EXPECTED REGULARIZED TOTAL VARIATION OF BROWNIAN MOTION

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ABSTRACT. We introduce a notion of regularized total variation on an interval for continuous functions with unbounded variation. The definition of regularized total variation is obtained from that of total variation by subtracting a penalty for the size of the partition used to estimate the variation. We present an explicit construction of a partition achieving the regularized total variation, and use this construction to estimate the expected regularized total variation of Brownian motion on an interval.

1. INTRODUCTION

For a continuous function $f : [a, b] \to \mathbb{R}$, it is standard to define the total variation

$$TV_{[a,b]}(f) = \sup_{k \geq 0} TV_{[a,b],k}(f),$$

where

$$TV_{[a,b],k}(f) = \max_{a = t_0 < t_1 < \cdots < t_{k+1} = b} \sum_{i=1}^{k+1} |f(t_i) - f(t_{i-1})|.$$  \hspace{1cm} (1)

Note that $TV_{[a,b],k}(f)$ is an increasing function of $k$, so we actually have

$$TV_{[a,b]}(f) = \lim_{k \to \infty} TV_{[a,b],k}(f).$$

If $TV(f) < \infty$, then one says that $f$ is of bounded variation, which is to say that one cannot make $TV_{[a,b],k}(f)$ arbitrarily large by increasing $k$. Conversely, if $f$ has unbounded variation, then $TV_{[a,b],k}(f)$ grows arbitrarily large as $k \to \infty$. To get a finer measurement of the oscillations of a function of unbounded variation, then, we can penalize the growth of $k$ in the optimization equation (1). This leads us to define, for each $\lambda > 0$, the regularized total variation

$$\Phi_{[a,b],\lambda}(f) = \sup_{k \geq 0} (TV_{[a,b],k}(f) - \lambda k) = \sup_{k \geq 0} \left( \max_{a = t_0 < t_1 < \cdots < t_{k+1} = b} \sum_{i=1}^{k+1} |f(t_i) - f(t_{i-1})| - \lambda k \right).$$  \hspace{1cm} (2)

We use the term “regularized” because, if we think of the optimization equation (1) as attempting to find the piecewise linear approximation to $f$ that captures as much of the oscillation of $f$ as possible (although the supremum may of course not be achieved), then the penalty term in the regularized equation (2) attempts to prevent “overfitting” $f$ by using too many piecewise-linear segments.

Brownian motion provides a standard example of a function with (almost surely) unbounded variation on a finite interval. The goal of the present paper is to evaluate $\mathbb{E}\Phi_{[0,1],\lambda}(W)$, where $\{W_t\}_{0 \leq t \leq 1}$ is a standard real-valued Brownian motion. We will prove the following.

**Theorem 1.** For each $\lambda > 0$, we have

$$0 \leq \mathbb{E}\Phi_{[0,1],\lambda}(W) - \frac{1}{\lambda} \leq \lambda,$$

and thus in particular $\mathbb{E}\Phi_{[0,1],\lambda}(W) = 1/\lambda + O(\lambda)$ as $\lambda \to 0$.

The proof will proceed in three steps. In Section 2, we give an explicit characterization of the optimal partition in (2). In Section 3, we use this characterization along with martingale methods and Brownian scaling to evaluate the asymptotic behavior of $\mathbb{E}\Phi_{[0,1],\lambda}(W)$ as $\lambda \to 0$. Finally, in Section 4, we use the Markov property of Brownian motion to establish the error bound stated in Theorem 1.

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We note in passing that the methods we use here, especially the characterization of the optimal partition, are quite specific to our particular notion of regularized total variation. In particular, one might wish to consider the quantity
\[
\sup_{k \geq 0} \max_{a=t_0 < t_1 < \cdots < t_{k+1} = b} \left( \sum_{i=1}^{k+1} |f(t_i) - f(t_{i-1})|^p - \lambda k \right)
\]
for some power \( p \), but our method of characterization of the optimal partition fails for this quantity. One essential difficulty is that the question of whether an interval \([t_{j-1}, t_j]\) of a partition should be “split” at \( s, s' \in [t_{j-1}, t_j] \) to improve the objective function (2) depends on the positions of \( f(s) \) and \( f(s') \) in the interval \( f([t_{j-1}, t_j]) \), not just on their difference \( f(s') - f(s) \), as is the case when \( p = 1 \) according to Proposition 6(2) below.

2. Characterizing the optimal partition

We begin by giving a characterization of a partition of the interval for which the outer supremum in (1) is achieved. (We establish the existence of such a partition in Corollary 5.) We begin by introducing some notation and terminology.

Definition 2. For a partition \( P = [a = t_0 < t_1 < \cdots < t_k < t_{k+1} = b] \) of \([a, b]\), define \( |P| = k \) and, for \( \lambda > 0 \), let \( \Phi_{I, \lambda, P}(f) = \sum_{i=1}^{k+1} |f(t_i) - f(t_{i-1})| - \lambda |P| \). (Thus \( \Phi_{I, \lambda}(f) = \max_P \Phi_{I, \lambda, P}(f) \).)

Definition 3. Let \( f \in C^0[a, b] \). Suppose \([x, y] \subset [a, b]\). We say that \([x, y] \) is an uptick (resp. downtick) for \( f \) if \( f(t_i) > f(t_{i-1}) \) (resp. \( f(t_i) < f(t_{i-1}) \)), and a \( \lambda \)-uptick (resp. \( \lambda \)-downtick) for \( f \) if \( f(t_i) \geq f(t_{i-1}) + \lambda \) (resp. \( f(t_i) \leq f(t_{i-1}) - \lambda \)).

Our first proposition is based on observations to the effect that a partition with certain properties cannot be optimal since moving or removing one or more of its points would increase \( \Phi_{I, \lambda, P}(f) \). In particular, this will allow us to derive the existence of an optimal partition as a corollary.

Proposition 4. Let \( \lambda > 0 \) and \( f \in C^0[a, b] \). Suppose that \( P = [a = t_0 < t_1 < \cdots < t_k < t_{k+1} = b] \) is a partition of \( I = [a, b] \) such that \( \Phi_{I, \lambda, P}(f) \geq \Phi_{I, \lambda, Q}(f) \) whenever \( |Q| \leq |P| \). Then

1. If \( 1 \leq j \leq k \), then \( f(t_j) - f(t_{j-1}) \) and \( f(t_j) - f(t_{j+1}) \) have the same nonzero sign. In other words, exactly one of \([t_{j-1}, t_j]\) and \([t_j, t_{j+1}]\) is an uptick and the other is a downtick.
2. (a) If \( a < t_{j-1} \) and \([t_{j-1}, t_j]\) is an uptick (resp. downtick), then \( f \) attains its minimum (resp. maximum) on \([t_{j-1}, t_j]\) at \( t_{j-1} \).
3. (b) If \( t_j < b \) and \([t_{j-1}, t_j]\) is an uptick (resp. downtick), then \( f \) attains its maximum (resp. minimum) on \([t_{j-1}, t_j]\) at \( t_j \).
4. For \( 1 \leq j \leq k \), if \([t_{j-1}, t_j]\) is an uptick (resp. downtick), then \( f \) attains its maximum (resp. minimum) on \([t_{j-1}, t_{j+1}]\) at \( t_j \).
5. If \( k \geq 1 \), then for each \( 1 \leq j \leq k+1 \), we have \( |f(t_j) - f(t_{j-1})| \geq \lambda/2 \).

Proof. We prove each part in turn.

1. Let \( Q = [a = t_0 < \cdots < t_{j-1} < t_{j+1} < \cdots < t_{k+1} = b] \), so \( |Q| = k - 1 \) and thus we have
\[
0 \leq \Phi_{I, \lambda, P}(f) - \Phi_{I, \lambda, Q}(f) = |f(t_{j+1}) - f(t_{j-1})| - |f(t_{j+1}) - f(t_j)| - |f(t_{j+1}) - f(t_{j-1})| - \lambda
\]
so \( |f(t_{j+1}) - f(t_j)| + |f(t_{j+1}) - f(t_{j-1})| \leq |f(t_{j+1}) - f(t_{j-1})| \). But if \( f(t_{j+1}) - f(t_j) \) and \( f(t_j) - f(t_{j-1}) \) were both nonpositive or nonnegative then equality would hold.

2. We prove 2a in the case when \([t_{j-1}, t_j]\) is an uptick. Fix \( x \in (t_{j-1}, t_j) \) and let \( Q = [a = t_0 < \cdots < t_{j-2} < x < t_j < \cdots < t_{k+1} = b] \), so \( |Q| = k \) and thus we have (using part 1)
\[
0 \leq \Phi_{I, \lambda, P}(f) - \Phi_{I, \lambda, Q}(f) = |f(t_j) - f(t_{j-1}) + f(t_{j-2}) - f(t_{j-1}) - |f(t_j) - f(x)| + |f(x) - f(t_{j-2})| |
\]
so \( f(x) \geq f(t_{j-1}) \).
(3) This is an immediate consequence of the first two statements.

(4) We prove the case when \([t_{j-1}, t_j]\) is an uptick. Let \(Q = [a = t_0 < \cdots < t_{j-2} < t_{j+1} < \cdots < t_{k+1} = b]\), so \(|Q| = k - 2\) and thus we have

\[
0 \leq \Phi_{I,\lambda,P}(f) - \Phi_{I,\lambda,Q}(f) = f(t_1) - f(t_0) + f(t_2) - f(t_1) + f(t_3) - f(t_2) - |f(t_3) - f(t_2)| - 2\lambda
\]

So, \(f(t_2) - f(t_1)| \leq 2\lambda\), hence \(f(t_2) - f(t_1) \leq 2\lambda\).

(5) Given part 4, it is sufficient to prove the cases \(j = 1\) and \(j = k + 1\). We will prove the case \(j = 1\); the case \(j = k + 1\) is the same. Suppose wlog that \(f(t_1) > f(t_0), f(t_2)\). Let \(Q = [a = t_0 < t_2 < \cdots < t_{k+1} = b]\), so \(|Q| = k - 1\) and

\[
0 \leq \Phi_{I,\lambda,P}(f) - \Phi_{I,\lambda,Q}(f) = f(t_1) - f(t_0) + f(t_2) - f(t_1) - |f(t_2) - f(t_1)| - \lambda
\]

\[
= 2f(t_1) - 2\max\{f(t_0), f(t_2)\} - \lambda
\]

So, \(f(t_1) - f(t_0) \geq \lambda/2\).

**Corollary 5.** Let \(\lambda > 0\) and \(f \in C^0[a,b]\). Then there is a \(K \geq 0\) such that if \(|P| > K\), then there is a partition \(Q\) with \(|Q| \leq |P|\) and \(\Phi_{I,\lambda,Q}(f) > \Phi_{I,\lambda,P}(f)\). In particular, there is a partition \(P\) (with \(|P| \leq K\)) so that \(\Phi_{I,\lambda,P}(f) = \Phi_{I,\lambda}(f)\).

**Proof.** Since \(f\) is uniformly continuous on \([a,b]\), there is a \(\delta > 0\) so that if \(|x - y| < \delta\), then \(|f(x) - f(y)| < \lambda/2\).

Let \(K = (b - a)/\delta\). If \(P = [a = t_0 < t_1 < \cdots < t_{k+1} = b]\) satisfies \(|P| > K\), then by the pigeonhole principle there is a \(j\) so that \(t_j - t_{j-1} < \delta\) and hence \(|f(t_j) - f(t_{j-1})| < \lambda/2\), so by statement 5 of Proposition 4 there is a \(Q\) with \(|Q| \leq |P|\) and \(\Phi_{I,\lambda,Q}(f) > \Phi_{I,\lambda,P}(f)\).

Let \(\tilde{P}_k\) be the space of partitions of size \(k\) with possibly-coincident points, which is to say partitions of the form \([a = t_0 \leq t_1 \leq \cdots \leq t_k \leq t_{k+1} = b]\). If we equip \(\tilde{P}_k\) with the usual topology, then \(\tilde{P}_k\) is compact for each \(k\) and \(\Phi_{I,\lambda,P}(f)\) depends continuously on \(P\). Therefore, the maximum of \(\Phi_{I,\lambda,P}(f)\) on \(\bigcup_{k=1}^r \tilde{P}_k\) is achieved whenever \(r < \infty\). But by the previous paragraph, the maximum cannot be achieved at a \(\tilde{P}\) with \(|P| > K\), and since \(\lambda > 0\), the maximum cannot be achieved at a partition with coincident points. Therefore, the maximum of \(\Phi_{I,\lambda,P}(f)\) over all partitions \(P\) (without coincident points) is achieved.

Now that we know that an optimal partition exists, we impose further conditions on such a partition, in addition to the ones we already have according to Proposition 4.

**Proposition 6.** Let \(\lambda > 0\) and \(f \in C^0[a,b]\). Suppose that \(P = [a = t_0 < t_1 < \cdots < t_k < t_{k+1} = b]\) is a partition of \(I = [a,b]\) such that \(\Phi_{I,\lambda,P}(f) = \Phi_{I,\lambda}(f)\), and moreover that \(|P|\) is maximal among all such \(P\). Then we have

1. (a) If \([a, t_1]\) is an uptick (resp. downtick) then \(\min_{a \leq x \leq t_1} f(x) > f(a) - \lambda/2\) (resp. \(\max_{a \leq x \leq t_1} f(x) < f(a) + \lambda/2\)).
2. (b) If \([t_k, b]\) is an uptick (resp. downtick) then \(\max_{t_k \leq x \leq b} f(x) < f(b) + \lambda/2\) (resp. \(\min_{t_k \leq x \leq b} f(x) > f(b) - \lambda/2\)).
3. (2) If \([t_{j-1}, t_j]\) is an uptick (resp. downtick), then \([t_{j-1}, t_j]\) contains no \(\lambda\)-downtick (resp. \(\lambda\)-uptick).

**Proof.** We prove each part in turn.

1. We prove 1a in the case when \([t_0, t_1]\) is an uptick. Fix \(x \in (t_0, t_1)\) and let \(Q = [a = t_0 < x < t_1 < \cdots < t_{k+1} = b]\), so \(|Q| = k + 1\) and, by the maximality of \(\Phi_{I,\lambda,P}(f)\) and \(|P|\), we have

\[
0 < \Phi_{I,\lambda,P}(f) - \Phi_{I,\lambda,Q}(f) = f(t_1) - f(a) - |[f(t_1) - f(x)] + |f(x) - f(a)|| + \lambda
\]

\[
= f(t_1) - f(a) - |f(t_1) - f(x) + f(a) - f(x)| + \lambda
\]

\[
= 2[f(x) - f(a)] + \lambda,
\]
so \( f(x) > f(a) - \lambda/2 \).

(2) We prove the case when \([t_{j-1}, t_j]\) is an uptick. Let \([x, y] \subset [t_{j-1}, t_j]\) and let \(Q = [a = t_0 < \cdots < t_{j-1} < x < y < t_j < \cdots < t_{k+1} = b]\), so \(|Q| = k + 2\) and, by the maximality of \(\Phi_{I,\lambda,P}(f)\) and \(|P|\), we have

\[
0 < \Phi_{I,\lambda,P}(f) - \Phi_{I,\lambda,Q}(f) = f(t_j) - f(t_{j-1}) - [f(t_j) - f(y)] + [f(y) - f(x)] + [f(x) - f(t_{j-1})] + 2\lambda
\leq f(t_j) - f(t_{j-1}) - [f(t_j) - f(y) + f(x) - f(y) + f(x) - f(t_{j-1})] + 2\lambda
= 2[f(y) - f(x) + \lambda],
\]
so \( f(y) > f(x) - \lambda \). This implies that \([t_{j-1}, t_j]\) cannot contain a \(\lambda\)-downtick. \(\Box\)

Having established all of the needed necessary conditions for a partition to be optimal, we now set out to write down a construction of an optimal partition. When \(f\) is taken to be a stochastic process, our construction will be in terms of stopping times for the natural filtration, which will allow us to use martingale methods in our later analysis.

**Definition 7.** Fix \(\lambda > 0\) and let \([a, b]\) be an interval. Let \(f \in C^0[a,b]\) and extend \(f\) to \([a, \infty)\) arbitrarily. We define a sequence of values \(\tau_0 < \tau_1 < \cdots \in [a, \infty)\) as follows. Let

\[
\tau_0^{\text{up}} = \min \left\{ t > a \mid f(t) \geq f(a) + \lambda/2 \right\} = \min \left\{ t > a \mid [a, t] \text{ is a } (\lambda/2)\text{-uptick} \right\},
\]
and

\[
\tau_0^{\text{down}} = \min \left\{ t > a \mid f(t) \leq f(a) - \lambda/2 \right\} = \min \left\{ t > a \mid [a, t] \text{ is a } (\lambda/2)\text{-downtick} \right\},
\]
Let \(\tau_0 = \tau_0^{\text{up}} \land \tau_0^{\text{down}}\). If \(f(\tau_0 \land b) > f(a)\), then we call \(\tau_0\) an upstop, while if \(f(\tau_0 \land b) < f(a)\), then we call \(\tau_0\) a downstop. Now we inductively define \(\tau_j\) for all \(j \geq 1\): if \(\tau_{j-1}\) is an upstop then define

\[
\tau_j = \min \left\{ t > \tau_{j-1} \mid [\tau_{j-1}, t] \text{ contains a } \lambda\text{-downtick} \right\} = \min \left\{ t > \tau_{j-1} \left| \max_{\tau_{j-1} \leq s \leq t} f(s) - f(t) \geq \lambda \right. \right\},
\]
and call \(\tau_j\) a downstop, while if \(\tau_{j-1}\) is a downstop then define

\[
\tau_j = \min \left\{ t > \tau_{j-1} \mid [\tau_{j-1}, t] \text{ contains a } \lambda\text{-uptick} \right\} = \min \left\{ t > \tau_{j-1} \left| f(t) - \min_{\tau_{j-1} \leq s \leq t} f(s) \geq \lambda \right. \right\},
\]
and call \(\tau_j\) an upstop. (We adopt the usual convention that \(\min \emptyset = \infty\).)

**Definition 8.** With setup as in Definition 7, let \(m_0 = f(a)\) and, for all \(j \geq 1\) such that \(\tau_{j-1} < b\), define

\[
m_j = m_j = \begin{cases} 
\max_{\tau_{j-1} \leq s \leq \tau_j \land b} f(s) & \text{if } \tau_j \text{ is a downstop, or} \\
\min_{\tau_{j-1} \leq s \leq \tau_j \land b} f(s) & \text{if } \tau_j \text{ is an upstop.}
\end{cases}
\]

See Figure 1 for a graphical interpretation of Definitions 7 and 8.

**Proposition 9.** With the setup as in Definition 8, if \(j \geq 1\) and \(\tau_j < b\), then \(m_j = f(\tau_j) + \lambda\) if \(\tau_j\) is a downstop and \(m_j = f(\tau_j) - \lambda\) if \(\tau_j\) is a downstop.

**Proof.** We prove the case in which \(\tau_j\) is a downstop. In this case we have

\[
\tau_j = \min \left\{ t > \tau_{j-1} \left| \max_{\tau_{j-1} \leq s \leq t} f(s) - f(t) \geq \lambda \right. \right\},
\]
so since \(f\) is continuous and \(\tau_j < b\), we see that \(m_j - f(\tau_j) = \max_{\tau_{j-1} \leq s \leq \tau_j} f(s) - f(\tau_j) = \lambda\). \(\Box\)

**Proposition 10.** With setup as in Definition 8, suppose that \(P = [a = t_0 < t_1 < \cdots < t_k < t_{k+1} = b]\) is a partition of \(I = [a, b]\) such that \(\Phi_{I,\lambda,P}(f) = \Phi_{I,\lambda}(f)\), and moreover that \(|P|\) is maximal over all such \(P\). For each \(j \geq 1\) such that \(\tau_{j-1} < b\), we have the following trichotomy.

1. If \(\tau_j < b\), then \(k \geq j\) and \(t_j \in [\tau_{j-1}, \tau_j]\), and \([\tau_{j-1}, t_j]\) is an uptick or downtick as \(\tau_{j-1}\) is an upstop or downstop, respectively.
2. If \(\tau_j \geq b\) and \(|m_j - f(b)| \geq \lambda/2\), then \(k = j\) and \(t_j \in [\tau_{j-1}, b]\).
3. If \(\tau_j \geq b\) and \(|m_j - f(b)| < \lambda/2\), then \(k = j - 1\).
Moreover, whenever $1 \leq j \leq k$, we have $f(t_j) = m_j$.

**Proof.** We prove case 1 by induction on $j$.

We first prove the base case, $j = 1$. Assume wlog that $\tau_0$ is an upstop. Proposition 4(5) implies that $t_1 \geq \tau_0$. Since we assume $\tau_0 < b$, we have $f(\tau_0) = f(a) + \lambda/2$. Proposition 6(1a) implies that $[t_0, t_1]$ cannot be a downtick, so it is an uptick. Proposition 6(2) then implies that $[t_0, t_1]$ cannot contain a $\lambda$-downtick, but (since $\tau_1 < b < \infty$) $[\tau_0, \tau_1]$ contains a $\lambda$-downtick, so $[t_0, t_1] \not⊇ [\tau_0, \tau_1]$, so $k \geq 1$ and $t_1 < \tau_1$. This proves the base case.

We now consider the inductive step, so assume that $t_{j-2} \leq \tau_{j-2} \leq t_{j-1} \leq \tau_{j-1} < \tau_j < b$ and that $[t_{j-2}, t_{j-1}]$ is an uptick if and only if $\tau_{j-2}$ is an upstop. Assume wlog that $\tau_{j-1}$ is a downstop, so $\tau_{j-2}$ is an upstop and $[t_{j-2}, t_{j-1}]$ is an uptick. Proposition 4(1) implies that $[t_{j-1}, t_j]$ is a downtick. Since $[\tau_{j-1}, \tau_j]$ contains a $\lambda$-uptick, Proposition 6(2) implies that $[t_{j-1}, t_j] \not⊇ [\tau_{j-1}, \tau_j]$, so $k \geq j$ and $t_j < \tau_j < b$ since $t_{j-1} \leq \tau_{j-1}$. Proposition 4(4) then implies that $[t_{j-1}, t_j]$ contains a $\lambda$-downtick, so $t_j \geq \tau_{j-1}$ since $t_{j-1} \geq \tau_{j-2}$ and $t_{j-1}$ is the first time $t$ such that $[t_{j-2}, t]$ contains a $\lambda$-downtick. This completes the proof of the inductive step for case 1.

We now consider cases 2 and 3, so assume that $\tau_{j-1} < b \leq \tau_j$. By case 1, we see that $k \geq j - 1$ and $t_{j-1} \leq \tau_{j-1}$. Assume wlog that $\tau_{j-1}$ is a downstop. In the same way as above, Proposition 4(4, 5) implies that $t_j \geq \tau_{j-1}$ if $t_j < b$, and if $t_j = b$ then of course $t_j \geq \tau_{j-1}$ as well. Since $\tau_j \geq b$ is the first upstop after $\tau_{j-1}$, we observe that $[\tau_{j-1}, b]$ does not contain a $\lambda$-uptick, so $k \leq j$ since if $k \geq j + 1$, then $[t_j, t_{j+1}]$ would have to be a $\lambda$-uptick by Proposition 4(4).

Under our wlog assumption, we have $m_j = \min_{\tau_{j-1} \leq s \leq b} f(s)$. If $m_j \leq f(b) - \lambda/2$ and $k = j - 1$, then $[t_{k-1}, t_k = b]$ is a downtick and we obtain a contradiction of Proposition 6(1b). Thus, in case 2, we have $k = j$, completing the proof for case 2. On the other hand, if $m_j > f(b) - \lambda/2$ and $k = j$, then $[t_j, t_{j+1} = b]$ is an uptick and $f(t_j) > f(b) - \lambda/2$, contradicting Proposition 4(5). This completes the proof for case 3.
Finally, we show that \( f(t_j) = m_j \) whenever \( 1 \leq j \leq k \). Assume wlog that \([t_{j-1}, t_j]\) is an uptick. We have shown that \( t_{j-1} \leq \tau_{j-1} \leq t_j \leq \tau_j \land b \leq t_{j+1} \), so \( f(t_j) \leq \max_{\tau_j \leq s \leq t_{j+1}} f(s) = m_j \), while Proposition 4(3) tells us that \( f(t_j) = \max_{t_{j-1} \leq s \leq t_{j+1}} f(s) \geq \max_{\tau_j \leq s \leq t_{j+1}} f(s) = m_j \). Therefore, \( f(t_j) = m_j \).

3. Estimating \( \Phi_{I, \lambda}(W) \) for small \( \lambda \)

Proposition 10 lets us algorithmically construct a partition \( P \) maximizing \( \Phi_{I, \lambda, P}(f) \) for any \( f \). More importantly for our purposes, however, it lets us estimate \( \Phi_{I, \lambda, P}(f) \) in terms of the \( \tau_j \)'s and \( f(\tau_j) \)'s, as the next proposition shows.

**Proposition 11.** Let \( \lambda > 0 \) and let \( I = [a, b] \) be an interval. Let \( f \in C^0[\cdot, b] \), define \( \tau_j \) as above, and let \( k' = 0 \lor \max \{ j \mid \tau_j < b \} \). Then there exists an \( \alpha \in \{0, 1\} \), depending only on the values of \( f \) on \([a, \tau_0 \land b] \), so that

\[
\Phi_{I, \lambda}(f) = \lambda k' + |f(\tau_0 \land b) - f(a)| + \sum_{j \geq 1} (-1)^{j+\alpha} (f(\tau_j \land b) - f(\tau_{j-1} \land b)) + \max \{0, 2|m_{k'+1} - f(b)| - \lambda\}.
\]

**Proof.** Choose a partition \( P = [a = t_0 < t_1 < \cdots < t_{k+1} = b] \) of \([a, b] \) so that \( \Phi_{I, \lambda, P}(f) = \Phi_{I, \lambda}(f) \) and \( |P| \) is maximal among all \( P \) with this property. Let \( 1_\geq = 1_{|m_{k'+1} - f(b)| \geq \lambda/2} \) and \( 1_< = 1_{|m_{k'+1} - f(b)| < \lambda/2} \). Note that \( k = k' + 1_\geq \). Applying Proposition 10, we can write

\[
\Phi_{I, \lambda}(f) = \Phi_{I, \lambda, P}(f) = \sum_{j=1}^{k+1} |f(t_j) - f(t_{j-1})| - \lambda k = \sum_{j \geq 1} |f(t_j) - f(t_{j-1})| 1_{j \leq k+1} - \lambda k
\]

\[
= |m_1 - f(a)| 1_{k' \geq 1} + \sum_{j=2}^{k'} |m_j - m_{j-1}| + (|m_{k'+1} - m_k| + |f(b) - m_{k'+1}|) 1_{\geq} + |f(b) - m_k| 1_< - \lambda(k' + 1_\geq).
\]

Let \( \alpha = 0 \) if \([t_0, t_1]\) is a downtick and \( \alpha = 1 \) if \([t_0, t_1]\) is an uptick, so \( j + \alpha \) is even if \([t_{j-1}, t_j]\) is an uptick and odd if \([t_{j-1}, t_j]\) is a downtick. Note that this definition of \( \alpha \) only depends on the values of \( f \) on \([a, \tau_0 \land b] \). For \( 0 \leq j \leq k' \), we have that \( j + \alpha \) is even if \( \tau_j \) is a downtick and odd if \( \tau_j \) is an uptick, so, by Proposition 9, we can write \( m_j = f(\tau_j) + (-1)^{j+\alpha} \lambda \) whenever \( 1 \leq j \leq k' \).

Now we can simplify each of the pieces of (3) in turn. We have

\[
|m_1 - f(a)| 1_{\tau_1 \land b} = (-1)^{\alpha+1}(m_1 - f(a)) 1_{k' \geq 1} = (-1)^{\alpha+1}(f(\tau_1) + (-1)^{\alpha+1} \lambda - f(a)) 1_{k' \geq 1} = (-1)^{\alpha+1}(f(\tau_1) - f(a)) + \lambda 1_{k' \geq 1}.
\]

We also have

\[
|f(b) - m_k| = (-1)^{k'+\alpha+1}(f(b) - m_k)
\]

\[
= (-1)^{k'+\alpha+1}(f(b) - f(\tau_k) + (1)^{k'+\alpha} \lambda 1_{k' \geq 1} - f(a) 1_{k' = 0})
\]

\[
= (-1)^{k'+\alpha+1}(f(b) - f(\tau_k)) 1_{k' \geq 1} + (f(b) - f(a)) 1_{k' = 0} + \lambda 1_{k' \geq 1}.
\]

Moreover, it is not hard to see that either \( m_k \leq f(b) \leq m_{k'+1} \) or \( m_k \geq f(b) \geq m_{k'+1} \). (If \( k' = 0 \) then this is true by the definitions. If \( k' > 0 \) and, for example, if \( \tau_{k'} \) is an uptick, then

\[
m_{k'+1} = \max_{\tau_{k'} \leq s \leq \tau_{k'+1}} f(s) = \max_{\tau_{k'} \leq s \leq \tau_{k'+1}} f(s) \geq f(b), \quad \text{and if } f(b) < m_{k'}, \text{ then } f(b) < m_{k'} \leq f(\tau_{k'}) - \lambda, \quad \text{contradicting the definition of } k'.
\]

This implies

\[
|m_{k'+1} - m_k| + |f(b) - m_{k'+1}| = 2|m_{k'+1} - f(b)| + |f(b) - m_k|.
\]

Therefore,

\[
(|m_{k'+1} - m_k| + |f(b) - m_{k'+1}|) 1_{\geq} + |f(b) - m_k| 1_< = (2|m_{k'+1} - f(b)| + |f(b) - m_k|) 1_{\geq} + |f(b) - m_k| 1_<
\]

\[
= 2|m_{k'+1} - f(b)| 1_{\geq} + (-1)^{k'+\alpha+1}(f(b) - f(\tau_k)) 1_{k' \geq 1} + (f(b) - f(a)) 1_{k' = 0} + \lambda 1_{k' \geq 1}.
\]
Finally, we can write
\[
\sum_{j=2}^{k'} |m_j - m_{j-1}| = \sum_{j=2}^{k'} (-1)^{j+\alpha}(m_j - m_{j-1}) = \sum_{j=2}^{k'} (-1)^{j+\alpha}(f(\tau_j) + (-1)^{j+\alpha} \lambda - (f(\tau_{j-1}) - (-1)^{j+\alpha} \lambda)) = 2\lambda(k' - 1 + \mathbf{1}_{k'=0}) + \sum_{j=2}^{k'} (-1)^{j+\alpha}(f(\tau_j) - f(\tau_{j-1}))\mathbf{1}_{k'\geq j}. \quad (6)
\]

Substituting (4), (5), and (6) into (3), we obtain
\[
\Phi_{I, \lambda}(f) = \left[\lambda + (-1)^{\alpha+1}(f(\tau_1) - f(a))\right] \mathbf{1}_{k'\geq 1} + 2\lambda(k' - 1 + \mathbf{1}_{k'=0}) + \sum_{j=2}^{k'} (-1)^{j+\alpha}(f(\tau_j) - f(\tau_{j-1}))\mathbf{1}_{k'\geq j}
\]
\[
+ 2|m_{k'+1} - f(b)|\mathbf{1}_2 + (-1)^{k'+\alpha+1}[(f(b) - f(\tau_{k'}))\mathbf{1}_{k'\geq 1} + (f(b) - f(a))\mathbf{1}_{k'=0}] + \lambda\mathbf{1}_{k'\geq 1} - \lambda(k' + \mathbf{1}_2)
\]
\[
= \lambda k' + (-1)^{\alpha+1}(f(\tau_1) - f(a))\mathbf{1}_{k'\geq 1} + \sum_{j=2}^{k'} (-1)^{j+\alpha}(f(\tau_j) - f(\tau_{j-1}))\mathbf{1}_{k'\geq j}
\]
\[
+ (2|m_{k'+1} - f(b)| - \lambda)\mathbf{1}_2 + (-1)^{k'+\alpha+1}[(f(b) - f(\tau_{k'}))\mathbf{1}_{k'\geq 1} + (f(b) - f(a))\mathbf{1}_{k'=0}]
\]
\[
= \lambda k' + (-1)^{\alpha+1}(f(\tau_1) - f(a))\mathbf{1}_{k'\geq 1} + (-1)^{k'+\alpha+1}(f(b) - f(\tau_{k'}))\mathbf{1}_{k'\geq 1} + (-1)^{\alpha+1}(f(b) - f(a))\mathbf{1}_{k'=0}
\]
\[
+ \sum_{j=2}^{k'} (-1)^{j+\alpha}(f(\tau_j) - f(\tau_{j-1}))\mathbf{1}_{\tau_j < \epsilon} + \max\{0, 2|m_{k'+1} - f(b)| - \lambda\}
\]
\[
= \lambda k' + (-1)^{\alpha+1}(f(\tau_1 \wedge b) - f(a)) + \sum_{j=2}^{k'} (-1)^{j+\alpha}(f(\tau_j \wedge b) - f(\tau_{j-1} \wedge b)) + \max\{0, 2|m_{k'+1} - f(b)| - \lambda\}
\]
\[
= \lambda k' + (-1)^{\alpha+1}(f(\tau_1 \wedge b) - f(a)) + \sum_{j=2}^{k'} (-1)^{j+\alpha}(f(\tau_j \wedge b) - f(\tau_{j-1} \wedge b)) + \max\{0, 2|m_{k'+1} - f(b)| - \lambda\}
\]
\[
= \lambda k' + \frac{1}{2}\sum_{j=2}^{k'} (-1)^{j+\alpha}(f(\tau_j \wedge b) - f(\tau_{j-1} \wedge b)) + \max\{0, 2|m_{k'+1} - f(b)| - \lambda\},
\]
as claimed.

Proposition 12. Let \(\lambda > 0\) and \(I = [a, b]\) be an interval. Let \(\{W_t\}_{t \geq a}\) be a standard Brownian motion defined on the time interval \([a, b]\). Let \(\{\mathcal{F}_t\}_{t \geq a}\) be the natural filtration of \(\{W_t\}\). Then \(\tau_0 < \tau_1 < \cdots\), defined as in Definition 7 for \(W\), are almost-surely finite stopping times with respect to the filtration \(\mathcal{F}_t\). Moreover, \(\{\tau_j - \tau_{j-1}\}_{j \geq 1}\) is an iid collection of random variables and \(\mathbf{E}(\tau_j - \tau_{j-1}) = \lambda^2\) for each \(j \geq 1\). Finally, \(\mathbf{E}\tau_0 = (\lambda/2)^2\).

Proof. The fact that each \(\tau_j\) is a stopping time is clear from the definition. That \(\{\tau_j - \tau_{j-1}\}_{j \geq 1}\) are iid follows immediately from the definition, given the strong Markov property of Brownian motion and the fact that the negative of a standard Brownian motion is another standard Brownian motion.

For \(j \geq 1\), to prove that \(\tau_j - \tau_{j-1}\) is almost-surely finite and to compute \(\mathbf{E}(\tau_j - \tau_{j-1})\), we note that, by the strong Markov property and the fact that a negative of a Brownian motion is another Brownian motion, \(\tau_j - \tau_{j-1}\) has the same distribution as the stopping time \(\rho = \min\{t > a \mid Y_t \geq \lambda\}\), where \(Y_t = \max_{a \leq s \leq t} W_s - W_t\).

Using the reflection principle, it can be shown that the process \(\{Y_t\}_{t \geq a}\) has the same finite-dimensional distributions as the process \(\{W_t\}_{t \geq a}\) [1, Problem 2.8.8 or Theorem 3.6.17]. Let \(\bar{\rho} = \min\{t > a \mid |W_t| \geq \lambda\}\). Since \(\{Y_t\}_{t \geq a}\) and \(\{W_t\}_{t \geq a}\) have continuous sample paths, \(\rho\) and \(\bar{\rho}\) are both measurable with respect to the \(\sigma\)-algebras generated by finite projections of \(\{Y_t\}_{t \geq a}\) and \(\{W_t\}_{t \geq a}\), respectively, and so the distributions of \(\rho\) and \(\bar{\rho}\) are the same since the finite-dimensional distributions of \(\{Y_t\}_{t \geq a}\) and \(\{W_t\}_{t \geq a}\) are the same. The fact that \(\bar{\rho}\) is almost-surely finite is the standard fact that Brownian motion is almost-surely unbounded, and the computation \(\mathbf{E}\bar{\rho} = \lambda^2\) is a standard application of the optional sampling theorem on the martingale \(W^2_t - t\). Therefore, \(\rho\) is almost-surely finite and \(\mathbf{E}\rho = \lambda^2\). We obtain \(\mathbf{E}\tau_0 = (\lambda/2)^2\) in the same way.
Lemma 13. With setup as in Proposition 12, the sequence

\[ \left\{ \sum_{j=1}^{N} (-1)^{j+\alpha}(W_{\tau_j \wedge b} - W_{\tau_{j-1}\wedge b}) \right\}_{N \geq 0} \]

is bounded in \( L^2 \) by \( b \) (and hence is uniformly integrable).

Proof. For each \( j \geq 1 \), we know that \( \tau_{j-1} \wedge b \) is a stopping time, so by the strong Markov property of Brownian motion, \( \{ (W_{t+\tau_{j-1}\wedge b} - W_{\tau_{j-1}\wedge b})^2 - t \}_{t \geq 0} \) is a martingale with respect to the filtration \( \{ \mathcal{F}_{t+\tau_{j-1}\wedge b} \}_{t \geq 0} \). Since \( \tau_j \wedge b - \tau_{j-1} \wedge b \) is a bounded stopping time with respect to this filtration, we have that
\[ \mathbb{E}(W_{\tau_j \wedge b} - W_{\tau_{j-1}\wedge b})^2 = \mathbb{E}(\tau_j \wedge b - \tau_{j-1} \wedge b) \] by the optional stopping theorem. Moreover, if \( j \neq j' \), then \( W_{\tau_j \wedge b} - W_{\tau_{j-1}\wedge b} \) and \( W_{\tau_{j'} \wedge b} - W_{\tau_{j'-1}\wedge b} \) have mean zero, and are independent by the strong Markov property. So

\[
\mathbb{E} \left[ \sum_{j=1}^{N} (-1)^{j+\alpha}(W_{\tau_j \wedge b} - W_{\tau_{j-1}\wedge b}) \right]^2 = \\
= \sum_{j=1}^{N} \mathbb{E}(W_{\tau_j \wedge b} - W_{\tau_{j-1}\wedge b})^2 + \sum_{1 \leq j \neq j' \leq N} (-1)^{j+j'} \mathbb{E} \left[ (W_{\tau_j \wedge b} - W_{\tau_{j-1}\wedge b})(W_{\tau_{j'} \wedge b} - W_{\tau_{j'-1}\wedge b}) \right] = \\
= \sum_{j=1}^{N} \mathbb{E}(\tau_j \wedge b - \tau_{j-1} \wedge b) = \mathbb{E}(\tau_N \wedge b - \tau_0 \wedge b) \leq b. \quad \square
\]

Lemma 14. With setup as in Proposition 12, we have \( \mathbb{E} \sum_{j \geq 1} (-1)^{j+\alpha}(W_{\tau_j \wedge b} - W_{\tau_{j-1}\wedge b}) = 0 \).

Proof. Note that \( \alpha \in \mathcal{F}_{\tau_0 \wedge b} \) by Proposition 11. Moreover, by the strong Markov property, \( W_{\tau_j \wedge b} - W_{\tau_{j-1}\wedge b} \) is independent of \( \mathcal{F}_{\tau_0 \wedge b} \) for each \( j \geq 1 \). So for each \( j \geq 1 \) we have
\[
\mathbb{E} (-1)^{j+\alpha}(W_{\tau_j \wedge b} - W_{\tau_{j-1}\wedge b}) = \mathbb{E} (-1)^{j+\alpha} \mathbb{E} [W_{\tau_j \wedge b} - W_{\tau_{j-1}\wedge b}] = 0.
\]

Combined with Lemma 13, this implies that \( \mathbb{E} \sum_{j \geq 1} (-1)^{j+\alpha}(W_{\tau_j \wedge b} - W_{\tau_{j-1}\wedge b}) = 0 \), as claimed. \( \square \)

Proposition 15. If \( \{W_t\}_{t \geq 0} \) is a standard Brownian motion, then
\[
\mathbb{E} \Phi_{[0,b],\lambda}(W) \sim \frac{b}{\lambda}
\]
as \( b \to \infty \).

Proof. Let \( I = [0,b] \), and set notation as in Proposition 11 with \( f(t) = W_t \). Proposition 11 then tells us that
\[
\mathbb{E} \Phi_{I,\lambda}(W) = \lambda \mathbb{E} k'(b) + \mathbb{E} |W_{\tau_0 \wedge b}| + \sum_{j \geq 1} (-1)^{j+\alpha}(W_{\tau_j \wedge b} - W_{\tau_{j-1}\wedge b}) + \mathbb{E} \max \{0, 2|m_{k'(b)+1} - W_b| - \lambda \},
\]
where \( k'(b) = 0 \vee \max \{j \mid \tau_j < b \} \). Elementary renewal theory and Proposition 12 then let us write
\[
\lim_{b \to \infty} \frac{\mathbb{E} k'(b)}{b} = \frac{1}{\mathbb{E}(\tau_j - \tau_{j-1})} = \frac{1}{\lambda^2}.
\]

Moreover, Lemma 14 says that \( \mathbb{E} \sum_{j \geq 1} (-1)^{j+\alpha}(W_{\tau_j \wedge b} - W_{\tau_{j-1}\wedge b}) = 0 \). We note that
\[
0 \leq \mathbb{E} |W_{\tau_0 \wedge b}| \leq \lambda/2,
\]
and
\[
|\mathbb{E} \max \{0, 2|m_{k'(b)+1} - W_b| - \lambda \}| \leq \mathbb{E} |\max \{0, 2|m_{k'(b)+1} - W_b| - \lambda \}| < \lambda
\]
by the definition of \( k' \). Therefore, we have
\[
\lim_{b \to \infty} \frac{1}{b} \mathbb{E} \Phi_{I,\lambda}(W) = \frac{1}{\lambda^2} = \frac{1}{\lambda},
\]
as claimed. \( \square \)
Proposition 16. Let \( b \geq 0 \) and \( \mu > 0 \), and let \( \{W_t\}_{t \geq 0} \) be a standard Brownian motion. Then \( \Phi_{[0, \mu b], \lambda}(W) \) has the same law as \( \sqrt{\mu} \Phi_{[0, b], \lambda/\sqrt{\mu}}(W) \). In particular, \( \mathbb{E} \Phi_{[0, \mu b], \lambda}(W) = \sqrt{\mu} \mathbb{E} \Phi_{[0, b], \lambda/\sqrt{\mu}}(W) \).

Proof. This follows from Brownian scaling by the simple computation

\[
\Phi_{[0, \mu b], \lambda}(W) = \max_{k \geq 0} \max_{0 = t_0 < \cdots < t_{k+1} = \mu b} \left( \sum_{i=1}^{k+1} |W_{t_i} - W_{t_{i-1}}| - \lambda k \right)
\]

\[
= \sqrt{\mu} \cdot \max_{k \geq 0} \max_{0 = s_0 < \cdots < s_{k+1} = b} \left( \sum_{i=1}^{k+1} \frac{1}{\sqrt{\mu}} |W_{s_i} - W_{s_{i-1}}| - \frac{\lambda}{\sqrt{\mu}} k \right)
\]

where \( \tilde{W}_t = W_{\mu t}/\sqrt{\mu} \). But by Brownian scaling, \( \tilde{W}_t \) is another standard Brownian motion, so \( \Phi_{[0, \mu b], \lambda}(W) \) has the same law as \( \sqrt{\mu} \Phi_{[0, b], \lambda/\sqrt{\mu}}(W) \).

Corollary 17. If \( \{W_t\}_{t \geq 0} \) is a standard Brownian motion, then \( \mathbb{E} \Phi_{[0, 1], \lambda}(W) \sim 1/\lambda \) as \( \lambda \downarrow 0 \).

Proof. An easy computation from the previous two propositions:

\[
\lim_{\lambda \downarrow 0} \lambda \mathbb{E} \Phi_{[0, 1], \lambda}(W) = \lim_{\lambda \downarrow 0} \lambda^2 \mathbb{E} \Phi_{[0, 1/\lambda^2], 1}(W) = \lim_{b \to \infty} \frac{1}{b} \mathbb{E} \Phi_{[0, b], \lambda}(W) = 1.
\]

4. Error analysis for fixed \( \lambda \)

Now that we have established the asymptotic behavior (as \( \lambda \to 0 \)) of \( \mathbb{E} \Phi_{[0, 1], \lambda}(W) \), we turn to estimating the error term. We use a simple argument based on subdividing intervals and applying Brownian scaling.

Proposition 18. Let \( f \in C^0[a, b] \). For any \( a < b < c \), we have \( \Phi_{[a, b], \lambda}(f) + \Phi_{[b, c], \lambda}(f) - \lambda \leq \Phi_{[a, c], \lambda}(f) \leq \Phi_{[a, b], \lambda}(f) + \Phi_{[b, c], \lambda}(f) \).

Proof. We compute

\[
\Phi_{[a, c], \lambda}(f) = \max_{k \geq 0} \max_{a = t_0 < \cdots < t_{k+1} = c} \left( \sum_{i=1}^{k+1} |f(t_i) - f(t_{i-1})| - \lambda k \right)
\]

\[
\geq \max_{k \geq 1} \max_{1 \leq \ell \leq k} \max_{a = t_0 < \cdots < t_{\ell} = b} \max_{b = t_{\ell+1} < \cdots < t_{k+1} = c} \left( \sum_{i=1}^{k+1} |f(t_i) - f(t_{i-1})| - \lambda k \right)
\]

\[
= \max_{k \geq 1} \max_{1 \leq \ell \leq k} \max_{a = t_0 < \cdots < t_{\ell} = b} \max_{b = t_{\ell+1} < \cdots < t_{k+1} = c} \left( \sum_{i=1}^{\ell} |f(t_i) - f(t_{i-1})| - \lambda(\ell - 1) \right)
\]

\[
+ \max_{m \geq 0} \max_{b = t_{\ell+1} < \cdots < t_{m+1} = c} \left( \sum_{i=0}^{m} |f(t_i) - f(t_{i-1})| - \lambda m \right) - \lambda
\]

Moreover,

\[
\Phi_{[a, c], \lambda}(f) = \max_{k \geq 0} \max_{a = t_0 < \cdots < t_{k+1} = c} \left( \sum_{i=1}^{k+1} |f(t_i) - f(t_{i-1})| - \lambda k \right)
\]

\[
= \max_{k \geq 0} \max_{a = t_0 < \cdots < t_{k+1} = c} \left( \sum_{i=1}^{\ell} |f(t_i) - f(t_{i-1})| + |f(b) - f(t_{\ell+1})| - \lambda k \right)
\]

\[
+ \max_{m \geq 0} \max_{b = t_{\ell+1} < \cdots < t_{m+1} = c} \left( \sum_{i=0}^{m} |f(t_i) - f(t_{i-1})| - \lambda m \right) - \lambda
\]
\[ \leq \max_{k \geq 0} \max_{1 \leq i < j < k} \left( \sum_{t=i+1}^{j} |f(t) - f(t_{i-1})| + |f(b) - f(t)| - \lambda \ell \right) \]

\[ + \left| f(t_{\ell+1}) - f(b) \right| + \sum_{i=\ell+2}^{k+1} \left| f(t_{i}) - f(t_{i-1}) \right| - \lambda(k - \ell) \right) \]

\[ = \Phi_{[a,\lambda]}(f) + \Phi_{[b,\lambda]}(f). \]

In summary, we have \( \Phi_{[a,\lambda]}(f) + \Phi_{[b,\lambda]}(f) - \lambda \leq \Phi_{[a,\lambda]}(f) \leq \Phi_{[a,\lambda]}(f) + \Phi_{[b,\lambda]}(f) \). \( \square \)

**Proposition 19.** If \( \{W_t\}_{t \geq 0} \) is a standard Brownian motion, then for any \( L \in \mathbb{N} \) we have

\[ \mathbb{E} \Phi_{[0,\lambda]}(W) = \sqrt{L} \mathbb{E} \Phi_{[0,\lambda]}(W - \epsilon_{\lambda,L}), \]

with \( 0 \leq \epsilon_{\lambda,L} \leq \lambda \).

**Proof.** By inductively applying Proposition 18 and the Markov property of Brownian motion, for any \( L \in \mathbb{N} \) we get the inequality

\[ L \mathbb{E} \Phi_{[0,\lambda]}(W) - \lambda \leq L \mathbb{E} \Phi_{[0,\lambda+1]}(W) - (L - 1) \lambda \leq \mathbb{E} \Phi_{[0,\lambda+1]}(W), \]

so we have

\[ \mathbb{E} \Phi_{[0,\lambda+1]}(W) = L \mathbb{E} \Phi_{[0,\lambda]}(W) - \epsilon_{\lambda,L}, \]

for some \( 0 \leq \epsilon_{\lambda,L} \leq \lambda \). By Proposition 16, we have \( \mathbb{E} \Phi_{[0,\lambda+1]}(W) = \sqrt{L} \mathbb{E} \Phi_{[0,\lambda]}(W - \epsilon_{\lambda,L}) \), so the result follows. \( \square \)

Our final proposition implies Theorem 1.

**Proposition 20.** With notation as in Proposition 19, we have

\[ \mathbb{E} \Phi_{[0,1]}(W) = \frac{1}{\lambda} + \alpha_{\lambda}, \]

where \( \alpha_{\lambda} = \lim_{r \to \infty} \epsilon_{\lambda,2^r} \) (and thus \( 0 \leq \alpha_{\lambda} \leq \lambda \)).

**Proof.** For ease of notation, we now put \( \xi(\lambda) = \mathbb{E} \Phi_{[0,1]}(f), \) where \( f(t) = W_t \), a standard Brownian motion. Note that \( \xi \) is decreasing. With this notation, and putting \( b = 1 \), the previous Proposition tells us that

\[ \xi(\lambda - \sqrt{L}) = \sqrt{L} \xi(\lambda) - \sqrt{L} \epsilon_{\lambda,L}. \]

For typographical convenience, put \( \zeta = \sqrt{2} \). Applying (7) three times, with \( L = 2^{r+1} \), \( L = 2 \), and \( L = 2^r \), we obtain

\[ \zeta^{r+1} \xi(\lambda) - \zeta^{r+1} \epsilon_{\lambda,2^r} = \xi(\lambda - \zeta) - \zeta \epsilon_{\lambda,\zeta,2^r} = \xi(\lambda) - \zeta \epsilon_{\lambda,\zeta,2^r}, \]

so

\[ \zeta^r \epsilon_{\lambda,2^r} = \xi(\lambda) - \zeta^{r+1} \epsilon_{\lambda,2^r} - \zeta \epsilon_{\lambda,\zeta,2^r}, \]

so in particular

\[ \epsilon_{\lambda,2^r} \geq \zeta \epsilon_{\lambda,\zeta,2^r}, \]

since \( \epsilon_{\lambda,\zeta,2^r} \geq 0 \). Thus for fixed \( \lambda \), the sequence \( \{ \epsilon_{\lambda,2^r} \} \), is nondecreasing and bounded above by \( \lambda \). Thus \( \alpha_{\lambda} = \lim_{r \to \infty} \epsilon_{\lambda,2^r} \) exists and lies in \([0,\lambda]\). From (7) again, we have

\[ \xi(\lambda) = \epsilon_{\lambda,2^r} + \lambda \xi(\lambda - \zeta), \]

so we can conclude, using Corollary 17, that

\[ 1 = \lim_{r \to \infty} \frac{\lambda}{\zeta^r} \xi(\lambda - \zeta) = \lambda \xi(\lambda) - \lambda \alpha_{\lambda}, \]

or \( \xi(\lambda) = 1/\lambda + \alpha_{\lambda} \), as claimed. \( \square \)
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