On the Lenstra constant associated to the Rosen continued fractions

by

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Abstract

The purpose of this paper is to describe the relation between the Legendre and the Lenstra constants. Indeed we show that they are equal whenever the Legendre constant exists; in particular, this holds for both Rosen continued fractions and \( \alpha \)-continued fractions. We also give the explicit value of the entropy of the Rosen map with respect to the absolutely continuous invariant probability measure.

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1 Introduction

Let \( x \) be an irrational number in \([0, 1]\). We denote by \( \frac{p_n}{q_n} \) the \( n \)-th principal convergent of \( x \) and recall the following.

**Theorem (Legendre)** Suppose \( p \) and \( q \) (> 0) are relatively prime integers and \( |x - \frac{p}{q}| < \frac{1}{2q^2} \). Then \( \frac{p}{q} \) is a principal convergent to \( x \). On the other hand, for any \( c > \frac{1}{2} \), there exist \( x \) and \( \frac{p}{q} \), which is not a principal convergent, such that \( |x - \frac{p}{q}| < c \frac{1}{q^2} \).

In this sense, we call \( \frac{1}{2} \) the Legendre constant of the regular continued fractions. Now we consider the error of the principal convergents. We put

\[
\Theta_n = q_n^2 |x - \frac{p_n}{q_n}|.
\]

The following fact was proved by [2].

**Theorem** (Bosma, Jager and Wiedijk, 1983) For a.e. \( x \),

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{\infty} \{ 1 \leq j \leq n : \Theta_j \leq t \}
\]

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exists for any \( t, 0 \leq t \leq 1 \) and the limit is equal to

\[
F(t) = \begin{cases} 
\frac{1}{\ln 2} & , \quad 0 \leq t \leq \frac{1}{2} \\
\frac{1}{\ln 2} (1 - t + \ln 2t) & , \quad \frac{1}{2} \leq t \leq 1 
\end{cases}
\]

We notice that \( F(t) \) is linear in \( 0 \leq t \leq \frac{1}{2} \) and not in \( \frac{1}{2} \leq t \leq 1 \). In this sense, we call \( \frac{1}{2} \) the Lenstra constant of regular continued fractions because this fact was conjectured by H. W. Lenstra in 1981. We can define Legendre constants and Lenstra constants for other types of continued fraction expansions in a similar manner (e.g. \( \alpha \)-expansions, [12], [2]). In general, it is not hard to show that the Lenstra constant exists and is at least as large than the Legendre constant for each type of continued fraction expansion whenever the Legendre constant exists. However, a number of examples indicate that these constants seem to be equal to each other.

Motivation of this paper is the metrical theory of Rosen continued fractions associated to Hecke groups. It is possible to define the Legendre constants in this case even though the convergents of Rosen continued fractions are not rational numbers. Indeed, in 1985 J. Lehner claimed that the Legendre constant of Rosen continued fractions is greater than or equal to \( \frac{1}{2} \) for Hecke group of any indices ([9]). However, the proof was not correct (see [10], actually the correct value is less than \( \frac{1}{2} \)) and a lower estimate was given by [17] (where constants depend on the indices of Hecke groups). On the other hand, the Lenstra constant was given by [3] (and [13] for even indices case). Also [13] claimed (without proof) that the Lenstra constant and the Legendre constant are the same for each Hecke group of even index. Here we note that Corollary 4.1. of [3] did not say that the constant is the best possible one (which means it is the Lenstra constant), it is not hard to see that it is the best possible. We refer [7] on this point. Indeed, the Lenstra constant for Rosen continued fraction is

\[
\begin{cases} 
\frac{1}{\ln 2} & , \quad q \text{ even} \\
\frac{1}{\ln 1} & , \quad q \text{ odd}
\end{cases}
\]

In the sequel, we show that the Lenstra constant is equal to the Legendre constant. In the next section, we introduce a generalized Diophantine approximation problem associated to a zonal Fuchsian group and give a law of large numbers for solutions of the Diophantine inequality. This assures the existence of the Lenstra constant under the existence of the Legendre constant and also implies that the Lenstra constant is at least as large than Legendre constant. In §3, we prove the equality of these two constants, mainly showing that the Lenstra constant can not be larger than the Legendre constant. As a corollary (of the proof), we get the explicit value of the entropy of the Rosen map. Finally we note that the same result holds for \( \alpha \)-continued fractions, \( 0 < \alpha \leq \frac{1}{2} \). For \( \frac{1}{2} \leq \alpha \leq 1 \), this result was shown by C. Kraaikamp [6] by a different way. Recently, R. Natsui [15] showed that the existence of the Legendre constant for any \( 0 < \alpha \leq \frac{1}{2} \) and thus we can apply the method of this paper to show the equality of these two constants. We stress the difference between concepts of two constants. The Legendre constant is determined by the property which holds for all \( x \), without exceptional point, on the other hand, the Lenstra constant comes from the metrical property which only holds for almost all points.
2 Generalized Diophantine Approximation

Let $\Gamma$ be a finitely generated Fuchsian group acting on the upper half complex plane $\mathbb{H}^2$, $L$ the set of limit points of $\Gamma$ and $P$ the set of parabolic points. We assume that $\infty \in P$.

An element $g \in \Gamma$ can be viewed as a $2 \times 2$ real matrix

$$
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
$$

of determinant 1. We write $a = a(g)$, $b = b(g)$, $c(g)$ and $d = d(g)$.

J. Lehner [8] proved that there exists a positive number $t$ depending on $\Gamma$ such that

$$
\# \{ g(\infty) : |x - g(\infty)| < \frac{t}{c^2(g)}, g \in \Gamma \} = \infty
$$

for any $x \in L \setminus P$. He also proved that if $\Gamma$ is of the first kind ($L = \mathbb{R}$), then for any sequence $\{ \varepsilon_n \}$ of positive numbers and a.e. $x \in L \setminus P$, there exists a sequence $\{ g_n \}$ in $\Gamma$ such that

$$
|x - g_n(\infty)| < \frac{\varepsilon_n}{c^2(g_n)}
$$

Moreover, Patterson [16] proved a kind of Khintchine theorem when $\Gamma$ is of the first kind: for example, his result implies that

$$
\# \{ g(\infty) : |x - g(\infty)| < \frac{\ln |c(g)|}{c^2(g)}, g \in \Gamma \} = \infty
$$

for a.e. $x \in \mathbb{R} \setminus P$.

We shall estimate the asymptotic number of solutions of

$$
g(\infty) : |x - g(\infty)| < \frac{t}{c^2(g)}, g \in \Gamma
$$

for some positive real number $t$ and a.e. $x \in \mathbb{R} \setminus P$. To do this, we consider a relation among the Diophantine inequality, geodesics of $\mathbb{H}^2$, and geodesics of $\mathbb{H}^2/\Gamma$. We show that the ergodicity of the geodesic flow on $\mathbb{H}^2/\Gamma$ with the hyperbolic measure is closely related to the quantitative theory of the Diophantine approximation on $\Gamma$, (see [18] for the qualitative theory). The relation between the Diophantine inequality and geodesics of $\mathbb{H}^2$ also have been considered by A. Haas [4] and A. Haas and C. Series [5] to discuss the Lagrange spectrum of the approximation on $\Gamma$. The “height” of the $\Gamma$-congruent family of geodesics plays an important role in their discussion.

This idea is also applicable to the theory of Diophantine approximations for complex numbers, where we have to consider $\mathbb{H}^3$ [14].

Since $\infty \in P$, there exists

$$
U_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \in \Gamma, \lambda \in \mathbb{R}_+
$$

such that

$$
\{ U_\lambda^k : k \in \mathbb{Z} \} = \Gamma_\infty
$$
where $\Gamma_\infty$ denotes the subgroup of $\Gamma$ that fixes $\infty$. We define the fundamental region $F$ of $\Gamma$ by

$$F = \{ z = x + iy : -\frac{\lambda}{2} < x < \frac{\lambda}{2}, y > 0 \} \cap \left( \cap_{g \in \Gamma \setminus \Gamma_\infty} \{ z : |c(g) \cdot z + d(g)| > 1 \} \right).$$

$F$ is a hyperbolic polygon and its each side is an arc of the isometric circle of an element $g \in \Gamma$. The image of this side by $g$ is also a side of $F$, which is an arc of the isometric circle of $g^{-1}$. We identify all such pairs and obtain a hyperbolic surface. It is well-known that the hyperbolic metric $ds = \sqrt{dx^2 + dy^2}$ and the hyperbolic measure $d\mu = \frac{dx \, dy}{y^2}$ on $\mathbb{H}^2$ are invariant under $\Gamma$-action over $\mathbb{H}^2$.

**Theorem 1.** Let

$$t_0 = \frac{1}{2} \min_{g \in \Gamma \setminus \Gamma_\infty} |c(g)|,$$

then we have

$$\lim_{N \to \infty} \frac{\sharp\{ g(\infty) : |x - g(\infty)| < \frac{t}{c^2(g)}, |c(g)| \leq N, g \in \Gamma \}}{\ln N} = \frac{4 \lambda \cdot t}{\pi \cdot \mu(F)}$$

for any $t$, $0 < t < t_0$, (a.e. $x \in \mathbb{R} \setminus \mathbb{P}$).

The proof of this theorem is basically the same as that of the main result in [14] for the imaginary quadratic field case with the hyperbolic upper half space. So we only give a sketch of the proof here. We start with some lemmas.

We denote by $\gamma(x, y)$ the geodesic curve with the initial point $x$ and the terminal point $y$ for $(x, \beta) \in (\mathbb{R} \cup \{\infty\})^2 \setminus \{\text{diagonal}\}$. We also denote by

$$F_t(g(\infty)), t > 0,$$

the circle which is tangent to the real line at $\frac{a(g)}{c(g)}$ with radius $\frac{t}{c^2(g)}$ for $g \notin \Gamma_\infty$ and

$$\{x + iy : y = \frac{1}{2t}\} \text{ for } g \in \Gamma_\infty.$$

It is possible to show the following:

**Lemma 1.** If we fix $t > 0$, then

$$h(F_t(g(\infty))) = F_t(hg(\infty))$$

for any $h$ and $g \in \Gamma$.

**Proof.** This follows from the fact that $F_t(g(\infty))$ is an image of $\{ x + iy : y = \frac{1}{2t} \}$, which makes an invariant family of circles under $\Gamma$-action. $\square$

**Lemma 2.** For any $k > 0$,

$$|x - g(\infty)| < \frac{t}{c^2(g)}$$

holds if and only if $\gamma(\infty, x)$ and $\cap F_t(g(\infty))$ do not cross to each other.

If $t < t_0$, then we see that $\{ F_t(g(\infty)) \}$ is a disjoint family of circles, that is,

$$F_t(g(\infty)) \cap F_t(h(\infty)) = \emptyset$$

if $g(\infty) \neq h(\infty)$. Thus we have the following:
Lemma 3. If $0 < t < t_0$, then every point of $F_t(g(\infty)) \setminus (\mathbb{R} \cup \{\infty\})$ is congruent to some point of $F_t(\infty) \cap (F \setminus \{\infty\})$. In particular, if $p \in F_t(g(\infty)) \setminus (\mathbb{R} \cup \{\infty\})$, then there exists $h \in \Gamma$ such that $h(p) = x + iy$, $\frac{\lambda}{2} < x < \frac{\lambda}{2}$ and $y = \frac{1}{2\pi}$.

Proof of the theorem. Let $T(H^2)$ and $T(F)$ be the unit tangent bundles of $H^2$ and $F$, respectively. We consider the geodesic flows $f_s$ and $\tilde{f}_s$ on $T(H^2)$ and $T(F)$, respectively. For $\omega^* \in T(H^2)$, there is a unique geodesic $(x, \beta)$ passing tangentially through $\omega^*$. If $x \neq \infty$ and $\beta \neq \infty$, then we denote by $s$ the (signed) hyperbolic length from the top of the geodesic arc $(x, \beta)$ to $\omega$, which is the base point of $\omega^*$. If $x = \infty$ (or $\beta = \infty$), then we denote by $s$ the hyperbolic length from the point $\beta + i$ or $(x + i)$ to $\omega$, respectively. Thus we can parameterize $\omega^* \in T(H^2)$ by $(x, \beta, s) \in ((\mathbb{R} \cup \{\infty\})^2 \setminus \{\text{diagonal}\}) \times \mathbb{R}$. So if $0 < t < t_0$, we see from Lemmas 2 and 3 that

$$\|\{g(\infty) : |x - g(\infty)| < \frac{t}{c^2(g)}, |c(g)| \leq N, g \in \Gamma\} -$$

$$\#\{s : f_s(\infty, x, -\ln(4t_0 + 1)) \text{ crosses a circle } F_t(g(\infty)) \text{ from outside at time } s, 0 < s \leq \ln(4t_0 + 1) - \ln t + 2 \ln N\} \leq 1$$

and

$$\#\{s : \tilde{f}_s(\omega^*) \text{ crosses } F_t(\infty) \text{ from below at time } s, 0 < s \leq \ln(4t_0 + 1) - \ln t + 2 \ln N\} = \#\{s : \tilde{f}_s(\omega^*) \text{ crosses } F_t(\infty) \text{ from below at time } s, 0 < s \leq \ln(4t_0 + 1) - \ln t + 2 \ln N\}$$

where $\omega^* \in T(F)$ is the point corresponding to $(\infty, x, -\ln(4t_0 + 1)) \in T(H^2)$.

Now we apply the individual ergodic theorem for $(T(F), \tilde{f}, \hat{\mu})$ to our problem. Here, the hyperbolic measure $\hat{\mu}$ on $T(F)$ induced from $\mu$ is defined by

$$\hat{\mu} = \frac{dx \, d\beta \, ds}{(x - \beta)^2}$$

if we parameterize a point in $T(F)$ by $(x, \beta, s)$.

Proposition 1. If we fix $t$, $0 < t < t_0$, then

$$\lim_{u \to \infty} \#\{s : f_s(\omega^*) \text{ crosses } F_t(\infty) \text{ from below, } 0 < s < u\} = \frac{\mu\{x + iy \in F : y > \frac{1}{2\pi}\}}{\pi \cdot \mu(F)}$$

for a.e. $\omega^* \in T(F)$.

Moreover, by using an approximation method, on $t$, we have

Proposition 2. For a.e. $\omega^* \in T(F)$,

$$\lim_{u \to \infty} \#\{s : f_s(\omega^*) \text{ crosses } F_t(\infty) \text{ from below, } 0 < s < u\} = \frac{\mu\{x + iy \in F : y > \frac{1}{2\pi}\}}{\pi \cdot \mu(F)}$$

for any $t$, $0 < t < t_0$. 

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Furthermore, it is possible to show that if \( \omega^* = (x, \beta, s) \in T(F) \) has the above property, then for any \( x' \in \mathbb{R} \cup \{\infty\} \) and \( s' \in \mathbb{R} \), \( \omega^{**} = (x', \beta, s') \) also has the same property. Since the hyperbolic length between \( x + (4t_0 + 1)i \) and \( x + \frac{1}{N}i \) is equal to \( \ln N + \ln(4t_0 + 1) \), we have

\[
\lim_{N \to \infty} \frac{\#\{g(\infty) : |x - g(\infty)| < \frac{t}{\sqrt{|g|}}, |c(g)| \leq N, g \in \Gamma\}}{\ln N} = 2 \cdot \mu\{x + iy \in F : y > \frac{1}{\pi}\}
\]

\[
= \frac{4\lambda \cdot t}{\pi \cdot \mu(F)}
\]

for any \( t, 0 < t < t_0 \) and a.e. \( x \in \mathbb{R} \setminus P \).

**Some remarks.** We apply the theorem to Hecke groups of index \( k \), \( G_k, 3 \leq k \leq \infty \), and its congruent subgroups \( G_k(m) \). Here \( G_k \) is the group generated by

\[
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & \lambda_k \\ 0 & 1 \end{pmatrix}
\]

where \( \lambda_k = 2 \cdot \cos \frac{\pi}{k} \) for \( k \geq 3 \) (and \( = 2 \) when \( k = \infty \)), and \( G_k(m) \) the subgroup of \( G_k \) defined by

\[
G_k(m) = \left\{ g \in G_k : g \equiv \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \mod (m \cdot \lambda_k) \right\}
\]

where \( (m \cdot \lambda_k) \) denotes the ideal generated by \( m \cdot \lambda_k \) with a positive integer \( m \).

A fundamental region \( F_k \) of \( G_k \) is given by

\[
F_k = \{ x + iy : -\cos \pi n < x \leq \cos \pi k, x^2 + y^2 > 1, y > 0 \}.
\]

Thus we see that \( G_k \) is of the first kind and

\[
P = P_k = G_k(\infty) = \{ g(\infty) : g \in G_k \}
\]

if \( k \neq \infty \). In this case, we have

\[
\lim_{N \to \infty} \frac{\#\{g(\infty) : |x - g(\infty)| < \frac{t}{\sqrt{|g|}}, |c(g)| \leq N, g \in G_k\}}{\ln N} = \frac{4 \cdot k \cdot \lambda_k \cdot t}{(k - 2) \cdot \pi^2}
\]

for any \( t, 0 < t < \frac{1}{4} \) and a.e. \( x \in \mathbb{R} \setminus P_k \). We can also apply Theorem 1 to \( G_k(m) \). Then, in this case, the set of parabolic points of \( G_k \) is divided into a finite number of disjoint sets. We put

\[
t_m = \frac{1}{2} \min_{g \in G_k(m) \setminus G_k(m)_{\infty}} |c(g)|.
\]

There exists a constant \( C > 0 \) such that

\[
\lim_{N \to \infty} \frac{\#\{g(\infty) : |x - g(\infty)| < \frac{t}{\sqrt{|g|}}, |c(g)| \leq N, g \in G_k(m)\}}{\ln N} = C \cdot t
\]
for any $t$, $0 < t < t_m$, and a.e. $x \in \mathbb{R}$. Since $G_k(m)$ is a subgroup of $G_k$, for each cusp of the fundamental region of $G_k(m)$ there exists $g_\eta \in G_k$ such that $g_\eta(\eta) = \infty$. It is obvious that $g_\eta \mathcal{F}$ is a fundamental region of $g_\eta G_k(m)g_\eta^{-1}$. Since $G_k(m)$ is normal, $g_\eta \mathcal{F}$ is a fundamental region of $G_k(m)$. This means the “width” of the cusp $\eta$ is the same as that of $\infty$. Thus we have

$$\lim_{N \to \infty} \frac{\sharp \{g(\infty) : |x - g(\infty)| < \frac{t}{c_g(y)}, |c(g)| \leq N, g \in G_k, \exists \tilde{g} \in G_k(m) \text{s.t. } g(\infty) = \tilde{g}(\eta)\}}{\ln N} = C \cdot t$$

for any $t$, $0 < t < t_m$, and a.e. $x \in \mathbb{R}$. Moreover, it turns out that $t_m \to \infty$ as $m \to \infty$. This shows the following:

**Corollary** For a.e. $x \in \mathbb{R}$

$$\lim_{N \to \infty} \frac{\sharp \{g(\infty) : |x - g(\infty)| < \frac{t}{c_g(y)}, |c(g)| \leq N, g \in G_k\}}{\ln N} = \frac{4 \cdot k \cdot \lambda_k \cdot t}{(k - 2) \cdot \pi^2}$$

for any $t > 0$.

**Remark.** The above proof (of this corollary) shows the equidistributed property (a.e.) of solutions associated to cusps. We refer R. Moeckel [11] for the original idea of this method.

### 3 Rosen Continued Fractions

Given any element of $G_k$ of the form

$$\begin{pmatrix} p \\ q \end{pmatrix},$$

we have $g(\infty) = p/q$ and, moreover for any $\tilde{g} \in G_k$ with $\tilde{g}(\infty) = p/q$ we have

$$\tilde{g} = \begin{pmatrix} p \\ q \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -p \\ -q \end{pmatrix}.$$

So, for any parabolic point of $G_k$, $p$ and $q > 0$ are uniquely determined.

We define the $\lambda_k$-nearest continued fraction transformation of $[-\frac{\lambda_k}{2}, \frac{\lambda_k}{2})$ onto itself by

$$S(x) = \begin{cases} \lfloor \frac{\lfloor w \rfloor - \lfloor \frac{\lfloor w \rfloor}{2} \rfloor}{\lambda_k} \rfloor & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

where $[w]_k = b \cdot \lambda_k$; $b \in \mathbb{Z}$, when $w \in [b - \frac{\lambda_k}{2}, b + \frac{\lambda_k}{2})$. We put

$$\varepsilon_n = \varepsilon_n(x) = \text{sgn} S^{n-1}(x)$$

and

$$a_n = a_n(x) = \lfloor \frac{1}{S^{n-1}(x)} \rfloor$$

for any $n \geq 1$ and have a continued fraction expansion:

$$x = \frac{\varepsilon_1}{a_1} + \frac{\varepsilon_2}{a_2} + \frac{\varepsilon_3}{a_3} + \cdots$$
We call this expansion the Rosen continued fraction expansion of $x$. In general, if a continued fraction, either finite or infinite, is given by some $x$ as its Rosen continued fraction expansion, we call it a Rosen continued fraction. We define the principal convergent $\frac{p_n}{q_n}$, $n \geq 0$, by

$$\begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_1 \\ 1 & a_1 \end{pmatrix} \begin{pmatrix} 0 & \varepsilon_2 \\ 1 & a_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & \varepsilon_n \\ 1 & a_n \end{pmatrix}$$

for $n \geq 0$.

It is easy to see that $q_n > 0$ for any $n \geq 0$.

**Lemma 4.** We have

$$\frac{1}{q_n (q_{n+1} + q_n)} \leq \left| x - \frac{p_n}{q_n} \right| \leq \begin{cases} \frac{1}{q_n^2 \left( 1 - \lambda \right)} & \text{if } k \text{ is even;} \\ \frac{1}{q_n^2 \left( R - \lambda \right)} & \text{otherwise,} \end{cases}$$

where $R$ is the positive root of $R^2 + (2 - \lambda)R - 1 = 0$.

**Proof.** We have

$$T^n(x) = \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix}^{-1} (x)$$

and

$$x = \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} (T^n x) = \frac{p_{n-1} T^n x + p_n}{q_{n-1} T^n x + q_n}.$$

Thus we see

$$\left| x - \frac{p_n}{q_n} \right| = \left| \frac{p_{n-1} T^n x + p_n}{q_{n-1} T^n x + q_n} - \frac{p_n}{q_n} \right| = \left| \frac{T^n x}{q_n (q_{n-1} T^n x + q_n)} \right| = \left| \frac{1}{q_n^2 \left( q_{n-1} + \frac{1}{T^n x} \right)} \right| = \left| \frac{1}{q_n^2 \left( q_{n-1} + \varepsilon_{n+1}(x) (r_{n+1} \lambda + T^{n+1} x) \right)} \right| = \frac{1}{q_n^2 \left( q_{n-1} + T^{n+1} x \right)}.$$
Since \( \frac{2n+1}{q_n} > 1 \) if \( k \) is even (and \( \frac{2n+1}{q_n} > \frac{1}{n} \) if \( k \) is odd, respectively), the result follows.

**Lemma 5.** For a.e. \( x \in I \),

\[
\lim_{n \to \infty} \frac{1}{n} \ln q_n
\]

exists and is equal to the half of the entropy \( h_k \) of the Rosen map w.r.t. the absolutely continuous invariant probability measure.

**Proof.** Let \( \mu \) be the absolutely continuous invariant probability measure for \( T \). Since \((T, \mu)\) is ergodic (see [3]), we have from Shannon-McMillan-Breiman's theorem that the entropy \( h_k \) of the Rosen map is given by

\[
h_k = \lim_{n \to \infty} \frac{1}{n} \ln \mu(\Delta[\varepsilon_1 : r_1, \ldots, \varepsilon_n : r_n]) \quad \text{a.e.}
\]

We can replace \( \mu \) to the normalized Lebesgue measure \( m \) because \( \mu \) has a positive density function bounded away from 0 and bounded from above, see [3], that is,

\[
h_k = \lim_{n \to \infty} \frac{1}{n} \ln m(\Delta[\varepsilon_1 : r_1, \ldots, \varepsilon_n : r_n]) \quad \text{a.e.}
\]

From Lemma 4, it turns out that

\[
\lim_{n \to \infty} \frac{1}{n} \ln m(\Delta[\varepsilon_1 : r_1, \ldots, \varepsilon_n : r_n]) = 2 \lim_{n \to \infty} \frac{1}{n} \ln q_n
\]

if the limit of the left hand side exists. Thus we have

\[
\lim_{n \to \infty} \frac{1}{n} \ln q_n = \frac{h_k}{2}
\]

for a.e. \( x \in I \). \( \square \)

Now we denote by \( Lg_k \) the Legendre constant of Rosen continued fractions of index \( k \), that is, the following hold:

1. for \( c \leq Lg_k \) and \( x \in [-\frac{\lambda_k}{2}, \frac{\lambda_k}{2}) \), and \( \left( \frac{p}{q} \cdot \right) \in G_k \), \( q \neq 0 \), if \( |x - \frac{p}{q}| < \frac{c}{q} \) holds, then \( \frac{p}{q} = \frac{p_n}{q_n} \) for some \( n \geq 0 \),

2. for \( c > Lg_k \), there exist \( x \in [-\frac{\lambda_k}{2}, \frac{\lambda_k}{2}) \) and \( \left( \frac{p}{q} \cdot \right) \in G_k \), \( q \neq 0 \), such that \( |x - \frac{p}{q}| < \frac{c}{q} \) and \( \frac{p}{q} \neq \frac{p_n}{q_n} \) for any \( n \geq 0 \).

As mentioned in the introduction, the existence of \( Lg_k \) was shown in [17].

On the other hand, we denote by \( Le_k \) the Lenstra constant of Rosen continued fractions of index \( k \). This means that for almost every \( x \in [-\frac{\lambda_k}{2}, \frac{\lambda_k}{2}) \),

\[
\lim_{N \to \infty} \frac{1}{N} \# \{ n : 1 \leq n \leq N, \Theta_n(x) < t \} = C_k \cdot t \quad \text{for any} \quad 0 < t \leq Le_k,
\]

where \( C_k \) is an absolute constant, which is given in [3], and

\[
\Theta_n = q_n^2 |x - \frac{p_n}{q_n}|.
\]

We will prove the following:
Theorem 2. For any \( k \geq 3 \), we have \( Lg_k = Le_k \).

To prove this theorem, we will show the following two propositions. The assertion of Theorem 2 is a direct consequence of these two.

**Proposition 3.** For any \( k \geq 3 \), we have \( Lg_k \leq Le_k \).

**Proof.** Suppose \( 0 \leq t \leq Lg_k \). From Corollary of §2, we have for a.e. \( x \in [-\frac{\lambda_k}{2}, \frac{\lambda_k}{2}) \)

\[
\lim_{N \to \infty} \frac{\sharp \{ g(\infty) : |x - g(\infty)| < \frac{t}{\pi(c)}, |c(g)| \leq q_n, g \in G_k, \text{ for some } 0 \leq n \leq N \}}{\ln q_N}
\]

\[
= \lim_{N \to \infty} \frac{\sharp \{ g(\infty) : |x - g(\infty)| < \frac{t}{\pi(c)}, |c(g)| \leq q_n, g \in G_k \}}{\ln q_N}
\]

\[
= \frac{4 \cdot k \cdot \lambda_k \cdot t}{(k-2) \cdot \pi^2}
\]

We note the following

\[
\lim_{N \to \infty} \frac{\sharp \{ g(\infty) : |x - g(\infty)| < \frac{t}{\pi(c)}, |c(g)| \leq q_n, g \in G_k, \text{ for some } 0 \leq n \leq N \}}{\ln q_N}
\]

\[
= \lim_{N \to \infty} \frac{\frac{1}{h} \sharp \{ g(\infty) : |x - g(\infty)| < \frac{t}{\pi(c)}, |c(g)| \leq q_n, g \in G_k, \text{ for some } 0 \leq n \leq N \}}{\frac{1}{h} \ln q_N}
\]

From Lemma 5, the denominator of the right hand side converges to \( h_k^2 \) (a.e.), we see that the numerator converges (a.e.) to

\[
\frac{4 \cdot k \cdot \lambda_k \cdot t \cdot h_k}{2(k-2) \cdot \pi^2}
\]

This means

\[
\lim_{N \to \infty} \frac{\frac{1}{N} \sharp \{ n : 1 \leq n \leq N, \Theta < t \}}{\ln q_N} = \frac{4 \cdot k \cdot \lambda_k \cdot t \cdot h_k}{2(k-2) \cdot \pi^2}
\]

for \( 0 \leq t \leq Lg_k \). \( \square \)

**Proposition 4.** For any \( k \geq 3 \), we have \( Lg_k \geq Le_k \).

**Proof.** Suppose that \( t > Lg_k \). Then there exist \( x \in [-\frac{\lambda_k}{2}, \frac{\lambda_k}{2}) \) and \( \left( \frac{p}{q} \right) \in G_k \) such that

\[
\left| x - \frac{p}{q} \right| < \frac{t}{q^2} \quad \text{and} \quad \frac{p}{q} \neq \frac{p_n}{q_n} \text{ for any } n \geq 0.
\]

From this inequality, there exists \( \varepsilon > 0 \) such that

\[
\left| x - \frac{p}{q} \right| < \frac{t}{q^2} - \varepsilon.
\]

If \( y \in [-\frac{\lambda_k}{2}, \frac{\lambda_k}{2}) \) is sufficiently close to \( x \), i.e.

\[
|x - y| < \frac{\varepsilon}{2},
\]

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then

\[ |y - \frac{p}{q}| < \frac{\varepsilon}{q^2} \tag{4} \]

holds. Moreover there exists a positive integer \( M_0 \) such that

\[ (\varepsilon_i(x), a_i(x)) = (\varepsilon_i(y), a_i(y)) \quad \text{for} \quad 1 \leq i \leq M_0 \]

implies that (2) holds.

Now we look at the expansion of \( \frac{p}{q} \) and \( x \). Since \( \frac{p}{q} \) is a parabolic point of \( G_k \), it is easy to see that \( \frac{p}{q} \) has a finite Rosen expansion, say,

\[ \frac{p}{q} = \hat{\varepsilon}_1 + \hat{\varepsilon}_2 + \cdots + \hat{\varepsilon}_n. \]

This means there exists \( m, 1 \leq m \leq n \) such that

\[ (\varepsilon_i, a_i) = (\hat{\varepsilon}_i, \hat{a}_i) \quad \text{for} \quad 1 \leq i \leq m - 1 \quad \text{and} \quad (\varepsilon_m, a_m) \neq (\hat{\varepsilon}_m, \hat{a}_m) \]

with

\[ x = \frac{\varepsilon_1}{a_1} + \frac{\varepsilon_2}{a_2} + \cdots + \frac{\varepsilon_m}{a_m} + \frac{\varepsilon_{m+1}}{a_{m+1}} + \cdots + \frac{\varepsilon_n}{a_n} + \cdots \]

Here we may assume that \( M_0 > n \). We choose a positive integer \( M_1 \) sufficiently large and define

\[ \hat{a}_0 = M_1 \cdot \lambda_k \]

We see that

\[ \frac{\hat{\varepsilon}_0}{\hat{a}_0} + \frac{\hat{\varepsilon}_1}{\hat{a}_1} + \frac{\hat{\varepsilon}_2}{\hat{a}_2} + \cdots + \frac{\hat{\varepsilon}_n}{\hat{a}_n} \]

is a Rosen continued fraction, where \( \hat{\varepsilon}_0 = +1 \). We fix any finite Rosen continued fraction

\[ \frac{\varepsilon'_1}{b_1} + \frac{\varepsilon'_2}{b_2} + \cdots + \frac{\varepsilon'_n}{b_n} \]

so that

\[ \frac{\varepsilon'_1}{b_1} + \frac{\varepsilon'_2}{b_2} + \cdots + \frac{\varepsilon'_n}{b_n} + \frac{\hat{\varepsilon}_0}{\hat{a}_0} + \frac{\hat{\varepsilon}_1}{\hat{a}_1} + \frac{\hat{\varepsilon}_2}{\hat{a}_2} + \cdots + \frac{\hat{\varepsilon}_n}{\hat{a}_n} \]

and

\[ \frac{\varepsilon'_1}{b_1} + \frac{\varepsilon'_2}{b_2} + \cdots + \frac{\varepsilon'_n}{b_n} + \frac{\hat{\varepsilon}_0}{\hat{a}_0} + \frac{\varepsilon_1}{a_1} + \cdots + \frac{\varepsilon_m}{a_m} + \frac{\varepsilon_{m+1}}{a_{m+1}} + \cdots + \frac{\varepsilon_n}{a_n} + \frac{\varepsilon_{M_0}}{a_{M_0}} + \cdots \]

are also Rosen continued fractions. Suppose that \( y_0 \in [-\frac{M_0}{2}, \frac{M_0}{2}] \) with the Rosen expansion of the form

\[ \frac{\varepsilon'_1}{b_1} + \frac{\varepsilon'_2}{b_2} + \cdots + \frac{\varepsilon'_n}{b_n} + \frac{\hat{\varepsilon}_0}{\hat{a}_0} + \frac{\hat{\varepsilon}_1}{\hat{a}_1} + \frac{\hat{\varepsilon}_2}{\hat{a}_2} + \cdots + \frac{\hat{\varepsilon}_n}{\hat{a}_n} + \frac{\varepsilon_{M_0}}{a_{M_0}} + (\text{free}) \]

and

\[ \frac{P}{Q} = \frac{\varepsilon'_1}{b_1} + \frac{\varepsilon'_2}{b_2} + \cdots + \frac{\varepsilon'_n}{b_n} + \frac{\hat{\varepsilon}_0}{\hat{a}_0} + \frac{\hat{\varepsilon}_1}{\hat{a}_1} + \frac{\hat{\varepsilon}_2}{\hat{a}_2} + \cdots + \frac{\hat{\varepsilon}_n}{\hat{a}_n} + \frac{\varepsilon_{M_0}}{a_{M_0}}. \]
We note that \( P \) and \( Q \) are uniquely determined by \( P \in G_k \) and \( Q > 0 \).

We also note that \( y := S^{l+1}(y_0) \) has Rosen continued fraction

\[
\varepsilon_1 \left\lfloor \frac{a_1}{} \right\rfloor + \cdots + \varepsilon_m \left\lfloor \frac{a_m}{} \right\rfloor + \varepsilon_{m+1} \left\lfloor \frac{a_{m+1}}{a_m} \right\rfloor + \cdots + \varepsilon_n \left\lfloor \frac{a_n}{a_{n-1}} \right\rfloor + \cdots + \varepsilon_{M_0} \left\lfloor \frac{a_{M_0}}{a_{M_0-1}} \right\rfloor + (\text{free})
\]

which satisfies (2). We put

\[
\begin{pmatrix} P_{n+l} & P_{n+l+1} \\ Q_{n+l} & Q_{n+l+1} \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon'_1 \\ 1 & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & \varepsilon'_l \\ 1 & b_l \end{pmatrix} \begin{pmatrix} 0 & \varepsilon_0 \\ 1 & \hat{a}_0 \end{pmatrix} \begin{pmatrix} 0 & \varepsilon_1 \\ 1 & \hat{a}_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & \varepsilon_n \\ 1 & \hat{a}_n \end{pmatrix},
\]

which implies \((P,Q) = (P_{n+l+1},Q_{n+l+1})\), and estimate

\[
|y_0 - \frac{P_{n+l+1}}{Q_{n+l+1}}|
\]

We also define

\[
\begin{pmatrix} P_{l-1} & P_l \\ Q_{l-1} & Q_l \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon'_1 \\ 1 & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & \varepsilon'_l \\ 1 & b_l \end{pmatrix}
\]

and

\[
\begin{pmatrix} P_l & P_{l+1} \\ Q_l & Q_{l+1} \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon'_1 \\ 1 & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & \varepsilon'_l \\ 1 & b_l \end{pmatrix} \begin{pmatrix} 0 & \varepsilon_0 \\ 1 & \hat{a}_0 \end{pmatrix}
\]

We denote by \( U \) the linear fractional transformation defined by \( \begin{pmatrix} P_l & P_{l+1} \\ Q_l & Q_{l+1} \end{pmatrix} \).

Then it is easy to see that

\[
U \left( \frac{p}{q} \right) = \frac{P}{Q} \quad \text{and} \quad U(y) = y_0
\]

Thus

\[
|y_0 - \frac{P_{n+l+1}}{Q_{n+l+1}}| = \left| U \left( \frac{p}{q} \right) - U(y) \right|
\]

and the following holds:

\[
\begin{align*}
& \left| U \left( \frac{p}{q} \right) - U(y) \right| \\
= & \left| \frac{P_l y + P_{l+1}}{Q_l y + Q_{l+1}} - \frac{P \frac{p}{q} + P_{l+1}}{Q \frac{p}{q} + Q_{l+1}} \right| \\
= & \left| \frac{Q_{l+1} (Q_l y + Q_{l+1})}{Q_{l+1} (Q_l y + Q_{l+1})} + \frac{\frac{p}{q} - \frac{p}{q}}{Q_{l+1} (Q_l y + Q_{l+1})} \right| \\
\leq & \left| \frac{y + \frac{p}{q}}{Q_{l+1} (Q_l y + Q_{l+1})} - \frac{\frac{p}{q}}{Q_{l+1} (Q_l y + Q_{l+1})} \right| + \left| \frac{\frac{p}{q}}{Q_{l+1} (Q_l y + Q_{l+1})} - \frac{\frac{p}{q}}{Q_{l+1} (Q_l y + Q_{l+1})} \right| \\
\leq & \frac{1}{Q_{l+1} (Q_l y + Q_{l+1})} \left( \frac{t}{q^2} - \frac{\varepsilon}{2} \right) + \frac{1}{Q_{l+1} (Q_l y + Q_{l+1})} \left( \frac{p}{q} \right) - \frac{1}{Q_{l+1} (Q_l y + Q_{l+1})} - \frac{1}{Q_{l+1} (Q_l y + Q_{l+1})}.
\end{align*}
\]
Finally we look at

\[ q_{l+n+1} = p \cdot Q_l + q \cdot Q_{l+1}. \]

Since \(|y| < 1, \left| \frac{p}{q} \right| \) cannot be large, \( Q_{l+1} = \hat{a}_0 Q_l + \hat{\varepsilon}_0 Q_{l-1} \), and \( \frac{Q_{l-1}}{Q_l} \) is bounded (see [3]), we see that

\[
\frac{Q_{l+1} y}{Q_{l+1}} \quad \text{and} \quad \frac{Q_{l+1} z}{Q_{l+1}}
\]

can be arbitrarily small and

\[
\frac{q \cdot Q_{l+1}}{Q_{l+n+1}}
\]

can be sufficiently close to 1 when we choose \( M_1 \) sufficiently large (note that \( \hat{a}_0 = M_1 \lambda_k \)).

In the above discussion, the choice of \( M_1 \) can be independent of \((\varepsilon'_1, b_1), (\varepsilon'_2, b_2), \ldots (\varepsilon'_r, b_r)\).

Thus we get

\[
|y_0 - P| < \frac{t}{Q^2}
\]

It is obvious from the construction that

\[
\frac{P}{Q} \neq \frac{p_u}{q_u}
\]

for any \( u \geq 0 \). Now we pick up a “generic point” \( w \in [-\frac{M_1}{2}, \frac{M_1}{2}] \). Then the ergodicity of \( S \) w.r.t. \( \mu_k \), we have

\[
\lim_{N \to \infty} \frac{1}{N} \mathbb{E} \{|1 \leq u \leq N : ((\varepsilon_n (w), a_n (w)) = (\hat{\varepsilon}_0, \hat{a}_0), (\varepsilon_{u+1} (w), a_{u+1} (w)) = (\varepsilon_1, a_1), \ldots, (\varepsilon_{u+n} (w), a_{u+n} (w)) = (\varepsilon_n, a_n), \ldots, (\varepsilon_{u+M_0} (w), a_{u+M_0} (w)) = (\varepsilon_{M_0}, a_{M_0}) \}
\]

\[
= \mu_k \{|w : (\varepsilon_1 (w), a_1 (w)) = (\hat{\varepsilon}_0, \hat{a}_0), (\varepsilon_2 (w), a_2 (w)) = (\varepsilon_1, a_1), \ldots, (\varepsilon_{M_0+1} (w), a_{M_0+1} (w)) = (\varepsilon_{M_0}, a_{M_0}) \}
\]

\[
> 0.
\]

Finally we look at

\[
\frac{1}{\ln Q} \mathbb{E} \{|1 \leq q \leq Q : |w - \frac{p}{q}| < \frac{t}{q^2} \}
\]

\[
= \frac{1}{\ln Q} \mathbb{E} \{|1 \leq q \leq Q : |w - \frac{p}{q}| < \frac{t}{q^2} \cdot \frac{p}{q} = \frac{p_n}{q_n} \text{ for some } n \geq 1 \}
\]

\[
+ \frac{1}{\ln Q} \mathbb{E} \{|1 \leq q \leq Q : |w - \frac{p}{q}| < \frac{t}{q^2} \cdot \frac{p}{q} = \frac{p_n}{q_n} \text{ for some } n \geq 1 \}.
\]

From the above discussion, the second term has a positive “liminf” and then the first term can not converge to \( \frac{48 M_1^2}{(k-2) \pi^2} \). Since this estimate holds for a.e. \( w \), \( t \) is larger than \( L e_k \).

Consequently we have shown the assertion of Theorem 2. The method of the proof in the above shows the following generalization:
Claim Suppose that $T$ is a map of an interval onto itself that induces continued fraction expansions for real numbers in the domain interval. Moreover we assume

(i) $T$ has an absolutely invariant probability measure.
(ii) There exists a real number $M > 0$ such that for any possible coefficient value $c$ larger than $M$ (or $|c| > M$), one can concatenate any admissible sequence after $c$ as an admissible sequence of continued fractions arising from $T$.
(iii) The Legendre constant of $T$ exists.
(iv) $t_0$ in Theorem 1 is larger than the Legendre constant.
Then the Lenstra constant exists and the Legendre and the Lenstra constants are equal.

From (1) in the proof of Proposition 3 together with Corollary 4.1 of [3], we have the explicit value of the entropy of the Rosen map.

Corollary The entropy of the Rosen map w.r.t. the absolutely continuous invariant probability measure is equal to

$$\frac{C \cdot (k - 2)\pi^2}{2k}$$

with

$$C = \begin{cases} \frac{1}{\ln((1 + \cos \frac{\pi}{k})/\sin \frac{\pi}{k})} & \text{if } k \text{ is even}, \\ \frac{1}{\ln(1 + R)} & \text{if } k \text{ is odd}. \end{cases}$$

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