Boltzmann or Bogoliubov?  
Approaches Compared in Gravitational Particle Production  
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Gravitational particle production is a minimal contribution to reheating the Universe after the end of inflation. To study this production channel, two different approaches have commonly been considered, one of which is based on the Boltzmann equation, and the other is based on the Bogoliubov transformation. Each of these has its pros and cons in practice. The collision term in the Boltzmann equation can be computed based on quantum field theory in the Minkowski spacetime, and thus many techniques have been developed so far. On the other hand, the Bogoliubov approach may deal with the particle production beyond the perturbation theory and is able to take into account the effect of the curved spacetime, whereas in many cases one should rely on numerical methods, such as lattice computation. We show by explicit numerical and analytical computations of the purely gravitational production of a scalar that these two approaches give consistent results for particle production with large momenta during reheating, whereas the Boltzmann approach is not capable of computing particle production out of vacuum during inflation. We also provide analytic approximations of the spectrum of produced scalar with/without mass for the low momentum regime obtained from the Bogoliubov approach.

I. INTRODUCTION

Reheating is the necessary phase that should be realized in any inflationary models of modern cosmology [1–7]. After the exponential expansion of space, the Universe is fulfilled by radiation, following the reheating era during which particles of the Standard Model are produced. The thermal plasma of the produced particles sets the stage of Big Bang Nucleosynthesis by requiring that the reheating temperature, $T_{RH}$, is greater than $\sim 1$ MeV [8–34]. If one supposes that the baryon asymmetry of the Universe is generated through lepton asymmetry [35–42], $T_{RH}$ needs to be greater than the critical temperature of the electroweak phase transition [43, 44].

For the Universe to be dominated by radiation after the reheating completes, a sufficient amount of energy stored in the inflaton sector should be transferred into the thermal bath. The energy transfer is most commonly realized in perturbative inflaton decay through a direct coupling between inflaton and matter particles. In general, the decay of inflaton does not occur instantaneously [61–65]. Such non-instantaneous inflaton decay leads to non-trivial evolution of the temperature and the equation of state until reaching the radiation-dominated era. Moreover, the time evolution of the inflaton energy density strongly depends on the shape of the inflaton potential during the reheating [66, 67] as well. The thermalization of the produced particles is often assumed to be instantaneous in the literature, whereas it is not always the case that the thermalization is achieved at once after particle production happens from, for instance, the inflaton decay [68–73].

Both reheating and subsequent thermalization may have direct impact on new physics beyond the Standard Model such as generating dark matter particles, especially a so-called freeze-in massive particle (FIMP) dark matter [74–91]. Among various portal couplings to produce FIMP out of thermal bath, the graviton exchange is the minimal contribution [92–98], namely, annihilation of the thermal bath particles to produce a pair of dark matter through a single graviton exchange. However, this contribution turns out to be subdominant, compared to the inflaton annihilation, instead of the thermal bath particles, to a pair of dark matter by exchanging a graviton [99–114].

In this context, gravitational particle production has gained renewed attention since the first study of particle creation in curved spacetime has been done long ago [115]. Since then, particle production has been studied based on quantum field theory in curved spacetime by utilizing the Bogoliubov transformation, which we will call the "Bogoliubov approach" in the following. So far, this is the most developed framework to keep track of the evolution of particle spectrum produced during inflation and reheating. The same approach has been applied to compute the gravitational dark matter production from inflaton oscillation during the reheating [99–107].

On the other hand, in the context of dark matter
production, it is more conventional to compute particle production based on the Boltzmann equation with the collision terms originally obtained in the framework of quantum field theory in flat spacetime, which we will call the "Boltzmann approach". This is because in many circumstances dark matter production takes place when the Universe undergoes power expansion (such as matter-dominated or radiation-dominated epoch), and hence quantum field theory in Minkowski spacetime becomes a good approximation. This framework has been used to compute the gravitational dark matter production [92–98, 108, 111, 113].

The Bogoliubov and Boltzmann approaches are supposed to give identical results in certain cases of particle production, especially when dark matter is dominantly produced. However, it is far from clear to what extent this claim is justified because these two approaches are based on different premises. In the Bogoliubov approach, particle production takes place since the vacuum is in general time-dependent in curved spacetime. By keeping track of the time evolution of the vacuum, one may observe the particle production out of the vacuum. It is also important to note that it is not always the case that the particle production is analytically computable for a given spacetime geometry.

In contrast to the Bogoliubov approach, the vacuum being a constant of time is a prerequisite in the Boltzmann approach. The particle production is instead described by scatterings of fields on Minkowskian geometry, and thus the analytic computation is much easier than the Bogoliubov approach in many cases. Therefore, at the first glance, there is no guarantee that these different approaches give the same physics result.

In this paper, we compare these two approaches by considering the gravitational production of a minimally coupled real scalar. We develop an analytic method to show that these approaches are certainly equivalent for the produced scalar particle whose momentum is greater than the inflaton mass. On the other hand, for smaller momentum ranges, the production rate of the particles in two formalisms are significantly different, because of large non-adiabacity of the regime. Through further analytic computation of Bogoliubov approach, we derive the power-law spectrum of the particle produced out of the vacuum during reheating, which is the contribution beyond the Boltzmann approach.

The paper is organized as follows. In section II we explain the setup to discuss the gravitational production of a scalar field during the reheating, in the Boltzmann approach in section III and in the Bogoliubov approach in section IV. After discussing the vacuum contribution to gravitational particle production in section V, we summarize our results and address possible applications in section VI.

II. SET-UP

Before getting into the detail of the comparison between the Boltzmann and Bogoliubov approaches, we set a stage for what we will consider of gravitational particle production in each approach. Our framework consists of two parts: the Einstein-Hilbert gravity (spin-2) and a minimally coupled real scalars \( \phi \) and \( \chi \) (spin-0), i.e., \( S = S_2 + S_0 \) where

\[
S_2 = -\frac{M_P^2}{2} \int d^4x \sqrt{-g} R, \quad (1)
\]

\[
S_0 = \int d^4x \sqrt{-g} \left[ \mathcal{L}_\phi + \mathcal{L}_\chi \right], \quad (2)
\]

\[
\mathcal{L}_\phi = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi), \quad (3)
\]

\[
\mathcal{L}_\chi = \frac{1}{2} g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - \frac{1}{2} m_\chi^2 \chi^2. \quad (4)
\]

Here, we introduced the reduced Planck mass \( M_P = 1/\sqrt{8\pi G} \approx 2.4 \times 10^{18} \text{ GeV} \) with \( G \) the Newton constant, and \( R \) is the Ricci scalar as a function of the Riemannian metric \( g_{\mu\nu} \), whose determinant is denoted by \( g \equiv \det(g_{\mu\nu}) \).2

In the spin-0 sector, \( \phi \) is the inflaton with potential \( V(\phi) \), and \( \chi \) is a spectator scalar field produced from the inflaton during the reheating. While our analytic results are quite independent to specific form of the potential, we take so-called T-model [116] potential as an explicit example for numerical results:

\[
V(\phi) = 6\lambda M_P^4 \tanh^2 \left( \frac{\phi}{\sqrt{6} M_P} \right). \quad (5)
\]

Note that after the end of inflation, \( \phi \) oscillates about \( \phi = 0 \) with a quadratic form of potential

\[
V(\phi \ll M_P) \approx \frac{1}{2} m_\phi^2 \phi^2 \quad (6)
\]

with \( m_\phi^2 \equiv 2\lambda M_P^2 \). The potential coupling \( \lambda \) is determined such that the amplitude of the curvature power spectrum \( A_S \) is obtained at a number of e-folds \( N \) through [67]

\[
\lambda \approx \frac{18\pi^2 A_S}{6N^2}. \quad (7)
\]

In our analysis, we take \( \ln(10^{10} A_S) = 3.044 \) for the pivot scale \( k = 0.05 \text{ Mpc}^{-1} \) [117] and assume \( N = 55 \), yielding \( \lambda \approx 2.1 \times 10^{-11} \) and \( m_\phi \approx 1.5 \times 10^{13} \text{ GeV} \).

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2 Throughout the paper we work in the sign convention of mostly-minus metric, i.e., \( \eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1) \) for the Minkowskian metric \( \eta_{\mu\nu} \).
III. PARTICLE PRODUCTION IN THE BOLTZMANN APPROACH

The Boltzmann approach (when computing the collision term) assumes that the background geometry is Minkowskian, in which the gravitational production of $\chi$ takes place through $\phi \phi \rightarrow \chi \chi$ by exchanging a single graviton. Note that this process exists during the reheating when $\phi$ coherently oscillates, but it disappears once the reheating completes as $\phi$ decays away. It should be emphasized that the initial state $\phi$ is not a single particle, but a coherently oscillating Bose-Einstein condensate. Namely, $\phi$ does not have spatial momentum as if it sits at its rest frame.

To consider the gravitational particle production, we should start with defining the graviton field $h_{\mu\nu}$ by linearizing the gravity sector as $g_{\mu\nu} \simeq \eta_{\mu\nu} + 2 h_{\mu\nu}/M_P$. By introducing the stress-energy-momentum tensor $T_{\mu\nu} \equiv (2/\sqrt{-g}) \delta S/\delta g^{\mu\nu}$, the action may be written as

$$S \simeq \int d^4x \left[ L_{h_{\phi \phi}} + L_{h_{\phi \chi}} + \frac{1}{M_P^2} h_{\mu\nu} (T_{\phi \phi}^\mu(T_{\chi \chi}^\nu) + T_{\phi \chi}^\mu(T_{\chi \chi}^\nu) \right],$$

(8)

where the graviton kinetic term $L_{h_{\phi \phi}}$ is given by

$$L_{h_{\phi \phi}} = -\frac{1}{2} h_{\mu\nu} P_{\mu\nu \alpha \beta} \eta^{\alpha \beta},$$

(9)

$$P_{\mu\nu \alpha \beta} \equiv \eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\beta} \eta_{\nu\alpha} - \eta_{\mu\nu} \eta_{\alpha\beta},$$

(10)

with $\square \equiv \eta^{\mu\nu} \partial_{\mu} \partial_{\nu}$, leading to the graviton propagator for a momentum $q$: $\Pi_{\mu\nu \alpha \beta}(q) = P_{\mu\nu \alpha \beta}/q^2$.

The stress-energy-momentum tensor for scalar $\chi$ is given by

$$T_{\chi \chi}^\mu = \partial_\mu \chi \partial_\nu \chi - \frac{1}{2} \eta_{\mu\nu} (\partial^\alpha \chi \partial_\alpha \chi - m_\chi^2 \chi^2).$$

(11)

For $T_{\phi \phi}^\mu$, since $\phi$ is a coherently oscillating condensate, it is convenient to define the Fourier series by

$$T_{\phi \phi}^\mu = \sum_{n=\pm \infty} T_{n,\mu \nu} e^{-i n \omega t},$$

(12)

where

$$T_{n,\mu \nu} = 2 K_n u_\mu u_\nu - \eta_{\mu\nu} (K_n - V_n),$$

(13)

$$u_\mu \equiv (1, 0, 0, 0)_\mu,$$

(14)

with $K_n$ and $V_n$ representing the Fourier coefficients for the kinetic and potential terms. Note that for $V = (1/2)m_\phi^2 \phi^2$, we have $\omega = m_\phi$.

For the $n$-th mode of the $\phi$ oscillation, we may write the transition amplitude of $\phi \phi \rightarrow h_{\mu\nu} \rightarrow \chi(p_A) \chi(p_B)$, $M_n$, as

$$M_n = \frac{1}{M_P^2} T_{\phi \phi}^\mu \Pi_{\mu\nu \alpha \beta} (n\omega) \times \left[ p_{A\alpha} p_{B\beta} + p_{A\beta} p_{B\alpha} - \eta_{\alpha\beta} (p_A \cdot p_B + m_\chi^2) \right] = \frac{V_n}{M_P^2} \left( 1 + \frac{2 m_\chi^2}{(n\omega)^2} \right).$$

(15)

Transition probability with the amplitude $M_n$ may be defined by

$$dP_{\phi \phi \rightarrow h_{\mu\nu}}^{(n)} \equiv \frac{d^4 p_A d^4 p_B}{(2\pi)^3 2 p_A^0 (2\pi)^3 2 p_B^0} |M_n|^2 \times (2\pi)^4 \delta(n\omega - p_A^0 - p_B^0) \delta(p_A + p_B).$$

(16)

Notice that $n \leq 0$ does not contribute due to the energy conservation. From this, for instance, the $\chi$ number density production rate through $\phi \phi \rightarrow \chi \chi$ can be computed by

$$R_{\phi \phi \rightarrow \chi \chi} = \int dP_{\phi \phi \rightarrow \chi \chi}^{(n)},$$

(17)

which is the right hand side of the Boltzmann equation given by

$$\dot{n}_\chi + 3 H n_\chi = R_{\phi \phi \rightarrow \chi \chi}^{(n)}$$

(18)

with $\dot{n}_\chi \equiv dn_\chi/dt$. Note that on the right-hand side of the above equation, we should multiply by 2 as two $\chi$ particles are produced per reaction and should multiply by $1/2$ of the symmetry factor as the final state $\chi$'s are identical particles, resulting in cancellation of these factors.

Since $V \propto \phi^2$, $\phi$ is identified as a simple harmonic oscillator whose solution may be written as $\phi = \phi_0 \cos(m_\phi t)$ with an amplitude $\phi_0$. Given that the energy density of $\phi$ is given by $\rho_\phi = V(\phi_0)$ and $V_n = \rho_\phi/4$ for $n = 2$, we end up with

$$R_{\phi \phi \rightarrow \chi \chi}^{(n)} = \frac{\rho_\phi^2}{128 \pi M_P^4} \left( 1 + \frac{m_\chi^2}{2 m_\phi^2} \right)^2 \sqrt{1 - \frac{m_\chi^2}{m_\phi^2}},$$

(19)

which is consistent with the results in Refs. [108, 111].

To find the distribution function of the final state particle $A$ produced through $\phi \phi \rightarrow A(p_A)B(p_B)$, we should solve the Boltzmann equation given by

$$\frac{\partial f_A}{\partial t} - H |\vec{p}_A| \frac{\partial f_A}{\partial |\vec{p}_A|} = C[f_A]$$

(20)

with the collision term

$$C[f_A] = \sum_{n=1}^{\infty} \frac{1}{2 p_A^0} \int \frac{d^3 p_B}{(2\pi)^3 2 p_B^0} |M_n|^2 \times (2\pi)^4 \delta(n\omega - p_A^0 - p_B^0) \delta(\vec{p}_A + \vec{p}_B),$$

(21)

$$= \sum_{n=1}^{\infty} \frac{\pi}{(n\omega)^2} \delta \left( \frac{|\vec{p}_A| - n \omega}{2 \beta_n} \right),$$

(22)

where

$$\beta_n = \sqrt{1 - \left( \frac{m_A + m_B}{(n\omega)^2} \right) \left( 1 - \frac{(m_A - m_B)^2}{(n\omega)^2} \right)}.\]
When the backward reaction $AB \rightarrow \phi \phi$ is negligible and $f_A, f_B \ll 1$, the Boltzmann equation has a generic solution given by

$$f_A(\vec{p}_A|, t) = \int_{t_r}^t dt' C[f_A](|\vec{p}_A| \times a(t)/a(t'), t'), \quad (23)$$

where $a$ is the scale factor and $t_r$ is an arbitrary reference time [60, 72, 118]. By plugging Eq. (15) into $C[f_A]$, $f_\chi$ is readily obtained as

$$f_\chi(|\vec{p}_\chi|, t) = \frac{9\pi}{64} \left( \frac{H_e}{m_\phi} \right)^3 \left( \frac{m_\phi}{|\vec{p}_\chi|(a/a_e)} \right)^{9/2} \times \left( 1 - \frac{m_\phi^2}{m_\phi^2} \right)^{5/4} \left( 1 + \frac{m_\phi^2}{2m_\phi^2} \right)^2, \quad (24)$$

where we have used $3M_P^2 H^2 \simeq \rho_\phi = \rho_e(a/a_e)^{-3}$ with $a_e$ the scale factor at which the inflation ends and $\rho_e$ is the energy density at $a = a_e$. In the same manner, $H_e$ is the Hubble parameter at $a = a_e$. Notice that $|\vec{p}_\chi|(a/a_e)$ is a comoving momentum and a constant of time.

### IV. THE HIGHER MODE SPECTRUM IN THE BOGOLIUBOV APPROACH

Instead of utilizing the transition probability to describe the particle production, in the Bogoliubov approach, one may deal with the time evolution of the wave function of the produced particle while keeping the effect of curved spacetime. To do so, it is convenient to introduce the conformal time $\eta$ as $d\eta = dt$ so that the metric is given as $g_{\mu\nu} = a^2 g_{\mu\nu}$.

Integrating by parts, we may rewrite the action of $\chi$ as

$$S_\chi = \int d^4x \left[ \frac{1}{2} (\nabla \chi')^2 - \frac{1}{2} \chi \omega^2 \chi \right], \quad (25)$$

$$\omega^2 \equiv -\nabla^2 + a^2 m_\chi^2 + \Delta, \quad (26)$$

$$\Delta \equiv \frac{a''}{a} = \frac{1}{6} a^2 R, \quad (27)$$

where $\chi$ is a rescaled field given by $\chi = a^{-1} \bar{\chi}$, and $' \equiv d/d\eta$. The equation of motion is thus given by

$$\chi'' + \omega^2 \chi = 0. \quad (28)$$

In this field definition, the conjugate momentum of $\chi$ is defined by $\pi \equiv \partial L_\chi/\partial \chi'$, so that the Hamiltonian is given by

$$H = \int d^3x [\pi \chi' - L] = \int d^3x \left[ \frac{1}{2} \pi^2 + \frac{1}{2} \chi \omega^2 \chi \right]. \quad (29)$$

From Eq. (26), it is clear that the Hamiltonian is changing with time through the time dependence in $\omega$.

Therefore, we cannot decompose $\bar{\chi}$ based on the positive/negative frequency in the Fourier space as is normally done in the flat spacetime quantum field theory. Instead, for the mode function $\bar{\chi}_k$ defined by

$$\bar{\chi}(x) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\vec{k} \cdot \vec{x}} \bar{\chi}_k, \quad (30)$$

with $a_\phi$ and $a_\phi^\dagger$ being annihilation and creation operators, respectively, we decompose $u_k$ as

$$u_k = \frac{A_k}{\sqrt{2\omega_k}} e^{-i f \omega_k d\eta} + \frac{B_k}{\sqrt{2\omega_k}} e^{i f \omega_k d\eta}, \quad (32)$$

where $\omega_k^2 = \Delta + 2m_\chi^2 + a^2 m_\chi^2 + \Delta$. Time-dependent coefficients introduced in this way (here $A_k$ and $B_k$) are called Bogoliubov coefficients. For latter convenience, we define $\alpha_k = A_k e^{-i f \omega_k d\eta}$, $\beta_k = B_k e^{i f \omega_k d\eta}$, (33) with which we parametrize the conjugate momentum as

$$\bar{\pi}(x) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\vec{k} \cdot \vec{x}} \bar{\chi}_k, \quad (34)$$

$$\bar{\pi}_k = a_\phi^\dagger u_k + a_\phi^\dagger v_k, \quad u_k = \frac{-i \omega_k \alpha_k}{\sqrt{2\omega_k}} + \frac{i \omega_k \beta_k}{\sqrt{2\omega_k}}. \quad (35)$$

To satisfy the equation of motion $\bar{\pi}' + \omega_\phi^2 \bar{\chi}_k = 0$ as well as the definition $\bar{\chi}_k = \bar{\chi}_k$, the coefficients are required to obey

$$\alpha'_k = -i \omega_k \alpha_k + \frac{\omega_k}{2\omega_k} \beta_k, \quad (36)$$

$$\beta'_k = i \omega_k \beta_k + \frac{\omega_k}{2\omega_k} \alpha_k. \quad (37)$$

Note that we have not assumed any slowly varying nature for $A_k$ and $B_k$ nor for $\alpha_k$ and $\beta_k$. Their time-dependence is fully taken into account and no approximation is made. Since the occupation number is given by $|\beta_k|^2$ (equivalent to $f_k(|\vec{p}_\chi|, t)$ in the Boltzmann approach), the particle production can be seen by solving these coupled differential equations for $\beta_k$. By setting $\alpha_k = 1$ and $\beta_k = 0$ as the initial condition, $\alpha_k(\eta) = e^{-i \omega_k \eta}$ and $\beta_k(\eta) = 0$ when $\eta_0 \equiv 0$, which corresponds to the case of a plane wave solution in the flat spacetime, namely, non-existence of particles in the initial state. In particular, if $|\beta_k|^2 \ll 1$ all the time (which is the case of the gravitational production with large $k$), then we may write

$$\beta_k \simeq \int_{\eta_0}^\eta d\eta' \frac{\omega_k'}{2\omega_k} e^{-2i\omega_k(\eta')}, \quad (38)$$

$$\Omega_k(\eta) \equiv \int_{\eta_0}^\eta d\eta' \omega_k(\eta'), \quad (39)$$

where $t_i$ is taken as some reference time where the initial condition is set.
As the gravitational effect is induced by the background field dynamics through $\Delta$, we need to take a closer look at the inflaton dynamics during reheating. By solving the equation of motion with the Hubble friction term,

$$\ddot{\phi} + 3H \dot{\phi} + m_\phi^2 \phi = 0,$$

we find a solution of the form

$$\phi(t) = \frac{\phi_e}{m_\phi t} \sin(m_\phi t)$$

(41)

where $\phi_e$ is the inflaton amplitude at the end of inflation, and we have assumed $H \approx 3/2t$. By identifying $\phi_0 = \phi_e/m_\phi t$, the slowly varying component of energy density, $\bar{\rho}_\phi$, is given by $\bar{\rho}_\phi = V(\phi_0)$, and accordingly, we define $3M_\Delta^2 H^2 \equiv \bar{\rho}_\phi$. It should be noted that the fast oscillating mode in $\rho_\phi$ quickly becomes subdominant as time increases, whereas in the pressure, it is always the dominant contribution as the slowly varying component gives zero pressure. This fast oscillating component in the pressure plays a central role in gravitational production.

At leading order, we obtain

$$\Delta = \frac{1}{6} a^2 R \approx -\frac{1}{2} (aH)^2 [1 - 3 \cos(2m_\phi t)],$$

(42)

$$\Delta' \approx -(aH)^3 \left[ \frac{\Delta}{(aH)^2} - \frac{3m_\phi}{H} \sin(2m_\phi t) \right],$$

(43)

which are used in evaluating Eq. (38) to find

$$\varphi^\pm(\eta_k) = 0$$

results in

$$a_k m_\phi = \omega_k(\eta_k) \approx \sqrt{k^2 + a_k^2 m_\chi^2} \Rightarrow a_k \approx \sqrt{\frac{k^2}{m_\phi^2 - m_\chi^2}},$$

(48)

where $a_k \equiv a(\eta_k)$. Note that we require that $\eta_k < \eta_k < \eta$, otherwise the integrand is highly oscillating and is strongly damped to yield null contribution. We also obtain

$$\varphi^+(\eta_k) \approx 2k^2 \frac{H_k}{m_\phi},$$

(49)

where $H_k \equiv \dot{H}(a_k)$. Notice that $e^{\varphi^+}$ highly oscillates except for the vicinity at $\eta = \eta_k$ when $\varphi^+(\eta_k) \gg 1/(\eta - \eta_k)^2$, and thus the integrand with $\eta$ far from $\eta_k$ cancels in the integration, leaving the integrand with $\eta = \eta_k$. Therefore, we obtain

$$\beta_{k \gg aH} \approx \frac{3}{8} (a_k H_k)^3 \frac{m_\phi / H_k}{k^2 + a_k^2 m_\chi^2} \left( 1 + \frac{m_\chi^2}{2m_\phi^2} \right) \left( \frac{k^2}{m_\phi^2 - m_\chi^2} \right)$$

(50)

With the normalization $a_e = 1$ and thus $H_k = H_e a_k^{-3/2}$, we end up with

$$|\beta_{k \gg aH}|^2 \approx \frac{9\pi}{64} \left( \frac{H_e}{m_\phi} \right)^3 \left( \frac{m_\phi}{k} \right)^{9/2} \left( \frac{m_\phi}{m_\chi} \right)^{5/4} \left( 1 + \frac{m_\chi^2}{2m_\phi^2} \right)^2,$$

(51)

This corresponds to taking the equation of state parameter $w \approx 0$. However, since the inflaton pressure is quickly oscillating with time as will be seen later, this assumption is not entirely a good approximation. Nevertheless, we have numerically checked that the solution given in Eq. (41) is in good agreement with numerical computation.

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$^3$ This corresponds to taking the equation of state parameter $w \approx 0$. However, since the inflaton pressure is quickly oscillating with time as will be seen later, this assumption is not entirely a good approximation. Nevertheless, we have numerically checked that the solution given in Eq. (41) is in good agreement with numerical computation.
which is identical to Eq. (24) by noticing the comoving momentum \( k = |\vec{p}_f|/(a/a_e) \). Therefore, we have confirmed that for \( k \gg a_e H_e \), both the Boltzmann and Bogoliubov approaches give the identical spectrum for the gravitational production of \( \chi \).

V. VACUUM CONTRIBUTION

In gravitational production, an important effect that cannot be caught by the Boltzmann approach is particle production due to the breaking down of the adiabaticity coming from the sudden change of the vacuum from the de-Sitter to the matter-dominated phase. It turns out that \( \chi \) with \( k \ll H_e \) may be produced by this phase transition, and this production rate differs from what is obtained in the Boltzmann approach. To compute \( |\beta_{k \ll H_e}|^2 \) for such modes, we need to develop the formalism further. In doing so, we follow closely Ref. [119]; up to Eq. (62) we is a review.

We set a reference time \( \eta_r \) after which the Hubble parameter evolves as

\[
H(a \geq a_r) = H_r \left( \frac{a}{a_r} \right)^{-\frac{3(1+w_r)}{2}},
\]

where \( a_r \equiv a(\eta = \eta_r) \), \( H_r \equiv H(\eta = \eta_r) \), and the equation of state parameter \( w_r \equiv w(\eta = \eta_r) \). Thus, the scale factor at \( \eta \geq \eta_r \) may be written as

\[
a(\eta \geq \eta_r) = c_r (\eta - \bar{\eta}_r)^{q_r},
\]

\[
q_r \equiv \frac{2}{1 + 3w_r}, \quad c_r \equiv a_r \left( \frac{a_r H_r}{q_r} \right)^{q_r}, \quad \bar{\eta}_r \equiv \eta_r - \frac{q_r}{a_r H_r},
\]

We also require that \( a(\eta) \) and its first derivative \( a'(\eta) \) are smooth at \( \eta = \eta_r \).

Supposing that for the mode function \( u_k(\eta) \) in Eq. (31) is given as \( u_{r-1} \) for the time interval \( \eta_{r-1} < \eta < \eta_r \) and as \( u_r \) for \( \eta_r < \eta < \eta_{r+1} \), the Bogoliubov coefficients can be computed by

\[
\alpha_k = -i W[u^*_r, u_r], \quad \beta_k = i W[u_{r-1}, u_r]
\]

with the Wronskian \( W[f,g] \equiv f'g - fg' \). Note that \( u_r \) is a solution of the equation of motion given by

\[
\frac{d^2}{d\eta^2} u_r + \omega_r^2 u_r = 0,
\]

\[
\omega_r^2 \equiv k^2 + c_r^2 m_X^2 (\eta - \bar{\eta}_r)^{2q_r} - \frac{q_r}{a_r H_r} (q_r - 1) (\eta - \bar{\eta}_r)^2.
\]

The simplest application of this method is for the case of \( m_X = 0 \). We take \( \eta_r \equiv \eta_e \) at which the transition from \( w_{r-1} = -1/2 \) (inflation) to \( w_r = 0 \) (matter-domination) takes place. Imposing the Bunch-Davies vacuum [120, 121] as the initial condition,

\[
u_{r-1}(\eta \rightarrow -\infty) \equiv \epsilon^{-ik\eta}/\sqrt{2k}, \quad (59)
\]

we find the solution for \( u_{r-1} \) during inflation given by

\[
u_{r-1} = e^{i\theta_{r-1}} \sqrt{\frac{\pi}{4k}} \sqrt{y} H_{l_r}^{(2)}(y), \quad (60)
\]

where \( H_{l_r}^{(2)}(y) \) is the Hankel function of the second kind, \( y \equiv k(\eta - \bar{\eta}_r) \), \( l_r \equiv q_{r-1} - 1/2 \), and \( \theta_{r-1} \) denotes an irrelevant phase in the particle production and is dropped in the following expressions. For \( \eta > \eta_r \), \( u_r \) has the same form as \( u_{r-1} \) but replacing \( y \rightarrow x \equiv k(\eta - \bar{\eta}_r) \), \( l_r \rightarrow m_r \equiv q_r - 1/2 \), and \( \theta_{r-1} \rightarrow \theta_r \). Evaluating Eq.(56) at \( \eta = \eta_r \), we obtain

\[
|\beta_{k \ll H_e}|^2 \equiv \frac{9}{64} \left( \frac{H_r}{k} \right)^6.
\]

Finally, we consider the case when \( m_X \) is not negligible compared to \( H_e \). In this case, for \( u_{r-1} \), the solution remains the same form with

\[
l_r \equiv -\sqrt{\frac{9}{4} - \frac{m_X^2}{H_r^2}}, \quad (64)
\]

For \( \eta > \eta_r \), we may approximate \( \omega_r^2 \simeq k^2 + c_r^2 m_X^2 (\eta - \bar{\eta}_r)^{2q_r} \) at later times as the third term in Eq. (58) is supposed to become negligible soon, compared to the second term for \( q_r = 2 \) (\( w_r = 0 \)). By noticing that \( \omega_r \) is monotonically increasing with \( \eta \), we may approximate the solution by the WKB-like form given as

\[
u_r \simeq \epsilon^{-i\Omega_r}/\sqrt{2\omega_r}, \quad \Omega_r(\eta) \equiv \int_{\eta_r}^{\eta} d\eta' \omega_r(\eta'), \quad (65)
\]

Again, using Eq. (56), we obtain
\[
\beta_k \simeq \frac{i}{2} \sqrt{\frac{\pi}{2k}} \left\{ \left( 1 + \frac{2i}{\sqrt{y_{r} \omega_{r}}} \right) + \frac{\sqrt{y_{r} \omega_{r}}}{2\sqrt{y_{r} \omega_{r}}} + i\sqrt{y_{r} \omega_{r}} \right\} H_{l_{r}}^{(2)}(y_{r}) - k \sqrt{y_{r} \omega_{r}} H_{l_{r}+1}^{(2)}(y_{r}) \right\}, \tag{66}
\]

where \(\omega_{r}\) is evaluated at \(x = x_{r}\). This is one of our main results. Note that irrelevant \(e^{-\Omega_{r}}\) factor is dropped in Eq. (66) as was done in the massless case. Thus, for \(k \ll H_{e}\), the asymptotic form of \(|\beta_{k}|^2\) becomes

\[
|\beta_{k}|^2 \simeq \frac{9}{32} \left( \frac{H_{e}}{m_{\chi}} \right) \left( \frac{H_{e}}{k} \right)^3. \tag{67}
\]

VI. SUMMARY AND DISCUSSION

Gravitational particle production is a minimal contribution to reheating the Universe after the end of inflation. To study this production channel, two different approaches have commonly been considered, one of which is based on the Boltzmann equation, and the other is based on the Bogoliubov transformation. However, their (in)equivalence in particle production is not obvious. We have explicitly shown by looking at the particle spectrum that for the higher \(k\)-modes, both approaches are equivalent. For the lower \(k\)-modes, however, the Boltzmann approach can not deal with the particle production. By developing a theoretical framework, we have obtained the analytic estimate of such contribution based on the Bogoliubov approach.

Our main results are summarized in Fig. 1 where the solid lines are obtained by numerically solving the equation of motion Eq. (28) and computing

\[
|\beta_{k}|^2 = \frac{1}{\omega_{k}} \sqrt{2 \omega_{k}} + \frac{\omega_{k}}{2} |\chi_{k}|^2 + \frac{i}{2} W[\chi^{*}, \chi] \tag{68}
\]

in the limit of \(n \to \infty\) at which the energy density is well defined (\(\omega_{k}^2 > 0\)). The initial condition for \(\chi_{k}\) is taken as the Bunch-Davies vacuum given in Eq. (59). For the numerical results, we take \(H_{e} = m_{\phi}\) which just sets the scale of the horizontal axis in the figure.

The spectrum at higher \(k\)-modes (\(k \gg H_{e}\)) is well explained by both Boltzmann and Bogoliubov approaches given in Eqs. (24) and (52), respectively. The analytic estimate is shown in the dashed line in the figure, while we take \(H_{e} = m_{\phi}/1.25\) instead of \(H_{e} = m_{\phi}\), to fit the numerical result. As one might notice, there is no proper way to define \(H_{e}\), as the transition from de Sitter to the matter-domination phase is smooth, rather than discrete. Therefore, the analytic estimate does not exactly coincide with the numerical one in general.

For the lower \(k\)-modes (\(k \ll H_{e}\)), the analytic estimate for \(m_{\chi} = 0\) given by Eq. (61) and \(m_{\chi} = m_{\phi}/10\) given by Eq. (66) are shown in dot-dashed and dotted lines, respectively. For the massless case, we again take \(H_{e} = m_{\phi}/1.25\), while \(H_{e} = m_{\phi}\) for the massive case. Both are in good agreement with the numerical results.

It should be noted that both Eqs. (61) and (66) cannot be applied for \(k \gg H_{e}\) since such modes always stay well inside the horizon, i.e. remain the plane wave as if the particles were in the Minkowski spacetime, and thus do not contribute particle production. In addition, for the massive case, we have shown only \(m_{\chi} = m_{\phi}/10\). This is because the approximation used in deriving Eq. (66) becomes worse when \(0 < m_{\chi} \ll m_{\phi}\). In such cases, in \(\omega_{r}^2\) given in Eq. (58), there exists a time scale where the third term is still dominant, though the second term (mass term) will dominate in the distant future. Therefore, to observe the finite mass effect, for instance, one may need to divide the time scales into three, namely, during inflation, the time scale when the mass term in \(\omega_{r}\) can still be negligible, and that when the mass term becomes dominant.

Our results can be used for various phenomenological studies. For instance, one may consider the particle production out of the vacuum when the inflaton potential is not quadratic during reheating. In such cases, the equation of state during reheating can be different than \(w = 0\), and thus the resulting spectrum can be modified. Also, we have not specified the particle nature of \(\chi\) for which one may identify a Higgs(-like) particle whose decay generates dark matter. In such scenarios, the spectrum of the parent particle is necessary to compute the dark matter abundance, which can be investigated in our framework. Furthermore, one may include the non-minimal coupling of \(\chi\) to the Ricci scalar, extend the potential of \(\chi\) beyond the quadratic, or even consider different spins of \(\chi\), which we will leave for future works.

**Note added:** After completion of the draft and while preparing for submission, a paper [122] appeared that partially overlapped with our analysis of gravitational particle production, in obtaining a numerical result from the Bogoliubov approach and an analytical result from the Boltzmann approach. In this work, we have obtained all the relevant results and have compared Boltzmann and Bogoliubov approaches both numerically and ana-
FIG. 1. Spectrum of gravitationally produced $\chi$ obtained by computing both numerically and analytically. The solid lines are obtained by numerical computation. The spectrum for $k \gg H_e$ is $|\beta_k|^2 \propto 1/k^{9/2}$ and well fitted by the analytic result in Eq. (24) [or equivalently Eq. (52)] shown in the dashed line. The dot-dashed line shows Eq. (61) which is valid only for $k \ll H_e$ and $m_\chi = 0$. The dotted line is obtained by Eq. (66) which is also valid only for $k \ll H_e$.

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