class of Singular Stochastic Control Problems*, Adv. Appl. Prob., 15(1983) pp. 225–254.

[19] I. Karatzas and S.E. Shreve, *Connections between optimal stopping and singular stochastic control II. Reflected follower problems*, SIAM J. Control Optim., 23(3)(1985) pp. 433–451.

[20] N. El Karoui, J.P. Lepeltier, and B. Marchal, *Optimal stopping of controlled Markov processes*, in Advances in Filtering and Optimal Stochastic Control, Lecture Notes in Control and Inform. Sci. 42, Springer, Berlin, pp. 106–112, 1982.

[21] J. Ma, *On the Principle of Smooth Fit for a Class of Singular Stochastic Control Problems for Diffusions*, SIAM J. Control Optim., 30(4) pp. 975–999, 1992.

[22] Z. Ma and M. Röckner, *Introduction to the theory of (non-symmetric) Dirichlet forms*, Springer-Verlag, Nov 19, 1992.

[23] V. Mackevicius, *Passing to the limit in the optimal stopping problems of Markov processes*, Liet. Mat. Rink., 13 pp. 115–128, 1973.

[24] J. F. Mertens, *Strongly supermedian functions and optimal stopping*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 26 pp. 119–139, 1973.

[25] H. Nagai, *On An Optimal Stopping Problem And A Variational Inequality*, J. Math. Soc. Japan, 30(1978) pp. 303–312.
[26] Y. Oshima, *On An Optimal Stopping Problem of Time Inhomogeneous Diffusion Processes*, SIAM J. Control Optim., 45(2) pp. 565–579, 2006.

[27] J. Palczewski and L. Stettner, *Finite Horizon Optimal Stopping of Time-discontinuous Functionals with Applications to Impulse Control with Delay*, SIAM J. Control Optim., 48(8) pp. 4874–4909, 2010.

[28] J. Palczewski and L. Stettner, *Stopping of functionals with discontinuity at the boundary of an open set*, Stochastic Processes and Their Applications, 121 pp. 2361–2392, 2011.

[29] G. Peskir, *On the American Option Problem*, Mathematical Finance, 15(1) (2005) pp. 169–181.

[30] H. Pham, *On the Smooth-fit Property for One-dimensional Optimal Switching Problem*, in S’eminaire de Probabilit’es XL, Lecture Notes in Math. 1899, Springer, Berlin, pp. 187–199, 2007.

[31] H. Pham, *Continuous-time Stochastic Control and Optimization with Financial Applications*, Stochastic Modelling and Applied Probability, Springer 2009.

[32] H.M. Soner and S.E. Shreve, *Regularity of the Value Function for a Two-Dimensional Singular Stochastic Control Problem*, SIAM J. Control and Optimization, 27(4) pp. 876–907, 1989.

[33] L. Stettner and J. Zabczyk, *Strong envelops of stochastic processes and a penalty method*, Stochastics, 4 pp. 267–280, 1981.

[34] L. Stettner, *Penalty Method for Finite Horizon Stopping Problems*, SIAM J. Control Optim., 49(3), pp. 1078–1099, 2011.

[35] C. Yang, L. Jiang and B. Bian, *Free boundary and American options in a jump-diffusion model*, European J. Appl. Math., 17 (2006) pp. 95–127.

[36] Yipeng Yang, *Multi-dimensional Stochastic Singular Control Via Dynkin Game and Dirichlet Form*, SIAM J. Control and Optim., to appear, arXiv:1209.2639, 2014.

[37] Yipeng Yang, *Refined Solutions of Time Inhomogeneous Optimal Stopping Problem and Zero-sum Game via Dirichlet Form*, Probability and Mathematical Statistics, to appear, 2014.

[38] J. Zabczyk, *Stopping problems in stochastic control*, in Proceedings of the International Congress of Mathematicians, Vol. II, PWN, Warsar, North-Holland, Amsterdam, pp. 1425–1437, 1984.

[39] J. Zabczyk, *Stopping Games for Symmetric Markov Processes*, Probab. Math. Statist., 4(2)(1984) pp. 185–196.