One-Dimensional Quasi-Exactly Solvable Schrödinger Equations

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Abstract

Quasi-Exactly Solvable Schrödinger Equations occupy an intermediate place between exactly-solvable (e.g. the harmonic oscillator and Coulomb problems etc) and non-solvable ones. Mainly, they were discovered in the 1980ies. Their major property is an explicit knowledge of several eigenstates while the remaining ones are unknown. Many of these problems are of the anharmonic oscillator type with a special type of anharmonicity. The Hamiltonians of quasi-exactly-solvable problems are characterized by the existence of a hidden algebraic structure but do not have any hidden symmetry properties. In particular, all known one-dimensional (quasi)-exactly-solvable problems possess a hidden \( \mathfrak{sl}(2, \mathbb{R}) \) – Lie algebra. They are equivalent to the \( \mathfrak{sl}(2, \mathbb{R}) \) Euler-Arnold quantum top in a constant magnetic field.

Quasi-Exactly Solvable problems are highly non-trivial, they shed light on the delicate analytic properties of the Schrödinger Equations in coupling constant, they lead to a non-trivial class of potentials with the property of Energy-Reflection Symmetry. The Lie-algebraic formalism allows us to make a link between the Schrödinger Equations and finite-difference equations on uniform and/or exponential lattices, it implies that the spectra is preserved. This link takes the form of quantum canonical transformation. The corresponding isospectral spectral problems for finite-difference operators are described. The underlying Fock space formalism giving rise to this correspondence is uncovered. For a quite general class of perturbations of unperturbed problems with the hidden Lie algebra property we can construct an algebraic perturbation theory, where the wavefunction corrections are of polynomial nature, thus, can be found by algebraic means.

In general, Quasi-Exact-Solvability points to the existence of a hidden algebra formalism which ranges from quantum mechanics to 2-dimensional conformal field theories.
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INTRODUCTION

Exact solutions of non-trivial problems can provide valuable information about the real and hidden properties of the problem. Very often such solutions lead to the discovery of unexpected features. The Schrödinger equation is the basic object of quantum mechanics and thus it is quite important to find as much as possible exact information about it. This motivates the search for exact solutions. Although everybody has its natural, intrinsic definition what does exact solution mean, the general, a widely accepted definition of “exact solution” (which would be rigorous enough from a mathematical point of view) was lacking until recently. Such a definition should possess a certain heuristic value giving us a chance to apply a mathematical formalism. This definition should also have a clear physical meaning.

Knowledge of the exact solutions often allows us to develop a constructive perturbation theory, where concrete feasible calculations of relevant quantities can be done.

In recent times a renewed interest in exact solutions was greatly inspired by discovery of a new class of quantum mechanical spectral problems, the so-called quasi-exactly-solvable problems. In these spectral problems a finite number of eigenstates can be found explicitly, by algebraic means. It has led to a certain definition of both the exact solution, and a solvable problem admitting an exact solution(s).

Perhaps, one of the most important and natural philosophical ideas for the present author is the idea about impossibility to learn everything about Nature

In one of the most famous Russian folklore books of the XIXth century "Koz’ma Prutkov Thoughts" written anonymously by several well-known Russian writers of 19th century it was repeated about hundred times ‘Nobody can embrace everything, do not trust anyone who claims that he can embrace everything’
take any normalizable (square-integrable) function \( \Psi_0(x) \) and find a potential in the Schrödinger operator for which this function is an eigenfunction

\[
\frac{\Delta \Psi_0(x)}{\Psi_0(x)} = V_0(x) - E_0 ,
\]

(0.1)

where without loss of generality one can place \( E_0 = 0 \).

Using this method one can generate a zillion potentials, where a single eigenstate is know explicitly.

A first example of this “non-trivial” procedure was proposed to the author by Felix A. Berezin in the mid-1970ies. Take

\[
\Psi_0(x) = e^{-\alpha x^4/4} .
\]

From (0.1) we find

\[
V_0(x) = \alpha^2 x^6 - 3\alpha x^2 , \ E_0 = 0 .
\]

Hence, if \( \alpha > 0 \) we know the ground state eigenfunction exactly in the potential \( V_0(x) \). Even on first sight, it is quite a non-trivial result, since we know an exact eigenfunction of the ground state for some anharmonic oscillator. In reality, this result is even more non-trivial. The Schrödinger operator with potential \( V_0(x) \) is Hermitian for any real \( \alpha \), and has infinitely-many bound states if \( \alpha \neq 0 \) and is analytic in \( \alpha \). For any \( \alpha > 0 \) the function \( \Psi_0 \) is the ground state eigenfunction with zero energy, but in the domain \( \alpha < 0 \) the function \( \Psi_0(x) \) is neither normalizable, nor the energy can be continued analytically from \( \alpha > 0 \). It can not be explained in the framework of the standard Stokes phenomenon approach. We do have two analytically disconnected spectral problems: one which is defined at \( \alpha > 0 \) and another one which is defined at \( \alpha < 0 \), see [5] and [75]. Chapter 2 will be devoted to this phenomenon. However, the non-triviality of this potential is not exhausted by the above-mentioned observation. The potential \( V_0(x) \) is a double-well potential with degenerate minima but \( \Psi_0(x) \) has a single maximum corresponding to a position of unstable equilibrium. Thus, the particle prefers to be near the maximum of potential contrary to our intuition. So, it is an explicit example for which neither a standard semiclassical analysis, nor instanton calculus can be applied straightforwardly.

Using the above procedure we can create as many potentials as we like for which one eigenstate can be found explicitly. This reasoning leads to a natural question: Can we find a potential in which we know two, three ... , a finite number of eigenstates constructively? In essence, the attempt to find an answer to this question has led to the discovery of a new class of spectral problems

\[\text{[2] In classical mechanics, the motion at } E = 0 \text{ is not periodic: for particle it takes infinite time to reach the tip of the barrier.}\]
– quasi-exactly-solvable problems. A straightforward attempt to solve this problem failed and a solution led to another discovery – a non-trivial link between the spectral theory and the representation theory of Lie algebras.

One of the goals of this Review is to describe this new connection between the representation theory of Lie algebras and linear differential equations.

The main idea of this connection is surprisingly simply. Let us consider a certain set of differential operators of the first order

\[ J^\alpha(x) = a^{\alpha,\mu}(x) \partial_\mu + b^\alpha(x), \quad \partial_\mu \equiv \frac{d}{dx^\mu}, \quad (0.2) \]

where \( \alpha = 1, 2, \ldots, k, x \in \mathbb{R}^n, \mu = 1, 2, \ldots, n \) and \( a^{\alpha,\mu}(x), b^\alpha(x) \) are certain functions on \( \mathbb{R}^n \). Assume that the operators form a basis of some Lie algebra \( g \) of dimension \( k = \dim g \). Now let us take a polynomial \( h \) in generators \( J^\alpha(x) \) and ask the question:

Does the differential operator \( h(J^\alpha(x)) \) have some specific properties, which distinguish this operator from a general linear differential operator?

Generically, the answer is negative – nothing special appears. Perhaps, it may be helpful to study the integrability of \( h \), to find operators commuting with \( h \) using the Lie-algebra properties. For instance, the Casimir operators (for the case of a reducible representation) are evident integrals. However, the situation becomes totally different if the algebra \( g \) is taken in a finite-dimensional representation. The answer becomes affirmative:

The differential operator \( h(J^\alpha(x)) \) begins to possess a finite-dimensional invariant subspace. This finite-dimensional invariant subspace coincides with a finite-dimensional representation space of the algebra \( g \) of the first order differential operators. If a basis of this finite-dimensional representation space can be constructed explicitly, the operator \( h \) can be presented in a block-diagonal form explicitly.

Such a differential operator having a finite-dimensional invariant subspace with an explicit basis in functions is called quasi-exactly-solvable. In general, a finite-dimensional invariant subspace with an explicit basis can not be connected with a representation space of some finite-dimensional Lie algebra.

The first explicit examples of quasi-exactly-solvable problems were published by Razavy [52,53] and by Singh-Rampal-Biswas-Datta [62]. In explicit form the general idea of quasi-exact-solvability had been formulated for the first time in [73,71], which led to a complete catalogue of one-dimensional, quasi-exactly-solvable Schrödinger operators based on spaces of polynomials [72].
The term \textit{quasi-exact-solvability} has been suggested in [90]. The connection between quasi-exact-solvability and the finite-dimensional representations of the $sl_2$ algebra was mentioned for the first time by Zaslavskii-Ulyanov [92]. Later, the idea of quasi-exact-solvability was generalized to multidimensional differential operators, matrix differential operators [60], finite-difference operators of different types (on different lattices [49]), \textit{mixed} operators containing differential operators, permutation operators [77], and Dunkl operators.

In [45] a connection was described between quasi-exact-solvability and two-dimensional conformal quantum field theories (see also Halpern-Kiritsis [28] and Tseytlin [68,69], and for a review [59,29]). A relationship with solid-state physics is given in [93] and very exciting results in this direction were found by Wiegmann-Zabrodin [97,98]. Survey of quasi-exact-solvability together with a formalism of invariant subspaces in functions of differential, finite-difference, matrix-differential operators and differential operators containing with reflection operators was done in [78,81]. Following this formalism, in the last 20 years, the (Lie)-algebraic nature of the Calogero-Moser-Sutherland models, see e.g. [50], was uncovered and explored [57,11,9,65] (see [86,87] for a review and references therein).

Since 1988 [72] hundreds of papers were published on quasi-exact-solvability - certainly, it is impossible to embrace all obtained results in a single review paper of a limited volume. Thus, the presentation will be limited to the basic facts and results which follow the taste of the present author. Chapter 1 contains a mathematical introduction, the general theory of the second order QES differential equations and a description of twelve (major) QES Schrödinger operators. Chapter 2 gives a description of one of the most important particular cases - the Quasi-Exactly-Solvable Anharmonic Oscillator. In Chapter 3 the Algebraic Perturbations of Exactly-Solvable Problems which are closely related to QES problems is briefly introduced. Chapter 4 is devoted to the Fock space formalism, the Lie-algebraic discretization, and the QES finite-difference equations on uniform and exponential lattices.

\footnote{In translation of [71] from Russian into English it appeared as \textit{quasi-solvability} which later was also used in different articles.}
Chapter 1.

Quasi-Exact Solvability - a general consideration

The goal of this Chapter is to describe a connection between the representation theory of Lie algebras and linear differential equations. Specifically, it will be taken the algebra \( \mathfrak{sl}(2, \mathbb{R}) \) realized by the first order differential operators in one variable and demonstrated a connection to the ordinary differential equations.

Thus, this Chapter is devoted to a general description of quasi-exactly-solvable operators acting on functions in one real (complex) variable.

1 Generalities

Let us take \( n \) linearly-independent functions \( f_1(y), f_2(y), \ldots, f_n(y) \) in one variable and form a linear space

\[
\mathcal{F}_n^{(1)} = \langle f_1(y), f_2(y), \ldots, f_n(y) \rangle,
\]

over complex (real) numbers. By construction the space \( \mathcal{F}_n \) has the dimension \( n \), \( \dim(\mathcal{F}_n) = n \). This space is defined “ambiguously” in a sense that functions \( f_i(y) \) admit a change of variable and multiplication of all of them by an arbitrary function. Thus, we have to introduce a notion of equivalence:

Two functional spaces \( \mathcal{F}_n^{(1)} \) and \( \mathcal{F}_n^{(2)} \) are called equivalent, if one space can be transformed into another through a change of variable and/or the multiplication by a function,

\[
\mathcal{F}_n^{(2)} = g(y)\mathcal{F}_n^{(1)}|_{y=y(x)} , \quad g(y) \neq 0 .
\]

Such a transformation is called gauge (similarity) transformation and the function \( g(y) \) is called gauge factor.

Choosing a space (1.1.1) and considering all possible changes of variable, \( y = y(x) \), and functions \( g(y) \), one can describe a family of equivalent spaces (orbit). In general, in the family there is a certain space which contains, as a subspace, the space of linear functions

\[
\mathcal{F}_n = \langle 1, x, \phi_1(x), \phi_2(x), \ldots, \phi_{n-2}(x) \rangle.
\]

A linear functional space containing a subspace of linear functions is called basic linear space.
If a space (1.1.1) is given, then in order to construct a basic linear space (1.1.3) we have to choose the gauge factor as \( g(y) = f_1(y)^{-1} \) and the variable \( x = f_2(y)/f_1(y) \). It allows us to introduce a standardization: hereafter, if it is not explicitly mentioned, only spaces of the form (1.1.3) are considered. In general, any other space of the family can be constructed by making a change of variable and by multiplication by an appropriate factor.

It is easy to see that if a linear differential operator \( h(y, \frac{d}{dy}) \) acts on a space \( \mathcal{F}_n^{(1)} \), then a similarity-transformed, linear differential operator

\[
\tilde{h}(y, \frac{d}{dy}) = g(x)h(x, \frac{d}{dx})g^{-1}(x)|_{x=x(y)},
\]

acts on an equivalent space \( \mathcal{F}_n^{(2)} \). Therefore, the operators \( \tilde{h}(y, \frac{d}{dy}) \) and \( h(x, \frac{d}{dx}) \) are gauge-equivalent.

Through this Chapter we focus on the only particular basic linear space – the linear space of polynomials of order not higher than \( n \) with real coefficients,

\[
\mathcal{P}_{n+1} = \langle 1, x, x^2, \ldots, x^n \rangle \equiv (x^k \mid 0 \leq k \leq n),
\]

where \( n \) is a non-negative integer and \( x \in \mathbb{R} \). Let

\[
\tilde{\mathcal{P}}_{n+1} = \left( \frac{1}{x^k} \mid 0 \leq k \leq n \right).
\]

It is worth noting an important property of the invariance:

\[
x^n \left( \mathcal{P}_{n+1}|_{x \to \frac{1}{x}} \right) = x^n \tilde{\mathcal{P}}_{n+1} = \mathcal{P}_{n+1}. \tag{1.1.6}
\]

The following stems from an evident feature of polynomials: if \( p_k(x) \in \mathcal{P}_{n+1} \) is a polynomial of degree \( k, \, k \leq n \), then \( x^k p_k(1/x) \in \mathcal{P}_{n+1} \) is a polynomial of degree \( k \) with inverse order of the coefficients in comparison to \( p_k(x) \). In general, in the space of polynomials \( \mathcal{P}_{n+1} \), the Möbius transformation acts,

\[
x \to \frac{ax+b}{cx+d}, \quad ad-bc \neq 0. \tag{1.1.7}
\]

It maps the real line to itself. The \( 2 \times 2 \) matrix \([a, b; c, d] \) is an element of the \( SL(2, \mathbb{R}) \) group. The inversion (1.1.6) is a particular case of the Möbius transformation which appears if \( a = d = 0 \) and \( b = c = 1 \). Another important transformation which acts in \( \mathcal{P}_{n+1} \) is a linear transformation

\[
x \to ax+b, \quad a \neq 0. \tag{1.1.8}
\]

It corresponds to the \( SL(2, \mathbb{R}) \) matrix \([a, b; 0, 1] \).
The spaces of polynomials can be ordered: \( P_1 \subset P_2 \subset P_3 \subset \cdots \subset P_n \subset \ldots \). They form an object which is called \textit{infinite flag (filtration)},

\[
P \equiv P_1 \subset P_2 \subset P_3 \subset \ldots \subset P_n \subset \ldots
\]  \hspace{1cm} (1.1.9)

It is worth mentioning that the linear transformation (1.1.8) preserves each particular subspace \( P_n \) of the flag. Hence, it preserves the flag \( P \). It is evident that the Möbius transformation does not preserve the flag \( P \). It preserves a single subspace \( P_n \) only.

2 Ordinary differential equations

2.1 General consideration

Take the linear space \( P_{n+1} \) of polynomials of degree not higher than \( n \) (1.1.5).

A linear differential operator of the \( k \)th order, \( T_k(x, \frac{d}{dx}) \), is called \textit{quasi-exactly-solvable}, if it preserves the linear space of polynomials \( P_{n+1} \),

\[
T_k(x, \frac{d}{dx}) : P_{n+1} \rightarrow P_{n+1} .
\]  \hspace{1cm} (1.2.1)

This operator is block-triangular in the basis of monomials \( x^m, m = 0, 1, \ldots \).

The operator \( E_k(x, \frac{d}{dx}) \) is called \textit{exactly-solvable}, if it preserves the infinite flag \( P \) of spaces of polynomials (1.1.9), implying that each individual space \( P_j \) is preserved

\[
E_k(x, \frac{d}{dx}) : P_j \rightarrow P_j , \ j = 0, 1, \ldots
\]  \hspace{1cm} (1.2.2)

This operator is block-triangular in the basis of monomials \( x^m, m = 0, 1, \ldots \).

It has been known since Sophus Lie that the algebra \( \mathfrak{sl}(2, \mathbb{R}) \) can be realized by first order differential operators in one variable

\[
J^+_n = x^2 \frac{d}{dx} - nx ,
\]

\[
J^0_n = x \frac{d}{dx} - \frac{n}{2} ,
\]

\[
J^-_n = \frac{d}{dx} .
\]  \hspace{1cm} (1.2.3)

where \( n \) is the mark of the representation \((j = \frac{n}{2} \) is called the spin of representation\). It is easy to check that for any \( n \in \mathbb{R} \) the generators (1.2.3) obey
the \( \mathfrak{sl}(2, \mathbb{R}) \) commutation relations:

\[
[J^\pm_n, J^0] = \pm J^\pm_n, \quad [J^+_n, J^-_n] = -2J^0_n.
\]

If the identity operator, \( X = \text{const} \) is added to (1.2.3), then the generators \( J^\pm_n, X \) span the algebra \( \mathfrak{gl}(2, \mathbb{R}) \). If in (1.2.3) the parameter \( n \) is an integer, \( n \in \mathbb{N} \), the representation becomes finite-dimensional and the operators \( J^\pm_n, X \) have the space \( \mathcal{P}_{n+1} \), (1.1.5) as a common invariant subspace, which is a finite-dimensional representation space

\[
J^\pm_n, X : \mathcal{P}_{n+1} \rightarrow \mathcal{P}_{n+1}.
\]

They realize the irreducible finite-dimensional representation: they act on \( \mathcal{P}_{n+1} \) irreducibly. Hence, the quadratic Casimir operator \( C_2 \),

\[
C_2 = \frac{1}{2}\{J^+_n, J^-_n\} - J^0_nJ^0_n = -\frac{n}{2}\left(\frac{n}{2} + 1\right), \quad (1.2.4)
\]

which commutes with all generators, becomes constant. Here \( \{a, b\} \equiv ab + ba \) is the anticommutator. Relation (1.2.4) holds for any real value of \( n \), hence, (1.2.4) can be considered as an artifact of realization of the algebra \( \mathfrak{sl}(2, \mathbb{R}) \) by the first order differential operators. It is worth mentioning that for any \( n \) the generators \( J^0_n \) form the Borel subalgebra, \( \mathfrak{b}_2 \subset \mathfrak{sl}(2, \mathbb{R}) \). It is convenient to introduce a notation \( J^0_{-n} \equiv J^0_{0-n} \).

One can prove a classification

**Lemma 2.1** [78]

(i) Suppose \( k < (n + 1) \). Any quasi-exactly-solvable operator \( T_k \) can be represented by a \( k \)th degree polynomial of the operators \( J^\pm_n \). If \( k \geq (n + 1) \), the part of the quasi-exactly-solvable operator \( T_k \) containing derivatives up to order \( n \) can be represented by an \( n \)th degree polynomial in the generators (1.2.3).

(ii) Conversely, any polynomial in (1.2.3) is quasi-exactly solvable.

(iii) Among quasi-exactly-solvable operators there exist exactly-solvable operators, \( E_k \). Any \( E_k \) can be represented by \( k \)th degree polynomial in generators \( J^0_{-n} \). Hence, \( E_k \) is a degeneration of \( T_k \) when terms which contain \( J^+_n \) are absent.

**Proof.** In order to prove the part (i) let us mention at first that the operators \( J^\pm_n \) act on the space \( \mathcal{P}_{n+1} \) irreducibly. This means there is no invariant subspace in \( \mathcal{P}_{n+1} \) under the action of \( J^\pm_n \). Thus, one can apply the Burnside

---

4 The representation (1.2.3) is one of the so-called ‘projectivized’ representations, see [71,73].
Let us call a universal enveloping algebra $U_{\mathfrak{sl}(2, \mathbb{R})}$ of a Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ the algebra of all ordered polynomials in generators $J_{n}^{\pm, 0}$. The notion ‘ordering’ means that in any monomial in $J_{n}^{\pm, 0}$ the generator $J_{n}^{+}$ is always placed to the left and the generator $J_{n}^{-}$ is placed to the right. Taking a realization of $\mathfrak{sl}(2, \mathbb{R})$ in terms of first order differential operators (1.2.3) we get a realization of the universal enveloping algebra in differential operators $U_{\mathfrak{sl}(2, \mathbb{R})}$. It is a subalgebra of the algebra of differential operators in one variable, $U_{\mathfrak{sl}(2, \mathbb{R})} \subset \text{diff}(1, \mathbb{R})$. Any element of $U_{\mathfrak{sl}(2, \mathbb{R})}$ preserves $P_{n+1}$. The converse is almost true: the algebra of differential operators which preserves $P_{n+1}$ is the infinite-dimensional algebra of differential operators generated by the operators (1.2.3) (which is $U_{\mathfrak{sl}(2, \mathbb{R})}$) plus annihilator $B \frac{d^{n+1}}{dx^{n+1}}$, where $B$ is any linear differential operator. The algebra $U_{\mathfrak{sl}(2, \mathbb{R})}$ depends on $n$. It is evident that two algebras characterized by different $n$’s are not isomorphic.

Comment 2.1 The notion - the universal enveloping algebra - allows us to state that at $k < n + 1$ a quasi-exactly-solvable operator $T_{k}$ of the order $k$ is simply an element of the universal enveloping algebra $T_{k} \in U_{\mathfrak{sl}(2, \mathbb{R})}$. However, if $k \geq n + 1$, then $T_{k}$ is represented as an element of $U_{\mathfrak{sl}(2, \mathbb{R})}$ plus $B \frac{d^{n+1}}{dx^{n+1}}$, where $B$ is any linear differential operator of order not higher than $(k - n - 1)$. Evidently, an operator $B \frac{d^{n+1}}{dx^{n+1}}$ annihilates the space (1.1.5). In other words, the algebra of differential operators acting on the space $P_{n+1}$ coincides with the algebra $U_{\mathfrak{sl}(2, \mathbb{R})}$ plus the annihilators.

Comment 2.2 As a consequence of the invariance (1.1.6) of the space $P_{n+1}$ the algebra $\mathfrak{sl}(2, \mathbb{R})$ generated by (1.2.3) is covariant with respect to conjugation
It is natural to introduce the notion of grading for the generators (1.2.3). It is easy to see that any \( \mathfrak{sl}(2, \mathbb{R}) \)-generator from (1.2.3) maps a monomial into monomial, \( J_\alpha^n x^p \mapsto x^{p+d_\alpha} \). Therefore, let us call \( d_\alpha \) the grading of the generator \( J_\alpha^n \): \( \deg(J_\alpha^n) = d_\alpha \). Following this definition \( \deg(J_\alpha^n) = +1 \), \( \deg(J_0^n) = 0 \), \( \deg(J_{-}^n) = -1 \), (1.2.6) and
\[
\deg[(J_\alpha^n)^{n_+} (J_0^n)^{n_0} (J_{-}^n)^{n_-}] = n_+ - n_- .
\] (1.2.7)

The notion of the grading allows us to classify the operators \( T_k \) in a Lie-algebraic sense.

**Lemma 2.2** A quasi-exactly-solvable operator \( T_k \subset U_{\mathfrak{sl}(2, \mathbb{R})} \) has no terms of positive grading, if and only if it is an exactly-solvable operator.

**Comment 2.3**. After transformation (1.2.5) the terms of zero grading remain of zero grading, ones of positive grading become of negative grading and, correspondingly, the terms of negative grading become of positive grading. Thus, any exactly-solvable operator having terms of negative and zero grading converts to the operator with terms of positive and zero grading under transformation (1.2.5). A quasi-exactly-solvable operator, however, always possesses terms of positive grading both in \( x \)-space representation and in \( z \)-space representation.

**Theorem 2.1** [75] Let \( n \) be a non-negative integer. Consider the eigenvalue problem for a linear differential operator of the \( k \)th order in one variable

\[
T_k(x, \frac{d}{dx}) \varphi(x) = \varepsilon \varphi(x) ,
\] (1.2.8)

where \( T_k \) is symmetric. The problem (1.2.8) has \( (n+1) \) linearly independent eigenfunctions in the form of a polynomial in variable \( x \) of order not higher than \( n \), if and only if \( T_k \) is quasi-exactly-solvable. The problem (1.2.8) has an infinite sequence of polynomial eigenfunctions, if and only if, the operator is exactly-solvable.

**Proof**. The “if” part of the first and the second statements is obvious. The “only if” part is a direct corollary of Lemma 2.1.
Theorem 2.1 gives a general classification of ordinary differential equations

\[ L_k \phi(x) \equiv \sum_{j=0}^{k} a_j(x) \frac{d^j \phi(x)}{dx^j} = \varepsilon \phi(x) , \tag{1.2.9} \]

having polynomial solution(s) in \( x \). The problem of finding spectra which corresponds to these polynomial eigenfunctions is reduced to the algebraic procedure of solving the system of \( (n+1) \) linear equations \(^6\). This leads to the solution of the secular (characteristic) equation which is a polynomial in \( \varepsilon \) of degree \( (n+1) \). In general, the spectral problem \((1.2.8)\) in addition to polynomial eigenfunctions possesses infinitely-many non-polynomial ones. Finding them is a non-algebraic procedure and is reduced to a diagonalization of a generic infinite-dimensional matrix.

A linear differential operator with polynomial coefficient functions is called algebraic. An operator which can be represented in terms of generators of some Lie algebra called Lie-algebraic.

According to the necessary condition formulated in Theorem 2.1 the coefficient functions \( a_j(x) \) in \((1.2.9)\) should be of the form

\[ a_j(x) = \sum_{i=0}^{k+j} a_{j,i} x^i . \tag{1.2.10} \]

Hence, the linear differential operator in the l.h.s. of \((1.2.9)\) should be algebraic of the Fuchs type, with Fuchs index \( k \). However, it is worth emphasizing that not every algebraic operator admits polynomial eigenfunctions or, in other words, the existence of an algebraic sector.

Sufficient condition provided by Theorem 2.1 requires that the coefficients \( a_{j,i} \) in \((1.2.10)\) for coefficient functions in \((1.2.9)\) are not arbitrary. They have to be of such a form that the linear differential operator in the l.h.s. of \((1.2.9)\) can be rewritten as a \( k \)th degree polynomial element of the universal enveloping algebra \( U_{\mathfrak{sl}(2,R)}^d \) taken in realization \((1.2.6)\) of spin \( n \). Therefore, the operator \( L \) should be not only algebraic but also Lie-algebraic. This generates a set of constraints and reduces the number of free parameters of the algebraic operator \( T \). The number of free parameters of the algebraic operator \( T \) defines the number of free parameters of the polynomial solutions in \((1.2.9)\). It can be obtained by counting the number of parameters which characterize a general \( k \)th degree polynomial element of the universal enveloping algebra \( U_{\mathfrak{sl}(2,R)}^d \) factorized over the Casimir operator. Thus, all elements containing \( J_n^0, J_n^0 \) (see \((1.2.4)\)) should be excluded from counting. A straightforward calculation leads

\(^6\) We will call this part of the spectra of the equation \((1.2.9)\) the algebraic sector.
to the following formula

$$par(T_k) = (k + 1)^2 + 1,$$  \hspace{1cm} (1.2.11)

where the number of free parameters of the quasi-exactly-solvable operator $T_k$ is denoted $par(T_k)$ where the mark of representation (1.2.6) $n$ is included. If the parameter $n$ is integer, the representation (1.2.6) becomes finite-dimensional. In this case the parameter $n$ defines a degree of a polynomial solution, it also appears in coefficients of polynomial eigenfunctions.

Comment 2.4. We can see that in the basis of monomials, $x^p, p = 0, 1, \ldots$, the Fuchs type operator $L_k$ with Fuchs index $k$, see (1.2.9) - (1.2.10), has the form of a $(2k + 1)$-diagonal infinite matrix with $k$ upper and $k$ lower sub-diagonals. In the quasi-exactly-solvable case the matrix $L_k$ additionally becomes upper (lower) block-triangular. In the exactly-solvable case the matrix $L_k$ additionally becomes upper (lower) triangular. By a suitable change of $x$-coordinate one sub-diagonal can be removed.

For the case of second order differential operators, the number of free parameters (1.2.11) is equal to $par(T_2) = 10$, while a generic second order differential operator $L_2$ in (1.2.9) with coefficients (1.2.10) is characterized by 15 free parameters \footnote{It is worth mentioning that the generic Fuchs type second-order differential operator with Fuchs index 2 is nothing but the celebrated Heun operator (see e.g. Kamke [94]).}. Imposing five conditions on the operator (1.2.9) with coefficients (1.2.10) (said differently, on a generic Heun operator) we get a second order differential operator which can be written in terms of the $\mathfrak{sl}(2, \mathbb{R})$ generators with the mark of representation equals to some integer $n$. Thus, the second order differential operator has the meaning of the Hamiltonian of the Euler-Arnold quantum top. A condition of quasi-exact solvability implies the finite-dimensionality of the representation of $\mathfrak{sl}(2, \mathbb{R})$. Eventually, it leads to six constraints such that, five parameters of the original Heun operator become fixed and the sixth parameter $n$ is restricted to an integer value. It will be discussed in details below, in Section 2.2.

In the case when the number of polynomial solutions is infinite, the expression (1.2.10) simplifies to

$$a_j(x) = \sum_{i=0}^{j} a_{j,i} x^i,$$ \hspace{1cm} (1.2.12)

thus, $k = 0$, in agreement with the results by Krall [36] (see also [40]). The number of free parameters is equal to

$$par(E_k) = \frac{(k + 1)(k + 2)}{2}.$$ \hspace{1cm} (1.2.13)
In this case the \( n \)th eigenfunction is a polynomial in \( x \) of degree \( n \). One can easily find the eigenvalue which corresponds to the eigenfunction given by the polynomial of degree \( n \),

\[
\varepsilon_n = \sum_{j=0}^{k} a_{jj} \frac{n!}{(n-j)!}.
\]

(1.2.14)

Thus, this eigenvalue is the polynomial in \( n \) of degree \( k \).

It can be shown that the operator \( T_k \) with the coefficients (1.2.12) preserves a finite flag

\[ P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_k \]

of spaces of polynomials. It can be easily verified that the preservation of such a finite flag of spaces of polynomials implies the preservation of an infinite flag of such spaces.

A class of spaces which are equivalent to the space of polynomials (1.1.5) is presented by

\[ \langle \alpha, \alpha \beta, \ldots, \alpha \beta^n \rangle, \]

(1.2.15)

where \( \alpha = \alpha(z) \), \( \beta = \beta(z) \) are arbitrary functions. A linear differential operator acting on the (1.2.15) is easily obtained from a quasi-exactly-solvable operator (1.2.8)–(1.2.10) (see Lemma 2.1) by making the change of variable

\[ x = \beta(z), \]

and the gauge (similarity) transformation

\[ \tilde{T} = \alpha(z) T \alpha(z)^{-1}. \]

It is worth noting that in the case of the Möbius transformation (1.1.7)

\[ x = \frac{az + b}{cz + d}, \quad \alpha(z) = (cz + d)^n, \]

the space (1.2.15) remains the space of polynomials and the operator \( \tilde{T} \) is an algebraic quasi-exactly-solvable operator. Explicitly, the operator of \( k \)th order has the form

\[
\tilde{T}_k = \alpha(z) \sum_{j=0}^{k} \left( \sum_{i=0}^{k-j} a_{ji} \beta(z)^i \right) \frac{d^j}{dx^j} \bigg|_{x=\beta(z)} \alpha(z)^{-1},
\]

(1.2.16)

where the coefficients \( a_{ji} \) are the same as in (1.2.10).

As a result the expression (1.2.16) gives a general form of the linear differential operator of \( k \)th order acting on a space (1.2.15) equivalent to (1.1.5). Since any one- or two-dimensional invariant functional space can be presented in
the form \((1.2.15)\) and thus can be reduced to \((1.1.5)\), the following general statement holds:

**Theorem 2.2** *There are no linear operators possessing one- or two-dimensional invariant sub-space with an explicit basis other than given by Lemma 2.1.*

Therefore, an eigenvalue problem \((1.2.8)\) for which a single eigenfunction can be found in an explicit form, is related to the operator

\[
T^{(1)} = B\left(x, \frac{d}{dx}\right) \frac{d}{dx} + q_0,
\]

where \(B(x, \frac{d}{dx})\) is an arbitrary linear differential operator. The operator \((1.2.17)\) has a constant eigenfunction with an eigenvalue \(\epsilon = q_0\).

It is worth mentioning that one can introduce a useful notion of *gauge* transformation of derivative. It corresponds to replacing the derivative by the *covariant* derivative,

\[
e^{-A(x)} \frac{d}{dx} e^{A(x)} = \frac{d}{dx} + A'(x) \equiv \mathcal{D}(A(x)),
\]

where \(A\) is called the gauge phase and \(A'(x)\) is a connection. It is evident that this transformation is canonical: the Lie bracket, \([\frac{d}{dx}, x]\) remains unchanged. The gauge transformation can be accompanied by a change of variable,

\[
e^{-A(x)} \frac{d}{dx} e^{A(x)} \bigg|_{z(z(x))} = \mathcal{D}(A(x)) \bigg|_{z(z(x))} = z'(x(z)) \frac{d}{dz} + A'_z(x(z)).
\]

Making the gauge transformation of \((1.2.17)\) we arrive at the gauge-transformed operator

\[
\tilde{T}^{(1)} = B(x, \mathcal{D}(A(x)))\mathcal{D}(A(x)) + q_0 = \tilde{B}(x, \frac{d}{dx})(\frac{d}{dx} + A'(x)) + q_0,
\]

which has an eigenfunction \(\varphi(x) = e^{-A(x)}\).

A general differential operator possessing two eigenfunctions is the Lie-algebraic operator: following Theorem 2.2 its explicit form is given by

\[
T^{(2)} = B\left(x, \frac{d}{dx}\right) \frac{d^2}{dx^2} + q_2(x) \frac{d}{dx} + q_1(x),
\]

where \(B(x, \frac{d}{dx})\) is again an arbitrary linear differential operator. The coefficients \(q_{1,2}(x)\) are the first- and second-order polynomials, respectively, with coefficients such that \(q_2(x) \frac{d}{dx} + q_1(x)\) can be expressed as a linear combination of the generators \(J_1^{\pm,0}\) (see \((1.2.3)\)). In general, the operator \((1.2.21)\) has two eigenfunctions in a form of linear function \(\varphi(x) = x + c\). Performing gauge transformation \((1.2.19)\) of \((1.2.21)\) we arrive at

\[
\tilde{T}^{(2)} = B(x, \mathcal{D}(A(x)))\mathcal{D}^2(A(x)) + q_2(x)\mathcal{D} + q_1(x)
\]
where two "algebraic" eigenfunctions are of the form \((x + c)e^{-A(x)}\).

### 2.2 Second-order differential equations

Second-order differential equations play an exceptionally important role in a vast majority of applications in different sciences. Due to their importance they certainly deserve a detailed and careful consideration. We aim to explore one particular aspect of a general theory: a description of the second-order differential equations (1.2.8) possessing polynomial solutions. This problem is reduced to studying the eigenvalue problem for the Heun operator, see e.g. [55]

\[
h_e = -P_4(x) \frac{d^2}{dx^2} + P_3(x) \frac{d}{dx} + P_2(x) ,
\]

(1.2.23)

where \(P_{4,3,2}(x)\) are polynomials of degrees 4, 3, 2, respectively, (hence, we study the Heun equation with constrained coefficients), which has polynomial eigenfunctions. Saying differently, we reduce this problem to a classification of the second order differential operators with finite-dimensional invariant subspace in polynomials. This, it is further reduced to a classification of the Heun operators admitting finite-dimensional invariant subspace in polynomials, in other words, operators which are quasi-exactly-solvable.

A general Heun operator with arbitrary coefficients in the basis of monomials is given by five-diagonal matrix. For a quasi-exactly-solvable (QES) Heun operator this five-diagonal matrix becomes block-triangular. However, in all known examples the QES Heun operator in the basis of monomials can be reduced to a tri-diagonal (Jacobi), block-triangular matrix.

Theorem 2.1 says that the second order differential operator which is quasi-exactly-solvable should be of the form

\[
T_2(J_n^\alpha, c_{\alpha\beta}, c_{\alpha}) = c_{++}J_n^+ J_n^+ + 2c_{++}J_n^+ J_n^0 + 2c_{+-}J_n^+ J_n^- + 2c_{0+}J_n^0 J_n^+ + c_- J_n^- J_n^-
\]

\[
+ c_+ J_n^+ + 2c_0 J_n^0 + c_- J_n^- + c ,
\]

(1.2.24)

where \(c_{\alpha\beta}, c_{\alpha}, c\) are arbitrary constants, the factor 2 in front of the crossterms is introduced for convenience, and \(J_n^\alpha\) are the generators of the algebra \(\mathfrak{sl}(2, \mathbb{R})\) (see (1.2.3)) in \((n+1)\)-dimensional representation. The number of free parameters is \(\text{par}(T_2) = 9\) with an extra (integer) parameter \(n\). The operator \(T_2\) (1.2.24) is simply the Hamiltonian of the \(\mathfrak{sl}(2, \mathbb{R})\) Euler-Arnold quantum top

\[
h_{E-A} = c_{++}J_n^+ J_n^+ + c_{+0}\{J_n^+, J_n^0\} + c_{+-}\{J_n^+, J_n^-\} + c_{0-}\{J_n^0, J_n^+\} + c_- J_n^- J_n^-
\]

\[
+ (c_+ + c_{+0})J_n^+ + 2(c_0 - c_{+-})J_n^0 + (c_- + c_{0-})J_n^- + c ,
\]

(1.2.25)
in a constant (imaginary) magnetic field,

\[ B = \left( (c_+ + c_{+0}), \ 2(c_0 - c_{+0}), \ (c_- + c_{0-}) \right). \]

Here \( c_{\alpha\beta} \) play the role of components of the tensor of inertia. If \( n \) takes an integer value, the operator \( T_2 \) has \((n + 1)\)-dimensional invariant subspace: it preserves the space \( P_{n+1} \).

As a consequence of the invariance (1.1.6) of the space \( P_{n+1} \), see also (1.2.5), the operator \( T_2 \) (1.2.24) is equivalent to

\[
T_2^{(e)}(J^\alpha_n, c_{\alpha\beta}, c_\alpha) = c_+ J_n^+ J_n^- + 2c_{+0} J_n^- J_n^0 + 2c_{+} J_n^- J_n^+ + 2c_{0-} J_n^0 J_n^+ + c_- J_n^+ J_n^- - c_- J_n^- J_n^+ + c.
\]

(1.2.26)

Substituting the representation (1.2.3) into the operator (1.2.24) we arrive at the Heun operator \( h_e \) (1.2.23), where the coefficient functions \( P_{4,3,2} \) are polynomials of degrees 4, 3, 2, respectively. The eigenvalue problem \( h_e \varphi = \varepsilon \varphi \) (cf. (1.2.8)) takes the form

\[
- P_4(x) \frac{d^2}{dx^2} \varphi(x) + P_3(x) \frac{d}{dx} \varphi(x) + P_2(x) \varphi(x) = \varepsilon \varphi(x),
\]

(1.2.27)

where the coefficients of \( P_j(x) \) are related to \( c_{\alpha\beta}, c_\alpha \) and \( n \). If the polynomial coefficients \( P_j(x) \) have arbitrary coefficients this equation becomes the well-known Heun equation (see e.g. [94]). This equation is characterized by four singular points and is considered to be of the next level of complexity after hypergeometrical (Riemann) equation. The general theory of the Heun equation is presented in the book by Ince [32] (see also [94, 55, 43] for concrete examples).

Theorem 2.1 provides the necessary and sufficient conditions for the spectral problem with the operator (1.2.24)-(1.2.27) to have \((n + 1)\) polynomial solutions of the form of polynomials in \( x \) of degree not higher than \( n \). It implies that the coefficient functions \( P_j(x) \) must be of the form

\[
-P_4(x) = c_{++} x^4 + 2c_{+0} x^3 + 2c_{+-} x^2 + 2c_{0-} x + c_{--},
\]

\[
P_3(x) = 2(1 - n) c_{++} x^3 + [(2 - 3n) c_{+0} + c_{+} ] x^2 - 2(n c_{+-} - c_0) x - n c_{0-} + c_-,\]

\[
P_2(x) = n^2 c_{++} x^2 + n^2 c_{+0} x - n c_{+} x - n c_0 + c. \]

(1.2.28)

In general, Theorem 2.1 states that there are no other algebraic operators of the second order beyond the Heun operator which generically admit polynomial eigenfunctions.

Comment 2.5. It is important to note that the reference point for coordinate \( x \) can be chosen by replacing \( x \to x + a \) in such a way that

\[
-P_4(a) = c_{++} a^4 + 2c_{+0} a^3 + 2c_{+-} a^2 + 2c_{0-} a + c_{--} = 0,
\]

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hence the coefficient function $P_4$ has no constant term and one of its singular points is located at the origin. It implies that in the operators $T_2$ (1.2.24) and $T_2^{(e)}$ (1.2.26) the coefficient $c_{--} = 0$, and the terms $J_n^--J_n^-$ and $J_n^+J_n^+$ are absent in these operators, respectively. In this case the eigenvalue problem for $T_2^{(e)}$ degenerates to

$$- \tilde{P}_3(z) \frac{d^2}{dz^2} \varphi(z) + \tilde{P}_2(z) \frac{d}{dz} \varphi(z) + \tilde{P}_1(x) \varphi(z) = \varepsilon \varphi(z) , \quad (1.2.29)$$

where $\tilde{P}_{3,2,1}$ are polynomials of degree 3,2,1, respectively. This is the so-called polynomial form of the celebrated Heun equation.

Let us proceed by exploring a few particular cases of the Heun operator.

**Lemma 2.3** If the operator (1.2.24) is such that

$$c_{++} = 0 \quad \text{and} \quad c_+ = (n - 2m)c_{+0} , \quad (1.2.30)$$

where $m$ is a non-negative integer, then the operator $T_2$ preserves both $\mathcal{P}_{n+1}$ and $\mathcal{P}_{m+1}$. In this case the number of free parameters is reduced, $\text{par}(T_2) = 7$.

The proof of Lemma 2.3 is based on the fact that under conditions (1.2.17), the operator $T_2(J_{\alpha}^n, c_{\alpha\beta}, c_{\alpha})$ can be rewritten in terms of the generators $J_{\alpha}^n$ like $T_2(J_{\alpha}^n, c_{\alpha\beta}', c_{\alpha}')$. As a consequence of Lemma 2.3 and Theorem 2.1 in general, among polynomial solutions of (1.2.9) there are some solutions given by polynomials of order $n$ and others given by polynomials of order $m$.

**Remark.** From the Lie-algebraic point of view Lemma 2.3 indicates the existence of representations of second-degree polynomials in the generators (1.2.3) possessing two invariant sub-spaces. In general, if $n$ in (1.2.3) is a non-negative integer, then among representations of $k$th degree polynomials in the generators (1.2.3), lying in the universal enveloping algebra, there exist representations possessing 1, 2, ..., $k$ invariant sub-spaces. Even starting from an infinite-dimensional representation of the original algebra (1.2.3) one can construct the elements of the universal enveloping algebra having finite-dimensional representation (for example, the parameter $n$ in (1.2.17) is non-integer, however, the operator $T_2$ has the invariant sub-space of dimension $(m + 1)$). Also this property implies the existence of representations of the polynomial elements of the universal enveloping algebra $U_{\mathfrak{sl}(2,\mathbb{R})}$, which can be obtained starting from different representations of the original algebra (1.2.3).

**Lemma 2.4** If the operator (1.2.24) is such that

$$c_{++} = c_{+-} = c_{--} = c_0 = c = 0 , \quad (1.2.31)$$

\[ n \text{ in (1.2.3) is not a non-negative integer} \]
then in the spectral problem (1.2.8) the \((n + 1)\) eigenvalues corresponding to polynomial eigenfunctions have a center of symmetry at \(\varepsilon = 0\).

The proof is based on the fact that under condition (1.2.31) the tridiagonal matrix \(T_2\), which describes the algebraic sector of the operator (1.2.24), has vanishing diagonal matrix elements. It can be easily verified that \(\text{tr}(T_2^{2k+1}) = 0\) for any \(k = 0, 1, 2\ldots\). Hence, the characteristic polynomial has the form

\[
S_{n+1}(\varepsilon) = \varepsilon^p s_{\frac{n+1}{2}}(\varepsilon^2),
\]

where \(p = 0\) or \(1\) depending on \((n + 1)\) is even or odd, respectively. ■

This phenomenon is called the energy-reflection symmetry [61,16]. There are several interesting potentials having this symmetry. The simplest of them is

\[
V(x) = x^6 - (3 + 2n)x^2, \quad n = 0, 1, 2, \ldots.
\]

It is worth mentioning that the presence of the energy-reflection symmetry simplifies the finding of eigenvalues.

A special situation occurs if the parameter \(n\) takes the value 0 or 1. For \(n = 1\) as a consequence of Theorem 2.2 a spectral problem for the operator which is more general than (1.2.24)−(1.2.27) arises

\[
- F_3(x) \frac{d^2}{dx^2} \varphi(x) + \tilde{P}_2(x) \frac{d}{dx} \varphi(x) + \tilde{P}_1(x) \varphi(x) = \varepsilon \varphi(x), \quad (1.2.32)
\]

where \(F_3(x)\) is an arbitrary function and

\[
\tilde{P}_2 = c_+ x^2 + 2c_0 x + c_- , \quad -\tilde{P}_1 = c_+ x + c_0 - c ,
\]

with arbitrary coefficients \(c\)'s. The problem (1.2.32) is characterized by two polynomial solutions

\[
\varphi_{\pm} = N_{\pm}(c_+ x + c_0 \pm \sqrt{c_0^2 - c_+ c_-}) , \quad \varepsilon_{\pm} = c \mp \sqrt{c_0^2 - c_+ c_-} , \quad (1.2.33)
\]

where \(N_{\pm}\) is a normalization factor. It is evident if the operator in the l.h.s. of (1.2.32) is symmetric, then the condition

\[
c_0^2 - c_+ c_- > 0 , \quad (1.2.34)
\]

will be fulfilled.

For the case \(n = 0\) (one polynomial solution, \(\varphi_0 = \text{const}\)) the spectral problem (1.2.8) becomes

\[
- F_2(x) \frac{d^2}{dx^2} \varphi(x) + F_1(x) \frac{d}{dx} \varphi(x) + c \varphi(x) = \varepsilon \varphi(x), \quad (1.2.35)
\]
(cf. (1.2.27)) with two arbitrary functions $F_{2,1}(x)$ and $c \in \mathbb{R}$. Here the eigenvalue $\varepsilon_0 = c$ which corresponds to $\varphi_0 = \text{const}$.  

A very important particular case arises if in the operator (1.2.24) some of the coefficients vanish: $c_{++} = c_{+0} = c_+ = 0$. The quasi-exactly-solvable operator $T_2$ becomes the exactly-solvable operator $E_2$ (Lemma 1.2.2)

$$E_2 = 2c_{+-} J_n J_n^- + 2c_0 J_n^0 J_n^- + c_- J_n^0 J_n^- + 2c_0 J_n^0 J_n^- + c_- J_n^- + c, \quad (1.2.36)$$

and the number of free parameters is reduced to $\text{par}(E_2) = 6$. Substituting the generator (1.2.3) into (1.2.36) and then into (1.2.9), we obtain the equation

$$-Q_2(x) \frac{d^2}{dx^2} \varphi(x) + Q_1(x) \frac{d}{dx} \varphi(x) + Q_0(x) \varphi(x) = \varepsilon \varphi(x), \quad (1.2.37)$$

where $Q_j(x)$ are polynomials of $j$th order

$$-Q_2(x) = 2c_{+-} x^2 + 2c_{0-} x + c_-, \qquad Q_1(x) = -2(nc_{+-} - c_0) x - nc_{0-} + c_-, \qquad Q_0(x) = -nc_0 + c. \quad (1.2.38)$$

Different values of $n$ correspond to a redefinition of parameters $c_{0-}$ and $c$, thus, without loss of generality, one can set $n = 0$. The equation (1.2.37) with coefficient functions (1.2.38) coincides with the hypergeometrical equation. The $c$-coefficients in (1.2.37) - (1.2.38) are arbitrary. However, effectively the number of free parameters has been reduced by three: (i) the parameter $c$ corresponds to a choice of the reference point for the energy and can be set equal to zero; Since the equation (1.2.37) is defined up to a linear change of variable (1.1.8) without a loss of generality, (ii) the parameter $c_{+-}$ or $c_0$ can be set equal to one and also (iii) the parameter $c_{0-}$ or $c_-$ can be set equal to zero.

As for exact-solvability, there exists families of equivalent, isospectral exactly-solvable operators $E_2$ whose spectrum coincide up to a reference point. Effectively, the number of free parameters of the exactly-solvable operator $E_2$ is equal to three, $\text{par}(E_2) = 3$.

The operator on the l.h.s. of (1.2.37) coincides with the generic hypergeometrical (Riemann) operator. It leads to

**Corollary 2.1.** The hypergeometrical operator

$$q = (2c_{+-} x^2 + 2c_{0-} x + c_-) \frac{d^2}{dx^2} + (2c_0 x + c_-) \frac{d}{dx} + c,$$

where $c$’s are arbitrary numbers, is the only exactly-solvable operator among second order differential operators.
In general, the eigenvalues of (1.2.37) are given by a quadratic polynomial in an integer number $k = 0, 1, 2, \ldots$ which enumerates eigenstates

$$
\varepsilon_k = 2c_{++}k^2 - 2(c_{+-} - c_0)k + c ,
$$

(1.2.39)

while the $(k + 1)$th eigenfunction is the polynomial in $x$ of order $k$. The parameter $c$ has the meaning of the reference point for eigenvalues, it can be set equal to zero, without loss of generality.
3 (Quasi)-Exactly-Solvable Schrödinger Equations

3.1 Generalities

The stationary Schrödinger equation

\[ \left( -\frac{d^2}{dz^2} + V(z) \right) \Psi(z) = E \Psi(z), \quad (1.3.1) \]

with \( L^2 \)-boundary conditions is the central equation of quantum physics. Hence, it is one of the most important types of second-order differential equations. Here the spectral parameter \( E \) is the energy and \( \Psi(z) \) must be a square-integrable (normalizable) function in the domain where the problem is defined. The operator in the l.h.s. of (1.3.1) is the Hamiltonian which is sometimes called the Schrödinger operator. Needless to say that any exact solution of (1.3.1) is of great importance revealing non-trivial properties, serving as a starting point for developing a perturbation theory, modeling physical phenomena, etc. Quasi-exact solvability is one method of creating exact solutions.

There are three domains where the equation (1.3.1) can be defined: a real line, semi-line or interval. Our task is two-fold: first to find (and classify) the quasi-exactly-solvable Schrödinger operators which preserve a finite-dimensional subspace of a Hilbert space. Second, to find the exactly-solvable Schrödinger operators which preserve the infinite flag of finite-dimensional subspaces of a Hilbert space.

One possible way to construct the (quasi)-exactly-solvable Schrödinger operators is to take the Lie-algebraic quasi-exactly-solvable \( T_2 \) (or exactly-solvable \( E_2 \)) operator acting on finite-dimensional space(s) of polynomials (1.1.5) and try to transform it into the Schrödinger type operator. This can always be done by making a change of a variable and a gauge transformation (see (1.1.2)) as a consequence of the one-dimensional nature of the differential equations we are studying. In practice, the realization of this transformation is nothing but a conversion of (1.2.27)-(1.2.37) into (1.3.1). All obtained solutions following such a procedure have a factorizable form of a polynomial in some variable multiplied by some factor. Usually, this factor is an entire function of \( z \) without zeroes at real \( z \) within the domain. For all the known quasi-exactly-solvable problems it has the meaning of the ground state eigenfunction of a primitive quasi-exactly-solvable problem at \( n = 0 \). One open question remains to be answered: whether obtained solutions of the equation (1.3.1) belong to square-integrable ones or not? From a general point of view, this question will be discussed in the end of this Section. In what follows, we restrict our consideration to particular examples being limited to real functions and real
variables only.

Let us consider the eigenvalue problem for the Heun operator (1.2.23), \( h_e \equiv T_2 \),

\[ T_2(x, \frac{d}{dx}) \varphi(x) = \varepsilon \varphi(x), \]

where \( h_e \) is the Lie algebraic operator of second degree (1.2.24) which is explicitly,

\[ -P_4(x) \frac{d^2}{dx^2} \varphi(x) + P_3(x) \frac{d}{dx} \varphi(x) + P_2(x) \varphi(x) = \varepsilon \varphi(x), \]

(see (1.2.27)-(1.2.28)). It is worth noting that at \( n = 0 \) the coefficient function \( P_2(x) \) becomes constant. Without loss of generality, this constant can be placed equal to zero by redefining \( \varepsilon \). By introduction of a new variable \( x = x(z) \) and a new function

\[ \Psi(z) = \varphi(x(z)) e^{-A(z)}, \quad (1.3.2) \]

one can always reduce the spectral problem for the Lie algebraic operator \( T_2 \) (1.2.27) to (1.3.1) with the potential \( V(z) = (A')^2 - A'' + P_2(x(z)) \).

(1.3.3)

Sometimes, \( A \) is called prepotential. Here

\[ A = \int \left( \frac{P_3}{P_4} \right) dx \log z', \quad z = \pm \int \frac{dx}{\sqrt{P_4}}, \quad (1.3.4) \]

Let us mention that in \( x \)-variable the derivative of \( A \) looks like a ratio of polynomials,

\[ A'(x) = \frac{P_3(x) - P'_4(x)}{2P_4(x)}. \]

It makes sense as the logarithmic derivative of a gauge factor (see (1.2.18)) - it is a very important object of the theory. The final form of the most general quasi-exactly-solvable potential in \( x \)-variable is given by the rational function

\[ V(x) = \frac{(P'_4(x) + 2P_3(x))(3P'_4(x) + 2P_3(x))}{16P_4(x)} - \frac{P''_4(x)}{4} - \frac{P'_3(x)}{2} + P_2(x), \quad (1.3.5) \]

where \( P_{4,3,2} \) are from (1.2.28). The Hamiltonian in \( x \)-variable is

\[ \mathcal{H}(x) = -\Delta_g + \]

\[ \frac{(P'_4(x) + 2P_3(x))(3P'_4(x) + 2P_3(x))}{16P_4(x)} - \frac{P''_4(x)}{4} - \frac{P'_3(x)}{2} + P_2(x), \quad (1.3.6) \]
where $\Delta_g$ is the one-dimensional Laplace-Beltrami operator with metric $g^{11}$,

$$
\Delta_g = \frac{1}{\sqrt{g}} \frac{d}{dx} g^{11} \sqrt{g} \frac{d}{dx} = g^{11} \frac{d^2}{dx^2} + \frac{g^{11}}{2} \frac{d}{dx}, 
\quad g^{11} = P_4(x),
$$

and determinant $g = g_{11} = \frac{1}{g^{11}}$. Changing the variable $x$ to $z$ (1.3.4) the Laplace-Beltrami operator in $x$ becomes the second derivative in $z$.

A similar procedure can be performed for the equation (1.2.32) with two known eigenfunctions. It requires

$$
A = \int \left( \frac{\tilde{P}_3}{F_3} \right) dx - \log z', \quad z = \pm \int \frac{dx}{\sqrt{F_3}},
$$

and leads to a potential

$$
V(z) = (A')^2 - A'' + \tilde{P}_1(x(z)),
$$

(cf. (1.3.3)). For the case of the equation (1.2.35) with one known eigenfunction

$$
A = \int \left( \frac{F_1}{F_2} \right) dx - \log z', \quad z = \pm \int \frac{dx}{\sqrt{F_2}},
$$

and

$$
V(z) = (A')^2 - A'' + c.
$$

where a new variable $z$ in (1.3.4)-(1.3.8) is found from the condition that the coefficient function in front of the second derivative in (1.3.1) is equal to one. If the functions (1.3.2) obtained after transformation belong to the $L^2(D)$-space\(^9\), we arrive at the quasi-exactly-solvable Schrödinger equations [73,71,72], where a finite number of eigenstates is found algebraically. In this case the spectral parameter $\varepsilon$ plays the role of energy $E$. We call these quasi-exactly-solvable equations the quasi-exactly-solvable equations of the first type.

There exists another type of quasi-exactly-solvable Schrödinger equations related with the generalized eigenvalue problem,

$$
(- \frac{d^2}{dz^2} + \tilde{V}(z)) \Psi(z) = \varepsilon \varrho(x(z)) \Psi(z),
$$

where a weight function $\varrho(z) = \varrho(x(z))$ is inserted in the r.h.s. (cf. (1.3.1)). Such a spectral problem occurs naturally if a new variable is found following the requirement that the coefficient in front of the second derivative in $z$ is equal to some function

$$
P_4(z')^2 = \frac{1}{\varrho(x)},
$$

\(^9\) Depending on the change of variable $x = x(z)$, the space $D$ can be the real line, a semi-line and a finite interval.
different from 1. In this case a new variable occurs

$$z = \pm \int \frac{dx}{\sqrt{\rho(x)P_4(x)}}.$$  \hspace{1cm} (1.3.10)

(cf. (1.3.4)). The equation which is obtained from (1.2.27), (1.2.32), (1.2.35) by taking the same gauge factor (1.3.2) has the form

$$\left\{ \frac{1}{\rho(x(z))} \left[ -\frac{d^2}{dz^2} + (A')^2 - A'' \right] + P_2(x(z)) \right\} \Psi(z) = 0.$$ \hspace{1cm} (1.3.11)

Multiplying both sides of this equation by $\rho(x(z))$ we arrive at

$$\left\{ -\frac{d^2}{dz^2} + (A')^2 - A'' + [P_2(x(z)) - \varepsilon] \rho(x(z)) \right\} \Psi(z) = 0,$$

which can be rewritten in the form (1.3.9) with a potential

$$\tilde{V}(z) = (A')^2 - A'' + P_2(x(z)) \rho(x(z)),$$ \hspace{1cm} (1.3.12)

this is a slight modification of (1.3.3). For all known quasi-exactly-solvable problems of the second type the weight factor is a monomial $\rho(x) = x^d$, where $d$ is equal to either $\pm 1$ or $\pm 2$. In the case of those problems the spectral parameter $\varepsilon$ looses its meaning of energy, while the energy enters explicitly (or implicitly) to the potential as a constant term. The meaning of the spectral problem (1.3.9) is rather unusual. We study a family of potentials such that the ground state energy in the first potential is equal to the energy of the first excited state in the second potential, which is itself equal to the energy of the second excited state in the third potential, etc. The eigenfunctions of these states have the form of a polynomial of a fixed degree multiplied by some factor. This procedure is widely used in atomic physics in the case of the Coulomb problem. Namely, instead of studying the spectral problem for the Coulomb Hamiltonian $(-\Delta - \frac{\alpha}{r}) \psi = E \psi$, we explore a modified spectral problem $(-\Delta - E) \psi = \frac{\alpha}{r} \psi$, or equivalently, $(-r \Delta - Er) \psi = \alpha \psi$, where the energy $E$ is fixed and quantization of the charge $\alpha$ is considered. It ends up with quantization formula $\alpha_k = \sqrt{-2E} k$, where $k = 1, 2, 3, \ldots$. If we assume now that the charge $\alpha$ is fixed this formula leads immediately to the familiar formula for the quantization of energy: $E_k = -\frac{1}{2k^2}$, $k = 1, 2, 3, \ldots$. From a physical point of view such an approach means that we are looking for the Coulomb problem of different charges which contains an eigenstate of a certain fixed energy $E$. Such a representation of the Schrödinger equation as a spectral problem is called the Sturm representation for the Coulomb problem. Our spectral problem is a generalization of the Sturm representation to other potentials. We will continue to call it the Sturm representation. This representation turned out to be very useful for studying the quasi-exactly-solvable problems [71,79].
Below, we will follow the catalogue given in [72]. A presentation of results is given the following sequence: First, we display the quadratic element $T_2$ of the universal enveloping algebra $\mathfrak{sl}(2, \mathbb{R})$ in the representation (1.2.3) and then its equivalent form of differential operator $T_2(x, \frac{d}{dx})$. Second, we present the corresponding potential $V(z)$ or $\tilde{V}(z)$ if the weight factor $\varrho(x) \neq 1$ and an explicit expression for the change of the variable $x = x(z)$, weight function $\varrho(z)$. Third, we present the explicit form of $y = -A'(z)$, which is the logarithmic derivative of the ground state eigenfunction for the potential at $n = 0$ with negative sign. Finally, the functional form of the eigenfunctions $\Psi(z)$ of the “algebraized” part of the spectra (the algebraic sector of the space of eigenfunctions) is given.

All known one-dimensional quasi-exactly-solvable Schrödinger equations degenerate either to well-known exactly-solvable ones, or the non-solvable special ones - like the Mathieu or Lame equations, being associated with them. In total, there exists ten types of the quasi-exactly-solvable, one-dimensional Schrödinger equations.

We begin our consideration with the quasi-exactly-solvable equations associated with the exactly-solvable Morse oscillator. It implies that at the limit, when the number of algebraic eigenstates $(n + 1)$ goes to infinity, the Morse oscillator is recovered. There are three quasi-exactly-solvable problems of this type which are given by Case I, II, III. Their potentials are finite pieces of the Laurent series in variable $e^{-\alpha z}$.

\subsection*{3.2 Morse-type potentials}

The Morse oscillator is one of the well-known exactly-solvable quantum-mechanical problems (see e.g. Landau and Lifschitz [37]). It is described by the Hamiltonian with the potential

$$V(z) = A^2 e^{-2\alpha z} - 2Ae^{-\alpha z}, \quad A > 0, \quad \alpha > 0,$$

which is called the Morse potential, see Fig. 1.1. In general, this equation is characterized by a finite number of bound states,

$$E_k = -\left[1 + \frac{\alpha}{2}(1 - 2k)\right]^2, \quad k = 0, 1 \ldots k_{max},$$

its number $k_{max} = \left[\frac{1}{\alpha} + \frac{3}{2}\right]$ depends on the parameter $\alpha$ only, and where $[a]$ denotes the integer part of $a$. The potential $V(z)$ is periodic with the imaginary

\footnote{The functions $p_n(x)$ occurring in the expressions for $\Psi(z)$ denote polynomials of the $n$th order. They are nothing but the polynomial eigenfunctions of the operator $T_2(x, \frac{d}{dx})$.}
Fig. 1.1. Morse potential (1.3.13) at $A = \alpha = 1$. It has a minimum at $z = 0$, $V_{\text{min}} = -1$. The discrete spectra consists of a single bound state (the ground state) with energy $E_0 = -\frac{1}{4}$, and eigenfunction $\Psi_0 = e^{-\frac{z}{2}}e^{-z}$.

period $i\frac{2\pi}{\alpha}$ and the Hamiltonian is translation-invariant, $T_c : z \to z + i\frac{2\pi}{\alpha}$. Taking the invariant $t_{\mp} = e^{\mp\alpha z}$ as a new variable (note, that $t_-t_+ = 1$), we get the Morse potential in polynomial form

$$V(t_-) = A^2t_-^2 - 2At_- , \quad V(t_+) = \frac{A^2}{t_+^2} - \frac{2A}{t_+} , \quad (1.3.14)$$

and the second derivative (1D Laplacian) becomes the Laplace-Beltrami operator with metric $g^{11} = \alpha^2t_{\mp}^2$,

$$\frac{d^2}{dz^2} = \alpha^2t_{\mp}^2 \frac{d^2}{dt^2} + \alpha^2t_{\mp} \frac{d}{dt} , \quad t = t_{\mp} .$$

Hence, the Laplace-Beltrami operator does not depend on the sign of $\alpha$ or, saying differently, it is invariant under the transformation $t \to 1/t$.

The Hamiltonian of the Morse oscillator in $t_-$-variable

$$\mathcal{H}(t_-) = -\alpha^2t_-^2 \frac{d^2}{dt_-^2} - \alpha^2t_- \frac{d}{dt_-} + A^2t_-^2 - 2At_- = \mathcal{H}(1/t_+) , \quad (1.3.15)$$

is the algebraic operator. It has the form of the Heun operator.

The Morse oscillator is widely used in molecular physics to model the interaction of the atoms in diatomic molecules.
Case I.

Consider the following bilinear combination of the \( sl(2,\mathbb{R}) \) generators (1.2.3) with index \( n \) (see [72])

\[
T_2 = -\alpha^2 J_n^+ J_n^- + 2\alpha a J_n^+ - \alpha [\alpha(n + 1) - 2b] J_n^0 - 2\alpha c J_n^- - \frac{\alpha n}{2} [\alpha(n + 1) - 2b],
\]

where \( \alpha \neq 0, a, b, c \) are real parameters. If \( a = 0 \) the term of positive grading in (1.3.16-1) disappears and \( T_2 \) becomes exactly-solvable. Since \( T_2 \) is proportional to \( \alpha \) it is convenient to divide it by \( \alpha \) and measure also the spectral parameter \( \varepsilon \) in units of \( \alpha \):

\[
\varepsilon \rightarrow \alpha \varepsilon.
\]

After the substitution of the generators (1.2.3) into (1.3.16-1) and division on \( \alpha \) we get an algebraic operator

\[
T_2(x, d_x) = -\alpha x^2 d_x^2 + [2ax^2 + (2b - \alpha) x - 2c] d_x - 2\alpha x .
\]

In the basis of monomials \( \{1, x, x^2, \ldots, x^k, \ldots x^n, \ldots\} \) this operator has the form of tri-diagonal, Jacobi matrix,

\[
t_{k,k-1} = -2c k , \quad t_{k,k} = (2b - \alpha k) k , \quad t_{k,k+1} = 2a(k - n) .
\]

If \( k = n \), the matrix element \( t_{n,n+1} = 0 \) and the Jacobi matrix \( T_2 \) becomes block-triangular. In this case the characteristic polynomial is factorized, \( \det(T_2 - \varepsilon) = P_n(\varepsilon)P_{\infty}(\varepsilon) \).

In general, this operator has \((n + 1)\) polynomial eigenfunctions. However, at \( a = 0 \) the operator \( T_2(x, d_x) \) becomes exactly-solvable: it has infinitely-many polynomial eigenfunctions. In this case the upper subdiagonal of \( T_2 \) vanishes, this matrix becomes lower-triangular.

As an illustration let us consider \( n = 1 \) in (1.3.16-2). In this case there are two polynomial eigenfunctions,

\[
\varphi_{\pm} = 4ax + (2b - \alpha) \mp \sqrt{(2b - \alpha)^2 + 16ac} ,
\]

with eigenvalues

\[
\varepsilon_{\pm} = \frac{(2b - \alpha) \pm \sqrt{(2b - \alpha)^2 + 16ac}}{2} ,
\]

(cf. (1.2.33)). Both eigenfunctions at fixed \( x \) as well as both eigenvalues form a two-sheeted Riemann surface in any of the parameters \( \alpha, a, b, c \) if the others
are kept fixed with square-root branch points at 
\((2b - \alpha) = \pm 4i(ac)^{1/2}\). In order for the spectra of \(1.3.16-1\) to be real
\[(2b - \alpha)^2 + 16ac > 0 ,\]
(cf. \(1.2.34\)), where the equality is excluded as a consequence of Hermiticity of \(1.3.16-2\) which prohibits the degeneracy of energy. Since the domain for \(1.3.16-2\) is \(x \in [0, +\infty)\) and also \(ac > 0\) (see below), \(\varphi_\pm\) describes the ground state (it has no zeroes (nodes) at \(x > 0\)) with energy \(\varepsilon_\pm\).

In the universal enveloping algebra \(\mathfrak{sl}(2, \mathbb{R})\) there exists an element which is equivalent to \(1.3.16-1\). It can be obtained from \(1.3.16-1\) by conjugation \(1.2.5\)
\[
T_2 = -\alpha^2 J_n^+ J_n^- + 2\alpha c J_n^+ + \alpha[a(n - 1) - 2b]J_n^0 - 2\alpha a J_n^- \quad (1.3.18-1)
\]
\[-\frac{\alpha n}{2}[\alpha(n + 1) - 2b] .
\]
In fact, the operator \(1.3.18-1\) coincides with \(1.3.16-1\) with renamed parameters (see below, \(1.3.18-3\)). So, this conjugation does not bring anything new when we are making a mapping \(1.3.16-1\) to itself. The operator \(1.3.18-1\) has \((n + 1)\) polynomial eigenfunctions with the same eigenvalues as \(1.3.16-1\). The polynomial eigenfunctions of the operators \(1.3.16-1\) and \(1.3.18-1\) are related with each other through the invariance condition \(1.1.6\). However, if \(c = 0\) a single term of positive grading in \(1.3.18-1\) disappears and the operator becomes exactly-solvable. Naturally, this exactly-solvable form does not occur for \(1.3.16-1\).

The algebraic operator, which corresponds to \(1.3.18-1\), after division by \(\alpha\) takes the form
\[
T_2(x, d_x) = -\alpha x^2 d_x^2 + [2cx^2 + (\alpha(2n-1) - 2b)x - 2a]d_x - 2cnx - n(\alpha n - 2b) ,
\]
(cf. \(1.3.16-2\)). This operator can be obtained from \(1.3.16-2\) through the following change of parameters
\[
a \leftrightarrow c ,
\]
\[
\alpha \rightarrow \alpha ,
\]
\[
b \rightarrow -b + \alpha n ,
\]
if we neglect constant terms in the operator, these terms can always be adjusted through a change of the reference point for eigenvalues.

The spectral problem for the operator \(1.3.16-1\) is reduced to the Schrödinger equation \(1.3.1\) by the procedure \(1.3.2)-(1.3.4)\). As a result the spectral problem \(1.3.9\) with the potential
\[
V_I(z) = a^2 e^{-2\alpha z} + a[2b - \alpha(2n + 1)] e^{-\alpha z} - c(2b + \alpha) e^{\alpha z} + c^2 e^{2\alpha z} \quad (1.3.19)
\]
Fig. 1.2. QES Morse potential I (1.3.19) at \(a = b = c = \alpha = 1\) and \(n = 1\): \(V_I(z) = e^{-2z} - e^{-z} - 3e^z + e^{2z}\). It has minimum at \(z \sim 1/2\), \(V_{\text{min}} \sim -2.2\). Algebraic discrete spectra consists of two bound states (ground state and the first excited state) with energies \(E_\pm = \frac{3}{2} \pm \frac{\sqrt{17}}{2}\), and eigenfunctions \(\Psi_\pm = (4e^{-z} + 1 \mp \frac{\sqrt{17}}{2})e^{-z}e^{z} + e^z\).

\[+b^2 - 2ac\]

occurs, where

\[x = e^{-\alpha z}\]

see Fig. 1.2. The constant part of the potential, \((b^2 - 2ac)\), can be absorbed into the definition of energy \(E = e - (b^2 - 2ac)\). Of course, a similar potential will occur if the operator (1.3.16-2) is reduced to the Schrödinger operator (1.3.1) by the procedure (1.3.2)-(1.3.4). This potential will correspond to a change of parameters in (1.3.19) by (1.3.18-3). The derivative of the gauge factor \(y = -A'\) is

\[y(z) = -ae^{-\alpha z} - b + ce^{\alpha z}\]

(1.3.20)

and has a meaning of the logarithmic derivative of the ground state eigenfunction at \(n = 0\) with negative sign, see (1.3.3). The Schrödinger equation is defined on the real line \(z \in (-\infty, +\infty)\), while the spectral problem for \(T_2\) is restricted to \(x \in [0, +\infty)\). For non-vanishing values \(a, c\) the potential (1.3.19) grows at \(|z| \to \infty\) and always has infinite discrete spectra. For \(n = 1\) it implies that one of the functions \(\varphi_\pm\) in (1.3.17) has either no nodes or one node, while another one has either one node or no nodes, respectively, in agreement with Sturm (oscillation) theorem \((ac \geq 0\), see below). Thus, the Schrödinger equation with the potential (1.3.19) is a quasi-exactly-solvable Schrödinger equation of the first type. A finite number of eigenfunctions and eigen-energies can be found through a linear algebra procedure.
The potential \( V(z) \) (1.3.19) is periodic with the imaginary period \( i \frac{2\pi}{\alpha} \) and the Hamiltonian is translationally-invariant, \( z \to z + i \frac{2\pi}{\alpha} \). The variable \( x \) has the meaning of the invariant \( t = x(= e^{-\alpha z}) \). Taking \( t \) as the new variable, we get the quasi-exactly-solvable potential (1.3.19) in rational form

\[
V(t) = a^2 t^2 + a[2b - \alpha(2n + 1)]t - \frac{c(2b + \alpha)}{t} + \frac{c^2}{t^2} \tag{1.3.21}
\]

The quasi-exactly-solvable Hamiltonian in \( t \)-variable is

\[
\mathcal{H} = -\alpha^2 t^2 \frac{d^2}{dt^2} + \alpha^2 t \frac{d}{dt} + a^2 t^2 + a[2b - \alpha(2n + 1)]t - \frac{c(2b + \alpha)}{t} + \frac{c^2}{t^2} \tag{1.3.22}
\]

The \((n + 1)\) polynomial solutions (up to a multiplicative factor) of the Schrödinger equation (1.3.1) define the *algebraic* part of the spectra; they have the form

\[
\Psi^{(k)}_n(z) = p^{(k)}_n(e^{-\alpha z}) \exp \left( -\frac{a}{\alpha} e^{-\alpha z} + b z - \frac{c}{\alpha} e^{\alpha z} \right), \ k = 0, 1, \ldots n \tag{1.3.23}
\]

where \( p^{(k)}_n(x) \) is the polynomial eigenfunction of the operator \( T_2 \). Multiplicative factor in (1.3.23)

\[
\Psi_0(z) = \exp \left( -\frac{a}{\alpha} e^{-\alpha z} + b z - \frac{c}{\alpha} e^{\alpha z} \right), \tag{1.3.24}
\]

if normalizable, defines the ground state. So far, these functions (1.3.23) are formal solutions of the Schrödinger equation. In order to answer the question whether these solutions have the meaning of eigenfunctions, we should check their normalizability. This will be performed later.

In the invariant variable \( t \), the eigenfunction (1.3.23) is

\[
\Psi(t) = p_n(t) t^{-\frac{k}{2}} \exp \left( -\frac{a}{\alpha} t - \frac{c}{\alpha} t \right). \tag{1.3.25}
\]

In order to find the algebraic eigenfunctions (1.3.23) we have to diagonalize the Jacobi matrix \( T_2 \) of size \((n + 1)\) by \((n + 1)\) with the matrix elements

\[
t_{k,k} = k(2b - \alpha k), \ t_{k,k+1} = -2a(n - k), \ t_{k+1,k} = -2c(k + 1),
\]

(cf. (1.3.16-3)), where \( k = 0, 1, 2, \ldots n \). Its eigenvalues are the energies of the corresponding algebraic eigenfunctions, \( E = \varepsilon / \alpha \) (hence, they are measured in units of \( \alpha \)) while its eigenfunctions define the coefficients of polynomials \( p_n \). Reality of the eigenvalues of \( J \) is guaranteed if \( t_{k+1,k} t_{k,k+1} > 0 \) (see e.g. [44], [44], [44].
The characteristic equations for energies \( E \) (measured in the units of \( \alpha \)) at \( n = 1, 2 \) have the form

\[
E^2 + (\alpha - 2b)E - 4ac = 0,
\]

\[
E^3 + (5\alpha - 6b)E^2 + 4[(\alpha - b)(\alpha - 2b) - 4ac]E - 32ac(\alpha - b) = 0,
\]

respectively. For a given \( n \) the characteristic equation has order \((n + 1)\), its eigenvalues (as well as eigenfunctions at fixed \( x \) or \( z \)) form a \((n + 1)\)-sheeted Riemann surface in any parameter when all the other parameters are kept fixed. One can easily see that when \( n \) tends to infinity, the parameter \( a \) tends to zero. One subdiagonal in the matrix \( t \) vanishes, the eigenvalues coincide with diagonal matrix elements and the Riemann surface splits into separate (disconnected) sheets.

Now we return to the question about normalizability of (1.3.23). If \( \alpha > 0 \), the eigenfunctions (1.3.23) are, in general, normalizable for any \( a, c > 0 \) and real \( b \), the potential has either one or two minima depending on the values of parameters.

If \( c = 0 \), then for \( a > 0, b < 0 \) the normalizability of (1.3.23) is guaranteed for any \( n \); the potential has a single minimum.

On the other hand, if \( a = 0 \) (which corresponds to the exactly-solvable Morse problem), the parameters \( c > 0 \) and \( b > 0 \). If the integer part \( \lfloor b/\alpha \rfloor = n \), the first \((n + 1)\) eigenstates of the potential (1.3.19) are characterized by normalizable eigenfunctions (1.3.23). Their energies are

\[
E_k = \alpha k(2b - \alpha k), \quad k = 0, 1, \ldots, n.
\]

It is worth noting that the operator \( T_2 \) (1.3.16-1) has infinitely many polynomial eigenfunctions with eigenvalues which are given by (1.3.26) with integer \( k \) running from 0 up to infinity. However, for \( k > n \) the corresponding eigenfunctions (1.3.23) are non-normalizable and, hence, they are irrelevant physically. It is known (see, for example, the textbook by Landau-Lifschitz) that the above eigenstates with \( k \leq n \) exhaust all bound states in the potential (1.3.19). If \( \alpha < 0 \), the eigenfunctions (1.3.23) are, in general, normalizable at \( a, c < 0 \) and real \( b \), though if \( c = 0 \), then \( a < 0, b > 0 \). Now, let us take

\[\text{11} \] It is worth noting that a rather amusing situation occurs at \( 2b = -\alpha < 0, a > 0 \) and \( c \equiv ic \) pure imaginary. The potential (1.3.19) is real and unbounded from below (bottomless), \( V(z) = a^2e^{-2\alpha z} - 2aa(n+1)e^{-\alpha z} - c^2e^{2\alpha z} \). All eigenfunctions (1.3.23) are normalizable. In such a potential, which is sometimes called inverted the time for a particle to travel to \(+\infty\) is finite (see for a discussion the book by Titchmarsh)

\[\text{12} \] The form in which this potential usually appears in the textbooks assumes that the parameter \( \alpha \) is negative.
\[ a = 0 \text{ (which again corresponds to the exactly-solvable problem); for normalizability the parameters } c > 0 \text{ and } b/\alpha > 0. \]

If the integer part \([b/\alpha] = n\), where \(n = 0, 1, 2, \ldots\), the first \((n+1)\) eigenstates of the potential (1.3.19) have normalizable eigenfunctions (1.3.23) and describe bound states with energies (1.3.26).

It is worth to emphasize how the limit to the exactly-solvable problem, \(a \rightarrow 0\), is taken. The procedure for energies is quite obvious but not at all for the eigenfunctions. Let us consider the case \(n = 1\) (and thus \(\varphi_+\)) as an example. At \(a \rightarrow 0\) the function \(\varphi_-\) tends to a constant, which is the lowest eigenfunction, and \(E_- \rightarrow 0\). As for the state \(\varphi_+\) one can see immediately that the energy \(E_+ \rightarrow (2b - \alpha)\) at \(a \rightarrow 0\). In order to get a meaningful eigenfunction \(\varphi_+\) at \(a \rightarrow 0\) we should introduce a normalization factor \(\propto 1/a\) and then take the limit \(a \rightarrow 0\). Eventually,

\[
\frac{1}{a} \varphi_+(x) \rightarrow 4x - \frac{8c}{(2b - \alpha)^2},
\]

or, in \(z\)-representation,

\[
\frac{1}{a} \varphi_+(z) \rightarrow 4e^{-\alpha z} - \frac{8c}{(2b - \alpha)^2}.
\]

Case II.

Let us take a bilinear combination of the \(\mathfrak{sl}(2, \mathbb{R})\) generators with index \(n\) (1.2.3) (see [72])

\[
T_2 = -\alpha J^0_n J^-_n + 2a J^+_n + 2c J^0_n - \left(\frac{n + 2}{2} \alpha + 2b\right) J^-_n + cn , \quad (1.3.27-1)
\]

(cf. (1.3.16-1)), where \(\alpha \neq 0, a, b, c\) are real parameters. As for the corresponding algebraic operator it looks like

\[
T_2(x, d_x) = -\alpha x d_x^2 - (2ax^2 + 2cx - 2b - \alpha)d_x + 2anx . \quad (1.3.27-2)
\]

In the basis of monomials \(\{1, x, x^2, \ldots, x^k, \ldots x^n, \ldots\\}\) this operator has the form of a tri-diagonal, Jacobi matrix,

\[
t_{k,k-1} = [2(b + \alpha) - \alpha k] k , \quad t_{k,k} = 2c k , \quad t_{k,k+1} = -2a(k - n) . \quad (1.3.27-3)
\]

If \(k = n\), the matrix element \(t_{n,n+1} = 0\) and the Jacobi matrix \(T_2\) becomes block-triangular. In this case, the characteristic polynomial can be factorized, 

\[
\det(T_2 - \varepsilon) = P_n(\varepsilon) P_{\infty}(\varepsilon).
\]

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Evidently, this operator has \((n + 1)\) polynomial eigenfunctions, which form the algebraic sector of eigenstates. For example, at \(n = 1\) there exist two polynomial eigenfunctions

\[ \varphi_{\pm} = (2b + \alpha)x - c \mp \sqrt{c^2 + 2a(2b + \alpha)} , \]

with eigenvalues

\[ \varepsilon_{\pm} = -c \pm \sqrt{c^2 + 2a(2b + \alpha)} . \]  

Both eigenfunctions and both eigenvalues form a two-sheeted Riemann surface in any of the parameters \(\alpha, a, b, c\) if the other parameters are fixed. Reality of the spectra of \((1.3.27-1)\) requires, in particular,

\[ c^2 + 2a(2b + \alpha) > 0 , \]

(cf. \((1.2.34)\)). Equality is excluded as a consequence of requirement of Hermiticity of \((1.3.27-2)\) which prohibits degeneracy in energy. Since the domain for \((1.3.27-2)\) is \([0, +\infty)\) and \(a(2b + \alpha) > 0\) (see below), \(\varphi_{-}\) describes the ground state (it has no zeroes at \(x > 0\)).

It is worth mentioning that in the universal enveloping algebra \(U_{\mathfrak{sl}(2, \mathbb{R})}\) there exists an element which is equivalent to \((1.3.27-1)\). It can be obtained from \((1.3.27-1)\) by conjugation \((1.2.5)\)

\[ T_2 = -\alpha J_0^+ J_n^+ + \left( \frac{n+2}{2} \alpha + 2b \right) J_n^+ - 2c J_0^+ - 2a J_n^- + cn , \]  

(see Case III, cf. \((1.3.36)\)). It has \((n + 1)\) polynomial eigenfunctions with the same eigenvalues as \((1.3.27-1)\). However, since \(\alpha \neq 0\) the terms of positive grading in \((1.3.28-1)\) never disappear; this operator never becomes exactly-solvable. It does not preserve the flag of polynomials. The limit \(\alpha \to 0\) is singular, the operators \((1.3.27-1)\) and \((1.3.28-1)\) loose their bilinearity in generators becoming linear in generators. The polynomial eigenfunctions of the operators \((1.3.27-1)\) and \((1.3.28-1)\) are related to each other through the invariance condition \((1.1.6)\). The algebraic operator which corresponds to \((1.3.28-1)\) has the form

\[ T_2(x, d_x) = -\alpha x^3 d_x^2 + [2(\alpha n + b)x^2 - 2cx - 2a - \alpha]d_x \]

\[ - [\alpha \left( \frac{n+3}{2} \right) + 2b]nx + c(n + 1) , \]

(cf. \((1.3.27-2)\)).

Rewriting the operator \((1.3.27-1)\) to the form of the Schrödinger operator, we arrive at a generalized spectral problem \((1.3.9)\) with the potential

\[ \tilde{V}(z)_{11} = a^2 e^{-4\alpha z} + 2ace^{-3\alpha z} + [c^2 - 2a(b + \alpha n + \alpha)]e^{-2\alpha z} - c(2b + \alpha)e^{-\alpha z} + b^2 , \]

(1.3.29)
Two potentials $V_{II}^{(±)}(z) = e^{-4z} + 2e^{-3z} + 6e^{-2z} - (3 + \varepsilon_{±})e^{-z}$, where the ground state energy for $\varepsilon_{+}$ (brown line) is equal to the 1st excited state energy for $\varepsilon_{-}$ (blue line) and is equal to $E = -1$; $\varepsilon_{±} = \pm \sqrt{7}$. Its eigenfunctions are

$$\Psi_{±} = (3e^{-z} - 1 \mp \sqrt{7})e^{-\frac{1}{2}e^{-2z} - e^{-z} - z}.$$ 

where

$$x = e^{-\alpha z},$$

and the weight factor is given by

$$\varrho = \frac{1}{\alpha} e^{-\alpha z}.$$ 

see Fig. 1.3. The constant part of the potential, $b^2$, can be absorbed into the definition of energy $E = \varepsilon - b^2$.

This potential (1.3.29) grows in one direction and tends to a constant $b^2$ in another direction. Since this constant can always be chosen as a reference point for the energy $E_{ref} = -b^2$, the new potential will vanish in that direction. The derivative of the gauge factor $y = -A'$ (logarithmic derivative of the ground state eigenfunction at $n = 0$ with sign minus, see (1.3.4)) is

$$y(z) = -ae^{-2\alpha z} - ce^{-\alpha z} + b.$$ 

(1.3.30)

Similar to Case I, the spectral problem (1.3.9) is defined again on the real line $z \in (-\infty, +\infty)$, while the spectral problem for $T_2$ is actually restricted to $x \in [0, +\infty)$. Thus, the quasi-exactly-solvable Schrödinger equation with potential (1.3.29) belongs to the second type. A finite number of eigenfunctions, each of them within its own potential, at a fixed energy, can be found through an algebraic procedure.
The potential $\tilde{V}(z)$ (1.3.29) is periodic with the imaginary period $i \frac{2\pi}{\alpha}$, hence, the Hamiltonian is translationally-invariant, $z \rightarrow z + i \frac{2\pi}{\alpha}$. The variable $x$ has a meaning of the invariant with respect to translations, $t = x(e^{-\alpha z})$. Taking $t$ as a new variable, we get the quasi-exactly-solvable potential (1.3.29) in polynomial form

$$\tilde{V}(t) = a^2 t^4 + 2a c t^3 + [c^2 - 2a(b + \alpha an + \alpha)] t^2 - c(2b + \alpha) t .$$

(1.3.31)

The quasi-exactly-solvable Hamiltonian in $t$-variable is

$$\mathcal{H} = -\alpha^2 t^2 \frac{d^2}{dt^2} + \alpha^2 t \frac{d}{dt} + a^2 t^4 + 2a c t^3 + [c^2 - 2a(b + \alpha an + \alpha)] t^2 - c(2b + \alpha) t .$$

(1.3.32)

The algebraic eigenfunctions in the potential (1.3.29) become

$$\Psi(z) = p_n(e^{-\alpha z}) \exp\left(-\frac{a}{2\alpha}e^{-2\alpha z} - \frac{c}{\alpha}e^{-\alpha z} - bz\right) .$$

(1.3.33)

and written in the invariant variable $t$, the eigenfunction (1.3.33) is

$$\Psi(z) = p_n(t) t^{\frac{b}{\alpha}} \exp\left(-\frac{a}{2\alpha} t^2 - \frac{c}{\alpha} t\right) .$$

(1.3.34)

For $\alpha > 0$ the eigenfunctions (1.3.33) are normalizable, if $a, b > 0$ and $c$ is arbitrary. If $a = 0$, which corresponds to an exactly-solvable situation (the Morse oscillator), the parameter $c$ should be positive, $c > 0$. For this case, the eigenvalues are

$$\varepsilon_k = -2ck , \ k = 0, 1, \ldots ,$$

(1.3.35)

and, thus, the Morse oscillator, treated in the Sturm representation, possesses an infinite discrete spectra. For $\alpha < 0$, normalizability of (1.3.33) occurs if both $a, b < 0$ and $c$ is arbitrary. If $a = 0$, the exactly-solvable situation appears again with the condition $c < 0$.

In order to find the algebraic eigenfunctions (1.3.33) we have to diagonalize the Jacobi matrix $T_2$ of size $(n + 1)$ by $(n + 1)$ with the matrix elements

$$t_{k,k} = -2ck , \ t_{k,k+1} = (k+1)[2b - \alpha(k-1)] , \ t_{k+1,k} = 2a(n-k) ,$$

where $k = 0, 1, 2, \ldots n$. Its eigenvalues do not have a meaning of the energies of the algebraic eigenfunctions. Instead they have a meaning of the coefficient in front of the term $e^{-\alpha z}$ in the potential. Its eigenfunctions define the coefficients of polynomials $p_n$. Reality of the eigenvalues of $T_2$ is guaranteed if $t_{k+1,k} t_{k,k+1} > 0$ (see e.g. [14], p. 28). The characteristic equations for $n = 1, 2$ have the form

$$\varepsilon^2 + 2c\varepsilon - 2a(2b + \alpha) = 0 ,$$

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respectively. For arbitrary $n$ the characteristic equation has the order $(n+1)$, its eigenvalues (as well as eigenfunctions) form a $(n+1)$-sheeted Riemann surface in any parameter when all others are kept fixed. One can easily see that when $n$ tends to infinity, the parameter $a$ tends to zero. One subdiagonal in the matrix $T_2$ vanishes, the eigenvalues coincide with diagonal matrix elements and the Riemann surface splits into separate (disconnected) sheets.

The spectral problem (1.3.27-1) has a quite outstanding property: once the parameter $c$ vanishes, $c = 0$. In this case, in the matrix $T_2$ all the diagonal matrix elements $J_{kk}$ vanish. Thus, the $(n+1)$ eigenvalues $\varepsilon$ from the algebraic sector (where eigenfunctions are polynomials) are distributed in such a way that they have a center of symmetry (see Lemma 1.2.4). The energy reflection symmetry occurs, see [61]. If the center of symmetry is chosen as the reference point for energy such that $\varepsilon_k = -\varepsilon_{n+1-k}$, it becomes evident that the characteristic equation of the algebraic sector actually depends on $\varepsilon^2$, when a possible zero eigenvalue is not taken into account. In particular,

$$n = 1 \quad , \quad \varepsilon^2 = 2a(2b + \alpha) ,$$

$$n = 2 \quad , \quad \varepsilon^2 = 4a(4b + \alpha) , \quad \varepsilon_0 = 0 ,$$

$$n = 3 \quad , \quad (\varepsilon^2)_\pm = 20ab \pm 2a\sqrt{64b^2 + 9\alpha^2} ,$$

$$n = 4 \quad , \quad (\varepsilon^2)_\pm = 10a(4b - \alpha) \pm 6a\sqrt{16b^2 - 8b\alpha + 9\alpha^2} , \quad \varepsilon_0 = 0 .$$
Case III.

Let us take a bilinear combination of the \(\mathfrak{sl}(2,\mathbb{R})\) generators with index \(n\) (see [72])

\[ T_2 = -\alpha J_n^+ J_n^0 + (2b - 3\alpha \frac{n}{2}) J_n^+ - 2a J_n^0 - 2c J_n^- - a n . \]  

(1.3.36)

(cf. (1.3.16-1), (1.3.27-1)), where \(\alpha \neq 0, a, b, c\) are real parameters. It can be immediately seen that after renaming the parameters

\[ a \leftrightarrow c , \]

\[ \alpha \rightarrow \alpha , \]

\[ 2b \rightarrow 2b + \alpha (2n + 1) , \]  

(1.3.37)

the operator (1.3.36) coincides with the operator (1.3.28-1). Therefore, if both parameters \(a, c \neq 0\), the quasi-exactly-solvable problems of Case II and Case III are related. As an algebraic operator the operator (1.3.36) looks like,

\[ T_2(x, d_x) = -\alpha x^3 d_x^2 + [(2b - \alpha)x^2 - 2ax - 2c]d_x + (\alpha n - 2b)nx , \]  

(cf.(1.3.28-2)). This operator has \((n + 1)\) polynomial eigenfunctions which can be found by linear algebra means. They form the algebraic sector of eigenstates.

In the \(L^2\)-space, the spectral problem for (1.3.36) leads to a generalized spectral problem (1.3.9) with the potential

\[ \tilde{V}(z)_{III} = c^2 e^{4\alpha z} + 2ac e^{3\alpha z} + [a^2 - 2c(b+\alpha)] e^{2\alpha z} - a(2b+\alpha) e^{\alpha z} + b^2 + \alpha n(\alpha n - 2b) , \]  

(1.3.38)

where

\[ x = e^{-\alpha z} , \]

and the weight factor

\[ \varrho = \frac{1}{\alpha} e^{\alpha z} . \]

This potential grows in one direction and tends to a constant in another direction. Hence, a new potential will vanish in that direction. The derivative of the gauge factor \(y = -A'\) (minus logarithmic derivative of the ground state eigenfunction at \(n = 0\), see (1.3.4)) is

\[ y(z) = ce^{2\alpha z} + ae^{\alpha z} - b . \]  

(1.3.39)

Similar to Cases I, II the spectral problem (1.3.9) with the potential (1.3.38) is defined on the real line \(z \in (-\infty, +\infty)\), while the spectral problem for \(T_2\) is actually restricted to the half-line \(x \in [0, +\infty)\). Thus, this is the quasi-exactly-solvable Schrödinger equation with potential (1.3.38) of the second type. A
finite number of eigenfunctions, each of them appears in its own potential with the same energy, can be found through an algebraic procedure.

The algebraic eigenfunctions become

\[ \Psi(z) = p_n(e^{-\alpha z}) \exp \left( -\frac{c}{2\alpha} e^{2\alpha z} - \frac{a}{\alpha} e^{\alpha z} + bz \right) \]  

(1.3.40)

For \( \alpha > 0 \), the eigenfunctions (1.3.40) are normalizable if \( b > 0, c \geq 0 \) and \( a \) is arbitrary.

In order to find the algebraic eigenfunctions (1.3.40) we have to diagonalize the Jacobi matrix \( T_2 \) of size \((n + 1) \times (n + 1)\) with the matrix elements

\[ t_{k,k} = -2ak \]  \[ t_{k,k+1} = -2c(k + 1) \]  \[ t_{k+1,k} = (k - n)[2b - \alpha(n + k)] \]

where \( k = 0, 1, 2, \ldots, n \). Its eigenvalues do not have a meaning of the energies of the algebraic eigenfunctions. Instead they have a meaning of the coefficient in front of the term \( e^{\alpha z} \) in the potential. Its eigenfunctions define the coefficients of polynomials \( p_n \). Reality of the eigenvalues of \( T_2 \) is guaranteed if \( t_{k+1,k} t_{k,k+1} > 0 \) (see e.g. [44], p. 28). The characteristic equations for \( n = 1, 2 \) have the form

\[ \varepsilon^2 + 2a\varepsilon - 2c(2b - \alpha) = 0 \],
\[ \varepsilon^3 + 6a\varepsilon^2 + 4[c(5\alpha - 4b) + 2a^2]\varepsilon + 32ac(\alpha + b) = 0 \],

respectively. For arbitrary \( n \), the characteristic equation has order \((n + 1)\), its eigenvalues (as well as eigenfunctions) form \((n + 1)\)-sheeted Riemann surface in any parameter when all others are kept fixed. One can easily see that when \( n \) tends to infinity, the parameter \( a \) tends to zero. One subdiagonal in the matrix \( J \) vanishes, the eigenvalues coincide with diagonal matrix elements and the Riemann surface splits into separate (disconnected) sheets.

The spectral problem (1.3.36) has a quite outstanding property: if the parameter \( a \) takes zero value, \( a = 0 \), in the matrix \( T_2 \) all the diagonal matrix elements \( t_{k,k} \) vanish. Thus, the \((n + 1)\) eigenvalues \( \varepsilon \) from the algebraic sector (where eigenfunctions are polynomials \( p_n \)) are distributed in such a way that they have a center of symmetry (see Lemma 1.2.4). The energy reflection symmetry occurs [61]. If the center of symmetry is chosen as the reference point for energy such that \( \varepsilon_k = -\varepsilon_{n+1-k} \), where \( k = 0, 1, \ldots, \left[ \frac{n+1}{2} \right] \) the characteristic equation of the algebraic sector takes the form \( \varepsilon^p P_{\frac{n+1}{2}}(\varepsilon^2) = 0 \), where \( p = \frac{1+(-1)^n}{2} \). If \( n \) is even, a zero eigenvalue occurs. For the particular cases,

\[ n = 1 \quad , \quad \varepsilon^2 - 2c(2b - \alpha) = 0 \],

\[ n = 2 \quad , \quad \varepsilon^2 - 4[c(4b - 5\alpha) - 2a^2] = 0 \], \( \varepsilon_0 = 0 \),
\[ n = 3 \quad \text{,} \quad (\varepsilon^2)_\pm = 20c(b - 2\alpha) \pm 2c\sqrt{64b^2 - 256\alpha b + 265\alpha^2} , \]

\[ n = 4 \quad \text{,} \quad (\varepsilon^2)_\pm = 10c(4b - 11\alpha) \pm 6c\sqrt{16b^2 - 88b\alpha + 129\alpha^2} \quad \varepsilon_0 = 0 . \]
Fig. 1.4. Pöschl-Teller potential (1.3.41) at $A = \alpha = 1$: $V(z) = -\frac{1}{\cosh^2 \alpha z}$, it has minimum at $z = 0$, $V_{\text{min}} = -1$. Discrete spectra consists of a single bound state (the ground state) with energy $E_0 = -\frac{1}{4}$, its eigenfunction $\Psi_0 = e^{-\frac{z}{2}} e^{-z}$.

Next two quasi-exactly-solvable potentials are associated to the (hyperbolic) Pöschl-Teller or one-soliton potential.

3.3 Pöschl-Teller-type potentials

The Pöschl-Teller potential or, in other words, the one-soliton potential describes a well-known exactly-solvable quantum-mechanical problem (see e.g. Landau and Lifschitz [37])

$$V(z) = -\frac{A^2}{\cosh^2 \alpha z}, \quad \alpha \neq 0,$$

(1.3.41)

see Fig. 1.4

This potential is defined on the real line $z \in \mathbb{R}$. The potential $V(z)$ is periodic with the imaginary period $i \frac{\pi}{\alpha}$ and has infinitely many second order poles distributed uniformly along the imaginary axis. The Hamiltonian is translationally-invariant, $T_c: z \rightarrow z + i \frac{\pi}{\alpha}$. Taking $T_c$-invariant

$$\tau = \cosh^2 \alpha z,$$

as a new variable, we get the Pöschl-Teller potential in a very simple, rational form,

$$V(\tau) = -\frac{A^2}{\tau},$$

(1.3.42)
and the second derivative (1D Laplacian) becomes the 1D Laplace-Beltrami operator with metric \( g^{11} = 4\alpha^2\tau(\tau - 1) \),

\[
\frac{d^2}{dz^2} = 4\alpha^2 \left( \tau(\tau - 1) \frac{d^2}{d\tau^2} + (\tau - \frac{1}{2}) \frac{d}{d\tau} \right),
\]

Hence, the Laplace-Beltrami operator does not depend on the sign of \( \alpha \), it changes the overall sign when \( \alpha \to i\alpha \).

Depending on the value of \( A \) the potential (1.3.41) has the finite number of bound states which can be found by linear algebra means. This potential has the unique property of reflectionless scattering. It is probably the simplest solution of the so called Korteweg-de Vries equation playing an important role in the inverse problem method (for detailed discussion see e.g. the book by V.E. Zakharov et al [48]).

The potential (1.3.41) can be generalized to

\[
V(z) = -\frac{A^2}{\cosh^2 \alpha z} + \frac{B}{\sinh^2 \alpha z}, \quad \alpha \neq 0,
\]

with \( z \in [0, \infty) \) preserving the property of exact-solvability. It is called a modified Pöschl-Teller potential. The potential (1.3.43) can be written in the form of \( BC_1 \)-hyperbolic potential

\[
V(z) = \frac{a}{\sinh^2 2\alpha z} + \frac{b}{\sinh^2 \alpha z}, \quad \alpha \neq 0,
\]

which occurs in the Hamiltonian Reduction method, see Olshanetsky-Perelomov [50].

Sometimes, trigonometric versions of (1.3.41) and (1.3.43) are considered by changing \( \alpha \to i\alpha \), for instance,

\[
V(z) = -\frac{A^2}{\cos^2 \alpha z} + \frac{B}{\sin^2 \alpha z}, \quad \alpha \neq 0,
\]

at \( z \in [0, \frac{\pi}{\alpha}] \). This potential has infinite discrete spectra which can be found by algebraic means. The potential (1.3.45) can be also written in the form of \( BC_1 \)-trigonometric potential

\[
V(z) = \frac{a}{\sinh^2 2\alpha z} + \frac{b}{\sinh^2 \alpha z}, \quad \alpha \neq 0,
\]

which occurs in the Hamiltonian Reduction method, see Olshanetsky-Perelomov [50].
Case IV.

Let us take the following bilinear combination of the $\mathfrak{sl}(2, \mathbb{R})$ generators with index $n$ \((1.2.3)\) (see \[72\])

$$T_2 = -4\alpha^2 J_n^+ J_0^+ + 4\alpha^2 J_n^+ J_n^+ - 2\alpha [(3n + 2p + 1)\alpha + 2c] J_n^+$$

$$+ 4\alpha [(n + 1)\alpha + c - a] J_n^0 + 4\alpha a J_n^- + 2\alpha n [(n + 1)\alpha + c - a] , \quad (1.3.30-1)$$

where $\alpha, a, c$ are real parameters, $p = 0, 1$. Since $T_2$ is proportional to $\alpha$ it is convenient to divide it by $\alpha$ and to measure the spectral parameter $\varepsilon$ in units of $\alpha$:

$$\varepsilon \to \alpha \varepsilon .$$

Since $\alpha \neq 0$ the terms of positive grading in \[1.3.30-1\] never disappear, this operator never becomes exactly-solvable. It never preserves the flag of polynomials.

After the substitution of the generators \((1.2.3)\) into \((1.3.30-1)\) and division by $\alpha$ we get an algebraic differential operator,

$$T_2(x, dx) = -4\alpha x^2(x - 1) d_x^2 - 2\{(2p + 3)\alpha + 2c\}x^2 - 2(\alpha - a + c)x - 2a\} dx$$

$$+ 2n[(2n + 2p + 1)\alpha + 2c] x . \quad (1.3.30-2)$$

In the basis of monomials \([1, x, x^2, \ldots, x^k, \ldots x^n, \ldots]\) this operator has the form of tri-diagonal, Jacobi matrix,

$$t_{k,k-1} = 4a k , \quad t_{k,k} = 4(-a+c+a k) k , \quad t_{k,k+1} = 2(n-k) [(2n+2p+2k+1)\alpha + 2c] . \quad (1.3.30-3)$$

If $k = n$, the matrix element $t_{n,n+1} = 0$ and the Jacobi matrix $T_2$ becomes block-triangular. The characteristic polynomial is factorized, $\det(T_2 - \varepsilon) = P_n(\varepsilon)P_\infty(\varepsilon)$.

This operator has $(n + 1)$ polynomial eigenfunctions which can be found by linear algebra means. They form the algebraic sector of eigenstates. As illustration of general situation we take $n = 1$ in \[1.3.30-2\]. There are two polynomial eigenfunctions,

$$\varphi_{\pm} = [(2p + 3\alpha) + 2c] x - \alpha + a - c \mp \sqrt{(\alpha + a + c)^2 + 2a(2p + 1)\alpha} ,$$

with eigenvalues

$$\frac{\varepsilon_+}{2} = \alpha - a + c \pm \sqrt{(\alpha + a + c)^2 + 2a(2p + 1)\alpha} . \quad (1.3.31)$$

Both eigenfunctions and both eigenvalues form two-sheeted Riemann surface in any of the parameters $\alpha, a, c$ if others are fixed. Reality of the spectra of
\[(\alpha + a + c)^2 + 2a(2p + 1)\alpha > 0,\]

(cf. \ref{1.2.34}) and equality is also excluded as a consequence of requirement of Hermiticity of \ref{1.3.30-2} which prohibits degeneracy of energy. Since the domain of \ref{1.3.30-2} is \([0, +\infty)\) (see below), \(\varphi_-\) describes the ground state (it does not vanish at \(x > 0\), hence, no nodes).

It is worth mentioning that in the universal enveloping algebra \(\mathfrak{sl}(2, \mathbb{R})\) there exists an element which is almost equivalent to \ref{1.3.30-1}. It can be obtained from \ref{1.3.30-1} by conjugation \ref{1.2.5}

\[
T_2 = 4\alpha^2 J_n^+ J_n^- - 4\alpha^2 J_n^0 J_n^- - 4\alpha (n - 1)\alpha J_n^+ - 4\alpha J_n^0 \tag{1.3.32-1}
\]

(cf. \ref{1.3.30-1} of Case V). It has \((n + 1)\) polynomial eigenfunctions with the same eigenvalues as \ref{1.3.30-1}. The limit \(\alpha \to 0\) is singular, the operators \ref{1.3.30-1} and \ref{1.3.32-1} loose their bilinearity in generators becoming linear in generators. Unlike \ref{1.3.32-1}, the term of positive grading in \ref{1.3.30-1} can be easily vanished at \(a = 0\), this operator becomes exactly-solvable. It preserves the flag of polynomials.

The polynomial eigenfunctions of the operators \ref{1.3.30-1} and \ref{1.3.32-1} are related to each other through the invariance condition \ref{1.1.6}. The algebraic differential operator which corresponds to \ref{1.3.32-1} has the form (after division by \(\alpha\))

\[
T_2(x, d_x) = 4\alpha x(x - 1) d_x^2 - 2\{2ax^2 + 2(n\alpha - c + a)x + [(2n + 2p + 3)\alpha + 2c]\} d_x \tag{1.3.32-2}
\]

(cf. \ref{1.3.30-2}).

The spectral problem for the operator \ref{1.3.30-1} is reduced to the Schrödinger equation \ref{1.3.1} by the above-described procedure \ref{1.3.2}-\ref{1.3.4}, where a new variable

\[
x = \cosh^{-2} \alpha z ,
\]

is introduced. Finally, the spectral problem \ref{1.3.9} with the potential

\[
V(z) = a^2 \cosh^4 \alpha z - a(a + 2\alpha - 2c) \cosh^2 \alpha z \tag{1.3.33}
\]

occurs. This is the quasi-exactly-solvable modification of the Pöschl-Teller potential of the first type, see Fig. 1.5. The derivative of the gauge factor \(y = -A'\) is

\[
y(z) = -c \tanh \alpha z + \frac{a}{2} \cosh 2\alpha z \tag{1.3.34}
\]
Fig. 1.5. Pöschl-Teller potential \( (1.3.33) \) at \( a = c = \alpha = n = 1 \): 
\[
V(z) = \cosh^4 z - \cosh^2 z - \frac{12}{\cosh^2 z};
\]
has minimum at \( z = 0 \), \( V_{\text{min}} = -12 \). Discrete spectra contains infinitely-many bound states, the ground state and the 2nd excited state are found algebraically, their energies \( E_{\pm} = 2 \mp 2\sqrt{11} \), their eigenfunctions \( \Psi_{\pm} = (\tanh \alpha z - 1 \mp \sqrt{11}) \cosh z \exp \left(-\frac{1}{4} \cosh 2z\right) \).

It has a meaning of logarithmic derivative of the ground state eigenfunction at \( n = 0 \) with negative sign, see (1.3.4).

The Schrödinger equation with the potential \( (1.3.33) \) is defined on the real line \( z \in (\infty, +\infty) \), while the spectral problem for \( T_2 \) is restricted to \( x \in [0, +\infty) \). For non-vanishing \( a \) the potential \( (1.3.33) \) grows at \( |z| \to \infty \) and always has infinite discrete spectra. Since the potential is even there are two families of eigenstates, one of positive and another one is of negative parity, \( m = (-1)^p \), \( p = 0, 1 \), respectively. Thus, the Schrödinger equation with the potential \( (1.3.33) \) is the quasi-exactly-solvable Schrödinger equation of the first type. A finite number of eigenfunctions and eigen-energies of definite parity \( m \) can be found through linear algebraic procedure.

The \( (n + 1) \) solutions of the Schrödinger equation \( (1.3.1) \) with the potential \( (1.3.33) \), possibly giving rise to an “algebraized” part of the spectra, have the form

\[
\Psi(z) = (\tanh \alpha z)^p p_n (\tanh^2 \alpha z)(\cosh \alpha z)^{-c/\alpha} \exp \left(-\frac{a}{4\alpha} \cosh 2\alpha z\right), \quad (1.3.35)
\]

where the polynomials \( \tilde{p}_n(x) = p_n(1-x) \) are the eigenfunctions of the operator \( T_2(x, d_x) \) \((1.3.30-2)\). In order to answer the question whether these solutions have a meaning of the eigenfunctions we should check their normalizability. It is evident if \( \frac{a}{\alpha} > 0 \) for any real \( c \) and integer \( n, p \) the function \( (1.3.35) \) is normalizable. If \( a = 0 \) the potential \( (1.3.33) \) becomes exactly-solvable. A
question of normalizability in this case which occurs at $\frac{c}{\alpha} + p > 0$ needs a certain clarification. It will be done later.

In order to find the algebraic eigenfunctions (1.3.35) we have to diagonalize the Jacobi matrix $T_2$ of the size $(n + 1)$ by $(n + 1)$ with the matrix elements

$$t_{k,k} = 4k(k\alpha-a+c), \ t_{k,k+1} = 4a(k+1), \ t_{k+1,k} = 2(n-k)(2n+2p+2k+1)\alpha+2c,$$

where $k = 0, 1, 2, \ldots, n$, see (1.3.30-3). Its eigenvalues have a meaning of the energies of the algebraic eigenfunctions. Reality of the eigenvalues of $T_2$ is guaranteed if $t_{k+1,k}t_{k,k+1} > 0$ is fulfilled (see e.g. [44], p. 28). In particular, the characteristic equation for $n = 1$ has the form

$$\varepsilon^2 - 4(\alpha - a + c)\varepsilon - 8a[(2p + 3)\alpha + 2c] = 0.$$

For arbitrary $n$ the characteristic equation has the order $(n + 1)$, its eigenvalues (as well as eigenfunctions) form $(n + 1)$-sheeted Riemann surface in any parameter when all others are kept fixed. If parameter $a$ tends to zero one subdiagonal in the matrix $T_2$ vanishes, the eigenvalues coincide with diagonal matrix elements and the Riemann surface splits into separate (disconnected) sheets.

**Case V.**

Let us take the following bilinear combination of the $\mathfrak{sl}(2, \mathbb{R})$ generators with the mark $n$ (1.2.3) (see [72])

$$T_2 = -4\alpha^2 J_n^- J_n^- + 4\alpha^2 J_0^0 J_n^- + 4\alpha b J_n^+ J_n^- - 2\alpha[\alpha(2n + 2p + 3) + 2a + 4b] J_n^0$$

$$+ 2\alpha[\alpha(n + 2) + 2a + 2b] J_n^- - \alpha[\alpha(n + 2p + 3) + 2an - 2bk], \quad (1.3.36-1)$$

where $\alpha, a, b$ are real parameters, $p = 0, 1$. Since $T_2$ is proportional to $\alpha$ it is convenient to divide it by $\alpha$ and to measure also the spectral parameter $\varepsilon$ in units of $\alpha$:

$$\varepsilon \rightarrow \alpha \varepsilon.$$

If $b = 0$ the term of positive grading in (1.3.36-1) disappears, this operator becomes exactly-solvable.

After the substitution of the generators (1.2.3) into (1.3.36-1) and division by $(2\alpha)$ we get an algebraic differential operator,

$$T_2(x, d_x) = -2\alpha x(x - 1)d_x^2 + [2bx^2 - (2a + 4b + 2p\alpha + 3\alpha)x + 2(\alpha + a + b)]d_x$$

$$- 2bx + b(2n + p). \quad (1.3.36-2)$$
In the basis of monomials \( \{1, x, x^2, \ldots, x^k, \ldots, x^n, \ldots \} \) this operator has the form of tri-diagonal, Jacobi matrix,

\[
t_{k,k-1} = 2(a + b + \alpha k) k , \quad t_{k,k} = -2\alpha k^2 - (2a + 4b + \alpha(2p + 1)) k + b(2n + p) ,
\]

\[
t_{k,k+1} = 2b(k - n) . \tag{1.3.36-3}
\]

If \( k = n \), the matrix element \( t_{n,n+1} = 0 \) and the Jacobi matrix \( T_2 \) becomes block-triangular. The characteristic polynomial is factorized, \( \det(T_2 - \varepsilon) = P_n(\varepsilon)P_\infty(\varepsilon) \).

The operator (1.3.36-2) has \((n + 1)\) polynomial eigenfunctions which can be found by linear algebra means. They form the algebraic sector of eigenstates. For instance, at \( n = 0 \) two polynomial eigenfunctions occur for different \( p \),

\[
\varphi_p = \text{const} ,
\]

with eigenvalue

\[
\varepsilon_p = b(p + 2) , \quad p = 0, 1 . \tag{1.3.37}
\]

The spectral problem for the operator (1.3.36-1) is reduced to the Schrödinger equation (1.3.1) by the above-described procedure (1.3.2)-(1.3.4). Finally, the spectral problem (1.3.9) with the potential

\[
V(z) = -b^2 \cosh^{-6} \alpha z + b[2a + 3b + \alpha(4n + 2p + 3)] \cosh^{-4} \alpha z \tag{1.3.38}
\]

\[
-[(a + 3b)(a + b + \alpha) + 2(2n + p)ab] \cosh^{-2} \alpha z + (a + b)^2 ,
\]

occurs, where

\[
x = \cosh^{-2} \alpha z , \quad \rho = \cosh^{-2} \alpha z .
\]

It is quasi-exactly-solvable Schrödinger equation of the second type. The derivative of the gauge factor \( y = -A' \) is

\[
y(z) = -c \tanh \alpha z + \frac{a}{2} \cosh 2\alpha z . \tag{1.3.39}
\]

has a meaning of logarithmic derivative of the ground state eigenfunction at \( n = 0 \) with negative sign, see (1.3.4).

The Schrödinger equation with the potential (1.3.38) is defined on the real line \( z \in (-\infty, +\infty) \), while the spectral problem for \( T_2 \) is restricted to \( z \in [0, +\infty) \). For non-vanishing \( a \) the potential (1.3.38) grows at \( |z| \to \infty \) and always has infinite discrete spectra. Since the potential is even there are two families of eigenstates of positive and negative parity, \( m = (-1)^p, p = 0, 1 \), respectively. Thus, the Schrödinger equation with the potential (1.3.38) is the quasi-exactly-solvable Schrödinger equation of the second type. Thus, a finite number of eigenfunctions and eigen-energies of definite parity \( m \) can be found through linear algebraic procedure.
The \((n + 1)\) solutions of the Schrödinger equation (1.3.1) with the potential (1.3.38), giving rise to an algebraized part of the spectra, have the form

\[
\Psi(z) = (\tanh \alpha z)^p p_n(\tanh^2 \alpha z)(\cosh \alpha z)^{-\frac{(a+b)}{\alpha}} \exp\left(\frac{b}{2\alpha} \tanh^2 2\alpha z\right), \quad (1.3.40)
\]

at \(\alpha > 0, (a + b) > 0\) and \(p = 0, 1\).

In order to find the algebraic eigenfunctions (1.3.40) explicitly we have to diagonalize the Jacobi matrix \(T_2\) of the size \((n + 1)\) by \((n + 1)\), see (1.3.36-3). Its eigenvalues have a meaning of the energies of the algebraic eigenfunctions. Reality of the eigenvalues of \(T_2\) is guaranteed if \(t_{k+1,k} t_{k,k+1} > 0\) is fulfilled (see e.g. [44], p. 28). It can be easily checked that for \(n = 0\) in the potential

\[
V(z) = -b^2 \cosh^{-6} \alpha z + b[2a + 3b + \alpha(2p + 3)] \cosh^{-4} \alpha z \quad (1.3.41)
\]

\[-[(a + 3b)(a + b + \alpha) + 2pab + b(2 + p)] \cosh^{-2} \alpha z ,
\]

the lowest eigenstate of parity \((-)^p\) is known

\[
\Psi_0(z) = (\tanh \alpha z)^p (\cosh \alpha z)^{-\frac{(a+b)}{\alpha}} \exp\left(\frac{b}{2\alpha} \tanh^2 2\alpha z\right), \quad (1.3.42)
\]

with the energy

\[
E_0 = -(a + b)^2 .
\]
3.4 Harmonic oscillator-type potentials

The next two quasi-exactly-solvable potentials are associated with the harmonic oscillator potential. Both of them are a particular kind of anharmonic oscillators.

Case VI.

The sextic polynomial potential was the first example of a potential for which the quasi-exactly-solvable Schrödinger operator has been found. As it usually happens this potential appeared in a number of different articles in various occasions at more or less the same time, for an (incomplete) history see \[90\]. It turns out that is the \textit{unique} quasi-exactly-solvable problem with polynomial potential \[72\]. It provides an enormous wealth of non-trivial properties, it can be used as a paradigm where different theoretical ideas can be tested. A separate Chapter 2 will be dedicated to this potential.

Let us take the following non-linear combination in the generators (1.2.3), see \[72\]

\[
T_2 = -4J_n^0J_n^- + 4aJ_n^+ + 4bJ_n^0 - 2(n + 1 + 2p)J_n^- + 2bn \quad (1.3.43)
\]

or as a differential operator,

\[
T_2(x,d_x) = -4xd_x^2 + 2(2ax^2 + 2bx - 1 - 2p)d_x - 4anx \ , \quad (1.3.44)
\]

where \( x \in \mathbb{R}^+ \) assuming \( a > 0, \forall b, \) or \( a \geq 0, b > 0 \) and \( p = 0, 1 \). In the basis of monomials \( \{1, x, x^2, \ldots, x^k, \ldots x^n, \ldots\} \) this operator has the form of tri-diagonal, Jacobi matrix,

\[
t_{k,k-1} = -2k(2k - 1 + 2p), \ t_{k,k} = 4bk, \ t_{k,k+1} = 4a(k - n) . \quad (1.3.45)
\]

If \( k = n \), the matrix element \( t_{n,n+1} = 0 \) and the Jacobi matrix \( T_2 \) becomes block-triangular. The characteristic polynomial is factorized, \( \det(T_2 - \varepsilon) = P_n(\varepsilon)P_\infty(\varepsilon) \). If \( b = 0 \) the principal diagonal of the Jacobi matrix vanishes, \( t_{k,k} = 0 \), and the energy-reflection symmetry occurs, see \[61\], and Lemma 2.4

\[
P_n(\varepsilon) = \varepsilon^p\tilde{P}_{\lfloor \frac{n}{2} \rfloor}(\varepsilon^2) ,
\]

where \( p = 0 \) or \( 1 \) depending on \( n \) being even or odd, respectively. For \( a = 0 \) this matrix becomes triangular.

Putting \( x = z^2 \) and choosing the gauge phase

\[
A = \frac{ax^2}{4} + \frac{bx}{2} - \frac{p}{2} \ln x ,
\]
Fig. 1.6. Sextic QES potential (1.3.45-2) with \( a = 1, b = 0 \) at \( n = 0, p = 0 \) (a) \( V_a = x^6 - 3x^2 \) with the known ground state (blue line), \( E_0 = 0, \Psi_0 = e^{-\frac{x^4}{4}} \) and at \( n = 2, p = 0 \) (b) \( V_b = x^6 - 7x^2 \) with two known states (red and blue lines), \( E_{\pm} = \pm 2\sqrt{2}, \Psi_{\pm} = (2z^2 \pm \sqrt{2})e^{-\frac{x^4}{4}} \).

we arrive at the spectral problem (1.2.35), following the above-described procedure (1.3.2)-(1.3.4), with the singular potential

\[
V(z) = a^2z^6 + 2abz^4 + [b^2 - (4n + 3 + 2p)a]z^2 - b(1 + 2p) + \frac{p(p-1)}{z^2}, \quad z \in \mathbb{R}^+. \tag{1.3.45-1}
\]

If \( p = 0, 1 \) the singular term disappears and the symmetric polynomial potential occurs, see [90],

\[
V(z) = a^2z^6 + 2abz^4 + [b^2 - (4n + 3 + 2p)a]z^2, \quad z \in \mathbb{R}. \tag{1.3.45-2}
\]

For \( a \geq 0 \) and \( p = 0 \) \((p = 1)\) the first \((n + 1)\) eigenfunctions, even \((\text{odd})\) with respect to reflection \( z \leftrightarrow -z \), can be found algebraically, by linear algebra means. Hence, the Schrödinger operator with the potential (1.3.45-2) is quasi-exactly-solvable. The number of those \textit{algebraized} eigenfunctions is nothing but the dimension of the irreducible representation of the algebra (1.2.3). Therefore, the \((n + 1)\) \textit{algebraized} eigenfunctions of (1.2.35) have the form

\[
\Psi^{(p)}(z) = z^p p_n(z^2) e^{-\frac{a^4}{4}-\frac{bz^2}{2}}, \tag{1.3.46}
\]

where \( p_n(y) \) is a polynomial of the \( n \)th degree. It is worth noting that if the parameter \( a \) goes to 0, the potential (1.3.43) becomes the harmonic oscillator potential and the polynomial \( z^p p_n(z^2) \) reconciles to the Hermite polynomial \( H_{2n+p}(z) \) (see discussion below). For illustration, two double-well potentials at \( b = 0 \) and \( n = 0, 1 \), respectively, are shown on Fig. 1.6.

If the parameter \( n \) in (1.3.45-2) is a real number, the potential becomes a generic sextic symmetric polynomial potential. Its energies are branches of an infinitely-valued analytic function in \( b \) (if \( a \) and \( n \) are fixed) with infinitely-
many square-root branch points on every Riemann sheet. Due to reflection symmetry $z \to -z$, two separate infinitely-sheeted Riemann surfaces occur: $E^{(+)}(b)$ describes even states while $E^{(-)}(b)$ describes odd ones. The branch points form pairs with complex-conjugated locations. If $n$ takes integer value, the $(n + 1)$-sheeted Riemann surface splits away into an infinitely-sheeted one. The $(n + 1)$-sheeted Riemann surface is determined by the characteristic equation of $(n + 1)$ degree of the Jacobi matrix (1.3.45). This matrix depends on parameters $a, b$ linearly, while the coefficients of the characteristic equation depend on them polynomially.

It is worth presenting several particular cases explicitly:

(i) Ground state (the lowest energy state of positive parity)

at $n = 0, p = 0$,

$E_0^{(+)} = b$, $\Psi_0^{(+)} = e^{-\frac{az^4}{4}} \frac{bz^2}{2}$,

in potential

$V(z) = a^2 z^6 + 2abz^4 + (b^2 - 3a)z^2$.

The Riemann sheet $E_0^{(+)}(b)$ of the ground energy splits away from the infinitely-sheeted Riemann surface $E^{(+)}(b)$.

(ii) Ground state (the lowest energy state of negative parity)

at $n = 0, p = 1$,

$E_0^{(-)} = 3b$, $\Psi_0^{(-)} = ze^{-\frac{az^4}{4}} \frac{bz^2}{2}$,

in potential

$V(z) = a^2 z^6 + 2abz^4 + (b^2 - 5a)z^2$.

The Riemann sheet $E_0^{(-)}(b)$ of the ground energy splits away from the infinitely-sheeted Riemann surface $E^{(-)}(b)$.

(iii) Ground state and the second excited state, both of positive parity,

at $n = 1, p = 0$,

$E_1^{(0/2)} = 3b \pm 2(b^2 + 2a)^{\frac{1}{2}}$, $\Psi_1^{(0/2)} = (2az^2 + b \pm (b^2 + 2a)^{\frac{1}{2}}) e^{-\frac{az^4}{4}} \frac{bz^2}{2}$,

in potential

$V(z) = a^2 z^6 + 2abz^4 + (b^2 - 7a)z^2$.

The two-sheeted Riemann surface $E_1^{(0/2)}(b)$ which describes the ground state energy and the second excited state splits away from the infinitely-sheeted Riemann surface $E^{(+)}(b)$. These levels intersect at $b = \pm i\sqrt{2a}$ where the square-root branch points occur, see Fig. [1.7] These are the Landau-Zener singularities: making the analytic continuation around any of them, starting
Fig. 1.7. $b$-Complex plane of the ground state energy at $n = 1, p = 0$ in the QES potential $V(z) = a^2 z^6 + 2abz^4 + (b^2 - 7a)z^2$. The square-root branch points are marked by bullets, branch cuts go along the imaginary axis to $\pm i\infty$ infinity, respectively.

from ground state energy at real $b$ (and fixed $a > 0$), we arrive at the energy of the 2nd excited state at the same $b$.

As an illustration, plots of the energy versus $b$ for the first three states are shown on Fig. 1.8, the energy of the first excited state is found in convergent perturbation theory \[70\],

$$E^1 \approx 3b - 2 \frac{\int_0^{\infty} z^4 e^{-\frac{z^4}{2} - b^2 z^2} dz}{\int_0^{\infty} z^2 e^{-\frac{z^4}{2} - b^2 z^2} dz}.$$  

The behavior of $E_1^{(0/2)}(b)$ demonstrates the effect of avoiding singularities: a minimal distance between these levels occurs at $b = 0$,

$$\Delta E_1 = E_1^{(2)} - E_1^{(0)} = 4(2a)^{\frac{1}{2}}.$$

(iv) The first and the third excited states, both of negative parity,

at $n = 1, p = 1$ (or, in other words, the ground state and the second excited state of negative parity),

$$E_1^{(1/3)} = 5b \pm 2(b^2 + 6a)^{\frac{1}{2}}, \quad \Psi_1^{(1/3)} = z (2az^2 + b \mp (b^2 + 6a)^{\frac{1}{2}}) e^{-\frac{az^4}{2} - b z^2},$$

in potential

$$V(z) = a^2 z^6 + 2abz^4 + (b^2 - 9a)z^2.$$  

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Fig. 1.8. Sextic QES potential (1.3.45-2) at \( a = 1, n = 1, p = 0 \): \( V(z) = z^6 + 2bz^4 + (b^2 - 7)z^2 - b \), it has maximum at \( z = 0 \), \( V_{\text{max}} = 0 \). Discrete spectra contains infinitely-many bound states, the ground state and the 2nd excited state are found algebraically, their energies \( E^{0,2} \equiv E^{(0/2)}_1 = 2b \pm 2\sqrt{b^2 + 2} \), their eigenfunctions \( \Psi_{\pm} = (2z^2 + b \mp (b^2 + 2)^{1/2}) e^{-z^2/4 - bz^2/2} \), the first excited state energy \( E^1 \) vs \( b \) is shown, in particular, \( E^1(b = 0) = -2\sqrt{2}\frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})} \) (see text).

The two-sheeted Riemann surface \( E^{(1/3)}_1(b) \) which describes the first and the third excited states splits away from the infinitely-sheeted Riemann surface \( E^{(-)}(b) \). These levels intersect at \( b = \pm i\sqrt{6a} \) where the square-root branch points occur, see e.g. Fig. 1.7. These are the Landau-Zener singularities: making the analytic continuation around any of this branch point, starting from the 1st excited state energy at real \( b \) (and fixed \( a > 0 \)), we arrive at the energy of the 3rd excited state at the same \( b \).

The behavior of \( E^{(1/3)}_1(b) \) demonstrates the effect of avoiding singularities: a minimal distance between these levels occurs at \( b = 0 \),

\[
\Delta E_1 = E^{(3)}_1 - E^{(1)}_1 = 4(6a)^{1/2}.
\]

Naively, the analytic continuation can be made from positive \( a \) to negative \( a \). The potential (1.3.45-2) at negative \( a \) remains confined and contains infinitely-many bound states, the operator (1.3.43) has \((n + 1)\) polynomial eigenfunctions, the only thing that happens is that, the eigenfunctions (1.3.46) become non-normalizable. A non-trivial fact is that for negative \( a \) a linear combination of (1.3.46) with the second linearly-independent solution (obtained by keeping the Wronskian equal to constant) can never be made normalizable! It implies the absence of analytical continuation from positive \( a \) to negative \( a \).
a. Thus, there exist two well-defined but analytically disconnected spectral problems with infinite discrete spectra: one for positive $a$, another one - for negative $a$ (see [5]).

Comment 3.3. The $d$-dimensional Laplacian

$$\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2},$$

written in spherical coordinates $(r, \Omega)$ has the well-known form

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r} + \frac{\Delta_{S^d}}{r^2},$$

where $\Delta_{S^d}$ is Laplacian on the sphere $S^d$. The eigenfunctions of $\Delta_{S^d}$ are the $d$-dimensional spherical harmonics $Y_{\{l\}}(\Omega)$ with a certain total angular momentum $l$,

$$\Delta_{S^d} Y_{\{l\}}(\Omega) = l(l + d - 2) Y_{\{l\}}(\Omega).$$

Separating out the angular variables leads to a spectral problem for finding the radial function of the operator

$$\Delta_r = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r} + \frac{l(l + d - 2)}{r^2},$$

which is sometimes called the radial Laplacian. Making different gauge rotations we arrive either to the spectral problem (1.2.35) on half-line $r \in \mathbb{R}^+$ with an effective singular potential,

$$r^{(d-1)/2} \Delta_r r^{-d/2} = \frac{\partial^2}{\partial r^2} + \frac{C}{r^2},$$

where $C$ is a constant, or at the spectral problem for the radial operator

$$r^{-l} \Delta_r r^l = \frac{\partial^2}{\partial r^2} + \frac{d + 2l - 1}{r} \frac{\partial}{\partial r},$$

defined also on the half-line, $r \in \mathbb{R}^+$. If the Hamiltonian with spherically symmetric potential $V(r)$ is considered,

$$\mathcal{H} = -\Delta + V(r),$$

the notion of the radial Hamiltonian can be introduced,

$$\mathcal{H}_r = -\frac{\partial^2}{\partial r^2} - \frac{d + 2l - 1}{r} \frac{\partial}{\partial r} + V(r).$$

(1.3.48)

This Hamiltonian is Hermitian with measure $\sim r^{d+2l-1}$. 56
Case VII.

Let us take the following non-linear combination in the generators (1.2.3), see [72],
\[ T_2 = -4J_n^0 J_n^- + 4a J_n^+ + 4b J_n^0 - 2(n + d + 2l - 2c) J_n^+ + 2bn , \] (1.3.49)
(c.f. (1.3.43) at \( 2p = d + 2l - 2c - 1 \)), or as the differential operator,
\[ T_2(x, d_x) = -4x d_x^2 + 2(2a x^2 + 2bx - d - 2l + 2c) d_x - 4anx , \] (1.3.50)
where \( a > 0, b, c, d, l, n \) are parameters. If the mark of representation (1.2.3) \( n \) is a non-negative integer, the operator (1.3.49) has finite-dimensional invariant subspace in polynomials in \( x \) of degree not higher than \( n \).

In the basis of monomials \( \{ 1, x, x^2, \ldots, x^k, \ldots x^n, \ldots \} \) this operator has the form of a tri-diagonal, Jacobi matrix,
\[ t_{k,k-1} = -2k(2k + d + 2l - 2c - 2) , \quad t_{k,k} = 4bk , \quad t_{k,k+1} = 4a(k - n) , \] (1.3.51)
(c.f. (1.3.45)). If \( k = n \), the matrix element \( t_{n,n+1} = 0 \) and the Jacobi matrix \( T_2 \) becomes block-triangular. The characteristic polynomial is factorized, \( \det(T_2 - \varepsilon) = P_n(\varepsilon) P_\infty(\varepsilon) \). If \( b = 0 \) the principal diagonal of the Jacobi matrix vanishes and the energy-reflection symmetry occurs, see [61], and Lemma 2.4
\[ P_n(\varepsilon) = \varepsilon^p \tilde{P}_{[n]}(\varepsilon^2) , \]
where the parameter \( p = 0 \) or \( 1 \) depending on whether the integer \( n \) is even or odd, respectively. For \( a = 0 \) this matrix becomes triangular. Its eigenvalues can be found explicitly, \( \varepsilon_k = 4bk, k = 0, 1, \ldots \).

Choosing the gauge phase
\[ A = \frac{ax^2}{4} + \frac{bx}{2} - \frac{D_c - 1}{4} \ln x , \]
where \( D_c \equiv (d + 2l - 2c) \) and following the above-described procedure (1.3.2)-(1.3.4) at
\[ x = r^2 , \quad \rho = 1 , \]
we arrive at the spectral problem for the radial Hamiltonian (1.3.48) with the potential, see [90],
\[ V(z) = a^2 r^6 + 2abr^4 + [b^2 - (4n + D_c + 2)a]r^2 - \frac{c(c + D_c - 2)}{r^2} , \] (1.3.52)
where
\[ \Psi(r) = P_n(r^2)e^{-\frac{a^2}{4} - \frac{b^2}{2}} , \]
at $a > 0, \forall b$ or $a \geq 0, b > 0$ and $(d + l - c) > 1$ at $z \in [0, \infty)$.

It corresponds to the radial part of a $d$-dimensional Schrödinger equation with angular momentum $l$, see (1.3.48). At $d = 1$ and $l = 0$ the radial operator coincides with the ordinary Schrödinger equation (1.3.1). The potential (1.3.52) becomes a generalization of the potential (1.3.45-2), with an additional singular term proportional to $r^{-2}$, see (1.3.45-1). Note that the operator (1.3.48) with potential (1.3.52) depends on a combination $D \equiv (d + 2l)$. It implies that the spectra for different $d, l$ but the same $D$ coincide (but not the multiplicities). In particular, for $d > 3$ the spectra of $S$-states in the $d$-dimensional case coincide with the spectra of $P$-states in the $(d - 2)$-dimensional case. There is no single feature which makes the physical dimension $d = 3$ special.

It is worth presenting two particular cases explicitly,

(i) Ground state, $n = 0$,

$$E_0^{(l)} = bDc, \quad \Psi_0 = r^{l-c} e^{-\frac{ar^4}{4} - \frac{br^2}{2}},$$

in potential

$$V(r) = a^2r^6 + 2abr^4 + [b^2 - (Dc + 2)a]r^2 - \frac{c(c + Dc - 2)}{r^2}.$$ 

The Riemann sheet $E_0^{(l)}(b)$ of the ground energy for given $l$ splits away from the infinitely-sheeted Riemann surface $E^{(l)}(b)$.

(ii) Ground state (marked by superscript -) and the first excited state (marked by superscript +), $n = 1$,

$$E_1^{(\pm)} = bDc + 2b\pm 2(b^2 + 2aDc)\frac{1}{2}, \quad \Psi_1^{(\pm)} = (2ar^2 + b\mp (b^2 + 2aDc)\frac{1}{2}) r^{l-c} e^{-\frac{ar^4}{4} - \frac{br^2}{2}},$$

in potential

$$V(r) = a^2r^6 + 2abr^4 + (b^2 - (Dc + 6)a)r^2 - \frac{c(c + Dc - 2)}{r^2}.$$ 

The two-sheeted Riemann surface $E_1^{(\pm)}(b)$ which describes the ground state energy and the first excited state splits away from the infinitely-sheeted Riemann surface $E(b)$. These levels intersect at $b = \pm i\sqrt{2aDc}$, where the square-root branch points occur, see e.g. Fig. 1.7.
3.5 Coulomb-type potentials

Next we consider two quasi-exactly-solvable potentials which are associated with the two-body Coulomb problem in \( d \)-dimensional space,

\[ V_c(r) = -\frac{\alpha}{r}, \quad x \in \mathbb{R}^d. \]

where \( r \) is the distance between two charged particles, \( \alpha \) is the parameter. Separating out the center-of-mass motion, we arrive at the Hamiltonian of relative motion of the form,

\[ \mathcal{H}(r) = -\Delta^{(d)} + V_c(r) . \]

Introducing the Euler coordinates \((r, \Omega)\) in \( \mathcal{H}(r) \) and separating out the angular degrees of freedom \( \{\Omega\} \), and then making a gauge rotation with \( r' \) as a gauge factor we arrive at the radial Hamiltonian \( \mathcal{H}_r \) and the eigenvalue problem for the radial motion

\[ \left( -\frac{\partial^2}{\partial r'^2} - \frac{d + 2l - 1}{r'} \frac{\partial}{\partial r'} + V_c(r) \right) \Psi(r) = E \Psi(r) , \quad (1.3.54) \]

see Comment 3.3. If we assume that the energy is fixed and negative,

\[ -E = E' \equiv k^2 \]

, and we look for the spectra of \( \alpha \), the equation for radial motion is changed,

\[ \left( -\frac{\partial^2}{\partial r^2} - \frac{d + 2l - 1}{r} \frac{\partial}{\partial r} + E' + \frac{\alpha}{r} \right) \Psi(r) = \alpha \Psi(r) . \]

Multiplying both sides by \( r \), we arrive at the spectral problem for radial motion

\[ H_r \Psi(r) = \left( -r \frac{\partial^2}{\partial r^2} - (d + 2l - 1) \frac{\partial}{\partial r} + E' r \right) \Psi(r) = \alpha \Psi(r) , \quad \Psi(r) \in L^2(\mathbb{R}^+) . \]

This eigenvalue problem has infinite discrete spectra. Such an approach to the Coulomb problem where we quantize the \( \alpha \)-parameter keeping the energy fixed is called the Sturm approach. The corresponding representation of the spectral problem is called the Sturm representation.

The operator \( H_r \) can be gauge-rotated with the gauge factor \( e^{-kr} \) to obtain

\[ h_r \equiv e^{kr} H_r e^{-kr} = -r \frac{\partial^2}{\partial r^2} + (2kr - d - 2l + 1) \frac{\partial}{\partial r} + k(d + 2l - 1) . \]

The resulting operator has infinitely-many finite-dimensional invariant subspaces in polynomials. Those subspaces form an infinite flag. In action on
monomials this operator is triangular, its spectra is linear in radial quantum number \( n_r \),
\[
\alpha_n = 2kn_r + k(d + 2l - 1) \equiv 2kn_r = 0, 1, \ldots
\]  
(1.3.55)
and its eigenfunctions are the Laguerre polynomials. At \( d = 3 \) the parameter \( n \) coincides with the principal quantum number, \( n = n_r + l + 1 \), see e.g. [37]. The operator \( h_r \) can be rewritten in terms of the generators \( J^0_\pm \equiv J^0_0 \) of the Borel subalgebra, \( b_2 \subset \mathfrak{sl}(2, \mathbb{R}) \), see (1.2.3),
\[
h_r = -J^0_0 J^- + 2kJ^0_0 - (d + 2l - 1)J^- + k(d + 2l - 1) .
\]

To summarize, we state that the existence of the Sturm representation implies that the two-body Coulomb problem can be considered as a (quasi)-exactly-solvable problem of the second type where the parameter in front of the Coulomb term is quantized while the energy plays the role of the external parameter.

Note that using (1.3.55) and assuming the parameter \( \alpha \) is kept fixed in (1.3.54), one can obtain the energy quantization
\[
k = \frac{\alpha}{2n} , \quad E = -\frac{\alpha^2}{4n^2} .
\]  
(1.3.56)
The Sturm representation can be constructed for a general spherical symmetrical potential \( V(r) \),
\[
\left( -\frac{\partial^2}{\partial r^2} - \frac{d + 2l - 1}{r} \frac{\partial}{\partial r} + V \right) \Psi (r) = E \Psi (r) .
\]  
(1.3.57)
which contains the singular, Coulomb-type potential term \( V_c(r) = -\alpha/r \). The analogue of the radial equation (1.3.54)
\[
\left( -r \frac{\partial^2}{\partial r^2} -(d+2l-1) \frac{\partial}{\partial r} + E'r + r(V-V_c) \right) \Psi (r) = \alpha \Psi (r) , \quad \Psi (r) \in L^2(\mathbb{R}^+) ,
\]  
(1.3.58)
defines a QES problem of the second type, c.f. (1.3.11). Also the Sturm representation can be constructed for the case when the potential \( V(r) \) contains the singular term \( V_s(r) = -\lambda/r^2 \). The analogous radial equation in this case is
\[
\left( -r^2 \frac{\partial^2}{\partial r^2} -(d+2l-1)r \frac{\partial}{\partial r} + E'r^2 + r^2(V-V_s) \right) \Psi (r) = \lambda \Psi (r) , \quad \Psi (r) \in L^2(\mathbb{R}^+) .
\]  
(1.3.59)

Case VIII.
Let us take the following bilinear combination of the \( \mathfrak{sl}(2, \mathbb{R}) \) generators with the mark \( n \) \( (1.2.3) \) (see [72])

\[
T_2 = -J_n^0 J_n^- + 2aJ_n^+ + 2bJ_n^0 - \left( \frac{n}{2} + d + 2l - 2c - 1 \right) J_n^- - bn , \tag{1.3.60}
\]
or, as the differential operator,

\[
T_2(x, d_x) = -xd_x^2 + (2ax^2 + 2bx + 2c - d - 2l + 1)dx - 2anx . \tag{1.3.61}
\]

where \( a > 0, b, c, d, l, n \) are parameters. If \( n \) is a non-negative integer, which plays the role of the mark of representation \( (1.2.3) \), the operator \( (1.3.60) \) has a finite-dimensional invariant subspace in polynomials of degree not higher than \( n \).

In the basis of monomials \( \{1, x, x^2, \ldots, x^k, \ldots x^n, \ldots\} \) this operator has the form of a tri-diagonal, Jacobi matrix,

\[
t_{k,k-1} = -k(k + d + 2l - 2c - 2) , \quad t_{k,k} = 2bk , \quad t_{k,k+1} = 2a(k - n) , \tag{1.3.62}
\]

(c.f. \( (1.3.51) \)). If \( k = n \), the matrix element \( t_{n,n+1} = 0 \) and the Jacobi matrix \( T_2 \) becomes block-triangular. The characteristic polynomial is factorizable, \( \det(T_2 - \alpha) = P_n(\alpha)P_\infty(\alpha) \). If \( b = 0 \) the principal diagonal of the Jacobi matrix vanishes and the energy-reflection symmetry occurs, see [61], and Lemma 2.4

\[
P_n(\alpha) = \alpha^p p_{\frac{n}{2}}(\alpha^2) ,
\]

where the parameter \( p = 0 \) or 1 depending on whether the integer \( n \) is even or odd, respectively. For \( a = 0 \) this matrix becomes triangular. Its eigenvalues can be found explicitly, \( \alpha_k = 2bk , k = 0, 1, \ldots . \)

Making a gauge rotation of \( (1.3.60) \) to \( (1.3.58) \) and a change of variables we arrive at the potential

\[
V(r) - V_c(r) = a^2r^2 + 2abr - \frac{b(D_c - 1)}{r} - \frac{c(D_c + c - 2)}{r^2}
+ b^2 - a(2n + D_c) , \tag{1.3.63}
\]

c.f. \( (1.3.11), (1.3.12) \), where

\[
x = r , \quad V_c = -\frac{\alpha}{r} ,
\]

and \( D_c \equiv (d + 2l - 2c) \), and the reference point for energy is chosen to be equal to zero. The eigenfunctions of \( (1.3.58) \) in the algebraic sector are

\[
\Psi(r) = p_n(r)r^{l-c-\frac{d}{2}r^2-br} , \tag{1.3.64}
\]

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at $a \geq 0, b > 0$ and $D_c > 2$, where $p_n(r)$ are eigenpolynomials of the operator (1.3.60). Eventually, we arrive at a family of Schrödinger equations,

$$(-\Delta^{(d)} + V(r))\Psi = E\Psi$$

(1.3.65)

where the $(n+1)$ radial excited states (the ground state of the 1st potential, the 1st excited state in the 2nd potential, $k$ excited state in the $(k+1)$ potential, ... , the $n$th excited state in the $(n+1)$ potential) can be found algebraically, solving the spectral problem (1.3.60). For a fixed $n$, the family of potentials is given by

$$V(r) = a^2r^2 + 2abr - \frac{b(D_c - 1) + \alpha}{r} - \frac{c(D_c + c - 2)}{r^2}.$$  

(1.3.66)

The parameter $\alpha$ takes the values of the eigenvalues of the finite-size matrix (1.3.62) ordered in the following manner:

$\alpha_0 > \alpha_1 > \ldots > \alpha_n$. The energies of all these states are equal to

$$E_n = a(2n + D_c) - b^2,$$

and the eigenfunctions are (1.3.64).

The potential (1.3.66) is a funnel-type potential which interpolates between the Coulomb potential at small distances and the harmonic oscillator potential at large distances, see Fig. 1.9. This QES potential appears in a number of applications: (i) the Hooke’s “pseudo-atom” - two electrons in the external harmonic oscillator potential [80], (ii) two electrons on a hypersphere [41], (iii) two charges on a plane in a constant magnetic field [89] - to mention a few.

Let us present two particular cases,

(i) Ground state, $n = 0$,

$$E_0^{(l)} = aD_c - b^2, \quad \Psi_0 = r^{l-c}e^{-\frac{\alpha^2}{2} - br},$$

in potential

$$V(r) = a^2r^2 + 2abr - \frac{b(D_c - 1)}{r} - \frac{c(D_c + c - 2)}{r^2},$$

see Fig. 1.9. The Riemann sheet $E_0^{(l)}(b)$ of the ground energy for a given $l$ and fixed $a, c$ splits away from the infinitely-sheeted Riemann surface $E^{(l)}(b)$.

(ii) Ground state (marked by superscript -) and the first excited state (marked by superscript +), $n = 1$. 

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Fig. 1.9. QES potential (1.3.66) in 3-dimensional space at \( l = 0 \): (a) Ground state in the potential \( V(r) = r^2 + 2r - \frac{2}{r} \) (blue line) with energy \( E_0 = 1 \) (red line) and the eigenfunction \( \Psi_0 = e^{-r^2/2} \); (b) Ground state (with superscript -, green line) and the first excited state (with +, blue line) in the potentials \( V_{\pm} = r^2 + 2r - 3\pm\sqrt{5}r \), respectively, with energy \( E = 4 \) (red line) and eigenfunctions \( \Psi_{\pm} = \left(2r + 1 \pm \sqrt{5}\right)e^{-r^2/2} \).

At energy
\[
E_1^{(l)} = a(D_c + 2) - b^2 ,
\]
in the potentials
\[
V_{\pm}(r) = a^2 r^2 + 2abr - \frac{b(\alpha_{\pm} + D_c - 1)}{r} - \frac{c(D_c + c - 2)}{r^2} ,
\]
see Fig. 1.9, where
\[
\alpha_{\pm} = b \pm \sqrt{b^2 + 2a(D_c - 1)} ,
\]
the eigenfunctions are
\[
\Psi_{1}^{\pm} = \left(2ar + b \pm \sqrt{b^2 + 2a(D_c - 1)}\right)r^{l-c} e^{-\frac{r^2}{2}} .
\]

Case IX.

Let us take the following bilinear combination of the \( \mathfrak{s}(2, \mathbb{R}) \) generators with the mark \( n \) (1.2.3) (see (72))
\[
T_2 = -J_n^+ J_n^- + 2aJ_n^+ - (n+d-1+2l-2c)J_n^0 + 2bJ_n^- - n(d+2l-1-2c) ,
\]
or as the differential operator,
\[
T_2(x, d_x) = -x^2 d_x^2 - \left[2ax^2 + (2c - d - 2l + 1)x - 2b \right]d_x - 2anx .
\]
If \( n \) is a non-negative integer, which plays the role of a mark of representation (1.2.3), the operator (1.3.49) has the finite-dimensional invariant subspace in polynomials of degree not higher than \( n \).

In the basis of monomials \( \{1, x, x^2, \ldots, x^k, \ldots x^n, \ldots\} \) this operator has the form of a tri-diagonal, Jacobi matrix,

\[
t_{k,k-1} = 2kb, \quad t_{k,k} = -k(k-d-2l+2c), \quad t_{k,k+1} = -2a(k-n), \quad (1.3.69)
\]

(c.f. (1.3.51)). If \( k = n \), the matrix element \( t_{n,n+1} = 0 \) and the Jacobi matrix \( T_2 \) becomes block-triangular. The characteristic polynomial is factorizable, \( \det(T_2 - \lambda) = P_n(\lambda)P_\infty(\lambda) \). If \( b = 0 \) this matrix becomes triangular. Its eigenvalues can be found explicitly, \( \lambda_k = -k(k-d-2l+2c) \), \( k = 0, 1, \ldots \).

Making a gauge rotation of (1.3.68) to (1.3.59) and a change of variables we arrive at the potential

\[
V(r) - V_s(r) = \frac{b^2}{r^4} + \frac{b(D_c - 3)}{r^3} - \frac{c(D_c + c - 2) + 2ab}{r^2} - \frac{a(2n + D_c - 1)}{r} + a^2, \quad (1.3.70)
\]

c.f. (1.3.11), (1.3.12), where

\[
x = r, \quad V_s = -\frac{\lambda}{r^2},
\]

and \( D_c \equiv (d + 2l - 2c) \), and the reference point for energy is chosen to be equal to zero. The eigenfunctions of (1.3.59) in the algebraic sector are of the form

\[
\Psi(r) = p_n(r)r^lce^{-ar-\lambda}b^{-1}, \quad (1.3.71)
\]
at \( a > 0, b \geq 0 \) and \( c \in \mathbb{R} \), and \( l, d > 0 \) are non-negative integers; if \( a = 0 \), the parameter \( b > 0 \) and \( d + l + n - c < 0 \). Here \( p_n(r) \) are eigenpolynomials of the operator (1.3.68). Eventually, we arrive at a family of Schrödinger equations (1.3.65)

\[
(-\Delta + V(r))\Psi = E\Psi,
\]

where the \( (n+1) \) radial excited states (the ground state of the 1st potential, the 1st radial excited state in the 2nd potential, \( k \)th radial excited state in the \( (k+1) \) potential, \( \ldots \), the \( n \)th radial excited state in the \( (n+1) \) potential) can be found algebraically, solving the spectral problem (1.3.67). For fixed \( n \), the family of potentials is given by

\[
V(r) = \frac{b^2}{r^4} + \frac{b(D_c - 3)}{r^3} - \frac{c(D_c + c - 2) + 2ab + \lambda}{r^2} - \frac{a(2n + D_c - 1)}{r}, \quad (1.3.72)
\]

see Fig. 1.10. The parameter \( \lambda \) takes the value of the eigenvalues of the finite-size matrix (1.3.69) ordered in the following manner:
Fig. 1.10. 3D-Potential \((1.3.72)\) at \(a = b = 1, c = 0, D_c = 3\) and \(n = 0, \lambda = 0\):
\[ V(r) = \frac{1}{r^4} - 2/r^2 - \frac{2}{r} ; \]
the ground state energy \(E = -1\) and the eigenfunction
\[ \Psi_0 = e^{-\frac{r^2}{2}} - \frac{1}{r} . \]
\(\lambda_0 > \lambda_1 > \ldots > \lambda_n\). The energies of all these states are equal to the same
\[ E_n = -a^2 , \]
and their eigenfunctions are \((1.3.71)\). For \(a = 0\) at \(d = 3\) the potential \((1.3.72)\)
was discovered by Korol [35].

Let us present a particular case,

(i) Ground state, \(n = 0\),
\[ E_0^{(l)} = -a^2 , \quad \Psi_0 = r^{l-c} e^{-\frac{ar^2}{2}} - \frac{1}{r} , \quad \lambda = 0 , \]
in potential
\[ V(r) = \frac{b^2}{r^4} + \frac{b(D_c - 3)}{r^3} - \frac{c(D_c + c - 2) + 2ab}{r^2} - \frac{a(D_c - 1)}{r} , \]
see Fig. 1.10. The Riemann sheet \(E_0^{(l)}(b)\) of the ground energy for given \(l\) and
fixed \(a, c\) splits away from the infinitely-sheeted Riemann surface \(E^{(l)}(b)\).
3.6 Non-singular Periodic potential

Now let us show the unique example of the non-singular periodic quasi-exactly-solvable potential associated with the Mathieu potential.

Comment 3.4 The Mathieu potential

\[ V(z) = A\alpha^2 \cos \alpha z , \]

which is sometimes called the sine-Gordon potential, is among the most important potentials in many branches of physics and engineering. Detailed description of the properties of the corresponding Schrödinger equation, which is called the Mathieu equation, can be found in Bateman-Erdélyi [3], Vol.3, also in Kamke [94], Equation 2.22 and in Whittaker-Watson [96].

The potential \( V(z) \) is periodic with period \( \frac{2\pi}{\alpha} \) and, hence, it has infinitely-many degenerate minima distributed uniformly along the real axis. It implies the existence of non-analytic, exponentially-small terms at \( \alpha = 0 \), see e.g. Dunne-Unsal [17]. The eigenvalues form four infinitely-sheeted Riemann surfaces in \( \alpha \), every sheet contains infinitely-many square-root branch points [31].

The Hamiltonian is translationally-invariant,

\[ T : z \rightarrow z + \frac{2\pi}{\alpha}. \]

Taking the simplest \( T \)-invariant

\[ \tau = \cos \alpha z , \]

as the new variable, we get the Mathieu potential in a very simple form,

\[ V(\tau) = A\alpha^2 \tau , \]

(1.3.73)

and the second derivative (1D Laplacian) becomes the one-dimensional Laplace-Beltrami operator with metric \( g^{11} = \alpha^2 (1 - \tau^2) \),

\[ \frac{d^2}{dz^2} |_{\tau = \cos \alpha z} = \alpha^2 \left( (1 - \tau^2) \frac{d^2}{d\tau^2} - \tau \frac{d}{d\tau} \right) , \]

(1.3.72-1)

which is an algebraic operator. This operator preserves the infinite flag of the spaces of polynomials \( \mathcal{P} \) (1.1.9); it is exactly solvable and can be rewritten in terms of the generators \( J^0_-, J^0_0 \) of the Borel subalgebra, \( \mathfrak{b}_2 \subset \mathfrak{sl}(2, \mathbb{R}) \), see (1.2.3),

\[ -\alpha^2 (J^0_0 J^0_0 - J^- J^-) , \]

hence, it has infinitely many polynomial eigenfunctions with eigenvalues

\[ \varepsilon_k = -\alpha^2 k^2 , \quad k = 0, 1, \ldots . \]
Note that there exist three gauge factors such that the gauge rotated Laplacian in $\tau$-variable remains an algebraic operator,

$$\sin^{-1} \alpha z \frac{d^2}{dz^2} \sin \alpha z \big|_{\tau = \cos \alpha z} = \alpha^2 (1-\tau^2)^{-\frac{1}{2}} \left( (1-\tau^2) \frac{d^2}{d\tau^2} - \tau \frac{d}{d\tau} \right) \left( 1-\tau^2 \right)^{\frac{1}{2}} =$$

$$\alpha^2 \left( (1-\tau^2) \frac{d^2}{d\tau^2} - 3\tau \frac{d}{d\tau} - 1 \right), \quad (1.3.72-2)$$

$$\sin^{-1} \left( \frac{\alpha z}{2} \right) \frac{d^2}{dz^2} \sin \left( \frac{\alpha z}{2} \right) \big|_{\tau = \cos \alpha z} = \alpha^2 (1-\tau) \frac{1}{2} \left( (1-\tau^2) \frac{d^2}{d\tau^2} - \tau \frac{d}{d\tau} \right) \left( 1-\tau^2 \right)^{\frac{1}{2}} =$$

$$\alpha^2 \left( (1-\tau^2) \frac{d^2}{d\tau^2} - (1+2\tau) \frac{d}{d\tau} - \frac{1}{4} \right), \quad (1.3.72-3)$$

$$\cos^{-1} \left( \frac{\alpha z}{2} \right) \frac{d^2}{dz^2} \cos \left( \frac{\alpha z}{2} \right) \big|_{\tau = \cos \alpha z} = \alpha^2 (1+\tau) \frac{1}{2} \left( (1-\tau^2) \frac{d^2}{d\tau^2} - \tau \frac{d}{d\tau} \right) \left( 1+\tau \right)^{\frac{1}{2}} =$$

$$\alpha^2 \left( (1-\tau^2) \frac{d^2}{d\tau^2} + (1-2\tau) \frac{d}{d\tau} - \frac{1}{4} \right). \quad (1.3.72-4)$$

Every operator (1.3.72-2) - (1.3.72-4) preserves the infinite flag of the spaces of polynomials $P(1.1.9)$: it is exactly solvable and can be rewritten in terms of the generators $J^0_\pm \equiv J^0_0$ of the Borel subalgebra, $b_2 \subset \mathfrak{sl}(2, \mathbb{R})$, see (1.2.3),

$$-\alpha^2 (J^0 J^0 - J^- J^+ + 2J^0 + 1),$$
$$-\alpha^2 (J^0 J^0 - J^- J^+ + J^0 + J^- + \frac{1}{4}),$$
$$-\alpha^2 (J^0 J^0 - J^- J^+ + J^0 - J^- + \frac{1}{4}),$$

respectively, hence, it has infinitely many polynomial eigenfunctions with eigenvalues

$$\varepsilon_k = -\alpha^2 (k + 1)^2, \ k = 0, 1, \ldots ,$$
$$\varepsilon_k = -\alpha^2 (k + \frac{1}{2})^2, \ k = 0, 1, \ldots ,$$
$$\varepsilon_k = -\alpha^2 (k + \frac{1}{2})^2, \ k = 0, 1, \ldots ,$$

respectively.

Finally, the Schrödinger operator in $\tau$-variable,

$$\mathcal{H} = -\frac{d^2}{dz^2} + A \alpha^2 \cos \alpha z = -\alpha^2 \left( (1-\tau^2) \frac{d^2}{d\tau^2} + \tau \frac{d}{d\tau} + A \tau \right),$$

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becomes an algebraic operator, which is the algebraic form of the Mathieu operator, see e.g. [3]. This algebraic operator is a particular form of the Heun operator.

**Case X.**

Let us take the following bilinear combination of the \( \mathfrak{sl}(2,\mathbb{R}) \) generators with mark \((n-\mu)\) \((1.2.3)\) (see \[72\])

\[
T_2 = \alpha^2 \left( J_{n-\mu}^+ J_{n-\mu}^- - J_{n-\mu}^- J_{n-\mu}^- + 2a J_{n-\mu}^+ + (n + \mu + 1) J_{n-\mu}^0 - 2a J_{n-\mu}^- + \frac{(n - \mu)(n + \mu + 1)}{2} J_{n-\mu}^- J_{n-\mu}^- \right),
\]

and rewrite it as a differential operator,

\[
T_2(x,d_x) = \alpha^2 \left( (x^2 - 1)d_x^2 + [2ax^2 + (1 + 2\mu)x - 2a]d_x - 2a(n-\mu)x \right) \equiv \alpha^2 t_2(x,d_x),
\]

(1.3.74)

where \(a \neq 0, \alpha, \mu, n\) are parameters. If \((n - \mu)\) is a non-negative integer, the differential operator (1.3.74) has a finite-dimensional invariant subspace in polynomials of degree not higher than \(n\).

In the basis of monomials \(\{1, x, x^2, \ldots, x^k, \ldots, x^n, \ldots\}\) the operator \(t_2(x,d_x)\) has the form of a four-diagonal matrix,

\[
t_{k,k-2} = -k(k-1), \ t_{k,k-1} = -2ak, \ t_{k,k} = k(k+2\mu), \ t_{k,k+1} = 2a(k-n+\mu).
\]

(1.3.75)

If \(k = n - \mu\), the matrix element \(t_{n-\mu,n-\mu+1} = 0\) and the matrix \(t_2\) becomes block-triangular. The characteristic polynomial is factorizable, \(\det(T_2 - \lambda) = P_{n-\mu}(\lambda)P_\infty(\lambda)\).

The transformation (1.3.2)-(1.3.4) leads eventually to the periodic potential

\[
V(z) = \alpha^2 \left( a^2 \sin^2 \alpha z - a(2n + 1) \cos \alpha z + \mu \right),
\]

(1.3.76)

where

\[
x = \cos \alpha z, \ \varrho = 1,
\]

\[
\Psi(z) = (\sin \alpha z)^\mu p_{n-\mu}(\cos \alpha z) e^{a \cos \alpha z},
\]

at \(a \neq 0, \alpha \geq 0\). Here \(\mu = 0, 1\) and \(n - \mu = 0, 1, \ldots\). For the (fixed) integer \(n\), the \((2n + 1)\) eigenstates have the meaning of the edges of bands (zones) and can be found algebraically. The quasi-exactly-solvable potential (1.3.76) was, perhaps, the first QES potential which was discovered in full generality,
it has a number of different names: the Whittaker-Hill potential \[96\], the Magnus-Winkler potential \[42\], the non-singular periodic QES potential \[71\], and the two term (trigonometric) potential \[14\]. The polynomials $p_{n-\mu}(\cos \alpha z)$ are sometimes called the Ince polynomials. The Hamiltonian corresponding to the QES potential \(1.3.76\) has the form

$$H(z) = -\frac{d^2}{dz^2} + \alpha^2 \left( a^2 \sin^2 \alpha z - a(2n + 1) \cos \alpha z + \mu \right). \quad (1.3.77)$$

Now let us take the following operator

$$T_2 = \alpha^2 \left( J_{n-1}^+ J_{n-1}^- - J_{n-1}^- J_{n-1}^+ \right) + 2a J_{n-1}^+ + (n + 1) J_{n-1}^- - [2a + (\nu_1 - \nu_2)] J_{n-1}^- + \left( \frac{n^2 - 1}{2} \right), \quad (1.3.78)$$

(c.f. \(1.3.73\)), or in the form of a differential operator,

$$T_2(x, d_x) = \alpha^2 \left( (x^2 - 1) d_x^2 + [2ax^2 + 2x - 2a(\nu_1 - \nu_2)] d_x - 2a(n - 1) x \right), \quad (1.3.79)$$

(c.f. \(1.3.79\)).

The transformation \(1.3.2\)-\(1.3.4\) leads to the periodic potential

$$V(z) = \alpha^2 \left( a^2 \sin^2 \alpha z - 2na \cos \alpha z + a(\nu_1 - \nu_2) - \frac{1}{4} \right), \quad (1.3.80)$$

where

$$x = \cos \alpha z, \quad \rho = 1,$$

$$\Psi(z) = \cos^{\nu_1} \frac{\alpha}{2} z \sin^{\nu_2} \frac{\alpha}{2} z P_{n-1}(\cos \alpha z) \ e^{a \cos \alpha z},$$

at $a \neq 0, \alpha \geq 0$. Here $\nu_{1,2} = 0, 1$, but $\nu_1 + \nu_2 = 1$. For the fixed $n$, $(2n + 1)$ eigenstates, which have the meaning of the edges of bands can be found algebraically. The polynomials $P_{n-1}(\cos \alpha z)$ are called the Ince polynomials. The Hamiltonian corresponding to the QES potential \(1.3.80\) has the form

$$\mathcal{H}(z) = -\frac{d^2}{dz^2} + \alpha^2 \left( a^2 \sin^2 \alpha z - 2na \cos \alpha z + a(\nu_1 - \nu_2) - \frac{1}{4} \right), \quad (1.3.81)$$

cf. \(1.3.77\).
3.7 Double-Periodic potentials

The last two one-dimensional quasi-exactly-solvable potentials we are going to present are double-periodic potentials, given by elliptic functions. They are written in terms of the Weierstrass function \( \wp(z) \equiv \wp(z|g_2, g_3) \) (see e.g. [96]),

\[
(\wp'(z))^2 = 4 \wp^3(z) - g_2 \wp(z) - g_3 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3),
\]

(1.3.82)

where \( g_{2,3} \) are its invariants and \( e_{1,2,3} \) are its roots, \( e_1 + e_2 + e_3 = 0 \). Both potentials appear in relation with the integrable quantum Calogero-Sutherland models with Weyl symmetry \( A_1 \) and \( BC_1 \), thus, respectively, they emerge in the Hamiltonian Reduction Method, for review see e.g. Olshanetsky-Perelomov [50].

The \( A_1 \)-quantum elliptic Calogero-Sutherland model describes a two-body system with pairwise interaction given by the elliptic potential. Its Hamiltonian is defined on the plane \( (z_1, z_2) \) as,

\[
H = -\frac{1}{2} \Delta^{(2)} + \frac{\kappa}{2} \wp(z_1 - z_2),
\]

where \( \kappa = m(m + 1) \) is the coupling constant. After center-of-mass separation \( Z = z_1 + z_2, \ z = z_1 - z_2 \) the relative motion is described by the Hamiltonian

\[
H_A = -d_z^2 + m(m + 1)\wp(z).
\]

(1.3.83)

It coincides with the celebrated Lamé operator, see e.g. [90]. Thus, it can be naturally called the \( A_1 \)-Lamé Hamiltonian or, simply, the Lamé Hamiltonian. In turn, the Hamiltonian of \( BC_1 \)-quantum elliptic Calogero-Sutherland model is defined as

\[
H_B = -\frac{1}{2} d_z^2 + \kappa_2 \wp(2z) + \kappa_3 \wp(z),
\]

(1.3.84)

see e.g. Olshanetsky-Perelomov [50], where \( \kappa_2, \kappa_3 \) are coupling constants. Certainly, it can be considered as the Hamiltonian of two-body relative motion. If one of the coupling constants vanishes, \( \kappa_2(\kappa_3) = 0 \), the Hamiltonian becomes \( A_1 \)-Lamé Hamiltonian. We will show that in variable \( \tau = \wp(z) \) both Hamiltonians take algebraic form becoming a particular case of the Heun operator (1.2.29). Each Hamiltonian can be rewritten in terms of the \( \mathfrak{sl}(2, \mathbb{R}) \) generators (1.2.3), hence, they are quadratic elements \( T_2 \) of the universal enveloping algebra \( U_{\mathfrak{sl}(2, \mathbb{R})} \). If the mark \( n \) of \( \mathfrak{sl}(2, \mathbb{R}) \) representation is an integer, the Hamiltonian has a finite-dimensional functional invariant subspace in polynomials. Furthermore, in the algebra \( \mathfrak{sl}(2, \mathbb{R}) \) both Hamiltonians have the remarkable property of factorization

\[
T_2 = T_1^a T_1^b,
\]
where \( T_1^{a(b)} = A^{a(b)} J_n^++B^{a(b)} J_n^0+C^{a(b)} J_n^-+D^{a(b)} \) are linear elements of \( U_{\mathfrak{sl}(2,\mathbb{R})} \). Note that such a factorization occurs for some other exactly-solvable problems as well (see below).

Both potentials we are going to study can be written in terms of the Weierstrass functions \( \wp(z|g_2,g_3) \), \( \wp(2z|g_2,g_3) \). The discrete symmetry of the Hamiltonian is \( \mathbb{Z}_2 \oplus T_r \oplus T_c \): It consists of reflection \( \mathbb{Z}_2(x \rightarrow -x) \) and two translations \( T_r: z \rightarrow z + P_r \) and \( T_c: z \rightarrow z + P_c \), where \( P_{r,c} \) are periods. Let us take the simplest invariant with respect to the above discrete symmetry group,

\[
\tau = \wp(z) ,
\]

as the new variable. The second derivative (1D Laplacian) becomes the one-dimensional Laplace-Beltrami operator \( \Delta_g \),

\[
\frac{d^2}{dz^2} \big|_{\tau = \wp(z)} \equiv g^{-1/2} \frac{\partial}{\partial \tau} g^{1/2} \frac{\partial}{\partial \tau} = (4\tau^3 - g_2 \tau - g_3) \partial^2_{\tau} + (6\tau^2 - \frac{g_2}{2}) \partial_{\tau} ,
\]

with flat metric

\[
g^{11} = (4\tau^3 - g_2 \tau - g_3) = \frac{1}{g} ,
\]

here \( g \) is its determinant with lower indices. It is the algebraic operator.

The operator (1.3.86) preserves the one-dimensional space of polynomials \( \mathcal{P}_0 \) (1.1.5); it is a primitive quasi-exactly solvable operator and can be rewritten in terms of the generators \( J_{+,0,-} \equiv J_{0,+,-} \) of the algebra \( \mathfrak{sl}(2,\mathbb{R}) \) (1.2.3) in one-dimensional representation,

\[
4J^+ J^0 - g_2 J^0 J^- - g_3 J^- J^- + 2J^+ - \frac{g_2}{2} J^- ,
\]

hence, it has a constant as the eigenfunction with eigenvalue

\[ \varepsilon_0 = 0 . \]

**Case XI. Lamé equation (or \( A_1 \)-quantum elliptic Calogero-Sutherland model)**

In this Section we consider one of the so-called \( m \)-zone Lamé equations

\[
- \frac{d^2}{dz^2} \Psi + m(m+1) \wp(z) \Psi = \varepsilon \Psi ,
\]

where \( \wp(z) \) is the Weierstrass function in standard notation (1.3.82), which depends on two free parameters, assuming that \( m = 1,2, \ldots \). It will be shown that it is a quasi-exactly-solvable Schrödinger equation [74].
Introducing the new variable $\xi = \varphi(z) + \frac{1}{3} \sum a_i$ in (1.3.87) (see, e.g. Kamke [94]), a new equation emerges

$$\eta'' + \frac{1}{2} \left( \frac{1}{\xi - a_1} + \frac{1}{\xi - a_2} + \frac{1}{\xi - a_3} \right) \eta' - \frac{m(m + 1)\xi + \varepsilon}{4(\xi - a_1)(\xi - a_2)(\xi - a_3)} \eta = 0,$$

(1.3.88)

where $\eta(\xi) \equiv \Psi(z)$. Without loss of generality $a_1 = 0$. Here the new parameters $a_i$ satisfy the system of linear equations $a_i - \frac{1}{3} \sum a_i = c_i$. Equation (1.3.88) is called the algebraic form for the Lamé equation. It is known that the equation (1.3.88) can have four types of solutions:

$$\eta^{(1)} = p_k(\xi),$$

(1.3.85-1)

$$\eta^{(2)}_i = (\xi - a_i)^{1/2} p_k(\xi), \quad i = 1, 2, 3$$

(1.3.85-2)

$$\eta^{(3)}_i = (\xi - a_{i_1})^{1/2} (\xi - a_{i_2})^{1/2} p_{k-1}(\xi), \quad l_1 \neq l_2; i \neq l_{1,2}; i, l_{1,2} = 1, 2, 3$$

(1.3.85-3)

$$\eta^{(4)} = (\xi - a_1)^{1/2} (\xi - a_2)^{1/2} (\xi - a_3)^{1/2} p_{k-1}(\xi)$$

(1.3.85-4)

where $p_r(\xi)$ are polynomials in $\xi$ of degree $r$. If the value of the parameter $m$ is fixed, there are $(2m + 1)$ linear independent solutions of the following form: if $m = 2k$ is even, then the $\eta^{(1)}(\xi)$ and $\eta^{(3)}(\xi)$ solutions arise, if $m = 2k + 1$ is odd we get solutions of the $\eta^{(2)}(\xi)$ and $\eta^{(4)}(\xi)$ types. Those eigenvalues have the meaning of the edges of the zones in the potential (1.3.87).

**Theorem 3.1** [74] The spectral problem (1.3.87) at $m = 1, 2, \ldots$ with polynomial solutions (1.3.85-1), (1.3.85-2), (1.3.85-3), (1.3.85-4) is equivalent to the spectral problem (1.2.24) for the operator $T_2$ (1.2.24) belonging to the universal enveloping $\mathfrak{sl}(2, \mathbb{R})$-algebra in the representation (1.2.3) with the coefficients

$$c_{++} = 4, \quad 2c_{+-} = -4 \sum a_i, \quad c_{0-} = 4 \sum a_ia_j, \quad c_{--} = a_1a_2a_3$$

(1.3.86)

before the terms quadratic in the generators and the following coefficients before the linear terms in the generators $J^\pm_1$:

1. For $\eta^{(1)}(\xi)$-type solutions at $m = 2k, r = k$

$$c_+ = -6k - 2, \quad c_0 = 4(k + 1) \sum a_i, \quad c_- = -2(k + 1) \sum a_i a_j$$

(1.3.87-1)

2. For $\eta^{(2)}_i(\xi)$-type solutions at $m = 2k + 1, r = k$

$$c_+ = -6k - 6, \quad c_0 = 4(k + 2) \sum a_i - a_i,$$

$$c_- = -2(k + 1) \sum a_ia_j - 4a_ia_{i_2}, \quad i \neq l_{1,2}, l_1 \neq l_2$$

(1.3.87-2)

3. For $\eta^{(3)}_i(\xi)$-type solutions at $m = 2k, r = k - 1$

$$c_+ = -6k - 4, \quad c_0 = 4(k + 1) \sum a_i + 4a_i,$$

(1.3.87-3)
\[ c_+ = -6k - 8, \quad c_0 = 4(k + 2) \sum a_i, \quad c_- = -2(k + 2) \sum a_i a_j \]  
(1.3.87-4)

Thus, each type of solution (1.3.85-1), (1.3.85-2), (1.3.85-3), (1.3.85-4) corresponds to the particular spectral problem (1.3.87) with a special set of parameters (1.3.86) plus (1.3.87-1), (1.3.87-2), (1.3.87-3), (1.3.87-4), respectively.

It can be easily shown that the calculation of eigenvalues \( \varepsilon \) of (1.3.87) corresponds to the solution of the characteristic equation for the four-diagonal matrix:

\[
C_{ll-1} = (l - 1 - 2j)[4(j + 1 - l) + c_+] , \\
C_{ll} = [l(2j + 1 - l)2c_+ + (l - j)c_0] , \\
C_{ll+1} = (l + 1)(j - l)c_0 + (l + 1)c_- , \\
C_{ll+2} = -(l + 1)(l + 2)c_- .
\]

(1.3.88)

where the size of this matrix is \((k + 1)\times(k + 1)\) and \(2j = k\) for (1.3.85-1), (1.3.85-2), and \(k \times k\) and \(2j = k - 1\) for (1.3.85-3), (1.3.85-4), respectively.

Since one of \(a\)'s can always be placed equal to zero, say, \(a_1 = 0\), the coefficient \(c_-\) vanishes and the matrix (1.3.88) becomes tri-diagonal, Jacobi matrix.

In connection to Theorem 3.1 one can prove the following theorem [74].

**Theorem 3.2** Let us fix all the parameters \(e^i\) (\(a^i\)) in (1.3.87) (or (1.3.88)) except for one, e.g. \(e_1(a_1)\). The first \((2m + 1)\) eigenvalues of (1.3.87) (or (1.3.88)) form a \((2m + 1)\)-sheeted Riemann surface in parameter \(e_1(a_1)\). This surface contains four disconnected pieces: one of them corresponds to \(\eta^{(1)}(\eta^{(4)})\) solutions and the others correspond to \(\eta^{(3)}(\eta^{(2)})\). At \(m = 2k\) the Riemann subsurface for \(\eta^{(1)}\) has \((k + 1)\) sheets and the number of sheets in each of the others is equal to \(k\). At \(m = 2k + 1\) the number of sheets for \(\eta^{(4)}\) is equal to \(k\) and for \(\eta^{(2)}\) each subsurface contains \((k + 1)\) sheets.

It is worth emphasizing that we cannot find a relation between the spectral problem for the two-zone potential

\[ V = -2(\sum_{k=1}^{3} \varphi(z - z_i) , \quad \sum_{i=1}^{3} z_i = 0 , \]  
(1.3.89)

(see [15]) and the spectral problem (1.2.8) for \(T_2\) with the parameters (1.3.86) and (1.3.87-1) or (1.3.87-3) at \(k = 1\). In this case the eigenvalues \(\varepsilon\) and also the eigenfunctions (1.3.88) but not (1.3.87) do not depend on parameters \(c_-\).

\(^{13}\)The potential (1.3.89) and the original Lamé potential (1.3.87) at \(m = 2\) are related via the isospectral deformation.
Comment 3.5. One can generalize the meaning of an isospectral deformation by saying we want to study a variety of potentials with the first several coinciding eigenvalues. They can be named quasi-isospectral (see below).

Now let us consider such a quasi-isospectral deformation of (1.3.87) at \( m = 2 \). It arises from the fact that the addition of the term \( c_{++}J_r^+ J_r^+ \) to the operator \( T_2 \) with the parameters (1.3.86) and (1.3.87-1) or (1.3.87-3) at \( k = 1 \) with appropriate \( r \) does not change the characteristic matrix (1.3.88). Making the reduction (1.3.2)-(1.3.3) from the equation (1.2.8) to the Schrödinger equation (1.2.39), we obtain

\[
V(x) = c_{++} \frac{2c_{++}\xi^6 - 2c_{+-}\xi^4 - 2c_{0-}\xi^3 - 3c_{--}\xi^2}{P_4^2(\xi)} + P_2(\xi),
\]

where

\[
P_4(\xi) = c_{++}\xi^4 + c_{0}\xi^3 + 2c_{+-}\xi^2 + c_{0-}\xi + c_{--}, \quad P_2(\xi) = -m(m+1)\xi + \frac{c_0}{2}.
\]

and \( \xi \) is defined via the equation

\[
z = \int \frac{d\xi}{\sqrt{P_4(\xi)}},
\]

In general, the potential (1.3.90) contains four double poles in \( x \) and does not reduce to (1.3.89). It is worth noting that the first five eigenfunctions in the potential (1.3.90) have the form

\[
\Psi(z) = \begin{cases} 
A\xi + B \\
(\xi - a_i)^{1/2}(\xi - a_j)^{1/2}
\end{cases}
\exp \left( -c_{++} \int \frac{\xi^3 d\xi}{P_4(\xi)} \right), \quad i \neq j, \quad i, j = 1, 2, 3.
\]

Here \( \xi \) is given by (1.3.92). These first five eigenvalues of the potential (1.3.90) do not depend on the parameters \( c_{--}, c_{++} \).
Case XII. \(BC_1\) Lamé equation (or \(BC_1\)-quantum elliptic Calogero-Sutherland model)

The goal of this Section is to show that the Schrödinger equation, which describes the \(BC_1\)-quantum elliptic Calogero-Sutherland model, is a quasi-exactly-solvable Schrödinger equation. This remarkable observation was made by Gomez-Ullate et al, [24] for the \(BC_1\)-quantum elliptic Calogero-Sutherland model, which was later extended by Brihaye-Hartmann [10]. In this presentation we mostly follow the paper [88].

The potential of the \(BC_1\) elliptic model, see (1.3.84), has the following interesting property: in \(\tau\)-variable (1.3.85) it is a simple rational function,

\[
V(z) = (\kappa_2 \wp(2z) + \kappa_3 \wp(z))|_{\tau=\wp(z)} = \frac{\kappa_2 + 4\kappa_3}{4} \tau + \frac{\kappa_2}{16} \frac{12g_2\tau^2 + 36g_3\tau + g_2^3}{4\tau^3 - g_2\tau - g_3},
\]

which is a superposition of the linear function with three simple poles situated at the roots of the Weierstrass function. The kinetic energy represented by the Laplace-Beltrami operator is the algebraic operator (1.3.86). Hence, the \(BC_1\) Hamiltonian (1.3.84) in \(\tau\)-variable

\[
\mathcal{H}_B = \left( -\frac{1}{2} \frac{d^2}{dz^2} + V(z) \right)|_{\tau=\wp(z)}
\]

has the very simple form of an algebraic operator with polynomial coefficient functions in front of the 1st and 2nd derivatives and with a rational function as the no-derivative term.

(I). It can be checked that the eigenvalue problem for the Hamiltonian (1.3.84) has a formal exact solution

\[
\Psi_0 = \left[ \wp'(x) \right]^{\mu},
\]

for coupling constants

\[
\kappa_2 = 2\mu(\mu - 1), \quad \kappa_3 = 2\mu(1 + 2\mu),
\]

where \(\mu\) is an arbitrary parameter, for which the eigenvalue is

\[
E_0 = 0.
\]

Let us parametrize the coupling constants as follows

\[
\kappa_2 = 2\mu(\mu - 1), \quad \kappa_3 = (2n + 1 + 2\mu)(n + 2\mu),
\]

where \(\mu\) and \(n\) are parameters. Making the gauge rotation

\[
h_B = -2(\Psi_0)^{-1} \mathcal{H}_B \Psi_0, \quad \Psi_0 = \left[ \wp'(x) \right]^{\mu},
\]

75
see (1.3.95), we arrive at the algebraic operator in variable \( \tau \),

\[
h_B(\tau) = (4\tau^3 - g_2\tau - g_3)\partial_\tau^2 + (1 + 2\mu)(6\tau^2 - \frac{g_2}{2})\partial_\tau - 2n(2n + 1 + 6\mu)\tau .
\]

(1.3.98)

It can be easily checked that if the parameter \( n \) is a non-negative integer, the operator \( h_B(\tau) \) (1.3.98) has the invariant subspace

\[
P_n = \langle \tau^p | 0 \leq p \leq n \rangle ,
\]

of dimension

\[
\dim P_n = (n + 1) ,
\]

namely,

\[
h_B : P_n \to P_n .
\]

The operator (1.3.98) can be rewritten in terms of \( \mathfrak{sl}(2,\mathbb{R}) \)-generators

\[
h_B = 4J^+(n)J^0(n) - g_2J^0(n)J^- - g_3J^-J^-
\]

\[
+ 2(4n + 1 + 6\mu)J^+(n) - g_2\left(n + \frac{1}{2} + \mu\right)J^- .
\]

(1.3.99)

Thus, it is the Hamiltonian of the \( \mathfrak{sl}(2,\mathbb{R}) \) quantum top in a constant magnetic field. This representation holds for any value of the parameter \( n \). Thus, the algebra \( \mathfrak{sl}(2,\mathbb{R}) \) is the hidden algebra of the \( BC_1 \) elliptic model with arbitrary coupling constants \( \kappa_{2,3} \) parameterized via (1.3.97).

If we parameterize in (1.3.82) the invariants \( g_2, g_3 \) as follows

\[
g_2 = 12(\lambda^2 - \delta) , \quad g_3 = 4\lambda(2\lambda^2 - 3\delta) ,
\]

(1.3.100)

where \( \lambda, \delta \) are parameters, then \( e = -\lambda \) becomes the root of the \( \wp \)-Weierstrass function, \( \wp'(-\lambda) = 0 \), see [65]. The defining equation for

\[
\bar{\wp}(z) \equiv \wp(z|\lambda, \delta) + \lambda = \wp(z|g_2, g_3) + \lambda
\]

takes the form,

\[
(\bar{\wp}(z))^2 = 4(\bar{\wp}(z))^3 - 12\lambda\bar{\wp}(z)^2 + 12\delta\bar{\wp}(z) .
\]

(1.3.101)

Now let us shift the variable \( \tau \) in (1.3.98),

\[
\bar{\tau} = \tau + \lambda ,
\]

and we arrive to the following operator,

\[
h_B(\bar{\tau}) = 4(\bar{\tau}^3 - 3\lambda\bar{\tau}^2 + 3\delta\bar{\tau})\partial_{\bar{\tau}}^2 + 6(1 + 2\mu)(\bar{\tau}^2 - 2\lambda\bar{\tau} + \delta)\partial_{\bar{\tau}} - 2n(2n + 1 + 6\mu)(\bar{\tau} - \lambda) ,
\]

(1.3.102)
c.f. (1.3.98). In space of monomials \( \tau^k \), \( k = 0, 1, 2, \ldots \) the operator \( \hat{h}_B(\tilde{\tau}) \) has the form of tri-diagonal, Jacobi matrix. In terms of \( \mathfrak{sl}(2, \mathbb{R}) \)-generators it reads

\[
\hat{h}_B = 4 J^+(n) J^0(n) - 12 \lambda J^0(n) J^+(n) + 12 \delta J^0(n) J^-
+ 2 (4n+1+6\mu) J^+(n) - 12 \lambda (n+2\mu) J^0(n) + 6 \delta (n+1+2\mu) J^- + \lambda n(n+2) .
\]

This is different Lie-algebraic form than (1.3.99) but equivalent. Again the operator \( \hat{h}_B \) has the meaning of the Hamiltonian of the \( \mathfrak{sl}(2, \mathbb{R}) \) quantum top in a constant magnetic field.

If \( n \) takes integer value, the hidden algebra \( \mathfrak{sl}(2, \mathbb{R}) \) (1.2.3) appears in a finite-dimensional representation, and the operator (1.3.98) has finite-dimensional invariant subspace \( \mathcal{P}_n \), which is the finite-dimensional representation space of the \( \mathfrak{sl}(2, \mathbb{R}) \)-algebra. It possesses a number of polynomial eigenfunctions

\[
\varphi_{n,i} = P_{n,i}(\tau; \mu) , \ i = 1, \ldots (n+1) .
\]

These polynomials are called \textit{BC} \textsubscript{1} Lamé polynomials of the first kind \cite{88}. Those polynomials degenerate either to the Lamé polynomials of the first kind at \( \mu = 0 \), see (1.3.85-1), or to the Lamé polynomials of the fourth kind at \( \mu = 1 \), see (1.3.85-4).

Few examples are presented below.

(i) For \( n = 0 \) at coupling constants

\[
\kappa_2 = 2\mu (\mu - 1) , \ \kappa_3 = 2\mu (1 + 2\mu) ,
\]

thus, for the potential

\[
V_0 = 2\mu (\mu - 1) \varphi(2z) + 2\mu (1 + 2\mu) \varphi(z) ,
\]

the single eigenstate is known

\[
E_{0,1} = 0 , \ P_{0,1} = 1 .
\]

(ii) For \( n = 1 \) at coupling constants

\[
\kappa_2 = 2\mu (\mu - 1) , \ \kappa_3 = 2(1 + 2\mu) (1 + \mu) ,
\]

thus, for the potential

\[
V_1 = 2\mu (\mu - 1) \varphi(2z) + 2\mu (1 + 2\mu) (1 + \mu) \varphi(z) ,
\]

two eigenstates are known

\[
E_{\mp} = \pm (1 + 2\mu) \sqrt{3g_2} , \ P_{1,\mp} = 2 \mp \sqrt{g_2^2} .
\]
As a function of $g_2$ both eigenvalues are branches of a double-sheeted Riemann surface. Note that if $\mu = -\frac{1}{2}$ the degeneracy occurs: both eigenvalues coincide, they are equal to zero; any linear combination of $P_{1,\mp}$ is an eigenfunction. If $g_2 = 0$ but $\mu \neq -\frac{1}{2}$, the Jordan cell occurs: both eigenvalues are equal to zero but there exists a single eigenfunction, $P = \tau$.

To summarize, it can be stated that for coupling constants (1.3.97) at integer $n$, the Hamiltonian (1.3.84) has $(n+1)$ eigenfunctions of the form

$$\Psi_{n,i} = P_{n,i}(\tau; \mu) [\psi'(x)]^\mu, \quad i = 1, \ldots (n+1),$$

(1.3.104)

where $P_{n,i}(\tau; \mu)$ is a polynomial in $\tau$ of degree $n$.

(II). It can be checked that the eigenvalue problem for the Hamiltonian (1.3.84) has a formal exact solution, other than (1.3.95),

$$\Psi_{0,k} = [\psi'(x)]^\mu \left(\psi(x) - e_k\right)^{\frac{1}{2}-\mu},$$

(1.3.105)

for coupling constants

$$\kappa_2 = 2\mu(\mu - 1), \quad \kappa_3 = (1+2\mu)(1-\mu),$$

(1.3.106)

where $\mu$ is an arbitrary parameter, for which the eigenvalue is

$$E_{0,k} = \frac{(4\mu^2 - 1)}{2}e_k, \quad k = 1, 2, 3,$$

here $e_k$ is the $k$th root of the Weierstrass function (1.3.82). It implies that for parameters (1.3.106) the Hamiltonian $H_B$ has one-dimensional invariant subspace.

Let us parametrize the coupling constants $\kappa_{2,3}$ as follows

$$\kappa_2 = 2\mu(\mu - 1), \quad \kappa_3 = (n+1+2\mu)(2n+1-\mu) - \mu n,$$

(1.3.107)

(cf. (1.3.97)). Making a gauge rotation of the Hamiltonian (1.3.84) with subtracted $E_{0,k}$ and changing variable to $\tau$,

$$h_k = -2(\Psi_{0,k})^{-1}(H_B - E_{0,k})\Psi_{0,k}|_{\tau = \varphi(z)},$$

we arrive at the algebraic operator

$$h_k(\tau) = (\tau - e_k)^{-\frac{1}{2}+\mu} (H_B(\tau) - 2E_{0,k}) (\tau - e_k)^{\frac{1}{2}-\mu} =$$

$$(4\tau^3 - g_2\tau - g_3)\partial^2_\tau + \left(2(5 + 2\mu)\tau^2 + 4(1 - 2\mu)e_k(\tau + e_k) - (3 - 2\mu)\frac{g_2}{2}\right)\partial_\tau - 2n(2n + 3 + 2\mu)\tau,$$

(1.3.108)
It can be checked that if the parameter \( n \) takes a non-negative integer value, the operator \( h_k(\tau) \) has the invariant subspace \( \mathcal{P}_n \). Furthermore, the operator \( (1.3.108) \) can be rewritten in terms of \( \mathfrak{sl}(2, \mathbb{R}) \)-generators \( (1.2.3) \) for any value of \( n \), cf. \( (1.3.99) \),

\[
h_k = 4 J^+(n) J^0(n) - g_2 J^0(n) J^- - g_3 J^- J^-
\]
\[+ 2(4n + 3 + 2\mu) J^+(n) + 4(1 - 2\mu) e_k (J^0(n) + n)
\]
\[+ \left( 4(1 - 2\mu) e_k^2 - (2n + 3 - 2\mu) \frac{g_2}{2} \right) J^-.
\]

Thus, it is \( \mathfrak{sl}(2, \mathbb{R}) \) quantum top in a constant magnetic field. This representation holds for any value of the parameter \( n \). Thus, the algebra \( \mathfrak{sl}(2, \mathbb{R}) \) is the hidden algebra of the \( BC_1 \) elliptic model with arbitrary coupling constants \( \kappa_{2,3} \) parameterized via \( (1.3.107) \).

If \( n \) takes a non-negative integer value, the hidden algebra \( \mathfrak{sl}(2, \mathbb{R}) \) \( (1.2.3) \) appears in a finite-dimensional representation, and the operator \( (1.3.108) \) has finite-dimensional invariant subspace \( \mathcal{P}_n \), which is the finite-dimensional representation space of the \( \mathfrak{sl}(2, \mathbb{R}) \)-algebra. It possesses a number of polynomial eigenfunctions

\[\varphi_{n,i} = P_{n,i}(\tau; \mu, e_k), \; i = 1, \ldots, (n + 1), \; k = 1, 2, 3 .\]

These polynomials are called \( BC_1 \) Lamé polynomials of the second kind \[88\]. Those polynomials degenerate either to the Lamé polynomials of the second kind at \( \mu = 0 \), see \( (1.3.85-2) \), or to the Lamé polynomials of the third kind at \( \mu = 1 \), see \( (1.3.85-3) \), to the Lamé polynomials of the fourth kind at \( \mu = 1/2 \), see \( (1.3.85-4) \).

For example, for \( n = 0 \) at couplings \( (1.3.107) \),

\[E_{0,k} = \frac{(4\mu^2 - 1)}{2} e_k, \; P_{0,1} = 1 .\]

In general, for \( n > 0 \), the eigenvalues are branches of \((n + 1)\)-sheeted Riemann surfaces in \( g_2 \).

To summarize, it can be stated that for coupling constants \( (1.3.107) \) with integer \( n = 0, 1, 2, \ldots \), the Hamiltonian \( (1.3.84) \) has \((n + 1)\) eigenfunctions of the form

\[\Psi_{n,i,k} = P_{n,i}(\tau; \mu, e_k) \Psi_{0,k}, \; i = 1, \ldots, (n + 1), \; k = 1, 2, 3 , \]

where \( \Psi_{0,k} \) is given by \( (1.3.105) \).
It can be verified that the eigenvalue problem for the Hamiltonian (1.3.84) has one more exact solution, other than (1.3.95) or (1.3.105),

\[ \Psi_{0, \tilde{k}} = [\varphi'(x)]^\nu \left[ (\varphi(x) - e_i)(\varphi(x) - e_j) \right]^{1/2 - \nu}, \quad (1.3.111) \]

where \( \tilde{k} \) is complement to \((i, j)\), for coupling constants

\[ \kappa_2 = 2\nu(\nu - 1), \quad \kappa_3 = \nu(3 - 2\nu), \quad (1.3.112) \]

where \( \nu \) is an arbitrary parameter, for which the eigenvalue is

\[ E_{0, \tilde{k}} = \frac{(1 - 2\nu)(3 - 2\nu)}{2} e_{\tilde{k}}, \]

here \( e_{\tilde{k}} \) is the \( \tilde{k} \)th root of the Weierstrass function (1.3.82). It implies that for parameters (1.3.112) the Hamiltonian \( H_B \) has one-dimensional invariant subspace. Note that if in (1.3.111) we replace \( \nu = 1 - \mu \) we arrive at the function (1.3.105).

Let us parametrize the coupling constants \( \kappa_{2,3} \) as follows

\[ \kappa_2 = 2\nu(\nu - 1), \quad \kappa_3 = n(2n + 5) - 2\nu n + \nu(3 - 2\nu), \quad (1.3.113) \]

(cf. (1.3.97) and (1.3.107)), where \( \nu \) and \( n \) are parameters.

Making a gauge rotation of the Hamiltonian (1.3.84) with subtracted \( E_{0, \tilde{k}} \),

\[ h_{\tilde{k}} = -2(\Psi_{0, \tilde{k}})^{-1}(H_B - E_{0, \tilde{k}})\Psi_{0, \tilde{k}} \]

and changing variable \( z \) to \( \tau \), we arrive at the algebraic operator

\[ h_{\tilde{k}}(\tau) = \left[ (\tau - e_i)(\tau - e_j) \right]^{1/2 + \nu} \left[ h^0(\tau) - 2E_{0, \tilde{k}} \right] \left[ (\tau - e_i)(\tau - e_j) \right]^{1/2 - \nu} = \]

\[ (4\tau^3 - g_2\tau - g_3)\partial_\tau^2 + 2(7 - 2\nu)\tau^2 + 2(2\nu - 1)e_{\tilde{k}}(\tau + e_{\tilde{k}}) - (5 + 2\nu)\frac{g_2}{4}\partial_\tau - 2n(2n + 5 - 2\nu)\tau, \quad (1.3.114) \]

(cf. (1.3.98)), where \( e_{\tilde{k}} \) is the \( \tilde{k} \)th root of the Weierstrass function, see (1.3.82).

It can be verified that if the parameter \( n \) takes a non-negative integer value, the operator \( h_{\tilde{k}}(\tau) \) has the invariant subspace \( P_n \). Furthermore, the operator (1.3.114) can be rewritten in terms of \( \mathfrak{sl}(2, \mathbb{R}) \)-generators (1.2.3) for any value of \( n \), cf. (1.3.99),

\[ h_{\tilde{k}} = 4J^+(n)J^0(n) - g_2J^0(n)J^- - g_3J^+J^- + 2(4n + 5 - 2\nu)J^+(n) + 4(2\nu - 1)e_{\tilde{k}}(J^0(n) + n) \]

\[ + 2\left(2(2\nu - 1)e_{\tilde{k}}^2 - (2n + 5 + 2\nu)\frac{g_2}{4}\right)J^- . \quad (1.3.115) \]
Thus, it is $\text{sl}(2, \mathbb{R})$ quantum top in constant magnetic field. This representation holds for any value of the parameter $n$. Thus, the algebra $\text{sl}(2, \mathbb{R})$ is the hidden algebra of the $BC_1$ elliptic model with arbitrary coupling constants $\kappa_{2,3}$ parameterized via (1.3.113).

If $n$ takes a non-negative integer value, the hidden algebra $\text{sl}(2, \mathbb{R})$ (1.2.3) appears in a finite-dimensional representation, and the operator (1.3.114) has a finite-dimensional invariant subspace $\mathcal{P}_n$, which is the finite-dimensional representation space of the $\text{sl}(2, \mathbb{R})$-algebra. It possesses a number of polynomial eigenfunctions

$$\varphi_{n,i} = \tilde{P}_{n,i}(\tau; \nu, e_{\tilde{k}}), \quad i = 1, \ldots, (n + 1), \quad \tilde{k} = 1, 2, 3.$$  

These polynomials are called $BC_1$ Lamé polynomials of the third kind [88]. The polynomials degenerate either to the Lamé polynomials of the third kind at $\nu = 0$, see (1.3.85-3), or to the Lamé polynomials of the second kind at $\nu = 1$, see (1.3.85-2), or to the Lamé polynomials of the fourth kind at $\nu = 1/2$, see (1.3.85-4). It is important to mention that the $BC_1$ Lamé polynomials of the second and third kind are related,

$$P_{n,i}(\tau; \mu, e_{\tilde{k}}) = \tilde{P}_{n,i}(\tau; 1 - \mu, e_{\tilde{k}}).$$

For example, for $n = 0$ at couplings (1.3.107),

$$E_{0,\tilde{k}} = \frac{(1 - 2\nu)(3 - 2\nu)}{2} e_{\tilde{k}}, \quad P_{0,1} = 1.$$  

In general, for $n > 0$, the eigenvalues are branches of $(n + 1)$-sheeted Riemann surfaces in $g_2$.

To summarize, it can be stated that for coupling constants (1.3.107) at integer $n = 0, 1, 2, \ldots$, the Hamiltonian (1.3.84) has $(n + 1)$ eigenfunctions of the form

$$\Psi_{n,i,e_{\tilde{k}}} = \tilde{P}_{n,i}(\tau; \nu, e_{\tilde{k}}) \Psi_{0,e_{\tilde{k}}}, \quad i = 1, \ldots, (n + 1), \quad e_{\tilde{k}} = 1, 2, 3, \quad (1.3.116)$$

where $\Psi_{0,e_{\tilde{k}}}$ is given by (1.3.111).

The most general one-dimensional elliptic potential which appears in literature is a superposition of four Weierstrass functions,

$$V(z) = \mu_0 \wp(z) + \mu_1 \wp(z + \omega_1) + \mu_2 \wp(z + \omega_2) + \mu_3 \wp(z + \omega_3),$$

where $\omega_{1,2,3}$ are half-periods and $\mu_{0,1,2,3}$ are coupling constants. The quantum model characterized by this potential is called $BC_1$ elliptic Inozemtsev model [33]. This is a generalization of $BC_1$-quantum elliptic Calogero-Sutherland model. Quasi-exact-solvability of this model was shown in [24,66], see also
Since the analysis of this model is very much similar to one presented above, we will not describe this here referring the reader to the original papers.

### 3.8 Exactly-solvable operators and classical orthogonal polynomials

For any exactly-solvable operator of the second order \( E_2 \) \((1.2.36)\)

\[
E_2 = c_{00}J^0J^0 + c_{0-}J^0J^- + c_-J^-J^- + c_0J^0 + c_-J^- ,
\]

where

\[
J^0 = xd_x , \quad J^- = dx ,
\]

the \( J^{0,-} \) span the Borel subalgebra, \( b_2 \subset \mathfrak{sl}(2, \mathbb{R}) \)-generators, the spectra of polynomial eigenfunctions is the second degree polynomial in quantum number \( n \),

\[
\epsilon_n = c_{00}n^2 + c_0n , \quad n = 0, 1, 2 \ldots ,
\]

c.f. \((1.2.39)\). Here \( n \) is a degree of the \( n \)th polynomial eigenfunction. It can be shown that the Hermite, Laguerre, Legendre and Jacobi operators give rise to four families of classical orthogonal polynomials as eigenfunctions for exactly-solvable operators, respectively. Hence, the eigenvalue problem \((1.2.8)\) for exactly-solvable operator \( E_2 \) (see \((1.2.36)\)) leads to the equations having the Hermite, Laguerre, Legendre and Jacobi polynomials as eigenfunctions \([73, 71]\). This is shown below. In the definition of these polynomials we follow the handbook by Bateman–Erdélyi \([3]\) as well as by Koekoek–Swarttouw \([34]\).

#### 1. Hermite polynomials

The Hermite polynomials \( H_{2n+p}(x), n = 0, 1, 2, \ldots , p = 0, 1 \) are the polynomial eigenfunctions of the operator

\[
E^{(Hermite)}(x) = -d_x^2 + 2xd_x , \quad \epsilon_{2n+p} = 4n + 2p ,
\]

hence, linear in \( n \) spectra, c.f. \((1.3.117)\). They can be rewritten in terms of the generators \((1.2.3)\) following the Lemma 2.1

\[
E^{(Hermite)} = -J^-J^- + 2J^0 .
\]

Although the operator \((1.3.118)\) can be factorized as a differential operator,

\[
E^{(Hermite)}(x) = -d_x(d_x - 2x) ,
\]

it can not be factorized in the \( \mathfrak{sl}(2, \mathbb{R}) \)-algebra representation, for a discussion see above eqs.\((1.3.83)-(1.3.84)\).

Let us note that the parameter \( p \) is related to the parity operator, those eigenvalues are \( P = (-1)^p \). The polynomial \( H_{2n+p} \) is characterized by the
definite parity $P$ being either even, $P = +1$, or odd, $P = -1$ with respect to reflection, $x \rightarrow -x$. Thus, for a polynomial $H_{2n+p}(x)$ there exists a remarkable representation

$$H_{2n+p}(x) = x^p L_n^{(-1/2+p)}(x^2), \quad (1.3.120)$$

where $L_n^{(-1/2+p)}$ is the associated Laguerre polynomial (see below). Here, the factor $x^p$ encodes the information about what representation of reflection group we are considering, even or odd. The remaining factor is already reflection-invariant.

Making a gauge transformation $x^{-p}E^{(Hermite)}(x, d_x)x^p$ and then changing the variable $x^2 = y$, we arrive at the operator having the Laguerre polynomial $L_n^{(-1/2+p)}(y)$ as the eigenfunction

$$\bar{E}(y) = -4yd_y^2 + 2(2y - 1 - 2p)d_y + 2p, \quad (1.3.121)$$

where $p = 0, 1$. It can be again rewritten in terms of the generators (1.2.3) but in variable $y$,

$$\bar{E} = -4J^0 J^- + 4J^0 - 2(1 + 2p)J^- + 2p, \quad (1.3.122)$$

(cf. (1.3.119)). Of course, these two representations (1.3.119) and (1.3.122) are equivalent, however, a quasi-exactly-solvable generalization can be reached for the second representation only (see Cases VI and VII in Section 3.1). If the operator (1.3.119) possesses reflection symmetry $x \rightarrow -x$, the operator (1.3.122) has no symmetries. Thus, the symmetry of the original operator $E^{(Hermite)}(1.3.118)$ is hidden in the variable $y$. This property is quite general – usually, a quasi-exactly solvable generalization exists for operators without a discrete symmetry.

Note that $\bar{E}(y)$ admits factorization as a differential operator

$$-4yd_y^2 + 2(2y - 1 - 2p)d_y = -(4yd_y - 2(2y - 1 - 2p))d_y,$$

while $(\bar{E} + 2(1 + p))$ admits factorization in the algebra $sl(2, \mathbb{R})$,

$$-4J^0 J^- + 4J^0 - 2(1 + 2p)J^- + 2(1 + 2p) = -2(2J^0 + (1 + 2p))(J^- - 1).$$

2. Laguerre polynomials

The associated Laguerre polynomials $L_n^{(a)}(x)$ occur as the polynomial eigenfunctions of the generalized Laguerre operator

$$E^{(Laguerre)}(x) = -xd_x^2 + (x - a - 1)d_x, \quad \epsilon_n = n, \quad (1.3.123)$$

where $a$ is any real, hence, with linear in $n$ spectra, c.f. (1.3.117). Of course, the operator (1.3.123) can be rewritten as

$$E^{(Laguerre)} = -J^0 J^- + J^0 - (a + 1)J^- . \quad (1.3.124)$$
If \( a = -1/2 + p \) the operator (1.3.124) coincides with \( \bar{E} \) in (1.3.121). Evidently, the operator \( E^{(\text{Laguerre})} \) admits factorization as differential operator as well as in the algebra \( \mathfrak{sl}(2, \mathbb{R}) \),
\[
E^{(\text{Laguerre})} = -J^0 J^- + J^0 - (a + 1) J^- + (a + 1) = -(J^0 + (a + 1))(J^- - 1) .
\]

(1.3.125)

3. Legendre polynomials

The Legendre polynomials \( P_{2n+p}(x) \), \( n = 0, 1, 2, \ldots, p = 0, 1 \) are the polynomial eigenfunctions of the operator
\[
E^{(\text{Legendre})}(x, d_x) = (1 - x^2)d_x^2 - 2xd_x , \quad \epsilon_{2n+p} = -(2n + p)(2n + p + 1) ,
\]

or, in \( \mathfrak{sl}(2, \mathbb{R}) \)-Lie-algebraic form
\[
E^{(\text{Legendre})} = -J^0 J^0 + J^- J^- - J^0 .
\]

(1.3.126)

Analogously to the Hermite polynomials, the Legendre polynomials possess the definite parity and can be represented as
\[
P_{2n+p}(x) = x^p P_n^{(-1/2+p,0)}(x^2) ,
\]

(1.3.128)

where \( P_n^{(a,b)} \) is the Jacobi polynomial (see below). Making a gauge transformation \( x^p E^{(\text{Legendre})}(x, d_x) \) and then changing the variable \( x^2 = y \), we arrive at the operator having the Jacobi polynomial \( P_n^{(-1/2+p,0)}(y) \) as the eigenfunction
\[
E^{(\text{Legendre})}(y) = 4y(1 - y)d_y^2 + 2[1 + 2p - (3 + 2p)y]d_y , \quad p = 0, 1 ,
\]

(1.3.129)

and, correspondingly,
\[
E^{(\text{Legendre})} = -4J^+ J^- + 4J^0 J^- - 2(3 + 2p)J^0 + 2(1 + 2p)J^- .
\]

(1.3.130)

The operator \( \bar{E} \) admits factorization as a differential operator as well as in the algebra \( \mathfrak{sl}(2, \mathbb{R}) \),
\[
\bar{E}^{(\text{Legendre})} = -4J^0 J^0 + 4J^0 J^- - 2(1 + 2p)J^0 + 2(1 + 2p)J^- =
-2(2J^0 + (1 + 2p))(J^0 - J^-) .
\]

(1.3.131)

4. Jacobi polynomials

The Jacobi polynomials \( P_n^{(a,b)} \) appear as the polynomial eigenfunctions of the Jacobi equation taken either in the symmetric form corresponding to the operator
\[
E^{(\text{Jacobi})}(x, d_x) = (1 - x^2)d_x^2 + [b - a - (a + b + 2)x]d_x , \quad \epsilon_n = -n(n+a+b+1) ,
\]

(1.3.132)
with the $\mathfrak{sl}(2, \mathbb{R})$-Lie-algebraic representation

\[ E^{(Jacobi)} = -J^0 J^0 + J^- J^- - (1 + a + b)J^0 + (b - a)J^- , \quad (1.3.133) \]

or in the asymmetric form (see e.g. the book by Murphy [46] or by Bateman–Erdélyi [3])

\[ E^{(Jacobi)}(x, dx) = x(1 - x)dx^2 + [1 + a - (a + b + 2)x]dx , \quad (1.3.134) \]

corresponding to

\[ \tilde{E}^{(Jacobi)} = -J^0 J^0 + J^0 J^- - (1 + a + b)J^0 + (a + 1)J^- . \quad (1.3.135) \]

It is not surprising that under a special choice of a general polynomial element of the universal enveloping algebra of Borel subalgebra $\mathfrak{b}_2 \subset \mathfrak{sl}(2, \mathbb{C})$ one can find Lie-algebraic forms of all known fourth-, sixth-, and eighth-order differential equations giving rise to infinite sequences of orthogonal polynomials (see e.g. the paper by Littlejohn [40] and other papers in this volume).

In [25] it was given the complete description of the second-order polynomial elements of $\mathcal{U}_{\mathfrak{sl}_2(\mathbb{R})}$ (1.2.24) at $c_{++} = c_{+0} = 0$ in the representation (1.2.3) leading after transformation (1.3.2)-(1.3.3) to the square-integrable eigenfunctions of the Sturm-Liouville problem (1.2.39). Similar classification for non-vanishing $c_{++} \neq 0$ and/or $c_{+0} \neq 0$ is still missing.

Consequently, for second-order ordinary differential equation (1.2.37) a combination of Theorems 2.1, 2.2 leads to the statement that a general solution of the problem of classification of equations, possessing the finite number of orthogonal polynomial solutions, is related to a chance to rewrite these equations in terms of $\mathfrak{sl}_2(\mathbb{R})$-generators in finite-dimensional representation.
Chapter 2.

Quasi-Exactly-Solvable Anharmonic Oscillator

In this Chapter we present a detailed description of the unique polynomial one-dimensional potential, which has the property of quasi-exact-solvability, see Case VI in Chapter 1:

\[ V(x) = a^2 x^6 + 2abx^4 + [b^2 - a(4n + 2k + 3)]x^2 - b(1 + 2k) , \quad (2.1) \]

where the parameter \( k = 0,1 \) is introduced for convenience, see below. If \( a, b \) and \( n \) are real parameters we have the general sextic, even polynomial potential with infinite discrete spectra. The property of the Quasi-Exact-Solvability (QES) takes place when two conditions are fulfilled: \( n \) is non-negative integer and \( a \) is a non-negative number, \( a \geq 0 \). In this case \((n + 1)\) eigenstates of parity \((-1)^k\) with square-integrable eigenfunctions can be found algebraically; we say that those states form the algebraic sector of eigenstates or, saying differently, those eigenstates are algebraic. In \( b \)-space the algebraic eigenvalues (eigenfunctions) form \((n + 1)\) sheeted-Riemann surface with the Landau-Zener square-root branch points. These branch points have the meaning of the points of level crossing. If \( a < 0 \), the algebraic eigenstates are absent, although the Schrödinger equation has \((n + 1)\) algebraic, non-square-integrable eigenfunctions.

A quantum system which is described by this potential can be called the anharmonic sextic quasi-exactly-solvable oscillator. For simplicity we call it the sextic QES oscillator. Since the potential \((2.1)\) is even, \( V(-x) = V(x) \), the eigenfunctions are characterized by the definite parity: they are either even or odd with respect to reflection \((x \rightarrow -x)\): \( \Psi(-x) = \pm \Psi(x) \). The parameter \( k \) takes value \( k = 0 \) for even eigenstates and \( k = 1 \) for odd eigenstates, correspondingly. From physical point of view the potential \((2.1)\) with the QES property does not seem very much different from generic sextic, even potential. For arbitrary \( n \) in \((2.1)\) the even (odd) eigenvalues in \( b \)-space are branches of infinitely-sheeted Riemann surface with square-root branch points (for a discussion see \([6,7,71,72,18]\)). If \( n \) takes an integer value, \((n + 1)\)-sheeted Riemann surface in \( b \)-space of even (odd) eigenstates splits away from the infinite-sheeted Riemann surface. It implies that zeroing of the residues of the branch points connecting the sheets of the \((n + 1)\)-sheeted Riemann surface with ones from the infinite-sheeted Riemann surface. Usually, there are infinitely-many such branch points. Thus, if \( n \) takes an integer value, infinitely-many conditions are fulfilled! In the same time for arbitrary \( n \) any eigenfunction is an entire function in \( x \). It has infinitely-many simple zeroes at complex \( x \) and finitely-many at real \( x \) (for a discussion see \([19]\)). If \( n \) takes an integer value,
for any algebraic eigenfunction infinitely-many zeroes disappear and the \((n+1)\) simple zeroes remain. Classically, it is not seen anything specific when \(n\) takes an integer value (for a discussion see \([17,12,19]\)).

The QES property gives us a unique chance to see exact solutions of non-trivial quantum problems. One of the important features of these solutions is a possibility to test results obtained in some approximation methods. To the best of our knowledge the sextic potential \((2.1)\) with the QES property had appeared in an explicit form for the first time in the paper by Peter Leach [39]. It is worth mentioning that there is no physical system with the QES property among the most popular quartic anharmonic oscillators

\[
V(x) = m^2 x^2 + gx^4,
\]

for a discussion see [4]. Hence, the limit \(a\) tends to zero in \((2.1)\) leads to destruction of the QES property. In the presentation of \((2.1)\) we mainly follow [90,5,75].

### 2.1 QES sextic potential

The first \((n + 1)\) even or odd eigenfunctions in \((2.1)\) are of the form

\[
\Psi_{n,i}^{(\text{QES})} = x^k P_n(x^2) e^{-\frac{b}{2} x^2 - \frac{a}{4} x^4}, \quad i = 0, 1, \ldots n, \tag{2.2}
\]

where \(P_n\) is a polynomial of the degree \(n\), it is an element of \((n+1)\)-dimensional representation space of the \(sl(2)\)-algebra. These eigenfunctions and corresponding eigenvalues can be found algebraically, by solving a system of \((n+1)\) linear homogeneous equations. Remaining eigenfunctions can not be written in a form of polynomial multiplied by the exponential, they are of non-algebraic nature. They are elements of infinite-dimensional representation space of the \(sl(2, \mathbb{R})\)-algebra.

A pattern of the potential curve \((2.1)\) depends on a relation between parameters \(a, b\) and corresponds to one-, two- and three-minima one (for illustration, see Fig. 2.1-2.3), respectively. It is worth mentioning that at \(b = 0\) and \(n = k = 0\) the potential \((2.1)\) looks like a standard double-well potential but the ground state eigenfunction is characterized by a single peak (!). This peak appears at a position of unstable equilibrium (see Fig. 2.2). It looks as a striking contradiction to a straightforward (naive) intuition based on (semi)classical picture. Similar contradiction occurs for \(b = -3\) (see Fig. 2.3): a triple-well potential with the two-peaked ground state eigenfunction. These results demonstrate that a semi-classical treatment of these cases is not yet applicable: the corresponding wells are not deep enough and the barriers are not large enough.
Let us consider the Hamiltonian corresponding to the potential (2.1)

\[ \mathcal{H} = -\frac{d^2}{dx^2} + a^2 x^6 + 2abx^4 + [b^2 - a(4n + 2k + 3)]x^2 - b(1 + 2k) , \]  

(2.3)

make a gauge rotation with a change of variable,

\[ h = (\Psi_0^{(qes)})^{-1} H \Psi_0^{(qes)} \big|_{y=x^2} \]

\[ = -4y \frac{d^2}{dy^2} + 2(2ay^2 + by - 1 - 2k) \frac{d}{dy} - 4any . \]  

(2.4)

The first \((n + 1)\)-eigenfunctions of the eigenvalue problem for the operator \(h\),

\[ h \varphi = \epsilon \varphi , \]  

(2.5)

are polynomials of degree \(n\): \(\varphi = P_n(y)\) (cf. (2.2)).

The gauge rotated Hamiltonian \(h\) (2.4) can be immediately rewritten as a quadratic combination in the \(\mathfrak{sl}(2, R)\)-generators (1.2.3) written in variable \(y\),

\[ J_n^+ = y^2 \frac{d}{dy} - ny , \]

\[ J_n^0 = y \frac{d}{dy} - \frac{n}{2} , \]

\[ J_n^- = \frac{d}{dy} , \]

as follows [72]

\[ h = -2\{J_n^0, J_n^-\} + 4aJ_n^+ + 4bJ_n^0 - 2(n + 2 + 2k)J_n^- + 2bn . \]  

(2.6)

This representation immediately reveals the meaning of the parameter \(n\): \(j = \frac{n}{2}\) is the spin of the representation (1.2.3) of the algebra \(\mathfrak{sl}(2, R)\). It shows that the one-dimensional quantum dynamics in a generic sextic, even potential is equivalent a quantum top with spin \(j = \frac{n}{2}\) in a constant magnetic field with a constraint

\[ \{J_n^+, J_n^-\} - 2J_n^0J_n^0 + n \left(\frac{n}{2} + 1\right) = 0 , \]

cf. (1.2.4), where \(\{,\}\) denotes anti-commutator. If the spin \(j\) is integer or half-integer, it occurs the irreducible finite-dimensional representation. In this case the generators (1.2.3) have a common invariant subspace \(P_n\) (1.1.5), which is a representation space of the finite-dimensional representation. Correspondingly, the operator (2.4) has this space as a finite-dimensional invariant subspace. Certainly, the number of those algebraic eigenfunctions (whose can be found algebraically) is nothing but the dimension of the irreducible finite-dimensional representation of the algebra (1.2.3).
Fig. 2.1. The potential (2.1) at $a = 1$, $n = k = 0$ and $b = 3$ and the ground state eigenfunction $\Psi = \Psi_0$ with zero eigenvalue, $E_0 = 0$.

Fig. 2.2. The potential (2.1) at $a = 1$, $n = k = 0$ and $b = 0$ and the ground state eigenfunction $\Psi = \Psi_0$ with zero eigenvalue, $E_0 = 0$.

Fig. 2.3. The potential (2.1) at $a = 1$, $n = k = 0$ and $b = -3$ and the ground state eigenfunction $\Psi = \Psi_0$ with zero eigenvalue, $E_0 = 0$.

One can find explicitly the algebraic eigenstates of (2.3) for $n = 0, 1$ at $a > 0$. 

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\* \* \* 

\* \( n = 0 \) 

\[ V(x) = a^2 x^6 + 2abx^4 + [b^2 - a(2k + 3)]x^2 - b(1 + 2k), \quad (2.8.1) \]

\[ E_0^{(k)} = 0, \quad \varphi_0^{(k)} = 1, \]

\[ \Psi_0^{(k)} = x^k e^{-\frac{b^2}{2} - \frac{a}{2}x^4}, \]

where the algebraic states are the ground state of the positive parity \((k = 0)\) and of the negative parity \((k = 1)\).

\* \( n = 1 \) 

\[ V(x) = a^2 x^6 + 2abx^4 + [b^2 - a(2k + 7)]x^2 - b(1 + 2k). \quad (2.8.2) \]

\[ E_\pm^{(k)} = 4(1 + k)b \pm 2 \left( b^2 + 2(1 + 2k)a \right)^{1/2}, \]

\[ \varphi_\pm^{(k)} = x^k \left( 2ax^2 + b \mp \left( b^2 + 2(1 + 2k)a \right)^{1/2} \right), \]

\[ \Psi_\pm^{(k)} = x^k \left( 2ax^2 + b \mp \left( b^2 + 2(1 + 2k)a \right)^{1/2} \right) e^{-\frac{b^2}{2} - \frac{a}{2}x^4}, \]

where the algebraic states are the ground/second-excited states of the positive parity \((k = 0)\) and of the negative parity \((k = 1)\).

\section*{2.2 “Phase transitions” in sextic QES potential}

Let us consider the harmonic oscillator

\[ V(x) = b^2 x^2, \quad x \in \mathbb{R}. \]

For any real \( b \neq 0 \) it has the infinite discrete spectra, which has a certain non-analytic behavior at \( b = 0 \) as a function of \( b \). For instance, the ground state is given by 

\[ E_0 = |b|, \quad \Psi_0 = e^{-|b| x^2}, \]

see Fig.2.4. The ground state energy as a function of \( b \) is continuous, but its derivative \( \frac{dE_0}{db} \) has a discontinuity at \( b = 0 \): it jumps from +1 at positive \( b \) to −1 at negative \( b \). Similar behavior occurs for any energy level. Of course, if the harmonic oscillator is placed in a box of the size \( 2L, \quad x \in [-L, L] \), such a non-analytic behavior disappears. Seemingly, at large \( L \) the ground state energy behaves like

\[ E_0 \sim \left( b^2 + \frac{\alpha}{L^4} \right)^{1/2}, \]
Fig. 2.4. Harmonic oscillator: the ground state energy $E_0$ vs. $b$.

where $\alpha$ is a positive number. Thus, we have two symmetric square-root branch points on the imaginary axis with branch cuts tending to $\pm i\infty$, respectively. In the limit $L \to \infty$ both branch points coincide and the non-analytic function $|b|$ occurs. It resembles a behavior typical for a second-order phase transition.

Now we turn to the sextic QES potential at $n = 0$ (2.8.1) and $k = 0$ (cf. (2.8.1)),
\[
V_0(x; a, b) = a^2 x^6 + 2ab x^4 + (b^2 - 3a) x^2 - b ,
\]
for which the ground state is known exactly if $a > 0$,
\[
E_0(a, b) = 0 ,
\]
and
\[
\Psi_0(x) = e^{-\frac{b}{2} x^2 - \frac{a}{4} x^4} .
\]
It can be easily shown that for $a < 0$ the lowest eigenvalue can not be equal to zero, $E_0 \neq 0$. In order to see it let us take $\Psi_0(x)$ which is a solution of the homogeneous Schrödinger equation with the potential $V_0(x; a, b)$ for any value $a$. Using the fact that the Wronskian should be a constant, one can construct the second, linearly-independent solution
\[
\Psi_1(x) = \Psi_0(x) \int_{-\infty}^{x} \Psi_0^{-2}(x')dx' ,
\]
and then consider a linear combination
\[
c_0 \Psi_0(x) + c_1 \Psi_1(x) .
\]
In order to fulfill the boundary conditions we are looking for a square-integrable solution of the Schrödinger equation. If $a > 0$ it corresponds to $c_1 = 0$. It can
be immediately seen that for $a < 0$, it does not exist $c_0, c_1$ for which a linear combination of $\Psi_{0,1}$ is square-integrable. A simple analysis shows that for $a < 0$, the lowest eigenvalue must be positive, $E_0 > 0$. It implies a non-analytic behavior at $a = 0$. It turns out that this behavior depends on sign of $b$.

It can be derived \[5,75] that at $a \to -\infty$ the ground state energy behaves asymptotically like

$$E_0 = 1.93556 |a|^{1/2} + \ldots$$

independently on the value of $b$. This result is exact for $b = 0$.

The QES potential (2.10) can be studied perturbatively, writing it as

$$V_0(x; a, b) = V_0 + aV_p + a^2V_{pp} = (b^2 x^2 - b) + a(2bx^4 - 3x^2) + a^2 x^6,$$

where $a$ is perturbation (formal) parameter, assuming that $b > 0$. It is evident that for the ground state energy

$$E_0 = \sum_{i=0}^{\infty} e_i(b) a^i,$$  \hspace{1cm} (2.11)

all perturbative coefficients vanish

$$e_i(b) = 0,$$

(see \[30] and also \[75]) \[14\]. In turn, the ground state eigenfunction has a non-trivial perturbative expansion,

$$\Psi_0(x) = e^{-\frac{1}{2}x^2} \left( \sum_{i=0}^{\infty} \frac{(-)^i}{i!} a^i x^{4i} \right).$$

It is evident that for $a > 0$ the sum (2.11) is equal to zero, $E_0 = 0$. On the other hand, we showed that for $a < 0$, the ground state energy is non-zero. Hence, the sum (2.11) is not equal to zero, $E_0 \neq 0$. It leads to a conclusion about the existence of exponentially-small terms at $a \to 0-$ for $b > 0$.

\[14\] This remarkable observation implies the existence of non-trivial identities involving matrix elements of $x^2, x^4$ and $x^6$, if the coefficients $e_i$ are treated as the coefficients in the Rayleigh-Schrödinger perturbation theory. On the other hand, the coefficients $e_i$ can be calculated using the vacuum Feynman diagrams \[6\]: the vanishing of these coefficients implies the existence of non-trivial relations between Feynman integrals. The present author is not aware that those identities and relations are profoundly investigated somewhere. A similar perturbation theory study can be carried out for other $n \neq 0$ sextic QES potentials for different algebraic states. The coefficients $e_i$ are always rational numbers. It is quite evident that one can find many potentials for which a perturbative theory analysis leads to similar conclusions. Perhaps, the most interesting class of such potentials is related to polynomial perturbations of the two-body Coulomb problem.
A simple analysis shows that there are three types of non-analytic behavior of the ground state energy at $a = 0$. If $b < 0$ the ground state energy is discontinuous at $a = 0$: it jumps from $E_0 = 0$ at $a \to 0^+$ to $E_0 = |b|$ at $a \to 0^-$. It looks like the 1st order phase transition. If $b = 0$ the ground state energy is continuous at $a = 0$ but the first derivative (and all others) is discontinuous: it jumps from $\frac{dE_0}{da} = 0$ at $a \to 0^+$ to $\frac{dE_0}{da} = \infty$ at $a \to 0^-$. It looks like the 2nd order phase transition. If $b > 0$ the ground state energy is continuous at $a = 0$ as well as all others being equal to zero. It looks like the infinite-order phase transition (the Kosterlitz-Thouless type phase transition). On Figs. 2.5, 2.6, 2.7 the behavior of the ground state energy in the potential $V_0$ vs. $a$ is illustrated for positive $b$, $b = 0$ and negative $b$, respectively. It is evident that similar discontinuities of the energy vs. $a$ at $a = 0$ will occur for any eigenstate in the QES problem (2.1).

Of course, if the sextic QES oscillator is placed in the box of the size $2L$, $x \in [-L, L]$, such a non-analytic behavior disappears: all curves on Figs. 2.5, 2.6, 2.7 begin to behave smoothly around $a = 0$. 

Fig. 2.5. Sextic QES oscillator at $n = 0$: the ground state energy $E_0$ vs. $a$ for positive $b$.

Fig. 2.6. Sextic QES oscillator at $n = 0$: the ground state energy $E_0$ vs. $a$ for $b = 0$. 

Fig. 2.7. Sextic QES oscillator at $n = 0$: the ground state energy $E_0$ vs. $a$ for negative $b$. 

Fig. 2.8. Sextic QES oscillator at $n = 0$: the ground state energy $E_0$ vs. $a$ for $b > 0$. 

Of course, if the sextic QES oscillator is placed in the box of the size $2L$, $x \in [-L, L]$, such a non-analytic behavior disappears: all curves on Figs. 2.5, 2.6, 2.7 begin to behave smoothly around $a = 0$. 

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Fig. 2.7. Sextic QES oscillator at $n = 0$: the ground state energy $E_0$ vs. $a$ for negative $b$. 
Chapter 3.

Algebraic Perturbations of Exactly-Solvable Problems

Perturbation theory is one of the most developed and powerful approaches in quantum mechanics. This approach is quite universal and can be applied, sometimes easily, to practically any problem. Many realistic problems of quantum mechanics can be naturally considered as perturbations of exactly-solvable problems. It explains the theoretical and practical importance of perturbative studies in quantum mechanics. In practice, the perturbation theory is usually employed in Rayleigh-Schrödinger form where construction of perturbative corrections is reduced to a calculation of matrix elements, given by certain integrals, and summation over intermediate states. However, there exists a large family of perturbations of exactly-solvable problems, where the perturbative corrections can be found by linear algebra means. We call such perturbations algebraic. The Lie-algebraic formalism described in Chapter 1 allows us to give a partial classification of the algebraic perturbations. In particular, it sheds light to a reason why a perturbation theory in powers $g$ for one-dimensional quartic anharmonic oscillator

$$V(x) = m^2 x^2 + g x^{2n}, \quad n = 2, 3, \ldots$$

developed in [6,7] is constructed by algebraic means through solving the recurrence relations.

In Chapter 1 a classification of the one-dimensional exactly-solvable linear differential operators was given. In particular, it was shown that the most general exactly-solvable differential operator of the second order is given by a general second-order element of the universal enveloping algebra of the Borel sub-algebra $b_2$ of the algebra $\mathfrak{sl}(2, \mathbb{R})$ of the differential operators of the first order,

$$E_2 = c_{00} J^0 J^0 + c_{0} J^0 J^- + c_{-} J^- J^- + c_{0} J^0 + c_{-} J^- + c,$$

with the number of free parameters equal to $\text{par}(E_2) = 6$. Here

$$J^0 \equiv J^0_0 = x \frac{d}{dx},$$

$$J^- \equiv J^-_0 = \frac{d}{dx},$$

(cf. (1.2.3)). Substituting $J^0, J^-$ into (3.2) we obtain the hypergeometrical operator

$$E_2(x) = -Q_2(x) \frac{d^2}{dx^2} + Q_1(x) \frac{d}{dx} + Q_0(x),$$
where the $Q_j(x)$ are polynomials of $j$th order

$$-Q_2(x) = c_{00} x^2 + c_{0-} x + c_-,$$

$$Q_1(x) = (c_{00} + c_0) x + c_-,$$

$$Q_0(x) = c.$$  \hspace{1cm} (3.4)

The parameter $c$ defines the reference point for energy and without loss of generality it can vanish, $c = 0$. For a sake of simplicity, we denote $c_{00} + c_0 = \tilde{c}_0$. It always can be chosen the reference point for the coordinate $x$ in such a way that $Q_2(0) = 0$. This, without a loss of generality the parameter $c_- = 0$.

Let us consider the spectral problem for the perturbed exactly-solvable operator $E_2$,

$$h = E_2(x) + gV_p(x),$$  \hspace{1cm} (3.5)

where $V_p(x)$ is a perturbation potential and $g$ is a perturbation parameter (the coupling constant),

$$h\varphi = \varepsilon \varphi.$$  \hspace{1cm} (3.6)

We develop perturbation theory for the equation (3.6) in powers of $g$ by following a standard procedure,

$$\varphi = \sum_{k=0}^{\infty} \varphi_k g^k, \quad \varepsilon = \sum_{k=0}^{\infty} \varepsilon_k g^k,$$  \hspace{1cm} (3.7)

assuming a solution of unperturbed problem

$$h_0 \varphi_0 = \varepsilon_0 \varphi_0,$$

is known. It seems clear that is nothing but the so-called Dalgarno-Lewis form of perturbation theory in quantum mechanics \[13\]. It is easy to derive an equation for the $k$th correction

$$(E_2 - \varepsilon_0) \varphi_k = \sum_{i=1}^{k} \varepsilon_i \varphi_{k-i} - V_p \varphi_{k-1},$$  \hspace{1cm} (3.8)

where the solution $\varphi_k(x)$ is looked for in the form of a polynomial. This requirement plays a role of a boundary condition. The important feature of this perturbation theory is related to a fact that perturbation corrections can be calculated for each eigenstate separately. It differs from a standard Rayleigh-Schrödinger perturbation theory, which is widely accepted, where the correction to eigenfunction is given by an expansion of the eigenfunctions of the unperturbed problem (see e.g. \[37\] and a discussion below).

In \[85\] it was proved a general theorem which states that if

*(i) unperturbed problem can be written as an element of universal enveloping algebra of an algebra of differential operators which preserves an infinite flag of finite-dimensional representation spaces and,*

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(ii) perturbation is an element of a finite-dimensional invariant subspace(s) in the flag, then, the perturbation theory is algebraic. It implies that any perturbation correction to the eigenfunction is an element of a finite-dimensional invariant subspace(s) in the flag. It can be found by linear algebra means.

Following this Theorem it is evident if \( h \) is exactly-solvable operator associated with the algebra \( \mathfrak{sl}(2, \mathbb{R}) \) and perturbation potential \( V_p \) is a polynomial of finite degree in \( x \), hence, it belongs a finite-dimensional invariant subspace of the algebra \( \mathfrak{sl}(2, \mathbb{R}) \), the perturbation theory \((3.7)\) should be algebraic. Thus, all perturbation corrections \( \varphi_k(x) \), which are found by means of solving \((3.8)\), are polynomials in \( x \) of a finite degree. Hence, the construction of perturbation theory is a linear algebraic procedure. In particular, all energy corrections should be rational functions of parameters of \( E_2 \) (see \((3.4)\)).

As an illustration let us consider a simplest non-trivial perturbation

\[
V_p = x ,
\]

of the operator

\[
h_0 \equiv E_2 = -(c_{00} x^2 + c_{0-} x) \frac{d^2}{dx^2} + (\bar{c}_0 x + c_-) \frac{d}{dx} ,
\]

cf. \((3.5)\), at \( c_{00} \neq 0 \), with spectra of polynomial eigenfunctions of the form

\[
\varepsilon^{(n)}_0 = -c_{00} n(n-1) + \bar{c}_0 n , \quad n = 0, 1, 2, \ldots .
\]

We focus on the ground state. It can be seen immediately, that the lowest eigenstate of the exactly-solvable operator \( E_2 \) is

\[
\varepsilon_0 = 0 , \quad \varphi_0 = 1 .
\]

It is a straightforward simple calculation by using \((3.8)\) to find the first several corrections, for example,

\[
\varepsilon_1 = -\frac{c_-}{\bar{c}_0} , \quad \varphi_1 = -\frac{x}{\bar{c}_0} ,
\]

and

\[
\varepsilon_2 = -\frac{c_-}{\bar{c}_0^2} \frac{c_0 - \bar{c}_0 - c_{00} c_-}{-c_{00} + \bar{c}_0} ,
\]

\footnote{At \( c_{00} = 0 \) the perturbed operator \((3.5)\) can be transformed to \( E_2 \) via canonical transformation \( \partial_x \to \partial_x + \alpha, x \to x \) with a certain parameter \( \alpha \).}
\[ \varphi_2 = \frac{x^2}{2c_0(-c_0 + \tilde{c}_0)} + \frac{x}{\tilde{c}_0} \frac{c_0\tilde{c}_0 - c_0c}{-c_0 + \tilde{c}_0}. \]

In general, the \( n \)th correction \( \varphi_n \) has the form of the \( n \)th degree polynomial without a constant term.

In a straightforward way the spectral problem for the operator (3.5) can be "lifted" to the Fock space,

\[ (E_2(b, a) + gV_p(b))\phi(b)|0> = \varepsilon\phi(b)|0>, \]

where the boundary conditions are converted to the search for polynomial \( \phi(b) \), here \( \varepsilon \) is the spectral parameter. Theorem 35 (see above) can be modified accordingly - in the case of a polynomial perturbation \( V_p(b) \) the corrections \( \phi_n(b) \) are polynomials, they can be found by algebraic means. Furthermore, these corrections \( \phi_n(b) \) remain the same as in the \( x \)-space as well as the corrections to energy \( \varepsilon_n \). Taking the realization of \( b, a \) in finite-difference operators we arrive at the spectral problem for a finite-difference operator, where the exactly-solvable finite-difference operator is perturbed by a finite-difference operator(!). In this case the perturbative corrections \( \varepsilon_n \) remain the same as in continuous case, the corrections \( \varphi_n = \phi_n(b)|0 > \) remain polynomials in \( x \) with coefficients changed accordingly. The case of the perturbed harmonic oscillator (3.1) (continuous, on uniform lattice and on exponential lattice) was studied in some details in 83,84.
Chapter 4.

Quasi-Exactly-Solvable finite-difference equations in one variable

4.1 Finite-difference equations in one variable (exponential lattice)

4.1.1 General consideration

Let us define a multiplicative finite-difference operator (derivative), or a shift operator, or the so-called Jackson symbol (derivative) (see e.g. Exton [20], Gasper and Rahman [23]),

\[ D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \]  

(4.1)

where \( q \in \mathbb{R} \) and \( f(x) \) is real function \( x \in \mathbb{R} \). All these names are used in literature for (4.1). The Leibnitz rule for the operator \( D_q \) is

\[ D_q(f(x)g(x)) = (D_q f(x))g(x) + f(qx)(D_q g(x)) , \]

it connects two neighbouring points in the lattice space. It is easy to check that

\[ D_q x^n = \frac{1 - q^n}{1 - q} x^n \equiv \{n\}_q x^n , \]  

(4.2)

where \( \{n\}_q \) is the so called \( q \)-number \( n \), at \( q \to 1, \{n\}_q \to n \). Finite-difference derivative \( D_q \) \( q \)-commutes with coordinate,

\[ [x, D_q]_q \equiv xD_q - qD_qx = 1 , \]  

(4.3)

where \( [A,B]_q \equiv AB - qBA \) sometimes called \( q \)-commutator, following David Fairlie suggestion. The expression (4.3) together with \( [1,x] = [1,D_q] = 0 \) defines the \( q \)-deformed Heisenberg algebra. This algebra like non-deformed one naturally acts in the space of monomials mapping monomial to monomial.

Now one can easily introduce a finite-difference analogue of the \( \mathfrak{sl}(2,\mathbb{R}) \)-algebra of the differential operators (4.2.3), where the operator \( D_q \) replaces the continuous derivative, see e.g. [49],

\[ J^+_n = x^2 D_q - \{n\}_q x , \]
\[ J^0_n = xD_q - \hat{n} , \]
\[ J^-_n = D_q , \]  

(4.4)
where \( \hat{n} \equiv \frac{\{n\}_{q}^{2n+1}}{\{2n+2\}_{q}} \). The operators (4.4) after multiplication by some factors

\[
\tilde{j}^0 = \frac{q^{-n}}{q+1} \{2n+2\}_{q} \tilde{j}^0_n , \quad \tilde{j}^\pm = q^{-n/2} \tilde{j}^\pm_n ,
\]

see [49], span a quantum algebra \( \mathfrak{sl}(2, \mathbb{R})_q \) with the following commutation (quommutation) relations

\[
q^2 \tilde{j}^0 \tilde{j}^- - \tilde{j}^- \tilde{j}^0 = -(q+1) \tilde{j}^0 , \quad q^2 \tilde{j}^+ \tilde{j}^- - \tilde{j}^- \tilde{j}^+ = -(q+1) \tilde{j}^0 , \quad (4.5)
\]

The parameter \( q \) does characterize the deformation of the commutators of the classical Lie algebra \( \mathfrak{sl}(2, \mathbb{R}) \). If \( q \to 1 \), the commutation relations (4.5) reduce to the standard \( \mathfrak{sl}(2, \mathbb{R}) \)-algebra ones. A remarkable property of generators (4.4) is that, if \( n \) is a non-negative integer, they form the finite-dimensional representation corresponding to the finite-dimensional representation space \( \mathcal{P}_{n+1} \) (1.1.3),

\[
\tilde{j}^{\pm, 0}_n : \mathcal{P}_{n+1} \to \mathcal{P}_{n+1} ,
\]

the same as of the non-deformed \( \mathfrak{sl}(2, \mathbb{R}) \) (see (1.2.3)). Note that, in general, for complex \( q \) other than prime root of unity, \( |q| \neq 1 \), this representation is irreducible.

Comment 4.1 The algebra (4.5) is known in the literature as the second Witten quantum deformation of the \( \mathfrak{sl}_2 \)-algebra, in the classification by C. Zachos [99].

Similarly as for differential operators one can introduce quasi-exactly-solvable \( \tilde{T}_k(x, D_q) \) and exactly-solvable \( \tilde{E}_k(x, D_q) \) finite-difference operators.

Lemma 4.1.1 [78]

(i) Suppose \( n > (k - 1) \). Any quasi-exactly-solvable operator \( \tilde{T}_k \), can be represented by a \( k \)th degree polynomial of the operators (4.4). If \( n \leq (k - 1) \), the part of the quasi-exactly-solvable operator \( \tilde{T}_k \) containing derivatives up to order \( n \) can be represented by a \( n \)th degree polynomial in the generators (4.4).

(ii) Conversely, any polynomial in (4.4) is quasi-exactly solvable.

(iii) Among quasi-exactly-solvable operators there exist exactly-solvable operators \( \tilde{E}_k \).

Comment 4.2 If we define an analogue of the universal enveloping algebra \( U_q \) for the quantum algebra \( \tilde{g} \) as an algebra of all ordered polynomials in generators, then a quasi-exactly-solvable operator \( \tilde{T}_k \) at \( k < n + 1 \) is simply an
element of the universal enveloping algebra $U_{\mathfrak{sl}(2,\mathbb{R})}^q$ of the algebra $\mathfrak{sl}(2,\mathbb{R})_q$ taken in representation (4.4). If $k \geq n+1$, then $\tilde{T}_k$ is represented as an element of $U_{\mathfrak{sl}(2,\mathbb{R})}^q$ plus $BD_{q}^n$, where $B$ is any linear difference operator of order not higher than $(k - n - 1)$.

Similar to $\mathfrak{sl}(2,\mathbb{R})$ (see Definition 2.3), one can introduce the grading of generators (4.4) of $\mathfrak{sl}(2,\mathbb{R})_q$ (cf. (1.2.3)) and, hence, the grading of monomials of the universal enveloping $U_{\mathfrak{sl}(2,\mathbb{R})}^q$ (cf. (1.2.7)).

**Lemma 4.1.2** A quasi-exactly-solvable operator $\tilde{T}_k \subset U_{\mathfrak{sl}(2,\mathbb{R})}^q$ has no terms of positive grading, iff it is an exactly-solvable operator.

**Theorem 4.1.1** [78]

Let $n$ be a non-negative integer. Take the eigenvalue problem for a linear difference operator of the $k$-th order in one variable

$$\tilde{T}_k(x, D_q) \varphi(x) = \varepsilon \varphi(x), \quad (4.6)$$

where $\tilde{T}_k$ is symmetric. The problem (4.6) has $(n+1)$ linearly independent eigenfunctions in the form of a polynomial in variable $x$ of order not higher than $n$, if and only if $\tilde{T}_k$ is quasi-exactly-solvable. The problem (4.6) has an infinite sequence of polynomial eigenfunctions, if and only if the operator is exactly-solvable $\tilde{E}_k$.

**Comment 4.3** Saying the operator $\tilde{T}_k$ is symmetric, we imply that, considering the action of this operator on a space of polynomials of degree not higher than $n$, one can introduce a positively-defined scalar product, and the operator $\tilde{T}_k$ is symmetric with respect to it.

This theorem gives a general classification of finite-difference equations

$$\sum_{j=0}^{k} \tilde{a}_j(x) D_q^j \varphi(x) = \varepsilon \varphi(x) \quad (4.7)$$

having polynomial solutions in $x$. The coefficient functions must have the form

$$\tilde{a}_j(x) = \sum_{i=0}^{k+j} \tilde{a}_{j,i} x^i. \quad (4.8)$$

In particular, this form occurs after substitution (4.4) into a general $k$th degree polynomial element of the universal enveloping algebra $U_{\mathfrak{sl}(2,\mathbb{R})}^q$. It guarantees the existence of at least a finite number of polynomial solutions. The coefficients $\tilde{a}_{j,i}$ are related to the coefficients of the $k$th degree polynomial element of the universal enveloping algebra $U_{\mathfrak{sl}(2,\mathbb{R})}^q$. The number of free parameters of the polynomial solutions is defined by the number of free parameters of
a general \( k \)-th order polynomial element of the universal enveloping algebra \( U_{\mathfrak{sl}(2,\mathbb{R})_q} \). A rather straightforward calculation leads to the following formula

\[
\text{par}(\tilde{T}_k) = (k + 1)^2 + 1
\]

(for the second-order finite-difference equation \( \text{par}(\tilde{T}_2) = 10 \)). For the case of an infinite sequence of polynomial solutions the formula (4.8) simplifies to

\[
\tilde{a}_j(x) = \sum_{i=0}^{j} \tilde{a}_{j,i} x^i
\]

(4.9)

and the number of free parameters is given by

\[
\text{par}(\tilde{E}_k) = \frac{(k + 1)(k + 2)}{2} + 1
\]

(for \( k = 2, \text{par}(\tilde{E}^2) = 7 \)). The increase in the number of free parameters compared to ordinary differential equations is due to the presence of the deformation parameter \( q \).

4.1.2 Second-order finite-difference exactly-solvable equations.

In [76] it is implemented a description in the present approach of the \( q \)-deformed Hermite, Laguerre, Legendre and Jacobi polynomials (for definitions of these polynomials, see Exton [20], Gasper-Rahman [23]). In order to reproduce the known \( q \)-deformed classical Hermite, Laguerre, Legendre and Jacobi polynomials (for the latter, there exists the \( q \)-deformation of the asymmetric form (1.3.134) only, see e.g. [20] and [23]), one should modify the spectral problem (4.6):

\[
\tilde{T}_k(x, D_q) \varphi(x) = \tilde{\varepsilon} \varphi(qx)
\]

(4.10)

by introducing the r.h.s. function the dependence on the argument \( qx \) (cf. (2.5) and (3.4)) as it follows from the book [20] (see also [23]).

Comment 4.4 The spectral problem (4.10) can be considered as a generalized spectral problem

\[
\tilde{T}_k(x, D_q) \varphi(x) = \tilde{\varepsilon} \left[ 1 - (q - 1)x D_q \right] \varphi(x)
\]

\[\text{for quantum } \mathfrak{sl}(2,\mathbb{R})_q \text{ algebra there are no polynomial Casimir operators (see, e.g. Zachos [99]). However, in the representation (4.10) the relationship between generators analogous to the quadratic Casimir operator}
q \tilde{j}_-^+ \tilde{j}_n^- - \tilde{j}_0^+ \tilde{j}_n^- + (\{n + 1\}_q - 2\hat{n}) \tilde{j}_0^+ = \hat{n}(\hat{n} - \{n + 1\}_q)
\]

appears. It reduces the number of independent parameters of the second-order polynomial element of \( U_{\mathfrak{sl}(2,\mathbb{R})_q} \). It becomes the standard Casimir operator at \( q \to 1 \).
It is evident that the operator in rhs is the element of $U_{\text{sl}(2, \mathbb{R})_q}$ in $(n+1)$-dimensional representation,

$$\left[1 - (q - 1)x D_q\right] = 1 - (q - 1)(\tilde{J}_n^0 + \tilde{n}) .$$

The corresponding $q$-difference operators having $q$-deformed classical Hermite, Laguerre, Legendre and Jacobi polynomials as eigenfunctions (see the equations (5.6.2), (5.5.7.1), (5.7.2.1), (5.8.3) in the book by Exton [20], respectively) are given by the combinations in the generators:

$$\tilde{E}_2 = \tilde{J}_0^- \tilde{J}_0^- - \{2\}_q \tilde{J}_0^0 , \quad \tilde{\varepsilon}_n = -\{2\}_{1/q} \{n\}_{1/q} , \quad (4.11.1)$$

$$\tilde{E}_2 = \tilde{J}_0^0 \tilde{J}_0^- - q^{-a-1} \tilde{J}_0^0 + (q^{-a-1} \{a+1\}_q) \tilde{J}_0^- , \quad \tilde{\varepsilon}_n = -q^{-a-2} \{n\}_{1/q} , \quad (4.11.2)$$

$$\tilde{E}_2 = -q \tilde{J}_0^0 \tilde{J}_0^- + \tilde{J}_- \tilde{J}_- + (q - \{2\}_q) \tilde{J}_0^0 , \quad (4.11.3)$$

$$\tilde{E}_2 = -q^{a+b-1} \tilde{J}_0^0 \tilde{J}_0^- + q^a \tilde{J}_0^0 \tilde{J}_0^- + [q^{a+b-1} - \{a\}_q \{b\}_q] \tilde{J}_0^0 + \{a\}_q \tilde{J}_0^- , \quad (4.11.4)$$

respectively.

**Lemma 4.1.3** If the operator $\tilde{T}_2$ (for the definition, see e.g. [1.2.24]) is such that

$$\tilde{c}_{++} = 0 \quad \text{and} \quad \tilde{c}_+ = (\tilde{n} - \{m\}) \tilde{c}_{+0} , \quad \text{at some} \quad m = 0, 1, 2, \ldots \quad (4.12)$$

then the operator $\tilde{T}_2$ preserves both $\mathcal{P}_{n+1}$ and $\mathcal{P}_{m+1}$, and polynomial solutions in $x$ with 8 free parameters occur.

(cf. Lemma 2.3).

As usual in quantum algebras, a rather outstanding situation occurs if the deformation parameter $q$ is equal to a primitive root of unity. For instance, the following statement holds.

**Lemma 4.1.4** If a quasi-exactly-solvable operator $\tilde{T}_k$ preserves the space $\mathcal{P}_{n+1}$ and the parameter $q$ satisfies to the equation

$$q^n = 1 , \quad (4.13)$$

then the operator $\tilde{T}_k$ preserves an infinite flag of polynomial spaces $\mathcal{P}_0 \subset \mathcal{P}_{n+1} \subset \mathcal{P}_{2(n+1)} \subset \cdots \subset \mathcal{P}_{k(n+1)} \subset \ldots$. 

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It is worth emphasizing that, in the limit as $q$ tends to one, Lemmas 4.1.1, 4.1.2, 4.1.3 and Theorem 4.1.1 coincide with Lemmas 2.1, 2.2, 2.3 and Theorem 2.1, respectively. Thus, the case of differential equations in one variable can be treated as a limiting case of finite-difference ones, of course, if $q$ is not a prime root of unity.

\section*{4.2 \textit{sl}(2, \mathbb{R})\text{-algebra in the Fock space}}

Take two operators (letters) $a$ and $b$ obeying the commutation relations

\begin{equation}
[a, b] \equiv ab - ba = 1 , \quad [a, 1] = [b, 1] = 0 ,
\end{equation}

with the identity operator on the r.h.s. – they span the three-dimensional Heisenberg algebra. In other words, $a$ and $b$ form canonical pair. By definition the universal enveloping algebra of the Heisenberg algebra is the algebra of all ordered polynomials in $a, b$: any monomial is taken to be of the form $b^k a^m$ \footnote{Sometimes this is called the Heisenberg-Weyl algebra}. If, besides the polynomials, all entire functions in $a, b$ are considered, then the \textit{extended} universal enveloping algebra of the Heisenberg algebra appears or, in other words, the extended Heisenberg-Weyl algebra. In the (extended) Heisenberg-Weyl algebra one can find the non-trivial embeddings of the Heisenberg algebra \footnote{This means that there exists a family of pairs of the non-trivial elements of the Heisenberg-Weyl algebra obeying the commutation relations \ref{4.14}}, whose can be treated as a certain type of quantum canonical transformations. We say that the (extended) Fock space appears if we take the (extended) universal enveloping algebra of the Heisenberg algebra and add to it the vacuum state $|0\rangle$ defined as follows

\begin{equation}
a |0\rangle = 0 .
\end{equation}

One can take a polynomial element of the Heisenberg-Weyl algebra $L(b, a)$ and define the eigenvalue problem in the Fock space \footnote{\textit{\[63\]}}

\begin{equation}
L(b, a) \phi(b) |0\rangle = \varepsilon \phi(b) |0\rangle .
\end{equation}

Here $\varepsilon$ is spectral parameter. As for boundary conditions: we are looking for eigenpolynomials in $b$, $\phi(b)$. Thus, technically the problem of finding eigen-polynomial $\phi(b)$ is reduced to reordering (the normal ordering) of $L(b, a) \phi(b)$ to superposition of monomials $b^p a^q$.

(i) One of the most important realizations of \ref{4.14} is the coordinate-momentum representation:

\begin{equation}
a = \frac{d}{dx} \equiv \partial_x , \quad b = x ,
\end{equation}

\footnote{\textit{\[17\]}}
Fig. 4.1. One-dimensional infinite uniform lattice \((-\infty, +\infty)\) with central point \(x\) and spacing \(\delta\).

where \(x\) stands for the multiplication operator in a space of functions \(f(x)\).

In this case the vacuum is a constant, without a loss of generality we put \(|0> = 1\). In this representation

\[
\phi_n \equiv b^n |0> = x^n .
\]  

(ii) Infinite uniform lattice

\[
\{\ldots , x - 2\delta , x - \delta , x , x + \delta , x + 2\delta , \ldots \}
\]  

is marked by \(x \in \mathbb{R}\) - a position of a central or reference point of the lattice and spacing \(\delta\), see for illustration Fig. 4.1. Finite-difference analogue of (4.16) on uniform lattice has been found since long time ago (see e.g. [63, 64]) and studied profoundly in e.g. [26, 27],

\[
a_\delta = D_+ , b_\delta = x(1 - \delta D_-) \equiv x_\delta ,
\]  

where

\[
D_{\pm} f(x) = \frac{f(x \pm \delta) - f(x)}{\pm \delta},
\]

is the finite-difference (shift) operator, \(\delta \in \mathbb{C}\) and \(D_+ \rightarrow D_-\), if \(\delta \rightarrow -\delta\). \(D_+\) makes sense of finite-difference derivative. Sometimes, it is called the Norlund derivative, it connects two neighbouring points in the lattice space. Its canonical partner \(x_\delta\) is a finite-difference analogue of a position operator. It also connects two neighbouring points in the lattice space. Interestingly, the Euler operator

\[
(b_\delta a_\delta) = x_\delta D_+ = x D_- ,
\]

depends on the lattice spacing \(\delta\). We define as "vacuum" \(|0> = 1\). All that gives rise to the so-called umbral calculus due to G.-C. Rota, see e.g. [54]. In this representation

\[
\phi_{n+1} \equiv b^{n+1} |0> = x(x - \delta) \ldots (x - n\delta) \equiv x^{(n+1)}(\delta) ,
\]  

where \(x^{(n)}\) is the Pochhammer symbol which is called in this context the quasi-monomial. It is worth noting that the commutator

\[
[D_+, x] = 1 + \delta D_+ ,
\]
Fig. 4.2. One-dimensional semi-infinite exponential lattice \([x, +\infty)\) with reference point \(x\) and dilation \(q\).

depends on \(D_+\) only.

(iii) Semi-Infinite exponential lattice

\[
\{ x, xq, xq^2, \ldots \},
\]  

(4.21)
is marked by \(x \in \mathbb{R}\) - a position of a reference point of the lattice, see Fig. 4.2.

Finite-difference analogue of (4.16) on exponential lattice has been found in [12]

\[
a_q = D_q, \quad b_q = x_q,
\]  

(4.22)

where

\[
D_q = \frac{1}{1-q}x^{-1}(1-q^A) = x^{-1}\{A\}_q = \frac{\{A+1\}_q}{A+1}\partial_x, \quad A = x\partial_x,
\]
cf. (4.1), \(q \in \mathbb{C}\), is the Jackson derivative and \(\{A\}_q = \frac{1-q^A}{1-q}\) is the so-called \(q\)-operator \(A\) (cf. with \(q\)-number (4.2)) \[19\]. It connects two neighbouring points in the lattice space, see (4.1). The canonical partner of \(D_q\) is

\[
x_q = \frac{A}{\{A\}_q}x = x\frac{A+1}{\{A+1\}_q}.
\]

Both \(D_q\) and \(x_q\) are pseudodifferential operators. It seems important to mention that \(D_q, x_q\) can be related to \(\partial_x, x\) via a similarity transformation \([12]\)

\[
D_q = U(A)^{-1}\partial_x U(A), \quad x_q = U(A)^{-1}xU(A), \quad A \equiv x\partial_x,
\]

where

\[
U(A) = \Gamma_q(A+1)/\Gamma(A+1),
\]

and \(\Gamma(x+1) = x\Gamma_q(x)\) is the gamma function, and \(\Gamma_q(x+1) = \{x\}\Gamma_q(x)\) the \(q\)-deformed gamma function (see, e.g. [20]).

\[19\] Since we move from quantum calculus associated with quantum algebra \(\mathfrak{sl}(2, \mathbb{R})_q\) to the same one but associated the Lie algebra \(\mathfrak{sl}(2, \mathbb{R})\) (see below), hereafter we change the notation for the Jackson derivative from \(D_q\) (see (4.1)) to \(D_q\) (see (4.22)). We do not expect any confusion due to that.
The Euler operator \((b_q a_q)\) does not depend on \(q\),

\[
b_q a_q = x_q D_q = x \partial_x = A ,
\]

hence, remains non-deformed. Hence, it is the local operator. Interestingly,

\[
x_q a^n = \frac{n+1}{\{n+1\}_q} x^{n+1} .
\]

The "vacuum" remains constant \(|0> = 1\). In this representation

\[
\phi_n \equiv b^n |0> = \frac{n!}{\{n\}_q!} x^n ,
\]

where \(n! = 1 \cdot 2 \cdot \ldots n\) is factorial and \(\{n\}_q! = \{1\}_q \cdot \{2\}_q \cdot \ldots \{n\}_q\) is the so-called \(q\)-factorial.

It is interesting to look at the classical limit of (4.22) when the commutators are replaced by the Poisson brackets. Indeed, with \(p, q\) satisfying \(\{p, q\} = 1\), where \(\{\cdot, \cdot\}\) is the Poisson bracket, one can easily verify that

\[
\{f(A)p, q f^{-1}(A)\} = 1 ,
\]

where \(A = qp\) and \(f(A)\) is a holomorphic function, giving rise to a wide class of classical canonical transformations. To the best of our knowledge such canonical transformations were introduced for the first time in [12].

(a). It is easy to check that if the operators \(a, b\) obey (4.14), then for any \(n \in \mathbb{C}\) the following three operators

\[
\hat{J}_n^+ = b^2 a - nb , \quad \hat{J}_n^0 = ba - \frac{n}{2} , \quad \hat{J}_n^- = a ,
\]

span the \(\mathfrak{sl}(2, \mathbb{R})\)-algebra with the commutation relations:

\[
[\hat{J}_n^0, \hat{J}_n^\pm] = \pm \hat{J}_n^\pm , \quad [\hat{J}_n^+, \hat{J}_n^-] = -2 \hat{J}_n^0 .
\]

For the representation (4.25) the quadratic Casimir operator is equal to

\[
C_2 \equiv \frac{1}{2} \{\hat{J}_n^+, \hat{J}_n^-\} - \hat{J}_n^0 \hat{J}_n^0 = -\frac{n}{2} \left(\frac{n}{2} + 1\right) ,
\]

where \(\{\,,\,\}\) denotes the anticommutator. If \(n\) is a non-negative integer, then (4.25) possesses a finite-dimensional, irreducible representation in the Fock space leaving invariant the space

\[
\mathcal{P}_n(b) = \langle 1, b, b^2, \ldots, b^n |0\rangle ,
\]
of dimension \( \dim \mathcal{P}_n = (n+1) \).

Substituting of (4.16) into (4.25) leads to a well-known realization of the \( \mathfrak{sl}(2, \mathbb{R}) \)-algebra as an algebra of differential operators of the first order

\[
J^+_n = x^2 \partial_x - nx, \\
J^0_n = x \partial_x - \frac{n}{2}, \\
J^- = \partial_x,
\]

see (1.23), where the finite-dimensional representation space (4.27) becomes the space of polynomials of degree not higher than \( n \)

\[
\mathcal{P}_n(x) = \langle 1, x, x^2, \ldots, x^n \rangle.
\]

(b-1). (Uniform lattice) The existence of a non-trivial embedding of the Heisenberg algebra into its extended universal enveloping algebra, namely, \( [\hat{a}(a, b), \hat{b}(a, b)] = [a, b] = 1 \) allows to construct different representations of the algebra \( \mathfrak{sl}(2, \mathbb{R}) \) by \( a \rightarrow \hat{a}, b \rightarrow \hat{b} \) in (4.25). In particular, such an embedding of the Heisenberg algebra into its extended universal enveloping algebra is realized by the following two operators \[82\],

\[
\hat{a} = \frac{(e^{\delta a} - 1)}{\delta}, \\
\hat{b} = b e^{-\delta a},
\]

where \( \delta \) can be any real (complex) number. If \( \delta \) goes to zero then \( \hat{a} \rightarrow a, \hat{b} \rightarrow b \).

In other words, (4.29) is a one-parameter quantum canonical transformation of the deformation type of the Heisenberg algebra (4.14). It is one of the (several) possible quantum analogies of a point-to-point canonical transformation. The substitution of the representation (4.29) into (4.25) results in the following representation of the \( \mathfrak{sl}(2, \mathbb{R}) \)-algebra

\[
J^+ = b \left( \frac{b}{\delta} - 1 \right)(1 - e^{-\delta a}) - n e^{-\delta a}, \\
J^0 = \frac{b}{\delta} (1 - e^{-\delta a}) - \frac{n}{2}, \\
J^- = \frac{1}{\delta} (e^{\delta a} - 1).
\]

(4.30)

If \( n \) is a non-negative integer, then (4.30) possesses a finite-dimensional irreducible representation of dimension \( \dim \mathcal{P}_n = (n+1) \) coinciding with (4.27). It is worth noting that the vacuum for (4.29) remains the same, for instance (4.15). Also the value of the quadratic Casimir operator for (4.30) coincides with that given by (4.26).
After substitution of (4.16) into (4.30) we arrive at a representation of the \( \mathfrak{sl}(2, \mathbb{R}) \)-algebra by finite-difference operators \([64,63]\),

\[
J_n^+ = x \left( \frac{x}{\delta} - 1 \right) (1 - e^{-\delta x}) - n \right) e^{-\delta x},
\]

\[
J_0^0 = \frac{x}{\delta} (1 - e^{-\delta x}) - \frac{n}{2},
\]

\[
J^- = \frac{1}{\delta} (e^{\delta x} - 1),
\]

(4.31)

or, equivalently,

\[
J_n^+ = -x \left( (x - \delta) \mathcal{D}_- - n \right) (1 - \delta \mathcal{D}_-),
\]

\[
J_0^0 = x \mathcal{D}_- - \frac{n}{2},
\]

\[
J^- = \mathcal{D}_+.
\]

(4.32)

The finite-dimensional representation space for (4.31)–(4.32) for integer values of \( n \) is again given by the space of polynomials of degree not higher than \( n \) \( \mathcal{P}_n(x) \) (4.28).

(b-2). (Exponential lattice) Another non-trivial embedding of the Heisenberg algebra into its extended universal enveloping algebra has been found (proved) in \([12]\)

\[
\hat{a} = \frac{\{ba + 1\} \_q}{ba + 1} a,
\]

\[
\hat{b} = b \frac{ba + 1}{\{ba + 1\} \_q} = \frac{ba}{\{ba\} \_q} b,
\]

(4.33)

(cf. (4.29)), where \( q \) is any real (complex) number and \( \{ba + 1\} \_q = \frac{1 - q^{ba+1}}{1 - q} \) is the \( q \) operator. If \( \delta \) goes to zero, then \( \hat{a} \rightarrow a, \hat{b} \rightarrow b \). In other words, (4.33) is a one-parameter quantum canonical transformation of the deformation type of the Heisenberg algebra (4.14). It is one of the (several) possible quantum analogies of a point-to-point canonical transformation. The substitution of the representation (4.33) into (4.25) results in the following representation of the \( \mathfrak{sl}(2, \mathbb{R}) \)-algebra

\[
J_n^+ = b \frac{ba + 1}{\{ba + 1\} \_q} (ba - n) = \frac{ba}{\{ba\} \_q} b (ba - n),
\]

\[
J_0^0 = ba - \frac{n}{2},
\]

\[
J^- = \frac{\{ba + 1\} \_q}{ba + 1} a.
\]

(4.34)

If \( n \) is a non-negative integer, then (4.30) possesses a finite-dimensional irreducible representation of dimension \( \text{dim} \mathcal{P}_n = (n + 1) \) coinciding with (4.27). It is worth noting that the vacuum for (4.29) remains the same, for instance (4.15). Also the value of the quadratic Casimir operator for (4.30) coincides with that given by (4.26).
After substitution of (4.16) into (4.34) we arrive at a representation of the \(\mathfrak{sl}(2, \mathbb{R})\)-algebra by finite-difference operators on exponential lattice, \([12]\)

\[
J^+_n = \frac{(1 - q)}{1 - q^{x\partial_x}} x (x\partial_x + 1) (x\partial_x - n) ,
\]

\[
J^0_n = x\partial_x - \frac{n}{2} ,
\]

\[
J^- = \frac{1 - q^{x\partial_x + 1}}{1 - q} \frac{1}{1 + x\partial_x} \partial_x ,
\]

or, equivalently,

\[
J^+_n = x_q (x\partial_x - n) ,
\]

\[
J^0_n = x\partial_x - \frac{n}{2} , J^- = \mathcal{D}_q .
\]

The finite-dimensional representation space for (4.35)–(4.36) for integer values of \(n\) is again given by the space of polynomials of degree not higher than \(n\) \(\mathcal{P}_n(x)\) \((4.28)\).

Taking into account that 

\[
\mathcal{D}_q x^k = \{k\}_q x^{k-1} , x_q x^k = \frac{k + 1}{\{k + 1\}_q} x^{k+1} ,
\]

one find the action of generators on monomial \(x^k\),

\[
J^+_n x^k = x_q (x\partial_x - n) x^k = \frac{k + 1}{\{k + 1\}_q} (k - n) x^{k+1} ,
\]

\[
J^0_n x^k = (x\partial_x - \frac{n}{2}) x^k = (k - \frac{n}{2}) x^k ,
\]

\[
J^- x^k = \mathcal{D}_q x^k = \{k\}_q x^{k-1} .
\]

(c). Another example of quantum canonical transformation is given by the oscillator representation

\[
\hat{a} = \frac{b + a}{\sqrt{2}} ,
\]

\[
\hat{b} = \frac{b - a}{\sqrt{2}} .
\]

Inserting (4.38) into (4.25) it is easy to check that the following three generators form a representation of the \(\mathfrak{sl}(2, \mathbb{R})\)-algebra,

\[
J^+_n = \frac{1}{2^{3/2}} [b^3 + a^3 - b(b + a)a - (2n + 1)(b - a) - 2b] ,
\]

\[
J^0_n = \frac{1}{2} (b^2 - a^2 - n - 1) ,
\]

\[
J^- = \frac{b + a}{\sqrt{2}} .
\]
where $n$ is any real (complex) number. In this case the vacuum state
\[
(b + a)|0 > = 0,
\]
differs from (4.15). If $n$ is a non-negative integer, then (4.39) possesses a finite-dimensional irreducible representation in a subspace of the Fock space
\[
\mathcal{P}_n(b) = \langle 1, (b-a), (b-a)^2, \ldots, (b-a)^n | 0 \rangle,
\]
of dimension $\dim \mathcal{P}_n = (n+1)$.

Taking $a, b$ in the realization (4.16) and substituting them into (4.39), we obtain
\[
J^+_n = \frac{1}{2n+1/2} [x^3 + \partial_x^3 - x(x + \partial_x)\partial_x - (2n+1)(x - \partial_x) - 2\partial_x],
\]
\[
J^0_n = \frac{1}{2}(x^2 - \partial_x^2 - n - 1),
\]
\[
J^- = \frac{x + \partial_x}{\sqrt{2}},
\]
which represents the $\mathfrak{sl}(2, \mathbb{R})$-algebra by means of differential operators of finite order (but not of first order as in (1.2.3)). The operator $J^0_n$ coincides with the Hamiltonian of the harmonic oscillator (with the reference point for eigenvalues changed). The vacuum state is
\[
|0 > = e^{-\frac{x^2}{2}},
\]
and the representation space is
\[
\mathcal{P}_n(x) = \langle 1, x, x^2, \ldots, x^n \rangle e^{-\frac{x^2}{2}},
\]
(cf. (4.41)).

(d). The following three operators
\[
J^+ = \frac{a^2}{2},
\]
\[
J^0 = -\frac{\{a, b\}}{4},
\]
\[
J^- = \frac{b^2}{2},
\]
are generators of the $\mathfrak{sl}(2, \mathbb{R})$-algebra and the quadratic Casimir operator for this representation is
\[
C_2 = \frac{3}{16}.
\]
This is the so-called metaplectic representation of the $sl_2$-algebra (see, for example, \[51\]). This representation is infinite-dimensional. Taking the realization (4.15) or (4.19) of the Heisenberg algebra we get the well-known representation

\[ J^+ = \frac{1}{2}\partial_x^2, \quad J^0 = -\frac{1}{2}(x\partial_x + \frac{1}{2}), \quad J^- = \frac{1}{2}x^2 \quad (4.46) \]

in terms of differential operators, sometimes it is called the oscillator representation, or

\[ J^+ = \frac{1}{2}D_+^2, \quad J^0 = -\frac{1}{2}(xD_ - + \frac{1}{2}), \]

\[ J^- = \frac{1}{2}x(x - \delta)(1 - 2\delta D_ - - \delta^2 D_ -^2), \quad (4.47) \]

in terms of finite-difference operators, correspondingly.

(e). For the sake of completeness let us take two operators $a$ and $b$ from the Clifford algebra $s_2$,

\[ \{a, b\} \equiv ab + ba = 0, \quad a^2 = b^2 = 1. \quad (4.48) \]

It can be shown that the operators

\[ J^1 = a, \quad J^2 = b, \quad J^3 = ab, \quad (4.49) \]

span the $sl_2$-algebra.

### 4.3 Spectral problem in the Fock space

Let $L(b, a)$ is a polynomial in $a, b$ - the generators of the Heisenberg algebra, $[a, b] = 1$. Define the eigenvalue problem in the Fock space, see e.g. \[63\], as

\[ L(b, a)\phi(b) |0> = \varepsilon\phi(b) |0>, \quad (4.50) \]

where $\varepsilon$ is spectral parameter. Instead of boundary conditions we impose a condition that we are looking for eigenpolynomials in $b$. Technically, the problem of finding eigenpolynomial is reduced to reordering (the normal ordering) of $L(b, a)\phi(b)$ to superposition of monomials $b^p a^q$,

\[ L(b, a)\phi(b) = \sum_0 L_i(b)a^i, \]

and then making a study of $L_0(b)$. In addition to standard relation,

\[ [a, b^k] = kb^{k-1}, \quad [a^k, b] = ka^{k-1}, \]
several identities can be useful \[21\[81\],
\[
(ab)\,^k = a^k b^k a^k , \quad (bab)\,^k = b^k a^k b^k ,
\]
\[
[a^k b^k, a^m b^m] = 0 ,
\]
where \(k, m\) are integer.

In general, if the operator \(L(b, a)\) can be represented in terms of the \(\mathfrak{sl}(2, \mathbb{R})\)-algebra generators \((4.25)\) at integer \(n\),
\[
L(b, a) = L_{qes}(\hat{J}_n^+, \hat{J}_n^0, \hat{J}_n^-) ,
\]
it is evident there exists \((n + 1)\) polynomial solutions of \((4.50)\), thus, the problem is quasi-exactly-solvable. If the generator \(\hat{J}_n^+\) is not present,
\[
L(b, a) = L_{es}(\hat{J}_n^0, \hat{J}_n^-) ,
\]
it is evident there exists infinitely-many polynomial solutions of \((4.50)\), thus, the problem is exactly-solvable. As an illustration let us calculate the spectra of polynomial solutions for the Cartan generator \(\hat{J}_0^0 \equiv \hat{J}_0^0\).

Example. Let
\[
L(b, a) = \hat{J}_0^0 = ba ,
\]
It is easy to check that
\[
\phi_k = b^k , \quad \varepsilon_k = k .
\]
It can be drawn the immediate conclusion: changing the operator \(L\) by adding \(J_n^- F(J_n^0, J_n^-)\), where \(F\) is any polynomial, is isospectral. The eigenpolynomial \(\phi_k\) is modified getting extra monomials of degrees less than \(k\).

Concrete realizations of the Heisenberg algebra in terms of differential or finite-difference operators leads to the spectral problem for differential or finite-difference operators with the same spectra!

1) in coordinate-momentum representation \((4.16)\) the operator \(J^0\) becomes the Euler operator
\[
L(x, \partial_x) = x \partial_x ,
\]
and
\[
\varphi_k(x) = \phi_k(b) |0 > = x^k , \quad \varepsilon_k = k .
\]
2) in \(\delta\)–representation \((4.19)\) on the uniform lattice the operator \(J^0\) becomes
\[ L(x_\delta, D_\delta) = x D_-, \]

while the spectral problem is two-point finite-difference equation,

\[ \frac{x}{\delta} \varphi(x) - \frac{x}{\delta} \varphi(x - \delta) = \varepsilon \varphi(x) \]

and with infinitely-many polynomial eigenfunctions,

\[ \varphi_k(x) = \phi_k(b) |0> = x^{(k)}(\delta) \equiv x(x - \delta)(x - 2\delta) \cdots (x - (k - 1)\delta), \varepsilon_k = k. \]

3) in \( q \)-representation (4.22) on the exponential lattice the operator \( J^0 \) becomes

\[ L(x, \partial_x) = x \partial_x, \]

and

\[ \varphi_k(x) = \phi_k(b) |0> = x^k, \varepsilon_k = k. \]

Concluding one can state the property of isospectrality of the spectral problem in the Fock space with respect to quantum canonical transformations, \([a, b] = [\hat{a}(a, b), \hat{b}(a, b)] = 1:\]

If vacuum \(|0>\) remains unchanged under quantum canonical transformations

\[ a|0> = \hat{a}|0> = 0, \]

the spectra of the operator \( L(b, a) \) with polynomial eigenfunctions \( \phi(b) \) coincides with the spectra of the operator \( L(\hat{b}, \hat{a}) \) with polynomial eigenfunctions \( \phi(\hat{b}) \).

Among known quantum canonical transformations realized in action on functions there are two special: one associated with shift operator,

\[ T_\delta f(x) = f(x + \delta), \]

see (4.29), and another one associated with dilatation operator,

\[ T_q f(x) = f(qx), \]

see (4.33), respectively. It gives a chance to make use of these transformation for isospectral discretization, connecting linear differential operators with finite-difference linear operators. It also allows us to connect, preserving isospectrality, different finite-difference operators,

\[ L(b, a) \leftrightarrow L(b_\delta, a_\delta) \leftrightarrow L(b_q, a_q). \]
It can be proved \cite{12} that if the eigenvalue problem
\[ L(x, \partial_x) \varphi(x) = \lambda \varphi(x), \]
has a polynomial eigenfunction
\[ \varphi(x) = \sum_{k} a_k x^k, \]
at a certain \( \lambda_N \), then the eigenvalue problem
\[ L(x_\delta, D_+) \varphi_\delta(x) = \lambda \varphi_\delta(x), \]
has a polynomial eigenfunction
\[ \varphi_\delta(x) = \sum_{k} a_k x^{(k)} , \]
for the same \( \lambda_N \), then the eigenvalue problem
\[ L(x_q, D_q) \varphi_q(x) = \lambda \varphi_q(x), \]
has a polynomial eigenfunction
\[ \varphi_q(x) = \sum_{k} a_k \frac{k!}{\{k\}_q!} x^k , \]
for the same \( \lambda_N \). Assume the operator \( L(b, a) \) can be rewritten in terms of the \( \mathfrak{sl}(2, \mathbb{R}) \)-algebra generators \((4.25)\), hence, it is the \( \mathfrak{sl}(2, \mathbb{R}) \)-Lie-algebraic operator. Then \( L(x, \partial_x) \) is the \( \mathfrak{sl}(2, \mathbb{R}) \)-Lie-algebraic differential operator as well as the operators \( L(x_\delta, D_+) \) and \( L(x_q, D_q) \) which are the \( \mathfrak{sl}(2, \mathbb{R}) \)-Lie-algebraic finite-difference operators. Transition from differential operator \( L(x, \partial_x) \) to \( L(x_\delta, D_+) \) or to \( L(x_q, D_q) \) is nothing but the \( \mathfrak{sl}(2, \mathbb{R}) \)-Lie-algebraic discretization for which polynomial eigenfunctions remains polynomial ones with the property isospectrality of the corresponding eigenvalues. Thus, for all (quasi)-exactly-solvable problems, see Cases I-XII, it can be constructed “polynomially-isospectral”, quasi-exactly-solvable discrete systems on uniform or exponential lattice: they have \((n + 1)\) polynomial eigenfunctions in a form of polynomial in \( x \) of the degree \( n \) with the same eigenvalues. It will be presented in the next Section.

\subsection*{4.4 Quasi-exactly-solvable finite-difference operators}

In this Section we give a list of the quasi-exactly-solvable operators of Cases I-X, XII in the Fock space; for some cases their representations on the uniform and exponential lattices are also given.
Case I.

Let us take the \( \mathfrak{sl}(2, \mathbb{R}) \)-Lie algebraic operator \( T_2/\alpha \) \((1.3.16-1)\). Then substitute the generators \((4.25)\) into \((1.3.16-1)\). We get the \( \mathfrak{sl}(2, \mathbb{R}) \)-Lie algebraic operator in the Fock space of Case I,

\[
T_2(\hat{b}, \hat{a}) = -\alpha \hat{b}^2 \hat{a}^2 + [2\alpha \hat{b}^2 + (2b - \alpha) \hat{b} - 2c] \hat{a} - 2an \hat{b} , \tag{4.51}
\]

cf. \((1.3.16-2)\), where \([\hat{a}, \hat{b}] = 1\). In \(\delta\)-representation \((4.19)\) on the uniform lattice, see Fig. 4.1, the operator \((4.51)\) becomes,

\[
T_2(x_\delta, D_\delta) = -(\alpha + 2a\delta)x(x - \delta)D_-D_- + [2ax + 2a\delta(n - 1) + 2b - \alpha]xD_- - 2cD_+ - 2anx , \tag{4.52}
\]

It is the four-point, quasi-exactly-solvable, finite-difference operator, which is polynomially-isospectral to \((1.3.16-1)\). In \(q\)-representation \((4.22)\) on the exponential lattice, see Fig. 4.2, the operator \((4.51)\) becomes,

\[
T_2(x_q, D_q) = -\alpha x \partial_x x \partial_x + 2ax_q x \partial_x + 2b \partial_x - 2cD_q - 2anx_q . \tag{4.53}
\]

It is the quasi-exactly-solvable differential-difference operator, which is polynomially-isospectral to \((1.3.16-1)\) and \((4.52)\).

Case II.

Let us take the \( \mathfrak{sl}(2, \mathbb{R}) \)-Lie algebraic operator \( T_2 \) \((1.3.27-1)\). Then substitute again the generators \((4.25)\) into \((1.3.27-1)\). We get the \( \mathfrak{sl}(2, \mathbb{R}) \)-Lie algebraic operator in the Fock space of Case II,

\[
T_2(\hat{b}, \hat{a}) = -\alpha \hat{b}^2 \hat{a}^2 + (2b - \alpha) \hat{b}^2 - 2c] \hat{a} - 2an \hat{b} , \tag{4.54}
\]

cf. \((1.3.27-2)\).

Case III.

Let us take the \( \mathfrak{sl}(2, \mathbb{R}) \)-Lie algebraic operator \( T_2 \) \((1.3.36)\) and substitute the generators \((4.25)\) into \((1.3.36)\). We get the \( \mathfrak{sl}(2, \mathbb{R}) \)-Lie algebraic operator in the Fock space of Case III,

\[
T_2(\hat{b}, \hat{a}) = -\alpha \hat{b}^3 \hat{a}^2 + [(2b - \alpha) \hat{b}^2 - 2a \hat{b} - 2c] \hat{a} + (\alpha n - 2b) \hat{b} , \tag{4.55}
\]

cf. \((1.3.27-2)\).

Case IV.

Let us take the \( \mathfrak{sl}(2, \mathbb{R}) \)-Lie algebraic operator \( T_2 \) \((1.3.30-1)\) and then substitute the generators \((4.25)\) into it. We get the \( \mathfrak{sl}(2, \mathbb{R}) \)-Lie algebraic operator
in the Fock space of Case IV,

\[ T_2(\hat{b}, \hat{a}) = -\alpha \hat{b}^3 \hat{a}^2 + [(2b - \alpha) \hat{b}^2 - 2a \hat{b} - 2c] \hat{a} + (\alpha n - 2b) \hat{b} , \quad (4.56) \]

cf. \(1.3.30-2\).

**Case V.**

Let us take the \(\mathfrak{sl}(2, \mathbb{R})\)-Lie algebraic operator \(T_2 \quad (1.3.36-1)\). After substitution of the generators \((4.25)\) into \((1.3.36-1)\), we get the \(\mathfrak{sl}(2, \mathbb{R})\)-Lie algebraic operator in the Fock space of Case V,

\[ T_2(\hat{b}, \hat{a}) = -2\alpha \hat{b}^2(\hat{b} - 1) \hat{a}^2 + [2b \hat{b}^2 - (2a + 4b + 2p\alpha + 3\alpha) \hat{b} + 2(\alpha + a + b)] \hat{a} - 2bn \hat{b} + b(2n + p) , \quad (4.57) \]

cf. \(1.3.36-2\).

**Case VI.**

Let us take the \(\mathfrak{sl}(2, \mathbb{R})\)-Lie algebraic operator \(T_2 \quad (1.3.43)\). Then substitute the generators \((4.25)\) into \((1.3.43)\). We get the \(\mathfrak{sl}(2, \mathbb{R})\)-Lie algebraic operator in the Fock space of Case VI,

\[ T_2(\hat{b}, \hat{a}) = -4\hat{b}^2 + 2(2a \hat{b}^2 + 2b \hat{b} - 1 - 2p) \hat{a} - 4an \hat{b} , \quad (4.58) \]

cf. \(1.3.44\), where \([\hat{a}, \hat{b}] = 1\). In \(\delta\)-representation \((4.19)\) on the uniform lattice, see Fig. 4.1, the operator \((4.58)\) becomes,

\[ T_2(x_\delta, D_\delta) = -4a\delta x(x - \delta) D_- D_- + 4x[a x + a\delta(n - 1) + b + \frac{1}{\delta}]D_- - 2(\frac{2x}{\delta} - 1 - 2p)D_+ - 4an x . \quad (4.59) \]

It is the four-point, quasi-exactly-solvable, finite-difference operator, which is polynomially-isospectral to \((1.3.43)\). In \(q\)-representation \((4.22)\) on the exponential lattice, see Fig. 4.2, the operator \((4.58)\) becomes,

\[ T_2(x_q, D_q) = -4x \partial_x D_q + 4ax_q x \partial_x + 4bx \partial_x - 2(1 + 2p) D_q - 4an x_q . \quad (4.60) \]

It is the quasi-exactly-solvable differential-difference operator, which is polynomially-isospectral to \((1.3.16-1)\) and \((1.59)\). It is also polynomially-isospectral to the Hamiltonian \((2.3)\) at \(a \geq 0\).

**Case VII.**

Let us take the \(\mathfrak{sl}(2, \mathbb{R})\)-Lie algebraic operator \(T_2 \quad (1.3.49)\) and substitute the generators \((4.25)\) into it. We get the \(\mathfrak{sl}(2, \mathbb{R})\)-Lie algebraic operator in the Fock
space of Case VII,

\[ T_2(\hat{b}, \hat{a}) = -4\hat{b}\hat{a}^2 + 2(2\hat{a}\hat{b}^2 + 2\hat{b}\hat{b} - d - 2l + 2c) \hat{a} - 4an\hat{b}, \quad (4.61) \]

cf. (1.3.50).

**Case VIII.**

Let us take the \(\mathfrak{sl}(2, \mathbb{R})\)-Lie algebraic operator \(T_2\) (1.3.60). After substitution the generators (4.25) into it we get the \(\mathfrak{sl}(2, \mathbb{R})\)-Lie algebraic operator in the Fock space of Case VIII,

\[ T_2(\hat{b}, \hat{a}) = -\hat{b}\hat{a}^2 + (2\hat{a}\hat{b}^2 + 2\hat{b}\hat{b} - d - 2l + 2c + 1) \hat{a} - 2an\hat{b}, \quad (4.62) \]

cf. (1.3.61).

**Case IX.**

Let us take the \(\mathfrak{sl}(2, \mathbb{R})\)-Lie algebraic operator \(T_2\) (1.3.67). Then substitute the generators (4.25) into (1.3.67). We get the \(\mathfrak{sl}(2, \mathbb{R})\)-Lie algebraic operator in the Fock space of Case IX,

\[ T_2(\hat{b}, \hat{a}) = -\hat{b}\hat{a}^2 - [2\hat{a}\hat{b}^2 + (c - d - 2l + 1) \hat{b} - 2b] \hat{a} - 2an\hat{b}, \quad (4.63) \]

cf. (1.3.68).

**Case X.**

Let us take the \(\mathfrak{sl}(2, \mathbb{R})\)-Lie algebraic operator \(T_2/\alpha^2\) (1.3.73). Then substitute the generators (4.25) into (1.3.73). We get the \(\mathfrak{sl}(2, \mathbb{R})\)-Lie algebraic operator in the Fock space of Case X,

\[ T_2(\hat{b}, \hat{a}) = (\hat{b}^2 - 1) \hat{a}^2 + [2a\hat{b}^2 + (1 + 2\mu) \hat{b} - 2a] \hat{a} - 2a(n - \mu) \hat{b}, \quad (4.64) \]

cf. (1.3.74). It is polynomially-isospectral to the Hamiltonian (1.3.77).

**Case XII.**

I.

Let us take the \(\mathfrak{sl}(2, \mathbb{R})\)-Lie algebraic operator \(h_B\) (1.3.99) and substitute the generators (4.25) into (1.3.99). We get the \(\mathfrak{sl}(2, \mathbb{R})\)-Lie algebraic operator in the Fock space of Case XII (I),

\[ h_B(\hat{b}, \hat{a}) = (4\hat{b}^3 - g_2\hat{b} - g_3) \hat{a}^2 + (1 + 2\mu)(6\hat{b}^2 - \frac{g_2}{2}) \hat{a} - 2n(2n + 1 + 6\mu) \hat{b}, \quad (4.65) \]
II.

Let us take the \( \mathfrak{sl}(2, \mathbb{R}) \)-Lie algebraic operator \( h_B \) (1.3.109). Then substitute the generators (4.25) into (1.3.109). We get the \( \mathfrak{sl}(2, \mathbb{R}) \)-Lie algebraic operator in the Fock space of Case XII (II),

\[
\hat{h}_{B,k}(\hat{b}, \hat{a}) = (4 \hat{b}^3 - g_2 \hat{b} - g_3) \hat{a}^2 + \\
(2(5+2\mu) \hat{b}^2 + 4(1-2\mu) c_k (\hat{b}+c_k) - (3-2\mu) \frac{g_2}{2}) \hat{a} - 2n (2n+3+2\mu) \hat{b},
\]

(4.66) cf. (1.3.108). At \( \mu = 0 \) the operator (4.66) degenerates to the Lame case, see Case XI.

4.5 Quasi-exactly-solvable 3-point finite-difference operators (Generalized Hahn Polynomials)

Typical second-order finite-difference equation relates an unknown function at three lattice points. In a form of the eigenvalue problem it is written as

\[
A(x)\varphi(x + \delta) - B(x)\varphi(x) + C(x)\varphi(x - \delta) = \lambda \varphi(x),
\]

(4.67) where \( A(x), B(x), C(x) \) are some functions and \( \lambda \) is spectral parameter. The equation (4.67) can also be rewritten in operator form

\[
\left( A(x) e^{\delta \partial_x} - B(x) + C(x) e^{-\delta \partial_x} \right) \varphi(x) = \lambda \varphi(x).
\]

(4.68) The celebrated Harper equation, which appears, in particular, in the Azbel-Hofstadter problem (see e.g. [97,98]) is of this type.

One can pose a natural question: what are the coefficient functions \( A(x), B(x), C(x) \) for which the equation (4.67) or (4.68) admits a finite number of polynomial eigenfunctions? It is evident if the operator

\[
H_\delta = A(x) e^{\delta \partial_x} - B(x) + C(x) e^{-\delta \partial_x},
\]

(4.69) in the r.h.s. of (4.68) can be rewritten in terms of generators of \( \mathfrak{sl}(2, \mathbb{R}) \)-algebra, realized as finite-difference operators (4.31), in finite-dimensional representation (hence, the \( n \) is an integer number) polynomial eigenfunctions can occur.

Instead of developing the general theory of polynomial eigenfunctions of the finite-difference operators we limit ourselves by a consideration of the (generalized) Hahn polynomials as the eigenfunctions of a certain operator which we
will call the *Hahn operator*. As for potential applications of quantum canonical transformations discussed in previous Sections, we present here various deformations of the Hahn operator and its eigenfunctions, the deformed Hahn polynomials. Needless to say that the Hahn polynomials are among the most important (and the most complicated) polynomials of discrete variable which appear in numerous applications (see e.g. [1,2,17] and references therein). Their degenerations contain the celebrated Meixner, Charlier, Tschebyschov and Krawtchouk polynomials.

Take the spectral problem

\[ \frac{1}{\delta^2} (A_1 x^2 + A_2 x + A_4 \delta) f(x + \delta) - \]

\[ \frac{1}{\delta^2} [2A_1 x^2 - (\delta A_1 - 2A_2 + A_3 \delta + A_4 \delta^2) x] f(x) + \]

\[ \frac{1}{\delta^2} [A_1 x^2 - (\delta A_1 - A_2 + A_3 \delta) x] f(x - \delta) = \lambda f(x) . \]  

(4.70)

where \( A_{1,2,3,4} \) and \( \delta \) are parameters, and \( \lambda \) is spectral parameter. It can be easily verified that the equation (4.70) has infinitely-many polynomial eigenfunctions with the eigenvalues

\[ \lambda_k = A_1 k^2 + A_3 k \quad , \quad k = 0, 1, 2, \ldots , \]  

(4.71)

which depend on \( k \) quadratically at large \( k \). It was shown in [63] that the operator (4.70) is the most general three-point finite-difference possessing infinitely-many polynomial eigenfunctions. The operator in the rhs of (4.70)

\[ H_\delta = \frac{1}{\delta^2} (A_1 x^2 + A_2 x + A_4 \delta) e^{\delta x} - \frac{1}{\delta^2} [2\delta A_1 x^2 - (\delta A_1 - 2A_2 + A_3 \delta + A_4 \delta^2) x] \]

\[ + \frac{1}{\delta^2} [A_1 x^2 - (\delta A_1 - A_2 + A_3 \delta) x] e^{-\delta x} , \]  

(4.72)

is called the *Hahn operator*.

Without a loss of generality we put spacing \( \delta = 1 \) and \( A_1 = -1 \), and parametrize other parameters as

\[ A_2 = N - 2 - \beta \ , \ A_3 = -\alpha - \beta - 1 \ , \ A_4 = (\beta + 1)(N - 1) , \]  

(4.73)

where \( N \) and \( \alpha, \beta \) are new parameters. We call the corresponding eigenfunctions the *Hahn polynomials of continuous argument* \( h_k^{(\alpha,\beta;\delta)}(x, N) \).

\[ \text{Besides that, if we choose} \]

\[ A_1 = 1 \ , \ A_2 = 2 - 2N - \nu \ , \ A_3 = 1 - 2N - \mu - \nu \ , \ A_4 = (N + \nu - 1)(N - 1) , \]

\[ \text{It must be emphasized that the Hahn polynomials of the continuous argument do not coincide to the so-called continuous Hahn polynomials known in literature [1]} \]
where $N$ and $\nu, \mu$ are new parameters, the so-called analytically-continued Hahn polynomials $\bar{h}_{k}^{(\nu, \mu; \delta)}(x, N)$, $k = 0, 1, 2 \ldots$ as the eigenfunctions appear, see e.g. [47].

Taking in (4.72)

$$A_1 = 0 \ , \ A_2 = -\mu \ , \ A_3 = \mu - 1 \ , \ A_4 = \gamma \mu ,$$

we reproduce the operator having the Meixner polynomials as the eigenfunctions. Furthermore, if

$$A_1 = 0 \ , \ A_2 = 0 \ , \ A_3 = -1 \ , \ A_4 = \mu ,$$

the operator (4.72) has the Charlier polynomials as the eigenfunctions (for the definition of the Meixner and Charlier polynomials see e.g. [47]). For a certain particular choice of the parameters, one can reproduce the equations having Tschebyschov and Krawtchouk polynomials as the solutions. It must be emphasized that if $x$ is the continuous argument, we will arrive at continuous analogues of all the above-mentioned polynomials. These polynomials are poorly studied in literature, for discussion see [64].

Since the operator (4.72) is the most general exactly-solvable finite-difference three-point operator, one can construct corresponding

$$H_{\delta} = A_1 J_0 J_0 (\delta J^- + 1) + A_2 J_0 J^- + A_3 J_0^2 + A_4 J^- , \quad (4.74)$$

which is the cubic polynomial in generators from the universal enveloping $\mathfrak{sl}(2, \mathbb{R})$-algebra leading to (4.70). Hence, the Hahn polynomials are related to the finite-dimensional representation of a cubic element of the universal enveloping $\mathfrak{sl}(2, \mathbb{R})$-algebra.

If $N$ is a positive integer number and the variable $x$ is restricted to a lattice $x = 0, 1, 2 \ldots, (N - 1)$, these Hahn polynomials of continuous argument coincide to the standard Hahn polynomials $h_{k}^{(\alpha, \beta)}(x, N)$ of the discrete argument (we use the notation of [47]). The Hahn polynomial of continuous argument can be represented as

$$h_{k}^{(\alpha, \beta; \delta)}(x, N) = \sum_{i=0}^{k} \gamma_i x^{(i)} , \quad (4.75)$$

where $x^{(i+1)} \equiv x(x - \delta) \ldots (x - i \delta)$ is the so-called $\delta$–quasi-monomial (see (4.20)) and $\gamma_i$ are known coefficients. If the parameter $N$ is integer and $x$ is continuous argument, then for the higher Hahn polynomials $k \geq N$ there is the property,

$$h_{k}^{(\alpha, \beta; \delta)}(x, N) = x^{(N)} p_{k-N}(x) , \quad (4.76)$$

where $p_{k-N}(x)$ is a Hahn polynomial of degree $(k - N)$. It explains why a finite number of the Hahn polynomials solely exists if continuous argument $x$ is restricted to the finite lattice $x = 0, 1, 2 \ldots, (N - 1)$. 

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In [63,64] it was proven that the operator $H_\delta$ in the r.h.s. (4.70) belongs to the Heisenberg-Weyl universal enveloping algebra, 

$$H_\delta = A_1 b_\delta a_\delta b_\delta a_\delta (\delta a_\delta + 1) + A_2 b_\delta a_\delta^2 + A_3 b_\delta a_\delta + A_4 a_\delta \, ,$$

(4.77) 

cf. (4.74), where

$$a_\delta = \delta^{-1}(e^{\delta Q} - 1) \, , \quad b_\delta = xe^{-\delta Q} \, ,$$

where $[a_\delta, b_\delta] = 1$ — this is the explicit $\delta$-realization of the quantum canonical commutation relations (4.19), (4.29).

Replacing in (4.77), $a_\delta, b_\delta \to a, b$ with $a = \partial, b = x$ we arrive at a linear differential operator isospectral to (4.77) and thus to (4.70),

$$H = \delta A_1 x^2 Q^3 + [(\delta A_1 + A_2) + A_1 x] x Q^2 + [A_4 + (A_1 + A_3) x] Q \, .$$

(4.78) 

This operator has infinitely-many polynomial eigenfunctions: they are simply related to the Hahn polynomials,

$$\tilde{h}_k^{(\alpha,\beta)}(x,N) = \sum_{i=0}^k \gamma_i x^i \, ,$$

(4.79) 

and, in particular,

$$\tilde{h}_N^{(\alpha,\beta)}(x,N) = \gamma_N x^N \, ,$$

(4.80) 

cf. (4.75). Explicitly, their eigenvalues are given by (4.71). Note that since the operator (4.78) is of the third order there is no continuous measure with respect of which the polynomials (4.79) are orthogonal.

Taking $q$—realization of the Heisenberg algebra, see (4.22) and (4.33), in the Fock space representation,

$$\hat{a} = \frac{ba + 1}{ba + 1}_q a \, ,$$

$$\hat{b} = b \cdot \frac{ba + 1}{\{ba + 1\}_q} = \frac{ba}{\{ba\}_q} b \, ,$$

where $[\hat{a}, \hat{b}] = 1$, and replacing in (4.77), $(a_\delta, b_\delta) \to (\hat{a}, \hat{b})$ and then by $(\partial_q, x_q)$ we get the remarkable, non-local, differential-difference operator

$$H_q = A_1 x \partial x Q (\delta Q + 1) + A_2 x \partial Q + A_3 x \partial + A_4 \partial Q \, ,$$

(4.81) 

which is (polynomially)-isospectral to both (4.77) and (4.78) for any $q$. The operator (4.81) has infinitely-many polynomial eigenfunctions and they are closely related to the Hahn polynomials,

$$h_k^{(\alpha,\beta;q)}(x,N) = \sum_{i=0}^k \gamma_i \frac{i!}{\{i\}_q} x^i \, ,$$

(4.82)
and
\[ h_N^{(\alpha,\beta;q)}(x, N) = \gamma_N \frac{N!}{\{N\}_q} x^N, \]  
(4.83)
(c.f. (4.79), (4.80)), where the corresponding eigenvalues are given by (4.71).

It seems the presence of the eigenfunction of the type (4.83) indicates that there is no continuous measure with respect of which the polynomials (4.82) are orthogonal. It also hints that the operator (4.81) is not self-adjoint.

It is evident that the differential operator (4.78) continues to have a finite-dimensional invariant subspace in polynomials even though the replacement \((\partial, x) \rightarrow (\partial_q, x)\) is made, which not a canonical transformation. Thus, the underlying Heisenberg algebra is replaced by \(q\)-Heisenberg algebra (4.3). In this case we get a non-local, finite-difference operator
\[ H = \delta A_1 x^2 \partial_q^3 + [(\delta A_1 + A_2) + A_1 x] x \partial_q^2 + [A_4 + (A_1 + A_3) x] \partial_q, \]  
(4.84)
(c.f. (4.81)). Certainly, this operator is not isospectral to (4.81), but still has infinitely-many polynomial eigenfunctions. Its eigenvalues are
\[ \tilde{\lambda}_k = A_1 \{k\}_q (\{k - 1\}_q + 1) + A_3 \{k\}_q, \quad k = 0, 1, 2, \ldots . \]  
(4.85)

The operator (4.84) does not support a special property of the eigenstate at \(k = N\) (see (4.80), (4.83)). This can be easily arranged modifying the coefficients \(A'\)s:
\[ A_1 = -1, \quad A_2 = \{N-2\}_q - \beta, \quad A_3 = -\alpha - \beta - 1, \quad A_4 = (\beta +1)(N-1), \]  
(4.86)
(c.f. (4.73)). The operator (4.84) with coefficients (4.86) still possesses infinitely-many polynomial eigenfunctions and
\[ \tilde{h}_N^{(\alpha,\beta;q)}(x, N) = \gamma_N x^N, \]  
(4.87)
(c.f. (4.80), (4.83)). The eigenvalues are continued to be given by the formula (4.85). So, by changing the \(A'\)s-coefficients we make an isospectral deformation (4.84), which now support an exceptional nature of the \(N\)th eigenstate.

Among the three-point equations (4.67) there also exist quasi-exactly-solvable equations possessing a finite number of polynomial eigenfunctions [63]. Corresponding operators in the r.h.s. of these equations are classified via the cubic polynomial element of the universal enveloping \(\mathfrak{sl}(2, \mathbb{R})\)-algebra taken in the representation (4.31)-(4.32), which is the explicit \(\delta\)-realization of the quantum canonical commutation relations (4.19), (4.29).

\[ \tilde{T} = A_+ (J_n^+ + \delta J_n^0 J_n^0) + A_1 J_n^0 J_n^0 (1 + \delta J_n^-) + A_2 J_n^0 J_n^- + A_3 J_n^0 + A_4 J_n^-; \]  
(4.88)
(cf. (4.74)), where the $A$’s are free parameters and $n$ takes integer value. This is nothing but the quasi-exactly-solvable generalization of the Hahn operator.

In terms of the Heisenberg algebra generators

$$\tilde{T} = A_+ (b_\delta^2 a_\delta - nb_\delta + \delta b_\delta a_\delta b_\delta a_\delta - n\delta b_\delta a_\delta + \frac{n^2}{4}\delta) +$$

$$A_1 b_\delta a_\delta b_\delta a_\delta (\delta a_\delta + 1) + A_2 b_\delta a_\delta^2 + A_3 b_\delta a_\delta + A_4 a_\delta , \quad (4.89)$$

where $[a_\delta, b_\delta] = 1$. Replacing in (4.89), $a_\delta, b_\delta \rightarrow a, b$ with $a = \partial, b = x$ we arrive at a linear differential operator, which is polynomially-isospectral to (4.88) or to (4.89),

$$\tilde{T}(x, \partial_x) = [A_4 (x + \delta A_+ x^2)\partial - \delta A_+ nx]. \quad (4.90)$$

It has $(n + 1)$ polynomial eigenfunctions in a form of polynomial of the degree $n$,

$$\phi_k^{(n)} = \sum_{i=0}^{n} a_{k,i} x^i, \quad k = 0, 1, \ldots n .$$

Replacing in (4.89), $(a_\delta, b_\delta) \rightarrow (a_q, b_q)$ with $a_q = D_q, b = x_q$ we arrive at a linear differential-difference operator, which is polynomially-isospectral to (4.88), (4.89) or to (4.90). It has $(n + 1)$ polynomial eigenfunctions in the form of polynomial of the degree $n$. 

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We have presented 12 families of quasi-exactly-solvable (QES) one-dimensional Schrödinger equations. Each family depends on a number of free parameters. Usually, QES problem represents a type of anharmonic oscillator with a specific anharmonicity: it is a perturbation of one of well-known exactly-solvable problems (the Harmonic oscillator, the Coulomb potential, the Morse oscillator, the Pöschl-Teller potential). The most prominent QES example is the sextic anharmonic oscillator. An outstanding feature of each QES family is that if one parameter takes an integer value, a finite number of eigenstates (eigenfunctions and eigenvalues) can be found algebraically, by linear algebra means. It implies that the corresponding QES Schrödinger operator has a finite-dimensional invariant subspace.

In general, any one-dimensional QES Hamiltonian can be transformed to a Heun operator,
\[ h_e(x) = P_3(x)\partial_x^2 + P_2(x)\partial_x + P_1(x) , \]
where \( P_{3,2,1}(x) \) are polynomials of degrees 3, 2, 1, respectively. This operator can always be rewritten in terms of the generators of the algebra \( sl_2 \) in a representation marked by spin \( \nu \) and realized by first order differential operators in one variable,
\[
J^+_\nu = x^2\partial_x - 2\nu x , \\
J^0_\nu = x\partial_x - \nu , \\
J^-_\nu = \partial_x .
\] (C.1)

It reveals a surprising connection between one-dimensional quantum dynamics and the \( sl_2 \)-algebra quantum top in a constant magnetic field \([72]\) - their Hamiltonian is
\[
h_e = g_{+0} \{ J^+_\nu, J^0_\nu \} + g_{+-} \{ J^+_\nu, J^-_\nu \} + g_{00} J^0_\nu J^0_\nu + g_{0-} \{ J^0_\nu, J^-_\nu \} \\
+ b_+ J^+_\nu + b_0 J^0_\nu + b_- J^-_\nu ,
\] (C.2)
with a constraint
\[
\{ J^+_\nu, J^-_\nu \} - 2J^0_\nu J^0_\nu + 2\nu (\nu + 1) = 0 ,
\]
where \( \{ , \} \) denotes the anti-commutator; \( g_{i,j} \) is the tensor of inertia and \( \vec{b} = (b_+, b_0, b_-) \) is a magnetic field. If the spin \( \nu \) is (half)-integer, \( \nu = \frac{n}{2}, n = 0, 1, \ldots \), the irreducible finite-dimensional representation occurs. In this case the generators \( J^{\pm,0}_\nu \) have a common invariant subspace \( \mathcal{P}_n \) (in polynomials),
which is a representation space of the finite-dimensional representation. Correspondingly, the operator \((C.2)\) has the space \(\mathcal{P}_n\) as a finite-dimensional invariant subspace. Certainly, the number of those algebraic eigenfunctions (those which can be found by linear algebra means) is nothing but the dimension of the irreducible finite-dimensional representation of the algebra \(sl_2\).

If \(g_{+0} \neq 0\) the quantum top corresponds to the \(A_1, BC_1\)-Calogero-Moser-Sutherland (Cases XI, XII in Chapter 1) and \(BC_1\)-Inozemtsev models. If \(g_{+0} = 0\) the quantum top corresponds to all other QES models (Cases I-X). If in addition to it the parameter \(b_+ = 0\), the quantum top corresponds to exactly-solvable problems (the Harmonic oscillator, the Coulomb potential, the Morse oscillator, the Pöschl-Teller potential).

The \(sl_2\)-algebra can be realized by finite-difference operators by replacing in \((C.1)\) the derivative by the Norlund (Jackson) derivative and the position operator by the canonical conjugate. In this case the \(sl_2\)-algebra quantum top in a constant magnetic field corresponds to a discrete system on uniform (exponential) lattice. If the spin \(\nu\) is (half)-integer, the discrete system becomes quasi-exactly-solvable: it has a finite-dimensional invariant subspace in polynomials. The QES discrete operator is polynomially-isospectral to the QES Schrödinger operator: the spectra of polynomial eigenfunctions coincides, they do not depend on spacing.

A natural extension of the idea of quasi-exact-solvability to multi-dimensional Schrödinger equations leads to the question: on what finite-dimensional space of multi-variate inhomogeneous polynomials can the differential operators act? Certainly, a natural candidate might be the space of finite-dimensional representation of a Lie algebra of differential operators of the first order. One of such algebras is the \(gl(d + 1)\)-algebra acting on functions in \(\mathbb{R}^d\).

\[
\begin{align*}
J_i^- &= \frac{\partial}{\partial \tau_i}, \quad i = 1, 2 \ldots d, \\
J_{ij}^0 &= \tau_i \frac{\partial}{\partial \tau_j}, \quad i, j = 1, 2 \ldots d, \\
J^0 &= \sum_{i=1}^{d} \tau_i \frac{\partial}{\partial \tau_i} - n, \\
J_i^+ &= \tau_i J^0 = \tau_i \left( \sum_{j=1}^{d} \tau_j \frac{\partial}{\partial \tau_j} - n \right), \quad i = 1, 2 \ldots d, \tag{C.3}
\end{align*}
\]

where \(n\) is an arbitrary number. The total number of generators is \((d + 1)^2\). If \(n\) takes the integer values, \(n = 0, 1, 2 \ldots\), the finite-dimensional irreps occur

\[
\mathcal{P}_n^{(d)} = \langle \tau_1^{p_1} \tau_2^{p_2} \ldots \tau_d^{p_d} | 0 \leq \sum p_i \leq n \rangle.
\]
It is a common invariant subspace for (C.3). It is shown that any quantum $A_d, BC_d, D_d$ rational, trigonometric, elliptic Calogero-Moser-Sutherland Hamiltonian has the finite-dimensional invariant subspace $\mathcal{P}^{(d)}_n$, see [57], [11], [86], [87], [24], [65]. For the rational and trigonometric cases the invariant subspace exists with any integer $n = 0, 1, \ldots$. All of these Hamiltonians are equivalent to $gl(d+1)$ quantum top (or, saying differently, $sl_{d+1}$ quantum top in a constant magnetic field). The quantum $G_2$ (rational, trigonometric, elliptic) Calogero-Moser-Sutherland Hamiltonian has the finite-dimensional invariant subspace

$$\mathcal{P}_{n}^{(2)} = \langle \tau_1^{p_1} \tau_2^{p_2} | 0 \leq p_1 + 2p_2 \leq n \rangle,$$

see [56, 91], [8], [65]. For the rational and trigonometric cases there exist infinitely-many finite-dimensional invariant subspaces, marked by integer $n = 0, 1, \ldots$. They form the infinite flag. The hidden algebra is not $gl(3)$ anymore: it is infinite dimensional, 10-generated algebra of differential operators in two variables with generalized Gauss decomposition property (for discussion see the reviews [86, 87]). For any quantum $F_4, E_{6,7,8}$ rational, trigonometric Calogero-Moser-Sutherland Hamiltonian there exist infinitely-many finite-dimensional invariant subspaces in polynomials of a special form [8], this is summarized in [9]. Analysis of the respectful hidden algebras is still incomplete. Quantum $F_4, E_{6,7,8}$ elliptic Calogero-Moser-Sutherland models were never studied.

It is known that the quantum $A_d, BC_d, D_d$ rational and trigonometric Calogero-Moser-Sutherland models admit the super-symmetric extension [22, 58], see for discussion [11] and references therein. All of these models are equivalent to the supersymmetric $gl(d + 1, d)$-superalgebra quantum top in a constant magnetic field [11].

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21For the $A_d$ elliptic Calogero-Moser-Sutherland Hamiltonian, this is demonstrated for $d = 2$ [65] and is conjectured for $d > 2$. 
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Historical reminiscence

There were three people who contributed significantly to establishing the subject. For many years I worked on different aspects of perturbation theory in quantum mechanics. One of essential elements of the approach I used was a construction of the potential by an arbitrary chosen, square-integrable function. For many years it was a puzzle how to choose several square-integrable functions which lead to sensible eigenfunctions in the same potential. When, finally, I succeeded to find non-trivial examples, A.B. (‘Sasha’) Zamolodchikov, who was the first to whom I told about my findings, immediately conjectured that the corresponding Schrödinger operators can be related to the algebra $sl_2$. From the very beginning I.M. Gel’fand expressed interest in the finding and postulated that a relation to the Lamé operator should exist – a study of this relation shed the light on the general construction. Later V.I. Arnold asked me to write for him a one page description of quasi-exact-solvability – after about two years of working I was able to do so, clarifying the general nature of the subject. I would like to express my deep gratitude to them.

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The story of this review paper comes back to 1990 when R.C. Slansky (Los Alamos) as the editor of Physics Reports has invited me to write a review for Physics Reports. I conditionally accepted saying that the subject is underdeveloped and it may take several years to write it up. Several times he reminded me on my promise. As a result it took 25 years. I thank Dick for the proposal and feel sorry that he had no chance to see the review.

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References

[1] R. Askey. Continuous Hahn polynomials. *Journal of Physics A*, 18:L1017–L1019, 1985.

[2] N. M. Atakishiev and S. K. Suslov. Hahn and Meixner polynomials of an imaginary argument and some their applications. *Journal of Physics A*, 18:1583–1596, 1985.

[3] H. Bateman and A. Erdélyi. *Higher transcendental functions, vol. 1, 2, 3.* McGraw-Hill Book Company, Inc., New York, Toronto, London, 1956. 'The Bateman Project'.

[4] C. M. Bender and S. Boettcher. Quasi-exactly solvable quartic potential. *J. Phys.*, A31:L273–L277, 1998.

[5] C. M. Bender and A. V. Turbiner. Analytic continuation of Eigenvalue problems. *Phys. Lett.*, A173:442–447, 1993. Preprint WU-HEP-92-13.

[6] C. M. Bender and T. T. Wu. Anharmonic Oscillator. *Phys.Rev.*, 184:1231–1260, 1969.

[7] C. M. Bender and T. T. Wu. Anharmonic Oscillator.II. *Phys.Rev. D*, 7:1620–1636, 1973.

[8] K. G. Boreskov, A. V. Turbiner, and J. C. Lopez Vieyra. Solvability of the Hamiltonians related to exceptional root spaces: rational case. *Comm. Math. Phys.*, 260:17–44, 2005.

[9] K. G. Boreskov, A. V. Turbiner, J. C. Lopez Vieyra, and M. A. Garcia Garcia. Sutherland-type trigonometric models, trigonometric invariants and multivariable polynomials. III. *E*8 case. *Intern. Journ. Mod. Phys. A*, 26:1399–1437, 2011.

[10] Y. Brihaye and B. Hartmann. Multiple algebraisations on elliptic Calogero-Sutherland model. *Journ.Math.Phys.*, 44:1576–1583, 2003.

[11] L. Brink, A. Turbiner, and N. Wyllard. Hidden algebras of the (super) Calogero and Sutherland models. *J. Math. Phys.*, 39:1285–1315, 1998.

[12] C. Chryssomalakos and A. V. Turbiner. Canonical commutation relation preserving maps. *J. Phys. A*, 34:10475–10483, 2001.

[13] A. Dalgarno and J. T. Lewis. The exact calculation of long-range forces between atoms by perturbation theory. *Proc. Royal Soc.*, A233:70, 1955.

[14] P. Djakov and B. Mityagin. Asymptotics of instability zones of the Hill operator with a two term potential. *Journ. of Funct. Anal.*, 242:157–194, 2007.

[15] B. A. Dubrovin and S. P. Novikov. Periodic and conditionally periodic analogs of the many-soliton solutions of the Korteweg-de Vries equation. *Zh.Eksp.Teor.Fiz.*, 67:2131–2144, 1974. *Sov.Physics–JETP* 40 (1974) 1058-1063 (English Translation).
[16] G. V. Dunne and M. A. Shifman. Duality and self-duality (energy reflection symmetry) of quasi-exactly solvable periodic potentials. *Annals Phys.*, 299:143–173, 2002.

[17] G. V. Dunne and M. Ünsal. Periodic and conditionally periodic analogs of the many-soliton solutions of the Korteweg-de Vries equation. *Phys. Rev. D*, 89:105009, 2014.

[18] A. Eremenko and A. Gabrielov. Analytic continuation of eigenvalues of a quartic oscillator. *Comm. Math. Phys.*, 287:431–457, 2009.

[19] A. Eremenko, A. Gabrielov, and B. Shapiro. Zeros of eigenfunctions of some anharmonic oscillators. *Ann. Inst. Fourier, Grenoble*, 58:603–624, 2008.

[20] H. Exton. *q-Hypergeometrical functions and applications*. Horwood Publishers, 1983. Chichester.

[21] N. Fleury and A. V. Turbiner. Polynomial relations in the Heisenberg algebra. *Journ.Math.Phys.*, 35:6144–6149, 1994.

[22] D. Z. Freedman and P. F. Mende. An exactly solvable $N$ particle system in supersymmetric quantum mechanics. *Nucl. Phys.*, B344:317–343, 1990.

[23] G. Gasper and M. Rahman. *Basic Hypergeometric Series*. Cambridge University Press, 1990. Cambridge.

[24] D. Gomez-Ullate, A. Gonzalez-Lopez, and M. A. Rodriguez. Exact solutions of a new elliptic Calogero-Sutherland model. *Phys. Lett. B*, 511:112–118, 2001.

[25] A. González-López, N. Kamran, and P. J. Olver. Normalizability of one-dimensional quasi-exactly-solvable Schroedinger operators. *Comm. Math. Phys.*, 153:117–143, 1993.

[26] A. Z. Gorski and J. Szmigielski. On pairs of difference operators satisfying $[d, x] = id$. *J. Math. Phys.*, 39:545–568, 1998.

[27] A. Z. Gorski and J. Szmigielski. Representations of the Heisenberg algebra by difference operators. *Acta Phys. Polon. B*, 31:789–799, 2000.

[28] M. B. Halpern and E. Kiritsis. General Virasoro construction on affine $g$. *Modern Phys.Lett.*, A4:1373–1380, 1989.

[29] M. B. Halpern, E. Kiritsis, N.A. Obers, and K. Clubok. Irrational conformal field theory. *Phys. Repts.*, 265:1–138, 1996.

[30] I. W. Herbst and B. Simon. Some remarkable examples in eigenvalue perturbation theory. *Phys. Lett.*, B78:304–306, 1978.

[31] C. Hunter and B. Guerrieri. The eigenvalues of Mathieu’s equation and their branch points. *Stud. Appl. Math.*, 64:113–141, 1981.

[32] E.L. Ince. *Ordinary differential equations*. Dover, 1956.

[33] V. I Inozemtsev. Lax representation with spectral parameter on a torus for integrable particle system. *Lett. Math. Phys.*, 17:11–17, 1989.
[34] R. Koekoek and R. F. Swarttouw. The Askey-scheme of hypergeometric orthogonal polynomials and its $q-$analogue. *Delft Univ. of Technology, Delft*, Report 94-05, 1994.

[35] E. N. Korol. Quantization of spherical pit $\frac{A}{r} - \frac{B}{r^3}$. *Ukrainian Journal of Physics*, 18:1885–1888, 1973.

[36] H. L. Krall. Certain differential equations for Chebyshev polynomials. *Duke Math. J.*, 4:705–718, 1938.

[37] L. D. Landau and E. M. Lifshitz. *Quantum Mechanics*. Pergamon Press (Oxford - New York - Toronto - Sydney - Paris - Frankfurt), 1977.

[38] S. Lang. *Algebra*. Addison–Wesley Publishing Company, Reading, Massachusets, 1965. Addison–Wesley Series in Mathematics.

[39] P.G.L. Leach. An exactly soluble Schrödinger equation with a bistable potential. *Journ. Math. Phys.*, 25:2974–2978, 1984.

[40] L. L. Littlejohn. Orthogonal polynomial solutions to ordinary and partial differential equations. In M. Alfaro et al., editor, *Orthogonal Polynomials and their Applications*, pages 98–124, Segovia, Spain, 1986, 1988. Proceedings of an International Symposium on Orthogonal Polynomials and their Applications, Springer-Verlag. Lecture Notes in Mathematics No.1329.

[41] P.-F. Loos and P. M. W. Gill. Two electrons on a hypersphere: A quasi-exactly solvable model. *Phys. Rev. Letters*, 103:123008, 2009.

[42] W. Magnus and S. Winkler. *Hill’s Equation*. Interscience Publishers: John Wiley & Sons, New York-London-Sydney, 1966.

[43] R. S. Maier. On reducing the Heun equation to the hypergeometric equation. *J. Differential Equations*, 213:171–203, 2005.

[44] A.P. Mishina and I.V. Proskuryakov. *Advanced Algebra*. Nauka, Moscow, 1962. in Russian.

[45] A. Yu. Morozov, A. M. Perelomov, A. A. Rosly, M. A. Shifman, and A. V. Turbiner. Quasi-exactly-solvable problems: One-dimensional analogue of rational conformal field theories. *Int.Journ.Mod. Phys. A*, 5:803–843, 1990.

[46] G. M. Murphy. *Ordinary differential equations and their solutions*. van Nostrand, New York, 1960.

[47] A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov. *Classical orthogonal polynomials of a discrete variable*. Springer-Verlag, 1991.

[48] Manakov S. V. Pitaevskii L. P. Zakharov V. E. Novikov, S. P. *Theory of Solitons: The Inverse Scattering Method*. Consultants Bureau, New York, 1984. Nauka, Moscow, 1984 (in Russian).

[49] O. Ogievetsky and A. V. Turbiner. $\mathfrak{sl}(2, \mathbb{R})_q$ and quasi-exactly-solvable problems. *Preprint CERN-TH: 6212/91*, 1991.
[50] M. A. Olshanetsky and A. M. Perelomov. Quantum integrable systems related to Lie algebras. *Phys. Repts.*, 94:313–393, 1983.

[51] A. M. Perelomov. *Generalized Coherent States and Their Applications*. Springer-Verlag, 1986. 320 pages.

[52] M. Razavy. An exactly soluble Schrödinger equation with a bistable potential. *Amer. J. Phys.*, 48:285–288, 1980.

[53] M. Razavy. A potential model for torsional vibrations of molecules. *Phys. Lett. A*, 82:7–9, 1981.

[54] S. M. Roman and G. C. Rota. The umbral calculus. *Advances in Mathematics*, 27:95–188, 1978.

[55] A. Ronveaux. *Heun Differential Equations*. Oxford University Press, Oxford, 1995.

[56] M. Rosenbaum, A.V. Turbiner, and A. Capella. Solvability of $G(2)$ integrable system. *Intern. Journ. Mod. Phys.*, A13:3885–3904, 1998.

[57] W. Rühl and A. V. Turbiner. Exact-solvability of the Calogero and Sutherland models. *Mod.Phys.Lett. A*, 10:2213–2222, 1995.

[58] B. S. Shastry and B. Sutherland. Superlax pairs and infinite symmetries in the $1/r^2$ system. *Phys. Rev. Lett.*, 70:4029–4033, 1993.

[59] M. A. Shifman. Quasi-exactly-solvable spectral problems and conformal field theory. *Lie Algebras, Cohomologies and New Findings in Quantum Mechanics*, Vol. 160:237–262, 1994. *Contemporary Mathematics*, AMS, N. Kamran and P. Olver (eds.), funct-an/9301001.

[60] M. A. Shifman and A. V. Turbiner. Quantal problems with partial algebraization of the spectrum. *Comm. Math. Phys.*, 126:347–365, 1989.

[61] M. A. Shifman and A. V. Turbiner. Energy reflection symmetry of Lie-algebraic problems: where the quasiclassical and weak coupling expansions meet. *Phys. Rev. A*, 59:1791–1799, 1999. hep-th/9806006.

[62] V. Singh, A. Rampal, S.N. Biswas, and K. Datta. A class of exact solutions for doubly anharmonic oscillators. *Lett. Math. Phys.*, 4:131–134, 1980.

[63] Yu. F. Smirnov and A. V. Turbiner. Lie algebraic discretization of differential equations. *Mod. Phys. Lett.*, A10:1795–1802, 1995.

[64] Yu. F. Smirnov and A. V. Turbiner. Hidden $sl(2)$ algebra of finite difference equations. *Proceedings of IV Wigner Symposium, World Scientific, N.M. Atakishiyev, T.H. Seligman and K.B. Wolf (Eds.)*, pages 435–440, 1996.

[65] V. V. Sokolov and A. V. Turbiner. Quasi-exact-solvability of the $A_2/G_2$ Elliptic model: algebraic forms, $sl(3)/g^{(2)}$ hidden algebra, polynomial eigenfunctions. *Journal of Physics A*, 48:155201, 2015. Corrigendum, *ibid*, 359501, ArXiv:1409.7439.
[66] K. Takemura. Quasi-exact solvability of Inozemtsev models. *J. Phys. A*, 35:8867–8881, 2002.

[67] E.C. Titchmarsh. *Eigenfunction expansions associated with second-order differential equations*. Clarendon Press, Oxford, 1962.

[68] A. A. Tseytlin. Conformal sigma models corresponding to gauged Wess-Zumino-Novikov-Witten theories. *Nucl. Phys. B*, 411:509–558, 1994.

[69] A. A. Tseytlin. On a ‘Universal’ class of WZW type conformal models. *Nucl. Phys.*, B418:173–194, 1994.

[70] A. V. Turbiner. The problem of spectra in quantum mechanics and the non-linearization procedure. *Soviet Phys. - Usp. Fiz. Nauk.*, 144:35–78, 1984.

[71] A. V. Turbiner. Quantum Mechanics: Problems Intermediate between Exactly-Solvable and Non-Solvable. *Zh. Eksp. Teor. Fiz.*, 94:33–44, 1988. *Sov. Phys.–JETP* **67** (1988) 230-236 (English Translation).

[72] A. V. Turbiner. Quasi-exactly-solvable problems and $\mathfrak{sl}(2,\mathbb{R})$ algebra. *Comm. Math. Phys.*, 118:467–474, 1988. (Preprint ITEP-197 (1987)).

[73] A. V. Turbiner. Spectral Riemannian surfaces of the Sturm-Liouville operators and Quasi-exactly-solvable problems. *Funk. Analysis i ego Prilozhenia*, 22:92–94, 1988. *Soviet Math. – Functional Analysis and its Applications* **22**, (1988) 163-166 (English Translation).

[74] A. V. Turbiner. Lame equation, $\mathfrak{sl}_2$ and isospectral deformation. *Journ. Phys. A*, 22:L1–L3, 1989.

[75] A. V. Turbiner. A new phenomenon of nonanalyticity and spontaneous supersymmetry 'breaking'. *Phys. Lett.*, B276:95–102, 1992. B291, 519 (corrigendum).

[76] A. V. Turbiner. On polynomial solutions of differential equations. *Journ. Math. Phys.*, 33:3989–3994, 1992.

[77] A. V. Turbiner. Hidden algebra of Calogero model. *Phys. Lett. B*, 320:281–286, 1994.

[78] A. V. Turbiner. Lie algebras and linear operators with invariant subspace. *Lie Algebras, Cohomologies and New Findings in Quantum Mechanics*, Vol. 160:263–310, 1994. *Contemporary Mathematics*, AMS, N. Kamran and P. Olver (eds.), funct-an/9301001.

[79] A. V. Turbiner. Two electrons in an external oscillator potential: the hidden algebraic structure. *Phys. Rev. A*, 50:5335–5337, 1994.

[80] A. V. Turbiner. Two electrons in an external oscillator potential: the hidden algebraic structure. *Phys. Rev. A*, 50:5335–5337, 1994.

[81] A. V. Turbiner. Invariant identities in the Heisenberg algebra. *Soviet Math. – Functional Anal. and its Appl.*, 29:291–294, 1995.
[82] A. V. Turbiner. Lie algebras in Fock space. Vol. 164 (Complex Analysis and Related Topics):265–284, 1999.

[83] A. V. Turbiner. Different faces of harmonic oscillator. In D. Levi and O. Ragnisco, editors, SIDE III—Symmetries and Integrability of Difference Equations, volume 25, pages 407–414, Montreal, Canada, 2000, 2000. CRM Proceedings and Lecture Notes, CRM Press and AMS. math-ph/9905006.

[84] A. V. Turbiner. Canonical discretization. I. Discrete faces of (an)harmonic oscillator. Int.Journ.Mod.Phys., A16:1579–1605, 2001. hep-th/0004175.

[85] A. V. Turbiner. Quantum many–body problems and perturbation theory. Physics of Atomic Nuclei, 65:1135–1143, 2002. hep-th/0108160.

[86] A. V. Turbiner. From quantum $A_n$ (Calogero) to $H_4$ (rational) model. SIGMA, 7:071, 2011.

[87] A. V. Turbiner. From quantum $A_n$ (Sutherland) to $E_8$ trigonometric model: space-of-orbits view. SIGMA, 9:003, 2013.

[88] A. V. Turbiner. The $BC_1$ quantum Elliptic model: algebraic forms, hidden algebra $sl(2)$, polynomial eigenfunctions. Journal of Physics A, 48:192002, 2015. ArXiv:1408.1610.

[89] A. V. Turbiner and M. A. Escobar-Ruiz. Two charges on a plane in a magnetic field: hidden algebra, (particular) integrability, polynomial eigenfunctions. Journ. Phys. A, 46:295204, 2013.

[90] A. V. Turbiner and A. G. Ushveridze. Spectral singularities and the quasi–exactly-solvable problems. Phys. Lett. A, 126:181–183, 1987.

[91] A.V. Turbiner. Hidden algebra of three-body integrable systems. Mod. Phys. Letters, A13:1473–1483, 1998.

[92] V. V. Ulyanov and O. B. Zaslavskii. New classes of exact solutions of the Schroedinger equation and a description of spin systems by means of potential fields. Zh.Eksp.Teor.Fiz., 87:179–251, 1992.

[93] V. V. Ulyanov and O. B. Zaslavskii. New methods in the theory of quantum spin systems. Phys. Repts., 216:179–251, 1992.

[94] E. von Kamke. Differentialgleichungen (losungsmethoden und losungen), I, Gewöhnliche Differentialgleichungen. Verbesserte Auflage, Leipzig, 1959.

[95] H. Weyl. The theory of Groups and Quantum mechanics. Dover Publications, New-York, 1931.

[96] E. T. Whittaker and G. N. Watson. A Course in Modern Analysis, 4th Edition. Cambridge University Press, 1927.

[97] P. B. Wiegmann and A. V. Zabrodin. Bethe-Anzatz for the Bloch electron in magnetic field. Phys. Rev. Lett., 72:1890–1893, 1992.
[98] P. B. Wiegmann and A. V. Zabrodin. Bethe-Anzatz solution for Azbel-Hofstadter problem. *Nucl. Phys. B*, 422:495–514, 1994.

[99] C. Zachos. *Elementary paradigms of quantum algebras*, volume 134. AMS series *Contemporary Mathematics*, 1991. in *Deformation Theory and Quantum Groups with Applications to Mathematical Physics*. 
