On Beilinson’s equivalence for $p$-adic cohomology

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Abstract

In this short note, we show a $p$-adic analogue of Beilinson’s equivalence comparing two derived categories: the derived category of holonomic modules and derived category of modules whose cohomologies are holonomic.

Introduction

Let $X$ be a smooth variety over $\mathbb{C}$. In $[\text{Be}]$, Beilinson establishes an equivalence of categories

$$D^b(\text{Hol}(X)) \xrightarrow{\sim} D^b_{\text{hol}}(X).$$

The aim of this short note is to prove a $p$-adic analogue of this equivalence.

Even though the proof of the equivalence is written in a way that it can be adopted for many cohomology theories, there are two points which we need to be clarified in the arithmetic $\mathcal{D}$-module theory:

1. What is a suitable definition of “holonomic modules” in the $p$-adic context?
2. How can we construct the theory of unipotent nearby cycles?

First, we notice that in order to work with triangulated categories, since categories of complexes with Frobenius structures are not triangulated, the naive answer for the first question might be to consider overholonomic complexes (without Frobenius structure) introduced by the second author. However, we do not know if this category is closed under taking tensor product when modules do not admit Frobenius structure. Thus, the category does not seem appropriate for the equivalence because Beilinson’s original proof uses the stability under Grothendieck six operations. Moreover, the full subcategory of overholonomic modules whose objects are endowed with some Frobenius structure is not thick. To solve these problems, in this paper, we construct some kind of smallest triangulated subcategory of the category of overholonomic complexes which contains modules with Frobenius structure. Its construction allows us to come down by “devissage” to the case of modules with Frobenius structure. To answer the second question, we follow the idea of $[\text{Be}]$. We need to show a certain finiteness, for which we use Kedlaya’s semistable reduction theorem applied to overconvergent isocrystals with Frobenius structure. An answer for the first question is the contents of the first section, and the second section is devoted for the second question.

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In this paper, we fix a complete discrete valuation ring $R$. Its residue field is denoted by $k$, and assume it to be perfect. We also fix a lifting $\sigma: R \xrightarrow{\sim} R$ of the $s$-th Frobenius automorphism of $k$. We put $q := p^s$, $K := \text{Frac}(R)$. If there is no ambiguity with $K$, we sometimes omit “$/K$” in the notation of some categories.
1. Overholonomic $\mathcal{D}_{X,\mathbb{Q}}^!$-modules

1.1 Lemma. — Let $\mathcal{P}$ be a smooth formal scheme over $R$. We denote by $\text{Ovhol}(\mathcal{P})$ the subcategory of the category $\text{Mod}(\mathcal{D}_{\mathcal{P},\mathbb{Q}}^!)$ of $\mathcal{D}_{\mathcal{P},\mathbb{Q}}^!$-modules consisting of overholonomic $\mathcal{D}_{\mathcal{P},\mathbb{Q}}^!$-modules. The category $\text{Ovhol}(\mathcal{P})$ is a thick abelian subcategory of $\text{Mod}(\mathcal{D}_{\mathcal{P},\mathbb{Q}}^!)$.

Proof. To check this, we need to show that kernel and cokernel are overholonomic. Let $E \to F$ be a homomorphism of overholonomic modules. Then these are holonomic by $[\text{Ca}2, \text{4.3}]$. Thus the kernel and cokernel are holonomic by $[\text{ibid.}, \text{2.14}]$. Since the functor $\mathbb{D}$ is exact on the category of holonomic modules, we get the overholonomicity of kernel and cokernel. The thickness can be seen easily. ■

1.2. A variety (i.e. a reduced scheme of finite type over $k$) $X$ is said to be realizable if there exists a smooth proper formal scheme $\mathcal{P}$ over $R$ such that $X$ can be embedded into $\mathcal{P}$. Since the cohomology theory does not change if we take the associated reduced scheme, in the following, we assume that schemes are always reduced. For any realizable variety $X$, choose $X \hookrightarrow \mathcal{P}$ an immersion with $\mathcal{P}$ a smooth proper formal scheme over $R$. Then by $[\text{Ca}1, \text{4.16}]$, the category of overholonomic $\mathcal{D}_{\mathcal{P},\mathbb{Q}}^!$-complexes $\mathcal{E}$ which is supported on $\overline{X}$ and $\mathbb{R}\Gamma^!_{\overline{X}\setminus X}(\mathcal{E}) = 0$ does not depend on the choice of $\mathcal{P}$. This category is denoted by $\mathcal{D}_{\text{ovhol}}^b(X/K)$.

Let $X$ be a realizable variety. From $[\text{AC}]$, we define a $t$-structure on $\mathcal{D}_{\text{ovhol}}^b(X/K)$, and its heart is denoted by $\text{Ovhol}(X/K)$.

1.3 Lemma. — Let $X$ be a realizable variety. Then for any overholonomic module $\mathcal{E} \in \text{Ovhol}(X/K)$, any ascending or descending chain of overholonomic submodules of $\mathcal{E}$ is stationary.

Proof. We need to show that any overholonomic module has finitely many constituents. We prove the claim using the induction on the dimension of the support. Let $\mathcal{E} \in \text{Ovhol}(X)$. From $[\text{Ca}2, \text{3.7}]$, there exists an open dense subscheme $U$ of $X$ such that $X \setminus U$ is a divisor and $G := \mathcal{E}|_U \in \text{Isoc}^{\dagger\dagger}(U)$. By induction hypothesis, it suffices to show that $j_+(G)$ has finitely many constituents. Take an irreducible submodule $G' \subset G$ in $\text{Ovhol}(U)$. From $[\text{AC} \text{1.4.7}]$, since $G'$ is irreducible then so is $j_+(G')$. Thus by induction hypothesis, $j_+(G')$ has finitely many constituents. Thus, it suffices to show that $j_+(G/G')$ has finitely many constituents. We conclude by using the induction on the generic rank (on the analytic rigid spaces). ■

Remark. — For a smooth formal scheme $\mathcal{P}$ (which may not be proper), we may also show that any overholonomic module on $\mathcal{P}$ satisfies the ascending and descending chain conditions. The proof is similar.

1.4 Corollary. — Let $\mathcal{E} \in \text{Ovhol}(X)$, and assume that $\mathcal{E}$ can be endowed with a $s'$-th Frobenius structure for an integer $s'$ which is a multiple of $s$. Then any constituents of $\mathcal{E}$ can be endowed with a $s''$-th Frobenius structure for some $s''$ a multiple of $s'$.

Proof. The verification is similar to $[\text{CM} \text{6.0-15}]$. ■

1.5. Let $X$ be a realizable variety. Let $\text{Hol}_F(X)'$ be the subset of $\text{Ob}(\text{Ovhol}(X))$ which can be endowed with $s'$-th Frobenius structure for some integer $s'$ which is a multiple of $s$, and let $\text{Hol}_F(X)$ be the thick abelian subcategory generated by $\text{Hol}_F(X)'$ in $\text{Ovhol}(X)$. We denote by $\mathcal{D}_{\text{hol}_F}^b(X)$ the triangulated full subcategory of $\mathcal{D}_{\text{ovhol}}^b(X)$ such that the cohomologies are in $\text{Hol}_F(X)$. By Lemma 1.1 and Corollary 1.4, we have:

Corollary. — Any object of $\text{Hol}_F(X)$ can be written as extensions of modules in $\text{Hol}_F(X)'$. 

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This corollary has the following consequences:

1.6 Theorem. — Let \( f : X \to Y \) be a morphism between realizable varieties.

1. If \( f \) is proper, \( f_+ \) induces the functor \( D^b_{\text{hol},F}(X) \to D^b_{\text{hol},F}(Y) \).

2. The functor \( f^! \) induces \( D^b_{\text{hol},F}(Y) \to D^b_{\text{hol},F}(X) \).

3. The functor \( \mathbb{D} \) induces the functor \( D^b_{\text{hol},F}(X) \to D^b_{\text{hol},F}(X) \) such that \( \mathbb{D} \circ \mathbb{D} \cong \text{id} \).

4. The functor \( \tilde{\otimes} \) (cf. [AC, 1.1.6 (ii)]) induces \( D^b_{\text{hol},F}(X) \times D^b_{\text{hol},F}(X) \to D^b_{\text{hol},F}(X) \).

Moreover, these functors satisfy the properties listed in [AC, 1.3.14].

We recall that when we take an embedding \( X \hookrightarrow P \) into a proper smooth formal scheme \( P \), then \( \tilde{\otimes} \) can be written as \( (-) \otimes^\mathbb{L} \mathcal{O}_P(-)[-\dim(P)] \), where \( \otimes^\mathbb{L} \mathcal{O}_P \) is the usual weakly completed tensor product. In the following, we introduce the functor \( \otimes : D^b_{\text{hol},F}(X) \times D^b_{\text{hol},F}(X) \to D^b_{\text{hol},F}(X) \) to be \( D(\mathbb{D}(-) \otimes \mathbb{D}(-)) \) as in [AC, 1.1.6 (iii)].

Remark. — Even if we replace \( D^b_{\text{hol},F} \) by \( D^b_{\text{ovhol}} \), the theorem holds except for 4, which have been checked by the second author.

1.7. Using this category, we can state our main theorem as follows:

Theorem. — Let \( X \) be a realizable variety. Then the canonical functor

\[ D^b(\text{Hol}_F(X/K)) \to D^b_{\text{hol},F}(X/K) \]

is an equivalence of categories.

Proof. With the aid of the next section, the proof of [Be1] can be adapted without any difficulties, so we only sketch the outline. We put \( M(X) := \text{Hol}_F(X/K) \) and \( D(X) := D^b_{\text{hol},F}(X/K) \). For a generic point \( \eta \in X \), we put \( D(\eta) := \lim\limits_{\eta \in U} D(U) \), and \( M(\eta) := \lim\limits_{\eta \in U} M(U) \). First we need to prove that the canonical functor \( D^b(M(\eta)) \to D(\eta) \) is an equivalence. For the proof, we need six functors formalism as we constructed in Theorem 1.6, and we can copy [Be1, 2.1]. We refer to [AC, A.5] for the relation between \( \text{Hom} \) and \( \text{Hom}_{D(X)} \).

Now let \( f : X \to \mathbb{A}^1 \) be a morphism, and put \( Y := f^{-1}(0) \) be a closed subscheme. The second thing we need to show is that for any integer \( i \) the canonical functor

\[ \text{Ext}^i_{M(Y)}(\mathcal{E}, \mathcal{F}) \to \text{Ext}^i_{M(X)}(\mathcal{E}, \mathcal{F}), \]

where \( \text{Ext}^i \) denotes the Yoneda’s Ext functor, is a bijection. For this we can copy [Be1, 2.2.1] except for the existence of the functors \( \Phi_f \) and \( \Xi_f \). These functors are defined and basic properties are shown in the next section (cf. Proposition 2.7).

Finally, we combine these two results to get the theorem. For the details, see [Be1, 2.2]. ■

Remark. — This theorem is a generalization of [AC, A.4]
2. Unipotent nearby cycle functor

2.1. Let $\Pi := \{(a, b) \in \mathbb{Z}^2; a \leq b\}$ be the partially ordered set such that $(a, b) \leq (a', b')$ if and only if $a \geq a'$, $b \geq b'$. For an abelian category $\mathfrak{A}$, we denoted by $\mathfrak{A}^\Pi$ the category of $\Pi$-shaped diagrams in $\mathfrak{A}$, i.e., the category whose objects are $\mathcal{E}^{\bullet, \bullet} = (\mathcal{E}^{a,b}, \alpha_{(a,b),(a',b')})$, where $(a, b), (a', b')$ runs through elements of $\Pi$ so that $(a', b') \leq (a, b)$, $\mathcal{E}^{a,b}$ belong to $\mathfrak{A}$, and $\alpha_{(a,b),(a',b')}: \mathcal{E}^{a',b'} \to \mathcal{E}^{a,b}$ are morphisms of $\mathfrak{A}$. We denote by $\mathfrak{A}_a^\Pi$ the full subcategory of $\mathfrak{A}^\Pi$ of objects $\mathcal{E}^{\bullet, \bullet} = (\mathcal{E}^{a,b}, \alpha_{(a,b),(a',b')})$ such that, for any $a \leq b \leq c$, the sequence $0 \to \mathcal{E}^{b,c} \to \mathcal{E}^{a,c} \to \mathcal{E}^{a,b} \to 0$ is exact. These objects are called admissible. Since this subcategory is closed under extension, this is an exact category so that the canonical functor $\mathfrak{A}_a^\Pi \to \mathfrak{A}^\Pi$ is exact.

Let $M$ be the set of order-preserving maps $\phi: \mathbb{Z} \to \mathbb{Z}$ such that $\lim_{i \to \pm \infty} \phi(i) = \pm \infty$. For any $\phi \in M$, we put $\phi^*(\mathcal{E}^{\bullet, \bullet}) := (\mathcal{E}^{\phi(a), \phi(b)}_{(a,b)}(a,b) \in \Pi)$. Let $S$ be the set of morphisms $f^{\bullet,*}: \mathcal{E}^{\bullet, \bullet} \to \mathcal{F}^{\bullet, \bullet}$ of $\mathfrak{A}^\Pi$ such that there exist $\phi \in M$ and a morphism $g_{\bullet}^{\bullet}: \phi^*(\mathcal{F}^{\bullet, \bullet}) \to \mathcal{E}^{\bullet, \bullet}$ of $\mathfrak{A}^\Pi$ so that the morphisms $f^{\bullet,*} \circ g_{\bullet}^{\bullet}$ and $g_{\bullet}^{\bullet} \circ \phi^*(f^{\bullet,*})$ of $\mathfrak{A}_{ab}^\Pi$ are the canonical morphisms. We denote by $S_a$ the elements of $S$ which are morphisms of $\mathfrak{A}_a^\Pi$ as well. Following [Be2, Appendix], we put $\lim\mathfrak{A} := \lim_{\to} \mathfrak{A}_{\Pi}$ and $\lim_{\to}^{ab} \mathfrak{A} := \lim_{\to} \mathfrak{A}_{ab}^\Pi$.

For any $\mathcal{E}^{\bullet,a}_{\bullet} \in \lim\mathfrak{A}$ and for any $\mathcal{E}^{\bullet,*}, \mathcal{F}^{\bullet,*} \in \lim_{\to}^{ab} \mathfrak{A}$ we have the equalities

$$\text{Hom}_{\lim\mathfrak{A}} (\mathcal{E}^{\bullet,a}_{\bullet}, \mathcal{F}^{\bullet,*}_{\bullet}) = \lim_{\phi \in M} \text{Hom}_{\mathfrak{A}_{\Pi}} (\phi^* \mathcal{E}^{\bullet,a}_{\bullet}, \mathcal{F}^{\bullet,*}_{\bullet}),$$

(2.1.1) $$\text{Hom}_{\lim_{\to}^{ab} \mathfrak{A}} (\mathcal{E}^{\bullet,*}, \mathcal{F}^{\bullet,*}) = \lim_{\phi \in M} \text{Hom}_{\mathfrak{A}_{\Pi}} (\phi^* \mathcal{E}^{\bullet,*}, \mathcal{F}^{\bullet,*}).$$

We get from (2.1.1) that the canonical functor $\lim\mathfrak{A} \to \lim_{\to}^{ab} \mathfrak{A}$ is fully faithful. This enables us to denote by $\lim_{\to} \mathfrak{A}_{\Pi} \to \lim\mathfrak{A}$ and $\lim_{\to} \mathfrak{A}_{ab}^\Pi \to \lim_{\to}^{ab} \mathfrak{A}$ the canonical functors.

Let $\mathcal{E} \in \mathfrak{A}$. For any $c \in \mathbb{R}$, we pose $\mathcal{E}^c = \mathcal{E}$ if $c < 0$ and $\mathcal{E}^c = \mathcal{E}$ if $c > 0$ otherwise. For any $(a, b) \in \Pi$, we set $\rho(\mathcal{E})^{a,b} := \mathcal{E}^a/\mathcal{E}^b$. We get canonically the object $\rho(\mathcal{E})^{\bullet, \bullet} \in \mathfrak{A}_{ab}^\Pi$. For simplicity, we put $\rho(\mathcal{E}) := \rho(\mathcal{E})^{\bullet, \bullet}$ and we get the fully faithful exact functor $\rho: \mathfrak{A} \to \mathfrak{A}_{ab}^\Pi$.

2.2 Lemma. — Let $N(\mathfrak{A})$ be the full subcategory of $\mathfrak{A}^\Pi$ whose objects are null in $\lim_{\to}^{ab} \mathfrak{A}$. Then, the category $N(\mathfrak{A})$ is a Serre subcategory of $\mathfrak{A}^\Pi$. Moreover, we have the equality $\mathfrak{A}^\Pi/N(\mathfrak{A}) = \lim_{\to}^{ab} \mathfrak{A}$. In particular, $\lim_{\to}^{ab} \mathfrak{A}$ is an abelian category.

Proof. This is identical to [Ca3, 1.2.4].

2.3. Let $U \to Y \to X$ be open immersions of realizable varieties. The exact functor $|_{(U,Y)}$ induces the exact functor

$$|_{(U,Y)}: \lim_{\to}^{ab} F\text{-Ovhol}(U, X/K) \to \lim_{\to}^{ab} F\text{-Ovhol}(U, Y/K).$$

Let $\mathcal{E}^{\bullet,*} \in \lim_{\to}^{ab} F\text{-Ovhol}(U, X/K)$. We remark that $\mathcal{E}^{\bullet,*} = 0$ if and only if $\mathcal{E}^{\bullet,*} \mid_{(U,Y)} = 0$. Let $\{U_i\}$ be an open covering of $U$. We notice that $\mathcal{E}^{\bullet,*} = 0$ if and only if $\mathcal{E}^{\bullet,*} \mid_{(U_i, U_i)} = 0$ for any $i$.

2.4. Set $\mathcal{O}_{\mathbb{P}^1_v} := \mathcal{O}_{\mathbb{P}^1_v}(\{0, \infty\})\mathbb{Q}$, and let $t$ be the coordinate of $\mathbb{P}^1_v$. Following Beilinson’s notation, for integers $a \leq b$, we put

$$T_{ab}^{s,l} := s^a \mathcal{O}_{\mathbb{P}^1_v}[s] \cdot t^s / s^b \mathcal{O}_{\mathbb{P}^1_v}[s] \cdot t^s.$$

Here, the $D$-module structure is defined so that for $x \in \mathcal{O}_{\mathbb{P}^1_v}$ and $l \in \mathbb{Z}$, we have

$$\partial_l (s^l x \cdot t^s) = s^l \partial_l (x) \cdot t^s + s^{l+1} x / t \cdot t^s.$$
We have an isomorphism
\[ \mathcal{I}_{G_k}^{a,b} \overset{\sim}{\rightarrow} F^*\mathcal{I}_{G_k}^{a,b} \]; \quad s^i x \cdot t^s \mapsto q^i x \otimes (s^i \cdot t^s) \]
with which \( \mathcal{I}_{G_k}^{a,b} \in F-\text{Isoc}^\dagger(G_k/K) \). We compute that the multiplication by \( s^n \) induces the isomorphism of \( F-\text{Isoc}^\dagger(G_k/K) \):

\[ \sigma^n: \mathcal{I}_{G_k}^{a,b} \overset{\sim}{\rightarrow} \mathcal{I}_{G_k}^{a+n,b+n}(-n). \]

Moreover, there is a non-degenerate pairing
\[ \mathcal{I}_{G_k}^{a,b} \otimes \mathcal{I}_{G_k}^{-a,b} \rightarrow \mathcal{O}_{G_k}(1) \]; \quad \left( x(s), g(s) \right) \mapsto \text{Res}_{s=0} f(s) \cdot g(-s). \]

We can check easily that this pairing is compatible with Frobenius structure. By using [3.12], the pairing induces an isomorphism

\[ \mathbb{D}(\mathcal{I}_{G_k}^{a,b}) \overset{\sim}{\rightarrow} \mathcal{I}_{G_k}^{-a,b}. \]

As a variant, we put \( \mathcal{I}_{G_k,\log}^{a,b} := s^a \mathcal{O}_{\hat{\mathbb{A}}_k^1}[s]t^s / s^b \mathcal{O}_{\hat{\mathbb{A}}_k^1}[s]t^s \). Then \( \mathcal{I}_{G_k,\log}^{a,b} \) is a convergent isocrystal on the formal log-scheme \( (\hat{\mathbb{A}}_k^1, \{0\}) \).

**Lemma.** — Let \( \mathcal{E} \in \text{Hol}_F(Y/K) \). The canonical morphism of \( \varinjlim \text{Hol}_F(X/K) \)
\[ \varinjlim j_!(\mathcal{E}^{\bullet\bullet}) \rightarrow \varinjlim j_+(\mathcal{E}^{\bullet\bullet}) \]
is an isomorphism.

**Proof.** We put \( d := \dim(Y) \). Using the five lemma, we may assume that \( \mathcal{E} \in F-\text{Ovhol}(Y/K) \). The proof is divided into several steps.

0) By [2.3], it is sufficient to check that the canonical homomorphism is an isomorphism over \( (Y, X) \). By abuse of notation in this proof, we still denote by \( \mathcal{E} := \mathcal{E}|(Y, X) \) and write \( j: (Y, X) \rightarrow (X, X) \) instead of \( (j, \text{id}) \).

1) We prove the lemma under the following hypotheses: "Let \( \mathcal{X} \) be a smooth formal \( V \)-scheme with local coordinates denoted by \( t_1, \ldots, t_d \) whose special fiber is \( X \). For any \( i = 1, \ldots, d \), we put \( Z_i = V(t_i) \). We suppose that there exists an open immersion \( U \hookrightarrow Y \) such that \( T := X \setminus U \) is a strict normal crossing divisor of \( X \) and an overconvergent \( F \)-isocrystal \( \mathcal{G} \) on \( (U, X)/K \) unipotent along \( T \) so that \( \mathcal{E} = i'_*(\mathcal{G}) \), where \( i': (U, X) \rightarrow (Y, X) \) is the induced morphism of couples. Fix \( 1 \leq r' \leq r \leq d \), and we denote by \( D \) the strict normal crossing divisor of \( \mathcal{X} \) whose irreducible components are \( Z_2, \ldots, Z_r \) and \( D = \emptyset \) if \( r = 1 \). We suppose that the special fiber of \( T := Z_1 \cup D \) (resp. \( Z := Z_1 \cup \cdots \cup Z_r \)) is \( T \) (resp. \( Z \))."
We proceed by induction on the number of irreducible components of $Z$. Consider the following commutative diagram of $F$-Ovhol$(X, X/K)^{II}$

\[
\begin{array}{cccc}
0 & \longrightarrow & H_{Z_1,j}(E^{**}) & \longrightarrow & j_!(E^{**}) & \longrightarrow & (\mathbb{I}^Z_1)_{(E^{**})} & \longrightarrow & H_{Z_1,j}^{II}(E^{**}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H_{Z_1,j_+}(E^{**}) & \longrightarrow & j_+(E^{**}) & \longrightarrow & (\mathbb{I}^Z_1)_{j_+}(E^{**}) & \longrightarrow & H_{Z_1,j_+}^{II}(E^{**})
\end{array}
\]

whose horizontal sequences are exact. By using the induction hypothesis and the homomorphism $\lim_{\mathbb{Z}} \mathbb{I}^Z_1 j_!(E^{**}) \rightarrow \lim_{\mathbb{Z}} \mathbb{I}^Z_1 j_+(E^{**})$ is an isomorphism. Since $\lim_{\mathbb{Z}} H_{Z_1,j_+}(E^{**}) = 0$, then it is sufficient to check that $\lim H_{Z_1,j_+}^{II}(E^{**}) = 0$, for any $i = 0, 1$ by the exactness of $\lim$.

We have a strict normal crossing divisor of $Z_1$ defined by $D_1 := \bigcup_{i=2}^{n} Z_1 \cap Z_i$. We put $U := X \setminus T$, and let $\iota : (U, X) \rightarrow (X, X)$ be the induced morphism of couples and $\iota_1 : Z_1 \rightarrow X$ be the canonical closed immersion. By hypotheses, there exists a convergent isocrystal $F$ on the log scheme $(X, T)$ so that $G \xrightarrow{\sim} (\mathbb{I})(F)$. We still denote by $f = t_1^{a_1} \cdots t_n^{a_n} \in \mathcal{O}_X$ ($a_i \in \mathbb{N}$), a lifting of $f$. We put

\[
I_{f, log}^{a,b} := (f^2)^*(I_{G, log}^{a,b}),
\]

where $f^2$ is the composition morphism of formal log-schemes $f^2 : (X, T) \rightarrow (X, T) \rightarrow (\widehat{A}_1, \{0\})$ whose last morphism is induced by $f$. We put $F^{a,b} := F \otimes_{\mathcal{O}_X} I_{f, log}^{a,b}$ which is a convergent isocrystal $F$ on the formal log scheme $(X, T)$ with nilpotent residues. We put $U_1 := Z_1 \setminus D_1$, and let $\iota_1 := (\ast, \id, \id) : (U_1, Z_1, Z_1) \rightarrow (Z_1, Z_1, Z_1)$ be the canonical morphism of frames. Let $N_{1,f,a,b}$ be the action induced by $\iota_1 \partial_1$ on $I_{f, log}^{a,b}$. We put

\[
J_{f}^{a,b} := \iota_1^+(I_{f}^{a,b}|_{(Y, X)}) = s^a \mathcal{O}_X(\mathbb{I}T)^Q[s] \cdot f^a/s^b \mathcal{O}_X(\mathbb{I}T)^Q[s] \cdot f^b.
\]

We have

\[
E^{a,b}[d] := \iota_1'(G) \otimes (I_{f}^{a,b}|_{(Y, X)}) \xrightarrow{\sim} \iota_1'(G \otimes J_{f}^{a,b}).
\]

We put $G^{a,b} := G \otimes J_{f}^{a,b}[-d] \in F$-Isoc$^{\dag}(U, X/K)$. Since $(\mathbb{I}T)(I_{f, log}^{a,b}) \xrightarrow{\sim} J_{f}^{a,b}$, we have $(\mathbb{I}T)(F^{a,b}) \xrightarrow{\sim} G^{a,b}$. By [K] 3.4.12, we get the isomorphisms

\[
H_{Z_1}^{II}(1) \circ \iota_1(G^{a,b}) \xrightarrow{\sim} i_1 \circ \alpha_{11} \circ (\mathbb{I}D_1) \circ (\ker N_{1,f,a,b}) , \quad H_{Z_1}^{I}(1) \circ \iota_1(G^{a,b}) \xrightarrow{\sim} i_1 \circ \alpha_{11} \circ (\mathbb{I}D_1) \circ (\ker N_{1,f,a,b}).
\]

Then, by functoriality, it is sufficient to check that $\lim \text{coker } N_{1,f,a,b} = 0$ and $\lim \text{ker } N_{1,f,a,b} = 0$. Since $N_{1,f,a,b} = N_{1,f} \otimes \id + \id \otimes N_{1,f,log}^{a,b}$, and since there exists an integer $n$ independent of $a, b$ such that $N_{1,f}^n = 0$, then we reduce to checking that $\lim \text{coker } N_{1,f,a,b} = 0$, which is obvious since $N_{1,f,log}^{a,b}$ is the multiplication by $s$.

2) Finally, let us reduce the lemma to 1). We proceed by induction on dim $X$. We can suppose that $j$ is dominant. Recalling that $Y$ being reduced, there exists a dominant open immersion $U \rightarrow Y$ such that $U$ is smooth and $G := \iota_1^+(E) \in F$-Isoc$^{\dag}(U, X/K)$, where $\iota : (U, X) \rightarrow (Y, X)$.

By the induction hypothesis, we can suppose that $E = \iota_1(G)$. Put $T := X \setminus U$. Then, we can suppose that $U, Y, X$ are integral and that $\iota$ is affine. By Kedlaya's semi-stable theorem [K], there exists a proper surjective generically finite and étale morphism $\alpha : \tilde{X} \rightarrow X$, such that $X$ is smooth and quasi-projective, $\tilde{X} := \alpha^{-1}(T)$ is a strict normal crossing divisor of $\tilde{X}$. We put $\alpha : (\tilde{X}, \tilde{X}) \rightarrow (X, X)$ (by abuse of notation), $\tilde{Y} := \alpha^{-1}(Y), \tilde{U} := \alpha^{-1}(U), \beta : (\tilde{Y}, \tilde{X}) \rightarrow (Y, X), \gamma : (\tilde{U}, \tilde{X}) \rightarrow (U, X), \tilde{t} : (\tilde{U}, \tilde{X}) \rightarrow (Y, \tilde{X}), \tilde{j} : (\tilde{Y}, \tilde{X}) \rightarrow (X, \tilde{X}), \tilde{G} := \gamma_1(G), \tilde{E} := \iota_1(\tilde{G})$. Since $G$ is a direct factor of $\mathcal{H}^0_\gamma(\tilde{G})$, then $E = \iota_1(G)$ is a direct factor of $\iota_1 \mathcal{H}^0_\gamma(\tilde{G}) \xrightarrow{\sim} \mathcal{H}^0_\beta \circ \iota_1(\tilde{G})$. Thus we are reduced to checking the lemma for $\mathcal{H}^0_\beta \circ \iota_1(\tilde{G})$. We have

\[
(*) \mathcal{H}^{d-b_1} \circ \iota_1(\tilde{G}) \otimes (I_{f}^{a,b}|_{(Y, X)}) \xrightarrow{\sim} \mathcal{H}^{d-b_1}(\iota_1(\tilde{G}) \otimes \beta^+(I_{f}^{a,b}|_{(Y, X)})) = \mathcal{H}^{d-b_1}(\tilde{E} \otimes I_{f}^{a,b}|_{(\tilde{Y}, \tilde{X})})
\]
where \( f = f \circ \alpha \) (the equality comes from \( \tilde{t}(G) = \tilde{E} \)) and \( \beta^+ (\mathcal{I}_f^n |_{(Y,X)}) = \mathcal{I}_f^{a,b} |_{(Y,\bar{X})} \). By applying the exact functor \( j_! \) (resp. \( j_+ \)) to the composition isomorphism of \( \tilde{t} \), we get the first isomorphisms of the following ones:

\[
\begin{align*}
& j_!(\mathcal{H} \circ \beta \circ \tilde{t}(G) \otimes (\mathcal{I}_f^{a,b} |_{(Y,X)})) \xrightarrow{\sim} j_! \circ \mathcal{H} \circ \beta \circ \tilde{t}(\tilde{E} \otimes \mathcal{I}_f^{a,b} |_{(Y,\bar{X})}) \xrightarrow{\sim} \mathcal{H} \circ \beta \circ \tilde{t}(\tilde{E} \otimes \mathcal{I}_f^{a,b} |_{(Y,\bar{X})}) \\
& j_+(\mathcal{H} \circ \beta \circ \tilde{t}(G) \otimes (\mathcal{I}_f^{a,b} |_{(Y,X)})) \xrightarrow{\sim} j_+ \circ \mathcal{H} \circ \beta \circ \tilde{t}(\tilde{E} \otimes \mathcal{I}_f^{a,b} |_{(Y,\bar{X})}) \xrightarrow{\sim} \mathcal{H} \circ \beta \circ \tilde{t}(\tilde{E} \otimes \mathcal{I}_f^{a,b} |_{(Y,\bar{X})}).
\end{align*}
\]

From 1), the canonical morphism \( \lim \tilde{j}_!(\tilde{E} \otimes \mathcal{I}_f^{a,b} |_{(Y,\bar{X})}[-d]) \to \lim \tilde{j}_+(\tilde{E} \otimes \mathcal{I}_f^{a,b} |_{(Y,\bar{X})}[-d]) \) is an isomorphism. Then so is \( \lim \mathcal{H} \circ \beta \circ \tilde{t}(\tilde{E} \otimes \mathcal{I}_f^{a,b} |_{(Y,\bar{X})}) \xrightarrow{\sim} \lim \mathcal{H} \circ \beta \circ \tilde{t}(\tilde{E} \otimes \mathcal{I}_f^{a,b} |_{(Y,\bar{X})}) \).

**2.6.** Let \( E \in \text{Hol}_F(Y/K) \). With the notation of 2.3, we put \( E_k^{a,b} := \mathcal{E}^{\max\{a,k\}, \max\{b,k\}} \) for any integer \( k \in \mathbb{Z} \). We get \( E_k^{a,b} \in \lim \mathcal{H} \text{Hol}_F(Y/K) \). Now, for \( E \in \mathcal{H} \text{Hol}_F(Y/K) \), we put

\[ \Pi^{a,b}_{i+}(E) := \lim \frac{j_!(E^{a,b})}{\lim j_!(E_k^{a,b})} \]

in \( \lim \mathcal{H} \text{Hol}_F(X/K) \). By Lemma 2.5, this is in fact in \( \mathcal{H} \text{Hol}_F(X/K) \), which yields a functor \( \Pi^{a,b}_{i+} : \mathcal{H} \text{Hol}_F(Y/K) \to \mathcal{H} \text{Hol}_F(X/K) \). The following properties can be checked easily:

1. By (2.4.2), we have \( D \circ \Pi^{a,b}_{i+} \cong (\Pi^{b,-a}_{i+} \circ D)(1) \).

2. The isomorphism \( \sigma^n \) of (2.4.1) induces an isomorphism \( \Pi^{a,b}_{i+} \cong \Pi^{a+n,b+n}_{i+}(-n) \).

We put \( \Psi_f^{(i)} := \Pi^{i,i}_{i+} \), \( \Xi_f^{(i)} := \Pi^{i+1,i+}_{i+} \), and put \( \Psi_f := \Psi_f^{(0)} \), \( \Xi_f := \Xi_f^{(0)} \). The isomorphisms

\[
\lim j_!(E^{a,b})/\lim j_!(E_k^{a,b}) \cong j_!(E(i)), \quad \lim j_+(E^{a,b})/\lim j_+(E_k^{a,b}) \cong j_+(E(i))
\]

induce exact sequences

\[
0 \to j_!(E(i)) \xrightarrow{\alpha} \Xi_f^{(i)}(E) \xrightarrow{\beta} \Psi_f^{(i)}(E) \to 0, \quad 0 \to \Psi_f^{(i+1)}(E) \xrightarrow{\beta} \Xi_f^{(i)}(E) \xrightarrow{\alpha} j_+(E(i)) \to 0.
\]

We define a functor \( \Phi_f : \mathcal{H} \text{Hol}_F(X/K) \to \mathcal{H} \text{Hol}_F(Z/K) \) as follows. Let \( E \in \mathcal{H} \text{Hol}_F(X/K) \), and put \( E_Y := j^+(E) \). Let \( \gamma_- : j_!(E_Y) \to E \) and \( \gamma_+ : E \to j_+(E_Y) \) be the adjunction homomorphisms. Consider the sequence

\[
\gamma_- j_!(E_Y) \xrightarrow{(\alpha_-, \gamma_-)} \Xi_f(E_Y) \oplus E \xrightarrow{(\alpha_+ - \gamma_+)} j_+(E_Y).
\]

The cohomology of this sequence is \( \Phi_f(E) \).

**2.7 Proposition.** — The functors \( \Pi^{a,b}_{i+} \) and \( \Phi_f \) are exact. When \( E \) is in \( \mathcal{H} \text{Hol}_F(Z/K) \), then \( E \cong \Phi_f(E) \) canonically.

**Proof.** The exactness of \( \Pi^{a,b}_{i+} \) follows by that of \( j_! \) and \( j_+ \). The exactness of \( \Phi_f \) follows since \( \alpha_- \) is injective and \( \alpha_+ \) is surjective. The last claim follows by definition.

**Remark.** — Since we do not use in the proof of the main theorem, we do not go into the details, but it is straightforward to get an analogue of [Be2, Prop 3.1], a gluing theorem of holonomic modules.
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