Affine Synthesis onto $L^p$ when $0 < p \leq 1$

R. S. Laugesen

ABSTRACT. The affine synthesis operator $S_c = \sum_{j>0} \sum_{k \in \mathbb{Z}^d} c_{j,k} \psi_{j,k}$ is shown to map the coefficient space $\ell^p(\mathbb{Z}_+ \times \mathbb{Z}^d)$ surjectively onto $L^p(\mathbb{R}^d)$, for $p \in (0, 1]$. Here $\psi_{j,k}(x) = |\det a_j|^{1/p} \psi(a_j x - b_k)$ for dilation matrices $a_j$ that expand, and the synthesizer $\psi \in L^p(\mathbb{R}^d)$ need satisfy only mild restrictions, for example, $\psi \in L^1(\mathbb{R}^d)$ with nonzero integral or else with periodization that is real-valued, nontrivial and bounded below.

An affine atomic decomposition of $L^p$ follows immediately:

$$\|f\|_p \approx \inf \left\{ \left( \sum_{j>0} \sum_{k \in \mathbb{Z}^d} |c_{j,k}|^p \right)^{1/p} : f = \sum_{j>0} \sum_{k \in \mathbb{Z}^d} c_{j,k} \psi_{j,k} \right\}.$$  

Tools include an analysis operator that is nonlinear on $L^p$.

1. Introduction

Many normed function spaces can be generated by discrete translates and dilates of just a single function. For example, Sobolev spaces can be decomposed by spline approximation or wavelet expansion. But in metric vector spaces that are not normed, the theory of such affine systems is much less developed. This article develops the affine theory of $L^p = L^p(\mathbb{R}^d), \ 0 < p \leq 1$.

Given a synthesizer $\psi \in L^p$, the affine synthesis operator is

$$c = \{c_{j,k}\} \mapsto \sum_{j>0} \sum_{k \in \mathbb{Z}^d} c_{j,k} \psi_{j,k} = Sc$$

where

$$\psi_{j,k}(x) = |\det a_j|^{1/p} \psi(a_j x - b_k), \quad x \in \mathbb{R}^d.$$  

Math Subject Classifications. Primary: 41A30, 46E30; secondary: 26B40, 42C30, 42C40.

Keywords and Phrases. Spanning, synthesis, analysis, nonlinear quasi-interpolation, Riesz basis, path connectedness.

Acknowledgements and Notes. Laugesen’s travel was supported by the NSF under Award DMS–0140481.
The dimension \( d \in \mathbb{N} \) and the exponent \( p \in (0, 1] \) are fixed. The dilation matrices \( a_j \) are invertible \( d \times d \) real matrices that are expanding, in the sense that their inverses contract to zero:
\[
\|a_j^{-1}\| \to 0 \quad \text{as } j \to \infty
\]
where \( \| \cdot \| \) denotes the operator norm of the matrix acting from \( \mathbb{R}^d \) to \( \mathbb{R}^d \). For example, one could take \( a_j = 2^j I \). The translation matrix \( b \) is an invertible \( d \times d \) real matrix, for example the identity. Note this article only uses \( j > 0 \), meaning the affine systems only use small scales.

Our first goal is to find the right domain for the synthesis operator, that is, to find a sequence space that \( S \) maps continuously into \( L^p \). Proposition 1 shows \( S \) is continuous from \( \ell^p(\mathbb{Z}_+ \times \mathbb{Z}^d) \) to \( L^p \), where \( \mathbb{Z}_+ = \{ j \in \mathbb{Z} : j > 0 \} \).

Our second goal is surjectivity: We want \( S \) to map \( \ell^p(\mathbb{Z}_+ \times \mathbb{Z}^d) \) onto \( L^p \), so that every function in \( L^p \) can be written as an infinite linear combination of the \( \psi_{j,k} \). Theorem 3 proves this surjectivity, by building on Hölder continuity of the nonlinear analysis operator as introduced in Theorem 1, and using an explicit \( L^p \)-approximation result in Theorem 2.

The underlying idea, roughly, is to quasi-interpolate via nonlinear analysis and then linear synthesis, at a very small scale, and then to apply the open mapping theorem.

To illustrate our result, observe that Theorem 3 and the sufficient condition in Proposition 2 combine to yield that if \( \psi \in L^p \cap L^1 \), \( p \in (0, 1) \), and either \( \int_{\mathbb{R}^d} \psi \, dx \neq 0 \) or else \( 0 \neq \sum_{k \in \mathbb{Z}^d} \psi(x - bk) \) is real-valued and bounded below, then \( S \) maps surjectively onto \( L^p \). Thus surjectivity holds for a large class of synthesizers \( \psi \). Indeed, Theorem 6 takes a global perspective and shows that surjectivity of the synthesis operator holds generically with respect to the choice of synthesizer \( \psi \in L^p \).

Interestingly, Strang-Fix conditions are not required in this article: The integer translates of the synthesizer need not form a partition of unity. But if these translates do sum up to 1, then the bounds in the surjectivity result Theorem 3 get better (because one can take \( \sigma = 0 \) and \( \lambda = 1 \) there).

Surjectivity of \( S \) onto \( L^p \) immediately implies an affine atomic decomposition (or metric equivalence) in Corollary 1, of the form
\[
\|f\|_p \approx \inf \{\|c\|_{\ell^p(\mathbb{Z}_+ \times \mathbb{Z}^d)} : f = Sc \}.
\]
When \( p = 1 \) this atomic decomposition was found earlier by Bruna [5, Theorem 4].

Corollary 2 localizes the atomic decomposition to \( L^p(\Omega) \), for domains \( \Omega \subset \mathbb{R}^d \).

Theorem 5 restricts to a single dilation scale \( j \), and states an atomic decomposition that does not need an “inf”. In other words, it proves when \( \psi \) has compact support that the \( \psi_{j,k} \), \( k \in \mathbb{Z}^d \), form a \( p \)-Riesz basis for their closed linear span in \( L^p \), or that the synthesis operator at scale \( j \) is bounded, injective and has closed range in \( L^p \). This result slightly extends some of Jia’s work [20] on \( L^p \)-stability of shift invariant subspaces.

The results of this article for \( 0 < p \leq 1 \) are contrasted with prior work on \( p \geq 1 \) in Section 3. Open problems are raised in Section 4, including Meyer’s Mexican hat spanning problem for \( L^p \) when \( 1 < p < \infty \), which this article resolves for \( 0 < p < 1 \).

**Discussion.**

This article shows that arbitrary \( L^p \) functions can be decomposed into linear combinations of discrete translates and dilates of the synthesizer \( \psi \), without requiring any particularly special properties of \( \psi \). This structural information about \( L^p \) has intrinsic mathematical interest, and might conceivably be useful in applications for which a particular shape of \( \psi \) is naturally preferred.
The central contribution of the article is its constructive method of $L^p$-controlled approximation via nonlinear analysis and linear synthesis (Theorem 2), which implies surjectivity of the synthesis operator (Theorem 3). The closest prior result for $p \in (0, 1)$ is due to Filippov and Oswald [16, 17], who proved for isotropic dilation matrices that every $L^p$ function can be written as $Sc$ for some sequence $c$, but unfortunately with no information on the size of $c$ or to what space $c$ might belong.

Incidentally, DeVore et al. [14] have proved that linear combinations of the $\psi_{j,k}$ provide good $L^p$-approximations to functions in Besov spaces, and an abstract framework for that work was developed in [11].

**Notation.**

$L^p = L^p(\mathbb{R}^d)$ denotes the class of complex valued functions with $\|f\|_p = (\int_{\mathbb{R}^d} |f|^p \, dx)^{1/p} < \infty$. It is a complete metric space with distance function

$$d_p(f, \tilde{f}) = \|f - \tilde{f}\|_p^p.$$

We write $f \equiv 0$ to mean $f = 0$ a.e., that is, $\|f\|_p = 0$.

A multi-scale, discrete analogue of $L^p$ is the space $\ell^p(\mathbb{Z}_+ \times \mathbb{Z}^d)$ consisting of doubly-indexed sequences $c = \{c_{j,k}\}_{j>0, k \in \mathbb{Z}^d}$ of complex numbers satisfying

$$\|c\|_{\ell^p(\mathbb{Z}_+ \times \mathbb{Z}^d)} = \left( \sum_{j>0} \sum_{k \in \mathbb{Z}^d} |c_{j,k}|^p \right)^{1/p} < \infty.$$

Clearly $\ell^p(\mathbb{Z}_+ \times \mathbb{Z}^d)$ is a complete metric space with distance function

$$d_{\ell^p}(c, \tilde{c}) = \|c - \tilde{c}\|_{\ell^p(\mathbb{Z}_+ \times \mathbb{Z}^d)}^p.$$

**Useful fact.**

The $\ell^p$-triangle inequality on the complex numbers says

$$\left| \sum_m z_m \right|^p \leq \sum_m |z_m|^p, \quad z_m \in \mathbb{C}, \quad p \in (0, 1].$$

In other words, $p$-th powers can be taken inside sums.

## 2. Results

Our first four results show that synthesis maps continuously into $L^p$, that nonlinear analysis is continuous on $L^p$, that synthesis and analysis can partially reconstruct every $L^p$ function, and hence that synthesis maps surjectively onto $L^p$. Next we deduce an affine atomic decomposition of $L^p$, and a metric equivalence via the analysis operator. Then we prove synthesis at each fixed dilation scale gives a $p$-Riesz basis. Our last result considers the class of all synthesizers for which the synthesis operator is surjective, and proves the class is dense, open and connected in $L^p$.

Recall from the introduction that the synthesis operator is

$$Sc = \sum_{j>0} \sum_{k \in \mathbb{Z}^d} c_{j,k} \psi_{j,k}.$$ 

(2.1)
Synthesis is Lipschitz continuous.

**Proposition 1** (Synthesis into $L^p$). Assume $\psi \in L^p$, $p \in (0, 1]$. Then $S : \ell^p(\mathbb{Z}^+ \times \mathbb{Z}^d_\ast) \rightarrow L^p$ is continuous and linear. More precisely, if $c \in \ell^p(\mathbb{Z}^+ \times \mathbb{Z}^d_\ast)$ then the series (2.1) for $Sc$ converges pointwise absolutely a.e., to a function in $L^p$ (and hence $Sc$ converges unconditionally in $L^p$), and

$$
\|Sc\|_p \leq \|\psi\|_p \|c\|_{\ell^p(\mathbb{Z}^+ \times \mathbb{Z}^d_\ast)} .
$$

(2.2)

The proof is in Section 6.1.

Next we develop our analysis operator. The traditional linear definition of analysis is clearly invalid on $L^p$ for $p < 1$, for if we tried to integrate $f \in L^p$ against an analyzing function then the integral might not even exist, since $f$ need not be locally integrable. We compensate for this lack of local integrability by applying a nonlinear radial stretch to $f$ before analyzing it, and undoing the stretch afterwards.

The radial stretch function

$$
\Theta(z) = |z|^{p-1}z, \quad z \in \mathbb{C} ,
$$

is a homeomorphism of the complex plane and satisfies $|\Theta(z)| = |z|^p$. It acts on complex-valued functions $f$ by

$$
(\Theta f)(x) = \Theta(f(x)) .
$$

Notice that if $f \in L^p$ then $\Theta f \in L^1$.

Define the **analysis operator at scale $j$** by

$$
T_j f = \left\{ |\det b|^{-1}(\Theta^{-1}(\Theta f, \phi(a_j \cdot -bk))) \right\}_{k \in \mathbb{Z}^d} ,
$$

(2.3)

where the notation $\langle \cdot , \cdot \rangle$ simply represents the integral: $\langle u, v \rangle = \int_{\mathbb{R}^d} u \tau \, dx$.

Roughly, $T_j$ maps a function $f$ to its sequence of sampled $\phi$-averages at scale $j$, except that radial stretching is applied to $f$ before the sampling, and then is undone at the end. The $|\det b|$ factor is for later convenience. We emphasize that

the analysis operator $T_j$ is **nonlinear**, and depends implicitly on the exponent $p$.

(The synthesis operator also depends on $p$, through the normalization of $\psi_{j,k}$.)

The next theorem shows the analysis operator is continuous. The hypotheses involve the **periodization** operator, defined on a function $g$ by

$$
Pg(x) = |\det b| \sum_{k \in \mathbb{Z}^d} g(x - bk) \quad \text{for } x \in \mathbb{R}^d .
$$

Clearly $Pg$ is periodic with respect to the lattice $b\mathbb{Z}^d$, provided it is well-defined a.e.

**Theorem 1** (Analysis on $L^p$). Assume $p \in (0, 1]$ and take $\phi \in L^\infty$ with $P|\phi| \in L^\infty$. Then for each $j$, the analysis operator $T_j : L^p \rightarrow \ell^p(\mathbb{Z}^d_\ast)$ is locally Hölder continuous, with

$$
d_{\ell^p}(T_j f, T_j g) \leq C[d_p(f, 0) + d_p(0, g)]^{1-p}d_p(f, g)^p , \quad f, g \in L^p .
$$

Here $C$ depends on the translation matrix $b$, the exponent $p$ and the analyzer $\phi$, but not on the dilation scale $j$. 
**Affine Synthesis onto** \( L^p \) **when** \( 0 < p \leq 1 \)

Section 6.2 has the proof. Recall the distance function on \( \ell^p(\mathbb{Z}^d) \) is \( d_{\ell^p}(s, t) = \| s - t \|_{\ell^p(\mathbb{Z}^d)}^p \) when \( p \in (0, 1] \). The hypothesis that the absolute value \(|\phi|\) of the analyzer have bounded periodization is easily satisfied, say if \( \phi \) is bounded with compact support or with rapid decay.

The nonlinear analysis operator \( T_j \) is also locally Hölder continuous when \( p > 1 \), with

\[
\|d_{\ell^p}(T_j f, T_j g)\| \leq C \left[ \|d_{\ell^p}(f, 0)\|_{\ell^p(\mathbb{Z}^d)}^{1/p} \|d_{\ell^p}(0, g)\|_{\ell^p(\mathbb{Z}^d)}^{1/p} \right]^{1/p},
\]

where for \( p > 1 \) the distance functions are defined by \( d_{\ell^p}(s, t) = \| s - t \|_{\ell^p(\mathbb{Z}^d)} \) and \( d_p(f, g) = \| f - g \|_p \). We omit the proof, since the rest of this article concerns \( p \in (0, 1] \).

Now we start to develop approximation results. Write

\[
S_j s = \sum_{k \in \mathbb{Z}^d} s_k \psi_{j,k}
\]

for the synthesis operator at scale \( j \), acting on sequences \( s = \{s_k\}_{k \in \mathbb{Z}^d} \). Notice \( S_j \) is continuous from \( \ell^p(\mathbb{Z}^d) \) to \( L^p \), with

\[
\|S_j s\|_{\ell^p(\mathbb{Z}^d)} \leq \|\psi\|_{\ell^p(\mathbb{Z}^d)} \|s\|_{\ell^p(\mathbb{Z}^d)}
\]

for \( p \in (0, 1] \) by Proposition 1.

The following approximation result will be used later in proving surjectivity of the synthesis operator. It uses \( C = [0, 1]^d \) to denote the unit cube in \( \mathbb{R}^d \).

**Theorem 2** (Affine quasi-interpolation). Assume \( \psi \in L^p, p \in (0, 1] \), and suppose \( \phi \in L^\infty \) with \( P|\phi| \in L^\infty \) and \( \int_{\mathbb{R}^d} \phi \, dx = 1 \).

Then

\[
\lim_{j \to \infty} \|S_j T_j f - f\|_p = \|P\psi(b \cdot) - 1\|_{L^p(C)} \|f\|_p, \quad f \in L^p.
\]

In particular, if \( \psi \) has constant periodization \( P\psi = 1 \) a.e., then \( S_j T_j f \to f \) in \( L^p \) as \( j \to \infty \).

Section 6.4 has the proof. Note the periodization series \( P\psi(bx) \) appearing in the theorem is well defined whenever \( \psi \in L^p \), because it converges absolutely a.e., and belongs to \( L^p(C) \):

\[
\int_C |P\psi(b x)|^p \, dx \leq |\det b|^p \int_C \sum_{k \in \mathbb{Z}^d} |\psi(b(x - k))|^p \, dx \quad \text{using} \ p \in (0, 1]
\]

\[
= |\det b|^p \|\psi(b \cdot)\|_p^p < \infty.
\]

The constant periodization condition \( P\psi = 1 \) says that the collection \( \{\det b|\psi(x - bk) : k \in \mathbb{Z}^d\} \) of translates of \( \psi \) is a partition of unity. Examples of such \( \psi \) (when \( b = I \)) include the indicator function \( 1_C \) and \( B \)-splines obtained by convolution with this indicator function.

When \( p \in [1, \infty) \) and \( \psi \) has constant periodization, the result that \( S_j T_j f \to f \) (with \( T_j \) being a linear analysis operator) has a long history in Strang-Fix approximation theory, summarized in [7, Section 3].
Next we show every $f \in L^p$ can be written as $Sc$ for some sequence $c \in \ell^p(Z_+ \times \mathbb{Z}^d)$, so that the synthesis operator is surjective.

**Theorem 3** (Synthesis onto $L^p$). Assume $\psi \in L^p$, $p \in (0, 1]$, and suppose that

$$\sigma := \|\lambda P\psi(b \cdot) - 1\|_{L^p(C)} < 1$$

(2.4)

for some $\lambda \in \mathbb{C}$.

Then $S : \ell^p(Z_+ \times \mathbb{Z}^d) \to L^p$ is open, and surjective. Indeed, if $f \in L^p$ and $\sigma' \in (\sigma, 1)$ then a sequence $c \in \ell^p(Z_+ \times \mathbb{Z}^d)$ exists such that $Sc = f$ and

$$\|c\|_{\ell^p(Z_+ \times \mathbb{Z}^d)} \leq (1 - \sigma')^{-1/p} |\lambda| |\det b|^{1-1/p} \|f\|_p.$$

The proof is in Section 6.5. Hypothesis (2.4) is discussed below.

I do not know any prior general work on surjectivity of the synthesis operator when $p \in (0, 1)$. The closest seems to be Filippov and Oswald’s construction in [16, 17] of “representation systems”, by which every $f \in L^p$ can be written as a convergent series $Sc = f$, provided the dilation matrices are real multiples of the identity. This construction looks similar to surjectivity, but the drawback is that their result yields no control over the size of coefficients in the sequence $c$, and thus it is unclear what the domain of the synthesis operator actually is. The achievement of Theorem 3 is to identify the sequence space $\ell^p(Z_+ \times \mathbb{Z}^d)$ as a domain from which $S$ maps onto $L^p$.

Filippov and Oswald’s method involves iterative approximation of simple functions, and thus is less concrete than our approach. Their article uses the open mapping theorem implicitly, whereas we use it explicitly. Interestingly, their key lemma is crucial to our proof too (Proposition 2 below), although we employ it differently.

**Noninjectivity of $S$**. The synthesis operator is certainly not injective, and has a very large kernel. For example, we could discard the dilation $a_1$ (discarding all terms with $j = 1$ in the sum defining $Sc$) and still show $S$ maps onto $L^p$, by applying Theorem 3 with the remaining dilations $\{a_2, a_3, \ldots\}$.

For Theorem 3 to be interesting, we need to exhibit examples of synthesizers $\psi$ satisfying hypothesis (2.4).

**Example**. If $\psi$ is supported in the cube $C$ with $\psi \geq 1$ there, and with $\|\psi - 1\|_{L^p(C)} < 1$ and $b = I$, then $\psi$ equals its own periodization on $C$, and so (2.4) holds with $\lambda = 1$. To be specific, in one dimension one could take $\psi(x) = (1 + Ax^{-\beta})\mathbb{1}_{[0,1)}$ for $\beta < 1/p$ and suitably small $A > 0$. This example shows $\psi$ can have a typical $L^p$-singularity at the origin.

Another singular example in one dimension is $\psi(x) = x^{-\beta} \mathbb{1}_{[0,1)}$ for $\beta \in (0, 2/(p + 1))$, which we prove in Section 6.6 satisfies (2.4) when $b = 1$.

On the other hand, some functions $\psi$ do not satisfy (2.4):

**Counterexample**. Tachev [27] proved that (2.4) with $b = 1$ fails in one dimension for $\psi(x) = x^{-\beta} \mathbb{1}_{[0,1)}$ when $\beta \in [2/(p + 1), 1/p)$ and $p \in (0, 1)$. We provide a proof in Section 6.6.

The next result contains several easy-to-check conditions that imply hypothesis (2.4).

**Proposition 2** (Sufficient conditions). Assume $\psi \in L^p$, $p \in (0, 1)$, and that one of the following conditions holds:

(a) $\psi \in L^1$ with $\int_{\mathbb{R}^d} \psi \, dx \neq 0$;
Affine Synthesis onto $L^p$ when $0 < p \leq 1$

(b) $\psi \in L^1$ with $\int_{\mathbb{R}} \psi \, dx = 0$, and $0 \not\equiv P\psi$ is real-valued and bounded either above or below;

(c) $P\psi \in L^2_{\text{loc}}$ with $p \int_{\mathbb{C}} |P\psi(b \cdot)|^2 \, dx < (2 - p) \int_{\mathbb{C}} (P\psi(b \cdot))^2 \, dx$.

Then for some $\lambda \in \mathbb{C}$ (in fact with $|\lambda| < 1$),
$$
\|\lambda P\psi(b \cdot) - 1\|_{L^p(\mathbb{C})} < 1.
$$

This conclusion holds also if $p = 1$ and $\psi$ satisfies condition (a).

Section 6.7 has the proof. Part (a) of the proposition is essentially due to Filippov and Oswald [16, Lemma 1], and so is part (c) when $\psi$ is real-valued. Part (b) is new.

**Example.** Condition (c) in the proposition holds if $\psi \in L^2$ is real valued with compact support (or with rapid enough decay to ensure $P\psi \in L^2_{\text{loc}}$) and with $P\psi \not\equiv 0$, because $p < 2 - p$. In particular, in one dimension with $a_j = 2^j$ and $b = 1$, condition (c) [and also condition (b)] covers the Haar wavelet $\psi = \mathbb{I}_{[0,1/2)} - \mathbb{I}_{[1/2,1)}$. Thus while the dyadic Haar system needs all scales $j \in \mathbb{Z}$ to span $L^2(\mathbb{R})$, the small scales $j > 0$ suffice to span $L^p(\mathbb{R})$ for $p \in (0,1)$, by Theorem 3.

This Haar example reminds us the integral of $\psi$ can equal 0, in parts (b) and (c). When $p = 1$, on the other hand, it is necessary for Proposition 2 that $\psi$ have nonzero integral, because otherwise
$$
\|\lambda P\psi(b \cdot) - 1\|_{L^1(\mathbb{C})} \geq \left| \int_{\mathbb{C}} (\lambda P\psi(b \cdot) - 1) \, dx \right| = 1.
$$

Theorem 3 also requires $\psi$ to have nonzero integral when $p = 1$, because otherwise $S$ can synthesize only the $L^1$ functions that have integral zero.

The next example shows it can be wise to ignore some of the translations.

**Under-synthesizing example.** Hypothesis (2.4) definitely fails if the periodization of $\psi$ vanishes identically. But if the periodization of $\psi$ with respect to some integer multiple of $b$ is nontrivial, then surjectivity of synthesis can still hold, as we explain. For simplicity, work in one dimension with $b = 1$ and suppose $\psi = \eta - \eta(-1)$ for some $\eta \in L^p[0,1]$, so that $\psi$ has the form of a unit step difference. (An illustrative example is the Haar-type function $\psi = \mathbb{I}_{[0,1)} - \mathbb{I}_{[1,2)}$.) The periodization of $\psi$ is $\eta - \eta \equiv 0$, and so Proposition 2 does not apply. But if we consider the same $\psi$ with $b = 2$, then the 2-periodization of $\psi$ equals $2\psi$ on the interval $[0,2)$. Hence, conditions (b) or (c) in Proposition 2 apply for $p \in (0,1)$, assuming $\psi \not\equiv 0$ is either bounded or is real valued and square integrable, respectively. Theorem 3 with $b = 2$ then tells us that every $f \in L^p(\mathbb{R})$ can be written as
$$
f = \sum_{j > 0} \sum_{k \in \mathbb{Z}} \tilde{c}_{j,k} |a_j|^{1/p} \psi(a_j x - 2k)
$$
for some $\tilde{c} \in \ell^p(\mathbb{Z}_+ \times \mathbb{Z})$. That is, $f = Sc$ where $c_{j,k} = \tilde{c}_{j,k}/2$ if $k$ is even, and $c_{j,k} = 0$ if $k$ is odd, and where $S$ denotes synthesis with $b = 1$. In other words, we have shown $S$ is surjective by under-synthesizing by a factor of 2, using only the even translates.

Under-synthesis leads also to the following variant of Theorem 3 and Proposition 2, proved in Section 6.8.

**Theorem 4 (More synthesis onto $L^p$).** Assume $\psi \in L^p \cap L^1 \setminus \{0\}$ for some $p \in (0,1)$, and that $\psi$ is real-valued and has negative part $\psi_-$ that is bounded with compact support.
Then $S : \ell^p(\mathbb{Z}_+ \times \mathbb{Z}^d) \to L^p$ is open, and surjective. Indeed, a constant $C = C(\psi, b, p)$ exists such that if $f \in L^p$ then there is a sequence $c \in \ell^p(\mathbb{Z}_+ \times \mathbb{Z}^d)$ with $Sc = f$ and $\|c\|_{\ell^p(\mathbb{Z}_+ \times \mathbb{Z}^d)} \leq C \|f\|_p$.

The conclusion holds also if the positive part of $\psi$ is bounded and has compact support, instead of the negative part.

**Example.** If $\eta \in L^\infty[0,1]$ is real-valued and $\eta \neq 0$, then the second difference function $\psi(x) = \eta(x+1) - 2\eta(x) + \eta(x-1)$ is covered by Theorem 4, but $\psi$ is not covered by Theorem 3 when $b = 1$ because $P\psi \equiv 0$.

Equivalence of the $L^p$ and $\ell^p$ metrics follows immediately from Proposition 1 and Theorem 3.

**Corollary 1** (Affine atomic decomposition of $L^p$). Assume $\psi \in L^p$, $p \in (0,1]$, and that $\psi$ satisfies hypothesis (2.4) in Theorem 3. Then for all $f \in L^p$,

$$\|f\|_p \approx \inf \{\|c\|_{\ell^p(\mathbb{Z}_+ \times \mathbb{Z}^d)} : f = Sc\}.$$

For $p = 1$, the corollary was proved by Bruna [5, Theorem 4] for $\psi \in L^1$ with $\int_{\mathbb{R}^d} \psi \, dx \neq 0$. His duality methods apply without our assumption that the translations lie in a lattice.

Next we localize the atomic decomposition to an open set $\Omega \subset \mathbb{R}^d$.

**Definition.** Say that a sequence $c = \{c_{j,k}\}_{j \geq 0, k \in \mathbb{Z}^d}$ is adapted to $\Omega$ and $\psi$ if $\text{spt}(\psi_{j,k}) \subset \Omega$ whenever $c_{j,k} \neq 0$, or in other words if $c_{j,k} = 0$ whenever $\text{spt}(\psi_{j,k}) \cap \Omega^c \neq \emptyset$.

The purpose of the definition is to ensure $Sc = 0$ on the complement of $\Omega$.

**Corollary 2** (Affine atomic decomposition of $L^p(\Omega)$). Assume $\Omega \subset \mathbb{R}^d$ is open and nonempty, take $p \in (0,1]$, and suppose $\psi \in L^p$ is compactly supported and satisfies hypothesis (2.4) in Theorem 3. Then for all $f \in L^p(\Omega)$,

$$\|f\|_{L^p(\Omega)} \approx \inf \{\|c\|_{\ell^p(\mathbb{Z}_+ \times \mathbb{Z}^d)} : f = Sc \text{ and } c \text{ is adapted to } \Omega \text{ and } \psi\}.$$

The constants in this metric equivalence are the same as in Corollary 1; thus they depend on $\psi, b$ and $p$ but are independent of $\Omega$. The corollary is proved in Section 6.9.

The full analysis operator puts the analysis sequences from all the different scales into a combined doubly-indexed sequence, by

$$Tf = \left\{ |\det b| \Theta^{-1}(\Theta f, \phi(a_j \cdot -bk)) \right\}_{j \geq 0, k \in \mathbb{Z}^d}.$$

It too yields a coefficient metric, as we prove in Section 6.10.

**Corollary 3** (Analysis metric for $L^p$). Assume $p \in (0,1]$, and take an analyzer $\phi \in L^\infty$ with $P|\phi| \in L^\infty$ and $\int_{\mathbb{R}^d} \phi \, dx = 1$.

Then for all $f \in L^p$,

$$\|f\|_p \approx \|Tf\|_{L^\infty(\ell^p)} = \sup_{j \geq 0} \left[ \sum_{k \in \mathbb{Z}^d} |(\Theta f, \phi(a_j \cdot -bk))| \right]^{1/p} |\det b|.$$
Synthesis at a single scale: Conditions for a $p$-Riesz basis.

This article concentrates on synthesizing $L^p$ by using all scales $j > 0$. But we divert briefly now from this mission to give a fuller account of the synthesis operator at a single scale $j$. We establish conditions for $S_j$ to be injective and have closed range, which is equivalent to “stability” or a $p$-Riesz basis condition, or an atomic decomposition at scale $j$.

Write $e_ξ(x) = e^{2πiξ·x}$, where $ξ ∈ \mathbb{R}^d$ is a row and $x ∈ \mathbb{R}^d$ is a column vector.

**Theorem 5** ($p$-Riesz basis at scale $j$). Assume $ψ ∈ L^p$, $p ∈ (0, 1]$, and that $P(e_ξ ψ) \not\equiv 0$ for each $ξ ∈ \mathbb{R}^d$. Let $j > 0$.

Then $S_j : ℓ^p(\mathbb{Z}^d) → L^p$ is injective.

If in addition $ψ$ has compact support, then $\|S_j s\|_p ≈ \|s\|_{ℓ^p(\mathbb{Z}^d)}$, or

$$\left\| \sum_{k ∈ \mathbb{Z}^d} s_k ψ_{j,k} \right\|_p ≈ \left( \sum_{k ∈ \mathbb{Z}^d} |s_k|^p \right)^{1/p}, \quad s ∈ ℓ^p(\mathbb{Z}^d).$$

(2.5)

That is, the collection of integer shifts $\{ψ_{j,k} : k ∈ \mathbb{Z}^d\}$ forms a $p$-Riesz basis for its span. Hence, the range $S_j(ℓ^p(\mathbb{Z}^d))$ is closed in $L^p$.

The constants in the norm equivalence (2.5) depend on $ψ$, $b$, and $p$, but are independent of $j$.

Theorem 5 is proved in Section 6.11. The periodization hypothesis $P(e_ξ ψ) \not\equiv 0$ says in the Fourier domain (when $ψ$ has a Fourier transform) that the sequence $\{ψ(ℓb^{-1} - ξ)\}_{ξ ∈ \mathbb{Z}^d}$ is nontrivial, because this sequence gives the Fourier coefficients of the $\mathbb{Z}^d$-periodic function $P(e_ξ ψ)(bx)$.

Ron [24] proved the injectivity conclusion in Theorem 5, and Jia [20] proved the Riesz basis conclusion (2.5), except that here we work directly with the periodization hypothesis $P(e_ξ ψ) \not\equiv 0$ and thus need not assume like Ron and Jia that $ψ$ can be Fourier transformed. (Note our $ψ ∈ L^p$ need not be locally integrable, or even a distribution.) Otherwise we follow Ron and Jia’s method exactly. Incidentally, Ron obtained a converse when $ψ$ has compact support and the restriction that $s ∈ ℓ^p(\mathbb{Z}^d)$ is dropped, saying that injectivity implies nontrivial periodizations, and Jia proved a converse saying that if (2.5) holds then $P(e_ξ ψ) \not\equiv 0$ for all $ξ ∈ \mathbb{R}^d$.

Properties of the class of synthesizers.

So far we have concentrated on individual synthesizers and analyzers. Now we broaden the view and consider the whole class

$$S^p = \{ ψ ∈ L^p : S_ψ maps ℓ^p(\mathbb{Z}_+ × \mathbb{Z}^d) onto L^p \}$$

of surjective synthesizers, where the $ψ$-dependence of the synthesis operator is emphasized by writing $S = S_ψ$.

**Theorem 6** (Most $L^p$ functions are surjective affine synthesizers). Let $p ∈ (0, 1]$. Then $S^p$ is dense, open, and path connected in $L^p$.

The proof is in Section 6.12.
3. Remarks on $L^p$ for $p \geq 1$, and on Hardy and Sobolev Spaces

This article treats affine synthesis in $L^p$ for $p \in (0, 1]$. It is helpful to contrast these results and methods with the corresponding work for $p \geq 1$.

Affine synthesis when $p \geq 1$ was treated in my article [10] with H.-Q. Bui. The domain of synthesis there is a mixed-norm sequence space $\ell^1(\ell^p)$ (meaning $\ell^p$ with respect to translations $k$ and then $\ell^1$ with respect to dilations $j$). The only situations I know where $p > 1$ and synthesis is bounded on the domain $\ell^p(\mathbb{Z}_+ \times \mathbb{Z}^d)$ like in this article (meaning $\ell^p$ with respect to both translation and dilation) are when the $\psi_{j,k}$ possess some “cancelation between scales”, such as for wavelets in $L^2$ where the $\psi_{j,k}$ form an orthonormal basis. This matter is discussed in [10, Section 5.1]. Of course, the two domain spaces coincide when $p = 1$.

Surjectivity of synthesis was established in [10] via $L^p$-controlled approximation, using linear analysis and linear synthesis. (A linear analysis operator could be used because $L^p$ functions are locally integrable when $p \geq 1$.) But rather than obtaining surjectivity of synthesis from Theorem 2 and Proposition 2(a) (which both still hold for $p \geq 1$ assuming $P|\psi| \in L^p_{\text{loc}}$), we used in [10] our scale-averaging technique from [7] to essentially take $\sigma = 0$, thereby improving the estimate on the norm of $c$ in the analogue of Theorem 3. Scale-averaging of this kind completely fails when $p \in (0, 1)$, because the unit ball of $L^p$ is nonconvex. Hence, we must make do in this article with the somewhat cruder approximation provided by Theorem 2.

Surjectivity of synthesis when $\psi$ has nonzero integral was proved earlier in [5, Theorem 4] for $p = 1$, and even earlier in [28, Theorem 2], [29], for $p \geq 1$. These works all proceed by studying the analysis operator (proving $\|Tf\|_{\ell^\infty(\ell^p)} \approx \|f\|_p$) and then invoking duality; thus they provide no constructive method of synthesis like we provide in [10] and in this article.

The theory of $p$-Riesz bases for $1 < p < \infty$ is described in Christensen’s book [12, Sections 7.2 and 17.4], and Theorem 5 should be read in that context.

A density result for synthesizers is known when $p > 1$, like in Theorem 6. But openness and path connectedness are not known: See [7, Section 4] for relevant remarks. Incidentally, some path connectedness results in the difficult wavelet and wavelet frame case, in $L^2$, can be found in [4, 32] and the references therein.

Scale-averaging and surjectivity results for the Hardy space $H^1$ and for Sobolev spaces are contained in [8, 9, 10]. Many smooth wavelet systems are known to span the Hardy space too [18, Theorem 5.6.19]. But note the Haar system does not span Hardy space: The closed span of the Haar system in $H^1(\mathbb{R})$ is the proper subspace $\{f \in H^1(\mathbb{R}) : \int_0^\infty f(x)dx = 0\}$, by [2, Theorem 2.1]; also see [1].

4. Open Problems

Obtain a Larger Class of Synthesizers?

The synthesizer $\psi \in L^p$ in Theorem 3 is assumed to satisfy

$$\|\lambda P \psi(b \cdot) - 1\|_{L^p(\mathbb{C})} < 1$$
for some $\lambda \in \mathbb{C}$.

Can this condition be weakened (or even eliminated), to obtain a larger class of synthesizers?
Affine Synthesis onto $L^p$ when $0 < p \leq 1$

Two facts advocate for caution here. First, our sufficient condition (4.1) fails for some $\psi \in L^p$, as Tachev [27] observed in one dimension (see Section 6.6 below). Second, if $\int_{\mathbb{R}} \psi \, dx = 0$ and $P \psi \in L^\infty$ satisfies the reverse of the inequality in Proposition 2(c), then the proof of that part of the proposition adapts to give the reverse of (4.1) for all small $\lambda \in \mathbb{C}, \lambda \neq 0$.

Employ Large Scales?

We have only used small scales $j > 0$ in this article. The large scales $j \leq 0$ are employed (for example) by wavelet systems in $L^p, p > 1$, and so it is natural to ask how our synthesis results for $p \in (0, 1)$ might be affected by the introduction of large scales.

For the sake of concreteness, consider dyadic dilations $a_j = 2^j I$ for $j \in \mathbb{Z}$. Then the synthesis operator $S : \ell^p(\mathbb{Z} \times \mathbb{Z}^d) \to L^p$ is bounded, since it makes no difference in the proof of Proposition 1 that the dilation parameter runs over all $j \in \mathbb{Z}$. The hope is to find a new approach to proving surjectivity of $S$ that somehow incorporates the large scales, and thereby yields synthesizers not already covered by Proposition 2 and Theorem 4.

In addition to introducing large scales, one might allow modulations (wavepacket theory, like in [13, 15, 22]) or multiplication by polynomials (like in Gausslet theory [30] and quarkonial theory [31]), and again ask for synthesizers that are surjective onto $L^p$ for $p \in (0, 1)$.

Establish a Rate of Convergence?

Theorem 2 says that if $\psi$ has constant periodization $P \psi \equiv 1$, then nonlinear quasi-interpolation at small scales will converge to the sampled function, meaning $S_j T_j f \to f$ in $L^p$.

How fast is this convergence to $f$? When $p \geq 1$, Strang-Fix theory [3, 9, 19, 26] provides very precise convergence rates, for synthesizers $\psi$ satisfying Strang-Fix conditions and signals $f$ belonging to a Sobolev space. The challenge is to develop similar approximation rate results when $p \in (0, 1)$, bearing in mind the nonlinearity of the analysis operator.

Mexican Hat Spanning Problem

It is an open problem of Y. Meyer [23, p. 137] to determine whether the affine system $\{\psi(2^j x - k) : j, k \in \mathbb{Z}\}$ spans $L^p(\mathbb{R})$ for each $p \in (1, \infty)$, when $\psi(x) = (1 - x^2)e^{-x^2/2}$ is the Mexican hat function (the second derivative of the Gaussian). The system is known to span $L^2$, where it forms a frame, but the question remains open for all other $p$-values between 1 and $\infty$.

Theorem 3 completely resolves the Mexican hat problem for $0 < p < 1$, because the Mexican hat $\psi$ belongs to $L^p$ and has periodization $P \psi \in L^\infty \setminus \{0\}$, so that it satisfies the hypotheses of Proposition 2(b) and hence of Theorem 3. (To see $P \psi \neq 0$, note the Fourier coefficients of $P \psi$ are given by the values of $\hat{\psi}$ at the integers, which are nonzero because $\hat{\psi}(\xi)$ equals a constant times $\xi^2$ times a Gaussian.) In fact, Theorem 3 shows the Mexican hat system spans $L^p$ using only the small scales $j > 0$, rather than all $j \in \mathbb{Z}$ as allowed by the problem.

More discussion of the Mexican hat problem can be found in [10, Section 8].
5. Radial Stretching

We will need some elementary continuity properties of the radial stretch function \( \Theta(z) = |z|^{p^{-1}}z \). Remember \( |\Theta(z)| = |z|^p \).

**Lemma 1** (Hölder continuity). If \( p \in (0, 1] \) then
\[
|\Theta(w) - \Theta(z)| \leq 2^{1-p}|w - z|^p, \quad w, z \in \mathbb{C}.
\]

**Proof of Lemma 1.** First reduce to the case \( w = 1 \) and \( z = re^{i\phi} \) with \( 0 < r \leq 1 \) and \( 0 \leq \phi \leq \pi \), by a rotation, dilation and reflection if necessary. Thus we want
\[
\left| \frac{1 - r^{p}e^{i\phi}}{2} \right| \leq \left| \frac{1 - e^{i\phi}}{2} \right|^p.
\]

This inequality holds for \( \phi = \pi \) just by concavity of \( r \to r^p \). To prove (5.1) for \( 0 \leq \phi < \pi \), we write \( R(\phi) \) for the ratio of the left side of (5.1) over the right side, and observe that
\[
\frac{\partial}{\partial \phi} \log R(\phi) = \sin \phi \left\{ r^p \left| 1 - r^p e^{i\phi} \right|^{-2} - pr \left| 1 - re^{i\phi} \right|^{-2} \right\} \\
\geq \sin \phi \left\{ r^p \left| 1 - r^p e^{i\phi} \right|^{-2} - r \left| 1 - re^{i\phi} \right|^{-2} \right\} \quad \text{since } p \leq 1 \\
= \sin \phi \left\{ (r^{-1} + r) - (r^{-p} + r^p) \right\} r^p \left| 1 - r^p e^{i\phi} \right|^{-2} r \left| 1 - re^{i\phi} \right|^{-2} \\
\geq 0
\]
since \( 0 < r \leq r^p \leq 1 \).

**Lemma 2** (Local Lipschitz continuity of the inverse). If \( p \in (0, 1] \) then
\[
|\Theta^{-1}(w) - \Theta^{-1}(z)| \leq \frac{1}{p} |w - z| \max(|w|, |z|)^{(1-p)/p}, \quad w, z \in \mathbb{C}.
\]

**Proof of Lemma 2.** Again we can reduce to the case \( w = 1 \) and \( z = re^{i\phi} \) with \( 0 \leq r \leq 1 \) and \( 0 \leq \phi \leq \pi \). Thus we want
\[
\left| 1 - r^{1/p} e^{i\phi} \right| \leq \frac{1}{p} \left| 1 - e^{i\phi} \right|.
\]

This inequality holds at \( \phi = 0 \) by the mean value theorem, since \( (1/p) \geq 1 \). To prove it for \( 0 < \phi \leq \pi \), we compute that
\[
\frac{\partial}{\partial \phi} |1 - r^{1/p} e^{i\phi}|^2 = 2r^{1/p} \sin \phi \leq 2p^{-2}r \sin \phi = \frac{\partial}{\partial \phi} \frac{1}{p^2} |1 - e^{i\phi}|^2,
\]
since \( p \leq 1 \) and \( 0 \leq r \leq 1 \).

Next we study radial stretching on \( L^p \) in a general measure space. These results apply not only to \( L^p(\mathbb{R}^d) \) but also to the counting measure space \( \ell^p(\mathbb{Z}^d) \).

**Lemma 3** (Lipschitz continuity of \( \Theta \)). Let \( (X, \mu) \) be a measure space. If \( p \in (0, 1] \) then \( \Theta : L^p(X) \to L^1(X) \) is Lipschitz continuous, with
\[
d_1(\Theta u, \Theta v) \leq 2^{1-p} d_p(u, v), \quad u, v \in L^p(X).
\]
Here $\Theta u$ is the function with values $(\Theta u)(x) = \Theta(u(x))$, and $d_p(u, v) = \|u - v\|_{L^p(X)}^p$ is the distance function on the metric space $L^p(X)$.

**Proof of Lemma 3.** Choose $w = u(x)$ and $z = v(x)$ in Lemma 1, and then integrate over $x$.

**Lemma 4** (Local Hölder continuity of $\Theta^{-1}$). Let $(X, \mu)$ be a measure space. If $p \in (0, 1]$ then $\Theta^{-1} : L^1(X) \to L^p(X)$ is locally Hölder continuous, with

$$d_p(\Theta^{-1}u, \Theta^{-1}v) \leq p^{-p} (\|u\|_{L^1(X)} + \|v\|_{L^1(X)})^{1-p} d_1(u, v)^p, \quad u, v \in L^1(X).$$

**Proof of Lemma 4.**

$$d_p(\Theta^{-1}u, \Theta^{-1}v)$$

$$= \int_X |\Theta^{-1}(u(x)) - \Theta^{-1}(v(x))|^p d\mu(x)$$

$$\leq p^{-p} \int_X |u(x) - v(x)|^p \max(|u(x)|, |v(x)|)^{1-p} d\mu(x) \quad \text{by Lemma 2}$$

$$\leq p^{-p} \left[ \int_X |u(x) - v(x)| d\mu(x) \right]^p \left[ \int_X \max(|u(x)|, |v(x)|) d\mu(x) \right]^{1-p}$$

by Hölder’s inequality

$$\leq p^{-p} d_1(u, v)^p (\|u\|_{L^1(X)} + \|v\|_{L^1(X)})^{1-p}.$$ 

### 6. Proofs

#### 6.1 Proof of Proposition 1 — Synthesis $\ell^p \to L^p$

We have

$$\|Sc\|_p^p \leq \int_{\mathbb{R}^d} \left( \sum_{j>0} \sum_{k \in \mathbb{Z}^d} |c_{j,k} \psi_{j,k}(x)| \right)^p dx$$

$$\leq \int_{\mathbb{R}^d} \sum_{j>0} \sum_{k \in \mathbb{Z}^d} |c_{j,k} \psi_{j,k}(x)|^p dx \quad \text{since } p \in (0, 1]$$

$$= \sum_{j>0} \sum_{k \in \mathbb{Z}^d} |c_{j,k}|^p \|\psi_{j,k}\|_{p}^p = \|c\|_{\ell^p(\mathbb{Z} \times \mathbb{Z}^d)}^p \|\psi\|_p^p$$

since $\|\psi_{j,k}\|_p = \|\psi\|_p$ for all $j, k$ by our normalization of $\psi_{j,k}$. Thus the series for $Sc$ converges a.e., to an $L^p$-function, which implies unconditional convergence of the series in $L^p$ (with the help of the dominated convergence theorem).

**Aside.**

Obviously all one really needs here, in order for the synthesis operator to be continuous, is that the synthesizing collection $\{\psi_{j,k}\}$ be bounded in $L^p$. 

6.2 Proof of Theorem 1 — Analysis $L^p \to \ell^p$ 

Define a linear analysis operator

$$\tau_j h = \{ \langle h, \phi(a_j \cdot - bk) \rangle \}_{k \in \mathbb{Z}^d}, \quad h \in L^1,$$

and note $\tau_j : L^1 \to \ell^1(\mathbb{Z}^d)$ is bounded (continuous) because

$$\|\tau_j h\| \ell^1(\mathbb{Z}^d) \leq \sum_{k \in \mathbb{Z}^d} \int_{\mathbb{R}^d} |h(y)| |\phi(a_j y - bk)| \, dy \leq |\det b|^{-1} \|P|\phi\|_{\infty} \|h\|_1. \quad (6.1)$$

The analysis operator $T_j$ defined in (2.3) can be decomposed as

$$T_j = |\det b|^{-1} \circ \tau_j \circ \Theta^{-1} \circ \Theta \quad (6.3)$$

where $\Theta : L^p \to L^1$ and $\tau_j : L^1 \to \ell^1(\mathbb{Z}^d)$ and $\Theta^{-1} : \ell^1(\mathbb{Z}^d) \to \ell^p(\mathbb{Z}^d)$.

The desired local Hölder continuity of $T_j$ now follows by combining the continuity estimates on $\Theta$ and $\Theta^{-1}$ in Lemmas 3 and 4 with the continuity of $\tau_j$ expressed by (6.2). One can check that the resulting Hölder constant is $C = 2^{p(1-p)} \|P|\phi\|_{\infty}$. For later reference we also note $T_j$ is bounded, satisfying

$$\|T_j f\| \ell^p(\mathbb{Z}^d) \leq |\det b|^{-1/p} \|P|\phi\|_{\infty}^{1/p} \|f\|_p \quad (6.4)$$

as one verifies by substituting $h = \Theta f$ into (6.2).

6.3 Continuity of the Analysis Operator, w.r.t. the Analyzer

The preceding section proved continuity of the analysis operator with respect to the signal $f$. Now we show analysis is continuous with respect to the analyzer $\phi$. Both results will be used in the proof of Theorem 2.

To emphasize that the analysis operator depends on $\phi$, we write $T_j = T_j,\phi$ in this section.

**Lemma 5.** Assume $p \in (0, 1)$, and that $f \in L^\infty$ has compact support.

Then for each $j$, the map $\phi \mapsto T_j,\phi f$ is locally Hölder continuous from $L^1$ to $\ell^p(\mathbb{Z}^d)$, with

$$d_\ell^p(T_j,\phi f, T_j,\psi f) \leq C \|f\|_\infty^p \|\phi\|_1 + \|\phi - \psi\|_1^1 - p \|\phi - \psi\|_1^p, \quad \phi, \psi \in L^1. \quad (6.5)$$

Here $C$ depends on the exponent $p$, the support of $f$ and on $\max_{j > 0} \|a_j^{-1} b\|$, but not on the dilation scale $j$.

**Proof of Lemma 5.** Suppose $h$ is bounded with compact support in $\mathbb{R}^d$. We will first show the linear map

$$\phi \mapsto \tau_j,\phi h$$

is bounded from $L^1$ to $\ell^1(\mathbb{Z}^d)$. (The subscript $\phi$ on the $\tau$ serves to remind us which analyzer is being used.) Indeed, for any $\phi \in L^1$ we have the bound

$$\|\tau_j,\phi h\| \ell^1(\mathbb{Z}^d) \leq |\det b|^{-1} \int_{\mathbb{R}^d} \phi(y) \, \left| \det a_j^{-1} b \right| \sum_{k \in \mathbb{Z}^d} |h(a_j^{-1} y + a_j^{-1} bk)| \, dy$$

by $y \mapsto a_j^{-1}(y + bk)$ in (6.1)

$$\leq |\det b|^{-1} \|\phi\|_1 \cdot C \|h\|_\infty. \quad (6.5)$$
Affine Synthesis onto $L^p$ when $0 < p \leq 1$

where the constant $C$ comes from estimating the Riemann sum on $h$. Clearly $C$ depends only on the diameter of the support of $h$ and on the "step size" $\|a_j^{-1}b\|$. 

Now take $f \in L^\infty$ with compact support, and put $h = \Theta f$ so that $h$ too is bounded with compact support. Then $T_j,\phi f = \det b \Theta^{-1}(\tau_j,\phi h)$ by the decomposition (6.3), so that $T_j,\phi f \in \ell^p(Z^d)$ by (6.5).

The Hölder continuity of $T_j,\phi f$ with respect to $\phi$, in Lemma 5, now follows from the continuity estimate on $\Theta^{-1} : \ell^1(Z^d) \to \ell^p(Z^d)$ in Lemma 4 and the continuity estimate (6.5) for the map $\phi \mapsto \tau_j,\phi h$ from $L^1$ to $\ell^1(Z^d)$. \qed

6.4 Proof of Theorem 2 — Affine Quasi-Interpolation

Step 1. Let $f \in L^p$. First we reduce to $f$ being continuous with compact support. Indeed, given $\epsilon > 0$ we can choose a continuous function $g$ with compact support and $\|f - g\|_p \leq \epsilon$. Then

$$d_p(S_j T_j f, S_j T_j g) \leq \|\psi\|_p d_p(T_j f, T_j g) \leq C(b, p, \phi, \psi)(\|f\|_1 + \|g\|_1)^{1-p} \|f - g\|_p$$

So if we prove $\lim_{j \to \infty} \|S_j T_j f - g\|_p = 0$ where

$$\sigma = \|P \psi(b \cdot) - 1\|_{L^p(C)}$$

then it follows that $\lim_{j \to \infty} \|S_j T_j f - f\|_p = \sigma \|f\|_p$ as desired, by taking $\epsilon$ arbitrarily small. Thus we may assume from now on that $f$ is continuous with compact support.

Step 2. Now we reduce to $\phi \in L^\infty$ having compact support. The analyzer $\phi$ is certainly integrable, since the periodization $P|\phi|$ is assumed to be bounded and hence is locally integrable. Therefore, given $\epsilon > 0$ we can choose $\varphi \in L^\infty$ with compact support and $\|\phi - \varphi\|_1 < \epsilon$ and $\int_{B^d} \psi dx = 1$. Proceeding analogously to the reduction in Step 1, we observe

$$d_p(S_j T_j,\phi f, S_j T_j,\varphi f) \leq C(b, p, \psi, f)(\|\phi\|_1 + \|\varphi\|_1)^{1-p} \|\phi - \varphi\|_1$$

Thus we need only prove $\lim_{j \to \infty} \|S_j T_j,\phi f - f\|_p = \sigma \|f\|_p$, because then taking $\epsilon$ arbitrarily small implies the corresponding limit with $\phi$ instead of $\varphi$. Thus we may assume from now on that $\phi \in L^\infty$ has compact support.

Step 3. Next we reduce to analyzing $f$ with pointwise sampling. Begin by uniformly sampling the continuous function $f$ at scale $j$, and recording the results in the sequence

$$U_j f = \{ \det a_j^{-1/p} \det b \} f(a_j^{-1}bk) \} \in \mathbb{Z}^d$$

That is, $U_j$ is a pointwise analysis operator at scale $j$. We aim to show average sampling and pointwise sampling are the same in the limit, or more precisely that

$$\|T_j f - U_j f\|_{\ell^p(Z^d)} \to 0 \quad \text{as } j \to \infty .$$

(6.6)
Take \( j \) large enough that
\[
|a_j^{-1}y| < 1 \quad \text{for all } y \in \text{spt}(\phi),
\]
using here that \( \phi \) is compactly supported and \( \|a_j^{-1}\| \to 0 \). Write \( F_r \) for the set of points within distance \( r > 0 \) of the support of \( f \), and let
\[
K(j) = \{ k \in \mathbb{Z}^d : a_j^{-1}bk \in F_1 \}.
\]
Then
\[
0 \neq (U_j f)_k \implies k \in K(j),
\]
because if \( f(a_j^{-1}bk) \neq 0 \) then \( a_j^{-1}bk \in \text{spt}(f) \subset F_1 \).

In view of (6.8) and (6.9), when proving (6.6) we need only sum over \( K(j) \). Thus
\[
\|T_j f - U_j f\|_{p(\mathbb{Z}^d)}^p = |\det b|^{-1} \|\Theta^{-1}((\Theta f, \phi(a_j \cdot -bk)) - f(a_j^{-1}bk))\|^p
\leq |\det b|^{-1} |\det a_j^{-1}b| \#K(j) \cdot M(j)^p
\]
where
\[
M(j) = \sup_{k \in \mathbb{Z}^d} \left| \Theta^{-1} \left( \int_{\mathbb{R}^d} \Theta f(a_j^{-1}(y + bk)) \phi(y) dy \right) - \Theta^{-1}(\Theta f(a_j^{-1}bk)) \right|.
\]
Since \( |\det a_j^{-1}b| \#K(j) \) is bounded by the volume of \( F_2 \), for all large \( j \), we can see that in order to prove (6.6) it suffices to show \( M(j) \to 0 \).

Notice the arguments of \( \Theta^{-1}(\ldots) \) in the definition of \( M(j) \) are bounded independently of \( j \) and \( k \), since \( f \) is bounded and \( \phi \) is integrable. Hence, the convergence of \( M(j) \) to 0 follows from local uniform continuity of \( \Theta^{-1} \), since the distance between the arguments converges to 0 uniformly with respect to \( k \), as follows:
\[
\sup_{k \in \mathbb{Z}^d} \left| \int_{\mathbb{R}^d} \Theta f(a_j^{-1}(y + bk)) \phi(y) dy - \Theta f(a_j^{-1}bk) \right| \\
\leq \int_{\mathbb{R}^d} \sup_{k \in \mathbb{Z}^d} \left| \Theta f(a_j^{-1}y + a_j^{-1}bk) - \Theta f(a_j^{-1}bk) \right| |\phi(y)| dy \\
\text{since } \int_{\mathbb{R}^d} \phi dx = 1 \\
\to 0
\]
as \( j \to \infty \), with dominated convergence justified by uniform continuity of the compactly supported function \( \Theta f \) and integrability of \( \phi \). This reasoning proves \( M(j) \to 0 \), and hence proves (6.6).

**Step 4.** We next derive the theorem with \( U_j \) in place of \( T_j \), in other words, we show
\[
\lim_{j \to \infty} \|S_j U_j f - f\|_p^p = \sigma \|f\|_p^p.
\]
Affine Synthesis onto $L^p$ when $0 < p \leq 1$

This equality implies the theorem because

$$\|S_j T_j f - S_j U_j f\|_p \leq \|\psi\|_p \|T_j f - U_j f\|_{\ell^p(\mathbb{Z}^d)}$$

by Proposition 1 → 0 by (6.6).

To prove (6.10), we start by decomposing

$$(S_j U_j f)(x) - f(x) = [P \psi(a_j x) - 1] f(x) + \text{Rem}_j(x),$$

where the remainder is

$$\text{Rem}_j(x) = |\det b| \sum_{k \in \mathbb{Z}^d} (f(aj^{-1}bk) - f(x))\psi(ax - bk).$$

The first term on the right of (6.11) has limit

$$\lim_{j \to \infty} \| (P \psi(a_j \cdot - 1) f \|_p^p = (\text{mean value of } |P \psi - 1|^p) \cdot \| f \|_p^p = \sigma \| f \|_p^p$$

by a Riemann-Lebesgue argument, since $|P \psi(a_j \cdot - 1)|^p$ oscillates rapidly around its mean value when $j$ is large. For details see [6, Lemma 26], for example, noting that $|P \psi - 1|^p$ is locally integrable and $|f|^p$ is bounded with compact support.

**Step 5.** To complete the proof of (6.10) we have only to show the remainder term $\text{Rem}_j$ vanishes in the limit, in $L^p$. We have

$$|\text{Rem}_j(x)|^p \leq |\det b|^p \sum_{k \in \mathbb{Z}^d} |f(aj^{-1}bk) - f(x)|^p |\psi(ax - bk)|^p,$$

and so after integrating with respect to $x$ and making the change of variable $x \mapsto a_j^{-1}(x + bk)$, we find

$$\|\text{Rem}_j\|_p^p \leq |\det b|^{p-1} \int_{\mathbb{R}^d} R_j(x)|\psi(x)|^p dx$$

where

$$R_j(x) = |\det a_j^{-1}b| \sum_{k \in \mathbb{Z}^d} |f(aj^{-1}bk) - f(a_j^{-1}(x + bk))|^p.$$

Formally, $\text{Rem}_j \to 0$ in $L^p$ because $R_j(x)$ is a Riemann sum that passes in the limit to the integral

$$\int_{\mathbb{R}^d} |f(z) - f(0 + z)|^p dz = 0.$$

To prove $\text{Rem}_j \to 0$ rigorously by dominated convergence, it is enough to show $R_j(x) \to 0$ pointwise and that $R_j$ is bounded independently of $x$ and $j$, for all large $j$. To get boundedness of $R_j$, we estimate that

$$|R_j(x)| \leq 2 \max_{z \in \mathbb{R}^d} |\det a_j^{-1}b| \sum_{k \in \mathbb{Z}^d} |f(z + aj^{-1}bk)|^p$$

for all $x \in \mathbb{R}^d$

$$\to 2\|f\|_p^p$$

as $j \to \infty$, by a Riemann sum argument applied to the continuous, compactly supported function $|f|^p$. Thus $R_j$ is bounded independently of $x$ and $j$, for all large $j.$
To get that $R_j(x) \to 0$ pointwise, we fix $x \in \mathbb{R}^d$ for the rest of the proof, and take $j$ to be large enough that $|a_j^{-1}x| < 1$. Then we need only sum over $K(j)$, when we evaluate $R_j(x)$:

$$R_j(x) = \left| \det a_j^{-1}b \right| \sum_{k \in K(j)} \left| f(a_j^{-1}bk) - f(a_j^{-1}(x+bk)) \right|^p \leq \left| \det a_j^{-1}b \right| \#K(j) \cdot N(j)^p$$

where

$$N(j) = \sup_{k \in \mathbb{Z}^d} \left| f(a_j^{-1}bk) - f(a_j^{-1}(x+bk)) \right|.$$

Since $|\det a_j^{-1}b| \#K(j)$ is bounded by the volume of $F_2$, for all large $j$, and $N(j) \to 0$ by uniform continuity of $f$, we deduce $R_j(x) \to 0$, which finishes the proof.

6.5 Proof of Theorem 3 — Synthesis onto $L^p$

Take $\lambda \in \mathbb{C}$ and $\sigma$ to be as in hypothesis (2.4), and choose $\sigma' \in (\sigma, 1)$, so that

$$\|\lambda P \psi(b \cdot) - 1\|_{L^p(\mathbb{C})} = \sigma < \sigma'.$$

Take the analyzer to be $\phi = |b \mathbb{C}|^{-1/2} \mathbb{1}_{b \mathbb{C}}$, a normalized indicator function which has constant periodization $P|\phi| \equiv 1$.

Then for each $f \in L^p$, $\|S_j(\lambda T_j f) - f\|_p \leq \sigma' \|f\|_p$ for some $j > 0$, by Theorem 2 applied to the function $\lambda \psi$ (instead of to $\psi$). The coefficient sequence here satisfies

$$\|\lambda T_j f\|_{\ell^p(\mathbb{Z}^d)} \leq |\lambda| \|\det b\|^{1-1/p} \|f\|_p$$

by formula (6.4) in the proof of Theorem 1.

Thus the open mapping theorem in Appendix A says $S : \ell^p(\mathbb{Z}^+ \times \mathbb{Z}^d) \to L^p$ is open and surjective, and that there exists $c \in \ell^p(\mathbb{Z}^+ \times \mathbb{Z}^d)$ with $Sc = f$ and $\|c\|_{\ell^p(\mathbb{Z}^+ \times \mathbb{Z}^d)} \leq (1 - \sigma')^{-1/p} |\lambda| \|\det b\|^{1-1/p} \|f\|_p$. The proof is complete.

6.6 Examples and Counterexamples for Hypothesis (2.4)

Consider the function $\psi(x) = x^{-\beta} \mathbb{1}_{[0,1]}$ in one dimension. We will show that if $\beta \in \left[2/(p+1), 1/p \right)$ then $\|1 - \lambda \psi\|_{L^p([0,1])} \geq 1$ for all $\lambda \in \mathbb{C}$, so that hypothesis (2.4) with $b = 1$ fails for this function. This counterexample for $p \in (0, 1)$ is due to Tachev [27].

Our proof below is different from Tachev’s. It yields also the positive result that the parameter range is sharp for $p \in (0, 1]$: If $\beta \in (0, 2/(p+1))$ then $\|1 - \lambda \psi\|_{L^p([0,1])} < 1$ for all small $\lambda > 0$. Tachev stated this sharpness for $\beta = 1$.

Assume $p \in (0, 1)$ and $\beta \in \left[2/(p+1), 1/p \right)$. To prove Tachev’s counterexample, we need only consider real, positive values $\lambda > 0$, since $\psi \geq 0$. Then after replacing $\lambda$ by $t^{-\beta}$ and defining

$$F(t) = \int_0^1 \left| 1 - (tx)^{-\beta} \right|^p \, dx,$$
we see we would like to prove \( F(t) > 1 \) for all \( t > 0 \). A change of variable gives

\[
F(t) = \frac{1}{t} \int_0^t |1 - y^{-\beta}|^p \ dy ,
\]

(6.12)

and hence \( F \) is decreasing for \( 0 < t \leq 1 \) because it equals the mean value over the interval \([0, t]\) of the decreasing positive function \((y^{-\beta} - 1)^p\). For \( t \geq 1 \) we have

\[
F(t) = 1 + \frac{1}{t} \int_0^t (|1 - y^{-\beta}|^p - 1) \ dy > 1 + \frac{1}{t} \int_0^\infty (|1 - y^{-\beta}|^p - 1) \ dy
\]

(6.13)

because \( |1 - y^{-\beta}|^p - 1 < 0 \) when \( y > 1 \). Note the last integral converges near infinity because \( \beta > 1 \). We now show this integral is nonnegative. Indeed,

\[
\int_0^\infty (|1 - y^{-\beta}|^p - 1) \ dy = \int_0^1 p(y^{-\beta} - 1)^{p-1} y^{-\beta} - \int_1^\infty p(1 - y^{-\beta})^{p-1} y^{-\beta} \ dy
\]

by parts on \((0, 1)\) and \((1, \infty)\), using that \( \beta p < 1 \) and \( \beta > 1 \),

\[
= \beta p \int_0^1 (1 - y^\beta)^{p-1} y^{-\beta} [1 - y^{(p+1)-2}] \ dy
\]

by putting \( y \mapsto y^{-1} \) in the second integral. Clearly the integrand is nonnegative in this last integral, since \( \beta(p + 1) \geq 2 \), and hence \( F(t) > 1 \) for all \( t > 0 \), as we wanted.

For the positive result when \( \beta \in (1, 2/(p + 1)) \), we simply use (6.13) to prove

\[
F(t) = 1 + \frac{1}{t} \int_0^\infty (|1 - y^{-\beta}|^p - 1) \ dy + o \left( \frac{1}{t} \right)
\]

as \( t \to \infty \),

and then note the integral is negative by the calculations above. Hence, \( F(t) < 1 \) for all large \( t \), which shows \( \|1 - \lambda \psi\|_{L^p(C)} < 1 \) for all small \( \lambda > 0 \). Next, when \( \beta \in (0, 1) \) we observe that for each fixed \( t > 1 \), the formula (6.12) for \( F(t) \) is increasing with respect to \( \beta \); therefore \( F(t) < 1 \) for all large \( t > 1 \) by the case \( \beta \in (1, 2/(p + 1)) \) just treated.

Lastly, the positive result when \( p = 1 \) and \( \beta \in (0, 1) \) can easily be proved directly.

### 6.7 Proof of Proposition 2 — Sufficient Conditions

Write \( \eta(x) = P \psi(bx) \), so that \( \eta \in L^p(C) \). Our goal is to prove \( \|1 - \lambda \eta\|_{L^p(C)} < 1 \) for some \( |\lambda| < 1 \).

**Part (a).** Suppose \( \psi \in L^1 \) and \( \int_{\mathbb{R}^d} \psi \ dx \neq 0 \). Then \( \eta \in L^1(C) \) with \( \int_C \eta \ dx = \int_{\mathbb{R}^d} \psi \ dx \neq 0 \). We will use below the elementary inequality that

\[
|1 - z| \leq 1 - \Re z + A|z|^2 ,
\]

for some positive constant \( A \). Given \( 0 < |\lambda| \leq 1/2 \), put

\[
E(\lambda) = \{ x \in C : |\eta(x)| \leq |\lambda|^{-1/3} \},
\]

and hence

\[
\int_E \psi \ dx > 0.
\]

For \( z \) in the interior of \( E \) we have

\[
|1 - \lambda \eta| \leq |1 - \lambda|^{-1/3}.
\]

Therefore, \( \|1 - \lambda \eta\|_{L^p(C)} < 1 \) for some \( |\lambda| < 1 \).
and notice that on \( E(\lambda) \) we have \(|\lambda \eta| \leq |\lambda|^{2/3} \leq 2^{-2/3} \). Then we see
\[
\|1 - \lambda \eta\|_{L^p(C)} \leq \|1 - \lambda \eta\|_{L^1(C)} \text{ by Jensen’s inequality, since } p \leq 1,
\]
\[
\leq \int_{E(\lambda)} (1 - \text{Re}(\lambda \eta(x))) + A|\lambda \eta(x)|^2 \, dx + \int_{C \setminus E(\lambda)} (1 + |\lambda \eta(x)|) \, dx
\]
\[
= 1 - \text{Re} \lambda \int C \eta(x) \, dx + A|\lambda|^{4/3} + o(|\lambda|) \quad \text{as } |\lambda| \downarrow 0,
\]
since \( E(\lambda) \uparrow C \). Thus we have only to choose \( \lambda \) with \( \lambda \int C \eta(x) \, dx > 0 \) and \( |\lambda| \) sufficiently small, in order to obtain \( \|1 - \lambda \eta\|_{L^p(C)} < 1 \) as desired.

Notice the above proof works for \( p = 1 \) as well.

Part (b). Suppose \( \psi \in L^1 \) (so that \( \eta \in L^1(C) \)) and that \( \eta \neq 0 \) is real valued and bounded above. (When \( \eta \) is bounded below, just change \( \lambda \) to \(-\lambda\) in what follows.) Suppose \( \int_{\mathbb{R}} \psi(x) \, dx = 0 \), so that \( \int C \eta(x) \, dx = 0 \). Then
\[
\|1 - \lambda \eta\|_{L^p(C)} < \|1 - \lambda \eta\|_{L^1(C)} \text{ by Jensen’s inequality}
\]
\[
= \int C (1 - \lambda \eta(x)) \, dx \quad \text{for all small } \lambda > 0, \text{ since } 1 - \lambda \eta > 0,
\]
\[
= 1. \]

Jensen’s inequality is strict here because \( p < 1 \) and \( \lambda \eta \) is nonconstant (indeed, \( \eta \) has mean value zero but is not identically zero).

Part (c). Assume \( \eta \in L^2(C) \) satisfies \( p \int C |\eta|^2 \, dx < (2 - p) \left| \int C \eta^2 \, dx \right| \). Without loss of generality we can assume \( \int C \eta^2 \, dx > 0 \), by multiplying \( \eta \) with a suitable complex constant. Then our assumption is equivalent to \( p \int C (\text{Im} \, \eta)^2 \, dx < (1 - p) \int C (\text{Re} \, \eta)^2 \, dx \), so that
\[
\alpha^{-1} \int C (\text{Im} \, \eta)^2 \, dx < \alpha (1 - p) \int C (\text{Re} \, \eta)^2 \, dx,
\]
for some \( \alpha < 1 \) sufficiently close to 1.

We will use below the binomial approximation that
\[
|1 - z|^p \leq 1 - p \text{Re} z + \alpha \frac{p(p - 1)}{2} (\text{Re} \, z)^2 + \alpha^{-1} \frac{p}{2} (\text{Im} \, z)^2, \quad |z| \leq B,
\]
where the small positive constant \( B \) depends on \( p \in (0, 1) \) and \( \alpha \in (0, 1) \). Putting
\[
F(\lambda) = \{ x \in C : |\eta(x)| \leq B/|\lambda| \},
\]
we deduce that
\[
\|1 - \lambda \eta\|_{L^p(C)}^p \leq \int_{F(\lambda)} \left( 1 - p \text{Re}(\lambda \eta(x)) + \alpha \frac{p(p - 1)}{2} (\text{Re} \, \lambda \eta(x))^2 + \alpha^{-1} \frac{p}{2} (\text{Im} \, \lambda \eta(x))^2 \right) \, dx
\]
\[
+ \int_{C \setminus F(\lambda)} (1 + |\lambda \eta(x)|^p) \, dx.
\]
Affine Synthesis onto $L^p$ when $0 < p \leq 1$

Averaging over $\lambda$ and $-\lambda$ (and noting $F(-\lambda) = F(\lambda)$) gives for small $\lambda > 0$ that

$$
\frac{1}{2} \left( \|1 - \lambda \eta\|_{L^p(C)}^p + \|1 + \lambda \eta\|_{L^p(C)}^p \right) 
\leq \int_{F(\lambda)} \left( 1 + \alpha \frac{p(p-1)}{2} \text{Re} \lambda \eta(x)^2 + \alpha^{-1} \frac{p}{2} \text{Im} \lambda \eta(x)^2 \right) dx
+ \int_{\mathcal{C} \setminus F(\lambda)} \left( 1 + |\lambda \eta(x)|^p \right) dx
= 1 + \lambda^2 \int \left( \alpha \frac{p(p-1)}{2} \text{Re} \eta(x)^2 + \alpha^{-1} \frac{p}{2} \text{Im} \eta(x)^2 \right) dx + o(\lambda^2)
$$

as $\lambda \downarrow 0$, where in the final step we used that $F(\lambda) \uparrow C$ and that $|\lambda \eta|^p < B^{p-2}|\lambda \eta|^2$ on $C \setminus F(\lambda)$ (because $|\lambda \eta|/B > 1$ there).

Hence, from (6.14) we conclude

$$
\frac{1}{2} \left( \|1 - \lambda \eta\|_{L^p(C)}^p + \|1 + \lambda \eta\|_{L^p(C)}^p \right) < 1
$$

for all small $\lambda > 0$, and thus by choosing either $\lambda$ or $-\lambda$ we complete the proof.

Aside.

Our proofs of parts (a) and (c) are modifications of Filippov and Oswald [16, Lemma 1]. They treated only real-valued functions $\eta$, which in the proof of part (c) above means they could choose $B = 1/2$ and $\alpha$ sufficiently close to 0, whereas we must choose $\alpha$ sufficiently close to 1 and then take $B$ sufficiently close to 0.

6.8 Proof of Theorem 4 — More Synthesis onto $L^p$

Let $\beta \in \mathbb{N}$ and consider the periodization of $\psi$ with respect to the translation matrix $b\beta$, that is,

$$
P_{b\beta} \psi(x) = |\det b\beta| \sum_{k \in \mathbb{Z}^d} \psi(x - b\beta k).
$$

After rescaling, we see $P_{b\beta} \psi(b\beta x)$ is integrable on the cube $C$ (since $\psi \in L^1$), and has Fourier coefficients $\hat{\psi}(m(b\beta)^{-1})$ for $m \in \mathbb{Z}^d$ (row vectors).

We claim $P_{b\beta} \psi$ is nontrivial for some $\beta$. For if $P_{b\beta} \psi = 0$ a.e., for each $\beta$, then the Fourier coefficients are zero too, so that $\hat{\psi}(m(b\beta)^{-1}) = 0$ for all $m \in \mathbb{Z}^d$ and all $\beta \in \mathbb{N}$. Then continuity of $\hat{\psi}$ forces $\hat{\psi} \equiv 0$, contradicting the hypothesis that $\psi \not\equiv 0$.

So fix a $\beta$ value for which $P_{b\beta} \psi$ is nontrivial. This periodization is real-valued (since $\psi$ is real valued), and is bounded below since $\psi_+$ is bounded and has compact support. Thus Proposition 2(a) or 2(b) applies, and says $\|\lambda P_{b\beta} \psi(b\beta \cdot) - 1\|_{L^p(C)} < 1$ for some $\lambda$.

Theorem 3 then provides a constant $C$ such that for each $f \in L^p$ there is a sequence $\tilde{c} \in \ell^p(\mathbb{Z}^+ \times \mathbb{Z}^d)$ with $\|\tilde{c}\|_{\ell^p(\mathbb{Z}^+ \times \mathbb{Z}^d)} \leq C\|f\|_p$ and

$$
f = \sum_{j>0} \sum_{k \in \mathbb{Z}^d} \tilde{c}_{j,k} |\det a_j|^{1/p} \psi(a_j x - b\beta k).
$$

That is, $f = Sc$ where $c_{j,k} = \tilde{c}_{j,\beta^{-1}k}$ if $k \in \beta \mathbb{Z}^d$ and $c_{j,k} = 0$ otherwise. The theorem follows, since $c$ and $\tilde{c}$ have the same $\ell^p$-norm.
6.9 Proof of Corollary 2 — Affine Atomic Decomposition of $L^p(\Omega)$

The “≤” direction of the corollary follows straight from Proposition 1.

For the “≥” direction, first define

$$L = \{ c \in \ell^p(\mathbb{Z}^+ \times \mathbb{Z}^d) : c \text{ is adapted to } \Omega \text{ and } \psi \}.$$ 

Clearly $L$ is a closed subspace of $\ell^p(\mathbb{Z}^+ \times \mathbb{Z}^d)$, and hence is a complete metric space under the $\ell^p$-metric. Take $\phi = |bC|^{-1/2} \mathbf{1}_{bC}$.

Consider an $f \in L^p$ that is continuous and compactly supported in $\Omega$. We claim the sequence $T_j f$ belongs to $L$, for each large $j$, or more precisely, that the sequence $c$ equalling $T_j f$ at level $j$ and zero at all other levels belongs to $L$. To see this fact, just notice

$$\text{spt}(\psi_{j,k}) \subset \Omega \text{ whenever } \langle \Theta f, \phi(a_j \cdot - bk) \rangle \neq 0 \text{ and } k \in \mathbb{Z}^d,$$

for all large $j$, because the support of $f$ lies at some positive distance from the boundary of $\Omega$, and $\psi$ and $\phi$ have compact support and $\|a_j^{-1}\| \to 0$. Thus $T_j f$ belongs to $L$.

The proof of Theorem 3 now applies word-for-word, except with $\ell^p(\mathbb{Z}^+ \times \mathbb{Z}^d)$ replaced by $L$. Admittedly we have verified the hypotheses of the open mapping theorem only for the dense class of continuous $f$ having compact support in $\Omega$, but a dense class in $L^p(\Omega)$ is enough, by the remark in Appendix A.

The conclusion of Theorem 3 with $c \in L$ gives the “≥” direction of Corollary 2.

6.10 Proof of Corollary 3 — Analysis Metric for $L^p$

By formula (6.4) in the proof of Theorem 1,

$$\sup_j \| T_j f \|_{\ell^p} \leq |\det b|^{1-1/p} \| P \phi \|_{L^1}^{1/p} \| f \|_p.$$ 

To prove the other direction of the metric equivalence, choose a synthesizer $\psi = |bC|^{-1/2} \mathbf{1}_{bC}$ that has constant periodization $P \psi = 1$. Then by Theorem 2, $S_j T_j f \to f$ in $L^p$ as $j \to \infty$. Therefore, Proposition 1 (bounded synthesis) implies that

$$\| f \|_p \leq \sup_j \| S_j T_j f \|_p \leq \| \psi \|_p \sup_j \| T_j f \|_{\ell^p(\mathbb{Z}^d)}.$$ 

6.11 Proof of Theorem 5 — $p$-Riesz Basis at Scale $j$

By a simple rescaling, we can suppose $a_j = I$ is the identity matrix.

To prove injectivity, take $s \in \ell^p(\mathbb{Z}^d)$ and suppose $S_j s = 0$, or

$$\sum_{k \in \mathbb{Z}^d} s_k \psi(x - bk) = 0 \text{ a.e.}$$ 

We will show $s = 0$.

Note the series $\sum_{k \in \mathbb{Z}^d} s_k \psi(x - bk)$ converges absolutely a.e., because $s \in \ell^p \subset \ell^\infty$ and

$$\left( \sum_{k \in \mathbb{Z}^d} |\psi(x - bk)| \right)^p \leq \sum_{k \in \mathbb{Z}^d} |\psi(x - bk)|^p \in L^1(bC).$$
Affine Synthesis onto $L^p$ when $0 < p \leq 1$

(Here we use that $p \in (0, 1]$.) Let $\xi \in \mathbb{R}^d$ and multiply the series by $e^{2\pi i \xi x}$ (where $\xi \in \mathbb{R}^d$ is arbitrary) to obtain

$$\sum_{k \in \mathbb{Z}^d} s_k e^{2\pi i \xi (x - bk)} \psi(x - bk) = 0 \text{ a.e.}$$

Replace $x$ by $x - b\ell$ and sum over $\ell \in \mathbb{Z}^d$ to obtain that

$$\sum_{k \in \mathbb{Z}^d} s_k e^{2\pi i \xi bk} \cdot P(e^{\xi \psi})(x) = 0 \text{ a.e.}$$

By hypothesis there is a set of positive measure on which $P(e^{\xi \psi})(x) \neq 0$, and hence $\sum_{k \in \mathbb{Z}^d} s_k e^{2\pi i \xi bk} = 0$ for each $\xi \in \mathbb{R}^d$. Since $s \in \ell^p \subset \ell^1$ we conclude $s_k = 0$ for all $k$, or $s = 0$, so that $S_j$ is injective.

Now suppose in addition that $\psi$ has compact support. We will prove the $p$-Riesz basis condition by following almost exactly the work of R.-Q. Jia [20, Section 3]. Our proof does present one new idea: Whereas Jia restricted his $\psi \in L^p$ to be a distribution, so that he could work with its Fourier transform, we avoid any such restriction by working directly with the periodization hypothesis.

Define

$$\psi^{(\ell)}(x) = \begin{cases} \psi(x + b\ell), & x \in bC, \\ 0, & \text{otherwise} \end{cases}$$

so that $\psi^{(\ell)}$ gives the value of $\psi$ on $b(\ell + C)$, translated to $bC$. Obviously $\psi$ can be reconstructed by summing up the pieces:

$$\psi = \sum_{\ell \in \mathbb{Z}^d} \psi^{(\ell)}(\cdot - b\ell).$$

(6.15)

Only finitely many of the $\psi^{(\ell)}$ are nontrivial, since $\psi$ has compact support, and so we can choose a maximal collection of them that are linearly independent in $L^p(bC)$. Denote this collection by $\{\psi^{(m)} : m \in M\}$ for some finite index set $M \subset \mathbb{Z}^d$.

For later use, write $v = \{v_m\}_{m \in M}$ for an arbitrary complex sequence supported on $M$, and observe that the function $f(v) = \| \sum_{m \in M} v_m \psi^{(m)} \|_{L^p(bC)}$ is continuous on the unit $p$-sphere $\{ v : \|v\|_{L^p(M)} = 1 \}$. Clearly $f$ cannot equal zero anywhere on this sphere, because the $\psi^{(m)}$ are linearly independent. Hence, $f$ attains a positive minimum value $C = C(\psi, b, p)$ on the unit $p$-sphere. (Finiteness of the index set $M$ is used here to ensure compactness of the unit sphere, and hence existence of a minimum for $f$.) Thus

$$\left\| \sum_{m \in M} v_m \psi^{(m)} \right\|_{L^p(bC)} \geq C \left( \sum_{m \in M} |v_m|^p \right)^{1/p}, \quad v_m \in \mathbb{C},$$

(6.16)

by homogeneity.

Each $\psi^{(\ell)}$ can be expressed as a linear combination

$$\psi^{(\ell)} = \sum_{m \in M} t_{\ell, m} \psi^{(m)}, \quad \ell \in \mathbb{Z}^d$$

(6.17)
for some coefficients $t_{\ell,m}$. Substituting this linear combination into the reconstruction formula (6.15) gives

$$
\psi(x) = \sum_{\ell \in \mathbb{Z}^d} \sum_{m \in M} t_{\ell,m} \psi^{(m)}(x - b\ell).
$$

Hence,

$$
S_j s(x) = \sum_{k \in \mathbb{Z}^d} s_k \psi(x - bk) \quad \text{(recalling that } a_j = 1) \]

by shifting the index $\ell \mapsto \ell - k$ and defining a sequence $t_m = \{t_{\ell,m}\}_{\ell \in \mathbb{Z}^d}$, for each $m \in M$. Convergence of the above multiple series is clear, because each sequence $t_m$ has only finitely many nonzero entries $t_{\ell,m}$ (noting $\psi^{(\ell)}$ is identically zero for all large $|\ell|$).

Since $\psi^{(m)}$ equals zero outside the cube $bC$, we deduce

$$
S_j s(x) = \sum_{m \in M} (s \ast t_m)_{\ell} \psi^{(m)}(x - b\ell), \quad x \in b(\ell + C), \quad \ell \in \mathbb{Z}^d.
$$

Therefore,

$$
\| S_j s \|^p_p = \sum_{\ell \in \mathbb{Z}^d} \left\| S_j s \right\|^p_{L^p(b(\ell + C))} \\
= \sum_{\ell \in \mathbb{Z}^d} \left\| \sum_{m \in M} (s \ast t_m)_{\ell} \psi^{(m)} \right\|^p_{L^p(bC)} \\
\geq C \sum_{\ell \in \mathbb{Z}^d} \sum_{m \in M} |(s \ast t_m)_{\ell}|^p \quad \text{by (6.16)} \\
= C \sum_{m \in M} \| s \ast t_m \|^p_{L^p(\mathbb{Z}^d)}. \quad (6.18)
$$

We must still bound the norm of $s \ast t_m$ from below in terms of the norm of $s$. To help do so, consider the trigonometric polynomial $\tau_m(\xi) = \sum_{\ell \in \mathbb{Z}^d} t_{\ell,m} e^{2\pi i \xi \ell}$. For each $\xi \in \mathbb{R}^d$, our periodization hypothesis guarantees that

$$
0 \neq P(e_\xi \psi)(x) = |\det b| \sum_{\ell \in \mathbb{Z}^d} e^{2\pi i \xi (x + b\ell)} \psi(x + b\ell) \\
= |\det b| e^{2\pi i \xi x} \sum_{\ell \in \mathbb{Z}^d} e^{2\pi i \xi b\ell} \psi^{(\ell)}(x) \quad \text{for } x \in bC \\
= |\det b| e^{2\pi i \xi x} \sum_{m \in M} \tau_m(\xi b) \psi^{(m)}(x)
$$

by substituting (6.17). We deduce that at least one of the values $\tau_m(\xi b), m \in M$, must be nonzero. Hence, $\sum_{m \in M} |\tau_m(\xi)|^2 > 0$ for all $\xi$, and so the reciprocal function

$$
\nu(\xi) = \left( \sum_{m \in M} |\tau_m(\xi)|^2 \right)^{-1},
$$
Affine Synthesis onto $L^p$ when $0 < p \leq 1$

is well defined, smooth and $\mathbb{Z}^d$-periodic. Write $u_\ell$ for its Fourier coefficients: $\sum_{\ell \in \mathbb{Z}^d} u_\ell e^{2\pi i \ell \cdot \xi} = \nu(\xi)$. These Fourier coefficients decay rapidly, since $\nu$ is smooth. And writing $\tilde{t}_{\ell,m} = t_{-\ell,m}$, we have from (6.18) the estimate

$$\|S_j s\|_p \geq C \sum_{m \in M} \| s * t_m \|_{\ell^p(\mathbb{Z}^d)} \| t_m * u \|_{\ell^p(\mathbb{Z}^d)}$$

$$\geq C \sum_{m \in M} \| s * \tilde{t}_m * \tilde{t}_m * u \|_{\ell^p(\mathbb{Z}^d)}$$

$$= C \| s \|_{\ell^p(\mathbb{Z}^d)}$$

since $\sum_{m \in M} t_m * \tilde{t}_m * u = \delta$, as one can check by taking the Fourier series: $\sum_{m \in M} t_m \tilde{t}_m \nu = 1$.

Thus we have proved the lower Riesz estimate for the theorem. The upper estimate $\|S_j s\|_p \leq C \| s \|_{\ell^p(\mathbb{Z}^d)}$ is immediate from Proposition 1. Now the range $S_j(\ell^p(\mathbb{Z}^d))$ must be complete in $L^p$, as one sees by considering Cauchy sequences in the range and using (2.5), and so the range is closed.

6.12 Proof of Theorem 6 — Most $L^p$ Functions are Surjective Affine Synthesizers

Density. The class $S^p$ contains every bounded function $\psi$ with compact support and nonzero integral, because every such function satisfies the hypotheses of Proposition 2(a) and hence of Theorem 3. These bounded functions are dense in $L^p$, and hence $S^p$ is dense in $L^p$.

Openness. Take $\psi \in S^p$. Then $S_\psi$ is a continuous linear mapping of the $F$-space $\ell^p(\mathbb{Z}_+ \times \mathbb{Z}^d)$ onto the $F$-space $L^p$, so that $S_\psi$ is open by [25, Corollary 2.12]. Hence, $A > 0$ exists such that for each $f \in L^p$, a sequence $c \in \ell^p(\mathbb{Z}_+ \times \mathbb{Z}^d)$ exists satisfying $S_\psi c = f$ and

$$d_p(0, c) \leq Ad_p(0, f)$$

We claim $S^p$ contains the $L^p$-ball of radius $1/A$ centered at $\psi$, from which it follows that $S^p$ is open in $L^p$.

So suppose $\psi_1 \in L^p$ with $d_p(\psi_1, \psi) = \delta / A$ for some $\delta \in (0, 1)$. Then

$$d_p(S_{\psi_1} c, f) = \|S_{\psi_1 - \psi} c\|_p$$

$$\leq \|\psi_1 - \psi\|_p \|c\|_{\ell^p(\mathbb{Z}_+ \times \mathbb{Z}^d)}$$

by Proposition 1

$$\leq \frac{\delta}{A} d_p(0, f) = \delta d_p(0, f)$$

by construction above.

Now the open mapping theorem in Appendix A guarantees that $S_{\psi_1}$ maps onto $L^p$, so that $\psi_1 \in S^p$ as desired.

Path connectedness. First we show path connectedness of

$$\{ \psi \in L^p : \|P \psi(b \cdot) - 1\|_{L^p(C)} < 1 \},$$

which is a subset of $S^p$ by Theorem 3. Consider the linear path

$$\psi_t = (1 - t)\psi + t|bC|^{-1}\mathbb{1}_b C, \quad t \in [0, 1],$$
which connects $\psi$ to the normalized indicator function $|bC|^{-1}1_{bC}$. This normalized indicator function has periodization identically equal to 1, and so

$$
\|P\psi_t(b \cdot) - 1\|_{L^p(C)} = \|P\psi(b \cdot) - 1\|_{L^p(C)}(1 - t) \\
\leq \|P\psi(b \cdot) - 1\|_{L^p(C)} < 1.
$$

This estimate proves path connectedness of the collection (6.19), as we wanted.

It follows immediately that the collection

$$
\{ \psi \in L^p : \|\lambda P\psi(b \cdot) - 1\|_{L^p(C)} < 1 \text{ for some } \lambda \in \mathbb{C}, \lambda \neq 0 \} \quad (6.20)
$$

is also path connected and lies in $S^p$, because $\lambda \psi$ belongs to the collection (6.19) and $\psi$ is path connected to $\lambda \psi$ within the collection (6.20), through an obvious path of rescalings.

Now consider an arbitrary $\tilde{\psi} \in S^p$. By openness, there exists an $L^p$-ball around $\tilde{\psi}$ that lies in $S^p$. This ball contains some bounded function $\psi$ having compact support and nonzero integral, and this $\psi$ belongs to the collection (6.20) by Proposition 2(a). We can connect $\tilde{\psi}$ to $\psi$ by a path lying in the ball, and so path connectedness of $S^p$ follows from path connectness of collection (6.20).

### Appendix

**A. The Open Mapping Theorem**

The open mapping theorem for metric spaces was used in the following form, in the proof of Theorem 3 (surjectivity of the synthesis operator).

**Proposition A.1.** Let $X$ and $Y$ be complete metric vector spaces with translation-invariant metrics $d_X$ and $d_Y$, respectively. Suppose $S : X \to Y$ is continuous and linear, take $\delta \in (0, 1)$ and $A > 0$, and assume for each $y \in Y$ that some $x \in X$ exists with

$$
\begin{align*}
&d_Y(Sx, y) \leq \delta d_Y(0, y), \\
&d_X(0, x) \leq Ad_Y(0, y).
\end{align*}
$$

Then $S$ is an open mapping, and $S(X) = Y$. Indeed, given $y \in Y$ there exists $x \in X$ with $Sx = y$ and $d_X(0, x) \leq (1 - \delta)^{-1}Ad_Y(0, y)$.

**Remark.** The hypothesis in Proposition A.1 can be weakened to assume only for some dense subset of $y$-values that $x$ exists satisfying (A.1), provided we are prepared to replace $\delta$ in the conclusion of the proposition by $\delta^* \in (\delta, 1)$ and $A$ by $A^* > A$.

**Proof of Proposition A.1.** Let $y_0 \in Y$. Choose $x_0 \in X$ according to (A.1) with $y = y_0$. Let $y_1 = y_0 - Sx_0$ and choose $x_1$ according to (A.1) with $y = y_1$. Let $y_2 = y_1 - Sx_1$, and continue this process, obtaining $x_0, x_1, x_2, \ldots \in X$ and $y_0, y_1, y_2, \ldots \in Y$ that satisfy

$$
\begin{align*}
y_{m+1} &= y_m - Sx_m, \\
d_Y(0, y_{m+1}) &\leq \delta d_Y(0, y_m), \\
d_X(0, x_m) &\leq Ad_Y(0, y_m),
\end{align*}
$$

for $m = 0, 1, 2, \ldots$ (The left-hand side of (A.3) uses the translation invariance of the $Y$-metric.)
Affine Synthesis onto $L^p$ when $0 < p \leq 1$

Now define $x = \sum_{m=0}^{\infty} x_m$, which converges in the complete, translation-invariant space $X$ because

$$\sum_{m=0}^{\infty} d_X(0, x_m) \leq A \sum_{m=0}^{\infty} d_Y(0, y_m) \quad \text{by (A.4)}$$

$$\leq A d_Y(0, y_0) \sum_{m=0}^{\infty} \delta^m \quad \text{by (A.3)}$$

$$= \frac{A}{1 - \delta} d_Y(0, y_0). \quad (A.5)$$

The continuity and linearity of $S$ imply that

$$Sx = \sum_{m=0}^{\infty} Sx_m = \sum_{m=0}^{\infty} (y_m - y_{m+1}) = y_0,$$

by (A.2) and telescoping, since $y_{m+1} \to 0$ by (A.3). Because $y_0$ was arbitrary, we have shown $S(X) = Y$. Further, (A.5) shows

$$d_X(0, x) \leq \frac{A}{1 - \delta} d_Y(0, y_0).$$

It follows for all $r > 0$ that $S(B_X(r)) \supset B_Y((1 - \delta)A^{-1}r)$, where “$B$” denotes an open ball, and thus $S$ is an open mapping. \hfill \Box

Finally, readers who wish to learn more about the functional analysis of $L^p$ and $\ell^p$ spaces when $0 < p < 1$ could consult the monograph of Kalton, Peck, and Roberts [21].

Acknowledgements

John Benedetto sparked this research during the International Conference on Harmonic Analysis and Applications, Villa de Merlo, Argentina (2006), by asking me about $L^p$-affine synthesis for $0 < p < 1$.

Qui Bui is my co-author on the articles [6]–[10]. The many fruitful discussions we enjoyed while writing those works have influenced the current article as well.

References

[1] Abu-Shammala, W. and Torchinsky, A. (2007). From dyadic $\Lambda_{\alpha}$ to $\Lambda_{\alpha}$, Illinois J. Math., to appear.
[2] Abu-Shammala, W., Shiu, J.-L., and Torchinsky, A. (2006). Characterizations of the Hardy space $H^1$ and $BMO$, preprint, arXiv(math/0510280v2).
[3] de Boor, C. and Jia, R.-Q. (1985). Controlled approximation and a characterization of the local approximation order, Proc. A.M.S. 95, 547–553.
[4] Bownik, M. Connectivity and density in the set of framelets, Math. Res. Lett., to appear.
[5] Bruna, J. (2006). On translation and affine systems spanning $L^1(\mathbb{R})$, J. Fourier Anal. Appl. 12(1), 71–82.
[6] Bui, H.-Q. and Laugesen, R. S. (2004). Spanning and sampling in Lebesgue and Sobolev spaces, University of Canterbury Research Report UCDMS2004/8, 64, www.math.uiuc.edu/~laugesen.
[7] Bui, H.-Q. and Laugesen, R. S. (2005). Affine systems that span Lebesgue spaces, J. Fourier Anal. Appl. 11(5), 533–556.
R. S. Laugesen

[8] Bui, H.-Q. and Laugesen, R. S. Approximation and spanning in the Hardy space, by affine systems, Constr. Approx., appeared online, http://dx.doi.org/10.1007/s00365-006-0672-1.

[9] Bui, H.-Q. and Laugesen, R. S. Sobolev spaces and approximation by affine spanning systems, Math. Ann., to appear, www.math.uiuc.edu/~laugesen.

[10] Bui, H.-Q. and Laugesen, R. S. Affine synthesis onto Lebesgue and Hardy spaces, Indiana Univ. Math. J., to appear, www.math.uiuc.edu/~laugesen.

[11] Canuto, C. and Tabacco, A. (1997). Multilevel decompositions of functional spaces, J. Fourier Anal. Appl. 3(6), 715–742.

[12] Christensen, O. (2003). An Introduction to Frames and Riesz Bases, Birkhäuser, Boston.

[13] Czaja, W., Kutyniok, G., and Speegle, D. (2006). The geometry of sets of parameters of wave packet frames, Appl. Comput. Harmon. Anal. 20, 108–125.

[14] DeVore, R. A., Jawerth, B., and Popov, V. (1992). Compression of wavelet decompositions, Amer. J. Math. 114, 737–785.

[15] Feichtinger, H. G. and Fornasier, M. (2006). Flexible Gabor-wavelet atomic decompositions for $L^2$-Sobolev spaces, Ann. Mat. Pura Appl. (4) 185, 105–131.

[16] Filippov, V. I. and Oswald, P. (1995). Representation in $L_p$ by series of translates and dilates of one function, J. Approx. Theory 82, 15–29.

[17] Filippov, V. I. (1998). On the completeness and other properties of some function systems in $L^p$, 0 < p < ∞, J. Approx. Theory 94, 42–53.

[18] Hernández, E. and Weiss, G. (1996). A First Course on Wavelets, CRC Press, Boca Raton, Florida.

[19] Holtz, O. and Ron, A. (2005). Approximation orders of shift-invariant subspaces of $W^s_2(\mathbb{R}^d)$, J. Approx. Theory 132, 97–148.

[20] Jia, R.-Q. (1998). Stability of the shifts of a finite number of functions, J. Approx. Theory 95, 194–202.

[21] Kalton, N. J., Peck, N. T., and Roberts, J. W. (1984). An $F$-space sampler, London Math. Soc. Lecture Note Ser. 89, Cambridge University Press, Cambridge.

[22] Labate, D., Weiss, G., and Wilson, E. (2004). An Approach to the Study of Wave Packet Systems. Wavelets, Frames and Operator Theory, 215–235, Contemp. Math., 345. Amer. Math. Soc., Providence, RI.

[23] Meyer, Y. (1992). Wavelets and Operators, Cambridge University Press, Cambridge.

[24] Ron, A. (1989). A necessary and sufficient condition for the linear independence of the integer translates of a compactly supported distribution, Constr. Approx. 5, 297–308.

[25] Rudin, W. (1991). Functional Analysis, 2nd ed., McGraw-Hill, New York.

[26] Strang, G. and Fix, G. (1973). A Fourier analysis of the finite element variational method, in Constructive Aspects of Functional Analysis, Geymonat, G., Ed., 793–840. C.I.M.E.

[27] Tachëv, G. T. (1995). A counterexample to the conjecture of W. Filippow and P. Oswald, Ann. Inst. Archit. Gënëve Civil Sofia, Fasc. II, Math. 37, (1993/94), 93–97.

[28] Terekhin, P. A. (1999). Inequalities for the components of summable functions and their representations by elements of a system of contractions and shifts, (Russian) Izv. Vyssh. Uchebn. ZavEd., Mat. 8, 74–81, 1999; translation in, Russian Math. (Iz. VUZ) 43(8), 70–77.

[29] Terekhin, P. A. (1999). Translates and dilates of function with nonzero integral, Mathematics Mechanics, (published by Saratov University) 1, 67–68, (Russian).

[30] Triebel, H. (2002). Towards a Gausslet Analysis: Gaussian Representations of Functions. Function Spaces, Interpolation Theory, and Related Topics (Lund, 2000), 425–449, De Gruyter, Berlin.

[31] Triebel, H. (2005). Spaces on sets, Uspekhi Mat. Nauk 60, 187–206, 2005; translation in Russian Math. Surveys 60, 1195–1215.

[32] The Wutam Consortium (1998). Basic properties of wavelets, J. Fourier Anal. Appl. 4(4-5), 575–594.

Received January 17, 2007
Revision received September 23, 2007
Department of Mathematics, University of Illinois
Urbana, IL 61801, U.S.A.
e-mail: Laugesen@uiuc.edu