Geodesic orbit Finsler metrics on Euclidean spaces

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Abstract

A Finsler space \((M, F)\) is called a geodesic orbit space if any geodesic of constant speed is the orbit of a one-parameter subgroup of isometries of \((M, F)\). In this paper, we study Finsler metrics on Euclidean spaces which are geodesic orbit. We will show that, in this case \((M, F)\) is a product of two factors which are both totally geodesic in the ambient manifold and they are geodesic orbit spaces themselves. One factor is a symmetric Finsler space of non-compact type, and the other is a nilmanifold whose step size is at most two. As an application, we obtain some rigidity results about non-positively curved geodesic orbit Finsler spaces.

1 Introduction

A homogeneous Riemannian or Finsler manifold is called a \textit{geodesic orbit space}, if any geodesic is the orbit of a one-parameter subgroup of isometries. The notion of a geodesic orbit space was introduced in Riemannian geometry by O. Kowalski and L. Vanhecke in 1991 \cite{15}, which is a generalization of the naturally reductive homogeneity. Geodesic orbit Riemannian manifolds have been studied rather extensively; see for example

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Meanwhile, Geodesic orbit Finsler spaces and their subclasses, for example, normal homogeneous and $\delta$-homogeneous Finsler spaces, were also studied in recent years; See \[16, 18, 19, 21, 22\].

In this paper, we study geodesic orbit Finsler spaces $(M, F)$ in the case that $M$ is diffeomorphic to a Euclidean space. We will show that $M$ is a product of two factors which are both totally geodesic in $(M, F)$ and are geodesic orbit spaces themselves. One factor is a symmetric homogeneous space of non-compact type, and the other is a nilmanifold whose step size is at most two. Our main theorem is the following.

**Theorem 1.1** Let $(M, F)$ be a geodesic orbit Finsler space such that $M$ is diffeomorphic to a Euclidean space. Then we have a decomposition $M = G_1 \times G_2 / H_1 \times H_2$ satisfying the following conditions:

1. $I_0(M, F) = G_1 \times G_2$, where $G_1$ is semisimple Lie group of non-compact type and $G_2$ is the semi-product of its maximal compact subgroup and its nilradical whose step size is at most two.

2. $H_1$ and $H_2$ are maximal compact subgroups in $G_1$ and $G_2$, respectively.

3. Both the factors are totally geodesic and are geodesic orbit Finsler spaces themselves.

We give several remarks for Theorem 1.1

1. The reason for us to assume that the smooth coset space $M$ is diffeomorphic to a Euclidean space is that in this case the manifold $M$ can be written as $M = G/H$, where $H$ is a maximal compact subgroup of $G$. Thus this can be changed to the condition that $M$ is homeomorphic or homotopic to a Euclidean space, or $M$ has some trivial topological invariants.

2. The $G_1/H_1$-factor in $M$ is referred to as $G_1/H_1 \times x_2 \subset M$ for any $x_2 \in G_2/H_2$. The submanifold metric $F|_{G_1/H_1 \times x_2}$ is $G_1$-invariant and independent of the choice of $x_2$. For the $G_2/H_2$-factor, we have a similar description. The coset space $M$ can be identified with the product of an Iwasawa subgroup in $G_1$ and the nilradical in $G_2$, such that $F$ is a left invariant Finsler metric.

3. By Theorem 3.3 in Section 3 the projection from $M$ to each $G_i/H_i$-factor can define (by submersion) a geodesic orbit metric. Generally speaking, this induced metric may be different from the submanifold metric.

Comparing Theorem 1.1 with the main theorem in [11], we see that, all the descriptions for geodesic orbit Riemannian metrics on a Euclidean spaces are valid in Finsler geometry. In this paper, we will apply the similar theory on general Lie groups as in [11].

In [11], the authors observed that the manifold $M$ in consideration may have different homogeneous presentations, and to prove their results they need to choose a suitable one. However, in this paper, we will fix $G$ to be the whole connected isometry group, and then prove Theorem 1.1 directly. This approach requires some new
technique which has been established recently by the authors. For example, Lemma 5.5, claiming that the non-compact factor $s_{nc}$ of the Levi subgroup and the radical $r$ commute, is the key step to prove Theorem 1.1. The proof in [11] does not work in Finsler context. Our approach uses Lemma 3.1 in [18] and a submersion technique (see Theorem 3.3).

Theorem 1.1 covers several special cases.

If the $G_1/H_1$-factor is trivial, then $(M, F)$ is a geodesic orbit Finsler nilmanifold. Theorem 1.3 in Section 4, which asserts that the step size is at most 2, was first given as Theorem 5.2 in [22]. However, there is a gap in the proof of [22]. Here we use another approach, which also gives a correction to the error of [22].

If the $G_2/H_2$-factor is trivial, then $(M, F)$ is a symmetric Finsler space of non-compact type, which has non-positive flag curvature. Conversely, Theorem 1.1 also tells us when a geodesic orbit Finsler space has non-positive flag curvature. In the simply connected case, we have

**Corollary 1.2** Let $(M, F)$ be a connected simply connected geodesic orbit Finsler space with non-positive flag curvature. Then there exists a subgroup $G$ of $I_0(M, F)$ acting transitively on $M$, and $M = G/H$ is a symmetric Finsler space which is the product of non-compact factors and an abelian factor.

In the non-simply connected case, we have the following interesting rigidity result:

**Corollary 1.3** A geodesic orbit Finsler space with non-positive flag curvature must be a Berwald space.

This paper is organized as follows. In Section 2, we summarize some fundamental facts on general Finsler geometry and homogeneous Finsler geometry which will be used in later discussion. In Section 3, we review the definition of a geodesic orbits Finsler space and give some fundamental results on such spaces. In Section 4, we consider the nilradical of $G$ when $G/H$ admits a $G$-geodesic orbit Finsler metric, in particular, we study the geometric properties of the geodesic orbit Finsler nilmanifolds. In Section 5, we complete the proofs of Theorem 1.1 and the corollaries.

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## 2 Preliminaries

### 2.1 Finsler metric and Minkowski norm

In this section, we give some fundamental facts about Finsler spaces. Throughout this paper, manifolds are always assumed to be connected and smooth.

A **Finsler metric** on an $n$-dimensional manifold $M$ is a continuous function $F : TM \to [0, +\infty)$, which satisfies the following properties [7]:

1. The restriction of $F$ to the slit tangent bundle $TM \setminus 0$ is a positive smooth function.
2. For any $\lambda \geq 0$, $F(x, \lambda y) = \lambda F(x, y)$. 

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(3) For any standard local coordinates \( x = (x^i) \in M \) and \( y = y^i \partial_{x^i} \in T_xM \), the Hessian matrix
\[
(g_{ij}(x, y)) = \left( \frac{1}{2} [F^2(x, y)]_{y^iy^j} \right)
\]
is positive definite.

We will call \((M, F)\) a Finsler manifold or a Finsler space. The restriction of \( F \) to a tangent space \( T_xM, x \in M \), is called a Minkowski norm. More generally, a Minkowski norm can be defined on any real linear space; See [6] for details.

Given a nonzero vector \( y \) in \( T_xM \), the Hessian matrix \((g_{ij}(x, y))\) defines an inner product \( \langle \cdot, \cdot \rangle_y^F \), such that for any \( u = u^i \partial_{x^i} \) and \( v = v^j \partial_{x^j} \) in \( T_xM \),
\[
\langle u, v \rangle_y^F = \frac{1}{2} \frac{\partial^2}{\partial r \partial s} F^2(\gamma + ru + sv)|_{r=s=0} = u^i v^j g_{ij}(x, y).
\]

For a Minkowski norm on a real vector space, the Hessian matrix \((g_{ij}(y))\) defines the inner product \( \langle \cdot, \cdot \rangle_y^F \) similarly.

2.2 Homogeneous Finsler space

A Finsler space \((M, F)\) is said to be homogeneous if its connected isometry group \( I_0(M, F) \) acts transitively. For any closed subgroup \( G \subset I_0(M, F) \) which acts transitively on \( M \), we can present \( M \) as \( M = G/H \), where \( H \) is the isotropy subgroup at \( o = eH \), and is a compact subgroup of \( G \). Denote \( \text{Lie}(H) = \mathfrak{h} \). Then an Ad\((H)\)-invariant linear decomposition \( g = \mathfrak{h} + \mathfrak{m} \) is called a reductive decomposition. The subspace \( \mathfrak{m} \) can be \( H \)-equivalently identified with the tangent space \( T_o(G/H) \), and a \( G \)-invariant metric \( F \) on \( G/H \) is completely determined by its restriction in \( T_o(G/H) \), i.e., an Ad\((H)\)-invariant Minkowski norm on \( \mathfrak{m} \) [8].

2.3 Geodesic and geodesic spray

On a Finsler space \((M, F)\), a smooth curve \( c = c(t) \) is called a geodesic if the curve \((c(t), \dot{c}(t)) \) on \( TM \) is the integration curve of the geodesic spray vector field \( \mathbf{G} \) on \( TM \setminus 0 \). With respect to a standard local coordinate system \( x = (x^i) \in M \) and \( y = y^j \partial_{x^j} \in T_xM \), the geodesic spray can be expressed as
\[
\mathbf{G} = y^i \partial_{x^i} - 2G^i \partial_{y^i},
\]
where \( G^i = \frac{1}{4}g^{il}(\{F^2\}_{x^k y^l} y^k - \{F^2\}_{x^l}) \). The equations defining the geodesic \( c = c(t) \) are
\[
\ddot{c}^i(t) + G^i(c(t), \dot{c}(t)) = 0, \quad \forall i.
\]

Notice that the notion of geodesic here implies that \( F(\dot{c}(t)) \equiv \text{const} > 0 \) [6], that is, in this paper, we will only consider geodesics of constant speed.

For a homogeneous Finsler space \((G/H, F)\), the geodesic spray \( \mathbf{G}(x, y) \) is completely determined by its value at \( o = eH \), as indicated by the following proposition.
Proposition 2.1 Let \((G/H, F)\) be a homogeneous Finsler space associated with a reductive decomposition \(\mathfrak{g} = \mathfrak{h} + \mathfrak{m}\), and \(\{u_1, \ldots, u_n\}\) be a basis of \(\mathfrak{m}\) with \([u_i, u_j]_m = c_{ij}^k u_k\). Then for \(y = y^i u_i \in T_o(G/H) = \mathfrak{m}\), we have

\[
G(o, y) = \tilde{y} - g^{il} c_{ij}^k [F^2]_{y^j} y^i \partial_{y^j},
\]

(2.1)

The vector \(\tilde{y} \in T_{(o,y)}(T(G/H))\) in Proposition 2.1 is defined as the following. Any \(y \in \mathfrak{m} \subset \mathfrak{g}\) defines a Killing vector field \(Y\) of \((G/H, F)\), and \(Y\) induces a vector field \(\tilde{Y}\) on \(T(G/H)\). Then \(\tilde{y}\) is the value of \(\tilde{Y}\) at \((o, y) \in T(G/H)\).

Proposition 2.1 is a reformulation of Theorem 3.1 in [17]. We omit the proof here.

The geodesic spray can also be determined by \(G(o, y)\) by the spray vector field \(\eta : m\backslash\{0\} \to m\) defined in [12]. Recall that, given \(y \in m\backslash\{0\}\), \(\eta(y)\) is determined by

\[
\langle \eta(y), v \rangle_y^F = \langle y, [v, y]_m \rangle_y^F, \quad \forall v \in m.
\]

With respect to the basis \(\{u_1, \ldots, u_n\}\) of \(m\), we have

\[
\eta(y) = \eta^i u_i = g^{il} c_{ij}^k g_{km} y^m y^j u_i = g^{il} c_{ij}^k [F^2]_{y^j} y^i u_i.
\]

Thus (2.1) can also be expressed as

\[
G(o, y) = \tilde{y} - \eta^i \partial_{y^i}.
\]

2.4 Totally geodesic submanifolds

An \(n\)-dimensional submanifold \(N\) of an \(m\)-dimensional Finsler space \((M, F)\) can be endowed with the induced submanifold Finsler metric \(F' = F|_N\). We call \(N\) a totally geodesic submanifold if any geodesic of \((N, F')\) is also a geodesic of \((M, F)\). By Theorem 3.1 in [17] (or Proposition 2.1), an equivalent condition for \(N\) to be totally geodesic in \((M, F)\) can be given by local tangent frames, i.e., around each point \(x \in N\) we can find local tangent frame \(X_i\) and the corresponding linear coordinates \(y = y^i X_i \in TM\), such that \(N\) is spanned by \(X_i\) with \(i \leq n = \dim N\). Moreover, at \(x\) the geodesic spray \(G(x, y) = y^i \dot{X}_i - 2G^i \partial_{y^i}\) satisfies

\[
G^i(x, y) = 0 \text{ when } y \in T_x N \text{ and } i > n.
\]

In the homogeneous case, the above observation gives the following criterion of totally geodesic homogeneous subspaces from Proposition 2.1

Lemma 2.2 Let \((G/H, F)\) be a homogeneous Finsler space with a reductive decomposition \(\mathfrak{g} = \mathfrak{h} + \mathfrak{m}\). Let \(G'\) be a closed subgroup of \(G\) whose Lie algebra \(\mathfrak{g}'\) satisfies the condition \(\mathfrak{g}' = \mathfrak{g}' \cap \mathfrak{h} + \mathfrak{g}' \cap \mathfrak{m}\). Then \(G'/G' \cap H\) is totally geodesic if and only if the spray vector field \(\eta(\cdot)\) of \(G/H\) satisfies

\[
\eta(y) \subset \mathfrak{g}' \cap \mathfrak{m}, \quad \forall y \in \mathfrak{g}' \cap \mathfrak{m}\backslash\{0\}.
\]

(2.3)

Proof. We first prove that \(G'/G' \cap H\) is totally geodesic when (2.3) is satisfied.

Notice that \(\mathfrak{g}' = \mathfrak{g}' \cap \mathfrak{h} + \mathfrak{g}' \cap \mathfrak{m}\) is a reductive decomposition for the coset space \(G'/G' \cap H\). It is easily seen that there is a basis \(\{u_1, \ldots, u_m\}\) of \(\mathfrak{m}\) such that the
elements $u_i$, $i \leq n \leq m$, span $g' \cap m$. This basis defines a local tangent frame of $G/H$ at $o \in G'/G' \cap H$. Then the assumption \ref{2.3} implies that \ref{2.2} is valid at $o$.

Since the replacement of $o$ with $g \in G'$ is just a Ad$(g)$-change for the spray vector field $\eta(\cdot)$, \ref{2.3} is still satisfied. This implies that \ref{2.2} is valid at any point $x \in G'/G' \cap H$. So $G'/G' \cap H$ is totally geodesic.

The same argument as above can also be used to prove \ref{2.3} when $G'/G' \cap H$ is totally geodesic. This completes the proof of the lemma.  

\section{Finslerian submersions}

A linear submersion $l : (V_1, F_1) \to (V, F_2)$ between two Minkowski spaces is a surjective linear map such that $l$ maps the $F_1$-unit ball in $V_1$ onto the $F_2$-unit ball in $V_2$.

Given $v_2 \in V_2$, there exists a unique $v_1 \in l^{-1}(v_2)$ such that

$$F_1(v_1) = \inf \{|F_1(v)|l(v) = v_2\}.$$  

We call $v_1$ the horizontal lift of $v_2$.

The smooth map $f : (M_1, F_1) \to (M_2, F_2)$ between two Finsler spaces is called a Finslerian submersion or simply a submersion, if for any $x_1 \in M_1$, the tangent map $f_* : (T_{x_1}M_1, F_1) \to (T_{x_2}M_2, F_2)$, where $x_2 = f(x_1)$, is a linear submersion [2].

Given a smooth map $f : M_1 \to M_2$ between two manifolds, and a Finsler metric $F_1$ on $M_1$, it is natural to ask if there exists a Finsler metric $F_2$ on $M_2$ such that $f$ is Finsler submersion. If such $F_2$ exists, we call it the induced metric defined by submersion from $F_1$ and $f$. The following lemma will be useful.

\begin{lemma}
Let $(M, F)$ be a smooth Finsler space, and $G$ a closed subgroup of $I_0(M, F)$ such that the quotient $G \setminus M$ is a smooth manifold. If the quotient map $\pi : M \to G \setminus M$ has surjective tangent maps everywhere, then there exists a unique induced metric defined by submersion from $F_1$ and $\pi$.

\end{lemma}

\begin{proof}
For any $x \in M$ and $\bar{x} \in G \setminus M$, there exists a unique Minkowski norm $F_2$ on $T_{\bar{x}}(G \setminus M)$, such that $\pi_* : (T_{\bar{x}}M_1, F_1) \to (T_{\bar{x}}(G \setminus M))$ is a linear submersion. It is clear that $F_2$ is smooth on $T_{\bar{x}}(G \setminus M)$. Therefore we only need to check that $F$ is well defined. Given $x_1$ and $x_2$ with $\bar{x} = \pi(x_1) = \pi(x_2)$, since

$$\pi_* T_{x_1}M \circ g_* T_{x_1}M = \pi_* T_{x_2}M,$$

for any $g \in G$ with $g \cdot x_1 = x_2$, the Minkowski norms at $x_1$ and $x_2$ define the same Minkowski norm at $\bar{x}$. This completes the proof of the lemma.  

A special case of Lemma 2.3 has been used in [19] and [20]. See Lemma 3.3 in [20].

\section{Geodesic orbit Finsler spaces}

Let $(M, F)$ be a Finsler space and $G$ a Lie group acting isometrically on $(M, F)$. Then we call $(M, F)$ a $G$-geodesic orbit space, if each geodesic of nonzero constant speed on $(M, F)$ is a homogeneous geodesic of $G$, i.e., it is the orbit of a one-parameter subgroup $\exp tu$ of $G$, where $u \in g = \text{Lie } G$. In the following, if the group $G$ is not specified, then it is automatically assumed that $G = I_0(M, F)$. 

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We remark here that in the above definition we do not assume the action of \( G \) on \( M \) to be transitive. However, this can be easily deduced from the connectedness of \( M \). On the other hand, it is not assumed that the \( G \)-action is effectiveness, or the closeness of the image of \( G \) in \( \mathcal{I}_0(M, F) \) is closed. However, this problem can be settled just by replacing \( G \) with the closure \( G' \) of the image \( G' \) of \( G \) in \( \mathcal{I}_0(M, F) \). It is easily seen that \((M, F)\) is also a \( G' \)-geodesic orbit space.

The following Proposition provides several equivalent definitions for a geodesic orbit Finsler space. Notice that by assuming \((M, F) = (G/H, F)\) to be a homogeneous Finsler space, we mean that \( G \) acts effectively on \((M, F)\).

**Proposition 3.1** Let \((G/H, F)\) be a homogeneous Finsler space, with a reductive decomposition \( g = h + m \), and denote \([·, ·]_m\) the \( m \)-factor in the bracket operation \([·, ·]\).

Then the following statements are equivalent:

1. \( F \) is a \( G \)-geodesic orbit metric.
2. For any \( x \in M \), and any nonzero \( y \in T_xM \), there exists a Killing vector field \( X \in g \) such that \( X(x) = y \) and \( x \) is a critical point for the function \( f(·) = F(X(·)) \).

3. For any nonzero vector \( u \in m \), there exists \( u' \in h \) such that
   \[
   \langle u, [u + u', m]_m \rangle_u = 0.
   \]  

4. The spray vector field \( \eta(·) : m \backslash \{0\} \to m \) is tangent to the \( \text{Ad}(H) \)-orbits.

For the proof, see [16].

Homogeneous totally geodesic subspace and Finslerian submersion provide important tools to study geodesic orbit spaces. Totally geodesic techniques in the Riemannian context can be naturally generalized to the Finsler situation. For example, we have the following lemma.

**Lemma 3.2** Let \((M, F)\) be a \( G \)-geodesic orbit Finsler space. For any subset \( L \) of isometries fixing \( x \in M \), we denote \( \text{Fix}_x(L) \) the connected component of the fixed point set \( \text{Fix}(L) \) of \( L \) containing \( x \), and \( C^0_G(L) \) the identity component of the centralizer \( C_G(L) \) of \( L \) in \( G \). Then \((\text{Fix}_x(L), F|_{\text{Fix}_x(L)})\) is a \( C^0_G(L) \)-geodesic orbit Finsler space.

**Proof.** Without loss of generality, we can assume that \( L \) is a compact Lie group. Then the submanifold \( \text{Fix}_x(L) \) is totally geodesic. We will prove that \( F|_{\text{Fix}_x(L)} \) is a \( C^0_G(L) \)-geodesic orbit metric.

Consider a geodesic \( \gamma \) of \((\text{Fix}_x(L), F|_{\text{Fix}_x(L)})\). Then \( \gamma \) is also a geodesic of \((M, F)\). By the geodesic orbit property of \((M, F)\), there exists a Killing vector field \( X \) such that \( X \) is defined by some vector \( v \in g \), and \( \gamma \) is an integration curve of \( X \). Since \( L \) is a compact Lie group, the average

\[
X' = \int_L \text{Ad}(g)Xd\text{Vol}/\text{Vol}(L),
\]

where \( d\text{Vol} \) is a fixed bi-invariant volume form, is also a Killing vector field of \((M, F)\), which is defined by

\[
v' = \int_L \text{Ad}(g)v d\text{Vol}/\text{Vol}(L).
\]
Restricted to the geodesic $\gamma$, $X'$ coincides with $X$, i.e., $\gamma$ is an integration curve of $X'$. It is also easy to see that

$$v' \in \mathfrak{c}_0(L) = \text{Lie}(C^0_G(L)).$$

So $(\text{Fix}_x(L), F|_{\text{Fix}_x(L)})$ is a $C^0_G(L)$-geodesic orbit Finsler space. ■

On the other hand, many submersion techniques for geodesic orbit Riemannian manifolds do not work in the Finsler context. However, we still have the following:

**Theorem 3.3** Let $(G/H, F)$ be a $G$-geodesic orbit Finsler space such that the $G$-action is effective. Suppose $H_1$ is a closed normal subgroup of $G$, and $H_2$ is the maximal normal subgroup of $G$ contained in $H_1$. Denote $G' = G/H_2$ and $H' = H_1H/H_2$. Then we have

1. $G'$ acts effectively on $G'/H'$.
2. There exists a unique $G'$-invariant metric $F'$ on $G'/H'$ defined by submersion from the metric $F$ and the natural projection from $G/H$ to

$$G/H_1H = (G/H_2)/(H_1H/H_2) = G'/H'.$$

3. $(G'/H', F')$ is a $G'$-geodesic orbit Finsler space.

**Proof.** Since $H_1$ is closed and normal in $G$, and $H$ is compact, the product $H_1H$ is a closed subgroup of $G$. The largest normal subgroup $H_2$ of $G$ contained in $H_1H$ coincides with $\cap_{g \in G} gH_1Hg^{-1} = \cap_{g \in G} H_1gHg^{-1}$. So $H_2$ contains $H_1$, and $H_2$ is closed in $G$. The subgroup $H_2$ consists of all the elements $g \in G$ which acts as the identity map on $G/H_1H$. Thus $G' = G/H_2$ acts effectively on $G/H_1H = (G/H_2)/(H_1H/H_2) = G'/H'$. This proves (1).

To prove (2), we first note that, since $H_1$ is a normal subgroup of $G$, we have $gH_1H = H_1gh$ for any $g \in G$. Then $G'/H'$ can be identified with $H_1\setminus G/H$, the orbit space of left $H_1$-actions on $G/H$. Meanwhile, the quotient map for $H_1\setminus G/H$, $\pi_1: G/H \to G_1\setminus G/H = G'/H'$ coincides with the projection map $\pi : G/H \to G/H_1H = G'/H'$. By Lemma 2.3 a unique Finsler metric $F'$ on $H_1\setminus G/H$ is well defined by submersion from $F$ and $\pi'$. It is the metric indicated in (2) of the lemma. The uniqueness of $F'$ is obvious.

On the other hand, the $G$-action on $G/H$ permutes the fibers of the projection $\pi$. It naturally induces a $G$-action on $G/H_1H = G'/H'$ which becomes effective through $G'$. By the uniqueness of $F'$, the $G'$-actions are isometries on $(G/H, F')$. This proves (2).

Finally, for any geodesic $\gamma'$ in $(G'/H', F')$, its horizontal lift $\gamma$ in $(G/H, F)$ is a geodesic in $(G/H, F)$ [2]. By the definition of geodesic orbit spaces, $\gamma$ is the orbit of a one-parameter subgroup, i.e., there exists a nonzero element $u \in \mathfrak{g}$ and $x \in G/H$ such that $\gamma(t) = \exp tu \cdot x$. Then it is obvious that $\gamma'$ is the orbit of $\exp tu'$, where $u'$ is the image of $u$ in $\mathfrak{g}' = \mathfrak{g}/h_2$, where $\mathfrak{g}'$ and $h_2$ are the Lie algebras of $G'$ and $H_2$, respectively. Therefore $(G'/H', F')$ is a $G'$-geodesic orbit space, which proves (3). ■

At the end of this section, we quote Lemma 3.1 in [18], which is an important technique in this paper.
Lemma 3.4 Let \((G/H, F)\) be a \(G\)-geodesic orbit Finsler space where \(G\) acts effectively on \(G/H\), and \(a\) be an abelian ideal of \(g = \text{Lie}(G)\) which has a trivial intersection with \(h = \text{Lie}(H)\). Then each vector in \(a\) defines a Killing vector field of constant length on \((G/H, F)\).

4 Geodesic orbit Finsler nilmanifold

We first recall some fundamental notions in general theory of Lie groups and Lie algebras.

The radical \(\text{Rad}(G)\) and the nil-radical \(\text{Nil}(G)\) of a connected Lie group \(G\) are the connected maximal solvable and nilpotent subgroups of \(G\), respectively. They are the unique closed subgroups, generated by the maximal solvable and nilpotent ideals of \(g\), respectively (see Definition 16.2.1 and Proposition 16.2.2 in [14]). We denote \(\text{rad}(g) = \text{Lie}(\text{Rad}(G))\) and \(\text{nil}(g) = \text{Lie}(\text{Nil}(G))\), and call them the radical and nil-radical of \(g\) [14].

In the case that \(G\) acts effectively, isometrically and transitively on a Finsler manifold, we have the following useful lemma.

Lemma 4.1 Let \((M, F)\) be a homogeneous Finsler space, and \(G\) a closed connected subgroup in \(I_0(M, F)\) which acts transitively on \(M\). Then \(\text{nil}(g)\) has zero intersection with the Lie algebra of any isotropy subgroup of \(G\).

Proof. Let \(H\) be the compact isotropy subgroup of \(G\) at a point \(x \in M\). Denote \(n = \text{nil}(g)\). Obviously \(\text{ad}(u) : g \to g\) is semisimple if \(u \in h\), and nilpotent if \(u \in n\). So \(h \cap n \subset c(g)\). Since \(G \subset I_0(M, F)\), its action on \(M\) is effective. Thus

\[ h \cap n = h \cap c(g) = 0, \]

which proves the lemma. ■

The following result is a generalization of a theorem of C. Gordon in [10].

Theorem 4.2 Let \((M, F)\) be a \(G\)-geodesic orbit Finsler space, where \(G\) is a closed connected subgroup of \(I_0(M, F)\). Then the step-size of the nil-radical \(\text{nil}(g)\) is at most two.

Proof. First write the manifold \(M\) as \(M = G/H\), where \(H\) is a compact subgroup of \(G\). Denote \(h = \text{Lie}(H)\) and \(n = \text{nil}(g)\). Then by Lemma 4.1 there is a reductive decomposition \(g = h + m\) such that \(n \subset m\).

Assume conversely that the step-size of \(n\) is \(m > 2\), i.e., its descending central series

\[ C^1(n) = n \text{ and } C^k(n) = [n, C^{k-1}(n)] \subset C^{k-1}(n), \]

satisfies \(C^m(n) \neq 0 \) and \(C^{m+1}(n) = 0\). Then it is easy to see that each \(C^k(n)\) is an ideal of \(g\) contained in \(m\). In particular, \(n' = C^{m-1}(n)\) is abelian with \(\dim n' > 1\). By Lemma 3.4 any nonzero vector \(u\) in \(n'\) defines a Killing vector field of constant length on \((M, F)\), that is,

\[ F(\text{pr}_m(\text{Ad}(g)u)) \equiv \text{const} \quad (4.5) \]
for \( g \in G \). Since \( n' \) is an ideal of \( g \), \( \text{Ad}(G)u \subset n' \subset m \). Setting \( g = \exp tv \) with \( v \in n \) in (4.5), and taking the differentiation at \( t = 0 \), we get

\[
\langle u, [u, v] \rangle_u^F = 0, \quad \forall u \in n', \ v \in n.
\] (4.6)

Now we claim that any \( v \in n \) commutes with \( n' \). Assume conversely that this is not true. Then \( \text{ad}(v)|_{n'} : n' \to n' \) is a nonzero nilpotent linear map. Meanwhile, on the linear space \( \text{End}(n') \), we can define two norms as the following. The first one, denoted as \( \| \cdot \|_1 \), is the \( l^2 \)-norm, namely, with respect to a fixed basis of \( n' \), any \( A \in \text{End}(n') \) corresponds to a matrix \((a_{ij})\), and

\[
\|A\|_1^2 = \sum_{i,j} a_{ij}^2.
\]

The second one, denoted as \( \| \cdot \|_2 \), is induced by the norm \( F|_{n'} \), i.e., for \( A \in \text{End}(n') \),

\[
\|A\|_2 = \sup\{F(Au)|u \in n', F(u) = 1\}.
\]

Then there exists a basis of \( n' \), such that the matrix of \( \text{ad}(v)|_{n'} \in \text{End}(n') \) under this basis is a nonzero Jordan form with zero diagonal entries. Now a direct calculation shows that

\[
\lim_{t \to \infty} \|\exp(t\text{ad}(v))|_{n'}\|_1 = +\infty.
\]

On the other hand, (4.6) implies that

\[
\|\exp(t\text{ad}(v))|_{n'}\|_2 = 1, \quad \forall t.
\]

This is a contradiction, since the norms \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) must be equivalent, i.e., there exist positive constants \( c_1 \) and \( c_2 \), such that \( c_1\| \cdot \|_1 \leq \| \cdot \|_2 \leq c_2\| \cdot \|_1 \).

To summarize, we have proved that

\[
C^m(n) = [n, n'] = 0,
\]

which is a contradiction with the assumption that the step size of \( n \) is \( m \). This completes the proof of the theorem. \( \blacksquare \)

As a corollary, we have the following theorem.

**Theorem 4.3** Let \( N \) be a nilpotent Lie group which admits a left invariant geodesic orbit Finsler metric. Then the step size of \( N \) is at most two.

**Proof.** Let \( F \) be a left invariant geodesic orbit Finsler metric on the nilpotent Lie group \( N \), \( G \) the connected isometry group \( I_0(N, F) \), and \( H \) the isotropy subgroup of \( G \) at \( e \in N \). Then \( H \) is a compact subgroup, \( N \) is a normal subgroup of \( G \), and \( G \) is the semi-product of \( N \) and \( H \). Obviously \( n = \text{Lie}(N) \) is a nilpotent ideal of \( g = \text{Lie}(G) \), so we have \( n \subset \text{nil}(g) \). By Lemma 4.1, \( \text{nil}(g) \cap h = 0 \). Thus \( n = \text{nil}(g) \). By Theorem 4.2, the step size of \( n \) is at most two. \( \blacksquare \)
5 Geodesic orbit Finsler metric on Euclidean spaces

In this section we will complete the proof of Theorem 1.1.

We first consider the case that \((M, F)\) is a \(G\)-geodesic orbit Finsler space diffeomorphic to a Euclidean space, where \(G\) is a closed connected subgroup of \(I_0(M, F)\). Then \(G\) acts on \(M\) transitively and effectively, and the isotropy subgroup \(H\) of \(G\) at a fixed \(x \in M\) is compact.

We assert that \(H\) is a maximal compact subgroup of \(G\). Assume conversely that \(H\) is not maximal. Then there is a maximal compact subgroup \(K\) of \(G\) such that \(H \subset K\). By the first manifold splitting theorem (Theorem 14.3.8 in [14]), \(G/K\) is also diffeomorphic to a Euclidean space, and \(K\) is connected. Then the assumption that \(H\) is not maximal implies that \(\dim K - \dim H = r > 0\). Since \(G/H\) is the total space of a \(K/H\)-bundle over \(G/K\), \(G/H\) is homotopic to \(K/H\). So we have

\[ H_r(G/H; \mathbb{Z}_2) = H_r(K/H; \mathbb{Z}_2) \neq 0, \]

which is a contradiction to the assumption that \(G/H\) is diffeomorphic to a Euclidean space.

Let \(B_g(\cdot, \cdot)\) be the Killing form of \(g\). Since \(G\) acts effectively on \(M = G/H\), the restriction \(B_g_{\vert h \times h}\) is non-degenerate. So we have a reductive decomposition \(g = h + m\) such that \(m\) is the \(B_g\)-orthogonal complement of \(h\).

Let \(g = r + s\) be the Levi decomposition of \(g\), i.e., \(r = \text{rad}(g)\) is the radical, and \(s = \text{lev}(g)\) is a Levi subalgebra of \(g\). Notice that the nilradical \(n = \text{nil}(g)\) is contained in \(r\).

Since all the Levi subalgebras of \(g\) are conjugate under the \(\text{Ad}(G)\)-action, we can choose a suitable Levi subalgebra \(s\) such that \(s \cap h\) is a maximal compact subalgebra of \(s\). Denote \(s_c\) and \(s_{nc}\) the compact and non-compact part of \(s\), respectively. Then we have the linear direct sum decomposition

\[ h = s_c \oplus (s_{nc} \cap h) \oplus (r \cap h). \]  \hspace{1cm} (5.7)

We now prove

**Lemma 5.1** The right side of (5.7) is also a direct sum of ideals.

**Proof.** To prove this, we only need to prove the bracket between any two different factors of (5.7) vanishes. Obviously, \([s_c, s_{nc} \cap h] = 0\). The brackets of the other two factors are all contained in

\[ [s, r \cap h] \subset [g, r] \subset n. \]

By Lemma 4.1, \(h \cap n = 0\). Thus all the brackets between different factors vanish.  

As a preparation for the first key steps, we also need to mention the following fundamental fact.

**Lemma 5.2** The nil-radical \(n\) is contained in the annihilation subspace of \(B_g\), thus \(n \subset m\).
Proof. To prove $B_\mathfrak{g}(n, \mathfrak{g}) = 0$, we only need to show

$$\begin{align*}
B_\mathfrak{g}(n, \mathfrak{s}) &= 0, \\
B_\mathfrak{g}(n, \mathfrak{r}) &= 0.
\end{align*}$$

(5.8) (5.9)

Since the Levi subalgebra $\mathfrak{s} = \text{lev}(\mathfrak{g})$ satisfies $[\mathfrak{s}, \mathfrak{s}] = \mathfrak{s}$, and the radical $\mathfrak{r} = \text{rad}(\mathfrak{g})$ satisfies $B_\mathfrak{g}([\mathfrak{g}, \mathfrak{g}], \mathfrak{r}) = 0$, (5.8) follows.

On the other hand, denote $\mathfrak{r} = \mathfrak{r} \otimes \mathbb{C}$, and $B_\mathfrak{r}(\cdot, \cdot)$ and $B_\mathfrak{r}^C(\cdot, \cdot)$ the Killing form of $\mathfrak{r}$ and $\mathfrak{r}^C$, respectively. Since $\mathfrak{r}$ is an ideal of $\mathfrak{g}$, we have

$$B_\mathfrak{g}(n, \mathfrak{r}) = B_\mathfrak{r}(n, \mathfrak{r}) = B_\mathfrak{r}^C(n, \mathfrak{r}).$$

By Lie's theorem, there exists a basis of $\mathfrak{r}^C$ such that $\text{ad}(u) : \mathfrak{r}^C \rightarrow \mathfrak{r}^C$ for any $u \in \mathfrak{r}$ corresponds to a upper triangular matrix. Furthermore, if $u \in n$, $\text{ad}(u) : \mathfrak{r}^C \rightarrow \mathfrak{r}^C$ is nilpotent, then the corresponding matrix is strictly upper triangular. Then a direct calculation shows that

$$B_\mathfrak{g}(n, \mathfrak{r}) = B_\mathfrak{r}^C(n, \mathfrak{r}) = 0,$$

which proves (5.9).

This proves the first statement, and the second one follows easily.

Now we turn to the proof of the main results of this paper. The following lemma is the first key step of our proof.

Lemma 5.3 Keeping all the assumptions and notations as above, we have $[\mathfrak{s}_{nc}, \mathfrak{r}] = 0$, and the Lie algebra decomposition

$$\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{s}_{nc},$$

(5.10)

where $\mathfrak{g}' = \mathfrak{s}_c + \mathfrak{r}$.

Proof. We only need to prove the first statement, because the second follows immediately.

We prove the lemma by induction on $\dim \mathfrak{r}$. When $\dim \mathfrak{r} = 0$, the lemma is obviously true. Assuming the lemma is valid for $\dim \mathfrak{r} < k$ with $k > 0$, we will prove that it is also true for $\dim \mathfrak{r} = k$.

Let $\mathfrak{a}$ be the maximal Abelian ideal of $\mathfrak{g}$. Obviously $\mathfrak{a}$ is contained in $\mathfrak{n}$. It is the Lie algebra of a connected Abelian normal subgroup $A$ of $G$. We claim $A$ is closed.

Notice that $A$ is contained in $N = \text{Nil}(G)$. Because $N$ is a closed subgroup of $G$, the closure $\bar{A}$ of $A$ is also connected Abelian normal subgroup of $G$ contained in $N$. Thus its Lie algebra must coincide with $A$ because of our assumptions for $\mathfrak{a}$, i.e. we have $A = \bar{A}$ is a closed subgroup of $G$.

By Lemma 5.2 we have $\mathfrak{a} \subset \mathfrak{n} \subset \mathfrak{m}$. Since $\mathfrak{a}$ contains $C^m(n) \neq 0$, where $m$ is the step size of $\mathfrak{n}$, we have $\mathfrak{a} \neq 0$.

By Lemma 3.4 each vector in $\mathfrak{a}$ defines a Killing vector field of constant length. So we have

$$\langle u, [u, v] \rangle^F_u = 0, \quad \forall u, v \in \mathfrak{s}_{nc}.$$  

The adjoint representation defines a Lie algebra endomorphism from $\mathfrak{s}_{nc}$ to the Lie algebra $\mathfrak{f}$ of the compact Lie group $SO(a, F|_a)$. If this endomorphism is not zero, then
there is a simple subalgebra \( s' \) of non-compact type in \( \mathfrak{k} \). The restriction of any bi-invariant inner product on \( \mathfrak{k} \) to \( \mathfrak{s}' \) coincides with the Killing form of \( \mathfrak{s}' \) up to a scalar. But this is a contradiction, since the Killing form of \( \mathfrak{s}' \) is indefinite. Therefore we have

\[
[\mathfrak{s}_{nc}, \mathfrak{a}] = 0. \tag{5.11}
\]

Now we apply Theorem 5.3 to the left \( A \)-action on \( M = G/H \). It defines by submersion a homogeneous Finsler metric \( F' \) on the quotient space \( M' = A \backslash G/H = G/AH \). Let \( A' \) be the maximal normal subgroup of \( G \) contained in \( AH \). Then \( A \subset A' \subset AH \), and \( G' = G/A' \) acts transitively on \( M' = G/AH = (G/A')/(AH/A') = G'/H' \). Moreover, \( (M', F') \) is a \( G' \)-geodesic orbit space.

The connected Abelian normal subgroup \( A \) cannot have a torus factor, otherwise \( H \) is not maximal compact. So the coset space \( M = G/H \) is a fiber bundle over \( M' = G'/H' \) such that each fiber is diffeomorphic to \( A \cong \mathfrak{a} \). Both the total space and the fibers are topologically trivial (i.e. all homology groups are trivial), so does the base manifold \( M' \). It follows that \( H' = AH/A' \) must be a maximal compact subgroup of \( G' \). Then by the first manifold splitting theorem, \( M' \) is also diffeomorphic to a Euclidean space.

Consider the Lie algebra \( g' \) of \( \text{Lie}(G') \). There is an induced Levi decomposition \( g' = r' + s' \), such that

\[
\mathfrak{h}' = \text{Lie}(H') = (\mathfrak{h} + \mathfrak{a})/\mathfrak{a}' = \mathfrak{h}/\mathfrak{a}' \cap \mathfrak{h}
\]

contains a maximal compact subalgebra of the Levi subalgebra \( \mathfrak{s}' \), and for the radical we have

\[
r' = (r + \mathfrak{a'})/\mathfrak{a}' = r/\mathfrak{a}.
\]

The compact factor \( s'_c \), and non-compact factor \( s'_{nc} \) of \( s' \) can be similarly described. In particular, the non-compact factor \( s'_{nc} = s_{nc} + \mathfrak{a}'/\mathfrak{a}' = s_{nc}/s_{nc} \cap \mathfrak{a}' \) is isomorphic to \( s_{nc} \), since \( s_{nc} \cap \mathfrak{a}' \) is an ideal of \( s_{nc} \) contained in the compact subalgebra \( s_{nc} \cap \mathfrak{h} \), and thus \( s_{nc} \cap \mathfrak{a}' = 0 \).

To summarize, \( (M', F') \) is a \( G' \)-geodesic orbit Finsler space diffeomorphic to a Euclidean space, and the dimension of the radical \( r' \) is smaller than \( k \). So by the inductive assumption, we get \( [s'_{nc}, r'] = 0 \), which implies that \( [s_{nc}, r] \subset \mathfrak{a}' \). On the other hand, since \( [s_{nc}, r] \subset \mathfrak{n} \), we also have

\[
[s_{nc}, r] \subset \mathfrak{a}' \cap \mathfrak{n} = \mathfrak{a}. \tag{5.12}
\]

Combining \( \tag{5.11} \) with \( \tag{5.12} \), we see that the adjoint representation of \( s_{nc} \) on \( r \) defines a Lie algebra endomorphism from \( s_{nc} \) to a nilpotent Lie algebra. This endomorphism must be zero. Thus \( s_{nc} \) commutes with \( r \), when \( \dim r = k \). By induction, the assertion is valid for all dimensions. This completes the proof of the first statement. The second follows easily. \( \square \)

Let \( \mathfrak{p} \) be the orthogonal complement of \( s_{nc} \cap \mathfrak{h} \) in \( s_{nc} \) with respect to the Killing form of \( s_{nc} \). The following lemma is a direct corollary of Lemma 5.3.

**Lemma 5.4** Keep all the notations and assumptions as above. Then \( \mathfrak{p} \subset \mathfrak{m} \).

We now go to the second key step. In the following lemma, we fix \( G = I_0(M, F) \).
Lemma 5.5  Keep all the notations and assumptions as above. Then we have a linear direct sum decomposition \( m = p + n \).

Proof. Let \( m' \) be the \( B_g \)-orthogonal complement of \( p \) in \( m \). Then \( m' \) is an \( Ad(H) \)-invariant subspace contained in the \( B_g \)-orthogonal complement of \( s \).

On the other hand, \( r \) is the \( B_g \)-orthogonal complement of \([g, g] = s + [g, r] \), and \([g, r] \subset n \) is contained in the annihilation of \( B_g \). So the \( B_g \)-orthogonal complement of \( s \) coincides with \( r \).

By Lemma 5.2 and Lemma 5.3, we have \( n \subset m' \). Thus

\[ n \subset m' \subset r. \]

To prove the lemma, we only need to prove \( m' = n \). Assume conversely that \( m' \neq n \). Considering the \( ad(h) \)-actions from \( m' \) to \( m' \), we have a linear direct sum decomposition

\[ m' = c(h) \cap m' + [h, m'], \]

in which the second term at the right side is contained in \([h, r] \subset n \). So the assumption \( m' \neq n \) indicates that there exists a vector \( v \in m' \setminus n \) commuting with \( h \).

Since \([v, h] = 0\), we have

\[ B_g([v, m], h) = B_g(m, [v, h]) = 0, \]

i.e., \([v, m] \subset m \). Denote \( g_t \) the one-parameter subgroup in \( G \) generated by \( v \). We assert that for any \( t \in \mathbb{R} \), the map \( Ad(g_t) : m \to m \) preserves the Minkowski norm \( F \) on \( m \). In fact, for any nonzero vector \( u \in m \), there exists \( u' \in h \), such that

\[ \langle u, [u + u', v] \rangle^F = \langle u, [u + u', v] \rangle^F = 0. \]

Since \([u', v] \in [h, v] = 0\), we have

\[ \langle u, [u, v] \rangle^F = \langle u, [u + u', v] \rangle^F = 0, \quad \forall u \in m, u \neq 0. \]

Thus \( g_t = \exp tv \) preserves \( F|_m \).

By this claim and some standard argument in homogeneous geometry, the diffeomorphisms \( \rho_t = L_{g_t} \circ R_{g_t} \) on \( M = G/H \), \( \rho_t(gH) = g_t g \rho_t^{-1} H \), provide a one-parameter subgroup in \( G = I_0(M, F) \). Since each \( \rho_t \) fixes \( eH \in M \), i.e., \( \rho_t \in H \), there exists a vector \( u \in h \), such that \( ad(u)|_m = ad(v)|_m \), i.e., \([u - v, m] = 0\). On the other hand, we must have \([u - v, h] \neq 0\), since otherwise \( u - v \in c(g) \subset n \), but the assumption that \( v \in m' \setminus n \) implies that \( h \), \( n \) and \( \mathbb{R}v \) are linearly independent.

Now it is easy to check by induction that the nonzero element \([u - v, h] = [u, h] \) generates a nonzero ideal

\[ h_0 = [u, h] + [[u, h], h] + [[[u, h], h], h] + \cdots \]

of \( h \) which commutes with \( m \). So \( h_0 \) is a nonzero ideal of \( g \) contained in \( h \) which only provides zero Killing vector fields on \( M \). This is a contradiction with the effectiveness of the action of \( G = I_0(M, F) \) on \( M \).

To summarize, we have proved that \( m' = n \). Thus \( m = p + n \) is a direct decomposition. ■
Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We keep all the notations and assumptions of this section, and fix $G = I_0(M, F)$. By Lemma 5.3 and Lemma 5.5 we have Lie algebra direct sum decompositions

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

and

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2,$$

where $\mathfrak{g}_1 = s_{nc}$, $\mathfrak{g}_2 = s_c + \tau$, $\mathfrak{h}_1 = \mathfrak{h} \cap s_{nc}$, and $\mathfrak{h}_2 = s_c + \tau \cap \mathfrak{h}$.

On the other hand, we also have the reductive decompositions

$$\mathfrak{g}_1 = \mathfrak{h}_1 + \mathfrak{m}_1,$$

and

$$\mathfrak{g}_2 = \mathfrak{h}_2 + \mathfrak{m}_2,$$

where $\mathfrak{m}_1 = \mathfrak{p}$, and $\mathfrak{m}_2 = \mathfrak{n}$.

Let $G_1$ and $G_2$ be the connected subgroups of $G$ corresponding to $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively. We now show that $I_0(M, F) = G_1 G_2 \cong G_1 \times G_2$. For this we only need to prove that $G_1 \cap G_2 = \{e\}$. Notice that each $G_2$-orbit in $M$ is a symmetric coset space of non-compact type. So for any $g \in G_1 \cap G_2$, $g$ is contained in $C(G_2)$, whose action restricted to each $G_2$-orbit is the identity map. Since $G_2 \subset I_0(M, F)$ acts effectively on $M$, $g \in G_1 \cap G_2$ must be the unit $e$. Hence $I_0(M, F) = G_1 \times G_2$.

The above argument shows that $M$ can be written as the coset space $M = G/H$, where $H$ is a maximal connected compact subgroup, with Lie algebra $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$. Therefore $H$ is also the product of two maximal compact connected subgroups $H_1$ and $H_2$ in $G_1$ and $G_2$, respectively.

By the effectiveness of the $G$-action on $M$, the intersection between $H_2 \subset H$ and the nil-radical $N = \text{Nil}(G) = \text{Nil}(G_2)$ is $\{e\}$. So by the reductive decomposition of $\mathfrak{g}_2$, $G_2$ is a semi-product of $H_2$ and $N$. By Theorem 4.2 the step size of the nilradical is at most two. This completes the proof of (1) and (2) of Theorem 1.1.

Now we prove (3). By Proposition 3.1 for any non-zero $u \in \mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2$, there exists a vector $v = v_1 + v_2 \in \mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$, such that $\eta(u) = [u, v]$. In particular, if $u \in \mathfrak{m}_2$, then we have

$$\eta(u) = [u, v_1 + v_2] = [u, v_2] \in \mathfrak{m}_2,$$

and

$$\langle \eta(u), v \rangle^F_u = \langle u, [v, u]_{\mathfrak{m}_2} \rangle^F_u, \quad \forall v \in \mathfrak{m}_2.$$

Now we make two observations based on the above argument. First, the restriction of the spray vector field $\eta(\cdot)$ to $\mathfrak{m}_2$ coincides with the spray vector field of $(o_1 \times G_2/H_2, F|_{o_1 \times G_2/H_2})$, where $o_1 = eH_1 \subset G_1/H_1$. Thus by Lemma 2.2, the Finsler submanifold $(o_1 \times G_2/H_2, F|_{o_1 \times G_2/H_2})$ is totally geodesic in $(M, F)$. Second, the spray vector field of $(o_1 \times G_2/H_2, F|_{o_1 \times G_2/H_2})$, which coincides with the restriction of $\eta(\cdot)$ to $\mathfrak{m}_2$, is tangent to $\text{Ad}(H_2)$-orbits in $\mathfrak{m}_2$. Then by Proposition 3.1 $(o_1 \times G_2/H_2, F|_{o_1 \times G_2/H_2})$ is a geodesic orbit Finsler space.

To summarize, we have shown that $(o_1 \times G_2/H_2, F|_{o_1 \times G_2/H_2})$ is totally geodesic in $(M, F)$ and it is a $G_2$-geodesic orbit Finsler space. By the homogeneity, the similar statement is valid for any other point $x_1 \in G_1/H_1$. 

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The same argument as above can also be used to prove that \((G_1/H_1 \times x_2, F|_{G_1/H_1 \times x_2})\) is totally geodesic in \((M, F)\) and it is also a \(G_1\)-geodesic orbit Finsler space. However, we will give a more direct argument for this case. Since \(G_1/H_1 \times o_2\) is a connected component of the fixed point set of \(G_2 \cap H\), i.e., \(G_1/H_1 \times o_2 = \text{Fix}_e(G_2 \cap H)\), where \(o = o_1 \times o_2 = eH\), it is totally geodesic in \((M, F)\). Notice that \(G_2 = C^0_e(G_1)\). So the \(G_2\)-geodesic orbit property for \(G_1/H_1 \times o_2\) follows Lemma 3.2 easily. This completes the proof of (3) of Theorem 1.1.

**Proof of Corollary 1.2** Since \(M\) is simply connected with non-positive flag curvature, by the Cartan-Hadamard theorem, \((M, F)\) is diffeomorphic to a Euclidean space. Therefore we can write \(M\) as \(G/H\), where \(H\) is a maximal compact connected subgroup of \(G\).

By Theorem 1.1, we have \(I_0(M, F) = G_1 \times G_2\) and \(M = G_1/H_1 \times G_2/H_2\), such that \(G_1/H_1\) is a symmetric homogeneous space of non-compact type and \((G_2/H_2, F|_{G_2/H_2})\) is a geodesic orbit space, which is totally geodesic in \((M, F)\). The quotient space \(G_2/H_2\) can be identified with the nil-radical \(N\) of \(G_2\), such that for any \(x \in G_1/H_1\), \(F|_{x \times G_2/H_2}\) defines a left invariant Finsler metric on \(N\). The subgroup \(N\) must be Abelian, otherwise by [13], the flag curvature of \((x \times G_2/H_2, F|_{x \times G_2/H_2})\) is positive somewhere. Moreover, the subgroup \(G' = G_1 \times N \subset I_0(M, F)\) acts transitively on \(M\), and \(M = G'/H'\) with \(H' = H_1 \times \{e\}\) can be associated with a Cartan decomposition \(g' = h' + m\) satisfying \([m, m] \subset h'\). Then \(M\) is a product \(M = G_1/H_1 \times N\), where \(G_1/H_1\) is the product of irreducible symmetric homogeneous spaces of non-compact type, and \(N\) is an abelian factor.

This ends the proof of Corollary 1.2.

**Proof of Corollary 1.3** Let \((M, F)\) be a non-positively curved \(G\)-geodesic orbit Finsler space. Then the universal covering \(\tilde{M}\) of \(M\), endowed with the metric \(\tilde{F}\) induced by \(F\), is also non-positively curved. Moreover, it is a geodesic orbit Finsler space with respect to the universal covering \(\tilde{G}\) of \(G\). By Corollary 1.2, \(\tilde{M}\) is a symmetric Finsler space. Hence the invariant Finsler metric \(\tilde{F}\) on \(\tilde{M}\) is a Berwald metric [8]. Since \(\tilde{F}\) and \(F\) are locally isometric, \(F\) is also a Berwald metric.

This ends the proof of Corollary 1.3.

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