Nuclear Density-Dependent Effective Coupling Constants in the Mean-Field Theory

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Abstract

It is shown that the equation of state of nuclear matter can be determined within the mean-field theory of $\sigma\omega$ model provided only that the nucleon effective mass curve is given. We use a family of the possible nucleon effective mass curves that reproduce the empirical saturation point in the calculation of the nuclear binding energy curves in order to obtain density-dependent effective coupling constants. The resulting density-dependent coupling constants may be used to study a possible equation of state of nuclear system at high density or neutron matter. Within the constraints used in this paper to $M^*$ of nuclear matter at saturation point and zero density, neutron matter of large incompressibility is strongly bound at high density while soft neutron matter is weakly bound at low density. The study also exhibits the importance of surface vibration modes in the study of nuclear equation of state.
I. INTRODUCTION

The relativistic $\sigma\omega$ model in the mean-field theory has been very successful in describing the properties of nuclear matter and finite nuclei [1–3]. It provides a useful framework to evaluate the one-baryon-loop vacuum fluctuation effects at finite density. In the mean-field approximation one can obtain the baryon self-energies in nuclear matter taking into account the Dirac sea modified due to the presence of valence nucleons. The energy density contains the modified negative energies, the correction that removes half of the interaction energy between the states in the Dirac sea, and the interaction energy between the valence particles and those in negative energy states.

Recently, Brockmann and Toki [6] reported a relativistic density-dependent Hartree calculation for finite nuclei, where the results agree very well with experiment. They employ $\sigma$ and $\omega$ exchange for the effective interaction without taking into account the vacuum fluctuation effects and adjust the density-dependent effective coupling constants such that the relativistic Hartree calculation reproduces the nuclear matter Dirac-Brueckner-Hartree-Fock (DBHF) results based on a realistic $NN$ interaction. The density-dependent coupling constants may effectively contain exchange term and contributions of iterated meson exchange. The effective scalar field for instance, contains large contribution from two pion exchange and the iterated exchange of $\pi$- and $\rho$-mesons contribute to the effective vector fields.

We show that the energy per nucleon curve can be calculated in nuclear matter if the nucleon effective mass is given in the relativistic Hartree approach (RHA) with the inclusion of the vacuum fluctuation effects. A brief summary about RHA with density-dependent effective coupling constants is given in Sections II and III. A resonable family of the nucleon effective mass curves is used to collect the density-dependent coupling constant curves that reproduce saturation properties of nuclear matter. These curves define the upper and lower bounds of the density-dependent coupling constants in RHA where one-baryon-loop vacuum effects are taken into account. The resulting range of the density-dependent coupling constant would restrict a possible equation of state of a neutron matter or a nuclear system at
high density. It also would limit a possible nuclear incompressibility at low density such as at the surface of finite nuclei which is important in study of giant resonances related with surface vibration.

II. THE APPLICATION OF THE THERMODYNAMIC LAW IN NUCLEAR MATTER

The nucleon self-energy in nuclear matter has the general form by assuming parity conservation, time-reversal invariance, and hermiticity [2]:

\[
\Sigma(k) = \Sigma^s(k, k^0, k_F) - \gamma^0 \Sigma^0(k, k^0, k_F) + \gamma \cdot k \Sigma^v(k, k^0, k_F),
\]

(1)

where \( k = (k^0, k) \) is the nucleon four momentum and \( \gamma^\mu \) is Dirac matrix. \( k_F \) is Fermi momentum of nucleon in nuclear matter and \( \vec{k} = |k| \), so that \( 0 < k \leq k_F \). The self-energy can be reduced within the Dirac equation to a potential

\[ U(k, k_F) = A(k, k_F) + \gamma^0 B(k, k_F) \]

(2)

where the scalar potential \( A \) and the vector potential \( B \) are given by

\[
A(k, k_F) = \frac{\Sigma^s - M \Sigma^v}{1 + \Sigma^v}
\]

(3)

\[
B(k, k_F) = \frac{E \Sigma^v - \Sigma^0}{1 + \Sigma^v}.
\]

(4)

Here, \( M \) is the free nucleon mass and \( E \) is the single particle energy of a nucleon in nuclear matter including the rest mass energy of a free nucleon. In a mean field approximation, \( A \) is independent of \( \vec{k} \) and define the nucleon effective mass \( M^*(k_F) \) as

\[
M^*(k_F) = M + A(k_F).
\]

(5)

The nucleon in nuclear matter behaves like a free particle with energy \( E^* \) and mass \( M^* \), where the relation

\[
E^* = E - B = \frac{E + \Sigma^0}{1 + \Sigma^v} = \sqrt{\vec{k}^2 + M^{*2}}
\]

(6)

\[ \]
holds in the Dirac equation.

In the thermodynamic treatment, we shall regard nuclear matter as the system containing the same kind of particles, namely single-component system. Then, the fundamental thermodynamic law is given by

\[ dE_{\text{int}} = TdS - PdV + \mu dN, \quad (7) \]

which describes the change in the internal energy \( E_{\text{int}} \) arising from small change in the entropy \( S \), the volume \( V \), and the number of nucleons \( N \) in \( V \). We confine ourselves to the ground state nuclear matter here, so the temperature \( T \) is zero. Furthermore, the Hugenholtz-Van Hove (HV) theorem states that the chemical potential \( \mu \) equals the highest occupied single particle energy \[ \text{[7]} \]. Therefore, we have

\[ \frac{P}{\rho_B} + E_A = e(M^*, k_F), \quad (8) \]

where \( e(M^*, k) \equiv E - M \) is the single particle energy, \( \rho_B = dN/dV \) the baryon density and \( E_A = dE_{\text{int}}/dN \) the energy per nucleon.

Using the relation

\[ P = \rho_B^2 \frac{dE_A}{d\rho_B} \quad (9) \]

and

\[ \rho_B = \frac{\gamma}{6\pi^2} k_F^3 \quad (10) \]

where \( \gamma \) is the spin-isospin degeneracy, it immediately follows from Eq. (8) that

\[ E_A + \frac{k_F dE_A}{3} dk_F = e(M^*, k_F). \quad (11) \]

On the other hand, from Eqs.(4) and (6) the single particle energy \( e(M^*, k) \) can be expressed as

\[ e(M^*, k) \equiv E - M = \sqrt{k^2 + M^*2} + B - M. \quad (12) \]

Substituting Eq. (12) for \( k = k_F \) into Eq. (11), we have

\[ E_A + \frac{k_F dE_A}{3} dk_F = \sqrt{k_F^2 + M^*2} + B - M. \quad (13) \]
III. THE NUCLEON BINDING ENERGY IN NUCLEAR MATTER

The role of quantum corrections is crucial in the application of hadronic fields theories to nuclear matter. We will use the relativistic Hartree approximation (RHA) which includes one-baryon-loop vacuum effects. The RHA involves divergent integrals over the occupied negative energy states. The divergences can be rendered finite by including the appropriate counterterms in the MFT Lagrangian and by defining a set of renormalization conditions. The vacuum fluctuation corrections to the binding energy per nucleon can be obtained by carrying out the renormalization procedure. The result of the binding energy per nucleon with the vacuum corrections is given by

\[ E_A = \frac{B}{2} - M + \frac{m_s^2}{2g_s^2} \left( M - M^* \right)^2 + \gamma \frac{\int_0^{k_F} d^3k \sqrt{k^2 + M^{*2}} + \Delta E_{VF}}{\rho_B} \]  

where

\[ \Delta E_{VF} = -\frac{3}{8k_F^3} \left[ M^{*4} \ln \left( \frac{M^*}{M} \right) + M^3(M - M^*) - \frac{7}{2} M^2(M - M^*)^2 + \frac{13}{3} M(M - M^*)^{3} - \frac{25}{12} (M - M^*)^{4} \right]. \]

No counterterm for vector self-energy to be finite is needed and therefore the RHA result for the vector self-energy term is identical to that of the MFT which is given by

\[ B = \frac{g_v^2}{m_v^2} \rho_B. \]

For given coupling constants, \((g_s/m_s)^2\) and \((g_v/m_v)^2\) as functions of \(k_F\), Eq. (14) is a function of the effective mass \(M^*\) and the Fermi momentum \(k_F\). The nuclear matter saturation is determined by minimizing \(E_A\) in \(k_F-M^*\) space.

For each value of \(k_F\), \(E_A\) given by Eq. (14) should take a minimum at the right value of \(M^*\), so that the partial derivative of \(E_A\) with respect to \(M^*\) vanishes:

\[ M - M^* = \frac{g_s^2}{m_s^2} \gamma \frac{\int_0^{k_F} d^3k \frac{M^*}{\sqrt{k^2 + M^{*2}}}}{(2\pi)^3} \]

\[ -\frac{g_v^2}{m_v^2} \frac{1}{4} \frac{\gamma}{\pi^2} \left[ M^{*3} \ln \left( \frac{M^*}{M} \right) - M^2(M^* - M) - \frac{5}{2} M(M^* - M)^2 - \frac{11}{6} (M^* - M)^3 \right]. \]
By using Eq. (17) the factor $m_s^2/g_v^2$ can be eliminated in Eq. (14) to have the expression of $E_A$ as

$$E_A = \frac{B}{2} - M + \frac{3}{4k_F^2} \left[ \left( k_F^2 + k_F M M^* - \frac{1}{2} k_F M^* s \right) \sqrt{k_F^2 + M^2} + \left( \frac{1}{2} M^* s - M M^* s \right) \ln \left( \frac{M^*}{M} \right) - \frac{1}{2} M^2 (M - M^*) + \frac{5}{4} M^2 (M - M^*)^2 + \frac{1}{3} M (M - M^*)^3 - \frac{19}{24} (M - M^*)^4 \right]$$

(18)

We combine Eq. (18) with Eq. (13) to eliminate $B$ (i.e., $g_v^2/m_v^2$) and obtain the following relation:

$$\frac{d}{dk_F} \left( \frac{E_A}{k_F} \right) = \frac{3M + 3\sqrt{k_F^2 + M^2}}{k_F^4} - \frac{9}{2k_F^2} \left[ \left( k_F^2 + k_F M M^* - \frac{1}{2} k_F M^* s \right) \sqrt{k_F^2 + M^2} + \left( \frac{1}{2} M^* s - M M^* s \right) \ln \left( \frac{M^*}{M} \right) - \frac{1}{2} M^2 (M - M^*) + \frac{5}{4} M^2 (M - M^*)^2 + \frac{1}{3} M (M - M^*)^3 - \frac{19}{24} (M - M^*)^4 \right]$$

(19)

One can solve Eq. (19) numerically for $M^*(k_0)$ for a given value of $E_A(k_0) = -15.75$ MeV by using the fact that $dE_A/dk_F = 0$ at $k_F = k_0$ where $k_0$ is the saturation density given by 1.42 fm$^{-1}$.

**IV. CALCULATION OF DENSITY-DEPENDENT COUPLING CONSTANTS**

The scalar meson coupling constant and the nucleon effective mass are related to each other through Eq. (17) with $\gamma = 4$. The density-dependent coupling constant curve for the scalar meson can be obtained immediately when $M^*(k_F)$ is given:

$$\frac{g_s^2}{m_s^2} = \frac{\pi^2 (M - M^*)}{F(k_F, M^*)}$$

(20)

where

$$F(k_F, M^*) = \left[ M^* k_F \sqrt{k_F^2 + M^2} - M^3 \ln \left( \frac{k_F + \sqrt{k_F^2 + M^2}}{M^*} \right) \right] - M^3 \ln \left( \frac{M^*}{M} \right) + M^2 (M^* - M) + \frac{5}{2} M (M^* - M)^2 + \frac{11}{6} (M^* - M)^3.$$  

(21)
Eq. (21) shows there is a discontinuity in $g_s^2$ changing from positive infinity to a negative infinity at some $k_F$ larger than $k_0$. This discontinuity results in the change of $g_s$ from real to a pure imaginary and from attractive n-n scalar coupling to a repulsive n-n scalar coupling. To describe a long range nuclear attraction and a short range nuclear repulsion within the Walecka’s $\sigma$-$\omega$ model, the coupling constants $g_s$ and $g_v$ must be real. If we assume that the same property of nuclear interaction, i.e., the attractive scalar interaction and the repulsive vector interaction holds at high density too, then $g_s$ and $g_v$ must stay real. As far as $g_s$ is real $F(k_F, M^*)$ should be positive as can be seen in Eq. (20). One can limit $M^*(k_F)$ such that $F(k_F, M^*) > 0$ by obtaining the solution of $F(k_F, M^*) = 0$ for $M^*(k_F)$ which plays a role for the boundary line. The boundary line is shown as the dash-double-dot line in Fig. 1. The boundary line is denoted by $M'(k_F)$ as a function of $k_F$. The region below the boundary line $M'(k_F)$ in Fig. 1 is forbidden for the nucleon effective mass. Eq. (19) can be integrated numerically to obtain $E_A(k_F)$ if $M^*(k_F)$ is known.

We choose the form of $M^*$ as a function of $k_F$ in the region above the boundary line $M'(k_F)$ in Fig. 1:

$$M^*(k_F) = \begin{cases} \frac{M}{1 + \frac{M - M^*(k_0)}{M^*(k_0)} \left(\frac{k_F}{k_0}\right)^\alpha}, & k_F < k_0 \\ \frac{1 + \beta}{1 + \beta \left(\frac{k_F}{k_0}\right)} \left[M^*(k_0) - M'(k_F)\right] + M'(k_F), & k_F > k_0 \end{cases}$$ (22)

where $\alpha$, $\beta$ and $\delta$ are related by

$$\delta = \frac{1}{M^*(k_0) - M'(k_0)} \left[\frac{2M^*(k_0)\left\{M - M^*(k_0)\right\}\alpha}{M} + 2k_0 \frac{dM'}{dk_F}igr|_{k_F=k_0}\right]$$

$$+ \frac{\{2M^*(k_0) - M\}\alpha}{M}$$ (23)

$$\beta = \frac{M^*(k_0)\left\{M - M^*(k_0)\right\}\alpha}{M\left\{M^*(k_0) - M'(k_0)\right\}\delta - M^*(k_0)\left\{M - M^*(k_0)\right\}\alpha}$$ (24)

where we choose $\alpha > 1$ such that $dM^*/dk_F = 0$ at $k_F = 0$. We note here that $dM^*/dk_F = -\infty$ if $\alpha < 1$. As discussed above we have chosen the $M^*$ curve in the region above the boundary line $M'(k_F)$ in the $M^* - k_F$ plane to confine our calculations to real $g_s$. Eq. (22) for $M^*(k_F)$ leads to the positive effective scalar coupling constants $g_s^2/m_s^2$ for all $\alpha > 1.0$ as
is required. Since $g_v$ must also be real, the region of the parameter $\alpha$ ($\alpha \gtrsim 3.5$) that leads to negative $g_v^2/m_v^2$, as will be shown, should be discarded. Therefore, for the possible $M^*$ curve we have chosen Eq. (22) with $1.0 < \alpha < 3.5$. The other parameters $\beta$ and $\delta$ have been related to $\alpha$ in such a way that $M^*$ curve is differentiable at $k_F = k_0$ up to the second order. Eq. (22) has been chosen such that $M^*(0) = M$ and $M^*(k_0)$ leads to the empirical saturation point ($k_0 = 1.42$ fm$^{-1}$ and $E_A = -15.75$ MeV). The $M^*$ curves are shown for each value of $\alpha$ in Fig. 1.

In numerical calculations, $M^*$ given by Eq. (22) for a fixed $\alpha$ is substituted in Eq. (19) and the integration with respect to $k_F$ is performed to obtain the energy per nucleon such that the $E_A$ curve passes through the empirical saturation point. The nucleon binding energy curves obtained with $M^*(k_F)$ for different $\alpha$'s are plotted in Fig. 2. The calculated incompressibility of nuclear matter increases with increasing $\alpha$ ($K^{-1} = 61, 313$ and $565$ MeV for $\alpha = 1.0, 2.25$ and $3.5$, respectively).

Once $M^*(k_F)$ is given by Eq. (22) for a parameter $\alpha$, the scalar effective coupling constant $g_s^2/m_s^2$ can be obtained from Eq. (20). The $g_s^2/m_s^2$ curves for different $\alpha$'s are shown in Fig. 3. The effective coupling constant for the vector meson can be expressed in terms of $k_F$, $E_A$ and $M^*$ by using Eqs. (16) and (18). The calculated curves for $g_v^2/m_v^2$ are depicted in Fig. 4. The negative value of $g_v^2$ at low densities for $\alpha > 3.5$ gives the upper bound of $\alpha$.

Using the density-dependent coupling constants we obtained the equations of state of neutron matter as shown in Fig. 5, where these are compared with the equation of state of neutron matter with constant coupling constants [8]. The saturation density and incompressibility of neutron matter increase with the increasing $\alpha$. The calculated results show that neutron matter is bound in the regions of $1 < \alpha < 1.65$ and $2.79 < \alpha < 3.5$. For small $\alpha$, the scalar coupling constant $g_s^2/m_s^2$ and the vector coupling constant $g_v^2/m_v^2$ are much stronger at low density and weaker at high $k_F$ than the fixed coupling constants. Thus the neutron matter is bound weakly with small $k_F$ for small $\alpha$. On the other hand, for large $\alpha$, both the scalar and vector couplings are weaker than the fixed coupling constants at low density (small $k_F$) and the vector coupling constant $g_v^2/m_v^2$ is the similar order as the
fixed coupling constant at high density while the scalar coupling constant $g_s^2/m^2_s$ becomes much stronger than the fixed coupling constant at high density. This density dependence of the coupling constants for large $\alpha$ leads the neutron matter to be strongly bound (11.1 MeV binding energy per neutron). In the mid $\alpha$ value region, the neutron matter is unbound. However, in the region near $\alpha = 2.3$, a neutron matter has a resonance state at $k_F = k_0 = 1.42$ fm$^{-1}$ because the coupling constants $(g_s^2/m^2_s, g_v^2/m^2_v)$ for all the cases ($\alpha$’s) are required to be the same at the density.

V. CONCLUSION

The results (Figs. 1 – 4) show that, at low density, the density-dependent coupling constants are very sensitive to the choice of the effective mass. However, the nuclear equation of state $E_A(k_F)$ is quite insensitive to the coupling constants at low density. This behavior of insensitivity reflects the fact that the inter-nucleon separation is large at low density. Due to the requirement of the empirical saturation in our calculation, all the choices of the effective mass, of course, give the same energy and coupling constants at the saturation point. However the nuclear compressibilities are all different.

We can see from the results that the equation of state is quite sensitive to the coupling constants at high density as we expected. In addition, the results show that the equation of state of a nuclear system is also sensitive to the values of the coupling constants at the intermediate density, around half the saturation density. This behavior suggests us that the detailed study of surface vibration modes of finite nuclei is very important in extrapolating the nuclear equation of state to a high density. To pursue a research in this direction, we need further investigation of our starting point of calculation whereas here we started with the effective mass functions $M^*(k_F)$ just for simplicity. The minimum requirement in this search is that all the fits should pass through at $k_F = 0$ and $k_F = k_0$, i.e., a zero density limit and a nuclear saturation point.

Fig. 5 also shows that the density-dependent coupling constants allow the existence of
neutron matter within RHA. The equations of state of neutron matter with the density-dependent coupling constants show saturation points with different incompressibilities of neutron matter. The saturation points for different $\alpha$’s are compared with that for constant coupling constants [8] in Fig. 5. The saturation state appears at higher density and the incompressibility is higher if $\alpha$ takes on the larger value. We note here that both the resonance states with positive saturation energy and the bound states with negative saturation energy appear in the neutron matter calculations with the density-dependent coupling constants. The saturation energies are positive in the coupling constant region $1.65 < \alpha < 2.79$ where RHA does not allow a stable neutron star. However, for the regions $1.0 < \alpha < 1.65$ and $2.79 < \alpha < 3.5$ RHA provides bound states of neutron matter, and for $\alpha = 3.5$ the binding energy appears to be 11.1 MeV the incompressibility to be 1258 MeV at $k_F = 1.94$ fm$^{-1}$.

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FIGURES

FIG. 1. Nucleon effective masses $M^*$ as functions of Fermi momenta $k_F$ for nuclear matter. The $M^*$ curves are chosen as Eq. (22) with a parameter $\alpha$ such that they lead to the empirical saturation point in nuclear matter.

FIG. 2. Binding energies per nucleon in nuclear matter calculated from the nucleon effective masses given by Eq. (22).

FIG. 3. Density-dependent effective coupling constant curves for the $\sigma$-meson exchange. The constants $g_s^2/m_s^2$ are obtained by using Eq. (21) with the nucleon effective masses given by Eq. (22).

FIG. 4. Density-dependent effective coupling constant curves for the $\omega$-meson exchange.

FIG. 5. Binding energies per nucleon for neutron matter ($\gamma = 2$) calculated from the density-dependent coupling constants $g_s^2/m_s^2$ in Fig. 3 and $g_v^2/m_v^2$ in Fig. 4. The maximum binding energy per neutron is 11.1 MeV for $\alpha = 3.5$. 