Renormalization-group Method for Reduction of Evolution Equations; invariant manifolds and envelopes

SHIN-ICHIRO EI, KAZUYUKI FUJII and TEIJI KUNIHIRO

1 Graduate School of Integrated Science, Yokohama City University, Yokohama 236-0027 Japan,
2 Department of Mathematical Sciences, Yokohama City University, Yokohama 236-0027 Japan,
3 Faculty of Science and Technology, Ryukoku University, Otsu, 520-2194 Japan

Abstract

The renormalization group (RG) method as a powerful tool for reduction of evolution equations is formulated in terms of the notion of invariant manifolds. We start with derivation of an exact RG equation which is analogous to the Wilsonian RG equations in statistical physics and quantum field theory. It is clarified that the perturbative RG method constructs invariant manifolds successively as the initial value of evolution equations, thereby the meaning to set $t_0 = t$ is naturally understood where $t_0$ is the arbitrary initial time. We show that the integral constants in the unperturbative solution constitutes natural coordinates of the invariant manifold when the linear operator $A$ in the evolution equation is semi-simple, i.e., diagonalizable; when $A$ is not semi-simple and has a Jordan cell, a slight modification is necessary because the dimension of the invariant manifold is increased by the perturbation. The RG equation determines the slow motion of the would-be integral constants in the unperturbative solution on the invariant manifold. We present the mechanical procedure to construct the perturbative solutions hence the initial values with which the RG equation gives meaningful results. The underlying structure of the reduction by the RG method as formulated in the present work turns out to completely fit to the universal one elucidated by Kuramoto some years ago. We indicate that the reduction procedure of evolution equations has a good correspondence with the renormalization procedure in quantum field theory; the counterpart of the universal structure of reduction elucidated by Kuramoto may be the Polchinski’s theorem for renormalizable field theories. We apply the method to interface dynamics such as kink-anti-kink and soliton-soliton interactions in the latter of which a linear operator having a Jordan-cell structure appears.

67 pages including the first two pages; no figures, no tables.
The Proposed running head:
Renormalization-group Method
Correspondence should be sent to
Prof. T. Kunihiro,
Faculty of Science and Technology, Ryukoku University,
Otsu, 520-2194 Japan
Tel. (+81)-77-543-7501
Fax. (+81)-77-543-7524
e-mail kunihiro@rins.ryukoku.ac.jp
1 Introduction

There is an ever growing interest in the renormalization groups (RG) \[1, 2, 3\] in various fields of science and mathematical physics since the work of Wilson\[4, 5, 6\]. The essence of the RG in quantum field theory (QFT) and statistical physics may be stated as follows: Let $\Gamma(\phi, g(\Lambda), \Lambda)$ be the effective action (or thermodynamical potential) obtained by integration of the field variable with the energy scale down to $\Lambda$ from infinity or a very large cutoff $\Lambda_0$. Here $g(\Lambda)$ is a collection of the coupling constants including the wave-function renormalization constant defined at the energy scale at $\Lambda$. Then the RG equation may be expressed as a simple fact that the effective action as a functional of the field variable $\phi$ should be the same, irrespective to how much the integration of the field variable is achieved, i.e.,

$$\Gamma(\phi, g(\Lambda), \Lambda) = \Gamma(\phi, g(\Lambda'), \Lambda').$$

(1.1)

If we take the limit $\Lambda' \to \Lambda$, we have

$$\frac{d\Gamma(\phi, g(\Lambda), \Lambda)}{d\Lambda} = 0,$$

(1.2)

which is the Wilson RG equation\[1\], or the flow equation in the Wegner’s terminology \[3\]; notice that Eq.(1.2) is rewritten as

$$\frac{\partial \Gamma}{\partial g} \cdot \frac{dg}{d\Lambda} = -\frac{\partial \Gamma}{\partial \Lambda}.$$  

(1.3)

If the number of the coupling constants is finite, the theory is called renormalizable. In this case, the functional space of the theory does not change in the flow given by the variation of $\Lambda$; one may say that the flow has an invariant manifold.

A notable aspect of the RG is that the RG equation gives a systematic tool for obtaining the infrared effective theories with fewer degrees of freedom than in the original Lagrangian relevant in the high-energy region. This is a kind of reduction of the dynamics. Finding effective degrees of freedom and extract the reduced dynamics of the effective variables in fact have constituted and still constitute the core of various fields of theoretical physics. In QCD written in terms of the fields of quarks and gluons, the low energy effective theories in which gluons are integrated out may be Nambu-Jona-Lasinio type lagrangians \[7\]; see for example \[8\]. If the quarks are further integrated out the effective theories are sigma models written solely with meson fields\[9\]. The flow equations, variants of the RG equation, may give a foundation on such sigma models as effective theories of QCD at low energies\[10, 11, 12\]; for a review, see \[13\]: In terms of notions in the theory of dynamical systems, the functional space represented by a sigma model may be an attractive submanifold of QCD.

Statistical physics may be said to be a collection of theories on how to reduce the dynamics of many-body systems to one with fewer variables, since the work of Boltzmann\[14\].
Bogoliubov showed that BBGKY (Bogoliubov-Born-Green-Kirkwood-Yvon) hierarchy can be reduced to Boltzmann equation with a single-particle distribution function for dilute gas systems. As indicated by Kuramoto, Bogoliubov seems to have claimed that the dilute-gas dynamics has an attractive manifold spanned by one-particle distribution function. Boltzmann equation in turn can be further reduced to the hydrodynamic equation (Navier-Stokes equation) by a perturbation theory like Chapman-Enskog method. Recent development of the theories of pattern formation with dissipative structures gives a good example how to reduce complicated ordinary and partial differential equations to simple equations with slow variables, such as Landau-Stuart equation, the time-dependent Ginzburg-Landau equation and so on. 

Some years ago, it was shown by an Illinois group and Bricmont and Kupiainen that the RG equations can be used for a global and asymptotic analysis of ordinary and partial differential equations, hence giving a reduction of evolution equations of some types. A unique feature of the Illinois group’s method is to start with the naive perturbative expansion and allow secular terms to appear; the secular terms correspond to the logarithmically divergent terms in QFT. Then introducing an intermediate time \( \tau \) and rewriting perturbative solutions in terms of renormalization constants reminiscent of those appearing in the perturbative renormalization theory in QFT, Gell-Mann-Low type RG equation is applied to obtain the evolution equations for the renormalization constants which are functions of \( \tau \). Finally, equating \( \tau \) with the time \( t \) appearing in the original perturbative solution, they obtained global solutions of differential equations. Bricmont and Kupiainen applied a scaling transformation (block transformation) to obtain asymptotic behavior of nonlinear diffusion equations in a rigorous manner.

Subsequently, one of the present authors (T.K.) formulated the Illinois group’s method in terms of the classical theory of envelopes: He indicated that the RG equation can be interpreted as the basic equation for constructing envelopes of a family of curves (or surfaces for partial differential equations). He also developed a short-cut prescription for the renormalization procedure without introducing an intermediate time \( \tau \) but utilizing an arbitrary initial time \( t_0 \). In latest papers, it was shown that the RG method can be well formulated as the method dealing with the initial values at arbitrary \( t = t_0 \); the initial values at \( t = t_0 \) are determined so that the unperturbative solutions which are valid only locally around \( t = t_0 \) are continued smoothly; this procedure is nothing but to construct the envelope of the perturbative solutions. He emphasized that the RG method is a powerful tool for reduction of evolution equations and demonstrated it by applying the method to obtain so-called the amplitude equations for systems of equations. He also suggested that if the phase equations describing slow motions in the system where a continuous symmetry is broken, which are classical counterpart of the Nambu-Goldstone modes in QFT, the RG method should be able to derive the phase equations.

The RG method developed by the Illinois group has been applied by many authors to quite a wide class of problems successfully. To mention some of them; Graham derived a rotationally invariant amplitude equation appearing in the problem of pattern formation. Some kinds of phase equations were also derived; Oono discussed the
interface dynamics relevant in the spinodal decomposition, Sasa derived a diffusion type phase equation, and Maruo et al. [31] derived Kuramoto-Sivashinsky equation [32]. The method was applied to analyze asymptotic behavior of the non-linear equations appearing in cosmology [33, 34]. Boyanovsky and de Vega also used the method to derive an anomalous transport coefficient [35] relevant in the non-equilibrium states in the early universe and QGP (quark-gluon plasma). Before that, the Boltzmann equation had been derived as an RG equation [36]. The RG method was also shown to be a powerful tool to resum divergent perturbation series appearing in problems of quantum mechanics [37, 27]. Tzenov applied the method to obtain global solutions appearing accelerator physics [38]. Possible relation between renormalizability and integrability of Hamilton systems was discussed by Yamaguchi and Nambu [39]. The method was proved to be applicable to discrete systems [40], too.

Although such extensive applications have been made, only few works are known which attempt to reveal the underlying reasons why the RG method works to some kinds of equations but not to others [25]. Nevertheless, it was indicated in [28, 40] that the RG method by the Illinois group works when the unperturbed solutions are neutrally stable solutions that are stationary (constant in time or stationary oscillation) hence do not decay nor blow up with time. A decade ago, Kuramoto revealed in an excellent paper [16] the universal underlying structure of all the existing perturbative methods for reduction of evolution equations; he noticed that when a reduction of evolution equation is possible, the unperturbed equation admits neutrally stable solutions, and succeeded in describing the reduction of dynamics in geometrical terms, i.e., attractive manifolds or invariant manifolds [24]. Although his actual presentation of the theory was based on the reductive perturbation theory and involves some ansatz on forms of the solutions, he emphasized that the universal structure which he revealed should not be dependent on the perturbation methods one employs. In the present paper, focusing on the aspect of the RG method as a powerful tool for reducing evolution equations, we shall present a comprehensive formulation of the perturbative RG method in terms of the notion of invariant manifolds, guided by this Kuramoto's work: One will see that his ansatz are derived naturally in the RG method. It may mean that his formulation of reduction of evolution equations, which is actually a natural extension of the asymptotic method by Krylov, Bogoliubov and Mitropolski for non-linear oscillators [11], is an RG theory although the term RG is not used [39].

We start with adapting the exact RG equations of Wilson type (flow equations) in quantum field theory [4, 5] to differential equations (evolution equations); one may recognize that the RG method described in terms of envelopes is best formulated in the framework of Wilson RG. Then confining ourselves to cases where a perturbative treatment is possible, we shall show that an invariant manifold exists when the RG method works, and that the RG method is a method to construct the invariant manifold and the reduced dynamics on it in a mechanical way. In this method, the initial values of the solution at an arbitrary time \( t = t_0 \) in the successive perturbation orders are determined so that the initial value make an invariant manifold, thereby the condition to set \( t_0 = t \) will be naturally emerged. We emphasize that the present formulation give a nice foundation
for the prescriptions adopted in [25, 26, 27].

The following should be mentioned here: (1) The relevance of Wilson type RG equation to their method was noted by Chen et al. [22] and Pashkov and Oono [36], although their formulation was totally based on the Gell-Mann-Low type perturbative RG method. (2) Shirkov [3] clarified that the RG equation concerns with the initial values and emphasized the Lie group structure of the RG method [42]; he extracted the notion of functional self-similarity (FSS) as the essence of the exact RG. He claims that the Wilson RG is an approximation to the Bogoliubov RG which is exact [3].

Once the underlying structure of the reduction of dynamics given by the RG method has become clear, one will recognize that the method could be applied to problems which have not been treated in the RG method so far although other methods are applied to them. One will also see that the RG method is simpler to apply to them than the previous methods. As such a problem, we take the problem to extract the interface interactions [43, 44, 45, 46], which are typical examples to be treated by the method of phase equations.

In §2, we formulate the RG method as a method of reduction of dynamics, starting from the non-perturbative flow equation (RG equation). Then on the basis of the perturbation theory, we show, guided by the presentation given in [16], the way how the invariant manifold which is supposed to exist to the evolution equation under consideration can be constructed in our method. We shall remark that the existence of an invariant manifold corresponds to the notion of the renormalizability in quantum field theory. In §3, some simple but typical examples including the Takens equation [49] are worked out to demonstrate how the RG method construct invariant manifolds and give the reduced dynamics on them. In §4, we show how to deal with generic systems which involve a linear operator $A$ having zero-eigenvalue where $A$ may or may not be diagonalizable; when $A$ is not diagonalizable, the eigenvalues are degenerate and $A$ is equivalent with a matrix having a Jordan cell $(0 1 0 0)$. As examples with a Jordan cell, we shall show that the RG equation gives the normal forms [50] of the two-dimensional equations including the Takens and the Bogdanov equation [51], and deal with an extended version of Takens equation with three-degrees of freedom. In §5, we apply the method to some problems such as the unstable motion in the Lotka-Volterra system and the Hopf bifurcation in the Brusselator [52]. Although the examples treated in §2-5 are simple ordinary differential equations, we believe that these examples should be instructive also for experts of the RG’s or flow equations in quantum field theory and/or statistical physics. In §6, we also apply the method to extract the interface dynamics of a kink-anti-kink interaction in the TDGL equation in one-dimension and the soliton-soliton interaction in the KdV equation. The final section is devoted to a brief summary and concluding remarks. In Appendix A, we summarize rules in a scheme of an operator method for constructing special solutions suitable for the RG method. In Appendix B, we present an elementary method different from the RG method to derive the approximate solution for the double-well potential discussed in subsection 3.3. In Appendix C, we show that the period of the Lotka-Volterra
system around the non-trivial fixed point obtained previously in the RG method coincides with that extracted by Frame in a quite different way.
2 Reduction of evolution equations with the RG method

In this section, we shall formulate the RG method in terms of the notion of invariant manifold.

2.1 The RG equation as the flow equation for initial values

2.1.1 Non-perturbative RG equation

Let us take the following $n$-dimensional dynamical system;
\[
\frac{dX}{dt} = F(X, t),
\]
where $n$ may be infinity. We remark that $F$ may depend on $t$ explicitly. We suppose that the equation is solved up to an arbitrary time $t = t_0$ from an initial time, say, at $t = 0$, to give $X(t) = W(t)$, and then we are trying to solve the equation with the initial condition at $t = \forall t_0$,
\[
X(t = t_0) = W(t_0),
\]
with $W(t_0)$ being unspecified yet. In fact, $W(t)$ as a function of $t$ is a solution of (2.1) by definition. The solution may be written as $X(t; t_0, W(t_0))$. We stress that the solution needs not be given by a perturbation method; if the solution is given non-perturbatively, the resultant equations remain non-perturbative ones.

Now, making use of $X(t; t_0, W(t_0))$ thus obtained, we determine $W(t_0)$ based on a simple fact of differential equations. When the initial point is shifted to $t'_0$, the resultant solution should be the same, i.e.,
\[
X(t; t_0, W(t_0)) = X(t; t'_0, W(t'_0)).
\]
Taking the limit $t'_0 \to t_0$, we have
\[
\frac{dX}{dt_0} = \frac{\partial X}{\partial t_0} + \frac{\partial X}{\partial W} \frac{dW}{dt_0} = 0.
\]
This equation gives the evolution equation or the flow equation of the initial value $W(t_0)$. This has the same form as that of the renormalization group (RG) equation in quantum field theory, hence the name of the RG method. We emphasize that the equation (2.4) is exact; we have not used any argument based on perturbation theories. This equation corresponds to the non-perturbative RG equations (flow equations) by Wilson[4], Wegner-Houghton[5] and so on in quantum field theory and statistical physics[13]: The reader should have recognized that $t_0$ corresponds to the logarithm of the energy scale to which
the integration is performed in the quantum field theory. The quantities corresponding to the coupling constants will be found to be the integration constants. One will also recognize that the existence of an invariant manifold of a dynamical system corresponds to the renormalizability. We also notice that one needs not to equate $t_0$ with $t$ at this stage.

2.1.2 Perturbative RG equation

So far, only the perturbative expansion method is available to make $X(t; t_0, W(t_0))$. In this case, $X(t; t_0, W(t_0))$ and $X(t; t'_0, W(t'_0))$ may be valid only for $t \sim t_0$ and $t \sim t'_0$. This condition is naturally satisfied if we restrict that $t_0 < t < t'_0$ (or $t'_0 < t < t_0$) because the limit $t'_0 \rightarrow t_0$ is to be taken. Thus when a perturbative expansion is used for constructing $X(t; t_0, W(t_0))$, we demand a more restrictive RG equation as given by

$$
\frac{dX}{dt_0} \bigg|_{t_0=t} = \frac{\partial X}{\partial t_0} \bigg|_{t_0=t} + \frac{\partial X}{\partial W} \frac{dW}{dt_0} \bigg|_{t_0=t} = 0. \tag{2.5}
$$

Notice that the demand to set $t_0 = t$ has naturally emerged.

We remark that a geometrical interpretation of this equation has been given on the basis of the classical theory of envelopes. When $t_0$ is varied, $X(t; t_0, W(t_0))$ gives a family of curves with $t_0$ being a parameter characterizing curves. Then Eq.(2.5) is the condition to construct the envelope of the family of curves which are valid only locally around $t \sim t_0$. The envelope is given by $X(t; t_0 = t) = W(t)$, i.e, the initial value. It is noteworthy that $W(t)$ satisfies the original equation (2.1) in a global domain up to the order with which $X(t; t_0)$ satisfies around $t \sim t_0$. In fact, one can see that

$$
\frac{dW}{dt} = \left. \frac{\partial X(t; t_0)}{\partial t} \right|_{t_0=t} + \left. \frac{\partial X(t; t_0)}{\partial t_0} \right|_{t_0=t}, \tag{2.6}
$$
on account of (2.5).

2.2 Invariant manifolds and renormalizability

In this section, we follow for notations. If the theory has an invariant manifold $M$ with the dimension less than $n$, we may have supposed that the initial point is on the manifold. Let the invariant manifold $M$ be represented by the coordinate $s$. The reduced dynamics of Eq.(2.1) on $M$ may be given in terms of a vector field $G$ by

$$
\frac{ds}{dt} = G(s), \tag{2.7}
$$
and the manifold \( M \) is represented by
\[
X = R(s).
\] (2.8)

Our task is to obtain the vector field \( G \) and the representation of the manifold \( R \) in a perturbation method. We consider a situation where the vector field \( F \) is composed of an unperturbed part \( F_0 \) and the perturbative one \( P \), i.e.,
\[
F = F_0(X) + \epsilon \cdot P(X, t).
\] (2.9)

Here notice that \( F_0 \) has no explicit \( t \)-dependence, while \( P(X, t) \) does. We assume that the unperturbed problem is solved and an attractive invariant manifold \( M_0 \) is easily found.

Now we try to solve Eqs. (2.1) and (2.9) by a perturbation theory with the initial condition
\[
X(t_0) = W(t_0),
\] (2.10)

at \( t = t_0 \). The decisive point of our method is to assume that
\[
W(t_0) = R(s(t_0)),
\] (2.11)

that is, the initial point is supposed to be on the invariant manifold \( M \) to be determined. Now we apply the perturbation theory, expanding
\[
X(t; t_0, W(t_0)) = X_0 + \sum \epsilon^n X_n(t; t_0, W(t_0)).
\] (2.12)

Here we have made it explicit that \( X \) is dependent on the initial condition. We should also expand the initial value,
\[
W(t_0) = W_0(t_0) + \rho(t_0),
\] (2.13)

with
\[
\rho(t_0) = \sum_{n=1}^{\infty} \epsilon^n W_n(t_0).
\] (2.14)

Now the unperturbed equation reads
\[
\frac{dX_0}{dt} = F_0(X_0).
\] (2.15)

As promised, we suppose that an attractive manifold is found for this equation as
\[
X_0(t) = R(s(t; C)),
\] (2.16)

where \( C \) is the integral constant vector with \( \dim C = m \) and may depend on \( t_0 \).
Here comes an important point of our method; we identify that

\[ s(t_0) = C(t_0), \]  

which gives a natural parameterization of the manifold \( M_0 \). This is a simple but a significant observation; notice that we need not to give any ansatz to the representation of the manifold because we only have to solve the unperturbed equation and the integral constants are trivially obtained.

The deformation of the manifold \( \rho \) will be determined perturbatively on the two principles, i.e.,

1. the function \( \rho \) should be independent of \( W_0 \), and
2. the resultant dynamics should be as simple as possible because we are interested to reduce the dynamics to a simpler one.

The choice of the \( \rho \) is intimately related to that of the forms of the perturbative solutions. For example, the first order equation reads

\[ \frac{dX_1}{dt} = F'_0(X_0)X_1 + P(X_0). \]  

The solution to this inhomogeneous equation is composed of a sum of the general solution of the homogeneous equation and the special solution of the inhomogeneous equation. If the unperturbed solution \( X_0(t) \) has a part of neutrally stable solution, there appear secular terms in the special solution as well as genuinely independent functions. It turns out that the secular terms can be utilized to renormalize out the homogeneous solutions at \( t = t_0 \) \cite{25,26}; this is a kind of renormalization conditions. Then the shift of the initial value is now determined as

\[ W_1(t_0) = X_1(t = t_0). \]  

Notice that the initial value is determined after solving the equation, hence the functional form of it as a function of \( t_0 \) is explicitly given. Of course, one can make \( W_1(t_0) = 0 \) by further adding unperturbed solutions; this prescription, however, will generally give a more complicated dynamics. It means that the choice of \( \rho \) has ambiguities, but the demand to obtain the simplest dynamics give it uniquely. In §3 and 4, we shall give a mathematical and mechanical procedure to select the initial values in accordance with the above rules using some examples.

We can repeat the procedure to any order of the perturbation. We remark that by this procedure the modification of the initial value \( \rho(t_0) \) and hence the total initial value \( W(t_0) \) are given solely in terms of \( C(t_0) \);

\[ W(t_0) = W_0[\mathcal{C}] + \rho[\mathcal{C}]. \]  

(2.20)
Now the dynamics of $C(t)$ is determined by the RG equation Eq. (2.5):

$$
\begin{align*}
\frac{dX}{dt_0} \bigg|_{t_0=t} &= \frac{\partial X}{\partial t_0} \bigg|_{t_0=t} + \frac{\partial X}{\partial C} \cdot \frac{dC}{dt_0} \bigg|_{t_0=t} = 0,
\end{align*}
$$

which is an evolution equation of $C(t)$. Then the manifold $M$ (more precisely, the trajectory on it) is represented as

$$
X = W(t) = W_0[C(t)] + \rho[C].
$$

Eq.’s (2.21) and (2.22) are essentially the basic postulates in Kuramoto’s theory of reduction of evolution equations\([16]\). Thus we have derived the Kuramoto’s basic equations in the RG method; in other words, Kuramoto’s theory is actually the RG theory for reduction of evolution equations. Eq. (2.22) means that the dynamics is renormalized to that for $C$. One can now see a correspondence between the renormalizability of the theory in quantum field theory\([9]\) and the existence of a finite dimensional invariant manifold in the theory of dynamical systems: $C = (C_1, C_2, \ldots, C_m)$ correspond to the collection of renormalizable coupling constants and $\rho$ to unrenormalizable operators; one may say that the fact that $\rho$ can be represented solely with $C$ is analogue of the Polchinski theorem\([6, 9]\) in quantum field theory. We remark also that Eq. (2.22) justifies the slaving principle by Haken\([56]\).

A comment is in order; When the unperturbed system is given by a linear operator with a Jordan cell, there will be a slight modification of the above scenario due to a technical complexity; see an example in the next section and the general argument given in §4.

### 3 Simple examples

In this section, we consider four simple equations to show our formulation of the RG method at work as a tool for reduction of evolution equations. We shall show how the initial values at $t = \forall t_0$ in higher orders are determined by the two principles that terms proportional to the unperturbed solution are suppressed and that possible fast motions disappear. The resultant forms of the unperturbed solutions composed of the secular terms which vanish at $t = t_0$ and the functions independent of the unperturbed solution. The final forms turn out to be the same as those given in the scheme adopted in \([25, 26, 27]\), where unstable manifolds and cases with a Jordan cell were not dealt with though. The resultant initial value represent the invariant manifold, and the RG equation gives the slow dynamics on the manifold.
3.1 A simple model with a reduction

We first consider the following simplest equation with a reduction [16]:

\[ \frac{dx}{dt} = \epsilon f(x, y), \quad \frac{dy}{dt} = -y + g(x). \]  

(3.1)

Writing \( u(x, y) = t(x, y) \) and we expanding \( u \) as \( u = u_0 + \epsilon u_1 + \cdots \), with \( u_n(x, y) = t(x_n, y_n) \) \((n = 0, 1, \ldots)\). We solve the equation with the initial value \( W(t_0) \) at \( t = t_0 \). We suppose that \( W(t_0) \) is on an attractive manifold \( M \).

The unperturbed equation reads

\[ \dot{x}_0 = 0, \quad \dot{y}_0 = -y_0 + g(x_0), \]  

(3.2)

the solution to which is readily obtained as \( x_0(t) = \text{const.} = C_0, \quad y_0(t) = g(C_0) + C_1 e^{-t} \), with \( C_i \) \((i = 0, 1)\) being the integral constants; we make it explicit that \( C_i \) may depend on the initial time \( t_0 \) as \( C_i = C_i(t_0) \). Since we are interested in the asymptotic behavior as \( t \to \infty \), we take the stationary solution putting \( C_1 = 0; \)

\[ x_0(t) = \text{const.} = C_0, \quad y_0(t) = g(C_0), \]  

(3.3)

accordingly,

\[ W_0(t_0) = t(C_0, g(C_0)), \]  

(3.4)

which suggests that the unperturbed invariant manifold \( M_0 \) is given by

\[ y = g(x), \]  

(3.5)

although the time dependence \( x(t) \) is not yet known.

The first order equation reads

\[ (\partial_t - A) \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} f(x_0, y_0) \\ 0 \end{pmatrix}, \]  

(3.6)

with

\[ A = \begin{pmatrix} 0 & 0 \\ g'(x_0) & -1 \end{pmatrix}. \]  

(3.7)

We notice that

\[ A U_1 = 0, \quad A U_2 = (-1) U_2; \quad U_1 = \begin{pmatrix} 1 \\ g' \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]  

(3.8)

Then the special solution with the initial value \( W_1(t_0) \) is obtained as follows:

\[ \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = e^{(t-t_0)A} W_1(t_0) + \int_{t_0}^t ds e^{(t-s)A} \begin{pmatrix} f(x_0, y_0) \\ 0 \end{pmatrix}, \]  

\[ = e^{(t-t_0)A} \left[ W_1(t_0) + f g' U_2 \right] + \{(t - t_0) f U_1 - f g' U_2 \}. \]  

(3.9)
Here we have used the relation that \( t'(1,0) = U_1 - g'U_2 \). For the solution to describe a slow motion, the first term should vanish; thus we are naturally led to the choice of the initial value as

\[
W_1(t_0) = -fg'U_2,
\]

(3.10)
to kill the fast motion. Thus we have also

\[
x_1(t; t_0) = f(C_0, g(C_0))(t - t_0), \quad y_1(t; t_0) = f(C_0, g(C_0))g'(C_0)(t - t_0) - 1.
\]

(3.11)

We notice that the solution and the initial value satisfy the rules given in the preceding subsection; \( W_1(t_0) \approx \rho(t_0) \) is independent of \( W_0(t_0) \).

Up to this order,

\[
u(t; t_0) = \begin{pmatrix} x(t; t_0) \\ y(t; t_0) \end{pmatrix} = \begin{pmatrix} C_0(t_0) + \epsilon f(C_0, g(C_0))(t - t_0) \\ g(C_0) - \epsilon g'(C_0)f(C_0, g(C_0))(1 + t_0 - t) \end{pmatrix},
\]

(3.12)

with

\[
W(t_0) = \begin{pmatrix} x(t_0) \\ y(t_0) \end{pmatrix} \simeq W_0(t_0) + \rho(t_0) = \begin{pmatrix} C_0 \\ g(C_0) - \epsilon g'(C_0)f(C_0, g(C_0)) \end{pmatrix}.
\]

(3.13)

Now the RG equation Eq.(2.5) applied to Eq.(3.12) gives the evolution equation for \( C_0(t) \);

\[
\frac{dC_0}{dt} = \epsilon f(C_0, g(C_0)).
\]

(3.14)

Since \( x(t) = C_0(t) \), Eq.(3.14) gives the reduced dynamics, and the slow manifold is given by \( u(t) = W(t) \), or in terms of the components,

\[
x(t) = C_0(t), \quad y(t) = g(C_0) - \epsilon g'(C_0)f(C_0, g(C_0)).
\]

(3.15)

One sees that the attractive manifold is given by

\[
y(x) = g(x) - \epsilon g'(x)f(x, g(x)),
\]

(3.16)

and the original two-dimensional evolution equation has been reduced to the one-dimensional equation Eq.(3.14).

In short, to obtain the invariant manifold and the slow dynamics on it, we have started with a neutrally stable solution with an integral constant, which constitutes a natural representation of the invariant manifold. The constant moves slowly by the perturbation and the whole dynamics is given through this slow variable. The evolution equation of the slow dynamics is given by the RG equation and the trajectory on the invariant manifold is given as the initial value in the RG method uniquely. The initial value in the higher order has been determined so that the fast motion disappears.

Generic systems which have a linear matrix having zero eigenvalue will be extensively analyzed in §4.
3.2 When the unperturbed solution is oscillatory

In the present subsection, we shall examine a case where the unperturbed solution has another type of neutral stability, i.e., is oscillatory. We shall treat a simplest equation of a damped oscillator with the geometrical terms putting an emphasis on the fact that the RG method concerns with the initial value. In a recent monograph, Nishiura [57] used this example to indicate that a careful identification of the initial values in the higher order terms is needed to construct the proper perturbative solutions for obtaining a meaningful result in the RG method, though he failed to give general principles for “a careful identification”. One will see that the general principles given in the foreword in this section determine the initial values uniquely, hence Nishiura’s concern is resolved. One will also see that the initial values thus obtained are of the same forms as obtained in [25].

The equation we deal with is

\[ \ddot{x} + \epsilon \dot{x} + x = 0, \]  \hspace{1cm} (3.17)

where \( 0 < \epsilon < 1 \). This system does not exhibit a decrease of the degrees of freedom, nevertheless the dynamics is reduced to a set of simpler equations for the amplitude and the phase, separately by the RG equation.

With the definition \( u = t(x, y) \), \( y = \dot{x} \), Eq. (3.17) is converted to

\[ (\partial_t - A)u = -\epsilon \begin{pmatrix} 0 \\ y \end{pmatrix}, \]  \hspace{1cm} (3.18)

with

\[ A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]  \hspace{1cm} (3.19)

We expand the dependent variable and the initial value in Taylor series as \( u(t) = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots \) and \( W(t_0) = W_0 + \epsilon W_1 + \epsilon^2 W_2 + \cdots \), with \( u_i = t(x_i, y_i) \). The lowest order solution reads with the (complex) integral constant \( C(t_0) \);

\[ u_0(t; t_0) = C e^{it} U_+ + C^* e^{-it} U_- \equiv \begin{pmatrix} x_0(t; t_0) \\ y_0(t; t_0) \end{pmatrix}, \]  \hspace{1cm} (3.20)

where \( U_\pm = t(1, \pm i) \); \( AU_\pm = \pm i U_\pm \). This is a neutrally stable solution. The initial value in this order reads,

\[ W_0(t_0) = z(t_0) U_+ + \text{c.c.}, \]  \hspace{1cm} (3.21)

with

\[ z(t_0) = C(t_0) e^{i\theta_0}. \]  \hspace{1cm} (3.22)
Here c.c. denotes the complex conjugate.

The first order equation reads
\[
(\partial_t - A)u_1 = -y_0(t) \frac{1}{2i}(U_+ - U_-),
\]  
(3.23)
with the initial condition \( u_1(t_0, t_0) = W_1(t_0) \) which is not yet known but to be determined. The equation is readily solved as follows;
\[
u_1(t; t_0) = e^{(t-t_0)A}W_1(t_0) - \frac{1}{2i} \int_{t_0}^{t} dse^{(t-s)A}y_0(s)(U_+ - U_-),
\]
\[
= e^{(t-t_0)A} \left[ W_1(t_0) - \frac{1}{2i} \{ C^* e^{-it_0}U_+ - Ce^{it_0}U_- \} \right] 
- \frac{1}{2} \left\{ (t - t_0)Ce^{it}U_+ - \frac{1}{2i}Ce^{it}U_- \} + c.c. \right\},
\]
(3.24)
which leads to the natural choice of the initial value
\[
W_1(t_0) = \frac{1}{2i} \{ z^*(t_0)U_+ - z(t_0)U_- \} = \rho_1[z, z^*],
\]
(3.25)
because otherwise the first term of\((3.24)\) gives rise to terms which could be renormalized away into the unperturbed solution with a redefinition of \(C\). We emphasize that a renormalization procedure enters here. Hence
\[
u_1(t; t_0) = -\frac{1}{2} \{ (t - t_0)C(t_0)e^{it}U_+ - \frac{1}{2i}C(t_0)e^{it}U_- \} + c.c. \equiv \left( x_1(t; t_0) \right). \]
(3.26)
Similarly, the second order solution is given by
\[
u_2(t; t_0) = e^{(t-t_0)A}W_2(t_0) - \frac{1}{2i} \int_{t_0}^{t} dse^{(t-s)A}y_1(s)(U_+ - U_-),
\]
\[
= e^{(t-t_0)A} \left[ W_2(t_0) - \left\{ -\frac{C}{16}e^{it_0}U_- + c.c. \right\} \right] 
+ \frac{iC}{8} \{ \{ (t - t_0)^2 + (t - t_0) \} U_+ + \{ t - t_0 + i/2 \} U_- \} e^{it} + c.c. \]
(3.27)
Thus we are led to the choice of the initial value
\[
W_2(t_0) = -\frac{1}{16}z(t_0)U_ - + c.c. = \rho_2[z, z^*],
\]
(3.28)
because of the same reason as in the first-order case. Hence
\[
u_2(t; t_0) = \frac{iC}{8} \{ \{ (t - t_0)^2 + (t - t_0) \} U_+ + \{ t - t_0 + i/2 \} U_- \} e^{it} + c.c. \]
(3.29)
Collecting all the terms up to the second order, we have
\[
u(t; t_0) = Ce^{it}U_+ - \frac{eC}{2} \{ (t - t_0)U_+ + \frac{i}{2}U_- \} e^{it},
\]
\[+ \frac{i e^2 C}{8} \{ \{ i(t - t_0)^2 + (t - t_0) \} U_+ + \{ t - t_0 + i/2 \} U_- \} e^{it} + c.c., \]
(3.30)
with the initial value
\[ W(t_0) = z(t_0)\{U_+ - i\frac{\epsilon}{4} U_+ - \frac{\epsilon^2}{16} U_-\} + \text{c.c.}. \] (3.31)

The RG equation Eq.(2.21) gives
\[ \dot{C} + \left(\frac{\epsilon}{2} + i\frac{\epsilon^2}{8}\right)C = 0. \] (3.32)

Parameterizing \( C \) as \( C = (A/2) \cdot \exp(i\theta) \), we have the equations governing the amplitude and the phase, respectively,
\[ \dot{A} + \frac{\epsilon}{2} \cdot A = 0, \quad \dot{\theta} = -\frac{\epsilon^2}{8}, \] (3.33)

which yields \( A(t) = A_0 e^{-\epsilon/2 \cdot t} \) and \( \theta(t) = -\epsilon^2 t/8 + \theta_0 \) with \( A_0 \) and \( \theta_0 \) being constants. Since \( u(t) = W(t) \), we have the final solution to the damped oscillator as
\[ x(t) = A_0 e^{-\epsilon/2 \cdot t}\{ (1 - \frac{\epsilon^2}{16}) \cos(\omega t + \theta_0) + \frac{\epsilon}{4} \sin(\omega t + \theta_0) \}, \] (3.34)

with \( \omega = 1 - \epsilon^2/8 \) being the angular velocity. The above expression is slightly different from that given in [25] where the equation is treated as a scalar equation; a redefinition of the constants \( A_0 \) and \( \theta_0 \) transforms the solutions to each other in this order. We see that although the number of the dimension of the equation is not changed, the dynamics is reduced to simpler equations (3.33) for the amplitude and the phase. The initial values in the higher order equations have been determined so that terms proportional to the unperturbed solution do not appear; such higher order terms have been "renormalized away" by a redefinition of the integral constant in the unperturbed solution.

A comment is in order: The above solution could be more efficiently obtained by using the operator method (implicitly) adopted in [25, 26] and fully accounted in Appendix A of the present article. For instance, the first order solution in the operator method reads
\[ u_1(t; t_0) = -\frac{1}{2}(\partial_t - A)^{-1}(Ce^{it} - C^*e^{-it})(U_+ - U_-), \] (3.35)

which is readily evaluated with the use of (A. 9) and (A. 10). The second order solution is also readily obtained with the use of the formulae given in Appendix A.

### 3.3 Unstable motion in the double-well potential

Next, we show that unstable manifolds are also constructed by the present method, using the double-well potential in mechanics;
\[ \ddot{x} = x - \epsilon x^3. \] (3.36)
Let us obtain the unstable motion around the origin by applying the RG method. This example is a peculiar case where the unperturbed solution is not neutrally stable but composed of a blowing and decaying function with the same exponent. We shall treat another example of this type in §5.2.

Putting \( \dot{x} = y \) and defining \( u = \tau(x, y) \), we have a system of equation
\[
\left( \frac{d}{dt} - A \right) u = \epsilon \left( \begin{array}{c} 0 \\ -x^3 \end{array} \right).
\tag{3.37}
\]

We solve the equation around an arbitrary \( t = t_0 \) with the initial value \( W(t_0) \). The solution is written as \( u = u(t; t_0, W(t_0)) \). We expand \( u = u_0 + \epsilon u_1 + \cdots \). Accordingly the initial value \( W(t_0) \) which is to be determined self-consistently is also expanded as \( W = W_0 + \epsilon W_1 + \cdots \).

The lowest order solution reads
\[
u_0(t; t_0) = C_+(t_0)e^{t_0}U_+ + C_-(t_0)e^{-t_0}U_-,
\tag{3.38}
\]
where we have suppressed the \( W \)-dependence of \( u_0 \) and \( U_\pm = \tau(1, \pm 1) \) are the eigenvectors of \( A \) belonging to the eigenvalues \( \pm 1 \), respectively. The initial value \( W_0(t_0) \) accordingly reads
\[
W_0(t_0) = u_0(t_0; t_0) = C_+(t_0)e^{t_0}U_+ + C_-(t_0)e^{-t_0}U_-,
\tag{3.39}
\]
which will imply that the trajectory is in a hyperbolic curve
\[
M_0 = \{ u = (x, y)|(x + y)(x - y) = \text{const} \} \tag{3.40}
\]
with the asymptotic lines \( y = \pm x \).

The first order equation reads
\[
\left( \frac{d}{dt} - A \right) u_1 = \frac{1}{2}(C_+(t_0)e^t + C_-(t_0)e^{-t})^3(U_+ - U_-),
\tag{3.41}
\]
which is solved with the initial value \( W_1(t_0) \) formally as
\[
u_1(t; t_0) = e^{(t-t_0)A}W_1(t_0)
- \frac{1}{2} \int_{t_0}^{t} ds e^{sA}(C_+(t_0)e^s + C_-(t_0)e^{-s})^3(U_+ - U_-),
= e^{(t-t_0)A}[W_1(t_0) + \frac{1}{2}(f_+(t_0; t_0)U_+ + f_-(t_0; t_0)U_-)]
- \frac{1}{2}(f_+(t; t_0)U_+ + f_-(t; t_0)U_-),
\tag{3.42}
\]
with
\[
f_+(t; t_0) = 3C_+^2C_-(t - t_0)e^t - \frac{3}{2}C_+ C_+^2 e^{-t} + \frac{1}{2}C_+^3 e^{3t} - \frac{1}{4}C_+^3 e^{-3t},
\tag{3.43}
\]
\[
f_-(t; t_0) = -3C_+ C_-^2(t - t_0)e^{-t} - \frac{3}{2}C_+^2 C_- e^{-t} + \frac{1}{2}C_-^3 e^{-3t} - \frac{1}{4}C_-^3 e^{3t}.
\tag{3.44}
\]
Thus the first-order initial value $W_1(t_0)$ is chosen to be

$$W_1(t_0) = -\frac{1}{2}(f_+(t_0; t_0)U_+ + f_-(t_0; t_0)U_-),$$

(3.45)

because otherwise the first term of (3.42) gives rise to terms proportional to the unperturbed solution which should be "renormalized away" with the redefinition of $C_\pm$. We notice that $W_1(t_0)$ is a function of $A_\pm(t_0) = C_\pm(t_0)e^{\pm t_0}$;

$$W_1(t_0) = \rho_1[A_+, A_-].$$

(3.46)

Applying the RG equation to $u(t; t_0) = u_0(t; t_0) + \epsilon u_1(t; t_0)$, one obtains

$$\left. \frac{du}{dt} \right|_{t_0} = \dot{C}_+ e^t U_+ + \dot{C}_- e^{-t} U_- + 3\frac{\epsilon}{2} \left\{ C_+^2 C_- e^t U_+ - C_+ C_-^2 e^{-t} U_- \right\} = 0,$$

(3.47)

which leads to

$$\dot{C}_+ = -\frac{3\epsilon}{2} C_+^2 C_-; \quad \dot{C}_- = \frac{3\epsilon}{2} C_+ C_-^2.$$ 

(3.48)

Noting that $C_+(t)C_-(t) = \text{const} \equiv c_+c_-$, one obtains the solution to the RG equation as follows;

$$C_+(t) = c_+e^{-3\epsilon c_+ c_- t/2}; \quad C_-(t) = c_-e^{3\epsilon c_+ c_- t/2}.$$  

(3.49)

Thus one finds that the global solution is given by

$$u(t) = W(t) = W_0[A_+, A_-] + \epsilon \rho_1[A_+, A_-],$$

$$= A_+(t) U_+ + A_-(t) U_-$$

$$-\frac{\epsilon}{8} \left\{ 2A_+^3(t) - 6A_+(t)A_-^2(t) - A_-^3(t) \right\} U_+$$

$$+ \left\{ -A_+^3(t) - 6A_+^2(t)A_-(t) + 2A_-^3(t) \right\} U_-,$$ 

(3.50)

where $A_\pm(t) = c_\pm \exp\{\pm \alpha t\}$ with $\alpha = 1 - 3\epsilon c_+ c_- / 2$. In this case, the unstable manifold is represented solely with $A_\pm$ and the dynamics on the manifold is given through the evolution of these variables.

To compare the result with the exact solution given in terms of an elliptic function, we first write down the first component $x(t)$ of $u(t)$ given in (3.50);

$$x(t) = (1 + \frac{3}{4}\epsilon \beta)(c_+ e^{\alpha t} + c_- e^{-\alpha t}) - \frac{\epsilon}{8}\{c_+^3 e^{3\alpha t} + c_-^3 e^{-3\alpha t}\},$$

(3.51)

with $\beta = c_+ c_-$. Writing $(1 + \frac{3}{4}\epsilon \beta)c_\pm$ as $c_\pm$, we have

$$x(t) = c_+ e^{\alpha t} + c_- e^{-\alpha t} - \frac{\epsilon}{8}\{c_+^3 e^{3\alpha t} + c_-^3 e^{-3\alpha t}\},$$

(3.52)
still with $\alpha = 1 - 3\epsilon c_+ c_- / 2$ up to $O(\epsilon^2)$. In fact, if one solved the equation in the scalar form without converting the equation to the vector one and applied the operator method adopted in [25, 26] and explained in Appendix A of the present paper, one would have directly reached the form given in (3.52). We stress that the method adopted in [25, 26] is more convenient in practice than that presented here.

To make the comparison with the exact solution easier, it is found convenient to consider the case with the initial condition $x(0) = 0$. This implies that $c_+ = -c_-$. Further putting $c_+(1 + 3\epsilon\beta/8)/2 = C$, one has

$$x(t) = C \sinh \alpha t - \frac{\epsilon}{8} C^3 \sinh^3 \alpha t,$$

with $\alpha = 1 + \frac{3}{8} \epsilon C^2$ up to $O(\epsilon^2)$.

Now the exact solution with the initial condition $x(0) = 0$ reads

$$x(t) = h \cnc(\bar{\alpha}(\epsilon)t + K(k), k),$$

$$= -hk'F(\bar{\alpha}(\epsilon)t, k),$$

where $F(\bar{\alpha}(\epsilon)t, k) = \sn(\bar{\alpha}(\epsilon)t, k)/\dn(\bar{\alpha}(\epsilon)t, k)$ with $\bar{\alpha}(\epsilon) = \sqrt{\epsilon/2k}/h$, $\cn(t, k), \sn(t, k)$ and $\dn(t, k)$ are Jacobi’s elliptic functions with modulus $k$, $K(k)$ the complete elliptic integral of the first kind and $k' = \sqrt{1-k^2}$ the complementary modulus. The constants $k$ and $h$ are functions of $\epsilon$ and the energy $E$ of the system, which is assumed to be positive here;

$$h = \sqrt{\frac{1 + \sqrt{1 + 4\epsilon E}}{\epsilon}} \simeq \sqrt{2/\epsilon(1 + \epsilon E/2)}, \quad k = h\sqrt{\epsilon/2(1 + 4\epsilon E)^{-1/4}} \simeq 1 - \frac{\epsilon E}{2},$$

and $\bar{\alpha}(\epsilon) \simeq 1 + \epsilon E$. Notice that $k$ approaches 1 as $\epsilon$ goes to 0. If one expands $F(\bar{\alpha}(\epsilon)t, k)$ w.r.t. $k$ around $k = 1$ with $\bar{\alpha}(\epsilon)$ fixed, one has

$$F(\bar{\alpha}(\epsilon)t, k) \simeq (1 + \frac{\epsilon}{4} E) \sinh(1 + \frac{3}{4} \epsilon E)t - \frac{\epsilon}{4} E \sinh^3(1 + \frac{3}{4} \epsilon E)t,$$

up to $O(\epsilon^2)$. Notice that the expansion of the elliptic functions at $k = 1$ is subtle; we have made the manipulation as follows;

$$\sinh u - \frac{\epsilon}{4} Eu \cosh u = \sinh u - \frac{\epsilon}{4} Eu \frac{d\sinh u}{du},$$

$$\simeq \sinh(1 - \frac{\epsilon}{4} E)u.$$ (3.57)

Then identifying $C = -hk'(1 + \frac{\epsilon}{4} E) = -\sqrt{2E}(1 + \frac{9}{8} \epsilon E)$, one reproduces the above result (3.53). In Appendix B, an elementary method is presented to derive the approximate formula obtained above.
3.4 An example with a Jordan cell; Takens equation

As examples which involve a linear operator with a Jordan cell, we take the Takens equation in the present subsections.

The Takens equation is given by
\[ \dot{x} = y + ax^2, \quad \dot{y} = bx^2. \] (3.58)

Since we are interested in a slow motion in the vicinity of the origin, we make a scale transformation;
\[ x = \epsilon^\alpha X, \quad y = \epsilon^\beta Y, \] (3.59)

where \( \epsilon \) is a small parameter. To make the equation to be balanced, one finds that \( \alpha = \beta \); we choose \( \alpha = \beta = 1 \) for simplicity. Then we end up with
\[ \dot{X} = Y + \epsilon aX^2, \quad \dot{Y} = \epsilon bX^2. \] (3.60)

Expanding as \( X = X_0 + \epsilon X_1 + \cdots \), and \( Y = Y_0 + \epsilon Y_1 + \cdots \), we first solve the equation around \( t \sim t_0 \) with the initial value \( W(t_0) = \{X(t_0), Y(t_0)\} = W_0(t_0) + \epsilon W_1(t_0) + \cdots \), where \( t_0 \) is arbitrary. The equations in the first few orders read
\[ \begin{align*}
\dot{X}_0 & = Y_0, \quad \dot{Y}_0 = 0, \quad (3.61) \\
\dot{X}_1 & = Y_1 + aX_0^2, \quad \dot{Y}_1 = bX_0^2, \quad (3.62)
\end{align*} \]
and so on.

We take the stationary solution as the lowest order one to describe a slow motion on an invariant manifold, in accordance with the previous treatment; namely,
\[ X_0(t; t_0) = \text{const} = C_0(t_0), \quad Y_0(t; t_0) = 0, \] (3.63)

\( C_0(t_0) \) is an integral constant. Accordingly, \( W_0(t_0) = \{C_0(t_0), 0\} \), i.e., the unperturbed manifold \( M_0 \) is the \( X \) axis.

Then the first order equation is solved successively from \( Y_1 \) to \( X_1 \) to yield
\[ \begin{align*}
Y_1(t; t_0) & = bC_0^2(t_0)(t - t_0) + C_1(t_0), \\
X_1(t; t_0) & = \frac{b}{2}C_0^2(t_0)(t - t_0)^2 + (C_1(t_0) + aC_0^2(t_0))(t - t_0), \quad (3.64)
\end{align*} \]
where \( C_1(t_0) \) is another integral constant. Accordingly, \( W_1(t_0) = \{0, C_1(t_0)\} \); namely, the modification of the invariant manifold is given in the \( Y \) direction;
\[ M_1 = \{(X, Y) | (X, Y) = (C_0, C_1)\}. \] (3.65)
Up to this order, we have
\[ \begin{align*}
X(t; t_0) &= C_0(t_0) + \epsilon \left\{ \frac{b}{2}C_0^2(t_0)(t-t_0)^2 + (C_1(t_0) + aC_0^2(t_0))(t-t_0) \right\}, \\
Y(t; t_0) &= \epsilon \{ bC_0^2(t_0)(t-t_0) + C_1(t_0) \},
\end{align*} \]
(3.66)
with \( W(t_0) = \psi(C_0(t_0), \epsilon C_1(t_0)) \).  

Now applying the RG equation to \( X(t; t_0) \) and \( Y(t; t_0) \) thus obtained, we have
\[ \begin{align*}
\dot{C}_0 &= \epsilon (C_1 + aC_0(t)), \\
\dot{C}_1 &= bC_0^2.
\end{align*} \]
(3.68)

The trajectory is given by
\[ \begin{align*}
X(t) = X(t; t) = C_0(t), \quad Y(t) = Y(t; t) = \epsilon C_1(t),
\end{align*} \]
(3.69)
which shows that the original Takens equation is reproduced. This means that Takens equation is "irreducible" and can not be reduced to a simpler equation.

A few comments are in order: In this case, the invariant manifold is represented with two variables in accordance with the dimension of the Jordan cell although the dimension of the unperturbed solution is one; namely, the dimension of the invariant manifold is increased from that of the unperturbed manifold \( M_0 \). The amplitude of the trajectory in the second direction, \( \epsilon C_1(t) \) is small compared with the amplitude in the first direction \( C_0 \), while the time dependence of the second variable is large in comparison with the first variable.

4 Generic systems with the linear operator having zero eigenvalues

In this section, we shall examine invariant manifolds and slow motions given by generic systems which have a linear operator having zero eigenvalues. We shall show how uniquely the initial values are chosen by using a simple formula for the special solutions to differential equations as in the previous section; the initial values are determined successively so that terms which give fast motions and those proportional to the unperturbed solution do not appear. We call these unwanted terms "dangerous ones". We shall also show a necessary condition on the type of equations for the RG method to be applicable, which condition is relevant when the linear operator has a Jordan cell.

We treat the following rather generic vector equations in this section:
\[ \partial_t \mathbf{u} = A \mathbf{u} + \epsilon \mathbf{F}(\mathbf{u}), \]
(4.1)
where \( \partial_t \mathbf{u} = \partial \mathbf{u} / \partial t \), \( A \) is a linear operator, \( \mathbf{F} \) a nonlinear function of \( \mathbf{u} \) and \( \epsilon \) is a small parameter (\( |\epsilon| < 1 \)). We assume that \( A \) has multiply degenerated zero eigenvalues and other eigenvalues of \( A \) have a negative real part.
We are interested in constructing the attractive manifold \( M \) at \( t \to \infty \) and the reduced dynamics on it. We try to construct solve the problem in the perturbation theory by expanding \( u \) as

\[
u(t; t_0) = u_0(t; t_0) + \epsilon u_1(t; t_0) + \epsilon^2 u_2(t; t_0) + \cdots,
\]

with the initial value \( W(t_0) \) at an arbitrary time \( t_0 \). We suppose that the equation has been solved up to \( t = t_0 \) and the solution has the value \( W(t_0) \) at \( t_0 \). Actually, the initial value must be determined by the perturbative solution self-consistently; indeed, \( u(t) = W(t) \) is the solution to (4.1) in the global domain. Therefore it should be also expanded as follows;

\[
W(t_0) = W_0(t_0) + \epsilon W_1(t_0) + \epsilon^2 W_2(t_0) + \cdots = W_0(t_0) + \rho(t_0),
\]

where \( \rho(t_0) \) is supposed to be an independent function of \( W_0 \). They are not yet known at present but will be determined so that the perturbative expansion becomes valid. One of the main purposes in this section is how sensibly the initial values can be determined order by order.

The equations in the first few orders read

\[
(\partial_t - A)u_0 = 0,
\]

\[
(\partial_t - A)u_1 = F(u_0),
\]

\[
(\partial_t - A)u_2 = F'(u_0)u_1,
\]

where

\[
(F'(u_0)u_1)_i = \sum_{j=1}^{n} \{ \partial(F'(u_0))_i / \partial(u_0)_j \} (u_1)_j,
\]

if \( u \) is an \( n \)-dimensional vector.

We treat the two cases separately where \( A \) has semi-simple 0 eigenvalues or a Jordan cell.

### 4.1 When \( A \) has semi-simple zero eigenvalues

In this subsection, we treat the case where \( A \) has semi-simple 0 eigenvalues. Let the dimension of \( \ker A \) be \( m \);

\[
AU_i = 0, \quad (i = 1, 2, \ldots, m).
\]

We suppose that other eigenvalues have negative real parts;

\[
AU_{\alpha} = \lambda_{\alpha}U_{\alpha}, \quad (\alpha = m + 1, m + 2, \ldots, n).
\]
where $\text{Re} \lambda_\alpha < 0$. One may assume without loss of generality that $\mathbf{U}_i$'s and $\mathbf{U}_\alpha$'s are linearly independent.

The adjoint operator $A^\dagger$ has the same eigenvalues as $A$ has;

$$
A^\dagger \mathbf{U}_i = 0, \quad (i = 1, 2, \ldots, m), \\
A^\dagger \mathbf{U}_\alpha = \lambda_\alpha^* \mathbf{U}_\alpha, \quad (\alpha = m + 1, m + 2, \ldots, n).
$$

(4.10)

Here we suppose that $\mathbf{U}_i$'s and $\mathbf{U}_\alpha$'s are linearly independent. Without loss of generality, one can choose the eigenvectors so that

$$
\langle \mathbf{U}_i, \mathbf{U}_\alpha \rangle = 0 = \langle \mathbf{U}_\alpha, \mathbf{U}_i \rangle,
$$

(4.11)

with $1 \leq i \leq m$ and $m + 1 \leq \alpha \leq n$.

We denote the projection operators by $P$ and $Q$ which projects onto the kernel of $A$ and the space orthogonal to ker$A$, respectively. The projection operators can be constructed in terms of $\mathbf{U}_i$ and $\mathbf{U}_i$ ($i = 1, 2, \ldots, m$) as follows: Let $\hat{U}_P$ be an $n \times m$ matrix defined by $\hat{U}_P = (\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_m)$ and $\hat{U}_P$ by $\hat{U}_P = (\hat{U}_1, \hat{U}_2, \ldots, \hat{U}_m)$, then

$$
P = \hat{U}_P (\hat{U}_P \hat{U}_P)^{-1} \hat{U}_P^*,
$$

(4.12)

and $Q = 1 - P$.

Since we are interested in the asymptotic state as $t \to \infty$, we may assume that the lowest-order initial value belongs to ker$A$:

$$
\mathbf{W}_0(t_0) = \sum_{i=1}^{m} C_i(t_0) \mathbf{U}_i = \mathbf{W}_0[\mathbf{C}].
$$

(4.13)

Thus trivially, $\mathbf{u}_0(t; t_0) = e^{(t-t_0)A} \mathbf{W}_0(t_0) = \sum_{i=1}^{m} C_i(t_0) \mathbf{U}_i$. We notice that a natural parameterization of the invariant manifold in the lowest order $M_0$ is given by the set of the integral constants $\mathbf{C} = \{C_1, C_2, \cdots, C_m\}$ being varied.

The first order equation (4.5) with the initial value $\mathbf{W}_1(t_0)$ which is not yet determined is formally solved to be

$$
\mathbf{u}_1(t; t_0) = e^{(t-t_0)A} \mathbf{W}_1(t_0) + \int_{t_0}^{t} ds e^{(t-s)A} \mathbf{F}(\mathbf{u}_0(s; t_0)).
$$

(4.14)

We remark that one may assume that the initial value $\mathbf{W}_1(t_0)$ is independent of $\mathbf{W}_0(t_0)$, namely $\mathbf{W}_1(t_0)$ belongs to the Q-space, because if $\mathbf{W}_1(t_0)$ had a component belonging to ker$A$, the component could be "renormalized away" into $\mathbf{W}_0$. Inserting the identity $I = P + Q$ between the two functions in the integral, we have

$$
\mathbf{u}_1(t; t_0) = e^{(t-t_0)A} [\mathbf{W}_1(t_0) + A^{-1} Q \mathbf{F}(\mathbf{W}_0(t_0))] \\
+ (t - t_0) PF(\mathbf{W}_0(t_0)) - A^{-1} Q \mathbf{F}(\mathbf{W}_0(t_0)).
$$

(4.15)
The first term has a possibility to give rise to a fast motion, which should be avoided and called "dangerous" term: The "dangerous" terms here are analogous with divergent terms in quantum field theory, which are subtracted away by counter terms, analogue to the initial values $W_i$ here, self-consistently. Indeed it is nice that the initial value $W_1(t_0)$ not yet determined can be chosen so as to cancel out the "dangerous" term as follows;

$$W_1(t_0) = - A^{-1} Q F(W_0(t_0)),$$

(4.16)

which satisfies $PW_1(t_0) = 0$ and is a function solely of $C(t_0)$. Thus we have for the first order solution

$$u_1(t; t_0) = (t - t_0) PF - A^{-1} Q F,$$

(4.17)

where the argument of $F$ is $W_0[C]$. Notice that Eq.(4.17) is consistent with (4.16).

We remark that what we have done is actually a simple thing; we have suppressed the unperturbed part that would be damped out as $t \rightarrow \infty$.

Now the invariant manifold is modified to $M_1$ given by

$$M_1 = \{ u | u = W_0 - \epsilon A^{-1} Q F(W_0) \}. \quad (4.18)$$

If one stops to this order, the approximate solution reads

$$u(t; t_0) = W_0 + \epsilon \{ (t - t_0) PF - A^{-1} Q F \}. \quad (4.19)$$

Then the RG equation $\partial u / \partial t_0 |_{t_0 = t} = 0$ gives

$$\dot{W}_0(t) = \epsilon P F(W_0(t)), \quad (4.20)$$

which is reduced to an $m$-dimensional coupled equation,

$$\dot{C}_i(t) = \epsilon \langle \tilde{U}_i, F(W_0[C]) \rangle, \quad (i = 1, 2, \ldots, m). \quad (4.21)$$

The global solution representing a trajectory on the invariant manifold up to this order is given by

$$u(t) = u(t; t_0 = t) = \sum_{i=1}^{m} C_i(t) U_i - \epsilon A^{-1} Q F(W_0[C]), \quad (4.22)$$

with $C(t)$ being the solution to (4.21).

In short, we have derived the invariant manifold as the initial value represented by (4.22) and the reduced dynamics (4.21) on it in the RG method in the first order approximation.

The second order solution can be obtained as follows. As has been done in the first order case, the second order solution is formally given by

$$u_2(t; t_0) = e^{(t-t_0) A} W_2(t_0) + \int_{t_0}^{t} dse^{(t-s) A} F'(u_0(s; t_0)) u_1(s; t_0). \quad (4.23)$$
We may assume again that the initial value \( W_2(t_0) \) belongs to the Q-space. A straightforward evaluation of the integral yields

\[
\mathbf{u}_2(t; t_0) = e^{(t-t_0)A} \left[ W_2(t_0) - \left\{ A^{-1}QF'A^{-1}QF - A^{-2}QF'PF \right\} \right] \\
+ A^{-1}QF'A^{-1}QF - A^{-2}QF'PF - (t-t_0) \left\{ PF'A^{-1}QF + A^{-1}QF'PF \right\} \\
+ \frac{1}{2} (t-t_0)^2 PF'PF, \tag{4.24}
\]

where the argument of \( F \) and \( F' \) is \( W_0[C] \). The initial value can be now determined so as to cancel out the "dangerous term", i.e., the fast moving part, as before;

\[
W_2(t_0) = A^{-1}QF'(W_0)A^{-1}QF(W_0) - A^{-2}QF'PF, \tag{4.25}
\]

which belongs to the Q-space. This implies that the invariant manifold is modified to \( M_2 \) represented by \( \mathbf{W} = W_0[C] + \rho[C]; \quad \rho \simeq \epsilon W_1 + \epsilon^2 W_2 \). Thus we obtain for the second order solution

\[
\mathbf{u}_2(t; t_0) = A^{-1}QF'A^{-1}QF - A^{-2}QF'PF - (t-t_0) \left\{ PF'A^{-1}QF + A^{-1}QF'PF \right\} \\
+ \frac{1}{2} (t-t_0)^2 PF'PF. \tag{4.26}
\]

Notice that Eq. (4.26) is consistent with (4.25). Thus the full expression of the solution up to the second order is given by

\[
\mathbf{u}(t; t_0) = W_0(t_0) + \epsilon \left\{ (t-t_0)PF - A^{-1}QF \right\} \\
+ \epsilon^2 \left\{ A^{-1}QF'A^{-1}QF - A^{-2}QF'PF - (t-t_0) \left\{ PF'A^{-1}QF + A^{-1}QF'PF \right\} \right\} \\
+ \frac{1}{2} (t-t_0)^2 PF'PF. \tag{4.27}
\]

The RG equation \( \partial \mathbf{u}/\partial t_{t_0=t} = 0 \) reads

\[
\dot{W}_0(t) - \epsilon PF - \epsilon A^{-1}QF'\dot{W}_0 + \epsilon^2 \left\{ PF'A^{-1}QF + A^{-1}QF'PF \right\} = 0. \tag{4.28}
\]

Operating the projections \( P \) and \( Q \) to the both sides of (4.28), respectively, we have

\[
\dot{W}_0(t) - \epsilon PF + \epsilon^2 PF'A^{-1}QF = 0, \tag{4.29}
\]

\[
- \epsilon A^{-1}QF'\dot{W}_0 + \epsilon^2 A^{-1}QF'PF = 0. \tag{4.30}
\]

Firstly, we notice that the last equation (4.30) is reduced to

\[
\epsilon A^{-1}QF'(-\dot{W}_0 + \epsilon PF) = 0,
\]

which is identically satisfied on account of (4.23) up to this order. Thus, we end up with the reduced equation given by

\[
\dot{W}_0(t) = \epsilon PF - \epsilon^2 PF'A^{-1}QF, \tag{4.31}
\]

26
which is further reduced to
\[ \dot{C}_i = \epsilon \langle \tilde{U}_i, F - \epsilon F' A^{-1} Q F \rangle, \]
for \( i = 1, 2, \ldots, m. \)

The global solution giving the trajectory on the invariant manifold is given by the initial value as
\[ u(t) = W(t) = W_0[C] + \rho[C] \]
\[ = W_0[C] - \epsilon A^{-1} Q F + \epsilon^2 \{ A^{-1} Q F' A^{-1} Q F - A^{-2} Q F' P F \}, \]
with \( C(t) \) being the solution to (4.32). Notice that the argument of \( F \) and \( F' \) in the above expression is all \( W_0 \), hence the r.h.s is a function of \( C \), i.e., \( u(t) = u[C] \). Recall that \( C \) was the integral constants of the unperturbed solution.

A couple of remarks are in order: (1) When the present formulation is applied to the Lorenz model\[58\] around the first bifurcation point which is simple one, the result coincides with that given in \[26\]; in other words, the present formulation gives a foundation to the prescription adopted in that paper. (2) The present formulation using the projection operators and the resultant RG equation (4.32) governing the slow motion resemble those by Mori theory \[59\] for stochastic motions.

### 4.2 When \( A \) has a Jordan cell

In this subsection, we treat the case where \( A \) has a Jordan cell. We assume the Jordan cell is two-dimensional for simplicity, and we define the normalized vectors \( U_1 \) and \( U_2 \) by
\[ A U_1 = 0, \quad A U_2 = U_1. \] (4.34)
The conjugate vectors \( \tilde{U}_1 \) and \( \tilde{U}_2 \) satisfy
\[ A^\dagger \tilde{U}_2 = 0, \quad A^\dagger \tilde{U}_1 = \tilde{U}_2, \] (4.35)
where \( A^\dagger \) is the conjugate of \( A \). The normalization condition is given by
\[ \langle \tilde{U}_1, U_1 \rangle = \langle \tilde{U}_2, U_2 \rangle = 1, \quad \langle \tilde{U}_1, U_2 \rangle = 0, \] (4.36)
where \( \langle \ , \ \rangle \) denotes the inner product. Note that \( \langle \tilde{U}_2, U_1 \rangle = 0 \) automatically holds.

We denote by \( P \) the projection operator to the subspace (P-space) spanned by \( U_1 \) and \( U_2 \); namely, for any vector \( u \),
\[ P u = \alpha U_1 + \beta U_2, \] (4.37)
with \( \alpha = \langle \tilde{U}_1, u \rangle \) and \( \beta = \langle \tilde{U}_2, u \rangle \).
Let $Q$ be the projection operator to the subspace (Q-space) complement of the P-space. Then one can verify that
\[ e^{tA}u = e^{tA}(P + Q)u, \]
\[ = (\alpha + \beta t)U_1 + \beta U_2 + e^{tA}Qu. \] (4.38)

So much for the preliminaries.

Now let us proceed to obtain the asymptotic solution to (4.4) as $t \to \infty$ by the perturbation method: Since we are interested in constructing the invariant manifold, let us take the stationary solution as the lowest order one;
\[ u_0(t; t_0) = C_0(t_0)U_1, \] (4.39)
accordingly, the initial value reads
\[ W_0(t_0) = C_0(t_0)U_1. \] (4.40)

Notice that we have not included a component in $U_2$ direction. The lowest order manifold is
\[ M_0 = \{u | u = C_0U_1\}. \] (4.41)

The first order solution is formally given by (4.14). The first order initial value is chosen to be independent of $W_0$;
\[ W_1(t_0) = C_1(t_0)U_2 + QW_1(t_0). \]

A simple evaluation of the integral in (4.14) gives the first order solution as
\[ u_1(t; t_0) = e^{(t-t_0)A}[QW_1(t_0) + A^{-1}QF] \]
\[ + \{C_1(t_0)(t - t_0) + \alpha_F(t - t_0) + \beta_F \frac{1}{2}(t - t_0)^2\}U_1 \]
\[ + \{C_1(t_0) + \beta_F(t - t_0)\}U_2 - A^{-1}QF + O((t - t_0)^{n+2}), \] (4.42)
where
\[ \alpha_F = \langle \tilde{U}_1, F \rangle, \quad \beta_F = \langle \tilde{U}_2, F \rangle. \] (4.43)

The argument of $F$ in the above expressions is $W_0[C_0]$. The initial value can be determined so as to cancel out the fast mode as before, namely,
\[ QW_1(t_0) = -A^{-1}QF, \] (4.44)
which implies that the invariant manifold is modified to
\[ M_1 = \{u | u = C_0U_1 + \epsilon C_1U_2 - \epsilon A^{-1}QF\}. \] (4.45)
Then the solution up to the first order is obtained as
\[
\mathbf{u}(t; t_0) = C_0(t_0)\mathbf{U}_1 + \epsilon \left\{ \alpha_F(t - t_0) + C_1(t_0)(t - t_0) + \frac{1}{2} \beta_F(t - t_0)^2 \right\} \mathbf{U}_1 \\
+ \{\beta_F(t - t_0) + C_1(t_0)\} \mathbf{U}_2 - A^{-1}QF.
\]  
(4.46)

The RG equation in this order is given by
\[
0 = \dot{C}_0\mathbf{U}_1 - \epsilon\{ (\alpha_F + C_1)\mathbf{U}_1 + (\beta_F - C_1)\mathbf{U}_2 + \frac{1}{A}QF'C_0\mathbf{U}_1, \\
\]
(4.47)
which leads to
\[
\dot{C}_0 = \epsilon \left( \langle \bar{U}_1, F \rangle + C_1 \right), \quad \dot{C}_1 = \langle \bar{U}_2, F \rangle.
\]  
(4.48)

We now see that the trajectory in the invariant manifold \( M_1 \) is given by
\[
\mathbf{u}(t) = W(t) \simeq W_0[C_0] + W_1[C_0], \\
= C_0(t)\mathbf{U}_1 + \epsilon C_1(t)\mathbf{U}_2 - \epsilon A^{-1}QF,
\]  
(4.49)
with \( C_0(t) \) and \( C_1(t) \) being governed by (4.48). Notice that \( \mathbf{u}(t) \) is a functional of \( C_0(t) \) and \( C_1(t) \).

The second order solution is obtained, similarly. The formal solution to the second order equation is given by Eq.(4.23). Let \( \mathbf{g} = C_1(t_0)\mathbf{U}_1 + \alpha_F\mathbf{U}_1 + \beta_F\mathbf{U}_2 \) and \( \mathbf{h} = -\frac{1}{A}QF + C_1\mathbf{U}_2 \). Then, a simple manipulation as before gives
\[
\mathbf{u}_2(t; t_0) = e^{(t-t_0)A}[W_2(t_0) + \{ A^{-1}QF'h + A^{-2}QF'g \}] \\
+ (t - t_0) \left\{ -A^{-1}QF'g + PF'h \right\} - \{ A^{-1}QF'h + A^{-2}QF'g \} \\
+ O((t - t_0)^2).
\]  
(4.50)

Thus the initial value is determined to be
\[
W_2(t_0) = - \left( A^{-1}QF'h + A^{-2}QF'g \right), \\
\]  
(4.51)
so that the fast modes disappear; notice that \( W_2 \) belongs to the Q-space. The invariant manifold is modified now in an apparent way; so we do not write the specification of the second order manifold \( M_2 \). Hence we have
\[
\mathbf{u}_2(t; t_0) = (t - t_0) \left\{ -A^{-1}QF'g + PF'h \right\} - \{ A^{-1}QF'h + A^{-2}QF'g \} \\
+ O((t - t_0)^2). \\
\]  
(4.52)
Applying the RG equation to \( \mathbf{u} = \mathbf{u}_0 + \epsilon\mathbf{u}_1 + \epsilon^2\mathbf{u}_2 \) thus obtained, we have
\[
\dot{C}_0\mathbf{U}_1 - \epsilon(\alpha_F + C_1)\mathbf{U}_1 + \epsilon(\dot{C}_1 - \beta_F)\mathbf{U}_2 - \epsilon A^{-1}QF'\dot{C}_0\mathbf{U}_1 \\
- \epsilon^2 \left\{ -A^{-1}QF'g + PF'h \right\} - \epsilon^2 \left\{ A^{-1}QF'\dot{C}_1\mathbf{U}_2 + A^{-2}QF'\dot{C}_1\mathbf{U}_1 \right\} = 0. \\
\]  
(4.53)
Operating $P$ and $Q$ to (4.53), we obtain

\begin{align}
0 &= \dot{C}_0 U_1 + \epsilon \dot{C}_1 U_2 - \epsilon g - \epsilon^2 P F' h, \\
0 &= -\epsilon^2 A^{-2} Q F' \dot{C}_1 U_1 - \epsilon A^{-1} Q F' \{ \dot{C}_0 U_1 + \epsilon \dot{C}_1 U_2 - \epsilon g \}. 
\end{align}

(4.54)

(4.55)

We remark that from (4.54) and (4.55),

\[ Q F' U_1 = 0 \]  

(4.56)

must hold as a compatibility condition, which gives a necessary condition for the RG method to work for higher approximations. With this condition taken for granted, $g$ is reduced to $g = \beta F U_2$.

Equating the components in the $U_1, U_2$ in (4.54), we have the reduced dynamics as follows,

\begin{align}
\dot{C}_0 &= \epsilon (\tilde{U}_1, F + \epsilon F' h) + C_1, \\
\dot{C}_1 &= \langle \tilde{U}_2, F + \epsilon F' h \rangle, 
\end{align}

(4.57)

(4.58)

where the argument of $F$ and $F'$ is $W_0(t) = C_0(t) U_1$.

The trajectory on the manifold $M_2$ is given by

\[ u(t) = W(t) = W_0(t) + \epsilon W_1(t) + \epsilon^2 W_2(t), \]

\[ = C_0(t) U_1 + \epsilon C_1(t) U_2 - \epsilon A^{-1} Q F - \epsilon^2 \{ A^{-1} Q F' h + \langle \tilde{U}_2, F \rangle A^{-2} Q F' U_2 \}. \]

(4.59)

Comments are in order: The solution is solely described by the coordinates $C_0$ and $C_1$ representing the P space as in the previous subsection. Conversely, the dynamics can not be described only by the coordinate in the zero-th manifold in this case; the dimension of the invariant manifold is increased from that of the unperturbed invariant manifold. We remark that the formulae obtained above are completely consistent with those given for the Takens equation in §3.

### 4.3 Practical way

We have formulated the RG method so that it is clarified that the method concerns with the initial values and the invariant manifold is constructed as the initial value perturbatively: The initial values are determined so that terms proportional to the unperturbed solution and those representing fast motions disappear in the perturbed solutions. In effect, the special solutions of the higher order equations are composed of secular terms which are proportional to the unperturbed solution and vanish at $t = t_0$ and solutions independent of the unperturbed solution. We remark that this way of construction of the perturbative special solutions have been adopted in [23, 24, 27]. In this subsection, having
known the above fact, we present the rules for constructing the special solutions in the form of an operator method; a detailed account of this method is given in Appendix A. This subsection will constitute a practical summary of the results obtained in the previous subsections.

When we try to obtain an invariant manifold and the reduced dynamics on it, the solution to the unperturbed equation (4.4) is given by a stationary one

$$u_0(t; t_0) = W_0(t_0).$$  \hspace{1cm} (4.60)

Then the first order solution is given by

$$u_1(t; t_0) = \frac{1}{\partial_t - A} F(u_0) = \frac{1}{\partial_t - A} (PF(u_0) + QF(u_0)), \hspace{1cm} (4.61)$$

Here (A.16) and (A.17) have been used and $A^{-1}$ is written as $\frac{1}{A}$.

Similarly, the second order solution is given by

$$u_2(t; t_0) = \frac{1}{\partial_t - A} F'(u_0)u_1,$$

$$= \frac{1}{\partial_t - A} (P + Q)F'(u_0)\{(t - t_0)PF + \frac{1}{-A}QF\},$$

$$= \frac{1}{2}(t - t_0)^2PF'PF + (t - t_0)PF'\frac{1}{-A}QF$$

$$- \{(t - t_0)\frac{1}{A} + \frac{1}{A^2}\}QF'PF + \frac{1}{-A}QF'\frac{1}{-A}QF,$$ \hspace{1cm} (4.62)

which coincides with (4.26). Here we have used the formulae (A.20), (A.21) and (A.22). The efficiency of the operator method is apparent.

Next, let us consider the case where $A$ has a two-dimensional Jordan cell. We take a stationary solution as the zeroth-order one;

$$u_0(t; t_0) = C_0(t_0)U_1.$$ \hspace{1cm} (4.63)

Notice that the kernel of $A$ is not yet fully spanned by the solution. Then the first order solution is given by a sum of the remaining component of $\ker A$ and the special solution;

$$u_1(t; t_0) = \frac{1}{\partial_t - A}\{0\} + \frac{1}{\partial_t - A} F(u_0),$$

$$= C_1(t_0)(t - t_0)U_1 + C_1(t_0)U_2 + \frac{1}{\partial_t - A}(PF(u_0) + QF(u_0)), $$

$$= C_1(t_0)(t - t_0)U_1 + C_1(t_0)U_2$$

$$+ \{(t - t_0)PF + \frac{1}{2}(t - t_0)^2(\tilde{U}_2, F)U_1\} + \frac{1}{-A}QF,$$ \hspace{1cm} (4.64)
which is found to coincide with Eq.(4.42). Here (A.17) and (A.27) have been used. The unperturbed solution \( \frac{1}{\partial_t - A} \{ 0 \} \) may be obtained in the following way: Putting \( \frac{1}{\partial_t - A} \{ 0 \} = a(t)U_1 + b(t)U_2 \), one has a coupled equation \( \dot{a} = b, \quad \dot{b} = 0 \), which is solved to yield \( a(t) = C_1(t_0)(t - t_0) \), \( b(t) = C_1(t_0) \).

The second order solution is given by

\[

u_2(t; t_0) = \frac{1}{\partial_t - A} F'(u_0)u_1(t; t_0),
\]

\[

= \frac{1}{\partial_t - A} (P + Q) F'(u_0)u_1(t; t_0). \quad (4.65)
\]

Then the r.h.s. is a sum of terms which are in the form \( \frac{1}{\partial_t - A} (t - t_0)^n PG(u_0) \) or \( \frac{1}{\partial_t - A} (t - t_0)^n QG(u_0) \), which are calculated in Appendix A. Actually, since terms proportional to \( (t - t_0)^n (n \geq 2) \) do not contribute to the RG equation nor to the resulting trajectory, one needs not calculate the terms of the form \( \frac{1}{\partial_t - A} (t - t_0)^n PG(u_0) \) with \( n \geq 1 \). Thus we easily reach the final result given in (4.52).

### 4.4 Normal form

As a simple example, we try to reduce the following two-dimensional evolution equation with a Jordan cell;

\[

\dot{x} = y + a_1 x^2 + b_1 xy + c_1 y^2, \quad \dot{y} = a_2 x^2 + b_2 xy + c_2 y^2, \quad (4.66)
\]

with \( a_i, b_i \) and \( c_i \) \((i = 1, 2)\) being constant. As in the Takens equation examined in §3, we make a scale transformation

\[

x = \epsilon X, \quad y = \epsilon Y. \quad (4.67)
\]

Then the equation is reduced to

\[

(\partial_t - A) u = \epsilon F(u), \quad u = t(X, Y), \quad (4.68)
\]

where

\[

A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F(u) = \begin{pmatrix} a_1 X^2 + b_1 XY + c_1 Y^2 \\ a_2 X^2 + b_2 XY + c_2 Y^2 \end{pmatrix}. \quad (4.69)
\]

Here the P-space is spanned by

\[

U_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad U_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (4.70)
\]

which satisfy \( AU_1 = 0 \) and \( AU_2 = U_1 \). And \( \bar{U}_i = U_i \) \((i = 1, 2)\).
Expanding as $u = u_0 + \epsilon u_1 + \cdots$, with $u_i = ^i(X_i, Y_i)$, we first solve the equation around $t \sim t_0$ with the initial value $W(t_0) = W_0(t_0) + \epsilon W_1(t_0) + \cdots$.

We take the stationary solution as the lowest order

$$u_0(t; t_0) = A_0(t_0) U_1,$$

where $A_0(t_0)$ is an integral constant. Accordingly, $W_0(t_0) = ^t(A_0(t_0), 0)$; the unperturbed manifold is the $X$ axis.

According to the general argument given in §4.2, we need to calculate the following quantities:

$$\alpha_F = \langle \dot{U}_1, F(u_0) \rangle = a_1 A_0^2(t_0), \quad \beta_F = \langle \dot{U}_2, F(u_0) \rangle = a_2 A_0^2(t_0).$$

Notice that $Q F(u_0) = 0$, identically. Then one has

$$u_1(t; t_0) = \{ A_1(t_0)(1 + a_1 A_1(t_0))(t - t_0) + \frac{a_2}{2} A_0^2(t_0)(t - t_0)^2 \} U_1 + \{ A_1(t_0) + a_2 A_0^2(t_0)(t - t_0) \} U_2,$$

where $A_1(t_0)$ is another integral constant. Accordingly, $W_1(t_0) = ^t(0, A_1(t_0))$. Then

$$M_1 = \{ u | u = ^t(A_0, \epsilon A_1) \}.$$  

If we stop at this order, the RG equation reads

$$\dot{A}_0 = \epsilon (A_1 + a_1 A_0^2), \quad \dot{A}_1 = a_2 A_0^2.$$  

This is the Takens equation; notice the trajectory is given by $x(t) = \epsilon X(t; t) = \epsilon A_0(t), y(t) = \epsilon Y(t; t) = \epsilon^2 A_1(t)$. It means that the RG method gives mechanically the normal form of the reduced equation. This is confirmed by proceeding to the second order.

In the present case, $g = 0$ and $h = A_1(t_0) U_2$, and

$$P F'(u_0) h = A_1 \{ b_1 A_0 U_1 + b_2 A_0 U_2 \}.$$  

Then the second order solution is found to be

$$u_2(t; t_0) = (t - t_0) A_1 \{ b_1 A_0 U_1 + b_2 A_0 U_2 \} + O((t - t_0)^{n+2}),$$

Accordingly, $W_2(t_0) = 0$.

Up to this order, we have $u = u_0 + \epsilon u_1 + \epsilon^2 u_2$. Then the RG equation reads

$$\dot{A}_0 = \epsilon A_1 + \epsilon(a_1 A_0^2 + \epsilon b_1 A_0 A_1),$$  

$$\dot{A}_1 = a_2 A_0^2 + \epsilon b_2 A_0 A_1,$$

which is the normal form when the linear matrix is of Jordan form. The trajectory is given by $x(t) = \epsilon A_0(t), y(t) = \epsilon^2 A_1(t)$. Thus one sees that the RG equation gives the normal form of the reduced evolution equation on the invariant manifold. We remark that when $a_1 = b_1 = 0$, the RG equation is nothing but the Bogdanov equation.

We remark that one needs the second order solution to give the Bogdanov equation, while the Takens equation is obtained in the first approximation.
4.5 An extended Takens equation

As the final example, we deal with an extension of the Takens equation to a system with three-degrees of freedom;

\[ \dot{x} = y + \epsilon ax^2, \quad \dot{y} = bx^2, \quad \dot{z} = -z + \epsilon f(x, y, z), \]

(4.79)

where \( f(x, y, z) \) is analytic function of \((x, y, z)\). We shall show the compatibility condition \([4.56]\), which becomes relevant only when the system has more than two-degrees of freedom, gives a restriction to the form of \( f(x, y, z) \).

With \( u = t'(x, y, z), [4.79] \) is converted to

\[(\partial_t - A)u = \epsilon F(u),\]

(4.80)

where

\[
A = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{pmatrix}, \quad F(u) = \begin{pmatrix}
ax^2 \\
 bx^2 \\
f(x, y, z)
\end{pmatrix}.
\]

(4.81)

Notice that \( A \) has a two-dimensional Jordan cell;

\[
AU_1 = 0, \quad AU_2 = U_1, \quad AU_3 = -U_3,
\]

(4.82)

where \( U_1 = t'(1, 0, 0), U_2 = t'(0, 1, 0), U_3 = t'(0, 0, 1) \). The projection operator to the subspace \( \{U_1, U_2\} \) is given by \( P = \text{diag}(1, 1, 0) \), while \( Q = 1 - P = \text{diag}(0, 0, 1) \). We also notice that \( \tilde{U}_i = U_i \) \((i = 1, 2)\) in this simple example. We are interested in the asymptotic behavior of the solution at \( t \to \infty \). We first solve \((4.80)\) around \( t \sim t_0 \) with the initial value \( W(t_0) \) at \( t = t_0 \) by the perturbation theory. The solution may be written as \( u(t; t_0, W(t_0)) \), which is expanded as \( u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots \). The initial value which is determined self-consistently with \( u \) is also expanded as \( W = W_0 + \epsilon W_1 + \epsilon^2 W_2 + \cdots \).

When \( t \to \infty \), we may take the stationary solution as an asymptotic one to the lowest order equation;

\[
u_0(t; t_0) = C_0(t_0)U_1.
\]

(4.83)

Accordingly the initial value reads \( W_0(t_0) = C_0(t_0)U_1 \), which implies that the unperturbed invariant manifold is given by

\[
M_0 = \{u | u = C_0U_1\},
\]

(4.84)

namely the \( x \) axis.

According to the general formulation given in the previous sections, to obtain the first order solution, we only have to evaluate

\[
\alpha_F = \langle \tilde{U}_1, F(u_0) \rangle = aC_0^2(t_0), \quad \beta_F = \langle \tilde{U}_2, F(u_0) \rangle = bC_0^2(t_0),
\]

\[
A^{-1}QF(u_0) = -f(u_0)U_3.
\]

(4.85)
Then we end up with

\[ u_1(t; t_0) = \{ aC_0^2(t_0)(t - t_0) + C_1(t_0)(t - t_0) + \frac{1}{2} bC_0^2(t_0)(t - t_0)^2 \} U_1 + \{ bC_0^2(t_0)(t - t_0) + C_1(t_0) \} U_2 + f(u_0) U_3. \] (4.86)

Accordingly,

\[ W_1(t_0) = C_1(t_0) U_2 + f(u_0) U_3, \] (4.87)

which implies that the modified invariant manifold is given by

\[ M_1 = \{ u | u = t(C_0, C_1, f(C_0, 0, 0)) \}. \] (4.88)

If we stop at this order, the full solution is given by \( u \simeq u_0 + \epsilon u_1 \). Applying the RG equation, we have

\[ \dot{C}_0 = \epsilon (aC_0^2 + C_1), \quad \dot{C}_1 = bC_0^2. \] (4.89)

And the trajectory on the manifold \( M \simeq M_1 \) is given by

\[ u(t) = C_0(t) U_1 + \epsilon C_1(t) U_2 + \epsilon f(C_0(t), 0, 0) U_3. \] (4.90)

To go to the second order, we first need to examine the compatibility condition;

\[ 0 = QF'(u_0) U_1 = \frac{\partial f(x, y, z)}{\partial x} \bigg|_{u=u_0} U_3, \] (4.91)

with \( u_0 = t(C_0, 0, 0) \), which means that when \( y = z = 0 \), \( f(x, y, z) \) does not depend on \( x \).

With this condition assumed, we can proceed to the second order. To obtain the second order RG equation, we notice the following: \( h = -A^{-1}QF(u_0) + C_1 U_2 = f(u_0) U_3 + C_1 U_2 \) and hence \( F'(u_0) h = 0 \). Thus the RG equation which gives the evolution equation of \( C_{0,1}(t) \) is not modified by the second order perturbation.

To obtain the second order correction to the trajectory, we need to evaluate the following;

\[ -A^{-1}QF' U_2 = (f(u_0) \frac{\partial f}{\partial z} + C_1 \frac{\partial f}{\partial y}) U_3, \]

\[ -A^{-2}QF' U_2 = \frac{\partial f}{\partial y} U_3, \] (4.92)

where the derivatives are evaluated at \( u = u_0 = t(C_0(t), 0, 0) \). Thus the second order correction of the initial value reads

\[ W_2(t_0) = (f(u_0) \frac{\partial f}{\partial z} + C_1 \frac{\partial f}{\partial y} - \beta_f \frac{\partial f}{\partial y}) U_3. \] (4.93)
Hence the trajectory in the second order approximation is given by

\[ u(t) = W(t) = C_0(t)U_1 + \epsilon C_1(t)U_2 + \epsilon f(u_0)U_3 \\
+ \epsilon^2 (f(u_0) \frac{\partial f}{\partial z} + C_1 \frac{\partial f}{\partial y} - \beta_f \frac{\partial f}{\partial y})U_3. \] (4.94)

Here \( C_{0,1}(t) \) are governed by the Takens equation \((4.89)\). We see that the higher order terms does not affect the dynamics but modifies the trajectory only in the \( U_2 \) and the Q-direction.

Similar discussions can be made for the following extended Bogdanov equation,

\[ \dot{x} = y, \quad \dot{y} = ax^2 + bxy, \quad \dot{z} = -z + \epsilon f(x, y, z), \] \( (4.95) \)

where \( f(x, y, z) \) is an analytic function.

### 4.6 Discussion

We have applied the naive perturbative expansion as the starting point of the RG method. However, the naive perturbation expansion is not always a good starting point. There are cases where a scaling transformation is needed to convert the unperturbed equation to a non-Jordan form before applying the perturbative expansion. Let us take the following example;

\[ \dot{u} = \begin{pmatrix} 0 & 1 - \epsilon \\ \epsilon & 0 \end{pmatrix} u, \] \( (4.96) \)

with \( u = t(x, y) \). The exact solution reads

\[ u(t) = Ae^{\lambda t}U_+ + Be^{-\lambda t}U_-, \] \( (4.97) \)

where \( \lambda = \sqrt{(1 - \epsilon)\epsilon} \) and \( U_\pm = t(1, \pm \sqrt{\epsilon/(1 - \epsilon)}); i.e., \)

\[ x(t) = Ae^{\lambda t} + Be^{-\lambda t}, \quad y(t) = \sqrt{\epsilon/(1 - \epsilon)} \cdot (Ae^{\lambda t} - Be^{-\lambda t}). \] \( (4.98) \)

Due to the appearance of the singular term \( \sqrt{\epsilon} \), the naive perturbation does not give a sensible result even in the RG method. In this case, we first try to convert the equation so that the converted equation has no Jordan nature in the unperturbed part. This can be performed by a scale transformation; \( x = \epsilon^\alpha \xi, \quad y = \epsilon^\beta \eta, \quad t = \epsilon^\nu \tau \). To make the unperturbed part to be a non-Jordan form, we choose that \( \beta - \alpha + \nu = 1 + \alpha - \beta + \nu = 0 \), which is satisfied with \( \alpha = 0, \quad \beta = \nu = 1/2 \), i.e., \( x = \xi, \quad y = \sqrt{\epsilon} \eta, \quad t = \tau/\sqrt{\epsilon} \). The converted equation reads

\[ \frac{d\xi}{d\tau} = \eta - \epsilon \eta, \quad \frac{d\eta}{d\tau} = \xi, \] \( (4.99) \)
which can be now solved by the RG method with no difficulty. The result obtained by the RG method up to $O(\epsilon^2)$ reads

$$x(t) = Ae^{\lambda t} + Be^{-\lambda t}, \quad y(t) = \sqrt{\epsilon}(1 + \epsilon/2)(Ae^{\lambda t} - Be^{-\lambda t}),$$

(4.100)

with $\lambda' = \sqrt{\epsilon}(1 - \epsilon/2)$. 
5 Applications I

In this section, we present examples of non-linear equations for which the unperturbed linear operator has no zero eigenvalues but a pair of eigenvalues $\lambda_i$ ($i = 1, 2$); (i) $\lambda_{1,2} = \pm i\omega$ and (ii) $\lambda_{1,2} = \pm \lambda$ where $\omega$ and $\lambda$ are real numbers. The first case is discussed in [25, 26] as well as in §3. We present it here for completeness.

5.1 Brusselator

This is an example of systems showing a Hopf bifurcation. An RG treatment of generic systems with a bifurcation has been given in [26] where an emphasis is put on the relation of the RG method with the envelope theory. Here we treat this interesting example in the present formulation emphasizing the aspect of the RG method as the one to construct attractive manifolds.

The Brusselator is given by

\[
\begin{align*}
\frac{\partial X}{\partial t} &= A - (B + 1)X + X^2Y + DX \frac{\partial^2 X}{\partial x^2}, \\
\frac{\partial Y}{\partial t} &= BX - X^2Y + DY \frac{\partial^2 Y}{\partial x^2},
\end{align*}
\]

(5.1)

where $A(>0), B(>0), DX$ and $DY$ are constant. We here treat a uniform system, hence the terms with the spatial derivatives vanish. The steady state is given by $(X_0, Y_0) = (A, B/A)$. Shifting the variables as $\xi = X - X_0, \eta = Y - Y_0$, and defining $u = ^t(\xi, \eta)$, we have

\[
\frac{d}{dt} u = \begin{pmatrix} (B - 1)\xi + A^2\eta \\ -B\xi - A^2\eta \end{pmatrix} + f(\xi, \eta) \begin{pmatrix} 1 \\ -1 \end{pmatrix},
\]

(5.2)

with

\[
f(\xi, \eta) = B/A \cdot \xi^2 + 2A\xi\eta + \xi^2\eta.
\]

The linear stability analysis shows that when $B$ exceeds the critical value $B_c = 1 + A^2$, there arises a bifurcation.

Now let us analyze the slow motion and the slow manifold around the bifurcation (critical) point. We define the following variables

\[
\mu = (B - B_c)/B_c, \quad \epsilon = \sqrt{\mu}, \quad \text{and} \quad \chi = \text{sgn}(\mu),
\]

(5.4)

accordingly, $\mu = \chi\epsilon^2$. We first expand $u$ and the initial value $u(t = t_0; t_0) = W(t_0)$ as Taylor series w.r.t $\epsilon$: $u = \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \cdots$, and $W = \epsilon W_1 + \epsilon^2 W_2 + \epsilon^3 W_3 + \cdots$. 

38
The first order equation reads

\[ (\partial_t - L_0)u_1 = 0, \]  

(5.5)

with

\[ L_0 = \begin{pmatrix} A^2 & A^2 \\ -(A^2 + 1) & -A^2 \end{pmatrix}, \]  

(5.6)

The solution is readily found to be

\[ u(t; t_0) = C(t_0)U e^{i\omega t} + \text{c.c.}, \]  

(5.7)

with \( \omega = A \) and

\[ U = \begin{pmatrix} 1 \\ i1+iA \\ A \end{pmatrix}. \]  

(5.8)

Here \( C \) is the (complex) integral constant. Accordingly,

\[ W_1(t_0) = C(t_0)U e^{i\omega t_0} + \text{c.c.} \]  

(5.9)

A simple manipulation using the formulae given in Appendix A gives the higher order terms as follows;

\[ u_2(t; t_0) = \{ C^2 V + e^{2i\omega t} + \text{c.c.} \} + |C|^2 V_0, \]  

(5.10)

where

\[ V_+ = \frac{1 + iA}{3A^3} \begin{pmatrix} -2iA \\ 1 + 2iA \end{pmatrix}, \quad V_0 = 2\frac{A^2 - 1}{A^3} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \]  

(5.11)

and

\[ u_3(t; t_0) = \left[ \frac{C_1}{2} \{(t - t_0)U + \frac{1}{2i\omega} U^*\} e^{i\omega t} + \frac{C_3}{4i\omega} (U + \frac{1}{2} U^*) e^{3i\omega t} \right] + \text{c.c.}, \]  

(5.12)

where

\[ C_1 = \chi B_c C + \{ -\frac{2 + A^2}{A^2} + i\frac{-4A^4 + 7A^2 - 4}{3A^3} \}|C|^2 C, \]  

(5.13)

\[ C_3 = \{ 2B_c V_{+\xi}/A + 2A(V_{+\eta} + V_{+\xi} \bar{U}_\eta + U_\eta) \} C^3, \]  

(5.14)

with \( V_{+\xi} \) being the \( \xi \)-component of \( V_+ \) and so on. Here the initial values have been chosen to be

\[ W_2(t_0) = \{ C^2 V + e^{2i\omega t_0} + \text{c.c.} \} + |C|^2 V_0, \]  

(5.15)

\[ W_3(t_0) = \left[ \frac{C_1}{4i\omega} U^* e^{i\omega t} + \frac{C_3}{4i\omega} (U + \frac{1}{2} U^*) e^{3i\omega t} \right] + \text{c.c.}. \]  

Here \( U^* \) is the complex conjugate of \( U \).
Collecting all the terms thus obtained to have the approximate $u(t; t_0)$ and applying
the RG equation to it, we have $dC/dt - \epsilon^2 C_1 = 0$, or

$$\frac{dC}{dt} = \alpha C + \beta |C|^2 C,$$

(5.16)

with

$$\alpha = \chi(1 + A^2), \quad \beta = -\frac{2 + A^2}{A^2} + \frac{i-4A^4 + 7A^2 - 4}{3A^3}.$$

(5.17)

The attractive manifold is given by the initial value as

$$u(t) = W(t) \approx \epsilon W_1(t) + \epsilon^2 W_2(t) + \epsilon^3 W(t),$$

$$= \epsilon\{C(t)Ue^{i\omega t} + \text{c.c.}\} + \epsilon^2[(C^2(t)V + e^{2i\omega t} + \text{c.c.}) + |C|^2 V_0]$$

$$+ \epsilon^3\left[\frac{C_1(t)}{4i\omega}U^* e^{i\omega t} + \frac{C_3(t)}{4i\omega}(U + \frac{1}{2}U^*)e^{3i\omega t}\right] + \text{c.c.}$$

(5.18)

These results coincide with those obtained in the reductive perturbation method[18].

5.2 Unstable motion in Lotka–Volterra system

The Lotka–Volterra (LV) equation[53] is known to be integrable, although the exact analytic solutions of it are not known. The equation was already treated in the RG method
by one of the present authors (TK), and an approximate solution was constructed explicitly
around the non-trivial fixed point[26]. A numerical comparison of the results with
the exact solution shows that the solution given by the RG method well approximates
the exact solution in a global domain even when the small parameter in the equation $\epsilon, \epsilon'$
(see below) are as large as 0.8[54]. The purpose of the present subsection is to apply the
RG method for analyzing the equation around the unstable fixed point (the origin).

Lotka–Volterra equation reads

$$\begin{cases}
\dot{x} = ax - \epsilon xy \\
\dot{y} = -by + \epsilon' xy
\end{cases}$$

(5.19)

Here, $x = x(t), y = y(t)$ and $a, b, \epsilon, \epsilon'$ are positive constants. In this work, we treat the case
where $0 < \epsilon < 1$ and $0 < \epsilon' < 1$ so that the perturbation theory can be applied.

There are two fixed points; (i) $x = y = 0$ and (ii) $x = b/\epsilon', \quad y = a/\epsilon$. One can see
that the fixed point (i) is asymptotically unstable. An approximate but globally valid
solution around the second fixed point was obtained in [26]. In this paper, we treat the
first fixed point.

With the new variables defined by $x = \frac{\epsilon}{a}\xi$, \quad $y = \eta$ (5.19), is converted to

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u} + \epsilon F(\mathbf{u}),$$

(5.20)
where
\[ u = \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad A = \begin{pmatrix} a & 0 \\ 0 & -b \end{pmatrix}, \quad F(u) = -\xi \eta \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \] (5.21)

One see that the fixed point (i) is asymptotically unstable in the linear approximation. Our aim here is to obtain the dynamics around the unstable fixed point and construct the unstable manifold using the RG method.

For later convenience, we introduce the normalized eigenvectors of \( A \):
\[ A U_1 = a U_1, \quad A U_2 = -b U_2, \] (5.22)
where \( U_1 = t(1,0) \) and \( U_2 = t(0,1) \).

Now let us apply the RG method to construct an approximate solution valid in a global domain. We suppose that the equation is solved up to arbitrary \( t = t_0 \) from the time, say \( t = 0 \), at which the genuine initial condition is imposed. With this up-to-date initial value \( W(t_0) \), we try to construct the solution to (5.20) around \( t \sim t_0 \) by the perturbation theory, expanding \( u \) as
\[ u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + o(\epsilon^3). \]

The equations to be solved are
\[ \frac{d}{dt} - A) u_0 = 0, \] (5.23)
\[ \frac{d}{dt} - A) u_1 = -\xi_0 \eta_0 (U_1 - U_2), \] (5.24)
\[ \frac{d}{dt} - A) u_2 = - (\xi_0 \eta_1 + \xi_1 \eta_0) (U_1 - U_2), \] (5.25)
and so on, where \( t(\xi_i, \eta_i) = u_i \) (\( i = 0, 1, 2 \ldots \)). It turns out that one needs to treat separately depending on whether \( a \neq b \) or \( a = b \).

**Case A: \( a = b \)**

This is the interesting case where secular terms appear and the RG method plays a role to construct an approximate solution in a global domain. The unperturbed solution reads
\[ u_0 = C_1(t_0) e^{at} U_1 + C_2(t_0) e^{-at} U_2, \] (5.26)
implying the initial condition \( W_0(t_0) = C_1(t_0) e^{at_0} U_1 + C_2(t_0) e^{-at_0} U_2 \). The first order solution now reads
\[ u_1 = -C_1 C_2 \frac{1}{\partial_t - A} (U_1 - U_2) = C_1 C_2 \left( \frac{1}{a} (U_1 + U_2) \right), \] (5.27)
which is constant and independent of \( u_0(t) \).
Using the formulae in Appendix A, the second order solution is also obtained. Collecting all the terms thus obtained, we have an approximate solution valid around $t \sim t_0$;

$$
\mathbf{u}(t : t_0) = C_1(t_0)e^{at}\mathbf{U}_1 + C_2(t_0)e^{-at}\mathbf{U}_2 + \epsilon \frac{C_1C_2}{a}(\mathbf{U}_1 + \mathbf{U}_2) \\
+ \epsilon^2 \left[ -\frac{C_1C_2}{a} + \frac{C_1C_2}{2a^2} e^{-at} \right] \mathbf{U}_1 \\
+ \{ \frac{C_1C_2}{2a^2} e^{-at} + \frac{C_1C_2}{a} (t - t_0) e^{-at} \} \mathbf{U}_2.
$$

(5.28)

Applying the RG equation $\partial \mathbf{u}/\partial t|_{t_0} = 0$ to (5.28), we have

$$
\dot{C}_1 + \epsilon^2 \frac{C_1C_2}{a} = 0, \quad \dot{C}_2 - \epsilon^2 \frac{C_1C_2}{a} = 0.
$$

(5.29)

Noting the constraint $C_1C_2 = \text{const.} = c$ given by (5.29), we reach

$$
C_1(t) = c_1 e^{-\frac{c_2}{a}t}, \quad C_2(t) = \frac{c}{c_1} e^{\frac{c_2}{a}t},
$$

(5.30)

with $c_1$ being a constant.

Our solution is given by the initial value $\mathbf{W}(t)$;

$$
\mathbf{u}(t; t_0) = \mathbf{W}(t) = C_1(t)e^{at}\mathbf{U}_1 + C_2(t)e^{-at}\mathbf{U}_2 + \epsilon \frac{C_1C_2}{a}(\mathbf{U}_1 + \mathbf{U}_2) \\
+ \epsilon^2 \{ C_1(t) e^{-at} \mathbf{U}_1 + \frac{C_1(t)}{2a^2} e^{-at} \mathbf{U}_2 \}.
$$

(5.31)

with $C_{1,2}(t)$ given by (5.30). Introducing $c_2$ by $c = c_1c_2$, the respective components are given by

$$
\xi(t; t) = c_1 e^{(a - \frac{c_2}{a})t} + \epsilon \frac{c_1c_2}{a} e^{-\left(\frac{c_2}{a} - \frac{c_2}{a} + \frac{c_2}{a} \right)t} + o(\epsilon^2),
$$

(5.32)

$$
\eta(t; t) = c_2 e^{-a - \frac{c_2}{a}}t + \epsilon \frac{c_1c_2}{a} e^{-\left(\frac{c_2}{a} - \frac{c_2}{a} + \frac{c_2}{a} \right)t} + o(\epsilon^2).
$$

(5.33)

Here, the constants $c_1$ and $c_2$ are determined by the initial condition imposed at $t = 0$.

Some remarks are in order: (1) Our solution (5.31) shows that the speed to approach to and to escape from the origin is shifted from $a$ to $a - \epsilon^2 c_1c_2/a \equiv \alpha$, which is dependent on the initial condition as expressed by $c_1$ and $c_2$. (2) The unstable manifold can be constructed explicitly from (5.32) and (5.33). Solving $e^{at}$ and $e^{-at}$ from these equations and making the product of them, we have

$$
c = (X - \theta Y)(Y - \theta X),
$$

(5.34)

where $X = \xi x - x_0$, $Y = y - y_0$, $\theta = \frac{\epsilon^2}{a}$ and $x_0 = y_0 = \epsilon^2 x$. (5.34) shows that the unstable manifold is the hyperbolic curve which has the non-orthogonal asymptotic lines coming out of the point $t(x_0, x_0)$ with the slopes $\theta$ and $1/\theta$, respectively.
Case B: $a \neq b$

This is a rather trivial case because no secular terms appear, and the RG method does not play any role. Therefore we only write down the result:

$$\begin{align*}
\xi(t) &= C_1 e^{at} + \epsilon \frac{C_1 C_2}{b} e^{(a-b)t} \\
&\quad + \epsilon^2 \left\{ \frac{C_1 C_2^2}{2b^2} e^{(a-2b)t} - \frac{C_1^2 C_2}{a(a - b)} e^{(2a-b)t} \right\} + 0(\epsilon^2), \\
\eta(t) &= C_2 e^{-bt} + \epsilon \frac{C_1 C_2}{a} e^{(a-b)t} \\
&\quad + \epsilon^2 \left\{ \frac{C_1^2 C_2^2}{2a^2} e^{(2a-b)t} + \frac{C_1 C_2^2}{b(a - b)} e^{(a-2b)t} \right\} + 0(\epsilon^2),
\end{align*}$$

(5.35)

where $C_1$ and $C_2$ are constant.
6 Applications II: Pulse interactions

In this section, we apply the RG method to obtain the dynamics of interacting pulses (or fronts) in one dimension. Some years ago, a systematic method was developed by Ei and Ohta [45] to this problem on the basis of the phase dynamics approach [19] which involves a solvability condition. We shall show how the dynamics of interacting pulses obtained by them can be derived mechanically in the present method virtually without any assumptions. In other words, we shall derive the phase equations describing the front and pulse interactions in the RG method for the first time, although interface dynamics in spinodal decomposition [29], a diffusion equation [30] and Kuramoto-Sivashinski equation [31] as the phase equations have been derived in the RG method by others. As representative examples of pulse dynamics, we take up the kink-anti-kink interactions in the time-dependent Ginzburg-Landau (TDGL) equation and the soliton-soliton interaction in the Kortweg-de Vries (KdV) equation.

6.1 Kink-anti-kink interaction in TDGL equation

The TDGL equation we study is given by

\[
\frac{\partial u}{\partial t} = \epsilon^2 \frac{\partial^2 u}{\partial x^2} + f(u),
\]

(6.1)

with

\[
f(u) = \frac{1}{2} u (1 - u^2),
\]

(6.2)

where \( u = u(x, t) \) is a real scalar function and \( 0 < \epsilon < 1 \). We remark that (6.1) has a stationary solution

\[
u(x, t) = \pm \tanh \frac{x - h}{2\epsilon} \equiv \pm U(x - h),
\]

(6.3)

where \( U(x) \) satisfies \( U(\pm \infty) = \pm 1 \) and \( U(0) = 0 \). \( U(x - h) (-U(x - h)) \) is called a kink (an anti-kink) at the position \( x = h \).

6.1.1 Interaction of one kink and anti-kink

We first consider the interaction of one kink and anti-kink. Suppose the initial data at \( t = 0 \) are given by

\[
U_0(x; x_1(0), x_2(0)) = U(x - x_1(0)) + \{-U(x - x_2(0)) - 1\},
\]

44
where \( x_2(0) - x_1(0) \gg \epsilon \); i.e., the kink (anti-kink) is located at \( x = x_1(0) \ (x = x_2(0)) \). Since \( U_0(x; x_1(0), x_2(0)) \) is not a solution to Eq. (6.1), the position of the kink \( x_1 \) and the anti-kink \( x_2 \) will move slowly. Our task is to find the equation governing the dynamics of \( x_1(t) \) and \( x_2(t) \) at \( t > 0 \).

To apply the RG method to solve this problem, we first identify the small parameter and the integral constants in the unperturbed solution, which will move by the perturbation. When \( x_2 - x_1 \gg \epsilon \), \( U_0(x; x_1, x_2) \) in the neighborhood of \( x \sim x_1 \) may be represented by

\[
U_0(x; x_1, x_2) = u_0(t; x_1)(x) + \delta s(x - x_2) + v(x, t),
\]

with \( v(x, t) \) as small as \( \delta \). Inserting (6.4) into (6.1), we have

\[
\partial_t v = Av + (f'(u_0) - f'(1))\delta s + O(\delta^2 + |v|^2),
\]

with

\[
Av \equiv \left( \epsilon^2 \frac{\partial^2}{\partial x^2} + f'(u_0) \right)v.
\]

Here, we have made use of the fact that \( u_0 \) and \( \delta s + 1 \) are stationary solutions to Eq. (6.1 ); we also note that \( f'(1) = f'(-1) \).

(6.3) is an equation of the type discussed in §§4.1. Indeed, owing to the translational invariance of the TDGL equation (6.1), the self-adjoint operator \( A \) has a zero eigenvalue with the eigenfunction \( U_1 \) together with the corresponding adjoint eigenfunction \( \tilde{U}_1 \)

\[
U_1(x) = \tilde{U}_1(x) = \partial_x U(x - x_1).
\]

Therefore, one can obtain the approximate solutions of (6.3) and the dynamics of \( x_1(t) \) according to the procedure developed in §§4.1.

The solution of the eigenvalue problem for the operator \( A \) may be seen in textbooks on quantum mechanics [47]. Let \( P \) be the projection operator onto the kernel of \( A \) and \( Q \) the one onto the subspace complement to the kernel \( A \); we call the respective subspaces \( P \)- and \( Q \)-space. We notice that

\[
P u = \frac{< U_1, u >}{< U_1, U_1 >} U_1(x),
\]

where \( < u, v > \) denotes the inner product defined by

\[
< u, v > = \int_{-\infty}^{\infty} u(x)v(x)dx.
\]
Now let us apply the RG method to solve \( v \), thereby obtain the dynamics of \( x_1(t) \). First we expand \( v \) as

\[
v = \delta v_1 + O(\delta^2). \tag{6.10}
\]

The equation for \( v_1 \) read

\[
\partial_t v_1 = Av_1 + (f'(u_0) - f'(1))s.
\]

In this case, \( W_0(t_0) \) in §§4.1 is \( U(x - x_1(t_0)) \). \( W_1(t_0) \) in \( Q \)-space is given by \((-A)^{-1}Q\{f'(u_0) - f'(1)\}s\) by (4.16) and \( v_1 \) is

\[
v_1(t; t_0)(x) = (t - t_0)P(f'(u_0) - f'(1))s + A^{-1}Q(f'(u_0) - f'(1))s. \tag{6.11}
\]

Thus, we have an approximate function

\[
u(t; t_0)(x) = U(x - x_1(t_0)) + \delta v_1(t; t_0)(x) + \delta s(x - x_2(t_0)).
\]

Applying the RG equation to \( u(t; t_0)(x) \), one has

\[
0 = \left. \frac{\partial u}{\partial t_0} \right|_{t_0=t} = -\dot{x}_1 \partial_x U(x - x_1) - P(f'(u_0) - f'(1))\delta s(x - x_2) + O(\delta^2),
\]

leading to the dynamics governing \( x_1 \),

\[
\dot{x}_1 = -\frac{\langle U_1, (f'(u_0) - f'(1))\delta s \rangle}{\langle U_1, U_1 \rangle} + O(\delta^2)
\]

\[
= -\frac{\langle \partial_x U(x - x_1), (f'(U(x - x_1)) - f'(1))\{U(x - x_2) - 1\} \rangle}{\langle U_1, U_1 \rangle} + O(\delta^2)
\]

\[
= -\frac{\langle \partial_x U(x), (f'(U(x)) - f'(1))\{U(x - (x_2 - x_1)) - 1\} \rangle}{\langle U_1, U_1 \rangle} + O(\delta^2)
\]

\[
= 12\epsilon e^{-\frac{x_2 - x_1}{\epsilon}} + O(\delta^2), \tag{6.12}
\]

which coincides with the result by Carr-Pego\[44\], Fuco-Hale\[46\] and Ei-Ohta\[45\] where an explicit evaluation of the inner products is also given.

### 6.1.2 Kink-anti-kink interaction in the presence of infinite kinks and anti-kinks

We next consider an initial value problem of Eq.(6.1) with the initial condition where infinite kinks (anti-kinks) are located periodically at \( x = h + 2n \ (x = -h + 2n) \) with \( n = 0, \pm 1, \pm 2... \). We suppose that the intervals of a kink and the neighboring anti-kinks are much larger than the width of a kink (anti-kink), i.e., \( h \gg \epsilon \). In this
situation, the problem may be formulated as an initial value problem with one kink at 
\( x = h \) in a finite domain \( 0 < x < 1 \) with a Neumann boundary condition, i.e.,

\[
\frac{\partial u}{\partial x}_{|x=0} = \frac{\partial u}{\partial x}_{|x=1}.
\]  

(6.13)

The initial profile in this case can be approximately represented by the function

\[
U_0(x; h) = U(x - h) + \{-U(x + h - 2) - 1\} + \{-U(x + h) + 1\},
\]

(6.14)

\[
\equiv u_0(x) + \delta_1 r(x + h - 2) + \delta_2 \ell(x + h)
\]  

(6.15)

where the second and the third terms denote the small effects coming from the anti-kink
at \( x = 2 - h \) and \( x = -h \), respectively; the smallness of the effects are represented by
the parameters \( \delta_1 \) and \( \delta_2 \), which are the orders of \( \exp(-2h/\epsilon) \) and \( \exp(-2(1-h)/\epsilon) \), respectively.

Let us represent the solution for \( t > 0 \) by

\[
u(x, t) = U_0(x; h) + v(x, t),
\]

(6.16)

where \( |v(x, t)| \) is supposed to be as small as \( \delta_1 \) and \( \delta_2 \). Inserting (6.16) into Eq.(6.1), we have

\[
\partial_t v = Av + (f'(u_0) - f'(1))(\delta_1 r + \delta_2 \ell) + O(\delta_1^2 + \delta_2^2 + |v|^2)
\]  

(6.17)

in a similar manner with (6.4). Here, \( Av \equiv (\epsilon^2 \frac{\partial^2}{\partial x^2} + f'(u_0))v \) and the eigenfunction
associated with zero eigenvalue in this case is \( U_1 = \partial_x U(x - h) \).

Let us apply the RG method to solve \( v \), thereby obtain the dynamics of \( h(t) \). First
we expand \( v \) as

\[
v = \delta_1 v_{1,0} + \delta_2 v_{0,1} + O(\delta_1^2 + \delta_2^2).
\]  

(6.18)

The equations for \( v_{1,0} \) and \( v_{0,1} \) read

\[
\partial_t v_{1,0} = Av_{1,0} + (f'(u_0) - f'(1))r,
\]

(6.19)

\[
\partial_t v_{0,1} = Av_{0,1} + (f'(u_0) - f'(1))\ell,
\]

(6.20)

respectively. These are of the same form as treated in §4.1, hence readily solved to be

\[
v_{1,0}(t; t_0)(x) = (t - t_0)P(f'(u_0) - f'(1))r + A^{-1}Q(f'(u_0) - f'(1))r,
\]

(6.21)

where \( r = r(x + h - 2) \) and \( h = h(t_0) \), by choosing the initial value

\[
v_{1,0}(t_0; t_0)(x) = -A^{-1}Q(f'(u_0) - f'(1))r(x + h(t_0) - 2),
\]

(6.22)

and \( v_{0,1}(x, t) \) with \( r \to \ell \). Thus, we have

\[
u(t; t_0)(x) = U(x - h(t_0)) + \delta_1 v_{1,0}(t; t_0)(x) + \delta_2 v_{0,1}(t; t_0)(x) + \delta_1 r + \delta_2 \ell.
\]  

47
Applying the RG equation to \( u(t; t_0)(x) \), one has

\[
0 = \left. \frac{\partial u}{\partial t_0} \right|_{t_0=t} = -\dot{h}U'(x-h) - P(f'(u_0) - f'(1))(\delta_1 r(x+h-2) + \delta_2 \ell(x+h)) + O(\delta_1^2 + \delta_2^2),
\]

leading to the dynamics governing \( h \),

\[
\dot{h} = -\epsilon \frac{< U_1, (f'(u_0) - f'(1))(\delta_1 r + \delta_2 \ell) >}{< U_1, U_1 >} + O(\delta_1^2 + \delta_2^2)
\]

\[
= -12\epsilon \left( e^{-2h} - e^{-2(1-h)} \right) + O(\delta_1^2 + \delta_2^2),
\]

which also coincides with the known result by Carr-Pego\cite{44} and Fusco-Hale\cite{46}.

### 6.2 Soliton-soliton interaction in KdV equation

The KdV equation reads

\[
\partial_t u + 6u \partial_x u + \partial_x^2 u = 0,
\]

which has a one-pulse solution given by

\[
u(x, t) = \frac{c}{2} \text{sech}^2 \left[ \frac{\sqrt{c}(x - ct)}{2} \right] \equiv \varphi(x - ct; c),
\]

with \( c \) being a velocity.

We consider the following problem: When two sufficiently separated pulses with almost the same velocities are located at \( x_1(0) \) and \( x_2(0) \) where \(|x_2(0) - x_1(0)| \gg \frac{1}{\sqrt{c}}\), how will the locations \( x_i(t) \) \((i = 1, 2)\) change at \( t > 0\)? To solve this problem, Ei and Ohta\cite{45} started with the ansatz

\[
u(x, t) = \varphi(x - ct - x_1; c + \dot{x}_1) + \varphi(x - ct - x_2; c + \dot{x}_2) + b(x - ct, t).
\]

In the present work, we shall apply the RG method without any ansatz to this problem and show that the same evolution equation as that obtained by Ei and Ohta is derived, thereby give a foundation of their treatment.

To study the problem, it is convenient to change the independent variables to

\[
t = t, \quad z = x - ct,
\]

namely to change to the co-moving frame with the pulse. Then the equation is converted to

\[
\partial_t u + F[u] = 0,
\]
with

\[ F[u] = -c \partial_z u + 6u \partial_z u + \partial_z^3 u. \]  
(6.30)

We remark that

\[ F[\varphi(z - b; c)] = 0, \]  
(6.31)

with \( b \) being an arbitrary constant.

Let us suppose that the solution around \( t = t_0 > 0 \) is given by

\[
\begin{align*}
  u(z, t) &= \varphi(z - z_1(t_0); c) + \varphi(z - z_2(t_0); c) + v(z, t), \quad \text{(6.32)} \\
  \equiv \varphi^{(1)} + \varphi^{(2)} + v, \quad \text{(6.33)}
\end{align*}
\]

where \(|z_2 - z_1|\) is sufficiently large. To study the effect coming from the other pulse, it is sufficient to consider the case of either \( z \sim z_1 \) or \( z \sim z_2 \). Then \( v \) is considered to be small.

Substituting Eq. (6.32) into (6.29), we have the equation governing \( v \) as

\[
\partial_t v + F'[\varphi^{(1)} + \varphi^{(2)}]v + 6\partial_z (\varphi^{(1)} \varphi^{(2)}) + O(|v|^2) = 0. \]  
(6.34)

Here we have used the identity \( F[\varphi^{(1)} + \varphi^{(2)}] = F[\varphi^{(1)} + \varphi^{(2)}] - F[\varphi^{(1)}] - F[\varphi^{(2)}] = 6\partial_z (\varphi^{(1)} \varphi^{(2)}) \) on account of Eq. (6.31).

Now let \( z \sim z_1 \), then \( \varphi^{(2)} \) is small and we may put

\[
\varphi^{(2)}(z - z_2(t_0)) = \delta g(z - z_2(t_0)), \]  
(6.35)

with a small parameter \( \delta \) whose order is \( e^{-\sqrt{c}(z_2 - z_1)} \). Then, (6.34) of \( v \) becomes

\[
\partial_t v = A^{(1)} v - 6\delta \partial_z (\varphi^{(1)} g) + O(\delta^2 + |v|^2), \]  
(6.36)

where

\[
A^{(1)} = -F'[\varphi^{(1)}] = c \partial_z - \partial_z^3 - 6(\partial_z \varphi^{(1)} + \varphi^{(1)} \partial_z). \]

Transforming \( z - z_1(t_0) \) to \( z' \), we see (6.36) becomes

\[
\partial_t v = Av - 6\delta \partial_z (\varphi^{(h)} g) + O(\delta^2 + |v|^2), \]  
(6.37)

where \( A = -F'(\varphi) \) and \( \delta g^{(h)}(z') = \varphi(z' - h) \) with \( h = z_2(t_0) - z_1(t_0) \).

(6.37) is an equation of the type discussed in §§4.3; the linear operator \( A \) has a Jordan cell reflecting the translational invariance of KdV equation and the arbitrariness of the velocity of a pulse,

\[
AU_1 = 0, \quad AU_2 = U_1, \]  
(6.38)

where

\[
U_1 = \partial_z \varphi, \quad U_2 = -\partial_c \varphi. \]  
(6.39)
The adjoint operator $A^\dagger$ of $A$ reads

$$A^\dagger = -c \partial_{z'} + \partial_{z'}^3 + 6\varphi \partial_{z'},$$  \hspace{1cm} (6.40)$$

which has also a two-dimensional Jordan cell. The zero mode $\tilde{U}_2$ of $A^\dagger$ is found to be

$$\tilde{U}_2(z') = \varphi(z'; c)$$  \hspace{1cm} (6.41)$$

while these eigenfunctions have not been normalized as (4.36) yet. It is known that there exists a function $\tilde{U}_1$ which satisfies

$$A^\dagger \tilde{U}_1 = \tilde{U}_2.$$  \hspace{1cm} (6.42)$$

The explicit form of it will be given later.

Let $P$ be the projection operator onto the subspace spanned by $U_1$ and $U_2$, and $Q$ onto the subspace compliment of the $P$-space.

Noting that $|v| \leq O(\delta)$ and expanding $v$ as

$$v = \delta v_1 + \delta^2 v_2 + \cdots,$$  \hspace{1cm} (6.43)$$

one has for $v_1$ from (6.37)

$$\partial_t v_1 = Av_1 - 6\partial_{z'}(\varphi g^{(h)}).$$  \hspace{1cm} (6.44)$$

Now we can proceed according to the general procedure given in §§4.3 as follows: Note that $u_0$ and hence $W_0(t_0)$ in §§4.3 correspond to $\varphi(z - z_1(t_0); c)$ because both of them comes from the translation invariance with respect to $z$. Let us solve the equation (6.44) with the initial value $W_1(t_0)$. Then, we have

$$v_1(t; t_0) = e^{(t-t_0)A} W_1(t_0) - 6 \int_{t_0}^t e^{(t-s)A} \partial_{z'}(\varphi g^{(h)}) ds,$$  \hspace{1cm} (6.45)$$

with

$$W_1(t_0) = C(t_0) U_2 + Q W_1(t_0)$$  \hspace{1cm} (6.46)$$

as discussed in §§4.3. Here we note that although $W_1(t_0)$ could have a $U_1$ component, its effect can be taken into account by a redefinition of $z_1(t_0)$ which is not yet determined. According to the discussion given in §§4.3, the $Q$-component of $W_1$ should be

$$Q W_1(t_0) = -A^{-1}Q(-6\partial_{z'}(\varphi g^{(h)})).$$  \hspace{1cm} (6.47)$$

Thus we have

$$v_1(t; t_0)(z') = \left\{ (t - t_0)C(t_0) + (t - t_0)\alpha + \frac{1}{2}(t - t_0)^2 \beta \right\} U_1$$

$$+ \{C(t_0) + (t - t_0)\beta\} U_2 - A^{-1}Q(-6\partial_{z'}(\varphi g^{(h)}))$$  \hspace{1cm} (6.48)$$
and the approximate function

\[ u(t; t_0)(z) = \varphi(z - z_1(t_0); c) + \delta v_1(t; t_0)(z - z_1(t_0)) + \delta g(z - z_2(t_0)), \]

where

\[ \alpha = \frac{\left< \bar{U}_1, -6 \partial_z(\varphi g^{(h)}) \right>}{\left< \bar{U}_1, U_1 \right>}, \quad \beta = \frac{\left< \bar{U}_2, -6 \partial_z(\varphi g^{(h)}) \right>}{\left< \bar{U}_2, U_2 \right>}. \]  \hspace{1cm} (6.50)

Applying the RG equation to \( u(t; t_0)(z) \), one has

\[ \frac{\partial u}{\partial t_{0 \mid t = t}} = -\dot{\varphi}_1 \partial_z \varphi^{(1)} + \delta \left[ \{ -\alpha \} U_1^{(1)} + \{ \dot{C} \} U_2^{(1)} \right] + O(\delta^2), \]  \hspace{1cm} (6.51)

where \( U_1^{(1)}(z) = U_1(z - z_1(t_0)) = \partial_z \varphi(z - z_1(t_0)) \) and \( U_2^{(1)} \) is similarly given. Since \( U_1^{(1)} = \partial_z \varphi^{(1)} \), (6.51) leads to

\[ \dot{z}_1 = -\delta C(t) - \delta \alpha, \quad \dot{C} = \beta. \]  \hspace{1cm} (6.52)

Recalling that

\[ \delta \alpha = \check{\alpha} = \frac{\left< \bar{U}_1, -6 \partial_z(\varphi g^{(h)}) \right>}{\left< \bar{U}_1, U_1 \right>} = \frac{\left< \bar{U}_1, -6 \partial_z(\varphi g^{(h)}) \right>}{\left< \bar{U}_1, U_1 \right>}, \]

and

\[ \delta \beta = \check{\beta} = \frac{\left< \bar{U}_2, -6 \partial_z(\varphi g^{(h)}) \right>}{\left< \bar{U}_2, U_2 \right>} = \frac{\left< \bar{U}_2, -6 \partial_z(\varphi g^{(h)}) \right>}{\left< \bar{U}_2, U_2 \right>}, \]

where \( \varphi^{(h)}(z') = \varphi(z' - h) \), and putting \( \delta C(t) = C_1(t) \), we finally obtain the equations of motion governing the position of the kink and the speed of it,

\[ \dot{z}_1 = -C_1 + \frac{\left< \bar{U}_1, -6 \partial_z(\varphi g^{(h)}) \right>}{\left< \bar{U}_1, U_1 \right>}, \quad \dot{C}_1 = \frac{\left< \bar{U}_2, -6 \partial_z(\varphi g^{(h)}) \right>}{\left< \bar{U}_2, U_2 \right>}. \]  \hspace{1cm} (6.53)

Similarly, one can readily obtain the equation for \( z_2 \). This is the main result in this subsection.

Eliminating \( C_1 \) from the above equations, one can compare our result with those given by Ei and Ohta[13]. We first notice that

\[ \frac{d}{dt} \check{\alpha} = \frac{d}{dt} \frac{\left< \bar{U}_1, -6 \partial_z(\varphi g^{(h)}) \right>}{\left< \bar{U}_1, U_1 \right>} = O(\delta^2), \]  \hspace{1cm} (6.54)

because \( \check{\alpha} \) depends only on \( h = z_2 - z_1 \) and \( \dot{h} = O(\delta) \). Then we have

\[ \ddot{z}_1 = \frac{\left< \bar{U}_2, 6 \partial_z^2(\varphi g^{(h)}) \right>}{\left< \bar{U}_2, U_2 \right>} = -16c^{5/2}e^{-\sqrt{c}(z_2 - z_1)} + O(\delta^2), \]  \hspace{1cm} (6.55)
and similarly
\[ \ddot{z}_2 \simeq 16c^{5/2}e^{-\sqrt{c}(z_2-z_1)}, \]  
(6.56)
which coincide with the result by Ei and Ohta\[15\]. In short, we have derived the equation describing the soliton-soliton dynamics in the KdV equation given by Ei and Ohta with fewer ansatz on the basis of the RG method.

As noticed above, the system (6.53) have more information for the soliton-soliton dynamics than (6.55) (or (6.56)) because (6.53) gives the the change of the velocity \( C_1 \) as well as the speed of the position \( z_1 \). It is intriguing to see an explicit form of (6.53), which will provide more detailed information of the dynamics of the solitons.

The right hand side of the equation for \( \dot{C}_1 \) in (6.53) is already given in (6.55). Let us calculate the r.h.s. of \( \dot{z}_1 \). The adjoint eigenfunction \( \tilde{U}_1 \) is given by
\[ \tilde{U}_1 = \int_{-\infty}^{z} \partial_c \varphi(s; c)ds + \theta \varphi, \]
where \( \theta = -M_1/M_2 \) with \( M_1 = 1/2 \cdot \left( \int_{-\infty}^{\infty} \partial_c \varphi ds \right)^2 \) and \( M_2 = \varphi, \partial_c \varphi \geq 18 \). Note that \( \langle \tilde{U}_1, U_2 \rangle = 0 \) holds. \( M_2 \) is evaluated to be \( \sqrt{c}/2 \)[13]. We here evaluate \( M_1 \). We first notice that \( \partial_c \varphi = \varphi/c + z \partial_z \varphi/2c \). Then by a partial derivative, we have
\[ \int_{-\infty}^{z} \partial_c \varphi ds = \frac{1}{2c} \left( \int_{-\infty}^{z} \varphi ds + z\varphi \right) \]
\[ \rightarrow \frac{1}{2c} \int_{-\infty}^{\infty} \varphi ds = \frac{1}{\sqrt{c}} \quad (z \rightarrow \infty), \]
hence \( M_1 = 1/2c \) and
\[ \theta = -1/c\sqrt{c}. \]  
(6.57)

Since \( \tilde{U}_1 \) is related with \( \tilde{U}_2 (= \varphi) \) through (6.42), we see that
\[ 6\varphi \partial_z \tilde{U}_1 = c \partial_z \tilde{U}_1 - \partial_z^2 \tilde{U}_1 + \varphi . \]

Then performing a partial derivative, we have
\[ \langle \tilde{U}_1, -6\partial_z (\varphi \varphi^{(h)}) \rangle = \langle c \partial_z \tilde{U}_1 - \partial_z^2 \tilde{U}_1 + \varphi, \varphi^{(h)} \rangle \]
\[ \simeq 2c e^{-\sqrt{c}h} \int_{-\infty}^{\infty} e^{\sqrt{c}z} \{ c \partial_z \tilde{U}_1 - \partial_z^2 \tilde{U}_1 + \varphi \} dz, \]  
(6.58)
where we have used the relation \( \varphi^{(h)}(z) \simeq 2c e^{\sqrt{c}(z-h)} \) for \( h \gg 1 \). The remaining integral can be performed using the the asymptotic forms of the derivatives of \( \tilde{U}_1 \),
\[ \partial_z \tilde{U}_1 \simeq (4 - \sqrt{c}z)e^{-\sqrt{c}z}, \partial_z^2 \tilde{U}_1 \simeq -\sqrt{c}(5 - \sqrt{c}z)e^{-\sqrt{c}z}. \]  
(6.59)
Thus, for $L \gg 1$ we have
\[ \int_{-\infty}^{L} e^{\sqrt{c} z} \partial_z \tilde{U}_1 dz = e^{\sqrt{c} L} \tilde{U}_1(L) - \sqrt{c} \int_{-\infty}^{L} e^{\sqrt{c} z} \tilde{U}_1 dz, \quad (6.60) \]
\[ \int_{-\infty}^{L} e^{\sqrt{c} z} \partial_z^3 \tilde{U}_1 dz = -\sqrt{c}(9 - 2\sqrt{c} + ce^{\sqrt{c} L} \tilde{U}_1(L) - c\sqrt{c} \int_{-\infty}^{L} e^{\sqrt{c} z} \tilde{U}_1 dz. \quad (6.61) \]

Finally, the third term of (6.58) is evaluated to be:
\[ \int_{-\infty}^{L} e^{\sqrt{c} z} \phi dz = \sqrt{c} \int_{L'}^{L} \int_{-\infty}^{\infty} e^{2z \text{sech}^2 z} dz \quad (L' = \sqrt{c} L/2), \]
\[ = 2\sqrt{c} \left( 2L' + \log(1 + e^{-2L'}) + \frac{1}{e^{2L'} + 1} - 1 \right), \]
\[ \approx 2\sqrt{c}(2L' - 1) = 2\sqrt{c}\sqrt{c}L - 1). \quad (6.62) \]

Inserting (6.60) ∼ (6.62) into (6.58), we have
\[ \langle \tilde{U}_1, -6\partial_z(\phi\varphi^{(h)}) \rangle \approx 2ce^{-\sqrt{c}} \lim_{L \to \infty} \int_{-\infty}^{L} e^{\sqrt{c} z} \{ c \partial_z \tilde{U}_1 - \partial_z^3 + \phi \} dz \]
\[ = 14\sqrt{c}. \quad (6.63) \]

The normalization integral is evaluated to be
\[ \langle \tilde{U}_1, U_1 \rangle = -\langle \int_{-\infty}^{L} \partial_c \phi dz, \partial_z \phi \rangle = -\langle \partial_c \phi, \phi \rangle = -\sqrt{c}/2. \]

Thus we finally obtain
\[ \langle \tilde{U}_1, -6\partial_z(\phi\varphi^{(h)}) \rangle \approx -28ce^{-\sqrt{c}}, \quad (6.64) \]
and hence the equations for $z_1$ and $C_1$ is
\[ \begin{cases} 
\dot{z}_1 &= -C_1 - 28ce^{-\sqrt{c}(z_2 - z_1)}, \\
\dot{C}_1 &= 10c^{5/2}e^{-\sqrt{c}(z_2 - z_1)}, \end{cases} \quad (6.65) \]

together with a similar equation for $z_2$. Notice that Eq.(6.63) is consistent with the order estimate of Eq.(6.53).
7 Brief summary and concluding remarks

We have formulated the RG method as a powerful tool for reduction of evolution equations in terms of the notion of invariant manifolds, starting from the exact Wilson RG equation. We have given an argument as to why $t_0$ should be set $t = 0$ in the perturbative RG method. We have shown that the perturbative RG method constructs invariant manifolds successively as the initial value of evolution equations; the integral constants in the unperturbative solution constitutes natural coordinates of the invariant manifold when the linear operator $A$ in the evolution equation has no Jordan cell. When $A$ has a Jordan cell, there is a slight complication because the dimension of the invariant manifold is to change by the perturbation. The RG equation determines the slow motion of the integral constants in the unperturbative solution on the invariant manifold. We have worked out several examples to demonstrate our formulation. We have emphasized that the underlying structure of the reduction by the RG method completely fits to the universal structure elucidated by Kuramoto [16] a decade ago. The prescription suggested by the present formulation has turned out to be the same as that adopted in [25, 26]. We have applied the method to interface dynamics such as kink-anti-kink and soliton-soliton interactions in the latter of which a linear operator having a Jordan-cell structure appears.

In the present work, actual calculations are all based on the perturbation theory, although we have started the formulation with the exact flow equation. Recently, variants of the exact RG equations or flow equations are applied to various problems such as the chiral symmetry breaking in QCD [3, 50], the Bose-Einstein condensation in Alkali atoms [51] and so on. There, in stead of the perturbation theory, a truncation of the functional spaces are usually employed as a practical method of calculations. One may thus imagine that a possible practical method for applying the non-perturbative RG method could be a scheme similar to, say, the Galerkin method [62]. Such a non-perturbative RG method should be most interesting for analyses of partial differential equations. An attempt to apply a kind of non-perturbative RG equation to partial differential equations is given in [63].

We have tried to formulate the RG method so that the mathematical structure of the method becomes as transparent as possible. Then the present work could be a basis for clarifying a possible relation between the RG method and another powerful theory for reduction of evolution equations called Whitham’s averaging method [64]: The latter method has been successfully used to extract the equations describing modulations of dispersive non-linear waves: Modulations of the phase function are given by slow variables which are governed by so-called Whitham equations. One may thus imagine that the modulations as described by Whitham equations may also be given by the RG method as it gives the amplitude and phase equations. Or more strongly, Whitham equations might be derived as RG ones [64]. Furthermore, Whitham gave a foundation of his equations on the basis of a variational principle for the actions, which is reminiscent of the fact that Wilsonian RG is formulated for effective actions which admit a variational principle.
Acknowledgement
We are grateful to the referee to have directed our attention to Whitham’s method.
Appendix A  An efficient operator method of solution suitable for the RG method

In this Appendix, we shall summarize rules for obtaining appropriate special solutions of non-homogeneous equations appearing in the higher orders in the perturbative RG method. The essential point of the present method is to consider the equations with initial conditions at $t = \forall t_0$; the initial values are determined so that the terms in the special solutions disappear which either could be ”renormalized away” by a redefinition of the integral constants in the unperturbed solution or are the ones describing a rapid motion. We shall show that the simple rules to write down the special solutions with such initial conditions can be summarized as an operator method.

Let us consider the special solution to the equation given by

\[(\partial_t - A)u(t; t_0) = F(t), \quad (A.1)\]

with the initial condition at $t = t_0$,

\[u(t_0, t_0) = W(t_0). \quad (A.2)\]

The solution reads

\[u(t; t_0) = e^{A(t-t_0)}W(t_0) + e^{At} \int_{t_0}^{t} ds e^{-As}F(s). \quad (A.3)\]

Let $A$ be a semi-simple matrix with eigenvalues $\lambda_\alpha (\alpha = 1, 2, ...)$;

\[AU_\alpha = \lambda_\alpha U_\alpha. \quad (A.4)\]

The case where $A$ has a Jordan cell will be considered later.

Let

\[F(t) = e^{\lambda_\alpha t}U_\alpha + e^{\lambda_\beta t}U_\alpha + e^{\lambda_\alpha t}U_\beta, \quad (A.5)\]

with $\lambda_\alpha \neq \lambda_\beta$, then the solution is evaluated to be

\[u(t; t_0) = e^{A(t-t_0)} \left[ W(t_0) - \frac{1}{\lambda_\beta - A} e^{\lambda_\beta t}U_\alpha - \frac{1}{\lambda_\alpha - A} e^{\lambda_\alpha t}U_\beta \right] + (t - t_0)e^{\lambda_\alpha t}U_\alpha + \frac{1}{\lambda_\beta - A} e^{\lambda_\beta t}U_\alpha + \frac{1}{\lambda_\alpha - A} e^{\lambda_\alpha t}U_\beta. \quad (A.6)\]

The first line suggests that the initial value should be chosen as

\[W(t_0) = \frac{1}{\lambda_\beta - A} e^{\lambda_\beta t}U_\alpha + \frac{1}{\lambda_\alpha - A} e^{\lambda_\alpha t}U_\beta, \quad (A.7)\]
where the first term corresponds to the one which could be renormalized away by a redefinition of the unperturbed solution and the second term to a rapid motion. Thus we end up with the special solution given by

\[ u(t; t_0) = (t - t_0) e^{\lambda_\alpha t} U_\alpha + \frac{1}{\lambda_\beta - A} e^{\lambda_\beta t} U_\alpha + \frac{1}{\lambda_\alpha - A} e^{\lambda_\alpha t} U_\beta. \]  

(A.8)

Then one may summarize the results as rules of an operator method for obtaining special solutions as follows;

\[
\frac{1}{\partial_t - A} e^{\lambda t} U_\alpha = \frac{1}{\lambda - \lambda_\alpha} e^{\lambda t} U_\alpha, \\
= \frac{1}{\lambda - \lambda_\alpha} e^{\lambda t} U_\alpha, \quad (\lambda \neq \lambda_\alpha), \\
= (t - t_0) e^{\lambda_\alpha t} U_\alpha. \\
\]  

(A.9)

\[
\frac{1}{\partial_t - A} e^{\lambda_\alpha t} U_\alpha = \frac{1}{\partial_t - \lambda_\alpha} e^{\lambda_\alpha t} U_\alpha, \\
= (t - t_0) e^{\lambda_\alpha t} U_\alpha. \\
\]  

(A.10)

Similarly, one can verify that

\[
\frac{1}{\partial_t - A} (t - t_0)^n e^{\lambda t} U_\alpha = \frac{1}{\partial_t - \lambda_\alpha} (t - t_0)^n e^{\lambda t} U_\alpha, \\
= \frac{1}{n + 1} (t - t_0)^{n+1} e^{\lambda t} U_\alpha. \\
\]  

(A.11)

Furthermore, when \( \lambda \neq \lambda_\alpha \),

\[
\frac{1}{\partial_t - A} (t - t_0)^n e^{\lambda t} U_\alpha = e^{\lambda t_0} \left. \frac{1}{\partial_\tau - A} \tau^n e^{\lambda \tau} U_\alpha \right|_{\tau = t - t_0}, \\
= e^{\lambda t_0} \partial_\lambda^n \left. e^{\lambda \tau} U_\alpha \right|_{\tau = t - t_0}. \\
\]  

(A.12)

Hence, for example,

\[
\frac{1}{\partial_t - A} (t - t_0) e^{\lambda t} U_\alpha = \frac{1}{\lambda - A} ((t - t_0) - \frac{1}{\lambda - A}) e^{\lambda t} U_\alpha, \\
\frac{1}{\partial_t - A} (t - t_0)^2 e^{\lambda t} U_\alpha = \frac{1}{\lambda - A} ((t - t_0)^2 - \frac{2}{\lambda - A} (t - t_0) + \frac{2}{(\lambda - A)^2}) e^{\lambda t} U_\alpha. \\
\]  

(A.13)

(A.14)

where \( A \) may be replaced with \( \lambda_\alpha \).

Next, we consider the case where \( A \) has a semi-simple zero eigenvalue;

\[ A U_0 = 0. \]  

(A.15)

Let \( P \) and \( Q \) be the projection operator onto the space spanned by \( U_0 \) and its orthogonal compliment, respectively. If \( G \) is a constant vector, then one can easily verify that

\[
\frac{1}{\partial_t - A} PG = \frac{1}{\partial_t} PG = \int_{t_0}^t ds PG = (t - t_0) PG, \\
\frac{1}{\partial_t - A} QG = \frac{1}{-A} QG. \\
\]  

(A.16)

(A.17)
Similarly,
\begin{equation}
\frac{1}{\partial_t - A} f(t) PG = \frac{1}{\partial_t} PG = \int_{t_0}^{t} ds f(s) PG, \tag{A.18}
\end{equation}
\begin{equation}
\frac{1}{\partial_t - A} f(t) QG = \frac{1}{-A} \sum_{n=0}^{\infty} (A^{-1} \partial_t)^n QG = - \sum_{n=0}^{\infty} f^{(n)}(t) \frac{1}{A^n} QG, \tag{A.19}
\end{equation}
with \( f^{(n)}(t) \) being the \( n \)-th derivative of \( f(t) \). Thus, for example,
\begin{equation}
\frac{1}{\partial_t - A} (t-t_0)^n PG = \frac{1}{n+1} (t-t_0)^{n+1} PG, \tag{A.20}
\end{equation}
\begin{equation}
\frac{1}{\partial_t - A} (t-t_0)QG = - \left[ (t-t_0) \frac{1}{A} + \frac{1}{A^2} \right] QG, \tag{A.21}
\end{equation}
\begin{equation}
\frac{1}{\partial_t - A} (t-t_0)^2 QG = - \left[ (t-t_0)^2 \frac{1}{A} + 2(t-t_0) \frac{1}{A^2} + \frac{1}{A^3} \right] QG. \tag{A.22}
\end{equation}

Finally we consider the case where \( A \) has a two dimensional Jordan cell;
\begin{equation}
AU_1 = 0, \quad AU_2 = U_1. \tag{A.23}
\end{equation}
The adjoint has also a Jordan cell;
\begin{equation}
A^\dagger \tilde{U}_1 = 0, \quad A^\dagger \tilde{U}_2 = \tilde{U}_1. \tag{A.24}
\end{equation}
The adjoint operator \( A^\dagger \) is defined by \( \langle V, AU \rangle = \langle A^\dagger V, U \rangle \), where \( \langle V, U \rangle \) is the Hermitian inner product. We define the projection operators \( P \) and \( Q \) onto the subspace \( \{U_1, U_2\} \) and its orthogonal compliment, respectively. We suppose that the following normalization condition is satisfied;
\begin{equation}
\langle \tilde{U}_2, U_1 \rangle = 1, \quad \langle \tilde{U}_1, U_2 \rangle = 1. \tag{A.25}
\end{equation}
A vector \( U \) in the P-space is decomposed as \( U = \langle \tilde{U}_2, U \rangle U_1 + \langle \tilde{U}_1, U \rangle U_2 \).

Let \( G \) be a constant vector, then one has,
\begin{equation}
\frac{1}{\partial_t - A} f(t) PG = \frac{1}{\partial_t} \sum_{n=0}^{\infty} (\partial_t^{-1} A)^n f(t) PG,
\end{equation}
\begin{equation}
= \frac{1}{\partial_t} \sum_{n=0}^{\infty} (\partial_t^{-n} f(t)) A^n \langle \tilde{U}_2, G \rangle U_1 + \langle \tilde{U}_1, G \rangle U_2],
\end{equation}
\begin{equation}
= \frac{1}{\partial_t} f(t) \{ \langle \tilde{U}_2, G \rangle U_1 + \langle \tilde{U}_1, G \rangle U_2 \} + \frac{1}{\partial_t} f(t) \langle \tilde{U}_1, G \rangle U_1],
\end{equation}
\begin{equation}
= \int_{t_0}^{t} ds f(s) PG + \int_{t_0}^{t} ds \int_{t_0}^{s} ds' f(s') \langle \tilde{U}_1, G \rangle U_1]. \tag{A.26}
\end{equation}
Thus, for example,
\begin{equation}
\frac{1}{\partial_t - A} PG = (t-t_0) PG + \frac{1}{2} (t-t_0)^2 \langle \tilde{U}_1, G \rangle U_1, \tag{A.27}
\end{equation}
\begin{equation}
\frac{1}{\partial_t - A} (t-t_0)^n PG = \frac{1}{n+1} (t-t_0)^{n+1} PG,
\end{equation}
\begin{equation}
+ \frac{1}{(n+2)(n+1)} (t-t_0)^{n+2} \langle \tilde{U}_1, G \rangle U_1. \tag{A.28}
\end{equation}
The formulae involving $QG$ are the same as those in the semi-simple case.

The extension to the case where $A$ has a higher dimensional Jordan cell is easy. For instance, when $A$ has a three-dimensional Jordan cell such as

$$AU_1 = 0, \quad AU_2 = U_1, \quad AU_3 = U_2,$$

one can easily verify that

$$\frac{1}{\partial_t - A} f(t)PG = \int_{t_0}^{t} ds f(s)PG + \int_{t_0}^{t} ds \int_{t_0}^{s} ds' f(s')\{\langle \tilde{U}_2, G \rangle U_1 + \langle \tilde{U}_1, G \rangle U_2\},$$

$$+ \int_{t_0}^{t} ds \int_{t_0}^{s} ds_1 \int_{t_0}^{s_1} ds_2 f(s_2)\langle \tilde{U}_1, G \rangle U_1,$$

where the adjoints satisfy

$$A^\dagger \tilde{U}_1 = 0, \quad A^\dagger \tilde{U}_2 = \tilde{U}_1, \quad A^\dagger \tilde{U}_3 = \tilde{U}_2.$$

The normalization condition reads $\langle \tilde{U}_3, U_1 \rangle = \langle \tilde{U}_2, U_2 \rangle = \langle \tilde{U}_1, U_3 \rangle = 1.$
Appendix B An elementary method to derive the approximate solution for the double-well potential

The first integral of the Newton equation with the initial condition \( x(0) = 0 \) for the double-well potential reads by

\[
t = \int_0^{x/\sqrt{2E}} \frac{dy}{1 + y^2 - \epsilon Ez^4}.
\]

(B.1)

Expanding the integral in a Taylor series, one readily obtains

\[
t = (1 - \frac{3}{4} \epsilon E) \text{Sinh}^{-1} X + \frac{\epsilon E X^3 + 3X}{4 \sqrt{X^2 + 1}},
\]

(B.2)

with \( X = x/\sqrt{2E} \) up to \( O(\epsilon^2) \). Here \( \text{Sinh}^{-1} X \equiv \ln |X + \sqrt{X^2 + 1}| \). The point of the present method is to notice that the above equation is equivalent to the following equation up to \( O(\epsilon^2) \);

\[
u = \text{Sinh}^{-1} X + \frac{\epsilon E X^3 + 3X}{4 \sqrt{X^2 + 1}},
\]

(B.3)

with \( u = (1 + \frac{3}{4} \epsilon E)t \). One may solve (B.3) perturbatively and obtains

\[
x(t) = \sqrt{2E}(1 - \frac{3}{4} \epsilon E) \sinh u - \frac{\epsilon}{8} (\sqrt{2E})^3 \sinh^3 u.
\]

(B.4)

Putting \( \sqrt{2E} \{ (1 - \frac{3}{4} \epsilon E) = C \), one ends up with

\[
x(t) = C \sinh \alpha t - \frac{\epsilon}{8} C^3 \sinh^3 \alpha t,
\]

(B.5)

with \( \alpha = 1 + \frac{3}{8} \epsilon C^2 \) up to \( O(\epsilon^2) \), which coincides with the result given in subsection 3.3 in the text.
Appendix C  The period of the Lotka-Volterra equation

The Lotka-Volterra equation admit periodic solutions. An approximate but globally valid periodic solution was explicitly constructed by the RG method by one of the present author. [26] The main purpose of this Appendix is to show that the solution constructed there gives the period which coincides with that obtained by Frame [55] in a quite different approach. To make the argument self-contained, however, we shall repeat the RG analysis but in a mathematically more simple way than that given in [26].

Introducing the new variable \( u = t(\xi, \eta) \) by
\[
x = (b + \epsilon \xi)/\epsilon', \quad y = a/\epsilon + \eta,
\]
Eq.(5.19) is reduced to the following one:
\[
\left( \frac{d}{dt} - L_0 \right) u = -\epsilon \xi \eta \left( \frac{1}{1} \right),
\]
where
\[
u = \left( \begin{array}{c} \xi \\ \eta \end{array} \right), \quad L_0 = \left( \begin{array}{cc} 0 & -b \\ a & 0 \end{array} \right).
\]

\( L_0 \) has the eigenvalues \( \lambda = \pm i\sqrt{ab} \equiv \pm i\omega \), with the corresponding eigenvectors given by \( U_1 = t(1, -i\omega/b) \) and \( U_2 = U_1^* \), respectively. Here \( A^* \) denotes the complex conjugate of \( A \). To make the following calculation as transparent as possible, we first transform the equation to the form where \( L_0 \) is diagonalized. Then one finds that the vector equation (C.2) is reduced to a scalar equation
\[
\left( \frac{d}{dt} - i\omega \right) z = \epsilon i\omega \alpha b \left( z^2 - \overline{z}^2 \right),
\]
where \( \alpha = 1/2 \cdot (1 - i\beta/\omega) \) and \( z(t) = 1/2 \cdot (\xi + i\beta/\omega \cdot \eta) \). This equation apparently has a much simpler form than that treated in [26].

Now we try to solve (C.4) around \( t \sim \forall t_0 \) by the perturbation theory by expanding \( z = z_0 + \epsilon z_1 + \epsilon^2 z_2 + o(\epsilon^2) \), with the initial condition \( z(t; t_0) = W(t_0) \). \( W(t_0) \) is also expanded as \( W(t_0) = W_0(t_0) + \epsilon W_1(t_0) + \epsilon^2 W_2(t_0) + o(\epsilon^2) \).

A simple manipulation gives the solution as follows;
\[
z(t; t_0) = C(t_0)e^{i\omega t} + \epsilon \frac{\alpha}{b} \left( C^2 e^{2i\omega t} + \frac{1}{3} \overline{C}^2 e^{-2i\omega t} \right)
+ \epsilon^2 \frac{\alpha^2}{b^2} \left( C^3 e^{3i\omega t} - \frac{1}{3} \overline{C} C^2 e^{-i\omega t} \right)
- \epsilon^2 \frac{|\alpha|^2}{b^2} \left\{ -\frac{1}{2} \overline{C} C e^{-3i\omega t} + \frac{2i\omega}{3} \overline{C} C^2 (t-t_0)e^{i\omega t} \right\}.
\]
Now the RG equation $\frac{dC}{dt_0}(t; t_0) \big|_{t_0=t} = 0$ gives

$$\frac{dC}{dt_0} = -\frac{i}{2} \frac{e^2}{6b^2} (1 + \frac{b^2}{\omega^2}) \omega |C|^2 C.$$  \hspace{1cm} (C.6)

Here we have used that $|\alpha|^2 = \frac{1}{4} (1 + \frac{b^2}{\omega^2})$. The general solution $z(t, t_0)$ is now given as the initial value $W(t)$ by construction $z(t) \equiv W(t)$.

Since $|C(t)|^2 = \text{const.}$ as easily verified from (C.6), one may put $C = \frac{A}{2 \pi} e^{i\theta}$, with $A$ being a real constant. Then (C.6) implies that $\dot{\theta} = -\frac{e^2 A^2}{24b^2 \cdot (1 + b^2/\omega^2)} \omega$, hence $\theta(t) = -\frac{e^2 A^2}{24b^2 \cdot (1 + b^2/\omega^2)} \omega t + \bar{\theta}$.

For a calculational convenience, let us define $\Theta$ by $\Theta(t) = \bar{\omega} t + \bar{\theta}$ with

$$\bar{\omega} = \left\{ 1 - \frac{e^2 A^2}{24b^2 (1 + \frac{b^2}{\omega^2})} \right\} \omega,$$  \hspace{1cm} (C.7)

which implies that $C(t)e^{i\omega t} = A/2 \cdot (\sin \Theta - i \cos \Theta)$. Then we have finally the components $t'(\xi(t), \eta(t))$,

$$\xi(t) = A \left(1 - e^2 \frac{\omega^2 - b^2 A^2}{4\omega^2 b^2} \right) \sin \Theta - \frac{e^2}{b \omega} \frac{A^3}{12} \cos \Theta - \frac{1}{\omega} \frac{A^2}{6} \sin 2\Theta - \frac{1}{b} \frac{A^2}{3} \cos 2\Theta$$

$$- \frac{e^2}{4\omega^2 b^2} \frac{3\omega^2 - b^2 A^2}{8} \sin 3\Theta + \frac{e^2}{\omega b} \frac{1}{8} \frac{A^3}{cos 3\Theta} + o(\epsilon^2),$$  \hspace{1cm} (C.8)

$$\eta(t) = A \frac{\omega}{b} \left\{ e^2 \frac{1}{b \omega} \frac{A^3}{24} \sin \Theta - A \left(1 + e^2 \frac{\omega^2 - b^2 A^2}{4\omega^2 b^2} \right) \cos \Theta - \frac{1}{b} \frac{A^2}{6} \sin 2\Theta + \frac{1}{\omega} \frac{A^2}{3} \cos 2\Theta$$

$$+ \frac{e^2}{\omega b} \frac{1}{8} \frac{A^3}{\cos 3\Theta} + e^2 \frac{3\omega^2 - 3b^2 A^2}{4\omega^2 b^2} \frac{A^3}{8} \cos 3\Theta \right\} + o(\epsilon^2).$$  \hspace{1cm} (C.9)

Although these expressions are seemingly different from those given in [26], they coincide with each other up to $o(\epsilon^2)$. In fact, by the redefinition of the constant variables $A(1 - e^2 \frac{\omega^2 - b^2 A^2}{4\omega^2 b^2}) \rightarrow A$ and $\bar{\theta} - e^2 \frac{1}{b \omega} \frac{A^3}{24} \rightarrow \bar{\theta}$, (C.8) and (C.9) reproduce the result in [26], up to $o(\epsilon^2)$.

It is remarkable that we have obtained the modified angular velocity $\bar{\omega}$ depending on the amplitude. A long ago, Frame [55] gave the period of the motion. Here, we shall show that the period $T_{RG} = 2\pi / \bar{\omega}$ coincides with that given by Frame up to $o(\epsilon^2)$.

Our period reads

$$T_{RG} = \frac{2\pi}{\omega - \frac{e^2 A^2}{24b} (\omega^2 + b^2)} = \frac{2\pi}{\sqrt{ab}} \left\{ 1 + \frac{e^2 A^2}{24b} \left( \frac{1}{a} + \frac{1}{b} \right) \right\} + o(\epsilon^2),$$  \hspace{1cm} (C.10)

where use has been made of $\omega = \sqrt{ab}$. On the other hand, the period $T_{Fr}$ given by Frame [57] reads in his notations

$$T_{Fr} = \frac{2\pi}{a_1 a_2} \left\{ 1 + \frac{(c_1 c)^2}{1!1!} + \frac{(c_1 c)^4}{2!2!} + \cdots \right\}.$$

\hspace{1cm} (C.11)
Here, \( c_1, c \) are identified with the variables expressed by our basic variables as follows:

\[
c_1^2 = \frac{a + b}{24}, \quad (C.12)
\]

\[
c^2 = \frac{2}{a} \left( \frac{\epsilon}{b} \xi - \log(1 + \frac{\epsilon}{b} \xi) \right) + \frac{2}{b} \left( \frac{\epsilon}{a} \eta - \log(1 + \frac{\epsilon}{a} \eta) \right). \quad (C.13)
\]

We remark that the correspondence between the variables of ours and Frame’s is summarized as follows:

- \( a \leftrightarrow a_1^2 \); \( b \leftrightarrow a_2^2 \); \( \epsilon \leftrightarrow a_{12} \); \( \epsilon' \leftrightarrow a_{21} \)
- \( x \leftrightarrow N_1 \); \( y \leftrightarrow N_2 \); \( \frac{b}{\epsilon'} \leftrightarrow n_1 \); \( \frac{a}{\epsilon} \leftrightarrow n_2 \).

It should be remarked here that \( c^2 \) is actually a constant because our system has a conserved quantity \( b \ln x - \epsilon' x + a \ln y - \epsilon y = \text{const.} \) In fact, one can easily verify that

\[
c^2 = \frac{\epsilon^2}{ab} \left( \frac{\xi^2}{b} + \frac{\eta^2}{a} \right) + O(\epsilon^2). \quad (C.14)
\]

Thus,

\[
T_{Fr} = \frac{2\pi}{\sqrt{ab}} \left\{ 1 + c_1^2 c^2 \right\} + O(\epsilon^2) = \frac{2\pi}{\sqrt{ab}} \left\{ 1 + \left[ \frac{a + b \epsilon^2}{24ab} \left( \frac{\xi^2}{b} + \frac{\eta^2}{a} \right) \right] \right\} + O(\epsilon^2). \quad (C.15)
\]

From (C.8) and (C.9), one sees that \( \xi = 0 + O(1), \quad \eta = \frac{\epsilon}{b} A + O(1). \) Hence,

\[
T_{Fr} = \frac{2\pi}{\sqrt{ab}} \left\{ 1 + \frac{\epsilon^2}{24} \left( \frac{1}{a} + \frac{1}{b} \right) A^2 \right\} + O(\epsilon^2)

= T_{RG} + O(\epsilon^2). \quad (C.16)
\]

This is what we wanted to show.
References

[1] E.C.G. Stueckelberg and A. Petermann, Helv. Phys. Acta 26(1953), 499; M. Gell-Mann and F. E. Low, Phys. Rev. 95 (1953), 1300. K. Wilson, Phys. Rev. D3(1971), 1818. S. Weinberg, in” Asymptotic Realms of Physics” (A. H. Guth et al. Ed.), MIT Press, 1983.

[2] As review articles, S.K. Ma, ”Modern Theory of Critical Phenomena”, W. A. Benjamin, New York, 1976. J. Zinn-Justin, Quantum Field Theory and Critical Phenomena, Clarendon Press, Oxford, 1989.

[3] D. V. Shirkov. hep-th/9602024; hep-th/9903073.

[4] K.G.Wilson and M.E.Fisher,Phys. Rev. Lett. 28 (1972),240; K.G. Wilson, Phys. Rev. Lett. 28(1972), 548; K.G. Wilson and J. Kogut,Phys. Rep. 12C (1974), 75.

[5] F. Wegner and A. Houghton,Phys. Rev. A8 (1973),401.

[6] J. Polchinski, Nucl. Phys. B231 (1984), 269; G. Keller and C. Kopper and M. Salmhofer, Helv. Phys. Acta 65 (1992), 32.

[7] Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122 (1961), 345.

[8] S. Klevanski, Rev. Mod. Phys. 64 (1992), 649; T. Hatsuda and T. Kunihiro, Phys. Rep. 247 (1994), 221; J. Bijnens, Phys. Rep. 265 (1996), 369.

[9] S. Weinberg,”The Quantum Theory of Fields II”, Cambridge U.P., 1996.

[10] C. Wetterich, Phys. Lett. B301 (1993), 90.

[11] T. R. Morris, Int. J. Mod. Phys. A 9 (1994), 2411.

[12] K.-I. Aoki, Prog. Theor. Phys. Suppl. 131 (1998), 129, and references cited therein.

[13] “Proceedings of 1997 Yukawa International Seminar, Non-Perturbative QCD — Structure of the QCD Vacuum—”, Prog. Theor. Phys. Suppl. 131 (1998) (K.-I. Aoki, O. Miyamura and T. Suzuki Ed.).

[14] L. Boltzmann, “Lectures on Gas Theory” University of California Press, Berkeley, 1964.

[15] N.N. Bogoliubov, in “Studies in Statistical Mechanics”, vol.1, (J. de Boer and G.E. Uhlenbeck Ed.)North-Holland.

[16] Y. Kuramoto, Prog. Theor. Phys. Suppl. 99 (1989), 244; Bussei Kenkyu 49(1987), 299(in Japanese).

[17] S. Chapman and T.G. Cowling, “The Mathematical Theory of Non-Uniform Gases” (3rd ed.) Cambridge U.P., 1970.
[18] P. Glansdorff and I. Prigogine, “Thermodynamic Theory of Structure, Stability, and Fluctuations” Wiley, London, 1971.

[19] Y. Kuramoto, “Chemical Oscillations, Waves, and Turbulence” Springer-Verlag 1984; P. Manneville, “Dissipative Structures and Weak Turbulence”, Academic Press, INC, 1990.

[20] See for example, J. Guckenheimer and P. Holmes, “Nonlinear Oscillators, Dynamical Systems, and Bifurcations of Vector Fields” Springer-Verlag, 1983.

[21] N. Goldenfeld, O. Martin and Y. Oono, J. Sci. Comp. 4(1989),4; N. Goldenfeld, O. Martin, Y. Oono and F. Liu, Phys. Rev. Lett. 64 (1990), 1361; N. D. Goldenfeld, “Lectures on Phase Transitions and the Renormalization Group,” Addison-Wesley, Reading, Mass., 1992.

[22] L. Y. Chen, N. Goldenfeld, Y. Oono and G. Paquette, Physica A 204 (1994)111.

[23] G. Paquette, L. Y. Chen, N. Goldenfeld and Y. Oono, Phys. Rev. Lett. 72 (1994)76; L. Y. Chen, N. Goldenfeld and Y. Oono, Phys. Rev. Lett. 73 (1994)1311; L. Y. Chen, N. Goldenfeld and Y. Oono, Phys. Rev. E 54 (1996), 376.

[24] J. Bricmont and A. Kupiainen, Commun. Math. Phys. 150 (1992), 193; J. Bricmont, A. Kupiainen and G. Lin, Cooun. Pure. Appl. Math. 47 (1994), 893; J. Bricmont and A. Kupiainen, chao-dyn/9411015.

[25] T. Kunihiro, Prog. Theor. Phys. 94 (1995), 503; (E) ibid., 95 (1996)835; Jpn. J. Ind. Appl. Math. 14 (1997), 51.

[26] T. Kunihiro, Prog. Theor. Phys. 97 (1997),179.

[27] T. Kunihiro, Phys. Rev. D57 (1998), R2035; Prog. Theor. Phys. Suppl. 131 (1998), 459; see also patt-sol/979003.

[28] R. Graham, Phys. Rev. Lett. 76 (1996) 2185; see also K. Matsuba and K. Nozaki, Phys. Rev. E56 (1997), R4926.

[29] Y. Oono, “Structural Stability of Spinodal Decomposition”, preprint (1996).

[30] S. Sasa, Physica D108 (1997), 45.

[31] T. Maruo, K. Nozaki and A. Yoshimori, Prog. Theor. Phys. 101 (1999), 243.

[32] Y. Kuramoto, Prog. Theor. Phys., 55 (1976), 356; G. I. Sivashinsky, Acta Astronautica 4 (1977), 1177.

[33] H.J. de Vega and J.F.J. Salgado, Phys.Rev. D56 (1997),6524.

[34] A. Taruya and Y. Nambu, Phys.Lett. B428 (1998), 37; Yasusada Nambu and Yoshiyuki Y. Yamaguchi, gr-qc/9904053.
[35] D. Boyanovsky, H.J. de Vega, R. Holman and M. Simionato, hep-ph/9809346; D. Boyanovsky and H.J. de Vega, Phys.Rev. D59 (1999), 105019.

[36] O. Pashko and Y. Oono, “The Boltzmann Equation is a Renormalization Group Equation”, preprint (1996).

[37] M. Frasca, Phys. Rev. A 56(1997), 1549; see also I. L. Egusquiza and M. A. Valle Basagoiti, hep-th/9611143.

[38] S.I. Tzenov and P.L. Colestock, preprint(FERMILAB-PUB-98-258); Stephan I. Tzenov, preprint(FERMILAB-PUB-98-275).

[39] Yoshiyuki Y. Yamaguchi and Yasusada Nambu, Prog.Theor.Phys. 100 (1998), 199.

[40] T. Kunihiro and J. Matsukidaira, Phys. Rev. E57 (1998), 4817.

[41] N.N. Bogoliubov and Y.A. Mitropolski, “Asymptotic Methods in the Theory of Nonlinear Oscillations”, Gordon and Breach, 1961.

[42] V.F. Kovalev, V.V. Pustovalov and D.V. Shirkov, J. Math. Phys. 39 (1998), 1170.

[43] K. Kawasaki and T. Ohta, Physica A116 (1982), 573.

[44] J. Carr and R. L. Pego, Commun. Pure Appl. Math. 42 (1989), 523.

[45] S-I. Ei and T. Ohta, Phys. Rev. E50 (1994), 4672, and references cited therein.

[46] G. Fusco and J. Hale, J. Dynam. Diff. Eq. 1 (1989), 75.

[47] L.D. Landau and E. M. Lifshitz, “ Quantum Mechanics,” Pergamon, Oxford, 1977.

[48] R. L. Pego and M. I. Weinstein, Commun. Math. Phys.164 (1994), 305-349.

[49] F. Takens, Publ. Math. IHES 43 (1974), 47.

[50] For example, see the text book, S. Wiggins, “Introduction to Applied Nonlinear Dynamical Systems and Chaos”, Springer-Verlag, New York, 1990.

[51] R. I. Bogdanov, Functional Anal. Appl. 9 (2) (1975),144.

[52] P. Glansdorff and I. Prigogine, “Thermodynamic Theory of Structure, Stability, and Fluctuations,” Wiley, London , 1971.

[53] A. J. Lotka, J. Amer. Chem. Soc. 42(1920)1595; V. Volterra, “Théorie mathématique de la lutte pour la vie,” Gauthier-Villars, Paris, 1931.

[54] T. Kunihiro, unpublished (1996).

[55] J. S. Frame, J. Theor. Biol. 43 (1974)73.

[56] H. Haken, Advanced Synergetics, 2nd ed. (Springer-Verlag, 1983).
[57] Y. Nishiura, *Nonlinear Problems I; mathematics of pattern formation* §1.4, (Iwanami syoten, 1998).

[58] E. N., Lorenz, *J. Atmos. Sci.* 20 (1963)130.

[59] H. Mori, *Prog. Theor. Phys.* 33,(1965), 423.

[60] J. Meyer, G. Papp, H.-J. Pirner and T. Kunihiro, nucl-th/9908019.

[61] For example, see , J. O. Andersen and M. Strickland, cond-mat/9811096 and references cited therein.

[62] For example, see, S. Garcia, G.S. Guralnik and J.W. Lawson, *Phys. Lett.* B333 (1994), 119.

[63] L-Y. Chen and N. Goldenfeld, *Phys. Rev.* 51(1995), 5577; cond-mat/9802103.

[64] G. B. Whitham, *Linear and Non-linear waves*, John Wiley and Sons, 1977. As for recent development in the modern context, see, for example, R. Carroll, hep-th/9712117 and references cited therein.

[65] K. Takasaki, talk presented at seminar in RIMS of Kyoto University on *Applications of RG Methods in Mathematical Sciences*, 21-23 July, 1999, to be published in the proceedings of the seminar.