Tail Probabilities in Queueing Processes

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Abstract

In the study of large scale stochastic networks with resource management, differential equations and mean-field limits are two key techniques. Recent research shows that the expected fraction vector (that is, the tail probability vector) plays a key role in setting up mean-field differential equations. To further apply the technique of tail probability vector to deal with resource management of large scale stochastic networks, this paper discusses tail probabilities in some basic queueing processes including QBD processes, Markov chains of GI/M/1 type and of M/G/1 type, and also provides some effective and efficient algorithms for computing the tail probabilities by means of the matrix-geometric solution, the matrix-iterative solution, the matrix-product solution and the two types of $RG$-factorizations. Furthermore, we consider four queueing examples: The M/M/1 retrial queue, the M(n)/M(n)/1 queue, the M/M/1 queue with server multiple vacations and the M/M/1 queue with repairable server, where the M/M/1 retrial queue is given a detailed discussion, while the other three examples are analyzed in less detail. Note that the results given in this paper will be very useful in the study of large scale stochastic networks with resource management, including the supermarket models and the work stealing models.

Keywords: Randomized load balancing; supermarket model; work stealing model; QBD Process; Markov chain of the GI/M/1 type; Markov chain of the M/G/1 type.
1 Introduction

We consider a discrete-time (resp. continuous-time) Markov chain whose transition probability matrix (resp. infinitesimal generator) is given by

\[
P = \begin{pmatrix}
P_{0,0} & P_{0,1} & P_{0,2} & P_{0,3} & \cdots \\
P_{1,0} & P_{1,1} & P_{1,2} & P_{1,3} & \cdots \\
P_{2,0} & P_{2,1} & P_{2,2} & P_{2,3} & \cdots \\
P_{3,0} & P_{3,1} & P_{3,2} & P_{3,3} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

where the size of the matrix \(P_{0,0}\) is \(m_0\), the size of the matrix \(P_{j,j}\) is \(m\) for \(j \geq 1\), and the sizes of other matrices can be determined accordingly. We assume that the Markov chain \(P\) is irreducible, aperiodic and positive recurrent. Let \(x = (x_0, x_1, x_2, x_3, \ldots)\) be the stationary probability vector of the Markov chain \(P\), where the size of the vector \(x_0\) is \(m_0\) while the size of the vector \(x_j\) is \(m\) for \(j \geq 1\). The main purpose of this paper is to discuss the tail probabilities: \(\pi_k = \sum_{j=k}^{\infty} x_j\) and to provide some efficient algorithms for computing the tail probabilities \(\pi_k\) for \(k \geq 1\).

Recent queueing literature indicates that the study of tail probabilities \(\{\pi_k, k \geq 0\}\) plays a key role in analyzing large scale stochastic networks with resource management, such as, the supermarket models and the work stealing models, e.g., see Vvedenskaya and Suhov \[39\] and Mitzenmacher \[27\]. When considering a large scale stochastic network with resource management, differential equations and mean-field limits are always two key techniques, while the tail probabilities play a key role in setting up mean-field differential equations. The detailed interpretation on the mean-field differential equations was given in Vvedenskaya et al \[38\], Mitzenmacher \[25\], Ethier and Kurtz \[10\] and Kurtz \[14\]. In the first two papers, the authors considered a supermarket model with \(N\) identical servers, where the service times are exponential with service rate \(\mu\), and the input flow is Poisson with arrival rate \(N\lambda\). Upon arrival, each customer chooses \(d \geq 1\) servers from the \(N\) servers independently and uniformly at random, and joins the one whose queue length is the shortest. Let \(n_k^{(N)}(t)\) denote the number of servers queued by at least \(k \geq 0\) customers at time \(t\), and \(u_k(t) = \lim_{N \to \infty} E\left[n_k^{(N)}(t)/N\right]\). If \(\rho = \lambda/\mu < 1\), then the supermarket model is stable, and

\[
\frac{d}{dt}u_k(t) = \lambda \left\{ [u_{k-1}(t)]^d - [u_k(t)]^d \right\} - \mu [u_k(t) - u_{k+1}(t)]
\]

(1)
with the boundary condition \( u_0(t) = 1 \). We write that \( \pi_k = \lim_{t \to +\infty} u_k(t) \) for \( k \geq 0 \). Then \( \pi_0 = 1 \) and for \( k \geq 1 \)

\[
\lambda \left( \pi_{k-1}^d - \pi_k^d \right) - \mu \left( \pi_k - \pi_{k+1} \right) = 0.
\]

(2)

This gives

\[
\pi_k = \rho \frac{d^k}{d!} , \quad k \geq 1.
\]

Specifically, \( \pi_1 = \rho \) is directly derived by

\[
\lambda \sum_{k=1}^{\infty} \left( \pi_{k-1} - \pi_k \right) - \mu \sum_{k=1}^{\infty} \left( \pi_k - \pi_{k+1} \right) = 0.
\]

If \( d = 1 \), then \( \pi_k = \rho^k \) for \( k \geq 0 \) are the tail probabilities of the M/M/1 queue.

Since the introduction of the expected fraction vector (or the tail probability vector) by Vvedenskaya et al [38] and Mitzenmacher [25], research on supermarket models and work stealing models has been greatly motivated by some practical applications such as computer networks, manufacturing systems and transportation networks. Subsequent papers have been published on this theme, among which, see, modeling more crucial factors by Mitzenmacher [26, 27], Jacquet and Vvedenskaya [12], Jacquet et al [13] and Vvedenskaya and Suhov [40]; studying fast Jackson networks by Martin and Suhov [24], Martin [23] and Suhov and Vvedenskaya [36]; discussing value of information by Mitzenmacher [28] and Mitzenmacher et al [29]; analyzing non-exponential server times and/or non-Poisson inputs by Mitzenmacher [25], Vvedenskaya and Suhov [39], Bramson [3], Bramson et al [4, 5, 6], Li et al [20], Li and Lui [19] and Li [17]. For a comprehensive analysis of supermarket models and work stealing models, readers may refer to Vvedenskaya and Suhov [39], Mitzenmacher et al [30] and Mitzenmacher and Upfal [31]. From those papers, it is seen that the tail probabilities \( \{ \pi_k, k \geq 0 \} \) is obtained from the mean-field differential equations as \( N \to \infty \) and \( t \to +\infty \), and also it is a key to analyze performance measures of the supermarket models and of the work stealing models.

During the last two decades considerable attention has been paid to studying QBD processes, which has been well documented, for example, by Chapter 3 of Neuts [33], Naoumov [32], Bright and Taylor [7, 8], Ramaswami [35], Latouche and Ramaswami [15] and Li and Cao [18]. For Markov chains of GI/M/1 type and Markov chains of M/G/1 type, readers may refer to four excellent books by Neuts [33, 34], Latouche and Ramaswami [15] and Li [16].
Some papers were published on asymptotic behavior of the stationary probability vectors for both queueing systems and Markov chains. Readers may refer to, such as, Markov chains of $GI/M/1$ type by Neuts [33]; Markov chains of $M/G/1$ type by Falkenberg [11], Abate et al [1], Choudhury and Whitt [9], Asmussen and Møller [2] and Takine [37]; and Markov chains of $GI/G/1$ type by Li and Zhao [21, 22].

The main purpose of this paper is to provide some novel and efficient algorithms for computing the tail probabilities in three classes of important Markov chains: QBD processes, Markov chains of $GI/M/1$ type and Markov chains of $M/G/1$ type. Note that the algorithms are based on the matrix-geometric solution, the matrix-iterative solution, the matrix-product solution and the two types of $RG$-factorizations. Also, we consider four queueing examples: The $M/M/1$ retrial queue, the $M(n)/M(n)/1$ queue, the $M/M/1$ queue with server multiple vacations and the $M/M/1$ queue with repairable server. Based on this, it is seen that the method of this paper can deal with more general queue examples, such as, the MAP/PH/1 queue, the GI/PH/1 queue and the BMAP/SM/1 queue. Therefore, the results of this paper are very useful in setting up the mean-field differential equations for large scale stochastic networks with resource management, including the supermarket models and the work stealing models.

The remainder of this paper is organized as follows. In Section 2, we analyze a continuous-time level-independent QBD process. When the QBD process is irreducible, aperiodic and positive recurrent, we apply the matrix-geometric solution and the two types of $RG$-factorizations to compute the tail probabilities in the stationary regime. In Section 3, we consider an continuous-time level-dependent QBD process, and compute the tail probabilities in the stationary regime. In Section 4, we discuss two classes of important Markov chains: Markov chains of $GI/M/1$ type and of $M/G/1$ type, and derive the tail probabilities in the stationary regime. In Section 5, we study four queueing examples, where the $M/M/1$ retrial queue is given a detailed discussion, while the other three queues are analyzed in less detail. Some concluding remarks are given in Section 6.

2 Level-Independent QBD Processes

In this section, we consider a continuous-time level-independent QBD process. When the QBD process is irreducible, aperiodic and positive recurrent, we apply the two types of $RG$-factorizations to compute the tail probabilities in the stationary regime. Furthermore, the
tail probabilities of the stationary probability vector is well related to the matrix-geometric solution when the UL-type $RG$-factorization is used.

We consider a continuous-time level-independent QBD process whose infinitesimal generator is given by

$$Q = \begin{pmatrix} B_1 & B_0 \\ B_2 & A_1 & A_0 \\ & A_2 & A_1 & A_0 \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

(3)

where the sizes of the two matrices $B_1$ and $A_1$ are $m_0$ and $m$, respectively; and the sizes of other matrices can be determined accordingly. We assume that this QBD process is irreducible, aperiodic and positive recurrent.

2.1 The matrix-geometric solution

Let $x = (x_0, x_1, x_2, \ldots)$ be the stationary probability vector of the QBD process, and $R$ and $G$ the minimal nonnegative solutions to the nonlinear equations $A_0 + RA_1 + R^2A_2 = 0$ and $A_0G^2 + A_1G + A_2 = 0$, respectively. Then

$$x_k = x_1 R^{k-1}, \quad k \geq 2,$$

(4)

where $x_0$ and $x_1$ are uniquely determined by the following system of linear equations

$$\begin{cases} 
    x_0B_1 + x_1B_2 = 0, \\
    x_0B_0 + x_1(A_1 + RA_2) = 0, \\
    x_0e + x_1(I - R)^{-1}e = 1,
\end{cases}$$

where $e$ is a column vector of ones.

We write

$$\pi_k = \sum_{j=k}^{\infty} x_j, \quad k \geq 1.$$

(5)

It follows from (4) that

$$\pi_k = x_1 (I - R)^{-1} R^{k-1}, \quad k \geq 1.$$

(6)

2.2 The UL-type $RG$-factorization

Now, we apply the UL-type $RG$-factorization to provide a novel method for deriving the tail probability vector $\pi = (\pi_1, \pi_2, \pi_3, \ldots)$. 

5
Note that $xQ = 0$, so we have

$$
\begin{align*}
&\begin{cases}
x_0 B_0 + x_1 A_1 + x_2 A_2 = 0, & k = 1, \\
x_{k-1} A_0 + x_k A_1 + x_{k+1} A_2 = 0, & k \geq 2.
\end{cases} \\
&\quad (7)
\end{align*}
$$

This gives

$$
\begin{align*}
&\begin{cases}
\pi_1 (A_0 + A_1) + \pi_2 A_2 = -x_0 B_0, & k = 1, \\
\pi_{k-1} A_0 + \pi_k A_1 + \pi_{k+1} A_2 = 0, & k \geq 2.
\end{cases}
\end{align*}
$$

Hence we obtain

$$
\pi Q = -(x_0 B_0, 0, 0, 0, \ldots),
$$

where

$$
Q = \begin{pmatrix}
A_0 + A_1 & A_0 \\
A_2 & A_1 & A_0 \\
A_2 & A_1 & A_0 \\
\vdots & \vdots & \vdots
\end{pmatrix}.
$$

Let

$$
\Phi_0 = (A_0 + A_1) + RA_2,
$$

$$
\Phi_k = \Phi = A_1 + RA_2, \quad k \geq 1.
$$

Then the UL-type $RG$-factorization of the matrix $Q$ is given by

$$
Q = (I - RU) U (I - GL),
$$

where

$$
RU = \begin{pmatrix}
0 & R \\
0 & R \\
0 & R \\
\vdots & \vdots
\end{pmatrix},
$$

$$
U = \text{diag} (\Phi_0, \Phi, \Phi, \ldots)
$$

and

$$
GL = \begin{pmatrix}
0 & G & 0 & \cdots \\
g & 0 & \cdots \\
g & 0 & \cdots \\
\vdots & \vdots & \ddots & \ddots
\end{pmatrix}.
$$
It follows from (8) and (9) that
\[
\pi = -(x_0 B_0, 0, 0, 0, \ldots) (I - G_L)^{-1} U^{-1} (I - R_U)^{-1}
\]
\[
= -(x_0 B_0 \Phi_0^{-1}, 0, 0, 0, \ldots) (I - R_U)^{-1}.
\]
Note that
\[
(I - R_U)^{-1} = \begin{pmatrix} I & R & R^2 & R^3 & \cdots \\ I & R & R^2 & \cdots \\ I & R & \cdots \\ \vdots & \end{pmatrix},
\]
and so we obtain
\[
\begin{cases}
\pi_1 = x_0 B_0 (-\Phi_0^{-1}), & k = 1, \\
\pi_k = x_0 B_0 (-\Phi_0^{-1}) R^{k-1}, & k \geq 2.
\end{cases}
\]
Comparing (10) with (6), we obtain
\[
x_0 B_0 (-\Phi_0^{-1}) = x_1 (I - R)^{-1}.
\]
This gives
\[
x_1 = x_0 B_0 (-\Phi_0^{-1}) (I - R).
\]

### 2.3 The LU-type RG-factorization

In what follows we apply the LU-type RG-factorization to provide a novel and effective method for deriving the tail probability vector \(\pi = (\pi_1, \pi_2, \pi_3, \ldots)\).

Let
\[
\Psi_0 = A_0 + A_1
\]
and for \(k \geq 1\)
\[
\Psi_k = A_1 + A_2 (-\Psi_{k-1}^{-1}) A_0.
\]
We write that for \(k \geq 1\)
\[
R_k = A_2 (-\Psi_{k-1}^{-1})
\]
and
\[
G_{k-1} = (-\Psi_{k-1}^{-1}) A_0.
\]
Then the LU-type RG-factorization of the matrix $Q$ is given by

$$Q = (I - R_L) U (I - G_U),$$  \hspace{1cm} (16)$$

where

$$R_L = \begin{pmatrix}
0 & & \\
R_1 & 0 & \\
& R_2 & 0 \\
& & R_3 & 0 \\
& & & \ddots & \ddots
\end{pmatrix},$$

$$U = \text{diag} (\Psi_0, \Psi_1, \Psi_2, \ldots)$$

and

$$G_U = \begin{pmatrix}
0 & G_0 & & \\
0 & & G_1 & \\
& & 0 & G_2 \\
& & & \ddots & \ddots
\end{pmatrix}.$$ 

Let

$$X_k^{(l)} = R_l R_{l-1} R_{l-2} \cdots R_{l-k+1}, \quad l \geq k \geq 1,$$

$$Y_k^{(l)} = G_l G_{l+1} G_{l+2} \cdots G_{l+k-1}, \quad k \geq 1, l \geq 0.$$ 

Then

$$-U^{-1} = \text{diag} (-\Psi_0^{-1}, -\Psi_1^{-1}, -\Psi_2^{-1}, -\Psi_3^{-1}, \ldots),$$

$$(I - R_L)^{-1} = \begin{pmatrix}
I & & \\
X_1^{(1)} & I & \\
X_2^{(2)} & X_1^{(2)} & I \\
X_3^{(3)} & X_2^{(3)} & X_1^{(3)} & I \\
& \ddots & \ddots & \ddots \\
& & & & \ddots
\end{pmatrix},$$

and

$$(I - G_U)^{-1} = \begin{pmatrix}
I & Y_1^{(0)} & Y_2^{(0)} & Y_3^{(0)} & \cdots \\
I & Y_1^{(1)} & Y_2^{(1)} & \cdots \\
I & Y_1^{(2)} & \cdots \\
& \ddots & \ddots
\end{pmatrix}.$$
It follows from (8) and (9) that
\[
\pi = -(x_0 B_0, 0, 0, 0, \ldots) (I - G_U)^{-1} U^{-1} (I - R_L)^{-1},
\]
this gives
\[
\pi_1 = x_0 B_0 \left[ (-\Psi_0^{-1}) + \sum_{k=1}^{\infty} Y_k^{(0)} (-\Psi_k^{-1}) X_k^{(k)} \right],
\]
and \(n \geq 2\)
\[
\pi_n = x_0 B_0 \left[ Y_n^{(0)} (-\Psi_{n-1}^{-1}) + \sum_{k=n}^{\infty} Y_k^{(0)} (-\Psi_k^{-1}) X_k^{(k)} \right].
\]

The expressions (17) and (18) for the tail probability vector \(\{\pi_k : k \geq 1\}\) seem complicated, but they can easily be computed by means of the iterative relations (12) to (15) through some simple matrix calculations.

### 3 Level-Dependent QBD Processes

In this section, we consider a continuous-time level-dependent QBD process. When the QBD process is irreducible, aperiodic and positive recurrent, we apply the LU-type RG-factorization to compute the tail probabilities in the stationary regime. Similarly, we can apply the UL-type RG-factorization to compute the tail probabilities without any difficulty.

We consider a continuous-time level-dependent QBD process whose infinitesimal generator is given by
\[
Q = \begin{pmatrix}
A_1^{(0)} & A_0^{(0)} \\
A_1^{(1)} & A_0^{(1)} \\
A_1^{(2)} & A_0^{(2)} \\
\vdots & \vdots & \ddots
\end{pmatrix},
\]
where the size of the matrix \(A_1^{(0)}\) is \(m_0\) while the size of the matrix \(A_1^{(k)}\) is \(m\) for \(k \geq 1\). We assume that this QBD process is irreducible, aperiodic and positive recurrent.

#### 3.1 The matrix-product solution

Let the matrix sequence \(\{R_l, l \geq 0\}\) be the minimal nonnegative solution to the system of nonlinear matrix equations
\[
A_0^{(l)} + R_l A_1^{(l+1)} + R_l R_{l+1} A_2^{(l+2)} = 0, \quad l \geq 0.
\]
Using Chapter 1 of Li [16], we have
\[ x_0 = \kappa v, \quad (20) \]
\[ x_k = \kappa v R_0 R_1 \cdots R_{k-1}, \quad k \geq 1, \quad (21) \]
where \( v \) is the stationary probability vector of the censored chain \( U_0 = A_1^{(0)} + R_0 A_2^{(1)} \) to level 0, and the constant \( \kappa \) is given by
\[ \kappa = \frac{1}{1 + v \left( \sum_{k=0}^{\infty} R_0 R_1 \cdots R_k \right) e} \]

Therefore, it follows from (20) and (21) that
\[ \pi_k = \sum_{j=k}^{\infty} x_j = \kappa v R_0 R_1 \cdots R_{k-1} \left( I + \sum_{l=0}^{\infty} R_k R_{k+1} \cdots R_{k+l} \right). \]

### 3.2 The LU-type \( RG \)-factorization

Here, we only provide a detailed analysis for applying the LU-type \( RG \)-factorization to compute the tail probabilities, while the UL-type \( RG \)-factorization can be used similarly for such an analysis.

Since
\[ \pi_k = \sum_{j=k}^{\infty} x_j, \quad k \geq 1, \]
it is easy to see that \( x_k = \pi_k - \pi_{k+1} \) for \( k \geq 1 \). Note that \( xQ = 0 \), we have
\[
\begin{align*}
\pi_1 A_1^{(1)} + \pi_2 \left[ A_2^{(2)} - A_1^{(1)} \right] - \pi_3 A_2^{(2)} &= -x_0 A_0^{(0)}, \\
\pi_{k-1} A_0^{(k-1)} + \pi_k \left[ A_1^{(k)} - A_0^{(k-1)} \right] + \pi_{k+1} \left[ A_2^{(k+1)} - A_1^{(k)} \right] - \pi_{k+2} A_2^{(k+1)} &= 0, \quad k \geq 2,
\end{align*}
\]
we obtain that \( \pi Q = - \left( x_0 A_0^{(0)}, 0, 0, 0, \ldots \right) \), where
\[
Q = \begin{pmatrix}
A_1^{(1)} & A_0^{(1)} \\
A_2^{(2)} - A_1^{(1)} & A_1^{(2)} - A_0^{(1)} & A_0^{(2)} \\
- A_2^{(3)} & A_2^{(3)} - A_1^{(2)} & A_1^{(3)} - A_0^{(2)} & A_0^{(3)} \\
- A_2^{(4)} & A_2^{(4)} - A_1^{(3)} & A_1^{(4)} - A_0^{(3)} & A_0^{(4)} \\
- A_2^{(5)} & A_2^{(5)} - A_1^{(4)} & A_1^{(5)} - A_0^{(4)} & A_0^{(5)} \\
& & & \ddots
\end{pmatrix}
\]

This gives
\[ \pi \left( \tilde{Q} - Q \right) = - \left( x_0 A_0^{(0)}, 0, 0, 0, \ldots \right), \quad (22) \]
where

$$
\hat{Q} = \begin{pmatrix}
A_1^{(1)} & A_0^{(1)} \\
A_2^{(2)} & A_1^{(2)} & A_0^{(2)} \\
A_2^{(3)} & A_1^{(3)} & A_0^{(3)} \\
& & & \ddots & \ddots & \ddots
\end{pmatrix}
$$

and

$$
\mathbf{0} = (0, 0, 0, 0, \ldots), \ 0 \text{ is an } m \times m \text{ zero matrix,}

Q = \begin{pmatrix}
\mathbf{0} \\
\hat{Q}
\end{pmatrix}.
$$

It follows from (22) that

$$
\pi \left( I - Q \hat{Q}^{-1}_{\text{max}} \right) = - \left( x_0 A_0^{(0)}, 0, 0, 0, \ldots \right) \hat{Q}^{-1}_{\text{max}},
$$

where \( \hat{Q}^{-1}_{\text{max}} \) is the maximal non-positive inverse of the infinitesimal generator \( \hat{Q} \). Hence, this gives

$$
\pi = - \left( x_0 A_0^{(0)}, 0, 0, 0, \ldots \right) \hat{Q}^{-1}_{\text{max}} \sum_{k=0}^{\infty} \left( Q \hat{Q}^{-1}_{\text{max}} \right)^k.
$$

Now, we apply the LU-type \( RG \)-factorization to provide the maximal non-positive inverse \( \hat{Q}^{-1}_{\text{max}} \) of the infinitesimal generator \( \hat{Q} \). To that end, we write

$$
\Psi_0 = A_1^{(1)}
$$

and for \( k \geq 1 \)

$$
\Psi_k = A_1^{(k+1)} + A_2^{(k+1)} (-\Psi_{k-1}^{-1}) A_0^{(k)}.
$$

It is easy to check that \( \Psi_l \) is the infinitesimal generator of an irreducible continuous-time Markov chain, and the Markov chain \( \Psi_l \) is transient. Thus the matrix \( \Psi_l \) is invertible for \( l \geq 0 \).

Based on the \( U \)-measure \( \{ \Psi_l \} \), for \( k \geq 1 \) we can respectively define the LU-type \( R \)- and \( G \)-measures as

$$
R_k = A_2^{(k+1)} (-\Psi_{k-1}^{-1})
$$

and

$$
G_{k-1} = (-\Psi_{k-1}^{-1}) A_0^{(k)}.
$$

Note that the matrix sequence \( \{ R_k : k \geq 1 \} \) is the unique nonnegative solution to the system of nonlinear matrix equations

$$
R_{k+1} A_0^{(k)} + R_{k+1} A_1^{(k+1)} + A_2^{(k+2)} = 0,
$$
with the boundary condition
\[ \mathbf{R}_1 = A_2^{(2)} (-\Psi_0^{-1}). \]

Hence we obtain
\[ \mathbf{R}_{k+1} = -A_2^{(k+2)} \left[ \mathbf{R}_k A_0^{(k)} + A_1^{(k+1)} \right]^{-1}. \]

Similarly, the matrix sequence \( \{ \mathbf{G}_k : k \geq 0 \} \) is the unique nonnegative solution to the system of nonlinear matrix equations
\[ A_0^{(k+1)} + A_1^{(k+1)} \mathbf{G}_k + A_2^{(k+1)} \mathbf{G}_{k-1} = 0, \]
with the boundary condition
\[ \mathbf{G}_0 = (-\Psi_0^{-1}) A_0^{(1)}. \]

Thus we obtain
\[ \mathbf{G}_k = -\left[ A_1^{(k+1)} + A_2^{(k+1)} \mathbf{G}_{k-1} \right]^{-1} A_0^{(k+1)}. \]

The LU-type RG-factorization of the QBD process \( \tilde{Q} \) is given by
\[ \tilde{Q} = (I - \mathbb{R}_L) U_D (I - \mathbb{G}_U), \tag{23} \]
where
\[ \mathbb{R}_L = \begin{pmatrix}
0 & 0 \\
\mathbf{R}_1 & 0 \\
\mathbf{R}_2 & 0 \\
\mathbf{R}_3 & 0 \\
& \ddots
\end{pmatrix}, \]
\[ U_D = \text{diag} (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \ldots), \]
\[ \mathbb{G}_U = \begin{pmatrix}
0 & \mathbf{G}_0 \\
0 & \mathbf{G}_1 \\
0 & \mathbf{G}_2 \\
& \ddots
\end{pmatrix}. \]

Let
\[ X_k^{(l)} = \mathbf{R}_l \mathbf{R}_{l-1} \mathbf{R}_{l-2} \cdots \mathbf{R}_{l-k+1}, \quad l \geq k \geq 1, \tag{24} \]
and
\[ Y_k^{(l)} = \mathbf{G}_l \mathbf{G}_{l+1} \mathbf{G}_{l+2} \cdots \mathbf{G}_{l+k-1}, \quad k \geq 1, l \geq 0. \tag{25} \]
Then

\[(I - R_L)^{-1} = \begin{pmatrix}
I & X_1^{(1)} & I \\
X_2^{(2)} & X_1^{(2)} & I \\
X_3^{(3)} & X_2^{(3)} & X_1^{(3)} & I \\
& \vdots & \vdots & \vdots & \ddots
\end{pmatrix}\]

and

\[(I - G_U)^{-1} = \begin{pmatrix}
I & Y_1^{(0)} & Y_2^{(0)} & Y_3^{(0)} & \cdots \\
I & Y_1^{(1)} & Y_2^{(1)} & \cdots \\
I & Y_1^{(2)} & \cdots \\
& \ddots
\end{pmatrix}.

Hence we obtain

\[
\pi = - \left( x_0 A_0^{(1)}, 0, 0, 0, \ldots \right) \hat{Q}_\text{max}^{-1} \sum_{k=0}^{\infty} \left( \hat{Q} \hat{Q}_\text{max}^{-1} \right)^{k}
\]

\[
= - \left( x_0 A_0^{(1)}, 0, 0, 0, \ldots \right) (I - G_U)^{-1} U_D^{-1} (I - R_L)^{-1} \sum_{k=0}^{\infty} \left[ \hat{Q} (I - G_U)^{-1} U_D^{-1} (I - R_L)^{-1} \right]^k.
\]

This can be calculated by some ordinary matrix computation.

### 4 Two Classes of Important Markov Chains

In this section, we consider two classes of important Markov chains: Markov chains of GI/M/1 type and of M/G/1 type, each of which is basic in the study of queueing processes, e.g., see Neuts [33, 34] for more details. We provide two different methods to derive the tail probabilities of stationary probability vectors of the two classes of Markov chains.
4.1 Markov chains of GI/M/1 type

We consider a discrete-time Markov chain $P$ of GI/M/1 type whose transition matrix is given by

$$
P = \begin{pmatrix}
B_1 & B_0 \\
B_2 & A_1 & A_0 \\
B_3 & A_2 & A_1 & A_0 \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
$$

where the sizes of the two matrices $B_1$ and $A_1$ are $m_0$ and $m$, respectively, while the sizes of other matrices can be determined accordingly. We assume that this Markov chain is irreducible, aperiodic and positive recurrent. Let the matrix $R$ be the minimal nonnegative solution to the nonlinear matrix equation $R = \sum_{k=0}^{\infty} R^k A_k$.

In what follows we provide two methods to derive the tail probabilities in the stationary regime.

(a) The matrix-geometric solution

Using Chapter 2 of Li [16], the stationary probability vector $x = (x_0, x_1, x_2, \ldots)$ is given by

$$
\begin{cases}
x_0 = \tau y_0, \\
x_k = x_0 R_1 R^{k-1}, & k \geq 1,
\end{cases}
$$

where

$$
R_1 = \left(I - \sum_{k=0}^{\infty} R^k B_{k+1}\right)^{-1} B_0
$$

and

$$
\Psi_0 = \sum_{k=0}^{\infty} R^k B_{k+1},
$$

$y_0$ is the stationary probability vector of the censored Markov chain $\Psi_0$ to level 0, and the scalar $\tau$ is determined by

$$
\tau = \frac{1}{1 + y_0 R_1 (I - R)^{-1} e}.
$$

Thus, for $k \geq 1$ we have

$$
\pi_k = \sum_{j=k}^{\infty} x_j = x_0 R_1 (I - R)^{-1} R^{k-1}.
$$

(27)
(b) The UL-type RG-factorization

Note that $x = xP$, we obtain

\[
\begin{align*}
\pi_1 &= \pi_1 (A_0 + A_1) + \sum_{k=2}^{\infty} \pi_k A_k + x_0 B_0, \quad k = 1, \\
\pi_k &= \sum_{j=0}^{\infty} \pi_{k-1+j} A_j, \quad k \geq 2,
\end{align*}
\]

this gives

\[\pi = \pi^P + (x_0 B_0, 0, 0, \ldots), \quad (28)\]

where

\[
\mathbb{P} = \begin{pmatrix}
A_0 + A_1 & A_0 \\
A_2 & A_1 & A_0 \\
A_3 & A_2 & A_1 & A_0 \\
A_4 & A_3 & A_2 & A_1 & A_0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

which is of GI/M/1 type. Then using Chapter 2 of Li [16], the $U$-measure is given by

\[\hat{\Psi}_0 = (A_0 + A_1) + \sum_{k=2}^{\infty} R_k R^{k-2} A_k\]

and for $k \geq 1$

\[\hat{\Psi} = \hat{\Psi}_k = \sum_{k=1}^{\infty} R^{k-1} A_k;\]

the $R$-measure is given by

\[R_k = R, \quad k \geq 1,\]

and the $G$-measure

\[G_{j;0} = \left( I - \hat{\Psi} \right)^{-1} \left( \sum_{k=j+1}^{\infty} R^{k-1} A_k \right), \quad j \geq 1,\]

and

\[G_j = \left( I - \hat{\Psi} \right)^{-1} \left( \sum_{k=j+1}^{\infty} R^{k-1} A_k \right), \quad j \geq 1.\]

Thus, the UL-type RG-factorization is given by

\[I - \mathbb{P} = (I - R_U) (I - \Phi_D) (I - G_L), \quad (29)\]
where
\[ R_U = \begin{pmatrix} 0 & R \\ 0 & R \\ \vdots & \vdots \end{pmatrix}, \]
\[ \Phi_D = \text{diag}(\hat{\Psi}_0, \hat{\Psi}, \hat{\Psi}, \ldots) \]
and
\[ GL = \begin{pmatrix} 0 \\ G_{1,0} & 0 \\ G_{2,0} & G_1 & 0 \\ G_{3,0} & G_2 & G_1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \]

It follows from (28) and (29) that
\[ \pi = (x_0B_0, 0, 0, 0, \ldots) (I - P)^{-1}_{\min} \]
\[ = (x_0B_0, 0, 0, 0, \ldots) (I - G_L)^{-1} (I - \Phi_D)^{-1} (I - R_U)^{-1} \]
\[ = \left( x_0B_0 \left( I - \hat{\Psi}_0 \right)^{-1}, 0, 0, 0, \ldots \right) (I - R_U)^{-1}, \]
where \((I - P)^{-1}_{\min} = \sum_{k=0}^{\infty} P^k\). Note that
\[ (I - R_U)^{-1} = \begin{pmatrix} I & R & R^2 & R^3 & \cdots \\ I & R & R^2 & \cdots \\ \vdots & I & R & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \]
we obtain
\[ \pi_k = x_0B_0 \left( I - \hat{\Psi}_0 \right)^{-1} R^{k-1}, \quad k \geq 1. \quad (30) \]

Comparing (30) with (27), we obtain
\[ x_0B_0 \left( I - \hat{\Psi}_0 \right)^{-1} = x_0R_1 (I - R)^{-1}. \]
4.2 Markov chains of M/G/1 type

We consider a discrete-time Markov chain $P$ of M/G/1 type whose transition matrix is given by

$$
P = \begin{pmatrix}
    B_1 & B_2 & B_3 & B_4 & \cdots \\
    B_0 & A_1 & A_2 & A_3 & \cdots \\
    & A_0 & A_1 & A_2 & \cdots \\
    & & A_0 & A_1 & \cdots \\
    & & & \ddots & \ddots
\end{pmatrix},
$$

(31)

where the sizes of the two matrices $B_1$ and $A_1$ are $m_0$ and $m$, respectively, while the sizes of other matrices can be determined accordingly. We assume that this Markov chain is irreducible, aperiodic and positive recurrent. Let the matrix $G$ be the minimal nonnegative solution to the nonlinear matrix equation $G = \sum_{k=0}^{\infty} A_k G^k$.

In what follows we provide two methods to derive the tail probabilities in the stationary regime.

(a) The matrix-iterative solution

Using Chapter 2 of Li [16], the $U$-measure is given by

$$
\Psi_0 = B_1 + \sum_{k=2}^{\infty} B_k G^{k-2} G_1
$$

and for $k \geq 1$

$$
\Psi = \Psi_k = \sum_{k=1}^{\infty} A_k G^{k-1};
$$

and the $R$-measure

$$
R_{0,j} = \left( \sum_{k=j+1}^{\infty} B_k G^{k-1} \right) (I - \Psi)^{-1}, \quad j \geq 1,
$$

and

$$
R_j = \left( \sum_{k=j+1}^{\infty} A_k G^{k-1} \right) (I - \Psi)^{-1}, \quad j \geq 1.
$$

The stationary probability vector $x = (x_0, x_1, x_2, \ldots)$ is given by

$$
\begin{cases}
    x_0 = \tau y_0, \\
    x_k = x_0 R_{0,k} + \sum_{i=1}^{k-1} x_i R_{k-i}, \quad k \geq 1,
\end{cases}
$$

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where $y_0$ is the stationary probability vector of the censored Markov chain $\Psi_0$ to level 0 and the scalar $\tau$ is determined by $\sum_{k=0}^{\infty} x_k e = 1$ uniquely. Thus, we obtain
\[ \pi_k = \sum_{j=k}^{\infty} x_j = x_0 \sum_{j=k}^{\infty} R_{0,j} + x_1 \sum_{j=k}^{\infty} R_{j-1}, \quad k \geq 1. \] (32)

(b) The UL-type RG-factorization

Note that $x = x_P$, for $k \geq 1$ we obtain
\[ \pi_k = x_0 \sum_{j=k+1}^{\infty} B_j + \sum_{i=2}^{k+1} \pi_i A_{k+1-i} + \pi_1 \sum_{j=k}^{\infty} A_j, \]
this gives
\[ \pi = \pi P + \left( x_0 \sum_{j=2}^{\infty} B_j, x_0 \sum_{j=3}^{\infty} B_j, x_0 \sum_{j=4}^{\infty} B_j, \ldots \right), \] (33)
where
\[
P = \begin{pmatrix}
\sum_{j=1}^{\infty} A_j & \sum_{j=2}^{\infty} A_j & \sum_{j=3}^{\infty} A_j & \cdots \\
A_0 & A_1 & A_2 & \cdots \\
A_0 & A_1 & \cdots \\
A_0 & \cdots \\
& \ddots
\end{pmatrix}.
\]
Using Chapter 2 of Li [16], the $U$-measure is given by
\[ \hat{\Psi}_0 = \sum_{j=1}^{\infty} A_j + \sum_{k=2}^{\infty} \sum_{j=k}^{\infty} A_j G^{k-2} G_1 \]
and for $k \geq 1$
\[ \hat{\Psi} = \hat{\Psi}_k = \sum_{i=1}^{\infty} A_i G^{i-1}; \]
and the $R$-measure
\[ R_{0,j} = \left( \sum_{k=j+1}^{\infty} B_k G^{k-1} \right) (I - \hat{\Psi})^{-1}, \quad j \geq 1, \]
and
\[ R_j = \left( \sum_{k=j+1}^{\infty} A_k G^{k-1} \right) (I - \hat{\Psi})^{-1}, \quad j \geq 1. \]
Thus, the UL-type RG-factorization is given by
\[ I - P = (I - R_U) (I - \Psi_D) (I - G_L), \] (34)
where

\[ R_U = \begin{pmatrix}
0 & R_{0,1} & R_{0,2} & R_{0,3} & \cdots \\
0 & R_1 & R_2 & \cdots \\
0 & R_1 & \cdots \\
0 & \cdots \\
\vdots & \vdots & \ddots & \ddots \\
\end{pmatrix}, \]

\[ \Psi_D = \text{diag} (\tilde{\Psi}_0, \tilde{\Psi}, \tilde{\Psi}, \ldots) \]

and

\[ G_L = \begin{pmatrix}
0 \\
G_1 & 0 \\
G & 0 \\
\vdots & \vdots \\
\end{pmatrix}. \]

It follows from \( \psi \approx 3 \) and \( \psi \approx 4 \) that

\[
\pi = \left( x_0 \sum_{j=2}^{\infty} B_j, x_0 \sum_{j=3}^{\infty} B_j, x_0 \sum_{j=4}^{\infty} B_j, \ldots \right) (I - \mathbf{P})_{\text{min}}^{-1}
\]

\[
= \left( x_0 \sum_{j=2}^{\infty} B_j, x_0 \sum_{j=3}^{\infty} B_j, x_0 \sum_{j=4}^{\infty} B_j, \ldots \right) (I - G_L)^{-1} (I - \Psi_D)^{-1} (I - R_U)^{-1}.
\]

5 Some Queueing Examples

In this section, we consider four queueing examples which indicate how to use our above results. We first provide a detailed discussion for the M/M/1 retrial queue with exponentially distributed retrial times. Then we simply analyze other three queueing examples: The M(n)/M(n)/1 queue, the M/M/1 queue with server multiple vacations, and the M/M/1 queue with repairable server.

5.1 The M/M/1 retrial queue

We consider an M/M/1 retrial queue with exponentially distributed retrial times, where the arrival, service and retrial rates are \( \lambda, \mu \) and \( \theta \), respectively. We denote by \( N (t) \) and \( C (t) \) the number of customers in the orbit and the state of server at time \( t \), respectively,
where $N(t) = 0, 1, 2, \ldots$ and $C(t) = W$ for the busy server or $I$ for the idle server. For $k \geq 0$, we write
\[ p_{W,k}(t) = P\{C(t) = W, N(t) = k\} \]
and
\[ p_{I,k}(t) = P\{C(t) = I, N(t) = k\} . \]

Hence, we obtain
\begin{align*}
\frac{d}{dt} p_{W,0}(t) &= -(\lambda + \mu) p_{W,0}(t) + \lambda p_{I,0}(t) + \theta p_{I,1}(t), \\
\frac{d}{dt} p_{W,k}(t) &= -(\lambda + \mu) p_{W,k}(t) + \lambda p_{I,k}(t) + (k + 1) \theta p_{I,k+1}(t) + \lambda p_{W,k-1}(t), \quad k \geq 1, \\
\frac{d}{dt} p_{I,0}(t) &= \mu p_{W,0}(t) - \lambda p_{I,0}(t), \\
\frac{d}{dt} p_{I,k}(t) &= \mu p_{W,k}(t) - \lambda p_{I,k}(t) - k \theta p_{I,k}(t), \quad k \geq 1.
\end{align*}

(35)

Let $\rho = \lambda/\mu < 1$. Then the M/M/1 retrial queue is stable. In this case, we write that for $k \geq 0$
\[ x_{W,k} = \lim_{t \to +\infty} p_{W,k}(t), \quad x_{I,k} = \lim_{t \to +\infty} p_{I,k}(t), \]
and
\[ \pi_{W,k} = \sum_{j=k}^{\infty} x_{W,j}, \quad \pi_{I,k} = \sum_{j=k}^{\infty} x_{I,j}. \]

Then it follows from (35) and (36) that
\begin{align*}
\mu \pi_{W,0} - \lambda \pi_{I,0} - \theta \sum_{j=1}^{\infty} \pi_{I,j} &= 0, \\
\pi_{W,0} + \pi_{I,0} &= 1,
\end{align*}

(37)

(38)

and for $k \geq 1$
\begin{align*}
\lambda (\pi_{W,k-1} - \pi_{W,k}) - \mu \pi_{W,k} + \lambda \pi_{I,k} + \theta \left[ (k + 1) \pi_{I,k+1} + \sum_{j=k+2}^{\infty} \pi_{I,j} \right] &= 0 \quad (39)
\end{align*}

and
\begin{align*}
\mu \pi_{W,k} - \lambda \pi_{I,k} - \theta \left[ k \pi_{I,k} + \sum_{j=k+1}^{\infty} \pi_{I,j} \right] &= 0. \quad (40)
\end{align*}

Let
\[ \Pi = (\Pi_1, \Pi_2, \Pi_3, \ldots) , \]
\[ \Pi_k = (\pi_{W,k}, \pi_{I,k}), \ k \geq 1; \]

\[ Q = \begin{pmatrix} A_1 & C \\ B_2 & A_2 & C \\ D & B_3 & A_3 & C \\ D & D & B_4 & A_4 & C \\ D & D & D & B_5 & A_5 & C \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (41) \]

and for \( k \geq 1 \)

\[ A_k = \begin{pmatrix} - (\lambda + \mu) & \mu \\ \lambda & - (\lambda + k\theta) \end{pmatrix}, \quad B_{k+1} = \begin{pmatrix} 0 & 0 \\ (k+1) \theta & -\theta \end{pmatrix}, \]

\[ C = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ \theta & -\theta \end{pmatrix}. \]

Note that \( \pi_{W,0} = \rho \) and \( \pi_{I,0} = 1 - \rho \), it follows from (39) and (40) that

\[ \Pi Q = (-\lambda \rho, 0, 0, \ldots). \quad (42) \]

To solve Equation (42), we need to construct a UL-type \( RG \)-factorization of the matrix \( Q \) in which the computational steps are similar to that in Subsection 2.2.3 of Li [16]. Here, we provide a sketch of the computation as follows. Let

\[ W_k = \begin{pmatrix} A_k & C \\ B_{k+1} & A_{k+1} & C \\ D & B_{k+2} & A_{k+2} & C \\ D & D & B_{k+3} & A_{k+3} & C \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \]

We denote by \( \left( \widehat{W}^{(k)}_{1,1}, \widehat{W}^{(k)}_{1,2}, \widehat{W}^{(k)}_{1,3}, \ldots \right) \) the first block-row of the matrix \( (-W_k)^{-1} \). Thus for \( k \geq 1 \) and \( j \geq 2 \),

\[ R_{k,k+1} = C \widehat{W}^{(k+1)}_{1,1} \overset{\text{def}}{=} R_k, \]

\[ R_{k,k+j} = 0; \]
and for $i \geq 2$ and $1 \leq j \leq i - 2$

$$G_{i,i-1} = \hat{W}_{1,1}^{(k)} B_i + \left[ \sum_{l=2}^{\infty} \hat{W}_{1,l}^{(k)} \right] D$$

$$G_{i,j} = \left[ \sum_{l=1}^{\infty} \hat{W}_{1,l}^{(k)} \right] D \overset{\text{def}}{=} G_i.$$ 

In what follows we provide some further interpretation on the $R$- and $G$-measures. Let the $R$-measure $\{R_k : k \geq 1\}$ be the minimal nonnegative solution to the following system of nonlinear equations

$$C + R_k A_k + R_k R_{k+1} B_{k+2} + R_k R_{k+1} \left( \sum_{l=2}^{\infty} R_{k+2} R_{k+3} \cdots R_{k+l} \right) D = 0, \quad k \geq 1.$$ 

Once the $R$-measure $\{R_k : k \geq 1\}$ is determined, we have

$$\Psi_k = A_k + R_k B_k + R_k \left( \sum_{l=1}^{\infty} R_{k+1} R_{k+2} \cdots R_{k+l} \right) D,$$

$$\hat{W}_{1,1}^{(k)} = (-\Psi_k)^{-1},$$

$$\hat{W}_{1,j}^{(k)} = \hat{W}_{1,1}^{(k)} R_k R_{k+1} \cdots R_{j-2} = (-\Psi_k)^{-1} R_k R_{k+1} \cdots R_{j-2}, \quad j \geq 2;$$

and for $i \geq 2$ and $1 \leq j \leq i - 2$

$$G_{i,i-1} = (-\Psi_k)^{-1} \left[ B_i + \left( \sum_{l=1}^{\infty} R_i R_{i+1} \cdots R_{i+l-1} \right) D \right],$$

$$G_{i,j} = (-\Psi_i)^{-1} \left( I + \sum_{l=1}^{\infty} R_i R_{i+1} \cdots R_{i+l-1} \right) D \overset{\text{def}}{=} G_i.$$ 

Thus the UL-type $RG$-factorization is given by

$$Q = (I - R_U) U_D (I - G_L), \quad (43)$$

where

$$R_U = \begin{pmatrix}
0 & R_1 \\
0 & R_2 \\
0 & R_3 \\
& \ddots
\end{pmatrix},$$

$$U_D = \text{diag} (\Psi_1, \Psi_2, \Psi_3, \Psi_4, \ldots)$$
and

\[
G_L = \begin{pmatrix}
0 & & & & \\
G_{2,1} & 0 & & & \\
G_3 & G_{3,2} & 0 & & \\
G_4 & G_4 & G_{4,3} & 0 & \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

It follows from (42) and (43) that

\[
\Pi = (-\lambda \rho, 0, 0, \ldots) (I - G_L)^{-1} U_D^{-1} (I - R_U)^{-1}
= ((-\lambda \rho, 0) \Psi_1^{-1}, 0, 0, \ldots) (I - R_U)^{-1}.
\]

Note that

\[
(I - R_U)^{-1} = \begin{pmatrix}
I & R_1 & R_1 R_2 & R_1 R_2 R_3 & \cdots \\
I & R_1 & R_1 R_2 & \cdots \\
I & R_1 & \cdots \\
I & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix},
\]

we obtain

\[
\pi_1 = (\lambda \rho, 0) \left( -\Psi_1^{-1} \right),
\]

\[
\pi_k = (\lambda \rho, 0) \left( -\Psi_1^{-1} \right) R_1 R_2 \cdots R_{k-1}, \quad k \geq 2.
\]

### 5.2 The M(n)/M(n)/1 queue

We consider an M(n)/M(n)/1 queue whose arrival and service rates depend on the number of customers in this system, denoted as \( \lambda_n \) and \( \mu_n \), respectively. We denote by \( N(t) \) the number of customers in this system at time \( t \). Then \( N(t) \in \{0, 1, 2, \ldots\} \). For \( k \geq 0 \), we write

\[
Q_k(t) = P\{N(t) \geq k\}
\]

and when the M(n)/M(n)/1 queue is stable,

\[
\pi_k = \lim_{t \to +\infty} Q_k(t).
\]

Then we obtain that for \( k \geq 1 \)

\[
\lambda_{k-1} (\pi_{k-1} - \pi_k) = \mu_k (\pi_k - \pi_{k+1})
\]
with the boundary condition \( \pi_0 = 1 \). Let \( \pi_1 = g \in (0, 1) \) and \( \rho_{k-1} = \lambda_{k-1}/\mu_k \) for \( k \geq 1 \). Then

\[
\pi_1 - \pi_2 = \rho_0 (\pi_0 - \pi_1) = \rho_0 (1 - g)
\]

and for \( k \geq 2 \)

\[
\pi_k - \pi_{k+1} = \rho_{k-1} (\pi_{k-1} - \pi_k) = \rho_{k-1} \rho_{k-2} \cdots \rho_1 \rho_0 (1 - g).
\]

We obtain

\[
g = (\rho_0 + \rho_1 \rho_0 + \rho_2 \rho_1 \rho_0 + \rho_3 \rho_2 \rho_1 \rho_0 + \cdots) (1 - g),
\]

from which follows

\[
g = \frac{\rho_0 + \rho_1 \rho_0 + \rho_2 \rho_1 \rho_0 + \rho_3 \rho_2 \rho_1 \rho_0 + \cdots}{1 + (\rho_0 + \rho_1 \rho_0 + \rho_2 \rho_1 \rho_0 + \rho_3 \rho_2 \rho_1 \rho_0 + \cdots)}.
\]

Thus for \( k \geq 1 \)

\[
\pi_k = \frac{\rho_{k-1} \rho_{k-2} \cdots \rho_1 \rho_0 + \rho_k \rho_{k-1} \rho_{k-2} \cdots \rho_1 \rho_0 + \rho_{k+1} \rho_k \rho_{k-1} \rho_{k-2} \cdots \rho_1 \rho_0 + \cdots}{1 + (\rho_0 + \rho_1 \rho_0 + \rho_2 \rho_1 \rho_0 + \rho_3 \rho_2 \rho_1 \rho_0 + \cdots)}.
\]

It is interesting to extend the above result to more general models such as the MAP(n)/M/1 queue and the M/PH(n)/1 queue. The more general queues can be analyzed by the level-dependent QBD processes, see Section 3.

### 5.3 The M/M/1 queue with server multiple vacations

We consider an M/M/1 queue with server multiple vacations, where the arrival, service and vacation rates are \( \lambda, \mu = 1 \) and \( \theta \). The vacation process is based on the multiple vacation policy: When there is no customer in the system, the server immediately proceeds on vacation and keeps taking vacations until it finds at least one customer waiting in the server or its buffer at the vacation completion instant. The arrival, service and vacation processes are independent of each other.

Let \( N(t) \) be the number of customers in the queueing system at time \( t \), and

\[
\xi(t) = \begin{cases} 
V, & \text{if the server is taking a vacation at time } t, \\
W, & \text{if the server is working at time } t.
\end{cases}
\]

Then \( \{(\xi(t), N(t)) : t \geq 0\} \) is a Markov chain on a state space \( E = \{(V, k), (W, l) : k \geq 0, l \geq 1\} \). We write

\[
Q_{V,l}(t) = P \{\xi(t) = V, N(t) \geq l\}, \quad l \geq 0,
\]
and

\[ Q_{W,k}(t) = P \{ \xi(t) = W, N(t) \geq k \}, \quad k \geq 1. \]

If \( 0 < \lambda < \mu = 1 \), then this queue is stable. We set

\[ \pi_{V,k} = \lim_{t \to +\infty} Q_{V,k}(t), \quad k \geq 0, \]
\[ \pi_{W,l} = \lim_{t \to +\infty} Q_{W,l}(t), \quad l \geq 1. \]

Then we obtain

\[ (\pi_{W,1} - \pi_{W,2}) - \theta \pi_{V,1} = 0, \quad (44) \]
\[ \lambda (\pi_{V,k-1} - \pi_{V,k}) - \theta \pi_{V,k} = 0, \quad k \geq 1, \quad (45) \]
\[ \lambda (\pi_{W,l-1} - \pi_{W,l}) - (\pi_{W,l} - \pi_{W,l+1}) + \theta \pi_{V,l} = 0, \quad l \geq 2. \quad (46) \]

Note that \( \pi_{V,0} = 1 - \lambda \) and \( \pi_{W,1} = \lambda \), thus it follows from (45) that

\[ \pi_{V,k} = \left( \frac{\lambda}{\lambda + \theta} \right)^k (1 - \lambda), \quad k \geq 0, \]

and from (44) that

\[ \pi_{W,2} = \lambda - \frac{\lambda \theta}{\lambda + \theta} (1 - \lambda). \]

Using \( \pi_{W,1} = \lambda \) and \( \pi_{W,2} = \lambda - \lambda \theta (1 - \lambda) / (\lambda + \theta) \), it follows from (46) that for \( k \geq 3 \)

\[ \pi_{W,k} = \pi_{W,k-1} - \lambda (\pi_{W,k-2} - \pi_{W,k-1}) - \theta \pi_{V,k-1}, \quad (47) \]

which can be computed iteratively.

Let

\[ Q = \begin{pmatrix}
- (1 + \lambda) & \lambda \\
1 & - (1 + \lambda) & \lambda \\
& 1 & - (1 + \lambda) & \lambda \\
& & \ddots & \ddots & \ddots
\end{pmatrix}. \]

Then using (47) we obtain

\[ (\pi_{W,2}, \pi_{W,3}, \pi_{W,4}, \ldots) = - (\lambda \pi_{W,1} + \theta \pi_{V,2}, \theta \pi_{V,3}, \theta \pi_{V,4}, \ldots). \quad (48) \]

Let

\[ R = \lambda, \quad G = 1. \]

Then

\[ U_k = -(1 + \lambda) + R = -1, \quad k \geq 0, \]
\[ U_D = \text{diag}(-1, -1, -1, -1, \ldots), \]

\[
R_U = \begin{pmatrix}
0 & \lambda \\
0 & \lambda \\
0 & \lambda \\
\vdots & \ddots
\end{pmatrix}
\]

and

\[
G_L = \begin{pmatrix}
0 \\
1 & 0 \\
1 & 0 \\
\vdots & \ddots
\end{pmatrix}
\]

Thus we obtain

\[(\pi_{W,2}, \pi_{W,3}, \pi_{W,4}, \ldots) = - (\lambda \pi_{W,1} + \theta \pi_{V,2}, \theta \pi_{V,3}, \theta \pi_{V,4}, \ldots) Q_{\max}^{-1}
= - (\lambda \pi_{W,1} + \theta \pi_{V,2}, \theta \pi_{V,3}, \theta \pi_{V,4}, \ldots) (I - G_L)^{-1} U_D^{-1} (I - R_U)^{-1}.\]

Note that

\[(I - G_L)^{-1} = \begin{pmatrix}
1 \\
1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

and

\[(I - R_U)^{-1} = \begin{pmatrix}
1 & \lambda & \lambda^2 & \lambda^3 & \ldots \\
1 & \lambda & \lambda^2 & \ldots \\
1 & \lambda & \ldots \\
1 & \ldots \\
\vdots & \ddots
\end{pmatrix},
\]

for \(k \geq 2\) we can obtain

\[
\pi_{W,k} = \lambda^{k-1} \pi_{W,1} + \theta \sum_{i=0}^{k-2} \lambda^i \sum_{l=k-i}^{\infty} \pi_{V,l}
= \lambda^k + (1 - \lambda) \frac{\lambda^2}{\theta} \left[ 1 - \left( \frac{\lambda}{\lambda + \theta} \right)^{k-1} \right].
\]
5.4 The M/M/1 queue with repairable server

We consider an M/M/1 queue with repairable server, where the arrival and service rates are \( \lambda \) and \( \mu \), respectively. The life time of the server is exponential with failure rate \( \alpha \). Once the server failed, it immediately is repaired, and the repair time is exponential with repair rate \( \beta \). The repaired server is the same as the new one. We assume that all the random variables defined above are independent of each other.

For this M/M/1 repairable queue, we denote by \( N(t) \) and \( C(t) \) the number of customers in this queuing system and the state of the server at time \( t \geq 0 \), respectively, where \( N(t) = 0, 1, 2, \ldots \), and \( C(t) = W \) for server working or \( R \) for server repair. It is easy to see that \( \{(N(t), C(t)) : t \geq 0\} \) is a Markov chain. For \( k \geq 0 \) and \( l \geq 1 \), we write

\[
Q_{W,k}(t) = P\{C(t) = W, N(t) \geq k\}
\]

and

\[
Q_{R,l}(t) = P\{C(t) = R, N(t) \geq l\}.
\]

If \( \rho = \frac{\lambda}{\mu} \left(1 + \frac{\alpha}{\beta}\right) < 1 \), then this queue is stable. Let

\[
\pi_{W,k} = \lim_{t \to +\infty} Q_{W,k}(t), \quad k \geq 0,
\]

and

\[
\pi_{R,l} = \lim_{t \to +\infty} Q_{R,l}(t), \quad l \geq 1.
\]

Then we obtain

\[
\pi_{W,0} + \pi_{R,1} = 1
\]

(49)

\[
- \alpha \pi_{W,1} + \beta \pi_{R,1} = 0,
\]

(50)

for \( k \geq 1 \)

\[
\lambda (\pi_{W,k-1} - \pi_{W,k}) - \mu (\pi_{W,k} - \pi_{W,k+1}) - \alpha \pi_{W,k} + \beta \pi_{R,k} = 0,
\]

(51)

for \( l \geq 2 \)

\[
\lambda (\pi_{R,l-1} - \pi_{R,l}) + \alpha \pi_{W,l} - \beta \pi_{R,l} = 0.
\]

(52)

It follows from (51) that

\[
\lambda \pi_{W,0} - \mu \pi_{W,1} - \alpha \sum_{k=1}^{\infty} \pi_{W,k} + \beta \sum_{k=1}^{\infty} \pi_{R,k} = 0
\]

(53)
and from (52) that
\[ \lambda \pi_{R,1} + \alpha \sum_{k=2}^{\infty} \pi_{W,k} - \beta \sum_{k=2}^{\infty} \pi_{R,k} = 0, \]
which, together with (50), leads to
\[ \lambda \pi_{R,1} + \alpha \sum_{k=1}^{\infty} \pi_{W,k} - \beta \sum_{k=1}^{\infty} \pi_{R,k} = 0. \] (54)

Using (53) and (54), we obtain
\[ \lambda \pi_{W,0} - \mu \pi_{W,1} + \lambda \pi_{R,1} = 0. \] (55)

It follows from (49), (50) and (55) that
\[ \pi_{W,0} = 1 - \frac{\lambda}{\mu} \frac{\alpha}{\beta}, \]
\[ \pi_{W,1} = \frac{\lambda}{\mu} \]
and
\[ \pi_{R,1} = \frac{\lambda}{\mu} \frac{\alpha}{\beta}. \]

It follows from (51) and (52) that for \( k \geq 2 \)
\[ \pi_{W,k} = \frac{\lambda + \mu + \alpha}{\mu} \pi_{W,k-1} - \frac{\lambda}{\mu} \pi_{W,k-2} - \frac{\beta}{\mu} \pi_{R,k-1} \] (56)
and
\[ \pi_{R,k} = \frac{\alpha}{\lambda + \beta} \pi_{W,k} + \frac{\lambda}{\lambda + \beta} \pi_{R,k-1}. \] (57)

Therefore, \( \pi_{W,k} \) and \( \pi_{R,k} \) for \( k \geq 2 \) can be computed iteratively.

To provide explicit expressions for \( \pi_{W,k} \) and \( \pi_{R,k} \) with \( k \geq 2 \), we write
\[ \Pi_k = (\pi_{W,k}, \pi_{R,k}), \ k \geq 2, \]
\[ \Pi = (\Pi_2, \Pi_3, \Pi_4, \Pi_5, \ldots), \]
\[ A = \left( \begin{array}{cc} -(\lambda + \mu + \alpha) & \alpha \\ \beta & -(\lambda + \beta) \end{array} \right), \]
\[ B = \left( \begin{array}{cc} \mu & 0 \\ 0 & 0 \end{array} \right), \]
\[ C = \left( \begin{array}{c} \lambda \\ \lambda \end{array} \right), \]
\[ Q = \left( \begin{array}{cccc} A & C \\ B & A & C \\ \vdots & \vdots & \vdots \end{array} \right). \]
It follows from (56) and (57) that
\[
\Pi Q = - \left( \left( \frac{\lambda^2}{\mu}, \frac{\lambda^2 \alpha}{\mu \beta} \right), 0, 0, 0, \ldots \right). 
\] (58)

Let \( R \) and \( G \) be the minimal nonnegative solutions to the nonlinear equations \( C + RA + R^2B = 0 \) and \( CG^2 + AG + B = 0 \), respectively. It is easy to see that the infinitesimal generator \( Q \) has the UL-type \( RG \)-factorization \( Q = (I - RU) U_D (I - GL) \), where

\[
U_D = \text{diag}(\Psi, \Psi, \Psi, \ldots), \quad \Psi = A + RB = A + CG, 
\]

\[
RU = \begin{pmatrix} 0 & R \\ 0 & R \\ 0 & R \\ \vdots & \vdots \\ \end{pmatrix}, \quad GL = \begin{pmatrix} 0 & 0 \\ G & 0 \\ G & 0 \\ \vdots & \vdots \\ \end{pmatrix}.
\]

Thus it follows from (58) that
\[
\Pi = \left( \left( \frac{\lambda^2}{\mu}, \frac{\lambda^2 \alpha}{\mu \beta} \right), 0, 0, 0, \ldots \right) (I - GL)^{-1} (-U_D^{-1}) (I - RU)^{-1}
\]
\[
= \left( \left( \frac{\lambda^2}{\mu}, \frac{\lambda^2 \alpha}{\mu \beta} \right) (-\Psi^{-1}), 0, 0, 0, \ldots \right) (I - RU)^{-1}.
\]

This gives
\[
\Pi_2 = \left( \frac{\lambda^2}{\mu}, \frac{\lambda^2 \alpha}{\mu \beta} \right) (-\Psi^{-1}), 
\]
\[
\Pi_k = \left( \frac{\lambda^2}{\mu}, \frac{\lambda^2 \alpha}{\mu \beta} \right) (-\Psi^{-1}) R^{k-2}, \quad k \geq 3.
\]

In fact, the minimal nonnegative solution \( R \) can be explicitly determined from the nonlinear 2-order matrix equation \( C + RA + R^2B = 0 \), here we omit the detail.

6 Concluding remarks

This paper discusses tail probabilities of queueing processes, such as, the QBD processes and Markov chains of GI/M/1 type and of M/G/1 type, and provides some efficient algorithms for computing the tail probabilities by means of the matrix-geometric solution, the matrix-iterative solution, the matrix-product solution and the two types of \( RG \)-factorizations. Also, we consider four queueing examples: The M/M/1 retrial queue, the
M(n)/M(n)/1 queue, the M/M/1 queue with server multiple vacations, and the M/M/1 queue with repairable server, where the M/M/1 retrial queue is given a detailed discussion, while for the other three queues, a sketch of the analysis is given. It is seen from the four queueing examples that the method of this paper can be applied to deal with more general queues including the MAP/PH/1 queue, the GI/PH/1 queue and the BMAP/SM/1 queue.

The results given in this paper are very useful in the study of large scale stochastic networks with resource management, such as, supermarket models and work stealing models. Also, it will open a new avenue to helpfully analyze the tail probabilities of many large scale stochastic networks when applying differential equations and mean-field limits.

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