FLAT COMPLEX VECTOR BUNDLES,  
THE BELTRAMI DIFFERENTIAL AND $W$–ALGEBRAS

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Abstract

Since the appearance of the paper by Bilal et al. in ’91, it has been widely assumed that $W$–algebras originating from the Hamiltonian reduction of an $SL(n, C)$-bundle over a Riemann surface give rise to a flat connection, in which the Beltrami differential may be identified.

In this letter, it is shown that the use of the Beltrami parametrisation of complex structures on a compact Riemann surface over which flat complex vector bundles are considered, allows to construct the above mentioned flat connection. It is stressed that the modulus of the Beltrami differential is necessarily less than one, and that solutions of the so-called Beltrami equation give rise to an orientation preserving smooth change of local complex coordinates. In particular, the latter yields a smooth equivalence between flat complex vector bundles. The role of smooth diffeomorphisms which induce equivalent complex structures is specially emphasized.

Furthermore, it is shown that, while the construction given here applies to the special case of the Virasoro algebra, the extension to flat complex vector bundles of arbitrary rank does not provide “generalizations” of the Beltrami differential usually considered as central objects for such non-linear symmetries.

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1 Introduction and Motivation

In a paper by A.M. Polyakov [2], diffeomorphism transformations for the 2-d conformal geometry are identified by a soldering procedure from \( SL(2) \) gauge transformations. Several further works, trying to interpret and generalize this ad hoc process, rely on the so-called Hamiltonian reduction. Among these papers, that by A. Bilal and coworkers in 1991, [1], pioneered the attempt to give a geometric interpretation to the non-linear \( W \)-symmetries. However, in spite of the abundance of new ideas and concepts presented in [1], it seems that the first part of the paper in fact concerns two disconnected topics.

On the one hand, in a self-contained introduction, the authors discuss both the notion of complex structures parametrized by Beltrami differentials, \( i.e. \) a \((-1,1)\)-differential \( \mu \) submitted to the condition \( |\mu| < 1 \), and the reinterpretation of the compatibility condition between complex and projective structures as the vanishing of the curvature 2-form of a connection. This connection has components which depend explicitly on \( \mu \).

On the other hand, in the part concerned with the Hamiltonian reduction, a general gauge scheme is provided as a possible way of generating \( W \)-algebras. But, when restricted to the Virasoro case, even if the above quoted connection is formally recovered, the condition \( |\mu| < 1 \) is not. This seems to be a technical detail. However, one ought to have expected that, especially in the simplest case of the Virasoro algebra, the requirement \( |\mu| < 1 \) should have been recovered. This lack of consistency is also encountered in other papers on the subject, see \textit{e.g.} [3, 4, 5] and references therein. In these articles, the identification of an “algebraically” obtained quantity carrying the same geometrical properties as a Beltrami differential is often made without paying any attention to the extra condition \( |\mu| < 1 \). If this observation that there are “Beltrami-like” objects is correct, then, clearly, it questions the geometric framework of [1] from which \( W \)-algebras are assumed to originate.

This question is important because the Beltrami differential is one of the central objects in the Lagrangian description of 2-d conformal models, [6, 7, 8, 9]. The requirement that \( |\mu| < 1 \) is also related to the positivity of the conformal classes of metrics on the surface, \( ds^2 \propto |dz + \mu d\bar{z}|^2 \), [10], and the fact that projective structures are parametrized by holomorphic projective connections [11, 12]. Following [1], the latter are readily identified with the well-known spin two “energy-momentum tensor” of bidimensional conformal models. Generalization of that concept to higher spin objects initiated by A.B. Zamolodchikov, [13], yields the so-called non-linear \( W \)-algebras as extensions of the 2-d conformal algebra, \textit{i.e.} the Virasoro algebra.

In this letter, it is shown that there is a consistent geometrical construction in which both the gauge aspect and the desired property of the Beltrami differential are reconciled. Two steps are required for a proper treatment of the geometry. In the first step, the gauge scheme approach of Bilal \textit{et al.} is reconsidered in the theory of flat complex vector bundles over a Riemann surface. In this holomorphic framework, as will be seen, special representatives for holomorphic connections turn out to have the so-called Drinfeld-Sokolov form [14]. In the second step, we give up the hamiltonian reduction method proposed in [1], in favour of the study of a smooth change of local
complex coordinates which implements the Beltrami parametrisation of complex structures. When
the latter is used, the action of smooth diffeomorphisms on the Riemann surface is well manageable.
It is recalled that the diffeomorphism group is the symmetry group for the 2-d Lagrangian conformal
models whose extensions are expected to be $W$–algebras.

The study of holomorphic vector bundles has been carried out intensively in the context of
$W$–algebras in [15, 16] where the role of the smooth change of complex coordinates has mostly
been overlooked. The flat complex vector bundles involved turn out to be jet bundles of the
sheaf of solutions of some conformally covariant complex differential equations. These results will
make contact with some of those obtained in [1]. Nevertheless, we shall restrict ourselves to the
geometric aspects of the construction without speculating on the possible physical interpretation
of the quantities involved, even if this deserves some careful attention.

In the case of the Virasoro algebra, the construction works perfectly. However, its generaliza-
tion to the $W$–case unfortunately does not lead to objects extending the notion of the Beltrami
differential [1, 17, 18, 19], even if the latter is truly involved, as will be shown in Section 5. This
suggests that the geometry of $W$–algebras, if any, is far from being understood. To be precise, there
is a clash between, on the one hand, the desire to retain the Beltrami parametrization of complex
structures which, as already said, is crucial for the study of bidimensional conformal models, and,
on the other hand, the desire to extend the conformal symmetry to the $W$–algebras.

The letter is organized as follows. In section two, we shall describe the basic material and general
results of Gunning’s work on flat complex vector bundles. This deals with the gauge aspects of
the construction. Section three is devoted to the equivalence between complex structures within
the Beltrami parametrization with special emphasis that the latter will implement a local smooth
change of complex coordinates on the Riemann surface. Infinitesimal features of the action of
diffeomorphisms will be briefly analyzed in the BRS framework. Section four is concerned with
$SL(2, C)$ complex vector bundles which happen to reproduce, when the action of diffeomorphisms
is considered, exactly the expected results for the case of the Virasoro algebra. In the fifth Section,
the question of extending the use of flat complex vector bundles is addressed. The study of the
$SL(3, C)$ complex vector bundles is carried out, with the Beltrami differential still involved. The
description of the $W_3$–case is only partially obtained. The letter concludes with a few words
regarding the geometric content of $W$–algebras.

2 A short review of Gunning’s results on flat complex
vector bundles

We will use Gunning’s notation in order to make contact with his results [11, 12, 20]. Gunning’s
results hold in the holomorphic case, i.e. both gauge group and connection will be holomorphic.

Consider a compact Riemann surface $\Sigma$ of genus greater than one with a complex coordinate
covering $\{(U_\alpha, Z_\alpha)\}$. Throughout this work, capital letters will label the geometry for the complex
structure corresponding to the local complex analytic coordinates $\{Z_\alpha\}$. Let $K$ be the canonical
bundle of $\Sigma$ defined by the holomorphic 1-cocycle $K_{\alpha\beta} = dZ_\beta/dZ_\alpha$. We furthermore consider over $\Sigma$ the holomorphic vector bundle $\Phi$ of rank $n$ determined by holomorphic transition functions $\Phi_{\alpha\beta}$ (1-cocycle with respect to the above covering). According to Gunning [20, 12], we define

**Definition 2.1** The endomorphisms of $\Phi$ are defined by the following set $\{G_\alpha\}$ of matrix-valued functions glueing as

$$\text{in } U_\alpha \cap U_\beta : G_\alpha \Phi_{\alpha\beta} = \Phi_{\alpha\beta} G_\beta .$$

(2.1)

At this stage we still have the freedom to choose either a smooth or a holomorphic gauge group according to the choice of smooth or holomorphic sections of the bundle $\Phi$.

**Definition 2.2** A connection $A$ on the vector bundle $\Phi$ is a collection of matrix-valued differential 1-forms, $\{A_\alpha\}$, defined by the following patching rules

$$\text{in } U_\alpha \cap U_\beta : d\Phi_{\alpha\beta} = \Phi_{\alpha\beta} A_\beta - A_\alpha \Phi_{\alpha\beta} .$$

(2.2)

Since the gauge group acts on the fibres of $\Phi$ it will also acts on connection by

**Definition 2.3** The gauge transformed of $A$ is locally defined by

$$\text{in } U_\alpha : (G A)_\alpha = G_\alpha A_\alpha G_\alpha^{-1} + G_\alpha dG_\alpha^{-1} .$$

(2.3)

First of all, the structure of connections can be analysed. The $(0,1)$-component $\overline{A}$ of any connection on a complex vector bundle over a Riemann surface is integrable so that there is a local change of the fibre coordinates for which the $(0,1)$-component may be put to zero [21, 22].

So, we have the following

**Theorem 2.4** At fixed complex structure on $\Sigma$, the compatible complex analytic structures on the bundle $\Phi$ viewed as a smooth vector bundle, are in one-to-one correspondence with the gauge equivalence classes of connections of type $(0,1)$.

Integrating the $(0,1)$-component of the connection means finding a collection of matrix-valued functions $\{M_\alpha\}$, defined up to an holomorphic matrix-valued function, such that locally

$$\text{in } U_\alpha : \overline{A}_\alpha = M_\alpha^{-1} \partial_{Z_\alpha} M_\alpha .$$

(2.4)

One obtains a new complex vector bundle equivalent to $\Phi$ and defined by the 1-cocycle $M_\alpha \Phi_{\alpha\beta} M_\beta^{-1}$. The latter defines new coordinates for which the $(0,1)$-component of any connection vanishes according to the local redefinition

$$\text{in } U_\alpha : A_\alpha = (M_\alpha A_\alpha - dM_\alpha) M_\alpha^{-1} .$$

(2.5)

This redefinition should not be confused with a gauge transformation [23]. One can show that $M_\alpha \Phi_{\alpha\beta} = N_\beta M_\beta$ in the overlap $U_\alpha \cap U_\beta$, where $N_\beta$ is a holomorphic matrix-valued function, and
thus means that the local solutions \( \{ M_\alpha \} \) are no longer endomorphisms of \( \Phi \). This new complex structure does not depend on the choice of the representative for \( \overline{A} \).

From now on, the coordinate system on the holomorphic vector bundle, with holomorphic gauge group, will be so chosen that \( \overline{A} \equiv 0 \), i.e. a connection on \( \Phi \) is then a collection of matrix-valued differential forms of type \((1,0)\) gluing according to (2.2). Note that the vanishing of the \((0,1)\) part of (2.2) makes sense thanks to the holomorphicity of \( \Phi_{\alpha\beta} \).

A flat bundle associated to the complex bundle \( \Phi \) will be a complex analytic bundle equivalent to \( \Phi \) with constant transition functions. Flat connections thus become holomorphic in those holomorphic coordinates.

**Theorem 2.5 (Gunning, [20])** The set of flat vector bundles associated to a complex vector bundle \( \Phi \) is in one-to-one correspondence with the gauge equivalent classes of holomorphic \((1,0)\)-connections. Moreover if \( \det \Phi = 1 \), the corresponding set of flat vector bundles is in one-to-one correspondence with the gauge equivalent classes of holomorphic \((1,0)\)-connections with null trace.

N.B. Since \( \overline{A} \equiv 0 \), one has \( F(A) = 0 \iff \overline{\partial} A = 0 \). If furthermore \( \det \Phi = 1 \) then \( \Phi \) is determined by an \( SL(n, \mathbb{C}) \) 1-cocycle.

This sums up the main results concerned with flat holomorphic vector bundles, i.e. the study of holomorphic \((1,0)\)-connections. Moreover, as a nice feature, there are always representatives of \( \Phi \) such that the matrices \( \Phi_{\alpha\beta} \) and \( G_\alpha \) are all upper triangular \([12]\). So, in the examples thereafter treated, we can limit ourselves with this special upper triangular form.

### 3 Equivalence of complex structures

In order to speak about the equivalence of complex structures, it is rather good to introduce the Beltrami parametrisation of complex structures, see e.g. \([10]\), subordinate to the smooth structure of the surface \( \Sigma \). It turns out that some complex structures can be defined to be equivalent in a sense which will be made precise down below.

In doing so, we need another diffeomorphic copy of \( \Sigma \) according to the unique differential structure on \( \Sigma \). This copy is turned into a Riemann surface, \( \Sigma_0 \), by choosing a fixed background complex structure given by the complex covering \( \{(U_\alpha, z_\alpha)\} \). We will denote by \( \kappa \) the canonical bundle of \( \Sigma \) defined by the 1-cocycle \( \kappa_{\alpha\beta} = dz_\beta/dz_\alpha \) with respect to this fixed local complex coordinates \( \{z_\alpha\} \), and we will set \( \partial_\alpha \equiv \partial/\partial z_\alpha \) and \( \overline{\partial}_\alpha \equiv \partial/\partial \overline{z}_\alpha \).

We introduce a Beltrami differential denoted by \( \mu \), namely a \((1,0)\)-vector valued \((0,1)\)-form with \( |\mu| < 1 \). Strickly speaking, the Beltrami differential \( \mu \) can be seen as a smooth section of the bundle \( \kappa^{-1} \otimes \kappa \)

\[
\text{in } U_\alpha \cap U_\beta : \mu_\alpha = \kappa_{\alpha\beta}^{-1} \kappa_{\alpha\beta} \mu_\beta , \quad (3.1)
\]

such that locally \( |\mu_\alpha| < 1 \). Let \( \mathcal{B}(\Sigma) \) denote the space of smooth Beltrami differentials on \( \Sigma \).
3.1 The smooth change of local complex coordinates

The Beltrami parametrization of complex structures over \( \Sigma \), consists in finding the holomorphic coordinates \( \{ Z_\alpha \} \) pertaining to the complex structure parametrized by \( \mu \equiv \{ \mu_\alpha \}, \, |\mu_\alpha| < 1 \). This amounts to solve locally the following Pfaff system, see e.g. [8],

\[
\text{in } U_\alpha : \quad dZ_\alpha = \lambda_\alpha (dz_\alpha + \mu_\alpha dz_\alpha) \quad \Rightarrow \quad \partial Z_\alpha = \frac{\partial_\alpha - \overline{\mu_\alpha} \overline{\partial_\alpha}}{\lambda_\alpha (1 - \mu_\alpha \overline{\mu_\alpha})} , \tag{3.2}
\]

with integrating factor \( \lambda_\alpha = \partial_\alpha Z_\alpha \) fulfilling

\[
\text{in } U_\alpha : \quad (d^2 = 0 \iff ) \quad (\overline{\partial_\alpha} - \mu_\alpha \partial_\alpha) \ln \lambda_\alpha = \partial_\alpha \mu_\alpha . \tag{3.3}
\]

Solving the Pfaff system (3.2) is equivalent to solving locally the so-called Beltrami equation

\[
\text{in } U_\alpha : \quad (\overline{\partial_\alpha} - \mu_\alpha \partial_\alpha) Z_\alpha = 0 . \tag{3.4}
\]

According to Bers, see e.g. [10], the Beltrami equation (3.4) always admits as a solution a quasiconformal mapping with dilatation coefficient \( \mu_\alpha \). One thus remarks that \( Z_\alpha \) is a holomorphic functional of \( \mu_\alpha \), which will be denoted for a while by \( Z_{\mu_\alpha} \), and will play the role of the complex coordinate introduced in the previous Section. Therefore, the solution of the Beltrami equation is a mapping on \( U_\alpha : (z_\alpha, \bar{z}_\alpha) \rightarrow (Z_\alpha(z_\alpha, \bar{z}_\alpha), \overline{Z_\alpha(z_\alpha, \bar{z}_\alpha)}) \) which preserves the orientation (the latter condition secures the requirement \( |\mu_\alpha| < 1 \) as noticed in [18]), and is thus locally invertible, so that \( Z_{\mu_\alpha}(z_\alpha, \bar{z}_\alpha) \) defines a new complex coordinate mapping on the open set \( U_\alpha \). Moreover, in the intersection \( U_\alpha \cap U_\beta \), it follows from the patching law of \( \mu \), that solving the Beltrami equation in the overlap shows that the transition function \( Z_{\mu_\alpha} \circ Z_{\mu_\beta}^{-1} \) is holomorphic in \( Z_{\mu_\beta} \) and depends holomorphically on \( \mu \). Hence, the covering \( \{(U_\alpha, Z_{\mu_\alpha})\} \) defines a new complex structure on \( \Sigma \), the one given by \( \mu \) and denoted by \( \Sigma_\mu \). The fibered complex manifold \( \mathcal{B}(\Sigma) \times \Sigma \), with local complex coordinates \( (\mu, Z_\mu) \), defines a complex analytic family of compact Riemann surfaces, see Chapter 2 of [23]. The reader is also referred for instance to Appendix D of [24]. We will say that the Riemann surface \( \Sigma_\mu \) is different from the Riemann surface \( \Sigma_0 \), \( (\mu \equiv 0 \) corresponds to the standard complex structure), with local complex coordinates \( \{ z_\alpha \} \). The appearance of the Beltrami differential simply stems from this non algebraic (!) process implemented by this (local) smooth change of local complex coordinates.

3.2 The action of smooth diffeomorphisms

One remark is in order. The Beltrami differential is a geometric object and as such, is transformed under the action of smooth diffeomorphisms by pull-back. Let \( \varphi \) be a smooth diffeomorphism of \( \Sigma \) homotopic to the identity map. Let \( \mu \) be a given smooth Beltrami differential on \( \Sigma \), represented in a chart, let say \( (U_\alpha, z_\alpha) \), by the smooth function \( \mu_\alpha \). Local complex coordinates on the inverse image of \( U_\alpha \) by \( \varphi, \varphi^{-1}(U_\alpha) \), are provided by intersection with a chart, \( (\varphi^{-1}(U_\alpha) \cap U_\beta, z_\beta) \). According to the fixed complex structure on \( \Sigma \), the local representative of \( \varphi \) reads

\[
(\varphi_{\alpha\beta}^z(z_\beta, \bar{z}_\beta), \varphi_{\alpha\beta}^z(z_\beta, \bar{z}_\beta)) = (z_\alpha \circ \varphi, \bar{z}_\alpha \circ \varphi) . \tag{3.5}
\]
The pull-back, $\mu^\varphi$, of $\mu$ by $\varphi$ is a new section of $\kappa^{-1} \otimes \pi$, with $|\mu^\varphi| < 1$ since $\varphi$ is homotopic to the identity, see for instance [25]. In terms of components and coordinates, $\mu^\varphi$ is given on $\varphi^{-1}(U_\alpha) \cap U_\beta$ by

$$
(\mu^\varphi)_\beta = \frac{\overline{\partial}_\beta \varphi^\varphi_{\alpha \beta} + \overline{\partial}_\beta \varphi^\varphi_{\alpha \beta} (\mu_\alpha \circ \varphi)}{\partial_\beta \varphi^\varphi_{\alpha \beta} + \partial_\beta \varphi^\varphi_{\alpha \beta} (\mu_\alpha \circ \varphi)}.
$$

We get for any point in $U_\alpha$ the following commutative diagram

$$
(\varphi^\varphi_{\alpha \beta}(z_\beta, \bar{z}_\beta), \varphi^\varphi_{\alpha \beta}(z_\beta, \bar{z}_\beta)) \xrightarrow{\mu_\alpha} (Z_{\mu_\alpha} \circ \varphi, \overline{Z}_{\mu_\alpha} \circ \varphi)
$$

by solving the Beltrami equation for both $\mu$ in $U_\alpha$ and $\mu^\varphi$ in $\varphi^{-1}(U_\alpha) \cap U_\beta$. Since the action of diffeomorphisms on Beltrami differentials is defined by the pull-back (3.6), it can be shown [10] that the right vertical arrow is a biholomorphic mapping and hence $Z(\mu^\varphi)_\beta$ can be chosen in $\varphi^{-1}(U_\alpha) \cap U_\beta$ as $Z(\mu^\varphi)_\beta = Z_{\mu_\alpha} \circ \varphi$. One can see that diffeomorphisms act holomorphically on the coordinates $(\mu, Z_\mu)$, and therefore $\mu$ and $\mu^\varphi$ define equivalent complex structures.

As will be seen later on by explicit computation, for a given complex structure $\mu$, the action of diffeomorphisms on geometric objects related to the holomorphic coordinates $Z_\mu$, (germs of holomorphic sections, abelian or holomorphic quadratic differentials, etc, and generically denoted here by $\Pi$) is easily worked out by pull-back

$$
(\varphi^*(\Pi))_{Z_{\mu^\varphi}}(Z_{\mu^\varphi}) = \Pi_{Z_\mu}(Z_\mu \circ \varphi).
$$

It is also worthwhile to consider the infinitesimal action of diffeomorphisms.

### 3.3 The infinitesimal action of diffeomorphisms

The infinitesimal action of diffeomorphisms on $\Sigma$ reduces down to representing the Lie algebra of diffeomorphisms connected to the identity and thus, preserving the orientation. The BRS formulation in terms of a nilpotent algebraic operation $s$ will be used for the (space) variations given by the Lie derivative, (with respect to the background complex coordinates $\{z_\alpha\}$), of geometric objects which will be involved in the exercise given below.

In this setting, the infinitesimal action of diffeomorphisms on a Beltrami differential $\mu$ is now well-known [4, 5]. But, another way to derive the variation which will be quite useful for our purpose can be directly obtained from the above diagram (3.7).

In the BRS algebraic set-up, we have $sd + ds = 0$, where $d$ is the exterior derivative. According to the complex coordinate covering $\{(U_\alpha, Z_\alpha)\}$ pertaining to the complex structure given by $\mu$ where “everything” will be taken to be holomorphic, one considers the action of smooth diffeomorphisms
homotopic to the identity map, \( \varphi_t = id_\Sigma + t c + o(t) \), where \( c = c^z \partial + c^{\bar{z}} \partial \bar{z} \) is the smooth Faddeev-Popov ghost associated to vector fields with respect to the background complex coordinates \( \{ z_\alpha \} \), \( s z_\alpha = 0 \). As said before, the action produces a new equivalent complex structure to that of \( \mu \) given by \( \mu^t \) and for which the complex covering can be written as \( \{(\varphi^{-1}(U_\alpha), Z_{\mu_\alpha} \circ \varphi_t)\} \), so that \( Z_\mu \) is transformed as \( Z_\mu \circ \varphi_t = Z_{\mu^t} \).

The infinitesimal action given by the (graded) Lie derivative, \( s \equiv L = i_c d - d i_c \) and, thanks to (3.2), writes locally in \( U_\alpha \),

\[
 sZ_\alpha = L_c Z_\alpha = i_c d Z_\alpha = \lambda_\alpha (c^{\bar{z}} + \mu_\alpha c^z) \equiv \lambda_\alpha C_\alpha \equiv \gamma Z_\alpha ,
\]

where \( c = \gamma Z \partial_Z + \gamma \bar{Z} \partial_{\bar{Z}} \) is the (smooth) Faddeev-Popov ghost in the holomorphic coordinates \( Z \), and \( C_\alpha \) reflects a change of basis in the Lie algebra of diffeomorphisms leading to the holomorphic factorization property of bidimensional conformal models [6, 7, 9].

So, with respect to the complex structure \( \mu \), the infinitesimal action of diffeomorphisms writes locally in \( U_\alpha \), with \( s^2 = 0 \),

\[
 sZ_\alpha \equiv \gamma \cdot Z_\alpha = \gamma Z_\alpha , \quad s\gamma_{\alpha} = 0 ,
\]

\[
 s d + d s = 0 \implies [s, \partial Z_\alpha] = - (\partial_{Z_\alpha} \gamma Z_\alpha) \partial Z_\alpha - (\partial Z_\alpha \gamma Z_\alpha) \partial_{\bar{Z}} .
\]

Furthermore, the infinitesimal version of the action of diffeomorphisms on (holomorphic) geometric objects defined by (3.8), is given by the Lie derivative which takes in the holomorphic coordinates \( Z \), the following very simple expression

\[
 s \Pi_\alpha = (L_c \Pi)_\alpha = \gamma Z_\alpha \partial Z_\alpha \Pi_\alpha .
\]

Next, by computing

\[
 s d Z_\alpha = -d\gamma Z_\alpha = s(\lambda_\alpha (dz_\alpha + \mu_\alpha d\bar{z}_\alpha)) ,
\]

see (3.2), together with the compatibility condition (3.3), gives rise to the following infinitesimal action of diffeomorphisms in terms of the nilpotent BRS operation, \( s^2 = 0 \),

\[
 s\mu_\alpha = (\partial_\alpha - \mu_\alpha \partial_\alpha + \partial_\alpha \mu_\alpha) C_\alpha , \quad sC_\alpha = C_\alpha \partial_\alpha C_\alpha , \quad s\ln \lambda_\alpha = \partial_\alpha C_\alpha + C_\alpha \partial_\alpha \ln \lambda_\alpha .
\]

Remark that, through the definition of \( C_\alpha \), the infinitesimal variation of \( \mu_\alpha \) given in (3.13) equals the infinitesimal action of diffeomorphisms directly computed from the formula (3.6).

At this stage, we have settled all the basic ingredients we need to deal with the Virasoro algebra.

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1Mathematically speaking, \( c \) is the generator of the Grassmann algebra of the dual of the Lie algebra of smooth diffeomorphisms of \( \Sigma \).
4 The $SL(2,\mathbb{C})$ complex vector bundle

To start with the usual Virasoro case, one considers over $\Sigma$, according to Section two, the holomorphic vector bundle $\Phi$ of rank 2 defined by the following holomorphic transition functions in $U_\alpha \cap U_\beta$: $\Phi_{\alpha\beta} = \begin{pmatrix} K_{\alpha\beta}^{1/2} & dK_{\alpha\beta}^{1/2} / dZ_\beta \\ 0 & K_{\alpha\beta}^{-1/2} \end{pmatrix}$. (4.1)

Note that $\det \Phi = 1$.

The endomorphisms of $\Phi$ are given by the following set $\{G_\alpha\}$ of holomorphic matrix-valued functions defined by (2.1). After using both the Riemann-Roch theorem, see for instance [11], and the Hawley and Schiffer results, [26], about holomorphic affine connections, we find that locally the gauge group is parametrized, up to an overall complex number, according to

$$\text{in } U_\alpha : G_\alpha = \begin{pmatrix} 1 & \Omega_\alpha \\ 0 & 1 \end{pmatrix}$$

(4.2)

where $\{\Omega_\alpha\}$ are the coefficients of an abelian differential $\Omega = \Omega_\alpha dZ_\alpha$, $\partial_Z^{\alpha} \Omega_\alpha = 0$. It is readily seen that $\text{End}(\Phi)$ is a subgroup of $SL(2,\mathbb{C})$ which is not the Borel subgroup, compare with [1].

Moreover, according to Gunning [11, 20], an endomorphism of $\Phi$ can also be seen as a sheaf homomorphism $G : \mathcal{O}(\Phi) \longrightarrow \mathcal{O}(\Phi)$, where $\mathcal{O}(\Phi)$ is the sheaf of germs of holomorphic sections of $\Phi$. Here, it turns out that $\Phi$ is represented by the first jet bundle $J_1(\mathcal{O}(K^{-1/2}))$, where $\mathcal{O}(K^{-1/2})$ is the sheaf of germs of holomorphic sections $\Psi$ of the complex line bundle $K^{-1/2}$ which are locally represented by germs of holomorphic $(-\frac{1}{2},0)$-differentials $\Psi (dZ)^{-1/2}$, $\partial_Z \Psi = 0$, such that

$$\text{in } U_\alpha \cap U_\beta : \Psi_\alpha = K_{\alpha\beta}^{-1/2} \Psi_\beta .$$

(4.3)

For a holomorphic $(1,0)$-connection, solving the glueing condition (2.2) yields the following local expression

$$\text{in } U_\alpha : A_\alpha = \begin{pmatrix} B_\alpha & \Xi_\alpha \\ 1 & -B_\alpha \end{pmatrix} dZ_\alpha .$$

(4.4)

where $\{B_\alpha\}$ are the coefficients of an abelian differential and $\{\Xi_\alpha\}$ are well defined holomorphic objects on $\Sigma$, see [20]. Note that $\text{Tr} A_\alpha = 0$. However, it is possible to parametrize locally these $\Xi$’s as $\Xi_\alpha = Q_\alpha - B_\alpha^2 - \frac{1}{2}P_\alpha - \partial_Z A_\alpha$, where $\{Q_\alpha\}$ are the coefficients of a holomorphic quadratic differential, and $\{P_\alpha\}$ are holomorphic functions defining a holomorphic projective connection, i.e. locally we have $\partial_Z P_\alpha = 0$ and

$$\text{in } U_\alpha \cap U_\beta : P_\beta = K_{\alpha\beta}^{-2} P_\alpha + \{Z_\alpha, Z_\beta\} ,$$

(4.5)

where $\{Z_\alpha, Z_\beta\}$ denotes the Schwarzian derivative characterizing the holomorphic projective transformations, i.e. $\{Z_\alpha, Z_\beta\} = -(d^2 \ln K_{\alpha\beta}/dZ_\beta^2 - \frac{1}{2}(d\ln K_{\alpha\beta}/dZ_\beta)^2)$. 

The gauge transformed of $A$ given by the local formula (2.3) together with the specification $\Omega_{\alpha} = -B_{\alpha}$ for $G$ gives rise to the following unique special representative of equivalence gauge classes of holomorphic $(1, 0)$-connections, see [12],

$$\in U_\alpha : \hat{A}_{\alpha} = \begin{pmatrix} 0 & -\frac{1}{2} H_{\alpha} \\ 1 & 0 \end{pmatrix} dZ_\alpha ,$$

(4.6)

where $H_{\alpha} = P_{\alpha} - 2Q_{\alpha}$ is thus a holomorphic projective connection; recall that holomorphic quadratic differentials are the ambiguities on the determination of projective connections. Of course the curvature $\hat{\Omega} = 0$. So, through this special representative (4.6) one may say that holomorphic projective connections parametrize the flat complex vector bundles of rank 2 associated to $\Phi$.

Next, following Gunning [20, 11], one restricts oneself to the subsheaf of $O(K^{-1/2})$ made of the complex analytic solutions of the conformally covariant differential equation

$$\Psi''_\alpha + \frac{1}{2} H_{\alpha} \Psi_\alpha = 0 ,$$

(4.7)

where the $'$ symbol stands for the holomorphic derivative with respect to the local coordinate $\{Z_\alpha\}$. This gives germs of holomorphic sections of the bundle $K^{3/2}$.

This concludes our review on $SL(2, \mathbb{C})$ complex holomorphic vector bundles, [11, 20]. But, nothing has been said yet about the action of smooth diffeomorphisms on the base Riemann surface and the importance of the Beltrami parametrisation of complex structures as described in Section 3. This will be the subject of the next two points.

### 4.1 The effect of the smooth change of local complex coordinates

For a given $\mu$ i.e. a given complex structure on our compact surface, the flat holomorphic complex vector bundle $\Phi$ is over the base Riemann surface $\Sigma_\mu$. The flat holomorphic complex vector bundle $\Phi$ will be “pulled-back” through the local smooth change of complex coordinates defined by eq.(3.2) to a smooth equivalent flat vector bundle over the Riemann surface $\Sigma \equiv \Sigma_0$.

In more details, expressing the holomorphic coordinates $\{Z_\alpha\}$ in terms of the underlying complex structure $\mu$, the sheaf $O(K^{-1/2})$ is pulled back to the sheaf $\mathcal{E}(K^{-1/2})$ of germs of smooth sections $\psi$ of $K^{-1/2}$ thanks to the local rescaling

$$\in U_\alpha : \Psi_\alpha = \lambda^{1/2}_\alpha \psi_\alpha .$$

(4.8)

This induces a local smooth change of coordinates on the fibres of the holomorphic vector bundle $\Phi$

$$M_{\alpha}(z_\alpha, \bar{z}_\alpha) = \begin{pmatrix} \lambda^{-1/2}_\alpha & \lambda^{-1/2}_\alpha \partial_\alpha \ln \lambda^{-1/2}_\alpha \\ 0 & \lambda^{1/2}_\alpha \end{pmatrix} ,$$

(4.9)

and yields a smooth vector bundle $\phi$ equivalent to $\Phi$ defined by the 1-cocycle with respect to the complex covering $\{(U_\alpha, z_\alpha)\}$,

$$\phi_{\alpha \beta} = M_\alpha^{-1} \Phi_{\alpha \beta} M_\beta ,$$

(4.10)
where, thanks to \( \lambda_\alpha K_{\alpha\beta} = \kappa_{\alpha\beta} \lambda_\beta \), the new holomorphic (in \( z \)) transition functions read

\[
\text{in } U_\alpha \cap U_\beta : \phi_{\alpha\beta} = \begin{pmatrix}
\frac{1}{2} \kappa_{\alpha\beta}^{1/2} \\
\frac{d \kappa_{\alpha\beta}^{1/2}}{dz_\beta} \\
0 \\
\frac{1}{2} \kappa_{\alpha\beta}^{-1/2}
\end{pmatrix}.
\]

(4.11)

Remark that \( \det \phi = 1 \) and \( \phi = J_1(\mathcal{E}(\kappa^{-1/2})) \) the 1-jet bundle of \( \mathcal{E}(\kappa^{-1/2}) \). The gauge group \( \text{End}(\phi) \) is thus defined according to the 1-cocycle \( \phi \), see (4.10), by

\[
\text{in } U_\alpha \cap U_\beta : \phi_{\alpha\beta} g_\beta = g_\alpha \phi_{\alpha\beta} \implies \text{in } U_\alpha : g_\alpha = M^{-1}_\alpha G_\alpha M_\alpha.
\]

(4.12)

Due to the holomorphicity of \( G \) and the compatibility condition (3.3), one finds

\[
\text{in } U_\alpha : g_\alpha = \begin{pmatrix}
1 & \omega_\alpha \\
0 & 1
\end{pmatrix}, \quad \partial_\alpha \Omega_\alpha = 0 \implies \left( \overline{\partial}_\alpha - \mu_\alpha \partial_\alpha \right) \omega_\alpha - \omega_\alpha \partial_\alpha \mu_\alpha = 0,
\]

(4.13)

where \( \lambda_\alpha^{-1} \omega_\alpha = \Omega_\alpha \), and \( \{\omega_\alpha\} \) are thus coefficients of a smooth \((1,0)\)-differential.

Accordingly, the flat connections on the new holomorphic vector bundle \( \phi \) are given by

\[
\hat{A}_\alpha = M^{-1}_\alpha \hat{A}_\alpha M_\alpha + M^{-1}_\alpha dM_\alpha.
\]

(4.14)

Note that this is the inverse of (2.3) and does not correspond to a gauge transformation: the connections \( \hat{A}_\alpha \) and \( \tilde{A}_\alpha \) are not defined on the same flat holomorphic vector bundle. After the use, once more, of eq.(3.3), one finds the remarkable smooth representative connection in more explicit form

\[
\hat{A}_\alpha = \begin{pmatrix}
0 & -\frac{1}{2} h_\alpha \\
1 & 0
\end{pmatrix} d\alpha + \begin{pmatrix}
-\frac{1}{2} \partial_\alpha \mu_\alpha & -\frac{1}{2} (\partial_\alpha^2 \mu_\alpha + \mu_\alpha h_\alpha) \\
\mu_\alpha & \frac{1}{2} \partial_\alpha \mu_\alpha
\end{pmatrix} d\overline{\alpha},
\]

(4.15)

where we have set

\[
h_\alpha(z_\alpha, \overline{z}_\alpha) = \lambda_\alpha^2 H_\alpha(Z_\alpha) + \{Z_\alpha, z_\alpha\},
\]

(4.16)

with \( \{Z_\alpha, z_\alpha\} = \partial_\alpha^2 \ln \lambda_\alpha - \frac{1}{2} (\partial_\alpha \ln \lambda_\alpha)^2 \) the Schwarzian derivative of \( Z \) with respect to \( z \). Moreover, thanks to eq.(3.3), one can show that

\[
(\overline{\partial}_\alpha - \mu_\alpha \partial_\alpha - 2 \partial_\alpha \mu_\alpha) \{Z_\alpha, z_\alpha\} = \partial_\alpha^3 \mu_\alpha,
\]

(4.17)

and thus, \( h_\alpha \) fulfils the equation

\[
\partial_{Z_\alpha} H_\alpha = 0 \iff \overline{\partial}_\alpha h_\alpha = (\partial_\alpha^3 + 2 h_\alpha \partial_\alpha + \partial_\alpha h_\alpha) \mu_\alpha.
\]

(4.18)

The latter corresponds exactly to the vanishing of the curvature 2-form \( \mathcal{F} \) associated to \( \hat{A} \). Indeed, we have locally \( \mathcal{F}_\alpha = M^{-1}_\alpha \hat{F}_\alpha M_\alpha = 0 \). Moreover, due to its definition (4.16), the set \( \{h_\alpha\} \) glues as a projective connection but it is not locally holomorphic with respect to the complex covering \( \{(U_\alpha, z_\alpha)\} \). Note that this non-holomorphic connection is local in \( \mu \).
Therefore, we have obtained smooth equivalent flat holomorphic vector bundles and the above equation \((4.18)\) expresses the compatibility condition between the complex structure \(\mu\) and the (smooth) projective connection \(h\) defined by \((4.16)\). This upshot was previously established in \([1]\).

Accordingly, the rescaled solution \(\psi\) has to satisfy locally

\[ \partial_{Z_\alpha} \Psi_\alpha = 0 \implies \overline{\partial}_\alpha \psi_\alpha = \mu_\alpha \partial_\alpha \psi_\alpha - \frac{1}{2} \psi_\alpha \partial_\alpha \mu_\alpha , \tag{4.19} \]

\[ \text{eq.(4.7)} \implies \partial_\alpha^2 \psi_\alpha + \frac{1}{2} h_\alpha \psi_\alpha = 0 . \]

This is how the results given in \([1]\) for the Virasoro case are recovered. The present derivation makes explicit use of the Beltrami parametrisation and of the relationship between field variables and the components of geometric objects via the integrating factor \(\lambda\) which is easily eliminated through eq.(3.3).

4.2 The action of diffeomorphisms

The Beltrami differential being explicitly introduced and involved in a local way, we are in position for treating the action of diffeomorphisms. The infinitesimal action of diffeomorphisms will reproduce the expected conformal Virasoro algebra. Hence, the “soldering” procedure, as initiated in \([2]\), to obtain the action of diffeomorphisms on \(\Sigma\) directly from a \(SL(2)\) gauge transformation on \(\Phi\) seems to remain at the level of a recipe.

The infinitesimal action of diffeomorphisms which happens to be worked out directly from the complex coordinates \(Z\) pertaining to the complex structure \(\mu\) is given locally in \(U_\alpha\), with \(s^2 = 0\), by, see (3.11),

\[ sH_\alpha = \gamma^{Z_\alpha} \partial_{Z_\alpha} H_\alpha , \quad s\Psi_\alpha = \gamma^{Z_\alpha} \partial_{Z_\alpha} \Psi_\alpha . \tag{4.20} \]

In order to compute the infinitesimal action of diffeomorphisms on the smooth projective connection \(\{h_\alpha\}\) defined in eq.(4.14), one uses in sequel, on the one hand, \(sH\) given in \((4.20)\) and the expression (3.2) for \(\partial_Z\), and on the other hand, \(s \ln \lambda\) given in \((3.13)\), eq.(3.3) and eq.(4.18). One finally finds

\[ sh_\alpha = \left( \partial_\alpha^2 + 2 h_\alpha \partial_\alpha + \partial_\alpha h_\alpha \right) C_\alpha , \tag{4.21} \]

which formally looks like the conformal transformation law for the energy-momentum tensor of the Virasoro algebra.

The question, which still remains to be solved, is whether such a \(h\) might be interpreted as the energy-momentum tensor of some theory according to eq.(4.18) which looks like the conformal Ward identity. The latter arises from the diffeomorphism symmetry.

The following transformation law for the solutions of eq.(4.19) can be added. Starting with \(s\psi_\alpha\) defined in \((4.20)\) and proceeding as above for \(h\) by using all together \((3.13)\), \((3.3)\) and \((4.19)\), one computes

\[ s\psi_\alpha = C_\alpha \partial_\alpha \psi_\alpha - \frac{1}{2} \psi_\alpha \partial_\alpha C_\alpha , \tag{4.22} \]

which shows that \(\{\psi_\alpha\}\) are the coefficients of a smooth \((-\frac{1}{2},0)\)-differential.
All these results fit perfectly together with what we expect regarding the Virasoro algebra. There are related with the compatibility between complex and projective structures on a Riemann surface, upon the use of the Beltrami parametrisation of complex structures.

5 On flat complex vector bundles of higher rank

It seems very nice now that, in the Beltrami parametrisation of complex structures, the dependence of the $SL(2, C)$-flat connections on the Beltrami differential actually emerges from the above local process generated by a smooth change of complex coordinates. Moreover, no constraint is imposed on the connection contrary to what is done in the Hamiltonian reduction framework, since the use of the holomorphic framework insures from the very beginning the vanishing of the curvature, see Section 2.

In view of the previous exercise, the question whether the construction extends to higher rank vector bundle is of utmost interest, that is, whether it works in the case of $W$–algebras. Indeed, the above appealing scheme (a special representative of holomorphic connections on a flat complex vector bundle together with the use of the Beltrami parametrisation of complex structures on a compact Riemann surface) fails in constructing “generalized Beltrami differentials” for flat complex vector bundles of higher rank. Down below, this fact will be illustrated in the example related to the $W_3$–algebra.

5.1 The $SL(3, C)$ complex vector bundle

Let $\Phi$ be the flat complex analytic vector bundle of rank 3 defined by the holomorphic 1-cocycle

$$
\begin{pmatrix}
K_{\alpha\beta} & \frac{dK_{\alpha\beta}}{dZ_{\beta}} & \frac{d^2K_{\alpha\beta}}{dZ_{\beta}^2} \\
0 & 1 & \frac{d\ln K_{\alpha\beta}}{dZ_{\beta}} \\
0 & 0 & K_{\alpha\beta}^{-1}
\end{pmatrix},
$$

with $\det \Phi = 1$. After solving the cocycle condition (2.1) according to (5.1) the endomorphisms of $\Phi$ are defined by the following set $\{G_{\alpha}\}$ of holomorphic matrix-valued functions parametrized as follows

$$
\begin{pmatrix}
1 & \Omega^1_{\alpha} & \Theta_{\alpha} + \Omega^2_{\alpha} - \Omega^1_{\alpha} \\
0 & 1 & \Omega^2_{\alpha} \\
0 & 0 & 1
\end{pmatrix},
$$

where $\{\Omega^i_{\alpha}\}, i = 1, 2$ are the coefficients of abelian differentials and $\{\Theta_{\alpha}\}$ are those of a holomorphic quadratic differential. For a holomorphic $(1, 0)$-connection on that bundle, solving the glueing
condition (2.2) for the $SL(3,\mathbb{C})$ 1-cocycle (5.1) gives rise to the following local parametrization for flat connections, in $U_\alpha$:

$$A_\alpha = \begin{pmatrix} B^1_\alpha & Q^1_\alpha + B^2_\alpha - B'^1_\alpha & C_\alpha - P'_\alpha + \frac{1}{2}(Q^2_\alpha - Q'^1_\alpha) - B'^2_\alpha - 2P_\alpha B^1_\alpha \\ 1 & B^2_\alpha & -2P_\alpha - B'^1_\alpha - 2B'^2_\alpha + Q^2_\alpha \\ 0 & 1 & -B'^1_\alpha - B'^2_\alpha \end{pmatrix} dZ_\alpha , \quad (5.3)$$

where $\{B^i_\alpha\}, i = 1, 2$, represents abelian differentials, $\{Q^i_\alpha\}, i = 1, 2$, holomorphic quadratic differentials, $\{C_\alpha\}$ are the coefficients of a holomorphic cubic differential and $\{P_\alpha\}$ are those of a holomorphic projective connection. Note that $\text{Tr} A_\alpha = 0$.

By choosing $\Omega^1_\alpha = -B^1_\alpha$, $\Omega^2_\alpha = -B^1_\alpha - B'^2_\alpha$, $\Theta_\alpha = B^1_\alpha B^2_\alpha - Q^1_\alpha + Q'^2_\alpha$ in (5.2), and by performing the corresponding gauge transformation on $A$ (see (2.3)) one gets the special representative for the gauge classes of holomorphic $(1,0)$-connections

$$\text{in } U_\alpha : A_\alpha = \begin{pmatrix} 0 & 0 & C_\alpha - H'_\alpha \\ 1 & 0 & -2H_\alpha \\ 0 & 1 & 0 \end{pmatrix} dZ_\alpha , \quad (5.4)$$

where $H_\alpha = P_\alpha + \frac{1}{2}(B^1_\alpha B^2_\alpha - (B^1_\alpha + B'^2_\alpha)^2 + Q^2_\alpha - Q'^1_\alpha)$ is the coefficient of a holomorphic projective connection. One may say that both the holomorphic projective connections and the cubic differentials parametrize the set of flat complex vector bundles of rank 3 associated to $\Phi$. Notice that this special representative takes the so-called Drinfeld-Sokolov form [14].

One can see that the 1-cocycle $\Phi_{\alpha\beta}$ defines the glueing rules of $J_2(\mathcal{O}(K^{-1}))$, the second order jet bundles of the sheaf of germs of holomorphic sections of the line bundle $K^{-1}$, which are then locally represented by germs of holomorphic vector fields $\Psi (dZ)^{-1}, \partial_{\bar{Z}}\Psi = 0$, satisfying the patching rules

$$\text{in } U_\alpha \cap U_\beta : \Psi_\alpha = K^{-1}_{\alpha\beta} \Psi_\beta . \quad (5.5)$$

As considered by Gunning in [20], one can restrict the sheaf $\mathcal{O}(K^{-1})$ to the subsheaf of complex analytic solutions of the conformally covariant differential equation

$$\Psi''_\alpha + 2H_\alpha \Psi'_\alpha + (H'_\alpha - C_\alpha) \Psi_\alpha = 0 . \quad (5.6)$$

Note that the left-hand-side of eq.(5.6) is a germ of holomorphic sections of the bundle $K^2$.

### 5.2 The Beltrami parametrisation and the action of diffeomorphisms

As before, the flat complex vector bundle $\Phi$ is pulled back by the smooth local change of coordinates on the fibres

$$M_\alpha (z_\alpha, \bar{z}_\alpha) = \begin{pmatrix} \lambda^{-1}_\alpha & \lambda^{-1}_\alpha \partial_\alpha \ln \lambda^{-1}_\alpha & (\lambda^{-1}_\alpha \partial_\alpha \ln \lambda^{-1}_\alpha)^2 + \lambda^{-1}_\alpha \partial^2_\alpha \ln \lambda^{-1}_\alpha \\ 0 & 1 & \partial_\alpha \ln \lambda^{-1}_\alpha \\ 0 & 0 & \lambda_\alpha \end{pmatrix} , \quad (5.7)$$
induced by the local rescaling

\[ \in U_\alpha : \Psi_\alpha = \lambda_\alpha \psi_\alpha. \]  

(5.8)

Using again \( \lambda_\alpha K_{\alpha\beta} = \kappa_{\alpha\beta} \lambda_\beta \), yields a smooth vector bundle \( \phi \) equivalent to \( \Phi \) defined by the holomorphic 1-cocycle \( \phi_{\alpha\beta} = M_\alpha^{-1} \Phi_{\alpha\beta} M_\beta \),

\[ \in U_\alpha \cap U_\beta : \phi_{\alpha\beta}(z_\beta) = \begin{pmatrix} \kappa_{\alpha\beta} & d\kappa_{\alpha\beta} & d^2\kappa_{\alpha\beta} \\
0 & 1 & d\ln \kappa_{\alpha\beta} \\
0 & 1 & \kappa_{\alpha\beta}^{-1} \end{pmatrix}. \]  

(5.9)

One has \( \det \phi = 1 \) so that \( \phi = J_2(\mathcal{E}(\kappa^{-1})) \). The gauge group for \( \phi \) is thus parametrized as

\[ \in U_\alpha : g_\alpha = M_\alpha^{-1} G_\alpha M_\alpha \implies g_\alpha = \begin{pmatrix} 1 & \omega_\alpha^1 & \theta_\alpha + \partial_\alpha \omega_\alpha^2 - \partial_\alpha \omega_\alpha^1 \\
0 & 1 & \omega_\alpha^2 \\
0 & 0 & 1 \end{pmatrix}, \]  

(5.10)

where \( \lambda_\alpha^{-1}\omega_\alpha^i = \Omega_\alpha^i, i = 1, 2, \lambda_\alpha^{-2}\theta_\alpha = \Theta_\alpha \), are, respectively, coefficients of smooth \((1,0)\)-differentials and of a smooth \((2,0)\)-differential with

\[ \partial_{\bar{z}} \Omega_\alpha^i = 0 \quad \implies \quad (\partial_\alpha - \mu_\alpha \partial_\alpha) \omega_\alpha^i - \omega_\alpha^i \partial_\alpha \mu_\alpha = 0; \]

\[ \partial_{\bar{z}} \Theta_\alpha = 0 \quad \implies \quad (\partial_\alpha - \mu_\alpha \partial_\alpha) \theta_\alpha - 2 \theta_\alpha \partial_\alpha \mu_\alpha = 0. \]  

(5.11)

Accordingly, the flat connections on the new holomorphic vector bundle \( \phi \) are given by

\[ \hat{\mathcal{A}}_\alpha = M_\alpha^{-1} \mathcal{A}_\alpha M_\alpha + M_\alpha^{-1} dM_\alpha . \]  

(5.12)

Note once more, this is not a gauge transformation but only a local change of coordinates. More explicitly, the remarkable smooth representative of the flat connection reads

\[ \hat{\mathcal{A}}_\alpha = \begin{pmatrix} 0 & 0 & W_\alpha^3 - \partial_\alpha h_\alpha \\
1 & 0 & -2h_\alpha \\
0 & 1 & 0 \end{pmatrix} dz_\alpha - \begin{pmatrix} \partial_\alpha \mu_\alpha & \partial_\alpha^2 \mu_\alpha & \partial_\alpha^3 \mu_\alpha + \mu_\alpha (\partial_\alpha h_\alpha - W_\alpha^3) \\
-\mu_\alpha & 0 & 2\mu_\alpha h_\alpha + \partial_\alpha^2 \mu_\alpha \\
0 & -\mu_\alpha & -\partial_\alpha \mu_\alpha \end{pmatrix} d\bar{z}_\alpha, \]  

(5.13)

where \( h_\alpha \) defined as in (4.16) and \( W_\alpha^3 \lambda_\alpha^{-3} = \mathcal{C}_\alpha \) fulfill respectively the following equations of vanishing curvature

\[ \partial_{\bar{z}} \mathcal{H}_\alpha = 0 \quad \iff \quad \bar{\mathcal{H}}_\alpha \mathcal{H}_\alpha = (\partial_\alpha^2 + 2h_\alpha \partial_\alpha + \partial_\alpha^3) \mu_\alpha; \]

\[ \partial_{\bar{z}} \mathcal{C}_\alpha = 0 \quad \iff \quad \bar{\mathcal{C}}_\alpha W_\alpha^3 = \mu_\alpha \partial_\alpha W_\alpha^3 + 3W_\alpha^3 \partial_\alpha \mu_\alpha. \]  

(5.14)

Notice that this special representative of flat smooth connections has a local dependence in \( \mu \).
Moreover, the rescaled solution $\psi$ has to satisfy locally

$$\partial Z_\alpha \Psi_\alpha = 0 \implies \bar{\partial}_\alpha \psi_\alpha = \mu_\alpha \partial_\alpha \psi_\alpha - \psi_\alpha \partial_\alpha \mu_\alpha ,$$

$$\text{eq.(5.6) } \implies (\partial^3 \alpha + 2h_\alpha \partial_\alpha + \partial_\alpha h_\alpha) \psi_\alpha - W^3_\alpha \psi_\alpha = 0 .$$

At the infinitesimal level, the action of diffeomorphisms is computed as before, and we have in addition to the variation (4.21),

$$s \psi_\alpha = C_\alpha \partial_\alpha \psi_\alpha - \psi_\alpha \partial_\alpha C_\alpha , \quad s W^3_\alpha = C_\alpha \partial_\alpha W^3_\alpha + 3W^3_\alpha \partial_\alpha C_\alpha ,$$

which shows that $\psi$ is a conformal field of weight $-1$ and that $W^3$ carries conformal weight 3. By comparison of the equations (5.14), (5.15) and (5.16) with those obtained in [19], one sees that the former reflect the failure to obtain a “generalized notion” of the Beltrami differential. Here, by a direct inspection in the $(0,1)$ component of the flat connection (5.13), these generalizations can be explicitly seen to be zero.

6 Conclusions

Despite the appearance of a Beltrami differential, and the fact that the object $W^3$ might be the good candidate for the expected spin 3 conformal tensor of the $W_3$–algebra, one can conclude that there is no “generalized” Beltrami differential to be read off from the row-column position 3-1 in the matrix of the $(0,1)$-component of $\check{\mathfrak{A}}$ in (5.13). We stress once more that the Hamiltonian reduction scheme does not lead to the appearance of a Beltrami differential. However, it does seem to provide a means of constructing, at least formally, equations related to the $W$–algebras.

Thus, having used the above holomorphic framework, one concludes that the flatness condition for complex vector bundles of higher rank is too restrictive since it enforces the vanishing of some components, $\text{i.e.} \mathfrak{T}$. Rather, one should start directly with a smooth connection with respect to a fixed complex structure on the base Riemann surface. In particular, this addresses the question on the way of parametrizing the complex structures on complex vector bundles of rank greater than two, namely, how to write down explicitly the $(0,1)$-part of a non holomorphic connection. Some attempts in that direction have been made, especially through the use of the so-called Hitchin connection for Higgs bundles, [27], [28] and references therein. However, also in these references, Beltrami differentials are ultimately identified without caring about the condition on their modulus to be less than one. Alternatively, another possibility based on the use of pseudogroups will be studied elsewhere [29]. But, one should keep in mind that the Beltrami differential originates from a smooth change of local complex coordinates on a compact Riemann surface. The latter gives rise to smooth equivalent flat vector bundles.

For the case of the Virasoro algebra, a geometric construction has been proposed in order to recover the very important Beltrami differential for 2-d conformal models. This construction is purely based on the Beltrami parametrisation of complex structures –a point which we wish
to emphasize—through a smooth change of local complex coordinates. The action of smooth diffeomorphisms, the symmetry group for those models, can be easily worked out and leads to the conformal variations computed in the physical literature. Furthermore, this result is expected to shed light on the issue of the possible geometric interpretation of $W$-algebras. The latter is highly non-trivial, since the Virasoro algebra remains only a very particular case and the conformal extended versions are not yet well understood.

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