Self-gravitating dark matter gets in shape

Jenny Wagner
j.wagner@uni-heidelberg.de
Universität Heidelberg, Zentrum für Astronomie,
Mönchhofstr. 12–14, 69120 Heidelberg, Germany
https://www.zah.uni-heidelberg.de/staff/jwagner/

May 20, 2020

Essay written for the
Gravity Research Foundation 2020 Awards for Essays on Gravitation.

Abstract
In our current best cosmological model, the vast majority of matter in the Universe is dark, consisting of yet undetected, non-baryonic particles that do not interact electro-magnetically. So far, the only significant evidence for dark matter has been found in its gravitational interaction, as observed in galaxy rotation curves or gravitational lensing effects. The inferred dark matter agglomerations follow almost universal mass density profiles that can be reproduced well in simulations, but have eluded an explanation from a theoretical viewpoint. Forgoing standard (astro-)physical methods, I show that it is possible to derive these profiles from an intriguingly simple mathematical approach that directly determines the most likely spatial configuration of a self-gravitating ensemble of collisionless dark matter particles.
Ubiquitous universality

Dark matter may be a mysterious form of matter, yet, its gravitational interaction can be reconstructed well in numerous examples like tracing the rotation curves of stars in a galaxy [1],[7], or observing the gravitational lensing effect of galaxies and galaxy clusters whose masses distort light bundles on their way from light-emitting background objects to us [3]. These observations show a high uniformity in the inferred dark matter mass density profiles. A small set of heuristic, parametric models fit a wide range of galaxy and galaxy cluster mass densities. N-body simulations that emulate the process of dark matter agglomeration with gravity as the only interaction corroborate these findings [4], [5]. But they also reveal deviations from universal mass density profiles with increasing resolution in length and mass scales [6].

Numerous ideas have been developed to derive the shape of dark matter mass density profiles from statistical mechanics as equilibrium configurations with maximum entropy in phase-space or energy-space [2], [8]. Although they greatly enhance our understanding, some fundamental questions remain. For instance, how is the innermost part of a dark matter structure, we call “dark matter halo”, shaped? Why does the outer halo region on galaxy scale show a steeper decrease in mass density than its galaxy-cluster-scale counterpart? Why does universality dissolve with increasing resolution? Why do halo shapes seem to be independent of their mass accretion history and the background cosmology?

As I show in the following and further detail in [9], all these questions find an answer in a simple mathematical approach that reverse engineers dark matter mass density profiles. Contrary to standard methods, it separates the morphological description of a halo from its dynamics and focuses on the spatial distribution of dark matter particles. The particle interactions are phenomenologically modelled by the mean gravitational field they generate themselves. This minimalistic approach does not require any definition of phase-space configurations, entropy, or the usage of the particle velocities.

Convincing characterisations

We track a finite amount of \( n_p \) dark matter particles building a halo of finite volume and assume that collisionless particles always keep a finite minimum distance. These limiting prerequisites prevent any divergences from occurring. Furthermore, we restrict our model to identical particles and spherical halo volumes, such that all equations are analytically solvable and the concept is clearly recognisable. Taking into account that the mean gravitational field of the ensemble is generated by applying Newton’s scale-free gravitation to all particle pairs, we assume that each particle follows a power-law probability density to be located at radius \( r \) inside the halo volume with maximum radius \( r_{\text{max}} \)

\[
p(r) = N(\alpha, r_\sigma, r_{\text{max}}) \left( 1 + \frac{r}{r_\sigma} \right)^{-\alpha}, \quad \alpha \geq 0 ,
\]

(1)

with power-law index \( \alpha \), and scale-radius \( r_\sigma \), introduced to obtain dimensionless quantities. \( N(\alpha, r_\sigma, r_{\text{max}}) \) normalises \( p(r) \), such that the probability of finding the particle in the halo volume equals one. Collisionless particles are independent of each other. Hence, the joint probability density to find the ensemble in a specific spatial configuration is given by multiplying the \( p(r_i) \) for all independent particles \( i = 1,...,n_p \).

Asking for extremal configurations with respect to \( \alpha \), the derivative of the logarithm of the joint probability density yields

\[
\frac{\partial_\alpha N(\alpha, r_\sigma, r_{\text{max}})}{N(\alpha, r_\sigma, r_{\text{max}})} = \frac{1}{n_p} \sum_{j=1}^{n_p} \ln \left( 1 + \frac{r_j}{r_\sigma} \right) = 0 .
\]

(2)

We note that \( \alpha \) enters via the normalisation, i. e. through the assumed halo geometry and its volume defined in \( N \). The sum-term containing the particle number and distribution of the ensemble accounts for resolution effects. Due to the choice of \( r_{\text{max}} \) and \( r_\sigma \), the particle
distribution is considered on a preferred length scale. Equation 2 is invariant for distinguishable and indistinguishable particles because the respective pre-factor in the joint probability density is independent of \( \alpha \).

Before solving Equation 2 to obtain \( \alpha \) for different physical approximations, we need to derive the continuous halo mass density, \( \rho(r) \), from the single-particle probability density function (Equation 1). This is easily achieved, because the number density \( n(r) \) for our spherical halo of collisionless particles is defined as the phase-space probability density function for a single particle after marginalising out the velocity. If we interpret Equation 1 as this spatial part of the single-particle phase-space probability density and multiply \( n(r) \) by the mass of a particle \( m \) we arrive at

\[
n(r) = n_p p(r) \quad \Rightarrow \quad \rho(r) = m n(r) = m n_p p(r).
\]

Hence, \( \rho(r) \) obeys the same power-law of Equation 1, which means that the slope of the mass density profiles can be directly related to \( \alpha \) for the extremum configurations of the particle ensemble determined by Equation 2.

Choosing \( r_p \) that \( r_j \gg r_p \) for all \( j \) simplifies Equation 2 to

\[
\frac{1}{\alpha - 3} = \frac{1}{n_p} \sum_{j=1}^{n_p} \ln \left( \frac{r_j}{r_{\text{max}}} \right) = \frac{1}{n_p} \sum_{j=1}^{n_p} \ln \left( 1 + \frac{r_j}{r_{\text{max}}} - 1 \right) \approx \frac{1}{n_p} \sum_{j=1}^{n_p} \frac{r_j}{r_{\text{max}}} - 1. \tag{4}
\]

The behaviour of \( \rho(r) \) thus depends on the particle distribution. The first term on the right-hand side can be interpreted as a scaled mean particle radius. For finite \( n_p \) and \( r_{\text{max}} \), the upper limit is \( \alpha = 3 \), the lower \( \alpha = 0 \) if all \( r_j \leq r_{\text{max}} \). Now, we can explain the shape of the most common density profiles:

Let \( r_{\text{max}} = r_{\text{core}} \) be the boundary of the core. Debates about the slope of \( \rho(r) \) in the core naturally arise because the particle number and their locations fix \( \alpha \). For a uniform particle distribution from 0 to \( r_{\text{max}} \), \( \alpha = 1 \). As simulations probe smaller radii, the slopes of heuristically fitted models become shallower towards \( r = 0 \). This trend is explained by Equation 4 when \( r_{\text{max}} \) of the probed part shrinks towards the radius of the first bin in the simulation, putting all particles at radii just below \( r_{\text{max}} \).

From now on, \( r_{\text{max}} \) is the boundary of the entire halo. Assuming the particle distribution becomes very dense, i.e. \( n_p \rightarrow \infty \), so that dark matter transfers into a homogeneous fluid. Then, Equation 4 yields \( \alpha = 2 \) for a most-likely ensemble configuration, which is interpreted as the stable, isothermal halo part every simulation and observation shows.

The last two approximations concern the choice of \( r_{\text{max}} \), i.e. our definition of the extension of a halo. Taking the limit \( r_{\text{max}} \rightarrow \infty \) and assuming that the particle farthest from the halo centre is at a much smaller, finite radius, we arrive at \( \alpha = 3 \) belonging to a least-likely ensemble configuration. Depending on the choice of \( r_{\text{max}} \), shallower slopes are also possible in this approximation that omits to assign particles to the halo which are far from the halo centre but still feel its gravitational influence.

Choosing \( r_{\text{max}} \) much smaller than the extent of the particle distribution, we arrive at \( \alpha = 4 \), assuming that, on the average, the particle radii scatter around \( 2 r_{\text{max}} \). This choice resembles models of galaxy luminosity profiles employing a half-light radius. The respective ensemble configuration is a local log-likelihood maximum and, depending on the choice \( r_{\text{max}} \), steeper slopes are also possible.

Considering these two boundary cases, the often found \( r^{-3} \)-decrease of galaxy-cluster mass densities can be explained, as well as the \( r^{-4} \)-decrease of galaxy mass densities. Due to the least- and most-likely configurations these power-law indices belong to, it is also clear that a large sample of simulated or observed galaxy clusters shows deviations from a universal behaviour with increasing resolution, while, on galaxy scale, universality may occur on average for a large amount of relaxed systems.

Summarising the results, we can decompose any dark matter halo mass density into three parts: an inner core, an isothermal region, and an outer boundary, as depicted in Figure 4.
Figure 1: Monotonically decreasing dark matter mass density profiles $\rho(r)$ for galaxy-cluster-scale halos (left) and galaxy-scale halos (right). The central region for both consists of dilute particle configurations with $\rho(r) \propto r^{-\alpha}$ and $\alpha \in [0, 2]$ depending on the particle positions. This transfers into an isothermal part consisting of a homogeneous dark matter fluid with $\alpha = 2$. At the outer bounds $\alpha \approx 3$ for galaxy-cluster-scale halos (left) and $\alpha \approx 4$ for galaxy-scale halos (right) due to the location of $r_{\text{max}}$ relative to the particle positions.

Remaining riddles

The approach presented here and detailed in [9] explains many dark matter halo properties found in simulations and observations in an astonishingly simple way. It only employs a minimum amount of necessary assumptions and finally answers the question why our heuristically inferred mass density profiles are good fits to artificial and real self-gravitating dark matter structures without resorting to any cosmological model, the assembly history of the structure, or its dynamics. Extensions to less symmetric halo shapes and the introduction of particle collisions are straightforward.

The approach shows that dark matter structures emerge from our halo shape definition and our findings are strongly dependent on the modelling prerequisites. For instance, the term “particles” refers to the smallest constituents of the system. In simulations, each particle is an entity of several sun masses, and, given the state-of-the-art quality of data acquisition, the same applies for observations. It thus remains an open question how gravity and potentially other interactions shape dark matter structures beyond our current analytical, numerical, and observational limits.

A second remaining riddle is the role of $r_{\sigma}$. Which property of dark matter does it represent? Is it the mean free path length of actually colliding dark matter particles or an auxiliary scaling parameter without meaning? Solving one mystery has entailed another. So, even in the 21st century, analysing the influence of Newton’s gravity on structure morphologies remains a challenging task.

Acknowledgements

I thank George F. R. Ellis and Carlo Rovelli for their inspiring works leading to this approach. In addition, I thank Xingzhong Er, Robert Grand, Jiaxin Han, Bettina Heinlein, Jens Hjorth, Sebastian Kapfer, Angela Lahee, Andrea Macciò, Christophe Pichon, Andrew Robertson, Björn Malte Schäfer, Johannes Schwinn, Volker Springel, Rüdiger Vaas, Gerd Wagner, and Liliya Williams for helpful comments, as well as the participants of the First Shanghai Assembly on Cosmology and Galaxy Formation 2019 for many helpful discussions and encouragement to further pursue this idea. I gratefully acknowledge the support by the Deutsche Forschungsgemeinschaft (DFG) WA3547/1-3.

Image credits for the “Smiling Lens” on the cover page: NASA & ESA
References

[1] de Blok, W.J.G., McGaugh, S.S., Bosma, A., Rubin, V.C.: Mass Density Profiles of Low Surface Brightness Galaxies. Astrophysical Journal Letters 552(1), L23–L26 (2001). DOI 10.1086/320262

[2] Hjorth, J., Williams, L.L.R.: Statistical Mechanics of Collisionless Orbits. I. Origin of Central Cusps in Dark-matter Halos. Astrophysical Journal 722(1), 851–855 (2010). DOI 10.1088/0004-637X/722/1/851

[3] Meylan, G., Jetzer, P., North, P. (ed.): Gravitational Lensing: Strong, Weak and Micro. Springer-Verlag, Berlin (2006)

[4] Navarro, J.F., Frenk, C.S., White, S.D.M.: The Structure of Cold Dark Matter Halos. Astrophysical Journal 462, 563 (1996). DOI 10.1086/177173

[5] Navarro, J.F., Frenk, C.S., White, S.D.M.: A Universal Density Profile from Hierarchical Clustering. Astrophysical Journal 490(2), 493–508 (1997). DOI 10.1086/304888

[6] Navarro, J.F., Ludlow, A., Springel, V., Wang, J., Vogelsberger, M., White, S.D.M., Jenkins, A., Frenk, C.S., Helmi, A.: The diversity and similarity of simulated cold dark matter haloes. Monthly Notices of the Royal Astronomical Society 402(1), 21–34 (2010). DOI 10.1111/j.1365-2966.2009.15878.x

[7] Rubin, V.C., Ford W. K., J., Thonnard, N.: Rotational properties of 21 SC galaxies with a large range of luminosities and radii, from NGC 4605 (R=4kpc) to UGC 2885 (R=122kpc). Astrophysical Journal 238, 471–487 (1980). DOI 10.1086/158003

[8] Salvador-Solé, E., Viñas, J., Manrique, A., Serra, S.: Theoretical dark matter halo density profile. Monthly Notices of the Royal Astronomical Society 423(3), 2190–2202 (2012). DOI 10.1111/j.1365-2966.2012.21066.x

[9] Wagner, J.: Cosmic structures from a mathematical perspective 1. Dark matter halo mass density profiles. arXiv e-prints arXiv:2002.00960 (2020)