CROSS PRODUCT QUANTISATION, NONABELIAN

COHOMOLOGY AND TWISTING OF HOPF ALGEBRAS

S. Majid

Department of Applied Mathematics & Theoretical Physics
University of Cambridge, Cambridge CB3 9EW, U.K.

ABSTRACT
This is an introduction to work on the generalisation to quantum groups of Mackey’s approach to quantisation on homogeneous spaces. We recall the bicrossproduct models of the author, which generalise the quantum double. We describe the general extension theory of Hopf algebras and the nonAbelian cohomology spaces $H^2(H, A)$ which classify them. They form a new kind of topological quantum number in physics which is visible only in the quantum world. These same cross product quantisations can also be viewed as trivial quantum principal bundles in quantum group gauge theory. We also relate this nonAbelian cohomology $H^2(H, C)$ to Drinfeld’s theory of twisting.

Keywords: Quantum mechanics – gravity – quantum group – non-Abelian cohomology – cocycle – anomaly – bicrossproduct – quantum double – non-commutative geometry – gauge theory – twisting

1 Introduction

This paper is concerned with quantum algebras of observables that happen to be Hopf algebras. Hopf algebras are commonly called ‘quantum groups’ because they occur as generalised symmetries in statistical mechanics and also in some quantum systems. The most famous quantum groups such as $U_q(g)$ as also related to Poisson structures etc[1] but they are not quantum algebras of observables in a physical sense. In contrast to this well-known line of development, there is in the literature a second independent origin or quantum groups or Hopf algebras where they really do arise as quantum algebras of observables. These are the bicrossproduct Hopf algebras[2][3][4][5] and we shall be concerned with these and recent work in the same line of development.

The physics behind these bicrossproduct Hopf algebras is that of quantum mechanics on homogeneous spaces. These are a class of curved spaces in which the quantisation scheme is perfectly clear and unambiguous. Namely, one can quantise a particle on a homogeneous space in a standard way by making a semidirect or cross product $C(M)\rtimes U(g)$ of the momentum group or Lie algebra $g$ acting on the position observables $C(M)$. Here $M$ is a manifold on which $g$ acts, from the right say, and the particle is constrained to lie on

---

1991 Mathematics Subject Classification 16W30, 17B37, 55R10, 57T10, 58B30, 81R50, 81T70
This paper is in final form and no version of it will be submitted for publication elsewhere
2Royal Society University Research Fellow and Fellow of Pembroke College, Cambridge
the orbits or homogeneous spaces for this action. The quantisation is characterised by the commutation relations

\[
\left\{ \hat{\xi}, \hat{f} \right\} = \hat{\xi}(f), \quad \hat{\xi}(f) = \frac{d}{dt}|_0 f(s \circ e^{t\xi}), \quad s \in M, \; \xi \in g, \; f \in C(M)
\]

where \(\hat{\xi}\) is the left-invariant vector field generated by \(\xi\). This is nothing more than Heisenberg’s commutation relations in a co-ordinate invariant form. All of this is quite standard. Sometimes, it also pays to work with the group \(G\) and make the semidirect product \(\mathbb{C}(M) \rtimes \mathbb{C}G\) by the group algebra \(\mathbb{C}G\) as a \(C^*\)-algebra or von Neumann algebra. At the algebraic level this is

\[
\hat{u} \hat{f} \hat{u}^{-1} = \hat{u} \triangleright f, \quad (\hat{u} \triangleright f)(s) = f(s \circ u), \quad u \in G, \; f \in \mathbb{C}(M)
\]

where the given right action of \(G\) on \(M\) induces the left action \(\triangleright\) on the position observables. Another version is to use Mackey’s system of imprimitivity, which is largely equivalent. Some early works on this topic are [6][7]. In fact, the idea of cross products as quantisation in this context is a conclusion reached independently by anyone (including the author) thinking about this topic. If we do not worry too much about the geometry then the cross product here is called a dynamical system.

We begin in Section 2 by reviewing the results in [2][5] in this context, but in a simplified and more accessible algebraic form given by working with finite groups or with Lie algebras. A reader who wants the full functional-analysis for these models should see our original paper[3]. The diagrammatic notion which we introduce is new. Basically, we ask when do the quantisations on homogeneous spaces as above give as a result of the quantisation a Hopf algebra? The physical implications of this when it happens are quite deep and formed our algebraic approach to the unification of quantum mechanics and geometry[3], see also [8]. The point is that not every homogeneous space or action \(\triangleleft\), gives a Hopf algebra. The constraint even in one dimension forces the action to be non-linear and the resulting motion far from rectilinear motion. In fact, the resulting motion has some similarities with black-holes and we called the constraint on the metric arising in this way a ‘toy version of Einstein’s equation’[3][2]. On the other hand, the famous quantum double\(D(G)\) on a group \(G\) also solves the condition and provides a more trivial example of one of these bicrossproduct Hopf algebras. In this case the orbits on which the particle moves are the conjugacy classes in \(G\).

Next, we observe that with a little care the role of momentum group or Lie algebra can be quite easily played by a Hopf algebra or quantum group \(H\). Some concrete examples of this are known, such as [8] for a particle on a \(q\)-deformed 2-sphere, and [11] where examples are once again provided by the quantum double, now in the general case \(D(H)\). In particular, \(D(U_q(su_2))\), which is also called the quantum Lorentz group appears now in a completely different light as the quantum algebra of observables of a particle moving on a mass-shell in \(q\)-Minkowski space. This is a recent result of the author[10] and we recall it breifly in Section 3. There are many interesting features and physical questions raised by \(q\)-deforming the above quantisations in this way. They are still quantum systems in so far as they form \(\ast\)-algebras, but ones for which the underlying classical geometry is a braided or \(q\)-deformed one.

In Section 4 we describe the extension theory of algebras by Hopf algebras. This technology includes such famous things as the central extension of the diffeomorphisms on a
circle (the Virasoro algebra) but also includes and generalises, with cocycles, all of the above cross products and bicrossproducts. The cocycles here can be considered modulo an equivalence and their classes form the nonAbelian cohomology spaces $H^2(H, A)$. An interesting point is that these Hopf algebras unify some quite different mathematical ideas into a single concept. Correspondingly different constructions in physics are likewise of the same type from this general point of view. A general theme in this area, which is also a theme of the paper, is that this unification is quite typically between constructions that are usually thought of as quantum mechanical and constructions that are usually thought of as geometric. For the author, this is the most important reason to study quantum groups or Hopf algebras.

In Section 5 we explore this duality principle by pointing out that these same cross products, which above are the quantum algebras of observables of quantum systems, can also be thought of in terms of non-commutative geometry as trivial quantum principal bundles. We use the general scheme of quantum group gauge theory introduced in [11]. In this context the cohomology $H^2(H, A)$ is a new kind of quantum number that exists even though the bundle is trivial from the usual geometrical point of view. So they do not correspond to any known geometrical cohomology of the base manifold etc, being a novel and purely quantum possibility.

Finally, in Section 6 we show that a special case of the cohomology, $H^2(H, \mathbb{C})$, describes exactly the kind of cocycles that are needed in a dual form of Drinfeld’s theory of twisting[12] if one wants to twist a Hopf algebra and remain as a Hopf algebra. This twisting theory gives an alternative cohomological way to go about quantisation, as we shall demonstrate on a formal example. This connection between Drinfeld’s twisting and nonAbelian cohomology is the mathematically new aspect of the present paper. Some of the material will also be described in rather more detail in my forthcoming book[13]. The present paper nevertheless provides a self-contained introduction for mathematical physicists to the topic.

2 Bicrossproduct Hopf algebras revisited

The Hopf algebra version of cross product quantisation is quite straightforward, and we recall it first. Thus, a Hopf algebra is an algebra $H$ equipped with a coproduct $\Delta : H \to H \otimes H$, a counit $\epsilon : H \to \mathbb{C}$ and an antipode $S : H \to H$ obeying various axioms. The main ones are that $\Delta$ is an algebra homomorphism and is coassociative in the same sense as the product is associative, but with arrows reversed. The antipode is like an inverse. If $G$ is a finite group then $H = \mathbb{C}G$ the algebra generated by 1 and the elements of the group (the group algebra) is a Hopf algebra with $\Delta u = u \otimes u$, $\epsilon u = 1$ and $Su = u^{-1}$ for all $u \in G$. The most unfamiliar thing about working with general Hopf algebras is the notation $\Delta h = h^{(1)} \otimes h^{(2)}$ for the explicit pieces of the tensor product element – a summation convention is to be understood here as there could be more than one term in the description of such tensor product elements. There are also some conventions associated with this to the effect that multiple subscripts here can be renumbered to the canonical form with counting in base ten. We refer to [14] for details. See also [15] for a general introduction to quantum groups. Over $\mathbb{C}$ it is also natural to ask that $H$ is a $*$-algebra, that $\Delta$ respects this and that $(S \circ \ast)^2 = \text{id}$ as in [16].

In this language it is easy enough to write down what is meant mathematically by a
Hopf algebra cross product quantisation. We have a Hopf $\ast$-algebra $H$ as a generalised
symmetry and a $\ast$-algebra $A$ to be viewed as like $\mathbb{C}(M)$, i.e. we require that $H$ acts on $A$, and preserves its product and $\ast$ in the form

$$h\triangleright(ab) = (h_{(1)}\triangleright a)(h_{(2)}\triangleright b), \quad h\triangleright 1 = \epsilon(h), \quad (h\triangleright a)^\ast = (Sh)^\ast \triangleleft a^\ast. \quad (3)$$

Given this, it is an easy exercise to see that we have a cross product $\ast$-algebra $A\triangleright\triangleleft H$
built on the vector space $A \otimes H$ with product and $\ast$-structure

$$(a \otimes h)(b \otimes g) = a(h_{(1)}\triangleright b) \otimes h_{(2)}g, \quad (a \otimes h)^\ast = (1 \otimes h^\ast)(a^\ast \otimes 1). \quad (4)$$

This solves the quantisation problem as above by defining $\widehat{h} = 1 \otimes h$ and $\widehat{a} = a \otimes 1$, for then in the cross product we have

$$\widehat{h_{(1)}} \triangleright \widehat{a} \widehat{S} \widehat{h_{(2)}} = h\triangleright a$$

which is clearly the correct generalisation of the familiar Lie group or Lie algebra cases \((1)\)–\((2)\) above. Thus we have formulated the quantisation of particles on orbits in algebraic

terms.

Let us now ask abstractly when this algebra $A\triangleright\triangleleft H$ is a Hopf algebra. Our construction
for these is based on the idea of keeping the input-output symmetry between the algebra
and the coalgebra, i.e. we introduce a coaction $\beta$ and use it to twist the coproduct. The
axioms of a coaction are like those of an action but with all the maps reversed.

**Theorem 2.1** \([4]\) Given a cross product as above, suppose further that $A$ is a Hopf algebra
and that there is a coaction $\beta : H \to A \otimes H$ of $A$ back on $H$, which we write explicitly as $\beta(h) = h^{(i)} \otimes h^{(2)}$. If we have the compatibility conditions

$$\epsilon(h\triangleright a) = \epsilon(h)\epsilon(a), \quad \Delta(h\triangleright a) = h_{(1)}^{(i)} \triangleright a_{(1)} \otimes h_{(1)}^{(2)}(h_{(2)}\triangleright a_{(2)})$$

$$\beta(1) = 1 \otimes 1, \quad \beta(gh) = g_{(1)}^{(i)}h^{(2)} \otimes g_{(2)}^{(1)}(g_{(2)}\triangleright h_{(2)}^{(2)})$$

$$h_{(2)}^{(i)} \otimes (h_{(1)}\triangleright a)h_{(2)}^{(2)} = h_{(1)}^{(i)} \otimes h_{(1)}^{(2)}(h_{(2)}\triangleright a)$$

for all $a, b \in A$ and $g, h \in H$, then the cross product is a Hopf algebra, the bicrossproduct $A\triangleright\triangleleft H$, with coproduct, counit and antipode

$$\Delta(a \otimes h) = a_{(1)} \otimes h_{(1)}^{(i)} \otimes a_{(2)}h_{(2)}^{(2)} \otimes h_{(2)}^{(2)}, \quad \epsilon(a \otimes h) = \epsilon(a)\epsilon(h)$$

$$S(a \otimes h) = (1 \otimes Sh^{(i)})(S(h^{(2)}) \otimes 1)$$

We content ourselves here with some concrete examples based on group factorisations. Also, we will emphasise the structure for the finite case – there is no problem for these
with $\ast$-structures and representation on Hilbert spaces needed for the quantum mechanical
picture. Note that the question of group or Lie algebra factorisations plays an important
role in many areas of mathematics and also in inverse scattering where the Lie case is
sometimes called a Manin-triple. On the other hand, the key idea for us is to express the
necessary data in an equivalent form as a pair of groups acting on each other. So, two
groups $(G, M)$ form a \textit{matched pair} if there is a right action $\lhd$ of $G$ on $M$ and a left action
\(\triangleright\) of \(M\) on \(G\) which are almost actions on each other by automorphisms. The full set of conditions on the maps are

\[
\begin{align*}
s \triangleleft e &= s, \quad (s \triangleleft u) \triangleleft v = s \triangleleft (uv); \quad e \triangleleft u &= e, \quad (st) \triangleleft u = (s \triangleleft (t \triangleright u)) (t \triangleleft u) \\
es \triangleright u &= u, \quad s \triangleright (t \triangleright u) = (st) \triangleright u; \quad s \triangleright e &= e, \quad s \triangleright (uv) = (s \triangleright u) ((s \triangleleft u) \triangleright v)
\end{align*}
\]

where the first two in each line says that we have an action, and the second says that the action is almost by automorphisms, but twisted by the other action. It is also useful to employ a graphical notation which expresses these matching conditions as the ability to sub-divide rectangles. Thus we adopt the notation that a square \(s \square \) has on its top boundary \(u\) transformed by the action of \(s\) and on its right boundary \(s\) transformed by the action of \(u\). Thus \(s \square \ = s \triangleright \triangleleft u\). In fact one can see that labelling any two adjacent edges is enough to uniquely determine the other two edges to conform to this convention. We also adopt the convention that any group elements labelling the same edge are to be read as multiplied in the usual way for horizontal edges and downwards for vertical edges. In this notation the matched pair conditions become

\[
\begin{align*}
\begin{array}{c}
e \square \ = \ e \triangleright \triangleleft u \\
n \square \ = n \triangleright \triangleleft e
\end{array}
\end{align*}
\]

On the top right the condition is that a box with edges \(st\) and \(u\) can be equally well viewed as a product of two boxes, one with edges \(t\) and \(u\) and the other with one edge \(s\) and the other edge the internal one labelled \(tv\). That the top edges agree on the two sides encodes the information that \(\triangleright\) is an action, and that the vertical edges agree (when multiplied going downwards) says the last condition in the first line of the matched pair conditions. The condition concerning the identity element \(e\) says that a box with edge labelled by \(e\) can be collapsed to one of zero height, which notation is consistent with the gluing property. Likewise for the second line. This diagrammatic notation is borrowed from \([17]\) where these notions were generalised to groupoids. We use it now for the original bicrossproduct Hopf algebras associated to matched pairs of groups as found in \([18]\) as a case of Theorem 2.1. We use \(\triangleleft\) to make a cross product and \(\triangleright\) to define the coaction \(\beta(u) = \sum_s s \triangleright \delta_s \otimes \delta_s\) for a cross coproduct. It turned out that this special case was also known from \([18]\) in another context.

**Proposition 2.2** If \((G,M)\) is a matched pair of groups then the induced action of \(C^*(G)\) on \(C^*(M)\) and coaction of \(C^*(M)\) on \(C^*(G)\) gives a left-right bicrossproduct Hopf algebra \(C^*(M) \triangleright \triangleright C^*(G)\) as in Theorem 2.1. In basis \(\{\delta_s \otimes u\}\) the structure is explicitly

\[
\begin{align*}
(\delta_s \otimes u)(\delta_t \otimes v) &= \delta_{s \circ t}\ (\delta_s \otimes uv), \quad \Delta(\delta_s \otimes u) = \sum_{ab=s} \delta_a \otimes b \triangleright u \otimes \delta_b \otimes u \\
1 &= \sum_s \delta_s \otimes e, \quad \epsilon(\delta_s \otimes u) = \delta_{s,e}, \quad S(\delta_s \otimes u) = \delta_{(s \triangleleft u)^{-1}} \otimes (s \triangleright u)^{-1}.
\end{align*}
\]
In the graphical notation this is

\[ \Delta(s \begin{array}{c} a \\ \hline b \\ \hline u \\ \hline v \end{array}) = \sum_{ab=s} a \begin{array}{c} a \\ \hline b \\ \hline u \\ \hline v \end{array}, \quad \epsilon(s \begin{array}{c} u \\ \hline v \end{array}) = \delta_{s,e} \]

\[ \begin{array}{c} s \\ \hline a \\ \hline b \\ \hline u \\ \hline v \end{array} = s \begin{array}{c} u \\ \hline v \end{array} \]

\[ \begin{array}{c} s \\ \hline a \\ \hline b \\ \hline u \\ \hline v \end{array} = s \begin{array}{c} u \\ \hline v \end{array} \]

Proof Here we want to provide a new and direct diagrammatic verification of the Hopf algebra axioms. Recall from our explanation above that two adjacent edges of a box determine the other two. We add to this the convention that two boxes can be multiplied by gluing as shown if the vertical edges match in their values. Otherwise the product is zero. This is the product in \( C(M) \triangleright \triangleleft C(G) \). We use the other dimension in a similar but dual way to make the coproduct. Thus the coproduct of a box is the sum over labelled boxes such that when glued vertically they would give the labelled box we began with. Our convention is to read vertical expressions from top to bottom, so the upper box is the first output of the coproduct and the lower box is the second. This explains the diagrammatic form of the Hopf algebra structure. The proof that \( \Delta \) is an algebra homomorphism is given in this language by

\[ \Delta(s \begin{array}{c} a \\ \hline b \\ \hline u \\ \hline v \end{array}) = \Delta(s \begin{array}{c} a \\ \hline b \\ \hline u \\ \hline v \end{array}) = \sum_{ab=s} a \begin{array}{c} a \\ \hline b \\ \hline u \\ \hline v \end{array}, \quad \epsilon(s \begin{array}{c} u \\ \hline v \end{array}) = \delta_{s,e} \]

\[ \begin{array}{c} s \\ \hline a \\ \hline b \\ \hline u \\ \hline v \end{array} = \begin{array}{c} s \\ \hline u \\ \hline v \end{array} \]

\[ \begin{array}{c} s \\ \hline a \\ \hline b \\ \hline u \\ \hline v \end{array} = \begin{array}{c} s \\ \hline u \\ \hline v \end{array} \]

We compute \( \Delta \) on a composite in the first line. The third equality decomposes each of the blocks of the coproduct into pieces. The subdivision picture of the matched pair conditions tells us exactly that this can be done in a way that the internal parallel edges in the fourth expression have matching values as needed for gluing. All the edges of all four boxes are full determined by \( a, b, u, v \) according to the above conventions. In particular, the values \( c, d \) for the left edges of the right-hand boxes must have product \( cd = (ab) \triangleright u \) which is the product of the right edges of the left-hand boxes. This is shown in the fifth expression. But written in this way, we have the product pairwise in two copies of the the Hopf algebra of the outputs of \( \Delta \) as required. Similarly for the other structures. \( \square \)

The matched pair obviously has a left-right symmetry, so not surprisingly there is another bicrossproduct Hopf algebra \( C(M) \triangleright \triangleleft C(G) \) constructed in a similar way but with \( \triangleright \) supplying the action and \( \triangleleft \) the coaction. It is built on \( C(M) \otimes C(G) \) with basis \( \{ s \otimes \delta_u \} \), say, and structure

\[ (s \otimes \delta_u)(t \otimes \delta_v) = \delta_{u,tv}(st \otimes \delta_{uv}), \quad \Delta(s \otimes \delta_u) = \sum_{vw=u} s \otimes \delta_v \otimes s \triangleright v \otimes \delta_w \]

\[ 1 = \sum_{u} e \otimes \delta_u, \quad \epsilon(s \otimes \delta_u) = \delta_{u,e}, \quad S(s \otimes \delta_u) = (s \triangleright u)^{-1} \otimes \delta_{(s \triangleright u)^{-1}} \]
One has that it is the dual Hopf algebra to the previous one by the evaluation pairing,
\[ \mathbb{C}M\bowtie\mathbb{C}(G) = (\mathbb{C}(M)\bowtie\mathbb{C}G)^*. \]  
(8)

One can also construct from our matched pair a double cross product group \( G\bowtie\bowtie M \) defined as the set \( G \times M \) with product and inverse
\[
(u, s)(v, t) = (u(s\bowtie v), (s\bowtie v)t), \quad (u, s)^{-1} = (s^{-1}\bowtie u^{-1}, s^{-1}u^{-1})
\]

If a group \( X \) factorises in the sense that \( G^i \hookrightarrow X \twoheadrightarrow M \) are two subgroups and the map \( G \times M \rightarrow X \) given by multiplication in \( X \) is a bijection, then \((G, M)\) are a matched pair and \( X \cong G\bowtie\bowtie M \). The actions are determined by
\[
j(s)i(u) = i(s\bowtie u)j(s\bowtie u). \]

This is why the matched pair data are in one-to-one correspondence with group factorisations. As such, they are abundant in nature. One can see [4] for some relevant examples according to our current physical interpretation. We content ourselves with the simplest one-dimensional one of these. So let \( G = M = (\mathbb{R}, +) \) with its additive group structure. Then the general solution of the matching conditions (7) in a neighbourhood of the origin has two parameters \( A, B \in \mathbb{R} \) and the form [3]

\[
s\bowtie u = \frac{1}{B} \ln(1 + (e^{Bs} - 1)e^{-Au}), \quad s\bowtie u = \frac{1}{A} \ln(1 + e^{-Bs}(e^{Au} - 1)).
\]
(9)

Plugging these into the \( C^* \) algebra or Hopf-von Neumann algebra version of Proposition 2.2 gives the quantum algebra of observables for this system [3]. Note that the general Hopf algebra version in Theorem 2.1 applies just as well to enveloping algebras as to group algebras, and at this level the same solution (9) gives gives the following bicrossproduct Hopf algebra.

**Example 2.3** [3] We take \( A = \mathbb{C}[g, g^{-1}] \) and \( H = U(\mathbb{R}) = \mathbb{C}[p] \). These form a bicrossproduct with action and coaction
\[
p\bowtie g = A(1 - g)g, \quad \beta(p) = p \otimes g.
\]

The cross product algebra and cross coproduct coalgebra are
\[
[p, g] = Ag(1 - g), \quad \Delta g = g \otimes g \quad \Delta p = p \otimes g + 1 \otimes p, \quad \epsilon g = 1, \quad \epsilon p = -p, \quad Sg = g^{-1}, \quad Sp = -pg^{-1}
\]
where we omit writing the quantisation maps \(^\wedge\).
Note that if we work formally over \( C[[B]] \) with \( x \) as the generator and \( g = e^{-Bx} \), and if we set \( \hbar = -\frac{A}{B} \) then
\[
p \ll x = \hbar (1 - e^{-Bx}), \quad [p, x] = \hbar (1 - e^{-Bx})
\]
is the cross product algebra. The stated action and coaction here are obtained by computing the action and coaction at the group level for the group actions \([5]\) and then differentiating with respect to the group \( G = \mathbb{R} \). This is because we are replacing it by the enveloping algebra \( U(\mathbb{R}) = C[p] \). We keep our position space \( M = \mathbb{R} \) in the additive form but work with its co-ordinate function \( x(s) = s \) or, more precisely, with the function \( s \mapsto e^{-Bs} \) as the abstract generator \( g \). After obtaining these formulas one then verifies them directly in our algebraic picture. Thus one can show that they extend uniquely to an action and coaction fulfilling the conditions of Theorem 2.1 for a bicrossproduct.

This Hopf algebra \( C[g, g^{-1}] \triangleright C[p] \) or \( C[x] \triangleright C[p] \) is one of the simplest non-commutative and non-cocommutative Hopf algebras, and was introduced along the lines above by the author. We now discuss in detail its physical meaning from \([3]\). Firstly, it is significant that this general class of constructions based on keeping the group structure of phase space in the quantum domain, allows us only two free parameters \( A, B \) with action and reaction as stated above. All possible quantisations of a particle in one dimension for which the quantum algebra of observables remains a Hopf algebra of self-dual type are classified by two parameters. We have already identified the combination \( \hbar = -\frac{A}{B} \) by looking at large \( xB \). To see the meaning of the remaining parameter \( B \) we consider how the particle moves classically. In our approach we keep \( p \) as a conserved momentum and keep Hamiltonian \( -\frac{p^2}{2m} \) so that the particle is in free-fall. The different motions of the particle are then controlled by changing the \( x, p \) commutation relations. It is more usual to keep the commutation relations fixed in a canonical form and vary the Hamiltonian, and indeed we could reformulate things this way except that the necessary change of variables would have to be singular. This is because the usual quantum mechanics algebra is not a Hopf algebra while our bicrossproduct one, is. Note also that in our conventions \( p \) is antihermitian, i.e. \(-ip\) is the physical momentum observable. Then
\[
\frac{dx}{dt} = \frac{i}{\hbar} - \frac{p^2}{2m}, [x] = (\frac{-ip}{m})(1 - e^{-Bx}) + O(\hbar), \quad \frac{dp}{dt} = 0
\]
as operators. Hence in representations where the system behaves like a particle, its classical trajectories will be of the form given here by the leading term. We identify \( \frac{-ip}{m} = v_\infty \), the velocity at \( x = \infty \). If we consider a particle falling in from \( \infty \) then we see that the particle approaches the origin \( x = 0 \) but does so more and more slowly. In fact, it takes an infinite amount of time to reach the origin, which therefore behaves in some ways like a black-hole event horizon. The present model is one dimensional but we can imagine that it is the radial part of some motion in spacetime, and can estimate the value of \( B \) on this basis. We find
\[
B = \frac{c^2}{MG}; \quad \frac{dx}{dt} = v_\infty \left( 1 - \frac{1}{\exp(\frac{c^2x}{2MG})} \right), \quad \text{cf.} \quad \frac{dx}{dt} = -c \left( 1 - \frac{1}{1 + \frac{c^2x}{2MG}} \right)
\]
where the comparison is with an in-falling photon at radial distance \( x \) from the event horizon in the Schwarzschild black-hole solution of mass \( M \). Here \( c \) is the speed of light and \( G \) the gravitational coupling constant. This analogy should not be pushed too far since our present treatment is in non-relativistic quantum mechanics, but it gives us at least one interpretation of the parameter \( B \) as being comparable to introducing the distortion in the geometry due to a gravitational mass \( M \).
Another, more mathematical way to reach the same conclusion is to take the limit \( \hbar \to 0 \). In this case our algebra becomes commutative, but the coalgebra remains non-cocommutative. In this case \( \mathbb{C}[x] \bowtie \mathbb{C}[p] \cong \mathbb{C}(X) \) where \( X = \{(s, u)\} \), \( (s, u)(t, v) = (s + t, ue^{-Bt} + v) \)

To see this, let \( p(s, u) = u \) and \( x(s, u) = s \) be the co-ordinate functions on \( X \). They commute, and with coproducts determined from

\[
(\Delta x)((s, u), (t, v)) = x((s, u)(t, v)) = s + t = (x \otimes 1 + 1 \otimes x)((s, u), (t, v))
\]

\[
(\Delta p)((s, u), (t, v)) = p((s, u)(t, v)) = ue^{-Bt} + v = (p \otimes e^{-Bx} + 1 \otimes p)((s, u), (t, v))
\]

they generate our algebraic model \( \mathbb{C}(X) \) of the functions on \( X \). This is clearly the limit of \( \mathbb{C}[x] \bowtie \mathbb{C}[p] \). This group \( \mathbb{R} \ltimes \mathbb{R} \) is therefore the underlying classical phase space of our system. The coproduct equips it with a nonAbelian group structure. In geometrical terms a nonAbelian group law corresponds to geometrical curvature. Note that since the group is not semisimple (it is solvable), its natural metric induced by the action is degenerate. So there are some subtleties here but the general principle is the same. In our case one can compute that the curvature on phase space is of the order of \( B^2 \). For dynamical models with a reasonable degree of symmetry one can expect that this should also be comparable to the curvature in position space. Comparing it with the curvature near a mass \( M \) for example would give the same estimate as above.

In summary, we see that the bicrossproduct model has two limits:

\[
\begin{align*}
\mathbb{C}[x] \bowtie \mathbb{C}[p] \quad \hbar \to 0 & \quad \text{usual quantum mechanics for } x > 0 \\
\mathbb{C}(X) \quad \hbar \to 0 & \quad \text{usual curved geometry}
\end{align*}
\]

This illustrates the first goal of [3] of unifying a quantum system and a geometrical one. The most general unification within this framework allows only two free parameters, which we have identified in general terms as \( \hbar, G \). Moreover, the existence of curvature in our geometrical system forces the dynamics of the quantum particle to be deformed from the usual one. This deformation forces, as we have seen, a structure not unlike a black-hole event horizon. With a little more care one can estimate also how small \( \hbar, G \) have to be in comparison to the other scales in the system. For example, when we take the two scales in the system as \( m \), the mass of the quantum test-particle moving in the curved background and \( M \), the active gravitational mass which we estimate as causing comparable curvature. The flat-space quantum-mechanical picture is valid if \( Bx \gg 1 \). If we consider a relativistic quantum particle then its position is only defined up to its Compton wavelength \( \hbar/mc \) so that the condition that the system is not detectably different from usual flat-space quantum mechanics in the region \( x > 0 \) is

\[
mM \ll m^2_{\text{Planck}}.
\]

On the other hand, the system appears classical if \( \hbar \ll \Delta p \Delta x \). If we estimate \( \Delta p \sim mc \) again and suppose that the length scale of interest obeys a natural curvature constraint \( \Delta x \gg \frac{1}{B} \) coming out of general relativity, then the condition that the quantum aspect of the algebra is not detectable appears as

\[
mM \gg m^2_{\text{Planck}}.
\]
Some of the general features here also hold for other bicrossproduct models associated to group factorisations. The action $\triangleright$ describes in general the flow or ‘metric’ under which the quantum particle moves. But not every action admits a back-reaction $\triangleleft$ forming a matched pair in the sense above. This is a genuine constraint, which one can think of as an integrated form of a second order differential equation for the action. Moreover, this constraint in the above examples, and also for more complicated examples in [11], does have qualitative similarities with the singularities forced by Einstein’s equation for the metric in the presence of matter. From this point of view, the matched pair conditions (7) are some kind of toy version of Einstein’s equations. For more complicated metrics one would need of course to leave the class of Hopf algebras and construct more complicated quantum geometries, but exhibiting perhaps some of these same features.

Next we note that by their construction, the bicrossproduct models are self-dual as we know from the symmetry in the matched pair $\triangleright,\triangleleft$. To see this more precisely one has to introduce some functional analysis and work with Hopf-von Neumann algebras, or else proceed with formal power-series. For our algebraic purposes we construct directly the corresponding Hopf algebra $C[\phi]\triangleright C[\psi]$ say, and show that it is dually paired. The construction follows this time the right action and left coaction version of Theorem 2.1 as,

$$\psi \triangleright \phi = h^{-1}(1 - e^{-A\psi}), \quad \beta(\phi) = e^{-A\psi} \otimes \phi, \quad [\psi, \phi] = h^{-1}(1 - e^{-A\psi})$$

$$\Delta \phi = \phi \otimes 1 + e^{-A\psi} \otimes \phi, \quad \Delta \psi = \psi \otimes 1 + 1 \otimes \psi$$

$$\epsilon \phi = \epsilon \psi = 0, \quad S \phi = -e^{A\psi} \phi, \quad S \psi = -\psi$$

We have followed here exactly the same steps as in Example 2.3 but with the roles of the two groups or the roles of $A, B$, interchanged. Next, we should think of $C[\phi]$ as the universal enveloping algebra $U(\mathbb{R})$ and hence dually paired with $C[x]$. This pairing is

$$\langle \phi^n, x^m \rangle = \delta_{n,m}^n!, \quad \text{i.e.,} \quad \langle \phi^n, f(x) \rangle = \frac{d^n}{dx^n} |0\rangle f.$$ 

In the same way, $C[p]$ is dually paired with $C[\psi]$. Along the same lines as (8), we conclude that $C[\phi]\triangleright C[\psi]$ is dually paired with $C[x]\triangleright C[p]$. Explicitly, it is

$$\langle \phi, \colon f(x, p) : \rangle = \frac{\partial f}{\partial x}(0, 0), \quad \langle \psi, \colon f(x, p) : \rangle = \frac{\partial f}{\partial p}(0, 0)$$

where $f(x, p) := \sum f_{n,m} x^n p^m$ is the normal-ordered form of a function in the two variables $x, p$. For a purely polynomial version, one should work with $e^{-Bx}$ and $e^{-A\psi}$ as the generators as explained above.

Now we note that linear functionals on a quantum Hopf algebra of observables, form themselves an ‘algebra of states’. The physical states are the positive ones among them [3]. In our model then, the algebra of states is generated by the linear functionals $\phi, \psi$. Moreover, we see that this algebra of states is exactly like a quantum system. It is natural to give $\phi$ the dimensions of inverse length, and $\psi$ the dimensions of inverse momentum. Moreover, if we consider

$$x' = h\psi, \quad p' = h e^{A\psi} \phi$$

we see that these have dimensions of length and momentum and have exactly the same Hopf algebra structure as $C[x]\triangleright C[p]$. Thus one could equally well regard this second Hopf
algebra as the algebra of observables, with \( p' \) as momentum and \( x' \) as position. One would have the same picture as above in terms of geometry and quantum mechanics. The possibility to make such a reinterpretation of the same algebraic structures is the second and more radical theme that the bicrossproduct models demonstrate. In this reinterpretation, the roles of observables and states become reversed. Hence also the roles of non-commutativity in the algebra (of quantum origin) and non-cocommutativity in the coalgebra (of geometrical origin as curvature on phase space), become reversed. This is a new kind of symmetry principle which one can propose as a speculative idea for the structure of Planck scale physics. This is the novel physical phenomenon underlying the bicrossproduct Hopf algebras. We refer to [2][3][21][2] for further details and discussion of its meaning.

3 Example of the quantum double

A simple case of Proposition 2.2 is when the back-reaction \( \triangleright \) is trivial. In this case the conditions of a matched pair become just that the remaining action of \( G \) on \( M \) is by group automorphisms. This is just what it takes then for \( C(M) \rtimes C_G \) to be a Hopf algebra with tensor product coalgebra structure.

A simple example of this type is when \( G \) acts on \( M = G \) by the right adjoint action. So

\[
s \triangleleft u = u^{-1} su, \quad s \triangleright u = u, \quad u, s \in G
\]

\[
(\delta_s \otimes u)(\delta_t \otimes v) = \delta_{u^{-1} su,t} (\delta_s \otimes uv), \quad \Delta(\delta_s \otimes u) = \sum_{ab=s} \delta_a \otimes u \otimes \delta_b \otimes v
\]

\[
\epsilon(\delta_s \otimes u) = \delta_{s,e}, \quad S(\delta_s \otimes u) = \delta_{u^{-1}s^{-1}u} \otimes u^{-1}
\]

which is the nowadays called the quantum double \( D(G) = C(G)_{\text{Ad}} \rtimes \mathbb{C}G \) of the group \( G \). According to our picture in [2] etc, as above, one can think of it as the quantum algebra of observables of a particle moving on conjugacy classes in \( G \).

Drinfeld in [1] introduced a more general construction of a quantum group \( D(H) \) for any finite-dimensional Hopf algebra \( H \). We do not want to recall it in detail – it is not manifestly a cross product of the type above, but rather an example of double cross product construction as introduced in [1] in a theory of Hopf algebra factorisations. Nevertheless, we have shown in [21][10] that if \( H \) is a real-quasitriangular Hopf \( * \)-algebra then the double is isomorphic to a \( * \)-algebra cross product

\[ D(H) \cong B \rtimes H \]

where \( B \) is a braided group of function algebra type[22]. On the other hand, if \( H \) is anti-real quasitriangular then it is the dual of the quantum double which is isomorphic to a \( * \)-algebra cross product based on the dual of \( B \) and the dual of \( H \). The two cases are

\[ R^{\otimes *} = R_{21}, \quad (\text{real}), \quad R^{\ast \otimes *} = R^{-1}, \quad (\text{anti} - \text{real}) \]

where \( R \) is the quasitriangular structure or universal R-matrix[1]. In one case we have that the quantum double is some quantum system in that it forms a \( * \)-algebra cross product; in the other case the dual of the quantum double is some other quantum system in this sense.

Thus we are in the general situation with observable-state symmetry as above. One big difference of course is that the generalised momentum group \( H \) and the position observables
$B$, etc, are now quantum or braided groups. But this does not prevent us from having an honest quantum system in the sense of a $C^*$-algebra or von Neumann algebra, after suitable functional analysis. It just means that we have a quantum system whose natural underlying ‘classical system’ is not usual geometry. In the example given in detail in [10], the classical system is rather a $q$-deformed geometry. Our point of view elevates the idea of an abstract quantum system expressed via an algebra of observables and states to first place. It is the starting point which may or may not have a familiar or recognisable classical limit. Too often in physics one begins with the classical system as something given which we must quantise, a view which we consider strictly unnecessary.

We content ourselves here with outlining the concrete example based on $q$-Minkowski space and $q$-rotations in [10], to which we refer the reader for details. Briefly, for our position observables we begin with the $\ast$-algebra $BH_q(2)$ of $2 \times 2$ braided-hermitian matrices

\[
\begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}
\]

The co-ordinate functions here do not commute but rather they form the algebra of $2 \times 2$ braided matrices $BM_q(2)$ introduced in [23].

\[qd + q^{-1}a \text{ central, } ba = q^2ab, \quad ac = q^2ca, \quad bc = cb + (1 - q^{-2})a(d - a); \quad q \in \mathbb{R}.\]

One can write these relations in the compact form $R_{21}u_1R_{21}u_2 = u_2R_{21}u_1R$ which is nowadays popular in a number of contexts, as explained in [21]. For the braided matrices in [23] these relations were introduced precisely as braided commutativity relations. This is even clearer if one uses the multi-index notation of [23] where $\{ u^i_j \}$ are regarded as a 4-vector. Then the relations can be written in the quantum-plane form $uu = uu\Psi'$ with suitable indices and with $\Psi'$ a solution of the quantum Yang-Baxter equations built from four copies of $R\{23\}$. From this point of view, $q$-Minkowski space is like a super-space in so far as its co-ordinates are commutative after allowing for some (braid) statistics. The $q$ here has to do with the statistics of our classical system and not to do with quantisation.

One can consider a particle moving on this $q$-deformed algebra of position observables. The $q$-Lorentz group in the form of a certain Hopf $\ast$-algebra (related in fact to the quantum double of $U_q(su_2)$ as symmetry quantum group) acts on it. In [10] we specialised further to the Lorentzian sphere $BS_q^3$ in $q$-Minkowski space by setting $BDET(u) \equiv ad - q^2cb = 1$. This is preserved by the $q$-rotation subgroup of the $q$-Lorentz group, of which the double cover is $U_q(su_2)$. So this acts on the algebra of functions on our $q$-Lorentzian sphere as angular momentum. The explicit action was given in [23] and is based on the quantum adjoint action. We consider then a particle moving on $A = BS_q^3$ as the position observables of the system and $H = U_q(su_2)$ as generalised momentum group (so a kind of $q$-deformed Lorentzian top). We then quantise the system according to the generalised Mackey scheme above and obtain a $\ast$-algebra cross product isomorphic to the quantum double. The overall situation is

\[
\begin{array}{c}
\mathbb{C}(S^3_{Lor} \otimes su_2) \overset{\text{deformation}}{\longrightarrow} BS_q^3 \otimes U_q(su_2) \\
\downarrow \text{quantisation} & \downarrow \text{quantisation} \\
\mathbb{C}(S^3_{Lor} \rtimes U(su_2)) \overset{\text{deformation}}{\longrightarrow} BS_q^3 \rtimes U_q(su_2)
\end{array}
\]

Note that at $q = 1$ we could just as well take the cross product $\mathbb{C}(SU(2)) \rtimes U(su_2)$, which is a particle moving on a Euclidean 3-sphere and which is a quantum double of the type
with the finite group replaced by \( SU(2) \). But this picture does not extend to the \( q \)-deformed case as a \( * \)-algebra cross product because of a breakdown of covariance for quantum matrices. Instead, to keep covariance one must use the braided matrices\(^{[23]}\), which then forces the Lorentzian signature. In summary, which is the important point for physics:

*the possibility of \( q \)-deformation in this context forces the Lorentzian signature of spacetime*

There is also a dual quantum system valid when \(|q| = 1\). It has position \( BSL_q(2, \mathbb{R}) \) and momentum quantum group \( U_q(su^*_2) \)\(^{[10]}\).

4 Quantum anomalies and nonAbelian cohomology

Now we are going to consider cocycle versions of some of the above constructions. Cocycles of the type we consider are associated in physics with quantum anomalies. Thus, a symmetry group or Lie algebra \( g \) may not be properly represented in our quantum system but rather only projectively represented. Instead then, the extended group or Lie algebra is represented. We recall that a Lie algebra extension means

\[
[\hat{\xi}, \hat{\eta}] = \hat{\xi}[\hat{\xi}, \hat{\eta}] + \chi(\hat{\xi}, \hat{\eta})
\]

where the extension is denoted by \( \hat{\cdot} \) and \( \chi \) is a Lie algebra 2-cocycle. At the group level this becomes

\[
\hat{u}\hat{v} = \hat{u}\hat{v}\chi(\hat{u}, \hat{v})
\]

for \( \chi \) a group 2-cocycle. This could have values in \( S^1 \) or more generally in any Abelian group \( M \). Then the extension \( M \hookrightarrow E \rightarrow G \) in this case can be built as a new group on the set \( E = G \times M \) with product

\[
(u, s)(v, t) = (uv, \chi(u, v)st), \quad \hat{u} = (u, e), \quad \hat{s} = (e, s).
\]

It is clear that \( M \) is a subgroup in the center. Some famous examples are the Virasoro algebra and loop group extensions.

Coming now to our original physical setting of cross products as quantisation, we ask under what conditions cocycles \( \chi \) might be observed. Recall that \( C(M)_\chi \rtimes CG \) is the quantisation of a system with \( G \) in the role of momentum group and \( M \) as position space. The point is that \( CG \) does not appear here as a subalgebra as soon as \( \chi \) is present. Rather, we have exactly the situation as above but as algebras and with the cocycle now having values in the algebra \( C(M) \) rather than in a group. Recall that \( M \) is the position space. So this time

\[
C(M) \hookrightarrow C(M)_\chi \rtimes CG, \quad \hat{u}\hat{f}\hat{u}^{-1} = \hat{u}\hat{f}, \quad \hat{u}\hat{v} = \hat{u}\hat{v}\chi(\hat{u}, \hat{v})
\]

is the cocycle cross product quantisation of particles on a homogeneous space, here at the group level. Similarly at the Lie algebra level. This is exactly the situation which arises in some models under the heading of a quantum anomaly. One does not often expect the momentum symmetry to be anomalous, but something similar does happen for the diffeomorphism group in conformal field theory, for example. Also, there is a standard notion of equivalence of cocycles up to coboundary and hence of group cohomology.
\[ \mathcal{H}(G, \mathbb{C}(M)) \] which controls the anomaly. The cohomologically trivial sector is the non-anomalous one.

We proceed now to extend all these ideas to the Hopf algebra setting, just as we did for ordinary cross products in Section 2. Apart from unifying the Lie algebra and group cases into a single setting, some purely quantum possibilities also arise as we shall see. We will end with the example of the quantum Weyl group as explained in [24]. The theory of general cocycle bicrossproduct Hopf algebras was introduced by the author [25]. The cohomology shows up even at the algebra level, where it was studied previously by Y. Doi [26] and subsequent authors. In fact, the main ideas here are already in earlier works of Hyeneman [28].

Consider then the same quantisation problem as in Section 2, but a bit more generally. We proceed now to extend all these ideas to the Hopf algebra setting, just as we did for 2-cocycles.

Let \( \gamma : H \to A \) be a convolution-invertible linear map with \( \gamma(1) = 1 \), then

\[ h_{(1)} \triangleright (g_{(1)} \triangleright a) = \chi(h_{(1)} \otimes g_{(2)})(h_{(2)} \triangleright a), \quad 1 \triangleright a = a \quad (15) \]

and moreover, this \( \chi \) is a 2-cocycle in the sense

\[ \chi(h_{(1)} \triangleright (g_{(1)} \otimes f_{(1)})) \chi(h_{(2)} \otimes g_{(2)} f_{(2)}) = \chi(h_{(1)} \otimes g_{(1)}) \chi(h_{(2)} g_{(2)} \otimes f) \]

\[ \chi(1 \otimes h) = \chi(h \otimes 1) = 1 \epsilon(h) \quad (16) \]

for all \( h, g, f \in H \) and \( a, b \in A \). The cocycle terminology here is justified by the following two propositions.

**Proposition 4.1** Let \( A \) be an algebra, \( H \) a bialgebra and \( (\chi, \triangleright) \) a cocycle-action as above. If \( \gamma : H \to A \) is a convolution-invertible linear map with \( \gamma(1) = 1 \), then

\[ \chi^\gamma(h \otimes g) = \gamma(h_{(1)})(h_{(2)} \triangleright \gamma(g_{(1)})) \chi(h_{(3)} \otimes g_{(2)}) \gamma^{-1}(h_{(4)} g_{(3)}) \]

\[ h \triangleright^\gamma a = \gamma(h_{(1)})(h_{(2)} \triangleright a) \gamma^{-1}(h_{(3)}) \]

is also cocycle-action. We say that \((\chi, \triangleright)\) and \((\chi^\gamma, \triangleright^\gamma)\) are cohomologous and denote the equivalence classes under such transformations by \( \mathcal{H}^2(H, A) \).

**Proof** This is known from previous work [27], but is also easy enough for us to prove directly. It is a straight verification from the definition of a cocycle. Convolution-invertible means that there is a map \( \gamma^{-1} : H \to A \) such that \( \cdot \circ (\gamma \otimes \gamma^{-1}) \circ \Delta = \epsilon \circ (\gamma^{-1} \otimes \gamma) \circ \Delta \).

Coassociativity gives at once that \( \triangleright^\gamma \) obeys (3) if \( \triangleright \) does. Next we compute

\[ h_{(1)} \triangleright (g_{(1)} \triangleright a) \chi^\gamma(h_{(2)} \otimes g_{(2)}) \]

\[ = \gamma(h_{(1)})(h_{(2)} \triangleright (g_{(1)})(g_{(2)} \triangleright a)\gamma^{-1}(g_{(3)})) \gamma^{-1}(h_{(3)}) \]

\[ \gamma(h_{(4)})(h_{(5)} \triangleright g_{(4)}) \chi(h_{(6)} \otimes g_{(5)}) \gamma^{-1}(h_{(7)} g_{(6)}) \]

\[ = \gamma(h_{(1)})(h_{(2)} \triangleright (g_{(1)}))(h_{(3)} \triangleright (g_{(2)} \triangleright a))(h_{(4)} \triangleright (\gamma^{-1}(g_{(3)}) \gamma(g_{(4)}))) \]

\[ \chi(h_{(5)} \otimes g_{(5)}) \gamma^{-1}(h_{(6)} g_{(6)}) \]

14
\[
\gamma(h_{(1)}(h_{(2)} \triangleright \gamma(g_{(1)})) \chi(h_{(3)} \otimes g_{(2)})((h_{(4)}g_{(3)}) \triangleright a) \gamma^{-1}(h_{(5)}g_{(4)})
\]
\[
= \gamma(h_{(1)}(h_{(2)} \triangleright \gamma(g_{(1)})) \chi(h_{(3)} \otimes g_{(2)}) \gamma^{-1}(h_{(4)}g_{(3)})
\]
\[
\gamma(h_{(5)}g_{(4)})((h_{(6)}g_{(5)}) \triangleright a) \gamma^{-1}(h_{(7)}g_{(6)})
\]
\[
= \chi\gamma(h_{(1)} \otimes g_{(1)})((h_{(2)}g_{(2)}) \triangleright \gamma a)
\]

as required for (15). We used for the third equality the same property for \(\chi, \triangleright\). Likewise, we verify

\[
h_{(1)} \triangleright \gamma \chi^\gamma(g_{(1)} \otimes f_{(1)}) \chi^\gamma(h_{(2)} \otimes g_{(2)} f_{(2)})
\]
\[
= \gamma(h_{(1)}(h_{(2)} \triangleright \gamma(g_{(1)})) \chi(g_{(3)} \otimes f_{(2)}) \gamma^{-1}(g_{(4)}f_{(3)})) \gamma^{-1}(h_{(3)})
\]
\[
\gamma(h_{(4)}(h_{(5)} \triangleright \gamma(g_{(5)}f_{(4)})) \chi(h_{(6)} \otimes g_{(6)} f_{(5)}) \gamma^{-1}(h_{(7)}g_{(7)}f_{(6)})
\]
\[
= \gamma(h_{(1)}(h_{(2)} \triangleright \gamma(g_{(1)})) \chi(h_{(3)} \otimes g_{(3)} \triangleright \gamma(f_{(1)})))
\]
\[
(h_{(4)} \triangleright \gamma(g_{(4)} \otimes f_{(2)}) \gamma^{-1}(h_{(5)}g_{(5)} f_{(4)})
\]
\[
= \gamma(h_{(1)}(h_{(2)} \triangleright \gamma(g_{(1)})) \chi(h_{(3)} \otimes g_{(2)})
\]
\[
((h_{(4)}g_{(3)}) \triangleright \gamma(f_{(1)})) \chi(h_{(5)}g_{(4)} \otimes f_{(2)}) \gamma^{-1}(h_{(6)}g_{(6)} f_{(3)})
\]
\[
= \chi\gamma(h_{(1)} \otimes g_{(1)}) \chi^\gamma(h_{(2)} g_{(2)} \otimes f)
\]

as required. The second equality uses (14) and makes cancellations of \(\gamma, \gamma^{-1}\) as before. The third uses the 2-cocycle condition (16) for \(\chi\). The fourth equality uses (15) for \(\chi, \triangleright\). The properties with respect to the unit are clear provided \(\gamma(1) = 1\). It is also easy to see that this transformation by \(\gamma\) gives a left action of the group of identity-preserving convolution-invertible maps \(H \rightarrow A\), i.e. \((\chi^\mu)\gamma = \chi^{\gamma \mu}\) and \((\triangleright^\mu)\gamma = \triangleright^{\gamma \mu}\) where \(\gamma \ast \mu = (\gamma \otimes \mu) \circ \Delta\) is the convolution product of two such maps. \(\square\)

Armed with such cocycle-actions we can make a (cocycle) cross product algebra \(A \prec \times H\). It is built on \(A \otimes H\) with product

\[
(a \otimes h)(b \otimes g) = a(h_{(1)} \triangleright b)\chi(h_{(2)} \otimes g_{(1)}) \otimes h_{(3)}g_{(2)}
\]

and the unit element is \(1 \otimes 1\). One can easily see that the cocycle conditions correspond to this being associative. Note also that \(H\) coacts on itself from the right and hence also on \(A \prec \times H\) by the algebra homomorphism \(\beta = \text{id} \otimes \Delta\). We will say that such two cocycle cross product algebras are regularly isomorphic or equivalent if they are isomorphic by a map which is covariant under this coaction. We require also that the isomorphism restricts to the identity on \(A\) as a subalgebra of the cross product.

**Proposition 4.2** Two cocycle cross products are regularly isomorphic iff their corresponding cocycle module algebra structures are cohomologous, i.e. the equivalence classes of cocycle cross products \(A \prec \times H\) are in one-one correspondence with elements of the non-Abelian cohomology \(H^2(H, A)\). In particular, \(A \prec \times H\) is regularly isomorphic to the tensor product algebra \(A \otimes H\) iff \((\chi, \triangleright)\) is a coboundary.
Proof Again, this is known from [23], but it is instructive to give a direct and self-contained proof. We begin by showing that any algebra map \( \tilde{\gamma} : A_{\chi} \triangleright H \to A_{\chi} \triangleright H \) covariant under the right coaction of \( H \) and restricting to the identity on \( A \) is necessarily of the form

\[
\tilde{\gamma}(a \otimes h) = a \gamma(h_{(1)}) \otimes h_{(2)}, \quad \gamma : H \to A.
\]

(18)

Just define \( \gamma(h) = (\text{id} \otimes \epsilon) \circ \tilde{\gamma}(1 \otimes h) \) and check that

\[
a \gamma(h_{(1)}) \otimes h_{(2)} = a \tilde{\gamma}(1 \otimes h_{(1)})^{(1)} \epsilon(\tilde{\gamma}(1 \otimes h_{(2)})^{(2)}) \otimes h_{(2)}
\]

\[
= a \tilde{\gamma}(1 \otimes h)^{(1)} \epsilon(\tilde{\gamma}(1 \otimes h)^{(2), (1)}) \otimes \tilde{\gamma}(1 \otimes h)^{(2), (2)} = (a \otimes 1) \tilde{\gamma}(1 \otimes h)
\]

where \( \tilde{\gamma} = \tilde{\gamma}^{(1)} \otimes \tilde{\gamma}^{(2)} \) is an explicit notation for the output of \( \tilde{\gamma} \) in \( A \otimes H \). We used the assumption of covariance under the right coaction of \( H \) for the second equality. Conversely, it is clear that any linear map \( \gamma : H \to A \) defines an \( H \)-comodule map \( \tilde{\gamma} \). Finally, note under this correspondence that \( \tilde{\gamma} \) is an isomorphism iff \( \gamma \) is convolution-invertible. Here \( \tilde{\gamma}^{-1} \) is provided in the same way as \( (18) \) with \( \gamma \) replaced by the convolution-inverse \( \gamma^{-1} \).

Next, evaluating the product in \( A_{\chi} \triangleright H \) and the image of the product in \( A_{\chi} \triangleright H \) we have respectively

\[
\gamma(a \otimes h)\gamma(b \otimes g) = (a \gamma(h_{(1)}) \otimes h_{(2)}) \cdot (b \gamma(g_{(1)}) \otimes g_{(2)})
\]

\[
= a \gamma(h_{(1)}) h_{(2)^{\triangleright}} (b \gamma(g_{(1)})) \chi(h_{(3)} \otimes g_{(2)}) \otimes h_{(4)} g_{(3)}
\]

\[
= a (h_{(1)^{\triangleright}} \gamma b) \chi^{\gamma}(h_{(2)} \otimes g_{(1)}) \gamma(h_{(3)} g_{(2)}) \otimes h_{(4)} g_{(3)}
\]

\[
\tilde{\gamma}((a \otimes h)(b \otimes g)) = \tilde{\gamma}(a (h_{(1)^{\triangleright}} b) \chi' h_{(2)} \otimes g_{(1)}) \otimes h_{(3)} g_{(2)}
\]

\[
= a (h_{(1)^{\triangleright}} b) \chi'(h_{(2)} \otimes g_{(1)}) \gamma(h_{(3)} g_{(2)}) \otimes h_{(4)} g_{(3)}
\]

Assuming these expressions coincide, we apply \( \text{id} \otimes \gamma^{-1} \) to both and multiply in \( A \), to conclude that

\[
(h_{(1)^{\triangleright}} b) \chi^{\gamma}(h_{(2)} \otimes g) = (h_{(1)^{\triangleright}} b) \chi'(h_{(2)} \otimes g).
\]

Setting first \( g = 1 \) and then \( b = 1 \), we conclude that \( (\chi', b') = (\chi^{\gamma}, b^{\gamma}) \) as required.

Conversely, it is clear that if this equality does hold then \( \tilde{\gamma} \) is an algebra homomorphism. It is clear also that \( \tilde{\gamma} \) preserves the unit \( 1 \otimes 1 \) iff \( \gamma(1) = 1 \). \( \square \)

From this cocycle cross product algebra, there are systematic techniques to generate all the right-handed and dual versions that we might need. For example, writing the constructions in terms of maps pointing downwards, a left-right reversal gives the formulae for a right-handed cocycle cross product algebra, while an up-down reflection gives the formulae for a left-handed cocycle cross coproduct coalgebra. This is one of the beautiful things one can do with Hopf algebras, because their axioms are symmetric under these reflections. Without going into details, it is clear that the necessary data for the dual construction is a cocycle coaction

\[
\beta : C \to H \otimes C, \quad \psi : C \to H \otimes H
\]

say where \( \psi \) is a cocycle in \( H \) with values from the coalgebra \( C \). There is the dual notion to \( (3) \), namely that \( \beta \) respects the coproduct of \( C \), and there are the axioms of a cocycle-coaction which come out as

\[
((\text{id} \otimes \beta) \circ \beta(c_{(1)}))(\psi(c_{(2)})) \otimes 1 = (\psi(c_{(1)}) \otimes \text{id})(\Delta \otimes \text{id}) \circ \beta(c_{(2)})
\]

\[
(\epsilon \otimes \text{id}) \circ \beta(c) = c
\]

(19)
((\text{id} \otimes \psi) \circ \beta(c_{(1)}))((\text{id} \otimes \Delta) \circ \psi(c_{(2)})) = (\psi(c_{(1)}) \otimes 1)((\Delta \otimes \text{id}) \circ \psi(c_{(2)}))
(\epsilon \otimes \text{id}) \circ \psi(c) = \epsilon(c) = (\text{id} \otimes \epsilon) \circ \psi(c)

\text{for all } c \in C. \text{ The resulting cocycle cross coproduct coalgebra } C^\psi \triangleright H \text{ is}

\Delta(c \otimes h) = h_{(1)} \otimes c_{(2)}^{(1)} \psi(c_{(3)})(h_{(1)} \otimes c_{(2)}^{(2)} \otimes \psi(c_{(3)})^{(2)}h_{(2)})

\text{and tensor product counit. The cohomology picture likewise goes through: The group of}
\text{convolution-invertible linear maps } \gamma : C \to H \text{ with } \epsilon \circ \gamma = \epsilon \text{ acts on the pair } (\psi, \beta) \text{ by}

\psi^\gamma(c) = (\gamma(c_{(1)}) \otimes 1)((\text{id} \otimes \gamma) \circ \beta(c_{(2)}))\psi(c_{(3)})(\Delta \gamma(c_{(4)}))
\beta^\gamma(c) = (\gamma(c_{(1)}) \otimes 1)\beta(c_{(2)})^{-1}(c_{(3)}) \otimes 1)

\text{and the equivalence classes under this are the elements of the nonAbelian cohomology}
\text{spaces } H^2(C, H). \text{ All of this is a straightforward dualisation of the above results for cross}
\text{product algebras. Now we combine the two constructions to obtain}

**Theorem 4.3** \cite{22, 23} \text{Let } H \text{ and } A \text{ be bialgebras, } (\chi, \triangleleft) \text{ a right handed}
\text{cocycle-action of } H \text{ on } A \text{ and } (\psi, \triangleright) \text{ a cocycle-coaction of } A \text{ on } H. \text{ If in addition we have}

\psi(h_{(1)})\Delta(a \triangleleft h_{(2)}) = ((a_{(1)} \triangleleft h_{(1)})h_{(2)}^{(1)} \otimes a_{(2)} \triangleleft h_{(2)}^{(2)}) \psi(h_{(3)})
\epsilon(a \triangleleft h) = \epsilon(a)\epsilon(h), \quad \beta(1) = 1 \otimes 1
\beta(h_{(1)}g_{(1)})(\chi(h_{(2)} \otimes g_{(2)}) \otimes 1) = \chi(h_{(1)} \otimes g_{(1)})(h_{(2)}^{(1)}g_{(2)})(g_{(3)}^{(1)} \otimes h_{(2)}^{(2)}g_{(3)}^{(2)})

h_{(1)}^{(1)}(a \triangleleft h_{(2)}) \otimes h_{(1)}^{(2)} = (a \triangleleft h_{(1)})h_{(2)}^{(1)} \otimes h_{(2)}^{(2)}

\text{and } (\chi, \psi) \text{ obey a compatibility condition}

\psi(h_{(1)}g_{(1)})\Delta\chi(h_{(2)} \otimes g_{(2)}) = (\chi(h_{(3)} \otimes g_{(1)})(h_{(2)}^{(1)}g_{(2)})(g_{(3)}^{(1)}\psi(h_{(3)}^{(1)}g_{(4)})(g_{(5)}^{(1)}
\otimes \chi(h_{(2)}^{(2)}g_{(3)}^{(2)})(\psi(h_{(3)}^{(2)}g_{(5)}^{(2)}))\psi(g_{(6)})
\epsilon(\chi(h \otimes g)) = \epsilon(h)\epsilon(g), \quad \psi(1) = 1 \otimes 1

\text{then the cocycle cross product algebra and cross coproduct coalgebra fit together to form the}
\text{cocycle bicrossproduct bialgebra } H^\psi \triangleright A \cdot \chi A.

\text{These Hopf algebra constructions may appear unfamiliar, but this is exactly the}
\text{machinery which recovers the familiar theory of central extensions and unifies them with other}
\text{cocycle constructions. The central extensions as in (12)–(13) are given in terms of the re-}
\text{spective group algebras by } \psi, \beta, \triangleleft \text{ all trivial and only } \chi : G \times G \to M \text{ non-trivial, where}
\text{M is an Abelian group and } A = \mathbb{C} M.

\text{On the other hand, we answer also the question of when the cocycle cross products}
\text{CG} \triangleleft \chi \mathbb{C}(M) \text{ corresponding to anomalous quantisations of homogeneous spaces, are Hopf}
\text{algebras. Thus, let } G \text{ be a finite group and } A \text{ an algebra. The condition that we have a}
\text{(right-handed) cocycle cross product algebra } CG \triangleleft \chi A \text{ is that we have maps } \chi : G \times G \to A,
\triangleleft : A \times G \to A \text{ obeying}

1 \triangleleft u = 1, \quad (ab) \triangleleft u = (a \triangleleft u)(b \triangleleft u)
\chi(u, v)((a \triangleleft u) \triangleleft w) = (a \triangleleft (uv))\chi(u, v), \quad a \triangleleft 1 = a
\chi(uv, w)(\chi(u, v) \triangleleft w) = \chi(u, vw)\chi(v, w), \quad \chi(e, u) = \chi(u, e) = 1

(20)
which, in the case $A$ commutative would just be the usual notion of a group two-cocycle in $Z^2(G, A)$. The resulting cross product algebra is

$$(u \otimes a)(v \otimes b) = uv \otimes \chi(u, v)(a \triangleright w)b, \quad 1 = e \otimes 1$$

It we set $A = \mathbb{C}(M)$ for example, we have exactly a right-handed version of the anomalous cross product quantisation (14) discussed at the start of the section.

So for this to become a Hopf algebra along the lines above, we need $A$ to be a Hopf algebra, and a cocycle with values from $G$ in the form of a map $\psi : G \to A \otimes A$. This means nothing other than

$$\psi(u)_{23}((id \otimes \Delta)\psi(u) = \psi(u)_{12}((id \otimes \Delta)\psi(u)$$

(26)

for all $u \in G$. To keep things simple, we take coaction $\beta$ to be trivial, so that we can concentrate on the cocycle $\psi$. The cocycle cross coproduct (21) then reduces to

$$\Delta(u \otimes a) = u \otimes \psi(u)^{(1)}a^{(1)} \otimes u \otimes \psi(u)^{(2)}a^{(2)}, \quad \epsilon(u \otimes a) = \epsilon(a).$$

(27)

Finally, we ask that these give a Hopf algebra, $\mathbb{C}G^{\psi,w}_A$. According to the general theory above, the conditions for this are 25 24

$$\epsilon(a \triangleright u) = \epsilon(a), \quad \psi(u)\Delta(a \triangleright u) = ((\Delta a)\triangleright (u \times u))\psi(u)$$

(28)

$$\psi(uv)\Delta\chi(u, v) = (\chi(u, v) \otimes \chi(u, v))(\psi(u)\triangleright (v \times v))\psi(v)$$

(29)

As well as describing interesting possibilities for quantum physics, this construction has important algebraic examples too. For the very simplest case we let $G = \mathbb{Z}_2$ the group with two elements $\{e, w\}$ say and $w^2 = 1$. Then our family (26) becomes just one 2-cocycle $\psi(w) \in A \otimes A$, our action $\triangleright$ just becomes one algebra automorphism $T = \triangleright w : A \to A$ and $\chi$ just becomes one element $x = \chi(w, w) \in A$. The cocycle conditions above become (29) and

$$\psi(\Delta T(a)) = ((T \otimes T)\Delta a)\psi, \quad \epsilon \circ T(a) = \epsilon(a)$$

$$x^{-1}ax = T^2(a), \quad T(x) = x, \quad \Delta x = (x \otimes x)((T \otimes T)(\psi))\psi, \quad \psi(x) = 1.$$ 

So this data defines a cocycle bicrossproduct Hopf algebra $\mathbb{C}\mathbb{Z}_2^{\psi,w}_A$ generated by $A$ as a subHopf algebra and one additional generator $w \equiv w \otimes 1$ with

$$w^{-1}aw = T(a), \quad w^2 = x, \quad \Delta w = (w \otimes w)\psi, \quad \epsilon w = 1, \quad Sw = wUSx$$

where $U = \psi^{(1)}S\psi^{(2)}$.

**Example 4.4** Let $A = U_q(sl_2)$ in the standard conventions and let $\psi = \mathcal{R}$ its quasitriangular structure, $x = \nu^{-1}$ the inverse of its ribbon element in the sense of 29, and $T$ the algebra automorphism

$$T(H) = -H, \quad T(X_{\pm}) = -q^{\pm \frac{1}{2}}X_{\mp}.$$ 

This data obeys the conditions above. The resulting cocycle bicrossproduct Hopf algebra is generated by $U_q(sl_2)$ and one element $w$ adjoined with relations

$$w^{-1}aw = T(a), \quad w^2 = \nu^{-1}, \quad \Delta w = (w \otimes w)\mathcal{R}, \quad \epsilon w = 1, \quad Sw = wq^{-H}.$$

This extended Hopf algebra is a variant of the quantum Weyl group of $U_q(sl_2)$ and $T$ is a variant of Lusztig’s automorphism 30. This example and the general situation for $U_q(g)$ are in 24. We have given here a new and more explicit version.
5 Remark on trivial quantum principal bundles

A fascinating aspect of quantum groups is that they can be interpreted either as like enveloping algebras, or (the same Hopf algebras), as like algebras of functions. This means that the cross product quantisations of Sections 2–4 in which the group or quantum group generates the momentum symmetry, can appear instead as like the algebra of functions on some geometrical object. We demonstrated exactly this for the $\hbar \to 0$ limit of our Example 2.3, where $C(x) \triangleright C(p)$ is non-commutative and so it cannot literally be the algebra of functions on any ordinary space. Rather, it is by definition the algebra of functions on some quantum space. In this section, we want to elucidate exactly what is this quantum geometrical structure. In fact, the general machinery which we need is that of quantum group gauge theory as introduced in [11] and this section is intended as an introduction to this work.

We will see from this point of view that the cross product algebras in Section 2, and in Section 4 with $\chi$ convolution-invertible are trivial quantum principal bundles. They are geometrically trivial according to any usual notion of topology from the base manifold etc, but we still know from Proposition 4.2 that they are classified by the non-Abelian cohomology $H^2(H, A)$, which appears now as a new kind of topological invariant present only when the principal bundle is a quantum space. These are the same cocycles $\chi$ which, from the quantisation point of view, relate to anomalies.

The explanation of our point of view is very simple. Recall that a usual trivial principal bundle $P$ over a manifold $M$ and with structure group $G$ is a presentation of the base manifold as a homogeneous space $M = P/G$ and a global group co-ordinate chart $P \to G$. This is surjective and intertwines the action on $P$ with the right action of $G$ on itself by multiplication. Along with the canonical projection, it provides the global trivialisation $P \cong M \times G$. Now it is easy to reformulate these notions in an algebraic way. Firstly, we require a right coaction $C(P) \to C(P) \otimes C(G)$ such that $C(M)$ appears as the fixed point subalgebra $C(P)^{C(G)}$. In addition, we need a global trivialisation map $j : C(G) \to C(P)$ which should be injective and an intertwiner for the right coaction of $C(P)$ and the right coaction of $C(G)$ on itself by comultiplication. For a non-trivial principal bundle one does not have this global trivialisation $j$ but other weaker conditions, such as freeness of the action and local exactness conditions [11].

Thus the abstract model of a trivial principal bundle is to replace $C(G)$ by $H$, a Hopf algebra; $C(P)$ by $E$, an algebra on which $H$ coacts by an algebra homomorphism $\beta : E \to E \otimes H$, and $C(M)$ by an algebra $A$ identified with the fixed point subalgebra

$$E^H = \{ e \in E \mid \beta(e) = e \otimes 1 \} \subseteq E.$$ 

This data is called an extension of an algebra $A$ by a Hopf algebra $H$. Next, the idea of a global trivialisation means a linear map $j : H \to E$ which should be covariant under the right coaction of $H$. We do not insist that it is an algebra homomorphism but it should be convolution-invertible and obey $j(1) = 1$. In this case, the extension is said to be left. On the other hand, this data $(A, E, H, j)$ is exactly what we have used in [11] as the starting point of quantum group gauge theory. This notion of extensions is also an important idea in Galois theory, where one is interested in extensions of fields. So this area of number-theory is also unified with the present considerations of quantum mechanics and gauge theory. Some other geometrically-motivated results at the Hopf algebraic level are in [3].
On the other hand, it is not hard to see that the cocycle cross products $E = A\chi\triangleleft H$ of Section 4 with convolution-invertible cocycle $\chi$ are always cleft extensions. Conversely, every cleft extension of an algebra $A$ by a Hopf algebra $H$ is isomorphic to a convolution-invertible cocycle cross product. One defines $\chi : H \otimes H \rightarrow E$ and $\triangleright : H \otimes A \rightarrow E$ by

$$
j(h_{(1)})j(g_{(1)})j^{-1}(h_{(2)}g_{(2)}) = \chi(h \otimes g)$$

$$
j(h_{(1)})i(a)j^{-1}(h_{(2)}) = h \triangleright a$$

(30)

where $i : A \rightarrow E$ explicitly denotes the inclusion of $A$ in $E$ as fixed point subalgebra. It is clear that $(\chi, \triangleright)$ form a trivial cocycle in $H^2(H,E)$ as the coboundary of $j$. On the other hand, $H$-covariance of $j$ gives at once that the images of these maps $\chi, \triangleright$ are in the fixed-point subalgebra, i.e. we they are actually maps $H \otimes H \rightarrow A$ and $H \otimes A \rightarrow A$ as required. They still obey the cocycle conditions but now viewed as an element of $H^2(H,A)$. As such they are not necessarily a coboundary since $j$ itself need not map to $A$. We then build the cocycle cross product $A\chi\triangleleft H$ on the vector space $A \otimes H$ and verify that $A \otimes H \rightarrow E$ given by $a \otimes h \mapsto i(a)j(h)$ is an isomorphism. In this way, the inequivalent cleft extensions are in one-one correspondence with the nonAbelian convolution-invertible cohomology classes in $H^2(H,A)$. This is all fairly standard by now; one can see [26] and also [32]. Note that the notion of equivalence of extensions used here is the same as in Proposition 4.2, namely an isomorphism which is covariant under the coaction of $H$ and restricts to the identity on $A$. In the gauge theory picture these are exactly gauge transformations $\gamma : H \rightarrow A$ in the sense described in [11].

Note that in the classical setting, we assume that $E$ is commutative and that $j$ is an algebra map. The first of these forces any possible action $\triangleright$ to be trivial and the second forces any possible cocycle $\chi$ to be trivial also. Hence $E \cong A \otimes H$ as an algebra in this classical situation. This is why our nonAbelian cohomology does not enter the scene in the usual classical theory of trivial principle bundles. On the other hand, as soon as we go beyond this conventional setting, we have the possibility of non-trivial cocycles and correspondingly new quantum numbers provided by the cohomology classes in $H^2(H,A)$.

6 NonAbelian cohomology and twisting

In this section we make a connection between the nonAbelian cohomology spaces $H^2(H,A)$ above and Drinfeld’s theory of twisting[12]. From a mathematical standpoint, this is the modest new result of the paper.

Drinfeld introduced his process of twisting for quasi-Hopf algebras. These are only coassociative up to conjugation by a 3-cocycle $\phi$, obeying

$$
(1 \otimes \phi)((\id \otimes \Delta \otimes \id)\phi)((\id \otimes \id \otimes \Delta)\phi) = ((\id \otimes \id \otimes \Delta)\phi)((\Delta \otimes \id \otimes \id)\phi)
$$

$$
(\id \otimes \epsilon \otimes \id)\phi = 1 \otimes 1.
$$

(31)

In this case, twisting consists of conjugating the coproduct by an arbitrary invertible element $\psi$, say, in $H \otimes H$. However, it is easy to see that if $\psi$ obeys a further condition

$$
(1 \otimes \psi)(\id \otimes \Delta)\psi = (\psi \otimes 1)(\Delta \otimes \id)\psi, \quad (\epsilon \otimes \id)\psi = 1
$$

(32)
them
\[ \Delta \psi = \psi(\Delta)\psi^{-1}, \quad R_\psi = \psi_2 \mathcal{R} \psi^{-1}, \quad S_\psi = U(S)U^{-1} \]
remains a quasitriangular Hopf algebra, which we denote \( H_\psi \). Here \( U = \psi^{(1)}(S\psi^{(2)}) \). Of course, the result works also if \( H \) is only a bialgebra. Recently cf.\[34\], O. Ogievetsky has pointed out that our bicrossproduct Hopf algebras in Example 2.3 provide a simple example of twisting:

**Example 6.1** The bicrossproduct Hopf algebra \( H = \mathbb{C}[x]_\triangleright \mathbb{C}[p] \) is triangular and is twisting-equivalent to \( U(b_\perp) \) where \( b_\perp \) is the two-dimensional solvable Lie algebra. The triangular structure and twisting cocycle are
\[ \psi = e^{\frac{p_\perp x}{\hbar}}, \quad R = e^{\frac{p_\perp x}{\hbar}}e^{-\frac{p_\perp x}{\hbar}}. \]

**Proof** We first change variables to \( X = e^{Bx} - 1 \) so that \( p_\perp \triangleright X = \hbar(1 - e^{-Bx})\frac{\partial}{\partial x}X = \hbar B(1 - e^{-Bx})e^{Bx} = \hbar B X \). In terms of this generator \( X \) the bicrossproduct structure in Example 2.3 becomes
\[ [p, X] = \hbar B X \]
\[ \Delta X = X \otimes 1 + 1 \otimes X + X \otimes X, \quad \Delta p = 1 \otimes p + p \otimes (1 + X)^{-1}. \]
The algebra here is just the enveloping algebra of the Lie algebra \( b_\perp \) with generators \( p, X \). Its coproduct is not that of an enveloping algebra. But since \([p \otimes x, p \otimes 1] = 0 = [p \otimes x, 1 \otimes X] \) while \([p \otimes x, 1 \otimes p] = -\hbar p \otimes X(1 + X)^{-1} \) and \([p \otimes x, X \otimes 1] = \hbar B X \otimes x \) we see at once that
\[ \psi(X \otimes 1 + 1 \otimes X)\psi^{-1} = \Delta X, \quad \psi(p \otimes 1 + 1 \otimes p)\psi^{-1} = \Delta p. \]
This means that the coalgebra is the twisting of \( U(b_\perp) \). One can easily check the assertion for the counit also. Moreover, since the latter Hopf algebra has the trivial quasitriangular structure \( 1 \otimes 1 \), we deduce that \( R = \psi_2 \psi^{-1} \) is a quasitriangular structure for the bicrossproduct Hopf algebra. It is triangular in the sense \( R_{21} R = 1 \).

As far as I know, [33] is perhaps the first place where the condition (32) was explicitly formulated (those in [35], for example, being stronger) while remaining of course no more than a special case of Drinfeld’s ideas in [12]. We observe now that this condition is exactly a special case of the cocycles \( \psi \) in Section 4, namely \( H^2(\mathbb{C}, H) \) in the notation there. Let us add to these the conditions of a 1-cocycle. Just as in the dual version of Proposition 4.1, this is now an element \( \gamma \in H \) such that
\[ \gamma \otimes \gamma = \Delta \gamma, \quad \epsilon \gamma = 1 \quad (33) \]
i.e., a group-like element. Then we have as a special case of (32):

**Proposition 6.2** Let \( H \) be a bialgebra or Hopf algebra. If \( \gamma \in H \) is an invertible element with \( \epsilon \gamma = 1 \) then \( \partial \gamma = (\gamma \otimes \gamma) \Delta \gamma^{-1} \) is a 2-cocycle in \( H \). We say that it is a coboundary. More generally, if \( \psi \) is a 2-cocycle then
\[ \psi^? = (\gamma \otimes \gamma) \psi \Delta \gamma^{-1} \]
is also a 2-cocycle. We say that it is cohomologous to \( \psi \). The non-Abelian cohomology space \( H^2(\mathbb{C}, H) \) is the 2-cocycles in \( H \) modulo such transformations.
Proof. For completeness, we give here a direct proof of the transformation of cocycles. Thus

\[(1 \otimes \psi^\gamma)(\text{id} \otimes \Delta)\psi^\gamma\]
\[= (1 \otimes (\gamma \otimes \gamma)\psi\Delta\gamma^{-1})(\text{id} \otimes \Delta)((\gamma \otimes \gamma)\psi\Delta\gamma^{-1})\]
\[= (\gamma \otimes \gamma \otimes \gamma)(1 \otimes \psi\Delta\gamma^{-1})(\text{id} \otimes \Delta)(1 \otimes \gamma\psi\Delta\gamma^{-1})\]
\[= (\gamma \otimes \gamma \otimes \gamma)((1 \otimes \psi)(\text{id} \otimes \Delta)\psi)(\text{id} \otimes \Delta)\Delta\gamma^{-1}\]
\[= (\gamma \otimes \gamma \otimes \gamma)((\psi \otimes 1)(\Delta \otimes \text{id})\psi)(\text{id} \otimes \Delta)\Delta\gamma^{-1}\]
\[= (\gamma \otimes \gamma \otimes 1)((\psi\Delta\gamma^{-1} \otimes 1)((\Delta \otimes \text{id})(\gamma \otimes \gamma)\psi)(\Delta \otimes \text{id})\Delta\gamma^{-1}\]
\[= (\psi^\gamma \otimes 1)(\Delta \otimes \text{id})\psi^\gamma\]

as required. The other facts are clear. \(\square\)

For example, \textbf{[33] Lemma 2.2} says in these terms that

\[\psi \sim (S \otimes S)(\psi_{21}^{-1}).\]

Likewise, \(\mathcal{R}\) itself regarded as a 2-cocycle is always cohomologous to \(\mathcal{R}_{21}^{-1}\). The twisting in these cases is clearly just the opposite Hopf algebra.

**Proposition 6.3** Let \(\psi, \psi'\) be 2-cocycles. The Hopf algebras given by twisting by them are isomorphic via an inner automorphism if \(\psi, \psi'\) are cohomologous. I.e. there is a map from \(\mathcal{H}^2(\mathbb{C}, H)\) to the set of twistings of \(H\) up to inner automorphism. In particular, if \(\psi\) is a coboundary then twisting by it can be undone by an inner automorphism.

**Proof** We suppose that \(\psi, \psi'\) are cohomologous in the sense of Proposition 6.2. This means \(\psi' = (\gamma \otimes \gamma)\psi\Delta\gamma^{-1}\) for some invertible \(\gamma \in H\). Then we have \(\Delta'_\psi(h) = \psi'(\Delta h)\psi'^{-1} = (\gamma \otimes \gamma)\psi(\Delta\gamma^{-1})(\Delta h)(\Delta\gamma)\psi^{-1}(\gamma^{-1} \otimes \gamma^{-1}) = (\gamma \otimes \gamma)(\Delta\psi(\gamma^{-1} h \gamma))(\gamma^{-1} \otimes \gamma^{-1}).\) As \(\gamma(\gamma^{-1})\) is an inner automorphism of the algebra structure, we see that it defines now a bialgebra isomorphism \(H'_\psi \to H_\psi\). Hence it is also a Hopf algebra isomorphism in the case where \(H\) has an antipode. One can also see this directly from the formulae given for the antipode after twisting. Finally, if \(H\) is quasitriangular then \(\mathcal{R}_\psi = \psi'_{21} \mathcal{R}\psi'^{-1} = (\gamma \otimes \gamma)\psi_{21}(\Delta^{op}\gamma^{-1})\mathcal{R}(\Delta\gamma)\psi^{-1}(\gamma^{-1} \otimes \gamma^{-1}) = (\gamma \otimes \gamma)\mathcal{R}_\psi(\gamma^{-1} \otimes \gamma^{-1})\) using the axioms of a quasitriangular structure. So the induced isomorphism maps the quasitriangular structures too if these are present. \(\square\)

This says that the process of twisting can only give a genuinely new Hopf algebra if the cocycle \(\psi\) used to twist is cohomologically non-trivial. For example, if \(\mathcal{H}^2\) is trivial for a Hopf algebra then all twists are isomorphic. On the other hand, this gloomy possibility does not seem to occur very often and the twisting process does generally enable one to obtain new quasitriangular Hopf algebras from old.

We remark that all of these 1-cocycles, 2-cocycles and also Drinfeld’s 3-cocycles are nicely described by the following general formulae for \(n\)-cocycles \(Z^n(\mathbb{C}, H)\). On the other hand, though a reasonable coboundary map \(\delta\) is defined, it obeys \(\delta^2 = 1\) only in a subtle sense in which the Hopf algebra etc is modified after the first application of \(\delta\). This is already evident in Drinfeld’s theory of quasi-Hopf algebras and only in the case of 1-cocycles.
and 2-cocycles does this ‘feedback’ not arise. With this caveat, we can still proceed with some reasonable formulae. Thus, let \( H \) be a bialgebra or Hopf algebra. We let

\[
\Delta_i : H^\otimes n \to H^\otimes n+1, \quad \Delta_i = \text{id} \otimes \cdots \otimes \Delta \otimes \cdots \otimes \text{id}
\]

(34)

where \( \Delta \) is in the \( i \)’th position. Here \( i = 1, \ldots, n \) and we add to this the conventions \( \Delta_0 = 1 \otimes ( ) \) and \( \Delta_{n+1} = ( ) \otimes 1 \) so that \( \Delta_i \) are defined for \( i = 0, \ldots, n + 1 \). We define an \( n \)-cochain \( \psi \) to be an invertible element of \( H^\otimes n \) and we define its coboundary as the \( n+1 \)-cochain

\[
\partial \psi = \left( \prod_{i=0}^{i\text{ even}} \Delta_i \psi \right) \left( \prod_{i=1}^{i\text{ odd}} \Delta_i \psi^{-1} \right)
\]

(35)

where the even \( i \) run 0, 2, \ldots, and the odd \( i \) run 1, 3, \ldots, and the products are each taken in increasing order. We also write \( \partial \psi \equiv (\partial_+ \psi)(\partial_- \psi^{-1}) \) for the separate even and odd parts.

An \( n \)-cocycle in a Hopf algebra or bialgebra is an invertible element \( \psi \) in \( H^\otimes n \) such that \( \partial \psi = 1 \). Finally, we assume that our cochains or cocycles are counital in the sense \( \epsilon_i \psi = 1 \) for all \( \epsilon_i = \text{id} \otimes \cdots \otimes \epsilon \otimes \cdots \otimes \text{id} \). One can also posit the formula

\[
\psi^\gamma = (\partial_+ \gamma) \psi (\partial_- \gamma^{-1})
\]

(36)

for transformation of a cocycle by a cochain of one degree lower, with the caveat about feedback mentioned above when \( n \geq 3 \).

These formulae recover of course the 1-, 2-, and 3-cocycles above and their transformation properties. Moreover, they reduce for \( H = \mathbb{C}(G) \) to the usual theory of group cocycles. We recall that a usual group \( n \)-cochain is a point-wise invertible function \( \psi : G \times G \cdots \times G \to \mathbb{C} \) and has coboundary

\[
(\partial \psi)(u_1, u_2, \ldots, u_{n+1}) = \prod_{i=0}^{n+1} \psi(u_1, \ldots, u_i u_{i+1}, \ldots, u_{n+1})(-1)^i
\]

where, by convention, the first \( i = 0 \) factor is \( \psi(u_2, \ldots, u_{n+1}) \) and the last \( i = n + 1 \) factor is \( \psi(u_1, \ldots, u_n)\pm1 \). One has \( \partial^2 = 1 \) and the group cohomology \( \mathcal{H}^n(G, \mathbb{C}) \) is then defined as the group of \( n \)-cocycles modulo multiplication by coboundaries of the form \( \partial( ) \). On the other hand, these ‘classical’ formulae are equally interesting when the commutative Hopf algebra \( \mathbb{C}(G) \) is replaced by something non-commutative, even the group algebra \( \mathbb{C}G \) or an enveloping algebra \( U(g) \).

For completeness, we now give the dual version of all these constructions, for cocycles \( Z^n(H, \mathbb{C}) \). Thus an \( n \)-cochain on a bialgebra or Hopf algebra \( H \) is a linear functional \( \chi : H^\otimes n \to \mathbb{C} \) which is invertible in the convolution algebra and unital in the sense \( \chi(h_1 \otimes \cdots \otimes 1 \otimes \cdots \otimes h_{n-1}) = \epsilon(h_1) \cdots \epsilon(h_{n-1}) \), for 1 in any position. The coboundary \( \partial \chi \) is an \( n+1 \)-cochain of the same form as (35), but with the product in the convolution algebra,

\[
\partial \chi = \left( \prod_{i=0}^{i\text{ even}} \chi \circ i \right) \left( \prod_{i=1}^{i\text{ odd}} \chi^{-1} \circ i \right)
\]

where \( \circ \) is the map that multiplies \( H \) in the \( i, i+1 \) positions. The convention is \( \chi \circ 0 = \epsilon \otimes \chi \) and \( \chi \circ n+1 = \chi \otimes \epsilon \). For example, a 1-coboundary is

\[
\partial \chi(h \otimes g) = \chi(g_{(1)}) \chi(h_{(1)}) \chi^{-1}(h_{(2)} g_{(2)})
\]
so that a 1-cocycle just means an algebra homomorphism $H \to \mathbb{C}$. 3-cocycles provide of course dual quasiHopf algebras as studied in [36]. Likewise, 2-cocycles provide the theory of twisting in the dual formulation. Thus if $\chi$ is a 2-cocycle on a Hopf algebra $H$ in the sense $\chi(h \otimes 1) = \epsilon(h) = \chi(1 \otimes h)$ and

$$\chi(g_{(1)} \otimes f_{(1)})\chi(h \otimes g_{(2)} f_{(2)}) = \chi(h_{(1)} \otimes g_{(1)})\chi(h_{(2)} g_{(2)} \otimes f)$$

then we obtain another Hopf algebra $H_{\chi}$ by twisting the product and antipode according to

$$h \cdot \chi g = \chi(h_{(1)} \otimes g_{(1)})h_{(2)}g_{(2)}\chi^{-1}(h_{(3)} \otimes g_{(3)})$$

$$S_{\chi}(h) = U(h_{(1)})Sh_{(2)}U^{-1}(h_{(3)}), \quad U(h) = \chi(h_{(1)} \otimes Sh_{(2)}).$$

If $H$ is dual-quasitriangular then

$$R_{\chi}(h \otimes g) = \chi(g_{(1)} \otimes h_{(1)})R(h_{(2)} \otimes g_{(2)})\chi^{-1}(h_{(3)} \otimes g_{(3)})$$

is a dual-quasitriangular structure on the twisted Hopf algebra.

We have seen in Section 4 that 2-cocycles provide on the one hand central extensions, as encountered in anomalous quantum theories, and on the other hand they provide also dual-twistings. It seems likely that this connection could be exploited in the study of anomalies. Moreover, we have the possibility too of even more general anomalies than those usually encountered and which we might expect to show up in $q$-deformed physics. On the mathematical side, one would like to see these nonAbelian cohomology spaces connected more closely with cohomological ideas in the abstract deformation theory of Hopf algebras and their generalisations, as started in [37][38][39] and elsewhere. A close connection could be expected since deformation theory and cohomology were part of the original background behind Drinfeld’s twisting construction.

References

[1] V.G. Drinfeld. Quantum groups. In A. Gleason, editor, *Proceedings of the ICM*, pages 798–820, Rhode Island, 1987. AMS.

[2] S. Majid. *Non-commutative-geometric Groups by a Bicrossproduct Construction*. PhD thesis, Harvard mathematical physics, 1988.

[3] S. Majid. Hopf algebras for physics at the Planck scale. *J. Classical and Quantum Gravity*, 5:1587–1606, 1988.

[4] S. Majid. Physics for algebraists: Non-commutative and non-cocommutative Hopf algebras by a bicrossproduct construction. *J. Algebra*, 130:17–64, 1990. From PhD Thesis, Harvard, 1988.

[5] S. Majid. Hopf-von Neumann algebra bicrossproducts, Kac algebra bicrossproducts, and the classical Yang-Baxter equations. *J. Funct. Analysis*, 95:291–319, 1991. From PhD Thesis, Harvard, 1988.

[6] G.W. Mackey. *Induced Representations*. Benjamin, New York, 1968.

[7] H.D. Doebner and J. Tolar. Quantum mechanics on homogeneous spaces. *J. Math. Phys.*, 16:975–984, 1975.

[8] S. Majid. Representation-theoretic rank and double Hopf algebras. *Comm. Algebra*, 18(11):3705–3712, 1990.
[9] I.L. Egusquiza. Quantum mechanics on the quantum sphere. *Preprint*, DAMTP/92-18, 1992.

[10] S. Majid. The quantum double as quantum mechanics, september 1992 (damtp/92-48). *J. Geom. Phys.* To appear.

[11] T. Brzeziński and S. Majid. Quantum group gauge theory on quantum space. *Comm. Math. Phys.*, 157:591–638, 1993.

[12] V.G. Drinfeld. QuasiHopf algebras. *Algebra i Analiz*, 1(6):2, 1989. In Russian. Translation in *Leningrad Math J.*

[13] S. Majid. *Foundations of Quantum Group Theory*. In preparation.

[14] M.E. Sweedler. *Hopf Algebras*. Benjamin, 1969.

[15] S. Majid. Quasitriangular Hopf algebras and Yang-Baxter equations. *Int. J. Modern Physics A*, 5(1):1–91, 1990.

[16] S.L. Woronowicz. Twisted SU(2)-group, an example of a non-commutative differential calculus. *Publ. RIMS (Kyoto)*, 23:117–181, 1987.

[17] K. Mackenzie. Double Lie algebroids and second-order geometry, I. *Adv. Math.*, 94:180–239, 1992.

[18] M. Takeuchi. Matched pairs of groups and bismash products of Hopf algebras. *Comm. Alg.*, 9:841, 1981.

[19] S. Majid. Matched pairs of Lie groups associated to solutions of the Yang-Baxter equations. *Pac. J. Math.*, 141:311–332, 1990. From PhD Thesis, Harvard, 1988.

[20] S. Majid. Principle of representation-theoretic self-duality. *Phys. Essays*, 4(3):395–405, 1991.

[21] S. Majid. Braided matrix structure of the Sklyanin algebra and of the quantum Lorentz group. *Comm. Math. Phys.*, 156:607–638, 1993.

[22] S. Majid. Braided groups. *J. Pure and Applied Algebra*, 86:187–221, 1993.

[23] S. Majid. Examples of braided groups and braided matrices. *J. Math. Phys.*, 32:3246–3253, 1991.

[24] S. Majid and Ya. S. Soibelman. Bicrossproduct structure of the quantum Weyl group, 1991. *J. Algebra*, To appear.

[25] S. Majid. More examples of bicrossproduct and double cross product Hopf algebras. *Isr. J. Math*, 72:133–148, 1990.

[26] Y. Doi. Equivalent crossed products for a Hopf algebra. *Comm. Algebra*, 17:3053–3085, 1989.

[27] Y. Doi and M. Takeuchi. Cleft comodule algebras and hopf modules. *Comm. Alg.*, 14:801–817, 1986.

[28] W. Singer. Extension theory for connected Hopf algebras. *J. Alg.*, 21:1–16, 1972.

[29] N.Yu Reshetikhin and V.G. Turaev. Ribbon graphs and their invariants derived from quantum groups. *Comm. Math. Phys.*, 127(1):1–26, 1990.

[30] G. Lusztig. Quantum groups at roots of 1. *Geometrica Dedicata*, 35:89–114, 1990.
[31] H-J. Schneider. Representation theory of Hopf galois extensions. *Isr. J. Math.*, 72:196–230, 1990.

[32] R.J. Blattner and S. Montgomery. Crossed products and galois extensions of Hopf algebras. *Pac. J. Math.*, 137:37–54, 1989.

[33] D.I. Gurevich and S. Majid. Braided groups of Hopf algebras obtained by twisting, 1991. *Pac. J. Math.* To appear.

[34] O. Ogievetsky. Seminar at the 13th Winter School of Geometry and Physics, Zdikov, 1993.

[35] N.Yu. Reshetikhin. Multiparameter quantum groups and twisted quasi-triangular Hopf algebras. *Lett. Math. Phys.*, 20:331–335, 1990.

[36] S. Majid. Tannaka-Krein theorem for quasiHopf algebras and other results. *Contemp. Maths*, 134:219–232, 1992.

[37] M. Gerstenhaber and S.D. Schack. Bialgebra cohomology, deformations and quantum groups. *Proc. Natl. Acad. Sci. USA*, 87:478–481, 1990.

[38] S. Shnider and S. Sternberg. The cobar resolution and a restricted deformation theory for Drinfeld algebras. *J. Algebra*, 1992. To appear.

[39] M. Markl and J.D. Stasheff. Deformation theory via deviations. *J. Algebra*. To appear.