An equality between entanglement and uncertainty

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In their seminal paper, Einstein, Podolsky and Rosen (EPR) show that an observer who is maximally entangled with the system to be measured can perfectly predict the outcome of two incompatible measurements. This feat stands in stark contrast to Heisenberg’s uncertainty principle which tells us that if the observer is not entangled with the system at all, then his ability to predict the outcomes of incompatible measurements such as position and momentum is extremely limited. The observations made by EPR and Heisenberg illustrate two extreme cases of the interplay between entanglement and uncertainty. On the one hand, no entanglement means that measurements will be maximally uncertain. Yet on the other hand, maximal entanglement means that there is no more uncertainty at all. Here we show that this apparent rift can be reconciled in that it is indeed possible to have an exact relation - an equality - between the amount of uncertainty and the amount of entanglement.

Heisenberg’s uncertainty principle forms one of the fundamental elements of quantum mechanics. Originally proven for measurements of position and momentum, it is one of the most striking examples of the difference between a quantum and a classical world [1]. Uncertainty relations today are probably best known in the form given by Robertson [2], who extended Heisenberg’s result to two arbitrary observables $X$ and $Z$. More precisely, Robertson’s relation states that when measuring the state $|\psi\rangle$ using either $X$ or $Z$, then

$$\Delta X \Delta Z \geq \frac{1}{2} \langle \psi | [X, Z] | \psi \rangle,$$

where $\Delta Y = \sqrt{\langle \psi | Y^2 | \psi \rangle - \langle \psi | Y | \psi \rangle^2}$ for $Y \in \{X, Z\}$ is the standard deviation resulting from measuring $|\psi\rangle$ with observable $Y$.

In the modern day literature, uncertainty is usually measured in terms of entropies [3, 4] (see [5] for a survey). One of the reasons this is desirable is that [1] makes no statement if $|\psi\rangle$ happens to give zero expectation on $[X, Z]$ [6]. To see how uncertainty can be quantified in terms of entropies, let us start with a simple example. Throughout, we let Alice ($A$) denote the system to be measured. For now, let us consider measuring a single qubit in the state $|\psi\rangle$ using two incompatible measurements given by the Pauli $\sigma_x$ or $\sigma_z$ eigenbases. We have from [6] that for any $\rho_A$

$$H(K|\Theta) = \frac{1}{2} \left[ H(K|\Theta = \sigma_x) + H(K|\Theta = \sigma_z) \right] \geq \frac{1}{2},$$

where $H(K|\Theta = \theta) = -\sum_k p_{k|\theta} \log p_{k|\theta}$ is the Shannon entropy of the probability distribution over measurement outcomes $k \in \{0, 1\}$ when we perform the measurement labeled $\theta$ on the state $\rho_A$, and each measurement is chosen with probability $p_\theta = 1/2$. To see that this is an uncertainty relation note that if one of the two entropies is zero, then [2] tells us that the other is necessarily non-zero, i.e., there is at least some amount of uncertainty. If we measure a $d_A$-dimensional system $A$ in two orthonormal bases $\theta_0 = \{ |x_0\rangle \}_{x=1}^{d_A}$ and $\theta_1 = \{ |x_1\rangle \}_{x=1}^{d_A}$ then the r.h.s. of (2) becomes $\log(1/c)$, where $c = \max_{x_0, x_1} |\langle x_0 | x_1 \rangle|^2$ [7]. The largest amount of uncertainty, i.e., the largest $\log(1/c)$, is thereby obtained when $|\langle x_0 | x_1 \rangle| = 1/\sqrt{d_A}$, that is, the two bases are mutually unbiased (MUB) [7].

When thinking about uncertainty, it is often illustrative to adopt the perspective of an “uncertainty game” [8, 9], commonly used in quantum cryptography [10]. In particular, we will think about uncertainty from the perspective of an observer called Bob holding a second system ($B$) whose task is to guess the outcome of the measurement on Alice’s system successfully. Bob thereby knows ahead of time what measurements could be made and the probability that a particular measurement setting is chosen. To help him win the game, Bob may even prepare $\rho_A$ himself, and Alice tells him which measurement she performed before he has to make his guess. The amount of uncertainty as measured by entropies can be understood as a limit on how well Bob can guess Alice’s measurement outcome - the more difficult
it is for Bob to guess the more uncertain Alice’s measurement outcomes are. If Bob is not entangled with \( A \) but only keeps classical information about the state, such as for example a description of the density operator \( \rho_A \), then (2) still holds even if we condition on Bob’s classical information \( B \) [11]. More precisely, we have \( H(K|\Theta_{B\text{classical}}) \geq 1/2 \) for any states or distribution of states that Bob may prepare.

### Uncertainty and entanglement

Another central element of quantum mechanics is the possibility of entanglement, and examples suggest that there is a strong interplay between entanglement and uncertainty. In particular, Einstein, Poldolsky and Rosen [9] observed, that if Bob is maximally entangled with \( A \) then his uncertainty can be reduced dramatically. To see this imagine that \( \rho_{AB} = |\Phi\rangle \langle \Phi| \) where \( |\Phi\rangle = (|00\rangle + |11\rangle)/\sqrt{2} \) is the maximally entangled state between \( A \) and \( B \). Since \( |\Phi\rangle \) is maximally correlated in both the \( \sigma_x \) and \( \sigma_z \) eigenbases, Bob can simply measure his half of the EPR pair in the same basis as Alice to predict her measurement outcome perfectly, winning the guessing game described above. This is precisely the effect observed in [9] and shows that the uncertainty relations of (1) and (2) are clearly inadequate to capture the interplay between entanglement and uncertainty. Fortunately, it is possible to extend the notion of uncertainty relations to take the possibility of entanglement into account [8, 12]. Such relations are known as uncertainty relations with quantum side information (here \( B \)). More precisely, it was shown [8] that if we measure \( A \) in two bases labeled \( \theta_0, \theta_1 \) then

\[
H(K|\Theta) = \frac{1}{2} [H(K|\Theta = \theta_0) + H(K|\Theta = \theta_1)] \geq \log(1/c) + H(A|B),
\]

where \( H(A|B) \) is the conditional von Neumann entropy of \( A \) given \( B \). If \( A \) and \( B \) are entangled, then \( H(A|B) \) can be negative. Indeed, \( H(A|B) = -\log d_A \) when \( \rho_{AB} \) is the maximally entangled state, in which case the lower bound in (3) becomes trivial. The uncertainty relation of (3) thus allows for the possibility that the uncertainty could be reduced in the presence of entanglement. It also provides us with a first clue to the relation between entanglement and uncertainty, that is, it tells us that a reduction in uncertainty implies the presence of entanglement. However, it does not tell us that the presence of entanglement really does lead to a significant reduction in uncertainty. Of course, if \( \rho_{AB} \) is close to the maximally entangled state then uncertainty is reduced by at least some amount, because two states which are close yield similar statistics when measured. Yet we will see below that this alone is insufficient for our purpose.

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2 More precisely, smoothing of the entropies is required for (4) to hold [10].
I. MAIN RESULT

Here, we prove that there exists an exact relation between entanglement and uncertainty. More precisely, we prove the following equality if we measure \( A \) in one of \( d_A + 1 \) possible mutually unbiased bases

\[
H_2(K|B\Theta) = \log(d_A + 1) - \log \left( 2^{-H_2(A|B)} + 1 \right),
\]

where \( H_2(A|B) = -\log \text{Tr} \left[ \rho_{AB}(\mathbb{1}_A \otimes \rho_B)^{-1/2} \rho_{AB}(\mathbb{1}_A \otimes \rho_B)^{-1/2} \right] \) is the conditional Rényi 2-entropy used in quantum cryptography [21], and \( K \) is the classical measurement outcome obtained by measuring \( A \) in the basis labeled \( \Theta = \theta \). For classical \( K \), it has an operational interpretation as \( H_2(K|B\Theta) = -\log P_{\text{guess}}^{\text{pg}}(K|B\Theta) \) given by the probability that Bob manages to guess Alice’s measurement outcome \( K \) using the pretty good measurement [22, 23] (see below) after he learns which measurement \( \Theta \) was made. For the general case we prove that

\[
H_2(A|B) = -\log[d_A F^{\text{pg}}(A|B)] \quad \text{with} \quad F^{\text{pg}}(A|B) := F(\Phi_{AA'}, \mathbb{1}_A \otimes \Lambda_{B\rightarrow A}^{\text{pg}}(\rho_{AB}))
\]

where \( \Phi_{AA'} \) is the maximally entangled state and \( \Lambda^{\text{pg}} \) is the pretty good recovery map [24]. Both the pretty good measurement and the pretty good recovery map get very close to the performance of the optimal processes [22, 24].

We note that the conditional Rényi 2-entropy also appears in the study of randomness extractors (especially with quantum side information - privacy amplification) [21, 25, 26], and in its quantum counterpart decoupling [26–28]. It is interesting to note that for the special case \( \rho_{AB} = \rho_A \otimes \rho_B \), (5) becomes an equation relating unconditional entropies that was discussed in [29, 30]. In addition, we also obtain relation (6) for a variant \( H_2(A|B) \approx H_{\text{min}}(A|B) \), where \( \approx \) means up to small error terms (see Appendix for details).

Our relation (5) establishes, for the first time, an equivalence between uncertainty \( H_2(K|B\Theta) \) and our ability to recover entanglement as given by \( H_2(A|B) \). It is deeply satisfying for us because it reconciles the apparently contradictory observations of EPR and Heisenberg into a single equation, and demonstrates that both effects were flip sides of the same coin.

II. DISCUSSION

Operational examples

To gain further intuition about (5), let us first return to the uncertainty game discussed earlier. In what follows we refer to \( H_2(A|B) \geq 0 \) as the Heisenberg regime and \( H_2(A|B) < 0 \) as the EPR regime. Note that in terms of the operational interpretations of \( H_2 \), we can rewrite (5) as

\[
P_{\text{guess}}^{\text{pg}}(K|B\Theta) = \frac{1}{d_A + 1} \sum_{\theta} P_{\text{guess}}^{\text{pg}}(K|B\Theta = \theta) = \frac{d_A \cdot F^{\text{pg}}(A|B) + 1}{d_A + 1}.
\]

In the game, Bob prepares a state \( \rho_{AB} \) and sends the \( A \) system to Alice. She measures \( A \) in one basis chosen uniformly at random from the complete set of \( d_A + 1 \) MUBs, and announces the basis (the index \( \theta \)) to Bob. Bob’s task is to guess Alice’s outcome using the pretty good measurement on \( B \). Equation (7) says that Bob’s ability to win or lose this game is quantitatively connected to the entanglement of \( \rho_{AB} \), as measured by \( F^{\text{pg}}(A|B) \).

Let us consider a number of special cases that illustrate this concept. If \( \rho_{AB} \) is the maximally entangled state, we have \( F^{\text{pg}} = 1 \) and Bob can guess Alice’s measurement outcome perfectly regardless of which measurement she performs, i.e., \( P_{\text{guess}}^{\text{pg}}(K|B\Theta = \theta) = 1 \) for all \( \theta \). That is, there is no uncertainty as expected. If Bob prepares \( \rho_{AB} \) with less than maximal entanglement, then \( F^{\text{pg}}(A|B) < 1 \) and there will be at least one basis for which Bob cannot perfectly guess the outcome. Thus, there is at least some amount of uncertainty expressed quantitatively as \( H_2(K|B\Theta) \).

If \( \rho_{AB} \) is separable, then Bob is stuck in the Heisenberg regime \( (F^{\text{pg}} \leq 1/d_A) \) and his ability to guess is very poor, constrained by the uncertainty relation \( P_{\text{guess}}^{\text{pg}}(K|B\Theta) \leq 2/(d_A + 1) \). This illustrates that entanglement is necessary for Bob to gain an advantage in the guessing game.

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3 We denote by \( \Phi_{AA'} \) the normalized maximally entangled state, and hence a factor \( d_A \) appears in [6].
Uncertainty and certainty relations

The curious reader may question why we must formulate our uncertainty equality \([5]\) using \(d_A + 1\) MUBs, can we not use fewer measurements? To answer this, it is instructive to study what kind of relations our main result \([3]\) implies. On the one hand, we can deduce regular uncertainty relations, and e.g., we get a relation in terms of the smooth conditional min-entropy similar to \([16]\).

\[
H^\epsilon_{\text{min}}(K|\Theta B) \geq \log(d_A + 1) - \log \left[2^{-H^\epsilon_{\text{min}}(A|B)} + 1\right] - 2\log(1/\epsilon) ,
\]

(8)

where \(\epsilon > 0\) denotes a small error term (see Appendix for details). Here, the l.h.s. has the operation meaning of minus the logarithm of Bob’s guessing probability (up to \(\epsilon\)), \(P_{\text{guess}}(K|\Theta B)\), when Alice measures in one of \(d_A + 1\) possible MUBs chosen uniformly at random. But on the other hand, we also get relations that upper bound the uncertainties of incompatible observables. In the literature these are known as certainty relations \([31, 32]\), and here we give the first such relations that allow for quantum memory. E.g., we get in terms of the conditional min-entropy (again up to a small error term \(\epsilon > 0\), see Appendix for details),

\[
H_{\text{min}}(K|\Theta B) \leq \log(d_A + 1) - \log \left[2^{-H_{\text{min}}(A|B)} + \frac{2}{\epsilon^2}\right] + 1 + 2\log(1/\epsilon) .
\]

(9)

This says that Bob’s certainty, i.e., his ability to guess Alice’s measurement outcome, must be high if he is highly entangled to Alice as measured by the smooth conditional min-entropy. Like our main result, \([5]\) implies that if Alice and Bob are maximally entangled, Bob has perfect certainty about Alice’s outcomes regardless of which measurement she performs (up to \(\epsilon\)).

Now there is a simple argument that considering fewer than \(d_A + 1\) measurements implies that only trivial certainty relations can hold. As uncertainty equalities as in \([5]\) imply non-trivial certainty relations, such equalities cannot hold for a smaller number of measurements. This is in sharp contrast to uncertainty relations, where non-trivial relations can be obtained for just two measurements. To see this, consider the case where \(\rho_A\) is just one qubit and we perform measurements in the \(\sigma_X\) and \(\sigma_Z\) eigenbases respectively. Let \(B\) be trivial, i.e., \(H_{\text{min}}(K|\Theta B) = H_{\text{min}}(K|\Theta)\). In this case, we are just considering the entropy of the outcome distribution of measuring \(\rho_A\) in one of the two bases. Clearly, when \(\rho_A\) is an eigenstate of \(\sigma_Y\), then the outcome distribution for both \(\sigma_X\) and \(\sigma_Z\) is uniform and hence \(H_{\text{min}}(K|\Theta) = 1\), which is the maximum value. The same argument shows that when measuring in less than \(d_A + 1\) mutually unbiased bases, \(H_{\text{min}}(K|\Theta) = \log d_A\) which is the maximum value that it can take and hence only the trivial upper bound/certainty relation holds. It is thus clear that equalities such as \([5]\) can only hold for sets of measurements which are sufficiently rich.

Bounds for less bases

Even though there does not exist an uncertainty equality for measuring in less than \(d_A + 1\) MUBs, we can still give lower and upper bounds for Bob’s uncertainty about \(1 \leq n \leq d_A\) MUBs on \(A\) in terms of the entanglement between \(A\) and \(B\). Moreover these inequalities are tight for all \(n\), that is, fixing the set of measurements and the entanglement to be constant, there exists states that achieve the upper and lower bounds. Our relations are again in terms of the conditional Rényi 2-entropy, and using \(P_{\text{guess}}(n)\) as a shorthand to denote Bob’s guessing probability \(P_{\text{guess}}(K|B\Theta)\) when Alice does \(n\) measurements, we find that the following bounds are tight in the Heisenberg regime \((F_{\text{Heiss}}(A|B) \leq 1/d_A)\),

\[
\frac{1}{d_A} \leq P_{\text{guess}}(n) \leq \frac{d_A}{n} \cdot F_{\text{Heiss}}(A|B) + \frac{n - 1}{n \cdot d_A} .
\]

(10)

Moreover, the following bounds are tight in the EPR regime \((F_{\text{Heiss}}(A|B) > 1/d_A)\),

\[
F_{\text{Heiss}}(A|B) \leq P_{\text{guess}}(n) \leq \frac{n - 1}{n} \cdot F_{\text{Heiss}}(A|B) + \frac{1}{n} .
\]

(11)

Here, the upper bounds (lower bounds) can be thought of as uncertainty relations (certainty relations). It is remarkable that we can derive these tight uncertainty relations for all \(n\) directly from our main result (see Appendix for derivation). Taken together, \([10]\) and \([11]\) completely characterize the allowable range of values that \(P_{\text{guess}}(n)\) can attain. As an example, consider \(d_A = 5\). Fig. [1] plots the tight upper and lower bounds as a function of \(F_{\text{Heiss}}(A|B)\). Notice that the tight lower bound does not change as \(n\) varies from 1 to 5; it only increases when we include the sixth basis. In
FIG. 1: Upper (UB) and lower (LB) bounds on $P_{\text{guess}}(n)$ as a function of $F_{pg}(A|B)$ for $d_A = 5$, for various values of $n$. For $n = 1, 2, 3, 4, 5$, the lower bound is given by the solid black line. The upper bounds for these values of $n$ are respectively the black, purple, blue, green, and orange dashed lines. For $n = 6$, the upper and lower bounds coincide, i.e., the allowed values are confined to lie on the red dashed line.

contrast, the tight upper bound steadily decreases with $n$, reflecting the complementarity between the different bases. Overall, as $n$ increases from 1 to 6, the area of the allowed range monotonically shrinks towards zero, and the allowed area becomes zero for $n = 6$ since the two quantities $P_{\text{guess}}(n)$ and $F_{pg}(A|B)$ are deterministically related by our main result.

From Fig. 1, one can also see that if Bob can guess two MUBs on $A$ well, then he can also guess $d_A + 1$ MUBs on $A$ fairly well. Conceptually, this follows from a two-step chain of reasoning: if Bob’s uncertainty is low for two MUBs, then he must be entangled to Alice, which in turn implies that he must have a low uncertainty for all bases. So entanglement provides the key link, from two MUBs to all bases. From the above results, it is straightforward to derive the following quantitative statement of this idea

$$P_{\text{guess}}(d_A + 1) \geq \frac{d_A \cdot (2P_{\text{guess}}(2) - 1) + 1}{d_A + 1},$$

which says that as $P_{\text{guess}}(2) \to 1$, then $P_{\text{guess}}(d_A + 1) \to 1$.

Applications in quantum information theory

We end this article with a discussion of the applications of our main result to quantum information processing tasks. Because entanglement is crucial for several quantum information technologies, the experimenter often needs a method to verify that their source is indeed producing entangled pairs, i.e., an “entanglement witness”. Following [8], our main result (5) offers a simple strategy for entanglement witnessing since it connects entanglement to uncertainty, which is experimentally measurable. In particular, Alice and Bob (in their distant labs, receiving $A$ and $B$ respectively) can sample from the source multiple times and communicate their results to gather statistics, say, regarding the $K_\theta$ observable on $A$ and the $L_\theta$ observable on $B$. Suppose they do this for a set of $n$ MUBs $\{K_\theta\}_{\theta=1}^n$ on $A$, with Bob measuring in a some arbitrary set of $n$ bases $\{L_\theta\}_{\theta=1}^n$ on $B$, and they estimate the joint probability distribution for each pair $\{K_\theta, L_\theta\}$ and hence can evaluate the classical entropies $H_2(K_\theta|L_\theta)$. Then according to our main result (see Appendix for details), their source is necessarily entangled if

$$\sum_{\theta=1}^n 2^{-H_2(K_\theta|L_\theta)} > 1 + \frac{n - 1}{d_A}.$$ 

Note that this method offers the flexibility of doing entanglement witnessing with $2 \leq n \leq d_A + 1$ observables. For $n = 2$, the same strategy based on the uncertainty relation [5] was discussed in [8] and implemented in [14, 15].
Our uncertainty equality \(5\) also gives insights for studying the monogamy of correlations. The basic idea of monogamy is that \(A\)'s entanglement with \(B\) limits the degree to which \(A\) can be entangled with a third system, \(E\). There have been several statements of monogamy in the literature; however, a nice aspect of our results is the potential to state monogamy as an equation rather than an inequality. By invoking a slight extension of our main result, we can show that for any tripartite pure state \(\rho_{ABE}\),

\[
D_0(\rho_{AE} \| \frac{\mathbb{1}_A}{d_A} \otimes \rho_E) = \log d_A - \log \left( (d_A + 1) \cdot 2^{-H'_2(\rho_{A|B\Theta}) - 1} \right),
\]

where \(D_0\) denotes the relative Rényi 0-entropy, and \(H'_2\) is a variant of the Rényi 2-entropy (see Appendix for details). This relation states that Bob’s uncertainty about a complete set of Alice’s MUBs is a quantitative measure of the distance of \(\rho_{AE}\) (Alice’s and Eve’s state) to the completely uncorrelated state. According to \(14\), Bob and Eve fight in a “zero-sum game” to be correlated to Alice, i.e., any gain of knowledge about Alice’s system by Bob forces Eve’s state to get closer to being uncorrelated with Alice, and conversely any gain of distance from the uncorrelated state by Eve forces Bob to lose knowledge.

We conclude by mentioning that the conditional Rényi 2-entropy is the relevant quantity in the study of classical and quantum randomness extractors against quantum side information. Randomness extractors are crucial objects in classical and quantum cryptography. Since our uncertainty equality \(5\) connects the conditional Rényi 2-entropy of the pre-measurement state (the entanglement term) to the conditional Rényi 2-entropy of the post-measurement state (the uncertainty term), one expects that our main result \(5\) shines some light on the relation between classical and quantum extractors. And indeed, our results can be used to get a new perspective on the results in \(16\), where security of the noisy storage model \(33\) was first linked to the quantum capacity.

III. METHODS

Quantifying entanglement

An important entropy measure studied in cryptography is the conditional min-entropy. It was originally defined in an abstract form \(21\), but was later given an intuitive operational meaning \(20\) in terms of entanglement, i.e., \(H_{\min}(A|B) = -\log[d_A F(A|B)]\) with

\[
F(A|B) = \max_{\Lambda_{B\rightarrow A'}} F(\Phi_{AA'}, (\mathcal{I}_A \otimes \Lambda_{B\rightarrow A'})(\rho_{AB})),
\]

where \(\Phi_{AA'}\) denotes the normalized maximally entangled state, the maximum is over all quantum operations \(\Lambda_{B\rightarrow A'}\) with \(A'\) a copy of \(A\), and \(F(\rho, \sigma) = (\text{Tr}\sqrt{\sqrt{\rho}\sqrt{\sigma}})^2\) denotes the fidelity.

A related entropy measure is the conditional Rényi 2-entropy, which is defined for a bipartite quantum state \(\rho_{AB}\) as

\[
H_2(A|B) = -\log \text{Tr} \left[ \rho_{AB}(\mathbb{1}_A \otimes \rho_B)^{-1/2} \rho_{AB}(\mathbb{1}_A \otimes \rho_B)^{-1/2} \right].
\]

Here we give an operational meaning for the conditional Rényi 2-entropy showing that, like the min-entropy, it is a natural measure of the entanglement between \(A\) and \(B\) in that \(H_2(A|B) = -\log[d_A F^{pg}(A|B)]\) with

\[
F^{pg}(A|B) = F(\Phi_{AA'}, \mathbb{1}_A \otimes \Lambda_{B\rightarrow A'}^{pg}(\rho_{AB})),
\]

and \(\Lambda_{B\rightarrow A'}^{pg}\) is the pretty good recovery map \(24\). To see this, we note that the pretty good recovery map can be written as

\[
\Lambda_{B\rightarrow A'}^{pg}(\cdot) = \frac{1}{d_A} \cdot \mathcal{E}_{B\rightarrow A'} (\rho_B^{-1/2}(\cdot)\rho_B^{-1/2}),
\]

where \(\mathcal{E}_{B\rightarrow A'}\) denotes the adjoint of the Choi-Jamikowski map of \(\rho_{AB}\),

\[
\mathcal{E}_{A\rightarrow B}(\cdot) = d_A \cdot \text{Tr}_A \left[ (\cdot)^T \otimes \mathbb{1}_B \right] \rho_{AB}.
\]

Putting this in \(16\) we arrive at \(17\). The map \(\Lambda_{B\rightarrow A'}^{pg}\) is pretty good in the sense that it is close to optimal for recovering the maximally entangled state, i.e., the following bound holds \(24\)

\[
F^2(A|B) \leq F^{pg}(A|B) \leq F(A|B).
\]
Quantifying uncertainty

When measuring a bipartite quantum state $\rho_{AB}$ on $A$ in some basis $K = \{|k\rangle\}$, we arrive at a so-called classical-quantum state

$$\rho_{KB} = \sum_k (|k\rangle\langle k| \otimes \mathbb{1}_B)\rho_{AB}(|k\rangle\langle k| \otimes \mathbb{1}_B) = \sum_k |k\rangle\langle k| \otimes \rho^k_B .$$

(21)

The conditional min-entropy of $\rho_{KB}$, using the formula for $F(K|B)$ from [15], translates to $H_{\text{min}}(K|B) = -\log P_{\text{guess}}(K|B)$ with [20]

$$P_{\text{guess}}(K|B) = \max_{\{E^k_B\}} \sum_k \text{Tr} [E^k_B \rho^k_B] ,$$

(22)

the probability for guessing $K$ correctly by performing the optimal measurement $\{E^k_B\}$ on the quantum side information $B$. The conditional min-entropy quantifies the uncertainty of $K$ in the exact sense of the uncertainty game, namely it quantifies the probability that Bob wins the uncertainty game.

The conditional Rényi 2-entropy of a classical-quantum state $\rho_{KB}$ (see (21)) is again defined as in (16). Furthermore, it was shown in [23] that its operational form (17) is given by

$$H_2(K|B) = -\log P_{\text{guess}}^\text{pg}(K|B) ,$$

(23)

where $P_{\text{guess}}^\text{pg}(K|B)$ denotes the probability of guessing $K$ by performing the “pretty good measurement” [22]. For the classical-quantum state (21) the pretty good measurement operators are defined as

$$\Pi^k_B = \rho_B^{-1/2} \rho^k_B \rho_B^{-1/2} .$$

(24)

By calculating $P_{\text{guess}}^\text{pg}(K|B) = \sum_k \text{Tr} [\Pi^k_B \rho^k_B]$, the equivalence of (23) to the definition of the conditional Rényi 2-entropy (16) can be seen. Hence, the conditional Rényi 2-entropy corresponds to the probability that Bob wins the uncertainty game by using the pretty good measurement. It is known that the pretty good measurement performs close to optimal, i.e., the following bound holds [22]

$$P_{\text{guess}}^2(K|B) \leq P_{\text{guess}}^\text{pg}(K|B) \leq P_{\text{guess}}(K|B) .$$

(25)

In the following we will not only measure in one fixed basis, but with equal probability in one of $d_A + 1$ MUBs. For that reason, we will work with the state

$$\rho_{KB\theta} = \frac{1}{d_A + 1} \cdot \sum_{\theta=1}^{d_A+1} \sum_{k=1}^{d_A} (|\theta_k\rangle\langle \theta_k| \otimes \mathbb{1}_B)\rho_{AB}(|\theta_k\rangle\langle \theta_k| \otimes \mathbb{1}_B) \otimes |\theta\rangle\langle \theta| ,$$

(26)

where the elements of the $d_A + 1$ MUBs $\theta$ are denoted by $\{|\theta_k\rangle\}$. It is straightforward to see that

$$P_{\text{guess}}^\text{pg}(K|B\theta) = \frac{1}{d_A + 1} \cdot \sum_{\theta} P_{\text{guess}}^\text{pg}(K|B = \theta) .$$

(27)

Main result

Here we prove our main result, the uncertainty equality [3]. For this we define $\tilde{\rho}_{AB} = (\mathbb{1}_A \otimes \rho_B^{-1/4})\rho_{AB}(\mathbb{1}_A \otimes \rho_B^{-1/4})$ and rewrite the fully quantum conditional Rényi 2-entropy as $H_2(A|B) = -\log \text{Tr} [\tilde{\rho}_{AB}^2]$. Similarly, we rewrite the classical-quantum conditional Rényi 2-entropy as

$$H_2(K|B\theta) = -\log \left( \frac{1}{d_A + 1} \cdot \sum_{\theta,k} \text{Tr}_B \left[ \text{Tr}_A [\tilde{\rho}_{AB}(|\theta_k\rangle\langle \theta_k| \otimes \mathbb{1}_B)]^2 \right] \right) .$$

(28)
Now we introduce the space $\mathcal{H}_{A'B'} \cong \mathcal{H}_{AB}$ as well as the state $\tilde{\rho}_{A'B'} \cong \hat{\rho}_{AB}$. We have
\[
(d_A + 1) \cdot 2^{-H_2(K|B^\Theta)} = \sum_{\theta, k} \text{Tr}_B\{\text{Tr}_A[(|\theta_k\rangle\langle\theta_k| \otimes \mathbb{1}_B)\hat{\rho}_{AB}] \text{Tr}_A[(|\theta_k\rangle\langle\theta_k| \otimes \mathbb{1}_B)\hat{\rho}_{AB}]\}
= \sum_{\theta, k} \text{Tr}_{BB'}\text{Tr}_{AA'}[(|\theta_k\rangle\langle\theta_k| \otimes |\theta_k\rangle\langle\theta_k|)(\tilde{\rho}_{AB} \otimes \tilde{\rho}_{A'B'})F_{BB'}]
= \text{Tr}_{BB'}\text{Tr}_{AA'}[(I_{AA'} + F_{AA'})(\tilde{\rho}_{AB} \otimes \tilde{\rho}_{A'B'})F_{BB'}]
= \text{Tr}_{BB'}\text{Tr}_{AA'}[\tilde{\rho}_{AB} \otimes \tilde{\rho}_{A'B'}] + \text{Tr}_{BB'}\text{Tr}_{AA'}[F_{AA'}(\tilde{\rho}_{AB} \otimes \tilde{\rho}_{A'B'})F_{BB'}]
= \text{Tr}_B[\tilde{\rho}_{AB}\text{Tr}_A(\tilde{\rho}_{AB})] + \sum_{t,s} \text{Tr}_{BB'}\text{Tr}_{AA'}[(|t\rangle\langle t| \otimes |s\rangle\langle s| \otimes \mathbb{1}_{BB'}) (\tilde{\rho}_{AB} \otimes \tilde{\rho}_{A'B'})F_{BB'}]
= 1 + \sum_{t,s} \text{Tr}_B\{\text{Tr}_A[(|t\rangle\langle t| \otimes \mathbb{1}_B)\tilde{\rho}_{AB}]\text{Tr}_A[(|s\rangle\langle t| \otimes \mathbb{1}_B)\tilde{\rho}_{AB}]\}
= 1 + \text{Tr}[\tilde{\rho}_{AB}']
\]
where $F_{AA'} = \sum_{t,s} |t\rangle\langle t| \otimes |s\rangle\langle s|$ is the operator that swaps $A$ and $A'$ (similarly for $F_{BB'}$). Here, the second line uses the “swap trick”, for operators $M$ and $N$, and swap operator $F$,\[
\text{Tr}(MN) = \text{Tr}[(M \otimes N)F],
\]and the third line invokes that a full set of MUBs generates a complex projective 2-design $[33]$, i.e.,\[
\sum_{\theta, k} |\theta_k\rangle\langle\theta_k| \otimes |\theta_k\rangle\langle\theta_k| = I_{AA'} + F_{AA'}.\]
In the appendix, we show in detail that our result also holds for other measurements as long as they form a 2-design. We note that some other extensions of our main result can also be found in the Appendix.

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and hence the result follows from The inequality step in this proof invoked the property that the fidelity decreases upon decreasing one of its arguments.

Proof. Let $K = \{|k\rangle\}$ be an orthonormal basis on some Hilbert space $\mathcal{H}_A$. Then, we have for any bipartite quantum state $\rho_{AB}$ that

$$P_{\text{guess}}^{\text{PS}}(K|B) \geq F_{\text{PS}}(A|B). \quad (A1)$$

Proof. For $\rho_{KB} = \sum_k (|k\rangle\langle k| \otimes \mathbb{1}_B)\rho_{AB}(|k\rangle\langle k| \otimes \mathbb{1}_B)$ we have that

$$P_{\text{guess}}^{\text{PS}}(K|B) = \text{Tr} \left[ \rho_{KB} \rho_B^{-1/2} \rho_{KB} \rho_B^{-1/2} \right]$$

$$= \text{Tr} \left[ \rho_{AB} \rho_B^{-1/2} \rho_{KB} \rho_B^{-1/2} \right]$$

$$= d_A \cdot \text{Tr} \left[ \Phi_{AA'}(\mathcal{I}_A \otimes \Lambda_{B \rightarrow A'}^{\text{PS}})(\rho_{KB}) \right]$$

$$= F(\Phi_{AA'}, d_A \cdot (\mathcal{I}_A \otimes \Lambda_{B \rightarrow A'})^{\otimes d_A})(\rho_{AB})$$

$$\geq F(\Phi_{AA'}, (\mathcal{I}_A \otimes \Lambda_{B \rightarrow A'}^{\text{PS}})(\rho_{AB}))$$

$$= F_{\text{PS}}(A|B). \quad (A2)$$

The inequality step in this proof invoked the property that the fidelity decreases upon decreasing one of its arguments, and hence the result follows from

$$d_A \cdot (\mathcal{I}_A \otimes \Lambda_{B \rightarrow A'}^{\text{PS}})(\rho_{KB}) \geq (\mathcal{I}_A \otimes \Lambda_{B \rightarrow A'}^{\text{PS}})(\rho_{AB}). \quad (A3)$$

To show $(A3)$, we denote the positive operator $\sigma_{AA'} = (\mathcal{I} \otimes \Lambda_{B \rightarrow A'}^{\text{PS}})(\rho_{AB})$, note that the measurement in $K$ on the $A$-system commutes with $\mathcal{I} \otimes \Lambda_{B \rightarrow A'}^{\text{PS}}$, and we have that

$$d_A \cdot (\mathcal{I}_A \otimes \Lambda_{B \rightarrow A'}^{\text{PS}})(\rho_{KB}) - (\mathcal{I}_A \otimes \Lambda_{B \rightarrow A'}^{\text{PS}})(\rho_{AB})$$

$$= d_A \cdot \sum_k (|k\rangle\langle k| \otimes \mathbb{1}_A') \sigma_{AA'}(|k\rangle\langle k| \otimes \mathbb{1}_A') - \sum_{k,k'} (|k\rangle\langle k| \otimes \mathbb{1}_A') \sigma_{AA'}(|k'\rangle\langle k'| \otimes \mathbb{1}_A')$$

$$= (d_A - 1) \sum_k (|k\rangle\langle k| \otimes \mathbb{1}_A') \sigma_{AA'}(|k\rangle\langle k| \otimes \mathbb{1}_A') - \frac{1}{d_A - 1} \sum_{k,k' \neq k} (|k\rangle\langle k| \otimes \mathbb{1}_A') \sigma_{AA'}(|k'\rangle\langle k'| \otimes \mathbb{1}_A')$$

$$= (d_A - 1) (\mathcal{F} \otimes \mathcal{I})(\sigma_{AA'}), \quad (A4)$$

where we set in the last line

$$\mathcal{F}(\cdot) = \frac{1}{d_A - 1} \sum_{m=1}^{d_A - 1} Z^m(\cdot)(Z^m)^\dagger, \quad Z = \sum_{k=0}^{d_A - 1} \omega^k |k\rangle\langle k|, \quad \omega = e^{2\pi i/d_A}. \quad (A5)$$

Since $\mathcal{F}$ is a CPTP map, the claim follows.
Lemma 3. MUBs on $\theta$ and this proves (10). The tightness of the bounds in (10) and (11) follows by construction. In the region $F(\theta) \leq \frac{1}{d_A}$, we have
\[
\sum_{\theta=1}^{n} P_{\text{guess}}(K|B\Theta = \theta) = (n - 1) \cdot F_{\theta}(A|B) + \left( d_A + 1 - n \right) \cdot F_{\theta}(A|B) - \sum_{\theta=n+1}^{d_A+1} P_{\text{guess}}(K|B\Theta = \theta) \leq (n - 1) \cdot F_{\theta}(A|B) + 1 ,
\]
and this proves (11). Similarly, we can invoke the immediate relation $P_{\text{guess}}(K|B) \geq 1/d_A$ to get
\[
\sum_{\theta=1}^{n} P_{\text{guess}}(K|B\Theta = \theta) = d_A \cdot F_{\theta}(A|B) + \frac{n - 1}{d_A} + \left( \frac{d_A + 1 - n}{d_A} - \sum_{\theta=n+1}^{d_A+1} P_{\text{guess}}(K|B\Theta = \theta) \right) \leq d_A \cdot F_{\theta}(A|B) + \frac{n - 1}{d_A} ,
\]
and this proves (10). The tightness of the bounds in (10) and (11) follows by construction. In the region $F_{\theta}(A|B) \geq 1/d_A$, the upper bound is achieved by a bipartite pure state whose Schmidt basis is one of the $\Theta$ bases appearing in the sum of guessing probabilities under consideration, and the lower bound is achieved by a bipartite pure state whose Schmidt basis is one of the $\Theta$ bases appearing to the same complete MUB set as the bases under consideration, but whose guessing probability was removed from the sum under consideration. In the region $F_{\theta}(A|B) \leq 1/d_A$, the upper bound is achieved by a tensor product state $\rho_A \otimes \rho_B$ such that $\rho_A$ is diagonal in one of the $\Theta$ bases appearing in the sum of guessing probabilities under consideration, and the lower bound is similarly achieved by such a tensor product state where $\rho_A$ is diagonal in one of the $\Theta$ bases that belongs to the same complete MUB set as the bases under consideration, but whose guessing probability was removed from the sum under consideration.

Appendix B: Entanglement witnessing

Here we show the origin of (13), our condition for witnessing entanglement. We will make use of the following lemma, which says that separable states cannot have a negative conditional entropy.

**Lemma 2.** Let $\rho_{AB} = \sum_j p_j \cdot \rho_A^j \otimes \rho_B^j$ be a separable state. Then, we have that
\[
H_2(A|B) \geq H_{\min}(A|B) \geq 0 .
\]

**Proof.** The inequality $H_2(A|B) \geq H_{\min}(A|B)$ holds in general for any state $\rho_{AB}$ since $\Lambda_{B\rightarrow A'}^{\text{guess}}$ in (17) is a particular CPTP map and $H_{\min}$ involves an optimization over all CPTP maps $\Lambda_{B\rightarrow A'}$ in (15). To prove $H_{\min}(A|B) \geq 0$ in the case of separable $\rho_{AB}$, note that any local operation on a separable state results in another separable state. Suppose $\sigma_{AA'} = (I_A \otimes \Lambda_{B\rightarrow A'})(\rho_{AB})$ is the separable state that achieves the optimization when evaluating $H_{\min}(A|B)$ for $\rho_{AB}$ (i.e., $\Lambda$ is the optimal channel in (15)). Then, we have that
\[
F(A|B) = F(\Phi_{AA'}, \sigma_{AA'}) \leq F(\Phi_{AA'}, \mathbb{I}_A \otimes \sigma_{A'}) = 1/d_A ,
\]
which follows because the fidelity increases upon increasing one of its arguments, and because for separable $\sigma_{AA'}$ we have $\sigma_{AA'} \leq \mathbb{I}_A \otimes \sigma_{A'}$ with $\sigma_{A'} = \text{Tr}_A(\sigma_{AA'})$. Using $H_{\min}(A|B) = -\log[d_A F(A|B)]$, the desired result follows.

The following lemma essentially proves our entanglement witnessing condition.

**Lemma 3.** Let $\rho_{AB} = \sum_j p_j \cdot \rho_A^j \otimes \rho_B^j$ be a separable state. Let $\{K_\theta\}_{\theta=1}^{n}$ be a subset (of size $n$) of a complete set of MUBs on $A$, and let $\{L_\theta\}_{\theta=1}^{n}$ be an arbitrary set of $n$ orthonormal bases on $B$. Then, for the state $\rho_{AB}$ we have
\[
\sum_{\theta=1}^{n} 2^{-H_2(K_\theta|L_\theta)} \leq 1 + \frac{n - 1}{d_A} ,
\]
where the classical joint probability distribution needed to evaluate $H_2(K_\theta|L_\theta)$ is obtained by decohering the state $\rho_{AB}$ in the $K_\theta = \{|K_\theta; p\rangle\}$ basis on $A$ and the $L_\theta = \{|L_\theta; q\rangle\}$ basis on $B$, i.e., $\rho_{K_\theta L_\theta} = \sum_{p,q} \langle K_\theta; p| K_\theta; p \rangle \otimes \langle L_\theta; q| L_\theta; q \rangle \rho_{AB}(|K_\theta; p\rangle \otimes |L_\theta; q\rangle)$.
Proof. The previous lemma indicates that $F_{\text{PS}}(A|B) \leq 1/d_A$ since $\rho_{AB}$ is separable. Combining this with the bound in [11] we have

$$\frac{n - 1}{d_A} + 1 \geq (n - 1) \cdot F_{\text{PS}}(A|B) + 1 \geq \sum_{\theta=1}^{n} P_{\text{guess}}(K_{\theta}|B) = \sum_{\theta=1}^{n} 2^{-H_2(K_{\theta}|B)} \geq \sum_{\theta=1}^{n} 2^{-H_2(K_{\theta}|L_{\theta})}, \tag{B4}$$

where the last inequality follows because $H_2$ satisfies the data-processing inequality [35], that is, decohering $B$ in some basis cannot reduce the uncertainty as measured by $H_2$.

Since separable states must satisfy [B3], it follows that if Alice and Bob can show that

$$\sum_{\theta=1}^{n} 2^{-H_2(K_{\theta}|L_{\theta})} > 1 + \frac{n - 1}{d_A}, \tag{B5}$$

then their state is non-separable, i.e., entangled.

Appendix C: More general entropies

Our main result [5] is in terms of the conditional Rényi 2-entropy. However, we can also prove it in terms of a more general continuous family of conditional 2-entropies. For $\nu \in [0, 1]$ and bipartite quantum states $\rho_{AB}$, we define

$$H_{2,\nu}(A|B) = -\log Tr \left[ \rho_{AB,\nu}^{\dag} \rho_{AB,\nu} \right], \tag{C1}$$

where $\rho_{AB,\nu} := (\mathbb{1}_A \otimes \rho_B^{-(1-\nu)/4}) \rho_{AB} (\mathbb{1}_A \otimes \rho_B^{-(1+\nu)/4})$. It is easily seen that $\nu = 0$ corresponds to the usual definition [16] in the main text. We state the generalization of our uncertainty equality [5] for a full set of MUBs, but note that it also holds for all other “informationally equivalent” measurements (cf. Appendix F).

Corollary 4. Let $\{\Theta\}_{\theta} \in \Theta$ be a complete set of MUBs on some Hilbert space $H_A$, and denote $\theta = \{|\theta_k\rangle\}_{k=1}^{d_A}$. Then, we have for any bipartite quantum state $\rho_{AB}$ that

$$H_{2,\nu}(K|B\Theta) = \log(d_A + 1) - \log \left( 2^{-H_{2,\nu}(A|B)} + 1 \right), \tag{C2}$$

where $\rho_{K\Theta} = \frac{1}{d_A+1} \sum_{\theta,k} (|\theta_k\rangle\langle\theta_k| \otimes \mathbb{1}_B) \rho_{AB} (|\theta_k\rangle\langle\theta_k| \otimes \mathbb{1}_B) \otimes |\theta\rangle\langle\theta|\Theta$.

The proof is obvious by just taking the proof for the conditional Rényi 2-entropy [29], and replacing $\tilde{\rho}_{AB}$ with $\rho_{AB,\nu}$. It is worth noting that (C2) applies (in addition to the $H_2$ entropy considered in the main text) to a variant, $H'_2(A|B) = H_{2,1}(A|B) = -\log Tr \left[ \rho_{AB}^{2} (\mathbb{1}_A \otimes \rho_B^{-1}) \right], \tag{C3}$

that is, the entropy obtained from setting $\nu = 1$ in (C1). The conditional entropy $H'_2$ is mathematically useful (see e.g. [36 37]), and is approximately equivalent to the smooth conditional min-entropy (see below).

Appendix D: Quantum cryptography

Here we show the uncertainty and certainty relations [8] and [9] in terms of the smooth conditional min-entropy as stated in the main text. For that, we first need to precisely define the smooth conditional min-entropy. For a bipartite quantum state $\rho_{AB}$ and smoothing parameter $\varepsilon > 0$, it is defined as

$$H_{\min}^\varepsilon(A|B)_{\rho} = \sup_{\rho_{AB}} H_{\min}(A|B)_{\rho}, \tag{D1}$$

where the supremum is over all sub-normalised states $\rho_{AB}$ on $AB$ that are $\varepsilon$-close to $\rho_{AB}$ in purified distance [17].

Now, the crucial point is that $H_{\min}^\varepsilon$ and $H_2$ are equivalent in the following sense: it holds for any bipartite quantum state $\rho_{AB}$ and $\varepsilon > 0$ that [17]

$$H_{\min}(A|B)_{\rho} \leq H'_2(A|B)_{\rho} \leq H_{\min}^\varepsilon(A|B)_{\rho} + \log \left( \frac{2}{\varepsilon^2} \right). \tag{D2}$$
Also, consider the extension of our main result (Corollary 4) evaluated for the case $\nu = 1$, 
\[
H'_2(K|B\Theta) = \log(d_A + 1) - \log \left(2^{-H'_2(A|B)} + 1\right), 
\]
(D3)

By combining (D2) with (D3), we obtain the uncertainty and certainty relation for the smooth conditional min-entropy as given in (8) and (9). We note that similar uncertainty relations have been derived in [16], and were used to analyze security in the noisy storage model [33].

Appendix E: Monogamy of correlations

Here we show (14) from the main text. The precise statement is as follows.

**Corollary 5.** Let $\{\Theta\}_{\theta \in \Theta}$ be a complete set of MUBs on some Hilbert space $\mathcal{H}_A$, and denote $\theta = \{\ket{\theta_k}\}_{k=1}^{d_A}$. Then, we have for any tripartite pure quantum state $\rho_{ABE}$ that 
\[
D_0(\rho_{AE}\|\frac{1}{d_A} \otimes \rho_E) = \log d_A - \log \left((d_A + 1) \cdot 2^{-H'_2(K|B\Theta)} - 1\right),
\]
(E1)

where $H'_2(K|B\Theta) = H_{2,1}(K|B\Theta)$ is defined in (C3), and $D_0(\rho_{AE}\|\frac{1}{d_A} \otimes \rho_E) = -\log \Tr \left[\rho_{AE}\left(\frac{1}{d_A} \otimes \rho_E\right)\right]$ denotes the relative Rényi 0-entropy.

**Proof.** First, let us rewrite (D3) as 
\[
H'_2(A|B) = -\log \left((d_A + 1) \cdot 2^{-H'_2(K|B\Theta)} - 1\right). 
\]

Now, for any conditional entropy that is invariant under local isometries on the conditioning system, one can define a so-called dual entropy. For some generic entropy $H_K$, the dual entropy $H'_K$ is defined by 
\[
H_K(A|B) = -H'_K(A|E)\rho, 
\]
(E3)

where $E$ is a system that purifies $\rho_{AB}$. Since $H'_2(A|B)$ is invariant under local isometries on $B$, the dual entropy is well defined, and it is known that [36] 
\[
-H'_2(A|B) = D_0(\rho_{AE}\|\frac{1}{d_A} \otimes \rho_E). 
\]

By the standard rewriting 
\[
D_0(\rho_{AE}\|\frac{1}{d_A} \otimes \rho_E) = D_0(\rho_{AE}\|\frac{1}{d_A} \otimes \rho_E) - \log d_A, 
\]
(E5)

the claim follows.

Appendix F: Main result - revisited

We have seen in the Methods section (Section III) that the proof of our uncertainty equality [5] crucially relies on the fact that a full set of MUBs generates a complex projective 2-design [34]. In general, a complex projective 2-design is a set $\{|\psi_y\rangle\}_{y \in Y}$ (of size $|Y|$) of vectors $|\psi_y\rangle$ lying in a Hilbert space $\mathcal{H}_A$ such that 
\[
\frac{1}{|Y|} \sum_{y \in Y} |\psi_y\rangle\langle\psi_y|^{\otimes 2} = \frac{1}{d_A \cdot (d_A + 1)} (\mathbb{1}_{AA'} + F_{AA'}), 
\]
(F1)

where $F_{AA'}$ denotes the swap operator, and $A'$ a copy of $A$. It turns out that there are other “informationally equivalent” measurements that generate a complex projective 2-design.

As an example we mention symmetric informationally complete positive operator valued measures (SIC-POVMs), such as the four states forming a tetrahedron on the Bloch sphere for qubits.

**Corollary 6.** Let $\{\frac{1}{d_A} \cdot |\psi_k\rangle\langle\psi_k|\}_{k=1}^{d_A^2}$ be a SIC-POVM on some Hilbert space $\mathcal{H}_A$. Then, we have for any bipartite quantum state $\rho_{AB}$ that 
\[
H_2(K|B) = \log[d_A(d_A + 1)] - \log \left(2^{-H_2(A|B)} + 1\right), 
\]
(F2)

where $\rho_{KB} = \sum_{k=1}^{d_A^2} |k\rangle\langle k| \otimes \Tr_A[(\frac{1}{d_A} \cdot |\psi_k\rangle\langle\psi_k| \otimes \mathbb{1})\rho_{AB}]$ is a classical-quantum state with $\{\ket{k}\}$ an orthonormal basis on $\mathcal{H}_K$. 


Notice that the $d_A$-dependent term on the r.h.s. of (F2) is slightly different from the corresponding term appearing in our main result (5), and indeed (F2) implies that $\log d_A \leq H_2(K|B) \leq 2 \log d_A$ for SIC-POVMs. Nevertheless, the proof of (F2) is identical to the proof of (5), with the appropriate version of (F1) substituted into the proof (29).

In addition, so-called unitary 2-designs are very related to complex projective 2-designs. A set $\{U_y\}_{y \in Y}$ (of size $|Y|$) of unitaries $U_y$ on some Hilbert space $\mathcal{H}_A$ forms a unitary two-design if

$$\frac{1}{|Y|} \sum_{y \in Y} (U_y \otimes U_y^\dagger)^{\otimes 2} = \int_{U(d_A)} (U \otimes U^\dagger)^{\otimes 2} dU,$$

where the integration is over all unitaries with respect to the Haar measure. Examples of unitary two-designs are the full unitary group, or the Clifford group for qubit systems. In fact, unitary 2-designs generate complex projective 2-designs.

**Lemma 7.** Let $\{U_\theta\}_{\theta \in \Theta}$ be a unitary two-design on some Hilbert space $\mathcal{H}_A$. Then, we have

$$\frac{1}{d_A \cdot |\Theta|} \sum_{\theta \in \Theta} \sum_{k=1}^{d_A} (U_\theta|k\rangle\langle k|U_\theta^\dagger)^{\otimes 2} = \frac{1}{d_A \cdot (d_A + 1)} (\mathbb{1}_{AA'} + F_{AA'}) ,$$

where $\{|k\rangle\}$ denotes some orthonormal basis of $\mathcal{H}_A$.

Hence, our main result also holds for unitary two-designs.

**Corollary 8.** Let $\{U_\theta\}_{\theta \in \Theta}$ be a unitary two-design on some Hilbert space $\mathcal{H}_A$. Then, we have for any bipartite quantum state $\rho_{AB}$ that

$$H_2(K|B\Theta) = \log(d_A + 1) - \log \left(2^{-H_2(A|B)} + 1\right) ,$$

where

$$\rho_{KB\Theta} = \frac{1}{|\Theta|} \sum_{\theta, k} |\theta_k\rangle\langle \theta_k| \otimes \mathbb{1}_B \rho_{AB} (|\theta_k\rangle\langle \theta_k| \otimes \mathbb{1}_B) \otimes |\theta\rangle\langle \theta| ,$$

and $|\theta_k\rangle = U_\theta|k\rangle$ for some orthonormal basis $\{|k\rangle\}$ of $\mathcal{H}_A$. 
