Notes on polynomials \((1 + X)^n + (-1)^n(X^n + 1)\) concerning the regularity problem for symmetric power sums in 3 variables

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Abstract

Let \(K\) be a field and \(f_n(X) = (X + 1)^n + (-1)^n(X^n + 1) \in K[X]\), for each \(n \in \mathbb{N}\). This note shows that the polynomials \(f_m(X)\) and \(f_{m'}(X)\) are relatively prime, for some distinct indices \(m\) and \(m'\) at most equal to 100, if and only if the product \(mm'\) is divisible by 6.

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1 Introduction

Let \(f_n(X) = (1 + X)^n + (-1)^n(X^n + 1)\), for each \(n \in \mathbb{N}\). Through this note, we assume that \(f_n(X)\), \(n \in \mathbb{N}\), are defined over a field \(K\) of characteristic zero. If the order \(n\) of \(f_n(X)\) is an even number, then the degree \(\deg(f_n)\) and the leading term of \(f_n(X)\) are equal to \(n\) and 2, respectively; when \(n\) is odd, \(\deg(f_n)\) and the leading term of \(f_n(X)\) are equal to \(n - 1\) and \(n\), respectively. In addition, it can be easily verified that \(f_n(X)\) is divisible by the polynomial \(X(X + 1)\), i.e. \(f_n(0) = f_n(-1) = 0\), if and only if \(n\) is odd. Similarly, one obtains by straightforward calculations that the polynomial \(X^2 + X + 1\) divides \(f_n(X)\) if and only if \(n\) is not divisible by 3. These observations prove the left-to-right implication in the following question:

(1) Find whether \(f_m(X)\) and \(f_n(X)\) are relatively prime, for a pair of distinct positive integers \(m\) and \(n\), if and only if \(mn\) is divisible by 6.

An affirmative answer to (1) would prove the following conjecture in the special case where \(a = 1\):

(2) Let \(a, b\) and \(c\) be a sequence of pairwise distinct positive integers with \(\gcd(a, b, c) = 1\). Then the symmetric polynomials \(X_1^a + X_2^a + X_3^a\), \(X_1^b + X_2^b + X_3^b\) and \(X_1^c + X_2^c + X_3^c\) form a regular sequence in the polynomial ring \(K[X_1, X_2, X_3]\) in three algebraically independent variables over the field \(K\) if and only if the product \(abc\) is divisible by 6.
Let us note that a set of $\sigma$ homogeneous polynomials in $\sigma$ algebraically independent variables is a regular sequence, if the associated polynomial system has only the trivial solution $(0, \ldots, 0)$. Conjecture (2) has been suggested in [1] (see also [2]). The purpose of this note is to show that the answer to (1) is affirmative, for polynomials of admissible degrees at most equal to 100. Formally, our main result can be stated as follows:

**Theorem 1.1.** Let $m$ and $n$ be distinct positive integers at most equal to 100, and let $K$ be a field with $\text{char}(K) = 0$. Then the polynomials $f_m(X), f_n(X) \in K[X]$ satisfy the equality $\gcd(f_m(X), f_n(X)) = 1$ if and only if $6 \mid mn$.

It is clearly sufficient to prove Theorem 1.1 and to consider (1) in the special case where $K$ is the field $\mathbb{Q}$ of rational numbers. Our notation and terminology are standard, and missing definitions can be found in [3].

## 2 Preliminaries

This Section begins with a brief account of some properties of the polynomials $f_m(X), f_n(X)$, where $m$ and $n$ are distinct positive integers. These properties are frequently used in the sequel without an explicit reference. Some of the most frequently used facts can be presented as follows:

(2.1) (a) Any complex root $\alpha_n$ of $f_n(X)$, for a given $n \in \mathbb{N}$, satisfies the following: $2\alpha_n$ is an algebraic integer, provided that $n$ is even; $n\alpha_n$ is an algebraic integer in case $n$ is odd;

(b) If $m$ and $n$ are positive integers of different parity, then the common complex roots of $f_m(X)$ and $f_n(X)$ (if any) are algebraic integers;

(c) 0 and 1 are simple roots of $f_n(X)$, provided that $n \in \mathbb{N}$ and $n$ is odd; the same applies to the reduction of $f_n(X)$ modulo 2;

(d) Given an integer $n > 0$ and a primitive cubic root of unity $\zeta_3$ (lying in the field $\mathbb{C}$ of complex numbers), we have $f_n(\zeta_3) = 0$ if and only if $n$ is not divisible by 3.

Note also that polynomial $f_n(X)$ has no real root, for any even integer $n$.

(2.2) The roots of $f_p(X)$ are algebraic integers, for every prime $p$.

Next we include a list of the polynomials $f_n(X)$, for some small values of $n$:

(2.3) (a) $f_{10}(X) = (X^2 + X + 1)^2 g_{10}(X)$, where $g_{10}(X) = 2X^6 + 6X^5 + 27X^4 + 44X^3 + 27X^2 + 6X + 2$;

(b) $f_9(X) = 3X(X + 1)g_9(X)$, where $g_9(X) = 3X^6 + 9X^5 + 19X^4 + 23X^3 + 19X^2 + 9X + 3$;

(c) $f_8(X) = 2(X^2 + X + 1)g_8(X)$, where $g_8(X) = X^6 + 3X^5 + 10X^4 + 15X^3 + 10X^2 + 3X + 1$;

(d) $f_7(X) = 7(X + 1)(X^2 + X + 1)^2$;

(e) $f_6(X) = 2X^6 + 6X^5 + 15X^4 + 20X^3 + 15X^2 + 6X + 2$;

(f) $f_5(X) = 5X(X + 1)(X^2 + X + 1)$;

(g) $f_4(X) = 2(X^2 + X^3 + X^2 + 2X + 1) = 2(X^2 + X + 1)^2$;

(h) $f_3(X) = 3X(X + 1)$.
(i) \( f_2(X) = 2(X^2 + X + 1) \).

The following lemma presents some well-known properties of Newton’s binomial coefficients that are frequently used in the sequel:

**Lemma 2.1.** Assume that \( p \) is a prime number, and \( n, s \) are positive integers, such that \( s \) does not divide \( p \). Then the binomial coefficients \( \binom{p^n}{j} \), \( j = 1, \ldots, p^n - 1 \), are divisible by \( p \); \( \binom{p^n}{s} \) is not divisible by \( p \).

**Proof.** The assertion is obvious if \( n = 1 \), so we assume further that \( j \geq p \) (and \( n \geq 2 \)). Suppose first that \( j \) is not divisible by \( p \) and denote by \( y \) the greatest integer divisible by \( p \) and less than \( j \). It is clear from the definition of Newton’s binomial coefficients that the maximal power of \( p \) dividing \( \binom{p^n}{j} \) is greater than the maximal power of \( p \) dividing \( \binom{p^n}{y} \); in particular, this ensures that \( p^2 \) divides \( \binom{p^n}{s} \). Now fix a positive integer \( u < p^{n-1} \) and denote by \( C[p^n, pu] \) the product of the multiples of the numerator of \( \binom{p^n}{s} \) that are divisible by \( p \), divided by the product of the multiples divisible by \( p \) of the denominator of \( \binom{p^n}{s} \). It is easily verified that \( C[p^n, pu] = \binom{p^{n-1}}{u} \). This allows to complete the proof of Lemma 2.1 arguing by induction on \( n \).

**Remark 2.2.** It follows from Lemma 2.1, applied to \( p = 2 \), that \( f_{2n+3}(X) \) decomposes over the field \( \mathbb{Q}_2 \) of 2-adic numbers into a product of three irreducible polynomials of degree \( 2^k \) each; one of these polynomials lies in the ring \( \mathbb{Z}_2[X] \) and is 2-Eisensteinian over the ring \( \mathbb{Z}_2 \) of 2-adic integers. In view of our irreducibility criterion, see Section 3, this means that if \( f_{2n+3}(X) \) is reducible over \( \mathbb{Q} \), then it decomposes into a product of 3 \( \mathbb{Q} \)-irreducible polynomials, say \( h_1(X), h_2(X) \) and \( h_3(X) \) (in fact \( h_j(X) \), \( j = 1, 2, 3 \), are irreducible even over \( \mathbb{Q}_2 \)). More precisely, the action of the symmetric group \( \text{Sym}_3 \) on the set of roots of \( f_{2n+3}(X) \) induces bijections \( y_j, j = 2, 3 \), from the set of roots of \( h_1(X) \) in \( \mathbb{Q}_2 \), onto the set of roots of \( h_j(X) \), for each index \( j \). It is therefore clear that if \( \gcd(f_{2n+3}(X), f_n(X)) \neq 1 \), for some \( n \in \mathbb{N} \), then \( f_n(X) \) and \( h_n(X) \) have a common root \( \zeta_n \in \mathbb{Q}_2 \), for each \( u \in \{1, 2, 3\} \). Thus it turns out that if \( \gcd(f_{2n+3}(X), f_n(X)) \neq 1 \), then \( f_{2n+3}(X) \mid f_n(X) \); in particular, \( f_n(X) \) has a complex root that is not an algebraic integer.

**Lemma 2.1** can be supplemented as follows:

**Lemma 2.3.** Let \( n \in \mathbb{N} \) and \( p \in \mathbb{P} \). Then the binomial coefficients \( \binom{p^n}{j} \) are divisible by \( p \), provided that \( j \in \mathbb{Z} \) and \( 1 \leq j < p^n \); in addition, if \( n \geq 2 \), then \( \binom{p^n}{j} \) is divisible by \( p^2 \) unless \( j = p^{n-1}j_0 \), \( 1 \leq j_0 \leq p - 1 \).

**Proof.** The former part of our assertion is a special case of Lemma 2.1 so we assume further that \( n \geq 2 \). Suppose first that \( j \) is not divisible by \( p \) and denote by \( y \) the greatest integer divisible by \( p \) and less than \( j \). It is clear from the definition of Newton’s binomial coefficients that the maximal power of \( p \) dividing \( \binom{p^n}{j} \) is greater than the maximal power of \( p \) dividing \( \binom{p^n}{y} \); in particular, this
ensures that \( p^2 \mid \binom{p^n}{j} \). Now fix a positive integer \( u < p^{n-1} \) and define \( C[p^n, pu] \) as in the proof of Lemma 2.1. It is easily verified that \( C[p^n, pu] = \binom{p^n-1}{u} \). This allows to complete the proof of Lemma 2.3 arguing by induction on \( n \).

\[ \square \]

3 Polynomials of even orders

This Section begins with a criterion for validity of the equality \( \gcd(f_m(X), f_n(X)) = 1 \) in the special case where \( m \) is divisible by 6 and \( f_m(X) \) is irreducible over \( \mathbb{Q} \).

**Proposition 3.1.** Let \( m \) and \( n \) be positive integers with \( 2 \mid m \) and \( 6 \mid mn \). Put \( f_n(X) = (1 + X)^n + (-1)^n(X^n + 1) \), and suppose that \( m < n \) and the polynomial \( f_m(X) = (1 + x)^m + X^m + 1 \) is \( \mathbb{Q} \)-irreducible or \( m = 3 \cdot 2^k \), for some \( k \in \mathbb{N} \).

Then \( \gcd(f_m(X), f_n(X)) = 1 \) except, possibly, under the following conditions:

(a) \( n \equiv 1 \pmod{m-1} \) and \( m \equiv n \pmod{2^k+1} \), where \( k \) is the greatest integer for which \( 2^k \) divides \( m \);

(b) If \( m \) is divisible by 4, then \( n/2 \equiv 1 \pmod{m/2-1} \).

**Proof.** Suppose that \( \gcd(f_m(X), f_n(X)) \neq 1 \), for some \( n \in \mathbb{N} \). This means that \( f_m(X) \) and \( f_n(X) \) have a common root \( \rho \in \mathbb{C} \). Note further that the irreducibility of \( f_m(X) \) over \( \mathbb{Q} \) and the assumption that \( m \) is even indicate that the complex roots of \( f_m(X) \) are not algebraic integers, so it follows from (2.1) (b) that \( n \equiv 0 \pmod{m} \). Observing also that \( f_m(X) \mid f_n(X) \) (in \( \mathbb{Z}[X] \)), one concludes that \( f_m(1) \mid f_n(1) \) (in \( \mathbb{Z} \)), i.e. \( 2^m + 2 | 2^n + 2 \). It is therefore clear that \( 2^m + 1 | 2^n + 1 \), and since \( m - 1 \) and \( n - 1 \) are odd, this requires that \( m - 1 \mid n - 1 \), proving the former part of Proposition 3.1 (a). The rest of the proof of Proposition 3.1 relies on Lemma 2.1, which allows to prove that \( f_m(X) \) and \( f_n(X) \) have unique divisors \( \theta_m(X) \) and \( \theta_n(X) \), respectively, over the field \( \mathbb{Q}_2 \), with the following properties: \( \theta_m \) and \( \theta_n \) are \( 2 \)-Eisensteinian polynomials over the ring \( \mathbb{Z}_2 \); the degree of \( \theta_m \) (\( X \)) equals the greatest power of 2 dividing \( m \), and the degree of \( \theta_n \) equals the greatest power of 2 dividing \( n \). Observing also that \( \theta_m(X) \) and \( \theta_n(X) \) can be chosen so that their leading terms be equal to 1, one obtains from the divisibility of \( f_n(X) \) by \( f_m(X) \) that \( \theta_m(X) = \theta_n(X) \). This result completes the proof of Proposition 3.1 (a). We turn to the proof of Proposition 3.1 (b), so we suppose further that \( 4 \mid m \). In view of Proposition 3.1 (a), this means that \( 4 \mid n \), which shows that \( f_m(\sqrt{-1}) = (2\sqrt{-1})^{m/2} + 2 = 2^{m/2} + 2 \) and \( f_n(\sqrt{-1}) = 2^{n/2} + 2 \). As \( f_m(X) \mid f_n(X) \), whence \( f_m(\sqrt{-1}) \mid f_n(\sqrt{-1}) \) (in the ring \( \mathbb{Z}[\sqrt{-1}] \) of Gaussian integers), our calculations lead to the conclusion that \( 2^{(m/2)-1} \mid 2^{(n/2)-1} + 1 \) (in \( \mathbb{Z} \)). Taking finally into account that \( (m/2) - 1 \) and \( (n/2) - 1 \) are odd positive integers, one obtains that \( (m/2) - 1 \mid (n/2) - 1 \). Proposition 3.1 is proved.

Our next result shows that the polynomials \( f_6(X), f_{12}(X), f_{18}(X), f_{30}(X), f_{50}(X), f_{54}(X), f_{64}(X) \) and \( f_{90}(X) \) are irreducible over \( \mathbb{Q} \).

**Proposition 3.2.** The polynomial \( f_m(X + 1) \) is 3-Eisensteinian relative to the ring \( \mathbb{Z} \) of integers, if \( m = 3^k + 3^l \), for some positive integers \( k \) and \( l \); in particular, this holds, for \( m = 6, 12, 18, 30, 36, 54, 84, 90 \).
Proof. Note first that the free term of the polynomial $t_m(X) = f_m(X + 1)$ is divisible by 3 but is not divisible by 9. Indeed, this term is equal to $2^m + 2$, and since $6 | m$, we have $2^m \equiv 1 \pmod{9}$ and $2^m + 2 \equiv 3 \pmod{9}$. Therefore, using Lemma 2.1 one sees that it suffices to show that the coefficient, say $a$, of the monomial $X^3$ in the reduced presentation of $t_m(X)$ is divisible by 3. The proof of this fact offers no difficulty because $a = (2^3 + 1)k(3^m)$ (the binomial coefficient $k(3^m)$ is a positive integer not divisible by 3 whereas $2^3 + 1$ is divisible by 3).

Our next result gives an affirmative answer to (1) in the special case where $m$ is a 2-primary number. It proves the validity of Theorem 1.1, under the condition that $m \in \{2, 4, 8, 16, 32, 64\}$.

**Proposition 3.3.** For any $k, n \in \mathbb{N}$, $\gcd(f_m(X), f_{3n}(X)) = 1$, where $m = 2^k$.

Proof. Our argument relies on the fact that $f_m(X) = 2g_m(X)$, $g_m(X)$ being a polynomial with integer coefficients, such that $g_m(X) = X^m - X^{m/2} - 1 = 2h_m(X)$, for some $h_m(X) \in \mathbb{Z}[X]$. This ensures that if $\rho$ is a complex root of $g_m$, $K = \mathbb{Q}(\rho)$ and $O_K$ is the ring of algebraic integers in $K$, then the coset $\rho + P$ is a cubic root of unity in the field $O/P$, for any prime ideal $P$ of $O_K$ of 2-primary norm (i.e. a prime ideal, such that $2 \in P$). The same holds whenever $K/K'$ is a finite extension, $O_K'$ is the ring of algebraic integers in $K'$, and $P'$ is a prime ideal in $O_K'$ of 2-primary norm. The noted property of $\rho$ indicates that $f_m(\rho) = 3 \equiv 1 \pmod{P}$ in case $n \in \mathbb{N}$ is divisible by 3, which proves the non-existence of a common root of $f_m(X)$ and $f_n(X)$, as claimed.

The following statement provides an affirmative answer to (1), under the hypothesis that $m/2$ or $m/4$ is an odd primary number not divisible by 3.

**Proposition 3.4.** For any prime number $p > 3$ and each pair of positive integers $k, n$, we have $\gcd(f_{2p^k}(X), f_{3n}(X)) = \gcd(f_{4p^k}(X), f_{3n}(X)) = 1$.

Proof. We proceed by reduction modulo $p$. Then $f_{2p^k}(X) = f_2(X)p = 2p(X^2 + X + 1)^p$ and $f_{4p^k}(X) = f_4(X)p = 2p(X^2 + X + 1)^{2p}$. This indicates that if $\hat{\rho}$ is a root of $f_{2p^k}(X)$ or $f_{4p^k}(X)$ in $(\mathbb{Z}/p\mathbb{Z})_{\text{sep}}$, then $\hat{\rho}$ is a cubic root of unity. Therefore, it is easily verified that $f_{3n}(\hat{\rho}) = -3 \neq 0$, provided that $n$ is odd. When $n$ is even, one obtains similarly that $f_{3n}(\hat{\rho}) = 3 \neq 0$. These calculations prove that $\gcd(f_{2p^k}(X), f_{3n}(X)) = \gcd(f_{4p^k}(X), f_{3n}(X)) = 1$, for each $n \in \mathbb{N}$. Our conclusion means that $\gcd(f_{2p^k}(X), f_{4p^k}(X), f_{3n}(X)) = 1$, which can be restated by saying that $u(X)f_{2p^k}(X)f_{4p^k}(X) + v(X)f_{3n}(X) = 1 + pw(X)$, for some $u(X), v(X), w(X) \in \mathbb{Z}[X]$. Suppose now that $f_{3n}(\beta) = f_{2p^k}(\beta)f_{4p^k}(\beta) = 0$, for some $\beta \in \mathbb{C}$, put $O = \{r \in \mathbb{Q} : 2^n(r) \in \mathbb{Z}, \text{ for some integer } n(r) \geq 0\}$, and denote by $O'_{\mathbb{Q}(\beta)}$ the integral closure in $\mathbb{Q}(\beta)$ of the ring $O$. Since $2\beta$ is an algebraic integer and $1 + pw(\beta) = 0$, one obtains consecutively that $\beta \in O'_{\mathbb{Q}(\beta)}$ and $p$ is an invertible element of $O'_{\mathbb{Q}(\beta)}$; in particular, this requires that $1/p \in O'_{\mathbb{Q}(\beta)}$. Since, however, $O$ is an integrally closed subring of $\mathbb{Q}$, the obtained result leads to the conclusion that $1/p \in O$ which is not the case. The obtained contradiction is due to the assumption that $f_{3n}(X)$ and $f_{2p^k}(X), f_{4p^k}(X)$ have a common root, so Proposition 3.4 is proved.
Let \( h(Z) \in (\mathbb{Z}/q\mathbb{Z})[Z] \) be the cubic polynomial defined so that \( h(X + X^{-1}) = g_8(X)/X^3 \). \( g_8(X) \in (\mathbb{Z}/q\mathbb{Z})[X] \) being the reduction of \( g_8(X) \in \mathbb{Z}[X] \) modulo a prime number \( q > 2 \) not dividing the discriminant \( d(h) \). It is not difficult to see that the discriminant \( d(h) \) is a non-square in \((\mathbb{Z}/q\mathbb{Z})^*\) if and only if \( h(X) \) has a unique zero lying in \( \mathbb{Z}/q\mathbb{Z} \). When this holds, \( g_8(X) \) decomposes over \( \mathbb{Z}/q\mathbb{Z} \) into a product of three (pairwise relatively prime) quadratic polynomials irreducible over \( \mathbb{Z}/q\mathbb{Z} \). For example, this applies to the case where \( q = 5 \) or \( q = 7 \), which is implicitly used for simplifying the proofs of the following two statements.

**Proposition 3.5.** The polynomials \( f_{8,5^k}(X) \) and \( f_n(X) \) satisfy the equality \( \gcd(f_{8,5^k}(X), f_n(X)) = 1 \), for each \( k \in \mathbb{N} \), and any \( n \in \mathbb{N} \) divisible by 3 and not congruent to 6 modulo 24.

**Proof.** It is easily verified that \( 2 + \sqrt{2}, 2 - \sqrt{2}, 1 + 2\sqrt{2}, 1 - 2\sqrt{2}, -2 + 2\sqrt{2} \) and \(-2 + 2\sqrt{2}\) are pairwise distinct roots of \( f_8(X) \), the reduction of \( f_8(X) \) modulo 5. These roots are contained in a field \( \mathbb{F}_{25} \) with 25 elements. None of them is a primitive cubic root of unity: \((3 + \sqrt{2})^3 = (2 + \sqrt{2})^3 = -\sqrt{2} \), \((1 + 2\sqrt{2})^3 = 2\sqrt{2}, (2 + 2\sqrt{2})^3 = 1; (-2 - 2\sqrt{2})^3 = -1, (-1 - 2\sqrt{2})^3 = -2\sqrt{2}.\) In other words, the noted elements are roots of \( g_8(X) \), the reduction modulo 5 of the polynomial \( g_8(X) = (1/2)f_8(X)/(X^2 + X + 1) \) \( g_8(X) \in \mathbb{Z}[X] \) is irreducible over \( \mathbb{Q} \). Observe that the latter two roots of \( g_8(X) \) are primitive 6-th roots of 1, whereas the remaining roots of \( g_8(X) \) are generators of the multiplicative group \( \mathbb{F}_{25}^\times \). Taking further into account that the elements \( a\sqrt{2}, a \in \mathbb{F}_5 \), are all primitive 8-th roots of unity in \( \mathbb{F}_{25} \) (\( \mathbb{F}_5 \) is the prime subfield of \( \mathbb{F}_{25} \)), and \( f_8(X) = 2(X^2 + X + 1)g_8(X) \), one concludes that \( f_8(X) \) and \( f_{3n}(X) \) do not possess a common root, for any odd \( n \in \mathbb{N} \). These calculations yield \( \gcd(f_{8,5^k}(X), f_{3n}(X)) = 1 \) which allows to deduce by the method of proving Proposition 3.4 that \( \gcd(f_{8,5^k}(X), f_{3n}(X)) = 1 \) whenever \( n \) is odd, and also, in the following two cases: \( 4 | n; n \equiv 6 \pmod{8} \). Thus Proposition 3.5 is proved.

**Proposition 3.6.** The polynomials \( f_{8,7^k}(X) \) and \( f_n(X) \) satisfy the equality \( \gcd(f_{8,7^k}(X), f_n(X)) = 1 \) whenever \( k \) and \( n \) are \( \mathbb{N} \), and \( n \) is divisible by 6.

**Proof.** The reductions \( \bar{f}_{8,7^k}(X) \) and \( \bar{f}_8(X) \) modulo 7 satisfy the equality \( \bar{f}_{8,7^k}(X) = f_8(X)^7 \), so it is sufficient to show \( \bar{f}_8(X) \) and \( f_n(X) \) do not possess a common root in \((\mathbb{Z}/7\mathbb{Z})_{\text{sep}}\). Our argument relies on the fact that \( \sqrt{-1} \notin \mathbb{Z}/7\mathbb{Z} \), and \( 3 + \sqrt{-1}, 3 - \sqrt{-1}, 1 + 2\sqrt{-1}, 1 - 2\sqrt{-1}, -2 + 2\sqrt{-1} \) and \(-2 + 2\sqrt{-1}\) are all roots in \((\mathbb{Z}/7\mathbb{Z})_{\text{sep}}\) of the reduction \( g_8(X) \) of \( g_8(X) \) modulo 7. This ensures that, for each of these roots, say \( \rho \), \( f_n(\rho) = \rho_1 + \rho_2 + 1 \) whenever \( n \in \mathbb{N} \) is fixed and divisible by 6. Here \( \rho_1 \) and \( \rho_2 \) are 8-th roots of unity in \((\mathbb{Z}/7\mathbb{Z})_{\text{sep}}\) depending on \( n \) and \( \rho \). We show that \( f_n(\rho) \neq 0 \). Consider an arbitrary primitive 8-th root of unity \( \varepsilon \in (\mathbb{Z}/7\mathbb{Z})_{\text{sep}} \). It is easily verified that \( \varepsilon \in (\mathbb{Z}/7\mathbb{Z})(\sqrt{-1}) \setminus \mathbb{Z}/7\mathbb{Z} \). More precisely, one obtains by straightforward calculations that \( \varepsilon = -2(\varepsilon_1 + \varepsilon_2 \sqrt{-1}) \), for some \( \varepsilon_j \in \{-1,1\}, j = 1,2 \). It is now easy to see that \( f_n(\rho) \neq 0 \), as claimed. Thus the assertion that \( \gcd(f_{8,7^k}(X), f_n(X)) = 1 \), for every admissible pair \( k, n \), becomes obvious, which completes the proof of Proposition 3.6.
Proposition 3.9 and our next result prove that $\gcd(f_{50}(X), f_n(X)) = 1$, for each $n \in \mathbb{N}$ divisible by 3. This, combined with Proposition 3.6 and Corollaries 3.6 and 5.7, proves the validity of Theorem 1.1 in the special case where $m$ is an even biprimary number (see also Remark 4.12 for the case of $m = 72$).

**Proposition 3.7.** The polynomials $f_{2q}(X)$ and $f_n(X)$ are relatively prime, provided that $k \in \mathbb{N}$, $q \in \{5, 7\}$ and $n \in \mathbb{N}$ is odd and divisible by 3.

**Proof.** Denote by $\bar{f}_{2q}(X)$ the reduction of $f_{2q}(X)$ modulo 2. It is not difficult to see that $f_{2q}(X) = f_q(X)^2$, where $f_q$ is the reduction of $f_q(X)$ modulo 2. This ensures that if $\alpha \in \mathbb{C}$ is an algebraic integer with $\bar{f}_{2q}(\alpha) = 0$, then $\bar{f}_q(\bar{\alpha}) = 0$, where $\bar{\alpha}$ is the residue class of $\alpha$ modulo any prime ideal $P$ of $\mathbb{Q}(\alpha)$, such that $2 \notin P$. In particular, this is the case where $\alpha$ is a common root of $f_{2q}(X)$ and $\bar{f}_q(X)$, for some odd $v \in \mathbb{N}$. Observing also that $\bar{f}_q(X) = X(X+1)(X^2+X+1)$ and $f_q(X) = X(X+1)(X^2+X+1)^2$, one concludes either $\bar{\alpha} \in \{0, -1\}$ or $\alpha$ is a primitive cubic root of unity. The latter possibility is clearly ruled out, if $3 \nmid \nu$.

At the same time, since $\nu$ is odd, 0 and $-1$ are simple roots of $f_q(X)$, so it is easy to see that $\gcd(\bar{f}_{2q}(X), \bar{g}_q(X)) = 1$, $\bar{g}_q(X)$ being the reduction modulo 2 of the polynomial $g_q(X) \in \mathbb{Z}[X]$ defined by the equality $f_q(X) = X(X+1)g_q(X)$. As 0 and $-1$ are not roots of $f_{2q}(X)$, the obtained result yields consecutively $\gcd(f_{2q}(X), g_q(X)) = \gcd(f_{2q}(X), f_q(X)) = 1$, as claimed. □

**Remark 3.8.** Let $f_{63}(X), f_9(X), f_{70}(X)$ and $f_{10}(X)$ be the reductions modulo 7 of the polynomials $f_{63}(X), f_9(X), f_{70}(X)$ and $f_{10}(X)$, respectively. It is easily verified that $f_9(X)$ and $f_{10}(X)$ are divisible in $\mathbb{Z}[X]$ by polynomials $g_9(X)$ and $g_{10}(X)$ both of degree 6, which are irreducible over $\mathbb{Q}$. One also sees that $g_9(X)$ decomposes over $\mathbb{Z}/7\mathbb{Z}$ to a product of two cubic polynomials irreducible over $\mathbb{Z}/7\mathbb{Z}$, whereas $g_{10}(X)$ is presentable as a product of three ($\mathbb{Z}/7\mathbb{Z}$)-irreducible quadratic polynomials. As $f_{63}(X) = f_9(X)^7$ and $f_{70}(X) = f_{10}(X)^7$, this yields $\gcd(f_{63}(X), f_{70}(X)) = 1$, which implies the existence of integral polynomials $u(X), v(X)$ and $h(X)$, such that $u(X)f_{63}(X) + v(X)f_{70}(X) = 1 + 7h(X)$. We prove that $\gcd(f_{63}(X), f_{70}(X)) = 1$, by assuming the opposite. Then $\mathbb{C}$ contains a common root $\beta$ of $f_{63}(X)$ and $f_{70}(X)$, and by (2.1) (b), $\beta$ must be an algebraic integer with $1 + 7\beta = 0$. This requires that 7 be invertible in the ring of algebraic integers in $\mathbb{Q}(\beta)$, a contradiction proving that $\gcd(f_{63}(X), f_{70}(X)) = 1$.

It is likely that one could achieve more essential progress in the analysis of Question (1) (up-to its full answer), by applying systematically other specializations of $f_m(X)$ and $f_n(X)$ than those used in the proof of Proposition 3.1. An example supporting this idea is provided by the proof of the following assertion.

**Proposition 3.9.** Let $n$ be a positive integer different from 6. Then $f_6(X)$ and $f_n(X)$ have no common root except, possibly, in the case of $n \equiv 6$ modulo 1200.

**Proof.** Our starting point is the fact that $f_6(X)$ is irreducible over $\mathbb{Q}$ (see Proposition 3.2), this means that $f_6(X)$ and $f_n(X)$ have a common root, for a given $n \in \mathbb{N}$, if and only if $f_6(X)$ divides $f_n(X)$. Note also that the leading term of
$f_6(X)$ is equal to 2, and that $f_6(X)$ is a primitive polynomial (i.e. its coefficients are integers and their greatest common divisor equals 1). These observations show that the complex roots of $f_6(X)$ are not algebraic integers. On the other hand, by (2.1) (b), if $n$ is odd and $r$ is a root of $f_n(X)$ and $f_6(X)$, then $r$ must be an algebraic integer. Therefore, one may assume in our further considerations that $n$ is even. Then it follows from Proposition 3.9 that if $m \in \mathbb{N}$ is even and $f_m(X)$ divides $f_n(X)$ in $\mathbb{Z}[X]$, then $m - 1 \mid n - 1$ and $4 \mid n - m$. Thus it becomes clear that $f_6(X) \mid f_n(X)$ except, possibly, in the case where $n \equiv 6 \pmod{20}$.

In order to complete the proof of Proposition 3.9 it remains to be seen that if $f_6(X) \mid f_n(X)$, then $n \equiv 6 \pmod{126}$ (by Proposition 3.9 $5 \mid n - 6$, so the divisibility of $n - 6$ by 126 would imply $n \equiv 6 \pmod{126}$). Hence, by the congruence $n \equiv 6 \pmod{126}$, $1260 \mid n - 6$, as claimed by Proposition 3.9.

Observe now that $f_6(3) = 4^6 + 3^6 + 1 = 4096 + 729 + 1 = 4826 = 2 \cdot 219 \cdot 127$. Note also that $2^7 = 128$ is congruent to 1 modulo 127, which implies $2^{7k'} \equiv 1 \pmod{127}$, for each $k' \in \mathbb{N}$. Now fix an integer $k \geq 0$ and put $S(k) = 4^k + 1$. It is verified by straightforward calculations that $S(0) = 2$, $S(1) = 5$, $S(2) = 17$, $S(3) = 65$, and $S(4)$, $S(5)$ and $S(6)$ are congruent to 3, 9 and 33, respectively, modulo 127. It is also clear that $S(l) \equiv S(k)(\pmod{127})$ whenever $l$ and $k$ are non-negative integers with $l - k$ divisible by 7. Thus it turns out that when $k$ runs across $\mathbb{N}$, $S(k)$ may take 7 possible values (in fact one value determined by the residue class of $k$ modulo 7).

The next step towards the proof of Proposition 3.9 is to show that 3 is a primitive root of unity modulo 127. Thereafter (in fact, almost simultaneously) we show that if $T(k) = 4^k + 3^k + 1$, for any integer $k \geq 0$, then $T(k)$ is divisible by 127 if and only if $k \equiv 6 \pmod{126}$. This particular fact allows us to take the final step towards our proof, as it shows that if $k$ is even and $f_6(X)$ divides $f_k(X)$, then $f_6(3)$ divides $f_k(3)$, which requires that $k \equiv 6 \pmod{126}$.

It is verified by direct calculations that $3^{33} - 1 \pmod{127}$ (apply the quadratic reciprocity law). Direct calculations also show that $3^6$, $3^7$, $3^{14}$, $3^{21}$, $3^{12}$ are congruent modulo 127 to $-33$, $28$, $22$ ($127$ divides $78422 = 762$), $-19$ ($28.22 = 616$ is congruent to $-19$ modulo 127), $-20$, respectively. Note also that $3^9$ and $3^{18}$ are congruent modulo 127 to $-2$ and 4, respectively. These calculations prove that 3 is a primitive root of unity modulo 127, as claimed.

The noted property of 3 means that the residue classes modulo 127 of the numbers $3^j$: $g = 1, \ldots, 126$, form a permutation of numbers 1, \ldots, 126. This ensures that for any $j = 1, \ldots, 7$, there exists a unique $s(j)$ modulo 126, such that $S(j) + 3^{s(j)}$ is divisible by 127. In order to take the final step towards our proof, it suffices to verify that $s(j)$ is not congruent to $j$ modulo 126, for any $j \neq 6$. The verification process specifies this as follows: $s(0) = 9$, $s(1) = 24$, $s(2) = 101$, $s(3) = 118$, $s(4) = 64$, $s(5) = 65$, $s(6) = 6$. The computational part of this process is facilitated by the observation that 17, $-11$, 5 and 16 are congruent modulo 127 to 144, $3^5 = 243$, 132 = $2^2 \cdot 3 \cdot 11$, and 143 = 11.13, respectively. Proposition 3.9 is proved.

\[\square\]
4 An irreducibility criterion for integral polynomials in one variable

The main result of this Section attracts interest in the question of whether the polynomials $f_n(X)$, $n \in \mathbb{N}$, are irreducible over $\mathbb{Q}$. It shows that this holds in several special cases (which, however, is crucial for the proof of Theorem 4.1).

**Proposition 4.1.** Let $f(X) \in \mathbb{Z}[X]$ be a polynomial of degree $n > 0$, and let $S$ be a finite set of prime numbers not dividing the discriminant $d(f)$ of $f$. For each $p \in S$, denote by $n_p$ the greatest common divisor of the degrees of the irreducible polynomials over the field with $p$ elements, which divide the reduction of $f(X)$ modulo $p$. Then every irreducible polynomial $g(X) \in \mathbb{Z}[X]$ over the field $\mathbb{Q}$ of rational numbers is divisible by the least common multiple, say $\nu$, of numbers $n_p$: $p \in S$; in particular, if $\nu = n$, then $f(X)$ is irreducible over $\mathbb{Q}$.

**Proof.** It is sufficient to observe that, by Hensel’s lemma, for each $p \in S$, there is a degree-preserving bijection of the set of $\mathbb{Q}_p$-irreducible polynomials dividing $f(X)$ in $\mathbb{Q}_p[X]$ upon the set of $(\mathbb{Z}/p\mathbb{Z})$-irreducible polynomials dividing the reduction of $f(X)$ modulo $p$ (in $(\mathbb{Z}/p\mathbb{Z})[X]$). One should also note that every $\mathbb{Q}$-irreducible polynomial dividing $f(X)$ in $\mathbb{Q}[X]$ is presentable as a product of $\mathbb{Q}_p$-irreducible polynomials dividing $f(X)$ in $\mathbb{Q}_p[X]$. \qed

Let $f(X) \in \mathbb{Z}[X]$ be a $\mathbb{Q}$-irreducible polynomial of degree $n$, and let $G_f$ be the Galois group of $f(X)$ over $\mathbb{Q}$. It is worth mentioning that if the irreducibility of $f(X)$ can be deduced from Proposition 4.1, then $n$ divides the period of $G_f$.

**Remark 4.2.** Using Proposition 4.1 and a computer program for mathematical calculations, Junzo Watanabe proved that the polynomials $f_{42}(X)$, $f_{60}(X)$, $f_{66}(X)$, $f_{73}(X)$, $f_{78}(X)$ are irreducible over $\mathbb{Q}$. This result, combined with Proposition 4.2, yields $\gcd(f_m(X), f_n(X)) = 1$, for every pair $m, n \in \mathbb{N}$ less than 100, such that $m$ is odd and 6 divides $n$. Similarly, he proved that $f_{88}(X) = (X^2 + 1)^2 g_{88}(X)$, where $g_{88}(X) \in \mathbb{Z}[X]$ has degree 84 and is irreducible over $\mathbb{Q}$. The obtained result indicates that the complex roots of $g_{88}(X)$ are not algebraic integers, which implies $\gcd(g_{88}(X), f_n(X)) = \gcd(f_{88}(X), f_n(X)) = 1$, for every odd $n \in \mathbb{N}$ divisible by 3.

It would be of interest to know whether the polynomials $f_{6m}(X)$, $n \in \mathbb{N}$, are $\mathbb{Q}$-irreducible, and whether this can be obtained by applying Proposition 4.1.

**Corollary 4.3.** The polynomials $f_{68}(X)$ and $f_n(X) \in \mathbb{Z}[X]$ are relatively prime, for each $n \in \mathbb{N}$ divisible by 3 and less than 100.

**Proof.** In view of (2.1) (b) and Remark 4.2 one may consider only the case of $6 \mid n$. Then our conclusion follows Propositions 5.1 and Remark 4.2. \qed

Statements (2.1) (d), (2.2) and Remark 4.2 combined with Propositions 5.1 and Remark 2.2 lead to the conclusion that $\gcd(f_{6m}(X), f_n(X)) = 1$, for every $n \in \mathbb{N}$ with $6 \mid n$ and $n \leq 100$, and for each prime $p < 100$. It is worth noting that there 26 prime numbers less than 100. The set of primary composite odd numbers consists of 9, 25, 27, 49 and 81.
5 Polynomials of odd orders

Our next step towards the proof of Theorem 1.1 aims at showing that \(\gcd(f_m(X), f_n(X)) = 1\), provided that \(m, n \leq 100\) and \(m\) is an odd primary number. In view of the observations at the end of Section 4, one may consider only the case where \(m\) is a power of a prime \(p \in \{3, 5, 7\}\). This part of our proof relies on (2.1) (b) and the following result.

Proposition 5.1. Let \(p\) be a prime number and \(\alpha_n\) a root of the polynomial \(f_p(X)\), for some \(n \in \mathbb{N}\). Suppose that \(\alpha_n\) is an algebraic integer and set \(\varphi_{p^n}(X) = p^{-1}f_{p^n}(X)\). Then \(\varphi_p(\alpha_n)^{p^{n-1}}\) lies in the ideal \(pO_{\mathbb{Q}(\alpha_n)}\) of the ring \(O_{\mathbb{Q}(\alpha_n)}\) of algebraic integers in \(\mathbb{Q}(\alpha_n)\).

Proposition 5.1 can be deduced from Lemma 2.3 and the following lemma.

Lemma 5.2. In the setting of Lemma 2.3, when \(n \geq 2\), the integers \(p^{-1}(\frac{p^n}{p^n})\) and \(p^{-1}(\frac{p^n}{p^{n-1}p^j})\) are congruent modulo \(p\), for each \(j_0 \in \mathbb{N}\), \(j_0 < p\).

Proof. It follows from the equality \(C[p^n, pu] = (p^{n-1}_u)\), where \(u\) is an integer with \(1 \leq u < p^{n-1}\), that \((p^n_u) = (p^{n-1}_u), u_{p,n}\), for some element \(u_{p,n}\) of the local ring \(\mathbb{Z}/p\) = \(\{r/s : r, s \in \mathbb{Z}, p\) does not divide \(s\}\), such that \(u_{p,n} - 1 \in p\mathbb{Z}/p\). This enables one to obtain step-by-step that \(p^{-1}(\frac{p^n}{p^n}) \equiv p^{-1}(\frac{p^{n-1}}{p^{n-1}p^j})(mod \ p\mathbb{Z}/p)\), \(n = 1, \ldots, n,\) and so to prove Lemma 5.2.

The proofs of the following results rely on the explicit definitions of the polynomials \(f_3(X)\), \(f_5(X)\) and \(f_7(X)\) (see (2.3)). We also need Proposition 5.1.

Corollary 5.3. We have \(\gcd(f_m(X), f_n(X)) = 1\) whenever \(m, n \in \mathbb{N}\) and \(2 \mid n\).

Proof. Let \(\alpha_m \in \mathbb{C}\) be a root of both \(f_m(X)\) and \(f_n(X)\), and \(P_3\) be a maximal ideal of \(O_{\mathbb{Q}(\alpha_n)}\), such that \(3 \in P_3\). Then \(\alpha_m \in O_{\mathbb{Q}(\alpha_n)}\), so it follows from Proposition 5.1 and equality (2.3) (h) that \(\alpha_m^2 + \alpha_m \in P_3\), whence, \(\alpha_m \in P_3\) or \(\alpha_m + 1 \in P_3\). On the other hand, it is easy to see that \(f_m(0)\) and \(f_n(-1)\) are integers not divisible by 3, which implies \(f_n(\alpha_m) \notin P_3\). Our conclusion, however, contradicts the assumption that \(f_n(\alpha_m) = 0\), so Corollary 5.3 is proved.

Corollary 5.4. The equalities \(\gcd(f_m(X), f_n(X)) = \gcd(f_{7m}(X), f_n(X)) = 1\) hold, if \(m, n \in \mathbb{N}\) and \(n\) is divisible by 6.

Proof. It is verified by straightforward calculations that \(f_n(0) = f_n(-1) = 2\) and \(f_n(\varepsilon_3) = 3\), for each primitive cubic root of unity \(\varepsilon_3 \in \mathbb{C}\). None of these values is divisible by 5 or 7, so it is not difficult to deduce (in the spirit of the proof of Corollary 5.3) from Proposition 5.1 and the definitions of \(f_3(X)\) and \(f_7(X)\) that \(f_n(X)\) has a common root neither with \(f_{5m}(X)\) nor with \(f_{7m}(X)\).
The following result proves the equality \( \gcd(f_m(X), f_n(X)) = 1 \) in the case where \( m \) is odd, \( m < 99 \), \( m \) has precisely two different prime divisors, and \( m \) is not divisible by 9. Here we note that 45, 63 and 99 are all odd numbers less than 100 and divisible by 9, which have exactly two different prime divisors.

**Corollary 5.5.** Let \( q \in \{3, 5, 7\} \) and \( p \) be a prime number different from 2, 3 and \( q \). Then \( \gcd(f_{q^\nu}(X), f_n(X)) = 1 \) whenever \( \nu \in \mathbb{N} \), \( n \in \mathbb{N} \) and 6 divides \( q^n \).

**Proof.** This can be obtained, proceeding by reduction modulo \( p \), and arguing as in the proofs of Corollaries 5.2 and 5.4. We omit the details.

The next two statements prove that \( \gcd(f_m(X), f_n(X)) = 1 \), if \( m \in \{40, 80\} \).

**Corollary 5.6.** The polynomials \( f_{8,5^k}(X) \) and \( f_n(X) \) satisfy the equality \( \gcd(f_{8,5^k}(X), f_n(X)) = 1 \) whenever \( k \in \mathbb{N} \), \( n \in \mathbb{N} \), \( 3 \mid n \) and \( n \leq 100 \).

**Proof.** Proposition 3.3 allows us to consider only the case of \( n \equiv 6(\text{mod } 24) \). This amounts to assuming that \( n \) equals 6, 30, 54 or 78. Then our calculations show that \( 2 + \sqrt{2}, 2 - \sqrt{2}, 1 + 2\sqrt{2}, 1 - 2\sqrt{2}, -2 - 2\sqrt{2} \) and \( -2 + 2\sqrt{2} \) are roots of \( f_n(X) \), which implies \( \gcd(f_{8,5^k}(X), f_n(X)) \neq 1 \). Since, however, \( f_n(X) \) is \( \mathbb{Q} \)-irreducible, for each admissible \( n \) (see Proposition 3.2 and Remark 4.2), it follows from Proposition 3.1 (a) that \( \gcd(f_{8,5^k}(X), f_n(X)) = 1 \), as required.

**Corollary 5.7.** The polynomials \( f_{80}(X) \) and \( f_n(X) \in \mathbb{Z}[X] \) are relatively prime, for every \( n \in \mathbb{N} \) divisible by 3 and less than 100.

**Proof.** By Proposition 3.4 we have \( \gcd(f_{80}(X), f_n(X)) = 1 \) in the case where \( n \in \mathbb{N} \) is odd and divisible by 3. Note further that the same equality holds, under the condition that \( n \in \mathbb{N} \), \( 6 \mid n \) and \( n < 100 \). If \( n > 80 \), i.e. \( n = 84, 90 \) or 96, this follows from Proposition 3.2 and Remark 4.2. When \( n \leq 80 \), our assertion can be deduced from Propositions 3.1, 3.2 and Remark 4.2.

Summing-up the obtained results, one concludes that Theorem 1.1 will be proved, if we show that \( \gcd(f_m(X), f_n(X)) = 1 \), provided that \( n \in \mathbb{N} \) is even, \( n \leq 100 \), and \( m \in \{45, 63, 99\} \). We achieve this goal on a case-by-case basis.

**Corollary 5.8.** For each even \( n \in \mathbb{N} \), \( \gcd(f_{45}(X), f_n(X)) = 1 \).

**Proof.** Let \( \bar{f}_n \) be the reduction of \( f_n \) modulo 5, for each \( n \in \mathbb{N} \). With these notation, we have \( f_{45}(X) = f_5(X)^5 \) and \( f_5(X) = X(X+1)(X-1)^2(X-2)^2(X-3)^2 \). On the other hand, it is easily verified that \( f_n(0) = f_n(-1) = 2 \) and \( f_n(j) \in \{-1, 1, 3\} \subseteq \mathbb{Z}/5\mathbb{Z}, j = 1, 2, 3 \).

**Corollary 5.9.** The equality \( \gcd(f_{63}(X), f_n(X)) = 1 \) holds, for any even number \( n > 0 \) at most equal to 100.
Proof. Suppose first that $6 \mid n$. Then our assertion can be proved, by using Remark 2.2 and combining Proposition 3.3(a) with Proposition 3.2 and Remark 3.1. It remains to consider the case where $3 \nmid n$. When $n = 70$, our conclusion follows from Remark 3.3, and in case $n \neq 70$, the equality $\gcd(f_{63}(X), f_n(X)) = 1$ is implied by Propositions 3.3, 3.4 and 3.7.

We are now in a position to complete the proof of Theorem 1.1. Note first that every $t \in \mathbb{N}$ with at least 4 distinct prime divisors is greater than 100; also, 70 is the unique natural number less than 100 and not divisible by 3, which has 3 distinct prime divisors. Therefore, the preceding assertions lead to the conclusion that it is sufficient to prove the equality $\gcd(f_{63}(X), f_n(X)) = 1$, for every even $n \in \mathbb{N}$, $n \leq 100$ (the question of whether $\gcd(f_{63}(X), f_n(X)) = 1$ whenever $n \in \mathbb{N}$ and $2 \mid n$, is open). Suppose first that $6 \mid n$. Then the claimed equality follows from the fact the roots of $f_n(X)$ are not algebraic integers, whereas the common roots of $f_{63}(X)$ and $f_n(X)$ (if any) must be algebraic integers. Henceforth, we assume that $n \leq 100$ and $n$ is not divisible by 3. Applying Propositions 3.3, 3.4 and 3.7, one reduces the rest of the proof of Theorem 1.1 to its implementation in the special case where $n = 70$. Denote by $f_m$ the reduction of $f_m$ modulo 7, for each $m \in \mathbb{N}$. Note that $f_{70}(X) = f_{10}(X)^7$ and $f_{10}(X) = (X^2 + X + 1)^2 \tilde{g}_{10}(X)$, where $\tilde{g}_{10}(X) \in \mathbb{Z}[X]$ is a degree 6 polynomial decomposing into a product of three pairwise distinct quadratic polynomials lying in $(\mathbb{Z}/7\mathbb{Z})[X]$ and irreducible over $\mathbb{Z}/7\mathbb{Z}$. Suppose now that $\gcd(f_{63}(X), f_{70}(X)) \neq 1$ and fix a common root $\beta \in \mathbb{C}$ of $f_{63}(X)$ and $f_{70}(X)$. Then $\beta$ is an algebraic integer, and by the preceding observations on $\tilde{g}_{10}(X)$, $\beta^{48} = 1$. Here $\beta$ stands for the residue class of $\beta$ modulo some prime ideal $P$ in the ring $O_{\mathbb{Q}(\beta)}$ of algebraic integers in $\mathbb{Q}(\beta)$, chosen so that $7 \in P$. It is clear from the equality $\beta^{48} = 1$ that $f_{63}(\beta) = (\beta + 1)^3 - \beta^3 - 1 = 3(\beta + 1)\beta$ unless $\beta \in \{0, -1\}$. On the other hand, the assumption that $f_{63}(\beta) = f_{70}(\beta) = 0$ requires that $f_{63}(\beta) = 0$, which is possible only in case $\beta \in \{0, -1\}$. The obtained contradiction is due to the hypothesis that $f_{63}(\beta) = f_{70}(\beta) = 0$. Thus it follows that $\gcd(f_{63}(X), f_{70}(X)) = 1$, which completes the proof of the equality $\gcd(f_{63}(X), f_n(X)) = 1$, for each even $n \in \mathbb{N}$, $n \leq 100$. Theorem 1.1 is proved.

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