What is a reduced boundary in general relativity?

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(Dated: November 17, 2020)

Abstract

The concept of boundary plays an important role in several branches of general relativity, e.g., the variational principle for the Einstein equations, the event horizon and the apparent horizon of black holes, the formation of trapped surfaces. On the other hand, in a branch of mathematics known as geometric measure theory, the usefulness has been discovered long ago of yet another concept, i.e., the reduced boundary of a finite-perimeter set. This paper proposes therefore a definition of finite-perimeter sets and their reduced boundary in general relativity. Moreover, a basic integral formula of geometric measure theory is evaluated explicitly in the relevant case of Euclidean Schwarzschild geometry, for the first time in the literature. This research prepares the ground for a measure-theoretic approach to several concepts in gravitational physics, supplemented by geometric insight. Moreover, such an investigation suggests considering the possibility that the in-out amplitude for Euclidean quantum gravity should be evaluated over finite-perimeter Riemannian geometries that match the assigned data on their reduced boundary.

PACS numbers: 04.20.Cv, 04.60.Ds

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I. INTRODUCTION

When Fermi was investigating the nature of mesons with his student Yang in the late forties, they conceived a title of their paper with a question mark at the end [1]. In the course of completing this research, Yang was getting skeptical, but Fermi encouraged his young student, pointing out that, when we are students, we have to solve problems, whereas, when we are researchers, we have to ask the right sort of questions [2].

It is precisely with this understanding that we have chosen the title for our paper. In order to help the general reader, we begin by recalling that surfaces have always played an important role in general relativity. The mathematical language of general relativity was indeed born at the time when Ricci-Curbastro was lecturing on the theory of surfaces at Padova University [3, 4]. Two- and three-dimensional surfaces have taught us many lessons on gravitational physics ever since. For example, event horizon and apparent horizon of a black hole have, both, two-sphere topology [5], and it is intriguing that the relation between area $A$ of the event horizon and black hole entropy $S$:

$$S = \frac{A}{4}, \quad (1.1)$$

can be obtained from the boundary term in the gravitational action of a Schwarzschild black hole, if the partition function is evaluated at tree level in Euclidean quantum gravity [6]. The consideration of the variational principle for classical general relativity [7–11] leads again to the boundary term used in Ref. [6], but also to another form of the boundary term, applied by Hartle and Hawking in quantum cosmology [12].

In a rather different framework, the mathematical community has studied over the centuries the problem of finding the surface of least area among those bounded by a given curve. In the attempt of building a rigorous theory of these minimal surfaces, it proved useful to regard a hypersurface in $\mathbb{R}^n$ as a boundary of a measurable set $E$, having characteristic function

$$\varphi(x, E) = 1 \text{ if } x \in E, \quad 0 \text{ if } x \in \mathbb{R}^n - E \quad (1.2)$$

whose distributional derivatives have finite total variation. What really matters is then the so-called reduced boundary $\mathcal{F}E$ of $E$. For every $x \in \mathcal{F}E$ one can define an approximate
normal vector \((A(x, \rho))\) being the open hypersphere of radius \(\rho\) centred at \(x\)

\[
\nu_{\rho}(x) = \frac{\int_{A(x, \rho)} \nabla \varphi(x, E)}{\int_{A(x, \rho)} \| \nabla \varphi(x, E) \|} \tag{1.3}
\]

where \(\|v\|\) without subscript denotes in our paper the Euclidean norm of a vector \(v\) in \(\mathbb{R}^n\), i.e.

\[
\|v\| = \sqrt{\sum_{i=1}^{n} (v_i)^2}.
\]

It was proved by De Giorgi that if, for some \(x \in \mathcal{F}E\) and some \(\rho > 0\), the vector \(\nu_{\rho}(x)\) has length close enough to 1, then the difference \(1 - \|\nu_{\rho}(x)\|\) approaches 0 as \(\rho \to 0\). This implies in turn that the reduced boundary is analytic in a neighbourhood of \(x\).

In general relativity, measure-theoretic concepts have been exploited to obtain very important results, e.g., the proof of the positive-mass theorem \([13–15]\) and of the Riemannian Penrose inequality \([16]\). Moreover, in recent years, the concepts and problems of geometric measure theory \([17–24]\) have been studied in non-Euclidean spaces and sub-Riemannian manifolds \([25]\), and the resulting framework is not only extremely elegant but also conceptually profound. Since a physics-oriented reader is not necessarily familiar with geometric measure theory, we begin with a pedagogical review of finite-perimeter sets and reduced boundary properties in Sec. II, while their counterpart for Riemannian manifolds is considered in Sec. III. Section IV evaluates the general formulae at the end of Sec. III in Euclidean Schwarzschild geometry. In Sec. V, we propose how to exploit the material of Secs. II and III in order to arrive at a definition of finite-perimeter set and reduced boundary in paracompact pseudo-Riemannian manifolds. Concluding remarks and open problems are presented in Sec. VI, while technical details are provided in the Appendices. Our presentation assumes only that the reader is familiar with the basic elements of Lebesgue’s theory of measure and integration \([26–30]\).

II. FINITE PERIMETER SETS IN \(\mathbb{R}^n\) AND THEIR REDUCED BOUNDARY

Research on geometric measure theory was initiated by Caccioppoli \([31, 32]\), who was aware that, in the early fifties, integral calculus was still lacking a theory of \(k\)-dimensional integration in a \(n\)-dimensional space \((k < n)\). He was aiming at a theory, of the same kind of generality of the Lebesgue theory of \(n\)-dimensional integration, relying upon simple and exhaustive notions of \(k\)-dimensional measure and integral, and culminating in an ultimate extension of integral theorems on differential forms. Until that time, there had been a
variety of efforts, but not inspired by a clear overall vision, i.e., several definitions of linear or superficial measure, and various partial extensions of the Gauss-Green formula. Such a program was carried out and completed successfully by De Giorgi \cite{19, 33, 34}, Federer and Fleming \cite{17, 18}. On denoting again by \( \varphi(x, E) \) the characteristic function of a set \( E \subset \mathbb{R}^n \) defined in Eq. (1.2), and by * the convolution product of two functions defined on \( \mathbb{R}^n \):

\[
f * g(x) \equiv \int f(x - \xi)g(\xi)d\xi,
\]

De Giorgi defined for all integer \( n \geq 2 \) and for all \( \lambda > 0 \) the function

\[
\varphi_{\lambda} : x \rightarrow \varphi_{\lambda}(x) \equiv (\pi \lambda)^{-\frac{n}{2}} \exp \left( \sum_{h=1}^{n} \frac{(x_h)^2}{\lambda} \right) * \varphi(x, E),
\]

and, as a next step, the perimeter of the set \( E \subset \mathbb{R}^n \)

\[
P(E) \equiv \lim_{\lambda \to 0} \int_{\mathbb{R}^n} \sqrt{\sum_{h=1}^{n} \left( \frac{\partial \varphi_{\lambda}}{\partial x_h} \right)^2} \, dx.
\]

The perimeter defined in Eq. (2.3) is not always finite. A necessary and sufficient condition for \( P(E) \) to be finite is the existence of a set function of vector nature completely additive and bounded, defined for any set \( B \subset \mathbb{R}^n \) and denoted by \( a(B) \), verifying the generalized Gauss-Green formula

\[
\int_E Dg \, dx = - \int_{\mathbb{R}^n} g(x) \, da.
\]

If Eq. (2.4) holds, the function \( a \) is said to be the distributional gradient of the characteristic function \( \varphi(x, E) \). A polygonal domain is every set \( E \subset \mathbb{R}^n \) that is the closure of an open set and whose topological boundary \( \partial E \) is contained in the union of a finite number of hyperplanes of \( \mathbb{R}^n \). The sets approximated by polygonal domains having finite perimeter were introduced by Caccioppoli \cite{31, 32} and coincide with the collection of all finite-perimeter sets \cite{19}. This is why finite-perimeter sets are said to be Caccioppoli sets.

The modern presentation of these concepts is even more refined. If \( u \) is a Lebesgue-summable function on an open set \( \Omega \) of \( \mathbb{R}^n \), \( u \) is said to have bounded variation in \( \Omega \) if its distributional derivative is representable by a measure in \( \Omega \), so that (cf. Eq. (2.4)) one can write \cite{22}

\[
\int_{\Omega} u \frac{\partial \phi}{\partial x^i} = - \int_{\Omega} \phi d(D_i u), \quad \forall \phi \in C^\infty_c(\Omega),
\]
where $Du = (D_1u, \ldots, D_nu)$ is a $\mathbb{R}^n$-valued measure. The variation of $u$ in $\Omega$ is a measure, denoted by $|Du|$, which, when evaluated on $\Omega$, gives:

$$|Du|(\Omega) \equiv \sup \left\{ \int_\Omega u \, \text{div} \, g \, dx : g \in [C^1_c(\Omega, \mathbb{R}^n)], \, \|g\|_{L^\infty} \leq 1 \right\},$$

where $\|g\|_{L^\infty}$ is the essential supremum norm on all components of $g$, i.e., $\|g\|_{L^\infty} = \inf \{c \geq 0 \mid |g_i(x)| \leq c \text{ for almost every } x \in \Omega, \forall i = 1, \ldots, n\}$. Note that integration by parts yields $\int_\Omega u \, \text{div} \, g = -\int_\Omega (\text{grad} \, u) \cdot g$. Hence, calculating the sup and applying Cauchy–Schwartz one obtains the simple but important relation $|Du|(\Omega) = \|\text{grad} \, u\|_{L^1(\Omega)}$.

The perimeter of a Lebesgue measurable set $E$ in $\Omega$ is the variation of the characteristic function $\varphi_E$ (hereafter we denote $\varphi(x,E)$ by $\varphi_E$ for simplicity of notation), i.e. (cf. Eq. (2.6))

$$P(\Omega, \varphi_E) = |D\varphi_E|(\Omega) = \sup \left\{ \int_\Omega \text{div} \, g \, dx : g \in [C^1_c(\Omega, \mathbb{R}^n)], \, \|g\|_{L^\infty} \leq 1 \right\},$$

while the associated reduced boundary, denoted by $\mathcal{FE}$, is the set of points of the topological boundary $\partial E$ satisfying the condition

there exists $\nu_E(x) \equiv \lim_{\rho \to 0^+} \frac{D\varphi_E(A(x,\rho))}{|D\varphi_E| (A(x,\rho))}$,\hspace{1cm} (2.8)

jointly with the unit Euclidean norm condition

$$\|\nu_E\| = 1.$$ \hspace{1cm} (2.9)

The unit norm of the generalized inner normal $\nu_E$ makes it possible to obtain the identity

$$\frac{1}{2|D\varphi_E|(A(x,\rho))} \int_{A(x,\rho)} |\nu_E(y) - \nu_E(x)|^2 \, d|D\varphi_E|(y)$$

$$= 1 - \left\langle \nu_E(x), \frac{D\varphi_E(A(x,\rho))}{|D\varphi_E| (A(x,\rho))} \right\rangle,$$ \hspace{1cm} (2.10)

By inserting the definition (2.8), and using the unit norm condition (2.9), one finds that

$$\left\langle \nu_E(x), \frac{D\varphi_E(A(x,\rho))}{|D\varphi_E| (A(x,\rho))} \right\rangle = \lim_{\rho \to 0^+} \left\langle \frac{D\varphi_E(A(x,\rho'))}{|D\varphi_E| (A(x,\rho'))}, \frac{D\varphi_E(A(x,\rho))}{|D\varphi_E| (A(x,\rho))} \right\rangle \to 1,$$ \hspace{1cm} (2.11)

as $\rho$ approaches 0, and therefore an equivalent definition of the inner normal is

$$\lim_{\rho \to 0} \frac{1}{|D\varphi_E|(A(x,\rho))} \int_{A(x,\rho)} |\nu_E(y) - \nu_E(x)|^2 \, d|D\varphi_E|(y) = 0,$$

$$\|\nu_E(x)\| = 1.$$ \hspace{1cm} (2.12)
The above relation will be crucial in the next section, because it will tell us how to define the concept of reduced boundary in Riemannian geometry. The theory of finite-perimeter sets was initiated in the works by Caccioppoli [31, 32], but it was De Giorgi who put on firm ground the brilliant ideas of Caccioppoli in an impressive series of theorems [33, 34].

In order to help the general reader, we give below a standard but very useful example of reduced boundary. For this purpose, given the finite-perimeter set

\[ E \equiv \{ (x, y) : 0 \leq x, y \leq 1 \} \cup \{ (x, 0) : -1 \leq x \leq 1 \} \subset \mathbb{R}^2, \]  

(2.13)

which represents (see Fig. 1) a square jointly with a line segment sticking out on the right, one finds that its perimeter \( P(E) = 4 \), which ignores the additional line segment. However, the topological boundary

\[ \partial E = \{ (x, 0) : -1 \leq x \leq 1 \} \cup \{ (x, 1) : 0 \leq x \leq 1 \} \]
\[ \cup \{ (x, y) : x \in \{0, 1\}, 0 \leq y \leq 1 \} \]

(2.14)

has one-dimensional Hausdorff measure equal to 5. Thus, the appropriate boundary should be a subset of the topological boundary, because the Hausdorff measure [23] of the latter overcompensates for the perimeter \( P(E) \).

III. REDUCED BOUNDARY IN RIEMANNIAN GEOMETRY

The passage to Riemannian geometry makes it necessary to take into account some additional concepts, for which we rely mainly on the work in Ref. [25]. Thus, we consider a smooth, oriented, connected, \( n \)-dimensional manifold \( M \) with tangent bundle \( TM \), endowed with a smooth \( n \)-form \( \omega \) and associated volume measure

\[ m(E) = \int_E \omega, \ E \subset M. \]  

(3.1)
If \( X : M \rightarrow TM \) is a smooth vector field and \( f \) is a function of class \( C^1 \), one has

\[
D_X f = (Xf)m. \tag{3.2}
\]

Since we are dealing with Riemannian geometry, we can exploit the existence of a Riemannian metric \( g \) to define the spaces

\[
\mathcal{D}(x) \equiv \{ v \in T_x M : g(v, v) < \infty \}, \tag{3.3}
\]

i.e., the family of all vectors belonging to the tangent space to \( M \) at \( x \) and having finite norm. If \( \Omega \) is an open set in \( M \), the space of smooth sections of \( \mathcal{D} \) is denoted by \( \Gamma(\Omega, \mathcal{D}) \) and is defined by

\[
\Gamma(\Omega, \mathcal{D}) \equiv \{ Y : Y \text{ is smooth in } \Omega, \ Y(x) \in \mathcal{D}(x), \ \forall x \in \Omega \}, \tag{3.4}
\]

where smoothness can be taken to be of class \( C^k \), up to \( k = \infty \). We shall need a subset of the space of smooth sections of \( \mathcal{D} \), defined as

\[
\Gamma^g(\Omega, \mathcal{D}) \equiv \{ Y \in \Gamma(\Omega, \mathcal{D}) : g(Y(x), Y(x)) \leq 1, \ \forall x \in \Omega \}. \tag{3.5}
\]

At this stage, we can say that if the set \( \Omega \subset M \) is open and the function \( u \) is Lebesgue summable on \((\Omega, m)\), such an \( u \) has bounded variation in \( \Omega \) if \( D_X u \) (see (3.2)) exists for all vector fields \( X \in \Gamma^g(\Omega, \mathcal{D}) \), and

\[
\sup \{ |D_X u|((\Omega) : X \in \Gamma^g(\Omega, \mathcal{D})} < \infty, \tag{3.6}
\]

where the variation of \( D_X u \) in \( \Omega \) is

\[
|D_X u|((\Omega) \equiv \sup \left\{ \int_{\Omega} u \text{ div}_\omega(fX)\omega : f \in C^\infty_c(\Omega), \ ||f||_{L^\infty} \leq 1 \right\}, \tag{3.7}
\]

where now \( ||f||_{L^\infty} \leq 1 \), for a scalar function \( f \), just means that \( |f(x)| \leq 1 \) almost everywhere. When we write \( \text{div}_\omega(fX)\omega \) we mean the divergence of the vector field \( fX \), which in Riemannian geometry can be defined as the Lie derivative with respect to \( X \) of the volume form: \( L_X(\omega) = (\text{div}_\omega X)\omega \).

If the condition (3.6) is fulfilled, one writes that the function \( u \) belongs to the functional space

\[
BV(\Omega, g, \omega).
\]
Once that an orthonormal frame \((X_1, \ldots, X_m)\) is fixed, one can define a measure \(|Dg\nu|\) which is the total variation of the vector measure \( Xu \equiv (D_{X_1}u, \ldots, D_{X_m}u) \). \(3.8\)

In other words, one has the local representation (see (3.7)) of the measure in the form

\[
|Dg\nu| = \sup \{ |DXu|(\Omega) : X \in \Gamma^g(\Omega, \mathcal{D}), \|X\|_{L^\infty} \leq 1 \},
\]

where the metric \(g\) in \(Dg\nu\) occurs in the definition because \(X \in \Gamma^g(\Omega, \mathcal{D})\).

The action of the vector field \(X\) on the characteristic function \(\varphi_E\) is expressed by the decomposition

\[
X\varphi_E = \nu^*_E |Dg\varphi_E|,
\]

where \(|Dg\varphi_E|\) is defined by Eq. (3.9) with \(u\) replaced by the characteristic function, and the dual normal to \(E\)

\[
\nu^*_E : \Omega \to \mathbb{R}^n
\]
is a Borel vector field (see Appendix A) with unit norm. Such a decomposition is said to be the polar decomposition.

The reduced boundary of \(E\), denoted by \(\mathcal{F}^*_gE\), is the set of all points in the support of \(|Dg\varphi_E|\) satisfying the conditions

\[
\lim_{r \to 0^+} \inf \frac{\min \{ m(A(x, r) \cap E), m(A(x, r) \setminus E) \}}{m(A(x, r))} > 0,
\]

\[
\lim_{r \to 0^+} \sup \frac{|Dg\varphi_E|(A(x, r))}{\frac{1}{2}m(A(x, r))} < \infty,
\]

and (cf. Eq. (2.12))

\[
\lim_{r \to 0^+} \frac{1}{|Dg\varphi_E|(A(x, r))} \int_{A(x, r)} \|\nu^*_E(y) - \nu^*_E(x)\|^2 d|Dg\varphi_E|(y) = 0,
\]

the squared norm in the integrand being now the squared Riemannian norm. Such a concept of reduced boundary is independent of the choice of orthonormal frame, unlike the dual normal \(\nu^*_E\). Equations (3.11)-(3.13) can be considered because the rectifiability properties of the reduced boundary are local, and hence can be formulated in terms of geodesic balls of small radius.

It should be stressed that the dual normal depends non-smoothly on \(x\) as \(x\) moves on the support of the measure \(|Dg\varphi_E|\). This is why Eq. (3.13) expresses a non-trivial property and deserves a careful check, that we perform in the next section. For further material on the concept of reduced boundary, we refer the reader to Appendix B.
IV. INFINITESIMAL PARALLEL TRANSPORT IN EUCLIDEAN SCHWARZSCHILD

The example at the end of Sec. II shows that, for a given finite-perimeter set, the concept of reduced boundary may be more relevant. Moreover, as far as we know, the general formulae at the end of Sec. III have been never evaluated in gravitational physics. Thus, we here consider a first example, provided by the so-called Euclidean Schwarzschild geometry, where the 4-metric is positive-definite and the $g_{00}$ component has therefore opposite sign with respect to the Lorentzian signature case of general relativity. With reference to Eqs. (3.11)-(3.13), our set $A$ will be a portion of Euclidean (or, more precisely, Riemannian) Schwarzschild geometry having non-empty intersection with the geodesic ball of radius $s$ centred at $x$ (see item (i) below Eq. (4.5)). The squared norm written in Eq. (3.13) makes it necessary to consider the scheme studied hereafter.

Upon performing parallel displacement of a vector $v$ from the point $P$, having coordinates $x$, to the point $Q$, having coordinates $x + \delta x$, the resulting variation of $v$ has components given by (we are here using coordinates since this section is devoted to explicit calculations)

$$
\delta v^\lambda = -\sum_{\mu, \rho=0}^3 \Gamma^\lambda_{\mu \rho}(x) \delta x^\mu v^\rho.
$$

Thus, in order to evaluate the infinitesimal parallel displacement, it is sufficient to evaluate the connection coefficients from the Euclidean Schwarzschild metric

$$
ds^2 = \left(1 - \frac{2M}{r}\right) dx_0^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\varphi^2\right),
$$

which can be viewed as the metric on the real Riemannian section of complexified Schwarzschild space-time \cite{35}. In turn, the squared norm in Eq. (3.13) can be obtained as we show below.

First, non-vanishing Christoffel symbols for the metric (4.2) are as follows (the only difference with respect to the case of Lorentzian signature of the metric is given by the sign
of the connection coefficient $\Gamma_{00}$):

$$
\begin{align*}
\Gamma^{0}_{01} &= -\Gamma^{1}_{11} = \frac{M}{r^2} \left( 1 - \frac{2M}{r} \right)^{-1}, \\
\Gamma^{1}_{00} &= \frac{M}{r^2} \left( \frac{2M}{r} - 1 \right), \\
\Gamma^{1}_{22} &= \frac{\Gamma_{33}^{1}}{\sin^2 \theta} = r \left( \frac{2M}{r} - 1 \right), \\
\Gamma^{2}_{12} &= \Gamma_{13}^{3} = \frac{1}{r}, \\
\Gamma^{2}_{33} &= -\cos \theta \sin \theta, \\
\Gamma^{3}_{23} &= \cot \theta. 
\end{align*}
$$

Hence we find, defining $v'^\mu = v^\mu + \delta v^\mu$,

$$
\begin{align*}
v'^0 &= v^0 - \frac{M}{r^2} \left( 1 - \frac{2M}{r} \right)^{-1} \left( \delta x^0 v^1 + \delta x^1 v^0 \right), \\
v'^1 &= v^1 + \frac{M}{r^2} \left( 1 - \frac{2M}{r} \right) \delta x^0 v^0 + \frac{M}{r^2} \left( 1 - \frac{2M}{r} \right)^{-1} \delta x^1 v^1 \\
&\quad + r \left( 1 - \frac{2M}{r} \right) \left( \delta x^2 v^2 + \sin^2 \theta \delta x^3 v^3 \right), \\
v'^2 &= v^2 - \frac{1}{r} \left( \delta x^1 v^2 + \delta x^2 v^1 \right) + \cos \theta \sin \theta \delta x^3 v^3, \\
v'^3 &= v^3 - \frac{1}{r} \left( \delta x^1 v^3 + \delta x^3 v^1 \right) - \cot \theta \left( \delta x^2 v^3 + \delta x^3 v^2 \right),
\end{align*}
$$

and therefore, when $y \equiv x + \delta x$ on the right-hand side of Eq. (3.13), we find

\textsuperscript{1}\textsuperscript{1}\textsuperscript{1}

Indeed, the values of the dual normal at different points are not always related by parallel transport, but we are not aware of obstructions in the particular case here studied.
\[ \|v_E^r(y) - v_E^r(x)\|^2 = \frac{M^2}{r^4} \left( 1 - \frac{2M}{r} \right)^{-1} \left( \delta x^0 v^1 + \delta x^1 v^0 \right)^2 \\
+ \frac{M^2}{r^4} \left( 1 - \frac{2M}{r} \right) \left[ \delta x_0 v^0 + \left( 1 - \frac{2M}{r} \right)^{-2} \delta x^1 v^1 \right]^2 \\
+ \left( 1 - \frac{2M}{r} \right) (\delta x^2 v^2 + \sin^2 \theta \delta x^3 v^3) \\
\times \left\{ \frac{r^2 (\delta x^2 v^2 + \sin^2 \theta \delta x^3 v^3)}{2M} \left[ \delta x^0 v^0 + \left( 1 - \frac{2M}{r} \right)^{-2} \delta x^1 v^1 \right] \right\} \\
+ \left( \delta x^1 v^2 + \delta x^2 v^1 - r \cos \theta \sin \theta \delta x^3 v^3 \right)^2 \\
+ \left[ \sin \theta (\delta x^1 v^3 + \delta x^3 v^1) + r \cos \theta (\delta x^2 v^3 + \delta x^3 v^2) \right]^2. \tag{4.5} \]

At this stage, our strategy for the evaluation of Eq. (3.13) in Euclidean Schwarzschild geometry is as follows.

(i) Recall from the end of Sec. III that, since rectifiability properties of the reduced boundary are local, we can formulate them in terms of geodesic balls with a little radius. The geodesic ball can be defined as follows. Let \( d(P_1, P_2) \) denote the geodesic distance between the points \( P_1, P_2 \) belonging to the spacetime manifold \( \mathcal{M} \) and defined as the arc length of the shortest geodesic curve from \( P_1 \) to \( P_2 \). The geodesic sphere \( S_{P_1}(\rho) \) centred at \( P_1 \) with radius \( \rho \) is defined by

\[ S_{P_1}(\rho) \equiv \{ P_2 \in \mathcal{M} : d(P_1, P_2) = \rho \}. \tag{4.6} \]

The body of \( S_{P_1}(\rho) \) is called geodesic ball. It is denoted by \( B_{P_1}(\rho) \) and can be defined as

\[ B_{P_1}(\rho) \equiv \{ P_2 \in \mathcal{M} : 0 \leq d(P_1, P_2) \leq \rho \}. \tag{4.7} \]

(ii) The author of Ref. [37] has exploited Riemann normal coordinates (see Refs. [38, 39]) to evaluate the \( n \)-dimensional volume of a small geodesic ball of a Riemannian manifold.

\(^{2}\text{Computation appearing in Ref. [39] are performed using an open access program called “CADABRA”. For an introduction to “CADABRA” see Ref. [41].}\)
Therefore, the denominator $|D_y \varphi_E|(A(x, s))$ is obtained from the theorem of Gray according to which, if $P$ is a point of an analytic and Ricci-flat $n$-dimensional Riemannian manifold, the volume $V_P(s)$ of the geodesic ball $B_P(s)$ centred at $P$ with radius $s$ is given by (hereafter, in this section, for simplicity of notation, repeated indices imply also summation from 0 to 3 for Greek indices) \[37, 42\]

$$V_P(s) = \alpha_n s^n \left\{ 1 - \frac{R^{\mu\nu\rho\sigma}(P) R_{\mu\nu\rho\sigma}(P) s^4}{120(n + 2)(n + 4)} + O(s^6) \right\}, \quad (4.8)$$

where the parameter $\alpha_n$ is the volume of the unit $n$-ball in $\mathbb{R}^n$, and is obtained from the $\Gamma$-function according to

$$\alpha_n \equiv \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2} + 1\right)}, \quad (4.9)$$

while the “Riemann square” term is\(^3\)

$$R^{\mu\nu\rho\sigma}(P) R_{\mu\nu\rho\sigma}(P) = \frac{48 M^2}{r^6}. \quad (4.10)$$

(iii) The squared norm in Eq. (3.13) is given by Eq. (4.5), and $\nu^*_E(x)$ in Eq. (3.13) has components here denoted by $v^\mu$.

(iv) The integration measure in Eq. (3.13) should be such that integration over $A(x, s)$ becomes

$$\int_I \sqrt{\det g(y)} \, \delta_D \left[ (y - x)^\lambda g_{\lambda\mu} v^\mu \right] \| \nu^*_E(y) - \nu^*_E(x) \|^2 d^4 y, \quad (4.11)$$

where $g_{\lambda\mu}$ are the components of the metric (4.2) and $v$ has unit norm with respect to such a metric: $g(v, v) = 1$, while $I$ is the geodesic ball centred at $P$ and having radius $s$ equal to the geodesic distance between the points $P$ and $Q$. The argument of the Dirac delta is

$$(y - x)^\lambda g_{\lambda\mu} v^\mu = g_{\mu\nu} \delta x^\mu v^\nu = \left( 1 - \frac{2M}{r} \right) v^0 \delta x^0 + \left( 1 - \frac{2M}{r} \right)^{-1} v^1 \delta x^1 + r^2 \left( v^2 \delta x^2 + \sin^2 \theta v^3 \delta x^3 \right), \quad (4.12)$$

\(^3\)The Riemann tensor components occurring in Eqs. (4.8) and (4.10) should be expressed in Riemann normal coordinates. The simplest way to obtain these formulae is by projection onto a local orthonormal tetrad. However, the “Riemann square” does not depend on the coordinates adopted and hence we write the factor $R^{\mu\nu\rho\sigma}(P) R_{\mu\nu\rho\sigma}(P)$ using Schwarzschild coordinates.
and the volume element can be written in the form
\[ \sqrt{\det g(y)} = (r + \delta x^1)^2 \sin (\theta + \delta x^2) . \] (4.13)

(v) In order to calculate (4.11), we will exploit Riemann normal coordinates. In Refs. [38–40] it is shown that the transformation rule between a generic set of coordinates (in our case, Schwarzschild coordinates) and Riemann normal coordinates \( z^\mu \), constructed locally around the point \( O \) (which represents the origin of Riemann normal coordinates), is given by
\[
x^\mu = x^\mu_O + \frac{1}{2} \Gamma^\mu_{\alpha\beta}(x_O)z^\alpha z^\beta \\
+ \frac{1}{6} \left[ 2\Gamma^\mu_{\alpha\tau}(x_O)\Gamma^\tau_{\beta\rho}(x_O) - \partial_\rho \Gamma^\mu_{\alpha\beta}(x_O) \right] z^\alpha z^\beta z^\rho + \mathcal{O}(\epsilon^3), \] (4.14)
where \( x^\mu_O \) denotes the Schwarzschild coordinates of the origin \( O \) of Riemann normal coordinates (note that we cannot employ Riemann normal coordinates at singular points \( r = 0 \) and \( r = 2M \)), the expressions for \( \Gamma^\mu_{\alpha\beta}(x_O) \) can be read from Eq. (4.3) and \( \epsilon \) denotes the typical length scale of the patch containing the origin \( O \) of Riemann normal coordinates. In Eq. (4.14) we have adopted the following notation:
\[
z^\alpha = e^\alpha_a z^a, \] (4.15)
where \( e^\alpha_a \) and \( e_\alpha^a \) are the components of an orthonormal tetrad at the point \( O \) and are defined by
\[
e^\alpha_a = \frac{\partial z^a}{\partial x^\alpha}(x_O), \quad e^\alpha_\alpha e^\alpha_\beta = \delta^\alpha_\beta, \quad e^\alpha_a e^b_\alpha = \delta^b_a. \] (4.16)
Furthermore, it should be stressed that Eq. (4.14) is valid if the Schwarzschild coordinates axes are aligned to those of Riemann normal coordinates (the most general formula containing also higher-order corrections can be found in Refs. [38, 39]).

(vi) As pointed out before, the geodesic ball centred at \( P \) will have radius \( s \) given by the geodesic distance between \( P \) and \( Q \). The great advantage of Riemann normal coordinates is that there exists, by assumption, a unique geodesic between \( P \) and \( Q \). Therefore, we can readily obtain the expression of the geodesic distance between this pair of points in terms

---

4 According to our notation, \( e^\alpha_\mu dx^\mu \) is a tetrad one-form, whereas \( e^\alpha_\mu \frac{\partial}{\partial x^\mu} \) is a tetrad vector field.
of their Riemann normal coordinates and the metric. This formula can be found in Refs. \cite{38, 39} and reads as

\[
[d(P, Q)]^2 = \delta_{\mu\nu} (\delta z^\mu) (\delta z^\nu) - \frac{1}{3} (^{(z)} R_{\mu\nu,\beta} (z^\alpha) (z^\beta) (\delta z^\mu) (\delta z^\nu) + O(\epsilon^3),
\]

(4.17)

where \( \delta_{\mu\nu} = \text{diag}(1, 1, 1, 1) \) represents the Euclidean metric and

\[
\delta z^\mu \equiv \tilde{z}^\mu - z^\mu,
\]

(4.18)

\( z^\mu, \tilde{z}^\mu \) denoting the Riemann normal coordinates of \( P \) and \( Q \), respectively. Moreover, \( ^{(z)} R_{\mu\nu,\beta} \) denotes the Riemann tensor components expressed in Riemann normal coordinates and evaluated at the origin \( O \) of Riemann normal coordinates (see footnote 3).

### A. Euclidean calculation

In this sub-section we perform a first simple calculation of the integral (4.11) by exploiting Eqs. (4.14) and (4.17) at the lowest order. Without loss of generality, we can consider now the special case

\[
v^\mu = \delta^\mu_0 \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}}.
\]

(4.19)

In this way, Eqs. (4.15) and (4.12) simplify somewhat and the integral (4.11) can be written as

\[
\int d^4y \sqrt{\text{det} g(y)} \, \delta_D \left[(y - x)^\lambda g_{\lambda\mu} v^\mu\right] \left[\nu_E^*(y) - \nu_E^*(x)\right]^2
\]

\[
= \int d^4(\delta x) \left\{ (r + \delta x^1)^2 \sin (\theta + \delta x^2) \, \delta_D \left[(1 - \frac{2M}{r}) v^0 \delta x^0\right]\right\}
\]

\[
\times \frac{M^2}{r^4} \left(1 - \frac{2M}{r}\right) (v^0)^2 \left[(1 - \frac{2M}{r})^{-2} (\delta x^1)^2 + (\delta x^0)^2\right]
\]

\[
= \frac{M^2}{r^4} (v^0) \int d^4(\delta x) \left\{ (r + \delta x^1)^2 \sin (\theta + \delta x^2) \, \delta_D (\delta x^0)\right\}
\]

\[
\times \left[(1 - \frac{2M}{r})^{-2} (\delta x^1)^2 + (\delta x^0)^2\right]
\]

\[
= \mathcal{G}(r) \int d^4(\delta x) \, \delta_D (\delta x^0) (r + \delta x^1)^2 \sin (\theta + \delta x^2) (\delta x^1)^2,
\]

(4.20)

where in the passage from the second to the third equality we have exploited the well-known relation

\[
\delta_D(ax) = \frac{1}{|a|} \delta_D(x),
\]

(4.21)
whereas in the last equality we have used that
\[ \int dx \delta_D(x)x^2 = 0, \quad (4.22) \]
and we have defined
\[ G(r) \equiv \frac{M^2(v^0)}{r^4 \left( 1 - \frac{2M}{r} \right)^2}. \quad (4.23) \]

If we perform a Taylor expansion of the term \( \sin (\theta + \delta x^2) \) for \( |\delta x^2| \ll 1 \), we obtain from \( (4.20) \)
\[ G(r) \int \int d^4 (\delta x) \delta_D (\delta x^0) \left\{ r^2 \sin \theta (\delta x^1)^2 + O[(\delta x^\mu)^3] \right\}. \quad (4.24) \]

At this stage, we transform the integration over coordinates \( \delta x^\mu \) into an integration over Riemann normal coordinates by exploiting \( (4.14) \) at the lowest order. Bearing in mind that Schwarzschild coordinates of the pair of points \( P \) and \( Q \) are \( x^\mu \) and \( y^\mu \equiv x^\mu + \delta x^\mu \), respectively, we obtain from Eq. \( (4.14) \)
\[ x^\mu = x^\mu_O + z^\mu + O(\epsilon) = x^\mu_O + z^\mu + O[(z^\mu)^2], \]
\[ y^\mu = x^\mu_O + \tilde{z}^\mu + O(\epsilon) = x^\mu_O + \tilde{z}^\mu + O[(\tilde{z}^\mu)^2], \quad (4.25) \]
where, as in Eq. \( (4.18) \), \( z^\mu \) and \( \tilde{z}^\mu \) are the Riemann normal coordinates of \( P \) and \( Q \), respectively. From the above equations, we simply get
\[ \delta x^\mu \equiv y^\mu - x^\mu = \delta z^\mu + O[(z^\mu)^2], \quad (\delta x^\mu)^2 = (\delta z^\mu)^2 + O[(z^\mu)^3], \quad (4.26) \]
where we have exploited the definition \( (4.18) \).

The Jacobian of the transformation is
\[ J = \det \left[ \frac{\partial (\delta x^\mu)}{\partial (\delta z^\lambda)} \right] = 1 + O(z^\mu). \quad (4.27) \]

Therefore, Eq. \( (4.24) \) becomes
\[ G(r) \int d^4 (\delta x) \delta_D (\delta x^0) \left\{ r^2 \sin \theta (\delta x^1)^2 + O[(\delta x^\mu)^3] \right\} \]
\[ = G(r) \int d^4 (\delta z) \delta_D \left\{ \delta z^0 + O[(z^\mu)^2] \right\} \left\{ r^2 \sin \theta (\delta z^1)^2 + O[(z^\mu)^3] \right\}, \quad (4.28) \]
where \( J \) is the portion of geodesic ball centred at \( P \) and having (squared) radius given by \( (4.17) \), which, to lowest order, gives
\[ s^2 = \delta_{\mu\nu} (\delta z^\mu)(\delta z^\nu) + O[(z^\mu)^4]. \quad (4.29) \]
If we enforce the Dirac delta term \( \delta_D \{ \delta z^0 + \mathcal{O} [(z^\mu)^2] \} \) appearing in (4.28), then Eq. (4.29) simply gives
\[
s^2 = (\delta z^1)^2 + (\delta z^2)^2 + (\delta z^3)^2 + \mathcal{O} [(z^\mu)^3],
\] (4.30)
i.e., the equation of the 2-sphere embedded in \( \mathbb{R}^3 \) (note the presence on the right-hand side of the above equation of the term \( \mathcal{O} [(z^\mu)^3] \)). In light of the above considerations, Eq. (4.28) becomes
\[
\mathcal{G}(r) \int_{J} d^4(\delta z) \delta D \{ \delta z^0 + \mathcal{O} [(z^\mu)^2] \} \left\{ r^2 \sin \theta (\delta z^1)^2 + \mathcal{O} [(z^\mu)^3] \right\}
= \mathcal{G}(r) \int_{B} d^3(\delta z) \left\{ r^2 \sin \theta (\delta z^1)^2 + \mathcal{O} [(z^\mu)^3] \right\},
\] (4.31)
where \( B \) is the geodesic 2-ball associated to the geodesic 2-sphere defined by (4.30).

The integral (4.31) can be evaluated by employing spherical coordinates
\[
\delta z^1 = R \sin \xi \cos \psi,
\delta z^2 = R \sin \xi \sin \psi,
\delta z^3 = R \cos \xi,
\]
\( R \in (0, s), \, \xi \in (0, \pi), \, \psi \in (0, 2\pi) \).
\] (4.32)

It is a simple task to show that (see Appendix C)
\[
\int_{B} d^3(\delta z) (\delta z^1)^2
= \int_{0}^{\xi_{0}} dR R^4 \int_{0}^{\psi_{0}} R_{0} \sin^3 \xi \int_{0}^{\psi_{0}} d\psi \cos^2 \psi
= \frac{s^5}{30} \left[ 2 - \cos \xi_{0} (2 + \sin^2 \xi_{0}) \right] (\psi_{0} + \cos \psi_{0} \sin \psi_{0}),
\] (4.33)
where \( \xi_{0} \in (0, \pi) \) and \( \psi_{0} \in (0, 2\pi) \).

Therefore, in conclusion we find that
\[
\int_{I} d^4y \sqrt{\text{det}g(y)} \delta_D \{(y - x)^\lambda g_{\lambda\mu} \nu_\mu \} \left[ \nu^E(y) - \nu^E(x) \right]^2
= \mathcal{G}(r) \int_{J} d^4(\delta z) \delta D \{ \delta z^0 + \mathcal{O} [(z^\mu)^2] \} \left\{ r^2 \sin \theta (\delta z^1)^2 + \mathcal{O} [(z^\mu)^3] \right\}
= \mathcal{G}(r) \int_{B} d^3(\delta z) \left\{ r^2 \sin \theta (\delta z^1)^2 + \mathcal{O} [(z^\mu)^3] \right\}
= \mathcal{F}(r, \theta) \frac{s^5}{30} \left[ 2 - \cos \xi_{0} (2 + \sin^2 \xi_{0}) \right] (\psi_{0} + \cos \psi_{0} \sin \psi_{0})
+ \mathcal{G}(r) \int_{J} d^4(\delta z) \delta D \{ \delta z^0 + \mathcal{O} [(z^\mu)^2] \} \left\{ \mathcal{O} [(z^\mu)^3] \right\},
\] (4.34)
where

\[ \mathcal{F}(r, \theta) \equiv \mathcal{G}(r) r^2 \sin \theta = \frac{M^2 (v^0) \sin \theta}{r^2 \left( 1 - \frac{2M}{r} \right)^2}. \quad (4.35) \]

The most important aspect of this calculation consists in proving that no terms proportional to \( s^4 \) occur in (4.34), the lowest-order terms being of order \( s^5 \). Indeed, if terms proportional to \( s^4 \) had been present in (4.34) (or, equivalently, in (4.11)), then upon dividing (4.34) by (4.8) (with \( n = 4 \)) we would have obtained an expression which does not vanish in the limit \( s \to 0 \), in contrast with the requirement of Eq. (3.13). However, it becomes crucial to assess the contribution given by the integral on the last line of (4.34), in order to establish if such a term is either of order \( s^5 \) or beyond. For this reason, in the following sub-section we will evaluate the integral (4.11) by employing Eqs. (4.14) and (4.17) at next-to-leading order.

**B. Riemannian calculation**

In this sub-section we carry out a next-to-leading-order calculation of the integral (4.11).

If we consider Eqs. (4.19) and (4.23), and perform a Taylor expansion of the term \( \sin (\theta + \delta x^2) \) for \( |\delta x^2| \ll 1 \), higher-order contributions to the integral (4.11) lead to the expression

\[
\int_I \sqrt{\text{det} g(y)} \, \delta_D \left[ (y - x)^\lambda g_\lambda \nu \mu \right] [\nu_E^\nu(y) - \nu_E^\nu(x)]^2 d^4 y
\]

\[
= \mathcal{G}(r) \int_I d^4 (\delta x) \delta_D (\delta x^0) \left\{ r^2 \sin \theta (\delta x^1)^2 + r^2 \cos \theta (\delta x^1)^2 (\delta x^2) + 2r \sin \theta (\delta x^1)^3 + \mathcal{O} [(\delta x^\mu)^4] \right\}. \quad (4.36)
\]

Furthermore, if we consider the next-to-leading-order correction occurring in Eq. (4.14) and bear in mind the definition (4.18), we obtain

\[
\delta x^\mu = \delta z^\mu - \frac{1}{2} \Gamma_{\alpha\beta}(x_O) \delta z^\alpha \left( \delta x^\beta + 2\delta z^\beta \right) + \mathcal{O} (\epsilon^2)
\]

\[
= \delta z^\mu - \frac{1}{2} \Gamma_{\alpha\beta}(x_O) \delta z^\alpha \left( \delta x^\beta + \delta z^\beta \right) + \mathcal{O} [(z^\mu)^3]. \quad (4.37)
\]

At this stage, we recall that the origin \( O \) of Riemann normal coordinates can be represented by any point of the spacetime manifold, provided that it does not coincide with a singular point. Therefore, our calculations can be greatly simplified if the origin \( O \) of Riemann
normal coordinates corresponds to the point $P$ (whose Schwarzschild coordinates are given by $x^\mu$). This means that we can set in all subsequent computations $z^\mu = 0$ and hence from (4.14) and (4.18) we get, respectively,

$$x^\mu = x^\mu_O, \quad (4.38a)$$

$$\delta z^\mu = \tilde{z}^\mu. \quad (4.38b)$$

By virtue of (4.38), Eq. (4.37) simply gives

$$\delta x^\mu = \tilde{z}^\mu - \frac{1}{2} \Gamma^\mu_{\alpha\beta}(x)\tilde{z}^\alpha \tilde{z}^\beta + \mathcal{O} \left[(z^\mu)^3\right], \quad (4.39)$$

which in turn leads to (cf. Eq. (4.3))

$$\delta x^0 = \tilde{z}^0 \left[1 + \frac{M}{r(2M - r)} \tilde{z}^1\right] + \mathcal{O} \left[(z^\mu)^3\right],$$

$$\delta x^1 = \tilde{z}^1 - \frac{1}{2} \left(2M - r\right) \left[\left(\tilde{z}^2\right)^2 + \frac{M}{r^3} \left(\tilde{z}^0\right)^2 \right] + \mathcal{O} \left[(z^\mu)^3\right],$$

$$\delta x^2 = \tilde{z}^2 - \frac{1}{2} \left[\frac{2}{r} \tilde{z}^1 \tilde{z}^2 - \sin \theta \cos \theta \left(\tilde{z}^3\right)^2\right] + \mathcal{O} \left[(z^\mu)^3\right],$$

$$\delta x^3 = \tilde{z}^3 - \left[\frac{1}{r} \tilde{z}^1 \tilde{z}^3 + (\cot \theta) \tilde{z}^2 \tilde{z}^3\right] + \mathcal{O} \left[(z^\mu)^3\right], \quad (4.40)$$

and to the following expression for the Jacobian

$$J = \det \left[\frac{\partial (\delta x^\mu)}{\partial \tilde{z}^\lambda}\right] = 1 - \frac{2}{r} \tilde{z}^1 - (\cot \theta) \tilde{z}^2 + \mathcal{O} \left[(z^\mu)^2\right]. \quad (4.41)$$

In order to compute Eq. (4.36), we need to evaluate the following three integrals:

$$\int_I d^4(\delta x) \delta_D (\delta x^0) \left\{ (\delta x^1)^2 \right\}, \quad (4.42a)$$

$$\int_I d^4(\delta x) \delta_D (\delta x^0) \left\{ (\delta x^1)^2 (\delta x^2) \right\}, \quad (4.42b)$$

$$\int_I d^4(\delta x) \delta_D (\delta x^0) \left\{ (\delta x^1)^3 \right\}. \quad (4.42c)$$

Each integral will be analyzed in the next sub-sections. In order to help the reader, we have summarized in Appendix C the set of integrals occurring in this calculation.
In this sub-section we will show in detail how to compute Eq. (4.42a). We thus have

\[
\int d^4(\delta x) \delta_D (\delta x^0) \left\{ (\delta x^1)^2 \right\} = \int d^4 \tilde{z} J \delta_D \left\{ \tilde{z}^0 \left[ 1 + \frac{M \tilde{z}^1}{r(2M - r)} \right] + \mathcal{O} \left[ (z^\mu)^3 \right] \right\} (\delta x^1(\tilde{z}))^2, \tag{4.43}
\]

where the Jacobian \( J \) is given by (4.41) and from (4.40)

\[
[\delta x^1(\tilde{z})]^2 \equiv (\tilde{z}^1)^2 - (2M - r) (\tilde{z}^1) \left( \tilde{z}^2 \right)^2 + \frac{M}{r^3} (\tilde{z}^0)^2 + \frac{M}{r(2M - r)^2} (\tilde{z}^1)^2 + \sin^2 \theta (\tilde{z}^3)^2 + \mathcal{O} \left[ (z^\mu)^4 \right], \tag{4.44}
\]

and \( J \) is the portion of geodesic ball having center at \( P \equiv O \) and (squared) radius given by (4.17), which in this case can be written as

\[
s^2 = \delta_{\mu\nu} \tilde{z}^\mu \tilde{z}^\nu + \mathcal{O} \left[ (z^\mu)^5 \right], \tag{4.45}
\]

where we have used that \( z^\mu = 0 \) and Eq. (4.38b). In Eq. (4.43), if we first perform the integration with respect to the \( \tilde{z}^0 \) variable, we can exploit (4.21) and hence we have

\[
\int d^3 \tilde{z} \int d\tilde{z}^0 \delta_D \{ \tilde{z}^0 + \mathcal{O} \left[ (z^\mu)^3 \right] \} \left\{ 1 - \frac{M \tilde{z}^1}{r(2M - r)} + \mathcal{O} \left[ (z^\mu)^2 \right] \right\} \times \left\{ 1 - \frac{2}{r} \tilde{z}^1 - (\cot \theta) \tilde{z}^2 + \mathcal{O} \left[ (z^\mu)^2 \right] \right\} \left[ \delta x^1(\tilde{z}) \right]^2
\]

\[
= \int d^3 \tilde{z} \left\{ (\tilde{z}^1)^2 - (2M - r) (\tilde{z}^1) (\tilde{z}^2)^2 - \frac{2(3M - r)}{r(2M - r)} (\tilde{z}^1)^3ight.
\]

\[
- \sin^2 \theta (2M - r) (\tilde{z}^1)(\tilde{z}^3)^2 - \cot \theta (\tilde{z}^1)^2 (\tilde{z}^2) + \mathcal{O} \left[ (z^\mu)^4 \right], \tag{4.46}
\]

where we have performed the integration over the Dirac-delta function and have employed Eq. (4.22). The integration region \( B \) has equation

\[
s^2 = (\tilde{z}^1)^2 + (\tilde{z}^2)^2 + (\tilde{z}^3)^2 + \mathcal{O} \left[ (z^\mu)^4 \right], \tag{4.47}
\]

which follows from the joint application of Eq. (4.45) and the Dirac-delta function \( \delta_D \{ \tilde{z}^0 + \mathcal{O} \left[ (z^\mu)^3 \right] \} \) occurring in (4.46) (note the presence of \( \mathcal{O} \left[ (z^\mu)^4 \right] \) in Eq. (4.47)).
Therefore, we can compute (4.46) by exploiting spherical coordinates (cf. Eq. (4.32))

\[
\begin{align*}
\tilde{z}^1 &= R \sin \xi \cos \psi, \\
\tilde{z}^2 &= R \sin \xi \sin \psi, \\
\tilde{z}^3 &= R \cos \xi, \\
R &\in (0, s), \quad \xi \in (0, \pi), \quad \psi \in (0, 2\pi).
\end{align*}
\]

(4.48)

Bearing in mind the trigonometrical integrals listed in Appendix C we obtain from Eq. (4.46), after an easy but lengthy calculation, the following result:

\[
\begin{align*}
\int_{\mathcal{I}} d^4(\delta x) \delta D(\delta x^0) \left\{ (\delta x^1)^2 \right\} \\
= I_1 - (2M - r) I_2 - \frac{2 (3M - r)}{r (2M - r)} I_3 \\
- \sin^2 \theta (2M - r) I_4 - \cot \theta I_5 \\
+ \int_{\mathcal{J}} d^4\tilde{z} \delta D \left\{ \tilde{z}^0 + \mathcal{O} \left[ (\tilde{z}^\mu)^3 \right] \right\} \left\{ \mathcal{O} \left[ (\tilde{z}^\mu)^4 \right] \right\},
\end{align*}
\]

(4.49)

where\(^5\)

\[
\begin{align*}
I_1 &\equiv \int_{\mathcal{B}} d^3 \tilde{z} (\tilde{z}^1)^2 \\
&= \frac{s^5}{30} \left[ 2 - \cos \xi_0 \left( 2 + \sin^2 \xi_0 \right) \right] (\psi_0 + \cos \psi_0 \sin \psi_0), \\
I_2 &\equiv \int_{\mathcal{B}} d^3 \tilde{z} (\tilde{z}^1) (\tilde{z}^2)^2 \\
&= \frac{s^6}{576} \left[ 12\xi_0 - 8 \sin (2\xi_0) + \sin (4\xi_0) \right] \sin^3 \psi_0, \\
I_3 &\equiv \int_{\mathcal{B}} d^3 \tilde{z} (\tilde{z}^1)^3 \\
&= \frac{s^6}{192} \left[ 12\xi_0 - 8 \sin (2\xi_0) + \sin (4\xi_0) \right] \left( 1 - \frac{\sin^2 \psi_0}{3} \right) \sin \psi_0, \\
I_4 &\equiv \int_{\mathcal{B}} d^3 \tilde{z} (\tilde{z}^1) (\tilde{z}^3)^2 \\
&= \frac{s^6}{192} \left[ 4\xi_0 - \sin (4\xi_0) \right] \sin \psi_0,
\end{align*}
\]

\(^5\)Note that Eq. (4.50) has already been exploited in Eq. (4.33).
\[ I_5 \equiv \int_B d^3 \tilde{z} (\tilde{z}^1)^2 (\tilde{z}^2) \]
\[ = \frac{s^6}{576} [12 \xi_0 - 8 \sin (2 \xi_0) + \sin (4 \xi_0)] (1 - \cos^3 \psi_0), \quad (4.54) \]

with \( \xi_0 \in (0, \pi) \) and \( \psi_0 \in (0, 2\pi) \).

2. Calculation of the integrals \((4.42b)\) and \((4.42c)\)

The evaluation of \((4.42b)\) can be performed as follows:

\[
\int_I d^4 (\delta x) \delta_D (\delta x^0) \left\{ (\delta x^1)^2 (\delta x^2) \right\} \\
= \int_J d^4 \tilde{z} J \delta_D \left\{ \tilde{z}^0 \left[ 1 + \frac{M}{r (2M - r)} \right] + \mathcal{O} [(z^\mu)^3] \right\} \times \left\{ (\tilde{z}^1)^2 (\tilde{z}^2) + \mathcal{O} [(z^\mu)^4] \right\}, \quad (4.55) \]

where the integration region \( J \) is defined by Eq. \((4.45)\), the Jacobian \( J \) can be read from \((4.41)\) and we have exploited Eqs. \((4.40)\) and \((4.44)\).

Along the same lines of the previous sub-section, we first deal with the integration with respect to the \( \tilde{z}^0 \) variable and apply Eq. \((4.21)\). After that, the resulting integration over the region \( B \), which is defined in Eq. \((4.47)\), can be carried out by means of spherical coordinates \((4.48)\). In this way, we end up with

\[
\int_I d^4 (\delta x) \delta_D (\delta x^0) \left\{ (\delta x^1)^2 (\delta x^2) \right\} \\
= I_5 + \int_J d^4 \tilde{z} J \delta_D \left\{ \tilde{z}^0 + \mathcal{O} [(z^\mu)^3] \right\} \left\{ \mathcal{O} [(z^\mu)^4] \right\}, \quad (4.56) \]

where \( I_5 \) is displayed in Eq. \((4.54)\).

By exploiting the same scheme, the evaluation process of \((4.42c)\) leads to

\[
\int_I d^4 (\delta x) \delta_D (\delta x^0) \left\{ (\delta x^1)^3 \right\} \\
= I_3 + \int_J d^4 \tilde{z} J \delta_D \left\{ \tilde{z}^0 + \mathcal{O} [(z^\mu)^3] \right\} \left\{ \mathcal{O} [(z^\mu)^4] \right\}, \quad (4.57) \]

\( I_3 \) being given by \((4.52)\).
3. Final result and higher-order calculation

Bearing in mind Eqs. (4.49), (4.56), and (4.57), the final form of (4.36) reads as

\[
\int \sqrt{\det g(y)} \delta_D [(y - x)^\lambda g_{\lambda\mu} v^\mu] \left[ \nu_E^+(y) - \nu_E^-(x) \right]^2 d^4y
\]

\[
= \mathcal{F}(r, \theta) \left[ I_1 - (2M - r) I_2 - \frac{2(3M - r)}{r(2M - r)} I_3 \right.
\]

\[\left. - \sin^2 \theta (2M - r) I_4 - \cot \theta I_5 \right] + \mathcal{H}(r, \theta) I_5 + \mathcal{R}(r, \theta) I_3
\]

\[+ \mathcal{G}(r) \int d^4\tilde{z} \delta_D \left\{ \tilde{z}^0 + \mathcal{O}(\tilde{z}^3) \right\} \left\{ \mathcal{O}(\tilde{z}^4) \right\}, \quad (4.58)\]

where (see Eq. (4.23))

\[
\mathcal{H}(r, \theta) \equiv \mathcal{G}(r) \left( r^2 \cos \theta \right) = \frac{M^2 (v^0) \cos \theta}{r^2 \left( 1 - \frac{2M}{r} \right)^2},
\]

\[
\mathcal{R}(r, \theta) \equiv \mathcal{G}(r) \left( 2r \sin \theta \right) = \frac{2M^2 (v^0) \sin \theta}{r^3 \left( 1 - \frac{2M}{r} \right)^2}. \quad (4.59)
\]

Since \( I_1 \) is proportional to \( s^5 \) and \( \{ I_2, \cdots, I_5 \} \) to \( s^6 \) (see Eqs. (4.50)–(4.54)), then the division of (4.58) by (4.8) (with \( n = 4 \)) leads to an expression which vanishes in the limit \( s \to 0 \), as required by Eq. (3.13).

Higher-order contributions to the integral (4.11) can be calculated by considering the corrections occurring in (4.14) beyond the next-to-leading order. The crucial aspect of this calculation is that if we consider the point \( P \), having Schwarzschild coordinates \( x^\mu \), as the origin \( O \) of Riemann normal coordinates, then the (squared) radius of the geodesic ball centered at \( P \) will be given by

\[
s^2 = \delta_{\mu\nu} \tilde{z}^\mu \tilde{z}^\nu, \quad (4.60)
\]

which amounts to an exact relation provided we know the exact form of the transformation between Schwarzschild coordinates and Riemann normal coordinates. Therefore, by considering higher terms of (4.14), Eq. (4.60) can be constructed order by order (in a way similar to our Eq. (4.45)) and hence the integral (4.11) can be evaluated by using the scheme set out in the previous (sub-)sections. In particular, the next-to-next-to-leading-order calculation can be worked out in the following way. First of all, upon considering (4.19), higher-order
contributions to the integral (4.11) lead to the expression

\[
\int_I \sqrt{\det g(y)} \frac{\delta_D}{\delta x^0} \left\{ r^2 \sin \theta \, (\delta x^1)^2 + r^2 \cos \theta \, (\delta x^1)^2 \, (\delta x^2)^3 \right\} \frac{1}{2^{1/2}} \delta_D \left[ (y-x) \right] \frac{1}{2} d^4 y = G(r)
\]

Furthermore, if we exploit in full Eq. (4.14) and bear in mind the definition (4.18), we obtain

\[
\delta x^\mu = \delta z^\mu - \frac{1}{2} \Gamma^\mu_{\alpha\beta} (x_O) \delta z^\alpha (\delta z^\beta + 2 z^\beta)
\]

\[
+ \frac{1}{3} \Gamma^\mu_{\alpha\tau} (x_O) \Gamma_{\beta\mu} (x_O) \left[ (z^\alpha + \delta z^\alpha) \left( z^\beta + \delta z^\beta \right) \delta z^\rho + \delta z^\alpha z^\beta \delta z^\rho \right]
\]

\[
- \frac{1}{6} \partial_\mu \Gamma^\mu_{\alpha\beta} (x_O) \left[ (2 z^\alpha + \delta z^\alpha) \delta z^\beta \delta z^\rho + z^\alpha z^\beta \delta z^\rho \right]
\]

\[
+ \mathcal{O} \left( (\delta x^\mu)^5 \right),
\]

(4.61)

If we rename dummy indices in the last expression, we obtain

\[
\delta x^\mu = \delta z^\mu - \frac{1}{2} \Gamma^\mu_{\alpha\beta} (x_O) \delta z^\alpha \left( z^\beta + \bar{z}^\beta \right)
\]

\[
+ \frac{1}{3} \Gamma^\mu_{\alpha\tau} (x_O) \Gamma_{\beta\mu} (x_O) \left[ \delta z^\alpha \left( z^\beta + \bar{z}^\beta \right) \delta z^\rho + \delta z^\beta \delta z^\rho \right]
\]

\[
- \frac{1}{6} \partial_\mu \Gamma^\mu_{\alpha\beta} (x_O) \left[ (z^\alpha + \delta z^\alpha) \delta z^\beta \delta z^\rho + z^\alpha z^\beta \delta z^\rho \right]
\]

\[
+ \mathcal{O} \left( (\delta x^\mu)^4 \right),
\]

(4.62)

which upon setting \( P \equiv O \) becomes (see Eq. (4.38))

\[
\delta x^\mu = \bar{z}^\mu - \frac{1}{2} \Gamma^\mu_{\alpha\beta} (x) \bar{z}^\alpha \bar{z}^\beta
\]

\[
+ \frac{1}{6} \left[ 2 \Gamma^\mu_{\rho\tau} (x) \Gamma_{\beta\mu} (x) - \partial_\rho \Gamma^\mu_{\beta\alpha} (x) \right] \bar{z}^\alpha \bar{z}^\beta \bar{z}^\rho
\]

\[
+ \mathcal{O} \left( (\delta x^\mu)^4 \right),
\]

(4.63)
which in turn leads to
\[
(\delta x^\mu)^2 = (\bar{z}^\mu)^2 - \Gamma^\mu_{\alpha\beta}(x) \bar{z}^\mu \bar{z}^\alpha \bar{z}^\beta + \frac{1}{4} \left[ \Gamma^\mu_{\alpha\beta}(x) \right]^2 (\bar{z}^\alpha)^2 (\bar{z}^\beta)^2 \\
+ \frac{1}{3} [2\Gamma^\mu_{\rho\tau}(x) \Gamma^\tau_{\beta\alpha}(x) - \partial_\rho \Gamma^\mu_{\beta\alpha}(x)] \bar{z}^\mu \bar{z}^\alpha \bar{z}^\beta \bar{z}^\rho \\
+ \mathcal{O} [(z^\mu)^5], \quad \text{(no sum over } \mu),
\]
(4.65)

\[
(\delta x^\mu)^3 = (\bar{z}^\mu)^3 - \frac{3}{2} \Gamma^\mu_{\alpha\beta}(x) (\bar{z}^\mu)^2 \bar{z}^\alpha \bar{z}^\beta \\
+ \mathcal{O} [(z^\mu)^5], \quad \text{(no sum over } \mu),
\]
(4.66)

\[
(\delta x^\mu)^4 = (\bar{z}^\mu)^4 + \mathcal{O} [(z^\mu)^5].
\]
(4.67)

The above equations can be invoked to perform the next-to-next-to-leading-order calculation of Eq. (4.11).

In conclusion, since the general formulae at the end of Sec. III have been here shown to be computable in a case of interest for gravitational physics and Euclidean quantum gravity, this technical result looks rather helpful for gaining familiarity with reduced-boundary calculations.

V. RIEMANNIAN METRICS IN PSEUDO-RIEMANNIAN GEOMETRY

The material presented so far is a measure-theoretic formulation of the concepts of normal vector, finite-perimeter sets and their boundary, and at first sight it might seem that no obvious counterpart can be conceived in pseudo-Riemannian geometry. However, we may recall a theorem [43] according to which a manifold \( M \) admits a positive-definite metric \( \gamma \) if and only if it is paracompact. The proof consists of first choosing, for each of a countable collection of coordinate patches that cover \( M \), a metric that is positive-definite in the interior of the patch and zero outside, and then taking the sum of these metrics, possibly after rescaling so that the sum converges to some metric on \( M \). Moreover, if \( M \) is endowed with a direction vector field \( \xi \), the metric with components

\[
g_{\alpha\beta} = \gamma_{\alpha\beta} - 2 \sum_{\rho,\sigma=0}^3 \frac{(\gamma_{\alpha\rho} \xi^\rho)(\gamma_{\beta\sigma} \xi^\sigma)}{\gamma(\xi,\xi)}
\]
(5.1)

is Lorentzian and is independent of the scaling of \( \xi \).
On a manifold there exists indeed an uncountable infinity of Riemannian metrics, and infinitely many partitions of unity \( \{ \rho_\alpha \} \) that can be used to glue such metrics and obtain a global metric \([44]\). Of course, this would be of little help without a strategy for choosing a definite Riemannian metric \( \gamma \). For this purpose, we consider again Eq. (5.1) and point out that, once a Lorentzian metric is given, we have to find the components of the timelike vector field that are consistent with the choice of \( \gamma_{\alpha\beta} \) and \( g_{\alpha\beta} \) in the equation

\[
\gamma_{\alpha\beta} - g_{\alpha\beta} = 2 \sum_{\rho,\sigma=0}^{3} \frac{(\gamma_{\alpha\rho}^\rho)(\gamma_{\beta\sigma}^\sigma)}{\gamma(\xi, \xi)}.
\]  

(5.2)

Two cases of diagonal metrics \( \gamma \) and \( g \) are here brought to the attention of the reader.

(i) **Euclidean metric in Minkowski space-time.** If we choose \( \gamma_{\alpha\beta} = \text{diag}(1, 1, 1, 1) \), \( g_{\alpha\beta} = \text{diag}(-1, 1, 1, 1) \),

Eq. (5.2) yields

\[
2 = 2 \frac{\xi^0}{\gamma(\xi, \xi)},
\]  

(5.3)

\[
0 = 2 \frac{\xi^k}{\gamma(\xi, \xi)} \implies \xi^k = 0, \quad \forall k = 1, 2, 3,
\]  

(5.4)

jointly with equations for \( \gamma_{0k} - g_{0k}, \gamma_{ij} - g_{ij} \) that are identically satisfied by virtue of Eq. (5.4). Hence we find

\[
\xi = \xi^0 \frac{\partial}{\partial x^0} \quad \forall \xi^0 \in \mathbb{R} - \{0\},
\]  

(5.5)

which is of course timelike in the Minkowski metric.

(ii) **Euclidean Schwarzschild metric in Lorentzian-signature Schwarzschild space-time.** If we assume that \( \gamma_{\alpha\beta} \) is the Riemannian metric with the components in Eq. \([1.2]\), while

\[
g_{\alpha\beta} = \text{diag} \left( -\left(1 - 2 \frac{M}{r}\right), \left(1 - 2 \frac{M}{r}\right)^{-1}, r^2, r^2 \sin^2 \theta \right),
\]

we find, from Eq. (5.5), the system of equations

\[
2 \left(1 - 2 \frac{M}{r}\right) = 2 \frac{(\gamma_{00}^0 \xi^0)^2}{\gamma(\xi, \xi)},
\]  

(5.6)

and (with no summation over \( k \))

\[
0 = 2 \frac{(\gamma_{kk} \xi^k)^2}{\gamma(\xi, \xi)} \implies \xi^k = 0, \quad \forall k = 1, 2, 3,
\]  

(5.7)

completed by equations for \( \gamma_{0i} - g_{0i}, \gamma_{ij} - g_{ij} \), that are identically satisfied by virtue of Eq. (5.7). Hence we find again the desired vector field in the form (5.5).
VI. CONCLUDING REMARKS AND OPEN PROBLEMS

Geometric measure theory can be viewed as a version of differential geometry studied with the help of measure theory, in order to deal with maps and surfaces that are not necessarily smooth, with applications to the calculus of variations [23, 45]. In our paper we have not worked in absence of smoothness, but we have focused on the task of evaluating the integral formula (3.13) for a finite-perimeter set that is a portion of Riemannian Schwarzschild geometry.

As far as we can see, in the course of considering a measure-theoretic approach to the boundary concept in pseudo-Riemannian geometry, one has to take into account the infinitely many Riemannian metrics $\gamma$ that are available, as we have shown in Sec. 5. If we agree that the metric $\gamma_B$ is equivalent to the metric $\gamma_A$ if a diffeomorphism turns $\gamma_A$ into $\gamma_B$, the resulting reduced boundary is actually an equivalence class. More precisely, to each representative of the same equivalence class of Riemannian metrics, we associate the same set of points that form the reduced boundary of a finite-perimeter set $S$. Two Riemannian metrics not related by a diffeomorphism belong instead to different equivalence classes of Riemannian metrics, and the associated reduced boundaries of $S$ are inequivalent.

The method suggested by our Eqs. (5.2)-(5.7) consists of associating a Riemannian metric to the original Lorentzian metric, whenever the physical space-time is endowed with a globally defined timelike vector field, and then exploiting our work in Sec. 4 in the Schwarzschild case, and the huge amount of rigorous results of Refs. [18–23] in the Euclidean case. This framework might have important implications for a functional integral approach to quantum gravity, since one might try to obtain the in-out amplitude [46] by integrating only over Riemannian manifolds with finite perimeter that are obtained from a Lorentzian counterpart in the sense of section 5 and have a non-empty reduced boundary, for which therefore the condition (3.13) is fulfilled. In other words, finite-perimeter sets are a good candidate for the functional integration because they are reasonably well understood, and their reduced boundary is what really matters, unlike the topological boundary (see again the simple example discussed at the end of Section 2).

The Riemannian nature of the concept we have defined is not necessarily a failure of our research. In this respect, we recall once more the profound result in Ref. [6], where the fundamental relation between black hole entropy and area of the event horizon was obtained.
from a tree-level evaluation of partition function in Euclidean quantum gravity.

Last but not least, our detailed calculations in Sec. IV show that the abstract concepts summarized in Secs. I-III lead to computable results in a gravitational background.

Acknowledgments

G. E. is grateful to Dipartimento di Fisica “Ettore Pancini” for hospitality and support, and to Luigi Ambrosio, Francesco D’Andrea and Francesco Maggi for enlightening correspondence. E. B. is very grateful to Leo Brewin for precious correspondence and for valuable discussions regarding the topic of Riemann normal coordinates. Special thanks are due to Flavio Mercati for scientific collaboration over the whole first year of this work.

Appendix A: Borel sets and Borel functions

Let us denote by \( \mathcal{P}(E) \) the set of parts of the set \( E \). A family \( \mathcal{B} \subset \mathcal{P}(E) \) is a Borel family if and only if it contains the empty set and the following conditions hold with \( E \in \mathcal{B} \):

\[
\begin{align*}
&i) \ A \in \mathcal{B} \implies E - A \in \mathcal{B}, \\
&ii) \ A_i \in \mathcal{B} \implies \bigcup \limits_{i=1}^{\infty} A_i \in \mathcal{B}, \\
&iii) \ A_i \in \mathcal{B} \implies \bigcap \limits_{i=1}^{\infty} A_i \in \mathcal{B}.
\end{align*}
\]

For each family \( S \subset \mathcal{P}(E) \) there exists a minimal Borel family containing \( S \) which is said to be the Borel family generated by \( S \), and is denoted by \( \mathcal{B}(S) \).

By definition, in a topological space, the sets belonging to the Borel family generated by the topology are said to be Borel sets. Thus, every set in a topological space that can be formed from open or closed sets by means of countable union, countable intersection, and complement, is a Borel set. A Borel function \([18]\) is a map \( f : X \rightarrow Y \) such that \( X \) and \( Y \) are topological spaces and the preimage \( f^{-1}(E) \) is a Borel subset of \( X \) whenever \( E \) is an open set of \( Y \). One proves that this is equivalent to having \( f^{-1}(E) \) as a Borel subset of \( X \) whenever \( E \) is a Borel subset of \( Y \). Modern developments study the role of Borel vector fields in solving ordinary differential equations with the associated transport equation \([47]\).
Appendix B: Density of points and reduced boundary

We here collect some definitions that are well known to expert research workers in geometric measure theory, but are hardly ever met by the majority of general relativists.

Given a subset $E$ of $\mathbb{R}^n$ and a point $x$ of $\mathbb{R}^n$, if the limit

$$\theta_n(E)(x) \equiv \lim_{r \to 0^+} \frac{m_L(E \cap A(x,r))}{\omega_n r^n}$$

exists, it is called the $n$-dimensional density of $E$ at $x$. Given a real number $t$ in the closed interval $[0, 1]$, the set of points of density $t$ of $E$ is defined as

$$E^{(t)} \equiv \{ x \in \mathbb{R}^n : \theta_n(E)(x) = t \}.$$  

A theorem ensures that every Lebesgue-measurable set is equivalent to the set of its points of density 1. The set $E^1$ is the measure-theoretic interior of $E$, while $E^0$ is the measure-theoretic exterior of $E$. The essential boundary is defined by

$$\partial^* E \equiv \mathbb{R}^n \setminus (E^0 \cup E^1),$$

and is the measure-theoretic boundary of $E$. The various concepts of boundary are related by the chain of inclusions

$$\mathcal{F} E \subset E^{1/2} \subset \partial^* E \subset \partial E.$$  

A point $x$ belongs to the reduced boundary of a Lebesgue-measurable set $E$ if, in an open neighbourhood of $x$, the topological boundary $\partial E$ of $E$ is of class $C^1$, and $x$ belongs to $\partial E$. On the contrary, a point $x$ does not belong to the reduced boundary of $E$ as soon as either its density differs from $\frac{1}{2}$, or its density equals $\frac{1}{2}$ but the blow-up of $E$ at $x$, defined as

$$\Phi_{x,\rho} \equiv \frac{(E - x)}{\rho},$$

is not a half-space. The latter two conditions are not necessary and sufficient, but help a lot in visualizing the reduced boundary of a measurable subset of $\mathbb{R}^n$.

Appendix C: Trigonometric integrals occurring in the calculation of Eq. (4.42)

In the evaluation process of Eq. (4.42), we have exploited the following trigonometric integrals:

$$\int_0^{\xi_0} d\xi \sin^3 \xi = \frac{1}{3} \left[ 2 - \cos \xi_0 \left( 2 + \sin^2 \xi_0 \right) \right].$$  

\[
\int_0^{\xi_0} d\xi \; \sin^4 \xi = \frac{1}{32} \left[ 12\xi_0 - 8 \sin (2\xi_0) + \sin (4\xi_0) \right],
\]
(C2)

\[
\int_0^{\xi_0} d\xi \; \sin^2 \xi \cos^2 \xi = \frac{1}{32} \left[ 4\xi_0 - \sin (4\xi_0) \right],
\]
(C3)

\[
\int_0^{\psi_0} d\psi \; \cos \psi = \sin \psi_0,
\]
(C4)

\[
\int_0^{\psi_0} d\psi \; \cos^2 \psi = \frac{1}{2} (\psi_0 + \sin \psi_0 \cos \psi_0),
\]
(C5)

\[
\int_0^{\psi_0} d\psi \; \cos^3 \psi = \sin \psi_0 \left( 1 - \frac{1}{3} \sin^2 \psi_0 \right),
\]
(C6)

\[
\int_0^{\psi_0} d\psi \; \sin \psi \cos^2 \psi = \frac{1}{3} \left( 1 - \cos^3 \psi_0 \right),
\]
(C7)

\[
\int_0^{\psi_0} d\psi \; \sin^2 \psi \cos \psi = \frac{1}{3} \sin^3 \psi_0,
\]
(C8)

where \( \xi_0 \in (0, \pi) \) and \( \psi_0 \in (0, 2\pi) \).

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