Geometry of Flat Directions in Scale-Invariant Potentials

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We observe that biquadratic potentials admit non-trivial flat directions when the determinant of the quartic coupling matrix of the scalar fields vanishes. This consideration suggests a new approach to the problem of finding flat directions in scale invariant theories, which noticeably simplifies the study of scalar potentials involving an arbitrary number of fields. The method generalizes to the case of arbitrary quartic potentials by requiring that the hyperdeterminant of the tensor of scalar couplings be zero. We demonstrate our approach with detailed examples that pertain to common scalar extensions of the Standard Model.

INTRODUCTION

During the last decades, scale-invariant models have taken the spotlight as possible solutions to an array of fundamental questions ranging from the hierarchy problem [1] to the origin of inflationary dynamics [2-5] and cosmological gravitational wave background [6-8].

Scale-invariant models are described by Lagrangians that depend purely on dimensionless parameters, then related to a dynamically generated scale. For weak couplings, such dimensional transmutation is provided by the Coleman-Weinberg (CW) mechanism [9]. In this scheme, quantum corrections in the scalar sector of the theory yield a non-trivial minimum of the effective potential which corresponds to the real vacuum state. Scalar fields then develop vacuum expectation values which source their mass terms and can achieve the spontaneous breaking of the symmetries of the action. A common feature of multi-scalar models based on the CW mechanism is the presence of a flat direction in the tree-level scalar potential [10] (alternative approaches to multi-scale potentials are presented in [11-17]). This is synonymous with a line in the field space along which the tree-level potential and its derivatives all vanish, resulting in a continuum of minima. The presence of a flat direction is required to ensure that quantum corrections alone shape the scalar potential via the CW mechanism.

In this Letter we investigate the appearance of flat directions with a new implementation of the Gildener-Weinberg approach based on the following observation: a scale-invariant scalar potential has a flat direction if the determinant of its quartic coupling matrix vanishes. In comparison to other methods present in the literature, our technique noticeably simplifies the study of scalar potentials involving an arbitrary number of fields or of a generic form. For instance, in the latter case, the problem is reduced to that of tensor eigenvalues [18, 19] previously employed for studying the stability and positivity of scalar potentials [20, 21].

We present simple examples of the method for the cases of a biquadratic two-field potential [22] (see also [23] and Refs. therein), a biquadratic three-field potential (see e.g. [24, 25]) and a general two-field potential [21].

GILDENER-WEINBERG APPROACH

We begin by reviewing the Gildener-Weinberg formalism [10] that generalizes the Coleman-Weinberg mechanism [9] to multi-field scale-invariant models. In our exposition we follow the conventions of Ref. [24].

Consider a general renormalizable gauge theory where the field vector $\Phi$ contains $n$ real scalar degrees of freedom. The most general scale-invariant quartic potential is of the form

$$V(\Phi) = \frac{1}{4!} \sum_{i,j,k,l} \lambda_{ijkl} \phi_i \phi_j \phi_k \phi_l,$$

where $\lambda_{ijkl}$ is the symmetric tensor of quartic couplings. At tree-level, all fields are massless, and the potential is bounded from below if $V(\Phi) \geq 0$ for all $\Phi$.

Suppose that the above potential admits a flat direction $\Phi = \varphi \mathbf{n}$ along the unit vector $\mathbf{n}$, where $\varphi$ is the radial distance from the origin of field space. We choose the renormalization scale $\mu$ such that the minimum of the potential on the unit sphere $N^2 \mathbf{N} = 1$ is zero. Therefore, if this minimum is obtained for a specific unit vector $\mathbf{N} = \mathbf{n}$, the potential along the whole flat direction $\Phi = \varphi \mathbf{n}$ also vanishes.

Because the flat direction includes only stationary points, the condition $\nabla_\mathbf{N} V(\mathbf{N})|_{\mathbf{N} = \mathbf{n}} = 0$ necessarily holds and the flat direction can be determined as a solution of

$$\sum_{j,k,l} \lambda_{ijkl} n_j n_k n_l = 0.

(2)$$

In order for the flat direction to comprise a continuous local minimum of the potential, we furthermore require that in the stationary point, the Hessian matrix

$$\left(P\right)_{ij} = \frac{\partial^2 V(\mathbf{N})}{\partial N_i \partial N_j} \bigg|_{\mathbf{N} = \mathbf{n}} = \frac{1}{2} \sum_{k,l} \lambda_{ijkl} n_k n_l

(3)$$

be positive-semidefinite.

Having found the flat direction, we can compute the main quantum corrections that shape the potential along it, obtaining

$$V^{(1)}(\varphi \mathbf{n}) = A(\mathbf{n}) \varphi^4 + B(\mathbf{n}) \varphi^4 \ln \frac{\varphi^2}{\mu^2}.

(4)$$
In the $\overline{\text{MS}}$ renormalization scheme, at the one-loop level, the constants

\[ A(n) = \frac{1}{64\pi^2 v^4} \left\{ \frac{\ln m_S^2}{v^2} \left( \ln m_S^2 - \frac{3}{2} \right) \right. \]

\[ + 3 \text{tr} \left[ m^4_i \frac{\ln m^2_i}{v^2} - \frac{5}{6} \right] \]

\[ - 4 \text{tr} \left[ m^4_F \left( \ln m^2_F - \frac{3}{2} \right) \right] \}, \]

\[ B(n) = \frac{1}{64\pi^2 v^4} \left( \text{tr} m^4_S + 3 \text{tr} m^4_F - 4 \text{tr} m^4_F \right) \]  

(6)

depend on the tree-level scalar, vector and fermion mass matrices that appear in the full theory, indicated, respectively, with $m^2_{S,V,F}$. These quantities are to be evaluated at the minimum of the potential, $\Phi = v_\varphi n$, with the vacuum expectation value $v_\varphi$ induced by quantum corrections along the flat direction. The tree-level scalar mass matrix is related to the Hessian via

\[ m^2_S = v^2_\varphi (P)_{ij}. \]  

(7)

The minimization of the Coleman-Weinberg potential with respect to the radial coordinate $\varphi$ shows that the full potential acquires a non-trivial stationary point at the renormalization scale $\mu = v_\varphi \exp(\frac{4}{27} n + \frac{1}{2})$. Then Eq. (4) becomes

\[ V^{(1)}(\varphi n) = B(n) \varphi^4 \ln \left( \frac{v^2_\varphi}{v^2_\varphi} - \frac{1}{2} \right). \]  

(8)

The masses squared of the scalars can then be computed as the eigenvalues of the matrix

\[ (m^2_S + \delta m^2_S)_{ij} = \frac{\partial^2 [V(\Phi) + V^{(1)}(\Phi)]}{\partial \phi_i \partial \phi_j} |_{\Phi = v_\varphi n}, \]  

(9)

revealing that the field along the flat direction – the pseudo-Goldstone of the classical scale invariance – obtains a mass $m^2_\varphi = 8Bv^2_\varphi$. The masses of the orthogonal fields receive only negligible corrections.

A NEW APPROACH TO FLAT DIRECTIONS IN SCALE-IN Variant POTENTIALS

We present an alternative mathematical method to implement the Gildener-Weinberg approach. We demonstrate that the appearance of a flat direction in a biquadratic scalar potential is ensured if the determinant of the quartic coupling matrix vanishes. The unit vector along the flat direction is obtained, via a matrix formalism, for any number of scalar fields. For generic potentials, the flat direction appears when the hyperdeterminant of the tensor of quartic couplings vanishes.

BIQUADRATIC POTENTIALS

A generic biquadratic potential of $n$ real scalar fields organized in the vector $\Phi$ is given by

\[ V(\Phi) = \sum_{i,j} \phi_i^2 \lambda_{ij} \phi_j^2 = (\Phi^{\circ 2})^T \Lambda \Phi^{\circ 2}, \]  

(10)

where $\Lambda$ is the symmetric matrix of the quartic couplings and the $\circ$ symbol indicates the Hadamard product, defined as the element-wise product of matrices of same dimensions: $(A \circ B)_{ij} = A_{ij}B_{ij}$. The Hadamard power is then simply given by $(A^{\circ n})_{ij} = A_{ij}^n$.

Note that the norm of $\Phi$ can be written in terms of its Hadamard square as

\[ \Phi^T \Phi = e^T \Phi^{\circ 2}, \]  

(11)

where $e = (1, \ldots , 1)^T$, the vector of ones, is an identity element of the Hadamard product. The relation (11) is quadratic in the field $\Phi$ but only linear in $\Phi^{\circ 2}$.

Instead of introducing spherical coordinates for the field space, we restrict the potential to the unit hyperspace by means of a Lagrange multiplier:

\[ V(N, \lambda) = (N^{\circ 2})^T \Lambda N^{\circ 2} + \lambda(1 - e^T N^{\circ 2}). \]  

(12)

The Euler-Lagrange equation for the Lagrange multiplier $\lambda$ then recovers

\[ e^T N^{\circ 2} = 1 \]  

(13)

with the vector $N$ lying on the unit hypersphere. The Hadamard square $N^{\circ 2}$, instead, lies on the unit simplex and the elements of $N^{\circ 2}$ are its barycentric coordinates. Extremizing the potential on the unit circle, as required to have a flat direction, is then equivalent to the problem of extremizing a quadratic function (of $N^{\circ 2}$ in this case) on the unit simplex. This problem is known within the theory of optimization as the standard quadratic program (see e.g. [27]).

In order for a minimum to exist, the potential must be bounded from below, for which a necessary and sufficient condition is that the matrix $\Lambda$ be copositive [28]. This can be ascertained via the Cottle-Habetler-Lenke theorem [29]: Suppose that the order $n - 1$ principal submatrices of a real symmetric matrix $\Lambda$ of order $n$ are copositive. In that case $\Lambda$ is copositive if and only if

\[ \text{det}(\Lambda) \geq 0 \quad \forall \text{ some element(s) of } \text{adj } \Lambda < 0. \]  

(14)

The adjugate $\text{adj}(\Lambda)$ of a matrix $\Lambda$ is defined through the relation $\Lambda \text{adj}(\Lambda) = \text{det}(\Lambda) I$. Notice that a semipositive-definite matrix is necessarily copositive, therefore a semipositive-definite $\Lambda$ is sufficient for the stability of the potential.

Proceeding with the explicit calculation, the first derivatives of the potential are given by the vector

\[ \nabla_N V = 4N \circ \Lambda N^{\circ 2} - 2\lambda N \circ e \]

\[ = 2N \circ (2\Lambda N^{\circ 2} - \lambda e) \]  

(15)
and stationary points on the hypersphere obey the usual relation \( \nabla_N V = 0 \).

At first, we assume that none of the elements of \( N^{\circ 2} \) vanish, postponing the discussion of these particular cases to the end of the derivation. Under this assumption

\[
2 \Lambda N^{\circ 2} = \lambda e, \tag{16}
\]

and, multiplying both sides of Eq. \[16\] from the left by \((N^{\circ 2})^T\) and using the constraint \[13\] yields

\[
\lambda = 2(N^{\circ 2})^T \Lambda N^{\circ 2} = 2V(N). \tag{17}
\]

Eq. \[16\] then gives

\[
\Lambda N^{\circ 2} = [(N^{\circ 2})^T \Lambda N^{\circ 2}] e \equiv V(N) e, \tag{18}
\]

which we need to solve in order to guarantee the presence of a flat direction. To this purpose, we make the ansatz that

\[
N^{\circ 2} = C \text{adj}(\Lambda) e, \tag{19}
\]

where \( C \) is a real normalization constant and \( \text{adj}(\Lambda) \) is the adjugate matrix of \( \Lambda \). Inserting Eq. \[19\] into Eq. \[18\], we then obtain

\[
C = \frac{1}{e^T \text{adj}(\Lambda) e}, \tag{20}
\]

which normalizes \( N \) to unity, as required. It is easy to see that \( \text{adj}(\Lambda) e \) is the vector of row sums of \( \text{adj}(\Lambda) \), and that \( e^T \text{adj}(\Lambda) e \) is the sum of all elements of \( \text{adj}(\Lambda) \). Consistency requires that all the elements of \( N^{\circ 2} \) be positive. \(^1\)

As a result, Eqs. \[18\], \[19\] and \[20\] show the existence of an extremum \( N \) on the unit hypersphere in field space such that

\[
V(N) = \frac{\text{det}(\Lambda)}{e^T \text{adj}(\Lambda) e} \tag{21}
\]

and, consequently, the determinant of \( \Lambda \) must be zero to have a flat direction along \( N = n \):

\[
\text{det}(\Lambda) = 0 \iff V(n) = 0. \tag{22}
\]

We see that the unit vector

\[
n^{\circ 2} = \frac{\text{adj}(\Lambda) e}{e^T \text{adj}(\Lambda) e} \tag{23}
\]

along the flat direction is then an eigenvector of \( \Lambda \) corresponding to a null eigenvalue.

So far we have shown that \( n \) is an extremal of the potential on the unit hypersphere, but for it to correspond to a stable flat direction we still have to determine whether the above solution is a minimum of the scalar potential on the hypersphere. To this purpose, we calculate the Hessian \[5\] of \( V(N, \lambda) \) as

\[
P = \nabla_N \nabla_N^T V(N, \lambda)|_{N=n} = \text{diag}[2(2n^{\circ 2} - \lambda e)] + 8\Lambda \circ (nn^T) \tag{24}
\]

where \( \text{diag}(v) \) designates a diagonal matrix whose diagonal is given by the vector \( v \), and we took into account that \( \lambda = 2V(n) = 0 \). The condition for the extremum on the unit hypersphere to be a local minimum is that the Hessian matrix \( P \) be positive on the space tangent to the hypersphere itself at \( N = n \). \(^{30}\) Because \( n \) corresponds to the eigenvector of \( P \) with zero eigenvalue, and given that any vector in the field space can be written as a linear combination of \( n \) and the tangent vectors, we then simply require that \( P \) be positive-semidefinite on all unit vectors \( N \).

Along the direction \( n \), the diagonal term in the expression \[24\] vanishes and the Hessian can be written as \( P = 8\Lambda \circ (nn^T) = 8 \text{diag}(n) \Lambda \text{diag}(n) \). Therefore, since then \( \text{det}(P) = \text{det}(\Lambda) \prod 8n_i^2 \), the Hessian is positive-semidefinite if and only if the coupling matrix \( \Lambda \) is positive-semidefinite. As discussed before, this condition also ensures that the potential is bounded from below.

We see that as \( \text{det}(\Lambda) \) goes to zero, the mass of the radial degree of freedom in the direction of \( n \) vanishes and a continuous minimum of the potential forms a stable flat direction, determined by \[19\]. Given a positive-semidefinite matrix \( \Lambda \), the condition \( \text{det}(\Lambda) = 0 \) therefore signals the presence of a flat direction.

We focus now on the cases where up to \( n - 1 \) components of the unit vector \( N \) vanish. In this case, Eq. \[16\] is restricted to the matrix of quartic couplings for the fields in the non-degenerate subspace and the vector \( e \) has a corresponding dimension.

Without loss of generality, the coupling matrix and the flat direction can be brought in the block form

\[
\Lambda = \begin{pmatrix}
\Lambda_{11} & \Lambda_{12} \\
\Lambda_{12}^T & \Lambda_{22}
\end{pmatrix}, \quad n = \begin{pmatrix}
n_1 \\
0
\end{pmatrix} \tag{25}
\]

by means of a suitable permutation of the fields.

The non-zero part \( n_1 \) of the flat direction is an eigenvector of the submatrix \( \Lambda_{11} \) with the zero eigenvalue and the mass matrix \[24\] takes the form

\[
P = \begin{pmatrix}
8\Lambda_{11} \circ (n_1n_1^T) & 0 \\
0^T & \text{diag}(4\Lambda_{12}^T n_1^{\circ 2})
\end{pmatrix}. \tag{26}
\]

In this case, \( n \) is a stable flat direction if \( \Lambda_{11} \) is positive-semidefinite and \( \Lambda_{12}^T n_1^{\circ 2} \) is a vector with positive elements. The only condition on the \( \Lambda_{22} \) submatrix is that

\(^1\) To this end, it is sufficient, but not necessary, that all the off-diagonal elements of \( \Lambda \) be negative.
of copositivity, imposed as the potential must be bounded from below.

We conclude by showing how to reverse-engineer our method in the cases where the desired field direction $n$ and masses of involved particles are known. A suitable coupling matrix $\Lambda$ can be found by taking the elementwise Hadamard inverse of the relation (24):

$$\Lambda = \frac{1}{8} P \circ (nn^T)^{\circ -1} = \frac{1}{8 v^2} m^T \circ (nn^T)^{\circ -1}. \quad (27)$$

Note that the mixing angles of the mass matrix are the same as the spherical angles of the flat direction. If some elements of $n$ are zero, then the Hadamard inverse in Eq. (27) can be calculated only as restricted to the submatrices $A_{11}$ and $n_1n_1^T$ in Eq. (25). The undetermined elements of $\Lambda$ must be chosen such as to keep the desired flat direction $n$ a local minimum in line with Eqs. (25) and (26).

**General Potentials**

The generic potential in Eq. (1), restricted to the unit hypersphere, can be written as

$$V(N, \lambda) = \frac{1}{4} N^4 + \lambda (1 - N^T N). \quad (28)$$

In our notation, a polynomial $f(x)$ of order $m$ and its coefficient tensor $\Lambda$ are related as $Ax^m = mf(x)$ and $Ax^{m-1} = \nabla f(x)$ with

$$\sum_{i_1, i_2, \ldots, i_m} A_{i_1 i_2 \ldots i_m} x_{i_1} x_{i_2} \ldots x_{i_m} = x^T A x^{m-1}. \quad (29)$$

The extrema conditions for the potential (28) are then given by

$$\Lambda N^3 = \lambda N, \quad (30)$$

$$N^T N = 1, \quad (31)$$

which we recognize as the $E$-eigenvalue equations for the $\Lambda$ tensor. In contrast to matrices, tensors possess several types of eigenvalues and corresponding eigenvectors [18, 19]. For instance, $N$-eigenvectors and $N$-eigenvalues are given by the solutions of $Ax^{m-1} = \lambda x^{m-1}$. It is the $E$-eigenvectors, however, which are normalized to the unit hypersphere and invariant under orthogonal transformations of the tensor $\Lambda$. We refer the reader to the book in Ref. [31] for further details on tensor eigenvalues.

The number of $E$-eigenvectors of a symmetric tensor of order $m$ in $\mathbb{R}^n$ is

$$d = \frac{(m - 1)^n - 1}{m - 2}. \quad (32)$$

A generic $E$-eigenvalue can be complex, but only real tensor eigenvectors and eigenvalues are physical solutions of Eqs. (30) and (31). Our analysis will then pertain only to real $E$-eigenvalues, which together with the associated real eigenvectors are called $Z$-eigenvalues and $Z$-eigenvectors.

We can eliminate $N_i$ from Eqs. (30) and (31), obtaining the characteristic polynomial $\phi_\lambda(\Lambda)$ of the tensor, with a degree given by Eq. (32). The multivariate resultant of a system of polynomial equations is a polynomial in their coefficients, which vanishes if and only if the equations have a common root. The free term of $\phi_\lambda(\Lambda)$ – the product of all $E$-eigenvalues – is given by the resultant $\text{res}_N(\Lambda N^3)$. For that reason, the resultant is also called the hyperdeterminant. Therefore, in order to have a zero tensor eigenvalue, we must have $\text{res}_N(\Lambda N^3) = 0$, implying that $\Lambda N^3 = 0$ has a non-trivial solution.

In addition, the tensor $\Lambda$ must be positive-semidefinite in order for the potential (1) to be bounded from below. That is, all of its eigenvalues and those of its principal subtensors – obtained by setting one or more fields to zero in $V(N)$ – must be non-negative. The Hessian matrix (3) must be positive-semidefinite as well.

Note that for a potential given by a quartic polynomial of two fields, the resultant is proportional to the discriminant of the polynomial obtained by setting one field to unity. Unfortunately, for a larger number of variables, it is, as a rule, impossible to calculate the resultant analytically and even a numerical computation can be prohibitively expensive.

Furthermore, unlike for biquadratic potentials, in general all the potential coefficients cannot be determined from the knowledge of the mass matrix.

**EXAMPLES**

**Biquadratic Two-Field Potential**

As a simple non-trivial example, we consider the scale-invariant biquadratic potential of two fields $\phi_1$ and $\phi_2$, given by

$$V = \lambda_1 \phi_1^4 + \lambda_2 \phi_1^2 \phi_2^2 + \lambda_2 \phi_2^4. \quad (33)$$

The associated matrix of couplings is determined via Eq. (10) as

$$\Lambda = \begin{pmatrix} \lambda_1 & \frac{1}{2} \lambda_{12} \\ \frac{1}{2} \lambda_{12} & \lambda_2 \end{pmatrix}, \quad (34)$$

and its adjugate matrix is

$$\text{adj}(\Lambda) = \begin{pmatrix} \lambda_2 & -\frac{1}{2} \lambda_{12} \\ -\frac{1}{2} \lambda_{12} & \lambda_1 \end{pmatrix}. \quad (35)$$

The extrema equations of $V(N, \lambda)$ on the unit simplex have three solutions, two corresponding to either of the
two fields set to zero and a solution where neither vanishes. Focusing for the moment on the last one, from Eqs. (19) and (20), we have

$$\mathbf{N}^{\text{e}2} = \frac{1}{\lambda_1 - \lambda_{12} + \lambda_2} \left( \lambda_2 - \frac{\lambda_{12}^2}{\lambda_1} \right).$$

(36)

The solution for $\mathbf{N}^{\text{e}2}$ is physical only if its components are non-negative and, as shown in our general discussion, the flat direction is stable if $\Lambda$ is positive-semidefinite. In addition to det $\Lambda \geq 0$, this requires that the determinants of all principal minors of $\Lambda$ be non-negative. In this case, as expected, this condition reduces to $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$.

The value of the potential along the extremal direction $\mathbf{N}$ is then

$$V(\mathbf{N}) = \frac{4\lambda_1\lambda_2 - \lambda_{12}^2}{4(\lambda_1 - \lambda_{12} + \lambda_2)}$$

(37)

and the flat direction $\mathbf{N} = \mathbf{n}$ appears when

$$\det \Lambda = \lambda_1\lambda_2 - \frac{1}{4}\lambda_{12}^2 = 0,$$

(38)

implying $V(\mathbf{n}) = 0$. Note that we must have a negative $\lambda_{12} = -2\sqrt{\lambda_1\lambda_2}$ in order for both elements of $\mathbf{n}^{\text{e}2}$ to be positive.

The Hessian matrix \cite{24} is given by

$$\mathbf{P} = \begin{pmatrix} 12\lambda_1N_1^2 + 2\lambda_{12}N_2^2 & 4\lambda_{12}N_1N_2 \\ 4\lambda_{12}N_1N_2 & 12\lambda_2N_2^2 + 2\lambda_{12}N_1^2 \end{pmatrix}$$

(39)

and, under the above condition, is automatically positive-semidefinite.

The two particular solutions corresponding to vanishing fields are given by $N_1 = 1$, $N_2 = 0$, in which case $V(\mathbf{N}) = \lambda_1$, and by $N_1 = 0$, $N_2 = 1$, yielding $V(\mathbf{N}) = \lambda_2$. According to our procedure, the flat direction is then obtained when the self-coupling of the non-zero field vanishes. Focusing, for example, on the first case, we have $\lambda_1 = 0$. The block form of the Hessian matrix in Eq. \cite{25} is then given by $\Lambda_{11} = (\lambda_1)$, $\Lambda_{12} = (\lambda_{12})$ and $\Lambda_{22} = (\lambda_2)$ and $\mathbf{n}_1 = (1)$, so the requirement that $\text{diag}(\Lambda_{12}^{\text{T}}\mathbf{n}_1^{\text{e}2}) = (\lambda_{12})$ be positive implies that $\lambda_{12} > 0$. The copositivity of $\Lambda_{22}$ requires $\lambda_2 > 0$.

In order to relate our formalism to the hyperspherical coordinates in field space commonly used, notice that the mixing angle between the fields in Eq. \cite{36} is given by

$$\tan^2 \theta = \frac{n_2^2}{n_1^2} = \frac{2\lambda_1 - \lambda_{12}}{2\lambda_2 - \lambda_{12}} = \sqrt{\frac{\lambda_1}{\lambda_2}}.$$  

(40)

with $\lambda_{12} = -2\sqrt{\lambda_1\lambda_2}$ from Eq. \cite{38}. If we rewrite the fields on the unit circle in polar coordinates, $N_1 = \cos \theta$ and $N_2 = \sin \theta$, the same angle is obtained from the minimization condition $dV/d\theta = 0$, corresponding to $\mathbf{N} = \mathbf{n}$. This equation has two other solutions, $\theta = 0$ and $\theta = \pi/2$, that correspond to the particular cases where the flat direction lies along a coordinate axis.

To conclude the example, let us assume a flat direction at a given angle $\theta$ and that the mass of the scalar field along the transverse direction is $M_2$. The mixing matrix acting on the fields $\phi_1$ and $\phi_2$ that span the field space is then given by

$$\mathbf{O} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

(41)

and the mass eigenstates $h_1$ and $h_2$, respectively along and orthogonal to the flat direction, are obtained upon the rotation

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \mathbf{O}^T \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$  

(42)

Taking into account that at tree-level $M_1 = 0$, the scalar mass matrix for the fields $\phi_1$ and $\phi_2$ is

$$\mathbf{m}^{2}_S = M_2^2 \begin{pmatrix} \sin^2 \theta & -\sin \theta \cos \theta \\ -\sin \theta \cos \theta & \cos^2 \theta \end{pmatrix}$$

(43)

and the relation \cite{27} then determines the quartic coupling matrix

$$\Lambda = \frac{1}{8} \frac{M_2^2}{v_\phi^2} \begin{pmatrix} \tan^2 \theta & -1 \\ -1 & \cot^2 \theta \end{pmatrix}.$$  

(44)

Biquadratic Three-Field Potential

For a biquadratic potential of three fields

$$V = \lambda_1 \phi_1^4 + \lambda_{12} \phi_1^2 \phi_2^2 + \lambda_2 \phi_2^4 + \lambda_{23} \phi_2^2 \phi_3^2 + \lambda_3 \phi_3^4 + \lambda_{13} \phi_1^2 \phi_3^2,$$

the matrix of couplings and its adjugate are given by

$$\Lambda = \begin{pmatrix} \lambda_1 & \frac{1}{2} \lambda_{12} & \frac{1}{2} \lambda_{13} \\ \frac{1}{2} \lambda_{12} & \lambda_2 & \frac{1}{2} \lambda_{23} \\ \frac{1}{2} \lambda_{13} & \frac{1}{2} \lambda_{23} & \lambda_3 \end{pmatrix}, \quad \text{adj}(\Lambda) = \begin{pmatrix} \lambda_2 \lambda_3 - \frac{1}{4} \lambda_{12}^2 & \frac{1}{2} \lambda_{13} \lambda_{23} - \frac{1}{2} \lambda_{12} \lambda_3 & \frac{1}{2} \lambda_{12} \lambda_3 - \frac{1}{2} \lambda_{13} \lambda_2 \\ \frac{1}{2} \lambda_{13} \lambda_{23} - \frac{1}{2} \lambda_{12} \lambda_3 & \lambda_1 \lambda_3 - \frac{1}{4} \lambda_{13}^2 & \frac{1}{2} \lambda_{13} \lambda_2 - \frac{1}{2} \lambda_{12} \lambda_3 \\ \frac{1}{2} \lambda_{12} \lambda_3 - \frac{1}{2} \lambda_{13} \lambda_2 & \frac{1}{2} \lambda_{13} \lambda_2 - \frac{1}{2} \lambda_{12} \lambda_3 & \lambda_1 \lambda_2 - \frac{1}{4} \lambda_{12}^2 \end{pmatrix}.$$  

(46)
The mixing angles are given by \( \theta \) and \( \lambda \) sin \( \theta \), which are real scalars coordinate planes or along one of the three axes.

In general, the flat direction can also lie along an axis or on a coordinate plane on the border of the simplex.

The Hadamard square of the unit vector along the flat direction is then given by Eq. (23) with \( \epsilon = (1, 1, 1)^T \), and lies on the unit simplex illustrated in Fig. 1. There are also six lower-dimensional solutions: the flat direction may lie on the border of the simplex on one of the three coordinate planes or along one of the three axes.

Writing the fields in spherical coordinates \( N_1 = \sin \theta \cos \phi \), \( N_2 = \sin \theta \sin \phi \), \( N_3 = \cos \theta \), we see that the mixing angles are given by

\[
\tan^2 \phi = \frac{n_2^0}{n_1^0}, \quad \cos^2 \theta = n_3^0.
\]

General Two-Field Potential

The most general renormalisable scalar potential of two real scalars \( \phi_1 \) and \( \phi_2 \) is given by

\[
V = \lambda_{40} \phi_1^4 + \lambda_{31} \phi_1^3 \phi_2 + \lambda_{22} \phi_1^2 \phi_2^2 + \lambda_{13} \phi_1 \phi_2^3 + \lambda_{04} \phi_2^4,
\]

where the indices of couplings count powers of the fields.

The tensor of the scalar couplings of the potential is given by

\[
A = \begin{pmatrix}
4 \lambda_{40} & \lambda_{31} \\
\lambda_{31} & \frac{2}{3} \lambda_{22} \\
\lambda_{31} & \frac{2}{3} \lambda_{22} \\
\frac{2}{3} \lambda_{22} & \lambda_{13}
\end{pmatrix}
\begin{pmatrix}
\lambda_{31} & \frac{2}{3} \lambda_{22} \\
\frac{2}{3} \lambda_{22} & \lambda_{13} \\
\frac{2}{3} \lambda_{22} & \lambda_{13} \\
\lambda_{13} & 4 \lambda_{04}
\end{pmatrix}
\]

that is, \( \lambda_{1111} = 4 \lambda_{40} \), \( \lambda_{2222} = 4 \lambda_{04} \), \( \lambda_{1112} = \lambda_{1121} = \lambda_{1211} = \lambda_{2111} = \lambda_{31} \) and so on.

The tensor eigenvalue equations are

\[
4 \lambda_{40} N_1^4 + 3 \lambda_{31} N_1^2 N_2 + 2 \lambda_{22} N_1 N_2^2 + \lambda_{13} N_2^3 = 4 \Lambda N_1,
\]

\[
\lambda_{31} N_1^3 + 2 \lambda_{22} N_1^2 N_2 + 3 \lambda_{13} N_1 N_2^2 + 4 \lambda_{04} N_2^3 = 4 \Lambda N_2,
\]

\[
N_1^2 + N_2^2 = 1.
\]

All tensor eigenvalues, satisfying Eqs. (51), corresponding to real tensor eigenvectors must be non-negative, but more succinct criteria for the potential (49) to be bounded from below also exist (20).

The resultant of the associated homogenous equations – the hyperdeterminant – therefore is

\[
\text{res}_\mathbf{N}(\mathbf{AN}^3) = 16|16 \lambda_{04} \lambda_{40} \lambda_{22} - 4(\lambda_{13}^2 \lambda_{40}) + \lambda_{04} \lambda_{22}^2 - (80 \lambda_{04} \lambda_{13} \lambda_{31} \lambda_{40}) + 128 \lambda_{04} \lambda_{40}^2 - \lambda_{13}^2 \lambda_{31} \lambda_{40}^2 + 18(\lambda_{04} \lambda_{13} \lambda_{31}^2 + \lambda_{13}^3 \lambda_{31} \lambda_{40} + 8 \lambda_{04} \lambda_{13}^2 \lambda_{31} \lambda_{40}^2) - 4 \lambda_{13}^3 \lambda_{31}^2 - 27 \lambda_{04}^3 \lambda_{13}^2
\]

\[
- 6 \lambda_{04} \lambda_{13}^2 \lambda_{31} \lambda_{40} - 27 \lambda_{13}^2 \lambda_{31}^2 - 192 \lambda_{04}^2 \lambda_{13} \lambda_{31} \lambda_{40}^2 + 256 \lambda_{04}^3 \lambda_{13} \lambda_{40}^2|
\]

which, in this example, is a quartic polynomial in \( \lambda_{22} \). We can then solve \( \text{res}_\mathbf{N}(\mathbf{AN}^3) = 0 \) for \( \lambda_{22} \) to ensure the existence of a flat direction.

The Hessian matrix of the potential (49) must be positive-semidefinite at the extremum.

CONCLUSIONS

We have proposed a novel technique for investigating the appearance of a flat direction in the scalar sector of a scale-invariant model. Our method builds on the observation that, in presence of a flat direction, the determinant of the matrix of quartic couplings of a biquadratic potential necessarily vanishes at a specific renormalization scale. The result is easily extended to more general potentials via the formalism of tensor eigenvalues. In comparison with the usual hyperspherical coordinate approach, we find that our matrix method noticeably simplifies the study of more involved scalar sectors, opening the way to phenomenological studies of less minimalistic scale-invariant models.

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