Color symmetric superconductivity in a phenomenological QCD model

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Abstract

In this paper, we construct a theory of the NJL-type where superconductivity is present, and yet the superconducting state remains, in the average, color symmetric. This shows that the present approach to color superconductivity is consistent with color singlet-ness. Indeed, quarks are free in the deconfined phase, but the deconfined phase itself is believed to be a color singlet. The usual description of the color superconducting state violates color singlet-ness. On the other hand, the color superconducting state here proposed, is color symmetric in the sense that an arbitrary color rotation leads to an equivalent state, with precisely the same physical properties.

1 Introduction

It is presently accepted that quark and gluon fields are the building blocks of hadronic matter, in the framework of quantum chromodynamics (QCD). The investigation of the phase structure of hadronic matter is a topic of great current interest. A diversity of phases is expected at high densities: chiral-symmetry restoration, deconfinement and color-superconductivity. Since, due to the complexity of the theory, it is extremely difficult, if not impossible, to obtain exact results directly from QCD when perturbation theory cannot be applied, effective models, such as the Nambu-Jona-Lasinio (NJL) model [1, 2, 3], must be employed and have been used with great success to investigate the properties of hadronic matter and to develop insight into its phase diagram [4].

Recently, the color superconducting phase in quark matter has been investigated by many authors [5], in the framework of the Bardeen-Cooper-Schrieffer (BCS) approach, which is familiar from condensed matter physics. For a recent review, see [6]. Quarks are free in the deconfined phase, but the deconfined phase itself is expected to be a color singlet. In Refs. [7, 8] it has been argued that in QCD the superconducting phase is automatically color symmetrical. Our basic assumption
is, therefore, the existence of globally color-symmetric superconducting phases, and our aim is to discuss how these phases may be described in terms of effective models with 4 fermion interactions, such as the NJL-like model considered in Refs. [5, 6], for which it is assumed that the gluon degrees of freedom have been integrated over.

A BCS state $|\Phi\rangle$ describes a physical state with zero net color charge if $N_1 = N_2 = N_3$, where $N_i$ denotes the average number of quarks of color $i$. This means that

$$
\langle \Phi | S_{\lambda_3} | \Phi \rangle = \langle \Phi | S_{\lambda_8} | \Phi \rangle = 0.
$$

(1)

Here, $S_{\lambda_k}$ denotes the $SU(3)$ generator associated with the Gell Mann matrix $\lambda_k$. However, the requirement (1), which is implemented in [9, 10], is not sufficient to insure that $|\Phi\rangle$ is physically acceptable. A stronger condition must then be imposed. Indeed, the $SU(3)$ symmetry, being a gauge symmetry, cannot be broken, according to the discussion in [7, 8], so that color rotated BCS states must be equivalent in the sense of the physics they describe. Let $U_c$ denote an arbitrary color rotation, i.e., $U_c = \exp \sum_{k=1}^{8} x_k S_{\lambda_k}$, the parameters $x_k$ being arbitrary and real. The BCS state $|\Phi\rangle$ must be equivalent to the state $U_c |\Phi\rangle$, for any $U_c$, as far as expectation values of physical observables are concerned. Therefore, the condition (1) must be replaced by

$$
\langle \Phi | U_c^\dagger S_{\lambda_k} U_c | \Phi \rangle = \langle \Phi | U_c^\dagger S_{\lambda_k} U_c | \Phi \rangle = 0,
$$

for an arbitrary $U_c$, and this implies

$$
\langle \Phi | S_{\lambda_k} | \Phi \rangle = 0, \quad \text{for} \quad k = 1, 2, \cdots, 8.
$$

(2)

This is the condition the BCS state $|\Phi\rangle$ must satisfy in order to be physically meaningful. If only the condition (1) is implemented, and not the condition (2), the BCS state $|\Phi\rangle$ is, in general, not equivalent to the state $U_c |\Phi\rangle$, so that it describes a state belonging to a representation of $SU(3)$ other than the singlet one, which is physically unacceptable. In [11] it is shown how the condition (2) may be easily implemented.

In the present paper, we apply to the NJL model the new BCS approach developed in Ref. [11] which uses the generalized Bogoliubov transformation and leads to a color symmetric BCS vacuum. In Section 2, a mean-field constrained Hamiltonian appropriate for the description of color superconductivity is presented. In section 3 we compare the new superconducting state which satisfies (2), with the usual one, which is not required to satisfy (2). In Section 4, we draw some conclusions.

## 2 Color symmetrical superconductivity

We wish to focus on the diquark condensate $\langle \bar{\psi}^C i \gamma_5 \lambda_j \tau_2 \psi \rangle$, $j \in \{2, 5, 7\}$, and, for simplicity, we will neglect the quark-antiquark condensate $\langle \bar{\psi} \psi \rangle$. The notation is the usual one, the superscript $C$ denoting charge-conjugation. In this sense, the model is not fully realistic, but is adequate for the present development. We assume that the pairing interaction is antisymmetric in the color indices and in the isospin indices, i.e., it involves the Gell-Mann matrices $\lambda_j$, $j \in \{2, 5, 7\}$ and the isospin matrix $\tau_2$ associated with flavor, when 2 is the number of flavors. Having in mind a Hamiltonian of the NJL [2] type, the mean-field constrained Hamiltonian reads:
\[ K_{MFA} = \int d^3x \left[ \bar{\psi} \left( \vec{p} \cdot \vec{\gamma} + M - \mu \gamma_0 \right) \psi + \frac{1}{2} \sum_{j \in \{2,5,7\}} (\Delta_j^* \bar{\psi}^C i \gamma_5 \tau_2 \lambda_j \psi + h.c.) \right] + \sum_{j \in \{2,5,7\}} \frac{|\Delta_j|^2}{4G_C}, \]

where
\[ \Delta_j = -2G_C (\bar{\psi}^C i \gamma_5 \tau_2 \lambda_j \psi), \]
denotes the BCS gap. We use the symbol \( \hat{K} \), instead of the more usual symbol \( \hat{H} \) to stress that this Hamiltonian is constrained in the sense that it fixes the Fermion number through the chemical potential \( \mu \), which behaves as a Lagrange multiplier. Thus the expectation value of \( \hat{K}_{MFA} \) is the thermodynamical potential which is equal to \( -PV \), where \( P \) is the pressure and \( V \) the volume, and determines the equation of state. By \( \langle X \rangle \) we denote the average of \( X \) in the BCS vacuum which will be specified in the following. The notation is essentially the same as in [12], slightly modified, as is required in order to treat the 3 colors on the same footing. In momentum space we have,

\[ \tilde{K}_{MFA} = \sum_{p,\eta,j,\tau} \left[ (\sqrt{p^2 + M^2} - \mu) c^\dagger_{p,\eta,j,\tau} c_{p,\eta,j,\tau} + (\sqrt{p^2 + M^2} + \mu) \bar{c}^\dagger_{p,\eta,j,\tau} \bar{c}_{p,\eta,j,\tau} \right] + \frac{1}{2} \sum_{l} \sum_{p,\eta,j,\tau,\tau'} (c^\dagger_{p,\eta,j,\tau} c^\dagger_{p,\eta,j,\tau'} + \bar{c}^\dagger_{p,\eta,j,\tau} \bar{c}_{p,\eta,j,\tau'}) \epsilon_{ijkl} \epsilon_{\tau\tau'} \zeta_{p,\eta} + h.c. + V \sum_{l} \frac{|\Delta_l|^2}{4G_C}, \]

where \( c^\dagger_{p,\eta,j,\tau} \) and \( \bar{c}^\dagger_{p,\eta,j,\tau} \) create, respectively, a quark and an antiquark of momentum \( p \), helicity \( \eta \), color index \( j \) and isospin index \( \tau \), \( \zeta_{p,\eta} = -\zeta_{-\eta} = -\zeta_{p,-\eta} = \zeta_{-\eta} \), \( |\zeta_{p,\eta}| = 1 \), and
\[ \Delta_l^* = -V^{-1} 2G_C \sum_{p,\eta,j,\tau,\tau'} \left[ \left( \langle c^\dagger_{p,\eta,j,\tau} c^\dagger_{p,\eta,j,\tau'} \rangle + \langle \bar{c}^\dagger_{p,\eta,j,\tau} \bar{c}_{p,\eta,j,\tau'} \rangle \right) \epsilon_{ijkl} \epsilon_{\tau\tau'} \zeta_{p,\eta} \right]. \]

To be precise, the index \( j \) in \( c^\dagger_{p,\eta,j,\tau} \) labels states of the 3 representation of \( su(3) \), while in \( \bar{c}^\dagger_{p,\eta,j,\tau} \) it labels states of the \( \bar{3} \) representation.

For convenience, we introduce the notation, \( m = (p, \eta, \tau) \), \( \bar{m} = (-p, \eta, -\tau) \). Clearly, \( \bar{m} = m \). The BCS vacuum \( |\Phi\rangle \) is defined as the state which is annihilated by the operators \( d_{jm} \) such that

\[ d_{1m} = c_{1m} - K_m (c_{2m}^\dagger - c_{3m}^\dagger), \quad \Omega' < m, \]
\[ d_{1m} = c_{1m}^\dagger + K_m (c_{2m} - c_{3m}), \quad m \leq \Omega', \]

that is, \( d_{jm} |\Phi\rangle = 0 \). The parameters \( K_m, K_m' \) real. The notation \( \Omega' < m \) means symbolically that \( m \) is such that \( \sqrt{p^2 + M^2} - \mu > 0 \), while \( m \leq \Omega' \) means that \( \sqrt{p^2 + M^2} - \mu \leq 0 \). The expressions for \( d_{2m}, d_{3m} \), are obtained by circular permutation of the indices 1, 2, 3. The transformation \( [17] \) is not canonical, since \( \{d_{im}, d_{jn}^\dagger\} \neq \delta_{ij} \delta_{mn} \), but the corresponding canonical transformation, which is not needed for the present purpose, may be easily obtained.
At this point, a short explanation may be in order, concerning the generalized Bogoliubov transformation, symmetric in the color indices, which has been proposed in eq. (25) of the first paper cited in [11], and which was supposed to “diagonalize” the pairing hamiltonian of the Bonn model. However, there is an error in that equation and the BCS vacuum associated with the Bogoliubov transformation it describes fails to produce a non-vanishing gap $\Delta$. That transformation is appropriate to diagonalize a pairing Hamiltonian which, although seemingly similar, is actually essentially different from the pairing Hamiltonian of the Bonn model, since that Hamiltonian is not invariant under $SU(3)$. That is the reason for the corrigendum in [11].

The color symmetrical BCS state reads

$$|\Phi\rangle = \exp \sum_{j=1}^{3} \left( \sum_{\Omega' \leq m} K_{m} A_{jm}^{\dagger} + \sum_{m \leq \Omega'} \tilde{K}_{m} A_{jm} \right) |0_{\Omega'}\rangle,$$

where

$$|0_{\Omega'}\rangle = \left( \prod_{j=1}^{3} \prod_{m \leq \Omega'} c_{jm}^{\dagger} c_{jm}^{\dagger} \right) |0\rangle,$$

and

$$A_{1m}^{\dagger} = c_{2m}^{\dagger} c_{3m}^{\dagger} + c_{3m}^{\dagger} c_{2m}^{\dagger},$$

$|0\rangle$ denoting the quark vacuum. The expressions for $A_{2m}^{\dagger}, A_{3m}^{\dagger}$, are obtained by circular permutation of the indices 1, 2, 3. For simplicity, pairing operators involving anti-quarks $\tilde{\tilde{c}}$ (negative energy states) are not shown, but, in principle, their contribution should be included. In conformity, we will not show the contribution of anti-quarks to color superconductivity, but it is straightforward to include that contribution.

We use the notation $\langle W \rangle = \langle \Phi | W | \Phi \rangle / \langle \Phi | \Phi \rangle$. We easily find

$$\langle c_{im}^{\dagger} c_{jm} \rangle = -\frac{K_{m}^{2}}{1 + 3K_{m}^{2}}, \quad i \neq j, \quad \langle c_{jm}^{\dagger} c_{jm} \rangle = \frac{2K_{m}^{2}}{1 + 3K_{m}^{2}}, \quad \Omega' < m. \quad (9)$$

On the other hand,

$$\langle c_{im}^{\dagger} c_{jm} \rangle = \frac{\tilde{K}_{m}^{2}}{1 + 3K_{m}^{2}}, \quad i \neq j, \quad \langle c_{jm}^{\dagger} c_{jm} \rangle = 1 - \frac{2\tilde{K}_{m}^{2}}{1 + 3K_{m}^{2}}, \quad m \leq \Omega'. \quad (10)$$

The derivation of of eqs. (9) and (10) is summarized in the Appendix.

The $U(3)$ generators read

$$S_{ij} = \sum_{m} c_{im}^{\dagger} c_{jm}^{\dagger}. \quad (11)$$

Clearly, the generators $S_{\lambda k}$ of $SU(3)$ considered in the Introduction are related to the generators $S_{kl}$. For instance, $S_{\lambda 1} = S_{12} + S_{21}, \quad S_{\lambda 2} = -i(S_{12} - S_{21}), \quad S_{\lambda 3} = S_{11} - S_{22}, \quad$ etc.

Since

$$\langle S_{ij} \rangle = -2 \sum_{m \leq \Omega'} \frac{K_{m}^{2}}{1 + 3K_{m}^{2}} + 2 \sum_{m \geq \Omega'} \frac{\tilde{K}_{m}^{2}}{1 + 3K_{m}^{2}}, \quad i \neq j, \quad (12)$$

we may obviously insure that $\langle S_{ij} \rangle = 0, \quad i \neq j$, which is the condition for color neutrality, by conveniently choosing $K_{m}, \tilde{K}_{m}$.
Next we compute the contractions $\langle c_2 m c_1 \Omega \rangle = \langle c_3 m c_2 \Omega \rangle = \langle c_1 m c_3 \Omega \rangle = \langle c_2 m c_1 \rangle = \langle c_3 m c_2 \rangle = \langle c_1 m c_3 \rangle = D_m$, where $D_m$ is real. We have,

\[
D_m = \frac{K_m}{1 + 3K_m^2}, \quad \Omega' < m; \quad D_m = \frac{\tilde{K}_m}{1 + 3\tilde{K}_m^2}, \quad m \leq \Omega',
\]  

(13)

the derivation of these relations being left to the Appendix.

We are now able to compute the expectation value of $\hat{K}_{MFA}$. We obtain

\[
\langle \hat{K}_{MFA} \rangle = 3 \sum_{\Omega' < m} \left( \varepsilon_m \frac{2K_m^2}{1 + 3K_m^2} + 2\Delta \frac{K_m^3}{1 + 3K_m^2} \right) + 3 \sum_{m \leq \Omega'} \left( \varepsilon_m \left( 1 - \frac{2K_m^2}{1 + 3K_m^2} \right) + 2\Delta \frac{\tilde{K}_m^3}{1 + 3\tilde{K}_m^2} \right) + V \frac{3\Delta^2}{4G_C},
\]  

(14)

\[
\Delta_1 = \Delta_2 = \Delta_3 = -\frac{2G_C}{V} \left( \sum_{\Omega' < m} \frac{K_m}{1 + 3K_m^2} + \sum_{m \leq \Omega'} \frac{\tilde{K}_m}{1 + 3\tilde{K}_m^2} \right) =: \Delta,
\]  

(15)

where $\varepsilon_m$ stands for $\sqrt{p^2 + M^2} - \mu$. It is convenient to define the angles $\theta_m$, $\tilde{\theta}_m$ such that $\sin \theta_m = \sqrt{3K_m}/\sqrt{1 + 3K_m^2}$, $\cos \theta_m = 1/\sqrt{1 + 3K_m^2}$, $\sin \tilde{\theta}_m = \sqrt{3\tilde{K}_m}/\sqrt{1 + 3\tilde{K}_m^2}$, $\cos \tilde{\theta}_m = 1/\sqrt{1 + 3\tilde{K}_m^2}$. Then, we have

\[
\langle \hat{K}_{MFA} \rangle = \sum_{\Omega' < m} \left( 2\varepsilon_m \sin^2 \theta_m + 2\sqrt{3}\Delta \sin \theta_m \cos \theta_m \right)
\]

\[
+ \sum_{m \leq \Omega'} \left( \varepsilon_m \left( 3 - 2\sin^2 \tilde{\theta}_m \right) + 2\sqrt{3}\Delta \sin \tilde{\theta}_m \cos \tilde{\theta}_m \right) + V \frac{3\Delta^2}{4G_C},
\]  

(16)

\[
\sqrt{3}\Delta = -\frac{2G_C}{V} \left( \sum_{\Omega' < m} \sin \theta_m \cos \theta_m + \sum_{m \leq \Omega'} \sin \tilde{\theta}_m \cos \tilde{\theta}_m \right).
\]  

(17)

Having in mind eq. (12), the color neutrality constraint $\langle S_{ij} \rangle = 0$ reduces to

\[
-\sum_{\Omega' < m} \sin^2 \theta_m + \sum_{m \leq \Omega'} \sin^2 \tilde{\theta}_m = 0.
\]  

(18)

The extremum condition reads (see Appendix)

\[
\cos 2\theta_m = \frac{\varepsilon_m - \lambda}{\sqrt{(\varepsilon_m - \lambda)^2 + 3\Delta^2}}, \quad \sin 2\theta_m = -\frac{\sqrt{3}\Delta}{\sqrt{(\varepsilon_m - \lambda)^2 + 3\Delta^2}}, \quad \Omega' < m,
\]

\[
\cos 2\tilde{\theta}_m = -\frac{\varepsilon_m - \lambda}{\sqrt{(\varepsilon_m - \lambda)^2 + 3\Delta^2}}, \quad \sin 2\tilde{\theta}_m = -\frac{\sqrt{3}\Delta}{\sqrt{(\varepsilon_m - \lambda)^2 + 3\Delta^2}}, \quad m \leq \Omega',
\]  

(19)

where $\lambda$ is the Lagrange multiplier which ensures the color neutrality constraint (18). The gap equation for color neutral superconductivity reads

\[
1 = \frac{G_C}{V} \left( \sum_{m \leq \Omega'} + \sum_{\Omega' < m} \right) \frac{1}{\sqrt{(\varepsilon_m - \lambda)^2 + 3\Delta^2}}.
\]  

(20)
By setting $\lambda = 0$, the gap equation for color neutral (not color symmetrical) superconductivity is obtained. Notice that we have been able to achieve color neutrality, eq. (11), independently of the $\lambda$ value, without introducing any extra Langrange multiplier, as opposed to what is done in [9, 10]. We stress, however, that the implementation of (17) is essential. Otherwise, the present BCS vacuum can be reduced to the usual one, discussed in the next section, by an appropriate color rotation.

At the extremum, the expectation value of $\hat{K}_{MFA}$ reduces to

$$
\langle \hat{K}_{MFA} \rangle = \sum_{m \leq \Omega'} (2\varepsilon_m - E_m) + \sum_{m > \Omega'} (\varepsilon_m - E_m) + V \frac{3\Delta^2}{4G_C},
$$

$$= 3 \sum_{m \leq \Omega'} \varepsilon_m + \sum_{m \leq \Omega'} (|\varepsilon_m| - E_m) + \sum_{\Omega' < m} (\varepsilon_m - E_m) + V \frac{3\Delta^2}{4G_C},
$$

(21)

where $E_m = \sqrt{(\varepsilon_m - \lambda)^2 + 3\Delta^2}$. In terms of proper canonical quasi particle operators $f_{im}$ such that $\{f_{im}, f^\dagger_{jn}\} = \delta_{ij}\delta_{mn}$, the constrained Hamiltonian reads

$$
\hat{K}_{MFA} = \sum_{m \leq \Omega'} \varepsilon_m + \sum_{m} (\varepsilon_m - E_m) + V \frac{3\Delta^2}{4G_C} + \sum_{m} \left( \sum_{j=1}^{2} E_m f^\dagger_{jm} f_{jm} + \varepsilon_m f^\dagger_{3m} f_{3m} \right).
$$

(22)

As explained in [11], the index $j$ in $f_{jm}$ does not specify a well defined color. Indeed,

$$f_{1m} = \kappa_1 (d_{1m} - d_{2m}), \quad f_{2m} = \kappa_2 (d_{1m} + d_{2m} - 2d_{3m}), \quad f_{3m} = \kappa_3 (d_{1m} + d_{2m} + d_{3m}),$$

where $\kappa_1, \kappa_2, \kappa_3$, are normalization constants.

### 3 Comparison with usual color superconductivity

In the usual approach to color superconductivity, which breaks color symmetry [5, 6], we have $\Delta_3 = \Delta \neq 0$, $\Delta_1 = \Delta_2 = 0$. The BCS transformation which diagonalizes $K_{MFA}$ reads

$$c_{1m} = \alpha_m d_{1m} + \beta_m d^\dagger_{2m}, \quad c_{2m} = \alpha_m d_{2m} - \beta_m d^\dagger_{1m},$$

with

$$\alpha_m^2 - \beta_m^2 = \frac{\varepsilon_m}{\sqrt{\varepsilon_m^2 + \Delta^2}}, \quad 2\alpha_m\beta_m = \frac{\Delta}{\sqrt{\varepsilon_m^2 + \Delta^2}}, \quad \alpha_m^2 + \beta_m^2 = 1.$$ 

The gap is expressed as

$$\Delta = 2 \frac{G_C}{V} \sum_m \alpha_m \beta_m,$$

so that, the gap equation reads

$$1 = \frac{G_C}{V} \sum_m \frac{1}{\sqrt{\varepsilon_m^2 + \Delta^2}},$$

(23)
The constrained mean field Hamiltonian reduces to

\[
\hat{K}_{MFA} = \sum_{m \leq \Omega'} \varepsilon_m + \sum_{m \leq \Omega'} (|\varepsilon_m| - E_m) + \sum_{\Omega' < m} (\varepsilon_m - E_m) + V \frac{\Delta^2}{4G_C} + \sum_m \left( E_m (d_{1m}^\dagger d_{1m} + d_{2m}^\dagger d_{2m}) + \varepsilon_m c_{3m}^\dagger c_{3m} \right).
\]

The gap $\Delta$ is only due to the pairing correlations between colors 1 and 2. The gap equation has essentially the same form in the color symmetric (eq. (20)) and in the color symmetry breaking (eq. (23)) description of the color superconductivity. However, in the first case we have automatically $\langle S_{11} \rangle = \langle S_{22} \rangle = \langle S_{33} \rangle$, while in the second case we have $\langle S_{11} \rangle = \langle S_{22} \rangle \neq \langle S_{33} \rangle$. In [4], it has been argued that the usual color superconductivity, where only two colors participate in the gap, softens the equation of state. It is expected that the participation of the three colors on the same footing will have significant consequences for the high density phases of QCD which exhibit color superconductivity. We do not discuss the important color-flavor-locking mechanism, which also leads to color neutrality, since our main concern is the two flavor case.

![Figure 1: Lowest energy in the BCS approximation for the schematic Bonn model [11] with $\Omega = 10$, versus the quark number, in arbitrary units. Thick line: the lowest energy of the color symmetric sector, according to the new method presented in Section 2; thin line: the result of the usual approach, which is summarized in Section 3, and describes the full groundstate energy of the model, including the “unphysical” sectors.](image)

4 Conclusions

Quarks in QCD are, in the confined phase, only allowed to form colorless states. However, at high temperature and density, quarks are expected to be free. It has been shown that in the deconfined phase, which then prevails, the tendency for the
formation of BCS pairs occurs \[5, 6\]. This gives rise to the so-called color superconducting phase, which, in the usual treatment \[3, 4, 5, 6, 9, 10\], violates color symmetry. In this paper, we show that color superconductivity is not incompatible with color symmetry. It will be interesting to compare, in a realistic model of the NJL-type, the properties of the phase described by the color symmetric version of color superconductivity presently considered, with the corresponding properties described by conventional color superconductivity, which drastically breaks color symmetry. This has been done for a schematic model in ref. \[11\], and, there, the difference found is important. In the Figure, we illustrate the performance of the new method here propose, and compare it with the usual approach to color superconductivity, in the context of the QCD inspired schematic model considered in \[11\].

This model, which is characterized by a parameter $\Omega$ measuring the level degeneracy, is admittedly unrealistic, but is quite useful to test approximation techniques, since it is analytically solvable. Applications to realistic situations are in progress.

An effective QCD theory, as is appropriate to describe e.g. the interior of neutron stars, will have vanishing confining force at high temperatures and densities, due to asymptotic freedom, but should also be consistent with color singlet-ness \[7, 8\]. The phase of superconducting color symmetric states is supposed to exist in the interior of neutron stars with high density, where the simplified NJL model becomes identical to an effective QCD field theory and, thus, realistic.

The present approach is in contrast with those followed in refs. \[13\] and \[9\], where the color neutrality problem was previously addressed. In \[13\], the authors resort to rather involved projection techniques to extract color neutral states out of BCS states which violate color symmetry. It should be pointed out that the correlations described by the present approach need not coincide with those arising within the framework of the projection technique. In \[9\], color neutrality is defined by the condition that the average or expectation value of some of the eight Gell-Mann operators vanishes, that is, color neutrality is implemented with the help of appropriate Lagrange multipliers. In connection with this question, a reference to a recent work of Buballa and Shovkovy \[10\] is appropriate. These authors observe that, if a common chemical potential $\mu$ is used for all colors, then the quark numbers $\langle S_{11}\rangle$, $\langle S_{22}\rangle$, $\langle S_{33}\rangle$, are not all the same, and hence will violate color neutrality. While, in the present approach, equality of the quark numbers for different colors is achieved with a common chemical potential, even if we set $\lambda = 0$, in \[10\] one advocates using different chemical potentials for different colors, to insure $\langle S_{11}\rangle = \langle S_{22}\rangle = \langle S_{33}\rangle$. However, in \[10\], it is already observed that this condition is not sufficient to insure color singlet-ness, due to its instability under color rotations. We have proposed a solution to the emerging problem.

The present article also provides theoretical “tools” for constructing phases with superconducting color-symmetric states.

Although our aim is not to discuss the QCD Meissner effect, we observe that this effect may be treated in terms of a Hamiltonian of the form

$$H = \int d^3x \left( \frac{1}{2} (B_a \cdot B_a + E_a \cdot E_a) + \vec{\Pi}^* \cdot \vec{\Pi} + (D \vec{\Delta})^* \cdot (D \vec{\Delta}) + V(\vec{\Delta}^* \cdot \vec{\Delta}) \right),$$

describing the color superconducting phase in interaction with the gluon field. In the definition of the covariant derivative $D$ one should keep in mind that $\vec{\Delta}$ belongs to the $\bar{3}$ representation. It is clear that the standard treatment goes through with our approach. We observe that our color symmetric BCS theory should lead to qualitatively the same masses for the gauge bosons as are found in the literature.
The only place where an effective model such as the NJL model comes in is in the estimation of the function $V$ which depends on $|\Delta_1|^2, |\Delta_2|^2, |\Delta_3|^2$. Our approach ensures that $V$ will depend only on the combination $|\Delta_1|^2 + |\Delta_2|^2 + |\Delta_3|^2$.

**Appendix**

**Derivation of eqs. (9) and (10).**

For $\Omega' < m$, we have

$$X_m := \langle c_{1m}^\dagger c_{2m} \rangle = -K^2 - K_m^2 \left( -\langle c_{3m}^\dagger c_{3m} \rangle - \langle c_{2m}^\dagger c_{2m} \rangle + \langle c_{3m}^\dagger c_{1m} \rangle + \langle c_{2m}^\dagger c_{1m} \rangle \right),$$

$$N_m := \langle c_{1m}^\dagger c_{1m} \rangle = 2K^2 - K_m^2 \left( \langle c_{3m}^\dagger c_{3m} \rangle + \langle c_{2m}^\dagger c_{2m} \rangle - \langle c_{3m}^\dagger c_{2m} \rangle - \langle c_{2m}^\dagger c_{3m} \rangle \right),$$

implying

$$X_m = -K_m^2 + K^2 (N_m - X_m), \quad N_m = 2K_m^2 - 2K_m^2 (N_m - X_m),$$

which leads to $X_m = -K_m^2 / (1 + 3K_m^2), \quad N_m = 2K_m^2 / (1 + 3K_m^2)$. The corresponding expressions for $m \leq \Omega'$ are similarly obtained.

**Derivation of eq. (13).**

For $\Omega' < m$ we have

$$\langle c_{3m}^\dagger c_{1m} \rangle = K_m - K_m^2 \left( \langle c_{2m}^\dagger c_{3m} \rangle + \langle c_{3m}^\dagger c_{1m} \rangle + \langle c_{1m}^\dagger c_{2m} \rangle - \langle c_{3m}^\dagger c_{2m} \rangle \right),$$

$$\langle c_{1m} c_{1m} \rangle = -K_m \left( \langle c_{2m}^\dagger c_{2m} \rangle + \langle c_{3m}^\dagger c_{3m} \rangle - \langle c_{2m} c_{2m} \rangle - \langle c_{2m}^\dagger c_{3m} \rangle \right),$$

which imply

$$D_m = K_m - 3K_m^2 D_m + K_m^2 P_m, \quad P_m = 2K_m^2 P_m,$$

where $P_m = \langle c_{1m} c_{1m} \rangle = \langle c_{2m} c_{2m} \rangle = \langle c_{3m} c_{3m} \rangle$, is also real. The procedure for $\Omega' < m$, is analogous. Finally, we find $P_m = 0$ and eq. (13) follows.

**Proof of eq. (19).**

Let

$$\Psi(\theta, \tilde{\theta}, \Delta, \lambda) = \langle \hat{K}_{MFA} \rangle + 2\lambda \left( -\sum_{\Omega' < m} \sin^2 \theta_m + \sum_{m \leq \Omega'} \sin^2 \tilde{\theta}_m \right),$$

where $\langle \hat{K}_{MFA} \rangle$ stands for its expression in eq. (16) and $\lambda$ is a Lagrange multiplier. We find

$$\partial_{\theta_m} \Psi = 4(\varepsilon_m - \lambda) \sin \theta_m \cos \theta_m + 2\sqrt{3}\Delta (\cos^2 \theta_m - \sin^2 \theta_m) = 0, \quad \Omega' < m,$$

$$\partial_{\tilde{\theta}_m} \Psi = -4(\varepsilon_m - \lambda) \sin \theta_m \cos \tilde{\theta}_m + 2\sqrt{3}\Delta (\cos^2 \tilde{\theta}_m - \sin^2 \tilde{\theta}_m) = 0, \quad m \leq \Omega'.$$

The condition $\partial_\Delta \Psi = 0$ yields eq. (17), and the desired result, eq. (19), follows now easily.
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