Governing Equations of Compressible Turbulence (The Revised)

Feng Wu

Department of Mechanics and Mechanical Engineering,
University of Science and Technology of China, Hefei 230026, China

By the nonstandard analysis theory of turbulence, the governing equations of compressible turbulence are given. The equations can hold at non-uniform points, in fact, are new kind of equations. There are three choices. In the choice one, the second-order infinitesimal quantities are neglected, the closed equations about the point(monad)-average quantities are obtained. In other two choices, the closed equations, in which the third-order infinitesimal quantities are omitted, are given and about the instantaneous, point-averaged and fluctuant quantities.

In the paper [1], a new approach, the nonstandard picture of turbulence, is presented. The essential idea in this picture is that a particle of fluid in a laminar field is uniform wholly, but in turbulence a particle of fluid is not uniform and has interior structure. By the nonstandard analysis, this picture can be described in mathematics. In the nonstandard analysis mathematics, an infinitesimal $\varepsilon$ is a certain number(nonstandard number) rather than a process of tending to zero. A particle of fluid is called as a monad and the dimension of a monad is an infinitesimal $\varepsilon$. By the concepts of the nonstandard analysis, the definition of “differential” can be given:

$$\frac{\partial f}{\partial t} = \frac{f(t + \varepsilon) - f(t)}{\varepsilon}, \quad \frac{\partial f}{\partial x} = \frac{f(x + \varepsilon) - f(x)}{\varepsilon}$$  \hspace{1cm} (1)

There is conceptual difference between this definition and that of the differential in the standard analysis theory.

In the nonstandard analysis theory of turbulence(NATT), there are six assumptions. They are:
Assumption 1: Global turbulent field is composed of standard points, and every standard point is yet a monad. Each monad possesses the internal structure, namely a monad is also composed of infinite nonstandard points (so called interior points of the monad).

Assumption 2: The flows in monad fields are controlled by the Navier-Stokes equations.

Assumption 3: Turbulent field is continuous.

Assumption 4: When a measurement at any point (monad) \((x_1, x_2, x_3, t)\) in a physical field is taken, the operation of the measurement will act randomly on one interior point (nonstandard point) of the point \((x_1, x_2, x_3, t)\).

Assumption 5: When a measurement at any point (monad) of a turbulent field is made, the operation of the measurement will act in equiprobability on various interior points of the monad. This assumption is called the equiprobability assumption.

Assumption 6: In both the value and structure of function, physical function, defined on the interior points of the monads of a turbulent field, is very close between two monads, when these two monads are infinitely close to each other.

By virtue of these assumptions, the fundamental equations for incompressible turbulence are obtained, also the closure problem is overcome in the paper [1]. These equations are based on the definition (1) and new kind of equations, which can hold at non-uniform points.
Now using the concepts mentioned above, we will give the governing equations for compressible turbulence.

The equations, the Navier-Stokes equations, governing the motion of laminar flows hold only at uniform points. The nonstandard points in a monad are uniform, therefore the Navier-Stokes equations hold in monad fields.

In a monad field, the governing equations of compressible flows are as follows.

\begin{equation}
\frac{\partial \rho}{\varepsilon^2 \partial t'} + \frac{\partial (\rho U_i)}{\varepsilon^2 \partial x_i} = 0
\end{equation}

\begin{equation}
\frac{\partial (\rho U_i)}{\varepsilon^2 \partial t'} + \frac{\partial (\rho U_i U_j)}{\varepsilon^2 \partial x_j} = -\frac{\partial P}{\varepsilon^2 \partial x_i} + \frac{\partial t_{ij}}{\varepsilon^2 \partial x_j}
\end{equation}

\begin{equation}
\frac{\partial}{\varepsilon^2 \partial x_j} \left[ \rho (e + U_i U_i) \right] + \frac{\partial}{\varepsilon^2 \partial x_j} \left[ \rho U_j (h + \frac{1}{2} U_i U_i) \right] = \frac{\partial}{\varepsilon^2 \partial x_j} (U_i t_{ij}) - \frac{\partial q_j}{\varepsilon^2 \partial x_j}
\end{equation}

Here the rule of summation over repeated indices is adopted, and $U_i$ is the velocity component in $i-$direction, $P$ the pressure, $\rho$ the density, $e$ the intrinsic energy per unit mass, $h$ the enthalpy per unit mass. $q_j$ is the heat conduction,

\begin{equation}
h = e + \frac{P}{\rho}
\end{equation}

\begin{equation}
P = f(\rho, T)
\end{equation}

\begin{equation}
q_j = -\kappa \frac{\partial T}{\varepsilon^2 \partial x_j}
\end{equation}

$t_{ij}$ is the stress tensor, $t_{ji} = t_{ij}$,

\begin{equation}
t_{ij} = \mu \left( \frac{\partial U_i}{\varepsilon^2 \partial x_j} + \frac{\partial U_j}{\varepsilon^2 \partial x_i} \right) - \frac{2}{3} \mu \delta_{ij} \frac{\partial U_d}{\varepsilon^2 \partial x_d}
\end{equation}

\begin{equation}
S_{ij} = \frac{1}{2} \left( \frac{\partial U_i}{\varepsilon^2 \partial x_j} + \frac{\partial U_j}{\varepsilon^2 \partial x_i} \right)
\end{equation}

Here $\mu$ is the dynamic viscosity, $\kappa$ the thermal conductivity, $T$ the absolute temperature, $\delta_{ij}$ the Kronecker delta.

The independent variables of all functions in these equations are
\[(x_1, x_2, x_3, t, x'_1, x'_2, x'_3, t')\]. \((x_1, x_2, x_3, t)\) is the coordinates set up in global field, while \((x'_1, x'_2, x'_3, t')\) is the coordinates set up in a monad \([\Pi]\).

Still there are the relations:

\[
U_i = \tilde{U}_i + u_i, \quad P = \tilde{P} + p, \quad \rho = \bar{\rho} + \rho_f, \quad T = \tilde{T} + T_f, \quad e = \bar{e} + e_f
\]

\[
h = \bar{h} + h_f, \quad q_j = \bar{q}_j + (q_j)_f, \quad t_{ij} = \tilde{t}_{ij} + (t_{ij})_f \quad \text{etc.} \quad (9)
\]

Here the lower-case \(u_i, p\) and the quantities with index \(f\) are called as fluctuation quantities, which have the order of magnitude \(0(\varepsilon)\).

Let “\(\sim\)” express the average operation over \((x'_1, x'_2, x'_3, t')\), “\(\sim^i\)” over \(x'_i\), “\(\sim\)” over \(t'\), etc., in a monad. Therefore,

\[
\tilde{U} = \frac{1}{L_t} \int_0^{L_t} dt' \frac{1}{L^3} \int_0^L \int_0^L U dx'_1 dx'_2 dx'_3 \quad (10)
\]

Here \(L_t\) and \(L\) are the infinite of time and space respectively.

Moreover, by use of the method used in the paper \([\Pi]\), the following equations are obtained,

\[
\begin{align*}
\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial (\bar{\rho} U_i)}{\partial x_i} &= 0 \quad (11) \\
\frac{\partial (\bar{\rho} U_i)}{\partial t} + \frac{\partial (\rho \tilde{U}_i U_j)}{\partial x_j} &= - \frac{\partial \bar{P}}{\partial x_i} + \frac{\partial \tilde{t}_{ij}}{\partial x_j} \quad (12) \\
\frac{\partial}{\partial t}[\rho (e + \frac{1}{2} U_i U_i)] + \frac{\partial}{\partial x_j} [\rho U_j (h + \frac{1}{2} U_i U_i)] &= \frac{\partial}{\partial x_j} (\bar{U}_i \tilde{t}_{ij}) - \frac{\partial \bar{q}_j}{\partial x_j} \quad (13)
\end{align*}
\]

The equations (11)-(13), in fact, are conservation equations of mass, momentum and energy, respectively, at one monad of global field. By the relations (9), we can obtained, from (11)-(13), the instantaneous and fluctuant equations. Clearly equation (11) can be written as

\[
\frac{\partial \rho}{\partial t} - \frac{\partial \rho_f}{\partial t} + \frac{\partial (\rho U_i)}{\partial x_i} - \frac{\partial (\rho U_i)_f}{\partial x_i} = 0 \quad (14)
\]
After splitting the equation (14) into two parts in different order of magnitude, it follows that:

The fluctuant continuity-equation in the order of magnitude 0(\(\varepsilon\)) is:

\[
\frac{\partial \rho_f}{\partial t} + \frac{\partial (\rho U_i)_f}{\partial x_i} = 0 \tag{15}
\]

The instantaneous continuity-equation in the order of magnitude 0(1) is:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho U_i)}{\partial x_i} = 0 \tag{16}
\]

Similarly, from (12) and (13), we have the instantaneous momentum-equations and energy-equation,

\[
\frac{\partial}{\partial t}(\rho U_i) + \frac{\partial}{\partial x_j}(\rho U_i U_j) = -\frac{\partial P}{\partial x_i} + \frac{\partial (t_{ij})}{\partial x_j} \tag{17}
\]

\[
\frac{\partial}{\partial t}\left[\rho(e + \frac{1}{2} U_i U_i)\right] + \frac{\partial}{\partial x_j}\left[\rho U_j\left(h + \frac{1}{2} U_i U_i\right)\right] = \frac{\partial}{\partial x_j}(U_i t_{ij}) - \frac{\partial q_j}{\partial x_j} \tag{18}
\]

also the fluctuant momentum-equations and energy-equation,

\[
\frac{\partial}{\partial t}(\rho U_i)_f + \frac{\partial}{\partial x_j}(\rho U_i U_j)_f = -\frac{\partial P_f}{\partial x_i} + \frac{\partial (t_{ij})_f}{\partial x_j} \tag{19}
\]

\[
\frac{\partial}{\partial t}\left[\rho(e + \frac{1}{2} U_i U_i)_f\right] + \frac{\partial}{\partial x_j}\left[\rho U_j\left(h + \frac{1}{2} U_i U_i\right)_f\right] = \frac{\partial}{\partial x_j}(U_i t_{ij})_f - \frac{\partial q_j}{\partial x_j} \tag{20}
\]

Now let \(P = R\rho T, \quad e = c_v T\) (in the case of perfect gas). Here \(R\) and \(c_v\) are, respectively, the gas constant and the specific heat at constant volume.

So there are the relations as follows.

\[
h = e + \frac{P}{\rho} = e + \frac{\bar{P}}{\bar{\rho}} \left[1 + \frac{p}{P} - \frac{\rho f}{P \bar{\rho}} - \frac{\rho f}{\bar{\rho}} + \frac{\rho f \rho f}{P \bar{\rho}}\right] + O(\varepsilon^3) \tag{21}
\]

\[
\bar{h} = \bar{e} + \frac{\bar{P}}{\bar{\rho}} \left[1 - \frac{(\bar{p} \rho f)}{P \bar{\rho}} + \frac{(\bar{\rho} \rho f)}{\bar{\rho}}\right] + O(\varepsilon^3) \tag{22}
\]

\[
h_f = h - \bar{h} = e_f + \frac{\bar{P}}{\bar{\rho}} \left[p - \rho f - \frac{\rho f}{\bar{\rho}} + \frac{\rho f \rho f}{P \bar{\rho}}\right] + O(\varepsilon^3) \tag{23}
\]

\[
P = R\rho T = R[\bar{\rho} \bar{T} + \rho_f \bar{T} + \rho_f T_f] \tag{24}
\]
\[
\bar{P} = R[\bar{\rho}\bar{T} + (\rho_f\bar{T}_f)]
\]  
(25)

\[
p = P - \bar{P} = R[\bar{\rho}T_f + \rho_f\bar{T} + \rho_fT_f - (\rho_f\bar{T}_f)]
\]  
(26)

Yet by Assumption 6, \(\frac{\partial}{\partial x_j}[u_iu_j - (\bar{u}_i\bar{u}_j)] \sim 0(\varepsilon^3)\). And so are the similar others. When the terms in order of magnitude \(\sim 0(\varepsilon^3)\) are, proximately, neglected, those will be omitted.

Then the expansion of \(\rho U_i\), \((\bar{\rho}\bar{U}_i)\) and \((\rho U_i)_f\) is, for example, given as follows. And so are the similar others.

\[
\rho U_i = (\bar{\rho} + \rho_f)(\bar{U}_i + u_i) = \bar{\rho}\bar{U}_i + \rho_f\bar{U}_i + \bar{\rho}u_i + \rho_fu_i
\]

\[(\bar{\rho}\bar{U}_i) = \bar{\rho}\bar{U}_i + \rho_f\bar{U}_i\]

\[(\rho U_i)_f = \rho U_i - (\bar{\rho}\bar{U}_i) = \bar{\rho}u_i + \rho_f\bar{U}_i + \rho_f u_i - (\rho_f\bar{U}_i)\]  
(27)

Now by the Assumption 6 and using the expansion like (27), from (11) - (13), we can write the mean equations:

\[
\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_i}(\bar{\rho}\bar{U}_i) + \frac{\partial}{\partial x_i}(\rho_f u_i) + 0(\varepsilon^3) = 0
\]  
(28)

\[
\frac{\partial}{\partial t}(\bar{\rho}\bar{U}_i) + \frac{\partial}{\partial x_j}(\bar{\rho}\bar{U}_j\bar{U}_i) + \frac{\partial}{\partial t}(\rho_f u_i) + \frac{\partial}{\partial x_j}(\bar{\rho}u_iu_j + \rho_f\bar{U}_i u_j + \rho_f u_i\bar{U}_j) = - \frac{\partial \bar{P}}{\partial x_i} + \frac{\partial (t_{ij})}{\partial x_j} + 0(\varepsilon^3)
\]  
(29)

\[
\frac{\partial}{\partial t} \left[ \bar{\rho}\bar{e} + \frac{1}{2}\bar{\rho}\bar{U}_i\bar{U}_i + \rho_f e_f + \frac{1}{2}\bar{\rho}u_iu_i + \rho_f\bar{U}_i u_i \right] +
\frac{\partial}{\partial x_j} \left[ \bar{\rho}\bar{u}_j\bar{h} + \frac{\bar{P}}{\bar{\rho}} + \frac{\rho_f \rho_f}{\rho \rho} \left( \bar{\rho}u_i u_i + \rho_f \bar{U}_i u_i \right) \right] + 
\frac{\partial}{\partial x_j} \left[ \frac{1}{2}\bar{\rho}\bar{U}_j\bar{U}_i u_i + \frac{1}{2}\bar{\rho}\bar{u}_i\bar{u}_i u_i + \frac{1}{2}\rho_f u_j\bar{U}_i\bar{U}_i + \bar{\rho}u_j\bar{U}_i u_i + \rho_f\bar{U}_j\bar{U}_i u_i \right] = \frac{\partial}{\partial x_j} \left[ \bar{U}_i t_{ij} + u_i (t_{ij})_f \right] - \frac{\partial \bar{q}_i}{\partial x_j} + 0(\varepsilon^3)
\]  
(30)
and, from (15), (19) and (20), the fluctuant equations:

\[
\frac{\partial \rho f}{\partial t} + \frac{\partial}{\partial x_i}(\tilde{\rho}u_i) + \frac{\partial}{\partial x_i}(\rho_f \tilde{U}_i) + 0(\varepsilon^3) = 0 \tag{31}
\]

\[
\frac{\partial}{\partial t}(\tilde{\rho}u_i) + \frac{\partial}{\partial x_j}(\rho_f \tilde{U}_i) + \frac{\partial}{\partial x_j}(\tilde{\rho}u_i \tilde{U}_j) + \rho_f \tilde{U}_i \tilde{U}_j \]

\[
= -\frac{\partial p}{\partial x_i} + \frac{\partial(t_{ij})}{\partial x_j} + 0(\varepsilon^3) \tag{32}
\]

\[
\frac{\partial}{\partial t} \left[ \tilde{\rho}e_f + \rho_f \tilde{\varepsilon} + \frac{1}{2} \rho_f \tilde{U}_i \tilde{U}_i + \tilde{\rho}u_i u_i \right] + \frac{\partial}{\partial x_j} \left[ \tilde{\rho} \tilde{U}_j h_f + \tilde{\rho}u_j h + \rho_f \tilde{U}_j \tilde{h} \right] + \frac{\partial}{\partial x_j} \left[ \frac{1}{2} \tilde{\rho}u_j \tilde{U}_i \tilde{U}_i + \frac{1}{2} \rho_f \tilde{U}_j \tilde{U}_i \tilde{U}_i + \tilde{\rho} \tilde{U}_j \tilde{U}_i u_i \right] \]

\[
= \frac{\partial}{\partial x_j} \left[ \tilde{U}_i (t_{ij}) + u_i \tilde{t}_{ij} \right] - \frac{\partial}{\partial x_j} (q_j) + 0(\varepsilon^3) \tag{33}
\]

Finally, from the equations (16)-(18), (28)-(30) and (31)-(33), the closed equations of turbulence in compressible fluid can be easily obtained. There are, like the case of incompressible turbulence, three choices.

Choice one: In the equations (28)-(30), the terms in the order of magnitude 0(\varepsilon^2) are omitted. We have

\[
\frac{\partial \tilde{P}}{\partial t} + \frac{\partial}{\partial x_i} (\tilde{\rho} \tilde{U}_i) = 0 \tag{34}
\]

\[
\frac{\partial}{\partial t} (\tilde{\rho} \tilde{U}_i) + \frac{\partial}{\partial x_j} (\tilde{\rho} \tilde{U}_i \tilde{U}_j) = -\frac{\partial \tilde{P}}{\partial x_i} + \tilde{t}_{ij} \tag{35}
\]

\[
\frac{\partial}{\partial t} \left[ \tilde{\rho} \tilde{c} + \frac{1}{2} \tilde{U}_i \tilde{U}_i \right] + \frac{\partial}{\partial x_j} \left[ \tilde{\rho} \tilde{U}_j (\tilde{h} + \frac{1}{2} \tilde{U}_i \tilde{U}_i) \right] = \frac{\partial}{\partial x_j} (\tilde{U}_i \tilde{t}_{ij}) - \frac{\partial}{\partial x_j} (q_j) \tag{36}
\]

\[
\tilde{P} = R \tilde{T}, \quad \tilde{h} = \tilde{c} + \frac{\tilde{P}}{\tilde{\rho}}, \quad \tilde{e} = c_v \tilde{T}, \quad \tilde{q}_j = -\kappa \frac{\partial \tilde{T}}{\partial x_j}
\]

\[
\tilde{t}_{ij} = \mu \left( \frac{\partial \tilde{U}_i}{\partial x_j} + \frac{\partial \tilde{U}_j}{\partial x_i} \right) - \frac{2}{3} \mu \delta_{ij} \frac{\partial \tilde{U}_d}{\partial x_d} \tag{37}
\]
Choice two: In the equations (31)-(33), the mean quantities are written as the differences between instantaneous and fluctuant quantities. Then the terms in the order of magnitude $0(\varepsilon^3)$ are omitted. It follows that

The instantaneous equations

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho U_i)}{\partial x_i} = 0 \tag{38}
\]

\[
\frac{\partial}{\partial t} (\rho U_i) + \frac{\partial}{\partial x_j} (\rho U_i U_j) = -\frac{\partial P}{\partial x_i} + \frac{\partial t_{ij}}{\partial x_j} \tag{39}
\]

\[
\frac{\partial}{\partial t} \left[ \rho (e + \frac{1}{2} U_i U_i) \right] + \frac{\partial}{\partial x_j} \left[ \rho U_j (h + \frac{1}{2} U_i U_i) \right] = \frac{\partial}{\partial x_j} (U_i t_{ij}) - \frac{\partial q_j}{\partial x_j} \tag{40}
\]

and the fluctuant equations

\[
\frac{\partial \rho_f}{\partial t} + \frac{\partial}{\partial x_i} (\rho_f U_i) + \frac{\partial}{\partial x_i} (\rho u_i) - 2 \frac{\partial}{\partial x_i} (\rho_f u_i) = 0 \tag{41}
\]

\[
\frac{\partial}{\partial t} (\rho u_i + \rho_f U_i - 2 \rho_f u_i) + \frac{\partial}{\partial x_j} (\rho U_i u_j + \rho u_i U_j + \rho_f U_i U_j)
\]

\[
-2 \frac{\partial}{\partial x_j} (\rho u_i u_j + \rho_f U_i u_j + \rho_f u_i U_j)
\]

\[
= -\frac{\partial p}{\partial x_i} + \frac{\partial (t_{ij})_f}{\partial x_j} \tag{42}
\]

\[
\frac{\partial}{\partial t} \left[ \rho e_f + \rho_f e - 2 \rho_f e_f + \frac{1}{2} \rho_f U_i U_i + \rho U_i u_i - \rho u_i u_i - 2 \rho_f U_i u_i \right] +
\]

\[
\frac{\partial}{\partial x_j} \left[ \rho U_j h_f + 2 (\rho U_j h) h_f - 2 \rho_f U_j h_f - 2 \rho_f U_j h + 2 \rho_f U_i h + 2 \rho_f U_j \left( \frac{\rho_p}{\rho} \frac{\rho_p}{\rho} \rho \right) \right] +
\]

\[
\frac{\partial}{\partial x_j} \left[ \frac{1}{2} \rho U_i U_i + \frac{1}{2} \rho_f U_i U_i + \rho U_j U_i u_i - \rho U_j u_i u_i - 2 \rho U_i u_i - 2 \rho_f U_j U_i u_i - \rho_f U_j U_i U_i \right]
\]

\[
- \frac{\partial}{\partial x_j} \left[ U_i (t_{ij})_f + u_i t_{ij} - 2 u_i (t_{ij})_f \right] - \frac{\partial (q_j)_f}{\partial x_j} \tag{43}
\]

Here,

\[
P = R\rho T, \quad \frac{\partial p}{\partial x_i} = R \frac{\partial}{\partial x_i} (\rho T_f + \rho_f T - 2 \rho_f T_f)
\]
\[ h = e + \frac{P}{\rho}, \quad h_f = e_f + \frac{P}{\rho} - \frac{P \rho_f}{\rho}, \quad q_j = -\kappa \frac{\partial T}{\partial x_j}, \quad (q_j)_f = -\kappa \frac{\partial T_f}{\partial x_j} \]

\[ e = c_v T, \quad e_f = c_v T_f, \quad t_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \mu \delta_{ij} \frac{\partial u_d}{\partial x_d} \]

\[ (t_{ij})_f = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \mu \delta_{ij} \frac{\partial u_d}{\partial x_d} \] (44)

Choice three: In the equations (28)-(30) and (31)-(33), the terms in the order of magnitude \(0(\varepsilon^3)\) are omitted. It is obtained that

The mean equations

\[ \frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_i} (\bar{\rho} \bar{U}_i) + \frac{\partial}{\partial x_i} (\rho_f u_i) = 0 \] (45)

\[ \frac{\partial}{\partial t} (\bar{\rho} \bar{U}_i) + \frac{\partial}{\partial x_j} (\bar{\rho} \bar{U}_i \bar{U}_j) + \frac{\partial}{\partial t} (\rho_f u_i) + \frac{\partial}{\partial x_j} (\bar{\rho} u_i u_j + \rho_f \bar{U}_i u_j + \rho_f u_i \bar{U}_j) = -\frac{\partial \bar{P}}{\partial x_i} + \frac{\partial (t_{ij})}{\partial x_j} \] (46)

\[ \frac{\partial}{\partial t} \left[ \bar{\rho} e + \frac{1}{2} \bar{\rho} \bar{U}_i \bar{U}_i + \rho_f e_f + \frac{1}{2} \bar{\rho} u_i u_i + \rho_f \bar{U}_i u_i \right] + \]
\[ \frac{\partial}{\partial x_j} \left[ \bar{\rho} \bar{U}_j (h + \frac{P}{\rho} \frac{\rho_f \rho_f}{\rho} - \frac{P \rho_f}{\rho}) + \bar{\rho} u_j h_f + \rho_f \bar{U}_j h_f + \rho_f u_j \bar{h} \right] + \]
\[ \frac{\partial}{\partial x_j} \left[ \frac{1}{2} \bar{\rho} \bar{U}_j \bar{U}_i \bar{U}_i + \frac{1}{2} \bar{\rho} u_j u_i u_i + \frac{1}{2} \rho_f u_j \bar{U}_i \bar{U}_i + \bar{\rho} u_j \bar{U}_i u_i + \rho_f \bar{U}_j \bar{U}_i u_i \right] \]
\[ = \frac{\partial}{\partial x_j} \left[ \bar{U}_i t_{ij} + u_i (t_{ij})_f \right] - \frac{\partial \bar{q}_j}{\partial x_j} \] (47)

and fluctuant equations

\[ \frac{\partial \rho_f}{\partial t} + \frac{\partial}{\partial x_i} (\bar{\rho} u_i) + \frac{\partial}{\partial x_i} (\rho_f \bar{U}_i) = 0 \] (48)

\[ \frac{\partial}{\partial t} (\bar{\rho} u_i) + \frac{\partial}{\partial t} (\rho_f \bar{U}_i) + \frac{\partial}{\partial x_j} (\bar{\rho} \bar{U}_i u_j + \bar{\rho} u_i \bar{U}_j + \rho_f \bar{U}_i \bar{U}_j) \]
\[- \frac{\partial p}{\partial x_i} + \frac{\partial (t_{ij})_f}{\partial x_j} \quad (49)\]

\[
\frac{\partial}{\partial t} \left[ \tilde{\rho} e_f + \rho_f \tilde{e} + \frac{1}{2} \rho_f \tilde{U}_j \tilde{U}_i + \tilde{\rho} \tilde{U}_i u_i \right] + \\
\frac{\partial}{\partial x_j} \left[ \tilde{\rho} \tilde{U}_j h_f + \tilde{\rho} u_j \tilde{h} + \rho_f \tilde{U}_j \tilde{h} + \frac{\partial}{\partial x_j} \left[ \frac{1}{2} \tilde{\rho} u_j \tilde{U}_i \tilde{U}_i + \frac{1}{2} \rho_f \tilde{U}_j \tilde{U}_i \tilde{U}_i + \tilde{\rho} \tilde{U}_j \tilde{U}_i u_i \right] \right] \\
= \frac{\partial}{\partial x_j} \left[ \tilde{U}_i (t_{ij})_f + u_i \tilde{t}_{ij} \right] - \frac{\partial (q_j)_f}{\partial x_j} \quad (50)
\]

Here,

\[
\frac{\partial \tilde{P}}{\partial x_i} = R \frac{\partial}{\partial x_i} [\tilde{\rho} \tilde{T} + \rho_f T_f], \quad \frac{\partial p}{\partial x_i} = R \frac{\partial}{\partial x_i} [\tilde{\rho} T_f + \rho_f \tilde{T}]
\]

\[
\tilde{h} = \tilde{e} + \frac{\tilde{P}}{\tilde{\rho}}, \quad h_f = e_f + \frac{p}{\rho} + \frac{\tilde{P} \rho_f}{\tilde{\rho} \rho}
\]

\[
\tilde{e} = c_v \tilde{T}, \quad e_f = c_v T_f, \quad \tilde{q}_j = -\kappa \frac{\partial \tilde{\rho}}{\partial x_j}, \quad (q_j)_f = -\kappa \frac{\partial T_f}{\partial x_j}
\]

\[
\tilde{t}_{ij} = \mu \left( \frac{\partial \tilde{U}_i}{\partial x_j} + \frac{\partial \tilde{U}_j}{\partial x_i} \right) - \frac{2}{3} \mu \delta_{ij} \frac{\partial \tilde{U}_d}{\partial x_d}
\]

\[
(t_{ij})_f = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \mu \delta_{ij} \frac{\partial u_d}{\partial x_d} \quad (51)
\]

We should note that these equations in the three choices are new kind of equations, which are based on the definition (1) and can hold at non-uniform points too.

Obviously, the number of unknown quantities equals to that of equations, therefore the equations in every Choice are closed. The instantaneous and fluctuant quantities are defined on nonstandard points \((x_1, x_2, x_3, t, x'_1, x'_2, x'_3, t')\). However, the mean quantities are the point(monad)-average values. Yet the average of the point-average values over certain time period or space range is once again taken. And the results of the again average could be compared with the measuring mean values over corresponding time period or space range.
References

[1] F.Wu, *Nonstandard Picture of Turbulence (The Second Revised)*, physics/0308012 (lanl.arXiv)