Valuation of asset and volatility derivatives using
decoupled time-changed Lévy processes*

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Abstract
In this paper we propose a general derivative pricing framework which employs decoupled
time-changed (DTC) Lévy processes to model the underlying asset of contingent claims. A
DTC Lévy process is a generalized time-changed Lévy process whose continuous and pure
jump parts are allowed to follow separate random time scalings; we devise the martingale
structure for a DTC Lévy-driven asset and revisit many popular models which fall under this
framework. Postulating different time changes for the underlying Lévy decomposition allows
the introduction of asset price models consistent with the assumption of a correlated pair
of continuous and jump market activity rates; we study one illustrative DTC model of this kind
basing on the the so-called Wishart process. The theory developed is applied to the problem
of pricing claims depending not only on the price or the volatility of an underlying asset,
but also to more sophisticated derivatives that pay-off on the joint performance of these
two financial variables, like the target volatility option (TVO). We solve the pricing problem
through a Fourier-inversion method; numerical computations validating our technique are
provided.

Keywords: Derivative pricing; time changes; Lévy processes; joint asset and volatility derivatives;
target volatility option; Wishart process

MSC: 91G20, 60G46

1 Introduction and motivation

The use of Lévy models in finance dates back to 1976 when Merton [35] proposed that the log-
price dynamics of a stock return should follow an exponential Brownian diffusion punctuated
by a Poisson arrival process of normally distributed jumps. For the first time then two of the
main shortcomings of the Black-Scholes model, continuity of the sample paths and normality of
returns, were addressed. Over the years, Lévy processes have proved to be a flexible and yet
mathematically tractable instrument for asset price modeling and sampling. One of the easiest
way of producing a Lévy process is using the principle of subordination of a Brownian motion
\( W_t \). If \( T_t \) is an increasing Lévy process independent of \( W_t \), then the subordinated process \( W_{T_t} \)
will still be of Lévy type. Subordination is the simplest example of a time change, that is, the
operation of considering the time evolution of a stochastic process as occurring at a random
time.

Return models depending on time-changed Brownian motions have been conjectured since
Clark [10]; further theoretical support to the financial use of time-changed models is given by
Monroe’s theorem, [36], asserting that any semimartingale can be viewed as a time change of
a Brownian motion. Consequently, any semimartingale representing the log-price process of an
asset can be considered as a re-scaled Wiener process. Empirical studies (Ané and Geman,
[11]) confirmed that normality of returns can be recovered in a new price density based on the
quantity and arrival times of orders, which justifies the interpretation of \( T_t \) as “business time”

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or “stochastic clock”; the instantaneous variation of \( T_t \) is hence the “activity rate” at which the market reacts to the arrival of information. Further advances were made by Carr and Wu, \([9]\) who demonstrated that much more general time changes are potential candidates for asset price modeling, and effectively re-interpreted a large amount of models from the literature as time changes.

However, not all the possibilities in time change modeling have been exhausted by the current research. For example, the stochastic volatility model with jumps (SVJ) treated by e.g. Bates \([2]\), and the stochastic volatility model with jumps and stochastic jump rate (SVJSJ) studied by Fang \([18]\), although retaining a time re-scaled structure, are not time-changed Lévy as understood in \([9]\). Indeed, in these two classes of models the jump component does not follow the same time scaling as the continuous Brownian part: in the SVJ model the discontinuities have stationary increments, whereas in the SVJSJ model the jump rate is allowed to follow a stochastic process of its own. In other words, price models for which the “stochastic clock” runs at different paces for the “small” and “big” market movements have already been proposed and tested. The statistical analyses of \([2, 18]\) confirm that these models are capable of an excellent data fitting, with particular regard to the SVJSJ model. As pointed out by Fang \([18]\), there are various other reasons for conjecturing a stochastic jump rate. If activity rates are to be interpreted as the frequencies of arrival of new market information, it seems unlikely that such a rate could be taken as a constant, as this would implicate a constant information flow. Moreover, a constant jump rate implies stationary jump risk premia, which also seems unreasonable. Another stylized fact potentially captured by a model with a stochastic jump rate is the slow convergence of returns to the normal distribution, which is not featured by a stationary jump model. Despite all these considerations, the idea of a stochastic jump rate has never really caught on.

On the other hand, if we want to exogenously model the market activity, the hypothesis of independence between the jump and the continuous instantaneous rates laid out in \([18]\) seems to be overly simplistic, as in reality the related information flows may very well influence each other. For example, a market crash or soaring certainly impacts the day-to-day volume of trading in the days following the event. Conversely, a sustained high activity trend over a long period, typically associated to the prices falling, may eventually lead to a panic-driven sudden plunge in the shares’ value. These and similar scenarios provide heuristic arguments for the assumption of a correlated pair of activity rates; nevertheless, to the best of this author’s knowledge, asset price models capturing this feature are not present in the literature yet.

Motivated by these arguments, the natural question arising is whether it is possible to manufacture consistent general time-changed price processes in which the continuous and discontinuous parts of the underlying Lévy model follow two different, possibly correlated, stochastic time changes. We shall show that the answer is affirmative. The family of stochastic processes we investigate is that obtained by time-modifying the continuous and jump parts of a given Lévy process \( X_t \) by two, in principle dependent, stochastic time scalings \( T_t \) and \( U_t \) satisfying a certain regularity condition (definition \( 3.1 \)). We call such processes decoupled time changes. In a formula:

\[
X_{T,U} := X_{T_t}^c + X_{U_t}^d,
\]

(1.1)

where \( X_t^c \) and \( X_t^d \) represent respectively the Brownian and jump components of \( X_t \). The DTC approach suggested allows to embed in a unitary mathematical framework many previously-known models or classes of models, so that the DTC theory offers a natural generalization of some of the standing asset modeling research. In addition, the assumption of a pair of dependent activity rates can be captured by making use of decoupled time changes. To our knowledge, this last feature is new to the asset modeling literature. In section 7 we shall illustrate a practical example of a model having this property by considering an explicit asset evolution based on a multivariate version of the square-root process known as the Wishart process (e.g. \([5, 23, 12, 13, 21]\)), which we use to model the instantaneous activity rates.

From the perspective of the valuation of financial derivatives, the aim of this work is to
gain some understanding of the impact on derivative pricing of the interactions between the
volatility and the price of the underlying. To give an example, a recent market innovation is
that of derivatives and investment strategies based on volatility-modified versions of plain vanilla
products. Such contracts are able to replicate classic European payoffs under a perfect volatility
foresight; at the same time, the component of the price due to a vega excess may be reduced by
using the realized volatility as a normalizing factor. One example of such a product is the target
volatility option. A target volatility (call) option (TVO) pays at maturity \( t \) the value:

\[
F(S_t, RV_t) = \frac{\sigma}{\sqrt{RV_t}} (S_t - K)^+,
\]

(1.2)

for a strike price \( K \) and a target volatility level \( \sigma \), a constant written in the contract. Intuitively,
the closer the realized volatility \( RV_t \) will be to \( \sigma \), the more this claim will behave like a call
option; however, the presence of \( RV_t \) at the denominator decreases the sensitivity of \( F \) to a
change in volatility. It can be shown (Di Graziano and Torricelli, [14]) that the price of an at-
the-money TVO is approximately that of an at-the-money Black-Scholes call of implied volatility
\( \sigma \), which thus represents the subjective volatility view of an investor, as opposed to the current
implied volatilities from the market.

In view of such a growing interaction between volatility and stock in the financial assets
available in the market, being able to efficiently price derivatives like the TVO and other similar
products is gaining relevance. The pricing problem of hybrid volatility/asset derivatives, with
special emphasis on the target volatility option, has been already addressed by Di Graziano
and the author in [14] for a 0-correlation stochastic volatility model, and by the author in [38],
for a general stochastic volatility model. However, to our knowledge, a comprehensive pricing
framework comparable to those available for plain vanilla derivatives (e.g [6, 33, 34, 9]) has not
been developed yet: this is one limitation we intend to overcome with this paper. The pricing
technique we use is a well-known approach yielding a semi-closed analytical formula for the
derivative price through an inverse Fourier integral. It should be apparent that in all the models
accounted, there is no particular reason not to consider mixed price and volatility payoffs as
the default input of pricing models e.g. for numerical implementation, as the introduction of
the realized volatility variable does not break down the Fourier-inversion technique. Clearly,
pricing both vanilla and pure volatility derivatives is still possible within this framework, since
the corresponding payoff types can be regarded as particular cases of our more general setting.

The remaining of the matter is organized as follows. In section 2 we lay out the assumptions
that we shall need; in section 3 martingale properties for a decoupled time-changed Lévy model
will be derived. Section 4 shows the fundamental relation linking the characteristic function of
the log-price and its quadratic variation and the joint Laplace transform of the time changes as
computed in an appropriate measure. Section 5 is dedicated to the derivation of a pricing formula
for products paying off on \( S_t \) and \( TV_t \). We devote section 6 to the interpretation of a number
of known models as decoupled time changes, and compute the joint characteristic function of
section 4 for each such model. In section 7 we introduce an exemplifying model of decoupled
time-changed type featuring correlation between the time changes/activity rates. Finally, in
section 8 we implement our formulae to valuate different asset and volatility derivatives under
various market conditions and asset price models, and briefly summarize the work done. The
more technical proofs have been placed in the appendix.

2 Assumptions and notation

As customary, our market is represented by a filtered probability space \( (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}) \) satisfying
the usual conditions. Throughout the paper we will assume that there exists a money market
account process paying a constant interest rate \( r \).
Let $S_t$ be a non-dividend-paying market asset. $\tilde{S}_t$ will denote its time-0 discounted value $e^{-rt}S_t$. The total realized variance on $[0, t]$ of $S_t$ is by definition the quadratic variation of the natural logarithm of $S_t$, that is:

$$TV_t := \langle \log S \rangle_t = \lim_{\pi \to 0} \sum_{t_i \in \pi} |\log S_{t_{i+1}} - \log S_{t_i}|^2.$$  \hspace{1cm} (2.1)

The limit runs over the supremum norm of all the possible partitions $\pi$ of $[t_0, t]$. The total realized volatility is $\sqrt{TV_t}$. The period realized variance and volatility (or realized variance/volatility tout court) are given respectively by $RV_t = TV_t/t$ and $\sqrt{RV_t}$. If $X_t = \log S_t$ is a semimartingale, by taking the limit in (2.1) it is an easy check that:

$$\langle X \rangle_t = X_t^2 - 2 \int_0^t X_u \, dX_u.$$  \hspace{1cm} (2.2)

The algebra of the square matrices of order $n$ with real entries is indicated $M_n(\mathbb{R})$ and that of symmetric real matrices $\text{Sim}_n(\mathbb{R})$. Matrix product is denoted by juxtaposition; the scalar product between vectors is either indicated by multiplying on the left with the transposed vector $\cdot^T$ or by the usual dot notation. The symbol $\text{Tr}$ stands for the trace operator.

If $J$ is an absolutely continuous random variable, we denote by $f_J(x)$ its probability density function and by $\phi_J(z)$ its characteristic function:

$$\phi_J(z) := \mathbb{E}[e^{iz^T J}].$$  \hspace{1cm} (2.3)

For a Fourier-integrable function $f : \mathbb{C}^n \to \mathbb{C}$ its Fourier transform will be denoted $\hat{f}$. For a complex-valued function or a complex plane subset, $\cdot^*$ indicates the complex conjugate function or set.

When we say that a process is a martingale we imply a martingale with respect to its natural filtration. The notation for the conditional expectation of a stochastic process $X_t$ at time $t_0 < t$ with respect to $\mathcal{F}_{t_0}$ is $\mathbb{E}_{t_0}[\cdot]$. When the distribution of a process $X_t$ depends on other state variables $x_t$ (as in the case of a Markov process) those are implicitly understood to be given at time $t_0$ by $x_{t_0}$. If $X_t$ is a process admitting conditional laws, the space of the integrable functions in the $t_0$-conditional distribution of $X_t$ at time $t_0 < t$ is indicated $L^1_{t_0}(X_t)$. The notation for the bilateral Laplace transform of the distribution of $X_t$ conditional at $t_0 < t$ is:

$$\mathcal{L}_X(z) = \mathbb{E}_{t_0}[e^{-z^T X_t}]$$  \hspace{1cm} (2.4)

where for brevity we drop the dependence on $t$ and $t_0$ in the left hand side. The process of the left limits of $X_t$ is indicated $X_{t-}$. The symbol $\Delta X_t$ stands for the difference $X_t - X_{t-}$ or $X_t - X_{t_0}$ for some prior time $t_0 < t$. Equalities are always understood to hold modulo almost-sure equivalence.

If $X_t$ is an $n$-dimensional Lévy process, the characteristic exponent of $X_t$ is the complex-valued function $\psi_X : \mathbb{C}^n \to \mathbb{C}$ such that:

$$\mathbb{E}[e^{i\theta^T X_t}] = e^{t\psi_X(i\theta)}$$  \hspace{1cm} (2.5)

where $\theta$ ranges in the subset of $\mathbb{C}^n$ where the left hand side is finite.

For a given choice of a truncation function $\epsilon(x)$ (that is, a bounded function which is $O(|x|)$ around 0) the characteristic exponent has the unique Lévy-Khintchine representation:

$$\psi_X(\theta) = i\mu^T \theta - \frac{\theta^T \Sigma \theta}{2} + \int_{\mathbb{R}^n} (e^{i\theta^T x} - 1 - i\theta^T \epsilon(x)) \nu(dx),$$  \hspace{1cm} (2.6)

where $\mu \in \mathbb{R}^n$, $\Sigma$ is a non-negative definite $n \times n$ matrix with real entries, and $\nu(dx)$ a Radon measure on $\mathbb{R}^n$ having a density function that is integrable at $+\infty$ and $O(|x|^2)$ around 0. We
shall make the standard choice \( \epsilon(x) = x \mathbb{1}_{|x| \leq 1} \) and drop the dependence of \( \mu \) on \( \epsilon \). The triplet \((\mu, \Sigma, \nu)\) is then called the characteristic triplet or the Lévy characteristics of \( X_t \).

A stochastic time change \( T_t \) is an \( \mathcal{F}_t \)-adapted càdlàg process, increasing and almost-surely finite, such that \( T_t \) is an \( \mathcal{F}_t \)-adapted stopping time for each \( t \). A time change of an \( n \)-dimensional Lévy process \( X_t \) according to \( T_t \) is the \( \mathcal{F}_{T_t} \)-adapted process \( Y_t := X_{T_t} \).

3 Definition, martingale relations and asset price dynamics

In this first section we introduce the notion of decoupled time-changed Lévy process and devise an exponential martingale structure naturally associated to it. This construct serves a twofold purpose. In first place it allows to formulate a DTC-based asset price evolution whose discounted value enjoys the martingale property. According to general theory, this in turn enables to postulate the existence of a risk-neutral measure that correctly prices the market securities. Secondly, it defines a class of complex measure change martingales which will be pivotal for the computations of the next section.

Let \( \mathcal{B} \) the space of the \( n \)-dimensional \( \mathcal{F}_t \)-supported Brownian motions with drift starting at 0, and \( \mathcal{J} \) the space of the \( \mathcal{F}_t \)-supported pure jump Lévy processes starting at 0, that is, the class of the càdlàg \( \mathcal{F}_t \)-adapted processes with stationary and independent increments orthogonal\(^1\) to all the elements of \( \mathcal{B} \). Every Lévy process \( X_t \) such that \( X_0 = 0 \) can be decomposed as the orthogonal sum:

\[
X_t = X_t^c + X_t^d, \tag{3.1}
\]

with \( X_t^c \in \mathcal{B} \) and \( X_t^d \in \mathcal{J} \). We shall refer to \( X_t^c \) and \( X_t^d \) respectively as the continuous and discontinuous parts of \( X_t \).

Time changes are fairly general objects, so we have to introduce some additional requirements in order for our discussion to proceed. One property we shall assume throughout is the so-called continuity with respect to the time change.

**Definition 3.1.** Let \( T_t \) be a time change on a filtration \( \mathcal{F}_t \). An \( \mathcal{F}_t \)-adapted process \( X_t \) is said to be \( T_t \)-continuous\(^2\) if it is almost-surely constant on all the sets \([T_t, T_t] \).

Obviously, a sufficient condition for \( T_t \)-continuity is the almost-sure continuity of \( T_t \). Hence, of particular relevance is the class of the absolutely continuous time changes, with respect to which every stochastic process is continuous. Given a pair of instantaneous rate of activity processes, that is, two exogenously-given càdlàg positive stochastic processes \((v_t, u_t)\), valid time changes are given by the pathwise integrals:

\[
T_t = \int_0^t v_s^- \, ds, \tag{3.2}
\]

\[
U_t = \int_0^t u_s^- \, ds. \tag{3.3}
\]

The processes \( v_t \) and \( u_t \) describe the instantaneous impact of market trading and information arrival on the price, and formalize the concept of “business activity” over time.

A decoupled time change of a Lévy process is the sum of the (ordinary) time changes of its continuous and discontinuous part.

**Definition 3.2.** Let \( X_t \) be an \( n \)-dimensional Lévy process and \( T_t, U_t \) two time changes such that \( T_t \) is almost-surely continuous and \( X_t^d \) is \( U_t \)-continuous. Then:

\[
X_{T,U} = X_{T_t}^c + X_{U_t}^d, \tag{3.4}
\]

\(^1\)Two processes \( X_t \) and \( Y_t \) are said to be orthogonal if \( \langle X, Y \rangle_t = 0 \).

\(^2\)In [27] Jacod uses \( T_t \)-adapted, and \( T_t \)-synchronized is sometimes found; however, \( T_t \)-continuous is also common in the literature, and in our view less ambiguous.
is the decoupled time change of \( X_t \) according to \( T_t \) and \( U_t \).

By Jacod [27], corollaire 10.12, a first important property of \( X_{T,U} \) is that it is an \( \mathcal{F}_{T,U} \) semi-martingale. In particular, it is an Ito semi-martingale (see Jacod and Protter, [28]). To avoid degenerate cases, in all that follows we always assume \( T_t \) and \( U_t \) to be such that \( X_{T_t}^{\theta} \) and \( X_{U_t}^{\theta} \) are Markov processes.

We can now define the class of exponential martingales canonically associated with \( X_{T,U} \). The following proposition represents the main theoretical tool of this paper:

**Proposition 3.3.** Let \( X_t^1 \) be an \( n \)-dimensional Brownian motion with drift and \( X_t^2 \) a pure jump Lévy process in \( \mathbb{R}^n \). Let \( T_t^1 \) and \( T_t^2 \) be two time changes such that \( X_t^1 \) and \( X_t^2 \) are respectively \( T_t^1 \)- and \( T_t^2 \)-continuous. Set \( X_t = X_t^1 + X_t^2 \) and \( T_t = (T_t^1, T_t^2) \); define \( X_t := X_t^{1+} + X_t^{2+} \) and denote by \( \Theta \) the domain of definition of \( \mathbb{E}[\exp(i\theta^T X_t^2)] \). The process:

\[
M_t(\theta, X_t, T_t) = \exp (i\theta^T X_{T_t^1} - T_t^1 \psi_{X^1}(\theta) - T_t^2 \psi_{X^2}(\theta))
\]

is a local martingale, and it is a martingale if and only if \( \theta \in \Theta_0 \), where:

\[
\Theta_0 = \{ \theta \in \Theta \text{ such that } \mathbb{E}[M_t(\theta, X_t, T_t)] = 1, \forall t \geq 0 \}.
\]

When \( T_t^1 = T_t^2 \), the exponential \( M_t \) reduces to an ordinary time change of the type discussed in [7]. Even in this simple case proposition 3.3 is not a consequence of Doob’s optional sampling theorem applied to the martingale \( Z_t(\theta) = \exp(i\theta^T X_t - t\psi_X(\theta)) \), because the latter is not necessarily uniformly integrable. Indeed, time-transforming a process always preserves the semi-martingale property, but the martingale property is guaranteed to be maintained for uniformly integrable martingales only; an actual example of an asset model of the form \( Z_{T_t} \) which is a strict supermartingale only has been given by Sini in [37]. This demonstrates that some choices of time changes are inherently unsuitable for time-changed asset price modeling. In the case of \( X_{T_t} \) being a one-dimensional Brownian integral, sufficient requirements for \( \Theta_0 \) to be satisfied are the well-known Novikov and Kazamaki conditions ([31], chapter 3). The set \( \Theta_0 \) is sometimes called the natural parameter set.

Having obtained martingale relations for a stochastic exponential involving \( X_{T,U} \), the risk-neutral dynamics for a DTC Lévy-driven asset are given by the natural choice. We have the following immediate corollary to proposition 3.3:

**Corollary 3.4.** Let \( X_t \) be a scalar Lévy process of characteristic triplet \( (\mu, \sigma^2, \nu) \) and \( (T_t, U_t) \) a pair of time changes such that the continuous and discontinuous parts of \( X_t \) are respectively \( T_t \) and \( U_t \)-continuous. For a spot price value \( S_0 \) let, for \( t > 0 \):

\[
S_t = S_0 \exp(r t + i\theta_0 X_{T_t,U} - T_t \psi_X(\theta_0) - U_t \psi_X^d(\theta_0)) = S_0 e^{rt} M_t(\theta_0, X_{T_t}^{1+}, X_t^{1+}, (T_t, U_t)) \quad (3.7)
\]

with \( \theta_0 \in \Theta_0 \) being such that \( \Theta_0 \) is a real number. The discounted process \( \tilde{S}_t \) is a martingale, and therefore \( S_t \) is a price process consistent with the no-arbitrage condition.

The stochastic process in (3.7) is the fundamental asset model we shall use throughout the rest of the paper.

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3In general, time changes of Markov processes are not Markovian; by using Dambis, Dubins and Schwarz’s theorem ([31], theorem 4.6) one can manufacture a large class of counterexamples by starting from any continuous martingale which is not a Markov process.
4 Characteristic functions and the leverage-neutral measure

Characteristic functions of state variables are the essential component of the Fourier-inverse pricing methodology, the reason being that state price densities are analytically available only for a small number of models; in contrast, characteristic functions are computable in closed form in many instances (e.g. exponential Lévy models, Ito diffusions). This effectively means that in order to compute expectations (prices), the standard approach is not integrating a payoff against a density function, but rather the payoff’s Fourier transform against the characteristic functions of the price transition densities. Famous examples include the FFT paper [6] by Carr and Madan, Lewis’s book [33] and following paper [34].

The transform we are interested in is one associated with the price process (3.7). Compared to the usual inverse Fourier/Laplace framework the characteristic function we shall consider is not that of the log(discounted)-price alone, but it will also incorporate the quadratic variation of the log-process. Indeed, just as the characteristic function of the log-price allows for the derivation of pricing formulae for contingent claims $F(S_t)$, the joint characteristic function of log($\tilde{S}_t$) and $TV_t$ permits the valuation of payoffs of the form $F(S_t, TV_t)$. This approach has been envisaged before by Carr and Sun, [7].

In the present section we compute this transform. There are normally two ways of computing characteristic functions/Laplace transforms of log-price densities. One way is the analytical approach, which is popular in e.g. affine models, when the problem is ultimately reduced to solving a certain system of ODEs. Another one is the probabilistic approach, in which the characteristic function of the log-price is linked with the Laplace transform of the integrated driving factors (where available) and then a change of measure is performed to keep track of correlations. As Carr and Wu show in [9] this technique is intimately connected with time-changed asset modeling; in what follows we extend it to the case of the underlying being modeled through a full DTC Lévy process.

First of all we must verify that the quadratic variation operator respects the additivity and time-changed structure of $X_{T,U}$. We have the following “linearity/commutativity property”, of independent interest:

**Proposition 4.1.** A decoupled time-changed Lévy process $X_{T,U}$ is such that $X^c_{T_t}$ and $X^d_{U_t}$ are orthogonal. Furthermore its quadratic variation satisfies:

$$\langle X_{T,U} \rangle_t = \langle X^c \rangle_{T_t} + \langle X^d \rangle_{U_t} = \Sigma_{T_t} + \langle X^d \rangle_{U_t}. \quad (4.1)$$

That is, the quadratic variation of $X_{T,U}$ is the sum of the time changes of the quadratic variations of its continuous and discontinuous part.

Crucially, the processes $X^c_{T_t}$ and $X^d_{U_t}$ are orthogonal but not independent. Without the $T_t$ and $U_t$-continuity assumption, this proposition would be false: a counterexample is provided in the appendix. Proposition 4.1 ensures that in presence of time continuity of the Lévy continuous and jump parts with respect to the corresponding time changes, the quadratic variation of a DTC Lévy process is itself of DTC-type.

Now, for $S_t$ as in (3.7) define:

$$\Phi_{t_0}(z, w) = \mathbb{E}_{t_0}[\exp(i z^T \log(\tilde{S}_t/S_{t_0}) + i w^T (TV_t - TV_{t_0})]. \quad (4.2)$$

For each $z, w$ in which the right hand side is finite, $\Phi_{t_0}(z, w)$ is the Fourier transform is the joint transition function from time $t_0$ to time $t$ of log($\tilde{S}_t$) and $TV_t$. The characteristic function $\Phi_{t_0}(z, w)$ can be completely characterized in terms of the Lévy triplet of $X_t = X^c_t + X^d_t$ and the joint $Q(z, w)$-distribution of $T_t$ and $U_t$ by virtue of the following proposition.
Proposition 4.2. Let $S_t$ be an asset evolution as in corollary $\text{3.4}$, and define the family of $\mathbb{P}$ absolutely-continuous measures $\mathbb{Q}(z, w) \ll \mathbb{P}$ having Radon-Nikodym derivative:

$$
\frac{d\mathbb{Q}(z, w)}{d\mathbb{P}} = M_t((iz\theta_0, i\omega\theta_0), C_t + D_t, (T_t, U_t)) \quad (4.3)
$$

where $C_t = (X_t^i, 0)$, $D_t = (X_t^i, i\theta_0(X)^4_t)$ and $M_t$ is given by $\text{(3.2)}$. For all $(z, w)$ such that $(iz\theta_0, i\omega\theta_0) \in \Theta_0$, the characteristic function in $\text{(4.2)}$ is given by:

$$
\Phi_{t_0}(z, w) = \mathcal{L}^\mathbb{Q}_{\Delta T, \Delta U}(\zeta(z, w, \mu, \sigma, \theta_0), \xi(z, w, \nu, \theta_0)), \quad (4.4)
$$

with the notation $\mathcal{L}^\mathbb{Q}_{\Delta T, \Delta U}(\cdot)$ indicating the bilateral Laplace transform of the conditional joint distribution of $T_t - T_{t_0}$ and $U_t - U_{t_0}$, taken under the measure $\mathbb{Q}(z, w)$, and

$$
\zeta(z, w, \mu, \sigma, \theta_0) = \theta_0 \mu(z - iz) - \theta_0^2 \sigma^2(z^2 + iz - 2i\omega)/2, \quad (4.5)
$$

$$
\xi(z, w, \nu, \theta_0) = iz\psi_z^\mathbb{Q}(\theta_0) - \psi_D(iz\theta_0, i\omega\theta_0). \quad (4.6)
$$

Notice that unlike the density processes used in the standard numéraire changes, the new distributions implied by $\text{(4.3)}$ also account for the quadratic variation as a factor. If we assume $T_t$ and $U_t$ to be pathwise integrals of the form $\text{(3.2)}$ and $\text{(3.3)}$, it is possible to interpret the Laplace transform $\text{(4.3)}$ as being the analogue of a bivariate bond pricing formula, where the short rates are replaced by the instantaneous activity rates, and the pricing measure is not given once and for all, but varies as an effect of the correlation of $(\nu_t, \omega_t)$ with the stock.

It is also of interest the interpretation of the measure $\mathbb{Q}(z, w)$. Let us consider the special case of $X_t$ being independent of $T_t$ and $U_t$. In such a case it is straightforward to prove, by using the laws of the conditional expectation, that one obtains $\text{(4.4)}$ with $\mathbb{Q}(z, w) = \mathbb{P}$. Therefore, whenever there is no dependence between the time changes and the underlying Lévy process, no change of measure is needed in order to extract the characteristic function $\Phi_{t_0}(z, w)$. In contrast, in presence of correlation between $X_t$ and $(T_t, U_t)$, the family $\mathbb{Q}(z, w)$ gives a measurement of the impact of leverage on the price densities. Furthermore, in some well-behaved cases this change of measure can be absorbed in the $\mathbb{P}$-dynamics of the asset by a suitable parameter alteration of the distributions of $T_t$ and $U_t$. In accordance with $\text{[9]}$, we call $\mathbb{Q}(z, w)$ the leverage-neutral measure and $\Phi_{t_0}(z, w)$ the leverage-neutral characteristic function. Just as prices in a risky market can be equivalently computed in a risk-neutral environment according to a different price distribution, valuations in presence of leverage can be performed in a different economy with no leverage by means of an appropriate distributional modification.

5 Pricing and price sensitivities

The characteristic function found in section $\text{3.4}$ is needed to obtain analytical formulae for the valuation of European-type derivatives having a sufficiently regular payoff $F$. In the present section we find a semi-analytical formula based on an inversion integral which extends the standard Fourier-inversion machinery to our multivariate context.

Recall that since all the involved processes are Markovian, it makes sense treating $\Phi_{t_0}(z, w)$ like a Gauss-Green integral kernel depending only on some given initial states at time $t_0$. The following proposition extends both theorem 1 of $\text{[34]}$ and proposition 3.1 of $\text{[38]}$.

Proposition 5.1. Let $Y_t = \log S_t$, with $S_t$ given in corollary $\text{3.4}$. Let $F(x, y) \in L^1_{t_0} (Y_t, (Y)^i_t)$ for all $t_0 < t$, be a positive payoff function having analytical Fourier transform $\hat{F}(z, w)$ in a
Suppose further that $\Phi_{t_0}(z, w)$ is analytical in

$$
\Sigma_\Phi = \{(z, w) \in \Theta, \, \gamma_1 < Im(z) < \gamma_2, \, \eta_1 < Im(w) < \eta_2, \, \gamma_1, \gamma_2, \eta_1, \eta_2 \in \mathbb{R}\}.
$$

We have that the time-$t_0$ value of the contingent claim $F$ maturing at time $t$ is given by:

$$
E_{t_0}[e^{-r(t-t_0)}F(Y_t, \{Y_t\}_t)] = \frac{e^{-r(t-t_0)}}{4\pi^2} \int_{ik_1-\infty}^{ik_1+\infty} \int_{ik_2-\infty}^{ik_2+\infty} e^{-iw(Y)_{t_0}} S_{t_0}^{-iz} e^{-r(t-t_0)iz} \Phi_{t_0}(-z, -w) \hat{F}(z, w) dz dw.
$$

It is clear that modifying the asset dynamics specifications only acts on $\Phi_{t_0}$, whereas changing the claim to be priced influences $\hat{F}$ only. Also, by setting either variable to 0, we are able to extract from (5.4) prices of both plain vanilla and pure volatility derivatives. For example, the pricing integrals in (5.3) and (5.4), are special cases of the equation above when $F$ does not depend on the realized volatility and $\Phi_{t_0}$ is either obtained from a diffusion or a Lévy process. Moreover, equation (3.10) of [38] is recovered when $S_t$ is assumed to follow a stochastic volatility model.

In addition, this representation is useful if we are interested in the sensitivities of the claim value with respect to the underlying state variables. Let us consider for instance the Delta (sensitivity with respect to the change in value of the underlying) and Gamma (sensitivity with respect to the rate of change in value of the underlying) of valuations performed through formula (5.4). Call $I(r, t_0, z, w)$ the integrand on the right hand side of (5.4); by differentiating (if possible) under integral sign and noting that $\Phi_{t_0}$ bears no dependence on $S_{t_0}$ we see that:

$$
\Delta_t := \frac{\partial}{\partial S} E_{t_0}[e^{-r(t-t_0)}F(Y_t, \{Y_t\}_t)] = \frac{e^{-r(t-t_0)}}{4\pi^2} \int_{ik_1-\infty}^{ik_1+\infty} \int_{ik_2-\infty}^{ik_2+\infty} \frac{i z}{S_{t_0}} I(r, t_0, z, w) dz dw,
$$

and

$$
\Gamma_t := \frac{\partial^2}{\partial S^2} E_{t_0}[e^{-r(t-t_0)}F(Y_t, \{Y_t\}_t)] = \frac{e^{-r(t-t_0)}}{4\pi^2} \int_{ik_1-\infty}^{ik_1+\infty} \int_{ik_2-\infty}^{ik_2+\infty} \frac{i z - z^2}{S_{t_0}^2} I(r, t_0, z, w) dz dw.
$$

Mutatis mutandis we can repeat this argument if we want to determine the price sensitivity with respect to the quadratic variation $\{V_t\}_t$. Finally, as $\Phi_{t_0}(z, w)$ could also depend on other variables (e.g. an instantaneous rate of activity $v_{t_0}$) known at time $t_0$, by calling $v$ one such variable we have that:

$$
\nu_t := \frac{\partial}{\partial v} E_{t_0}[e^{-r(t-t_0)}F(Y_t, \{Y_t\}_t)] = \frac{e^{-r(t-t_0)}}{4\pi^2} \int_{ik_1-\infty}^{ik_1+\infty} \int_{ik_2-\infty}^{ik_2+\infty} e^{-iw(Y)_{t_0}} S_{t_0}^{-iz} e^{-r(t-t_0)iz} \frac{\partial \Phi_{t_0}}{\partial v}(-z, -w) \hat{F}(z, w) dz dw.
$$

This is especially well-suited to the case in which $\Phi_{t_0}(z, w)$ is exponentially-affine in $v$, i.e.

$$
\Phi_{t_0}(z, w) = \exp(A(z, w, t-t_0)v_{t_0} + B(z, w, t-t_0)),
$$
when we have:

\[
\frac{\partial \Phi_{t_0}}{\partial \nu}(-z, -w) = A(-z, -w, t-t_0)\Phi_{t_0}(-z, -w).
\]  

(5.9)

In the next section we calculate explicitly \( \Phi_{t_0} \) for a number of decoupled time-changed models.

6 Specific model analysis

We now determine the DTC Lévy structure (3.7) of various popular asset price processes, and find for each of them the corresponding leverage-neutral characteristic function \( \Phi_{t_0}(z, w) \). Such a derivation allows for the full implementation of equation (5.4) for the pricing of joint asset and volatility derivatives in all of the accounted cases. What the discussion below should make apparent is that DTCs offer a natural unifying framework for a priori different strains of financial asset models (e.g., continuous/jump diffusions, jump diffusions with stochastic volatility, Lévy processes). By classifying models through their DTC structure it is possible to recognize a “nesting” pattern linking different models, in which some can be considered particular cases of some others. This is of use for numerical purposes: as we shall see in section 8 one single implementation of equation (5.4) can produce values for several models, each one obtained by using a different instantiation of the code. Four categories of asset models are discussed: standard Lévy processes, stochastic volatility models, decoupled time-changed jump diffusions and general exponentially-affine asset models.

6.1 Lévy processes

In case of the Lévy process the decoupled time changed structure coincides with the underlying Lévy process. To determine \( \Phi_{t_0}(z, w) \) no change of measure is necessary, so this function represents the joint conditional characteristic function of the log-price and its quadratic variation as given in the risk-neutral measure. Following are the calculations for some popular models.

6.1.1 Black-Scholes model

The classic SDE with constant parameters \( \sigma, r \) driven by a Brownian motion \( W_t \):

\[
dS_t = rS_t dt + \sigma S_t dW_t
\]

(6.1)
can be trivially recovered from (3.7) by setting the triplet for the underlying Lévy process \( X_t = X_t^c \) to be \((0, \sigma, 0)\) and letting \( T_t = t, U_t = 0 \), so that \( X_{T,U} = X_t \). From (4.4), we have immediately:

\[
\Phi_{t_0}(z, w) = \exp(- (t-t_0) \sigma^2 (z^2 + i z - 2 i w)/2).
\]

(6.2)

6.1.2 Jump diffusion models

In their classic works, Merton and Kou [35, 32] proposed to model the log-price dynamics as a finite-activity jump diffusion. The risk-neutral asset dynamics are given by:

\[
dS_t = rS_t dt + \sigma S_t dW_t + S_t (\exp(J) - 1) dN_t - \kappa \lambda S_t^{-} dt
\]

(6.3)

where \( W_t \) is a standard Brownian motion, \( N_t \) is a Poisson counter of intensity \( \lambda \), and \( J \) is the jump size distribution. \( N_t \) and \( W_t \) are assumed to be independent, and the compensator \( \kappa \) equals \( \phi_J(-i) - 1 \). For the discounted price \( \tilde{S}_t \) to be a true martingale, conditions on the asymptotic behavior of \( f_J(x) \) must be imposed (see e.g. [11]). In the Merton model \( J \) is normally distributed \( J \sim N(m, \delta^2) \), whereas Kou assumed it an asymmetrically skewed double-
observe that \( f \) where
\[
\text{and the integral converges for } \text{Im}(t \Phi) = \alpha > 0 \text{ and } \beta > 0 \text{ and } p + q = 1.
\]
In these models no time change is involved, so \( X_{T,U} \) coincides with the underlying Lévy process \( X_t \) having characteristic triplet \((0, \sigma^2, \lambda f_J(x)dx)\). To completely characterize \( \Phi_{0,J_0}(z, w) \), observe that \((X^d_t, X^{d^2}_t)\) is just a bivariate compound Poisson process of joint jump density \( f_{J,J^2}(x, y) \) and intensity \( \lambda \), whence:
\[
\psi_D(z, w) = \lambda (\phi_{J,J^2}(z, w) - 1),
\]
where \( \phi_{J,J^2}(z, w) \) is the joint characteristic function of \( J \) and \( J^2 \). We conclude from (6.4) that \( \Phi_{0,J_0} \) has the exponential structure:
\[
\Phi_{0,J_0}(z, w) = \exp(-(t - t_0)(\sigma^2(z^2/2 + iz/2 - 2iw)/2 + \lambda (iz\kappa - \phi_{J,J^2}(z, w) + 1)).
\]
Now for the Merton model we have
\[
\phi_{J,J^2}(z, w) = \frac{\exp\left(\frac{imz - \delta^2 z^2/2 + im^2w}{1 - 2i\delta^2 w}\right)}{\sqrt{1 - 2i\delta^2 w}},
\]
and the integral converges for \( \text{Im}(w) > -1/2\delta^2 \). For the Kou model we can write:
\[
\phi_{J,J^2}(z, w) = \phi_{J_+,J^2_+}(z, w) + \phi_{J_-,J^2_-}(z, w);
\]
the characteristic function of the positive and negative parts are:
\[
\begin{align*}
\phi_{J_+,J^2_+}(z, w) &= \alpha p \sqrt{\pi} e^{-\frac{(\alpha - iz)^2}{4\delta^2 w}} \left( \frac{\text{Erfc}\left(\frac{\alpha + iz}{2\sqrt{-iw}}\right)}{2\sqrt{-iw}} \right), \\
\phi_{J_-,J^2_-}(z, w) &= \beta q \sqrt{\pi} e^{-\frac{(\alpha - iz)^2}{4\delta^2 w}} \left( \frac{\text{Erfc}\left(\frac{\beta - iz}{2\sqrt{-iw}}\right)}{2\sqrt{-iw}} \right),
\end{align*}
\]
which both converge for \( \text{Im}(w) > 0 \).

6.1.3 Tempered Lévy stable and CGMY

Another way of obtaining Lévy distributions for the asset price is that of directly specifying an infinite activity Lévy measure \( \nu(dx) \). In such a case we have \( X_{T,U} = X_t = X^d_t \), with \( X_t \) being a pure jump Lévy process of Lévy measure \( \nu(dx) \). The two instances we analyze here are the tempered Lévy stable process and the CGMY model. Both of these are obtained as an exponential smoothing of stable-type distributions; the latter can be viewed as a generalization of the former allowing for an asymmetrical skew between the distribution of positive and negative jumps. The Lévy density for a CGMY process is:
\[
\frac{d\nu(x)}{dx} = \frac{c_+ e^{-\beta_+ |x|}}{|x|^{1+\alpha_+}} \mathbb{1}_{\{x < 0\}} + \frac{c_+ e^{-\beta_- x}}{x^{1+\alpha_-}} \mathbb{1}_{\{x \geq 0\}}.
\]
Which is well defined for all \( c_+, c_- \), \( \beta_+, \beta_- > 0 \), \( \alpha_+, \alpha_- < 2 \). When \( \alpha_+ = \alpha_- \) one has the tempered stable process. For simplicity in what follows we assume \( \alpha_+, \alpha_- \neq 0,1 \); for such
values the involved characteristic functions still exist, but lead to particular cases. Being:

\[ \Phi_{t\theta}(z, w) = \exp((t - t_0)\xi(z, w, \nu(x)dx, -i)) \]  

(6.12)
to fully characterize \( \Phi_{t\theta}(z, w) \) we only need to determine \( \psi_D^d(\theta) \) and \( \psi_D(z, w) \). Letting \( \gamma_1 = \int_{-1}^x \psi(x)dx \), the exponent \( \psi_X(\theta) \) is given by the standard theory ([11], proposition 4.2) as:

\[
\psi_X^d(\theta) = \gamma_1 + \Gamma(-\alpha_+)|\beta_+|c_+ \left( \left(1 - \frac{i\theta}{\beta_+}\right)_{\alpha} - 1 + \frac{i\theta\alpha_+}{\beta_+} \right) + \\
\Gamma(-\alpha_-)|\beta_-|c_- \left( \left(1 + \frac{i\theta}{\beta_-}\right)_{\alpha} - 1 - \frac{i\theta\alpha_-}{\beta_-} \right). 
\]

(6.13)

Set \( \gamma_2 = \int_{-1}^1 x^2\nu(x)dx \); the positive part \( \psi_D^+ \) of \( \psi_D \) is computed as:

\[
\psi_D^+(z, w) = iz\gamma_1 + i\nu\gamma_2 + \int_0^{+\infty} (e^{iwx + \nu x^2} - 1 - (iwx + \nu x^2))c_+e^{-\beta_+} x^2 + \alpha \frac{dx}{w} = iz\gamma_1 + i\nu\gamma_2 + \\
ic_+\beta_+^+ \left( -\frac{\Gamma(2 - \alpha_+)}{2i\beta_+} + \frac{\Gamma(1 - \alpha_+)}{i} + i\Gamma(-\alpha_+) \right) - c_+ \left( \frac{i(\beta_+ - i)^2}{w} \right)^{-\alpha_+/2} \\
\left( \sqrt{i\beta_+ - iz} \right) \frac{1}{w} \Gamma \left( \frac{1}{2} - \frac{\alpha_+}{2} \right) \left[ \frac{1}{2} - \frac{3}{4} \right] \\
\Gamma \left( 1 - \frac{\alpha_+}{2} \right) \frac{1}{w} \left[ \frac{1}{2} - \frac{3}{4} \right] \\
\Gamma \left( -\frac{\alpha_+}{2} \right) \frac{1}{w} \left[ \frac{1}{2} - \frac{3}{4} \right]. 
\]

(6.14)

Here \( \Gamma \) is the Euler Gamma function and \( \gamma_1 \) the confluent hypergeometric function. The multi-strip of convergence of \( (6.13) \) is the set \( \Sigma_\Phi = \{(z, w), \text{Im}(w) > 0, \text{Im}(z) > -\beta_+\} \). The determination \( \psi_D^+ \) has a similar expression.

### 6.2 Stochastic volatility and the Heston model

In a stochastic volatility model the asset process is given, in a risk neutral-measure, by the SDE:

\[ dS_t = rS_t dt + \sqrt{\nu_t} S_t dW_t^1 \]  

(6.15)

where \( \nu_t \) is some continuous stochastic variance process. By Dubins-Schwarz theorem ([31], chapter 3, theorem 4.6), any continuous martingale \( M_t \) can be written as \( M_t = W(M_t) \) for a certain Brownian motion \( W_t \), which implies that the DTC structure of a stochastic volatility model corresponds to a standard Brownian motion \( W_t \) time-changed by \( T_t \) as in (6.2). In order to explicitly express the characteristic function \( \Phi_{t\theta}(z, w) \) we must make a specific choice for the dynamics (6.15). For instance, we can make the popular choice of selecting a square-root (CIR) equation for the instantaneous variance:

\[ d\nu_t = \alpha(\theta - \nu_t)dt + \eta\sqrt{\nu_t}dW^2. \]  

(6.16)

for positive constants \( \alpha, \theta, \eta \) and a Brownian motion \( W^2_t \) linearly correlated with \( W^1_t \), having correlation coefficient \( \rho \). For \( \bar{S}_t \) to be well-defined, the parameters \( \alpha, \theta, \eta > 0 \) need to satisfy the Feller condition \( 2\alpha\theta \geq \eta^2 \). The system of SDEs \( (6.15)-(6.16) \) is the model by Heston in [26]. As we change to the measure \( \mathbb{Q}(z, w) \), application of the complex-plane version of Girsanov’s theorem and a simple algebraic manipulation reveals that the leverage-neutral dynamics \( \nu_t^c \) of \( \nu_t \) are of the same form of \( (6.16) \), but with parameters:

\[ \alpha^2 = \alpha - i\rho z \eta; \]  

(6.17)

\[ \theta^2 = \alpha\theta/\alpha^2, \]  

(6.18)
(see also Carr and Wu [29]). Using equation (4.4), we determine \( \Phi_{t_0} \) as:

\[
\Phi_{t_0}(z, w) = L^{zT}_{\Delta T}(z^2/2 + iz/2 - iw),
\]

where \( L^{zT}_{\Delta T} \) indicates the transform with respect to \( v_t \) and it is well-known analytically (Dufresne, [16]). The case \( T_t = t \) reverts back to the Black-Scholes model, when (6.19) collapses to (6.2) with \( \sigma^2 = v_0 \).

Other choices for \( v_t \) are clearly possible, yielding different stochastic volatility models (the 3/2 model, GARCH, etc.). It is clear from the arguments above that for an analytical expression for \( \Phi_{t_0} \) to exist it suffices that the Laplace transform of \( T_t \) is known in closed form\(^4\) and that \( v_t \) belongs to a class of models stable under the Girsanov transformation.

6.3 DTC jump diffusions

When the underlying Lévy process is represented by a finite activity jump diffusion, operating a decoupled time change amounts to either introducing a stochastic volatility coefficient in the continuous Brownian part, making the intensity of the compound Poisson process \( X^d_t \) stochastic, or both. Models carrying this structure have been prominently discussed by D.S. Bates in [2] and by H. Fang in [18].

6.3.1 Stochastic volatility with jumps

The stochastic volatility model with jumps provides us with a first instance of a decoupled time change not otherwise obtainable as an ordinary time change. The SVJ model is in fact a Lévy decoupled time change with a time-changed continuous part and a time-homogeneous jump part.

The dynamics for the asset price are given by the exponential jump diffusion:

\[
dS_t = rS_t dt + \sqrt{v_t}S_t dW^1_t + S_t (\exp(J) - 1) dN_t - \kappa \lambda S_t dt;
\]

for some Brownian motion \( W^1_t \), stochastic variance process \( v_t \), Poisson process \( N_t \) and jump size \( J \) having compensator \( \kappa \). The underlying DTC structure of the Bates model is given by \( X^T_{T,U} = X^d_{t_0} + X^d_t \) with the characteristic triplet for \( X_t \) being \((0, 1, \lambda f_J(x) dx)\) and \( T_t \) of the form (3.2). By taking as a jump distribution a Normal random variable, and as a process for the variance a square-root equation

\[
dv_t = \alpha(\theta - v_t) dt + \eta \sqrt{v_t} dW^2_t.
\]

we have the model by Bates, [2]. For the discounted asset value to be a martingale, the parameters of the driving stochastic volatility and jump process must be subject to the requirements of both subsection 6.2 and subsection 6.1.2. It is straightforward to see that \( \Phi_{t_t}(z, w) \) decomposes as:

\[
\Phi_{t_0}(z, w) = \Phi_{t_0}^c(z, w)\Phi_{t_0}^d(z, w),
\]

where \( \Phi_{t_0}^c(z, w) \) and \( \Phi_{t_0}^d(z, w) \) are given respectively by (6.19) and (6.6)-(6.7). Therefore:

\[
\Phi_{t_0}(z, w) = L^{zT}_{\Delta T}(z^2/2 + iz/2 - iw) \exp(-(t - t_0) \lambda (iz - \phi_{J,J2} + 1)).
\]

So far, we have encountered either exponential Lévy models, or exponentially affine functions arising as solutions of a PDE problem. Here we have a mixture of the two: a time-homogeneous jump factor, modeled as a compound Poisson process, and a continuous diffusion factor, whose

\(^4\)In [38], this author has independently found \( \Phi_{t_0} \) in the Heston model by augmenting the SDE system associated to the problem (6.15)-(6.16) with the equation \( dt_t = v_t dt \), and solved the associated Fourier-transformed parabolic equation via the usual Feynman-Kac argument. As it has to be, the two approaches coincide.

\(^5\)See e.g. Lewis, [34] chapter 2, for the Laplace transform of the cited models.
characteristic function solves a diffusion problem. The degenerate case \( T_t = t \), yields a Merton jump diffusion with diffusion coefficient \( \sqrt{v_t} \).

### 6.3.2 Stochastic volatility with jumps and a stochastic jump rate

Another way of obtaining a DTC model is obtained by introducing a stochastic jump frequency in the jump diffusion of the log-price. A jump process with stochastic volatility and stochastic jump rate has been suggested and empirically studied by H. Fang in [18]. For a time change \( U_t \), we assume \( N_t \) to be a pure jump process of finite activity such that conditionally on \( U_t \), \( N_t \) is distributed like a Poisson random variable of parameter \( U_t \), and is independent of every other involved process. We let \( \lambda_t \) be another continuous stochastic process; with the remaining notation as in subsection [6.3.1] define the asset price dynamics as:

\[
dS_t = rS_t \, dt + \sqrt{v_t} S_t \, dW^1_t + S_t \,(\exp(J) - 1) \, dN_t - \kappa \lambda_t S_t \, dt;
\]

This model has a clear DTC Lévy structure \( X_{T,U} \) given by \( T_t, U_t \) as in [6.2] and [6.3] with \( u_t = \lambda_t \), and the characteristic triplet \((0,1,f_J(x) \, dx)\). The model by Fang is obtained by setting:

\[
dv_t = \alpha (\theta - v_t) \, dt + \eta \sqrt{v_t} \, dW^2_t; \quad d\lambda_t = \alpha \lambda (\theta - \lambda_t) \, dt + \eta \lambda \sqrt{\lambda_t} \, dW^3_t.
\]

As usual we impose \( \langle W^1_t, W^2_t \rangle = \rho \, dt \); in contrast, the Brownian motion \( W^3_t \) is assumed to be independent of all the other random variables. If both of the diffusion parameter sets obey Feller’s condition and the density of \( J \) decays sufficiently fast, \( \tilde{S}_t \) is a martingale. Like in the Bates model, the jumps \( J \) are normally distributed. The function \( \Phi_{t_0} \) is then given by:

\[
\Phi_{t_0}(z, w) = \mathcal{L}^z_\Delta \lambda (z^2/2 + i z / 2 - i w) \mathcal{L}^\Delta (iz - \phi_{J,F}(z, w) + 1).
\]

Again we recognize that we can decompose \( \Phi_{t_0}(z, w) = \Phi_{t_0}^e(z, w) \Phi_{t_0}^d(z, w) \), where \( \Phi_{t_0}^e \) is the leverage-neutral characteristic function of a Heston process of variance \( v_t \), and \( \Phi_{t_0}^d \) that of a compound Poisson process time-changed with \( U_t \), whose argument has been computed in subsection [6.1.2]. The Laplace transforms of the integrated-square root processes arising from \( v^2_t \) and \( \lambda_t \) are known, and the leverage-neutral version \( v^2_t \) of \( v_t \) has been given in subsection [6.2]. Observe that there is no leverage effect in the jump part because of the assumptions on \( W^3_t \).

Finally, notice that the case \( U_t = t \) reduces to the Bates model where the jump activity rate equals \( \lambda_0 \).

### 6.4 General theory exponentially-affine activity rate models

A general theory of affine models for the discounted asset dynamics has been laid out by Duffie et al. [15], and Filipović [19], as well as from others. We briefly illustrate how this ties in with decouple time-changed processes. Suppose we have a Markov process given by the stochastic differential equation:

\[
dY_t = \mu(Y_t) \, dt + \sigma(Y_t) \, dW_t + dN_t
\]

where \( W_t \) is an \( n \)-dimensional Brownian motion, \( N_t \) is an \( n \)-dimensional pure jump process of intensity \( \lambda(Y_t) \) and joint jump size distribution \( F(x_1, \ldots, x_m) \) on \( \mathbb{R}^n \). We fix a discount functional \( \tilde{R}(x) = r_0 + r_1 \cdot x \), \((r_0, r_1) \in \mathbb{R} \times \mathbb{R}^n \) and assume for the coefficients the linear structure:

\[
\mu(x) = m_0 + m_1 x, \quad (m_0, m_1) \in \mathbb{R}^n \times \mathcal{M}_n(\mathbb{R})
\]

\[
\sigma \cdot \sigma^T(x) = \Sigma_0 + \Sigma_1 x, \quad (\Sigma_0, \Sigma_1) \in \text{Sym}_n(\mathbb{R}) \times \text{Sym}_n(\mathbb{R})
\]

\[
\lambda(x) = l_0 + l_1 x, \quad (l_0, l_1) \in \mathbb{R} \times \mathbb{R}^n.
\]

For some one-dimensional DTC process \( X_{T,U} \), let \( M_t \) be the change of measure martin-
gale in (4.3) and assume $Y_t$ to be two-dimensional, so that the marginals of $Y_t$ represent the instantaneous activity rates $v_t$ and $u_t$.

The leverage-risk-neutral characteristic function $\Phi_{t_0}$ can be recovered as follows. By taking the Itô differential of $\log M_t$ one sees that $M_t$ is itself a jump diffusion; we can thus define the three-dimensional augmented process $\tilde{Y}_t = (Y_t, \log M_t)$ having some associated extended parameters $\tilde{m}_0, \tilde{m}_1, \tilde{\Sigma}_0, \tilde{\Sigma}_1, l_0, l_1$ in (6.28). Furthermore, we can rewrite $M_t$ as:

$$M_t = \exp(b \cdot \tilde{Y}_t)$$

(6.29)

where $b = (0, 0, 1)^T$. Now, according to the results of [15], appendix C, under the measure $Q = Q(z, w)$ having Radon-Nikodym derivative $M_t$, it is:

$$\Psi_{t_0}^Q(u) := \mathbb{E}_{t_0}^Q \left[ \exp \left( - \int_{t_0}^t R(\tilde{Y}_s)ds \right) e^{\alpha \tilde{Y}_t} \right] = e^{-\alpha(t_0) - \beta(t_0)\tilde{Y}_{t_0}}$$

(6.30)

for all $u$ for which (6.30) is defined, and $\alpha(t), \beta(t)$ following the Riccati system of ODEs:

$$\beta(\tau)' = r_1 + (\tilde{m}_T^T + \tilde{\Sigma}_1)b(\tau) - \frac{1}{2} \beta(\tau)^T \tilde{\Sigma}_1 \beta(\tau) - \tilde{l}_1 (\mathcal{L}_F(\beta(\tau) + b) - \mathcal{L}_F(b))$$

(6.31)

$$\alpha(\tau)' = r_0 + (\tilde{m}_0 + \tilde{\Sigma}_0)b(\tau) - \frac{1}{2} \beta(\tau)^T \tilde{\Sigma}_0 \beta(\tau) - \tilde{l}_0 (\mathcal{L}_F(\beta(\tau) + b) - \mathcal{L}_F(b))$$

(6.32)

for $\tau \leq t$, with boundary conditions $\beta(t) = u$ and $\alpha(t) = 0$. By choosing

$$r_0 = 0, \quad r_1 = (\zeta(z, w, \mu, \sigma, \theta_0), \zeta(z, w, \nu(dx), \theta_0), 0),$$

(6.33)

one notices that:

$$\Psi_{t_0}^Q(0) = \Phi_{t_0}(z, w).$$

(6.34)

The solvability of equations (6.31)-(6.32) is discussed and characterized in Grasselli and Tebaldi [22]. What we have just shown is that the class of the exponential affine processes and that of the DTC Lévy processes intersect in the class of the DTC processes whose instantaneous activity rates are given by affine jump diffusions.

We remark that $\tilde{Y}_t$ implicitly defines a price process $S_t$ through the instantaneous activity rates and the change of measure martingale $M_t$ accounting for the dependence structure between the time changes and the underlying Lévy process. The augmented diffusion $\tilde{Y}_t$ is an exponentially-affine decoupled time change; all the models reviewed so far fall under this category. Another example of a model that can be represented in this form is the “double jump model” of [15], given by a jump diffusion having stationary jump intensity, whose stochastic volatility is itself a jump diffusion process having same intensity as the stock.

7 A theoretical multifactor DTC jump diffusion

In this section we illustrate a theoretical model in the DTC framework admitting a closed formula for $\Phi_{t_0}$. The price evolution we consider has several attractive features: it is a DTC jump diffusion and therefore allows for the presence of a stochastic jump rate and a stochastic volatility; in addition, both of these processes are given through a multifactor specification. Regarding the correlation modeling, the dynamics we assume carry the usual linear correlation between the stochastic volatility and the Brownian motion driving the stock, as well as a dependence structure between the instantaneous rates of activity, given in matrix form. Thus, the hypothesis of correlated market jump and continuous activity finds room in this model.

The case for multifactor volatility has been raised by a number of authors. As pointed out in [3], a volatility specification of such kind overcomes the inability of single factor models to
fit the current market skew while at the same time predicting the future evolution of implied volatilities consistently with historical data, which is of particular relevance in the pricing of certain forward-starting derivatives like the reverse cliquet option. Furthermore multifactor models make possible a long-term volatility specification which accounts for the slow decay of the autocorrelation function of the variance (Gallant, Hsu and Tauchen [20]), as opposed to single factor models, whose autocorrelation decays exponentially.

The price process we analyze links to a modern and currently very active strain of research, which makes use for financial modeling of the so-called Wishart process. The Wishart process is a matrix-valued affine process, studied foremostly by M.F. Bruin [5], that can be thought as a multivariate extension of the CIR process. It has been used to model the driving factors of term structures and price processes by e.g. da Fonseca et al. [12, 13], and Gouriéroux and Sufana [24, 25], among the others.

For \( n \times n \) matrices \( Q \) and \( M \), with \( Q \) invertible and \( M \) negative definite (to capture mean-reversion), a Wishart process \( \Sigma_t \) is defined by the SDE:

\[
d\Sigma_t = \sqrt{\Sigma_t} dB_t Q + Q^T dB_t^T \sqrt{\Sigma_t} + (M \Sigma_t + \Sigma_t M^T + cQ^T Q) dt. \tag{7.1}
\]

The Wishart process is thus a symmetric matrix-valued process. The matrix \( M \) must satisfy the further constraint \( c \geq n - 1 \) for some \( c > 0 \); \( B_t \) is here an \( n \times n \) matrix of Brownian motions.

We can use \( \Sigma_t \) to build a one-dimensional DTC jump diffusion model as follows. We choose \( N_t \) being a finite activity jump process like in subsection 6.3.2, which we further assume it to be independent of both \( W_t \) and \( B_t \). As usual, the jump distribution \( J \) is set to be independent of every other variable. Denoting by \( \sigma_t \) the positive-definite matrix square root of \( \Sigma_t \), we can define the risk-neutral dynamics of the log-price process \( Y_t = \log(S_t/S_0) \) as:

\[
dY_t = (r - \Sigma_t^{1,1}/2 - \Sigma_t^{2,2} \kappa) dt + \sigma_t^{1,1} dW_t^1 + \sigma_t^{1,2} dW_t^2 + JdN_t, \quad Y_0 = 0 \tag{7.2}
\]

where \( \kappa \) equals \( \phi_J(-i) - 1 \). The process \( S_t \) is seen to be a local martingale of the form \([37]\) by assuming the time changes in proposition 3.3 to be like in equations 3.2 and 3.3 and letting:

\[
dx_t^c = \frac{\sigma_t^{1,1}}{\sqrt{\Sigma_t^{1,1}}} dW_t^1 + \frac{\sigma_t^{1,2}}{\sqrt{\Sigma_t^{1,1}}} dW_t^2, \quad x_t^d = \sum_{i=0}^{M_t} j_i, \quad v_t = \Sigma_t^{1,1}, \quad u_t = \Sigma_t^{2,2}, \quad \theta_0 = (-i, -i). \tag{7.3}
\]

where \( M_t \) is a Poisson process of intensity 1. Multifactoriality is reflected in the fact that the equations \( \Sigma_t^{i,j} \) form a system of mutually dependent stochastic processes. In particular, the covariation between \( \Sigma_t^{1,1} \) and \( \Sigma_t^{2,2} \) gives the correlation between the activity rates. Let \( w_t^1 \) be the scalar Brownian motion driving \( \Sigma_t^{1,1} \): it can be proved that

\[
d\langle w^1, w^2 \rangle_t = \frac{\Sigma_t^{1,2}(Q^{1,1}Q^{1,2} + Q^{2,1}Q^{2,2})}{\sqrt{\Sigma_t^{1,1}((Q^{1,1})^2 + (Q^{2,1})^2) \sqrt{\Sigma_t^{2,2}((Q^{1,2})^2 + (Q^{2,2})^2)}} dt. \tag{7.4}
\]

Observe that this correlation is stochastic. The correlation of \( Y_t \) with its instantaneous variance \( \Sigma_t^{1,1} \) is instead determined by the interplay between \( \rho \) and \( Q \); we have:

\[
d\langle w^1, x^c \rangle_t = \frac{\rho Q^{1,1}}{\sqrt{(Q^{1,1})^2 + (Q^{2,1})^2}}. \tag{7.5}
\]

By applying the Girsanov’s transformation, we see that the \( Q(z, w) \)-dynamics of (7.1) are given by the complex-valued Wishart process:

\[
d\Sigma_t^c = \sqrt{\Sigma_t} dB_t Q + Q^T dB_t^T \sqrt{\Sigma_t} + (M^2 \Sigma_t + \Sigma_t (M^2)^T + cQQ^T) dt. \tag{7.6}
\]
where
\[ M^z = M + izQ^TR, \quad R = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix}, \] (7.7)
whence:
\[ \Phi_{t_0}(z,w) = \mathcal{L}^\Delta T,\Delta U(z^2/2 + iz/2 - iw, iz\kappa - \phi_{J,J}(z, w) + 1). \] (7.8)

Notably, the Laplace transform \( \mathcal{L}^\Delta T,\Delta U(\cdot) \) for \( v_t \) and \( u_t \) like in (7.3) can be derived in closed form (see the appendix), since it is a particular case of well-studied transforms of the Wishart process.

It is therefore possible to price, and find price sensitivities of, joint price/volatility contingent claims on an asset whose log-price process follows \( Y_t \). The model just presented is a particular DTC jump diffusion featuring not only the usual leverage effect between the underlying jump diffusion and the continuous/jump market activity, given by (7.5), but also a correlation structure between the rates of activities themselves, as implicitly captured by the matrices \( Q \) and \( M \).

This asset pricing model provides an example of how non-trivial (i.e. achieved by using dependent time changes) DTC modeling might work in practice. As a general approach, one could start from a multivariate stochastic process whose integrated marginals have known joint Laplace transform, and use these as time changes for the continuous and discontinuous parts of some given Lévy process. The underlying Lévy triplet will only appear as an argument of such a transform, and the characteristic function of the process is then completely determined up to a measure change. This or similar models could be the subject of further research.

8 Numerical testing and final remarks

8.1 Implementation of the pricing formula

For validation purposes we numerically implemented equation (5.4) on MATHEMATICA® for various models and payoffs, and compared the analytical prices found to a MATLAB® simulation following an Euler scheme. The results confirm the consistence of the pricing formula with the risk-neutral valuation theory.

We have analyzed three different contingent claims; one on \( S_t \), one on \( TV_t \), and one joint derivative on \( S_t \) and \( TV_t \). Namely, we accounted for three different kinds of options: a vanilla call option, a call option on the realized volatility, and a call TVO.

For a plain call option of maturity \( t \) and strike \( K \) the function \( F \) and its Fourier transform \( \hat{F} \) to be used in (5.4) are:
\[ F(z) = (e^z - K)^+, \quad \hat{F}(z) = \frac{K^{1+iz}}{(iz - z^2)^3}; \] (8.1)

the function \( \hat{F} \) exists and is analytical for \( \text{Im}(z) > 1 \).

A possible volatility investment is to write a call option using as an underlying the total realized volatility \( \sqrt{TV_t} \) of an asset, or buying a call option directly on a volatility index like the VIX. Hence, we would like to price the contingent claim paying \( (\sqrt{TV_t} - Q)^+ \) at time \( t \) for some strike realized volatility level \( Q \). In our equation we would then need to take:
\[ F(w) = (\sqrt{w} - Q)^+, \quad \hat{F}(w) = \frac{\sqrt{\pi} \text{Erfc}(Q\sqrt{-iw})}{2(-iw)^{3/2}}; \] (8.2)

the Fourier transform is well-defined and holomorphic in \( \text{Im}(w) > 0 \).

The target volatility option mentioned in the introduction is a natural candidate for testing mixed-claim structures, being to our knowledge an instance of currently traded joint asset/volatility derivatives. The payoff function \( F \) and the Fourier transform a call target volatility
option of strike $H$, maturing at $t$ and having target volatility level $\sigma$ are:

$$F(z, w) = \sigma \sqrt{\frac{\tau}{w}} (e^z - H)^+, \quad \hat{F}(z, w) = \sigma (1 + i) \sqrt{\frac{\pi t}{2w}} i^{-1 + iz} (iz - z^2).$$  \tag{8.3}

Observe that unlike the previous contracts, the payoff $F$ of a TVO shows explicit dependence on the expiry $t$. The domain of holomorphy of $\hat{F}$ is the strip $\Sigma_F = \{(z, w) \in \mathbb{C}^2, \text{Im}(z) > 1, \text{Im}(w) > 0\}$.

These derivatives have been numerically tested using five different stochastic models for the underlying asset processes: namely, the Black-Scholes, Heston, Merton, Bates and Fang models. All the prices have been produced with a single implementation of (5.4) with $\Phi_{t_0}$ given by (6.26). All we had to do is changing/voiding the relevant parameters, and replacing the module for $\hat{F}$ whenever switching payoff.

The parameter estimates have been taken from the S&P 500 fits of [18] and are illustrated in table 1. Tables 2 to 6 summarize the result obtained for five different sets of observable market conditions $(r, t_0, S_{t_0}, TV_{t_0})$ and contract parameters $t, K, Q, H, \sigma$. For each given $t_0$, the maturity $t > t_0$ is the same for all the three options considered; a TVO is always compared to a vanilla call having same strike, and the target volatility is set to be constant across all the data sets.

We have simulated 100,000 paths of step size $(t - t_0)/1000$. The figures show a good overall match of the analytical value (AV) with the Monte Carlo value (MC); in parentheses is the relative error $|\text{AV} - \text{MC}| / \text{MC}$. For the call option on the volatility in some cases we almost attain four-digit precision. On the other hand, for some models and data sets the integrands for the TVO valuation remain highly oscillatory around the maximum integration range; when this occurs a certain loss of accuracy is observed.

### 8.2 Conclusions

In this paper we suggested a theoretical pricing framework which easily instantiates to popular settings, but whose full model and payoff generalities were not allowed by the previous theory. This was made possible by the introduction of the concept of decoupled time change, and by considering as default target contingent claims payoffs on an asset and its accrued volatility.

The DTC theory provides a common time-changed representation for many models from the standing literature, and helps to explain possible dependence relationships between the continuous and the jump market activities. We obtained martingale relations for stochastic exponentials based on DTC Lévy processes, basing on which we defined an asset price’s dynamics. We then linked the joint characteristic function of the log-price dynamics and the quadratic variation to the joint Laplace transform of the time changes. As a by-product, we have extended the measure change technique of [9] to the class of DTC Lévy processes. In the DTC setup, we have rigorously posed and solved the valuation problem of a derivative paying off on an asset $S_t$ and its realized volatility, by means of an inverse-Fourier integral relation which extends previously known formulae.

Several stochastic models and contingent claims have been analyzed. In all the accounted cases we outlined the underlying DTC structure and found the leverage-neutral characteristic function. In particular, the SVJ and SVJSJ models are seen to have their own time-changed Lévy structure.

Finally, we have introduced a novel DTC Lévy multifactor model which provides insight of how equity modeling could benefit from the idea of decoupled time changes.

For numerical comparison and validation, we focused on specific instances from the three payoff classes allowed by our equation: plain vanilla claims, volatility claims, and joint asset/volatility claims. The results confirm the validity of our method. From a computational standpoint, a single software implementation can output prices for several different combinations of models and payoffs.
9 Appendix: proofs

We begin from recalling some basic definitions from the semimartingale representation theory. The \( \text{Doléans-Dade exponential} \) of an \( n \)-dimensional semimartingale \( X_t \) starting at 0 is:

\[
E(X_t) = e^{X_t - (X_t^c)^{1/2} \prod_{s \leq t} (1 + \Delta X_s)} e^{-\Delta X_s},
\]

where \( X_t^c \) denotes the continuous part of \( X_t \) and the infinite product converges uniformly. This is known to be the solution of the SDE \( dY_t = Y_{t-} dX_t, \ Y_0 = 1 \).

Let \( \epsilon(x) \) be a truncation function and \( (\alpha_t, \beta_t, \rho(dt \times dx)) \) be a triplet of predictable processes which are well-behaved in the sense of [29], chapter 2, equations (2.12)-(2.14). For \( \theta \in \mathbb{C}^n \), associate with \( (\alpha_t, \beta_t, \rho(dt \times dx)) \) the following complex-valued functional:

\[
\Psi_t(\theta) = i \theta^T \alpha_t - \theta^T \beta_t \theta/2 + \int_0^t \int_{\mathbb{R}^n} (e^{i \theta^T x} - 1 - i \theta^T x \epsilon(x)) \rho(ds \times dx).
\]

This functional is well-defined on:

\[
\mathcal{D} = \left\{ \theta \in \mathbb{C}^n \text{ such that } \int_0^t \int_{\mathbb{R}^n} e^{i \theta^T x} \epsilon(x) \rho(ds \times dx) < +\infty \text{ almost surely} \right\}
\]

and because of the assumptions made it is also predictable and of finite variation.

Let \( X_t \) be an \( n \)-dimensional semimartingale. The \textit{local characteristics} of \( X_t \) are the unique predictable processes \( (\alpha_t, \beta_t, \rho(dt \times dx)) \) as above, such that \( E(\Psi_t(\theta)) \neq 0 \) and \( \exp(i \theta^T X_t)/E(\Psi_t(\theta)) \) is a local martingale for all \( \theta \in \mathcal{D} \). The process \( \Psi_t^X(\theta) \) in (9.2) arising from the local characteristics of \( X_t \) is called the \textit{cumulant process} of \( X_t \), and it is independent of the choice of \( \epsilon(x) \).

It is clear that the local characteristics of a Lévy process \( X_t \) of Lévy triplet \( (\mu, \Sigma, \nu) \) are \( (\mu t, \Sigma t, \nu dt) \).

If \( \mathcal{B} \) is a Borel space, the time change of a random measure \( \rho(dt \times dx) \) on the product measure space \( \Omega \times \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^n) \) is the random measure:

\[
\rho(dt \times dx)(\omega, [0, t] \times B) = \rho(dt \times dx)(\omega, [0, T_t(\omega)] \times B)
\]

for \( \omega \in \Omega, \ t \geq 0 \) and all sets \( B \in \mathcal{B}(\mathbb{R}^n) \).

A semimartingale \( X_t \) is said to be \textit{quasi-left-continuous} if its associated random measure \( \rho \) is such that \( \rho(dt \times dx)(\omega, \{t\} \times B) = 0 \) for all \( t \geq 0 \), Borel sets \( B \) in \( \mathbb{R}^n \), and \( \omega \in \Omega \). Quasi-left-continuity intuitively means that the discontinuities of the process cannot occur at fixed times.

The following theorem clarifies the importance of continuity under time changing, i.e. that stochastic integration and integrating with respect to a random measure “commute” with the time changing operation.

\textbf{Theorem A.} Let \( T_t \) be a time change with respect to some filtration \( \mathcal{F}_t \).

(i) Let \( X_t \) be a \( T_t \)-continuous semimartingale. For all \( \mathcal{F}_t \)-predictable integrands \( H_t \), we have that \( H_{T_t} \) is \( \mathcal{F}_{T_t} \)-predictable, and:

\[
\int_0^{T_t} H_s dX_s = \int_0^t H_{T_{s-}} dX_{T_s};
\]
(ii) Let \( \rho(dt \times dx) \) be a \( T_t \)-adapted random measure on \( \Omega \times \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^n) \). For all measurable random functions \( W(t, \omega, x) \) and \( \omega \in \Omega \) it is:

\[
\int_0^{T_t} \int_{\mathbb{R}^n} W(s, \omega, x)\rho(ds \times dx)(\omega) = \int_0^t \int_{\mathbb{R}^n} W(T_s- (\omega), \omega, x)\rho(dT_s- \times dx)(\omega). \tag{9.6}
\]

Proof. Jacod, [27], théorème 10.19, (a), part (i), and théorème 10.27, (a), part (ii).

It is essentially a consequence of theorem A that under the assumption of continuity with respect to \( T_t \) the local characteristic of a time-changed semimartingale are well-behaved, in the sense of the next theorem.

**Theorem B.** Let \( X_t \) be a semimartingale having local characteristics \( (\alpha_t, \beta_t, \rho(dx \times dt)) \) and cumulant process \( \Psi^X_t(\theta) \) with domain \( \mathcal{D} \), and let \( T_t \) be a time change such that \( X_t \) is \( T_t \)-continuous. Then the time-changed semimartingale \( Y_t = X_{T_t} \) has local characteristics \( (\alpha_{T_t}, \beta_{T_t}, \rho(dT_t \times dx)) \) and the cumulant process \( \Psi^Y_t(\theta) \) equals \( \Psi^X_{T_t}(\theta) \), for all \( \theta \in \mathcal{D} \).

Proof. Kallsen and Shiryaev, [30], lemma 2.7.

**Proof of proposition 4.3.** Let \( (\mu, \Sigma, 0) \) and \( (0, 0, \nu) \) be the Lévy triplets of \( X^1_t \) and \( X^2_t \). Because of the \( T_t \)- and \( T^{2}_t \)-continuity assumption, we can apply theorem B and we immediately see that the local characteristics of \( X^1_{T_t} \) and \( X^2_{T_t} \) are respectively \( (T^1_t \mu, T^1_t \Sigma, 0) \) and \( (0, 0, T^2_t \nu) \). By a result on the linear transformation of semimartingales, these two sets of local characteristics above are additive (in [17], proposition 2.4, take \( U \) to be the juxtaposition of two \( n \times n \) identity blocks and let \( H = (X^1_{T_t}, X^2_{T_t})^T \) ), so that \( X_{T_t} \) has local characteristics \( (T^1_t \mu, T^1_t \Sigma, dT^2_t \nu) \).

Let \( \Psi_t(\theta) \) be the cumulant process of \( X_{T_t} \); by definition the exponential \( \mathcal{E}(\Psi_t(\theta)) \) is well-defined if and only if \( \theta \in \Theta \). After applying theorem A part (ii), a direct verification shows that \( X_{T_t} \) is quasi-left-continuous, which in turn implies that \( \Psi_t(\theta) \) is continuous ([29], chapter 3, theorem 7.4). Therefore, since \( \Psi_t \) is of finite variation, we have that \( \mathcal{E}(\Psi_t(\theta)) = \exp(\Psi_t(\theta)) \); in particular, this means that \( \mathcal{E}(\Psi_t(\theta)) \) never vanishes. By definition of local characteristics, we have then that \( M_t(\theta, X_t, T_t) \) is a local martingale for all \( \theta \in \Theta \), and thus it is a martingale if and only if \( \theta \in \Theta_0 \).

**Proof of proposition 4.4.** An immediate consequence of theorem B, is that in the present assumptions, the class of continuous and pure jump martingales are closed under time changing, so that orthogonality follows. Therefore:

\[
\langle X_{T,U} \rangle_t = \langle X^0_t \rangle_t + \langle X^d_t \rangle_t. \tag{9.7}
\]

The fact that \( \langle X^0_t \rangle_t = \langle X^c_t \rangle_t = \Sigma T_t \) is a consequence of Dubins and Schwarz theorem; recalling that

\[
\langle X^d_t \rangle_t = \sum_{s < t}(\Delta X_s)^2 = \int_0^t x^2 \rho(ds \times dx) \tag{9.8}
\]

we have that \( \langle X^d_t \rangle_t = \langle X^d \rangle_{U_t} \) follows from theorem A, part (ii).

**Counterexample to proposition 4.1.** Let \( X^c_t \) be a standard Brownian motion, and let \( T_t \) be an inverse Gaussian subordinator of parameters \( \alpha > 0 \) and \( 1 \), independent of \( X^c_t \). The process \( X^e_t \) is a normal inverse Gaussian process of parameters \( (\alpha, 0, 0, 1) \) and is a pure jump process (Barndorff-Nielsen [1]). Therefore by letting \( X^d_t = X^e_{U_t} \) and \( U_t = t \) we have \( X^e_{U_t} = X^d_t \) so that orthogonality does not hold; moreover \( \langle X_{T,U} \rangle_t = 2\langle X^d_t \rangle_t \) while the left hand side of (4.1) equals \( \langle T_t \rangle + \langle X^0_t \rangle_t \).

**Proof of proposition 4.2.** Since \( T_t \) and \( U_t \) are of finite variation, the total realized variance of an asset as in ([37]) satisfies \( TV_t = -\theta^2_0 \langle X_{T,U} \rangle_t \), so that by proposition 4.1 we have:

\[
TV_t = -\theta^2_0 (\sigma^2 T_t + \langle X^d \rangle_{U_t}). \tag{9.9}
\]
Application of proposition 3.3 to $C_t + D_t$ guarantees that $Q(z, w)$ is a martingale for all $z, w \in \mathbb{C}$ such that $(iz\theta_0, iw\theta_0) \in \Theta_0$. By using relation (6.20) and operating the change of measure entailed by (13.3) we have:

$$
\Phi_{t_0}(z, w) = E_{t_0}[\exp(i z \log(\tilde{S}_t/S_{t_0}) + i w (TV_t - TV_{t_0})]
$$

$$
= \mathbb{E}_{t_0}[\exp(i z(\Delta X^d_{t_0} + \Delta X^d_{t_1}) - \Delta T_t\psi_X(\theta_0) - \Delta U_t\psi_X^2(\theta_0)) - i w\theta_0^2(\sigma^2\Delta T_t + \Delta X^d_{t_0})])
$$

$$
= \mathbb{E}_{t_0}[\exp(i z(\theta_0, i w\theta_0) \cdot (\Delta C_t + \Delta D_t) - \Delta T_t(i z\psi_X^2(\theta_0) + i w\theta_0^2\sigma^2) - \Delta U_t i z\psi_X(\theta_0))]
$$

$$
= \mathbb{E}_{t_0}[\exp(-\Delta T_t(\theta_0m(z - iz) - \theta_0^2\sigma^2(z^2 + iz - 2iw)/2) - \Delta U_t(i z\psi_X(\theta_0) - \psi_D(i z\theta_0, iw\theta_0))).
$$

To fully characterize $\Phi_{t_0}$ all that is left is expressing $\psi_D$ in terms of $\nu$. Since

$$
\psi_D(z, w) = \log E \left[ \exp \left( \sum_{s < t} iz\Delta X^d_s + i w(\Delta X^d_s)^2 \right) \right],
$$

we have that:

$$
\psi_D(z, w) = \int_{\mathbb{R}} \left( e^{izx + iwz^2} - 1 - i(zx + wz^2)I_{|z|\leq 1} \right) \nu(dx).
$$

which finishes the proof. \qed

**Proof of proposition 3.4.** We follow the proof by Lewis [34], theorem 3.2, lemma 3.3 and theorem 3.4. By writing the expectation as an inverse-Fourier integral (which can be done by the assumptions on $F$ and because $\Phi_{t_0}$ is a characteristic function) and passing the expectation under the integration sign we have:

$$
\mathbb{E}_{t_0}[e^{-r(t-t_0)}F(Y_t, Y_{t_0})] = \mathbb{E}_{t_0} \left[ \frac{e^{-r(t-t_0)}}{4\pi^2} \int_{ik_1 - \infty}^{ik_1 + \infty} \int_{ik_2 - \infty}^{ik_2 + \infty} S_t^{-iz}e^{-iw(Y_t)\cdot\hat{F}(z, w)}dzw \right]
$$

$$
= \frac{e^{-r(t-t_0)}}{4\pi^2} \int_{ik_1 - \infty}^{ik_1 + \infty} \int_{ik_2 - \infty}^{ik_2 + \infty} e^{-iw(Y_t)\cdot\hat{F}(z, w)} \hat{F}(z, w)dzw.
$$

(9.12)

All that remains to be proven is that Fubini’s theorem application is justified. Let $N_t = M_t(\theta_0, X_t, (T_t, U_t))$ be the discounted, normalized log-price; define the probability transition densities $p_t(x, y) = \mathbb{P}(N_t < x, (N_t) < y) | t_0, N_{t_0}, (N)_{t_0})I_{x \in \mathbb{R}, y \geq (N)_{t_0}}$ and let $\hat{p}_t(z, w)$ be their characteristic functions. For all $(z, w) \in L_{k_1, k_2}$ we have:

$$
\int_{ik_1 - \infty}^{ik_1 + \infty} \int_{ik_2 - \infty}^{ik_2 + \infty} \left[ e^{-iw(Y_t)\cdot\hat{F}(z, w)} \right] dzw = \int_{ik_1 - \infty}^{ik_1 + \infty} \int_{ik_2 - \infty}^{ik_2 + \infty} \hat{p}_t(-z, w) \hat{F}(z, w)dzw.
$$

(9.13)

and

$$
\int_{R^2} \hat{p}_t(-z + ik_1, -w + ik_2) \hat{F}(z + ik_1, w + ik_2)dzw.
$$

(9.14)

For $x \in \mathbb{R}, y \geq 0$, set $f(x, y) = e^{-k_1x-k_2y}F(x, y)$ and $g(x, y) = e^{k_1x+k_2y}p_t(x, y)$. We see that the integrand in the right-hand side of (9.13) equals $\hat{g}^*(z, w)\hat{f}(z, w)$. But now $f$ is $L^1(dx \times dy)$ because $F$ is Fourier-integrable in $\Sigma_F$ (for $(z, w) \in \Sigma_F$ take $Re(z) = Re(w) = 0$); similarly, $\hat{g}^*$ is $L^1(dx \times dw)$ because of the $L^1$ assumption on $\Phi_{t_0}$. Therefore, application of Parseval’s formula yields:

$$
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \hat{p}_t(-z + ik_1, -w + ik_2) \hat{F}(z + ik_1, w + ik_2)dzw = 4\pi^2 \mathbb{E}_{t_0}[F(N_t, (N)_{t_0})] < +\infty,
$$

(9.15)

since $F \in L_{k_0}^1(N_t, (N)_{t_0})$. \qed
Proof of the equations of section\[^{[2]}\] We can endow $Y_t$ with a correlation structure as follows. Let $Z_t$ be a two-dimensional matrix Brownian motion independent of $W_t$. The matrix process:

$$B_t = \begin{pmatrix} \rho W_t^1 + \sqrt{1 - \rho^2} Z_t^{1,1} & Z_t^{1,2} \\ \rho W_t^2 + \sqrt{1 - \rho^2} Z_t^{1,1} & Z_t^{2,2} \end{pmatrix}$$

(9.16)

is also a matrix Brownian motion enjoying the property that $\langle W_j, B^{j,1} \rangle_t = \rho t$ and $W_t$ is independent of $B_t^{j,2}$ for $j = 1, 2$. Since $\Sigma_t^{i,j} = (\sigma_t^{i,j})^2 + (\sigma_t^{i,2})^2$, we have that $X_t^i$ is indeed a Brownian motion and the activity rates are connected through the element $\sigma_t^{i,2}$.

To verify equations (7.4) and (7.5), observe that for some bounded variation processes $S_t^j$ it is

$$d\Sigma_t^{i,j} = S_t^j dt + 2\sigma_t^{i,j}(Q^{1,j} dB_t^{1,1} + Q^{2,j} dB_t^{1,2}) + 2\sigma_t^{i,2}(Q^{1,j} dB_t^{2,1} + Q^{2,j} dB_t^{2,2}),$$

(19.17)

whence:

$$dw_t^j := \frac{d\Sigma_t^{i,j} - S_t^j dt}{2\sqrt{\Sigma_t^{i,j}((Q^{1,j})^2 + (Q^{2,j})^2)}} = \frac{\sigma_t^{i,j}(Q^{1,j} dB_t^{1,1} + Q^{2,j} dB_t^{1,2}) + \sigma_t^{i,2}(Q^{1,j} dB_t^{2,1} + Q^{2,j} dB_t^{2,2})}{\sqrt{\Sigma_t^{i,j}((Q^{1,j})^2 + (Q^{2,j})^2)}}$$

(19.18)

By taking the quadratic variation on the right hand side we see that $w_t^j$ are two Brownian motions such that $d\Sigma_t^{i,j} = S_t^j dt + 2\sqrt{\Sigma_t^{i,j}((Q^{1,j})^2 + (Q^{2,j})^2)} dw_t^j$; equations (7.4) and (7.5) then follow from a direct computation.

Since $X_t^i$ is orthogonal to $\Sigma_t$, the change in the dynamics of $\Sigma_t$ under $Q(z, w)$ is only due to the correlation between $X_t^i$ and $B_t$. Hence, for $(z, w) \in \Theta$, the Radon-Nikodym derivative $M_t$ to be considered in (5.3) reduces to

$$M_t = \mathcal{E} \left( iz \int_0^t \sqrt{\Sigma_s^{i,1}} dX_s^z \right).$$

(9.19)

Furthermore, for $j = 1, 2$ we have:

$$d \left( \int_0^t \sqrt{\Sigma_s^{i,1}} dX_s^z, B_t^{j,1} \right)_t = \rho \sigma_t^{i,j} dt$$

(9.20)

$$d \left( \int_0^t \sqrt{\Sigma_s^{i,1}} dX_s^z, B_t^{j,2} \right)_t = 0$$

(9.21)

so that application of Girsanov’s theorem tells us that

$$dB_t = dB_t - iz \rho \left( \begin{array}{cc} \sigma_t^{i,1} dt & 0 \\ \sigma_t^{i,2} dt & 0 \end{array} \right)$$

(9.22)

is a $Q(z, w)$-matrix Brownian motion. Solving for $B_t$ and substituting in (7.1) yields (7.6). Equation (7.8) then follows from (4.1).

Finally we give the formula for $\mathcal{L}_{T_i, t_i}(\cdot)$. For $\tau > 0$ and $n > 1$ consider the transform:

$$\phi_\Sigma(z) = \mathbb{E} \left[ \exp \left( -\int_0^\tau \sum_{j=1}^n z_j \Sigma_s^{i,j} ds \right) \right]$$

(9.23)

for every vector of complex numbers $z = (z_1, \ldots, z_n)$ such that the above expectation is finite. The function $\phi_\Sigma(z)$ is exponentially-affine of the form

$$\phi_\Sigma(z) = \exp(-a(\tau) - Tr(A(\tau) \Sigma_0)),$$

(9.24)

since it is a particular case of the transforms studied in e.g. Grasselli and Tebaldi, \[^{[22]}\] and
Gnoatto and Grasselli [21]. The ODEs for $A(\tau), a(\tau)$ are given by:

\[
A(\tau)' = A(\tau)M + M^T A(\tau) - 2A(\tau)Q^T QA(\tau) + D, \quad A(0) = 0 \tag{9.25}
\]

\[
a(\tau)' = Tr(cQ^T QA(\tau)), \quad a(0) = 0. \tag{9.26}
\]

Here $D$ is the diagonal matrix having the values $z_1, \ldots, z_n$ on the diagonal. The solution of (9.25)-(9.26) is obtainable by a linearization procedure which entails doubling the dimension of the problem, yielding:

\[
A(\tau) = \left( A^{1,2}(\tau), A^{2,1}(\tau) \right) \tag{9.27}
\]

\[
a(\tau) = \frac{c}{2} Tr(\log (A^{2,2}(\tau)) + M^T \tau) \tag{9.28}
\]

\[
\left( \begin{array}{cc} A^{1,1}(\tau) & A^{1,2}(\tau) \\ A^{2,1}(\tau) & A^{2,2}(\tau) \end{array} \right) = \exp \left( \tau \left( \frac{M}{2} 2Q^T Q \right) \right), \tag{9.29}
\]

see e.g. [21], proposition 9, or [22], section 3.4.2. The formula for $\mathcal{L}_{\Delta T, \Delta U}$ follows from (9.27)-(9.29) after having chosen $n = 2$, $(z_1, z_2) = (z, w)$ in (9.23), and set $\tau = t - t_0$ and $\Sigma_0 = \Sigma_{t_0}$ in (9.24).

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Tables

Table 1: Parameters from the S&P estimations of Fang [13], section 4.

| Parameters | Black-Scholes | Heston | Merton | Bates | Fang |
|------------|---------------|--------|--------|-------|------|
| $\sigma_{t_0}$ | 0.14 | 0.15 | 0.12 | 0.15 | 0.14 |
| $\alpha$ | 4.57 | 8.93 | 6.5 |
| $\theta$ | 0.0306 | 0.0167 | 0.0104 |
| $\eta$ | 0.48 | 0.22 | 0.2 |
| $\rho$ | -0.82 | -0.58 | -0.48 |
| $\lambda_0$ | 1.42 | 0.39 | 0.41 |
| $\delta$ | 0.0894 | 0.1049 | 0.2168 |
| $\kappa$ | -0.075 | -0.11 | -0.21 |
| $\alpha\lambda$ | 5.06 |
| $\theta\lambda$ | 0.13 |
| $\eta\lambda$ | 1.069 |

Table 2: $S_{t_0} = 100, K = H = 80, Q = 0.05, t_0 = 0, t = 1, r = 0.06, \bar{r} = 0.1, TV_{t_0} = 0.$

| Model | Vanilla Call | Volatility Call | TVO Call |
|-------|--------------|----------------|---------|
|       | AV | MC | AV | MC | AV | MC |
| B-S   | 24.7627 | 24.7775(0.05%) | 0.0847 | 0.0848(0.12%) | 17.5441 | 17.6982(0.87%) |
| Heston | 25.3893 | 25.3710(0.07%) | 0.1088 | 0.1084(0.37%) | 17.2248 | 17.6044(2.16%) |
| Merton | 25.3243 | 25.2290(0.38%) | 0.1192 | 0.1194(0.17%) | 17.7529 | 17.7922(0.22%) |
| Bates  | 25.1166 | 25.0889(0.11%) | 0.1002 | 0.1005(0.30%) | 18.5980 | 18.7480(0.80%) |
| Fang   | 25.5686 | 25.6508(0.32%) | 0.0907 | 0.0892(1.68%) | 24.0494 | 24.0764(0.11%) |
Table 3: $S_0 = 100, K = H = 120, Q = 0.1, t_0 = 0.5, t = 4, r = 0.039, \sigma = 0.1, TV_0 = 0.018.$

| Model | Vanilla Call | Volatility Call | TVO Call |
|-------|--------------|-----------------|---------|
|       | AV | MC | AV | MC | AV | MC |
| B-S   | 8.4801 | 8.4784 (0.02%) | 0.1672 | 0.1695 (1.36%) | 5.7622 | 5.6957 (1.17%) |
| Heston | 10.3063 | 10.3023 (0.04%) | 0.2167 | 0.2172 (0.23%) | 6.3815 | 6.7080 (4.87%) |
| Merton | 11.5845 | 11.5713 (0.11%) | 0.2357 | 0.2356 (0.04%) | 7.4564 | 7.4239 (0.44%) |
| Bates | 9.8607 | 9.8371 (0.24%) | 0.2002 | 0.2001 (0.05%) | 6.8180 | 6.9085 (1.31%) |
| Fang | 8.8630 | 8.8737 (0.12%) | 0.1827 | 0.1828 (0.05%) | 7.4173 | 7.5046 (1.16%) |

Table 4: $S_0 = 100, K = H = 100, Q = 0.25, t_0 = 1.25, t = 1.5, r = 0.072, \sigma = 0.1, TV_0 = 0.23.$

| Model | Vanilla Call | Volatility Call | TVO Call |
|-------|--------------|-----------------|---------|
|       | AV | MC | AV | MC | AV | MC |
| B-S   | 3.7627 | 3.7346 (1.02%) | 0.2300 | 0.2305 (0.22%) | 0.9771 | 0.9437 (3.54%) |
| Heston | 4.1390 | 4.1304 (0.21%) | 0.2318 | 0.2320 (0.09%) | 1.0480 | 1.0451 (0.28%) |
| Merton | 4.4169 | 4.4435 (0.60%) | 0.2348 | 0.2320 (0.21%) | 1.1254 | 1.1235 (0.17%) |
| Bates | 4.1842 | 4.1687 (0.37%) | 0.2327 | 0.2328 (0.04%) | 1.0593 | 1.0544 (0.46%) |
| Fang | 4.3219 | 4.3420 (0.46%) | 0.2362 | 0.2362 (0.00%) | 1.0919 | 1.0987 (0.62%) |

Table 5: $S_0 = 100, K = H = 60, Q = 0.2, t_0 = 3, t = 5, r = 0.0225, \sigma = 0.1, TV_0 = 0.19.$

| Model | Vanilla Call | Volatility Call | TVO Call |
|-------|--------------|-----------------|---------|
|       | AV | MC | AV | MC | AV | MC |
| B-S   | 42.6506 | 42.6452 (0.01%) | 0.2670 | 0.2665 (0.19%) | 19.7252 | 19.9181 (0.96%) |
| Heston | 42.9595 | 43.0010 (0.10%) | 0.2859 | 0.2858 (0.03%) | 19.8454 | 19.6512 (0.99%) |
| Merton | 42.8984 | 42.8580 (0.09%) | 0.2955 | 0.2954 (0.03%) | 19.4192 | 19.3975 (0.11%) |
| Bates | 42.7768 | 42.7928 (0.04%) | 0.2804 | 0.2802 (0.07%) | 19.8042 | 19.8318 (0.14%) |
| Fang | 43.0039 | 43.0252 (0.05%) | 0.2793 | 0.2791 (0.07%) | 20.5992 | 20.5998 (0.01%) |

Table 6: $S_0 = 100, K = H = 130, Q = 0.015, t_0 = 1, t = 2.5, r = 0.087, \sigma = 0.1, TV_0 = 0.009.$

| Model | Vanilla Call | Volatility Call | TVO Call |
|-------|--------------|-----------------|---------|
|       | AV | MC | AV | MC | AV | MC |
| B-S   | 2.3393 | 2.3080 (1.36%) | 0.1590 | 0.1588 (0.13%) | 1.9535 | 1.8622 (4.90%) |
| Heston | 2.5095 | 2.5071 (0.11%) | 0.1852 | 0.1862 (0.54%) | 2.2190 | 2.1317 (4.10%) |
| Merton | 3.7078 | 3.6843 (0.64%) | 0.1983 | 0.1981 (0.10%) | 3.0330 | 3.0165 (0.55%) |
| Bates | 3.7416 | 3.7380 (0.13%) | 0.1767 | 0.1769 (0.11%) | 2.3727 | 2.3798 (0.30%) |
| Fang | 1.9814 | 1.9410 (2.08%) | 0.1664 | 0.1668 (0.24%) | 1.9455 | 1.9167 (1.49%) |