COHOMOLOGIES OF LANDAU-GINZBURG MODELS

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Abstract. Let $V$ be a holomorphic bundle over a complex manifold $M$, and $s$ be a holomorphic section of $V$. We study different types of cohomology associated to the Koszul complex induced by $s$. When $M$ is complete, these cohomologies are isomorphic to each other and have self duality.

1. Introduction

The properties of the cohomologies associated to the Landau-Ginzburg model $(X, W)$ had been widely studied in the following papers [1–3, 5–8], where $X$ is a noncompact complex manifold, and $W$ is a holomorphic function on $X$. In this paper we consider the properties of cohomology for more general cases. Let $M$ be a complex manifold (usually noncompact), $V$ be a holomorphic bundle over $M$ with $\text{rk} V = \dim M = n$ and $s$ be a holomorphic section of $V$ with compact zero loci $Z = \{s = 0\}$. Let $V^*$ be the dual bundle of $V$, $s$ induced the following Koszul complex

$$0 \to \wedge^n V^* \xrightarrow{\iota_s} \cdots \xrightarrow{\iota_s} \wedge^2 V^* \xrightarrow{\iota_s} V^* \xrightarrow{\iota_s} C \to 0,$$

where $\iota_s$ is the contraction operator defined by $s$.

Let $A_{i,j}(\wedge^l V^*)$ be the sheaf of smooth $(i,j)$ forms on $M$ with value in $\wedge^l V^*$. Let $\Omega_{i,j}(\wedge^l V^*) := \Gamma(M, A_{i,j}(\wedge^l V^*))$ and assign its element $\alpha$ to have degree $\sharp \alpha = i + j - l$. Then

$$B := \bigoplus_{i,j,l} \Omega_{i,j}(\wedge^l V^*)$$

is a graded commutative algebra with the (wedge) product uniquely extending wedge products in $\Omega^\bullet$, $\wedge^\bullet V^*$ and mutual tensor products. Denote $E_k M = \bigoplus_{i-j=k} E_{i,j} M$ with $E_{i,j} M := \Omega^{0,i}(\wedge^j V^*) = \Gamma(M, A_{0,i}(\wedge^j V^*))$, and $E_{c,k} M = \bigoplus_{i-j=k} E_{c,i,j} M$ with $E_{c,i,j} M := \{\alpha \in E_{i,j} M \mid \alpha \text{ has compact support}\}$.

Let $E_M := \bigoplus_k E^k M$ and $E_{c,M} := \bigoplus_k E^k_{c,M}$. For $\alpha \in E_M$ we denote $\alpha_{i,j}$ to be its component in $E_{i,j} M$. Clearly, $E_M$ is a bi-graded $C^\infty(M)$-module. Under the operations

$$\overline{\partial} : E^{i,j}_M \to E^{i+1,j}_M \quad \text{and} \quad \iota_s : E^{i,j}_M \to E^{i-1,j}_M$$

the space $E_{\bullet,\bullet} M$ becomes a double complex and $E_{c,\bullet,\bullet} M$ is a subcomplex. We shall study the cohomology of $E_{\bullet,\bullet} M$ and $E_{c,\bullet,\bullet} M$ with respect to the following coboundary operator

$$\overline{\partial} := \overline{\partial} + \iota_s.$$

One checks $\overline{\partial}^2 = 0$ using Leibniz rule of $\overline{\partial}$ and $\overline{\partial} s = 0$. Denoted by

$$H^k(M, s) := H^k(E_{\bullet,\bullet} M),$$

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and
\[ H^k_c(M, s) := H^k_c(M, s). \]

Because \( Z \) is compact and (1.1) is exact on \( U := M \setminus Z \), the hypercohomology \( H^*(M, s) \) are finite dimension over \( \mathbb{C} \). In section two we proved that

**Theorem 1.1.** For \( -n \leq k \leq n \),
\[ H^k_c(M, s) \cong H^k_c(M, s). \]

Let \( \psi \in \Gamma(M, \det V \otimes \det \Omega_M) \) be a holomorphic section, where \( \Omega_M \) is the cotangent bundle of \( M \). We can defined the following trace map
\[ \text{tr} : E_{c,M} \rightarrow \mathbb{C}, \quad \text{tr}(\alpha) := \int_M \psi \lrcorner \alpha, \]
where \( \lrcorner \) is a contraction operator defined by (4.1). By definition we have \( \text{tr}(\partial \alpha) = 0 \) and \( \text{tr}(\iota_s \alpha) = 0 \), which imply that the trace map is well defined on the cohomology \( \text{tr} : H^*_c(M, s) \rightarrow \mathbb{C}. \)

It induces a pairing \( \langle -, - \rangle : H^*_c(M, s) \times H^*_c(M, s) \rightarrow \mathbb{C} \) defined by
\[ (\alpha, \beta) \mapsto \langle \alpha, \beta \rangle := \text{tr}(\alpha \wedge \beta). \]

When \( M \) is a complete Kähler manifold, by introducing the space of square integral forms \( F_{2,q}^0(\wedge^l V^*) \) and the Hodge star operator \( *_V : \wedge^{0,p} \Omega_M \otimes \wedge^q V^* \rightarrow \wedge^{0,n-p} \Omega_M \otimes \wedge^{n-q} V^* \), we proved the following theorem in section four.

**Theorem 1.2.** Let \( M \) be a complete Kähler manifold, \( V \) be a holomorphic bundle and \( s \) be a holomorphic section of \( V \) with compact zero loci. Assume that \( \psi \in \Gamma(M, \det V \otimes \det \Omega_M) \) is nowhere vanishing. Then the above pairing \( \langle -, - \rangle \) is non-degenerate. Therefore for \( -n \leq k \leq n \),
\[ H^k_c(M, s) \cong H^{-k}_c(M, s)^\vee. \]

An interesting application of the above theorem is the follow vanishing theorem.

**Theorem 1.3.** Let \( M \) be a Stein manifold, \( V \) be a holomorphic bundle and \( s \) be a holomorphic section of \( V \) with compact zero loci. Assume that \( \psi \in \Gamma(M, \det V \otimes \det \Omega_M) \) is nowhere vanishing. Then
\[ H^k(M, s) = 0, \quad k \neq 0. \]

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2. Cohomology with compact support

In this section we study the cohomology with compact support. Let us introduce more operators. Fix a Hermitian metric \( h_V \) on \( V \). For \( s \) is nonzero on \( U := M \setminus Z \), we can form the following smooth section
\[ \tilde{s} := \frac{(s, s)_{h_V}}{(s, s)_{h_V}} \in \Gamma(U, A^{0,0}(V^*)). \]

It associates a map
\[ \tilde{s} \wedge : \Gamma(U, A^{0,i}(\wedge^j V^*)) \rightarrow \Gamma(U, A^{0,i}(\wedge^{j+1} V^*)). \]

To distinguish it in later calculation, we denote \( T_s := \tilde{s} \wedge : E_U^{\bullet, \bullet} \rightarrow E_U^{\bullet, \bullet + 1}, \) where \( E_U^{\bullet, \bullet} := \Gamma(U, A^{0, \bullet}(\wedge^{\bullet} V^*)). \)
The injection $j : U \to M$ induces the restriction $j^* : E_M^{\bullet,\bullet} \to E_U^{\bullet,\bullet}$. Let $\rho$ be a smooth cut-off function on $M$ such that $\rho|_{U_1} \equiv 1$ and $\rho|_{M \setminus U_2} \equiv 0$ for some relatively compact open neighborhoods $U_1 \subset \overline{U_1} \subset U_2$ of $Z$ in $M$.

We define the degree of an operator to be its change on the total degree of elements in $E_M(E_U)$. Then $\overline{\partial}$ and $T_s$ are of degree 1 and $-1$ respectively, and $[\overline{\partial}, T_s] = \overline{\partial} T_s + T_s \overline{\partial}$ is of degree 0. Consider two operators introduced in [3, (3.1), (3.2)] or [11, page 11]

\begin{align}
(2.1) \quad T_\rho : E_M \to E_{c,M} \quad T_\rho(\alpha) := \rho \alpha + (\overline{\partial} \rho) T_s \frac{1}{1 + [\overline{\partial}, T_s]} (j^* \alpha) \\
\text{and} \quad (2.2) \quad R_\rho : E_M \to E_M \quad R_\rho(\alpha) := (1 - \rho) T_s \frac{1}{1 + [\overline{\partial}, T_s]} (j^* \alpha).
\end{align}

Here as an operator

$$
\frac{1}{1 + [\overline{\partial}, T_s]} := \sum_{k=0}^{\infty} (-1)^k [\overline{\partial}, T_s]^k
$$

is well-defined since $[\overline{\partial}, T_s]^k(\alpha) = 0$ whenever $k > n$. Clearly $T_\rho$ is of degree zero and $R_\rho$ is of degree $-1$. Also $R_\rho(E_{c,M}) \subset E_{c,M}$ by definition.

**Lemma 2.1.** $[\overline{\partial}_s, R_\rho] = 1 - T_\rho$ as operators on $E_M$.

**Proof.** It is direct to check that

\begin{equation}
(2.3) \quad [\iota_s, T_s] = 1 \quad \text{on } E_U.
\end{equation}

Moreover,

$$
[P, [\overline{\partial}, T_s]] = 0
$$

for $P$ being $\iota_s, \overline{\partial}$ or $T_s$. Therefore, we have

\begin{align*}
[\overline{\partial}_s, R_\rho] &= [\overline{\partial}_s, 1 - \rho] T_s \frac{1}{1 + [\overline{\partial}, T_s]} j^* + (1 - \rho) [\overline{\partial}_s, T_s] \frac{1}{1 + [\overline{\partial}, T_s]} j^* \\
&= - (\overline{\partial} \rho) T_s \frac{1}{1 + [\overline{\partial}, T_s]} j^* + (1 - \rho) j^* \\
&= - (\overline{\partial} \rho) T_s \frac{1}{1 + [\overline{\partial}, T_s]} j^* + (1 - \rho) = 1 - T_\rho.
\end{align*}

\hfill \Box

**Proposition 2.2.** The embedding $(E_{c,M}, \overline{\partial}_s) \to (E_M, \overline{\partial}_s)$ is a quasi-isomorphism.

**Proof.** By Lemma 2.1 $H^*(E_M/E_{c,M}, \overline{\partial}_s) \equiv 0$, and thus the proposition follows. \hfill \Box

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1 As a notation convention, we always denote $[,]$ for the graded commutator, that is for operators $A, B$ of degree $|A|$ and $|B|$, the bracket is given by

$$
[A, B] = AB - (-1)^{|A||B|} BA.
$$
In this section, we assume that $M$ is a Kähler manifold with metric $h_M$. Let $h_V$ be a Hermitian metric on $V$. Let us denote by $F^{p,q}(\wedge^l V^*)$ the space of all measurable $(p,q)$ forms on $M$ evaluate in $\wedge^l V^*$. The global $L^2$ inner product of $(p,q)$-forms $\eta, \xi \in F^{p,q}(\wedge^l V^*)$ is defined as:

$$(\eta, \xi) := \int_M \langle \eta, \xi \rangle dv_M$$

Where $dv_M$ is the volume form on $M$ and $\langle, \rangle$ is induced by $h_M$ and $h_V$. Denote by $F^{p,q}_2(\wedge^l V^*) := \{ \alpha \in F^{p,q}(\wedge^l V^*) | (\alpha, \alpha) < \infty \}$ the space of square integral forms. $F^{p,q}_2(\wedge^l V^*)$ is a Hilbert space. Let

$$F^k := \bigoplus_{i-j=k} F^{0,i}_2(\wedge^j V^*).$$

The inner products of $F^{p,q}(\wedge^l V^*)$ induce an inner product on $F^k$. The operator $\partial_s$ can be extended to a closed subspace of $F^k$. We defined the formal adjoint of $\partial_s$:

$$\partial_s^* : F^k \rightarrow F^{k-1}$$

by the formula

$$(\partial_s^* \alpha, \beta) = (\alpha, \partial_s \beta),$$

where $\alpha \in E^k_{c,M}$ and $\beta \in E^{k-1}_{c,M}$. Then we consider the space of closed $L^2$ closed forms

$$Z^k_2 := \{ \alpha \in F^k | \partial_s \alpha = 0 \}$$

where it is understood that the equation $\partial_s \alpha$ holds weakly that is to say $\forall \beta \in E^{k+1}_{c,M}, (\alpha, \partial_s \beta) = 0$.

Thus we have:

$$Z^k_2 = (\partial_s E^{k+1}_{c,M})^\perp,$$

hence $Z^k_2$ is a closed space of $F^k$. We define

$$H^k(M, s) = (\partial_s E^{k+1}_{c,M})^\perp \cap (\partial_s E^{k-1}_{c,M})^\perp = Z^k_2 \cap \{ \alpha \in F^k | \partial_s \alpha = 0 \} = \{ \alpha \in F^k | \partial_s \alpha = 0, \partial_s^* \alpha = 0 \}.$$

Because the operator $\partial_s + \partial_s^*$ is elliptic, we have by elliptic regularity: $H^k(M, s) \subset E^k_{c,M}$. For $\alpha \in E^{k-1}_{c,M}$, $\beta \in E^{k+1}_{c,M}$, $(\partial_s \alpha, \partial_s^* \beta) = (\partial_s \partial_s^* \alpha, \beta) = 0$, hence

$$\partial_s E^{k+1}_{c,M} \perp \partial_s^* E^{k-1}_{c,M},$$

and we get a Hodge-de Rham decomposition of $F^k$

$$F^k = H^k(M, s) \oplus \partial_s E^{k-1}_{c,M} \oplus \partial_s^* E^{k+1}_{c,M},$$

where the closures are taken for the $L^2$ topology.

Denote by $\Box_s = \partial_s \partial_s^* + \partial_s^* \partial_s$.

**Proposition 3.1.** When $M$ is a complete Kähler manifold, then

$$H^k(M, s) = \{ \alpha \in F^k | \partial_s \alpha = 0 \}$$
Proof. Clearly we only need to check the inclusion:
\[ \{ \alpha \in \mathcal{F}^k | \square_s \alpha = 0 \} \subset \mathcal{H}^k(M, s). \]
Because \( M \) is a complete Kähler manifold, there exists an exhaustive sequence \( K_j \) of compact sets of \( M \) and function \( \chi_j \in C^\infty(M) \) such that, \( \chi_j = 1 \) on a neighborhood of \( K_j \), support of \( \chi_j \subset K_{j+1}^0, 0 \leq \chi_j \leq 1 \) and \( |d\chi_j| \leq \frac{1}{2^j} \), see [4, Lemma 2.4, page 366].

\[
\begin{align*}
& (\chi_j \overline{\partial_s} \alpha, \chi_j \overline{\partial_s} \alpha) + (\chi_j \overline{\partial_s} \alpha, \chi_j \overline{\partial_s} \alpha) \\
= & (\chi_j \overline{\partial_s} \alpha, \overline{\partial_s} \alpha) + (\alpha, \overline{\partial_s}(\chi_j \overline{\partial_s} \alpha)) \\
= & (\overline{\partial_s}(\chi_j \overline{\partial_s} \alpha), \overline{\partial_s} \alpha) + (\alpha, \chi_j \overline{\partial_s} \alpha) \\
& - 2(\chi_j \overline{\partial_s} \alpha, \overline{\partial_s} \alpha) + 2(\alpha, \chi_j \overline{\partial_s} \alpha) \\
= & (\chi_j \overline{\partial_s} \alpha, \square_s \alpha) - 2(\chi_j \overline{\partial_s} \alpha, \overline{\partial_s} \alpha) + 2(\alpha, \chi_j \overline{\partial_s} \alpha) \\
\leq & (\chi_j \overline{\partial_s} \alpha, \square_s \alpha) - 2(\chi_j \overline{\partial_s} \alpha, \overline{\partial_s} \alpha) + 2(\alpha, \chi_j \overline{\partial_s} \alpha) + 2(\chi_j \overline{\partial_s} \alpha, \overline{\partial_s} \alpha) + \frac{1}{2^j - 1}(\alpha, \alpha).
\end{align*}
\]
Thus we have
\[
(3.3) \quad (\chi_j \overline{\partial_s} \alpha, \chi_j \overline{\partial_s} \alpha) \leq \frac{1}{1 - 2^{-j}} \left( (\chi_j \overline{\partial_s} \alpha, \square_s \alpha) + 2^{1-j}(\alpha, \alpha) \right).
\]
If \( \square_s \alpha = 0 \), then by dominated convergence theorem, taking limit as \( j \to \infty \) in (3.3), we see \( \overline{\partial_s} \alpha = 0 \) and \( \overline{\partial_s} \alpha = 0 \).

\[ \square \]

**Theorem 3.2.** When \( M \) is a complete Kähler manifold, then
\[
(3.4) \quad \mathbb{H}^k_c(M, s) \cong \mathcal{H}^k(M, s)
\]

**Proof.** Because \( \mathcal{E}_c^k \subset \mathcal{F}^k \), it induced a map \( \Phi : \mathbb{H}^k_c(M, s) \to \mathcal{H}^k(M, s) \). If \( \alpha \in \mathcal{E}_{c,M}^k \) is \( \overline{\partial}_s \) closed and \( \Phi(\alpha) = 0 \in \mathcal{H}^k(M, s) \), by the decomposition (3.2), \( \alpha \in \mathcal{E}_{c,M} \), thus \( \alpha \) is zero in \( \mathbb{H}^k_c(M, s) \). Therefore \( \Phi \) is injective. Because \( \mathcal{H}^k(M, s) \subset \mathcal{E}_c^k \) by elliptic regularity, the surjective is from Lemma 2.1. \( \square \)

4. **Duality**

In this section we assume that there exists a nowhere vanishing holomorphic section \( \psi \in \Gamma(M, det V \otimes det \Omega_M) \). It induced an isomorphism from \( det V^* \to det \Omega_M \). Choosing a Hermitian metric \( h_V \) on \( V \) such that the induced metric \( | \cdot |_n \) on \( det V \otimes det \Omega_M \) satisfies \( |\psi|_n = 1 \). Recall in section one \( B \) is a graded commutative algebra extending the wedge products of \( \Omega^* \) and \( \wedge^* \).

The degree of \( \alpha \in \Omega^{i,j}(\wedge^* V^*) \) is \( \varepsilon \alpha := i + j - l \).

Given \( u \in \Omega^{i,j}(\wedge^k V) \) and \( k \geq l \), we define
\[
(4.1) \quad (u \wedge \theta, \nu^*) = (-1)^{ij(l+q)(p+q)}\langle u, \theta \wedge \nu^* \rangle, \quad \forall \nu^* \in A^p(\wedge^{k-l} V^*).
\]

The properties of the contraction can be found in the Appendix of [3]. Then we can defined the following map
\[
* : \wedge^0 \Omega_M \otimes \wedge^q V^* \to \wedge^{0, -p} \Omega_M \otimes \wedge^{n-q} V^*
\]
by

\[ \langle \alpha, \beta \rangle dv_M = \psi_J(\alpha \wedge \ast V \beta). \]

This \( \ast_V \) operator was first introduced by Feng and Ma in [9, Section 3] when \( M \) is compact. Let \( \{\omega^i\}_{i=1}^n \) and \( \{\mu^i\}_{i=1}^n \) be the orthonormal base of \( \Omega_M \) and \( V^* \) for a local chart of \( M \), and \( \{\mu_i\}_{i=1}^n \) be the dual orthonormal base of \( V \), then

\[ dv_M = (-1)^{(n+1)/2}(\sqrt{-1})^n \omega^1 \wedge \cdots \wedge \omega^n \wedge \bar{\omega}^1 \wedge \cdots \wedge \bar{\omega}^n, \]

and

\[ \psi = f \omega^1 \wedge \cdots \wedge \omega^n \wedge \mu_1 \wedge \cdots \wedge \mu_n \]

with \(|f| = 1\). If

\[ \beta = \bar{\omega}^1 \wedge \cdots \wedge \bar{\omega}^p \wedge \mu_1 \wedge \cdots \wedge \mu^q, \]

then

\[ \ast_V \beta = (-1)^{(n-p)q}(\sqrt{-1})^n f^{-1} \bar{\omega}^{p+1} \wedge \cdots \wedge \bar{\omega}^n \wedge \mu^{q+1} \wedge \cdots \wedge \mu^n. \]

Thus \( \ast_V \ast_V \beta = (-1)^{p-q} \beta \), for all \( \beta \in E_2^{p,q}(\wedge q V^*). \)

Proposition 4.1.

\[ \overline{\partial}^* = \ast_V \overline{\partial} \ast; \iota^*_s = \ast_V \iota_s \ast V; \ast V \square_s = \square_s \ast V. \]

Proof. For \( \alpha \in E_{c,M}^{p,q+1} \) and \( \beta \in E_{c,M}^{p,q} \), by [3, Lemma 5.1]

\[ (\iota_s \alpha, \beta) = \int_M \psi_J(\iota_s \alpha \wedge \ast V \beta) \]

\[ = \int_M \psi_J(\iota_s (\alpha \wedge \ast V \beta)) - (-)^{2\alpha} \int_M \psi_J(\alpha \wedge \iota_s \ast V \beta) \]

\[ = \int_M \psi_J(s \wedge (\alpha \wedge \ast V \beta)) - (-)^{2\alpha} \int_M \psi_J(\alpha \wedge \iota_s \ast V \beta) \]

\[ = (-)^{\alpha+1} \int_M \psi_J(\alpha \wedge \iota_s \ast V \beta) \]

\[ = (-)^{\alpha+1+p-q} \int_M \psi_J(\alpha \wedge \ast V \iota_s \ast V \beta) \]

\[ = (\alpha, \ast_V \iota_s \ast V \beta). \]

Thus \( \iota^*_s = \ast_V \iota_s \ast V. \)

For \( \alpha \in E_{c,M}^{p,q+1} \) and \( \beta \in E_{c,M}^{p,q} \), by [3, Lemma 5.2]

\[ (\overline{\partial} \alpha, \beta) = \int_M \psi_J(\overline{\partial} \alpha \wedge \ast V \beta) \]

\[ = \int_M \psi_J(\overline{\partial} (\alpha \wedge \ast V \beta)) - (-)^{2\alpha} \int_M \psi_J(\alpha \wedge \overline{\partial} \ast V \beta) \]

\[ = \int_M \overline{\partial} (\psi_J(\alpha \wedge \ast V \beta)) - (-)^{2\alpha} \int_M \psi_J(\alpha \wedge \overline{\partial} \ast V \beta) \]

\[ = (-)^{\alpha+1} \int_M \psi_J(\alpha \wedge \overline{\partial} \ast V \beta) \]

\[ = (-)^{\alpha+1+p-q} \int_M \psi_J(\alpha \wedge \ast V \overline{\partial} \ast V \beta) \]

\[ = (\alpha, \ast_V \overline{\partial} \ast V \beta). \]
Thus $\overline{\partial} = *_V \overline{\partial} *_V$. By definition we have
\[ *_V \Box_s = *_V \overline{\partial}_s *_V \overline{\partial}_s + *_V \overline{\partial}_s *_V + *_V *_V *_V *_V, \]
and
\[ \Box_s *_V = \overline{\partial}_s *_V + \overline{\partial}_s ^* *_V = \overline{\partial}_s *_V + *_V *_V + *_V *_V *_V. \]
For $\alpha \in E^0_2 (\wedge^q V^*)$, $*_V *_V *_V *_V \alpha = *_V *_V *_V *_V \alpha$, then
\[ *_V *_V = *_V *_V. \]

**Theorem 4.2.** The bilinear paring
\[ \mathcal{H}^k (M, s) \times \mathcal{H}^{-k} (M, s) \to \mathbb{C}, \quad (\alpha, \beta) \to \int_M \psi \lrcorner (\alpha \wedge \beta) \]
is non-degenerate.

**Proof.** Let $\beta \in F^k$ be a $\overline{\partial}_s$-closed form, $\alpha \in F^{-k-1}$ and $\overline{\partial}_s \alpha \in F^{-k}$, then
\[ \int_M \psi \lrcorner (\overline{\partial}_s \alpha \wedge \beta) = \int_M \psi \lrcorner (\overline{\partial}_s (\alpha \wedge \beta)) = \int_M \overline{\partial} \left( \psi \lrcorner (\alpha \wedge \beta) \right) = 0, \]
the last equality is obtained by Stokes theorem [10, p141, Thm]. The above bilinear pairing is well defined because it doesn’t depend on the representation form of the cohomology class. By Proposition 4.1, $\alpha \in \mathcal{H}^k (M, s)$ if and only if $*_V \alpha \in \mathcal{H}^{-k} (M, s)$. The theorem is then a consequence of the fact that the integral $(\alpha, \alpha) = \int_M \psi \lrcorner (\alpha \wedge *_V \alpha)$ does not vanish unless $\alpha = 0$. \[ \square \]

**Corollary 4.3.** Let $M$ be a complete Kähler manifold, $V$ be a holomorphic bundle and $s$ be a holomorphic section of $V$ with compact zero loci. Assume that $\psi \in \Gamma (M, \det V \otimes \det \Omega_M)$ is nowhere vanishing. Then the pairing $(1.3)$ is non-degenerate.

**Proof.** Choosing the Hermitian metric $h_V$ on $V$ such that the induced metric $| \cdot |_{ln}$ on $\det V \otimes \det \Omega_M$ satisfies $| \psi |_{ln} = 1$. Then the result is obtained by Theorem 3.2, Proposition 3.1 and Theorem 4.2. \[ \square \]

**Theorem 4.4.** Let $M$ be a Stein manifold, $V$ be a holomorphic bundle and $s$ be a holomorphic section of $V$ with compact zero loci. Assume that $\psi \in \Gamma (M, \det V \otimes \det \Omega_M)$ is nowhere vanishing. Then
\[ H^k (M, s) = 0, \quad k \neq 0. \]

**Proof.** $E^\bullet \bullet_M$ is a double complex with horizontal operator $\iota_s$ and vertical operator $\overline{\partial}$. We consider the spectral sequence associated to the descending filtration
\[ \tilde{\mathcal{E}}^k (E^\bullet \bullet_M) = \bigoplus_{i \leq n-k} E^{\bullet, i}_M. \]
Since $M$ is Stein manifold, the $E_1$-term is given by holomorphic sections $\bigoplus_{i=0}^n \mathbb{H}^0(M, \wedge^i V^*)$ with the differential $\iota_s$. Thus

$$\mathbb{H}^k(M, s) = 0, \quad k > 0.$$  

Then applying Theorem 4.3 we prove

$$\mathbb{H}^k(M, s) = 0, \quad k \neq 0.$$  

□

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